Superintegrability for super partition function hierarchies with $W$-representations

Rui Wang$^a$, Fan Liu$^b$, Min-Li Li$^b$, Wei-Zhong Zhao$^b$

$^a$Department of Mathematics, China University of Mining and Technology, Beijing 100083, China

$^b$School of Mathematical Sciences, Capital Normal University, Beijing 100048, China

Abstract

We present the super extensions for the $\beta$-deformed Hurwitz-Kontsevich partition function. Based on the $W$-representation of these super extended models, we construct the super partition function hierarchies with $W$-representations and derive their character expansions with respect to the Jack superpolynomials.

Keywords: Matrix Models, Superintegrability

1 Introduction

$W$-representation of matrix model realizes the partition function by acting on elementary functions with exponents of the given $W$-operator [1]. A considerable amount is already known about $W$-representations for matrix models. As the generalizations of matrix models from matrices to tensor, much interest has been then attributed to $W$-representations of tensor models. For the Gaussian tensor model [2] and (fermionic) rainbow tensor models [3, 4], it was shown that they can be realized by the $W$-representations. In addition, there have also been attempts to investigate $W$-representations of supereigenvalue models. It was noted that there is the $W$-representation for the supereigenvalue model in the Ramond sector [5]. However, for the Gaussian and chiral supereigenvalue models in the Neveu-Schwarz sector, they can only be expressed as the infinite sums of the homogeneous operators acting on the elementary functions [6]. In spite of this, a remarkable property is that there are still the compact expressions of the correlators in these two supereigenvalue models.

The superintegrability for matrix models has attracted much attention (see [7] and the references therein, [8]–[15]). $W$-representation is useful in analyzing the structures of matrix models, such as the superintegrability and calculations of the correlators. For a wide range of superintegrable matrix models, their character expansions can be derived from the corresponding $W$-representations [9, 10, 13]. The Hurwitz-Kontsevich (HK) matrix model with $W$-representation is an important superintegrable matrix model which can be used to describe the Hurwitz numbers and Hodge integrals over the moduli space of complex curves [1, 16, 17]. Its character expansions with respect to the Schur functions can be easily derived from the $W$-representation. The superintegrable $\beta$-deformed HK partition function is given by [13]

$$Z_0\{p\} = e^{tW_0}\beta^{p_1/e^tN} = \sum_{\lambda} e^{t\sum_{(i,j)\in \lambda} c(i,j)} \langle J_\lambda, J_\lambda \rangle_\beta J_\lambda\{p_k = e^{-tN} \delta_{k,1}\} J_\lambda\{p\},$$

References

[1] Wang R., Liu F., Li M.-L., Zhao W.-Z., Superintegrability for super partition function hierarchies with $W$-representations, J. Phys. A: Math. Theor. 55 (2022) 445201.

[2] Itzykson C., Zuber J.-B., Matrix Models and Topological Field Theories, in: Phase Transitions and Critical Phenomena, Vol. 12, A. Aharony, D. Friedan, C. Itzykson, J.-B. Zuber (Eds.), Academic Press, London, 1987.

[3] Ambjorn J., Bergström P., Parke S., Tye S.-H.O., The Rainbow Tensor Model, JHEP 05 (1998) 003.

[4] Okounkov A., Vafa C., Gaussian fluctuations of the rainbow tensor model, Commun. Math. Phys. 279 (2008) 351–382.

[5] Kostov P., Superintegrability of the supereigenvalue models, J. Phys. A: Math. Theor. 43 (2010) 265201.

[6] Kostov P., Supercorrelators of the supereigenvalue models in the Neveu-Schwarz sector, J. Phys. A: Math. Theor. 43 (2010) 425201.

[7] Kostov P., Superintegrability of the supereigenvalue models in the Ramond sector, J. Phys. A: Math. Theor. 43 (2010) 265201.

[8] Kostov P., Supercorrelators of the supereigenvalue models in the Neveu-Schwarz sector, J. Phys. A: Math. Theor. 43 (2010) 425201.

[9] Kostov P., Supercorrelators of the supereigenvalue models in the Neveu-Schwarz sector, J. Phys. A: Math. Theor. 43 (2010) 425201.

[10] Kostov P., Supercorrelators of the supereigenvalue models in the Neveu-Schwarz sector, J. Phys. A: Math. Theor. 43 (2010) 425201.

[11] Kostov P., Supercorrelators of the supereigenvalue models in the Neveu-Schwarz sector, J. Phys. A: Math. Theor. 43 (2010) 425201.

[12] Kostov P., Supercorrelators of the supereigenvalue models in the Neveu-Schwarz sector, J. Phys. A: Math. Theor. 43 (2010) 425201.

[13] Kostov P., Supercorrelators of the supereigenvalue models in the Neveu-Schwarz sector, J. Phys. A: Math. Theor. 43 (2010) 425201.

[14] Kostov P., Supercorrelators of the supereigenvalue models in the Neveu-Schwarz sector, J. Phys. A: Math. Theor. 43 (2010) 425201.

