A GENERALIZATION OF THE GRAPH PACKING THEOREMS
OF SAUER–SPENCER AND BRANDT

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Received July 9, 2020
Revised October 31, 2021
Online First October 10, 2022

We prove a common generalization of the celebrated Sauer–Spencer packing theorem and a theorem of Brandt concerning finding a copy of a tree inside a graph. This proof leads to the characterization of the extremal graphs in the case of Brandt’s theorem: If \( G \) is a graph and \( F \) is a forest, both on \( n \) vertices, and \( 3\Delta(G) + \ell^*(F) \leq n \), then \( G \) and \( F \) pack unless \( n \) is even, \( G = \frac{n}{2}K_2 \) and \( F = K_{1,n-1} \); where \( \ell^*(F) \) is the difference between the number of leaves and twice the number of nontrivial components of \( F \).

1. Introduction

Given two graphs \( G \) and \( H \) both on \( n \) vertices, we say that \( G \) and \( H \) pack if there is a bijection \( f : V(G) \to V(H) \) such that for every \( uv \in E(G) \), \( f(u)f(v) \notin E(H) \); in other words, edge-disjoint copies of \( G \) and \( H \) can be found in \( K_n \), or equivalently, \( G \) is isomorphic to a subgraph of the complement of \( H \). This concept leads to a natural generalization of a number of problems in extremal graph theory, such as existence of a fixed subgraph, equitable colorings, and Turán-type problems. The study of packing of graphs was started in the 1970s by Bollobás and Eldridge [3], Sauer and Spencer [16], and Catlin [6]. See the surveys by Kierstead et al. [15], Wozniak [18], and Yap [19] for later developments in this field. In the following, we will use \( \Delta(G) \) (resp., \( \delta(G) \)) to denote the maximum (resp., minimum) degree of a graph \( G \).

Mathematics Subject Classification (2010): 05C35
The major conjecture in graph packing is that of Bollobás and Eldridge [3], and independently by Catlin [7], from 1978, that
$$\left(\Delta(G) + 1\right)\left(\Delta(H) + 1\right) \leq n + 1$$
is sufficient for $G$ and $H$ to pack. Some partial results are known, e.g. [11,13,1,2,8,9,17].

In 1978, Sauer and Spencer proved the following celebrated result.

**Theorem 1 (Sauer, Spencer [16]).** Let $G, H$ be graphs on $n$ vertices such that $2\Delta(G)\Delta(H) < n$. Then $G$ and $H$ pack.

Kaul and Kostochka [12] strengthened the result by characterizing the extremal graphs: if $2\Delta(G)\Delta(H) = n$ and $G$ and $H$ fail to pack, then $n$ is even, one of the graphs is $n/2K_2$, and the other is either $K_{n/2,n/2}$ (with $n/2$ odd) or contains $K_{n/2+1}$. Let $\ell(F)$ denote the number of leaves in a forest $F$. In 1994, Brandt [5] proved that if $G$ is a graph and $T$ is a tree, both on $n$ vertices, and $3\Delta(G) + \ell(T) - 2 < n$, then $G$ contains a copy of $T$. This can be rephrased in terms of packing.

**Theorem 2 (Brandt [5]).** If $G$ is a graph and $T$ is a tree, both on $n$ vertices, and
$$3\Delta(G) + \ell(T) - 2 < n,$$
then $G$ and $T$ pack.

We need a generalization of this theorem to a forest $F$, which is straightforward and motivates the following definition.

**Definition.** The *excess leaves* of a forest $F$, denoted $\ell^*(F)$, is
$$\sum_{v \in V(F)} \max\{d(v) - 2, 0\}.$$Note that linear forests are precisely the forests with zero excess leaves. We also have that $\ell^*(F)$ equals the number of leaves of $F$ minus twice the number of nontrivial components of $F$ (those having at least two vertices), and that for a tree $T$, $\ell^*(T) = \ell(T) - 2$.\(^1\)

**Corollary 3.** If $G$ is a graph and $F$ is a forest, both on $n$ vertices, and $3\Delta(G) + \ell^*(T) < n$, then $G$ and $F$ pack.

\(^1\) For these statements, consider the sum $\sum_{i \geq 0} (i - 2)n_i$, where $n_i$ is the number of vertices with degree $i$. 
Proof. Iteratively add edges joining leaves of distinct nontrivial components of $F$; each such addition does not change $\ell^*$. When there is only one nontrivial component left, iteratively add edges from any leaf to the remaining (isolated) vertices; again $\ell^*$ is preserved. Now we have a tree, for which $\ell^* = \ell - 2$. Brandt’s theorem now applies, so that $G$ and the new tree pack, and deleting the added edges gives a packing of $G$ with $F$.

