Abstract: In this paper, we introduce the new fully degenerate poly-Bernoulli numbers and polynomials and investigate some properties of these polynomials and numbers. From our properties, we derive some identities for the fully degenerate poly-Bernoulli numbers and polynomials.

Keywords: Fully degenerate poly-Bernoulli polynomial, Fully degenerate poly-Bernoulli number, Umbral calculus

MSC: 11B75, 11B83, 05A19, 05A40

1 Introduction

It is well known that the Bernoulli polynomials are defined by the generating function

\[ \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n (x) \frac{t^n}{n!}, \quad (\text{see } [1–21]). \]  

(1)

When \( x = 0 \), \( B_n = B_n (0) \) are called the Bernoulli numbers. From (1), we note that

\[ B_n (x) = \sum_{l=0}^{n} \binom{n}{l} B_l x^{n-l}, \quad (n \geq 0), \]

(2)

and

\[ B_0 = 1, \quad B_n (1) - B_n = \delta_{1,n}, \quad (n \in \mathbb{N}), \quad (\text{see } [1, 19]). \]

(3)

where \( \delta_{n,k} \) is the Kronecker’s symbol.

In [3], L. Carlitz considered the degenerate Bernoulli polynomials which are given by the generating function

\[ \frac{t}{1 + \lambda t} \sum_{n=0}^{\infty} \beta_{n,\lambda} (x) \frac{t^n}{n!} = \lim_{\lambda \to 0} \left( \frac{t}{1 + \lambda t} \right)^x. \]

(4)

When \( x = 0 \), \( \beta_{n,\lambda} = \beta_{n,\lambda} (0) \) are called the degenerate Bernoulli numbers. From (1) and (4), we note that

\[ \frac{t}{e^t - 1} e^{xt} = \lim_{\lambda \to 0} \frac{\frac{t}{1 + \lambda t} \sum_{n=0}^{\infty} \beta_{n,\lambda} (x) \frac{t^n}{n!}}{(1 + \lambda t)^x}. \]

(5)

Thus, by (5), we get

\[ B_n (x) = \lim_{\lambda \to 0} \beta_{n,\lambda} (x), \quad (\text{see } [3, 15]). \]

(6)
By (4), we get
\[
\sum_{n=0}^{\infty} \left( \beta_{m,\lambda} (1) - \beta_{m,\lambda} \right) \frac{t^n}{n!} = t.
\] (7)
and
\[
\beta_{n,\lambda} (x) = \sum_{l=0}^{n} \binom{n}{l} \beta_{l,\lambda} \lambda^{n-l} \left( \frac{x}{\lambda} \right)^{n-l}.
\] (8)
From (7), we have
\[
\beta_{n,\lambda} (1) - \beta_{n,\lambda} = \delta_{1,n}, \quad (n \geq 0), \quad \beta_{0,\lambda} = 1.
\] (9)
Now, we consider the degenerate Bernoulli polynomials which are different from the degenerate Bernoulli polynomials of L. Carlitz as follows:
\[
\log \left( 1 + \lambda t \right)^{\frac{1}{\lambda}} \left( 1 + \lambda t \right)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} b_{n,\lambda} (x) \frac{t^n}{n!}.
\] (10)
When \( x = 0 \), \( b_{n,\lambda} = b_{n,\lambda} (0) \) are called the degenerate Bernoulli numbers.

**Note.** The degenerate Bernoulli polynomials are also called Daehee polynomials with \( \lambda \)-parameter (see [13]).

From (10), we note that
\[
\frac{t}{e^t - 1} e^{xt} = \lim_{\lambda \to 0} \frac{\log (1 + \lambda t)^{\frac{1}{\lambda}} \left( 1 + \lambda t \right)^{\frac{x}{\lambda}}}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} = \sum_{n=0}^{\infty} b_{n,\lambda} (x) \frac{t^n}{n!}.
\] (11)
By (1) and (11), we see that
\[
B_n (x) = \lim_{\lambda \to 0} b_{n,\lambda} (x), \quad (n \geq 0).
\]
The classical polylogarithm function \( \text{Li}_k (x) \) is defined by
\[
\text{Li}_k (t) = \sum_{n=1}^{\infty} \frac{t^n}{n^k}, \quad (k \in \mathbb{Z}), \quad \text{see [10, 11]}.
\] (12)
It is known that the poly-Bernoulli polynomials are defined by the generating function
\[
\frac{\text{Li}_k \left( 1 - e^{-t} \right)}{1 - e^{-t}} e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)} (x) \frac{t^n}{n!}, \quad \text{see [9, 10, 12]}.
\] (13)
When \( k = 1 \), we have
\[
\sum_{n=0}^{\infty} B_n^{(1)} (x) \frac{t^n}{n!} = \frac{t}{e^t - 1} e^{xt} = \frac{t}{e^t - 1} e^{(x+1)t} = \sum_{n=0}^{\infty} B_n (x + 1) \frac{t^n}{n!}.
\] (14)
By (14), we easily get
\[
B_n^{(1)} (x) = B_n (x + 1), \quad (n \geq 0).
\]
Let \( x = 0 \). Then \( B_n^{(k)} = B_n^{(k)} (0) \) are called the poly-Bernoulli numbers.

