Superconductors that do not expel magnetic flux

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Abstract
An ultrathin superconducting bilayer creates a coreless fractional vortex when only the second layer has a hole. The quantization is broken by the hole, and the normal core disappears. The magnetic flux is no longer confined near the normal core, and its density profile around the hole becomes similar to that of a crème caramel: the divergence of the magnetic flux density is truncated around the center. We propose basic design of a practical device to realize a coreless fractional vortex.
1. Introduction

In a superconductor, a vortex concentrates its self-energy around its center [1]. For example, the region within a 2-µm-diameter circle holds more than 80% of self-energy, assuming a coherent length of 40 nm and a magnetic penetration depth of 80 nm for a 20-nm-thick niobium thin film [2-4]. This condition occurs because of quantization, in which the superconducting phase should rotate by $2\pi$ radians around the center, which provides a large amount of velocity to the superconducting pair near the center. If we can eliminate the singular point of the superconducting phase at the center (which corresponds to the normal core), we can expect a large reduction in the self-energy. This elimination can be made possible by introducing an inter-band phase difference soliton [5-32], which makes the phases of the first and second bands different and creates a fraction vortex [33-67]. The superconducting quantum phase does not rotate in one band while rotating by $2\pi$ radians in another band in the fractional vortex [30,33,43,65,66,68]. If we remove the second band around the center of the fractional vortex, the singular point is also removed. Elimination of the second band can be realized using a mimic two-band superconductor composed of an ultrathin superconducting bilayer [7, 69]. (The thickness of the layer is lesser than the penetration depth, and a zero-thickness limit yields a pure two-band superconductor with a substantially small inter-band Josephson interaction.) The first and second layers correspond to the first and second bands, respectively. The formation of a hole in the second layer (as shown in Fig. 1) corresponds to the elimination of the superconducting component in the second band. When a superconducting current streamline crosses the edge of the hole, as shown in Fig. 1(b), the quantization in the second layer is not constrained (because the path is not closed in the second layer). The first-layer phase still rotates by 0 radian along the streamline, which implies that if the center of the vortex is on the edge, as shown Fig. 1(b), the quantization constraint no longer forces the velocity of the superconducting pair to be large, and the velocity can thus be zero. To realize energy gain by the elimination of the core with the creation of a phase difference between the layers, some quantity must be traded off. One trade-off is the use of Josephson energy. Another one is the use of kinetic energy accompanied with an inter-layer circulating current that flows in the second layer, flows back to the first layer, and passes through the Josephson barrier [indicated by the yellow arrows in Fig. 1 (a)]. The magnetic induction of the inter-layer circulating current is mutually cancelled and can be ignored in the mimicked two-band superconductor limit. When these factors are satisfied because of the gain resulting from the elimination of the vortex core, a fractional vortex appears.

In this report, we present a basic design of a practical device to realize a
coreless fractional vortex. In addition, we calculate the magnetic-flux-density profile, which reveals that the magnetic flux spreads around the hole and is no longer confined to a small area, as shown in Fig. 2.

2. Model and calculation method

We consider two specially overlapped thin layers, as shown in Fig. 1(a). These layers are coupled to each other through the Josephson interaction (where the critical Josephson current density is \( j_c \)), which is the limit of an ultrathin bilayer and is regarded as a two-band superconductor [7,69,70]. The current flows within a two-dimensional sheet. In each sheet, the current density does not vary along the direction perpendicular to the surface, as used in the approximation analyzing a Pearl vortex in a thin superconductor film [71-76]. However, the direction of the current in one layer can differ from that in the other layer [7, 69,70].

