A Review of the Real Space Renormalization Group Analysis of the Two-Dimensional Coulomb Gas, the Kosterlitz-Thouless-Berezinskii Transition, and Extensions to a Layered Vortex Gas

Stephen W. Pierson

Department of Physics, Worcester Polytechnic Institute, Worcester, MA 01609-2280

(March 24, 2022)

Fundamental properties of the Kosterlitz-Thouless-Berezinskii (KTB) transition which occurs in systems in the universality class of the two-dimensional X-Y model are reviewed here with an emphasis on the real-space renormalization group analysis used to derive the recursion relations. A derivation of the unique temperature dependence of the correlation length of this system will be presented and the signatures of the KTB transition in electronic transport measurements will be explained. Extensions of this model and technique to layered superconductors will also be presented. The size of the 3D critical region, the nature of the 3D crossover, and the effect of a finite electric current will be discussed.

I. INTRODUCTION

The phase transition of the systems in the universality class of the two-dimensional (2D) X-Y model, known as the Kosterlitz-Thouless-Berezinskii (or some permutation of this) transition (Berezinskii 1971; Kosterlitz and Thouless 1973; Kosterlitz 1974), is a fascinating one. And, even though the basic details of this transition were worked out in the early 1970’s, interest in this transition remains keen with more than 200 articles published yearly on the topic. Indeed, one can find references to the “KTB” transition in the literature on topics ranging from layered materials (such as the high-temperature superconductors (HTSC’s)) to superfluid films. Given its wide-spread importance, an understanding of the Kosterlitz-Thouless-Berezinskii transition and the derivation of its critical behavior properties is essential for a student of condensed matter and statistical physics. Renormalization group (RG) techniques (Wilson 1971) are also an important tool for a theoretical physicist and can be used for a variety of problems in a number of ways. [For example, see the article by N. Andrei in this same issue.] It was originally applied to the 2D X-Y model by Kosterlitz (1974) to study its critical behavior. In this paper, we will review his real-space RG analysis of the 2D X-Y model, describe the quintessential properties of the critical behavior, and discuss recent results on layered superconductors derived using the same mathematical formulation.

II. THE 2D X-Y MODEL AND THE REAL SPACE RENORMALIZATION GROUP ANALYSIS

Falling into the universality class of the 2D X-Y model are 2D superconductors and superfluids as well as 2D two-component magnetic systems. More generally, one can say that this universality class covers 2D systems with a two-component order parameter. Systems in this universality class are believed to undergo the Kosterlitz-Thouless-Berezinskii phase transition. In this section, we will show how the 2D X-Y model can be mapped to the 2D Coulomb gas and then review Kosterlitz’s RG study of that system which illuminates many of the principal features of the critical behavior. (For pedagogical arguments of the phase transition see Kosterlitz et al., 1973 or Halperin (1979).) The RG study culminates in the 2D recursion relations which can be used to derive the temperature dependence of the correlation length. In the last subsection we will describe how KTB behavior can be detected in superconducting films via electronic transport measurements.

One might guess that the physics of the 2D X-Y model is uninteresting since it has been shown that there is no long-range order at finite temperature in magnetic systems (Mermin and Wagner 1966) nor in superfluids (Hohenberg 1967). Yet hints of novel behavior were provided by Stanley and Kaplan (Stanley and Kaplan 1966) who found a divergence of the susceptibility based on a high-temperature series expansion. Wegner (Wegner 1967) went on to show that the susceptibility was infinite at low temperatures. Further suggestions of a phase transition were provided by Berezinskii (Berezinskii 1971) who found a power-law decay of the spatial correlation function at low temperatures while others had ascertained the exponential decay at high temperatures.

It was Kosterlitz and Thouless (Kosterlitz and Thouless 1973) who formulated the transition explicitly in terms of vortices and, to be more precise, as a vortex/anti-vortex pair unbinding transition. In this zero-field transition, vortices are created spontaneously in pairs at finite temperatures and have an energy which goes as the logarithm of their separation. As the temperature increases, the vortex pairs increase in size and in density. As this happens, screening
of their interactions becomes important and enables a phase transition at a temperature \( T_{KTB} \). At this temperature two things happen: vortex pairs start to unbind and spontaneous creation of single free vortices becomes possible. Being driven by topological excitations is one of the properties of this transition that makes it so unique. Another is its correlation length which diverges according to an exponential rather than a power-law at \( T_{KTB} \). As we will explain, it is this latter property that makes detection of this phase transition difficult to detect by thermodynamic measurements because there is no singularity in the free energy. Evidence for the transition can be accumulated however through electronic transport measurements for the case of superconductors.

