Emergence of supersymmetry on the surface of three-dimensional topological insulators

Pedro Ponte\textsuperscript{1,2,4} and Sung-Sik Lee\textsuperscript{1,3}

\textsuperscript{1} Perimeter Institute for Theoretical Physics, Waterloo, ON N2L 2Y5, Canada
\textsuperscript{2} Centro de Física do Porto e Departamento de Física e Astronomia, Faculdade de Ciências da Universidade do Porto, Rua do Campo Alegre 687, 4169-007 Porto, Portugal
\textsuperscript{3} Department of Physics and Astronomy, McMaster University, Hamilton, ON L8S 4M1, Canada
\textsuperscript{4} Author to whom any correspondence should be addressed.

E-mail: pponte@perimeterinstitute.ca

Received 7 October 2013, revised 10 December 2013
Accepted for publication 12 December 2013
Published 23 January 2014

New Journal of Physics \textbf{16} (2014) 013044
doi:10.1088/1367-2630/16/1/013044

Abstract

We propose two possible experimental realizations of a (2+1)-dimensional spacetime supersymmetry at a quantum critical point on the surface of three-dimensional topological insulators. The quantum critical point between the semi-metallic state with one Dirac fermion and the s-wave superconducting state on the surface is described by a supersymmetric conformal field theory within the $\epsilon$-expansion. We predict the exact voltage dependence of the differential conductance at the supersymmetric critical point.

For the last 40 years, supersymmetry has been studied intensively in high-energy physics because of its attractive features, e.g. as a possible solution to the hierarchy problem [1]. Although there is so far no experimental evidence for our universe to be supersymmetric, there is some expectation that supersymmetry may be revealed in the large hadron collider in the near future. Condensed matter systems provide alternative ways to realize supersymmetry in nature through emergence [2]. Namely, supersymmetry can dynamically emerge in the low-energy limit of some condensed matter systems although the microscopic Hamiltonians

\textsuperscript{4} Author to whom any correspondence should be addressed.
explicitly break it. Because supersymmetry is a symmetry between boson and fermion, it is essential to have the same number of low-energy modes for boson and fermion in order to realize supersymmetry. Although there are examples of emergent spacetime supersymmetry in 1 + 1 dimensions [3–5], where the distinction between boson and fermion is rather obscure, it is not easy to realize supersymmetry in lattice models in higher dimensions [6]. Because of the fermion doubling problem, which is actually an intrinsic feature rather than a ‘problem’ in condensed matter systems, there are usually more fermionic degrees of freedom than bosonic degrees of freedom unless there is a special symmetry or dynamical mechanism that protects multiple gapless bosonic modes at low energies [7–9]. In contrast, there is no such problem in the continuum model [10].

The topological insulator [11–20] is a topological phase of matter where gapless edge or surface modes are protected by time-reversal symmetry [21–23]. In topological insulators, there is no fermion doubling problem because the second set of fermionic modes is located on the other edge or surface of a sample. For example, on the surface of a semi-infinite three-dimensional (3D) topological insulator one can have only one (two + one)-dimensional Dirac fermion, which is worth one complex boson in terms of counting the number of propagating modes. Therefore, the topological insulator provides a platform to realize interesting critical states [24], including states with emergent supersymmetry. In this paper, we consider a superconducting quantum critical point on the surface of a 3D topological insulator. It is likely that the critical point exhibits an emergent supersymmetry because there are the same number of propagating modes for bosons and fermions which are strongly mixed with each other at low energies.