In this paper, we introduce the new fully degenerate poly-Bernoulli numbers and polynomials and investigate some properties of these polynomials and numbers. From our investigation, we derive some identities for the fully degenerate poly-Bernoulli numbers and polynomials.
2 Fully degenerate poly-Bernoulli polynomials

For $k \in \mathbb{Z}$, we define the fully degenerate poly-Bernoulli polynomials which are given by the generating function

$$
\frac{\text{Li}_k \left(1 - (1 + \lambda t)^{-\frac{1}{k}}\right)}{1 - (1 + \lambda t)^{-\frac{1}{k}}} (1 + \lambda t)^{\frac{1}{k}} = \sum_{n=0}^{\infty} \beta_{n,\lambda}^{(k)} (x) \frac{n!}{n!}.
$$

(15)

When $x = 0$, $\beta_{n,\lambda}^{(k)} = B_{n,\lambda}^{(k)} (0)$ are called the fully degenerate poly-Bernoulli numbers.

From (13) and (15), we have

$$
\frac{\text{Li}_k \left(1 - e^{-t}\right) e^{xt}}{1 - e^{-t}} e^{x t} = \lim_{\lambda \to 0} \frac{\text{Li}_k \left(1 - (1 + \lambda t)^{-\frac{1}{k}}\right)}{1 - (1 + \lambda t)^{-\frac{1}{k}}} (1 + \lambda t)^{\frac{1}{k}} = \sum_{n=0}^{\infty} \lim_{\lambda \to 0} \beta_{n,\lambda}^{(k)} (x) \frac{n!}{n!}.
$$

(16)

Thus, we get

$$
\lim_{\lambda \to 0} \beta_{n,\lambda}^{(k)} (x) = B_{n}^{(k)} (x), \quad (n \geq 0).
$$

(17)

By (15), we get

$$
\sum_{n=0}^{\infty} \beta_{n,\lambda}^{(k)} (x) \frac{n!}{n!} = \frac{\text{Li}_k \left(1 - (1 + \lambda t)^{-\frac{1}{k}}\right)}{1 - (1 + \lambda t)^{-\frac{1}{k}}} (1 + \lambda t)^{\frac{1}{k}} = \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} \beta_{l,\lambda}^{(k)} (x) \frac{x^{n-l}}{\lambda^{n-l}} \lambda^{n-l}\right) \frac{n!}{n!}.
$$

(18)

Thus, from (18), we have

$$
\beta_{n,\lambda}^{(k)} (x + y) = \sum_{l=0}^{n} \binom{n}{l} \left(\frac{y}{\lambda}\right)_{n-l} \lambda^{n-l} \beta_{l,\lambda}^{(k)} (x), \quad (n \geq 0),
$$

(19)

and

$$
\beta_{n,\lambda}^{(k)} (x) = \sum_{l=0}^{n} \binom{n}{l} \left(\frac{x}{\lambda}\right)_{n-l} \lambda^{n-l} \beta_{l,\lambda}^{(k)}.
$$

(20)

Therefore, by (17) and (19), we obtain the following theorem.

**Theorem 2.1.** For $k \in \mathbb{Z}$, $n \geq 0$, we have

$$
\beta_{n,\lambda}^{(k)} (x + y) = \sum_{l=0}^{n} \binom{n}{l} \left(\frac{y}{\lambda}\right)_{n-l} \lambda^{n-l} \beta_{l,\lambda}^{(k)} (x), \quad (n \geq 0),
$$

(20)

and

$$
\lim_{\lambda \to 0} \beta_{n,\lambda}^{(k)} (x) = B_{n}^{(k)} (x),
$$

where $(x)_n = x (x-1) \cdots (x-n) = \sum_{l=0}^{n} S_1 (n, l) x^l$.

From (15), we can derive the following equation:

$$
\sum_{n=0}^{\infty} \beta_{n,\lambda}^{(k)} (x) \frac{n!}{n!} = \frac{\text{Li}_k \left(1 - (1 + \lambda t)^{-\frac{1}{k}}\right)}{1 - (1 + \lambda t)^{-\frac{1}{k}}} \frac{(x + 1)^{\frac{1}{k}} - 1}{(x + 1)^{\frac{1}{k}}}.
$$

(21)

Thus, by (21), we get

$$
\sum_{n=0}^{\infty} \left|\beta_{n,\lambda}^{(k)} - \beta_{n,\lambda}^{(k)} (-1)^n\right| \frac{n!}{n!} = \frac{\text{Li}_k \left(1 - (1 + \lambda t)^{-\frac{1}{k}}\right)}{1 - (1 + \lambda t)^{-\frac{1}{k}}} \frac{(x + 1)^{\frac{1}{k}} - 1}{(x + 1)^{\frac{1}{k}}}.
$$

(22)
Therefore, by (25), we obtain the following theorem.

