WEIGHTS AND NILPOTENT SUBGROUPS
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Abstract. In a finite group $G$, we consider nilpotent weights, and prove a $\pi$-version of the Alperin Weight Conjecture for certain $\pi$-separable groups. This widely generalizes an earlier result by I. M. Isaacs and the first author.

1. Introduction

Let $G$ be a finite group and let $p$ be a prime. The celebrated Alperin Weight Conjecture asserts that the number of conjugacy classes of $G$ consisting of elements of order not divisible by $p$ is exactly the number of $G$-conjugacy classes of $p$-weights. Recall that a $p$-weight is a pair $(Q, \gamma)$, where $Q$ is a $p$-subgroup of $G$ and $\gamma \in \text{Irr}(N_G(Q)/Q)$ is an irreducible complex character with $p$-defect zero (that is, such that the $p$-part $\gamma(1)_p = |N_G(Q)/Q|_p$).

In the main result of this paper, we replace $p$ by a set of primes $\pi$ as follows:

Theorem A. Let $G$ be a $\pi$-separable group with a solvable Hall $\pi$-subgroup. Then the number of conjugacy classes of $\pi'$-elements of $G$ is the number of $G$-conjugacy classes of pairs $(Q, \gamma)$, where $Q$ is a nilpotent $\pi$-subgroup of $G$ and $\gamma \in \text{Irr}(N_G(Q)/Q)$ has $p$-defect zero for every $p \in \pi$.

Recall that a finite group is called $\pi$-separable if all its composition factors are $\pi$-groups or $\pi'$-groups. Let us restate Theorem A in the (presumably trivial) case where $G$ itself is a (solvable) $\pi$-group. In this case, there is only one conjugacy class of $\pi'$-elements of $G$. On the other hand, if $Q$ is a nilpotent subgroup of $G$, then $\gamma \in \text{Irr}(N_G(Q)/Q)$ has $p$-defect zero for every $p \in \pi$ if and only if $N_G(Q) = Q$. Amazingly enough, there is only one conjugacy class of self-normalizing nilpotent subgroups: the Carter subgroups of $G$ (see p. 281 in [R]).

Of course, if $\pi = \{p\}$, then Theorem A is the $p$-solvable case of the Alperin Weight Conjecture (AWC). As a matter of fact, AWC was proven for $\pi$-separable groups with a nilpotent Hall $\pi$-subgroup by Isaacs and the first author [IN]. Now we realize that

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the nilpotency hypothesis can be dropped if one counts nilpotent weights instead. The solvability hypothesis is still needed, as shown by $G = A_5$ and $\pi = \{2, 3, 5\}$.

There is a price to pay, however. The proof in [IN] relied on the so called Okuyama–Wajima argument, a definitely non-trivial but accessible tool involving extensions of Glauberman correspondents. In order to prove Theorem A, however, we shall need to appeal to a deeper theorem of Dade and Puig (which uses Dade’s classification of the endo-permutation modules).

As it is often the case in a “solvable” framework, the equality of cardinalities in Theorem A has a hidden structure which we are going to explain now. For sake of convenience we interchange from now on the roles of $\pi$ and $\pi'$ (of course, $\pi$-separable is equivalent to $\pi'$-separable). Recall that in a $\pi$-separable group $G$, the set $I_\pi(G)$ of irreducible $\pi$-partial characters of $G$ is the exact $\pi$-version (when $\pi$ is the complement of a prime $p$) of the irreducible Brauer characters $\text{IBr}(G)$ of a $p$-solvable group (see next section for precise definitions). Each $\varphi \in I_\pi(G)$ has canonically associated a $G$-conjugacy class of $\pi'$-subgroups $Q$, which are called the vertices of $\varphi$. If $I_\pi(G|Q)$ is the set of irreducible $\pi$-partial characters with vertex $Q$, unless $\pi = p'$, it is not in general true that $|I_\pi(G|Q)| = |I_\pi(N_G(Q)|Q)|$. Instead we will prove the following theorem.

**Theorem B.** Suppose that $G$ is $\pi$-separable with a solvable Hall $\pi$-complement. Let $R$ be a nilpotent $\pi'$-subgroup of $G$ and let $Q$ be the set of $\pi'$-subgroups $Q$ of $G$ such that $R$ is a Carter subgroup of $Q$. Then

$$\left| \bigcup_{Q \in Q} I_\pi(G|Q) \right| = |I_\pi(N_G(R)|R)|.$$

Since $|I_\pi(N_G(R)|R)|$ is just the number of $\pi'$-weights with first component $R$ (see Lemma 6.28 of [I2]), Theorem B implies Theorem A.

As happens in the classical case where $\pi = p'$, and following the ideas of Dade, Knörr and Robinson, one can define chains of $\pi'$-subgroups and relate them with $\pi$-defect of characters. This shall be explored elsewhere. Similarly, one can attach every weight to a $\pi'$-block $B$ of $G$ by using Slattery’s theory [S]. In this setting we expect that the number of $\pi$-partial characters belonging to $B$ equals the number of nilpotent weights attached to $B$.

