On the Logarithmic Mean of Accretive Matrices

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Abstract. In this paper, we define the logarithmic mean of two accretive matrices and study its basic properties. Among other results, we show that if \( A, B \) are accretive matrices, then

\[
\Re L(A, B) \geq L(\Re A, \Re B),
\]

where \( L(A, B) \) is the logarithmic mean of \( A \) and \( B \), and \( \Re A \) means the real part of \( A \). This complements a recent result of Lin and Sun.

1. Introduction

The logarithmic mean of two positive numbers \( a \) and \( b \), which is of interest in geometry, statistics, and thermodynamics, is defined as

\[
L(a, b) = \frac{a - b}{\log a - \log b} = \int_0^1 a^{1-t}b^t \, dt.
\]

It is well known that

\[
\sqrt{ab} \leq L(a, b) \leq \frac{a + b}{2}.
\]

(1)

The logarithmic mean has also been defined for positive definite matrices or operators; see for example [6], in which comparison with various other means are studied. In the sequel, we let \( M_n \) be the set of \( n \times n \) complex matrices. The conjugate transpose of \( A \in M_n \) is denoted by \( A^* \). Every \( A \in M_n \) has a unique Cartesian decomposition

\[
A = \Re A + i\Im A,
\]

where \( \Re A = \frac{A + A^*}{2} \) and \( \Im A = \frac{A - A^*}{2i} \) are called the real and imaginary part of \( A \), respectively. If \( \Re A \) is positive definite, then we say \( A \) is accretive. This class of matrices and its subclass, viz, accretive-dissipative matrices, are receiving much attention over the past few years; see [4, 11–16, 19].
The geometric mean of two accretive matrices $A, B \in M_n$ was first brought in by Drury [3], who defined

$$A^\# B = \left( \frac{2}{\pi} \int_0^\infty \left( sA + s^{-1}B \right)^{-1} ds \right)^{-1}.$$ 

However, to define the logarithmic mean of accretive matrices, a weighted geometric mean seems essential. Raissouli, Moslehian and Furuichi [17] recently defined the following weighted geometric mean of two accretive matrices $A, B \in M_n$,

$$A^\#_t B = \frac{\sin t\pi}{\pi} \int_0^\infty s^{t-1} \left( A^{-1} + sB^{-1} \right)^{-1} ds,$$

where $t \in [0, 1]$. It could be verified that $A^\#_1 B = A^\# B$. We summarize some basic properties of the weighted geometric mean in the following proposition.

**Proposition 1.1.** [17] Let $A, B \in M_n$ be accretive. Then

1. $A^\#_t B$ is accretive;
2. $A^\#_t B = B^\#_{1-t} A$;
3. for any nonsingular $P \in M_n$, $(PAP^*)^\#(PBP^*) = P(A^\# B)P^*$;
4. in particular, the definition of $A^\#_t B$ coincides with the regular definition of weighted geometric mean when $A$ and $B$ are positive definite.

With the weighted geometric mean of two accretive matrices, we are able to define the logarithmic mean of accretive matrices $A, B \in M_n$ as

$$L(A, B) = \int_0^1 A^\#_t B \, dt.$$

In this paper, we intend to study some basic properties of the logarithmic mean (2) and compare it with other matrix means. To enrich our study, we need to define a sector $S_\theta$ on the complex plane

$$S_\theta = \{ z \in \mathbb{C} : \Re z > 0, |\Im z| \leq (\Re z) \tan \theta \},$$

where $\theta \in [0, \pi/2)$ is fixed.

Recall that the numerical range (see, e.g., [5]) of $A \in M_n$ is defined as the set on the complex plane

$$W(A) = \{ x^* Ax : x \in \mathbb{C}^n, x^* x = 1 \}.$$

In [9], if $W(A) \subset S_\theta$, then $A$ is called a sector matrix. Clearly, if $W(A) \subset S_\theta$, then $\Re A$ is positive definite. Some recent studies of sector matrices can be found in [2, 9, 18, 20].

### 2. Auxiliary Results

In this section, we present some auxiliary results which motivate and facilitate the proofs of the main results in the next section.

For two Hermitian matrices $A, B$, we write $A \geq B$ to mean that $A - B$ is positive semidefinite. The following remarkable property about the geometric mean of accretive matrices was proved by Raissouli, Moslehian and Furuichi.