A. Mapping the 2D X-Y Model to the 2D Coulomb Gas

The 2D X-Y model is defined by the Hamiltonian:

\[
H = -J \sum_{<i,j>} \mathbf{s}_i \cdot \mathbf{s}_j = -JS^2 \sum_{<i,j>} \cos(\phi_i - \phi_j),
\]

where \(<i,j>\) are all nearest-neighbor spins and \(\phi_i\) is the angle that the spin vector \(\mathbf{s}_i\) makes with an arbitrary axis. In the low temperature and continuous limit, one has

\[
H = E_0 + \frac{JS^2}{2} \int d^2 r |\nabla \phi|^2,
\]

where \(E_0\) is a constant. One can immediately see from Eq. (2) the similarity to a Ginzburg-Landau free energy functional in which the amplitude variations are neglected. We now proceed to express Eq. (2) in terms of the energy for a 2D Coulomb gas. This is accomplished (Halperin 1979) by first splitting the angle \(\phi\) into two parts, \(\phi = \phi_v + \phi_{sw}\), corresponding to a vortex part and a spin wave part where

\[
\nabla \times \nabla \phi_{sw} = 0,
\]

\[
\nabla \cdot \nabla \phi_v = 0.
\]

This substitution leads to three terms in Eq. (2) including a cross term \(\int d^2 r \nabla \phi_{sw} \cdot \nabla \phi_v\) which can be shown to be equal to zero after an integration by parts leaving the vortex and spin wave contributions independent of one another:

\[
H = E_0 + \frac{JS^2}{2} \int d^2 r |\nabla \phi_v|^2 + \frac{JS^2}{2} \int d^2 r |\nabla \phi_{sw}|^2.
\]

It is the vortex term that interests us because, as we will show, it contains the part that drives the phase transition.

The next step in getting to the 2D Coulomb gas is to assume that vortices are present in the system which means that the line integral of the change in the field \(\phi_v\) around any contour \(c\) is \(2\pi N\) where \(N\) is any integer:

\[
\oint_c \nabla \phi_v \cdot d\mathbf{r} = 2\pi N.
\]

The simplest, non-trivial case is when the contour encloses exactly one vortex of vorticity one in which case \(N = 1\). The right hand side (RHS) of Eq. (6) can be written in terms of a vortex density \(\sigma(r)\) and the left hand side (LHS) can be rewritten using Stoke’s theorem yielding

\[
\hat{\mathbf{z}} \cdot (\nabla \times \nabla \phi_v) = 2\pi \sigma(r).
\]

By using vector identities and defining \(\nabla \phi'_v = \nabla \times \nabla \phi_v\), Eq. (8) becomes Poisson’s equation:

\[
\nabla^2 \phi'_v = 2\pi \sigma(r).
\]

Using a Green’s function method to solve Eq. (8), one arrives at

\[
\phi'_v(r_1) = \int d^2 r_2 \sigma(r_2) \ln |r_1 - r_2|/\xi_0
\]

where \(\xi_0\) is an infrared cutoff associated with the size of the core of the vortex. Finally, since \(\nabla \phi_v \cdot \nabla \phi_v = \nabla \phi'_v \cdot \nabla \phi'_v\) and \(\nabla \cdot (\phi'_v \nabla \phi'_v) = \nabla \phi'_v \cdot \nabla \phi'_v + \phi'_v \nabla^2 \phi'_v\), one gets the energy for a configuration of vortices to be
\[ H = E_0 + \frac{J s^2}{2} \int \! d^2 r |\nabla \varphi(r)|^2 + 2\pi \frac{J s^2}{2} \int \! d^2 r \int \! d^2 r' \sigma(r)\sigma(r') \ln |r - r'|/\xi_0 \]

\[ + 2\pi \frac{J s^2}{2} \int \! d^2 r \sigma(r) \int \! d^2 r' \sigma(r') \ln(\xi L/\xi_0), \]  

(10)

where \( L \) is the size of the system. The first terms in Eq. (10) contain the spin wave terms and the energy associated with the vortex core energies. The third term is an interaction term between the vortices and the last term which diverges as the size of the system ensures system neutrality.

We can now neglect the spin-wave term, assume vortex neutrality, and write the partition function for the 2D Coulomb gas,

\[
Z = \sum_N y^{2N} \frac{1}{(N!)^2} \int_{D_1} d^2 r_1 \int_{D_2} d^2 r_2 \cdots \int_{D_{2N}} d^2 r_{2N} \times \exp \left[ -\frac{\beta}{2} \sum_{i \neq j} p_i p_j V(|r_i - r_j|) \right]
\]

(11)

where \( 2N \) is the total number of particles, \( N \) of which have a positive (negative) charge \( p_i = +p \) (\( p_i = -p \)), and \( r_i \) are the coordinates of the \( i \)th charge. \( \beta^{-1} = k_B T \) where \( T \) is the temperature and \( k_B \) is the Boltzmann constant. \( y = \exp(\beta \mu)/\tau^2 \) is the fugacity, where \( \mu = -E_c \) and \( E_c \) is the “core energy,” that is, the energy needed to create a normal core in a superconducting fluid (in our case). \( V(R) = -\ln(R/\tau) \) is expressed in the units \( p^2 \). Note that the notation \( r_{ij} = r_i - r_j \) will be used. The integrals are over an area \( D_i \) which is all of the area except for disks \( d_i(j) \) of radius \( \tau \) around the charges \( j < i \). The area \( D_i \) is written, where \( A \) is the total area,

\[
D_i = A - \sum_{j<i} d_i(j).
\]

