An $\omega$-categorical structure with amenable automorphism group

A. Ivanov *

Abstract. We analyse $\omega$-categorical precompact expansions of particular $\omega$-categorical structures from the viewpoint of amenability of their automorphism groups.

2010 Mathematics Subject Classification: 03C15, 03E15

Keywords: Amenable groups, Countably categorical structures.

0 Introduction

A group $G$ is called amenable if every $G$-flow (i.e. a compact Hausdorff space along with a continuous $G$-action) supports an invariant Borel probability measure. If every $G$-flow has a fixed point then we say that $G$ is extremely amenable. Let $M$ be a relational countably categorical structure which is a Fraïssé limit of a Fraïssé class $\mathcal{K}$. In particular $\mathcal{K}$ coincides with $\text{Age}(M)$, the class of all finite substructures of $M$. By Theorem 4.8 of [9] the group $\text{Aut}(M)$ is extremely amenable if and only if the class $\mathcal{K}$ has the Ramsey property and consists of rigid elements. Here the class $\mathcal{K}$ is said to have the Ramsey property if for any $k$ and a pair $A < B$ from $\mathcal{K}$ there exists $C \in \mathcal{K}$ so that each $k$-coloring

$$\xi : \binom{C}{A} \rightarrow k$$

*The author is supported by Polish National Science Centre grant DEC2011/01/B/ST1/01406
is monochromatic on some \((B'_A)\) from \(C\) which is a copy of \((B)\), i.e.
\[ C \rightarrow (B)_{k}^{A}. \]

We remind the reader that \((A)\) denotes the set of all substructures of \(C\) isomorphic to \(A\). This result has become a basic tool to amenability of automorphism groups. To see whether \(\text{Aut}(M)\) is amenable one usually looks for an expansion \(M^{*}\) of \(M\) so that \(M^{*}\) is a Fraïssé structure with extremely amenable \(\text{Aut}(M^{*})\). Moreover it is usually assumed that \(M^{*}\) is a precompact expansion of \(M\), i.e. every member of \(K\) has finitely many expansions in \(\text{Age}(M^{*})\), see [9], [10], [12], [1] and [13]. Theorem 9.2 from [11] and Theorem 2.1 from [13] describe amenability of \(\text{Aut}(M)\) in this situation. The question if there is a countably categorical structure \(M\) with amenable automorphism group which does not have expansions as above was formulated by several people. We mention very similar Problems 27, 28 in [2] where precompactness is replaced by \(\omega\)-categoricity and finite homogeneity.

We think that in order to construct a required example one can use the ideas applied in [7] where we construct an \(\omega\)-categorical structure so that its theory is not \(G\)-compact and it does not have AZ-enumerations. These ideas develop ones applied in slightly different forms in [8] and [6] for some other questions. Moreover Casanovas, Pelaez and Ziegler suggest in [3] a general method which simplifies and generalises our approach from [6], [7] and [8]. The basic object of this construction is a particular theory \(T_{E}\) of equivalence relations \(E_{n}\) on \(n\)-tuples. The paper [3] pays attention to several model-theoretic properties of \(T_{E}\).

Below we study \(T_{E}\) from the viewpoint of (extreme) amenability of its expansions. Then we apply our results to a construction of a family of concrete candidates for an example of an \(\omega\)-categorical structure with amenable automorphism group and without \(\omega\)-categorical precompact expansions with extremely amenable automorphism groups. We will in particular show that these structures have the following unusual combination of properties:

- the automorphism group is amenable;
- it does not satisfy Hrushovski’s extension property;
- it does not have an order expansion with the Ramsey property.

In fact we will show a slightly stronger version of the latter property.

1 Equivalence relations

We start with a very interesting reduct of the structure from [7]. This is \(T_{E}\) mentioned in the introduction. It has already deserved some attention in model-theoretic community, see [3].

Let \(L_{0} = \{E_{n} : 0 < n < \omega\}\) be a first-order language, where each \(E_{n}\) is a relational symbol of arity \(2n\). Let \(K_{0}\) be the class of all finite \(L_{0}\)-structures \(C\) where each relation
\( E_n(x, y) \) determines an equivalence relation on the set (denoted by \( \binom{C}{n} \)) of unordered \( n \)-element subsets of \( C \). In particular for every \( n \) the class \( K_0 \) satisfies the sentence

\[
\forall \bar{x} \bar{y} (E_n(x_1, ..., x_n, y_1, ..., y_n) \rightarrow \bigwedge \{ E_n(y_1, ..., y_n, x_{\sigma(1)}, ..., x_{\sigma(n)}) : \sigma \in Sym(n) \}).
\]

Note that for \( C \in K_0 \), \( E_n \) is not satisfied by \( \bar{a}, \bar{b} \) if one of these tuples has a repetition. Thus for \( n > |C| \) we put that no \( 2n \)-tuple from \( C \) satisfies \( E_n(\bar{x}, \bar{y}) \). It is easy to see that \( K_0 \) is closed under taking substructures and the number of isomorphism types of \( K_0 \)-structures of any size is finite.

