Influence of Material Damage on the Rayleigh Wave Propagation along Half-Space Boundary

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Abstract—At present, mechanics of damage media, which studies both the stress–strain state of media and the damage accumulation in materials, is being actively developed. In this study, a self-consistent problem, including the dynamic equation of the theory of elasticity and the kinetic equation of damage accumulation in a material, is formulated for an isotropic elastic half-space with damage in the material. It is assumed that damage is distributed uniformly over the medium. The surface-wave propagation along the free boundary of damaged half-space has been investigated. The wave propagates horizontally and decays in the vertical direction. All processes are assumed to be homogeneous along the third axis. It is shown that a self-consistent system with boundary conditions expressing the absence of stress at the half-space boundary is reduced to a complex dispersion equation in this case. In the limiting case (damage-free material), the obtained dispersion equation is reduced to the classical dispersion equation for a Rayleigh wave in the polynomial form (the surface wave propagates along the half-space boundary without dispersion and attenuation). If damage is present in the medium, the surface wave attenuates along the propagation direction, and low-frequency disturbances have frequency-dependent dissipation and dispersion. It is shown that dispersion has abnormal character. It is established that, in the high-frequency region, the phase and group velocities increase and decrease, respectively, with an increase in the damage coefficient. At low frequencies, both velocities increase with a decrease in the damage coefficient.

Keywords: damped surface wave, Rayleigh wave, half-space, damaged medium, complex dispersion equation, low-frequency dispersion

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1. INTRODUCTION

Damage is generally considered as the decrease in the elastic response of body due to reduction of the effective area transferring internal forces from one part of body to another. The decrease in the effective area is caused in turn by the occurrence and development of a stray field of microdefects (microcracks for elasticity, dislocations for plasticity, micropores for creep, and surface microcracks for fatigue) [1–3].

The fundamental studies by Kachanov (summarized in [4]) and Rabotnov (summarized in [5]) gave an impetus to intense development of the mechanics of damaged continuum. In conventional calculations, the measure of damage during development of deformations is taken to be the scalar damage parameter \( \psi(x, t) \), which characterizes the relative density of microdefects scattered uniformly throughout a unit volume. This parameter is zero in the absence of damage and close to unity at the instant of destruction.

In the mechanics of deformed solids, problems of dynamics of damaged materials are generally considered separately from the problems related to damage accumulation. When developing solution methods, it is generally accepted to postulate beforehand that the elastic-wave velocity is a specified function of damage and then determine experimentally the proportionality factors. The phase-wave velocity and wave damping are generally considered as power-law functions of frequency and linear functions of damage [6]. Despite some evident advantages (e.g., simplicity), this approach has a number of drawbacks (as well as any approach that is not based on mathematical models of processes and systems).
In [7–9], the problem was considered as self-consistent and including, along with the damage-evolution equation, the dynamic equation of the elasticity theory. Some problems of the wave dynamics of damaged materials and design elements [7–18] were solved within this statement; in particular, design elements exposed to an external magnetic field affecting the formation and propagation of elastic waves were considered in [17, 18].

2. THREE-DIMENSIONAL SELF-CONSISTENT PROBLEM OF DAMAGE DYNAMICS

In the three-dimensional statement, a self-consistent problem is described by the Lame equations and kinetic equation of damage accumulation:

\[
\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \left( K + \frac{1}{3} G \right) \nabla \nabla \cdot \mathbf{u} + G \Delta \mathbf{u} - \beta_1 \nabla \nabla \psi, \\
\frac{\partial \psi}{\partial t} + \alpha \psi - \beta_2 \left( K + \frac{4}{3} G \right) \nabla \cdot \mathbf{u} = 0.
\]

Here, \( \mathbf{u} \) is the displacement vector; \( K \) is the bulk modulus; \( G \) is the shear modulus; \( \rho \) is the material density; \( t \) is time; \( \alpha = 1/\tau_\alpha \), \( \beta_1 \), and \( \beta_2 \) are constant parameters characterizing the material damage and relationship between the cyclic and damage accumulation processes (\( \alpha > 0 \)); and \( \tau_\alpha \) is the relaxation time.

3. RELATIONSHIP BETWEEN THE DAMAGED-MATERIAL AND HEREDITARY-MEDIUM MODELS

If a material sample is shaped as a rod, along which a longitudinal elastic wave may propagate, it is assumed that the rod is subjected to cyclic tests and that damage may accumulate in the rod material. In this case, the following system of equations is valid [7]:

\[
\frac{\partial^2 \mathbf{u}}{\partial t^2} - C_0 \frac{\partial^2 \mathbf{u}}{\partial x^2} = \beta_1 \frac{\partial \psi}{\partial x}, \\
\frac{\partial \psi}{\partial t} + \alpha \psi = \beta_2 \frac{\partial u}{\partial x}.
\]

Here, \( u(x,t) \) is the displacement of particles on the rod center line; \( C_0 = \sqrt{E/\rho} \) is the velocity of longitudinal-wave propagation in the absence of damage in material; \( E = \frac{9KG}{3K+G} \) is Young’s modulus; and \( \rho \) is the material density.