The groups described in Theorem A are sometimes called $\pi$-solvable. We did not find a counterexample in the wider class of so-called $\pi$-selected groups. Here, $\pi$-selected means that the order of every composition factor is divisible by at most one prime in $\pi$. P. Hall [H] has shown that these groups still have solvable Hall $\pi$-subgroups. Since every finite group is $p$-selected for every prime $p$, this version of the conjecture includes the full AWC.

Unfortunately, Theorem A does not hold for arbitrary groups even if they possess nilpotent Hall $\pi$-subgroups. It is not so easy to find a counterexample, though.
The fourth Janko group $G = J_4$ has a cyclic Hall $\pi$-subgroup of order 35 (that is, $\pi = \{5, 7\}$). The normalizers of the non-trivial $\pi$-weights are contained in a maximal subgroup $M$ of type $2^{3+12} \cdot (S_5 \times L_3(2))$. However, $l(G) - k_0(G) = 25 \neq 30 = l(M) - k_0(M)$, where $l(G)$ denotes the number of $\pi'$-conjugacy classes and $k_0(G)$ is the number of $\pi$-defect zero characters of $G$. (The fact that $J_4$ was a counterexample for the $\pi$-version of the McKay conjecture for groups with a nilpotent Hall $\pi$-subgroup was noticed by Pham H. Tiep and the first author.)

We take the opportunity to thank the developers of [GAP]. Without their tremendous work the present paper probably would not exist.

The paper is organized as follows: In the next section we review $\pi$-partial characters which were introduced by Isaacs. In Section 3 we present two general lemmas on characters in $\pi$-separable groups. Afterwards we prove Theorem B. In the final section we construct a natural bijection explaining Theorem B in the presence of a normal Hall $\pi$-subgroup.

2. Review of $\pi$-theory

Isaacs’ $\pi$-theory is the $\pi$-version in $\pi$-separable groups of the $p$-modular representation theory for $p$-solvable groups. When $\pi = p'$, the complement of a prime, then $I_{\pi}(G) = IBr(G)$ and we recover most of the well-known classical results. In what follows $G$ is a finite $\pi$-separable group, where $\pi$ is a set of primes. All the references for $\pi$-theory can now be found together in Isaacs’ recent book [I2]. For the reader’s convenience, we review some of the main features. If $n$ is a natural number and $p$ is a prime, recall that $n_p$ is the largest power of $p$ dividing $n$. If $\pi$ is a set of primes, then $n_\pi = \prod_{p \in \pi} n_p$. The number $n$ is a $\pi$-number if $n = n_\pi$.

If $G$ is a $\pi$-separable group, then $G^0$ is the set of elements of $G$ whose order is a $\pi$-number. A $\pi$-partial character of $G$ is the restriction of a complex character of $G$ to $G^0$. A $\pi$-partial character is irreducible if it is not the sum of two $\pi$-partial characters. We write $I_{\pi}(G)$ for the set of irreducible $\pi$-partial characters of $G$. Notice that if $\mu \in I_{\pi}(G)$ by definition there exists $\chi \in Irr(G)$ such that $\chi^0 = \mu$, where $\chi^0$ denotes the restriction of $\chi$ to the $\pi$-elements of $G$. Also, it is clear by the definition, that every $\pi$-partial character is a sum of irreducible $\pi$-partial characters. Notice that if $G$ is a $\pi$-group, then $I_{\pi}(G) = Irr(G)$.

**Theorem 2.1** (Isaacs). Let $G$ be a finite $\pi$-separable group. Then $I_{\pi}(G)$ is a basis of the space of class functions defined on $G^0$. In particular, $|I_{\pi}(G)|$ is the number of conjugacy classes of $\pi$-elements of $G$.

**Proof.** This is Theorem 3.3 of [I2].
We can induce and restrict $\pi$-partial characters in a natural way. If $H$ is a subgroup of $G$ and $\varphi \in \text{I}_\pi(G)$, then $\varphi_H = \sum_{\mu \in \text{I}_\pi(H)} a_\mu \mu$ for some uniquely defined nonnegative integers $a_\mu$. We write $\text{I}_\pi(G|\mu)$ to denote the set of $\varphi \in \text{I}_\pi(G)$ such that $a_\mu \neq 0$.

A non-trivial result is that Clifford’s theory holds for $\pi$-partial characters. If $N \triangleleft G$, it is then clear that $G$ naturally acts on $\text{I}_\pi(N)$ by conjugation.

**Theorem 2.2** (Isaacs). Suppose that $G$ is $\pi$-separable and $N \triangleleft G$.

(a) If $\varphi \in \text{I}_\pi(G)$, then $\varphi_N = e(\theta_1 + \cdots + \theta_t)$, where $\theta_1, \ldots, \theta_t$ are all the $G$-conjugates of some $\theta \in \text{I}_\pi(N)$.