We consider a 3D topological insulator which has a gap in the bulk and a gapless Dirac fermion at the \( \Gamma \)-point of the surface Brillouin zone. For example, Bi\(_2\)Se\(_3\) has the desired properties [19, 25]. This is an ideal material due to the large band gap in the bulk (0.3 eV) and the possibility of manipulating the Fermi energy of the bulk and the surface by chemical modifications [26]. We consider the case where the chemical potential is tuned to the Dirac point. Since the dispersion relation is linear near the Dirac point, the low-energy excitations are described by a two-component massless Dirac fermion

\[
\mathcal{L} = i \tilde{\psi} (\gamma_0 \partial_t + c_f \gamma_i \partial_i) \psi,
\]

where \( c_f \) is the Fermi velocity, \( \gamma_0 \equiv \sigma_3, \gamma_1 \equiv \sigma_1, \gamma_2 \equiv \sigma_2 \), \( \tilde{\psi} = -i \psi^\dagger \gamma_0 \), and \( \tau \) is the imaginary time.

The two-dimensional electrons on the surface are subject to the electrostatic Coulomb repulsion and the attractive interaction mediated by phonons. If the attractive interaction is strong enough, the semi-metallic state can become unstable, undergoing a quantum phase transition to a superconducting state. In order to access the critical point by tuning the strength of the Coulomb repulsion, we consider a substrate placed at a distance \( d \) above the topological insulator (figure 1). The substrate consists of a two-dimensional metal with dispersion relation \( \epsilon = \frac{p^2}{2m} - \mu_F \), where \( \mu_F \) is the chemical potential. The long-range Coulomb repulsion between electrons on the surface of the topological insulator is screened by the metallic substrate. The strength of the residual short-range repulsion can be controlled by changing \( d \), which can be used to drive the system to the critical point. Here we assume that the attractive interaction due to phonon is sufficiently strong, so that the semi-metallic state is unstable to the s-wave superconducting state without the Coulomb repulsion. Now we examine how the
screened Coulomb interaction depends on the distance between the substrate and the topological insulator.

The system composed of the topological insulator and the substrate is described by the partition function

$$Z = \int D\bar{\psi} D\psi D\Psi D\bar{a}_0 e^{-S_{\text{TI}}[\bar{\psi}, \psi] - S_s[\Psi^\dagger, \Psi] - S_{\text{int}}[\bar{\psi}, \psi, \Psi, \bar{a}_0]},$$

where

$$S_{\text{TI}} = \int d\tau dx dy \ i\bar{\psi} (\gamma_0 \partial \tau + c_1 \gamma_1 \partial x + c_1 \gamma_2 \partial y) \psi,$$

$$S_s = \sum_\sigma \int d\tau dx dy \ \Psi^\dagger_\sigma \left( \partial_\tau - \frac{1}{2m}(\partial_x^2 + \partial_y^2) - \mu_F \right) \Psi_\sigma,$$

$$S_{\text{g}} = -\frac{1}{2} \int d\tau dx dy dz \ a_0(\partial_x^2 + \partial_y^2 + \partial_z^2) a_0,$$

$$S_{\text{int}} = -i \int d\tau dx dy \left[ e \Psi^\dagger \psi a_0(\tau, x, y, 0) + \sum_\sigma e \Psi^\dagger_\sigma \Psi_\sigma a_0(\tau, x, y, d) \right].$$

Here $a_0$ is the temporal component of the three + one-dimensional electromagnetic field. We choose to work in the Coulomb gauge $\nabla \cdot a = 0$, and neglect the spatial components of the electromagnetic gauge field whose contribution is down by $c_f/c$, where $c$ is the speed of light. The $z$-coordinates of the surface of the topological insulator and the substrate are 0 and $d$, respectively. We neglect tunneling between the substrate and the topological insulator.

By integrating out the fermions of the substrate, we obtain the effective action for the gauge field

$$S_0 = \frac{1}{2(2\pi)^2} \int d^3 p \ dp_z dp'_z a_0(p, p_z) G^{-1}_{\text{0}}(p, p_z) a_0(-p, p'_z),$$

where

$$G^{-1}_{\text{0}}(p, p_z) = G_0^{-1}(p, p_z)(2\pi)\delta(p'_z + p_z) - \Pi^0(p) e^{i p_z d} e^{i p'_z d}.$$ 

