A computer assisted proof for 100,000 years stability of the solar system

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Abstract

We present an analytical proof assisted by computer calculations for the dynamical stability of the eight main planets and Pluto for the next 100,000 years. It means that the semi-major axes of the planets will not change significantly during this period. Also the eccentricities and inclinations of the orbits will remain sufficiently small. A standard linear four-step numerical method is used to integrate approximately the orbits of Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, Neptune and Pluto. Written in orbital elements, the dynamics of the nine planets manifests a system of 54 first-order ordinary differential equations. The step-size of the numerical method – about six days, has been performed 6,290,000 times. We estimate the total accumulation of rounding-off errors, deviations related to possible uncertainty in the astronomical data and the accuracy of the computer calculations.

Keywords: celestial mechanics, numerical analysis, solar system, stability.
1 Introduction

Important questions in celestial mechanics (Hagihara, 1972) are: “Will the present configuration of the solar system be preserved without radical changes for a long interval of time? What is the interval of time, at the end of which the configuration of the planetary orbits deviates from the present by a given small amount?” Next Hagihara claims that “present mathematics hardly permit this question to be answered satisfactory for the actual solar system.”

On the other hand, after some simplifying assumptions in the secular perturbation theory of Laplace and Lagrange, the motion of the eight main planets – from Mercury to Neptune, becomes integrable and the solar system would be stable. However, these simplifications are quite restrictive: one neglects the products of the planetary masses and assumes constant semi-major axes.

In fact, it turns out that the exact dynamics of the planets is highly sophisticated. That is why almost all studies on the question use computers to carry out numerical integration either of the full equations of motion, or numerical integration of the averaged equations, see comments in (Murray & Dermott, 1999). Most of the computer simulations cover hundreds of millions or billions of years (Sussman & Wisdom, 1992, Laskar, 1996), but without any analytical evaluations about possible deviations of the obtained results from the real dynamics of the planets.

Our algorithm for numerical integration is described in detail in section 3. It is possible without substantial additional difficulties to improve considerably the accuracy of the numerical method. However this is not necessary because the biggest problems arise when one evaluates the deviations caused by small changes in the initial conditions, the masses of the planets or sun.

For the calculations we use HP xw9400 workstation. The computer code performs 6,290,000 steps; each step-size is about six days, i.e. about 63 steps per year. Thus the integration covers 100,000 years.

Our main result is the following

**Theorem.** The configuration of the osculating ellipses on which the planets move around the Sun will remain stable at least 100,000 years in the sense that the semi-major axis of each planet varies within or less than 1%. The maximal values of the eccentricities and inclinations of the orbits remain bounded.
In the conditions of the Theorem, we also suppose possible uncertainties in the data from astronomical observations: up to $\pm 10^{-4}$ absolute error in the orbital elements and the masses of the planets for the J2000 epoch. By exception only for the masses of Mercury, Mars and Pluto we assume up to $\pm 10\%$ possible relative deviations.

The paper has the following structure.

In Section 2 we define three sets of generalized co-ordinates and the equations of motion, corresponding to the third set.

In section 3 we define additional helpful variables to enable the work of the computer code. Certain details for this code are discussed.

In section 4 we discuss the results of the numerical integration. The maximal deviations of the semi-major axes, eccentricities and inclinations of the orbits are listed.

In section 5 we prove that for any orbital element the error of the numerical integration does not exceed $10^{-9}$ per step. Thus the final result would not be differ significantly from the real dynamics of any planet.

Next we change slightly one by one the 54 initial conditions and the masses of the Sun and the planets. The obtained numerical results show that the solar system dynamics is stable under sufficiently small changes of initial conditions, masses and sufficiently small additional perturbations.

In section 6 we make our conclusions.

2 Equations of motion

In a heliocentric coordinate system, any planetary orbit can be described by six time-depending functions. We shall need three different such six-tuples:

$(x, y, z, \dot{x}, \dot{y}, \dot{z})$ – the usual rectangular coordinates and velocities in $\mathbb{R}^3$, dot denotes a differentiation with respect to the time $t$,

$(a, e, i, l, g, \theta)$ – six orbital elements, $a$ is the semi-major axis, $e$ is the eccentricity, $i$ is the inclination of the orbit, $l$ is the mean anomaly, $g$ is the argument of perihelion and $\theta$ is the longitude of ascending node,

