Moduli Spaces and Brane Solitons for M-Theory Compactifications on Holonomy $G_2$ Manifolds

J. Gutowski

Department of Physics, Queen Mary College, Mile End
London, E1 4NS

and

G. Papadopoulos

Department of Mathematics, King’s College London, Strand
London, WC2R 2LS

ABSTRACT

We investigate the local geometry on the moduli space of $G_2$ structures that arises in compactifications of M-theory on holonomy $G_2$ manifolds. In particular, we determine the homogeneity properties of couplings of the associated $N = 1, D = 4$ supergravity under the scaling of moduli space coordinates. We then find some brane solitons of $N = 1, D = 4$ supergravity that are associated with wrapping M-branes on cycles of the compact space. These include cosmic strings and domain walls that preserve 1/2 of supersymmetry of the four-dimensional theory, and non-supersymmetric electrically and magnetically charged black holes. The geometry of some of the black holes is that of non-extreme M-brane configurations reduced to four-dimensions on a seven torus.
1. Introduction

Compactifications of M-theory on manifolds with $G_2$ holonomy provide a way of constructing four-dimensional effective theories which have a realistic amount of supersymmetry. These effective theories are $N = 1, D = 4$ supergravities with field content which is determined by the Betti numbers of the compact space. In particular it has been shown that the associated $N = 1, D = 4$ supergravity has $b_2$ vector and $b_3$ chiral multiplets [1]. Reducing $N = 1, D = 11$ supergravity on compact $G_2$ holonomy manifolds, one can also determine the couplings of the four-dimensional theory as a function of the various moduli fields [1, 2]. These couplings are naturally interpreted in terms of the geometry of the moduli space of $G_2$ structures in a similar way to that of Calabi-Yau compactifications of string theory, see for example [3]. Many compactifications of M-theory on holonomy $G_2$ manifolds have been investigated, see for example [4], using the examples of holonomy $G_2$ manifolds constructed by Joyce in [5].

Some of the solutions of D=4 and D=5 supergravity theories which arise from compactifications of strings and M-theory, like black holes, strings and domain walls, can be associated with branes wrapped on the homology cycles of the compact manifold. This correspondence between solutions of lower dimensional supergravity theories and ten- and eleven-dimensional brane configurations has been very fruitful, for example it has led to the microscopic computation of the black hole entropy for a certain class of extreme black holes [6]. Another application is the use of the behaviour of the black hole solutions of $N = 2, D = 4$ supergravity theories [7] to provide evidence for the existence of calibrated representatives for certain homology cycles of Calabi-Yau manifolds [8].

In this paper, we shall examine the couplings of the $N = 1, D = 4$ supergravity theories that arise from compactifications of $N = 1, D = 11$ supergravity on compact holonomy $G_2$ manifolds. In particular we shall show that the components of the metric of the sigma model manifold, which is $TM$, of $N = 1, D = 4$ theory are homogeneous of degree $−2$ under the scaling of certain coordinates of the moduli
space $\mathcal{M}$ of $G_2$ structures. For this we shall use a result obtained by Hitchin [9] that the volume of the compact $G_2$ manifold is homogeneous of degree $7/3$ under the scaling of some coordinates of $\mathcal{M}$. It turns out that the metric on the moduli space of $G_2$ structures is invariant under this scaling transformation; the isometry group of the sigma model metric on $T\mathcal{M}$ is generated by the same scaling transformation and the translations along the fibres. In addition, we shall show that the Kähler potential of the sigma model manifold can be expressed in terms of the logarithm of the volume of the compact holonomy $G_2$ manifold, see also [2]. The couplings of the vector multiplets to the scalars are linear in some natural complex coordinates on the sigma model manifold.

Having established the homogeneity properties of the couplings of $N = 1, D = 4$ supergravity associated with compactifications of $N = 1, D = 11$ supergravity on holonomy $G_2$ manifolds, we shall explore the various solutions of the four-dimensional theory that arise by wrapping M-branes on the homology cycles of the compact manifold. We shall find that the $N = 1, D = 4$ supergravity admits string solutions which preserve $1/2$ of supersymmetry. These are associated with M5-branes wrapped on coassociative cycles of the compact manifold. The form of these string solutions is that of cosmic string solutions of [10, 11]. However the string solutions associated with $G_2$ compactifications have infinite tension because the sigma model manifold $T\mathcal{M}$ is non-compact. We shall also describe the M-theory origin of domain wall solutions of $N = 1, D = 4$ supergravity which preserve $1/2$ of the supersymmetry. The Killing spinor equations for such domain walls are given in an appendix. Next we shall explore the electric and magnetic black hole solutions of $N = 1, D = 4$ supergravity that arise from wrapping M2-branes and M5-branes on 2- and 5-cycles of $N$, respectively. We shall show that such solutions are not supersymmetric as expected. We shall mainly focus in the case where the only non-vanishing modulus field is that corresponding to the overall scale of the moduli space coordinates. We shall call such solutions “dilatonic”; we justify this terminology in an appendix. We shall find a class of extreme dilatonic solutions of the $N = 1, D = 4$ theory which have the same spacetime geometry as two
supersymmetric orthogonally intersecting M-branes, e.g., two M2-branes intersecting on a 0-brane or two M5-branes intersecting on a 3-brane [12], reduced to four-dimensions on seven-dimensional torus. Such four-dimensional solutions exhibit a naked singularity. We shall also present some dilatonic black hole solutions. These have the same spacetime geometry as that of two non-extreme orthogonally intersecting M-branes reduced to four-dimensions again on a seven-dimensional torus found in [13].

We remark that our dilatonic domain wall and black hole solutions depend only on the homogeneity properties of the couplings of the $N = 1, D = 4$ supergravity. So they will remain solutions of the effective theory of $G_2$ compactifications after perturbative or non-perturbative corrections to the couplings are taken into account which preserve these homogeneity properties.

The organization of this paper is as follows: In section two, we present the action, Killing spinor equations and the geometry associated with the couplings $N = 1, D = 4$ supergravity with scalar and vector multiplets. In section three, we give the couplings of $N = 1, D = 4$ supergravity that arise from the compactification of $N = 1, D = 11$ supergravity on holonomy $G_2$ manifolds. We then present two approaches in the investigation of the geometry of the moduli space of $G_2$ structures. One is based on the Kähler geometry and the other is based on the symplectic geometry. We also express the metric on the $G_2$ moduli space, that arises in these compactifications, in terms of the volume of the compact holonomy $G_2$ manifold. In section four, we summarize some of the results on calibrating cycles in holonomy $G_2$ manifolds. We also give the number of supersymmetries preserved by M-brane probes wrapping such cycles. In section five, we present our string solutions of $N = 1, D = 4$ supergravity associated with $G_2$ compactification. In section six, we describe various domain walls. In section seven, we give various black hole solutions associated with M2- and M5-branes wrapped on homology 2- and 5-cycles of the compact manifold. Finally in the appendices, we give our spinor conventions, analyze the Killing spinor equations of $N = 1, D = 4$ supergravity in connection to strings and domain walls that arise in $G_2$ compactifications, and
give the bosonic action that describes the dilatonic black hole system.

2. $N = 1 \, D = 4$ Supergravity

2.1. Supergravity Action and Killing Spinor Equations

The geometric data that determine the couplings of $N = 1$ supergravity in four-dimensions with $n$ vector and $m$ chiral multiplets that we shall use in this paper are the following:

(i) A Kähler-Hodge manifold $M$ of complex dimension $m$ with Kähler potential $K$.

(ii) A vector bundle $E$ over $M$ of rank $n$ for which its complexified symmetric product admits a holomorphic section $h$.

(iii) A locally defined holomorphic function $f$ on $M$.

(iv) Sigma model maps, $z$, from the four-dimensional spacetime $\Sigma$ into the manifold $M$.

(v) A principal bundle $P$ on the four-dimensional spacetime $\Sigma$ with fibre the abelian group $U(1)^n$ such that the pull back of $E$ with respect to $z$ is isomorphic to $P \times_{U(1)^n} \mathcal{L}U(1)^n$, where $\mathcal{L}U(1)^n$ is the Lie algebra of $U(1)^n$.

Given these data, the bosonic part of $N = 1, D = 4$ supergravity action [14, 15, 16] is

$$L = \sqrt{-g} \left[ \frac{1}{2} R(g) - \frac{1}{4} \text{Re} h_{ab} F^a_{MN} F^{bMN} + \frac{1}{4} \text{Im} h_{ab} F^a_{MN} \ast F^{bMN} - \gamma_{ij} \partial_M z^i \partial^M \bar{z}^j - V \right]$$

(2.1)

where

$$V = e^K [\gamma^{ij} D_i f D_j \bar{f} - 3|f|^2] + \frac{1}{2} D_a D^a ,$$

(2.2)

$$F^a_{MN} = \partial_M A^a_N - \partial_N A^a_M ,$$

(2.3)
\[ D_i f = \partial_i f + \partial_i K f \quad , \tag{2.4} \]

\( A^a_N \) are \( U(1) \) (Maxwell) gauge potentials and the \( D_a \) are constants associated to a Fayet-Iliopoulos term. We remark that the gauge indices \( a, b = 1, \ldots, n \) are raised and lowered with \( \text{Re} h_{ab}; \ i, j = 1, \ldots, m \) and \( M, N = 0, \ldots, 3 \) are holomorphic sigma model manifold and spacetime indices, respectively. Clearly \( h_{ab} \) are the gauge couplings, \( f \) is a superpotential and \( M \) is the sigma model manifold.

The above action may also describe the coupling of \( \ell \) linear multiplets to \( N = 1, D = 4 \) supergravity [17]. This is because the two-form gauge potentials of the linear multiplets can be dualized to scalars in four dimensions. The resulting action depends only on the spacetime derivatives of dual scalars. Therefore it is invariant under \( \mathbb{R}^\ell \) acting on these scalars with constant shifts.

In what follows some of the solutions of the \( N = 1, D = 4 \) supergravity that we shall consider will be supersymmetric. To explore their properties, we need the Killing spinor equations of (2.1) which are the vanishing conditions of the supersymmetry transformations of the fermions of the theory. These are most conveniently expressed in terms of a real 4-component Majorana spinor \( \epsilon \) as

\[ 2(\partial_M + \frac{1}{4} \omega_{MAB} \Gamma^{AB}) \epsilon - (\text{Im}(K_i \partial_M z^i) + e^\frac{K}{2} (\text{Re} f - \text{Im} f \Gamma^5) \Gamma_M) \epsilon = 0 \quad , \tag{2.5} \]

\[ ( - \frac{1}{2} F^a_{MN} \Gamma^{MN} + \Gamma^5 D^a ) \epsilon = 0 \quad \tag{2.6} \]

and

\[ (\text{Re}(\partial_M z^i) - \Gamma^5 \text{Im}(\partial_M z^i)) \Gamma^M \epsilon - e^\frac{K}{2} (\text{Re}(\gamma^i \dot{D}_j \dot{f}) - \Gamma^5 \text{Im}(\gamma^i \dot{D}_j \dot{f})) \epsilon = 0 \quad , \tag{2.7} \]

where underlined indices \( A, B \) denote tangent frame indices and \( \Gamma^5 = \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 \).

For our spinor conventions see the appendix.

