Linear nonlocal problem for the abstract time-dependent Schrödinger equation

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Abstract
A nonlocal-in-time problem for the abstract Schrödinger equation is considered. By exploiting the linear nature of nonlocal condition we derive an exact representation of the solution operator under assumptions that the spectrum of Hamiltonian is contained in the horizontal strip of complex plane. The derived representation permits us to establish the necessary and sufficient conditions for the problem’s well-posedness and existence of its mild, strong solutions. Furthermore, we present new sufficient conditions which extend the existing results to the case when some nonlocal parameters are unbounded. Two examples are provided.

Keywords: nonlocal problem, abstract time-dependent Schrödinger equation, well-posedness, solution operator, Dunford-Cauchy formula, zeros of polynomial, driven quantum systems.

1 Introduction

In the abstract setting the evolution of quantum system is governed by differential equation

\[ i\psi_t' - H\psi = v(t), \quad t \in T, \ T \subseteq \mathbb{R}, \]

(1)
which is called time-dependent Schrödinger equation. Standard axiomatic approach to the quantum mechanics ensures that the state of the system described by a wave function \( \psi(t) \in X \), is uniquely determined by (1) and a given initial state \( \psi_0 \)
\[
\psi(0) = \psi_0.
\] (2)

This is achieved by requiring that the linear operator (Hamiltonian) \( H : X \to X \) is self-adjoint in the Hilbert space \( X \) and its domain \( D(H) \subseteq X \) is dense. Stone theorem states that in such case there exists a strongly continuous unitary group \( U(t) = e^{-iHt} \) with generator \( H \) [4]. The function \( \psi(t) \) is called a mild solution of (1), (2) if
\[
\psi(t) = U(t - t_0)\psi(t_0) + \int_{t_0}^{t} U(t - s)v(s)ds, \quad t_0 \in T.
\] (3)

Substitution of the initial data from (2) into this general solution representation leads to the usual propagator formula \( \psi(t) = U(t)\psi_0 \) for the solution of (1) – (2) with \( v(t) \equiv 0 \).

In this work we consider a nonlocal generalization of condition (2)
\[
\psi(0) + \sum_{k=1}^{n} \alpha_k \psi(t_k) = \psi_1.
\] (4)

For the fixed state \( \psi_1 \in X \) this condition is determined by the set of parameters \( 0 < t_1 < t_2 < \ldots < t_n \in T, \alpha_k \in \mathbb{C} \) which will be called the parameters of nonlocal condition. Aside from the standard initial condition (2) it generalizes other important condition types, such as periodic conditions \( \psi(0) = \psi(t_1) \) and Bicadze-Samarskii conditions \( \psi(0) + \alpha_1 \psi(t_1) = \alpha_2 \psi(t_2) \). Formula (4) can be also viewed as approximation to a more general nonlinear condition \( \psi(0) + g(t_1, \ldots, t_k, \psi(\cdot)) = 0 \) for a suitably defined function \( g(t_1, \ldots, t_k, \cdot) : X \to X \). Nonlocal problem (1), (4) is essential to the theory of driven quantum systems. Where one is interested in a way to recapture specific nonlocal behavior of solution \( \psi(0) = \alpha_1 \psi(t_1) + \psi_1 \) by changing the properties of driving potential \( p(t) \) from the Hamiltonian \( H = H_0 + p(t) \).

To stay withing the classical formulation (1), (2) this theory operates upon assumption that \( p(t) \) is periodic [3], [18], [9]. Then a predictable nonlocal-in-time behavior of the system follows from the Floquet theorem. The case of non-periodic \( p(t) \) is much harder to treat, since the Floquet theory can not be applied. It is our belief that the nonlocal formulation is a viable alternative to other proposed non-periodic generalization of quantum driving that are currently under research [19], [20].
Mentioned two-point nonlocal condition with $\psi_1 \neq 0$ can be also thought of as a generalization of the renowned Rabi problem used in the modern quantum computing for state preparation and information processing [10].

