This paper studies angle-based sensor network localization (ASNL) in a plane, which is to determine locations of all sensors in a sensor network, given locations of partial sensors (called anchors) and angle constraints based on bearings measured in the local coordinate frame of each sensor. We firstly show that a framework with a non-degenerate bilateration ordering must be angle fixable, which implies that it can be uniquely determined by angles between edges up to translation, rotation, scaling, and reflection. Then we prove that an ASNL problem has a unique solution if and only if its grounded framework is angle fixable and anchors are not all collinear. Subsequently, ASNL is solved under centralized and distributed approaches, respectively. The noise-free centralized ASNL is formulated as a rank-constrained optimization problem, and shown to be equivalent to a linear semi-definite program (SDP) when the grounded framework is acute-triangulated. In order to deal with large and sparse ASNL, a decomposition strategy is proposed to recast the centralized ASNL as an SDP with semi-definite constraints on multiple submatrices in reduced sizes. The centralized ASNL considering noise is investigated via a maximum likelihood formulation, which can be formulated as a rank-constrained SDP with multiple semi-definite constraints. A distributed protocol based on inter-sensor communications is also proposed, which solves ASNL in finite time when the grounded framework has a non-degenerate bilateration ordering. Finally, simulation examples are provided to validate effectiveness of our theoretical analysis.

Index Terms
Network localization, angle rigidity, rank-constrained optimization, non-convex optimization, chordal decomposition

I. INTRODUCTION

Localization of wireless sensor networks is to seek locations of all sensors when locations of partial sensors (called anchors) and relative measurements between some pairs of sensors are available. It has been intensively investigated since the location information of sensors is frequently required in many scenarios, e.g., fusion of sensor measurements according to locations, searching sensors in specified areas, and tracking a moving target [1], [2], [3].

In the literature, depending on sensing capabilities of sensors, the sensor network localization (SNL) problems have been studied via relative position-based [4], range-based [1], [5], [6], [7], [8], [9], [10] and bearing (angle of arrival)-based [11], [12], [13], [14], [15], [16], [17] approaches. Among them, bearing-based SNL is a popular topic in recent years due to the advantage that bearing measurements can be easily obtained by vision sensors [15]. Nevertheless, in bearing-based SNL, usually each sensor should be able to obtain bearings with respect to the global coordinate frame, which can only be realized by either equipping each sensor with high cost devices (e.g., GPS, compass) [12], [13] or developing coordinate frame alignment algorithms [16], [17] for sensors via inter-sensor communications. Although the authors in [14] proposed an algorithm based on bearings measured in local coordinate frames, the sensing graph required for solvability of SNL should have more edges compared to the one in localization via global bearing measurements (e.g., [12], [13]). In addition, a lot of efforts have been carried out on range-based SNL, in which range measurements are independent of the global coordinate frame. Unfortunately, usually the graph condition for solvability of range-based SNL is more stringent than that for solvability of bearing-based SNL [1], [7], and range measurements are more costly [3].

In SNL problems, it is important to distinguish what kind of sensor network is localizable given available anchor locations and measurements captured by sensors. This problem is usually tackled by checking whether the shape of the grounded graph can be uniquely determined by constraints on measurements. In bearing-based and range-based SNL, bearing rigidity theory [15] and distance rigidity theory [18] are employed to give conditions for localizability, respectively. In [19], the authors proposed an angle-based shape determination approach (namely, angle rigidity theory), where the number of edges required for shape determination is the same as that for infinitesimal bearing rigidity, which is necessary for localizability of bearing-based SNL. We notice that an angle between two edges joining one sensor can always be computed by bearings corresponding to these two edges measured in the local coordinate system of the sensor, hence is independent of the global coordinate frame. In practice, bearing (angle) measurements in a local coordinate system are usually low cost, reliable, and can be captured easily by vision sensors (e.g., monocular pinhole cameras [21]). In recent years, formation control via angle constraints has attracted growing interests [20], [21], [22], [19], [23], [24]. Nevertheless, the application of angle constraints to SNL has not been fully explored. We hence investigate the topic in this paper. More specifically, given locations of anchors and bearings measured in the local coordinate system of each sensor, we use angle constraints to determine locations of the sensors other than anchors. SNL based on angle constraints is named angle-based sensor network localization (ASNL). The main advantages of ASNL are...
two-fold: (i) Compared to bearing-based SNL, each sensor does not need bearing information in the global coordinate frame; (ii) Compared to the range-based SNL, the graphical condition for ASNL to have a unique solution is milder.

In this paper, we propose the concept of angle fixability based on angle rigidity theory in [19], [23] to characterize the property of a network that can be determined by angles uniquely up to translations, rotations, scalings and reflections. By establishing connections between angle fixability and angle localizability, the results on angle fixability are applied to ASNL problems. ASNL will be studied in centralized and distributed frameworks, respectively. A preliminary version of the centralized case has been presented in [25]. This paper expands on the work in the conference paper by presenting chordal decomposition, noisy ASNL, distributed ASNL and detailed proofs absent in [25]. In both centralized and distributed frameworks, each sensor is considered to be capable of measuring relative bearing measurements from neighbors with respect to its own local coordinate system. In the centralized framework, all sensors are able to transmit its measured information to a common central unit, while communications between different sensors are not required. For the central unit, the ASNL problem is modeled as a rank-constrained semi-definite programming problem that has been widely studied in the literature [26], [10], [27], [28]. Since a rank-constrained optimization problem is generally NP-hard, we further give a graph condition for removing the rank constraint by exploiting particular properties of ASNL. Although SDP with or without rank constraints have been solved in existing literature, it is time consuming to solve an optimization problem subject to a semi-definite constraint, especially for large scale SDPs. To overcome this challenge, we exploit the sparsity of the formulated SDP problem when the network consists of a large amount of sensor nodes. Under certain graph conditions, we further propose a decomposition approach for ASNL. Centralized ASNL in a noisy environment is also studied via a maximum likelihood formulation. In the distributed framework, there is no central unit required to receive information from all sensors. However, to cooperatively solve the ASNL problem, neighboring sensors should be able to communicate with each other.

Our main contributions are summarized as follows:

- We propose equivalent algebraic conditions (Lemmas [2], [3]) and a sufficient graphical condition (Theorem [1]) for angle fixability in a plane. The graphical condition is milder than the one proposed in [19], and it implicitly contains an approach to constructing angle fixable frameworks (Definition [4]), which can also be used to construct angle localizable sensor networks. We show that the angle fixability property of a framework with a non-degenerate bilateration ordering is invariant to space dimensions, see Lemma [8].
- We formulate the ASNL problem as a quadratically constrained quadratic program (QCQP) problem which is equivalent to a rank-constrained optimization problem. Based on analysis of angle fixability, it is shown that if a grounded framework has a non-degenerate bilateration ordering and anchors are not all collinear, then ASNL has a unique solution (Lemma [10]). Furthermore, if the grounded framework is acute-triangulated, then ASNL is equivalent to a linear semi-definite program (SDP), which can be solved in polynomial time; see Theorem [4].
- To handle large scale ASNL problems, we formulate ASNL as an SDP with two unknown matrices (problem [8]). When the grounded graph has a bilateration ordering, the first unknown matrix can be decomposed via chordal decomposition (Theorem [6]); when the grounded framework is acute-triangulated, the second unknown matrix can also be decomposed into multiple submatrices in reduced sizes according to maximal cliques in a sparsity pattern (Theorem [7]).
- In a noisy environment, from the maximum likelihood estimation perspective, we model ASNL as an SDP with multiple rank-1 constraints and semi-definite constraints, which can be solved by algorithms in [27], [28].
- Considering communications between adjacent sensors, we propose a distributed protocol (Protocol 1) which solves ASNL with guaranteed convergence if the grounded framework has a non-degenerate bilateration ordering. We further prove that the upper bound of the convergence steps is the number of sensors to be localized.

The outline of this paper is as follows. Section [II] provides preliminaries of angle rigidity theory and chordal decomposition. Section [III] introduces the concept of angle fixability and provides criteria for angle fixability and relevant properties. Section [IV] formulates ASNL as a QCQP and gives the necessary and sufficient conditions for ASNL to have a unique solution. Section [V] solves the noise-free and noisy ASNL using a centralized framework. Section [VI] proposes a distributed protocol via inter-sensor communications for ASNL. Section [VII] exhibits several simulation examples. The concluding remarks and future work are addressed in Section [VIII].

Notations: Throughout the paper, $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ denotes an undirected graph, where $\mathcal{V}$ and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ denote the vertex set and edge set, respectively. The neighbor set of each vertex $i$ is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. A $m \times n$ zero matrix is denoted by $0_{m \times n}$, where “$m \times n$” may be omitted if the dimension of the zero matrix can be observed. Given sets $A$ and $B$, $|A|$ is the cardinality of $A$, $A \setminus B$ is the set of elements in $A$ but not in $B$. The $d$-dimensional orthogonal group is written as $O(d)$. Given a matrix $X$, $\text{rank}(X)$ is the rank of $X$, $X \succeq 0$ implies that $X$ is positive semi-definite, $\det(X)$ denotes the determinant of $X$. A vector $p = (p_1^T, ..., p_p^T)\top$ is degenerate if $p_1, ..., p_p$ are collinear. We use $K$ to represent a complete graph with appropriate number of vertices, $I_d$ to denote the $d \times d$ identity matrix, $\otimes$ to denote the Kronecker product, $X_{a:b,c:d}$ is the submatrix of $X$ consisting of elements from $a$-th to $b$-th rows and $c$-th to $d$-th columns of $X$. 
II. PRELIMINARIES OF GRAPH THEORY

A. Angle Rigidity Theory

In [19], angle rigidity theory is proposed to study what kind of geometric shapes can be uniquely determined by angles subtended in the graph only. Similar to distance rigidity theory in range-based SNL and bearing rigidity theory in bearing-based SNL, angle rigidity theory plays an important role in solving ASNL. In [24], the authors presented their angle rigidity theory by taking the sign of each angle into account, which implies that all angles are defined in a common counterclockwise direction. Different from [24], the angle considered in this paper does not have a specific sign. As a result, different sensors are allowed to have different definitions about the rotational direction. In this subsection, we will briefly review several definitions regarding angle rigidity theory proposed in [19] that will be used later.

A graph $G = (V, E)$ with $|V| = n$ can be embedded in a plane by giving each vertex $i$ a position $p_i \in \mathbb{R}^2$. The vector $p = (p_1^T, \ldots, p_n^T)^T \in \mathbb{R}^{2n}$ is called a configuration. $(G, p)$ is called a framework. Each angle we use to determine the framework shape is an angle between two edges joining one common vertex, and the cosine of this angle will be constrained. For example, for the angle between edge $(i, j)$ and $(i, k)$, $g_{ij} g_{ik}$ will be constrained, where $g_{ij} = \frac{p_i - p_j}{||p_i - p_j||}$ is the bearing between vertices $i$ and $j$. The set of angle constraints in a graph $G$ can be denoted by \(\{g_{ij} g_{ik} = a_{ijk} : a_{ijk} \in [-1, 1], (i, j, k) \in T_G\}\), $T_G = \{(i, j, k) \in V^3 : (i, j), (i, k) \in E, j < k\}$. Let $\theta_{ijk}$ denote the angle between $p_i - p_j$ and $p_i - p_k$, when $a_{ijk}$ is given, we can obtain a unique $\theta_{ijk} = \arccos a_{ijk} \in [0, \pi]$. That is, each angle constraint actually constrains an angle within the range $[0, \pi]$. Similar settings are considered in [20], [21], [22], [23], [24]. Note that when an angle is defined under a specified counterclockwise direction, it should be within the range $[0, 2\pi)$ [24].

The angle rigidity function of a framework $(G, p)$ is defined as

$$f_G(p) = (\ldots, g_{ij}(p)g_{ik}(p), \ldots)^T, (i, j, k) \in T_G. \tag{1}$$

A framework $(G, p)$ is globally angle rigid if $f^{-1}_G(f_G(p)) = f^{-1}_K(f_K(p))$, it is infinitesimally angle rigid if all the infinitesimal angle motions are trivial. Here, the infinitesimal angle motion is a motion of the framework such that all angles in the framework (i.e., $f_G(p)$) are invariant, a motion is trivial if it is a combination of translations, rotations, and scalings. An alternative condition for infinitesimal angle rigidity in $\mathbb{R}^2$ is rank($\frac{\partial f}{\partial p}$) = $2n - 4$. Three examples are presented in Fig. 1 to illustrate these definitions. In Fig. 1 frameworks (a) and (e) are nonrigid. In framework (a), vertices 1, 2, 3 and 4 can move simultaneously to deform the shape while maintaining all subtended angles. In framework (e), vertices 4 and 5 can move freely along the line between 1 and 4 and the line between 2 and 5, respectively. Frameworks (b), (c) and (d) are globally and infinitesimally angle rigid because the angles in each framework are sufficient to determine the entire shape uniquely; In Fig. 1(f), since the graph is complete, the framework is globally angle rigid. It is not infinitesimally angle rigid because vertex 2 can move freely along the line between vertices 1 and 3.