$(a, \epsilon, h, k, p, q)$ – an alternative system of six orbital elements, $\epsilon$ is the modified mean longitude at epoch [Brower & Clemence, 1961], $h$ and $k$ are eccentric elements, while $p$ and $q$ are oblate elements. The first and the second coordinate systems have been related by [Poincaré, 1905].
\[ x = (\cos g \cos \theta - \sin g \sin \theta \cos i) X - (\sin g \cos \theta + \cos g \sin \theta \cos i) Y, \]
\[ y = (\cos g \sin \theta + \sin g \cos \theta \cos i) X - (\sin g \sin \theta - \cos g \cos \theta \cos i) Y, \]  
\[ z = \sin i \sin g X + \sin i \sin g Y, \]

where [Wintner, 1947]

\[
X = a \left[ \cos l - e - \sum_{s=1}^{\infty} \frac{e^s}{s!} \frac{ds^{-1} \sin^{s+1} l}{dl^{-1}} \right],
\]
\[
Y = a \sqrt{1 - e^2} \left[ \sin l + \sum_{s=1}^{\infty} \frac{e^s}{s!} \frac{ds^{-1} (\sin^s l \cos l)}{dl^{-1}} \right],
\]

and the mean anomaly \( l \) and the mean motion \( n \) are connected by the Kepler’s third law:

\[ l = n (t - t_0), \]
\[ n := \sqrt{1 + \mu a^{-3/2}}, \]
\[ \mu := \frac{\text{mass of the planet}}{\text{mass of the Sun}}, \]

\( t_0 \) is a time of perihelion passage.

Analogous to (1) equations hold for \( \dot{x}, \dot{y} \) and \( \dot{z} \), just replace \( X \) by \( n \partial_l X \) and \( Y \) by \( n \partial_l Y \). By \( \partial_w \) we shall denote the partial derivative with respect to the variable \( w \).

On the other hand,

\[ \lambda := l + g + \theta, \quad \epsilon(t) := \lambda(t) - \int_0^t n(s) \, ds, \]
\[ h := e \sin (g + \theta), \quad k := e \cos (g + \theta), \]
\[
\begin{aligned}
p &:= 2 \sin \frac{i}{2} \sin \theta, \\
q &:= 2 \sin \frac{i}{2} \cos \theta,
\end{aligned}
\]

relates the above two systems of orbital elements. The mean longitude \( \lambda \) is a fast variable while the modified mean longitude at epoch \( \epsilon \) is a slow variable.

We shall consider the Sun and planets as ten point masses, indexed by \( j = 0, 1, 2, \ldots, 9 \), located at their position vectors \( r_j \) and moving with 3-velocity \( \dot{r}_j \),

\[ r_j := (x_j, y_j, z_j)^t, \quad \dot{r}_j := (\dot{x}_j, \dot{y}_j, \dot{z}_j)^t. \]
Choose the unit of mass to be the mass of the Sun and the unit of distance to be the mean distance between the Earth and the Sun, i.e. one astronomical unit (AU). Then the relative mass $\mu_j$ of the $j$th planet satisfies (3) and time $t = 2\pi$ corresponds to one Julian year, i.e. 365.25 days. Time $t = 0$ corresponds to the year A.D. 2000.

According to the Newton’s theory of gravitation, the 3-accelerations of the planets

$$\ddot{r}_j = -(1 + \mu_j) \frac{\dot{r}_j}{r_j^3} + \partial_t R_j, \quad j = 1, \ldots, 9.$$ 

Here $r_j^2 = x_j^2 + y_j^2 + z_j^2$ is the squared distance between the Sun and $j$th planet. The disturbing function

$$R_j := \sum_{s \neq j} \mu_s \left[ \frac{1}{(r_j r_j + r_s r_s - 2 r_j r_s)^{1/2}} - \frac{r_j r_s}{r_s^3} \right],$$

where $r_j r_s = x_b x_c + y_b y_c + z_b z_c$ is the Euclidean dot product. Also, the gradient

$$\partial_t R_j = (\partial_{x_j} R_j, \partial_{y_j} R_j, \partial_{z_j} R_j)^t.$$ 

Now we apply the classical Lagrange’s brackets transform [Brower & Clemence, 1961] from coordinates $(x, y, z, \dot{x}, \dot{y}, \dot{z})$ to $(a, \epsilon, h, k, p, q)$. This leads to the system of $9 \cdot 6 = 54$ first order ODE’s

$$\dot{a} = \frac{2}{na} \partial_a R,$$
$$\dot{\epsilon} = -\frac{2}{na} \partial_\epsilon R + h \partial_\epsilon R + k \partial_k R + p \partial_p R + q \partial_q R - \frac{2na^2E}{na^2(1 + E^{-1})},$$
$$\dot{h} = \frac{E}{na^2} \partial_h R - \frac{h \partial_\epsilon R}{na^2(1 + E^{-1})} + k \frac{p \partial_p R + q \partial_q R}{2na^2E},$$
$$\dot{k} = -\frac{E}{na^2} \partial_k R - \frac{k \partial_\epsilon R}{na^2(1 + E^{-1})} - h \frac{p \partial_p R + q \partial_q R}{2na^2E},$$
$$\dot{p} = \frac{E^{-1}}{na^2} \partial_p R - p \frac{\partial_j R + k \partial_h R - h \partial_k R}{2na^2E},$$
$$\dot{q} = -\frac{E^{-1}}{na^2} \partial_q R - q \frac{\partial_j R + k \partial_h R - h \partial_k R}{2na^2E},$$

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where $E := \sqrt{1 - h^2 - k^2}$. For simplicity, all the running from 1 to 9 indexes $j$ have been omitted and one should understand $a_j$ instead of $a$, $\epsilon_j$ instead of $\epsilon$, ..., $R_j$ instead of $R$.