(b) If $\theta \in \text{I}_\pi(N)$ and $T = G_\theta$ is the stabilizer of $\theta$ in $G$, then induction defines a bijection $\text{I}_\pi(T|\theta) \to \text{I}_\pi(G|\theta)$.

**Proof.** See Corollary 5.7 and Theorem 5.11 of [I2]. □

In part (b) of Theorem 2.2, if $\mu^G = \varphi$, where $\mu \in \text{I}_\pi(T|\theta)$, then $\mu$ is called the **Clifford correspondent** of $\varphi$ over $\theta$, and sometimes it is written $\mu = \varphi_\theta$.

It is not a triviality to define vertices for $\pi$-partial characters (a concept that in classical modular representation theory has little to do with character theory). This was first accomplished in [IN] (generalizing a result of Huppert on Brauer characters of $p$-solvable groups).

**Theorem 2.3.** Suppose that $G$ is $\pi$-separable, and let $\varphi \in \text{I}_\pi(G)$. Then there exist a subgroup $U$ of $G$ and $\alpha \in \text{I}_\pi(U)$ of $\pi$-degree such that $\alpha^G = \varphi$. Furthermore, if $Q$ is a Hall $\pi$-complement of $U$, then the $G$-conjugacy class of $Q$ is uniquely determined by $\varphi$.

**Proof.** This is Theorem 5.17 of [I2]. □

The uniquely defined $G$-class of $\pi'$-subgroups $Q$ associated to $\varphi$ by Theorem 2.3 is called the set of **vertices** of $\varphi$. If $Q$ is a $\pi'$-subgroup of $G$, then we write $\text{I}_\pi(G|Q)$ to denote the set of $\varphi \in \text{I}_\pi(G)$ which have $Q$ as a vertex. By definition, notice in this case that $\varphi(1)_{\pi'} = |G : Q|_{\pi'}$.

Our last important ingredient is the Glauberman correspondence.

**Theorem 2.4** (Glauberman). Let $S$ be a finite solvable group acting via automorphisms on a finite group $G$ such that $(|S|, |G|) = 1$. Then there exists a canonical bijection, called the **$S$-Glauberman correspondence**, $\text{Irr}_S(G) \to \text{Irr}(C), \quad \chi \mapsto \chi^*$, where $\text{Irr}_S(G)$ is the set of $S$-invariant irreducible characters of $G$ and $C = C_G(S)$. Here, $\chi^*$ is a constituent of the restriction $\chi_C$. Also, if $T \triangleleft S$, then the $T$-Glauberman correspondence is an isomorphism of $S$-sets. Moreover, if $X$ acts by automorphism on $G \rtimes S$ fixing $S$ setwise, then the $S$-Glauberman correspondence commutes with the action of $X$. 
Proof. See Theorem 13.1 of [I]. The last claim is proven in Lemma 2.10 of [N] under the assumption that $S$ is a $p$-group. The general case can be obtain by induction on $|S|$.

3. Preliminaries

If $G$ is a finite group, $\pi$ is a set of primes, and $\chi \in \text{Irr}(G)$, then we say that $\chi$ has $\pi$-defect zero if $\chi(1)_{\pi} = |G|_{\pi}$.

Lemma 3.1. If $\chi \in \text{Irr}(G)$ has $\pi$-defect zero, then $O_{\pi}(G) = 1$.

Proof. Let $N \triangleleft G$, and let $\theta \in \text{Irr}(N)$ be under $\chi$. Then we have that $\theta(1)$ divides $|N|$ and $\chi(1)/\theta(1)$ divides $|G : N|$ by Corollary 11.29 of [I]. Thus $\chi(1)_{\pi} = |G|_{\pi}$ if and only if $\theta(1)_{\pi} = |N|_{\pi}$ and $(\chi(1)/\theta(1))_{\pi} = |G : N|_{\pi}$. The result is now clear applying this to $N = O_{\pi}(G)$.

We shall use the following notation. Suppose $G$ is $\pi$-separable, $N \triangleleft G$, $\tau \in I_{\pi}(N)$ and $Q$ is a $\pi'$-subgroup of $G$. Then 

$$I_{\pi}(G|Q, \tau) = I_{\pi}(G|Q) \cap I_{\pi}(G|\tau).$$

Lemma 3.2. Suppose $G$ is $\pi$-separable and that $N \triangleleft G$. Let $Q$ be a $\pi'$-subgroup of $G$.

(a) Suppose that $\mu \in I_{\pi}(G|Q)$. Then there is a unique $N_{G}(Q)$-orbit of $\tau \in I_{\pi}(N)$ such that $\mu_{\tau} \in I_{\pi}(G_{\tau}|Q)$, where $\mu_{\tau}$ is the Clifford correspondent of $\mu$ over $\tau$. Every such $\tau$ is $Q$-invariant.

