ON THE CHOW GROUP OF ZERO-CYCLES OF A GENERALIZED KUMMER VARIETY

by

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Résumé. — For a generalized Kummer variety $X$ of dimension $2n$, we will construct for each $0 \leq i \leq n$ some co-isotropic subvarieties in $X$ foliated by $i$-dimensional constant cycle subvarieties. These subvarieties serve to prove that the rational orbit filtration introduced by Voisin on the Chow group of zero-cycles of a generalized Kummer variety coincides with the induced Beauville decomposition from the Chow ring of abelian varieties. As a consequence, the rational orbit filtration is opposite to the conjectural Bloch-Beilinson filtration for generalized Kummer varieties.

1 Introduction

The motivation of this work comes from the study of the Chow group of zero-cycles of a hyper-Kähler manifold. Based on the existence of the Beauville-Voisin zero-cycle in a projective $K3$ surface [4], which up to a scalar multiple is the intersection of any two divisor classes, Beauville asked in [3] for a projective hyper-Kähler manifold $X$, whether the Bloch-Beilinson filtration $F^\cdot_{BB}$ on the Chow ring of $X$ with rational coefficients $CH^\cdot(X)$, if exists, admits a multiplicative splitting as in the case of abelian varieties [2]. While the existence of the Bloch-Beilinson filtration is still largely conjectural, Beauville also formulated his weak splitting conjecture [3] which is predicted by the existence and the splitting of $F^\cdot_{BB}$, but does not rely on the existence of $F^\cdot_{BB}$. The reader is referred to [3, 20, 8, 16] for the partial results of this conjecture.

Another way to approach the splitting question, at least for $CH_0(X)$, is based on the following observation due to Voisin [21, Lemma 2.2]. If $C$ is a curve in a projective $K3$ surface $S$ such that all points in $C$ are rationally equivalent in $S$, then the Beauville-Voisin zero-cycle is the class of any point in $C$. Such a curve and its higher dimensional analogue in a hyper-Kähler manifold led Huybrechts to introduce the following definition:

**Definition 1.1 ([11])**. — A constant cycle subvariety $Y \subset X$ is a subvariety such that all points in $Y$ are rationally equivalent in $X$. For $0 \leq i \leq n$, a $C^i$-subvariety $Y$ of $X$ is a subvariety of dimension $2n - i$ such that $\dim O_x \geq i$ for every $x \in Y$.

Constant cycle subvarieties are used by Voisin in [22] to introduce the rational orbit filtration $S_\cdot CH_0(X)$ as follows.

**Definition 1.2 ([22])**. — For any integer $p$, the subgroup $S_p CH_0(X)$ is generated by the classes of points $x \in X$ supported on a constant cycle subvariety of dimension $p$. 
A priori this filtration is indexed by $0 \leq p \leq 2n := \dim X$, but since constant cycle subvarieties are isotropic with respect to any holomorphic symplectic two-form on $X$, by Mumford-Roitman’s theorem [19 Proposition 10.24], $S_pCH_0(X)$ vanishes for $p > n$. As for $F^*_BBCH_0(X)$, since $H^0(X, \Omega_X^{2p-1})$ vanishes for all integer $p$, from the axioms of $F^*_BB$ [19 Chapter 11] we see that $F^{2p-1}_BBCH_0(X) = F_{BB}^{2p}CH_0(X)$. The following conjecture was formulated in [22].

**Conjecture 1.3** (22). — The rational orbit filtration $S_\ast CH_0(X)$ is opposite to the Bloch-Beilinson filtration $F^*_BBCH_0(X)$. Equivalently, the restriction to $S_pCH_0(X)$ of the quotient map $CH_0(X) \rightarrow CH_0(X)/F^{2n-2p+1}BBCH_0(X)$ is an isomorphism.

This conjecture has been verified by Voisin [22] for a punctual Hilbert scheme of a K3 surface and a Fano variety of lines on a cubic fourfold, for which we know explicitly some candidate for the Bloch-Beilinson filtration [17]. We refer to [22] for details and further discussions of this conjecture.

Motivated by the approach described above to Beauville’s splitting conjecture, the goal of this article is to study two natural filtrations defined on the Chow group of zero-cycles $CH_0(X)$ for a generalized Kummer variety $X$. The first one is the rational orbit filtration $S_\ast CH_0(X)$ introduced above, and the second filtration comes from the abelian nature of a generalized Kummer variety. Let $A$ be an abelian surface and $X$ the generalized Kummer variety of dimension $2n$ defined by $A$. If $A^{n+1}_0$ denotes the kernel of the sum map $\mu : A^{n+1} \rightarrow A$, then $X$ is a desingularization of $A^{n+1}_0/\Xi_{n+1}$ where $\Xi_{n+1}$ acts as permutations of factors. So the quotient map $A^{n+1}_0 \rightarrow A^{n+1}_0/\Xi_{n+1}$ induces an isomorphism between $CH_0(X)$ and $CH_0(A^{n+1}_0)/\Xi_{n+1}$, where $CH_0(A^{n+1}_0)/\Xi_{n+1}$ is the $\Xi_{n+1}$-invariant part of $CH_0(A^{n+1}_0)$. As the $\Xi_{n+1}$-action on $CH_0(A^{n+1}_0)$ is compatible with the Beauville decomposition $\bigoplus_{0\leq s\leq 2n}CH_0(A^{n+1}_0)$, the last decomposition induces a decomposition $\bigoplus_{0\leq s\leq 2n}CH_0(X)_s$ of $CH_0(X)$, which defines the second filtration.

Contrary to the case of $S_pCH_0(X)$ where the splitting property is conjectural, since the Beauville decomposition of $CH_0(A^{n+1}_0)$ splits the Bloch-Beilinson filtration on $CH_0(A^{n+1}_0)$ [15 §2.5], by functoriality the aforementioned induced Beauville decomposition $\bigoplus_{0\leq s\leq 2n}CH_0(X)_s$ actually defines a natural splitting of $F^*_BBCH_0(X)$. As before, the axioms that $F^*_BB$ should satisfy implies the vanishing of $CH_0(X)_s$ for any odd number $s$ since $H^0(X, \Omega_X^s) = 0$. In fact, this vanishing property can be proven unconditionally (cf. Sections 4 and 5 for two different proofs).

**Theorem 1.4.** — For any odd number $s$, $CH_0(X)_s = 0$.

The main result of this paper is to show that on $CH_0(X)$, the rational orbit filtration coincides with the induced Beauville decomposition.

**Theorem 1.5.** — If $X$ is a generalized Kummer variety of dimension $2n$, then

$$S_pCH_0(X) = \bigoplus_{2p \leq 2n-2p} CH_0(X)_s.$$  

In particular, Theorem 1.5 implies that the rational orbit filtration $S_\ast CH_0(X)$ is opposite to the Bloch-Beilinson filtration $F^*_BBCH_0(X)$ (assumed to exist), thus proving Conjecture 1.3 in the case of generalized Kummer varieties.

The outline of the proof of Theorem 1.5 is as follows. Let $A^{n+1}_0 \subset A^{n+1}$ be the kernel of the sum map $\mu : A^{n+1} \rightarrow A$ and let $K_{(0)} \subset A^{(n+1)}$ be the image of $A^{n+1}_0$ under the quotient map $q_{n+1} : A^{n+1} \rightarrow A^{(n+1)}$. Thus $K_{(0)}$ can be defined as

$$A^{n+1}_0/\Xi_{n+1},$$

(1.1)
where $\Xi_{n+1}$ is the symmetric group which acts naturally on $A_{0}^{n+1}$. Since the Hilbert-Chow morphism $\nu : X \to K_{(0)}$ induces an isomorphism $\text{CH}_2(X) \cong \text{CH}_0(K_{(0)})$ (where the Chow groups are defined with rational coefficients), we will be working with $K_{(0)}$ instead of $X$. Suppose that $X$ is defined by the jacobian of a smooth curve of genus two $C$. Using the symmetric products of $C$ and observing that the abelian sum $C^{(0)} \to A := J(C)$ is generically a $\mathbb{P}^{2n-2}$-fibration, we will first construct for all $0 \leq i \leq n$ and $k \in \mathbb{Z}_{>0}$, subvarieties $V_{i,k} \subset K_{(0)}$ of dimension $2n - i$, subject to the following property:

**Proposition 1.6** (Corollary 3.13). — $V_{i,k}$ is swept out by constant cycle subvarieties of dimension $i$ and for any constant cycle subvariety $Y \subset K_{n}$ of dimension $i$, there exist $k \in \mathbb{Z}_{>0}$ and $p \in V_{i,k}$ such that $p$ is rationally equivalent to any point in $v(Y)$. In other words, $v_{*} CH_{0}(K_{n})$ is supported on $\cup_{k>0} V_{i,k}$.

Next, we will prove that

**Proposition 1.7** (Lemma 2.1 + Proposition 2.3). — The rational equivalence class of a zero-cycle in $\cup_{k>0} V_{i,k}$ lies in $CH_{0}(K_{(0)})_{L_{2n-2i}} := \oplus_{p \in \mathbb{Z}_{>0}} CH_{0}(K_{(0)}).$ Conversely, $CH_{0}(K_{(0)})_{L_{2n-2i}}$ is supported on $V_{i,k}$.

