A COUNTER-EXAMPLE TO THE EQUIVARIANCE STRUCTURE ON
SEMI-UNIVERSAL DEFORMATION

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Abstract. We provide a counter-example to the the $G$-equivariant structure on semi-universal deformation in the case that $G$ is nonreductive.

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Introduction

Let $X$ be an algebraic variety. Due to Schlessinger’s work in [5], the existence of a formally semi-universal deformation (unique up to non-canonical isomorphism), which contains all the information of small deformations of $X$, is guaranteed provided that $H^1(X, \mathcal{T}_X)$ and $H^2(X, \mathcal{T}_X)$ are finite dimensional vector spaces. These conditions realise for example, if $X$ is a complete scheme over the ground field or an affine scheme with at most isolated singularities (see [6, Corollary 2.4.2]). Now, we equipe $X$ with an action of a group $G$. One question arisen naturally is whether there exists a formally semi-universal deformation $\pi : \mathcal{X} \to S$ of $X$, on which we can provide a $G$-action extending the given action on $X$. The answer is positive in the case that $G$ satisfies some vanishing condition on its cohomology groups, i.e $H^1(G, -) = 0$ and $H^2(G, -) = 0$ for a class of $G$-modules determined by $X$. In particular, these vanishing conditions hold for linearly reductive groups (see [4]). However, we do not know if there exists a non-reductive group whose action on $X$ does not extend to the formally semi-universal deformation of $X$. Therefore, we wish to give an example which illustrates this phenomenon. More precisely, we prove that the action of the automorphism group of the second Hirzebruch surface $\mathbb{F}_2$, denoted by $G$, does not extend to its formally semi-universal deformation.
Our proof goes as follows. First, we find a nice presentation of $G$ and then construct a formally semi-universal deformation of $\hat{X}$ of $\mathbb{F}_2$. It turns out that $G$ is non-reductive and that the Lie algebra of $G$ is a 7-dimensional vector space. In particular, we obtain seven vector fields on $\mathbb{F}_2$ with Lie bracket relations induced by those in Lie$(G)$. Next, we describe general form of formal vector fields on $\hat{X}$. Finally, we conclude the paper by means of contradiction. Suppose that the $G$-action on $\mathbb{F}_2$ does extend to a $G$-action on $\hat{X}$ then we also have seven formal vector fields on $\hat{X}$ whose restriction on the central fiber is nothing but our former vector fields on $\mathbb{F}_2$. By manipulation on these vector fields with a filtration $F$ given by the vanishing order at 0, we obtain the existence of a 3-dimensional abelian Lie subalgebra in $\mathfrak{sl}_2(\mathbb{C}) \times \mathfrak{sl}_2(\mathbb{C})$, where $\mathfrak{sl}_2(\mathbb{C})$ is the special linear group, which is not the case. A remark is in order. Since $\mathbb{F}_2$ does not have a space of moduli, another possible way to obtain a contradiction is to use Wavrik’s criterion (see [7, Theorem 4.1]) but the calculations are somewhat more complicated.

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1. Formal schemes and formal deformations

We begin this sections by recalling some basic definitions of formal schemes. For more details, the readers are referred to [2, Chapter III. 9].

**Definition 1.1.** Let $X$ be a noetherian scheme and let $Y$ be a closed subscheme defined by a sheaf of ideals $J$. Then we define the formal completion of $X$ along $Y$, denoted $(\hat{X}, \mathcal{O}_{\hat{X}})$, to be the following ringed space. We take the topological space $Y$, and on it the sheaf of rings $\mathcal{O}_{\hat{X}} = \lim_n \mathcal{O}_X / I^n$. Here we consider each $\mathcal{O}_X / I^n$ as sheaf of rings on $Y$

**Definition 1.2.** A noetherian formal scheme is a locally ringed space $(X, \mathcal{O}_X)$ which has a finite open cover $\{U_i\}$ such that for each $i$, the pair $(U_i, \mathcal{O}_X |_{U_i})$ is isomorphic, as a locally ringed space, to the completion of some noetherian scheme $X_i$ along a closed subscheme $Y_i$. A morphism of noetherian formal schemes is a morphism as locally ringed spaces.

