The ubiquitous ‘c’: from the Stefan–Boltzmann law to quantum information

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Abstract. I discuss various aspects of the role of the conformal anomaly number c in two- and one + one-dimensional critical behaviour: its appearance as the analogue of Stefan’s constant, its fundamental role in conformal field theory, in the classification of 2d universality classes, and as a measure of quantum entanglement, among other topics.

Keywords: conformal field theory, critical exponents and amplitudes (theory), entanglement in extended quantum systems (theory)
1. The Stefan–Boltzmann law

It seems appropriate to honour the founder of our subject, after whom this award is named, by recalling one of his most elegant pieces of work. In 1879 Stefan had published his famous law [1] stating that the power per unit area radiated by an ideal black body is proportional to the fourth power of its absolute temperature $T$. In terms of the energy density $u$ in the radiation field, this may be written as

$$u = \left(\frac{4\pi}{c}\right)\sigma T^4,$$

where $\sigma$ is Stefan’s constant, and $c$ here is not the conformal anomaly number of my title, but the (even more ubiquitous) speed of light. Stefan apparently based his empirical law on the analysis of experimental data of Tyndall [2], a Victorian scientist now more famous for his work in magnetism as well as glaciology.\(^1\)

Stefan was Boltzmann’s advisor [3], and it must have been a great pleasure to him when his former student produced [4] a theoretical derivation of this law based on thermodynamic reasoning. It is worth recalling Boltzmann’s derivation, as it illustrates the power, as well as the potential pitfalls, of combining ideas from two different branches of physics. He imagined a box, filled with black-body radiation, of which one wall is a piston which is moved by the pressure of the radiation. According to classical electromagnetism, the pressure $P$ is related to the energy density by $P = \frac{1}{3}u$. The heat-energy balance equation then reads

$$T \, dS = d(uV) + P \, dV = u'(T)V \, dT + u \, dV + \frac{1}{3}u \, dV = u'(T)V \, dT + \frac{4}{3}u \, dV.$$

Dividing through by $T$ and using the fact that $dS$ is a complete differential,

$$\frac{\partial}{\partial V} \left(\frac{u'(T)V}{T}\right) = \frac{\partial}{\partial T} \left(\frac{4u}{3T}\right),$$

which simplifies, after some algebra, to $u'(T) = 4u(T)/T$, that is, $u(T) \propto T^4$.\(^2\)

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\(^1\) As well as Alpine mountaineering—among many famous climbs he made the first ascent of the Weisshorn in the Swiss Alps. The strong correlation between physicists and climbers was evident even at that time.

\(^2\) Note that in $d$ space dimensions an analogous argument leads to $u(T) \propto T^{d+1}$. 

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This simple yet elegant argument was praised by Lorentz in 1907 [5] as ‘a jewel of theoretical physics’. But, with hindsight, we see that Boltzmann was either fortunate, or, more likely, inspired, in juxtaposing results of classical electromagnetism with the idea that radiation behaves like a fluid. For the immediate question is then ‘a fluid of what kind of particle?’— and of course this was not answered, even heuristically, until Planck made his hypothesis, and, more systematically, until the quantization of the radiation field. Indeed, as every student knows, \( \sigma = \frac{\pi^4 k^4}{60 \hbar^3 c^2} \), and, once again with hindsight, a little dimensional analysis would have shown Boltzmann that, if Stefan’s constant depends on fundamental constants, one of these must contain the dimensions of mass and therefore be something which at the time was unknown in the classical physics of the pure radiation field.

In more modern terms, the equation \( P = \frac{1}{3} u \), which was the starting point of Boltzmann’s argument, is equivalent to the statement that the energy–momentum tensor \( T_{\mu\nu} \) is traceless:

\[
 u - 3P = T_{00} - \sum_{j=1}^{3} T_{jj} = T_{\mu\mu} = 0.
\]

This equation is true for the classical Maxwell field, but Boltzmann was implicitly assuming that it also holds for the quantized field. Yet nowadays we know many examples of field theories for which the energy–momentum tensor is traceless at the classical level, but not when the theory is quantized properly. Examples are electrodynamics coupled to (massless) particles, with non-trivial vacuum polarization effects (so in fact Stefan’s law does not hold in QED at high temperatures), and non-Abelian gauge theories.

In fact it is now understood that in a quantum field theory \( T_{\mu\mu} \propto \beta(g) \), where \( \beta(g) \) is the renormalization group (RG) beta-function\(^3\). This means that, even when fluctuations (either quantum or thermal) are taken into account, the trace \( T_{\mu\mu} \) vanishes at RG fixed points, that is, in a critical theory at which the correlation length diverges.