(b) Suppose that $\tau \in I_{\pi}(N)$ is $Q$-invariant. Let $U$ be a complete set of representatives of the $G_{\tau}$-orbits on the set $\{Q^{g} \mid g \in G, Q^{g} \subseteq G_{\tau}\}$. Then

$$|I_{\pi}(G|Q, \tau)| = \sum_{U \in U} |I_{\pi}(G_{\tau}|U, \tau)|.$$

Thus, if $G_{\tau}N_{G}(Q) = G$, then

$$|I_{\pi}(G|Q, \tau)| = |I_{\pi}(G_{\tau}|Q, \tau)|.$$

Proof. (a) Let $\nu \in I_{\pi}(N)$ be under $\mu$, and let $\mu_{\nu} \in I_{\pi}(G_{\nu}|\nu)$ be the Clifford correspondent of $\mu$ over $\nu$. If $R$ is a vertex of $\mu_{\nu}$, then $R$ is a vertex of $\mu$, by Theorem 2.3. Therefore $R = Q^{g}$ for some $g \in G$. If $\tau = \nu^{g^{-1}}$, then we have that $\mu_{\tau}$ has vertex $Q$. Suppose now that $\rho \in I_{\pi}(N)$ is under $\mu$ such that $\mu_{\rho}$ has vertex $Q$. By Theorem 2.2(a), there exists $g \in G$ such that $\tau^{g} = \rho$. Thus $Q^{g}$ is a vertex of $\mu_{\rho}$. Then there is $x \in G_{\rho}$ such that $Q^{gx} = Q$. Since $\tau^{gx} = \rho$, the proof of part (a) is complete.

(b) We have that induction defines a bijection $I_{\pi}(G_{\tau}|\tau) \rightarrow I_{\pi}(G|\tau)$. Notice that

$$\bigcup_{U \in U} I_{\pi}(G_{\tau}|U)$$
is a disjoint union. It suffices to observe, again, that if \( \xi \in I_\pi(G_\tau|\tau) \) has vertex \( U \), then \( \xi^G \) has vertex \( U \). \( \square \)

4. Proofs

The deep part in our proofs comes from the following result.

**Theorem 4.1.** Suppose that \( L \) is a normal \( \pi \)-subgroup of \( G \), \( Q \) is a solvable \( \pi' \)-subgroup of \( G \) such that \( LQ \triangleleft G \). Suppose that \( M \subseteq Z(G) \) is contained in \( L \) and that \( \varphi \in \text{Irr}(M) \). Then \( |I_\pi(G|\varphi)| = |I_\pi(N_G(Q)|Q,\varphi)| \).

**Proof.** Let \( A \) be a complete set of representatives of \( N_G(Q)\)-orbits on \( \text{Irr}_Q(L|\varphi) \), the \( Q \)-invariant members of \( \text{Irr}(L|\varphi) \). Using Lemma 3.2, we have that \( I_\pi(G|\varphi) = \bigcup_{\tau \in A} I_\pi(G|\tau) \) is a disjoint union. Let \( A^* \) be the set of the \( Q \)-Glauberman correspondents of the elements of \( A \). Notice that \( A^* \) is a complete set of representatives of \( N_G(Q)\)-orbits on \( \text{Irr}(C_L(Q)|\varphi) \). Moreover, \( N_{G,\tau}(Q) = N_{G,\tau^*}(Q) \). Then, as before,

\[
I_\pi(N_G(Q)|Q,\varphi) = \bigcup_{\tau \in A} I_\pi(N_G(Q)|Q,\tau^*)
\]

is a disjoint union. Thus

\[
|I_\pi(N_G(Q)|Q,\varphi)| = \sum_{\tau \in A} |I_\pi(N_G(Q)|Q,\tau^*)|.
\]

Thus we need to prove that

\[
|I_\pi(G_\tau|\varphi)| = |I_\pi(N_{G,\tau}(Q)|Q,\tau^*)|.
\]

We may assume that \( \tau \) is \( G \)-invariant.

Now, since \( LQ \triangleleft G \) and \( \tau \) is \( G \)-invariant, by Lemma 6.30 of [I2], we have that \( Q \) is contained as a normal subgroup in some vertex of \( \theta \), whenever \( \theta \in \text{I}_G(G|\tau) \) lies over \( \tau \). Therefore \( \theta \in \text{I}_G(G|\tau) \) has vertex \( Q \) if and only if \( \theta(1)_{\pi'} = |G : Q|_{\pi'} \). Similarly, \( \theta \in \text{I}_G(N_G(Q)|\tau^*) \) has vertex \( Q \) if and only if \( \theta(1)_{\pi''} = |N_G(Q) : Q|_{\pi''} = |G : Q|_{\pi''} \) since \( G = L N_G(Q) \) by the Frattini argument and the Schur–Zassenhaus theorem.

