ON CONFIGURATION SPACES OF STABLE MAPS

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ABSTRACT. We study here some aspects of the topology of the space of smooth, stable, genus 0 curves in a Riemannian manifold $X$, i.e. the Kontsevich stable curves, which are not necessarily holomorphic. We use the Hofer-Wysocki-Zehnder polyfold structure on this space and some natural characteristic classes, to show that for $X = BU$, the rational homology of the spherical mapping space injects into the rational homology of the space of stable curves. We also give here a definition of what we call $q$-complete symplectic manifolds, which roughly speaking means Gromov-Witten theory captures all information about homology of the space of smooth stable maps.

1. INTRODUCTION

The space of smooth stable curves (unparametrized stable maps) of a Riemann surface into a Riemannian manifold appears naturally in the context Gromov-Witten theory in symplectic geometry, particularly in the context of the beautiful polyfold approach of Hofer-Wysocki-Zehnder [7].

The topology of the configuration space of stable curves in a general Riemannian manifold seems very interesting on its own merit. For example we show that the space of based stable curves has the structure of an $H$-space. This is interesting as the space of unparametrized based spheres in $X$ does not have an $H$-space structure.

Moreover, this configuration space may also be very natural in the study of gradient flow for the energy functional on the space of smooth maps of say a Riemann sphere into a Riemannian manifold $X$. It was first observed by Sacks-Uhlenbeck [12] that the flow lines of the resulting parabolic flow often do not converge to smooth maps but rather the associated maps develop bubbling phenomena. This of course presents problems for Morse theory considerations. For example, Eells and Wood [2] show that in a simply connected Kahler manifold $X$, the only critical points of the energy functional on the mapping space of a Riemann sphere are (anti)-holomorphic maps, which are also absolutely energy minimizing. If Morse theory worked as expected we could conclude that the topology of the mapping space coincides with the topology of the space of absolute minima, i.e. with the space of (anti)-holomorphic maps, which is usually the wrong conclusion. However in a series of remarkable papers [1], [6], [5], [8], [4] (sorry for incomplete list) it is shown that the conclusion becomes essentially correct after a suitable process of stabilization. One possibility for treating or at least understanding this problem is to partially compactify the space by adding all appropriate stable maps, in such a way that bubbling becomes built in. One may then hope that the gradient flow

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on the enlarged space satisfies some version of Palais-Smale condition, after appropriate completion. It would be most exciting to see if the polyfold theory helps in this.

Our main result concerns injectivity of the map from rational homology of the spherical mapping space into the rational homology of the space of stable curves in the case of $X = BU(r)$. This at least tells us the space of stable curves in that case is topologically non-trivial. Although we are far from understanding the topology of this space.

Finally, we define a notion of $q$-complete symplectic manifolds, inspired by work Cohen-Segal-Jones [1] studying the spherical mapping space into an almost Kahler manifold by means of certain stabilization of holomorphic spherical mapping spaces.

1.1. The space of smooth stable maps. Let $\mathcal{P}^A_{0,n} X$ denote the space whose elements are continuous maps into a Riemannian manifold $X, g$, with total homology class $A$, from a nodal connected Riemann surface $\Sigma$ with genus 0, with complex structure $j$ and $n$ marked points. These maps are assumed smooth on each component of $\Sigma$, and satisfy some additional conditions:

- If the restriction $u_\alpha$ of $u$ to a component $\alpha$ is non-constant then $u_\alpha$ is not null-homologous.
- If $u_\alpha$ is constant then the number of special points is at least 3.

We will distinguish one marked point by $z_0$. A pair of stable maps

$$u_1 : (\Sigma_1, j_1, \{z_1^1\}) \to X, u_2 : (\Sigma_2, j_2, \{z_2^1\}) \to X$$

are equivalent if there is a continuous map $\phi : \Sigma_1 \to \Sigma_2$, which is a diffeomorphism on each component, satisfies $\phi^* j_2 = j_1$ maps marked points to marked points and satisfies $u_2(z_2^1) = u_1(z_1^1)$. The quotient by the equivalence relation will be denoted by $\overline{\Omega}_n^A X$, and its elements are called stable curves in $X$.

The topology on $\mathcal{P}^A_{0,n} X$ we consider is the standard Gromov topology, (taking into account energy of maps for the metric $g$) and the arguments of [10, Chapter 5] can be adopted to show that this induces a metrizable topology on $\overline{\Omega}_n^A X$. From now on we will suppress mention of the axillary metric $g$, since topology of our spaces is independent of $g$. This topology should be equivalent to the topology considered in [7, Section 3.2]. (At any rate the topology in [7] is the one we need for Theorem 1.3.)

