COMPLETELY MONOTONIC DEGREE OF A FUNCTION INVOLVING THE TRI- AND TETRA-GAMMA FUNCTIONS

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ABSTRACT. In the paper, the author show that the completely monotonic degree of the function $[\psi'(x)]^2 + \psi''(x)$, which involves the tri-gamma function $\psi'(x)$ and the tetra-gamma function $\psi''(x)$, is four and supply a proof for the result that a function $f(x)$ is strongly completely monotonic if and only if the function $xf(x)$ is completely monotonic. Moreover, the author surveys the history and motivation of this topic and poses a conjecture.

1. Introduction

A function $f$ is said to be completely monotonic on an interval $I$ if $f$ has derivatives of all orders on $I$ and

$$0 \leq (-1)^{k-1} f^{(k-1)}(x) < \infty$$

for $x \in I$ and $k \in \mathbb{N}$, where $f^{(0)}(x)$ means $f(x)$ and $\mathbb{N}$ is the set of all positive integers. See [24, Chapter XIII], [48, Chapter 1], and [52, Chapter IV].

Let $f(x)$ be a nonnegative function and have derivatives of all orders on $(0, \infty)$. A number $r \in \mathbb{R} \cup \{\pm \infty\}$ is said to be the completely monotonic degree of $f(x)$ with respect to $x \in (0, \infty)$ if $x^rf(x)$ is a completely monotonic function on $(0, \infty)$ but $x^{r+\varepsilon}f(x)$ is not for any positive number $\varepsilon > 0$. For convenience, we use the notation $\text{deg}_{x \text{cm}}[f(x)]$ to denote the completely monotonic degree $r$ of $f(x)$ with respect to $x \in (0, \infty)$. For simplicity, in what follows, we sometimes just say that $\text{deg}_{x \text{cm}}[f(x)]$ is the degree of $f(x)$. For more information on this notion, see [10, 30, 45] and related references therein.

The classical Euler’s gamma function $\Gamma(x)$ may be defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt.$$

The logarithmic derivative of $\Gamma(x)$, denoted by $\psi(x) = \Gamma'(x)/\Gamma(x)$, is called the psi or di-gamma function, the derivatives $\psi'(x)$ and $\psi''(x)$ are respectively called the tri- and tetra-gamma functions. As a whole, hereafter, the derivatives $\psi^{(i)}(x)$ for $i \geq 0$ are called polygamma functions.

The purpose of this paper is to compute the completely monotonic degree of the function

$$\Psi(x) = [\psi'(x)]^2 + \psi''(x) \quad (1.1)$$

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with respect to \( x \in (0, \infty) \). We may state our main result as the following theorem.

**Theorem 1.1.** The completely monotonic degree of \( \Psi(x) \) defined by (1.1) with respect to \( x \in (0, \infty) \) is 4, that is,

\[
\deg_{cm}^{x}[\Psi(x)] = 4. \tag{1.2}
\]

**Remark 1.1.** Our Theorem 1.1 strengthens and improves related results in [19, 20].

In Section 3, we will prove our Theorem 1.1. In the last section, among other things, we will survey something to do with the function \( \Psi(x) \) and introduce the motivation of this paper.

# 2. Lemmas

In order to smoothly prove our Theorem 1.1, we need the following lemmas.

**Lemma 2.1** ([1, p. 260, 6.4.1]). For \( n \in \mathbb{N} \) and \( x > 0 \),

\[
\psi^{(n)}(x) = (-1)^{n+1} \int_{0}^{\infty} \frac{t^{n}}{1-e^{-xt}} e^{-st} \, dt. \tag{2.1}
\]

**Lemma 2.2** ([51]). Let \( f_{i}(t) \) for \( i = 1, 2 \) be piecewise continuous in arbitrary finite intervals included in \((0, \infty)\) and suppose that there exist some constants \( M_{i} > 0 \) and \( c_{i} \geq 0 \) such that \( |f_{i}(t)| \leq M_{i} e^{c_{i} t} \) for \( i = 1, 2 \). Then

\[
\int_{0}^{\infty} \left[ \int_{0}^{t} f_{1}(u)f_{2}(t-u) \, du \right] e^{-st} \, dt = \int_{0}^{\infty} f_{1}(u)e^{-su} \, du \int_{0}^{\infty} f_{2}(v)e^{-sv} \, dv. \tag{2.2}
\]