Thus we expect a generalized Stefan–Boltzmann law to hold at all quantum critical points which have a relativistic dispersion law \( E \sim v |k| \) at low energies, where, however, \( v \) does not have to be the speed of light, but could, for example, be the Fermi velocity or the speed of sound. In 1+1 dimensions, with which we shall henceforth be concerned, there are many such examples: free fermions at finite density, Luttinger liquids, quantum Hall edge states, and many critical quantum spin chains. In that case, the one + one-dimensional analogue of the Stefan–Boltzmann law takes the form

\[
 u = \frac{\pi c}{6 \hbar v} (kT)^2, \tag{1}
\]

where now \( c \) is not the speed of light, but, as a simple calculation in quantum statistical mechanics shows,

\[
 c = \begin{cases} 
 1 : & \text{single free boson;} \\
 N : & N \text{ free bosons;} \\
 \frac{1}{2} : & \text{a spinless fermion.}
\end{cases}
\]

\(^3\) The appearance of a non-zero value for the trace after quantization is often (confusingly) referred to as the ‘conformal anomaly’. However this is not the anomaly related to \( c \) which is the object of the subsequent discussion.
However, in general, \( c \) is fractional, and indeed is the first appearance of the ubiquitous conformal anomaly number.

2. Why is the conformal anomaly anomalous?

In order to understand how \( c \) arises in conformal field theory, let us consider the simplest possible example of a single scalar field \( \phi(r) \) with action

\[
S = \int (\nabla \phi)^2 \, d^2 x.
\]

In two space dimensions, this might represent the energy in the electrostatic field, or of spin waves at low temperatures. Many important models of statistical physics in 2 or 1 + 1 dimensions can be ‘bosonized’ into this simple form. In order to understand why this is conformally invariant, it is useful to define so-called complex coordinates

\[
z = x_1 + i x_2 \quad \bar{z} = x_1 - i x_2,
\]

in terms of which

\[
S \propto \int (\partial_z \phi)(\partial_{\bar{z}} \phi) \, d^2 z.
\]

In two dimensions conformal mappings correspond locally to analytic functions \( z \to z' = f(z) \), and we can see that, classically, \( S \) is indeed invariant under these, since \( \partial_z = f'(z) \partial_{z'} \), \( \bar{z} = \overline{f'(z)} \partial_{\bar{z}'} \), and \( d^2 z = |f'(z)|^{-2} \, d^2 z' \). Indeed, the trace \( T_{\mu}^{\mu} \), as calculated by Noether’s theorem, vanishes identically. The non-zero components, in complex coordinates, are

\[
T = T_{zz} = -(\partial_z \phi)^2 \quad \text{and} \quad \bar{T} = T_{\bar{z}\bar{z}} = -(\partial_{\bar{z}} \phi)^2.
\]

Under a conformal mapping \( z \to z' = f(z) \), we see that classically \( T \) transforms simply:

\[
T(z) = (f'(z))^2 T(z'),
\]

and similarly for \( \bar{T} \).

However, once the fluctuations are taken into account, this is no longer the case. In fact \( T \) as defined above is divergent, since \( \langle \phi_k \phi_{-k} \rangle \propto 1/k^2 \), so \( \langle T \rangle \sim \int k^2 (d^2 k/k^2) \). One way to define it properly is by point-splitting:

\[
T(z) = \lim_{\delta \to 0} \left[ \partial_z \phi \left( z + \frac{\delta}{2} \right) \partial_z \phi \left( z - \frac{\delta}{2} \right) - \frac{1}{2\delta^2} \right]
\]

where \( \delta \) is a short-distance cut-off of the order of the lattice spacing. However this subtraction does not in general commute with conformal mappings, because in the \( z' \)-plane we should still subtract off \( 1/2\delta^2 \) rather than \( 1/2(\|f'(z)\|\delta)^2 \). The result is the appearance of an anomalous term in the transformation law for \( T \):

\[
T(z) = f'(z)^2 T(f(z)) - \frac{c}{12} \{ f, z \}, \tag{2}
\]

where \( \{ f, z \} = (f''' f' - \frac{3}{2} f''^2)/f'^2 \) (the Schwartzian derivative), and in this case \( c = 1 \). This is a classic example of the appearance of an anomaly in quantum field theory, when a symmetry (in this case, conformal symmetry) is not respected by the necessary regularization procedure. In fact the form of the last term on the rhs of (2), although complicated, is completely fixed by the requirement that it hold under a general iterated
sequence of conformal mappings. The only arbitrariness is in the coefficient $c$, which is thereby a fixed parameter characterizing the particular CFT or universality class.

Although (2) might appear rather technical, from this result in fact flow all the various ubiquitous physical manifestations of $c$.

