Global well-posedness and scattering of the two dimensional cubic focusing nonlinear Schrödinger system

Xing Cheng∗, Zihua Guo∗∗, Gyeongha Hwang∗∗∗, and Haewon Yoon∗∗∗∗

February 23, 2022

Abstract

In this article, we prove the global well-posedness and scattering of the cubic focusing infinite coupled nonlinear Schrödinger system on $\mathbb{R}^2$ below the threshold in $L^2_h(\mathbb{R}^2 \times \mathbb{Z})$. We first establish the variational characterization of the ground state, and derive the threshold of the global well-posedness and scattering. Then we show the global well-posedness and scattering below the threshold by the concentration-compactness/rigidity method, where the almost periodic solution is excluded by adapting the argument in the proof of the mass-critical nonlinear Schrödinger equations by B. Dodson. As a byproduct of the scattering of the cubic focusing infinite coupled nonlinear Schrödinger system, we obtain the scattering of the cubic focusing nonlinear Schrödinger equation on the small cylinder, this is the first large data scattering result of the focusing nonlinear Schrödinger equations on the cylinders. In the article, we also show the global well-posedness and scattering of the two dimensional $N$-coupled focusing cubic nonlinear Schrödinger system in $(L^2(\mathbb{R}^2))^N$.

Keywords: Nonlinear Schrödinger system, ground state, scattering, almost periodic solution, focusing.

Mathematics Subject Classification (2010) Primary: 35Q55; Secondary: 35P25, 58J37

1 Introduction

In this article, we consider the cubic focusing nonlinear Schrödinger system on $\mathbb{R}^2$:

$$
\begin{cases}
   i\partial_t \vec{u} + \Delta_{\mathbb{R}^2} \vec{u} = -\vec{F}(\vec{u}), \\
   \vec{u}(0) = \vec{u}_0,
\end{cases}
$$

(1.1)
where \( \tilde{u}_j = \{ u_j : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C} \}_{j \in \mathbb{Z}_N}, \) \( \tilde{u}_0 = \{ u_{0,j} : \mathbb{R}^2 \to \mathbb{C} \}_{j \in \mathbb{Z}_N}, \) and the nonlinear term \( \tilde{F}_j(\tilde{u}) = \{ \tilde{F}_j(\tilde{u}) \}_{j \in \mathbb{Z}_N} \),

\[
\tilde{F}_j(\tilde{u}) := \sum_{(j_1,j_2,j_3) \in \mathcal{R}(j)} u_{j_1}\tilde{u}_{j_2}u_{j_3} = 2\left( \sum_{k \in \mathbb{Z}_N} |u_k|^2 \right) u_j - |u_j|^2 u_j,
\]

(1.2)

with

\[
\mathcal{R}(j) = \{ (j_1,j_2,j_3) \in \mathbb{Z}_N^3 : j_1 - j_2 + j_3 = j, j_1^2 - j_2^2 + j_3^2 = j^2 \}.
\]

Here \( \mathbb{Z}_N := \{ 0, 1, \ldots, N - 1 \} \), for any \( N \in \mathbb{N}_+ \), with the convention that \( \mathbb{Z}_\infty = \mathbb{Z} \).

The nonlinear Schrödinger system (1.1) enjoys the following conservation laws:

\textbf{mass:} \( \mathcal{M}_{a,b,c}(\tilde{u}(t)) = \int_{\mathbb{R}^2} \sum_{j \in \mathbb{Z}} \left( a + bj + cj^2 \right) |u_j(t,x)|^2 \, dx \), where \( a, b, c \in \mathbb{R} \),

and

\textbf{energy:} \( \mathcal{E}(\tilde{u}(t)) = \int_{\mathbb{R}^2} \sum_{j \in \mathbb{Z}} \left( \frac{1}{2} |\nabla u_j(t,x)|^2 - \frac{1}{4} \left( \tilde{u}_j \tilde{F}_j(\tilde{u}) \right)(t,x) \right) \, dx \)

\[= \int_{\mathbb{R}^2} \sum_{j \in \mathbb{Z}} \frac{1}{2} |\nabla u_j(t,x)|^2 - \frac{1}{4} \sum_{j \in \mathbb{Z}, \ n \in \mathbb{N}} \sum_{j_1,j_2,j_3} (u_{j_1}\tilde{u}_{j_2})(t,x) |u_{j_3}|^2 \, dx. \]

For \( N \in \mathbb{N}_+ \), the nonlinear Schrödinger system (1.1) is exactly the \( N \)-coupled nonlinear Schrödinger system:

\[
\begin{cases} 
i \partial_t u_j + \Delta u_j = -|u_j|^2 u_j - 2 \sum_{k \neq j} |u_k|^2 u_j, \\
u_j(0,x) = u_{0,j}(x), \ j = 0, 1, \ldots, N - 1,
\end{cases}
\]

(1.3)

where \( u_j : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C} \) for \( j = 0, 1, \ldots, N - 1 \) and \( N \in \mathbb{N}_+ \). This kind of finite coupled nonlinear Schrödinger system has applications in nonlinear optics, see [1] and the references therein. It is a good approximation describing the propagation of self-trapped mutually incoherent wave packets in nonlinear optics, with \( u_j(t) \) denoting the \( j \)-th component of the beam. It also has application in the Bose-Einstein condensates, see [44, 57] and the references therein. We also refer to [54] for the study of other type coupled nonlinear Schrödinger system. There are many interesting research works on the coupled nonlinear Schrödinger system, especially the ground states, we refer to [55, 52].

On the other hand, for the infinite coupled nonlinear Schrödinger system, that is \( N = \infty \), the nonlinear Schrödinger system appears in the nonlinear approximate of the cubic focusing nonlinear Schrödinger equations on the cylinder \( \mathbb{R}^2 \times \mathbb{T} \) in [9], where it is called the resonant nonlinear Schrödinger system therein. The equation in the defocusing case was studied in [55], a similar nonlinear Schrödinger system is derived also in the study of the nonlinear Schrödinger equation with partial harmonic potentials in [6]. We also refer to [25] for a different view point. Although there are a lot of works on the global well-posedness

---

*In fact, \( \| \tilde{u} \|_{L^2_t L^4_x} \) is conserved for any \( k \in \mathbb{N} \), and therefore the nonlinear Schrödinger system (1.1) has infinite conservation laws, which is very interesting. This is pointed out by S. Kwon.*
and scattering of the defocusing nonlinear Schrödinger equations on the cylinder $\mathbb{R}^2 \times T$ (see [9, 10, 51]), there are very few results on the long time behavior of the solutions of the focusing nonlinear Schrödinger equations on the cylinders. We refer to [49] for a result on the orbital stability in the focusing case, and global well-posedness result in [56].

There are some work on other type nonlinear Schrödinger systems. We refer to the work of T. Chen, Y. Hong, and N. Pavlović [3, 4]. They studied a infinite system of cubic nonlinear Schrödinger equations in $\mathbb{R}^d$ when $d \geq 2$, which arises in the mean field quantum fluctuation dynamics for a system of infinitely many fermions with delta pair interactions in the vicinity of an equilibrium solution at zero temperature. In [3], they proved the global well-posedness of the infinite system of cubic nonlinear Schrödinger equations in $d = 2$ or $3$. Later, they studied the dynamics of the system in $d \geq 3$ near thermal equilibrium and proved scattering in the case of small perturbation around equilibrium in a certain generalized Sobolev space of density operators [4]. The last author together with Y. Hong and S. Kwon [27] constructed an extremizer for the Lieb-Thirring energy inequality by developing the concentration-compactness technique for operator valued inequality, and gave the global well-posedness versus finite time blowup dichotomy for the infinite system of focusing cubic nonlinear Schrödinger equations in $\mathbb{R}^3$ where each wave function is restricted to be orthogonal.

In the same time, there are some progress on the study of the quadratic nonlinear Schrödinger system in $\mathbb{R}^d$, which is mass-critical when $d = 4$. In [26], N. Hayashi, T. Ozawa, and K. Tanaka studied the equation in $d \leq 6$, and proved the existence of ground states. In [29], T. Inui, N. Kishimoto, and K. Nishimura studied scattering problem of the quadratic nonlinear Schrödinger system in $\mathbb{R}^4$ when the equation satisfies mass-resonance condition or does not satisfy the mass-resonance condition. In [30], they studied the equation in $d \leq 6$, and proved finite time blow-up in the radial case in $d = 5, 6$ when the equation does not satisfy mass-resonance condition and prove blow-up or grow-up in $d = 4$. In [23], M. Hamano, T. Inui, and K. Nishimura proved scattering below the standing wave solution in the radial case when the equation does not satisfy mass-resonance condition.

In this article, we will mainly study the long time behavior of the solution to the nonlinear Schrödinger system (1.1). Because the behavior of the solution to the nonlinear coupled Schrödinger system is slightly different between $N = \infty$ and $N < \infty$, we divide the discussion of the main result into two subsections.

### 1.1 Infinite coupled nonlinear Schrödinger system

We now present the global well-posedness and scattering of the cubic focusing infinite coupled nonlinear Schrödinger system in $L^2_h(\mathbb{R}^2 \times \mathbb{Z})$.

**Theorem 1.1** (Global well-posedness and scattering of the cubic focusing infinite coupled nonlinear Schrödinger system). For any initial data $\tilde{u}_0 \in L^2_h(\mathbb{R}^2)$ satisfying $\|\tilde{u}_0\|_{L^2_h} < \frac{1}{\sqrt{2}}\|Q\|_{L^2_h}$, where $Q$ is the ground state of $\Delta_{\mathbb{R}^2}Q - Q = -Q^3$, there exists a global solution $\tilde{u} = \{u_j\}_{j \in \mathbb{Z}}$ to (1.1), satisfying

$$\|	ilde{u}\|_{L^4_{t,x}L^1(\mathbb{R}^2 \times \mathbb{Z})} \leq C,$$

for some constant $C$ depends only on $\|\tilde{u}_0\|_{L^2_h}$. Furthermore, the solution scatters in $L^2_h$ in the sense
that there exists \( \{u^\pm_j\}_{j \in \mathbb{Z}} \in L^2_x h^1 \) such that

\[
\left\| \left( \sum_{j \in \mathbb{Z}} (j)^2 u_j(t) - e^{it \Delta_{\mathbb{R}^2} u_j^0} \right)^{\frac{1}{2}} \right\|_{L^2(\mathbb{R}^2)} \to 0, \text{ as } t \to \pm \infty.
\]

**Remark 1.2.** The threshold of scattering is sharp in the sense that if the mass is greater than \( \frac{1}{2} \|Q\|_{L^2}^2 \), we have finite time blow up. We believe when the mass is equal to the threshold, the solution still scatters, we refer to the similar results on the 2-D cubic-quintic NLS, see \([2, 5, 37]\). This will be discussed in our future project, \([7]\).

As a consequence of Theorem 1.1 by the argument in the proof of Theorem 3.9 in \([9]\), we can obtain the global well-posedness and scattering of the large-scale solution of the focusing cubic NLS on \( \mathbb{R}^2 \times \mathbb{T} \) in \( L^2_x H^1_y \), where \( \mathbb{T} = \mathbb{R}/2\pi \mathbb{Z} \).

**Theorem 1.3** (GWP & scattering of the large-scale solution of the focusing cubic NLS on the cylinder). Let \( \phi \in L^2_x H^1_y(\mathbb{R}^2 \times \mathbb{T}) \) with \( \|\phi\|_{L^2_x} < \frac{1}{\sqrt{2}} \|Q\|_{L^2} \) be given, then there is \( \lambda_0 = \lambda_0(\phi) \) sufficiently large such that for \( \lambda \geq \lambda_0 \), we have a unique global solution \( U_\lambda \in C^0_t L^2_x H^1_y(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{T}) \) of

\[
\begin{align*}
&i \partial_t U_\lambda + \Delta_{\mathbb{R}^2 \times \mathbb{T}} U_\lambda = -|U_\lambda|^2 U_\lambda, \\
&U_\lambda(0, x, y) = \frac{1}{\lambda} \phi \left( \frac{x}{\lambda}, y \right).
\end{align*}
\]

Moreover, for \( \lambda \geq \lambda_0 \), we have

\[
\|U_\lambda\|_{L^\infty_t L^2_x H^1_y(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{T})} \lesssim \|\phi\|_{L^2_x}^{\frac{1}{2}} H^1_y.
\]

As a consequence, \( U_\lambda \) scatters in \( L^2_x H^1_y \) in the sense that there exist \( \{u^\pm_\lambda\} \in L^2_x H^1_y \) such that

\[
\left\| U_\lambda(t) - e^{it \Delta_{\mathbb{R}^2 \times \mathbb{T}}} u^\pm_\lambda \right\|_{L^2_x H^1_y} \to 0, \text{ as } t \to \pm \infty.
\]

Let

\[
\tilde{U}_\lambda(t, x, y) = \lambda U_\lambda(\lambda^2 t, \lambda x, y),
\]

we have

\[
\begin{align*}
&i \partial_t \tilde{U}_\lambda + \Delta_{\mathbb{R}^2 \times \mathbb{T}_{\lambda^{-1}}} \tilde{U}_\lambda = -|\tilde{U}_\lambda|^2 \tilde{U}_\lambda, \\
&\tilde{U}_\lambda(0, x, y) = \tilde{\phi}(x, y),
\end{align*}
\]

where \( \tilde{\phi} \) is a modification of the periodic function \( \phi \) with respect to \( y \) on \( [-\pi, \pi] \), such that \( \tilde{\phi}(x, y) \) is periodic with respect to \( y \) on \( \mathbb{T}_{\lambda^{-1}} := \lambda^{-1} \mathbb{T} = [-\lambda^{-1} \pi, \lambda^{-1} \pi] \), and \( \tilde{\phi} = \phi, a.e. \)

Thus, we get the global well-posedness and scattering in \( L^2_x H^1_y(\mathbb{R}^2 \times \mathbb{T}_{\lambda^{-1}}) \) of the focusing cubic NLS on the small cylinder \( \mathbb{R}^2 \times \mathbb{T}_{\lambda^{-1}} \), where \( \lambda \) is sufficient large constant.
Theorem 1.4 (GWP & scattering of the focusing cubic NLS on the small cylinder). Let $\mathbf{\phi} \in L^2_x H^1_y(\mathbb{R}^2 \times T)$ with $\|\mathbf{\phi}\|_{L^2_y} < \frac{1}{\sqrt{2}} \|Q\|_{L^2}$ be given, then there is $\lambda_0 = \lambda_0(\mathbf{\phi})$ sufficiently large such that for $\lambda \geq \lambda_0$, we have a unique global solution $U_\lambda \in C^0_t L^2_x H^1_y(\mathbb{R} \times \mathbb{R}^2 \times T_{\lambda^{-1}})$ of

$$i\partial_t U_\lambda + \Delta_{\mathbb{R}^2 \times T_{\lambda^{-1}}} U_\lambda = -|U_\lambda|^2 U_\lambda,$$

with $U_\lambda(0, x, y) = \tilde{\mathbf{\phi}}(x, y)$, where $\tilde{\mathbf{\phi}}$ is a periodic modification of $\mathbf{\phi}$. Moreover, for $\lambda \geq \lambda_0$,

$$\|U_\lambda\|_{L^\infty_t L^2_x H^1_y(\mathbb{R} \times \mathbb{R}^2 \times T_{\lambda^{-1}})} \lesssim |\mathbf{\phi}|_{L^2_y H^1_x}^{1/2}.$$

As a consequence, $U_\lambda$ scatters to the solution of the linear equation $i\partial_t V_\lambda + \Delta_{\mathbb{R}^2 \times T_{\lambda^{-1}}} V_\lambda = 0$ in $L^2_x H^1_y(\mathbb{R}^2 \times T_{\lambda^{-1}})$ when $\lambda$ is sufficiently large.

Remark 1.5. In the above theorem, we only get the global well-posedness and scattering of the cubic focusing nonlinear Schrödinger equation on the small cylinder. It seems difficult to give the global well-posedness and scattering of the focusing nonlinear Schrödinger equation on the general cylinder. The main obstacle is a lack of knowledge on the threshold of the global well-posedness, which is closely related to the sharp Gagliardo-Nirenberg inequality on the cylinder.

1.2 Finite couple nonlinear Schrödinger system

The argument in the proof of Theorem 1.1 also works for the $N$–coupled cubic focusing nonlinear Schrödinger system, where $N$ is any positive integer.