System of equations (2a) is derived from (1) if the wave process is one-dimensional; the hypothesis of flat cross sections \( u(x, y, z, t) = u(x, t) \) is valid; and the condition of uniaxial deformed state \( u_2(x, y, z, t) = -v \frac{\partial}{\partial x} u(x, t), u_3(x, y, z, t) = -v \frac{\partial}{\partial x} u(x, t) \), which is characteristic of rod extension (compression), is satisfied. Here, \( u_{1,2,3} \) are the displacement-vector components; \( v = \frac{3K - 2G}{2K - 2G} \) is the Poisson ratio; and \( x = x_1, y = x_2, z = x_3 \).

Equation (2b) can be rewritten in the equivalent form:

\[
\psi(x,t) = \beta_2 E \frac{\partial}{\partial x} u(x, \xi) e^{(\xi - t)/\tau} \int_0^\tau \frac{d\xi}{\tau} = \beta_2 E R(t) \frac{\partial}{\partial x} u(x, t),
\]

where the asterisk indicates convolution of the relaxation function \( R(t) \),

\[
R(t) = e^{-t/\tau},
\]

with the deformation function \( \frac{\partial u}{\partial x} \).

Equation (3) describes the process of damage growth in dependence of the deformation history; in this context, one may state that the constant \( \tau > 0 \) is the relaxation time [9]. It is assumed that the damage accumulation history begins at the instant \( \tau = 0 \), because damage is absent in the rod material at the initial moment.
instant \((\psi = 0)\). At a time \(t \gg \tau\) one can use Eq. (3) to derive a dependence determining the damage growth in the case of slow change in stress:

\[
\psi = \tau \beta E \frac{\partial u}{\partial x}.
\]  

(5)

Using Eq. (3), we will rewrite Eq. (2a) as follows:

\[
\rho \frac{\partial^2 u}{\partial t^2} = E \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} - \rho \beta \beta_2 \frac{\partial u}{\partial x} \right).
\]  

(6)

Taking into account the classical equation of motion

\[
\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma}{\partial x},
\]  

(7)

where \(\sigma\) is the longitudinal stress-tensor component, we arrive at the following governing equation for damaged medium:

\[
\sigma = E (1 - \rho \beta \beta_2 R^*) \frac{\partial u}{\partial x} = E \left( \frac{\partial u}{\partial x} - \rho \beta \beta_2 \int_0^t \frac{\partial}{\partial x} u(x, \xi) e^{(\xi - t)/\tau} d\xi \right).
\]  

(8)

The materials described by this equation, in which the current stress–strain state depends on the deformation history, are referred to as materials with memory [5].

A model of a material with memory can be constructed by replacing the constant elastic parameters of solids with time-dependent operators. For example, the elastic response of internal constraints to an external effect decreases in a damaged material (damage makes the material more readily deformed). This phenomenon can be taken into account by replacing constant Young’s modulus with the operator

\[
\tau = \tau(t).
\]  

Then Eq. (8) can be rewritten in the form

\[
\sigma(\varepsilon) = E_0 (t) \varepsilon.
\]  

(10)

It follows from Eq. (10) that, if \(t \ll \tau\), the governing equation for rapid loading can be written as

\[
\sigma = E \frac{\partial u}{\partial x} = E_d \frac{\partial u}{\partial x}.
\]  

(11)

Here, \(E_d\) is dynamic Young’s modulus.

On the contrary, in the case of slow loading \((t \gg \tau)\), the governing equation is

\[
\sigma = E_d (1 - \tau \rho \beta \beta_2) \frac{\partial u}{\partial x} = \delta E_d \frac{\partial u}{\partial x} = E_s \frac{\partial u}{\partial x},
\]  

(12)

where \(E_s = \delta E_d\) is static Young’s modulus and \(\delta = 1 - \tau \rho \beta \beta_2\) is a parameter characterizing the material damage (it lies in the range of \([0;1]\)). In the absence of damage \(\delta = 1\), whereas for a fractured material \(\delta = 0\).

