Dissipation in mesoscale superfluids

Adrian Del Maestro\textsuperscript{1,*} and Bernd Rosenow\textsuperscript{2}

\textsuperscript{1}Department of Physics, University of Vermont, Burlington, VT 05405, USA
\textsuperscript{2}Institut für Theoretische Physik, Universität Leipzig, D-04103, Leipzig, Germany

We investigate the maximum speed at which a driven superfluid can flow through a narrow constriction with a size on the order of the healing length. Considering dissipation via the thermal nucleation of quantized vortices, we calculate the critical velocity for superfluid $^4$He and ultracold atomtronic circuits, identify fundamental length and velocity scales, and are thus able to present results obtained in widely different temperature and density ranges in a universal framework. For ultra-narrow channels we predict a drastic reduction in the critical velocity as the energy barrier for flow reducing thermally activated phase slip fluctuations is suppressed.

The flow of dissipationless atomic supercurrents in neutral superfluids is one of the most dramatic manifestations of macroscopic quantum coherence \cite{1,3,41–44}, with applications to matter wave interferometry \cite{2,36}. Recently, there has been increased interest in dimensionally confined superfluids, due to progress in manufacturing nanoscale channels and fountain effect devices for studying the flow of superfluid helium \cite{7–24} and the availability of trapped non-equilibrium atomic Bose-Einstein condensates \cite{25–39}. Common to these experiments in vastly different density and interaction regimes is an observed increase in dissipation for highly confined systems.

In general, superflow is possible at speeds less than a superfluid critical velocity set by the Landau criterion $v_c \leq \min \varepsilon(p)/p$ below which there are no accessible excitations $\varepsilon(p)$ with momentum $p$ \cite{20}. Among the different types of excitations in superfluids, quantized vortices \cite{1,11,14} give rise to the smallest $v_c$. For flow through a cylindrical channel, if the total kinetic energy is converted into vortex rings with the size $a$ of the constriction, the Landau criterion predicts a critical velocity $v_{c,F} \sim (\kappa/a) \ln(a/\xi_0)$ \cite{22} where $\kappa = h/m$ is the quantum of circulation for condensed bosons of mass $m$ and $\xi_0$ is a characteristic length scale of the superfluid. This prediction (due to Feynman) has been born out by nearly a half-century worth of superfluid massflow observations with temperature independent critical velocities \cite{22,23}. However, it must ultimately break down as the constriction radius approaches $\xi_0$. Moreover, any observed temperature dependence of $v_c$ can only be described by the existence of an energy activation barrier for the creation of vortices.

In this letter, we consider confined mesoscale superflow through quasi-one-dimensional (1d) constrictions with a characteristic size $a$ approaching the temperature ($T$) dependent correlation (healing) length $\xi(T)$, and find a strong increase in dissipation when $a/\xi(T)$ approaches one. Going beyond previous studies \cite{22,36}, we (i) quantitatively predict the temperature, size, and drive dependence of the critical velocity without adjustable parameters, (ii) use a paradigmatic orifice geometry to model the enhancement of vortex creation in spatially inhomogeneous flow near a sharp boundary, which significantly lowers critical velocities, (iii) point out the universality between high density $^4$He \cite{3,13} and low density atomic condensates \cite{23,27,28,35}, by characterizing constrictions via the dimensionless length $a/\xi$ and measuring velocities in units of $v_0 = \kappa/(4\pi\xi_0)$, and (iv) describe the crossover to the purely 1d limit, a Luttinger liquid in the thermal regime. Predictions are expected to be logarithmically accurate in the critical regime while corrections of order unity may arise when extrapolating to lower $T$.

We begin by considering superflow between reservoirs with a chemical potential difference $\Delta\mu$ (pressure differ-

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{a) A current of superfluid atoms driven through a long channel with a homogeneous (radius independent) flow profile. b) Flow through a narrow orifice formed from the surface of revolution of a hyperbola around the flow axis. c) Velocity field $v_s$ for the potential flow through an orifice in units of the average flow speed $v_f$. The flow direction is indicated by black lines, with the magnitude diverging as a power law at the orifice boundary \cite{46}.}
\end{figure}

