Remarks on the analysis of the reaction $e^+e^- \rightarrow \Sigma^0\bar{\Sigma}^0$

Göran Fäldt

Department of physics and astronomy,
Uppsala University, Box 516, S-751 20 Uppsala, Sweden
(Dated: May 22, 2020)

Abstract

We investigate roads for evaluating model-independent cross-section distributions for the sequential hyperon decay $\Sigma^0 \rightarrow \Lambda\gamma; \Lambda \rightarrow p\pi^-$ and its corresponding antihyperon decay. The hyperons are produced in the reaction $e^+e^- \rightarrow J/\psi \rightarrow \Sigma^0\bar{\Sigma}^0$. Cross-section distributions are calculated using the folding technique.

PACS numbers:

*Electronic address: goran.faldt@physics.uu.se
I. INTRODUCTION

The BESIII experiment [1] is exploring new venues into hyperon physics, based on $e^+e^-$ annihilation into hyperon-antihyperon pairs. In a recent paper [2], we investigated in some detail the reaction $e^+e^- \rightarrow J/\psi \rightarrow \Sigma^0 \bar{\Sigma}^0$ and its associated decay chains $\Sigma^0 \rightarrow \Lambda \gamma; \Lambda \rightarrow p\pi^-$ and $\bar{\Sigma}^0 \rightarrow \bar{\Lambda} \gamma; \bar{\Lambda} \rightarrow \bar{p} \pi^+$. By measuring this process in the vicinity of the $J/\psi$-vector-charmonium state, one gains information on the strong baryon-antibaryon-decay process of the $J/\psi$-vector-charmonium state and also, it offers a model-independent way of measuring weak-decay-asymmetry parameters, that in turn can probe $CP$ symmetry [3].

The diagram for the basic reaction $e^+e^- \rightarrow J/\psi \rightarrow \Sigma^0 \bar{\Sigma}^0$ is graphed in Fig.1. Its structure is governed by two vertices. The strength of the lepton-vertex function is determined by a single parameter, the electromagnetic-fine-structure constant $\alpha_e$, but two complex form factors $G^\psi_M(s)$ and $G^\psi_E(s)$ are needed for the baryonic-vertex function. However, we shall not work with the form factors themselves but with certain combinations thereof: the strength of form factors $D^\psi(s)$; the ratio of form-factor magnitudes $\eta^\psi(s)$; and the relative phase of form factors $\Delta \Phi^\psi(s)$. These form-factor combinations are defined in Appendix A.

The theoretical description of the annihilation reaction of Fig.1 can be found in Ref.[4]. Accurate experimental results for the form-factor parameters $\eta^\psi$ and $\Delta \Phi^\psi$ and the weak-interaction parameters $\alpha_\Lambda(\alpha_{\bar{\Lambda}})$ for the $J/\psi$ annihilation process are all reported in Ref.[3]. In addition, the graph can be generalized to include hyperons that decay sequentially.

Our analysis of the cross-section-distribution function for the annihilation reaction $e^+e^- \rightarrow J/\psi \rightarrow \Sigma^0 \bar{\Sigma}^0$, followed by its subsequent hyperon decays, starts from the master formula of Ref.[2], and which is reproduced in the following section. The purpose of
our investigation is to find out which coordinate choice would be most convenient when evaluating the master formula, and at the same time being able to compare our result to those of others.

II. MASTER FORMULA

In several previous publications we studied $e^+e^-$ annihilation into hyperon pairs $Y\bar{Y}$ and the subsequent decays of those pairs. Photon as well as charmonium induced annihilation was considered. In the present investigation we limit ourselves to the hyperon-decay chain $\Sigma^0 \to \Lambda \gamma; \Lambda \to p\pi^-$, and its corresponding antihyperon-decay chain $\bar{\Sigma}^0 \to \bar{\Lambda} \gamma; \bar{\Lambda} \to \bar{p}\pi^+$, again when simultaneously occurring in the reaction $e^+e^- \to J/\psi \to \Sigma^0\bar{\Sigma}^0$.

In Ref.[2] it was shown that the cross-section-distribution function for a $J/\psi$ induced joint production and subsequent decay of a $\Sigma^0\bar{\Sigma}^0$ pair can be summarized in the master formula

$$d\sigma = d\sigma(e^+e^- \to J/\psi \to \Sigma^0\bar{\Sigma}^0) \left[ W(\xi) \right] \frac{d\Phi(\Sigma^0, \Lambda, p; \bar{\Sigma}^0, \bar{\Lambda}, \bar{p})}{R}, \quad (II.1)$$

As can be seen the master formula involves three factors, describing the annihilation of a lepton pair into a hyperon pair; the folded product of spin densities $W(\xi)$ representing hyperon production and decay; and the phase space element of sequential hyperon decays. Each event is specified by a nine-dimensional vector $\xi = (\theta, \Omega_\Lambda, \Omega_p, \Omega_{\bar{\Lambda}}, \Omega_{\bar{p}})$, with $\theta$ the scattering angle in the $e^+e^- \to \Sigma^0\bar{\Sigma}^0$ subprocess.

Following Refs. [1] and [2] we write the cross-section-distribution function for the $J/\psi$ induced annihilation reaction $e^+e^- \to J/\psi \to \Sigma^0\bar{\Sigma}^0$ as

$$\frac{d\sigma}{d\Omega_{\Sigma^0}}(e^+e^- \to J/\psi \to \Sigma^0\bar{\Sigma}^0) = \frac{\alpha_\psi\alpha_g}{k} \frac{\alpha_\psi\alpha_g}{(s - m_\psi^2)^2 + m_\psi^2\Gamma(m_\psi)} D_\psi(s) R, \quad (II.2)$$

where the strength function $D_\psi(s)$ is defined in Appendix A, and the structure function $R$ in Appendix B. The electromagnetic-coupling constant $\alpha_\psi$ is determined by the electromagnetic-decay width $\Gamma(J/\psi \to e^+e^-)$, and the hadronic-coupling constant $\alpha_g$ similarly by the hadronic-decay width $\Gamma(J/\psi \to \Sigma^0\bar{\Sigma}^0)$.

The differential-spin-distribution function $W(\xi)$ of Eq.(II.1) is obtained by folding a product of five spin densities,

$$W(\xi) = \left\langle S(n_{\Sigma^0}, n_{\bar{\Sigma}^0}) G(n_{\Sigma^0}, n_\Lambda) G(n_\Lambda, n_p) G(n_{\bar{\Sigma}^0}, n_{\bar{\Lambda}}) G(n_{\bar{\Lambda}}, n_{\bar{p}}) \right\rangle_n, \quad (II.3)$$
in accordance with the prescription of Ref.\cite{5} and of Eq.\( (V.20) \). The folding operation \( \langle ... \rangle_n \) applies to each of the six hadron spin vectors, \( n_{\Sigma^0}, ..., n_{\bar{p}} \).