Here $p = (\omega, \mathbf{p})$ denotes a (two + one)-dimensional energy–momentum vector. The bare propagator and the polarization (in the limit $|\mathbf{p}| \ll p_F$) are given by

$$G_0^{-1}(p, p_z) = |\mathbf{p}|^2 + p_z^2,$$

$$\Pi^0(0, \mathbf{p}) = -\frac{e^2 m}{\pi}.$$ 

**Figure 1.** A metallic substrate placed above a 3D topological insulator.
It is noted that the presence of the substrate breaks the translation invariance along the \( z \) direction, and the momentum along this direction is not conserved. Inverting the dressed propagator, we obtain the two-dimensional screened Coulomb repulsion between electrons on the surface of the topological insulator

\[
V(p) = \int \frac{dp_1}{2\pi} \frac{dp_2}{2\pi} G(p, p_1, p_2) = \frac{e^2}{4|p|} \left( 1 + \frac{\Pi^0(p)e^{-2ipd}}{2|p| - \Pi^0(p)} \right).
\]

The static effective potential is not singular as \( |p| \to 0 \). It is given by

\[
V_0 = \frac{e^2}{2} \left( d + \frac{1}{\epsilon N(\mu_F)} \right),
\]

where \( N(\mu_F) \) is the density of states of the substrate at the Fermi energy.

If \( N(\mu_F) \) is large and \( d \) is small, the Coulomb repulsion can be made weak so that the attractive interaction mediated by phonon dominates. If the strength of the attractive interaction is sufficiently strong, one can tune the system across the superconducting phase transition by changing \( d \).

In the presence of one Dirac point located at the \( \Gamma \) point, one can, in general, have a pairing of the form \( \Delta_{\alpha,\beta}(k) \psi_{\alpha,k} \psi_{\beta,-k} \), where \( \alpha, \beta = 1, 2 \) are pseudospin indices and \( \Delta_{\alpha,\beta}(k) = -\Delta_{\beta,\alpha}(-k) \). The gap function can be decomposed as \( \Delta_{\alpha,\beta}(k) = \epsilon_{\alpha,\beta} \Delta_{s}(k) + [\sigma_i \tilde{\sigma}]_{\alpha,\beta} \Delta_{t}(k) \), where \( \Delta_{s}(k) \) and \( \Delta_{t}(k) \) are pseudospin singlet and triplet order parameters, respectively. It is expected that the triplet state is energetically less favorable than the singlet state because the gap vanishes at \( k = 0 \) for the triplet state. However, this ultimately depends on the microscopic details of the systems. Here we proceed with the assumption that the s-wave singlet superconducting state is the dominant instability channel in the presence of the strong attractive interaction.

Suppose that electrons are interacting through a net attractive interaction

\[
U = -v_0 \psi^\dagger \psi \psi^\dagger \psi.
\]

To decouple the four fermion interactions we introduce a complex boson field through the Hubbard–Stratonovich transformation

\[
-\frac{v_0}{2} (\psi^T \epsilon \psi)^\dagger (\psi^T \epsilon \psi) \rightarrow (\psi^T \epsilon \psi)^\dagger \phi + \phi^* (\psi^T \epsilon \psi) + \frac{2|\phi|^2}{v_0}.
\]

Here \( \epsilon \) is the \( 2 \times 2 \) antisymmetric matrix with \( \epsilon_{12} = -\epsilon_{21} = 1 \). Although the complex boson (Cooper pair) has no bare kinetic term, it is generated by fermions at low energies. The low-energy effective theory becomes

\[
\mathcal{L} = i \bar{\psi} \left( \gamma_0 \partial_t + c_l \gamma_i \partial_i \right) \psi + |\partial_t \phi|^2 + c_b^2 |\partial_t \phi|^2 + m^2 |\phi|^2 + \lambda |\phi|^4 + h (\phi^* \psi^T \epsilon \psi - \phi \bar{\psi} \epsilon \psi^T),
\]

where \( c_b \) is the velocity of bosons which may be different from \( c_l \). Note that the dynamics of bosons is guaranteed to be relativistic with the dynamical critical exponent \( z = 1 \) as far as fermions are relativistic because Cooper pairs are formed out of the relativistic Dirac fermions.

