Discrete-Aware Matrix Completion via Proximal Gradient

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Abstract—We present a novel algorithm for the completion of low-rank matrices whose entries are limited to a finite discrete alphabet such as recommender systems. The proposed method is based on the proximal gradient (PG) framework borrowed from optimization theory, which is applied here to solve a regularized formulation of the completion problem that includes a term enforcing the discrete-alphabet membership of the matrix entries as well as a low-rank regularizer. Simulation results demonstrate the superior performance of the proposed method in comparison with recently proposed state-of-the-art alternatives. Also, possible applications of the proposed method are listed in the conclusion.

I. INTRODUCTION

With fair-winds of big data and internet of things (IoT), modern signal and information processing applications such as information filtering systems, networking, machine learning, and wireless communications often face a structured low-rank matrix completion (LRMC) problem, which intends to infer a low-rank matrix \( \mathbf{X} \in \mathbb{R}^{m \times n} \) given a partially observed incomplete matrix \( \mathbf{O} \in \mathbb{R}^{m \times n} \) [1]–[3]. Matrix completion (MC) has therefore attracted much attention from both academic and industrial researchers, and has been applied to many different applications including recommender systems, localization, image compression and restoration, massive multiple-input multiple-output (MIMO) and millimeter wave channel estimation, and phase retrieval, to name a few.

To address this challenge, effective strategies based on convex relaxation have been well-studied in the literature [1], [4], [5] in terms of theoretical performance and complexity guarantees, of which crux is to replace the intractable non-convex rank function with its convex envelope (i.e., the nuclear norm (NN)). To cite several milestones, one of the earliest works [4] proposed to convert such a nuclear-norm-based optimization problem into semidefinite program (SDP), which however is not suitable to large-scaled problems as seen in practical scenarios due to the fact that SDP solvers require at least the cubic order complexity. To circumvent this issue, the singular value thresholding (SVT) as a proximal minimizer of the NN function was proposed in [5], which has been later extended to its low-complexity alternative via the Lanczos algorithm. These methods in addition to other state-of-the-arts will be technically reviewed in Section II.

In spite of intractability, structured non-convex optimization frameworks to address low-rankness have numerically shown successful performance improvements against its convex counterparts [6], which have recently been guaranteed to possess lower complexities from a theoretical point of view [2]. Indeed, as recently shown in [7]–[9], non-convex approaches outperformed the state-of-the-art convex methods in terms of mean square error (MSE) regardless of observation ratios.

Despite such intensive developments over the last decade, most of the LRMC algorithms have been designed for general MC problems at the cost of missing use of the most of the problem structure, leaving potential of further performance improvements. To elaborate, many existing MC algorithms including ones mentioned above or in Section II have assumed randomness or continuity of entries of the low-rank matrix \( \mathbf{X} \), albeit in many practical situations those entries must belong to a certain finite discrete alphabet set.

In this article, we therefore introduce an additive discrete-aware regularizer that can be adopted for many different state-of-the-art LRMC algorithms, proposing a discrete-aware variate of Soft-Impute, one of the state-of-the-art methods for large-scaled LRMC problems, so as to illustrate the effectiveness of the proposed regularizer. Simulation results confirm the superior performance of the proposed method.

II. PRIOR WORK

In this section we briefly review major LRMC techniques studied over the last decade, which intend to recover unknown entries of a targeted low-rank matrix from partial observations, facilitating introduction to our proposed discrete-aware MC framework. To this end, we start with the original MC optimization problem, which can be written as the following intractable rank minimization problem:

\[
\begin{align*}
\argmin_{\mathbf{X} \in \mathbb{R}^{m \times n}} & \quad \text{rank} (\mathbf{X}) \\
\text{s.t.} & \quad P_0 (\mathbf{X}) = P_0 (\mathbf{O}),
\end{align*}
\]

where \( \text{rank} (\cdot) \) denotes the rank of a given input matrix and \( P_0 (\cdot) \) indicates the mask operator (i.e., projection) defined as

