Abstract

The structure of the Euler-Lagrange equations for a general Lagrangian theory (e.g. singular, with higher derivatives) is studied. For these equations we present a reduction procedure to the so-called canonical form. In the canonical form the equations are solved with respect to highest-order derivatives of nongauge coordinates, whereas gauge coordinates and their derivatives enter in the right hand sides of the equations as arbitrary functions of time. The reduction procedure reveals constraints in the Lagrangian formulation of singular systems and, in that respect, is similar to the Dirac procedure in the Hamiltonian formulation. Moreover, the reduction procedure allows one to reveal the gauge identities between the Euler-Lagrange equations. Thus, a constructive way of finding all the gauge generators within the Lagrangian formulation is presented. At the same time, it is proven that for local theories all the gauge generators are local in time operators.

1 Introduction

At present increasingly complicated gauge models are used in field and string theory. Generally a comprehensive analysis of their structure is not a simple task. In the Lagrangian formulation the problem includes, in particular, finding generators of gauge symmetries and their algebra, revealing the hidden structure of the equations of motion and so on. One ought to say that in the Hamiltonian formulation there exists a relatively well-developed scheme of constraint finding (Dirac procedure [1]) and reorganization [2, 3, 4]. The constraint structure can be, in principle, related to the symmetry properties of the initial gauge theory in the Lagrangian formulation [5]. However, in the general case, this relation cannot be considered as a constructive method to study the Lagrangian symmetries (it is indirect and complicated). Moreover, the modern tendency is to avoid the non-covariant hamiltonization step and to use the Lagrangian quantization [6] for constructing quantum theory. Such an approach incorporates all the Lagrangian structures (in particular, the total gauge algebra). That is why it seems important to develop a reduction procedure
within the Lagrangian formulation – in a sense similar to the Dirac procedure in the Hamiltonian formulation – that may allow one in a constructive manner to reveal the hidden structure of the Euler-Lagrange equations (ELE) of motion and to find all the gauge identities and therefore the generators of all the gauge transformations. An idea of such a procedure was first mentioned in publications of the authors (D.G and I.T) \([7, 8]\) (see also Appendix C in [2]), but was not appropriately elaborated and some important points where not revealed.

In the present paper we return to this idea studying the structure of the ELE for a general Lagrangian theory (singular, with higher derivatives, and with external fields). In Sect. II we introduce some notation and definitions. In Sect. III, we reduce the ELE of nonsingular theories to the so called canonical form (in the canonical form the equations are solved with respect to highest-order derivatives of nongauge coordinates, whereas, gauge coordinates and their derivatives enter in the right hand sides of the equations as arbitrary functions of time, see below). In Sect. IV we formulate the reduction procedure for the singular case. In a sense, the reduction procedure reveals constraints in the Lagrangian formulation of singular systems and, in that respect, is similar to the Dirac procedure in the Hamiltonian formulation. In Sect. V we demonstrate how the reduction procedure reveals the gauge identities between the ELE. Thus, a constructive way of finding all the gauge generators within the Lagrangian formulation is presented. At the same time it is proven that for local theories all the gauge generators are local in time operators. In the Appendix we collect some Lemmas being useful for our consideration.

2 General ELE

2.1 Notation, definitions, and conventions

We consider a system with finite degrees of freedom (classical mechanics). These degrees of freedom are described by the generalized coordinates \(q^a, a = 1, ..., n\), which depend on the time \(t\). The following notation is used:

\[
\dot{q}^a = \frac{dq^a}{dt}, \quad \ddot{q}^a = \frac{d^2q^a}{dt^2}, \quad \dddot{q}^a, \quad \text{or} \quad q^{a[l]} = \frac{d^l q^a}{dt^l}, \quad l = 0, 1, ..., \quad (q^{a[0]} = q^a) .
\]

(1)

The coordinates \(q^a = q^{a[0]}\) are called sometimes velocities of zeroth order; the velocities \(\dot{q}^a = q^{a[1]}\) are called velocities of the first order; the accelerations \(\ddot{q}^a = q^{a[2]}\) are called velocities of second order, and so on. The space of all the velocities is often called the jet space, see [2].

As local functions (LF) we call those functions that are defined on the jet space and depend on the velocities \(q^{a[l]}\) up to some finite orders \(N_a \geq 0\) \((l \leq N_a)\). Further, we call \(N_a\) the order of the coordinate \(q^a\) in the LF. For the LF we use the following notation:\[1]\:

\[
F (q^a, \dot{q}^a, \ddot{q}^a, ...) = F (q^{a[0]}, q^{a[1]}, q^{a[2]}, ...) = F (q^{[0]}), \quad q^{[l]} = (q^{a[l]}, \quad 0 \leq l \leq N_a),
\]

or sometimes:

\[
F (q^{[l]}) = F (\cdots q^{a(N_a)}) .
\]

(2)

In the latter form, we indicate only the highest-order derivatives in the arguments of the LF.

\[1\] The functions \(F\) may depend on time explicitly, however, we do not include \(t\) in the arguments of the functions.
The following notation is often used: \([a]\) is the number of the indices \(a\), namely, if \(a = 1, \ldots, n\), then \([a] = n\). Similarly, suppose \(F_a(\eta)\), \(a = 1, \ldots, n\) are some functions, then \([F]\) is the number of these functions, \([F] = n\), etc. However differently, writing \(q^{a[l]}\) we denote by \([l]\) the order of the time derivatives, see (II).

On the jet space, we define local operators (LO) to be matrix operators \(\hat{U}\) of the form

\[
\hat{U}_{AA} = \sum_{k=0}^{K} u_{AA}^{k} \left( \frac{d}{dt} \right)^{k},
\]

where \(K\) is a finite number and \(u_{AA}^{k}\) are some LF. The LO act on columns of LF \(f_{a}\) producing columns of LF \(F_A = \hat{U}_{AA} f_{a}\) as well. We define the transposed operator to \(\hat{U}\) as

\[
\left(\hat{U}^{T}\right)_{aA}^{b} = \sum_{k=0}^{K} \left( -\frac{d}{dt} \right)^{k} u^{k}_{AA} = \sum_{k=0}^{K} \sum_{l=0}^{k} \left( \begin{array}{c} k \\ l \end{array} \right) u^{k-l}_{AA} \left( \frac{d}{dt} \right)^{k-l}.
\]

Then the following relation holds true

\[
F_A \hat{U}_{AA} f_{a} = \left[ \left(\hat{U}^{T}\right)_{aA} F_A \right] f_{a} + \frac{d}{dt} Q,
\]

where \(F_A\), \(f_{a}\), and \(Q\) are LF. The LO \(\hat{U}_{ab}\) is symmetric (+) or skew-symmetric (−) whenever the relation \(\left(\hat{U}^{T}\right)_{ab} = \pm \hat{U}_{ab}\) holds true.

Suppose a set of LF \(F_A(\cdots q^{a[N_A]}), \) or a set of equations \(F_A(\cdots q^{a[N_A]} ) = 0\), be given. In the general case the orders \(N_a^{A}\) of the coordinates \(q^a\) in the functions \(F_a\) (in the equations \(F_A = 0\)) are different, i.e. these orders depend both on \(a\) and \(A\). The number \(N_a = \max_A N_a^{A}\) is called the order of the coordinate \(q^a\) in the set of the functions \(F_a\) (in the set of the equations \(F_A = 0\)).

Whenever, a subset \(F_{A'} = 0\), \(A' \subset A\) has orders \(N'_{a}\) of the coordinates less than the corresponding orders of the complete set, namely, \(\forall a : N'_{a} < N_{a}\), we call this subset the constraint equations.

Generally two sets of equations, \(F_A(\cdots q^{[l]} ) = 0\) and \(f_{a}(\cdots q^{[l]} ) = 0\) are equivalent whenever they have the same set of solutions. In what follows we denote this fact as: \(F = 0 \iff f = 0\).

Suppose that two sets of LF \(F_A(\cdots q^{[l]} )\) and \(\chi_A(\cdots q^{[l]} )\), \([F] = [\chi]\), are related by some LO,

\[
F = \hat{U}\chi , \quad \chi = \hat{V}F , \quad \hat{U}\hat{V} = 1 .
\]

Then we call such functions equivalent and denote this fact as: \(F \sim \chi\). Obviously,

\[
F \sim \chi \implies F = 0 \iff \chi = 0 .
\]

If (II) holds true, we will call the equations \(F_A = 0\) and \(f_{a} = 0\) strong equivalent.

In what follows we often meet the case where

\[
\chi_A = \begin{pmatrix} f_{a} \\ 0_{G} \end{pmatrix} , \quad A = (\alpha, G) ; \quad \forall G : 0_{G} \equiv 0 .
\]

Here the equivalence \(F \sim \chi\) implies the equivalence of the equations \(F = 0\) and \(f = 0\) and the existence of the identities \(\hat{V}_{GA} F_{A} \equiv 0\). Namely,

\[
F \sim \chi \implies \begin{cases} F = 0 \iff f = 0 \\ \hat{V}_{GA} F_{A} \equiv 0 \end{cases} .
\]
2.2 ELE

Below we restrict our consideration to the Lagrange functions $L$ that are LF on the jet space, and depend on some external coordinates (fields) $u^\mu$ (we call the coordinates $u^\mu$ external ones in contrast to the coordinates $q^a$, which we call inner coordinates) which are some given functions of time. Thus,

$$L = L (\cdots q^{a[N_a]}, u^\mu), \quad a = 1, \ldots, n, \quad N_a \geq 0. \quad (10)$$

The orders $N_a$ of the inner coordinates $q^a$ in the Lagrange function will be called further the proper orders of the coordinates. Coordinates $q^a$ with the proper orders $N_a = 0$, we call the degenerate coordinates [10].

Equations of motion of a Lagrangian theory (the ELE) follow from the action principle $\delta S = 0$, $S = \int L \, dt$, and have the form (merely the inner coordinates have to be varied):

$$\frac{\delta S}{\delta q^a} = \sum_{l=0}^{N_a} \left(-\frac{d}{dt}\right)^l \frac{\partial L}{\partial q^a[l]} = 0, \quad a = 1, \ldots, n. \quad (11)$$

Following [10], we classify the Lagrangian theories as nonsingular ($M \neq 0$) and singular ($M = 0$) ones by the help of the generalized Hessian $M = \det |M_{ab}|$, where

$$M_{ab} = \frac{\partial^2 L}{\partial q^a[N_a] \partial q^b[N_b]} \quad (12)$$

is the generalized Hessian matrix.

In what follows the ELE of a nonsingular (singular) theory will be called the nonsingular (singular) ELE.

Sometimes, it is convenient to enumerate the inner coordinates and organize them into groups such that $q^a = (q^{a_0}, \ldots, q^{a_I})$, where $a_i$ are groups of indices that enumerate coordinates having the same proper orders, $N_{a_i} = n_k$. Besides, we organize these groups such that $n_I > n_{I-1} \cdots > n_0 = 0$ (max $N_a = N_{a_I} = n_I$, and $q^{a_0}$ are the degenerate coordinates, $N_{a_0} = n_0 = 0$). Thus,

$$a = (a_k, \quad k = 0, 1, \ldots, I), \quad [a] = \sum_i [a_i], \quad [a_i] \geq 0, \quad n_I > n_{I-1} \cdots > n_0 = 0. \quad (13)$$

Taking into account the notation (13), we may write the Lagrange function and the ELE as:

$$L = L (\cdots q^{a_k[N_k]}, u^\mu), \quad k = 0, 1, \ldots, I; \quad (14)$$

$$F_{a_k} (\cdots q^{b[N_b+n_k]}, \cdots u^{\mu[n_k]}) = 0, \quad (15)$$

$$F_{a_k} = \left\{ \begin{array}{l} M_{ab} q^{b[N_b+n_k]} + K_{a_k} (\cdots q^{b[N_b+n_k-1]}, \cdots u^{\mu[n_k]}), \quad k = 1, \ldots, I \\ M_{a_0} (\cdots q^{[N_0]}, u^\mu) = \partial L / \partial q^{a_0} \end{array} \right. \quad (16)$$

Here $M_{ab}$ is the generalized Hessian matrix and $K_{a_k}$ and $M_{a_0}$ are some LF of the indicated arguments.