As \( d \) is tuned, the mass of the boson is changed. In the superconducting state with \( m^2 < 0 \), the boson is condensed, and the Dirac fermion becomes gapped. At a critical distance \( d_c \), bosons are massless and the theory flows to an interacting fixed point in the low-energy limit. The field theory which has two copies of the present theory has been studied, where each set of modes describes one Dirac fermion and one complex boson defined at one of the two distinct
momentum points ($K$ and $K'$) on the honeycomb lattice [8]. In the $\epsilon$ expansion, it has been shown that the theory flows to a supersymmetric critical point with four emergent supercharges where two sets of modes are decoupled in the low-energy limit. Therefore the same conclusion can be drawn for the present case. In the low-energy limit, the critical point is described by the $\mathcal{N} = 2$ Wess–Zumino theory with one chiral multiplet [27].

In order for an intrinsic superconducting state to be stable, one needs to have a sufficiently strong electron–phonon interaction, which may or may not be the case for real materials. In cold-atom systems, the strength of interaction can be easily tuned. It will be of interest to realize 3D topological insulators and the supersymmetric quantum critical point in cold-atom systems by tuning attractive interaction between particles [30, 31]. Here we propose a second scenario which realizes supersymmetry in the low-energy limit. The system consists of a Josephson junction array (JJA) deposited on the top of a topological insulator (figure 2). A JJA consists of a regular network of superconducting islands coupled by tunnel junctions. The Dirac electrons of the topological insulator can tunnel to the JJA to form Cooper pairs and vice versa. The JJA and the Dirac fermion are also described by the same Lagrangian in equation (13) at low energies if the average number of Cooper pairs within each island is tuned to be an integer. As the Josephson coupling is tuned, the JJA can undergo a phase transition to a superconducting state. The quantum critical point is again described by the $\mathcal{N} = 2$ Wess–Zumino theory with one chiral multiplet.

At the critical point, the scaling dimensions of the chiral primary fields, including the Dirac fermion and the Cooper pair field, are constrained by the superconformal algebra [28]. As a result, the exact anomalous dimensions of the fermion and boson are given by $\eta_\phi = \eta_\psi = \frac{1}{3}$ [29]. It is of interest to provide a clear experimental signature for the emergent supersymmetry. Here we consider tunneling spectroscopy that measures the local density of states. The single-particle Green’s function of electron at the supersymmetric critical point is given by

$$G(p) \sim \frac{(p \sigma)}{(p^2)^{5/6}},$$

where $\sigma = (\sigma_3, c_1 \sigma_1, c_2 \sigma_2)$. Integrating the spectral function over momentum, we obtain the local density of states

$$\rho(\omega) \sim |\omega|^{4/3}.$$  

As a result, we expect the differential conductance $\frac{dI}{dV} \sim V^{4/3}$ at the supersymmetric critical point. This provides a clear experimental signature of the supersymmetric state. Note that the exact exponent is predicted thanks to the superconformal symmetry, although the critical point is described by the strongly interacting theory.

In summary, we propose that a (two + one)-dimensional superconformal field theory can be realized at the quantum critical point between the semi-metallic state and the s-wave
superconducting state on the surface of 3D topological insulators. The critical point is described by the $\mathcal{N} = 2$ Wess–Zumino model with one chiral multiplet. At the critical point, the local density of states obeys the power-law behavior $\rho(\omega) \sim \omega^{4/3}$, which can be measured by scanning tunneling microscopes. Recently, it has been shown that the supersymmetric critical point is not stable in the presence of disorder [32]. However, the critical behavior governed by the putative supersymmetric critical point can be observed within a finite temperature range in the weak disorder limit.