Let us verify the amalgamation property for \( K_0 \). Given \( A, B_1, B_2 \in K_0 \) with \( B_1 \cap B_2 = A \), define \( C \in K_0 \) as \( B_1 \cup B_2 \) with the finest equivalence relations among those which obey the following rules. When \( n \leq |B_1 \cup B_2| \) and \( \bar{a} \in \binom{B_1}{n} \cup \binom{B_2}{n} \) we put that the \( E_n \)-class of \( \bar{a} \) in \( C \) is contained in \( \binom{B_1}{n} \cup \binom{B_2}{n} \). We also assume that all \( n \)-tuples meeting both \( B_1 \setminus B_2 \) and \( B_2 \setminus B_1 \) are pairwise equivalent with respect to \( E_n \). In particular if \( n \geq \max(|B_1|, |B_2|) \) we put that all \( n \)-element \( n \)-tuples from \( C \) are pairwise \( E_n \)-equivalent.

It is easy to see that this amalgamation also works for the joint embedding property.

Let \( M_0 \) be the countable universal homogeneous structure for \( K_0 \). It is clear that in \( M_0 \) each \( E_n \) defines infinitely many classes and each \( E_n \)-class is infinite. Let \( T_E = Th(M_0) \).

Theorem 1.2 which we prove below, shows that \( M_0 \) cannot be treated by the methods of [9]. It states that the group \( Aut(M_0) \) is amenable but the structure \( M_0 \) does not have a linear ordering so that the corresponding age has the order property and the Ramsey property.

It is worth noting that this statement already holds for the \( \{E_1, E_2\} \)-reduct of \( M_0 \), see the proof below. Thus our theorem also gives some interesting finitely homogeneous examples. On the other hand amenability of \( Aut(M_0) \) is a harder task than the corresponding statement in the reduct’s case.

The statement that \( Aut(M_0) \) is amenable is a consequence of a stronger property, namely \textit{Hrushovski’s extension property} for partial isomorphisms. This is defined for Fraïssé limits as follows.

\textbf{Definition 1.1} A universal ultrahomogeneous structure \( U \) satisfies Hrushovski’s extension property if for any finite family of finite partial isomorphisms between substructures of \( U \) there is a finite substructure \( F < U \) containing these substructures so that any isomorphism from the family extends to an automorphism of \( F \).

Proposition 6.4 of [11] states that the structure \( U \) has Hrushovski’s extension property if and only if \( Aut(U) \) has a dense subgroup which is the union of a countable chain of compact subgroups. The latter implies amenability by Theorem 449C of [4].

\textbf{Theorem 1.2} (a) The structure \( M_0 \) satisfies Hrushovski’s extension property. In particular the group \( Aut(M_0) \) is amenable.
(b) The structure $M_0$ does not have any expansion by a linear order so that $Th(M_0, <)$ admits elimination of quantifiers and the age of $(M_0, <)$ satisfies the Ramsey property.

The proof uses some material from [5]. We now describe it.

Let $L$ be a finite relational language. We say that an $L$-structure $F$ is irreflexive if for any $R \in L$, any tuple from $F$ satisfying $R$ consists of pairwise distinct elements. An irreflexive $L$-structure $F$ is called a link structure if $F$ is a singleton or $F$ can be enumerated $\{a_1, \ldots, a_n\}$ so that $(a_1, \ldots, a_n)$ satisfies a relation from $L$.

Let $\mathcal{S}$ be a finite set of link structures. Then an $L$-structure $N$ is of link type $\mathcal{S}$ if any substructure of $N$ which is a link structure is isomorphic to a structure from $\mathcal{S}$.

An $L$-structure $F$ is packed if any pair from $F$ belongs to a link structure which is a substructure of $F$.

If $\mathcal{R}$ is a finite family of packed irreflexive $L$-structures, then an $L$-structure $F$ is called $\mathcal{R}$-free if there does not exist a weak homomorphism (a map preserving the predicates) from a structure from $\mathcal{R}$ to $F$.

Proposition 4 and Theorem 5 of [5] state that for any family of irreflexive link structures $\mathcal{S}$ and any finite family of irreflexive packed $L$-structures $\mathcal{R}$ the class of all irreflexive finite $L$-structures of link type $\mathcal{S}$ which are $\mathcal{R}$-free, has the free amalgamation property and Hrushovski’s extension property for partial isomorphisms.

We will use a slightly stronger version of this statement concerning permorphisms. A partial mapping $\rho$ on $U$ is called a $\chi$-permorphism, if $\chi$ is a permutation of symbols in $L$ preserving the arity and for every $R \in L$ and $\bar{a} \in Dom(\rho)$ we have
\[
\bar{a} \in R \Leftrightarrow \rho(\bar{a}) \in R^\chi.
\]

The following statement is a version of Lemma 6 from [5].