Consider the orders of the inner coordinates in the complete set of the ELE. These orders are $N_a = N_a + n_I$. One can see that these orders are, in fact, defined by a subset of (13) with $k = I$. In any subset of the equations (13) with $k < I$ the orders of the
coordinates are less than in the complete set. Then according to the above definition, all the ELE with \( k < I \) are constraints. The set (13) has the following specific structure: In each equation of the complete set the order of a coordinate \( q^a \) is the sum of the proper order \( N_a \) and of the order \( n_k \). The latter is the same for all the coordinates and depends only on the number \( a_k \) of the equation.

2.3 Canonical form

Let a set of equations

\[
F_A (\cdots q^{a[N_a]}) = 0 ,
\]

be given. Suppose that these equations can be transformed to the following equivalent form:

\[
q^{\alpha[l_\alpha]} = \phi^\alpha (\cdots q^{\alpha[l_\alpha-1]}; \cdots q^{g[l_g]}) , \quad a = (\alpha, g) , \quad l_\alpha \leq N_a .
\]

The equation (18) presents the canonical form of the initial set (17). In the canonical form the equations are solved with respect to the highest-order time derivatives \( q^{\alpha[l_\alpha]} \) of the coordinates \( q^\alpha \). The coordinates \( q^g \) (if they exist) and their derivatives \( q^{g[l_g]} \) enter into the set (18) as arbitrary functions of time. In fact, there are no equations for these coordinates. In what follows we call these coordinates the gauge coordinates whereas \( q^\alpha \) we call the nongauge coordinates. The orders of the coordinates in the canonical forms may be less than those in the initial set. In the general case, one and the same set of equations can have different canonical forms. Generally there are many canonical forms of the same set of equations.

Below, we are going to formulate a general procedure of reducing the ELE to the canonical form (in what follows it is called the reduction procedure). Our consideration is always local in a vicinity of a given consideration point \( q^{a[l]} \) (in the jet space), which is on shell w.r.t. the ELE. We consider theories and coordinates where the consideration point could be selected as zero point. Thus, we suppose that the zero point is on shell. Further we always suppose that the ranks of the encountered Jacobi matrices\(^2\) are constant in a vicinity of the consideration point. Such suppositions we call ”suppositions of the ranks”. Saying that some suppositions hold true in the consideration point, we always suppose that they hold true in a vicinity of the consideration point. In course of the reduction procedure we perform several typical transformations with LF or with the corresponding equations. Each of such transformations lead to equivalent sets of equations or to equivalent sets of LF (definitions of such equivalences are given above). The proof of these equivalences is based on two Lemmas which are presented in the Appendix. Any statement of the form ”the following equivalence holds true” can be easily justified by these Lemmas.

3 Canonical form of nonsingular ELE

3.1 A particular case

Consider theories without external coordinates and with only two different proper orders of the inner coordinates. In such a case all the indices \( a \) can be divided into two groups:

\(^2\)A retangular matrix with elements \( \partial A^\alpha / \partial x^i \) is often denoted as \( \partial A / \partial x \) and called the Jacobi matrix.
\( a = (a_1, a_2) \), such that \( N_{a_2} = n_2 > N_{a_1} = n_1 \), \( L = L(\ldots q^{a_2[n_2]} \ldots q^{a_1[n_1]}) \). Consider first the case \( n_1 > 0 \). Then Eqs. (13) can be written as:

\[
\begin{align*}
F_{a_2} &= M_{a_2} q^{a_2[N_{a_2}]} + K_{a_2} (\ldots q^{b[N_{b}+n_2-1]} \ldots) = 0, \\
F_{a_1} &= M_{a_1} q^{a_1[N_{a_1}]} + K_{a_1} (\ldots q^{b[N_{b}+n_1-1]} \ldots) = 0.
\end{align*}
\]  

The equations (20) are constraints. Consider the set

\[
M_{a_1} q^{a_1[N_{a_1}]} + K_{a_1}^{(1)} (\ldots q^{b[N_{b}+n_2-1]} \ldots) = 0,
\]  

obtained from the constraints after being \( n_2 - n_1 \) times differentiated with respect to the time \( t \). Since \( M \neq 0 \), the rectangular matrix \( M_{a_1} \) has a maximal rank, that is why there exists another division of the indices:

\[
a = (a_{i}), \quad [a_{i}] = [a_{i}], \quad i = 1, 2, \quad \det M_{a_1 b_1} \neq 0.
\]  

Remark that

\[
a_{i} = (a_{i|1}, a_{i|2}), \quad a_{i|1} = (a_{i|1}, a_{i|2}), \quad [a_{1|2}] = [a_{2|1}].
\]  

The set (21) can be solved with respect to the derivatives \( q^{a_1[N_{a_1}+n_2]} \) as follows:

\[
q^{a_1[N_{a_1}+n_2]} = - (M_{1}^{-1})^{a_1 a_1} \left( (M_{3})_{a_1 a_1} q^{a_2[N_{a_2}+n_2]} + K_{a_2}^{(1)} (\ldots q^{b[N_{b}+n_2-1]} \ldots) \right).
\]  

Here the matrices \( M_{1} \) and \( M_{3} \) are defined by the following block representation of the matrix \( M \):

\[
M_{ab} = \begin{pmatrix} (M_{2})_{a_2 b_1} & (M_{4})_{a_2 b_2} \\ (M_{1})_{a_1 b_1} & (M_{3})_{a_1 b_2} \end{pmatrix}, \quad \det M_{a_1 b_1} \neq 0 \implies \det M_1 \neq 0.
\]  

Excluding the derivatives \( q^{a_1[N_{a_1}+n_2]} \) from Eqs. (19) by the help of (24), we get the equations

\[
(M_{5})_{a_2 b_2} q^{b_2[N_{b_2}+n_2]} + K_{a_2}^{(2)} (\ldots q^{b[N_{b}+n_2-1]} \ldots) = 0.
\]  

Taking into account an useful relation

\[
\det M = \det \begin{pmatrix} M_{2} & M_{4} \\ M_{1} & M_{3} \end{pmatrix} = \det \begin{pmatrix} 0 & M_{4} - M_{2} M_{1}^{-1} M_{3} \\ M_{1} & M_{3} \end{pmatrix}
\]

\[
= \det M_{1} \det (M_{4} - M_{2} M_{1}^{-1} M_{3}),
\]

which is related to the Gaussian reduction of matrices [14], we get:

\[
\begin{align*}
\det M_{1} \neq 0 \quad & \implies \det M_{5} \neq 0, \quad M_{5} = M_{4} - M_{2} M_{1}^{-1} M_{3}.
\end{align*}
\]  

Therefore, (23) can be solved with respect to the highest-order derivatives \( q^{a_2[N_{a_2}+n_2]} \) as:

\[
q^{a_2[N_{a_2}+n_2]} = - (M_{5}^{-1})^{a_2 a_2} \left[ K_{a_2} (\ldots q^{b[N_{b}+n_2-1]} \ldots) - (M_{2})_{a_2} M_{1}^{-1} a_{1} K_{a_1}^{(1)} (\ldots q^{b[N_{b}+n_2-1]} \ldots) \right]
\]

\[
= \varphi^{a_2} (\ldots q^{b_2[N_{b_2}+n_2-1]} \ldots q^{b_1[N_{b_1}+n_2-1]}). \]  

(28)
Thus, we get a set
\[
\psi^{a_2|2} = q^{a_2|2} \left[ N_{a_2|n_2} + n_2 \right] - \Phi^{a_2|2} \left( \ldots q^{b_2|2} \left[ N_{b_2|n_2-1} + n_2 \right], \ldots q^{b_1|1} \left[ N_{b_1|n_2-1} + n_2 \right] \right) = 0, \tag{29}
\]
\[
F_{a_1} = (M_1)_{a_1 a_1} q^{a_1|1} \left[ N_{a_1|n_1} + n_1 \right] - K_{a_1} \left( \ldots q^{b_2|2} \left[ N_{b_2|n_1} + n_1 \right], \ldots q^{b_1|1} \left[ N_{b_1|n_1} + n_1 \right] \right) = 0, \tag{30}
\]
which is strong equivalent to the initial ELE by virtue of the Lemma 1.

Due to the condition \( \det M_1 \neq 0 \), the equations (30) can be solved with respect to \( q^{a_1|1} \left[ N_{a_1|n_1} + n_1 \right] \) and we obtain:
\[
q^{a_1|1} \left[ N_{a_1|n_1} + n_1 \right] = - (M_1^{-1})^{a_1 a_1} \left[ (M_3)_{a_1 a_1} q^{a_2|2} \left[ N_{a_2|n_2} + n_2 \right] + K_{a_1} \left( \ldots q^{b_2|2} \left[ N_{b_2|n_1} + n_1 \right], \ldots q^{b_1|1} \left[ N_{b_1|n_1} + n_1 \right] \right) \right]
\equiv f^{a_1} \left( \ldots q^{b_2|2} \left[ N_{b_2|n_1} + n_1 \right], \ldots q^{b_1|1} \left[ N_{b_1|n_1} + n_1 \right] \right). \tag{31}
\]

Eqs. (29) and (31) are not of canonical form since the functions \( \Phi^{a_2|2} \) contain derivatives \( q^{b_1|1} \left[ N_{b_1|n_2-1} + n_2 \right] \) exceeding the "allowed" order \( \left[ N_{b_1|n_1} + n_1 - 1 \right] \). Now we exclude all the surplus derivatives \( q^{a_1|1} \left[ N_{a_1|n_1} + n_1 \right], \ldots q^{a_1|1} \left[ N_{a_1|n_1} + n_1 - 1 \right] \) from the right hand side of (29) by the help of (31) and corresponding derivatives of it. To this end we need to differentiate (31) not more than \( n_2 - n_1 - 1 \) times. Finally, we obtain the following strong equivalent form (the equivalence is justified by the Lemma 1) of the ELE:
\[
q^{a_2|2} \left[ N_{a_2|n_2} + n_2 \right] = f^{a_2|2} \left( \ldots q^{b_2|2} \left[ N_{b_2|n_2-1} + n_2 \right], \ldots q^{b_1|1} \left[ N_{b_1|n_2-1} + n_2 \right] \right),
\]
\[
q^{a_1|1} \left[ N_{a_1|n_1} + n_1 \right] = f^{a_1|1} \left( \ldots q^{b_2|2} \left[ N_{b_2|n_1} + n_1 \right], \ldots q^{b_1|1} \left[ N_{b_1|n_1} + n_1 \right] \right). \tag{32}
\]

It is just the canonical form. Taking into account the division of the indices w.r.t. proper orders of the coordinates, one gets:
\[
q^{a_2|2} \left[ 2n_2 \right] = f^{a_2|2} \left( \ldots q^{b_2|2} \left[ 2n_2-1 \right], \ldots q^{b_1|2} \left[ 2n_2-1+n_1 \right], \ldots q^{b_1|1} \left[ 2n_2-1+n_1 \right] \right),
\]
\[
q^{a_1|2} \left[ n_1+n_2 \right] = f^{a_1|2} \left( \ldots q^{b_2|2} \left[ 2n_1-1 \right], \ldots q^{b_1|2} \left[ 2n_1-1+n_2 \right], \ldots q^{b_1|1} \left[ 2n_1-1+n_2 \right] \right),
\]
\[
q^{a_1|1} \left[ 2n_1 \right] = f^{a_1|1} \left( \ldots q^{b_2|2} \left[ 2n_2-1 \right], \ldots q^{b_1|2} \left[ 2n_2-1+n_1 \right], \ldots q^{b_1|1} \left[ 2n_2-1+n_1 \right] \right). \tag{33}
\]

Remark that the number of the initial data is equal to \( 2 \sum a \cdot N_a \). Indeed,
\[
\left[ a_2|2 \right] (n_2 + n_2) + \left[ a_1|2 \right] (n_1 + n_2) + \left[ a_2|1 \right] (n_2 + n_1) + \left[ a_1|1 \right] (n_1 + n_1)
= 2 [a_2|2] n_2 + 2 [a_1|2] n_1 = 2 \sum_a N_a.
\]

One ought to mention that the canonical form (33) was obtained in [12]. However, the procedure that was used for that purpose did not provide the proof of the equivalence between the initial ELE and the form (33).
Suppose now that the Lagrange function contains degenerate coordinates \( q^{a_0} \), \( a = (a_0, a_1) \). Thus, \( L = L (q^{a_0}, \ldots q^{a_1[n_1]}) \) and the ELE read:

\[
F_{a_1} \equiv M_{a_1} \partial L = K_{a_1} = 0 \tag{34}
\]

\[
F_{a_0} = \frac{\partial L}{\partial q^{a_0}} = M_{a_0} = 0 \quad \tag{35}
\]

Despite these equations are formally different from the above case, the whole procedure of reductions goes through without any essential change. In fact, differentiating Eq. (35) \( n_1 \) times, one obtains

\[
M_{a_0} \partial L = K_{a_0} = 0 \tag{36}
\]

and all the previous steps may be done as before. Namely, one obtains

\[
q^{a_1} [N_{a_0} + n_1] = \varphi^{a_1} \left( \ldots q^{b_1} [N_{b_0} + n_0 - 1] \right), \tag{37}
\]

and, since \( \det |(M_1)_{a_0 a_0}| \neq 0 \), Eqs. (33) can be solved with respect to the variable \( q^{a_0} [N_{a_0}] \) as follows:

\[
q^{a_0} [N_{a_0}] = f^{a_0} \left( \ldots q^{b_0} [N_{b_0}], \ldots q^{b_0} [N_{b_0} - n_0 - 1] \right). \tag{38}
\]

Finally, after eliminating the “bad” derivatives in the right hand side of (37) for \( q^{a_1} [N_{a_0} + n_1] \) one ends up again with Eqs. (33) but now with \( n_2 \to n_1, \quad n_1 \to 0 \) (by convention: \( q^{b_1} [-1] \equiv 0 \)).

