PRIMARY SINGULARITIES OF VECTOR FIELDS ON SURFACES

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ABSTRACT. Unless another thing is stated one works in the $C^\infty$ category and manifolds have empty boundary. Let $X$ and $Y$ be vector fields on a manifold $M$. We say that $Y$ tracks $X$ if $[Y, X] = fX$ for some continuous function $f: M \to \mathbb{R}$. A subset $K$ of the zero set $Z(X)$ is an essential block for $X$ if it is non-empty, compact, open in $Z(X)$ and its Poincaré-Hopf index does not vanishes. One says that $X$ is non-flat at $p$ if its $\infty$-jet at $p$ is non-trivial. A point $p$ of $Z(X)$ is called a primary singularity of $X$ if any vector field defined about $p$ and tracking $X$ vanishes at $p$. This is our main result: Consider an essential block $K$ of a vector field $X$ defined on a surface $M$. Assume that $X$ is non-flat at every point of $K$. Then $K$ contains a primary singularity of $X$. As a consequence, if $M$ is a compact surface with non-zero characteristic and $X$ is nowhere flat, then there exists a primary singularity of $X$.

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1. INTRODUCTION

Whether a family of vector fields has a common singularity is a classical issue in dynamical systems. For instance, on a compact surface with
non-vanishing Euler characteristic there always exists a common zero pro-
vided that the vector fields commute (Lima [9]) or if they span a finite-
dimensional nilpotent Lie algebra (Plante [10]). On the existence of a com-
mon singularity for a family of commuting vector fields in dimension ≥ 3
several interesting results are due to Bonatti [2] (analytic in dimension 3
and 4) and Bonatti & De Santiago [3] (dimension 3). For a complementary
discussion on the existence of a common zero the reader is referred to the
introduction of [6].

In this paper one shows that on surfaces every essential block of a nowhere
flat vector field \( X \) includes a point at which all vector fields tracking \( X \) van-
ish (see Theorem 1.1 below).

Throughout this work manifolds (without boundary) and their associated
objects are real \( C^\infty \) unless another thing is stated. Consider a tensor \( T \)
on a manifold \( P \). Given \( p \in P \) the principal part of \( T \) at \( p \) means \( j^n_p T \) if
\( j^{n-1}_p T = 0 \) but \( j^n_p T \neq 0 \), or zero if \( j^n_p T = 0 \). The order of \( T \) at \( p \) is \( n \) in
the first case and \( \infty \) in the second one. One will say that \( T \) is flat at \( p \) if its
order at this point equals \( \infty \), and non-flat otherwise.

In coordinates about \( p \) the principal part is identified to the first signifi-
cant term of the Taylor expansion of \( T \) at \( p \). Given a function \( f \) such that
\( f(p) \neq 0 \), the principal part of \( fT \) at \( p \) equals that of \( T \) multiplied by \( f(p) \).

\( Z(T) \) denotes the set of zeros of \( T \) and \( Z_n(T) \), where \( n \in \mathbb{N}' \) and \( \mathbb{N}' := \mathbb{N} \cup \{ \infty \} \), the set of zeros of order \( n \). (Here \( \mathbb{N} \) is the set of positive integers.)
Notice that \( Z(T) = \bigcup_{k \in \mathbb{N}'} Z_k(T) \) where the union is disjoint.

Consider a vector field \( Y \) on \( P \). \( Y \) tracks \( T \) provided \( L_Y T = fT \) for some
continuous function \( f: P \to \mathbb{R} \), referred to as the tracking function. (When
\( T \) is also a vector field this means \( [Y, T] = fT \).) A set \( \mathcal{A} \) of vector fields
on \( P \) tracks \( T \) provided each element of \( \mathcal{A} \) tracks \( X \).
A point \( p \in Z(T) \) is a primary singularity of \( T \) if every vector field defined about \( p \) that tracks \( T \) vanishes at \( p \). Obviously isolated singularities are primary. The notion of primary singularity is the fundamental new concept of this work.

Let \( X \) be a vector field on \( P \). Consider an open set \( U \) of \( P \) with compact closure \( \overline{U} \) such that \( Z(X) \cap (\overline{U} \setminus U) = \emptyset \). The index of \( X \) on \( U \), denoted by \( i(X, U) \in \mathbb{Z} \), is defined as the Poincaré-Hopf index of any sufficiently close approximation \( X' \) to \( X|\overline{U} \) (in the compact open topology) such that \( Z(X') \) is finite. Equivalently: \( i(X, U) \) is the intersection number of \( X|U \) with the zero section of the tangent bundle (Bonatti [1]). This number is independent of the approximation, and is stable under perturbation of \( X \) and replacement of \( U \) by smaller open sets containing \( Z(X) \cap U \).

A compact set \( K \subset Z(X) \) is a block of zeros for \( X \) (or an \( X \)-block) provided \( K \) is non-empty and relatively open in \( Z(X) \), that is to say provided \( K \) is non-empty and \( Z(X) \setminus K \) is closed in \( P \). Observe that a non-empty compact \( K \subset Z(X) \) is a \( X \)-block if and only if it has a precompact open neighborhood \( U \subset P \), called isolating for \((X, K)\), such that \( Z(X) \cap \overline{U} = K \) (manifolds are normal spaces). This implies \( i(X, U) \) is determined by \( X \) and \( K \), and does not depend on the choice of \( U \). The index of \( X \) at \( K \) is \( i_K(X) := i(X, U) \).