B. Chordal Graphs and Chordal Decomposition

A graph is said to be chordal if each cycle with more than three vertices in this graph has a chord. Here a chord is an edge between two nonconsecutive vertices in the cycle. A clique $C$ of a graph $G = (V, E)$ is a subset of $V$ such that each pair of vertices in $C$ are adjacent. In Fig. 1 graphs (a) and (b) are not chordal, graphs (c)-(f) are all chordal. We say a clique $C$ is a
s-point clique if $|C| = s$. A clique $C$ is said to be a maximal clique if there is no other clique containing this clique. In Fig. 1 (c), there are 3 maximal cliques: $C_1 = \{1, 2, 3\}$, $C_2 = \{1, 3, 4\}$, $C_3 = \{2, 3, 5\}$. Given a maximal clique $C$, we define a matrix $Q_C \in \mathbb{R}^{n \times n}$ such that $Q_C\eta = (\eta_{C(1)}, \ldots, \eta_{C(|C|)})^\top$, where $\eta = (\eta_1, \ldots, \eta_n)^\top \in \mathbb{R}^n$, $C(i)$ denotes the $i$-th element of $C$. It is easy to see that an element of $Q_C$ is in the following form:

$$(Q_C)_{ij} = \begin{cases} 1, & j = C(i), \\ 0, & \text{otherwise}. \end{cases}$$  \hspace{1cm} (2)

The following lemma will be used in this paper.

Lemma 1: [29] Given $G = (V, E)$ as a chordal graph and a matrix $X \in \mathbb{R}^{|V| \times |V|}$, let $\{C_1, C_2, \ldots, C_p\}$ be the set of its maximal clique sets. Then, $X \succeq 0$ if and only if $Q_{C_k}XQ_{C_k}^\top \succeq 0$, $k = 1, \ldots, p$.

### III. Angle Fixability

To better understand what kind of geometric shapes can be uniquely determined by angles, we introduce the notion of angle fixability in this section. The formal definition of angle fixability is given below.

**Definition 1:** A framework $(G, p)$ is angle fixable in $\mathbb{R}^d$ if $f_G^{-1}(f_G(p)) = \mathcal{S}_p$, where

$$\mathcal{S}_p = \{q \in \mathbb{R}^d : q = c(I_n \otimes \mathcal{R})p + 1_n \otimes \xi, \mathcal{R} \in O(d), c \in \mathbb{R} \setminus \{0\}, \xi \in \mathbb{R}^d\}$$

is a $\left(\frac{d(d+1)}{2} + 1\right)$-dimensional manifold.

From Definition 1, we can see that the set $\mathcal{S}_p$ actually defines a set of configurations forming the same shape as the one formed by $p$. That is, if $q \in \mathcal{S}_p$, then $q$ can be obtained from $p$ by a combination of rotation, translation, scaling and reflection.

In this paper, we mainly focus on angle fixability in $\mathbb{R}^2$. Since angle fixability in $\mathbb{R}^d$ with $d \geq 3$ will also be used in particular cases, the general definition is given above.

#### A. Equivalent Conditions for Angle Fixability in $\mathbb{R}^2$

The following two lemmas give two necessary and sufficient conditions for angle fixability in $\mathbb{R}^2$.

**Lemma 2:** In $\mathbb{R}^2$, $(G, p)$ is angle fixable if and only if it is globally and infinitesimally angle rigid.

**Proof:** The sufficiency has been proven in [19], next we prove the necessity. Suppose that $(G, p)$ is angle fixable but not infinitesimally angle rigid, then $\text{rank}(f_G^{-1}(f_G(p))) = s < 2n - 4$. As a result, there exists a neighborhood of $p$ in which $f_G^{-1}(f_G(p))$ is a 2$n - s > 4$ dimensional manifold, this conflicts with the fact that $f_G^{-1}(f_G(p)) = \mathcal{S}_p$ is a 4-dimensional manifold. For global angle rigidity, since it always holds that $f_K^{-1}(f_K(p)) \subset f_G^{-1}(f_G(p))$, it suffices to prove $f_G^{-1}(f_G(p)) \subset f_K^{-1}(f_K(p))$. For any $q \in f_G^{-1}(f_G(p))$, we have $q \in \mathcal{S}_p$, then $g_{ij}(q)g_{ik}(q) = g_{ij}(p)g_{ik}(p)$ for all $i, j, k \in V$. That is, $(G, p)$ is globally angle rigid.

**Lemma 3:** In $\mathbb{R}^2$, $(G, p)$ is angle fixable if and only if it is globally angle rigid and $p$ is non-degenerate.

**Proof:** Since the configuration of an infinitesimally angle rigid framework can never be degenerate, the necessity is easy to obtain by Lemma 2. Next we prove sufficiency. Global angle rigidity implies that $f_G^{-1}(f_G(p)) = f_K^{-1}(f_K(p))$, hence we only have to show that there exists some subgraph $G'$ of $K$ such that $(G', p)$ is angle fixable. Without loss of generality, let 1, 2, 3 be three vertices not lying collinear. We start with the complete graph with vertices 1, 2, 3. It is easy to see that for any $4 \leq i \leq n$, there always exist two vertices $j, k \in \{1, 2, 3\}$ such that $p_i - p_j$ and $p_i - p_k$ are not collinear. By adding vertex $i$ and edges $(i, j), (i, k)$ for $i = 4, \ldots, n$ iteratively, we can obtain a graph $G'$ with a bilateration ordering. Moreover, at each step during the generation, the conditions in Lemma 7 are satisfied. Thus $(G', p)$ is angle fixable.

Combining Lemma 2 and Lemma 3, it is straightforward to obtain the following result.

**Lemma 4:** Consider a globally angle rigid framework $(G, p)$ in $\mathbb{R}^2$, the following statements are equivalent:

(i) $p$ is non-degenerate;

(ii) $(G, p)$ is infinitesimally angle rigid;

(iii) $(G, p)$ is angle fixable.

#### B. Generic Angle Fixability

In [19], the authors showed that both infinitesimal angle rigidity and global angle rigidity are generic properties of the graph. That is, given a graph $G$, either for all generic configurations $p \in \mathbb{R}^{2n}$, $(G, p)$ is infinitesimally (globally) angle rigid, or none of them is. Therefore, angle fixability in $\mathbb{R}^2$ is also a generic property of the graph. We give the following definition and result.

**Definition 2:** A graph $G$ is generically angle fixable in $\mathbb{R}^2$ if $(G, p)$ is angle fixable for any generic configuration $p \in \mathbb{R}^{2n}$.

**Lemma 5:** If $(G, p)$ is angle fixable for some configuration $p \in \mathbb{R}^{2n}$, then $G$ is generically angle fixable in $\mathbb{R}^2$.

$^1$A configuration $p = (p_0^\top, \ldots, p_n^\top)^\top \in \mathbb{R}^{2n}$ is generic if its $2n$ coordinates are algebraically independent [19].
We also note that all generic configurations in \( \mathbb{R}^2 \) form a dense space. Therefore, for a generically angle fixable graph \( G \), the set of all configurations \( p \in \mathbb{R}^{2n} \) such that \( (G, p) \) is not angle fixable is of measure zero. Lemma 5 together with Lemma 3 implies that for a framework with a generic configuration \( p \in \mathbb{R}^{2n} \), angle fixability and global angle rigidity are equivalent. We summarize this result in the following lemma.

**Lemma 6:** A graph \( G \) is generically angle fixable in \( \mathbb{R}^2 \) if and only if it is generically globally angle rigid in \( \mathbb{R}^2 \).

### C. Recognizing Angle Fixable Frameworks

In this subsection, we present a graphical approach to recognizing angle fixable frameworks. Before showing that, we firstly give the following result for angle fixable frameworks.

**Lemma 7:** Given an angle fixable framework in \( \mathbb{R}^2 \), after adding a node and two non-collinear edges connecting this node to two existing nodes, the induced framework is still angle fixable in \( \mathbb{R}^2 \).

**Proof:** Let \( (G, p) \) be the angle fixable framework with \( n \) vertices, \( n + 1 \) be the added node, \((n + 1,u) \) and \((n + 1, v) \) be the two added edges, \((G', p') \) be the induced framework. We only need to verify \( f_{G'}^{-1}(f_G(p')) = \mathcal{SP}_p' \) in order to prove that \((G', p') \) is still angle fixable. Since it always holds that \( \mathcal{SP}_p \subset f_{G'}^{-1}(f_G(p')) \), it suffices to show \( f_{G'}^{-1}(f_G(p')) \subset \mathcal{SP}_p' \).

For each \( q' \in f_{G'}^{-1}(f_G(p')) \), it is easy to see that \( q' = (q^T, q_{n+1}')^T \in \mathbb{R}^{2n+2} \), where \( q \in \mathcal{SP}_p \), \( q_{n+1}' \) satisfies \( g_i'^T g_{ij}'(q') = g_i'^T g_{ij}(q) \), \( i \in \{u, v\}, j \in N_i \). Next we show that \( u_{n+1} \) can be uniquely determined by \( q \) and \( f_G(p') \). Lemma 2 shows that \((G, p) \) is infinitesimally angle rigid. Then vertex \( u \) must have at least two neighbors \( j_1, j_2 \) such that \( p_u - p_{j_1} \) and \( p_u - p_{j_2} \) are not collinear. Denote \( A = (g_{uj_1}, g_{uj_2}) \in \mathbb{R}^{2 \times 2} \), then \( \text{rank}(A) = 2 \). Note that if we regard \( u_{n+1} = x = (x_1, x_2)^T \in \mathbb{R}^2 \) as unknown variables, we then have \( A' x = b \), where \( b = (g_u, g_{u_{n+1}}(p') g_{uj_1}(p'), g_{u_{n+1}}(p') g_{uj_2}(p')) \). Hence \( u_{n+1}(p') \) can be uniquely determined by \( q \) and \( f_G(p') \).

Similarly, \( g_{u_{n+1}} \) can be uniquely determined by \( q \) and \( f_G(p') \). Since \( u_{n+1} \) and \( g_{u_{n+1}} \) are not collinear, they have only one intersection point. As a result, \( q_{n+1}' \) can be uniquely determined. Note that there must exist \( \tilde{q} = (q^T, q_{n+1}'\tilde{g})^T \in \mathcal{SP}_p' \) such that \( \tilde{q} \in f_{G'}^{-1}(f_G(p')) \), we then have \( q' = \tilde{q} \in \mathcal{SP}_p' \).

In two-dimensional (2D) space, it is well known that any minimally rigid framework is embedded by a Laman graph \([30]\), which can be obtained by Henneberg constructions \([31]\). At each step of Henneberg construction, either one vertex and two new edges are added (named vertex addition), or one vertex and three new edges are added, while an existing edge is removed (named edge splitting). By Lemma 7 it is straightforward that the specified Henneberg vertex additions preserve angle fixability in 2D space. In \([8]\), a graph containing a subgraph induced by Henneberg vertex addition is said to have a **bilateral ordering**, whose definition is as follows.

**Definition 3:** (Bilateration Ordering) A graph \( G_b(n) \) with \( n \geq 3 \) vertices is said to have a bilateral ordering if it can be generated in the following procedure: Let \( G_b(3) \) be a complete graph with 3 vertices, \( G_b(i + 1) \) is obtained by adding one vertex to \( G_b(i) \) and at least two edges connecting this vertex to some existing vertices in graph \( G_b(i) \).

For frameworks generated by graphs with a bilateral ordering, we define the non-degenerate bilateral ordering as follows.

**Definition 4:** (Non-degenerate Bilateral Ordering) A framework \((G_b(n), p(n)) \) is said to have a **non-degenerate bilateral ordering** if it can be generated by the following procedure: Starting with the 3-vertex framework \((G_b(3), p(3)) \) where \( p(3) \) is non-degenerate, \((G_b(i + 1), p(i + 1)) \) is obtained from \((G_b(i), p(i)) \) by adding one vertex \( l \) and \( s \geq 2 \) edges connecting \( l \) to existing vertices \( l_1, \ldots, l_s \) such that \( p_l - p_{l_j} \), \( j \in \{1, \ldots, s\} \) are not all collinear.