4. SPECIFIC FEATURES OF SURFACE-WAVE PROPAGATION

Let us consider the propagation of a Rayleigh surface wave in an isotropic elastic half-space, provided that the material occupying the half-space is damaged (we restrict ourselves to a two-dimensional case, where all processes occurring in the direction of the \(x_3\) axis are homogeneous). In this case, the system of equations (1) becomes two-dimensional:

\[
\rho \frac{\partial^2 u_1}{\partial t^2} - \left( \lambda + 2\mu \right) \frac{\partial^2 u_1}{\partial x_1^2} - \mu \frac{\partial^2 u_1}{\partial x_1 \partial x_2} - \left( \lambda + \mu \right) \frac{\partial^2 u_1}{\partial x_2^2} = -\beta \frac{\partial \psi}{\partial x_1},
\]  

(13)

\[
\rho \frac{\partial^2 u_2}{\partial t^2} - \left( \lambda + 2\mu \right) \frac{\partial^2 u_2}{\partial x_2^2} - \mu \frac{\partial^2 u_2}{\partial x_1^2} - \left( \lambda + \mu \right) \frac{\partial^2 u_2}{\partial x_1 \partial x_3} = -\beta \frac{\partial \psi}{\partial x_3},
\]  

(14)
where \( u_1(x_1, x_3, t) \) and \( u_3(x_1, x_3, t) \) are the components of the displacement vector along the \( x_1 \) and \( x_3 \) axes, respectively, and \( \lambda \) and \( \mu \) are the Lame constants.

The system of equations (13)–(15) must be supplemented with the following boundary conditions, which determine the absence of stress at the half-space boundary:

\[
\left[ \frac{\partial u_1}{\partial x_1} + \frac{\partial u_3}{\partial x_3} \right]_{x_3=0} = 0,
\]

\[
\left[ \frac{\partial u_3}{\partial x_3} + \left(1 - 2\frac{c_i^2}{c_l^2}\right)\frac{\partial u_1}{\partial x_1} \right]_{x_3=0} = 0,
\]

where \( c_l = \sqrt{(\lambda + 2\mu)/\rho} \) and \( c_i = \sqrt{\mu/\rho} \) are, respectively, the velocities of dilatation and shift waves in an infinite medium.

Having excluded the damage function from the system of equations (13)–(15), we obtain the following two equations with respect to the longitudinal and transverse displacements:

\[
\frac{\partial^2 u_1}{\partial t^2} - c_i^2 \left(1 - \frac{\beta \beta_2}{\alpha} \right) \frac{\partial^2 u_1}{\partial x_1^2} = c_i^2 \frac{\partial^2 u_3}{\partial x_3^2} - \left(c_m^2 - c_i^2 \frac{\beta \beta_2}{\alpha} \right) \frac{\partial^2 u_3}{\partial x_1 \partial x_3},
\]

\[
\frac{\partial^2 u_3}{\partial t^2} - c_i^2 \left(1 - \frac{\beta \beta_2}{\alpha} \right) \frac{\partial^2 u_1}{\partial x_3^2} = c_i^2 \frac{\partial^2 u_1}{\partial x_1^2} - \left(c_m^2 - c_i^2 \frac{\beta \beta_2}{\alpha} \right) \frac{\partial^2 u_1}{\partial x_1 \partial x_3},
\]

where \( c_m = \sqrt{(\lambda + \mu)/\rho} \) (\( c_i < c_m < c_l \) and \( c_m^2 = c_i^2 - c_i^2 \)).

Let us consider a disturbance propagating along the \( x_3 = 0 \) boundary and decaying in the direction of the \( x_1 \) axis. The displacement–vector components can be written as

\[
u_1 = \frac{\partial \varphi}{\partial x_1} - \frac{\partial \theta}{\partial x_3}, \quad u_3 = \frac{\partial \varphi}{\partial x_3} + \frac{\partial \theta}{\partial x_1},
\]

where \( \varphi \) is the scalar potential (\( \text{curl grad} \varphi = 0 \)) and \( \theta = \{0, \theta, 0\} \) is the nonzero component of the vector potential (\( \text{div curl} \theta = 0 \)). These potentials have the form

\[
\varphi = A_1 \exp(-q_1 x_3 + i (k x_1 - \omega t)),
\]

\[
\theta = A_2 \exp(-q_2 x_3 + i (k x_1 - \omega t)),
\]

where \( \omega \) is frequency, \( k \) is the wave number, \( q_1 \) and \( q_2 \) are positive parameters characterizing the attenuation of disturbance propagating into the half-space bulk, and \( A_1 \) and \( A_2 \) are arbitrary constants.

