A New Signal Representation Using Complex Conjugate Pair Sums

Shaik Basheeruddin Shah, Student Member, IEEE,
Vijay Kumar Chakka, Senior Member, IEEE, and Arikatla Satyanarayana Reddy

Abstract—This letter introduces a real valued summation known as Complex Conjugate Pair Sum (CCPS). The space spanned by CCPS and its one circular downshift is called Complex Conjugate Subspace (CCS). For a given positive integer \( N \geq 3 \), there exists \( \frac{\varphi(N)}{2} \) CCPSs forming \( \frac{\varphi(N)}{2} \) CCSs, where \( \varphi(N) \) is the Euler’s totient function. We prove that these CCSs are mutually orthogonal and their direct sum form a \( \varphi(N) \) dimensional subspace \( s_N \) of \( C^N \). We propose that any signal of finite length \( N \) is represented as a linear combination of elements from a special basis of \( s_N \), for each divisor \( d \) of \( N \). This defines a new transform named as Complex Conjugate Periodic Transform (CCPT). Later, we compared CCPT with DFT (Discrete Fourier Transform) and RPT (Ramanujan Periodic Transform). It is shown that, using CCPT we can estimate the period, hidden periods and frequency information of a signal. Whereas, RPT does not provide the frequency information. For a complex valued input signal, CCPT offers computational benefit over DFT. A CCPT dictionary based method is proposed to extract non-divisor period information.

Index Terms—Complex Conjugate Pair Sum, Complex Conjugate Subspace, CCPT, CCPT Dictionary.

I. INTRODUCTION

FINITE length signal representation is a fundamental concept in signal processing, which has been studied for years. There are some techniques like DFT, DCT (Discrete Cosine Transform) and RPT [1], [2]. Here DFT and DCT are very well-known techniques, whereas RPT is newly introduced by P.P. Vaidyanathan, which is widely used in period estimation applications [3], [4], [5]. Recently Srikanth V. Tenneti and P.P. Vaidyanathan introduced a specific kind of signal representation using Nested Periodic Matrices (NPMs) [6], which is useful for period estimation. RPT and DFT are members of NPMs family. The signal representation proposed in this letter is inspired from NPMs and subspace-based signal representation [7], [8], [9].

For a given positive integer \( N \geq 3 \), there exist \( \frac{\varphi(N)}{2} \) pairs of complex conjugate exponential sequences, having period exactly equal to \( N \) [10]. By using each pair, a real valued summation known as complex conjugate pair sum is introduced. Later, some of its properties like symmetricity, periodicity and orthogonality are studied. It is shown that, each proposed summation and its one circular downshift act as a basis for two-dimensional complex conjugate subspace. So, there are total \( \frac{\varphi(N)}{2} \) complex conjugate pair sums leading to \( \frac{\varphi(N)}{2} \) complex conjugate subspaces for a given positive integer \( N \geq 3 \).

II. COMPLEX CONJUGATE PAIR SUM

In DFT, an \( N \)-finite length signal is represented by using the following complex exponential basis,

\[ s_{N,k}(n) = e^{i2\pi kn/N}, \quad 0 \leq n, k \leq N - 1. \]  

For a given \( k \), we denote the period of \( s_{N,k}(n) \) by \( P_k \). It is known that \( P_k = \frac{N}{\text{gcd}(N,k)} \). Hence \( P_k | N \). Let \( D_N = \{ d \in \mathbb{N} | d | N \} \) be the set of positive divisors of \( N \). In order to count the number of \( k \in \{0, 1, \ldots, N-1\} \) having a particular period \( d \in D_N \), it is sufficient to count the number of \( k \) such that \( (k, N) = \frac{N}{d} \), i.e., \#\( H_d \), where \( H_d = \{ k \in \mathbb{N} | 0 \leq k \leq N - 1, (k, N) = \frac{N}{d} \} \). Here it is easy to see that the sets \( H_d \) for each \( d \in D_N \) form a partition of \( \{0, 1, \ldots, N-1\} \). And the set \( H_d \) is equivalent to the set \( \{ x \in \mathbb{N} | 0 \leq x \leq d - 1, (x, d) = 1 \} \). As a consequence, we have \#\( H_d = \varphi(d) \) and \( \sum_{d | N} \varphi(d) = N \). From this discussion the following result is trivial.

