TWO IDENTITIES RELATING EISENSTEIN SERIES ON CLASSICAL GROUPS

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Abstract. In this paper we introduce two general identities relating Eisenstein series on split classical groups, as well as double covers of symplectic groups. The first identity can be viewed as an extension of the doubling construction introduced in [CFGK17]. The second identity is a generalization of the descent construction studied in [GRST11].

0. Introduction

For simplicity, in order to avoid general notation in this introduction, we describe the two identities in the case of symplectic groups.

Let \( F \) be a number field, and let \( \mathbb{A} \) be its ring of Adeles. Let \( \tau \) be an irreducible, automorphic, cuspidal representation of \( \text{GL}_n(\mathbb{A}) \). Consider, for a positive integer \( \ell \), the Speh representation \( \Delta(\tau, \ell) \) of \( \text{GL}_{n\ell}(\mathbb{A}) \). Let \( m, i \) be positive integers, and assume that \( m \) is even. Consider, for a complex number \( s \), the parabolic induction \( \rho_{\Delta(\tau, m+i),s} = \text{Ind}_{\text{Sp}_{2n}(m+i)(\mathbb{A})}^{\text{Sp}_{2n}(m+i)(\mathbb{A})} \Delta(\tau, m+i) | \det \cdot|^s \), where \( \text{Sp}_{2n}(m+i) \) is the Siegel parabolic subgroup of \( \text{Sp}_{2n}(m+i) \). Let \( E(\rho_{\Delta(\tau, m+i),s}) \) be the Eisenstein series on \( \text{Sp}_{2n}(m+i)(\mathbb{A}) \), corresponding to a smooth, holomorphic section \( f_{\Delta(\tau, m+i),s} \) of \( \rho_{\Delta(\tau, m+i),s} \).

For our first identity, we consider a Fourier coefficient \( F_{\psi}(E(\rho_{\Delta(\tau, m+i),s})) \) of our Eisenstein series, along the unipotent radical \( U_{m-1} \) of the standard parabolic subgroup \( Q_{m-1} \), whose Levi part is isomorphic to \( \text{GL}_m^n \times \text{Sp}_{2(m+n)} \). The corresponding character of \( U_{m-1}(\mathbb{A}) \) is written explicitly in (1.2), and denoted there by \( \psi_H \), with \( H = \text{Sp}_{2n}(m+i) \). This character is attached to the symplectic partition \( (2n-1)m, 12ni+m) \). It is stabilized by a subgroup \( D_H \), where \( D \) isomorphic to \( \text{Sp}_m \times \text{Sp}_{2ni+m} \). See [CM93], Theorem 6.1.3. For \( g \in \text{Sp}_m(\mathbb{A}), h \in \text{Sp}_{2ni+m}(\mathbb{A}) \), denote by \( t(g,h) \) the corresponding element in \( D_H \). We consider the Fourier coefficient above as a function on \( \text{Sp}_m(\mathbb{A}) \times \text{Sp}_{2ni+m}(\mathbb{A}) \) (it is automorphic),

\[
F_{\psi}(E(\rho_{\Delta(\tau, m+i),s}))(g,h) = \int_{U_{m-1}(\mathbb{A})} E(f_{\Delta(\tau, m+i),s}, ut(g,h)) \psi_H^{-1}(u) du.
\]

Now we use \( F_{\psi}(E(\rho_{\Delta(\tau, m+i),s}))(g,h) \) as a kernel function and integrate it against cusp forms \( \varphi_\sigma \) in a space of an irreducible, automorphic, cuspidal representation \( \sigma \)

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of $\text{Sp}_m(\mathbb{A})$, thus obtaining automorphic functions on $\text{Sp}_{2n_1+m}(\mathbb{A})$,
\begin{equation}
E(f_{\Delta(\tau,m+i),s},\varphi_\sigma)(h) = \int_{\text{Sp}_m(F)\backslash \text{Sp}_m(\mathbb{A})} F_\psi(E(f_{\Delta(\tau,m+i),s}))(g,h)\varphi_\sigma(g)dg.
\end{equation}

**Theorem A.** $E(f_{\Delta(\tau,m+i),s},\varphi_\sigma)$ is an Eisenstein series on $\text{Sp}_{2n_1+m}(\mathbb{A})$, corresponding to the parabolic induction from $\Delta(\tau,i)\mid \text{det-}^s \times \sigma^i$, where $\sigma^i$ is a certain outer conjugation of $\sigma$ by an element of order 2. There is an explicit meromorphic section $\Lambda(f_{\Delta(\tau,m+i),s},\varphi_\sigma)$ of the last parabolic induction, such that for $\text{Re}(s)$ sufficiently large,
\begin{equation}
E(f_{\Delta(\tau,m+i),s},\varphi_\sigma)(h) = \sum_{\gamma \in \mathcal{Q}_n(F)\backslash \text{Sp}_{2n_1+m}(F)} \Lambda(f_{\Delta(\tau,m+i),s},\varphi_\sigma)(\gamma h).
\end{equation}

The right hand side of (0.3) continues to a meromorphic function in the whole plane. Denote it by $E(\Lambda(f_{\Delta(\tau,m+i),s},\varphi_\sigma))$. Then, as meromorphic functions on $\text{Sp}_{2n_1+m}(\mathbb{A})$,
\begin{equation}
E(f_{\Delta(\tau,m+i),s},\varphi_\sigma) = E(\Lambda(f_{\Delta(\tau,m+i),s},\varphi_\sigma)).
\end{equation}

We can prove a similar identity with normalized Eisenstein series, outside a finite set of places of $F$, containing the Archimedean ones, outside which $\tau$ and $\sigma$ are unramified. We also prove analogous identities for metaplectic groups and split orthogonal groups. The nice thing about this identity is that it expresses an Eisenstein series on $\text{Sp}_{2n_1+m}(\mathbb{A})$, induced from $\Delta(\tau,i)\mid \text{det-}^s \times \sigma^i$, in a uniform way, for all $\sigma$, in terms of an Eisenstein series on $\text{Sp}_{2n_1(m+i)}(\mathbb{A})$, induced from $\Delta(\tau,m+i)\mid \text{det-}^{s+i}$. For example, when $i = 1$, we get an expression of an Eisenstein series on $\text{Sp}_{2n_1+m}(\mathbb{A})$, induced from the cuspidal representation $\tau\mid \text{det-}^s \times \sigma^i$, in terms of an Eisenstein series on $\text{Sp}_{2n_1(m+1)}(\mathbb{A})$, induced from $\Delta(\tau,m+1)\mid \text{det-}^{s+i}$. This shows that Eisenstein series induced from Speh representations play a prominent role. We can use them to produce Eisenstein series induced from cuspidal representations of Levi parts of maximal parabolic subgroups.

We note that the section $\Lambda(f_{\Delta(\tau,m+i),s},\varphi_\sigma)$ has a nice explicit integral expression, and that it is related to the integrals of the generalized doubling method for $\text{Sp}_m \times \text{GL}_n$. It is interesting that the identity above exhibits a tower, as in towers of theta lifts, with the tower parametrized by $i$. It will be easy to note that our proofs are valid for $i = 0$ as well. In this case, the proof amounts to the Euler product expansion of the generalized doubling integrals for $\text{Sp}_m \times \text{GL}_n$, representing the partial $L$-function $L^2(\sigma \times \tau, s)$. In this case, $E(f_{\Delta(\tau,m+i),s},\varphi_\sigma)$ defines an element in $\sigma^i$. Thus, the tower starts with the cuspidal representation $\sigma^i$ on $\text{Sp}_m(\mathbb{A})$, and moves up to the Eisenstein series representations on $\text{Sp}_{2n_1+m}(\mathbb{A})$ induced from $\Delta(\tau,i)\mid \text{det-}^s \times \sigma^i$, for $i = 1, 2, ..., n$. We also allow the case $n = 1$, where $\tau$ is simply a character of $F^* \backslash \mathbb{A}^*$, and then $\Delta(\tau,m+i) = \tau \circ \text{det}_{\text{GL}_{m+i}}, \rho_{\Delta(\tau,m+i),s} = \text{Ind}_{\mathbb{Q}_n}^{\mathbb{A}}(\tau(\cdot))\mid \text{det-}^s$. In this case, $U_{m-1}$ is trivial, and the function $F_\psi(E(f_{\Delta(\tau,m+i),s}))(g,h)$ is the restriction of the Eisenstein series $E(f_{\text{ro det}_{\text{GL}_{m+i}},s})$ to $\text{Sp}_m(\mathbb{A}) \times \text{Sp}_{2i+m}(\mathbb{A})$. In this case, the Eisenstein series $E(f_{\text{ro det}_{\text{GL}_{m+i}},s},\varphi_\sigma)$ of Theorem A, corresponds to the section $\Lambda(f_{\text{ro det}_{\text{GL}_{m+i}},s},\varphi_\sigma)$ of the parabolic induction from $(\tau \circ \text{det}_{\text{GL}})\mid \text{det-}^s \times \sigma$ to $\text{Sp}_{2i+m}(\mathbb{A})$. This was considered in [GPSR97], Sec. 1, for (arbitrary) orthogonal groups and $i = 1$ and generalized by Moeglin in [M97], for any $i$ and $H$ symplectic, or even orthogonal.
In our second identity, we consider, for \( i \geq 1 \), the descent of \( E(\Delta_{(\tau, i+1), s}) \) from \( \text{Sp}_{2n(i+1)}(\mathbb{A}) \) to the metaplectic group \( \text{Sp}^{(2)}_{2ni}(\mathbb{A}) \), that is we apply to \( E(\Delta_{(\tau, i+1), s}) \) a Fourier-Jacobi coefficient corresponding to the partition \((2n, 1^{2ni})\). Denote this coefficient by \( \mathcal{D}^{(2), \phi}_{\psi, ni}(E(\Delta_{(\tau, i+1), s})) \), and view it as an automorphic function of \( \tilde{\psi} \in \text{Sp}^{(2)}_{2ni}(\mathbb{A}) \). Here \( \phi \in \mathcal{S}(\mathbb{A}^{ni}) \) is a Schwartz function entering in the theta series on the semi-direct product of the Heisenberg group in \( 2ni + 1 \) variables and \( \text{Sp}^{(2)}_{2ni}(\mathbb{A}) \). We consider in a similar way the analogous descent of \( E(\Delta_{(\tau, i+1), s}) \) from \( \text{Sp}^{(2)}_{2n(i+1)}(\mathbb{A}) \) to \( \text{Sp}_{2ni}(\mathbb{A}) \). Here, \( \gamma_{\psi} \) is the Weil factor attached to \( \psi \). Denote the corresponding Fourier-Jacobi coefficient by \( \mathcal{D}^{\phi}_{\psi, ni}(E(\Delta_{(\tau, i+1), s})) \), and view it as an automorphic function of \( \psi \in \text{Sp}_{2ni}(\mathbb{A}) \). We note that the case \( i = 0 \) makes sense. Here, the Fourier-Jacobi coefficients above are simply \( \psi\)-Whittaker coefficients of our Eisenstein series, which are well known. The second main theorem in this paper is

**Theorem B.** \( \mathcal{D}^{(2), \phi}_{\psi, ni}(E(\Delta_{(\tau, i+1), s})) \) is an Eisenstein series on \( \text{Sp}^{(2)}_{2ni}(\mathbb{A}) \), corresponding to the parabolic induction from \( \Delta_{(\tau, i+1), s} \) to \( \text{Sp}^{(2)}_{2ni}(\mathbb{A}) \). There is an explicit meromorphic section \( \Lambda(f_{\Delta_{(\tau, i+1), s}, \phi}) \) of the last parabolic induction, such that for \( \text{Re}(\phi) \) sufficiently large,

\[
(0.4) \quad \mathcal{D}^{(2), \phi}_{\psi, ni}(E(\Delta_{(\tau, i+1), s}))(\tilde{\psi}) = \sum_{\gamma \in Q_{ni}(F)} \Lambda(f_{\Delta_{(\tau, i+1), s}, \phi})(\gamma \tilde{\psi}).
\]

The right hand side of (0.4) continues to a meromorphic function in the whole plane. Denote it by \( E(\Lambda(f_{\Delta_{(\tau, i+1), s}, \phi}))(\tilde{\psi}) \). Then, as meromorphic functions on \( \text{Sp}^{(2)}_{2ni}(\mathbb{A}) \),

\[
(0.5) \quad \mathcal{D}^{(2), \phi}_{\psi, ni}(E(\Delta_{(\tau, i+1), s})) = E(\Lambda(f_{\Delta_{(\tau, i+1), s}, \phi})).
\]

We can prove a similar identity with normalized Eisenstein series, outside a finite set of places of \( F \), containing the Archimedean places, outside which \( \tau \) is unramified. We prove a similar theorem for the descent \( \mathcal{D}^{\phi}_{\psi, ni}(E(\Delta_{(\tau, i+1), s})) \) from \( \text{Sp}^{(2)}_{2n(i+1)}(\mathbb{A}) \) to \( \text{Sp}_{2ni}(\mathbb{A}) \), expressing it as an Eisenstein series on \( \text{Sp}_{2ni}(\mathbb{A}) \), corresponding to the parabolic induction from \( \Delta(\tau, i)(\text{det} \cdot)^{s} \). We also prove similar identities for split orthogonal groups.

As in Theorem A, we allow the case \( n = 1 \), where \( \tau \) is a character of \( F^{*} \backslash \mathbb{A}^{*} \). In this case, Theorem B, for \( H_{A} = \text{Sp}_{2i+2}(\mathbb{A}) \), is a special case of a theorem of Ikeda [94].

There are several extensions and applications to the above two identities. First, we intend to extend these identities to metaplectic covering groups of split classical groups. In recent years there is a growing interest in the structure of automorphic representations of these groups. There is also a growing realization that many constructions which work for the linear groups can be extended to metaplectic covering groups. For example, as described in [CFGK16], the doubling construction for linear groups given in [CFGK17], extends in a natural way to metaplectic covering groups. Thus, as in the above case when \( i = 1 \), we expect to express every Eisenstein series on a metaplectic covering group, parabolically induced from cuspidal data \( \tau \) and \( \sigma \), in terms of an Eisenstein series involving the representation \( \tau \) only. In other words, we expect that a similar identity to identity (0.2) will hold also for
metaplectic covering groups. Here, $\tau$ and $\sigma$ are irreducible cuspidal representations of certain metaplectic covering groups. In general, we expect that Theorems A and B extend to metaplectic covering groups of split classical groups.

A second extension which we intend to study is a variation of the first identity. This new identity, which we refer to as the dual identity, uses the fact that the embedding of $\text{Sp}_m(\mathbb{A}) \times \text{Sp}_{2n+1}(\mathbb{A})$ inside $\text{Sp}_{2n+m+1}(\mathbb{A})$ given by $t(g,h)$ is not symmetric with respect to $g$ and $h$. We give some details in the case when $i = 1$. Instead of starting with a Fourier coefficient corresponding to the unipotent orbit $((2n-1)^m,1^{2n+m})$ of the group $\text{Sp}_{2n(m+1)}$, we start with a Fourier coefficient corresponding to the orbit $((2n-1)^{2n+m},1^m)$ of the group $\text{Sp}_{2n(2n+m-1)}$. Recall that $m$ is an even number. In both cases the stabilizer contains the same group $\text{Sp}_m(\mathbb{A}) \times \text{Sp}_{2n+m}(\mathbb{A})$. However, for $g \in \text{Sp}_m(\mathbb{A})$, and $h \in \text{Sp}_{2n+m}(\mathbb{A})$, the embedding now is given by $t(h,g)$. As in equation (0.1) we can form the Fourier coefficient

$$F_\psi(E(f_{\Delta(\tau,2n+m-1),s}))(h, g) = \int_{U(F)\backslash U(\mathbb{A})} E(f_{\Delta(\tau,2n+m-1),s}, ut(h,g)) \psi^{-1}_H(u) du.$$ 

Here $U$ is a certain unipotent subgroup of $\text{Sp}_{2n(2n+m-1)}$, and $\psi_H$ is a character of $U(\mathbb{A})$. Pairing this Fourier coefficient against a function $\varphi_\sigma(g)$ as above, we obtain the identity

$$E(f_{\Delta(\tau,m+1),s},\varphi_\sigma)(h) = \int_{\text{Sp}_m(F)\backslash \text{Sp}_m(\mathbb{A})} F_\psi(E(f_{\Delta(\tau,2n+m-1),s}))(h, g) \varphi_\sigma(g) dg.$$ 

In other words, we expect two different ways to obtain the same Eisenstein series $E(f_{\Delta(\tau,m+1),s},\varphi_\sigma)$. First by using Theorem A, and the second by interchanging the roles of $g$ and $h$ and forming a similar construction.

There are also certain applications to the identities given by Theorems A and B, which we intend to study. The first is the study of poles of the various Eisenstein series involved in these identities. Thus, if $E(\Lambda(f_{\Delta(\tau,m+i),s},\varphi_\sigma))$ has a pole at $s = s_0$, then $E(f_{\Delta(\tau,m+i),s})$ must have a pole at $s = s_0$, and then we can relate the leading terms in the Laurent expansions via the identity. We note that the poles of $E(f_{\Delta(\tau,m+i),s})$ are well studied in [JLZ13]. Of course, the pole $s = s_0$ of $E(\Lambda(f_{\Delta(\tau,m+i),s},\varphi_\sigma))$ might come from a pole of the section $\Lambda(f_{\Delta(\tau,m+i),s},\varphi_\sigma)$, and then we have to separate these poles out. Moreover, we hope to use the above two identities to study the Fourier coefficients which the residue representations support. In many constructions which use certain small representations, a crucial ingredient is the knowledge of the top unipotent orbit which these representations have. Even for an Eisenstein series induced from cuspidal data attached to a maximal parabolic subgroup, this information is not known in general. For example, consider the Eisenstein series $E(\Delta(\tau,m+1),s,\varphi_\sigma)$. This Eisenstein series on $\text{Sp}_{2n+m}(\mathbb{A})$ associated with the parabolic induction from $\tau|\det|^s \times \sigma^s$. It is well known that this Eisenstein series has a simple pole if the partial $L$ function $L^S(\tau \times \sigma, s)L^S(\tau, \lambda^2, 2s)$ has a simple pole at $s = 1/2$. It is conjectured, that the unipotent orbit associated with this residue is $((2n + m), 2n)$. See Conjecture (FC) in [GJR04]. We hope to use Theorem A to study this conjecture.

In a forthcoming work of ours, we will use the identities as in Theorem B to analyze the maximal unipotent orbits supported by the two Eisenstein series.
$E(f_{\Delta(t,i+1),s}^\tau)$, and $E(f_{\Delta(t,i+1),s}^\tau)$ at their various poles. For example, given a Fourier coefficient, which is nontrivial on the residue at a given pole $s_0$ of the Eisenstein series on the r.h.s. of (6), and corresponds to a unipotent orbit for $Sp_{2n}$, we get a corresponding unipotent orbit for $Sp_{2n(i+1)}$ supported by the residue at $s_0$ of $E(f_{\Delta(t,i+1),s}^\tau)$. We will also combine both identities of Theorems A, B to find explicit relations of various Rankin-Selberg integrals to those of the doubling method.

In the next section, we will state precisely the main theorems of this paper. Sections 2-4 deal with Theorem A (and its analogues for metaplectic groups and split orthogonal groups), and Sections 5-8 deal with Theorem B (and its analogues).

1. Statement of the main theorems

In this section, we consider Eisenstein series on a split classical group $H$, or a metaplectic double cover of a symplectic group, induced from a Speh representation. We will introduce an explicit kernel function obtained by considering a certain Fourier coefficient of such an Eisenstein series. In the linear case, it turns out to be an automorphic function on a product of two classical groups $L \times G$, where $L$ is a direct factor of the Levi part of a maximal parabolic subgroup of $G$. In case of metaplectic groups, $G$ and $L$ should be replaced by their double covers. Then we pair this kernel function against cusp forms on $L$. Our first main theorem will identify such a kernel integral as an Eisenstein series on $G$. Later on, we will consider another Fourier coefficient, or a Fourier-Jacobi coefficient on the Eisenstein series above (on $H$). It is of the same form as the ones studied in [GRS11] (descent). Viewed as automorphic functions on the group $H'$ stabilizing the character defining the Fourier coefficient, our second main theorem will identify this descent, as a similar type Eisenstein series on $H'$, induced from a similar Speh representation, but of length shorter by 1.

1. The groups

Let $F$ be a number field and $\mathbb{A}$ its ring of Adeles. We will consider symplectic groups $Sp_{2k}$ and split orthogonal groups $SO_k$ over $F$. We will realize these groups as matrix groups in the following standard way. Let $w_k$ denote the $k \times k$ permutation matrix which has 1 along the main anti-diagonal. Then the corresponding matrix algebraic groups are

$$Sp_{2k} = \{ g \in GL_{2k} \mid t^g \left( \begin{array}{cc} -w_k & w_k \\ w_k & -w_k \end{array} \right) \} = \left( \begin{array}{cc} -w_k & w_k \\ w_k & -w_k \end{array} \right),$$

$$SO_k = \{ g \in SL_k \mid t^g w_k g = w_k \}.$$

We will oftentimes denote such a group by $H_\ell$, where $\ell$ is the number of variables of the corresponding anti-symmetric, or symmetric form. We let $H_\ell$ act on the column space over $F$ of dimension $\ell$. We will consider also double covers of symplectic groups over the local fields $F_v$, or over the Adele ring $\mathbb{A}$. We will denote these groups by $Sp_{2k}(F_v)$, $Sp_{2k}(\mathbb{A})$, and realize them using the normalized Rao cocycle, corresponding to the standard Siegel parabolic subgroup ([Rao93]). In order to unify notation, let $\epsilon = 1, 2$. We will consider the groups $H_\ell^{(1)} = H_\ell$ and
the groups $H^{(2)}_i$ only when $H_\ell$ is symplectic. In both cases we will use the notation $H^{(c)}_i$ with the agreement that when $H_\ell$ is orthogonal then $\epsilon = 1$. In the metaplectic case, we apply the language of algebraic groups by saying that the $F_v$ - points of $H^{(2)}_i$ is the group $H^{(2)}_i(F_v)$, and the $A$ - points of $H^{(2)}_i$ is the group $H^{(2)}_i(\hat{A})$.

We fix positive integers $n, m$, and a nonnegative integer $i$. We are mainly interested in the case where $i$ is positive, but the definitions and results make sense also for $i = 0$. As will be seen, the case $i = 0$ corresponds to the generalized doubling method of [CFGK17]. We will consider soon certain Eisenstein series on $H^{(c)}_{2n(m+i)}$. In case this group is symplectic, or metaplectic, we assume that $m$ is even. When convenient, we will shorten our notation by putting $H = H^{(c)}_{2n(m+i)}$ and $r_H = 2n(m+i)$. We let $\delta_H = 1$ when $H$ is orthogonal, and $\delta_H = -1$, when $H$ is symplectic, or metaplectic. When $H$ is orthogonal, and $m$ is even, we will assume that $m \geq 4$. (The reason is that we will soon consider cuspidal representations on $H_{n}(\hat{A})$. The case of split $SO_2$ is problematic, since it doesn’t have nontrivial unipotent radicals.) When $H$ is orthogonal and $m$ is odd, we will allow $m = 1$. As will be clear in the sequel, all arguments work here nicely.

We denote the standard basis of $F^{2n(m+i)}$ by

$$\{e_1, \ldots, e_{n(m+i)}, e_{-n(m+i)}, \ldots, e_{-1}\}.$$ 

Assume that $H$ is linear. For positive integers, $k_1, \ldots, k_\ell$, such that $k_1 + \cdots + k_\ell \leq \frac{rm}{2}$, let $Q_{k_1, \ldots, k_\ell} = Q^{(1)}_{k_1, \ldots, k_\ell} = Q^{H}_{k_1, \ldots, k_\ell}$ denote the standard parabolic subgroup of $H$, whose Levi part is isomorphic to $GL_{k_1} \times \cdots \times GL_{k_\ell} \times H'$, where $H'$ is a classical group of the same type as $H$. Denote the corresponding Levi part by $M_{k_1, \ldots, k_\ell} = M^{H}_{k_1, \ldots, k_\ell}$, and unipotent radical by $U_{k_1, \ldots, k_\ell} = U^{H}_{k_1, \ldots, k_\ell}$. When $k_1 = \cdots = k_\ell = k$, we will simply denote $Q_{k, \ell}, M_{k, \ell}, U_{k, \ell}$. When $H$ is metaplectic, we consider the analogous subgroups $Q^{(2)}_{k_1, \ldots, k_\ell} = Q^{H}_{k_1, \ldots, k_\ell}$, obtained as the inverse image in $H$ of the similar parabolic subgroup of the corresponding symplectic group.

When $H$ is linear and $j \leq \frac{rm}{2}$, denote, for $a \in GL_j$, by $\hat{a}$ the following element of $H$,

$$\hat{a} = \begin{pmatrix} a & I_{r_H-2j} \\ & a^* \end{pmatrix},$$

where $a^* = w_j^t a^{-1} w_j$.

Given positive integers $k_1, \ldots, k_\ell$, such that $k_1 + \cdots + k_\ell = k$, we denote by $P_{k_1, \ldots, k_\ell}$ the standard parabolic subgroup of $GL_k$, consisting of upper triangular block matrices, with diagonal of the form $\text{diag}(g_1, \ldots, g_\ell)$, where $g_i \in GL_{k_i}$, for $1 \leq i \leq \ell$. We denote the corresponding unipotent radical by $V_{k_1, \ldots, k_\ell}$.
Consider the parabolic subgroup $Q_{m^{n-1}}$. The elements of $U_{m^{n-1}}$ have the form

$$(1.1)\quad u = \begin{pmatrix}
I_m & x_1 & \ast & \ast \\
& \ddots & \ast & \ast \\
I_m & x_{n-2} & y_1 & y_2 & y_3 & \ast \\
& & I_m & \ast & \ast \\
& & & I_{2(ni+\lfloor \frac{n-1}{2} \rfloor)} & \ast \\
& & & & I_{2(ni+\lfloor \frac{n-1}{2} \rfloor)+1} \\
& & & & & I_{m} & x_{n-2} & \cdots & \ast \\
& & & & & & \ddots & \ast \\
& & & & & & & & x_1' \\
& & & & & & & & I_m
\end{pmatrix},$$

and, of course, $u$ should lie in $H$. In case $H$ is metaplectic, we identify $U_{m^{n-1}}$ as a subgroup of $H$, over local fields, or Adeles, by the embedding $u \mapsto (u, 1)$. We fix a nontrivial character $\psi$ of $F \backslash \mathbb{A}$. It defines the following character $\psi_H = \psi|_{U_{m^{n-1}(\mathbb{A})}}$ trivial on $U_{m^{n-1}}(F)$. Its value on the element $u$ of the form $(1.1)$, with Adele coordinates, is

$$(1.2)\quad \psi_H(u) = \psi(tr(x_1 + \cdots + x_{n-2})) \psi(tr((y_1, y_2, y_3)A_H),$$

where $A_H$ is the following matrix. When $m$ is even,

$$A_H = \begin{pmatrix}
I_{\frac{n}{2}} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & I_{\frac{n}{2}}
\end{pmatrix}.$$

When $m$ is odd (and hence $H$ is orthogonal),

$$(1.3)\quad A_H = \begin{pmatrix}
I_{\frac{n}{2}} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & I_{\frac{n}{2}}
\end{pmatrix},$$

where the second and fifth block rows of zeroes contain each $ni + \lfloor \frac{n}{2} \rfloor$ rows. We note that $\psi_H$ corresponds to the nilpotent orbit in the Lie algebra of $H$ corresponding to the partition $((2n-1)m, 1^{2ni+m})$. See [MWS7, GRS03].

Assume that $H$ is linear. The stabilizer of $\psi_H$ in $M_{m^{n-1}}(\mathbb{A})$ is the Adele point of an algebraic group over $F$, which we denote by $D = D_{\psi_H}$. It is isomorphic to $H_m \times H_{2ni+m}$. Note that when $m = 2m'$ is even and $H$ is symplectic, this is $\text{Sp}_{2m'} \times \text{Sp}_{2(ni+m')}$. When $H$ is orthogonal, this is the group $\text{SO}_{2m'} \times \text{SO}_{2(ni+m')}$. Assume that $m = 2m' - 1$ is odd, so that $H$ is orthogonal. Then $D$ is isomorphic to $\text{SO}_{2m'-1} \times \text{SO}_{2ni+2m'-1}$. See [CM93, Theorem 6.1.3]. The elements of $D$ are realized as

$$(1.5)\quad t(g, h) = \text{diag}(g^{\Delta_{n-1}}, j(g, h), (g^*)^{\Delta_{n-1}}),\quad g \in H_m, h \in H_{2ni+m},$$

where $g^{\Delta_{n-1}} = \text{diag}(g, \ldots, g)$ ($n - 1$ times), and $j(g, h)$ is as follows.