The function \( S(n_{\Sigma^0}, n_{\bar{\Sigma}^0}) \) represents the spin-density distribution for the \( \Sigma^0\bar{\Sigma}^0 \) hyperon pair. This function also depends on the unit vectors \( l_{\Sigma^0} \) and \( l_{\bar{\Sigma}^0} \), which are unit vectors in the directions of motion of the \( \Sigma^0 \) and \( \bar{\Sigma}^0 \) hyperons in the center-of-momentum (c.m.) frame of the event. The four remaining spin-density-distribution functions \( G(n_{Y_1}, n_{Y_2}) \) represent spin-density distributions for the hyperon decays \( \Sigma^0 \rightarrow \Lambda \gamma; \) or \( \Lambda \rightarrow p\pi^-; \) or their antihyperon counterparts.

The spin-decay-distribution functions \( G(n_{Y_1}, n_{Y_2}) \) are normalized to unity, which means their spin independent terms are unity. However, for convenience the spin-density-distribution function \( S(n_{\Sigma^0}, n_{\bar{\Sigma}^0}) \) is normalized to \( R \).

The phase-space factor, \( d\Phi(\Sigma^0, \Lambda, p; \bar{\Sigma}^0, \bar{\Lambda}, \bar{p}) \) of the master equation, describes the normalized phase-space element for the sequential decays of the two baryons \( \Sigma^0 \) and \( \bar{\Sigma}^0 \),

\[
d\Phi(\Sigma^0, \Lambda, p; \bar{\Sigma}^0, \bar{\Lambda}, \bar{p}) = \frac{n(\Sigma^0 \rightarrow \Lambda \gamma) d\Omega_{\Lambda}}{3\pi n(\Sigma^0 \rightarrow all)} \times \frac{n(\Lambda \rightarrow p\pi^-) d\Omega_{\bar{p}}}{3\pi n(\Lambda \rightarrow all)} \times \frac{n(\bar{\Sigma}^0 \rightarrow \bar{\Lambda} \gamma) d\Omega_{\bar{\Lambda}}}{3\pi n(\bar{\Sigma}^0 \rightarrow all)} \times \frac{n(\bar{\Lambda} \rightarrow \bar{p}\pi^-) d\Omega_{\bar{\bar{p}}}}{3\pi n(\bar{\Lambda} \rightarrow all)}.
\]

The widths are defined in the usual way. For \( n(\Sigma^0 \rightarrow \Lambda \gamma) \) this means forming an average over the \( \Sigma^0 \) spin directions, and summing over the Lambda and gamma spin directions. The angles \( \Omega_{\Lambda} \) define the direction of motion of the \( \Lambda \) hyperon in the \( \Sigma^0 \) rest system; the angles \( \Omega_{p} \) the direction of motion of the \( p \) baryon in the \( \Lambda \) rest system, and so on.

### III. \( e^+e^- \) ANNIHILATION INTO \( \Sigma^0\bar{\Sigma}^0 \) PAIRS

The cross-section-distribution function for \( e^+e^- \) annihilation into a \( \Sigma^0\bar{\Sigma}^0 \) pair appears in two places in the master formula of Eq.\( (II.1) \). The unpolarized-cross-section-distribution function is a prefactor in the master formula, and the hyperon-spin-density-distribution function enters as a factor in the spin-density-distribution function of Eq.\( (II.3) \).

The cross-section distribution for polarized-final-state hyperons was derived in Refs.\cite{2} and \cite{4},

\[
\frac{d\sigma}{d\Omega_{\Sigma^0}} (e^+e^- \rightarrow J/\psi \rightarrow \Sigma^0\bar{\Sigma}^0) = \frac{p}{4k} \frac{\alpha\alpha_D(s)}{(s - m_\psi^2)^2 + m_\psi^2} S(n_{\Sigma^0}, n_{\bar{\Sigma}^0}),
\]
where $D_{\psi}(s)$ is the strength function of Eq.(A.1), $\mathbf{n}_{\Sigma^0}$ and $\mathbf{n}_{\bar{\Sigma}^0}$ the spin vectors of the $\Sigma^0$ and $\bar{\Sigma}^0$ hyperons, and $S(\mathbf{n}_{\Sigma^0}, \mathbf{n}_{\bar{\Sigma}^0})$ the spin-density-distribution function for the final-state hyperons. This spin-density-distribution function is normalized so that its spin-independent part equals $\mathcal{R}$, with

$$\mathcal{R} = 1 + \eta_\psi \cos^2 \theta,$$  \hspace{1cm} (III.6)

according to Eq.(B.1). Consequently, summing over the final-state-hyperon polarizations gives the unpolarized cross-section-distribution function

$$\frac{d\sigma}{d\Omega_{\Sigma^0}}(e^+ e^- \rightarrow J/\psi \rightarrow \Sigma^0 \bar{\Sigma}^0) = \frac{p}{k} \frac{\alpha_\psi \alpha_g}{(s - m_{\psi}^2)^2 + m_{\psi}^2 \Gamma(m_{\psi})} D_{\psi}(s) \mathcal{R},$$  \hspace{1cm} (III.7)

For a spin-one-half baryon of four-momentum $\mathbf{p}$, the four-vector spin $s(p)$ is related to the three-vector-spin direction $\mathbf{n}$, the spin in the rest system, by

$$s(\mathbf{p}, \mathbf{n}) = \frac{n_{\parallel}}{M}(|\mathbf{p}|, E\hat{\mathbf{p}}) + (0, \mathbf{n}_{\perp}).$$  \hspace{1cm} (III.8)

Longitudinal and transverse directions of vectors are relative to the $\hat{\mathbf{p}}$ direction.

In the global c.m. system kinematics simplifies. There, three-momenta $\mathbf{p}$ and $\mathbf{k}$ are defined such that

$$\mathbf{p}_{\Sigma^0} = -\mathbf{p}_{\bar{\Sigma}^0} = \mathbf{p},$$  \hspace{1cm} (III.9)

$$\mathbf{k}_{e^+} = -\mathbf{k}_{e^-} = \mathbf{k},$$  \hspace{1cm} (III.10)

and the scattering angle $\theta$ such that $\cos \theta = \hat{\mathbf{p}} \cdot \hat{\mathbf{k}}$. For the $\Sigma^0$ and $\bar{\Sigma}^0$ unit vectors $\mathbf{l}_{\Sigma^0}$ and $\mathbf{l}_{\bar{\Sigma}^0}$, we have $\mathbf{l}_{\Sigma^0} = -\mathbf{l}_{\bar{\Sigma}^0} = \hat{\mathbf{p}}$.

