On the high-accuracy approach to flow simulation around the airfoils by using vortex method

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Abstract. Some problems connected to vortex methods development for 2D incompressible flow simulation around airfoils are discussed. In the numerical schemes and algorithms which are normally used in the vortex methods, the airfoil is approximated by a polygon consists of rectilinear panels, and the the vortex sheet intensity, which is placed on the airfoil surface line and simulates the airfoil’s influence the flow, is assumed to be piecewise-constant or piecewise-linear function. The most accurate mathematical models and numerical algorithms, based on such approaches, provide the estimation for the order of the error between $O(h)$ and $O(h^2)$ for the average vortex sheet intensity over the panels ($h$ is the maximal panel length). In the present research a new approach is developed, where the vortex sheet intensity is assumed to be discontinuous piecewise-linear or piecewise-quadratic function and the curvature of the panels is taken into account. In order to compute the coefficients of the main system of the algebraic equations, the least squares method is used instead of commonly used no-through or no-slip conditions at separate control points or over the panels in the average. It is shown, that the developed approach is much more accurate in comparison with previously known ones: for some test problems (flows around a circular airfoil, an elliptical airfoil and Zhukovsky airfoil) it permits to obtain numerical solution which has error of order $O(h^4)$.

1. Introduction

The vortex methods which refer to the wide class of the meshless Lagrangian CFD methods, are a powerful tool in number of engineering applications, for example, in coupled hydroelastic problems, when it is possible to consider media to be incompressible, but the flow domain changes due to motion of the airfoil which is determined, in turn, by unsteady hydrodynamic loads.

For 2D problems there are some approaches to numerical solution of the Navier — Stokes equations by using vortex methods [1–4]; the Viscous Vortex Domains (VVD) method [5] seems to be the most suitable in practice. In this method the so-called “diffusive velocity” field is introduced in order to simulate vorticity evolution in viscous incompressible flow [3, 6]. The accuracy of the flow simulation and hydrodynamic loads computation depends on number of factors:

- the accuracy of the airfoil surface line approximation;
- the accuracy of computation of the vortex sheet intensity distribution over the surface line;
- the accuracy of the vortex wake approximation and its evolution simulation.

In the present paper we consider some possible ways how to raise the accuracy of the simulation due to
the first and the second factors, because they seem to be the most important: the vortex sheet being
generated on the surface line of the airfoil, is the source for the vortex wake formation in the flow.
Normally in previously known implementations of the vortex methods developed for numerical
simulation in 2D hydrodynamic and hydroelastic problems, the vortex sheet intensity is found from the
singular boundary integral equation of the 1-st kind [1]. However, such approach sometimes leads to
significant errors and even to qualitatively wrong solution [7, 8].
The alternative approach is also known, which makes it possible to bring the problem of the vortex sheet
intensity computation to solution of the Fredholm-type integral equation of the 2-nd kind [9]. The
efficiency of such method in practice has been discussed in [7, 8, 10, 11]; there some numerical schemes
are also described, which are based on this approach, and it has been shown, that it provides much more
accurate solutions for wide class of airfoils.
In order to raise the accuracy of the flow simulation in the vortex methods, first of all, the vortex
sheet intensity computation procedure should be improved. In the well-known numerical schemes it
is restricted within the low quality of the shape approximation of the surface line. In the present
research new algorithm is suggested and the corresponding quadrature formulae are derived, which
make it possible to develop main ideas of the approach [9] taking into account the curvature of the
surface line. In the framework of this method the unknown vortex sheet intensity distribution can be
 discontinuous piecewise-constant, piecewise-linear or piecewise-quadratic function over the
curvilinear panels, which approximate the shape of the surface line (in traditional approach vortex
sheet intensity normally is considered to be piecewise-constant or piecewise-linear function over
rectilinear panels [12]).
The mentioned ways for the accuracy improvement at first sight seem to be similar to the
approaches, which are used extensively in the so-called “panel methods” [13], but there are
significant differences between them. Firstly, in the suggested method the solution is not assumed to
be continuous (and differentiable even less so) over the surface line of the airfoil — it is important
for correct simulation of the flow around the airfoils having sharp edges and/or angle points. So in
order to compute an integrals over the curvilinear parts of the surface line, the Gaussian quadrature
formulae are used instead of integrating of the power series expansions. Secondly, in order to obtain
the main linear system which approximates the boundary integral equation, the ideas of the least
squares method are used: the true solution should minimize the integral over the surface line from the
squared difference between the flow velocity (which depends on unknown vortex sheet intensity
distribution) and the velocity of the surface line of the airfoil itself.
The described approach and the corresponding numerical algorithm allow to raise the accuracy of the
vortex sheet intensity computation in the vortex methods significantly.