Let

$$\left( L^2(\mathbb{R}^2) \right)^N := \underbrace{L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times \cdots \times L^2(\mathbb{R}^2)}_{N\text{-copy}},$$

we have

Theorem 1.6 (GWP & scattering of the cubic focusing $N$–coupled nonlinear Schrödinger system). For any initial data $\bar{u}_0 \in \left( L^2_2 \right)^N$ satisfying $\|\bar{u}_0\|_{\left( L^2_2 \right)^N} \leq \sqrt{\frac{N}{2^{N-1}}} \|Q\|_{L^2_2}$, where $Q$ is the ground state of $\Delta_{\mathbb{R}^2} Q = -Q^3$, there exists a global solution $\bar{u} = \{u_j\}_{j \in \mathbb{Z}^N}$ to (1.1) satisfying

$$\|\bar{u}\|_{L^4_{t,x}(\mathbb{R} \times \mathbb{R}^2)} \leq C,$$

for some constant $C$ depends only on $\|\bar{u}_0\|_{\left( L^2_2 \right)^N}$. In addition, the solution scatters in $\left( L^2_2 \right)^N$ in the sense that there exists $\{u_j^\pm\}_{j \in \mathbb{Z}^N} \in \left( L^2_2 \right)^N$ such that

$$\left( \sum_{j \in \mathbb{Z}^N} |u_j(t) - e^{it\Delta_{\mathbb{R}^2}} u_j^\pm| \right)^{1/2} \rightarrow 0, \text{ as } t \rightarrow \pm \infty.$$

Remark 1.7. When $\|\bar{u}(0)\|_{L^2_2}^2 = \frac{N}{2^{N-1}} \|Q\|_{L^2_2}^2$, we expect a rigidity theorem, and refer to the recent work on the mass-critical NLS when the mass is equal to the threshold of scattering, see [19–21].
When $N = 2$, the system (1.1) degenerates into the 2–coupled nonlinear Schrödinger system:

$$\begin{align*}
  i\partial_t u_0 + \Delta_{\mathbb{R}^2} u_0 &= -|u_0|^2 u_0 - 2|u_1|^2 u_0, \\
  i\partial_t u_1 + \Delta_{\mathbb{R}^2} u_1 &= -2|u_0|^2 u_1 - |u_1|^2 u_1,
\end{align*}$$

(1.6)

with $u_j(0, x) = u_{0,j}(x)$ for $j = 0, 1$. (1.6) is also the non-relativistic limit of the complex-valued cubic focusing nonlinear Klein-Gordon equation in $\mathbb{R}^2$, we refer [8][36].

As a straightforward consequence of Theorem 1.6, we have

**Corollary 1.8** (Global well-posedness and scattering of the 2–coupled focusing nonlinear Schrödinger system). (1.6) is globally well-posed for $\bar{u}(0) \in L^2 \times L^2$ satisfying $\|\bar{u}(0)\|_{L^2}^2 < \frac{2}{3} \|Q\|_{L^2}^2$, and $\bar{u}(t)$ scatters to a free solution as $t \to \pm\infty$.

In this article, we first establish the variational characteristic of the corresponding ground state of (1.1) for both $N = \infty$ and $N < \infty$. First, we consider the focusing mass-critical nonlinear Schrödinger equation

$$i\partial_t u + \Delta_{\mathbb{R}^2} u = -|u|^2 u.$$  

The corresponding ground state is given by a solution to the following elliptic equation

$$\Delta_{\mathbb{R}^2} Q - Q = -Q^3$$

which is known to be unique up to modulo space translations and multiplication by $e^{i\theta}$, see [33]. Hence, the threshold of the scattering is also unique [53]. However, for (1.1), there are many positive solutions up to modulo space translations and multiplication by $e^{i\theta}$ of the corresponding elliptic equation

$$\Delta_{\mathbb{R}^2} Q_j - Q_j = - \sum_{(j_1,j_2,j_3) \in \mathcal{R}(j)} Q_{j_1} Q_{j_2} Q_{j_3}. $$

We refer to [11, 28, 35, 45, 52] for the finite coupled case, this reflects the complex behaviour of the nonlinear Schrödinger system. However, we find the best constant of the sharp Gagliardo-Nirenberg inequality is unique and thus the threshold of the scattering is also unique, which is $\frac{1}{2} \|Q\|_{L^2}^2$.

**Remark 1.9.** We have

$$\partial_t \left( \int |x|^2 \sum_{j \in \mathbb{Z}} |u_j(t, x)|^2 \, dx \right) = 4 \sum_{j \in \mathbb{Z}} \int x \cdot \text{Im}(\bar{u}_j \nabla u_j) \, dx$$

and

$$\partial_t \left( 4 \sum_{j \in \mathbb{Z}} \int x \cdot \text{Im}(\bar{u}_j \nabla u_j) \, dx \right) = 16 E(\bar{u}).$$

Therefore, when $\|\bar{u}\|_{L^2}^2 < \frac{1}{2} \|Q\|_{L^2}^2$, we see

$$\partial_t^2 \left( \int |x|^2 \sum_{j \in \mathbb{Z}} |u_j(t, x)|^2 \, dx \right) = \partial_t \left( 4 \sum_{j \in \mathbb{Z}} \int x \cdot \text{Im}(\bar{u}_j \nabla u_j) \, dx \right)$$

$$= 16 E(\bar{u}) = 8 \|\nabla \bar{u}\|_{L^2}^2 - 4N(\bar{u}) \geq 8 \left( \|\nabla \bar{u}\|_{L^2}^2 - \frac{1}{2} \|Q\|_{L^2}^2 \|\bar{u}\|_{L^2}^2 \right) > 0.$$
On the other hand, for any \( \varepsilon > 0 \), there exists \( \tilde{u}(0) \) which satisfies \( \left\| \tilde{u}(0) \right\|_{L^2} = \frac{1}{2} \left\| Q \right\|_{L^2} + \varepsilon \), with \( E(\tilde{u}(0)) < 0, \int |x|^2 \tilde{u}(0, x) dx < \infty \), and \( \sum_{j \in \mathbb{Z}} \int x \cdot \text{Im}(\tilde{u}_j(t, x) \nabla_x \tilde{u}_j(t, x)) dx < \infty \). Then the solution of (1.1) blows up in finite time for such initial data \( \tilde{u}(0) \). Thus the threshold of global well-posedness and scattering for (1.1) is sharp.

Once finding the exact threshold of global well-posedness and scattering, we will show the global well-posedness and scattering below the threshold. We will focus on the cubic focusing infinite coupled nonlinear Schrödinger system because this case is more difficult than the cubic focusing finite coupled nonlinear Schrödinger system, in fact, the finite case is a trivialization of the infinite case. We now explain the idea of the proof of the global well-posedness and scattering below the threshold. To prove the global well-posedness and scattering below the threshold, it suffices to show the finiteness of the \( L^4_{t,x} L^2 \) norm of the solution. By using the concentration-compactness/rigidity method, we get an almost periodic solution which is almost periodic modulo the scaling, Galilean transformation and spatial translation. So we only need to exclude the almost periodic solution. To exclude the almost periodic solution, we divide the almost periodic solution into two different kinds according to whether \( \int_0^\infty N(t)^3 dt \) is finite or not. If \( \int_0^\infty N(t)^3 dt \) is finite, the almost periodic solution has higher regularity and belongs to \( H^3_{x} \). So we can exclude the almost periodic solution by exploiting the conservation of energy. Otherwise, i.e. if \( \int_0^\infty N(t)^3 dt \) is infinite, the exclusion of the almost periodic solution relies on the argument of B. Dodson [18]. We first modify the scale function \( N(t) \) so that it oscillates less, and then we exclude the solution by exploiting the low frequency localized interaction Morawetz estimate.

The remaining part of the paper is organized as follows. We will focus on the proof of Theorem 1.1 from Section 2 to Section 5 precisely, we give the local well-posedness theory and small data scattering in Section 2. In Section 3, we establish the variational estimate of the ground state. In Section 4, we reduce the non-scattering to the existence of the almost periodic solution. In Section 5, we preclude the almost periodic solution by using the method of B. Dodson developed in [18]. In Section 6, we give a sketch of the scattering of the \( N \)-coupled NLS.

1.3 Notation and Preliminaries

We will use the notation \( X \lesssim Y \) whenever there exists some constant \( C > 0 \) so that \( X \leq CY \). Similarly, we will use \( X \sim Y \) if \( X \lesssim Y \lesssim X \).

For the vector-valued function \( \tilde{f}(t, x) = \{f_j(t, x)\}_{j \in \mathbb{Z}} \), we denote

\[
\| \tilde{f} \|_{L^p_t L^q_x h^s} := \left\| \left( \sum_{j \in \mathbb{Z}} (j)^{2s} |f_j(t, x)|^2 \right)^{\frac{1}{2}} \right\|_{L^p_t L^q_x},
\]

where \( 0 \leq s \leq 1 \). When \( s = 0 \), we write \( L^p_t L^q_x h^0 \) to be \( L^p_t L^q_x L^2 \).

We define the discrete nonisotropic Sobolev space. For \( \tilde{\phi} = \{\phi_k\}_{k \in \mathbb{Z}} \) a sequence of real-variable functions, we define

\[
H^{s_1, s_2}_{x} := \left\{ \tilde{\phi} = \{\phi_k\}_{k \in \mathbb{Z}} : \| \tilde{\phi} \|_{H^{s_1, s_2}_{x}} = \left\| \left( \sum_{k \in \mathbb{Z}} (k)^{2s_2} |(\nabla_x)^{s_1} \phi_k(x)|^2 \right)^{\frac{1}{2}} \right\|_{L^2_x} < \infty \right\},
\]
where $s_1, s_2 \geq 0$. In particular, when $s_1 = 0$, we denote the space $H_x^{s_1} h^{s_2}$ to be $L^2_x h^{s_2}$.

We also define

$$
\mathcal{N}(\tilde{f}) = \int_{\mathbb{R}^2} N(\tilde{f}) \, dx := \int_{\mathbb{R}^2} \sum_{j \in \mathbb{Z}} (\tilde{f}_j \tilde{F}_j(\tilde{f}))(x) \, dx.
$$

We also have the following important observation.

**Lemma 1.10** (The equivalence of space-time norms).

$$
\mathcal{N}(\tilde{u}) \sim \|\tilde{u}\|_{L^4_x t^2}^4.
$$

**Proof.** By the elementary inequality

$$
\left(\sum_{j \in \mathbb{Z}} |u_j|^4\right)^{\frac{1}{2}} \leq \left(\sum_{j \in \mathbb{Z}} |u_j|^2\right)^{\frac{1}{2}},
$$

we have

$$
\left(\sum_{j \in \mathbb{Z}} |u_j|^2\right)^{\frac{1}{2}} \leq 2 \left(\sum_{j \in \mathbb{Z}} |u_j|^2\right)^{\frac{1}{2}} - \sum_{j \in \mathbb{Z}} |u_j|^4 \leq 3 \left(\sum_{j \in \mathbb{Z}} |u_j|^2\right)^{\frac{1}{2}}.
$$

Thus

$$
\sum_{j \in \mathbb{Z}} \sum_{(j_1, j_2, j_3) \in \mathcal{R}(j)} \tilde{u}_{j_1} u_{j_1} \tilde{u}_{j_2} u_{j_3} = 2 \left(\sum_{j \in \mathbb{Z}} |u_j|^2\right)^{\frac{1}{2}} - \sum_{j \in \mathbb{Z}} |u_j|^4 \sim \left(\sum_{j \in \mathbb{Z}} |u_j|^2\right)^{\frac{3}{2}} \sim \|\tilde{u}\|_{t^2}^4.
$$

Therefore, we get (1.7).

\[ \Box \]

**Lemma 1.11.** By Minkowski’s inequality, interpolation, and Hölder’s inequality, we have

$$
\|\tilde{v}\|_{L^2_x t^2} \lesssim \left(\sum_{j \in \mathbb{Z}} \|v_j(x)\|_{L^2_x} \|v_j(x)\|_{\dot{H}^1_x}\right)^{\frac{1}{2}} \lesssim \|\tilde{v}\|^{\frac{1}{2}}_{L^2_x t^2} \|\tilde{v}\|^{\frac{1}{2}}_{\dot{H}^1_x t^2}.
$$

(1.8)

\section{Local well-posedness and small data scattering}

In this section, we will review the local wellposedness, small data scattering and the stability theory, for detailed exposition, we refer to [9, 10, 55].

**Definition 2.1** (Strichartz admissible pair). We call a pair $(q, r)$ is Strichartz admissible pair if $2 < q \leq \infty$, $2 \leq r < \infty$, and $\frac{1}{q} + \frac{1}{r} = \frac{1}{2}$.

**Theorem 2.2** (Strichartz estimate, [9, 50, 51, 55]). For any $\alpha = 0, 1,$

$$
\left\| e^{it\Delta} \tilde{f} \right\|_{L^q_t L^r_x} \lesssim \left\| \tilde{f} \right\|_{L^2_x} \quad \text{and} \quad \left\| \int_0^t e^{i(s-s') \Delta} \tilde{F}(s, x) \, ds \right\|_{L^q_t L^r_x} \lesssim \left\| \tilde{F} \right\|_{L^{\bar{q}}_t L^{\bar{r}}_x},
$$

where $(q, r)$ and $(\bar{q}, \bar{r})$ are Strichartz admissible.
By standard arguments with Theorem 2.2, we obtain

**Proposition 2.3** (Local wellposedness and small data scattering). Let \( \hat{u}(0) = \{u_p(0)\}_{p} \in L^2 \) satisfying
\[
\|\hat{u}_0\|_{L^2} \leq E \text{ for some } E > 0.
\]
Then there exists an open interval \( I \ni 0 \) and a unique solution \( \hat{u}(t) \) of (1.1) in \( C^0_t L^2_x (\mathbb{R}^2 \times \mathbb{Z}) \cap L^4_t L^4_x (\mathbb{R}^2 \times \mathbb{Z}) \). In addition, if \( \hat{u}(0) \in H^1_x \alpha(\mathbb{R}^2 \times \mathbb{Z}) \) for some \( \alpha \geq 1 \) and \( k \geq 0 \), then \( \hat{u}(t) \in C^0_t H^k_x \alpha(\mathbb{R}^2 \times \mathbb{Z}) \). Moreover, there exists \( \delta_0 > 0 \) such that if \( E \leq \delta_0 \), then \( \hat{u}(t) \) is global and scatters in positive and negative infinite time.

**Theorem 2.4** (Scattering norm). If the solution \( \hat{u} \) of (1.1) satisfies
\[
\|\hat{u}\|_{L^4_t L^4_x (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{Z})} < \infty,
\]
we have scattering in \( L^2_x \), that is, there exist \( \hat{u}_\pm \in L^2_x \) such that
\[
\|\hat{u}(t) - e^{it\Delta} \hat{u}_\pm\|_{L^2_x} \to 0, \text{ as } t \to \pm \infty.
\]

**Theorem 2.5** (Stability theorem). For any \( \alpha = 0, 1 \), let \( I \) be a compact interval and \( \vec{e} = \{\vec{e}_j\}_{j \in \mathbb{Z}} \),
\[
\vec{e}_j = i \partial_t u_j + \Delta u_j + \vec{F}_j(\hat{u}).
\]
with
\[
\|\vec{u}\|_{L^4_t L^4_x h^\alpha (I \times \mathbb{R}^2)} \leq A,
\]
for \( A > 0 \). Then for any \( \epsilon > 0 \), there is \( \delta > 0 \), such that if
\[
\|\vec{e}\|_{L^4_t L^4_x h^\alpha} \leq \delta \text{ and } \|\hat{u}(t_0) - \vec{v}_0\|_{L^2_x} \leq \delta,
\]
then (1.1) has a solution \( \vec{v} \in C^0_t L^2_x (\mathbb{R}^2 \times \mathbb{Z}) \cap L^4_t L^4_x (\mathbb{R}^2 \times \mathbb{Z}) \) with initial data \( \vec{v}(t_0) = \vec{v}_0 \). Furthermore,
\[
\|\hat{u} - \vec{v}\|_{L^4_t L^4_x h^\alpha} \leq \epsilon.
\]

### 3 Variational characterization of the ground state

In this section, we study the sharp constant \( C_{res} \) of the following Gagliardo-Nirenberg inequality:
\[
2 \int_{\mathbb{R}^2} \left( \sum_{j \in \mathbb{Z}} |u_j|^2 \right)^2 - \int_{\mathbb{R}^2} \sum_{j \in \mathbb{Z}} |u_j|^4 \leq C_{res} \left( \int_{\mathbb{R}^2} \sum_{j \in \mathbb{Z}} |u_j|^2 \right) \left( \int_{\mathbb{R}^2} \sum_{j \in \mathbb{Z}} |\nabla u_j|^2 \right),
\]
(3.1)
where \( \vec{u} \in H^1_x \alpha(\mathbb{R}^2 \times \mathbb{Z}) \) when \( N = \infty \) and \( \vec{u} \in (H^1_x)^N \) when \( N \in \mathbb{N} \).