**Example 1.1.** If $X$ is any noetherian scheme, and $Y$ a closed subscheme then the formal completion $\hat{X}$ of $X$ along $Y$ is a formal scheme.

**Example 1.2.** For $X = \mathbb{C}^1 = \text{Spec}(\mathbb{C}[t])$ and $Y = \{0\}$, the formal scheme $\hat{X}$ is the locally ringed space $(Y, \mathcal{O}_{\hat{X}})$, where the structure sheaf $\mathcal{O}_{\hat{X}}$ is $\mathbb{C}[[t]]$. We denote by $\text{Specf}(\mathbb{C}[t]) := (Y, \mathcal{O}_{\hat{X}})$.

Now, we come to the notion of formal deformation. Let $X$ be an algebraic scheme and $A$ be a complete local noetherian $\mathbb{C}$-algebra with residue field $\mathbb{C}$ and the maximal ideal $m$.

**Definition 1.3.** A formal deformation of $X$ over $A$ is a sequence $\{\nu_n\}$ of infinitesimal deformations of $X$, in which $\nu_n$ is represented by a deformation
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\[
\begin{array}{ccc}
X & \xrightarrow{f_n} & X_n \\
\downarrow & & \downarrow \pi_n \\
\text{Spec}(\mathbb{C}) & \longrightarrow & \text{Spec}(A_n)
\end{array}
\]

where \( A_n = A/m^n + 1 \), such that for all \( n \geq 1 \), \( \nu_n \) induces \( \nu_{n-1} \) by pullback under the natural inclusion \( \text{Spec}(A_{n-1}) \to \text{Spec}(A_n) \), i.e. \( \nu_{n-1} \) is also represented by the deformation

\[
\begin{array}{ccc}
X & \xrightarrow{f_{n-1}} & X_n \times_{\text{Spec}(A_n)} \text{Spec}(A_{n-1}) \\
\downarrow & & \downarrow \pi_{n-1} \\
\text{Spec}(\mathbb{C}) & \longrightarrow & \text{Spec}(A_{n-1})
\end{array}
\]

In the language of formal schemes, we can write \( \{ \nu_n \} \) as the morphism of formal schemes

\[ \hat{\pi} : \hat{X} \to \text{Spec}(A) \]

where

\[ \hat{X} = (X, \lim \mathcal{O}_{X_n}) \text{ and } \hat{\pi} = \lim \pi_n. \]

Here, \( \mathcal{O}_{X_n} \) is the structure sheaf on \( X_n \) and \( \text{Spec}(A) \) is the formal scheme obtained by completing \( \text{Spec}(A) \) along its closed point which corresponds to the unique maximal ideal of \( A \).

We end this section by introducing the definition of formal scheme associated to a given deformation. Let \( X \) be a projective scheme and \( \nu \) be a deformation represented by

\[
\begin{array}{ccc}
X & \xrightarrow{f} & X \\
\downarrow & & \downarrow \pi \\
\text{Spec}(\mathbb{C}) & \longrightarrow & (S, s)
\end{array}
\]

where \( S = \text{Spec}(B) \) for some \( \mathbb{C} \)-algebra \( B \) and \( s \) is a \( \mathbb{C} \)-rational point in \( S \).

**Definition 1.4.** The formal deformation associated to \( \nu \) is defined to be the sequence of deformations \( \{ \nu_n \} \) where each \( \nu_n \) is the pullback of \( \nu \) under the natural closed embedding

\[ \text{Spec}(\mathcal{O}_{S,s}/m_s^{n+1}) \to S \]

where \( m_s \) is the unique maximal ideal of the local ring \( \mathcal{O}_{S,s} \).