Fig. 2 shows an example of the non-degenerate bilateral ordering. From Lemma 7 it is straightforward to have the following result.

**Theorem 1:** In \( \mathbb{R}^2 \), if a framework has a non-degenerate bilateral ordering, then it is angle fixable.

A strongly non-degenerate triangulated framework \((G_t, p) \) is a kind of framework with a non-degenerate bilateral ordering where at each step when a vertex \( l \) is added to \((G_t(i), p(i)) \), two non-collinear edges connecting \( l \) to \( j \) and \( k \) such that \((j, k) \in E_t \)
are added accordingly. In \cite{19}, (\(G_n, p\)) is shown to be angle fixable in \(\mathbb{R}^2\). One can realize that a strongly non-degenerate triangulated framework always has a non-degenerate bilateration ordering, but not vice versa. Fig. 1(b) shows a framework with a non-degenerate bilateration ordering, while it is not a triangulated framework, because vertices 3 and 5 are not adjacent.

By generic property of angle fixability, the following result holds.

**Corollary 1:** A graph with a bilateration ordering is generically angle fixable in \(\mathbb{R}^2\).

An interesting fact is that if we place the framework generated by Definition 4 in \(\mathbb{R}^2\) into a higher dimensional space, it will not lose its angle fixability property, which is stated in the following lemma.

**Lemma 8:** Given a framework \((G_n(n), p)\) with a non-degenerate bilateration ordering in \(\mathbb{R}^2\), for any integer \(d \geq 3\), \((G_n(n), \bar{p})\) is angle fixable in \(\mathbb{R}^d\), where \(\bar{p} = (\bar{p}_1, ..., \bar{p}_n)^T\), \(\bar{p}_i = (p_i^T, 0_1 \times (d-2))^T \in \mathbb{R}^d\).

**Proof:** We prove the result by showing that in \(\mathbb{R}^d\), every \(q \in f_{G_n(n)} f_{G_n(n)}(\bar{p})\) satisfies \(q \in \mathcal{S}_p\). Note that 1, 2 and 3 always form a non-degenerate triangle, and a triangle is always angle fixable in any dimensional space (The shape of a non-degenerate triangle constrained by angles is invariant to the dimension of the space). Hence, there must hold that \(q_i = c \bar{p}_i + \xi\) for some appropriate \(c, \mathcal{R}\) and \(\xi, i = 1, 2, 3\).

Suppose originally we have \((G_n(i), q(i))\), where \(q(i) \in \mathcal{S}_{p(i)}\) for \(c, \mathcal{R}\) and \(\xi\). Next, we prove that in the following Henneberg vertex addition introduced in Definition 4, the position of the new vertex to be added can be uniquely determined by angle constraints and positions of existing vertices. Let \(l\) be the added vertex, \((l, j)\) and \((l, k)\) be the two non-collinear added edges, \(j_1\) and \(j_2\) are two neighbors of \(j\) in \(G_n(i)\) and \(q_j - q_{j_1}\) is not collinear with \(q_j - q_{j_2}\). From the angle constraints involving \(j\), we have \(g_{ji}(q)g_{jj}(q) = c_{ji}, j\) and \(g_{ji}(q)g_{jj}(q) = c_{ji}, j\). Next we show \(g_{j}\) can be uniquely determined. Note that \(g_{ji}(q)g_{jj}(q) = g_{ji}(q)g_{jj}(p), g_{ji}(q)g_{jj}(q) = g_{ji}(q)g_{jj}(p), \bar{p}_s = (p_s^T, 0_1 \times (d-2))^T\) for \(s = j, j_1, j_2\). Denote \(\mathcal{R}^\top g_{ji}(q)\) by \(b = (b_1, ..., b_d)^T \in \mathbb{R}^d\), \(g_{ji}(p)\) by \(\eta = (\eta_1, \eta_2, 0_1 \times (d-2))^T\) and \(g_{jj}(p)\) by \(\eta = (\eta_1, \eta_2, 0_1 \times (d-2))^T\). Then we have \(\mathcal{R}^\top c = c_{ji}, \eta^\top b = c_{ji}\), which is equivalent to \(\begin{pmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = c_{ji}\). Since \(g_{jj}(p)\) and \(g_{jj}(p)\) are not collinear, \(b_1\) and \(b_2\) can be uniquely determined. It is easy to see \((b_1, b_2, 0_1 \times (d-2))^\top = q_i(p)\). Then \(b_1^2 + b_2^2 = 1\). Since \(|b| = 1\), we have \(b_s = 0, s = 3, ..., d\). Then \(g_{ji}(q) = \mathcal{R}^\top(p)\) is uniquely determined. Similarly, \(g_{ik}(q)\) is also unique. Recall that \(g_{j}g_{j}\) and \(g_{j}g_{j}\) are not collinear, they have a unique intersection point, i.e., \(q_i\). This completes the proof.

The above lemma implies that the angle fixability of a framework with a non-degenerate bilateration ordering in \(\mathbb{R}^2\) is invariant to space dimensions. However, it does not mean that any framework generated by Definition 4 in \(\mathbb{R}^d\) is angle fixable. It may be true but requires further rigorous proof, we leave it as a future work.

IV. ANGLE-BASED SENSOR NETWORK LOCALIZATION

A. Problem Formulation

1) **Sensor network modeling:** Given a network of sensors indexed by \(V = \{1, ..., n\} = A \cup S\), where \(A = \{1, ..., n_a\}\), \(S = \{n_a + 1, ..., n_a + n_s\}\), sensors in \(A\) are called anchors, while sensors in \(S\) are called unknown sensors. An undirected graph \(\mathcal{G} = (V, \mathcal{E})\) is used as the sensing graph interpreting the interaction relationships between sensors. Each sensor has the capability of sensing bearing measurements from other neighboring sensors \(j \in \mathcal{N}_i\). An ASNL problem in \(\mathbb{R}^d\) is to determine \(x_i, i \in \mathcal{S}\) when \(\{x_i \in \mathbb{R}^d : i \in A\}\) and all the angles between edges are available. In this paper, we will focus on the case with \(d = 2\).

Let \(x = (x_1, ..., x_n)^T\), the localization problem is to realize framework \((\mathcal{G}, x)\) with known inter-edge angles and coordinates of partial vertices (anchor sensors). Note that for any two anchors \(i, j\), even if \((i, j) \notin \mathcal{E}\), we can still obtain \((x_i - x_j)/||x_i - x_j||\) since we know the accurate values of \(x_i\) and \(x_j\). Therefore, it is more reasonable to consider realization of framework \((\mathcal{G}, x)\), with \(\mathcal{G} = (V, \mathcal{E})\), \(\mathcal{E} = \{i, j \in V : i, j \in A\}\) as the equivalent problem to SNL. In this paper, we call \((\mathcal{G}, x)\) the grounded framework, and use \(\mathcal{N} = (\mathcal{G}, x, A)\) to denote a sensor network to be localized.

A simple example of ASNL is shown in Fig. 3. It is easy to see that the position of sensor 4 can be uniquely determined by angles \(\angle A, \angle B\) and \(\angle C\), while the position of sensor 5 can be uniquely determined by angles \(\angle D, \angle E\) and \(\angle F\). Note that the framework in Fig. 3 is minimally infinitesimally bearing rigid and not globally distance rigid, while infinitesimal bearing rigidity and global distance rigidity are necessary for localizability of bearing-based SNL and range-based SNL, respectively. Therefore, the condition for localizability of ASNL is milder.

2) **Sensing capability:** In this paper, we consider that each sensor only has to sense relative bearing measurements from its immediate neighbors, which can be easily captured by vision-based sensors \cite{21}, \cite{15}. Moreover, we consider a GPS-denied environment, in which each sensor has an independent coordinate system. As a result, each pair of sensors may have different understandings about their relative bearing. Note that in the literature of bearing-based network localization, e.g., \cite{12}, \cite{13}, bearings have to be measured in the global coordinate frame in general.

In fact, it is interesting to observe that for each sensor \(i\), the angle between any pair of edges joining \(i\) is invariant to different coordinate frames. In addition, we find that such angle can always be computed by bearings measured in the local coordinate frame. To explain this phenomenon mathematically, let \(x_j^i\) denote the coordinate of sensor \(j\) expressed in the local coordinate frame of \(i\), \(x_i\) be the coordinate of \(i\) in the global coordinate frame. Then \(x_j^i = \mathcal{R}_i x_j + \xi_i\) for some \(\mathcal{R}_i \in \text{O}(2), \xi_i \in \mathbb{R}^2\). It
follows that

\[
\frac{(x^i_i - x^i_j)^\top (x^i_k - x^i_j)}{|x^i_i - x^i_j||x^i_k - x^i_j|} = \frac{(x^i_i - x^i_j)^\top (x^i_j - x^i_k)}{|x^i_i - x^i_j||x^i_j - x^i_k|} = \frac{(x^i_j - x^i_k)^\top (x^i_j - x^i_k)}{|x^i_i - x^i_j||x^i_j - x^i_k|}
\]

3) A QCQP formulation: Given a sensor network \((\hat{G}, x, A)\) in \(\mathbb{R}^2\), let \(p = (p_1^\top, \ldots, p_n^\top) \in \mathbb{R}^{2n}\) be the real locations of sensors. The ASNL problem can be modeled as the following feasibility problem:

\[
\text{find } x, d_{ij}, (i, j) \in \hat{E}
\]

\[
\text{s.t. } (x_i - x_j)^\top (x_i - x_k) = a_{i,j} d_{ij} d_{ik}, (i, j, k) \in T_{\hat{G}}
\]

\[
||x_i - x_j||^2 = d_{ij}^2, \quad (i, j) \in \hat{E}
\]

\[
x_i = p_i, \quad i \in A
\]

where \(a_{i,j,k} = \frac{(p_i - p_j)^\top (p_i - p_k)}{|p_i - p_j| |p_i - p_k|}\) is the angle information obtained from bearing measurements, \(T_{\hat{G}} = \{ (i, j, k) \in \mathcal{V}^3 : (i, j), (i, k) \in \hat{E}, j < k \}\) is the angle index set determining all the angles subtended in the framework, \(d_{ij}\) is the distance between sensors \(i\) and \(j\) for \((i, j) \in \hat{E}\). The known quantities in (4) include: \(a_{i,j,k} \in \mathbb{R}\) for \((i, j, k) \in T_{\hat{G}}\), \(p_i \in \mathbb{R}^2\) for \(i \in A\); the unknown variables in (4) are: \(x_i \in \mathbb{R}^2\) for \(i \in S\), \(d_{ij} \in \mathbb{R}\) for \((i, j) \in \hat{E}\), \(i\) or \(j\) \(\in S\).

**Remark 1:** In (4), all the angles in a sensor network are taken into account to localize unknown sensors. From the example in Fig.3 only partial angles are required to determine positions of unknown sensors. That is, there may have redundant angle information for network localization. Therefore, \((i, j, k) \in T_{\hat{G}}\) in (4) can be reduced to \((i, j, k) \in T_{\hat{G}}\), where \(T_{\hat{G}} \subset T_{\hat{G}}\). When \((\hat{G}, x)\) is strongly non-degenerate triangulated, [19] Theorem 7] gives a minimal set of angle constraints for determining angle fixability of \((\hat{G}, x)\), which is also sufficient for network localization.

We make the following assumption for problem (4).

**Assumption 1:** Problem (4) is feasible and all sensors do not collide with each other. All the results in this paper will be established in the premise that Assumption [valid is valid.

**B. Angle Localizability**

To distinguish what kind of sensor network can be uniquely localized, we give the following definition and provide the corresponding illustrations in Fig.4

**Definition 5:** A sensor network is angle localizable if there is a unique feasible solution to (4).

**Lemma 9:** If the sensor network is angle localizable in \(\mathbb{R}^2\), then anchors are not all collinear.

**Proof:** Suppose that anchors are all collinear. We discuss the following two cases:

Case 1. All sensors are collinear. Then given any sensor \(i \in S\), there must exist a constant \(\delta > 0\), such that for \(y_i = x_i - \delta \frac{x_i - x_j}{|x_i - x_j|}\), it holds that \(\frac{y_i - x_j}{|y_i - x_j||x_i - x_j|} = \frac{x_i - x_j}{|x_i - x_j|}\) for any \(j \in V\). That is, all the unknown sensors cannot be uniquely localized.