Theorem 1: The number of sequences in \( \{s_{N,k}(n)\} \) with period exactly equal to \( d \) is \( \varphi(d) \), where \( d | N \).
For every \( s_{N,k}(n) \) there exists a complex conjugate sequence \( s_{N,-k}(n) \) as \( (k,N) = (N-k,N) \). These two sequences form a complex conjugate pair. So, for \( N \geq 3 \), there are \( \varphi(N) \) complex conjugate pair sequences, having period exactly equal to \( N \). If \( N = 1 \) (or 2), the sequence \( s_{N,k}(n) \) itself is a complex conjugate, because \( \varphi(1) = \varphi(2) = 1 \). Given any \( N \in \mathbb{Z}^+ \), the Complex Conjugate Pair Sum (CCPS) is defined as,

\[
c_{N,k}(n) = M(s_{N,k}(n) + s_{N,-k}(n)) = 2M\cos\left(\frac{2\pi kn}{N}\right).
\]

(2)

where \( M = \begin{cases} \frac{1}{2}, & \text{if } N = 1 \text{ or } 2, \\ 1, & \text{if } N \geq 3 \end{cases} \), \( k \in A_N = \{a \in \mathbb{N}|1 \leq a \leq \left\lfloor \frac{N}{2} \right\rfloor, (a,N) = 1\} \) if \( N \geq 3 \) and \( k = 1 \) otherwise. Notice that CCPSs are real, even symmetric \( (c_{N,k}(n) = c_{N,k}(N-n)) \) and periodic with period \( N \). Using the orthogonality between the complex exponentials, one can easily verify the following theorem.

**Theorem 2: Orthogonality:** Any two CCPSs and their circular shifts are orthogonal to each other, i.e.,

\[
\sum_{n=0}^{N-1} c_{N,1}(n-l_1)c_{N,2}(n-l_2) = 2MN^2\cos\left(\frac{2\pi k_1(l_1 - l_2)}{N}\right)\delta(N_1 - N_2)\delta(k_1 - k_2)
\]

(3)

where \( N = \text{lcm}(N_1, N_2) \), \( l_1 \in \mathbb{Z} \) and \( l_2 \in \mathbb{Z} \).

### III. A New Basis For Complex Conjugate Subspaces

For each \( k \in A_N \), we can construct an \( N \times N \) circulant matrix \( D_{N,k} \) as given below:

\[
D_{N,k} = \begin{bmatrix}
  c_{N,k}(0) & c_{N,k}(N-1) & \cdots & c_{N,k}(1) \\
  c_{N,k}(1) & c_{N,k}(0) & \cdots & c_{N,k}(2) \\
  \vdots & \vdots & \ddots & \vdots \\
  c_{N,k}(N-1) & c_{N,k}(N-2) & \cdots & c_{N,k}(0)
\end{bmatrix}.
\]

(4)

Here each column in \( D_{N,k} \) is a circular downshift of the previous column. Using (2), \( D_{N,k} \) can be written as

\[
D_{N,k} = BB^H,
\]

(5)

where

\[
B^H = \begin{bmatrix}
  e^{-\frac{2\pi k_0 i}{N}} & e^{-\frac{2\pi k_1 i}{N}} & \cdots & e^{-\frac{2\pi k_{N-1} i}{N}} \\
  e^{-\frac{2\pi k_1 i}{N}} & e^{-\frac{2\pi k_2 i}{N}} & \cdots & e^{-\frac{2\pi k_{N-1} i}{N}} \\
  \vdots & \vdots & \ddots & \vdots \\
  e^{-\frac{2\pi k_{N-1} i}{N}} & e^{-\frac{2\pi k_{N-2} i}{N}} & \cdots & e^{-\frac{2\pi k_0 i}{N}}
\end{bmatrix}_{2 \times N}.
\]

(6)

This property of decomposing \( D_{N,k} \) into \( BB^H \) is known as factorization. Since \( \text{rank}(D_{N,k}) = \text{rank}(BB^H) = \text{rank}(B) \) \([13]\) and columns of \( B \) are orthogonal \([1]\), the \( \text{rank}(D_{N,k}) \) is always equal to 2, whatever the values of \( N \) and \( k \). Further, it is easy to check that first two columns of \( D_{N,k} \) are linearly independent.