**Proposition 2.1.** [17, Theorem 2.4] Let $A, B \in M_n$ be accretive and let $t \in [0, 1]$. Then

$$\Re(A^\#_t B) \geq (\Re A)^\#_t (\Re B).$$

We remark that when $t = 1/2$, the previous result was observed by Lin and Sun in [10]. Our Proposition 3.2 in the next section complements Lin and Sun’s result.
Proposition 2.2. Let \( A, B \in M_n \) be positive definite. Then
\[
A \# B \leq L(A, B) \leq \frac{A + B}{2}.
\] (4)

Proof. This is a known result (e.g. [1, Eq. (17)]), but we mention a simple proof here. The key observation is the simultaneous diagonalization of two positive definite matrices, that is, there is a nonsingular \( P \in M_n \) such that \( PAP^* \) and \( PBP^* \) are diagonal; see [7, Theorem 7.6.1]. Then (4) reduces to the case where the underlying matrices are positive diagonal, which is essentially the scalar inequality (1).

Lemma 2.3. [8, Lemma 2.4] Let \( A \in M_n \) be accretive. Then
\[
(\Re A)^{-1} \geq \Re A^{-1}.
\]

A reverse inequality of Lemma 2.3 is as follows.

Lemma 2.4. [9, Lemma 3] Let \( A \in M_n \) with \( W(A) \subset S_\theta \). Then
\[
(\Re A)^{-1} \leq (\sec \theta)^2 \Re A^{-1}.
\]

The next lemma is known as the Ostrowski-Taussky inequality.

Lemma 2.5. [7, p. 510] If \( A \in M_n \) is accretive, then it holds
\[
\det(\Re A) \leq |\det A|.
\]

The following lemma gives a reverse of the Ostrowski-Taussky inequality.

Lemma 2.6. [8, Lemma 2.6] If \( A \in M_n \) such that \( W(A) \subset S_\theta \), then it holds
\[
|\det A| \leq \sec^\alpha(\theta) \det(\Re A).
\]

3. Main Results

Some basic properties about the logarithmic mean are included in the following proposition.

Proposition 3.1. Let \( A, B \in M_n \) be accretive. Then
1. \( L(A, B) \) is accretive;
2. \( L(A, B) = L(B, A) \);
3. for any nonsingular \( P \in M_n \), \( L(PAP^*, PBP^*) = PL(A, B)P^* \).

Proof. Since we know from [17] that \( A^t_t B \) is accretive for all \( t \in [0, 1] \), it follows
\[
\Re L(A, B) = \Re \int_0^1 A^t_t B \ dt = \int_0^1 \Re (A^t_t B) \ dt
\]
is positive definite. That is, \( L(A, B) \) is accretive. To show the second item, notice that \( A^t_t B = B^t_{1-t} A \), then
\[
L(A, B) = \int_0^1 A^t_t B \ dt = \int_0^1 B^t_{1-t} A \ dt = \int_0^1 B^t_{1-t} A \ ds = L(B, A),
\]
in which the third equality by change of variable. To show the third item, notice that \( (PAP^*)^t_t (PBP^*) = P(A^t_t B)^t_t P^* \), then
\[
L(PAP^*, PBP^*) = \int_0^1 (PAP^*)^t_t (PBP^*) \ dt
= \int_0^1 P(A^t_t B)^t_t P^* \ dt = PL(A, B)P^*.
\]
This completes the proof.
The next result provides an analogue of Proposition 2.1.

**Proposition 3.2.** Let $A, B \in \mathbb{M}_n$ be accretive. Then

$$\Re L(A, B) \geq L(\Re A, \Re B).$$

**Proof.** We compute

$$\Re L(A, B) = \int_0^1 \Re (A \#_t B) dt$$

$$\geq \int_0^1 (\Re A \#_t (\Re B)) dt$$

$$= L(\Re A, \Re B),$$

in which the inequality is by Proposition 2.1. \(\square\)

Under the assumption that $A, B$ are sector matrices, we could derive a reverse inequality. We need a new lemma.