Now we use the Dade–Puig theory on the character theory above Glauberman correspondents, which is thoroughly explained in [T]. By Theorem 6.5 of [T], in the language of Chapter 11 of [I1] (see Definition 11.23 of [I1]), we have that the character triples \( (G, L, \tau) \) and \( (N_G(Q), C_L(Q), \tau^*) \) are isomorphic. Write \( * : \text{Irr}(G|\tau) \to \text{Irr}(N_G(Q)|\tau^*) \) for the associated bijection of characters. By Lemma 6.21 of [I2], there exists a unique bijection

\[
* : I_\pi(G|\tau) \to I_\pi(N_G(Q)|\tau^*)
\]
such that if $\chi^0 = \phi \in I_\pi(G|\tau)$ and $\chi \in \text{Irr}(G)$ (which necessarily lies over $\tau$), then $(\chi^*)^0 = \phi^*$. Since $\chi(1)/\tau(1) = \chi^*(1)/\tau^*(1)$ (by Lemma 11.24 of [I1]), it follows that $\chi(1)_{\pi^*} = \chi^*(1)_{\pi^*}$. We deduce that
\[ |I_\pi(G|Q, \tau)| = |I_\pi(\mathcal{N}_G(Q)|Q, \tau^*)| , \]
as desired. \hfill $\Box$

In order to prove Theorem B, we argue by induction on the index of a normal $\pi$-subgroup $M$ of $G$. Theorem B follows from the special case $M = 1$.

**Theorem 4.2.** Suppose that $G$ is $\pi$-separable with a solvable Hall $\pi$-complement. Let $R$ be a nilpotent $\pi'$-subgroup of $G$. Let $M \trianglelefteq G$ be a normal $\pi$-subgroup, and let $\varphi \in \text{Irr}(M)$ be $G$-invariant. Let $\mathcal{Q}$ be the set of $\pi'$-subgroups $Q$ of $G$ such that $R$ is a Carter subgroup of $Q$. Then
\[ \left| \bigcup_{Q \in \mathcal{Q}} I_\pi(G|Q, \varphi) \right| = |I_\pi(M\mathcal{N}_G(R)|R, \varphi)| . \]

**Proof.** We argue by induction on $|G : M|$.

By Lemma 3.11 of [I2], let $(G^*, M^*, \varphi^*)$ be a character triple isomorphic to $(G, M, \varphi)$, where $M^*$ is a central $\pi$-subgroup of $G^*$. If $Q$ is a $\pi'$-subgroup of $G$, notice that we can write $(QM)^* = M^* \times Q^*$, for a unique $\pi'$-subgroup $Q^*$ of $G^*$. If $R$ is contained in a $\pi'$-subgroup $Q$, then $R$ is a Carter subgroup of $Q$ if and only if $RM/M$ is a Carter subgroup of $QM/M$, using that $Q$ is naturally isomorphic to $QM/M$. This happens if and only if $(RM)^*$ is a Carter subgroup of $(QM)^*$, which again happens if and only if $R^*$ is a Carter subgroup of $Q^*$. Notice further that if $R$ is a Carter subgroup of $Q$, then $R$ is a Carter subgroup of every Hall $\pi$-complement $Q_1$ of $QM$ that happens to contain $R$ (again using the isomorphism between $QM/M$ and $Q$). We easily check now that the set of $\pi'$-subgroups of $G^*$ that contain $R^*$ as a Carter subgroup is exactly $Q^* = \{Q^*|Q \in \mathcal{Q}\}$.

By the Frattini argument and the Schur–Zassenhaus theorem, notice that $\mathcal{N}_G(MR) = MN_G(R)$. By Lemma 6.21 and the proof of Lemma 6.32 of [I2], there is a bijection $*: I_\pi(G|\varphi) \to I_\pi(G^*|\varphi^*)$ such that $\eta$ has vertex $Q$ if and only if $\eta^*$ has vertex $Q^*$. From all these arguments, it easily follows that we may assume that $M$ is central. In particular, $M \leq \mathcal{N}_G(R)$.

Let $K = O_{\pi'}(G)$. Suppose that there exists some $\mu \in I_\pi(G|Q, \varphi)$ for some $Q \in \mathcal{Q}$. By Lemma 6.30 of [I2] (in the notation of that lemma, $K$ is 1 and $Q$ is $K$), we have that $K$ is contained in $Q$. Hence, it is no loss if we only consider $Q \in \mathcal{Q}$ such that $K \subseteq Q$.