The spin-density-distribution function $S(\mathbf{n}_{\Sigma^0}, \mathbf{n}_{\bar{\Sigma}^0})$ is a sum of seven mutually orthogonal contributions [6],

$$S(\mathbf{n}_{\Sigma^0}, \mathbf{n}_{\bar{\Sigma}^0}) = \mathcal{R} + \mathcal{S} \mathbf{N} \cdot \mathbf{n}_{\Sigma^0} + \mathcal{S} \mathbf{N} \cdot \mathbf{n}_{\bar{\Sigma}^0} + \mathcal{T}_1 \mathbf{n}_{\Sigma^0} \cdot \hat{\mathbf{p}} \mathbf{n}_{\bar{\Sigma}^0} \cdot \hat{\mathbf{p}}$$

$$+ \mathcal{T}_2 \mathbf{n}_{\Sigma^0 \perp} \cdot \mathbf{n}_{\bar{\Sigma}^0 \perp} + \mathcal{T}_3 \mathbf{n}_{\Sigma^0 \perp} \cdot \hat{\mathbf{k}} \mathbf{n}_{\bar{\Sigma}^0 \perp} \cdot \hat{\mathbf{k}} / \sin^2 \theta$$

$$+ \mathcal{T}_4 \left( \mathbf{n}_{\Sigma^0} \cdot \hat{\mathbf{p}} \mathbf{n}_{\bar{\Sigma}^0 \perp} \cdot \hat{\mathbf{k}} + \mathbf{n}_{\bar{\Sigma}^0} \cdot \hat{\mathbf{p}} \mathbf{n}_{\Sigma^0 \perp} \cdot \hat{\mathbf{k}} \right) / \sin \theta,$$  \hspace{1cm} (III.11)

where $\mathbf{N}$ is normal to the scattering plane,

$$\mathbf{N} = \frac{1}{\sin \theta} \hat{\mathbf{p}} \times \hat{\mathbf{k}}.$$  \hspace{1cm} (III.12)

The six structure functions $\mathcal{R}$, $\mathcal{S}$, and $\mathcal{T}$ of Eq.(III.11) depend on the scattering angle $\theta$, the ratio function $\eta_\psi(s)$, and the phase function $\Delta \Phi_\psi(s)$. For their definitions we refer to Appendix [B], but be careful, our original definitions were slightly different [6].
IV. ASSORTED SPIN DENSITIES

To be able to calculate the differential-distribution function of Eq. (II.3) we need in addition to the spin-density-distribution function for the $\Sigma^0\Sigma^0$ final-state pair, the spin-density-distribution functions for the decays $\Sigma^0 \rightarrow \Lambda\gamma$ and $\Lambda \rightarrow p\pi^-$, and their antiparticle conjugate decays.

Weak decays of spin-one-half baryons, such as $\Lambda \rightarrow p\pi^-$, involve both S- and P-wave amplitudes, and the spin-density-decay distribution is commonly parametrized by three parameters, denoted $\alpha\beta\gamma$, and which fulfill a relation

$$\alpha^2 + \beta^2 + \gamma^2 = 1.$$  \hspace{1cm} (IV.13)

Details of this description can be found in Refs. [8] and [2].

The spin-density-distribution function $G(n_\Lambda, n_p)$, describing the decay $\Lambda \rightarrow p\pi^-$, is a scalar, which we choose to evaluate in the rest system of the $\Lambda$ hyperon, to get

$$G(n_\Lambda, n_p) = 1 + \alpha_\Lambda n_\Lambda \cdot l_p + \alpha_\Lambda n_p \cdot l_p + n_\Lambda \cdot L_\Lambda(n_p, l_p),$$  \hspace{1cm} (IV.14)

with the vector-valued function $L_\Lambda(n_p, l_p)$ defined as

$$L_\Lambda(n_p, l_p) = \gamma_\Lambda n_p + [(1 - \gamma_\Lambda) n_p \cdot l_p] l_p + \beta_\Lambda n_p \times l_p.$$  \hspace{1cm} (IV.15)

Here, $n_\Lambda$ and $n_p$ are the spin vectors of the $\Lambda$ hyperon and the $p$ baryon, and $l_p$ a unit vector in the direction of motion of the proton in the rest system of the $\Lambda$ hyperon. The $\Lambda$ indices remind us the parameters refer to a $\Lambda$ decay. An important aspect of the spin-density-distribution function is its normalization. The spin-independent term is unity.

The spin-density-distribution function $G(n_\Lambda, n_\bar{\Lambda})$ for the antiparticle-conjugate decay $\bar{\Lambda} \rightarrow \bar{p}\pi^+$ has exactly the same functional structure as $G(n_\Lambda, n_p)$, but the decay parameters take other numerical values. For $CP$ conserving interactions the asymmetry parameters of the $\Lambda$-hyperon decay are related to those of the $\bar{\Lambda}$-hyperon decay by [9, 10]

$$\alpha_\Lambda = -\alpha_{\bar{\Lambda}}, \quad \beta_\Lambda = -\beta_{\bar{\Lambda}}, \quad \gamma_\Lambda = \gamma_{\bar{\Lambda}}.$$  \hspace{1cm} (IV.16)

Next, we turn to the electromagnetic M1 transition $\Sigma^0 \rightarrow \Lambda\gamma$. It is caused by a transition-magnetic moment, of strength

$$\mu_{\Sigma\Lambda} = e F_2(0)/(m_\Sigma + m_\Lambda).$$  \hspace{1cm} (IV.17)
The normalized-spin-density-distribution function for a $\Sigma^0 \to \Lambda\gamma$ transition to a final state of fixed photon helicity $\lambda\gamma$ is, according to Ref.\[2\],

$$G_\gamma(n_{\Sigma^0}, n_\Lambda; \lambda\gamma) = 1 - n_{\Sigma^0} \cdot l_\gamma l_\gamma \cdot n_\Lambda + \lambda\gamma(n_{\Sigma^0} \cdot l_\gamma - n_\Lambda \cdot l_\gamma), \quad (IV.18)$$

where $l_\gamma$ is a unit vector in the direction of motion of the photon, and $l_\Lambda = -l$, a unit vector in the direction of motion of the $\Lambda$ hyperon, both in the rest system of the $\Sigma^0$ baryon. The photon helicities $\lambda\gamma$ take on the values $\pm 1$.

We notice that when both hadron spins are parallel or anti-parallel to the photon momentum, then the decay probability vanishes, a property of angular-momentum conservation.

Summing, in Eq.\[IV.19\], the contributions from the two photon-helicity states gives the normalized-spin-density-distribution function

$$G(n_{\Sigma^0}, n_\Lambda) = 1 - n_{\Sigma^0} \cdot l_\gamma l_\gamma \cdot n_\Lambda. \quad (IV.19)$$

The normalized-spin-density-distribution function for the conjugate transition, $\bar{\Sigma}^0 \to \bar{\Lambda}\gamma$, is obtained by replacing, in expression \[IV.19\], the particle spin vectors $n_{\Sigma^0}$ and $n_\Lambda$ by the antiparticle-spin vectors $n_{\bar{\Sigma}^0}$ and $n_\bar{\Lambda}$.

