STRONG OPENNESS CONJECTURE AND RELATED PROBLEMS FOR PLURISUBHARMONIC FUNCTIONS

QI’AN GUAN AND XIANGYU ZHOU

Abstract. In this article, we solve the strong openness conjecture on the multiplier ideal sheaves for the plurisubharmonic functions posed by Demailly. We prove two conjectures about the growth of the volumes of the sublevel sets of plurisubharmonic functions related to the complex singularity exponents and quasi-plurisubharmonic functions related to the jumping numbers, which were posed by Demailly-Kollár and Jonsson-Mustatá respectively. We give a new proof of a lower semicontinuity conjecture posed by Demailly-Kollár without using the ACC conjecture. Other applications by combining with well-known results are also mentioned.

1. Introduction

Plurisubharmonic functions have been fundamental in several complex variables and complex geometry since they were introduced by Oka and Lelong in 1940’s. For a nice survey on the theory of psh functions, the reader is referred to [25] by Kiselman. The philosophy behind the Levi problem, $L^2$ method for solving $\bar{\partial}$ equation and vanishing theorems on multiplier ideal sheaves is that construction of a specific holomorphic function or section could be reduced to construction of a specific psh function. Singularities of the psh functions play an important role in such a construction. In the present paper, we discuss the properties related to the singularities of the psh functions.

1.1. Outline of the main results and organizations.

In this article, we establish a strong openness property for plurisubharmonic functions. We obtain estimates about the growth of the volumes of the sublevel sets of plurisubharmonic functions and quasi-plurisubharmonic functions. We also obtain a lower semicontinuity property of plurisubharmonic functions.

We establish a strong openness property on multiplier ideal sheaves for plurisubharmonic functions. We obtain estimates about the growth of the volumes of the sublevel sets of plurisubharmonic functions and quasi-plurisubharmonic functions. We also obtain a lower semicontinuity property of plurisubharmonic functions. The paper is organized as follows. In the rest of this section, we present our main theorems and their corollaries, among others, solutions of the strong openness conjecture posed by Demailly and two related conjectures posed by Demailly-Kollár and Jonsson-Mustatá. In Section 2, we recall or give some preliminary lemmas used in the proofs of the main theorems. In Section 3, we introduce three propositions

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used in the proofs of the main theorems. In Section 4, we give the proofs of the propositions stated in the last Section. In Section 5, we give the detailed proofs of the main theorems.

1.2. Strong openness conjecture.

Let $X$ be a complex manifold with dimension $n$ and $\varphi$ be a plurisubharmonic function on $X$. Following Nadel [34], one can define the multiplier ideal sheaf $\mathcal{I}(\varphi)$ to be the sheaf of germs of holomorphic functions $f$ such that $|f|^2e^{-\varphi}$ is locally integrable (see also [42], [43], [6], etc.). Let

$$\mathcal{I}_+(\varphi) := \bigcup_{\varepsilon > 0} \mathcal{I}((1 + \varepsilon)\varphi).$$

In [2], Berndtsson gave a proof of the openness conjecture of Demailly and Kollár in [9]:

**Openness conjecture:** Let $\varphi$ be a plurisubharmonic function on $X$. Assuming that $\mathcal{I}(\varphi) = \mathcal{O}_X$. Then

$$\mathcal{I}_+(\varphi) = \mathcal{I}(\varphi).$$

The dimension two case of the Openness conjecture was proved by Favre and Jonsson in [12] (see also [11]).

In the present article, we discuss more general conjecture—the strong openness conjecture about multiplier ideal sheaves for plurisubharmonic functions which was posed by Demailly in [5] and [6] (see also [8], [10], [22], [23], [29], [30], [22], [20], [11], [23], [31], [32], etc.):

**Strong openness conjecture:** Let $\varphi$ be a plurisubharmonic function on $X$. Then

$$\mathcal{I}_+(\varphi) = \mathcal{I}(\varphi).$$

For $\dim X \leq 2$, the strong openness conjecture was proved in [22] by studying the asymptotic jumping numbers for graded sequences of ideals.

It is not hard to see that the truth of the strong openness conjecture is equivalent to the following theorem:

**Theorem 1.1.** [18] Let $\varphi$ be a negative plurisubharmonic function on the unit polydisc $\Delta^n \subset \mathbb{C}^n$, suppose $F$ is a holomorphic function on $\Delta^n$, which satisfies

$$\int_{\Delta^n} |F|^2e^{-\varphi}d\lambda_n < +\infty,$$

where $d\lambda_n$ is the Lebesgue measure on $\mathbb{C}^n$. Then there exists a number $p > 1$, such that

$$\int_{\Delta^p_r} |F|^2e^{-p\varphi}d\lambda_n < +\infty,$$

where $r \in (0, 1)$.

1.3. A conjecture of Demailly and Kollár.

In [9], Demailly and Kollár posed a conjecture for the growth of the volumes of the sublevel sets of plurisubharmonic functions related to the complex singularity exponents (see also [12], [11], [22] and [23], etc.):
**Conjecture D-K:** Let $\varphi$ be a plurisubharmonic function on $\Delta^n \subset \mathbb{C}^n$, and $K$ be compact subset of $\Delta^n$. If $c_K(\varphi) < +\infty$, then

$$\frac{1}{r^{2c_K(\varphi)}} \mu(\{ \varphi < \log r \})$$

has a uniform positive lower bound independent of $r \in (0,1)$, where $c_K(\varphi) = \sup \{ c \geq 0 : \exp^{-2c\varphi} \text{ is } L^1 \text{ on a neighborhood of } K \}$, and $\mu$ is the Lebesgue volumes on $\mathbb{C}^n$.

The above conjecture is a more precise form of the openness conjecture.

By Theorem 1.1 and the fact that $c_K(\varphi) = \min_{z \in K} c_{\{z\}}(\varphi)$, i.e. there exists $z \in K$, such that $c_{\{z\}}(\varphi) = c_K(\varphi)$ (see [9]), then $e^{-2c_K(\varphi)\varphi}$ is not integrable on any neighborhood of $K$.

In the following theorem, we obtain an estimate for the sublevel sets of plurisubharmonic functions:

**Theorem 1.2.** Let $\varphi$ be a plurisubharmonic function on $\Delta^n \subset \mathbb{C}^n$. Let $F$ be a holomorphic function on $\Delta^n$. Assume that $|F|^2 e^{-\varphi}$ is not locally integrable near $o$, Then

$$\int_{\Delta^n} |\{ -R < \varphi < -R+1 \}| F|^2 e^{-\varphi} \, d\lambda_n$$

has a uniform positive lower bound independent of $R >> 0$. Especially, if $F = 1$, then

$$e^R \mu(\{ -(R+1) < \varphi < -R \})$$

has a uniform positive lower bound independent of $R >> 0$.

In particular, and replacing $2c_K(\varphi)$ by $\varphi$, and $-2c_K \log r$ by $R$, we solve the Conjecture D-K:

**Corollary 1.3.** Conjecture D-K holds

For $n \leq 2$, the above corollary was proved by Favre and Jonsson in [12] (see also [11])

1.4. A conjecture of Jonsson and Mustată.

Let $I$ be an ideal of $O_{\Delta^n,o}$, which is generated by $\{ f_j \}_{j=1, \ldots, l}$. In [23], Jonsson and Mustată posed the following conjecture about the volumes growth of the sublevel sets of quasi-plurisubharmonic functions (see also [22]):

**Conjecture J-M:** Let $\psi$ be a plurisubharmonic function on $\Delta^n \subset \mathbb{C}^n$. If $c_j^I(\psi) < +\infty$, then

$$\frac{1}{r^2} \mu(\{ c_j^I(\psi) \psi - \log |I| < \log r \})$$

has a uniform positive lower bound independent of $r \in (0,1)$, where

$$\log |I| := \log \max_{1 \leq j \leq l} |f_j|,$

$c_j^I(\psi) = \sup \{ c \geq 0 : |I|^2 e^{-2c\psi} \text{ is } L^1 \text{ on a neighborhood of } o \}$ is the jumping number in [23], and $\mu$ is the Lebesgue measure on $\mathbb{C}^n$.

For $n \leq 2$, the above conjecture was proved by Jonsson and Mustată in [22].

In the following theorem, we give an estimate for the sublevel sets of quasi-plurisubharmonic functions:
Theorem 1.4. Let $\psi$ be a plurisubharmonic function on $\Delta^n$, and $F$ be a holomorphic function on $\Delta^n$. Assume that $|F|^2e^{-\psi}$ is not locally integrable near $o$. Then

$$e^R \frac{1}{B_0} \mu(\{-R - B_0 < \psi - \log |F|^2 < -R\})$$

has a uniformly positive lower bound independent of $R >> 0$ and $B_0 \in (0, 1]$. 

In particular, we obtain:

$$e^R \mu(\{\psi - \log |F|^2 < -R\})$$

has a uniform positive lower bound independent of $R \in (0, +\infty)$.

By Theorem 1.1, it follows that $|I|^2e^{-2c_Io(\psi)}$ is not integrable on any neighborhood of $o$. Replacing $\psi$ by $2c_Io(\psi)$, and $R$ by $-2 \log r$ in equality 1.1, we solve conjecture J-M:

Corollary 1.5. Conjecture J-M holds.

1.5. A lower semicontinuity conjecture.

In [9], Demailly and Kollár conjectured that:

For every nonzero holomorphic function $f$ on $X$, there is a number $\delta = \delta(f, K, L) > 0$, such that for any holomorphic function $g$ on $X$ with

$$\sup_L |g - f| < \delta \Rightarrow c_K(\log |g|) \geq c_K(\log |f|),$$

where the compact set $K$ contained in an open subset $L$ of complex manifold $X$.

In [9], the authors proved that the above conjecture is implied by the ACC conjecture (see [37] or [26]). The ACC conjecture was proved by Hacon, McKernan and Xu in [21].

It is not hard to see that the just mentioned conjecture posed by Demailly and Kollár is equivalent to the following conjecture:

Let $\{g_m\}_{m=1,2,\ldots}$ be a sequence of holomorphic functions on $\Delta^n$, which are uniformly convergent to holomorphic function $f$ on $\Delta^n$. Assume that $|g_m|^{-2c}$ is not integrable near $o \in \Delta^n$ for any $m = 1, 2, \ldots$, where $c$ is a positive constant. Then $|f|^{-2c}$ is not integrable near $o \in \Delta^n$.

Note that $c \log |f|$ is a plurisubharmonic function, then we replace $c \log |f|$ by general plurisubharmonic functions, and obtain the following lower semicontinuous property of plurisubharmonic functions:

Proposition 1.6. Let $\{\phi_m\}_{m=1,2,\ldots}$ be a sequence of negative plurisubharmonic functions on $\Delta^n$, which is convergent to a negative Lebesgue measurable function $\phi$ on $\Delta^n$ in Lebesgue measure. Assume that $e^{-\phi_m}$ are all not integrable near $o$. Then $e^{-\phi}$ is not integrable near $o$.

Replacing $c \log |f|$ by any plurisubharmonic functions in the above conjecture, using Proposition 1.6 we obtain a generalization of the conjecture as follows, which is a new proof of the conjecture without using the ACC conjecture:

Let $\{\phi_m\}_{m=1,2,\ldots}$ be a sequence of plurisubharmonic functions on $\Delta^n$, such that $e^{\phi_m}$ are uniformly convergent to $e^\phi$ on $\Delta^n$, where $\phi$ is a plurisubharmonic function on on $\Delta^n$. Assume that $e^{-\phi_m}$ is not integrable near $o \in \Delta^n$ for any $m = 1, 2, \ldots$. Then $e^{-\phi}$ is not integrable near $o \in \Delta^n$. 


1.6. Some applications of the strong openness conjecture.

In the present subsection, combining our Theorem 1.1 (the truth of the strong openness conjecture) with some known results, we obtain some direct conclusions.

1.6.1. Singular metric with minimal singularities.

Let \( L \) be a line bundle on a smooth projective complex variety \( X \), whose Kodaira-Iitaka dimension \( \kappa(X, L) \geq 0 \) (see [27, 28, 8, 29]). Then the asymptotic multiplier ideal \( \mathcal{J}(|L|) \) can be defined as the maximal member of the family of ideals \( \{ \mathcal{J}( \frac{1}{k} \cdot |kL|) \} (k \text{ large}) \) (see Definition 1.7 in [8], see also [28]).

Demailly has shown that if \( L \) is any pseudo-effective divisor, then up to equivalence of singularities, \( \mathcal{O}_X(L) \) has a unique singular metric \( h_{\text{min}} \) with minimal singularities having non-negative curvature current (see [6], see also [28]).

Let \( \mathcal{J}(h_{\text{min}}) \) be the associated multiplier ideal sheaf of \( h_{\text{min}} \) (see [6]).

In [8, 28], the authors conjectured the following:

For big line bundle \( L \), the equality
\[
\mathcal{J}(|mL|) = \mathcal{J}(h_{\text{min}}^m)
\] (1.2)
holds for every \( m > 0 \).

Note that \( mL \) is big and \( \kappa(X, mL) \geq 0 \) for any \( m \). Let \( \hat{h}_{\text{min}}^m \) be the singular metric with minimal singularities on \( L^m \). By the uniqueness of the singular metric with minimal singularities, it follows that \( \hat{h}_{\text{min}}^m \) is a function with uniformly positive upper and lower bound on \( X \). Therefore \( \mathcal{J}(\hat{h}_{\text{min}}^m) = \mathcal{J}(h_{\text{min}}^m) \). Then it suffices to consider the case \( m = 1 \).

In [29], the author conjectured the following analogue of the above conjecture:

Let \( X \) be a smooth projective complex variety and \( L \) be a pseudo-effective \( \mathbb{R} \)-divisor on \( X \). Then
\[
\mathcal{J}(T_{\text{min}}) \subseteq \mathcal{J}_\sigma(L),
\] (1.3)
where \( T_{\text{min}} \) is a current of minimal singularities in the numerical class of \( L \), and \( \mathcal{J}_\sigma(L) \) is the diminished ideal in [29].

By the arguments after Theorem 1.2 in [29], the above conjecture can be proved by the strong openness conjecture.

In [29], it was shown that
\[
\mathcal{J}_\sigma(L) = \mathcal{J}(|L|)
\]
when \( L \) is a big line bundle (see Corollary 6.12 in [29]). Note that \( \mathcal{J}(h_{\text{min}}) \) in [8] is just \( \mathcal{J}(T_{\text{min}}) \) in [29]. Using inequality 1.3, one has
\[
\mathcal{J}(h_{\text{min}}) \subseteq \mathcal{J}(|L|).
\]

In [8], it was shown that
\[
\mathcal{J}(|L|) \subseteq \mathcal{J}(h_{\text{min}}).
\]

Then one has equality 1.2 holds.
1.6.2. Kawamata-Viehweg-Nadel type vanishing theorem.

Let \((L, \varphi)\) be a pseudo-effective line bundle on a compact Kähler manifold \(X\) of dimension \(n\), and \(nd(L, \varphi)\) be the numerical dimension of \((L, \varphi)\) defined in \([4]\).

In \([4]\), Cao obtained a Kawamata-Viehweg-Nadel type vanishing theorem for \(I_+(\varphi)\) on any compact Kähler manifold:

\[
H^p(X, K_X \otimes L \otimes I_+(\varphi)) = 0
\]

holds for any \(p \geq n - nd(L, \varphi) + 1\).

In \([4]\), Cao asked whether the Kawamata-Viehweg-Nadel type vanishing theorem holds for \(I(\varphi)\) on any compact Kähler manifold, i.e. does

\[
H^p(X, K_X \otimes L \otimes I(\varphi)) = 0
\]

hold for any \(p \geq n - nd(L, \varphi) + 1\)?

Using Theorem 1.1 and the above Kawamata-Viehweg-Nadel type vanishing theorem for \(I_+(\varphi)\), one can answer the just mentioned question of Cao as follows:

**Corollary 1.7.** Let \((L, \varphi)\) be a pseudo-effective line bundle on a compact Kähler manifold \(X\) of dimension \(n\), then

\[
H^p(X, K_X \otimes L \otimes I(\varphi)) = 0,
\]

for any \(p \geq n - nd(L, \varphi) + 1\).

When \(X\) is a projective manifold, some similar result can be referred to \([33]\).

1.6.3. Multiplier ideal sheaves with analytic singularities.

It is known that \(I_+(\varphi)\) is essentially with analytic singularities (see \([4]\)) using Demailly’s approximation of plurisubharmonic functions (see \([6]\)). Then it follows from Theorem 1.1 that \(I(\varphi)\) is essentially with analytic singularities, that is to say,

**Corollary 1.8.** There is a plurisubharmonic function \(\varphi_A\) with analytic singularities, such that \(I(\varphi) = I(\varphi_A)\).

2. Some lemmas used in the proof of main theorems

In this section, we will show some results used in the proof of main theorem.

2.1. \(L^1\) integrable function.

Let \(G\) be a positive Lebesgue measurable and integrable function on a domain \(\Omega \subset \subset \mathbb{C}^n\), which is

\[
\int_{\Omega} Gd\lambda_n < +\infty.
\]

Consider the function

\[
F_G(t) := \sup\{a|\lambda_n(\{G \geq a\}) \geq t\},
\]

\(t \in (0, \lambda_n(\Omega)]\).

We first consider the finiteness of \(F_G(t)\):

If for some \(t_0\), \(F_G(t_0) = +\infty\), then \(\lambda_n(G \geq A_j) \geq t_0\), where \(A_j\) is a number sequence tending to \(+\infty\), when \(j \to +\infty\). Since \(G\) is \(L^1\) integrable, we have

\[
t_0A_j \leq A_j\lambda_n(\{G \geq A_j\}) \leq \int_{\{G \geq A_j\}} Gd\lambda_n \leq \int_{\Omega} Gd\lambda_n < +\infty,
\]
Letting \( A_j \to +\infty \), we thus obtain a contradiction. Therefore \( F_G(t) < +\infty \) for any \( t \).

Secondly, we consider the decreasing property of \( F_G(t) \):

Note that \( \{A|\lambda_n(G \geq A) \geq t_1 \} \supset \{A|\lambda_n(G \geq A) \geq t_2 \} \), when \( t_1 \leq t_2 \). Then we have \( F_G(t_1) \geq F_G(t_2) \), when \( t_1 \leq t_2 \).