In spite of the gaining importance a surprisingly little is known about the solution of (1), (4). The problem was studied in the work of Ashyraliev and Sirma [1], using the Hilbert space methods. For self-adjoint $H$ they proved that the condition

$$\sum_{k=1}^n |\alpha_k| < 1$$

is sufficient for the existence of solution to (1), (4), when $\psi_1$ is smooth. The same condition appeared earlier in [3], where a more general nonlocal problem for the first order equation with sectorial operator coefficient in Banach space was considered. In the current work we are more interested in the generalization of the above condition

$$\sum_{k=1}^n |\alpha_k| e^{dt_k} \leq 1. \quad (5)$$

developed in [14]. Here $d$ is a height of the strip containing the spectrum of an operator $H$ defined in the Banach space $X$. By the end of the work it became obvious that inequality (5) represents only a fraction of the parameter space where problem (1), (4) is well-posed and have a mild solution defined by (3). More generally we establish the necessary and sufficient conditions for the existence of solution to (1), (4) which can be verified for any given set of $\alpha_k, t_k$ from (4). In addition to that we derive new sufficient conditions for the solvability of given nonlocal problem which extend the region of admissible $\alpha_k$ outside the manifold governed by (5).

The paper is organized as follows. In Section 2 we introduce a notion of strip-type operators $H$ acting on Banach space $X$ and review the functional calculus of such operators. Our aim is to specify the class of $H$ such that the propagator $U(t)$ is well-defined and can be represented via the Dunford-Cauchy formula. Section 3 is devoted to the solution’s existence analysis. We start with reduction of nonlocal problem (1), (4) to classical Cauchy problem (1), (2). Then apply the operator calculus of section 2 to study the obtained solution operator of nonlocal problem. Theorem 1 gives the necessary and sufficient conditions for the existence and uniqueness of mild solution to (1), (4). Corollaries 1 and 2 concern the existence of strong solution and the well-posedness of the given problem. Apart from the simple cases, the mentioned in theorem 1 conditions on $\alpha_k, t_k$ can be verified only for a specified set of these nonlocal parameters. In section 4 we further adopt...
the technique from [11], which, when suited with the properly chosen conformal mapping (adjusted to the spectral parameters $\rho, \sigma$), permits us reduce the question of solution’s existence to the question about the location of roots for a certain polynomial associated with the nonlocal condition. This, in turn, enables us to finally get conditions for the existence and uniqueness of the solution to (1), (4) stated in terms of the constrains on $t_k, \alpha_k$ (theorems [2] [3]). In the end we compare newly derived conditions against (5), using the three-point nonlocal problem as a model example.

2 Functional calculus of strip-type operators

With intent to study problem (1), (4) in a Banach space setting, in this section, we review holomorphic functional for operators with the spectrum in a horizontal strip [6]. A densely defined closed linear operator $H$ with the domain $D(H) \subseteq X$, whose spectrum
\[ \Sigma_d = \{ z = x + iy | x, y \in \mathbb{R}, |y| \leq d \}, \]
and the resolvent $R(z, H) = (zI - H)^{-1}$:
\[ \|R(z, H)\| \leq \frac{M}{|\Im z| - d}, \quad z \in \Omega \setminus \Sigma, \Sigma \subset \Omega, \]
is called a strip-type operator of height $d > 0$. Next we define the rule to interpret operator functions. Let $f(z)$ be a complex valued function analytic in the neighborhood $\Omega$ of the spectrum $\Sigma(H) \subset \mathbb{C}$ and $|f(z)| < c_f (1 + |z|)^{-1-\delta}$, for $\delta > 0$. Suppose that there exists a closed set $\Phi \subset \Omega$ with the boundary $\Gamma$ consisting of a finite number of rectifying Jourdan curves, then the operator function $f(H)$ can be defined as follows
\[ f(H)x = \frac{1}{2\pi i} \int_{\Gamma} f(z)R(z, H)x dz. \]