**Remark 1.1.** Note that \( \{ \nu_n \} \) is formal because of the isomorphism

\[ \mathcal{O}_{S,s}/m_s^{n+1} \cong \hat{\mathcal{O}}_{S,s}/\hat{m}_s^{n+1} \]

for all \( n \).
2. The second Hirzebruch surface and its automorphism group

We always assume that our ground field is the field of complex numbers \( \mathbb{C} \). The general linear group \( \text{GL}(2, \mathbb{C}) \) has an obvious linear action on \( \mathbb{C}^2 \). This induces an action on the \( \mathbb{C} \)-vector space of polynomials in two variables \( \mathbb{C}[X, Y] \). Since the subspace of homogeneous polynomials of degree 2, denoted by \( \mathbb{C}[X, Y]_2 \), is \( \text{GL}(2, \mathbb{C}) \)-invariant then we have a \( \text{GL}(2, \mathbb{C}) \)-action on \( \mathbb{C}[X, Y]_2 \). More precisely, for \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathbb{C}) \) and \( f = a_0X^2 + a_1XY + a_2Y^2 \in \mathbb{C}[X, Y]_2 \), the action of \( g \) on \( f \) is given by the linear substitution

\[
\begin{pmatrix} X \\ Y \end{pmatrix} := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},
\]

i.e.

\[
g.f = a_0(aX + bY)^2 + a_1(aX + bY)(cX + dY) + a_2(cX + dY)^2
= (a^2a_0 + ac + c^2a_2)X^2 + (2aba_0 + (ad + bc)a_1 + 2cd)XY + (b^2a_0 + bda_1 + d^2a_2)Y^2.
\]

Identifying \( \mathbb{C}[X, Y]_2 \) with \( \mathbb{C}^3 \), the corresponding action on \( \mathbb{C}^3 \) can be written as

\[
g.(a_0, a_1, a_2) = \begin{pmatrix} a^2 & ac & c^2 \\ ab & ad + bc & 2cd \\ b^2 & bd & d^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}.
\]

This action gives rise to an algebraic group \( H \) which is the semi-product of \( \mathbb{C}^3 \) and \( \text{GL}(2, \mathbb{C}) \), i.e.

\[
H := \mathbb{C}^3 \times \text{GL}(2, \mathbb{C}).
\]

This is a non-reductive linear group. Recall that an algebraic group \( K \) is reductive if \( R_u(K) \) of \( K \) is trivial, where \( R_u(K) \) is the unipotent radical of \( K \), i.e. the greatest connected normal subgroup of \( K \). In our case, \( R_u(H) = \mathbb{C}^3 \).

Next, we recall the definition of the second Hirzebruch surface \( \mathbb{F}_2 \).

**Definition 2.1.** The second Hirzebruch surface \( \mathbb{F}_2 \) is defined to be the projectivization of \( \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1} \), i.e. \( \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}) \), where \( \mathcal{O}_{\mathbb{P}^1} \) is the structure sheaf of the projective space \( \mathbb{P}^1 \).

An equivalent definition of \( \mathbb{F}_2 \) is given in the following.

**Proposition 2.1.** The second Hirzebruch surface \( \mathbb{F}_2 \) is isomorphic to the variety

\[
\{( [x : y : z], [u : v]) \in \mathbb{P}^2 \times \mathbb{P}^1 | yv^2 = zu^2 \}.
\]

**Proof.** Let \( \sigma : \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}) \to \mathbb{P}^1 \) be the canonical projection of the projectivization \( \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}) \), let \( U = \text{Spec}(\mathbb{C}[v]) \) and \( U' = \text{Spec}(\mathbb{C}[v']) \) such that \( v'v = 1 \) on \( U \cap U' \). Then \( \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}) \) has the following presentation

\[
\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}) = \sigma^{-1}(U) \cup \sigma^{-1}(U') = (U \times \mathbb{P}^1) \cup (U' \times \mathbb{P}^1),
\]

so that on the intersection of the affine open sets \( V = \text{Spec}(\mathbb{C}[v, y]) \subset U \times \mathbb{P}^1 \) and \( V' = \text{Spec}(\mathbb{C}[v', y']) \subset U' \times \mathbb{P}^1 \), we have

\[
\begin{cases}
vv' = 1 \\
y' = yv^2
\end{cases}.
\]
So, we have an open covering of $F_2$ given by the following open embeddings

$$
\rho_1 : U \times \mathbb{P}^1 \to F_2
$$

$$(v, [x : y]) \mapsto ([x : y : yv^2], [1 : v])$$

and

$$
\rho_2 : U' \times \mathbb{P}^1 \to F_2
$$

$$(v', [x' : y']) \mapsto ([x' : y'v'^2 : y'], [v' : 1]),$$

which yields an isomorphism $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}) \to F_2$ by gluing. 

