Mutual Information and Boson Radius in $c = 1$ Critical Systems in One Dimension

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(Dated: March 5, 2009)

We study the generic scaling properties of the mutual information between two disjoint intervals, in a class of one-dimensional quantum critical systems described by the $c = 1$ bosonic field theory. A numerical analysis of a spin-chain model reveals that the mutual information is scale-invariant and depends directly on the boson radius. We interpret the results in terms of correlation functions of branch-point twist fields. The present study provides a new way to determine the boson radius, and furthermore demonstrates the power of the mutual information to extract more refined information of conformal field theory than the central charge.

PACS numbers: 03.67.Mn, 05.70.Jk, 75.10.Pq

Conformal field theory (CFT) provides a powerful framework to study one-dimensional (1D) quantum many-body systems. One can recast the low-energy degrees of freedom of both the bosonic and fermionic gases into a simple bosonic field theory, known as the Tomonaga-Luttinger liquid (TLL) \cite{1,2}. Thanks to the recent technological advances, a precise correspondence between the TLL predictions and various (quasi-)1D systems, such as carbon nanotubes \cite{3}, antiferromagnetic chains \cite{4}, and cold Bose gases in 1D traps \cite{5} is currently being investigated.

Given a microscopic model, an important and often nontrivial issue is how to obtain the effective field theory controlling its long-distance behavior. The notion of quantum entanglement, or more specifically, the entanglement entropy, has been extensively applied as a measure of quantum entanglement, or more specifically, the entropy controlling its long-distance behavior. The notion of quantum entanglement, or more specifically, the entanglement entropy, has been extensively applied as a measure of quantum entanglement. In this Letter, we demonstrate that the entanglement entropy can achieve this task if we consider two disjoint intervals, $A = [x_1, x_2]$ and $B = [x_3, x_4]$. We analyze the scaling of the mutual information defined as

$$I_{A:B} := S_A + S_B - S_{A∪B}. \quad (2)$$

This measures the amount of information shared by two subsystems $A$ and $B$. A numerical analysis of a spin-chain model reveals a robust relation between $I_{A:B}$ and $R$, irrespective of microscopic details. We compare the result with the general prediction of Calabrese and Cardy (CC) \cite{9}, and find a relevant correction to their result.

Roughly speaking, the mutual information \cite{8} may be regarded as a region-region correlator. It is known that $I_{A:B}$ is non-negative, and becomes zero if $\rho_{A∪B} = \rho_A \otimes \rho_B$, i.e., in a situation of no correlation \cite{9}. A motivation to consider $I_{A:B}$ comes from that microscopic details at short-range scales, which are often obstacles when analyzing point-point correlators, can be smoothed out over regions. As we enlarge the region sizes, we expect that $I_{A:B}$ detects essential features of the correlations emerging in the coarse-grained limit. When there is a long-range order in local operators, we have $I_{A:B} \neq 0$ for finite local regions $A$ and $B$, even in the limit of large separation \cite{9}. In a critical system with power-law decaying correlations, $I_{A:B}$ goes to zero if $A$ and $B$ are far apart in comparison with their lengths, $r_A$ and $r_B$. However, if $r_A$ and $r_B$ are of the order of the separation, $I_{A:B}$ can remain finite, which is the situation we examine here.

First, suppose we treat the mutual information \cite{8} following the prediction of Calabrese and Cardy \cite{9}, which will turn out in our analysis to correspond to the $SU(2)$-symmetric case. For an infinite chain, the entanglement entropy on double intervals $A∪B$ was predicted to be \cite{9}

$$S_{A∪B} = \frac{c}{3} \log \left( \frac{x_{21}x_{32}x_{43}x_{41}}{x_{31}x_{42}} \right) + 2s_1, \quad (3)$$

with $x_{ij} = x_i - x_j$. Here the constant term $2s_1$ is deter-
we now consider a division (conserved quantity, we can label the ground-states using the Calabrese-Cardy result (4).

FIG. 1: (color online) The mutual information for fixed divisions \( r_A: r_C: r_B: r_D = 1:1:1:1 \) and 1:2:1:2, versus \( \eta = 2\pi R^2 \). We set the magnetization at \( M = \frac{1}{L} \) with \( k = 0, 1, \ldots, \frac{L}{2} - 3 \) for \(-1 < \Delta \leq 1\) and with \( k = 1, \ldots, \frac{L}{2} - 3 \) for \( 1 < \Delta \), so that the system is inside the critical phase. Black and green points correspond to the larger \((L = 28,30)\) and smaller \((L = 24)\) systems, respectively. Horizontal red lines indicate the Calabrese-Cardy result (4).

