OPTIMAL LIABILITY RATIO AND DIVIDEND PAYMENT STRATEGIES UNDER CATASTROPHIC RISK

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Abstract. This paper investigates the optimal strategies for liability management and dividend payment in an insurance company. The surplus process is jointly determined by the reinsurance policies, liability levels, future claims and unanticipated shocks. The decision maker aims to maximize the total expected discounted utility of dividend payment in infinite time horizon. To describe the extreme scenarios when catastrophic events occur, a jump-diffusion Cox-Ingersoll-Ross process is adopted to capture the substantial claim rate hikes. Using dynamic programming principle, the value function is the solution of a second-order integro-differential Hamilton-Jacobi-Bellman equation. The subsolution–supersolution method is used to verify the existence of classical solutions of the Hamilton-Jacobi-Bellman equation. The optimal liability ratio and dividend payment strategies are obtained explicitly in the cases where the utility functions are logarithm and power functions. A numerical example is provided to illustrate the methodologies and some interesting economic insights.

1. Introduction. Managing the surplus and designing dividend payment policies have long been an important issue in finance and actuarial sciences. The dividend payment plan released in the financial report for public companies represents an important signal about a firm’s future growth opportunities and profitability. On the one hand, if dividend payments are too high, firms’ retained profits may become

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low, thereby weakening their financial stability or undermining growth opportunities. On the other hand, the decision of dividend payment strategies is so important for a public company’s financial strength because the company’s share price is very sensitive to the information of dividend plans. Dividend payment strategies also significantly influence the firms’ financing structure and investment decisions. Companies’ stocks may become less popular due to low dividend payment rate. In addition, since dividends are funded by internal capital, which is considered as a source of inexpensive fund raising comparing with external capital.

For insurance companies, because of the nature of their products, insurers tend to accumulate relatively large amounts of cash, cash equivalents, and investments in order to pay future claims and avoid insolvency. The payment of dividends to shareholders may reduce an insurer’s ability to survive adverse investment and underwriting experience. However, due to the undergone pressure of managing balance sheets and distributing the surplus for public insurance companies, one natural objective for insurers is to optimize the management of surplus and sustained stream of dividend payments. A practitioner will manage the reserve and dividend payment against various financial risks so that the company can satisfy its minimum capital requirement in the long run. The study of optimal dividend payment and liability management of an insurance company has become a high priority task.

In previous work, [22] studied the relationship between a company’s dividend policy and the valuation of its shares, there have been increasing effort to study the dividend payment policy and its impact on the firms’ value. [23] argued that dividends can signal the firms’ future profitability. This argument was supported by empirical evidence in early years (see [1]). Unlike companies in other sectors, insurance companies collected premiums and reserve the capital to pay for the future possible claims. Considering the speciality of insurance companies’ cashflow, [10] analyzed optimal dividend payment strategies in risk models. Since then, researchers have developed various advanced methods to study the optimal dividend policy for insurers. The majority of research are conducted in finding optimal dividend payment policies to maximize the present value of the cumulative dividend payment in targeted time horizon. Regular controls, singular controls, and impulse controls are involved in various scenarios. [15] studied the dividend and risk control with a diffusion where the drift is quadratic. [16] studied the mixed control of dividend, proportional reinsurance and financing for the model. [20] solved the optimal dividend and issuance of equity policies in the presence of bankruptcy risk. [2] and [6] studied the impulse dividend control problem for a rather general linear diffusion model in which some growth and smoothness conditions are imposed on. [4] addressed the dividend and reinvestment control in a spectrally negative Lévy process. [5] analyzed the problem of the maximization of total discounted dividend payment for an insurance company. [19] investigated the optimal dividend control with affine penalty at ruin for a spectrally negative Lévy process using singular control. [27] addressed the regular control problem under a regime switching Brownian motion model and [28] studied the problem of optimal financing and dividend distribution in a general diffusion model with regime switching.

To protect insurance companies against the impact of claim volatilities, reinsurance is a standard tool with the goal of reducing and eliminating risk. The primary insurance carrier pays the reinsurance company a certain part of the premiums. In return, the reinsurance company is obliged to share the risk of large claims. Proportional reinsurance is one of such reinsurance policies. Using such an approach, the
reinsurance company covers a fixed percentage of losses and the premium of reinsurance is determined. The most common nonproportional reinsurance policy is the so-called excess-of-loss reinsurance, within which the cedent (primary insurance carrier) will pay all of the claims up to a pre-given level of amount (termed retention level). Related work in both reinsurance schemes under various framework can be found in [3], [21] [18], [8], [26], [7] and references therein.

Recently, leverage tools have been widely used to increase the leverage of financial status for both individual investors and public companies. When the market is highly leveraged, the undertaken liabilities become very risky for the risk takers, and the financial system also becomes vulnerable since the small change in asset values will significantly influence the financial institutions’ surplus status. The insurance companies, providing protections for the whole financial system, are also vulnerable when they overtake liabilities, and no one could survive in the turmoil. The determination of optimal liability level has its short- and long-run effect on the insurance company’s performance. More importantly, it is a crucial issue to the vulnerabilities of the whole financial system. The failure of AIG has taught us a lesson and leads us to address the key issues in liability management of insurance companies and tackle the public concerns on how to avoid the risks of either total collapse of our financial system and economy or inject trillions of taxpayer dollars into the financial system as US did in 2008 financial crisis (see [25]). Following the formulation in [17], the liability ratio, defined as the ratio of liability and surplus \( \frac{L_t}{X_t} \), measures the leverage level of the insurance companies. As the actual liability ratio exceeds the optimal liability ratio, the probability of default rises.

In [17], the authors proposed an asset and liability model with liability constraint. The insurance company manages the surplus and designs the dividend payment strategies taking into account the liability capacity. The claim rate, which is essential for the optimal dividend payment and policy written strategies, is assumed to follow the stochastic differential equation as the following

\[
\begin{align*}
\frac{dc(t)}{dt} &= g(c(t))dt + \sigma^2dW_2(t), \\
c(0) &= c.
\end{align*}
\]

The optimal strategies are obtained correspondingly under this framework. However, in view of (1), [17] failed to capture certain key features in the claim rate process such as mean-reverting and nonnegativity. The Cox-Ingersoll-Ross (CIR) process, which was firstly introduced in [9] to describe the random evolution of short rate, guarantees the features of the mean-reverting and nonnegativity. As an extension of the CIR process, we adopt the jump-diffusion CIR process to describe the stochastic claim rate. Not only does the jump-diffusion CIR process provide features of mean-reverting and nonnegativity for the claim rate, but also it captures the claim hikes when certain catastrophic event occurs. Catastrophic risks, which are rarely and discrete events, usually lead major consequences and incur substantial losses. The claims rate hikes, caused by the discrete catastrophic risks, are described as the positive jumps in the jump-diffusion CIR process. Hence, the jump-diffusion CIR process is more versatile and realistic to model the claim rate process. For the study of jump-diffusion CIR process, please refer to [11], [14] and [12].