**Theorem 2.2.** For \( k \in \mathbb{Z}, n \geq 1 \), we have

\[
\beta_{n,\lambda}^{(k)} - \beta_{n,\lambda}^{(k)} (-1) = \sum_{l=1}^{n} \sum_{m=0}^{l-1} \frac{m! (-1)^{l-m-1} \lambda^{n-l} S_2(l, m + 1) S_1(n, l)}{(m + 1)^k}. 
\]

From (12), we can easily derive the following equation:

\[
\text{Li}_k'(t) = \frac{d}{dt} \text{Li}_k(t) = \frac{1}{t} \text{Li}_{k-1}(t). \tag{23}
\]

Thus, by (23), the generating function of the fully degenerate poly-Bernoulli numbers is also written in terms of the following iterated integral:

\[
\frac{(1 + \lambda t)^{\frac{1}{k}}}{(1 + \lambda t)^{\frac{1}{k}} - 1} \int_0^t \frac{1}{(1 + \lambda t)^{\frac{1}{k}} - 1} \frac{1}{(1 + \lambda t)} \frac{1}{(1 + \lambda t)^{\frac{1}{k}} - 1} \cdots \frac{1}{(1 + \lambda t)^{\frac{1}{k}} - 1} \frac{\log (1 + \lambda t)^{\frac{1}{k}}}{(1 + \lambda t)^{\frac{1}{k}} - 1} \frac{dt}{(1 + \lambda t)^{\frac{1}{k}} - 1} 
\times \frac{dt}{(1 + \lambda t)^{\frac{1}{k}} - 1} \cdots \frac{dt}{(1 + \lambda t)^{\frac{1}{k}} - 1} \frac{dt}{(1 + \lambda t)^{\frac{1}{k}} - 1} = \sum_{n=0}^{\infty} \beta_{n,\lambda}^{(2)} \frac{t^n}{n!}. \tag{24}
\]

For \( k = 2 \), we have

\[
\sum_{n=0}^{\infty} \beta_{n,\lambda}^{(2)} \frac{t^n}{n!} = \frac{(1 + \lambda t)^{\frac{1}{2}}}{(1 + \lambda t)^{\frac{1}{2}} - 1} \int_0^t \frac{\log (1 + \lambda t)^{\frac{1}{2}}}{(1 + \lambda t)^{\frac{1}{2}} - 1} \frac{(1 + \lambda t)^{-\frac{1}{2}} dt}{(1 + \lambda t)^{\frac{1}{2}} - 1} = \frac{(1 + \lambda t)^{\frac{1}{2}}}{(1 + \lambda t)^{\frac{1}{2}} - 1} \int_0^t \sum_{m=0}^{\infty} b_{m,\lambda} (-\lambda) \frac{1}{m!} \frac{t^m dt}{(1 + \lambda t)^{\frac{1}{2}} - 1} 
\times \frac{t^m dt}{(1 + \lambda t)^{\frac{1}{2}} - 1} \frac{t^m dt}{(1 + \lambda t)^{\frac{1}{2}} - 1} \frac{t^m dt}{(1 + \lambda t)^{\frac{1}{2}} - 1} \frac{t^m dt}{(1 + \lambda t)^{\frac{1}{2}} - 1} \frac{t^m dt}{(1 + \lambda t)^{\frac{1}{2}} - 1} \frac{t^m dt}{(1 + \lambda t)^{\frac{1}{2}} - 1} \frac{t^m dt}{(1 + \lambda t)^{\frac{1}{2}} - 1} = \sum_{n=0}^{\infty} \sum_{l=0}^{n} \frac{n}{l} \beta_{l,\lambda} (1) \frac{b_{n-l,\lambda} (-\lambda)}{n-l+1} \frac{t^n}{n!}. \tag{25}
\]

Therefore, by (25), we obtain the following theorem.
Theorem 2.3. For \( n \geq 0 \), we have

\[
\beta^{(2)}_{n, \lambda} = \sum_{l=0}^{n} \binom{n}{l} \beta_{l, \lambda} (1) \frac{b_{n-l, \lambda} (-\lambda)}{n-l+1}.
\]

Note that

\[
B^{(2)}_n = \lim_{\lambda \to 0} \beta^{(2)}_{n, \lambda} = \sum_{l=0}^{n} \binom{n}{l} B_l (1) \frac{B_{n-l}}{n-l+1}.
\]

From (15), we have

\[
\sum_{n=0}^{\infty} \beta^{(k)}_{n, \lambda} \frac{t^n}{n!} = \frac{\text{Li}_k \left( 1 - (1 + \lambda t)^{-\frac{1}{\lambda}} \right)}{1 - (1 + \lambda t)^{-\frac{1}{\lambda}}} \tag{26}
\]

\[
= \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)^k} \left( 1 - (1 + \lambda t)^{-\frac{1}{\lambda}} \right)^m
\]

\[
= \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)^k} \sum_{l=m}^{\infty} S_2 (l, m) \left( -\frac{1}{\lambda} \right)^l \frac{(\log (1 + \lambda t))^l}{l!}
\]

\[
= \sum_{l=0}^{\infty} \sum_{m=0}^{l} \frac{(-1)^{m+l}}{(m+1)^k} \frac{m!}{l!} S_2 (l, m) \lambda^{-l} \sum_{n=0}^{l} \frac{1}{l!} \sum_{n=l}^{\infty} S_1 (n, l) \lambda^{n-l} \left( \frac{t^n}{n!} \right)
\]

Therefore, by (26), we obtain the following theorem.