The field equations of the supergravity action (2.1) are the following:
(1) The Einstein equations are:
\[
G_{MN} - \text{Re} h_{ab} F^a_{ML} F^b_{N L} - 2 \gamma_{i j} \partial_{(M} z^i \partial_{N)} z^j \\
+ g_{MN} \left( \frac{1}{4} \text{Re} h_{ab} F^a_{LP} F^b_{LP} + \gamma_{i j} \partial_L z^i \partial^L z^j + V \right) = 0.
\]

(2) The Maxwell field equations are:
\[
\partial_M \left[ \sqrt{-g} \left( \text{Re} h_{ab} F^{bMN} - \text{Im} h_{ab} F^{bMN} \right) \right].
\]

(3) The scalar equations; varying \( z^\ell \) gives the equation
\[
- \frac{1}{8} \partial_{\ell} h_{ab} F^a_{MN} F^{bMN} - \partial_{\ell} V - \frac{i}{8} \partial_{\ell} h_{ab} F^a_{MN} F^{bMN} \\
+ \gamma_{i j} \left( \nabla_M \partial_M z^j + \Gamma^j_{i k} \partial_M z^i \partial_M z^k \right) = 0,
\]
where \( \nabla_M \) is the covariant derivative with respect to the Levi-Civita connection of the spacetime metric and
\[
\partial_{\ell} V = \partial_{\ell} \left( e^K \gamma^i j D_i f \right) D_j \bar{f} - 2 e^K \bar{f} D_{\ell} f + \frac{1}{2} \partial_{\ell} \left( D_a D^a \right).
\]

Taking the conjugate of this equation, one obtains the field equation for \( \bar{z}^\ell \).

### 3. Supergravity Actions from \( G_2 \) Compactifications

#### 3.1. Compactification Ansatz

The bosonic part of the \( D = 11, N = 1 \) supergravity Lagrangian [18] that we shall consider is
\[
\mathcal{L} = \frac{1}{2} \sqrt{\mathfrak{h} R} - \frac{1}{4} F \wedge \ast F + \frac{1}{12} C \wedge F \wedge F
\]
where \( \mathfrak{h} \) is the eleven-dimensional metric, \( F \) is the 4-form field strength and \( C \) is the 3-form gauge potential, \( F = dC \).

\(*\) Our form conventions are \( \chi = \frac{1}{k!} \chi_{I_1...I_k} dx^{I_1} \wedge ... \wedge dx^{I_k}, \ (\chi, \psi) = \frac{1}{k!} \chi_{I_1...I_k} \psi^{I_1...I_k} \) and \( (\chi, \psi) = \chi \wedge \ast \psi \).
To derive the $N = 1, D = 4$ supergravity action that arises from the compactification of $N = 1, D = 11$ supergravity on a holonomy $G_2$ manifold $N$, we introduce a basis of harmonic forms $\{\phi_i; i = 1, \ldots, m = b_3\}$ in $H^3(N, \mathbb{R})$ and similarly a basis $\{\omega_a; a = 1, \ldots, n = b_2\}$ in $H^2(N, \mathbb{R})$. Repeating the analysis in [1, 2], we write the compactification ansatz for the eleven-dimensional metric $ds^2$ and the three-form gauge potential $C$ as

$$ds^2 = g_{MN}(x)dx^M dx^N + G_{IJ}(y, s(x))dy^I dy^J$$

$$C = \sum_{a=1}^n A^a(x) \wedge \omega_a(y) + \sum_{i=1}^m p^i(x) \wedge \phi_i(y),$$

(3.2)

where $G_{IJ}$ is the metric on $N$ with holonomy $G_2$ depending on the real coordinates $\{s^i; i = 1, \ldots, b_3\}$ of the moduli space $\mathcal{M}$ of the $G_2$ structures which have been promoted to four-dimensional scalar fields; $I, J = 1, \ldots, \text{dim}N$. In addition, $A^a$ and $p^i$ are the one-form gauge potentials and real scalars of the four-dimensional theory, respectively. The fields $g, A^a, s^i, p^i$ describe the small fluctuations of the $G_2$ manifold $N$ within the $N = 1, D = 11$ supergravity. To solve the field equations of $N = 1, D = 11$ supergravity at the linearized level, the basis of harmonic forms $\{\phi_i; i = 1, \ldots, n = b_3\}$ in $H^3(N, \mathbb{R})$ is chosen with respect to the $G_{IJ}$ metric and similarly for the basis $\{\omega_a; a = 1, \ldots, n = b_2\}$ in $H^2(N, \mathbb{R})$. So far the coordinates $s^i$ on the moduli space $\mathcal{M}$ have been chosen in an arbitrary way. However below for the investigation of the couplings of the $N = 1, D = 4$ supergravity theory a special choice will be made.

The compactification of $N = 1, D = 11$ supergravity on holonomy $G_2$ manifolds preserves four real supercharges. So it is expected that the four-dimensional action that describes the dynamics of the small fluctuations of such background will exhibit $N = 1, D = 4$ supersymmetry. Some of the couplings of the four-dimensional effective action can be easily deduced from the eleven dimensional
supergravity action and are as follows:
\[ ds^2 = \gamma_{ij}dz^i d\bar{z}^j = k_{ij}(s)dz^i ds^j + m_{ij}(s)dp^i dp^j \]
\[ m_{ij}(s) = \frac{1}{4 \int_Y \sqrt{g} d^7 y (\phi_i, \phi_j)} \]
\[ \text{Re} h_{ab}(s) = \frac{1}{2} \int_Y \sqrt{g} d^7 y (\omega_a, \omega_b) = \frac{1}{2} \int \omega_a \wedge \ast \omega_b = -\frac{1}{2} \int \omega_a \wedge \omega_b \wedge \phi \]
\[ \text{Im} h_{ab}(p) = -\frac{1}{2} p^i \int \omega_a \wedge \omega_b \wedge \phi_i = -\frac{1}{2} p^i C_{iab}, \]

where \( \phi \) is the parallel 3-form associated with the \( G_2 \) structure on \( N \) and we have used \( G \) to also denote the determinant of the metric \( G_{IJ} \) with holonomy \( G_2 \). In an adapted frame \( \{e_1, \ldots, e_7\} \) of the \( G_2 \) structure the 3-form \( \phi \) can be written as
\[ \phi = (e_1 \wedge e_2 - e_3 \wedge e_4) \wedge e_5 + (e_1 \wedge e_3 - e_4 \wedge e_2) \wedge e_6 + (e_1 \wedge e_4 - e_2 \wedge e_3) \wedge e_7 + e_5 \wedge e_6 \wedge e_7. \]

The last equality in the third equation in (3.3) can be established using \( G_2 \) representation theory, see for example [5, 27] and next section. We remark that the intersection numbers \( C_{iab} \) are topological and so they do not depend on the moduli space coordinates \( s \) of \( G_2 \) structures or the choice of harmonic representatives. In addition, we remark that the couplings in (3.3) do not depend on the harmonic representatives chosen for the basis \( \omega_a \) in \( H^2(N, \mathbb{R}) \) and the basis \( \phi_i \) in \( H^3(N, \mathbb{R}) \).

To express the four-dimensional couplings as above, we have rescaled the four-dimensional metric \( g \rightarrow \Theta^{-1} g \) with the volume \( \Theta \) of the compact space in order to bring the \( D = 4 \) action in the Einstein frame. Observe that the sigma model metric \( ds^2 \) above is invariant under the action of \( \mathbb{R}^{b_3} \) acting with constant shifts on \( p \).

Since the \( G_2 \) compactifications preserve four real supercharges, the effective theory has \( N = 1, D = 4 \) supersymmetry. In particular the couplings (3.3) should obey the conditions described in the previous section for the couplings of \( N = 1, D = 4 \) supergravity. There are two ways to describe this depending on the way we choose coordinates on the moduli space which we shall now describe below.
3.2. *G_2* Moduli Space: A Kähler Approach

The sigma model metric $ds^2$ in (3.3) is required by $N = 1, D = 4$ supersymmetry to be Kähler. In addition the action of the group $\mathbb{R}^3$ on $p$ leaves the sigma model metric $ds^2$ invariant and also commutes with the supersymmetry transformations of the scalars. This is because, as for the D=4 effective action, the supersymmetry transformations depend only on the spacetime derivatives of the fields $p$. This implies that $\mathbb{R}^3$ acts with holomorphic isometries on the sigma model target space $M$ which can be identified with the tangent space $TM$ of the moduli space $M$ of $G_2$ structures. The typical fibre of $TM$ is $H^3(N, \mathbb{R})$.

To continue the investigation of the moduli space geometry, it is convenient to choose coordinates on the moduli space $M$ so that the parallel form is

$$\phi = s_i \phi_i . \quad (3.5)$$

In fact, the basis $\phi_i$ of harmonic 3-form with respect to the $G_2$ metric depends on the choice of $G_2$ structure and so on the coordinates $s$. We take the origin of the coordinate system to be $s_i = s^i_o \neq 0$. However this dependence does not contribute in the couplings (3.3) of four-dimensional theory because, as we have mentioned in the previous section, they do not depend on the choice of harmonic representatives for $\phi_i$. Next one can introduce holomorphic coordinates $z^i = -\frac{1}{2}(s^i + ip^i)$ on $TM$ such that $\mathbb{R}^3$ acts on $z^i$ with shifts along the imaginary directions. In these coordinates, the sigma model metric on $TM$ is

$$ds^2 = \gamma_{ij} dz^i d\bar{z}^j = \partial_i \partial_j K(Rez) dz^i d\bar{z}^j , \quad (3.6)$$

and the Kähler form is

$$\Omega = idz^i \wedge d\bar{z}^j \partial_i \partial_j K(Rez) . \quad (3.7)$$
Comparing the metric (3.6) with that in (3.3), we find that

\[ 4k_{ij} = 4m_{ij} = \partial_i \partial_j K = \frac{1}{\int_N d^7 y \sqrt{G}} \int_N \sqrt{G} d^7 y (\phi_i, \phi_j), \quad (3.8) \]

where \( K \) is the Kähler potential.

We turn now to investigate the couplings of the vector multiplets. Using the holomorphicity of \( h_{ab} \) required by supersymmetry and the expression given in (3.3), we find that

\[ h_{ab} = z^i C_{iab}. \quad (3.9) \]

The coupling \( h_{ab} \) can be though as a holomorphic section of a bundle with fibre \( S^2 H^2(N, \mathbb{R}) \otimes \mathbb{C} \) over the sigma model manifold \( TM \).

It remains now to find the Kähler potential of the metric on the sigma model manifold \( TM \). We shall show that the Kähler potential \(^*\) is

\[ K = -\frac{3}{7} \log \left[ \int_N \phi \wedge * \phi \right] = -\frac{3}{7} \log \left[ \int_N d^7 y \sqrt{G} \right]. \quad (3.10) \]

For this, we shall use the relation shown by Hitchin in [9] that

\[ \hat{K} = \int_N \phi \wedge * \phi = \int_N [\text{det} B]^{\frac{1}{2}}, \quad (3.11) \]

where \([\text{det} B]^{\frac{1}{2}}\) is a top-form constructed taking the determinant of

\[ B_{IJ} = -\frac{1}{6!} \phi_{IK_1 K_2 K_3 K_4} \phi_{K_5 K_6 K_7} dy^{K_1} \wedge \ldots \wedge dy^{K_7}. \quad (3.12) \]

It is clear from this that \( \hat{K} \) is homogeneous of degree \( 7/3 \) in the \( s \) coordinates \(^\dagger\).

The metric on the \( G_2 \) holonomy manifold is given by \( G_{IJ} = (\text{det} B)^{-\frac{1}{2}} B_{IJ} \). Using

\(^*\) A similar expression for the Kähler potential, but with different normalization factor, was given in [2].