V. SEQUENTIAL DECAY OF HYPERONS

A factor of our master formula for hyperon production and decay, Eq.\[II.1\], is the differential-spin-distribution function $W(\xi)$ of Eq.\[II.3\], which is obtained by folding a product of five spin densities. The folding prescription is especially adapted to spin one-half baryons. A folding operation implies forming an average over intermediate-spin directions $n$ according to the prescription of Refs.\[5\] and \[7\]

$$\langle 1 \rangle_n = 1, \quad \langle n \rangle_n = 0, \quad \langle n \cdot k \cdot l \rangle_n = k \cdot l. \quad (V.20)$$

The spin-density distribution $W(n_{\Sigma^0}, n_\rho)$ for the decay chain $\Sigma^0 \to \Lambda\gamma; \Lambda \to p\pi^-$ is obtained by folding the product of the spin density distributions in the decay chain. We obtain

$$W(n_{\Sigma^0}, n_\rho) = \left\langle G(n_{\Sigma^0}, n_\Lambda)G(n_\Lambda, n_\rho) \right\rangle_{n_\Lambda}, \quad (V.21)$$
where the two spin-density-distribution functions on the right-hand side are defined in Eqs.\textcolor{red}{(IV.19)} \textcolor{red}{and (IV.14)}. Performing the folding operation gives

\[
W(n_{\Sigma^0}, n_p) = U_{\Sigma^0} + n_{\Sigma^0} \cdot V_{\Sigma^0},
\]

\[
U_{\Sigma^0} = 1 + \alpha \Lambda n_p \cdot l_p,
\]

\[
V_{\Sigma^0} = -l_\gamma [\alpha \Lambda l_\gamma \cdot l_p + n_p \cdot L_\Lambda(l_\gamma, -l_p)],
\]

and ditto for \(W(n_{\bar{\Sigma}^0}, n_{\bar{p}})\).

\section*{VI. PRODUCTION AND DECAY OF \(\Sigma^0\bar{\Sigma}^0\) PAIRS}

Now, we come to our final task; production and decay of \(\Sigma^0\bar{\Sigma}^0\) pairs. The starting point is the reaction \(e^+e^- \rightarrow \Sigma^0\bar{\Sigma}^0\), the spin-density-distribution function of which was calculated in Sect. 3, and named \(S(n_{\Sigma^0}, n_{\Sigma^0})\). The spin-density-distribution function \(W(n_{\Sigma^0}, n_p)\) which represents the decay chain \(\Sigma^0 \rightarrow \Lambda \gamma; \Lambda \rightarrow p\pi^-\) was calculated in Sect. 5; and so for the anti-chain-decay function \(W(n_{\bar{\Sigma}^0}, n_{\bar{p}})\).

The final-state-angular distributions are obtained by folding the spin distributions for production and decay, according to prescription \textcolor{red}{(V.20)}. Invoking Eq.\textcolor{red}{(II.11)} for the production step and Eq.\textcolor{red}{(V.22)} and its anti-distribution for the decay steps, we get the differential-spin-
The density-distribution function

\[
\mathcal{W}(\xi) = \left\langle S(n_{\Sigma^0}, n_{\bar{\Sigma}^0}) W(n_{\Sigma^0}, n_p) W(n_{\bar{\Sigma}^0}, n_p) \right\rangle_{n_{\Sigma^0}, n_{\bar{\Sigma}^0}} 
\]

\[
= RU_{\Sigma^0}U_{\bar{\Sigma}^0} + SU_{\Sigma^0} N \cdot V_{\Sigma^0} + SU_{\bar{\Sigma}^0} N \cdot V_{\bar{\Sigma}^0} 
+ \mathcal{T}_1 V_{\Sigma^0} \cdot \hat{p} V_{\bar{\Sigma}^0} \cdot \hat{p} + \mathcal{T}_2 V_{\Sigma^0 \perp} \cdot V_{\bar{\Sigma}^0 \perp} 
+ \mathcal{T}_3 V_{\Sigma^0 \perp} \cdot \hat{k} V_{\Sigma^0 \perp} \cdot \hat{k} / \sin^2 \theta 
+ \mathcal{T}_4 \left( V_{\Sigma^0} \cdot \hat{p} V_{\Sigma^0 \perp} \cdot \hat{k} + V_{\Sigma^0} \cdot \hat{p} V_{\Sigma^0 \perp} \cdot \hat{k} \right) / \sin \theta. \quad (VI.25)
\]

The functions \(U_{\Sigma^0}\) and \(V_{\Sigma^0}\) are defined in Sect. 5, and

\[
U_{\Sigma^0} = 1 + \alpha_L n_p \cdot l_p, \quad (VI.26)
\]

\[
V_{\Sigma^0} = -l_\gamma \left[ \alpha_L l_\gamma \cdot l_p + n_p \cdot L_\Lambda(l_\gamma, -l_p) \right]. \quad (VI.27)
\]

We observe that \(U_{\Sigma^0}\) depends on the weak interaction parameter \(\alpha_L\), whereas \(V_{\Sigma^0}\) in addition depends on the parameters \(\beta_\Lambda\) and \(\gamma_\Lambda\) through the vector function \(L_\Lambda\), of Eq.(IV.15).

The angular distributions of Eq.(VI.25) are the most general ones, and still depend on the spin vectors \(n_p\) and \(n_{\bar{p}}\) which are difficult to measure. If we are willing to consider proton- and anti-proton-spin averages, then variables \(U\) and \(V\) simplify,

\[
U_{\Sigma^0} = 1, \quad V_{\Sigma^0} = -\alpha_L l_\Lambda \cdot l_p l_\Lambda,
\]

\[
U_{\bar{\Sigma}^0} = 1, \quad V_{\bar{\Sigma}^0} = -\alpha_\bar{\Lambda} l_\bar{\Lambda} \cdot l_\bar{p} l_\bar{\Lambda}. \quad (VI.28)
\]

Since \(U_{\Sigma^0} = U_{\bar{\Sigma}^0} = 1\) the effect of the folding is to make, in the spin-density function \(S(n_{\Sigma^0}, n_{\Sigma^0})\) of Eq.(III.11), the replacements \(n_{\Sigma^0} \rightarrow V_{\Sigma^0}\) and \(n_{\bar{\Sigma}^0} \rightarrow V_{\bar{\Sigma}^0}\). We notice that the \(U\) and \(V\) variables are independent of the weak-asymmetry parameters \(\beta_\Lambda\) and \(\gamma_\Lambda\). Their dependence is hidden in the vector function \(L_\Lambda(l_\gamma, -l_p)\) of Eq.(V.22), and which is absent in Eq.(VI.25).

Inserting the expressions of Eq.(VI.28) into the spin-density function of Eq.(VI.25), we get

\[
\mathcal{W}(\xi) = \mathcal{R} - \alpha_L S N \cdot l_\Lambda l_\Lambda \cdot l_p - \alpha_\bar{\Lambda} S N \cdot l_\bar{\Lambda} l_\bar{\Lambda} \cdot l_\bar{p}
+ \alpha_L \alpha_\bar{\Lambda} l_\Lambda \cdot l_\bar{p} l_\Lambda \cdot l_\bar{p} \left[ \mathcal{T}_1 l_\Lambda \cdot \hat{p} l_\Lambda \cdot \hat{p} 
+ \mathcal{T}_2 l_{\Lambda \perp} \cdot l_{\bar{\Lambda} \perp} + \mathcal{T}_3 l_{\Lambda \perp} \cdot \hat{k} l_{\bar{\Lambda} \perp} \cdot \hat{k} / \sin^2 \theta 
+ \mathcal{T}_4 \left( l_\Lambda \cdot \hat{p} l_{\Lambda \perp} \cdot \hat{k} + l_\Lambda \cdot \hat{p} l_{\bar{\Lambda} \perp} \cdot \hat{k} \right) / \sin \theta \right]. \quad (VI.29)
\]
Thus, this is the angular distribution obtained when folding the product of spin densities for production and decay. These results were previously reported in Refs. [2] and [7].