\*Adrian.DelMaestro@uvm.edu
The energy cost for creating a quantized vortex is due to the change of coordinates from the superfluid phase $\Phi$ to the condensed phase $\Phi = \Phi_0 e^{i \Psi}$, where $\Psi$ is the condensate order parameter by flow lines leading to a change of phase of the superfluid mass density for a transition temperature $T_c$ and $\xi(T) = \xi_0 (1 - T/T_c)^{-\nu}$ and $\nu$ is the correlation length critical exponent. We numerically checked that the Josephson relation is valid to within 20% down to $T/T_c \approx 0.7$, for details see [46]. For flow through an orifice with speed $v_J$, we obtain the energy barrier for a line vortex located a distance $x$ from its center:

$$\beta_c E_{\text{line}}(x) = 2\pi \frac{a}{\xi} \sqrt{1 - \left(\frac{x}{a}\right)^2} \left[ \ln \left(1 - \frac{|x|}{a}\right) + \ln \frac{a}{\xi} + \alpha \right]$$

and that for a centered ring vortex with radius $R$:

$$\beta_c E_{\text{ring}}(R) = 2\pi^2 \frac{R}{\xi} \left( \ln \frac{R}{\xi} + \alpha \right) - \frac{v_J}{v_0} \left( \frac{a}{\xi} \right)^2 \frac{\pi^2 \xi}{\xi_0} \left[ 1 - \left( 1 - \left( \frac{R}{a} \right)^2 \right) \right]$$

where $\beta_c = 1/(k_B T_c)$. From these expressions (and those for channel flow derived in the supplementary material [46]) we observe the emergence of natural length ($\xi_0$) and velocity ($v_0 = \kappa / (4\pi \xi_0)$) scales that are essential for constructing a universal theory of dissipative superfluids.

The velocity of superflow at finite $T$ is limited by the thermal nucleation of quantized vortices which traverse the flow lines leading to a change of phase of the superfluid order parameter by $\pm 2\pi$, (see e.g. Ref. [33]). The decay of a persistent current is then governed by vortex energetics via:

$$\Gamma = \Gamma_0 \left[ e^{-\beta E_{\text{max}}(v_J)} - e^{-\beta E_{\text{max}}(-v_J)} \right] ,$$

where $h \Gamma = \Delta \mu = m \Delta P / \rho_s$ drives total mass flow $J = \rho_s \mathcal{J} \left[ v_s \cdot dS \right] \equiv \pi a^2 \rho_s v_J$ and $E_{\text{max}} \equiv \max_{\mathcal{L}} E$ is the saddle point of the vortex excitation energy over the domain of the channel or orifice. The difference of rates in Eq. (5) corresponds to the contributions from vortices which reduce and increase the superflow, respectively. The attempt rate $\Gamma_0$ is related to the phase space available for vortex excitations and is geometry dependent:

$$\Gamma_0 = \frac{1}{\tau_{\text{GL}} \xi^2} \left\{ \begin{array}{ll} \frac{\pi a}{2} & \text{vortex ring} \\ \frac{\pi \kappa}{2\pi} & \text{vortex line} \end{array} \right.$$
to the radius $R$ of the vortex ring or the location $x$ of the vortex line, and a contribution from integration over the negative eigenvalue mode. As previously discussed [51, 54], the Jacobian is $\sqrt{S_o/2\pi}$ for the zero modes, and the negative eigenvalue mode contributes a factor of similar magnitude. We have verified that the combination of all these factors is of order unity and thus neglect them. Other modifications to the pre-factor in Eq. (5) could result when considering the microscopic details of dynamics and vortex evolution inside the constriction [50, 61] and would introduce quantitative logarithmic corrections to the nucleation theory.

Evaluation of the critical velocity from Eq. (5) proceeds as follows. For a given flow profile and vortex type, we maximize the vortex energy as a function of $\pm v_J$ over the spatial domain of possible configurations. This leads to critical vortex positions $x^*(\pm v_J) \in (-a, a)$ for line vortices and critical radii $R^*(\pm v_J) \in (0, a)$ for ring vortices. Vortices with a length smaller than the critical one will tend to collapse, or annihilate at boundaries, while those larger can proliferate, leading to dissipation. For a fixed constriction radius $a/\xi_0$, temperature $T/T_c$ and external drive potential $\Gamma_0/\Gamma$, Eq. (5) can be numerically solved self-consistently giving the critical velocity when $v_J = v_c$.

When $a/\xi_0 \gg 1$, the boundaries of the constriction no longer play an important role and only uniform channel flow is relevant. Due to the resulting large critical velocities, energy increasing fluctuations can be neglected and the saddle point energies can be found analytically. The resulting pair of transcendental equations

$$\frac{R^*}{\xi} = \frac{1}{\pi^2} \frac{T}{T_c} \ln \frac{\Gamma_0}{\Gamma} \left( \ln \frac{R^*}{\xi} + \alpha + 1 \right)$$

$$\frac{v_c}{v_0} = \frac{\xi_0}{R^*} \left( \ln \frac{R^*}{\xi} + \alpha - 1 \right)$$

can be solved numerically for $R^*$ and $v_c$ [66].

