Gaussian Assignment Process

M.A. Lifshits, A.A. Tadevosian

Abstract

We define Gaussian assignment process, determine the asymptotic behavior of its maximum’s expectation and suggest an explicit strategy that attains the corresponding asymptotics.

1 Introduction

We consider the following random assignment problem. Let \((X_{ij})\) be an \(n \times n\) random matrix with i.i.d. random entries having a common distribution \(P\). Let \(S_n\) denote the group of permutations \(\pi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}\). For every \(\pi \in S_n\) let

\[
S(\pi) = \sum_{i=1}^{n} X_{i \pi(i)}.
\]

We are interested in the study of \(\min_{\pi \in S_n} S(\pi)\) or \(\max_{\pi \in S_n} S(\pi)\) and in finding the optimal permutation \(\arg \min_{\pi \in S_n} S(\pi)\), resp. \(\arg \max_{\pi \in S_n} S(\pi)\).

We refer to [6, 15] for many applications of assignment problem in various fields of mathematics.

The setting with \((X_{ij})\) uniformly distributed on \([0, 1]\) was studied by Steele [15] and Mézard and Parisi [12], where the authors proved that

\[
\mathbb{E} \min_{\pi \in S_n} S(\pi) = \zeta(2) - \frac{\zeta(2) + 2\zeta(3)}{n} + O\left(\frac{1}{n^2}\right), \quad \text{as } n \to \infty,
\]

\(\zeta(\cdot)\) being Riemann’s zeta function. Mézard and Parisi [13] also conjectured that in the exponential case (\(P = \text{Exp}(1)\)) it is true that

\[
\mathbb{E} \min_{\pi \in S_n} S(\pi) \to \zeta(2) = \frac{\pi^2}{6}.
\]

Using replica method from statistical physics [7], they provided an heuristical argumentation in favor of this conjecture.

Later Parisi [14] conjectured the following explicit expression for every fixed \(n\)

\[
\mathbb{E} \min_{\pi \in S_n} S(\pi) = \sum_{k=1}^{n} \frac{1}{k^2}.
\]
and confirmed it for \( n = 1, 2 \) and for \( n \to \infty \). Dotsenko [8] also investigated the precise solution in the exponential case.

In [3, 5, 6] similar problems were investigated for rectangular matrices.

Aldous [1] gave a rigorous proof of Mézard–Parisi conjecture (1). His approach is based on the assignment analysis of a graph with edges provided with exponentially distributed weights, see [2].

We are interested here in the case when the distribution \( P \) is the standard normal, i.e. \( P = \mathcal{N}(0, 1) \). By convenience reasons, for such symmetric distributions it is more natural to study the maximum of random assignment.

We stress that the support of Gaussian distribution is unbounded which essentially changes the results. Now the expectation of maximum does not tend to a finite limit, as \( n \to \infty \), but increases to infinity, although quite slowly.

Our main result is the following theorem:

**Theorem 1** Let \( \{S(\pi), \pi \in S_n\} \) be a Gaussian process given by

\[
S(\pi) = \sum_{i=1}^{n} X_{\pi(i)}, \quad \pi \in S_n,
\]

where \( X_{ij} \ (1 \leq i, j \leq n) \) are i.i.d. standard Gaussian random variables. Then it is true that

\[
\lim_{n \to \infty} \frac{\mathbb{E} \max_{\pi \in S_n} S(\pi)}{n\sqrt{2 \log n}} = 1.
\]

In the following we call \( \{S(\pi), \pi \in S_n\} \) defined in (2) a Gaussian assignment process. It seems to be quite an interesting object worth of detailed studies in its own right. Note that \( S \) is stationary with respect to the group structure of \( S_n \).

We also show that this asymptotic behavior of the maximum is attained at an explicitly constructed greedy random permutation \( \pi^* \) that turns out to be asymptotically optimal in terms of expectation, i.e.

\[
\lim_{n \to \infty} \frac{\mathbb{E} S(\pi^*)}{n\sqrt{2 \log n}} = 1.
\]

In addition to the expectation study, we provide a central limit theorem for \( S(\pi^*) \), namely, the following is true.

**Theorem 2** We have

\[
\frac{S(\pi^*) - A_n}{B_n} \xrightarrow{n \to \infty} \mathcal{N}(0, 1),
\]

where

\[
A_n = n \sqrt{2 \log n} + O \left( \frac{n \log \log n}{\sqrt{\log n}} \right),
\]

\[
B_n^2 = \frac{\pi^2}{12} n \log n + o \left( \frac{n}{\log n} \right).
\]
Moreover,
\[ \sup_{r \in \mathbb{R}} \left| \mathbb{P} \left( \frac{S(\pi^*) - A_n}{B_n} \leq r \right) - \Phi(r) \right| = O \left( \frac{1}{\sqrt{n}} \right), \]
where \( \Phi(\cdot) \) denotes the standard normal distribution function.