Minimized so that \( S_{A:B} = S_A + S_B \) in the limit \( x_{31}, x_{43} \ll x_{31}, x_{42} \). For a finite chain of length \( L \), one replaces \( x_j \) by the cord distance \( \frac{L}{2} \sin \frac{\pi x_j}{L} \) in Eqs. (1) and (2). We now consider a division \((r_A, r_C, r_B, r_D)\) of a finite chain as depicted in the inset of Fig. 1. Then the CC formula for the mutual information reads

\[
I_{A:B}^{CC} = \frac{c}{3} \log \left[ \frac{\sin \frac{\pi (r_A + r_C)}{L}}{\sin \frac{\pi r_A}{L}} \frac{\sin \frac{\pi (r_B + r_C)}{L}}{\sin \frac{\pi r_B}{L}} \right]. \tag{4}
\]

Note that the UV-divergent constant \( c \) has been canceled out in the mutual information, and the resultant (4) is invariant under global scale transformations. Similar ideas of eliminating the UV-divergence have been suggested by Casini and Huerta (4) and have also been exploited in the context of topological entropy in higher dimensions. Henceforth, lengths of (sub)systems are measured in units of the lattice spacing.

Now we turn to numerical analyses of the mutual information in a spin chain, based on Lanczos diagonalization of finite systems up to \( L = 30 \). We consider a spin-\( \frac{1}{2} \) XXZ chain in a magnetic field,

\[
H := \sum_{j=1}^{L} \left( S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z \right) - h \sum_{j=1}^{L} S_j^z. \tag{5}
\]

Since the magnetization per site, \( M := \frac{1}{L} \sum_j S_j^z \), is a conserved quantity, we can label the ground-states using \( M \). A critical phase extends over a wide region in \( \Delta > -1 \) (4). The boson radius \( R \) controls the nature of critical correlations. Indeed, the exponents for the leading algebraic decay of spin correlations, \( \langle S_j^z S_{j+1}^z \rangle \) and \( \langle S_j^z S_{j+1}^z \rangle - M^2 \), are given by \( \eta := 2\pi R^2 \) and \( \min(1/\eta, 2) \), respectively. Henceforth we frequently use \( \eta = 2\pi R^2 \) instead of \( R \). For \(-1 < \Delta \leq 1 \) and \( h = 0 \), we have \( \eta = 1 - (1/\pi) \arccos \Delta \). For \( \Delta > 1 \), the system is in a gapped Néel phase at \( h = 0 \) and enters the critical phase at a critical field with \( \eta = 2 \). For general \( h \neq 0 \), \( \eta \) can be determined by numerically solving the integral equations obtained from the Bethe ansatz (4, 5, 20). When increasing \( h \), \( \eta \) monotonically increases \((-1 < \Delta < 0)\) or decreases \((0 < \Delta)\) to \( 1/2 \) at the saturation. Summaries of the value of \( \eta \) in the \( M-\Delta \) and \( h-\Delta \) phase diagrams can be found in e.g. Refs. (4) and (5).

We first evaluate \( I_{A:B} \) for fixed divisions \((r_A, r_C, r_B, r_D) = \frac{1}{3}(1,1,1,1) \) and \( \frac{1}{2}(1,2,1,2) \). Figure 2 shows a plot of \( I_{A:B} \) against \( \eta = 2\pi R^2 \) for various \((M, \Delta)\) in the critical phase. Remarkably, the data points almost form a single curve for each type of division. The collapse of a two-dimensional \( M-\Delta \) plane onto these two curves strongly indicates a direct dependence of \( I_{A:B} \) on \( R \). Agreement with the CC formula (4) can be observed only around \( \eta = 1 \) (\( SU(2) \)-symmetric case). One can also observe that \( I_{A:B} \) is symmetric under \( \eta \to 1/\eta \), which might reflect the duality in the effective theory.