2. The governing equations
The flow of a viscous incompressible media is described by the Navier — Stokes equations
\[
\nabla \cdot \mathbf{V} = 0, \quad \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} = -\nabla p + \frac{\nu}{\rho} \nabla^2 \mathbf{V}.
\]
Here, \( \mathbf{V}(\mathbf{r}, t) \) is the velocity field; \( p(\mathbf{r}, t) \) is the pressure distribution in the flow; \( \rho = \text{const} \) is the
density of the media; \( \nu \) is the kinematic viscosity coefficient.
Now we consider the problem of an outer flow simulation, so we should add the no-slip boundary
condition on the surface line of the airfoil \( K \) and the perturbation decay conditions on infinity:
\[
\mathbf{V}(\mathbf{r}, t) = \mathbf{V}_K(\mathbf{r}, t), \quad \mathbf{V}(\mathbf{r}, t) \rightarrow \mathbf{V}_\infty, \quad p(\mathbf{r}, t) \rightarrow p_\infty, \quad |\mathbf{r}| \rightarrow \infty.
\]
Here, \( \mathbf{V}_K(\mathbf{r}, t) \) is the velocity of the surface line of the airfoil; we consider it to be known.
The Navier — Stokes equations can be written down in Helmholtz-type form in terms of the vorticity
distribution $\bar{\Omega}(\vec{r}, t) = \nabla \times \vec{V}(\vec{r}, t)$:

$$\frac{\partial \bar{\Omega}}{\partial t} + \nabla \times (\bar{\Omega} \times \vec{U}) = 0.$$  

(1)

Here, $\vec{U}(\vec{r}, t) = \vec{V}(\vec{r}, t) + \vec{W}(\vec{r}, t)$; $\vec{W}(\vec{r}, t)$ is the so-called “diffusive velocity”, which is proportional to the viscosity coefficient and plays the key role for correct simulation of vorticity evolution in viscous media [3, 5]:

$$\vec{W}(\vec{r}, t) = \nu \frac{(\nabla \times \bar{\Omega}) \times \bar{\Omega}}{|\bar{\Omega}|^2}.$$  

It follows from (1), that the vorticity, which exists in the flow domain at the initial time moment, moves with the velocity $\vec{U}$, while “new” vorticity is generated only on flow domain boundary, i.e., on the surface line of the airfoil. We consider the initial vorticity distribution $\bar{\Omega}(\vec{r}, t)$ to be known.

The airfoil influences the flow in the same way as the superposition of the influences of the attached vortex sheet with intensity $\gamma_{att}(\vec{r}, t)$, the attached source sheet with intensity $q_{att}(\vec{r}, t)$ and the free vortex sheet of unknown intensity $\gamma(\vec{r}, t)$. These sheets are placed on the surface line of the airfoil; the attached vortex and source sheets simulate the motion and deformation (if they take place) of the airfoil, so their intensities are determined by the velocities of the points on the surface line:

$$\gamma_{att}(\vec{r}, t) = \vec{V}_K(\vec{r}, t) \cdot \vec{n}(\vec{r}, t), \quad q_{att}(\vec{r}, t) = \vec{V}_K(\vec{r}, t) \cdot \vec{t}(\vec{r}, t), \quad \vec{r} \in K,$$

where $\vec{n}(\vec{r}, t)$ and $\vec{t}(\vec{r}, t)$ are normal and tangent unit vectors, respectively [5].

In the present paper, for simplicity we consider the airfoil to be immovable, so $\gamma_{att}(\vec{r}, t) = 0$, $q_{att}(\vec{r}, t) = 0$. However, such assumption is non-essential and it can be omitted. We also consider the incident flow velocity to be constant vector: $\vec{V}_\infty = \text{const}$.  

When vorticity distribution in the flow is known, the velocity field can be reconstructed by using the Biot — Savart law:

$$\vec{V}(\vec{r}, t) = \vec{V}_\infty + \frac{1}{2\pi} \int_S \frac{\bar{\Omega}(\vec{s}, t) \times (\vec{r} - \vec{s})}{|\vec{r} - \vec{s}|^2} dS + \frac{1}{2\pi} \int_K \frac{\vec{r}(\vec{s}, t) \times (\vec{r} - \vec{s})}{|\vec{r} - \vec{s}|^2} dL.$$

(2)

Here, $S$ is the flow domain; $K$ is the surface line of the airfoil; $\vec{r} = \gamma \vec{k}$ and $\bar{\Omega} = \Omega \vec{k}$ are the vectors of the vortex sheet intensity on the surface line and the vorticity field distribution in the flow domain, respectively; $\vec{k}$ is the unit vector orthogonal to the 2D flow domain ($\vec{n}(\vec{r}) \times \vec{t}(\vec{r}) = \vec{k}$).

The vortex sheet intensity on the surface line $\gamma(\vec{s}, t)$ can be found form the no-slip boundary condition on the surface line of the immovable airfoil:

$$\vec{V}(\vec{r}, t) = \vec{0}, \quad \vec{r} \in K.$$

We consider the simplest model problem: without vorticity in the flow domain ($\Omega(\vec{r}, t) = 0$). It is necessary to determine vortex sheet intensity distribution over the surface line of the airfoil. From mathematical point of view, this problem is equivalent to the ideal (inviscid) incompressible flow simulation around the given airfoil when total circulation of the velocity field around the airfoil is known. When the media is viscous, the vortex sheet becomes the source of vorticity in the flow domain, and all the vorticity from it becomes part of the vortex wake at every time step. Thus, the described problem of the vortex sheet intensity computation (in the presence of vorticity in the flow) is also being solved at every time step.
3. Boundary integral equation for vortex sheet intensity

Taking into account, that unknown vortex sheet intensity $\gamma(s,t)$ corresponds to the so-called “free” vorticity, which is the part of the vortex wake in the flow domain, according to (2), it is possible to show, that the limit value of the flow velocity on the airfoil is equal to (hereinafter the dependencies of all values on time are omitted)

$$
\widetilde{V}_-(\vec{r}) = \vec{V}_\infty + \frac{1}{2\pi} \int_K \left( \frac{\vec{\gamma}(\vec{s}) \times (\vec{r} - \vec{s})}{|\vec{r} - \vec{s}|^2} \right) dS - \frac{1}{2\pi} \int_K \frac{\vec{\Omega}(\vec{s},t) \times (\vec{r} - \vec{s})}{|\vec{r} - \vec{s}|^2} dS, \quad \vec{r} \in K.
$$