[15] Kostov P., Supercorrelators of the supereigenvalue models in the Neveu-Schwarz sector, J. Phys. A: Math. Theor. 43 (2010) 425201.

[16] Kostov P., Supercorrelators of the supereigenvalue models in the Neveu-Schwarz sector, J. Phys. A: Math. Theor. 43 (2010) 425201.

[17] Kostov P., Supercorrelators of the supereigenvalue models in the Neveu-Schwarz sector, J. Phys. A: Math. Theor. 43 (2010) 425201.

1 wangrui@cumtb.edu.cn

2 liufan-math@cnu.edu.cn

3 liml@cnu.edu.cn

4 Corresponding author: zhaowz@cnu.edu.cn
where

$$\begin{align*}
W_0 &= \frac{1}{2} \sum_{k,l=1}^{\infty} \left( \beta(k + l) p_k p_l \frac{\partial}{\partial p_{k+l}} + k l p_{k+l} \frac{\partial}{\partial p_k} \frac{\partial}{\partial p_l} \right) \\
&\quad + \frac{1}{2} \sum_{k=1}^{\infty} \left( (1 - \beta)(k - 1) + 2 \beta N \right) k p_k \frac{\partial}{\partial p_k},
\end{align*}$$

(2)

c(i, j) = j - 1 + \beta(N - i + 1), \text{ and } J_\lambda \{p \} \text{ are the Jack polynomials.}

In terms of the operator $W_0$, $p_1$ and $\frac{\partial}{\partial p_1}$, we have constructed the superintegrable partition function hierarchies in the previous paper [13]. The goal of this paper is to make a step towards the case of super partition functions. We shall extend the $\beta$-deformed HK partition function to the super cases and construct the superintegrable super partition function hierarchies with $W$-representations.

2 Super $\beta$-deformed HK partition function

A superpartition $\Lambda \mapsto (n|m)$ of bosonic degree $n = |\Lambda| = \sum_{i=1}^{l(\Lambda)} \Lambda_i$ and fermionic degree $m$ is a pair of partitions written as [15], [20]

$$\Lambda = (\Lambda^a; \Lambda^s) = (\Lambda_1, \Lambda_2, \ldots, \Lambda_m; \Lambda_{m+1}, \Lambda_{m+2}, \ldots, \Lambda_{l(\Lambda)})$$

where $\Lambda_i, i = 1, \ldots, l(\Lambda)$ are integers, $\Lambda_1 > \Lambda_2 > \cdots > \Lambda_m \geq 0$ and $\Lambda_{m+1} \geq \Lambda_{m+2} \geq \cdots \geq \Lambda_{l(\Lambda)} > 0$.

Note that the superpartitions of degree $(n|0)$ are the regular partitions. In the following, we denote $\Lambda^* = (\Lambda^a; \Lambda^s)^+$ as the partition obtained by reordering the concatenation of the entries of $\Lambda^a$ and $\Lambda^s$ in non-increasing order, $\Lambda^s = (\Lambda^a + 1^m; \Lambda^s)^+$, where $\Lambda^a + 1^m$ is the partition obtained by adding one to each entry of $\Lambda^a$. The diagram of $\Lambda$ is obtained by drawing the diagram of $\Lambda^s$ and then replacing the cells of $\Lambda^\circ$ by circles.

It is known that the Jack superpolynomials $J_\lambda$ are eigenfunctions of the following superoperators [21], [22]

$$\begin{align*}
D &= \frac{1}{2} \sum_{i=1}^{N} x_i^2 \frac{\partial^2}{\partial x_i^2} + \beta \sum_{1 \leq i \neq j \leq N} \frac{x_i x_j}{x_i - x_j} \left( \frac{\partial}{\partial x_i} - \frac{\theta_i - \theta_j}{x_i - x_j} \frac{\partial}{\partial \theta_i} \right), \\
\Delta &= \sum_{i=1}^{N} x_i \theta_i \frac{\partial}{\partial x_i} \frac{\partial}{\partial \theta_i} + \beta \sum_{1 \leq i \neq j \leq N} \frac{x_i \theta_j + x_j \theta_i}{x_i - x_j} \frac{\partial}{\partial \theta_i},
\end{align*}$$

(3)

where $x_i$ and $\theta_i$, $i = 1, \ldots, N$, are the bosonic and fermionic variables, respectively. The corresponding eigenvalues are given by $\sum_{j=1}^{l(\Lambda^s)} (j - 1) \Lambda_j^s - \beta \sum_{i=1}^{l(\Lambda^s)} (i - 1) \Lambda_i^s$ and $|\Lambda^a| - \beta |\Lambda^m|$, where $\Lambda'$ is the conjugate of $\Lambda$ whose diagram is obtained by reflecting the diagram of $\Lambda$ with respect to the main diagonal.