The \( X \)-block \( K \) is essential provided \( i_K(X) \neq 0 \), which implies \( K \neq \emptyset \), and inessential otherwise.

If \( P \) is compact, it is isolating for every vector field on \( P \) and its set of zeros. Therefore, in this case, \( i_{Z(X)}(X) = i(X, P) = \chi(P) \).

This is our main result, which will be proved in the Section 2.1.

**Theorem 1.1.** Consider an essential block \( K \) of a vector field \( X \) defined on a surface \( M \). Assume that \( X \) is non-flat at every point of \( K \). Then \( K \) contains a primary singularity of \( X \).

As a straightforward consequence:
Corollary 1.2. On a compact connected surface $M$ with $\chi(M) \neq 0$ consider a vector field $X$. Assume that $X$ is nowhere flat. Then there exists a primary singularity of $X$.

Moreover, four examples illustrating these results are given in Section 3.

Remark 1.3.

(a) The hypothesis on the non-flatness of Theorem 1.1 and Corollary 1.2 cannot be omitted as the following example shows. On $S^2 \subset \mathbb{R}^3$ consider the vector field $X = \varphi(x_3)(-x_2 \partial/\partial x_1 + x_1 \partial/\partial x_2)$ where $\varphi(0) = 1$ and $\varphi(\mathbb{R} \setminus (-1/2, 1/2)) = 0$. Then the vector fields $Y = -x_2 \partial/\partial x_1 + x_1 \partial/\partial x_2$ and $V = \psi(x_3)(-x_3 \partial/\partial x_1 + x_1 \partial/\partial x_3)$ where $\psi(1) = \psi(-1) = 1$ and $\psi([-3/4, 3/4]) = 0$ track $X$ and $Z(Y) \cap Z(V) = \emptyset$. Therefore $X$ has no primary singularity.

(b) Two particular cases of Theorem 1.1 were already known, namely: if $X$ and $K$ are as in the foregoing theorem and $\mathcal{G}$ is a finite-dimensional Lie algebra of vector fields on $M$ that tracks $X$, then the the elements of $\mathcal{G}$ have a common singularity in $K$ provided that $\mathcal{G}$ is supersolvable (Theorem 1.4 of [5]) or $\mathcal{G}$ and $X$ are analytic (real case of Theorem 1.1 of [6]). Thus these two results are generalized here.

For general questions on Differential Geometry readers are referred to [8], and for those on Differential Topology to [4].

2. Other results

One will need:

Lemma 2.1. On a manifold $P$ of dimension $m \geq 1$ consider a vector field $X$ of finite order $n \geq 1$ at a point $p$. Then for almost every $v \in T_p P$ there exists a vector field $U$ defined around $p$ such that $U(p) = v$ and the $n$-times iterated bracket $[U, [U, \ldots [U, X] \ldots ]]$ does not vanish at $p$.

Proof. It suffices to prove the result for $0 \in \mathbb{R}^m$ and a non-vanishing $n$-homogeneous polynomial vector field $X = \sum_{i=1}^n Q_i \partial/\partial x_i$. Up to a change of the order of the coordinates, we may suppose $Q_1 \neq 0$. 
Given \( a = (a_1, \ldots, a_m) \in \mathbb{R}^m \) set \( U_a = \sum_{\ell=1}^{m} a_\ell \partial/\partial x_\ell \). It suffices to show that for almost any \( a \in \mathbb{R}^m - \{0\} \) one has \( (U_a \cdot U_a \cdot Q_1)(0) \neq 0 \), which is equivalent to show that the restriction of \( Q_1 \) to the vector line spanned by \( a \) does not vanish identically. But this last assertion is obvious. \( \Box \)

Given a vector field \( V \) on a manifold \( P \), a set \( S \subset P \) is \( V \)-invariant if it contains the orbits under \( V \) of its points.

**Proposition 2.2.** Consider two vector fields \( X, Y \) on a surface \( M \). Assume that \( Y \) tracks \( X \) with tracking function \( f \). Then each set \( Z_n(X), n \in \mathbb{N}', \) is \( Y \)-invariant.

Moreover \( f \) is differentiable on the open set

\[
[M \setminus Z(X)] \cup [(Z(X) \setminus Z_\infty(X)) \cap (M \setminus Z(Y))].
\]

This result is a consequence of the following two lemmas.

**Lemma 2.3.** Under the hypotheses of Proposition 2.2 consider \( p \in Z_n(X), n < \infty \), such that \( Y(p) \neq 0 \). One has:

(a) \( f \) is differentiable around \( p \).

(b) Let \( \gamma: (a, b) \to M \) be an integral curve of \( Y \) with \( \gamma(t_0) = p \) for some \( t_0 \in (a, b) \). Then there exists \( \epsilon > 0 \) such that \( \gamma(t_0 - \epsilon, t_0 + \epsilon) \subset Z_n(X) \).

**Proof.** Around \( p \) consider a vector field \( U \) as in Lemma 2.1 such that \( U(p), Y(p) \) are linearly independent. Then there are coordinates \((x_1, x_2)\) about \( p \equiv 0 \), whose domain \( D \) can be identified to a product of two open intervals \( J_1 \times J_2 \), such that \( Y = \partial/\partial x_1 \) and \( U = \partial/\partial x_2 + x_1 V \).

