Remarks Concerning Polyakov’s Conjecture for the 3D Ising Model and the Hierarchical Approximation

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ABSTRACT

We consider the possibility of using the hierarchical approximation to understand the continuum limit of a reformulation of the 3D Ising model initiated by Polyakov. We introduce several new formulations of the hierarchical model using dual or fermionic variables. We discuss several aspects of the renormalization group transformation in terms of these new variables. We mention a reformulation of the model closely related to string models proposed by Zabrodin.
1. Introduction

There has been a long-standing effort to describe the large distance behavior of
gauge theories in terms of a string theory. This is obviously not an easy task and it
would be very interesting to have at hand a simple example where all details can be
carried out explicitly. It seems worth trying with a 3D $Z(2)$ gauge theory, because
this model is dual to the nearest neighbor 3D Ising model for which we have a
reasonable understanding of the critical behavior. Polyakov has shown that there
exist linear relations among the averages of products of order and disorder variables
taken along contours. These reveal the existence of fermionic excitations “held
together” by gauge-invariance. Subsequently, these contour averages have been
reexpressed as sums over “equipped surfaces” by Polyakov and Dotsenko. These
results strongly suggest that near the critical temperature, the 3D Ising model can
be described in terms of some free fermionic string.

Despite significant efforts, we do not yet have a clear understanding of the con-
tinuum limit of the reformulation of Polyakov and Dotsenko. For this reason, we
have suggested considering this problem in the context of the hierarchical approx-
imation. This approximation has been justified by Wilson and used to calculate
the critical exponents of the 3D Ising model with good precision. The practi-
cal advantage of the hierarchical approximation is that the renormalization group
procedure reduces to the study of a simple integral equation (“the approximate
recursion formula”). The fixed points of this recursion formula and the relevant
eigenvectors of the linearized transformation have been studied in great detail. The
critical exponents have a very simple form in terms of the corresponding eigenval-
ues.

The hierarchical approximation holds exactly in the case of the hierarchical
model (HM in the following) which will be the main object of the present study.
For definiteness, this model is briefly reviewed in section 2. We then introduce
several new formulations of the HM. The model dual to the HM is described in
section 3. Various fermionizations are discussed in section 4. Due to the fact that
the model has long range interactions, specific methods were necessary to obtain these results. These methods indeed apply to a larger class of models and are reported with due mathematical care in a separate article. Finally, we mention in the conclusion that that the model can be rewritten as a nearest neighbors model on a “branch” by introducing auxiliary (gaussian) fields. This construction is related, but distinct on several points, to string models proposed by Zabrodin.

The main question we address is how the extraordinary simplification provided by the renormalization group method (we only need the relevant directions) takes place when, instead of the order variables, we use the new variables discussed above. In all the cases, the renormalization group transformation appears to be more involved than in the usual formulation. This can be understood from the fact that in the HM, the number of links grows proportionally to the square of the number of sites. However, preliminary results indicate that when the symmetries of the model are taken into account, the procedure can be reduced to a size comparable to the usual one. The hard work (fixed points, linearization) remains to be done. A proper understanding of this question may shed a new light on the continuum limit of the sum over equipped surfaces discussed above.

2. The Hierarchical Model

For definiteness recall a few facts concerning the HM. More details and references can be found in the lecture notes of Collet and Eckmann or Gawedzki and Kupiainen. The hamiltonian of a hierarchical model with \( q^n \) sites can be written as

\[
H = -\frac{1}{2} \sum_{l=1}^{n} \left( \frac{c}{q^2} \right)^l \sum_{i_1, \ldots, i_{l+1}} \left( \sum_{i_1, \ldots, i_1} \sigma_{(i_1, \ldots, i_1)} \right)^2
\]  

(1)

All the indices \( i_j \) run from 1 to \( q \), an integer which controls the size of the boxes used during the renormalization group transformation. We assume that \( 1 \leq c < q \). It is suggestive to write \( c = q^{1-d_h} \) where \( d_h \) is the Hausdorff dimension of the random walk associated with \( H \). If the spins are integrated with a gaussian measure,
the model has the same critical exponents as an ordinary gaussian model in $D$ dimensions, provided that $d_h = \frac{2}{D}$. A non-gaussian continuum limit exists for $1 > d_h > 1/2$.

In the next two sections, we use the short notation $i$ instead of $(i_n, \ldots, i_1)$ and we define a function $v(i,j)$ which is equal to $l$ if $i_l$ is the first index (starting from the left) differing in $i$ and $j$. Another way of saying it that the smallest box containing $i$ and $j$ contains $q^{v(i,j)}$ sites. The partition function reads

$$Z = \sum_{\{\sigma_i = \pm 1\}} \sum_{i<j} K_{ij} \sigma_i \sigma_j .$$

with

$$K_{ij} = \beta (1 - \frac{c}{q^2})^{-1} (\frac{c}{q^2})^{v(i,j)} - (\frac{c}{q^2})^{n+1} .$$

The notation $i < j$ in the sum means that $i$ is distinct from $j$ and that the pairs of distinct elements which can be obtained by interchange are only counted once.

