Statistical Mechanics of Violent Relaxation in Stellar Systems

Pierre-Henri Chavanis

Laboratoire de Physique Quantique,
Université Paul Sabatier,
118 route de Narbonne,
31062 Toulouse, France

Abstract. We discuss the statistical mechanics of violent relaxation in stellar systems following the pioneering work of Lynden-Bell (1967). The solutions of the gravitational Vlasov-Poisson system develop finer and finer filaments so that a statistical description is appropriate to smooth out the small-scales and describe the “coarse-grained” dynamics. In a coarse-grained sense, the system is expected to reach an equilibrium state of a Fermi-Dirac type within a few dynamical times. We describe in detail the equilibrium phase diagram and the nature of phase transitions which occur in self-gravitating systems. Then, we introduce a small-scale parametrization of the Vlasov equation and propose a set of relaxation equations for the coarse-grained dynamics. These relaxation equations, of a generalized Fokker-Planck type, are derived from a Maximum Entropy Production Principle (MEPP). We make a link with the quasilinear theory of the Vlasov-Poisson system and derive a truncated model appropriate to collisionless systems subject to tidal forces. With the aid of this kinetic theory, we qualitatively discuss the concept of “incomplete relaxation” and the limitations of Lynden-Bell’s theory.

1 Introduction

It has long been realized that galaxies, and self-gravitating systems in general, follow a kind of organization despite the diversity of their initial conditions and their environment. This organization is illustrated by morphological classification schemes such as the Hubble sequence and by simple rules which govern the structure of individual self-gravitating systems. For example, elliptical galaxies display a quasi-universal luminosity profile described by de Vaucouleurs’ $R^{1/4}$ law and most of globular clusters are well fitted by the Michie-King model. The question that naturally emerges is, what determines the particular configuration to which a self-gravitating system settles. It is possible that their present configuration crucially depends on the conditions that prevail at their birth and on the details of their evolution. However, in view of their apparent regularity, it is tempting to investigate whether their organization can be favoured by some fundamental physical principles like those of thermodynamics and statistical physics. We ask therefore if the actual states of self-gravitating systems are not simply more probable than
any other possible configuration, i.e. if they cannot be considered as maximum entropy states. This thermodynamical approach may be particularly relevant for globular clusters and elliptical galaxies which are described by a distribution function that is almost isothermal 8. In the case of globular clusters, the relaxation proceeds via two-body encounters and this collisional evolution is well-described by kinetic equations of a Fokker-Planck-Landau type for which a H-theorem \(^1\) is available. By contrast, for elliptical galaxies, two-body encounters are completely negligible (the corresponding relaxation time \(t_{\text{coll}}\) exceeds the age of the universe by many orders of magnitude) and the galaxy dynamics is described by the Vlasov equation, i.e. collisionless Boltzmann equation 8. Since the Vlasov equation rigorously conserves entropy, a relaxation towards an isothermal distribution looks at first sight relatively surprising. Yet, the inner regions of elliptical galaxies appear to be isothermal and this fact stemmed as a mystery for a long time.

In a seminal paper, Lynden-Bell 43 argued that the violently changing gravitational field of a newly formed galaxy leads to a redistribution of energies between stars and provides a mechanism analogous to a relaxation in a gas. The importance of this form of relaxation had previously been stressed by a number of authors including Hénon and King but Lynden-Bell showed for the first time the relevance of a statistical description. He argued that the Vlasov-Poisson system develops an intricate mixing process in phase space associated with the heavily damped oscillations of a protogalaxy initially far from mechanical equilibrium and collapsing under its own gravity. As a result, the solutions of the Vlasov equation are not smooth but involve intermingled filaments at smaller and smaller scales. In this sense, there is no convergence towards equilibrium but rather the formation of a fractal-like structure in phase space. However, if we introduce a macroscopic level of description and make a local average of the distribution function over the filaments, the resulting “coarse-grained” distribution function is smooth and is expected to reach a maximum entropy state (i.e. most mixed state) on a very short time scale of the order of the dynamical time \(t_D\). This process is called violent relaxation and is acknowledged to account for the regularity of elliptical galaxies or other collisionless self-gravitating systems. Lynden-Bell predicted that the equilibrium state should be described by a Fermi-Dirac distribution function or a superposition of Fermi-Dirac distributions. Here, degeneracy is due not to quantum mechanics but to the Liouville theorem that prevents the smooth distribution function from exceeding the maximum of its initial value. In the non degenerate limit, the Fermi-Dirac distribution functions reduce to Maxwellsians. The prediction of an isothermal distribution

\(^1\) The H-theorem, proved by Boltzmann for an ideal gas, states that entropy is a monotonically increasing function of time. Boltzmann’s kinetic theory of gases therefore provides a direct justification of the second principle of thermodynamics. The generalization of this theorem for self-gravitating systems rests on simplifying assumptions which are difficult to rigorously justify 8.
for collisionless stellar systems was considered as a triumph in the 1960’s and laid the foundation of a new type of statistical mechanics. Of course, the validity of the theory is conditioned by a hypothesis of ergodicity which may not be completely fulfilled. This is the complicated problem of “incomplete relaxation” which limits the power of prediction of Lynden-Bell’s approach. However, as we shall see, these difficulties should not throw doubt on the importance of this statistical description. A similar relaxation process is at work in two-dimensional turbulence (described by the 2D Euler equation) and can explain the organization and maintenance of coherent vortices, such as the Great Red Spot of Jupiter, which are common features of large-scale geophysical or astrophysical flows [53,46,23,52,26,6,7]. The mathematical relevance of this statistical description has been given by Robert [50] introducing the concept of Young measures. The formal analogy between two-dimensional vortices and stellar systems has been discussed by Chavanis [11,14,17,27].

This paper is organized as follows. In section 2, we introduce the gravitational Vlasov-Poisson system and list its main properties. In section 3, we present the statistical approach of Lynden-Bell [43] to the problem of violent relaxation. In section 4, we show that the Fermi-Dirac equilibrium distribution predicted by Lynden-Bell is not entirely satisfactory since it has an infinite mass. We must therefore invoke incomplete relaxation and introduce truncated models. In the artificial situation in which the system is enclosed within a spherical box, we can calculate the Fermi-Dirac spheres explicitly and prove the existence of a global entropy maximum for each value of energy. For low energies, this equilibrium state has a degenerate core surrounded by a dilute atmosphere, as calculated by Chavanis & Sommeria [25]. More generally, we determine the complete equilibrium phase diagram and discuss the nature of phase transitions in self-gravitating systems. In section 5, we describe the coarse-grained relaxation of collisionless stellar systems towards statistical equilibrium in terms of a generalized Fokker-Planck equation. This relaxation equation is derived from a phenomenological Maximum Entropy Production Principle (MEPP) and involves a diffusion in velocity space compensated by a nonlinear friction. In section 6, we present a quasilinear theory of the Vlasov-Poisson system and show that it leads to a kinetic equation of a Landau type. When the system is close to equilibrium (so that a thermal bath approximation can be implemented) this equation reduces to the Fokker-Planck equation of the thermodynamical approach and the diffusion coefficient can be explicitly evaluated. This provides a new, self-consistent, equation for the “coarse-grained” dynamics of stellar systems where small scales have been smoothed-out in an optimal way. In section 7, we use this kinetic model to derive the distribution function of a tidally truncated collisionless stellar system. This truncated model preserves the main features of Lynden-Bell’s distribution (including degeneracy) but has a finite mass, avoiding the artifice of a spherical container. Other truncated models attempting to take into account incomplete relaxation are discussed.
2 The gravitational Vlasov-Poisson system

For most stellar systems, the encounters between stars are negligible and the galaxy dynamics is described by the self-consistent Vlasov-Poisson system

\[
\frac{\partial f}{\partial t} + \mathbf{v} \frac{\partial f}{\partial \mathbf{r}} + \mathbf{F} \frac{\partial f}{\partial \mathbf{v}} = 0, \tag{1}
\]

\[
\Delta \Phi = 4\pi G \int f d^3v. \tag{2}
\]

Here, \(f(r, v, t)\) denotes the distribution function (defined such that \(f d^3r d^3v\) gives the total mass of stars with position \(r\) and velocity \(v\) at time \(t\)), \(\mathbf{F}(r, t) = -\nabla \Phi\) is the gravitational force (by unit of mass) experienced by a star and \(\Phi(r, t)\) is the gravitational potential related to the star density \(\rho(r, t) = \int f d^3v\) by the Newton-Poisson equation (2). The Vlasov equation (1) simply states that, in the absence of encounters, the distribution function \(f\) is conserved by the flow in phase space. This can be written \(Df/Dt = 0\) where \(D/ Dt = \partial/\partial t + U_6 \nabla_6\) is the material derivative and \(U_6 = (v, F)\) is a generalized velocity field in the 6-dimensional phase space \((r, v)\) [by definition, \(\nabla_6 = (\partial/\partial r, \partial/\partial v)\) is the generalized nabla operator]. Since the flow is incompressible, i.e. \(\nabla_6 U_6 = 0\), the hypervolume of a “fluid” particle is conserved. Since, in addition, a fluid particle conserves the distribution function, this implies that the total mass (or hypervolume) of all phase elements with phase density between \(f\) and \(f + \delta f\) is conserved. This is equivalent to the conservation of the Casimir integrals

\[
I_h = \int h(f) d^3r d^3v, \tag{3}
\]

for any continuous function \(h(f)\) (they include in particular the total mass \(M = \int f d^3r d^3v\)). It is also straightforward to check that the Vlasov-Poisson system conserves the total energy (kinetic + potential)

\[
E = \frac{1}{2} \int v^2 d^3r d^3v + \frac{1}{2} \int f\Phi d^3rd^3v = K + W, \tag{4}
\]

the angular momentum

\[
\mathbf{L} = \int f(r \wedge v) d^3r d^3v, \tag{5}
\]

and the impulse

\[
\mathbf{P} = \int f\mathbf{v} d^3r d^3v. \tag{6}
\]

In the following, we shall work in the barycentric frame of reference and assume that the system is non rotating so that the conservation of \(\mathbf{L}\) and \(\mathbf{P}\) can be ignored (see Refs. [43,27] for a generalization).
3 Lynden-Bell’s approach of violent relaxation

When the initial condition is far from equilibrium, the Vlasov-Poisson system develops a complicated mixing process in phase space and generates intermingled filaments due to stirring effects. Therefore, a deterministic description of the flow in phase space requires a rapidly increasing amount of information as time goes on. For that reason, it is appropriate to undertake a probabilistic description in order to smooth out the small-scales (fine-grained) and concentrate on the locally averaged (coarse-grained) quantities. This statistical analysis has been considered a long time ago by Lynden-Bell [43]. The statistical mechanics of continuous systems is not as firmly established as in the usual case of N-body systems but Robert [50] has recently developed a mathematical justification of this procedure in terms of Young measures. We shall describe below the argumentation of Lynden-Bell, which is more intuitive in a first approach.