Assume that $m = 2m'$ is even. Then $g \in H_{2m'}$ and $h \in H_{2(ni+m')}$, and $H$ is either
symplectic, or orthogonal. Write \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \), where \( a, \ldots, d \) are \( m' \times m' \) matrices. Then

\[
(1.6) \quad j(g, h) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad g \in H_{2m'}, h \in H_{2(ni + m')}. \]

Let \( X^\pm_{m'} \) be the subspaces spanned by

\[
e_{\pm}(2m'(n-1)+1), e_{\pm}(2m'(n-1)+2), \ldots, e_{\pm}(2m'(n-1)+m'),
\]

and let \( Y^\pm_{ni+m'} \) be the subspace spanned by

\[
e_{\pm}(2m'(n-1)+m'+1), e_{\pm}(2m'(n-1)+m'+2), \ldots, e_{\pm}(2m'(n-1)+m'+ni).
\]

Denote

\[
X_m = X^+_m + X^-_m, \quad Y_{2ni+m} = Y^+_m + Y^-_{ni+m}.
\]

Then \( j(g, h) \) acts on \( X_m \) according to \( g \), and acts on \( Y_{2ni+m} \) according to \( h \). Assume that \( m = 2m'-1 \) is odd. Then \( H = SO_{2(2m'-1)+1} \); \( g \in SO_{2m'-1} \), \( h \in H_{2ni+2m'-1} \cong SO_{2ni+2m'-1} \). In this case, we will write the elements of \( H_{2ni+2m'-1} \) with respect to the symmetric matrix \( w'_{2ni+2m'-1} \), where

\[
(1.7) \quad w'_{2ni+2m'-1} = \begin{pmatrix} 0 & w_{ni+m'}-1 \\ w_{ni+m'}-1 & 0 \end{pmatrix},
\]

so that

\[
H_{2ni+2m'-1} = \{ g \in SL_{2ni+2m'-1} \mid {}^t gu'_{2ni+2m'-1}g = w'_{2ni+2m'-1} \}.
\]

Write

\[
g = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}, \quad h = \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix},
\]

where the first and third block rows (resp. columns) of \( g \) contain each \( m' \) - 1 rows (resp. columns), and similarly, the first and third block rows (resp. columns) of \( h \) contain each \( ni + m' - 1 \) rows (resp. columns). Then

\[
(1.8) \quad j(g, h) = \begin{pmatrix} a_1 & 0 & \frac{1}{2}b_1 \\ 0 & A_1 & \frac{1}{2}B_1 \\ \frac{1}{2}a_2 & -\frac{1}{2}A_2 & \frac{1}{4}(b_2 + B_2) \end{pmatrix},
\]

\[
\begin{pmatrix} b_1 & 0 \\ -B_1 & C_1 \\ \frac{1}{2}b_2 & B_2 \end{pmatrix},
\]

\[
\begin{pmatrix} a_3 & 0 \\ 0 & A_3 \\ \frac{1}{2}b_3 & B_3 \end{pmatrix},
\]

\[
\begin{pmatrix} 0 & \frac{1}{2}B_3 \\ -B_3 & C_2 \\ \frac{1}{2}c_2 \end{pmatrix},
\]

\[
\begin{pmatrix} c_1 & 0 \\ 0 & \frac{1}{2}c_3 \\ c_2 \end{pmatrix},
\]

Let \( X^\pm_{m'-1} \) be the subspaces spanned by

\[
e_{\pm}(2m'-1)(n-1)+1), e_{\pm}(2m'-1)(n-1)+2), \ldots, e_{\pm}(2m'(n-1)+m'-1),
\]

and let \( Y^\pm_{ni+m'-1} \) be the subspaces spanned by

\[
e_{\pm}(2m'-1)(n-1)+m'), e_{\pm}(2m'-1)(n-1)+m'+1), \ldots, e_{\pm}(2m'(n-1)+ni+2m'-2).
\]

Denote

\[
X_m = X^+_m + F(e_{n(m+i)} + \frac{1}{2}e_{-n(m+i)}) + X^-_{m'-1},
\]

\[
Y_{2ni+m} = Y^+_m + F(e_{n(m+i)} - \frac{1}{2}e_{-n(m+i)}) + Y^-_{ni+m'-1}.
\]

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Then \( j(g, h) \) acts on \( X_m \) according to \( g \), and it acts on \( Y_{2n+i,m} \) according to \( h \). More precisely, for \( x^\pm \in X_m^{\pm 1} \), and \( z \in F \),

\[
j(g, h)(x^+ + z(e_{n(m+i)} + \frac{1}{2}e_{-n(m+i)}) + x^-) = (x')^+ + z'(e_{n(m+i)} + \frac{1}{2}e_{-n(m+i)}) + (x')^-,
\]

where \((x')^\pm \in X_m^{\pm 1} \), \( z' \in F \), such that

\[
g \begin{pmatrix} x^+ \\ z \\ x^- \end{pmatrix} = \begin{pmatrix} (x')^+ \\ z' \\ (x')^- \end{pmatrix},
\]

where we identify \( x^\pm \) with its column vector of \( m' - 1 \) coordinates according to the basis above of \( X_m^{\pm 1} \). Similarly, for \( y^\pm \in Y_{ni+m'm'}^{\pm 1} \), and \( z \in F \),

\[
j(g, h)(y^+ + z(e_{n(m+i)} - \frac{1}{2}e_{-n(m+i)}) + y^-) = (y')^+ + z'(e_{n(m+i)} - \frac{1}{2}e_{-n(m+i)}) + (y')^-,
\]

where \((y')^\pm \in Y_{ni+m'm'}^{\pm 1} \), \( z' \in F \), such that

\[
h \begin{pmatrix} y^+ \\ z \\ y^- \end{pmatrix} = \begin{pmatrix} (y')^+ \\ z' \\ (y')^- \end{pmatrix},
\]

where we identify \( y^\pm \) with its column vector of \( ni + m' - 1 \) coordinates according to the basis above of \( Y_{ni+m'm'}^{\pm 1} \). Assume that \( H \) is metaplectic. Then the analogue of \( (1.5) \) is given by the homomorphism (over \( F_v \))

\[
t^{(2)} : \text{Sp}^{(2)}_{2m'}(F_v) \times \text{Sp}^{(2)}_{2n+i+2m'}(F_v) \to \text{Sp}^{(2)}_{2n(2m'+i)}(F_v)
\]
given by

\[
t^{(2)}((g, \alpha), (h, \beta)) = (t(g, h), \alpha \beta(x_1(g), x_2(h))), \tag{1.9}
\]

where \( t(g, h) \) is given by \( (1.5) \), \( \alpha, \beta = \pm 1 \), and \( x_1, x_2 \) are the Rao \( x \)-functions on \( \text{Sp}_{2m'}(F_v), \text{Sp}_{2n+i+2m'}(F_v) \), respectively. They take values in \( F_v^*/(F_v^*)^2; (x_1(g), x_2(h)) \) is the Hilbert symbol. The kernel of the homomorphism \( (1.9) \)

\[
\{((I_{2m'}, \epsilon), (I_{2n+i+2m'}, \epsilon)) \mid \epsilon = \pm 1\}.
\]

See, for example, [JS07], Sec. 2.1. In order to unify and ease our notations, we will re-denote, when convenient, \( t(g, h), j(g, h) \), in the linear case, by \( t^{(1)}(g, h), j^{(1)}(g, h) \). Similarly, we will re-denote, in the linear case, the stabilizer of \( \psi_H \) by \( D^{(1)} \), and in the metaplectic case, we will denote by \( D^{(2)}(F_v) \) the image of the homomorphism \( (1.9) \). When there is no risk of confusion, we will simply denote \( t(g, h) = t^{(\epsilon)}(g, h), D = D^{(\epsilon)} \), \( \epsilon = 1, 2 \).

2. Eisenstein series

Let \( \tau \) be an irreducible, automorphic, cuspidal representation of \( \text{GL}_n(\mathbb{A}) \). Assume that \( \tau \) has a unitary central character. Let \( \Delta(\tau, m + i) \) be the Speh representation of \( \text{GL}_{n(m+i)}(\mathbb{A}) \). This representations is spanned by the residual Eisenstein series corresponding to the parabolic induction from

\[
\tau | \det \left| \frac{m+i-1}{9} \right| \times \tau | \det \left| \frac{m+i-3}{9} \right| \times \cdots \times \tau | \det \left| \frac{m+i-1}{9} \right|.
\]
We consider the Eisenstein series on $H$, $E(f_{\Delta(\tau,m+i)\gamma^0_{\psi},s})$, corresponding to a smooth, holomorphic section $f_{\Delta(\tau,m+i)\gamma^0_{\psi},s}$ of the parabolic induction

$$
\rho_{\Delta(\tau,m+i)\gamma^0_{\psi},s} = \text{Ind}_{Q_{n(m+i)}(\mathbb{A})}^{H} \Delta(\tau,m+i)_{\gamma^0_{\psi}} \det \cdot^s.
$$

Here, $\epsilon = 1, 2$, according to whether $H$ is linear, or metaplectic. When $\epsilon = 1$, $H$ is linear and $\gamma^0_{\psi} = \gamma_{\psi}$, double cover of $\text{GL}_{n}$. When $\epsilon = 2$, $H_\mathbb{A} = \text{Sp}_{2n(2m'+i)}(\mathbb{A})$, and $\gamma^0_{\psi} = \gamma_{\psi} \circ \det$ is the Weil factor attached to $\psi$, composed with the determinant. This is a character of the double cover of $\text{GL}_{n(2m'+i)}(\mathbb{A})$. We will denote the value at $h$ of our Eisenstein series by $E(f_{\Delta(\tau,m+i)\gamma^0_{\psi},s}, h)$. Consider the Fourier coefficient of $E(f_{\Delta(\tau,m+i)\gamma^0_{\psi},s})$ along $U_{m-1}$ with respect to the character $\psi_H$, and view it as a function on $D_\mathbb{A} = D_{\psi_H}(\mathbb{A})$.

$$
\mathcal{F}_\psi(E(f_{\Delta(\tau,m+i)\gamma^0_{\psi},s}))(g,h) = \int_{U_{m-1}(F) \backslash U_{m-1}(\mathbb{A})} E(f_{\Delta(\tau,m+i)\gamma^0_{\psi},s}, ut(g,h))\psi_H^{-1}(u)du,
$$

(1.10)

where $g \in H^e_{m}(\mathbb{A})$, $h \in H^{e}_{2ni+m}(\mathbb{A})$. Since $D_\mathbb{A}$ stabilizes $\psi_H$, the function (1.10) is automorphic on $D_\mathbb{A}$.

Let $\sigma$ be an irreducible, automorphic, cuspidal representation of $H^e_{m}(\mathbb{A})$. One of the main objects of study in this paper is the following kernel integral,

$$
\mathcal{E}(f_{\Delta(\tau,m+i)\gamma^0_{\psi},s}, \varphi_{\sigma})(h) = \int_{H^e_{m}(F) \backslash H^e_{m}(\mathbb{A})} \mathcal{F}_\psi(E(f_{\Delta(\tau,m+i)\gamma^0_{\psi},s}))((g,h)\varphi_{\sigma}(g))dg,
$$

where $\varphi_{\sigma}$ is in the space of $\sigma$, and $h \in H_{2ni+m}^{e}(\mathbb{A})$. This is an automorphic function on $H_{2ni+m}^{e}(\mathbb{A})$. Note the precise form of (1.11), in the metaplectic case.

$$
\mathcal{E}(f_{\Delta(\tau,m+i)\gamma^0_{\psi},s}, \varphi_{\sigma})(h, \beta) =
$$

$$
\int_{\text{Sp}_{2m'}(F) \backslash \text{Sp}_{2m'}(\mathbb{A})} \mathcal{F}_\psi(E(f_{\Delta(\tau,m+i)\gamma^0_{\psi},s}))((g,1), (h, \beta)\varphi_{\sigma}((g,1)))dg.
$$

(1.12)

Our first main goal is to identify $\mathcal{E}(f_{\Delta(\tau,m+i)\gamma^0_{\psi},s}, \varphi_{\sigma})$ as an Eisenstein series on $H_{2ni+m}^{e}(\mathbb{A})$, parabolically induced from $\Delta(\tau,i)\gamma^0_{\psi} \det \cdot^s \times \sigma^t$, where $\sigma^t$ is the following representation obtained from $\sigma$. Assume, first, that $H$ is linear. Then $\sigma^t$ is the following outer conjugation. Let

$$
J_0 = \begin{pmatrix} I_m & I_{m'} \\ -\delta_H I_{m'} & I_{m'} \end{pmatrix}
$$

and define, for $b \in H_m(\mathbb{A})$,

$$
b' = J_0^{-1}bJ_0, \quad \sigma^t(b) = \sigma(b').
$$

(1.13)

In the metaplectic case, let $m = 2m'$, and consider the matrix $J_0$ as above. Note that now,

$$
J_0 = \begin{pmatrix} I_{m'} & I_{m'} \\ 0 & 1 \end{pmatrix}.
$$
and consider, as in \((1.13)\), the automorphism of \(H_m = \text{Sp}_{2m'}\) given by \(b \mapsto b' = J_{0}^{-1}bJ_{0}\). We will soon lift this automorphism, over each local field, to \(\text{Sp}_{2m'}(F_v)\).

We use the same notation, as in \((1.13)\), and similarly define \(\sigma'\). We will show in Sec. 3 that, for \(Re(s)\) sufficiently large,

\[
(1.14) \quad \mathcal{E}(f_{\Delta(\tau, m+i)\gamma_{\psi}^{(\tau, \sigma)}, \rho})(h) = \sum_{h' \in Q_n(F) \setminus H_{2n+i+m}(F)} \Lambda(f_{\Delta(\tau, m+i)\gamma_{\psi}^{(\tau, \sigma)}, \rho})(h'h),
\]

where

\[
\Lambda(f_{\Delta(\tau, m+i)\gamma_{\psi}^{(\tau, \sigma)}, \rho})(h) = \int_{H_m(\mathbb{A})} \varphi_{\omega}(g) d\mathcal{U}(\mathbb{A}) f_{\Delta(\tau, m+i)\gamma_{\psi}^{(\tau, \sigma)}, \rho}(\delta_0 u t(g, h)) \psi^{-1}_H(u) dudg.
\]

Here, \(U'_{n(n-1)}\) is a certain subgroup of \(U_{n(n-1)}\), \(\delta_0\) is a certain element in \(H(F)\), and the upper \(\psi\) on the section denotes a composition of the section with a Fourier coefficient on \(\Delta(\tau, m + i)\). This Fourier coefficient is along \(V_{n_1, n_2}\), with respect to the character

\[
\begin{pmatrix}
I_{n_1} & * & * & \cdots & * \\
I_m & x_1 & * & \cdots & * \\
I_m & x_2 & \cdots & * \\
& \ddots & & \ddots & \ddots \\
I_m & x_{n-1} & & * & I_m
\end{pmatrix}
\rightarrow \psi(tr(x_1 + x_2 + \cdots + x_{n-1})).
\]

We prove in Sec. 3 that \(\Lambda(f_{\Delta(\tau, m+i)\gamma_{\psi}^{(\tau, \sigma)}, \rho})(h)\) is a smooth, meromorphic section of

\[
(1.16) \quad \text{Ind}_{Q_{n+1}(\mathbb{A})}^{H_{2n+i+m}(\mathbb{A})} \Delta(\tau, i)\gamma_{\psi}^{(\tau, \sigma)} | \det - \cdot |^s \times \sigma'.
\]

Note the case \(i = 0\). Here, in \((1.14)\) there is no summation. It reads

\[
\mathcal{E}(f_{\Delta(\tau, m)\gamma_{\psi}^{(\tau, \sigma)}, \rho})(h) = \Lambda(f_{\Delta(\tau, m)\gamma_{\psi}^{(\tau, \sigma)}, \rho})(h) = \int_{H_m(\mathbb{A})} \varphi_{\omega}(g) d\mathcal{U}(\mathbb{A}) f_{\Delta(\tau, m)\gamma_{\psi}^{(\tau, \sigma)}, \rho}(\delta_0 u t(g, h)) \psi^{-1}_H(u) dudg,
\]

and \((1.16)\) says that \(\Lambda(f_{\Delta(\tau, m)\gamma_{\psi}^{(\tau, \sigma)}, \rho})(h)\) is smooth, meromorphic and takes values in \(\sigma'\). Let us write the precise form of \(\sigma'\) in \((1.16)\) in the metaplectic case. For this, we first write the lift to \(\text{Sp}_{m}(\mathbb{A})\) of the automorphism \(b \mapsto b'\) of \(\text{Sp}_{m}(\mathbb{A})\), \(m = 2m'\). Let

\[
\bar{u}_0 = \left(\begin{array}{c}
I_m \\
I_m
\end{array}\right) \in \text{Sp}_{4m'}(F).
\]

We denote by \(c\) the normalized Rao cocycle, without mentioning the rank of the corresponding symplectic group. This will always be clear from the context. Define, for each local field \(F_v\), for \((b, \epsilon) \in \text{Sp}_{m}(F_v)\),

\[
(b, \epsilon)^{c} = (b', \epsilon c(\bar{u}_0, j(b'), b))(x(b'), x(b)).
\]

Here, \(x\) is the Rao \(x\)-function on \(\text{Sp}_{m}(F_v)\). Note that when we write

\[
b = \left(\begin{array}{c}
b_1 \\
b_2 \\
b_3 \\
b_4
\end{array}\right),
\]

and consider, in \((1.13)\), the automorphism of \(H_m = \text{Sp}_{2m'}\) given by \(b \mapsto b' = J_{0}^{-1}bJ_{0}\). We will soon lift this automorphism, over each local field, to \(\text{Sp}_{2m'}(F_v)\).
with \( b_i \) being \( m' \times m' \) blocks, then
\[
j(b', b) = \begin{pmatrix} b_4 & b_3 \\ b_1 & b_2 \\ b_3 & b_4 \\ b_2 & b_1 \end{pmatrix}.
\]

It is an exercise to check that (1.17) defines an automorphism. For \((b, \epsilon) \in \text{Sp}_{2m'}(\mathbb{A})\), let \(((b_v, \epsilon_v))_v\) be in its inverse image in the restricted tensor product of the groups \(\text{Sp}^{(2)}_{2m'}(F_v)\). Then we keep denoting by \((b, \epsilon)'\), the element in \(\text{Sp}^{(2)}_{2m'}(\mathbb{A})\), which is the image of \(((b_v, \epsilon_v)')_v\). We define
\[
\sigma'((b, \epsilon)) = \sigma((b, \epsilon')).
\]

The inducing representation in (1.16) in the metaplectic case is
\[
(\begin{pmatrix} a & b \\ b & a^* \end{pmatrix}, \alpha) \mapsto \alpha \gamma_{\psi}(\text{det}(a))((\text{det}(a), x(b))| \text{det}(a)|^\alpha \Delta(\tau,i)(a) \otimes \sigma'((b,1)),
\]
for \( a \in \text{GL}_{m'}(\mathbb{A}), b \in \text{Sp}_{2m'}(\mathbb{A}), \alpha = \pm 1.\)

Our identity can be formulated in terms of normalized Eisenstein series. Let \( S \) be a finite set of places of \( F \), containing the infinite ones, outside which \( \sigma, \tau \) and \( \psi \) are unramified. Assume that our section is decomposable, unramified outside \( S \), and normalized in a way which we don’t specify now. Let us multiply our given Eisenstein series on \( H(\mathbb{A}) \) by its normalizing factor outside \( S \), \( d_{\tau}^{H+S}(s) \). Note that
\[
(1.19)
\]
\[
d_{\tau}^{\text{Sp}_{2m}(2j+1)}(s) = L^S(\tau,s+j+1)\prod_{k=1}^{j+1} L^S(\tau,\wedge^2,2s+2k-1)\prod_{k=1}^{j} L^S(\tau,\text{sym}^2,2s+2k);
\]
\[
(1.20)
\]
\[
d_{\tau}^{\text{Sp}_{4m}(2j)}(s) = L^S(\tau,s+j+\frac{1}{2})\prod_{k=1}^{j} L^S(\tau,\wedge^2,2s+2k)L^S(\tau,\text{sym}^2,2s+2k-1);
\]
\[
(1.21)
\]
\[
d_{\tau}^{\text{Sp}_{4m}(2j+1)}(s) = \prod_{k=1}^{j} L^S(\tau,\wedge^2,2s+2k)\prod_{k=1}^{j+1} L^S(\tau,\text{sym}^2,2s+2k-1);
\]
\[
(1.22)
\]
\[
d_{\tau}^{\text{SO}_{4m}(2j)}(s) = \prod_{k=1}^{j} L^S(\tau,\wedge^2,2s+2k-1)L^S(\tau,\text{sym}^2,2s+2k);
\]
\[
(1.23)
\]
\[
d_{\tau}^{\text{SO}_{4m}(2j+1)}(s) = \prod_{k=1}^{j+1} L^S(\tau,\wedge^2,2s+2k-1)\prod_{k=1}^{j} L^S(\tau,\text{sym}^2,2s+2k);
\]
\[
(1.24)
\]
\[
d_{\tau}^{\text{SO}_{4m}}(s) = \prod_{k=1}^{j} L^S(\tau,\wedge^2,2s+2k)L^S(\tau,\text{sym}^2,2s+2k-1).
\]

Denote
\[
E_{\psi}(f_{\Delta(\tau,m+i)}^{(\epsilon)}(\tau),h) = d_{\tau}^{H+S}(s)E(f_{\Delta(\tau,m+i)}^{(\epsilon)}(\tau),s,h).
\]
This is a normalized Eisenstein series.
Let \( d_{\sigma,\tau}^{\text{SO}(2j+1)+m}(s) \) be the normalizing factor, outside \( S \), corresponding to the global induced representation \( \text{1.16} \). Explicitly,

\[
d_{\sigma,\tau}^{\text{Sp}(2n+j+m),S}(s) = L^S(\sigma \times \tau, s + j + \frac{1}{2}) \prod_{k=1}^{j+1} L^S(\tau, \wedge^2, 2s + 2k - 1) \prod_{k=1}^{j} L^S(\tau, \text{sym}^2, 2s + 2k);
\]

\[
d_{\sigma,\tau}^{\text{Sp}(2n+j+m),S}(s) = L^S(\sigma \times \tau, s + j + \frac{1}{2}) \prod_{k=1}^{j} L^S(\tau, \wedge^2, 2s + 2k) L^S(\tau, \text{sym}^2, 2s + 2k - 1);
\]

\[
d_{\sigma,\tau}^{\text{Sp}(2n+j+m),S}(s) = L^S(\sigma \times \tau, s + j + \frac{1}{2}) \prod_{k=1}^{j} L^S(\tau, \wedge^2, 2s + 2k - 1) L^S(\tau, \text{sym}^2, 2s + 2k);
\]

\[
d_{\sigma,\tau}^{\text{SO}(2n+j+m),S}(s) = L^S(\sigma \times \tau, s + j + \frac{1}{2}) \prod_{k=1}^{j} L^S(\tau, \wedge^2, 2s + 2k) L^S(\tau, \text{sym}^2, 2s + 2k - 1);
\]

\[
d_{\sigma,\tau}^{\text{SO}(2n+j+m),S}(s) = L^S(\sigma \times \tau, s + j + \frac{1}{2}) \prod_{k=1}^{j} L^S(\tau, \wedge^2, 2s + 2k - 1) L^S(\tau, \text{sym}^2, 2s + 2k);
\]

Recall that in the symplectic, or metaplectic cases \( m \) is even. Also, in the metaplectic case, \( \sigma \) is a genuine representation of \( \text{Sp}^{(2)}(\mathbb{A}) \), and \( L^S_{\psi}(\sigma \times \tau, s) \) depends on \( \psi \). We will denote in general \( L^S_{\psi}(\sigma \times \tau, s) \), so that when \( \epsilon = 1 \), this is just \( L^S(\sigma \times \tau, s) \), and when \( \epsilon = 2 \), this is \( L^S_{\psi}(\sigma \times \tau, s) \). Let

\[
\lambda_s(f_{\Delta,\tau}^{(m+1)\gamma_{\psi}^{(\epsilon)},s}, \psi) = \frac{dH_{S}(s)}{d_{\tau}^{(m+1)}} \Lambda(f_{\Delta,\tau}^{(m+1)\gamma_{\psi}^{(\epsilon)},s}, \psi).
\]

We prove in Sec. 4 that \( \lambda(f_{\Delta,\tau}^{(m+1)\gamma_{\psi}^{(\epsilon)},s}, \psi) \) is decomposable, and it is normalized, so that its component at \( v \) outside \( S \) is such that its value at 1 is a pre-chosen
unramified vector in \( \Delta(\tau, i) \otimes \sigma_v \). Consider then (1.11) in a normalized form

\[
E^*_S(f_{\Delta(\tau, m+i)} \gamma^{(s)}_v, s', \varphi_\sigma)(h) = \int_{H_m(F) \backslash H_m(\mathbb{A})} \mathcal{F}_\psi(E^*_S(f_{\Delta(\tau, m+i)} \gamma^{(s)}_v, s)(g, h)) \varphi_\sigma(g) dg.
\]

Then (1.14) says that for \( \text{Re}(s) \) sufficiently large,

\[
E^*_S(f_{\Delta(\tau, m+i)} \gamma^{(s)}_v, s', \varphi_\sigma)(h) = \sum_{h' \in Q_n(F) \backslash H_{2n+m}(F)} \lambda_S(f_{\Delta(\tau, m+i)} \gamma^{(s)}_v, s', \varphi_\sigma)(h'h).
\]

The right hand side of (1.34) is the normalized Eisenstein series (outside \( S \))

\[
E^*_S(\lambda_S(f_{\Delta(\tau, m+i)} \gamma^{(s)}_v, s', \varphi_\sigma)) \text{ on } H_{2n+m}^c(\mathbb{A}) \text{ corresponding to the normalized section }
\]

\[
\lambda_S(f_{\Delta(\tau, m+i)} \gamma^{(s)}_v, s', \varphi_\sigma(\text{det} \cdot |^s \times \sigma^t). \text{ Our first main theorem then is the identity }
\]

**Theorem 1.1.**

\[
E^*_S(f_{\Delta(\tau, m+i)} \gamma^{(s)}_v, s', \varphi_\sigma) = E^*_S(\lambda_S(f_{\Delta(\tau, m+i)} \gamma^{(s)}_v, s', \varphi_\sigma)).
\]

The left hand side is given by the kernel integral (1.11), normalized by \( d^H_\lambda, S(s) \). The right hand side is the normalized Eisenstein series on \( H_{2n+m}^c(\mathbb{A}) \) corresponding to the section \( \lambda(f_{\Delta(\tau, m+i)} \gamma^{(s)}_v, s', \varphi_\sigma) \) of \( \text{Ind}_{Q_n^c(\mathbb{A})}^{H_{2n+m}^c(\mathbb{A})} \Delta(\tau, i) \gamma^{(s)}_v \). This section is spherical and normalized outside \( S \).

The proof of this theorem will occupy the next three sections. As we remarked in the introduction, we allow the case \( n = 1 \), where \( \tau \) is simply a character of \( F^* \backslash \mathbb{A}^* \),

\[
\Delta(\tau, m+i) = \tau \circ \text{det}_{\text{GL}_{m+i}} \text{ and } \rho_{\Delta(\tau, m+i)}^{(s)} = \text{Ind}_{Q_n^c(\mathbb{A})}^{H_{2n+m}^c(\mathbb{A})} \tau(\text{det} \cdot |^s) \gamma^{(s)}_v \text{ det } |^s. \]

Since now \( U_{m+n-1} \) is trivial, the function (1.10) is the restriction of the Eisenstein series \( E(f_{\text{det}^{(c)}(\gamma^{(s)}_v, s')}) \) to \( H_{2n}^c(\mathbb{A}) \) \( \times H_{2n+m}^c(\mathbb{A}) \). The Eisenstein series obtained from (1.11) corresponds to a section of the parabolic induction from \( (\tau \circ \text{det}_{\text{GL}_n}) \gamma^{(s)}_v \text{ det } |^s \times \sigma \) to \( H_{2n+m}^c(\mathbb{A}) \). This section is the analytic continuation of the following integral, which converges absolutely for \( \text{Re}(s) \) sufficiently large,

\[
\Lambda(f_{\text{det}^{(c)}(\gamma^{(s)}_v, s')}) \text{ det } |^s \text{ to } H_{2n}^c(\mathbb{A}) \text{ \( \times \) } H_{2n+m}^c(\mathbb{A}) \text{. The integral (1.35) was considered in [GPSR97], Sec. 1, for (arbitrary) orthogonal groups and } i = 1 \text{ and generalized by Moeglin in [M97] for any } i \text{ and } H \text{ linear.}
\]

Our second main theorem states roughly that the "descent" of the Eisenstein series parabolically induced from \( \Delta(\tau, i+1) \text{ det } |^s \) is an Eisenstein series parabolically induced from \( \Delta(\tau, i) \text{ det } |^s \). More precisely, let \( H \) be one of the groups \( SO_{2n(i+1)} \), \( SO_{2n(i+1)+1} \), \( Sp_{2n(i+1)} \), \( Sp_{2n(i+1)}^{(2)} \). Consider the Eisenstein series on \( H_k \),

\[
E(f_{\Delta(\tau, i+1)} \gamma^{(s)}_v, s'). \text{ Let } j_0 = 0, \text{ when } H \text{ is odd orthogonal, and } j_0 = 1, \text{ when } H \text{ is even orthogonal, symplectic, or metaplectic. Consider the Fourier coefficient along}
\]
\( U_{1^n - j_0} \) with respect to the character (on Adele points)

\[
\psi_{n-j_0} \left( \begin{pmatrix} z & x & y \\ I_{2n_i+1+j_0} & x' \\ z \end{pmatrix} \right) = \psi \left( \sum_{r=1}^{n-j_0-1} z_{r,r+1} \right) \psi(x_{n-j_0} \cdot e_0).
\]

Here, \( z \in Z_{n-j_0}(A) \), where, in general, \( Z_k \) denotes the standard maximal unipotent subgroup of \( GL_k \); \( x_{n-j_0} \) is the last row of \( x \) and \( e_0 \) is the following column vector in \( F^{2n_i+1+j_0} \). When \( H \) is even orthogonal,

\[
e_0 = \begin{pmatrix} 0_{n_i} \\ 1 \\ \frac{1}{2} \\ 0_{n_i} \end{pmatrix}.
\]

When \( H \) is odd orthogonal,

\[
e_0 = \begin{pmatrix} 0_{n_i} \\ 1 \\ 0_{n_i} \end{pmatrix}.
\]

When \( H \) is symplectic, or metaplectic,

\[
e_0 = \begin{pmatrix} 1 \\ 0_{2n_i+1} \end{pmatrix}.
\]

This character is stabilized by the Adele points of the following subgroup of elements:

\[
t(h) = \begin{pmatrix} I_{n-j_0} \\ h \\ I_{n-j_0} \end{pmatrix} \in H_{2n(i+1)+1-j_0}, \ h e_0 = e_0.
\]

In case \( H \) is orthogonal, this is isomorphic to \( H_{2n_i+j_0} \). In case \( H \) is symplectic, we get the semi-direct product of \( H_{2n_i}^{(s)} \) and the following unipotent subgroup, isomorphic to the Heisenberg group in \( 2n_i + 1 \) variables \( H_{2n_i+1} \). It consist of the elements \( t(u) \), where

\[
u = \begin{pmatrix} 1 & x & t \\ I_{2n_i} & x' \\ 1 \end{pmatrix} \in Sp_{2n_i+2}.
\]

When \( H \) is orthogonal, define, for \( h \in H_{2n_i+j_0}(A) \),

\[
D_{\psi,ni}(E(f_{\Delta(\tau,i+1),s}))(h) = \int_{U_{1^n-j_0}(F) \backslash U_{1^n-j_0}(A)} \psi(x_{n-j_0} - 1)(u) \psi_{n-j_0}(u) du.
\]

This is an automorphic function on \( H_{2n_i+j_0}(A) \). It is defined by applying a Bessel coefficient to \( E(f_{\Delta(\tau,i+1),s}) \). The nilpotent orbit corresponding to this Fourier coefficient is associated to the partition \( (2n-1,1^{2n_i+1}) \), when \( H \) is even orthogonal, and to the partition \( (2n+1,1^{2n_i}) \), when \( H \) is odd orthogonal.

Assume that \( H \) is symplectic, or metaplectic. In this case, we consider the Fourier-Jacobi coefficient on \( E(f_{\Delta(\tau,i+1),s}^{(sp)}) \), corresponding to the partition \( (2n,1^{2n_i}) \). Thus, for \( H = Sp_{2n(i+1)} \), we define, for \( \tilde{h} \in Sp_{2n_i}^2(A) \), projecting to \( h \in Sp_{2n_i}(A) \).