Corollary 3 is sharp when $n$ is even, with $G = \frac{n}{2}K_2$ and $F = K_{1,n-1}$. We will prove that this is the only pair of extremal graphs, strengthening Brandt’s result as follows.

**Theorem 4.** If $G$ is a graph and $F$ is a forest, both on $n$ vertices, and

$$3\Delta(G) + \ell^*(F) \leq n,$$

then $G$ and $F$ pack unless $n$ is even, $G = \frac{n}{2}K_2$, and $F = K_{1,n-1}$.

To accomplish this, we will first prove the following theorem, which generalizes both the Sauer–Spencer and Brandt packing theorems.

**Theorem 5.** Let $G$ be a graph and $H$ a $c$-degenerate graph, both on $n$ vertices. Let $d_1^{(G)} \geq d_2^{(G)} \geq \cdots \geq d_n^{(G)}$ be the degree sequence of $G$, and similarly for $H$. If

$$\sum_{i=1}^{\Delta(G)} d_i^{(H)} + \sum_{j=1}^{c} d_j^{(G)} < n,$$

then $G$ and $H$ pack.

This strengthens Sauer–Spencer, since $c \leq \Delta(H)$.
This also strengthens Brandt’s theorem: if $H$ is a tree, then $c = 1$, so the second summation is just $\Delta(G)$. For the first summation,

$$\sum_{i=1}^{\Delta(G)} d_i^{(H)} = 2\Delta(G) + \sum_{i=1}^{\Delta(G)} (d_i^{(H)} - 2) \leq 2\Delta(G) + \ell(H) - 2.$$

It is easy to construct examples of graphs $G$ and $H$ for which conditions in the Sauer–Spencer theorem, Brandt’s theorem, or even the Bollobás, Eldridge, and Catlin conjecture are not true, but Theorem 5 does apply.

The proof of Theorem 5 generally follows that of Sauer–Spencer. In the special setting of Brandt’s theorem, the proof can be analyzed more closely to show that the only sharpness example is the one mentioned above.

Theorem 5 is sharp itself, with several sharpness examples. It retains all the Sauer–Spencer sharpness examples (with $n$ even) mentioned earlier:
• $H = \frac{n}{2} K_2$ and $G \supseteq K_{n/2+1}$
• $H = \frac{n}{2} K_2$ and $G = K_{n/2,n/2}$, with $n/2$ odd
• $H \supseteq K_{n/2+1}$ and $G = \frac{n}{2} K_2$
• $H = K_{n/2,n/2}$ and $G = \frac{n}{2} K_2$, with $n/2$ odd.

And it has an additional family of sharpness examples:
• $H = K_{s,n-s}$ and $G = \frac{n}{2} K_2$, with $n$ even and $s$ odd (in particular, $H = K_{1,n-1}$ and $G = \frac{n}{2} K_2$).

We do not know whether these are all the sharpness examples, even if we restrict to the case that $H$ is a forest.

**Question 6.** What are the extremal graphs for Theorem 5? Do the above listed families of graphs include all the extremal graphs for Theorem 5 when $H$ is a forest?

Note that Theorem 4 shows that the only extremal graphs for that theorem have $n$ even and $\Delta(G) = 1$. So, it is natural to ask:

**Question 7.** By Theorem 4, $3\Delta(G) + \ell^*(F) < n+1$ is a sufficient condition for packing of a graph $G$ and a forest $F$ on $n$ vertices when $n$ is odd or $\Delta(G) \geq 2$. Is this statement sharp? If yes, what are all its sharpness examples?