Starting from some arbitrary initial condition, the distribution function is stirred in phase space but conserves its values \( \eta \) (levels of phase density) and the corresponding hypervolumes \( \gamma(\eta)d\eta \) as a property of the Vlasov equation. Let us introduce the probability density \( \rho(r,v,\eta) \) of finding the level of phase density \( \eta \) in a small neighborhood of the position \( r,v \) in phase space. This probability density can be viewed as the local area proportion occupied by the phase level \( \eta \) and it must satisfy at each point the normalization condition

\[
\int \rho(r,v,\eta)d\eta = 1. \tag{7}
\]

The locally averaged distribution function is then expressed in terms of the probability density as

\[
\bar{f}(r,v) = \int \rho(r,v,\eta)\eta d\eta, \tag{8}
\]

and the associated (macroscopic) gravitational potential satisfies

\[
\Delta \Phi = 4\pi G \int \bar{f}d^3v. \tag{9}
\]

Since the gravitational potential is expressed by space integrals of the density, it smoothes out the fluctuations of the distribution function, supposed at very fine scale, so \( \Phi \) has negligible fluctuations. It is then possible to express the conserved quantities of the Vlasov equation as integrals of the macroscopic fields. These conserved quantities are the global probability distributions of phase density \( \gamma(\eta) \) (i.e., the total hypervolume occupied by each level \( \eta \) of phase density)

\[
\gamma(\eta) = \int \rho(r,v,\eta)d^3r d^3v, \tag{10}
\]
and the total energy

\[ E = \int \frac{1}{2} v^2 d^3 r d^3 v + \frac{1}{2} \int F d^3 r d^3 v. \]  
(11)

As discussed above the gravitational potential can be considered as smooth, so we can express the energy in terms of the coarse-grained functions \( \overline{F} \) and \( \Phi \) neglecting the internal energy of the fluctuations.

To determine the equilibrium distribution of the system, we need to introduce an entropy functional like in ordinary statistical mechanics. As is customary, Lynden-Bell defines the mixing entropy as the logarithm of the number of microscopic configurations associated with the same macroscopic state (characterized by the probability density \( \rho(\mathbf{r}, \mathbf{v}, \eta) \)). After introducing a counting “à la Boltzmann”, he arrives at the expression [43]:

\[ S = - \int \rho(\mathbf{r}, \mathbf{v}, \eta) \ln \rho(\mathbf{r}, \mathbf{v}, \eta) d\eta d^3 \mathbf{r} d^3 \mathbf{v}, \]  
(12)

where the integral extends over phase space and over all the levels of phase elements. A mathematical justification of this entropy has been given by Robert [50]. The most likely distribution to be reached at equilibrium is then obtained by maximizing the mixing entropy (12) subject to the constraints (10)(11) and the normalization condition (7). This variational problem is treated by introducing Lagrange multipliers so that the first variations satisfy

\[ \delta S - \beta \delta E - \int \alpha(\eta) \delta \gamma(\eta) d\eta - \int \zeta(\mathbf{r}, \mathbf{v}) \delta \left( \int \rho d\eta \right) d^3 \mathbf{r} d^3 \mathbf{v} = 0, \]  
(13)

where \( \beta \) is the inverse temperature and \( \alpha(\eta) \) the “chemical potential” of species \( \eta \). The resulting optimal probability density is a Gibbs state which has the form [43,27]:

\[ \rho(\mathbf{r}, \mathbf{v}, \eta) = \frac{e^{-\alpha(\eta) - \beta \eta(\frac{v^2}{2} + \Phi)}}{\int e^{-\alpha(\eta) - \beta \eta(\frac{v^2}{2} + \Phi)} d\eta}. \]  
(14)

The previous analysis gives a well defined procedure to compute the statistical equilibrium states. The gravitational field is obtained by solving the Poisson equation (2) with the distribution function (6) determined by the Gibbs state (14). The solution depends on the Lagrange multipliers \( \beta \) and \( \alpha(\eta) \) which must be related to the conserved quantities \( E \) and \( \gamma(\eta) \) by equations (13)(11). This procedure determines critical points of entropy. Whether these critical points are maxima or not is decided by the sign of the second order variations of entropy.

Following Lynden-Bell [43], we consider a particular situation that presents interesting features and for which the previous problem can be studied in detail. Keeping only two levels \( f = \eta_0 \) and \( f = 0 \) is convenient to simplify the
discussion and is probably representative of more general cases. Within this “patch” approximation, the mixing entropy reduces to

\begin{equation}
S = - \int \left\{ \frac{\bar{f}}{\eta_0} \ln \frac{\bar{f}}{\eta_0} + \left(1 - \frac{\bar{f}}{\eta_0} \right) \ln \left(1 - \frac{\bar{f}}{\eta_0} \right) \right\} d^3r d^3v, \tag{15}
\end{equation}

and the equilibrium coarse-grained distribution \( \bar{f} = \rho(r, v, \eta_0) \eta_0 \) takes explicitly the form

\begin{equation}
\bar{f} = \frac{\eta_0}{1 + \lambda e^{\beta \epsilon}}, \tag{16}
\end{equation}

where \( \epsilon = \frac{v^2}{2} + \Phi \) is the energy of a star (per unit of mass) and \( \lambda \equiv e^{\alpha(\eta_0) - \alpha(0)} > 0 \) an equivalent Lagrange multiplier. Equation (16) is, apart from a reinterpretation of the constants, the distribution function for the self-gravitating Fermi-Dirac gas. Here, the exclusion principle \( \bar{f} \leq \eta_0 \) is due to the incompressibility constraint (7) not to quantum mechanics. Because of the averaging procedure, the coarse-grained distribution function can only decrease by internal mixing, as vacuum is incorporated into the patch, and this results in an “effective” exclusion principle. Lynden-Bell’s distribution therefore corresponds to a 4-th type of statistics since the “fluid” particles are distinguishable but subject to an exclusion principle. However, formally, this distribution coincides with the Fermi-Dirac distribution. In the fully degenerate case, this equilibrium has been studied extensively in connection with white dwarf stars \( ^4 \). The structure of equilibrium may crucially depend on the degree of degeneracy (see section 4), but Lynden-Bell \( ^{13} \) gives arguments according to which stellar systems would be non degenerate \( ^2 \). In that limit, \( \bar{f} \ll \eta_0 \), and equation (16) reduces to the Maxwell-Boltzmann statistics

\begin{equation}
\bar{f} = Ae^{-\beta \epsilon}, \tag{17}
\end{equation}

This was in fact the initial goal of Lynden-Bell in 1967: his theory of “violent relaxation” was able to justify a Maxwell-Boltzmann equilibrium distribution, without recourse to collisions, on a short time scale \( \sim t_D \ll t_{\text{coll}} \) consistent with the age of ellipticals. In addition, the individual mass of the stars never appears in his theory based on the Vlasov equation. Therefore, the equilibrium state (17) does not lead to a segregation by mass in contrast with a collisional relaxation. This is in agreement with the observed light distributions in elliptical galaxies that would show greater colour differences if a marked segregation by mass was established.

### 4 Computation of Fermi-Dirac spheres

The first problem to be tackled is obviously the computation of self-gravitating Fermi-Dirac spheres. Let us first consider the non degenerate limit \( \bar{f} \ll \eta_0 \)

\(^2\) In fact, his arguments do not apply to galactic nuclei where his type of degeneracy may be important.
corresponding to a dilute system. In that case, the mixing entropy (15) reduces to the ordinary Boltzmann entropy

\[ S = -\int f \ln f \, d^3r \, d^3v, \tag{18} \]

whose maximization at fixed mass and energy leads to the Maxwell-Boltzmann statistics (17). Substituting this optimal distribution in the Poisson equation (2), we find that the gravitational potential is determined at equilibrium by the differential equation

\[ \Delta \Phi = 4\pi G A' e^{-\beta \Phi}, \tag{19} \]

where the Lagrange multipliers \( A' \) and \( \beta \) have to be related to the total mass and total energy of the system. The Boltzmann-Poisson equation (19) has been studied extensively in the context of isothermal gaseous spheres [9] and in the case of collisional stellar systems such as globular clusters [45]. We can check by direct substitution that the distribution

\[ \Phi_s(r) = \frac{1}{\beta} \ln(2\pi G \beta Ar^2), \quad \rho_s(r) = \frac{1}{2\pi G \beta r^2}, \tag{20} \]

is an exact solution of equation (19) known as the singular isothermal sphere [8]. Since \( \rho \sim r^{-2} \) at large distances, the total mass of the system \( M = \int_0^\infty \rho 4\pi r^2 \, dr \) is infinite! More generally, we can show that any solution of the Boltzmann-Poisson equation (19) behaves like the singular sphere as \( r \to +\infty \) and has therefore an infinite mass. This means that no equilibrium can exist in an unbounded domain: the density can spread indefinitely while conserving energy and increasing entropy [8].

In practice, the infinite mass problem does not arise if we realize that the relaxation of the system is necessarily incomplete [43]. There are two major reasons for incomplete relaxation: (i) The mean field relaxation process is dependent on the strength of the variations in potential. As these die out, the relaxation ceases and it is likely that the system may find a stable steady state before the relaxation process is completed. Therefore, the relaxation is effective only in a finite region of space (roughly the main body of the galaxy) and during a finite period of time (while the galaxy is dynamically unsteady). Orbits which lie partly outside the relaxing region and have periods longer than the time for which the galaxy is unsteady will not acquire their full quotas of stars. (ii) In practice, the galaxy will not be isolated but will be subject to the tides of other systems. Therefore, high energy stars will escape the system being ultimately captured by the gravity of a nearby object. These two independent effects have similar consequences and will produce a modification of the distribution function at high energies. Some truncated models can be obtained by developing a kinetic theory of encounterless relaxation (sections 5-7). This kinetic approach will provide a precise framework...
to understand what limits relaxation and why complete equilibrium is not reached in general.