VII. DIFFERENTIAL-SPIN DISTRIBUTIONS

A closer inspection of the differential-spin-density-distribution function of Eq. (VI.29) shows that the weak-interaction parameters $\alpha_{\Lambda}$ and $\alpha_{\bar{\Lambda}}$ always come in the combinations $\alpha_{\Lambda}l_{\Lambda}\cdot l_{p}$ or $\alpha_{\bar{\Lambda}}l_{\bar{\Lambda}}\cdot l_{\bar{p}}$. Therefore, it is convenient to define the following functions;

$$\lambda_{\Lambda}(\theta_{\Lambda p}) = \alpha_{\Lambda}l_{\Lambda}\cdot l_{p} = \alpha_{\Lambda} \cos(\theta_{\Lambda p}),$$  \hspace{1cm} (VII.30)

$$\lambda_{\bar{\Lambda}}(\theta_{\bar{\Lambda} p}) = \alpha_{\bar{\Lambda}}l_{\bar{\Lambda}}\cdot l_{\bar{p}} = \alpha_{\bar{\Lambda}} \cos(\theta_{\bar{\Lambda} p}).$$  \hspace{1cm} (VII.31)

Then, the differential-spin-density-distribution function of Eq. (VI.29) can be rewritten as

$$W(\boldsymbol{\xi}) = R - \left[ \lambda_{\Lambda}Q_{\Lambda} + \lambda_{\bar{\Lambda}}Q_{\bar{\Lambda}} \right] S$$

$$+ \lambda_{\Lambda}\lambda_{\bar{\Lambda}} \left[ Q_{1}T_{1} + Q_{2}T_{2} + Q_{3}T_{3} + Q_{4}T_{4} \right],$$  \hspace{1cm} (VII.32)

with the argument $\boldsymbol{\xi}$ a nine-dimensional vector $\boldsymbol{\xi} = (\theta, \Omega_{\Lambda}, \Omega_{p}, \Omega_{\bar{\Lambda}}, \Omega_{\bar{p}})$ representing the scattering angle and four directional-unit vectors of particle motion.

The six structure functions $R$, $S$, and $T$ are functions of the scattering angle $\theta$ and the ratio of form factors $\eta_{\phi}$. The six kinematic $Q$ functions are functions of $l_{\Lambda}$ and $l_{\bar{\Lambda}}$. Their dependencies on the unit vectors $l_{p}$ and $l_{\bar{p}}$ reside solely in the functions $\lambda_{\Lambda}$ and $\lambda_{\bar{\Lambda}}$ of Eqs. (VII.30) and (VII.31).

The analytic expressions for the six functions $Q(l_{\Lambda}, l_{\bar{\Lambda}})$ are obtained by comparing Eqs. (VI.29) and (VII.32);

$$Q_{\Lambda} = N \cdot l_{\Lambda},$$

$$Q_{\bar{\Lambda}} = N \cdot l_{\bar{\Lambda}},$$

$$Q_{1} = l_{\Lambda} \cdot \hat{p}l_{\bar{\Lambda}} \cdot \hat{p},$$

$$Q_{2} = l_{\Lambda\perp} \cdot l_{\bar{\Lambda}\perp},$$

$$Q_{3} = l_{\Lambda\perp} \cdot \hat{k}l_{\bar{\Lambda}\perp} \cdot \hat{k}/\sin^{2}\theta,$$

$$Q_{4} = \left[ l_{\Lambda} \cdot \hat{p}l_{\bar{\Lambda}\perp} \cdot \hat{k} + l_{\bar{\Lambda}} \cdot \hat{p}l_{\Lambda\perp} \cdot \hat{k} \right] / \sin \theta.$$  \hspace{1cm} (VII.33)
Here, longitudinal and transverse components of vectors are defined relative to $\mathbf{p}$, the direction of motion of the $\Sigma^0$ hyperon.

The differential-spin-density distribution of Eq. (VII.32), and the angular functions above, depend on a number of unit vectors; $\mathbf{p}$ and $-\mathbf{p}$ are unit vectors along the directions of motion of the $\Sigma^0$ and the $\bar{\Sigma}^0$ in the c.m. system; $\mathbf{k}$ and $-\mathbf{k}$ are unit vectors along the directions of motion of the incident electron and positron in the c.m. system; $\mathbf{l}_\Lambda$ and $\mathbf{l}_{\bar{\Lambda}}$ are unit vectors along the directions of motion of the $\Lambda$ and $\bar{\Lambda}$ in the rest systems of the $\Sigma^0$ and the $\bar{\Sigma}^0$; $\mathbf{l}_p$ and $\mathbf{l}_{\bar{p}}$ are unit vectors along the directions of motion of the $p$ and the $\bar{p}$ in the rest systems of the $\Lambda$ and the $\bar{\Lambda}$.

VIII. GLOBAL ANGULAR FUNCTIONS

The differential-spin-density distribution (VI.29) is a function of several unit vectors. In order to handle them we need a common coordinate system, which we call global and define as follows. The scattering plane of the reaction $e^+e^- \rightarrow \Sigma^0\bar{\Sigma}^0$ is spanned by the unit vectors $\mathbf{p} = l_{\Sigma^0}$ and $\mathbf{k} = l_e$, as measured in the c.m. system. The scattering plane makes up the $xz$-plane, with the $y$-axis along the normal to the scattering plane. We choose a right-handed coordinate system with basis vectors

$$
\mathbf{e}_z = \mathbf{p},
\mathbf{e}_y = \frac{1}{\sin \theta} (\mathbf{p} \times \mathbf{k}),
\mathbf{e}_x = \frac{1}{\sin \theta} (\mathbf{p} \times \mathbf{k}) \times \mathbf{p},
$$

(VIII.34)

and where the initial-state-lepton momentum is decomposed as

$$
\mathbf{k} = \sin \theta \mathbf{e}_x + \cos \theta \mathbf{e}_z.
$$

(VIII.35)

The reason we call this coordinate system global is that we use it whenever studying a sub-process of the $e^+e^-$ annihilation.

In spherical $xyz$ coordinates the unit vectors $\mathbf{l}_\Lambda$ and $\mathbf{l}_{\bar{\Lambda}}$ associated with the directions of motion of the $\Lambda$ and $\bar{\Lambda}$ hyperons are,

$$
l_\Lambda = (\cos \phi_\Lambda \sin \theta_\Lambda, \sin \phi_\Lambda \sin \theta_\Lambda, \cos \theta_\Lambda),
$$

$$
l_{\bar{\Lambda}} = (\cos \phi_{\bar{\Lambda}} \sin \theta_{\bar{\Lambda}}, \sin \phi_{\bar{\Lambda}} \sin \theta_{\bar{\Lambda}}, \cos \theta_{\bar{\Lambda}}).
$$