Combining Proposition 1.6 and 1.7, Theorem 1.5 follows easily.

The paper is organized as follows. We will first construct subvarieties $V_{i,k} \subset K_{(0)}$ in Section 2 and then prove Proposition 1.6 in Section 3. The proof of Theorem 1.4 can be found in Section 4, after introducing the induced Beauville decomposition. Finally in Section 5, we will prove Proposition 1.7 hence Theorem 1.5.

### 1.1 Conventions and notations

In this paper, all varieties are defined over the field of complex numbers $\mathbb{C}$. The Chow rings appearing in this paper are with rational coefficients.

For $n \in \mathbb{Z}_{>0}$, the $n$-th symmetric product $X^{n}/\Xi_n$ is denoted by $X^{[n]}$. We write $q_{n} : X^{n} \to X^{[n]}$ the quotient map. The Hilbert scheme of points of length $n$ on $X$ is denoted by $X^{[n]}$ and $v_{n} : X^{[n]} \to X^{(n)}$ (or simply $\nu$ if there is no ambiguity) the associated Hilbert-Chow morphism.

We use additive notation for the group operation in an abelian variety $A$. Each integer $n \in \mathbb{Z}$ defines a multiplication-by-$n$ map $[n] : A \to A$, whose kernel is denoted by $A[n]$. Since we will use the same notation for the addition of algebraic cycles in $A$, when a subvariety $V$ in $A$ is considered as an algebraic cycle, it will be systematically denoted by $[V]$ in order to avoid any confusion.

### 2 Constructing $C^{i}$-subvarieties in generalized Kummer varieties

#### 2.1 Definitions

Let $X$ be a variety. For any $x \in X$, let $O_{x}$ denote the set of points in $X$ which are rationally equivalent to $x$. Since $O_{x}$ is a countable union of Zariski closed subsets in $X$, we can define $\dim O_{x}$ to be the maximum of the dimension of its irreducible components.

**Definition 2.1** ([22]). — A $C^{i}$-subvariety $Y$ of $X$ is a subvariety of dimension $2n - i$ such that $\dim O_{x} \geq i$ for every $x \in Y$.

When $X$ is an algebraic hyper-Kähler manifold of dimension $2n$, for instance a generalized Kummer variety, a constant cycle subvariety is isotropic with respect to any holomorphic symplectic two-form on $X$ by Mumford-Roitman’s theorem, so its dimension is at most $n$. As another application of Mumford-Roitman’s theorem, any subvariety in $X$ covered by constant cycle subvarieties of dimension $i$ is of dimension at most $2n - i$ [22 Theorem 1.3]. It is also proved in [22] that any $C^{i}$-subvariety is swept out by constant
cycle subvarieties of dimension \( i \). In other words, \( C^i \)-subvarieties are exactly the subvarieties of maximal dimension sharing this property.

The rational orbit filtration \( S_i CH_0(X) \), defined for \( i \in \mathbb{Z} \), is the subgroup of \( CH_0(X) \) generated by the classes of points supported on some constant cycle subvariety of dimension \( \geq i \). The goal of Section 2 is to construct some \( C^i \)-subvarieties in an (algebraic) generalized Kummer variety \( X \) which support zero-cycles in \( S_i CH_0(X) \). Below we recall their definition and set up some conventions.

Let \( A \) be an abelian surface. For each \( n \in \mathbb{N} \), let \( \mu_{n+1} : A^{[n+1]} \to A \) denote the sum map. We will use the same notation \( \mu_n \) to denote other sum maps like \( A^{(n)} \to A \) and \( A^n \to A \). A generalized Kummer variety is defined to be one of the fibers of the iso-trivial fibration \( \mu : A^{[n+1]} \to A \) and is denoted by \( K_n(A) \) or \( K_n \) if there is no ambiguity.

**Remark 2.2.** — In general, if \( f : X \to Y \) is a morphism between quotient varieties of non-singular varieties by some finite group action, then by [9 Example 16.1.13] the pullback map \( f^* : CH^*(Y) \to CH^*(X) \) (where we recall that the Chow groups are defined with rational coefficients) is well-defined. If \( f \) is birational, then \( f^* : CH_0(Y) \to CH_0(X) \) is an isomorphism. In particular, applying this to the Hilbert-Chow morphism \( \nu : K_0 \to K_0(\sigma) \), we conclude that \( \nu_* CH_0(K_0(\sigma)) \to CH_0(K_0) \) is an isomorphism.

It also follows that if \( Z \) is a \( C^i \)-subvariety in \( Y \), then the proper transformation of \( Z \) under \( f^{-1} \) is a \( C^i \)-subvariety in \( X \). We see that the \( C^i \)-subvarieties \( V_{i,k} \) constructed above lift to \( C^i \)-subvarieties in \( K_n \).

The following result will be useful.

**Lemma 2.3.** — If \( f : A' \to A \) is an isogeny, then \( f \) induces an isomorphism of Chow groups \( CH_0(K_n(A')) \cong CH_0(K_n(A)) \).

**Proof.** — First using Remark 2.2, it suffices to prove that the natural morphism \( CH_0(K_0(\sigma)) \to CH_0(K_0(\sigma)(\sigma')) \) is an isomorphism. Using formula (1.1), this last fact follows from the fact that the morphism \( CH_0(A_0^{[n+1]}) \to CH_0(A_0^{[n+1]}) \) is an isomorphism, since \( A_0^{[n+1]} \to A_0^{[n+1]} \) is an isogeny of abelian varieties [5].

Thanks to Lemma 2.3 we can suppose that \( A \) is a principally polarized abelian surface \( (A, C) \). So either \( (A, C) \) is the Jacobian variety \( J(C) \) of a genus 2 curve \( C \) together with an Abel-Jacobi embedding \( C \hookrightarrow A \) defining the theta divisor, or is the product of two elliptic curves \( E \times E' \) with \( C = E \times \{o'\} \cup \{o\} \times E' \) where \( o \in E \) and \( o' \in E' \) are the origins of \( E \) and \( E' \). We assume that the origin of \( A \) is a Weierstrass point of \( C \) in the former case, and \( (o, o') \) in the latter case.

### 2.2 Construction

Now we construct for all \( 0 \leq i \leq n \) and \( k \in \mathbb{Z}_{>0} \), a \( C^i \)-subvariety \( V_{i,k} \) in \( K_n(\sigma) \) where \( X \) is the generalized Kummer variety defined by a principally polarized abelian surface \( (A, C) \). These \( V_{i,k} \)'s will be used in Section 3 to prove that \( S_i CH_0(X) \) and the induced Beauville filtration are the same.

Set
\[
\tau_{n+1} : A^{[n+2]} \to A^{[n+1]} \quad \text{(2.1)}
\]
\[
(a_0, \ldots, a_n, a) \mapsto \tau_{n+1}(a) := (a_0 + a, \ldots, a_n + a);
\]
We will omit the index \( n + 1 \) when there is no ambiguity. For \( Z_1 \subset A^{[n+1]} \) and \( Z_2 \subset A \), we also define
\[
\tau(Z_1, Z_2) := \tau(Z_1 \times Z_2). \quad \text{(2.2)}
\]
For \( k \in \mathbb{Z}_{\geq 0} \), let \( C_k \) be the pre-image of the multiplication-by-\( k \) map \([k] : A \to A\) of the theta divisor \( C \subset A\); in the case where \( A = E \times E'\), \( C_k \) is the union of all \( \tau_{(a,a')}(C) = E \times \{a'\} \cup \{a\} \times E' \) as \( a \in E \) and \( a' \in E' \) run through all \( k \)-torsion points.

For \( 0 \leq i \leq n \), let

\[ E_{ik} := C_{k}^{i+2} \times A^{n-i} \subset A^{n+1} \times A, \]

and set

\[ V_{ik} := q_{n+1}(\tau(E_{ik})) \cap K_{(0)} \]

where we recall that \( q_{n+1} : A^{n+1} \to A^{(n+1)} \) is the quotient map.

**Lemma 2.4.** — The rational equivalence class of a point \( z = \sum_{j=1}^{i+2} (a + c_j) + \sum_{j=i+3}^{n+1} (a_j) \) in \( K_{(0)} \) is independent of \( c_1, \ldots, c_{i+2} \in C_k \) whenever \( \sum_j c_j \) is fixed in \( A \). Similarly, the rational equivalence class of \( z \) as a zero-cycle in \( A \) is also independent of \( c_1, \ldots, c_{i+2} \in C_k \) whenever \( \sum_j c_j \) is fixed.

**Proof.** — The fibers of the sum map \( \mu_2 : C^{(2)} \to A \) are CH\(_0\)-trivial varieties. When \( C \) is smooth, recall that for any \( l > 2 \), the abelian sum map \( \mu_l : C^{(l)} \to A \) is a \( \mathbb{P}^{l-2} \)-fibration. So if \( (A, C) \) is any principally polarized abelian surface, the fibers of \( \mu_l : C^{(l)} \to A \) are also CH\(_0\)-trivial varieties since \( \mu_l \) is a specialization of a family of \( \mathbb{P}^{l-2} \)-fibrations.