Let \( X = g_1 \partial/\partial x_1 + g_2 \partial/\partial x_2 \). Then

\[
\frac{\partial g_k}{\partial x_1} = f g_k, \quad k = 1, 2.
\]

Since \( f \) is continuous the general solution to the equation above is:

\[
g_k(x) = h_k(x_2) e^{f}, \quad k = 1, 2,
\]

where \( \partial \varphi/\partial x_1 = f \) and \( \varphi([0] \times J_2) = 0 \).
From the Taylor expansion at \( p \) of \( X \) and \( U \) it follows that

\[
[U, [U, \ldots [U, X] \ldots] ](0) = \left[ \frac{\partial}{\partial x_2}, \left[ \frac{\partial}{\partial x_2}, \ldots \left[ \frac{\partial}{\partial x_2}, X \right] \ldots \right] \right](0)
\]

for the \( n \)-times iterated bracket.

Note that

\[
\left[ \frac{\partial}{\partial x_2}, \left[ \frac{\partial}{\partial x_2}, \ldots \left[ \frac{\partial}{\partial x_2}, X \right] \ldots \right] \right](0) = \frac{\partial^n g_1}{\partial x_2^n}(0) \frac{\partial}{\partial x_1} + \frac{\partial^n g_2}{\partial x_2^n}(0) \frac{\partial}{\partial x_2},
\]

since on \( \{0\} \times J_2 \) each \( g_k = h_k \) finally one has

\[
\frac{\partial^n h_1}{\partial x_2^n}(0) \frac{\partial}{\partial x_1} + \frac{\partial^n h_2}{\partial x_2^n}(0) \frac{\partial}{\partial x_2} = [U, [U, \ldots [U, X] \ldots]](0) \neq 0,
\]

which implies the existence of two differentiable functions \( \tilde{h}_1(x_2) \) and \( \tilde{h}_2(x_2) \) such that \( h_k = x_2^n \tilde{h}_k(x_2), k = 1, 2, \) and \( \tilde{h}_1(0) + \tilde{h}_2(0) > 0. \)

Therefore by shrinking \( D \) if necessary, we may suppose that at least one of these function, say \( \tilde{h}_\ell \), does not have any zero. Observe that \( f \) will be differentiable if \( \tilde{h}_\ell e^{\phi} \) is differentiable because \( \tilde{h}_\ell \) is differentiable without zeros and \( \partial \phi / \partial x_1 = f \).

As \( g_\ell = x_2^n \cdot (\tilde{h}_\ell e^{\phi}) \), it follows that \( g_\ell \) is divisible by \( 1, x_2, \ldots, x_2^n \) and the respective quotient functions are at least continuous. Moreover \( g_\ell / x^r, r = 1, \ldots, n - 1, \) vanish if \( x_2 = 0 \), that is to say on \( J_1 \times \{0\} \).

The Taylor expansion of \( g_\ell \) transversely to \( J_1 \times \{0\} \) leads

\[
g_\ell = \sum_{r=0}^{n-1} x_2^r \mu_r(x_1) + x_2^n \mu_n(x_1, x_2)
\]

where each \( \mu_k, k = 1, \ldots, n \) is differentiable.

Now since \( g_\ell(J_1 \times \{0\}) = 0 \) one has \( \mu_0 = 0 \).

In turn as \( g_\ell / x_2 \) equals zero on \( J_1 \times \{0\} \) it follows \( \mu_1 = 0 \), and so one.

Hence \( \mu_0 = \cdots = \mu_{n-1} = 0 \), which implies \( g_\ell = x_2^n \mu_n(x_1, x_2) \). Therefore \( \tilde{h}_\ell e^{\phi} = \mu_n \) is differentiable, which proves (a).

On the other hand, as \( e^{\phi} \) is differentiable and positive, \( X \) and

\[
X' = e^{-\phi} X = x_2^n \left( \tilde{h}_1 \frac{\partial}{\partial x_1} + \tilde{h}_2 \frac{\partial}{\partial x_2} \right)
\]
have the same order everywhere. Thus $X$ has order $n$ at every point of $J_1 \times \{0\}$ and \textit{(b)} becomes obvious. □

\textbf{Lemma 2.4.} \textit{Under the hypotheses of Proposition 2.2 consider $p \in Z_\infty(X)$ with $Y(p) \neq 0$. Let $\gamma: (a, b) \to M$ be an integral curve of $Y$ passing through $p$ for some $t_0 \in (a, b)$. Then there exists $\varepsilon > 0$ such that $\gamma(t_0 - \varepsilon, t_0 + \varepsilon) \subset Z_\infty(X)$.}

\textit{Proof.} Around $p \equiv 0$ consider coordinates $(y_1, y_2)$, whose domain $E$ can be identified to a product of two open intervals $K_1 \times K_2$, such that $Y = \partial/\partial y_1$ and $X = a_1(y_2) e^{\rho} \partial/\partial y_1 + a_2(y_2) e^{\rho} \partial/\partial y_2$ where $\partial \rho / \partial y_1 = f$ and $\rho(\{0\} \times K_2) = 0$. These coordinates exist by the same reason as in the proof of Lemma 2.3.