(12)

B. RG Study of the Neutral 2D Coulomb Gas

The renormalization group technique (Wilson 1971) is useful for studying second order phase transitions which are scale invariant at the transition temperature. RG incorporates the essential idea of scaling and extends it. Scaling says that near a second order phase transition, one length scale, the correlation length, dominates the critical behavior and all other length scales are thrown out. RG does not eliminate the short length scales altogether as scaling does, but incorporates their contribution into the renormalization of the system parameters. There are two steps in an RG analysis: the first is to coarse-grain the system which amounts to taking out the small-scale structure and incorporating it into a renormalization of the parameters of the system. This can be done both in real space and in momentum space. (Wilson and Kogut 1974). The second step is to rescale the system so that the infrared cutoff is restored to its original value. The connection to the physics of the system is achieved by studying how the parameters of the system change as one goes through the RG iterations. The relationship between the old values and the “renormalized” values are called recursion relations. For a thorough description of the renormalization group, the reader should consult any of the many textbooks on the subject (e.g. Binney, Dowrick, Fisher, and Newman 1995; or Ma 1976). Below we sketch the details of the real space renormalization group study of the 2D Coulomb gas (Kosterlitz 1974). A manuscript containing the complete details of this RG analysis and the derivation of the recursion relations is available from the author. The reader is also encouraged to see any of the excellent reviews of the KTB transition (Halperin 1979; Minnhagen 1987; Suzuki 1979).

The first step in the RG calculation is to integrate out small scale structure. In our case, this amounts to increasing the minimum separation between the vortices \( \tau \) to \( \tau + d\tau \) and determining the effect of the vortex pairs with a separation in that range on the interactions of other vortices. This is realized by first increasing slightly the allowed size of the disks around each vortex \( D_i \) in Eq. (12) from \( \tau \) to \( \tau + d\tau \):

\[
\int_{D_i} d^2 r_i = \int_{D_i'} d^2 r_i + \sum_{n<i} \int_{s_n(i)} d^2 r_i.
\]

(13)

When substituted into the integrals of Eq. (11), one gets,
\[ \int_{D_1} d^2r_1 \cdots \int_{D_{2N}} d^2r_{2N} = \int_{D_1'} d^2r_1 \cdots \int_{D_{2N}} d^2r_{2N} + \frac{1}{2} \sum_{i \neq j} \int_{D_1'} d^2r_1 \cdots \int_{D_{2N}} d^2r_{i-1} \]
\[ \times \int_{D_{i+1}'} d^2r_{i+1} \cdots \int_{D_{j-1}'} d^2r_{j-1} \cdots \int_{D_{i+1}' \cdots} d^2r_{2N} \int_{D_{j}'} d^2r_j \int_{\delta_{(j)}} d^2r_i + O(d^2). \] (14)

It is the last two integrals of the second term of the right hand side of Eq. (14) which concern us. By doing these integrals, the effect of the small pairs is integrated out. Physically, the integral over \( \delta_{(j)} \) puts the charge \( i \) in an annulus around charge \( j \) to form a pair of smallest separation. The integral over \( D_j' \) then move this pair through all possible positions in the system. To do the integrals over \( r_i \) and \( r_j \), all of the terms in the partition function that include \( i \) and \( j \) are isolated:

\[ \int_{D_j'} d^2r_j \int_{\delta_{(j)}} d^2r_i \exp \left[ -\beta \sum_k p_j p_k V(r_{jk}) - \beta \sum_k p_i p_k V(r_{ik}) \right]. \] (15)

One can then assume that only vortices of opposite charge can form pairs so that \( p_i = -p_j \),

\[ \int_{D_j'} d^2r_j \int_{\delta_{(j)}} d^2r_i \exp \left[ -\beta p_j \sum_k p_k \left[ -\ln(r_{jk}/\tau) + \ln(r_{ik}/\tau) \right] \right]. \] (16)

Integrating over \( r_i \), one gets \( r_i = r_j + \vec{r} \) and, \[
\tau \int_{D_j'} d^2r_j \int_0^{2\pi} d\theta \exp \left[ -\beta p_j \sum_k p_k \frac{1}{2} \ln \left( \frac{r_{jk} + 2r_{jk} \cdot \vec{r} + \vec{r}^2}{r_{jk}^2} \right) \right]. \] (17)