\[
[P_0 (\mathbf{A})]_{ij} = \begin{cases} 
[A]_{ij} & \text{if } (i,j) \in \Omega \\
0 & \text{otherwise}
\end{cases},
\]

with \([\cdot]_{ij}\) being the \((i,j)\)-th element of a given matrix and \(\Omega\) denoting the observed index set.
Although the global solution of equation (1) corresponds to a matrix that has the lowest rank and matches observations corresponding to indexes belonging to the indicator set \( \Omega \), naively solving the above rank minimization problem is known to be non-deterministic polynomial-time (NP)-hard due to the non-convexity of the rank operator \( \cdot \). Taking advantage of the idea that the \( \ell_0 \)-norm function can be replaced by its convex surrogate \( \ell_1 \)-norm in compressed sensing (CS)-related problems, the above rank minimization problem can be relaxed by introducing the NN \( \| A \|_* \), \( i.e., \) the sum of the singular values of \( A \) \[1\], namely,

\[
\begin{align*}
\arg\min_{X \in \mathbb{R}^{m \times n}} & \quad \| X \|_* \\
\text{s.t.} & \quad P_0(X) = P_0(O),
\end{align*}
\]

where NN is known to be the tightest convex lower bound of the rank operator \[10\].

Among various numerical optimization algorithms solving equation (3), one of the landmark attempts has been proposed in literature [4], which recasts equation (3) as an SDP [11]:

\[
\begin{align*}
\arg\min_{X, W_1, W_2} & \quad \text{Tr} ( W_1 ) + \text{Tr} ( W_2 ) \\
\text{s.t.} & \quad P_0(X) = P_0(O),
\end{align*}
\]

which can be solved by interior point methods available at various convex optimization solvers including SDPT3 [12], MOSEK [13], and SeDuMi [14].

Since the aforementioned SDP solvers suffer from prohibitive time and random access memory (RAM) complexity due to the nature of second-order methods, however, the above approaches are only suitable for small-sized problems in spite of the fact that we are often interested in scenarios where the dimension of \( X \) is large. Aiming at reducing the computational burden while relaxing the equality constraint (3b) for cases where the observations contain noise or the target matrix to be recovered may only be regarded as approximately low-rank, various prior works including ones proposed in [5], [15]–[22] can be categorized as a solution to either the problem:

\[
\begin{align*}
\arg\min_{X \in \mathbb{R}^{m \times n}} & \quad \| X \|_* \\
\text{s.t.} & \quad \frac{1}{2} \| P_0(X - O) \|_F^2 \le \varepsilon,
\end{align*}
\]

or its regularized form

\[
\begin{align*}
\arg\min_{X \in \mathbb{R}^{m \times n}} & \quad f(X) + \lambda \| X \|_* ,
\end{align*}
\]

or with the rank information

\[
\begin{align*}
\arg\min_{X \in \mathbb{R}^{m \times n}} & \quad f(X) \\
\text{s.t.} & \quad \text{rank} ( X ) \le s,
\end{align*}
\]

where \( f(\cdot) \) is implicitly defined for notational convenience.

Although a great amount of efforts has been made to efficiently tackle the aforementioned convex problems\[1\], there is growing progress on developing non-convex optimization algorithms for LRMC via first-order methods, which have shown via numerical studies remarkable success in practice and is based on non-convex regularizers such as the capped \( \ell_1 \)-norm, the truncated nuclear norm (TNN), and the log-sum-penalty (LSP). To cite a few examples, [7] proposed a proximal gradient (PG) algorithm for general non-convex and non-smooth optimization, named nonmonotone accelerated proximal gradient (nmAPG), which computes gradient steps in a forward-backward fashion and is further extended in [6] to its accelerated variate, dubbed as nonconvex inexact accelerated proximal gradient (niAPG). The authors in [9] study the strong duality of non-convex matrix factorization problems, proving that under certain dual conditions, the global optimality of such non-convex MC problems can be achieved by solving its convex bi-dual problem, while [24] paves the way towards a theoretical guarantee for non-convex optimization frameworks to properly learn the targeted underlying low-rank matrix. For more information, please refer to a recent comprehensive survey [2] on non-convex MC problems and solutions.
aim at solving an optimization problem similar to equation (6) and therefore to equation (5). To elaborate, Soft-Impute consists of the following recursion