As usual, in the expression (5) for the disturbing function $R_j$ we use the mean longitudes $\lambda_j$ or $\lambda_s$ instead of the modified longitudes at epoch $\epsilon_j$ or $\epsilon_s$. In the equations of motion (6), under differentiating with respect to $\epsilon$ we understand differentiating with respect to $\lambda$ viz.,

$$\partial \epsilon_j = \partial \lambda_j, \quad \partial \epsilon_s = \partial \lambda_s.$$ 

Note a technical difference in (6) with respect to the equations of motion used in (Brower & Clemence, 1961) or (Murray & Dermott, 1999). Namely, we prefer the multiplier $2 \sin \frac{i}{2}$ when define the oblate elements $p$ and $q$ in (6), in contrast to the corresponding multipliers $\tan i$ or $\sin i$.

### 3 Numerical code

In order to solve numerically the system of 54 ODE’s (6) we apply the simplest four-step method (Butcher, 2008). For any variable $\sigma = a_1, \lambda_1, \ldots, p_9$ or $q_9$ the iteration formula reads

$$\sigma(\tau s) \approx \sigma_s,$$

$$\sigma_{s+1} := \sigma_s + \frac{\tau}{24} \left(55\dot{\sigma}_s - 59\dot{\sigma}_{s-1} + 37\dot{\sigma}_{s-2} - 9\dot{\sigma}_{s-3}\right)$$

for $s$ from 0 to 6, 290, 000 and step-size $\tau = 0.1$.

The error $|\sigma(\tau s) - \sigma_s| = O(\tau^5)$ will be specified in section 5.

For initial conditions in (6) we cite (Standish, 1992; NASA). Before starting the iterations with $s = 0$, it is necessary to carry out 300 single backward Euler steps with step-sizes $\tau = 10^{-3}$ to find $\dot{\sigma}_{-1}$, $\dot{\sigma}_{-2}$ and $\dot{\sigma}_{-3}$. Approximately, each step-time equals to six days, 63 steps cover one year and 6, 290, 000 steps cover more than 100, 000 years.

The numerical code is organized as follows.

First we define the intermediate eccentric variables

$$H := e \sin l = -h \cos \lambda + k \sin \lambda, $$

$$K := e \cos l = h \sin \lambda + k \cos \lambda.$$ 

(8)
and calculate the expansions (2) of $X$ and $Y$ up to order $\leq 10$ in the eccentricity $e$. It turns out that the new variables

$$U := (X \cos l + Y \sin l) \cos \lambda + (X \sin l - Y \cos l) \sin \lambda$$

$$= a \left( 1 - K - H^2 + \cdots \right) \cos \lambda + a H \left( 2 + \frac{1}{2} K + \cdots \right) \sin \lambda + \text{Err}_{11}(U)$$

and

$$V := (X \cos l + Y \sin l) \sin \lambda - (X \sin l - Y \cos l) \cos \lambda$$

$$= a \left( 1 - K - H^2 + \cdots \right) \sin \lambda - a H \left( 2 + \frac{1}{2} K + \cdots \right) \cos \lambda + \text{Err}_{11}(V)$$

do not depend explicitly on the mean anomaly $l$. Inside the $\text{Err}_{11}(U)$ and $\text{Err}_{11}(V)$ are included all terms of degree $\geq 11$ in $H$ and $K$.

Neglecting the eleventh and higher order eccentricities in the code, the approximate coordinates $x$ and $y$ become polynomials of degree twelve:

$$x = \left( 1 - \frac{1}{2} p^2 \right) U + \frac{1}{2} pq V,$$

$$y = \left( 1 - \frac{1}{2} q^2 \right) V + \frac{1}{2} pq U,$$

$$z = \left[ 1 - \frac{1}{4} (p^2 + q^2) \right]^{1/2} (-pU + qV).$$

Remark that these formulae look much convenient compared with (1). In view of (3) one can easily calculate all the derivatives of $x$, $y$ and $z$ with respect to $\lambda$, $h$ and $k$. Also, there is no problem to take derivatives with respect to $a$, $p$ or $q$.

Let us now briefly explain the structure of our numerical code. Each $\tau$-step requires as initial conditions five $(9 \times 6)$-matrices $S$, $dS$, $dS_1$, $dS_2$ and $dS_3$.