**Lemma 1.3** Let $L$ be a finite language, $\chi_1, \ldots, \chi_n$ be arity preserving permutations of $L$ and $\mathcal{S}$ be a finite $\{\chi_i\}_{i \leq n}$-invariant family of irreflexive link structures. Let $\mathcal{R}$ be a finite family of finite irreflexive packed $L$-structures of link type $\mathcal{S}$ so that $\mathcal{R}$ is invariant under all $\chi_i$. Let $A$ be a finite structure which belongs to the class, say $K$, of $L$-structures of link type $\mathcal{S}$ which are $\mathcal{R}$-free. Let $\rho_i, i \leq n$, be partial $\chi_i$-permorphisms of $A$.

Then there is a finite $B \in K$ containing $A$ so that each $\rho_i$ extends to a permutation of $B$ which is a $\chi_i$-permorphism.

**Proof of Theorem 1.2** (a) For each $n > 0$ enumerate all $E_n$-classes. Consider the expansion of $M_0$ by distinguishing each $E_n$-class by a predicate $P_{n,i}$ according the enumeration. Let $L^*$ be the language of all predicates $P_{n,i}$ and let $M^*$ be the $L^*$-structure defined on $M_0$. For every finite sublanguage $L' \subseteq L^*$ let $M^*(L')$ be the $L'$-reduct of $M^*$ defined by these interpretations.

We denote by $\mathcal{K}(L')$ the class of all finite $L'$-structures with the properties that for any arity $l$ represented by $L'$:
• any $l$-relation is irreflexive and invariant with respect to all permutations of variables,

• any two relations of $L'$ of arity $l$ have empty intersection.

Let $S(L')$ be the set of all link structures of $K(L')$ satisfying these two properties. Thus $K(L')$ is of link type $S(L')$.

Claim 1. For every finite sublanguage $L' \subseteq L^*$ the structure $M^*(L')$ is a universal structure with respect to the class $K(L')$.

It is easy to see that any structure from $K(L')$ can be expanded to a structure from $K_0$ so that $L'$-predicates become classes of appropriate $E_n$'s.

Claim 2. For every finite sublanguage $L' \subseteq L^*$ the structure $M^*(L')$ is an ultrahomogeneous structure.

Let $f$ be an isomorphism between finite substructures of $M^*(L')$. We may assume that $\text{Dom}(f)$ contains tuples representing all $M^*(L')$-predicates of $L'$ (some disjoint tuples can be added to $\text{Dom}(f)$ in a suitable way). Then $f$ extends to an automorphism of $M_0$ fixing the classes of appropriate $E_n$'s which appear in $L'$. Thus this automorphism is an automorphism of $M^*(L')$ too.

Claim 3. For each finite sublanguage $L' \subseteq L^*$ let $R(L')$ be the family of all packed $L'$-structures of the form $\{(a_1, ..., a_n), P_{n,i}, P_{n,j}\}$, where $i \neq j$, $P_{n,i} = \{(a_1, ..., a_n)\}$ and $P_{n,j} = \{\sigma(a_1, ..., a_{\sigma(n)})\}$ for some permutation $\sigma$. Then the class $K(L')$ is the class of all irreflexive finite $L'$-structures of link type $S(L')$, which are $R(L')$-free.

The claim is obvious. By Proposition 4 and Theorem 5 of [5] we now see that $K(L')$ is closed under substructures, has the joint embedding property, the free amalgamation property, Hrushovski’s extension property and its version for permorphisms, i.e. the statement of Lemma 1.3.

By Claim 1 and Claim 2 the structure $M^*(L')$ is the universal homogeneous structure of $K(L')$. In particular any tuple of finite partial isomorphisms (permorphisms) of $M^*(L')$ can be extended to a tuple of automorphisms (permorphisms) of a finite substructure of $M^*(L')$.

Note that the same statement holds for the structure $M^*$. To see this take any tuple $f_1, ..., f_k$ of finite partial isomorphisms (resp. $\chi_i$-permorphisms) of $M^*$ (assuming that $\chi_i$ are finitary). Let $r$ be the size of the union $\bigcup_{i \leq k} \text{Dom}(f_i)$ and $L'$ be the minimal (resp. $\{\chi_i\}_{i \leq k}$-invariant) sublanguage of $L^*$ of arity $r$ containing all relations of $M^*$ which meet any tuple from $\bigcup_{i \leq k} \text{Dom}(f_i)$. Then there is a finite substructure $A$ of $M^*(L')$ containing $\bigcup_{i \leq k} \text{Dom}(f_i)$ so that each $f_i$ extends to an automorphism (resp. $\chi_i$-permorphism) of $A$.