### 3.2 General nonsingular ELE

Consider the general nonsingular ELE. Here the Lagrange function may contain some degenerate inner coordinates, higher derivatives of some inner coordinates, and, moreover, \( \varphi^{a_1} \) may depend on some external coordinates, \( L = L (\ldots q^{a_1[n_1]}, u^i) \), \( N_a \geq 0 \). Thus, we are going to deal with the nonsingular ELE of the form (15). Our aim is to present these equations in an equivalent canonical form.

**Theorem 1**: The nonsingular ELE (15) can be transformed to the following equivalent canonical form:

\[
f^{a_{i|k}} = q^{a_{i|k} [n_1 + n_k]} - \varphi^{a_{i|k}} \left( \ldots q^{b_{j|k} [n_j + n_{k_+} - 1]}, \ldots q^{b_{j|k} [n_j + n_k]}, \ldots u^{i[n_k]} \right) = 0, \tag{38}
\]

where the indices of the coordinates are divided into groups as follows: \( a = (a_i) \) is the division of the indices w.r.t. the proper orders of the coordinates, and besides

\[
a_i = (a_{i|k}, \quad i, k = 0, 1, \ldots, I), \quad [a_{i|k}] \geq 0, \quad \sum_k [a_{i|k}] = \sum_k [a_{k|j}] = [a_i] = [a_i].
\]

8
Moreover, the equivalence \( F \sim f \) between the corresponding LF holds true. That implies

\[
F_a = \hat{U}_{ab} f^b, \quad f^b = \hat{V}^{ba} F_a, \quad \hat{U}_{ab} \hat{V}^{bc} = \delta_a^c,
\]

where \( \hat{U} \) and \( \hat{V} \) are LO. Besides, that implies the strong equivalence between the ELE and their canonical form (38).

The proof of the Theorem 1 may be considered, in fact, as the general reduction procedure to the canonical form for the nonsingular ELE.

It is reasonable to divide the reduction procedure into two parts. These parts may be called conditionally "the preliminary resolution", and "the subordination procedure".

**Preliminary resolution**

Let us introduce the notation \( a = (a, a_I), \; a = (a_k, k = 0, 1, \ldots, I - 1), \; N_\bar{a} < n_I \), such that the ELE read:

\[
F_{a_1} \left( \cdots q_{b[N_\bar{a}+n_I]; \cdots} u^{\mu[n_I]} \right) = M_{a_1 b} q_{b[N_\bar{a}+n_I]} + K_{a_1} (\cdots q_{b[N_\bar{a}+n_I-1]; \cdots} u^{\mu[n_I]} \right) = 0, \quad (39)
\]

\[
F_{\bar{a}} \left( \cdots q_{b[N_\bar{a}+n_\bar{a}]; \cdots} u^{\mu[n_\bar{a}]} \right) = 0. \quad (40)
\]

Recall that the equations (40) can be considered as constraints.

The first step of the procedure is the following: We consider the consistency conditions of the constraints. Namely, we consider the equations that are obtained from the constraints by differentiating them \( n_I - n_\bar{a} \) times,

\[
F_{\bar{a}}^{[n_I-n_\bar{a}]} = M_{\bar{a} b} q_{b[N_\bar{a}+n_I]} + K_{\bar{a}}^{(1)} (\cdots q_{b[N_\bar{a}+n_I-1]; \cdots} u^{\mu[n_I]} \right) = 0. \quad (41)
\]

Here \( K_{\bar{a}}^{(1)} \) are some LF of the indicated arguments. Remark that the orders of all the coordinates in the set (41) coincide with the ones in the complete set. For \( M \neq 0 \), the matrix

\[
\frac{\partial F_{\bar{a}}^{[n_I-n_\bar{a}]}}{\partial q_{b[N_\bar{a}+n_I]}} = \frac{\partial^2 L}{\partial q^{[n_\bar{a}]}} = M_{ab} \quad (42)
\]

is invertible. At the same time, the rectangular matrix \( M_{\bar{a} a} \) has the maximal rank \([\bar{a}]\). Therefore, there exists a division of the indices \( a \) such that:

\[
a = (\bar{a}, a_I), \; [\bar{a}] = [\bar{a}], \; [a_I] = [a_I], \; \det M_{\bar{a} \bar{a}} \neq 0. \quad (43)
\]

Thus, the division (13) of the indices \( a \) w.r.t. the coordinate proper orders becomes more detailed,

\[
a_i = (\bar{a}_i, a_{i I}), \; \bar{a} = (\bar{a}_i), \; a_I = (a_{i I}), \; [a_{i I}] \geq 0,
\]

\[
\sum_i [a_{i I}] = [a_I] = [a_I], \; \sum_i [\bar{a}_i] = [\bar{a}] = [\bar{a}].
\]

Due to (13), the set (41) can be solved with respect to the derivatives \( q^{b[N_\bar{a}+n_I]} \) as:

\[
q_{b[N_\bar{a}+n_I]} = - (M_1^{-1})_{\bar{a} \bar{a}} \left( (M_2)_{\bar{a} b_{i I}} q_{b[i I]}^{[N_I]} + K_{\bar{a}}^{(1)} (\cdots q_{b[N_\bar{a}+n_I-1]; \cdots} u^{\mu[n_I]} \right), \quad (44)
\]
Then the set (48) can be written as:

\[ \phi \text{ where } (M_1)_{a \bar{b}'} (M_3)_{a b'(\mu)} = 0, \]

where \( K \) are some LF of the indicated arguments. The set (48) can be solved with respect to its highest-order derivatives \( q^{a_{[N_k]}} \) as:

\[ q^{a_{[N_k]}} = \phi^{a_{[N_k]}} \left( \cdots q^{b_{[N_{k-1}]}} \cdots q^{b_{[N_1]}} \right), \]

where \( \phi^{a_{[N_k]}} \) are some LF. Thus, after the first step we get a set of equations

\[ \psi^{a_{[N_k]}} = q^{a_{[N_k]}} - \phi^{a_{[N_k]}} \left( \cdots q^{b_{[N_{k-1}]}} \cdots q^{b_{[N_1]}} \right) = 0, \]

where \( \psi^{a_{[N_k]}} \) are some LF. Thus, after the first step we get a set of equations

\[ F_{a \bar{b}'} (\cdots q^{a_{[N_k+N_\bar{a}]}} \cdots u^{[N_\bar{a}]} ) = 0, \quad a = (a_k, k = 0, 1, \ldots, I - 1), \quad N_\bar{a} < N_I, \]

which are strong equivalent to the initial ELE by virtue of the Lemma 1 from the Appendix.

At the second step we turn to the subset (18). We remark that this subset has the same structure as the complete initial set of the ELE if one considers the coordinates \( q^{\bar{a}} \) as inner ones and the variables \( q^{a_{[N_k]}} \) as external ones. Namely, let us denote

\[ \frac{1}{F_{a \bar{b}'}} (\cdots q^{a_{[N_k+N_\bar{a}]}} \cdots u^{[N_\bar{a}]} ) = 0, \quad a = (a_k, k = 0, 1, \ldots, I - 1), \quad N_\bar{a} < N_I, \]

Then the set (18) can be written as:

\[ \frac{1}{F_{a \bar{b}'} (\cdots q^{a_{[N_k+N_\bar{a}]}} \cdots u^{[N_\bar{a}]} )} = 0, \quad a = (a_k, k = 0, 1, \ldots, I - 1), \quad N_\bar{a} < N_I, \]

where

\[ \frac{1}{F_{a \bar{b}'} (\cdots q^{a_{[N_k+N_\bar{a}]}} \cdots u^{[N_\bar{a}]} )} = \frac{\partial F_{a \bar{b}'} (\cdots q^{a_{[N_k+N_\bar{a}]}} \cdots u^{[N_\bar{a}]} )}{\partial q^{a_{[N_k+N_\bar{a}]}}}. \]

Here \( q^{\bar{a}} \) are the inner coordinates, and \( u^{a_{[N_k]}} \) are the external coordinates. The order of the set (19) is \( 2n_{l-1} \). Furthermore, by virtue of (13), the matrix

\[ \frac{1}{F_{a \bar{b}'} (\cdots q^{a_{[N_k+N_\bar{a}]}} \cdots u^{[N_\bar{a}]} )} = M_{\bar{a} b}. \]

(50)
is invertible. Thus, the structure (15,16) is repeated completely.

At the same time, the number of the inner variables, the number of the equations, and the order of the set (19) are less than those of the initial set of the ELE (15,16).

Now, we apply the same procedure as in the first step to the reduced ELE (19). That will be the second step of the reduction procedure. It will produce equations of similar structure with less inner variables and of lower order. After the last \((I+1)\)-th step the ELE (13) may be written in the following strong equivalent form:

\[
q^{a_{i|k}[n_i+n_k]} = \varphi^{a_{i|k}} \left( \cdots q^{b_{i|k_+}[n_j+n_k]}, \cdots q^{b_{i|k_-}[n_j+n_k-1]], \cdots u^{u[n_k]}, \right),
\]

\[
I \geq k_+ \geq k + 1, \quad k \geq k_- \geq 0, \quad I \geq i, j \geq 0,
\]

where \(\varphi^{a_{i|k}}\) are some LF of the indicated arguments (the arguments \(\cdots q^{b_{i|k_+}[n_j+n_k]}\) result from those coordinates that intermediately have been considered as external ones), and the indices \(a_i\) are divided into the following groups:

\[
a_i = \left( a_{i|k} \right), \quad \left[ a_{i|k} \right] \geq 0, \quad \sum_k \left[ a_{i|k} \right] = \sum_k \left[ a_{k|i} \right] = \left[ a_i \right], \quad i, k = 0, 1, \ldots, I.
\]

The set (51) is still not the canonical form of the ELE. The reason is that the right hand sides of the set contain derivatives of orders that may exceed the orders \(n_i + n_k\) of the (highest) derivatives \(q^{a_{i|k}[n_i+n_k]}\) appearing on the left hand side of the set. We recall that by the definition in the canonical form there is a subordination of derivative orders, namely, the orders of all the derivatives in the right hand sides have to be less than the ones on the left hand side. Explicitly, this subordination would require that the following inequalities should hold:

\[
n_j + n_{k_+} > n_j + n_k,
\]

\[
n_j + n_{k_-} > n_j + n_k - 1,
\]

which, because of the inequalities \(n_I > n_{I-1} > \cdots n_1 > n_0\), is true for the first line and the case \(k_- = k\) of the second line, and it is definitely not true for the cases \(k_- < k\).