The matrix $S$ fixes the numerical values of $a_j, \epsilon_j, h_j, k_j, p_j$ and $q_j$ at its $j$th row. In terms of (1), each entry of $S$ corresponds to $\sigma_s$.

On the other hand, the matrices $dS$, $dS_1$, $dS_2$ and $dS_3$ remember the values of the derivatives

$$\dot{a}_j, \dot{\epsilon}_j, \dot{h}_j, \dot{k}_j, \dot{p}_j, \dot{q}_j, \quad 1 \leq j \leq 9$$

at the $s$th, $(s-1)$th, $(s-2)$th and $(s-3)$th steps correspondingly. In terms of (4) these last four matrices define $\dot{\sigma}_s$, $\dot{\sigma}_{s-1}$, $\dot{\sigma}_{s-2}$ and $\dot{\sigma}_{s-3}$.
We define three procedures to perform the \( s \)th step. The procedure “\( \text{make}_{\text{xyz}} \)” computes \( x, y, z \) and their derivatives with respect to \( a, \epsilon, h, k, p, q \) with the help of the above defined additional variables \( K, H, U \) and \( V \).

The procedure “\( \text{make}_2 \)” calculates all dot products

\[
\mathbf{r}_j \mathbf{r}_s , \; (\partial_{a_j} \mathbf{r}_j) \mathbf{r}_s , \; (\partial_{\lambda_j} \mathbf{r}_j) \mathbf{r}_s , \; (\partial_{h_j} \mathbf{r}_j) \mathbf{r}_s , \; (\partial_{k_j} \mathbf{r}_j) \mathbf{r}_s , \; (\partial_{p_j} \mathbf{r}_j) \mathbf{r}_s , \; (\partial_{p_j} \mathbf{r}_j) \mathbf{r}_s ,
\]

for \( j \) and \( s \) ranging from 1 to 9. Next “\( \text{make}_2 \)” computes all mutual distances \( \Delta_{j,s} \) between the planets:

\[
\Delta_{j,s}^2 = \mathbf{r}_j \mathbf{r}_j + \mathbf{r}_s \mathbf{r}_s - 2 \mathbf{r}_s \mathbf{r}_s ,
\]

and finally computes the partial derivatives

\[
\partial_{a_j} R_j , \; \partial_{\lambda_j} R_j , \; \partial_{h_j} R_j , \; \partial_{k_j} R_j , \; \partial_{p_j} R_j , \; \partial_{q_j} R_j
\]

of the disturbing functions \( [5] \).

Given 54 values of these last derivatives, the third procedure “\( \text{make}_{\text{EQ}} \)” finds a \( (9 \times 6) \)-matrix \( dS \) of the new values of the velocities \( \dot{a}_j, \dot{\epsilon}_j, \dot{h}_j, \dot{k}_j, \dot{p}_j, \dot{q}_j \) according to the equations of motion \( [6] \).

Finally, every single step executes:

(i) the procedure “\( \text{make}_{\text{xyz}} \)”,
(ii) the procedure “\( \text{make}_2 \)”,
(iii) the procedure “\( \text{make}_{\text{EQ}} \)”,
(iv) the numerical scheme

\[
S := S + \frac{0.1}{24} \left( 55 \ dS - 59 \ dS_{-1} + 37 \ dS_{-2} - 9 \ dS_{-3} \right) ,
\]

which defines the new values of \( a_j, \epsilon_j, h_j, k_j, p_j \) and \( q_j \),

(v) compute the new mean longitudes \( \lambda \):

\[
\lambda_{s+1} = \lambda_s + \epsilon_{s+1} - \epsilon_s + \frac{0.1}{24} \left( \frac{55}{2} \hat{n}_s - \frac{59}{3} \hat{n}_{s-1} + \frac{37}{4} \hat{n}_{s-2} - \frac{9}{5} \hat{n}_{s-3} \right) ;
\]

see the second equation in \( [4] \),

(vi) \( dS \) becomes \( dS_{-1} \), \( dS_{-1} \) becomes \( dS_{-2} \), \( dS_{-2} \) becomes \( dS_{-3} \).

After 6,290,000 times repeating these (i)–(vi) points we obtain the final matrix \( S \) and so the approximated position of each planet after 100,000 years.

Remark: The implemented code for numerical integration in the present paper can be made available upon request to the authors.
4 Results and discussion

The numerical integration of the equations of motions (6) for the next $10^5$ years shows more or less expected results.

The semi-major axes $a_j$ oscillate within the periodic perturbations, see the next table.