Let $r'$ be the size of $A$. Let $L''$ be a sublanguage of $L^*$ so that $L' \subseteq L''$ and for each arity $l \leq r'$ the sublanguage $L'' \setminus L'$ contains exactly one $l$-relation, say $P_{l,n_i}$ (fixed by $\{\chi_i\}_{i \leq k}$). Since $M^*$ is the universal homogeneous structure of $K(L'')$ the substructure $A$ can be chosen so that any $l$-subset of $A$ which does not satisfy any relation from $L'$, does satisfy $P_{l,n_i}$.

As a result any automorphism (permorphism) of $A$ preserves the relations of $L'' \setminus L'$ for any $L'' \subseteq L^*$ containing $L''$. Thus it extends to an automorphism (permorphism) of $M^*(L'')$. In particular it extends to an automorphism (permorphism) of $M^*$.
As in Proposition 6.4 of [11] we see that $Aut(M^*)$ has a dense subgroup which is the union of a countable chain of compact subgroups. In particular we arrive at the following statement.

**Claim 4.** $Aut(M^*)$ is amenable.

Since each automorphism of $M_0$ is a peromorphism of $M^*$ and vice versa, we also see that $Aut(M_0)$ has a dense subgroup which is the union of a countable chain of compact subgroups. In particular $Aut(M_0)$ is amenable.

(b) Consider a linearly ordered expansion $(M_0, <)$ together with the corresponding age, say $K^<$. Assume that $K^<$ has the Ramsey property.

Note that $K^<$ does not contain any three-element structure of the form $a < b < c$, where $a$ and $c$ belong to the same $E_1$-class which is distinct from the $E_1$-class of $b$. Indeed, otherwise repeating the argument of Theorem 6.4 from [9], we see that in any larger structure from $K^<$ we can colour two-elements structures $a < b$ with $\neg E_1(a, b)$, so that there is no monochromatic three-element structure of the form above.

As a result we see that any $E_1$-class of $(M_0, <)$ is convex. We now claim that the following structure $B$ can be embedded into $(M_0, <)$.

Let $B = \{a_1 < a_2 < a_3 < a_4 < b_1 < b_2\}$, where the $E_1$-classes of all elements are pairwise distinct, but the pairs $\{a_1, a_2\}$ and $\{b_1, b_2\}$ are $E_2$-equivalent. We assume that in all other cases any two distinct pairs from $B$ belong to distinct $E_2$-classes. Moreover we assume that for each $k = 3, 4, 5$ all $k$-subsets from $B$ belong to the same $E_k$-class. In particular the ordered structures defined on $\{a_1, a_2, a_3, a_4\}$ and $\{a_3, a_4, b_1, b_2\}$ are isomorphic. Let $A$ represent this isomorphism class.

Since $M_0$ is the universal homogeneous structure with respect to $K_0$, taking any tuple $a_1' < a_2' < a_3' < a_4' < b_1' < b_2'$ with pairwise distinct $E_1$-classes we can find $B$ in $M_0$ as a half of a copy of a structure from $K_0$ consisting of 12 elements where each $E_1$-class is represented by a pair $(a_i', a_i)$ or $(b_i', b_i)$.

To show that the Ramsey property does not hold for the age of $(M_0, <)$ take any finite substructure $C$ of this age which extends $B$. Fix any enumeration of $E_2$-classes occurring in $C$. Then colour a copy of $A$ red if the class of the first two elements is enumerated before the class of the last pair. Otherwise colour such a copy green. It is clear that $C$ does not contain a structure isomorphic to $B$ so that all substructures of type $A$ are of the same colour. □

**Remark 1.4** It is worth noting that the class $K^<_0$ of all linearly ordered members of $K_0$ has JEP and AP, i.e. there is a generic expansion of $M_0$ by a linear ordering. To see AP we just apply the amalgamation described above together with the standard amalgamation of orderings.

## 2 Adding dense linear orders

In order to obtain a structure with the properties as in Section 1, but without Hrushovski’s extension property we use a general approach from [3]. In fact our starting point is Corollary 2.8 from [3] that sets $(\frac{M_0}{E_n})/E_n$ (definable in $Th^{eq}(M_0)$) are
stably embedded in $M_0$.

We remind the reader that a 0-definable predicate $P$ of a theory $T$ is called **stably embedded** if every definable relation on $P$ is definable with parameters from $P$. If $M$ is a saturated model of $T$ then $P$ is stably embedded if and only if every elementary permutation of $P(M)$ extends to an automorphism of $M$ (see remarks after Definition 2.4 in [3]). We now formulate Lemma 3.1 from [3].

Let $T$ be a complete theory with two sorts $S_0$ and $S_1$. Let $\tilde{T}_1$ be a complete expansion of $T \upharpoonright S_1$. Assume that $S_1$ is stably embedded. Then

1. $T = T \cup \tilde{T}_1$ is a complete theory;
2. $S_1$ is stably embedded in $\tilde{T}$ and $\tilde{T} \upharpoonright S_1 = \tilde{T}_1$;
3. if $T$ and $\tilde{T}_1$ are $\omega$-categorical, then $\tilde{T}$ is also $\omega$-categorical.