In this study, we choose logarithm and power utility functions to describe the risk aversion level for different types of insurers to find the optimal capital requirement or leverage that balances risk against expected growth and return. The
value function depends on the surplus and claim rate. By dynamic programming principle, the value function obeys a second-order Hamilton-Jacobi-Bellman (HJB) equation. The existence of classical solutions of the HJB equation is verified by the subsolution–supersolution method introduced in [13]. Optimal liability ratio and dividend payment rate are obtained correspondingly. In simulation results, sensitivity analysis is provided to illustrate the ideas and methodologies in the cases of both logarithm and power utility functions. The impact of key parameters on the liability management and dividend payment policies are clearly obtained from the analytical solutions of optimal controls.

The rest of the paper is organized as follows. A general formulation of asset value, surplus, insurance liabilities, claim rates, dividend strategies, and assumptions is presented in Section 2. Section 3 deals with optimal debt ratio and dividend payment strategies in power utilities. The subsolution–supersolution method is introduced, and the existence of classical solution of HJB equation is proved. The verification theorem of optimal value function is presented in Section 3.3. Section 4 deals with optimal liability ratio and dividend payment strategies in the case of logarithm utility. A numerical example is provided in Section 5 and the impact of key parameters are considered. Finally, additional remarks are provided in Section 6.

2. Formulation. Let \( K(t) \) and \( L(t) \) be the asset value and liabilities of an insurer at time \( t \), respectively. Then the surplus process \( X(t) \) is described as the difference between the asset value and liabilities, i.e. \( X(t) = K(t) - L(t) \).

For an insurer issuing protections for reference assets, claims are the collateral calls for the notional values. The notional values are determined in the insurance policies as the insured liabilities. Denote by \( a \) the premium rate, which represents the cost of protection per dollar of insurance liabilities. The asset value increases from the premiums collected during the time period \([t, t + dt]\). The increment is denoted as \( aL(t)dt \). Moreover, in order to protect insurance companies against the impact of claim volatilities, the insurer buy reinsurance protections. We assume that proportional reinsurance is adopted by the primary insurance company in our model. Let \( \theta \in (0, 1] \) be an exogenous retention level for the reinsurance policy. Denote by \( h(\theta) \) be reinsurance charge rate that is the cost of reinsurance protection per dollar of reinsured liabilities. From a practical view of point, the cost of reinsurance protection per dollar of reinsured liabilities should be nonnegative. Denote \( c(t) \) be the claim rate. We assume that \( c(t) \) is risky and can be described as a stochastic process in (3). In a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\), the dynamics of \( K(t) \) and \( c(t) \) are given by

\[
\frac{dK(t)}{K(t)} = \mu(t)dt + \sigma_1(t)dW_1(t),
\]

and

\[
dc(t) = (p - qc(t))dt + \sigma_2\sqrt{c(t)}dW_2(t) + d\sum_{i=1}^{N(t)} Y_i, \quad c_0 > 0,
\]

where \( \mu(t) \) is the varying drift of the asset and \( \sigma_1(t) \) is the corresponding volatility. \( p - qc(t) \) is the drift of the claim rate, where \( p \) and \( q \) are positive constants representing the mean-reverting property of the claim rates. \( \sigma_2 \) is a positive constant, and represents the volatility of the claim rate. To guarantee the non-negativity of \( c(t) \), we assume that \( \sigma_2^2 > 2p \). \( W_1(t), W_2(t) \) are two Brownian motions, \( N(t) \) is a
Poisson process with intensity $\lambda$ and $\{Y_i, i = 1, 2, \ldots\}$ are i.i.d random variables with distribution $G(dy) = ke^{-k_y}I_{[y \geq 0]}dy$. $I(\cdot)$ is the indicator function. In view of (3), the nonegativity and mean-reversion is guaranteed by the CIR type process. The claim rate hikes due to the rare catastrophic events that are described by the compound Poisson process. We assume that $W_1(t), W_2(t), N(t), Y_i$ are mutually stochastic independent and $\mu(t), \sigma_1(t)$ are bounded. $\mathcal{F}_t$ is the $\sigma$-algebra generated by $\{W_1(s), W_2(s), N(s), Y_i : 0 \leq s \leq t, i = 1, 2, \cdots\}$ and $\{\mathcal{F}_t\}$ is the filtration satisfying the usual conditions.

Let $\pi(t) = L(t)/X(t)$ be the liability ratio of the insurance company. Then, the leverage, which is described as the ratio between asset values and surplus, can be written as $K(t)/X(t) = 1 + \pi(t)$. To avoid the insurance liabilities being too large, the insurers will decide the optimal liability ratio to manage the sale of insurance policies. Furthermore, let $D(t)$ be the dividend payment rate. Assume that $D(t)$ is a linear function of $X(t)$, that is,

$$D(t) = z(t)X(t),$$

where $z(t) \in [0, 1]$ denotes the proportion of the surplus of the insurance company at $t$ that is distributed as dividend. Then the insurer’s surplus process follows

$$dX(t) = dK(t) + (a - h(\theta) - c(t))L(t)dt - z(t)X(t)dt, \quad x_0 > 0.$$

**Definition 2.1.** A strategy $u(\cdot) := (\pi(\cdot), z(\cdot))$ is said to be admissible if

(i) $\pi(\cdot)$ is an $\mathbb{F}$-predictable process, such that $\pi(t) \geq 0$ and $\mathbb{E}[\int_0^\infty \pi^2(t)dt] < \infty$;

(ii) $z(\cdot)$ is an $\mathbb{F}$-predictable process, such that $z(t) \geq 0$ and $\mathbb{E}[\int_0^\infty z^2(t)dt] < \infty$;

(iii) the wealth equations associated with $u(\cdot)$ has a unique strong solution.

Let $\mathcal{A}$ be the space of all admissible strategies.