Table 1: Minimum and maximum values of the semi-major axis (in AU). Displacement (in degrees) of the modified mean longitude at epoch $\Delta \epsilon$ for $10^5$ years. Mean motions $n$ for the next $10^5$ years.

|          | $a_{min}$ | $a_{max}$ | $\Delta \epsilon$ | $n$  |
|----------|-----------|-----------|---------------------|------|
| Mercury  | 0.3870    | 0.3871    | $-187^\circ$        | 4.1524 |
| Venus    | 0.7233    | 0.7234    | $-288^\circ$        | 1.6258 |
| Earth    | 1         | 1.0001    | $50^\circ$          | 1.0001 |
| Mars     | 1.5232    | 1.5238    | $-237^\circ$        | 0.5318 |
| Jupiter  | 5.1997    | 5.2038    | $-14^\circ$         | 0.0844 |
| Saturn   | 9.4670    | 9.5426    | $3122^\circ$        | 0.0342 |
| Uranus   | 19.034    | 19.280    | $1177^\circ$        | 0.0120 |
| Neptune  | 29.766    | 30.247    | $599^\circ$         | 0.0061 |
| Pluto    | 38.614    | 40.116    | $408^\circ$         | 0.0041 |

The small oscillations $a_{1,2,3,4,5}$ of the inner planets and Jupiter are consequence of the choice of heliocentric reference frame. The relatively large oscillations of $a_{6,7,8,9}$ are due to the indirect parts

$$- \sum_{s<j} \mu_s \frac{r_j r_s}{r_s^3}$$

of the perturbations $R_j$ combined with the following near resonant mean motions:

$$n_5 : n_6 \approx 5 : 2, \quad n_6 : n_7 \approx 20 : 7, \quad n_7 : n_8 \approx 2 : 1, \quad n_8 : n_9 \approx 3 : 2.$$ 

The mean motions $n_j$ for the next $10^5$ years remain almost the same as to “instant mean motions $n_j$ at the epoch $J2000$”, see 4th column of table 1.

The eccentric and oblate variables manifest some unanticipated results, see the next two tables.
Table 2: Minimum and maximum values of the eccentricities. Initial and final values of the longitude of perihelion.

|        | \(e_{\text{min}}\) | \(e_{\text{max}}\) | \(\tilde{\omega}_{2000}\) | \(\tilde{\omega}_{6\cdot10^5}\) |
|--------|-----------------|-----------------|-----------------|-----------------|
| Mercury| 0.195           | 0.208           | 77\(^\circ\)    | 222\(^\circ\)   |
| Venus  | 0.002           | 0.022           | 132\(^\circ\)   | 98\(^\circ\)    |
| Earth  | 0.002           | 0.017           | 103\(^\circ\)   | 476\(^\circ\)   |
| Mars   | 0.086           | 0.121           | -24\(^\circ\)   | 471\(^\circ\)   |
| Jupiter| 0.025           | 0.060           | 15\(^\circ\)    | 160\(^\circ\)   |
| Saturn | 0.010           | 0.088           | 92\(^\circ\)    | 863\(^\circ\)   |
| Uranus | 0.028           | 0.055           | 171\(^\circ\)   | 280\(^\circ\)   |
| Neptune| 0.001           | 0.018           | 45\(^\circ\)    | 83\(^\circ\)    |
| Pluto  | 0.237           | 0.266           | 224\(^\circ\)   | 226\(^\circ\)   |

Table 3: Minimum and maximum values of the inclination. Initial and final values of the longitudes of ascending node.

|        | \(i_{\text{min}}\) | \(i_{\text{max}}\) | \(\theta_{2000}\) | \(\theta_{6\cdot10^5}\) |
|--------|-----------------|-----------------|-----------------|-----------------|
| Mercury| 5.8\(^\circ\)   | 7\(^\circ\)     | 48\(^\circ\)    | -143\(^\circ\)  |
| Venus  | 0.05\(^\circ\)  | 3.4\(^\circ\)   | 77\(^\circ\)    | -882\(^\circ\)  |
| Earth  | 0\(^\circ\)     | 3.3\(^\circ\)   | 0\(^\circ\)     | -244\(^\circ\)  |
| Mars   | 0.4\(^\circ\)   | 4.5\(^\circ\)   | 50\(^\circ\)    | -561\(^\circ\)  |
| Jupiter| 1.2\(^\circ\)   | 2\(^\circ\)     | 101\(^\circ\)   | 109\(^\circ\)   |
| Saturn | 0.7\(^\circ\)   | 2.6\(^\circ\)   | 114\(^\circ\)   | 97\(^\circ\)    |
| Uranus | 0.6\(^\circ\)   | 1.4\(^\circ\)   | 74\(^\circ\)    | 151\(^\circ\)   |
| Neptune| 0.05\(^\circ\)  | 3.4\(^\circ\)   | 132\(^\circ\)   | 125\(^\circ\)   |
| Pluto  | 16.6\(^\circ\)  | 17.2\(^\circ\)  | 110\(^\circ\)   | 95\(^\circ\)    |

In addition to the above arguments for the indirect parts and near-resonances there are two main reasons for the discrepancy with the Laplace–Lagrange secular theory, as well as with the Brouwer & van Woerkom secular theory [Brouwer & van Woerkom, 1950]. In contrast of the numerical integration these theories does not take into account the fourth order (in the eccentricities and inclinations) secular terms.