Let $p_k = \sum_i x_i^k$, $k = 1, 2, \ldots$, and $\tilde{p}_l = \sum_i \theta_i x_i^l$, $l = 0, 1, 2, \ldots$, be the bosonic and fermionic power sums, respectively. In terms of the variables $\{p\} = \{p_1, p_2, \ldots\}$ and $\{\tilde{p}\} = \{\tilde{p}_0, \tilde{p}_1, \tilde{p}_2, \ldots\}$, the operator $D + \beta N \sum_{i=1}^{N} \frac{\partial}{\partial x_i}$ can be written as

$$W_0 = \frac{1}{2} \sum_{k,l=1}^{\infty} \left( \beta(k + l) p_k p_l \frac{\partial}{\partial p_{k+l}} + k l p_{k+l} \frac{\partial}{\partial p_k} \frac{\partial}{\partial p_l} \right)$$

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When we neglect the fermionic variables \( \{ \tilde{p} \} \) in (23), it reduces to the operator (22), thus we call \( \mathcal{W}_0 \) the super \( \beta \)-deformed Hurwitz operator.

In terms of \( \mathcal{W}_0 \), we may extend the \( \beta \)-deformed HK partition function (11) to the super case,

\[
\mathcal{Z}_0\{p, \tilde{p}\} = e^{i\mathcal{W}_0} e^{e^{-tN}(p_1 + \tilde{p}_1 \theta)},
\]

where \( \theta \) is a constant fermionic parameter.

Let us take \( g_k = e^{-tN} \delta_{k,1} \) and \( \tilde{g}_k = e^{-tN} \theta \delta_{k,1} \) in the Cauchy formula for the Jack superpolynomials (23)

\[
e^{\beta e^{\sum_{k=1}^{\infty} \left( \frac{p_k \tilde{p}_k}{k} + k \tilde{p}_{k+1} \frac{\partial}{\partial \tilde{p}_k} \frac{\partial}{\partial p_k} \right)}} + \frac{1}{2} \sum_{k=1}^{\infty} \left( (1 - \beta)(k - 1) + 2\beta N \right) k \left( \frac{\partial}{\partial p_k} \frac{\partial}{\partial \tilde{p}_k} + \frac{\partial}{\partial \tilde{p}_k} \frac{\partial}{\partial p_k} \right),
\]

Then we have

\[
\mathcal{W}_0 J_\Lambda = \sum_{(i,j) \in \Lambda^*} c(i,j) J_\Lambda.
\]

When we neglect the fermionic variables \( \{ \tilde{p} \} \) in (24), it reduces to the operator (22), thus we call \( \mathcal{W}_0 \) the super \( \beta \)-deformed HK partition function (11). In this case, (6) becomes the super HK partition function

\[
\mathcal{Z}_0\{p, \tilde{p}\} = e^{i\mathcal{W}_0} e^{e^{-tN}(p_1 + \tilde{p}_1 \theta)},
\]

where \( \theta \) is a constant fermionic parameter.

Let us take \( g_k = e^{-tN} \delta_{k,1} \) and \( \tilde{g}_k = e^{-tN} \theta \delta_{k,1} \) in the Cauchy formula for the Jack superpolynomials (23)

\[
e^{\beta e^{\sum_{k=1}^{\infty} \left( \frac{p_k \tilde{p}_k}{k} + k \tilde{p}_{k+1} \frac{\partial}{\partial \tilde{p}_k} \frac{\partial}{\partial p_k} \right)}} + \frac{1}{2} \sum_{k=1}^{\infty} \left( (1 - \beta)(k - 1) + 2\beta N \right) k \left( \frac{\partial}{\partial p_k} \frac{\partial}{\partial \tilde{p}_k} + \frac{\partial}{\partial \tilde{p}_k} \frac{\partial}{\partial p_k} \right),
\]

Then we have

\[
\mathcal{W}_0 J_\Lambda = \sum_{(i,j) \in \Lambda^*} c(i,j) J_\Lambda.
\]

When \( \beta = 1 \), the Jack superpolynomials \( \prod_{s \in \mathcal{B}_\Lambda} h^\Lambda_{\alpha}(s)^{-1} J_\Lambda \) reduce to the Schur-Jack superpolynomials \( s^{Jack}_\Lambda \) (24). In this case, (6) becomes the super HK partition function

\[
\mathcal{Z}_0\{p, \tilde{p}\} = e^{i\mathcal{W}_0} e^{e^{-tN}(p_1 + \tilde{p}_1 \theta)} = \sum_{\Lambda} e^{i \sum_{(i,j) \in \Lambda^*} c(i,j) \frac{1}{\langle J_\Lambda, J_\Lambda \rangle_\beta} J_\Lambda\{p, \tilde{p}\} J_\Lambda\{g_k = e^{-tN} \delta_{k,1}, \tilde{g}_k = e^{-tN} \theta \delta_{k,1}\}},
\]

where

\[
\mathcal{W}_0 = \frac{1}{2} e^{\sum_{k,l=1}^{\infty} \left( \left( k + l \right) p_k p_l \frac{\partial}{\partial p_{k+l}} + k l p_{k+l} \frac{\partial}{\partial p_k} \frac{\partial}{\partial p_l} \right)} + \frac{1}{2} \sum_{k,l=1}^{\infty} \left( \left( k + l \right) \tilde{p}_k \tilde{p}_l \frac{\partial}{\partial \tilde{p}_{k+l}} + k l \tilde{p}_{k+l} \frac{\partial}{\partial \tilde{p}_k} \frac{\partial}{\partial \tilde{p}_l} \right) + N \sum_{k=1}^{\infty} \left( k p_k \frac{\partial}{\partial p_k} + k \tilde{p}_k \frac{\partial}{\partial \tilde{p}_k} \right),
\]