In Fig. 2 we plot \( I_{A:B} \) as a function of \( \eta \) for divisions \((r_A, r_C, r_B, r_D) = (r, \frac{1}{2} - r, r, \frac{1}{2} - r) \), in comparison with CC formula (4). For each \( \Delta \), the results from \( L = 28 \) and 24 obey a single curve, indicating the scale invariance of \( I_{A:B} \). The curve for \( \Delta = 1 \) agrees well with the CC formula (4). In other cases, the curves run above the CC formula. We can confirm that \( I_{A:B} \) approaches zero in the limit \( \frac{1}{\eta} \to 0 \), as expected for systems without long-range order. If we subtract the CC formula (see black circles in Fig. 1), we find that the curves are symmetric under \( \frac{1}{\eta} \to \frac{1}{\eta} - \frac{1}{2} \) and have maxima at \( \frac{1}{\eta} = \frac{1}{4} \).
The von Neumann entropy

\[ S_A = \log(\text{Tr} \rho_A^n) \]

This, it is expected that

\[ R_{\text{ent}}(n) = \frac{1}{n} \log(\text{Tr} \rho_A^n). \]

The von Neumann entropy \( S_A \) can be reached in the limit \( n \to 1 \). Following Calabrese and Cardy [21], one can derive the following expression (originally found in Ref. [21]) for a single interval \( A = [x_1, x_2] \) in an infinite chain:

\[ R_A^{(n)} = \frac{1 + n}{6n} c \log x_{21} + s_n, \]

where \( s_n \) is again a UV-divergent constant. Likewise, within CC argument, the translation from von Neumann to Rényi can be done via replacements \( \frac{1}{n} \to \frac{1 + n}{6n}, \) \( s_1 \to s_n \). We define the “Rényi” mutual information as \( I_{A:B}^{(n)} := R_A^{(n)} + R_B^{(n)} - R_{A:B} \).

In Fig. 3, we plot the deviation of the “Rényi” mutual information \( I_{A:B}^{(n)} \) from the CC prediction \( I_{A:B}^{\text{CC}(n)} \) for divisions \( (r, \frac{r}{2} - r, r, \frac{r}{2} - r) \). Different symbols correspond to \( n = 1, 2, 3, 4 \).

we relate it to a correlation function of twist fields \( T \) and \( \tilde{T} \) with conformal dimensions \( \Delta_n = \Delta_n = \frac{1}{2} \left( n - \frac{1}{2} \right) \) [23]. For double intervals \( A \cup B = [x_1, x_2] \cup [x_3, x_4] \) in an infinite chain, we can write it down as

\[ \text{Tr} \rho_{A:B}^{n} \propto \langle T(x_1) \tilde{T}(x_2) T(x_3) \tilde{T}(x_4) \rangle. \]

The SL(2, \( \mathbb{C} \)) covariance property requires this four-point function to have the following form:

\[ \frac{\bar{x}_{31} x_{42}}{x_{21} x_{32} x_{43} x_{41}} 2\Delta_n \]

with \( x_i = \bar{x}_i \). Here, \( F_n(x, \bar{x}; \eta) \) is a function of the cross ratios \( x := \frac{\bar{x}_{31} x_{42}}{x_{21} x_{32}} \) and \( \bar{x} := \frac{x_{43} x_{41}}{x_{21} x_{32}} \), normalized as \( \lim_{\eta \to 0} F_n(x, \bar{x}; \eta) = 1 \), and should be determined by \( \eta = 2\pi R^2 \) as suggested by Fig. 3. The power function \( \langle \cdots \rangle\Delta_0 \langle \cdots \rangle\Delta^+ \) corresponds to the CC prediction [23], and the function \( F_n \) gives an additional contribution \( \frac{1}{n+1} \log F_n(x; \eta) := f_{n}(x; \eta) \) to the Rényi entropy \( I_{A:B}^{(n)} \). The mutual information detects this new part:

\[ I_{A:B}^{(n)} - I_{A:B}^{\text{CC}(n)} = f_n(x; \eta), \]

\[ I_{A:B}^{(n)} - I_{A:B}^{\text{CC}(n)} = \lim_{n \to 1} f_n(x; \eta) =: f(x; \eta). \]

As a check of this result, we plot \( I_{A:B} - I_{A:B}^{\text{CC}} \) as a function of the cross ratio \( x \) in Fig. 4. For a finite chain, the cross ratio is given by

\[ x = \frac{\sin \frac{x_{43}}{L}}{\sin \frac{x_{32}}{L}} \frac{\sin \frac{x_{41}}{L}}{\sin \frac{x_{21}}{L}} \]
The crossing-invariant solution gives the correlation function (8) of four twist fields with these squared lines. The formula agrees relatively well with the data of At a special point for $\Delta = 0$, and runs slightly above the data. This small disagreement might be due to finite-size effects, to a subtle difference between lattice systems and continuum descriptions, or to some missing factor in Eq. (13).