\(^\dagger\) In [9], the metric on the moduli space \( \mathcal{M} \) is taken to be the Hessian of \( \hat{K} \) which differs from the metric that arises in the compactifications we are investigating. The metric associated with the Hessian of \( \hat{K} \) has Lorentzian signature.
this one can show that

\[ \frac{\partial}{\partial s^i} \int_N \phi \wedge *\phi = \frac{7}{3} \int_N \phi_i \wedge *\phi . \quad (3.13) \]

To proceed we remark that the \( \Lambda^3(\mathbb{R}^7) \) representation of \( SO(7) \) of dimension 35 can be decomposed under the action of \( G_2 \) as \( 1 + 7 + 27 \). The first is the direction along the \( G_2 \) invariant form \( \phi \). The representation 7 is associated with the vector arising from the inner product of a 3-forms with \( *\phi \). If the three-form is harmonic with respect to the \( G_2 \) metric, as it is the case here, then this part vanishes due to a standard argument about harmonic one-forms on irreducible Ricci-flat spaces. Therefore \( \phi_i \) can be written as \( \phi_i = \pi_1(\phi_i) + \pi_{27}(\phi_i) \), where \( \pi_1 \) and \( \pi_{27} \) are the obvious projections. In addition it is known [5, 9] that

\[ \frac{\partial}{\partial s^i} *\phi = \frac{4}{3} *\pi_1(\phi_i) - *\pi_{27}(\phi_i) . \quad (3.14) \]

Using (3.13) and (3.14) it is straightforward to verify (3.10). In conclusion, we find

\[ k_{ij} = m_{ij} = -\frac{3}{28} \partial_i \partial_j \log \int_N \phi \wedge *\phi . \quad (3.15) \]

The numerical normalization factors that appear in the expressions for the moduli metric and Kähler potential are important in the investigation of the various brane solitons that arise in these compactifications.

We remark that the components \( k_{ij} \) and \( m_{ij} \) of the sigma model metric are homogeneous of degree −2 under the scaling of the \( s \) coordinates of the moduli space \( \mathcal{M} \). This follows in a straightforward manner from the homogeneity properties of the volume of \( G_2 \) manifolds that we have explained above. We find that (3.6) is invariant under scaling \( s^i \to \ell s^i \) and \( p^i \to \ell p^i \), where \( \ell \in \mathbb{R} - \{0\} \). So this scaling transformation is an isometry. The isometry group of the metric (3.6) on \( T\mathcal{M} \) is the semi-direct product of \( \mathbb{R} - \{0\} \) with \( \mathbb{R}^{b_3} \) the group of translations along the \( p^i \) coordinates.
3.3. \textbf{G}_2 \textbf{ Moduli Space: A symplectic Approach}

An alternative way to describe the geometry of the moduli space is to use symplectic geometry. For this we consider the cotangent bundle $T^*\mathcal{M}$ of the moduli space $\mathcal{M}$, a typical fibre of which can be identified with $H^4(N, \mathbb{R})$. There is a symplectic pairing between $H^3(N, \mathbb{R})$ and $H^4(N, \mathbb{R})$ given by Poincaré duality. Of course $M = T\mathcal{M}$ is isomorphic to $T^*\mathcal{M}$. Choose now coordinates on $T^*\mathcal{M}$ such that the Kähler form is

$$\Omega = du^i \wedge dr_i . \quad (3.16)$$

Next we write the metric on $T^*\mathcal{M}$ as

$$ds^2 = n_{ij}(u)du^i du^j + q^{ij}(s)dr_i dr_j . \quad (3.17)$$

Given the symplectic form and the metric, one can introduce the (almost) complex structure

$$I^s_i = n^{u^r k} \Omega_{u^r r_j} = n^{ij} \quad \text{(3.18)}$$

where $n^{ij}$ and $q_{ij}$ are the inverse matrices of $n_{ij}$ and $q^{ij}$, respectively. Requiring that $I^2 = -1$, we find that

$$n_{ij} = q_{ij} . \quad (3.19)$$

It remains to investigate the integrability of the above complex structure. For this define the (1,0) forms

$$e_i = idr_i - n_{ij} du^j \quad \hat{e}^i = idu^i + n^{ij} dr_j . \quad (3.20)$$

Observe that $e_i = in_{ij} \hat{e}^j$. Requiring that $de_i$ and $d\hat{e}^i$ do not contain a (0,2) part,
we find that
\[ \partial_i n_{jk} = 0 \] (3.21)
which in turn implies that
\[ n_{ij} = \partial_i \partial_j \tilde{K}(u) \] (3.22)
for some function \( \tilde{K}(u) \). A set of complex coordinates with respect to the above complex structure is
\[ \tilde{z}_i = -\partial_i \tilde{K} + ir_i . \] (3.23)
To make connection with the Kähler approach to the geometry of the moduli space, define the coordinates
\[ v_i = -\partial_i \tilde{K}, \quad r_i = r_i . \] (3.24)
Then the sigma model metric becomes
\[ ds^2 = n_{ij} dv_i dv_j + n_{ij} dr_i dr_j . \] (3.25)
Next observe from (3.24) that \( du^i = -n^{ij} dv_j \). Taking the exterior derivative of this equation, we find that
\[ \frac{\partial}{\partial v_i} n^{jk} - \frac{\partial}{\partial v_j} n^{ik} = 0 \] (3.26)
which in turn implies that \( n^{ij} \) can be expressed as two \( v \)-derivatives on a scalar. Setting \( v_i = s^i, r_i = p^i \) and \( n^{ij} = k_{ij} \), we establish the relation between the Kähler and symplectic approaches to the geometry of \( T \mathcal{M} \). The Kähler geometry on \( T \mathcal{M} \) will be used in the sections below to construct solutions for the \( N = 1, D = 4 \) supergravity that arise from compactifications of M-theory on holonomy \( G_2 \) manifolds and are associated with M-branes wrapped on cycles in the compact space.
3.4. Potentials

As we have seen, potentials do not arise in the four-dimensional effective theory associated with the (direct) compactification of $N = 1, D = 11$ supergravity on $G_2$ holonomy manifolds. However several mechanisms have been proposed for generating a potential. One such mechanism involves compactifications in the presence of a non-vanishing 4-form field strength $F_4^0$ along the directions of the compact manifold. This is a Scherk-Schwarz type of mechanism which has been recently adapted in the context of $G_2$ compactifications of string theory in [19, 20, 21]. It turns out though that the presence of non-vanishing $F_4^0$ in the context of $G_2$ compactifications of M-theory is not consistent with the compactness of the internal manifold [22]. Nevertheless, adapting the formalism proposed in [19, 20, 21] we find that the superpotential $f$ associated with such $F^0$ is

$$\text{Re}f = \int \phi \wedge F^0.$$  \hspace{1cm} (3.27)

The imaginary part of $f$ is determined by holomorphicity.

Other mechanisms of generating a potential in the low energy effective action in four dimensions involve instanton effects which arise from wrapping M2-branes on associative 3-cycles of the $G_2$ manifold. Such cycles exist in some $G_2$ holonomy manifolds and the associated instantons induce a scalar potential. In particular it has been found in [2] that a probe homology 3-sphere instanton M2-brane wrapping the cycle $C$ generates the superpotential

$$f(z) = \mu e^{k_i z^i},$$  \hspace{1cm} (3.28)

where $\mu$ is a constant and $k_i \sim \int_C \phi_i$. Such cycles exist in special holonomy $G_2$ manifolds but they may not exist for generic ones (see next section). In the case that such a superpotential (3.28) appears it easy to see using the results of section three that the only supersymmetric vacuum is at $|s| \to \infty$. The theory may have other vacua but they are not supersymmetric.
Because of the above mentioned difficulties for generating a potential for generic $G_2$ compactifications of M-theory, we shall mostly focus in the investigation of the solutions of the four-dimensional action with couplings described in section three and without a potential. However from the perspective of the general $N = 1, D = 4$ supergravity, one can do a more general analysis which we shall present in an appendix.

4. Cycles and Wrapped Branes

4.1. Calibrations and Supersymmetric Cycles

On $G_2$ holonomy manifolds there are two calibrations that are associated with supersymmetric cycles. One is of degree three (associative) calibration and the other of degree four (coassociative) calibration associated with the parallel three- and four-forms on these manifolds [23]. We shall refer to them as supersymmetric calibrations. There may be other calibrations on $G_2$-manifolds but they will not be supersymmetric. To see this, the supersymmetry condition which is deduced from $\kappa$-symmetry is [24,25]

$$\Gamma \eta = \eta ,$$

(4.1)

where $\Gamma$ is the $\kappa$-symmetry projector and $\eta$ is the $N = 1, D = 11$ supersymmetry parameter which should be parallel with respect to the Levi-Civita connection of the compact $G_2$ holonomy manifold. Since from such a parallel spinor $\eta$, one can construct the parallel three- and four-forms, only calibrations associated to these forms are supersymmetric.

Even though supersymmetric calibration forms exist on $G_2$ manifolds, it is not apparent that there always exist (calibrated) supersymmetric representatives of the homology 3- and 4- classes of the $G_2$ manifold, respectively. It is known that if supersymmetric (associative) 3-cycles exist, they do not have moduli [26]. Thus such 3-cycles are isolated. In fact it has been conjectured that they do not exist
for generic $G_2$ manifolds, though one expects to find them in special cases\cite{27}. Supersymmetric, coassociative, 4-cycles, $X$, have moduli in $G_2$ manifolds with dimension $^* b_2^+(X)$ \cite{26}. So one expects to have locally smooth moduli space for coassociative calibrations.

There are several homology cycles on $G_2$ manifolds on which one can wrap M-theory branes. We shall be mainly concerned with two-cycles, three-cycles, four-cycles and five-cycles. As we have seen, two- and five-cycles cannot be supersymmetric. This however does not mean that none of them is calibrated, with respect to a non-supersymmetric calibration, or there is no minimal submanifold in the homology class of these cycles. Three- and four-cycles can be supersymmetric, but as it has been mentioned above this does not necessarily imply that every three- and four-cycle has a supersymmetric calibrated submanifold representing its homology class.

4.2. Wrapping M-branes on Homology Cycles

The brane solitons in four dimensions that one expects to find by wrapping M-branes on homology cycles of $G_2$ holonomy manifold $N$ which are represented by a minimal submanifold are as follows:

(i) Wrapping M2-branes on two-cycles leads to non-supersymmetric 0-branes in four dimensions.

(ii) Wrapping M5-brane on two-cycles, three-cycles, four-cycles and five-cycles leads to non-supersymmetric spacetime filling 3-branes, supersymmetric 2-branes, supersymmetric 1-branes and non-supersymmetric 0-branes, respectively.

All brane configurations above that arise from wrapping M-branes to four-dimensions should be described by solutions of the effective $N = 1, D = 4$ effective

\footnote{$^*$ We use conventions similar to \cite{27}.}
supergravity theory of this compactification. Since spacetime filling supersymmetric or non-supersymmetric 3-branes are characterized by 3+1-dimensional Poincaré invariance, the associated supergravity solutions are those of flat Minkowski spacetime with vanishing gauge potentials and constant scalars. Such solutions are of course the (supersymmetric) vacua of the theory.

Some non-supersymmetric 0-brane solutions can be identified with the black hole solutions of the supergravity theory. Typically the electrically charged black holes are associated with wrapped M2-branes, and magnetically charged ones with wrapped M5-branes. The 2-cycles and 5-cycles in the $G_2$ holonomy manifold are Poincaré dual to each other. It is well known that the electrically and magnetically charged black holes in four dimensions are dual to each other via electromagnetic duality. So one can view the electro-magnetic duality in four-dimensions as consequence of the Poincaré duality on $G_2$ manifolds.