The first lemma is about the sublevel sets of \( F_G \):

**Lemma 2.1.** We have

\[
\mu_R(\{t|F_G(t) \geq a\}) = \lambda_n(\{G \geq a\}),
\]

for any \( a > 0 \), where \( \mu_R \) is the Lebesgue measure on \( \mathbb{R} \). Moreover, we have

\[
\mu_R(\{t|F_G(t) > a\}) = \lambda_n(\{G > a\}).
\]

**Proof.** Since

\[
\mu_R(\{t|F_G(t) > a\}) = \lim_{k \to +\infty} \mu_R(\{t|F_G(t) \geq a + \frac{1}{k}\}),
\]

and

\[
\lambda_n(\{G > a\}) = \lim_{k \to +\infty} \lambda_n(\{G \geq a + \frac{1}{k}\}),
\]

we only need to prove

\[
\mu_R(\{t|F_G(t) \geq a\}) = \lambda_n(\{G \geq a\}),
\]

for any \( a > 0 \),

Note that

\[
\sup\{a_1|\lambda_n(\{G \geq a_1\}) \geq \lambda_n(\{G \geq a\})\} \geq a.
\]

Then we have

\[
\lambda_n(\{G \geq a\}) \in \{t|\sup\{a_1|\lambda_n(\{G \geq a_1\}) \geq t\} \geq a\},
\]

therefore

\[
\{t|\sup\{a_1|\lambda_n(\{G \geq a_1\}) \geq t\} \geq a\} \supset \{t|\lambda_n(\{G \geq a\}) \geq t\},
\]

where \( t > 0 \).

If \( \sup \) in the above relation is strictly \( \sup \), then there exists \( t_0 \), such that

1. \( \sup\{a_1|\lambda_n(\{G \geq a_1\}) \geq t_0\} \geq a \);
2. \( t_0 > \lambda_n(\{G \geq a\}) \).

Let

\[ a_0 := \sup\{a_1|\lambda_n(\{G \geq a_1\}) \geq t_0\} \geq a. \]

Note that

\[ \lambda_n(\cap_{a_1 < a_0}\{G \geq a_1\}) = \inf_{a_1 < a_0} \lambda_n(\{G \geq a_1\}). \]

Then we have \( \lambda_n(\{G \geq a_0\}) \geq t_0 \).

As \( t_0 > \lambda_n(\{G \geq a\}) \), we have

\[ \lambda_n(\{G \geq a\}) > \lambda_n(\{G \geq a_0\}), \]

which is a contradiction to

\[ a_0 \geq a. \]

Then the following holds:

\[ \{t|\sup\{a_1|\lambda_n(\{G \geq a_1\}) \geq t\} \geq a\} = \{t|\lambda_n(\{G \geq a\}) \geq t\}, \]

where \( t > 0 \).
According to the definition of $F_G$ and equality 2.2 it follows that
\[ \{ t | F_G(t) \geq a \} = \{ t | \sup \{ a_1 | \mu(\{ G \geq a_1 \}) \geq t \} \geq a \} = \{ t | \lambda_n(\{ G \geq a \}) \geq t \}, \]
where $t > 0$.
Note that
\[ \mu_{\mathbb{R}}(\{ t | F_G(t) \geq a \}) = \mu_{\mathbb{R}}(\{ t | \lambda_n(\{ G \geq a \}) \geq t \}) = \lambda_n(\{ G \geq a \}), \]
for $t > 0$. We have thus proved the present lemma. \[ \square \]

Denote by
\[ s(y) := y^{-1}(-\log y)^{-1}, \]
where $y \in (0, e^{-1})$. It is clear that $s$ is strictly decreasing on $(0, e^{-1})$.

We define a function $u$ by
\[ u(s(y)) = y^{-1}, \]
where $u \in C^\infty((e, +\infty))$. It is clear that $u$ is strictly increasing on $(e, +\infty)$.

The second lemma is about the measure of the level set of $G$:

**Lemma 2.2.** For $A > e$, we have
\[ \lim_{A \to +\infty} \lambda_n(\{ G > A \})u(A) = 0, \]
where $\lambda_n$ is the Lebesgue measure of $\mathbb{C}^n$. Especially, \[ \lim_{A \to +\infty} \frac{A}{u(A)} = 0. \]

**Proof.** According to the definition of Lebesgue integration and Lemma 2.1 it follows that
\[ \int_0^{\mu(t)} F_G(t)dt = \int_\Omega Gd\lambda_n < +\infty. \]

Then we have
\[ \lim_{t \to 0} \inf \frac{F_G(t)}{t^{-1}(-\log t)^{-1}} = 0, \]
which implies that there exists $t_j \to 0$, when $j \to +\infty$, such that
\[ \lim_{j \to +\infty} \frac{F_G(t_j)}{t_j^{-1}(-\log t_j)^{-1}} = 0. \quad (2.3) \]

Using Lemma 2.1 we have
\[ \lambda_n(\{ G > F_G(t_j) \}) = \mu_{\mathbb{R}}(\{ t | F_G(t) > F_G(t_j) \}) \leq \mu_{\mathbb{R}}(\{ 0, t_j \}) = t_j. \]

We now want to prove that $u(F_G(t_j)) = o(t_j^{-1})$ by contradiction: if not, there exists $\varepsilon_0 > 0$, such that $u(F_G(t_j)) \leq \varepsilon_0 t_j^{-1}$. However,
\[ u(F_G(t_j)) \leq \varepsilon_0 t_j^{-1} = u(\frac{1}{\frac{\varepsilon_0}{\log}(-\log t_j)}). \]

According to the strictly increasing property of $u$, it follows that $F_G(t_j) \leq \frac{1}{\frac{\varepsilon_0}{\log}(-\log t_j)}$, which is contradict to equality 2.3 because \[ \lim_{t_j \to 0} \frac{t_j(-\log t_j)}{\frac{\varepsilon_0}{\log}(-\log t_j)} = \varepsilon_0. \]

Now we obtain $u(F_G(t_j)) = o(t_j^{-1})$.

Then we have
\[ \lim_{j \to +\infty} \mu(\{ G > F_G(t_j) \})u(F_G(t_j)) \leq \lim_{j \to +\infty} t_j o(t_j^{-1}) = 0. \]

Note that if $F_G(t_j)$ is bounded above, when $t_j$ to 0, then $G$ has positive upper bound. Therefore $\mu(\{ G > A \}) = 0$, for $A$ large enough.
Then we have proved
\[\liminf_{A \to +\infty} \lambda_n(\{G > A\})u(A) = 0.\]
As \(\frac{1}{t(-\log t)}\) is strictly decreasing on \((0, e^{-1})\), then for any \(A > e\), there exists \(t_A\), such that
1). \(\frac{1}{t_A(-\log t_A)} = A;\)
2). \(t_A\) goes to zero, when \(A\) goes to \(+\infty.\)
As
\[\frac{A}{u(A)} = \frac{1}{u(t_A(-\log t_A))} = \frac{1}{t_A(-\log t_A)} = \frac{1}{-\log t_A},\]
then we obtain
\[\lim_{A \to +\infty} \frac{A}{u(A)} = 0,\]
by the above property 2) of \(t_A.\)
The present lemma is thus proved. \(\square\)

2.2. Estimation of integration of holomorphic functions on singular Riemann surfaces.

Lemma 2.3. Let \(h \not\equiv 0\) be a holomorphic function on the disc \(\Delta_r\) in \(\mathbb{C}\). Let \(f_a\) be a holomorphic function on \(\Delta_r\), which satisfies \(f|_o = 0\) and \(f_a(b) = 1\) for any \(b^k = a\) \((k\) is a positive integer), then we have
\[\int_{\Delta_r} |f_a|^2|h|^2d\lambda_1 > C_1|a|^{-2},\]
where \(a \in \Delta_r,\) whose norm is small enough, \(k\) is a positive integer, \(C_1\) is a positive constant independent of \(a\) and \(f_a.\)

Proof. As \(h \not\equiv 0\), we may write \(h = z^ih_1\) near \(o\), where \(h_1|_o \neq 0\). Then there exists \(r' < r,\) such that \(h_1|_{\Delta_{r'}} \geq C_0 > 0.\) Therefore it suffices to consider the case that \(h = z^i\) on \(\Delta_{r'.}\)

By Taylor expansion at \(o\), we have
\[f(z) = \sum_{j=1}^{\infty} c_j z^j.\]
As \(f(b) = 1,\) then
\[\sum_{j=1}^{\infty} c_j b^j = \frac{1}{k} \sum_{1 \leq l \leq k} \sum_{j=1}^{\infty} c_j b^j = 1\]
where \(b^k = a,\) and \(\sum_{1 \leq j \leq k} b^j = 0\) when \(0 < j < k.\)

It is clear that
\[\int_{\Delta_{r'}} |f_a|^2|h|^2d\lambda_1 = \int_{\Delta_{r'}} |f_a|^2|z^i|^2d\lambda_1 = 2\pi \sum_{j=1}^{\infty} |c_j|^2 \frac{r'^{2j+2i+2}}{2j+2i+2} \quad (2.4)\]
By Schwartz Lemma, we have
\[
(\sum_{j=1}^{\infty} |c_j|^2 \frac{r'^{2j+2i+2}}{2j+2i+2}) \left( \sum_{j=1}^{\infty} \frac{2kj + 2i + 2}{r'^{2kj+2i+2}} |a|^2j \right) 
\geq \left( \sum_{j=1}^{\infty} |c_j|^2 \frac{r'^{2kj+2i+2}}{2kj + 2i + 2} \right) \left( \sum_{j=1}^{\infty} \frac{2kj + 2i + 2}{r'^{2kj+2i+2}} |a|^2j \right) 
\geq \sum_{j=1}^{\infty} c_{kj}a^j|^2 = 1.
\]
(2.5)

Note that
\[
\sum_{j=1}^{\infty} \frac{2kj + 2i + 2}{r'^{2kj+2i+2}} |a|^2j = \frac{a}{r'^2} \left( (2i + 2) \frac{r'^{-2i-2}}{1 - |a/r'^{-2i-2}|^2} + 2k \frac{r'^{-2i-2}}{(1 - |a/r'^{-2i-2}|^2)} \right),
\]
and \((2i + 2) \frac{r'^{-2i-2}}{1 - |a/r'^{-2i-2}|^2} + 2k \frac{r'^{-2i-2}}{(1 - |a/r'^{-2i-2}|^2)}\) has uniform upper bound independent of \(a\), when \(|a| < r'\). The Lemma thus follows.

\(\square\)

Let recall the local parametrization theorem:

**Theorem 2.4.** (see [7]) Let \(\mathcal{J}\) be a prime ideal of \(\mathcal{O}_n\) and let \(\mathcal{C} = V(\mathcal{J})\) be an analytic curve at \(o\). Then the ring \(\mathcal{O}_n/\mathcal{J}\) is a finite integral extension of \(\mathcal{O}_d\); let \(q\) be the degree of the extension. There exists a local coordinates \((z'; z'') = (z_1; z_2, \cdots, z_n)\), such that if \(\Delta'_{r'}\) and \(\Delta''_{r''}\) are polydisks of sufficient small radii \(r'\) and \(r''\) and if \(r' \leq r''\) with \(C\) large, the projective map \(\pi' : \mathcal{C}' \cap (\Delta'_{r'} \times \Delta''_{r''}) \to \Delta'_{r'}\) is a ramified covering with \(q\) sheets, whose ramification locus is contained in \(S = \{o'\} \subset \Delta'_{r'}\). This means that

a) the open set \(C'_S := \mathcal{C}' \cap (\Delta'_{r'} \setminus S) \times \Delta''_{r''}\) is a smooth 1-dimensional manifold, dense in \(\mathcal{C}' \cap (\Delta'_{r'} \times \Delta''_{r''})\);

b) \(\pi' : C'_S \to \Delta'_{r'} \setminus S\) is an unramified covering;

c) the fibre \(\pi'^{-1}(z')\) have exactly \(q\) elements if \(z' \in \Delta'_{r'} \setminus S\) and at most \(q\) if \(z' \in S\).

Moreover, \(C'_S\) is a connected covering of \(\Delta'_{r'} \setminus S\), and \(\mathcal{C}' \cap (\Delta'_{r'} \times \Delta''_{r''})\) is contained in a cone \(|z''| \leq C_0|z'|\).

Let \(\Delta'\) and \(\Delta''\) be unit disc with coordinates \((z_1)\) and unit polydisc with coordinates \((z_2, \cdots, z_n)\) respectively. Let
\[
\pi : \Delta' \times \Delta'' \to \Delta'
\]
be the projective map which is given by
\[
\pi(z_1; z_2, \cdots, z_n) = z_1.
\]
We have the following Remark of Theorem 2.4.

**Remark 2.5.** Let \(\mathcal{J}\) be a prime ideal of \(\mathcal{O}_n\) and let \(\mathcal{C} = V(\mathcal{J})\) be an analytic curve at \(o\). Then the ring \(\mathcal{O}_n/\mathcal{J}\) is a finite integral extension of \(\mathcal{O}_d\); let \(q\) be the degree of the extension, there exists a biholomorphic map \(j\) from a neighborhood of \(\Delta' \times \Delta''\) to a neighborhood \(U_o\) of \(o\), such that the projective map \(\pi|_{\mathcal{C}' \cap (\Delta' \times \Delta'')} \to \Delta'\) is a ramified covering with \(q\) sheets, whose ramification locus is contained in \(S = \{o'\} \subset \Delta'\) where
\[
\mathcal{C} := j^{-1}(\mathcal{C}').
\]
This means that

\( a \), the open set \( C_S := C \cap (\Delta' \setminus S) \times \Delta'' \) is a smooth 1-dimensional manifold, dense in \( C \cap (\Delta' \times \Delta'') \);

\( b \), \( \pi|_{C_S} : C_S \to \Delta' \setminus S \) is an unramified covering;

\( c \), the fibre \( \pi^{-1}(z') \) have exactly \( q \) elements if \( z' \in \Delta' \setminus S \) and at most \( q \) if \( z' \in S \).

Moreover, \( C_S \) is a connected covering of \( \Delta' \setminus S \), and \( C \cap (\Delta' \times \Delta'') \) is contained in a cone \( |z''| \leq \frac{1}{6}|z'| \).

Using Lemma 2.3 and Remark 2.5, we obtain the following singular version of Lemma 2.3.

**Lemma 2.6.** Let \( h \) be a holomorphic function on an analytic curve \( C \) as in Remark 2.3. Let \( f_a \) be a holomorphic function on \( C \), which satisfies \( f(a) = 0 \) and \( f_a(\pi^{-1}(a) \cap C) = 1 \), then we have

\[
\int_{C_S} |f_a|^2 |h|^2 (\pi|_{C_S})^* d\lambda_{\Delta'} > C_2 |a|^{-2},
\]

when \( a \in \Delta' \) and \( |a| \) is small enough, where \( C_2 \) is a positive constant independent of \( a \) and \( f_a \).

**Proof.** As \((C, o)\) is irreducible and locally irreducible, then there is a normalization \( j_{nor} : (\Delta, 0) \rightarrow (C, o) \), denoted by

\[ j_{nor}(t) = (g_1(t), \cdots, g_n(t)), \]

where \( t \) is the coordinate of \( \Delta \). As \( \pi_C \) is a covering, then \( g_1 \neq 0 \).

Without loss of generality, we assume \( g_1(t) = t^{i_1} \) on \( \Delta_{r_0} \), for small enough \( r_0 \in (0, 1) \).

There is a given \( r > 0 \), which is small enough, such that

\( (C \cap \Delta^n_{r_0}) \supset \{(t^{i_1}, g_2(t), \cdots, g_n(t)) | t \in \Delta_r \} \),

where \( i_1 \geq 1 \), and \( g_i \) (\( i \geq 2 \)) are holomorphic functions on \( \Delta_r \), satisfying \( |g_i| \leq \frac{1}{6}|t^{i_1}| \).

For given \( r' < r \) small enough, we have

\[
\int_{C_S} |f_a|^2 |h|^2 (\pi|_{C_S})^* d\lambda_{\Delta'} \geq i_1^2 \int_{\Delta_r} |j_{nor}^* f_a(t)|^2 |j_{nor}^* h(t)|^2 |t|^{2(i_1-1)} d\lambda_{\Delta} \]

\[
= i_1^2 \int_{\Delta_r} |j_{nor}^* f_a(t)(t^{i_1-1} j_{nor}^* h(t))|^2 d\lambda_{\Delta}, \tag{2.6}
\]

for any \( a \) satisfying \( |a|^{1/2} \in \Delta_{r'} \).

As

\( j_{nor}^* f_a(b) = f_a(b^{i_1}, g_2(b), \cdots, g_n(b)) \)

and

\( (b^{i_1}, g_2(b), \cdots, g_n(b)) \subset (\pi^{-1}(b^{i_1}) \cap C) \),

then we have

\( j_{nor}^* f_a(b) = 1 \),

for any \( b^{i_1} = a \).

Using Lemma 2.3, we have

\[
\int_{\Delta_{r'}} |j_{nor}^* f_a(t)(t^{i_1-1} j_{nor}^* h(t))|^2 d\lambda_{\Delta} \geq C_1 |a|^{-2},
\]
where $C_1$ is independent of $a$ and $f_a$.  

Combining with inequality (2.4) we thus obtain the present lemma.  

As $C \cap (\Delta' \times \Delta'')$ is contained in a cone $|z''| \leq \frac{1}{6}|z'|$, using the submean value property of plurisubharmonic function, we obtain the following lemma:

**Lemma 2.7.** For any holomorphic function $F$ on $\Delta' \times \Delta''$, we obtain an approximation of the $L^2$ norm of $F$:

$$\int_{\Delta' \times \Delta''} |F|^2 d\lambda_n \geq C_3 \int_{C_S} |F|_{c_S}^2 (\pi|c_S|)^* d\lambda_{\Delta'},$$

where $C_3$ is a positive constant independent of $F$. Here all symbols $C_S$, $\Delta'$ and $\pi$ are the same as in Remark 2.5.

**Proof.** Using the Fubini Theorem,

$$\int_{\Delta' \times \Delta''} |F|^2 d\lambda_n = \int_{\Delta'} (\int_{\delta(z') \times \Delta''} |F|^2 d\lambda_{n-1}) d\lambda_{\Delta'},$$

and the submean value inequality of plurisubharmonic function, we have

$$\int_{\delta(z) \times \Delta''} |F|^2 d\lambda_{n-1} \geq \left(\frac{\pi}{3}\right)^{n-1} |F(z', z'')|^2,$$

for $|z''| \leq \frac{1}{6}$.  