We can confirm that for a given $\Delta$, and for various divisions $(r_A, r_C, r_B, r_D)$, the additional contribution to the CC result can be fit by a single curve with good accuracy, strongly supporting Eq. (11).

For $n = 2$, two twist fields, $T$ and $\tilde{T}$, are identical, and have conformal dimensions $\Delta_2 = \Delta_3 = 1/16$. The correlation function (3) of four twist fields with these dimensions (Ramond fields) has been derived previously [23, 24]. The crossing-invariant solution gives

$$f^R_2(x; \eta) = \log \frac{\theta_2(\eta \tau) \theta_3(\eta^{-1} \tau)}{[\theta_3(\tau)]^2},$$

(13)

where $\tau$ is pure imaginary, and is related to $x$ via $x = [\theta_2(\tau)/\theta_3(\tau)]^4$. Here $\theta_2$ and $\theta_3$ are Jacobi theta functions:

$$\theta_2(\tau) := \sum_{m \in \mathbb{Z}} e^{i \pi (m+1/2)^2}, \quad \theta_3(\tau) := \sum_{m \in \mathbb{Z}} e^{i \pi m^2}.$$  

(14)

At a special point $\eta = 1/2$, Eq. (13) can be simplified as $f^R_2(x; 1/2) = \log (1 + (1-x)^i)/(1 + (1-x)^i)/2$. One can check that $f^R_2(x; \eta)$ satisfies all the aforementioned properties (i)-(iii). When two intervals of small lengths $x_{21} = x_{43} =: r$ are separated far apart by a distance $x_{31} = x_{42} =: d(\gg r)$, Eq. (13) reduces to $f^R_2 \approx 2 \left(\frac{x_{21}}{d}\right)^{2\min(1/\eta, \eta)}$. This scales as the dominant spin correlation squared.

In Fig. 3, the formula $f^R_2(x; \eta)$ is drawn as smooth blue lines. The formula agrees relatively well with the data of $f^{(2)}_{A:B} - f^{(2)}_{CC(2)}$ for $\Delta = 0$, and runs slightly above the data for $\Delta = -0.6$. In Fig. 3, we plot $f^{(2)}_{A:B}$ for the 1:1:1:1 division, in comparison with $f^R_2 + f_{CC(2)}$. For $\eta \geq 0.5$, $f^{(2)}_{A:B}$ contains strong oscillations, and the formula goes inside these oscillations. For $\eta \lesssim 0.5$, oscillations are small, but the formula goes slightly above the data. This small disagreement might be due to finite-size effects, to a subtle difference between lattice systems and continuum descriptions, or to some missing factor in Eq. (13).

To summarize, we have shown that the scaling of the mutual information $I_{A:B}$ is controlled directly by the boson radius $R$. This result can be used as a new method to determine $R$ from the ground state, complementary to the standard spectroscopic method [2] based on the Drude weight and the compressibility. To obtain an analytical expression of $f_{n}(x; \eta)$ for general $n$ and especially its $n \rightarrow 1$ limit $f(x; \eta)$ remains a challenging and intriguing open problem. In general, we expect that every CFT has its characteristic function $f(x)$ in the mutual information. This can be used as a fingerprint for distinguishing different CFTs, as originally suggested in Ref. [16].

The authors are grateful to J. Cardy, A. Furusaki, D. Ivanov, H. Katsura, G. Misguich, B. Nienhuis, M. Oshikawa, S. Ryu, and Masahiro Sato for stimulating discussions. The collaboration of the authors was initiated in the Workshop of “Topological Aspects of Solid State Physics” at ISSP, Univ. of Tokyo.

Note added. After the preprint of this paper was posted on arXiv, Calabrese and Cardy have added a note to their paper [1]. Another analysis of the double-interval entropy has been done in parallel by Caraglio and Gliozzi [25].

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