Arranging the equations (51) (for fixed value of \(i\)) in descending order w.r.t. \(k\), and the arguments in the functions \(\varphi\) (for fixed value of \(j\)) also in descending order w.r.t. the value of \(k_+\) and \(k_-\), we get, when disregarding the common value \(n_j\), a quadratic matrix whose main diagonal (i.e. elements with \(k = k_-\)) contains the entries \(n_k - 1\), whereas the entries to the left of that diagonal are equal to \(n_k\), and to the right of that diagonal are equal to \(n_k - 1\). Therefore, below the main diagonal occur “good” derivatives, and above it occur “bad” derivatives not obeying the subordination requirement.

**Subordination procedure**

One can see that these ”bad” derivatives can be excluded from the right hand sides by the help of corresponding ”lower” equations of the set and their differential consequences (compare Eqs. (29) and (31) for the simple case \(I = 2\)). In what follows we call such an exclusion the subordination procedure.

In order to be more definite let us write down two arbitrary lines, \(\ell > k\), of the right hand sides of the set of equations (51) (for the highest derivatives only):

\[
\varphi^{a_{\ell|k}} \left( q^{b_{\ell|\ell}[n_j+n_k]}, \cdots q^{b_{\ell|\ell+1}[n_j+n_k]}, q^{b_{\ell|\ell}[n_j+n_{\ell-1}]} \cdots, q^{b_{\ell|k+1}[n_j+n_{\ell-1}]}, q^{b_{\ell|k}[n_j+n_{\ell-1}]}, \cdots q^{b_{\ell|0}[n_j+n_{\ell-1}]}, \right),
\]

\[
\varphi^{a_{i|k}} \left( q^{b_{i|\ell}[n_j+n_k]}, \cdots q^{b_{i|\ell+1}[n_j+n_k]}, q^{b_{i|\ell}[n_j+n_k]}, \cdots, q^{b_{i|k+1}[n_j+n_k]}, q^{b_{i|k}[n_j+n_{k-1}]}, \cdots q^{b_{i|0}[n_j+n_{k-1}]}, \right).
\]
Obviously, because \( n_\ell > n_k \) all the derivatives of the equation for \( q_{a|\ell[i|n_\ell+n_k]} \) with 
\( k \geq \ell_- \geq 0 \) are “bad” with respect to the derivatives \( q_{a|\ell[i|n_\ell+n_k]} \) (remind \( \ell \geq \ell_- \geq 0 \)). However, these “bad” derivatives can be eliminated by the equations for the latter ones, \( q_{a|\ell[i|n_\ell+n_k]} \), and their differential consequences up to the order \( n_\ell - n_k - 1 \). Thereby, the function \( \phi_{a|\ell} \) changes into some new function \( \tilde{\phi}_{a|\ell} \). One can see that doing this we do not change the highest orders of derivatives of the other coordinates, both proper and external ones, in the right hand side of the equation for \( q_{a|\ell[i|n_\ell+n_k]} \). (Remind, that the derivatives of the external coordinates are \( u^{\mu[n_\ell]} \) and \( u^{\mu[n_k]} \), respectively.)

This subordination procedure, starting with \( \ell = I \) may be done for any \( k < I \), thereby “cleaning” every entry on the right hand side of equations for \( q_{a|\ell[i|n_\ell+n_i]} \). Namely, the highest orders of derivatives on the r.h.s. become \( q_{b|j|k_-[n_\ell+n_k]} \) with \( I \geq k_- \geq 0 \) (for the case \( \ell = I \) no \( k_+ \) appears). Then the procedure will be applied to the equations for \( q_{a|\ell-1[i|n_\ell+n_{i-1}]} \), and so forth, up to \( q_{a|0[i|n_\ell+n_0]} \), where nothing is to be changed.

After having eliminated all the “bad” derivatives, we transformed the set \( (51) \), and therefore the initial ELE, to the following strong equivalent (the equivalence is justified by the Lemma 1) canonical form

\[
q_{a|i|k} = \varphi_{a|i|k} \left( \ldots q_{b|j|k_+[n_j+n_k]}, \ldots q_{b|j|k_-[n_j+n_k-1]}, \ldots ; u^{\mu[n_k]} \right),
\]

\[
I \geq k_+ \geq k + 1, \ k \geq k_- \geq 0, \ i, j, k = 0, 1, \ldots, I,
\]

where \( \varphi_{a|i|k} \) are some LF of the indicated arguments. This proves the Theorem 1.

We see that there are no gauge coordinates in the nonsingular ELE.

The number of the initial data is equal to \( 2 \sum a N_a \). Indeed,

\[
\sum_{i,k} [a_{i|k}] (N_i + N_k) = \sum_i \left( N_i \sum_k [a_{i|k}] \right) + \sum_k \left( N_k \sum_i [a_{i|k}] \right) = 2 \sum a N_a.
\]

One ought to remark that in the general case there exist many different canonical forms of the nonsingular ELE. This uncertainty is related to the possibility of different choices of nonzero minors of a matrix with a given rank (different divisions of the indices \( a_i \) in course of the reduction procedure). However, as it was demonstrated above, the number of the equations in the canonical form (which is equal to the number of the ELE in the nonsingular case) and the number of the initial data is the same for all possible canonical forms.

4 Canonical form of singular ELE

Studying the canonical form of nonsingular ELE, we have demonstrated that the equations in the canonical form are solved with respect to the highest-order derivatives \( q_{a|i|k[n_i+n_k]} \), where \( n_i \) are the proper orders of the coordinates \( q_{a|i} \). However, considering specific examples, one can see that this is not always true for singular ELE. Namely, in the canonical form of the latter case, the highest orders of the derivatives \( q_{a|i|\ell} \) may take on all the values from zero to \( n_i + I \). The reduction procedure to the canonical form for the general singular ELE is considered below. In the singular case, already after the first step of the reduction procedure, the ELE cease to have their initial specific structure (\( 15,16 \)). Namely, the simple structure of terms with highest-order derivatives in the equations may
be lost. That is why in the singular case it is more convenient to formulate the reduction procedure for a more general set of ordinary differential equations, which contains the ELE as a particular case. Namely, further we are going to consider a set of the form

\[ F_{A\mu} (\cdots q^{ai[i+\mu]}) = 0 ; \quad i = 0, 1, \ldots, I , \quad \mu = 0, \ldots, J . \]  

(52)

Here \( F_{A\mu} (\cdots q^{ai[i+\mu]}) \) are some LF. Via \( a_i \) and \( A\mu \) are denoted sets of indices, \([a_i] \geq 0, [A\mu] \geq 0 \), and the complete set of the inner coordinates in Eqs. (52) is \( q^\rho = (q^{a_0}, \ldots, q^{a_I}) \), \( a = (a_i, \quad i = 0, 1, \ldots, I) \). The indices \( A = (A\mu) \) enumerate the equations. In the general case the number of the indices \( A \) (the number of all the equations) is not equal to the number of the indices \( a \) (the number of the coordinates). The division of the indices \( A \) into the groups is not related to the division of the indices \( a \) into the groups. The orders of the coordinates \( q^{a_i} \) in the complete set (52) are: \( N_{a_i} = i + J \). In fact, these orders are defined by a subset of (52) with \( \mu = J \). In all the other equations with \( \mu < J \) the coordinates \( q^{a_i} \) have the orders less than \( i + J \). Thus, the latter equations are constraints.

Similar to the ELE (15), the set (52) has the following specific structure: In each equation of the set the order of a coordinate \( q^{a_i} \) is the sum of the proper order \( i \) and of the order \( \mu \). The latter is the same for all the coordinates and is related to the number of the equation in the set.

Below we consider the reduction procedure to the canonical form for the equations (52). In fact, this reduction procedure is formulated in the proof of the Theorem 2 given below. The Theorem 2 holds true under certain suppositions of the structure of the functions \( F_{A\mu} \). These suppositions are formulated as suppositions of the ranks of some Jacobi matrices involving the functions \( F_{A\mu} \). First of all, the complete matrix

\[ M_{A\mu a_i} = \frac{\partial F_{A\mu}}{\partial q^{a_i[i+\mu]}} = \frac{\partial F^{[J-\mu]}_{A\mu}}{\partial q^{a_i[i+J]}} ; \]  

(53)

has to have a constant rank in a vicinity of the consideration point (one can see that the matrix \( M_{A\mu a_i} \) coincides with generalized Hessian matrix if the set (52) is the Lagrangian one).

Theorem 2: Under certain suppositions of the ranks, the equations (52) can be transformed to the following equivalent canonical form:

\[ f^{a_i[\sigma]} = q^{a_i[\sigma][i+\sigma]} - \varphi^{a_i[\sigma]} (\cdots q^{a_j[\sigma][j+\sigma-1]} \cdots q^{a_j[\sigma][j+\sigma]} ) = 0 , \]

\[ i, j = 0, 1, \ldots, I , \quad \sigma = -I, \ldots, J , \quad -I \leq \sigma_- \leq \sigma , \quad \sigma + 1 \leq \sigma_+ \leq J + 1 , \]  

(54)

where all the indices \( a \) are divided into groups as follows:

\[ a_i = (a_i[\sigma]) , \quad [a_i[\sigma]] \geq 0 , \quad \sigma = -I, \ldots, J + 1 , \quad ([a_i[\sigma]] = 0 \text{ if } i + \sigma < 0) , \]  

(55)

and it is thought that negative powers of the time derivatives do not exist, that is: \( [q^{a[\rho]}] = 0 \) for \( p < 0 \).
Moreover, the following equivalence between the corresponding LF holds true:

\[ F_A \sim \bar{F}_A = \left( \begin{array}{c} f_{a_i|\sigma} \\ 0_G \end{array} \right), \quad A = (a_{i|\sigma}, G), \quad i = 0, 1, ..., I, \quad \sigma = -I, ..., J, \]

\[ 0_G \equiv 0 \quad \forall G, \quad [G] = [A] - [a] + \sum_i [a_{i|J+1}]. \quad (56) \]

That implies

\[ F_A = \hat{U}_B^A \bar{F}_B, \quad \bar{F}_B = \hat{V}_A^B F_A, \quad \hat{U}_A^B \hat{V}_B^C = \delta_A^C, \quad (57) \]

where \( \hat{U} \) and \( \hat{V} \) are LO.

Let us make some comments to the Theorem 2. The canonical form (54) of the singular ELE differs from that (38) of the nonsingular ELE. As was demonstrated in the previous Sect., in the latter case the spectrum of the orders of the variable \( q^{a_i} \) in the canonical form extends from \( i + \mu_{\text{min}} \) to \( i + J \). In the singular case, we have to admit (and one can see this on specific examples) the spectrum extends from 0 to \( i + J \). Under such a supposition we can justify by the induction the structure (54) of the canonical form. One can see from (55) that each group of the indices \( a_i \) is divided in subgroups \( a_i \rightarrow a_{i|\sigma}, \sigma = -I, ..., J + 1 \). In the canonical form the singular ELE are solved with respect to the highest-order derivatives \( q^{a_{i|J+1}} \). These coordinates enter the set (54) as arbitrary functions of time. They are gauge coordinates according to the general definition. As in the nonsingular case, it is supposed that no coordinate \( q^{a_{i|\sigma}} \) in the function \( \varphi^{a_{i|\sigma}} \) has an order greater than \( k + \sigma \) (the proper order plus \( \sigma \)). Besides, the order of the coordinates \( q^{b_{|J+1}} \) in the function \( \varphi^{a_{i|\sigma}} \) has to be less than \( k + \sigma \).

We are going to prove the Theorem 2 by induction w.r.t. \( N = I + J \). To this end, we consider first equations of lower orders, then we use an induction to prove the general case.

4.1 Equations of lower orders

Remark that the case \( N = 0 \) implies \( I = J = 0 \) and the set (52) is reduced to form

\[ F_A (q) = 0, \quad q = (q^a). \quad (58) \]

Here the Theorem 2 holds true by virtue of the Lemma 3 from the Appendix.