**Lemma 2.3** ([34, p. 631, Lemma 2.1]). Let \( f(x, t) \) is differentiable in \( t \) and continuous for \((x, t) \in \mathbb{R}^{2}\). Then

\[
\frac{d}{dt} \int_{x_{0}}^{t} f(x, t) \, dx = f(t, t) + \int_{x_{0}}^{t} \frac{\partial f(x, t)}{\partial t} \, dx. \tag{2.3}
\]

**Lemma 2.4** ([23, p. 374]). If \( f_{i}, 1 \leq i \leq n, \) are nonnegative Lebesgue square integrable functions on \([0, a] \) for all \( a > 0 \), then for all \( n \geq 2 \) and \( x \geq 0 \),

\[
f_{1} \ast \cdots \ast f_{n}(x) \geq \frac{x^{n-1}}{(n-1)!} \exp \left[ \sum_{j=1}^{n} \ln f_{j}(x) \right], \tag{2.4}
\]

where \( f_{i} \ast f_{j}(x) \) denotes the convolution \( \int_{0}^{x} f_{i}(t)f_{j}(x-t) \, dt \).

**Lemma 2.5** ([46, p. 241, Corollary 1] and [50]). Let \( f : [a, b] \to \mathbb{R} \) be a twice differentiable mapping on \([a, b] \) and suppose that \( m \leq f''(t) \leq M \) for \( t \in (a, b) \). Then we have

\[
\frac{2m - 3S_{2}}{12} (b-a)^{2} \leq \frac{1}{b-a} \int_{a}^{b} f(t) \, dt - \frac{f(a) + f(b)}{2} \leq \frac{2M - 3S_{2}}{12} (b-a)^{2} \tag{2.5}
\]

and

\[
\frac{3S_{2} - M}{24} (b-a)^{2} \leq \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_{a}^{b} f(t) \, dt \leq \frac{3S_{2} - m}{24} (b-a)^{2}, \tag{2.6}
\]

where \( S_{2} = \frac{f(b) - f'(a)}{b-a} \).

**Remark 2.1.** There have been some new refinements and generalizations of the famous Hermite-Hadamard inequality. See [4, 5, 46, 53, 54, 56] and cited references therein.
Lemma 2.6. If $f(x)$ is differentiable and logarithmically concave (or logarithmically convex respectively) on $(-\infty, \infty)$, then the product $f(x)f(\lambda - x)$ for any fixed number $\lambda \in \mathbb{R}$ is increasing (or decreasing respectively) with respect to $x \in \left( -\infty, \frac{\lambda}{2} \right)$ and decreasing (or increasing respectively) with respect to $x \in \left( \frac{\lambda}{2}, \infty \right)$.

Proof. Taking the logarithm of $f(x)f(\lambda - x)$ and differentiating give

$$\left\{ \ln[f(x)f(\lambda - x)] \right\}' = \frac{f'(x)}{f(x)} - \frac{f'(\lambda - x)}{f(\lambda - x)}.$$ 

In virtue of the logarithmic concavity of $f(x)$, it follows that the function $\frac{f'(x)}{f(x)}$ is decreasing and $\frac{f'(\lambda - x)}{f(\lambda - x)}$ is increasing on $(-\infty, \infty)$. From the obvious fact that $\left\{ \ln[f(x)f(\lambda - x)] \right\}'|_{x = \lambda/2} = 0$, it is deduced that $\{\ln[f(x)f(\lambda - x)]\}' < 0$ for $x > \frac{\lambda}{2}$ and $\{\ln[f(x)f(\lambda - x)]\}' < 0$ for $x < \frac{\lambda}{2}$. Hence, the function $f(x)f(\lambda - x)$ is decreasing for $x > \frac{\lambda}{2}$ and increasing for $x < \frac{\lambda}{2}$.

For the case of $f(x)$ being logarithmically convex, it may be proved similarly. □

Remark 2.2. The techniques in the proof of Lemma 2.6 has been utilized in the papers [12, 32, 41, 59] and closely related references therein.