Our main results for the critical velocity in both the channel and orifice flow profiles are shown as lines in Fig. 3. When employing the scales $v_0$ and $\xi_0$, mass flow measurements in drastically different density, interaction, and temperature regimes are well bounded by the vortex nucleation theory, and experiments on confined superfluid 4He and low-dimensional Bose-Einstein condensates can be directly compared. For both flow profiles, line vortices have lower activation energies than ring vortices, giving larger velocities and a lower bound on $v_c$. An absolute upper bound is provided by ring vortices in the orifice flow profile. In the limit $a \gg \xi_0$, where the geometry approaches that of bulk flow, we recover the intrinsic superfluid velocity due to the nucleation of vortex rings analyzed by Langer and Fisher [13]. For 4He, we have used a vortex core size determined from specific heat measurements [46, 63], and have thus been able to correct a long-standing inconsistency of 27 orders of magnitude in the applied pressure difference employed in Ref. [43] to obtain agreement with experiments. For tight constrictions, both the lower and upper bound turn towards smaller velocities indicative of enhanced dissipation.

While details of additional experiments are discussed in the supplemental material [16], Fig. 3 includes data from two 4He studies whose critical velocities display a clear temperature dependence – a signature of activated behavior. Harrison et al. [15] measured flow in networks of imperfect pores of varying radii which should provide a lower bound on flow speeds, behavior consistent with our results. Recent measurements on single nanopores by Duc et al. [13] exhibit drastically different behavior: large critical velocities that decrease as the radius approaches the correlation length. While the microscopic details of the flow profiles are not known, $v_c$ for the narrowest pore is bounded by the channel prediction, consistent with the reported aspect ratio of 10:1 and suggesting a crossover to strongly dissipative quasi-1d flow.

Fig. 3 also includes results from analogous neutral atomtronic circuits using ultracold bosonic and fermionic condensates [8]. Here, the “orifice” can be replaced with a quantum point contact between two resonantly coupled Fermi gases [37] or channel-like flow can be driven by the discharge of an bosonic atom capacitor [29] or by dragging an optical potential through a simply [27, 48] or multiply connected [28, 30, 53] Bose-Einstein condensate. For the latter, our nucleation theory yields $v_c \approx 100 \mu m/s$ for the drag and $v_c \approx 1 \text{ mm/s}$ for the toroidal flow in remarkable agreement with measurements and more microscopic theoretical analysis [33, 35, 64].
Intuition for the dissipation in narrow pores with radius $a \approx \xi$ can be gained by considering the unoptimized energy of line vortices with position $x = 0$ at the center of the channel. This approximation is qualitatively correct since narrow pores can be expected to be in the channel flow regime with line vortex activation energies comparable to temperature. The vortex energy is found from the sum of Eqs. (1) and (2) with $x = 0$:

$$\beta E_{\text{line}}(x = 0) = 2\pi \frac{a}{\xi} \left( \ln \frac{a}{\xi} + \alpha \right) - \frac{\pi^2 \xi v_0}{2\xi_0 v_0} \left( \frac{a}{\xi} \right)^2, \quad (9)$$

with resulting critical velocity

$$\frac{v_c}{v_0} = \frac{2 \xi_0}{\pi^2 a^2 T_c} \sinh^{-1} \left\{ \frac{\pi^2}{64\pi a L} \left( 1 - \frac{T}{T_c} \right) \right\} \times \exp \left[ \frac{2\pi a T_c}{\xi} \left( \ln \frac{a}{\xi} + \alpha \right) \right]. \quad (10)$$

As $a/\xi \rightarrow 1^+$, the activation energy in the exponent of Eq. (10) decreases rapidly, and the small multiplier of the exponential is no longer compensated. As a consequence, the critical velocity drops by several orders of magnitude.

In the quasi-1d ($a \lesssim \xi$) limit there are no transverse degrees of freedom, and the system can be described in analogy to fluctuating superconducting wires by computing the resistance due to thermally activated phase slips \[65, 66\]. Translated into the language of 1d superfluidity, the line vortex activation energy in the exponent of $E_{\text{line}}$ is $\xi_0^2/T_c$, and the resulting bounds they place on the critical velocity for a channel ($L = 10^4 \xi_0$) and orifice ($L = 10 \xi_0$) with $a = 10 \xi_0$. As $T \rightarrow T_c$, the correlation length $\xi$ diverges, thus reducing the effective channel width $a/\xi(T)$ and lowering $v_c$. The shaded gray bars demarcates the radii where $1 \leq a/(4\xi(T)) \leq 3/2$ and in this region the upper bound due to ring vortices is no longer expected to be relevant.