Taking into account (20)–(22), we derive the dispersion equation from the system (16)–(19)

\[
-c_i^2 (k^2 - q_1^2)(k^2 + q_3^2) + 2c_i^2 k^2 (k^2 + q_3^2 - 2q_1 q_2) = 0,
\]

in which the parameters \( q_1 \) and \( q_2 \) are determined as

\[
q_1 = k^2 - \frac{(i \omega - \alpha) \omega^2}{(i \omega - \alpha + \beta \beta_2)c_i^2}, \quad q_2 = k^2 - \frac{\omega^2}{c_i^2}.
\]

It follows from relations (23) and (24) that the wave number \( k \) is complex: \( k = k_t + i k_z \). Substituting this expression into the complex dispersion equation and isolating the real and imaginary parts, we obtain a system of two nonlinear algebraic equations.
In the limiting case, where the material is free of damage, this system of algebraic equations is reduced to the dispersion equation for a Rayleigh wave [19, 20], and the frequency dependence of the wave number can be written as

$$\omega^6 - 8k_1^2\omega^4 + 8k_1^4(3 - 2a_1^2)\omega^2 - 16k_1^6(1 - a_1^2) = 0,$$

(25)

where $a_1 = c_1/c_0$. Dispersion equation (25) is written in dimensionless variables $\bar{\omega}$ and $\bar{k}$ ($\omega = \alpha\bar{\omega}$, $k = (\alpha/c_1)\bar{k}$; the bar symbols are omitted in (25) and the expressions below). The dimensionless Rayleigh wave velocity is determined as $c_\alpha = \omega/k_1$. It is known that a classical Rayleigh wave propagates along the free boundary of a half-space without damping and dispersion and that its velocity is constant for any material [19]. It follows from (23) and (24) that a Rayleigh wave decays in a damaged material.

The frequency dependences of the real and imaginary parts of the wave number at different values of the damage parameter and fixed parameter $a_0$ are shown in Figs. 1 and 2. The parameter $a_0$ is expressed in terms of the Poisson ratio ($\nu$) as $a_0 = \sqrt{(1 - 2\nu)/(2 - 2\nu)}$, and its value changes within the range $0 \leq a_0 \leq \sqrt{2}/2$. The dimensionless parameter $a_1 = (\beta_1\beta_2)/\alpha$ characterizes the material damage, and its sign depends on the signs of the initial parameters $\beta_1$ and $\beta_2$ (in most cases, $a_2 < 0$). The dispersion curves for positive and negative values of this parameter differ radically.

At $a_2 \neq 0$, the dispersion relations of the real and imaginary parts have two branches: positive and negative $k_1$ values correspond to positive and negative $k_2$ values, respectively. Therefore, these dependences can be considered in only the first quarter. Figure 1 shows the functions $k_1(\omega)$ at different $a_2$ values (Fig. 1a) and two dependences clearly demonstrating the existing deviations of the curve under consideration from the dispersion straight line [19] corresponding to the classical Rayleigh wave (Fig. 1b). One can see in Fig. 2 that the dependence $k_2(\omega)$ has a horizontal asymptote at $\omega \to +\infty$, and damping coefficient $\gamma = k_2/k_1$ tends to zero at large $\omega$ values (see Fig. 3). At $a_2 = 0$, the frequency dependence of the real part of the wave number is a straight line; the imaginary part is absent.

The frequency dependences of the phase ($\nu_{ph} = \omega/k_1$) and group ($\nu_{gr} = d\omega/dk_1$) velocities are shown in Fig. 4. One can see that the curve for the group velocity is located above the phase-velocity curve at a fixed nonzero parameter $a_2$; dispersion is obvious at low frequencies. With a decrease in the damage coef-
Influence of material damage

The results of studying the Rayleigh wave propagation at frequencies of 2.5 and 10 MHz along the boundary of a St3ps steel sample in two states, unloaded and extended to plastic deformation threshold ($\sigma_{02} < \sigma < \sigma_B$, where $\sigma_B = 470$ MPa is the strength limit and $\sigma_{02} = 300$ MPa is the conditional yield strength) were reported in [21]. It was established that the Rayleigh wave velocity is independent

Fig. 2. Frequency dependences of the imaginary part of wave number, $k_2(\omega)$, at different $a_2$ values: $a_2 = -1$ (solid line) and $-1 < a_2 < 0$ (dashed line).

Fig. 3. Frequency dependences of the damping coefficient $\gamma(\omega)$ at different $a_2$ values: $a_2 = -1$ (solid line) and $-1 < a_2 < 0$ (dashed line).
of frequency (dispersion is absent) in unloaded samples and under stresses below the elasticity limit. At the same time, rather strong frequency dependence of velocity was observed in the plastic deformation range.

5. CONCLUSIONS

In the case of damaged medium, surface waves decay when propagating along a half-space boundary and exhibit dispersion. The presence of damage facilitates dispersion in the low-frequency range (the smaller the damage coefficient, the lower the dispersion is). The dispersion has an anomalous character. In the case of damage-free medium, surface waves propagate without dispersion and damping.

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