Brandt and Clem proposed in 2004 a calculation method for the current-density and magnetic-flux-density distributions of a thin circular disk using the London limit in which the superconducting pair density is constant [77] (Brandt presented another method in 2005 for arbitrary shapes with two-dimensional mesh [74]. However, for circular shapes, the Brandt-Clem method [77] with one-dimensional mesh gives more accurate results with higher performance (reduction of computer resources). Thus, we adopted the Brandt-Clem method. First, to implement their method, we separate the current density into two components [68-70,78]: a circulating current density around the center of the vortex (\( j_i^A \)), which induces a magnetic field and corresponds to the current density of the Brandt-Clem method, and an inter-layer circular current density (\( j_i^B \)), which does not induce a magnetic field because of the mutual cancellation of the contributions from the two layers and can be ignored in the Brandt-Clem method. This approximation is practically the same as those adopted in the Brandt-Clem method, in which the sheet current density obtained by accumulating the local current density perpendicular to the film surface, determined the vector potential on the surface of the film in the zero-thickness limit [71,74,77]. The sheet current density of the circulating current around the center is assumed to be constant on the streamline according to the definition of the streamline of constant flow of incompressible fluid (In another view, it is Kirchhoff’s first law). Second, the current streamline is approximated by a circle, and the hole is shifted from the center of the disk to the perimeter of the hole to calculate the coreless vortex, as shown in Fig. 3, because the circulating current around the center is axially symmetric in the Brandt-Clem method. This is a variational function yielding required properties in the form of fractional and coreless vortex on the edge under our
discussion. When this function provides lower energy than that of the conventional vortex at the normal location, and not on the edge, we can conclude with certainty that the coreless fractional vortex appears. An additive refinement, e.g., an ellipsoidal streamline, might reduce the energy further. However, the basic properties that are coreless and possess a fractional quantum, are expected to be retained.

By applying the London approximation to each layer indexed by $i$ [68-70,78], the gradient of the superconducting quantum phase ($\nabla \theta_i$) can be expressed using the velocity ($v_i$), mass ($m$), and charge ($e^*$) of a pair [68-70,78].

$$\hbar \nabla \theta_i = mv_i + e^* A$$  \hspace{1cm} (1)

where $A$ is the vector potential and $\hbar = \frac{h}{2\pi}$, in which $h$ is the Plank constant. We define the two velocities for each layer as follows:

$$j^a_i \equiv ne^*v^a_i$$  \hspace{1cm} (2a)

$$j^b_i \equiv ne^*v^b_i$$  \hspace{1cm} (2b)

where $n$ is the pair density and $v^a_1 = v^a_2$ corresponds to the velocity of the circular current in the two layers. Eq. (1) then becomes

$$\hbar \nabla \theta_i = mv^a_i + mv^b_i + e^* A.$$  \hspace{1cm} (3)

The quantization condition for the first layer along the superconducting streamline yields [5,6,10,12,17,26,30,33,39-41,43,47,48,55,57,61,66,68,69]

$$0 = \oint \nabla \theta_1 ds = \frac{1}{\hbar} \oint mv^a_1 ds + \oint \frac{e^*}{\hbar} A ds + \frac{1}{\hbar} \oint mv^b_1 ds.$$  \hspace{1cm} (4)

The final term gives the phase shift due to the interlayer circulating current.

$$\hbar \theta_s \equiv \oint mv^b_1 ds$$  \hspace{1cm} (5)

$v^b_1 = 0$ under the hole. By introducing penetration depth $\lambda = \sqrt{\frac{1-m}{\mu_0 n(e^*)^2}}$, where $\mu_0$ is the magnetic permeability of vacuum, Eq. (4) becomes

$$0 = \oint ne^*v^a_1 ds + \oint \frac{\Phi}{\mu_0 \lambda^2} ds - \frac{\Phi_0}{\mu_0 \lambda^2} \frac{\theta_s}{2\pi}.$$  \hspace{1cm} (6)

$\Phi_0 \equiv \frac{h}{|e^*|}$ is the magnetic-flux quantum. The total sheet current density that contributes to the magnetic induction is $J = ne^*(d_1v^a_1 + d_2v^b_2) = ne^*dv^a_1$, where $d_i$ is the layer thickness. We set $d_1 = d_2$ in this study. Then $d = 2d_1$ when two layers are present, and $d = d_1$ when only one layer is present. $J$ is constant on the streamline, and $\oint nev_1^a ds = J \oint d ds$. Following the approximation of the streamline of $J$ by the circle, we introduce
the effective thickness \( d_{\text{eff}} = \frac{\oint \frac{ds}{2\pi r}}{\oint ds} \), where \( r \) is the radius of the circle of the streamline. Provided that the vector potential is parallel to the tangential direction of the streamline and its norm is constant, we obtain

\[
\mu_0 A_{\text{eff}} = -\left[ \frac{\Theta_s \Phi_0}{2\pi 2\pi r} + A \right].
\]