When \( N = 1 \), we have the following know result.

**Lemma 3.1** (Sharp Gagliardo-Nirenberg inequality for the scalar function).
\[
\|u\|_{L^4_x}^4 \leq \frac{2}{\|Q\|_{L^2}^2} \frac{2}{\|u\|_{L^2_x}^2} \|\nabla u\|_{L^2_x}^2,
\]
and \( Q \) is the ground state of \( \Delta Q - Q = \bar{Q}^3 \), the equality holds if and only if \( u(x) = c\bar{Q}(x) \), where \( c \) is a constant and \( \bar{Q} \) is \( Q \) under the action of the group of translation and dilation.
When \( 1 < N < \infty \), we denote \( \bar{u} = (u_0, u_1, \cdots, u_{N-1}) \). It is obviously that \( \left( \sqrt{\frac{1}{2N-1}} Q, \cdots, \sqrt{\frac{1}{2N-1}} Q \right) \) is a solution of the corresponding elliptic equation system

\[
\Delta Q_j - Q_j = -|Q_j|^2 Q_j - 2 \sum_{k \neq j} |Q_k|^2 Q_j, \quad \text{for } j = 0, 1, \cdots, N - 1,
\]

(3.2)

where \( Q \) is the ground state solution of \( \Delta_{\mathbb{R}^2} Q - Q = -Q^2 \). When \( N = 2, \) B. Sirakov [45] conjectured that under spatial translation and rotation that \( \left( \sqrt{\frac{1}{3}} Q, \sqrt{\frac{1}{3}} Q \right) \) is the unique positive solution to (3.2). This conjecture was proved by J. Wei and W. Yao [52] (see also [28]). These arguments can be extended to the general finite \( N > 1 \) case, and thus under spatial translation and rotation that \( \left( \sqrt{\frac{1}{2N-1}} Q, \cdots, \sqrt{\frac{1}{2N-1}} Q \right) \) is the unique positive solution to (3.2). N. V. Nguyen, R. Tian, B. Deconinck, and N. Sheils [38] showed the ground state is connected with the best constant for the vector valued sharp Gagliardo-Nirenberg inequality for \( \vec{f} = (f_0, f_1, \cdots, f_{N-1}) \),

\[
\mathcal{N}(\vec{f}) \leq C_{op} \left\| \nabla_x \vec{f} \right\|_{L^2_x(\mathbb{R}^2)}^2 \left\| \vec{f} \right\|_{L^2_x(\mathbb{R}^2)}^2,
\]

(3.3)

where

\[
\left\| \nabla_x \vec{f} \right\|_{L^2_x} = \left( \sum_{j=0}^{N-1} \int_{\mathbb{R}^2} \left| \nabla f_j(x) \right|^2 \, dx \right)^{1/2}, \quad \left\| \vec{f} \right\|_{L^2_x} = \left( \sum_{j=0}^{N-1} \int_{\mathbb{R}^2} \left| f_j(x) \right|^2 \, dx \right)^{1/2},
\]

and \( \mathcal{N}(\vec{f}) = \sum_{j=0}^{N-1} \int_{\mathbb{R}^2} \left( \left| f_j(x) \right|_{L^4}^4 + 2 \sum_{0 \leq k \leq N-1, k \neq j} \left| f_k(x) \right|^2 \left| f_j(x) \right|^2 \right) \, dx \).

We will now study the best constant of the Gagliardo-Nirenberg inequality (3.1) for \( N \in \mathbb{N} \cup \{ \infty \} \). Let \( J \subseteq \mathbb{Z} \) be the index set whose size can be finite or (countably) infinite, for any nontrivial \( \bar{u} = \{u_j\}_{j \in \mathbb{Z}, N} \subseteq H^1_x L^2(\mathbb{R}^2 \times J) \), we define the Weinstein functional

\[
W(\bar{u}) := \frac{2 \int_{\mathbb{R}^2} \left( \sum_{j \in J} |u_j|^2 \right)^2 - \int_{\mathbb{R}^2} \sum_{j \in J} |u_j|^4}{\left( \int_{\mathbb{R}^2} \sum_{j \in J} |u_j|^2 \right) \left( \int_{\mathbb{R}^2} \sum_{j \in J} |\nabla u_j|^2 \right)};
\]

where the size of \( J \) can be finite or countably infinite, i.e., \( |J| \in \{1, 2, \cdots \} \cup \{ \infty \} \). To emphasize the role of the size of \( |J| \), we denote \( W_{|J|}(\bar{u}) = W(\bar{u}) \), and

\[
C_{|J|} = \sup \left\{ W_{|J|}(\bar{u}) : \bar{u} = \{u_j\}_{j \in J} \in H^1_x L^2(\mathbb{R}^2 \times J), \bar{u} \neq \bar{0} \right\}.
\]

Note that the Weinstein functional \( W(\bar{u}) \) is invariant under homogeneity and scaling symmetry, that is, \( W(\bar{u}) = W(\bar{u}^\mu) \) where \( u_j^{\lambda \mu}(\cdot) := \mu u_j(\lambda \cdot) \) for any \( \mu, \lambda > 0 \) and \( j \in J \). By standard variational argument, a maximizer \( \bar{Q} = \{Q_j\}_{j \in J} \) of the Weinstein functional \( W \) weakly solves the system of Euler-Lagrange equations

\[
\Delta_{\mathbb{R}^2} Q_j - Q_j + 2 \left( \sum_{j \in J} |Q_j|^2 \right) Q_j - |Q_j|^2 Q_j = 0, \quad j \in J,
\]

(3.4)

if it exists.
Remark 3.2 (Properties of maximizer). If there exists a maximizer $\bar{Q} = \{Q_j\}_{j \in J}$, then it satisfies the following properties:

1. By switching every $u_j$ by $|u_j|$, we may assume that the maximizer $\bar{Q}$ is non-negative. Then by standard argument by maximum principle, each component $Q_j$ of such non-negative maximizer is indeed strictly positive.

2. For $j \in J$, denote $Q^*_j$ as a symmetric decreasing rearrangement (Schwarz symmetrization) [34] of strictly positive function $Q_j \in H^1(\mathbb{R}^2)$. Let $\bar{Q}^* = \{Q^*_j\}_{j \in J}$. Then, it is well-known that

$$\int_{\mathbb{R}^2} Q^2_i Q^2_j \, dx \leq \int_{\mathbb{R}^2} (Q^*_i)^2 (Q^*_j)^2 \, dx, \quad i, j \in J.$$  

Especially, by summing over $i, j \in J$, it follows that

$$2 \int_{\mathbb{R}^2} \left( \sum_{j \in J} Q^2_j \right)^2 - \int_{\mathbb{R}^2} \sum_{j \in J} Q^4_j \leq 2 \int_{\mathbb{R}^2} \left( \sum_{j \in J} (Q^*_j)^2 \right)^2 - \int_{\mathbb{R}^2} \sum_{j \in J} (Q^*_j)^4.$$  

Now, we can take every $Q_j$ to be radial, i.e., $Q_j(x) = Q_j(|x|)$, since we get $W(\bar{Q}) \leq W(\bar{Q}^*)$ from Polya-Szegő inequality.

3. From the simple observation (as in [52] for instance), by multiplying $i$-th equation of (3.4) by $Q_j$ and vice versa, we have

$$\int_{\mathbb{R}^2} \nabla Q_i \cdot \nabla Q_j + \int_{\mathbb{R}^2} Q_i Q_j - \int_{\mathbb{R}^2} \left( \sum_{j \in J} |Q_j|^2 \right) Q_i Q_j + \int_{\mathbb{R}^2} |Q_j|^2 Q_i Q_j = 0,$$

$$\int_{\mathbb{R}^2} \nabla Q_i \cdot \nabla Q_j + \int_{\mathbb{R}^2} Q_i Q_j - \int_{\mathbb{R}^2} \left( \sum_{j \in J} |Q_j|^2 \right) Q_i Q_j + \int_{\mathbb{R}^2} |Q_i|^2 Q_j Q_j = 0.$$  

In particular, these implies that

$$\int_{\mathbb{R}^2} (Q^2_i - Q^2_j) Q_i Q_j = \int_{\mathbb{R}^2} (Q_i - Q_j)(Q_i + Q_j) Q_i Q_j = 0.$$  

By strict positivity of each $Q_i$, we have $Q_i = Q_j$.

When $|J| < \infty$, simply following the compactness argument of Weinstein [53], the maximizer $\bar{Q} = \{Q_j\}_{j \in J}$ exists. From the third observation in the Remark 3.2, we can let $Q_j = \phi$, and the Euler-Lagrange equation (3.4) will be decoupled and written as single equation $\Delta_{\mathbb{R}^2} \phi - \phi + |\phi|^2 \phi = 0$. Now, we have

$$C_{|J|} = W_{|J|}(\bar{Q}) = \frac{2 |J| - 1}{|J|} \cdot \frac{\|Q\|_{L^4}^4}{\|\nabla Q\|_{L^2}^2 \|Q\|_{L^2}^2} = \frac{2 (2 |J| - 1)}{|J|} \cdot \frac{1}{\|Q\|_{L^2}^2}, \quad (3.5)$$

where $Q$ is the ground state solution of

$$\Delta_{\mathbb{R}^2} Q - Q + Q^3 = 0. \quad (3.6)$$

Therefore, for $N < \infty$, we have
Theorem 3.3 (Sharp Gagliardo-Nirenberg inequality for the vector function). \( \forall \tilde{u} \in (H^1_x)^N \), we have

\[
\mathcal{N}(\tilde{u}) \leq \frac{2(2N-1)}{N} \|Q\|_{L^2}^2 \|\nabla_x \tilde{u}\|_{L^2_x}^2 \|\tilde{u}\|_{L^2_x}^2,
\]

the equality holds at the positive radial function \( \tilde{Q} = \left(\sqrt{\frac{1}{2N-1}} Q, \ldots, \sqrt{\frac{1}{2N-1}} Q\right) \in (H^1)^N \cap (C^\infty)^N \), where \( Q \) is the ground state of the elliptic equation

\[ \Delta_{R^2} Q - Q = -Q^3. \]

We now turn to the infinite case when \( J = \mathbb{Z} \). In this case, we can construct a maximizing sequences that converges to \( C_\infty \) from the case when \( N \) is finite using simple compactness argument. To see this, we first fix \( 0 < \epsilon \ll 1 \) and choose \( \tilde{\tilde{Q}} = \{\tilde{q}_j\}_{j \in \mathbb{Z}} \) such that \( C_\infty - W_\infty (\tilde{\tilde{Q}}) < \frac{\epsilon}{2} \). In particular, from homogeneity and scaling invariance of \( W \), we may assume that

\[
\int_{R^2} \sum_{j \in \mathbb{Z}} |\tilde{q}_j|^2 = \int_{R^2} \sum_{j \in \mathbb{Z}} |\nabla \tilde{q}_j|^2 = 1.
\]

Also, we can choose \( N = N(\epsilon) \) so that

\[
2 \int_{R^2} \left( \sum_{j=-N}^{N} |\tilde{q}_j|^2 \right)^2 - \int_{R^2} \sum_{j=-N}^{N} |\tilde{q}_j|^4 > \left\{ 2 \int_{R^2} \left( \sum_{j \in \mathbb{Z}} |\tilde{q}_j|^2 \right)^2 - \int_{R^2} \sum_{j \in \mathbb{Z}} |\tilde{q}_j|^4 \right\} - \frac{\epsilon}{2}.
\]

Let \( \tilde{Q}_N = \{\tilde{q}_{j,k}\}_{|j| \leq N} \) where \( \tilde{q}_{j,k} := \tilde{q}_j \) for \( |j| \leq N \). Then

\[ C_\infty - \epsilon < W_\infty (\tilde{\tilde{Q}}) < W_{2N+1}(\tilde{Q}_N) \leq C_{2N+1} \leq C_\infty. \]

So we can conclude that \( C_{2N+1} \nearrow C_\infty \leq \infty \) as \( N \to \infty \). (Indeed, one can easily see that \( C_N \nearrow C_\infty \) as \( N \to \infty \). Finally, from (3.5), we have

\[ C_\infty = \frac{4}{\|Q\|_{L^2}^2}, \]

and the sharp Gagliardo-Nireberg interpolation inequality (without equality sign) can be written precisely as

\[
2 \int_{R^2} \left( \sum_{j \in \mathbb{Z}} |u_j|^2 \right)^2 - \int_{R^2} \sum_{j \in \mathbb{Z}} |u_j|^4 < \frac{4}{\|Q\|_{L^2}^2} \left( \int_{R^2} \sum_{j \in \mathbb{Z}} |u_j|^2 \right) \left( \int_{R^2} \sum_{j \in \mathbb{Z}} |\nabla u_j|^2 \right).
\]

Therefore, we obtain

Theorem 3.4 (Sharp Gagliardo-Nirenberg inequality for the infinite vector function). \( \forall \tilde{u} \in H^1_x \), we have

\[
\mathcal{N}(\tilde{u}) \leq \frac{4}{\|Q\|_{L^2}^2} \|\nabla_x \tilde{u}\|_{L^2_x}^2 \|\tilde{u}\|_{L^2_x}^2,
\]

the constant is optimal in the sense: \( \exists \tilde{u}^k \in H^1_x \), s.t.

\[
\mathcal{N}(\tilde{u}^k) \xrightarrow{\|\nabla_x \tilde{u}^k\|_{L^2_x}^2 \|\tilde{u}^k\|_{L^2_x}^2} \frac{4}{\|Q\|_{L^2}^2}, \quad \text{as } k \to \infty.
\]
As a direct consequence of Theorem 3.4 whenever \( \|\tilde{\varphi}\|_{L^2_t L^2_x}^2 < \frac{1}{2} \|Q\|_{L^2_x}^2 \), we have
\[
E(\tilde{\varphi}) \geq \frac{1}{2} \left( 1 - \frac{\|\tilde{\varphi}\|_{L^2_t L^2_x}^2}{\frac{1}{2} \|Q\|_{L^2_x}^2} \right) \|\nabla_x \tilde{\varphi}\|_{L^2_t L^2_x}^2 \geq \|\nabla_x \tilde{\varphi}\|_{L^2_t L^2_x}^2.
\] (3.7)

On the other hand, by Theorem 3.4 we have
\[
E(\tilde{\varphi}) \leq \|\nabla_x \tilde{\varphi}\|_{L^2_t L^2_x}^2 + \|\tilde{\varphi}\|_{L^2_t L^2_x}^2 \|\nabla_x \tilde{\varphi}\|_{L^2_t L^2_x}^2.
\]

4 Reduction to the almost periodic solution

In this section, we show the non-scattering is equivalent to the existence of almost periodic solution. This reduction is standard, we refer to \([5,32,47,48]\) and also \([10,55]\) for the argument of the resonant nonlinear Schrödinger system in the defocusing case.