The wealth process is given by

$$dX(t) = \left[(1 + \pi(t))X(t)\mu(t) - z(t)X(t) + \pi(t)X(t)(a - h(\theta)) - \pi(t)X(t)\theta c(t)\right]dt$$

$$+ (1 + \pi(t))X(t)\sigma_1(t)dW_1(t).$$

(4)

The representative decision-maker is risk averse. The objective is to maximize the expectation of the discounted value of the utility of dividend payment until financial ruin. The value function is given by

$$V(x, c) = \sup_{(\pi, z) \in \mathcal{A}} E_{x,c} \left[ \int_0^\tau e^{-rt}U(z(t)X(t))dt \right],$$

where $\tau := \inf\{t > 0; X(t) < 0\}$ is the ruin time. $E_{x,c}$ denotes the expectation conditioned on $X(0) = x$ and $c(0) = c$.

It is worthy noting that, under the model setting in this paper, the ruin probability should be 0, i.e. $P(\tau = \infty) = 1$. Then the value function can be rewritten as

$$V(x, c) = \sup_{(\pi, z) \in \mathcal{A}} E_{x,c} \left[ \int_0^\infty e^{-rt}U(z(t)X(t))dt \right].$$

We firstly derive the HJB equation that the value function satisfies in this problem.

For simplicity of notation in the following, we let $\mu := \mu(t), \sigma_1 := \sigma_1(t)$, where appropriate. Denote the number of jumps during time interval $[0, \varepsilon]$ by $N(\varepsilon)$, where
\( \varepsilon > 0 \) denotes a small time period. The derivation of HJB equation follows

\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \sup_{(\pi, z) \in A} \left\{ E_{x,c}\left[ \int_0^\varepsilon e^{-rt}U(z(t)X(t))dt + e^{-r\varepsilon}V(X_\varepsilon, c_\varepsilon) \right] - V(x, c) \right\}
\]

\[
= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \sup_{(\pi, z) \in A} \left\{ E_{x,c}\left[ \left( \int_0^\varepsilon e^{-rt}U(z(t)X(t))dt + e^{-r\varepsilon}V(X_\varepsilon, c_\varepsilon) \right) I_{N(\varepsilon)=0} \right] - V(x, c)I_{N(\varepsilon)=0} \right\}
\]

\[
+ \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \sup_{(\pi, z) \in A} \left\{ E_{x,c}\left[ \left( \int_0^\varepsilon e^{-rt}U(z(t)X(t))dt \right) I_{N(\varepsilon)=1} \right] - V(x, c)I_{N(\varepsilon)=1} \right\}
\]

\[
= U(zx) + \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \sup_{(\pi, z) \in A} \left\{ E_{x,c}\left[ e^{-r\varepsilon}V(X_\varepsilon, c_\varepsilon) I_{N(\varepsilon)=0} - V(x, c)I_{N(\varepsilon)=0} \right] \right\}
\]

\[
+ \lambda \int_0^\infty V(x, c+y)G(dy) - \lambda V(x, c).
\]

Note that

\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \sup_{(\pi, z) \in A} \left\{ E_{x,c}\left[ \int_0^\varepsilon e^{-rt}U(z(t)X(t))dt I_{N(\varepsilon)=0} \right] \right\} = U(zx),
\]

\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \sup_{(\pi, z) \in A} \left\{ E_{x,c}\left[ \int_0^\varepsilon e^{-rt}U(z(t)X(t))dt I_{N(\varepsilon)=1} \right] \right\} = 0,
\]

\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \sup_{(\pi, z) \in A} \left\{ E_{x,c}\left[ e^{-r\varepsilon}V(X_\varepsilon, c_\varepsilon) I_{N(\varepsilon)=1} \right] \right\} = \lambda \int_0^\infty V(x, c+y)G(dy).
\]

The HJB equation is given by

\[
\sup_{\pi \in A} \left\{ \left[ (1 + \pi)x\mu + \pi x(a - h(\theta)) - \pi xc\theta \right] V_x + \frac{1}{2} \left[ (1 + \pi)^2 x^2 \sigma_1^2 \right] V_{xx} \right\}
\]

\[
+ \lambda \int_0^\infty V(x, c+y)G(dy) + \frac{1}{2} \sigma_2^2 c V_{cc} + (p - q) V_c - (\lambda + r) V(x, c)
\]

\[
+ \sup_{z \in A} \left\{ U(zx) - zx V_x \right\} = 0.
\]

3. **Power utility.** Consider power utility function \( U(x) = \frac{x^\gamma}{\gamma} \), \( 0 < \gamma < 1 \). Assume the solution \( V(x, c) = \frac{x^{\gamma}}{\gamma} Y(c) \), \( Y(c) > 0 \). We obtain

\[
V_x = x^{\gamma-1} Y(c), \quad V_{xx} = (\gamma - 1)x^{\gamma-2} Y(c); \quad V_c = \frac{x^{\gamma}}{\gamma} Y'(c), \quad V_{cc} = \frac{x^{\gamma}}{\gamma} Y''(c).
\]

Then HJB equation (6) turns to be

\[
\sup_{\pi \in A} \left\{ \frac{1}{\gamma} z\gamma - z Y(c) \right\} + \sup_{\pi \in A} \left\{ \left[ (1 + \pi)x\mu + \pi x(a - h(\theta)) - \pi xc\theta \right] Y(c) \right\}
\]

\[
- \frac{1}{2} \left( (1 + \pi)^2 \sigma_1^2 (1 - \gamma) Y(c) \right) + \lambda \int_0^\infty Y(c+y)k e^{-ky} dy
\]

\[
+ \frac{1}{2} \sigma_2^2 c \frac{1}{\gamma} Y''(c) + (p - q) \frac{1}{\gamma} Y'(c) - (\lambda + r) \frac{1}{\gamma} Y(c) = 0.
\]
Firstly we consider the optimal $z:\]
\[
\arg\max_z \left\{ \frac{1}{\gamma} z^\gamma - zY(c) \right\} = Y_{\pi^*}(c), \quad \sup_{z \in A} \left\{ \frac{1}{\gamma} z^\gamma - zY(c) \right\} = (\frac{1}{\gamma} - 1) Y_{\pi^*}(c).
\]
Define $\Lambda(\pi) := (1 + \pi)\mu + \pi(a - h(\theta)) - \pi c\theta - \frac{1}{2}(1 + \pi)^2 \sigma^2(1 - \gamma)$, we have
\[
\Lambda'(\pi) = \mu + a - h(\theta) - c\theta - (1 + \pi)\sigma^2(1 - \gamma),
\]
\[
\Lambda''(\pi) = -\sigma^2(1 - \gamma) < 0.
\]
To obtain the optimal $\pi^*$, we discuss the positivity of $\Lambda'$ in the following subsections. The optimal strategies will be given in different scenarios in Section 3.1 and Section 3.2. The verification theorem will be provided in Section 3.3.