\[
c_\Lambda = \sum_{(i,j) \in \Lambda^*} (N - i + j) \text{ and } \langle s^{Jack}_\Lambda, s^{Jack}_\Lambda \rangle = \left( -1 \right)^{m(m-1)/2} \prod_{s \in \mathcal{B}_\Lambda} a^{\Lambda^*(s) + l_{\Lambda^*}(s) + 1} \text{ with } m \text{ the fermionic degree of } \Lambda.
\]
3 Super partition function hierarchies with $W$-representations

Let us define the operator

$$E_1 = [W_0, p_1] = \sum_{n=1}^{\infty} (np_{n+1} \frac{\partial}{\partial p_n} + n\hat{p}_{n+1} \frac{\partial}{\partial \hat{p}_n}) + \beta N p_1,$$

such that

$$E_1 J_\Lambda = \sum_{\Omega, (i,j) \in \Omega^*/\Lambda^*} c(i,j) B_{\Lambda\Omega} J_\Omega,$$

where the sum is over the superpartitions $\Omega$ satisfying $\Lambda^* \subset \Omega^*$, $\Lambda^o \subset \Omega^o$ and $|\Omega^*/\Lambda^*| = |\Omega^o*/\Lambda^o*| = 1$, $B_{\Lambda\Omega}$ are the coefficients in the action

$$p_1 J_\Lambda = \sum_{\Omega} B_{\Lambda\Omega} J_\Omega.$$

For the operator

$$W_{-1} = [W_0, E_1],$$

we have

$$W_{-1} J_\Lambda = \sum_{\Omega, (i,j) \in \Omega^*/\Lambda^*} c^2(i,j) B_{\Lambda\Omega} J_\Omega.$$

Let us introduce a series of operators

$$W_{-n} = \frac{1}{(n-1)!} [W_{-1}, [W_{-1}, \ldots, [W_{-1}, E_1] \ldots]], \quad n \geq 2.$$

There are the actions

$$W_{-n} J_\Lambda = \sum_{\Omega} \prod_{(i,j) \in \Omega^*/\Lambda^*} c(i,j)^\alpha B_{\Lambda\Omega} J_\Omega, \quad n \geq 1,$$

where $\alpha = 1 + \delta_{n,1}$, the sum is over the superpartitions $\Omega$ satisfying $\Lambda^* \subset \Omega^*$, $\Lambda^o \subset \Omega^o$ and $|\Omega^*/\Lambda^*| = |\Omega^o*/\Lambda^o*| = n$, $B_{\Lambda\Omega}$ are the coefficients in

$$p_n J_\Lambda = \sum_{\Omega} B_{\Lambda\Omega} J_\Omega.$$

We introduce the super partition function hierarchy with $W$-representations

$$Z_{-n,m}(p,\hat{p}) = e^{W_{-n}/n J_{\Lambda_m}}, \quad n \geq 1, m \geq 0,$$

where $\Lambda_m = (\delta_m; 0)$ and $\delta_m = (m-1, m-2, \ldots, 0)$.

When $m = 0$, (20) gives rise to the partition function hierarchy defined in Ref. [13].

$$Z_{-n,0}(p) = e^{W_{-n}/n} \cdot 1 = \sum_{\lambda} \beta^{|\lambda|/(n-1)} n \left( \frac{J_\lambda\{p_k = N\}^2}{J_\lambda\{p_k = \delta_{k,1}\}} \right) \langle J_\lambda, J_\lambda \rangle \beta J_\lambda\{p\}.$$

Taking $n = 1, 2$ in (21), it gives the $\beta$-deformed rectangular complex (with $N_1 = N_2$) matrix model [25]

$$Z_{-1,0}(p) = \sum_{\lambda} \beta^{|\lambda|} \frac{J_\lambda\{p_k = N\}^2}{J_\lambda\{p_k = \delta_{k,1}\}} \langle J_\lambda, J_\lambda \rangle \beta J_\lambda\{p\}.$$
and Gaussian hermitian matrix model \[25\]

\[
Z_{-2,0}(p) = \prod_{i=1}^{\infty} \int_{-\infty}^{\infty} dz_i \Delta(z)^{2\beta} e^{\sum_{k=1}^{\infty} \sum_{s=1}^{\infty} \beta \frac{p_k^2}{s_i^4} + \frac{1}{4} \sum_{k=1}^{\infty} z_i^2}
\]

\[
= \sum_{\lambda} \beta^{\frac{\lambda}{2}} J_\lambda \{ p_k = N \} J_\lambda \{ p_k = \delta_{k,2} \} \frac{J_\lambda \{ p_k = \delta_{k,1} \} (J_\lambda, J_\lambda)_{\beta}}{(J_\lambda, J_\lambda)_\beta} J_\lambda \{ p \}.
\]  