Case 2. Not all the sensors are collinear. Let \(y \in \mathbb{R}^2\) be a unit vector perpendicular to the line determined by anchors. It is easy to verify that \((\bar{x}_i - \bar{x}_j)^\top (\bar{x}_i - \bar{x}_k) = (x_i - x_j)^\top (x_i - x_k)\) for all \((i, j, k) \in T_{\hat{G}}\) and \(|\bar{x}_i - \bar{x}_j||x_i - x_j|\) for all \((i, j) \in \hat{E}\), where \(\bar{x}_i = p_i\) for \(i \in A\), \(\bar{x}_i = H_y x_i\) for \(i \in S\), \(H_y = I_y - 2yy^\top\) is the Householder transformation.

**Lemma 9** implies that at least 3 anchors are required for ASNL in \(\mathbb{R}^2\). Note that for SNL based on bearings measured in the global coordinate frame [12, 13, 15, 16], the minimum number of anchors required for unique localizability in \(\mathbb{R}^2\) is 2. This
is because the angle between two global bearings can be determined as clockwise or counterclockwise in the global coordinate frame. When each sensor has its own local coordinate system, different sensors may have different understandings about the rotational direction. Hence, each angle computed by bearings measured in a local coordinate frame cannot be recognized as clockwise or counterclockwise in the global coordinate frame. When each sensor has its own local coordinate system, different sensors may have different understandings about the rotational direction. Hence, each angle computed by bearings measured in a local coordinate frame cannot be recognized as clockwise or counterclockwise in the global coordinate frame.

The following lemma shows a connection between angle localizability and angle fixability.

**Lemma 10:** A sensor network \( N = (\hat{G}, x, A) \) is angle localizable in \( \mathbb{R}^2 \) if and only if \((\hat{G}, x)\) is angle fixable and anchors are not all collinear.

**Proof:** Sufficiency. Let \( y = (y_1^T, ..., y_n^T)^T \in \mathbb{R}^{2n} \) be a solution to (4). Since \((\hat{G}, x)\) is angle fixable, we have \( y = c(I_n \otimes \hat{A})x + 1_n \otimes \xi \) for some \( c \in \mathbb{R} \setminus \{0\}, \hat{A} \in \text{O}(2), \xi \in \mathbb{R}^2 \). Lemma 9 implies that there exist at least 3 anchors not staying collinear. Without loss of generality, let 1, 2, 3 be the three anchors. Then \( y_i = x_i, i = 1, 2, 3 \) and \( x_1 - x_2 = y_1 - y_2 = c\hat{A}(x_1 - x_2) \). Since \( ||\hat{A}|| = 1 \), we have \( |c| = 1 \). Therefore, \( \hat{A}' \triangleq c\hat{A} \in \text{O}(2) \). Similarly, we have \( x_1 - x_3 = \hat{A}'(x_1 - x_3) \).

Let \( A = (x_1 - x_2, x_1 - x_3) \in \mathbb{R}^{2 \times 2} \), \( A \) must be of full rank. Then we can obtain \( \hat{A}' = I_2 \) from \( A = \hat{A}'A \). Since \( x_1 = Y_{12} = \hat{A}'x_1 + \xi \), we have \( \xi = 0 \). As a result, \( y = x \).

Necassity. Lemma 9 has shown that anchors are not all collinear, we next prove angle fixability of \((\hat{G}, x)\). Consider \( y = (y_1^T, ..., y_n^T)^T \in \mathbb{R}^{2n} \) such that \( y \in f_{\hat{G}}^{-1}(f_{\hat{G}}(x)) \), it suffices to prove \( y \in \mathcal{F}_x \). Note that the subgraph \( \hat{G}_a \) composed by vertices \( \{1, ..., n_a\} \) and related edges is complete. Let \( x_a = (x_1^T, ..., x_{n_a}^T)^T, y_a = (y_1^T, ..., y_{n_a}^T)^T \). From Lemma 9 \((\hat{G}_a, x_a)\) is angle fixable, then \( y_a \in \mathcal{F}_{x_a} \). Angle localizability implies that given \( y_a \), the rest of coordinates of \( y \) such that \( f_{\hat{G}}(x) = f_{\hat{G}}(y) \) can be uniquely determined. Since a suitable \( y \) is an element in \( \mathcal{F}_x \), hence there must hold \( y \in \mathcal{F}_x \).

The condition for angle localizability proposed in Lemma 10 remains invariant in both centralized and distributed approaches proposed below to solve ASNL.

V. CENTRALIZED ASNL: AN OPTIMIZATION PERSPECTIVE

In this section, a centralized optimization framework is proposed to solve the ASNL problem. In the centralized ASNL, the information sensed by all sensors (including anchors) will be collected in a central unit, and the ASNL problem will be solved by this central unit. An example to illustrate the centralized approach to ASNL is presented in Fig. 5, where each sensor is able to transmit information to the central unit. However, each sensor is not necessary to have the computational capability, and inter-sensor communications are not required.
A. An SDP Formulation

Let \( X = (x_{n+1}, \ldots, x_{n+n}) \in \mathbb{R}^{2 \times n_x} \), \( Y = \left( \frac{I_2}{X^T} \right) \in \mathbb{R}^{(2+n_x) \times (2+n_x)} \), \( m = |\hat{E}| \), \( \hat{d} = (\ldots, d_{ij}, \ldots)^T \in \mathbb{R}^m \), \( D = \hat{d}\hat{d}^T \in \mathbb{R}^{m \times m} \), \( e_i \in \mathbb{R}^{n_x} \) and \( E_i \in \mathbb{R}^m \) be a \( n_x \)-dimensional and a \( m \)-dimensional unit vector with the i-th entry being 1, respectively. Then problem (4) is equivalently converted into a SDP

\[
\begin{array}{ll}
\min_{Y, D} & 0 \\
\text{s.t.} & (f_i - f_j)^T Y (f_i - f_k) = a_{ijk} E_{t_{ij}}^T D E_{t_{ik}}, (i, j, k) \in T_{\hat{g}}, \\
& (f_i - f_j)^T Y (f_i - f_j) = E_{t_{ij}}^T D E_{t_{ij}}, (i, j) \in \hat{E}, \\
& Y_{1:2,1:2} = I_2, \quad Y \succeq 0, \quad D \succeq 0,
\end{array}
\]

(5)

where \( Y_{1:2,1:2} \) is the second leading principal submatrix of \( Y \), and

\[
f_i = \begin{cases}
(p_i^T, 0_{1 \times n_x})^T, & i \in A, \\
(0_{1 \times 2}, e_{i - n_x})^T, & i \in S.
\end{cases}
\]

Let \( Q_{ijk} = (f_i - f_k)(f_i - f_j)^T \in \mathbb{R}^{(n_x+2) \times (n_x+2)} \), \( Q_{ij} = (f_i - f_j)(f_i - f_j)^T \in \mathbb{R}^{(n_x+2) \times (n_x+2)} \), \( R_{ijk} = E_{t_{ij}} E_{t_{ik}}^T \in \mathbb{R}^{m \times m} \), \( R_{ij} = E_{t_{ij}} E_{t_{ij}}^T \in \mathbb{R}^{m \times m} \), then (5) can be equivalently rewritten as

\[
\begin{array}{ll}
\min_{Y, D} & 0 \\
\text{s.t.} & \langle Q_{ijk}, Y \rangle = a_{ijk} \langle R_{ijk}, D \rangle, \quad (i, j, k) \in T_{\hat{g}}, \\
& \langle Q_{ij}, Y \rangle = \langle R_{ij}, D \rangle, \quad (i, j) \in \hat{E}, \\
& Y_{1:2,1:2} = I_2, \quad Y \succeq 0, \quad D \succeq 0.
\end{array}
\]

(6)

Note that \( Q_{ijk} \) and \( R_{ijk} \) are not symmetric, but it holds that \( \langle Q_{ijk}, Y \rangle = \langle Q_{ijk}^T, Y \rangle \) and \( \langle R_{ijk}, D \rangle = \langle R_{ijk}^T, D \rangle \). Hence, we replace \( Q_{ijk} \) and \( R_{ijk} \) with \( \frac{Q_{ijk} + Q_{ijk}^T}{2} \) and \( \frac{R_{ijk} + R_{ijk}^T}{2} \), respectively, the equalities still hold. Let \( Z = \left( \begin{array}{cc} Y & 0 \\ 0 & D \end{array} \right) \), \( \Phi_{ijk} = \left( \begin{array}{cc} Q_{ijk} + Q_{ijk}^T & 0 \\ 0 & 0 \end{array} \right) \), \( \Psi_{ij} = \left( \begin{array}{cc} Q_{ij} & 0 \\ 0 & 0 \end{array} \right) \), \( \bar{\Phi}_{ijk} = \left( \begin{array}{cc} 0_{(n_x+2) \times (n_x+2)} & 0 \\ 0 & R_{ijk} + R_{ijk}^T \end{array} \right) \), \( \bar{\Psi}_{ij} = \left( \begin{array}{cc} 0_{(n_x+2) \times (n_x+2)} & 0 \\ 0 & 0 \end{array} \right) \). Then (5) can be written as

\[
\begin{array}{ll}
\min_Z & 0 \\
\text{s.t.} & \langle \Phi_{ijk}, Z \rangle = a_{ijk} \langle \bar{\Phi}_{ijk}, Z \rangle, \quad (i, j, k) \in T_{\hat{g}}, \\
& \langle \Psi_{ij}, Z \rangle = \langle \bar{\Psi}_{ij}, Z \rangle, \quad (i, j) \in \hat{E}, \\
& Z_{1:2,1:2} = I_2, \\
& Z_{1:n_x+2, n_x+3:n_x+2+m} = 0, \\
& Z_{n_x+3:n_x+2+m,1:n_x+2} = 0, \\
& Z \succeq 0.
\end{array}
\]

(7)

**Lemma 11:** Let \( Z \) be a solution to (7), then \( \text{rank}(Z) \geq 3 \).

**Proof:** From the definition of \( Z \), it is easy to see \( \text{rank}(Z) = \text{rank}(Y) + \text{rank}(D) \). Since \( Y \) is symmetric, and \( Y_{1:2,1:2} = I_2 \), we have \( Y = \left( \begin{array}{cc} I_2 & Y_{12} \\ Y_{12}^T & Y_{22} \end{array} \right) \) with \( Y_{12} \in \mathbb{R}^{2 \times n_x} \) and \( Y_{22} \in \mathbb{R}^{n_x \times n_x} \). Then \( \text{rank}(Y) = \text{rank}(I_2) + \text{rank}(Y_{22} - Y_{12}^T Y_{12}) \geq 2 \). For matrix \( D \), Assumption 1 implies that \( D_{t_{ij}, t_{ij}} \) is not a zero matrix. Therefore, \( \text{rank}(D) \geq 1 \). In conclusion, \( \text{rank}(Z) \geq 3 \). □

In (7), there is no rank constraint. From \( Z \succeq 0 \), we can only obtain \( Y_{22} \succeq Y_{12}^T Y_{12} \), and \( D \) is nontrivial. It is possible that the rank of a solution \( Z \) is greater than 3, then \( Z \) does not correspond to a solution to (4). The following lemma shows a rank constraint for (7), which makes it equivalent to (4).

**Lemma 12:** If the rank of every solution to (7) is 3, then (7) is equivalent to (4).

**Proof:** Given \( X \) as a solution to (4), it is obvious that the induced \( Z \) is a solution to (7). Next we prove that given a solution \( Z \) (with rank 3) to (7), we can find a unique solution \( X \) corresponding to \( Z \) such that \( X \) is a solution to (4). Again, consider \( Y = \left( \begin{array}{cc} I_2 & Y_{12} \\ Y_{12}^T & Y_{22} \end{array} \right) \) with \( Y_{12} \in \mathbb{R}^{2 \times n_x} \) and \( Y_{22} \in \mathbb{R}^{n_x \times n_x} \). Since \( \text{rank}(Z) = 3 \), from the proof of Lemma 11 we have \( \text{rank}(Y) = 2 \), implying \( \text{rank}(Y_{22} - Y_{12}^T Y_{12}) = 0 \), then \( Y_{22} = Y_{12}^T Y_{12} \). Moreover, \( \text{rank}(D) = 1 \), then there exists some vector \( \bar{d} \in \mathbb{R}^{|E|} \) such that \( D = \bar{d} \bar{d}^T \). Since \( Z \) and \( D \) are in the desired forms, the constraints in (7) are equivalent to those in (4). Let \( X = Y_{12} \), \( X \) must be the solution to (4). □
Lemma 12 implies that we can equivalently reformulate the ASNL problem as the SDP in (7) with rank constraint \( \text{rank}(Z) = 3 \), which is a non-convex optimization problem and generally NP-hard. Although multiple methods for rank-constrained optimization have been proposed in the literature, e.g., [20, 10, 27], they can only guarantee local convergence for general cases. That is, the initial state given to the algorithm has to be sufficiently close to a local optimum solution. In the next subsection, by utilizing some inherent properties of ASNL, we will derive a condition for removing the rank constraint on \( Z \).