Now let \( k \in \mathbb{Z}_{\geq 0} \). Since an isogeny \( B \to B' \) between abelian varieties induces a natural isomorphism \( \text{CH}_0(B) \cong \text{CH}_0(B') \) \([5]\), and since the push-forward of a zero-cycle in \( A^{(0)} \) supported on a fiber \( F_k \) of the sum map \( C_{k}^{(0)} \to A \) under the isogeny \([k] : A \to A\) is supported on \([k](F_k)\), which is a fiber of \( \mu_l : C^{(l)} \to A \) so constant cycle in \( A^{(0)} \), we conclude that \( F_k \) is a constant cycle subvariety in \( A^{(0)} \). Thus if \( l = i + 2 \), then the image of \( F_k \) under the map \( A^{(0)} \to K_{(0)} \) sending \( z \) to \( z + \sum_{j=i+3}^{n+1} (a_j) \) is a constant cycle subvariety, which proves the first assertion. Since the push-forward of points in \( F_k \) under the incidence correspondence \( \text{CH}_0(K_n) \to \text{CH}_0(A) \) has constant rational equivalence class, the second assertion follows.

\( \square \)

**Proposition 2.5.** — \( V_{ik} \) is a \( C^i \)-subvariety of dimension \( 2n - i \) in \( K_{(0)} \).

**Proof.** — Fix \( k \in \mathbb{Z}_{\geq 0} \). Since the sum map \( \mu_2 : C^{(2)} \to A \) is birational, it is easy to see that \( \tau|_{E_{ik}} \) is generically finite. So dim \( V_{ik} \geq 2n - i \).

When \( i < n \), note that \( V_{ik} \) is covered by subvarieties

\[ F_b := \left\{ \sum_{j=0}^{i+1} (c_j + a) + \sum_{j=i+2}^{n} (a_j + a) \in K_{(0)} \mid \sum_{j=0}^{i+1} c_j + a \in A, \sum_{j=0}^{i+1} c_j = b \right\} \]

for all \( b \in A \), which are constant cycle subvarieties of dimension \( i \) by Lemma 2.4. We conclude by 22 Theorem 1.3 that dim \( V_{ik} = 2n - i \), so \( V_{ik} \) is a \( C^i \)-subvariety.

In the case \( i = n \), let \( z = \sum_{j=0}^{n} (c_j + c) \in V_{nk} \) where \( c, c_0, \ldots, c_n \in C_k \). Since

\[ \sum_{j=0}^{n} c_j = (n + 1) \cdot (-c), \]

\( z \) is rationally equivalent to \( (n + 1) \cdot [0] \) in \( K_{(0)} \) by Lemma 2.4. Hence \( V_{nk} \) is a constant cycle subvariety of dimension \( n \).

\( \square \)
We terminate this section by the following result which is a direct consequence of Lemma \ref{lem:main}. This gives simple representatives of classes of points supported on \(V_{i,k}\) modulo rational equivalence in \(K_{(n)}\).

**Lemma 2.6.** — If \(i < n\), every \(z \in V_{i,k}\) is rationally equivalent in \(K_{(n)}\) to

\[ i \cdot |a| + |a + c| + |a + c'| + \sum_{j=i+3}^{n+1} |a_j| \]

for some \(a, a_{i+3}, \ldots, a_{n+1} \in A\) and \(c, c' \in C_k\) such that \((i + 2) a + c + c' + \sum_{j=i+3}^{n+1} a_j = 0\).

**Proof.** Suppose \(z = \sum_{j=1}^{i+2} |a + c_j| + \sum_{j=i+3}^{n+1} |a_j|\) for some \(c_1, \ldots, c_{i+2} \in C_k\). The cycle \(z\) is rationally equivalent to \(i \cdot |a| + |a + c| + |a + c'| + \sum_{j=i+3}^{n+1} |a_j|\) where \(c, c'\) are elements in \(C_k\) such that \(c + c' = \sum_{j=1}^{i+2} c_j\) by Lemma \ref{lem:main}. \(\mathbf{\Box}\)

\[ \square \]

\[ \text{ON THE CHOW GROUP OF ZERO-CYCLES OF A GENERALIZED KUMMER VARIETY} \]

\[ \text{6} \]

\[ \text{3} \]

**The support of \(S_6 \mathrm{CH}_0(X)\)**

The subvarieties \(V_{i,k}\) that we constructed in the previous section have the following property, whose proof will occupy the whole Section 3. Recall that \(\nu : K_n \to K_{(n)}\) is the Hilbert-Chow morphism.

**Theorem 3.1.** — If \(Z \subset K_n \subset A^{[n+1]}\) is a subvariety of dimension \(i\) such that the zero-cycles in \(A\) parameterized by \(Z\) are rationally equivalent in \(A\) to each other, then for some \(k \in Z_{>0}\), there exist \(x \in \nu^{-1}(V_{i,k})\) and \(z \in Z\) such that \(x\) and \(z\) are rationally equivalent in \(K_n\).

**Remark 3.2.** — Naively, since

\[ \dim Z + \dim V_{i,k} = \dim K_n, \]

the subvarieties \(Z\) and \(\nu^{-1}(V_{i,k})\) are expected to have nonempty intersection, which would imply Theorem 3.1. Part of the argument in the proof establishes directly this nonemptiness in some situations (cf. Subsection 3.4). See also Remark 3.7 below.

**Proof of Theorem 3.1.** — The structure of the proof is inspired by Voisin’s proof of \cite{voisin} Theorem 2.1. Up to taking an irreducible component of \(Z\), we suppose that \(Z\) is irreducible. The case \(i = 0\) is trivial; below we will assume \(i > 0\).

\[ \text{3.1 Reduction to the open multiplicity-stratum} \]

**Lemma 3.3.** — It suffices to treat the case where a general element \(z\) in \(Z\) lies in the open multiplicity-stratum \(A^{[n+1]}_{\text{red}}\) parameterizing reduced subschemes of \(A\).

**Proof.** Assume the conclusion of Theorem 3.1 for all subvarieties in \(K_n\) parameterizing zero-cycles in \(A\) of constant class modulo rational equivalence and satisfying the condition in Lemma 3.3. Let \(Z\) be a subvariety of \(K_n\) as in the theorem. Suppose that a general element \(z\) in \(Z\) lies in the multiplicity-stratum \(A^{[n+1]}_{\mu}\) for some partition

\[ \mu = 1^{n_1} \cdots (n + 1)^{n_{n+1}} \]

of \(n + 1\). Consider

\[ Z_1 := \left\{ \sum_{j=1}^{n+1} \sum_{1 \leq p \leq n_j} \sum_{q=1}^{j} [c_{jp,q} + a_{jp}] \in K_{(n)} \bigg| a_{jp} \in A, c_{jp,q}, c_{jp,q,p} \in C, \sum_{q=1}^{j} c_{jp,q} = j \cdot c_{jp}, \sum_{j=1}^{n+1} \sum_{1 \leq p \leq n_j} j(c_{jp} + a_{jp}) \in \nu(Z) \right\}, \]
where we recall that the sum of elements within (resp. without) curly brackets is the sum of zero-cycles (resp. defined by the group law in $A$). By Lemma 2.3, $Z_1$ parameterizes the same class of zero-cycles in $A$ as $Z$ parameterizes. On one hand, it is easy to see that

$$\dim Z_1 = \dim \nu(Z) + \sum_{j=1}^{n+1} \alpha_j(j-1).$$

On the other hand, we see by [6] that if $z$ is a general element in $\nu(Z)$, then

$$\dim \nu^{-1}(z) = \sum_{j=1}^{n+1} \alpha_j(j-1).$$

So if $\tilde{Z}_1$ denotes the strict transform of $Z_1$ under $\nu : K_n \to K_{(n)}$, then $\dim \tilde{Z}_1 = \dim Z = i$ and a general element in $\tilde{Z}_1$ lies in the open multiplicity-stratum $A_{\text{red}}^{[n+1]}$. By assumption, there exist $x \in \nu^{-1}(V_{i,k})$ and $z \in \tilde{Z}_1$ such that $x$ and $z$ are rationally equivalent in $K_n$. Finally by Lemma 2.4 and the definition of $Z_1$, there exists $z' \in Z$ which is rationally equivalent to $z$ in $K_n$, hence to $x$.

\[\square\]

### 3.2 Setups

By virtue of Lemma 3.3, we may and we will assume that a general element $z$ in $Z$ lies in the open multiplicity-stratum $A_{\text{red}}^{[n+1]}$. In particular, $\dim Z = \dim \nu(Z)$. Since the Hilbert-Chow morphism $\nu : K_n \to K_{(n)}$ induces an isomorphism $\nu_* : \text{CH}_0(K_n) \xrightarrow{\sim} \text{CH}_0(K_{(n)})$, Theorem 3.1 it suffices to prove the following analogue version of Theorem 3.1 in $K_{(n)}$.