Figure 4. a The critical velocity in a quasi-1d superfluid channel of length $L = 10^4 \xi_0$ as a function of radius for $T/T_c = 3/4$, $h \Gamma = k_B T_c$ and $\xi(3T_c/4) \approx 2.5 \xi_0$. The solid and dashed blue lines extending over the full domain of the plot were computed via the vortex nucleation theory, while the green line is for 1d, Eq. (11). b The temperature dependence of the critical velocity for a channel ($L = 10^4 \xi_0$) and orifice ($L = 10 \xi_0$) with $a = 10 \xi_0$. As $T \rightarrow T_c$, the correlation length $\xi$ diverges, thus reducing the effective channel width $a/\xi(T)$ and lowering $v_c$. The shaded gray bars demarcates the radii where $1 \leq a/(4\xi(T)) \leq 3/2$ and in this region the upper bound due to ring vortices is no longer expected to be relevant.

In summary, we considered two characteristic confined flow geometries and the thermal activation of representative low energy excitations, ring and line vortices, inside them. The resulting bounds they place on the critical velocity of neutral superflow through narrow constrictions agree with a large body of measurements on confined superfluid $^4$He and low-dimensional ultracold gases. As the confinement radius approaches the healing length, we find an exponential suppression of the critical velocity of three orders of magnitude. The experimental observation of this dramatic reduction would be a clear signal of entering the strongly fluctuating mesoscale regime.

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Supplementary material for “Dissipation in mesoscale superfluids”

VELOCITY FLOW PROFILES

As illustrated in Fig. 1a-b of the main text, we consider two different superfluid flow profiles through long narrow channels, and hole-like orifices.

Channel Flow

For a long cylindrical channel of radius \( a \) and length \( L \gg a \) oriented with its axis along the \( z \)-direction we neglect any acceleration of the fluid at the entry and exit to obtain a spatially independent velocity field

\[
v_s = v_J \hat{z} = \frac{J}{\pi a^2 \rho_s} \hat{z}
\]  

(S1)

where \( J \) is the total mass flow rate and \( \rho_s \) is the superfluid mass density.

Orifice Flow

The geometry of an orifice of radius \( a \) oriented in the \( x-y \) plane centered at the origin can be conveniently described in oblate spheroidal coordinates \((\zeta, \eta, \phi)\) by the surface \( \eta = 0 \) where

\[
x = a \cos \phi \sqrt{(1 + \zeta^2)(1 - \eta^2)}
y = a \sin \phi \sqrt{(1 + \zeta^2)(1 - \eta^2)}
z = a \zeta \eta .
\]  

(S2)

In general, \( \phi \in [0, 2\pi) \), \( \zeta \in [0, \infty) \) and \( \eta \in [-1, 1) \) with the surface \( |\eta| = \eta_0 \) corresponding to a hyperboloid of revolution about the \( y \)-axis (see Fig. 1b). Steady incompressible flow through the orifice with total rate \( J \) can be studied by defining a potential field \( \Phi(\zeta, \eta, \phi) \) such that \( v_s = (\hbar/m) \nabla \Phi \) and solving \( \nabla^2 \Phi = 0 \) by requiring that \( \Phi \) is continuous at \( \zeta = 0 \), and subject to the boundary condition that the velocity component perpendicular to the surface defined by \( \eta = 0 \) vanishes outside the orifice. This yields the velocity field: \[S1\] \[S3\]

\[
v_s = \frac{v_J}{2} \frac{1}{\sqrt{(\zeta^2 + \eta^2)(\zeta^2 + 1)}} \hat{\zeta}
\]  

(S3)

which is plotted in the \( y-z \) plane in Fig. 1c. Inside the orifice at \( \zeta = 0 \) this expression simplifies to:

\[
v_s(r, z = 0) = \frac{v_J}{2\sqrt{1 - r^2/a^2}} \hat{z}
\]  

(S4)

which diverges near the boundary as \( r \to a \).

VOXET ENERGIES

Employing Eqs. (1)–(2) in the main text in combination with the spatial dependence of \( v_s \) we may now obtain the energy cost of nucleating line and ring vortices.