3.1. Case 1: $\Lambda'(0) > 0$. If $\Lambda'(0) > 0$, i.e. $0 < c < \frac{1}{\theta} [\mu + a - h(\theta) - \sigma^2(1 - \gamma)]$, it follows
\[
\pi^* = \frac{1}{\sigma^2(1 - \gamma)} \left( \mu + a - h(\theta) - c\theta - \sigma^2(1 - \gamma) \right).
\]
Substituting $\pi^*$ into (7) yields
\[
\begin{align*}
\left[ h(\theta) + c\theta - a + \frac{1}{2\sigma^2(1 - \gamma)} (\mu + a - h(\theta) - c\theta)^2 - (\lambda + r) \frac{1}{\gamma} \right] Y(c) \\
+ \frac{\lambda}{\gamma} \int_0^\infty Y(c + y) ke^{-ky}dy + \frac{1}{2} \frac{1}{\gamma} \sigma^2 Y''(c) + (p - qc) \frac{1}{\gamma} Y'(c) \\
+ \left( \frac{1}{\gamma} - 1 \right) Y_{\pi^*}(c) = 0.
\end{align*}
\]
Denote the left hand of (8) by $F(Y(c), c)$. To obtain the classical solution of (8), we will use the subsolution and supersolution method in [13]. The definition of subsolution and supersolution is given as follows.

**Definition 3.1.** A solution $Y_{\text{sub}}(c)$ is said to be a subsolution of (8) iff $\forall c \in R, Y_{\text{sub}}(c) \in C^2(R)$ and $Y_{\text{sub}}(c)$ satisfies
\[
F(Y_{\text{sub}}(c), c) \geq 0;
\]
A solution $Y_{\text{sup}}(c)$ is said to be a supersolution of (8) iff $\forall c \in R, Y_{\text{sup}}(c) \in C^2(R)$ and $Y_{\text{sup}}(c)$ satisfies
\[
F(Y_{\text{sup}}(c), c) \leq 0.
\]
Moreover, if $\forall c \in R, Y_{\text{sub}}(c) \leq Y_{\text{sup}}(c)$. We say $Y_{\text{sub}}(c)$ and $Y_{\text{sup}}(c)$ are an ordered pair of subsolution and supersolution.

The existence of a classical solution for (8) can be proved by Theorem 5.2 in Chapter 7 in [24]. To proceed, we need an assumption:

**Assumption 1.**
\[
B := \mu + \frac{1}{2\sigma^2(1 - \gamma)} \left( \mu + a - h(\theta) - \sigma^2(1 - \gamma) \right)^2 - \frac{1}{2} \sigma^2(1 - \gamma) - \frac{r}{\gamma} < 0.
\]
The assumption is satisfied when $\gamma$ is sufficiently small. That is, the decision-maker is sufficiently risk-averse. In this case, significant penalty to the accumulated utilities will be allocated when dividend payment amount is small.
Theorem 3.2. Let
\[ Y_{\text{sub}}(c) = \left[ -\frac{1}{A} \right]^{1-\gamma}, \]
and
\[ Y_{\text{sup}}(c) = \left[ -\frac{1}{B} \right]^{1-\gamma}, \]
where
\[ A := \mu - \frac{1}{2} \sigma^2 (1 - \gamma) - \frac{r}{\gamma}. \]
Then under assumption 1, \( Y_{\text{sub}}(c) \) and \( Y_{\text{sup}}(c) \) are the subsolution and supersolution of (8), respectively.

Proof. Under Assumption 1 and the constraint \( 0 < \gamma < 1 \), we have
\[ A \leq B < 0, \quad 0 \leq \left[ -\frac{1}{A} \right]^{1-\gamma} \leq \left[ -\frac{1}{B} \right]^{1-\gamma}. \]
It is not hard to find that
\[ F(Y_{\text{sup}}(c), c) \leq 0, \quad F(Y_{\text{sub}}(c), c) \geq 0. \]

3.2. Case 2: \( \Lambda'(0) \leq 0 \). If \( \Lambda'(0) \leq 0 \), i.e. \( c \geq \frac{1}{\theta} [\mu + a - h(\theta) - \sigma^2 (1 - \gamma)] \), it follows \( \pi^* = 0 \).

Substituting \( \pi^* \) into (7) yields
\[ \left( \frac{1}{\gamma} - 1 \right) Y \overset{\pi^*}{=} (c) + \left[ \mu - \frac{1}{2} \sigma^2 (1 - \gamma) - \left( \lambda + r \right) \frac{1}{\gamma} \right] Y(c) + \frac{\lambda}{\gamma} \int_0^\infty Y(c + y) ke^{-ky} dy + \frac{1}{2} \sigma^2 c \frac{1}{\gamma} Y''(c) + (p - qc) \frac{1}{\gamma} Y'(c) = 0. \]
The solution can be given by
\[ Y(c) = \left[ -\frac{1}{A} \right]^{1-\gamma} > 0. \]
Thus
\[ z^* = -\frac{A}{\frac{1}{\gamma} - 1}. \]

3.3. Verification theorem. In this subsection, we focus on the verification theorem. To begin with, we provide a lemma that will be used in verification theorem.