The 1-brane configurations can be identified with strings. As we shall see the solutions are in fact similar to those of cosmic strings [10, 11]. The 2-brane solitons are the domain wall solutions of $N = 1, D = 4$ supergravity theory.

5. Strings

5.1. Coassociative Cycles and M5-branes

As we have mentioned the string solutions of $N = 1, D = 4$ supergravity can be thought off as M5-branes wrapped on coassociative cycles of the $G_2$ manifold. It is known that the supersymmetry conditions [28, 29] associated with such a cycle in the directions 123457 are

\[
\begin{align*}
\Gamma_{1346} \epsilon &= \epsilon \\
\Gamma_{2356} \epsilon &= \epsilon \\
\Gamma_{4567} \epsilon &= \epsilon .
\end{align*}
\] (5.1)

Now suppose that we place a M5-brane extended in the directions 081346 associated
with the projection

$$\Gamma_{081346}\epsilon = \epsilon .$$  \hspace{1cm} (5.2)

The string directions are taken to be 08. It is easy to see that the above projectors lead to a configuration that preserves two supersymmetries, i.e., it preserves 1/2 of supersymmetry of \( N = 1, D = 4 \) theory.

There are two simple cases of coassociative cycles to consider. One is that of coassociative cycles with the topology of the torus \( T^4 \) and the other is of coassociative cycles with the topology of a \( K_3 \) surface. In both cases the dimension of the moduli space is three. So one expects to find string solutions of \( N = 1, D = 4 \) supergravity associated with the wrapping of M5-branes on these coassociative cycles with topology \( T^4 \) and \( K_3 \). The tension of corresponding strings will be equal to the tension of the M5-brane times the volume of the coassociative cycles.

### 5.2. \( G_2 \) Strings

To investigate the string solutions to the supergravity field equations it is convenient to use the Kähler parameterisation of the moduli space of \( G_2 \) structures. The \( G_2 \) strings are a special case of the cosmic strings for which the sigma model manifold is the space \( T\mathcal{M} \). The solution is

$$ds^2 = ds^2(\mathbb{R}^{1,1}) + e^{-K}dwd\bar{w}$$

$$z^i = z^i(w)$$

$$A^a = 0 ,$$

where \( w \) is a complex coordinate of spacetime, and the Kähler potential \( K \) and the complex coordinates \( z^i \) are given in section three. The \( G_2 \) strings above do not have finite tension because \( T\mathcal{M} \) is not compact. However, it is known that the fibre directions of \( T\mathcal{M} \) can be compactified to a torus by dividing with \( H^1(T^{b_2},\mathbb{Z}) \) which is thought of as a group of large gauge transformations. This leaves the base \( \mathcal{M} \) of \( T\mathcal{M} \) which is an open set in \( H^3(N,\mathbb{R}) \). It is expected that certain points of \( \mathcal{M} \)
should be identified by large diffeomorphisms of the compact $G_2$ holonomy manifold $N$. However to our knowledge it is not known how such large diffeomorphisms act on $\mathcal{M}$ and whether they are sufficient to allow for string solutions with finite tension adapting a similar construction in [11]. The identification of the $G_2$ strings with wrapped M5-branes on coassociative cycles provides some indirect evidence that $G_2$ strings with finite tension exist in the case when coassociative cycles are present. We note that there are supersymmetric string solutions even in the presence of a Fayet-Iliopoulos term [30].

6. $G_2$ Domain Walls

The supersymmetry projections of domain walls that arise from a M2-brane in the directions 089 in the background of the $G_2$ manifold in the directions 1234567, are those of (5.1) and

$$\Gamma_{089} \epsilon = \epsilon . \quad (6.1)$$

So the supersymmetry preserved is $1/16$ of M-theory, ie $1/2$ of that of $N = 1, D = 4$ supergravity. The domain wall is in the directions 089.

It is straightforward to write the supergravity solution of a brane that is located in a Ricci-flat manifold. For the case of interest here, the transverse space of the M2-brane is $\mathbb{R} \times N$, where $N$ is the holonomy $G_2$ compact space. The solution is

$$ds^2 = h^{-\frac{2}{3}} ds^2(\mathbb{R}^{1,2}) + h^{\frac{1}{3}} (ds_{(7)}^2 + dy^2)$$

$$F_4 = \frac{1}{2} dvol(\mathbb{R}^{1,2}) \wedge dh^{-1} , \quad (6.2)$$

where $h$ is harmonic in $\mathbb{R} \times N$. Additional fluxes $F^0$ can be added in the solution along $\mathbb{R} \times N$. However in this case $h$ is not harmonic but rather obeys the equation $\Delta h = |F^0|^2$. For $F^0 = 0$, $h$ can be chosen to be harmonic in $\mathbb{R}$, ie $h$ is piece-wise linear function of the coordinate $y \in \mathbb{R}$, see [31, 32]. This solution has electric fluxes and it cannot be described from the four-dimensional perspective using the
linearized compactification ansatz of section three. We shall not pursue this point further here.

Alternatively, domain walls can arise by wrapping M5-branes on associative 3-cycles of the compact space. If the $G_2$ manifold is in the directions 1234567 and the M5-brane is in the directions 012389, then the projections are as in (5.1) and in addition

$$\Gamma_{012389} \epsilon = \epsilon .$$

These lead again to a configuration preserving $1/16$ of M-theory supersymmetry, ie $1/2$ of supersymmetry of $N = 1, D = 4$ supergravity. From the perspective of $N = 1, D = 4$ supergravity, these domain walls may arise from a superpotential generated by a M2-brane instanton wrapping the associative cycle. The investigation of the Killing spinor equations of $G_2$ domain walls that preserve $1/2$ of supersymmetry will be given in an appendix.

7. Black Holes

Black Holes as Wrapped M2-branes

As we have mentioned the electrically charged black holes of $N = 1, D = 4$ supergravity action that arise from compactifying M-theory on holonomy $G_2$ manifolds can be viewed as wrapped M2-branes on the two-cycles of the compact space $N$. It is known that there are minimal sphere representatives of every homotopy class $\pi_2(N)$ (see for example chapter VI [33]). The mass and the charges of such black holes are given by $M = T_2 Vol(C)$ and $Q_a = T_2 \omega_a[C]$, respectively, where $T_2$ is the M2-brane tension and $C$ is the two-cycle.

Since two cycles in holonomy $G_2$ manifolds are not supersymmetric, it is not expected to find relation between the mass and the charges of the black holes. This is despite the fact that the mass and the charge per-unit volume of the associated M2-brane are equal. Although it is not apparent that there are non-supersymmetric degree two calibrations on holonomy $G_2$ manifolds, suppose that
there is one associated with the two-form $\lambda$. Then $\lambda = \xi^a \omega_a$, for some constants $\xi^a$, and $M = T_2 \text{Vol}(\Sigma) = T_2 \int_C \phi = T_2 \xi^a Q_a$ which can be interpreted as an extremality relation for the black hole. As we shall see there are extreme solutions $N = 1, D = 4$ supergravity associated with $G_2$ compactifications which however exhibit a naked singularity.

### 7.1. Electric $G_2$ Black Holes

**Ansatz and Field Equations**

In order to find black hole solutions of $N = 1, D = 4$ supergravity associated with $G_2$ compactifications of M-theory, we consider the ansatz

\[
d s^2 = -A^2(r) d t^2 + B^2(r) (d r^2 + r^2 d s^2(\Sigma^2))
\]

\[A^a = d t C^a(r)
\]

\[s^i = s^i(r)
\]

\[p^i = p^i(r).
\]

(7.1)

We recall from section 3 that $\gamma^i_j dz^i dz^j = k_{ij} (d s^i ds^j + d p^i d p^j)$ with $k_{ij} = \partial_i \partial_j \Phi(s)$, where $\Phi = -\frac{3}{28} \log(\Theta(s))$, $\Theta$ is the volume of the compact space and $\partial_i = \frac{\partial}{\partial s^i}$; so $\partial_i \Phi$ is homogeneous of degree $-1$ in $s$. Furthermore we shall take the scalar potential of $N = 1, D = 4$ supergravity to vanish $V \equiv 0$. It is straightforward to observe from the Killing spinor equations that all electrically charged solutions cannot be supersymmetric.

Substituting the ansatz (7.1) into the Maxwell equations we find

\[B A^{-1} r^2 \Re h_{ab} \partial_r C^b = H_a,
\]

where $H_a$ are constants. To obtain the scalar equations we vary $s^f$ and $p^f$. Recalling that $\Im h_{ab} = -\frac{1}{2} C_{iab} p^i$ and $\Re h_{ab} = -\frac{1}{2} C_{iab} s^i$, we find that the field equations for
\( s^i \) and \( p^i \) are

\[
\sqrt{|g|} \left( \frac{1}{8} C_{ab} F^{a}_{MN} F^{b}_{MN} - \partial \ell k_{ij} \left( \partial_{M} s^{i} \partial_{M} s^{j} + \partial_{M} p^{i} \partial_{M} p^{j} \right) \right) + 2\partial_{M} (\sqrt{|g|} k_{ij} \partial_{M} s^{j}) = 0
\]

(7.2)

and

\[
-\frac{1}{8} \sqrt{|g|} C_{ab} F^{a}_{MN} F^{b}_{MN} + 2\partial_{M} (\sqrt{|g|} k_{ij} \partial_{M} p^{j}) = 0 ,
\]

(7.3)

respectively.

It is convenient to define

\[
\psi = AB \quad N = r B .
\]

(7.4)

Then the Maxwell equations may be rewritten as

\[
N^2 \psi^{-1} \text{Re} h_{ab} \partial_{r} C^{b} = H_{a} .
\]

(7.5)

On substituting the black hole ansatz into the scalar equations, and eliminating \( \partial_{r} C^{a} \) using (7.5), one obtains

\[
-\psi r^{2} \partial \ell k_{ij} \left( \partial_{r} s^{i} \partial_{r} s^{j} + \partial_{r} p^{i} \partial_{r} p^{j} \right)
- \frac{1}{4} \psi N^{-2} C_{ab} \text{Re} h^{ac} H_{c} \text{Re} h^{bd} H_{d} + 2\partial_{r} (\psi r^{2} k_{ij} \partial_{r} s^{j}) = 0
\]

(7.6)

\[
\partial_{r} (\psi r^{2} k_{ij} \partial_{r} p^{j}) = 0 .
\]

Lastly we consider the Einstein equations. We adopt the notation \( \dot{B} = \frac{dB}{d\ell} \), \( \dot{A} = \frac{dA}{d\ell} \), \( \ddot{B} = \frac{d^2 B}{d\ell^2} \), \( \ddot{A} = \frac{d^2 A}{d\ell^2} \). The non-vanishing components of the Einstein tensor
are given by

\[
G_{00} = -\frac{A^2}{rB^4} (4BB' - rB'^2 + 2rB\dot{B})
\]

\[
G_{rr} = \frac{1}{rAB^2} (2r\dot{A}B + 2AB\dot{B} + rA\dot{B}^2 + 2B^2A)
\]

\[
G_{\theta\theta} = \frac{r}{AB^2} (B^2 \dot{A} + BAB + AB\dot{B} + rB^2A - rA\dot{B}^2)
\]

\[
G_{\phi\phi} = \frac{r\sin^2\theta}{AB^2} (B^2 \dot{A} + BAB + AB\dot{B} + rB^2A - rA\dot{B}^2).
\]

Eliminating \(C^a\) from the above equations using (7.5) and utilizing the definitions (7.4), we find that the independent Einstein equations can be expressed as

\[
\frac{d^2}{dr^2} (r^{-\frac{3}{2}} \psi) = \frac{3}{4} r^{-\frac{1}{2}} \psi
\]