Theorem 2.4. For \( n \geq 0 \), we have

\[
\beta^{(k)}_{n, \lambda} = \sum_{l=0}^{n} \sum_{m=0}^{l} \frac{(-1)^{m+l}}{(m+1)^k} \frac{m!}{l!} S_2 (l, m) S_1 (n, l) \lambda^{n-l}.
\]

Note that

\[
B^{(k)}_n = \lim_{\lambda \to 0} \beta^{(k)}_{n, \lambda} = \sum_{m=0}^{n} \frac{(-1)^{m+n}}{(m+1)^k} S_2 (n, m).
\]

From (23), we have

\[
\frac{d}{dt} \frac{\text{Li}_k \left( 1 - (1 + \lambda t)^{-\frac{1}{\lambda}} \right)}{1 - (1 + \lambda t)^{-\frac{1}{\lambda}}} = \frac{1}{1 - (1 + \lambda t)^{-\frac{1}{\lambda}}} \text{Li}_{k-1} \left( 1 - (1 + \lambda t)^{-\frac{1}{\lambda}} \right) \tag{27}
\]

\[
= (1 + \lambda t)^{-\frac{1}{\lambda}-1} \sum_{n=0}^{\infty} \beta^{(k-1)}_{n, \lambda} \frac{t^n}{n!}.
\]

On the other hand,

\[
\frac{d}{dt} \left( \text{Li}_{k} \left( 1 - (1 + \lambda t)^{-\frac{1}{\lambda}} \right) \right) \tag{28}
\]

\[
= \frac{d}{dt} \left( \left( 1 - (1 + \lambda t)^{-\frac{1}{\lambda}} \right) \frac{1}{1 - (1 + \lambda t)^{-\frac{1}{\lambda}}} \text{Li}_k \left( 1 - (1 + \lambda t)^{-\frac{1}{\lambda}} \right) \right)
\]
Thus, by (29), we have

\[
(1 + \lambda t)^{-\lambda - 1} \frac{1}{1 - (1 + \lambda t)^{-\lambda}} \text{Li}_k \left( (1 - (1 + \lambda t)^{-\lambda}) + (1 - (1 + \lambda t)^{-\lambda}) \frac{d}{dt} \left( \sum_{n=0}^{\infty} \frac{\beta_n^{(k)}}{n!} t^n \right) \right).
\]

By (27) and (28), we get

\[
\sum_{n=0}^{\infty} \frac{\beta_n^{(k-1)}}{n!} t^n = \sum_{n=0}^{\infty} \frac{\beta_n^{(k)}}{n!} + (1 + \lambda t)^{\frac{1}{\lambda} - 1} \sum_{n=1}^{\infty} \frac{\beta_n^{(k)}}{n!} t^{n-1},
\]

By (27) and (28), we get

\[
\sum_{n=0}^{\infty} \frac{\beta_n^{(k-1)}}{n!} t^n = \sum_{n=0}^{\infty} \frac{\beta_n^{(k)}}{n!} + (1 + \lambda t)^{\frac{1}{\lambda} - 1} \sum_{m=0}^{\infty} \frac{\beta_m^{(k)}}{m!} t^m
\]

\[
+ \lambda \left( (1 + \lambda t)^{\frac{1}{\lambda} - 1} \sum_{m=0}^{\infty} \frac{\beta_m^{(k)}}{m!} t^m \right)
\]

\[
= \sum_{n=0}^{\infty} \frac{\beta_n^{(k)}}{n!} + \left( \sum_{l=1}^{\infty} \frac{1}{\lambda} \right) \left( \sum_{m=0}^{\infty} \frac{\beta_m^{(k)}}{m!} t^m \right)
\]

\[
= \sum_{n=0}^{\infty} \frac{\beta_n^{(k)}}{n!} + \sum_{n=1}^{\infty} \left( \frac{1}{\lambda} - 1 \right) \sum_{m=0}^{n-1} \frac{\beta_{n-m}^{(k)}}{(n-m)!} \sum_{m=0}^{\infty} \frac{\beta_m^{(k)}}{m!} t^m
\]

\[
+ \lambda \sum_{n=1}^{\infty} \left( \frac{1}{\lambda} - 1 \right) \sum_{m=0}^{n-1} \frac{\beta_{n-m}^{(k)}}{(n-m)!} \sum_{m=0}^{\infty} \frac{\beta_m^{(k)}}{m!} t^m
\]

\[
= \sum_{n=0}^{\infty} \frac{\beta_n^{(k)}}{n!} + \sum_{n=1}^{\infty} \left( \frac{1}{\lambda} - 1 \right) \sum_{m=0}^{n-1} \frac{\beta_{n-m}^{(k)}}{(n-m)!} \sum_{m=0}^{\infty} \frac{\beta_m^{(k)}}{m!} t^m
\]