By Theorem 2.4, to prove \((1.1)\) is globally well-posed and scatters in \(L^2_t h^1\) for \(\tilde{u}_0 \in L^2_t h^1\) satisfying \(\|\tilde{u}_0\|_{L^2_t L^2_x} < \frac{1}{2} \|Q\|_{L^2_x}\), it suffices to prove that if \(\tilde{u}\) is a solution of \((1.1)\), then
\[
\|\tilde{u}\|_{L^4_t L^4_x} < \infty.
\]

Define
\[
\Lambda(m) = \sup \left\{ \|\tilde{u}\|_{L^4_t L^4_x(I \times \mathbb{R}^2 \times \mathbb{Z})} : \|\tilde{u}(0)\|_{L^4_t L^4_x(\mathbb{R}^2 \times \mathbb{Z})} \leq m, \text{ and } \tilde{u}(0) \in L^2_t h^1(\mathbb{R}^2 \times \mathbb{Z}) \right\}
\]

where \(I\) is the maximal lifespan interval. Let
\[
m_0 = \sup \left\{ m : \Lambda(m') < \infty, \forall m' < m \right\}.
\]

If we can prove \(m_0 = \frac{1}{\sqrt{2}} \|Q\|_{L^2_x}\), then the global well-posedness and scattering are established. Assume \(m_0 < \frac{1}{\sqrt{2}} \|Q\|_{L^2_x}\), by following the standard concentration-compactness/rigidity arguments in \([32,47,48]\), we obtain

**Theorem 4.1 (Existence of the minimal blowup solution).** Suppose \(m_0 < \frac{1}{\sqrt{2}} \|Q\|_{L^2_x}\), there exists a solution \(\tilde{u} \in C^0_t L^2_x h^1(I \times \mathbb{R}^2 \times \mathbb{Z}) \cap L^4_t L^4_x(I \times \mathbb{R}^2 \times \mathbb{Z})\) of \((1.1)\) with \(\|\tilde{u}\|_{L^2_t L^2_x} = m_0\), which is almost periodic in the sense that there exists \((x(t), \xi(t), N(t)) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^+\) such that for any \(\eta > 0\), there exists \(C(\eta) > 0\) satisfying for any \(t \in I\),
\[
\int_{|x-x(t)| \leq C(\eta)} \|\tilde{u}(t,x)\|_{h^1}^2 \, dx + \int_{|\xi-\xi(t)| \geq C(\eta) N(t)} \|\tilde{u}(t,\xi)\|_{h^1}^2 \, d\xi < \eta. \tag{4.1}
\]

Here \(I\) is the maximal lifespan interval. Moreover, we can take \(N(0) = 1\), \(x(0) = \xi(0) = 0\), \(N(t) \leq 1\) on \(I\), and
\[
|N'(t)| + |\xi'(t)| \leq N(t)^3. \tag{4.2}
\]

By using the argument in \([15,17,32]\), we have the following results of the almost periodic solution in the above theorem:
Lemma 4.2. (1) There exists $\delta(\bar{u}) > 0$ such that for any $t_0 \in I$,
\[
\|\bar{u}\|_{L^4_{t,x}((t_0, t_0 + \frac{\delta}{N(t_0)}) \times \mathbb{R}^2 \times \mathbb{Z})} \sim \|\bar{u}\|_{L^4_{t,x}((t_0 - \frac{\delta}{N(t_0)}, t_0)) \times \mathbb{R}^2 \times \mathbb{Z})} \sim 1.
\]

(2) If $J$ is a characteristic interval which is defined to be an interval satisfying $\|\bar{u}\|_{L^4_{t,x}(J \times \mathbb{R}^2 \times \mathbb{Z})} = 1$, then for any $t_1, t_2 \in J$, we have $N(t_1) \sim_{m_0} N(t_2)$, and $|\xi(t_1) - \xi(t_2)| \lesssim N(J)$, where $N(J) := \sup_{t \in J} N(t)$. In addition,
\[
N(J) \sim \int_J N(t)^3 \, dt \sim \inf_{t \in J} N(t).
\]

By the Strichartz estimates and (4.1), we have

Lemma 4.3. On the characteristic interval $J$, the following inequality holds
\[
\|P_{|\xi| \geq RN(t)}\bar{u}\|_{L^4_{t,x}(J \times \mathbb{R}^2 \times \mathbb{Z})}^4 + \int_J \int_{|x-x(t)| \geq RN(t)} \|\bar{u}(t, x)\|_{L^2_x}^2 \, dx \, dt \lesssim o_R(1).
\]

By the result in [15–17, 55], we have

Theorem 4.4. Suppose $\bar{u}(t, x)$ is the almost periodic solution to (1.1) in Theorem 4.1 with $\int_0^\infty N(t)^3 \, dt = K < \infty$. Then for $0 \leq s < 3$,
\[
\|\bar{u}\|_{L^\infty_t H^s_x([0, \infty) \times \mathbb{R}^2 \times \mathbb{Z})} \lesssim K^s.
\]

The errors arising from the Fourier truncation can be estimated for a variety of potentials. We refer to Theorem 7.1 in [55] and also Theorem 5.3 in [16] for a proof of $a(t, x) = \frac{1}{|x|}$, which can be extended to the more general potentials.

Theorem 4.5. Suppose $\bar{u}$ is the almost periodic solution of (1.1) with $\int_0^T N(t)^3 \, dt = K$, and there exists a constant $R$ such that
\[
|a(t, x)| \leq R, \ |\nabla_x a(t, x)| \leq \frac{R}{|x|}, \ a(t, x) = -a(t, -x), \text{ and } \|\partial_t a(t, x)\|_{L^1_t L^2_x} \leq R.
\]

Then the Fourier truncation error arising from $P_{\leq CK} \tilde{F}(\bar{u}) - \tilde{F}(P_{\leq CK} \bar{u})$ is bounded by $Ro(K)$.

5 Exclusion of the almost periodic solution

In this section, we will exclude the almost periodic solution in Theorem 4.1. We consider the cases (i) $\int_0^\infty N(t)^3 \, dt$ is finite or (ii) infinite separately. If it is infinite, we exclude the almost periodic solution by exploiting the interaction Morawetz estimate, otherwise, we exclude the almost periodic solution by exploiting the conservation of energy.
5.1 Exclusion of the almost periodic solution when $\int_0^{\infty} N(t)^3 \, dt = \infty$

In this subsection, we exclude the case when $\int_0^{\infty} N(t)^3 \, dt = \infty$ by the frequency localized interaction Morawetz estimate. First, we can replace the frequency scale function $N(t)$ by a slowly varying frequency scale function of the almost periodic solution in Theorem 4.1. Following the argument in [18], we can use a smoothing algorithm developed by B. Dodson, and replace $N(t)$ with a slowly varying $\tilde{N}(t)$ and $\tilde{N}(t) \leq N(t)$. Furthermore, by the construction, we can make sure

$$\frac{|\tilde{N}'(t)|}{\tilde{N}(t)^3} \lesssim 1, \forall t > 0,$$

(5.1)

and if $\tilde{N}'(t) \neq 0$, then $\tilde{N}(t) = N(t)$.

By applying the argument in [18], we get

**Lemma 5.1.** For any $\delta > 0$, we can take a smoother $\tilde{N}(t)$ such that

$$\liminf_{T \to \infty} \frac{\int_0^T |\tilde{N}'(t)| \, dt}{\int_0^T \tilde{N}(t) N \left( \frac{P_{\leq CK} u}{t} \right)(t) \, dt} \leq \delta.$$

(5.2)

In the following, we still take $N(t)$ as $\tilde{N}(t)$. And therefore, the $N(t)$ satisfies (5.1) and (5.2), and we will exclude the case when $\int_0^{\infty} N(t)^3 \, dt = \infty$ by the interaction Morawetz estimate. The interaction Morawetz estimate is developed by J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao [12], which is used to prove the scattering of the nonlinear Schrödinger equation [13, 15–17, 47] in the non-radial case. By direct calculation, we obtain the following lemma on the interaction Morawetz estimate for (1.1).

**Lemma 5.2.** For a weight function $a : \mathbb{R}^2 \to \mathbb{R}$, let

$$M(t) = 2 \sum_{j,j' \in \mathbb{Z}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |u_{j'}(t,y)|^2 \nabla_x a(x-y) \cdot \text{Im}(\bar{u}_j \nabla_x u_j)(t,x) \, dx \, dy,$$

we have

$$M'(t) = \sum_{j,j' \in \mathbb{Z}} \sum_{1 \leq k,l \leq 2} 4 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \partial_{x_k} \partial_{x_l} a(x-y) \text{Re}(\partial_{x_k} u_j \partial_{x_l} \bar{u}_j)(t,x)|u_{j'}(t,y)|^2 \, dx \, dy$$

$$- \sum_{j,j' \in \mathbb{Z}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \Delta_x a(x-y)|u_j(t,x)|^2|u_{j'}(t,y)|^2 \, dx \, dy$$

$$- \sum_{j,j' \in \mathbb{Z}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \Delta_x a(x-y)|u_j(t,y)|^2 \sum_{R(j)} (\bar{u}_j u_j \bar{u}_j u_j)(t,x) \, dx \, dy$$

$$- 4 \sum_{j,j' \in \mathbb{Z}} \sum_{1 \leq k,l \leq 2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \partial_{x_l} a(x-y) \text{Im}(\bar{u}_j \partial_{x_l} u_j)(t,x) \partial_{y_k} \text{Im}(\bar{u}_{j'} \partial_{y_k} u_{j'})(t,y) \, dx \, dy.$$

For any $T > 0$, define

$$K(T) = \int_0^T N(t)^3 \, dt.$$
Theorem 5.3. If \( \bar{u} \) is the almost periodic solution to (1.1) satisfying \( \int_0^\infty N(t)^3 \, dt = \infty \) in Theorem 4.7 then \( \bar{u} = 0. \)

Proof. Let \( \varphi \) be a \( C^\infty_0 \) radial function with

\[
\varphi(x) = \begin{cases} 1, & |x| \leq R - \sqrt{R}, \\ 0, & |x| \geq R. \end{cases}
\]

Let

\[
\phi(x) = \frac{1}{2\pi R^2} \int_{\mathbb{R}^2} \varphi(|x-s|)\varphi(|s|) \, ds,
\]

and define

\[
\psi_{RN(t)}^{-1}(r) = \frac{1}{r} \int_0^r \phi \left( \frac{N(t)s}{R} \right) \, ds.
\]

We have

\[
\left| \psi_{RN(t)}^{-1}(r) \right| \lesssim \min \left( (RN(t)^{-1})^{-\frac{3}{2}}, (RN(t)^{-1})^{-\frac{3}{2}} r, r^{-1} \right). \tag{5.3}
\]

Define the frequency localized interaction Morawetz action

\[
M(t) = \sum_{j,j' \in \mathbb{Z}} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \psi_{RN(t)^{-1}}(|x-y|) N(t)(x-y) \cdot \nabla_x \text{Re} \left( \frac{\overline{u_j(t,x)} \nabla_x u_j(t,x)}{u_{j'}(t,y) \nabla_y u_{j'}(t,y)} \right) |u_{j'}(t,y)|^2 \, dx \, dy,
\]

where \( I = P_{\mathbb{C} K}. \)

Notice that \( \psi_{RN(t)^{-1}}(|x-y|) N(t)(x-y) \) satisfies the conditions of Theorem 4.5 with \( R \) replaced by \( \mathbb{R}^2. \) By Lemma 5.2 and Theorem 4.5 we obtain

\[
M'(t) = \sum_{j,j' \in \mathbb{Z}} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \psi_{RN(t)^{-1}}(|x-y|) N(t)(x-y) \cdot \nabla_x \text{Re} \left( \frac{\overline{u_j(t,x)} \nabla_x u_j(t,x)}{u_{j'}(t,y) \nabla_y u_{j'}(t,y)} \right) |u_{j'}(t,y)|^2 \, dx \, dy \tag{5.4}
\]

\[
- 4 \sum_{j,j' \in \mathbb{Z}} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \psi_{RN(t)^{-1}}(|x-y|) N(t)(x-y) \cdot \nabla_x \Delta_x \left( |u_j(t,x)|^2 \right) |u_{j'}(t,y)|^2 \, dx \, dy \tag{5.5}
\]

\[
+ \sum_{j,j' \in \mathbb{Z}} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \psi_{RN(t)^{-1}}(|x-y|) N(t)(x-y) \cdot \nabla_x \left( \frac{u_j(t,x) \overline{u_j(t,x)}}{u_{j'}(t,y) \nabla_y u_{j'}(t,y)} \right) (t,y) |u_{j'}(t,y)|^2 \, dx \, dy \tag{5.6}
\]

\[
+ \sum_{j,j' \in \mathbb{Z}} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \psi_{RN(t)^{-1}}(|x-y|) N(t)(x-y) \cdot \nabla_x \left( \frac{u_j(t,x) \overline{u_j(t,x)}}{u_{j'}(t,y) \nabla_y u_{j'}(t,y)} \right) (t,x) |u_{j'}(t,y)|^2 \, dx \, dy \tag{5.7}
\]

\[
+ \sum_{j,j' \in \mathbb{Z}} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{d}{dt} \left( \psi_{RN(t)^{-1}}(|x-y|) N(t)(x-y) \right) \cdot \text{Im} \left( \frac{u_j(t,x) \overline{u_j(t,x)}}{u_{j'}(t,y) \nabla_y u_{j'}(t,y)} \right) |u_{j'}(t,y)|^2 \, dx \, dy + \mathcal{E}(t), \tag{5.8}
\]
with

\[ \int_0^T \mathcal{E}(t) \, dt \lesssim R^2 o(K). \]  

(5.9)

In (5.4), \( \otimes \) represents \( \sum_{k,k' \in \mathbb{Z}} (x_{k'} - y_{k'}) \partial_k \left( \overline{\partial_{k} I u_j} \partial_{k'} I u_j \right) \). Integrating by parts, we have

\[ (5.7) = -2 \sum_{j,j' \in \mathbb{Z}} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \psi_{R(t)}^{-1} (|x - y|) N(t) \sum_{\mathcal{R}(j)} \left( \overline{I u_j I u_{j_1}} \overline{I u_{j_2}} \right) (t, x) |I u_j(t, y)|^2 \, dx \, dy \]

(5.10)

\[ - \sum_{j,j' \in \mathbb{Z}} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \psi'_{R(t)}^{-1} (|x - y|) N(t) |x - y| \sum_{\mathcal{R}(j)} \left( \overline{I u_j I u_{j_1}} \overline{I u_{j_2}} \right) (t, x) |I u'_j(t, y)|^2 \, dx \, dy, \]

and

\[ (5.6) = - \sum_{j,j' \in \mathbb{Z}} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \Delta \left( \psi_{R(t)}^{-1} (|x - y|) + \phi \left( \frac{N(t)|x - y|}{R} \right) \right) |I u_j(t, x)|^2 |I u'_j(t, y)|^2 \, dx \, dy. \]  

(5.11)

The gradient vector can be decomposed into a radial component and an angular component. Now let \( \nabla_{r,y} \) be the radial derivative with respect to \( y \), i.e. \( \nabla_{r,y} = \frac{x - y}{|x - y|} \nabla_x \) and \( \nabla_y \) be the angular component of \( \nabla \). Then by integrating by parts, we have

\[ (5.4) + (5.5) \]

\[ = 4 \sum_{j,j' \in \mathbb{Z}} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \phi \left( \frac{|x - y| N(t)}{R} \right) N(t) |\nabla I u_j(t, x)|^2 |I u'_j(t, y)|^2 \, dx \, dy \]

\[ - 4 \sum_{j,j' \in \mathbb{Z}} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \phi \left( \frac{|x - y| N(t)}{R} \right) N(t) \text{Im} \left( \overline{I u_j(t, x)} \nabla_x I u_j(t, x) \right) \cdot \text{Im} \left( \overline{I u'_j(t, y)} \nabla_y I u'_j(t, y) \right) \, dx \, dy \]

(5.12)

\[ + 4 \sum_{j,j' \in \mathbb{Z}} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \left( \psi_{R(t)}^{-1} (|x - y|) - \phi \left( \frac{|x - y| N(t)}{R} \right) \right) N(t) |\nabla_y I u_j(t, x)|^2 |I u'_j(t, y)|^2 \, dx \, dy \]

\[ - 4 \sum_{j,j' \in \mathbb{Z}} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \left( \psi_{R(t)}^{-1} (|x - y|) - \phi \left( \frac{|x - y| N(t)}{R} \right) \right) N(t) \text{Im} \left( \overline{I u_j(t, x)} \nabla_y I u_j(t, x) \right) \]

\[ \cdot \text{Im} \left( \overline{I u'_j(t, y)} \nabla_x I u'_j(t, y) \right) \, dx \, dy. \]  

(5.13)

Because \( \psi_R \) and \( \phi \) are radial functions, we have (5.13) \( \geq 0 \).