(3)

In the framework of the “classical” approach, which is normally used in the implementations of the vortex methods, unknown function $\gamma(\vec{r})$ can be found from the satisfaction of the boundary condition for normal component of the limit value of velocity on the surface line of the airfoil:

$$
\widetilde{V}_-(\vec{r}) \cdot \vec{n}(\vec{r}) = 0, \quad \vec{r} \in K.
$$

(4)

Using (3), it is easy to obtain the boundary integral equation of the 1-st kind for unknown $\gamma(\vec{r})$, which is singular and the integral in it should be computed as the Cauchy principal value [1]. The numerical solution of such equation requires specific quadrature formulae usage and can lead to significant error, and in some cases to incorrect results in the whole. The alternative approach is also known [9], and according to it the boundary condition (4) is equivalent to the boundary condition, written down in the form

$$
\widetilde{V}_-(\vec{r}) \cdot \vec{\tau}(\vec{r}) = 0, \quad \vec{r} \in K,
$$

(5)

which means the equality to zero of the tangent component of the velocity limit value on the surface line of the airfoil. It can be shown that the boundary condition (5) leads to the Fredholm-type boundary integral equation of the 2-nd kind:

$$
\frac{1}{2\pi} \int_K \frac{\vec{h}(\vec{r}) \cdot (\vec{r} - \vec{s})}{|\vec{r} - \vec{s}|^2} \gamma(\vec{s}) dS = \frac{\gamma(\vec{r})}{2} - \widetilde{V}_\infty \cdot \vec{\tau}(\vec{r}) - \frac{1}{2\pi} \int_K \frac{\vec{h}(\vec{r}) \cdot (\vec{r} - \vec{s})}{|\vec{r} - \vec{s}|^2} \vec{\Omega}(\vec{s},t) dS, \quad \vec{r} \in K.
$$

(6)

The kernel of the obtained equation is a bounded function in case of smooth surface line of the airfoil:

$$
\lim_{|\vec{s} - \vec{x}| \to 0} \frac{\left| \vec{h}(\vec{r}) \cdot (\vec{r} - \vec{s}) \right|}{|\vec{r} - \vec{s}|^2} = \frac{1}{2},
$$

where $\gamma(\vec{r})$ is the curvature at the surface line at the corresponding point.

The boundary integral equation (6), as well as the singular integral equation following from (4), has an infinite set of solutions. In order to select the unique solution, each of them should be solved together with an additional condition, which set the value of total circulation $\Gamma$ of the velocity field around the airfoil. This value is equal to total “quantity of vorticity” and usually is known from the physical problem statement:

$$
\int_K \gamma(\vec{s}) dS = \Gamma.
$$

(7)

For the airfoils of the simplest shape (circular, elliptical, Zhukovsky airfoils) it is possible to use the conformal mappings technique in order to solve the corresponding problem analytically and obtain the exact solution for the vortex sheet intensity distribution over the surface line [14]. These exact solutions are used for the accuracy estimation of the developed numerical schemes.
4. Numerical scheme for the vortex sheet intensity computation

Now we consider two numerical schemes for computation of the vortex sheet intensity distribution over the surface line of the airfoil.

4.1. Numerical scheme with rectilinear panels

As it was mentioned earlier, in the vortex methods the surface line of the airfoil is usually approximated with a polygon which legs (called “panels”) have lengths $L_i$, $i = 1, \ldots, N$, and the vortex sheet intensity over the panels is assumed to be constant or linear. Here we consider the numerical scheme with piecewise-constant vorticity distribution, which however provides nearly the same accuracy as piecewise-linear schemes with respect to average values of the vortex sheet intensity over the panels.

The vortex wake is normally simulated with separate vortex elements, whose positions and circulations are assumed to be known, so the vorticity distribution in the flow domain is approximated by a linear combination of the following form

$$
\Omega(\vec{r}) = \sum_{s=1}^{N_v} \gamma_s \delta(\vec{r} - \vec{r}_s^v),
$$

where $\delta(\vec{r})$ is the 2D Dirac delta-function. In the framework of these assumptions, the integral in the left-hand side in (6) can be replaced by a sum of the integrals over the individual panels, which are proportional to the corresponding intensities. The integral in the right-hand side in (6) is equal to a sum of the influences of separate vortices in the flow. The accuracy of such approach can be raised significantly if we satisfy the boundary condition not only at the centers of the panels (which are usually called “control points”), but on average over the panels:

$$
\int_{L_i} \gamma_i \left( \frac{\vec{n}_s \cdot (\vec{r} - \vec{s})}{|\vec{r} - \vec{s}|} \right) \, dr = \frac{1}{2\pi} \int_{L_i} \frac{\vec{n}_s \cdot (\vec{r} - \vec{r}_s^v)}{|\vec{r} - \vec{r}_s^v|^2} \, dr, \quad i = 1, \ldots, N.
$$

The coefficients of these equations and right-hand side values can be calculated analytically; the exact formulae are given in [11, 12].

The numerical experiments show that the described scheme provides the accuracy order between $O(h)$ and $O(h^2)$ for different types of airfoils. The error means the maximal difference between the average vortex sheet intensity over the $i$-th panel computed numerically and by integrating of the exact solution over this panel ($i = 1, \ldots, n$). Here, $h$ is the maximal panel length (figure 1).