\[
+ \lambda \sum_{n=1}^{\infty} \left( \frac{1}{\lambda} - 1 \right) \sum_{m=0}^{n-1} \frac{\beta_{n-m}^{(k)}}{(n-m)!} \sum_{m=0}^{\infty} \frac{\beta_m^{(k)}}{m!} t^m
\]

where

\[
\left( \frac{1}{\lambda} \right)_n = \left( \frac{1}{\lambda} \right) \left( \frac{1}{\lambda} - 1 \right) \cdots \left( \frac{1}{\lambda} - n + 1 \right) = \sum_{l=0}^{n} S_l (n, l) \lambda^{-l}, \quad (n \geq 0).
\]

Thus, by (29), we have

\[
\beta_{n, \lambda}^{(k-1)} = \beta_{n, \lambda}^{(k)} + \sum_{m=0}^{n-1} \binom{n}{m} \left( \frac{1}{\lambda} \right)_{n-m} \left( \frac{1}{\lambda} - n + 1 \right) = \sum_{l=0}^{n} S_l (n, l) \lambda^{-l}, \quad (n \geq 0).
\]

Therefore, by (30), we obtain the following theorem.
Theorem 2.5. For \( n \geq 1 \), we have
\[
\beta_{n, \lambda}^{(k)} = \frac{1}{n+1} \left\{ \beta_{n, \lambda}^{(k-1)} - \sum_{m=1}^{n-1} \binom{n}{m} \frac{1}{(n-m+1)} \right\} \lambda^{n-m+1}
\]
Note that
\[
B_n^{(k)} = \lim_{\lambda \to 0} \beta_{n, \lambda}^{(k)} = \frac{1}{n+1} \left\{ \beta_{n}^{(k-1)} - \sum_{m=1}^{n-1} \binom{n}{m} B_m^{(k)} \right\}.
\]
Now, we observe that
\[
\sum_{n=0}^{\infty} \left( 1 - (1 + \lambda t)^{-\frac{1}{2}} \right)^n (n+1)^{-k} = \sum_{n=1}^{\infty} \left( 1 - (1 + \lambda t)^{-\frac{1}{2}} \right)^n \frac{1}{n^k} \frac{1}{1 - (1 + \lambda t)^{-\frac{1}{2}}}
\]
\[
= \frac{1}{1 - (1 + \lambda t)^{-\frac{1}{2}}} \log \left( 1 - (1 + \lambda t)^{-\frac{1}{2}} \right)
\]
\[
= \sum_{n=0}^{\infty} \beta_{n, \lambda}^{(k)} \frac{t^n}{n!}.
\]
By (31), we get
\[
\sum_{k=0}^{\infty} \left( \sum_{n=0}^{\infty} \beta_{n, \lambda}^{(k)} \frac{x^n}{n!} \right) y^k = \sum_{k=0}^{\infty} \left( \sum_{m=0}^{\infty} \left( 1 - (1 + \lambda x)^{-\frac{1}{2}} \right)^m (m+1)^k \right) \frac{y^k}{k!}
\]
\[
= \sum_{m=0}^{\infty} \left( 1 - (1 + \lambda x)^{-\frac{1}{2}} \right)^m \sum_{k=0}^{\infty} (m+1)^k \frac{y^k}{k!}
\]
\[
= \sum_{m=0}^{\infty} \left( 1 - (1 + \lambda x)^{-\frac{1}{2}} \right)^m e^{(m+1)y}
\]
\[
= \sum_{j=0}^{\infty} (-1)^j \left( e^{-\frac{1}{2} \log(1+\lambda x)} - 1 \right)^j e^{(j+1)y}
\]
\[
= \sum_{j=0}^{\infty} (-1)^j j! \sum_{m=j}^{\infty} S_2(m, j) \frac{1}{m!} \frac{\log(1+\lambda x)^m}{m!} e^{(j+1)y}
\]
\[
= \sum_{m=0}^{\infty} \sum_{j=0}^{m} (-1)^{j+m} j! S_2(m, j) \lambda^{-m} \sum_{n=m}^{\infty} S_1(n, m) \lambda^n \frac{x^n}{n!} e^{(j+1)y}
\]
\[
= \sum_{n=0}^{\infty} \sum_{m=0}^{n} (-1)^{j+m} j! S_2(m, j) \lambda^{n-m} S_1(n, m) e^{(j+1)y} \frac{x^n}{n!}
\]
\[
= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} (-1)^{j+m} j! \lambda^{n-m} S_2(m, j) S_1(n, m) (j+1)^k \frac{x^n}{n!} \frac{y^k}{k!}.
\]
Therefore, by (32), we obtain the following theorem.