Let \( N = 1 \). That implies either \( I = 1, \quad J = 0 \) or \( I = 0, \quad J = 1 \). Consider, for example, the first case. Here \( (i = 0, 1, \mu = 0) \) and the set (52) reads

\[ F_A (q^{a_0}, q^{a_1}, q^{a_1}) = 0, \quad [a_1] > 0, \quad [a_0] \geq 0. \quad (59) \]

In the case under consideration the supposition (53) reads:

\[ \frac{\partial F_A}{\partial q^{a_i}} = r. \quad (60) \]

Then there exists a division of the indices: \( A = (A_1, A_2), \quad a_i = (a_{i/1}, a_{i/2}), \quad [A_1] = [a_{0/1}] + [a_{0/1}] = r, \) such that
\[
\det \left| \frac{\partial F_{A/1}}{\partial q^{a_{i/1}[i]}} \right| \neq 0 .
\]

Thus, we may solve the equations \( F_{A/1} = 0 \) with respect to \( q^{a_{i/1}[i]} \),
\[
F_{A/1} = 0 \iff q^{a_{i/1}[i]} = \phi^{a_{i/1}} \left( q^{b_{i/1}[i-1]}, q^{b_{i/2}[i-1]}, q^{b_{i/2}[i]} \right) .
\] (61)

Then we exclude the arguments \( q^{a_{i/1}[i]} \) from the functions \( F_{A/2} \) by the help of (61),
\[
\bar{F}_{A/2} = F_{A/2} \bigg|_{q^{a_{i/1}[i]} = \phi^{a_{i/1}}} = \bar{F}_{A/2} \left( q^{a_{1}} \right) .
\]

By virtue of the Lemma 2 from the Appendix, the functions \( \bar{F}_{A/2} \) depend on \( q^{a_{1}} \) only . Thus, we have the equivalence\( ^4 \)
\[
F_A \sim \bar{F}_A = \left( \begin{array}{c}
F_{A/1} \left( q^{a_0}, q^{a_{1}}, \bar{q}^{a_{1}} \right) \\
\bar{F}_{A/2} \left( q^{a_{1}} \right)
\end{array} \right) . \] (62)

Now we suppose that the matrix \( \partial \bar{F}_{A/2} / \partial q^{a_{1}} \) has a constant rank. Therefore (see Lemma 3)
\[
\bar{F}_{A/2} \sim \left( \begin{array}{cc}
q^{a_{1}} - \varphi^{a_{1}} (\bar{q}^{a_{1}}) \\
0_{G_1}
\end{array} \right) , \ a_1 = (a_1, \bar{a}_1) .
\]

Let us exclude the arguments \( q^{\bar{a}_{1}}, \bar{q}^{\bar{a}_{1}} \) from the functions \( F_{A/1} \), by the help of the equations \( q^{\bar{a}_{1}} = \varphi^{\bar{a}_{1}} (q^{a_{1}}) \),
\[
\left. \frac{1}{F_{A/1}} \right|_{\varphi^{\bar{a}_{1}} = \varphi^{\bar{a}_{1}}} = F_{A/1} \bigg|_{q^{\bar{a}_{1}} = \varphi^{\bar{a}_{1}}} .
\]

Then the equivalence holds true:
\[
F \sim \frac{1}{F} = \left( \begin{array}{c}
\left. \frac{1}{F_{A/1}} \right|_{q^{\bar{a}_{1}} = \varphi^{\bar{a}_{1}}} \\
q^{\bar{a}_{1}} - \varphi^{\bar{a}_{1}} (q^{a_{1}}) \\
0_{G_1}
\end{array} \right) , \ a = (a_0, a_1) , \ a_1 = (a_1, \bar{a}_1) . \] (63)

The set of functions \( \frac{1}{F} \) has the same structure as the initial set \( F \). However, the number of the nonzero functions \( \frac{1}{F} \) is less than the number of the functions \( F \). Moreover, some of the functions \( \frac{1}{F} \) depend linearly on a part of the variables. That is why the supposition of the type (60) for the functions \( F^{(1)} \) is reduced to the supposition about the rank of the matrix \( \partial \frac{1}{F_{A/1}} / \partial (q^{a_0}, \bar{q}^{a_{1}}) \). Accepting the latter supposition we apply the above reduction procedure to the functions \( \frac{1}{F} \) and so on. After the \( i \)-th stage we have the following equivalence:
\[
F \sim \frac{i}{F} = \left\{ \begin{array}{c}
\left. \frac{i}{F_{A/1}} \right|_{q^{\bar{a}_{i}} = \varphi^{\bar{a}_{i}} (q^{a_{i}})} \\
q^{\bar{a}_{i}} - \varphi^{\bar{a}_{i}} (q^{a_{i}}) \\
0_{G_i}
\end{array} \right) , \ a = (a_0, a_1) , \ a_1 = (a_i, \bar{a}_i) .
\]

\( ^4 \)Here, and in what follows, we use Lemma 1 to justify the equivalence.
4.2 Equations of arbitrary orders

The procedure ends at a $k$-th stage when

$$\operatorname{rank} \frac{\partial F_{A/k}}{\partial (\dot{q}^{a_1}, q^{a_0})} = [A_k] .$$

Then there exists a division of the indices $\bar{a}_k = (a_1|0, a_{g_1})$, $a_0 = (a_{0|0}, a_{g_0})$, $[a_1] + [a_0|0] = [A_k]$, such that

$$\det \frac{\partial F_{A/k}}{\partial (\dot{q}^{a_1}, q^{a_0})} \neq 0 \implies F_{A/k} \sim \left( \begin{array}{c} \dot{q}^{a_1} - \varphi^{a_1} (q^{a_1}, q^{a_0}, q^{a_0}, \dot{q}^{a_1}) \\ q^{a_0} - \varphi^{a_0} (q^{a_1}, q^{a_0}, q^{a_0}, \dot{q}^{a_1}) \\ \dot{q}^{a_1} - \varphi^{a_1} (q^{a_1}, q^{a_0}) \\ 0_G \end{array} \right).$$

Denoting $a_k \equiv a_{i-1}$, $G = G_k$, such that $a = (a_{i-1}, a_{0|0}, a_{1|0}, a_g)$, and $a_g = (a_{g_0}, a_{g_1})$, $[G] = [A] - [a] + [a_g]$, we get finally the equivalence:

$$F \sim \left( \begin{array}{c} \dot{q}^{a_1} - \varphi^{a_1} (q^{a_1}, q^{a_0}, q^{a_0}, \dot{q}^{a_1}) \\ q^{a_0} - \varphi^{a_0} (q^{a_1}, q^{a_0}, q^{a_0}, \dot{q}^{a_1}) \\ \dot{q}^{a_1} - \varphi^{a_1} (q^{a_1}, q^{a_0}) \\ 0_G \end{array} \right).$$

(64)

Here $q^{a_g} = (q^{a_{g_0}}, q^{a_{g_1}})$ are gauge coordinates. Thus, the Theorem 2 holds true in this case.

The case $I = 0$, $J = 1$ ($i = 0$, $\mu = 0, 1$) corresponds to the equations of the form

$$F_{A_1} (\dot{q}^{a_1}, q^{a_1}) = 0 , \quad F_{A_0} (q^{a_1}) = 0 .$$

(65)

Such equations present a particular case ($[a_0] = 0$) of the equations $F_A = 0$ with the LF $F_A$ defined in (62). The reduction procedure for the latter case was considered above. It leads to the following equivalence

$$F \sim \left( \begin{array}{c} \dot{q}^{a_1} - \varphi^{a_1} (q^{a_1}, q^{a_0}, \dot{q}^{a_0}) \\ q^{a_0} - \varphi^{a_0} (q^{a_1}, \dot{q}^{a_0}) \\ 0_G \end{array} \right), \quad a = (a_{0|0}, a_{1|0}, a_g) , \quad [G] = [A] - [a] + [a_g].$$

Here $q^{a_g}$ are the gauge coordinates. Thus, the Theorem holds true in this case as well.

4.2 Equations of arbitrary orders

We have verified that the Theorem 2 holds true for $N = 0, 1$. Now we are going to prove the theorem for $N = I + J = K$ (where $K$ is some fixed number) supposing that the theorem holds true for any $N < K$.

At the first step we consider the set

$$F^{[J-\mu]}_{A_\mu} (\cdots q^{a_{i+J}}) = 0 , \quad i = 0, 1, ..., I , \quad \mu = 0, ..., J ,$$

(66)

which is obtained from the initial set (52) by substituting the constraints by the corresponding consistency conditions (conditions obtained from the constraints $F_{A_\mu}$ by $J - \mu$ time differentiations). According to the supposition (53), there exists a division of the indices $A_\mu$ and $a_i$ as: $A_\mu = (A_{\mu/1}, A_{\mu/2})$, $a_i = (a_{i/1}, a_{i/2})$, $\sum_{\mu} [A_{\mu/1}] = \sum_i [a_{i/1}] = r$, such that:

$$\det \frac{\partial F^{[J-\mu]}_{A_{\mu/1}}}{\partial q^{a_{i/1}[i+J]}} \neq 0 .$$

(67)
Thus, we may solve the equations $F_{A_{p+1}}^{-[J-p]} = 0$ with respect to the derivatives $q^{a_i/[i,j]}$. Namely,

$$F_{A_{p+1}}^{-[J-p]} = 0 \iff q^{a_i/[i,j]} = \varphi^{a_i} (\ldots q^{b_j/[j+J-1]} \ldots q^{b_j/[j+J]}).$$

(68)

Now we pass from the functions $F_{A_{J/2}}$ to the ones $\bar{F}_{A_{J/2}}$ excluding the arguments $q^{b_j/[i,j]}$ from the former,

$$\bar{F}_{A_{J/2}} = F_{A_{J/2}} \bigg|_{f=0} = \bar{F}_{A_{J/2}} (\ldots q^{b_j/[i+j-1]}).$$

(69)

The fact that the functions $\bar{F}_{A_{J/2}}$ do not depend on both $q^{b_j/[i,j]}$ and $q^{b_j/[i,j]}$ is based on the Lemma 2 from the Appendix. Thus, we have the equivalence (see Lemma 1 from the Appendix)

$$F_A \sim \begin{pmatrix} F_{A_{J/2}} \\ F_{A_{J/2}} \end{pmatrix} \sim \begin{pmatrix} F_{A_{J/2}} \\ F_{A_{J/2}}' \end{pmatrix},$$

(70)

where

$$F'_{A_{J/2}} = (F'_{A_{J/2}}, \nu = 0, \ldots, J - 1) = \begin{cases} F_{A_{J/2}} (\ldots q^{b_j/[i+j]}), & \nu = 0, \ldots, J - 2 \\ F'_{A_{J/2}} (\ldots q^{b_j/[i+j-1]}) = \begin{pmatrix} F_{A_{J/2}} \\ F_{A_{J/2}} \end{pmatrix}. \end{cases}$$

(71)

Let us turn to the functions $F'_{A_{J/2}}$. They have the same structure as in (72) and correspond to the case $N = I + J < K$. In accordance with the induction hypothesis, supposing, in particular, that the matrix

$$M'_{A_{J/2}} \frac{\partial F'_{A_{J/2}}}{\partial q^{a_i/[i,j]}}$$

has a constant rank in the consideration point the following equivalence holds true:

$$F'_{A_{J/2}} \sim \begin{pmatrix} q^{a_i/[i+j]} - \varphi^{a_i} \left( q^{b_j/[j+J-1]} \ldots q^{b_j/[j+J]} \right) \\ 0_{G'} \end{pmatrix},$$

$$i, j = 0, 1, \ldots, I, \quad [G'] = [A'] - [a] + \sum_i [a_i], \quad \sigma = -I, \ldots, J - 1, \quad -I \leq \sigma_- \leq \sigma, \quad \sigma + 1 \leq \sigma_+ \leq J - 1.$$ 

(72)

Taking into account (71), we obtain

$$F \sim \begin{pmatrix} q^{a_i/[i+j]} - \varphi^{a_i} \left( q^{b_j/[j+J-1]} \ldots q^{b_j/[j+J]} \right) \\ 0_{G'} \end{pmatrix},$$

$$i, j = 0, 1, \ldots, I, \quad \sigma = -I, \ldots, J - 1, \quad -I \leq \sigma_- \leq \sigma, \quad \sigma + 1 \leq \sigma_+ \leq J - 1.$$ 