### B. Relation to SDP Relaxation

In the literature, e.g., [5, 6, 9], SDP relaxation has been widely used to solve range-based SNL. However, usually a solution to a relaxed problem may not correspond to a solution to the original problem. In this subsection, we will establish connections between the relaxed formulation in (7) and the original nonconvex QCQP in (4).

The following theorem shows that under a specific graph condition, the rank of \( Z \) can be efficiently constrained once the rank of \( D \) is constrained.

**Theorem 2:** Let \( Z = \begin{pmatrix} Y & 0 \\ 0 & D \end{pmatrix} \) be a solution to (7) and \( \text{rank}(D) = 1 \), then \( \text{rank}(Z) = 3 \) must hold if and only if

1. (i) anchors are not all collinear;
2. (ii) \((\hat{G}, x)\) is angle fixable in \( \mathbb{R}^2 \) and its angle fixability is invariant to space dimensions.

**Proof:** Sufficiency. Suppose that there is a solution \( \bar{Z} \) such that \( \text{rank}(\bar{Z}) > 3 \). Consider \( \bar{Y} = \begin{pmatrix} I_2 & Y_{12} \\ Y_{12}^\top & Y_{22} \end{pmatrix} \) as a part of the solution \( \bar{Z} \). Then there must hold \( Y_{22} \geq Y_{12}^\top Y_{12} \) and \( Y_{22} \neq Y_{12}^\top Y_{12} \). Hence, there exists some nontrivial \( Y_{12} \in \mathbb{R}^{n \times n} \), such that \( Y_{22} = Y_{12}^\top Y_{12} + Y_{12}^\top Y_{22} \). It is easy to see that anchors’ locations \( P = (p_1, \ldots, p_n) \in \mathbb{R}^{2n \times 2n} \), if \( Y_{12} \in \mathbb{R}^{2 \times n} \) is a feasible set of locations for sensors, then anchors’ locations \( (P^T, 0_{n \times r})^T \in \mathbb{R}^{(2+r) \times n} \) in \( \mathbb{R}^{2+r} \), \( (Y_{12}^\top, Y_{12}^\top)^T \in \mathbb{R}^{(2+r) \times n} \) is also a feasible set of locations for sensors. Note that \( (Y_{12}^\top, 0_{n \times r})^T \in \mathbb{R}^{(2+r) \times n} \) is also a solution. Let \( \hat{p} = (\hat{p}_1, \ldots, \hat{p}_n)^T \), \( \hat{x} = (\hat{x}_1, \ldots, \hat{x}_n)^T \in \mathbb{R}^{2n} \), \( \hat{x} = (\hat{x}_1, \ldots, \hat{x}_n)^T \in \mathbb{R}^{2n} \), \( \hat{x}_i \) be the \((i + n_d)\)-th column of \((Y_{12}^\top, 0_{n \times r})^T \) and \( \hat{x}_i \) be the \((i + n_d)\)-th column of \((Y_{12}^\top, Y_{12}^\top)^T \). The condition \( \text{rank}(D) = 1 \) implies that \((\hat{p}^\top, \hat{x}^\top)^T \) and \((\hat{p}^\top, \hat{x}^\top)^T \) are two different feasible realizations of framework \((\hat{G}, x)\). Since anchors are not all collinear, \( \hat{p} \) is non-degenerate. Then \((\hat{p}^\top, \hat{x}^\top)^T \) can never be obtained by a trivial motion from \((\hat{p}^\top, \hat{x}^\top)^T \). That is, angle fixability of \((\hat{G}, x)\) is not preserved in \( \mathbb{R}^{2+r} \), which is a contradiction.

Necessity. We first prove that \((\hat{G}, x)\) is angle fixable in \( \mathbb{R}^2 \). Due to Lemma 10, it suffices to show that (4) has a unique solution. Suppose this is not true, by Lemma 12, (7) also has multiple solutions. Let \( Z_1 = \begin{pmatrix} Y_1 & D_1 \end{pmatrix} \) with \( Y_1 = \begin{pmatrix} I_2 & X_1 \\ X_1^\top & X_2 \end{pmatrix} \) and \( Z_2 = \begin{pmatrix} Y_2 \\ D_2 \end{pmatrix} \) with \( Y_2 = \begin{pmatrix} I_2 \\ X_2^\top \end{pmatrix} \) be two different solutions to (7), then it is easy to see that \( Z_3 = \frac{1}{2}Z_1 + \frac{1}{2}Z_2 \) is also a solution to (7). As a result, \( \frac{1}{2}Y_1 + \frac{1}{2}Y_2 = \begin{pmatrix} I_2 \\ \frac{1}{2}X_1^\top + \frac{1}{2}X_2^\top \end{pmatrix} \) is a solution to (7). Since \( \frac{1}{2}X_1^\top X_1 + \frac{1}{2}X_2^\top X_2 = \frac{1}{2}(X_1 + X_2)^\top (X_1 + X_2) = \frac{1}{2}X_1^\top X_1 + \frac{1}{2}X_2^\top X_2 \). It follows that \( ||X_1 - X_2|| = 0 \). Since \( \text{rank}(D) = 1 \) and all the diagonal elements of \( D \) can be determined by \( X \), \( D \) is uniquely determined by \( X \). Then we have \( D_1 = D_2 \). Accordingly, \( Z_1 = Z_2 \), which is a contradiction. Hence, \((\hat{G}, x)\) is angle fixable. By Lemma 9, anchors must be not all collinear.

To show that the angle fixability of \((\hat{G}, x)\) is invariant to space dimensions, we note that from the proof of sufficiency, if \((\hat{G}, x)\) is not angle fixable in \( \mathbb{R}^{2+r} \), we can always accordingly find a solution to (4) with rank 3 + r. Hence the proof is completed.

**Remark 2:** The condition for \((\hat{G}, x)\) in Theorem 3 seems to be conservative. In simulation experiments, by solving the SDP formulation in (7), all unknown sensors can always be correctly localized when \((\hat{G}, x)\) is strongly non-degenerate triangulated. We will make further efforts to prove this in future. However, when \((\hat{G}, x)\) has a non-degenerate bilateration ordering but is not triangulated, the solution to (7) may correspond to incorrect localization results. An example (Example 9) will be given in Section V.

By virtues of Theorems 2 and Lemma 8, it is straightforward to give the following result.

**Theorem 4:** Given a sensor network \( N = (\hat{G}, x, A) \), if \((\hat{G}, x)\) contains an acute-triangulated subframework, and anchors are not all collinear, then
That is, a randomly generated network satisfies the condition in Theorem 5 with probability 1.

Here we use where

of large scale, matrices $\Phi_{ijk}$, $\Phi_{ijk}$, $\Psi_{ij}$ and $\Psi_{ij}$ in (7) are usually large and sparse. In this subsection, we will recognize some special features of ASNL and transform a large and sparse ASNL to a linear SDP with multiple semi-definite cone constraints for small matrices.

C. Decomposition for Large-Scale ASNL

The formulation in (7) is a standard linear SDP, thus can be globally solved by the interior-point methods in polynomial time. However, when the dimension of $Z$ is very large, due to high computational costs of solving the positive semi-definite constraint, the interior-point algorithms may be time-consuming or even cannot find the solution within reasonable computational time. However, large scale networks are ubiquitously encountered in practice. From intuitive observation, when the sensor network is of large scale, matrices $\Phi_{ijk}$, $\Phi_{ijk}$, $\Psi_{ij}$ and $\Psi_{ij}$ in (7) are usually large and sparse. In this subsection, we will recognize some special features of ASNL and transform a large and sparse ASNL to a linear SDP with multiple semi-definite cone constraints for small matrices.

One can observe that compared to (7), the problem in (6) has a smaller size and fewer constraints. Therefore, we will focus on problem (6) directly. By Theorem 2, the original ASNL in (4) is equivalent to the following optimization problem,

$$
\min_{Y,D} 0
$$

s.t. \quad (A_i,Y)+(B_i,D) = c_i, \quad i = 1, \ldots, s,

$$
Y, D \succeq 0, \quad \text{rank}(D) = 1,
$$

where $A_i \in \mathbb{R}^{(n_s+2) \times (n_s+2)}$, $B_i \in \mathbb{R}^{m \times n}$, $s = |T_g| + |\hat{E}| + 4$.

1) Decomposition for $Y$: Due to the definition of the inner product for matrices, an entry in $Y$, e.g., $Y_{ij}$, is constrained by some equality constraint if and only if the entry in the same position of some $A_i$ is nonzero. Let $A = \sum_{i=1}^{s} \text{abs}(A_i)$, where $\text{abs}(A_i)$ is the element-wise absolute operation. Then $Y_{ij}$ is constrained by some equality constraint if and only if $A_{ij} \neq 0$.

Here we use $E(A)$ to denote the sparsity pattern of $A$, where

$$
E(A) = \{(i,j) \in V(A) \times V(A) : A_{ij} \neq 0, i \neq j\},
$$

$V(A) = \{1, \ldots, n_s + 2\}$. Denote graph $G(A) = (V(A), E(A))$, we have the following result.

Theorem 5: If $G$ has a bilateration ordering and for any $i \in A$ connecting to some sensor $k \in S$, there exists another anchor $j \in A$ such that $(p_i - p_j)_x(p_i - p_j)_y \neq 0$, then graph $G(A)$ is chordal.

Proof: We will show that if $(i,j), (i,k) \in E(A)$ and $j \neq k$, then $(j,k) \in E(A)$, i.e., $A_{jk} > 0$. Note that $i,j,k \in \{1, \ldots, n_s + 2\}$ and

$$
A = \frac{1}{2} \sum_{(i,j,k) \in T_g} |[(f_i - f_k)(f_i - f_j)^\top + (f_i - f_j)(f_i - f_k)^\top]| + \sum_{(i,j) \in \hat{E}} |(f_i - f_j)(f_i - f_j)^\top|.
$$

Without loss of generality, we consider the following cases:

Case 1. $i,j \in \{1,2\}$, $k > 2$. Let $k' = k - 2 + n_s$, then $k' \in S$. It is easy to see that $A_{jk} \neq 0$ only if there exists at least one anchor $i'$ such that $(i',k') \in E$. Let $j'$ be another anchor distinct to $i'$ such that $(p_{i'} - p_{j'})_x(p_{i'} - p_{j'})_y \neq 0$, then

$$
M = (f_{i'} - f_{j'})(f_{i'} - f_{k'})^\top = \begin{pmatrix} p_{i'} - p_{j'} & p_{j'} \\ 0_{n_s \times 1} & p_{i'} - p_{j'} \end{pmatrix} \begin{pmatrix} p_{i'} - p_{j'} \\ 0_{n_s \times 2} \end{pmatrix} = \left(\begin{array}{c} (p_{i'} - p_{j'})_x \\ (p_{i'} - p_{j'})_y \end{array}\right) e_{k,2}. 
$$

It can be computed that $M_{jk} = (p_{i'} - p_{j'})_x$ if $j = 1$ and $M_{jk} = (p_{i'} - p_{j'})_y$ if $j = 2$. As a result, $A_{jk} \geq \frac{1}{2} |M_{jk}| > 0$.

Case 2. $i \in \{1,2\}$, $j,k > 2$. $(i,j), (i,k) \in E(A)$ implies that there exist $i' \in A$, $j' = j - 2 + n_s \in S$ and $k' = k - 2 + n_s \in S$ such that $(i,j), (i,k) \in E$. It follows that $M = (f_{i'} - f_{j'})(f_{i'} - f_{k'})^\top = \begin{pmatrix} p_{i'} \quad -e_{j,2} \\ -e_{j,2} & -e_{k,2} \end{pmatrix}$. Since $M_{jk} = 1$, we have $A_{jk} \geq \frac{1}{2} |M_{jk}| = \frac{1}{2}$.