Assume the existence of a $q \in K_1 \times \{0\}$ of finite order $n$. Since $p \in Z_\infty$ and $e^\rho$ equals 1 on $\{0\} \times K_2$, it follows that $j_0^\infty a_1 = j_0^\infty a_2 = 0$. Therefore $a_k(y_2) = y_2^{q+1} b_k(y_2)$, $k = 1, 2$, where each $b_k$ is differentiable. Hence there exists a continuous vector field $X_n$ such that $X = y_2^{q+1} X_n$; that is to say $X$ is continuously divisible by $y_2^{q+1}$.

In turn one can find coordinates $(x_1, x_2)$ around $q \equiv 0$ whose domain $D$ can be identify to $J_1 \times J_2$ as in the proof of Lemma 2.3, which implies that $X = x_2^q e^\rho \left( \tilde{h}_1(x_2) \frac{\partial}{\partial x_1} + \tilde{h}_2(x_2) \frac{\partial}{\partial x_2} \right)$ where $\tilde{h}_1 \partial / \partial x_1 + \tilde{h}_2 \partial / \partial x_2$ has no zero on $D$.

By shrinking $D$ if necessary, we may suppose $D \subset E$. Then, regarded both sets in $M$, $J_1 \times \{0\}$ is a subset of $K_1 \times \{0\}$ since they are traces of integral curves of $Y$ with $q$ as common point.

On the other hand as $y_2$ vanishes on $K_1 \times \{0\}$ but its derivative never does, on $D$ one has $y_2 = x_2 c(x_1, x_2)$ where $c$ has no zero. This fact implies that $X$ on $D$ is continuously divisible by $x_2^{q+1}$ because it was continuously divisible by $y_2^{q+1}$.

But clearly from the expression of $X$ in coordinates $(x_1, x_2)$ it follows the non-divisibility by $x_2^{q+1}$, contradiction. In short the order of $X$ at each point of $K_1 \times \{0\}$ is infinite. □
Remark 2.5. Under the hypotheses of Proposition 2.2 the tracking function $f$ can be not differentiable around a flat point. For instance, on $\mathbb{R}^2$ set $Y = x_1^4 \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}$ and $X = g(x_1) \frac{\partial}{\partial x_1}$, where $g(x_1) = e^{-1/x_1}$ if $x_1 > 0$, $g(x_1) = e^{-1/x_1}$ if $x_1 < 0$ and $g(0) = 0$. Then $f(x) = x_1^4 - 4x_1^3$ if $x_1 > 0$, $f(x) = 2x_1 - 4x_1^3$ if $x_1 < 0$ and $f(\{0\} \times \mathbb{R}) = 0$, which is not differentiable on $\{0\} \times \mathbb{R}$.

Proof of Proposition 2.2: Let us prove the first assertion. Consider a non-constant integral curve of $Y$ (the constant case is clear) $\gamma: (a, b) \to M$. By Lemmas 2.3 and 2.4, $\gamma^{-1}(Z(X))$ is open in $(a, b)$. As this set is closed too one has $\gamma^{-1}(Z(X)) = \emptyset$ or $\gamma^{-1}(Z(X)) = (a, b)$. The first case is obvious; in the second one $(a, b) = \bigcup_{\text{netr}X} \gamma^{-1}(Z(X))$ where each term of this union is open. Therefore a single term of this disjoint union is non-empty since $(a, b)$ is connected.

For the second assertion apply (a) of Lemma 2.3 taking into account that $f$ is always differentiable on $M \setminus Z(X)$ because, on this set, the quotient $[Y, X]/X$ has a meaning. □

Proposition 2.6. On a surface $M$ consider a vector field $X$ such that $Z(X) \neq \emptyset$ but $Z_{\infty}(X) = \emptyset$. Then at least one of the following assertions holds:

(1) $Z(X)$ is a regular (embedded) 1-submanifold.

(2) There exists a primary singularity of $X$.

Proof. Assume the non-existence of primary singularities.

Consider any $p \in Z(X)$ and a vector field $Y$ defined around $p$ with $Y(p) \neq 0$ that tracks $X$. Let $U$ be a second vector field about $p$ as in Lemma 2.1 such that $U(p), Y(p)$ are linearly independent. Then there exist coordinates $(x_1, x_2)$, about $p \equiv 0$, whose domain $D$ can be identified to a product of two open intervals $J_1 \times J_2$ such that $Y = \partial/\partial x_1$ and $U = \partial/\partial x_2 + x_1 V$.

The same reasoning as in the proof of Lemma 2.3 allows to suppose that

$$X = x_2^2 e^\epsilon \left( \tilde{h}_1 \frac{\partial}{\partial x_1} + \tilde{h}_2 \frac{\partial}{\partial x_2} \right)$$

with $\tilde{h}_1^2 + \tilde{h}_2^2 > 0$ everywhere.
Therefore \( Z(X) \cap D \) is given by the equation \( x_2 = 0 \), which implies that \( Z(X) \) is a regular 1-submanifold. □

**Theorem 2.7.** Consider a vector field \( X \) on a surface \( M \). Assume that:

(a) \( Z_\infty(X) = \emptyset \).

(b) There is a connected component of \( Z(X) \) that is not included in a single \( Z_n(X) \).

Then there exists a primary singularity of \( X \).

**Proof.** Assume there is no primary singularity. By Proposition 2.6 \( Z(X) \) is a regular 1-submanifold of \( M \). By hypothesis there are a connected component \( C \) of \( Z(X) \) and two different natural numbers \( m \) and \( n \) such that \( C \) meets \( Z_m(X) \) and \( Z_n(X) \).