Because (5.12) is Galilean invariant, we can take a Galilean transform to eliminate the second term in (5.12). For any \( s \in \mathbb{R}^2 \), taking \( \xi(s) \in \mathbb{R}^2 \) such that

\[ \int_{\mathbb{R}^2} \phi \left( \frac{|N(t)x|}{R} - s \right) \text{Im} \left( e^{i x \xi(s)} \overline{I u_j(t)} \nabla \left( e^{-i x \xi(s)} I u_j(t) \right) \right) (t, x) \, dx = 0. \]  

(5.14)
Then, we get
\[
\int_0^T M'(t) \, dt \\
\geq 4 \int_0^T \sum_{j,j' \in \mathbb{Z}} \iint \frac{1}{2\pi R^2} \left\{ \phi \left( \left| \frac{xN(t)}{R} - s \right| \right) \cdot \left| \nabla \left( e^{-i\alpha \xi(s)} Iu_j(t, x) \right) \right|^2 \cdot \left| Iu_{j'}(t, y) \right|^2 \right\} \, dx \, dy \, ds \, dt \\
- \int_0^T \sum_{j,j' \in \mathbb{Z}} \iint \Delta \left( \psi_{RN(t)-1} \left( |x - y| \right) + \phi \left( \left| \frac{x - y}{R} \right| \right) \right) N(t) \left| Iu_j(t, x) \right|^2 \left| Iu_{j'}(t, y) \right|^2 \, dx \, dy \, dt \\
- 2 \int_0^T \sum_{j,j' \in \mathbb{Z}} \iint \psi'_{RN(t)-1} \left( |x - y| \right) N(t) \left| x - y \right| \sum_{j' \in \mathbb{Z}} \left( Iu_j Iu_{j_1} Iu_{j_2} Iu_{j_3} \right) (t, x) \left| Iu_{j'}(t, y) \right|^2 \, dx \, dy \, dt \\
+ \int_0^T \sum_{j,j' \in \mathbb{Z}} \iint \frac{d}{dt} \left( \psi_{RN(t)-1} \left( |x - y| \right) N(t) \left( x - y \right) \right) \text{Im} \left( Iu_j(t, x) \nabla_x Iu_j(t, x) \right) \left| Iu_{j'}(t, y) \right|^2 \, dx \, dy \, dt \\
+ \int_0^T \mathcal{E}(t) \, dt.
\] (5.15)

We will first consider the estimate of (5.16). We note for \( r = |x| \), we have
\[
\psi'_{RN(t)-1}(r) = \frac{\phi \left( \frac{N(t)\xi}{R} \right) - \psi_{RN(t)-1}(r)}{r},
\]
and
\[
\psi''_{RN(t)-1}(r) = \frac{N(t)}{R} \phi' \left( \frac{N(t)\xi}{R} \right) - 2\psi'_{RN(t)-1}(r).
\]

Therefore, we have
\[
\Delta \left( \psi_{RN(t)-1}(|x|) + \phi \left( \frac{N(t)|x|}{R} \right) \right) \\
= \psi''_{RN(t)-1}(r) + \frac{N(t)}{R^2} \phi'' \left( \frac{N(t)r}{R} \right) + \frac{1}{r} \psi'_{RN(t)-1}(r) + \frac{1}{r} \frac{N(t)}{R} \phi' \left( \frac{N(t)r}{R} \right) \\
= \frac{1}{r} \left( \frac{N(t)}{R} \phi' \left( \frac{N(t)r}{R} \right) - 2\psi'_{RN(t)-1}(r) \right) + \frac{N(t)}{R^2} \phi'' \left( \frac{N(t)r}{R} \right) + \frac{1}{r} \psi'_{RN(t)-1}(r) + \frac{1}{r} \frac{N(t)}{R} \phi' \left( \frac{N(t)r}{R} \right).
\] (5.21)

By
\[
\left| \phi' \left( \frac{N(t)r}{R} \right) \right| \leq \min \left( R^{-\frac{3}{2}} N(t)^{\frac{3}{2}}, R^{-\frac{5}{2}} N(t)^{\frac{5}{2}} r \right),
\]
Therefore, by (5.21), (5.22), (5.23), and (5.25), we have for a large enough,\[ \frac{N(t)}{rR} \left| \phi' \left( \frac{N(t)r}{R} \right) \right| \leq \frac{N(t)}{rR} R^{-\frac{a}{2}} N(t)^{\frac{a}{2}} r \leq N(t)^{\frac{a}{2}} R^{-\frac{a}{2}}. \] (5.22)

By \[ \phi'' \left( \frac{N(t)r}{R} \right) \right| \leq R^{-\frac{a}{2}} N(t)^{\frac{a}{2}}, \] we have\[ \left| \frac{N(t)^2}{R^2} \phi'' \left( \frac{N(t)r}{R} \right) \right| \leq \frac{N(t)^2}{R^2} R^{-\frac{a}{2}} N(t)^{\frac{a}{2}} \sim R^{-\frac{a}{2}} N(t)^{\frac{a}{2}}. \] (5.23)

By\[ \left| \psi_{RN(t)}^{-1}(r) \right| \leq \min \left( R^{-\frac{a}{2}} N(t)^{\frac{a}{2}}, R^{-\frac{a}{2}} N(t)^{\frac{a}{2}} r, r^{-1} \right), \] we have\[ \left| \frac{1}{r} \psi_{RN(t)}^{-1}(r) \right| \leq \frac{1}{r} R^{-\frac{a}{2}} N(t)^{\frac{a}{2}} r \sim R^{-\frac{a}{2}} N(t)^{\frac{a}{2}}. \] (5.25)

Therefore, by (5.21), (5.22), (5.23), and (5.25), we have for \( R \) large enough,\[ \left| \Delta \left( \psi_{RN(t)}^{-1}(r) + \phi \left( \frac{N(t)r}{R} \right) \right) \right| \leq N(t)^{\frac{a}{2}} R^{-\frac{a}{2}} + R^{-\frac{a}{2}} N(t)^{\frac{a}{2}} + R^{-\frac{a}{2}} N(t)^{\frac{a}{2}} \leq R^{-\frac{a}{2}} N(t)^{\frac{a}{2}}. \] (5.26)

Then, by (5.26) and \( N(t) \leq 1 \), we have\[ \int_0^T \left| - \sum_{j,j' \in \mathbb{Z}} \int \Delta \left( \psi_{RN(t)}^{-1}(|x-y|) + \phi \left( \frac{N(t)|x-y|}{R} \right) \right) N(t)|Iu_j(t,x)|^2 |Iu_{j'}(t,y)|^2 \, dx \, dy \right| dt \leq \int_0^T R^{-\frac{a}{2}} N(t)^{\frac{a}{2}} \int \sum_{j} |Iu_j(t,x)|^2 \, dx N(t) \int \sum_{j'} |Iu_{j'}(t,y)|^2 \, dy \, dt \leq R^{-\frac{a}{2}} \| \tilde{u} \|_{L^2_{x,t}}^4 \int_0^T N(t)^{\frac{a}{2}} \, dt \sim R^{-\frac{a}{2}} \| \tilde{u} \|_{L^2_{x,t}}^4 \int_0^T N(t)^{\frac{a}{2}} \, dt \sim R^{-\frac{a}{2}} \| \tilde{u} \|_{L^2_{x,t}}^4 K. \] (5.27)

We now turn to the estimate of (5.18). On the characteristic interval \( J_k \) of \([0,T] \), we have\[ \int_{J_k} \left| \int \psi_{RN(t)}^{-1}(|x-y|) N(t)|x-y| \sum_{j,j' \in \mathbb{Z}} \left( \overline{Iu_j Iu_{j'}} Iu_{j\pm 1} \right)(t,x) |Iu_{j'}(t,y)| \, dx \, dy \right| dt \leq \int_{J_k} \left( \int \int |x-x(t)| \leq \frac{R}{N(t)} + \int |y-y(t)| \leq \frac{R}{N(t)} + \int |x-x(t)| \leq \frac{R}{N(t)} \right) \psi_{RN(t)}^{-1}(|x-y|) N(t)|x-y| \sum_{j,j' \in \mathbb{Z}} \left( \overline{Iu_j Iu_{j'}} Iu_{j\pm 1} \right)(t,x) |Iu_{j'}(t,y)| \, dx \, dt. \] (5.28)
On the integral domain \( \left\{|x-x(t)| \leq \frac{R}{N(t)}, |y-x(t)| \leq \frac{R}{N(t)} \right\} \), by (5.24), we have

\[
\int_{J_k} \int_{|x-x(t)| \leq \frac{R}{N(t)}} \int_{|y-x(t)| \leq \frac{R}{N(t)}} \psi'_{RN(t)}^{-1} (|x-y|) N(t)|x-y| \sum_{\mathcal{R}(j)} (\overline{U_{u_{j1}}U_{u_{j2}}U_{u_{j3}}}) (t, x)|U_{u_j}(t, y)|^2 \, dx \, dy \, dt \\
\leq \int_{J_k} \int_{|x-x(t)| \leq \frac{R}{N(t)}} \int_{|y-x(t)| \leq \frac{R}{N(t)}} N(t) \frac{2}{5} R^{-\frac{2}{5}} |x-y| \sum_{\mathcal{R}(j)} \sum_{j, j' \in \mathcal{R}(j)} (\overline{U_{u_{j1}}U_{u_{j2}}U_{u_{j3}}}) (t, x)|U_{u_{j'}}(t, y)|^2 \, dx \, dy \, dt \\
\leq R^{-\frac{2}{5}} R \int_{J_k} \int_{|x-x(t)| \leq \frac{R}{N(t)}} \int_{|y-x(t)| \leq \frac{R}{N(t)}} N(t) \frac{2}{5} R^{-\frac{2}{5}} |x-y| \sum_{\mathcal{R}(j)} \sum_{j, j' \in \mathcal{R}(j)} (\overline{U_{u_{j1}}U_{u_{j2}}U_{u_{j3}}}) (t, x)|U_{u_{j'}}(t, y)|^2 \, dx \, dy \, dt \\
\leq R^{-\frac{2}{5}} \int_{J_k} \sum_{j, j' \in \mathcal{R}(j)} \int_{|x-x(t)| \leq \frac{R}{N(t)}} \int_{|y-x(t)| \leq \frac{R}{N(t)}} (\overline{U_{u_{j1}}U_{u_{j2}}U_{u_{j3}}}) (t, x)|U_{u_{j'}}(t, y)|^2 \, dx \, dy \, dt \\
\leq R^{-\frac{2}{5}} \|\bar{u}\|_{L^{2}_{y,x}}^{2} \int_{J_k} \sum_{j, j' \in \mathcal{R}(j)} N(J_k) \leq R^{-\frac{2}{5}} \|\bar{u}\|_{L^{2}_{y,x}}^{2} \sum_{J_k \in [0,T]} \int_{J_k} N(t)^{3} \, dt \sim R^{-\frac{2}{5}} \|\bar{u}\|_{L^{2}_{y,x}}^{2} K. \tag{5.29}
\]

We now consider

\[
\int_{J_k} \int_{|x-x(t)| \leq \frac{R}{N(t)}} \psi'_{RN(t)}^{-1} (|x-y|) N(t)|x-y| \sum_{j, j' \in \mathcal{R}(j)} (\overline{U_{u_{j1}}U_{u_{j2}}U_{u_{j3}}}) (t, x)|U_{u_{j'}}(t, y)|^2 \, dx \, dy \, dt.
\]

in (5.28). By the fact \( |\psi'_{RN(t)}^{-1} (r)| \leq r^{-1} \), we have

\[
|\psi'_{RN(t)}^{-1} (|x-y|)| N(t)|x-y| \leq |x-y|^{-1} N(t)|x-y| \sim N(t), \tag{5.30}
\]

Taking the summation of the characteristic interval \( J_k \) in the above estimate, we have by \( N(t) \leq 1 \) and Lemma 4.2,

\[
\int_{0}^{T} \left| - \sum_{j, j' \in \mathcal{R}(j)} \int_{|x-x(t)| \leq \frac{R}{N(t)}} \int_{|y-x(t)| \leq \frac{R}{N(t)}} \psi'_{RN(t)}^{-1} (|x-y|) N(t)|x-y| \sum_{\mathcal{R}(j)} (\overline{U_{u_{j1}}U_{u_{j2}}U_{u_{j3}}}) (t, x)|U_{u_{j'}}(t, y)|^2 \, dx \, dy \, dt \\
= \sum_{J_k \in [0,T]} \int_{J_k} \left| - \sum_{j, j' \in \mathcal{R}(j)} \int_{|x-x(t)| \leq \frac{R}{N(t)}} \int_{|y-x(t)| \leq \frac{R}{N(t)}} \psi'_{RN(t)}^{-1} (|x-y|) N(t)|x-y| \\
\cdot \sum_{\mathcal{R}(j)} (\overline{U_{u_{j1}}U_{u_{j2}}U_{u_{j3}}}) (t, x)|U_{u_{j'}}(t, y)|^2 \, dx \, dy \, dt \\
\right| \right| \leq \sum_{J_k \in [0,T]} R^{-\frac{2}{5}} \|\bar{u}\|_{L^{2}_{y,x}}^{2} \int_{J_k} \sum_{j, j' \in \mathcal{R}(j)} (\overline{U_{u_{j1}}U_{u_{j2}}U_{u_{j3}}}) (t, x)|U_{u_{j'}}(t, y)|^2 \, dx \, dy \, dt \\
\leq R^{-\frac{2}{5}} \|\bar{u}\|_{L^{2}_{y,x}}^{2} \sum_{J_k \in [0,T]} N(J_k) \leq R^{-\frac{2}{5}} \|\bar{u}\|_{L^{2}_{y,x}}^{2} \sum_{J_k \in [0,T]} \int_{J_k} N(t)^{3} \, dt \sim R^{-\frac{2}{5}} \|\bar{u}\|_{L^{2}_{y,x}}^{2} K. \tag{5.29}
\]
then by (5.30) and Lemma 4.3, we have
\[
\int_{J_k} \int_{|x-x(t)| \geq \frac{R}{N(t)}} \sum_{j,j' \in \mathbb{Z}} \psi_{RN(t)}'(|x-y|) N(t)|x-y| \sum_{\mathcal{R}(j)} (\overline{U_{j} I u_{j_1} I u_{j_2} I u_{j_3}})(t,x) |I u_{j'}(t,y)|^2 \ dx \ dy \ dt \\
\leq \int_{J_k} \int_{|x-x(t)| \geq \frac{R}{N(t)}} \sum_{j,j' \in \mathbb{Z}} N(t) \sum_{\mathcal{R}(j)} (\overline{U_{j} I u_{j_1} I u_{j_2} I u_{j_3}})(t,x) |I u_{j'}(t,y)|^2 \ dx \ dy \\
\leq \int_{J_k} \sum_{j,j' \in \mathbb{Z}} \int |I u_{j'}(t,y)|^2 \ dy \cdot \int_{|x-x(t)| \geq \frac{R}{N(t)}} N(t) \sum_{j,j' \in \mathcal{R}(j)} (\overline{U_{j} I u_{j_1} I u_{j_2} I u_{j_3}})(t,x) \ dx \ dt \\
\leq \|\tilde{u}\|_{L^2_t L^4_x}^2 \int_{J_k} \int_{|x-x(t)| \geq \frac{R}{N(t)}} N(t) \sum_{j,j' \in \mathcal{R}(j)} (\overline{u_{j} u_{j_1} u_{j_2} u_{j_3}})(t,x) \ dx \ dt \\
+ \|\tilde{u}\|_{L^2_t L^4_x}^2 \left( \int_{J_k} \int \sum_{j,j' \in \mathbb{Z}} |P_{z \leq C K} u_{j}(t,x)|^4 \ dx \ dt \right)^{\frac{1}{4}} \|\tilde{u}\|_{L^4_t L^4_x(J_k)} N(J_k) \\
\leq \|\tilde{u}\|_{L^2_t L^4_x}^2 \int_{J_k} \int_{|x-x(t)| \geq \frac{R}{N(t)}} N(t) \sum_{j,j' \in \mathcal{R}(j)} (\overline{u_{j} u_{j_1} u_{j_2} u_{j_3}})(t,x) \ dx \ dt + \|\tilde{u}\|_{L^2_t L^4_x}^2 \|P_{z \leq C K} \tilde{u}\|_{L^4_t L^4_x(J_k)} N(J_k) \\
\leq \|\tilde{u}\|_{L^2_t L^4_x}^2 N(J_k) \int_{J_k} \int_{|x-x(t)| \geq \frac{R}{N(t)}} \sum_{j,j' \in \mathcal{R}(j)} (\overline{u_{j} u_{j_1} u_{j_2} u_{j_3}})(t,x) \ dx \ dt + \|\tilde{u}\|_{L^2_t L^4_x}^2 \|P_{z \leq C K} \tilde{u}\|_{L^4_t L^4_x(J_k)} N(J_k) \\
\leq o_R(1) N(J_k).
\]