The structure of the work is as follows. In Section 2 we provide an upper bound for maximum’s expectation. In Section 3 we describe the greedy strategy (Section 3.1) and provide the lower bound for its outcome thus proving its optimality (Section 3.2). Finally, the corresponding central limit theorem (Theorem 2) is proved in Section 3.3.

2 An upper bound

We use the following standard estimate for the maximum of Gaussian random variables, see [11, p.180].

**Lemma 3** Let \( \{X_j\}_{j=1}^N \) be a family of centered Gaussian random variables such that \( \max_{1 \leq j \leq N} \mathbb{E} X_j^2 \leq \sigma^2 \). Then
\[ \mathbb{E} \max_{1 \leq j \leq N} X_j \leq \sqrt{2 \log N} \sigma. \quad (5) \]

We stress that no assumptions on the dependence are required in this statement.

Since \( \mathbb{E} S(\pi)^2 = n \) for all \( \pi \in \mathcal{S}_n \) and \( |\mathcal{S}_n| = n! \), we obtain from (5) the necessary upper bound
\[ \mathbb{E} \max_{\pi \in \mathcal{S}_n} S(\pi) \leq \sqrt{2 \log(n!)} n = n \sqrt{2 \log n} + O(n), \quad \text{as } n \to \infty. \]

3 The greedy strategy and its properties

3.1 Definition

Consider the following greedy strategy for constructing a random permutation \( \pi^* \) providing an asymptotically optimal (in average) value of the assignment process. Let \( [i] := \{1, 2, \ldots, i\} \). Define
\[ \pi^*(1) := \arg \max_{j \in [n]} X_{1j}, \]
and for all \( i = 2, \ldots, n \)
\[ \pi^*(i) := \arg \max_{j \notin \pi^*(i-1)} X_{ij}. \]

It is natural to call this strategy greedy, because on every step we consider the line \( i \), take the maximum of its available elements (without considering the
influence of this choice on subsequent steps) and then forget the line $i$ and the corresponding column $\pi^*(i)$.

Due to the simple structure, the summands in the representation

$$S(\pi^*) = \sum_{i=1}^{n} X_{i\pi^*(i)}$$

are independent.

### 3.2 A lower bound

Our main goal in this subsection is summarized in the following statement.

**Proposition 4** It is true that

$$\mathbb{E} S(\pi^*) \geq n \sqrt{2 \log n (1 + o(1))}, \quad \text{as } n \to \infty.$$  

**Proof:**

For every fixed $i$ introduce the index set

$$\beta_{n,i} := [n] \setminus \pi^*([i-1]).$$

and denote $m = m(n,i) := |\beta_{n,i}| = n - i + 1$. Notice that $m$ random variables

$$\{X_{ij}, j \in \beta_{n,i}\}$$

are still i.i.d. standard normal although the index set $\beta_{n,i}$ is itself random.

Denote $\mathcal{A} := \{ \max_{j \in \beta_{n,i}} X_{ij} \geq 0 \}$ and let $\mathcal{A}^c$ be its complement.

The expectation $\mathbb{E} X_{i\pi^*(i)}$ can be split into two parts:

$$\mathbb{E} X_{i\pi^*(i)} = \mathbb{E} \max_{j \in \beta_{n,i}} X_{ij} = \mathbb{E} \max_{j \in \beta_{n,i}} X_{ij} 1_{\{\mathcal{A}\}} + \mathbb{E} \max_{j \in \beta_{n,i}} X_{ij} 1_{\{\mathcal{A}^c\}}.$$  

The second term is small because for every $k \in \beta_{n,i}$

$$\left| \mathbb{E} \max_{j \in \beta_{n,i}} X_{ij} 1_{\{\mathcal{A}^c\}} \right| \leq \mathbb{E} \left( |X_{ik}| 1_{\{X_{ij} \leq 0, j \in \beta_{n,i}, j \neq k\}} \right) = \left( \frac{1}{2} \right)^{m-1} \cdot \mathbb{E} |X_{11}| = o(1).$$

Consider the first term

$$\mathbb{E} \max_{j \in \beta_{n,i}} X_{ij} 1_{\{\mathcal{A}\}} = \int_0^\infty \mathbb{P} \left\{ \max_{j \in \beta_{n,i}} X_{ij} 1_{\{\mathcal{A}\}} > t \right\} dt.$$  