Theorem 2.6. For \( k \in \mathbb{Z} \) and \( n \geq 0 \), we have
\[
\beta_{n, \lambda}^{(-k)} = \sum_{m=0}^{n} \sum_{j=0}^{m} (-1)^{j+m} j! \lambda^{n-m} (j+1)^k S_2(m, j) S_1(n, m).
\]
Note that
\[ B_n^{(-k)} = \lim_{\lambda \to 0} \beta^{(-k)}_{n, \lambda} = \sum_{j=0}^{n} (-1)^{j+n} j! (j + 1)^k S_2(n, j). \]

From Theorem 2.1, we have
\[
\frac{d}{dx} \beta^{(k)}_{n, \lambda}(x) = \sum_{l=0}^{n} \binom{n}{l} \beta^{(k)}_{l, \lambda} \frac{d}{dx} \left( \prod_{i=0}^{n-l-1} (x - i\lambda) \right)
= \sum_{l=0}^{n} \binom{n}{l} \frac{\beta^{(k)}_{l, \lambda}}{n-l} \sum_{j=0}^{n-l-1} \frac{1}{(x - j\lambda)} \prod_{i=0}^{n-l-1} (x - i\lambda).
\]

The generalized falling factorial \((x \mid \lambda)_n\) is given by
\[ (x \mid \lambda)_n = x (x - \lambda) (x - 2\lambda) \cdots (x - (n - 1)\lambda), \quad (n \geq 0). \quad (33) \]

As is well known, the Bernoulli numbers of the second kind are defined by the generating function
\[ \frac{t}{\log(1 + t)} = \sum_{n=0}^{\infty} b_n \frac{t^n}{n!}. \quad \text{(see [20])}. \quad (34) \]

We observe that
\[
\int_{0}^{1} (1 + \lambda t)^{\frac{x}{n}} dx = \frac{\lambda^n}{n!} \frac{\lambda^n t^n}{n!} = \sum_{n=0}^{\infty} b_n (1 \mid \lambda)_{n+1} \frac{t^n}{n!} \quad (35)
\]

On the other hand,
\[
\int_{0}^{1} (1 + \lambda t)^{\frac{x}{n}} dx = \frac{\lambda}{\log(1 + \lambda t)} \left( (1 + \lambda t)^{\frac{1}{n}} - 1 \right)
= \frac{\lambda t}{\log(1 + \lambda t)} \left( \frac{(1 + \lambda t)^{\frac{1}{n}} - 1}{t} \right)
= \left( \sum_{m=0}^{\infty} b_m \lambda^m \frac{t^m}{m!} \right) \left( \sum_{l=0}^{\infty} \frac{(1 \mid \lambda)_{l+1} t^l}{l!} \right)
= \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \frac{(1 \mid \lambda)_{l+1}}{l} \lambda^{n-l} b_{n-l} \frac{t^n}{n!} \right). \quad (36)
\]

From (35) and (36), we have
\[
\int_{0}^{1} (x \mid \lambda)_n dx = \sum_{l=0}^{n} \frac{n}{l} \lambda^{n-l} b_{n-l} \frac{t^n}{l+1}. \quad (n \geq 0). \quad (37)
\]

By Theorem 2.1, we get
\[
\int_{0}^{1} \beta^{(k)}_{n, \lambda}(x) dx = \sum_{l=0}^{n} \binom{n}{l} \beta^{(k)}_{l, \lambda} \int_{0}^{1} \left( x \mid \lambda \right)_{n-l} \lambda^{n-l} dx
= \sum_{l=0}^{n} \binom{n}{l} \beta^{(k)}_{l, \lambda} \int_{0}^{1} (x \mid \lambda)_l dx
= \sum_{l=0}^{n} \left( \sum_{m=0}^{l} \frac{l}{m} \lambda^{l-m} b_{l-m} \frac{(1 \mid \lambda)_{m+1}}{m+1} \right) \binom{n}{l} \beta^{(k)}_{l, \lambda}. \quad (38)
\]
3 Further remarks

Let $\mathbb{C}$ be complex number field and let $\mathcal{F}$ be the set of all formal power series in the variable $t$ over $\mathbb{C}$ with

$$\mathcal{F} = \left\{ f (t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \mid a_k \in \mathbb{C} \right\}. \tag{38}$$

Let $\mathcal{P}$ be the algebra of polynomials in a single variable $x$ over $\mathbb{C}$ and let $\mathcal{P}^*$ be the vector space of all linear functionals on $\mathcal{P}$. The action of linear functional $L \in \mathcal{P}^*$ on a polynomial $p(x)$ is denoted by $(L \mid p(x))$, and linearly extended as

$$\{cL + c'L \mid p(x)\} = c(L \mid p(x)) + c'(L' \mid p(x)),$$

where $c, c' \in \mathbb{C}$.

For $f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!}$, we define a linear functional on $\mathcal{P}$ by setting

$$\{ f(t) \mid x^n \} = a_n \quad \text{for all } n \geq 0. \tag{39}$$

Thus, by (39), we get

$$\{t^k \mid x^n\} = n! \delta_{n,k}, \quad (n, k \geq 0). \tag{40}$$

For $f_L(t) = \sum_{k=0}^{\infty} (L \mid x^k) \frac{t^k}{k!}$, by (40), we get $(f_L(t) \mid x^n) = (L \mid x^n)$. In addition, the mapping $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathcal{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ denotes both the algebra of the formal power series in $t$ and the vector space of all linear functionals on $\mathcal{P}$ and so an element $f(t)$ of $\mathcal{F}$ can be regarded as both a formal power series and a linear functional. We refer to $\mathcal{F}$ umbral algebra. The umbral calculus is the study of umbral algebra (see [5, 15, 20]). The order $o(f(t))$ of the non-zero power series $f(t)$ is the smallest integer $k$ for which the coefficient of $t^k$ does not vanish.