However, for the present, we shall avoid the infinite mass problem by confining artificially the system within a box of radius $R$. The calculation of finite isothermal spheres has been carried out by Antonov [1], Lynden-Bell & Wood [12], Katz [32], Padmanabhan [17,18], de Vega & Sanchez [40] and Chavanis [18] (see Chavanis [21] for an extension in general relativity). These studies were performed in the framework of gaseous stars or collisional stellar systems (e.g., globular clusters) but they extend in principle to violently relaxed collisionless stellar systems described by the distribution function (17). Therefore, we shall give a brief summary of these classical results before considering the case of the Fermi-Dirac distribution (16). The phase diagram is represented in Fig. 1 where the inverse temperature is plotted as a function of minus the energy. It is possible to prove the following results: (i) there is no global maximum for the Boltzmann entropy. (ii) there are not even critical points for the Boltzmann entropy if $\Lambda = -ER/GM^2 > 0.335$. (iii) local entropy maxima (LEM) exist if $\Lambda = -ER/GM^2 < 0.335$; they have a density contrast $\mathcal{R} = \rho(0)/\rho(R) < 709$ (upper branch). (iv) critical points of entropy with density contrast $\mathcal{R} > 709$ are unstable saddle points SP (spiraling curve). Conclusions (i) and (ii) have been called “gravothermal catastrophe” or “Antonov instability”. When $\Lambda = -ER/GM^2 > 0.335$, there is no hydrostatic equilibrium and the system is expected to collapse and overheat. This is a natural evolution (in a thermodynamical sense) because a self-gravitating system can always increase entropy by taking a “core-halo” structure and by making its core denser and denser (and hotter and hotter) [8]. As discussed by Lynden-Bell & Wood [45], this instability is probably related to the negative specific heats of self-gravitating systems: by losing heat, the core grows hotter and evolves away from equilibrium.

The “gravothermal catastrophe” picture has been confirmed by sophisticated numerical simulations of globular clusters that introduce a precise description of heat transfers between the “core” and the “halo” using moment equations [11], orbit averaged Fokker-Planck equation [29] or fluid equations for a thermally conducting gas [14]. In these studies, the collapse proceeds self-similarly (with power law behaviours) and the central density becomes infinite in a finite time. This singularity has been known as “core collapse” and many globular clusters have probably experienced core collapse. This is certainly the most exciting theoretical aspect of the collisional evolution of stellar systems. In practice, the formation of hard binaries can release sufficient energy to stop the collapse and even drive a reexpansion of the system [32]. Then, a series of gravothermal oscillations should follow but it takes times much larger than the age of globular clusters [8]. It has to be noted that, although the central density tends to infinity, the mass contained in the core tends to zero so, in this sense, the gravothermal catastrophe is a rather unspectacular process [8]. However, the gravothermal catastrophe may also
be at work in dense clusters of compact stars (neutron stars or stellar mass black holes) embedded in evolved galactic nuclei and it can presumably initiate the formation of supermassive black holes via the collapse of such clusters. Indeed, when the central redshift (related to the density contrast) exceeds a critical value $z_c \sim 0.5$, a dynamical instability of general-relativistic origin sets in and the star cluster undergoes a catastrophic collapse to a black hole on a dynamical time [49]. It has been suggested that this process leads naturally to the birth of supermassive black holes of the right size to explain quasars and AGNs [56].

Since collisionless stellar systems undergoing violent relaxation are described by an isothermal distribution function of the Fermi-Dirac type [16], it is particularly relevant to ask whether galaxies in their birth stages can undergo a form of gravothermal catastrophe and if so what they do about it. The collapse of galaxies was described qualitatively by Lynden-Bell & Wood [45]: “...The centers will then begin to separate into a core - a sort of separate body which might even be called a nucleus. This will cease to shrink when it becomes degenerate in Lynden-Bell’s sense. The system will then have a core-halo structure which is an equilibrium of an isothermal Fermi-Dirac gas sphere. These will show a variety of different nuclear concentrations depending on the degeneracy parameter. It is evident that theory developed along
these lines has the chance of making sense of the variety of different galactic nuclei."

The computation of isothermal Fermi-Dirac spheres has been performed only recently by Chavanis & Sommeria [25]. It can be proved that a global entropy maximum exists for all accessible values of energy [25,51]. Therefore, degeneracy has a stabilizing role and is able to stop the "gravothermal catastrophe". The equilibrium diagram is represented in Figs. 2-3 and now depends on the degeneracy parameter $\mu = \eta_0/\langle f \rangle \equiv \eta_0 \sqrt{512 \pi^3 G^4 M R^5}$, where $\eta_0$ is the maximum allowable value of the distribution function and $\langle f \rangle$ its typical average value in the box of radius $R$. We see that the inclusion of degeneracy has the effect of unwinding the spiral (dashed line). When $\mu$ is small (Fig. 2), there is only one critical point of entropy for each value of energy and it is a global entropy maximum (GEM). For small values of $\Lambda$ (high energies) the solutions almost coincide with the classical, non degenerate, isothermal spheres. When $\Lambda$ is increased (low energies) the solutions take a "core-halo" structure with a partially degenerate core surrounded by a dilute Maxwellian atmosphere. It is now possible to overcome the critical energy $\Lambda_c = 0.335$ and the critical density contrast $R \approx 709$ found by Antonov for a classical isothermal gas. In that region, the specific heat $C = dE/dT$ is negative. As
energy decreases further, more and more mass is concentrated into the nucleus (which becomes more and more degenerate) until a minimum accessible energy, corresponding to $\Lambda_{\text{max}}(\mu)$, at which the nucleus contains all the mass. In that case, the atmosphere has been “swallowed” and the system has the same structure as a cold white dwarf star [9]. This is a relatively singular limit since the density drops to zero at a finite radius whereas for partially degenerate systems, the density decays like $r^{-2}$ at large distances. When the degeneracy parameter $\mu$ is large (Fig. 3), there are now several critical points of entropy for each single value of energy in the range $\Lambda_*(\mu) < \Lambda < \Lambda_c$. The solutions on the upper branch (points A) are non degenerate and have a smooth density profile; they form the “gaseous” phase. The solutions on the lower branch (points C) have a “core-halo” structure with a degenerate nucleus and a dilute atmosphere; they form the “condensed” phase. According to the theorem of Katz [39], they are both entropy maxima while the intermediate solutions, points B, are unstable saddle points (SP). These points are similar to points A, except that they contain a small embryonic nucleus (with small mass and energy) which plays the role of a “germ” in the language of phase transitions. For small values of $\Lambda$, the non degenerate solutions (upper branch) are global entropy maxima (GEM). They become local entropy maxima (metastable states) at a certain $\Lambda_t \geq \Lambda_*$ depending on the degeneracy parameter. At that $\Lambda_t$, the degenerate solutions (lower branch) that were only local entropy
maxima (LEM) suddenly become global entropy maxima. We expect therefore a phase transition to occur between the “gaseous” phase (upper branch) and the “condensed” phase (lower branch) at \( \Lambda_t \). For \( \Lambda_t(\mu) < \Lambda < \Lambda_c \), the non degenerate solutions (points A) are metastable but we may suspect that they are long-lived so that they are physical. It is plausible that these metastable states will be selected by the dynamics even if states with higher entropy exist. This depends on a complicated notion of “basin of attraction” as studied by Chavanis, Rosier and Sire [22] with the aid of a simple model of gravitational dynamics. Therefore, the true phase transition will occur at \( \Lambda_c = 0.335 \) at which the gaseous phase completely disappears: at that point, the gravothermal catastrophe sets in but, for collisionless stellar systems (or for fermions), the core ultimately ceases to shrink when it becomes degenerate. In that case, the system falls on to a global entropy maximum which is the true equilibrium state for these systems (point D). This global entropy maximum has a “core-halo” structure with a degenerate core and a dilute atmosphere. A simple analytical model of these phase transitions has been proposed in Ref. [20] and provides a fairly good agreement with the full numerical solution. A particularity of self-gravitating systems, which are in essence non-extensive, is that the statistical ensembles (microcanonical and canonical) are not interchangeable. Therefore, the description of the equilibrium diagram is different whether the system evolves at fixed energy of fixed temperature. A discussion of this interesting phenomenon can be found in the review of Padmanabhan [48] and in Chavanis [20].

For astrophysical purposes, it is still a matter of debate to decide whether collisionless stellar systems like elliptical galaxies are degenerate (in the sense of Lynden-Bell) or not. Since degeneracy can stabilize the system without changing its overall structure at large distances, we have suggested that degeneracy could play a role in galactic nuclei [25]. The recent simulations of Leeuwin and Athanassoula [12] and the theoretical model of Stiavelli [57] are consistent with this idea especially if the nucleus of elliptical galaxies contains a primordial massive black hole. Indeed, the effect of degeneracy on the distribution of stars surrounding the black hole can explain the cusps observed at the center of galaxies [57]. This form of degeneracy is also relevant for massive neutrinos in Dark Matter models where it competes with quantum degeneracy [40]. In fact, the Fermi-Dirac distribution of massive neutrinos in Dark Matter models (which form a collisionless self-gravitating system) might be justified more by the process of “violent relaxation” [43] than by quantum mechanics [44]. As shown by Chavanis & Sommeria [25], violent relaxation can lead to the formation of a dense degenerate nucleus with a small radius and a large mass. This massive degenerate nucleus could be an alternative to black holes at the center of galaxies [4, 5, 25].
5 The Maximum Entropy Production Principle

Basically, a collisionless stellar system is described in a self-consistent mean field approximation by the Vlasov-Poisson system \( \text{(1)}(\text{2}) \). In principle, these coupled equations determine completely the evolution of the distribution function \( f(r,v,t) \). However, as discussed in section \( \text{3} \), we are not interested in practice by the finely striated structure of the flow in phase space but only by its macroscopic, i.e. smoothed-out, structure. Indeed, the observations and the numerical simulations are always realized with a finite resolution. Moreover, the “coarse-grained” distribution function \( f \) is likely to converge towards an equilibrium state contrary to the exact distribution \( f \) which develops smaller and smaller scales.

If we decompose the distribution function and the gravitational potential in a mean and fluctuating part \( (f = \overline{f} + \tilde{f}, \Phi = \overline{\Phi} + \tilde{\Phi}) \) and take the local average of the Vlasov equation \( \text{(1)} \), we readily obtain an equation of the form

\[
\frac{\partial \overline{f}}{\partial t} + v \frac{\partial \overline{f}}{\partial r} + \nabla \cdot \overline{J} = -\frac{\partial J_f}{\partial v},
\]

for the “coarse-grained” distribution function with a diffusion current \( J_f = \overline{f \tilde{F}} \) related to the correlations of the “fine-grained” fluctuations.