(73)

Now we pass from the functions $F_{A_{J/1}}$ to the ones $\bar{F}_{A_{J/1}}$ excluding the arguments $q^{a_i/[i,j]}$, $p_i \geq i + \sigma, \sigma = -I, \ldots, J - 1$ from the former. As a result, the following equivalence takes place:

$$F \sim \bar{F} = \begin{pmatrix} q^{a_i/[i+j]} - \varphi^{a_i} \left( q^{b_j/[j+J-1]} \ldots q^{b_j/[j+J]} \right) \\ 0_{G'} \end{pmatrix}.$$ 

(74)
The functions \( \tilde{F} \) have the same structure as in (52), however, they depend linearly on a part of highest-order derivatives. Here the supposition of the rank for the matrix

\[
\frac{\partial \tilde{F}}{\partial (q^{a_{i|J}[i+J]}, q^{a_{i|\sigma}[i+\sigma]})}, \quad A = (A_{J/1}, a_{i|\sigma}, G')
\]

(75)

is equivalent to the same supposition for the matrix

\[
\frac{\partial \tilde{F}_{A_{J/1}}}{\partial q^{b_{i|J}[i+J]}},
\]

(76)

Let this rank be equal to \([A_{J/1}] \). In this case there exists a final division of indices,

\[a_{i|J} \rightarrow (a_{i|J}, a_{i|J+1}) \text{ with } [a_{i|J}] = [A_{J/1}],\]

such that the equations \( \tilde{F}_{A_{J/1}} = 0 \) can be solved with respect to the derivatives \( q^{a_{i|J}[i+J]} \) and we obtain, instead of the two first lines of (74), the following expressions:

\[
q^{a_{i|J}[i+J]} - \varphi^{a_{i|J}} \left( \ldots q^{b_{j|J}[j+J-1]}, \ldots q^{b_{j|\sigma}[j+\sigma-1]}, \ldots q^{b_{j|J+1}[j+J]} \right),
\]

\[
q^{a_{i|\sigma}[i+\sigma]} - \varphi^{a_{i|\sigma}} \left( \ldots q^{b_{j|\sigma}[j+\sigma-1]}, \ldots q^{b_{j|\sigma+1}[j+\sigma]}, \ldots q^{b_{j|J}[j+\sigma]} \right),
\]

\[i, j = 0, 1, \ldots, I, \quad \sigma = -I, \ldots, J - 1, \quad -I \leq \sigma_- \leq \sigma, \quad \sigma + 1 \leq \sigma_+ \leq J - 1.\]

Now, let us put together the first two entries of \( \varphi^{a_{i|J}} \) as \( \ldots q^{b_{j|\sigma}[j+\sigma-1]} \), \(-I \leq \sigma \leq J \) and remind that for \( \sigma = J \) no corresponding \( \sigma_+ \) occurs. Furthermore, let us replace the last entry of \( \varphi^{a_{i|\sigma}} \) as follows: \( \ldots q^{b_{j|J}[j+\sigma]} \rightarrow \ldots q^{b_{j|J}[j+\sigma]} \ldots q^{b_{j|J+1}[j+\sigma]} \), \(-I \leq \sigma \leq J - 1 \), then we get the missing contribution to \( \sigma_+ \) for the case under consideration. So, we end up exactly with Eq. (54) and the Theorem 2 is proved.

If the rank is less than \([A_{J/1}] \) then the above procedure is applied to the functions \( \tilde{F}_{A_{J/1}} \). Doing that we lower the number of the equations that are not yet reduced to the canonical form (the equations of the type \( \tilde{F}_{A_{J/1}} = 0 \)). Remark that such a diminution does not happen at the first stage if \([A_{J/2}] = 0 \). At a certain stage the procedure does not lower the number of the above mentioned equations. This can happen when the rank of the matrix of the type (76) is maximal, i.e. is equal to the number of the functions of the type \( \tilde{F}_{A_{J/1}} \). In such a case we may reduce them to the canonical form as was said above. This can also happen when we do not obtain the functions of the type \( F_{A_{J/4}} \) in the reduction procedure. That means that already at the previous step the set is reduced to the case \( N = K - 1 \), i.e. the possibility of the reduction to the canonical form is proved.

Finally we stress that the reduction procedure is formulated for sets of equations of the type (52) (the ELE are a particular case of such sets). The procedure holds true under certain suppositions of ranks. These suppositions demand various Jacobi matrices of the type \( \partial F_s / \partial q^{a[l]} \) to have constant ranks in the vicinity of the consideration point. Here \( F_s = 0 \) are equations obtained to a given stage of the procedure and \( q^{a[l]} \) are highest-order derivatives in these equations. It is important to realize that proving the equivalence (58) we prove at the same time the locality of the operators \( \hat{U} \) and \( \hat{V} \) from (57). In fact, the latter proof is provided by the applicability of the Lemmas from the Appendix.
5 Gauge identities and action symmetries

It was demonstrated above that in the general case of singular ELE the number of the equations in the canonical form is less than the number of the equations in the initial set of the differential equations. This reduction is related to the fact that in the canonical form we retain the independent equations only, whereas the initial equations may be dependent. The dependence of the equations in the initial set may be treated as the existence of some identities between the initial equations. The identities between the ELE imply the existence of gauge transformations of the corresponding action. Below we discuss this interrelationship in detail.

First, we introduce some relevant definitions: The relation of the form

$$\hat{R}^a F_a \equiv 0,$$

(77)

where $\hat{R}^a$ are some LO, and $F_a (q^{[l]}$) are some LF, is called the identity between the equations $F_a (q^{[l]} = 0$. The identity sign $\equiv$ means that the left hand side of (77) is zero for any arguments $q^{[l]}$.

Any set $\hat{R} = (\hat{R}^a)$ of LO that obeys the relation (77) is called the generator of an identity. Whenever $\hat{R}$ is a generator than $\hat{n} \hat{R}$ with some LO $\hat{n}$ is a generator as well. Any linear combination $\hat{n}_i \hat{R}_i$ of some generators $\hat{R}_i$ with operator coefficients $\hat{n}_i$ is a generator.

A generator $\hat{R}$ will be called nontrivial if the relation $\hat{n} \hat{R} = \hat{O} (F)$ can only be provided by a LO $\hat{n}$ of the form $\hat{n} = \hat{O} (F)$.

A set of generators $\hat{R}_i$ will be called independent if the relation $\hat{n}_i \hat{R}_i = \hat{O} (F)$ can only be provided by $\hat{n}_i$ of the form $\hat{n}_i = \hat{O} (F)$. Identities generated by independent generators will be called independent.

Note that for any set of LF $F_a$, there always exist trivial generators. Namely, the generators $\hat{R}_{\text{triv}} = (\hat{R}_{\text{triv}}^a)$ of the form

$$\hat{R}_{\text{triv}}^a = \sum_{k,l} F_b^{[k]} u^{bk|al} \frac{d^l}{dt^l}, \ u^{nk|al} = -u^{al|bk},$$

(78)

with arbitrary antisymmetric LF $u^{bk|al}$ obviously lead to the identities (77). These identities are not, however, connected to the mutual dependence of the functions $F_a$.

An independent set of generators $\hat{R}_g$ is complete whenever any generator $\hat{R}$ can be represented in the form $\hat{R} = \hat{\lambda}^g \hat{R}_g + \hat{R}_{\text{triv}}$ with some LO $\hat{\lambda}^g$. Any two complete sets of independent generators $\hat{R}_g$ and $\hat{R}_g'$ are related as $\hat{R}_g' = \hat{U}^{g'} \hat{R}_g + \hat{R}_{\text{triv}}$, where $\hat{U}$ is an invertible LO.

Supposing now that $F_a$ in Eq. (77) are functional derivatives of an action, $F_a = \delta S/\delta q^a$, such that $F_a = 0$ are ELE. Let the functions $F_a$ obey all the necessary suppositions of ranks such that ELE can be reduced to the canonical form (54). Let us write here this canonical form as follows:\footnote{By $\hat{O}(F)$ we denote LO of the form $O(F)$ with all the LF $u^k_a = O(F)$, where $O(F)|_{F=0} = 0$.}

$$f^a = q^{a[l_\alpha]} - \varphi^a (\cdots q^{a[l_{\alpha-1}]}; \cdots q^{a[l_\beta]}) = 0, \ a = (\alpha, g),$$

(79)

\footnote{Here, we do not distinguish possible different proper orders of the coordinates.}
where \( q^g \) are gauge coordinates. Moreover, according to the Theorem 2, there exists the equivalence

\[
F_a \sim \tilde{F}_a = \begin{pmatrix} f^a_0 \\ f^a_g \end{pmatrix} \implies F_a = \hat{U}_a^b \tilde{F}_b, \quad \tilde{F}_a = \hat{V}_a^b F_b, \quad \hat{U}_a^b \hat{V}_b^c = \delta^c_a, \tag{80}
\]

where \( \hat{U} \) and \( \hat{V} \) are LO. Now we may consider the identity \((77)\) as an equation for finding the general form for the generator \( \hat{R} \). Using \((80)\) we transform this problem to the one for finding the operators \( \hat{\xi} \),

\[
\hat{\xi}^a \tilde{F}_a \equiv 0, \quad \hat{\xi}^b \hat{V}_b^a. \tag{81}
\]

Using the explicit form \((80)\) of the functions \( \tilde{F}_a \), we get \( \hat{\xi}^a = \begin{pmatrix} \hat{\xi}^\alpha \\ \hat{\xi}^g \end{pmatrix}, \quad a = (\alpha, g), \)

where \( \hat{\xi}^\alpha \) obey the equation

\[
\hat{\xi}^\alpha f^\alpha \equiv 0, \tag{82}
\]

and \( \hat{\xi}^g \) is a set of arbitrary LO. Since the functions \( f \) have the canonical form \((80)\), any solution of the equation \((82)\) is presented by trivial generators of the form

\[
\hat{\xi}^\alpha = \hat{\xi}_{\text{triv}} = \sum_{k,l} \left( f^k_{\alpha} \right) u^{l\alpha'|k\alpha} q^k \frac{d}{dt^k}, \quad u^{l\alpha'|k\alpha} = -u^{k\alpha|l\alpha'}, \tag{83}
\]

where \( u^{l\alpha'|k\alpha} \) are arbitrary antisymmetric LF. To demonstrate that we present the generators \( \hat{\xi}^\alpha \) as \( \hat{\xi}^\alpha = \sum_{k=0}^K \xi^{ak} d^k/dt^k \), where \( \xi^{ak} \) are some LF. Then, in the equation \((84)\), we pass from the variables \( q^{[k]} \), \( q^{[l]} \), \( k, l = 0, 1, ... \) to ones \( q^{[k\alpha]} \), \( f^l_{\alpha} \), \( q^{[l]} \), \( k_{\alpha} = 0, 1, ... , l_{\alpha} - 1, \quad l = 0, 1, ... \). Such a variable change is not singular. In terms of the new variables, the equation \((82)\) reads

\[
\sum_{k=0}^K \xi^{ak} f^k_{\alpha} = 0, \quad K < \infty.
\]

Its general solution is well known

\[
\xi^{ak} = \sum_l f^l_{\alpha} u^{l\alpha'|k\alpha}, \quad u^{l\alpha'|k\alpha} = -u^{k\alpha|l\alpha'}. \]

Now we can write the general solution of the equation \((81)\) as:

\[
\hat{\xi}^a = \hat{\xi}^g \delta^g_a + \hat{\xi}_{\text{triv}}, \quad \hat{\xi}_{\text{triv}} = \sum_{k,l} \left( f^k_b \right) u^{l\alpha'|k\alpha} d^k dt^k, \quad u^{l\alpha'|k\alpha} = -u^{k\alpha|l\alpha'}. \tag{84}
\]