Suppose that $\mathcal{N}_K(R)$ is not contained in $R$. Then there cannot be weights $(R, \gamma)$, where $\gamma \in \text{Irr}(\mathcal{N}_G(R)/R)$ has $\pi$-defect zero by Lemma 3.1. So the right hand side is zero. Suppose that there exists some $\mu \in I_\pi(G|Q, \varphi)$ for some $Q \in \mathcal{Q}$ (with $K \subseteq Q$). Since $R$ is a Carter subgroup of $Q$, then $R$ is a Carter subgroup of $KR$, and therefore
\(N_K(R)\) is contained in \(R\). Therefore may assume that \(N_K(R)\) is contained in \(R\). We claim that \(R\) is a Carter subgroup of \(Q\) if and only if \(RK/K\) is a Carter subgroup of \(Q/K\). One implication is known (see 9.5.3 in [R]). Suppose that \(RK/K\) is a Carter subgroup of \(Q/K\). Since \(N_Q(R)\) normalizes \(RK\), it is contained in \(RK\). Hence \(N_Q(R) = N_{KR}(RK) = R\), and \(R\) is a Carter subgroup of \(Q\). In this situation the Frattini argument yields \(N_G(R)K = N_G(RK)\).

Next, we will replace \(G\) by \(G/K\). By Lemma 6.31 of [I2] (the roles of \(K\) and \(M\) are interchanged in that lemma),

\[|\mathbf{I}_\pi(G/Q, \varphi)| = |\mathbf{I}_\pi(G/K|Q/K, \hat{\varphi})|,\]

where \(\hat{\varphi} \in \text{Irr}(MK/K)\) corresponds to \(\varphi\) via the natural isomorphism. Similarly,

\[|\mathbf{I}_\pi(N_Q(R)|R, \varphi)| = |\mathbf{I}_\pi(N_Q(R)K/K|RK/K, \hat{\varphi})| = |\mathbf{I}_\pi(N_G(RK)/K|RK/K, \hat{\varphi})|.

Hence, for the remainder of the proof we may assume that \(O_p(G) = K = 1\).

Suppose now that \(L = O_\pi(G)\). Let \(A\) be a complete set of \(N_G(R)\)-representatives of the \(R\)-invariant characters in \(\text{Irr}(L|\varphi)\). If \(L = M\), then \(L = G\) by the Hall-Higman Lemma 1.2.3, and \(G\) is a \(\pi\)-group. In this case, \(R = 1\), and there is nothing to prove. Thus, we may assume that \(|G:L| < |G:M|\).

For each \(\tau \in A\), let \(Q_\tau\) be the set of \(\pi'\)-subgroups \(Q\) of \(G_\tau\) such that \(R\) is a Carter subgroup of \(Q\). By induction,

\[\left| \bigcup_{Q \in Q_\tau} \mathbf{I}_\pi(G_\tau|Q, \tau) \right| = |\mathbf{I}_\pi(LN_G(R)|R, \tau)|.
\]

Since \(L\) is a \(\pi\)-group and \(R\) is a \(\pi'\)-subgroup, we have that \(LN_G(R) = N_G(LR)\). Also, \(N_G(LR)_\tau = LN_G(R)_\tau\). By Lemma 3.2, we have that

\[|\mathbf{I}_\pi(LN_G(R)|R, \tau)| = |\mathbf{I}_\pi(LN_G(R)|R, \tau)|.
\]

Also,

\[|\mathbf{I}_\pi(LN_G(R)|R, \varphi)| = \sum_{\tau \in A} |\mathbf{I}_\pi(LN_G(R)|R, \tau)|,
\]

by the first paragraph of the proof of Theorem 4.1. By Theorem 4.1,

\[|\mathbf{I}_\pi(LN_G(R)|R, \varphi)| = |\mathbf{I}_\pi(N_G(R)|R, \varphi)|.
\]

Therefore,

\[\sum_{\tau \in A} \left| \bigcup_{Q \in Q_\tau} \mathbf{I}_\pi(G_\tau|Q, \tau) \right| = |\mathbf{I}_\pi(N_G(R)|R, \varphi)|.
\]

We are left to show that

\[\left| \bigcup_{Q \in Q} \mathbf{I}_\pi(G|Q, \varphi) \right| = \sum_{\tau \in A} \left| \bigcup_{Q \in Q_\tau} \mathbf{I}_\pi(G_\tau|Q, \tau) \right|.
\]
Let $R$ be a complete set of representatives of $\mathbf{N}_G(R)$-orbits in $Q$, and notice that
\[ \bigcup_{Q \in \mathcal{Q}} I_\pi(G|Q, \varphi) = \bigcup_{Q \in R} I_\pi(G|Q, \varphi) \]
is a disjoint union. Indeed, if $\mu \in I_\pi(G|Q_1, \varphi) \cap I_\pi(G|Q_2, \varphi)$ for $Q_i \in \mathcal{Q}$, then we have that $Q_1 = Q_2^g$ for some $g \in G$ by the uniqueness of vertices. Hence $R^\omega$ and $R$ are Carter subgroups of $Q_1$, and therefore $R^{xx} = R$ for some $x \in Q_1$. It follows that $Q_1 = Q_2^{gt}$ are $\mathbf{N}_G(R)$-conjugate.

