Probes in fluxbranes and supersymmetry breaking through Hodge-duality

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ABSTRACT

In this letter we consider what happens to an M2-brane probe when Minkowski space is dimensionally reduced to a fluxbrane solution of IIA supergravity. Given that fluxbrane reductions generally break supersymmetry, we look at how supersymmetry is realised on the D2-brane probe after dualisation. We also show how to extend this to more general non-linear sigma models.

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1 Introduction

It is standard lore that, at the level of the low energy effective action, dimensional reduction and T-duality can break supersymmetry \[1, 2\]: two supergravity solutions related via T-duality or via dimensional reduction do not have in general the same number of supersymmetries. Probably the best-known example is the one obtained by dimensionally reducing Minkowski space, written in spherical coordinates, over an angular Killing vector \(\partial_\phi\). Minkowski space is of course maximally supersymmetric, but the obtained solution has no surviving supersymmetries. Many other examples were given in \[1, 2\]. The breaking of supersymmetry in these cases is due to the fact that, although the (bosonic) solution admits an isometry, the Killing spinors are not invariant under its action. In order for supersymmetry to be preserved in the dualisation or dimensional reduction of supergravity solutions, it is therefore necessary that the Killing spinors are independent of the isometry direction that is considered \[2\].

This is of course an artefact of the fact we are only dealing with the low energy effective action. If one would take in account the full string spectrum with all higher-order Kaluza-Klein modes, then it is clear that supersymmetry is preserved, since dimensional reduction and dualisations are merely rewritings and canonical transformations \[3\]. Indeed, it has been shown \[4, 5, 6, 7\] that in the dualisation local realisations of supersymmetry can be replaced by non-local realisations, due to non-local world-sheet effects, but this will never break supersymmetry at the level of the conformal field theory. Only while looking at the lowest-mode approximation, supersymmetry can be lost in the truncation.

There are some well known examples where specific use is made of dimensional reduction along non-trivial Killing vectors (involving both translations and rotations) in order to obtain new solutions. In \[8, 9, 10\] it was realised that a dilatonic version of the Melvin universe \[11\] can be obtained from dimensional reduction of flat space-time, if the higher-dimensional solution has some non-trivial identifications \[12, 13, 14\]. The lower-dimensional solution describes a fluxtube, with the vector potential being the Kaluza-Klein vector from the reduction. In \[15\] a ten-dimensional version of this fluxtube (F7-brane in modern terminology) was used to study the Green-Schwarz string action in backgrounds with RR-fields, in \[16\] to point out a duality between Type IIA and Type 0A, and in \[17, 18, 19\] to construct supergravity solutions of D-branes and F-strings undergoing the dielectric effect \[20\]. These solutions are in general non-sypersymmetric due to the dependence of the Killing spinors on the compactified coordinate.

However, recently in \[21, 22, 23\] it was found that in some specific cases, involving several carefully chosen (magnetic) parameters, some amount of supersymmetry can be preserved in the dimensional reduction. In \[21\] a systematic study of these cases was made: it turned out that dimensional reduction of flat space along a Killing vector consisting of a translation and a rotation will preserve some fraction of the supersymmetry if the rotation lays in the isotropy algebra of the Killing spinor, and these isotropy algebras were classified. In this way, the Killing vector will leave some Killing spinors invariant and the amount of preserved supersymmetry is given by the number of Killing spinor components that are independent of the coordinate associated to the isometry direction. In this way several supersymmetric fluxbranes have been constructed.