Lemma 2.7 ([1, p. 260, 6.4.12, 6.4.13 and 6.4.14]). As $z \to \infty$ in $|\arg z| < \pi$,

$$\psi'(z) \sim \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} - \frac{1}{30z^5} + \frac{1}{42z^7} - \frac{1}{30z^9} + \cdots,$$

$$\psi''(z) \sim -\frac{1}{z^2} - \frac{1}{2z^3} + \frac{1}{6z^4} + \frac{1}{6z^6} - \frac{1}{10z^8} - \frac{3}{6z^9} + \frac{5}{6z^{12}} + \cdots,$$

$$\psi^{(3)}(z) \sim \frac{2}{z^3} + \frac{3}{2z^4} + \frac{3}{5z^5} - \frac{1}{z^6} - \frac{1}{3z^8} - \frac{4}{3z^9} - \frac{3}{2z^{11}} + \frac{10}{z^{13}} + \cdots.$$ 

Remark 2.3. The formulas listed in Lemma 2.7 are special cases of [1, p. 260, 6.4.11].

3. Proof of Theorem 1.1

Now we are in a position to compute the completely monotonic degree of the function $\Psi(x)$.

Using the integral representation (2.1) and the formula (2.2) gives

$$\Psi(x) = \left[ \int_{0}^{\infty} \frac{t}{1-e^{-t}} e^{-xt} \, dt \right]^2 - \int_{0}^{\infty} \frac{t^2}{1-e^{-t}} e^{-xt} \, dt$$

$$= \int_{0}^{\infty} \left[ \int_{0}^{t} \frac{s(t-s)}{1-e^{-s}} \left[ 1 - e^{-(t-s)} \right] ds - \frac{t^2}{1-e^{-t}} \right] e^{-xt} \, dt$$

$$= \int_{0}^{\infty} q(t) e^{-xt} \, dt,$$

where

$$q(t) = \int_{0}^{t} \sigma(s) \sigma(t-s) \, ds - t \sigma(t) \quad \text{and} \quad \sigma(s) = \begin{cases} \frac{s}{1-e^{-s}}, & s \neq 0 \\ 1, & s = 0. \end{cases}$$

By Lemma 2.3 and integration-by-part, a straightforward computation yields

$$q'(t) = \int_{0}^{t} \sigma(s) \sigma'(t-s) \, ds + \sigma(0) \sigma(t) - [t \sigma'(t) + \sigma(t)]$$
It is easy to see that $q(0) = q'(0) = q''(0) = 0$ and $q^{(3)}(0) = \frac{1}{12}$. Direct calculations reveal

\[
\sigma'(s) = \frac{(e^s - s - 1)e^s}{(e^s - 1)^2},
\]
\[
\sigma''(s) = \frac{[(s - 2)e^{2s} + s + 2]e^s}{(e^s - 1)^3},
\]
\[
\sigma^{(3)}(s) = \frac{[(s - 3)e^{2s} + 4se^s + s + 3]e^s}{(e^s - 1)^4},
\]
\[
\sigma^{(4)}(s) = \frac{[(s - 4)e^{3s} + (11s - 12)e^{2s} + (11s + 12)e^s + s + 4]e^s}{(e^s - 1)^5},
\]
\[
\ln\sigma''(s) = -\frac{(s - 3)e^{2s} + 4se^s + s + 3}{[(s - 2)e^s + s + 2][(e^s - 1)^3]},
\]
\[
\ln\sigma''(s)'' = -\frac{e^{4s} - 4(s^2 - 3s + 4)e^{3s} - (4s^2 - 30)e^{2s} - 4(s^2 + 3s + 4)e^s + 1}{(e^s - 1)^2[(s - 2)e^s + s + 2]^2},
\]
\[
\ln\sigma''(s)'' = \frac{h_1(s)}{(e^s - 1)^2[(s - 2)e^s + s + 2]^2},
\]
\[
h_1(s) = 4[e^{3s} - (3s^2 - 7s + 9)e^{2s} - (2s^2 + 2s - 15)e^s - s^2 - 5s - 7],
\]
\[
h_2(s) = 3e^{3s} - (6s^2 - 8s + 11)e^{2s} - (2s^2 + 6s - 13)e^s - 2s - 5,
\]
\[
h_3(s) = 9e^{3s} - 2(6s^2 - 2s + 7)e^{2s} - (2s^2 + 10s - 7)e^s - 2,
\]
\[
h_4(s) = 27e^{2s} - 8e^s(3s^2 + 2s + 3) - 2s^2 - 14s - 3, e^s
\]
\[
h_5(s) = 54e^{2s} - 8(3s^2 + 8s + 5)e^s - 2(2s + 7),
\]
\[
h_6(s) = 4[27e^{2s} - 2(3s^2 + 14s + 13)e^s - 1],
\]
for $s \in (0, \infty)$, and
\[ h_3''(0) = h_3(0) = h_2^{(3)}(0) = h_2''(0) = h_2(0) = h_1''(0) = h_1(0) = 0. \]
This means that
\[ h_3''(s) > 0, \quad h_3'(s) > 0, \quad h_3(s) > 0, \quad h_2^{(3)}(s) > 0, \quad h_2''(s) > 0, \quad h_2'(s) > 0, \quad h_1''(s) > 0, \quad h_1'(s) > 0. \]
Therefore, the derivative $\ln \sigma''(s)$ is negative on $(0, \infty)$, and the function $\sigma''(s)$ is logarithmically concave on $(0, \infty)$.