Now, the algebraic group $H$ acts on the second Hirzebruch surface

$$F_2 = \{([x : y : z], [u : v]) \in \mathbb{P}^2 \times \mathbb{P}^1 | yv^2 = zu^2\}$$

in the following manner: for $p = ([x : y : z], [u : v]) \in F_2$ and $g = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in H,

$$
g(p) = \left\{ \begin{array}{ll}
([xu^2 + y(a_0v^2 + a_1uv + a_2u^2) : y(au + bv)^2 : z(au + bv) : z(au + bv : cu + dv)^2], [au + bv : cu + dv]) & \text{if } u \neq 0 \\
([xv^2 + z(a_0v^2 + a_1uv + a_2u^2) : z(au + bv)^2 : z(au + bv : cu + dv)^2], [au + bv : cu + dv]) & \text{if } v \neq 0
\end{array} \right.

The following theorem is well-known (see [1, Section 6.1]).

**Theorem 2.1.** The group of automorphisms of $F_2$ is exactly the quotient of $H$ by the subgroup $I$ consisting of diagonal matrices of the form $\begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}$ where $\mu \in \mathbb{C}$ such that $\mu^2 = 1$.

3. A formally semi-universal deformation of $F_2$ and formal vector fields on it

3.1. **Construction of the semi-universal deformation of $F_2$.** We shall following the construction given in [4, Example 1.2.2.(iii)]. Consider two copies of $\mathbb{C} \times \mathbb{C} \times \mathbb{P}^1$ given by $W := \text{Proj}(\mathbb{C}[t, v, x, y])$ and $W' := \text{Proj}(\mathbb{C}[t', v', x', y'])$ (note that these two rings are graded with respect to $x, y$ and $x', y'$). Take two affine subsets of $W$ and $W'$ given by $\text{Spec}(\mathbb{C}[t, v, y])$ and $\text{Spec}(\mathbb{C}[t', v', y'])$, respectively and then glue along the open subsets

$$\text{Spec}(\mathbb{C}[t, v, v^{-1}, y]) \subset \text{Spec}(\mathbb{C}[t, v, y])$$

and

$$\text{Spec}(\mathbb{C}[t', v', v'^{-1}, y']) \subset \text{Spec}(\mathbb{C}[t', v', y'])$$

by the rules

$$(3.1) \begin{cases}
vv' = 1 \\
y' = yv^2 - tv \\
t' = t
\end{cases}$$

Hence, this gives a gluing of $W$ and $W'$, which we call $W$, along

$$\text{Proj}(\mathbb{C}[t, v, v^{-1}, x, y]) \text{ and } \text{Proj}(\mathbb{C}[t', v', v'^{-1}, x', y']).$$

Now, let $\pi : W \to \mathbb{C}$ be the morphism induced by the projections.
Theorem 3.1. The family \( \pi : \mathcal{W} \to \mathbb{C} \) is a semi-universal deformation of \( \mathbb{F}_2 \). Moreover,

\[
\pi^{-1}(t) = \begin{cases} 
\mathbb{F}_2 & \text{if } t = 0 \\
\mathbb{P}^1 \times \mathbb{P}^1 & \text{otherwise}.
\end{cases}
\]

Proof. The map \( \pi \) is obviously surjective by construction. Since \( \pi \) is locally a projection, it is a flat morphism. Moreover, by Proposition 2.1, \( W_0 = \pi^{-1}(0) = \mathbb{F}_2 \). Then \( \pi : \mathcal{W} \to \mathbb{C} \) is a deformation of \( \mathbb{F}_2 \). To see that \( \pi^{-1}(t) = \mathbb{P}^1 \times \mathbb{P}^1 \) for \( C \setminus \{0\} \), we give new coordinates on \( \text{Spec}(\mathbb{C}[t, v, v^{-1}, y]) \) and \( \text{Spec}(\mathbb{C}[t', v', (v')^{-1}, y']) \) by the following transformation:

\[
r = \frac{vy - t}{ty}
\]

and

\[
r' = \frac{y'}{t'v'y' + t'^2},
\]

respectively. The gluing (3.1) gives the relation

\[
r' = \frac{y'}{t'v'y' + t'^2} = \frac{yv^2 - tv}{t'v'(yv^2 - tv) + t'^2} = \frac{yv^2 - tv}{t'vy} = \frac{yv - t}{ty} = r.
\]

This is nothing but the gluing process to obtain \( \mathbb{P}^1 \times \mathbb{P}^1 \).