3. The Duality Transformation

As we have seen, the Kramers-Wannier duality\[3\] plays an important role in Polyakov’s approach. In the case of the HM, a dual spin $S_{ijk}$ is associated with any triangle $ijk$. These triangles can be used to build surfaces whose boundary is a contour appearing in the high temperature expansion of the model. The fact that tetrahedrons are closed surfaces provides the principle underlying the gauge-invariance of the dual formulation. All we need to check is that all the terms of the high temperature expansion appear with the same “gauge-multiplicity”. This is proven elsewhere.\[10\] In the following, we only state the main results.

If we define the dual couplings $\tanh K_{ij} = e^{-2D_{ij}}$, we can write the partition
function of Eq.(2) as

$$Z = \left( \prod_{i<j} \cosh K_{ij} \right) 2^{N-(N-1)/3} \sum_{\{S_{ijk}=\pm 1\}} e^{\sum_{i<j}(D_{ij}(\prod_{k \neq i,j} S_{(ijk)}-1))}.$$  

(4)

This reformulation has a local invariance. For any four distinct sites $i, j, k$ and $l$, the dual interaction energy is invariant under the simultaneous changes of sign of $S_{(ijk)}, S_{(ijl)}, S_{(ikl)}$ and $S_{(jkl)}$. A possible gauge-fixing condition is $S_{(ijk)} = 0$ if none of $i, j$ and $k$ are equal to a given site. In general, if we start with $q^n$ sites, the dual formulation has $(1/2)(q^n - 1)(q^n - 2)$ physical degrees of freedom. This number can be obtained directly by subtracting the $q^n - 1$ independent constraints on the parity of the number of visits at each site from the total number of links.

We insist on the fact that the symmetries of the model are crucial to deal with this proliferation. In order to fix the ideas, let us just compute the high temperature expansion of the partition function “hierarchically”. We perform successive integrations over the links variables inside boxes of size $q^l$, starting with $l = 1$ and proceeding similarly for higher $l$. At first sight it looks like we have to keep track of a prohibitively large number of quantities, namely $2^{q^l}$, since each site can be visited an even or an odd number of times and we need this information when we integrate over the links in larger boxes. At the end of the calculation (say for $l = n$), we only retain those contributions where the number of visits at each site is even. This indeed amounts to calculate all the correlation functions for $q^l$ sites for $l = 1, 2.., n - 1$. Fortunately, when the symmetries of the model are taken into account, the number of correlation functions to calculate is reduced logarithmically. It seems thus, in principle possible to use the renormalization group procedure in the dual formulation. Nevertheless, for practical reasons, it seems advantageous to first represent the high temperature expansion as an integral over Grassmann variables.
4. Hierarchical Fermions

A compact formulation of the high temperature expansion of the HM can be obtained\textsuperscript{[10]} by associating to each link $ij$ the four Grassmann variables $\psi^i_j, \psi^j_i, \chi^i_j$ and $\chi^j_i$. The final result for the partition function is

$$Z = (\prod_{i<j} \cosh K_{ij}) 2^N \int [d\psi d\chi] e^{\sum_{i<j} ((\text{th} K_{ij}) \psi^i_j \psi^j_i - \chi^i_j \chi^j_i)} \prod_{i=1}^{N} z(\sum_{j \neq i} \psi^j_i \chi^i_j)$$  \hspace{1cm} (5)$$

where $[d\psi d\chi] = \prod_{i<j} d\chi^i_j d\psi^i_j d\psi^j_i$ and $z(x)$ is equal to $\sinh(x)$ (resp. $\cosh(x)$) if $q$ is even (resp. odd). The average value of products of order or disorder variables and the Schwinger-Dyson equations are easily obtainable from Eq.(5). Note that, unlike in the 3D Ising model, these equations are highly non-linear. Note also that the number of Grassmann variables introduced is not minimal. We have doubled the number of fermions in order to avoid the splitting of the partition function into an uncontrollably large number of terms. The details concerning the above results can be found in Ref.[10].

This representation seems well-suited for the renormalization group approach. We can proceed in two steps. First, we integrate over the variables corresponding to the links inside the boxes of size $q$. Second, we make a linear transformation among the $4q^2$ variables associated to the links joining any pair of boxes of size $q$ in such a way that the average over upper and lower indices for the four type of variables are among the new variables. We then perform a gaussian integration over the remaining ones. After an appropriate rescaling, the “kinetic term” keeps its original form. What is transformed is the function $z(x)$ appearing in Eq.(5). We intend to study this transformation with the methods used for the recursion formula mentioned in the introduction.

Note also that we can construct a fermionic representation of the low temperature expansion. To each dual site $ijk$ we associate the three Grassmann variables
\[ \psi_{ij}^k, \psi_{ik}^j, \psi_{jk}^i \] and their \( \chi \) counterpart. The kinetic term contains a part of the form

\[ \sum_{i<j} \theta h D_{ij} \prod_{k: k \neq i,j} \psi_{ij}^k \] (6)

and the method based on gaussian integration is not applicable.