3. Stefan–Boltzmann in $1+1$ dimensions

Consider a critical one + one-dimensional quantum system (with a linear dispersion relation $E \sim v|k|$ at low energies) of length $L$, at low but finite temperature $T$. As Feynman taught us, the partition function $\text{Tr} e^{-\beta \hat{H}}$ is given by the path integral in imaginary time with periodic boundary conditions modulo $\beta \hbar$, where $\beta = 1/kT$. That is, it is equivalent to a two-dimensional classical system on a cylinder (see figure 1) of circumference $\beta \hbar$. This is conformally related to the plane by the mapping $z \rightarrow z' = (\beta \hbar/2\pi) \log z$. Applying (2) and using the fact that $\langle T \rangle_{\text{plane}} = 0$, we find the result for the energy density

$$u = \langle T_{00} \rangle = (1/2\pi) (\langle T \rangle_{\text{cylinder}} + \langle \bar{T} \rangle_{\text{cylinder}}) = \frac{\pi c}{6\hbar} (kT)^2,$$

(3)

the one + one-dimensional version (1) of the Stefan–Boltzmann law, in units where $v = 1$. Note that this corresponds to a linear specific heat. In principle, this can be compared with experiment, although this requires a separate determination of $v$.

Equivalently, we can think of the coordinate along the cylinder as representing imaginary time, in which case we have a one + one-dimensional quantum theory defined on a finite spatial interval $\ell = \beta \hbar$, with periodic boundary conditions. In that case (3) gives the finite-size corrections to the ground state energy, $E_0 \sim O(\ell) - (\pi c/6\ell)$, once again in units where $v = 1$. This is one the most effective ways of determining $c$ from numerical or analytic diagonalization of the Hamiltonian $\hat{H}$.

4. The conformal periodic table

As with any quantum theory, it is advantageous to realize the symmetries of CFT in terms of generators acting on the Hilbert space of states of the theory. In this case these are the so-called Virasoro generators $L_n = (1/2\pi i) \oint z^{n+1} T(z) \, dz$ (and analogously $\bar{L}_n$). The transformation law (2) is completely equivalent to the Virasoro algebra

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{1}{12} cn(n^2 - 1) \delta_{n,-m}.$$
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**Table 1.** The first few elements in the conformal periodic table.

| \( c \) | Scaling dimensions | Universality class |
|---------|-------------------|-------------------|
| \( \frac{1}{2} \) | \( 0, \frac{1}{4}, 1 \) | Critical Ising |
| \( \frac{7}{10} \) | \( 0, \frac{3}{80}, \frac{3}{10}, \frac{7}{16}, \frac{3}{5}, \frac{3}{7} \) | Tricritical Ising |
| \( \frac{4}{5} \) | \( 0, \frac{1}{20}, \frac{1}{5}, \frac{1}{2}, \frac{21}{30}, \frac{2}{5}, \frac{7}{5}, \frac{13}{30}, \frac{3}{7} \) | Tetracritical Ising |
| \( \frac{5}{6} \) | \( 0, \frac{1}{15} \times 2, \frac{3}{5}, \frac{2}{5} \times 2, \frac{7}{5}, \frac{3}{7} \) | Critical three-state Potts |
| \( \vdots \) | \( \vdots \) | \( \vdots \) |

In particular \( L_0 \) and \( \overline{L}_0 \) generate scale transformations and rotations, and the scaling fields of the CFT correspond to eigenstates of these operators, with eigenvalues giving all the critical exponents. Thus the study of the representation theory of the Virasoro algebra gives a way of classifying all possible CFTs and thereby universality classes in 2d. The breakthrough in this direction came following the seminal 1984 paper of Belavin, Polyakov and Zamolodchikov (BPZ) [8] in which they showed that, for certain special rational values of \( c < 1 \), the CFT closes with only a finite number of representations of the Virasoro algebra, and, for these cases, all the critical exponents and multi-point correlation functions are calculable. Shortly thereafter Friedan \textit{et al} [9] showed that unitary CFTs (corresponding to local, positive definite Boltzmann weights) are a subset of this list, with \( c = 1 - 6/m(m+1) \) and \( m \) an integer \( \geq 3 \). This gives rise to what might be termed the ‘conformal periodic table’ (table 1). The first few examples may be identified with well-known universality classes. The ‘hydrogen atom’ of CFT is the scaling limit of the critical Ising model, ‘helium’ is the tricritical Ising model, and so on. Note, however, that at the next value of \( c = \frac{4}{5} \) two possible ‘isotopes’ arise. In the second, corresponding to the critical three-state Potts model, not all the scaling dimensions allowed by BPZ in fact occur, but some of those that do actually appear twice. In fact the constraint of unitarity is not sufficient to determine exactly which representations actually occur in a given CFT. The answer to this is provided by demanding consistency of the theory on a torus [10], by interchanging the interpretations of space and imaginary time similar to the case of the cylinder mentioned above. For the torus, this is a modular transformation, and the requirement of modular invariance has become another powerful tool in classifying CFTs, completely solved in the case \( c < 1 \) by Cappelli \textit{et al} [11].