Another effect arises when some eccentricity or inclination is very small. Then the corresponding longitude of perihelion or node becomes chaotic behavior, with possible \(\pm360^\circ\) errors in their numerical integration.
For instance, it turns out that the longitude of perihelion of Venus for the next 100,000 years moves clockwise from 132° to 90°. Another example are the counterclockwise movements of Jupiter and Uranus nodes.

5 Proof of the Theorem

Let \( u^* \) be the explicitly obtained after numerical integration of the equations of motion \( u \) approximate solution. Suppose also that \( v = v(t) \) be some exact solution, which initial conditions \( v(0) \) are close to \( u^*(0) \).

Then the proof of the theorem requires to show that the deviation

\[
\delta(t) := v(t) - u^*(t)
\]  

will be sufficiently small for the next 100,000 years.

So, we shall represent and estimate \( \delta \) as a superposition of the error of the numerical integration method and the linear and non-linear variations due to the difference \( v(0) \neq u^*(0) \).

5.1 The maximal error of a single step

The calculations are made with accuracy of \( 10^{-10} \) for each step of the algorithm, which gives an accumulation less than \( 10^{-4} \) for the entire period.

Also, the estimation of the errors

\[
|Err_{11}(U)| + |Err_{11}(V)| \leq \frac{e^{11}}{1-e} \exp \frac{11}{2}
\]

follows from an estimation of the derivatives in \( [2] \) for \( j \geq 11 \). In the calculations, we have ignored these errors which gives an error less than \( 10^{-10} \) per step. With accumulation this would yield deviations less than \( 6 \cdot 10^{-4} \) in the final values of the action variables \( a, e \) and \( i \).

Next, let \( \sigma \) be an element of the \( j \)th planetary orbit, that is \( \sigma \in \{a_j, \epsilon_j, h_j, k_j, p_j, q_j\} \).

During the procedure of numerical integration of \( \sigma \) we have neglected the fifth time–derivative \( \sigma^{(5)}(t) \) and now the error coming from the numerical method will be estimated. It will be done for every \( \tau \)-step and afterwards accumulated for all 6,290,000 \( \tau \)-steps.

Thus one should apply the operator

\[
\frac{d}{dt} = \sum_{m=1}^{9} \left( a_m \partial_a + \epsilon_m \partial_{\epsilon} + h_m \partial_{h_m} + k_m \partial_{k_m} + p_m \partial_{p_m} + q_m \partial_{q_m} \right)
\]  

(10)
four times over the $\dot{\sigma}$ right side in [9]. Since $\dot{a}, \dot{h}, \dot{k}, \dot{p}$ and $\dot{q}$ are of order $10^{-3}$ (or even smaller) and $\dot{e} = \dot{\lambda} + n$, we can consider that

$$\frac{d}{dt} \approx \sum_{m=1}^{9} n_m \partial_{\lambda m}$$

and consequently

$$\frac{d^4}{dt^4} \approx \sum_{m=1}^{9} \sum_{b=1}^{9} \sum_{c=1}^{9} \sum_{d=1}^{9} n_m n_b n_c n_d \partial_{\lambda m} \partial_{\lambda b} \partial_{\lambda c} \partial_{\lambda d}.$$  \hspace{1cm} (11)

It is easily to see that the approximated $\frac{d^4}{dt^4}$ differs from the exact fourth degree of $\frac{d}{dt}$ from (10) with no more than 10%.

The right side of (11) is a sum of $9^4$ differential operators but only 129 of them do not nullify the right side of the equation for $\dot{\sigma}$. These are those for which among the indexes $j, m, b, c, d$ no more than two are different, see the equation (5) for the disturbing function.

The mean motions $n_{5,6,7,8}$ of the big planets are relatively small and so the differentiations with respect to $\partial_{\lambda 5,6,7,8}$ compensate the relatively large masses $\mu_{5,6,7,8}$,

$$\mu_{1,2,\ldots,9} \approx \begin{bmatrix} 1/6023600 & 1/408523 & 1/328900 & 1/3098708 \\ 1/1047.35 & 1/3497.89 & 1/22903 & 1/19412 & 1/1.35 \cdot 10^8 \end{bmatrix}.$$

Reversely, the large values of $n_{1,2,3,4}$ are compensated by the small values of $\mu_{1,2,3,4}$. The perturbations caused by Pluto are entirely negligible.

Straightforward calculations show that for all $\sigma$

$$|\sigma^{(5)}(t)| < \frac{1}{200}. \hspace{1cm} (12)$$

Such an estimate takes place only for $\sigma = \epsilon_1$. For the remaining 53 orbital elements the evaluation is $|\sigma^{(5)}| < \frac{1}{1000}$.