\[ X_t = \text{SVT}_\lambda(X_{t-1} + P_\Omega(O - X_{t-1})), \quad (8) \]

where we utilized the fact that \( f(X) \) is a convex function with 1-Lipschitz constant, \( \lambda \) denotes the iteration index and the SVT function is given by [5, Theorem 2.1] as

\[ \text{SVT}_\lambda(A) \triangleq U \left( \Sigma - \lambda I \right)_+ V^T, \quad (9) \]

with \( A \triangleq U \Sigma V^T \) and \((\cdot)_+\) being the positive part of the input.

It has recently been shown that Soft-Impute can be categorized as a PG algorithm [19], and therefore, the well-known Nesterov-type momentum acceleration technique can be employed without loss of convergence guarantee [36, 37], leading to

\[ X_t = \text{SVT}_\lambda(Y_t + P_\Omega(O - Y_t)), \quad (10) \]

with \( Y_t \triangleq (1 + \beta_t)X_{t-1} + \beta_tX_{t-2} \) where \( \beta_t \) is the momentum weight.

**B. Discrete-Aware Matrix Completion**

Assuming that entries of the matrix to be recovered belong to a certain finite discrete alphabet set \( A \triangleq \{a_1, a_2, \ldots \} \) (e.g., integers in case of recommendation systems), we intend to tackle a variety of the following regularized minimization problem

\[ \arg\min_{X \in \mathbb{R}^{m \times n}} f(X) + \lambda g(X) + \xi r(X|p), \quad (11) \]

where \( g(X) \) denotes a non-smooth (possibly non-convex) low-rank regularizer [8], \( \xi \geq 0 \), and

\[ r(X|p) \triangleq \sum_{k=1}^{|A|} \| \text{vec}_{\ell_1}(X) - a_k 1 \|_p \quad (12) \]

where \( r(X|p) \) is the discrete-space regularizer\(^2\) with \( 0 \leq p \leq 2 \), \( \text{vec}_{\ell_1}(X) \) denotes vectorization of entries of \( X \) corresponding to a given index set \( \Omega^c \), and \( \Omega^c \) being the complementary set of \( \Omega \).

Although non-convex scenarios where either \( g(X), r(X|p) \) or both are non-convex regularizer(s) can be considered, we hereafter focus on the convex scenario (i.e., \( g(X) = \|X\|_F \)) and \( r(X|1) = \sum_{k=1}^{\|\Omega\|} \| \text{vec}_{\ell_1}(X) - a_k 1 \|_1 \) for the sake of simplicity and because of space constraints. The accelerated PG algorithm for a discrete-aware convex variant of Soft-Impute as shown in equation (11), which hold the convergence rate \( O(\frac{1}{t^2}) \), can be summarized as the following recursion:

\[ Y_t = (1 + \beta_t)X_{t-1} + \beta_tX_{t-2} \quad (13a) \]
\[ Z_t = \text{prox}_{\xi r}^p(Y_t) \quad (13b) \]
\[ X_t = \text{SVT}_\lambda(P_{\Omega^c}(Z_t) + P_\Omega(O)) \quad (13c) \]

where \( \text{prox}_{\xi r}^p(Y) \) is the proximal operator given by

\[ \text{prox}_{\xi r}^p(Y) \triangleq \arg\min_U \| U \|_1 + \frac{1}{2\xi} \| \text{vec}_{\ell_1}(U - Y) \|_2^2. \quad (14) \]