Acknowledgments

We would like to thank Patrick Lee for helpful discussions. This work is a result of the research essay performed as a part of the Perimeter Scholars International (PSI) program. This research was supported in part by NSERC and ERA. Research at the Perimeter Institute is supported in part by the Government of Canada through Industry Canada, and by the Province of Ontario through the Ministry of Research and Information. After the completion of the draft, we became aware of a recent preprint [33] by Tarun Grover and Ashvin Vishwanath, who reached the same conclusion by studying various instabilities on the surface of topological insulators in a more general context. We thank the authors for their helpful feedbacks on our draft.

References

[1] Weinberg S 2000 The quantum theory of fields Supersymmetry vol 3 179
[2] Thomas S 2005 Emergent supersymmetry KITP Talk http://online.kitp.ucsb.edu/online/qpt-c05/thomas/
[3] Friedan D, Qui Z and Shenkar S 1985 Phys. Lett. B 151 37
[4] Fendley P, Schoutens K and Boer J 2003 Phys. Rev. Lett. 90 120402
[5] Fendley P, Nienhuis B and Schoutens K 2003 J. Phys. A: Math. Gen. 36 12399
[6] Roy B, Juricic V and Herbut I 2013 Phys. Rev. B 87 041401
[7] Balents L, Fisher M P A and Nayak C 1998 Int. J. Mod. Phys. B 12
[8] Lee S-S 2007 Phys. Rev. B 76 075103
[9] Yu Y and Yang K 2010 Phys. Rev. Lett. 105 150605
[10] Antipin O, Mojaza M, Pica C and Sannino F 2011 Magnetic fixed points and emergent supersymmetry (arXiv:1105.1510)
[11] Kane C L and Mele E J 2005 Phys. Rev. Lett. 95 146802
[12] Bernevig B A, Hughes T L and Zhang S C 2006 Science 314 1757
[13] Fu L and Kane C L 2007 Phys. Rev. B 76 045302
[14] Fu L, Kane C L and Mele E J 2007 Phys. Rev. Lett. 98 106803
[15] Moore J E and Balents L 2007 Phys. Rev. B 75 121306
[16] König M, Wiedmann S, Brüne C, Roth A, Buhmann H, Molenkamp L W, Qi X L and Zhang S C 2007 Science 318 766
[17] Hsieh D, Qian D, Wray L, Xia Y Q, Hor Y S, Cava R J and Hasan M Z 2008 Nature 452 970
[18] Xia Y et al 2009 Nature Phys. 5 398
[19] Hsieh D et al 2009 Nature 460 1101
[20] Chen Y L et al 2009 Science 325 178
[21] Xu C and Moore J E 2006 Phys. Rev. B 73 045322
[22] Wu C, Bernevig B A and Zhang S C 2006 Phys. Rev. Lett. 96 106401
[23] Schnyder A P, Ryu S, Furusaki A and Ludwig A W W 2008 Phys. Rev. B 78 195125
[24] Xu C 2010 Phys. Rev. B 81 020411
[25] Zhang H, Liu C X, Qi X L, Dai X, Fang Z and Zhang S C 2009 Nature Phys. 5 438
[26] Hasan M Z and Kane C L 2010 Rev. Mod. Phys. 82 3045
[27] Wess J and Zumino B 1974 Nucl. Phys. B 70 39
[28] Minwalla S 1998 Adv. Theor. Math. Phys. 2 781
[29] Aharony O, Hanany A, Intriligator K, Seiberg N and Strassler M J 1997 Nucl. Phys. B 499 67
[30] Bermudez A, Mazza L, Rizzi M, Goldman N, Lewenstein M and Martin-Delgado M A 2010 Phys. Rev. Lett. 105 190404
[31] Lin Y J, Jimenez-Garcia K and Spielman I B 2011 Nature 471 83
[32] Nandkishore R, Maciejko J, Huse D A and Sondhi S L 2013 Phys. Rev. B 87 174511
[33] Grover T and Vishwanath A 2012 Quantum criticality in topological insulators and superconductors: emergence of strongly coupled Majoranas and supersymmetry (arXiv:1206.1332)