Degeneracy versions of the Sauer–Spencer packing theorem have been studied before, in [4] and [14]. If we think of the condition in Sauer–Spencer as the sum of two terms: $\Delta(G) \Delta(H) + \Delta(H) \Delta(G) < n$, then Theorem 5 can be thought of as replacing $\Delta(H)$ by the degeneracy $c(H)$ in one of the terms (in addition to other degree sequence related improvements). The result in [4] replaces $\Delta(G)$ by $c(G)$ in one term and $\Delta(H)$ by $\log \Delta(H)$ in the other. In [14], $\Delta(G)$ is replaced by $(gcol(G) - 1)$ in one term and $\Delta(H)$ by $(gcol(H) - 1)$ in the other, where $gcol$ denotes the game coloring number and $gcol(G) - 1$ lies in between the degeneracy and the maximum degree (see [14] for precise definition and details). It is natural to ask for improvements or extensions of Theorem 5 by considering degree-sum conditions that interplay between maximum degree, degeneracy, and game coloring number. For example, does $\sum_{i=1}^{gcol(G)-1} d_i^{(H)} + \sum_{j=1}^{gcol(H)-1} d_j^{(G)} < n$ suffice for a packing of $G$ and $H$ under the set-up of Theorem 5? Or, does $c_1 \sum_{i=1}^{[\log \Delta(G)]} d_i^{(H)} < n$ and $c_2 \sum_{j=1}^{c(H)} d_j^{(G)} < n$ for some fixed constants $c_1$ and $c_2$ suffice?

2. Proofs

Throughout, we think of a bijective mapping $f : V(G) \to V(H)$ as the multi-graph with vertices $V(G)$ and edges labelled by “$G$” or “$H$”. We speak of $H$-
and $G$-edges, $H$- and $G$-neighbors of vertices, and $H$-cliques, $H$-independent sets, etc. A link is a copy of $P_3$ with one $H$-edge and one $G$-edge, and a $uv$-link is a link with endpoints $u$ and $v$; a $GH$-link from $u$ to $v$ is a link with endpoints $u,v$ whose edge incident to $u$ is from $G$; similarly, we have $HG$-links. From a given mapping $f$, a $uv$-swap results in a new mapping $f'$ with $f'(u) = f(v)$, $f'(v) = f(u)$, and $f' = f$ otherwise. A quasipacking of $G$ with $H$ is a mapping $f$ whose multigraph is simple except for a single pair of vertices joined by both an $H$-edge and a $G$-edge; this pair is called the conflicting edge of the quasipacking.

Consider a pair of graphs $(G,H)$, with $H$ being $c$-degenerate, each on $n$ vertices, that do not pack; furthermore assume that $H$ is edge-minimal with this property. Thus for any edge $e$ in $H$, $G$ and $H - e$ pack, and so there is a quasipacking of $H$ and $G$ with conflicting edge $e$.

Let $u'$ be a vertex of minimum positive degree in $H$, let $x' \in N_H(u')$, and consider a quasipacking $f$ of $G$ with $H$ with conflicting edge $u'x'$. Let $u = f^{-1}(u')$ and $x = f^{-1}(x')$. We will now consider the set of links from $u$ to each vertex.

Consider a $y \in V(G) \setminus \{u,x\}$. Perform a $uy$-swap: since $G$ and $H$ do not pack, there must be some conflicting edge, and such a conflict must involve an $H$-edge incident to either $u$ or $y$; together with the conflicting $G$-edge, we have a $uy$-link in the original quasipacking. There are two links from $u$ to itself, using the parallel edges $ux$ in each order. Thus, there are at least $n$ links from $u$ in the original quasipacking $f$.

The number of $GH$-links from $u$ is at most $\sum_{y \in N_G(u')} \deg_H(f(y))$. The number of $HG$-links from $u$ is at most $\sum_{z' \in N_H(u')} \deg_G(f^{-1}(z'))$. Hence we have

$$n \leq \# \text{ links from } u \leq \sum_{y \in N_G(u')} \deg_H(f(y)) + \sum_{z' \in N_H(u')} \deg_G(f^{-1}(z'))$$

This establishes Theorem 5.

To prove Theorem 4, suppose additionally that $H$ is a forest, henceforth called $F$, and that $3\Delta(G) + \ell^*(F) = n$. (So, we still assume that $G$ and $F$ do not pack, and that $F$ is edge-minimal with this property.)

If $\Delta(G) = 1$, then it is easy to show that $n$ is even, $G = \frac{n}{2}K_2$, and $F = K_{1,n-1}$. (In fact, such a $G$ will pack with any bipartite graph that is not complete bipartite.) So we henceforth assume that $\Delta(G) > 1$, and seek a contradiction.
Lemma 8. For any leaf $u'$ of $F$ and $x'$ its neighbor, and a quasipacking $f$ of $G$ with $F$ with $f(u) = u'$ and $f(x) = x'$ and conflicting edge $ux$, we have the following.

1. For every $y \in V(G) \setminus \{u, x\}$, there is a unique link from $u$ to $y$; there is no link from $u$ to $x$; and there are two links from $u$ to itself.

2. $\deg_G(x) = \deg_G(u) = \Delta(G)$.