(23)

By a straightforward calculation of the power of \( W \) acting on \( \Lambda \), we obtain

\[
\mathcal{W}_{\Gamma_n} J_{\Lambda_m} = \sum_{\Lambda} \prod_{(i,j) \in \Lambda^*/\delta_m} c(i,j)^{\alpha} b(\Lambda) J_{\Lambda}, \quad n \geq 1,
\]

(24)

where

\[
b(\Lambda) = \frac{\langle p_\Lambda, J_{\Lambda_m}, J_{\Lambda} \rangle_{\beta}}{(J_\lambda, J_\lambda)_{\beta}} = (-1)^{m(m-1)/2} r_\Lambda^\Gamma \beta^{-m} \frac{J_{\Lambda/m} \{ p_k = \delta_{k,n} \}}{(J_\lambda, J_\lambda)_{\beta}},
\]

(25)

\( J_{\Lambda/m} \) is the skew Jack superpolynomial \[22\].

Using the evaluation formula \[22\]

\[
E_{N,m-1}(J_\lambda) = \beta^{-\frac{\lambda}{2}(\lambda - \frac{m(m-1)}{2})} \prod_{(i,j) \in \Lambda^*/\delta_m} c(i,j), \quad m \geq 1,
\]

(26)

where \( m \) is the fermionic degree of \( \Lambda \), \( J_\lambda = ((-1)^{m-1} \partial_{\theta_{N+1}} J_\lambda)_{x_{N+1}=0} \), for a homogeneous symmetric superpolynomial

\[
f(x_1, \cdots, x_N, \theta_1, \cdots, \theta_N) = \sum_{1 \leq i_1 < \cdots < i_m \leq N} \theta_{i_1} \cdots \theta_{i_m} f_{i_1, \cdots, i_m}(x),
\]

(27)

the evaluation map \( E_{N,m}(f) \) is given by

\[
E_{N,m}(f) = \left[ \prod_{1 \leq i < j \leq m} (x_i - x_j)^{-1} f_{i_1, \cdots, i_m}(x) \right]_{x_1=\cdots=x_N=1},
\]

(28)

we further obtain

\[
Z_{-n,m}(p, \bar{p}) = \sum_{\lambda} (-1)^{\frac{m(m-1)}{2}} \beta^{(\frac{\lambda}{2}(\lambda - \frac{m(m-1)}{2}))(n\alpha - 1)/n} (E_{N,m-1}(J_\lambda))^\alpha
\]

\[
\frac{J_{\Lambda/m} \{ p_k = \delta_{k,n} \}}{(J_\lambda, J_\lambda)_{\beta}} J_\lambda \{ p, \bar{p} \}, \quad m \geq 1.
\]

(29)

Let us turn to construct the operator

\[
\mathcal{W}_1 = [\mathcal{W}_0, E_{-1}],
\]

(30)

where the operator \( E_{-1} \) is given by

\[
E_{-1} = \beta^{-1}[\mathcal{W}_0, \frac{\partial}{\partial p_1}] = - \sum_{n=1}^{\infty} (n + 1) p_n \frac{\partial}{\partial p_{n+1}} - \sum_{n=1}^{\infty} n \bar{p}_n \frac{\partial}{\partial \bar{p}_{n+1}} - N \frac{\partial}{\partial p_1}.
\]

(31)

The operator \( \mathcal{W}_1 \) acting on \( J_\lambda \) is

\[
\mathcal{W}_1 J_\lambda = \sum_{\Gamma', (i,j) \in \Lambda^*/\Gamma^*} c^2(i,j) B_{\lambda \Gamma} J_{\Gamma},
\]

(32)
where the sum is over the superpartitions $\Gamma$ satisfying $\Gamma^* \subset \Lambda^*$, $\Gamma^\circ \subset \Lambda^\circ$ and $|\Lambda^* / \Gamma^*| = |\Lambda^\circ / \Gamma^\circ| = 1$, $B_{\Lambda \Gamma}$ are the coefficients in
\begin{equation}
\beta^{-1} \frac{\partial}{\partial p_1} J_{\Lambda} = \sum_{\Gamma} B_{\Lambda \Gamma} J_{\Gamma}.
\end{equation}

Let us introduce the operators
\begin{equation}
W_n = \frac{(-1)^n}{(n-1)!} \left[ W_1, \ldots, W_n, E_{-1} \right], \quad n \geq 2.
\end{equation}

There are the actions
\begin{equation}
W_n J_{\Lambda} = \sum_{\Gamma} \prod_{(i,j) \in \Lambda^*/\Gamma^*} c(i,j)^n B_{\Lambda \Gamma} J_{\Gamma}, \quad n \geq 1,
\end{equation}
where the sum is over the superpartitions $\Gamma$ satisfying $\Gamma^* \subset \Lambda^*$, $\Gamma^\circ \subset \Lambda^\circ$ and $|\Lambda^* / \Gamma^*| = |\Lambda^\circ / \Gamma^\circ| = n$, $B_{\Lambda \Gamma}$ are the coefficients in
\begin{equation}
\beta^{-1} n \frac{\partial}{\partial p_n} J_{\Lambda} = \sum_{\Gamma} B_{\Lambda \Gamma} J_{\Gamma}.
\end{equation}