Channel Flow

For channels, we use the velocity profile in Eq. \[S1\].
Line Vortices

For a line vortex attached to the walls of a channel of radius $a$ offset a distance $x$ from the center with length $L(x) = 2\sqrt{a^2 - x^2}$ as seen in Fig. 2, the total energy is the sum of $E_{\text{tension}} + E_{\text{flow}}$ and given by:

$$E_{\text{line}}(x) = \frac{\kappa^2 \rho_s}{2\pi} \sqrt{a^2 - x^2} \left( \ln \frac{a - |x|}{\xi} + \alpha \right)$$

$$- \kappa \rho_s v_J \left[ a^2 \cos^{-1} \left( \frac{x}{a} \right) - x \sqrt{a^2 - x^2} \right].$$

Now, the superfluid density and correlation length $\xi$ can be related via the Josephson scaling relation in three dimensions [S4]

$$\xi(T) = \frac{4\pi^2 k_B T_c}{\kappa^2 \rho_s(T)}$$

where $\xi(T) = \xi_0 (1 - T/T_c)^{-\nu}$. Fig. S1 shows the accuracy of this relation for superfluid $^4\text{He}$ [S5] and a weakly interacting Bose gas [S6] down to $T/T_c \approx 0.7$. All temperature dependence now enters expressions through the correlation length and the resulting dimensionless vortex energy is given by:

$$\beta_c E_{\text{line}}(x) = 2\pi \frac{a}{\xi} \sqrt{1 - \left( \frac{x}{a} \right)^2} \left[ \ln \left( 1 - \frac{|x|}{a} \right) + \ln \frac{a}{\xi} + \alpha \right]$$

$$- \frac{v_J}{v_0} \left( \frac{a}{\xi} \right)^2 \frac{\pi \xi}{\xi_0} \left[ \cos^{-1} \left( \frac{x}{a} \right) - \frac{x}{a} \sqrt{1 - \left( \frac{x}{a} \right)^2} \right]$$

with $\beta_c \equiv 1/(k_BT_c)$ and we have identified the fundamental velocity scale $v_0 = \kappa/(4\pi \xi_0)$. This expression simplifies to Eq. (9) in the main text when the vortex line is located at the center of the channel $x = 0$.

Ring Vortices

For a ring vortex of radius $R$, with length $L = 2\pi R$ the energy can be written as:

$$E_{\text{ring}}(R) = \frac{1}{2} \kappa^2 \rho_s R \left( \ln \frac{R}{\xi} + \alpha \right) - \kappa \rho_s \pi R^2 v_J$$

(S8)
yielding

$$\beta_c E_{\text{ring}}(R) = 2\pi^2 \frac{R}{\xi} \left( \ln \frac{R}{\xi} + \alpha \right) - \frac{v_J}{v_0} \left( \frac{R}{\xi} \right)^2 \frac{\pi^2 \xi}{\xi_0}. \quad (S9)$$

**Orifice Flow**

For the orifice flow profile, the velocity field now has the spatial dependence seen in Eq. (S4) and Fig. 1c with a divergence at the boundary. While the effective line tension $E_{\text{tension}}$ (first term) in Eqs. (S5) and (S8) are unchanged, a modified spatial integral in $E_{\text{flow}}$ needs to be computed.

**Line Vortices**

The energy cost for flow captured by a line vortex at position $x$ is given by

$$E_{\text{flow, line}}(x) = \frac{\pi}{2} \kappa \rho_s v_J a^2 \left( 1 - \frac{x}{a} \right)^2 \quad (S10)$$

leading to the total dimensionless vortex energy:

$$\beta_c E_{\text{line}}(x) = 2\pi a \sqrt{1 - \left( \frac{x}{a} \right)^2} \left[ \ln \left( 1 - \left( \frac{x}{a} \right) \right) + \ln \frac{a}{\xi} + \alpha \right] - \frac{v_J}{v_0} \left( \frac{a}{\xi} \right)^2 \frac{\pi^2 \xi}{2\xi_0} \left( 1 - \frac{x}{a} \right). \quad (S11)$$

**Ring Vortices**

For ring vortices, the flow integral is given by

$$E_{\text{flow, ring}}(R) = \kappa \rho_s v_J \pi a^2 \left[ 1 - \sqrt{1 - \left( \frac{R}{a} \right)^2} \right] \quad (S12)$$

which leads to

$$\beta_c E_{\text{ring}}(R) = 2\pi^2 \frac{R}{\xi} \left( \ln \frac{R}{\xi} + \alpha \right) - \frac{v_J}{v_0} \left( \frac{a}{\xi} \right)^2 \frac{\pi^2 \xi}{\xi_0} \left[ 1 - \sqrt{1 - \left( \frac{R}{a} \right)^2} \right]. \quad (S13)$$

**APPLICATION TO MASS FLOW EXPERIMENTS**

In this section we provide a more complete analysis of neutral bosonic mass flow experiments that were discussed in Fig. 3 of the main text.