The problem in hand consists in determining a closed form for the diffusion current \( J_f \). Its detailed expression \( \overline{f \tilde{F}} \) depends on the subdynamics and is therefore extremely difficult to capture from first principles. Indeed, the “violent relaxation” is a very nonlinear and very chaotic process ruling out any attempt to implement perturbative methods if we are far from equilibrium. For that reason, there is a strong incentive to explore variational methods which are considerably simpler and give a more intuitive understanding of the problem. We propose to describe the relaxation of the coarse-grained distribution function \( \overline{f} \) towards the Gibbs state \( \text{(14)} \) with a Maximum Entropy Production Principle \( \text{(27)} \). This thermodynamical principle assumes that: “during its evolution, the system tends to maximize its rate of entropy production \( \dot{S} \) while satisfying all the constraints imposed by the dynamics”. There is no rigorous justification for this principle and it is important therefore to confront the MEPP with kinetic theories (when they are available) in order to determine its range of validity. In any case, the MEPP can be considered as a convenient tool to build relaxation equations which are mathematically well-behaved and which can serve as numerical algorithms to calculate maximum entropy states. It is also “perfectible” in the sense that any new information about the dynamics of the system can be incorporated. This principle is reminiscent of Jaynes \( \text{(36)} \) ideas and is a clear extension of the well-known principle of statistical mechanics according to which: “at equilibrium, the system is in a maximum entropy state consistent with all the constraints”. This is also the most natural extension of Lynden-Bell’s theory out of equilibrium.
Like for equilibrium states, the evolution of the flow in phase space is described by a set of local probabilities \( \rho(r, v, \eta, t) \) and the locally averaged gravitational potential \( \Phi(r, t) \) is obtained by the integration of equation (2) where \( f(r, v, t) \) is replaced by \( \int f(r, v, t \, \eta \, d\eta \). The phase elements are thus transported in phase space by the corresponding averaged velocity field \( \mathbf{U}_6 = (v, -\nabla \Phi) \) and we suppose that, in addition, they undergo a diffusion process. This diffusion occurs only in velocity space, due to the fluctuations of the gravitational field \( \Phi \). There is no diffusion in position space since the velocity \( v \) is a pure coordinate and does not fluctuate. As a result, the probability densities will satisfy a convection-diffusion equation of the form

\[
\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial r} + \frac{\partial \rho}{\partial \mathbf{\nabla} \Phi} = -\frac{\partial J}{\partial v},
\]

where \( J(r, v, \eta, t) \) is the diffusion current of the phase element \( \eta \). This equation conserves the total hypervolume (in phase space) occupied by each phase element. Multiplying equation (22) by \( \eta \) and integrating over the levels of phase density returns equation (21) with \( J_f = \int J(r, v, \eta, t \, \eta \, d\eta. \)

To apply the MEPP, we need first to compute the rate of change of entropy during the convection-diffusion process. Taking the time derivative of equation (12), expressing \( \frac{\partial \rho}{\partial t} \) by equation (22) and noting that \( \rho \ln \rho \) is conserved by the advective term, we get

\[
\dot{S} = -\int J \frac{\partial \ln \rho}{\partial v} d^3r d^3v d\eta.
\]

We now determine \( J \) such that, for a given field \( \rho \) at each time \( t \), \( J \) maximizes the entropy production \( \dot{S} \) with appropriate dynamical constraints, which are:

- the conservation of the local normalization condition (\[ \int J(r, v, \eta, t) d\eta = 0, \]

- the conservation of energy expressed from equations (11) and (21) as

\[
\dot{E} = \int J_f v d^3r d^3v = 0,
\]

- a limitation on the eddy flux \( |J| \), characterized by a bound \( C(r, v, t) \), which exists but is not specified

\[
\int \frac{J^2}{2\rho} d\eta \leq C(r, v, t).
\]

This variational problem can be solved by introducing (at each time \( t \)) Lagrange multipliers \( \zeta, \beta, 1/D \) for the three respective constraints. It can be shown by a convexity argument that reaching the bound (26) is always favorable for increasing \( \dot{S} \), so that this constraint can be replaced by an equality.
The condition
\[ \delta \dot{S} - \beta \delta E - \int \frac{1}{D} \delta \left( \int \frac{J^2}{2\rho} d\eta \right) d^3r d^3v - \int \zeta \delta \left( \int J d\eta \right) d^3r d^3v = 0, \quad (27) \]
yields an optimal current of the form \[\text{(27)}\]:
\[ J = -D(\mathbf{r}, \mathbf{v}, t) \left[ \frac{\partial \rho}{\partial \mathbf{v}} + \beta (\eta - \bar{f}) \rho \mathbf{v} \right]. \quad (28) \]

The Lagrange multiplier \( \zeta \) has been eliminated, using the condition (24) of local normalization conservation. At equilibrium, the diffusion currents must vanish and we can check that this yields the Gibbs state (14) with \( \beta \) the corresponding inverse temperature \[\text{(27)}\]. During the evolution, this quantity varies with time and is determined by the condition of energy conservation (25). Introducing (28) in the constraint (25), we get
\[ \beta(t) = -\frac{\int D \frac{\partial f}{\partial \mathbf{v}} \mathbf{v} d^3r d^3\mathbf{v}}{\int D(\mathbf{f}^2 - \bar{f}^2) \mathbf{v}^2 d^3r d^3\mathbf{v}}. \quad (29) \]

where \( \mathbf{f}^n = \int \rho \eta^n d\eta \) are the local moments of the fine-grained distribution function. We have thus obtained a complete set of relaxation equations which exactly conserve the distribution of phase levels and energy. The rate of entropy production can be put in the form \[\text{(27)}\]:
\[ \dot{S} = \int \frac{J^2}{D\rho} d^3r d^3\mathbf{v} d\eta, \quad (30) \]
so the diffusion coefficient \( D \) must be positive for entropy increase. Except for its sign, the diffusion coefficient is not determined by the thermodynamical approach as it is related to the unknown bound \( C(\mathbf{r}, \mathbf{v}, t) \).

These relaxation equations (22)(28)(24) can be simplified in the case of a single non zero density level \( \eta_0 \) and provide a new non linear equation for the evolution of the coarse-grained distribution function \( \bar{f} = \rho \eta_0 \):\[\frac{\partial \bar{f}}{\partial t} + \mathbf{v} \frac{\partial \bar{f}}{\partial \mathbf{r}} + \mathbf{F} \frac{\partial \bar{f}}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}} \left[ D \left( \frac{\partial \bar{f}}{\partial \mathbf{v}} + \beta \eta_0 \bar{f} \right) \right]. \quad (31) \]

In the non degenerate limit \( \bar{f} \ll \eta_0 \), equation (31) takes the form of a Fokker-Planck equation
\[ \frac{\partial \bar{f}}{\partial t} + \mathbf{v} \frac{\partial \bar{f}}{\partial \mathbf{r}} + \mathbf{F} \frac{\partial \bar{f}}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}} \left[ D \left( \frac{\partial \bar{f}}{\partial \mathbf{v}} + \beta \eta_0 \bar{f} \right) \right]. \quad (32) \]

This Fokker-Planck equation (sometimes called the Kramers-Chandrasekhar equation) is well-known in the case of collisional stellar systems (without the bar on \( f \)!) and is usually derived from a Markov hypothesis and a stochastic
Langevin equation \[10\]. It can also be obtained from the following argument: because of close encounters, the stars undergo brownian motion and diffuse in velocity space (responsible for the first term in the right hand side). However, under the influence of this diffusion, the kinetic energy per star will diverge as \( <v^2/2 > \sim 3Dt \) and one is forced to introduce an “ad hoc” dynamical friction (second term in the right hand side) with a friction coefficient \( \xi = \beta D \eta_0 \) (Einstein relation) in order to compensate this divergence and recover the Maxwellian distribution of velocities at equilibrium. From these two considerations (which express the fluctuation-dissipation theorem) results the ordinary Fokker-Planck equation \(32\). The Fokker-Planck equation is justified here by an argument of a very wide scope that does not directly refer to the subdynamics: it appears to maximize the rate of entropy production at each time with appropriate constraints. Therefore, it applies equally well to collisional or (coarse-grained) collisionless stellar systems. In this description, the diffusion term and the “dynamical friction” directly result from the variational principle and are associated with the variations of \( S \) and \( E \) respectively. Furthermore, the inverse temperature \( \beta \) appears as a Lagrange multiplier associated with the conservation of energy and the Einstein relation is automatically satisfied. In addition, our procedure can take into account the degeneracy of collisionless stellar systems, keeping equation \(31\) instead of the non degenerate limit \(32\). In the degenerate case, the friction is non linear in \( f \) so that equation \(31\) is not a Fokker-Planck equation in a strict sense. This nonlinearity is necessary to recover the Fermi-Dirac distribution function at equilibrium (while a linear friction drives the system towards the Maxwell-Boltzmann distribution). This generalization is important because degeneracy is specific to collisionless systems and may be crucial for the existence of an equilibrium state (see section 4).

In equations \(31\) \(32\), \( \beta \) is not constant but evolves with time so as to conserve the total energy. In the non degenerate limit, and assuming that \( D \) is constant, we get after a part integration

\[
\beta(t) = -\frac{\int \frac{\partial T}{\partial v} v d^3r d^3v}{\int \eta_0 f v^2 d^3r d^3v} = \frac{3M}{2 \eta_0 K(t)} \tag{33}
\]

This equation relates the formal Lagrange multiplier \( \beta(t) \) to the inverse of the average kinetic energy. This is of course satisfying on a physical point of view. An alternative Fokker-Planck equation involving a local temperature \( T(r, t) \) instead of \( \beta(t)^{-1} \) has been proposed by Clemmow & Dougherty \(28\) in the case of collisional systems. The energy is assumed to be locally conserved by the collisions, which is valid when the mean free path is much smaller than the size of the system. By contrast, this hypothesis does not seem to be justified for the violent relaxation of a collisionless system, which is rather a global process.

Let us briefly review the main properties of our relaxation equations. First of all, they rigorously satisfy the conservation of energy and phase space hy-
pervolumes like the Vlasov-Poisson system (the conservation of impulse and angular momentum can also be satisfied by introducing appropriate Lagrange multipliers \[27\]). Moreover, they guarantee the increase of entropy at each time \(\dot{S} \geq 0\) with an optimal rate. Of course, this H-theorem is true for the coarse-grained entropy \(S = -\int \rho \ln \rho d^3r d^3v d\eta\) and not for the fine-grained entropy \(S_{f,g} = -\int f \ln f d^3r d^3v\) which is constant, as the integral of any function of \(f\). The source of irreversibility is due to the coarse-graining procedure that smoothes out the small-scales and erases the microscopic details of the evolution. Accordingly, the relaxation equations \((22)(28)(29)\) are likely to drive the system towards an equilibrium state (the Gibbs state \((14)\) contrary to the Vlasov equation that develops finer and finer scales. In mathematical terms, this means that the distribution function \(f\) converges to an equilibrium state \(f\) in a weak sense. In fact, the situation is more complicated since the Gibbs state does not exist in an infinite domain (section \(4\)). We shall see, however, in section \(6\) that the diffusion coefficient is proportional to the fluctuations of the distribution function (equation \((52)\)). Now, in physical situations, these fluctuations vanish before the systems had time to relax completely. As a result, the relaxation stops and the system remains frozen in a subdomain of phase space. It is only in this subdomain (corresponding to the main body of the galaxy) that the Gibbs state is justified. This provides a physical mechanism for confining galaxies and justifying truncated models. Alternatively, if the galaxy is not isolated but subject to the tides of a neighboring object, a tidally truncated model can be explicitly derived from these relaxation equations (section \(7\)). It has a finite mass while preserving the essential features of Lynden-Bell’s distribution function.