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Fourier coefficients corresponding to nilpotent orbits.

\[ D_{\psi,n}(E(f_{\Delta(\tau,i+1),s})(h)) = \int_{U_1(F)\backslash U_1(A)} E(f_{\Delta(\tau,i+1),s}, ut(h))\psi_{n-1}^{-1}(u)\theta_\psi^{-1}(j(u)h)du, \]

where \( \theta_\psi^{-1} \) is the theta series corresponding to \( \phi \in \mathcal{S}(\mathbb{A}^n) \), and \( j(u) \) denotes the projection of \( U_1 \) onto \( \mathcal{H}_{2ni+1} \). For \( H = \text{Sp}^{(2)}_{2n+1} \), we define, for \( h \in \text{Sp}_{2n+1}(A) \), which is the projection of \( \tilde{h} \in \text{Sp}^{(2)}_{2n+1}(A) \),

\[ D_{\psi,n}(E(f_{\Delta(\tau,i+1),\gamma\psi,s})(h)) = \int_{U_1(F)\backslash U_1(A)} E(f_{\Delta(\tau,i+1),\gamma\psi,s}, ut(h))\psi_{n-1}^{-1}(u)\theta_\psi^{-1}(j(u)h)du. \]

We regard \( D_{\psi,n}(E(f_{\Delta(\tau,i+1),\gamma\psi,s})) \) as an automorphic function on \( \text{Sp}^{(2)}_{2n+1}(A) \), or \( \text{Sp}_{2n+1}(A) \), respectively.

When \( H \) is even orthogonal, the fact that \( \text{lcm}^{(2)}_{\psi} = 1 \) is an Eisenstein series on \( SO_{2ni+1}(A) \), parabolically induced from \( \gamma_{\psi} \) (resp. \( \gamma_{\psi}^{-1} \)). They both define smooth meromorphic sections of \( \text{Ind}_{Q_n(A)}^{SO_{2ni+1}(A)}(\Delta(\tau,i)) \). Moreover, \( E_{\psi}(E_{\Delta(\tau,i+1),s}) \) is the sum of the two normalized (outside \( S \)) Eisenstein series on \( SO_{2ni}(A) \), corresponding to the above two sections.

**Theorem 1.2.** Assume that \( H \) is odd orthogonal. Then \( D_{\psi,n}(E(f_{\Delta(\tau,i+1),s})) \) is a sum of two Eisenstein series on \( SO_{2ni}(A) \). The corresponding sections \( \Lambda^{\pm}(f_{\Delta(\tau,i+1),s}) \) are given by explicit unipotent Adelic integrations of \( f_{\Delta(\tau,i+1),s} \), similar to \( (1.37) \) (without the \( dg \)-integration). They both define smooth meromorphic sections of \( \text{Ind}_{Q_n(A)}^{SO_{2ni}(A)}(\Delta(\tau,i)) \) \( \gamma^{-1} \). Moreover, \( E_{\psi}(E_{\Delta(\tau,i+1),s}) \) is the sum of the two normalized (outside \( S \)) Eisenstein series on \( SO_{2ni}(A) \), corresponding to the above two sections.

**Theorem 1.3.** Assume that \( H \) is symplectic (resp. metaplectic). Then \( D_{\psi,n}(E(f_{\Delta(\tau,i+1),\gamma\psi,s})) \) is an Eisenstein series on \( \text{Sp}^{(2)}_{2n+1}(A) \) (resp. \( \text{Sp}_{2n+1}(A) \)). The corresponding section \( \Lambda(f_{\Delta(\tau,i+1),\gamma\psi,s}, \phi) \) is given by an explicit unipotent Adelic integration of \( f_{\Delta(\tau,i+1),\gamma\psi,s} \). It defines a smooth meromorphic section of \( \text{Ind}_{Q_n(A)}^{\text{Sp}^{(2)}_{2n+1}(A)}(\Delta(\tau,i)) \) \( \gamma^{-1} \) \( \phi \) (resp. \( \text{Ind}_{Q_n(A)}^{\text{Sp}_{2n+1}(A)}(\Delta(\tau,i)) \) \( \gamma^{-1} \) \( \phi \)). Moreover, \( E_{\psi}(E_{\Delta(\tau,i+1),\gamma\psi,s}) \) is the normalized (outside \( S \)) Eisenstein series on \( \text{Sp}^{(2)}_{2n+1}(A) \) (resp. \( \text{Sp}_{2n+1}(A) \)), corresponding to the section \( \Lambda(f_{\Delta(\tau,i+1),\gamma\psi,s}, \phi) \).

As we remarked in the introduction, we allow the case \( n = 1 \), where \( \tau \) is a character of \( F^* \backslash A^* \), \( \Delta(\tau,i+1) = \tau \circ \det_{GL_{i+1}} \). Note that when \( H = SO_{2i+2} \), \( (1.37) \) is simply the restriction of \( \Delta(\tau,i+1) \) to \( SO_{2i+1}(A) \) and then \( D_{\psi}(E(f_{\Delta(\tau,i+1),s})) \) is an Eisenstein series on \( SO_{2i+1}(A) \) corresponding to a section (explicit) of the parabolic induction from \( \tau \circ \det_{GL_i} \). When \( n = 1 \), Theorem 1.3 for \( H \) is symplectic is a special case of a theorem of Ikeda. See [194].

In the sequel, we will need the following proposition. Recall again the notion of Fourier coefficients corresponding to nilpotent orbits.
Proposition 1.4. Let $O$ be a nilpotent orbit in $gl_n$, corresponding to a partition $P$ of $n$. Assume that $\Delta(\tau, j)$ admits a nontrivial Fourier coefficient with respect to $O$. Then

$$P \leq (n^2).$$

Moreover, let $O(\Delta(\tau, j))$ denote the set of maximal nilpotent orbits, whose corresponding Fourier coefficients are supported by $\Delta(\tau, j)$. Then

$$O(\Delta(\tau, j)) = (n^2).$$

This is Prop. 5.3 in [G06]. See [JL13] for a detailed proof.

2. Analysis of the Fourier coefficient $F_\psi(E(f_{\Delta(\tau, m+i)}(\gamma^{(r)}, s^*)^s))(g, h)$

In this section, we unfold the Fourier coefficient $F_\psi(E(f_{\Delta(\tau, m+i)}(\gamma^{(r)}, s^*)^s))(g, h)$.

We will analyze the contributions to the Fourier coefficient of the various double cosets in $Q_{n(m+i)} \setminus H/DU_{m(n-1)}$, and we will show that except the open double coset all others contribute zero.

Assume that $\text{Re}(s)$ is sufficiently large. Then, for $x \in H_A$, our Eisenstein series is given by the following absolutely convergent series

$$E(f_{\Delta(\tau, m+i)}(\gamma^{(r)}, s^*)^s)(\gamma x).$$

Here, we abbreviated $Q_{n(m+i)}(F)$ to $Q_{n(m+i)}$ etc. In the metaplectic case, we know that we can embed $Sp_{2n(m+i)}(F)$ as a subgroup of $Sp_{2n(2m+i)}(A)$ by $\gamma \mapsto (\gamma, 1)$, and we identify $\gamma$ and $(\gamma, 1)$. We factor the summation modulo $Q_{m(n-1)}$ from the right.

$$E(f_{\Delta(\tau, m+i)}(\gamma^{(r)}, s^*)^s, x) = \sum_{\gamma \in Q_{n(m+i)} \setminus H} \sum_{\alpha \in Q_{n(m+i)} \setminus Q_{m(n-1)}} f_{\Delta(\tau, m+i)}(\gamma^{(r)}, s^*)^s(\alpha \gamma x),$$

where the second summation is over $\gamma \in Q_{m(n-1)} \cap \alpha^{-1}Q_{n(m+i)}A \setminus Q_{m(n-1)}$. The representatives $\alpha$ in (2.1) are described in [GRS11], Sec. 4.2. They are parameterized by integers $0 \leq r \leq m(n-1)$,

$$\alpha_r = \begin{pmatrix} I_r & \alpha_r' \\ \alpha_r' & I_r \end{pmatrix},$$

where

$$\alpha_r' = \begin{pmatrix} 0 & I_{m+i} & 0 & 0 \\ 0 & 0 & I_{m(n-1)-r} & 0 \\ \delta H I_{m(n-1)-r} & 0 & 0 & 0 \\ 0 & 0 & I_{m+i} & 0 \end{pmatrix} \omega_H^{m(n-1)-r}.$$

Here, $\omega_H = I$, unless $H$ is orthogonal, where we need its presence to guarantee that the determinant of $\alpha_r$ is 1. We will choose $\omega_H$ (when $H$ is orthogonal), as follows. When $m$ is even,

$$\omega_H = \text{diag}(I_{(m+i)n-1}, w_2, I_{(m+i)n-1}).$$
Recall that \( w_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \). When \( m \) is odd,
\[
\omega_H = \text{diag}(I_{(m+i)n-1}, \begin{pmatrix} 1/2 \\ 2 \end{pmatrix}, I_{(m+i)n-1}).
\]
Since \( H \) will always be clear from the context, we will write \( \omega_0 \) instead of \( \omega_H \). We will use the same notation \( \omega_0 \) for the similar element in a group of the same type as \( H \), but in a different number of variables, that is when we replace \( I_{(m+i)n-1} \) by an identity matrix of a different size. The elements of \( Q_{m(n-1)} \cap \alpha_r^{-1}Q^{(r)}_{n(m+i)} \), which we will denote, for short, by \( Q^{(r)} \), have the form
\[
\begin{pmatrix}
  a_1 & a_2 & y_1 & z_1 & z_2 \\
  0 & a_4 & 0 & y_2 & z_4 \\
  c & v & y'_1 & y'_2 \\
  c^* & 0 & y'_1 & y'_2 \\
  a_4 & a_2 & a_2 & a_1 \\
\end{pmatrix}^{w_0^{m(n-1)-r}} \in H,
\]
where \( a_1 \in \text{GL}_r, \ a_4 \in \text{GL}_{m(n-1)-r}, \ c \in \text{GL}_{m+n} \). The element \((2.3)\) is conjugated by \( \alpha_r \) to the element
\[
\begin{pmatrix}
  a_1 & y_1 & z_1 & a_2 & y_2 & z_2 \\
  c & y'_1 & 0 & v & y'_2 \\
  a_4 & a_2 & a_2 & a_1 \\
\end{pmatrix}^{\text{def}}.
\]
Apply the Fourier coefficient \((2.4)\). Then by \((2.1)\)
\[
\mathcal{F}_\psi(E(f_{\Delta(\tau,m+i)\gamma_\psi,s}))(x) =
\sum_{r=0}^{m(n-1)} \int_{U_{m-1}(F)\backslash U_{m-1}(A)} \sum_{\gamma \in Q^{(r)}(m(n-1))} f_{\Delta(\tau,m+i)\gamma_\psi,s}(\alpha_r \gamma uz) \psi^{-1}_H(u) \ du.
\]
Note that, according to our notation, when \( H \) is metaplectic, \((2.5)\) has the following form.
\[
\mathcal{F}_\psi(E(f_{\Delta(\tau,m+i)\gamma_\psi,s}))(x, \beta) =
\sum_{r=0}^{2m'(n-1)} \int_{\gamma \in Q^{(r)}(2m'(n-1))} f_{\Delta(\tau,2m' + i)\gamma_\psi,s}(\alpha_r \gamma, (u, 1)(x, \beta)) \psi^{-1}_H(u) \ du,
\]
where \( x \in \text{Sp}_{2m'(2m'+i)}(A), \ \beta = \pm 1, \) and the \( du \)-integration is over \( U_{2m' -1}(F) \backslash U_{2m' -1}(A) \). Note that in \((2.6)\), \( (\alpha_r \gamma, 1) = (\alpha_r, 1)\gamma, 1) \). Denote the summand corresponding to \( r \) in \((2.5)\) by \( \mathcal{F}_{\psi,r}(f_{\Delta(\tau,m+i)\gamma_\psi,s}) \).

**Theorem 2.1.** For all \( 0 < r \leq m(n-1) \), and all smooth holomorphic sections \( f_{\Delta(\tau,m+i)\gamma_\psi,s} \),
\[
\mathcal{F}_{\psi,r}(f_{\Delta(\tau,m+i)\gamma_\psi,s}) = 0.
\]
and we specify the forms of each block. First, for

\( n = 4 \) and \( \gamma = 4 \), we may take the elements

\( \theta = \left( \begin{array}{ccc} A & B & C \\ h & B' & A^* \end{array} \right) \) \in H, A \in \text{GL}(m,n-1)(F), \)

and we specify the forms of each block. First, \( h \in Q_{m,n-1}^{(r)} \), that is \( h \) lies in the

\( \omega_0^{m(n-1)-r} \)-conjugate of the standard parabolic subgroup of \( H_2(n,m) \), whose Levi
part is isomorphic to $GL_{ni+m}$. Next,

$$A = \begin{pmatrix} g_1 & x_{1,2} & \cdots & x_{1,n-1} \\ g_2 & \cdots & & x_{2,n-1} \\ \vdots & & & \vdots \\ g_{n-2} & x_{n-2,n-1} \\ g_{n-1} \end{pmatrix},$$

where each block is of size $m \times m$, such that, for $1 \leq i, j \leq n - 1$, $g_i \in P_{r_i, \ell_i}$, and

$$x_{i,j} = \begin{pmatrix} x_{i,j}^{(1)} \\ x_{i,j}^{(2)} \\ \vdots \\ x_{i,j}^{(4)} \end{pmatrix}.$$

Note that $x_{i,j}^{(1)}$ is of size $r_i \times r_j$, $x_{i,j}^{(2)}$ is of size $r_i \times \ell_j$, and $x_{i,j}^{(4)}$ is of size $\ell_i \times \ell_j$.

$$B = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{pmatrix},$$

where each block is of size $m \times 2(ni + m)$, such that, for $1 \leq i \leq n - 1$,

$$x_i = \begin{pmatrix} x_i^{(1)} \\ x_i^{(2)} \\ \vdots \\ x_i^{(4)} \end{pmatrix} \in GL_{m(n-1)-r}.$$

Note that $x_i^{(1)}$, $x_i^{(2)}$ are of size $r_i \times (ni + m)$, and $x_i^{(4)}$ is of size $\ell_i \times (ni + m)$. Finally, write $C$ as a $m(n-1) \times m(n-1)$ block matrix, with all blocks $c_{i,j}$ of size $m \times m$, $1 \leq i, j \leq n - 1$. Then

$$c_{i,j} = \begin{pmatrix} c_{i,j}^{(1)} & c_{i,j}^{(2)} \\ 0_{\ell_i \times \ell_{n-j}} & c_{i,j}^{(4)} \end{pmatrix}.$$

For the element $Q_{m-1} \cap \tilde{\alpha}_\tilde{r} Q(r) \tilde{\alpha}_\tilde{r}$, let us describe $\tilde{\alpha}_\tilde{r} \theta \tilde{\alpha}_\tilde{r}^{-1}$. For this, write, for $1 \leq i \leq n - 1$,

$$g_i = \begin{pmatrix} g_i^{(1)} \\ g_i^{(2)} \\ \vdots \\ g_i^{(4)} \end{pmatrix}.$$

Write $\tilde{\alpha}_\tilde{r} \theta \tilde{\alpha}_\tilde{r}^{-1}$ in the form $Q_{m-1} \cap \tilde{\alpha}_\tilde{r} Q(r) \tilde{\alpha}_\tilde{r}$, with the same notation. Then, for $t = 1, 2, 4$,

$$a_t = \begin{pmatrix} g_1^{(t)} & x_1^{(t)} & \cdots & x_{1,n-1}^{(t)} \\ g_2^{(t)} & \cdots & & x_{2,n-1}^{(t)} \\ \vdots & & & \vdots \\ g_{n-2}^{(t)} & x_{n-2,n-1}^{(t)} \\ g_{n-1}^{(t)} \end{pmatrix} ;$$

$$y_t = \begin{pmatrix} x_1^{(t)} \\ x_2^{(t)} \\ \vdots \\ x_{n-1}^{(t)} \end{pmatrix} ;$$

(2.15)
\[
(2.18) \quad z_t = \begin{pmatrix}
\frac{(t)}{t} c_{1,1} & \cdots & \frac{(t)}{t} c_{1,n-1} \\
\frac{(t)}{t} c_{n-1,1} & \cdots & \frac{(t)}{t} c_{n-1,n-1}
\end{pmatrix}.
\]

Note the form of (2.4) when applied to $\tilde{a}_F \tilde{a}_F^{-1}$.

Let us factor the summation in $\eta$ in (2.14) modulo $U_{m-1} = U_{m-1} F$ from the right. From the description (2.13)-(2.15) of $Q_{m-1} \cap a_F^{-1} Q^{(r)} a_F$, we may take representatives of $Q_{m-1} \cap a_F^{-1} Q^{(r)} a_F \backslash Q_{m-1} / U_{m-1}$ of the form
\[
(2.19) \quad h_{\eta,\gamma} = \text{diag}(\eta_1, \ldots, \eta_{n-1}, \gamma, \eta_{n-1}', \ldots, \eta_1'),
\]
where $\eta_i \in P_{r_i, \xi_i} \backslash \text{GL}_m(F)$, for $1 \leq i \leq n-1$, and $\gamma \in Q_{m-1}^{m(n-1)-r} H_{2(ni+m)}$.

Put $Q^{(r)}(\eta, \gamma) = h_{\eta,\gamma}^{-1} Q_{m-1} \cap a_F^{-1} Q^{(r)} a_F h_{\eta,\gamma}$. The summation (2.7) becomes
\[
(2.20) \quad \sum_{h_{\eta,\gamma}} \sum_{u' \in Q^{(r)}(\eta, \gamma) \cap U_{m-1} \backslash U_{m-1}} f_{\Delta(\tau, m+1)}(\gamma, \alpha_r a_F h_{\eta,\gamma} u') dx.
\]

Substitute (2.20) in (2.5) to get, for $0 \leq r \leq m(n-1),$ $x$
\[
F_{\psi, r}(f_{\Delta(\tau, m+1)}(\gamma, \alpha_r a_F h_{\eta,\gamma} u')) =
\]
\[
(2.21) \quad \sum_{\eta} \sum_{\gamma} \int_{U_{m-1} \backslash U_{m-1} (K)} f_{\Delta(\tau, m+1)}(\gamma, \alpha_r a_F h_{\eta,\gamma} u') dx \psi_H^{-1}(u) du.
\]

Factor the summation in $\gamma$ modulo $j(H_m \times H_{2ni+m})$ from the right. The sum over $\gamma \in Q_{m(n-1)-r}^{m(n-1)-r} H_{2(ni+m)} H_{2ni+m}$ is followed by a sum over $(g', h') \in j^{-1}(\gamma^{-1} Q_{m(n-1)-r}^{m(n-1)-r} \gamma \cap (H_m \times H_{2ni+m}) \backslash H_m \times H_{2ni+m})$. In (2.21), $h_{\eta,\gamma}$ is now replaced by $h_{\eta,\gamma}(g', h')$, and similarly for $Q^{(r)}(\eta, \gamma)$.

Now, change variable in $\eta, \eta_i \mapsto \eta' g'$, so that $\eta \mapsto \eta'(g', h')$. Note that $h_{(g', h')}^{-1} \gamma = t(g', h')$. Since $t(g', h')$ normalizes $U_{m-1}(F), U_{m-1}(K)$ and preserves $\psi_H$ (the same in the metaplectic case), (2.21) becomes
\[
(2.22) \quad \sum_{\eta} \sum_{\gamma} \sum_{(g', h')} \sum_{\eta'} \int f_{\Delta(\tau, m+1)}(\gamma, \alpha_r a_F h_{\eta,\gamma} u'(g', h') x) \psi_H^{-1}(u) du,
\]
where $\gamma$ is summed over $Q_{m(n-1)-r}^{m(n-1)-r} \backslash H_{2(ni+m)} H_{2ni+m}$, $(g', h')$ is summed over $j^{-1}(\gamma^{-1} Q_{m(n-1)-r}^{m(n-1)-r} \gamma \cap (H_m \times H_{2ni+m}) \backslash H_m \times H_{2ni+m})$, $\eta = (\eta_1, \ldots, \eta_{n-1})$ is summed over $\prod_{i=1}^{n-1} P_{r_i, \xi_i} \backslash \text{GL}_m(F)$. Finally, $u$ is integrated along $Q^{(r)}(\eta, \gamma) \cap U_{m-1} \backslash U_{m-1}(K)$.

We will now write a set of representatives of
\[
Q_{m(n-1)-r}^{m(n-1)-r} \backslash H_{2(ni+m)} H_{2ni+m}.
\]

Note that $j(H_m \times H_{2ni+m})$ is invariant to conjugation by $\omega_0$. Thus, it is enough to find a set of representatives of $Q_{m(n-1)-r}^{m(n-1)-r} \backslash H_{2ni+m} H_{2ni+m}$. Let us realize $Q_{m(n-1)-r}^{m(n-1)-r} H_{2ni+m}$ as the variety $Z$ of all (maximal) $ni + m$-dimensional isotropic subspaces of the space in which $H_{2ni+m}$ acts. We consider the action of $j(H_m \times H_{2ni+m})$ on $Z$. Given an orbit $O$ of this action, it is clear that $\dim(Z \cap X_m) = d_O$ is independent of $Z$, for any $Z \in O$. (See right after (1.6) for the definition of $X_m$.)

The following lemma is proved exactly as Lemma 3.1 in [GJS15].
We note that we must have that $\eta$. Thus, by (2.25), we get in (2.22) an inner integration of the character (2.26). Hence (2.27)

\[ \eta \eta^{-1} \]

We may now take in (2.22), $\gamma = \gamma_e$. Note that $\gamma_e$ commutes with $\omega_0$.

**Proposition 2.3.** Let $\bar{r} = (r_1, \ldots, r_{n-1})$ be as in (2.8) and $e$ as above. If the summand corresponding to $\bar{r}, \gamma_e$, in (2.22), is nonzero, then

\[ r_1 \leq r_2 \leq \cdots \leq r_{n-1} \leq e \leq \left[ \frac{m}{2} \right]. \]

**Proof.** Consider the following element $v$ of the form (2.12), with $B = 0$, $C = 0$, $h = I_{2(ni+m)}$, $A$ with $g_1 = \cdots = g_{n-1} = I_m$, and, for $1 \leq i < j \leq n - 1$, $\gamma_r = \gamma_{i,j} = \eta^{-1}_{i,j} (0 \ y_{i,j}^{(2)} \ 0) \eta_j$, where the last matrix has Adele coordinates and its block division is of sizes as in (2.13). It follows that $v \in Q^{(r,\bar{g},\gamma)}(\mathbb{A}) \cap U_{m^{n-1}}(\mathbb{A})$. By (2.16),

\[ \hat{a}_r h_{\bar{g},\gamma} v h_{\bar{g},\gamma}^{-1} \hat{a}_r^{-1} = \begin{pmatrix} I_r & y \\ 0 & I_{m(n-1)-r} \end{pmatrix}, \]

(this is an element of the form (2.3)): the matrix $y$ is the following

\[ y = \begin{pmatrix} 0 & y_{1,2}^{(2)} & \cdots & y_{1,n-1}^{(2)} \\ 0 & y_{2,n-1}^{(2)} \\ \vdots \\ 0 & y_{n-2,n-1}^{(2)} \\ 0 \end{pmatrix}. \]

By (2.4), the element on the r.h.s. of (2.24) is conjugated by $\alpha_r$ into $U_{(m+i)n}(\mathbb{A})$. Thus, in (2.22),

\[ f_{\Delta_{(r,m+i)\gamma^{(e)},s}}(\alpha_r \hat{a}_r h_{\bar{g},\gamma} v u t(g', h')x) = f_{\Delta_{(r,m+i)\gamma^{(e)},s}}(\alpha_r \hat{a}_r h_{\bar{g},\gamma} u t(g', h')x). \]

We note that

\[ \psi_H(v) = \prod_{i=2}^{n-1} \psi(tr(0 \ 0 \ \eta_{\bar{r},i-1}^{-1})(\eta_{\bar{r},i-1}^{-1})). \]

Thus, by (2.25), we get in (2.22) an inner integration of the character (2.26). Hence we must have that $\eta_{\bar{r},i-1}^{-1}$ is of the following form (or else, we get zero)

\[ \eta_{\bar{r},i-1}^{-1} = \begin{pmatrix} \mu^{(1)}_{i-1} & \mu^{(2)}_{i-1} & \mu^{(4)}_{i-1} \\ 0_{i, i-1} & 0_{i, i-1} & 0_{i, i-1} \end{pmatrix}, \]

\[ \psi_{\bar{r},i-1}(v) \]
Note that $\mu_{i-1}^{(1)} \in M_{ri \times r_{i-1}}(F)$. Since $\eta_i \eta_{i-1}^{-1} \in \text{GL}_n(F)$, we must have $r_{i-1} \leq r_i$, for $2 \leq i \leq n - 1$. It remains to show that $r_{n-1} \leq e$. The proof is similar when we take the following element $v$ of the form (2.12), with Adele coordinates, so that

$$A = I_{m(n + m)}; h = I_{2(n + m)}.$$ 

where the last matrix has block division of sizes as in (2.14). The matrix $C$ is with

$$c_{i,j} = \eta_i^{-1} \left( \begin{array}{cc} 0 & \gamma_i^{(2)} \\ 0 & 0 \end{array} \right) \left( \eta_{n-j}^{-1} \right)^{-1},$$

where the last matrix has block division of sizes as in (2.16). Then $v \in Q^{(F; \eta; \gamma)}(\mathbb{A}) \cap U_{m \times -1}(\mathbb{A})$, and by (2.17), (2.18).

(2.28)

\[
\hat{a}_H \eta_i \gamma_i \hat{h}_{i,j}^{-1} \hat{a}_H^{-1} = \begin{pmatrix} I_r & 0 & 0 & \gamma_i^{(2)} \\ 0 & I_{m(n + 1)} & 0 & 0 \\ 0 & 0 & I_{ni + m} & 0 \\ 0 & 0 & 0 & I_r \end{pmatrix},
\]

(this is an element of the form (2.3)); the matrix $y$ is the following

$$y = \begin{pmatrix} \gamma_i^{(2)} \\ \gamma_i^{(2)} \\ \gamma_i^{(2)} \\ \gamma_i^{(2)} \end{pmatrix},$$

and $\zeta$ is the matrix of all blocks $\gamma_i^{(2)}$. By (2.4), the element on the r.h.s. of (2.28) is conjugated by $\alpha_v$ into $U_{m \times (i + m)}(\mathbb{A})$. Thus we get (2.25), with the present element $v$ and $\gamma = \gamma_e$. Now we note that (since $\omega \eta \gamma_e = \gamma_e \omega$ and $\omega \hat{a}_H = A_H$

(2.29)

$$\psi_H(v) = \psi(\text{tr} \left( \begin{pmatrix} 0 & \gamma_i^{(2)} \\ 0 & 0 \end{pmatrix} \gamma_e \hat{a}_H \eta_i^{-1} \right)).$$

Thus, we get in (2.22) an inner integration of the character (2.29), and we conclude that

(2.30)

$$\gamma_e \hat{a}_H \eta_i^{-1} = \begin{pmatrix} \nu^{(1)} & \nu^{(2)} \\ 0_{(ni + m) \times r_{n-1}} & \nu^{(4)} \end{pmatrix}.$$ 

Note that $\nu^{(1)} \in M_{(ni + m) \times r_{n-1}}(F)$. It follows that

(2.31)

$$\eta_{n-1}^{-1} = \begin{pmatrix} \mu_{n-1}^{(1)} & \mu_{n-1}^{(2)} \\ 0_{(m-e) \times r_{n-1}} & \mu_{n-1}^{(4)} \end{pmatrix}.$$ 

Since $\eta_{n-1}^{-1} \in \text{GL}_m(F)$, we must have $r_{n-1} \leq e$. This proves the proposition. \qed

Note that from (2.30), (2.31), it follows that

(2.32) 

$$\nu^{(1)} = \begin{pmatrix} \mu_{n-1}^{(1)} \\ 0_{(ni + m-e) \times r_{n-1}} \end{pmatrix}.$$
It also follows that for \( m \) even,

\[
\nu^{(4)} = \begin{pmatrix}
0_{(ni+e) \times \ell_{n-1}} \\
\mu^{(4)}_{n-1}
\end{pmatrix},
\]

and for \( m \) odd, if we write

\[
\mu^{(4)}_{n-1} = \begin{pmatrix}
d_1 \\
d_2 \\
d_3
\end{pmatrix}, \quad d_1 \in M_{(\bar{\nu}^2)_{-c} \times \ell_{n-1}}(F), \quad d_2 \in M_{1 \times \ell_{n-1}}(F), \quad d_3 \in M_{(\bar{\nu}) \times \ell_{n-1}}(F),
\]

then

\[
\nu^{(4)} = \begin{pmatrix}
\frac{1}{2}d_2 \\
\frac{1}{2}d_2 \\
d_1 \\
d_3
\end{pmatrix}.
\]

Consider now \( Q^{(\bar{\gamma}^2)}(H) \cap U_{m-n-1}(H) \). Its elements and their conjugates by \( a^{h_{\bar{\gamma}}} \) can be read from (2.12)-(2.18), and further conjugation by \( \alpha_\ell \) can be read from (2.23). Using these, we see that factoring the \( du \)-integration in (2.22), modulo \( Q^{(\bar{\gamma}^2)}(H) \cap U_{m-n-1}(H) \), and carrying out the conjugations above, gives the following inner integration, for fixed \( \bar{r} \) as in Prop. 2.3 \( \bar{\eta} \) as in (2.27)-(2.31), \( \gamma = \gamma_c, (g', h') \),

\[
\int_{E_r(F) \backslash E_r(H)} f_{\Delta(x,m)} g(\psi) \langle \nu \alpha_\ell a^{h_{\bar{\gamma}}} ut(g', h') x \rangle^{-1} \mathbf{E}_{\bar{\eta},\bar{\gamma},c}(v) dv,
\]

where \( E_r \) is a semi-direct product of two unipotent subgroups. One is a subgroup of \( U_{(m+i)n} \), and the other is the following unipotent subgroup \( V_{n}^n \subset \mathbf{GL}_{(m+i)n} = M_{(m+i)n} \). Its elements have the form

\[
\hat{v} = \begin{pmatrix}
I_{r_1} & x_{1,2} & \cdots & x_{1,n-1} & b_1 & * & \cdots & * \\
I_{r_2} & x_{2,n-1} & \cdots & x_{2,n-1} & b_2 & * & \cdots & * \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots \\
I_{r_{n-1}} & b_{n-1} & * & \cdots & * \\
I_{m+i} & c_{n-1} & \cdots & c_{n-1} \\
I_{\ell_{n-1}} & y_{n-2,n-1} & \cdots & y_{1,n-2} \\
\vdots & \vdots & \ddots & \vdots \\
y_{1,2} & I_{\ell_1}
\end{pmatrix}.
\]

Note that \( V_{\bar{r}} = V_{r_1,\ldots,r_{n-1},m+i,\ell_{n-1},\ldots,\ell_1} \). The character \( \psi_{E_r,\bar{\eta},c} \) is trivial on \( E_r(H) \cap U_{(m+i)n}(H) \), and takes the following value on \( v \) in (2.36), with Adele coordinates,

\[
\prod_{i=1}^{n-2} \psi(tr(\mu^{(1)}_i x_{i,i+1})) \psi(tr(\mu^{(4)}_i y_{i,i+1})) \psi(tr(\nu^{(1)}_i b_{n-1})) \psi(tr(\nu^{(4)}_i c_{n-1})),
\]

where \( \mu, \nu \) are as in (2.27), (2.30), and we define

\[
\hat{\mu}^{(4)}_i = -w_{\ell_i} t \mu^{(4)}_i, \quad \hat{\nu}^{(4)}_i = -w_{\ell_{n-1}} t \nu^{(4)}_i w_{m+i}.
\]
Let us view (2.37) as a character of $V_{\ell}(\mathbb{A})$ and denote it by $\psi_{V_{\ell},\bar{\eta},e}$. For an automorphic form $\xi$ in the space of $\Delta(\tau, m+i)$, consider the following Fourier coefficient

$$
(2.38) \quad \xi^{\psi,\bar{r},\bar{\eta},e}(a) = \int_{V_{\ell}(F) \setminus V_{\ell}(\mathbb{A})} \xi(va)\psi_{V_{\ell},\bar{\eta},e}^{-1}(v)dv.
$$

Then (2.35) is the application of the Fourier coefficient (2.38) to the automorphic form on $\text{GL}_{(m+i)n}(\mathbb{A})$, evaluated at the identity, $a \mapsto f^0_{\Delta(\tau, m+i)\gamma_{\psi}^{(s)},s}(\tilde{ah})$, (where in (2.35), $h = \alpha_r \hat{a}_r \gamma_{\psi} \gamma_{\nu} \gamma_v x$). We denote this Fourier coefficient, evaluated at $I_{(m+i)n}$, by $f^{\psi,\bar{r},\bar{\eta},e}_{\Delta(\tau, m+i)\gamma_{\psi}^{(s)},s}(h)$. Denote $U^{\bar{r},\bar{\eta},e}_{m^{n-1}} = Q^{(\bar{r},\bar{\eta} ; \gamma_v)} \cap U_{m^{n-1}}$. Then (2.22) becomes

$$
(2.39) \quad \sum_{r} \sum_{e} \sum_{(g',h')} \sum_{\bar{r}} \int_{U^{\bar{r},\bar{\eta},e}_{m^{n-1}}(\mathbb{A}) \setminus U_{m^{n-1}}(\mathbb{A})} f^{\psi,\bar{r},\bar{\eta},e}_{\Delta(\tau, m+i)\gamma_{\psi}^{(s)},s}(\alpha_r \hat{a}_r \gamma_{\psi} \gamma_{\nu} \gamma_v x)\psi_{\tilde{\nu},\bar{h}}^{-1}(u)du.
$$

Here the various summations are as in (2.22), with $\bar{r}, \bar{\eta}$ as in Prop. 2.3 and its proof. The last step in the proof of Theorem 2.1 will be to show

**Proposition 2.4.** Assume that $r_{n-1} > 0$. Then the Fourier coefficient (2.38) is identically zero on $\Delta(\tau, m+i)$.