(VIII.36)
However, in order to make our formulas more transparent we introduce the notations \( \Lambda = E = (E_x, E_y, E_z) \) and \( \bar{\Lambda} = F = (F_x, F_y, F_z) \). In this Cartesian notation, the expressions for kinematic functions \( Q(\Lambda, \bar{\Lambda}) \) of Eq.(VII.33) are,

\[
\begin{align*}
Q_{\Lambda} &= E_y, & Q_{\bar{\Lambda}} &= F_y, \\
Q_1 &= E_z F_z, & Q_2 &= E_x F_x + E_y F_y, \\
Q_3 &= E_x F_x, & Q_4 &= E_x F_z + E_z F_x.
\end{align*}
\] (VIII.37)

Inserting these expressions for the \( Q \) functions into Eq.(VII.32), the definition of the differential-spin-density-distribution function, gives

\[
W(\xi(\Omega)) = 1 + \eta \psi \cos^2 \theta \\
- \sqrt{1 - \eta^2} \sin(\Delta \Phi \psi) \sin \theta \cos \theta \left[ \lambda_{\Lambda} E_y + \lambda_{\bar{\Lambda}} F_y \right] \\
+ \lambda_{\Lambda} \lambda_{\bar{\Lambda}} \left[ (1 + \eta \psi) E_z F_z + \sin^2 \theta (E_x F_x - E_z F_z - \eta \psi E_y F_y) \right] \\
+ \sqrt{1 - \eta^2 \cos^2(\Delta \Phi \psi) \sin \theta \cos \theta} \left( E_x F_z + E_z F_x \right) .
\] (VIII.38)

The phase-space-angular variables are hidden inside the \( \mathbf{E}(\theta_{\Lambda}, \phi_{\Lambda}) \) and \( \mathbf{F}(\theta_{\bar{\Lambda}}, \phi_{\bar{\Lambda}}) \) functions.

The differential-spin-density-distribution function \( W(\xi) \) of Eq.(VIII.38) involves two parameters related to the \( e^+e^- \rightarrow \Sigma^0\bar{\Sigma}^0 \) reaction that can be determined by data: the ratio of form factors \( \eta \psi \), and the relative phase of form factors \( \Delta \Phi \psi \). In addition, the distribution function \( W(\xi) \) depends on the weak-asymmetry parameters \( \alpha_{\Lambda} \) and \( \alpha_{\bar{\Lambda}} \) of the two Lambda-hyperon decays. The dependencies on the weak-asymmetry parameters \( \beta \) and \( \gamma \) drop out, when final-state-proton and antiproton spins are not measured.

An important conclusion to be drawn from the differential distribution of Eq.(VIII.38) is that when the phase \( \Delta \Phi \psi \) is small, the parameters \( \alpha_{\Lambda} \) and \( \alpha_{\bar{\Lambda}} \) are strongly correlated and therefore difficult to separate. In order to contribute to the experimental precision value of \( \alpha_{\Lambda} \) and \( \alpha_{\bar{\Lambda}} \) a non-zero value of \( \Delta \Phi \psi \) is required.

**IX. HELICITY ANGULAR FUNCTIONS**

In the helicity-coordinate system, the scattering plane of the reaction \( e^+e^- \rightarrow \Sigma^0\bar{\Sigma}^0 \) is still spanned by the unit vectors \( \hat{p} = l_{\Sigma^0} \) and \( \hat{k} = l_e \), as measured in the c.m. system, and with scattering angle \( \cos \theta = \hat{k} \cdot \hat{p} \). The scattering plane makes up the \( x'z' \)-plane, and with
the \( y' \)-axis normal to this plane. We choose a right-handed-coordinate system with basis vectors

\[
\begin{align*}
\mathbf{e}'_x &= \hat{k}, \\
\mathbf{e}'_y &= \frac{1}{\sin \theta}(\hat{k} \times \hat{p}), \\
\mathbf{e}'_z &= \frac{1}{\sin \theta}(\hat{k} \times \hat{p}) \times \hat{k}.
\end{align*}
\]

(IX.39)

In the helicity-coordinate system the final-state-hyperon momentum can be decomposed as

\[
\hat{p} = \sin \theta \mathbf{e}'_x + \cos \theta \mathbf{e}'_z,
\]

(IX.40)

and \( \mathbf{N} = -\mathbf{e}'_y \) normal to the scattering plane, for \( \mathbf{N} \) defined in Eq. (III.12).

In spherical \( x'y'z' \) coordinates the unit vectors \( \mathbf{l}_\Lambda \) and \( \mathbf{l}_{\bar{\Lambda}} \) associated with the directions of motion of the \( \Lambda \) and \( \bar{\Lambda} \) hyperons are,

\[
\begin{align*}
\mathbf{l}_\Lambda &= (\cos \phi'_\Lambda \sin \theta'_\Lambda, \sin \phi'_\Lambda \sin \theta'_\Lambda, \cos \theta'_\Lambda), \\
\mathbf{l}_{\bar{\Lambda}} &= (\cos \phi'_{\bar{\Lambda}} \sin \theta'_{\bar{\Lambda}}, \sin \phi'_{\bar{\Lambda}} \sin \theta'_{\bar{\Lambda}}, \cos \theta'_{\bar{\Lambda}}),
\end{align*}
\]

(IX.41)

and similarly for the unit vectors \( \mathbf{l}_p \) and \( \mathbf{l}_{\bar{p}} \).

As in the previous section we introduce a short-hand notation for vectors expressed in helicity coordinates, \( \mathbf{l}_\Lambda = \mathbf{E}' = (E'_x, E'_y, E'_z) \) and \( \mathbf{l}_{\bar{\Lambda}} = \mathbf{F}' = (F'_x, F'_y, F'_z) \). In order to determine the spin-density-distribution function in terms of the helicity angles we need the six kinematic functions \( Q(\mathbf{l}_\Lambda, \mathbf{l}_{\bar{\Lambda}}) \) of Eq. (VII.33) in terms of the helicity angles of Eqs. (IX.41). In principle, this is straightforward but it turns out to be more involved than for the global case, since some of the \( Q(\mathbf{l}_\Lambda, \mathbf{l}_{\bar{\Lambda}}) \) functions will depend on the scattering angle \( \theta \).

The basis vectors of Eqs. (IX.39) and (VIII.34) are related by

\[
\begin{align*}
\mathbf{e}_x &= -\cos \theta \mathbf{e}'_x + \sin \theta \mathbf{e}'_z, \\
\mathbf{e}_y &= -\mathbf{e}'_y, \\
\mathbf{e}_z &= \sin \theta \mathbf{e}'_x + \cos \theta \mathbf{e}'_z.
\end{align*}
\]

(IX.42)

From this relation one derives a corresponding relation for the global-vector components \( F_k \), and helicity-vector components \( F'_k \), of the directional unit vector \( \mathbf{l}_{\bar{\Lambda}} \) associated with the \( \bar{\Lambda} \)
hyperon, 

\[
F_x = -\cos \theta F'_x + \sin \theta F'_z, \\
F_y = -F'_y, \\
F_z = \sin \theta F'_x + \cos \theta F'_z,
\]

(IX.43)

and ditto for the \(\Lambda\) hyperon case.