As \( C \) is a regular 1-submanifold, Proposition 2.2 and Lemma 2.3 imply that each \( C \cap Z_r(X), r \in \mathbb{N} \), is open in \( C \). Therefore \( C \) is a disjoint union of a family of non-empty open sets with two or more elements hence not connected, contradiction. □

### 2.1. Proof of Theorem 1.1

It consists of three steps.

1. Assume that there is no primary singularity in \( K \). From Proposition 2.6 applied to an isolating open set it follows that \( K \) is a compact 1-submanifold. Notice that at least one of its connected component is an essential block. Therefore one may suppose that \( K \) is diffeomorphic to \( S^1 \) and, by shrinking \( M \), that \( Z(X) = K \).

Consider a Riemannian metric \( g \) on \( M \). Given \( p \in K \) by reasoning as before one can find coordinates \((x_1, x_2)\) such that \( p \equiv 0 \) and

\[
X = x_2^\alpha e^{\alpha} \left( h_1 \frac{\partial}{\partial x_1} + h_2 \frac{\partial}{\partial x_2} \right)
\]

where \( h_1 \partial/\partial x_1 + h_2 \partial/\partial x_2 \) has no zero. Therefore around \( p \) there exists an 1-dimensional vector subbundle \( E \) of the tangent bundle that is orthogonal to \( X \). Such a vector subbundle is unique because clearly it exists and is
unique outside \( K \). Thus, gluing together the local constructions gives rise to an 1-dimensional vector subbundle \( E \) of \( TM \) that is orthogonal to \( X \).

2. If \( E \) is trivial there exists a nowhere singular vector field \( V \) such that \( g(V, X) = 0 \). Let \( \varphi : M \to \mathbb{R} \) be a function with a sufficiently narrow compact support such that \( \varphi(K) = 1 \). Set \( X_\delta := X + \delta \varphi V, \delta > 0 \). Then \( X_\delta \) approaches \( X \) as much as desired and \( Z(X_\delta) = \emptyset \), so \( K \) is an inessential block.

3. Now assume that \( E \) is not trivial. There always exists a 2-folding covering space \( \pi : M' \to M \) such that the pull-back \( E' \subset TM' \) of the vector subbundle \( E \) is trivial.

Consider the vector field \( X' \) on \( M' \) defined by \( \pi_*(X') = X \). Then \( Z(X') = \pi^{-1}(K) \) and \( X' \) is nowhere flat. Moreover \( E' \) is orthogonal to \( X' \) with respect to the pull-back of \( g \). Now the same reasoning as in the foregoing step shows that \( i_{Z(X')}(X') = 0 \). But clearly \( i_{Z(X')}(X') = 2i_k(X) \) and hence \( K \) is inessential.

3. Examples

Example 3.1. In this example one shows two facts. First, primary singularities can exist even if the index of \( X \) is not definable. Second, being nowhere flat is a weaker hypothesis than being analytic.

Consider a proper closed subset \( C \) of \( \mathbb{R} \) and a function \( \varphi : \mathbb{R} \to \mathbb{R} \) such that \( \varphi^{-1}(0) = C \). Set \( X := x_2^2 \partial/\partial x_1 + x_1 \varphi(x_2) \partial/\partial x_2 \). Then \( Z(X) = \{0\} \times \mathbb{R} \), \( Z_1(X) = \{0\} \times (\mathbb{R} \setminus C) \), \( Z_2(X) = \{0\} \times C \) and \( Z_n(X) = \emptyset \) for \( n \neq 1, 2 \), so \( X \) is nowhere flat. By Theorem 2.7 the vector field \( X \) has primary singularities.

More exactly the set \( S_a \) of primary singularities of \( X \) equals \( \{0\} \times (C \setminus \hat{C}) \). Indeed:

1. \( \varphi(x_2) \partial/\partial x_2 \) tracks \( X \) and does not vanish on \( \{0\} \times (\mathbb{R} \setminus C) \).
2. \( \partial/\partial x_2 \) tracks \( X \) on \( \mathbb{R} \times \hat{C} \).

Therefore \( S_a \subset \{0\} \times (C \setminus \hat{C}) \).
Take \( p = (0, c) \in \{0\} \times (C \setminus \overset{\circ}{C}) \). Assume the existence around this point of a vector field \( Y \) with \( Y(p) \neq 0 \) that tracks \( X \). Then from Proposition 2.2 and Lemma 2.3 it follows the existence of \( \varepsilon > 0 \) such that the order of \( X \) at every point of \( \{0\} \times (c - \varepsilon, c + \varepsilon) \) is constant and hence \( c \) belongs to the interior of \( \mathbb{R} \setminus C \) or to that of \( C \). Therefore \( c \notin C \setminus \overset{\circ}{C} \) contradiction.

In short, each element of \( \{0\} \times (C \setminus \overset{\circ}{C}) \) is a primary singularity and \( S_a = \{0\} \times (C \setminus \overset{\circ}{C}) \).

Finally observe that if \( C \) is a Cantor set, then \( X \) is not analytic for any analytic structure on \( \mathbb{R}^2 \) since \( Z_2(X) = \{0\} \times C \) is never an analytic set.

**Example 3.2.** In this example one gives a vector field on \( S^2 \), which is analytic so with no flat points, whose zero set is a circle just with two primary singularities.