**Lemma 3.3.** Let $A, B \in \mathbb{M}_n$ with $W(A), W(B) \subset S_\theta$. Then

$$\Re (A \#_t B)^{-1} \leq \left( \Re A^{-1} + t \Re B^{-1} \right)^{-1}.$$  

On the other hand, by Lemma 2.4 we have

$$\Re A^{-1} + t \Re B^{-1} \geq (\cos \theta)^2 \left( (\Re A)^{-1} + t(\Re B)^{-1} \right).$$

Thus

$$\Re (A^{-1} + tB^{-1})^{-1} \leq (\sec \theta)^2 \left( (\Re A)^{-1} + t(\Re B)^{-1} \right)^{-1}.$$  

Combining previous two inequalities gives

$$\Re (A \#_t B) = \frac{\sin t \pi}{\pi} \int_0^\infty s^{-1} \Re \left( A^{-1} + sB^{-1} \right)^{-1} ds$$

$$\leq \frac{\sin t \pi}{\pi} \int_0^\infty s^{-1} (\sec \theta)^2 \left( (\Re A)^{-1} + s(\Re B)^{-1} \right)^{-1} ds$$

$$= (\sec \theta)^2 \Re (A \#_t B).$$

The proof is complete. \(\square\)

**Proposition 3.4.** Let $A, B \in \mathbb{M}_n$ with $W(A), W(B) \subset S_\theta$. Then

$$\Re L(A, B) \leq (\sec \theta)^2 L(\Re A, \Re B).$$

**Proof.** By Lemma 3.3, we could estimate

$$\Re L(A, B) = \int_0^1 \Re (A \#_t B) dt$$

$$\leq (\sec \theta)^2 \int_0^1 (\Re A \#_t (\Re B)) dt$$

$$= (\sec \theta)^2 L(\Re A, \Re B).$$

This completes the proof. \(\square\)
In the next theorem, we establish an analogue of Proposition 2.2.

**Theorem 3.5.** Let $A, B \in \mathbb{M}_n$ with $W(A), W(B) \subset S_\theta$. Then

$$
(cos \theta)^2 \Re(A\#B) \leq \Re L(A, B) \leq (sec \theta)^2 \Re \frac{A + B}{2}.
$$

(5)

**Proof.** By Lemma 3.3,

$$
\Re(A\#B) \leq (sec \theta)^2 (\Re(A)\#(\Re(B)).
$$

Then by the first inequality of (4), we have

$$
(\Re(A)\#(\Re(B) \leq L(\Re(A, \Re(B).
$$

Combing with Proposition 3.2 gives

$$
\Re(A\#B) \leq (sec \theta)^2 \Re L(A, B),
$$

which is the first inequality of (5). To show the second inequality of (5), we estimate

$$
\Re L(A, B) \leq (sec \theta)^2 L(\Re(A, \Re(B)) \\
\leq (sec \theta)^2 \Re \frac{A + B}{2} \\
= (sec \theta)^2 \Re \frac{A + B}{2},
$$

in which the first inequality is by Proposition 3.4 and the second inequality is by (4).

Note that if $A \geq B \geq 0$, then $\det A \geq \det B \geq 0$. Thus we have an immediate corollary of Theorem 3.5.

**Corollary 3.6.** Let $A, B \in \mathbb{M}_n$ with $W(A), W(B) \subset S_\theta$. Then

$$
(cos \theta)^2 \det \Re(A\#B) \leq \det \Re L(A, B) \leq (sec \theta)^2 \det \Re \frac{A + B}{2}.
$$

(6)

The next result shows the first inequality of (6) could be considerably improved.

**Proposition 3.7.** Let $A, B \in \mathbb{M}_n$ with $W(A), W(B) \subset S_\theta$. Then

$$
(cos \theta)^n \det \Re(A\#B) \leq \det \Re L(A, B).
$$

**Proof.** By Lemma 2.5,

$$
\det \Re(A\#B) \leq |\det(A\#B)| = \sqrt{\det A \| \det B|,
$$

in which the equality is by [3, Theorem 3.4] since $A\#B = A^{1/2} (A^{-1/2}BA^{-1/2})^{1/2} A^{1/2}$. Then by Lemma 2.6,

$$
\sqrt{\det A} \| \det B \leq (sec \theta)^n \sqrt{(\det(\Re(A)(\det \Re(B) = (sec \theta)^n \det(\Re(A)\#(\Re(B).
$$

It follows by the first inequality of (4) and Proposition 3.2 that

$$
\det \Re(A\#B) \leq (sec \theta)^n \det(\Re(A)\#(\Re(B)) \\
\leq (sec \theta)^n \det L(\Re(A, \Re(B) \\
\leq (sec \theta)^n \det \Re L(A, B).
$$

This proves the assertion.

It would be interesting to know whether the second inequality of (6) could be similarly improved. We leave it as a question for future research.
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