Case 3. $i,j,k > 2$. Let $i' = i - 2 + n_s$, $j' = j - 2 + n_s$ and $k' = k - 2 + n_s$, we have $M = (f_{i'} - f_{j'})(f_{i'} - f_{k'})^\top = \begin{pmatrix} 0_{2 \times 1} \\ e_{i,2} - e_{j,2} \end{pmatrix} \begin{pmatrix} 0_{1 \times 2} \\ e_{i,2} - e_{j,2} \end{pmatrix} = \begin{pmatrix} -e_{j,2}p_{i'} - e_{k,2}p_{i'} \end{pmatrix}$. Similar to Case 2, $A_{jk} \geq \frac{1}{2} |M_{jk}| = \frac{1}{2}$.

The condition in Theorem 5 implies that the edges between several pairs of anchors are not parallel to both $x$-axis and $y$-axis of the global coordinate frame. Note that $\{q \in \mathbb{R}^{2n_s} : (q_i - q_j)_x(q_i - q_j)_y = 0, i,j \in \{1, \ldots, n_s\}\}$ is of measure zero. That is, a randomly generated network satisfies the condition in Theorem 5 with probability 1.

When graph $G(A)$ is chordal, by Lemma 1, the constraint $Y \succeq 0$ in (6) can be replaced by positive semidefinite constraints $Y_i = Q_{c_i}YQ_{c_i}^\top \succeq 0$, where $Y_i \in \mathbb{R}^{|G_i| \times |G_i|}$, $c_i$ is the set of vertices corresponding to the $i$-th maximal clique of $G(A)$. 

(8)
However, if sensor $k \in S$ has no anchor neighbors, then it is easy to see that $A_{1,k} = A_{2,k'} = 0$, where $k' = k - n_a + 2$. As a result, $Y_{1,k'}$ and $Y_{2,k'}$ are not constrained in the converted optimization problem. In the ASNL problem, we hope to find $X = Y_{1,2;3;n_a+2} = [x_{n_a+1}, \ldots, x_{n_a+n_a}]$, which contains position information of all unknown sensors. If $Y_{1,k'}$ and $Y_{2,k'}$ are not constrained, then we cannot obtain the correct position of sensor $k$ by solving the decomposed optimization problem directly. In [5], to obtain the solution to the original undecomposed problem, a positive semi-definite matrix completion problem should be addressed. In this paper, to avoid solving the matrix completion problem, we extend graph $G(A)$ by adding edges such that positions of all unknown sensors can be constrained.

Let $\bar{A} = A + \begin{pmatrix} 0 & 1_{2n_a} \\ 0 & 0 \end{pmatrix}$, and decompose matrix $Y$ according to the sparsity pattern $E(\bar{A})$. Then $Y_{1;2;3;n_a+2}$ is always constrained. By following similar lines of proofs for Theorem 5, the following result can be obtained.

Lemma 13: If $\hat{G}$ has a bilateration ordering, then $G(\hat{A}) = (V(\hat{A}), \mathcal{E}(\hat{A}))$ is chordal.

Together with Lemma 14, we have the following decomposition law.

Theorem 6: If $\hat{G}$ has a bilateration ordering, then $Y \succeq 0$ is equivalent to $Y_i = Q_{\mathcal{C}_i(\hat{A})} Y Q_{\mathcal{C}_i(\hat{A})}^\top \succeq 0$, where $\mathcal{C}_i(\hat{A})$ is the set of vertices corresponding to the $i$-th maximal clique of graph $G(\hat{A})$.

2) Decomposition for $D$: Similar to $G(A)$, we can obtain graph $G(B) = (V(B), E(B))$, where $V(B) = \{1, \ldots, m\}$, $E(B)$ is the aggregate sparsity pattern for $B_i$, $i = 1, \ldots, s$, $B = \sum_{i=1}^s \text{abs}(B_i)$. Unlike $G(A)$, graph $G(B)$ is more sparse and can never be chordal. By observing the form of (6), one can see that only partial elements of $D$ are constrained. Although the desired $D$ should be of rank 1, since our final goal is to find $Y$, we only require all the constrained elements of $D$ to satisfy the rank 1 constraint.

Theorem 7: Suppose that $\hat{G}$ is acute-triangulated, the solution $Y$ to (3) remains invariant if constraints “$D \succeq 0$” and “rank($D$) = 1” are relaxed to “$D_i = Q_{\mathcal{C}_i(B)} Y Q_{\mathcal{C}_i(B)}^\top \succeq 0$”, where $\mathcal{C}_i(B)$ is the set of vertices corresponding to the $i$-th maximal clique of graph $G(B)$.

Proof: From the proof of Theorem 3 one can realize that in the absence of the rank constraint on $D$, if the $3 \times 3$ submatrix corresponding to a triangle (e.g., composed of $i, j$ and $k$) is positive semi-definite, then constraints on angles in this triangle are exact (being equalities rather than inequalities). Moreover, if the $3 \times 3$ submatrix corresponding to a pair of angles sharing a common edge is positive semi-definite, then the corresponding three angle constraints are exact. For example, suppose $(i, j), (i, k), (i, h) \in \mathcal{E}$, and $l_{ij} < l_{ik} < l_{ih}$, if the third order principal submatrix of $D$ corresponding to $l_{ij}, l_{ik}, l_{ih}$ is positive semi-definite, then $\frac{(x_i - x_j)^\top (x_i - x_k)}{|x_i - x_j| |x_i - x_k|} = a_{ijk}$, $\frac{(x_i - x_j)^\top (x_i - x_h)}{|x_i - x_j| |x_i - x_h|} = a_{ikh}$, $\frac{(x_i - x_j)^\top (x_i - x_k)}{|x_i - x_k| |x_i - x_k|} = a_{ikh}$. This implies that if angles within each triangle are exactly constrained, then angles between edges in different triangles can also be exactly constrained. Note that for any $(i, j), (i, k), (i, h) \in \mathcal{E}$, $l_{ij}, l_{ik}$ and $l_{ih}$ must be adjacent to each other in graph $G(B)$. Moreover, the three edges of each triangle are also adjacent to each other in graph $G(B)$. Hence, we only require the third order principal submatrix of $D$ corresponding to each $3$-point clique in $G(B)$ to be positive semi-definite, which must hold if $D_i = Q_{\mathcal{C}_i(B)} D Q_{\mathcal{C}_i(B)}^\top \succeq 0$ for all maximal cliques $\mathcal{C}_i(B)$ of graph $G(B)$.

3) Decomposed ASNL: Combining Theorems 6 and 7, we obtain the following result.

Theorem 8: If $\hat{G}$ is acute-triangulated, then the ASNL problem in (4) is equivalent to the following optimization problem,

$$
\min_{Y, D} 0 \\
\text{s.t.} \quad \langle A_i, Y \rangle + \langle B_i, D \rangle = c_i, \quad i = 1, \ldots, s, \\
Y_i = Q_{\mathcal{C}_i(\hat{A})} Y Q_{\mathcal{C}_i(\hat{A})}^\top, \quad i = 1, \ldots, \xi, \\
D_i = Q_{\mathcal{C}_i(B)} Y Q_{\mathcal{C}_i(B)}^\top, \quad i = 1, \ldots, \zeta, \\
Y_i \succeq 0, \ i = 1, \ldots, \xi, \\
D_i \succeq 0, \ i = 1, \ldots, \zeta
$$

(9)

where $A_i, B_i$ and $s$ are the same as those in (8), $\xi$ is the number of maximal cliques of $G(\hat{A})$, and $\zeta$ is the number of third order principal submatrices of $G(B)$.

Let $\text{vec}(X) \in \mathbb{R}^{\sigma^2}$ denote the vector stacking columns of matrix $X \in \mathbb{R}^{\sigma \times \sigma}$ in the order that they appear in $X$, and $\text{mat}(x)$ be a matrix such that $x = \text{vec}(\text{mat}(x))$. By denoting $\mathbf{a}' = (\text{vec}(A_1), \ldots, \text{vec}(A_s))^\top \in \mathbb{R}^{s \times (n_a+2)^2}$, $\mathbf{B} = (\text{vec}(B_1), \ldots, \text{vec}(B_s))^\top \in \mathbb{R}^{s \times m^2}$, $\mathbf{M} = [\mathbf{a}', \mathbf{B}] \in \mathbb{R}^{s \times ((n_a+2)^2+m^2)}$, $\mathbf{y} = \begin{pmatrix} \text{vec}(Y^1) \\ \text{vec}(D) \end{pmatrix} \in \mathbb{R}^{(n_a+2)^2+m^2}$, $\mathbf{z}_i = \text{vec}(Y_i) \in \mathbb{R}^{\mathcal{C}_i(\hat{A})^2}$, $J_i = [Q_{\mathcal{C}_i(\hat{A})} \otimes Q_{\mathcal{C}_i(\hat{A})}, \mathbf{0}_{|\mathcal{C}_i(\hat{A})^2| \times m^2}]$, $i = 1, \ldots, \xi$, $z_{ix+i} = \text{vec}(D_i) \in \mathbb{R}^{9}$, $J_{ix+i} = [\mathbf{0}_{9 \times ((n_a+2)^2)} , Q_{\mathcal{C}_i(B)} \otimes Q_{\mathcal{C}_i(B)} ]$.

A matrix completion problem is to recover missing entries of a matrix from a set of known entries [29].
\[ i = 1, \ldots, \zeta, \quad c = (c_1, \ldots, c_\zeta)^\top, \quad \text{problem } (9) \text{ can be rewritten as} \]
\[
\min_{y, z_1, \ldots, z_{\zeta + \zeta}} 0 \quad \text{s.t. } A_i y = c,
\]
\[
z_i = J_i y, i = 1, \ldots, \zeta + \zeta, \quad \text{mat}(z_i) \succeq 0, i = 1, \ldots, \zeta + \zeta. \tag{10}
\]

The SDP formulation in (10) has been solved in multiple references [33], [34], [35]. In [34], a distributed algorithm for (10) has been proposed. In [35], an algorithm based on fast alternating direction method of multiplies (ADMM) has been developed to solve (10), which can also be implemented distributively. In practice, the convergence speed for solving (10) may depend on the sensing graph of a specific network.

**Remark 3:** Theorems [3] and [7] indicate that the condition for decomposing \( D \) is more demanding than that for decomposing \( Y \). When \((\hat{G}, x)\) has a bilateration ordering but is not acute-triangulated, we can decompose the positive semi-definite constraint on \( Y \) only and keep the rank constraint and the positive semi-definite constraint on \( D \). In this case, \( D \) can still be decomposed by using the rank-1 constraint, and solved by an iterative algorithm proposed in [27]. In [28], the authors proposed another approach for solving the rank-constrained optimization via chordal decomposition and a reweighted heuristic. By extending graph \( \hat{G}(B) = (V(B), E(B)) \) to a chordal graph, the approach in [28] can be implemented to solve ASNL.

**D. ASNL in a Noisy Environment**

In practice, measurements obtained by sensors are usually inexact. When the angle measurements contain some noise, we can replace \( a_{ijk} \) by \( \bar{a}_{ijk} = a_{ijk} + n_{ijk} \), where \( \bar{a}_{ijk} \) is the actual cosine value of the angle between \( x_i - x_j \) and \( x_i - x_k \), \( n_{ijk} \) denotes the measurement noise effect. Now assume that we only have \( \bar{a}_{ijk} \) available, \( a_{ijk} \) is an unknown variable to be determined. Let \( a = (\ldots, a_{ijk}, \ldots)^\top \in \mathbb{R}^{\lvert T \rvert_{\bar{G}}} \) and \( \bar{a} = (\ldots, \bar{a}_{ijk}, \ldots)^\top \in \mathbb{R}^{\lvert T \rvert_{\bar{G}}} \). Inspired by [9], we model the ASNL with noise as

\[ f(a) = -\sum_{(i,j,k) \in T_{\bar{G}}} \ln P_{ijk}(a_{ijk}|\bar{a}_{ijk}), \]

where \( P_{ijk}(a_{ijk}|\bar{a}_{ijk}) \) is the sensing probability density function, which depends on the property of noise \( n_{ijk} \). When \( P_{ijk}(a_{ijk}|\bar{a}_{ijk}) \) is a log-concave function of \( a_{ijk} \), \( f(a) \) is always convex. Now we simply consider the Gaussian zero-mean white noise, i.e., \( n_{ijk} \sim \mathcal{N}(0, \sigma_{ijk}^2) \), the objective function becomes

\[ f(a) = \sum_{(i,j,k) \in T_{\bar{G}}} \frac{(a_{ijk} - \bar{a}_{ijk})^2}{\sigma_{ijk}^2}. \tag{12} \]

Note that (11) is no longer a QCQP since \( a_{ijk} \) becomes a variable. However, by introducing new variables \( d_{ijk} \) and constraints \( d_{ijk} = a_{ijk} - \bar{a}_{ijk}, (11) \) can be converted to a QCQP again.