The sphere \( S^2 \) can be regarded as the leaves space of the 1-dimensional foliation on \( \mathbb{R}^3 \setminus \{0\} \) associated to the vector field \( V = \sum_{k=1}^{3} x_k \partial / \partial x_k \), while the canonical projection \( \pi : \mathbb{R}^3 \setminus \{0\} \to S^2 \) is given by \( \pi(x) = x / \| x \| \).

Every linear vector field \( U' \) commutes with \( V \) and can be projected by \( \pi \) on a vector field \( U \) on \( S^2 \). Moreover \( U(a) = 0 \), where \( a = (a_1, a_2, a_3) \in S^2 \), if and only if \( a \) is an eigenvector of \( U' \) regarded as an endomorphism of \( \mathbb{R}^3 \), that is to say if and only if

\[
\left[ \sum_{k=1}^{3} a_k \frac{\partial}{\partial x_k}, U' \right] = \lambda \sum_{k=1}^{3} a_k \frac{\partial}{\partial x_k}
\]

for some scalar \( \lambda \).

Set \( X : = \pi_*(x_1 \partial / \partial x_2) \). Then \( Z(X) = \{x \in S^2 : x_1 = 0\} \) is an essential block of index two since \( \chi(S^2) = 2 \). By Corollary 1.2 the set \( S_a \) of primary singularities of \( X \) is not empty.

For determining it consider the vector field \( Y : = \pi_*(x_3 \partial / \partial x_2) \). Then \([X, Y] = 0 \) because \([x_1 \partial / \partial x_2, x_3 \partial / \partial x_2] = 0 \). Moreover \( Z(Y) = \{x \in S^2 : x_3 = 0\} \).

As \( Y \) tracks \( X \), the vector field \( Y \) is tangent to \( Z(X) \). On the other hand \( Z(X) \cap Z(Y) = \{(0, 1, 0), (0, -1, 0)\} \), so \( S_a \subset \{(0, 1, 0), (0, -1, 0)\} \). Since
$F, X = X$, where $F$ is the antipodal map, one has $F(S_a) = S_a$ and hence $S_a = \{(0, 1, 0), (0, -1, 0)\}$.

**Example 3.3.** Let $M$ be a connected compact surface of non-vanishing Euler characteristic. As it is well known, on $M$ there always exist two vector fields $X, Y$ with no common zero such that $[Y, X] = X$ (Lima [9], Plante [10]; see [1, 12] as well). Therefore there is no primary singularity of $X$, but there always exists a periodic regular trajectory of $Y$ included in $Z_0(X)$.

Indeed, by Corollary 1.2 and Proposition 2.2 the set $Z_0(X)$ is non-empty and $Y$-invariant. Since $Z_0(X)$ is compact, there always exists a minimal set $S \subset Z_0(X)$ of (the action of) $Y$.

As $Z(X) \cap Z(Y) = \emptyset$, a generalization of the Poincaré-Bendixson theorem [11] implies that $S$ is homeomorphic to a circle. In other words, there exists a non-trivial periodic trajectory of $Y$ consisting of flat points of $X$.

More generally, given a vector field $\tilde{X}$ on $M$ let $\mathcal{A}$ be the real vector space of those vector fields on $M$ that track $\tilde{X}$. Assume that $Z(\tilde{X}) \neq M$ and $Z(\tilde{X}) \cap (\bigcap_{V \in \mathcal{A}} Z(V)) = \emptyset$. Then by Corollary 1.2 the compact set $Z_0(\tilde{X})$ is not empty and contains a minimal set $\tilde{S}$ of $\mathcal{A}$ (more exactly of the group of diffeomorphisms of $M$ spanned by the flows of the elements of $\mathcal{A}$).

Clearly $\tilde{S}$ is not a point. A second generalization of the Poincaré-Bendixson theorem [7] shows that $\tilde{S}$ is homeomorphic to a circle.

Even more, in our case $\tilde{S}$ is a regular 1-submanifold and hence diffeomorphic to a circle. Let us see it. Take $p \in \tilde{S}$; then there is $V \in \mathcal{A}$ with $V(p) \neq 0$. Consider coordinates $(x_1, x_2)$ around $p \equiv 0$ whose domain $D$ is identified in the natural way to a product $(-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon)$ such that $V = \partial / \partial x_1$.

Let $\gamma: (-\delta, \delta) \to M$ be an integral curve of $V$ with initial condition $\gamma(0) = p$. Then $\gamma(-\delta, \delta) \subset \tilde{S}$. Moreover, if $\delta$ is sufficiently small $\gamma(-\delta, \delta)$ is a relatively open subset of $\hat{S}$. Indeed, $\gamma: (-\delta, \delta) \to \hat{S}$ will be injective so open because $\hat{S}$ is a 1-dimensional topological manifold (actually $S^1$). Now by shrinking $D$ and $(-\delta, \delta)$ if necessary, we may suppose that $\gamma(-\delta, \delta) \subset D$, $\delta = \varepsilon$ and $\gamma(t) = (t, 0)$. Thus $(-\varepsilon, \varepsilon) \times \{0\} = \gamma(-\delta, \delta)$ is relatively open in $\hat{S}$.
and there exists an open set \( E \) of \( M \) such that \( E \cap \widehat{S} = (-\varepsilon, \varepsilon) \times \{0\} \). Hence \( \widehat{S} \cap (D \cap E) \) is defined by the equation \( x_2 = 0 \) in the system of coordinates \((D \cap E, (x_1, x_2))\).