(7)

where \( A_{\text{eff}} = \frac{\lambda^2}{d_{\text{eff}}} \). This equation corresponds to Eq. (2) of the Brandt–Clem method [77]. The current coupling to the electromagnetic field \( (J) \) can be calculated when \( \Theta_s \) is given.

The phase shift of the inter-layer phase difference between two edges (\( P_S \) and \( P_E \) in Fig. 3) is \( 2\Theta_s \). The inter-layer phase difference at the center is \( \pi \) radians [as shown in Fig. 1(b)], increases to \( 2\pi \) radians at one edge (denoted by \( P_E \) in Fig. 3), and shrinks down to 0 radian at another edge (denoted by \( P_S \)) with the increase in the radius \( (r \) in Fig. 3) of the streamline [Fig. 1(b)]. The inter-layer phase difference ranges from 0 to \( 2\pi \) radians when the path does not cross the edge of the hole after \( P_S \) and \( P_E \) meet at \( P_1 \) (Fig. 3). When the length of the current streamline becomes longer than that of an inter-layer phase difference soliton [in our case, its length \( (L_{\text{soliton}} = \frac{\pi}{2\lambda} \sqrt{\frac{d\Phi_0}{\pi\mu_0 J}}) \) is approximated to be 200 \( \mu\text{m} \)], an inter-layer phase difference soliton is formed; this inter-layer phase difference only occurs in this soliton [5·32,68·70,77·79]. \( L_{\text{soliton}} \) is obtained by dividing the slope of the inter-band phase difference at the center of the soliton by \( 2\pi \) radian [69]. The phase shift between two edges is defined as a function of the radius of the streamline. If we want to avoid a divergent singularity of the current at the center or assign a change in the current flow direction near the center, the phase shift should be proportional to the square of the radius of the streamline near the center. Higher order functions can be mixed when the phase shift approaches \( 2\pi \) radians. However, although it could connect the phase slip more smoothly, the electromagnetic energy increases. Therefore, we set the square function of the radius of the streamline between 0 and \( 2\pi \) radians as a trial function. The phase variation in the streamline is estimated to be linear when the length of the streamline is less than that of the soliton. Beyond the length of the soliton, the phase variation is estimated to be composed of a linear part with the same slope as that at \( \pi \) radians of the soliton function and a flat part with 0 or \( 2\pi \) radians [5,18,30]. This approximation means that a phase difference exists in the double layer for the whole path when the streamline length is shorter than \( L_{\text{soliton}} \). When the streamline length is longer than \( L_{\text{soliton}} \), an inter-layer phase difference is present in the limited fragment of
the path, and the length of this fragment is $L_{soliton}$. With these considerations, we express the $r$ dependence of $\Theta$ as $\Theta = \pi \frac{r^2}{r_A^2}$ for $r < r_A$ and $\pi$ for $r > r_A$. $r_A$ is the hole diameter ($r_A = r_0$, see Fig. 3) for $r_0 < \frac{L_{soliton}}{2\pi}$. Otherwise, $r_A = \frac{L_{soliton}}{2\pi}$.