Another useful representation of \( \mathcal{W} \) is given as follows.

Proposition 3.1. The scheme \( \mathcal{W} \) is isomorphic to the surface

\[
\mathcal{X} := \{([x : y : z], [u : v], t) \in \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{C} \mid yv^2 - zu^2 - txuv = 0\}.
\]
Proof. We have an open covering of $\mathbb{F}_2$ given by the following open embeddings

$$\rho_1 : \mathbb{C} \times \mathbb{C} \times \mathbb{P}^1 \to X$$

$$(t, v, [x : y]) \mapsto ([x : y : yv^2 - tv], [1 : v], t)$$

and

$$\rho_2 : \mathbb{C} \times \mathbb{C} \times \mathbb{P}^1 \to X$$

$$(t', v', [x' : y']) \mapsto ([x' : y'v^2 + t'v' : y'], [v' : 1], t)$$

which glue to give an isomorphism $\mathcal{W} \to X$. □

Remark 3.1. Because of the equivalence, from now on, we use interchangeably between $\mathbb{F}_2, X$ and $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}), W$, respectively.

3.2. Formal vector fields on the formally semi-universal deformation of $\mathbb{F}_2$. The formal deformation associated to $X$, $\tilde{\pi} : \tilde{X} \to \text{Specf}(\mathbb{C}[[t]])$ is a formally semi-universal $\tilde{X}$ of $\mathbb{F}_2$ (here $\mathbb{C}[[t]]$ is the ring of formal power series in $t$). We will give explicit description of a formal vector fields on $\tilde{X}$. Consider the covering $\{W, W'\}$ where $W := \text{Proj}(\mathbb{C}[t, v, x, y])$ and $W' := \text{Proj}(\mathbb{C}[t', v', x', y'])$ as before. A formal vector field on $W$ is of the form

$$(3.2) \quad g_1(v, t) \frac{\partial}{\partial v} + (\alpha_1(v, t)y^2 + \beta_1(v, t)y + \gamma_1(v, t)) \frac{\partial}{\partial y} + k_1(t) \frac{\partial}{\partial t}$$

where $g_1, \alpha_1, \beta_1, \gamma_1, k_1(t)$ are formal power series in $t$. Likewise, a formal vector field on $W'$ is of the form

$$(3.3) \quad g_2(v', t') \frac{\partial}{\partial v'} + (\alpha_2(v', t')y'^2 + \beta_2(v', t')y' + \gamma_2(v', t')) \frac{\partial}{\partial y'} + k_2(t') \frac{\partial}{\partial t'}$$

where $g_2, \alpha_2, \beta_2, \gamma_2, k_2$ are formal power series in $t'$. Therefore, a vector field on $\tilde{X}$ which is of the form (3.2) on $W$ and (3.3) on $W'$ must satisfy the relation

$$(3.4) \quad g_1(v, t) \frac{\partial}{\partial v} + (\alpha_1(v, t)y^2 + \beta_1(v, t)y + \gamma_1(v, t)) \frac{\partial}{\partial y} + k_1(t) \frac{\partial}{\partial t} = g_2(v', t') \frac{\partial}{\partial v'} + (\alpha_2(v', t')y'^2 + \beta_2(v', t')y' + \gamma_2(v', t')) \frac{\partial}{\partial y'} + k_2(t') \frac{\partial}{\partial t'}$$

on the overlapping open set $W \cap W'$.