One then makes the approximation that the density of vortices is small and consequently that \( r_{jk} \) is large meaning that the probability that another vortex is close to the pair \( i-j \) is small. An expansion in \( \tau/r_{jk} \) can then be performed,

\[ 2\pi \tau d\tau \int_{D_j'} d^2r_j \left( 1 + \int_0^{2\pi} d\theta \left( \frac{\beta p r_{jk}}{2} \right)^2 \sum_{k \neq l} p_k p_l \left[ \frac{\tau r_{jk} \cos \theta \tau r_{jl} \cos(\phi - \theta)}{r_{jk}^2} - \frac{\tau^2 \cos^2 \theta}{r_{jk}^2} \right] \right). \] (18)

After the straightforward integration over \( \theta \), one has,

\[ 2\pi \tau d\tau \int_{D_j'} d^2r_j \left( 1 + \frac{(\beta pr_{jk})^2}{4} \sum_{k \neq l} p_k p_l \left[ \frac{r_{jk} \cdot r_{jl}}{r_{jk}^2 r_{jl}^2} - \frac{1}{r_{jk}^2} \right] \right). \] (19)

The integration over \( r_j \) is also straightforward and one finds,

\[ 2\pi \tau d\tau \left[ A - 2\pi \tau^2 \beta^2 \frac{p^2}{4} \sum_{k \neq l} p_k p_l \ln \frac{r_{kl}}{\tau} \right]. \] (20)

We can now write the second term of the RHS of Eq. (14): \[ \sum_{N=1}^{\infty} \frac{y^{2N}}{(N!)^2} \frac{1}{2} \sum_{i \neq j} \int_{D_1'} d^2r_1 \cdots \int_{D_{2N}} d^2r_{i-1} \cdots \int_{D_{2N}} d^2r_{i+1} \cdots \int_{D_{2N}} d^2r_j \cdots \int_{D_{2N}} d^2r_{2N} \]
\[ \times \int_{D_{2N}} d^2r_{2N} \exp \left[ \frac{\beta}{2} \sum_{k \neq i \neq j} p_k p_l \ln \frac{r_{kl}}{\tau} \right] \left( 2\pi \tau d\tau \left[ A - 2\pi \tau^2 \beta^2 \frac{p^2}{4} \sum_{k \neq l} p_k p_l \ln \frac{r_{kl}}{\tau} \right] \right). \] (21)

After a change of variables in Eq. (21) \((N' = N - 1)\), this equation can be added to the first term of the right hand side of Eq. (14):

\[ \sum_{N} \frac{y^{2N}}{(N!)^2} \int_{D_1'} d^2r_1 \cdots \int_{D_{2N}} d^2r_{2N} \]
\[ \left[ 1 + 2\pi y^2 \tau d\tau \left( A - 2\pi \tau^2 \beta^2 \frac{p^2}{4} \sum_{k \neq l} p_k p_l \ln \frac{r_{kl}}{\tau} \right) \right] \exp \left[ -\frac{\beta}{2} \sum_{i \neq j} p_i p_j V(r_{ij}) \right]. \] (22)
Exponentiating the terms in the square bracket,

\[
Z = \exp[2\pi y^2 \tau A] \sum_N y^{2N} \frac{1}{(N!)^2} \int_{D'_1} d^2 r_1 ... \\
\times \int_{D'_{2N}} d^2 r_{2N} \exp \left\{ -\frac{\beta}{2} \sum_{i \neq j} p_i p_j \left[ -\left( 1 - (2\pi y^2)^2 \frac{\beta p^2}{2} \frac{d\tau}{\tau} \right) \ln \frac{r_{ij}}{\tau} \right] \right\}. \quad (23)
\]

Finally, we can rescale the lengths in Eq. (23) so that the limits in the integral are the same as those of the original partition function using \( r' = r/(1 + d\tau/\tau) \),

\[
Z = \exp[2\pi y^2 \tau A] \sum_N \left( y \left[ 1 + d\tau/\tau(2 - \beta p^2/2) \right] \right)^{2N} \frac{1}{(N!)^2} \\
\int_{D'_1} d^2 r_1 ... \int_{D'_{2N}} d^2 r_{2N} \exp \left\{ -\frac{\beta}{2} \sum_{i \neq j} p_i p_j \left[ -\left( 1 - (2\pi y^2)^2 \frac{\beta p^2}{2} \frac{d\tau}{\tau} \right) \ln \frac{r_{ij}}{\tau} \right] \right\}. \quad (24)
\]

One can see that this expression has the same form as the original except that the parameters have been renormalized:

\[
(\beta p^2)' = \beta p^2 (1 - (2\pi y^2)^2 \beta p^2 d\tau/2\tau),
\]

\[
y' = y(1 + (2 - \beta p^2/2)d\tau/\tau) \quad (25)
\]