To implement the Brandt–Clem procedure, we introduce a non-equidistant grid. The streamline radius ($r_k$) is given as follows:

$$
\begin{align*}
    r_k &= r_a + \frac{(n_k - r_a)f(u_k)}{f(1)}, \\
    f(u_k) &= 30u_k^7 - 70(1 + b)u_k^6 + 42(2b + (1 + b)^2)u_k^5 - 105b(1 + b)u_k^4 + 70b^2u_k^3, \\
    u_k &= \left( k - \frac{1}{2} \right) / 52
\end{align*}
$$

where $r_a$ is zero for a coreless vortex and is the coherent length of a conventional vortex. $b$ is the ratio of the minimum radius ($r_0$) where the streamline in the second layer is close to the disk radius. Its value is 0.25 for a coreless vortex and 0.125 for a conventional vortex. $k$ ranges from 1 to 52. These equations yield a fine grid near the center and edge of the disk and near $r_0$. This grid was determined, as the separation of the grid becomes dense near $r_a$, $r_b$ and $r_0$. In other words, $f(u) \propto \frac{\partial}{\partial u}(u^2(u - 1)^2(u - b)^2)$. This is an extension of the grid that Brandt and Clem adopted [77].

The kinetic energy of the pair is given by

$$
E_k = \iiint \left( \frac{nd}{2}m(v_1^A + v_1^B)^2 + \frac{nd}{2}m(v_2^A + v_2^B)^2 \right) d^3x.
$$

Because the condition in which the interlayer circulating current is cancelled ($j_1^B + j_2^B = 0$) eliminates the cross terms of $v_1^A$ and $v_1^B$, the kinetic energy from $v_1^A$ and that from $v_1^B$ are separated as follows [41,43,68,78,79]:

$$
\begin{align*}
    E_k^A &= \iiint \left( \frac{nd}{2}m((v_1^A)^2 + (v_2^A)^2) \right) d^3x, \\
    E_k^B &= \iiint \left( \frac{nd}{2}m((v_1^B)^2 + (v_2^B)^2) \right) d^3x.
\end{align*}
$$

The electromagnetic energy is given by

$$
E_m = \frac{\mu_0}{2} \iiint H^2 d^3x = \frac{1}{2} \iint JA d^2x,
$$

where $H$ is the magnetic field strength. $E_k^A + E_m$ corresponds to the usual self-energy of the disk with the vortex [77].

$$
E_k^A + E_m = \frac{\phi_0}{2} \int \frac{\Theta_2(r)}{2\pi} f(r) dr.
$$
Under the applied field \( (B_a) \), the electromagnetic energy contribution can be described following the Brandt–Clem method [74,77].

\[
E^A_k + E_m = -\frac{1}{2} (M_1 + M_2) B_a + \frac{\Phi_0}{2} \int \frac{\theta_s(r)}{2\pi} (J_1 + J_2) dr,
\]

(13)

where \( M_1 \) is the moment induced by the vortex and \( M_2 \) is that induced by the applied field. \( J_1 \) is the sheet current density due to the vortex, and \( J_2 \) is that due to the applied field. For the interlayer circulating current, we derive

\[
m v^p_1 = -m v^p_2 = \frac{\hbar \theta_s}{l_s}.
\]

(14)

\( l_s \) is the length of the path where a phase difference exists between two layers in the double-layered region. Equations (10b) and (14) give the kinetic energy of the pair of the inter-layer circulating currents. To build up the inter-layer circulating current, the phase difference \( (\phi) \) ranges from \( \pi - \theta_s \) to \( \pi + \theta_s \). We approximate the phase as linearly varying along the streamline. The Josephson interaction can be given as

\[
E_j = \frac{\Phi_0}{2\pi j_c} \iint dS (1 - \cos \phi),
\]

(15)

where \( j_c \) is the Josephson critical current density. The total energy of the disk with the vortex can be estimated as follows:

\[
E_{Disk} = E^A_k + E_m + E^B_k + E_j.
\]

(16)

We estimate that \( -\frac{1}{2} M_2 B_a \) is the same for three cases—a disk with a conventional vortex, that with a coreless vortex, and that without any vortex. This estimation is based on the fact that the magnetic moment induced by the current in the film becomes the sum of the moment by the current accompanying the flux of the vortex and the moment by the current induced by the external field, as mentioned by Brandt [74,77]. We shift the center of the hole so that the center of the vortex is at the center of the disk when we estimate the energy, magnetic-flux-density distribution, and current-density distribution under the axially symmetric approximation.