Remark 1.1. Note that under these conditions for a compact Riemannian manifold $X$ a bound on Riemannian energy of $u$ gives a bound on the number of smooth components. As by assumption that each $u_\alpha$ is not null-homologous, it's area and consequently energy is bounded by some constant $\hbar g$ depending only on $g$. This is an elementary application of the main compactness theorem for rectifiable currents in geometric measure theory, [3]. This brings up a question: are closed bounded subsets in $\overline{\Omega}_n^A X$ compact for a compact $X$?

Set

$$\overline{\Omega}_n^A X = \text{colim} \overline{\Omega}_n^A X,$$

the colimit with the maps in the directed system

$$i_n : \overline{\Omega}_n \to \overline{\Omega}_{n+1}$$

(1.1)
defined as follows. For \( u \in \mathcal{P}^i_{0,n} X \), \( u : (\Sigma, j, \{ z_i \}) \to X \), \( i_n(u) \) is an equivalence class of a map \( u' \) from
\[
\Sigma' = \Sigma \sqcup (\mathbb{C}\mathbb{P}^1, 0, z'_1, z'_2) \sim z'_1.
\]
The map \( i_n(u) \) is \( u \) on the component \( \Sigma \) and constant on the new component. The new distinguished marked point \( z'_0 \) on \( \Sigma' \) is the marked point 0.

It will later be necessary to remember more structure and consider the spaces \( \overline{\Omega}_n X \) as polyfolds, which is an orbifold version of M-polyfolds, or with even more structure as polyfold groupoids. In this case instead of saying that stable maps are smooth, we should instead require some regularity on the underlying continuous structure as polyfold groupoids. In this case instead of saying that stable maps are homotopy equivalent by a similar argument. Consequently there is an induced equivalence class of a map from \( \Sigma' = \Sigma \sqcup \Omega_n X \to \overline{\Omega}_{n+m-1} X \to \overline{\Omega}_{n+m+1} X \) defined as an equivalence class of a map from
\[
\Sigma' = \Sigma_1 \sqcup \Sigma_2 \sqcup (\mathbb{C}\mathbb{P}^1, 0, z'_1, z'_2) \sim z'_1, z'_2.
\]
which is \( u_1 \) respectively \( u_2 \) on the components \( \Sigma_1 \), respectively \( \Sigma_2 \) and is constant on the new \( \mathbb{C}\mathbb{P}^1 \) component. The new distinguished marked point for \( \Sigma' \) is 0. In other words we concatenate \( u_1, u_2 \) with a ghost bubble as intermediary.

This multiplication is homotopy associative, since the main ghost components of domains for \( (u_1 \ast u_2) \ast u_3 \), and \( u_1 \ast (u_2 \ast u_3) \), where concatenation takes place correspond to a pair of points in \( \overline{\mathbb{M}}_{0,4} \), which we may connect by a path. In other words it is conceptually the same argument as the argument for associativity of quantum multiplications. Also the maps
\[
\overline{\Omega}_n X \times \overline{\Omega}_m X \to \overline{\Omega}_{n+m-1} X \to \overline{\Omega}_{n+m+1} X
\]
\[
\overline{\Omega}_n X \times \overline{\Omega}_m X \to \overline{\Omega}_{n+1} X \times \overline{\Omega}_{m+1} X \to \overline{\Omega}_{n+m+1} X,
\]
are homotopy equivalent by a similar argument. Consequently there is an induced map in the homotopy category \( \overline{\mathbb{M}} X \times \overline{\mathbb{M}} X \to \overline{\mathbb{M}} X \).

If we ask that our maps \( u \) are based, i.e. map the distinguished marked point \( z_0 \) to \( x_0 \in X \), then the corresponding space \( \overline{\Omega}_{x_0} X \) is a homotopy associative H-space. Consequently homology of \( \overline{\Omega}_{x_0} X \) is a ring with Pontryagin product.

**Notation 1.2.** From now on we will be in the above based situation and the subscript \( x_0 \) in \( \overline{\Omega}_{x_0} \) will be dropped.

Our main observation in this paper is this:

**Theorem 1.3.** The natural map \( H_* (\Omega^2 BSU(r), \mathbb{Q}) \to \text{Cob}^{orb}_*(\overline{\Omega} BSU(r), \mathbb{Q}) \), is injective for \( * \leq 2r - 2 \).