These equations can be put in differential form,

\[
\frac{d(\beta p^2)}{(\beta p^2)^2} = -\frac{d(\beta p^2)^{-1}}{2\tau} = -\frac{(2\pi y^2)^2 d\tau}{2\tau},
\]

\[
dy = \frac{y\beta p^2}{2} \left( \frac{4}{\beta p^2} - 1 \right) d\tau/\tau. \quad (27)
\]

and the definitions \( \epsilon = \ln(\tau/\xi_0), y' = 2\pi y^2, \) and \( K = \beta p^2 \) can be made yielding,

\[
dK/d\epsilon = -y^2 K^2/2, \quad (29)
\]

\[
dy/d\epsilon = y(4 - K)/2, \quad (30)
\]

where we have dropped the primes. These equations are frequently written in terms of \( x = 4/K - 1 \):

\[
dx/d\epsilon = 2y^2, \quad (31)
\]

\[
dy/d\epsilon = 2yx, \quad (32)
\]

where we have linearized the equations in \( x \). These equations can be integrated to find \( x(\epsilon) \) (or \( K(\epsilon) \)) and \( y(\epsilon) \).

C. Analysis of the Recursion Relations

The recursion relations Eqs. (31)-(32) have a fixed line at \( y = 0 \) along which the parameters of the system do not change. But it is the fixed point \( [x = 0, y = 0] \) that is of more interest to us as we shall see. How these parameters change as one goes through the RG iterations are represented as “flows” in phase space. By inspecting Eq. (31), one sees that \( x \) can only increase under a RG iteration but that the increments can become infinitesimally small as \( y \) goes to zero. Eq. (32) on the other hand tells us that \( y \) increases when \( x \) is positive and decreases when \( x \) is negative. This behavior is reflected in Fig. 1a where the RG flows are plotted in \( x \)-\( y \) space for various initial values of \( x \) and \( y \). One can see that the lines end on the \( x \) axis, tend toward \( [x = \infty, y = \infty] \), or, in one case, flow to \( [x = 0, y = 0] \). This reflects the tendency of the RG iterations to drive the system to its “pure” limits: the low-temperature limit of no vortices \( (y = 0) \) or the high-temperature limit of a high vortex density. The last case where the flows go to
the fixed point represents a sepathrix or, in RG parlance, a critical surface. This surface is associated with the critical temperature (where the system is scale-invariant) since it does not flow to either the low or high temperature limits. Plotted in Fig. 1b are the RG flows for Eqs. (29)-(30) with initial values of $K$ and $y$ corresponding to those used in Fig. 1a. The same behavior is reflected.

The first integral of Eqs. (31)-(32) is

$$y^2 - x^2 = c,$$

where $c$ is a constant. Eq. (33) is the equation for a hyperbola and the flows of Fig. 1 represent a family of hyperbolae corresponding to different values of $c$. Based on our discussion in the above paragraph, it is natural to make the assumption,

$$c \propto (T - T_{KTB})/T_{KTB}.$$  

For $c > 0$, the system is at a temperature above the critical one and the flows go to the high-temperature limit. For $c = 0$, $T = T_{KTB}$ and the RG flow goes to the origin. For $c < 0$, one is in the low-temperature limit.

With Eq. (33) and Eq. (34), one can derive the temperature dependence for the vortex correlation length. Substituting Eq. (33) into Eq. (31), one has $dx/dc = 2(x^2 + c)$. Integrating, one gets

$$\frac{1}{\sqrt{c}} \arctan \left( \frac{x}{\sqrt{c}} \right) = 2 \epsilon,$$

where $x_i$ is the bare or initial value of $x$ and $x_{\text{max}}$ is the final value. To be more precise, one integrates the recursion relations until there are no longer valid which is when the vortex density becomes large (i.e., when $y$ becomes of $O(1)$).

The value of $\epsilon$ at that point is $\epsilon_{\text{max}}$. Because the correlation length $\xi$ is dominating the critical behavior and because the only length scale in Eq. (35) is $\tau = \xi_0 \exp[\epsilon_{\text{max}}]$, one must make the association $\xi = \tau$. Then, for small $c$, one finds

$$\xi(T) \propto \exp[\sqrt{b/(T - T_{KTB})}].$$

This result is valid for $T > T_{KTB}$ and for $T < T_{KTB}$ one can show that the correlation length is infinite (Kosterlitz 1974), which agrees with the result mentioned earlier that the susceptibility is infinite at small temperatures. This unique temperature dependence of the correlation length is a distinguishing property of the KTB transition. Typically, one has a power law divergence of the correlation length. The correlation length can be thought of as a screening length for the Coulombic interaction. For separations less than it, the interaction is logarithmic but approaches a one has a power law divergence of the correlation length. The correlation length can be thought of as a screening length for the Coulombic interaction.