**Proof.** We note that $\text{rank}(\mu^{(1)}_i) = r_i$, $\text{rank}(\mu^{(4)}_i) = \ell_{i+1}$, for $1 \leq i \leq n-2$. Also, $\text{rank}(\nu^{(1)}) = \text{rank}(\mu^{(1)}_{n-1}) = r_{n-1}$, $\text{rank}(\nu^{(4)}) = \text{rank}(\mu^{(4)}_{n-1}) = m - e$. These follow from the last proof. Due to the form of $\nu^{(1)}, \nu^{(4)}$, (2.32)-(2.34), it is not hard to see that one can conjugate the character $\psi_{V_{\ell},\bar{\eta}}$, given by (2.37), by an element $\text{diag}(a_1, ..., a_{n-1}, b, \alpha_{n-1}, ..., \alpha_1) \in \text{GL}_{(m+i)n}(F)$, where $a_i \in \text{GL}_r(F)$, $\alpha_i \in \text{GL}_r(F)$, for $1 \leq i \leq n-1$, and $b \in \text{GL}_{ni+m}(F)$, to a character of the same form (as (2.37)), as follows. The matrix $\mu^{(1)}_i$ is replaced by

$$
R_i = \begin{pmatrix} I_{r_i} & 0 \end{pmatrix}, \quad 1 \leq i \leq n-2.
$$

The matrix $\nu^{(1)}$ is replaced by

$$
R_{n-1} = \begin{pmatrix} I_{r_{n-1}} & 0 \end{pmatrix}.
$$

The matrix $\tilde{\mu}^{(4)}_i$ is replaced by

$$
L_i = \begin{pmatrix} I_{\ell_{i+1}} & 0 \end{pmatrix}, \quad 1 \leq i \leq n-2.
$$

Finally, $\nu^{(4)}$ is replaced by

$$
S = \begin{pmatrix} I_{m-e} & 0_{(m-e) \times (ni+e)} \\ 0_{(e-r_{n-1}) \times (m-e)} & I_{(e-r_{n-1}) \times (ni+e)} \end{pmatrix}.
$$

After conjugation, the character $\psi_{V_{\ell},\bar{\eta}}$ becomes the character

$$
(2.40) \quad \prod_{i=1}^{n-2} \psi(tr(R_i x_{i,i+1}))\psi(tr(L_i y_{i,i+1}))\psi(tr(R_{n-1} b_{n-1}))\psi(tr(S e_{n-1})).
$$
The character (2.30) corresponds to the nilpotent orbit \((g_{(m+i)\ell}(F))\) of the matrix

\[
A = \begin{pmatrix}
0_{r_1} & 0_{r_2} & \ddots & & \\
R_1 & \eta & & & \\
0_{r_{n-2}} & R_{n-2} & 0_{r_{n-1}} & 0_{n_{i+m}} & S \\
& R_{n-1} & 0_{n_{i+m}} & & \\
& & S & 0_{\ell_{n-1}} & \\
& & & L_{n-1} & 0_{\ell_{n-2}} \\
& & & & \ddots \\
& & & & L_2 \\
& & & & 0_{r_1}
\end{pmatrix}.
\]

The orbit of \(A\) corresponds to a partition of the form \(((2n-1)^r, \ldots)\). By Prop. 1.4, we must have \(r_1 = 0\), unless \(n = 1\), in which case the proposition is clear, since \(n - 1 = 0\). Assume that \(n > 1\). Consider the matrix (2.41), with \(r_1 = 0\). Then, similarly, its orbit corresponds to a partition of the form \(((2n-2)^r, \ldots)\). By Prop. 1.4, we must have \(r_2 = 0\), unless \(n = 2\), in which case the proposition is clear, since \(n - 1 = 1\). If \(n > i - 1\), and we proved that \(r_{i-1} = 0\), then we get that the orbit of \(A\) corresponds to a partition of the form \(((2n-i)^r, \ldots)\). By Prop. 1.4 we must have \(r_1 = 0\), unless \(n = i\), in which case the proposition is clear, since \(n - 1 = i - 1\). This proves the proposition.

The last proposition implies that if \(F_\psi, r(f_{(\Delta(m+i)\gamma)}^{(\psi)})\) is nonzero, then \(r_1 = \cdots = r_{n-1} = 0\) (due to Prop. 2.3). By (2.3), \(r = 0\). This completes the proof of Theorem 2.1.

3. The contribution of the open orbit to \(E(f_{(\Delta(m+i)\gamma)}^{(\psi, s)}(s), \bar{\varphi}_s)\)

Theorem 2.1 (2.5) and (2.39) imply that

\[
F_\psi(E(f_{(\Delta(m+i)\gamma)}^{(\psi, s)})) = F_\psi,0(E(f_{(\Delta(m+i)\gamma)}^{(\psi, s)})) = \frac{1}{|\mathfrak{g}|} \sum_{e=0} \sum_{(g', h')} \sum_{\bar{\eta}} \int_{L_{m-1}^0 \setminus \text{GL}_m(F)} f_{(\Delta(m+i)\gamma)}^{(\psi, \bar{\eta}, e)}(\alpha_0 \hat{a}_0 h_{\bar{\eta}} \gamma) \psi^{-1}(u) du.
\]

Let us describe the ingredients of (3.1) in the present case (i.e. \(r = 0\)). Recall that the summation on \(\bar{\eta} = (\eta_1, \ldots, \eta_{n-1})\) runs over \(\prod_{i=1}^{n-1} P_{r_i} \setminus \text{GL}_m(F)\). Since \(r_i = 0\), for \(1 \leq i \leq n - 1\), we may take \(\eta_i = I_m\), for all \(i\), and so there is no further summation on \(\eta\) in (3.1). Let us denote the element \(h_{\bar{\eta}, \gamma}\), in this case by \(h_{\gamma}\). Note also that \(a_0 = I_{m(n-1)}\), and that

\[
\alpha_0 = \begin{pmatrix}
0 & I_{n_{i+m}} & 0 & 0 \\
0 & 0 & 0 & I_{m(n-1)} \\
\delta_H I_{m(n-1)} & 0 & 0 & 0 \\
0 & 0 & I_{n_{i+m}} & 0
\end{pmatrix} \omega_{0}^{m(n-1)}.
\]
Note that $\omega_0^{m(n-1)} = I$ when $m$ is even. When $m$ is odd, i.e. $H_m$ is odd orthogonal, $\omega_0^{m(n-1)} = \omega_0^{n-1}$. We denote, for $\bar{\eta}$ as above, $f^\psi,\bar{\eta},e_{\Delta(\tau,m+i)\gamma_\psi^\prime,s} = f^\psi,\eta_{\Delta(\tau,m+i)\gamma_\psi^\prime,s}$. Note that $V_0^* = V_{ni+m,mn-1}$, and the character $\psi_V,\bar{\eta},e$ is now the following character $\psi'$ of $V_{ni+m,mn-1}$,

$$
\psi'_e\left(
\begin{pmatrix}
I_{ni+m} & c_{n-1} & \cdots & c_1 \\
I_m & y_{n-2,n-1} & \cdots & y_{1,n-2} \\
& & \ddots & \\
y_{1,2} & & & I_m
\end{pmatrix}
\right) = \prod_{i=1}^{n-2} \psi^{-1}(tr(y_{i,i+1})),
$$

where in case $m$ is even,

$$
\nu = \left(
\begin{array}{ccc}
I_{m-e} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right),
$$

and in case $m = 2m' - 1$ is odd,

$$
\nu = \left(
\begin{array}{ccc}
I_{m'-1} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right),
$$

Next, the subgroup $U_{mn-1}^\bar{\eta},e, which we denote for short $U_{mn-1}^e$, consists of the elements

$$
\left(
\begin{array}{ccc}
\nu & y & 0 \\
I_{2(ni+m)} & y' & 0 \\
0 & 0 & 0
\end{array}\right) \in U_{mn-1},
$$

where $\nu \in V_{mn-1}$ and $y$ has the form

$$
y = (0_m \times (m-e), y_1, 0_m \times (n_i + (m-2[m^n])e), y_2, \tilde{y}_1, y_3),
$$

where $y_1$ is of size $m(n-1) \times 2(\frac{m^n}{2^n} - e)$, $y_2$ is of size $m(n-1) \times (ni + (m-2[m^n])e)$, and $y_3$ is of size $m(n-1) \times e$, and $y_1 = y_1 \cdot \text{diag}(I_{m-2[e]}^{-\gamma^e_{\psi_0,ni+m}}, -\delta H I_{[\frac{m}{2}]-e})$. Finally, the summation in $(g', h')$ in (3.1) runs over $j^{-1}(\gamma_{e}^{-1} Q_{ni+m}^{m(n-1)} - e) \cap (H_m \times H_{2ni+m}) \backslash H_m \times H_{2ni+m}$. See (2.22). The elements of $\gamma_{e}^{-1} Q_{ni+m}^{m(n-1)} - e \cap (H_m \times H_{2ni+m})$ have the following form.

$$
\left(
\begin{array}{ccc}
a & x & y \\
b & x' & a^* \\
a^* & & A^*
\end{array}\right), \left(
\begin{array}{ccc}
A & X & Y \\
\frac{1}{\psi_0,ni+m} & \bar{\psi}_0,ni+m & e_b X' \\
A & & A^*
\end{array}\right) \in H_m \times H_{2ni+m},
$$

where $a \in \text{GL}_e$, $A \in \text{GL}_{ni+e}$, and, for $b \in H_{m-2e}$,

$$
\bar{e}_b = J_e b J_{-e}^{-1}, J_e = \left(
\begin{array}{ccc}
I_{m-2[e]} & I_{[\frac{m}{2}]-e} \\
-\delta H I_{[\frac{m}{2}]-e} & I_{[\frac{m}{2}]-e}
\end{array}\right);
$$

$$
\psi_{[\frac{m}{2}]-e,ni+e} = \left(I_{ni+e} \frac{I_{[\frac{m}{2}]-e}}{27}
\right) : = \beta_e.
$$
Note that \( b \mapsto \psi b \) is an outer conjugation of \( H_{2n+i} \). Now (3.1) becomes
\[
\mathcal{F}_\psi(E(f_{\Delta(\tau,m+i)\gamma(y)}))(x) = \sum_{a=0}^{127} \sum_{h' \in Q_{n+i+e}} \sum_{g' \in Q^0_{m}} \int_{V_{n+i+1}(\mathbb{A})} f'_{\Delta(\tau,m+i)\gamma(y)}(a_0 h \gamma_e u(t', \beta h') x) \psi^{-1}(u) du.
\]
(3.5)

Here, \( Q_{n+i+e} = Q^{H_{2n+i}}_{n+i+e} \), and \( Q^0_{m} \) is the subgroup of \( Q^H_{m} \) consisting of elements as the first matrix in (3.4), with \( b = I_{m-2e} \). The Fourier coefficient \( f'_{\Delta(\tau,m+i)\gamma(y)} \) is obtained by applying the \( \psi' \) coefficient (3.3) to \( a \mapsto f_{\Delta(\tau,m+i)\gamma(y)}(I, a) \) (and then we evaluate at \( a_0 h \gamma_e u(t', \beta h') x \)). As in (2.38), consider the Fourier coefficient applied to an automorphic form \( \xi \) in the space of \( \Delta(\tau, m + i) \),
\[
\xi_{\psi'}(a) = \int_{V_{n+i,m,n-1}(F) \setminus V_{n+i,m,n-1}(\mathbb{A})} \xi(va) \psi^{-1}(v) dv.
\]
(3.6)

It will be convenient to conjugate the character \( \psi_e' \) as follows. Define the following element \( w_0 \in \text{GL}_{(m+i)n}(F) \). Assume that \( m \) is even, then
\[
w_0 = \text{diag} \left( I_m, I_{n(m-1)} \right).
\]
Assume that \( m = 2m' - 1 \) is odd. Then
\[
w_0 = \text{diag} \left( I_{m'-1}, I_{n} \right) \cdot I_{(2m'-1)(n-1)}.
\]
We will use the following conjugate of \( \psi_e' \): \( \psi_e(v) = \psi_e'(w_0^{-1}v w_0) \). Then
\[
\xi_{\psi_e}(w_0 a) = \int_{V_{n+i,m,m-1}(F) \setminus V_{n+i,m,m-1}(\mathbb{A})} \xi(v w_0 a) \psi_e^{-1}(v) dv = \xi_{\psi_e}(a).
\]
(3.7)

Explicitly, the character \( \psi_e \) is given by
\[
\psi_e \left( \begin{array}{cccccc}
I_n & 0 & y_1 & \cdots & y_{n-1} \\
I_m & x_1 & \cdots & \ast \\
& \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots \\
& & & \ddots & \ddots & \ddots \\
& & & & I_m & x_{n-1} \\
& & & & & I_m
\end{array} \right) = \psi^{-1}(tr(a_c x_1) \prod_{i=2}^{n-1} \psi^{-1}(tr(x_i)),
\]
where \( a_c = \left( \begin{array}{cc} I_{m-e} & 0 \\
0 & 0_{e \times e} \end{array} \right) \).

Proposition 3.1. For any automorphic form \( \xi \) in the space of \( \Delta(\tau, m + i) \), and any \( y \in M_{n \times (m \times \mathbb{A})} \), \( \xi_{\psi_e} \) is left invariant to \( n_y = \text{diag} \left( \begin{array}{cc} I_n & y \\
I_m & I_m \end{array} \right), I_{m(n-1)} \).

Proof. Consider, for fixed \( a \), the following function on \( M_{n \times (m \times \mathbb{A})} \),
\[
\phi_\xi(y) = \xi_{\psi_e}(n_y a).
\]

Note that, for \( y \in M_{n \times (m \times \mathbb{A})} \), \( n_y \) normalizes the rational and Adele points of \( V_{m+n, m-1} \), and it preserves \( \psi_e \). Consider the Fourier expansion of \( \phi_\xi \) along the
compact Abelian group $M_{n_1 \times m}(F) \backslash M_{n_1 \times m}(A)$. We claim that the Fourier coefficients of $\phi_{\xi}$ with respect to all nontrivial characters are zero. Indeed, such a Fourier coefficient is, in fact, a Fourier coefficient of $\xi$ along the unipotent radical $V_{n_1, m}$ with respect to a character of the form

$$\sigma = \left( \begin{array}{cccc} I_{n_1} & y_0 & y_1 & \cdots & y_{n-1} \\ I_m & x_1 & \cdots & * \\ \vdots & & & \\ I_m & x_{n-1} \\ I_m \end{array} \right)$$

(3.9)

$$\psi_{A, n}(\sigma) = \psi^{-1}(tr(Ay_0))\psi^{-1}(tr(a_ex_1)) \prod_{i=2}^{n-1} \psi^{-1}(tr(x_i)),$$

where $A \in M_{m \times n_1}(F)$ is nonzero. It is easy to see that such a Fourier coefficient corresponds to a partition of the form $(n+1, \ldots)$, which by Prop. 1.4 must be trivial. Thus, $\phi_{\xi}$ is equal to its constant term along $M_{n_1 \times m}(F) \backslash M_{n_1 \times m}(A)$. This proves the proposition. \hfill \Box

Consider the character (3.9) $\tilde{\psi}_{0,e}$ (i.e. $A = 0$). Denote

$$\xi_{\psi_{0,e}}(a) = \int_{V_{n_1, m}(F) \backslash V_{n_1, m}(A)} \xi(va)\psi_{0,e}^{-1}(v)dv.$$

(3.10)

Then the last proposition and (3.7) show that, for any automorphic form $\xi$ in the space of $\Delta(\tau, m + i)$,

$$\xi_{\psi_{0,e}}(a) = \xi_{\psi_{0,e}}(\tilde{w}_0a).$$

(3.11)

Denote $\tilde{\alpha}_0 = \tilde{w}_0\alpha_0$. By (3.11) and Prop. 3.1, we can rewrite (3.5) as

$$F_{\psi}(E(f_{\Delta(\tau, m+i)\gamma_e^{(\epsilon)})})(x) =$$

(3.12)

$$\sum_{e=0}^{[\Gamma]} \sum_{h'} \sum_{g'} \int_{U_{m-1}(A) \backslash U_{m-1}(A)} f_{\psi_{0,e}}^{\Delta(\tau, m+i)\gamma_e^{(\epsilon)}}(\tilde{\alpha}_0h_x, ut(g', \beta_h)x)\psi_H^{-1}(u)du.$$

Here $h'$ runs over $Q_{n_1+i} \backslash H_{2m}$. At this point, we go back to our kernel integral (1.11), and bring into our considerations, for the first time, the cuspidal representation $\sigma$ of $H_m(A)$. Using (3.12), we have

$$E(f_{\Delta(\tau, m+i)\gamma_e^{(\epsilon)}}\varphi_{\sigma})(h) =$$

(3.13)

$$\sum_{e=0}^{[\Gamma]} \sum_{h' \in Q_{n_1+i} \backslash H_{2m}} \Lambda_{e}(f_{\Delta(\tau, m+i)\gamma_e^{(\epsilon)}}\varphi_{\sigma})(h'h),$$

where

$$\Lambda_{e}(f_{\Delta(\tau, m+i)\gamma_e^{(\epsilon)}}\varphi_{\sigma})(h) =$$

(3.14)

$$\int_{\Delta(\tau, m+i)\gamma_e^{(\epsilon)}} \varphi_{\sigma}(g) \int_{U_{m-1}(A) \backslash U_{m-1}(A)} f_{\psi_{0,e}}^{\Delta(\tau, m+i)\gamma_e^{(\epsilon)}}(\tilde{\alpha}_0h_x, ut(g, \beta_h))\psi_H^{-1}(u)dudg.$$

Denote by $M_0^0$ the reductive part of $Q_0^0$ ($M_0^0 = GL_n^\vee$). The unipotent radical of $Q_0^0$ is $U_c = U_{c}^{H_m}$. We factor the $dg$-integration in (3.14) modulo $U_c(A)$. The subgroup
elements of $V_ξ$ proof works similarly for $g$ to $η$ Consider, for $η$ $g$ proof is the same one used in the proof of Prop. 3.1, that is when we take a $(3.18) ψ$ $y$ where the diagonally repeated matrix, denote it by $V_ξ$ $(3.17)$. For any $x \in H_ξ$ and all $f_{\Delta(τ,m+i)γ^s_ψ}^{ψ_0,e}(x)$, have the following form $(3.19)$. See Lemma 4.1 in [JL13]. Let $ψ_0,e$ be the following character of $V_m^\psi(ξ)$, $(3.18)$ $ψ_e(\begin{pmatrix}
 I_m & x_1 & \cdots & * \\
 I_m & x_2 & \cdots & \\
 \vdots & \ddots & \ddots & \\
 I_m & x_{n-1} & \cdots & I_m
\end{pmatrix}) = \psi^{-1}(tr(a_ξx_1) \prod_{i=2}^{n-1} \psi^{-1}(tr(x_i))$. Consider, for $η$ in the space of $\Delta(τ,m)$, the Fourier coefficient $η^{ψ_e}$ of $η$ with respect to $ψ_e$, along $V_m^\psi$. Then we have to prove that for $g_1, g_2$, as in $(3.17)$, and $a \in GL_m(ξ)$, $(3.19)$ $η^{ψ_e}(diag(g_1, g_2^\Delta^m) a) = η^{ψ_e}(a)$.
Fourier coefficient of $u \mapsto \eta^{\psi}(u a)$, along an abelian unipotent subgroup $X$ of the stabilizer above of $\psi$, with respect to a nontrivial character, it follows that if this Fourier coefficient is nontrivial, then we get a resulting nontrivial Fourier coefficient on $\Delta(\tau, m)$, corresponding to a partition, which is strictly larger than $(n^m)$. By Prop. [13] we will get a contradiction. Hence only the trivial character of $X_\Delta$ appears in the Fourier expansion of $\eta^{\psi}$ along $X$, that is $\eta^{\psi}$ is left-$X_\Delta$ invariant. Here is an example sketching how the proof goes. Take above $g_1 = I_m$ and $g_2 = \nu_c(z) = \left( I_{m-e} \ z \ 0 \right)$.

Consider a Fourier coefficient

$$\psi^{-1}(v')^{-1}(\text{tr}(Bz))dv' dz_2 dz.$$  

Next, we exchange roots in (3.22) by zero. We conclude that if (3.20) is nontrivial (as $\eta$ varies) then the following Fourier coefficient is not identically zero,

$$\psi^{-1}(v')^{-1}(\text{tr}(Bz))dv' dz_2 dz.$$  

Now we carry out the process of exchanging roots, ([GRS11], Sec. 7.1), as follows. Write in the matrix (3.18), for $2 \leq j \leq n - 1$,

$$x_j = \begin{pmatrix} x_{j,1} & x_{j,2} \\ x_{j,3} & x_{j,4} \end{pmatrix},$$

where $x_{j,1}$ is of size $(m - e) \times (m - e)$. Now it is easy to check that one can exchange (in the sense of Sec. 7.1 in [GRS11]), $x_{2,3}$ with $\text{diag}(I_{m, \nu_c(z_1)}, I_{(n-2)m})$. Denote by $V$ the subgroup of $V_m$ obtained by replacing in (3.18) $x_{2,3}$ by zero. We conclude that if (3.20) is nontrivial (as $\eta$ varies) then the following Fourier coefficient is not identically zero,

$$\int_{M(m-e)\times \mathbb{F}\setminus M(m-e)\times \mathbb{F}} \int_{V_1(F)\setminus V_1(\mathbb{A})} \eta(\text{diag}(I_{m}, \nu_c(z_1), I_{\Delta(\tau, m - 2)}))^{-1}(v')^{-1}(\text{tr}(Bz))dv' dz_2 dz.$$  

Next, we exchange roots in (3.22) by zero. We conclude that the following Fourier coefficient is not identically zero,

$$\psi^{-1}(v')^{-1}(\text{tr}(Bz))dv' dz_2 dz,$$

where $V$ is the subgroup of $V_m$ obtained by replacing in (3.18) $x_{2,3}$ and $x_{3,3}$ by zero. We continue in this way, exchanging $x_{j,3}$ with $\text{diag}(I_{j,m}, \nu_c(z_j), I_{(n-j-1)m})$, step by step, up to $j = n - 1$, and finally we get that the following Fourier coefficient is not identically zero,

$$\int_{M(m-e)\times \mathbb{F}\setminus M(m-e)\times \mathbb{F}} \int_{V_1(F)\setminus V_1(\mathbb{A})} \eta(\text{diag}(I_{m}, \nu_c(z_1), \ldots, \nu_c(z_{n-1})))^{-1}(v')^{-1}(\text{tr}(Bz))dv' dz_2 \cdots dz_{n-1}.$$
where $V'$ is the subgroup of $V_{m^n}$ obtained by replacing in (3.18) $x_{j,3}$ by zero, for $2 \leq j \leq n - 1$. Note that the product of $V'$ and the subgroup of all diag$(I_m, \nu_\epsilon(z_1), \ldots, \nu_\epsilon(z_{n-1}))$ is the unipotent radical $V_{m,m-e,m^n-2,e}$. Now one checks that the Fourier coefficient (3.24) on $\Delta(\tau, m)$ corresponds to a partition of $mn$, of the form $(n + 1, \ldots)$, and, by Prop. 3.4 (5.24), is identically zero, for all nonzero $B$. A similar argument works for proving the left invariance of $\eta^\psi$ under the Adele points of elements of the form above with

$$g_1 = \begin{pmatrix} I_e & a & 0 \\ I_{m-2e} & 0 & 0 \\ I_e & \end{pmatrix}, \quad g_2 = \begin{pmatrix} I_e & a & b \\ I_{m-2e} & 0 & 0 \\ I_e & \end{pmatrix}.$$

\[ \square \]

**Corollary 3.3.** For all $1 \leq e \leq \lfloor \frac{m}{2} \rfloor$, 

$$\Lambda_e(\mathbf{f}_e(\gamma^\psi, s, \sigma)) = 0.$$

**Proof.** Note that when $H_m$ is even orthogonal, we assume that $m \geq 4$, so that $U_e$ is a nontrivial unipotent radical of $H_m$, when $e \geq 1$.

By Prop. 3.2 and 3.10,

$$\Lambda_e(\mathbf{f}_e(\gamma^\psi, s, \sigma))(h) = \int_{M^0(F)U_e(\mathfrak{A})/H_m(\mathfrak{A})} \varphi^U_{\sigma}(g) \phi_{\sigma}^{(\psi)}(\Delta(\tau, m+i)\gamma^\psi, s, \sigma) \mathbf{f}_{\psi}(g, (\hat{\gamma}_\epsilon h)) \psi_H^{-1}(u) du dg,$$

where the $du$-integration is along $U_{m,n-1}(\mathfrak{A})/U_{m,n-1}(\mathfrak{A})$, and $\varphi^U_{\sigma}$ denotes the constant term of $\varphi_{\sigma}$ along the unipotent radical $U_e = U_{m,n-1}$. Since $\sigma$ is cuspidal, this constant term is zero identically on $\sigma$. \[ \square \]

Now, we can rewrite (3.13) as

\begin{equation}
(3.25) \quad \mathcal{E}(\mathbf{f}_e(\gamma^\psi, \sigma))(h) = \sum_{h' \in Q_m} \Lambda(\mathbf{f}_e(\gamma^\psi, s, \sigma))(h'h),
\end{equation}

where

$$\Lambda(\mathbf{f}_e(\gamma^\psi, s, \sigma))(h) = \Lambda_0(\mathbf{f}_e(\gamma^\psi, s, \sigma))(h) = \delta_0(I_m, \mathfrak{A}_0) \phi^{(\psi)}_{\Delta(\tau, m+i)\gamma^\psi, s, \sigma} = \phi^{(\psi)}_{\Delta(\tau, m+i)\gamma^\psi, s, \sigma},$$

\begin{equation}
(3.26) \quad \int_{H_m(\mathfrak{A})} \varphi_{\sigma}(g) \phi^{(\psi)}(g, (\hat{\gamma}_\epsilon h)) \psi_H^{-1}(u) du dg.
\end{equation}

Here, $\delta_0 = \hat{\beta} \mathfrak{A}_0(I_m, \hat{\beta}_{0})$, $\phi^{(\psi)}_{\Delta(\tau, m+i)\gamma^\psi, s, \sigma} = \phi^{(\psi)}_{\Delta(\tau, m+i)\gamma^\psi, s, \sigma}$, and $U_{m(n-1)}'$ is the following subgroup of $U_{m(n-1)}$, realizing the quotient $U_{m(n-1)}' = U_{m(n-1)} \setminus U_{m(n-1)},$

\begin{equation}
(3.27) \quad U_{m(n-1)}' = \{ u_{x,y} = \begin{pmatrix} I_{m(n-1)} & x & 0 & y \\ I_{m+ni} & 0 & 0 \\ I_{m+ni} & x' & 0 \\ I_{m(n-1)} & \end{pmatrix} \} \in H.
\end{equation}

Note that $\delta_0 = e^0 h_{\gamma_0}$, with

$$\delta_0 = \begin{pmatrix} 0 & W_0 & 0 & 0 \\ 0 & 0 & 0 & I_{m(n-1)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & W_0^* & 0 \end{pmatrix} \omega_0^{m(n-1)},$$

where $m = 2m'$ even,

$$W_0 = \begin{pmatrix} 0 & I_{ni} & 0 & 0 \\ I_{m' - 1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & I_{m' - 1} & 0 \end{pmatrix};$$

and for $m = 2m' - 1$ odd,

$$W_0 = \begin{pmatrix} 0 & I_{ni} & 0 & 0 \\ I_{m' - 1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & I_{m' - 1} & 0 \end{pmatrix};$$

$h_{\gamma_0} = \text{diag}(I_{m(n-1)}, \psi_0^{m(n-1)}), \text{ with}$

$$\gamma_0 = \begin{pmatrix} I_{[\frac{m}{2}]} & I_{ni} & I_{[\frac{m-1}{2}]} \\ I_{ni} & I_{[\frac{m}{2}]} & I_{[\frac{m-2}{2}]} \\ I_{[\frac{m}{2}]} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\delta_0 I_{[\frac{m}{2}]} \\ I_{ni} & I_{[\frac{m}{2}]} & I_{[\frac{m-2}{2}]} \end{pmatrix}.$

Note again that $\omega_0^{m(n-1)} = I$, unless $m$ is odd and $n$ is even, where $\omega_0^{m(n-1)} = \omega_0$, and in this case $\omega_0$ commutes with $h_{\gamma_0}$.