The relaxation equations \((22)(28)(29)\) can be simplified in two particular limits when: (i) the initial condition is approximated by a single level of phase density \(\eta_0\) surrounded by vacuum (ii) degeneracy, in Lynden-Bell’s sense, is neglected. These simplifications lead to the Fokker-Planck equation \((32)\). If the galaxy has sufficiently large energy, this equation will drive the system towards a Maxwell-Boltzmann equilibrium state \((17)\). However, if the energy falls below a critical value, the Fokker-Planck equation does not reach any equilibrium state anymore and the system can achieve ever increasing values of entropy by developing core collapse (Antonov instability). For collisionless stellar systems, this “gravothermal catastrophe” should stop when the center of the galaxy becomes degenerate (see section \(4\)). In that case, the Fokker-Planck equation \((32)\) is not valid anymore and must be replaced by the degenerate relaxation equation \((51)\). This equation converges towards the Fermi-Dirac equilibrium state \((16)\), which exists for all values of energy.

In summary, the MEPP is able to yield relevant relaxation equations for the coarse-grained dynamics of collisionless stellar systems experiencing violent relaxation. This relatively elegant and simple variational principle shows that the global structure of the relaxation equations is determined by purely thermodynamical arguments. All explicit reference to the subdynamics
is encapsulated in the diffusion coefficient which cannot be captured by the MEPP (it appears as a Lagrange multiplier related to an unknown bound on the diffusion flux). It must be therefore calculated with a kinetic model such as the quasilinear theory described in the next section.

6 The quasilinear theory

The quasilinear theory of the Vlasov-Poisson system was first considered by Kadomtsev & Pogutse [37] for a homogeneous Coulombian plasma and extended by Severne & Luwel [55] for an inhomogeneous gravitational system. This theory was further discussed by Chavanis [13] in an attempt to make a link with the Maximum Entropy Production Principle. We shall give here a simple account of the quasilinear theory. Further details can be found in Refs. [37,55,13]. Our objective is to obtain an expression for the diffusion current $J$ by working directly on the Vlasov-Poisson system, i.e. without assuming that the entropy increases as is done in the thermodynamical approach.

Since the diffusion current $J_f = \tilde{f}F$ is related to the fine-grained fluctuations of the distribution function, any systematic calculation starting from the Vlasov equation (1) must necessarily introduce an evolution equation for $\tilde{f}$. This equation is simply obtained by subtracting equation (21) from equation (1). This yields

$$\frac{\partial \tilde{f}}{\partial t} + v \frac{\partial \tilde{f}}{\partial r} + F \frac{\partial \tilde{f}}{\partial v} = -\tilde{F} \frac{\partial f}{\partial v} - F \frac{\partial \tilde{f}}{\partial v} + F \frac{\partial \tilde{f}}{\partial v}.$$  (34)

The essence of the quasilinear theory is to assume that the fluctuations are weak and neglect the nonlinear terms in equation (34) altogether. In that case, equations (21) and (34) reduce to the coupled system

$$\frac{\partial \tilde{f}}{\partial t} + L\tilde{f} = -\frac{\partial \tilde{f}}{\partial v} \tilde{F},$$  (35)

$$\frac{\partial \tilde{f}}{\partial t} + L\tilde{f} = -\tilde{F} \frac{\partial \tilde{f}}{\partial v},$$  (36)

where $L = v \frac{\partial}{\partial r} + F \frac{\partial}{\partial v}$ is the advection operator in phase space. Physically, these equations describe the coupling between a subdynamics (here the small scale fluctuations $\tilde{f}$) and a macrodynamics (described by the coarse-grained distribution function $\tilde{F}$).

Introducing the Greenian

$$G(t_2, t_1) \equiv \exp\left\{-\int_{t_1}^{t_2} dt L(t)\right\},$$  (37)

we can immediately write down a formal solution of equation (36), namely

$$\tilde{f}(r, v, t) = G(t, 0) \tilde{f}(r, v, 0) - \int_0^t ds G(t, t-s) \tilde{F}(r, t-s) \frac{\partial \tilde{f}}{\partial v}(r, v, t-s).$$  (38)
Although very compact, this formal expression is in fact extremely complicated. Indeed, all the difficulty is encapsulated in the Greenian $G(t, t - s)$ which supposes that we can solve the smoothed-out Lagrangian flow

$$\frac{dx}{dt} = v, \quad \frac{dv}{dt} = F,$$

between $t$ and $t - s$. In practice, this is impossible and we will have to make some approximations.

The objective now is to substitute the formal result (38) back into equation (35) and make a closure approximation in order to obtain a self-consistent equation for $f(r, v, t)$. If the fluctuating force $\tilde{F}$ were external to the system, we would simply obtain a diffusion equation

$$\frac{\partial \tilde{f}}{\partial t} + L\tilde{f} = \frac{\partial}{\partial v^\mu} \left( D^{\mu\nu} \frac{\partial \tilde{f}}{\partial v^\nu} \right),$$

with a diffusion coefficient given by a Kubo formula

$$D^{\mu\nu} = \int_0^t ds \tilde{F}^\mu(r, t) \tilde{F}^\nu(r, t - s).$$

However, in the case of the Vlasov-Poisson system, the gravitational force is in fact produced by the distribution of matter itself and this coupling will give rise to a friction term in addition to the pure diffusion. Indeed, we have

$$\tilde{F}(r, t) = \int F(r' \to r) \hat{f}(r', v', t) d^3r' d^3v',$$

where

$$F(r' \to r) = G \frac{r' - r}{|r' - r|^3},$$

represents the force created by a (field) star in $r'$ on a (test) star in $r$ (Newton’s law). Therefore, considering equations (38) and (42), we see that the fluctuations of the distribution function $\tilde{f}(r, v, t)$ are given by an iterative process: $\tilde{f}(t)$ depends on $\tilde{F}(t - s)$ which itself depends on $\tilde{f}(t - s)$ etc... We shall solve this problem perturbatively in an expansion in powers of the gravitational constant $G$. This is the equivalent of the “weak coupling approximation” in plasma physics. To order $G^2$, we obtain after some rearrangements

$$\frac{\partial \tilde{f}}{\partial t} + L\tilde{f} = \frac{\partial}{\partial v^\mu} \int_0^t ds \int d^3r' d^3v' d^3r'' d^3v'' F^{\mu}(r' \to r) G'(t, t - s) G(t, t - s) \left\{ F^{\nu}(r'' \to r) \hat{f}(r', v', t - s) \hat{f}(r'', v'', t - s) \frac{\partial \tilde{f}}{\partial v^\nu}(r, v, t - s) + F^{\nu}(r'' \to r') \hat{f}(r, v, t - s) \hat{f}(r'', v', t - s) \frac{\partial \tilde{f}}{\partial v^\nu}(r', v', t - s) \right\}.$$ 

(44)
In this expression, the Greenian $G$ refers to the fluid particle $\mathbf{r}(t)$, $\mathbf{v}(t)$ and the Greenian $G'$ to the fluid particle $\mathbf{r}'(t)$, $\mathbf{v}'(t)$. To close the system, it remains for one to evaluate the correlation function $\overline{f(\mathbf{r}, \mathbf{v}, t)f(\mathbf{r}', \mathbf{v}', t)}$. We shall assume that the mixing in phase space is sufficiently efficient so that the scale of the kinematic correlations is small with respect to the coarse-graining mesh size. In that case,

$$\overline{f(\mathbf{r}, \mathbf{v}, t)f(\mathbf{r}', \mathbf{v}', t)} = \epsilon_r^3 \epsilon_v^3 \delta(\mathbf{r} - \mathbf{r}') \delta(\mathbf{v} - \mathbf{v}') \overline{f^2(\mathbf{r}, \mathbf{v}, t)}, \quad (45)$$

where $\epsilon_r$ and $\epsilon_v$ are the resolution scales in position and velocity respectively. Now,

$$\overline{f^2} = (f - \overline{f})^2 = \overline{f^2} - \overline{f}^2. \quad (46)$$

We shall assume for simplicity that the initial condition in phase space consists of a patch of uniform distribution function ($f = \eta_0$) surrounded by vacuum ($f = 0$). This is the two-levels approximation already considered in sections 8 and 9. In that case $\overline{f^2} = \eta_0 \times f = \eta_0 \overline{f}$ and, therefore,

$$\overline{f(\mathbf{r}, \mathbf{v}, t)f(\mathbf{r}', \mathbf{v}', t)} = \epsilon_r^3 \epsilon_v^3 \delta(\mathbf{r} - \mathbf{r}') \delta(\mathbf{v} - \mathbf{v}') \overline{f(\eta_0 - \overline{f})}. \quad (47)$$

Substituting this expression in equation (44) and carrying out the integrations on $\mathbf{r}''$ and $\mathbf{v}''$, we obtain

$$\frac{\partial \overline{f}}{\partial t} + L \overline{f} = \epsilon_r^3 \epsilon_v^3 \frac{\partial}{\partial \nu^\nu} \int_0^t ds \int d^3 \mathbf{r}' d^3 \mathbf{v}' F^\nu(\mathbf{r}' \rightarrow \mathbf{r})_t F'^\nu(\mathbf{r}' \rightarrow \mathbf{r})_{t-s} \times \left\{ \overline{f(\eta_0 - \overline{f})} \frac{\partial \overline{f}}{\partial \nu^\nu} - \overline{f(\eta_0 - \overline{f})} \frac{\partial \overline{f}}{\partial \nu^\nu} \right\}_{t-s}. \quad (48)$$

We have written $\overline{f}_{t-s} \equiv \overline{f(\mathbf{r}'(t-s), \mathbf{v}'(t-s), t-s)}$, $\overline{f}_{t-s} \equiv \overline{f(\mathbf{r}(t-s), \mathbf{v}(t-s), t-s)}$, $F^\nu(\mathbf{r}' \rightarrow \mathbf{r})_t \equiv F^\nu(\mathbf{r}'(t) \rightarrow \mathbf{r}(t))$ and $F'^\nu(\mathbf{r}' \rightarrow \mathbf{r})_{t-s} \equiv F'^\nu(\mathbf{r}'(t-s) \rightarrow \mathbf{r}(t-s))$ where $\mathbf{r}(t-s)$ and $\mathbf{v}(t-s)$ are the position and velocity at time $t-s$ of the stellar fluid particle located in $\mathbf{r} = \mathbf{r}(t)$, $\mathbf{v} = \mathbf{v}(t)$ at time $t$. It is determined by the characteristics (99) of the smoothed-out Lagrangian flow.