On the right hand side we have orbifold bordism groups which are to be defined.
1.2. Complete symplectic manifolds. Suppose now $(X, \omega)$ is a symplectic manifold. Let $a_i : D_i \to X$, $a_\xi : D_\xi \to \overline{M}_{0,n}$ be smooth maps of closed oriented smooth manifolds, with $\overline{M}_{0,n}$ denoting the moduli space of stable genus 0 Riemann surfaces with $n$ marked points.

Under suitable conditions, for example if $(X, \omega)$ is semi-positive we have natural cycles $gw : \overline{M}_n(A, \{a_i\}, a_\xi) \to \overline{\Omega}X$, defined as follows. Consider the diagram below:

$$
\begin{tikzcd}
(\prod_i D_i) \times D_\xi \\
\overline{M}_{0,n}(X, J) \\
\end{tikzcd}
$$

with $\overline{M}_{0,n}(X, J)$ denoting the compactified moduli space of genus zero, class $A$, $J$-holomorphic curves in $X$, for a regular $\omega$-tamed $J$. After perturbing the maps to be transverse, we define $\overline{M}_{0,n}(X, \{a_i\}, \xi)$ as the pull-back of this diagram (oriented fibre product) and the cycle $gw$ is defined to be the composition of the projection of $\overline{M}_{n}(X, \{a_i\}, \xi)$ to $\overline{M}_{0,n}(X; J)$, with the tautological map $\overline{M}_{0,n}(X, J) \to \overline{\Omega}X$.

For a completely general symplectic manifold $(M, \omega)$ the homology class of the cycle $gw$ is defined via the homology pushforward of the orbifold virtual fundamental class of $\overline{M}_n(A, \{a_i\}, \xi)$. We shall call cycles $gw$: Gromov-Witten cycles. We will also call all 0-dimensional cycles into $\overline{\Omega}X$ Gromov-Witten cycles.

One of the motivations we had for undertaking study of configuration space of smooth stable maps, is so we could make the following definition, we say more in the remark below.

**Definition 1.4.** We will say that a symplectic manifold $(X, \omega)$ is **q-complete** if homology of $\overline{\Omega}X$ is multiplicatively generated over $\mathbb{Q}$ by homology classes of Gromov-Witten cycles.

**Remark 1.5.** This is the homological version of homotopy approximation of $\Omega^2X$ by holomorphic mapping spaces from $\mathbb{C}P^1$, which was studied for example in [1], [6], [5], [8], [4]. Given a symplectic manifold $X, \omega$, the basic question is when does

$$
\Omega^2X \simeq Hol^+(\mathbb{C}P^1, X, \omega, jX)),
$$

where $Hol^+$ denotes the group completion of the topological monoid of based $j_X$-holomorphic maps of $\mathbb{C}P^1$ into $X$, under the gluing operation. Remarkably, this is known to be the case for example, for complex projective spaces, generalized flag manifolds and toric manifolds. Unfortunately (1.3) only makes sense for a fixed complex structure $j_X$ i.e. it is a priori not a symplectic property. We wanted a purely symplectic notion, and q-complete symplectic manifolds is one possibility.

**Question 1.6.** Is $\mathbb{C}P^n, \omega_{st}$ q-complete?

This appears to be at the moment a difficult and interesting question. Some interesting and possibly related work is done by Miller [11].

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2. Main argument

2.1. Quantum classes. We are going to use the basic notation and definitions of [14]. We will be concerned with stable quantum classes, which are classes

\[(2.1) \quad q_c^\infty \in H^{2k}(\Omega^2 BU(r) \simeq \Omega SU(r), QH(\mathbb{C}P^{r-1})),\]

with \(2k \leq 2r - 2\). The quantum homology ring \(QH(\mathbb{C}P^{r-1})\) is taken with \(\mathbb{Q}\) coefficients, so that additively it is just the rational homology of \(\mathbb{C}P^{r-1}\). The above classes are interesting in the present context because they canonically “extend” to \(\Omega^2 BSU(r)\), while such an extension is not apparent for Chern classes especially since they are not \textit{intrinsically} defined on \(\Omega^2 BSU(r)\). Extend here means that they are pull-backs of some orbifold cohomology classes on \(\Omega^2 BSU(r)\), via the natural map \(\Omega^2 BSU(r) \to \Omega^2 BSU(r)\). Although orbifold cohomology groups for us will just be the space of \(\text{QH}(\mathbb{C}P^\infty)\) valued linear functionals on certain orbifold bordism groups \(\text{Cob}^n_{sc}(\Omega BSU(r))\), which contain too much information to be really practical, and in principle it would be good to cut this down, but it is not obvious how to do this in a way that still allows definition of our classes. In particular it is at the moment unclear how and if these are related to Chen-Ruan orbifold cohomology groups (with suitable coefficients). A map \(X \to \Omega BSU(r)\) by definition factors through a map \(X \to \Omega_n BSU(r)\) for some \(n\), composed with the universal map

\[
\Omega_n BSU(r) \to \Omega BSU = \text{colim}_n \Omega_n BSU(r).
\]