**Proposition 3.4.** Let $q \in Q_{\text{m+2m}}^H(\mathcal{A})$ be of the form

$$q = \begin{pmatrix} a & x & y \\ b & x' & \ast \\ \ast & \ast & \ast \end{pmatrix},$$

where $a \in \text{GL}_{m}(\mathcal{A}), b \in H_m(\mathcal{A})$. Assume that $H$ is linear. Then

$$\Lambda(f_{\Delta(\sigma,\text{m+1})}, \varphi)(qh) =$$

$$|\det(a)|^{-m(n-1)} \int_{H_m(\mathcal{A})} \varphi(b'g) \int_{U_{\text{m+1}}(\mathcal{A})} f_{\Delta(\sigma,\text{m+1})}^{\psi}(\hat{a} \delta_0 t(g, h)) \psi_H^{-1}(u)du dg,$$

where $b' = J_0^{-1} b J_0$ (see [13.23, 33.1], Here, $\hat{a} = \text{diag}(a, I_{2m}, \ast^\ast).$)

**Proof.** Write in (3.26), with $qh$ instead of $h$, $(g, qh) = (b', q)((b')^{-1} g, h)$ (see 3.31). Change variable, in (3.26), $g \rightarrow b'g$. We get

$$\Lambda(f_{\Delta(\sigma,\text{m+1})}, \varphi)(qh) =$$

$$\int_{H_m(\mathcal{A})} \varphi(b'g) \int_{U_{\text{m+1}}(\mathcal{A})} f_{\Delta(\sigma,\text{m+1})}^{\psi}(\delta_0 t(b', q) t(g, h)) \psi_H^{-1}(u)du dg.$$
Now, it is straightforward to check that the inner $du$-integral is equal to
\[ |\det(a)|^{-m(n-1)} \int_{U_{m(n-1)}(\mathbb{A})} f_\psi^{\Delta(\tau, m+i)\gamma_0^{(\epsilon)}, s} (\text{diag}(a, (b^\delta)^{\Delta_n})^\gamma \delta_0 ut(g, h)) \psi_H^{-1}(u) du. \]
The factor $|\det(a)|^{-m(n-1)}$ comes from a change of variables in $u$. The main part of the calculation is to check that
\[ \delta_0 t(b', q) \delta_0^{-1} = \text{diag}(a, (b^\delta)^{\Delta_n}, ((b^\delta)^\epsilon)^{\Delta_n}, a^*) u, \]
where $u \in U_{m+i}(\mathbb{A})$. By (3.17),
\[ f_\psi^{\Delta(\tau, m+i)\gamma_0^{(\epsilon)}, s} (\text{diag}(a, (b^\delta)^{\Delta_n})^\gamma \delta_0 ut(g, h)) = f_\psi^{\Delta(\tau, m+i)\gamma_0^{(\epsilon)}, s} (\delta_0 ut(g, h)). \]
This proves the proposition.

Let us write the version of the last proposition in the metaplectic case. Let $q$ be the matrix as in the proposition, and $h \in \text{Sp}_{m+2n}(\mathbb{A})$. Here $m = 2m'$ is even. Then the analogue of (3.29) is
\[
\Lambda(f_{\Delta(\tau, m+i)\gamma_0^{(\epsilon)}}, \varphi_\sigma)((q, (\det(a), x(b)))h) = (3.30) |\det(a)|^{-m(n-1)} \int \varphi_\sigma((b, 1)^t(g, 1)) f_{\Delta(\tau, m+i)\gamma_0^{(\epsilon)}}((\hat{u}, 1) \delta_0 ut(g, h)) \psi_H^{-1}(u) du dx,
\]
Here, $g$ is integrated over $H_m(\mathbb{A}) = \text{Sp}_m(\mathbb{A})$ and $u$ is integrated over $U_{m(n-1)}(\mathbb{A})$; $(b, 1)^t$ is given by (1.17). The main point here is to carry out precisely in the metaplectic group the following conjugation (notation as above)
\[ (\delta_0, 1)(t(b', q), (x(b'), \det(a)x(b))(\delta_0, 1)^{-1}, \]
and show that it is equal to
\[ (\delta_0 t(b', q), c(\bar{u}_0, j(b', b))(x(b'), x(b))(\det(a), x(b))). \]
Let $\sigma'$ be the cuspidal representation of $H_m^{\omega_1}(\mathbb{A})$ realized in the space of functions $b \mapsto \varphi_\sigma(b')$, as $\varphi_\sigma$ varies in the space of $\sigma$. In the metaplectic case, $\sigma'$ is the representation of $\text{Sp}_{2m'}(\mathbb{A})$, given by $\sigma'(b, \epsilon) = \sigma((b, \epsilon)^t)$.

**Theorem 3.5.** The function on $H_{m+2ni}^{(\epsilon)}(\mathbb{A}), \Lambda(f_{\Delta(\tau, m+i)\gamma_0^{(\epsilon)}}, \varphi_\sigma)(h)$, defined for $\text{Re}(s)$ sufficiently large by the integral (3.29), admits an analytic continuation to a meromorphic function of $s$ in the whole plane. It defines a smooth meromorphic section of
\[ \rho_{\Delta(\tau, i)\gamma_0^{(\epsilon)}}, \sigma', s = \text{Ind}_{H_{m+i}^{\omega_1}(\mathbb{A})}^{H_{m+2n}(\mathbb{A})} \Delta(\tau, i)\gamma_0^{(\epsilon)} |\det|^{s} \times \sigma'. \]
Thus, by (3.26), $\mathcal{E}(f_{\Delta(\tau, m+i)\gamma_0^{(\epsilon)}}, \varphi_\sigma)(h)$ is the Eisenstein series on $H_{m+2n}(\mathbb{A})$, corresponding to the section $\Lambda(f_{\Delta(\tau, m+i)\gamma_0^{(\epsilon)}}, \varphi_\sigma)$ of $\rho_{\Delta(\tau, i)\gamma_0^{(\epsilon)}}, \sigma', s$.

**Proof.** Consider the r.h.s. of (3.29), for $\text{Re}(s)$, sufficiently large, so that the integral converges absolutely. It is clearly smooth in $h$ and of moderate growth. Fix $h$ in the inverse image of $K_{H_{m+2ni}}$, the standard maximal compact subgroup of $H_{m+2ni}(\mathbb{A})$, inside $H_{m+2ni}(\mathbb{A})$, and then consider the integral as a function of $(a, b) \in \text{GL}_m^{(\epsilon)}(\mathbb{A}) \times H_{m}^{(\epsilon)}(\mathbb{A})$. It is left invariant to $\text{GL}_m^{(\epsilon)}(F) \times H_{m}(F)$. As a function of $b$, it lies in $\sigma'$. For any fixed $h_0 \in H_{\mathbb{A}}$, the following function on $\text{GL}_{(m+i)n}(\mathbb{A})$,
A \mapsto f^\psi_{\Delta(t,m+i)\gamma^{(c)}_t}\hspace{0.5em}(\hat{A}h_0)\text{ factors through the constant term along the unipotent radical }V_{ni,mn}\text{ (of the parabolic subgroup }P_{ni,mn}\text{ of }\text{GL}_{(m+i)n}).\text{ In the metaplectic case, we mean the function on }\text{GL}_{(m+i)n}(\mathbb{A}),\text{ }A \mapsto f^\psi_{\Delta(t,m+i)\gamma_{n,s}}((\hat{A}, 1)h_0).\text{ This constant term is, in fact, applied to }\Delta(t, m + i)\text{. Hence the following function on }\text{GL}_{ni}(\mathbb{A}),\text{ }a \mapsto f^\psi_{\Delta(t,m+i)\gamma^{(c)}_{s}(\hat{a}h_0)}\text{ lies in }|\det(a)|^{\frac{(m+i)n-4}{2}}\text{ times a function obtained as }\phi(\text{diag}(a, I_{mn})),\text{ where }\phi\text{ lies in the constant term of }\Delta(t, m + i)\text{ along }V_{ni,mn}.\text{ Altogether, taking into account the factor }|\det|^{-m(n-1)}\text{ in front of }\text{(3.29)},\text{ it is easy to check that, as a function of }a,\text{ we get an element of }

\left(\frac{\delta}{Q_{ni}^{H_{m+2ni}}}|\det|^{-s}\Delta(t, i)\gamma^{(c)}_t\right).

Now it is clear that the r.h.s. of \text{(3.29)} defines an element of }

\left(\frac{\delta}{Q_{ni}^{H_{m+2ni}}}|\det|^{-s}\sigma^t\right).

Since the function \(q \mapsto \Lambda(f_{\Delta(t,m+i)\gamma^{(c)}_{s},\varphi})(qh),\) for \(q \in Q^{(c)}_{ni}H_{m+2ni}(\mathbb{A})\) is invariant to the unipotent radical, we get what we want, for \(\text{Re}(s)\) sufficiently large. In particular, \(\text{(3.29)}\) exhibits, (for \(\text{Re}(s)\) sufficiently large) \(\mathcal{E}(f_{\Delta(t,m+i)\gamma^{(c)}_t},\varphi)(h)\) as an Eisenstein summation corresponding to the section \(\Lambda(f_{\Delta(t,m+i)\gamma^{(c)}_t},\varphi)(h)\) of \(\rho_{\Delta(t,i)\gamma^{(c)}_t,\sigma^t,s^t}\). Hence the usual calculation of the constant term of the series \(\text{(3.29)},\) along the unipotent radical of \(Q^{H_{m+2ni}}_{ni}\), works, that is

\[\mathcal{E}(f_{\Delta(t,m+i)\gamma^{(c)}_t},\varphi)Q_{ni}(\hat{a}_t)\mathcal{E}(f_{\Delta(t,m+i)\gamma^{(c)}_t},\varphi)Q_{ni}(\hat{a}_t) = \Lambda(f_{\Delta(t,m+i)\gamma^{(c)}_t},\varphi) + M(\Lambda(f_{\Delta(t,m+i)\gamma^{(c)}_t},\varphi)),\]

where the second term is obtained by applying the associated intertwining operator to \(\Lambda(f_{\Delta(t,m+i)\gamma^{(c)}_t},\varphi)\). Let \(a_t = tI_{ni},\) for \(t \in \mathbb{A}^\times\). Then it follows that

\[\begin{align*}
\delta_{Q_{ni}^{H_{m+2ni}}}(\hat{a}_t)\mathcal{E}(f_{\Delta(t,m+i)\gamma^{(c)}_t},\varphi)Q_{ni}(\hat{a}_t)h &= \\
(3.31) \quad \alpha_s(t)\Lambda(f_{\Delta(t,m+i)\gamma^{(c)}_t},\varphi)(h) + \beta_s(t)M(\Lambda(f_{\Delta(t,m+i)\gamma^{(c)}_t},\varphi))(h),
\end{align*}\]

where \(\alpha_s(t) = \gamma^{(c)}_t(t^{ni})w(t)|t|^{n\text{is}},\) \(\beta_s(t) = \gamma^{(c)}_t(t^{-ni})w^{-1}(t)|t|^{-n\text{is}}\). Again, in the metaplectic case, we replace \(\hat{a}_t\) by \((\hat{a}_t, 1)\). Consider \(3.31\) for two different values \(|t_1|, |t_2|\), and solve the resulting \(2 \times 2\) system of linear equations. By Cramer’s rule, we get that

\[\Lambda(f_{\Delta(t,m+i)\gamma^{(c)}_t},\varphi)(h) = \]

\[\begin{align*}
(3.32) \quad \theta_1(s)\mathcal{E}(f_{\Delta(t,m+i),t\gamma^{(c)}_t},\varphi)Q_{ni}(\hat{a}_{t_1}h) + \theta_2(s)\mathcal{E}(f_{\Delta(t,m+i),t\gamma^{(c)}_t},\varphi)Q_{ni}(\hat{a}_{t_2}h),
\end{align*}\]

where \(\theta_1(s), \theta_2(s)\) are explicit easy meromorphic functions. Since \(\mathcal{E}(f_{\Delta(t,m+i)\gamma^{(c)}_t},\varphi)Q_{ni}(h)\) is holomorphic (for any \(h\)), away from poles of the Eisenstein series \(E(f_{\Delta(t,m+i)\gamma^{(c)}_t},\varphi),\) \(3.32\) gives the meromorphic continuation of \(\Lambda(f_{\Delta(t,m+i)\gamma^{(c)}_t},\varphi)(h).\) This proves the theorem. \(\square\)
4. The section $\Lambda(f_{\Delta(\tau, m+i)\gamma^{(\psi)}_{\psi}, \varphi_\sigma})$ and its relation to the doubling integrals for $\sigma \times \tau$

Our main goal here is to examine the section $\Lambda(f_{\Delta(\tau, m+i)\gamma^{(\psi)}_{\psi}, \varphi_\sigma})$ as a meromorphic function. In the linear case, we will consider the automorphic form in $\Delta(\tau, i) \otimes \sigma^\dagger$,

\[(a, g) \mapsto \delta_{Q_{m+2n1}^{-1}\hat{a}}(\hat{a})|\det(a)|^{-s} \Lambda(\Delta(\tau, m+i), \varphi_\sigma)((a \ g \ a^*))\]

Here, $a \in \text{GL}_{m1}(\mathbb{A})$, $g \in H_m(\mathbb{A})$. In the metaplectic case, we modify this to

\[\delta_{Q_{m+2n1}^{-1}\hat{a}}(\hat{a})|\det(a)|^{-s\psi}(\det(a))(\det(a), x(g)) \Lambda(\Delta(\tau, m+i)\gamma_{\psi}, \varphi_\sigma)((a \ g \ a^*), 1)\]

We first apply to $\Delta(\tau, i)$ the Fourier coefficient along $V_{\Delta^i} \subset \text{GL}_{m1}$, with respect to the following character of $V_{\Delta^i}(\mathbb{A})$,

\[(4.1) \quad \psi_{V_{\Delta^i}}^{-1}\left(\begin{array}{cccc} I_i & x_1 & \cdots & \ast \\ \vdots & & & \\ x_{n-1} & \vdots & & I_i \end{array}\right) = \psi^{-1}(tr(x_1) + \cdots + tr(x_{n-1})).\]

Next, we take an $L^2$-pairing along $H_{m}^{(\psi)}(\mathbb{A})$ against a cusp form $g \mapsto \xi_\sigma(g^\psi)$, where $\xi_\sigma \in \sigma$. We get

\[\mathcal{L}(f_{\Delta(\tau, m+i)\gamma^{(\psi)}_{\psi}, \varphi_\sigma, \xi_\sigma}) = \int_{H_m(\mathbb{A})} <\sigma(g)\varphi_\sigma, \xi_\sigma> \int_{U_{m+2n1}(\mathbb{A})} f_{W, \psi}^{\Delta(\tau, m+i)\gamma^{(\psi)}, \sigma}(\delta_{0ut}(g, l_{m+2n1}))\psi_{H}^{-1}(u)du\, \text{d}ug.\]

We explain the notations:

\[<\sigma(g)\varphi_\sigma, \xi_\sigma> = \int_{H_m(F)\setminus H_m(\mathbb{A})} \varphi_\sigma(bg)\overline{\xi_\sigma(b)}\, \text{d}b,\]

and, for $y \in H_{\mathbb{A}}$,

\[f_{W, \psi}^{\Delta(\tau, m+i)\gamma^{(\psi)}, \sigma}(y) = \int_{V_{\Delta}(F)\setminus V_{\Delta}(\mathbb{A})} f_{W, \psi}^{\Delta(\tau, m+i)\gamma^{(\psi)}, \sigma}(\hat{y})\psi_{V_{\Delta^i}}(v)\, \text{d}v.\]

Recall that in the metaplectic case, unipotent elements such as $\hat{v} \in V_{\Delta^i}(\mathbb{A})$, are identified with $(\hat{v}, 1)$. The integral (4.2) is Eulerian in the sense that for decomposable data it is a product of similar local integrals

\[\mathcal{L}(f_{\Delta(\tau, m+i)\gamma^{(\psi)}, \varphi_\sigma, \xi_\sigma}) = \prod_{v} \mathcal{L}_{v}(f_{\Delta(\tau, m+i)\gamma^{(\psi)}, \sigma, \varphi_\sigma, \xi_\sigma}).\]

The notations are as follows.
\( \mathcal{L}_v(\Delta_{(\tau, i, m)}\varphi_{\sigma_v}, \xi_{\sigma_v}) = \)

\[
\int < \sigma_v(g)\varphi_{\sigma_v}, \xi_{\sigma_v}> \int f_{\Delta_{(\tau, i, m)}(\varphi_{\psi_v})}(\delta_0 u i(g, I_{m+2ni}); I_{ni}, I_{mni})\psi_{H_v}^{-1}(u)du dg;
\]

The \( dg \)-integration is over \( H_m(F_v) \) and the \( du \)-integration is over \( U'_{m(n-1)}(F_v) \).

Here, for a place \( v \), we choose a realization \( V_{\sigma_v} \) of the space of \( \sigma_v, \varphi_{\sigma_v} \in V_{\sigma_v} \), and \( \xi_{\sigma_v} \in \hat{V}_{\sigma_v} \).

Let

\[
\rho_{\Delta_{(\tau, i, m)}(\varphi_{\psi_v})} = \text{Ind}_{Q_{ni,mn}(F_v)}^{H(F_v)}(\Delta(\tau, i)\gamma_{\psi_v}(\det \cdot)^{-\frac{m}{2}} \times \Delta(\tau, m)\gamma_{\psi_v}(\det \cdot)^{\frac{m}{2}}),
\]

where \( \Delta(\tau, i) \) is realized in its model with respect to \( \psi_{V_{\psi_v}}^{-1} \) and \( \Delta(\tau, m) \) is realized in its model with respect to \( \psi_{V_{\psi_v}}^{-1} \). These are the models which show up in \[\text{CFGK17}\], where they are called Whittaker-Speh-Shalika models. For \( \Delta(\tau, i) \), this is a space of smooth functions on \( GL_n(F_v) \), which translate from the left, with respect to \( V_{\psi_v}(F_v) \), according to the character \( (1, 1) \); similarly for \( \Delta(\tau, m) \).

Then, in the linear case, \( f_{\Delta_{(\tau, i, m)}(\varphi_{\psi_v})} \) is a section of \( \rho_{\Delta_{(\tau, i, m)}(\varphi_{\psi_v})} \), and we view it as a function on \( H(F_v) \times GL_{ni}(F_v) \times GL_{mn}(F_v) \), such that for a fixed element in \( H(F_v) \), the function in the two other variables lies in the tensor product of the two models above of \( \Delta(\tau, i) \) and \( \Delta(\tau, m) \). It will be convenient to simplify notation and to re-define \( f_{\Delta_{(\tau, i, m)}(\varphi_{\psi_v})} \) as a function on \( \text{Sp}_{2n(m+i)}(F_v) \times GL_{ni}(F_v) \times GL_{mn}(F_v) \), where we use the homomorphism

\[
GL_{ni}(F_v) \times GL_{mn}(F_v) \rightarrow GL_{n(m+i)}(F_v)
\]

given by

\[
((a_1, \alpha), (a_2, \beta)) \mapsto \left( \begin{array}{c} a_1 \\ a_2 \end{array} \right), \alpha \beta(\det(a_1), \det(a_2)).
\]

We fix a finite set of places \( S \), containing the Archimedean places, outside which \( \tau, \sigma \) and \( \psi \) are unramified, \( f_{\Delta_{(\tau, i, m)}(\varphi_{\psi_v})} \) is spherical and normalized, as well as the matrix coefficient \( g \mapsto < \sigma_v(g)\varphi_{\sigma_v}, \xi_{\sigma_v}> \).

The integral \[\text{Eq}(5)\] converges absolutely for \( \text{Re}(s) \) sufficiently large. The technicalities of the proof of this convergence are standard and similar to those of the convergence of the local integrals in \[\text{CFGK17}\]. Consider the inner integral in \[\text{Eq}(5)\],

\[
f_{\Delta_{(\tau, i, m)}(\varphi_{\psi_v})}(g) = \int f_{\Delta_{(\tau, i, m)}(\varphi_{\psi_v})}(\delta_0 u i(g, I_{m+2ni})); \psi_{H_v}^{-1}(u)du.
\]

Write \( g = k_1k_2 \) in the Cartan decomposition, with \( a = \text{diag}(a_1, \ldots, a_1(\overline{\overline{m}})), |a_1| \leq \cdots \leq |a_1(\overline{m})| \leq 1; k_1, k_2 \in K_{H_m}(F_v) \). As in the end of the proof of Prop. \[\text{Eq}(3.3)\].
\[ \ell_{\psi_v}(f_{\Delta(\tau_0,i;m)}^{(s)})(g) = \]

\[ \ell_{\psi_v}(\rho_{\Delta(\tau_0,i;m)}^{(s)}(\tau_2, k_2, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + k_1)) f_{\Delta(\tau_0,i;m)}^{(s)}(\hat{a}), \]

(4.6)

where \( k_1 = J_0 k_0^{-1} J_0^{-1} \) (see (1.13)). (We know that in the metaplectic case, the double cover splits over \( \text{Sp}_m(\mathbb{O}_v) \).) Fixing \( k_1, k_2, k_3 \), we may assume that \( g = \hat{a} \). Note that outside \( S \), \( \ell_{\psi_v}(f_0^{(s)}(\Delta(\tau_0,i;m))^{(s)}(\hat{a})) = \ell_{\psi_v}(f_0^{(s)}(\Delta(\tau_0,i;m))^{(s)}(\hat{a})) \).

Write the elements of \( U''_{m(n-1)}(\mathbb{F}_v) \) in the form \( u_{x,y}' \), as in (5.27), with \( x = (x_1, x_2, x_3) \), where \( x_1 \) has \( \left\lfloor \frac{m}{2} \right\rfloor \) columns, \( x_2 \) has \( ni \) columns, and \( x_3 \) has \( m - \left\lfloor \frac{m}{2} \right\rfloor \) columns. Let

\[ U''_{m(n-1)}(\mathbb{F}_v) = \{ u_{x,y} \in U''_{m(n-1)}(\mathbb{F}_v) \mid x_2 = 0 \}. \]

Recall that in the metaplectic case, we identify the elements \( (u_{x,y}', 1) \) with \( u_{x,y} \).

**Proposition 4.1.** Let \( v \) be a place of \( F \). There is a section \( f'_{\Delta(\tau_0,i;m)}^{(s)} \), which depends on (the smoothness of) \( f_{\Delta(\tau_0,i;m)}^{(s)} \), such that for all diagonal \( a \),

\[ \ell_{\psi_v}(f_{\Delta(\tau_0,i;m)}^{(s)})(\hat{a}) = \int_{U''_{m(n-1)}(\mathbb{F}_v)} f'_{\Delta(\tau_0,i;m)}^{(s)}(\hat{a}, I_{m+2ni})(\psi_v^{-1}(u) du. \]

If \( f_{\Delta(\tau_0,i;m)}^{(s)} \) is spherical, then \( f'_{\Delta(\tau_0,i;m)}^{(s)} = f_{\Delta(\tau_0,i;m)}^{(s)} \). In general, \( f'_{\Delta(\tau_0,i;m)}^{(s)} \) is obtained from \( f_{\Delta(\tau_0,i;m)}^{(s)} \) by a finite sequence of convolutions against certain Schwartz functions (described in the proof).

**Proof.** Assume that \( v \) is a finite place. We assume, for simplicity, that \( \psi_v \) is normalized. Denote, for \( x_2 \in M_{m(n-1)\times ni}(\mathbb{F}_v) \),

\[ \ell_{\psi_v}(f_{\Delta(\tau_0,i;m)}^{(s)})(\hat{a}, x_2) = \]

\[ \int_{U''_{m(n-1)}(\mathbb{F}_v)} f_{\Delta(\tau_0,i;m)}^{(s)}(\delta u''_{0, x_2, 0, 0})(\hat{a}, I_{m+2ni})(\psi_v^{-1}(u') du''. \]

(4.8)

We will show that (4.8) has support in \( x_2 \) in a set of the form \( r_0 \Omega \), where \( \Omega \) is a compact set, independent of \( a \), and \( r_0 = (\hat{a})^{\Delta n-1} = \text{diag}(\hat{a}, \ldots, \hat{a}) \), \( n-1 \) times. In case \( v \) is outside \( S \), \( \Omega = M_{m(n-1)\times ni}(\mathbb{O}_v) \), where \( \mathbb{O}_v \) is the ring of integers in \( \mathbb{F}_v \). We let \( \mathcal{P}_v \subset \mathbb{O}_v \) denote the prime ideal. Put \( d_a = l(\hat{a}, I_{m+2ni}) \). Note that in (4.8),

\[ d_a^{-1} u_{0, x_2, 0, 0} d_a = u_{0, x_2, 0, 0}. \]

(4.8)

Let \( z \in M_{m(n-1)\times ni}(\mathbb{F}_v) \). Take \( k \) to be so large that the matrix

\[ c_z = \begin{pmatrix} I_{m(n-1)} \\ I_{\frac{m}{2}} \\ z \\ I_{ni} \end{pmatrix} \]

fixes \( f_{\Delta(\tau_0,i;m)}^{(s)} \). For \( v \) outside \( S \), we take \( k = 0 \). Note that \( d_a c_z d_a^{-1} = c_{z a^{-1}} \). Note also that \( c_{z a^{-1}} \) commutes with \( \omega_0, h_0 \) and \( h_0 \) commutes with the elements \( u_{x,y} \). Conjugate \( c_{z a^{-1}} u_{x,y} c_{z a^{-1}} = u_{x+1, x+2 a^{-1}, x+3, y} \). Change variable \( x_1 \mapsto x_1 - \)
\[ x_2 a^{-1} \text{ in } (4.8), \text{ where we write } u'' = u_{x_1,0,x_3,y}. \text{ Note that } \psi_1^{-1}(u'') = \psi_H^{-1}(u'') \psi(\text{tr}(a^{-1} x_2 (n-1) z)) \text{, where we write} \\
\begin{align*}
x_2 = \begin{pmatrix}
    x_2^{(1)} \\
    \vdots \\
    x_2^{(n-1)}
\end{pmatrix},
x_2^{(j)} = \begin{pmatrix}
    x_2^{(j,1)} \\
    x_2^{(j,2)}
\end{pmatrix}; x_2^{(j,1)} \in M_{[\mathbb{F}_q]}(F_v), x_2^{(j,2)} \in M_{(m-|\mathbb{F}_q|)}(F_v).
\end{align*}
\]

Next, conjugation by \( e^0 \) takes \( c_{2a^{-1}} \) into the unipotent radical \( U_{Q_{a^{-1}}(F_v)} \), and hence
\[
\ell'_{\psi_v}(f_{\Delta(\tau_i;m) \gamma_v}^{(e)})(\hat{a}, x_2) = \psi(\text{tr}(a^{-1} x_2 (n-1) z)) \ell'_{\psi_v}(f_{\Delta(\tau_i;m) \gamma_v}^{(e)})(\hat{a}, x_2).
\]
This implies that \( x_2^{(n-1,1)} \in a M_{[\mathbb{F}_q]}(P_{v^{-k}}) \), or else \((4.8)\) is zero. In case our section is spherical, this means that \( a^{-1} x_2 (n-1,1) \) has integral coordinates. (We assume that \( \psi_v \) is normalized for \( v \notin S \).) Thus, we can express \((4.9)\)
\[
\ell'_{\psi_v}(f_{\Delta(\tau_i;m \gamma_v}^{(e)})(\hat{a}, x_2) = \sum_{j=1}^N \phi^j(\hat{a}, x_2 (n-1,1)) \ell'_{\psi_v}(f_{\Delta(\tau_i;m \gamma_v}^{(e)})(\hat{a}, x_2 (n-1,1)),
\]
where \( \phi^j \) are characteristic functions of small neighborhoods of a finite set of elements; \( N \) and the sections \( f_{\Delta(\tau_i;m) \gamma_v}^{(e)} \) depend only on \( f_{\Delta(\tau_i;m) \gamma_v}^{(e)} \)\(a^{-1} x_2 (n-1,1) \) by zero. Again, in the spherical case, \( N = 1 \), \( \phi^1 \) is the characteristic function of the corresponding matrices with integral coordinates and \( f_{\Delta(\tau_i;m) \gamma_v}^{(e)} = f_{\Delta(\tau_i;m) \gamma_v}^{(e)} \). We may now continue and examine \( \ell'_{\psi_v}(f_{\Delta(\tau_i;m) \gamma_v}^{(e)})(\hat{a}, x_2 (n-1,1)) \). Thus we may consider \((4.8)\) with \( x_2 (n-1,1) \) instead of \( x_2 \).