In this letter, we will use probe techniques to study the (partial) breaking of supersymmetry in the construction of these new fluxbrane solutions. More specifically, we will study a simple toy model and see what happens with an M2-brane probe in eleven-dimensional Minkowski space, when the space is dimensionally reduced to a fluxbrane solution. The reduction is transverse to the probe such that the M2-brane turns into a D2-brane probe in the fluxbrane background. Given that the lower-dimensional background has less supersymmetry than the original Minkowski space, it is to be expected that also the world-volume theory living on the D2-brane probe will be less supersymmetric then the one living on the M2-brane at the level of the non-linear sigma model. Indeed, it is clear from \[11, 2\] that (some) supersymmetry will be broken due to explicit dependence of the Killing spinor on the isometry direction. We will show that, as in \[24\], with specific choices of the rotational parameters, the world-volume theory on the D2-brane will preserve a certain amount of supersymmetry. The difference with the construction of \[24\] lays in the fact that here, in the world-volume theory, it is not the dimensional reduction directly that is responsible for the partial supersymmetry breaking, but the Hodge-dualisation of one of the embedding scalars of the M2-brane into the Born-Infeld vector of the D2-brane \[25\]. The amount of supersymmetry on the world-volume after dualisation will be given by the number of supersymmetry parameters that are independent of the dualised coordinate.

This paper is organised as follows: in Section 2 we revise shortly the construction of supersymmetric...
fluxbranes of [24] and discuss the conditions for supersymmetry to be preserved in these solutions. In Section 3 we consider the world-volume theories that live on the M2- and D2-brane probes, performing a dimensional reduction of the type done in [24], we see how the Hodge-dualisation of the isometric scalar has to be performed for the M2-brane world-volume action to be reduced to the D2-brane action, first in the bosonic case and then in the supersymmetric case. By looking at the supersymmetry algebra and the invariance of the action in the reduced theory, we show that the same conditions for supersymmetry preservation hold for the world-volume fields as for the backgrounds. In Section 4 we argue that this procedure is quite general and that the Hodge-dualisation of a scalar in any non-linear sigma-model is compatible with (some part of) the supersymmetry-algebra, as long as the Killing vector associated to the dualised scalar commutes with some subgroup of the super-Poincaré algebra. Finally, our conventions on the spinors of the different groups appearing in this letter are given in the Appendix.

2 Reductions of Minkowski space

It is well known that the ten-dimensional dilatonic Melvin universe [8, 9, 10, 15, 16] can be obtained via a dimensional reduction of eleven-dimensional Minkowski space with some non-trivial identifications [12, 13, 14]. After a reduction over a Killing vector

$$\xi = \partial_z + B_i x^i \partial_j, \quad (2.1)$$

a so-called fluxbrane is obtained, where the Kaluza-Klein vector gives rise to a flux tube in the lower-dimensional solution. It has been pointed out in [21, 22, 23, 24] that for specific choices of the rotation $B$, the obtained fluxbranes could preserve some amount of supersymmetry. In this Section we will follow closely the construction presented in [24].

The first step in the dimensional reduction is choosing a coordinate system of Minkowski space adapted to the Killing vector, such that $\xi = \partial_z$. We start from the standard coordinates on eleven-dimensional Minkowski space. Firstly, we write the Killing vector (2.1) as

$$\xi = U^{-1} \partial_z U, \quad U = e^z B. \quad (2.2)$$

The coordinate system adapted to the Killing vector $\xi$ is then defined by

$$\tilde{y} = U^{-1} \tilde{x}, \quad (2.3)$$

since in these coordinates $\{y^i, z\}$ we have $\xi \tilde{y} = 0$. In these coordinates Minkowski space is given by

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + \left( \delta_{ij} - \Lambda^{-1} B_{ik} y^k B_{jl} y^l \right) dy^i dy^j + \Lambda \left( dz - \Lambda^{-1} B_{ij} y^i dy^j \right)^2, \quad (2.4)$$

where

$$\Lambda = 1 - y^i B_{ij} y^j. \quad (2.5)$$

Using the standard Kaluza-Klein Ansatz, the corresponding Type IIA solution is given by

$$ds^2 = \Lambda^2 \eta_{\mu\nu} dx^\mu dx^\nu + \Lambda^2 \left( \delta_{ij} + \Lambda^{-1} B_{ik} y^k B_{jl} y^l \right) dy^i dy^j$$