If $(z', z'') \in (\pi^{-1}(z') \cap C_S)$, then $|z''| \leq \frac{1}{6}$.  

As

$$\int_{\Delta' \setminus \{0\}} \sum_{w \in (\pi^{-1}(z') \cap C_S)} |F(w)|^2(z') d\lambda_{\Delta'} = \int_{C_S} |F|_{c_S}^2 (\pi|c_S|)^* d\lambda_{\Delta'},$$

it follows that

$$q \int_{\Delta' \times \Delta''} |F|^2 d\lambda_n = q \int_{\Delta' \setminus \{0\}} (\int_{\delta(z') \times \Delta''} |F|^2 d\lambda_{n-1}) d\lambda_{\Delta'}$$

$$\geq \left(\frac{\pi}{3}\right)^{n-1} \int_{\Delta' \setminus \{0\}} \left(\sum_{w \in (\pi^{-1}(z') \cap C_S)} |F(w)|^2(z') d\lambda_{\Delta'} \right)$$

$$= \left(\frac{\pi}{3}\right)^{n-1} \int_{C_S} |F|_{c_S}^2 (\pi|c_S|)^* d\lambda_{\Delta'},$$

(2.7)

where $q$ is the degree of the covering map $\pi|c_S|$.  

2.3. $L^2$ extension theorem with negligible weight.

We state the optimal constant version of the Ohsawa’s $L^2$ extension theorem with negligible weight (3.3) as follows:

**Theorem 2.8.** [13] Let $X$ be a Stein manifold of dimension $n$. Let $\varphi + \psi$ and $\psi$ be plurisubharmonic functions on $X$. Assume that $w$ is a holomorphic function on $X$ such that $\text{sup}(\psi + 2 \log |w|) \leq 0$ and $dw$ does not vanish identically on any branch of $w^{-1}(0)$. Put $H = w^{-1}(0)$ and $H_0 = \{x \in H : \text{dw}(x) \neq 0\}$. Then there exists a uniform constant $C = 1$ independent of $X$, $\varphi$, $\psi$ and $w$ such that, for any holomorphic $(n - 1)$-form $f$ on $H_0$ satisfying

$$c_{n-1} \int_{H_0} e^{-\varphi - \psi} f \wedge \bar{f} < \infty,$$

$$\int_{H_0} e^{-\varphi - \psi} f \wedge \bar{f} < \infty,$$
Lemma 2.10. \( (\text{see Lemma}) \) such that 

There exists \( \Delta \) vanishing at \( 0 \) for every germ of analytic curve \( \gamma \) for any germ of analytic subvariety of \( Y \) through \( 0 \). 

Proof. Assume that for any given neighborhood of \( 0 \), \( \{f = 0\} \) is biholomorphic to \( \Delta \), which contains \( r \), such that \( r \) is irreducible. Then \( Y \) contains \( \Delta \). We give the existence of some kind of germs of analytic curves, which will be used.

Lemma 2.9. Let \( (Y, o) \) be a germ of irreducible analytic subvariety in \( \mathbb{C}^n \), and \( (A, o) \) be a germ of analytic subvariety of \( (Y, o) \), such that \( \dim A < \dim Y \). Then there exists a germ of holomorphic curve \( (\gamma, o) \), such that \( \gamma \subset Y \), and \( \gamma \not\subset A \).

Proof. Note that \( (Y, o) \) is locally Stein. Then using Cartan’s Theorem A, we obtain the lemma. \( \square \)

Now we recall the curve selection lemma stated as follows:

Lemma 2.10. \( (\text{see [3]} \) Let \( f, g_1, \cdots, g_s \in \mathcal{O}_n \) be germs of holomorphic functions vanishing at \( 0 \). Then we have \( |f| \leq C|g| \) for some constant \( C \) if and only if for every germ of analytic curve \( \gamma \) through \( 0 \) there exists a constant \( C_\gamma \) such that \( |f \circ \gamma| \leq C_\gamma |g \circ \gamma| \).

In order to obtain some uniform properties of \( \gamma \), we need to consider the following Lemma which was contained in the proof of Lemma 2.10 in [6].

Lemma 2.11. \( (\text{see [3]} \) Let \( f, g_1, \cdots, g_s \in \mathcal{O}_n \) be germs of holomorphic functions vanishing at \( o \). Assume that for any given neighborhood of \( o \), \( |f| \leq C|g| \) doesn’t hold for any constant \( C \), where \( g = (g_1, \cdots, g_s) \). Then there exists a germ of analytic curve \( \gamma \) through \( o \), satisfying \( \gamma \cap \{f = 0\} = o \), such that \( \tilde{A} f |\gamma \) is holomorphic on \( \gamma \setminus o \) with \( \tilde{f} \gamma(0) = 0 \), for any \( i \in \{1, \cdots, s\} \), where \( \tilde{\gamma} \) is the holomorphic extension of \( \gamma \) from \( \gamma \setminus o \) to \( \gamma \).

Proof. There exists \( \Delta \), such that \( g_1, \cdots, g_s, f \in \mathcal{O}(\Delta) \). We define a germ of analytic set \( (Y, o) \subset (\Delta, o) \) by \( g_j(z) = f(z)z^{n+j}, \quad 1 \leq j \leq s \).

Let \( p \) be a projection \( p : \Delta \times \mathbb{C} \rightarrow \Delta \), such that \( p((z_1, \cdots, z_n), (z_{n+1}, \cdots, z_{n+s})) = (z_1, \cdots, z_n) \). Then \( Y \cap p^{-1}(\Delta \setminus \{f = 0\}) \) is biholomorphic to \( \Delta \setminus \{f = 0\} \), which is irreducible. As every analytic variety has an irreducible decomposition, then \( Y \) contains an irreducible component \( Y_f \) which contains \( Y \cap p^{-1}(\Delta \setminus \{f = 0\}) \).

Since \( Y_f \) is closed, then \( Y_f = \overline{Y \cap p^{-1}(\Delta \setminus \{f = 0\})} \).
By assumption, for any given neighborhood of \( o \), \( |f| \leq C|g| \) doesn’t hold for any constant \( C \), then there exists a sequence of positive numbers \( C_\nu \) which goes to \( +\infty \) as \( \nu \to \infty \), and a sequence of points \( \{ z_\nu \} \) in \( \Delta^o \) convergent to \( o \) when \( \nu \to \infty \), such that \( |f(z_\nu)| > C_\nu |g(z_\nu)| \).

Then \( (z_\nu, \frac{g(z_\nu)}{f(z_\nu)}) \) converging to \( 0 \) as \( \nu \) tends to \( +\infty \), with \( f(z_\nu) \neq 0 \).

As \( (z_\nu, \frac{g(z_\nu)}{f(z_\nu)}) \in Y_f \), then \( Y_f \) contains \( o \).

It follows from Lemma \( \ref{lem:2.15} \) that there exists a germ of analytic curve \( (\gamma, \gamma_{n+}) \subset Y_f \) through \( o \) satisfying \( \gamma \cap \{ f = 0 \} = o \), such that \( \overline{\frac{g_1}{f}} \mid _\gamma \) is holomorphic on \( \gamma \setminus o \) for each \( i \in \{ 1, \ldots, s \} \).

By the Riemann removable singularity theorems, it follows that \( \overline{\frac{g_1}{f}} \mid _\gamma \setminus o \) can be extended to \( \gamma \), and
\[
\overline{\frac{g_1}{f}} \mid _\gamma (0) = 0,
\]
for any \( i \in \{ 1, \ldots, s \} \).

\( \square \)

**Remark 2.12.** Let \( g_1, \ldots, g_s \in O_o \) be germs of holomorphic functions vanishing at \( o \), and \( f(o) \neq 0 \). Then there exists a germ of analytic curve \( \gamma \setminus o \) through \( o \) satisfying \( \gamma \cap \{ f = 0 \} = o \), and \( \overline{\frac{g_i}{f}} \mid _\gamma \) is holomorphic on \( \gamma \setminus o \) for any \( i \in \{ 1, \ldots, s \} \).

Let’s recall a strong Noetherian property of coherent sheaves as follows:

**Lemma 2.13.** (see \( \ref{lem:2.13} \)) Let \( \mathcal{F} \) be a coherent analytic sheaf on a complex manifold \( M \), and let \( \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \) be an increasing of coherent subsheaves of \( \mathcal{F} \). Then the sequence \( (\mathcal{F}_k) \) is stationary on every compact subset of \( M \).

Let \( \varphi \) be a negative plurisubharmonic function on \( \Delta^n \subset C^n \), and \( \{ \psi_j \}_{j=1,2,\ldots} \) be a sequence of plurisubharmonic functions on \( \Delta^n \), which is increasingly convergent to \( \varphi \) on \( \Delta^n \), when \( j \to \infty \).

**Remark 2.14.** By Lemma \( \ref{lem:2.13} \), it is clear that \( \cup_{j=1}^\infty \mathcal{I}(\psi_j) \) is a coherent subsheaf of \( \mathcal{I}(\varphi) \); actually for any open \( V_1 \subset M \), there exists \( j_1 \in \{ 1, 2, \ldots \} \), such that \( \cup_{j=1}^\infty \mathcal{I}(\psi_j)|_{V_1} = \mathcal{I}(\psi_{j_1}) \).

By Remark \( \ref{rem:2.14} \), we derive the following proposition about the generators of the coherent sheaf \( \cup_{j=1}^\infty \mathcal{I}(\psi_j) \):

**Proposition 2.15.** Assume that \( f \in O_o \) is a holomorphic function on neighborhood \( V_0 \) of \( o \), which is not a germ of \( \cup_{j=1}^\infty \mathcal{I}(\psi_j) \). Let \( g_1, \ldots, g_s \in \cup_{j=1}^\infty \mathcal{I}(\psi_j) \) be germs of holomorphic functions on some neighborhood \( V_1 \subset V_0 \) of \( o \), such that \( g_1, \ldots, g_s \) generate \( \cup_{j=1}^\infty \mathcal{I}(\psi_j)|_{V_1} \). Then there exists a germ of analytic curve \( \gamma \) through \( o \) satisfying \( \gamma \cap \{ f = 0 \} \subset \{ o \} \), such that \( \overline{\frac{g_1}{f}} \mid _\gamma \) is holomorphic on \( \gamma \setminus o \) for any \( i \in \{ 1, \ldots, s \} \), and
\[
\overline{\frac{g_i}{f}} \mid _\gamma (0) = 0,
\]
where \( \overline{\frac{g_1}{f}} \mid _\gamma \) is the holomorphic extension of \( \overline{\frac{g_1}{f}} \mid _\gamma \) from \( \gamma \setminus o \) to \( \gamma \). Moreover, for any germ \( g \) of \( \mathcal{I}(\varphi + \varepsilon_0 \varphi) \), \( \overline{\frac{g}{f}} \mid _\gamma \) is holomorphic on \( \gamma \setminus o \), and
\[
\overline{\frac{g}{f}} \mid _\gamma (0) = 0,
\]
where \( \overline{\frac{g}{f}} \mid _\gamma \) is the holomorphic extension of \( \overline{\frac{g}{f}} \mid _\gamma \) from \( \gamma \setminus o \) to \( \gamma \).
Proof. By Remark 2.14, it follows that there exists \( j_1 \in \{1, 2, \cdots \} \), such that \( g_1, \cdots, g_s \in I(\psi_{j_1})(V_1) \).

As \( f \) is not a germ of \( (\cup_{j=1}^{\infty} I(\psi_{j}))_a = I(\psi_{j_1})_a \), then for any neighborhood of \( o \), \( |f| \leq C(\sum_{1 \leq j \leq s} |g_j|^2)^{1/2} \) doesn’t hold for any constant \( C \).

According to Lemma 2.11 and Remark 2.12, it follows that there exists a germ of analytic curve \( \gamma \) through \( o \) such that \( \gamma \cap \{ f = 0 \} \subseteq \{ o \} \), and \( \tilde{g} \frac{\partial \gamma}{f \circ \gamma} \) is holomorphic on \( \gamma \) for any \( i \), and

\[
\left. \frac{\tilde{g}_i \circ \gamma}{f \circ \gamma} \right|_0 = 0,
\]

where \( \tilde{g} \frac{\partial \gamma}{f \circ \gamma} \) is the holomorphic extension of \( \tilde{g} \frac{\partial \gamma}{f \circ \gamma} \) from \( \gamma \setminus o \) to \( \gamma \). \( \square \)

Remark 2.16. Let \( (f, o) \) be a germ of holomorphic function on \( \Omega \ni o \), such that \( (f, o) \notin I(\varphi)_o \). Assume \( \{ F_i \}_{i=1, 2, \cdots} \) is a sequence of holomorphic functions on \( \Omega \), such that \( F_i \) is uniformly convergent to a holomorphic function \( F \) on any compact subset, and \( (F_i - f, o) \notin I(\varphi)_o \). Then \( F \neq 0 \).

Proof. As \( (f, o) \notin I(\varphi)_o \), let \( \psi_j = \varphi \) for any \( j \in \{1, 2, \cdots, \} \), it follows from Proposition 2.14 that there exists a germ of analytic curve \( \gamma \) through \( o \), such that for any germ \( F_i - f \) of \( I(\varphi)_o \), \( \frac{(F_i - f) \circ \gamma}{\tilde{f} \circ \gamma} \) is holomorphic on \( \gamma \setminus o \), and

\[
\left. \frac{(F_i - f) \circ \gamma}{\tilde{f} \circ \gamma} \right|_0 = 0,
\]

where \( \tilde{g} \frac{\partial \gamma}{f \circ \gamma} \) is the holomorphic extension of \( \tilde{g} \frac{\partial \gamma}{f \circ \gamma} \) from \( \gamma \setminus o \) to \( \gamma \).

Therefore

\[
\left. \frac{F_i \circ \gamma}{\tilde{f} \circ \gamma} \right|_0 = \left. \frac{\tilde{f} \circ \gamma}{\tilde{f} \circ \gamma} \right|_0 = 1.
\]

As \( F_i \) is uniformly convergent to \( F \), it follows that \( F|_\gamma \neq 0 \), which implies \( F \neq 0 \). \( \square \)

2.5 \( L^2 \) estimates for some \( \bar{\partial} \) equations.

For the sake of completeness, we recall some lemmas on \( L^2 \) estimates for some \( \bar{\partial} \) equations, and \( \bar{\partial}^* \) means the Hilbert adjoint operator of \( \bar{\partial} \).

Lemma 2.17. (see [20], see also [1]) Let \( \Omega \subset \subset \mathbb{C}^n \) be a domain with \( C^\infty \) boundary \( b\Omega, \Phi \in C^\infty(\overline{\Omega}) \). Let \( \rho \) be a \( C^\infty \) defining function for \( \Omega \) such that \( |d\rho| = 1 \) on \( b\Omega \). Let \( \eta \) be a smooth function on \( \overline{\Omega} \). For any \((0, 1)\)-form \( \alpha = \sum_{j=1}^n \alpha_j d\bar{z}^j \in Dom_\Omega(\bar{\partial}^*) \cap C^\infty(\Omega_{\rho,1})(\overline{\Omega}) \),

\[
\int_\Omega \eta |\bar{\partial} \alpha|^2 e^{-\Phi} d\lambda_n + \int_\Omega \eta |\bar{\partial} \alpha|^2 e^{-\Phi} d\lambda_n = \sum_{i,j=1}^n \int_{\Omega} \eta |\partial_j \alpha_i|^2 e^{-\Phi} d\lambda_n + \sum_{i,j=1}^n \int_{\Omega} \eta |\partial_i \alpha_j|^2 e^{-\Phi} d\lambda_n + \sum_{i,j=1}^n \int_{\Omega} \eta (\partial_i \partial_j \rho) \alpha_i \alpha_j e^{-\Phi} dS + \sum_{i,j=1}^n \int_{\Omega} \eta (\partial_i \partial_j \Phi) \alpha_i \alpha_j e^{-\Phi} d\lambda_n + \sum_{i,j=1}^n \int_{\Omega} \eta (\partial_i \partial_j \eta) \alpha_i \alpha_j e^{-\Phi} d\lambda_n + 2\text{Re}(\bar{\partial}^* \alpha, \alpha_\nu (\bar{\partial} \eta)^2)_{\Omega,\Phi},
\]

where \( d\lambda_n \) is the Lebesgue measure on \( \mathbb{C}^n \), and \( \alpha_\nu (\bar{\partial} \eta)^2 = \sum_j \alpha_j \partial_j \eta \).
The symbols and notations can be referred to [15]. See also [40], [41], or [44].

Lemma 2.18. (see [1], see also [15]) Let $\Omega \subset \subset \mathbb{C}^n$ be a strictly pseudoconvex domain with $C^\infty$ boundary $\partial \Omega$ and $\Phi \in C^\infty(\Omega)$. Let $\lambda$ be a $\bar{\partial}$ closed smooth form of bidgree $(n, 1)$ on $\Omega$. Assume the inequality

$$|(\lambda, \alpha)_{\Omega, \Phi}|^2 \leq C \int_{\Omega} |\bar{\partial}_{\Phi}\alpha|^2 \frac{e^{-\Phi}}{\mu} d\lambda_n < \infty,$$

where $\frac{1}{\mu}$ is an integrable positive function on $\Omega$ and $C$ is a constant, holds for all $(n, 1)$-form $\alpha \in \text{Dom}_{\Omega}(\bar{\partial}) \cap \text{Ker}(\bar{\partial}) \cap C^\infty_{(n, 1)}(\Omega)$. Then there is a solution $u$ to the equation $\bar{\partial} u = \lambda$ such that

$$\int_{\Omega} |u|^2 e^{-\Phi} d\lambda_n \leq C.$$ 

3. Some propositions on multiplier ideal sheaves

3.1. A proposition used in the proof of Conjecture D-K.

We prove Theorem 1.2 and 1.6 by the following proposition:

**Proposition 3.1.** Let $D_v$ be a strongly pseudoconvex domain relatively compact in $\Delta^a$ containing $o$. Let $F$ be a holomorphic function on $\Delta^a$. Let $\varphi$ be a negative plurisubharmonic function on $\Delta^a$, such that $\varphi(o) = -\infty$. Then there exists a holomorphic function $F_{v, t_0}$ on $D_v$, such that,

$$(F_{v, t_0} - F, o) \in I(\varphi)_o$$

and

$$\int_{D_v} |F_{v, t_0} - (1 - b_{t_0}(\varphi))F|^2 d\lambda_n \leq \int_{D_v} (\mathbb{I}_{-t_0 < t < t_0} \circ \varphi)|F|^2 e^{-\varphi} d\lambda_n,$$

where $b_{t_0}(t) = \int_{-\infty}^t \mathbb{I}_{(-t_0 < s < -t_0)} ds$.