Thus, taking summation of the characteristic interval $J_k$ of $[0,T]$, by Lemma 4.2, we have
\[
\int_{0}^{T} \left[ - \sum_{j,j' \in \mathbb{Z}} \int_{|x-x(t)| \geq \frac{R}{N(t)}} \psi_{RN(t)}'(|x-y|) N(t)|x-y| \sum_{\mathcal{R}(j)} (\overline{U_{j} I u_{j_1} I u_{j_2} I u_{j_3}})(t,x) |I u_{j'}(t,y)|^2 \ dx \ dy \right] \ dt \\
= \sum_{J_k \in [0,T]} \int_{J_k} \left[ - \sum_{j,j' \in \mathbb{Z}} \int_{|x-x(t)| \geq \frac{R}{N(t)}} \psi_{RN(t)}'(|x-y|) N(t)|x-y| \sum_{\mathcal{R}(j)} (\overline{U_{j} I u_{j_1} I u_{j_2} I u_{j_3}})(t,x) |I u_{j'}(t,y)|^2 \ dx \ dy \right] \ dt \\
\leq o_R(1) \sum_{J_k \in [0,T]} N(J_k) \sim o_R(1) \sum_{J_k \in [0,T]} \int_{J_k} N(t)^3 \ dt \sim o_R(1) K. \tag{5.31}
\]

In the meantime, on the characteristic interval $J_k$, by (5.30), (4.1), and Lemma 4.3, we have
\[
\int_{J_k} \int_{|y-x(t)| \geq \frac{R}{N(t)}} \psi_{RN(t)}'(|x-y|) N(t)|x-y| \sum_{j,j' \in \mathbb{Z}} (\overline{U_{j} I u_{j_1} I u_{j_2} I u_{j_3}})(t,x) |I u_{j'}(t,y)|^2 \ dx \ dy \\
\leq \int_{J_k} N(t) \int_{|y-x(t)| \geq \frac{R}{N(t)}} \sum_{j,j' \in \mathbb{Z}} (\overline{U_{j} I u_{j_1} I u_{j_2} I u_{j_3}})(t,x) |I u_{j'}(t,y)|^2 \ dx \ dy \\
\leq \int_{J_k} \sum_{j,j' \in \mathcal{R}(j)} (\overline{U_{j} I u_{j_1} I u_{j_2} I u_{j_3}})(t,x) \ dx \ dt \cdot \sup_{t \in J_k} \int_{|y-x(t)| \geq \frac{R}{N(t)}} \sum_{j,j'} |I u_{j'}(t,y)|^2 \ dy \\
\leq \sup_{t \in J_k} \int_{|y-x(t)| \geq \frac{R}{N(t)}} \sum_{j,j' \in \mathbb{Z}} |P_{z \leq C K} u_{j'}(t,y)|^2 \ dy \\
\leq \sup_{t \in J_k} \left( \int_{|y-x(t)| \geq \frac{R}{N(t)}} \sum_{j,j'} |u_{j'}(t,y)|^2 \ dy + \int_{j,j' \in \mathbb{Z}} |P_{z \leq C K} u_{j'}(t,y)|^2 \ dy \right) \\
\leq \sup_{t \in J_k} \int_{|y-x(t)| \geq \frac{R}{N(t)}} \sum_{j,j'} |u_{j'}(t,y)|^2 \ dy + \sup_{t \in J_k} \int_{|y-x(t)| \geq \frac{R}{N(t)}} \sum_{j,j'} |P_{z \leq C K} u_{j'}(t,y)|^2 \ dy \leq o_R(1) N(J_k).
\]
Taking the summation over the characteristic interval \( J_k \) of \([0, T]\), by Lemma 4.2, we have
\[
\int_0^T \iint_{|y-x(t)| \leq \frac{R}{N(t)}} |\psi'_{RN(t)}^{-1}(|x-y|)| N(t)|x-y| \sum_{j,j' \in \mathbb{Z}} \sum_{R(j)} \left( \overline{U_{j,1}} U_{j,1} \overline{U_{j,2}} U_{j,2} \right)(t, x) |U_{j'}(t, y)|^2 \, dx \, dy \, dt
\]
\[
= \sum_{j \in [0, T]} \int_{J_k} \iint_{|y-x(t)| \leq \frac{R}{N(t)}} |\psi'_{RN(t)}^{-1}(|x-y|)| N(t)|x-y| \cdot \sum_{j', R(j)} \left( \overline{U_{j,1}} U_{j,1} \overline{U_{j,2}} U_{j,2} \right)(t, x) |U_{j'}(t, y)|^2 \, dx \, dy \, dt
\]
\[
\leq o_R(1) \sum_{J_k \in [0, T]} N(J_k) \sim o_R(1) \sum_{J_k \in [0, T]} \int_{J_k} N(t)^3 \, dt \sim o_R(1) \int_0^T N(t)^3 \, dt = o_R(1)K. \tag{5.32}
\]

Thus, by (5.29), (5.31), and (5.32), we have
\[
\int_0^T \left| - \sum_{j,j' \in \mathbb{Z}} \int_{|y-x(t)| \leq \frac{R}{N(t)}} |\psi'_{RN(t)}^{-1}(|x-y|)| N(t)|x-y| \sum_{R(j)} \left( \overline{U_{j,1}} U_{j,1} \overline{U_{j,2}} U_{j,2} \right)(t, x) |U_{j'}(t, y)|^2 \, dx \, dy \right| \, dt
\]
\[
\leq R^{-\frac{3}{2}} \| \tilde{u} \|^2_{L^2_y(x, t)} K + o_R(1)K, \tag{5.33}
\]
and therefore this completes the estimate of (5.18).

We now turn to the estimate of (5.17). By simple calculation, we see
\[
-2 \int_0^T \sum_{j,j'} \iint_{|y-x(t)| \leq \frac{R}{N(t)}} |\psi'_{RN(t)}^{-1}(|x-y|)| N(t) \sum_{R(j)} \left( \overline{U_{j,1}} U_{j,1} \overline{U_{j,2}} U_{j,2} \right)(t, x) |U_{j'}(t, y)|^2 \, dx \, dy \, dt
\]
\[
= -2 \int_0^T \sum_{j,j'} \iint_{|y-x(t)| \leq \frac{R}{N(t)}} \left( |\psi'_{RN(t)}^{-1}(|x-y|)| - \frac{|x-y|}{R} \right) N(t) \sum_{R(j)} \left( \overline{U_{j,1}} U_{j,1} \overline{U_{j,2}} U_{j,2} \right)(t, x) |U_{j'}(t, y)|^2 \, dx \, dy \, dt
\]
\[
= -2 \int_0^T \sum_{j,j'} \iint_{|y-x(t)| \leq \frac{R}{N(t)}} \frac{N(t)|x-y|}{R} N(t) \sum_{R(j)} \left( \overline{U_{j,1}} U_{j,1} \overline{U_{j,2}} U_{j,2} \right)(t, x) |U_{j'}(t, y)|^2 \, dx \, dy \, dt
\]
\[
= 2 \int_0^T \sum_{j,j'} \iint_{|y-x(t)| \leq \frac{R}{N(t)}} |x-y| |\psi'_{RN(t)}^{-1}(|x-y|)| N(t) \sum_{R(j)} \left( \overline{U_{j,1}} U_{j,1} \overline{U_{j,2}} U_{j,2} \right)(t, x) |U_{j'}(t, y)|^2 \, dx \, dy \, dt \tag{5.34}
\]
\[
- 2 \int_0^T \iint_{|y-x(t)| \leq \frac{R}{N(t)}} \phi \left( \frac{N(t)|x-y|}{R} \right) N(t) \sum_{j,j' \in \mathbb{Z}} \sum_{R(j)} \left( \overline{U_{j,1}} U_{j,1} \overline{U_{j,2}} U_{j,2} \right)(t, x) |U_{j'}(t, y)|^2 \, dx \, dy \, dt \tag{5.35}
\]
we see the term (5.34) has been estimated previously in (5.18), while the estimate of the term (5.35) will be incorporated into the estimate of the term (5.15) in the following.

We now turn to (5.15). We will estimate this term together with the remainder (5.35) in the estimate of (5.17). We consider
\[
4 \sum_{j,j' \in \mathbb{Z}} \iint \frac{1}{\sqrt{2\pi R^2}} \int \varphi \left( \left| \frac{x}{R} N(t) - s \right| \right) \varphi \left( \left| \frac{y}{R} N(t) - s \right| \right) N(t) \left| \nabla \left( e^{-i\xi(t)} I_{j}(t, x) \right) \right|^2 |U_{j'}(t, y)|^2 \, dx \, dy \, ds
\]
\[
- 2 \sum_{j,j' \in \mathbb{Z}} \iint \phi \left( \frac{N(t)|x-y|}{R} \right) N(t) \sum_{R(j)} \left( \overline{U_{j,1}} U_{j,1} \overline{U_{j,2}} U_{j,2} \right)(t, x) |U_{j'}(t, y)|^2 \, dx \, dy.
\]
Let $\chi \in C^\infty_0$ such that
\[
\chi(x) = \begin{cases} 1, |x| \leq R - 2\sqrt{R}, \\ 0, |x| \geq R - \sqrt{R}, \end{cases}
\]
we have $|\chi''(x)| \lesssim \frac{1}{R}$, $|\chi'(x)| \lesssim \frac{1}{\sqrt{R}}$, and $\varphi(x) \geq \chi^2(x)$. Thus, we have
\[
\sum_{j \in \mathbb{Z}} \int \chi^2 \left( \left| \frac{xN(t)}{R} - s \right| \right) |\nabla \left( e^{-ix\xi(s)} Iu_j(t,x) \right)|^2 \, dx = \sum_{j \in \mathbb{Z}} \int \left| \nabla \left( \chi \left( \left| \frac{xN(t)}{R} - s \right| \right) e^{-ix\xi(s)} Iu_j(t,x) \right) \right|^2 \, dx
\]
\[
+ \frac{N(t)^2}{R^2} \sum_{j \in \mathbb{Z}} \int \chi \left( \left| \frac{xN(t)}{R} - s \right| \right) (\Delta \chi) \left( \left| \frac{xN(t)}{R} - s \right| \right) |Iu_j(t,x)|^2 \, dx
\]
\[
\geq \sum_{j \in \mathbb{Z}} \int \left| \nabla \left( \chi \left( \left| \frac{xN(t)}{R} - s \right| \right) e^{-ix\xi(s)} Iu_j(t,x) \right) \right|^2 \, dx - \frac{cN(t)^2}{R^2} \|\tilde{u}\|_{L^2_t L^2_x}^2. \tag{5.36}
\]

We now consider
\[
\frac{2}{2\pi R^2} \int_0^T \sum_{j,j' \in \mathbb{Z}} \int \left( \chi \left( \left| \frac{xN(t)}{R} - s \right| \right) \right)^4 - \varphi \left( \left| \frac{xN(t)}{R} - s \right| \right) \varphi \left( \left| \frac{yN(t)}{R} - s \right| \right) \chi \left( \left| \frac{xN(t)}{R} - s \right| \right) \varphi \left( \left| \frac{yN(t)}{R} - s \right| \right) N(t) \, ds
\]
\[
\cdot \sum_{R(j)} (Iu_j Iu_{j_1} Iu_{j_2} Iu_{j_3}) (t,x) |Iu_{j'}(t,y)|^2 \, dx \, dy \, dt.
\]

Direct calculation shows for $x, y$ satisfy $|x - y| \leq \frac{R}{\chi(t)}$,
\[
\frac{1}{2\pi R^2} \int_{\mathbb{R}^2} \varphi \left( \left| \frac{N(t)x}{R} - s \right| \right) - \chi \left( \left| \frac{N(t)x}{R} - s \right| \right) \varphi \left( \left| \frac{N(t)y}{R} - s \right| \right) ds \lesssim \frac{1}{\sqrt{R}}. \tag{5.37}
\]
then by (5.37) and Lemma 4.2, we have
\[
\int_0^T \frac{2}{2\pi R^2} \sum_{j,j' \in \mathbb{Z}} \int_{|x-y| \leq \frac{R}{\chi(t)}} \varphi \left( \left| \frac{xN(t)}{R} - s \right| \right)^4 - \varphi \left( \left| \frac{xN(t)}{R} - s \right| \right) \varphi \left( \left| \frac{yN(t)}{R} - s \right| \right) N(t) \, ds
\]
\[
\cdot \sum_{R(j)} (Iu_j Iu_{j_1} Iu_{j_2} Iu_{j_3}) (t,x) |Iu_{j'}(t,y)|^2 \, dx \, dy \, dt
\]
\[
\lesssim \frac{1}{\sqrt{R}} \sum_{j,j' \in \mathbb{Z}} \sum_{j,k \in [0,T]} N(J_k) \int_{J_k} \int \sum_{R(j)} (Iu_j Iu_{j_1} Iu_{j_2} Iu_{j_3}) (t,x) |Iu_{j'}(t,y)|^2 \, dx \, dy \, dt
\]
\[
\lesssim \frac{1}{\sqrt{R}} \|\tilde{u}\|_{L^2_t L^2_x}^2 \int_0^T N(t)^3 \, dt \lesssim \frac{1}{\sqrt{R}} \|\tilde{u}\|_{L^2_t L^2_x}^2 K. \tag{5.38}
\]
We now consider when \(|x - y| \geq \frac{R^2}{4N(t)}\), then

\[
\int_0^T \sum_{j,j' \in \mathbb{Z}} \iint_{|x-y|^2 \leq \frac{R^2}{4N(t)}} \left( \left| \left( \frac{xN(t)}{R} - s \right) \right|^4 - \varphi \left( \left| \frac{xN(t)}{R} - s \right| \right) \varphi \left( \left| \frac{yN(t)}{R} - s \right| \right) \right) N(t) \, ds \\
\cdot \sum_{\mathcal{R}(j)} \left( \mathcal{I}_{u_j^{-1}} \mathcal{I}_{j+1} \mathcal{I}_{j+1} \right) (t, x) \left| \mathcal{I}_{u_j}(t, y) \right|^2 \, dx \, dy \, dt \\
\leq \int_0^T \sum_{j,j' \in \mathbb{Z}} \iint_{|x-y|^2 \leq \frac{R^2}{4N(t)}} \left( \left| \left( \frac{xN(t)}{R} - s \right) \right|^4 - \varphi \left( \left| \frac{xN(t)}{R} - s \right| \right) \varphi \left( \left| \frac{yN(t)}{R} - s \right| \right) \right) N(t) \, ds \\
\cdot \sum_{\mathcal{R}(j)} \left( \mathcal{I}_{u_j^{-1}} \mathcal{I}_{j+1} \mathcal{I}_{j+1} \right) (t, x) \left| \mathcal{I}_{u_j}(t, y) \right|^2 \, dx \, dy \, dt \leq \frac{K}{\sqrt{R}} + o_R(1) K. \tag{5.39}
\]

Therefore, by (5.40) and (5.39), we have

\[
\int_0^T \sum_{j,j' \in \mathbb{Z}} \iint_{|x-y|^2 \leq \frac{R^2}{4N(t)}} \left( \left| \left( \frac{xN(t)}{R} - s \right) \right|^4 - \varphi \left( \left| \frac{xN(t)}{R} - s \right| \right) \varphi \left( \left| \frac{yN(t)}{R} - s \right| \right) \right) N(t) \, ds \\
\cdot \sum_{\mathcal{R}(j)} \left( \mathcal{I}_{u_j^{-1}} \mathcal{I}_{j+1} \mathcal{I}_{j+1} \right) (t, x) \left| \mathcal{I}_{u_j}(t, y) \right|^2 \, dx \, dy \, dt \leq o_R(1) K. \tag{5.40}
\]