![Figure 1](image-url)
4.2. Numerical scheme for the airfoils when the curvilinearity is taken into account

While developing of a new high-order numerical scheme, we should take into account explicitly the curvilinearity of the surface line of the airfoil, and also raise the quality of approximation of the vortex sheet intensity distribution. We assume that such distribution is linear or quadratic over each panel. If we take into account only one of these two factors, it will provide only a slight improvement of the scheme, but the accuracy order will remain the same.

It is known [1], that vortex sheet intensity becomes unbounded in the neighborhood of “outer” angular points of the airfoil, so it is necessary to approximate smooth parts of the surface line with smooth curves when constructing high-order schemes. We consider that the shape of the surface line of the airfoil is known and it is described by the parametric piecewise-smooth dependencies \( x = x(t) \), \( y = y(t) \), \( t \in [0, 2\pi] \), so we can exactly compute not only positions of particular points (which correspond to the endings of the panels), but also the directions of tangent lines (or their limit values) at these points.

We demand that the curve which approximates the surface line of the airfoil, passes through the chosen points and the direction of the tangent line for the curve at these points should be the same as for the original surface line.

Let’s denote the beginning and the ending of the \( i \)-th chord of the surface line by \( C_i \) and \( C_{i+1} \), respectively (these points correspond to parameter values \( t = t_i \) and \( t = t_{i+1} \) in the parametric equations of the surface line) and let \( \vec{t}_i^0 \) be a tangent unit vector, which direction coincides with the vector \( \overrightarrow{C_iC_{i+1}} \); \( \vec{n}_i^0 \) is a unit vector, which is orthogonal to the vector \( \vec{t}_i^0 \) (figure 2). Hereinafter when we say “chord” we mean the rectilinear line segment between points \( C_i \) and \( C_{i+1} \), which in general case only common ending with the original surface lines.

On each panel a local orthogonal coordinate system is introduced \( C_i\xi_i\eta_i \), so that the points \( C_i \) and \( C_{i+1} \) have local coordinates \( (0,0) \) and \( (L_i,0) \), respectively, where \( L_i = C_iC_{i+1} \) is the length of the \( i \)-th chord. In order to construct an interpolation curve, we firstly calculate tangents of inclination angles \( \phi_i \) and \( \psi_i \) for the tangent lines to the panels (figure 2) by using the following formulae:

\[
\tan \phi_i = -\frac{x'_i(t_i)\sin \theta_i - y'_i(t_i)\cos \theta_i}{x'_i(t_i)\cos \theta_i + y'_i(t_i)\sin \theta_i}, \quad \tan \psi_i = \frac{x'_i(t_{i+1})\sin \theta_i - y'_i(t_{i+1})\cos \theta_i}{x'_i(t_{i+1})\cos \theta_i + y'_i(t_{i+1})\sin \theta_i}.
\]

Here \( \theta_i \) is an angle between the \( i \)-th panel and \( Ox \) axis; \( x'_i, x'_i, y'_i, y'_i \) denote right-side and left-side derivatives of the parametric dependencies \( x(t) \) and \( y(t) \), being calculated at the
corresponding points; $\phi_i$ and $\psi_j$ can have values from the range $\left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$.

The equation of the interpolation curve (which is the cubic spline in the local coordinates), which approximates the surface line of the airfoil in the local coordinates over the $i$-th panel has the following form:

$$p_i(\xi) = \frac{\xi(\xi - L_i)}{L_i} \left( a_i + b_i \frac{\xi - L_i/2}{L_i} \right),$$

where the conditions $p_i(0) = 0$, $p_i(L_i) = 0$ are satisfied automatically, the values of the coefficients $a_i$ and $b_i$ can be found from the conditions $p_i'(0) = \tan \phi_i$ and $p_i'(L_i) = -\tan \psi_i$:

$$a_i = -\frac{1}{2} \left( \tan \phi_i + \tan \psi_i \right), \quad b_i = \tan \phi_i - \tan \psi_i.$$

The constructed interpolation curve over the $i$-th chord of the surface line of the airfoil we call hereinafter “curvilinear panel”, or simply “panel”. So the position of an arbitrary point $M$, which lies on the curvilinear panel and has the local coordinate (abscise) $\xi$, is given by a radius vector

$$\overrightarrow{OM}(\xi) = \overrightarrow{OC} + \xi \overrightarrow{e}_i + p_i(\xi) \overrightarrow{n}_i^0.$$

If the part of the surface line of the airfoil over the $i$-th panel is a smooth curve of $C^4$ class, then the approximation error has the order $O(L_i^4)$. This results can be easily checked by writing down the corresponding Taylor expansion.

Then we approximate the unknown vortex sheet intensity distribution over the $i$-th panel with a quadratic function:

$$\gamma_i(\xi) = \alpha_i + \beta_i \frac{\xi}{L_i} + \delta_i \frac{\xi^2}{L_i^2}.$$

The values of the coefficients $\alpha_i$, $\beta_i$, $\delta_i$, $i = 1, \ldots, N$, should be found from the integral equation (6) with the additional condition (7).