Lemma 3.3. Define
\[ G := \gamma (1 + \pi^*) \mu - \gamma z^* + \gamma \pi^* (a - h(\theta)) - \gamma \pi^* \theta c - \frac{1}{2} \left( \gamma - \gamma^2 \right) (1 + \pi^*)^2 \sigma^2 - r, \]
then \( G < 0 \).
Proof. If $c \geq \frac{1}{\theta} [\mu + a - h(\theta) - \sigma_1^2 (1 - \gamma)]$, we have

$$G = \gamma \mu - \gamma z^* - \frac{1}{2} \left(\gamma - \gamma^2\right) \sigma_1^2 - r$$

$$= \gamma \left(\mu + \gamma \mu - \frac{1}{2} \sigma_1^2 \gamma (1 - \gamma) - r \frac{1}{\gamma} - \frac{1}{2} (1 - \gamma) \sigma_1^2\right)$$

$$= \frac{\gamma}{1 - \gamma} \left(\mu - \frac{1}{2} \gamma (1 - \gamma) \sigma_1^2 - r \frac{1}{\gamma} - \frac{1}{2} (1 - \gamma) \sigma_1^2\right)$$

$$= A\gamma \frac{A}{1 - \gamma} < 0.$$

In addition, if $0 < c < \frac{1}{\theta} [\mu + a - h(\theta) - \sigma_1^2 (1 - \gamma)]$, we have

$$G = \gamma \left\{ (1 + \pi^*) \mu - z^* + \pi^* (a - h(\theta)) - \pi^* \theta c - \frac{1}{2} (1 - \gamma) (1 + \pi^*) \sigma_1^2 - r \frac{1}{\gamma}\right\}$$

$$= \gamma \left\{ \mu - z^* + \frac{r}{\gamma} - \frac{1}{2} (1 - \gamma) \sigma_1^2 + \frac{1}{2} (1 - \gamma) \sigma_1^2 (\pi^*)^2\right\}$$

$$+ \pi^* \left\{ \mu + a - h(\theta) - \theta c - (1 - \gamma) \sigma_1^2 - \pi^* (1 - \gamma) \sigma_1^2\right\}$$

$$= \gamma \left\{ \mu - z^* + \frac{r}{\gamma} - \frac{1}{2} (1 - \gamma) \sigma_1^2 + \frac{1}{2} (1 - \gamma) \sigma_1^2 (\pi^*)^2\right\}$$

$$= \gamma \left\{ \mu + \frac{1}{2 \sigma_1^2 (1 - \gamma)} \left( \mu + a - h(\theta) - \sigma_1^2 (1 - \gamma) - c \theta\right)^2 - \frac{1}{2} \sigma_1^2 (1 - \gamma) - \frac{r}{\gamma} - z^*\right\}$$

$$\leq \gamma (B - z^*) \leq 0.$$

\[\square\]

**Theorem 3.4.** Suppose there exists a $Y_{\text{sub}}(c) \leq Y^*(c) \leq Y_{\text{sup}}(c)$ and $Y^*(c)$ solves (9) and (8). Let $V(x, c) = \frac{2}{\gamma} Y^*(c)$, then

1. For all admissible pairs of control policies $u = (\pi, z)$,

$$V(x, c) \geq J(x, c, u) = \mathbb{E}_{x, c} \left[ \int_0^\infty e^{-rt} \frac{(z(t) X(t))\gamma}{\gamma} dt \right].$$

2. If $u^* = (\pi^*, z^*)$ is given as follows:

$$\pi^* = \begin{cases} 0, & c \geq C_0 \\ \tilde{\sigma}, & c < C_0 \end{cases}$$

$$z^* = (Y^*)^{\frac{1}{\gamma}}(c),$$

where

$$C_0 := \frac{1}{\theta} [\mu + a - h(\theta) - \sigma_1^2 (1 - \gamma)],$$

$$\tilde{\sigma} := \frac{1}{\sigma_1^2 (1 - \gamma)} \left( \mu + a - h(\theta) - c \theta - \sigma_1^2 (1 - \gamma)\right).$$

Then

$$V(x, c) = J(x, c, u^*) = \mathbb{E}_{x, c} \left[ \int_0^\infty e^{-rt} \frac{(z^*(t) X^*(t))\gamma}{\gamma} dt \right].$$
Proof. Applying Itô’s lemma to $e^{-rt}V(X(t), c(t))$, we have
\[ e^{-rt}V(X(t), c(t)) = V(x, c) + \int_0^t e^{-rs} \left[ L^u V(X(s), c(s)) - rV(X(s), c(s)) \right] ds + M(t), \]
where
\[ L^u V(x, c) := [(1 + \pi) x \mu + \pi x (a - h(\theta)) - \pi x \theta c - z x] V_x + \frac{1}{2} [(1 + \pi)^2 x^2 \sigma_1^2] V_{xx} \]
\[ + (p - qc) V_c + \frac{1}{2} \sigma_2^2 c \sigma c + [V(x, c + \Delta c) - V(x, c)], \]
\[ M(t) := \int_0^t V_z (1 + \pi(s)) X(s) \sigma_1(s) dW_1(s) + \int_0^t X_c \sigma_2 \sqrt{c(s)} dW_2(s). \]
From HJB equation (6), we have
\[ E[e^{-rT}V(X(T), c(T))] - V(x, c) \leq -E \left[ \int_0^T e^{-rt} U(z(t)X(t)) dt \right]. \]
Therefore,
\[ V(x, c) \geq E \left[ \int_0^T e^{-rt} U(z(t)X(t)) dt \right] + E[e^{-rT}V(X(T), c(T))] \]
\[ = E \left[ \int_0^T e^{-rt} \frac{(z(t)X(t))^\gamma}{\gamma} dt \right] + E \left[ e^{-rT} X^\gamma(T) \gamma Y(c(T)) \right]. \]
(10)
Since $X(t)$ and $Y(c(t))$ are non-negative, we have
\[ V(x, c) \geq E \left[ \int_0^T e^{-rt} \frac{(z(t)X(t))^\gamma}{\gamma} dt \right]. \]
Under the control $u^* = (\pi^*, z^*)$, we have
\[ X^*(T) = x \exp \left( \int_0^T \left( 1 + \pi^*(t) \right) \mu(t) - z^*(t) + \pi^*(t) \left( a - h(\theta) \right) - \pi^*(t) \theta c(t) \right) \]
\[ - \frac{1}{2} (1 + \pi^*(t))^2 \sigma_1^2(t) dt + \int_0^T (1 + \pi^*(t)) \sigma_1(t) dW_1(t) \right). \]
Thus,
\[ E \left[ e^{-rT} \frac{(X^*(T))^\gamma}{\gamma} Y^* (c(T)) \right] \]
\[ = \frac{x^\gamma}{\gamma} \left[ \exp \left( \int_0^T \left[ (1 + \pi^*(t)) \mu(t) - z^*(t) + \gamma \pi^*(t) \left( a - h(\theta) \right) - \gamma \pi^*(t) \theta c(t) \right) \right. \]
\[ - \frac{1}{2} \gamma (1 + \pi^*(t))^2 \sigma_1^2(t) - \frac{1}{2} \gamma^2 (1 + \pi^*(t))^2 \sigma_1^2(t) - r \) dt + \int_0^T (1 + \pi^*(t)) \sigma_1(t) dW_1(t) \right) \]
\[ = \frac{x^\gamma}{\gamma} \left[ \exp \left( \int_0^T \left[ (1 + \pi^*(t)) \mu(t) - z^*(t) + \gamma \pi^*(t) \left( a - h(\theta) \right) - \gamma \pi^*(t) \theta c(t) \right) \right. \]
\[ - \frac{1}{2} (\gamma - \gamma^2) (1 + \pi^*(t))^2 \sigma_1^2(t) - \frac{1}{2} \gamma^2 (1 + \pi^*(t))^2 \sigma_1^2(t) - r \) dt \]
\[ + \int_0^T (1 + \pi^*(t)) \sigma_1(t) dW_1(t) \right) \]
\[ Y^* (c(T)) \right]. \]
By using the Hölder inequality, we have
\[
E \left[ e^{-rT} \frac{X^\gamma(T)}{\gamma} Y^\ast(c(T)) \right] \\
\leq \frac{x^\gamma}{\gamma} \mathbb{E}^* \left[ \exp \left( \hat{p} \int_0^T \left( \gamma(1 + \pi^\ast(t)) \mu(t) - \gamma z^\ast(t) + \gamma \pi^\ast(t)(a - h(\theta)) - \frac{1}{2}(\gamma - \gamma^2)(1 + \pi^\ast(t))^2 \sigma_1^2(t) - \gamma \pi^\ast(t) \theta c(t) - r \right) dt \right) \right] \\
\mathbb{E}^+ \left[ \exp \left( \int_0^T \hat{q} \gamma(1 + \pi^\ast(t)) \sigma_1(t) dW_1(t) - \frac{1}{2} \hat{q}^2 \gamma^2(1 + \pi^\ast(t))^2 \sigma_1^2(t) dt \right) \right] \\
\exp \left( \int_0^T \frac{1}{2}(\hat{q}^2 - \hat{q}) \gamma^2(1 + \pi^\ast(t))^2 \sigma_1^2(t) dt \right) \left[ - \frac{1}{\gamma} - 1 \right]^{1 - \gamma},
\]
where \( \hat{p} \geq 1, \hat{q} \geq 1 \) and \( \frac{1}{\hat{p}} + \frac{1}{\hat{q}} = 1 \).