**Theorem 3.1.** — If $Z \subset K_{(n)} \subset A^{(n+1)}$ is a subvariety of dimension $i$ such that the zero-cycles in $A$ parameterized by $Z$ are rationally equivalent in $A$ to each other, then for some $k \in Z_{>0}$, there exist $x \in V_{i,k}$ and $z \in Z$ such that $x$ and $z$ are rationally equivalent in $K_{(n)}$.

We will prove Theorem 3.1 by induction on $n \geq 1$. For $n = 1$, the only case to prove is that of $i = 1$. Note that $K_2 \to A/\langle a \rangle$ associating $(a, -a) \in K_2$ to the class of $a$ under the involution action is an isomorphism. Via this isomorphism, $V_{1,1}$ is the image of the theta divisor $C \subset A$ under the quotient map $A \to A/\langle a \rangle$, so $V_{1,1}$ is ample. As $\dim Z \geq 1$, $Z \cap V_{1,1}$ is not empty, which proves Theorem 3.1 in this case.

From now on we assume $n > 1$. Let $Z'$ be one of the irreducible components of $\nu_{n+1}^{-1}(Z)$ where we recall that $\nu_{n+1} : A^{n+1} \to A^{n+1}$ denotes the quotient map. Let (H) denote the assumption

**There exists an integer $j$ such that the image of $Z'$ under the $j$-th projection $A^{n+1} \to A$ is $A$.**

(H)

### 3.3 Proof of Theorem 3.1 under induction hypothesis and assumption (H)

In this paragraph, we assume that $Z'$ verifies (H). If we define

$$p_j : A^{n+1} \to A^2$$

$$(a_1, \ldots, a_{n+1}) \mapsto \left( a_j, \sum_{l \neq j} a_l \right),$$

(3.1)
assumption (H) implies that \( p_{|Z'} : Z' \to A^3_0 \) is surjective; without loss of generality we can assume \( j = 1 \).

For simplicity, \( p_1 \) will be denoted by \( p \) from now on until the end of the proof. Define the map

\[
\hat{p} : A^{n+1} \to A^n
\]

\[
(a, a_1, \ldots, a_n) \mapsto (a + n \cdot a_1, \ldots, a + n \cdot a_n).
\]

\textbf{Lemma 3.4. —} In the situation above, the map \( \hat{p}_{|Z'} \) is generically finite.

\textbf{Proof. —} First of all, let

\[
\Gamma := \{(a_1, \ldots, a_{n+1}, a) \mid a_1, \ldots, a_{n+1} \in A, \ a = a_i \text{ for some } i \} \subset A^{n+1} \times A
\]
denote the incidence correspondence. Since \( Z' \subset A^{n+1} \) parameterizes zero-cycles of constant class in \( \text{CH}_0(A) \), we see by [19] Proposition 10.24 that for all \( \alpha, \beta \in H^0(A, \Omega^1_A) \),

\[
\Gamma^* \alpha = \sum_{i=1}^{n+1} (\text{pr}_i^* \alpha)_{|Z'} = 0 \quad \text{and} \quad \Gamma^* (\alpha \wedge \beta) = \sum_{i=1}^{n+1} (\text{pr}_i^* (\alpha \wedge \beta))_{|Z'} = 0,
\]

where \( \text{pr}_i : A^{n+1} \to A \) is the \( i \)-th projection.

Next by definition of \( \hat{p} \), if \( \sigma := \sum_{i=1}^n \text{pr}_i^* (\alpha \wedge \beta) \in H^0(A^n, \Omega^2_A) \), then elementary computations show that

\[
\hat{p}^* \sigma = \sum_{i=2}^{n+1} (\text{pr}_i^* (\alpha \wedge \beta) + n^2 \cdot \text{pr}_i^* (\alpha \wedge \beta) + n \cdot \text{pr}_i^* \alpha \wedge \text{pr}_i^* \beta - n \cdot \text{pr}_i^* \beta \wedge \text{pr}_i^* \alpha).
\]

The above formula together equations (3.3) yield

\[
\hat{p}^* \sigma_{|Z'} = (n - n^2) \cdot \text{pr}_1^* (\alpha \wedge \beta)_{|Z'}.
\]

Here we recall that \( n > 1 \), so \( n - n^2 \neq 0 \).

Now choose \( \alpha, \beta \in H^0(A, \Omega^1_A) \) so that \( \alpha \wedge \beta \in H^0(A, \Omega^2_A) \) is non-degenerated. Recall that \( \text{pr}_1_{|Z'} \) is surjective by assumption (H). It follows that if \( z \in Z' \) is a smooth general point, the differential \( (\text{pr}_1)_{|Z'} \), is surjective at \( z \) and thus the kernel of the two-form \( (\text{pr}_1)_{|Z'}^* (\alpha \wedge \beta) \) is equal to \( \ker((\text{pr}_1)_{|Z'}) \). On the other hand, formula (3.4) shows that if \( u \in T_{Z', z} \) is annihilated by \( \hat{p}^* \), then \( u \in \ker((\text{pr}_1^* (\alpha \wedge \beta)_{|Z'}) \), since \( u \in \ker(\hat{p}^* \sigma_{|Z'}) \). Therefore \( u \in \ker(\hat{p}_z) \cap \ker((\text{pr}_1^* \sigma)_{|Z'}) = \{0\} \), hence \( \hat{p}_{|Z'} \) is generically finite.

\[\square\]

Before we continue, let us prove a general formula.

\textbf{Lemma 3.5. —} For \( a, a_1, \ldots, a_n \in A \), the following equality holds in \( \text{CH}_0(A) \):

\[
\sum_{j=1}^n [a_j + a] = \left(a + \sum_{j=1}^n a_j\right) + \left(\sum_{j=1}^n a_j\right) - \left(\sum_{j=1}^n a_j\right) + (n - 1) ([a] - [0]).
\]

\textbf{Proof. —} Let us recall for convenience the following formula due to Bloch [5] Theorem (0.1), case \( n = 2 \). If \( a, b, c \in A \) where \( A \) is an abelian surface with origine \( o \), then the following holds in \( \text{CH}_0(A) \):

\[
|o| - |a| - |b| - |c| + |a + b| + |b + c| + |c + a| - |a + b + c| = 0.
\]

We will prove equality (3.5) by induction starting from \( n = 1 \) and 2. When \( n = 1 \), there is noting to prove. A direct application of Bloch’s formula (3.5) yields the case \( n = 2 \), from which we deduce the following
equality for $n > 1$ in $\text{CH}_0(A)$:

$$[a_n + a] + \left( a + \sum_{j=1}^{n-1} a_j \right) = \left( a + \sum_{j=1}^{n} a_j \right) + [a_n] - \left( \sum_{j=1}^{n} a_j \right) + \{(a) - \{0\}. \]

Lemma 3.5 thus follows easily from induction hypothesis. □

**Lemma 3.6.** — Let $\Delta^w_{c_i} := \{(c_i, -c) \in A^2 \mid c \in C_i\}$. The subvariety $Z_{c_i} := \phi^{\prime} p^{-1}(\Delta^w_{c_i}) \cap Z'$ parameterizes effective zero-cycles of degree $n$ in $A$ of constant class modulo rational equivalence.

**Proof.** — Every equality appearing in this proof holds in $\text{CH}_0(A)$. Let $(c, a_1, \ldots, a_n), (c', a'_1, \ldots, a'_n) \in p^{-1}(\Delta^w_{c_i}) \cap Z'$. Note that by formula (3.5)

$$\sum_{j=1}^{n} [n \cdot a_j + c] = [c - n \cdot c] + \left( \sum_{j=1}^{n} [n \cdot a_j] + n \cdot c \right) - \{(n \cdot c) + \{n - c\} + (n - 1) \cdot \{c\} - \{0\}. \]$$

(3.7)

Since $c \in C_i$, one has

$$\{c - n \cdot c\} + (n - 1) \cdot \{c\} - \{0\} = 0 \]$$

(3.8)

For the same reason,

$$\{n \cdot c\} + \{n - c\} = \{n \cdot c'\} + \{-n \cdot c'\}. \]$$

(3.9)

Since $n \cdot Z$ parameterizes zero-cycles in $A$ of constant rational equivalence class, we see that

$$\{n \cdot c\} + \sum_{j} [n \cdot a_j] = \{n \cdot c'\} + \sum_{j} [n \cdot a'_j]. \]$$

(3.10)

Combining identities (3.7), (3.8), (3.9), and (3.10), we deduce that

$$\sum_{j=1}^{n} [n \cdot a_j + c] = \sum_{j=1}^{n} [n \cdot a'_j + c']. \]$$

□

**Proof of Theorem 3.7 under the assumption (H).** — Recall that assumption (H) says that $p|Z' : Z' \to A^2_{v_0}$ is surjective. On one hand, since $p|Z'$ is generically finite by Lemma 3.4 and since the union $\cup_{k \in Z} \Delta^w_{c_k}$ is Zariski dense in $A^2_{v_0}$, there exists $l \in Z_{>0}$ such that $Z_{C_l}$ is of dimension $l - 1$. On the other hand $Z_{C_l} \subset A^n_{v_0}$ and by Lemma 3.6 $Z_{C_l}$ parameterizes effective zero-cycles of degree $n$ in $A$ of constant class modulo rational equivalence, we can apply induction hypothesis on $Z_{C_l}$.