3.1. **An example from the blowup process.** In this subsection one constructs a homogeneous polynomial vector field on \( \mathbb{R}^2 \) whose trajectories but a finite number, let us call them *exceptional*, have the origin both as \( \alpha \)- and \( \beta \)-limit. Then by blowing up the origin one obtains a new vector field on a Moebius band whose number of primary singularities equals half that of exceptional trajectories of the first vector field.

Thus a global property on the trajectories of a vector field becomes a semi-local property on the primary singularities of another vector field.

First some technical facts. Denote by \( \tilde{\mathbb{R}}^2 \) the surface obtained by blowing up the origin of \( \mathbb{R}^2 \) and by \( \tilde{p}: \tilde{\mathbb{R}}^2 \to \mathbb{R}^2 \) the canonical projection. Recall that \( \tilde{\mathbb{R}}^2 \) is a Moebius band. If \( X \) is a vector field on \( \mathbb{R}^2 \) that vanishes at the origin, the blowup process gives rise to a vector field \( \tilde{X} \) on \( \tilde{\mathbb{R}}^2 \) such that \( \tilde{p}_* \tilde{X} = X \).

When the origin is an isolated singularity of index \( k \) and the order of \( X \) at this point is \( \geq 2 \), then \( \tilde{p}^{-1}(0) \) is a \( \tilde{X} \)-block of index \( k - 1 \).

Now identify \( \mathbb{C} \) to \( \mathbb{R}^2 \) by setting \( z = x_1 + ix_2 \). Then each complex vector field \( z^n \partial / \partial z, \ n \geq 2 \), can be considered as a vector field \( X_n = P_n \partial / \partial x_1 + Q_n \partial / \partial x_2 \) on \( \mathbb{R}^2 \) where \( z^n = (x_1 + ix_2)^n = P_n(x_1, x_2) + iQ_n(x_1, x_2) \). Our purpose will be to show that \( Z(\tilde{X}_n) = \tilde{p}^{-1}(0) \) contains \( n - 1 \) primary singularities of \( \tilde{X}_n \). (Recall that the origin is a singularity of \( X_n \) of index \( n \) and hence \( \tilde{p}^{-1}(0) \) is a \( \tilde{X}_n \)-block of index \( n - 1 \).)

3.1.1. \( \tilde{\mathbb{R}}^2 \) from another point of view. Consider the map \( \varphi: \mathbb{R} \times S^1 \to \mathbb{R}^2 \) given by \( \varphi(r, \theta) = (rcos \theta, rsin \theta) \). Then \( \varphi: \mathbb{R}_+ \times S^1 \to \mathbb{R}^2 \setminus \{0\} \) and \( \varphi: \mathbb{R}_- \times S^1 \to \mathbb{R}^2 \setminus \{0\} \) are diffeomorphisms, and \( \varphi(r, \theta) = \varphi(r', \theta') \) with \( (r, \theta), (r', \theta') \in (\mathbb{R} \setminus \{0\}) \times S^1 \) if and only if \( (r, \theta) = (r', \theta') \) or \( (r', \theta') = (-r, \theta + \pi) \).
Let \( \sim \) be the equivalence relation on \( \mathbb{R} \times S^1 \) defined by \((r, \theta) \sim (r', \theta')\) if and only if \((r, \theta) = (r', \theta')\) or \((r', \theta') = (-r, \theta + \pi)\). Then the quotient space \( M_s: = (\mathbb{R} \times S^1)/\sim \) is a Moebius strip and the canonical projection \( p: \mathbb{R} \times S^1 \to M_s \) is a (differentiable) covering space with two folds. Moreover the map \( \tilde{\varphi}: M_s \to \mathbb{R}^2 \) given by \( \tilde{\varphi}(p(r, \theta)) = \varphi(r, \theta) \) is well defined and differentiable.

Recall that \( \tilde{p}^{-1}(0) = \mathbb{R}P^1 \) is the space of vector lines in \( \mathbb{R}^2 \) and \( \tilde{p}: \mathbb{R}^2 \setminus \tilde{p}^{-1}(0) \to \mathbb{R}^2 \setminus \{0\} \) a diffeomorphism. Now one defines \( \Psi: M_s \to \mathbb{R}^2 \) as follows:

(a) \( \Psi(p(r, \theta)) = \tilde{p}^{-1}(\varphi(r, \theta)) \) if \( r \neq 0 \),

(b) \( \Psi(p(r, \theta)) \) equals the vector line of \( \mathbb{R}^2 \) spanned by \((\cos \theta, \sin \theta)\) if \( r = 0 \).

It is easily checked that \( \Psi: M_s \to \mathbb{R}^2 \) is a diffeomorphism and \( \tilde{p} \circ \Psi = \tilde{\varphi} \). Therefore \( \tilde{p}: \mathbb{R}^2 \to \mathbb{R}^2 \) and \( \tilde{\varphi}: M_s \to \mathbb{R}^2 \) can be identified in this way. For sake of simplicity in what follows \( \tilde{p}: \mathbb{R}^2 \to \mathbb{R}^2 \) will replace by \( \tilde{\varphi}: M_s \to \mathbb{R}^2 \) in our computations. Thus if \( X \) is a vector field on \( \mathbb{R}^2 \) that vanishes at the origin, then \( \tilde{X} \) will be the single vector field on \( M_s \) such that \( \tilde{\varphi}_* \tilde{X} = X \).