To convert (11) into an SDP with a reasonable scale, we introduce new 3 x 3 matrix variables \( \Lambda_{t_{ijk}} = \lambda_{t_{ijk}} \Lambda_{t_{ijk}}^T \), where \( \lambda_{t_{ijk}} = (a_{ijk}, d_{ijk}, 1)^\top \in \mathbb{R}^3 \). Similar to (8), the noisy ASNL is equivalent to the following SDP,

\[
\min_{Y, D, A_i} \sum_{i=1}^{\lvert T_{\bar{G}} \rvert} \langle F_i(\hat{a}), A_i \rangle \quad \text{s.t. } \langle A_i^T Y + (B_i^T D) + \sum_{j=1}^{\lvert T_{\bar{G}} \rvert} \langle C_i^T A_j \rangle = c_i, i = 1, \ldots, s',}
\]
\[
Y, D \succeq 0, \quad \text{rank}(D) = 1,
\]
\[
\Lambda_j(3, 3) = 1, \quad \Lambda_j \succeq 0,
\]
\[
\text{rank}(\Lambda_j) = 1, \quad j = 1, \ldots, \lvert T_{\bar{G}} \rvert,
\]
\[
\text{here } F_i(\hat{a}) \text{ is determined by } f(a) \text{ in (12), } \Lambda_j(3, 3) \text{ represents the element in the 3rd row and 3rd column of } \Lambda_j, \quad Y \in \mathbb{R}^{(n_x + 2) \times (n_x + 2)}, \quad D \in \mathbb{R}^{m \times m} \text{ are in the same sense as those in (8), but } A_i^T \in \mathbb{R}^{(n_x + 2) \times (n_x + 2)}, \quad B_i^T \in \mathbb{R}^{m \times m}, \quad c_i \in \mathbb{R} \text{ and } s' \text{ are different from } A_i, \ B_i, \ c_i \text{ and } s \text{ in (8). More specifically, } s' = 2 \lvert T_{\bar{G}} \rvert + m + 4.\]
Let $A' = \sum_{i=1}^{s} \text{abs}(A_i')$, $B' = \sum_{i=1}^{s} \text{abs}(B_i')$, it is easy to see that the sparsity patterns of $A'$ and $B'$ are the same as those of $A$ and $B$ in (8), thus semi-definite constraints for $Y$ and $D$ in (13) can still be decomposed in the way described in the last subsection. Also the rank constraint for $D$ can be removed when $(\mathbf{g}, x)$ is acute-triangulated. Efficient algorithms for solving (13) can be found in [27], [28]. If we simply ignore rank constraints, the resulting ASNL relaxation is a linear SDP, which can be solved by a SDP solver, e.g., CVX [32], directly.

VI. DISTRIBUTED ASNL VIA INTER-SENSOR COMMUNICATIONS

Solving ASNL in a centralized manner requires that all sensors are capable of transmitting information to a common central unit, which generates high computation and communication load in practice. Although the algorithms in [34], [35], [27], [28] can solve decomposed ASNL in a distributed fashion, all the required data should be collected in a central unit beforehand. Moreover, the algorithms in [34], [35], [27], [28] cannot be distributively executed by assigning each subtask to a sensor node.

In this section, we propose a fully distributed algorithm for ASNL, where each sensor computes its own position by using only local information obtained from its neighbors. The sensing measurements between each pair of neighboring sensors are still relative bearings in their own local coordinate frames. Similar to most of the existing distributed optimization references, we assume that each sensor is able to communicate with its neighbors. Note that it would be impossible for a sensor to localize itself if it has no access to exact positions of neighbors, while such information, could be obtained via communication from neighbors that have been localized.

A. Bilateralation Localization

Given an angle localizable sensor network $(\mathbf{G}, x, A)$, which has already been localized. Now we show that after placing a new sensor $k$ being a common neighbor of $i$ and $j$ in the network such that $x_i - x_k$ and $x_j - x_k$ are not collinear, $x_k$ can be uniquely determined by two angles subtended at $i$ and two angles subtended at $j$. An example is shown in Fig. 6. Note that since the sensor network is angle localizable, $(\mathbf{G}, x)$ is angle fixable. Then both $i$ and $j$ must have two neighboring nodes not lying collinear. Without loss of generality, let $i_1$ and $i_2$ be the two neighbors of $i$, $j_1$ and $j_2$ be the two neighbors of $j$ (It is possible that $i_1$ or $i_2 = j_1$, $j_1$ or $j_2 = i_1$). Since sensors $i$, $i_1$ and $i_2$ have already been localized, the bearings $g_{i i_1}$ and $g_{i i_2}$ with respect to the global coordinate system can both be obtained. In addition, both $\cos \angle 1$ and $\cos \angle 2$ can be computed by sensor $i$ using bearings $g_{i i_1}^i$, $g_{i i_2}^i$ and $g_{i k}^i$ measured in its local coordinate frame. Let $g_{i k}$ be the bearing between $i$ and $k$ in the global coordinate frame, then we have

$$g_{i i_1}^T g_{i k} = g_{i i_2}^T g_{i k}, \quad g_{i i_1}^T g_{i k} = g_{i i_2}^T g_{i k}.$$

Recall that $g_{i i_1}$ and $g_{i i_2}$ are not collinear, $g_{i k}$ can be uniquely solved. For simplicity, we denote $F_y$ as the function to compute $g_{i k}$, i.e., $g_{i k} = F_y(g_{i i_1}, g_{i i_2}, g_{i i_1}^i, g_{i i_2}^i, g_{i k}^i)$. Similarly, $g_{j k}$ can also be obtained. It is easy to see that

$$\det(x_i - x_k, g_{i k}) = 0, \quad \det(x_j - x_k, g_{j k}) = 0.$$

Since $g_{i k}$ and $g_{j k}$ are linearly independent, $x_k$ can be uniquely solved. We denote $F_x$ as the function to compute $x_k$, i.e.,

$$x_k = F_x(x_i, x_j, g_{i k}, g_{j k}).$$

We consider that each sensor has two modes: localized and unlocalized. Each anchor is in the localized mode. Only the localized sensors transmit information to their neighbors, while all sensors are always able to sense relative bearings from neighbors. As a result, each sensor can determine if a neighbor is in the localized mode by checking if it could receive information from this neighbor. Now we give a distributed protocol called “Bilateration Localization Protocol (BLP)”. The pseudo codes of BLP are shown in Protocol 1.
Fig. 7 illustrates the procedure of localizing a sensor network by implementing BLP. The black nodes denote anchors, red nodes are sensors in localized modes, white sensors are in unlocalized modes. It is shown that all the sensors are localized at step 3.

**Theorem 9:** Given a sensor network $N = (\hat{G}, x, A)$, if $(\hat{G}, x)$ has a non-degenerate bilateration ordering, and anchors are not all collinear, then BLP solves ASNL within $n_s$ steps.

**Proof:** We only have to prove that at each step, at least one unknown sensor can be localized. Since $(\hat{G}, x)$ has a non-degenerate bilateration ordering, we can label all the sensors as $\{1, \ldots, na, na + 1, \ldots, na + ns\}$ according to the order they appear in non-degenerate bilateration ordering, where $\{1, \ldots, na\}$ denotes the set of anchors. It is easy to see that sensor $na + k$ must be localized within $k$ steps.

**Protocol 1** The bilateration localization protocol for ASNL

**Configuration:** Each sensor has two modes: localized and unlocalized; Each sensor is able to transmit and receive information to/from neighbors, and sense relative bearings from neighbors in its local coordinate frame.

**Sensor $i$ in the localized mode:**

**Available information:** Position $x_i$, position $x_j$ received from localized neighbor $j$, bearings $g_{ij}$, $j \in N_i$ sensed from neighbors. Denote $N_{il}$ and $N_{iu}$ as the sets of localized and unlocalized neighbors of $i$, respectively.

**Protocol:**

1) for all $k \in N_{il}$ do
2) Arbitrarily choose distinct $i_1$ and $i_2$ from $N_{il}$ such that $x_i - x_{i_1}$ and $x_i - x_{i_2}$ are not collinear
3) Compute $g_{ik} = \mathcal{F}(g_{i_1}, g_{i_2}, g_{i_1}^i, g_{i_2}^i, g_{ik}^i)$ by solving the linear equations in (14)
4) Transmit $x_i, g_{ik}$ to sensor $k$
5) end for
6) for all $k \in N_{il}$ do
7) Transmit $x_i$ to sensor $k$
8) end for

**Sensor $k$ in the unlocalized mode:**

**Available information:** Positions $x_i$ and bearings $g_{ik}$ received from localized neighbors $i \in N_{kl}$, bearings $g_{ik}^i$ sensed from neighbors $i \in N_k$ in its local coordinate frame.

**Protocol:**

1) If positions from more than two neighbors received and these positions are not lying collinear then
2) Arbitrarily choose distinct $i$ and $j$ from $N_{kl}$ such that $x_i - x_k$ and $x_j - x_k$ are not collinear
3) Compute $x_k = \mathcal{F}(x_i, x_j, g_{ik}, g_{jk})$ by solving the linear equations in (15)
4) Switch to localized mode
5) end if
When \((\hat{G}, x)\) has a non-degenerate bilateration ordering such that the \(i\)-th vertex has exactly two neighbors with one of them being the \((i-1)\)-th vertex, BLP solves ASNL by \(n_s\) steps.

**Remark 4:** Observe that when implementing BLP, the accuracy in localizing an unknown sensor depends on the accuracy of the information received from its localized neighbors. If the neighbors of a sensor are inaccurately localized, then this sensor will be inaccurately localized accordingly. As a result, when the sensor network is in a noisy environment, namely, sensing measurements are all inaccurate, the position estimation errors will accumulate during the implementation of BLP. The later a sensor is localized, the greater error its estimated position has. In conclusion, although BLP accomplishes the localization task with a fast speed, it requires high accuracy of sensed measurements.

**VII. SIMULATION EXAMPLES**

In this section, we present four simulation examples. The first two examples show that the equivalence between ASNL \((4)\) and the decomposed linear SDP \((10)\) holds if the grounded framework \((\hat{G}, x)\) is acute-triangulated, but may not hold when \((\hat{G}, x)\) has a non-degenerate bilateration ordering. The third case shows the ASNL solution considering noisy measurements. The last example shows that ASNL can be solved by the distributed protocol BLP within finite steps when the grounded framework has a non-degenerate bilateration ordering.

**A. Simulations for Centralized ASNL**

1) **Noise-free ASNL:**

**Example 1:** Consider a sensor network \((\hat{G}, x, A)\) with \(n = 30\) sensors and \(n_a = 3\) anchors among them randomly distributed in the unit box \([0, 1]^2\), and \((\hat{G}, x)\) is acute-triangulated. The sensor network is shown in Fig. 8(a). By solving \((10)\) via CVX/SeDuMi \([32]\), we obtain the results plotted in Fig. 8(b). It is observed that the locations of unknown sensors estimated by CVX/SeDuMi closely match the real locations. Note that the framework in Fig. 8(a) is minimally infinitesimally bearing rigid and not globally distance rigid.