Taking into account the fact that the proximal operator of a sum of convex regularizers can be computed from a sequence of individual proximal operators [36], we readily obtain

\[ \text{prox}_{\xi r}^p(Y_t) = \text{prox}_{\xi r_1}^p \left( \left( \text{prox}_{\xi r_2}^p \left( \cdots \text{prox}_{\xi r_{|\Omega|}}^p(Y_t) \right) \right) \right). \quad (15) \]

where \( r_k(Y_t) \triangleq \| \text{vec}_{\ell_1}(Y_t) - a_k 1 \|_1 \) for \( k \in \{1, 2, \ldots, |\Omega|\} \). To this end, each proximal operator can be written as

\[ \text{prox}_{\xi r_k}(Y_t) \triangleq \arg\min_U \| u - a_k 1 \|_1 + \frac{1}{2\xi} \| u - Y_t \|_2^2, \quad (16) \]

with \( u \triangleq \text{vec}_{\ell_1}(U) \) and \( y_t \triangleq \text{vec}_{\ell_1}(Y_t) \), which can be compactly written element-by-element as

\[ \arg\min_{y_{\ell}} |u_{\ell} - y_{\ell}| + \frac{1}{2\xi} (u_{\ell} - y_{\ell})^2, \quad (17) \]

where \( u_{\ell} \triangleq |y_{\ell}| \pm a_k, \quad y_{\ell} \triangleq |y_{\ell}| - a_k, \quad u \triangleq [u_1, u_2, \ldots, u_{|\Omega|}]^T, \quad Y_t \triangleq [y_{\ell_1}, y_{\ell_2}, \ldots, y_{\ell_{|\Omega|}}]^T \) and \( \ell \in \{1 \leq \ell \leq |\Omega|\} \).

One readily notice that equation (17) has a closed form solution \( (i.e., \text{soft-thresholding function}) \) given by

\[ \bar{u} = \text{sign}(y_{\ell}) \odot (|y_{\ell}| - \xi 1)_+, \quad (18) \]

where \( \odot \) is the Hadamard product and \text{sign}(\cdot) denotes the (element-wise) sign function.

Notice that in equation (18), \( |y_{\ell}| \) performs the element-wise absolute operation. Finally we recover \( u \) by

\[ u = \bar{u} + a_k 1, \quad (19) \]

and \( U \) by mapping \( u \) onto the unobserved indexes, namely,

\[ U = \text{vec}_{\ell_1}^{-1}(u), \quad (20) \]

where \( \text{vec}_{\ell_1}^{-1}(\cdot) \) denotes the inverse function of \( \text{vec}_{\ell_1}(\cdot) \).

**IV. NUMERICAL EVALUATION**

In this section, we perform numerical experiments on discrete-valued real-world data sets to evaluate the proposed discrete-aware MC algorithm. To this end, we adopt the MovieLens-100k data set\(^3\) for recommender systems, one of the popular data sets utilized in MC literature for performance evaluations, which is composed of integer ratings (from 1 to 5) associated with many different user-movie pairs and possesses a low-rank nature due to the inter-user correlation in preferred movies. To evaluate the robustness jointly with the recovery performance, we vary the observed ratio from 20% to 60%, while NMSE is utilized as the performance metric, which is given by

\[ \text{NMSE} \triangleq \frac{\| P_{\Omega^c}(X - O) \|_F^2}{\| P_{\Omega^c}(O) \|_F^2}. \quad (21) \]

Besides the Soft-Impute algorithm [18], we compare our proposed algorithm with other state-of-the-art methods such as

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\(^2\)Although it has been shown in the literature [25]-[29], [38], [39] that the base of the norm function is set to be \( p = 0 \) or \( p = 1 \) to enhance the discreteness of the inputs, the base \( p \) can be any positive number in principle.

\(^3\)https://grouplens.org/datasets/movielens/
Fig. 1. NMSE performance evaluations of the proposed discrete-aware MC algorithms (red) and other first-order state-of-the-art methods (gray and black) with respect to different observation ratios.

Fig. 2. NMSE performance behavior on the MovieLens-100k data set as a function of algorithmic iterations at 20% observed ratio.

AIS-Impute [19], an accelerated variate of Soft-Impute, niAPG [6], a non-convex variate of Soft-Impute with the LSP non-convex regularizer.