(7.8)

\[
\frac{\psi}{N} \frac{d}{dr} (\frac{\dot{N}}{\psi}) = -k_{ij} (\partial_r s^i \partial_r s^j + \partial_r p^i \partial_r p^j)
\]

(7.9)

\[
r^\frac{1}{2} N^3 \psi^{-1} \frac{d^2}{dr^2} (N^{-1} r^\frac{3}{2} \psi) + N \left[ \frac{5}{4} N - r \dot{N} - r^2 \ddot{N} - \text{Re} h_{a} h_{b} = 0 .
\]

(7.10)

A useful identity implied by the scalar equations (or the Einstein equations) is

\[
-\frac{1}{2} \psi r^2 N^{-2} \partial_r (\text{Re} h_{a} h_{b}) + \partial_r (\psi^2 r^4 k_{ij} [\partial_r s^i \partial_r s^j + \partial_r p^i \partial_r p^j]) = 0 .
\]

(7.11)

Using the Einstein equation (7.8), we determine \(\psi\) as

\[
\psi = \beta + \frac{\alpha}{r^2}
\]

(7.12)

for constants \(\alpha, \beta\). Then (7.10) may be simplified to

\[
2\psi - 2 \frac{d}{dr} (\psi r^2 N^{-1} \dot{N}) = \psi N^{-2} \text{Re} h_{a} h_{b} .
\]

(7.13)

In addition, from the \(p^\ell\) scalar equation, \(p^\ell\) must satisfy

\[
\partial_r p^\ell = \psi^{-1} r^{-2} k^{\ell j} \theta_j
\]

(7.14)

for some real constants \(\theta_j\), and here \(k^{\ell j}\) is the inverse of the Hessian of \(\Phi\).
A solution to the above system of equations is the Schwarzschild black hole. In this case the Maxwell gauge potential vanishes and the scalars are constant. This black hole cannot be thought of as a M2-brane wrapped on a homology 2-cycle of the compact $G_2$ holonomy manifold because it does not carry electric charges. Generically, to find electrically charged black holes, one has to take some of the scalars to be non constant*. So we take $\partial_s s^i \neq 0$. On eliminating $p^i$ from the scalar equation for $s^\ell$ and making use of the fact that $k_{ij}$ is the Hessian of $\Phi$, one obtains

$$-\frac{1}{4} \psi N^{-2} C_{\ell a b} \text{Re} h^{a c} \text{Re} h^{b d} H_c H_d + \psi r^2 \partial_r k_{i j} \partial_r s^i \partial_r s^j + \psi^{-1} r^{-2} \partial_k k^{ij} \theta_i \theta_j + 2 k_{\ell j} \partial_r (\psi r^2 \partial_r s^j) = 0.$$  

(7.15)

To solve the field equations, we shall make use of the homogeneity properties of $\partial_t \Phi$. In particular, contracting (7.15) with $s^\ell$ and setting $\theta_i = 0$, one finds the identity

$$\frac{1}{2} \psi N^{-2} \text{Re} h^{a b} H_a H_b - 2 \partial_r (\psi r^2 \partial_r \Phi) = 0.$$  

(7.16)

From this it follows that in order to have charged solutions, we shall require $\partial_r (\psi r^2 \partial_r \Phi) \neq 0$. Substituting (7.16) into (7.13) with $\psi = 1 - \frac{\mu^2}{r^2}$ for $\mu \neq 0$, one obtains

$$N = \kappa r \psi e^{-2\Phi} \left(\frac{r^2 + \mu}{r - \mu}\right)$$  

(7.17)

for constants $\kappa$ and $\delta$. In the special case when $\psi = 1$ one obtains

$$N = \kappa r e^{-2\Phi} e^{\delta r}.$$  

(7.18)

The only remaining independent equations are the Einstein equation (7.9) together with (7.15). We have been unable to find general solutions to the field equations, and our general analysis ends here.

* In special cases this depends on the properties of the intersection numbers $C_{i a b}$ but we shall not pursue this further here.
Dilatonic $G_2$ Electric Black Holes

To find an explicit solution to the equation (7.9) and (7.15), we shall now consider the special case where

$$p^i = 0 \quad s^i = s(r)c^i \quad (7.19)$$

for some constants $c^i$. Then the homogeneity of $\partial_i \Phi$ implies that

$$\partial_i \Phi = \lambda_i s^{-1}$$

$$k_{ij} = \partial_i \partial_j \Phi = \lambda_{ij} s^{-2} \quad (7.20)$$

$$\partial_i k_{ij} = \partial_i \partial_j \partial_\ell \Phi = \lambda_{ij\ell} s^{-3}$$

where $\lambda_i$, $\lambda_{ij}$ and $\lambda_{ij\ell}$ are constants related by

$$c^i \lambda_i = -\frac{1}{4}$$

$$c^i \lambda_{ij} = -\lambda_j \quad (7.21)$$

$$c^i \lambda_{ij\ell} = -2 \lambda_{j\ell}$$

It is also convenient to define $\mathcal{H}_{ab} = -\frac{1}{2} c^\ell C_{\ell ab}$ so that $R_{ab} = s \mathcal{H}_{ab}$. Then the $s^\ell$ scalar equation (7.15) implies that

$$-\frac{1}{2} C_{\ell ab} \mathcal{H}^{ac} \mathcal{H}^{bd} H_c H_d = \lambda \lambda_\ell \quad (7.22)$$

for some constant $\lambda$, and

$$\frac{\lambda}{2} \psi N^{-2} s^{-2} + 2 \psi r^2 s^{-3} (\partial_r s)^2 - 2 s^{-2} \partial_r (\psi r^2 \partial_r s) = 0 \quad (7.23)$$

Note that on contracting (7.22) with $c^\ell$ and using the homogeneity properties of $\Theta$, one obtains

$$-\frac{\lambda}{4} = \mathcal{H}^{ab} H_a H_b \quad (7.24)$$

Then using the above identities, it follows that for $\partial_r s \neq 0$ the scalar equations are
implied by the Einstein equations. It therefore suffices to solve

$$\frac{\psi}{N} \frac{d}{dr} \left( \frac{\dot{N}}{\psi} \right) = -\frac{1}{4} s^{-2} (\partial_r s)^2$$

(7.25)

$$2\psi - 2 \frac{d}{dr} (\psi r^2 N^{-1} \dot{N}) = -\frac{1}{4} \psi N^{-2} s^{-1} \lambda .$$

To find solutions to these equations which are asymptotically Minkowskian, we shall set \( \psi = 1 - \frac{\mu^2}{r^2} \), see (7.12). The simplest way to solve (7.25) is to eliminate \( s(r) \) from the first equation by making use of the second equation. This equation for \( N(r) \) may be simplified further by defining \( H(r) \) according to

$$\frac{\dot{N}}{N} = r^{-2} \psi^{-1} \left( H + r + \frac{\mu^2}{r} \right) .$$

(7.26)

Then \( H \) must satisfy

$$(r^2 - \mu^2)^2 \ddot{H} + 4r(r^2 - \mu^2) \dot{H} \ddot{H} + 4(r^2 - \mu^2) H \ddot{H} \dot{H} + 4(r^2 - \mu^2) \dot{H}^3 + 4\dot{H}^2 \left[ r^2 + 2rH - 4\mu^2 + 2H^2 \right] = 0 .$$

(7.27)

We discard the solution \( H = \text{const} \) because it is inconsistent with (7.25) for \( \lambda \neq 0 \).

To simplify this equation even further, define \( X = H \) and define implicitly

$$f(X) = \left( (r^2 - \mu^2) \dot{H} \right)^{-1} .$$

(7.28)

Then

$$\ddot{H} = -\frac{1}{(r^2 - \mu^2)^2} \left( 2r f^{-1} + f^{-3} f' \right) ,$$

(7.29)

where \( f' = \frac{df}{dX} \). Substituting these expressions into (7.27) , one obtains

$$4f^3 + 8f^4 X^2 - 16\mu^2 f^4 + (f')^2 - 4f^2 X f' = 0 .$$

(7.30)

On setting \( f(X) = (L(X) - X^2)^{-1} \), this simplifies to give

$$4L - 16\mu^2 + (L')^2 = 0 .$$

(7.31)
There are two possible solutions to this equation,

\[ L(X) = 4\mu^2 \]  \quad (7.32)

or

\[ L(X) = 4\mu^2 - (X + \tau)^2 \]  \quad (7.33)

for some constant \( \tau \). However, \( L(X) = 4\mu^2 \) leads to \( s \) constant and so the Schwarzschild black hole. We therefore take

\[ L(X) = 4\mu^2 - (X + \tau)^2 . \]  \quad (7.34)

Hence \( H \) must satisfy

\[ (r^2 - \mu^2) \dot{H} = 4\mu^2 - 2H^2 - 2\tau H - \tau^2 . \]  \quad (7.35)

For \( \mu \neq 0 \) there are three cases to consider, according as to whether \( \tau^2 = 8\mu^2 \), \( \tau^2 > 8\mu^2 \) or \( \tau^2 < 8\mu^2 \). We shall focus on the cases \( \tau = \mu = 0 \) and \( \tau^2 < 8\mu^2 \).

For \( \mu = \tau = 0 \), we find that

\[
\begin{align*}
 ds^2 &= -(1 + \frac{2\kappa^2}{r})^{-1}dt^2 + (1 + \frac{2\kappa^2}{r})dx^2 \\
 F^a &= -\frac{16\kappa^4}{\lambda(r + 2\kappa^2)^2}H_{ab}H_bdr \wedge dt \\
 s^i &= -\frac{\lambda}{16\kappa^r}(2 + \kappa^{-2}r)c^i
\end{align*}
\]  \quad (7.36)

for constant \( \kappa > 0 \). This solution is asymptotically Minkowski, and the charges, mass and asymptotic values of the scalar are

\[
\begin{align*}
 Q^a &= s_\infty^{-1}H_{ab}H_b \\
 M &= \kappa^2 \quad (7.37) \\
 s_\infty &= -\frac{\lambda}{16\kappa^4} ,
\end{align*}
\]

respectively.
The spacetime geometry of the solution (7.36) is that of two M-brane configuration reduces to four dimensions on a 7-torus, for example two orthogonally intersecting M2-branes on a 0-brane [12]. The above solution exhibits a naked singularity at $r = 0$.

For $\tau^2 < 8\mu^2$, we find that

$$H = -\frac{1}{2}(\tau + \sqrt{8\mu^2 - \tau^2}) + \frac{\sqrt{8\mu^2 - \tau^2}}{1 - \rho(r + \mu)(\tau^2 - \tau^2)}$$

(7.38)

which in turn implies that the solution is

$$ds^2 = -r^2\psi^2N^{-2}dt^2 + r^{-2}N^2[dr^2 + r^2ds^2(S^2)]$$

$$F^a = \psi N^{-2}s^{-1}H^{ab}H_b \wedge dt$$

$$s^i = s(r)c^i$$

(7.39)

where

$$N = \frac{\kappa}{r}(r^2 - \mu^2)\left(\frac{r + \mu}{r - \mu}\right)\frac{1}{4\mu}(\tau - \sqrt{8\mu^2 - \tau^2}) \sqrt{-1 + (1 + \kappa^2)(\frac{r + \mu}{r - \mu})\frac{\sqrt{8\mu^2 - \tau^2}}{\rho}}$$

$$s = -\frac{\lambda}{16\rho\kappa^2(\tau^2 - 8\mu^2)}\left[\left(\frac{r + \mu}{r - \mu}\right) - \frac{1}{4\mu}(\tau + \sqrt{8\mu^2 - \tau^2}) - (1 + \kappa^2)(\frac{r + \mu}{r - \mu}) - \frac{1}{4\mu}(\tau - \sqrt{8\mu^2 - \tau^2})\right]$$

(7.40)

for constant $\kappa > 0$, and we have set $\rho = 1 + \kappa^{-2}$ in (7.38).