Let \( b = (\alpha', g') \), \( a = (\alpha, g) \) in \((84)\). Then \( u^{l\alpha'|k\alpha}, \quad u^{l\alpha'|k\alpha} = -u^{k\alpha|l\alpha'} \) and \( u^{l\alpha'|kg} \) are arbitrary LF (e.g., they can be selected to be zero). Indeed, the functions \( u^{l\alpha'|k\alpha} \) and \( u^{l\alpha'|kg} \) do not enter the expressions for the generators \( \hat{\xi}^a \). Besides, terms with \( u^{l\alpha'|kg} \) affect only the generators \( \hat{\xi}^g \), which are arbitrary by the construction. Respectively, the general solution of the equation \((77)\) reads:

\[
\hat{R} = \hat{\xi}^g \hat{R}_g + \hat{R}_{\text{triv}}, \quad \hat{R}_g = (\hat{F}^a_g = \delta^a_g \hat{V}_b^a = \hat{V}^a_g), \tag{85}
\]

20
and
\[ \hat{R}_{\text{triv}}^a = \xi_{\text{triv}}^b \dot{V}_b^a = \sum_{k,l} \left[ \frac{d^l}{dt^l} \left( \dot{V}_c^e F_{c}^{e} \right) \right] u^{(b)kd} \frac{d^k}{dt^k} \dot{V}_d^a = \sum_{k,l} \left( \frac{d^l}{dt^l} F_b \right) T^{(b)ka} \frac{d^k}{dt^k}, \]

where \( T^{(b)ka} = -T^{(k)ab} \) are some LF. The set of the generators \( \hat{R}_g = (\hat{R}_a^g = \hat{V}_a^g) \) is complete and it is presented by LO. Moreover, these generators are independent. Indeed, multiplying the equation \( \hat{R}_a^g \dot{R}_a^g = \hat{O}(F) \) from the right by \( \hat{U}_b^a \), we get:
\[ \hat{R}_a^g \delta q^a = \hat{O}(F) = \Rightarrow \hat{R}_a^g = \hat{O}(F). \]

Thus, there exist the following nontrivial identities between the ELE:
\[ \hat{R}_a^g \delta S_{\delta q^a} \equiv 0, \quad g = 1, \ldots, r, \quad (86) \]

with generators \( \hat{R}_g \) that are LO. These identities are called the gauge identities. As is well known (see for example, [2, 3]), the existence of the gauge identities (86) implies the existence of infinitesimal gauge transformations of the form
\[ \hat{q}^a \rightarrow \hat{q}^a + \delta \hat{q}^a, \quad \delta \hat{q}^a = \left( \hat{R}^T_g \right)^a_g \delta q^a, \quad (87) \]

where \( r \) parameters \( \epsilon^a = \epsilon^a(t) \) are arbitrary functions of time \( t \). Remark that \( \hat{R}^T \) are LO as well.

Thus, it was demonstrated that for theories that obey appropriate suppositions of the ranks there exists a constructive procedure of revealing the gauge generators. For such theories all the generators are LO. The number of the independent generators and, therefore, the number of the independent gauge transformations is equal to the number of the gauge coordinates in the ELE.

As a simple mechanical example, consider the action of the form\[ S = \int L dt, \quad L = \frac{1}{2} (\dot{x} - y)^2 + \frac{a}{2} (y^2 - x^2). \quad (88) \]

The corresponding ELE are:
\[ F_1 = \ddot{x} - \dot{y} + ax = 0, \quad F_2 = \dot{x} - (1 + a) y = 0, \quad (89) \]

where \( F_2 = 0 \) is a constraint. The generalized Hessian reads:
\[ M = \begin{vmatrix} \frac{\partial^2 L}{\partial y^2} = 1 & \frac{\partial^2 L}{\partial x \partial y} = -1 \\ \frac{\partial^2 L}{\partial y \partial x} = -1 & \frac{\partial^2 L}{\partial y^2} = a + 1 \end{vmatrix} = a. \quad (90) \]

Let \( a \neq 0, \ M \neq 0 \). In such a nonsingular case the reduction procedure looks as follows:
By the help of the consistency condition \( \dot{F}_2 = 0 \Rightarrow \ddot{x} = (1 + a) \dot{y}, \) we eliminate \( \dot{x} \) from the first ELE. Thus, we get an equivalent set, which has the canonical form,
\[ \dot{y} = -x, \quad \dot{x} = (1 + a) y. \quad (91) \]

Another canonical form
\[ \ddot{x} = -(1 + a) x, \quad y = (1 + a)^{-1} \dot{x}, \quad (92) \]

\[ \text{At } a \neq 0 \text{ we have a finite-dimensional analog of the Proca action, and at } a = 0 \text{ we have the analog of the Maxwell action}. \]

21
we obtain eliminating $\dot{y}$ from the equation $F_1 = 0$ by the help of the consistency condition $F_2 = 0 \implies \dot{y} = \ddot{x} / (1 + a)$.

Let $a = 0$. The case is singular, $M = 0$, and rank of the Hessian matrix is equal to 1. One can easily see that the equivalence

$$\left( \begin{array}{c} F_1 \\ F_2 \end{array} \right) = \hat{U} \left( \begin{array}{cc} \dot{x} - y \\ 0 \end{array} \right), \quad \hat{U} = \left( \begin{array}{cc} d/dt & 1 \\ 1 & 0 \end{array} \right), \quad \hat{U}^{-1} = \left( \begin{array}{cc} 0 & 1 \\ 1 & -d/dt \end{array} \right)$$

holds true. Then the canonical form of the ELE reads $\dot{x} = y$ and there is a gauge identity

$$\hat{R}^a F_a \equiv 0, \quad \hat{R}^1 = 1, \quad \hat{R}^2 = -d/dt.$$ 

The operators transposed to $\hat{R}^a$ are $(\hat{R}^T)^a = \left( \begin{array}{cc} (\hat{R}^T)^1 = 1, & (\hat{R}^T)^2 = d/dt \end{array} \right)$. Thus, at $a = 0$, the action \[\text{(88)}\] is invariant under the gauge transformation $x \to x + \epsilon$, $y \to y + \dot{\epsilon}$. In the case under consideration, the ELE have two canonical forms: $\dot{x} = y$ and $y = \dot{x}$.

6 Concluding remarks

We have formulated the reduction procedure which allows one to transform the ELE to the canonical form as well as to establish possible gauge identities between the equations. The latter part of the procedure can be considered as a constructive way of finding all the gauge generators within the Lagrangian formulation. At the same time, it is proven that, for local theories, all the gauge generators are local in time operators. The canonical form of the ELE reveals their hidden structure, in particular, it presents the spectrum of possible initial data, and it allows one to separate coordinates into nongauge and gauge ones. One ought also to remark that the reduction procedure can be, in particular, treated as a procedure of finding constraints in the Lagrangian formulation.

In that respect one can compare the reduction procedure with the well-known Dirac procedure in the Hamiltonian formulation of constrained systems \[\text{(1, 2, 3)}\]. Recall that the Dirac procedure is applicable to the Hamilton equations with primary constraints, namely to equations of the form

$$F(\eta, \dot{\eta}) = \dot{\eta} - \{\eta, H^{(1)}\} = 0, \quad \Phi^{(1)}(\eta) = 0, \quad H^{(1)} = H(\eta) + \lambda \Phi^{(1)}(\eta). \quad (93)$$

Here $\eta = (q^a, p_a)$ are phase-space variables; $\Phi^{(1)}(\eta) = 0$ are primary constraints, $\lambda$’s are Lagrange multipliers to the primary constraints, and $H^{(1)}$ is the total Hamiltonian. Via $\{\cdot, \cdot\}$ the Poisson bracket is denoted. The aim of the procedure is to eliminate as many as possible $\lambda$’s from the equations, to find all the constraints in the theory. The procedure is based on the consistency conditions $\dot{\Phi}^{(1)} = 0$. Using the equations $F(\eta, \dot{\eta}) = 0$, we may transform any consistency condition to the following form:

$$\dot{\Phi}^{(1)} = \{\Phi^{(1)}, H^{(1)}\} = 0.$$ 

From these equations one can define some $\lambda$’s as functions of $\eta$ and reveal some new constraints. Then the procedure has to be applied to the latter constraints and so on.

The equations \[\text{(93)}\] present a particular case of differential equations considered in the present article (indeed, these equations are ELE for a Hamiltonian action). Thus,
our reduction procedure may be applied to these equations. Namely, first one has to consider the equations $F_A = 0$, $\Phi^{(1)} = 0$ and select independent w.r.t. $\bar{\eta}$ equations. Since equations of the primary constraints are independent by construction, we pass to the next step and solve the constraint equations $\Phi^{(1)} = 0$ with respect to a part of the variables $\eta$, as $\hat{\Phi}^{(1)} = 0 \rightarrow \eta_1 - \varphi_1(\eta_2) = 0$. Then we exclude $\eta_1$ and $\dot{\eta}_1$ from the equations $F = 0$. Thus, we get $F = 0 \rightarrow \bar{F}_A(\eta_2, \dot{\eta}_2) = 0$. Then one has to select independent w.r.t. $\dot{\eta}_2$ functions $\bar{F}_{A/2}$. At the same time one finds new constraints $\bar{F}_{A/2}(\eta_2) = 0$ and so on (see the Subsect. "First order equations").

We see that the Dirac procedure differs from our reduction procedure. Indeed, as was mentioned above, in the Dirac procedure one excludes all the derivatives $\dot{\eta}$ by the help of the equations $\bar{F} = 0$ from the consistency conditions $\hat{\Phi}^{(1)} = 0$. Thus, one gets equations for the Lagrange multipliers and new constraints. Besides, one of the aim of the Dirac procedure is to maintain the canonical Hamiltonian structure of the equations $F = 0$. The possibility of the Dirac reduction is due to the specific structure of the equations $(93)$. Namely, here the consistency conditions never involve $\dot{\lambda}$ and rank $\partial F/\partial \dot{\eta} = [F] = [\eta]$. Besides, one ought to mention the work [14] where it was proposed an alternative (to the Dirac procedure) way of reducing the equations of motion for theories with actions of the form $S = \int \left[ \varphi_A(\eta) \dot{\eta}^A - V(\eta) \right] dt$. One can verify that, in fact, the procedure of that work, in a part (the procedure does not reveal the gauge identities), is similar to our reduction procedure in the case of the first order equations (see Sect. IV).

However, the reduction procedure proposed in the present article is formulated for a wider class of Lagrangian systems (differential equations). It does not need the introduction of new variables such as momenta and Lagrange multipliers, and it is defined in the framework of the initial Lagrangian formulation. Moreover, its aim is twofold: to reduce ELE to their canonical form and to reveal the gauge identities between the ELE equations.

The consideration in the present article is restricted by finite-dimensional systems. Its application to field theories (theories with infinite number degrees of freedom) demands additional study. We hope to present the corresponding formulation in futures publications. However, in simple cases, one can apply the present reduction procedure with some natural modifications in the infinite-dimensional case. Consider the Maxwell action $S = -(1/4) \int F_{\mu\nu} F^{\mu\nu} dx$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ as a common example of a singular field theory. The ELE read:

$$F^i = \frac{\partial S}{\partial A_i} = \partial_\nu F^{\nu\mu} = \dot{A}^i + \partial_i \dot{A}^0 - \Delta A^i + \partial_i \varphi = 0,$$  \hspace{1cm} (94)  

$$F^0 = -\frac{\partial S}{\partial A^0} = \partial_\nu F^{\nu 0} = \varphi + \Delta A^0 = 0, \hspace{0.5cm} \varphi = \partial_k A^k.$$  \hspace{1cm} (95)  

The equation $F^0 = 0$ is a constraint. Following the reduction procedure, we have to consider the set $F_i = 0$, $\dot{F}_0 = 0$. The Jacobi matrix $\partial F^{\mu}/\partial \dot{A}_\nu$ has the constant rank 3. We can, for example, select the equations $(94)$ as independent with respect to the derivatives $\dot{A}^i$. The equation $\dot{F}_0 = 0$ is their consequence. No more constraints appear. Now we exclude $A^0$ and $\dot{A}^0$ from $(94)$ by the help of $(93)$. That creates the equivalence

$$\begin{pmatrix} F^i \\ F^0 \end{pmatrix} = \begin{pmatrix} \delta^i_k \\ 0 \end{pmatrix} \begin{pmatrix} \dot{A}^k \\ F^0 \end{pmatrix}, \hspace{1cm} \dot{F}^k = F^k \bigg|_{F^0=0} = \Box (A^k + \partial_k \varphi).$$  \hspace{1cm} (96)  

23
Now we discover that the functions $\vec{F}_k$ are dependent, $\partial_k \vec{F}_k \equiv 0$. In our terms that reads, for example, as the following equivalence

$$
\begin{pmatrix}
\vec{F}_1 \\
\vec{F}_2 \\
\vec{F}_3 
\end{pmatrix}
=
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-\partial_3^{-1} \partial_1 & -\partial_3^{-1} \partial_2 & 1 
\end{pmatrix}
\begin{pmatrix}
\vec{F}_1 \\
\vec{F}_2 \\
0 
\end{pmatrix}.
\tag{97}
$$

The equations $\vec{F}_1 = 0, \vec{F}_2 = 0, \vec{F}_3 = 0$ present one of the canonical forms of the Maxwell equations. The identity that follows from the presence of the zero in the right column of (97) reads as $\partial_\mu F^\mu = 0$ in terms of the initial functions $F^\mu$ and implies the invariance of the Maxwell action under gradient gauge transformations.