Applying Lemma 2.4 to $f = \sigma''$ and $n = 2$ and utilizing Lemma 2.5 lead to
\[
\int_0^t \sigma''(s)(t-s)\,ds \geq t \exp \left[ \frac{2}{t} \int_0^t \ln \sigma''(u)\,du \right] \geq t \exp \left[ \ln \sigma''(0) + \ln \sigma''(t) \right] = \frac{1}{6} t \sigma''(t).
\]
Combining this with Lemma 2.6 leads to
\[
q^{(4)}(t) \geq \frac{1}{6} t \sigma''(t) + \frac{2}{3} \sigma''(t) - 2 \sigma^{(3)}(t) - t \sigma^{(4)}(t) \triangleq -\frac{e^t \vartheta(t)}{6(e^t - 1)^2},
\]
where
\[
\vartheta(t) = (5t^2 - 40t + 48)e^{3t} + 67t^2 - 108t - 72)e^{2t} + t(67t + 120)e^t + 5t^2 + 28t + 24
\]
\[
= \frac{1}{6} \sum_{k=5}^{\infty} \left[ 6(66k^2 + 35k - 78) + 3(33k^2 - 148k + 12)2^k 
\quad + 2(2k^2 - 31k + 66)3^k \right] \frac{t^k}{k!}
\]
\[
\triangleq \frac{1}{6} \sum_{k=5}^{\infty} Q(k) \frac{t^k}{k!}
\]
for $t \in (0, \infty)$. It is not difficult to obtain that the larger roots of equations
\[ 66x^2 + 35x - 78 = 0, \quad 33x^2 - 148x + 12 = 0, \quad 2x^2 - 31x + 66 = 0 \]
are
\[
\frac{\sqrt{21817} - 35}{132} = 0.8 \ldots, \quad \frac{2(37 + \sqrt{1270})}{33} = 4.4 \ldots, \quad \frac{31 + \sqrt{433}}{4} = 12.9 \ldots
\]
respectively. This implies that $Q(k) > 0$ for $k \geq 13$. Moreover,
\[
Q(5) = 840, \quad Q(6) = 4968, \quad Q(7) = 16296, \quad Q(8) = 39888,
Q(9) = 104040, \quad Q(10) = 472824, \quad Q(11) = 2962344, \quad Q(12) = 17643744.
\]
Consequently, we have $Q(k) > 0$ for all $k \geq 5$. As a result, the functions $\vartheta(t)$ and $q^{(4)}(t)$ are positive on $(0, \infty)$.

Integrating by part successively results in
From Lemma 2.7, it follows that the function $q(t) = 1 + t + \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{4}$ is completely monotonic on $(0, \infty)$. In other words,
\[\deg_{\text{cm}}^{x} [\Psi(x)] \geq 4.\] (3.2)

Suppose that the function
\[f_{\alpha}(x) = x^{\alpha} \Psi(x)\] (3.3)
is completely monotonic on $(0, \infty)$. Then
\[f'_{\alpha}(x) = x^{\alpha-1} \left\{ \alpha \Psi(x) + x \left[ 2\psi'(x)\psi''(x) + \psi^{(3)}(x) \right] \right\} \leq 0\] on $(0, \infty)$, that is,
\[\alpha \leq -\frac{x \left[ 2\psi'(x)\psi''(x) + \psi^{(3)}(x) \right]}{\Psi}(x) \triangleq \phi(x), \quad x > 0.\] (3.4)

From Lemma 2.7, it follows that
\[\lim_{x \to \infty} \phi(x) = -\lim_{x \to \infty} \frac{x}{\left( \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{3x^3} + \frac{1}{4x^4} + O\left(\frac{1}{x^4}\right) \right)^2 + \left[ -\frac{1}{2x^2} - \frac{1}{3x^3} + O\left(\frac{1}{x^4}\right) \right]}
\times \left[ 2\left( \frac{1}{x} + \frac{1}{2x^2} + O\left(\frac{1}{x^4}\right) \right) \left( \frac{1}{x^2} - \frac{1}{x^3} + O\left(\frac{1}{x^4}\right) \right) \right]
+ \left( \frac{2}{x^3} + \frac{3}{x^4} + O\left(\frac{1}{x^4}\right) \right) \right]}
= 4.