Taking into account the coefficient $\frac{0.15}{5!} \frac{1}{200}$ in front of the remainder in Taylor series expansion of $\sigma$ we conclude that the exponentially accumulated error (Butcher, 2008) of the numerical integration for $10^5$ years does not exceed

$$\left[1 + \frac{0.15}{5!} \frac{1}{200} \right]^{6290000} - 1 < \frac{1}{380}.$$
In practice, the real estimate is much better since the signs of $\sigma^{(5)}(t)$ do alternate for different $t$ and largely compensate each other.

### 5.2 Variations due to different initial conditions

Let us extend (6) with nine new differential equations $\dot{\mu}_j = 0$ and rewrite this new system of 63 ODE’s as

$$\dot{v} = f(v), \quad v := (a_1, \epsilon_1, \ldots, q_1, \mu_1, \ldots, a_9, \epsilon_9, \ldots, q_9, \mu_9).$$

Introduce next a metric

$$||v|| := \max \left ( a_1, a_2, a_3, a_4, \frac{a_5}{5}, \frac{a_6}{20}, \frac{a_7}{30}, \frac{a_8}{40}, \frac{a_9}{50}, \max_{1 \leq j \leq 9} (0.3|\epsilon_j|, |h_j|, |k_j|, |p_j|, |q_j|), 10^5 \mu_1, \mu_2, \mu_3, 10^5 \mu_4, \mu_5, \mu_6, \mu_7, \mu_8, 10^5 \mu_9 \right ),$$

where all $\epsilon_j \in (-\pi, \pi]$. Roughly speaking, the solutions of (6) do not exceed 2 by norm.

We shall suppose that the varied initial conditions $||\delta(0)|| = ||v(0) - u^*(0)|| < 10^{-4}$.

The explicitly obtained numerical solution $u^*$ has initial conditions $u^*(0)$ set in [Standish, 1992; NASA] and satisfies a system of 63 ODE’s

$$\dot{u}^* = f^*(u^*),$$

where the vector function $f^* \approx f$.

To be precise, as consequence of (12)

$$||f^*(v) - f(v)|| < \frac{6 \cdot 10^{-5}}{5!} \cdot \frac{1}{200} = 5 \cdot 10^{-10}$$

provided

$$||\delta(t)|| = ||v(t) - u^*(t)|| < 10^{-3} \quad \text{for all} \quad t \leq 2\pi \cdot 10^5.$$  \hspace{1cm} (14)

Indeed, the velocities $\dot{a}_1, \ldots, \dot{q}_9$ in (6) are small as well as their first partial derivatives and so the Taylor’s expansion of $f$ says that a multiplier $\frac{6}{5}$ is sufficient to verify (14).
Define next the $(63 \times 63)$ matrix

$$A(t) := \frac{Df(u^*(t))}{Du^*}, \quad A_{ms}(t) = \frac{\partial f_m(u^*(t))}{\partial u^*}, \quad m, s = 1, \ldots, 63,$$

as well as the fundamental solution $\Phi = \Phi(t)$ for the linear system of $63^2$ ODE's

$$\dot{\Phi} = A(t)\Phi, \quad \Phi(0) = I,$$

$I$ is the identity matrix. Remark that $A(t)$ has been already known and $\Phi$ can be computed approximately as follows. We start-up the program 64 times independently of one another. At the $m$th start-up we change only the $m$th initial condition to be

$$(u_1^*(0), \ldots, u_{m-1}^*(0), u_m^*(0) + 10^{-5}, u_{m+1}^*(0), \ldots, u_{63}^*(0))$$

and thus we find the $m$th column of $\Phi$. The change in the mass of the Sun gives the final $\Phi$.

We perform these additional numerical integrations and conclude that $\Phi$ remains close to $\Phi(0)$:

$$\|\Phi(t) - I\| < \frac{1}{5}$$

for all $t < 2\pi 10^5$ and the matrix norm associated with the vector norm (13).

Now we compute

$$\dot{\delta}(t) = f(v(t)) - f^*(u^*(t))$$

$$= f^*(u^* + \delta) - f^*(u^*) + f(v) - f^*(v)$$

$$= A(t)\delta(t) + F(t) + G(t),$$

where the functions

$$F(t) := \dot{u}^*(t) - A(t)u^*(t),$$

$$G(t) := f(v(t)) - f^*(v(t)),$$

will be estimated in the next subsection.

The weakly non-linear system (17) satisfies (Cesari, 2008) the non-linear Volterra integral equation

$$\delta(t) = \Phi(t)\left[\delta(0) + \int_0^t \Phi^{-1}(\alpha)[F(\alpha) + G(\alpha)] d\alpha \right]$$

and our aim is to prove that $\|\delta(t)\|$ is sufficiently small for all $0 \leq t \leq 2\pi \cdot 10^5$. 