$$\phi = \frac{3}{4} \ln \Lambda$$

$$C_i = \Lambda^{-1} B_{ij} y^j. \quad (2.5)$$

It is interesting to know for what choices of $B$ the solution (2.5) preserves some supersymmetry. In general, the amount of supersymmetry after dimensional reduction over a Killing vector $\xi$ is given by the number of supersymmetry parameters that are invariant along the flow of the vector field. This is saying that the Lie derivative of the (parallel) spinor $\epsilon$ along $\xi$ should vanish:

$$\mathcal{L}_\xi \epsilon = \xi^A D_A \epsilon + \frac{1}{4} \partial_A \xi_B \gamma^{AB} \epsilon = 0, \quad (A, B = 0, \ldots, 10). \quad (2.6)$$

3The Greek indices $\mu$ run from 0 to 2 and the Latin indices $i$ from 1 to 7. This specific splitting of the eleven-dimensional indices is motivated by the fact that we will later put an M2- and D2-brane probe in the $x^\mu$-directions. We therefore restricted the rotation on the Killing vector deliberately to the subspace transverse to the $x^\mu$s. This choice will of course limit the number of possible supersymmetric solutions.
Table 1: The number of supercharges $Q$ of the Type IIA fluxbrane solution (2.5) preserved for different choices of $B$. The group $g$ denotes the subgroup of $so(7)$ in which the rotation is contained.

| $\alpha = \beta = \gamma = 0$ | $g$ | $Q$ |
|--------------------------------|-----|-----|
| $\alpha = -\beta, \gamma = 0$ | $sp(1)$ | 16 |
| $\alpha + \beta + \gamma = 0$ | $su(3)$ | 8 |
| $\alpha + \beta + \gamma \neq 0$ | $so(7)$ | 0 |

In coordinates adapted to the Killing vector, this implies that the spinor does not depend on the compact coordinate: $\partial_z \epsilon = 0$.

In our setting, there are 32 parallel eleven-dimensional spinors components $\epsilon$ which are constants in natural coordinates. The condition for preserved supersymmetries in IIA then reduces to

$$B \epsilon = 0.$$  \hspace{1cm} (2.7)

In other words, the rotation $B$ in our Killing vector should leave some subset of the parallel spinors invariant. Without loss of generality, $B$ can always be chosen of the following form:

$$B = \begin{pmatrix}
 i \alpha \sigma_2 & 0 & 0 & 0 \\
 0 & i \beta \sigma_2 & 0 & 0 \\
 0 & 0 & i \gamma \sigma_2 & 0 \\
 0 & 0 & 0 & 0
\end{pmatrix}, \quad \text{where} \quad \sigma_2 = \begin{pmatrix}
 0 & -i \\
 i & 0
\end{pmatrix},$$  \hspace{1cm} (2.8)

i.e. we can always choose the rotation $B$ to lie in the Cartan subalgebra of $so(7)$.\(^5\) Therefore, it should be contained in $su(3)$ to preserve some of the supersymmetries. In the parametrisation (2.8) this means

$$\alpha + \beta + \gamma = 0.$$  \hspace{1cm} (2.9)

The number of preserved supersymmetries for different choices of these parameters are given in the Table I where $Q$ denotes the number of preserved supersymmetries and $g$ is the subgroup of $so(7)$ in which the rotation is contained. Note that if we add an M2- or D2-brane probe, half of the supersymmetries will be broken by the probe.

### 3 Hodge–duality on the probes

In the present Section, we will study this reduction process from the viewpoint of an M2/D2-brane probe. We will simplify the theory on the probe by gauge fixing the world-volume diffeomorphism invariance and the $\kappa$-symmetry and Taylor expanding the membrane (or Born-Infeld) action up to terms quadratic in the derivatives of the fields. In that way, we obtain non-linear sigma models, and we will point out what happens when a scalar in the $\mathcal{N} = 8$ theory, corresponding to the M2-brane, is dualised into a vector in a theory corresponding to the D2-brane. In particular we expect this vector theory not to have $\mathcal{N} = 8$ supersymmetry anymore.