5. Conclusions and Perspectives

We have introduced several reformulations of the hierarchical model. The renormalization group method seems an appropriate tool to handle at least one of them. We think that a study of the fixed points and the relevant directions of the new transformation will provide a better understanding of the continuum limit of the reformulation of the 3D Ising model proposed by Polyakov and Dotsenko. This issue may also be clarified by a more systematic understanding of the hierarchical approximation. This could be done by using a complete set of multiplicative characters and taking the degree of ramification as the order of perturbation.

Note also that the HM can be reformulated as a nearest-neighbor model on a “branch”. The hamiltonian reads

\[ H = \sum_{l=1}^{n} \sum_{i_n, \ldots, i_{l+1}} \left( \frac{a_l}{2} (\sigma_{l-1}^{(l)}(i_n, \ldots, i_{l+1}))^2 - \frac{c_{l/2}}{q} \sigma_{l-1}^{(l)}(i_n, \ldots, i_{l+1}) \sum_{i_l} \sigma_{l-1}^{(l-1)}(i_n, \ldots, i_l) \right) \] (7)

with \( a_1 = 1 \) and \( a_l = 1 + \frac{c}{q} \) for \( l \geq 2 \). The \( \sigma_{l}^{(0)} \) are the spins variables of the HM as in section 2. The new variables \( \sigma_{l}^{(l)} \), for \( l \geq 2 \) are integrated with \( (2\pi\beta)^{-1/2} \int_{-\infty}^{+\infty} d\sigma_{l}^{(l)} \).

This construction has been inspired by Zabrodin’s models\(^{[11]}\) designed to reproduce the \( p \)-adic generalization\(^{[14]}\) of the tachyon amplitudes of string theory. Our reformulation differs on several points (discreteness of the boundary, the local couplings are not necessarily the same) from these models, however, it might be worth exploiting their resemblances. The “vertex operator” \( V_k = \sum_i Exp(k\sigma_{l}^{(0)}) \) can be
used to generate the average value of functions of the total spin. Note however, that in the gaussian case, the $SL_2$ invariance\textsuperscript{[14,11]} applies only for $d_h=1$ i.e $D = 2$. Otherwise, a “mass term” spoils the transformation properties and we cannot use the Koba-Nielsen trick.

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**REFERENCES**

1. Detailed lists of references can be found e.g. in A. Migdal, *Phys. Rep.* 102 (1983) 199 or in A. Neveu, Les Houches 1982, J. Zuber and R. Stora, Editors.

2. A. Polyakov, *Phys. Lett.* 82B (1979) 247 and *Phys. Lett.* 103B (1981) 211.

3. H. Kramers and G. Wannier, *Phys. Rev.* 60 (1941) 252; F. Wegner, *Jour. Math. Phys.* 12 (1971) 2259; L. Kadanoff and H. Ceva, *Phys. Rev. B* 3 (1971) 3918; D. Merlini and C. Gruber *Jour. Math. Phys.* 13 (1972) 1822; R. Balian, J.M. Drouffe and C. Itzykson *Phys. Rev. D* 11 (A.) 2098; Polyakov, *Nucl. Phys.* 120 (1977) 429; G. ’t Hooft, *Nucl. Phys.* 138 (1978) 1; E. Fradkin and L. Susskind *Phys. Rev. D* 17 (1978) 2637.

4. V. Dotsenko and A. Polyakov, in *Advanced Studies in Pure Mathematics*, 16, 1988.

5. V. Dotsenko, *Nucl. Phys.*

  B285 (1987) 45; A. Kavalov and A. Sedrakyan, *Nucl. Phys.* 285 (1987) 264; P. Orland *Phys. Rev. Lett.* 59 (1987) 2393; J. Distler, preprint PUPT 1324.
6. Y. Meurice, Proceedings of the International Lepton-Photon Symposium 1991, p.114.

7. K. Wilson, *Phys. Rev. B* 4 (1971) 3174; *ibid.* 3184. see also K. Gawedzki and A. Kupiainen, Les Houches 1985, K. Osterwalder and R. Stora, Editors.

8. P. Bleher and Y. Sinai, *Comm. Math. Phys.* 45 (1975) 247; P. Collet and J. P. Eckmann, *Comm. Math. Phys.* 55 (1977) 67 and references therein.

9. F. Dyson, *Comm. Math. Phys.* 12 (1969) 91.

10. Y. Meurice, Univ. of Iowa Preprint.

11. A. Zabrodin, *Comm. Math. Phys.* 123 (1989) 463.

12. Y. Meurice, *Phys. Lett.* 265B (1991) 377; J.L. Lucio and Y. Meurice, *Mod. Phys. Lett.* A6 (1991) 1199.

13. S. Samuel, *Jour. Math. Phys.* 21 (1980) 2806; E. Fradkin, M. Srednicki and L. Susskind, *Phys. Rev. D* D21 (1980) 2885; C. Itzykson, *Nucl. Phys.* 210 (1982) 477.

14. P. Freund and M. Olson, *Phys. Lett.* 199B (1987) 186; P. Freund and E. Witten *Phys. Lett.* B199 (1987) 191.