Now fix $Q \in R$. For each $\mu \in I_\pi(G|Q, \varphi)$, we claim that there is a unique $\tau \in \mathcal{A}$ such that $\mu_\tau \in I_\pi(G_\tau|Q^\tau, \tau)$, for some $x \in \mathbf{N}_G(R)$. We know that there is $\nu \in \text{Irr}(L|\varphi)$ such that $\mu_\nu \in I_\pi(G_\nu|Q, \nu)$ by Lemma 3.2(a). Now, $\nu^x = \tau$ for some $x \in \mathbf{N}_G(R)$ and $\tau \in \mathcal{A}$, and it follows that $\mu_\tau \in I_\pi(G_\tau|Q^\tau, \tau)$. Suppose that $\mu_\tau \in I_\pi(G_\tau|Q^\tau, \tau)$, for some $y \in \mathbf{N}_G(R)$ and $\tau \in \mathcal{A}$. Now, $\epsilon = \tau^y$ for some $g \in G$, by Clifford’s theorem. Thus $Q^{xgy} = Q^\tau$ for some $t \in G_\epsilon$, by the uniqueness of vertices. Thus $xgty^{-1} \in \mathbf{N}_G(Q)$. Since $R$ is a Carter subgroup of $Q$, by the Frattini argument we have that $xgty^{-1} =qv$, where $q \in Q$ and $v \in G$ normalizes $Q$ and $R$. Since $Q^\tau$ fixes $\tau$, then $Q$ fixes $\tau^{x^{-1}}$. Now
\[ \epsilon^{y^{-1}} = (\tau^gy)^{-1} = \tau^{x^{-1}}xgty^{-1} = \tau^{x^{-1}}qv = \tau^{x^{-1}v}. \]
So $\epsilon$ and $\tau$ are $\mathbf{N}_G(R)$-conjugate, and thus they are equal.

Now we define a map
\[ f : \bigcup_{Q \in R} I_\pi(G|Q, \varphi) \to \bigcup_{\tau \in \mathcal{A}} \left( \bigcup_{Q \in \mathcal{Q}_\tau} I_\pi(G_\tau|Q, \tau) \right) \times \{ \tau \} \]
given by $f(\mu) = (\mu_\tau, \tau)$, where $\tau \in \mathcal{A}$ is the unique element in $\mathcal{A}$ such that $\mu_\tau \in I_\pi(G_\tau|Q^\tau, \tau)$, for some $x \in \mathbf{N}_G(R)$. Since $\mu^G_\mu = \mu$, we have that $f$ is injective. If we have that $\gamma \in \bigcup_{Q \in \mathcal{Q}_\tau} I_\pi(G_\tau|Q, \tau)$ then $\gamma^{G_\tau} \in \bigcup_{Q \in \mathcal{Q}_\tau} I_\pi(G|Q, \varphi)$, so $f$ is surjective.

Some of the difficulties in Theorem 4.2 are caused by the fact that Clifford correspondence does not necessarily respect vertices, even in quite restricted situations. Suppose that $N$ is a normal $p'$-subgroup of $G$, $\tau \in \text{Irr}(N)$, $Q$ is a $p$-subgroup of $G$ and $\tau$ is $Q$-invariant. Then it is not necessarily true that induction defines a bijection $\text{IBr}(G_\tau|Q, \tau) \to \text{IBr}(G|Q, \tau)$. For instance, take $p = 2$ and $G = \text{SmallGroup}(216,158)$. This group has a unique normal subgroup $N$ of order 3. The Fitting subgroup $F$ of $G$ is $F = N \times M$, where $M$ is a normal subgroup of type $C_3 \times C_3$, and $G/F = D_8$. Let $1 \neq \tau \in \text{Irr}(N)$. Then $G_\tau \cdot G$ has index 2, and $G_\tau/N = S_3 \times S_3$. Now $\tau$ has a unique extension $\tilde{\tau} \in \text{IBr}(G_\tau)$. The group $G_\tau$ has three conjugacy classes of subgroups $Q$ of order 2. Take $Q_1$ that corresponds to $C_2 \times 1$ and $Q_2$ that corresponds to $1 \times C_2$. Then $Q_1$ and $Q_2$ are not $G_\tau$-conjugate but $G$-conjugate. So $|\text{IBr}(G_\tau|Q_1, \tau)| = 1$ and $|\text{IBr}(G|Q_1, \tau)| = 2$.\[ \]
5. A canonical bijection

If $G$ has a normal Hall $\pi$-subgroup, then we have a canonical bijection in Theorem 4.2. This seems worth exploring.

Lemma 5.1. Suppose that $G = NH$ where $N$ is a normal $\pi$-subgroup and $H$ is a $\pi'$-subgroup. Then $N_G(Q) = C_N(Q)N_H(Q)$ for every $Q \leq H$.