This formula yields an algebra homomorphism between the mentioned class of holomorphic functions and the algebra of bounded operators on $X$. Besides, any two valid functions of the same operator commute. Unfortunately, Dunford-Cauchy integral (8) can not be used straight away to define the propagator, because $|e^{-iz}|$ will not vanish as $z \to \infty$ on $\Gamma$. Assume that there exists a so-called regularizer function $\varepsilon(z)$ such, that both $e(H)$ and $ef(H)$ are well defined in terms of (8).
and \( e(H) \) is injective. Then the formula

\[
f(A) = e^{-1}(H)ef(H)
\]

is used to define \( f(H) \) for a class of functions wider than the natural function calculus defined by (8) alone. By setting \( e(z) = (\lambda - z)^{-1-\delta} \) with \( |\Im \lambda| > d \) we ensure that \( f(H)x \) is well defined and bounded, whenever \( f(z) \) is bounded in \( \Omega \) and \( e^{-1}(H)x \) exists. In other words, the propagator \( U(t) \) is bounded linear operator with the domain \( x \in D(H^{1+\delta}) \). Using the closed graph theorem \( U(t) \), \( t \in \mathbb{R} \) can be extended to the bounded operator on \( X \) when \( D(H^{1+\delta}) \) is dense.

### 3 Reduction of nonlocal problem to classical Cauchy problem

We depart from the general solution formula (9), with \( t_0 = 0 \) and \( \psi(0) \) supplied by (4)

\[
\psi(t) = U(t) \left( \psi_1 - \sum_{k=1}^{n} \alpha_k \psi(t_k) \right) + \int_0^t U(t-s)v(s)ds,
\]

that is valid for the strip-type operator \( H \) under assumptions of section 2. To get the exact representation for \( \psi(t) \) one needs to factor out the unknown \( \psi(t_k), k = 1, n \) from the above formula. We define \( w = \sum_{k=1}^{n} \alpha_k \psi(t_k) \) and then formally evaluate this expression using (10) as a representation for \( \psi(t) \). It leads to the equation

\[
w = -\sum_{i=1}^{n} \alpha_i U(t_i)w + \sum_{i=1}^{n} \alpha_i U(t_i)\psi_1 + \sum_{i=1}^{n} \alpha_i \int_0^{t_i} U(t_i-s)v(s)ds.
\]

By denoting \( B = I + \sum_{i=1}^{n} \alpha_i U(t_i) \) we rewrite this equation as follows

\[
Bw = B\psi_1 - \psi_1 + \sum_{i=1}^{n} \alpha_i \int_0^{t_i} U(t_i-s)v(s)ds.
\]

At this point it is clear that equation (11) can be solved for \( w \) with any right-hand side if and only if the operator function \( B \) posses an inverse \( B^{-1} \). In such
case the substitution

\[ w = \psi_1 - B^{-1} \psi_1 + B^{-1} \sum_{i=1}^{n} \alpha_i \int_{0}^{t_i} U(t_i - s)v(s)ds \]

into (10) yields a representation of the general (mild) solution to nonlocal problem (1), (4)

\[ \psi(t) = U(t) \left( B^{-1} \psi_1 - B^{-1} \sum_{i=1}^{n} \alpha_i \int_{0}^{t_i} U(t_i - s)v(s)ds \right) + \int_{0}^{t} U(t - s)v(s)ds. \quad (12) \]

Now we can formalize our previous analysis as a theorem.

**Theorem 1.** Let \( H \) be a closed linear operator with the spectrum \( \Sigma \) contained in strip (6) and the domain \( D(H^\delta) \) is dense in \( X \) for some \( \delta > 1 \). The mild solution of nonlocal problem (1), (4) exists for any \( \psi_1 \in X, v \in L^1((0; T), X) \) and is equivalent to the solution of classical initial value problem (1), (2) represented by (3), with

\[ \psi_0 = B^{-1} \psi_1 - B^{-1} \sum_{i=1}^{n} \alpha_i \int_{0}^{t_i} U(t_i - s)v(s)ds, \quad (13) \]

if and only if all the zeros of entire function \( b(z) \) associated with (4),

\[ b(z) = 1 + \sum_{k=1}^{n} \alpha_k e^{-it_kz}, \quad (14) \]

are contained in the interior of the set \( \mathbb{C} \setminus \Sigma \).