Lemma 3.1. A global formal vector field on $\tilde{X}$ whose restriction on $W$ is

$$g_1(v, t) \frac{\partial}{\partial v} + (\alpha_1(v, t)y^2 + \beta_1(v, t)y + \gamma_1(v, t)) \frac{\partial}{\partial y} + k_1(t) \frac{\partial}{\partial t}$$

must satisfy the following

$$(3.5) \quad \begin{cases} g_1(v, t) = A(t)v^2 + B(t)v + C(t) \\ \alpha_1(v, t) = a(t)v^2 + b(t)v + c(t) \\ \beta_1(v, t) = -2(a(t)t + A(t))v + e(t) \\ \gamma_1(v, t) = t^2a(t) + tA(t) \end{cases}$$

where $A, B, C, a, b, c, e$ are formal power series of $t$ with a relation

$$(3.6) \quad b(t)t^2 + e(t)t + B(t)t - k(t) = 0.$$
Proof. By (3.1), we have
\[
\begin{aligned}
&y = v^2y' + tv' \\
v = \frac{1}{tv} \\
t = t' \\
\partial_v = -v^2\partial_v' + (2y'v' + t)\partial_y' \\
\partial_y = \frac{1}{v^2}\partial_y' \\
\partial_t = -\frac{1}{v}\partial_y' + \partial_v.
\end{aligned}
\]
Substituting these into the left hand side of (3.4) and equalizing gives us
\[
(3.7) \begin{cases}
g_2(v', t') = -v'^2g_1(\frac{1}{v'}, t') \\
\alpha_2(v', t') = v'^2\alpha_1(\frac{1}{v'}, t') \\
\beta_2(v', t') = 2t'v'\alpha_1(\frac{1}{v'}, t') + \beta_1(\frac{1}{v'}, t') + 2v'g_1(\frac{1}{v'}, t') \\
\gamma_2(v', t') = t'^2\alpha_1(\frac{1}{v'}, t') + \beta_1(\frac{1}{v'}, t') + \frac{1}{v^2}\gamma_1(v', t') + t'g_1(\frac{1}{v'}, t') - \frac{k_1(t')}{v}.
\end{cases}
\]
From these above equations, we have that
\[
\begin{aligned}
g_1(v, t) &= A(t)v^2 + B(t)v + C(t) \\
\alpha_1(v, t) &= a(t)v^2 + b(t)v + c(t) \\
\beta_1(v, t) &= -2(a(t)t + A(t))v + e(t) \\
\gamma_1(v, t) &= t^2a(t) + tA(t),
\end{aligned}
\]
where $A, B, C, a, b, c, e$ are formal power series of $t$ with a relation
\[
b(t)t^2 + e(t)t + B(t)t - k_1(t) = 0.
\]
This constraint comes from the coefficient of $\frac{1}{v}$ in the fourth equation in (3.7). \qed

Remark 3.2. If $t = 0$ then (3.5) becomes
\[
\begin{aligned}
g_1(v) &= Av^2 + Bv + C \\
\alpha_1(v) &= av^2 + bv + c \\
\beta_1(v) &= -2Av + e \\
\gamma_1(v, t) &= 0
\end{aligned}
\]
which agrees with Kodaira’s calculation of vector fields on $\mathcal{X}_0 = \mathbb{F}_2$ (see [3, Page 75]). In particular, we have seven linearly independent vector fields on $\mathbb{F}_2$. If $t$ is non-zero and fixed then we have six linearly independent vector fields on the fiber $\hat{\mathcal{X}}_t$ which is due the existence of the relation (3.6).
4. The non-existence of $G$-equivariant structure on the formally semi-universal deformation

The Lie algebra of $G := \text{Aut}(\mathbb{P}_2)$, i.e. Lie(G) := $\mathbb{C}^3 \times M(2, \mathbb{C})$ is 7-dimensional. Take a $\mathbb{C}$-basis of Lie(G) given by the following elements

\[
\begin{align*}
    e_1 &= (1, 0, 0) \times \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\
    e_2 &= (0, 0, 1) \times \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\
    e_3 &= (0, 0, 0) \times \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\
    e_4 &= (0, 1, 0) \times \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \\
    e_5 &= (0, 0, 0) \times \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \\
    e_6 &= (0, 0, 0) \times \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \\
    e_7 &= (0, 0, 0) \times \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\end{align*}
\]