In order to find an approximate solution of this equation for the given dependency $\gamma_i(\xi)$, we use the least squares method and solve an unconstrained minimization problem for the following function

$$\Psi = \frac{1}{2\pi} \int_K \left( \frac{1}{2\pi} \int_{K_s} \overrightarrow{n}(\overrightarrow{r}) \cdot (\overrightarrow{r} - \overrightarrow{s}) \gamma(\overrightarrow{s}) d\overrightarrow{s} - \frac{\gamma(\overrightarrow{r})}{2} + \overrightarrow{V}_\infty \cdot \overrightarrow{r}(\overrightarrow{r}) + \frac{1}{2\pi} \sum_{j=1}^{N} \overrightarrow{n}(\overrightarrow{r}) \cdot (\overrightarrow{r} - \overrightarrow{r}_j^\pm) \Gamma_s \right) d\overrightarrow{r} - \lambda \int_K \gamma(\overrightarrow{r}) d\overrightarrow{r} - \Gamma \rightarrow \min$$

for all possible values of the parameters $\alpha_i$, $\beta_i$, $\delta_i$ and the Lagrangian coefficient $\lambda$.

Both, the inner and the outer integrals, in (9) can be replaced with sums of the integrals over the curvilinear panels:

$$\Psi = \sum_{i=1}^{N} \int_{K_i} \left( \frac{1}{2\pi} \sum_{j=1}^{N} \overrightarrow{n}(\overrightarrow{r}) \cdot (\overrightarrow{r} - \overrightarrow{s}) \gamma(\overrightarrow{s}) d\overrightarrow{s} - \frac{\gamma(\overrightarrow{r})}{2} + \overrightarrow{V}_\infty \cdot \overrightarrow{r}(\overrightarrow{r}) + \frac{1}{2\pi} \sum_{j=1}^{N} \overrightarrow{n}(\overrightarrow{r}) \cdot (\overrightarrow{r} - \overrightarrow{r}_j^\pm) \Gamma_s \right) d\overrightarrow{r} - \lambda \left( \sum_{i=1}^{N} \int_{K_i} \gamma(\overrightarrow{r}) d\overrightarrow{r} - \Gamma \right) \rightarrow \min .$$

All the integrals are computed in the local coordinates, so for the $i$-th and the $j$-th panels we obtain
\[ \vec{\tau} = \vec{\tau}_0(\xi) \quad \text{and} \quad \vec{s} = \vec{s}_j(\xi) : \]

\[ \vec{r}_i(\xi) = \overrightarrow{OC}_i + \xi \vec{r}_0^i + p_i(\xi) \vec{n}_0^i, \quad \vec{s}_j(\xi) = \overrightarrow{OC}_j + \xi \vec{s}_0^j + p_j(\xi) \vec{n}_0^j. \]

The Jacobian determinant of the coordinates transformation from global to local ones has the following form:

\[ J_j(\xi) = \left. \frac{dL_j}{d\xi} \right|_{\xi = \xi_j(\xi)} = \sqrt{1 + \left( \frac{p_j'(\xi)}{J_j(\xi)} \right)^2}, \]

and as the result, denoting for the simplicity \( \gamma_j(\xi) = \gamma_i(\xi), \quad \vec{n}_i(\xi) = \vec{n}_j(\xi), \quad \vec{\tau}_i(\xi) = \vec{\tau}_j(\xi) \), the function to be minimized (10) can be written down in the following form:

\[ \Psi = \sum_{k=1}^N \int_{\xi_k}^{\xi_k'} \left( \frac{1}{2\pi} \sum_{j=1}^N \sum_{i=1}^N \alpha_j \int_{\xi_j}^{\xi_j'} \frac{\vec{n}_i(\xi) \cdot (\vec{r}_i(\xi) - \vec{s}_j(\xi))}{|\vec{r}_i(\xi) - \vec{s}_j(\xi)|^2} J_j(\xi) d\xi + \beta_j \int_{\xi_j}^{\xi_j'} \frac{\vec{n}_i(\xi) \cdot (\vec{r}_i(\xi) - \vec{s}_j(\xi))}{|\vec{r}_i(\xi) - \vec{s}_j(\xi)|^2} \frac{\xi}{L_j} J_j(\xi) d\xi + \right) \]

\[ + \delta \int_{\xi_j}^{\xi_j'} \frac{\vec{n}_i(\xi) \cdot (\vec{r}_i(\xi) - \vec{s}_j(\xi))}{|\vec{r}_i(\xi) - \vec{s}_j(\xi)|^2} \frac{\xi^2}{L_j^2} J_j(\xi) d\xi - \frac{1}{2} \left( \alpha_i + \beta_i \frac{\xi}{L_i} + \delta_i \frac{\xi^2}{L_i^2} \right) + \vec{V}_0 \cdot \vec{\tau}_j(\xi) + \vec{\tau}_i(\xi) \rightarrow \min. \]  

(11)

According to above mentioned assumptions, the necessary information about the shape of the surface line, which is needed for the numerical scheme construction, is limited by positions of the nodes (the panel end points) and the tangent line directions at these points. So the unit tangent vector \( \vec{\tau}_i(\xi) \) in formula (11) should be calculated for the panel instead of the initial surface line:

\[ \vec{\tau}_i(\xi) = \frac{\vec{r}_i^0 + p_i'(\xi) \vec{n}_0^i}{|\vec{r}_i^0 + p_i'(\xi) \vec{n}_0^i|}. \]

The approximation error for the tangent unit vector has the order \( O(L_j^2) \) for \( C^4 \)-class curves. The normal unit vector \( \vec{n}_i(\xi) \) should be chosen orthogonal to the \( \vec{r}_i(\xi) \).