We introduce the super partition function hierarchy with $W$-representations
\begin{equation}
Z_n\{g, \tilde{g}|p, \tilde{p}\} = e^{W_n/n} e^{\beta \sum_{k=1}^{\infty} \frac{p_k \tilde{p}_k}{k^2}}, \quad n \geq 1.
\end{equation}

When we neglect the fermionic variables $\{\tilde{p}\}$ and $\{\tilde{g}\}$ in [37], it reduces to the partition function hierarchy [133]
\begin{equation}
Z_n\{g|p\} = e^{W_n/n} e^{\beta \sum_{k=1}^{\infty} \frac{p_k \tilde{p}_k}{k^2}} \sum_{\lambda, \mu} \beta^{\lambda(\lambda+1)/2} \alpha^{\mu(\mu+1)/2} \langle J_{\lambda}, J_{\mu} | p_k = \delta_{k,n} \rangle \cdot J_{\lambda}(g).
\end{equation}

Similar to [24], we have
\begin{equation}
\frac{1}{n^{r+1}} W_n J_{\Lambda} = \sum_{\Gamma \rightarrow ([\Lambda] - nr|m)} (-1)^{m(m-1)/2} \beta^{r(\alpha-1)} \frac{E_{N,m-1}(J_{\Lambda})}{E_{N,m-1}(J_{\Gamma})} \frac{\alpha^{J_{\Lambda}/\Gamma} \delta_{k,n}}{(J_{\Gamma}, J_{\Gamma})_{\beta}} J_{\Gamma},
\end{equation}
where $m > 0$. Hence there are the character expansions for the super partition function hierarchy [37]
\begin{equation}
Z_n\{g, \tilde{g}|p, \tilde{p}\} = \sum_{\lambda, \mu} \beta^{\lambda(\lambda+1)/2} \alpha^{\mu(\mu+1)/2} \langle J_{\lambda}, J_{\mu} | p_k = \delta_{k,n} \rangle \cdot J_{\lambda}(g)
\end{equation}
\begin{equation}
\cdot \sum_{\Lambda, \Gamma} (-1)^{m(m-1)/2} \beta^{\Lambda(\Lambda+1)/2} \alpha^{\Gamma(\Gamma+1)/2} \langle J_{\Lambda}, J_{\Gamma} | p_k = \delta_{k,n} \rangle \cdot J_{\Gamma}(p) \cdot J_{\lambda}(g),
\end{equation}
where $\Lambda$ and $\Gamma$ are superpartitions of fermionic degree $m > 0$. 

6
Generalized super $\beta$-deformed HK partition function and super partition function hierarchies with $W$-representations

In terms of the variables $\{p\} = \{p_1, p_2, \cdots\}$ and $\{\tilde{p}\} = \{\tilde{p}_0, \tilde{p}_1, \tilde{p}_2, \cdots\}$, the operator $\Delta + \beta N \sum_{i=1}^{N} \theta_i \frac{\partial}{\partial \theta_i}$ can be written as

$$\bar{\Delta} = \sum_{n \geq 0} ((1 - \beta)n + \beta N) \tilde{p}_n \frac{\partial}{\partial \tilde{p}_n} + \sum_{m \geq 0, n > 0} (n \tilde{p}_{m+n} \frac{\partial}{\partial \tilde{p}_m} + \beta p_n \tilde{p}_m \frac{\partial}{\partial \tilde{p}_{m+n}}).$$  \hfill (41)

Define the operator $\bar{W}_0 = W_0 + \bar{\Delta}$, then we have

$$\bar{W}_0 J_{\Lambda} = \sum_{(i,j) \in \Lambda^\ast} c(i,j) J_{\Lambda}. \hfill (42)$$

We call $\bar{W}_0$ the generalized super $\beta$-deformed Hurwitz operator. In terms of $\bar{W}_0$, we can construct the generalized super $\beta$-deformed HK partition function

$$\bar{Z}_0(p, \tilde{p}) = e^{\bar{W}_0} e^{\beta e^{-tN}(p_{1} + \tilde{p}_{1})} = \sum_{\Lambda} e^{t \sum_{(i,j) \in \Lambda^\ast} c(i,j) \frac{1}{(J_{\Lambda}, J_{\Lambda})_{\beta}} J_{\Lambda}(p, \tilde{p}) \cdot J_{\Lambda}(g_k = e^{-tN} \delta_{k,1}, \tilde{g}_k = e^{-tN} \theta \delta_{k,1}).} \hfill (43)$$