It is well known that a simplification to the level of non-linear sigma models could accidentally enhance the number of supersymmetries. A famous example being an M2-brane probing orthogonal to an eight-dimensional hyper-Kähler manifold. The world-volume theory has $\mathcal{N} = 3$ although the non-linear sigma model has $\mathcal{N} = 4$ \cite{20}. We will show though, that such an enhancement does not appear in our setting.

\(^4\)Note that this is still a condition on the eleven-dimensional spinors $\epsilon$. The condition in terms of the ten-dimensional supersymmetry parameters is much more involved. For later convenience it turns out to be easier to keep explicitly working with the eleven-dimensional parameters for the rest of the paper.

\(^5\)In order for the Killing vector not to have fixed points and the reduced solution to be everywhere non-singular, the rotation $B$ has to lie in the $SO(7)$ subgroup of $SO(8)$. 

4
3.1 Probing the bosonic background

Following the above procedure for an M2-brane probe in eleven-dimensional Minkowski space, we find that the resulting non-linear sigma model reads

\[ \mathcal{L}_{M2} = -\frac{1}{2} \partial_\mu \phi^i \partial^\mu \phi^i - \frac{1}{2} \partial_\mu Z \partial^\mu Z. \]  

We thus trivially find eight non-interacting scalars. The field \( Z \) corresponds to the position of the brane in the \( z \)-direction, while the \( \phi^i \) denote degrees of freedom in the \( x^i \)-directions. On the other hand, probing the background with a D2-brane along \( x^0, x^1, x^2 \), gauge fixing and Taylor expanding yields

\[ \mathcal{L}_{D2} = -\frac{1}{2} \partial_\mu \chi^i \left[ \delta_{ij} + \Lambda^{-1} B_{ik} \chi^k \chi^B_{ij} \right] \partial^\mu \chi^j - \frac{1}{2} \Lambda^{-1} F_{\mu \nu} F^{\mu \nu} - \frac{1}{2} \epsilon^{\mu \nu \kappa} \Lambda^{-1} \chi^i B_{ij} \partial_\mu \chi^j F_{\nu \kappa}, \]  

where \( F_{\mu \nu} = 2 \partial_{[\mu} A_{\nu]} \) is the field strength of the Born-Infeld vector \( A_\mu \) and the scalars \( \chi^i \) denote the degrees of freedom transverse to the D2. \( \Lambda \) is again given by \( \Lambda = 1 - \chi^i B_{ij} \chi^j \).

It is well known that an M2-brane action transforms into a D2-brane Born-Infeld theory by dualising the scalar corresponding to the compact direction in the M2-brane action into the Born-Infeld vector of the D2-brane. In the same spirit, the theory (3.1) is transformed into (3.2) by dualising the coordinate \( \xi \) parametrising the flow along the Killing vector \( \xi \) of (2.1). In order to see this, we should first change to coordinates in the target space of (3.1) adapted to \( \xi \), meaning that we need the coordinate transformation (3.3). The corresponding field redefinition is obviously

\[ \tilde{\chi} = e^{-ZB} \tilde{\phi}, \]  

and in terms of these new fields, the M2-probe (3.1) reads

\[ \mathcal{L}_{M2} = -\frac{1}{2} \Lambda \partial_\mu Z \partial^\mu Z - \partial_\mu \chi^i B_{ij} \chi^j \partial^\mu Z - \frac{1}{2} \partial_\mu \chi^i \partial^\mu \chi^i. \]  

In order to dualise \( Z \), we substitute \( L_\mu = \partial_\mu Z \) and enforce the Bianchi identity \( \partial_\mu L_\nu | = 0 \) by adding a Lagrange multiplier \( A_\mu \) to the M2-brane action (3.3):

\[ \mathcal{L} = \mathcal{L}_{M2} - \epsilon^{\mu \nu \kappa} A_\mu \partial_\nu L_\kappa. \]  

Eliminating \( L_\mu \) with its equation of motion yields the dualisation condition for \( Z \)

\[ \partial_\mu Z = -\frac{1}{2} \epsilon^{\mu \nu \kappa} \Lambda^{-1} F_{\nu \kappa} - \Lambda^{-1} \partial_\mu \chi^i B_{ij} \chi^j, \]  

which filled in in the action (3.5), yields the D2-brane action (3.2).