Further insights into the KTB transition can be gained by looking at the length scale dependence of $K$ and $y$ as derived from Eqs. (29)-(30). In Fig. 1b we have plotted $K(\epsilon)$ and $y(\epsilon)$ for the same initial values of $K$ and $y$ as those used in Fig. 1a. From the length scale dependence of $y(\epsilon)$, one can see that there are two types of behavior as in Fig. 1b, one corresponding to $T > T_{KTB}$ and the other to $T < T_{KTB}$. In the latter, $y(\epsilon)$ falls monotonically to zero meaning that the density of vortices with a separation $R = \xi_0 \ln \epsilon$ becomes negligible at large separations. Above $T_{KTB}$, this quantity initially decreases meaning again that the density of vortices falls with increasing separation, but then starts to increase. This increase marks the onset of unbinding of vortex pairs at the larger separations and the spontaneous creation of free vortices. This behavior is reflected in $K(\epsilon)$ depicted in Fig. 2b. For temperatures less than $T_{KTB}$, $K$ decreases due to the screening of vortex pairs but saturates at a finite value. The length scale at which it saturates corresponds to the length beyond which vortex pairs with the corresponding separation are negligible. The effective interaction is proportional to $K(R) \ln R/\xi_0$ which remains largely logarithmic. This is in contrast to the $T > T_{KTB}$ case where one can see that $K$ does not saturate but decreases to zero. This is because vortices are present at all length scales. In fact, it is not just $K$ that changes but also the logarithmic dependence of the interaction energy due to free vortices. For further information, see the discussion by Minnhagen (1987).

### D. Observing the KTB transition in superconducting films

An intriguing consequence of the KTB temperature dependence is that there is no singularity in the free energy of this system. Using a scaling analysis, one finds that the singular part of the free energy density scales as

$$f \propto \xi^{-2}$$

(37)
which is analytic as $T \rightarrow T_{KTB}$. The specific heat and the magnetization, both thermodynamic derivatives of the free energy, are also analytic at the critical temperature leaving detection of this transition difficult.

In superconducting films, it is primarily through electronic transport measurements that KTB critical behavior is verified. Here we will briefly sketch the arguments leading to the current-voltage characteristics and the temperature dependence of the resistance $R(T)$. In the presence of a current, a force (whose magnitude is linear with current and whose sign is opposite for vortices and anti-vortices,) is exerted on a vortex in a direction perpendicular to the current. Free vortices can therefore dissipate energy. The resistance in the limit of zero current should be proportional to the density of free vortices which in turn is inversely proportional to the square of the correlation length:

$$R(T) \propto n_F \propto \xi^{-2}.$$  \hspace{1cm} (38)

Minnhagen (1987) has generalized this formula to include the effects of the underlying superfluid to obtain $R(T) \propto \exp\left(\sqrt{\frac{b}{T_{c0} - T}}/(T - T_{KTB})\right)$ where $T_{c0}$ is the mean-field transition temperature.

The effect of a current on a pair is to pull them apart with a force which is independent of their separation. Thus, the bare energy of a vortex pair in the presence of a current is

$$E(R) = 2E_c + p^2 \log(R/\xi_0) - IR/\xi_0$$  \hspace{1cm} (39)

where we have expressed the current in units of energy. At large $R$, the current repulsion will always dominate and $E(R)$ will have a peak at $R_c = \xi_0 p^2/I$. Thermal energy will activate hops over this barrier providing a mechanism for vortex pair unbinding with a rate (Huberman, Myerson, and Doniach 1978)

$$\Gamma \propto \exp[-E_B/k_B T],$$  \hspace{1cm} (40)

where $E_B = 2E_c - p^2 + p^2 \log(p^2/I)$ is the height of the barrier. The density of free vortices in this case can be found using the kinetic equation,

$$dn_F/dt = \Gamma - n_F^2,$$  \hspace{1cm} (41)

where the second term on the right hand side account for vortex/anti-vortex pairing or annihilation. In the steady state, the RHS is zero so that $n_F = \Gamma^{1/2}$. This results in a non-linear $I$-$V$ relation:

$$V \propto I^{\alpha(T)+1},$$  \hspace{1cm} (42)

where, neglecting renormalization effects, $\alpha(T) = p^2/2k_B T$. In the limit of weak currents, $\alpha(T)$ is decreasing linearly to a value, $\alpha(T_{KTB}) = 2$, as one increases the temperature to $T_{KTB}$ at which point it jumps discontinuously to zero.

The $I$-$V$ signatures and the temperature dependence of $R(T)$ that characterize KTB behavior, Eqs. (38) and (42), have been observed in superconducting films (Minnhagen 1987) providing strong evidence for the influence of vortex pair unbinding in these systems. They have also been observed in layered superconductors with a generalized form of Eq. (42) (Jensen and Minnhagen 1991) indicating that vortices also play a major role there. We now proceed to discuss the layered system.