After substitution of the expression (8) for unknown vortex sheet intensity distribution into (11), we obtain the function to be minimized, which depends only on unknown coefficients \( \alpha_k, \beta_k, \delta_k, \]

\( k = 1, \ldots, N \), and Lagrangian coefficient \( \lambda \):

\[ \Psi = \sum_{i=1}^N \int_{\xi_i}^{\xi_i'} \left( \frac{1}{2\pi} \sum_{j=1}^N \sum_{i=1}^N \alpha_j \int_{\xi_j}^{\xi_j'} \frac{\vec{n}_i(\xi) \cdot (\vec{r}_i(\xi) - \vec{s}_j(\xi))}{|\vec{r}_i(\xi) - \vec{s}_j(\xi)|^2} J_j(\xi) d\xi + \beta_j \int_{\xi_j}^{\xi_j'} \frac{\vec{n}_i(\xi) \cdot (\vec{r}_i(\xi) - \vec{s}_j(\xi))}{|\vec{r}_i(\xi) - \vec{s}_j(\xi)|^2} \frac{\xi}{L_j} J_j(\xi) d\xi + \right) \]

\[ + \delta \int_{\xi_j}^{\xi_j'} \frac{\vec{n}_i(\xi) \cdot (\vec{r}_i(\xi) - \vec{s}_j(\xi))}{|\vec{r}_i(\xi) - \vec{s}_j(\xi)|^2} \frac{\xi^2}{L_j^2} J_j(\xi) d\xi - \frac{1}{2} \left( \alpha_i + \beta_i \frac{\xi}{L_i} + \delta_i \frac{\xi^2}{L_i^2} \right) + \vec{V}_0 \cdot \vec{\tau}_i(\xi) + \vec{\tau}_i(\xi) \rightarrow \min. \]  

(12)

After introducing the denotations as follows

\[ \int_{\xi_i}^{\xi_i'} \frac{\vec{n}_i(\xi) \cdot (\vec{r}_i(\xi) - \vec{s}_j(\xi))}{|\vec{r}_i(\xi) - \vec{s}_j(\xi)|^2} \frac{\xi}{L_j} J_j(\xi) d\xi = I_{i,j}^{(1)}(\xi); \quad \int_{\xi_i}^{\xi_i'} \frac{\vec{n}_i(\xi) \cdot (\vec{r}_i(\xi) - \vec{s}_j(\xi))}{|\vec{r}_i(\xi) - \vec{s}_j(\xi)|^2} \frac{\xi^2}{L_j^2} J_j(\xi) d\xi = I_{i,j}^{(2)}(\xi); \quad \frac{\vec{n}_i(\xi) \cdot (\vec{r}_i(\xi) - \vec{s}_j(\xi))}{|\vec{r}_i(\xi) - \vec{s}_j(\xi)|^2} = Q_i(\xi), \]

the equation (12) takes more compact form:

\[ \Psi = \sum_{i=1}^N \int_{\xi_i}^{\xi_i'} \left( \frac{1}{2\pi} \sum_{j=1}^N \left( \alpha_j I_{i,j}^{(0)}(\xi) + \beta_j I_{i,j}^{(1)}(\xi) + \delta_j I_{i,j}^{(2)}(\xi) \right) - \frac{1}{2} \left( \alpha_i + \beta_i \frac{\xi}{L_i} + \delta_i \frac{\xi^2}{L_i^2} \right) + \right) \]
+ \tilde{V}_i \cdot \tilde{\tau}_i(\xi) + \frac{1}{2\pi} \sum_{s=1}^{N} \Gamma_s Q_s(\xi) \right)^2 \int J_i(\xi) d\xi - \lambda \left( \sum_{i=1}^{N} \left( \alpha_j J_i^{(0)} + \beta_j J_i^{(1)} + \delta J_i^{(2)} \right) - \Gamma \right) \rightarrow \min. \]

The minimal value of this function is achieved when all the partial derivatives are equal to zero:

\[
\frac{\partial \Psi}{\partial \alpha_k} = \sum_{i=1}^{N} \int_J 2 \left( \frac{1}{2\pi} \sum_{j=1}^{N} \left( \alpha_j J_i^{(0)}(\xi) + \beta_j J_i^{(1)}(\xi) + \delta_j J_i^{(2)}(\xi) \right) - \frac{1}{2} \left( \alpha_i + \beta_i \frac{\xi}{L_i} + \delta_i \frac{\xi^2}{L_i^2} \right) + \tilde{V}_i \cdot \tilde{\tau}_i(\xi) + \frac{1}{2\pi} \sum_{s=1}^{N} \Gamma_s Q_s(\xi) \right) \int J_i^{(0)}(\xi) \frac{d\xi}{2\pi} - \frac{1}{2} \right) J_i(\xi) d\xi - \lambda J_i^{(0)} = 0,
\]

\[
\frac{\partial \Psi}{\partial \beta_k} = \sum_{i=1}^{N} \int_J 2 \left( \frac{1}{2\pi} \sum_{j=1}^{N} \left( \alpha_j J_i^{(0)}(\xi) + \beta_j J_i^{(1)}(\xi) + \delta_j J_i^{(2)}(\xi) \right) - \frac{1}{2} \left( \alpha_i + \beta_i \frac{\xi}{L_i} + \delta_i \frac{\xi^2}{L_i^2} \right) + \tilde{V}_i \cdot \tilde{\tau}_i(\xi) + \frac{1}{2\pi} \sum_{s=1}^{N} \Gamma_s Q_s(\xi) \right) \int J_i^{(1)}(\xi) \frac{d\xi}{2\pi} - \frac{1}{2} \right) J_i^{(1)} = 0,
\]