3.2 Probing supersymmetric backgrounds

We will now extend this procedure to the supersymmetric case and show that dualising a scalar into a vector can break supersymmetry, as can be predicted by looking at the background.

As Minkowski space is maximally supersymmetric, the theory on the M2-brane probe has 16 supersymmetries. This fixes the (non-)linear sigma model of (3.1) to the following form \( (a = 1, \ldots, 8)^6 \)

\[ \mathcal{L}_{M2} = \frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi^a + \tilde{\psi} \Gamma^a \tilde{\phi}^a, \]  

while the \( N = 8 \) supersymmetry transformation rules are

\[ \delta \phi^a = \frac{1}{2} \Gamma^a \tilde{\phi}^a \tilde{e}, \]

\[ \delta \tilde{\psi} = -\frac{1}{2} \Gamma^a \phi^a \tilde{e}. \]  

The R-symmetry algebra is \( so(8) \) and we used triality to put the scalars into the vector representation, while the spinors \( \psi \) transform as a spinor of both \( so(8) \) and \( so(2, 1) \) and the parameters for supersymmetry \( \tilde{e} \) as a cospinor.

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6Our conventions on fermions and gamma matrices are explained in the appendix.

7Note that this is an on-shell algebra.
To find the non-linear sigma model corresponding to the D2-brane probe in the fluxbrane background, we have to repeat the dualisation procedure in the presence of fermions. As we first have to find target space coordinates adapted to $\xi$, we have to single out one scalar, just as in (3.1). This breaks the R-symmetry to $\mathfrak{so}(7)$. Therefore, we have to choose a special base for our $\mathfrak{so}(8)$ gamma matrices 

$$
\Gamma_\ast = \sigma_3 \otimes 1_8, \quad C = 1_2 \otimes C, \quad \Gamma^i = i \sigma_2 \otimes \Gamma^i, \quad \Gamma^8 = -i \sigma_1 \otimes 1_8,
$$

(3.9)

where $\sigma_1, \sigma_2$ and $\sigma_3$ are the Pauli matrices. Furthermore we decompose scalars and the spinors as

$$
\phi^a = (\phi^i, Z), \quad \bar{\epsilon} = \begin{pmatrix} 1 & 0 \end{pmatrix} \otimes \epsilon, \quad \bar{\psi} = \begin{pmatrix} 0 & 1 \end{pmatrix} \otimes \psi,
$$

(3.10)

where $\psi$ and $\epsilon$ are Majorana spinor of $\mathfrak{so}(7)$ and $\mathfrak{so}(2,1)$. The M2-brane action and the supersymmetry transformations can then be rewritten as

$$
\mathcal{L}_{M2}' = -\frac{1}{2} \partial_\mu \phi^i \partial^\mu \phi^i - \frac{1}{2} \partial_\mu Z \partial^\mu Z - \bar{\psi} \psi,
$$

(3.11)

and

$$
\delta \phi^i = \bar{\epsilon} \tilde{\Gamma}^i \psi, \quad \delta Z = -i \bar{\epsilon} \psi, \quad \delta \psi = -\frac{1}{2} \tilde{\Gamma}^i \phi^i \epsilon - \frac{1}{4} i \bar{\phi} \phi \epsilon.
$$

(3.12)

We subsequently have to do the field redefinition on the fermions and the $\mathfrak{so}(7)$ gamma matrices as well:

$$
\bar{\chi} = e^{-ZB} \bar{\psi}, \quad \theta = e^{-\frac{1}{2} Z B} \psi, \quad \gamma^i = e^{-\frac{1}{2} Z B} (e^{-ZB})^i_j \tilde{\Gamma}^j e^{\frac{1}{2} Z B}, \quad \epsilon = e^{-\frac{1}{2} Z B} \bar{\epsilon}
$$