Let \( z \in M_{[\mathbb{F}_q]}(P_{v^k}), \) with \( k \) large, so that the following element fixes \( f_{\Delta(\tau_i;m) \gamma_v}^{(e)} \)\(,\)
\[
c_z = \text{diag}(I_{m(n-1)}, 0, 0, z, 0, z', 0, 0), I_{n_i}, I_{2(m-|\mathbb{F}_q|)}(\hat{a}, x_2 (n-1,1)),
\]
In the spherical case, we take \( k = 0 \). Note that \( d_{a} e_{a} a^{-1} = e_{a}, \text{ and that conjugation by } \delta_0 \text{ takes } e_{a} \text{ into the unipotent radical } U_{Q_{a^{-1}}(F_v)}. \)

Next,
\[
e_{a^{-1}} u_{x, y} e_{a} = \begin{pmatrix}
    I_{m(n-1)} x & v(x_1 a_z, x_2 z'(a^*)^{-1}) & y \\
    I_{m+n} & 0 & v(x_1 a_z, x_2 z'(a^*)^{-1})' \\
    I_{m+n} & 0 & x'
\end{pmatrix},
\]
where \( v(x_1 a_z, x_2 z'(a^*)^{-1}) = (0, m(n-1), x_1 a_z, x_2 z'(a^*)^{-1}). \) Denote the last matrix by \( u_{x, y} v(x_1 a_z, x_2 z'(a^*)^{-1}); y \). Thus, we get that
where $u'' = u_{x_1,0,0,0}$, and $x_2 = x_2(n-1,1)$. We have

$$f_{\Delta(\tau_v,i,m)}(\gamma_{\psi_v})^0(\delta_0u_{0,v(x_1;az,x_2z'(a^*)^{-1})};0u''u_{0,0,0,0})\psi_{\psi_v}^{-1}(u'')du'',$$

We conclude that

$$
\ell'_v(f_{\Delta(\tau_v,i,m)}(\gamma_{\psi_v})^0)(\hat{a},x_2) = \psi(tr(x_2^{(n-1,1)}z'(a^*)^{-1}))\ell'_v(f_{\Delta(\tau_v,i,m)}(\gamma_{\psi_v})^0)(\hat{a},x_2),
$$

for all $z \in M_{[\mathbb{F}]^{n_i}}(\mathcal{P}_v^k)$. Here, $x_2^{(n-1,1)} = x_2^{(n-1,2)}$, in case $m$ is even. In case $m$ is odd, $x_2^{(n-1,3)}$ is obtained from $x_2^{(n-1,2)}$ by deleting its first row. This implies that if

$$\ell'_v(f_{\Delta(\tau_v,i,m)}(\gamma_{\psi_v})^0)(\hat{a},x_2(n-1,1)) \neq 0,$$

then $(a^*)^{-1}x_2^{(n-1,3)} \in M_{[\mathbb{F}]^{n_i}}(\mathcal{P}_v^k)$. When $m = 2m' - 1$ is odd, we still need to show that we have a similar compact support in the first row of $x_2^{(n-1,2)}$. For this we use right translations by the $\omega_0^{m(n-1)}$-conjugate of

$$
\begin{pmatrix}
I_{m(n-1)} & 0 & z \\
0 & I_{m-1} & z \\
1 & 0 & 1
\end{pmatrix}^{\wedge},
$$

for column vectors $z \in (\mathcal{P}_v^k)^{n_i}$ and $k$ large. The argument is similar. We showed, so far, that $\ell'_v(f_{\Delta(\tau_v,i,m)}(\gamma_{\psi_v})^0)(\hat{a},x_2)$ is supported such that $x_2^{(n-1)} \in \hat{a}M_{m|n_i}(\mathcal{P}_v^{-k})$, with $k = 0$ in the spherical case, and then (for $x_2^{(n-1)} \in \hat{a}M_{m|n_i}(\mathcal{O}_v)$)

$$
\ell'_v(f_{\Delta(\tau_v,i,m)}(\gamma_{\psi_v})^0)(\hat{a},x_2) = \ell'_v(f_{\Delta(\tau_v,i,m)}(\gamma_{\psi_v})^0)(\hat{a},x_2(n-1)),
$$

where $x_2(n-1)$ is obtained from $x_2$ by replacing $x_2^{(n-1)}$ by zero. For $v \in S$ and finite, we conclude that there is an expression similar to [1.3] with $x_2(n-1)$ instead of $x_2(n-1,1)$, the Schwartz functions are now on $M_{m|n_i}(\mathcal{F}_v)$, evaluated on $\hat{a}^{-1}x_2(n-1)$. Thus, we may continue and consider [1.8] with $x_2(n-1)$ instead of $x_2$. The argument is similar, using right translations by

$$
\begin{pmatrix}
I_{m(n-2)} & 0 & z \\
0 & I_{m-1} & z \\
1 & 0 & 1
\end{pmatrix}^{\wedge},
$$

for $z \in M_{m|n_i}(\mathcal{P}_v^k)$, and $k$ large so that all such elements fix $f_{\Delta(\tau_v,i,m)}(\gamma_{\psi_v})^0$. We conclude that in the support, $x_2^{(n-2)} \in \hat{a}M_{m|n_i}(\mathcal{P}_v^{-k})$. In general, let $x_2(n-j+1)$ be the matrix obtained from $x_2$ by replacing $x_2^{(n-j)}$, $x_2^{(n-j-1)}$, ..., $x_2^{(n-j+1)}$ by zero. Then we show that $\ell'_v(f_{\Delta(\tau_v,i,m)}(\gamma_{\psi_v})^0)(\hat{a},x_2(n-j+1))$ is supported such that

$$x_2^{(n-j-1)} \in \hat{a}M_{m|n_i}(\mathcal{P}_v^{-k}),$$

for $k$ large. For this, we apply a similar argument by using right translations by the following elements fixing $f_{\Delta(\tau_v,i,m)}(\gamma_{\psi_v})^0$, for all $z \in [\mathbb{F}]^{n_i}$.
By the Dixmier-Malliavin Lemma, \[ \text{DM78}. \] We show how to adapt the first step of the proof in case unipotent subgroups above, and apply at each step the Dixmier-Malliavin Lemma \[ \text{DM78}. \] When \[ z \] close to zero, we use convolutions against Schwartz functions on the unipotent subgroups above, and apply at each step the Dixmier-Malliavin Lemma \[ \text{DM78}. \] We show how to adapt the first step of the proof in case \[ v \] is Archimedean. By the Dixmier-Malliavin Lemma, \( f_{\Delta(\tau_v,i;m)}g_{\gamma_{\psi_v},s} \) is a finite sum of sections of the form \( \phi' \ast \varphi_{\Delta(\tau_v,i;m)}g_{\gamma_{\psi_v},s} \), where \( \phi' \) is Archimedean, the proof is the same, except that instead of right translations by the unipotent elements above, with \( z \) close to zero, we use convolutions against Schwartz functions on the unipotent subgroups above, and apply at each step the Dixmier-Malliavin Lemma \[ \text{DM78}. \] When \[ v \] is Archimedean, the proof is the same, except that instead of right translations by the unipotent elements above, with \( z \) close to zero, we use convolutions against Schwartz functions on the unipotent subgroups above, and apply at each step the Dixmier-Malliavin Lemma \[ \text{DM78}. \] Then the calculations that we did in this case show that

\[ \ell'_{\psi_v}(\phi' \ast \varphi_{\Delta(\tau_v,i;m)}g_{\gamma_{\psi_v},s})(\hat{a},x_2) = \hat{\phi}(a^{-1}x_2^{(n-1,1)})\ell'_{\psi_v}(\varphi_{\Delta(\tau_v,i;m)}g_{\gamma_{\psi_v},s})(\hat{a},x_2), \]

where \( \hat{\phi} \) is the Fourier transform

\[ \hat{\phi}(a^{-1}x_2^{(n-1,1)}) = \int_{M_{ni}(\mathbb{A})} \phi(z) \psi(tr(a^{-1}x_2^{(n-1,1)}z))dz. \]

The proof carries over in complete analogy with the non-Archimedean case. This completes the proof of the proposition. \[ \square \]

We can now compute the value of \( 4.3 \), for unramified normalized data.

**Theorem 4.2.** Let \( v \) be a place outside \( S \). Then

\[ \mathcal{L}_v(f^0_{\Delta(\tau_v,i;m)}g_{\gamma_{\psi_v},s},f^0_{\sigma_v},f^0_{\tau_v}) = \frac{L_{\tau_v,\psi_v}(\sigma_v \times \tau_v, s + \frac{i+1}{2})}{D_{\tau_v}^H(s)}. \]

When \( H = \text{Sp}_{2n(2m'+i)} \),

\[ D_{\tau_v}^H(s) = L(\tau_v, s + m' + \frac{i + 1}{2}) \prod_{k=1}^{m'} L(\tau_v, \wedge^2, 2s + 2k + i)L(\tau_v, \text{sym}^2, 2s + 2k + i - 1). \]

When \( H = \text{Sp}_{2n(2m'+i)}^{(2)} \),

\[ D_{\tau_v}^H(s) = \prod_{k=1}^{m'} L(\tau_v, \wedge^2, 2s + 2k + i)L(\tau_v, \text{sym}^2, 2s + 2k + i) \]

When \( H = \text{SO}_{2n(2m'+i)} \),

\[ D_{\tau_v}^H(s) = \prod_{k=1}^{m'} L(\tau_v, \wedge^2, 2s + 2k + i)L(\tau_v, \text{sym}^2, 2s + 2k + i - 1). \]
When $H = \text{SO}_{2n(2m'-1+i)}$, 
\[
D^H_{\tau_v}(s) = \prod_{k=1}^{m'} L(\tau_v, \lambda^2, 2s + 2k + i - 1) \prod_{k=1}^{m'-1} L(\tau_v, \text{sym}^2, 2s + 2k + i).
\]

Proof. By Prop. 4.1 and (4.6), the integral (4.3) is equal to (for $\text{Re}(s)$ sufficiently large)
\[
\mathcal{L}_v(f_0^0, \tau_v, \iota; m)\gamma_v(t)\delta_\sigma, \varphi_\sigma, \xi_\sigma) = \int <\sigma_v(t), \varphi_\sigma, \xi_\sigma, \psi_v(t)>(\delta_0 t u^0 I_{m+2m})(e^0)^{-1}\psi_{H_u}(u) du. \tag{4.10}
\]

Here, the $dg$-integration is along $H_m(F_v)$, and the $du$-integration is along $U''_{m(n-1)}$.

Write $u \in U''_{m(n-1)}(F_v)$ in the form $u_{x_1,0,x_3:y} = u_{x_1,0,x_3:y}$, as before. We have
\[
\epsilon_0 = \begin{pmatrix} 0 & I_m & 0 & 0 \\ 0 & 0 & 0 & I_{m(n-1)} \\ 0 & 0 & I_m & 0 \\ \delta_H I_{m(n-1)} & 0 & 0 & 0 \end{pmatrix}.
\]

Put $x = (x_1, x_3)$, viewed as a $m(n-1) \times m$ matrix, and re-define $u'_{x_1,x_3:y} = u'_{x:3:y}$.

Then
\[
u'_{x:y} = \begin{pmatrix} I_m & 0 & y \\ 0 & I_m & 0 \\ x' & I_m & 0 \\ I_m & 0 & I_{m(n-1)} \end{pmatrix}^{-1} \omega_0^{m(n-1)}
\]

\[
h_{\gamma_0} = \text{diag}(I_m, I_m, \gamma_0'. I_{m(n-1)}).
\]

Finally, we explicate $t'(g, I_m)$. Assume that $m = 2m'$ is even, and write $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where the blocks are $m' \times m'$ matrices. Then
\[
t'(g, I_m) = \text{diag}(g^{\Delta_{n-1}}, \begin{pmatrix} a & 0 & b \\ 0 & I_m & 0 \\ c & 0 & d \end{pmatrix}, (g^*)^{\Delta_{n-1}});
\]

In the metaplectic case, we modify this as in (1.9).

Assume that $m = 2m' - 1$ is odd. Write $g = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$, where the first and
third block rows (resp. columns) contain $m' - 1$ rows (resp. columns). Then

$$t'(g, I_m) = \text{diag}(g^\Delta_{n-1}, \frac{1}{2}b_1 - 1, b_1 0 0 0^T) \cdot (g^*)^\Delta_{n-1})$$

Denote, for $h \in H_{2nm}(F_v)$,

$$f^0_\Delta(\tau_v, m)\gamma_{v,s}^{(\tau)}(h) = f^0_\Delta(\tau_v, i;m)\gamma_{v,s}^{(\tau)}(\text{diag}(I_{mi}, h, I_{mi})).$$

This is the spherical, normalized element of

$$\rho_{\Delta(\tau_v, m)\gamma_{v,s}^{(\tau)}} = \text{Ind}_{Q_{2nm}(F_v)}^{H_{2nm}(F_v)}(t_{\Delta(\tau_v, m)\gamma_{v,s}^{(\tau)}}) \cdot |\text{det}(.)|^{s+\frac{1}{2}},$$

where $\Delta(\tau_v, m)$ is realized in its Whittaker-Speh-Shalika model, as before. Put $\delta'_0 = c_0 h^{-1}_0$, and let $U_{m(n-1)}(F_v)$ denote the subgroup of all elements of the form $w_{x_0}$. Consider the character $\psi_{H_{2nm},v}$ of $U_{m(n-1)}(F_v)$ defined similarly to $\psi_{H_v}$, only that in [1.2], we take the following matrix $A_{H_{2nm}}$ instead of $A_H$. When $m = 2m'$ is even,

$$A_{H_{2nm}} = \begin{pmatrix} I_{m'} & 0 & 0 \\ 0_{m\times m'} & 0_{m\times m'} & I_{m'} \end{pmatrix}.$$ 

When $m = 2m' - 1$ is odd (and hence $H_{2nm} = SO_{2m(2m'-1)}$),

$$A_{H_{2nm}} = \begin{pmatrix} I_{m'-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{m'-1} \end{pmatrix},$$

where the second and fifth block rows of zeroes contain each $m' - 1$ rows. Note that the stabilizer of $\psi_{H_{2nm},v}$ inside $M_{H_{2mn}}(F_v)$ is isomorphic to $H_m(F_v) \times H_m(F_v)$. This is the subgroup of all elements

$$t'(g, h) = \text{diag}(g^\Delta_{n-1}, f'(g, h), (g^*)^\Delta_{n-1}),$$

where $g, h \in H_m(F_v) \times H_m(F_v)$, $t'(g, I_m)$ is described above, and $t'(I_m, h)$ is given as follows. When $m = 2m'$ is even,

$$t'(I_m, h) = \text{diag}(I_{m'}, h, I_{m'}).$$

Again, in the metaplectic case, we use [1.9].

When $m = 2m' - 1$ is odd, write

$$h = \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix},$$
where the corner blocks are matrices of size $(m' - 1) \times (m' - 1)$ Then, as in (1.18),

\[ t'(I_m, h) = \begin{pmatrix} I_{m-1} & 0 & 0 & 0 & 0 \\ 0 & A_1 & \frac{1}{2} B_1 & -B_1 & C_1 \\ 0 & A_2 & \frac{1}{2} (1 + B_2) & 1 - B_2 & C_2 \\ 0 & -\frac{1}{2} A_2 & \frac{1}{2} (1 - B_2) & \frac{1}{2} (1 + B_2) & -\frac{1}{2} C_2 \\ 0 & A_3 & \frac{1}{2} B_3 & -B_3 & C_3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \]

This is the set-up of the generalized doubling construction [CFGK17], corresponding to the $L$-function for $\sigma_v \times \tau_v$, except that $s$ is shifted to $s + \frac{i}{2}$. From (1.11), the integral (4.10) becomes

\[
\mathcal{L}_v(f_0^{\Delta(\tau_v,i,m)\gamma(s),s}) = \int_{H_m(F_v)} \frac{\sigma_v(g)\varphi_{\sigma_v,s}^{\psi_0} \xi_{\sigma_v}}{\delta(\tau_v,m)\gamma(s),s + \frac{i}{2}} \cdot \psi_{H_v}^{-1}(u) \, du \, dg.
\]

This is the unramified integral of [CFGK17] for $H_{m}^{(c)} \times \GL_n$, with $s$ shifted by $\frac{i}{2}$. Now, we use the unramified computation of [CFGK17] and get the theorem. □

The arguments of the last proof and of the proof of Prop. 4.11 show in general, at any place $v$

\[ \textbf{Proposition 4.3.} \textit{Given a smooth, holomorphic section } f_0^{\Delta(\tau_v,i,m)\gamma(s),s} \textit{of } \rho^{\Delta(\tau_v,i,m)\gamma(s),s}, \textit{there is a smooth, holomorphic section } \tilde{f}_0^{\Delta(\tau_v,m)\gamma(s),s + \frac{i}{2}} \textit{of } \rho^{\Delta(\tau_v,m)\gamma(s),s + \frac{i}{2}}, \textit{which depends on the given section, such that, for Re}(s) \textit{sufficiently large,}
\]

\[
\mathcal{L}_v(f_0^{\Delta(\tau_v,i,m)\gamma(s),s}) = \int_{H_m(F_v)} < \sigma_v(g)\varphi_{\sigma_v,s}^{\psi_0} \xi_{\sigma_v}> \int_{H_{m-1}(F_v)} \tilde{f}_0^{\Delta(\tau_v,m)\gamma(s),s + \frac{i}{2}} \cdot \psi_{H_v}^{-1}(u) \, du \, dg.
\]

The last integral is the local integral arising from the generalized global doubling method corresponding to $\sigma \times \tau$.

Recall the normalizing factor $d_\tau^{H,c}$ of $E(f_\Delta(\tau_v,i,m)\gamma(s),s)$ on $H(\mathbb{A})$. See (1.12) - (1.18). Let $d_\tau^{H,c}$ be its local analog at $v$ outside $S$. Note that in Theorem 4.2 $D_\tau^{H,c}(s) = d_\tau^{H,c}(s + \frac{i}{2})$. Let $d_{\sigma_\tau,v}^{H,c}(s)$ be the normalizing factor, outside $S$, corresponding to the global induced representation

\[
\rho^{\Delta(\tau_v,i,m)\gamma(s),s} = \text{Ind}_{\mathbb{A}_1}^{\mathbb{A}} \Delta(\tau_v,i)^{\gamma(s),s} \cdot \text{det}(\cdot)^{s} \otimes \sigma'.
\]

Similarly, let $d_{\sigma_\tau,v}^{H,c}(s)$ be the local analog at $v$ outside $S$.

\[ \textbf{Proposition 4.4.} \textit{In the notation above, for } v \textit{outside } S,
\]

\[
d_\tau^{H,c}(s)\mathcal{L}_v(\sigma_v \times \tau_v, s + \frac{i + 1}{2}) = d_\tau^{H,c}(s + \frac{i}{2})d_{\sigma_\tau,v}^{H,c}(s).
\]
Proof. The proof is by a straightforward verification. Let us carry this out in one example. Take $H = \text{Sp}_{2n(m+i)}$ when $i = 2i' + 1$ is odd. Here, $m = 2m'$ is even (as always in the symplectic, or metaplectic cases). Then, as in (1.19),

$$d_{\tau_v}^{\text{Sp}_{2n(m+i)}}(s) =$$

(4.14) \[ L(\tau_v, s + m' + i' + 1) \prod_{k=1}^{m'+i'+1} L(\tau_v, \wedge^2, 2s + 2k - 1) \prod_{k=i'+2}^{m'+i'} L(\tau_v, \text{sym}^2, 2s + 2k). \]

We have, as in (1.20),

$$D_{\tau_v}^{\text{Sp}_{2n(m+i)}}(s) = d_{\tau_v}^{\text{Sp}_{4n'm'}}(s + \frac{i}{2}) =$$

(4.15) \[ L(\tau_v, s + m' + i' + 1) \prod_{k = i'+2}^{i'+1} L(\tau_v, \wedge^2, 2s + 2k - 1) \prod_{k = i'+1}^{m'+i'} L(\tau_v, \text{sym}^2, 2s + 2k). \]

Thus, by (4.14), (4.15),

(4.16) \[ d_{\tau_v}^{\text{Sp}_{2n(m+i)}}(s) = d_{\tau_v}^{\text{Sp}_{4n'm'}}(s + \frac{i}{2}) \prod_{k=1}^{i'+1} L(\tau_v, \wedge^2, 2s + 2k - 1) \prod_{k=1}^{i'} L(\tau_v, \text{sym}^2, 2s + 2k). \]

We note that

(4.17) \[ d_{\sigma_v, \tau_v}^{\text{Sp}_{2n}+m}(s) = L(\sigma_v \times \tau_v, s + i' + 1) \prod_{k=1}^{i'+1} L(\tau_v, \wedge^2, 2s + 2k - 1) \prod_{k=1}^{i'} L(\tau_v, \text{sym}^2, 2s + 2k). \]

Now, (4.16), (4.17) imply that

$$d_{\tau_v}^{\text{Sp}_{2n(m+i)}}(s)L(\sigma_v \times \tau_v, s + i' + 1) = d_{\tau_v}^{\text{Sp}_{4n'm'}}(s + \frac{i}{2})d_{\sigma_v, \tau_v}^{\text{Sp}_{2n}+m}(s),$$

which is (4.13) in this case. All other cases are checked similarly. \qed

Denote in Theorem 4.2

$$f^{0,*, e}_{\Delta(\tau_v, i'; m)\gamma_{\psi_v}^{(e)}, s} = d_{\tau_v}^{H}(s)f^{0}_{\Delta(\tau_v, i; m)\gamma_{\psi_v}^{(e)}, s}. \quad (4.18)$$

Then Theorem 4.2 and (4.13) imply

$$L_v(f^{0,*, e}_{\tau_v, \Delta(\tau_v, i'; m)\gamma_{\psi_v}^{(e)}, s}, \varphi_{\sigma_v}^0, \xi_{\sigma_v}^0) = d_{\sigma_v, \tau_v}^{H}(s). \quad (4.18)$$

This is the normalizing factor at $v$ corresponding to an Eisenstein series associated to a section of $\rho_{\Delta(\tau, i)\gamma_{\psi}^{(e)}, \sigma, \gamma, s}$, say a decomposable section, such that at a place $v$, outside $S$, the local section is normalized to be a tensor product of two unramified vectors in the spaces of $\Delta(\tau_v, i), \sigma_v$, which we fix outside $S$. For $\Delta(\tau_v, i)$, we may use its model with respect to $\psi_{\psi_v}^{(e)}, s$, and take its normalized, unramified element. By (4.18), these properties are satisfied by the section

$$\lambda_{S}(f_{\Delta(\tau_v, m+i)\gamma_{\psi_v}^{(e)}, s}, \varphi_{\sigma}^*) = \frac{d_{H,S}(s)}{d_{\sigma_v, \tau_v}^{H}(s)}\Lambda_{S}(f_{\Delta(\tau_v, m+i)\gamma_{\psi_v}^{(e)}, s}, \varphi_{\sigma}^*). \quad (4.18)$$
Consider the Eisenstein series on $H_{2n+1}^{(s)}(\mathbb{A})$, corresponding to this section. Denote it by $E(\lambda_S(f_{\Delta(\tau,m+i)}\gamma_{\psi}^{(s)},\varphi_\sigma))$, and in normalized form,

$$E_S^*(\lambda_S(f_{\Delta(\tau,m+i)}\gamma_{\psi}^{(s)},\varphi_\sigma)) = d_{\alpha}^{H_{2n+1}}(s)E(\lambda_S(f_{\Delta(\tau,m+i)}\gamma_{\psi}^{(s)},\varphi_\sigma)).$$

Now we get the identity of Theorem 1.1

(4.19) $$E_S^*(f_{\Delta(\tau,m+i)}\gamma_{\psi}^{(s)},\varphi_\sigma) = E_S^*(\lambda_S(f_{\Delta(\tau,m+i)}\gamma_{\psi}^{(s)},\varphi_\sigma)),$$

where the left hand side of (4.19) is our kernel integral (1.11), with the normalized Eisenstein series on $H(\mathbb{A})$, $E_S^*(f_{\Delta(\tau,m+i)}\gamma_{\psi}^{(s)},\varphi_\sigma) = d_{\alpha}^{H_{2n+1}}E(\lambda_S(f_{\Delta(\tau,m+i)}\gamma_{\psi}^{(s)},\varphi_\sigma))$. See (4.19).

5. APPLICATION OF BESSEL COEFFICIENTS TO $E(f_{\Delta(\tau,i+1),s})$ ON $SO_{2n(1+i)}(\mathbb{A})$: DESCENT TO $SO_{2ni+1}(\mathbb{A})$

Let $H = SO_{2n(i+1)}$. The previous theorems are valid here when we take $m = 1$. Thus, in this case, $H_1 = SO_1$ is the trivial group, and $H_{m+2ni}$ is $SO_{2ni+1}$. Let us write the group $U_{m+1}$ and the character $\psi_H$ in this case. The elements of $U_{1-n}$ have the form

(5.1) $$u = \begin{pmatrix} z & x \\ I_{2ni+2} & y \\ x' & z' \end{pmatrix} \in H, \ z \in \mathbb{Z}_{n-1}$$

where $\mathbb{Z}_k$ denotes the upper unipotent subgroup of $GL_k$. The character $\psi_H$ of $U_{1-n}(\mathbb{A})$, which we re-denote by $\psi_{n-1}$ is given on $u$, with Adele coordinates, as follows. Let $\psi_{n-1}$ be the standard Whittaker character of $\mathbb{Z}_{n-1}(\mathbb{A})$, corresponding to $\psi$, that is

$$\psi_{n-1}(z) = \psi(z_{1,2} + z_{2,3} + \cdots + z_{n-2,n-1}).$$

Then

(5.2) $$\psi_{n-1}(u) = \psi_{n-1}(z)\psi(x_{n-1,ni+1} + \frac{1}{2}x_{n-1,ni+2}).$$

The character $\psi_{n-1}$ is stabilized by $SO_{2ni+1}(\mathbb{A})$ realized as the subgroup of elements $\text{diag}(I_{n-1},h,I_{n-1})$, with $h \in SO_{2ni+2}(\mathbb{A})$ satisfying

$$h \begin{pmatrix} 0_{ni} \\ 1 \\ \frac{1}{2} \\ 0_{ni} \end{pmatrix} = \begin{pmatrix} 0_{ni} \\ 1 \\ \frac{1}{2} \\ 0_{ni} \end{pmatrix}.$$ 

Let $j$ denote the isomorphism from $SO_{2ni+1}$ to this stabilizer, given by (1.8), and denote

$$t(h) = \begin{pmatrix} I_{n-1} \\ j(h) \\ I_{n-1} \end{pmatrix}, \ h \in SO_{2ni+1}.$$ 

We considered the Eisenstein series $E(f_{\Delta(\tau,i+1),s})$ on $SO_{2n(i+1)}(\mathbb{A})$, corresponding to the section $f_{\Delta(\tau,i+1),s}$, and we took its Fourier coefficient along $U_{1-n}$, with respect to $\psi_{n-1}$. We re-denote this Fourier coefficient by $D_{\psi,ni}(E(f_{\Delta(\tau,i+1),s}))$, and, as before, we view it as an automorphic function on $SO_{2ni+1}(\mathbb{A})$, realized as
above. This is the Bessel coefficient used in automorphic descent. See [GRS11]. Thus,

\[ (5.3) \]

\[ \mathcal{D}_{\psi,n_i}(E(f_{\Delta(\tau+1,i)}))(h) = \int_{U_{n-1}(F)/U_{n-1}(\mathbb{A})} f_{\Delta(\tau+1,i)}(u)(\omega_{n-1}^{-1}(u)du. \]

We don’t need to further integrate along \( H_1 \) against \( \sigma \), as \( H_1 \) is trivial. Now (3.25), Theorem 3.5 and (4.19) become in this case

**Theorem 5.1.** For \( \text{Re}(s) \) sufficiently large and \( h \in \text{SO}_{2n+1}(\mathbb{A}) \),

\[ \mathcal{D}_{\psi,n_i}(E(f_{\Delta(\tau+1,i)}))(h) = \sum_{h' \in Q_{n_i}\setminus\text{SO}_{2n+1}} \Lambda(f_{\Delta(\tau+1,i)})(h'h), \]

where

\[ (5.4) \]

\[ \Lambda(f_{\Delta(\tau+1,i)})(h) = \int_{U_{n-1}(\mathbb{A})} f_{\Delta(\tau+1,i)}^\psi(\delta_0(\omega_{n-1}^{-1}(u)dudg. \]

This function \( \Lambda(f_{\Delta(\tau+1,i)})(h) \), defined for \( \text{Re}(s) \) sufficiently large by the last integral, admits an analytic continuation to a meromorphic function of \( s \) in the whole plane. It defines a smooth meromorphic section of

\[ \rho_{\Delta(\tau,i)} = \text{Ind}_{Q_{n_i}(\mathbb{A})}^{\text{SO}_{2n+1}(\mathbb{A})}(\Delta(\tau,i)|\det|)^s. \]

Thus, \( \mathcal{D}_{\psi,n_i}(E(f_{\Delta(\tau+1,i)}))(h) \) is the Eisenstein series on \( \text{SO}_{2n+1}(\mathbb{A}) \), corresponding to the section \( \Lambda(f_{\Delta(\tau+1,i)})(h) \) of \( \rho_{\Delta(\tau,i)} \). Moreover, when we normalize (outside \( S \), as before) \( E(f_{\Delta(\tau+1,i)})(h) \) by

\[ E^*_S(f_{\Delta(\tau+1,i)})(h) = d^{|\text{SO}_{2n+1}(\mathbb{A})\setminus\text{SO}_{2n}(\mathbb{A})|}_r \sigma E(f_{\Delta(\tau+1,i)})(h), \]

then \( \mathcal{D}_{\psi,n_i}(E^*_S(f_{\Delta(\tau+1,i)}))(h) \) is an Eisenstein series on \( \text{SO}_{2n+1}(\mathbb{A}) \), corresponding to \( \rho_{\Delta(\tau,i)} \), and it is normalized outside \( S \).

For completeness, let us specify in this case the elements which appear in (5.4). \( U_{n-1}' \) is the subgroup

\[ U_{n-1}' = \{ u_{x,y} = \begin{pmatrix} I_{n-1} & x & 0 & y \\ I_{n+1} & 0 & 0 \\ I_{n+1} & x & I_{n-1} \end{pmatrix} \}, \quad \psi_{n-1}(u_{x,y}) = \psi(x_{n-1,n+1}); \]

\[ \delta_0 = \begin{pmatrix} 0 & I_{n_i} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{n-1} \\ I_{n-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{n_i} \end{pmatrix} \omega_0^{n-1}. \]

Finally

\[ f_{\Delta(\tau+1,i)}^\psi(x) = \int_{V_{n_i,n}(F)/V_{n_i,n}(\mathbb{A})} \int_{Z_n(F)/Z_n(\mathbb{A})} f_{\Delta(\tau+1,i)}(\tilde{v}(I_{n_i}) x) \psi_{n_i}(z) dz dv. \]
6. Application of Bessel coefficients to $E(f_{\Delta(\tau,i+1)},s)$ on $SO_{2n(i+1)+1}(\mathbb{A})$: descent to $SO_{2n}(\mathbb{A})$

In this section, we will consider a Fourier coefficient similar to the one in the previous section applied to an Eisenstein series on $SO_{2n(i+1)+1}(\mathbb{A})$, parabolically induced from $\Delta(\tau,i+1)$. Here, we cannot use the work of the previous sections, since there the group $H$ is even orthogonal (when $H$ is orthogonal). However, the treatment is similar, and in fact, most of the details can be derived from [GRS11], Chapter 5 (Theorems 5.1, 5.2, 5.3).