Consequently, we may define a map from a smooth manifold \(f : X \to \Omega BSU(r)\) as sc-smooth if factors through an sc-smooth map to \(\Omega_n BSU(r)\). Note that \(\text{BU}(r) = \lim_{n \to \infty} GGr_C(r, \mathbb{C}^n)\), and we of course only need a polyfold structure on \(\Omega_n GGr_C(r, \mathbb{C}^n)\) for every \(n\), for all our arguments. To simplify notation we will still just refer \(\Omega BSU(r)\) as a polyfold, although this of course only means that it is a direct limit of polyfolds.

The group \(\text{Cob}_{sc}^{orb}(\Omega BSU(r))\), is the group of equivalence classes of sc-smooth orbifold maps \(f : X^k \to \Omega BSU(r)\), with \(X^k\) closed oriented smooth orbifold with corners, with composition given by disjoint union, where the equivalence relation is \((f_1, X_1) \sim (f_2, X_2)\) if there is an sc-smooth map of an orbifold with boundary:

\[F : B^{k+1} \to \Omega BSU(r),\]

with \((\partial F, \partial B) = (f_1, X_1^\text{op}) \sqcup (f_2, X_2)\), where \(X_1^\text{op}\) denotes \(X_1\) with the opposite orientation. From the groupoid point of view of orbifolds an orbifold \(X\) is some coarse equivalence class of an étale, proper, stable, smooth groupoid, also known as orbifold groupoid, \(\mathcal{X}\), see for example [9]. The orbifold map \(f\) above is determined by data of an sc-smooth functor \(\tilde{f} : \mathcal{X} \to \Omega BSU(r)\), with the right side considered as a polyfold groupoid, also called ep-groupoid, but we only use the former name. Let \(E \to BSU(r)\) denote the projectivization of the universal \(\mathbb{C}^r\)-bundle.

We have a natural polyfold groupoid fibration \(\mathcal{E} \to \Omega BSU(r)\), with fiber over \(x \in \Omega BSU(r)\), \([x : \Sigma_x \to BSU(r)]\) the space of “stable sections” i.e. section class stable maps \(\sigma : \Sigma_x^\prime \to E_x = x^*E\). These are defined as follows: \(\Sigma_x^\prime\) has some components labeled as principal, and some components labeled as vertical. The principal components of \(\Sigma_x^\prime\) are identified with components of \(\Sigma_x\) and \(\sigma\) is a “smooth” (continuous section with appropriate regularity [7, Section 1.1]) section of \(E_x\) over these components, while the vertical components of \(\Sigma_x^\prime\) are mapped into
the fibers of $E_x$. The restriction of $\sigma$ to the collection of all vertical components is required to be a stable map as defined in Section 1.1.

This definition requires a bit more explanation as $x$ is only some equivalence class of a map. To this end consider the large category $\mathcal{P}_{0,n}^A BSU(r)$ with morphisms given by label preserving reparametrizations, (see Section 1.1). Then clearly we have an analogously defined fibration of topological categories $\mathcal{E} \to \mathcal{P}_{0,n}^A BSU(r)$ (and now there is no ambiguity in the definition). On the other hand the groupoid $\overline{\Omega} BSU(r)$ is actually constructed as a refinement of $\mathcal{P}_{0,n}^A BSU(r)$ and this refinement also refines the above categorical fibration to a polyfold groupoid fibration $pr : \mathcal{E} \to \overline{\Omega} BSU(r)$. In what follows we keep applying this implicit understanding.

Fix a unitary (in other words $PU(r)$) connection $\mathcal{A}$ on $E \to BSU(r)$. Then the restriction $\mathcal{A}_x$ of $\mathcal{A}$ to $E_x = x^*E$ induces almost complex structures $\{J_x\}$ in the following standard way.

- The natural map $\pi : (E_x, J_x) \to (\Sigma_x, j_x)$ is holomorphic.
- $J_x$ preserves the horizontal subbundle of $TE_x$ induced by $\mathcal{A}_x$.
- $J_x$ preserves the vertical tangent bundle $T_{vert}E_x$ of $\mathbb{CP}^{r-1} \to E_x \to \Sigma_x$, and restricts to the standard complex structure on the fibers $\mathbb{CP}^{r-1}$. (That is to say the fibers are identified with $\mathbb{CP}^{r-1}$ up to action of $PU(r)$, which preserves this complex structure.)