In view of Lemma 3.3 and the boundedness of the parameters, we have
\[
E \left[ e^{-rT} \frac{X^\gamma(T)}{\gamma} Y^\ast(c(T)) \right] \\
\leq \frac{x^\gamma}{\gamma} \left[ - \frac{1}{\hat{p} + 1} \right]^{1 - \gamma} \mathbb{E}^+ \left[ \exp \left( \hat{p} \int_0^T G dt \right) \right] \\
\mathbb{E}^+ \left[ \exp \left( \int_0^T \frac{1}{2}(\hat{q}^2 - \hat{q}) \gamma^2(1 + \pi^\ast(t))^2 \sigma_1^2(t) dt \right) \right] \left[ - \frac{1}{\gamma} - 1 \right]^{1 - \gamma}
\]
Let \( \hat{q} \to 1 \) and \( T \to \infty \), and combine \( G < 0 \), we can obtain
\[
\lim_{T \to \infty} E \left[ e^{-rT} \frac{X^\gamma(T)}{\gamma} Y^\ast(c(T)) \right] = 0.
\]
Then,
\[
V(x, c) = E \left[ \int_0^\infty e^{-rt} \frac{(z^\ast(t) X^\ast(t))}{\gamma} dt \right].
\]

4. Logarithm utility. Consider the logarithm utility function \( U(x) = \ln x \). Assume the solution \( V(x, c) = b \ln x + Y(c) \), \( b > 0 \). We obtain
\[
V_x = b \frac{x}{x}, \quad V_{xx} = -b \frac{x}{x^2}, \quad V_c = Y^\prime(c), \quad V_{cc} = Y^{\prime\prime}(c).
\]

Then HJB equation (6) turns to be
\[
\sup_{z(t) \in \mathcal{A}} \left\{ \ln z - b z \right\} + \sup_{\pi \in \mathcal{A}} \left\{ \left( (1 + \pi) \mu + \pi (a - h(\theta)) - \pi \theta c \right) b - \frac{b}{2} (1 + \pi)^2 \sigma_1^2 \right\} \\
+ \lambda \int_0^\infty Y(c + y) k e^{-ky} dy + \frac{1}{2} \sigma_2^2 \varepsilon Y^{\prime\prime}(c) + (p - q c) Y^\prime(c) - (\lambda + r) Y(c) \\
+ \left( \ln x - rb \ln x \right) = 0.
\]

(11)

Apparently, it must have \( b = \frac{1}{r} > 0 \) and we obtain the optimal \( z \) :
\[
\arg \max_{z} \left\{ \ln z - b z \right\} = r, \quad \sup_{z(t) \in \mathcal{A}} \left\{ \ln z - \frac{z}{r} \right\} = \ln r - 1.
\]
Define \( \Lambda(\pi) := (1 + \pi)\mu + \pi(a - h(\theta)) - \pi c\theta - \frac{1}{2}(1 + \pi)^2\sigma_1^2 \), we have
\[
\begin{align*}
\Lambda'(\pi) &= \mu + a - h(\theta) - c\theta - (1 + \pi)\sigma_1^2, \\
\Lambda''(\pi) &= -\sigma_1^2 < 0.
\end{align*}
\]

To obtain the optimal \( \pi^* \), we discuss the positivity of \( \Lambda' \) in the following subsections. The optimal strategies will be given in different scenarios in Section 4.1 and Section 4.2. The verification theorem will be provided in Section 4.3.