**Superfluid Helium-4**

A recent review by Varoquaux [S7] includes a compilation of superfluid critical velocity results from a diverse set of experiments (both dependent and independent of temperature) that are reproduced in Fig. S2 along with theoretical upper and lower bounds computed within the vortex nucleation theory using parameters relevant for superfluid helium. The experimental data from Ref. [S7] can be grouped into two distinct regions. For larger channels, a temperature
Figure S2. A compilation of experimental results on the critical superfluid velocity \( v_c \) of \(^4\)He under pressure driven flow through constrictions with radius \( a \) from Varoquaux (temperature independent and dependent) \([S7]\), Harrison et al. \([S8]\) and Duc et al. \([S9]\). The temperature independent results for larger channels are qualitatively described by the Feynman critical velocity \( v_{c,F} \sim \frac{\kappa}{4\pi a} \ln(2a/\xi_0) \) where \( \xi_0 \approx 3.45\,\text{Å} \) is the zero temperature coherence length. For smaller pores with larger temperature dependent critical velocities, the data is well-bounded by the predictions of the vortex nucleation theory for ring and line vortices in the orifice and channel flow profiles with \( L = 30\,\text{nm} \) and \( \Gamma = 4\,\text{GHz} \) at \( T = 1.5\,\text{K} \), values consistent with those in the experiment \([S9]\), and representative for the other experiments.

Table I. Critical velocity and related velocity, length and temperature scales extracted from four neutral bosonic mass flow experiments in ultracold atomic and molecular condensates.

| Reference   | System | \( v_c \) (mm/s) | \( v_0 \) (mm/s) | \( a \) (µm) | \( \xi_0 \) (µm) | \( T/T_c \) |
|-------------|--------|------------------|------------------|-------------|----------------|-------------|
| Neely et al. \([S10]\) | \(^{87}\)Rb | 0.2 | 1.2 | 47 | 0.3 | 0.6 |
| Ramanathan et al. \([S11]\) | \(^{23}\)Na | 0.9 | 0.7 | 7 | 0.51 | 0.2 |
| Raman et al. \([S12]\) | \(^{23}\)Na | 1.6 | 4.6 | 15 | 0.3 | 0.8 |
| Weimer et al. \([S13]\) | \(^6\)Li (molecule) | 1.7 | 3.1 | 10 | 0.85 | 0.5 |

A second group of temperature dependent critical velocities in pores with radii \( a \sim 100\,\text{nm} - 10\,\text{µm} \) have considerably larger velocities and are in a thermally activated dissipation regime that is well bounded by the vortex nucleation theory. Additionally, Fig. S2 includes a systematic set of experiments performed by Harrison and Mendelssohn \([S8]\) (green circles) employing the fountain effect to drive superfluid mass flow through arrays of \( 10^4 - 10^7 \) \( L = 5\,\text{µm} \) pores etched in irradiated mica using HF acid with radii \( a = 20, 50, 80, 120 \) and 200 nm. These results, taken at \( T = 1.5\,\text{K} \), should be considered as representing a lower bound on the critical velocity as all pores were assumed to be open in the analysis and both a variation in radii and a taper along the channel were observed.

Experiments by Duc et al. \([S9]\) shown in Fig. S2 are in the interesting mesoscopic regime where \( a/[\xi_0(1 - T/T_c)^{-\nu}] \sim \text{O}(1) \). For superfluid helium, \( \xi_0 \approx 3.45\,\text{Å} \), \( T_c = T_\lambda \approx 2.1768\,\text{K} \) and \( \nu \approx 0.6717 \). They observed a decrease in the critical velocity as the pore radius was reduced. In these experiments, \(^4\)He mass flow is studied through single pores nanofabricated using a transmission electron beam incident on a \( L \approx 30\,\text{nm} \) thick silicon nitride wafer resulting in smooth constrictions with radii \( a \approx 3.14, 7.81 \) and 20 nm. Critical velocities were reported for \( T = 1.5\,\text{K} \), and flow was driven via pressure differences between \( \Delta P = 250 - 830\,\text{mbar} \) which corresponds to an external drive frequency \( \Gamma \approx 2 - 7\,\text{GHz} \sim k_BT_c/h \).
Ultracold Gases