When $\varphi$ is a polar function (see [36], see also [14, 15, 16, 17]), various versions of Proposition 3.1 were used to prove the main results in [36], [14], [15], [16], [17], etc.

3.2. A proposition used in the proof of Conjecture J-M.

Let

$$\varphi := 2 \max\{\psi, \log |F|^2\},$$

and

$$\Psi := \min\{\psi - \log |F|^2, 0\} - 1.$$ 

Then $\Psi + \varphi$ and $2\Psi + \varphi$ are both plurisubharmonic functions on $\Delta^a$.

Note that

$$e^{-1} \mathbb{I}_{\{\psi \leq \log |F|^2\}} e^{-\Psi} e^{-\Psi} \leq |F|^2 e^{-\psi} \leq e^{-1} e^{-\Psi},$$

and

$$e^{-1} e^{-\Psi} - e^{-1} \mathbb{I}_{\{\psi \leq \log |F|^2\}} e^{-\Psi} = e^{-1} \mathbb{I}_{\{\psi > \log |F|^2\}} e^{-\Psi} = e^{-1} \mathbb{I}_{\{\psi > \log |F|^2\}},$$

then we have the following three statements are equivalent:
1. $|F|^2 e^{-\Psi}$ is not locally integrable near $o$
2. $e^{-\Psi}$ is not locally integrable near $o$
3. $\mathbb{I}_{\{\psi \leq \log |F|^2\}} e^{-\Psi}$ is not locally integrable near $o$.

Note that

$$|F|^4 e^{-\varphi} \mathbb{I}_{\{\psi \leq \log |F|^2\}} = e^{-\max\{\psi - \log |F|^2, 0\}} |\{\psi \leq \log |F|^2\}| = 1,$$

then we have:

**Remark 3.2.** The following two statements are equivalent:
1. $|F|^2 e^{-\Psi}$ is not locally integrable near $o$;
2. $|F|^4 e^{-\varphi - \Psi}$ is not locally integrable near $o$.

We prove Theorem 1.4 by the following proposition:

**Proposition 3.3.** Let $D_v$ be a strongly pseudoconvex domain relatively compact in $\Delta^n$ containing $o$.

Then there exists a holomorphic function $F_{v,t_0}$ on $D_v$, satisfying:

$$(F_{v,t_0} - F^2, o) \in \mathcal{I}(\varphi + \Psi)_o$$

and

$$\int_{D_v} |F_{v,t_0} - (1 - b_{t_0}(\Psi))F^2| e^{-\varphi} d\lambda_n$$

$$\leq 2 \int_{D_v} \frac{1}{B_0} \mathbb{I}_{\{-t_0 - B_0 < t < -t_0\}}(\o) |F^2|^2 e^{-\varphi} e^{-\Psi} d\lambda_n,$$

where

$$b_{t_0} := \int_{-\infty}^t \frac{1}{B_0} \mathbb{I}_{\{-t_0 - B_0 < t < -t_0\}} ds,$$

and $t_0 \geq 0$.

### 3.3 A smooth form of Proposition 3.3

Let $\{v_{t_0,\varepsilon}\}_{t_0 \in \mathbb{R}, \varepsilon \in (0, \frac{1}{4} B_0)}$ be a family of smooth increasing convex functions on $\mathbb{R}$, which are continuous functions on $\mathbb{R} \cup \{-\infty\}$, such that:

1. $v_{t_0,\varepsilon}(t) = t$ for $t \geq -t_0 - \varepsilon$, $v_{t_0,\varepsilon}(t) = \text{constant}$ for $t < -t_0 - B_0 + \varepsilon$;
2. $v''_{t_0,\varepsilon}(t)$ are pointwise convergent to $\frac{1}{B_0} \mathbb{I}_{\{-t_0 - B_0, -t_0\}}$, when $\varepsilon \to 0$, and $0 \leq v''_{t_0,\varepsilon}(t) \leq 2$ for any $t \in \mathbb{R};$
3. $v''_{t_0,\varepsilon}(t)$ are pointwise convergent to $b_{t_0}(t) = \int_{-\infty}^t \frac{1}{B_0} \mathbb{I}_{\{-t_0 - B_0, -t_0\}} ds$ ($b_{t_0}$ is also a continuous function on $\mathbb{R} \cup \{-\infty\}$), when $\varepsilon \to 0$, and $0 \leq v''_{t_0,\varepsilon}(t) \leq 1$ for any $t \in \mathbb{R}$.

One can construct the family $\{v_{t_0,\varepsilon}\}_{t_0 \in \mathbb{R}, \varepsilon \in (0, \frac{1}{4} B_0)}$ by the setting

$$v_{t_0,\varepsilon}(t) := \int_{-\infty}^t \left( \int_{-\infty}^{t_1} \frac{1}{1 - 4\varepsilon B_0} \mathbb{I}_{\{-t_0 - B_0 + 2\varepsilon, -t_0 - 2\varepsilon\}}(\rho_{\frac{1}{4} \varepsilon}) ds \right) dt_1$$

$$- \int_{-\infty}^0 \left( \int_{-\infty}^{t_1} \frac{1}{1 - 4\varepsilon B_0} \mathbb{I}_{\{-t_0 - B_0 + 2\varepsilon, -t_0 - 2\varepsilon\}}(\rho_{\frac{1}{4} \varepsilon}) ds \right) dt_1,$$

where $\rho_{\frac{1}{4} \varepsilon}$ is the kernel of convolution satisfying $\text{supp}(\rho_{\frac{1}{4} \varepsilon}) \subset \left(-\frac{3}{4} \varepsilon, \frac{1}{4} \varepsilon\right)$. Then it follows that

$$v''_{t_0,\varepsilon}(t) = \frac{1}{1 - 4\varepsilon B_0} \mathbb{I}_{\{-t_0 - B_0 + 2\varepsilon, -t_0 - 2\varepsilon\}} \rho_{\frac{1}{4} \varepsilon}(t),$$
and
\[ v'_{t_0}(t) = \int_{-\infty}^t \left( \frac{1}{1 - 4\varepsilon} \frac{1}{B_0} \frac{1}{(-t_0 - B_0 + 2\varepsilon, -t_0 - 2\varepsilon)} * \rho_{\frac{1}{2}\varepsilon}(s) \right) ds. \]

We prove Proposition 3.4 by the following proposition:

**Proposition 3.4.** Let \( D_v \) be a strongly pseudoconvex domain relatively compact in \( \Delta^n \) containing \( o \). Then there exists a holomorphic function \( F_{v, t_0, \varepsilon} \) on \( D_v \), satisfying:

\[ (F_{v, t_0, \varepsilon} - F^2, o) \in \mathcal{I}(\varphi + \Psi)_o, \]

and

\[ \int_{D_v} |F_{v, t_0, \varepsilon} - (1 - v'_{t_0}(\Psi)) F^2|^2 e^{-\varphi} d\lambda_n \leq 2 \int_{D_v} \frac{1}{B_0} |v''_{t_0}(\Psi)| F^2 e^{-\varphi} d\lambda_n, \]

where \( t_0 \geq 0 \).

### 4. Proofs of the propositions

#### 4.1. Proof of Proposition 3.1

For the sake of completeness, let’s recall some step in our proof in [15] (see also [17]) with some modifications in order to prove Proposition 3.1.

Let \( \{v_{t_0, \varepsilon} \}_{t_0 \in \mathbb{R}, \varepsilon \in (0, \frac{1}{2})} \) be a family of smooth increasing convex functions on \( \mathbb{R} \), which are continuous functions on \( \mathbb{R} \cup \{-\infty\} \), such that:

1. \( v_{t_0, \varepsilon}(t) = t \) for \( t \geq -t_0 - \varepsilon, v_{t_0, \varepsilon}(t) = \text{constant} \) for \( t < -t_0 - 1 + \varepsilon \);
2. \( v'_{t_0, \varepsilon}(t) \) are pointwise convergent to \( \|_{(-t_0 - 1, -t_0)} \), when \( \varepsilon \to 0 \), and \( 0 \leq v''_{t_0, \varepsilon}(t) \leq 2 \) for any \( t \in \mathbb{R} \);
3. \( v'_{t_0, \varepsilon}(t) \) are pointwise convergent to \( b_{t_0}(t) = \int_{-\infty}^t \|_{(-t_0 - 1, -t_0)} ds \) (\( b_{t_0} \) is also a continuous function on \( \mathbb{R} \cup \{-\infty\} \), when \( \varepsilon \to 0 \), and \( 0 \leq v'_{t_0, \varepsilon}(t) \leq 1 \) for any \( t \in \mathbb{R} \).

One can construct the family \( \{v_{t_0, \varepsilon} \}_{t_0 \in \mathbb{R}, \varepsilon \in (0, \frac{1}{2})} \) as equality 3.4 by taking \( B_0 = 1 \). Then it follows that

\[ v''_{t_0, \varepsilon}(t) = \left( \frac{1}{1 - 4\varepsilon} \|_{(-t_0 - 1 + 2\varepsilon, -t_0 - 2\varepsilon)} * \rho_{\frac{1}{2}\varepsilon}(t) \right), \]

and

\[ v'_{t_0, \varepsilon}(t) = \int_{-\infty}^t \left( \frac{1}{1 - 4\varepsilon} \|_{(-t_0 - 1 + 2\varepsilon, -t_0 - 2\varepsilon)} * \rho_{\frac{1}{2}\varepsilon} \right) ds. \]

As \( D_v \subset \Delta^n \subset C^n \), then there exist negative smooth plurisubharmonic functions \( \{\varphi_m \}_{m=1,2,...} \) on a neighborhood of \( \overline{D_v} \), such that the sequence \( \{\varphi_m \}_{m=1,2,...} \) is decreasingly convergent to \( \varphi \) on a smaller neighborhood of \( \overline{D_v} \), when \( m \to +\infty \).

Let \( \eta = s(-v_{t_0, \varepsilon} \circ \varphi_m) \) and \( \phi = u(-v_{t_0, \varepsilon} \circ \varphi_m) \), where \( s \in C^\infty((0, +\infty)) \) satisfies \( s \geq 0 \), and \( u \in C^\infty((0, +\infty)) \), satisfies \( \lim_{t \to +\infty} u(t) = 0 \), such that \( u'' s - s'' > 0 \), and \( s' - u's = 1 \).

Let \( \Phi = \varphi_m + \phi \).
Now let $\alpha = \sum_{j=1}^{n} \alpha_j d\bar{z}^j \in \text{Dom}_{D_v}(\bar{\partial}^*) \cap \text{Ker}(\bar{\partial}) \cap C^\infty_{(0,1)}(D_v)$. By Cauchy-Schwarz inequality, it follows that

$$2\text{Re}(\bar{\partial}^* g, \alpha, (\partial \eta)^2)_{\Omega, \Phi} \geq -\int_{D_v} g^{-1} |\bar{\partial}^* g|^2 e^{-\Phi} d\lambda_n + \sum_{j,k=1}^{n} \int_{D_v} (-g(\partial_j \eta \bar{\partial}_k \eta) \alpha_j \alpha_k e^{-\Phi} d\lambda_n. \tag{4.1}$$

Using Lemma 2.17 and inequality 4.1, since $s \geq 0$ and $\varphi_m$ is a plurisubharmonic function on $D_v$, we get

$$\int_{D_v} (\eta + g^{-1}) |\bar{\partial}^* g|^2 e^{-\Phi} d\lambda_n \geq \sum_{j,k=1}^{n} \int_{D_v} (-\partial_j \bar{\partial}_k \eta + \eta \partial_j \bar{\partial}_k \Phi - g(\partial_j \eta \bar{\partial}_k \eta) \alpha_j \alpha_k e^{-\Phi} d\lambda_n \tag{4.2}$$

$$\geq \sum_{j,k=1}^{n} \int_{D_v} (-\partial_j \bar{\partial}_k \eta + \eta \partial_j \bar{\partial}_k \Phi - g(\partial_j \eta \bar{\partial}_k \eta) \alpha_j \alpha_k e^{-\Phi} d\lambda_n,$n

where $g$ is a positive continuous function on $D_v$. We need some calculations to determine $g$.

We have

$$\partial_j \bar{\partial}_k \eta = -s'(-v_{t_0, \e} \circ \varphi_m) \partial_j \bar{\partial}_k (v_{t_0, \e} \circ \varphi_m) + s''(-v_{t_0, \e} \circ \varphi_m) \partial_j \bar{\partial}_k (v_{t_0, \e} \circ \varphi_m), \tag{4.3}$$

and

$$\partial_j \bar{\partial}_k \phi = -u'(-v_{t_0, \e} \circ \varphi_m) \partial_j \bar{\partial}_k (v_{t_0, \e} \circ \varphi_m) + u''(-v_{t_0, \e} \circ \varphi_m) \partial_j \bar{\partial}_k (v_{t_0, \e} \circ \varphi_m), \tag{4.4}$$

for any $j, k \ (1 \leq j, k \leq n)$.

We have

$$\sum_{1 \leq j, k \leq n} (-\partial_j \bar{\partial}_k \eta + \eta \partial_j \bar{\partial}_k \phi - g(\partial_j \eta \bar{\partial}_k \eta) \alpha_j \alpha_k$$

$$= (s' - su') \sum_{1 \leq j, k \leq n} \partial_j \bar{\partial}_k (v_{t_0, \e} \circ \varphi_m) \alpha_j \alpha_k$$

$$+ ((u'' s - s'') - gs') \sum_{1 \leq j, k \leq n} \partial_j (-v_{t_0, \e} \circ \varphi_m) \bar{\partial}_k (-v_{t_0, \e} \circ \varphi_m) \alpha_j \alpha_k \tag{4.5}$$

$$= (s' - su') \sum_{1 \leq j, k \leq n} ((v_{t_0, \e} \circ \varphi_m) \partial_j \bar{\partial}_k \varphi_m + (v_{t_0, \e} \circ \varphi_m) \partial_j (\varphi_m) \bar{\partial}_k (\varphi_m)) \alpha_j \alpha_k$$

$$+ ((u'' s - s'') - gs') \sum_{1 \leq j, k \leq n} \partial_j (-v_{t_0, \e} \circ \varphi_m) \bar{\partial}_k (-v_{t_0, \e} \circ \varphi_m) \alpha_j \alpha_k.$$

We omit composite item $(-v_{t_0, \e} \circ \varphi_m)$ after $s' - su'$ and $(u'' s - s'') - gs'$ in the above equalities.

Let $g = \frac{u'' s - s''}{s'} \alpha(-v_{t_0, \e} \circ \varphi_m)$. It follows that $\eta + g^{-1} = (s + \frac{s'}{u'' s - s''}) \circ (-v_{t_0, \e} \circ \varphi_m)$.
Because of \( v'_{t_0, \varepsilon} \geq 0 \) and \( s' - su' = 1 \), using inequalities 4.2 we have
\[
\int_{D_v} (\eta + g^{-1})(\bar{\partial}_v^* \alpha)^2 e^{-\Theta} d\lambda_n \geq \int_{D_v} (v''_{t_0, \varepsilon} \circ \varphi_m) |\alpha_v(\bar{\partial} \bar{\varphi}_m)^2| e^{-\Theta} d\lambda_n. \tag{4.6}
\]
Let \( \lambda = \bar{\partial}(1 - v'_{t_0, \varepsilon}(\varphi_m)|F|) \). By the definition of contraction, Cauchy-Schwarz inequality, and inequality 4.0 it follows that
\[
|\langle \lambda, \alpha \rangle_{D_v, \varphi} |^2 = |\langle (v''_{t_0, \varepsilon} \circ \varphi_m) \bar{\partial} \bar{\varphi}_m F, \alpha \rangle_{D_v, \varphi} |^2 \\
\leq \int_{D_v} (v''_{t_0, \varepsilon} \circ \varphi_m)^2 e^{-\varphi} d\lambda_n \int_{D_v} (v''_{t_0, \varepsilon} \circ \varphi_m) |\alpha_v(\bar{\partial} \bar{\varphi}_m)^2| e^{-\Theta} d\lambda_n \\
\leq \int_{D_v} (v''_{t_0, \varepsilon} \circ \varphi_m)^2 e^{-\varphi} d\lambda_n (\eta + g^{-1}) |\alpha_v(\bar{\partial} \bar{\varphi}_m)^2| e^{-\Theta} d\lambda_n. \tag{4.7}
\]
Let \( \mu := (\eta + g^{-1})^{-1} \). Using Lemma 2.11b we have locally \( L^1 \) function \( u_{v, t_0, m, \varepsilon} \) on \( D_v \) such that \( \bar{\partial} u_{v, t_0, m, \varepsilon} = \lambda \), and
\[
\int_{D_v} |u_{v, t_0, m, \varepsilon}|^2 (\eta + g^{-1})^{-1} e^{-\Theta} d\lambda_n \leq \int_{D_v} (v''_{t_0, \varepsilon} \circ \varphi_m)|F|^2 e^{-\varphi} d\lambda_n. \tag{4.8}
\]
Let \( \mu_1 = e^{v''_{t_0, \varepsilon} \varphi_m}, \tilde{\mu} = \mu_1 e^\varphi \). Assume that we can choose \( \eta \) and \( \phi \) such that \( \bar{\partial} \tilde{\mu} \leq \mathcal{C}(\eta + g^{-1})^{-1} = \mu_1 \), where \( \mathcal{C} = 1 \).
Note that \( v''_{t_0, \varepsilon}(\varphi_m) \geq \varphi_m \). Then it follows that
\[
\int_{D_v} |u_{v, t_0, m, \varepsilon}|^2 d\lambda_n \leq \int_{D_v} |u_{v, t_0, m, \varepsilon}|^2 \mu_1 e^\varphi e^{-\varphi_m - \varphi} d\lambda_n = \int_{D_v} |u_{v, t_0, m, \varepsilon}|^2 \tilde{\mu} e^{-\varphi_m} d\lambda_n. \tag{4.9}
\]
Using inequalities 4.8 and 4.9 we obtain that
\[
\int_{D_v} |u_{v, t_0, m, \varepsilon}|^2 d\lambda_n \leq \mathcal{C} \int_{D_v} (v''_{t_0, \varepsilon} \varphi_m)|F|^2 e^{-\Theta} d\lambda_n,
\]
under the assumption \( \tilde{\mu} \leq \mathcal{C}(\eta + g^{-1})^{-1} \).
As \( -v''_{t_0, \varepsilon} \varphi_m(D_v) \subset \subset (0, t_0 + 1) \) and \( \{\varphi_m\}_{m=1,2,...} \) is decreasing, then it is clear that
\[
-v''_{t_0, \varepsilon} \varphi_m(D_v) \subset \subset K_{t_0} \subset \subset (0, t_0 + 1) \tag{4.10}
\]
where \( K_{t_0} \) is independent of \( m \) and \( \varepsilon \in (0, \frac{1}{2}) \). As \( u \) is positive and smooth on \((0, +\infty)\), it follows that \( \phi \) is uniformly bounded on \( D_v \) independent of \( m \).
As \( \text{Supp}(v''_{t_0, \varepsilon}) \subset \subset (-t_0 - 1, -t_0) \), then it is clear that \( (v''_{t_0, \varepsilon} \varphi_m)|F|^2 e^{-\varphi_m} \) are uniformly bounded on \( D_v \) independent of \( m \). Therefore \( \int_{D_v} (v''_{t_0, \varepsilon} \varphi_m)|F|^2 e^{-\Theta} d\lambda_n \) are uniformly bounded independent of \( m \), for any given \( v, t_0, \varepsilon \).
By weakly compactness of the unit ball of \( L^2(D_v) \) and dominated convergence theorem, when \( m \to +\infty \), it follows that the weak limit of some weakly convergent subsequence of \( \{u_{v, t_0, m, \varepsilon}\}_m \) gives an \( (n, 0) \)-form \( u_{v, t_0, \varepsilon} \) on \( D_v \) satisfying
\[
\int_{D_v} |u_{v, t_0, \varepsilon}|^2 d\lambda_n \leq \frac{\mathcal{C}}{e^{-A_{t_0}}} \int_{D_v} (v''_{t_0, \varepsilon} \varphi_m)|F|^2 e^{-\varphi} d\lambda_n, \tag{4.11}
\]
where \( A_{t_0} := \inf_{t \geq t_0} \{u(t)\} \).
As \( \varphi_m \) is decreasingly convergent to \( \varphi \) on \( \Delta^n \), and \( \varphi(o) = -\infty \), then for any given \( t_0 \) there exists \( m_0 \) and a neighbourhood \( U_0 \) of \( o \in D_v \) on \( \Delta^n \), such that for any \( m \geq m_0 \) and \( \varepsilon < 1 \), \( \nu_{t_0,\varepsilon} \circ \varphi_m|_{U_0} = 0 \). It follows that