**Example 2:** Consider a sensor network \((\hat{G}, x, A)\) randomly distributed in the unit box \([0, 1]^2\), \((\hat{G}, x)\) has a bilateration ordering but is not triangulated, which is shown in Fig. 9(a). The localization results obtained by solving \((10)\) via CVX/SeDuMi are depicted in Fig. 9(b), from which we observe that not all unknown sensors can be localized.
Example 3: Consider a sensor network \((\mathcal{G}, x, A)\) with 3 anchors and 5 unknown sensors randomly distributed in the unit box \([0, 1]^2\) (as shown in Fig. 10 (a)) suffering a Gaussian white noise, and \((\mathcal{G}, x)\) is acute-triangulated. Then the rank constraint on \(D\) in (13) can be removed. An additive zero-mean white noise with a uniform standard deviation \(\sigma\) is applied to each angle measurement, i.e., \(a_{ijk} = a_{ijk} + n_{ijk}, \ n_{ijk} \sim N(0, \sigma^2)\). Similar to [26], [27], we solve (13) by an iterative rank minimization approach, which is to solve a series of linear SDPs as follows:

\[
\min_{Y, D, \Lambda^i_j, r_l} \sum_{i=1}^{\vert \mathcal{T}_G \vert} \langle F_i(\bar{a}), \Lambda^i_j \rangle + w_lr_l \\
\text{s.t.} \quad \langle A^i_i, Y \rangle + \langle B^i_i, D \rangle + \sum_{j=1}^{\vert \mathcal{T}_G \vert} \langle C^i_j, \Lambda^i_j \rangle = c^i_i, i = 1, ..., s',
\]

(16)

where \(w_l = \alpha^l w_0\) is set as an increasing positive sequence, i.e., \(\alpha > 1, w_0 > 0, V_j^l = (v^l_j, 1)^T \in \mathbb{R}^{2 \times 3}, v_j^l\) and \(v_j^2\) are two eigenvectors corresponding to the two smallest eigenvalues of \(\Lambda^{-1}_j\), which is obtained by solving the SDP formulation in (16) at step \(l - 1\). The initial state of each \(\Lambda^i_j\), i.e., \(\Lambda^0_j\), is obtained by solving (13) without considering the rank constraints.

We solve a sequence of SDPs (16) by CVX/SeDuMi successively until \(r_l < \epsilon\) at some step \(l^*\), where \(\epsilon\) is a positive scalar close to 0. The solution \((Y, D, \Lambda^i_j)\) to (16) at step \(l^*\) is regarded as the solution to (13). Note that the selections of \(w_0\) and \(\alpha\) are quite important for convergence of the iterative rank minimization algorithm. In [27], a convolutional neural network (CNN) is designed to seek appropriate \(w_0\) and \(\alpha\).

Now we consider \(\sigma = 0.005\), and set \(w_0 = 1, \alpha = 1.3\), by solving noisy ASNL with random white noise 100 times, the localization results are depicted in Fig. 10 (b). We observe that the unknown sensor whose two neighbors are both anchors can be localized with a small error, while the unknown sensor with two different types of neighboring sensors is localized with a relatively larger error.

B. Simulations for Distributed ASNL

Example 4: Consider three sensor networks with 100, 500, 1000 sensors in the plane, positions of sensors are randomly generated by Matlab such that each network has a non-degenerate bilateration ordering. Moreover, each network has only 3 anchor nodes among all the sensors. By implementing the distributed protocol BLP shown in Subsection VI-B, the three ASNL problems are solved, respectively. Fig. 11 (a) shows evolution of the percentage of unlocalized sensors with respect to all unknown sensors. We can observe that as the network size grows, the required number of iterations increases slowly.

Fig. 11 (b) depicts the history of the volume of sensors localized along each step. It is shown that during the implementation of BLP, the number of sensors localized per step increases at the beginning, and usually decreases sharply after half of total iteration steps.

VIII. CONCLUSIONS

This paper presented comprehensive analysis and approaches for angle-based sensor network localization (ASNL). We first proposed a notion termed angle fixability to recognize frameworks that can be uniquely determined by angles up to translations,
rotations, scalings and reflections. It has been proven that any framework with a non-degenerate bilateration ordering is angle fixable. The ASNL problem, which aims to determine locations of all sensors via anchor locations and angle constraints, was shown to have a unique solution if and only if the grounded framework is angle fixable. ASNL has been solved in centralized and distributed approaches, respectively. The centralized ASNL in a noise-free environment was modeled as a rank-constrained SDP, which is proven to be equivalent to a linear SDP when the grounded framework is acute-triangulated. A decomposition strategy was proposed to efficiently solve large-scale ASNL problems. The centralized ASNL in a noisy environment was studied via a maximum likelihood formulation, and was also formulated as an SDP with multiple rank constraints and semi-definite constraints. Distributed ASNL was realized by using a bilateration localization approach based on inter-sensor communications. Simulation results have been presented to illustrate effectiveness of the centralized formulations and the distributed protocol.

Future work will focus on the following aspects: (i) The graphical condition for angle fixability proposed in this paper is sufficient but not necessary, we will further seek a milder graphical condition; (ii) Numerical experiments show that ASNL is always equivalent to a linear SDP when the grounded framework is strongly non-degenerate triangulated, which is a proposition we would like to prove in the future; (iii) We observe that the ASNL problem in 3- or higher-dimensional space can be always equivalent to a linear SDP when the grounded framework is acute-triangulated. As a result, we have $\bar{a}^2 \bar{b} \bar{c}$ is positive semi-definite, we have $\det(M) \geq 0$. Then we can derive that $a^2(M_{23}^2 - 2bcM_{23} + b^2c^2) \leq 0$. Together with $a^2 \neq 0$, we have $M_{23} = bc$.

**Proof of Theorem 3** Without loss of generality, suppose $Y = \begin{pmatrix} I_2 & X \\ X^\top & X^\top X \end{pmatrix}$, where $X = (\bar{x}_1, ..., \bar{x}_{n+2}) \in \mathbb{R}^{(2+r) \times (n+2)}$, $r \geq 0$ is an integer. It follows from (6) that
\[
(\bar{x}_i - \bar{x}_j)^\top (\bar{x}_i - \bar{x}_k) = a_{ijk} D_{ijk}, (i, j, k) \in \mathcal{T}_G,
\]
\[
||\bar{x}_i - \bar{x}_j||^2 = D_{ij}, (i, j) \in \mathcal{E}.
\]
For any $(i, j, k) \in \mathcal{T}_G$ such that $(j, k) \in \mathcal{E}$, since the angle between $(i, j)$ and $(i, k)$ is acute, $a_{ijk} > 0$. From $D \geq 0$, we have $D_{l,ijkl} \leq D_{ijkl} D_{ijkl}$. Then
\[
(\bar{x}_i - \bar{x}_j)^\top (\bar{x}_i - \bar{x}_k) \leq a_{ijk} ||\bar{x}_i - \bar{x}_j|| ||\bar{x}_i - \bar{x}_k||.
\]
Let $\theta_1$ be the angle between $\bar{x}_i - \bar{x}_j$ and $\bar{x}_i - \bar{x}_k$. Then $\theta_1 \geq \arccos a_{ijk}$. Similarly, let $\theta_2$ and $\theta_3$ be the angles between $\bar{x}_j - \bar{x}_i$ and $\bar{x}_k - \bar{x}_i$ and $\bar{x}_k - \bar{x}_j$, respectively. It holds that $\theta_2 \geq \arccos a_{ijk}$ and $\theta_3 \geq \arccos a_{kij}$. Note that $\arccos a_{ijk} + \arccos a_{ij} + \arccos a_{kij} = \pi$, and $\theta_1 + \theta_2 + \theta_3 = \pi$, it follows that $\theta_1 = \arccos a_{ijk}$, $\theta_2 = \arccos a_{ijk}$ and $\theta_3 = \arccos a_{kij}$. As a result, $D_{ijkl}^2 = D_{ijl} D_{ijkl}$. 

**IX. APPENDIX**

To prove Theorem 3 the following lemma will be used.

**Lemma 14:** Consider a positive semi-definite matrix $M \in \mathbb{R}_{++}^{3 \times 3}$ with positive diagonal entries and one missing non-diagonal entry. If each $2 \times 2$ principal submatrix associated with available elements is of rank 1, then $M$ is uniquely completable.

**Proof:** Without loss of generality, let $M_{23}$ be the missing entry, then $M_1 = \begin{pmatrix} M_{11} & M_{12} \\ M_{12} & M_{22} \end{pmatrix}$ and $M_2 = \begin{pmatrix} M_{11} & M_{13} \\ M_{13} & M_{33} \end{pmatrix}$ are both positive semi-definite and of rank 1. Suppose that $M_1 = (a \ b)^\top (a \ b)$, $M_2 = (a \ c)^\top (a \ c)$. As a result, $M = \begin{pmatrix} a^2 & ab & ac \\ ab & b^2 & M_{23} \\ ac & M_{23} & c^2 \end{pmatrix}$. Since $M$ is positive semi-definite, we have $\det(M) \geq 0$. Then we can derive that $a^2(M_{23}^2 - 2bcM_{23} + b^2c^2) \leq 0$. Together with $a^2 \neq 0$, we have $M_{23} = bc$. ■

**Proof of Theorem 3** Without loss of generality, suppose $Y = \begin{pmatrix} I_2 & X \\ X^\top & X^\top X \end{pmatrix}$, where $X = (\bar{x}_1, ..., \bar{x}_{n+2}) \in \mathbb{R}^{(2+r) \times (n+2)}$, $r \geq 0$ is an integer. It follows from (6) that
\[
(\bar{x}_i - \bar{x}_j)^\top (\bar{x}_i - \bar{x}_k) = a_{ijk} D_{ijkl}, (i, j, k) \in \mathcal{T}_G,
\]
\[
||\bar{x}_i - \bar{x}_j||^2 = D_{ij}, (i, j) \in \mathcal{E}.
\]
For any $(i, j, k) \in \mathcal{T}_G$ such that $(j, k) \in \mathcal{E}$, since the angle between $(i, j)$ and $(i, k)$ is acute, $a_{ijk} > 0$. From $D \geq 0$, we have $D_{ijkl} \leq D_{ijl} D_{ijkl}$. Then
\[
(\bar{x}_i - \bar{x}_j)^\top (\bar{x}_i - \bar{x}_k) \leq a_{ijk} ||\bar{x}_i - \bar{x}_j|| ||\bar{x}_i - \bar{x}_k||.
\]
Let $\theta_1$ be the angle between $\bar{x}_i - \bar{x}_j$ and $\bar{x}_i - \bar{x}_k$. Then $\theta_1 \geq \arccos a_{ijk}$. Similarly, let $\theta_2$ and $\theta_3$ be the angles between $\bar{x}_j - \bar{x}_i$ and $\bar{x}_k - \bar{x}_i$ and $\bar{x}_k - \bar{x}_j$, respectively. It holds that $\theta_2 \geq \arccos a_{ijk}$ and $\theta_3 \geq \arccos a_{kij}$. Note that $\arccos a_{ijk} + \arccos a_{ij} + \arccos a_{kij} = \pi$, and $\theta_1 + \theta_2 + \theta_3 = \pi$, it follows that $\theta_1 = \arccos a_{ijk}$, $\theta_2 = \arccos a_{ijk}$ and $\theta_3 = \arccos a_{kij}$. As a result, $D_{ijkl}^2 = D_{ijl} D_{ijkl}$. 

Fig. 11. (a) Evolution of percentage of unlocalized sensors; (b) Evolution of percentage of localized sensors per step.
By Lemma \[\text{[14]}\] if \((i, j, k) \in T_\mathcal{G}, (j, k) \in \mathcal{E}\) and \((i, j, h) \in T_\mathcal{G}, (j, h) \in \mathcal{E}\), then \(D_{ij,h}^2 = D_{ij,li}D_{li,h}\). Without loss of generality, let \(k < h\), \(l_{ij} < l_{ik} < l_{ih}\), \(y = D_{li,h}\). From \(D \geq 0\), we have
\[
\det \begin{pmatrix}
D_{ij,j} & D_{ij,k} & D_{ij,h} \\
D_{ij,k} & D_{ij,k} & D_{ij,h} \\
D_{ij,h} & D_{ij,h} & y \\
\end{pmatrix} \geq 0.
\]
Together with \(D_{ij,ik}^2 = D_{ij,li}D_{li,k}\) and \(D_{ij,ik}^2 = D_{ij,li}D_{li,h}\), we can derive that \(y = D_{li,h}D_{li,k}\).

By Lemma \[\text{[14]}\] we can obtain that for any three edges in the graph, e.g., \((i, j), (k, h)\) and \((u, v)\), since \(D_{ij,kh}^2 = D_{ij,li}D_{li,kh} > 0\) and \(D_{l_{ik},uv}^2 = D_{l_{ik},h}D_{l_{ik},uv} > 0\), then there must hold that \(D_{ij,ku}^2 = D_{ij,li}D_{l_{ih},uv} > 0\) since \((\mathcal{G}, x)\) is triangulated, and the anchors are not all collinear, we have \(D_{ij}^2 = D_{ii}D_{jj} > 0\) for all \(i, j \in \{1, ..., m\}\). That is, \(\text{rank}(D) = 1\).

\[\Box\]

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