From (3), the column space of \( D_{N,k} \) (denoted as \( v_{N,k} \)) is the same as the column space of \( B \) \([13]\), which consists of periodic signals having a particular frequency \( \frac{2\pi k}{N} \). In \([10]\), S. W. Deng et al., introduced a subspace, spanned by the columns of \( B \) known as Complex Conjugate Subspace (CCS). So, the subspace \( v_{N,k} \) is also termed as CCS. This results in a new real basis for CCS. Any \( x_{N,k}(n) \in v_{N,k} \) can be expressed as

\[
x_{N,k}(n) = \sum_{l=0}^{1} \beta_{lk} c_{N,k}(n-l). \tag{7}
\]

Since \( #A_N = \frac{\varphi(N)}{2} \), we can construct \( \frac{\varphi(N)}{2} \) circulant matrices, corresponding to the total \( \frac{\varphi(N)}{2} \) CCSs. As CCSPs and its circular shifts are mutually orthogonal, whenever \( N_1 = N_2 \) and \( k_1 \neq k_2 \) (refer Theorem 2), it follows that any two CCSs \( v_{N,k_1} \) and \( v_{N,k_2} \) are orthogonal, \( \forall i \neq j \). In summary, for a given \( N \geq 3 \), there are \( \frac{\varphi(N)}{2} \) two-dimensional orthogonal CCSs, each having CCSPs and its one circular downshift as a basis.

### IV. Finite Length Signal Representation

From the previous section, the direct sum of \( \frac{\varphi(N)}{2} \) orthogonal CCSs form a subspace \( s_N \) of dimension \( \varphi(N) \) i.e.,

\[
s_N = v_{N,k_1} \oplus v_{N,k_2} \oplus \ldots \oplus v_{N,k_{\frac{\varphi(N)}{2}}}.
\]

(8)

Since \( c_{N,k}(n) \) is an \( N \) periodic sequence, any \( x(n) \in s_N \) is also an \( N \) periodic sequence and it can be represented as,

\[
x(n) = \sum_{k=1}^{N} x_{N,k}(n) = \sum_{k=1}^{N} \sum_{l=0}^{1} \beta_{lk} c_{N,k}(n-l), \tag{9}
\]

where \( x_{N,k}(n) \in v_{N,k} \) and 0\( \leq n \leq N-1 \). Since \( \sum_{d|N} \varphi(d) = N \), an \( N \) length sequence \( x(n) \) is represented as a linear combination of sequences \( x_d \in s_d \), where \( d \in D_N \). Let \( D_N = \{p_1, p_2, \ldots, p_m\} \), where \( m = \#D_N \), then

\[
x(n) = \sum_{p_1|N} x_{p_1}(n) = \sum_{p_1|N} \sum_{\substack{k=1 \\text{\tiny (k,p_1)=1}}}^{\#D_{p_1}} \beta_{k} c_{p_1,k}(n-l), \tag{10}
\]

where \( 0 \leq l \leq N-1 \). Equation (10) written in matrix form as,

\[
x = T_N^T \beta = [R_{p_1} \ R_{p_2} \ldots \ R_{p_m}]_{N \times N} \beta, \tag{11}
\]

where \( \beta \) is a transform coefficient vector and \( R_{p_1} \) is the basis matrix for the subspace \( s_{p_1} \), which is given below

\[
R_{p_1} = \begin{bmatrix}
  \hat{c}_{p_1,k_1} & \hat{c}_{p_1,k_1}^2 & \cdots & \hat{c}_{p_1,k_1}^{(N-1)} \\
  \hat{c}_{p_1,k_2} & \hat{c}_{p_1,k_2}^2 & \cdots & \hat{c}_{p_1,k_2}^{(N-1)} \\
  \vdots & \vdots & \ddots & \vdots \\
  \hat{c}_{p_1,k_{N-1}} & \hat{c}_{p_1,k_{N-1}}^2 & \cdots & \hat{c}_{p_1,k_{N-1}}^{(N-1)}
\end{bmatrix}_{N \times \varphi(p_1)}, \tag{12}
\]

This property of \( \hat{c}_{p_1,k_1} \) is an \( N \times 1 \) sequence, obtained by repeating \( c_{p_1,k_1} \) periodically \( N \) times and \( c_{p_1,k_{N-1}}^{(N-1)} \) indicates one circular downshift of \( \hat{c}_{p_1,k_1} \). Since \( c_{p_1,k_1} \) is not orthogonal to \( c_{p_1,k_{N-1}} \) (refer Theorem 2), the matrix \( T_N \) is not an orthogonal matrix. Now we show that \( T_N \) is an NPM. For the properties of NPM refer \([6]\) and \([14]\).