It is obvious that

$$\mathbb{P} \left\{ \max_{j \in \beta_{n,i}} X_{ij} 1_{\{\mathcal{A}\}} \leq t \right\} \leq \mathbb{P} \left\{ \max_{j \in \beta_{n,i}} X_{ij} \leq t \right\} = \Phi^m(t).$$
Therefore, for every $r > 0$ it is true that
\[
\mathbb{E} \max_{j \in \beta_{n,i}} X_{ij} 1_{\{A\}} \geq \int_0^r (1 - \Phi^m(t)) dt \geq r (1 - \Phi^m(r)).
\]

We have the following upper bound for $\Phi^m(r)$:
\[
\Phi^m(r) = (1 - \hat{\Phi}(r))^m \leq \exp \left( -m \hat{\Phi}(r) \right),
\]
where $\hat{\Phi}$ is the tail of the standard normal law. By using inequality
\[
\hat{\Phi}(r) \geq \frac{1}{\sqrt{2\pi}} \left( \frac{1}{r} - \frac{1}{r^3} \right) e^{-r^2/2}
\]
for $r := r(m) = \sqrt{2 \log m} - 1$ we have
\[
\exp (-m \hat{\Phi}(r)) = o(1),
\]
hence,
\[
\mathbb{E} \max_{j \in \beta_{n,i}} X_{ij} 1_{\{A\}} \geq \sqrt{2 \log m} (1 + o(1)),
\]
and we arrive at
\[
\mathbb{E} X_{i\pi^*(i)} \geq \sqrt{2 \log m} (1 + o(1)) + o(1).
\]

Therefore we have a lower bound
\[
\begin{align*}
\mathbb{E} S(\pi^*) &= \sum_{i=1}^n \mathbb{E} X_{i\pi^*(i)} \\
&\geq \sum_{i=1}^n \sqrt{2 \log m(n, i)} (1 + o(1)) \\
&= \sum_{m=1}^n \sqrt{2 \log m} (1 + o(1)) \\
&= n \sqrt{2 \log n} (1 + o(1)).
\end{align*}
\]

\[\square\]

Taken together, the upper bound (5) and Proposition 4 yield the chain of bilateral estimates:
\[
n \sqrt{2 \log n} (1 + o(1)) \leq \mathbb{E} S(\pi^*) \leq \mathbb{E} \max_{\pi \in \mathcal{S}_n} S(\pi) \leq n \sqrt{2 \log n} (1 + o(1)), \quad \text{as } n \to \infty,
\]
which proves both (3) and (4).

We conclude that the greedy strategy is asymptotically optimal for maximization of expectation of the Gaussian assignment process.
3.3 Central Limit Theorem

Proof of Theorem 2

For proving a central limit theorem for sums of independent variables, it is sufficient to prove that the corresponding Lyapunov fraction tends to zero. Recall that each term $X_{\pi^*(i)}$ of the sum in the representation (6) is a maximum of $m = m(n, i) := n - i + 1$ independent standard normal random variables. It is well known that properly centered and scaled Gaussian maxima converge weakly to Gumbel distribution, namely,

$$X_{\pi^*(i)} - a_m b_m \xrightarrow{d} G,$$

where $G$ has the distribution function $F_G(x) = \exp(-e^{-x})$, and

$$a_m := \sqrt{2 \log m} - \frac{\log \log m + \log 4\pi}{2\sqrt{2\log m}},$$
$$b_m := (2\log m)^{-\frac{1}{3}}.$$

see, e.g. [9, Sect. 2.3.2].

It is known (see. [10]) that all moments of $G$ are finite. In particular,

$$\mathbb{E} G = \gamma, \quad \text{Var} G = \zeta(2),$$

where $\gamma$ is Euler constant and $\zeta(\cdot)$ is Riemann’s zeta function.

Furthermore, an elementary calculation shows that the family of random variables $\left\{ \left| \frac{X_{\pi^*(i)} - a_m}{b_m} \right|^3 \right\}_{n, i}$ is uniformly integrable. Therefore, the weak convergence implies convergence of moments, see [4, Sect. 1.5].

$$\mathbb{E} \frac{X_{\pi^*(i)} - a_m}{b_m} \xrightarrow{m \to \infty} \mathbb{E} G;$$
$$\mathbb{E} \left( \frac{X_{\pi^*(i)} - a_m}{b_m} \right)^2 \xrightarrow{m \to \infty} \mathbb{E} G^2;$$
$$\mathbb{E} \left| \frac{X_{\pi^*(i)} - a_m}{b_m} \right|^3 \xrightarrow{m \to \infty} \mathbb{E} |G|^3.$$