Proof. First note that

$$Q = Q(N \cap H) = QN \cap H = NN_G(Q) \cap H \leq N_H(Q).$$

Let $xh \in N_G(Q)$ where $x \in N$ and $h \in H$. Then $h = x^{-1}(xh) \in NN_G(Q) \cap H \leq N_H(Q)$. This shows $N_G(Q) = N_N(Q)N_H(Q) = C_N(Q)N_H(Q)$. □

Lemma 5.2. Suppose that $G = NH$ where $N$ is a normal $\pi$-subgroup and $H$ is a solvable $\pi'$-subgroup. Let $R \leq H$, and let $\tau \in \operatorname{Irr}(C_N(R))$ be such that $N_G(R)_{\tau} = C_N(R) \times R$. Let $\gamma \in \operatorname{Irr}_R(N)$ be the Glauberman correspondent of $\tau$. Then $R = N_{H,\gamma}(R)$.

Proof. Suppose that $R < S \leq H,\gamma$, where $R \triangleleft S$. Then $S$ acts on the $R$-Glauberman correspondence. Since $S$ fixes $\gamma$, therefore it fixes $\gamma^* = \tau$. But this gives the contradiction $S \subseteq N_G(R)_{\tau} = C_N(R) \times R$. □

Theorem 5.3. Suppose that $G = NH$ where $N$ is a normal $\pi$-subgroup and $H$ is a solvable $\pi'$-subgroup. Let $R$ be a nilpotent subgroup of $H$. Let $Q$ be the set of subgroups $Q \subseteq H$ such that $R$ is a Carter subgroup of $Q$. Then there is a natural bijection

$$\bigcup_{Q \in Q} I_\pi(G|Q) \rightarrow I_\pi(N_G(R)|R).$$

Proof. Let $Q \in Q$. By the Frattini argument, notice that $N_G(Q) = Q(N_G(Q) \cap N_G(R))$, and that $Q \cap (N_G(Q) \cap N_G(R)) = R$.

Let $\phi \in I_\pi(G|Q)$. By Lemma 3.2, there exists a $Q$-invariant $\theta \in \operatorname{Irr}(N)$ under $\phi$. Then $T = G_\theta = QN$ using Corollary 8.16 in [11] for instance. If $\theta_1$ is another such choice, then $\theta_1 = \theta^g$ for some $g \in N_G(Q)$. Thus, we may assume that $g \in N_G(Q) \cap N_G(R)$. Let $\theta^* \in C_N(R)$ be the $R$-Glauberman correspondent of $\theta$. Now, by Lemma 5.1 applied in $T$, we have that $N_T(R) = C_N(R)N_Q(R) = C_N(R) \times R$. We claim that $N_T(R)$ is the stabilizer of $\theta^*$ in $N_G(R)$. If $x \in N_G(R)$ fixes $\theta^*$, then $x$ fixes $\theta$, and thus $x \in N_T(R)$, as claimed. Now $\phi^* := (\theta^* \times 1_R)^{N_G(R)}$ is irreducible,
and belongs to \( I_\pi(N_G(R)|R) \). Since \( \theta_1 \) is \( N_G(Q) \cap N_G(R) \)-conjugate to \( \theta \), \( \phi^* \) is independent of the choice of \( \theta \).

Suppose that \( \phi^* = \mu^* \), where \( \phi \in I_\pi(G|Q_1) \) and \( \mu \in I_\pi(G|Q_2) \), where \( R \) is a Carter subgroup of \( Q_1 \) and \( Q_1 \subseteq H \). Suppose that we picked \( \theta \) for \( \phi \) and \( \epsilon \) for \( \mu \), so that \( \phi^* = (\theta^* \times 1_R)^{N_G(R)} \) and \( \mu^* = (\epsilon^* \times 1_R)^{N_G(R)} \). Then \( N_G(R) = C_N(R)N_H(R) \), and \( \theta^* \) and \( \epsilon^* \) are \( N_H(R) \)-conjugate, say \( (\theta^*)^x = \epsilon^* \). Then \( \theta^* = \epsilon \). By replacing \((Q_1, \theta)\) by \((Q_1^x, \theta^x)\), we may assume that \( \theta = \epsilon \). But then \( Q_1 = H_\theta = H_\epsilon = Q_2 \). Since \( \pi \)-partial character are determined on the \( \pi \)-elements, we must have \( \phi = \mu \) now.

Suppose conversely that \( \tau \in I_\pi(N_G(R)|R) \). Then \( \tau \) is induced from \( C_N(R) \times R \). Let \( \mu \in \text{Irr}(C_N(R)) \) such that \( \mu \times 1_R \) induces \( \tau \). Then the stabilizer of \( \mu \) in \( N_G(R) \) is \( C_N(R) \times R \). If \( \rho \in \text{Irr}_R(N) \) is the \( R \)-Glauberman correspondent of \( \mu \), then by Lemma 5.2 we know that \( R \) is a Carter subgroup of \( Q = H_\rho \), where \( Q \) is the stabilizer in \( H \) of \( \rho \). Thus with the notation of the first part of the proof we obtain \( \tau = \mu^* \) where \( \mu \) is induced from \( G_\rho = QN \).

\[ \square \]

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