Therefore, by (5.36) and (5.40), we have

\[
4 \sum_{j,j' \in \mathbb{Z}} \int_0^T \iint \frac{1}{2\pi R^2} \int \varphi \left( \left| \frac{xN(t)}{R} - s \right| \right) \varphi \left( \left| \frac{yN(t)}{R} - s \right| \right) N(t) \\
\cdot \nabla \left( e^{-ix\xi(s)} \mathcal{I}_{u_j}(t, x) \right)^2 \left| \mathcal{I}_{u_j}(t, y) \right|^2 \, ds \, dx \, dy \, dt \\
-2 \sum_{j,j' \in \mathbb{Z}} \int_0^T \iint \varphi \left( \left| \frac{N(t)x-y}{R} \right| \right) N(t) \sum_{\mathcal{R}(j)} \left( \mathcal{I}_{u_j^{-1}} \mathcal{I}_{j+1} \mathcal{I}_{j+1} \right) (t, x) \left| \mathcal{I}_{u_j}(t, y) \right|^2 \, dx \, dy \, dt \\
\geq 2 \sum_{j,j' \in \mathbb{Z}} \int_0^T \iint \nabla \left( \left| \frac{xN(t)}{R} - s \right| \right) e^{-ix\xi(s)} \mathcal{I}_{u_j}(t, x) \right|^2 \, dx \cdot \varphi \left( \left| \frac{yN(t)}{R} - s \right| \right) \left| \mathcal{I}_{u_j}(t, y) \right|^2 N(t) \, dy \, ds \\
- \frac{1}{\pi R^2} \sum_{j,j' \in \mathbb{Z}} \int_0^T \iint \left| \nabla \left( \left| \frac{xN(t)}{R} - s \right| \right) \right|^4 \varphi \left( \left| \frac{yN(t)}{R} - s \right| \right) N(t) \, ds \\
\cdot \sum_{\mathcal{R}(j)} \left( \mathcal{I}_{u_j^{-1}} \mathcal{I}_{j+1} \mathcal{I}_{j+1} \right) (t, x) \left| \mathcal{I}_{u_j}(t, y) \right|^2 \, dx \, dy \, dt \\
- C \frac{K}{R^4} \| \tilde{\mu} \|^4_{L^2(E)} - o_R(1) K.
\]
Then by Theorem 3.4 and the fact $\| \hat{u}(0) \|_{L^2} < \frac{1}{2} \| Q \|_{L^2}$, one can find $\eta = \eta \left( \| \hat{u}(0) \|_{L^2} \right) > 0$ such that

$$\sum_{j \in \mathbb{Z}} \int \left| \nabla \left( \left| \frac{xN(t)}{R} - s \right| e^{-ix\xi(s)} Iu_j(t, x) \right) \right|^2 \, dx$$

$$- \frac{1}{2} \sum_{j \in \mathbb{Z}} \int \left| \frac{xN(t)}{R} - s \right|^4 \sum_{\mathcal{R}(j)} (\overline{Iu_j} Iu_{j1} Iu_{j2} Iu_{j3}) (t, x) \, dx$$

$$\geq \frac{1}{2} \left( \frac{\sqrt{2} \| Q \|_{L^2}}{\sum_{j \in \mathbb{Z}} \left| \left( \frac{xN(t)}{R} - s \right) e^{-ix\xi(s)} Iu_j(t, x) \right|^2} \right)^2 \cdot \sum_{j \in \mathbb{Z}} \left| \frac{xN(t)}{R} - s \right|^4 \sum_{\mathcal{R}(j)} (\overline{Iu_j} Iu_{j1} Iu_{j2} Iu_{j3}) (t, x) \, dx$$

$$- \frac{1}{2} \sum_{j \in \mathbb{Z}} \int \left| \frac{xN(t)}{R} - s \right|^4 \sum_{\mathcal{R}(j)} (\overline{Iu_j} Iu_{j1} Iu_{j2} Iu_{j3}) (t, x) \, dx$$

$$\geq \eta \sum_{j \in \mathbb{Z}} \int \left| \frac{xN(t)}{R} - s \right|^4 \sum_{\mathcal{R}(j)} (\overline{Iu_j} Iu_{j1} Iu_{j2} Iu_{j3}) (t, x) \, dx.$$ 

By

$$\frac{1}{2\pi R^2} \int \chi^4 \left| \frac{xN(t)}{R} - s \right| \varphi \left( \frac{yN(t)}{R} - s \right) \, ds \geq \frac{1}{4},$$

we have

$$\frac{4}{2\pi R^2} \sum_{j \in \mathbb{Z}} \int \left| \frac{xN(t)}{R} - s \right|^4 \sum_{\mathcal{R}(j)} (\overline{Iu_j} Iu_{j1} Iu_{j2} Iu_{j3}) (t, x) \, dx$$

$$\cdot \int \varphi \left( \frac{yN(t)}{R} - s \right) \, N(t) |Iu_j(t, y)|^2 \, dy \, ds$$

$$\geq c \sum_{j \in \mathbb{Z}} \int \left| \frac{xN(t)}{R} - s \right|^4 \sum_{\mathcal{R}(j)} (\overline{Iu_j} Iu_{j1} Iu_{j2} Iu_{j3}) (t, x) N(t) |Iu_j(t, y)|^2 \, dx \, dy$$

$$= c \sum_{j} \int \sum_{\mathcal{R}(j)} \left( \overline{u}_j u_{j1} \overline{u}_{j2} u_{j3} \right) (t, x) \, dx \, N(t) \| \hat{u} \|_{L^2}^2$$

$$- c \sum_{j \in \mathbb{Z}} \int \sum_{\mathcal{R}(j)} (P_{\geq CK} u_{j1} u_{j2} u_{j3}) (t, x) N(t) \| \hat{u} \|_{L^2}^2$$

$$- c \sum_{j \in \mathbb{Z}} \int \sum_{\mathcal{R}(j)} \left( \overline{u}_j u_{j1} \overline{u}_{j2} u_{j3} \right) (t, x) \left( \overline{u}_{j1} u_{j2} u_{j3} \right) (t, x) |P_{\geq CK} u_j(t, y)|^2 \, N(t) \, dx \, dy$$

$$- c \sum_{j \in \mathbb{Z}} \int \sum_{\mathcal{R}(j)} \left( \overline{u}_j u_{j1} \overline{u}_{j2} u_{j3} \right) (t, x) \left( \overline{u}_{j1} u_{j2} u_{j3} \right) (t, x) |P_{\geq CK} u_j(t, y)|^2 \, N(t) \, dx \, dy.$$ 

We consider (5.41). Integrating on $[0, T]$, by Lemma 4.2, we obtain

$$\int_{T}^{T} \sum_{j} \int \sum_{\mathcal{R}(j)} \left( \overline{u}_j u_{j1} \overline{u}_{j2} u_{j3} \right) (t, x) \, dx \, dt$$

$$\geq \| \hat{u} \|_{L^2}^2 \sum_{j \in [0, T]} N(J_k) \| \hat{u} \|_{L^2}^2 \sum_{j \in [0, T]} \int \left( N(t) \right)^3 \, dt \sim \| \hat{u} \|_{L^2}^2 \int_{T}^{T} N(t) \, dt \sim \| \hat{u} \|_{L^2}^2 K.$$ 

(5.45)
We now consider (5.43). By (4.1), we have
\[ \|P_{\xi - \xi(t) \geq C(\eta)N(t)} \tilde{u}\|_{L^2_t}^2 < \eta. \]

Since \( \int_0^T |\xi'(t)| \, dt \ll CK \), we have
\[ \|P_{\xi - \xi(t) \geq C(\eta_0)N(t)} \tilde{u}\|_{L^2_t}^2 \leq \|P_{\xi - \xi(t) \geq C(\eta_0)N(t)} \tilde{u}\|_{L^2_t}^2 \leq \eta_0, \]
where we take \( C >> C(\eta) \). Integrating (5.43) on \([0, T]\), we obtain
\[
\int_0^T \sum_{j \in \mathbb{Z}} \sum_{R(j)} (\bar{u}_j u_{j_1} \bar{u}_{j_2} u_{j_3}) (t, x) N(t) \|P_{\xi - \xi(t) \geq C(\eta_0)N(t)} \tilde{u}\|_{L^2_t}^2 \, dt
\leq \eta_0 \int_0^T N(t) \sum_{j \in \mathbb{Z}} \sum_{R(j)} (\bar{u}_j u_{j_1} \bar{u}_{j_2} u_{j_3}) (t, x) \, dx \, dt
\sim \eta_0 \sum_{J_k \subset [0, T]} N(J_k) \sim \eta_0 K. \tag{5.46}
\]

We now consider (5.42). Integrating on \([0, T]\), we obtain
\[
\int_0^T \sum_{j \in \mathbb{Z}} \sum_{R(j)} (P_{\xi - \xi(t) \geq C(\eta_0)N(t)} u_j \bar{u}_{j_1} \bar{u}_{j_2} u_{j_3}) (t, x) N(t) \|\tilde{u}\|_{L^2_t}^2 \, dx \, dt
\leq \|\tilde{u}\|_{L^2_t}^2 \int_0^T \sum_{j \in \mathbb{Z}} \sum_{R(j)} (P_{\xi - \xi(t) \geq C(\eta_0)N(t)} u_j \bar{u}_{j_1} \bar{u}_{j_2} u_{j_3}) (t, x) N(t) \, dx \, dt
\leq o_{C(\eta_0)}(1) K. \tag{5.47}
\]

We consider (5.44). By Lemma 4.3, we have on the characteristic interval \( J_k \subset [0, T] \),
\[ \int_{J_k} \int_{|x-x(t)| \geq \frac{R^2}{8N(t)}} \|\bar{u}(t, x)\|_{L^2}^4 \, dx \, dt \leq o_{R^2}(1), \tag{5.48} \]
and
\[ \int_{J_k} \int_{|x-x(t)| \geq \frac{R^2}{8N(t)}} \|P_{\xi - \xi(t) \geq C(\eta_0)N(t)} \tilde{u}(t, x)\|_{L^2}^4 \, dx \, dt \leq \int_{J_k} \int \|P_{\xi - \xi(t) \geq C(\eta_0)N(t)} \tilde{u}(t, x)\|_{L^2}^4 \, dx \, dt
\leq \int_{J_k} \int \|P_{\xi - \xi(t) \geq C(\eta_0)N(t)} \tilde{u}(t, x)\|_{L^2}^4 \, dx \, dt
\leq o_{C(\eta_0)}(1), \tag{5.49} \]
where we use \( |\xi'(t)| \leq N(t)^3 \) in the last but one inequality.

By (4.1), we also have
\[ \int_{|y-y(t)| \geq \frac{R^2}{8N(t)}} \|\tilde{u}(t, y)\|_{L^2}^2 \, dy \leq o_{R^2}(1), \tag{5.50} \]
and
\[ \int_{|y-y(t)| \geq \frac{R^2}{8N(t)}} \|P_{\xi - \xi(t) \geq C(\eta_0)N(t)} \tilde{u}(t, y)\|_{L^2}^2 \, dy \leq \int \|P_{\xi - \xi(t) \geq C(\eta_0)N(t)} \tilde{u}(t, y)\|_{L^2}^2 \, dy \leq \int \|P_{\xi - \xi(t) \geq C(\eta_0)N(t)} \tilde{u}(t, y)\|_{L^2}^2 \, dy \leq \eta_0. \tag{5.51} \]
Integrating on $[0,T]$, by (5.48), (5.49), (5.50), and (5.51), we have

$$\int_0^T \iint_{|x-y| \geq \frac{R^2}{2N(x)}} \sum_{j, j' \in \mathcal{Z}(R)} (\mathcal{U}_j \mathcal{U}_{j_1} \mathcal{U}_{j_2} \mathcal{U}_{j_3}) (t, x) |\mathcal{U}_{j'}(t, y)|^2 N(t) \, dx \, dy \, dt$$

$$\leq \int_0^T \iint_{|x-x(t)| \geq \frac{R^2}{8N(x)}} \sum_{j \in \mathcal{Z}(R)} (\mathcal{U}_j \mathcal{U}_{j_1} \mathcal{U}_{j_2} \mathcal{U}_{j_3}) (t, x) N(t) \, dx \, dt \cdot \sup_{t \in [0,T]} \int_{|y-x(t)| \geq \frac{R^2}{8N(x)}} |\mathcal{U}_{j'}(t, y)|^2 \, dy$$

$$\leq \int_0^T N(t) \int \|\tilde{u}(t, x)\|^2 \, dx \, dt \cdot \sup_{t \in [0,T]} \int_{|y-x(t)| \geq \frac{R^2}{8N(x)}} \left( \|\tilde{u}(t, y)\|^2 + \|P_{\geq CK} \tilde{u}(t, y)\|^2 \right) \, dy \lesssim o_R(1) K.$$

Thus, we get

$$\int_0^T \iint_{|x-y| \geq \frac{R^2}{2N(x)}} \sum_{j, j' \in \mathcal{Z}(R)} (\mathcal{U}_j \mathcal{U}_{j_1} \mathcal{U}_{j_2} \mathcal{U}_{j_3}) (t, x) N(t) |\mathcal{U}_{j'}(t, y)|^2 \, dx \, dy \, dt \lesssim o_R(1) K. \quad (5.52)$$

Collecting the above estimates (5.15)-(5.20), (5.27), (5.33), (5.34), (5.35), (5.45), (5.46), (5.47), (5.52), and (5.59), we obtain

$$\int_0^T M'(t) \, dt$$

$$\geq c_\eta K - o_R(1) K - R^2o(K) \quad (5.53)$$

$$+ \int_0^T \sum_{j, j' \in \mathcal{Z}} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{d}{dt} \left( \psi_{RN(t)}(t, x-y) N(t) \right) \cdot \text{Im} \left( (\mathcal{U}_j(t, x) \nabla_x \mathcal{U}_j(t, x)) \cdot (x - y) \right) |\mathcal{U}_{j'}(t, y)|^2 \, dx \, dy \, dt. \quad (5.54)$$

We now consider (5.54). We note

$$\frac{d}{dt} \left( \psi_{RN(t)}(t, x-y) N(t) \right) = N'(t) \frac{1}{2\pi R^2} \int_{\mathbb{R}^2} \varphi \left( \left| \frac{x N(t)}{R} - s \right| \right) \varphi \left( \left| \frac{y N(t)}{R} - s \right| \right) (x - y) \, ds.$$

Notice that

$$N'(t) \sum_{j, j' \in \mathcal{Z}} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \varphi \left( \left| \frac{x N(t)}{R} - s \right| \right) \varphi \left( \left| \frac{y N(t)}{R} - s \right| \right) \text{Im} \left( (\mathcal{U}_j(t, x) \nabla_x \mathcal{U}_j(t, x)) \cdot (x - y) \right) |\mathcal{U}_{j'}(t, y)|^2 \, dx \, dy$$

is also invariant under the Galilean transformation $\mathcal{U}(t, x) \mapsto e^{-ix\xi} \mathcal{U}(t, x)$. Then for any $\epsilon > 0$, by the Cauchy-Schwarz inequality, we have

$$N'(t) \sum_{j, j' \in \mathcal{Z}} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \varphi \left( \left| \frac{x N(t)}{R} - s \right| \right) \varphi \left( \left| \frac{y N(t)}{R} - s \right| \right) \text{Im} \left( (\mathcal{U}_j \nabla_x (e^{-ix\xi} \mathcal{U}_j)) (t, x) \cdot (x - y) \right) |\mathcal{U}_{j'}(t, y)|^2 \, dx \, dy$$

$$\leq \epsilon N(t) \sum_{j, j' \in \mathcal{Z}} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \varphi \left( \left| \frac{x N(t)}{R} - s \right| \right) \varphi \left( \left| \frac{y N(t)}{R} - s \right| \right) |\nabla_x (e^{-ix\xi} \mathcal{U}_j) (t, x)|^2 |\mathcal{U}_{j'}(t, y)|^2 \, dx \, dy$$

$$+ C(\epsilon) \frac{(N'(t))^2}{N(t)^3} \sum_{j, j' \in \mathcal{Z}} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \varphi \left( \left| \frac{x N(t)}{R} - s \right| \right) \varphi \left( \left| \frac{y N(t)}{R} - s \right| \right) |\mathcal{U}_j(t, x)|^2 |\mathcal{U}_{j'}(t, y)|^2 \, dx \, dy. \quad (5.55)$$
For \( \epsilon > 0 \) small, the contribution of (5.55) will be absorbed into (5.13), which in turn being absorbed into the first term in (5.53). To estimate (5.56), we recall that \( N(t) \) satisfies (5.1) and (5.2). Thus, taking \( T \) sufficiently large, by (5.1), (4.2), the conservation of mass, and (5.2), we have

\[
\int_0^T \frac{5.56}{\text{integral}} \, dt \leq C(\epsilon) \int_0^T |N'(t)| \, dt \leq \frac{C(\epsilon)}{m} \int_0^T N(t) \int_{\mathbb{R}^2} \sum_{j \in \mathbb{Z}} \sum_{R(j)} (T_{uj} u_{uj} T_{uj} u_{uj}) (t, x) \, dx \, dt \leq \frac{c\eta}{2} K. \tag{5.57}
\]

Therefore, (5.53), (5.54), (5.55), and (5.57) yield

\[
\int_0^T M'(t) \, dt \geq \frac{c\eta}{2} K - R^2 o(K) - o_R(1) K.
\]

On the other hand, by (4.1), we have

\[
\sup_{t \in [0,T]} |M(t)| \lesssim R^2 o(K).
\]

Choosing \( R(\eta) \) sufficiently large, by Lemma 4.3 and (4.1), since we can take \( T \) large enough and make \( K \) be arbitrarily large, we conclude that \( \vec{u} = 0 \).

\[\square\]

5.2 Exclusion of the almost periodic solution when \( \int_0^{\infty} N(t)^3 \, dt < \infty \)

In this subsection, we exclude the case when \( \int_0^{\infty} N(t)^3 \, dt < \infty \) by using the conservation of mass and energy.