As a result, we have
\[\deg_{\text{cm}}^{x} [\Psi(x)] \leq 4.\] (3.5)

Combining (3.2) with (3.5) yields (1.2). The proof of Theorem 1.1 is completed.

4. Remarks

In this section we demonstrate the motivation of this paper by retrospecting the history and mentioning some known results related to our Theorem 1.1.

Remark 4.1. In [2, p. 208, (4.39)], it was established that the inequality
\[\Psi(x) > \frac{p(x)}{900x^4(x + 1)^{10}}\] (4.1)
holds for $x > 0$, where
\[p(x) = 75x^{10} + 900x^9 + 4840x^8 + 15370x^7 + 31865x^6 + 45050x^5 + 44101x^4 + 29700x^3 + 13290x^2 + 3600x + 450.\] (4.2)
It is clear that the inequality
\[ \psi(x) > 0 \] (4.3)
for \( x > 0 \) is a weakened version of the inequality (4.1). This inequality was deduced and recovered in [6, Theorem 2.1] and [8, Lemma 1.1]. The inequality (4.3) was also employed in [3, Theorem 4.3], [6, Theorem 2.1], [8, Theorem 2.1], and [40, Theorem 2]. This inequality has been generalized in [3, Lemma 4.6], [7, Lemma 1.2 and Remark 1.3], and [9, 28, 35, 38, 39]. For more information about the history and background of this topic, please refer to the expository and survey articles [26, 42] and plenty of references therein.

In [16], it was proved that, among all functions \([\psi^{(m)}(x)]^2 + \psi^{(n)}(x)\) for \( m, n \in \mathbb{N} \), only the function \( \psi(x) \) is nontrivially completely monotonic on \((0, \infty)\).

In [57, Theorem 1] and [58, Theorem 2], the functions
\[ \frac{x + 12}{12x^4(x + 1)} - \psi(x) \text{, } \psi(x) - \frac{x^2 + 12}{12x^4(x + 1)^2} \text{, and } \psi(x) - \frac{p(x)}{900x^4(x + 1)^{10}} \] (4.4)
are proved to be completely monotonic on \((0, \infty)\). From this, we obtain
\[ \max \left\{ \frac{x^2 + 12}{12x^4(x + 1)^2}, \frac{p(x)}{900x^4(x + 1)^{10}} \right\} < \psi(x) < \frac{x + 12}{12x^4(x + 1)}, \quad x > 0. \] (4.5)

In [43, Theorem 1], the function
\[ h_\lambda(x) = \psi(x) - \frac{x^2 + \lambda x + 12}{12x^4(x + 1)^2} \] (4.6)
was proved to be completely monotonic on \((0, \infty)\) if and only if \( \lambda \leq 0 \), and so is \(-h_\lambda(x)\) if and only if \( \lambda \geq 4 \); Consequently, the double inequality
\[ \frac{x^2 + \mu x + 12}{12x^4(x + 1)^2} < \psi(x) < \frac{x^2 + \nu x + 12}{12x^4(x + 1)^2} \] (4.7)
holds on \((0, \infty)\) if and only if \( \mu \leq 0 \) and \( \nu \geq 4 \). The inequality (4.7) refines and sharpens the right-hand side inequality in (4.5).

Motivated by the above results, we naturally pose the following problems:

1. Is the function \( f_\delta(x) \) defined by (3.3) completely monotonic on \((0, \infty)\)?
2. Is \( \alpha \leq 4 \) the necessary and sufficient condition for the function (3.3) to be completely monotonic on \((0, \infty)\)?

In other words, is the number 4 the completely monotonic degree of the function \( \psi(x) \) with respect to \( x \in (0, \infty) \)?

Our Theorem 1.1 of this paper affirmatively answers the above questions.