If $i < n$, induction hypothesis shows that there exist $a_i', a_{i+2}', \ldots, a'_n \in A$ and some $k \in Z_{>0}$ such that each element in $q_n(Z_{C_l})$ is rationally equivalent in $K_{(n-1)}$ to

$$\sum_{j=1}^{i+1} [a' + c_j] + \sum_{j=i+2}^{n} [a' + a'_j],$$

for all $c_1, \ldots, c_{i+1} \in C_k$ such that $n \cdot a' + \sum_{j=1}^{i} c_j + \sum_{j=i+2}^{n} a'_j = 0$. As $\phi : C_k \times C_l \to A$ defined by $\phi(c, c') \mapsto c$ is surjective, there exist $c_0 \in C_k$ and $c \in C_l$ such that $n \cdot c = c_0 - c + a'$. Therefore for any $(c, a_1, \ldots, a_n) \in p^{-1}(\Delta_{C_l}) \cap Z'$, whose existence is due to the surjectivity of $p|Z' : Z' \to A^2_{v_0}$, the following equality holds in $\text{CH}_0(K_{(0)})$:

$$\{n \cdot c\} + \sum_{j=1}^{n} [n \cdot a_j] = \sum_{j=0}^{i+1} [(a' - c) + c_j] + \sum_{j=i+2}^{n} [(a' - c) + a'_j].$$
Thus if $z \in Z$ satisfies $v(n \cdot z) = [n \cdot c] + \sum_{j=1}^{n} [n \cdot a_j]$, there exists $z' \in V_{\nu, k}$ such that $v(z) \sim_{rat} z'$ in $K_{(0)}$.

For the remaining case $i = n$, applying induction hypothesis as before, there exists $c' \in C_k$ for some $k \in \mathbb{Z}_{>0}$ such that every point in $q_n(Z_C)$ is rationally equivalent in $K_{(n-1)}$ to $\sum_{j=1}^{n} [c' + c_j]$ for all $c_1, \ldots, c_n \in C_k$ such that $n \cdot c' + \sum_{j=1}^{n} c_j = 0$. The same argument above replacing $a'$ with $c'$ allows to conclude. □

3.4 General case

Now assume that $Z'$ does not verify (H). Then there exist curves $D_1, \ldots, D_{n+1} \subset A$ such that

$$Z' \subset \prod_{j=1}^{n+1} D_j.$$  \hspace{1cm} (3.11)

As $Z'$ is irreducible, up to removing some irreducible components of $D_j$, we can suppose that $D_j$ is irreducible for all $j$.

Remark 3.7. — One would expect to prove Theorem 3.1 in this case by showing directly that

$$\tau(Z', A) \cap \left(C_k^{n+1} \times A^{n-1-i}\right) \neq \emptyset$$  \hspace{1cm} (3.12)

where $\tau$ is defined in (2.2), merely under the assumption (3.11) just by positivity arguments as in Voisin’s proof of [21] Theorem 2.1. However the following example shows that the above expectation fails in some cases. Take for example $i = 1, n = 2$ and set

$$Z' := \{ (c, c + a, c + a') : c \in C_k \} \subset A^3$$

for some fixed $a, a' \in A$. If $a$ and $a'$ are generically chosen, then there is no $c \in C_k$ such that $c + a, c + a' \in C_k$. In other words, $\tau(Z', A) \cap C_k = \emptyset$.

We will not prove the non-emptiness (3.12) for any $Z'$ which does not satisfy hypothesis (H). Instead, for those $Z'$ such that (3.12) might fail, we will reduce the proof of Theorem 3.1 to the situation where hypothesis (H) is verified.

3.4.1 Case $i = n$

Under hypothesis (3.11), we will first prove Theorem 3.1 for the Lagrangian case, that is for $i = \dim Z' = n$. Since $Z' \subset \prod_{j=1}^{n+1} D_j \cap A^{n+1}_0$, we see that $\dim A^{n+1}_0 \cap \prod_{j=1}^{n+1} D_j = n$.

Lemma 3.8. — If $\dim A^{n+1}_0 \cap \prod_{j=1}^{n+1} D_j = n$ and the image of $\prod_{j=1}^{n+1} D_j$ under the sum map $\mu : A^{n+1} \to A$ is $A$, then $n = 1$.

Proof. — Suppose that $n > 1$. Since $\dim A^{n+1}_0 \cap \prod_{j=1}^{n+1} D_j = n$, there exists a projection from $\prod_{j=1}^{n+1} D_j$ onto a product of $n - 1$ factors whose restriction (denoted $r$) to $\prod_{j=1}^{n+1} D_j \cap A^{n+1}_0$ is surjective; without loss of generality we can suppose $r$ to be the projection $\prod_{j=1}^{n-1} D_j \cap A^{n-1}_0 \to \prod_{j=1}^{n-1} D_j$. Since a general fiber of $r$ is one-dimensional, for a general $(n - 1)$-uple $(c_1, \ldots, c_{n-1}) \in \prod_{j=1}^{n-1} D_j$ and for any $c \in D_n$, there exists $c' \in D_{n+1}$ such that $c + c' + \sum_{j=1}^{n-1} c_j = 0$,

which is impossible unless $n = 1$. □

Lemma 3.9. — If the image of $\prod_{j=1}^{n+1} D_j$ under the sum map $\mu : A^{n+1} \to A$ is of dimension $< 2$, then there exists some $k \in \mathbb{Z}_{>0}$ such that,

$$Z' \cap \tau(C_k^{n+1}, C_k) \neq \emptyset.$$
In particular, Theorem 3.1 holds when \( i = n \) and \( Z' \) does not satisfy hypothesis (H).

**Proof.** — By the assumption of Lemma 3.2, for all \( 1 \leq j \leq n+1 \) and \( (c_1, \ldots, c_{n+1}) \in \prod_{j=1}^{n+1} D_j \),

\[
\sum_{j=1}^{n+1} D_j = \left( \sum_{j=1}^{n} c_j \right) + D_j.
\]

It follows that there exists an elliptic curve \( E_0 \subset A \) such that the \( D_j \)'s are translations of \( E_0 \). Thus \( A_0^{n+1} \cap \prod_{j=1}^{n+1} D_j \) is irreducible, so \( Z' = A_0^{n+1} \cap \prod_{j=1}^{n+1} D_j \).

If \( A = E \times E' \), we can suppose without loss of generality that \( \tau_{a}(E_0) \cap (E \times \{0\}) \neq \emptyset \) for all \( a \in A \); choose \( c_j \in D_j \cap (E \times \{0\}) \) for all \( 0 \leq j \leq n \). Since \( 0 \in \sum_{j=1}^{n+1} D_j = \left( \sum_{j=1}^{n} c_j \right) + D_{n+1} \), there exists \( c_{n+1} \in D_{n+1} \) such that \( \sum_{j=1}^{n+1} c_j = 0 \). Since \( c_1, \ldots, c_n \in E \times \{0\} \), we see that \( c_{n+1} \in E \times \{0\} \), hence \( Z \cap C^{n+1} \neq \emptyset \).

In the case where \( A \) is the Jacobian of a smooth curve, \( C_k \) is not contained in any of the translates of \( E_0 \). Accordingly, since

\[
F := \left\{ -c - \sum_{j=1}^{n} (c + c_j) \left| c \in C_k, c + c_j \in (c + C_k) \cap D_j \right. \right\} \subset C_k + E'_0
\]

where \( E'_0 \) is some translation of \( E_0 \), \( F \) is of dimension \( > 0 \). So \( F \cap C_k \) is non-empty since \( C_k \) is ample. Therefore there exist \( c, c_1, \ldots, c_n \in C_k \) such that \( c + c_j \in (c + C_k) \cap D_j \) and \( c_{n+1} := -c - \sum_{j=1}^{n} (c + c_j) \in C_k \), so

\[
(c + c_1, \ldots, c + c_{n+1}) \in A_0^{n+1} \cap \prod_{j=1}^{n+1} D_j = Z'.
\]

Hence \( Z' \cap \tau(C_k^{n+1}, C_k) \neq \emptyset \).

Finally, choose \( z \in Z' \cap \tau(C_k^{n+1}, C_k) \). Since \( q_{n+1}(Z') = Z \subset K(n) \) and \( V_{nk} = q_{n+1}(\tau(C_k^{n+1}, C_k)) \cap K(n) \), we see that \( q_{n+1}(z) \in Z \cap V_{nk} \). Hence Theorem 3.1 is proven in this case.

\[\square\]

### 3.4.2 Case \( i < n \)

The following lemma allows to conclude the proof of Theorem 3.1 for the remaining case.

**Lemma 3.10.** — If \( i < n \) and \( Z' \) satisfies hypothesis (3.11), then either the non-emptiness (3.12) holds, or there exists \( Z'' \subset A_0^{n+1} \) of dimension \( i \) such that \( Z'' \) verifies hypothesis (H) and all points of \( Z'' \) represent zero-cycles in \( A \) of the same rational equivalence class in \( A \) as zero-cycles parameterized by \( Z' \).