3. For every $w \in N_G(u)$, $\deg_F(f(w)) \geq 2$.

4. For every $w \notin N_G(u)$, $\deg_F(f(w)) \leq 2$.

Proof. Note that we now have $\deg_F(u') = 1$, so $\sum_{z' \in N_F(u')} \deg_G(f^{-1}(z')) = \deg_G(x)$. In this case we can expand on (1):

\[(2)\]
\[n \leq \# \text{ links from } u \leq \sum_{y \in N_G(u)} \deg_F(f(y)) + \deg_G(x)\]

\[(3)\]
\[\leq \sum_{y \in N_G(u)} (\deg_F(f(y)) - 2) + 2\Delta(G) + \Delta(G)\]

\[(4)\]
\[\leq \sum_{y \in N_G(u)} \max\{\deg_F(f(y)) - 2, 0\} + 3\Delta(G)\]

\[(5)\]
\[\leq \sum_{i=1}^{n} \max\{d_i^{(F)} - 2, 0\} + 3\Delta(G) = \ell^*(F) + 3\Delta(G) = n,\]

so we have equality throughout. Conclusion $i$ follows from having equality in line $(i + 1)$ above, for $i \in [4]$. 


For a vertex $v$ in a graph $H$, we write $N_H[v]$ for the closed neighborhood, i.e., $N_H(v) \cup \{v\}$. For a set $S$ of vertices, $N_H(S) = \bigcup_{v \in S} N_H(v) - S$.

Lemma 9. For any leaf $u'$ of $F$ and $x'$ its neighbor, and a quasipacking $f$ of $G$ with $F$ with $f(u) = u'$ and $f(x) = x'$ and conflicting edge $ux$, we have the following.

1. $N_G[u] = N_G[x]$.

2. Let $Q = N_G[u]$. Then $G[Q]$ is a clique component.

Proof. Proof of part 1.

Let $A = N_G(u) - N_G[x]$, $B = N_G(u) \cap N_G(x)$, $C = N_G(x) - N_G[u]$. Also, let $N_A = N_F(f(A))$, $N_B = N_F(f(B))$, $N_C = N_F(f(C))$, and $N_x = N_F(x')$.

We will show that $A = N_A = N_C = C = \emptyset$.

Note that $B \cup C \cup \{u\}$ is precisely the set of vertices with an $FG$-link from $u$. By Lemma 8(1), there are no $F$-edges from $A$ to $x$, else $x$ would
have a GF-link; and there are no F-edges from A to B∪C∪Nx, else such an endpoint in B∪C∪Nx would have two links. So for each vertex of A to have exactly one link, F[f(A)] must be a perfect matching. Furthermore, the F-edges incident to A only have endpoints in A∪NA. Each vertex of NA must have exactly one F-edge from A (to have one link). And by Lemma 8(3), each vertex of A has at least one F-neighbor in NA. Note that we thus have |NA| ≥ |A|. The vertices of NA, NB, Nx all have GF-links by definition, and to have exactly one, these sets must be disjoint. Thus we have that \{u,x\}, A, B, C, NA, NB, Nx is a partition of V(G). See the left side of Figure 1.

Now perform a ux-swap. In Figure 1, we visualize with V(G) fixed, so just the F-edges adjacent to u′ and x′ move; roughly speaking, we just interchange the roles of u and x and those of A and C. The result is again a quasipacking with ux the only conflicting edge. The F-neighbors of u are precisely Nx. Repeating the arguments of the last paragraph, for each vertex of NC to have exactly one link, we must have |NC| ≥ |C|. Suppose that A ≠ ∅. Then since |NA| ≥ |A|, NA ≠ ∅ as well. Now, the only possible links (from x) to vertices in NA are GF-links through C; hence NC = NA, and the F-edges incident to C have endpoints in C∪NA. Furthermore, since NC = NA ≠ ∅, C ≠ ∅ as well; so each vertex of NA has F-degree at least 2 (one edge from A and one from C). But also, by Lemma 8(3) applied to the original and also this new quasipacking, every vertex of A and C has F-degree at least 2, with F-edges entirely in A∪C∪NA. So F[f(A∪C∪NA)] has minimum degree at least 2, contradicting that it is a forest, unless A = C = NA = ∅.

Note that this implies that NG[u] = NG[x] = \{u, x\} ∪ B.

Proof of part 2.