Equation (48) is a non Markovian integrodifferential equation: the value of $\overline{f}$ in $\mathbf{r}$, $\mathbf{v}$ at time $t$ depends on the value of the whole field $\overline{f(\mathbf{r}', \mathbf{v}', t-s)}$ at earlier times. If the decorrelation time $\tau$ is short, we can make a Markov approximation and replace the bracket at time $t-s$ by its value taken at time $t$. Noting furthermore that the integral is dominated by the contribution of field stars close to the star under consideration (i.e. when $\mathbf{r}' \rightarrow \mathbf{r}$), we shall make a local approximation and replace $\overline{f(\eta_0 - \overline{f})}$ and $\frac{\partial \overline{f}}{\partial \nu^\nu}$ by their values taken at $\mathbf{r}$. In that case, the foregoing equation simplifies in

$$\frac{\partial \overline{f}}{\partial t} + L \overline{f} = \epsilon_r^3 \epsilon_v^3 \frac{\partial}{\partial \nu^\nu} \int_0^t ds \int d^3 \mathbf{r}' d^3 \mathbf{v}' F^\nu(\mathbf{r}' \rightarrow \mathbf{r})_t F'^\nu(\mathbf{r}' \rightarrow \mathbf{r})_{t-s} \times \left\{ \overline{f(\eta_0 - \overline{f})} \frac{\partial \overline{f}}{\partial \nu^\nu} - \overline{f(\eta_0 - \overline{f})} \frac{\partial \overline{f}}{\partial \nu^\nu} \right\}_{t-s} \quad (49)$$
where, now, $\bar{f}' = \bar{f}(r, v', t)$. The explicit reference to the past evolution of the system is only retained in the memory function

$$\int_0^t ds \int d^2 r' F^\mu(r' \to r) f'(r' \to r)_{s,t}.$$  

This function can be calculated explicitly if we assume that, between $t - s$ and $t$, the stars follow linear trajectories, so that $v(t - s) = v$ and $r(t - s) = r - vs \,[2]$. This leads to the generalized Landau equation

$$\frac{\partial \bar{f}}{\partial t} + L \bar{f} = \frac{\partial}{\partial v^\mu} \int d^3 v' K_{\mu
u} \left\{ \bar{f} (\eta_0 - \bar{f}) \frac{\partial \bar{f}}{\partial v^\nu} - \bar{f} (\eta_0 - \bar{f}) \frac{\partial \bar{f}}{\partial v'^\nu} \right\}, \quad (50)$$

where $K_{\mu\nu}$ is the tensor

$$K_{\mu\nu} = 2\pi G^2 \epsilon^2 \epsilon^3 \ln \Lambda \left( \frac{\delta_{\mu\nu}}{u} - \frac{u^\mu u^\nu}{u^2} \right), \quad (51)$$

and $u = v' - v$, $\ln \Lambda = \ln(R/\epsilon_r)$. This equation applies to inhomogeneous systems but, as a result of the local approximation, the effect of inhomogeneity is only retained in the advective term. Equation (50) is very similar to the well-known Landau equation of collisional self-gravitating systems and electric charges \,[38]. There are nevertheless two important differences: (i) The friction term is non linear in $\bar{f}$, accounting for the degeneracy discovered by Lynden-Bell at equilibrium. (ii) The diffusion coefficient is proportional to the mass $\eta_0 \epsilon^3 \epsilon^3 r \ln \Lambda$ of a macrocell completely filled by the phase fluid instead of the mass $m$ of an individual star in the ordinary Landau equation. In general $\eta_0 \epsilon^3 \epsilon^3 \gg m$ so that the relaxation by phase mixing is much more rapid than the collisional relaxation. From the above theory, we find that the collisionless relaxation operates on a time scale $\sim t_D$, the dynamical time, whereas the collisional relaxation operates on a time scale $t_{coll} \sim N \ln N t_D \gg t_D$ where $N \sim 10^{12}$ is the number of stars in the galaxy. Therefore, the relaxation by phase mixing really corresponds to a “violent relaxation” \,[43].

It can be shown that the generalized Landau equation (50) conserves mass and energy \,[2]. In fact, as a result of the local approximation, the energy is conserved locally, as if the system were homogeneous, and we have

$$\int \left( \frac{\partial \bar{f}}{\partial t} \right)_{c.g.} \frac{v^2}{2} d^3 v = \int J_f v d^3 v = 0, \quad \forall r. \quad (52)$$

It is also easy to show that equation (52) satisfies a H-theorem ($\dot{S} \geq 0$) for the Fermi-Dirac entropy (15). When a stationary state is reached $\dot{S} = 0$ and the Fermi-Dirac distribution (16) is obtained, in agreement with Lynden-Bell’s statistical theory \,[13]. This provides therefore another way of justifying his results from a dynamical point of view which does not explicitly rely on a maximization of entropy (the $H$-theorem is not assumed but derived from
the kinetic theory). It can be noted that these properties result from the symmetry of the Landau collision term and not from Lagrange multipliers like in the thermodynamical approach. This is more satisfactory on a physical point of view.

It is important to stress, however, that this quasilinear theory cannot describe the early, very nonlinear and very chaotic, stages of the “violent relaxation”. Indeed, the detailed study of Severne & Luwel [55] reveals that the various approximations introduced in the theory make equation (50) applicable only for $t \gg t_D$, where $t_D$ is the dynamical time. Since the relaxation time is precisely of order $t_D$, the quasilinear theory is only marginally applicable and can describe, at most, the late quiescent phases of the relaxation, when the fluctuations have weaken. By contrast, it is plausible that equation (51) is more general and more appropriate to the context of “violent relaxation” since it results from a thermodynamical approach which exploits at best the chaoticity of the system and the complete lack of information that we have to face at small scales. As a clear difference, it should be noted that the MEPP takes into account only the global conservation of energy (the Lagrange multiplier $\beta(t)$ is a kind of inverse average temperature determined by an integration over the whole system) whereas the approximations introduced in the kinetic model lead to a local conservation of energy. This strong locality cannot account for the rather collective processes which are involved in the violent relaxation and may unveil a failure of the quasilinear theory. However, in section 7, we show that the two approaches are consistent if we are close to equilibrium and we use the quasilinear theory as a model to determine an explicit expression for the diffusion coefficient that appears in equation (31).

7 Truncated models for collisionless stellar systems

The equations provided by the MEPP and by the quasilinear theory have a very different mathematical structure. The relaxation equation (31) is a partial differential equation whereas the generalized Landau equation (50) is an integrodifferential equation: the value of $f(v)$ at time $t + dt$ depends explicitly on the whole distribution of velocities $f(v')$ at time $t$ through an integration over $v'$. The usual way to transform an integrodifferential equation into a partial differential equation is to make a guess for the function appearing in the integral and refine the initial guess by successive iterations. In practice, we simply make one sensible guess. Therefore, if we are close to equilibrium (and this is in fact dictated by the conditions of validity of the quasilinear theory), it seems natural to replace the distribution function $f$ by the Fermi-Dirac distribution

$$
\bar{f} = \frac{\eta_0}{1 + \lambda e^{\beta \eta_0 v^2}}.
$$

(53)
In more physical terms, this amounts to a “thermal bath approximation”: the stars have not yet relaxed completely, but when we focus on the relaxation of a given stellar fluid particle (described by $f$) we can consider, in a first approximation, that the rest of the system (described by $f'$) is at equilibrium. With this thermal bath approximation, the generalized Landau equation (50) reduces to the nonlinear partial differential equation (54):

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial r} + F \frac{\partial f}{\partial v} = \frac{\partial}{\partial v} \left[ D \left( \frac{\partial f}{\partial v} + \beta f (\eta_0 - f) v \right) \right].$$

with a diffusion coefficient

$$D = \frac{16 \sqrt{2\pi^2 G^2 \epsilon^3 \epsilon_r^3}}{\eta_0^{1/2} \beta^{5/2} \nu^3} \ln A \int_0^{\beta \eta_0 \epsilon^2} \sqrt{x} \frac{1}{1 + \lambda e^x} dx.$$  

Equation (54) is precisely the equation derived from the MEPP. Together with the explicit expression of the diffusion coefficient (55) it provides a self-consistent model for the “coarse-grained” dynamics of collisionless stellar systems experiencing violent relaxation. More general equations including anisotropic effects can also be obtained from this formalism (13).

We shall now assume that the galaxy is not isolated but subject to the tides of other systems. In that case, high energy stars that have elongated orbits are removed by the gravity of these objects. We seek therefore a stationary solution of equation (54) satisfying the boundary condition $f(\epsilon_m) = 0$, where $\epsilon_m$ is the escape energy above which $f = 0$. This solution will provide a truncated distribution function with a finite mass (13). In fact, this problem was already tackled by Lynden-Bell in his seminal paper (43). He proposed to describe the evolution of the coarse-grained distribution function $f$ by the ordinary Fokker-Planck equation (32) with the heuristic argument that the fluctuations of the gravitational potential during violent relaxation play the same role as collisions. With the additional (heuristic) argument that $D \sim \frac{1}{v^3}$ for large velocities, he could obtain a stationary solution of a Michie-King type (8):

$$f = \begin{cases} A(e^{-\beta \eta_0 \epsilon} - e^{-\beta \eta_0 \epsilon_m}) & \epsilon \leq \epsilon_m, \\ 0 & \epsilon \geq \epsilon_m. \end{cases}$$

Since stars with $\epsilon \geq \epsilon_m$ are removed by the tidal field, this distribution function provides a depletion of high energy states and solves the infinite mass problem.

Our present approach justifies the two phenomenological arguments of Lynden-Bell since equation (54) reduces to the Fokker-Planck equation if we assume $f \ll \eta_0$ (no degeneracy) and equation (55) gives a diffusion coefficient $D \sim \frac{1}{v^3}$ for large velocities. There is however a kind of “gap” in Lynden-Bell’s approach since equation (56) does not reduce to the Fermi-Dirac statistics (16) in the limit of low energies. Using the more general equation (54) which
properly accounts for degeneracy effects, we now try to build up a “one piece”
distribution function which makes the bridge between Lynden-Bell’s statistics
\( \epsilon \ll \epsilon_m \) and the Michie-King model \( \epsilon \sim \epsilon_m \).