If $o(f(t)) = 1$ (respectively, $o(f(t)) = 0$), then $f(t)$ is called a delta (respectively, an invertible) series (see [20]). For $o(f(t)) = 1$ and $o(g(t)) = 0$, there exists a unique sequence $s_n(x)$ of polynomials such that

$$\{g(t) f(t)^k \mid s_n(x)\} = n! \delta_{n,k}, \quad (n, k \geq 0).$$

The sequence $s_n(x)$ is called the Sheffer sequence for $(g(t), f(t))$, and we write $s_n(x) \sim (g(t), f(t))$ (see [20]).

Let $f(t) \in \mathcal{F}$ and $p(x) \in \mathcal{P}$. Then, by (40), we get

$$f(t) = \sum_{k=0}^{\infty} \{f(t) \mid x^k\} \frac{t^k}{k!}, \quad p(x) = \sum_{k=0}^{\infty} \{t^k \mid p(x)\} \frac{x^k}{k!}. \tag{41}$$

From (41), we have

$$p^{(k)}(0) = \{t^k \mid p(x)\} = 1 \{p^{(k)}(x)\}, \tag{42}$$

where $p^{(k)}(x) = \frac{d^k}{dx^k} p(x)$, (see [11, 14, 20]).

By (42), we easily get

$$t^k p(x) = p^{(k)}(x), \quad e^{yt} p(x) = p(x + y), \quad \text{and} \quad \{e^{yt} \mid p(x)\} = p(y). \tag{43}$$

From (43), we have

$$\frac{e^{yt} - 1}{t} p(x) = \int_{x}^{x+y} p(u) \, du, \quad \{e^{yt} - 1 \mid p(x)\} = p(y) - p(0).$$

Let $f(t)$ be the linear functional such that

$$\{ f(t) \mid p(x) \} = \int_{0}^{y} p(u) \, du. \tag{44}$$
for all polynomials $p(x)$. Then it can be determined by (41) to be

$$
   f(t) = \sum_{k=0}^{\infty} \left( \frac{f(t)}{x^k} \right) t^k = \sum_{k=0}^{\infty} \frac{y^{k+1}}{(k+1)!} t^k = \frac{1}{t} (e^{yt} - 1).
$$

(45)

Thus, for $p(x) \in \mathbb{P}$, we have

$$
   \left\{ \frac{e^{yt} - 1}{t} \right\} p(x) = \int_0^y p(u) \, du.
$$

(46)

It is known that

$$
   s_n(x) \sim (g(t), f(t)) \iff \frac{1}{g(f(t))} e^{xT(t)} = \sum_{k=0}^{\infty} s_k(x) \frac{t^k}{k!} \quad (x \in \mathbb{C})
$$

(47)

where $\mathcal{F}(t)$ is the compositional inverse of $f(t)$ such that $f(\mathcal{F}(t)) = f(f(t)) = t$ (see [11, 20]).

From (15), we note that

$$
   \beta_{n,\lambda}^{(k)}(x) \sim \left( \frac{1 - e^{-t}}{\text{Li}_k(1 - e^{-t})} \right) \left( \frac{1}{\lambda} (e^{\lambda t} - 1) \right).
$$

(48)

That is,

$$
   \sum_{n=0}^{\infty} \beta_{n,\lambda}^{(k)}(x) \frac{t^n}{n!} = \frac{\text{Li}_k \left( 1 - (1 + \lambda t)^{-\frac{1}{\lambda}} \right)}{1 - (1 + \lambda t)^{-\frac{1}{\lambda}}} (1 + \lambda t)^{\frac{x}{\lambda}}.
$$

Thus, by (48),

$$
   \frac{1}{\lambda} (e^{\lambda t} - 1) \beta_{n,\lambda}^{(k)}(x) = n \beta_{n-1,\lambda}^{(k)}(x).
$$

(49)

On the other hand,

$$
   (e^{\lambda t} - 1) \beta_{n,\lambda}^{(k)}(x) = \beta_{n,\lambda}^{(k)}(x + \lambda) - \beta_{n,\lambda}^{(k)}(x).
$$

(50)

Therefore, by (49) and (50), we obtain the following theorem.