This solution is again asymptotically Minkowski as $r \to \infty$, and the electric charges $Q^a$, the mass $M$ and asymptotic values $s_\infty$ of the moduli scalar are

$$Q^a = s_\infty^{-1}H^{ab}H_b$$

$$M = \frac{1}{2}(\tau + (1 + 2\kappa^2)\sqrt{8\mu^2 - \tau^2})$$

$$s_\infty = \frac{\lambda}{16\kappa^2(1 + \kappa^2)(\tau^2 - 8\mu^2)}$$

(7.41)

respectively. To examine this spacetime geometry it is particularly useful to consider the metric in the neighbourhood of $r = \mu$. We define $W = r - \mu$. Then as
\[ W \to 0^+, \]
\[ N \to \rho W^{1 - \frac{1}{4\mu}(\tau + \sqrt{8\mu^2 - \tau^2})} \]

where \( \rho \) is determined by
\[ \rho = \frac{\kappa}{\mu} \sqrt{1 + \kappa^{-2}(2\mu)^{1 + \frac{1}{4\mu}(\tau + \sqrt{8\mu^2 - \tau^2})}. \] (7.43)

The leading order behaviour of the metric in the neighbourhood of \( W = 0 \) is given by
\[ ds^2 = -\frac{1}{4\mu}W^{\frac{1}{4\mu}(\tau + \sqrt{8\mu^2 - \tau^2})}dt^2 + \mu^{-2}W^{\frac{1}{4\mu}(\tau + \sqrt{8\mu^2 - \tau^2})}(dW^2 + \mu^2 ds^2(S^2)) \] (7.44)

From this metric it is straightforward to show that if there exist geodesics along which in-falling particles may pass through \( r = \mu \) in a finite proper time then it is necessary to impose the constraint \( \frac{1}{4\mu}(\tau + \sqrt{8\mu^2 - \tau}) > 0 \). This condition is sufficient to ensure the positivity of the mass given in (7.41). Furthermore, the Ricci scalar associated with (7.44) is given by
\[ R = 2\rho^{-2}\left[\mu^2(\delta^2 - 1) - W^2\right]W^{2\delta - 4}, \] (7.45)

where \( \delta = \frac{1}{4\mu}(\tau + \sqrt{8\mu^2 - \tau}) \). In order for this to remain bounded as \( W \to 0 \), one must impose the condition \( \delta = 1 \), i.e. \( \tau = 2\mu \). Hence, in order for there to be a horizon at \( r = \mu \) one requires \( \tau = 2\mu \). For this special case, it is most convenient to change co-ordinates by setting
\[ v = \frac{1}{r}(r + \mu)^2, \] (7.46)

so that the metric can be written as
\[ ds^2 = -L(v)F(v)dt^2 + L(v)^{-1}\left[F(v)^{-1}dv^2 + v^2 ds^2(S^2)\right] \] (7.47)
where
\[ F(v) = 1 - \frac{4\mu}{v}, \quad (7.48) \]
and
\[ L(v) = \left(1 + \frac{4\kappa^2 \mu}{v}\right)^{-1} . \quad (7.49) \]

This four-dimensional spacetime geometry has been considered before in [13] where it was obtained via compactification of two intersecting non-extreme M2-branes on a 7-torus, where the two M2-branes have the same electric charge. The solution (7.47) has finite horizon area.

### 7.2. Magnetic $G_2$ Black Holes

To find magnetic black hole solutions which are associated with M5-branes wrapped on 5-cycles of the compact space, we take the ansatz for the metric and moduli scalars as in the electric case (7.1) but for the Maxwell field we write
\[ F^a = \mu^a \sin \theta d\theta \wedge d\phi \quad (7.50) \]
where $\mu^a$ are constants and $\phi, \theta$ are the standard angular coordinates on a two sphere. We shall see that the electric and the magnetic solutions are related as expected because of electro-magnetic duality in four-dimensions. So we shall not elaborate in the description of this case.

First it is straightforward to see that the gauge field equations are satisfied provided that \( p^i = \text{const} \). Taking this to be the case, the scalar equations of \( p^\ell \) are automatically satisfied. Given this, the \( s^\ell \) scalar equations are given by
\[ \frac{1}{4} \psi N^{-2} C_{ab} \mu^a \mu^b + \psi r^2 \partial \ell k_{ij} \partial_r s^i \partial_r s^j + 2k_{ij} \partial_r (\psi r^2 \partial_r s^j) = 0 , \quad (7.51) \]
where \( N, \psi \) are defined as in the electric case. The Einstein field equations with
vanishing scalar potential imply

\[
\frac{d^2}{d r^2}(r^3 \psi) = \frac{3}{4} r^{-\frac{1}{2}} \psi
\]

\[
\frac{\psi}{N} \frac{d}{d r} \left( \frac{\dot{N}}{\psi} \right) = -k_{ij} \partial_r s^i \partial_r s^j
\]

\[
2 \psi - 2 \frac{d}{d r} \left( \psi r^2 N^{-1} \dot{N} \right) = \psi N^{-2} \text{Re} h_{ab} \epsilon^a \mu^b.
\]

The first two of these equations are identical to the electric black hole case. In particular, the first equation implies \( \psi = \beta + \frac{\alpha}{r^2} \) for real constants \( \alpha, \beta \).

Next write \( s^i = s(r)c^i \) for some constants \( c^i \). Using (7.20) and (7.21) and the definition of \( H_{ab} \) as in the electric case, the \( s^i \) scalar equation implies that

\[
-\frac{1}{2} C_{\ell ab} \epsilon^a \mu^b = \lambda \lambda_{\ell}
\]

for some constant \( \lambda \). So it follows that \( H_{ab} \epsilon^a \mu^b = -(1/4) \lambda \). Then just as for the electric case, by making use of the identity

\[
-\frac{1}{2} \psi^2 r^2 N^{-2} \partial_r (\text{Re} h_{ab} \epsilon^a \mu^b) + \partial_r (\psi^2 r^4 k_{ij} \partial_r s^i \partial_r s^j) = 0
\]

obtained from the Einstein equations, one may see that the scalar equations are implied by the Einstein equations. Hence, it suffices to solve the remaining Einstein equations

\[
\frac{\psi}{N} \frac{d}{d r} \left( \frac{\dot{N}}{\psi} \right) = -\frac{1}{4} s^{-2} (\partial_r s)^2
\]

\[
2 \psi - 2 \frac{d}{d r} \left( \psi r^2 N^{-1} \dot{N} \right) = -\frac{1}{4} \psi N^{-2} s \lambda.
\]

Clearly, these equations are equivalent to (7.25) under the transformation \( s \rightarrow s^{-1} \). Hence it follows that the spacetime geometry of these magnetic black holes is identical to the electric black hole case, and \( s_{\text{magnetic}} = s_{\text{electric}}^{-1} \). For example the M-theory interpretation of the analogue of the solution (7.36) is that of two M5-branes orthogonally intersecting on a 3-brane [12].

32
**Acknowledgments:** G.P. thanks ITP and the Physics Department of University of Stanford for hospitality. G.P. is supported by a University Research Fellowship from the Royal Society. J.G. is supported by a EPSRC postdoctoral grant. This work is partially supported by SPG grant PPA/G/S/1998/00613.

**APPENDIX A: Spinor Notation**

It is most convenient to present the supersymmetry transformations in terms of a 4-component Majorana spinor $\epsilon$ with real components. We take $\sigma^M = (\sigma^M_{\alpha\beta})$ to be the Pauli matrices;

\[
\begin{align*}
\sigma^0 &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & \sigma^1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\sigma^2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} & \sigma^3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{align*}
\]

(A.1)

We set $\eta_{MN} = \text{diag}(-1, 1, 1, 1)$, $\epsilon^{12} = \epsilon_{21} = 1$, and to perform the supersymmetry calculations we define explicitly

\[
\begin{align*}
\Gamma_\bar{x} &= \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix} & \Gamma_y &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\
\Gamma_\bar{\alpha} &= \begin{pmatrix} 0 & -i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix} & \Gamma_z &= \begin{pmatrix} 0 & -\sigma^3 \\ -\sigma^3 & 0 \end{pmatrix}
\end{align*}
\]

(A.2)

so that

\[
\Gamma^5 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

With these definitions, the gamma matrices satisfy the Clifford algebra

\[
\Gamma_M \Gamma_N + \Gamma_N \Gamma_M = 2\eta_{MN}.
\]

(A.3)
APPENDIX B: String Solitons and Supersymmetry

In order to examine string and domain wall solutions, we shall consider the following ansatz;

\[ ds^2 = A^2(u, \bar{u})ds^2(\mathbb{R}^{1,1}) + ds^2_{(2)} \]
\[ z^i = z^i(w, \bar{w}) \]
\[ A^a = 0 \]  \hspace{1cm} (B.1)

where \( ds^2_{(2)} \) is a metric on the manifold spanned by \( w, \bar{w} \) where \( w = x + iy \) and \( \bar{w} = x - iy \) for \( x, y \in \mathbb{R} \). Without loss of generality, we shall take

\[ ds^2_{(2)} = B^2(x, y)(dx^2 + dy^2) \]  \hspace{1cm} (B.2)

to be diagonal, using the fact that any metric on a Riemann surface is locally conformally flat.

Substituting this ansatz into the Killing spinor equations, we find that

\[ \partial_x A \Gamma_x \epsilon + \partial_y A \Gamma_y \epsilon + AB e^{K/2} (\text{Re} f + \text{Im} f \Gamma^5) \epsilon = 0 \]  \hspace{1cm} (B.3)

together with

\[ 2\partial_x \epsilon - \partial_x \log A \epsilon - \partial_y \log B \Gamma_x \Gamma_y \epsilon - \Gamma^5 \text{Im}(K_i \partial_x z^i) \epsilon = 0 \]
\[ 2\partial_y \epsilon - \partial_y \log A \epsilon - \partial_x \log B \Gamma_x \Gamma_y \epsilon - \Gamma^5 \text{Im}(K_i \partial_y z^i) \epsilon = 0 \]  \hspace{1cm} (B.4)

and

\[ (\text{Re} \partial_x z^i - \Gamma^5 \text{Im} \partial_x z^i) \Gamma_x \epsilon + (\text{Re} \partial_y z^i - \Gamma^5 \text{Im} \partial_y z^i) \Gamma_y \epsilon \]
\[ - Be^{K/2} (\text{Re} (\gamma^{ij} D_j \bar{f}) - \Gamma^5 \text{Im} (\gamma^{ij} D_j \bar{f})) \epsilon = 0 \]  \hspace{1cm} (B.5)

There are two cases to consider. Firstly, if the the scalar potential vanishes, then the first Killing spinor equation above with \( f = 0 \) implies that \( A \) is constant,
and we set \( A = 1 \). The remaining Killing spinor equations can be solved by taking

\[
\begin{align*}
  z^i &= z^i(w) \\
  B &= e^{-\frac{K}{2}} 
\end{align*}
\]

(B.6)
i.e. the \( z^i \) are holomorphic. The solution preserves \( \frac{1}{2} \) of the supersymmetry; \( \epsilon \) is a constant spinor satisfying

\[
\Gamma^5 \Gamma_x \Gamma_y \epsilon = -\epsilon 
\]

(B.7)

More generally, one may construct solutions for which \( f \neq 0 \). In particular, we may begin by examining (B.5). If we work in a real basis, so that

\[
\epsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}
\]

(B.8)

with \( \epsilon_1, \epsilon_2 \) real; then (B.5) implies

\[
\begin{align*}
  (\sigma^1 - 1) &\left[ 2i\partial_w z^i \bar{\eta} + Be^{\frac{K}{2}} \gamma^{ij} D_j \bar{f} \eta \right] = 0 \\
  (\sigma^1 + 1) &\left[ -2i\partial_{\bar{w}} z^i \bar{\eta} + Be^{\frac{K}{2}} \gamma^{ij} D_j \bar{f} \eta \right] = 0
\end{align*}
\]

(B.9)

where \( \eta = \epsilon_1 + i\epsilon_2 \). Suppose now \( \exists \ i \) such that \( \gamma^{ij} D_j \bar{f} = 0 \). Then for these \( i \), these equations may be solved by taking \( z^i \) constant. Alternatively, one may have \( z^i = z^i(w) \) non-constant holomorphic with \( \Gamma^5 \Gamma_x \Gamma_y \epsilon = -\epsilon; \) or \( z^i = z^i(\bar{w}) \) non-constant anti-holomorphic with \( \Gamma^5 \Gamma_x \Gamma_y \epsilon = \epsilon \) (however if there exists more that one value of \( i \) such that \( \gamma^{ij} D_j \bar{f} = 0 \) then one cannot have a supersymmetric solution with a mixture of corresponding non-constant holomorphic and anti-holomorphic complex scalars).