We now describe our **variations** of $M_0$. Let us fix $S_n = (\mathcal{M}_n)/E_n$, $n \in \omega$, and consider them as a sequence of stably embedded sorts in $Th^{eq}(M_0)$ (this is Corollary 2.8 of [3]). We can distinguish relations $\{a_1, ..., a_n\} \in e$, where $e \in S_n$ is an $E_n$-class, $n \in \omega$.

We also fix a subset $P \subset \omega \setminus \{1, 2\}$ and consider the language

$$L^S_P = \{E_n : 0 < n \in \omega\} \cup \{S_n, <_{S_n} : n \in P\},$$

where $<_{S_n}$ are binary relations on $S_n$. Let $\tilde{T}_1$ be the theory of sorts $\{S_n : n \in \omega\}$, where for every $n \in P$ the relation $<_{S_n}$ is a dense linear order without ends. When $n \notin P$ the sort $S_n$ is considered as a pure set. This is an $\omega$-categorical theory for each $S_n$. Applying Lemma 3.1 from [3] we define the complete theory $T^S_P = T_E \cup \tilde{T}_1$ which is $\omega$-categorical and every sort $S_n$ is stably embedded into $T^S_P$.

We now define an one-sorted version of $T^S_P$. Its countable model will be the example anounced in Introduction.

Let $L_P = \{E_n : 0 < n \in \omega\} \cup \{<_n : n \in P\}$ be a first-order language, where each $E_n$ and $<_n$ is a relational symbol of arity $2n$. The $L_P$-structure $M$ is built by the Fraïssé’s construction. Let us specify a class $K_P$ of finite $L_P$-structures, which will become the class of all finite substructures of $M$.

Assume that in each $C \in K_P$ each relation $E_n(\bar{x}, \bar{y})$ determines an equivalence relation on the set (denoted by $\binom{C}{n}$) of unordered $n$-element subsets of $C$. As before for $C \in K_P$ and $n > |C|$ we put that no $2n$-tuple from $C$ satisfies $E_n(\bar{x}, \bar{y})$.

For $n \in P$ the relations $<_n$ are irreflexive and respect $E_n$:

$$\forall \bar{x}, \bar{y}, \bar{u}, \bar{w}(E_n(\bar{x}, \bar{y}) \land E_n(\bar{u}, \bar{w}) \land <_n (\bar{x}, \bar{u}) \to <_n (\bar{y}, \bar{w})).$$

Every $<_n$ is interpreted by a linear order on the set of $E_n$-classes. Therefore we take the corresponding axioms (assuming below that tuples consist of pairwise distinct elements):

$$\forall \bar{x}, \bar{y}(<_n (\bar{x}, \bar{y}) \to \neg E_n(\bar{x}, \bar{y}));$$

$$\forall \bar{x}, \bar{y}, \bar{z}(<_n (\bar{x}, \bar{y}) \wedge <_n (\bar{y}, \bar{z}) \to <_n (\bar{x}, \bar{z}));$$

$$\forall \bar{x}, \bar{y}(\neg E_n(\bar{x}, \bar{y}) \to <_n (\bar{x}, \bar{y}) \lor <_n (\bar{y}, \bar{x})).$$
Lemma 2.1  (1) The class $\mathcal{K}_P$ satisfies the joint embedding property and the amalgamation property.

(2) Let $M$ be the generic structure of $\mathcal{K}_P$. For every $n > 0$ let $M_n = (\binom{n}{3})/E_n$. Then $\text{Th}(M)$ is $\omega$-categorical, admits elimination of quantifiers, and $<_n$ is a dense linear ordering on $M_n$ without ends (when $n \in P$). The structure $M$ is an expansion of $M_0$.

(3) Let $\rho_i$, $i \leq k$, be a sequence of finitary maps on $M_i$ which respect $<_i$ for $i \in P$. Then there is an automorphism $\alpha \in \text{Aut}(M)$ realising each $\rho_i$ on its domain.

Proof. (1) Given $A, B_1, B_2 \subseteq C$ with $B_1 \cap B_2 = A$, define $C \subseteq K$ as $B_1 \cup B_2$. The relations $E_n, <_n, n \leq |B_1 \cup B_2|$, are defined so that $C \subseteq K, B_1 < C, B_2 < C$ and the following conditions hold. Let $n \leq |B_1 \cup B_2|$. We put that all $n$-element $n$-tuples meeting both $B_1 \setminus B_2$ and $B_2 \setminus B_1$ are pairwise equivalent with respect to $E_n$. We additionally demand that they are equivalent to some tuple from some $B_i, i \in \{1, 2\}$, if $n \leq \max(|B_i|, |B_2|)$. If for some $i \in \{1, 2\}$, $|(\binom{n}{3})/E_n| = 1$, then we put that all $n$-tuples $\bar{c} \in B_1 \cup B_2$ meeting $B_i$ are pairwise $E_n$-equivalent. We additionally arrange that they are equivalent to some tuple from $B_{3-i}$ if $n \leq |B_{3-i}|$. If $n \geq \max(|B_1|, |B_2|)$ then all $n$-element $n$-tuples from $C$ are pairwise $E_n$-equivalent. We take $E_n$ to be the minimal equivalence relation satisfying the conditions above. In particular if $n$-tuples $\bar{b}_1$ and $\bar{b}_2$ are $E_n$-equivalent to the same $n$-tuple from $A$, then $E_n(\bar{b}_1, \bar{b}_2)$.