**Proof.** To prove the necessity we simply need to look at (4), with \( v(t) = 0 \)

\[ B\psi(0) = \psi_1. \]

By definition the operator \( B \) is compact, as a finite sum of compact operators. Suppose that the function \( b(z) \) has a root \( z_0 \in \Sigma \) which belongs to the point spectrum of \( H \), with \( \phi \neq 0 \) being the corresponding eigenstate. Now, if \( \psi_1 \to \phi \) via some sequence of states from \( D(H^\delta) \) (which always exists since the domain of \( H^\delta \) is dense in \( X \)), such that \( \lambda \phi \) does not belong to this sequence, \( \forall \lambda \in \mathbb{R} \). Due to the
compactness the sequence of corresponding $\psi(0)$ should have a finite limit. To show the opposite we evaluate $B\varphi$ via the Dunford-Cauchy integral

$$B\varphi = \frac{1}{2\pi i} \int_\Gamma \frac{b(z)}{z-z_0} R(z,H) \varphi dz$$

and then use the general inequality $\|B^{-1}\| \geq \frac{1}{\|B\|}$ to get

$$\lim_{\psi_1 \to \varphi} \|B^{-1}\| \geq \lim_{\psi_1 \to \varphi} \frac{1}{\|B\|} = \infty.$$ 

Next we prove the sufficiency. Assume that all the zeros of $b(z)$ belong to the interior of $\mathbb{C}\setminus\Sigma$. Using the operator function calculus from section 2 we define

$$B^{-1} \varphi = \frac{1}{2\pi i} \int_\Gamma \frac{1}{b(z)} R(z,H) \varphi dz,$$

for any $\varphi \in X$ in a sense discussed in section 2. The contour $\Gamma$ satisfying the requirements of (8) always exists, since $1/b(z)$ is holomorphic in the neighborhood of $\Sigma$. Formula (15) and Lemma 5.2 from [4] guaranty that the initial state $\psi_0$ given by (13) and $\psi(t)$ (12) are well-defined for any $v(t)$, $\psi_1$ fulfilling the theorem’s assumptions. Formula (12) is nothing but the general solution formula (3) with $\psi(0)$ represented by (13). This proves the equivalence of nonlocal and Cauchy problems. The validity of nonlocal condition (4) is verified by the direct substitution of (3) into (4).

The correspondence between the solution of nonlocal problem and the solution of the classical problem permits us to establish other important properties of (1), (4).

**Corollary 1.** Assume that in addition to the requirements of theorem 1 on $H$, both $b(z)$, $\psi_1$ belong to $D(H)$ and either one of two following conditions is satisfied:

a) $v(t) \in D(H)$ and $v(t)$, $Hv(t)$ are continuous on $[0,T]$, or

b) $v(t)$ is continuously differentiable on $[0,T]$.

Then (12) is a strong (genuine) solution of nonlocal problem (1), (4).
Proof. We proceed by reducing the proof to the corresponding results on the genuine solution of classical Cauchy problem \cite[Lemma 5.1]{4}. In order to achieve that it is enough to show that the theorem’s assumptions imply $\psi_0 \in D(H)$ or, which is the same, that $H\psi_0$ is well defined. Depart from (13) and use the properties of function calculus described above:

$$H\psi_0 = HB^{-1}\left(\psi_1 - \sum_{i=1}^{n} \alpha_i \int_{0}^{t_{i_2}} U(t_{i_1} - s)v(s)ds\right)$$

$$= B^{-1}H\psi_1 - B^{-1}\sum_{i=1}^{n} \alpha_i \int_{0}^{t_{i_2}} U(t_{i_1} - s)Hv(s)ds.$$  

The first term in the last formula is well defined because $\psi_1 \in D(H)$ and there always exists a sequence of states from $D(H^\delta)$ with $\psi_1$ as a limit, such that $B^{-1}H$ is bounded on the elements of that sequence. By the same token we can show the well-definiteness of the second term under assumption that a) is true. The case of b) as well as the rest of the proof literally repeats the proof of the mentioned Lemma 5.1 from \cite{4}, and thus will be omitted here.

The conditions necessary for the existence of the strong solution are closely related to the well-posedness of (1), (4). The evolution problem is called uniformly well-posed in $t \in T$ \cite[Section 1.2]{4} if and only if the strong solution exists for a dense subspace of the initial data and the solution operator is uniformly bounded in $t \in T$ on the compact subsets of $T$.