Let $\tilde{\mathcal{E}}$ denote the pull-back of $\mathcal{E}$ to $\mathcal{X}$, via $\tilde{f} : \mathcal{X} \to \overline{\Omega} BSU(r)$. Then over $\tilde{\mathcal{E}}$ we have a natural strong Polyfold Banach bundle $\mathcal{W}$. For an element $\sigma \in E_x$, $\sigma : \Sigma'_x \to E_x$, the fiber over $\sigma$ consists (after appropriate completion) of the space of continuous, and smooth over smooth components $J_x$ anti complex linear 1-forms on $\Sigma'_x$ with values in $\sigma^*T_{vert}E_x$. By essentially identical arguments to [7, Section 1.2] over the whole $\mathcal{E}$ this can be given the structure of a strong polyfold Banach bundle. And we have a Fredholm sc-section of $\mathcal{W}$: the Cauchy-Riemann section, whose total degree is $\langle c_1(T_{vert}E_x), \sigma \rangle \cdot \frac{1}{r}$.

Let $\mathcal{M}$ denote the polyfold groupoid consisting of pairs $(\sigma, x)$, $x \in \mathcal{X}$ with $\sigma \in \tilde{\mathcal{E}}$ in the 0-set of the above constructed Cauchy-Riemann section, whose total degree is $d$, and which intersects $I_x(\mathbb{CP}^1)$. Where $d$ here is defined as

$$\langle c_1(T_{vert}E_x), \sigma \rangle \cdot \frac{1}{r}.$$ 

This is an integer, since by assumption $\tilde{f}$ maps to the identity component of $\overline{\Omega} BU(r)$ which implies that $P_x \simeq \mathbb{CP}^{r-1} \times \Sigma_x$, as a topological bundle. But then
the vertical Chern number of a section is \( n \) times the degree of the projection of the section to \( \mathbb{CP}^{r-1} \).

The virtual dimension of \( \mathcal{M}(P, [\mathbb{C}P^l], d) \), is \( 2dr + 2k + 2l \). Note that if \( 0 < 2k \leq 2r - 2 \), then
\[
2dr + 2k + 2l < 0 \text{ unless } d \geq -1.
\]

On the other hand \( d > 0 \) results in too high virtual dimension, and \( d = 0 \) only contributes to degree 0 class as we will see below. Under this condition definition of quantum classes is particularly simple:
\[
\langle qc_k, [\tilde{f}] \rangle = b \in QH(\mathbb{CP}^{r-1}),
\]
where \( b \in H_*(\mathbb{CP}^{r-1}, \mathbb{Q}) \) is defined by duality:
\[
b \cdot [\mathbb{C}P^l] = \# \mathcal{M}(P, [\mathbb{C}P^l], -1) \in \mathbb{Q},
\]
where the left side is the usual intersection product and the right side is the orbifold Gromov-Witten invariant counting elements of \( \mathcal{M}(P, [\mathbb{C}P^l], -r) \), which is zero unless the expected dimension is zero. Of course we also have to show that our definition of \( qc_k \) is well defined, i.e. that (2.2) is independent of the choice of representative \( \tilde{f} \). This is worked out in [13], and readily generalizes to polyfold setting, as it is just the usual cobordism of moduli space argument, similarly with independence of choices of almost complex structures, (connections).

The pull-back of the above defined classes to \( \Omega^2 BSU(n) \) are exactly the stable quantum classes considered in [14], except of course we did not need orbifold cycles but worked with cycles that are maps of closed oriented manifolds. We paraphrase the main theorem of [14] as follows:

**Theorem 2.1** ([14]). If \( 2k \leq 2r - 2 \), then \( 0 \neq a \in H_{2k}(\Omega^2 BSU(r), \mathbb{Q}) \) if and only if for some \( \{ \beta_i, \alpha_i \} \)
\[
0 \neq \langle \prod_i qc_{\beta_i}^{\alpha_i}, a \rangle, \text{ where } \sum_i 2\beta_i \cdot \alpha_i = k.
\]

**Proof of Theorem 1.3.** By Milnor-Moore, Cartan-Serre theorems the rational homology of \( \Omega^2 BSU(r) \simeq \Omega SU(r) \) is multiplicatively generated by spherical classes, in particular by maps of smooth manifolds. Consequently the theorem follows immediately by Theorem 2.1 and by existence of extension of classes \( qc_k \) to \( \Omega BSU(r) \)
described above. \( \square \)

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