On the other hand \( X' \) will denote the pull-back by \( p \) of \( \tilde{X} \). Clearly \( \varphi_*X' = X \). Moreover with respect to \( X' \) the index of \( \{0\} \times S^1 \) and the number of primary singularities included in it are twice those of \( \tilde{\varphi}^{-1}(0) \) relative to \( \tilde{X} \).

As a consequence, in the case of \( X_n \) it will suffice to show that \( Z(X'_n) = \{0\} \times S^1 \) contains \( 2n - 2 \) singularities of \( X'_n \).

3.1.2. Computation of the primary singularities of \( X'_n \). As \( \varphi: (\mathbb{R} \setminus \{0\}) \times S^1 \to \mathbb{R}^2 \setminus \{0\} \) is a covering space any vector field on \( \mathbb{R}^2 \setminus \{0\} \) can be lifted up. Denote by \( \partial'/\partial x_k, k = 1, 2, \) the lifted vector field of \( \partial/\partial x_k \). Then

\[
\frac{\partial'}{\partial x_1} = \cos \theta \frac{\partial}{\partial r} - r^{-1} \sin \theta \frac{\partial}{\partial \theta} \quad \text{and} \quad \frac{\partial'}{\partial x_2} = \sin \theta \frac{\partial}{\partial r} + r^{-1} \cos \theta \frac{\partial}{\partial \theta}.
\]

Since \((r \cos \theta + ir \sin \theta)^n = r^n \cos(n\theta) + ir^n \sin(n\theta) \) one has \( P_n \circ \varphi = r^n \cos(n\theta) \) and \( Q_n \circ \varphi = r^n \sin(n\theta) \). Observe that on \((\mathbb{R} \setminus \{0\}) \times S^1 \) the vector field \( X'_n \) is the lifted one of \( X_n \), so \( X'_n = r^n \cos(n\theta) \partial'/\partial x_1 + r^n \sin(n\theta) \partial'/\partial x_2 \). Finally,
developing the foregoing expression of $X'_n$ and extending it by continuity to $\mathbb{R} \times S^1$ yields:

$$X'_n = r^{n-1} \left( r \cos((n-1)\theta) \frac{\partial}{\partial r} + \sin((n-1)\theta) \frac{\partial}{\partial \theta} \right)$$

The vector field $Y = r \cos((n-1)\theta) \frac{\partial}{\partial r} + \sin((n-1)\theta) \frac{\partial}{\partial \theta}$ tracks $X'_n$ with tracking function $(n-1)\cos((n-1)\theta)$. Therefore the set $S_a$ of primary singularities of $X'_n$ is included in $\{0\} \times T_n$ where $T_n = \{ \theta \in S^1 : \sin((n-1)\theta) = 0\}$.

On the other hand, the order of $X'_n$ at the points of $\{0\} \times (S^1 \setminus T_n)$ is $n-1$ and strictly greater than $n-1$ at the points of $\{0\} \times T_n$. As $T_n$ is finite, more exactly it has $2n-2$ elements, Proposition 2.2 and Lemma 2.3 imply that all the points of $\{0\} \times T_n$ are primary singularities. In short $S_a = \{0\} \times T_n$ and hence $Z(\tilde{X}_n) = \tilde{p}^{-1}(0)$ contains $n-1$ primary singularities.

3.1.3. The geometric meaning of the primary singularities of $\tilde{X}_n$. When $n \geq 2$ the complex flow of $z^n \frac{\partial}{\partial z}$ is

$$\Phi(z, t) = z \left[ (1-n)t \frac{z^{n-1}}{z} + 1 \right]$$

with initial condition $\Phi(z, 0) = z$.

(Fixed $z \neq 0$ consider as domain of the variable $t$ the open set $D_z = \mathbb{C} \setminus R_z$ where $R_z = \{ (s(n-1))^{-1} \tilde{z}^{1-n} : s \in [1, \infty) \}$. Note that $D_z$ is star shaped with respect to the origin. Since $D_z$ is simply connected, the initial condition $\Phi(z, 0) = z$ defines a single continuous and hence holomorphic map $\Phi(z, \cdot) : D_z \to \mathbb{C}$. Thus the apparent ambiguity introduced by the root of order $n-1$ is eliminated.)

On the other hand considering, in the foregoing expression of $\Phi$, real values of $t$ only and identifying $z$ with $(x_1, x_2)$ yield the real flow of $X_n$. Therefore given $(x_1, x_2) \in \mathbb{R}^2 \setminus \{0\}$ if $z^{n-1} = (x_1 + ix_2)^{n-1}$ is not a real number, its $X_n$-trajectory is defined for any $t \in \mathbb{R}$ and has the origin both as $\alpha$ and $\omega$-limit.
On the contrary when \( z^{n-1} = (x_1 + ix_2)^{n-1} \) is a real number, the \( X_n \)-trajectory of \((x_1, x_2)\), as set of points, equals the open half-line spanned by the vector \((x_1, x_2)\) and hence one of its limits is the origin and the other one the infinity.

It is easily checked that the set of \((x_1, x_2) \in \mathbb{R}^2\) such that \((x_1 + ix_2)^{n-1} \in \mathbb{R}\) consists of \(n - 1\) vector lines each of them including two exceptional trajectories. These lines regarded as elements of \(\mathbb{R}P^1 = \tilde{p}^{-1}(0)\) are the primary singularities of \(\tilde{X}_n\).

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