**Corollary 2.** Let $H$ be an operator satisfying the assumptions of Theorem \cite{7}. The solution of nonlocal problem (1), (4) is uniformly well-posed in $t \in \mathbb{R}$ for any $t_k$, $\alpha_k \in \mathbb{C}$ if and only if all the zeros of $b(z)$ defined by (14) are separated from $\Sigma$.

**Proof.** In corollary \cite{1} we’ve already identified the dense subset $D(H)$ of $X$ such that for any $\psi_1 \in D(H)$ there exists a genuine solution of (1), (4). Assumptions on the parameters of nonlocal condition imply the boundedness of $B^{-1}$. In section \cite{2} we showed that $U(t)$ is bounded, as well, thus the solution operator from (12) is bounded. To conclude the proof it is enough to point out that the propagator $U(t)$ forms the group for $t \in \mathbb{R}$, therefore the bounded solution operator is also uniformly bounded \cite[Theorem 2.1]{4}.

**Example 1.** Let us consider the two point version of nonlocal problem (1), (4). In such simple case (4) takes the form

$$u(0) + \alpha_1 u(t_1) = u_0, \quad 0 < t_1 \leq T.$$  (16)
Here we assume that $H$ has all the properties mentioned in theorem. To find out the location of zeros of $b(z)$ we need to solve the equation

$$1 + \alpha_1 e^{-z_{it_1}} = 0,$$

assuming that $\alpha_1 \in \mathbb{C}$, $t_1$ are given. It has an infinite number of solutions $z_m$

$$z_m = -\frac{1}{it_1} \ln \left( -\frac{1}{\alpha_1} \right) = -\frac{1}{t_1} \left[ \text{Arg} \left( \frac{1}{\alpha_1} \right) + 2\pi m + i\ln \left| \frac{1}{\alpha_1} \right| \right], \quad m \in \mathbb{Z} \quad (17)$$

Here $\text{Arg}(\cdot)$ stands for the principal value of argument. The zeros $z_m$ are situated on the line, where the imaginary part $\Im z = \ln |1/\alpha_1|/t_1$ is constant. They will belong to $\mathbb{C} \setminus \Sigma$ if $|\Im z|$ is greater than spectral height $d$ introduced by (6). Consequently, the solution of (1), (16) exists if and only if

$$|\alpha_1| < e^{-t_1d}, \quad \text{or} \quad |\alpha_1| > e^{t_1d}. \quad (18)$$

Given nonlocal problem is well-defined for any $\alpha_1 \in \mathbb{C}$, except the values lying in the annulus $e^{-t_1d} \leq |\alpha_1| \leq e^{t_1d}$.

It is important to note that constraints (18) enforce $|\alpha_1| \neq 1$. That requirement can be relaxed for some $\psi_1, v(t)$ if the spectrum of $H$ is disjoint in the neighborhood of $\mathbb{R}$. Another unique feature of the two-point problem (1), (16) expressed by ones ability to write the closed-form solution (17) without specifying $\alpha_1$ beforehand. It becomes impossible for the general case of multi-point nonlocal condition (4), where one must rely on the numerical procedures to solve $b(z) = 0$ and for that reason predefine the parameters of nonlocal condition. For many applications of (1), (4) with $n > 1$ this is not enough as one still would like to have some a priori information about the admissible set of $\alpha_k$ rather than simply check the existence of solution for a fixed sequence $\alpha_k, k = 1, n$.