The new set of the six \(Q(1_{\Lambda}, 1_{\bar{\Lambda}})\) functions of Eq.(VII.33) is obtained by replacing global-vector components by helicity-vector components, to give

\[
Q_{\Lambda} = -E'_y, \\
Q_{\bar{\Lambda}} = -F'_y, \\
Q_1 = (\sin \theta E'_x + \cos \theta E'_z)(\sin \theta F'_x + \cos \theta F'_z), \\
Q_2 = E'_y F'_y, \\
Q_3 = (\cos \theta E'_x - \sin \theta E'_z)(\cos \theta F'_x + \sin \theta F'_z), \\
Q_4 = (\cos \theta E'_x + \sin \theta E'_z)(\cos \theta F'_x - \sin \theta F'_z) + (\sin \theta E'_x + \cos \theta E'_z)(-\cos \theta F'_x - \sin \theta F'_z). \\
\]

(IX.44)

This set of helicity-angular-dependent functions has a decidedly more complex dependence on the scattering angle \(\theta\) than the global-angular set of Eq.(VIII.37), which is independent of the scattering angle. Helicity coordinates are e.g. used by the BES group, \cite{1,3}, and by \cite{11}.

The differential-spin-density distribution is defined in Eq.(VII.32). For the application to helicity coordinates it takes the form

\[
\mathcal{W}(\xi(\Omega')) = 1 + \eta_\psi \cos^2 \theta \\
+ \sqrt{1 - \eta_\psi^2} \sin(\Delta \Phi_\psi) \sin \theta \cos \theta \left[ \lambda_{\Lambda} E'_y + \lambda_{\bar{\Lambda}} F'_y \right] \\
+ \lambda_{\Lambda} \lambda_{\bar{\Lambda}} \left( (1 + \eta_\psi) Q_1 + \sin^2 \theta \left( (Q_3 - Q_1) + \eta_\psi (Q_3 - Q_2) \right) \right) \\
+ \sqrt{1 - \eta_\psi^2} \cos(\Delta \Phi_\psi) \sin \theta \cos \theta Q_4, \\
\]

(IX.45)

with the \(Q\) functions as defined in Eqs.(IX.44). The argument \(\xi(\Omega')\) of the function \(\mathcal{W}(\xi(\Omega'))\) remind us we work in the helicity-coordinate system.
X. CROSS-SECTION DISTRIBUTIONS

In view of its simplicity, we propose evaluating the cross-section distribution for each event in the global $xyz$ coordinate system of Eq. (VIII.34). The expression for the differential-spin-density distribution $W(\xi(\Omega))$ in this coordinate system is already known, and displayed in Eq. (VIII.38), where the symbol $\Omega$ refers to spherical angles, $\Omega = (\phi, \theta)$, in the $xyz$ coordinate system.

It might be remembered we introduced the notation $E = l_{\Lambda}$ and $F = \bar{l}_{\Lambda}$, with Cartesian components as defined in Eq. (VIII.36). A unit vector such as $l_{\Lambda}$, which is a unit vector in the direction of motion of the $\Lambda$ hyperon in the rest system of the $\Sigma^0$ hyperon, can be expressed in either Cartesian $xyz$ or spherical-angular variables,

$$l_{\Lambda} = (l_{\Lambda x}, l_{\Lambda y}, l_{\Lambda z}) = (\cos \phi_{\Lambda} \sin \theta_{\Lambda}, \sin \phi_{\Lambda} \sin \theta_{\Lambda}, \cos \theta_{\Lambda}). \quad \text{(X.46)}$$

The decomposition into spherical coordinates needs to be known since the phase-space element $d\Omega_{\Lambda}$ is expressed in terms of spherical-angular variables.

It was already noticed in Sect. VII that the angular variables $\Omega_p$ and $\Omega_{\bar{p}}$ only appear in the multiplicative parameters $\lambda_{\Lambda}(\theta_{\Lambda p})$ and $\lambda_{\bar{\Lambda}}(\theta_{\bar{\Lambda} p})$ of Eqs. (VII.30) and (VII.31). Averages over $\Omega_p$ and $\Omega_{\bar{p}}$ give

$$\langle \lambda_{\Lambda}(\theta_{\Lambda p}) \rangle = \int \frac{d\Omega_p}{4\pi} \alpha_{\Lambda} \cos(\theta_{\Lambda p}) = \frac{1}{3} \alpha_{\Lambda}, \quad \text{(X.47)}$$

$$\langle \lambda_{\bar{\Lambda}}(\theta_{\bar{\Lambda} p}) \rangle = \int \frac{d\Omega_{\bar{p}}}{4\pi} \alpha_{\bar{\Lambda}} \cos(\theta_{\bar{\Lambda} p}) = \frac{1}{3} \alpha_{\bar{\Lambda}}, \quad \text{(X.48)}$$

where $\alpha_{\Lambda}$ and $\alpha_{\bar{\Lambda}}$ are the weak-interaction-decay parameters for the $\Lambda$ and $\bar{\Lambda}$ hyperons.

Thus ends our exposition of the factors making up the master formula, Eq. (II.1), for the normalized cross-section distribution for production and decay of $\Sigma^0\bar{\Sigma}^0$ pairs in $e^+e^-$ annihilation.

XI. SUMMARY

This is a study of joint production and simultaneous sequential decay of $\Sigma^0\bar{\Sigma}^0$ pairs produced in $e^+e^-$ annihilation. It starts from a master formula which is a product of three factors, describing: the annihilation of a lepton pair into a hyperon pair; the spin-density distribution $W(\xi)$ representing the spin dependence in hyperon production and decay; and
the phase-space element in sequential hyperon decay. Each measured event is specified by a nine-dimensional vector $\xi = (\theta, \Omega_\Lambda, \Omega_p, \Omega_{\bar{p}}, \Omega_{\bar{\Lambda}})$, with $\theta$ the scattering angle in the $e^+e^- \rightarrow \Sigma^0\Sigma^0$ subprocess.

The dynamics of the process is described by four unit-three vectors $l_p, l_\Lambda, l_{\bar{p}}, l_{\bar{\Lambda}}$, directed along the directions of motion of the final state baryons ($\Omega_p, \Omega_\Lambda, \Omega_{\bar{p}}, \Omega_{\bar{\Lambda}}$). We have arranged so that the spin-density-distribution function can be written as

$$W(\xi) = \mathcal{R} - \left[ \lambda_\Lambda Q_\Lambda + \lambda_{\bar{\Lambda}} Q_{\bar{\Lambda}} \right] S + \lambda_\Lambda \lambda_{\bar{\Lambda}} \left[ Q_1 T_1 + Q_2 T_2 + Q_3 T_3 + Q_4 T_4 \right].$$

(XI.49)

Here, the six functions $\mathcal{R}, S$, and $T$ are functions of the scattering angle $\theta$ and the ratio of form factors $\eta_\psi$, whereas the six functions $Q$ are functions of $l_\Lambda$ and $l_{\bar{\Lambda}}$, and of $\hat{p} = k_{\Sigma^0}$ and $\hat{k} = l_e$. The unit vectors $l_p$ and $l_{\bar{p}}$ only enter the weak-asymmetry functions $\lambda_\Lambda$ and $\lambda_{\bar{\Lambda}}$ of Eqs. (VII.30) and (VII.31).