We can now define the linear orderings $<_n$ on $C/E_n$ for $n \in P$. There is nothing to do if $|(\binom{n}{3})/E_n| = 1$. In the case when for some $i = 1, 2$, $|(\binom{n}{3})/E_n| = 1$, the relation $<_n$ is defined by its restriction to $B_{3-i}$. When $|(\binom{n}{3})/E_n| \neq 1 \neq |(\binom{n}{3})/E_n|$ and $V_1, V_2$ is a pair of two $<_n$-neighbours among $E_n$-classes having representatives both in $B_1$ and $B_2$, we amalgamate the $<_n$-linear orderings between $V_1$ and $V_2$ assuming that all elements of $(\binom{n}{3})/E_n \cap [V_1, V_2]$ are less than those from $(\binom{n}{3})/E_n \cap [V_1, V_2]$.

We appropriately modify this procedure for intervals open from one side. It is clear that this defines $<_n$-ordering on $(\binom{n}{3})/E_n$.

(2) The statement that $\text{Th}(M)$ admits elimination of quantifiers and is $\omega$-categorical, follows from (1). This also implies that $M$ is a natural expansion of $M_0$.

To see the second statement of this part of the lemma it is enough to show that for $n \in P$ and any two sequences $V_1 <_n ... <_n V_k$ and $V'_1 <_n ... <_n V'_k$ from $M_n$ there is an automorphism of $M$ taking each $V_i$ to $V'_i$. To see this we use the fact that $M$ is the Fra"issé limit of $\mathcal{K}_P$. This allows us to find pairwise disjoint representatives of classes $V_1, ..., V_k, \bar{a}_1, ..., \bar{a}_k$, and classes $V'_1, ..., V'_k, \bar{a}'_1, ..., \bar{a}'_k$, so that for every $m \neq n$ all $m$-tuples of the substructures $\bar{a}_1 \cup ... \cup \bar{a}_k$ and $\bar{a}'_1 \cup ... \cup \bar{a}'_k$ are $E_m$-equivalent. Moreover all $n$-tuples meeting at least two $\bar{a}_s, \bar{a}_t$ or $\bar{a}'_s, \bar{a}'_t$ also belong to a single $E_n$-class. Taking an appropriate isomorphism induced by these representatives we extend it to a required automorphism.

(3) We develop the argument of (2). For each $\rho_i$ find a sequence $\bar{a}_1, ..., \bar{a}_t$ of pairwise disjoint tuples from $M$ representing the $E_i$-classes of the domain and of the range of $\rho_i$. We may assume that for any $j \neq i$ all $j$-tuples of the union $\Omega_i = \bar{a}_1 \cup ... \bar{a}_t$ belong to the same $E_j$-class. Moreover all $i$-tuples meeting at least two $\bar{a}_i, \bar{a}_m$ also form a single $E_i$-class. Thus $\rho_i$ can be realised by a partial map on $\Omega_i$. We may
arrange that all $\Omega_i$ are pairwise disjoint and do not have common $E_n$-classes. Thus all $\rho_i$ can be realised by a partial isomorphism on the union of these $\Omega_i$. Since $M$ is ultrahomogeneous, this partial isomorphism can be extended to an automorphism of $M$. □

Let us consider $M$ in the language $L_P^S$, i.e.

$$(M, E_1, ..., E_n, ...) \cup (M_1, *_1) \cup ... \cup (M_n, *_n) \cup ... ,$$

where $*_n = <_n$ for $n \in P$ and disappears for $n \notin P$. By Lemma 2.1(3) the structure of all sorts \{ $M_n : n \in \omega$ \} coincides with the theory $\tilde{T}_1$ of sorts \{ $S_n : n \in \omega$ \} of the theory $T_P^S$. This implies the following corollary.

**Corollary 2.2** The theory of $M$ in the language $L_P^S$ coincides with $T_P^S$. In particular the sets $M_n$ are stably embedded into $M$.

We see that for $n \in P$ any automorphism of $(M_n, <_n)$ can be realized by an automorphism of $M$. Assume that $2n \notin P$. Let us consider automorphisms $\alpha$ of $M_n$ which are increasing, i.e. for any $V \in M_n$, $V <_n \alpha(V)$.