4 Zeros of $b(z)$ and associated problem for polynomials

To find a way around the direct solution of $b(z) = 0, n > 1$ we start with a general observation suggested by Example. The function $b(z)$ can be arbitrary closely
approximated by a periodic function $b^*(z) \equiv 1 + \sum_{k=1}^n \alpha_k e^{(-i	au_k z)}$, where $\tau_k$ is the rational approximation to the corresponding real number $t_k$, $k = \overline{1,n}$. The function $b^*(z)$ better suits our needs than $b(z)$ because the equation $b^*(z) = 0$ can always be reduced to the polynomial root finding problem. Let

$$t_k = \frac{\lambda_k}{\mu_k}, \quad \lambda_k \in \mathbb{Z}, \quad \mu_k \in \mathbb{N},$$

set $c_k = \frac{Q_k}{\mu_k}$, where

$$Q = \frac{\text{LCM}(\mu_1, \mu_2, \ldots, \mu_n)}{\text{GCD}(\lambda_1, \lambda_2, \ldots, \lambda_n)}$$

is the ratio of the least common multiple (LCM) and the greatest common divisor (GCD) of the numerators and denominators of $t_k$ correspondingly. A substitution

$$\Phi : u = \exp(-iz/Q)$$

(19)

turns the original problem about the location of zeros of $b(z)$ in $\mathbb{C} \setminus \Sigma$ to the problem about the location of zeros of a polynomial

$$r(u) = 1 + \sum_{k=1}^n \alpha_k u^{c_k}$$

(20)

in the exterior of an annulus

$$\Upsilon : e^{-d/Q} \leq |u| \leq e^{d/Q}, \quad u \in \mathbb{C}.$$ 

The polynomial root finding problem for $r(u) = 0$ is extensively studied (see [12, 13], as well as [16, 7]). Polynomial $r(u)$ has exactly $c_n$ roots $u_k$ over $\mathbb{C}$. Their closed form representation exists for $c_n \leq 4$. So now, we, technically, can write the exact solvability conditions for (1), (4) in terms of $\alpha_k$ for $k$ up to 4. More importantly, it is possible to avoid the full solution of $r(u) = 0$ altogether whilst checking $u_k \in \mathbb{C} \setminus \Upsilon$:

$$|u_k| < e^{-d/Q} \lor |u_k| > e^{d/Q}, \quad k = \overline{1,c_n}. \quad (21)$$

The shape of $\Upsilon$ suggest us to focus on the subclass of available root finding methods with results stated in the form of bounds (21). Among those, we choose to list here tree effective complex root bounds (see [2, 15, 17] for the discussion and comparisons) and ordered them by the increase of computational complexity. Each of the following bounds has been reformulated as a double estimate to better fit (21).
Lemma 1. ([12], Theorem 2.4) The zeros of $P(u)$ satisfy the inequalities

$$|u| \leq \left( 1 + \left( \frac{M_s}{|a_N|} \right)^q \right)^{1/q}, \quad |u| \geq \frac{|a_0|}{(|a_0| + M_s^q)^{1/q}},$$

$$M_s = \left( \sum_{k=1}^{N} |a_k|^s \right)^{1/s}, \quad s, q \in \mathbb{R}_{>1}, \quad \frac{1}{s} + \frac{1}{q} = 1.$$

Next estimate is due to M. Fujiwara [5]. It is the nearly optimal homogeneous bound in the space of polynomials [2]:

Lemma 2. All zeros of $P(u)$ satisfy the inequalities

$$|u| \leq 2 \max \left\{ \frac{|a_0|}{2d_N} \left| \frac{1}{|a_N|} \right|^{1/N}, \left| \frac{a_1}{a_N} \right|^{1/(N-1)}, \ldots, \left| \frac{a_{N-1}}{a_N} \right| \right\},$$

$$|u| \geq \frac{1}{2} \min \left\{ \frac{2d_N}{a_0} \left| \frac{1}{|a_N|} \right|^{1/N}, \left| \frac{a_1}{a_1} \right|^{1/(N-1)}, \ldots, \left| \frac{a_{N-1}}{a_{N-1}} \right| \right\},$$

here $1/0 = +\infty$.

The fourth estimate, originally proved by H. Linden, gives bounds on the real and imaginary part of zeros separately. It has been adapted in [17] to fit within the framework studied here.