### III. CRITICAL BEHAVIOR OF VORTICES IN LAYERED SYSTEMS

In this Section, the model and techniques of Section II are generalized to a weakly coupled layered system. The various properties of the critical behavior of vortices in layered systems deduced using the methods reviewed above will be emphasized but it should be noted that many authors (Chattopadhyay and Shenoy 1994; Fischer 1993; Friesen 1995, Horovitz 1991; Korshunov 1990) have made significant contributions to the field using other approaches. We will begin with the expected behavior of a layered system and the basic model used for such systems and discuss how the bare interactions of the vortices are modified in a layered system. The recursion relations for the interacting layered vortex gas derived from a RG analysis will then be given and an analysis of these recursion relations will be used to study the three-dimensional to two-dimensional crossover. Finally, the effect of a uniform electric current $I$ on the system and the $I$-$T$ phase diagram will be considered.

An intuitive understanding of the conventional wisdom (Leggett 1989) of the critical behavior of vortices in layered systems can be achieved by considering the correlation length which diverges at the critical temperature $T_c$. When the correlation length is smaller than the distance $d$ separating the layers, 2D behavior of the vortices is expected. When the correlation length becomes larger than the interlayer separation, the behavior should be 3D. Since the latter condition is met as the correlation length diverges near the transition, 3D behavior is expected in a small
interactions have been derived with the intralayer interaction \( V \). The interaction is to strengthen the intralayer vortex interaction and to introduce interlayer vortex interactions. These (Lawrence and Doniach, 1971) in which the coupling between the layers is taken to be the same as that between two superconductors separated by an insulator: Josephson coupling. The effect of the Josephson coupling on the vortex interactions is to strengthen the intralayer vortex interaction and to introduce interlayer vortex interactions. These interactions have been derived with the intralayer interaction \( V(R,0) \) (Cataudella and Minnhagen 1990) being
\[
V(R,0) = \left\{ \begin{array}{ll}
-\ln(R/\tau) + (\lambda R^2/4\tau^2) \ln(\lambda R^2/\tau^2), & \tau \ll R \ll R_{\lambda} \\
-(\pi\sqrt{\lambda R/\tau^{3/2}}), & R \gg R_{\lambda}
\end{array} \right.
\]
and the interlayer interaction \( V(R,1) \) (Bulaevski, Meshkov, and Feinberg, 1991) being
\[
V(R,1) \propto \left\{ \begin{array}{ll}
-\lambda R^2/\tau^2 \ln(\lambda R^2/\tau^2), & R \ll R_{\lambda} \\
\sqrt{\lambda R/\tau}, & R \gg R_{\lambda}
\end{array} \right.
\]
where \( R_{\lambda} = \xi_0/\sqrt{\lambda} \). For generality, one can introduce the effect of the current through the term,
\[
H = \sum_i p_i \mathbf{r}_i \cdot \mathbf{J},
\]
where \( \mathbf{r}_i \) is the in-plane Cartesian coordinate of the \( i \)th particle, \( p_i = \pm \sqrt{\tau/2}\Phi_0/2\pi\Lambda \) is its charge, and \( J \) is related to the current \( I \) by \( J = \Phi_0 dI/pec \) where \( d \) is the distance between the layers, \( \Phi_0 \) is the superconducting flux quantum, \( p \) is the characteristic cross-sectional area that relates the current to the current density \( I/a \), \( \Lambda \) is the London penetration depth, and \( c \) is the speed of light.

The partition function for the layered vortex gas (Pierson 1994; Pierson 1995b) in the presence of a current is,
\[
Z = \sum_N y^{2N} \frac{1}{(N!)^2} \int_{D_1} d^2 r_1 \int_{D_2} d^2 r_2 \ldots \int_{D_{2N}} d^2 r_{2N} \exp \left[ -\beta \sum_{i \neq j} p_i p_j V(\mathbf{r}_i - \mathbf{r}_j, l_i - l_j) + \beta \sum_i p_i \mathbf{r}_i \cdot \mathbf{J} \right],
\]
where \( l_i \) is the index of the layer in which the \( i \)th particle is located. Interactions between vortices separated by more than one layer are neglected. A renormalization group analysis has been carried out on this partition function and the following recursion relations were found, (Pierson 1995a; Pierson 1995b)
\[
dx/d\epsilon = 2y^2[1 - \lambda/16 + J^2/(1 + x)],
\]
\[
dy/d\epsilon = 2[x + (1/2)\lambda \ln \lambda y/(1 + x)] + Jy/\sqrt{1 + x},
\]
\[
dl/\epsilon = 2\lambda \{1 - 4y^2(1 + J^2/(1 + x))/(1 + x)\},
\]
\[
dJ/\epsilon = 0
\]
where a factor of \( \sqrt{\lambda} \) is absorbed into \( J \). Note that these recursion relations reduce to Eqs. (11)-(12) for \( J = 0 \) and \( \lambda = 0 \). One can see that in Eqs. (45)-(46) the current tends to counteract the effect of the Josephson coupling. This is expected because the current introduces a repulsion between vortices of opposite sign while the Josephson interaction strengthens the interaction.