The NMSE performance results comparing our proposed discrete-aware variates of Soft-Impute and the aforementioned state-of-the-art methods as a function of ratio of observed ratings for training are shown in Figure 1, where the red lines correspond to our proposed methods and black or gray lines are associated with the state-of-the-arts. For the sake of clarity, the NMSE performance gaps due to discreteness-awareness is highlighted by annotation arrows. It can be observed from the figure that most of the algorithms are able to successfully achieve less than 0.1 in terms of NMSE for a wide range of observed ratios, albeit non-convex algorithms with LSP can reduce the performance degradation in a severe scenario, where only a few number of entries of the matrix can be observed. More interestingly, even in case of convex algorithms, the discreteness-awareness considerably decrease increment of the NMSE curve at the low observed ratio range, which indicates the robustness of the proposed discrete-aware regularizer.

In Figure 2, the NMSE convergence behavior of the algorithms with respect to the number of algorithmic iterations is presented, where we can perceive that most of the algorithms converge within 100 iterations in case with the convex NN regularizer and 180 iterations in case with the non-convex LSP regularizer, respectively. Furthermore, the figure illustrates the accelerated convergence of the proposed algorithm with the convex NN regularizer. According to this observation, it may be concluded that the discreteness-awareness can not only improve the NMSE performance but also contribute to finding the optimality condition. However, the latter benefit is not necessary in case of non-convex scenarios due to multiple local minima, which rather results in slightly slower convergence.

Besides the above, we remark that the additional complexity due to the discreteness-aware regularizer in equation (12) with $p = 1$ is linear with respect to the cardinality of the unknown index set (i.e., $|\Omega^c|$) as one may readily observe from the element-by-element operation in equations (18)–(20). Therefore, one may conclude that the most expensive part of the algorithm in terms of complexity is the same as that of the state-of-the-art methods, i.e., SVT, indicating that the proposed algorithm maintains the same complexity order.

In case of $p = 0$, however, the regularizer may affect the convergence or the complexity of the proposed PG algorithm due to many different reasons such as expansiveness of $r_0(X)$ [40] or successive convex approximation to relax the $\ell_0$-norm function. Taking into account the aforementioned issues, it is an open problem to develop a fully non-convex algorithm (i.e., a non-convex low-rank regularizer and a discrete-aware regularizer with $p < 1$) and analyse its convergence property.

In light of all the above, we conclude from the numerical performance evaluations that our proposed discreteness-aware MC algorithm may further accelerate the convergence and improve the completion performance in case of adopting convex functions for both regularizers (i.e., $g(\cdot) = \|\cdot\|$, and $r_1(\cdot)$), while enjoying the uniqueness of the solution due to the convexity of equation (11). In case of non-convex low-rank regularizer (i.e., LSP) while maintaining convex discreteness-aware regularizer (i.e., $r_1(\cdot)$), it has been shown that at the expense of slower convergence, the NMSE performance can be enhanced as shown in Figure 2.

V. CONCLUSION AND REMARKS

In this article, we proposed a novel discrete-aware LRMC algorithm for structured practical MC problems where entries of the matrix to be recovered is subject to a certain finite discrete alphabet set such as recommender systems. To tackle this open problem indicated by a recent comprehensive survey [3], we introduce a discrete-aware additive regularizer that has
been recently considered in signal processing and wireless communication literature. Performance evaluations via software simulations demonstrate the superior performance of the proposed methods due to the awareness to such a specific structure of the target matrix. We conclude this article by providing some possible applications of the proposed MC algorithm as follows.

- The most important and obvious application is recommender systems for Netflix, Amazon, and so on with discrete scores, e.g., 1 of 5.
- Another application of the proposed MC algorithm is an estimation of connections among users in networks such as social networks, large wireless ad-hoc networks, etc where 0 means not-connected and 1 denotes connected. Although it is not impossible to obtain the whole adjacency matrix from the network, it would cost tons of resources for that. For example, if the proposed MC precisely estimates the whole matrix of the wireless ad-hoc network with the partial information, it enables the network to perform the optimal routing, network coding, distributed coding, and so on with even less overhead, which results in significant improvement of the throughput.
- An interesting field to which discreteness-aware MC algorithms can be applied is an index coding problem in a broadcast channel [41], where a single source communicates with multiple users over a rate-limited channel.

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