Suppose now we consider \( i \) for which \( \gamma^{ij} D_j \bar{f} \neq 0 \). Define

\[
\begin{align*}
  \psi^i &= 2i\partial_w z^i \left[ Be^{\frac{K}{2}} \gamma^{ij} D_j \bar{f} \right]^{-1} \\
  \tau^i &= -2i\partial_{\bar{w}} z^i \left[ Be^{\frac{K}{2}} \gamma^{ij} D_j \bar{f} \right]^{-1}
\end{align*}
\]

(B.10)
Then one requires for these $i$;

\[
(1 - \sigma^1)(\psi^i \bar{\eta} + \eta) = 0 \tag{B.11}
\]
\[
(1 + \sigma^1)(\tau^i \bar{\eta} + \eta) = 0
\]

There are several possibilities. Firstly, note that one cannot have a supersymmetric solution with both $\psi^i = \tau^i = 0$. If $\psi^i = 0$ then it turns out that $\Gamma^5 \Gamma_5 \Gamma_5 \epsilon = -\epsilon$. If $\tau^i = 0$, however, then $\Gamma^5 \Gamma_5 \Gamma_5 \epsilon = \epsilon$. Alternatively, one may have $\psi^i, \tau^i$ both nonzero. It turns out that if both $|\psi^i| \neq 1$ and $|\tau^i| \neq 1$ then the solution cannot be supersymmetric. If however, $|\psi^i| \neq 1$ but $|\tau^i| = 1$ then one has $\Gamma^5 \Gamma_5 \Gamma_5 \epsilon = -\epsilon$.

Another possibility is to take $|\tau^i| \neq 1$ and $|\psi^i| = 1$; then $\Gamma^5 \Gamma_5 \Gamma_5 \epsilon = \epsilon$.

We shall however concentrate on the remaining possibility, in which we take $|\tau^i| = |\psi^i| = 1$ (but we do not necessarily require $\tau^i = \psi^i$). Writing then $\psi^i = e^{i\theta^i}$, $\tau^i = e^{i\phi^i}$ for real $\theta^i$, $\phi^i$, the supersymmetry constraint (B.5) for $|\tau^i| = 1$ and $|\psi^i| = 1$ is satisfied by taking

\[
\epsilon_1 = \sin \phi^i \begin{pmatrix} \lambda^i \\ \lambda^i \end{pmatrix} + \sin \theta^i \begin{pmatrix} -\mu^i \\ \mu^i \end{pmatrix} \tag{B.12}
\]
\[
\epsilon_2 = -(1 + \cos \phi^i) \begin{pmatrix} \lambda^i \\ \lambda^i \end{pmatrix} - (1 + \cos \theta^i) \begin{pmatrix} -\mu^i \\ \mu^i \end{pmatrix}
\]

Analogous reasoning may be used to consider (B.3). In particular, (B.3) may be written as

\[
(\sigma^1 - 1) \left[ 2i \partial_w A \bar{\eta} - AB e^{\frac{K}{2}} \bar{f} \eta \right] = 0
\]
\[
(\sigma^1 + 1) \left[ -2i \partial_{\bar{w}} A \bar{\eta} - AB e^{\frac{K}{2}} \bar{f} \eta \right] = 0 \tag{B.13}
\]

Defining

\[
\Omega = -2i \partial_w A (ABe^{\frac{K}{2}} \bar{f})^{-1}
\]
\[
\Lambda = 2i \partial_{\bar{w}} A (ABe^{\frac{K}{2}} \bar{f})^{-1}
\]

we note that (B.3) is equivalent to

\[
(\sigma^1 - 1)(\Omega \bar{\eta} + \eta) = 0
\]
\[
(\sigma^1 + 1)(\Lambda \bar{\eta} + \eta) = 0 \tag{B.15}
\]
Hence the reasoning used to determine the various possible values of $\psi^i$, $\tau^i$ also applies to $\Omega$ and $\Lambda$.

To summarize then, neglecting the cases for which $\Gamma^5 \Gamma_x \Gamma_y \epsilon = \pm \epsilon$, (B.3) and (B.5) imply that $\Gamma^5 \Gamma_x \Gamma_y \epsilon \neq \pm \epsilon$. Furthermore, if $\Gamma^5 \Gamma_x \Gamma_y \epsilon \neq \pm \epsilon$, $\epsilon$ is given by (B.12).

It is also necessary to examine (B.4). This constraint may be rewritten as

$$4 \partial_{\bar{w}} \epsilon - 2i \partial_{\bar{w}} \log \frac{B}{A} \Gamma_x \Gamma_y \epsilon + i \Gamma^5 (-\partial_{\bar{w}} K + 2K_i \partial_{\bar{w}} z^i) \epsilon = 0 .$$

where $\epsilon = A^{-1/2} \epsilon$. In this case $\tau^i = \Lambda$ and $\psi^i = \Omega$ imply (for $f \neq 0$)

$$-\partial_{\bar{w}} z^i = A^{-1} \partial_{\bar{w}} A \bar{f} \Gamma_{\bar{f}}\Gamma \epsilon,$$

and we solve the supersymmetry constraints by taking $\Lambda = e^{i\phi}$, $\Omega = e^{i\theta}$, for $\theta, \phi \in \mathbb{R}$ with

$$\hat{\epsilon}_1 = \sin \phi \left( \begin{array}{c} \hat{\lambda} \\ \hat{\lambda} \end{array} \right) + \sin \theta \left( \begin{array}{c} -\hat{\mu} \\ \hat{\mu} \end{array} \right),$$

$$\hat{\epsilon}_2 = -(1 + \cos \phi) \left( \begin{array}{c} \hat{\lambda} \\ \hat{\lambda} \end{array} \right) - (1 + \cos \theta) \left( \begin{array}{c} -\hat{\mu} \\ \hat{\mu} \end{array} \right) .$$

where $\hat{\lambda}, \hat{\mu} \in \mathbb{R}$. Then (B.4) implies that

$$4 \partial_{\bar{w}} (\hat{\lambda} \sin \phi) + i(1 + \cos \phi)(\partial_{\bar{w}} (K + 2 \log \frac{B}{A}) - 2K_i \partial_{\bar{w}} z^i) \hat{\lambda} = 0$$

$$4 \partial_{\bar{w}} ((1 + \cos \phi)\hat{\lambda}) + i \sin \phi(\partial_{\bar{w}} (K + 2 \log \frac{B}{A}) - 2K_i \partial_{\bar{w}} z^i) \hat{\lambda} = 0$$

$$4 \partial_{\bar{w}} (\hat{\mu} \sin \theta) + i(1 + \cos \theta)(\partial_{\bar{w}} (K - 2 \log \frac{B}{A}) - 2K_i \partial_{\bar{w}} z^i) \hat{\mu} = 0$$

$$4 \partial_{\bar{w}} ((1 + \cos \theta)\hat{\mu}) + i \sin \theta(\partial_{\bar{w}} (K - 2 \log \frac{B}{A}) - 2K_i \partial_{\bar{w}} z^i) \hat{\mu} = 0 .$$
This is solved by taking
\[
\hat{\lambda} = \frac{\xi}{\sqrt{1 + \cos \phi}} \quad \hat{\mu} = \frac{\zeta}{\sqrt{1 + \cos \theta}}
\]  
for constant \( \xi, \zeta \in \mathbb{R} \) and \( B, \phi \) and \( \theta \) are determined by
\[
\partial_{\bar{w}} (2\phi + i(K + 2\log \frac{B}{A})) = 2iK_{\bar{w}} z^i
\]
\[
\partial_{\bar{w}} (-2\theta - i(K + 2\log \frac{B}{A})) = -2iK_{\bar{w}} z^i
\]  
We note that these solutions generically preserve \( \frac{1}{2} \) of the supersymmetry.

It is straightforward to check that these conditions ensure that the scalar and Einstein field equations hold.

APPENDIX C: \( G_2 \) Domain Walls

C.1. Ansatz and Killing Spinor Equations

To find domain wall solutions of \( N = 1, D = 4 \) supergravity, we shall use the ansatz
\[
ds^2 = B^2(y)[dy^2 + ds^2(\mathbb{R}^{1,2})] \\
z^i = z^i(y) \\
A^a = 0 ,
\]  
where \( B \) and \( z^i \) will be determined by the field equations. Properties of domain walls in supergravity have been reviewed in [34]. We shall also assume that they are associated with some superpotential \( f \) which we shall not specify. Substituting this ansatz to the Killing spinor equations of section three, one finds
\[
\partial_y B \Gamma_y \epsilon + B^2 e^{\frac{K}{2}} (\text{Ref} + \text{Im} f \Gamma_5) \epsilon = 0 \\
2\partial_y \epsilon - \Gamma_5 \text{Im}(K_{\bar{w}} z^i) \epsilon + B e^{\frac{K}{2}} \Gamma_y (\text{Ref} + \text{Im} f \Gamma_5) \epsilon = 0 \\
(\text{Re} \partial_y z^i - \text{Im} \partial_y z^i \Gamma_5) \epsilon - B e^{\frac{K}{2}} (\text{Re}(\gamma^{ij} D_j f) - \text{Im}(\gamma^{ij} D_j f) \Gamma_5) \epsilon = 0 .
\]  
We are seeking solutions that preserve two real supercharges. This leads to
consider a set of Killing spinor equations which arise as a special case of the analysis presented in the previous appendix. In particular for the domain wall solution we take (for those $z^i$ directions such that $\gamma^{ij} D_j \bar{f} \neq 0$) $\psi^i = \tau^i = \Omega = \Lambda$ with $|\Lambda| = 1$. The angles $\theta$, $\phi$ defined in the appendix therefore satisfy $\theta = \phi$. Then the Killing spinors are given by

$$\epsilon_1 = B^{1/2} \sin \phi \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) \frac{\sqrt{1 + \cos \phi}} {\sqrt{1 + \cos \phi}}$$

$$\epsilon_2 = -B^{1/2} \sqrt{1 + \cos \phi} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right)$$

for real constants $\alpha$, $\beta$. Clearly the solution preserves $1/2$ of the supersymmetry of $N = 1, D = 4$ theory. The conditions which are implied from the Killing spinor equations on the fields are $\Lambda = e^{i\phi}$, i.e.

$$\partial_y B = -e^{i\phi} B^2 e^{\frac{K}{2}} \bar{f},$$

(C.4)

together with

$$\partial_y z^i = -B^{-1} \partial_y B \bar{f}^{-1} \gamma^{ij} D_j \bar{f}$$

(C.5)

and

$$\partial_y [2\phi + iK] = 2iK_i \partial_y z^i.$$  

(C.6)

C.2. Dilatonic $G_2$ Domain Walls

So far we have considered the general case of domain walls of $N = 1, D = 4$ supergravity associated with a superpotential $f$ which preserve $1/2$ of supersymmetry. Now we shall consider the special case of domain walls in the context of $G_2$ compactifications. In the beginning of the analysis, we shall keep the superpotential $f$ arbitrary but to give some explicit solutions of domain walls, we shall later consider some special cases. We recall that the Kähler potential is given by $K = -\frac{2}{3} \log \Theta$, where $\Theta$ is the volume of the compact $G_2$-manifold, and $\Theta$ is homogeneous of degree $\frac{2}{3}$ in $s$ coordinates of the moduli space $\mathcal{M}$. 