Denote in this section $H = SO_{2n(i+1)+1}$ (written with respect to $w_{2n(i+1)+1}$). Consider the unipotent radical $U_{1n}$ consisting of the elements

$$u = \begin{pmatrix} z & x & y \\ I_{2n+1} & x' & z' \\ & & z^* \end{pmatrix} \in H, \ z \in Z_n$$

and the character $\psi_{ni}$ of $U_{1n}(\mathbb{A})$ given by

$$\psi_n(u) = \psi_{\mathbb{Z}_n}(z)\psi(x_{n,ni+1}).$$

This character is stabilized by $SO_{2n}(\mathbb{A})$ realized as the subgroup of elements $diag(I_n,h,I_n)$, with $h \in SO_{2n+1}(\mathbb{A})$ satisfying

$$h \begin{pmatrix} 0_{ni} \\ 1 \\ 0_{ni} \end{pmatrix} = \begin{pmatrix} 0_{ni} \\ 1 \\ 0_{ni} \end{pmatrix}.$$  

The isomorphism of this stabilizer and $SO_{2ni}$ is given by

$$j(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{pmatrix},$$

where $a,b,c,d \in M_{ni \times ni}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SO_{2ni}$. For $h \in SO_{2n}$, denote

$$t(h) = \begin{pmatrix} I_n \\ j(h) \\ I_n \end{pmatrix}.$$  

Denote by $Q_{n(i+1)}$ the standard parabolic subgroup of $H$, whose Levi part $M_{n(i+1)}$ is isomorphic to $GL_{n(i+1)}$. We will use the following notation, similar to our previous one,

$$M_{n(i+1)} = \{ \hat{a} = \begin{pmatrix} a \\ 1 \\ a^* \end{pmatrix} | a \in GL_{n(i+1)} \}.$$  

Consider the parabolic induction

$$\rho_{\Delta(\tau,i+1),s} = \text{Ind}_{Q_{n(i+1)}(\mathbb{A})}^{SO_{2n(i+1)+1}(\mathbb{A})} \Delta(\tau,i+1) | \det| ^s.$$  

Let $f_{\Delta(\tau,i+1),s}$ be a smooth, holomorphic section of $\rho_{\Delta(\tau,i+1),s}$ in [GRS], consider the corresponding Eisenstein series $E(f_{\Delta(\tau,i+1),s})$, and apply to it the Fourier coefficient along $U_{1n}$ in [GRS], with respect to the character [GRS2]. Denote this Fourier coefficient by $D_{\psi,ni}(E(f_{\Delta(\tau,i+1),s}))$, and, as before, we view it as an automorphic
function on \( SO_{2n}(\mathbb{A}) \), realized via \([\text{[13]}]\). This is the Bessel coefficient used in automorphic descent. See \([\text{[GRS11]}]\). Thus,

\[
(6.5) \quad D_{\psi,n}(E(f_{\Delta(\tau,i+1),s})(h)) = \int_{U_{1n}(F) \backslash U_{1n}(\mathbb{A})} E(f_{\Delta(\tau,i+1),s}, ut(h))\psi_n^{-1}(u)du.
\]

The analog of \((3.25)\) in this case is a little more involved. It turns out that \((6.5)\) is a sum of two Eisenstein series on \( SO_{2n}(\mathbb{A}) \), with the second term obtained from the first by an outer conjugation. In order to write the precise identity, we need to introduce some more notations. Let \( \alpha_0 \in H \) be the following element

\[
(6.6) \quad \alpha_0 = \begin{pmatrix}
0 & I_{ni} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I_n \\
0 & 0 & (-1)^n & 0 & 0 \\
I_n & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I_{ni} & 0
\end{pmatrix};
\]

Denote

\[
(6.7) \quad U_n = \{ux_1, x_2, y = \begin{pmatrix} I_n & x_1 & x_2 & 0 & y \\
I_{ni} & 0 & 0 & 0 & 1 \\
1 & 0 & x_2' & I_{ni} & x_1' \\
I_n & I_{ni} & 0 & 0 & 0
\end{pmatrix} \in H \};
\]

\[
(6.8) \quad \omega_0 = \text{diag}(I_{n(i+1)}, -1, I_{n(i+1)}); \quad \omega_0' = \text{diag}(I_{ni-1}, \begin{pmatrix} 1 & 0 \\
0 & 1 \end{pmatrix}), I_{ni-1}) \in O_{2n}.
\]

We extend \( t \) in \((6.2)\) to \( O_{2ni} \), so that \( t(\omega_0') \in O_{2n(i+1)+1} \). Denote \( \omega_0'' = \omega_0 t(\omega_0') \). Let \( \psi_{V_{ni,1}^n} \) be the following character of \( V_{ni,1}^n(\mathbb{A}) \), similar to \((3.9)\)

\[
(6.9) \quad \psi_{V_{ni,1}^n}(\begin{pmatrix} I_{ni} & y \\
z & z \end{pmatrix}) = \psi_{Z_n}(z), \quad z \in Z_n(\mathbb{A}),
\]

and let, for \( g \in H_\mathbb{A} \),

\[
(6.10) \quad f_{\Delta(\tau,i+1),s}(g) = \int_{V_{ni,1}^n(F) \backslash V_{ni,1}^n(\mathbb{A})} f_{\Delta(\tau,i+1),s}(\hat{g})\psi_{V_{ni,1}^n}(v)dv.
\]

**Theorem 6.1.** For \( Re(s) \) sufficiently large, \( h \in SO_{2n}(\mathbb{A}) \),

\[
D_{\psi,n}(E(f_{\Delta(\tau,i+1),s}))(h) = \sum_{\gamma \in Q_{ni} \setminus SO_{2n}} \Lambda^+(f_{\Delta(\tau,i+1),s})(\gamma h\omega_0''),
\]

where

\[
\Lambda^+(f_{\Delta(\tau,i+1),s})(h) = \int_{U_{ni}^n(\mathbb{A})} f_{\Delta(\tau,i+1),s}(\alpha_0 ut(h))\psi_n^{-1}(u)du;
\]

\[
\Lambda^-(f_{\Delta(\tau,i+1),s})(h) = \int_{U_{1n}^n(\mathbb{A})} f_{\Delta(\tau,i+1),s}(\alpha_0 ut(h)\omega_0'')\psi_n(u)du;
\]

In the sum \((6.11)\), \( Q_{ni} = Q_{ni}^{SO_{2n}} \).
Proof. The proof is similar to that of Theorem 1.1. Many details appear in [GRS11]. We sketch it here for the convenience of the reader. As in (2.1), we factorize (for $Re(s)$ large)

$$E(f_{\Delta(\tau, i+1), s}, x) = \sum_{\alpha \in Q_n(i+1) \setminus H/Q_n} \sum_{\gamma} f_{\Delta(\tau, i+1), s}(\alpha \gamma x),$$

where, as before, $Q_k = Q_k^H$ is the standard parabolic subgroup of $H = SO_{2n(i+1)+1}$, with Levi part isomorphic to $GL_k \times SO_{2(n(i+1) - k) + 1}$ ($1 \leq k \leq n(i+1)$); the second summation in (6.12) is over $\gamma \in Q_n \cap \alpha^{-1}Q_n(i+1)\alpha \setminus Q_n$. The representatives $\alpha$ in (6.12) are described in [GRS11], Sec. 4.2. They are parameterized by integers $0 \leq r \leq n$,

$$\alpha_r = \begin{pmatrix} I_r & \alpha_r' \\ \alpha_r' & I_r \end{pmatrix},$$

where

$$\alpha_r' = \begin{pmatrix} 0 & I_{ni} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{n-r} \\ 0 & 0 & (-1)^{n-r} & 0 & 0 \\ I_{n-r} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{ni} & 0 \end{pmatrix}.$$ 

The elements of $Q^{(r)} = Q_n \cap \alpha_r^{-1}Q_n(i+1)\alpha_r$ have the form

$$\begin{pmatrix} a & x & y_1 & y_2 & y_3 & z_1 & z_2 \\ 0 & b & 0 & 0 & y_4 & 0 & z_1' \\ c & u & v & y_4' & y_3' & 1 & u' \end{pmatrix} = \begin{pmatrix} 1 & u' & 0 & y_2' \\ c^* & 0 & y_3' \\ b^* & x^* \end{pmatrix},$$

where $a \in GL_r$, $b \in GL_{n-r}$, $c \in GL_{ni}$. The element (6.14) is conjugated by $\alpha_r$ to

$$\begin{pmatrix} a & y_1 & z_1 & y_2 & x & y_3 & z_2 \\ c & y_4' & u & 0 & v & y_3' & b^* \end{pmatrix} \omega_0^{n-r} \begin{pmatrix} b^* & 0 & 0 & 0 & x_2^* \\ 1 & 0 & u' & z_1' & y_1' \\ c^* & y_1' & a^* \end{pmatrix}.$$

Thus, for $h \in SO_{2ni}(A)$,

$$D^{(r)}_{\psi, ni}(E(f_{\Delta(\tau, i+1), s}))(h) =$$

$$\sum_{r=0}^{n} \int_{U_1^{ni}(F) \setminus U_1^{ni}(A)} \sum_{\gamma \in Q^{(r)} \setminus Q_n} f_{\Delta(\tau, i+1), s}(\alpha_r \gamma u \tau(h))\psi_n^{-1}(u)du.$$ 

The analogue of Theorem 2.1 is valid here, namely for all $1 \leq r \leq n$, the summand in (6.15), corresponding to $r$ is (identically) zero. The proof follows similar steps as that of Theorem 2.1 and it is easily read from the proof of Prop. 5.1 in [GRS11].
One only needs to carry the easy translation from the language of twisted Jacquet modules to that of Fourier coefficients. Thus, (6.15) becomes

\[ D_{\psi,ni}(E(f_{\Delta(\tau,i+1)}))(h) = \]

\[
\int_{U_1^*(F) \backslash U_1(F)} \sum_{\gamma \in Q^{(0)} \cap Q_n} f_{\Delta(\tau,i+1),s}(\alpha_0 \gamma ut(h)) \psi^{-1}_n(u) du.
\]

Factor the sum in (6.16) modulo \( Q_1^n \) from the right. As before, \( Q_1^n = Q_1^H \) denotes the standard parabolic subgroup of \( H = SO_{2n(i+1)+1} \), with Levi part isomorphic to \( GL_n \times SO_{2ni+1} \). Note that \( Q_n = Q^{(0)} Q_1^n \). Hence (6.16) becomes

\[ D_{\psi,ni}(E(f_{\Delta(\tau,i+1)}))(h) = \]

\[
\int_{U_1^*(F) \backslash U_1(F)} \sum_{\eta \in Q^{(0)} \cap Q_2^n \cap Q_1^n} f_{\Delta(\tau,i+1),s}(\alpha_0 \eta ut(h)) \psi^{-1}_n(u) du,
\]

The subgroup \( Q^{(0)} \cap Q_1^n \) consists of the elements

\[
\begin{pmatrix}
  b & 0 & 0 & y & 0 \\
  c & u & v & y' & 0 \\
  1 & u' & 0 \\
  c^* & 0 & b^*
\end{pmatrix},
\]

where \( c \in GL_{ni} \) and \( b \in B_{GL_n} \) - the standard Borel subgroup of \( GL_n \). Factor the sum in (6.17) modulo \( U_1^n \) from the right. We get

\[ D_{\psi,ni}(E(f_{\Delta(\tau,i+1)}))(h) = \]

\[
\sum_{\gamma \in Q_n^{SO_{2ni+1}} \backslash SO_{2ni+1}} \int_{U_1^n \cap h^{-1}_\gamma Q^{(0)} \cap Q_n \cap h_\gamma \backslash U_1^n} f_{\Delta(\tau,i+1),s}(\alpha_0 h_\gamma \gamma ut(h)) \psi^{-1}_n(u) du,
\]

where, for \( \gamma \in SO_{2ni+1} \), \( h_\gamma = diag(I_n, \gamma, I_n) \). Now, factor the sum in (6.18) modulo \( SO_{2ni} \) from the right (realized via the embedding \( j \)). The set \( Q_n^{SO_{2ni+1}} \backslash SO_{2ni+1} / SO_{2ni} \) consists of three elements. See Prop. 4.4 in [GRS11]. Here are representatives for the three double cosets: one is the identity \( I_{2ni+1} \), another is \( \gamma_0 \), such that \( h_{\gamma_0} = \omega_0'' \), and the third is

\[
\gamma_1 = diag(I_{ni-1}, \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, I_{ni-1}),
\]
Let us show that the contribution of \( \gamma_1 \) to \((6.18)\) is zero. For this, we note that the subgroup \( U_{1n}^1 = U_{1n} \cap h_{\gamma_1}^{-1}(Q(0) \cap Q_{1n})h_{\gamma_1} \) consists of the elements

\[
\begin{pmatrix}
z & 0 & -\frac{y_2}{2} & 0 \\
I_{ni-1} & y_1 & y_2 & 0 \\
0 & 0 & 0 & y_2' \\
1 & 0 & 0 & y_2'
\end{pmatrix}
\in H, \ z \in Z_n.
\]

(6.19)

When we conjugate the element \((6.19)\) by \( \alpha_0 h_{\gamma_1} \), we get

\[
\begin{pmatrix}
I_{ni-1} & 0 & y_2' \\
1 & y_2' \\
z'
\end{pmatrix}
\]

(6.20)

From \((6.19, 6.20)\), it follows that the contribution of \( \gamma_1 \) to \((6.18)\) is

\[
\sum_{\gamma \in SO_{2n} \cap \Psi_{ni,1\infty} \backslash SO_{2n}} \int_{U_{1n}^1(A) \backslash U_{1n}(A)} I_\psi(f_{\Delta(\tau,i+1),s})(\alpha_0 h_{\gamma_1} ut(\gamma h))\psi_n^{-1}(u)du,
\]

where

\[
I_\psi(f_{\Delta(\tau,i+1),s})(x) = \int_{V_{ni,1\infty}(F) \backslash V_{ni,1\infty}(A)} f_{\Delta(\tau,i+1),s}(\tilde{v}x)\tilde{\psi}_{V_{ni,1\infty}}(v)dv,
\]

where

\[
\tilde{\psi}_{V_{ni,1\infty}}\left(\begin{pmatrix}
I_{ni} & y \\
1 & z
\end{pmatrix}\right) = \psi_{Z_n}(z)\psi(y_{ni,1}).
\]

The Fourier coefficient \((6.22)\) is obtained by applying the Fourier coefficient with respect to \( \psi_{V_{ni,1\infty}} \) to the elements of \( \Delta(\tau, i+1) \) obtained as \( a \mapsto f_{\Delta(\tau,i+1),s}(\tilde{a}x) \). This Fourier coefficient corresponds to the partition \((n+1, 1^{ni-1})\), and by Prop. \([4]\), this Fourier coefficient is zero on \( \Delta(\tau, i+1) \).

It remains to examine the contributions of \( I_{2ni+1} \) and \( \gamma_0 \) to \((6.18)\), and it is straightforward to check that we get the two terms of \((6.11)\).

Next, let us show that the two summands in \((6.11)\) are Eisenstein series.

**Theorem 6.2.** Both functions \( \Lambda^\pm(f_{\Delta(\tau,i+1),s}) \), defined for \( \Re(s) \) sufficiently large by the integrals in Theorem \((6.1)\) admit analytic continuations to meromorphic functions of \( s \) in the whole plane. They define smooth meromorphic sections of

\[
\rho_{\Delta(\tau,i),s} = \text{Ind}_{Q_{ni}^1(A)}^{SO_{2n}(A)} \Delta(\tau,i)|\det|^{-s}.
\]

Thus, \( D_{\psi,ni}(E(f_{\Delta(\tau,i+1),s})) \) is the sum of two Eisenstein series on \( SO_{2ni}(A) \); the first corresponds to the section \( \Lambda^+(f_{\Delta(\tau,i+1),s}) \) of \( \rho_{\Delta(\tau,i),s} \), and the second corresponds to the section \( h \mapsto \Lambda^-(f_{\Delta(\tau,i+1),s})(h\omega_0^\dagger) \) of

\[
\rho_{\Delta(\tau,i),s} = \text{Ind}_{Q_{ni}^1(A)}^{SO_{2n}(A)} \Delta(\tau,i)|\det|^{-s}.
\]

**Proof.** We will show that for \( \Re(s) \) sufficiently large, \( \Lambda^\pm \) indeed define elements of \( \rho_{\Delta(\tau,i),s} \), and then the proof of meromorphic continuation is as in the proof of Theorem \((3.5)\). Since \( \Lambda^\pm \) is defined in almost the same way as \( \Lambda^+ \), it suffices to show
this for $\Lambda^+(f_{\Delta(\tau,r+1),s}, h)$.

Let $q = \begin{pmatrix} a & x \\ x & a^* \end{pmatrix} \in Q_{ni}(\mathbb{A}) \subset SO_{2ni}(\mathbb{A})$. Note that
\[
\alpha_0 t(q) \alpha_0^{-1} = \begin{pmatrix} a & 0 & x \\ I_{2n+1} & 0 & 0 \\ a^* & & \end{pmatrix}
\]

Hence, for $Re(s)$ large,
\[
(6.23) \quad \Lambda^+(f_{\Delta(\tau,r+1),s}, qh) = |\det(a)|^{-n} \int_{U_n'(\mathbb{A})} f^\psi_{\Delta(\tau,r+1),s}(\alpha_0 q u t(h)) \psi^{-1}_n(u) du
\]
\[
= |\det(a)|^{-n} \int_{U_n'(\mathbb{A})} f^\psi_{\Delta(\tau,r+1),s}(\begin{pmatrix} a \\ I_n \end{pmatrix}^\wedge \alpha_0 q u t(h)) \psi^{-1}_n(u) du.
\]

As in the proof of Theorem 3.5, for any fixed $x_0$, the function on $GL_{ni}(\mathbb{A})$, $a \mapsto f^\psi_{\Delta(\tau,r+1),s}(\begin{pmatrix} a \\ I_n \end{pmatrix} x_0)$ lies in $|\det(a)|^{s + \frac{n(n+1)}{2}}$ times a function obtained as $\psi(diag(a, I_n))$, where $\psi$ lies in the constant term of $\Delta(\tau, r+1)$ along $V_{ni,n}$, followed by taking a Whittaker coefficient with respect to $\psi_{Z_n}$. All in all, it is now easy to see that, as a function of $a \in GL_{ni}(\mathbb{A})$, the second integral in (6.23) defines an element of $|\det|^{s + \frac{n(n+1)}{2}} \Delta(\tau, r) = \delta_{Q_{ni}}^{\psi_{Z_n}} \cdot |\det|^{s} \Delta(\tau, i)$.

The last thing that we want to show in this case is the compatibility with normalization. We will see that $\Lambda^+(f_{\Delta(\tau,r+1),s})$ is related to the Langlands-Shahidi integral on $SO_{2n+1}(\mathbb{A})$, corresponding to the symmetric-square $L$-function of $\tau$. (In the previous section, we got a relation to the exterior-square $L$-function of $\tau$, which follows from Section 4, as a special case.) Of course, the same holds for $\Lambda^-(f_{\Delta(\tau,r+1),s})$ since it has almost the same structure. Thus, let us apply to $\Delta(\tau, r)$ the Fourier coefficient along $V_{n} \subset GL_{ni}$, with respect to the character $\psi_{V_n}^{-1}$ (4.1), and consider, as in (6.24), the integral
\[
(6.24) \quad \mathcal{L}(f_{\Delta(\tau,r+1),s}) = \int_{U_n'(\mathbb{A})} f^W_{\Delta(\tau,r+1),s}(\alpha_0 u) \psi_n^{-1}(u) du.
\]

The integral (6.24) is Eulerian in the sense that for decomposable data it is a product of similar local integrals
\[
\mathcal{L}(f_{\Delta(\tau,r+1),s}) = \prod_v \mathcal{L}_v(f_{\Delta(\tau,v,1),s}),
\]
where
\[
(6.25) \quad \mathcal{L}_v(f_{\Delta(\tau,v,1),s}) = \int_{U_n'(F_v)} f_{\Delta(\tau,v,1),s}(\alpha_0 u; I_{ni}, I_n) \psi_n^{-1}(u) du.
\]

Here, $f_{\Delta(\tau,v,1),s}$ is a section of
\[
\rho_{\Delta(\tau,v,1)} = \text{Ind}_{Q_{ni,n}(F_v)}^{SO_{2n+1}(F_v)}(\Delta(\tau,v,1) |\det|^{-\frac{s}{2}} \times \tau_v |\det|^{-\frac{s+1}{2}}).
\]

As before, $Q_{ni,n}$ is the standard parabolic subgroup of $H = SO_{2n+1}(F_v)$, whose Levi part is isomorphic to $GL_{ni} \times GL_n$. The representation $\Delta(\tau,v,1)$ is realized in its model with respect to $\psi_{V_n}^{-1}$ and $\tau_v$ is realized in its $\psi_{Z_n}$-Whittaker model. As
in (4.2), we view \( f_{\Delta(\tau,\iota;1),s} \) as a function on \( H(F_v) \times \operatorname{GL}_m(F_v) \times \operatorname{GL}_n(F_v) \), such that for a fixed element in \( H(F_v) \), the function in the two other variables lies in the tensor product of the two models above of \( \Delta(\tau,\iota) \) and \( \tau \). We simplify notation and re-denote \( f_{\Delta(\tau,\iota;1),s}(y) = f_{\Delta(\tau,\iota;1),s}(y;|n_i,I_n|) \).

We fix a finite set of places \( S \), containing the Archimedean places, outside which \( \tau \) is unramified, \( \psi \) is normalized, and \( f_{\Delta(\tau,\iota;1),s} = f_{\Delta(\tau,\iota;1),s}^{00} \) is spherical and normalized. The integral (6.28) converges absolutely for \( \text{Re}(s) \) sufficiently large. Again, the proof is standard and we skip it. (The proof of the next proposition implies that the integral stabilizes for large compact subgroups of \( U_n'(F_v) \), when \( v \) is finite, and an appropriate analog in case \( v \) is infinite.) Write the elements of \( U_n'(F_v) \) in the form \( u_{x_1,x_2,y} \) as in (4.7). Let

\[
U_n''(F_v) = \{ u_{x_1,x_2,y} \in U_n'(F_v) | x_1 = 0 \}.
\]

We have the following analog of Prop. 4.1.

**Proposition 6.3.** Let \( v \) be a place of \( F \). There is a section \( f'_{\Delta(\tau,\iota;1),s} \), which depends on (the smoothness of) \( f_{\Delta(\tau,\iota;1),s} \), such that

\[
(6.26) \quad L_v(f_{\Delta(\tau,\iota;1),s}) = \int_{U_n''(F_v)} f'_{\Delta(\tau,\iota;1),s}(\alpha_0 u)\psi^{-1}_{n,v}(u)du.
\]

If \( f_{\Delta(\tau,\iota;1),s} \) is spherical, then \( f'_{\Delta(\tau,\iota;1),s} = f_{\Delta(\tau,\iota;1),s} \). In general, \( f'_{\Delta(\tau,\iota;1),s} \) is obtained from \( f_{\Delta(\tau,\iota;1),s} \) by a finite sequence of convolutions against certain Schwartz functions (described in the proof).

**Proof.** The proof is similar to that of Prop. 4.1. We explain the case where \( v \) is finite and indicate the main steps. We assume, for simplicity that \( \psi_v \) is normalized. Write \( u_{x_1,x_2,y} = v(x_1)u_n(x_2;y) = u_n(x_2;y)v(x_1) \), where \( v(x_1) = u_{x_1,0,0} \), \( u_n(x_2;y) = u_{0,x_2,y} \). Note that

\[
(6.27) \quad \alpha_0 u_n(x_2;y)\alpha_0^{-1} = \text{diag}(I_{n_i}, \begin{pmatrix} I_n & x_1 \\ x_2 & I_n \end{pmatrix}, I_{n_i}).
\]

Let

\[
(6.28) \quad L'_v(f_{\Delta(\tau,\iota;1),s})(x_1) = \int_{U_n''(F_v)} f_{\Delta(\tau,\iota;1),s}(\alpha_0 v(x_1)u)\psi^{-1}_{n,v}(u)du.
\]

We will show that the last function has compact support in \( M_{n \times n}(F_v) \), and in case of a spherical section this support is \( M_{n \times n}(O_v) \). Let

\[
e_{a,b} = \text{diag}(I_n, \begin{pmatrix} I_n & a \\ 1 & a' \\ I_{n_i} \end{pmatrix}, I_{n_i}) \in H,
\]

where \( a, b \) have coordinates sufficiently close to zero, such that right translation by \( e_{a,b} \) fixes \( f_{\Delta(\tau,\iota;1),s} \). If the section is spherical, then we may take \( a, b \) to have coordinates in \( O_v \). It is straightforward to check that

\[
\alpha_0 u_{x_1,x_2,y} e_{a,b} = u'\alpha_0 u_{x_1,x_2+x_1a,y},
\]

where \( u' \) is in the unipotent radical of \( Q_{n_i,n}(F_v) \). Changing variables \( x_2 \mapsto x_2 - x_1a \) in (6.28) gives

\[
L'_v(f_{\Delta(\tau,\iota;1),s})(x_1) = L'_v(\rho(e_{a,b})f_{\Delta(\tau,\iota;1),s})(x_1) = \psi((x_1)_n)a) L'_v(f_{\Delta(\tau,\iota;1),s})(x_1).
\]
Here, $\rho(c_{a,b})$ denotes right translation by $c_{a,b}$, and $(x_1)_n$ denotes the last row of $x_1$.
We conclude that $(x_1)_n$ is supported in a compact set (of the form $(P_{v}^{k})^{n_1} \subset F_{v}^{n_1}$), and in case the section is spherical, then $(x_1)_n$ is supported in $O_{v}^{n_1}$. Now, we may assume that $(x_1)_n = 0$, and show that the $(n-1)$-th row of $x_1$ must lie in a compact set (in order to be in the support of $L'_{v}(f\Delta(\tau_0,i;1),s)$). Then we may assume that the last two rows of $x_1$ are zero and move on to show compact support in row $(x_1)_{n-2}$, and so on. In general, assume that $(x_1)_r = 0$, for $n - \ell \leq n$, $0 \leq \ell \leq n - 2$. Let

$$c_{\ell,z} = \begin{pmatrix} I_{n-\ell-1} & 1 \\ 0 & I_{\ell} \\ z & 0 \end{pmatrix} \in SO_{2n+1}(F_v),$$

with $z \in (P_{v}^{k})^{n_1}$, where $k$ is sufficiently large, such that right translation by $c_{\ell,z}$ fixes $f\Delta(\tau_0,i;1),s$. As before, for $x_1$ as above, we get

$$L'_{v}(f\Delta(\tau_0,i;1),s)(x_1) = L'_{v}(\rho(c_{\ell,z})f\Delta(\tau_0,i;1),s)(x_1) = \psi((x_1)_{n-\ell-1}z)L'_{v}(f\Delta(\tau_0,i;1),s)(x_1).$$

We conclude that $(x_1)_{n-\ell-1}$ is supported in $(P_{v}^{k})^{n_1}$, and in case the section is spherical, it is supported in $O_{v}^{n_1}$. \hfill $\square$

We conclude

**Theorem 6.4.** Let $v$ be a place outside $S$. Then

$$L'_{v}(f^{0}_{\Delta(\tau_0,i;1),s}) = \frac{1}{L(\tau_v,\text{sym}^2,2s+i+1)}.$$  

*Proof.* Denote, for $h \in SO_{2n+1}(F_v)$,

$$f^{0}_{\tau_v,i+\frac{\ell}{2}}(h) = f^{0}_{\Delta(\tau_0,i;1),s}(\text{diag}(I_{ni},h,I_{ni})).$$

This is the spherical, normalized element of

$$\rho_{\tau_v,i+\frac{\ell}{2}} = \text{Ind}_{Q_{v}^{n+1}(F_v)}^{SO_{2n+1}(F_v)} \tau_v | \det \cdot |^{s+i+\frac{\ell}{2}}.$$  

Prop. 6.3 and 6.27 show that, for $Re(s)$ sufficiently large,

$$L'_{v}(f^{0}_{\Delta(\tau_0,i;1),s}) = \int_{U_{v}^{n}SO_{2n+1}(F_v)} f^{0}_{\tau_v,i+\frac{\ell}{2}}( \begin{pmatrix} I_{n} & x' & 1 \\ x & y & x \end{pmatrix} ) \psi^{-1}(x_{n})du.$$  

This is the Jacquet integral applied to $f^{0}_{\tau_v,i+\frac{\ell}{2}}$, with respect to the standard $\psi_{\tau,v}$-Whittaker character. Now, we get the theorem by the Casselman-Shalika formula. \hfill $\square$

Let us multiply our Eisenstein series $E(f\Delta(\tau_0,i;1),s)$ on $SO_{2n(i+1)+1}(A)$ by its normalizing factor outside $S$, $d_{S}^{SO_{2n(i+1)+1};S}(s)$. We have

$$d_{S}^{SO_{2n(2i)+1};S}(s) = \prod_{k=1}^{j} L^{S}(\tau,\lambda^{2},2s+2k) \prod_{k=1}^{j+1} L^{S}(\tau,\text{sym}^2,2s+2k-1);$$

$$d_{S}^{SO_{2n(i)+1};S}(s) = \prod_{k=1}^{j} L^{S}(\tau,\lambda^{2},2s+2k-1)L^{S}(\tau,\text{sym}^2,2s+2k).$$

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One sees immediately from \((6.30), (6.31)\), that (see \((1.23), (1.24)\))

\[(6.32)\]

\[
\frac{L^S(\tau, \mathrm{sym}^2, 2s + i + 1)}{E^i_S(f_{\Delta(\tau, i+1), s})} = d_{\tau}^{\SO_{2ni+1}, S}(s).
\]

Let

\[
E^i_{\psi}(f_{\Delta(\tau, i+1), s}) = d_{\tau}^{\SO_{2ni+1}, S}(s)E(f_{\Delta(\tau, i+1), s}).
\]

We conclude from Theorem \((6.2), (6.4)\) and \((6.32)\),

**Theorem 6.5.** The descent to \(\SO_{2ni}(\mathbb{A})\) of the normalized Eisenstein series \(E^i_{\psi}(f_{\Delta(\tau, i+1), s})\) is the sum of two normalized (outside \(S\)) Eisenstein series on \(\SO_{2ni}(\mathbb{A})\); the first corresponds to the section \(\Lambda^+(d_{\tau}^{\SO_{2ni+1}, S}(s)f_{\Delta(\tau, i+1), s})\) and the second corresponds to the section \(h \mapsto \Lambda^-(d_{\tau}^{\SO_{2ni+1}, S}(s)f_{\Delta(\tau, i+1), s})(h_{\psi})\).