(3.13)

The resulting superalgebra is given by

$$
\delta \chi^i = i B_{ij} \chi^j \bar{\epsilon} \theta + \bar{\epsilon} \gamma^i \theta, \quad \delta Z = -i \bar{\epsilon} \theta,
$$

(3.14)

$$
\delta \theta = -\frac{1}{2} \gamma^i \partial \chi^i \epsilon - \frac{1}{4} i \bar{\theta} \partial Z \epsilon + \frac{1}{4} i B \theta \bar{\epsilon} \theta - \frac{1}{2} \gamma^i B_{ij} \chi^j \partial Z \epsilon,
$$

and the new action Lagrangian

$$
\mathcal{L}_{M2} = -\frac{1}{2} \Lambda \partial_\mu B_{ij} \partial^\mu B_{ij} - \frac{1}{2} \partial_\mu \chi^i \partial^\mu \chi^i - \bar{\theta} \partial \theta - \frac{1}{4} \bar{\theta} \partial Z \partial \theta,
$$

(3.15)

is still $\mathcal{N} = 8$ supersymmetric, provided that we take in account that

$$
\delta_1 \varepsilon_2 = -\frac{1}{4} \bar{\varepsilon}_1 \theta \bar{B} \varepsilon_2.
$$

(3.16)

To dualise the action, we again write $L_\mu = \partial_\mu Z$, add a Lagrange multiplier to enforce the Bianchi identity and subsequently eliminate the $L_\mu$ from through its equation of motion, which is the duality condition

$$
\partial_\mu Z = -\frac{1}{4} \bar{\varepsilon}_1 \theta \bar{B} \varepsilon_2.
$$

(3.17)

The resulting dualised action now reads

$$
\mathcal{L}_{D2} = -\frac{1}{2} \partial_\mu \chi^i \partial^\mu \chi^i - \frac{1}{4} \Lambda^{-1} \partial_\mu \chi^i B_{ij} \chi^j \partial^\mu \chi^i - \frac{1}{4} \Lambda^{-1} F^2 - \frac{1}{2} \epsilon_{\mu \nu \kappa} \Lambda^{-1} \chi^i B_{ij} \partial_\mu \chi^j F_{\nu \kappa} - \bar{\theta} \partial \theta - \frac{1}{2} \Lambda^{-1} \bar{\theta} \partial F \partial \theta - \frac{1}{4} \Lambda^{-1} B_{ij} \partial_\mu \chi^j \partial^\mu \chi^i \partial \theta - \frac{1}{2} \Lambda^{-1} \tau_{\mu} \partial \theta \bar{\theta} \partial \theta + \frac{1}{2} \Lambda^{-1} \tau_\mu \partial \theta \bar{\theta} \partial \theta.
$$

(3.18)

Since the dualised action has become independent of $Z$, we interpret it as the action of an D2-brane probe in the background of the fluxbrane. The amount of supersymmetry preserved after the dualisation depends on the supersymmetry preserved by the background and is given by the rotation $B$ in the Killing vector. In analogy with the action, we demand the supersymmetry rules of (3.15) to be related to

\footnote{Note that strictly speaking $\varepsilon$ is not a parameter, due to the $Z$-dependence.}
Table 2: The number of supercharges $Q$ of the D2-brane probe preserved for different choices of $B$. The group $\mathfrak{g}$ denotes the subgroup of $\mathfrak{so}(7)$ in which the rotation is contained.

(3.14) Actually, we ask that the transformation rules for $\chi^i$ and $\theta$ remain the same, but substituting the $\partial_\mu Z$-factors by their duality condition (3.17). Consistency requires however that all $Z$-dependence should drop out of the transformation rules, in particular also the $Z$-dependence in the supersymmetry parameter $\varepsilon$. We therefore also have to impose the following condition on $\varepsilon$:

$$\partial_Z \varepsilon = 0 \implies B \varepsilon = 0, \quad \varepsilon = \varepsilon.$$

We thus see that only those components of $\varepsilon$ survive that are annihilated by $B$, which reduces the amount of supersymmetry. Note that this is precisely the condition (2.7) we found for the background (see Section 4 for a discussion here on).