39
To proceed motivated by (6.2), we shall consider solutions for which

\[ s^i = s(y)c^i \]  

(C.7)

for some constants \( c^i \), and \( p^j = 0 \), ie the only modulus field is the volume of the compact manifold. Then (C.6) implies that \( \phi = \phi_0 \), where \( \phi_0 \in \mathbb{R} \) is constant. Defining \( f = e^{i\phi_0}F \), we may write (C.4) and (C.5) as

\[ \partial_y B = -B^2 e^{\frac{K}{2}} \bar{F} \]  

(C.8)

and

\[ \partial_y z^i = B e^{\frac{K}{2}} \gamma^{ij} D_j \bar{F}, \]  

(C.9)

respectively. Furthermore because \( \partial_i K \) is homogeneous of degree \(-1\), we write

\[ K_i = \lambda_i s^{-1} \]  

(C.10)

for some constants \( \lambda_i \). But we also know from section three that

\[ K_i = \frac{3}{7} \Theta^{-1} \partial \Theta / \partial s^i. \]  

(C.11)

It then follows that \( c^i\lambda_i = 1 \). Now (C.9) may be written as

\[ \frac{1}{2} \partial_y K_i = B e^{\frac{K}{2}} (FK_i + \partial F / \partial z^i). \]  

(C.12)

Motivated by this we shall consider \( F \) of the form \( F = F(-2z^i\lambda_i) \), so that

\[ \left( \frac{\partial F}{\partial z^i} \right)_{p=0,s^i=se^i} = -2\lambda_i F' \]  

(C.13)

where \( ' \) denotes differentiation with respect to \( s \). So (C.9) simplifies to

\[ -\frac{1}{4} s^{-\frac{7}{3}} \partial_y (s^{-1}) = B e^{\frac{K}{2}} (s^{-\frac{7}{3}} F)' . \]  

(C.14)

In addition, because \( \Theta \) is homogeneous of degree \( \frac{7}{3} \) in \( s^i \) we have

\[ \Theta = \lambda s^{\frac{7}{3}} \]  

(C.15)

for constant \( \lambda \). Hence \( e^{\frac{K}{2}} = \lambda^{-\frac{3}{14}} s^{-\frac{1}{7}} \). Eliminating \( y \) from (C.8) and (C.14), we
can determine the component $B$ of the metric in terms of the superpotential as

$$B = e^{-\frac{1}{4} \int \frac{W}{\pi W} ds} \quad (C.16)$$

where $W = s^{-\frac{1}{2}}F$. To find the full solution, it remains to substitute (C.16) in (C.14) and solve for $s$. However the resulting equation is rather involved for a general superpotential. To find explicit solutions additional information is needed to describe the superpotential.

**APPENDIX D: G$_2$ and Dilatonic Black Holes**

It is convenient to make an explicit connection between the $N = 1$, $D = 4$ supergravity theory with couplings fixed by $G_2$ compactification of $D = 11$ supergravity and the standard form of Einstein-Maxwell-Dilaton supergravity such as that used in [35]. This supergravity theory has Lagrangian (in 3+1 dimensions)

$$L = \sqrt{-g} \left[ R - 2 |\nabla \varphi|^2 - e^{-2a\varphi} F^2 \right] \quad (D.1)$$

where $\varphi$ is the dilaton, $a$ is the constant dilaton coupling, $F$ is the Maxwell field strength, and the Chern-Simons term has been neglected as we are considering only purely electrically (or magnetically) charged solutions. We recall that the portion of the $N = 1$, $D = 4$ supergravity action containing the curvature and scalar terms $s^i$ is

$$\tilde{L} = \sqrt{-g} \left[ R - 2 k_{ij} \partial_M s^i \partial^M s^j \right]. \quad (D.2)$$

Suppose we consider the special case where the only modulus field is the volume of the compactified $G_2$ manifold, so that $s^i = sc^i$ for constants $c^i$. Then the truncated theory has

$$\tilde{L} = \sqrt{-g} \left[ R - \frac{1}{2} s^{-2} \partial_M s \partial^M s \right]. \quad (D.3)$$

Then setting $s = e^{-2\varphi}$ in (D.3) one obtains the curvature and scalar portions of (D.1) and matching the gauge field couplings of the $N = 1$, $D = 4$ supergravity ac-
tion with (D.1) we observe that the $N = 1, D = 4$ supergravity action is equivalent to (D.1) on taking $a = 1$.

REFERENCES

1. G. Papadopoulos and P.K. Townsend, Compactification of $D = 11$ Supergravity on Spaces of Exceptional Holonomy, Phys. Lett B357 (1995) 300: hep-th/9506150.

2. J. A. Harvey and G. Moore, Superpotentials and Membrane Instantons hep-th/9907020.

3. P. Candelas and X. de la Ossa, Moduli Space of Calabi-Yau Manifolds, Nucl. Phys. B355 (1991) 455.

4. H. Partouche and B. Pioline, Rolling Among $G_2$ Vacua, JHEP 0103:005, 2001; hep-th/0011130.

5. D.D. Joyce, Compact Riemannian 7-Manifolds with Holonomy $G_2$, I, II, J. Diff. Geom. 43 (1996) 291 and 329.

6. C.G. Callan, Jr. and J. Maldacena, D-Brane Approach to Black Hole Quantum Mechanics, Nucl. Phys. B472 (1996) 591; hep-th/9602043.

7. S. Ferrara, R. Kallosh and A. Strominger, $N = 2$ Extremal Black Holes, Phys. Rev. D52 (1995) 5412; hep-th/9508072.

8. G. Moore, Attractors and Arithmetic, hep-th/9807056.

9. N. Hitchin, The geometry of three-forms in six and seven dimensions, math.DG/0010054.

10. A. Comtet and G.W. Gibbons, Bogomolny Bounds for Cosmic Strings Nucl.Phys. B299:719, 1988.

11. B. R. Greene, A. Shapere, C. Vafa, S-T Yau, Stringy Cosmic Strings and Noncompact Calabi-Yau Manifolds, Nucl.Phys. B337:1, 1990.
12. G. Papadopoulos and P.K. Townsend, *Intersecting M-branes*, Phys. Lett. **B380** (1996) 273; hep-th/9603087.

13. M. Cvetič and A.A. Tseytlin, *Non-extreme Black Holes from Non-extreme Intersecting M-Branes*, Nucl. Phys. **B478** (1996) 181; hep-th/9606033.

14. E. Cremmer, S. Ferrara, L. Girardello, A. Van Proeyen, *Yang-Mills Theories with Local Supersymmetry: Lagrangian, Transformation Laws and Superhiggs Effect*, Nucl.Phys. **B212**:413,1983.

15. E. Witten, J. Bagger, *Quantization of Newton’s Constant in Certain Supergravity Theories*, Phys.Lett.**B115**:202,1982.

16. J.A. Bagger and J. Wess, *Supersymmetry and Supergravity*, Princeton University Press, (1992).

17. P. Howe, A. Karhede, U. Lindström and M. Rocek, *The Geometry of Duality*, Phys. Lett. **168B** (1986) 98.

18. E. Cremmer, B. Julia and J. Scherk, *Supergravity in Eleven-Dimensions* Phys. Lett. **B76** (1978) 403.

19. S. Gukov, *Solitons, Superpotentials and Calibrations*, Nucl. Phys. **B574** (2000) 169; [hep-th/9911011].

20. R. Hernandez, *Calibrated Geometries and Non Perturbative Superpotentials in M-Theory*, Eur. Phys. J. **C18** (2001) 619; [hep-th/9912022].

21. B. S. Acharya and B. Spence, *Flux, Supersymmetry and M Theory on 7-Manifolds*, [hep-th/0007213].

22. K. Becker, *Compactifying M-theory to Four-Dimensions*, [hep-th/0010282].

23. R. Harvey and H.B. Lawson, *Calibrated Geometries*, Acta Mathematica **148** (1982) 47.

24. K. Becker, M. Becker and A. Strominger, *Five-Branes, Membranes and Non-Perturbative String Theory*, Nucl. Phys. **B456** (1995) 130; [hep-th/9507158].
25. E. Bergshoeff, R. Kallosh, T. Ortin and G. Papadopoulos, *Kappa Symmetry, Supersymmetry and Intersecting Branes*, Nucl. Phys. **B502** (1997) 149; [hep-th/9705040](http://arxiv.org/abs/hep-th/9705040).

26. R.C. McLean, *Deformations of Calibrated Submanifolds*, Commun. in Analysis and Geometry, **6** (1998) 705.

27. D.D. Joyce, *Compact Manifolds with Special Holonomy* Oxford University Press, (2000).

28. G.W. Gibbons and G. Papadopoulos, *Calibrations and Intersecting Branes*, Commun. Math. Phys. **202** (1999) 593; [hep-th/9803163](http://arxiv.org/abs/hep-th/9803163).

29. J. P. Gauntlett, N. D. Lambert and P.C. West, *Branes and Calibrated Geometries*, Commun. Math. Phys. **202** (1999) 571; [hep-th/9803210](http://arxiv.org/abs/hep-th/9803210).

30. J. Gutowski and G. Papadopoulos, *Magnetic Cosmic Strings of N=1, D = 4 Supergravity with Cosmological Constant*; [hep-th/0102165](http://arxiv.org/abs/hep-th/0102165).

31. J. Polchinski, E. Witten *Evidence for Heterotic - Type I String Duality*, Nucl.Phys. **B460**:525-540,1996; [hep-th/9510169](http://arxiv.org/abs/hep-th/9510169).

32. E. Bergshoeff, M. de Roo, M.B. Green, G. Papadopoulos, P.K. Townsend, *Duality of Type II 7 Branes and 8 Branes*, Nucl.Phys. **B470**:113-135,1996; [hep-th/9601150](http://arxiv.org/abs/hep-th/9601150).

33. R. Schoen and S.T. Yau, *Lectures on Harmonic Maps* Conference Proceedings and Lecture Notes in Geometry and Topology, Vol II, International Press (1997).

34. M. Cvetič and H. H. Soleng, *Supergravity Domain Walls*, Phys. Rep. **282** (1997) 159; [hep-th/9604090](http://arxiv.org/abs/hep-th/9604090).

35. K. Shiraishi, *Moduli Space Metric for Maximally-Charged Dilaton Black Holes*, Nucl. Phys. **B402** (1993) 399.