In systems of neutral ultracold atoms, flow is driven by dynamically shaping an optical potential which causes a local imbalance in the chemical potential (see Ref. [S14] for current experimental designs). For dilute gases, the prefactor $\xi_0$ which appears in the critical scaling relation for the correlation length can be related to the zero temperature healing length $\xi_h = 1/\sqrt{8\pi a_n \hbar}$ via known universal results for the weakly interacting Bose gas [S6]. Here $n$ is the number density and $a_n$ is the scattering length and provided that $na^3 \ll 10^{-5}$ the superfluid density can be written in terms of a universal scaling function $f_s$:

$$\rho_s(t) = \frac{16\pi^3 mn}{\xi (\frac{3}{2})^{4/3}} (na^3)^{1/3} (1 - t)^2 f_s (t, na^3) \quad \text{(S14)}$$

where $m$ is the mass, $t = 1 - T/T_c$ is the reduced temperature and the Riemann zeta function, $\zeta(3/2)$, appears through the use of the critical temperature of the non-interacting Bose gas: $k_B T_c = (2\pi \hbar^2/m)[n/\zeta(3/2)]^{2/3}$. The function $f_s$ is known from high precision Monte Carlo calculations [S6, S15] and when $t \ll 1$, $f_s \propto (t/\sqrt{na^3})^\nu$. Writing the temperature dependent correlation length in the critical region as $\xi(T) = (\xi_0/\xi_h) \xi_h t^{-\nu}$, we can use the Josephson relation in Eq. (S6) to determine

$$\frac{\xi_0}{\xi_h} = A (na^3)^{(2\nu-1)/6} \quad \text{(S15)}$$

where $A \approx 3.8$ is a universal number. Thus, for experimentally accessible weakly interacting Bose gases $\xi_0 \approx B \xi_h$ with $B \approx 1 - 2$. Away from the critical region, the full scaling function can be used to test the accuracy of the Josephson relation with the result shown in Fig. S1.

Thus, to analyze non-equilibrium mass flow experiments employing ultracold Bose-Einstein condensates, we approximate $\xi_0 \approx \xi_h \sim \mu\text{m}$, which yields the critical velocity scale $v_0 = \hbar/(4\pi m \xi_0) \sim 1 \text{mm/s}$. In Fig. 3 of the main text, we have shown results for the critical velocity from three ultracold gas experiments using the parametrization shown in Table II. We have not included the low temperature data point from Ref. [S11] where the errors in our critical theory are difficult to estimate. We find that the other experiments are in the same flow regime as tightly confined superfluid helium.

**INTRINSIC CRITICAL VELOCITY OF BULK SUPERFLUIDS**

In the $a \gg \xi_0$ limit we can directly find the critical ring vortex radius $R^*$ that maximizes the energy barrier in the constant channel flow profile in Eq. (S8) as

$$\frac{R^*}{\xi} = \frac{v_0 \xi_0}{v J} \left( \ln \frac{R^*}{\xi} + \alpha + 1 \right). \quad \text{(S16)}$$

In this bulk regime, one only needs to consider the effects of vortices which reduce the total energy and Eq. (5) gives

$$\ln \frac{\Gamma_0}{\Gamma} = \beta E_{\text{ring}}(R^*) \quad \text{(S17)}$$

$$= T_c \left[ 2\pi^2 \frac{R^*}{\xi} \left( \ln \frac{R^*}{\xi} + \alpha \right) - \frac{v_c}{v_0} \left( \frac{R^*}{\xi} \right)^2 \frac{\pi^2 \xi}{\xi_0} \right]$$

which can be solved for the critical velocity $v_c$. Combining Eq. (S17) with Eq. (S16) with $v_J = v_c$ gives the transcendental equation

$$\frac{R^*}{\xi} = 2 \left[ \ln \frac{R^*}{\xi} + \alpha \right] + \left[ \frac{\ln \xi_0}{\xi_0} \right] \frac{\Gamma_0}{\Gamma} \quad \text{(S18)}$$

which can be simplified to yield Eq. (7). Replacing $\ln \Gamma_0/\Gamma$ in Eq. (S17) via Eq. (S18) we obtain the critical velocity in the large radius limit in Eq. (8).