then we obtain 7 vector fields $E'_1, \ldots, E'_7$ on $\mathbb{P}_2$ with the relations

\[
\begin{align*}
    [E'_1, E'_2] &= 0, \\
    [E'_1, E'_3] &= -2E'_4, \\
    [E'_1, E'_4] &= 0, \\
    [E'_1, E'_5] &= 0, \\
    [E'_1, E'_6] &= -2E'_1, \\
    [E'_1, E'_7] &= 0, \\
    [E'_2, E'_3] &= 0, \\
    [E'_2, E'_4] &= 0, \\
    [E'_2, E'_5] &= -2E'_2, \\
    [E'_2, E'_6] &= 0, \\
    [E'_2, E'_7] &= -2E'_4, \\
    [E'_3, E'_4] &= E'_2, \\
    [E'_3, E'_5] &= -E'_3, \\
    [E'_3, E'_6] &= E'_3, \\
    [E'_3, E'_7] &= E'_5 - E'_6, \\
    [E'_4, E'_5] &= -E'_4, \\
    [E'_4, E'_6] &= -E'_4, \\
    [E'_4, E'_7] &= -E'_1, \\
    [E'_5, E'_6] &= 0, \\
    [E'_5, E'_7] &= -E'_7, \\
    [E'_6, E'_7] &= E'_7.
\end{align*}
\]

Now, we come to the main result of this paper. Suppose that the $G$-action extends on $\hat{X}$. This implies that we also have 7 formal vector fields $E_1, E_2, E_3, E_4, E_5, E_6, E_7$ on $\hat{X}$ with the following Lie bracket constraints

\[
\begin{align*}
    [E_1, E_2] &= 0, \\
    [E_1, E_3] &= -2E_4, \\
    [E_1, E_4] &= 0, \\
    [E_1, E_5] &= 0, \\
    [E_1, E_6] &= -2E_1, \\
    [E_1, E_7] &= 0, \\
    [E_2, E_3] &= 0, \\
    [E_2, E_4] &= 0, \\
    [E_2, E_5] &= -2E_2, \\
    [E_2, E_6] &= 0, \\
    [E_2, E_7] &= -2E_4, \\
    [E_3, E_4] &= E_2, \\
    [E_3, E_5] &= -E_3, \\
    [E_3, E_6] &= E_3, \\
    [E_3, E_7] &= E_5 - E_6, \\
    [E_4, E_5] &= -E_4, \\
    [E_4, E_6] &= -E_4, \\
    [E_4, E_7] &= -E_1, \\
    [E_5, E_6] &= 0, \\
    [E_5, E_7] &= -E_7, \\
    [E_6, E_7] &= E_7.
\end{align*}
\]

These vector fields form a Lie subalgebra of the Lie algebra of formal vector fields on $\hat{X}$, which we denote by $g$. Of course, the restriction of $E_i$ on the central fiber are nothing but $E'_i$ ($i = 1, \ldots, 7$).
From the previous section, we can assume that our seven vector fields are of the form

\[ E_i = g_i(v, t) \frac{\partial}{\partial v} + (\alpha_i(v, t)y^2 + \beta_1(v, t)y + \gamma_i(v, t)) \frac{\partial}{\partial y} + k_i(t) \frac{\partial}{\partial t}, \]

where \( A, B, C, a, b, c, e \) are formal power series of \( t \) (\( i = 1, \ldots, 7 \)).

**Remark 4.1.** The general fibre \( \hat{X} \) of the deformation \( \hat{X} \) is \( P := \mathbb{P}^1 \times \mathbb{P}^1 \) whose automorphism group is the product of two projective linear group: \( \text{PGL}(2, \mathbb{C}) \times \text{PGL}(2, \mathbb{C}) \). The Lie algebra of this group is nothing but \( \mathfrak{sl}_2(\mathbb{C}) \times \mathfrak{sl}_2(\mathbb{C}) \) where \( \mathfrak{sl}_2(\mathbb{C}) \) is the special linear group. Moreover, the Lie algebra of vector fields on \( \mathbb{P}^1 \times \mathbb{P}^1 \) is isomorphic to this Lie algebra.