Let us introduce a series of operators

$$\bar{W}_{-n} = \frac{1}{(n-1)!} [\bar{W}_{-1}, [\bar{W}_{-1}, \cdots, [\bar{W}_{-1}, E_1] \cdots]], \quad n \geq 2, \hfill (44)$$

where

$$E_1 = [\bar{W}_0, p_1] = \sum_{n=1}^{\infty} n p_{n+1} \frac{\partial}{\partial p_n} + \sum_{n=0}^{\infty} (n+1) \tilde{p}_{n+1} \frac{\partial}{\partial \tilde{p}_n} + \beta N p_1,$$

$$\bar{W}_{-1} = [\bar{W}_0, E_1]. \hfill (45)$$

There are the actions

$$\bar{W}_{-n} J_{\Lambda} = \sum_{\Omega} \prod_{(i,j) \in \Omega^\ast/\Lambda^\ast} c(i,j)^\alpha B_{\Omega,\Omega'} J_{\Omega}, \quad n \geq 1, \hfill (46)$$

where the sum is over the superpartitions $\Omega$ satisfying $\Lambda^\ast \subset \Omega^\ast$, $\Lambda^\ast \subset \Omega^\ast$ and $|\Omega^\ast/\Lambda^\ast| = |\Omega^\ast/\Lambda^\ast| = n$.

Similar to (21), we introduce the super partition function hierarchy with $W$-representations

$$\bar{Z}_{-n,m}(p, \tilde{p}) = e^{\bar{W}_{-n/m} J_{\Lambda_{m}}}, \quad n \geq 1, m \geq 0. \hfill (47)$$

When $m = 0$ in (47), it is indeed equivalent to the partition function hierarchy (21). There are the character expansions for (47)

$$\bar{Z}_{-n,m}(p, \tilde{p}) = \sum_{\Lambda} (-1)^{\frac{m(m-1)}{2} \beta (|\Lambda| - \frac{m(m-1)}{2}) (na - 1)/n} (E_{N,m}(J_{\Lambda}))^\alpha.$$
\[
\left. J_{\Lambda/\Lambda_m} \{ p_k = \delta_{k,n} \} \right|_{\langle J_{\Lambda}, J_{\Lambda} \rangle} \beta J_{\Lambda} \{ p, \tilde{p} \}, \quad m \geq 0,
\]
where we have used the evaluation formula [22]
\[
E_{N,m}(J_{\Lambda}) = \beta^{-\frac{m(m-1)}{2}} \prod_{(i,j) \in \Lambda^\oplus/\Lambda_m^\oplus} c(i,j)
\]
rather than [26] due to the coefficients in the action [49].

In similarity with [37], we may introduce the super partition function hierarchy with \( W \)-representations
\[
\bar{Z}_n \{ g, \tilde{g} | p, \tilde{p} \} = e^{\bar{W}_n / n} e^{\beta \sum_{k=1}^{\infty} \left( \frac{\rho_k}{k} + \rho_{k-1} \right)} \\
= \sum_{\Lambda, \Gamma} (-1)^{m(m-1)/2} \beta^{|\Lambda|/\Gamma} (n\alpha-1) / n \left( \frac{E_{N,m}(J_{\Lambda})}{E_{N,m}(J_{\Gamma})} \right)^{\alpha} \\
\left. J_{\Lambda/\Gamma} \{ p_k = \delta_{k,n} \} \right|_{\langle J_{\Lambda}, J_{\Lambda} \rangle} \beta (J_{\Gamma}, J_{\Gamma}) \beta J_{\Gamma} \{ p \} J_{\Lambda} \{ g \},
\]
where \( \Lambda \) and \( \Gamma \) are superpartitions of fermionic degree \( m \geq 0 \), the operators \( \bar{W}_n \) are defined as
\[
\bar{W}_n = \frac{(-1)^n}{(n-1)!} [ \bar{W}_1, [ \bar{W}_1, \ldots, [ \bar{W}_1, E_{-1} ], \ldots ] ], \quad n \geq 2,
\]
\( E_{-1} \) and \( \bar{W}_1 \) are respectively given by
\[
E_{-1} = \beta^{-1} \left[ \bar{W}_0, \frac{\partial}{\partial p_1} \right] \\
= - \sum_{n=1}^{\infty} (n+1) p_n \frac{\partial}{\partial p_{n+1}} - \sum_{n=0}^{\infty} (n+1) \tilde{p}_n \frac{\partial}{\partial \tilde{p}_{n+1}} - N \frac{\partial}{\partial p_1}, \\
\bar{W}_1 = [ \bar{W}_0, E_{-1} ].
\]

It is obvious that when we neglect the fermionic variables \( \{ \tilde{p} \} \) and \( \{ \tilde{g} \} \) in (50), it reduces to the partition function hierarchy (38).

5 Summary

We have constructed the (generalized) super \( \beta \)-deformed HK partition functions which are expressed as the exponents of the (generalized) super \( \beta \)-deformed Hurwitz operators acting on the super function \( e^{\beta e^{-tN}(p_1 + \tilde{p}_1)} \). We have derived their character expansions with respect to the Jack superpolynomials. It was noted that these two super partition functions are superintegrable, i.e., \( \langle \text{character} \rangle \sim \text{character} \). In terms of \( p_1, \frac{\partial}{\partial p_1} \) and the (generalized) super \( \beta \)-deformed Hurwitz operators \( \bar{W}_0 \) and \( \bar{W}_0 \), we have constructed the super partition function hierarchies with \( W \)-representations. Moreover, by providing their character expansions with respect to the Jack superpolynomials, the superintegrability for these super partition function hierarchies has been confirmed. Our results provide new insights into the supereigenvalue models. For further research, it would be interesting to search for the super integral representations for the super partition function hierarchies.
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References

[1] A. Morozov and Sh. Shakirov, Generation of matrix models by $\hat{W}$-operators, J. High Energy Phys. 04 (2009) 064, [arXiv:0902.2627].