Eq. (7) in the main text is equivalent to Eq. (13) in Langer and Fisher [S16] (LF), noting the change of variables $\eta_c = \ln 8R^*/\xi$, their use of a rigid vortex core with $\alpha = \ln 8 - 7/4$ and setting $\ln \Gamma_0/\Gamma \approx 83$. The un-physically large
Figure S3. The intrinsic critical velocity scale $v_{c0}$ of superfluid $^4$He as a function of the effective zero temperature core size $\xi_0$ and external drive frequency $\ln \Gamma_0/\Gamma$. In order to obtain a physically meaningful value of the critical velocity $v_c = v_{c0}(1 - T/T_c)^{\nu}$ when using $\xi_0 = 2\,\text{Å}$, Langer and Fisher [S16] had to consider an extremal value for the external drive as indicated by the cross (×). When utilizing a larger core size of $\xi_0 = 3.45\,\text{Å}$ as experimentally determined in Ref. [S17], $\ln \Gamma_0/\Gamma$ can be reduced to more physically meaningful values as indicated by the plus (+).

value of $\ln \Gamma/\Gamma_0$ employed by LF was required to ensure that their computed value of $v_c = v_{c0}(1 - T/T_c)^{\nu}$ in the scaling regime was below the absolute upper bound given by the Landau criterion. In order to understand the origin of this long-standing discrepancy, we have investigated $v_{c0}$ as a function of both the external drive $\Gamma_0/\Gamma$ and vortex core details $\xi_0$ with the results shown in Fig. S3.

**THE ONE-DIMENSIONAL LIMIT**

To derive Eq. (11) in the main text, we begin with a free energy functional near $T_c$:

$$f = f_0 + \frac{\hbar^2}{2m} |\nabla \Psi|^2 + A |\Psi|^2 + \frac{B}{2} |\Psi|^4$$  \hspace{1cm} (S19)

where $f_0$ is a condensate energy density and the wave function is given by: $\Psi(\mathbf{r}) = \sqrt{n(\mathbf{r})} e^{i \Phi(\mathbf{r})}$ with $n(\mathbf{r})$ the number density. In a spatially homogeneous superfluid, the free energy is minimized when $|\Psi|^2 \equiv n_0 = -A/B$. Substituting this value for the field into Eq. (S19):

$$f - f_0 = - \frac{A^2}{2B} = - \frac{H_c^2}{8\pi} ,$$ \hspace{1cm} (S20)

where the last equality is schematic and in analogy to superconductors where $H_c$ is the critical field. Identifying $-A/B$ with the density and performing the usual rescaling: $\tilde{\Psi} = \Psi/\sqrt{n_0}$ one introduces the correlation length $\xi(T)$ such that $\xi^2(T) = \hbar^2/2mA(T)$, and we can now write $f - f_0 = \rho_s\kappa^2/16\pi^2\xi(T)$. According to Refs. [S18,S19], the free energy barrier of a phase slip excitation inside a quasi-one-dimensional channel of cross-sectional area $\pi a^2$ is given by

$$\Delta F_0 = \frac{8\sqrt{2}}{3} \frac{H_c^2}{8\pi} A\xi = \frac{1}{6\pi} \rho_s\kappa^2 \xi \left( \frac{a}{\xi} \right)^2 .$$  \hspace{1cm} (S21)

The energy reduction due to a superflow $J$ is $\delta F = \kappa \rho_s v_J \pi a^2$, and adding the free energy barrier Eqs. (S21) one obtains the total phase slip energy in Eq. (11) of the main text. The total phase slip rate is $\Gamma = 2\Gamma_0 e^{-\Delta F_0/k_B T} \sinh(\delta F/k_B T)$, and solving for $v_c$ we find

$$\frac{v_c}{v_0} = \frac{1}{\pi} \left( \frac{\xi_0}{a} \right) \frac{T}{T_c} \left( 1 - \frac{T}{T_c} \right)^{-\nu} \times \sinh^{-1} \left[ \frac{\Gamma}{2\Gamma_0} \exp \left( \frac{4\pi}{3\sqrt{2}} \frac{a^2}{\xi_0^2} \frac{T}{T_c} \left( 1 - \frac{T}{T_c} \right)^{2\nu} \right) \right] ,$$ \hspace{1cm} (S22)
where \( \Gamma_0 = \frac{aL}{\xi_0^2} \left( 1 - \frac{T}{T_c} \right)^{2\nu+1} \left( \frac{2\sqrt{2}\pi T_c}{3T} \right)^{1/2} \frac{8k_B T_c}{\pi \hbar} \).

Eq. (S22) is shown in Fig. 4 of the main text.