Take an orbit of $\alpha$ of the following form:

$$... \rightarrow \bar{a}_{-1} \rightarrow \bar{a}_0 \rightarrow \bar{a}_1 \rightarrow \bar{a}_2 \rightarrow \bar{a}_3 \rightarrow \bar{a}_4 \rightarrow ...$$

and consider $E_{2n}$-classes of tuples $\bar{a}_i \bar{a}_{i+1}$. Applying ultrahomogenity and the choice of $n$ it is easy to see that $\alpha$ can be taken so that there are four $E_{2n}$-classes, say $V_1, V_2, V_3, V_4$, represented by consecutive pairs of tuples $\bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4, \bar{a}_5, \bar{a}_6$ and $\alpha$ acts on them by $\mathbb{Z}/4\mathbb{Z}$:

if $\bar{a}_1 \bar{a}_2 \in V_1$, then $\bar{a}_2 \bar{a}_3 \in V_2$, $\bar{a}_3 \bar{a}_4 \in V_3$ and $\bar{a}_4 \bar{a}_5 \in V_4$,

where $\bar{a}_1 \bar{a}_2$ and $\bar{a}_5 \bar{a}_6$ are $E_{2n}$-equivalent.

Slightly generalizing this situation we will say that a sequence $\bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4, \bar{a}_5, \bar{a}_6$ is $<_n$-**increasing of type** $\mathbb{Z}/4\mathbb{Z}$ if the following conditions are satisfied:

- tuples $\bar{a}_1 \bar{a}_2, \bar{a}_2 \bar{a}_3$ and $\bar{a}_3 \bar{a}_4$ are of the same isomorphism type,
- tuples $\bar{a}_1 \bar{a}_2 \bar{a}_3 \bar{a}_4$ and $\bar{a}_3 \bar{a}_4 \bar{a}_5 \bar{a}_6$ are of the same isomorphism type and
- tuples $\bar{a}_1 \bar{a}_2$ and $\bar{a}_5 \bar{a}_6$ are $E_{2n}$-equivalent but not $E_{2n}$-equivalent to $\bar{a}_3 \bar{a}_4$.

Let $L'$ be an extension of $L_P$ and $M' = (M, \bar{r})$ be an $L'$-expansion of $M$ with quantifier elimination. We do not demand that $\bar{r}$ is finite, we only assume that $M'$ is a precompact expansion. It is clear that $M'$ induces a subgroup of $Aut(M_n, <_n)$.

We will say that a sequence $\bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4, \bar{a}_5, \bar{a}_6$ is $<_n$-**increasing of type** $\mathbb{Z}/4\mathbb{Z}$ in $M'$ if the definition above holds under the assumption that the isomorphism types appeared in the definition are considered with respect to the relations of $M'$.
Theorem 2.3 Let $M$ be the generic structure of $K_P$ where $P \neq \emptyset$. Then the group $G = \text{Aut}(M)$ is amenable, $M$ does not satisfy Hrushovski’s extension property and does not have an extremely amenable ultrahomogeneous expansion by a linear ordering.

Let $M'$ be a precompact expansion of $M$ with quantifier elimination. If $\text{Aut}(M')$ is extremely amenable, then for any $n \in P$ with $2n \notin P$ the structure $M'$ does not have an $<_n$-increasing sequence of type $\mathbb{Z}/4\mathbb{Z}$.

The main point of this theorem is that although in different arities the structures induced by $M$ are completely independent, any expansion $M'$ as in the formulation simultaneously destroys $M$ in all arities $n \in P$ with $2n \notin P$.

The proof below uses the proof of Theorem 1.2.

Proof of Theorem 2.3. For each $n > 1$ enumerate all $E_n$-classes. Consider the expansion of $M$ by distinguishing each $E_n$-class by a predicate $P_{n,i}$ according the enumeration. Let $L^*$ be the language of all predicates $P_{n,i}$ and let $M^*$ be the $L^*$-structure defined on $M$. By Claims 1 - 4 of the proof of Theorem 1.2 the structure $M^*$ has Hrushovski’s extension property and $\text{Aut}(M^*)$ is amenable.

Let us consider the structure $(M_n, <_n)$, where $n \in P$. As it is isomorphic to $(\mathbb{Q}, <)$, the group $\text{Aut}(M_n, <_n)$ is extremely amenable (9).

Since each automorphism of $M$ preserves all $<_i$, $i \in P$, it is easy to see that there is a natural homomorphism from $\text{Aut}(M)$ to the product

$$\prod_{i \in P} \text{Aut}(M_i, <_i) \times \prod_{i \notin P} \text{Sym}(M_i)$$

and $\text{Aut}(M^*)$ is the kernel of it. By Corollary 2.2 this homomorphism is surjective. Now by Theorem 449C of [1] we have the following claim.