**Proof.** — Since \( C_k \) is ample and \( Z' \subset \prod_{j=1}^{n+1} D_j \), we see that for all \( a \in A \),

\[
[Z'] \cdot \left( \sum_{h < i} \prod_{l=1}^{i} \pi_j^\prime \tau_{a}(C_k) \right) = [Z'] \cdot \left( \sum_{j=1}^{n+1} \pi_j^\prime \tau_{a}(C_k) \right)^i \neq 0,
\]

where \( \pi_j : A^{n+1} \to A \) denotes the \( j \)-th standard projection. Thus for all \( a \in A \), up to permutation of factors,

\[
Z' \cap \tau_{a}(C_k^i \times A^{n+1}) \neq \emptyset
\]

Next consider

\[
Z'' := \left\{ (a_1, \ldots, a_{n-i+1}) \in A^{n-i+1} \left| \tau_{a}(c_1, \ldots, c_i, a_1, \ldots, a_{n-i+1}) \in Z' \cap \tau_{a}(C_k^i \times A^{n+1}) \right. \right\} \text{ for some } a, c_1, \ldots, c_i \in A.
\]
Since \( Z' \subset \prod_{j=1}^{m+1} D_j \) by (3.13) the first projection of \( \pi : Z'' \to A \) is not constant; in particular \( \dim Z'' > 0 \). Accordingly \( \pi^{-1}(C_k) \subset Z'' \) has codimension \( \leq 1 \). Thus if \( \dim Z'' \geq 2 \), one of the standard projection \( \pi^{-1}(C_k) \to A \) has positive dimension hence must intersect \( C_k \). It follows that up to permutation of factors,

\[
Z' \cap \pi \left( C_k^{i+2} \times A^{n-i-1}, A \right) \neq \emptyset.
\]

So if we choose \( z \in Z' \cap \pi \left( C_k^{i+2} \times A^{n-i-1}, A \right) \), since \( q_{n+1}(Z') = Z \subset K(0) \) and \( V_{ijk} = q_{n+1}(\pi(C_k^{i+2} \times A^{n-i-1}, A)) \cap K(0) \), we see that \( q_{n+1}(z) \in Z \cap V_{ijk} \). Thus Theorem 3.1 is proven in this case.

If \( \dim Z'' = 1 \), then by (3.13), for all \( a := (d_1, \ldots, d_{n-1}) \in Z'' \), there exists a curve \( D_a \subset A \) such that for all \( a \in D_a \) there exist \( c_1, \ldots, c_i \in C_k \) such that

\[
\tau_a(c_1, \ldots, c_i, a_1, \ldots, a_{n-1}) \in Z' \cap \tau_a \left( C_k^i \times A^{n-i+1} \right).
\]

It follows that for all \( 1 \leq j \leq n-i+1 \),

\[
a_j + D_a = D_{i+j}.
\]

(3.14)

On the other hand, as \( \tau_a(c_1, \ldots, c_i, a_1, \ldots, a_{n-1}) \in Z'' \subset A_0^{n+1} \),

\[
(n+1) \cdot a + \sum_{j=1}^{i} c_j + \sum_{j=1}^{n-i} a_j = 0.
\]

(3.15)

for all \( a \in D_a \). So if \( i = 1 \), we see that a translation of \( -(n+1) \cdot D_a \) is contained in \( C_k \). Since \( D_a \) does not depend on \( k \) by (3.14), we deduce that if \( A \) is a jacobian of a smooth curve, there exists \( k \in Z_{\geq 0} \) such that \( \dim Z'' \neq 1 \). Hence \( Z' \cap \pi \left( C_k^i \times A^{n-i+1}, A \right) \neq \emptyset \) for such a \( k \), so Theorem 3.1 is proven in this situation by the same argument above. Still in the case where \( i = 1 \), if \( A = E' \), without loss of generality we can suppose that \( D_a \) is a translation of \( E \times \emptyset \). So for each \( 1 \leq j \leq n+1 \), \( D_j \) is also a translation of \( E \times \emptyset \). During \( \dim Z'' > 0 \) and \( Z'' \subset A_0^{n+1} \), there exists \( (x_1, \ldots, x_{n+1}) \in Z'' \subset A^{n+1} \) such that the projection of \( x_i - x_l \in A \) to \( E \) is \( k \)-torsion for some \( j \) and \( l \); without loss of generality we can assume \( j = 2 \) and \( l = 3 \). If \( y \) denotes the projection of \( x_2 \in A \) onto \( E \), we see that

\[
\tau_{x_1, \ldots, x_{n+1}}(x_1, \ldots, x_{n+1}) \in C_k^3 \times A^{n-2}.
\]

Hence we also have \( Z' \cap \pi \left( C_k^3 \times A^{n-2}, A \right) \neq \emptyset \).

There remains the case where \( i > 1 \). Since \( C_k \) is ample, without loss of generality there exists \( a := (d_1, \ldots, d_{n-1}) \in Z'' \) such that \( a_1 \in C_k \). Define

\[
Z''_i := \left\{ \tau_a(c_1', \ldots, c_i', a_2, \ldots, a_{n-1}) \in A^{n+1} \left| \begin{array}{c}
\sum_{j=1}^{i} c_j' = a_1 + \sum_{j=1}^{i} c_j, \text{ for some } c_1', \ldots, c_i' \in C_k \text{ such that } \\
\tau_a(c_1, \ldots, c_i, a_1, \ldots, a_{n-1}) \in Z' \cap \tau_a \left( C_k^i \times A^{n-i+1} \right) \text{ for some } a \in D_a
\end{array} \right. \right\}
\]

On one hand, by Lemma 2.3 the zero-cycles in \( A \) parameterized by \( Z''_i \subset A_0^{n+1} \) have the same class in \( CH_0(A) \) as the one parameterized by \( Z' \). On the other hand, it is easy to see that \( \dim Z''_i = i \). If \( i > 1 \), then the image of \( Z''_i \) under some standard projection \( A^{n+1} \to A \) is \( A \). We conclude that \( Z''_i \) satisfies hypothesis (H).

Remark 3.11. — The properties (3.14) and (3.15) imply that for all \( a \in D_a \) and \( 1 \leq j \leq i \), \( c_j \in C_k \cap (D_j - a) \) and \( \sum_{j=1}^{i} c_j \) belongs to a translation of \( (n+1) \cdot D_a \). Given these restrictive conditions, it would be possible to conclude directly as in the case \( i = 1 \) that there exists \( k \in Z_{>0} \) such that \( \dim Z'' \neq 1 \).
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Remark 3.12. — The reason why we distinguish the cases \( i = n \) and \( i < n \) in the proof is essentially because \( V_{i,k} = q_{n+1}(\tau(C_i \times A^{n-i}, A)) \cap K(0) \) for \( i < n \) and \( V_{n,k} = q_{n+1}(\tau(C_k, C_k) \cap K(0) \) have different form: \( V_{i,k} \) is first of all the union of all translates by elements in \( A \) of \( q_{n+1}(C_i \times A^{n-i}) \) then intersected with \( K(0) \), whereas \( V_{n,k} \) is only the union of all translates by elements in \( C_k \) of \( q_{n+1}(C_k) \), then intersected with \( K(0) \).

Corollary 3.13. — Lemmata 3.9 and 3.10.

The following result is an important corollary of Theorem 3.1'.

Theorem 3.1' now results from the combination of the proof of Theorem 3.1' under hypothesis (H) and Lemmata 3.9 and 3.10.

□

The induced Beauville decomposition on generalized Kummer varieties

We define in this section another filtration on \( \text{CH}_0(K_n) \) coming from the Beauville decomposition of an abelian variety.

4.1 Description of the Beauville decomposition

Recall in [2] that for any abelian variety \( B \), the Chow ring (with rational coefficients) of \( B \) has a canonical ring grading called the Beauville decomposition

\[
\text{CH}^p(B) = \bigoplus_{s=0}^{p} \text{CH}^s(B),
\]

for \( 0 \leq p \leq g := \dim B \), where

\[
\text{CH}^s(B) := \{ z \in \text{CH}^s(B) \mid [m]z = m^{p-s}z \text{ for all } m \in \mathbb{Z} \}.
\]

Based on [2] [7] [13] [18], the Künneth decomposition of the cohomological class of the diagonal \( [\Lambda] = \sum_{j=1}^{2g} [\Lambda_j] \in H^{2g}(B \times B, \mathbb{Q}) \) (where \( [\Lambda_j] \) is the component inducing the identity map on \( H^j(B, \mathbb{Q}) \)) lifts to a decomposition \( \Delta = \sum_{j=1}^{2g} \Lambda_j \in \text{CH}^g(B \times B) \) such that \( \Lambda_j \) acts as the projector \( \text{CH}^s(B) \to \text{CH}^s(B)_{2p-j} \) for all \( p \) [15 §2.5]. Such a decomposition of \( \Delta \in \text{CH}^g(B \times B) \) is called a Chow-Künneth decomposition of \( B \). As the Beauville decomposition is multiplicative, by [12] Theorem 5.2 if the Bloch-Beilinson filtration \( F_{\text{BB}}^* \) on \( \text{CH}^* (B) \) exists, then the Beauville decomposition would give a splitting of \( F_{\text{BB}}^* \text{CH}^* (B) \).