[2] H. Itoyama, A. Mironov and A. Morozov, Complete solution to Gaussian tensor model and its integrable properties, Phys. Lett. B 802 (2020) 135237, [arXiv:1910.03261].

[3] B. Kang, L.Y. Wang, K. Wu, J. Yang, W.Z. Zhao, $W$-representation of rainbow tensor model, J. High Energy Phys. 05 (2021) 228, [arXiv:2104.01332].

[4] L.Y. Wang, R. Wang, K. Wu and W.Z. Zhao, $W$-representations of the fermionic matrix and Aristotelian tensor models, Nucl. Phys. B 973 (2021) 115612, [arXiv:2110.14269].

[5] Y. Chen, R. Wang, K. Wu and W.Z. Zhao, Correlators in the supereigenvalue model in the Ramond sector, Phys. Lett. B 807 (2020) 135563, [arXiv:2006.11013].

[6] R. Wang, S.K. Wang, K. Wu and W.Z. Zhao, Correlators in the Gaussian and chiral supereigenvalue models in the Neveu-Schwarz sector, J. High Energy Phys. 11 (2020) 119, [arXiv:2009.02929].

[7] A. Mironov and A. Morozov, Superintegrability summary, [arXiv:2201.12917].

[8] A. Mironov, A. Morozov and Z. Zakirova, New insights into superintegrability from unitary matrix models, Phys. Lett. B 831 (2022) 137178, [arXiv:2203.03869].

[9] R. Wang, C.H. Zhang, F.H. Zhang and W.Z. Zhao, CFT approach to constraint operators for ($\beta$-deformed) hermitian one-matrix models, [arXiv:2203.14578].

[10] V. Mishnyakov and A. Oreshina, Superintegrability in $\beta$-deformed Gaussian Hermitian matrix model from $W$-operators, Eur. Phys. J. C 82 (2022) 548, [arXiv:2203.15675].

[11] A. Morozov and N. Tselousov, Differential expansion for antiparallel triple pretzels: the way the factorization is deformed, [arXiv:2205.12238].

[12] A. Mironov and A. Morozov, Bilinear character correlators in superintegrable theory, [arXiv:2206.02045].

[13] R. Wang, F. Liu, C.H. Zhang and W.Z. Zhao, Superintegrability for ($\beta$-deformed) partition function hierarchies with $W$-representations, [arXiv:2206.13038].

[14] A. Bawane, P. Karimi and P. Sułkowski, Proving superintegrability in $\beta$-deformed eigenvalue models, [arXiv:2206.14763].

[15] A. Mironov and A. Morozov, Superintegrability as the hidden origin of Nekrasov calculus, [arXiv:2207.08242].

[16] I. Goulden and D. Jackson, Transitive factorization into transpositions and holomorphic mappings on the sphere, Proc. Amer. Math. Soc. 125 (1997) 51.
[17] A. Mironov and A. Morozov, Virasoro constraints for Kontsevich-Hurwitz partition function, J. High Energy Phys. 02 (2009) 024, arXiv:0807.2843.

[18] P. Desrosiers, L. Lapointe and P. Mathieu, Supersymmetric Calogero-Moser-Sutherland models and Jack superpolynomials, Nucl. Phys. B 606 (2001) 547, arXiv:hep-th/0103178.

[19] S. Corteel and J. Lovejoy, Overpartitions, Trans. Am. Math. Soc. 356 (2004) 1623.

[20] P. Desrosiers, L. Lapointe and P. Mathieu, Classical symmetric functions in superspace, J. Alg. Comb. 24 (2006) 209, arXiv:math/0509408.

[21] P. Desrosiers, L. Lapointe and P. Mathieu, Jack polynomials in superspace, Commun. Math. Phys. 242 (2003) 331, arXiv:hep-th/0209074.

[22] P. Desrosiers, L. Lapointe and P. Mathieu, Evaluation and normalization of Jack superpolynomials, Int. Math. Res. Not. 23 (2012) 5267, arXiv:1104.3260.

[23] P. Desrosiers, L. Lapointe and P. Mathieu, Orthogonality of Jack polynomials in superspace, Adv. Math. 212 (2007) 361, arXiv:math-ph/0509039.

[24] O. Blondeau-Fournier, P. Desrosiers, L. Lapointe and P. Mathieu, Macdonald polynomials in superspace: conjectural definition and positivity conjectures, Lett. Math. Phys. 101 (2012) 27, arXiv:1112.5188.

[25] A. Morozov, On W-representations of \( \beta \)- and \( q,t \)-deformed matrix models, Phys. Lett. B 792 (2019) 205, arXiv:1901.02811.