\[
\frac{\partial \Psi}{\partial \delta_k} = \sum_{i=1}^{N} \int_J 2 \left( \frac{1}{2\pi} \sum_{j=1}^{N} \left( \alpha_j J_i^{(0)}(\xi) + \beta_j J_i^{(1)}(\xi) + \delta_j J_i^{(2)}(\xi) \right) - \frac{1}{2} \left( \alpha_i + \beta_i \frac{\xi}{L_i} + \delta_i \frac{\xi^2}{L_i^2} \right) + \tilde{V}_i \cdot \tilde{\tau}_i(\xi) + \frac{1}{2\pi} \sum_{s=1}^{N} \Gamma_s Q_s(\xi) \right) \int J_i^{(2)}(\xi) \frac{d\xi}{2\pi} - \frac{1}{2} \right) J_i^{(2)} = 0,
\]

\[
\frac{\partial \Psi}{\partial \lambda} = \sum_{i=1}^{N} \left( \alpha_j J_i^{(0)} + \beta_j J_i^{(1)} + \delta_j J_i^{(2)} \right) - \Gamma = 0.
\]

The derived equations can be written down in much more simple form if we additionally introduce the following designations:

\[
J_{mnk}^{\text{(p,q)}} = \int_0^L I_{mn}^{(p)}(\xi) I_{mk}^{(q)}(\xi) J_m(\xi) d\xi, \quad J_{mn}^{\text{(p,r)}} = \int_0^L I_{mn}^{(p)}(\xi) \frac{\xi^r}{L_m} J_m(\xi) d\xi,
\]

and also denote unknown values as \( \gamma_j^{(u)} \), \( u = 0, 1, 2, \) where

\[
\gamma_j^{(0)} = \alpha_j, \quad \gamma_j^{(1)} = \beta_j, \quad \gamma_j^{(2)} = \delta_j, \quad j = 1, \ldots, N.
\]

Now the derived linear system of algebraic equations can be written down in the most compact form:

\[
\sum_{j=1}^{2} \sum_{u=0}^{2} \int \left( \frac{1}{2\pi} \sum_{j=1}^{N} \left( J_{mn}^{(u,r)} \frac{L_j}{L_m} - \frac{1}{2\pi} \sum_{j=1}^{N} J_{ij}^{(u,r)} \frac{L_j}{L_i} \right) - \lambda J_{mn}^{(r)} \right) = \left. \sum_{j=1}^{2} \int \left( \frac{1}{2\pi} \sum_{j=1}^{N} \left( \tilde{V}_i \cdot \tilde{\tau}_i(\xi) + \frac{1}{2\pi} \sum_{s=1}^{N} \Gamma_s Q_s(\xi) \right) \left( \frac{1}{\pi} \int J_{ik}^{(r)}(\xi) - \frac{\xi^r}{L_k} \right) d\xi \right) \right|_{k=1, \ldots, N}, \quad r = 0, 1, 2,
\]

\[
\sum_{j=1}^{2} \sum_{u=0}^{2} J_{mn}^{(u)} = \Gamma.
\]
\( n_{gp} \) nodes, and according to them
\[
\int_{a}^{b} f(x) dx \approx \frac{b-a}{2} \sum_{k=1}^{n_{gp}} \omega_k f \left( \frac{a+b}{2} + \frac{b-a}{2} x_k \right).
\]
Here, the values of the weight coefficients \( \omega_k \) and the positions of the Gaussian points \( x_k \), \( k = 1, \ldots, n_{gp} \), are chosen in standard way [15].

In the present research the value \( n_{gp} = 7 \) have been used. In the cases, when the Gaussian approximate formulae do not provide the necessary accuracy of the integrals computation (this usually occurs for the neighboring panels), an additional splitting procedure can be implemented to the chords, according to which these chords are divided into \( N_{add} = 4 \) sub-chords of smaller length (only for numerical calculations of the integrals by using the Gaussian quadrature formulae). If it is necessary, these “small” panels should be split once again.

Numerical experiments show, that taking into account of the Jacobian determinant \( J_\gamma(\xi) \) value only slightly influences the accuracy of the solution when number of the panels \( N \) is sufficiently high, so in practice we can consider \( J_\gamma(\xi) \equiv 1 \); this simplification makes the computational procedure easier.

It should be noted, that if the described numerical algorithm is implemented in some special mathematical software (MatLAB, Mathematica, etc.), built-in algorithms for numerical integration can be used, of course. It makes the development of the corresponding subroutine much easier in comparison with C++ or Fortran, but time of quadratures computation, however, becomes much higher in general case.

5. Numerical experiment
We consider three model problems of a flow simulation around the following airfoils: the circular airfoil of unit radius \( (R = 1) \), elliptical airfoil with the semiaxes \( a_1 = 1.0 \), \( b_1 = 0.5 \) and the Zhukovsky airfoil with parameters \( a = 3.5 \), \( d = 0.4 \), \( h = 0.3 \) at the angle of incidence \( \beta = \pi/6 \).