During the stage of violent relaxation, the stars extract their energy from
the rapid fluctuations of the gravitational field. By this process, some stars
may acquire very high energies and escape from the system (being ultimately
captured by the gravity of other systems). For these stars, \( D = \frac{K}{\epsilon^3} \) is a good
approximation and equation (54) reads

\[
\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial r} + F \frac{\partial f}{\partial v} = \frac{\partial}{\partial v} \left[ K \frac{\partial f}{\partial v} + \beta \eta_0 f (\eta_0 - f) v \right].
\]

We seek a stationary solution of the form \( f = f(\epsilon) \). Using the identity
\( \frac{\partial}{\partial v} (v v^3) = 0 \) (valid for sufficiently large \( |v| \)), we obtain

\[
0 = \frac{d}{d\epsilon} \left[ K \left( \frac{df}{d\epsilon} + \beta \eta_0 f (\eta_0 - f) \right) \right],
\]

or, equivalently,

\[
\frac{df}{d\epsilon} + \beta \eta_0 f (\eta_0 - f) = -J,
\]

where \( J \) is a constant of integration. If \( J = 0 \), we recover Lynden-Bell’s
distribution function \( 14 \). If \( J \neq 0 \), equation (59) accounts physically for an
escape of stars at a constant rate \( J \). The system is therefore not truly static
since it looses gradually stars but we may consider that the galaxy passes by
a succession of quasi-stationary states which are solution of equation (59). As
stated previously, this equation is valid only for high energy stars. For lower
energies, the system has settled down in a pure equilibrium state and \( J = 0 \),
leading to Lynden-Bell’s distribution.

Our goal, now, is to solve equation (59). Put under the form

\[
\frac{df}{d\epsilon} + \beta \eta_0 f - \beta f^2 + J = 0,
\]

we recognize a Riccatti equation \( 33 \). With the change of variables \( \bar{f} = -\frac{1}{2} \frac{df}{d\epsilon} \),
it can be converted into a linear partial differential equation of second order

\[
\frac{d^2 u}{d\epsilon^2} + \eta_0 \beta \frac{du}{d\epsilon} - J \beta u = 0.
\]

The associated characteristic polynomial \( x^2 + \eta_0 \beta x - J \beta = 0 \) has a strictly
positive discriminant \( \Delta = \eta_0^2 \beta^2 + 4 J \beta > 0 \). Therefore, the general solution of
equation (61) is

\[
u = e^{-\eta_0 \frac{\beta}{2}} (A e^{\frac{\beta}{2} \epsilon} + B e^{-\frac{\beta}{2} \epsilon}),
\]

(62)
where $\delta = \sqrt{\Delta}$ and $A, B$ are constants of integration. The solution of equation (60) is therefore

$$f = \frac{1}{2\beta} A(\eta_0 \beta - \delta) e^{\frac{\delta}{2}\epsilon} + B(\eta_0 \beta + \delta) e^{-\frac{\delta}{2}\epsilon} + Ae^{\delta \epsilon} + Be^{-\delta \epsilon}. \quad (63)$$

Setting $\lambda = A/B$ ($B \neq 0$ otherwise $f$ would be constant), it can be rewritten

$$f = \frac{1}{2\beta} \lambda(\eta_0 \beta - \delta) + (\eta_0 \beta + \delta) e^{-\delta \epsilon} \quad (64)$$

This distribution function vanishes at the escape energy $\epsilon = \epsilon_m$ defined by

$$\lambda(\eta_0 \beta - \delta) + (\eta_0 \beta + \delta) e^{-\delta \epsilon_m} = 0. \quad (65)$$

With this new variable, we obtain the result

$$f = \lambda \eta_0 \frac{e^{-\delta \epsilon} - e^{-\delta \epsilon_m}}{(\lambda - e^{-\delta \epsilon_m})(\lambda + e^{-\delta \epsilon})}. \quad (66)$$

where $\delta$ is a solution of

$$\delta = \eta_0 \beta \frac{\lambda + e^{-\delta \epsilon_m}}{\lambda - e^{-\delta \epsilon_m}}. \quad (67)$$

Now, for the cases of physical interest $\lambda \gg e^{-\beta \eta_0 \epsilon_m}$, which means that degeneracy is negligible for energies close to the escape energy $\epsilon_m$. In that case, $\delta \approx \eta_0 \beta$ and we obtain the final result [13]:

$$f = \eta_0 \frac{e^{-\beta \eta_0 \epsilon} - e^{-\beta \eta_0 \epsilon_m}}{\lambda + e^{-\beta \eta_0 \epsilon}}. \quad (68)$$

The previous calculation is justified as long as $D \neq 0$, corresponding to relatively strong fluctuations. When the fluctuations die down at the end of the relaxation, the diffusivity $D$ and therefore the diffusion current $|J|$ go to zero. There is no more evaporation but the distribution function $\mathcal{F}$ is maintained as a stationary solution of the Vlasov equation. When $\epsilon \sim \epsilon_m$, we recover the Michie-King model [3] and when $\epsilon \ll \epsilon_m$ equation (68) reduces to the Fermi-Dirac distribution function [4]. Therefore, the distribution function (68) connects continuously the two limits considered by Lynden-Bell [13] and can serve as a relevant model for (possibly degenerate) collisionless stellar systems. In particular, this distribution function could describe galactic halos limited in extension as a consequence of tidal interactions with other systems [34]. It could also be used to calculate realistic equilibrium states of collisionless stellar systems without the artifice of a material box. However, the main results of the box model [25] should not be dramatically altered.

The previous model describes self-gravitating systems subject to tidal forces. This form of confinement was first introduced in the case of globular clusters tidally truncated by a nearby galaxy [3]. This model can also describe
a fraction of elliptical galaxies living in a rich environment. However, for the majority of elliptical galaxies, tidal forces are weak and the galaxy can be considered as isolated. Now, for isolated systems, other processes can account for “incomplete relaxation” and lead to different truncated models. Starting from equation (61), we can show that the diffusion coefficient is proportional to the fluctuations of the distribution function integrated over the velocity:

$$D_{\mu\nu} = 2\pi G^2 \epsilon_i^3 \epsilon_{i'}^3 \ln A \int \frac{1}{u} \left( \delta_{\mu\nu} - \frac{u^\mu u^\nu}{u^2} \right) (f - f^2) d^3 v'. \quad (69)$$

Since these fluctuations rapidly decay as we depart from the relatively well-mixed central region of the galaxy, the diffusion current decreases also and this results in a confinement of the distribution function. It is expected therefore that the relaxation will be effective only in a limited region of space where $f - f^2$ is sufficiently large. On the other hand, as the system develops finer and finer filaments during the mixing process, the diffusion coefficient is expected to decrease in time. The diffusion current takes therefore increasingly small values and the relaxation is slowed down. This decay may be quite rapid and a quantitative treatment of this effect would require a better understanding of the correlation function $\tilde{f}(r,v,t) \tilde{f}(r',v',t)$ whose expression was simply postulated in section 5. The development of these ideas will lead to other truncated models probably closer to those introduced phenomenologically by, e.g., Stiavelli & Bertin [58], Tremaine [59] and Hjorth & Madsen [31]. In particular, the above discussion is quite consistent with the scenario of incomplete violent relaxation developed by Hjorth & Madsen. These authors introduce a two-step process: (i) in a first step, they assume that violent relaxation proceeds to completion in a finite spatial region, of radius $r_{\text{max}}$, which represents roughly the core of the galaxy (where the fluctuations are important). At this stage, the escape energy represents no special threshold so that negative as well as positive energy states are populated in that region. (ii) After the relaxation process is over, positive energy particles leave the system and particles with $\Phi(r_{\text{max}}) < \epsilon < 0$ move in orbits beyond $r_{\text{max}}$, thereby changing the distribution function to something significantly ‘thinner’ than a Boltzmann distribution. The crucial point to realize is that the differential energy distribution $N(\epsilon)$, where $N(\epsilon) d\epsilon$ is the number of stars with energy between $\epsilon$ and $\epsilon + d\epsilon$, will be discontinuous at the escape energy $\epsilon = 0$ since there is a finite number of particles within $r_{\text{max}}$ after the relaxation process. It can be shown that this discontinuity implies necessarily that $f(\epsilon) \sim (-\epsilon)^{5/2}$ for $\epsilon \to 0^-$. Therefore, the truncated model consistent with this scenario is [31]:

$$f = \begin{cases} 
A e^{-\beta \epsilon} & \epsilon < \epsilon'; \\
B (-\epsilon)^{5/2} & \epsilon' \leq \epsilon < 0; \\
0 & \epsilon \geq 0
\end{cases} \quad (70)$$

This truncated model corresponds formally to a composite configuration made of an isothermal core and a polytropic envelope with index $n = 4$. 

Violent Relaxation in Stellar Systems 27
Of course, if the core is degenerate, the Boltzmann distribution for $\epsilon < \epsilon'$ must be replaced by the Fermi-Dirac one. This truncated model gives very good fit with elliptical galaxies and can reproduce the $R^{1/4}$ law [31]. The box model of section 4 can be considered as a simple approximation of this more realistic model, the “box” playing the role of the envelope. In fact, the scenario developed by Hjorth & Madsen is almost equivalent to complete violent relaxation in a finite container with the container removed after the relaxation. This scenario solves the infinite mass problem and rehabilitates the use of statistical mechanics to understand the structure of elliptical galaxies.

8 Conclusion

We have described in this paper the process of violent relaxation in stellar systems from the viewpoint of statistical mechanics, as originally introduced by Lynden-Bell [43]. Lynden-Bell proposed that the structure of galaxies could result from a law of chaos: there is a total lack of information at small scales since the stars have complicated orbits, but the exciting phenomenon is that microscopic disorder leads to macroscopic order. The same ideas of statistical mechanics have been introduced in two-dimensional turbulence described by the Euler equation [3,4,2,3] to explain the formation and maintenance of large scale coherent vortices like the Great Red Spot of Jupiter or the cyclones and anticyclones that populate the earth atmosphere [2,3]. The formation of self-organized vortices in two-dimensional turbulence can also have applications in the context of planet formation where large-scale vortices present in the solar nebula could efficiently trap dust particles to form the planetesimals and the planets (see a complete list of references in Chavanis [15]). In fact, the statistical mechanics of the 2D Euler equation is equivalent to the theory of Lynden-Bell applying to the Vlasov equation. In a sense, the Vlasov equation is just the Euler equation for a “fluid” evolving in a six-dimensional phase space. In this analogy [1,4,7], the vorticity and the stream function in 2D turbulence play the same role as the distribution function and the gravitational field in galaxies. This analogy concerns not only the equilibrium states (the formation of large-scale structures) but also the relaxation towards equilibrium [27] and the statistics of fluctuations [24].