**Proposition 1:** For a given \( N \in \mathbb{Z}^+ \), \( T_N \) is an NPM.

**Proof:** For each \( i \in \{1,2,\ldots,m\} \) the columns of \( R_{p_i} \) are periodic with period \( p_i \) and form a basis for \( s_{p_i} \), consequently the rank of \( R_{p_i} \) is \( \varphi(p_i) \). From Theorem 2, it is easy to verify that \( R_{p_i}^T R_{p_j} = 0, \ \forall \ p_i \neq p_j \). Hence the result follows by invoking \( \sum_{p_i|N} \varphi(p_i) = N \).
As mentioned above, $R^T_{p_i}R_{p_j} = 0$, $\forall p_i \neq p_j$, i.e., the column space of $R_{p_i}$ is orthogonal to column space of $R_{p_j}$. Whenever these subspaces are orthogonal, they can be uniquely determined as Ramanujan subspaces (RSs) (refer Theorem 2 in [6]). So the subspace defined in [8] is known as RS. One of the important properties of RS, which is useful in period estimation is:

- Consider a $p$-periodic signal $x(n) = \sum_{k=1}^{m} x_{p_i}(n)$, where the period of $x_{p_i}(n)$ is equal to $p_i$. In general $p|\text{lcm}(p_1, \ldots, p_m)$, but if $x_{p_i}(n) \in s_{p_i}$, then $p = \text{lcm}(p_1, \ldots, p_m)$ [12].

In the above theoretical framework, we considered $N \geq 3$, if $N = 1 \text{ or } 2$, the $\text{rank}(D_{N,k}) = 1$, so, the one independent column is the sequence itself. In this case, the value of $l$ (in equation (7)) corresponding to circular downshift is chosen as 0. An example of representing a 5 length sequence $x(n)$ using (11) and (12), having divisors $(p_i)$ 1 and 5 is given below:

$$x = \begin{bmatrix}
\alpha_{1}(0) \\
\alpha_{1}(1) \\
\alpha_{1}(2) \\
\alpha_{1}(3) \\
\alpha_{1}(4)
\end{bmatrix}, \quad \text{where } \beta = T_N^{-1}x$$

(13)

The RS is spanned by Ramanujan sums and its circular shifts in Ramanujan Periodic Representation (RPR) [2]. Whereas, RS is spanned by CCPSSs and their one circular downshift in the proposed representation. Analogous to RPR the proposed representation is termed as Complex Conjugate Periodic Representation (CCPR). The transformation from $x$ to $\beta$, i.e., $\beta = T_N^{-1}x$ is termed as Complex Conjugate Periodic Transform (CCPT).

V. APPLICATION OF CCPT

A. Period and Frequency Estimation Using CCPT

In general, signals can be either real or complex valued. There are many fields like communication, electromagnetics, acoustics etc., where people work with complex valued signals [15], [16]. To extract the maximum benefit of the proposed concept, two complex valued sequences are considered as examples.

The first example is:

$$y_1(n) = x_{12}(n) + x_{13}(n) + x_{14}(n)$$

$$= e^{j2\pi(\frac{n}{36})} + e^{j2\pi(\frac{n}{36})} + e^{j2\pi(\frac{n}{36})},$$

(14)

where $y_1(n)$ is a 72 length periodic sequence with period 36. It is easy to see that the periods of $x_{12}(n)$, $x_{13}(n)$ and $x_{14}(n)$ are 36, 9 and 36 respectively. That is, $y_1(n)$ is decomposed into signals whose periods known as hidden periods which are less than or equal to the period of $y_1(n)$.

Fig 1 (a)-(b) depicts the absolute values of transform coefficients obtained after applying CCPT and RPT techniques accordingly on $y_1(n)$. Out of 72 transform coefficients, the coefficient indices 13 to 18 belongs to $s_9$ and 37 to 48 belongs to $s_{36}$ respectively. One can observe that the significant transform coefficient strength is present in both $s_9$ and $s_{36}$. This implies the period of $y_1(n)$ is equal to $\text{lcm}(9, 36)$. So, we can find out the period information in a signal by using the proposed transform and RPT.