This time perform a uy-swap for some vertex y ∈ Q \ {u, x} to get ˜f. The result is again a quasipacking with yx the only conflicting edge, with ˜f(y) = u′. By part 1, NG[y] = NG[x] = Q. Since this holds for every y ∈ Q \ {u, x}, we have that Q is a clique; and since degG(x) = Δ(G) by Lemma 8(2), G[Q] is a clique component of G.
Let $u'$ be a leaf in $F$, and let $x'$ be its neighbor. Consider a quasipacking $f$ of $G$ with $F$ with $f(u) = u'$ and $f(x) = x'$ and conflicting edge $ux$. (Such exists by the extremal choice of $F$, as in the proof of Theorem 5.)

Let $G[Q]$ be the clique component of $G$ given in Lemma 9(2). Let $z$ be a vertex of $Q$ with smallest $F$-degree larger than 1 (such a choice is possible, as $\deg_F(x') \geq 2$ by Lemma 8(3)), and let $z' = f(z)$. Let $z_1, z_2 \in V(G)$ be two $F$-neighbors of $z$.

In $f$, $z_1$ and $z_2$ each have exactly one $F$-edge into $Q$ and at most one other $F$-edge, by Lemma 8(1,4). So $z_1, z_2$ have no $F$-neighbors inside $Q$ except $z$. From this and that $Q$ is a $G$-clique in the quasipacking, the set $Q \cup \{z_1, z_2\} \setminus \{u, z\}$ is $F$-independent. Let $X = f(Q \cup \{z_1, z_2\} \setminus \{u, z\})$.

Let $g : V(G) \to V(F)$ be a bijection such that $g(Q) = X$. Since $G[Q]$ is a clique component and $X$ is independent, $g$ is a packing if and only if $g|_{G-Q}$ is a packing of $G-Q$ with $F-X$.

Claim. $\deg_F(z') \geq 4$.

Suppose to the contrary that $\deg_F(z') \leq 3$. We have taken two of the neighbors of $z$ into $X$, so $\deg_{F-X}(z') \leq 1$. And $z'$ is the only vertex of $F-X$ that may have degree larger than 2, by Lemma 8(4). That is, $F-X$ is a linear forest. We have that $\Delta(G-Q) \leq \Delta(G) \leq \frac{n}{3}$, so

$$\delta(G-Q) = |V(G-Q)| - 1 - \Delta(G-Q)$$

$$= n - (\Delta(G) + 1) - 1 - \Delta(G-Q)$$

$$\geq n - \frac{3}{2} \Delta(G) - \frac{1}{2} \Delta(G) - 2$$

$$\geq \frac{1}{2} n - \frac{1}{2} \Delta(G) - 2$$

$$= \frac{1}{2} |V(G-Q)|,$$

and so Dirac’s condition for Hamiltonicity applies ([10]). Since $G-Q$ contains a Hamiltonian cycle, it also contains the linear forest $F-X$, i.e., $F-X$ and $G-Q$ pack, a contradiction. This completes the proof of the Claim.

This Claim, together with having $z' \notin X$ but its two neighbors $z_1, z_2 \in X$, gives us the inequality

$$\ell^*(F-X) = \sum_{v \in V(F-X)} \max\{\deg_{F-X}(v) - 2, 0\}$$

$$\leq -2 + \sum_{v \in V(F-X)} \max\{\deg_F(v) - 2, 0\}$$
\[
= -2 + \sum_{v \in V(F)} \max\{\deg_F(v) - 2, 0\} - \sum_{v \in X} \max\{\deg_F(v) - 2, 0\}
\]

From Lemma 8(3), every vertex of \( f(Q - u) \) has \( F \)-degree at least two; and since \( z' \) was chosen to have smallest \( F \)-degree among the non-leaves of \( f(Q) \), the Claim gives that they must in fact have degree at least four. All these vertices except \( z' \) are in \( X \), so we have at least \( \Delta(G) - 1 \) vertices of \( X \) with degree at least 4. Hence

\[2 + \sum_{v \in X} \max\{\deg_F(v) - 2, 0\} \geq 2 \Delta(G) > \Delta(G) + 1,\]

so

\[3 \Delta(G - Q) + \ell^*(F - X) \leq 3 \Delta(G) + \ell^*(F) - 2\]
\[= n - 2 - \sum_{v \in X} \max\{\deg_F(v) - 2, 0\}
< n - \Delta(G) - 1
= |V(G - Q)|.\]

Thus, by Theorem 5, \( G - Q \) and \( F - X \) pack, a contradiction. This completes the proof of Theorem 4.

Acknowledgment. The authors thank the anonymous referees for their helpful suggestions for improving the exposition.

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