\[
\tilde{\partial}u_{v,t_0,m,\varepsilon}|_{U_0} = \lambda|_{U_0} = \tilde{\partial}[(1 - \nu_{t_0,\varepsilon}(\varphi_m))F]|_{U_0} = -(\nu_{t_0,\varepsilon} \circ \varphi_m)F\tilde{\partial}\varphi_m|_{U_0} = 0.
\]

That is to say \( u_{v,t_0,m,\varepsilon}|_{U_0} \) are all holomorphic. Therefore \( u_{v,t_0,\varepsilon}|_{U_0} \) is holomorphic.

Recall that the integrals \( \int_{D_v}|u_{v,t_0,m,\varepsilon}|^2d\lambda_n \) have a uniform bound independent of \( m \), then we can choose a subsequence with respect to \( m \) from the chosen weakly convergent subsequence of \( u_{v,t_0,m,\varepsilon} \), such that the subsequence is uniformly convergent on any compact subset of \( U_0 \), and we still denote the subsequence by \( u_{v,t_0,m,\varepsilon} \) without ambiguity.

By above arguments, it follows that the right hand side of inequality \( \text{[1.3]} \) are uniformly bounded independent of \( m \) and \( \varepsilon \in (0, \frac{1}{2}) \). By inequality \( \text{[1.3]} \) it follows that \( \int_{D_v}|u_{v,t_0,m,\varepsilon}|^2(\eta + g^{-1})^{-1}e^{-\varphi_m}d\lambda_n \) are uniformly bounded independent of \( m \) and \( \varepsilon \in (0, \frac{1}{2}) \).

Using inequality \( \text{[4.10]} \) we obtain that

\[
(\eta + g^{-1})^{-1} = (s(-\nu_{t_0,\varepsilon} \circ \varphi_m) + \frac{s^2}{u''s - s''} \circ (-\nu_{t_0,\varepsilon} \circ \varphi_m))^{-1}
\]

and \( e^{-\phi} = e^{-u(\nu_{t_0,\varepsilon} \circ \varphi_m)} \) have positive uniform lower bounds independent of \( m \) and \( \varepsilon \in (0, \frac{1}{2}) \). Then the integrals

\[
\int_{K_0}|u_{v,t_0,m,\varepsilon}|^2e^{-\varphi_m}d\lambda_n
\]

have a uniform upper bound independent of \( m \) and \( \varepsilon \in (0, \frac{1}{2}) \) for any given compact set \( K_0 \subset \subset U_0 \cap D_v \), and

\[
\tilde{\partial}u_{v,t_0,m,\varepsilon}|_{U_0} = 0.
\]

As \( \varphi_{m'} \leq \varphi_m \) where \( m' \geq m \), it follows that

\[
|u_{v,t_0,m',\varepsilon}|^2e^{-\varphi_m} \leq |u_{v,t_0,m',\varepsilon}|^2e^{-\varphi_{m'}}.
\]

Then for any given compact set \( K_0 \subset \subset U_0 \cap D_v \), \( \int_{K_0}|u_{v,t_0,m',\varepsilon}|^2e^{-\varphi_m}d\lambda_n \) have a uniform bound independent of \( m \) and \( m' \). It is clear that for any given compact set \( K_0 \subset \subset U_0 \cap D_v \), the integrals \( \int_{K_0}|u_{v,t_0,\varepsilon}|^2e^{-\varphi_m}d\lambda_n \) have a uniform bound independent of \( m \) and \( \varepsilon \in (0, \frac{1}{2}) \). Therefore the integrals \( \int_{K_0}|u_{v,t_0,\varepsilon}|^2e^{-\varphi}d\lambda_n \) have a uniform bound independent of \( \varepsilon \in (0, \frac{1}{2}) \), for any given compact set \( K_0 \subset \subset U_0 \cap D_v \) containing \( o \).

In summary, we have \( |u_{v,t_0,\varepsilon}|^2e^{-\varphi} \) is integrable near \( o \), and \( \tilde{\partial}u_{v,t_0,\varepsilon} = 0 \) near \( o \). That is to say

\[
(u_{v,t_0,\varepsilon}, o) \in \mathcal{I}(\varphi)_o.
\]

Let \( F_{v,t_0,\varepsilon} := (1 - \nu_{t_0,\varepsilon} \circ \varphi)F - u_{v,t_0,\varepsilon} \). By inequality \( \text{[4.11]} \) and \( (u_{v,t_0,\varepsilon}, o) \in \mathcal{I}(\varphi)_o \), it follows that \( F_{v,t_0,\varepsilon} \) is a holomorphic function on \( D_v \) satisfying \( (F_{v,t_0,\varepsilon} - F, o) \in \mathcal{I}(\varphi)_o \), and

\[
\int_{D_v}|F_{v,t_0,\varepsilon} - (1 - \nu_{t_0,\varepsilon} \circ \varphi)F|^2d\lambda_n
\]

\[
\leq C \int_{D_v}(\nu_{t_0,\varepsilon} \circ \varphi)|F|^2e^{-\varphi}d\lambda_n.
\]
Given $t_0$ and $D_v$, it is clear that $(v'_{t_0,\varepsilon} \circ \varphi)|F|^2 e^{-\varphi}$ have a uniform bound on $D_v$ independent of $\varepsilon$. Then the integrals $\int_{D_v} (v'_{t_0,\varepsilon} \circ \varphi)|F|^2 e^{-\varphi} d\lambda_n$ have a uniform bound independent of $\varepsilon$, for any given $t_0$ and $D_v$. As $|(1 - v'_{t_0,\varepsilon} \circ \varphi)|F|^2$ have a uniform bound on $D_v$ independent of $\varepsilon$, it follows that the integrals $\int_{D_v} (1 - v'_{t_0,\varepsilon} \circ \varphi)|F|^2 d\lambda_n$ have a uniform bound independent of $\varepsilon$, for any given $t_0$ and $D_v$.

As
\[
\int_{D_v} |F_{v,t_0,\varepsilon}|^2 d\lambda_n 
\leq \int_{D_v} |F_{v,t_0,\varepsilon} - (1 - v'_{t_0,\varepsilon} \circ \varphi)|F|^2 d\lambda_n + \int_{D_v} |(1 - v'_{t_0,\varepsilon} \circ \varphi)|F|^2 d\lambda_n \quad (4.13)
\]
\[
\leq \frac{C}{e^{A t_0}} \int_{D_v} (v'_{t_0,\varepsilon} \circ \varphi)|F|^2 e^{-\varphi} d\lambda_n + \int_{D_v} |(1 - v'_{t_0,\varepsilon} \circ \varphi)|F|^2 d\lambda_n,
\]
then $\int_{D_v} |F_{v,t_0,\varepsilon}|^2 d\lambda_n$ have a uniform bound independent of $\varepsilon$.

As $\partial F_{v,t_0,\varepsilon} = 0$ when $\varepsilon \to 0$ and the unit ball of $L^2(D_v)$ is weakly compact, it follows that the weak limit of some weakly convergent subsequence of $\{F_{v,t_0,\varepsilon}\}$ gives us a holomorphic $(n, 0)$-form $F_{v,t_0}$ on $\Delta^n$. Then we can also choose a subsequence of the weakly convergent subsequence of $\{F_{v,t_0,\varepsilon}\}$, such that the chosen sequence is uniformly convergent on any compact subset of $D_v$, denoted by $\{F_{v,t_0,\varepsilon}\}$ without ambiguity.

For any given compact subset $K_0$ on $D_v$, $F_{v,t_0,\varepsilon}$, $(1 - v'_{t_0,\varepsilon} \circ \varphi)|F|$ and $(v''_{t_0,\varepsilon} \circ \varphi)|F|^2 e^{-\varphi}$ have uniform bounds on $K_0$ independent of $\varepsilon$.

As the integrals $\int_{K_0} |u_{v,t_0,\varepsilon}|^2 e^{-\varphi} d\lambda_n$ have a uniform bound independent of $\varepsilon \in (0, \frac{1}{8})$, for any given compact set $K_0 \subset \subset U_0 \cap D_v$ containing $o$, it follows that
\[
(F_{v,t_0} - (1 - b_{t_0}(\varphi))F, o) \in \mathcal{I}(\varphi)_o.
\]

Using the dominated convergence theorem on any compact subset $K$ of $D_v$ and inequality $4.12$ we obtain
\[
\int_K |F_{v,t_0} - (1 - b_{t_0}(\varphi))F|^2 d\lambda_n 
\leq \frac{C}{e^{A t_0}} \int_{D_v} (\mathbb{I}_{-t_0 - 1 < t < -t_0} \circ \varphi)|F|^2 e^{-\varphi} d\lambda_n. \quad (4.14)
\]

It suffices to find $\eta$ and $\phi$ such that $|\eta + g^{-1}| \leq C e^{-\varphi_m} e^{-\phi} = C \mu^{-1}$ on $D_v$.

As $\eta = s(-v_{t_0,\varepsilon} \circ \varphi_m)$ and $\phi = u(-v_{t_0,\varepsilon} \circ \varphi_m)$, we have $(\eta + g^{-1})e^{v_{t_0,\varepsilon} \circ \varphi_m} e^{\phi} = (s + \frac{s^2}{u'' e - s''}) e^{-t} e^u \circ (-v_{t_0,\varepsilon} \circ \varphi_m)$.

Summarizing the above discussion about $s$ and $u$, we are naturally led to a system of ODEs:
\[
\begin{align*}
1). \quad & (s + \frac{s^2}{u'' s - s''}) e^{u-t} = C, \\
2). \quad & s' - su' = 1,
\end{align*}
\]
where $t \in [0, +\infty)$, and $C = 1$.

It is not hard to solve the ODE system $4.13$ (details see the following Remark) and get $u = -\log(1 - e^{-t})$ and $s = \frac{t}{1 - e^{-t}} - 1$. It follows that $s \in C^\infty((0, +\infty))$ satisfies $s \geq 0$, $\lim_{t \to +\infty} u(t) = 0$ and $u \in C^\infty((0, +\infty))$ satisfies $u'' s - s'' > 0$. 

As $u = -\log(1 - e^{-t})$ is decreasing with respect to $t$, then
\[
\frac{C}{e^{A_{t_0}}} = \frac{1}{\exp \inf_{t \geq t_0} u(t)} = \sup_{t \geq t_0} \frac{1}{e^{u(t)}} = \sup_{t \geq t_0} (1 - e^{-t}) = 1,
\]
therefore we are done. Now we obtain Proposition 3.1.

**Remark 4.1.** In fact, we can solve the equation 4.15 as follows:

By 2) of equation 4.15, we have $su'' - s'' = -su'$. Then 1) of equation 4.15 can be change into
\[
(s - s')e^{u-t} = C,
\]
which is
\[
\frac{su' - s'}{u'}e^{u-t} = C.
\]
By 2) of equation 4.15, we have
\[
C = \frac{su' - s'}{u'}e^{u-t} = \frac{-1}{u'}e^{u-t},
\]
which is
\[
\frac{de^{-u}}{dt} = -u' e^{-u} = \frac{e^{-t}}{C}.
\]
Note that 2) of equation 4.15 is equivalent to $\frac{d(se^{-u})}{dt} = e^{-u}$. As $s \geq 0$ and $\lim_{t \to +\infty} u = 0$, it follows that the solution
\[
\begin{cases}
   u = -\log(1 - e^{-t}), \\
   s = \frac{t+e^{-t}-1}{1-e^{-t}},
\end{cases}
\]

4.2. Proof of Proposition 3.4.

For the sake of completeness, we recall our proof in [15] (see also [17]) with some modifications.

As $D_v \subset \Delta^n \subset C^n$, then there exist negative smooth plurisubharmonic functions $\{\varphi_m\}_{m=1,2,\ldots}$ and smooth functions $\{\Psi_m\}_{m=1,2,\ldots}$ on a neighborhood of $\overline{D_v}$, such that

1. $\{\varphi_m + \Psi_m\}_{m=1,2,\ldots}$ and $\{\varphi_m + 2\Psi_m\}_{m=1,2,\ldots}$ are negative smooth plurisubharmonic functions;

2. the sequence $\{\varphi_m\}_{m=1,2,\ldots}$ is decreasingly convergent to $\varphi$;

3. the sequence $\{\varphi_m + \Psi_m\}_{m=1,2,\ldots}$ is decreasingly convergent to $\varphi + \Psi$;

on a smaller neighborhood of $\overline{D_v}$, when $m \to +\infty$.

Let $\eta = s(-v_{l_0,\epsilon} \circ \Psi_m)$ and $\phi = u(-v_{l_0,\epsilon} \circ \Psi_m)$, where $s \in C^\infty((0, +\infty))$ satisfies $s \geq 1$, and $u \in C^\infty((0, +\infty))$, such that $u''s - s'' > 0$, and $s' - u's = 1$.

Let $\Phi := \varphi_m + \Psi_m + \phi$.

Now let $\alpha = \sum_{j=1}^{n} \alpha_j d\bar{z}^j \in \text{Dom}_{D_v}(\partial) \cap \text{Ker}(\partial) \cap C^\infty((0,1))(\overline{D_v})$. By Cauchy-Schwarz inequality, it follows that
\[
2\text{Re}(\partial \Phi, \alpha_l)(\partial \eta)_{l,\Phi} \geq -\int_{D_v} g^{-1}|\partial \Phi|^2 e^{-\Phi} d\lambda_n
\]
\[
+ \sum_{j,k=1}^{n} \int_{D_v} (-g(\partial_j \eta)\partial_k \eta)\alpha_j \overline{\alpha_k} e^{-\Phi} d\lambda_n.
\]
(4.16)
Using Lemma 2.17 and inequality 4.16 since \( s \geq 0 \) and \( \varphi_m \) is a plurisubharmonic function on \( \mathbb{T}_v \), we get

\[
\int_{D_v} (\eta + g^{-1})|\bar{\partial}\alpha|^2e^{-\Phi}d\lambda_n \\
\geq \sum_{j,k=1}^n \int_{D_v} (-\partial_j \partial_k \eta + \eta \partial_j \partial_k \Phi - g(\partial_j \eta)\partial_k \eta)\alpha_j \alpha_k e^{-\Phi}d\lambda_n \\
= \sum_{j,k=1}^n \int_{D_v} (-\partial_j \partial_k \eta + \eta \partial_j \partial_k \eta + \eta \partial_j \partial_k (\Psi_m + \varphi_m) - g(\partial_j \eta)\partial_k \eta)\alpha_j \alpha_k e^{-\Phi}d\lambda_n,
\]

(4.17)

where \( g \) is a positive continuous function on \( D_v \). We need some calculations to determine \( g \).

We have

\[
\partial_j \partial_k \eta = -s'(-v_{t_0,\varepsilon} \circ \Psi_m)\partial_j \partial_k (v_{t_0,\varepsilon} \circ \Psi_m) \\
+ s''(-v_{t_0,\varepsilon} \circ \Psi_m)\partial_j (v_{t_0,\varepsilon} \circ \Psi_m)\partial_k (v_{t_0,\varepsilon} \circ \Psi_m),
\]

(4.18)

and

\[
\partial_j \partial_k \Phi = -u'(-v_{t_0,\varepsilon} \circ \Psi_m)\partial_j \partial_k v_{t_0,\varepsilon} \circ \Psi_m \\
+ u''(-v_{t_0,\varepsilon} \circ \Psi_m)\partial_j (v_{t_0,\varepsilon} \circ \Psi_m)\partial_k (v_{t_0,\varepsilon} \circ \Psi_m),
\]

(4.19)

for any \( j, k (1 \leq j, k \leq n) \).