4.1. **Case 1:** \( \Lambda'(0) \leq 0 \). If \( \Lambda'(0) \leq 0 \), i.e. \( c \geq \frac{1}{\theta}(\mu + a - h(\theta) - \sigma_1^2) \), we have
\[
\pi^* = 0.
\]

Substituting \( \pi^* \) into (11) yields
\[
\frac{1}{2}\sigma_1^2 cY''(c) + (p - qc)Y'(c) - (\lambda + r)Y(c) + \lambda \int_0^\infty Y(c + y)ke^{-ky}dy + \ln r - 1 + \frac{\mu}{r} - \frac{\sigma_1^2}{2r} = 0.
\]

The solution can be given by
\[
Y(c) = \frac{\ln r - 1 + \frac{\mu}{r} - \frac{\sigma_1^2}{2r}}{r}.
\]

4.2. **Case 2:** \( \Lambda'(0) > 0 \). If \( \Lambda'(0) > 0 \), i.e. \( 0 < c < \frac{1}{\theta}(\mu + a - h(\theta) - \sigma_1^2) \). If \( \mu + a - h(\theta) - \sigma_1^2 \leq 0 \), we only need to consider the first case. We assume \( \mu + a - h(\theta) - \sigma_1^2 > 0 \) in the following, and then
\[
\pi^* = \frac{\mu + a - h(\theta) - c\theta - \sigma_1^2}{\sigma_1^2}.
\]

Substituting \( \pi^* \) into (11) yields
\[
\begin{align*}
\frac{1}{2}\sigma_1^2 cY''(c) + (p - q c)Y'(c) - (\lambda + r)Y(c) &+ \lambda \int_0^\infty Y(c + y)ke^{-ky}dy \\
+ \ln r - 1 + \frac{\mu}{r} - \frac{\sigma_1^2}{2r} + \frac{\sigma_1^2}{2r}(\pi^*)^2 &= 0.
\end{align*}
\]

Denote the left hand of (13) by \( F(Y(c), c) \). In view of the definitions of subsolution and supsolution in Section 3.1, the subsolution and supsolution can be chosen by
\[
\begin{align*}
Y_{\text{sub}}(c) &= \frac{\ln r - 1 + \frac{\mu}{r} - \frac{\sigma_1^2}{2r}}{r}, \\
Y_{\text{sup}}(c) &= \frac{\ln r - 1 + \frac{\mu}{r} - \frac{\sigma_1^2}{2r} + \frac{(\mu + a - h(\theta) - \sigma_1^2)^2}{2r\sigma_1^2}}{r}.
\end{align*}
\]

4.3. **Verification theorem.** We state the verification theorem in the following.

**Theorem 4.1.** Suppose there exists a \( Y_{\text{sub}}(c) \leq Y^*(c) \leq Y_{\text{sup}}(c) \) and \( Y^*(c) \) solves (12) and (13). Let \( V(x, c) = b \ln x + Y^*(c) \), then

1. For all admissible pairs of control policies \( u = (\pi, z) \),
\[
V(x, c) \geq J(x, c, u) = E_{x,c}\left[ \int_0^\infty e^{-rt} \ln (z(t)X(t))dt \right].
\]
2. If \( u^* = (\pi^*, z^*) \) is given as following:

\[
\pi^* = \begin{cases} 
0, & c \geq C_1 \\
\hat{\pi}, & c < C_1 
\end{cases}
\]

\[
z^* = r,
\]

where

\[
C_1 := \frac{1}{\theta} (\mu + a - h(\theta) - \sigma_1^2),
\]

\[
\hat{\pi} := \frac{\mu + a - h(\theta) - c\theta - \sigma_1^2}{\sigma_1^2}.
\]

Then

\[
V(x, c) = J(x, c, u^*) = E_{x, c} \left[ \int_0^\infty e^{-rt} \ln (z^*(t)X(t))dt \right].
\]

Proof. Applying Itô’s lemma to \( e^{-rt}V(X(t), c(t)) \), we have

\[
e^{-rt}V(X(t), c(t)) = V(x, c) + \int_0^t e^{-rs} \left[ rV(X(s), c(s)) - rV(X(s), c(s)) \right] ds + M(t),
\]

where

\[
\mathcal{L}^u V(x, c) := [((1 + \pi) x\mu + \pi x(a - h(\theta)) - \pi x\theta c - zx)V_x + \frac{1}{2} [(1 + \pi)^2 x^2 \sigma_1^2] V_{xx}
\]

\[
+ (p - qc)V_c + \frac{1}{2} \sigma_2^2 c^2 V_{cc} + [V(x, c + \Delta c) - V(x, c)],
\]

\[
M(t) := \int_0^t V_x (1 + \pi(s)) X(s) \sigma_1(s) dW_1(s) + \int_0^t V_c \sigma_2 \sqrt{c(s)} dW_2(s).
\]

From HJB equation (6), we have

\[
E \left[ e^{-rT} V(X(T), c(T)) \right] - V(x, c) \leq -E \left[ \int_0^T e^{-rt} U(z(t)X(t)) dt \right].
\]

Hence,

\[
V(x, c) \geq E \left[ \int_0^T e^{-rt} U(z(t)X(t)) dt \right] + E \left[ e^{-rT} V(X(T), c(T)) \right]
\]

\[
= E \left[ \int_0^T e^{-rt} \ln (z(t)X(t)) dt \right] + E \left[ e^{-rT} \frac{1}{r} \ln X(T) + e^{-rT} Y^*(c(T)) \right].
\]

(14)

Applying Itô’s lemma to \( \ln X(t) \), we have

\[
d[\ln X(t)] = [(1 + \pi(t)) \mu(t) - z(t) + \pi(t)(a - h(\theta)) - \pi(t)\theta c(t)] dt
\]

\[
+ (1 + \pi(t)) \sigma_1(t) dW_1(t) - (1 + \pi(t))^2 \sigma_1^2(t) dt.
\]

Thus,

\[
E[e^{-rT} \ln X(T)] = e^{-rT} \left\{ \ln x_0 + \int_0^T \left[ (1 + \pi(t)) \mu(t) - z(t) - \pi(t)\theta c(t)
\right.
\]

\[
+ \pi(t)(a - h(\theta)) - (1 + \pi(t))^2 \sigma_1^2(t) dt \right\}.
\]

Due to the boundedness of the parameters, it has

\[
\lim_{T \to \infty} E[e^{-rT} \ln X(T)] = 0.
\]
It is not hard to find that
\[
\lim_{T \to \infty} E e^{-rT} Y^* (c(T)) = 0.
\]
Hence,
\[
V(x, c) \geq E \left[ \int_0^\infty e^{-rt} \ln (z(t)X(t)) dt \right].
\]
The value function follows
\[
V(x, c) = J(x, c, u^*) = E_{x,c} \left[ \int_0^\infty e^{-rt} \ln (z^*(t)X(t)) dt \right].
\]

5. **A numerical example.** In this section, we analyze the impact of key parameters on the optimal liability ratio. We assume that the parameters are constant. Set the initial value as \(\mu = 0.05, \sigma = 0.25, r = 0.01, a = 0.08, \theta = 0.5, c = 0.03, \gamma = 0.1\). We will respectively analyze the impact of \(c, \mu, \sigma, \theta\) on the optimal liability ratio for both logarithmic utility function and power utility function, and the impact of \(\gamma\) on the optimal liability ratio for power utility function. We assume that the insurance premium and reinsurance premium are determined by the net premium principle. That is, \(h(\theta) = (1 - \theta)a\).