The group $\text{Aut}(M)$ is amenable.

To see that $M$ does not satisfy Hrushovski’s extension property take $n \in P$ and let us consider any triple of pairwise disjoint $n$-tuples $a, b, c$ representing pairwise distinct elements of $M_n$ so that $a <_n b <_n c$. Then the map $\phi$ fixing $a$ and taking $b$ to $c$ cannot be extended to an automorphism of a finite substructure of $M$.

Consider a linearly ordered expansion $(M, <)$ with quantifier elimination. To see that $\text{Aut}(M, <)$ is not extremely amenable just apply the argument of statement (b) of Theorem 1.2. Since at arity 2 the structure $M$ coincides with $M_0$ it works without any change.

To prove the second part of the theorem we slightly modify that argument.

Let $n \in P$ and $2n \notin P$. Let a structure $B$ consist of $6n$ elements forming a sequence

$$\bar{a}_1 <_n \bar{a}_2 <_n \bar{a}_3 <_n \bar{a}_4 <_n \bar{b}_1 <_n \bar{b}_2,$$

where the tuples $\bar{a}_1 \bar{a}_2$ and $\bar{b}_1 \bar{b}_2$ are $E_{2n}$-equivalent but not of the same $E_{2n}$-class with $\bar{a}_3 \bar{a}_4$. We assume that the tuples $\bar{a}_1 \bar{a}_2$, $\bar{a}_2 \bar{a}_3$, and $\bar{a}_3 \bar{a}_4$ are of the same isomorphism class in $M'$ and the substructure $\bar{a}_1 \bar{a}_2 \bar{a}_3 \bar{a}_4 < M'$ is isomorphic to $\bar{a}_3 \bar{a}_4 \bar{b}_1 \bar{b}_2 < M'$. Since
$Aut(M')$ is extremely amenable, these structures are rigid and the corresponding isomorphisms are uniquely defined on these tuples.

Let $A$ represent the isomorphism class of $\bar{a}_1\bar{a}_2\bar{a}_3\bar{a}_4$ in $M'$. Let us show that the Ramsey property does not hold for the age of $M'$. Take any finite substructure $C$ of this age which extends $B$. Fix any enumeration of $E_{2n}$-classes occuring in $C$. Then colour a copy of $A$ red if the class of the first two $n$-tuples is enumerated before the class of the last pair. Otherwise colour such a copy green. It is clear that $C$ does not contain a structure isomorphic to $B$ so that all substructures of type $A$ are of the same colour. $\square$

References

[1] O.Angel, A.Kechris and R.Lyone, *Random orderings and unique ergodicity of automorphism groups*, to appear in Jour. Europ. Math. Soc., ArXiv: 1208.2389

[2] M.Bodirsky, M.Pinsker and T.Tsankov, *Decidability of definability*, In: Proceedings of the 26-th Annual IEEE Symposium on Logic in Computer Science (LICS’11), IEEE Computer Society, 2011, 321 - 328.

[3] E.Casanovas, R.Pelaez and M.Ziegler, *On many-sorted $\omega$-categorical theories*, Fund Math. 214(2011), 285 - 294.

[4] D.H.Fremlin, *Measure Theory, vol 4. Topological measure spaces*. Part I,II. Torres Fremlin, Colchester, 2006

[5] B. Herwig, *Extending partial isomorphisms for the small index property of many $\omega$-categorical structures*, Israel J. Math. 107 (1998), 93 - 123.

[6] A.Ivanov, *Automorphisms of homogeneous structures*, Notre Dame Journal of Formal Logic, 46 (2005), no.4, 419 – 424.

[7] A.Ivanov, *A countably categorical theory which is not $G$-compact*, Siberian Adv. in Math. 20(2010), 75 - 82.

[8] A.Ivanov and H.D.Macpherson, *Strongly determined types*, Ann. Pure and Appl. Logie 99(1999), 197 - 230.

[9] A.Kechris, V.Pestov and S.Todorcevic, *Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups*, GAFA, 15(2005), 106 -189.

[10] A.Kechris and M.Sokić, *Dynamical properties of the automorphism groups of the random poset and random distributive lattice*, Fund Math. 218(2012), 69 - 94.

[11] A.Kechris and Ch.Rosendal, *Turbulence, amalgamation, and generic automorphisms of homogeneous structures*, Proc. London Math. Soc. (3) 94(2007), 302 - 350.
[12] L.N. van Th´e, More on KeCHRIS-PeStov-ToDorCEVic cORRESPONDENCE: preCOMPACT expansions, Fund. Math. 222(2013), 19 - 47.

[13] A.Zucker, Amenability and unique ergodicity of automorphism groups of Fraïssé structures, arXiv:1304.2839

Institute of Mathematics, University of Wroclaw, pl.Grunwaldzki 2/4, 50-384 Wroclaw, Poland,
E-mail: ivanov@math.uni.wroc.pl