5.1. Exact analytical solution
Exact analytical solution for elliptical airfoils and Zhukovsky wing airfoils, which will be used for the numerical schemes accuracy estimations, can be constructed by using the conformal mappings techniques [14, 16], so for vortex sheet intensity distribution over the surface line of the airfoil the formulae can be used
\[
\gamma^*(t) = \left( 2V_0 \sin(\phi + \beta - t) + (\Gamma + W(t))/(\pi R) \right) \left( 1 - \frac{a^2}{(R e^{i(t-\phi)} + H)^2} \right)^{-1}, \quad H = i h - d e^{-i\phi}.
\] (14)

Here, parameter \( t \in [0,2\pi) \) corresponds to the position of the point on the surface line; the parametric dependencies \( x(t) = \text{Re} z(t), \quad y(t) = \text{Im} z(t) \) determine the shape of the airfoil, where
\[
z(t) = x(t) + i y(t) = \frac{1}{2} \left( \chi(t) + \frac{a^2}{\chi(t)} \right), \quad \chi(t) = R e^{i(t-\phi)} + H;
\] (15)
the term \( W \) expresses the influence of vortex elements (point vortices) in the flow:
\[
W(t) = \sum_{i=1}^{N_v} \frac{\cos(t - \theta_i) - d_i^x}{\cos(t - \theta_i) + (d_i^x + d_i^y)/2}.
\]

In order to compute the values \( d_i^x, \ d_i^y \) and \( \theta_i \) we should consider the vortices on the complex plane, then their positions will be \( z_s = x_s + i y_s, \quad s = 1, \ldots, N_v, \)
\[ d_i = |w_i - H| / R, \quad d_i^* = d_i^{-1}, \quad \phi_i = \phi + \text{Arg}(w_i - H), \]

where \( w_i = z_i + (z_i^2 - a^2)^{1/2} \), and that branch of the square root function should be chosen, that provides the condition \( d_i > 1 \) satisfaction.

For the elliptical airfoil we should choose

\[ a = (a_1^2 - b_1^2)^{1/2}, \quad R = a_1 + b_1, \quad \phi = 0, \quad h = d = 0, \]

where \( a_1 \) and \( b_1 \) are the semiaxes of the ellipse; for the Zhukovsky airfoil

\[ R = |H - a|, \quad \phi = \arctan(h/a), \]

and the parameters \( a, d \) and \( h \) determine length, width and curvature of the airfoil, respectively.

The value of total vorticity \( \Gamma \) for the elliptical airfoil can be chosen arbitrary, for our test problem we assume \( \Gamma = 0 \) for any angle of incidence; for the Zhukovsky airfoil \( \Gamma \) can be chosen from the condition, that velocity at the sharp edge is bounded: its value is proportional to the incident flow velocity and depends on the shape of the airfoil and the angle of incidence, as well as on the vortex elements positions in the flow:

\[ \Gamma = -2\pi v \sin(\beta + \phi)((h^2 + a^2)^{1/2} + d) \sum_{\alpha=1}^{N} \frac{\cos \theta_i - d_i^*}{\cos \theta_i - (d_i + d_i^*)/2}. \]

5.2. Numerical solution and the developed scheme accuracy estimation

Firstly, by using the developed numerical scheme with curvilinear panels, the vortex sheet intensity distribution had been computed (the values of the coefficients \( \alpha_i, \beta_i \) and \( \delta_i \) for all the panels). Then this distribution was integrated over the curvilinear panels, and the average value was found by dividing on the panel length:

\[ \gamma_i = \frac{1}{\Delta t} \int_{t_{i-1}}^{t_i} \left( \alpha_i + \beta_i \frac{\xi}{L_i} + \delta_i \frac{\xi^2}{L_i^2} \right) J_i(\xi)d\xi / \int_{t_{i-1}}^{t_i} J_i(\xi)d\xi, \quad i = 1, \ldots,N. \]

The similar values \( \gamma_i^* \) were obtained by integration of the exact solution (14):

\[ \gamma_i^* = \int_{t_{i-1}}^{t_i} \gamma^*(t)(x'(t)^2 + y'(t)^2)^{1/2} dt / \int_{t_{i-1}}^{t_i} (x'(t)^2 + y'(t)^2)^{1/2} dt, \quad i = 1, \ldots,N. \]

where the parameter values \( t_i \) and \( t_{i+1} \) correspond to the beginning and the ending of the \( i \)-th panel, the dependencies \( x(t) \) and \( y(t) \) can be found from (15).

The error of the numerical solution was computed as the maximal absolute difference between the average vortex sheet intensities over the panels, computed for the exact and numerical solutions:

\[ \Delta \gamma = \max_i |\gamma_i - \gamma_i^*|. \]

The results show nearly the same value of the error for the considered test problems in cases of piecewise-linear and piecewise-quadratic vorticity distributions over the panels, however, it should be noted that in case of piecewise-quadratic approximation the algebraic system (13) becomes ill-conditioned.

In all cases (for the circular, elliptical and Zhukovsky wing airfoils) the error of the numerical solution has order \( O(h^4) \), where \( h \) is the maximal panel length (figure 3).

6. Conclusion

A new approach and the corresponding high-order numerical algorithm are developed for the vortex sheet intensity computation on the surface line of the airfoil in incompressible flow. In order to achieve high order of accuracy, the curvature of the airfoil is taken into account as well as the piecewise-linear or
piecewise-quadratic vorticity distribution in the vortex sheet over each panel. The developed numerical scheme permits to simulate the solution discontinuities between the panels; this makes it possible to simulate airfoils with sharp edges, for example, the Zhukovsky wing airfoil. The numerical experiments proved the properties of the developed scheme and showed that it provides the 4-th order of accuracy.

![Figure 3. The error in total vorticity over the panels (in logarithmic scale) for the developed numerical scheme (13): (a) — circular airfoil; (b) — elliptical airfoil; (c) — Zhukovsky airfoil; dashed line everywhere corresponds to $O(h^4)$ error level](image)

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