![Figure 1. Optimal liability ratio values versus c](image)

In Figure 1, we analyze the optimal liability ratio versus the claim rate. It is shown that the optimal liability ratio is a decreasing function of \(c\) for both logarithmic utility function and power utility function. To insurance companies, higher claim rate leads to higher probability of ruin. With a higher claim rate, the risk-averse decision maker will choose to lower the liability level. That is, insurance companies will reduce the volume of the policies and decrease the financial leverage to avoid the risks. In addition, we observe that in both utility cases, the optimal liability ratio converges to zero beyond a certain claim rate threshold. The thresholds can be considered as the boundaries of the extremely high risk regimes. With high frequencies of the abnormal catastrophic risks, insurance companies would rather not write any protections for financial safety.

In Figure 2, we study the relationship between the optimal liability ratio and financial market performance. The optimal liability ratio responds positively with respect to the return rate of the financial market. As we know, insurance companies rely on the investment return of assets to fund the future claims. The financial market cycle has a substantial impact on insurance business cycle. With higher asset return, insurance companies have the ability to lower the premium rate and
perform aggressively in the policy-writing market. While the asset return is in the downturn, insurance companies will escalate the premium rates to counteract the losses from the asset investment. The volume of the written policies is restrained, leading to a lower optimal liability ratio. From Figure 2, it is shown that optimal liability level gradually increases when asset return is improved. When the financial market is bear, insurance companies choose to write policies as less as possible to control the liability volume. The insurance companies adopt similar strategies with the ones when claim rate is high.

In Figure 3, we investigate the impact of market volatility on the optimal liability ratio. It is shown that the optimal liability level is negatively related the market volatility. In a volatile market, insurance companies reveals a risk-averse attitude and optimize their financial structures by limiting the volume of policies. Unlike the linearity of the responses with respect to claim rate and asset return, the optimal liability ratio behaves certain level of convexity respect to the volatility. Hence, the optimal liability ratio is more sensitive to the market volatility when the market volatility is low. Further, the decision makers with a logarithm utility is more risk-averse than the one with power utility functions. This is consistent with the lower optimal liability ratios for decision makers with a logarithm utility depicted in Figure 3.

Power utility function is a type of utility function with constant relative risk aversion. $\gamma$ denotes the insurer’s risk preference. Smaller $\gamma$ leads to more curvature in the utility functions and higher risk aversion. Hence, a lower $\gamma$, which represents a more risk-averse decision maker, will lead to a lower level of liability. In view...
of Figure 4, the optimal liability ratio is monotonically increasing with respect to \( \gamma \). Intuitively, the less risk-averse decision makers, with a higher \( \gamma \) in their utility functions, are more flexible in taking risks. A risk-taking strategy is represented by aggressive policy writing in our formulation, leading to a higher liability level. Thus, the optimal liability ratio will increase with larger \( \gamma \).

We will further study the expected premium principle. Assume that the risk loadings for the insurer and reinsurer are \( \rho_1 \) and \( \rho_2 \), respectively. Then \( h(\theta) = a(1 - (1 - \theta)\frac{1+\rho_2}{1+\rho_1}) \). We plot the relationship between the retention rate \( \theta \) and the optimal liability ratio \( \pi^* \) under three cases: \( \rho_1 > \rho_2, \rho_1 = \rho_2, \) and \( \rho_1 < \rho_2 \). When \( \rho_1 = \rho_2 \), the expression of \( g(\theta) \) for the expected premium principle is the
same with the expression for the net premium principle. Taking the values $\rho_1 = 0.1, \rho_2 = 0.05, 0.1, 0.15$, we obtain the optimal liability ratios for logarithm and power utility functions in Figure 5 and Figure 6, respectively. In both figures, we observe that higher safely loadings for reinsures lead decreases in optimal liability ratio. This is consistent with our expectation. If a reinsurer set a higher safe loading, a more expensive reinsurance premium is charged. Hence, it becomes more costly for insurers to write new policies. To avoid the additional financial burden from the reinsurance policy, insurance companies is optimally to maintain a relatively low liability level, demonstrated by lower optimal liability ratios in Figure 5 and Figure 6.

6. **Concluding remarks.** In this paper, we derive the optimal liability ratio and dividend optimization of an insurance company taking into account the catastrophic risks. Reinsurance tools are adopted to mitigate the large discrete claims because of the catastrophic risks. The nonnegative claim rate is assumed to be risky and unpredictable. A jump-diffusion CIR process is used to capture the claim hikes when some catastrophic event occurs. Incorporating the impact of reinsurance on the financial status of the insurance companies, we aim to maximize the total expected discounted utility of dividend in the infinite time horizon in the logarithm and power utility cases, respectively. By using the dynamic programming approach, we derive the associated integro-differential HJB equation. Furthermore, by using the subsolution-supersolution method, the existence of classical solutions for the HJB equations is proved. The explicit classical solution of the corresponding optimal liability ratio and dividend strategies are obtained under simple conditions. Sensitivity of key parameters are presented in the numerical example. The economic insights provide guidance for decision makers in government or industries to manage the leverage level and dividend policies.

Current model is limited to capture discrete movements and certain insurance risks such as random environment, market trends, interest rate risk, and insurance business cycles, to better reflect the market complexity, we plan to model and analyze the asset and liability by using stochastic hybrid systems which model both continuous dynamics and discrete events in the systems. Thus, the model becomes more versatile but more complicated. Solving the coupled system of HJB equations analytically is very difficult. Nevertheless, we can always refer to Markov approximation method to provide a numerical approximation.

Further, this model can be enriched to incorporate investment in the asset and net premium. It is natural for insurance companies to invest the asset and collected premium in the financial market. Optimal investment management will translate into profitable growth and an increase in shareholder, hence leads to a powerful advantage in the marketplace for the insurance company. Incorporating additional control variable will make the stochastic control problem more versatile. Although the HJB equation will be derived in routine and subsolution-supersolution can be used to solve for the value function, we need overcome the difficulty that ordered pair of subsolution and supersolution may not have simple explicit expressions, which will add difficulties to find the analytic solutions.

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