A kinetic theory of two-dimensional vortices can be developed in analogy with stellar dynamics [19]. In this kinetic theory, a point vortex experiences a diffusion process and a “systematic drift”. This “systematic drift” [12] is the counterpart of the “dynamical friction” [10] experienced by a star as a result of close encounters. Relaxation equations analogous to equations (31) and (50) have been derived in the context of vortex dynamics [1,2,10,11]. In addition, the problem of “incomplete relaxation” is also encountered in 2D turbulence to explain the confinement of a vortex (e.g., a dipole or a tripole) that forms after a rapid merging [2,3,5,2]. It has been demonstrated explicitly in two-dimensional turbulence (where the numerical simulations are
easier to implement) that the relaxation equations derived from the MEPP and including a space dependant diffusion coefficient related to the fluctuations of the vorticity (analogous to Eq. (69)) can account for this kinetic confinement [52]. We believe that the relaxation equations proposed in the stellar context should work as well. The statistical mechanics of continuous vorticity fields also predicts a Fermi-Dirac distribution at equilibrium with the same interpretation of the degeneracy as in Lynden-Bell’s theory. Although this degeneracy is hard to evidence for galaxies (and remains controversial), it has been vindicated by various numerical simulations and laboratory experiments of two-dimensional fluids. This suggests that the degenerate version of the theory should also be used in the stellar context. If all these effects are taken into account correctly, it is plausible that the statistical mechanics of 2D vortices and self-gravitating systems has a chance to account for the fascinating process of self-organization in nature.

References

1. V.A. Antonov, Vest. Leningr. Gos. Univ. 7, 135 (1962).
2. R. Balescu, Statistical Mechanics of Charged Particles, Interscience, New York (1963).
3. E. Bettwieser and D. Sugimoto, “Post-collapse evolution and gravothermal oscillation of globular clusters”, Mon. Not. R. astr. Soc. 208, 493 (1984).
4. N. Billic and R.D. Viollier, “Gravitational phase transition of heavy neutrino matter”, Phys. Lett. B 408, 75 (1997).
5. N. Billic, R.J. Lindebaum, G.B. Tupper and R.D. Viollier, “On the formation of degenerate heavy neutrino stars”, Phys. Lett. B 515, 105 (2001).
6. F. Bouchet and J. Sommeria, “Emergence of intense jets and Jupiter Great Red Spot as maximum entropy structures”, J. Fluid. Mech. 464, 165 (2002).
7. F. Bouchet, P.H. Chavanis and J. Sommeria, “Statistical mechanics of Jupiter’s Great Red Spot in the shallow water model”, in preparation.
8. J. Binney and S. Tremaine, Galactic Dynamics (Princeton Series in Astrophysics, 1987).
9. S. Chandrasekhar, An Introduction to the Theory of Stellar Structure (Dover 1958).
10. S. Chandrasekhar, “Stochastic problems in physics and astronomy”, Rev. Mod. Phys. 15, 1 (1943).
11. P.H. Chavanis, Contribution à la mécanique statistique des tourbillons bidimensionnels. Analogie avec la relaxation violente des systèmes stellaires, Thèse de doctorat, Ecole Normale Supérieure de Lyon (1996).
12. P.H. Chavanis “Systematic drift experienced by a point vortex in two-dimensional turbulence”, Phys. Rev. E 58, R1199 (1998).
13. P.H. Chavanis “On the coarse-grained evolution of collisionless stellar systems”, Mon. Not. R. astr. Soc. 300, 981 (1998).
14. P.H. Chavanis, “From Jupiter’s Great Red Spot to the structure of galaxies: statistical mechanics of two-dimensional vortices and stellar systems”, Annals of the New York Academy of Sciences 867, 120 (1998).
15. P.H. Chavanis “Trapping of dust by coherent vortices in the solar nebula”, Astron. Astrophys. 356, 1089 (2000).
16. P.H. Chavanis “Quasilinear theory of the 2D Euler equation”, Phys. Rev. Lett. 84, 5512 (2000).
17. P.H. Chavanis “On the analogy between two-dimensional vortices and stellar systems”, Proceedings of the IUTAM Symposium on Geometry and Statistics of Turbulence (2001), T. Kambe, T. Nakano and T. Miyauchi Eds. (Kluwer Academic Publishers).
18. P.H. Chavanis “Gravitational instability of finite isothermal spheres”, Astron. Astrophys. 381, 340 (2002).
19. P.H. Chavanis “Kinetic theory of point vortices: diffusion coefficient and systematic drift”, Phys. Rev. E 64, 026309 (2001).
20. P.H. Chavanis “Phase transitions in self-gravitating systems. Self-gravitating fermions and hard sphere models”, Phys. Rev. E 65, 056123 (2002).
21. P.H. Chavanis “Gravitational instability of finite isothermal spheres in general relativity. Analogy with neutron stars.”, Astron. Astrophys. 381, 709 (2002).
22. P.H. Chavanis, C. Rosier and C. Sire “Thermodynamics of self-gravitating systems”, Phys. Rev. E. 66, 036105 (2002).
23. P.H. Chavanis and J. Sommeria, “Classification of robust isolated vortices in two-dimensional hydrodynamics”, J. Fluid Mech. 356, 259 (1998).
24. P.H. Chavanis and C. Sire, “The statistics of velocity fluctuations arising from a random distribution of point vortices: the speed of fluctuations and the diffusion coefficient”, Phys. Rev. E 62, 490 (2000).
25. P.H. Chavanis and J. Sommeria, “Degenerate equilibrium states of collisionless stellar systems”, Mon. Not. R. astr. Soc. 296, 569 (1998).
26. P.H. Chavanis and J. Sommeria, “Statistical mechanics of the shallow water system”, Phys. Rev. E 65, 026302 (2002).
27. P.H. Chavanis, J. Sommeria and R. Robert, “Statistical mechanics of two-dimensional vortices and collisionless stellar systems”, Astrophys. J. 471, 385 (1996).
28. P. Clemmow and J. Dougherty, Electrodynamics of Particles and Plasmas (New-York: Addison-Wesley).
29. H. Cohn, “Late core collapse in star clusters and the gravothermal instability”, Astrophys. J. 242, 765 (1980).
30. H.J. de Vega, N. Sanchez, “Statistical mechanics of the self-gravitating gas: I. Thermodynamical limit and phase diagrams”, Nucl. Phys. B 625, 409 (2002).
31. J. Hjorth and J. Madsen, “Statistical mechanics of galaxies”, Mon. Not. R. astr. Soc. 265, 237 (1993).
32. S. Inagaki and D. Lynden-Bell, “Self-similar solutions for post-collapse evolution of globular clusters”, Mon. Not. R. astr. Soc. 205, 913 (1983).
33. E.L. Ince, Ordinary Differential Equations, Dover, New-York (1956).
34. G. Ingrosso, M. Merafina, R. Ruffini and F. Strafella, “System of self-gravitating semidegenerate fermions with a cutoff of energy and angular momentum in their distribution function”, A&A 258, 223 (1992).
35. W. Jaffe, 1987, in de Zeeuw T., ed., Proc. IAU Symp. 127, Structure and Dynamics of Elliptical Galaxies, Reidel, Dordrecht, pp. 511.
36. E.T. Jaynes, in The Minimum Entropy Production Principle. Kluwer. Dordrecht.
37. B.B. Kadomtsev and O.P. Pogutse, “Collisionless relaxation in systems with Coulomb interactions”, Phys. Rev. Lett. 25, 1155 (1970).
38. H.E. Kandrup, “A generalized Landau equation for a system with a self-consistent mean field: derivation from an N-particle Liouville equation”, Astrophys. J. 244, 316 (1981).

39. J. Katz, “On the number of unstable modes of an equilibrium”, Mon. Not. R. astr. Soc. 183, 765 (1978).

40. A. Kull, R.A. Treumann and H. Böringer, “Violent relaxation of indistinguishable objects and neutrino hot dark matter in clusters of galaxies”, Astrophys. J. Lett. 466, L1 (1996).

41. R.B. Larson, “A method for computing the evolution of star clusters”, Mon. Not. R. astr. Soc. 147, 323 (1970).

42. F. Leeuwin and E. Athanassoula, “Central cusp caused by a supermassive black hole in axisymmetric models of elliptical galaxies”, Mon. Not. R. astr. Soc. 417, 79 (2000).

43. D. Lynden-Bell, “Statistical mechanics of violent relaxation in stellar systems”, Mon. Not. R. astr. Soc. 136, 101 (1967).

44. D. Lynden-Bell and P.P. Eggleton, “On the consequences of the gravothermal catastrophe”, Mon. Not. R. astr. Soc. 191, 483 (1980).

45. D. Lynden-Bell and R. Wood, “The gravothermal catastrophe in isothermal spheres and the onset of red-giants structure for stellar systems”, Mon. Not. R. astr. Soc. 138, 495 (1968).

46. J. Miller, “Statistical mechanics of the Euler equation in two dimensions”, Phys. Rev. Lett. 65, 2137 (1990).

47. T. Padmanabhan, “Antonov instability and the gravothermal catastrophe revisited”, Astrophys. J. Supp. 71, 651 (1989).

48. T. Padmanabhan, “Statistical mechanics of gravitating systems”, Phys. Rep. 188, 285 (1990).

49. F. Rasio, S.L. Shapiro and S.A. Teukolsky, “Solving the Vlasov equation in general relativity”, Astrophys. J. 344, 146 (1989).

50. R. Robert, “A maximum entropy principle for two-dimensional Euler equations”, J. Stat. Phys. 65, 531 (1991).

51. R. Robert, “On the gravitational collapse of stellar systems”, Class. Quantum Grav. 15, 3827 (1998).

52. R. Robert and C. Rosier, “The modelling of small scales in 2D turbulent flows: A statistical mechanical approach”, J. Stat. Phys. 86, 481 (1997).

53. R. Robert and J. Sommeria, “Statistical equilibrium states for two-dimensional flows”, J. Fluid Mech. 229, 291 (1991).

54. R. Robert and J. Sommeria, “Relaxation towards a statistical equilibrium state in two-dimensional perfect fluid dynamics”, Phys. Rev. Lett. 69, 2776 (1992).

55. G. Severne and M. Luwel, “Dynamical theory of collisionless relaxation”, Astrophys. & Space Sci. 72, 293 (1980).

56. S.L. Shapiro and S.A. Teukolsky, “The collapse of dense star clusters to supermassive black holes: the origin of quasars and AGNs”, Astrophys. J. 292, L41 (1995).

57. M. Stiavelli, “Violent relaxation around a massive black hole”, Astrophys. J. Lett. 495, L91 (1998).

58. M. Stiavelli and G. Bertin, “Statistical mechanics and equilibrium sequences of ellipticals”, Mon. Not. R. astr. Soc. 229, 61 (1987).

59. S. Tremaine, 1987, in de Zeeuw T., ed., Proc. IAU Symp. 127, Structure and Dynamics of Elliptical Galaxies, Reidel, Dordrecht, pp. 367.