Lemma 3. All zeros of $P(u)$ satisfy the double estimate $\max\{V_1^{-1}, V_2^{-1}\} \leq |u| \leq \min\{V_1', V_2'\}$, where

$$V_1 = \cos \frac{\pi}{N+1} + \frac{|\alpha_N|}{2|\alpha_0|} \left( \left| \frac{\alpha_1}{\alpha_N} \right| + \sqrt{1 + \sum_{k=1}^{N-1} \left| \frac{\alpha_k}{\alpha_N} \right|^2} \right)$$

$$V_2 = \frac{1}{2} \left( \left| \frac{\alpha_1}{\alpha_0} \right| + \cos \frac{\pi}{N} \right)$$

$$+ \frac{1}{2} \left[ \left( \left| \frac{\alpha_1}{\alpha_0} \right| - \cos \frac{\pi}{N} \right)^2 + \left( 1 + \left| \frac{\alpha_N}{\alpha_0} \right| \sqrt{1 + \sum_{k=2}^{N-1} \left| \frac{\alpha_k}{\alpha_N} \right|^2} \right) \right]^{1/2}$$

and $V_i'$ is obtained from $V_i$ by the substitution $a_k = a_{N-k}, k = 0, N, i = 1, 2.$
Theorem 2. Suppose that operator $H$ from (1) satisfy the assumptions of theorem 7 and all $t_k$ in (4) are rational numbers. If at least one bound from Lemmas 1 - 3 for polynomial (20) induce (21), then the nonlocal problem (1), (4) has the following properties:

1. it is uniformly well-posed in $t \in \mathbb{R}$;
2. for any $\psi_1 \in X$, $v \in L^1((0; T), X)$ there exists mild solution (12) with the characteristics mentioned in theorem 7;
3. solution (12) will also be strong if $\psi, v(t)$ satisfy either of the requirements a) or b) from corollary 1.

Proof. If the zeros $u_k$ of (20) obey (21), their images
$$z_k = \Phi^{-1}(u_k) = Q[\text{Arg}(u_k) + 2\pi m + i \ln |u_k|],$$
are clearly in the interior of $\mathbb{C} \setminus \Sigma$ no matter what is the value of $m \in \mathbb{Z}$. The application of theorem 7 and corollaries 1 concludes the proof.

The result of theorem 2 can be turned into criteria by enforcing the necessary and sufficient conditions for the validity of (21) derived via the Schur-Cohn algorithm [7, p. 493]. For a given polynomial $r(u)$ the algorithm produces a set of up to $2c_n$ inequalities, polynomial in $\alpha_k$, $k = 1, n$ which needs to be valid simultaneously in order for the Schur-Cohn test to pass [7, Thm. 6.8b]. The precise result is stated as follows.

Theorem 3. Suppose that operator $H$ from (1) satisfy the assumptions of theorem 7 and all $t_k$ in (4) are rational numbers. Nonlocal problem (1), (4) has properties 1-3 of theorem 2 if and only if the polynomials $b(e^{d/Q}u)$, $u^{c_n}b(e^{-d/Q}u)$ pass the Schur-Cohn test for the given set of parameters $\alpha_k \in \mathbb{C}$, $k = 1, n$ from (4).

Proof. The substitution $u = e^{d/Q}u'$ ($u = u'^{c_n}b(e^{-d/Q}u)$) transforms right (left) inequality from (21) into $u_k' > 1$. In both cases the validity of the last inequality is checked by the Schur-Cohn test [7, Thm. 6.8b]. "If" clause of Theorem 1 along with corollaries 1 assure the sufficiency. Mapping $\Phi$ is a bijection of the vertical strip $|z| \leq \pi GCD(\mu_1, \mu_2, ..., \mu_n) \over LCM(\lambda_1, \lambda_2, ..., \lambda_n)$ onto $\mathbb{C}$. The strip’s height equals to the period of $b(z)$. This fact guaranty the necessity via application of ”only if” clause of Theorem 1 and corollaries 1.
It remains to study the question: what happens when some of $t_k$ are irrational? Consider an approximation $b^*(z)$ of $b(z)$ mentioned above. If $t^*_k \to t_k$, $k = \overline{1,n}$ the function $b^*(z)$ uniformly converges to $b(z)$ on the compact subsets of the open set containing $\Sigma$. Hurwitz theorem [7 Corollary 4.10f] provides the means to claim that all zeros of $b(z)$ lies in the interior of $\mathbb{C} \setminus \Sigma$, if that is true for $b^*(z) : t^*_k \to t_k$. The degree of a polynomial $r^*(u)$ corresponding to $b^*(z)$ grows to $\infty$ when $t^*_k \to t_k$. But, its coefficients $\alpha_k$ are not affected by the increase of $c^*_n$. This keeps the root estimates from lemmas 1-3 meaningful. We’ve conducted the proof of the following.