Let \( A_0^{n+1} \) denote the kernel of the sum map \( \mu: A^{n+1} \to A \). The symmetric group \( \Sigma_{n+1} \) acts on \( A_0^{n+1} \) and the resulting quotient variety is \( K(0) \). Let \( q: A_0^{n+1} \to K(0) \) denote the quotient map. Since \([m]: A_0^{n+1} \to A_0^{n+1}\) commutes with the action of \( \Sigma_{n+1} \) permuting the factors for each \( m \in \mathbb{Z} \),

\[
q^*: \text{CH}_0(K(0)) \sim \bigoplus_{s=0}^{2n} \text{CH}_0(A_0^{n+1})_{s}^{\Sigma_{n+1}} = \bigoplus_{s=0}^{2n} \text{CH}_0(A_0^{n+1})_{s}^{\Sigma_{n+1}}.
\]
Again, we recall that throughout this article the Chow groups are defined with rational coefficients. Since \( q \circ q' : CH_0(K_0) \to CH_0(K_0) \) is the multiplication by \((n+1)! \) [Example 1.7.6], hence bijective, the restriction to \( \text{Im}(q') \) of the map

\[
q : \bigoplus_{s=0}^{2n} CH_0(A_0^{n+1}) \to CH_0(K_0)
\]

is also bijective. Therefore we obtain the following decomposition

\[
CH_0(K_0) = \bigoplus_{s=0}^{2n} CH_0(K_0)_s,
\]

(4.3)

where

\[
CH_0(K_0)_s := q_s CH_0 \left( A_0^{n+1} \right) \cong q_s CH_0 \left( A_0^{n+1} \right).
\]

The decomposition (4.3) of \( CH_0(K_0) \) is called the induced Beauville decomposition. Since the Hilbert-Chow morphism induces an isomorphism \( \nu \cdot : CH_0(K_0) \to CH_0(K_0) \) (see Remark 2.2), this also defines a decomposition on \( CH_0(K_0) \).

4.2 The vanishing of \( CH_0(K_n)_{\text{odd}} \)

The goal of this subsection is to prove Theorem 1.4, which is a consequence of the following

**Theorem 4.1.** — The involution of \( A_0^{n+1} \) acts trivially on \( CH_0(A_0^{n+1}) \).

**Proof.** — Instead of studying \( CH_0(A_0^{n+1}) \), we will show that the involution of \( A^n \) acts trivially on \( CH_0(A^n) \), where the action of \( \mathcal{E}_{n+1} \) on \( A^n \) is given by the action of \( \mathcal{E}_{n+1} \) on \( A_0^{n+1} \) via the isomorphism

\[
A_0^{n+1} \xrightarrow{\sim} A^n \left( a_1, \ldots, a_{n+1}, \sum_{j=1}^{n} a_j \right) \mapsto (a_1, \ldots, a_n).
\]

Explicitly, if we identify \( \mathcal{E}_n \) with the permutation group of \( \mathbb{Z} \cap [1, m] \) for each \( m \) so that \( \mathcal{E}_n \) is considered as a subgroup of \( \mathcal{E}_{n+1} \), then the action of \( \mathcal{E}_{n+1} \) on \( A^n \) is determined by the action of \( \mathcal{E}_n \subset \mathcal{E}_{n+1} \) on \( A^n \) permuting the factors, and by the action of any transposition \( t_i \) exchanging \( i \) and \( n+1 \) for \( 1 \leq i \leq n \) defined by

\[
t_i \cdot (z_1, \ldots, z_n) := \left( z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n, z_i \right).
\]

For \( 1 \leq j \leq n \), let \( p_j : A^n \to A \) denote the \( j \)-th projection. Since every zero-cycle \( z \in CH_0(A^n) \) can be decomposed as a sum of zero-cycles of the form \( p_j^* z_1 \cdots p_j^* z_n \) where for each \( j, z_j \in CH_0(A) \) is either \( i \)-invariant or \( i \)-anti-invariant, Theorem 4.1 is a consequence of the following

**Lemma 4.2.** — Let \( z_1, \ldots, z_n \in CH_0(A) \) be as above. If there exists \( i \) such that \( z_i \) is \( i \)-anti-invariant, then

\[
\sum_{s \in \mathcal{E}_{n+1}} s \cdot \left( \prod_{j=1}^{n} p_j^* z_j \right) = 0.
\]

Before proving Lemma 4.2, we note that

**Lemma 4.3.** — If \( z_i \) is \( i \)-anti-invariant, then \( \mu^* z_i = \sum_{j=1}^{n} p_j^* z_j \).
Proof. — Since $z_i$ is $\iota$-anti-invariant, $z_i \in CH_0(A_1)$. Let $L_i := \mathcal{F}(z_i) \in \text{Pic}^0(A)$ be the Fourier-Mukai transform of $z_i$ [14]. It follows from the Seesaw theorem [14, page 54] that $\mu' L_i = \sum_{j=1}^{n} p'_j L_j$. Applying Fourier-Mukai transform $\mathcal{F}$ on both sides of the preceding identity yields the result. \hfill \Box

Proof of Lemma 4.2 — By definition, $p_i \circ \iota_i = -\mu_i$ and $p_j \circ \iota_i = p_j$ for all $i \neq j$. Together with Lemma 4.3 we see that

$$t'_i \left( \prod_{j=1}^{n} p'_j z_i \right) = \prod_{j=1}^{n} t'_j \left( p'_j z_i \right) = (-\mu)_i z_i \cdot \prod_{j=1}^{n} p'_j z_j = -\prod_{j=1}^{n} p'_j z_j. \tag{4.5}$$

Hence

$$\sum_{\sigma \in \mathfrak{S}_{n+1}} \sigma' \left( \prod_{j=1}^{n} p'_j z_j \right) = \sum_{\sigma \in \mathfrak{S}_n} \sigma' \left( \prod_{j=1}^{n} p'_j z_j + t'_i \left( \prod_{j=1}^{n} p'_j z_j \right) \right) = 0,$$

where $\mathfrak{S}_{n+1} < \mathfrak{S}_{n+1}$ stands for the alternating subgroup of $n + 1$ elements. \hfill \Box

Proof of Theorem 4.1 — By definition of the induced Beauville decomposition, the $\iota$-anti-invariant part of $CH_0(K_n)$ is identified with $\mathfrak{F}_{\iota \mu} CH_0(K_n)_{2p+1}$. Thus by Theorem 4.1, $CH_0(K_n)_{2p+1}$ vanishes for all $p \in \mathbb{Z}$. \hfill \Box

Remark 4.4. — Let $\Sigma$ denote the quotient of $A$ under the involution $\iota$ and $\Sigma$ the Kummer $K3$ surface defined by $A$. The action (4.4) of $\mathfrak{S}_{n+1}$ on $A^n$ descends to an action on $\Sigma^n$. The quotient map $A \to \Sigma$ induces a morphism

$$CH_0(\Sigma^{(n)}) \to CH_0(\Lambda^{(n)})$$

whose restriction to $CH_0(\Sigma^{(n)})^{\mathfrak{S}_{n+1}}$ is isomorphic to the $\iota$-invariant part of $CH_0(A^{(n)})^{\mathfrak{S}_{n+1}} \simeq CH_0(K_n)$ (so the whole $CH_0(\Sigma^{[n]})$ and that of $CH_0(K_n)$ does not come from a morphism $f : \Sigma^{[n]} \to K_n$, which is why we cannot compare the rational orbit filtration of $CH_0(\Sigma^{[n]})$ and that of $CH_0(K_n)$).

We finish this section by stating the Chow-Künneth decomposition for zero-cycles on $K_n$, which is a direct consequence of the existence of Chow-Künneth decomposition for abelian varieties.

Proposition 4.5. — There exist $\pi_1, \ldots, \pi_n \in CH^{2n}(K_n \times K_n)$ such that for all $1 \leq j \leq n$,

i) $\pi_j$ acts as the projection $CH_0(K_n) \to CH_0(K_n)_{2j}$ with respect to decomposition (4.2);

ii) $\pi_j^*$ acts as the identity map on $H^0(K_n, O_{K_n})$ if $l = 2j$ and as 0 otherwise.

Proof. — As we recalled at the beginning of this section, there exist $\Lambda_1, \ldots, \Lambda_n \in CH^{2n}(A_0^{p+1} \times A_0^{p+1})$ such that $\Lambda_j$ acts as the projection $CH_0(A_0^{p+1}) \to CH_0(A_0^{p+1})_{4n-2j}$ with respect to the Beauville decomposition and that $[\Lambda] = \sum_{j=1}^{2n} [\Lambda_j]$ in $H^{2n}(A_0^{p+1}, C)$ is the Künneth decomposition. If $\pi_1, \ldots, \pi_n \in CH^{2n}(K_n \times K_n)$ verify

$$(v \times v) \pi_j = (q_{n+1} \times q_{n+1})_* \Lambda_{4n-2j}$$

where we recall that $v : K_n \to K_n$ is the Hilbert-Chow map and $q_{n+1} : A_0^{n+1} \to K_n$ is the quotient map, then $\pi_1, \ldots, \pi_n$ satisfy the properties listed in Proposition 4.5. \hfill \Box
5 The rational orbit filtration and the induced Beauville decomposition coincide

The last part of this paper is devoted to the comparison between the rational orbit filtration and the induced Beauville decomposition of a generalized Kummer variety.