**Theorem 5.4.** There is no almost periodic solution to (1.1) with \( \int_0^{\infty} N(t)^3 \, dt = K < \infty \) in Theorem 4.1.

**Proof.** By Theorem 4.4 we have for \( 0 \leq s < 3 \),

\[
\|\vec{u}\|_{L_t^\infty H_x^s(\mathbb{R}^2)} \lesssim_{m_0} K^s. \tag{5.58}
\]

Also by (4.2), there exists \( \xi_\infty \in \mathbb{R}^2 \) with \( |\xi_\infty| \lesssim K \) such that

\[
\lim_{t \to \infty} \xi(t) = \xi_\infty.
\]

Taking

\[
\vec{v}(t, x) = e^{-it|\xi_\infty|^2} e^{-ix\xi_\infty} \vec{u}(t, x + 2t\xi_\infty),
\]

then we have

\[
\|\vec{v}\|_{H_x^{s/2}} \lesssim K^s.
\]

By the interpolation inequality and (4.1), we have for any \( \eta > 0 \),

\[
\lim_{t \to \infty} \inf \|\vec{v}(t, x)\|_{H_x^{s/2}} \lesssim \lim_{t \to \infty} \inf \frac{C(\eta)N(t)^2 + \eta^{\frac{1}{2}} K}{\eta^{\frac{1}{2}} K}. \]

Therefore, by the conservation of energy, we have
\[ E(\vec{v}) = 0. \]  
(5.59)

By (3.7), we have
\[ E(\vec{v}(t)) \geq c \| \vec{v}(t, x) \|^2_{L^2_x}. \]  
(5.60)

where \( c \) is some constant depending on \( \| \vec{v}(0) \|_{L^2_x} \). Therefore, by (5.59), (5.60), and (1.8), we have \( \vec{v} = 0 \) and therefore \( \vec{u} = 0 \).

6 Sketch of the proof of Theorem 1.6

In this section, we give a sketch of the main idea of the proof of the global well-posedness and scattering of the \( N \)-coupled focusing cubic nonlinear Schrödinger system (1.3).

The local well-posedness and stability theory of (1.3) can be established by following the argument of the nonlinear Schrödinger equations, see for example, [46]. The variational estimate of (1.3) is given in Section 3, and we will rely on Theorem 3.3. The reduction of the non-scattering to the almost periodic solution as in Theorem 4.1 is similar to the argument of the two dimensional cubic nonlinear Schrödinger equations, we refer to [48]. The exclusion of the almost periodic solution is similar to the argument in Section 5, but is easier because the couple of the nonlinear terms is finite, and we do not need to care the interchange of the Lebesgue norms.

Acknowledgments. X. Cheng has been supported by “the Fundamental Research Funds for the Central Universities” (No. B210202147), and he thanks the MATRIX for hosting him and some parts of this article was preparing when Xing Cheng was in MATRIX and Monash University. Z. Guo was supported by the ARC project (No. DP170101060). G. Hwang was supported by the 2019 Yeungnam University Research Grant. H. Yoon was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Science and ICT (NRF-2020R1A2C4002615). The authors are grateful to Professor S. Kwon, S. Masaki, G. Xu, and Z. Zhao for helpful discussion on the nonlinear Schrödinger equations on the cylinders and also the nonlinear Schrödinger system.

References

[1] N. Akhmediev and A. Ankiewicz, Partially coherent solitons on a finite background, Phys. Rev. Lett. 82 (1999), 2661-2664.

[2] R. Carles and C. Sparber, Orbital stability vs. scattering in the cubic-quintic Schrödinger equation, Rev. Math. Phys. 33 (2021), no. 3, Paper No. 2150004, 27 pp.

[3] T. Chen, Y. Hong, and N. Pavlović, Global well-posedness of the NLS system for infinitely many fermions, Arch. Ration. Mech. Anal. 224 (2017), no. 1, 91-123.
[4] T. Chen, Y. Hong, and N. Pavlović, *On the scattering problem for infinitely many fermions in dimensions $d \geq 3$ at positive temperature*, Ann. Inst. H. Poincaré Anal. Non Linéaire 35 (2018), no. 2, 393-416.

[5] X. Cheng, *Scattering for the mass super-critical perturbations of the mass critical nonlinear Schrödinger equations*, Illinois J. Math. 64 (2020), no. 1, 21-48.

[6] X. Cheng, C.-Y. Guo, Z. Guo, X. Liao, and J. Shen, *Scattering of the three-dimensional cubic nonlinear Schrödinger equation with partial harmonic potentials*, arXiv:2105.02515.

[7] X. Cheng, Z. Guo, G. Hwang, and H. Yoon, *Threshold behavior of solution of the nonlinear Schrödinger system*, preprint.

[8] X. Cheng, Z. Guo, and S. Masaki, *Scattering for the two dimensional cubic complex-valued nonlinear Klein-Gordon equation*, preprint.

[9] X. Cheng, Z. Guo, K. Yang, and L. Zhao, *On scattering for the cubic defocusing nonlinear Schrödinger equation on the waveguide $\mathbb{R}^2 \times \mathbb{T}$*, Rev. Mat. Iberoam. 36 (2020), no. 4, 985-1011.

[10] X. Cheng, Z. Guo, and Z. Zhao, *On scattering for the defocusing quintic nonlinear Schrödinger equation on the two-dimensional cylinder*, SIAM J. Math. Anal. 52 (2020), no. 5, 4185-4237.

[11] T. Colin and M. I. Weinstein, *On the ground states of vector nonlinear Schrödinger equations*, Ann. Inst. H. Poincaré Phys. Théor. 65 (1996), no. 1, 57-79.

[12] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, *Global existence and scattering for rough solutions of a nonlinear Schrödinger equation on $\mathbb{R}^3$*, Comm. Pure Appl. Math. 57 (2004), no. 8, 987-1014.

[13] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, *Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in $\mathbb{R}^3$*, Ann. of Math. 167 (2008), 767-865.

[14] J. Colliander, M. Grillakis, and N. Tzirakis, *Tensor products and correlation estimates with applications to nonlinear Schrödinger equations*, Comm. Pure Appl. Math. 62(2009), no. 7, 920-968.

[15] B. Dodson, *Global well-posedness and scattering for the defocusing, $L^2$-critical nonlinear Schrödinger equation when $d \geq 3$*, J. Amer. Math. Soc. 25 (2012), no. 2, 429-463.

[16] B. Dodson, *Global well-posedness and scattering for the defocusing, $L^2$-critical, nonlinear Schrödinger equation when $d = 2$*, Duke Math. J. 165 (2016), no. 18, 3435-3516.

[17] B. Dodson, *Global well-posedness and scattering for the defocusing, $L^2$-critical, nonlinear Schrödinger equation when $d = 1$*, Amer. J. Math. 138 (2016), no. 2, 531-569.

[18] B. Dodson, *Global well-posedness and scattering for the mass critical nonlinear Schrödinger equation with mass below the mass of the ground state*, Adv. Math. 285 (2015), 1589-1618.

[19] B. Dodson, *The $L^2$ sequential convergence of a solution to the mass-critical NLS above the ground state*, arXiv:2101.09172.
[20] B. Dodson, A determination of the blowup solutions to the focusing NLS with mass equal to the mass of the soliton, arXiv:2106.02723.

[21] C. Fan, The $L^2$ weak sequential convergence of radial focusing mass critical NLS solutions with mass above the ground state, Int. Math. Res. Not. IMRN 2021, no. 7, 4864-4906.

[22] L.G. Farah and A. Pastor, Scattering for a 3D coupled nonlinear Schrödinger system, J. Math. Phys. 58 (2017), no. 7, 071502, 33 pp.

[23] M. Hamano, T. Inui, K. Nishimura, Scattering for the quadratic nonlinear Schrödinger system in $\mathbb{R}^5$ without mass-resonance condition, arXiv: 1903.05880.

[24] Z. Hani and B. Pausader, On scattering for the quintic defocusing nonlinear Schrödinger equation on $\mathbb{R} \times \mathbb{T}^2$, Comm. Pure Appl. Math. 67 (2014), no. 9, 1466-1542.

[25] E. Haus and M. Procesi, KAM for beating solutions of the quintic NLS, Comm. Math. Phys. 354 (2017), no. 3, 1101-1132.

[26] N. Hayashi, T. Ozawa, and K. Tanaka, On a system of nonlinear Schrödinger equations with quadratic interaction, Ann. Inst. H. Poincaré Anal. Non Linéaire 30 (2013), no. 4, 661-690.

[27] Y. Hong, S. Kwon, and H. Yoon, Global existence versus finite time blowup dichotomy for the system of nonlinear Schrödinger equations, J. Math. Pures Appl. (9) 125 (2019), 283-320.

[28] N. Ikoma, Uniqueness of positive solutions for a nonlinear elliptic system, NoDEA Nonlinear Differential Equations Appl. 16 (2009), no. 5, 555-567.

[29] T. Inui, N. Kishimoto, and K. Nishimura, Scattering for a mass critical NLS system below the ground state with and without mass-resonance condition, Discrete Contin. Dyn. Syst. 39 (2019), no. 11, 6299-6353.

[30] T. Inui, N. Kishimoto, and K. Nishimura, Blow-up of the radially symmetric solutions for the quadratic nonlinear Schrödinger system without mass-resonance, arXiv:1810.09153.

[31] C. E. Kenig and F. Merle, Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case, Invent. Math. 166 (2006), no. 3, 645-675.

[32] R. Killip and M. Visan, Nonlinear Schrödinger equations at critical regularity. Proceedings for the Clay summer school “Evolution Equations”, Eidgenössische technische Hochschule, Zürich, 2008.

[33] M. K. Kwong, Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^n$, Arch. Rational Mech. Anal. 105 (1989), no. 3, 243-266.

[34] E. H. Lieb and M. Loss, Analysis. Second edition. Graduate Studies in Mathematics, 14. American Mathematical Society, Providence, RI, 2001. xxii+346 pp.

[35] T.-C. Lin and J. Wei, Ground state of $N$ coupled nonlinear Schrödinger equations in $\mathbb{R}^n$, $n \leq 3$, Comm. Math. Phys. 255 (2005), no. 3, 629-653.
[36] N. Masmoudi and K. Nakanishi, *From nonlinear Klein-Gordon equation to a system of coupled nonlinear Schrödinger equations*, Math. Ann. **324** (2002), no. 2, 359-389.

[37] J. Murphy, *Threshold scattering for the 2D radial cubic-quintic NLS*, Comm. Partial Differential Equations **46** (2021), no. 11, 2213-2234.

[38] N. V. Nguyen, R. Tian, B. Deconinck, and N. Sheils, *Global existence for a coupled system of Schrödinger equations with power-type nonlinearities*, J. Math. Phys. **54** (2013), no. 1, 011503, 19 pp.

[39] M. Ohta, *Stability of solitary waves for coupled nonlinear Schrödinger equations*, Nonlinear Anal. **26** (1996), no. 5, 933-939.

[40] F. Oliveira and A. Pastor, *On a Schrödinger system arising in nonlinear optics*, Anal. Math. Phys. **11** (2021), no. 3, Paper No. 123, 38 pp.

[41] T. Ozawa and H. Sunagawa, *Small data blow-up for a system of nonlinear Schrödinger equations*, J. Math. Anal. Appl. **399** (2013), no. 1, 147-155.

[42] A. Pastor, *On three-wave interaction Schrödinger systems with quadratic nonlinearities: global well-posedness and standing waves*, Commun. Pure Appl. Anal. **18** (2019), no. 3, 2217-2242.

[43] F. Planchon and L. Vega, *Bilinear virial identities and applications*, Ann. Sci. Éc. Norm. Supér. (4) **42** (2009), no. 2, 261-290.

[44] Y.-H. Qin, L.-C. Zhao, and L. Ling, *Non-degenerate bound state solitons in multi-component Bose-Einstein condensates*, Phys. Rev. E **100** (2019), 022212.

[45] B. Sirakov, *Least energy solitary waves for a system of nonlinear Schrödinger equations in \( \mathbb{R}^n \)*, Comm. Math. Phys. **271** (2007), no. 1, 199-221

[46] T. Tao, *Nonlinear dispersive equations: local and global analysis*, CBMS Regional Conference Series in Mathematics, 106. American Mathematical Society, Providence, R.I., 2006.

[47] T. Tao, M. Visan and X. Zhang, *Global well-posedness and scattering for the defocusing mass-critical nonlinear Schrödinger equation for radial data in high dimensions*, Duke Math. J. **140** (2007), no. 1, 165-202.

[48] T. Tao, M. Visan and X. Zhang, *Minimal-mass blowup solutions of the mass-critical NLS*, Forum Math. **20** (2008), no. 5, 881-919.

[49] S. Terracini, N. Tzvetkov, and N. Visciglia, *The nonlinear Schrödinger equation ground states on product spaces*, Anal. PDE **7** (2014), no. 1, 73-96.

[50] N. Tzvetkov and N. Visciglia, *Small data scattering for the nonlinear Schrödinger equation on product spaces*, Comm. Partial Differential Equations. **37** (2012), no. 1, 125-135.

[51] N. Tzvetkov and N. Visciglia, *Well-posedness and scattering for NLS on \( \mathbb{R}^d \times \mathbb{T} \) in the energy space*, Rev. Mat. Iberoam. **32** (2016), no. 4, 1163-1188.
[52] J. Wei and W. Yao, *Uniqueness of positive solutions to some coupled nonlinear Schrödinger equations*, Commun. Pure Appl. Anal. **11** (2012), no. 3, 1003-1011.

[53] M. I. Weinstein, *Nonlinear Schrödinger equations and sharp interpolation estimates*, Comm. Math. Phys. **87** (1982/83), no. 4, 567-576.

[54] G. Xu, *Dynamics of some coupled nonlinear Schrödinger systems in $\mathbb{R}^3$*, Math. Methods Appl. Sci. **37** (2014), no. 17, 2746-2771.

[55] K. Yang and L. Zhao, *Global well-posedness and scattering for mass-critical, defocusing, infinite dimensional vector-valued resonant nonlinear Schrödinger system*, SIAM J. Math. Anal. **50** (2018), no. 2, 1593-1655.

[56] X. Yu, H. Yue, and Z. Zhao, *Global Well-posedness for the focusing cubic NLS on the product space $\mathbb{R} \times T^3$*, SIAM J. Math. Anal. **53** (2021), no. 2, 2243-2274.

[57] L.-C. Zhao and J. Liu, *Rogue-wave solutions of a three-component coupled nonlinear Schrödinger equation*, Phys. Rev. E **87** (2013), 013201.