We have

\[
\sum_{1 \leq j, k \leq n} (-\partial_j \partial_k \eta + \eta \partial_j \partial_k \Phi - g(\partial_j \eta)\partial_k \eta)\alpha_j \alpha_k \\
= (s' - su') \sum_{1 \leq j, k \leq n} \partial_j \partial_k (v_{t_0,\varepsilon} \circ \Psi_m)\alpha_j \alpha_k \\
+ ((u''s - s'') - gs'') \sum_{1 \leq j, k \leq n} \partial_j (-v_{t_0,\varepsilon} \circ \Psi_m)\partial_k (-v_{t_0,\varepsilon} \circ \Psi_m)\alpha_j \alpha_k \\
= (s' - su') \sum_{1 \leq j, k \leq n} ((v'_{t_0,\varepsilon} \circ \Psi_m)\partial_j \partial_k \Psi_m + (v''_{t_0,\varepsilon} \circ \Psi_m)\partial_j (\Psi_m)\partial_k (\Psi_m))\alpha_j \alpha_k \\
+ ((u''s - s'') - gs'') \sum_{1 \leq j, k \leq n} \partial_j (-v_{t_0,\varepsilon} \circ \Psi_m)\partial_k (-v_{t_0,\varepsilon} \circ \Psi_m)\alpha_j \alpha_k.
\]

(4.20)

We omit composite item \( (-v_{t_0,\varepsilon} \circ \Psi_m) \) after \( s' - su' \) and \( (u''s - s'') - gs'' \) in the above equalities.

Since \( \varphi_m + \Psi_m \) and \( \varphi_m + 2\Psi_m \) are plurisubharmonic on \( \mathbb{T}_v \) and \( 0 \leq v'_{t_0,\varepsilon} \circ \Psi_m \leq 1 \), we have

\[
(1 - v'_{t_0,\varepsilon} \circ \Psi_m)\sqrt{-1}\bar{\partial}(\varphi_m + \Psi_m) + (v'_{t_0,\varepsilon} \circ \Psi_m)\sqrt{-1}\bar{\partial}(\varphi_m + 2\Psi_m) \geq 0,
\]

(4.21)

on \( \mathbb{T}_v \), which means that

\[
\sqrt{-1}\bar{\partial}(\varphi_m + \Psi_m) + (v'_{t_0,\varepsilon} \circ \Psi_m)\sqrt{-1}\bar{\partial}\Psi_m \geq 0,
\]

(4.22)
on $\overline{D_v}$.

Let $g = \frac{u''}{u'' - u'} \circ (-v_{t_0, \varepsilon} \circ \Psi_m)$. It follows that $\eta + g^{-1} = (s + \frac{\varepsilon^2}{u'' - u'}) \circ (-v_{t_0, \varepsilon} \circ \Psi_m)$.

Because of $v_{t_0, \varepsilon} \geq 0$ and $s' - su' = 1$, using inequalities (4.17), (4.22) and (4.20) we have

$$
\int_{D_v} (\eta + g^{-1})|\overline{\partial}^* 2 - \Phi|e^\Phi d\lambda_n \geq \int_{D_v} (v''_{t_0, \varepsilon} \circ \Psi_m)|\alpha_t (\overline{\partial} \Psi_m)^2| e^{-\Phi} d\lambda_n. \quad (4.23)
$$

Let $\lambda = \overline{\partial} (1 - v''_{t_0, \varepsilon} (\Psi_m))^2 |F^2|$. By the definition of contraction, Cauchy-Schwarz inequality and inequality (4.23) it follows that

$$
|\lambda, \alpha)_{D_v, \Phi}|^2 = |((v''_{t_0, \varepsilon} \circ \Psi_m) \Phi F^2, \alpha)_{D_v, \Phi}|^2
$$

$$
\leq (\int_{D_v} (v''_{t_0, \varepsilon} \circ \Psi_m)|F^2|^2 e^{-\Phi} d\lambda_n) \int_{D_v} (v''_{t_0, \varepsilon} \circ \Psi_m)|\alpha_t (\overline{\partial} \Psi_m)^2| e^{-\Phi} d\lambda_n. \quad (4.24)
$$

Let $\mu := (\eta + g^{-1})^{-1}$. Using Lemma (2.18) we have locally $L^1$ function $u_{v, t_0, m, \varepsilon}$ on $D_v$ such that $\overline{\partial}_v u_{v, t_0, m, \varepsilon} = \lambda$, and

$$
\int_{D_v} |u_{v, t_0, m, \varepsilon}|^2 (\eta + g^{-1})^{-1} e^{-\Phi} d\lambda_n \leq \int_{D_v} (v''_{t_0, \varepsilon} \circ \Psi_m)|F^2|^2 e^{-\Phi} d\lambda_n. \quad (4.25)
$$

Let $\mu_1 = e^{v_{t_0, \varepsilon} \circ \Psi_m}$, $\tilde{\mu} = \mu_1 e^\Phi$. Assume that we can choose $\eta$ and $\phi$ such that $\tilde{\mu} \leq C (\eta + g^{-1})^{-1} = \mu$, where $C = 1$

Note that $v_{t_0, \varepsilon} (\Psi_m) \geq \Psi_m$. Then it follows that

$$
\int_{D_v} |u_{v, t_0, m, \varepsilon}|^2 e^{-\Phi} d\lambda_n \leq \int_{D_v} |u_{v, t_0, m, \varepsilon}|^2 \mu_1 e^\phi e^{-\Psi_m - \varphi_m - \phi} d\lambda_n
$$

$$
= \int_{D_v} |u_{v, t_0, m, \varepsilon}|^2 \tilde{\mu} e^{-\Phi} d\lambda_n. \quad (4.26)
$$

Using inequalities (4.25) and (4.26) we obtain that

$$
\int_{D_v} |u_{v, t_0, m, \varepsilon}|^2 e^{-\Phi} d\lambda_n \leq C \int_{D_v} (v''_{t_0, \varepsilon} \circ \Psi_m)|F^2|^2 e^{-\Phi} d\lambda_n,
$$

under the assumption $\tilde{\mu} \leq C (\eta + g^{-1})^{-1}$.

As $-v_{t_0, \varepsilon} \circ \Psi_m (D_v) \subset (-\infty, t_0 + 1)$, then it is clear that

$$
-v_{t_0, \varepsilon} \circ \Psi_m (\overline{D_v}) \subset (-\infty, K_{t_0}) \quad (4.27)
$$

where $K_{t_0}$ is independent of $m$ and $\varepsilon \in (0, \frac{1}{2} B_0)$. As $u$ is positive and smooth on $(-\infty, +\infty)$, it follows that $\phi$ is uniformly bounded on $\overline{D_v}$ independent of $m$.

As $\text{Supp}(v''_{t_0, \varepsilon}) \subset (-t_0 - B_0, -t_0)$ and $|F^2|^2 e^{-\Psi_m} \leq |F^2|^2 e^{-\phi} \leq 1$, then it is clear that $(v''_{t_0, \varepsilon} \circ \Psi_m)|F^2|^2 e^{-\Psi_m - \varphi_m}$ are uniformly bounded on $\overline{D_v}$ independent of $m$. Therefore the integrals $\int_{D_v} (v''_{t_0, \varepsilon} \circ \Psi_m)|F^2|^2 e^{-\Phi} d\lambda_n$ are uniformly bounded independent of $m$, for any given $v$, $t_0$, $\varepsilon$.

By weakly compactness of the unit ball of $L^2_p (D_v)$ and dominated convergence theorem, when $m \to +\infty$, it follows that the weak limit of some weakly convergent
subsequence of \( \{u_{v,t_0,m,\varepsilon}\}_m \) gives function \( u_{v,t_0,\varepsilon} \) on \( D_v \) satisfying
\[
\int_{D_v}|u_{v,t_0,\varepsilon}|^2 e^{-\varphi}d\lambda_n \leq \frac{C}{\varepsilon^{A_0}} \int_{D_v}(v_{t_0,\varepsilon}'(\Psi_m))|F|^2e^{-\varphi}\Psi d\lambda_n, \tag{4.28}
\]
where \( A_0 := \inf_{t \geq t_0} \{u(t)\} \).

Let \( F_{v,t_0,m,\varepsilon} := (1 - v_{t_0,\varepsilon}'(\Psi_m))F^2 - u_{v,t_0,m,\varepsilon} \), which is a holomorphic function on \( D_v \).

As \( |F|^2e^{-\varphi_m} \leq |F|^2e^{-\varphi} \leq 1 \), then the integrals \( \int_{D_v}(1 - v_{t_0,\varepsilon}'(\Psi_m))F^2e^{-\varphi_m}d\lambda_n \) have a uniform bound independent of \( m \). Recall that the integrals
\[
\int_{D_v}|u_{v,t_0,m,\varepsilon}|^2 e^{-\varphi_m}d\lambda_n
\]
have a uniform bound independent of \( m \), then the integrals
\[
\int_{D_v}|F_{v,t_0,m,\varepsilon}|^2 e^{-\varphi_m}d\lambda_n
\]
have a uniform bound independent of \( m \). Therefore we can choose a subsequence of \( \{F_{v,t_0,m,\varepsilon}\}_{m=1,2,...} \) from the chosen weakly convergent subsequence of
\[
(1 - v_{t_0,\varepsilon}'(\Psi_m))F^2 - u_{v,t_0,m,\varepsilon},
\]
such that the subsequence is uniformly convergent on any compact subset of \( D_v \), and we still denote the subsequence by \( \{F_{v,t_0,m,\varepsilon}\}_{m=1,2,...} \) without ambiguity. Let
\[
F_{v,t_0,\varepsilon} := \lim_{m \to \infty} F_{v,t_0,m,\varepsilon}.
\]

By above arguments, it follows that the right hand side of inequality \ref{2.26} are uniformly bounded independent of \( m \) and \( \varepsilon \in (0, \frac{1}{8}B_0) \). By inequality \ref{4.25} it follows that the integrals
\[
\int_{D_v}|u_{v,t_0,m,\varepsilon}|^2 (\eta + g^{-1})^{-1}e^{-\phi - \varphi_m - \Psi_m}d\lambda_n
\]
are uniformly bounded independent of \( m \) and \( \varepsilon \in (0, \frac{1}{8}B_0) \).

Using inequality \ref{4.27}, we obtain that
\[
(\eta + g^{-1})^{-1} = (s(-v_{t_0,\varepsilon} \circ \Psi_m) + \frac{s^2}{u'^2 - s^2} \circ (-v_{t_0,\varepsilon} \circ \Psi_m))^{-1}
\]
and \( e^{-\phi} = e^{-u(-v_{t_0,\varepsilon} \circ \Psi_m)} \) have positive uniform lower bounds independent of \( m \) and \( \varepsilon \in (0, \frac{1}{8}B_0) \). Then the integrals
\[
\int_{K_0}|u_{v,t_0,m,\varepsilon}|^2 e^{-\varphi_m - \Psi_m}d\lambda_n
\]
have a uniform upper bound independent of \( m \) and \( \varepsilon \in (0, \frac{1}{8}B_0) \) for any given compact set \( K_0 \subset \subset D_v \).

As
\[
\text{Supp}(v_{t_0,\varepsilon}'(\Psi_m)) \subset \{\Psi_m > -t_0 - 1\},
\]
it follows that
\[
|v_{t_0,\varepsilon}'(\Psi_m)|^2 e^{-\Psi_m} \leq e^{t_0 + 1}.
\]
Furthermore, as
\[
|F|^4 e^{-\varphi_m} \leq |F|^4 e^{-\varphi} = e^{-2\max\{|\varphi - \log |F|^2|, 0\}} \leq 1
\]
then the integrals
\[
\int_{K_0} |v'_{t_0,\varepsilon}(\Psi_m)|F^2 e^{-\varphi_m} d\lambda_n
\]
have a uniform upper bound independent of \(m\) and \(\varepsilon \in (0, \frac{1}{8} B_0)\) for any given compact set \(K_0 \subset \subset D_v\). Therefore the integrals
\[
\int_{K_0} |F_{v,t_0,m,\varepsilon} - F^2 e^{-\varphi_m} d\lambda_n
\]
have a uniform upper bound independent of \(m\) and \(\varepsilon \in (0, \frac{1}{8} B_0)\) for any given compact set \(K_0 \subset \subset D_v\).

As \(\varphi_m + \Psi_m \leq \varphi_m + \Psi_m\) where \(m' \geq m\), it follows that
\[
|F_{v,t_0,m',\varepsilon} - F^2 e^{-\varphi_m+\Psi_m}| \leq |F_{v,t_0,m,\varepsilon} - F^2 e^{-\varphi_m+\Psi_m}|.
\]
By inequality (4.28) it follows that for any given compact set \(K_0 \subset \subset D_v\), the integrals
\[
\int_{K_0} |F_{v,t_0,m',\varepsilon} - F^2 e^{-\varphi_m+\Psi_m}| d\lambda_n
\]
have a uniform bound independent of \(m\) and \(m'\). Therefore for any given compact set \(K_0 \subset \subset \cap D_v\), the integrals \(\int_{K_0} |F_{v,t_0,\varepsilon} - F^2 e^{-\varphi_m+\Psi_m}| d\lambda_n\) have a uniform bound independent of \(m\) and \(\varepsilon \in (0, \frac{1}{8} B_0)\). It is clear that the integrals
\[
\int_{K_0} |F_{v,t_0,\varepsilon} - F^2 e^{-\varphi_m} d\lambda_n
\]
have a uniform upper bound independent of \(\varepsilon \in (0, \frac{1}{8} B_0)\) for any given compact set \(K_0 \subset \subset D_v\).

In summary, we have \(|F_{v,t_0,\varepsilon} - F^2 e^{-\varphi} - \Psi|\) is integrable near \(o\). That is to say
\[
(F_{v,t_0,\varepsilon} - F^2, o) \in \mathcal{I}(\varphi + \Psi)\).
\]
By inequality (4.28) it follows that \(F_{v,t_0,\varepsilon}\) is a holomorphic function on \(D_v\) satisfying \((F_{v,t_0,\varepsilon} - F^2, o) \in \mathcal{I}(\varphi)\), and
\[
\int_{D_v} |F_{v,t_0,\varepsilon} - (1 - v'_{t_0,\varepsilon} \circ \Psi) F^2 e^{-\varphi} d\lambda_n
\]
\[
\leq \frac{C}{e^{\lambda_m}} \int_{D_v} (v'_{t_0,\varepsilon} \circ \Psi)|F^2 e^{-\varphi} d\lambda_n.
\]

It suffices to find \(\eta\) and \(\phi\) such that \((\eta + g^{-1}) \leq C e^{-\Psi_m} e^{-\phi} = C e^{-\mu_1}\) on \(D_v\). As \(\eta = s(\nu_{t_0,\varepsilon} \circ \Psi_m)\) and \(\phi = u(\nu_{t_0,\varepsilon} \circ \Psi_m)\), we have \((\eta + g^{-1}) e^{\nu_{t_0,\varepsilon} \circ \Psi_m} e^{\phi} = (s + \frac{s^2}{s^2 + s^2}) e^{-t} e^{u(\nu_{t_0,\varepsilon} \circ \Psi_m)}\).

Summarizing the above discussion about \(s\) and \(u\), we are naturally led to a system of ODEs:
\[
1). \quad (s + \frac{s^2}{s^2 + s^2}) e^{u-t} = C,
\]
\[
2). \quad s' - su' = 1,
\]
where \(t \in [0, +\infty)\), and \(C = 1\).

It is not hard to solve the ODE system (4.32) (details see the following Remark) and get \(u = -\log(2 - e^{-t})\) and \(s(t) = \frac{2t + e^{-t}}{2 - e^{-t}}\). It follows that \(s \in C^\infty((0, +\infty))\) satisfies \(s \geq 1, u' \leq 0\) and \(u \in C^\infty((0, +\infty))\) satisfies \(u''s - s'' > 0\).
As \( u = - \log(2 - e^{-t}) \) is decreasing with respect to \( t \), then
\[
\frac{C}{e^{t_{t_0}}} = \frac{1}{\exp \inf_{t \geq t_0} u(t)} = \sup_{t \geq t_0} \frac{1}{e^{u(t)}} = \sup_{t \geq t_0} (2 - e^{-t}) = 2, 
\]
for any \( t_0 \geq 0 \), therefore we are done. Now we obtain Proposition 3.3.

**Remark 4.2.** In fact, we can solve the equation \((4.32)\) as follows:

By 2) of equation \((4.32)\), we have \( su'' - s'' = -s'u' \). Then 1) of equation \((4.32)\) can be change into
\[
(s - s')e^{u-t} = C, 
\]
which is
\[
\frac{su' - s'}{u'}e^{u-t} = C. 
\]
By 2) of equation \((4.32)\), we have
\[
C = \frac{su' - s'}{u'}e^{u-t} = -\frac{1}{u'}e^{u-t}, 
\]
which is
\[
\frac{de^{-u}}{dt} = -u'e^{-u} = \frac{e^{-t}}{C}. 
\]

Note that 2) of equation \((4.32)\) is equivalent to \( \frac{d(e^{-u})}{dt} = -e^{-u} \). As \( s \geq 1 \) and \( \lim_{t \to +\infty} u = 0 \), it follows that the solution
\[
\begin{cases} 
  u = -\log(2 - e^{-t}), \\
  s = \frac{2t + e^{-t}}{2 - e^{-t}}, 
\end{cases} 
\]

### 4.3. Proof of Proposition 3.3

Given \( t_0 \) and \( D_v \), it is clear that \( (v''_{t_0,\varepsilon} \circ \Psi)|F^2|e^{-\varphi-\Psi} \) have a uniform bound on \( D_v \), independent of \( \varepsilon \). Then the integrals \( \int_{D_v} (v''_{t_0,\varepsilon} \circ \Psi)|F^2|e^{-\varphi-\Psi}d\lambda \) have a uniform bound independent of \( \varepsilon \), for any given \( t_0 \) and \( D_v \).

As \( (v''_{t_0,\varepsilon} \circ \Psi)|F^2|e^{-\varphi} \) have a uniform bound on \( D_v \), independent of \( \varepsilon \), it follows that the integrals \( \int_{D_v} (1 - v'_{t_0,\varepsilon} \circ \Psi)|F^2|e^{-\varphi}d\lambda \) have a uniform bound independent of \( \varepsilon \), for any given \( t_0 \) and \( D_v \).