Apart from period information, the proposed CCPT has an additional advantage over RPT. To understand this, consider the following question: is it possible to separate both $x_{11}(n)$ and $x_{13}(n)$ having same period 36 with different frequencies, from CCPT and RPT coefficients belongs to $s_{36}$?

- In RPT, $s_{36}$ is spanned by Ramanujan sum $c_{36}(n)$ and its circular shifts. Note that, $c_{36}(n)$ is generated by adding all $\varphi(36)$ complex exponentials, having period 36 with different frequencies. Due to this, it is not possible to separate any particular frequency component information from RPT coefficients. This is the reason all $\varphi(36)$ coefficients corresponding to $s_{36}$ are non-zero in RPT.

- Whereas in CCPT, $s_{36}$ is further decomposed as a direct sum of $\varphi(36)$ orthogonal CCSs (refer equation (8)). These are $v_{36,1}$, $v_{36,5}$, $v_{36,7}$, $v_{36,11}$, $v_{36,13}$ and $v_{36,17}$, where each subspace consist of frequency $10Hz$, $50Hz$, $70Hz$, $110Hz$, $130Hz$ and $170Hz$ respectively. Since these subspaces are orthogonal, the information related to each frequency component is separable. As $y_1(n)$ has frequencies $10Hz$ and $50Hz$, the first four coefficients of $s_{36}$, corresponding to CCSs $v_{36,1}$ and $v_{36,5}$ are non-zero.

So, it is possible to get the frequency information of a signal using CCPT, whereas it is not possible using RPT.

The second example is:

$$y_2(n) = x_{21}(n) + x_{22}(n), \quad 0 \leq n \leq 99,$$

(15)

where, periods of $y_2(n)$, $x_{21}(n)$ and $x_{22}(n)$ are 35, 5 and 7 respectively. One period data of $x_{21}(n)$ and $x_{22}(n)$ follow Gaussian distribution with mean zero and variance one.

Now, CCPT coefficients are calculated for $y_2(n)$. For each $p_i|100$, the absolute square sum of the $\varphi(p_i)$ CCPT coefficients corresponding to $s_{p_i}$, will give the strength of periodic component $p_i$ in the signal. Fig. 2 (a) depicts the strength of each $p_i$ present in $y_2(n)$ and it is easy to see that all $p_i$’s are having significant strength, because the period of $y_2(n)$ is not a divisor of the signal length. Whereas period of $y_1(n)$ is a divisor of signal length in the previous example. These results are obvious as we are considering only the divisor subspaces in finite length signal representation. Therefore, we are able to identify the divisor periods of a signal using CCPT.

As a summary, using CCPT, DFT (complex basis) and RPT (integer basis), we are able to find only the divisor period information [2]. TABLE I summarizes the comparison between DFT, RPT and CCPT in different aspects.

B. Non-Divisor Period Estimation

In general, period and hidden periods of a signal are not divisors of signal length ($N$). In literature [2], [6], [14], [17].
there are two approaches used for both DFT and RPT to find out non-divisor periods. Here we applied the same two approaches for CCPT and compared it with DFT and RPT.

**TABLE I: COMPARISON OF DIFFERENT TRANSFORMATION TECHNiques**

| Period | Non-divisor Period | Frequency | Complexity |
|--------|-------------------|-----------|------------|
| DFT    | ✓                 | ✓         | 4N^2 + 1/2 ×-Real multiplications |
| RPT    | ✓                 | ✓         | 2N^2 - Real multiplications      |
| CCPT   | ✓                 | ✓         | 2N^2 - Real multiplications      |

1) First Approach: As proposed in [2], the transform coefficients are computed for the range of lengths \([N_1, N]\), where \(N_1 < N\). In this way the signal is projected in every divisor subspace of \(N_i\), where \(N_1 \leq N_i \leq N\). Then, by comparing the projection strength in each subspace we can find out the hidden periods in a signal.