**5. \( c \) and entanglement entropy**

In recent years, \( c \) has been playing a new role in quantifying the degree of quantum entanglement in the ground state of a one + one-dimensional critical system (for example, a quantum spin chain). There is by now a huge literature on this [12], but here I will concentrate on the simplest possible scenario: a long system, of length \( L \), near a quantum critical point which is divided into two halves A and B at the origin. We assume that the degrees of freedom in A are accessible only to observer A, and conversely those in B only to B. In the ground state, A’s observations are entangled with those of B: if A performs a measurement with a certain outcome, this can restrict the possible outcomes of measurements B can subsequently make. A very useful way to measure the entanglement

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Figure 2. The surface on which the path integral giving the Rényi entropy for $n = 2$ is evaluated.

between A and B is through A’s reduced density matrix

$$\rho_A = \text{Tr}_B |0\rangle \langle 0|,$$

and the so-called Rényi entropies

$$S_A^{(n)} = (1 - n)^{-1} \log \text{Tr}_A \rho_A^n.$$ 

The limit $n \to 1$ gives the well-known von Neumann entropy—$\text{Tr}_A \rho_A \log \rho_A$. If the state $|0\rangle$ is unentangled, $\rho_A$ has a single eigenvalue = 1, so that $S_A^{(n)} = 0$. However, a maximally entangled state will give rise to $O(e^{\text{const.}L})$ eigenvalues all of the same order, so that $S_A^{(n)} = O(L)$.

It turns out that $\text{Tr}_A \rho_A^n$ is given by the path integral, or partition function, on an $n$-sheeted surface $\mathcal{R}_n$ with a branch cut running from the origin to the boundary, as shown for the case $n = 2$ in figure 2. This, however, is related the full plane by the conformal mapping $z \to z' = z^{1/n}$. Using once again (2), we then find

$$\langle T \rangle_{\mathcal{R}_n} = \frac{c(1-n^{-2})}{12 z^2}.$$ 

This behaviour means that the branch point behaves like the insertion of a scaling operator with dimension $x_n = (c/12)(n - n^{-1})$, and thus the partition function on $\mathcal{R}_n$ goes like $L^{-x_n}$ at the critical point, or $\xi^{-x_n}$ away from it, where $\xi$ is the correlation length. Thus

$$S_A^{(n)} \sim \frac{c}{12} (1 + n^{-1}) \log L,$$

for $L \gg \xi$, with $L$ replaced by $\xi$ in the opposite limit. This result [13, 14] has been verified by numerous analytic computations in exactly solvable models, and has become the gold standard for measuring $c$ numerically, using, for example, density matrix RG methods. However, it is but the tip of the iceberg in results of this type: taking, for example, A to consist of two disjoint intervals gives access to the entire spectrum of scaling dimensions of the CFT, as well as the operator product expansion coefficients [12].
6. Other appearances of \( c \)

These are too numerous to list exhaustively, but I will give some of my favourites.

- If we put a CFT on a manifold of Euler character \( \chi \) and linear size \( L \), the free energy has the asymptotic form as \( L \to \infty \) [15]

\[
F \sim AL^2 + BL - \frac{1}{6}c\chi \log L + \ldots .
\]

This works even for a disc, with \( \chi = 1 \). If there are corners on the boundary, however, the coefficient is modified in a known manner.

- One cannot give a survey of the role of \( c \) without mentioning Zamolodchikov’s beautiful \( c \)-theorem [16]: there exists a function \( C(g) \) of the coupling constants \( \{g\} \) which is decreasing along RG flows and is stationary at RG fixed points, where it equals \( c \). This implies that RG flows, at least in 2d, go ‘downhill’ and rules out (at least for unitary theories) exotic behaviour such as limit cycles. Remarkably, no analogous result has been shown in higher dimensions (except in the case of a high degree of supersymmetry), and it remains an open question as to whether it is in fact true.

- One nice consequence of the above is the \( c \)-theorem sum rule [17]: suppose that we move slightly away from the critical point by adding, for example, a magnetic field \( H \). Then in that case the \( q \)-dependent susceptibility \( \chi(q) \) is analytic at wavevector \( q = 0 \), and its curvature there, by scaling, goes like \( H^{-2} \). In fact the coefficient is universal:

\[
c = 3\pi^2 \left( \frac{\delta}{\delta + 1} \frac{H}{k_B T_c} \right)^2 \chi''(q)|_{q=0},
\]

where \( \delta \) is the usual critical exponent in \( M \sim H^{1/\delta} \).

This is only a very selective list. Indeed, one may truthfully say that, at least in two or \( 1+1 \) dimensions, \( c \) is everywhere!

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