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5.3 End of the proof of the main Theorem

According to (14), \( ||G|| < 5 \cdot 10^{-10} \) provided \( ||v - u^*|| < 10^{-3} \) for all \( t \).

More complicated is to estimate the vector function \( \mathbf{F} \). To do this we analyze the second order reminder in the Taylor’s expansion of \( f^*(u^* + \delta) \), by definition equal to \( \mathbf{F} \):

\[
F_m(t) := \frac{1}{2} \sum_{b=1}^{54} \sum_{c=1}^{54} \frac{\partial^2 f_m^*(\xi(t))}{\partial u_b^* \partial u_c^*} \delta_b \delta_c , \tag{19}
\]

where \( F_m \) are the components of the vector \( \mathbf{F} \), \( 1 \leq m \leq 54 \) and \( ||\xi - u^*|| < ||v - u^*|| \) for all \( t \).

Next we estimate the second partial derivatives in (19) by

\[
F_{m; b, c} := \sup_{t} \left| \frac{\partial^2 f_m^*(\xi(t))}{\partial u_b^* \partial u_c^*} \right|.
\]

Every index \( m, b \) or \( c \) corresponds to an orbital element or a relative mass. We shall distinguish the following three types of \( F_{m; b, c} \):

(i) vanishing \( F_{m; b, c} \) if \( m, b \) and \( c \) correspond to three different planet, or if some of them corresponds to some relative mass,

(ii) negligible small \( F_{m; b, c} \) if \( m, b \) or \( c \) corresponds to an element of any inner planet or Pluto (their masses are too small); or if \( m, b \) or \( c \) corresponds to some oblate orbital element,

(iii) \( F_{m; b, c} \) is neither of type (i) nor of type (ii). For any fixed \( m \) there exist 16 such orbital elements: \( a, \epsilon, h \) and \( k \) of Jupiter, Saturn, Uranus and Neptune, and always

\[
F_{m; b, c} < \frac{1}{3000} .
\]

All these examinations together enable us to conclude that

\[
||\mathbf{F}|| < \frac{||\delta||^2}{1000} .
\]

Compared with (16), (18) and the already made estimation of \( \mathbf{G} \) this proves
that

\[ \|\delta(t)\| < \frac{3}{2} \int_0^t \left(\|\delta(t)\|^2 + 5 \cdot 10^{-10}\right) dt \]

= \frac{\sqrt{2}}{2000} \tan \left( \frac{3\sqrt{2} t}{4 \cdot 10^6} + \arctan \frac{\sqrt{2}}{10} \right).

Hence \(\|\delta(t)\|\) is sufficiently small for \(0 \leq t \leq 2\pi 10^5\), satisfies the requirement \((15)\) and the theorem has been finally proved.

6 Conclusions

Throughout the proof of the Theorem, we have not used the fact that for long periods of time the perturbations of the orbital elements are canceling almost entirely.

With simple additional reasonings and evaluations, while preserving the numerical integration algorithm, the Theorem could be proven for one million years.

Presumably under more detailed additional evaluations and availability of powerful computer, the stability of the solar system could be proved for the next \(5 \cdot 10^9\) years.

We will note also that the evaluations which are done about the deviation between the numerical integration and the real dynamics are a little rough. Hence it is possible to include in them additional real existing details such as:

- relativistic effects
- the fact that the planets are not point masses
- passing of external sufficiently small celestial bodies across the solar system.

However accounting the influence of the known small bodies in the solar system demands huge extra labor.

The numerical integration of the solar system dynamics indicates that the fourth order secular terms should be taken into consideration in the purely analytical approaches.
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References

Brower D., Clemence J., Methods in Celestial Mechanics, Academic Press, New York, 1961

Brower D., van Woerkom J., 1950, Astron. Pap. Am. Eph., 13, 81

Butcher J., Numerical Methods for Ordinary Differential Equations, John Wiley & Sons Ltd, 2008

Cesari L., Asymptotic Behavior and Stability Problems in Ordinary Differential Equations, Springer, 1963

Hagihara Y., Celestial Mechanics, MIT Press, 1972

Laskare J., 1996, Cel. Mech. Dyn. Astr., 64, 115-162

Murray C., Dermott S., Solar System Dynamics, Cambridge Univ. Press, 1999

NASA, https://ssd.jpl.nasa.gov/?planet_pos

Poincaré H., Lecons de Mecanique Celeste, Gauthier-Villars, 1905

Standish E., Newhall X., Williams J., Yeomans D., Explanatory Supplement to the Astronomical Almanac, University Science Books, Mill Valley, CA 1992

Sussman G., Wisdom J., 1992, Science, Chaotic Evolution of the Solar System, 257

Wintner A., The Analytical Foundations of Celestial Mechanics, Princeton University Press, Princeton, N. J., 1947