**Corollary 3.** Assume that for every $k = 1, \ldots, n$ the sequence of rational numbers $\{t^*_l\}_{l=1}^\infty$ is such that $\lim_{l \to \infty} t^*_l = t_k$. If the conditions of theorems 2, 3 against the previously known condition (5). The rest of theorem’s statement remains valid for $t_k \in \mathbb{R}$.

We would like to compare the conditions on $\alpha_1, \alpha_2 \in \mathbb{R}$ obtained with help of theorems 2, 3 against the three point nonlocal condition

$$u(0) + \alpha_1 u(t_1) + \alpha_2 u(t_2) = 0, \quad t_1, t_2 > 0.$$ \tag{22}

For simplicity we set $t_1 = 1$, $t_2 = 2$ and consider the non-zero spectral height $d = \pi/40$. Then, the equation $b(z) = 0$ is reduced to $1 + \alpha_1 u + \alpha_2 u^2 = 0$. As shown in Fig. 7b, the exact conditions on $\alpha_1, \alpha_2$ calculated by theorem 3

$$\begin{cases}
|a_2|^2 < e^{4d}, \\
 e^{4d}|a_1|^2|a_2|^2 - e^{6d}|a_2|^4 - 2e^{4d}|a_2|(|a_1|^2 - |a_2|) + |a_1|^2 < e^{-2d} \\
|a_2|^2 > e^{4d}, \\
 e^{-4d}|a_1|^2|a_2|^2 - e^{-6d}|a_2|^4 - 2e^{-2d}|a_2|(|a_1|^2 - |a_2|) + |a_1|^2 > e^{2d}
\end{cases} \tag{23}
$$

lead to a considerably wider class of admissible pairs $(\alpha_1, \alpha_2)$ than those obtained by (5). In fact, the second system of inequalities from (23) gives rise to the unbounded region (union of two unbounded sets depicted in Fig. 7b) in the space of parameters $\alpha_1, \alpha_2$, meanwhile the solutions of (5) are strictly bounded in $|\alpha_1|, |\alpha_2|$ (the interior of the rhombic region in Fig. 7b). They lay within the isosceles triangle which acts as graphical solution of the first system of inequalities in (23). The gap between this triangle and the two other regions containing the solutions of (23) shortens when $d \to 0$, and in the limit is described by
\[ \alpha_1, \alpha_2 \in \mathbb{R} \] from (4) where problem (1), (4) is well-posed, \( d = \pi/40, t_1 = 1, t_2 = 2 \) (color on-line).

a) Application of theorem 2 and root estimates from: lemma 1 with \( s = q = 2 \) – dark gray (red); lemma 2 – middle gray (green); lemma 3 – light gray; b) The complete set of feasible \( (\alpha_1, \alpha_2) \) via the application of theorem 3 – gray, and set of pairs based on the estimate (5) – dark grey (magenta).

Comparison of generalized Ashyraliev condition (5) and the sufficient conditions provided by theorem 2 (depicted in Fig. 1 b) unveils that (5) performs better than the inner circle estimates of lemmas 1–3. Therefore, when it comes to the a priori estimates on the parameters of nonlocal condition, we advice to use the combination of (5) and the part of theorem 2 which imply \( |u_k| > e^{d/Q} \).

Conclusions

We established exact dependence of the solution to (1), (4) on the parameters of nonlocal condition, derived the well-posedness criteria and proved the theorems regarding existence of mild(strong) solution. The obtained solution’s existence conditions generalize other available results [1, 14] to the case of periodic problem and beyond. Our method of analysis remains adequate upon a non-self-adjoint perturbation of the equation’s Hamiltonian, which may be due to the discretization of self-adjoint \( H \) by a nonconservative numerical scheme, for instance. Future research will be focused on the extension of the proposed technique to other linear nonlocal problems.
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