**Theorem 4.1.** The action of \( G \) on \( \mathbb{F}_2 \) does not extend to the formally semi-universal deformation \( \hat{X} \) where \( G \) is the automorphism group of \( \mathbb{F}_2 \).

**Proof.** We denote by \( \mathfrak{g} \) the Lie algebra of formal vector fields in one variable \( t \). Consider a map \( \delta : \mathfrak{g} \rightarrow \mathfrak{v} \) which sends

\[ g_i(v, t) \frac{\partial}{\partial v} + (\alpha_i(v, t)y^2 + \beta_1(v, t)y + \gamma_i(v, t)) \frac{\partial}{\partial y} + k_i(t) \frac{\partial}{\partial t} \]

to \( k_i(t) \frac{\partial}{\partial t} \) for \( i = 1, \ldots, 7 \). Since, the first two components \( \frac{\partial}{\partial v} \) and \( \frac{\partial}{\partial y} \) contribute nothing to the component \( \frac{\partial}{\partial t} \) in the Lie bracket then \( \delta \) is a well-defined Lie homomorphism. Set \( F_i := \delta(E_i) = k_i(t) \frac{\partial}{\partial t} \) (\( i = 1, \ldots, 7 \)).

Note that \( \mathfrak{v} \) can be equipped with a filtration \( F \) given by the vanishing order at 0 and we have two well-known facts

\[ [F^p \mathfrak{v}, F^q \mathfrak{v}] \subset F^{2p} \mathfrak{v}, \text{ and } [F^p \mathfrak{v}, F^q \mathfrak{v}] \subset F^{p+q-1} \mathfrak{v} \]

for \( p, q \geq 1 \). Also, the vanishing order of all \( k_i \) at 0 is at least 1. Let \( k_i(t) = \sum_{j=1}^{\infty} a_j^i t^j \) (\( i = 1, 2, 4, 5 \)). Using the first fact and Lie relations induced by \( \delta \): \( [F_1, F_0] = -2F_1, [F_2, F_3] = -2F_2, \text{ and } [F_1, F_3] = -2F_4 \), we obtain \( a_1^1 = a_2^3 = a_4^1 = 0 \). Suppose that \( k_4(t) \) is not identically zero, then there exists \( j^* \geq 2 \) such that \( a_j^4 \) is nonzero. By computing explicitly the lie relation \( [F_4, F_3] = -F_4 \) in terms of power series in \( t \) and equalizing coefficient, we get that

\[ a_j^4[(j - 1)a_1^5 - 1] = 0 \]

for all \( j \geq 2 \). Thus, \( a_j^5 = \frac{1}{j - 1} \), which is clearly nonzero. A similar computation for the relation \( [F_1, F_5] = 0 \) gives

\[ (j - 1)a_1^1 a_5^5 = 0 \]

for all \( j \geq 2 \). Hence, all \( a_j^1 = 0 \) so that \( k_1(t) = 0 \). By the relation \( [F_1, F_3] = -2F_4 \), we deduce that \( k_4(t) = 0 \), a contradiction. Therefore, \( k_4(t) = 0 \). From the relations \( [F_3, F_4] = F_2 \text{ and } [F_4, F_7] = -F_1 \), we have in turn that \( k_2(t) = 0 \) and \( k_1(t) = 0 \). Thus, \( E_1, E_2, \text{ and } E_4 \) are all vertical.

In other words, \( E_1, E_2, \text{ and } E_4 \) are vector fields on the fibers \( \mathbb{P}^1 \times \mathbb{P}^1 \). This means that there exists a 3-dimensional abelian Lie subalgebra of \( \mathfrak{sl}_2(\mathbb{C}) \times \mathfrak{sl}_2(\mathbb{C}) \). The image of that subalgebra under one of the two canonical projections of the product \( \mathfrak{sl}_2(\mathbb{C}) \times \mathfrak{sl}_2(\mathbb{C}) \) provides a 2-dimensional abelian Lie subalgebra in \( \mathfrak{sl}_2(\mathbb{C}) \). This is a contradiction since \( \text{rank(}\mathfrak{sl}_2(\mathbb{C})\text{)} \) is only 1. \( \Box \)
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