3. Results

We briefly introduce the result for one example which is the most feasible one from the viewpoint of current device technology, which we can utilize as the first step to realize the coreless fractional vortex based on our model. It is one model case to demonstrate the effectiveness of our basic design. We can modify the design with varying penetration depth by changing the impurity doping to achieve more ideal thin-film limit. However, we observed that the current parameter is sufficient to realize the
fractional vortex in preliminary experimental results [80].

Fig. 4 shows the energy difference between the disks with a vortex (conventional case), that with a fractional vortex, and that without a vortex where we approximate the diamagnetic magnetization induced by the external field to be equal. The diameters of the disk and hole are 80 and 10 μm, respectively. The thickness of one layer is 20 nm, and the Josephson current density is set to 10^5 A/cm². The coherent length and penetration depth are set to 40 and 80 nm, respectively [3, 4]. The energy of the conventional vortex includes the energy of the normal core [69], which is approximately 8% of the total energy at 0 A/m. The self-energy of the coreless vortex is less than half of that of the conventional vortex at 0 A/m, which validates the discussions in the previous paragraphs. By increasing the field, the interaction between the magnetic moment of the fractional vortex and applied field decreases the energy, which becomes lower than that of the disk with or without a conventional vortex between 18 and 68 A/m.

The selected parameters are determined considering the practical situation. In principle, a large Josephson current and a large sample are unfavorable for creating a fractional vortex, because we should pay a large cost to create the i-soliton spanning a fractional vortex to the outer rim of the sample. The formation of the fractional vortex is more easily achieved with lower Josephson current density and a small sample. However, there is a limit coming from the current technology. To fabricate a reliable device, the Josephson current density should be large [81]. To obtain a reliable magnetic image by a scanning SQUID microscope, the size of the device should be certainly large also [82,83]. The selected conditions satisfy these requirements entirely.

4. Discussion

In this section, we discuss which experimental condition is appropriate to create a coreless fractional vortex in the model case. Then we make a remark on a noticeable knowledge in quantum science demonstrated by this experiment.

To realize a coreless vortex, the sample is cooled down under an applied field from a temperature higher than the transition temperature. At low temperatures, the coreless vortex is stable at the applied field and continues to exist after the external field is turned off [69,79].

Fig. 5 shows the magnetic-flux-density profile. A flat magnetic-flux-density profile can be realized when no constraint is present to force phase rotation. The current density reaches zero at the center. We note that the hole is still filled by the superconducting condensation of the first layer. In this model, the magnetic field also
spreads outside the hole where two layers exist. The Higgs mechanism [84,85] still functions; however, the given ‘Higgs mass’ cannot confine the magnetic flux. (The superconducting condensation corresponds to the Higgs scalar field in superconductors.) In the absence of the quantization condition, the superconducting condensation cannot expel the magnetic field, although expulsion of the magnetic field is a fundamental superconducting property, known as the Meissner–Ochsenfeld effect [86]. London required quantization to explain the Meissner–Ochsenfeld effect [87]. When a sample does not have a hole, the London gauge \( j = -\frac{A}{\mu_0} \) assures that a winding number of superconducting phase going around any closed path on the surface of the sample becomes zero. W. F. Edwards tried to explain the Meissner–Ochsenfeld effect without quantization in 1981 [88,89], but his conclusion was completely negated by J. B. Taylor and others in 1982 [90-94]. They discussed if Meissner–Ochsenfeld effect is really a quantum effect, by using a classical system [88-96]. Our study shows that the Meissner–Ochsenfeld effect is missed even in charged quantum condensations by damaging the boundary condition. For technical understanding, we may explain this phenomenon as follows. The current in the Lagrangian for Maxwell formalism cannot be replaced by the vector potential as seen in Eq. (7), though this replacement (with multiplying an appropriate constant) based on London consideration [87] usually reproduces Higgs’ Lagrangian [84,85]. The reason why the vector potential cannot replace the current is that there is another current accompanying the soliton. The path for this soliton current is disconnected and not closed. Because of the phase shift due to the soliton current, \( \frac{1}{2\pi} \int \nabla \theta_2 \, ds \) is not quantized to neither unit one nor a specific fractional value but varies continuously with increasing radius of streamline. It is noteworthy that two types of currents can be considered as reminiscent of the non-abelian current of two-band superconductors [47, 97].