**Theorem 3.1.** For $n \in \mathbb{N}$, we have

$$
   \lambda \beta_{n-1,\lambda}^{(k)}(x) = \frac{1}{n} \left\{ \beta_{n,\lambda}^{(k)}(x + \lambda) - \beta_{n,\lambda}^{(k)}(x) \right\}.
$$

By (46), we get

$$
   \left\{ \frac{e^{yt} - 1}{t} \right\} \beta_{n,\lambda}^{(k)}(x) = \int_x^y \beta_{n,\lambda}^{(k)}(u) \, du.
$$

(51)

From (51), we have

$$
   \left\{ \frac{e^{yt} - 1}{t} \right\} \beta_{n,\lambda}^{(k)}(x) = \int_0^y \beta_{n,\lambda}^{(k)}(u) \, du.
$$

(52)

Thus, by (52), we get

$$
   \left\{ \frac{e^t - 1}{t} \right\} \beta_{n,\lambda}^{(k)}(x) = \int_0^y \beta_{n,\lambda}^{(k)}(u) \, du = \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{l}{m} \binom{n}{l} \lambda^{l-m} b_{l-m} \beta_{n-1,\lambda}^{(k)}(1 \mid \lambda)^{m+1} \frac{m+1}{m+1}.
$$

(53)

Therefore, by (53), we obtain the following theorem.

**Theorem 3.2.** For $n \geq 0$, we have

$$
   \left\{ \frac{e^t - 1}{t} \right\} \beta_{n,\lambda}^{(k)}(x) = \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{l}{m} \binom{n}{l} \lambda^{l-m} b_{l-m} \beta_{n-1,\lambda}^{(k)}(1 \mid \lambda)^{m+1} \frac{m+1}{m+1}.
$$

Note that
\[
\left\langle \frac{e^t - 1}{t} \right| B_n^{(k)}(x) = \lim_{\lambda \to 0} \left\langle \frac{e^t - 1}{t} \right| \beta_{n,\lambda}^{(k)}(x) = \lim_{\lambda \to 0} \int_0^1 \beta_{n,\lambda}^{(k)}(u) \, du = \sum_{l=0}^n \binom{n}{l} B_{n-l}^{(k)} \frac{1}{l+1}
\]

Let
\[
P_n = \{ p(x) \in \mathbb{C}[x] | \deg p(x) \leq n \}, \quad (n \geq 0).
\]

For \( p(x) \in P_n \) with \( p(x) = \sum_{m=0}^n a_m \beta_{m,\lambda}^{(k)}(x) \), we have
\[
\left\langle \frac{1 - e^{-t}}{\text{Li}_k (1 - e^{-t})} \left( \frac{1}{\lambda} (e^{\lambda t} - 1) \right)^m \middle| p(x) \right\rangle = \sum_{l=0}^n a_l \left\langle \frac{1 - e^{-t}}{\text{Li}_k (1 - e^{-t})} \left( \frac{1}{\lambda} (e^{\lambda t} - 1) \right)^m \middle| \beta_{l,\lambda}^{(k)}(x) \right\rangle
\]
(54)

From (48), we note that
\[
\left\langle \frac{1 - e^{-t}}{\text{Li}_k (1 - e^{-t})} \left( \frac{1}{\lambda} (e^{\lambda t} - 1) \right)^m \middle| \beta_{l,\lambda}^{(k)}(x) \right\rangle = l! \delta_{l,m}.
\]
(55)

By (54) and (55), we get
\[
a_m = \frac{1}{m!} \left\langle \frac{1 - e^{-t}}{\text{Li}_k (1 - e^{-t})} \left( \frac{1}{\lambda} (e^{\lambda t} - 1) \right)^m \middle| p(x) \right\rangle, \quad (m \geq 0).
\]
(56)

Therefore, by (56), we obtain the following theorem.

**Theorem 3.3.** For \( p(x) \in P_n \), we have
\[
p(x) = \sum_{m=0}^n a_m \beta_{m,\lambda}^{(k)}(x),
\]
where
\[
a_m = \frac{1}{m!} \left\langle \frac{1 - e^{-t}}{\text{Li}_k (1 - e^{-t})} \left( \frac{1}{\lambda} (e^{\lambda t} - 1) \right)^m \middle| p(x) \right\rangle.
\]

For example, let us take \( p(x) = B_n^{(k)}(x) \) \((n \geq 0)\). Then, by Theorem 3.3, we have
\[
B_n^{(k)}(x) = \sum_{m=0}^n a_m \beta_{m,\lambda}^{(k)}(x),
\]
(57)

where
\[
a_m = \frac{1}{m!} \left\langle \frac{1 - e^{-t}}{\text{Li}_k (1 - e^{-t})} \left( \frac{1}{\lambda} (e^{\lambda t} - 1) \right)^m \middle| B_n^{(k)}(x) \right\rangle
\]
(58)

\[
= \frac{\lambda^{-m}}{m!} \left\langle (e^{\lambda t} - 1)^m \middle| x^n \right\rangle = \lambda^{-m} \sum_{l=m}^\infty S_2(l, m) \frac{\lambda^l}{l!} \left\langle t^l \middle| x^n \right\rangle
\]

\[
= \lambda^{n-m} S_2(n, m).
\]

From (57) and (58), we have
\[
B_n^{(k)}(x) = \sum_{m=0}^n \lambda^{n-m} S_2(n, m) \beta_{m,\lambda}^{(k)}(x).
\]
(59)
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