Finally, we still have to find the transformation of the vector, with the condition that the supersymmetry algebra should close and the action (3.18) should be invariant. This can be done by asking that the dualisation condition (3.17) is invariant under supersymmetry (using the equation of motion for the fermions). Indeed we find that for the supersymmetry transformation rules

$$\delta \chi^i = i B_{ij} \chi^j \bar{\varepsilon} \theta + \bar{\varepsilon} \gamma^i \theta,$$

$$\delta A_\mu = i \bar{\varepsilon} \tau_\mu \theta - \bar{\varepsilon} \tau_\mu \gamma^i \theta B_{ij} \chi^j,$$

$$\delta \theta = -\frac{i}{2} \gamma^i \phi \chi^i \epsilon + \frac{1}{2} i \Lambda^{-1} \gamma^i \phi \chi^i B_{ij} \chi^j B_{kl} \chi^k + \frac{i}{2} \Lambda^{-1} \gamma^i \phi \chi^i B_{ij} \chi^j - \frac{i}{2} \Lambda^{-1} \gamma^i \phi \chi^i B_{ij} \chi^j$$

the supersymmetry algebra closes on-shell and the D2-brane probe action (3.18) is invariant up to terms proportional to $B\epsilon$, which are set to zero. The total number $Q$ of supercharges preserved is thus determined by $B$ and can be found in Table 2.

In summary, from a $D = 3, \mathcal{N} = 8$ theory with eight scalars, we have constructed via the dualisation of one of the scalars into a vector, the Lagrangian (3.18), which is still invariant under the supersymmetry transformations (3.20). The obtained theory with seven scalars and one vector is however in general no longer $\mathcal{N} = 8$ supersymmetric, but some of the supersymmetries get broken, due to the dependence of the supersymmetry parameter on the dualised coordinate.

4 General considerations

In the previous Section, we have shown that it is possible to dualise a scalar in the $\mathcal{N} = 8$ multiplet in such a way that the dualised theory has less supersymmetry. This was to be expected as we were probing with a non-linear sigma model a supergravity background which was losing supersymmetry while dimensionally reducing. We will now show that this breaking by dualisation is quite general.

The theory (3.47-3.48) is maximally supersymmetric and realises the extended super-Poincaré algebra. We now look for an extra $u(1)$ or $\mathbb{R}$ isometry $G$, which acts as

$$\delta_G \phi^a = \lambda \xi^a, \quad \delta_G \psi = \lambda \cdot \psi,$$
where the parameter $\lambda$ is a constant, $\xi^a$ is a Killing vector of target space,\footnote{I.e. it leaves the action invariant.} and $t$ is the representation of this isometry on the fermions, whose explicit form will be determined shortly, but that consists of an even number of gamma matrices to preserve the $\mathfrak{so}(8)$ chirality of the fermions under the transformations $G$: 

$$t = t_0 + t_{abc} \Gamma^{abc}.$$  

We would like to find for which $\xi^a$ and $t$ (part of the) supersymmetry is realised. The only non-trivial condition will come from the commutator of the isometry transformations with the supersymmetry.

In the most general case, the commutator of an external symmetry and a soft supersymmetry algebra can give a new supersymmetry transformation with a field dependent parameter $\eta$:

$$[\delta_G, \delta_Q] = \delta_Q(\eta).$$  

When we try to realise this for the isometry (4.1) and the supersymmetry transformations (3.8), we notice that such a realisation is trivial: the only transformations (3.8) that preserve all the supersymmetry are the translations

$$\delta_G \phi^a = C^a, \quad \delta_G \psi = 0,$$  

($C^a$ is a constant vector) and they necessarily commute with the supersymmetry transformations (3.8). This is of course the well known case of standard reduction and dualisation in an isometry direction that preserves all supersymmetry.