5.1 Proof of Theorem 1.5

Recall that we want to prove $S_p CH_0(K_n) = CH_0(K_n)_{p \leq 2n-2p}$. We first prove one inclusion

**Lemma 5.1.** — For all $0 \leq p \leq n$,

\[^{2}\] \[S_p CH_0(K_n) \subset CH_0(K_n)_{p \leq 2n-2p}.\]

**Proof.** — By Corollary 3.13, it suffices to show that for all $0 \leq p \leq n$ and $k \in \mathbb{Z}_{>0}$,

\[\text{Im}(CH_0(V_{p,k}) \rightarrow CH_0(K_{(n)})) \subset CH_0(K_n)_{p \leq 2n-2p}.\]

Let $z \in V_{p,k}$. If $p = n$, then $z \sim_{\text{rat}} (n + 1) \cdot [0]$ in $K_{(n)}$. Since $\{(0,\ldots,0)\} \in CH_0(A_n^{n+1})$, we see that

\[|z| = q_*((0,\ldots,0)) \in CH_0(K_{(n)})\]

where $q : A_n^{n+1} \rightarrow K_{(n)}$ stands for the quotient map.

Now assume that $p < n$. By Lemma 2.6, $z$ is rationally equivalent in $K_{(n)}$ to some element of the form

\[p \cdot |a| + |a + c| + |a + c'| + \sum_{j=p+3}^{n+1} |a + a_j|\]

for some $c, c' \in C_k$. Hence by Lemma 5.2 below, in $CH_0(K_{(n)})$ we have

\[(n + 1)^2 \cdot |z| = q_* i' \tau_* \left( \prod_{j=1}^{p} \pi_j' |0| \cdot \pi_{p+1}' |c| \cdot \pi_{p+2}' |c'| \cdot \prod_{j=p+3}^{n+1} \pi_j' |a_j| \right)\]

where $\pi_j : A^{n+2} \rightarrow A$ denotes the $j$-th projection, $i : A_n^{n+1} \hookrightarrow A^{n+1}$ the inclusion map, and $\tau : A^{n+2} \rightarrow A^{n+1}$ was defined as (2.7). Since $[0] \in CH_0(A_n)$ and $|c|, |c'| \in CH_0(A)_{\leq 1}$, we conclude that $|z| \in CH_0(K_{(n)})_{p \leq 2n-2p}$ by [2] Proposition 2.\[\square\]

Using the same notations as in the proof of Lemma 5.1, we have the following easy

**Lemma 5.2.** — Let $z \in CH_0(A_n^{n+1})$ be a zero-cycle supported on $A_n^{n+1}$, then

\[(n + 1)^2 \cdot z = i' (\tau_* (z \times A)).\]

**Proof.** — Let $z \in A_n^{n+1} \subset A^{n+1}$, then

\[A_n^{n+1} \cap \tau(z, A) = \{\tau_a(z) \mid (n + 1) \cdot a = 0\}.\]

It follows that as zero-cycles,

\[i' (\tau_* (z \times A)) = \sum_{a \in A^{n+1}} \tau_a^* z = (n + 1)^2 \cdot z\]

where the last equality follows from [10] Theorem 1.\[\square\]

**Proposition 5.3.** — For all $0 \leq p \leq n$,

\[CH_0(K_{(n)})_{p \leq 2n-2p+1} \subset \text{Im}(CH_0(V_{p,1}) \rightarrow CH_0(K_{(n)})).\]
Proof. — Given \( z \in CH_0(\mathbb{A}_{\mathbb{Q}}^{n+1}) \otimes_{2n-2p+1}, \) so \( l_z \in CH_0(\mathbb{A}_{\mathbb{Q}}^{n+1}) \otimes_{2n-2p+1}, \) then by \([2]\) Proposition 4] applying to the symmetric ample divisor \( \sum_{j=1}^{n+1} \pi_j^*C, \) \( l_z \) is supported on

\[
\bigcup_{j|z \sim l_z} W_{j,l_z}
\]

where \( W_{j,l_z} \) denotes the orbit of \([0]^l \times C \times A^{a_{n+1} - j} \subset A^{n+1} \) under the permutation of factors. Let \( z' \) be a zero-cycle supported on \( W_{j,l_z}. \) Since \( q_z \) is proportional to \( q_z \) \( (z' \times A) \) in \( CH_0(K_{(a)}) \) by Lemma 5.2, it suffices to show that the later is supported on \( V_{p,1} \) to finish the proof.

By definition of the \( V_{p,1} \)'s, if \( l > 1 \) then \( q_z \) \( (z' \times A) \) is supported on \( V_{p,1}. \) Assume that \( l = 1 \) then there exist \( c \in C \) and \( a, a_{p+2}, \ldots, a_{n+1} \in A \) such that \( q_z \) \( (z' \times A) \) is rationally equivalent to a sum (as 0-cycles) of elements in \( K_{(a)} \) of the form \( z'' := p \cdot [a] + [a + c] + \sum_{m=p+2}^{n+1} [a_m]. \)

**Lemma 5.4.** — For any \( k \in \mathbb{Z}_{\geq 0} \) and \( c' \in C_k, \)

\[
p \cdot [-c'] + [p \cdot c'] = (p + 1) \cdot [0]
\]

in \( CH_0(K_{(a)}). \)

Proof. — We prove Lemma 5.4 by induction on \( p \geq 0. \) The case \( p = 0 \) is obviously true. Lemma 5.4 holds also for \( p = 1 \) since \( c' \in C_k. \) Suppose that \( p \geq 2, \) since \( 2 \cdot [0] = ((p - 1)c') + (-(p - 1)c') \in CH_0(K_{(a)}), \) we see that in \( CH_0(K_{(a)}), \)

\[
p \cdot [-c'] + [p \cdot c'] = (p - 2) \cdot [-c'] + \tau_{c'} ((p - 1)c') + (-(p - 1)c')) + [p \cdot c']
\]

\[
= (p - 2) \cdot [-c'] + ((p - 2)c') + (-(p - 2)c') + [p \cdot c'] = (p + 1) \cdot [0]
\]

where the last equality results from induction hypothesis and the fact that \([-p c'] + [p \cdot c'] = 2 \cdot [0]. \)

Back to the proof of Proposition 5.3, let \( c' \in C_{p+1} \) such that \( (p + 1) \cdot c' = c. \) By Lemma 5.4 we see that \( z'' \) is rationally equivalent to \( (p + 1) \cdot [a + c] + \sum_{m=p+2}^{n+1} [a_m]. \) If \( p = n, \) then \( a' + c' \) is a torsion point so \( z'' \sim_{\text{rat}} (n + 1) \cdot [0] \) by \([10], \) Theorem 1. \) If \( p < n, \) since there exist \( a' \in A \) and \( c_1, c_2 \in C \) such that \( a' + c_1 = a + c' \) and \( a' + c_2 = a_{p+2}, \)

\( z'' \) is rationally equivalent to \( (p + 1) \cdot [a' + c_1] + [a' + c_2] + \sum_{m=p+3}^{n+1} [a_m]. \) In either case, we conclude that \( z'' \) is supported on \( V_{p,1}. \)

**Proof of Theorem 1.3** — Since \( \nu' \text{Im}(CH_0(V_{p,1}) \to CH_0(K_{(a)})) \subset SpCH_0(K_a), \) by Proposition 4.5 it follows from Lemma 5.1 and Proposition 5.3 that

\[
SpCH_0(K_a) \subset CH_0(K_a)_{\leq 2n-2p} \subset CH_0(K_a)_{\leq 2n-2p+1} \subset \nu' \text{Im}(CH_0(V_{p,1}) \to CH_0(K_{(a)})) \subset SpCH_0(K_a).
\]

5.2 Final remarks

i) Note that the chain of inclusions 5.2 gives a second proof of Theorem 1.4. If we are only interested in proving Theorem 1.5 instead of proving Proposition 5.3, we could have just shown that \( CH_0(K_{(a)})_{\leq 2n-2p} \subset \text{Im}(CH_0(V_{p,1}) \to CH_0(K_{(a)})) \) and used Theorem 1.4 to conclude.

ii) Combining Theorem 1.5 and Proposition 4.5, we obtain a positive answer of [22 Conjecture 0.8] for generalized Kummer varieties.
Finally we note that the chain of inclusions (5.2) also implies that $S_p\text{CH}_0(K_n)$ is supported on a subvariety of codimension $p$, while in Corollary 3.13, $S_p\text{CH}_0(K_n)$ is only proved to be supported on a countable union of subvarieties of codimension $p$:

**Corollary 5.5.** — For all $0 \leq p \leq n$,

$$\text{Im} \left(\text{CH}_0(V_{p,1}) \rightarrow \text{CH}_0(K_{(p)})\right) = v_\ast S_p\text{CH}_0(K_n).$$

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