As
\[
\int_{D_v} |F_{v, t_0, \varepsilon}|^2 e^{-\varphi} d\lambda_n 
\]
\[
\leq \int_{D_v} |F_{v, t_0, \varepsilon} - (1 - v'_{t_0, \varepsilon} \circ \Psi)|F^2|e^{-\varphi}d\lambda_n + \int_{D_v} |(1 - v'_{t_0, \varepsilon} \circ \Psi)|F^2|e^{-\varphi}d\lambda_n 
\]
\[
\leq \frac{C}{e^{A_{t_0}}} \int_{D_v} (v''_{t_0, \varepsilon} \circ \Psi)|F^2|e^{-\varphi-\Psi}d\lambda_n + \int_{D_v} |(1 - v'_{t_0, \varepsilon} \circ \Psi)|F^2|e^{-\varphi}d\lambda_n, 
\]

then the integrals \( \int_{D_v} |F_{v, t_0, \varepsilon}|^2 e^{-\varphi} d\lambda_n \) have a uniform bound independent of \( \varepsilon \).

As \( \partial F_{v, t_0, \varepsilon} = 0 \) when \( \varepsilon \to 0 \) and the unit ball of \( L^2(D_v) \) is weakly compact, it follows that the weak limit of some weakly convergent subsequence of \( \{F_{v, t_0, \varepsilon}\}_\varepsilon \) gives us a holomorphic function \( F_{v, t_0} \) on \( \Delta^n \). Then we can also choose a subsequence of the weakly convergent subsequence of \( \{F_{v, t_0, \varepsilon}\}_\varepsilon \), such that the chosen sequence is uniformly convergent on any compact subset of \( D_v \), denoted by \( \{F_{v, t_0, \varepsilon}\}_\varepsilon \) without
ambiguity. For any given compact subset $K_0$ on $D_v$, $F_{v,t_0,\varepsilon}$, $|(1-v'_t\circ \Psi)F^2|e^{-\varphi}$ and $(v''_t\circ \Psi)|F^2|e^{-\varphi-\psi}$ have uniform bounds on $K_0$ independent of $\varepsilon$.

By inequality (4.30) it follows that

$$(F_{v,t_0}-F^{2},o)\in I(\varphi + \Psi).$$

Using the dominated convergence theorem on any compact subset $K$ of $D_v$ and Proposition 3.4, we obtain

$$\int_{K}|F_{v,t_0} - (1 - b_{t_0}(\Psi))F^2|e^{-\varphi}d\lambda_n \leq 2\int_{D_v}(|t_{-t_0-1<t<-t_0}\circ \Psi)|F^2|e^{-\varphi-\psi}d\lambda_n.$$  \hspace{1cm} (4.34)

Then Proposition 3.3 has thus been proved.

5. PROOFS OF THE MAIN RESULTS

5.1. Proof of Theorem 1.1

We will prove Theorem 1.1 by the methods of induction and contradiction, and by using dynamically $L^2$ extension theorem with negligible weight.

Let $\{p_j\}_{j=1,2,\cdots}$ be a sequence of positive numbers which are strictly decreasingly convergent to 1, when $j$ goes to infinity.

5.1.1. Step 1: Theorem 1.1 for dimension 1 case.

We first consider Theorem 1.1 for dimension 1 case, which is elementarily but revealing.

We choose $r_0$ small enough, such that $\{F = 0\} \cap \Delta_{r_0} \subset \{o\}$.

As $\int_{\Delta}|F|^2e^{-\varphi}d\lambda_1 < +\infty$, by Lemma (2.2) we have

$$\lim_{A \to +\infty} \mu(\{|p_j\|^2 \geq A\} u(A)) = 0.$$  \hspace{1cm} (5.1)

It is clear that, for any given $B > 0$, there exists $A > B$ and $z_A \in \Delta_{u(A)^{-1/2}}$, such that $e^{-\varphi(z_A)}|F(z_A)|^2 \leq A$. We can assume that $A > 10$.

Let $\psi = -\log 2$, then $\log |z' - z_A| + \psi < 0$.

Using Theorem (2.3) on $\Delta$, we obtain holomorphic function $F_A$ on $\Delta$ for each $A$ and $p_jA\varphi$ $(j_A \in \{1,2,\cdots\})$, such that $F_A|_{z_A} = F(z_A)$, and

$$\int_{\Delta}|F_A|^2e^{-p_jA\varphi}d\lambda_1 < 8\pi A.$$  \hspace{1cm} (5.1)

By the negativeness of $\varphi$ and $p_jA\varphi$, it follows that

$$\int_{\Delta}|F_A|^2d\lambda_1 < 8\pi A.$$  \hspace{1cm} (5.2)

Assume Theorem 1.1 for $n = 1$ is not true. Therefore

$$\int_{\Delta}|F|^2d\lambda = +\infty,$$

for any $r > 0$ and any $j \in \{1,2,\cdots\}$.

Since $\{F = 0\} \cap \Delta_{r_0} \subset \{o\}$, then it follows from inequality (4.1) that one can derive that $F/F_A$ is unbounded. Otherwise, the boundedness would imply the finiteness
of the integral of $|F|^2 e^{-p_j \varphi}$, according to inequality \ref{ineq:5.1}. This contradicts to the assumption. Then there exists a holomorphic function $h_A$ on $\Delta_{r_0}$, such that

1). $F_A|_{\Delta_{r_0}} = F|_{\Delta_{r_0}} h_A$;
2). $h_A(o) = 0$;
3). $h_A(z_A) = 1$.

By Lemma \ref{lem:2.3} it follows that

$$\int_{\Delta_{r_0}} |F_A|^2 d\lambda_1 > C_1 u(A),$$

where $C_1$ is independent of $A$.

It contradicts to

$$\int_{\Delta} |F_A|^2 d\lambda_1 < 8 \pi A.$$

We have thus proved Theorem \ref{thm:1.1} for $n = 1$.

5.1.2. Step 2: Theorem \ref{thm:1.1} for $n = k$.

Assume Theorem \ref{thm:1.1}

$$\int_{\Delta_r^k} |F|^2 e^{-p_j \varphi} d\lambda_k < +\infty,$$

for some $r > 0$, and

$$\int_{\Delta_r^k} |F|^2 e^{-p_j \varphi} d\lambda_k = +\infty,$$

for any $r > 0$ and $j \in \{1, 2, \cdots\}$.

Then the germ of the holomorphic function $F$ is in $I(\varphi)$ but is not in $(\cup_{j=1}^\infty I(p_j \varphi))_o$.

Using Proposition \ref{prop:2.15} we have a germ of analytic curve $\gamma$ through $o$ satisfying $\{F|_{\gamma} = 0\} = \{o\}$, such that for any germ of holomorphic function $g$ in $\cup_{j=1}^\infty I(p_j \varphi)$, and we also have a holomorphic function $h_g$ on $\gamma$ satisfying

$$h_g|_o = 0,$$

such that

$$g|_{\gamma} = F|_{\gamma} h_g.$$ \hfill (5.3)

Then we can choose biholomorphic map $i$ from a neighborhood of $\overline{\Delta^k \times \Delta^l}$ to a neighborhood $V_o \subset \Delta^k$ of $o$, which is small enough, with origin keeping $i(o) = o$, such that

1). $i^{-1}(\gamma)$ is a closed analytic curve of the neighborhood of $\overline{\Delta^k \times \Delta^l}$;
2). $i^{-1}(\gamma)$ satisfies the parametrization property as analytic curve $C$ in Remark \ref{rem:2.5}.

Note that

$$\int_{V_o} |F|^2 e^{-p_j \varphi} d\lambda_n = \int_{\Delta^r \times \Delta^l} |i^* (F)|^2 e^{-i^* (p_j \varphi)} i^* (d\lambda_n).$$

Then

$$\int_{\Delta_k^r} |F|^2 e^{-p_j \varphi} d\lambda_n = +\infty,$$

for any $r > 0$ and any $j \in \{1, 2, \cdots\}$, is equivalent to

$$\int_{\Delta^r \times \Delta^l} |i^* (F)|^2 e^{-i^* (p_j \varphi)} i^* (d\lambda_n) = +\infty,$$

for any $r > 0$ and any $j \in \{1, 2, \cdots\}$. 

As \( \int_{\Delta' \times \Delta''} |t^*(F)|^2 e^{-t^*(\varphi)} d\lambda_k < +\infty \), it follows from Lemma 2.2 that

\[
\liminf_{A \to +\infty} \mu(\{ z_1 \mid \int_{\pi^{-1}(z_1)} |t^*(F)|^2 e^{-t^*(\varphi)} d\lambda_{k-1} > A \}) u(A) = 0,
\]

where \( \pi \) is the projection in Remark 2.5.

It is clear that for any given \( B > 0 \), there exists \( A > B \), such that

\[
\{ z_1 \mid \int_{\pi^{-1}(z_1)} |t^*(F)|^2 e^{-t^*(\varphi)} d\lambda_{k-1} > A \}
\]

cannot contain \( \Delta_{u(A)}^{-1/2} \).

As \( \lambda_n(\{ t^*(p_1 \varphi) = -\infty \}) = 0 \), then \( \lambda_1(\{ z' | t^*(p_1 \varphi) |_{\pi^{-1}(z')} \equiv -\infty \}) = 0 \). Then for any given \( B > 0 \), there exists \( A > B \) and \( z_A \in \Delta_{u(A)}^{-1/2} \), satisfying

\[
\int_{\pi^{-1}(z_A)} |t^*(F)|^2 e^{-t^*(\varphi)} d\lambda_{k-1} \leq A,
\]

such that

\[
t^*(p_1 \varphi) |_{\pi^{-1}(z_A)} \not\equiv -\infty.
\]

We assume that \( A > e^{10} \).

5.1.3. Using dynamically \( L^2 \) extension theorems with negligible weight.

As Theorem 1.1 for \( n = k - 1 \) holds, and \( t^*(p_1 \varphi) |_{\pi^{-1}(z_A)} \not\equiv -\infty \), then there exists \( j_A \in \{ 1, 2, \cdots \} \), such that

\[
\int_{\pi^{-1}(z_A)} |t^*(F)|^2 e^{-t^*(p_{1A} \varphi)} d\lambda_{k-1} < 2A.
\]

Let \( \psi = -\log 2 \), then \( \log |z' - z_A| + \psi < 0 \). By Theorem 2.8 on \( \Delta' \times \Delta'' \), we obtain a holomorphic function \( F_A \) on \( \Delta' \times \Delta'' \) for each \( A \), such that \( F_A |_{\pi^{-1}(z_A)} = t^*(F) |_{\pi^{-1}(z_A)} \), and

\[
\int_{\Delta' \times \Delta''} |F_A|^2 e^{-t^*(p_{jA} \varphi)} d\lambda_k < 8\pi A.
\]

It follows form equality (5.3) that there exists a holomorphic function on \( \gamma \), denoted by \( h_A \), such that

\[
t_A F_A |_{\gamma} = F |_{\gamma} h_A,
\]

therefore,

\[
F_A |_{\pi^{-1}(\gamma)} = t^*(F) |_{\pi^{-1}(\gamma)} t^*(h_A),
\]

where \( h_A(0) = 0 \), \( t^*(h_A)(\pi^{-1}(\gamma) \cap \pi^{-1}(z_A)) = 1 \).

By the negativeness of \( \varphi \), it is clear that

\[
\int_{\Delta' \times \Delta''} |F_A|^2 d\lambda_1 < 8\pi A.
\]

Using equality (5.3) the condition \( |z_A| < u(A)^{-\frac{1}{2}} \), and Lemma 2.6 we have

\[
\int_{\pi^{-1}(\gamma)} |F_A|^2 \pi^* d\lambda_{\Delta'} > C_1 u(A),
\]

where \( C_1 > 0 \) is independent of \( A \) and \( F_A \). In our use of Lemma 2.6 \( z_A \) corresponds to \( a \) in Lemma 2.6, the function \( t^*(h_A) \) corresponds to \( f_a \) in Lemma 2.6, which does not correspond to \( h \) in Lemma 2.6 (actually \( t^*(F) |_{\pi^{-1}(\gamma)} \) corresponds to \( h \) in Lemma 2.6).
Using Lemma 2.7, we obtain
\[ \int_{\Delta' \times \Delta''} |F_A|^2 d\lambda_k \geq C_3 \int_{\gamma} |F_A|^2 |e_S|^2 d\lambda_{\gamma}, \]
where \( C_3 > 0 \) is independent of \( A \) and \( F_A \).

Therefore
\[ \int_{\Delta' \times \Delta''} |F_A|^2 d\lambda_k \geq C_1 C_3 u(A), \]
which contradicts to
\[ \int_{\Delta' \times \Delta''} |F_A|^2 d\lambda_k < 8\pi A, \]
for \( A \) large enough.

We have thus proved Theorem 1.1 for \( n = k \).

The proof of Theorem 1.1 is thus complete.

5.1.4. Some remarks of Theorem 1.1

Let \( \varphi \) be a negative plurisubharmonic function on \( \Delta^n \subset \mathbb{C}^n \), and \( \psi_j \) be a sequence of plurisubharmonic functions on \( \Delta^n \), which is increasingly convergent to \( \varphi \) on \( \Delta^n \), when \( j \to \infty \).

Without loss of generality, one can assume that \( \psi_1 \neq -\infty \).

In the proof of Theorem 1.1 replacing \( p_j \varphi \) by \( \psi_j \), and \( p_j A \varphi \) by \( \psi_j A \), one can obtain:

Let \( F \) be a holomorphic function on \( \Delta^n \), such that
\[ \int_{\Delta^n} |F|^2 e^{-\varphi} d\lambda_n < +\infty. \]

Then there exists a number \( j_0 \geq 1 \), such that
\[ \int_{\Delta^n} |F|^2 e^{-\psi_{j_0}} d\lambda_n < +\infty, \]
for some \( r \in (0, 1) \).

That is to say:
\[ \bigcup_{j=1}^{\infty} \mathcal{I}(\psi_j) = \mathcal{I}(\varphi). \]
(5.6)

In particular, let \( \psi_j = \varphi + \frac{1}{j} \varphi_0 \) in equality 5.6 then we get the following modified version of the strong openness conjecture which was conjectured in [24]:

Let \( \varphi \) be a negative plurisubharmonic function on \( \Delta^n \subset \mathbb{C}^n \), and \( \varphi_0 \neq -\infty \) be a negative plurisubharmonic function on \( \Delta^n \). Then
\[ \cup_{\varepsilon > 0} \mathcal{I}(\varphi + \varepsilon \varphi_0) = \mathcal{I}(\varphi). \]

5.2. Proof of Theorem 1.2

We prove Theorem 1.2 by contradiction: if Theorem 1.2 is not true, then there exists \( t_j \to +\infty \) (\( j \to +\infty \)), such that
\[ \lim_{j \to \infty} \int_{\Delta^n} \mathbb{I}_{\{-t_j-1 < \varphi < -t_j\}} |F|^2 e^{-\varphi} d\lambda_n = 0. \]
(5.7)
Let $D_v$ be a strongly pseudoconvex domain relatively compact in $\Delta^n$ containing $o$, it follows that
\[
\int_{\Delta^n} |F|_\nu e^{-\varphi} d\lambda_n \geq \int_{D_v} |F|_\nu e^{-\varphi} d\lambda_n. \tag{5.8}
\]
According to equality \[5.18\] and inequality \[5.21\] it follows that
\[
\lim_{j \to +\infty} \int_{D_v} |F|_\nu e^{-\varphi} d\lambda_n = 0. \tag{5.9}
\]
By Proposition \[3.1\] it follows that there exists $F_{v,t_j}$, which is a holomorphic function on $D_v$ satisfying:
\[
\int_{D_v} |F_{v,t_j} - (1 - b_{t_j}(\varphi))F|_\nu^2 d\lambda_n \leq \int_{D_v} |F|_\nu e^{-\varphi} d\lambda_n, \tag{5.10}
\]
and
\[
(F_{v,t_j} - F,o) \in I(\varphi,o). \]
As $|(1 - b_{t_j}(\varphi))F|$ on $D_v$ have a uniform bound independent of $j$, then $\int_{D_v} |(1 - b_{t_j}(\varphi))F|_\nu^2 d\lambda_n$ have a uniform bound independent of $j$.
According to equality \[5.18\] and inequality \[5.21\] it follows that $\int_{D_v} |F_{v,t_j} - (1 - b_{t_j}(\varphi))F|_\nu^2 d\lambda_n$ have a uniform bound independent of $j$.
Using
\[
(\int_{D_v} |F_{v,t_j}|_\nu^2 d\lambda_n)^\frac{1}{2} \leq (\int_{D_v} |F_{v,t_j} - (1 - b_{t_j}(\varphi))F|_\nu^2 d\lambda_n)^\frac{1}{2} + (\int_{D_v} |(1 - b_{t_j}(\varphi))F|_\nu^2 d\lambda_n)^\frac{1}{2}, \tag{5.11}
\]
we have $\int_{D_v} |F_{v,t_j}|_\nu^2 d\lambda_n$ have a uniform bound independent of $j$. Then there is a subsequence of $F_{v,t_j}$ denoted by $F_{v,t_j}$ without ambiguity, which is convergent to a holomorphic function $F_v$ uniformly on any compact subset of $D_v$. Then for any $K \subset \subset D_v$, we have
\[
\int_K |F_v|_\nu^2 d\lambda_n \leq \liminf_{j \to +\infty} \int_{D_v} |F_{v,t_j}|_\nu^2 d\lambda_n. \tag{5.12}
\]
Therefore
\[
\int_{D_v} |F_v|_\nu^2 d\lambda_n \leq \liminf_{j \to +\infty} \int_{D_v} |F_{v,t_j}|_\nu^2 d\lambda_n. \tag{5.13}
\]
As $\{(1 - b_{t_j}(\varphi))F\}_{j=1,2,\ldots}$ goes to zero when $j \to +\infty$ and $|(1 - b_{t_j}(\varphi))F|$ on $D_v$ have a uniform bound independent of $j$, by the Lebesgue dominated convergence theorem it follows that
\[
\lim_{j \to +\infty} \int_{D_v} |(1 - b_{t_j}(\varphi))F|_\nu^2 d\lambda_n = 0. \tag{5.14}
\]
According to equality \[5.22\] inequality \[5.23\] it follows that
\[
\lim_{j \to +\infty} \int_{D_v} |F_{v,t_j} - (1 - b_{t_j}(\varphi))F|_\nu^2 d\lambda_n = 0. \tag{5.15}
\]
Using equality 5.27, equality 5.28 and inequality 5.24, we have
\[
\liminf_{j \to +\infty} \int_{D_n} |F_{v,t_j}|^2 d\lambda_n = 0. \tag{5.16}
\]
According to inequality 5.29 and inequality 5.26, it follows that
\[
\int_{D_n} |F_t|^2 d\lambda_n \leq 0. \tag{5.17}
\]
As \((F_{v,t_j} - F_v, o) \in \mathcal{I}\langle o \rangle_n\), and by Remark 2.16, we have \(F_v \neq 0\), which contradicts to inequality 5.30. Then Theorem 1.2 here has thus been proved.