**Acknowledgement 1** B. Geyer thanks the foundations FAPESP and DAAD for support and Institute of Physics USP for hospitality; D. Gitman is grateful to the foundations FAPESP, CNPq, DAAD for support as well as to the Lebedev Physics Institute (Moscow) and to the Institute of Theoretical Physics (University of Leipzig) for hospitality; I. Tyutin thanks INTAS 00-00262, RFBR 02-02-16944, 00-15-96566 for partial support.

## 7 Appendix

Here we present three Lemmas which are used in the reduction procedure to justify equivalence of equations and LF. In this respect it is useful to recall here the relevant definitions from the Sect. II. Namely:

Two sets of equations, $F_A (q^{[l]}) = 0$ and $f_a (q^{[l]}) = 0$ are equivalent $F = 0 \iff f = 0$ whenever they have the same set of solutions. If two sets of LF $F_A (q^{[l]})$ and $\chi_A (q^{[l]})$, $[F] = [\chi]$, are related by some LO $\hat{U}$ and $\hat{V}$ as $F = \hat{U} \chi$, $\chi = \hat{V} F$, $\hat{U} \hat{V} = 1$, then we call such LF equivalent and denote this fact as: $F \sim \chi$. In this case the corresponding equations are strongly equivalent.

** Lemma 1**

Let a set of equations

$$
\Phi_\mu (x, y^{[l]}) = 0, \ F_a (x, y) = 0, \ x = (x^\mu), \ y = (y^a),
$$

be given, where $\Phi$ are some LF. And let $\det \partial F_a / \partial y^b |_{x_0, y_0} \neq 0$, where the consideration point $(x_0, y_0)$ is on shell. Then:

a) The equations $F_a (x, y) = 0$ can be solved w.r.t. $y$ as: $y^a = \varphi^a (x)$, where $\varphi^a (x)$ are some single-valued functions of $x$ in the vicinity of the point $x_0$. In other words, there is the equivalence

$$
F_a (x, y) \sim y^a - \varphi^a (x),
$$

which implies the strong equivalence between the equations $F_a (x, y) = 0$ and $y^a = \varphi^a(x)$.

b) The following equivalence between the LF holds true:

$$
\begin{pmatrix}
\Phi_\mu (x, y^{[l]}) \\
F_a (x, y) 
\end{pmatrix}
\sim
\begin{pmatrix}
\Phi_\mu (x^{[l]}) \\
\Phi_\mu (y^a - \varphi^a (x))
\end{pmatrix}, \quad \Phi_\mu = \Phi_\mu |_{y^a = \varphi^a}.
\tag{100}
$$
The first statement is, in fact, the well-known implicit function theorem \[13\]. Taking into account (99), we have: \( F_a(x, y) = u_{ab}(y^a - \varphi^a(x)) \), \( \det u_{x_0,y_0} \neq 0 \). On the other side one can write \( \Phi = \Phi + \hat{V}_{\mu a} [y^a - \varphi^a(x)] \), where \( \hat{V}_{a} \) is a LO. Thus,

\[
\left( \begin{array}{c} \Phi \\ F \end{array} \right) = \hat{U} \left( \begin{array}{c} \Phi \\ y - \varphi \end{array} \right), \quad \hat{U} = \left( \begin{array}{cc} 1 & \hat{V} \\ 0 & u \end{array} \right), \quad \hat{U}^{-1} = \left( \begin{array}{cc} 1 & -\hat{V} \\ 0 & u^{-1} \end{array} \right),
\]

and the equivalence (100) is justified.

**Lemma 2**

Let a set of equations \( F_A(q, z) = 0, \, q = (q^a), \, z = (z^i), \, A = 1, \ldots, m, \, a = 1, \ldots, n, \, i = 1, \ldots, l, \) be given. And let the Jacobi matrix \( \partial F_A/\partial q^a \) have a constant rank in a vicinity \( D_0 \) of the consideration point \( (q_0, z_0) \), which is on shell \( (F_A(q_0, z_0) = 0) \),

\[
\text{rank} \frac{\partial F_A}{\partial q^a} \bigg|_{q,z \in D_0} = r. \tag{102}
\]

Then there exists an equivalence

\[
F_A \sim \bar{F}_A = \left( \begin{array}{c} y^\mu - \varphi^\mu(x, z) \\ \Omega_G(z) \end{array} \right), \quad q^a = (x^g, y^\mu), \quad A = (\mu, G), \quad [\mu] = r. \tag{103}
\]

We begin the proof with the remark that due to (102), there exists a division of the indices \( A = (\mu, G), \quad a = (\mu, g), \quad [\mu] = r, \quad q^a = (x^g, y^\mu) \), such that

\[
\det \frac{\partial F^\mu}{\partial y^\nu} \bigg|_{q_0, z_0} \neq 0. \tag{104}
\]

Then by virtue of the Lemma 1 we can write

\[
F_\mu = u_{\mu\nu} f^\nu, \quad f^\nu = y^\nu - \varphi^\nu(x, z), \quad \det u_{q_0, z_0} \neq 0. \tag{105}
\]

Let us present the functions \( F_G \) in the form \( F_G(x, y, z) = \Omega_G(x, z) + \Pi_G f^\mu(x, y, z) \), where \( \Omega_G(x, z) = F_G|_{y=\varphi(z,x)} \), such that \( \Omega_G(x_0, z_0) = 0 \). Then

\[
F_A = \left( \begin{array}{c} F^\mu \\ F_G \end{array} \right) = U_{AB} \chi_B, \quad \chi_B = \left( \begin{array}{c} f^\mu \\ \Omega_G \end{array} \right), \quad U = \left( \begin{array}{cc} u & 0 \\ \Pi & 1 \end{array} \right), \quad \det U|_{q_0, z_0} \neq 0. \tag{106}
\]

In virtue of (102) and (106)

\[
\text{rank} \frac{\partial \chi_A}{\partial q^a} \bigg|_{q, z \in D_0} = r. \tag{107}
\]

The Jacobi matrix \( \partial \chi_A/\partial q^a \) has the following structure:

\[
\frac{\partial \chi_A}{\partial q^a} = \frac{\partial (f^\mu, \Omega_G)}{\partial (y^\nu, x^g)} = \left( \begin{array}{cc} \delta^\mu_\nu & -\partial \varphi^\mu/\partial x^g \\ 0 & \partial \Omega_G/\partial x^g \end{array} \right).
\]
Therefore,
\[
\text{rank } \frac{\partial \Omega_G}{\partial x^g} \bigg|_{x \in D_0} = 0 \implies \frac{\partial \Omega_G}{\partial x^g} \bigg|_{x, z \in D_0} = 0.
\] (108)

Eq. (108), together with the relation \(\Omega_G(x_0, z_0) = 0\), implies
\[
\Omega_G|_{x, z \in D_0} = \Omega_G(z), \quad \Omega_G(z_0) = 0.
\]

Finally, we may write
\[
F_A = U_{AB} \chi_B, \quad \chi_B = \left( f^\mu(x, y, z) \begin{pmatrix} \Omega_G(z) \\ \Omega_G(z_0) \end{pmatrix} \right), \quad \text{det } U|_{q_0, z_0} \neq 0,
\] (109)

Thus, the equivalence (103) is justified.

**Lemma 3**

As a consequence of the Lemma 2 the following Lemma holds true:

Let a set of equations
\[
F_A(q^a) = 0, \quad A = 1, \ldots, m, \quad a = 1, \ldots, n,
\]
be given. And let the Jacobi matrix \(\partial F_A/\partial q^a\) have a constant rank in a vicinity \(D_0\) of the consideration point \(q_0\) which is on shell \((F(q_0) = 0)\),
\[
\text{rank } \frac{\partial F_A}{\partial q^a} \bigg|_{q \in D_0} = r.
\]

Then there exists an equivalence
\[
F_A \sim \bar{F}_A = \left( y^\mu - \varphi^\mu(x) \right), \quad A = (\mu, G), \quad 0_G \equiv 0 \quad \forall G, \quad [\mu] = r.
\] (110)

The proof of this Lemma follows the one of the Lemma 2 if one selects there \(z = z_0\).

**References**

[1] P.A.M. Dirac, *Lectures on Quantum Mechanics*, (Belfer Graduate School of Science, Yeshiva University, New York 1964)

[2] D.M. Gitman and I.V. Tyutin, *Quantization of Fields with Constraints*, (Springer-Verlag, Berlin 1990)

[3] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems* (Princeton University Press, Princeton 1992)

[4] D.M. Gitman, and I.V. Tyutin, *Constraint reorganization consistent with Dirac procedure*, hep-th/0112103; Michael Marinov Memorial Volume: *Multiple Facets of Quantization and Supersymmetry*, (World Publishing, Singapore 2002)
[5] V.A. Borochov and I.V. Tyutin, Yadernaya Fizika, 61 (1998) 1715 (Physics of Atomic Nuclei, 61 (1998) 1603); ibid 62 (1999) 1137 (Physics of Atomic Nuclei, 62 (1999) 1070)

[6] I.A. Batalin and G.A. Vilkovisky, Phys. Lett. B102 (1981) 27; Phys. Rev. D28 (1983) 2567

[7] D.M. Gitman and I.V. Tyutin, *Canonical quantization of singular theories*, Izv. VUZov Fizika 5 (1983) 3 (Sov. Phys. Journ. 5 (1983) 423)

[8] D.M. Gitman and I.V. Tyutin, *The structure of gauge theories in the Lagrangian and Hamiltonian formalisms*, In Quantum field theory and quantum statistics v. I, pp. 143–164, Ed. by Batalin, Isham and Vilkovisky (Adam Hilger, Bristol 1987)

[9] P. Olver, *Applications of Lie groups to differential equations*, Graduated Texts in Mathematics, 107 (Springer-Verlag, Berlin 1986); B.A. Kupershmidt, *Geometry of jet bundles and structure of Lagrangian and Hamiltonian formalism*, 162-218, Lecture Notes in Mathematics 775 (Springer-Verlag, Berlin 1980); I.S. Krasilshchik, V.V. Lychagin, and A.M. Vinogradov, *Geometry of jet spaces and nonlinear partial differential equations*, Adv. Studies Contemp. Math., (Gordon and Breach, New York 1986)

[10] D.M. Gitman, and I.V. Tyutin, Nucl. Phys. B630 (2002) 509

[11] F.R. Gantmacher, *The theory of matrices*, Vol. 1 (Chelsea Publ. Co., New York 1959); *Matrizentheorie*, (Dt. Verlag Wiss., Berlin 1986)

[12] D.M. Gitman, S.L. Lyakhovich, and I.V. Tyutin, Izv. VUZov Fizika 8 (1983) 61 (Sov. Phys. Journ. 8 (1983) 730)

[13] J. Dieudonné, *Foundations of Modern Analysis*, (Academic Press, NY, London 1968); G.M. Fichtenholz, *Differential- und Integralrechnung*, Vol. I (Dt. Verlag Wiss., Berlin 1989)

[14] L. Faddeev and R. Jackiw, Phys. Rev. Lett. 60 (1988) 1692