Considering first the zero current recursion relations, the critical behavior of vortices in a layered system can be addressed. This is done by examining the correlation length which is defined in the same way as in Section 11C, \( \xi(T) = \xi_0 \exp[\epsilon_{max}] \). The correlation length will diverge like the 2D dependence sufficiently far above the transition temperature \( T_c \) but will cross over to a power-law temperature dependence closer to the transition temperature. \( T_c \)
is easily derived from this quantity since it peaks at this temperature. By varying $\lambda$ one can determine the functional dependence of $T_c$ on the interlayer coupling to find (Pierson 1995a)

$$T_c \propto 1/(\ln(\lambda))^2$$

(49)

in agreement with the results of Hikami and Tsuneto (1980). By comparing the correlation length to the 2D correlation length, insights into the size of the 3D critical region can be found. One will find that sufficiently far from $T_c$ the two correlation lengths are nearly identical and can be compared to the layered system behavior two-dimensionally in that region. Closer to $T_c$ there is a slow bifurcation which is taken to indicate crossover to the 3D region and the size of this region can be studied as a function of $\lambda$. It is convenient to divide the 3D temperature window into two parts, one above $T_c$: $\tau_{3D}^+$, and one below $T_c$: $\tau_{3D}^-$. It is found (Pierson 1995a) that while $\tau_{3D}^-(\lambda)$ has the same logarithmic dependence on $\lambda$ as previously found for $\tau_{3D}$, $\tau_{3D}^+$ differs markedly: $\tau_{3D}^+ \propto \lambda^{1/4}$. A power-law dependence on $\lambda$ was also found subsequently by Friesen (1995): $\tau_{3D}^+ \propto \lambda^{3/4}$.

How the system crosses over the 3D to 2D behavior can also be addressed (Pierson 1995c) by studying Eq. (47), the recursion relation for $\lambda$. Below $T_c$, where $\lambda$ is small, $\lambda$ grows to the isotropic value. This means that as the size of the vortex pairs becomes larger, 3D effects become more important. Above $T_c$, $\lambda$ grows initially but then starts to decrease at larger length scales due to vortex screening represented in the term proportional to $y^2$. The length scale at which $\lambda$($y$) peaks is the interlayer screening length $l_{3D/2D}$ and it is found to get larger as one lowers the temperature towards $T_c$ presumably diverging there as the intralayer screening length (i.e. the correlation length) does. The implication of this for the nature of 3D to 2D crossover is that the layer decoupling occurs first at the largest length scales just above $T_c$ and then proceeds to smaller length scales as the temperature is increased. As a consequence, the interaction between two vortices in neighboring layers separated by a distance larger than $l_{3D/2D}$ will be strongly screened giving the effect of decoupled layers. The linear (or 3D-like) interaction between two vortices in the same layers separated by a distance larger than interlayer screening length will also be screened resulting in the same effect. The screening of 3D effects at large length scales has interesting effects on the effective dimensionality of vortices where the 3D effects are felt primarily at those same lengths (Pierson 1995c).

IV. SUMMARY

In this paper, we have reviewed the critical behavior of the 2D $X$-$Y$ model universality class with an emphasis on the derivation of the recursion relations through a renormalization group analysis. We reviewed the derivation of the temperature dependence of the correlation length and the methods of verifying the transition in superconductors. Finally those methods were generalized to a layered superconductor system where the vortex interactions are modified due to the Josephson coupling between the layers. The power and usefulness of the real space RG method become readily apparent here as evidenced by the calculation of a number of important properties of the critical behavior including the dependence of the transition temperature and the size of the 3D critical region on the strength of the interlayer coupling.

ACKNOWLEDGMENTS

This work was supported by the Office of Naval Research.
FIG. 1. (a) The RG flows for the recursion relations Eqs. (31)-(32) for various initial values of \( x \) and \( y \) \([x_i = -0.5, y_i = 0.44, 0.46, \ldots, 0.56]\); and (b) The RG flows for Eqs. (29)-(30) for the initial values: \( K_i = 8.0, y_i = 0.56, 0.58, \ldots, 0.68 \). Because Eqs. (31)-(32) have been linearized around \( x = 0 \), the initial values in (a) are not the same as those in (b). In both figures, there are two classes of flows, one for which \( y \) goes to zero and one where \( y \) becomes large, separated by a curve corresponding to \( T = T_{KT B} \).

FIG. 2. The length scale dependence of (a) \( y \) and (b) \( K \) plotted versus \( \epsilon = \ln(\tau/\xi_0) \). In the low temperature phase, \( K(\epsilon) \) saturates to a finite value at a length corresponding to the separation beyond which the vortex pair density becomes negligible. Above \( T_{KT B} \) where the number of vortices increases at larger lengths, \( K(\epsilon) \) is renormalized towards zero.