Remark 4.2. In [19, p. 2273, Corollary 3] and [20], among others, it was deduced that the functions \( f_2(x) \) and \( f_3(x) \) defined by (3.3) are completely monotonic on \((0, \infty)\), which means that \( \deg_{x_m}^c \psi(x) \geq 3 \). Hence, our Theorem 1.1 strengthens and improves these results and inversely hints or implies that the main results in [19, 20] can be further improved, developed, or amended.

Remark 4.3. It was remarked in [42, p. 137, Remark 2.7] that a divided difference version of the inequality (4.3) has been obtained in [18] implicitly. The divided difference form of the function \( \psi(x) \) and related functions have been investigated in the papers [9, 31, 35, 38, 44] and closely related references therein.
Remark 4.4. The function $\sigma$ defined in (3.1) is a special case of

$$
g_{a,b}(s) = \begin{cases}
    \frac{s}{b^s - a^s}, & s \neq 0, \\
    \frac{1}{\ln b - \ln a}, & s = 0,
\end{cases}
$$

where $a, b$ are positive numbers. We remark that some cases of the function $g_{a,b}(s)$ and its reciprocal have been investigated and applied in many articles such as [11, 13, 14, 15, 17, 22, 25, 27, 29, 30, 32, 33, 36, 37, 45, 47, 55] and closely-related references therein.

Remark 4.5. Recall from [49] that a function $f$ is said to be strongly completely monotonic on $(0, \infty)$ if it has derivatives of all orders and

$$
(-1)^n x^{n+1} f^{(n)}(x)
$$

is nonnegative and decreasing on $(0, \infty)$ for all $n \geq 0$. In [21, p. 34, Proposition 1.1], it was claimed that a function $f(x)$ is strongly completely monotonic if and only if the function $xf(x)$ is completely monotonic. So any completely monotonic function on $(0, \infty)$ of degree not less than 1 with respect to $x$ must be strongly completely monotonic, and the degree with respect to $x$ of any strongly completely monotonic function is not less than 1.

Since not finding a proof for [21, p. 34, Proposition 1.1], we here provide in detail a nontrivial and complete proof for it.

If $xf(x)$ is completely monotonic on $(0, \infty)$, then

$$
(-1)^k [xf(x)]^{(k)} = (-1)^k [xf^{(k)}(x) + kf^{(k-1)}(x)]
$$

$$
= \frac{(-1)^k x^{k+1} f^{(k)}(x) - k[(-1)^{k-1}x^k f^{(k-1)}(x)]}{x^k}
$$

$$
geq 0
$$
on $(0, \infty)$ for all integers $k \geq 0$. From this and by induction, we obtain

$$
(-1)^k x^{k+1} f^{(k)}(x) \geq k[(-1)^{k-1}x^k f^{(k-1)}(x)] \geq \cdots \geq k!xf(x) \geq 0
$$
on $(0, \infty)$ for all integers $k \geq 0$. So $f(x)$ is strongly completely monotonic on $(0, \infty)$.

Conversely, if $f(x)$ is strongly completely monotonic on $(0, \infty)$, then

$$
(-1)^k x^{k+1} f^{(k)}(x) \geq 0
$$

(4.8)

and its derivative

$$
[((-1)^k x^{k+1} f^{(k)}(x))]' = \frac{(k + 1)[(-1)^k x^{k+1} f^{(k)}(x)] - (-1)^{k+1}x^{k+2} f^{(k+1)}(x)}{x}
$$

$$\leq 0
$$
on $(0, \infty)$ for all integers $k \geq 0$. So it follows that $xf(x) \geq 0$ and $(-1)^{k+1} [xf(x)]^{(k+1)}$ on $(0, \infty)$ for all integers $k \geq 0$. As a result, the function $xf(x)$ is completely monotonic on $(0, \infty)$. The proof of [21, p. 34, Proposition 1.1] is complete.

Remark 4.6. Finally we conjecture that the completely monotonic degrees with respect to $x \in (0, \infty)$ of the functions $h_\lambda(x)$ and $-h_\mu(x)$ defined by (4.6) are all 4 if and only if $\lambda \leq 0$ and $\mu \geq 4$. Namely,

$$
\deg_{cm}^x[h_\lambda(x)] = \deg_{cm}^x[-h_\mu(x)] = 4
$$

(4.10)

if and only if $\lambda \leq 0$ and $\mu \geq 4$. 
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