5.3. Proof of Theorem 1.4

We prove Theorem 1.4 by contradiction: if Theorem 1.4 is not true, then there exists \(t_j \to +\infty (j \to +\infty)\) and \(B_j \in (0, 1]\), such that
\[
\lim_{j \to +\infty} \int_{\Delta \setminus B_j \{ -t_j - B_j < \Psi < -t_j \}} e^{-\Psi + \log |F|^2} d\lambda_n
= \lim_{j \to +\infty} \int_{\Delta \setminus B_j \{ -t_j - B_j < \Psi < -t_j \}} e^{-\Psi + \log |F|^2} d\lambda_n = 0. \tag{5.18}
\]
Let \(t_j > 10\), for all \(j\). Note that
\[
e^{-\max\{\psi - \log |F|^2, 0\}} \{ -t_j - B_j < \Psi < -t_j \}
= e^{-\max\{\psi - \log |F|^2, 0\}} \{ -t_j - B_j < \min\{\psi - \log |F|^2, 0\} - 1 < -t_j \}
= e^{-\max\{\psi - \log |F|^2, 0\}} \{ -t_j - B_j < \psi - \log |F|^2 - 1 < -t_j \}
= e^{-\max\{\psi - \log |F|^2, 0\}} \{ -t_j - B_j + 1 < \psi - \log |F|^2 < -t_j + 1 \}
\leq e^0 = 1
\]
then
\[
\lim_{j \to +\infty} \int_{\Delta \setminus B_j \{ -t_j - B_j < \Psi < -t_j \}} |F|^4 e^{-\Psi - \Psi} d\lambda_n
\leq \lim_{j \to +\infty} \int_{\Delta \setminus B_j \{ -t_j - B_j < \Psi < -t_j \}} e^{-2 \max\{\psi - \log |F|^2, 0\} - \min\{\psi - \log |F|^2, 0\} + 1} d\lambda_n
\leq \lim_{j \to +\infty} \int_{\Delta \setminus B_j \{ -t_j - B_j < \Psi < -t_j \}} e^{-\max\{\psi - \log |F|^2, 0\} + \psi - \log |F|^2 + 1} d\lambda_n
\leq \lim_{j \to +\infty} \int_{\Delta \setminus B_j \{ -t_j - B_j < \Psi < -t_j \}} e e^{-\psi + \log |F|^2} d\lambda_n = 0. \tag{5.20}
\]

Let \(D_v\) be a strongly pseudoconvex domain relatively compact in \(\Delta^n\) containing \(o\), it follows that
\[
\int_{\Delta \setminus B_j \{ -t_j - B_j < \Psi < -t_j \}} |F|^4 e^{-\Psi - \Psi} d\lambda_n \geq \int_{D_v} \frac{1}{B_j} |F|^4 e^{-\Psi - \Psi} d\lambda_n. \tag{5.21}
\]
According to equality 5.18 and inequality 5.21, it follows that
\[
\lim_{j \to +\infty} \int_{D_v} \frac{1}{B_j} |F|^4 e^{-\Psi - \Psi} d\lambda_n = 0. \tag{5.22}
\]
By Proposition 3.3, it follows that there exists $F_{v,t_j}$, which is a holomorphic function on $D_v$ satisfying:

$$\int_{D_v} |F_{v,t_j} - (1 - b_{t_j} (\Psi)) F^2 |^2 e^{-\varphi} d\lambda_n \leq \int_{D_v} \frac{1}{B_j} \|(-t_j - B_j < \Psi < t_j)|F|^{4}e^{-\varphi}d\lambda_n,$$

and

$$(F_{v,t_j} - F^2, o) \in \mathcal{I}(\varphi + \Psi)_o.$$

As $|(1 - b_{t_j} (\Psi)) F^2|$ on $D_v$ have a uniform bound independent of $j$, then $\int_{D_v} |(1 - b_{t_j} (\Psi)) F^2 |^2 e^{-\varphi} d\lambda_n$ have a uniform bound independent of $j$.

According to equality (5.22) and inequality (5.23), it follows that $\int_{D_v} |F_{v,t_j} - (1 - b_{t_j} (\Psi)) F^2 |^2 e^{-\varphi} d\lambda_n$ have a uniform bound independent of $j$.

Using equality (5.27), equality (5.28) and inequality (5.24), we have

$$\int_{D_v} |F_{v,t_j}|^2 e^{-\varphi} d\lambda_n$$

have a uniform bound independent of $j$. Then there is a subsequence of $F_{v,t_j}$ denoted by $F_{v,t_j}$ without ambiguity, which is convergent to a holomorphic function $F_v$ uniformly on any compact subset of $D_v$. Then for any $K \subset \subset D_v$, we have

$$\int_K |F_v|^2 e^{-\varphi} d\lambda_n \leq \liminf_{j \to +\infty} \int_{D_v} |F_{v,t_j}|^2 e^{-\varphi} d\lambda_n.$$

Therefore

$$\int_{D_v} |F_v|^2 e^{-\varphi} d\lambda_n \leq \liminf_{j \to +\infty} \int_{D_v} |F_{v,t_j}|^2 e^{-\varphi} d\lambda_n.$$

As $\{(1 - b_{t_j} (\varphi)) F^2\}_{j=1,2, \ldots}$ goes to zero when $j \to +\infty$ and $|1 - b_{t_j} (\varphi)) F^2|$ on $D_v$ have a uniform bound independent of $j$, by the Lebesgue dominated convergence theorem it follows that

$$\lim_{j \to +\infty} \int_{D_v} |(1 - b_{t_j}) (\Psi)) F^2 |^2 e^{-\varphi} d\lambda_n = 0.$$

According to equality (5.22), inequality (5.23) it follows that

$$\lim_{j \to +\infty} \int_{D_v} |F_{v,t_j} - (1 - b_{t_j} (\Psi)) F^2 |^2 e^{-\varphi} d\lambda_n = 0.$$

Using equality (5.27), equality (5.28) and inequality (5.24), we have

$$\liminf_{j \to +\infty} \int_{D_v} |F_{v,t_j}|^2 e^{-\varphi} d\lambda_n = 0.$$

According to inequality (5.29) and inequality (5.26), it follows that

$$\int_{D_v} |F_v|^2 e^{-\varphi} d\lambda_n \leq 0.$$

As $(F_{v,t_j} - F^2, o) \in \mathcal{I}(\varphi + \Psi)_o$, and by Remark 3.2 and 2.16, we have $F_v \neq 0$, which contradicts to inequality (5.30). Then Theorem 1.4 here has thus been proved.
5.4. Proof of Proposition 1.6

We prove Proposition 1.6 by contradiction. If $e^{-\phi}$ is integrable near $o \in \Delta^n$, then there exists a strong pseudoconvex domain $\Omega \subset \Delta^n$, such that $e^{-\phi}$ is $L^1$ integrable on $\Omega$.

Without losing of generality, we assume that $\Omega = \mathbb{B}(o, r)$, where $r > 0$ small enough.

As $e^{-\phi}$ is $L^1$ integrable on $\Omega$, then

$$\lim_{R \to +\infty} e^R \mu(\{\phi < -R\}) = 0.$$  \hspace{1cm} (5.31)

Therefore there exists $t_1 > 0$, such that

$$\mu(\{\phi < -t_1 + 1\}) < \frac{1}{6} \mu(\Omega).$$  \hspace{1cm} (5.32)

As $\{\phi_m\}_{m=1,2,...}$ is convergent to $\phi$ in Lebesgue measure, then there exists $m_0 > 0$, such that for any $m \geq m_0$,

$$\mu(\{|\phi_m - \phi| \geq 1\}) < \frac{1}{12} \mu(\Omega).$$  \hspace{1cm} (5.33)

Note that

$$\{|\phi_m < -t_1\} \setminus \{|\phi_m - \phi| \geq 1\} \subset \{|\phi < -t_1 + 1\},$$

for any $m \geq m_0$. Therefore

$$\mu(\phi_m < -t_1) \leq \mu(\phi < -t_1 + 1) + \mu(\{|\phi_m - \phi| \geq 1\}) < \frac{1}{4} \mu(\Omega),$$  \hspace{1cm} (5.34)

for any $m \geq m_0$.

In Proposition 3.1 let $F \equiv 1$ and $\varphi = \phi_m$, then there exists a holomorphic function $F_{v,t_0}$ on $\Omega$, satisfying:

$$F_{v,t_0}|_o = F = 1$$  \hspace{1cm} (5.35)

and

$$\int \Omega |F_{v,t_0} - (1 - b_{t_0}(\phi_m))F|^2 d\lambda_n \leq \int \Omega (1_{\{t_0-1 < t < t_0\}} \circ \phi_m)|F|^2 e^{-\phi_m} d\lambda_n.$$  \hspace{1cm} (5.36)

By submean inequality of plurisubharmonic function, it follows from 5.33 that

$$\int \Omega |F_{v,t_0}|^2 d\lambda_n \geq \mu(\Omega).$$  \hspace{1cm} (5.37)

It follows from inequality 5.34 and equality 5.32, and $b_{t_0}(t)|_{t \geq -t_0} = 1$, that

$$\int \Omega (1 - b_{t_0}(\phi_m))F|^2 d\lambda_n = \int \{\phi < -t_0\} \cap \Omega |(1 - b_{t_0}(\phi_m))F|^2 d\lambda_n \leq \int \{\phi < -t_0\} \cap \Omega |F|^2 d\lambda_n = \int \{\phi < -t_0\} \cap \Omega d\lambda_n \leq \mu(\{\phi < -t_0\}) < \frac{1}{4} \mu(\Omega)$$  \hspace{1cm} (5.38)
Then we have for any $t_0 > t_1$ and any $m \geq m_0$.

It follows from inequalities 5.36 and 5.39 that

$$
\int_{\Omega} |F_{v,t_0} - (1 - b_{t_0}(\phi_m))F|^2 d\lambda_n \geq \mu(\Omega)^{1/2} - 2^{-1}\mu(\Omega)^{1/2} = 2^{-1}\mu(\Omega)^{1/2},
$$

for any $t_0 > t_1$ and any $m \geq m_0$.

Remark 5.1. Using the same method as in the above proof with more subtle bounds in inequalities 5.32 and 5.33, one can obtain inequality 5.41 with a lower bound $\mu(\Omega)^{1/2}$.

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REFERENCES

[1] B. Berndtsson, The extension theorem of Ohsawa-Takegoshi and the theorem of Donnelly-Fefferman, Ann. L’Inst. Fourier (Grenoble) 46 (1996), no. 4, 1083–1094.

[2] B. Berndtsson, The openness conjecture for plurisubharmonic functions, arXiv:1305.5781.

[3] S. Boucksom, C. Favre, M. Jonsson, Valuations and plurisubharmonic singularities, Publ. Res. Inst. Math. Sci. 44 (2008), no. 2, 449–494.

[4] J.Y. Cao, Numerical dimension and a Kawamata-Viehweg-Nadel type vanishing theorem on compact Kähler manifolds, arXiv:1210.5692.

[5] J-P. Demailly, Multiplier ideal sheaves and analytic methods in algebraic geometry. School on Vanishing Theorems and Effective Results in Algebraic Geometry (Trieste, 2000), 1–148, ICTP Lect. Notes, 6, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2001.

[6] J.-P. Demailly, Analytic Methods in Algebraic Geometry, Higher Education Press, Beijing, 2010.

[7] J.-P. Demailly, Complex analytic and differential geometry, electronically accessible at http://www-fourier.ujf-grenoble.fr/~demailly/books.html.

[8] J.-P. Demailly, L. Ein, and R. Lazarsfeld, A subadditivity property of multiplier ideals, Michigan Math. J. 48 (2000), 137–156.

[9] J.-P. Demailly, J. Kollár, Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds. Ann. Sci. École Norm. Sup. (4) 34 (2001), no. 4, 525–556.

[10] J.-P. Demailly, T. Peternell, A Kawamata-Viehweg vanishing theorem on compact Kähler manifolds. J. Differential Geom. 63 (2003), no. 2, 231–277.

[11] C. Favre and M. Jonsson, Valuative analysis of planar plurisubharmonic functions, Invent. Math. 162 (2005), no. 2, 271–311.

[12] C. Favre and M. Jonsson, Valuations and multiplier ideals, J. Amer. Math. Soc. 18 (2005), no. 3, 655–684.

[13] Q.A. Guan and X.Y. Zhou, Optimal constant problem in the $L^2$ extension theorem, C. R. Acad. Sci. Paris. Ser. I. 350 (2012), no. 15–16, 753–756.

[14] Q.A. Guan and X.Y. Zhou, Generalized $L^2$ extension theorem and a conjecture of Ohsawa, C. R. Acad. Sci. Paris. Ser. I. 351 (2013), no. 3–4, 111–114.

[15] Q.A. Guan and X.Y. Zhou, Optimal constant in $L^2$ extension and a proof of a conjecture of Ohsawa, submitted.

[16] Q.A. Guan and X.Y. Zhou, An $L^2$ extension theorem with optimal estimate, C. R. Acad. Sci. Paris. Ser. I. (2014), no. 2, 137–141.

[17] Q.A. Guan and X.Y. Zhou, A solution of an $L^2$ extension problem with optimal estimate and applications, arXiv:1310.7169.

[18] Q.A. Guan and X.Y. Zhou, Strong openness conjecture for plurisubharmonic functions, arXiv:1311.3781.

[19] Q.A. Guan, X.Y. Zhou, and L.F. Zhu, On the Ohsawa-Takegoshi $L^2$ extension theorem and the twisted Bochner-Kodaira identity, C. R. Acad. Sci. Paris. Ser. I. 349 (2011), no. 13–14, 797–800.

[20] H. Guenancia, Toric plurisubharmonic functions and analytic adjoint ideal sheaves, Math. Z. 271 (2012), no. 3–4, 1011–1035.

[21] C. Hacon, J. McKernan and C. Xu, ACC for log canonical thresholds, arXiv:1208.4150v1.

[22] M. Jonsson and M. Mustată, Valuations and asymptotic invariants for sequences of ideals, Annales de l’Institut Fourier 62 (2012), no.6, pp. 2145–2209.

[23] M. Jonsson and M. Mustată, An algebraic approach to the openness conjecture of Demailly and Kollár, J. Inst. Math. Jussieu (2013), 1–26.

[24] D. Kim, The exactness of a general Skoda complex, arXiv:1007.0551.

[25] C.O. Kiselman, Plurisubharmonic functions and potential theory in several complex variables. Development of mathematics 1950-2000, 655-714, Birkhäuser, Basel, 2000.

[26] J. Kollár (with 14 coauthors): Flips and Abundance for Algebraic Threefolds; Astérisque Vol. 211 (1992).
R. Lazarsfeld, Positivity in algebraic geometry. I. Classical setting: line bundles and linear series. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, 48. Springer-Verlag, Berlin, 2004. xviii+387 pp.

R. Lazarsfeld, Positivity in algebraic geometry. II. Positivity for vector bundles, and multiplier ideals. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, 49. Springer-Verlag, Berlin, 2004. xviii+385 pp.

B. Lehmann, Algebraic bounds on analytic multiplier ideals, arXiv:1109.4452v3 [math.AG].

Y. Li, Théorèmes d’extension et métriques de Kähler-Einstein Généralisées, PhD thesis.

Y. Li, An Ohsawa-Takegoshi theorem on compact Kähler manifolds, to appear in Science China Mathematics, doi: 10.1007/s11425-013-4656-3.

S. Matsumura, A Nadel vanishing theorem for metrics with minimal singularities on big line bundles, arXiv:1306.2497.

S. Matsumura, An injectivity theorem with multiplier ideal sheaves of singular metrics with transcendental singularities, arXiv:1308.2033.

A. Nadel, Multiplier ideal sheaves and Kähler-Einstein metrics of positive scalar curvature. Ann. of Math. (2) 132 (1990), no. 3, 549–596.

T. Ohsawa, On the extension of $L^2$ holomorphic functions. III. Negligible weights. Math. Z. 219 (1995), no. 2, 215–225.

T. Ohsawa, On the extension of $L^2$ holomorphic functions. V. Effects of generalization. Nagoya Math. J. 161 (2001), 1–21. Erratum to: “On the extension of $L^2$ holomorphic functions. V. Effects of generalization” [Nagoya Math. J. 161 (2001), 1–21]. Nagoya Math. J. 163 (2001), 229.

V. Shokurov: 3-fold log flips; Izv. Russ. Acad. Nauk Ser. Mat. Vol. 56 (1992) 105–203.

Nessim Sibony: Quelques problèmes de prolongement de courants en analyse complexe. (French) [Some extension problems for currents in complex analysis] Duke Math. J. 52 (1985), no. 1, 157–197.

Y.T. Siu, Analyticity of sets associated to Lelong numbers and the extension of closed positive currents. Invent. Math. 27 (1974), 53–156.

Y.T. Siu, The Fujita conjecture and the extension theorem of Ohsawa-Takegoshi, Geometric Complex Analysis, Hayama. World Scientific (1996), 577–592.

Y.T. Siu, Extension of twisted pluricanonical sections with plurisubharmonic weight and invariance of semipositively twisted plurigenera for manifolds not necessarily of general type. Complex geometry (Göttingen, 2000), 223–277, Springer, Berlin, (2002).

Y.T. Siu, Multiplier ideal sheaves in complex and algebraic geometry. Sci. China Ser. A 48 (2005), suppl., 1–31.

Y.T. Siu, Dynamic multiplier ideal sheaves and the construction of rational curves in Fano manifolds. Complex analysis and digital geometry, 323–360, Acta Univ. Upsaliensis Skr. Uppsala Univ. C. Organ. Hist., 86. Uppsala Universitet, Uppsala, 2009.

E. Straube, Lectures on the $L^2$-Sobolev Theory of the $\bar{\partial}$-Neumann Problem. ESI Lectures in Mathematics and Physics. Zrich: European Mathematical Society (2010).

L.F. Zhu, Q.A. Guan and X.Y. Zhou, On the Ohsawa-Takegoshi $L^2$ extension theorem and the Bochner-Kodaira identity with non-smooth twist factor, J. Math. Pures Appl. (9) 97 (2012), no. 6, 579–601.