7. **Application of Fourier-Jacobi coefficients to \(E(f_{\Delta(\tau, i+1), s})\) on \(\Sp_{2ni+1}(\mathbb{A})\): descent to \(\Sp_{2ni}(\mathbb{A})\)**

In this section and the next one, we will carry out for symplectic groups and metaplectic groups the descent, via Fourier-Jacobi coefficients, analogous to the previous two sections. Denote in this section \(H = \Sp_{2ni+1}\). We will use the conventions of \([\text{GRS11}]\). Consider the unipotent radical \(U_{1_{n-1}}\) consisting of the elements

\[(7.1)\]

\[
\begin{pmatrix}
  x & y & z \\
  I_{2ni+2} & y \\
  z^* & z
  \end{pmatrix} \in H, \ z \in Z_{n-1}
\]

and the character \(\psi_{n-1}\) of \(U_{1_{n-1}}(\mathbb{A})\) given by

\[(7.2)\]

\[
\psi_{n-1}(u) = \psi_{Z_{n-1}}(z)\psi(x_{n-1,n}).
\]

This character is stabilized by the semi-direct product of \(\Sp_{2ni}(\mathbb{A})\) and \(\mathcal{H}_{2ni+1}(\mathbb{A})\), where \(\mathcal{H}_{2ni+1}\) is the Heisenberg group corresponding to the symplectic space of dimension \(2ni\). The group \(\Sp_{2ni}\) is realized inside \(H\) by

\[
t(h) = \mathrm{diag}(I_n, h, I_n), \ h \in \Sp_{2ni}(\mathbb{A}),
\]

and the group \(\mathcal{H}_{2ni+1}\) is realized inside \(H\) by

\[(7.3)\]

\[
t((x, e)) = \mathrm{diag}(I_{n-1}, \begin{pmatrix} 1 & x & e \\ 0 & I_{2ni} & x' \\ 0 & 0 & I_{n-1} \end{pmatrix}), \ n \in H.
\]

We take \(\mathcal{H}_{2ni+1}(F)\) as the group of pairs \((x, t)\), with \(x\) a row vector of \(2ni\) coordinates, and \(t \in F\), with multiplication defined by

\[
(x, e) \cdot (y, z) = (x + y, e + z + x \begin{pmatrix} -w_{ni} & w_{ni} \\ n_{ni} & n_{ni} \end{pmatrix} t y).
\]

Note that \(U_1 = U_{1_{n-1}} \times t(\mathcal{H}_{2ni+1})\), and denote by \(j\) the natural projection of \(U_1\) on \(\mathcal{H}_{2ni+1}\). We extend the character \((7.2)\) to \(U_1(\mathbb{A})\) by making it trivial on \(t(\mathcal{H}_{2ni+1}(\mathbb{A}))\). We continue to denote this extension by \(\psi_{n-1}\).

Let \(\omega^{(2ni)}_\psi\) be the Weil representation of \(\mathcal{H}_{2ni+1}(\mathbb{A}) \times \Sp_{2ni}(\mathbb{A})\), associated to \(\psi\), i.e. the elements \((0, z)\) of the center of \(\mathcal{H}_{2ni+1}(\mathbb{A})\) act by multiplication by \(\psi(z)\). We
Let $\omega^{(2n_i)}_\psi$ act on the space of Schwartz functions $S(\mathbb{A}^n)$. For $\phi \in S(\mathbb{A}^n)$, we let $\theta^\phi_\psi$ denote the corresponding theta series, viewed as a function on $\mathcal{H}_{2n+1}(\mathbb{A}) \rtimes \text{Sp}_{2n}^2(\mathbb{A})$.

Consider the parabolic induction

\begin{equation}
\rho_\Delta(\tau, i + 1, s) = \text{Ind}_{Q_{n+1}(\mathbb{A})}^{\text{Sp}_{2n+1}(\mathbb{A})} \Delta(\tau, i + 1)| \det \cdot|^s.
\end{equation}

Let $f_\Delta(\tau, i + 1, s)$ be a smooth, holomorphic section of $\rho_\Delta(\tau, i + 1, s)$ \eqref{7.3}. Consider the corresponding Eisenstein series $E(f_\Delta(\tau, i + 1, s))$, and apply to it the following Fourier-Jacobi coefficient

\begin{equation}
D_{\psi, ni}^\phi(E(f_\Delta(\tau, i + 1, s))((h, \epsilon))
\end{equation}

\begin{equation}
\int_{U_{1, n}(F) \setminus U_{1, n}(\mathbb{A})} E(f_\Delta(\tau, i + 1, s), ut(h))\psi^{-1}(u)\theta^\phi_{\psi^{-1}}(j(u)(h, \epsilon))du.
\end{equation}

Here, $h \in \text{Sp}_{2n}(\mathbb{A})$, and $\epsilon = \pm 1$. Let us state the identity analogous to \eqref{3.25} and apply to it the following Fourier-Jacobi coefficient

\begin{equation}
\alpha_0 = \begin{pmatrix}
0 & I_{ni} & 0 & 0 \\
0 & 0 & 0 & I_n \\
-I_n & 0 & 0 & 0 \\
0 & 0 & I_{ni} & 0
\end{pmatrix}.
\end{equation}

Denote

\begin{equation}
U'_{n} = \{ u'_{x, y} = \begin{pmatrix} I_n & x & 0 & y \\
I_{ni} & 0 & 0 & 0 \\
I_{ni} & x' & I_n & 0 \end{pmatrix} \in H \}.
\end{equation}

As in \eqref{6.10}, let, for $g \in H_{n}$,

\begin{equation}
f_\Delta(\tau, i + 1, s)(g) = \int_{V_{ni, 1, n}(F) \setminus V_{ni, 1, n}(\mathbb{A})} f_\Delta(\tau, i + 1, s)(\bar{g})\psi_{V_{ni, 1, n}}(v)dv,
\end{equation}

where $\psi_{V_{ni, 1, n}}$ is given by \eqref{6.9}.

**Theorem 7.1.** For $\text{Re}(s)$ sufficiently large, $h \in \text{Sp}_{2n}(\mathbb{A})$, $\epsilon = \pm 1$,

\begin{equation}
D_{\psi, ni}^\phi(E(f_\Delta(\tau, i + 1, s))((h, \epsilon)) = \sum_{\gamma \in Q_{n, 1, n} \setminus \text{Sp}_{2n, 1}} \Lambda(f_\Delta(\tau, i + 1, s), \phi)(\gamma, 1)(h, \epsilon),
\end{equation}

where

\begin{equation}
\Lambda(f_\Delta(\tau, i + 1, s), \phi)(h, \epsilon) = \int_{U'_{n}(\mathbb{A})} \omega_{\psi^{-1}}(j(u)(h, \epsilon))\phi(0)f_\Delta(\tau, i + 1, s)(\alpha_0ut(h))du.
\end{equation}

In the sum \eqref{7.4}, $Q_{n, 1, n} = Q_{n, 1, n}^{\text{Sp}_{2n, 1}}$. The function $\Lambda(f_\Delta(\tau, i + 1, s), \phi)$, defined for $\text{Re}(s)$ sufficiently large, by the last integral, admits analytic continuation to a meromorphic function of $s$ in the whole plane. It defines a smooth meromorphic section of $\rho_\Delta(\tau, i + 1, s) = \text{Ind}_{Q_{n, 1, n}^{(2n)}} \Delta(\tau, i + 1)| \det \cdot|^s$.

Thus, $D_{\psi, ni}^\phi(E(f_\Delta(\tau, i + 1, s))$ is the Eisenstein series on $\text{Sp}_{2n}^2(\mathbb{A})$ corresponding to the section $\Lambda(f_\Delta(\tau, i + 1, s), \phi)$ of $\rho_\Delta(\tau, i + 1, s)$.
Proof. The proof follows the same steps as in the last sections. The details can easily be read from Chapter 6 in [GRS11]. We start with unfolding the Eisenstein series in (7.5), and factor the summation on $Q_{n(i+1)}(F) \setminus Sp_{2n(i+1)}(F)$ modulo $Q_n(F)$ from the right. The easy analog of Prop. 6.2 in [GRS11] shows that only the open double coset in $Q_{n(i+1)}(F) \setminus Sp_{2n(i+1)}(F)/Q_n(F)$ contributes to (7.10). The element $\alpha_0$ in (7.10) is a representative of the open double coset. (See Sec. 4.2 in [GRS11] for the representatives of all double cosets.) The subgroup $Q^{(0)} = Q_n \cap \alpha_0^{-1}Q_{n(i+1)}\alpha_0$ consists of the elements

\[
(7.10) \quad \begin{pmatrix} a & 0 & y & 0 \\ b & c & y' & 0 \\ b^* & 0 & 0 & a^* \\ \end{pmatrix} \in H,
\]

where $a \in GL_n$, $b \in GL_{ni}$. The element (7.10) is conjugated by $\alpha_0$ to

\[
\begin{pmatrix} b & y' & 0 & c \\ a^* & 0 & 0 & a \\ \end{pmatrix}.
\]

As in (6.17), we get (for $Re(s)$ sufficiently large)

\[
\mathcal{D}_{\psi,n}^{\phi}(E(f_{\Delta(\tau,i+1),s}))(h, \epsilon) \\
(7.11) \quad = \int_{U_n(\mathbb{A}) \setminus U_n(F)} \theta_{\psi^{-1}}^{\phi}(j(u)(h, \epsilon)) \sum_{\eta \in Q^{(0)} \cap Q_n \setminus Q_n} f_{\Delta(\tau,i+1),s}(\alpha_0 \eta u t(h))\psi^{-1}_{n-1}(u) du.
\]

Let $U_n^{(0)} = U_n \cap Q^{(0)}$. The elements of $U_n^{(0)}$ have the form (7.10), with $a = z \in Z_n$, $b = I_n$, $c = 0$. Note, that for such an element $v$ in $U_n^{(0)}$, we have

\[
(7.12) \quad j(v) = ((0, y_n), 0), \quad \alpha_0 v \alpha_0^{-1} = \left( I_n, y', z^* \right) ^\wedge.
\]

Factoring the summation in $\eta$ in (7.11) modulo $U_n$ from the right, we get

\[
\mathcal{D}_{\psi,n}^{\phi}(E(f_{\Delta(\tau,i+1),s}))(h, \epsilon) \\
(7.13) \quad = \sum_{\gamma \in Q_n \setminus \text{Sp}_{2n}(F)} \int_{U_n^{(0)}(F) \setminus U_n(\mathbb{A})} \theta_{\psi^{-1}}^{\phi}(j(u)(\gamma, 1)(h, \epsilon)) f_{\Delta(\tau,i+1),s}(\alpha_0 u t(\gamma h))\psi^{-1}_{n-1}(u) du.
\]

Here, $Q_n = Q_n^{\text{Sp}_{2n}}$. Note that $U_n = U'_n \times U_n^{(0)}$, and, for $u = u'_{x,y} \in U'_n$, written in the form (7.12), $j(u) = ((x_n, 0), y_{n,1})$. Let us fix $\gamma$ in (7.13). Then the corresponding summand becomes

\[
(7.14) \quad \int_{U'_n(\mathbb{A})} \int_{U_n^{(0)}(F) \setminus U_n^{(0)}(\mathbb{A})} \theta_{\psi^{-1}}^{\phi}(j(vu)(\gamma, 1)(h, \epsilon)) f_{\Delta(\tau,i+1),s}(\alpha_0 v u t(\gamma h))\psi^{-1}_{n-1}(v) dv du.
\]

Consider in (7.14) the inner integration of the $dv$-integration, when we integrate, in the notation of (7.12), along $Z_n$, and along $y$ with zero last row. Using (7.12),
this inner integral is

\[ \int_{\mathcal{V}_{n+1,n-1}(F)} f_{\Delta(\tau,i+1),s}(v^\gamma g) \tilde{\psi}_{\mathcal{V}_{n+1,n-1}}(v) dv, \]

where \( g = v_c^\gamma \alpha \mu(u)(\gamma h), \)

\[ v_c = \begin{pmatrix} I_{n_0} & c & 0 \\ 1 & 0 & 0 \\ I_{n-1} \end{pmatrix}, \quad \tilde{\psi}_{\mathcal{V}_{n+1,n-1}}\left( \begin{pmatrix} I_{n_0+1} & c \\ e & z \end{pmatrix} \right) = \psi_{Z_{n-1}}(z)\psi(e_{n+1,1}). \]

By Prop. 3.1 the integral (7.15), as a function of \( g \in H_{\mathbb{A}}, \) is left invariant to \( v_c^\gamma, \)
for any \( c \in \mathbb{A}^{n_0}. \) Then (7.14) becomes

\[ \int_{U_{\gamma}(F)} \int_{F^{n_1}\backslash H_{\mathbb{A}}^{n_1}} \theta_{\psi^{-1}}^{\phi'}((0, x), 0) \psi^{-1}(v) \phi'(e) = \sum_{e \in F^{n_1}} \psi^{-1}(2e w_{n_0}^t x) \phi'(e). \]

Carrying out the \( dx \)-integration in (7.16), we get

\[ \int_{U_{\gamma}(F)} \omega_{\psi^{-1}}(j(u)(\gamma, 1)(h, \epsilon)) \phi(0) f_{\Delta(\tau,i+1),s}(\alpha \mu(u)(\gamma h)) du = \Lambda(f_{\Delta(\tau,i+1),s}, \phi)((\gamma, 1)(h, \epsilon)). \]

This proves (7.9). The rest of the theorem is proved similarly to Theorem 6.2 and we skip it.

Now, we show compatibility with normalization. As in the last section (and the ones before), we need to consider the local integrals, analogous to (6.25).

(7.17) \( \mathcal{L}_v(f_{\Delta(\tau,v),s}, \phi_v) = \int_{U_v(F_v)} \phi_v(x_n)f_{\Delta(\tau,v),s}(\alpha \mu u; I_{n_0}, I_{n})\psi_v^{-1}(y_{n,1}) du. \)

Let us explain the notations. As in (6.25), \( f_{\Delta(\tau,v),s} \) is a section of

\[ \rho_{\Delta(\tau,v),s} = \text{Ind}_{Q_{n_0,n}(F_v)}^{\text{Sp}_{2n_0}(F_v)}(\Delta(\tau_v,i)|\det \cdot^{1/2} \times \tau_v|\det \cdot^{1/2}) \]

with \( Q_{n_0,n} \) denoting the standard parabolic subgroup of \( H = \text{Sp}_{2n_0}(F_v), \) whose Levi part is isomorphic to \( \text{GL}_{n_0} \times \text{GL}_n. \) The representation \( \Delta(\tau_v, i) \) is realized in its model with respect to \( \psi_{\mathbb{A}}^{-1} \) and \( \tau_v \) is realized in its \( \psi_{\mathbb{A}}^{-1} \)-Whittaker model. As before, we simplify notation and re-denote \( f_{\Delta(\tau_v,i),s}(y) = f_{\Delta(\tau_v,i),s}(y; I_{n_0}, I_{n}). \)

Finally, we wrote in (7.17) an element \( u = u_{x,y} \in U_v(F_v) \) in the form (7.10), so that \( x_n \) is the last row of \( x \) and \( y_{n,1} \) is the \( (n, 1) \)-coordinate of \( y. \)

Let \( S \) be a finite set of places of \( F_v \), containing the infinite ones, such that for a place \( v \) outside \( S, \) \( \tau_v \) is unramified, \( \psi_v \) is normalized, \( \phi_v = \phi_v^0 \) is the characteristic function of \( O_v \), and \( f_{\Delta(\tau_v,i),s} = f_{\Delta(\tau_v,i),s}^0 \) is spherical and normalized. Write the elements of \( U_v(F_v) \) in the form \( u_{x_1,x_2:y} \) as in (7.7). Let

\[ U''_v(F_v) = \{ u_{0,y}^0 = \begin{pmatrix} I_n & 0 & y \\ I_{2n_0} & 0 & I_n \end{pmatrix} \in H(F_v) \}. \]
Proposition 7.2. Let \( v \) be a place of \( F \). There is a section \( f'_{\Delta(\tau_v,i;1),s} \), which depends on (the smoothness of) \( f_{\Delta(\tau_v,i;1),s} \) and \( \phi_v \), such that

\[
L_v(f_{\Delta(\tau_v,i;1),s}, \phi_v) = \int_{U''_n(F_v)} f'_{\Delta(\tau_v,i;1),s}(\alpha_0u)\psi_{v}^{-1}(u_{n,n+2n+1})du.
\]

If \( f_{\Delta(\tau_v,i;1),s} \) and \( \phi_v \) are spherical, then \( f'_{\Delta(\tau_v,i;1),s} = f_{\Delta(\tau_v,i;1),s} \). In general, \( f'_{\Delta(\tau_v,i;1),s} \) is obtained from \( f_{\Delta(\tau_v,i;1),s} \) by a finite sequence of convolutions against certain Schwartz functions.

Proof. The proof is similar to that of Prop. 4.1 [6.3]. We explain the case where \( v \) is finite and indicate the main steps. We assume, for simplicity that \( \psi_v \) is normalized. Write \( u'_{x,y} = v(x)u_n(y) \), where \( v(x) = u'_{x,0} \), \( u_n(y) = u_{0,y} \). Note that

\[
\alpha_0u_n(y)\alpha_0^{-1} = \text{diag}(I_{n_1}, \begin{pmatrix} I_n & 0 \end{pmatrix}, I_{n_1}) - \text{diag}(I_{n_1}, \begin{pmatrix} I_n & I_{n_1} \end{pmatrix}, I_{n_1}).
\]

We will show that the last function has compact support in \( M_{n\times n}(F_v) \), and in case of a spherical section this support is \( M_{n\times n}(O_v) \). First, the support of \( L'_v(f_{\Delta(\tau_v,i;1),s}, \phi_v)(x) \) in \( x_n \) is compact, since \( \phi_v \) is in \( S(F_v^{n_1}) \). Thus, \( L'_v(f_{\Delta(\tau_v,i;1),s}, \phi_v)(x) \) is a finite linear combination of integrals of the form (we keep denoting the section by \( f_{\Delta(\tau_v,i;1),s} \))

\[
l_v(f_{\Delta(\tau_v,i;1),s})(x(n)) = \int_{U''_n(F_v)} f_{\Delta(\tau_v,i;1),s}(\alpha_0v(x(n))u)\psi_v^{-1}(u_{n,n+2n+1})du,
\]

where \( x(n) \) is obtained from \( x \) by replacing \( x_n \) with 0. Assume that we have proved that the support of \( l_v(f_{\Delta(\tau_v,i;1),s}, \phi_v)(x(n)) \) in \( x_r \), for \( n-\ell \leq r \leq n, 0 \leq \ell \leq n-2 \) is compact. Thus, we may assume that \( x_r = 0 \), for \( n-\ell \leq r \leq n \). Let

\[
c_{\ell,z} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_{\ell} & 0 \\ z & 0 & I_{n_1} \end{pmatrix} \in \text{Sp}_{2n+i+1}(F_v),
\]

with \( z \in (\mathcal{P}^k)^{n_1} \), where \( k \) is sufficiently large, such that right translation by \( c_{\ell,z} \) fixes \( f_{\Delta(\tau_v,i;1),s} \). As in the proof of Prop. 6.3 we get, for \( x \) as above,

\[
l_v(f_{\Delta(\tau_v,i;1),s})(x) = l_v(\rho(c_{\ell,z})f_{\Delta(\tau_v,i;1),s})(x) = \psi((x)_{n-\ell-1})l_v(f_{\Delta(\tau_v,i;1),s})(x).
\]

We conclude that \( x_n-\ell-1 \) is supported in \( (\mathcal{P}^{-k})^{n_1} \), and in case the section is spherical, it is supported in \( O_v^{n_1} \). \( \square \)

Now we get the analog of Theorem 6.4.

Theorem 7.3. Let \( v \) be a place outside \( S \). Then

\[
L_v(f_{\Delta(\tau_v,i;1),s}, \phi_v^0) = \frac{1}{L(\tau_v, s + \frac{1}{2} + 1)L(\tau_v, \wedge^2 2s + i + 1)}.
\]
Proof. Denote, for \( h \in \operatorname{Sp}_{2n}(F_\nu) \),
\[
 f^0_{\tau, s + \frac{1}{2}}(h) = f^0_\Delta(\tau, i + 1, s)(\text{diag}(In_i, h, In_i)).
\]
This is the spherical, normalized element of
\[
 \rho_{\tau, s + \frac{1}{2}} = \text{Ind}_{Q^n(F_\nu)}^{\operatorname{Sp}_{2n}(F_\nu)}(\tau_v)\det \cdot |^s + \frac{1}{2}.
\]
Prop. \[8.2\] and \[8.4\] show that, for \( Re(s) \) sufficiently large,
\[
 (7.22) \quad L_v(f^0_{\Delta(\tau, i + 1, s)}; \phi^0_v) = \int_{U^{0,2}_{2n}(F_\nu)} f^0_{\tau, s + \frac{1}{2}}((I_n \ y \ I_n)) \psi^{-1}(y_{n, 1}) du.
\]
This is the Jacquet integral applied to \( f^0_{\tau, s + \frac{1}{2}} \) with respect to the standard \( \psi_v \)-Whittaker character. Now, we get the theorem by the Casselman-Shalika formula. 

Let us multiply our Eisenstein series \( E(f_\Delta(\tau, i + 1, s)) \) on \( \operatorname{Sp}_{2n(i + 1)}(F_\nu) \) by its normalizing factor outside \( S \), \( d^\nu_{\tau} \operatorname{Sp}_{2n(i + 1)}(S)(s) \). See \[1.19\], \[1.20\]. By \[1.21\], \[1.22\], we check that
\[
 (7.23) \quad \frac{d^\nu_{\tau} \operatorname{Sp}_{2n(i + 1)}(S)(s)}{L(\tau_v, s + \frac{1}{2}, 2s + i + 1)} = d^\nu_{\tau} \operatorname{Sp}_{2n(i + 1)}(S)(s).
\]
Let
\[
 E_S^\nu(f_\Delta(\tau, i + 1, s)) = d^\nu_{\tau} \operatorname{Sp}_{2n(i + 1)}(S)(s)E(f_\Delta(\tau, i + 1, s))
\]
as in Theorem \[6.5\] we get

**Theorem 7.4.** The descent to \( \operatorname{Sp}_{2n}(F_\nu) \) of the normalized Eisenstein series \( E_S^\nu(f_\Delta(\tau, i + 1, s)) \), \( D^\nu_{\psi, ni}(E_S^\nu(f_\Delta(\tau, i + 1, s))) \), is the normalized (outside \( S \)) Eisenstein series on \( \operatorname{Sp}_{2n}(F_\nu) \) corresponding to the section \( \Lambda(d^\nu_{\tau} \operatorname{Sp}_{2n(i + 1)}(S)(s)f_\Delta(\tau, i + 1, s), \phi) \).

8. **Application of Fourier-Jacobi Coefficients to \( E(f_\Delta(\tau, i + 1, s)) \) on \( \operatorname{Sp}_{2n(1 + \frac{1}{2})}(F_\nu) \): Descent to \( \operatorname{Sp}_{2n}(F_\nu) \)**

We keep the notations of the previous section. We consider the parabolic induction
\[
(8.1) \quad \rho_\Delta(\tau, i + 1, \gamma_{\psi, s}) = \text{Ind}_{Q^n(F_\nu)}^{\operatorname{Sp}_{2n(i + 1)}(F_\nu)}(\Delta(\tau, i + 1, s))\det \cdot |^s.
\]
Let \( f_\Delta(\tau, i + 1, \gamma_{\psi, s}) \) be a smooth, holomorphic section of \( \rho_\Delta(\tau, i + 1, \gamma_{\psi, s}) \) in \[8.1\], consider the corresponding Eisenstein series \( E(f_\Delta(\tau, i + 1, \gamma_{\psi, s})) \), and apply to it the Fourier-Jacobi coefficient similar to \[7.5\]:
\[
 D^\nu_{\psi, ni}(E(f_\Delta(\tau, i + 1, \gamma_{\psi, s}))(h))
\]
\[
 (8.2) \quad \int_{U_1(F_\nu) \setminus U_{1n}(F_\nu)} E(f_\Delta(\tau, i + 1, \gamma_{\psi, s}), \tilde{u}i(h)) \psi^{-1}(u) \theta^{\phi}_\psi(\tilde{t}(u)(h, 1)) du.
\]
Here, \( h \in \operatorname{Sp}_{2n}(F_\nu) \), and \( \tilde{t}(h) = (t(h), 1) \in \operatorname{Sp}_{2n(1 + \frac{1}{2})}(F_\nu) \). As usual, we identify \( u \in U_{1n}(F_\nu) \) with \( (u, 1) \). Define the function on \( \operatorname{Sp}_{2n(1 + \frac{1}{2})}(F_\nu) \), \( f^\nu_\Delta(\tau, i + 1, \gamma_{\psi, s}) \), in a similar way to \[7.8\]. The proof of the following theorem is the same as that of Theorem \[7.1\] with obvious modifications. We just remark that in the proof that the section
\( \Lambda(f_{\Delta(\tau,i+1)\gamma_\psi,s}, \phi) \) below corresponds to the parabolic data \((Q_{ni}, \Delta(\tau,i) \mid \det \cdot |^s) \), we use the fact that
\[
\gamma_\psi(t) \gamma_\psi^{-1}(t) = \gamma_\psi(t)^2 (t,-1) = (t,-t) = 1,
\]
where \((t,t')\) denotes the Hilbert symbol.

**Theorem 8.1.** For \(\text{Re}(s)\) sufficiently large, \(h \in \text{Sp}_{2n}(\mathbb{A})\),
\[
(8.3) \quad D_{\psi,m}^\phi(E(f_{\Delta(\tau,i+1)\gamma_\psi,s}))(h) = \sum_{\gamma \in Q_n \backslash \text{Sp}_{2n}} \Lambda(f_{\Delta(\tau,i+1)\gamma_\psi,s}, \phi)(\gamma h),
\]
where
\[
\Lambda(f_{\Delta(\tau,i+1)\gamma_\psi,s}, \phi)(h) = \int_{U_n^2(h)} \omega_{\psi^{-1}}(j(u)(h,1)) \phi(0) f_{\Delta(\tau,i+1)\gamma_\psi,s}(\alpha_0 u \bar{u}(h)) du.
\]
The function \(\Lambda(f_{\Delta(\tau,i+1)\gamma_\psi,s}, \phi)\), defined, for \(\text{Re}(s)\) sufficiently large, by the last integral, admits analytic continuation to a meromorphic function of \(s\) in the whole plane. It defines a smooth meromorphic section of
\[
\rho_{\Delta(\tau,i),s} = \text{Ind}_{\text{Q}_n}^{\text{Sp}_{2n}(\mathbb{A})} \Delta(\tau,i) \mid \det \cdot |^s.
\]
Thus, \(D_{\psi,m}^\phi(E(f_{\Delta(\tau,i+1)\gamma_\psi,s}))\) is the Eisenstein series on \(\text{Sp}_{2n}(\mathbb{A})\) corresponding to the section \(\Lambda(f_{\Delta(\tau,i+1)\gamma_\psi,s}, \phi)\) of \(\rho_{\Delta(\tau,i),s}\).

At a place \(v\) of \(F\), consider, as in (7.17), with the same notation,
\[
(8.4) \quad L_v(f_{\Delta(\tau,i;1)\gamma_\psi,s}, \phi_v) = \int_{U_n^v(F_v)} \phi_v(x_n) f_{\Delta(\tau,i;1)\gamma_\psi,s}(\alpha_0 u; I_{ni}, I_n) \psi_v^{-1}(y_{n,1}) du,
\]
where \(f_{\Delta(\tau,i;1)\gamma_\psi,s}\) is a section of
\[
\rho_{\Delta(\tau,i;1)\gamma_\psi,s} = \text{Ind}_{Q_{ni,m}}^{\text{Sp}_{2n}(F_v)} \Delta(\tau,i) \gamma_\psi_v \mid \det \cdot |^{s-\frac{1}{2}} \times \tau_v \gamma_\psi_v \mid \det \cdot |^{s+\frac{1}{2}}.
\]
See also (4.4). Now, the same proof as that of Prop. 7.2 shows that there is a section \(f_{\Delta(\tau,i;1)\gamma_\psi,s}\), which depends on the smoothness of \(f_{\Delta(\tau,i;1)\gamma_\psi,s}\) and \(\phi_v\), such that
\[
(8.5) \quad L_v(f_{\Delta(\tau,i;1)\gamma_\psi,s}, \phi_v) = \int_{U_n^v(F_v)} f_{\Delta(\tau,i;1)\gamma_\psi,s}(\alpha_0 u; I_{ni}, I_n) \psi_v^{-1}(u_{n,n+2ni+1}) du;
\]
and if \(f_{\Delta(\tau,i;1)\gamma_\psi,s}\) and \(\phi_v\) are spherical, then \(f_{\Delta(\tau,i;1)\gamma_\psi,s} = f_{\Delta(\tau,i;1)\gamma_\psi,s}\). Thus, for such a finite place \(v\), where all data are spherical and normalized, define, for \((h,\epsilon) \in \text{Sp}_{2n}^2(F_v)\),
\[
f_{\tau_v \gamma_\psi_v, s+\frac{1}{2}}((h,\epsilon)) = f_{\Delta(\tau,i;1)\gamma_\psi,s}((\text{diag}(I_{ni}, h, I_n), \epsilon)).
\]
This is the spherical, normalized element of
\[
\rho_{\tau_v \gamma_\psi_v, s+\frac{1}{2}} = \text{Ind}_{Q_n^{2}(F_v)}^{\text{Sp}_{2n}(F_v)} \tau_v \gamma_\psi_v \mid \det \cdot |^{s+\frac{1}{2}}.
\]
By (8.5), we have, for \(\text{Re}(s)\) sufficiently large,
\[
(8.6) \quad L_v(f_{\Delta(\tau,i;1)\gamma_\psi,s}, \phi_v^0) = \int_{U_n^{\text{Sp}_{2n}(F_v)}} f_{\tau_v \gamma_\psi_v, s+\frac{1}{2}}((\begin{pmatrix} I_n & 1 \\ y & I_n \end{pmatrix}, 1)) \psi_v^{-1}(y_{n,1}) du.
\]
This is the Jacquet integral applied to $f_{\tau,i}^{0}$, with respect to the standard \( \psi_{\tau} \)-Whittaker character. The analog of the Casselman-Shalika formula (BFH91) in this case gives then the following.

**Theorem 8.2.**

\[ \mathcal{L}_{v}(f_{\tau,i}^{0} \gamma_{0}^{*},s) \phi^{0}(v) = \frac{L(\tau_{0},s + \frac{1}{2})}{L(\tau_{0},sym^{2},2s + i + 1)}. \]

Finally, let us multiply our Eisenstein series \( E(f_{\Delta(\tau,i+1)\gamma_{0}^{*}}) \) on \( Sp_{2n}(\mathbb{A}) \) by its normalizing factor outside \( S \), \( d_{\tau}^{Sp_{2n}(\mathbb{A})}(s) \). See (1.21), (1.22). By (1.19), (1.20), we check that

\[ \frac{d_{\tau}^{Sp_{2n}(\mathbb{A})}(s)L(\tau_{0},s + \frac{1}{2})}{L(\tau_{0},sym^{2},2s + i + 1)} = d_{\tau}^{Sp_{2n}(\mathbb{A})}(s). \]

Let

\[ E_{S}(f_{\Delta(\tau,i+1)\gamma_{0}^{*}}) = d_{\tau}^{Sp_{2n}(\mathbb{A})}(s)E(f_{\Delta(\tau,i+1)\gamma_{0}^{*}}). \]

As in Theorem 1.4 we get

**Theorem 8.3.** The descent to \( Sp_{2n}(\mathbb{A}) \) of the normalized Eisenstein series \( E_{S}(f_{\Delta(\tau,i+1)\gamma_{0}^{*}}) \), \( D_{\gamma_{0}}^{\phi}(E_{S}(f_{\Delta(\tau,i+1)\gamma_{0}^{*}})) \), is the normalized (outside \( S \)) Eisenstein series on \( Sp_{2n}(\mathbb{A}) \) corresponding to the section \( \Lambda(d_{\tau}^{Sp_{2n}(\mathbb{A})}(s)f_{\Delta(\tau,i+1)\gamma_{0}^{*}}) \).

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