We obtain asymptotic expressions for expectations and variances of $X_{\pi^*(i)}$, namely:

$$\mathbb{E} X_{\pi^*(i)} = a_m + b_m \mathbb{E} G(1 + o(1)) = a_m + b_m \gamma(1 + o(1)),$$

$$\text{Var} X_{\pi^*(i)} = b_m^2 \left[ \mathbb{E} \left( \frac{X_{\pi^*(i)} - a_m}{b_m} \right)^2 - \left( \mathbb{E} \frac{X_{\pi^*(i)} - a_m}{b_m} \right)^2 \right]$$
$$= b_m^2 \left[ \mathbb{E} G^2 - (\mathbb{E} G)^2 \right] (1 + o(1))$$
$$= b_m^2 \text{Var} G (1 + o(1)) = (2\log m)^{-1} \zeta(2) (1 + o(1)).$$

By using [4] we also obtain the bound

$$|a_m - \mathbb{E} X_{\pi^*(i)}| = O \left( b_m \right).$$
Finally, let us evaluate the third absolute central moment of $X_{i\pi^*(i)}$:

$$
\begin{align*}
\mathbb{E} \left[ X_{i\pi^*(i)} - \mathbb{E} X_{i\pi^*(i)} \right]^3 &= \mathbb{E} \left[ X_{i\pi^*(i)} - a_m + a_m - \mathbb{E} X_{i\pi^*(i)} \right]^3 \\
&\leq 4 \mathbb{E} \left| X_{i\pi^*(i)} - a_m \right|^3 + 4 \left| a_m - \mathbb{E} X_{i\pi^*(i)} \right|^3 \\
&= O(b_m^3) = O \left( (\log m)^{-3/2} \right).
\end{align*}
$$

It follows that

$$
\sum_{i=1}^{n} \mathbb{E} \left| X_{i\pi^*(i)} - \mathbb{E} X_{i\pi^*(i)} \right|^3 \leq \sum_{i=1}^{n} O \left( (\log m(n,i))^{-3/2} \right)
$$

$$
= \sum_{m=1}^{n} O \left( (\log m)^{-3/2} \right)
$$

$$
= O \left( n (\log n)^{-3/2} \right).
$$

Consider now the variance of $S(\pi^*)$

$$
B^2_n := \text{Var} \, S(\pi^*) = \sum_{i=1}^{n} \text{Var} \, X_{i\pi^*(i)}.
$$

We have

$$
B^2_n = \sum_{m=1}^{n} \zeta(2) (2 \log m)^{-1} (1 + o(1))
$$

$$
= \frac{\zeta(2)}{2} \sum_{m=1}^{n} \frac{1 + o(1)}{\log m} = \frac{\pi^2}{12} \frac{n (1 + o(1))}{\log n}.
$$

Therefore, for Lyapunov fraction we have a bound

$$
L_n := B^{-3/2}_n \sum_{i=1}^{n} \mathbb{E} \left[ X_{i\pi^*(i)} - \mathbb{E} X_{i\pi^*(i)} \right]^3 = O(n^{-1/2}), \quad \text{as } n \to \infty.
$$

Now Lyapunov’s Central Limit Theorem yields

$$
\frac{S(\pi^*) - A_n}{B_n} \xrightarrow{n \to \infty} \mathcal{N}(0,1).
$$

with $A_n := \mathbb{E} S(\pi^*)$ and $B^2_n = \text{Var} \, S(\pi^*)$ as defined above. Moreover, we have Berry–Esseen bound for the convergence rate: there exists a numerical constant $C > 0$ such that for all $n$ it is true that

$$
\sup_{r \in \mathbb{R}} \left| \mathbb{P} \left\{ \frac{S(\pi^*) - A_n}{B_n} \leq r \right\} - \Phi(r) \right| \leq C L_n = O(n^{-1/2}).
$$
Finally, we have the following asymptotic expression for $A_n$.

$$A_n = \sum_{i=1}^{n} E[X_{\pi^*(i)}] = \sum_{m=1}^{n} (a_m + b_m \gamma (1 + o(1)))$$

$$= \sum_{m=1}^{n} \left[ \sqrt{2 \log m} + O \left( \frac{\log \log m}{\sqrt{\log m}} \right) + \frac{\gamma (1 + o(1))}{(2 \log m)^{1/2}} \right]$$

$$= n \sqrt{2 \log n} + O \left( \frac{n \log \log n}{\sqrt{\log n}} \right).$$

□

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