The coreless vortex can appear when we completely remove the superconductor passing through a hole. It does not conflict with the Higgs mechanism because there is no Higgs scalar field inside the hole and the connectivity of the superconducting quantum phase is not required inside the hole. The vortex has a unit flux quantum, because the phase must be connected outside the hole. When the superconductor remains as seen in a blind hole in a single layer, any coreless fractional vortex does not appear [98-105]. Other cases should also be mentioned. The amount of the magnetic flux can be smaller than a unit flux quantum for a mesoscopic sample with a finite circulating current at its outer rim [106]. It is
rather trivial that a ring has a zero or finite current when the amount of the magnetic flux inside the ring becomes a unit or fractional flux quantum. The Little-Parks experiment is based on this principle where the transition temperature or the current oscillates depending on the total magnetic flux inside the ring [12,19,107,108]. The period of the oscillation is quantized in this case. The “fractional vortex” in a mesoscopic ring is rather categorized into the extension of the Little-Parks experiment where a ring has a finite current. The fractional quantization can also be achieved by the externally applied current [109,110]. These currents directly couple to the electromagnetic field through the vector potential and these electromagnetically active currents rotate the phase. In our case, these currents are not required. Instead, a reminiscent of the non-abelian current (current accompanying the soliton and not directly coupling to the electromagnetic field) compensates the phase shift by the vector potential. The mechanism of fractionalization is rather similar to that discussed in spin-triplet \( p \)-wave quantum condensates [111, 112, 113], multi-component Bose-Einstein condensates [114,115] and particle physics [29,61,68,116]. There are theoretical reports on the fractional vortex for layered systems and multiband superconductors [10,12,16,24,31,33, 37, 38, 40, 42-44, 47, 48, 50-60, 65-69, 117-119], for example, inset 1 of Fig. 11 of ref. 53. Our report suggests that a partial suppression or elimination of one component enhanced these effects.

5. Conclusion

This letter may propose one method of studying quantum physics. Abrikosov predicted the quantized vortex concentrating the magnetic field around its center based on a phenomenological theory without experimental knowledge [1]. Our model sheds light on how to disperse the magnetic field concentrated by an Abrikosov quantized vortex. The ultrathin superconducting bilayer becomes a new platform to break the quantization condition without a quantum-to-classical crossover. The realized state resembles that of a perfect conductor with infinite conductivity but does not possess quantum properties [120]. Because the superconducting condensation is considered a Higgs scalar field [81], we speculate if a large object could possibly exist in a low-energy density in particle physics, such as dark energy [121], which corresponds to a coreless vortex with low magnetic-flux density.
Figure 1. Basic design of a device to promote a coreless fractional vortex. (a) Perspective view. The structure is composed of ultrathin bilayers where one layer has a hole, but the other layer does not. These two layers are coupled to each other through an inter-layer Josephson interaction. The red arrows indicate a circulating current around the center of the coreless vortices. The yellow arrows indicate the inter-layer circulating current through the Josephson barrier. (b) Top view. $\theta_1$ and $\theta_2$ are the superconducting phases of the first and second layers, respectively.
Figure 2. Schematic illustration of the magnetic flux. A sharply confined flux (denoted by red allows) for a single-layer becomes a broad flux (with low density) after attaching to another layer with a hole.
Figure 3. Illustration of the approximation of the calculation. The streamline is composed of $\Gamma_I$ and $\Gamma_{II}$ when it crosses the edges ($P_2$ and $P_3$) of the hole. When the radius of the circular current is larger than the diameter ($r_0$) of the hole, the current streamline does not cross the edges, as shown by $\Gamma_{III}$. 
Figure 4. Energy difference between the disks with and without a vortex. The contribution of the interaction between the diamagnetic moment and applied field is approximated to be equal.
Figure 5. Magnetic-flux-density and current-density distributions in the disk. The current density is estimated at the location where two layers are present. The illustration of the flux density for the coreless vortex is magnified $10^4$ times.

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