It remains to connect the four kinematic unit vectors to measured quantities. To this end we imbed Cartesian-coordinate systems in our events. Then, with the Lambda hyperon as an example,

$$l_\Lambda = (l_{\Lambda x}, l_{\Lambda y}, l_{\Lambda z}) = (\cos \phi_\Lambda \sin \theta_\Lambda, \sin \phi_\Lambda \sin \theta_\Lambda, \cos \theta_\Lambda).$$

(XI.50)

Our preferred coordinate system is named global and has the $xz$-plane as scattering plane, and $\hat{p}$ along the $z$-direction. In global coordinates the building blocks of the spin-density-distribution function $W(\xi)$ in Eq. (XI.49) have the simple structure mentioned above. In particular, the six $Q$ functions are independent of the scattering angle $\theta$.

An alternative to global coordinates is helicity coordinates, when the $x'z'$-plane is the scattering plane, and $\hat{k}$ directed along the $z'$ axis. Several of the $Q$ function now depend on the scattering angle $\theta$ in a complex way, even though the two coordinate systems are related by a rotation.

Appendix A: Baryon form factors

The diagram in Fig.1 describes the annihilation reaction $e^- (k_1) e^+ (k_2) \rightarrow Y(p_1) \bar{Y}(p_2)$ and involves two vertex functions; one of them leptonic, the other one baryonic. The strength of the lepton-vertex function is determined by the fine-structure constant $\alpha_e$, but two complex
form factors $G_M^\psi(s)$ and $G_E^\psi(s)$ are needed for a proper parametrization of the baryonic vertex function, as of Ref.\cite{4}. The values of these form factors vary with energy, $s = (p_1 + p_2)^2$.

The strength of the baryon form factors is measured by the function $D_\psi(s)$,

$$D_\psi(s) = s \left| G_M^\psi \right|^2 + 4M^2 \left| G_E^\psi \right|^2,$$

with the $M$-variable representing the hyperon mass. The ratio of form factors is measured by $\eta_\psi(s)$,

$$\eta_\psi(s) = \frac{s \left| G_M^\psi \right|^2 - 4M^2 \left| G_E^\psi \right|^2}{s \left| G_M^\psi \right|^2 + 4M^2 \left| G_E^\psi \right|^2},$$

with $\eta_\psi(s)$ satisfying $-1 \leq \eta_\psi(s) \leq 1$. The relative phase of form factors is measured by $\Delta \Phi_\psi(s)$,

$$\frac{G_E^\psi}{G_M^\psi} = e^{i \Delta \Phi_\psi(s)} \left| \frac{G_E^\psi}{G_M^\psi} \right|.$$

**Appendix B: Structure functions**

The six structure functions $R$, $S$, and $T$ of Eq.\cite{III.11} depend on the scattering angle $\theta$, the ratio function $\eta_\psi(s)$, and the phase function $\Delta \Phi_\psi(s)$. To be specific \cite{4,6},

$$R = 1 + \eta_\psi \cos^2 \theta,$$

$$S = \sqrt{1 - \eta_\psi^2} \sin \theta \cos \theta \sin(\Delta \Phi_\psi),$$

$$T_1 = \eta_\psi + \cos^2 \theta,$$

$$T_2 = -\eta_\psi \sin^2 \theta,$$

$$T_3 = (1 + \eta_\psi) \sin^2 \theta,$$

$$T_4 = \sqrt{1 - \eta_\psi^2} \sin \theta \cos \theta \cos(\Delta \Phi_\psi).$$

The parameters $\eta_\psi$ and $\Delta \Phi_\psi$ are defined in Eqs.\cite{A.2} and \cite{A.3}. The function $T_3$ of Eq.\cite{B.5} differs from the corresponding function $T_3$ of Ref.\cite{2} by the $\sin^2 \theta$ factor. Similarly, the function $T_4$ of Eq.\cite{B.6} differs from the corresponding function $T_4$ of Ref.\cite{2} by the $\sin \theta$ factor.
Appendix C: Finding angular variables

The angular functions $Q(l_\Lambda, l_\bar{\Lambda})$ of Eq.(VII.33) and the $\lambda$ parameters of Eqs.(VII.30) and (VII.31) are expressed in terms of unit vectors such as $l_p$ and $l_\Lambda$, which are not directly measurable but which must be calculated. We suggest the following approach.

For each event we imbed the particle momenta in its c.m. system and with coordinate axes as defined in Eq.(VIII.34). For the $\Sigma^0$ hyperon the components of the momentum are, by definition,

$$\hat{p}_{\Sigma^0} = (0, 0, 1).$$  \hspace{1cm} (C.1)

Then, let us consider the proton and the hyperon of the final state, with momenta $p_p$ and $p_\Lambda$ in the c.m. system. In the rest system of the Lambda hyperon, $L_p$ denotes the proton momentum, which is given by the expression

$$L_p = p_p + B_{\Lambda p} p_\Lambda,$$  \hspace{1cm} (C.2)

$$B_{\Lambda p} = \frac{1}{m_\Lambda} \left[ \frac{1}{E_\Lambda + m_\Lambda} p_\Lambda \cdot p_p - E_\Lambda \right].$$  \hspace{1cm} (C.3)

Now, the length of the vector $L_p$ is well-known, being the momentum in the hyperon decay $\Lambda \to \pi N$, and therefore

$$|L_p| = \frac{1}{2m_\Lambda} \left[ (m_\Lambda^2 + m_\pi^2 - m_N^2)^2 - 4m_\Lambda^2 m_\pi^2 \right]^{1/2}.$$  \hspace{1cm} (C.4)

Hence, the unit vector $l_p$ appearing in our equations should be

$$l_p = L_p/|L_p|,$$  \hspace{1cm} (C.5)

$$= (\cos \phi_p \sin \theta_p, \sin \phi_p \sin \theta_p, \cos \theta_p).$$  \hspace{1cm} (C.6)

Also, the equation for $l_\Lambda$ in the decay $\Sigma^0 \to \Lambda \gamma$ is easily written down, as are the corresponding equations for the antiparticles, $\bar{p}$ and $\bar{\Lambda}$.

Acknowledgments

I would like to thank Karin Schöning for informative discussions

[1] M. Ablikim et al. (BESIII), Phys. Rev. D 95, 052003 (2017).
[2] G. FälDt and K. SchönnIng, Phys. Rev. D 101, 033001 (2020).
[3] M. Ablikim et al. (BESIII), Nat. Phys. 15, 631 (2019)
[4] G. FälDt and A. Kupsc, Phys. Lett. B 772, 16 (2017).
[5] G. FälDt, Eur. Phys. J. A 51, 74 (2015).
[6] G. FälDt, Eur. Phys. J. A 52, 141 (2016).
[7] G. FälDt, Sequential hyperon decays (Lecture notes, Uppsala, June 2017).
[8] T. D. Lee and C. N. Yang, Phys. Rev. 108, 1645 (1957).
[9] John F. Donoghue and Sandip Pakvasa, Phys. Rev. Lett. 55, 162 (1985).
[10] John F. Donoghue, Xiao-Gang He, and Sandip Pakvasa, Phys. Rev. D34, 833 (1986).
[11] H. Czyż, A. Grzelińska, and J.H. Kühn, Phys. Rev. D75, 074026 (2007).