Now, this method is applied on \(y_2(n)\) by considering the range of \(N_i\) as [70, 100]. TABLE II shows the hidden periods having significant strength in \(y_2(n)\), for few \(N_i\) values. One important observation of the results is that, if any one of the hidden periods is a divisor of \(N_i\), then the significant period strength exists in both \(N_i\) and hidden period. Otherwise, the period strength is distributed on most of the divisors of \(N_i\). From TABLE II it is also evident that along with hidden periods, it gives spurious hidden periods. There are two more drawbacks to this approach along with this limitation.

- There is a huge possibility of divisors overlaps. This leads to the projection strength computation of a particular subspace multiple times.
- Computational Complexity: Using CCPT, DFT and RPT basis requires \(2 \left( N^2 - N_1^2 \right) + 2N_1^2 + N - N_1 \) number of real, complex and real multiplications respectively.

Since \(N_1 \leq N_i \leq N\), the radix-2 FFT algorithm does not help in reducing the computational complexity of DFT. Due to these drawbacks, the following method is suggested to estimate non-divisor periods.

2) Second Approach: In this approach, a signal \(x\) is projected into each and every subspace from \(s_1\) to \(s_{p_{max}}\), where \(p_{max}\) is the maximum possible hidden period exists in \(x\). This leads to define the synthesis dictionary model [14, 17] as,

\[
x = Ab = \begin{bmatrix} R_1, & R_2, & \cdots, & R_{p_{max}} \end{bmatrix}_{N \times \hat{N}} b_{\hat{N} \times 1},
\]

where \(\hat{N} = \sum_{i=1}^{p_{max}} \varphi(i)\), \(b\) is the transform coefficient vector and \(A\) is known as CCPT dictionary (fat matrix). Since \(\hat{N} \gg N\), there exist multiple solutions for \(b\). Now, considering the computational time as the main criteria, the non-divisor period estimation using dictionary is formulated as a data fitting problem. To get the best fit of the given signal with the signals having smaller periods, an optimization problem is formulated as follows [14, 17]:

\[
\min ||Db||^2 \text{ s.t. } x = Ab.
\]

If \(p_i\) is the period of \(i^{th}\) column in \(A\) and \(f(p_i) = p_i^2\), then the diagonal matrix \(D\) (known as penalty matrix) consist of \(f(p_i)\) as \(i^{th}\) diagonal entry. Now finding the optimal solution \((\hat{b})\) to the above problem leads to the following closed form expression:

\[
\hat{b} = D^{-2} A^T (AD^{-2} A^T)^{-1} x.
\]

The \(\varphi(p_i)\) coefficients in \(\hat{b}\), gives the strength of period \(p_i\) in the signal. Fig. 2(b) shows the hidden periods present in \(y_2(n)\) by using the synthesis CCPT dictionary model with \(p_{max} = 80\). These are 1, 5, 7 and period of the signal is \(lcm(1, 5, 7)\). Refer [6, 14] and [17] for the detailed theory of dictionary based method and for the result of synthesis dictionary model using DFT (Farey) and RPT dictionaries.

So, the proposed CCPT dictionary can be used to estimate non-divisor period and hidden periods of a signal. Moreover, if the given signal consists a particular frequency, then we can estimate it from \(\hat{b}\), computed by using CCPT and Farey dictionaries, but it is not possible by using RPT dictionary.

**TABLE III: COMPARISON OF DIFFERENT DICTIONARY TECHNIQUES**

| Period | Non-divisor Period | Frequency | Complexity |
|--------|-------------------|-----------|------------|
| Farey  | ✓                 | ✓         | 2L - Real multiplications |
| RPT    | ✓                 | ✓         | L - Real multiplications |
| CCPT   | ✓                 | ✓         | L - Real multiplications |

Since \(N_1 \leq N_i \leq N\), the maximum possible hidden period exists in \(x\). This leads to define the synthesis dictionary model [14, 17] as, the given signal is real valued, then the complexity of both CCPT and DFT is almost same due to the conjugate symmetry property of DFT basis/dictionary coefficients.

**VI. CONCLUSION**

In this work, the problem of a finite length signal representation is solved by using the signals belongs to CCSs. Later, we have shown that the proposed transformation (CCPT) allows us to get both divisor and non-divisor period information of a signal. We compared the proposed method with DFT and RPT for a given complex (or) real valued input sequence. Although this letter introduces CCPT, there are many other properties that require a detailed analysis. We will aspire to work on this in the near future.

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