It is however possible to look for more general cases, where only a subgroup of the Poincaré algebra commutes with the isometry transformations (4.1), which will in general break supersymmetry. When we try to close the commutator of (4.1) and the supersymmetry (3.8) on the scalars and the fermions, we find the conditions

$$[\delta_G, \delta_Q] \phi^a = -\lambda \bar{\psi} \left( \Gamma^b \partial_b \xi^a - (t_{(0)} - t_{(2)} + t_{(4)}) \Gamma^a \right) \epsilon = 0,$$

$$[\delta_G, \delta_Q] \psi = -\frac{1}{2} \lambda \left( \Gamma^b \partial_b \xi^a + (t_{(0)} + t_{(2)} + t_{(4)}) \Gamma^a \right) \bar{\psi} \phi^a \epsilon = 0,$$  

where we took $\partial_a t = 0$ because it is the only commutator term in (4.5) quadratic in the fermions. Equating (4.4) and (4.5) then yields

$$t_{(0)} = t_{abcd} = 0,$$

$$\left[ \Gamma^b \partial_b \xi^a + t_{bc} \left( \Gamma^a \Gamma^{bc} - 4 \delta^{ab} \Gamma^c \right) \right] \epsilon = 0.$$  

The latter condition has a simple solution of the form

$$t_{ab} = -\frac{1}{4} \partial_a \xi_b, \quad \Gamma^{ab} t_{ab} \epsilon = 0.$$  

Hence, isometry transformations of the form (4.1) will commute with the supersymmetry algebra and preserve that part of the supersymmetry for which $t \epsilon = 0$. Note that this coincides with the condition (2.7) and (3.19). Dualisation of a scalar along the Killing vector $\xi$ is precisely what we have done in Section 3.

It is now obvious that these considerations are quite general. In dimensional reduction, the (super)isometry algebra of the reduced solution is that part of the isometry algebra of the higher-dimensional field configuration that commutes with the Killing vector used during the reduction. Therefore, if we look at the theory on the probes, the part of the supersymmetry algebra commuting with the Killing vector will in general be realised after the dualisation.

## 5 Conclusion

We have shown how the world-volume supersymmetry can be broken in the Hodge-dualisation involved in transforming the world-volume effective action of an M2-brane into a D2-brane, at the level of the non-linear
sigma model. In particular, we have looked at an M2-brane probe in eleven-dimensional Minkowski space and reduced this over a non-trivial Killing vector (involving both translations and rotations) to a D2-brane probe in a fluxbrane background. Dualising one of the 8 scalars in the $D = 3$, $\mathcal{N} = 8$ theory living on the world-volume of the M2-brane, we obtain a $D = 3$ theory with seven scalars and one vector on the D2-brane, that has less then $\mathcal{N} = 8$ supersymmetry. At the level of the background, the supersymmetry is broken through the dependence of the Killing spinor on the Killing vector. At the level of the M2-brane world-volume theory, this translates into the fact that the algebra has explicit dependence on $Z$ (not only via its derivatives). Consistency of the Kaluza-Klein picture requires this dependence to vanish in the D2-brane theory, such that only the components of the supersymmetry parameter survive that are invariant under the action of the Killing vector. This technique is similar to the well-known feature of supersymmetry breaking in dimensional reduction in supergravity solutions, as pointed out in [1, 2], but to our knowledge it is the first time that it is performed at the level of the non-linear sigma models of brane effective actions.

As in the case of dimensional reduction of supergravity solutions, this supersymmetry breaking is a consequence of the fact that we are limiting ourselves to the level of the non-linear sigma model. If we would take in account the full Born-Infeld effective actions of the M2- and the D2-brane, the spinor components that do depend on the compactified and dualised coordinate would still enter in the game as higher-order Kaluza-Klein modes and supersymmetry would the completely preserved, analogously to what happens at the level of the string world-sheet in the case of T-duality [4, 5, 6, 7]. However, since the non-linear sigma-model is a consistent truncation of the full membrane effective action, our results hold in this range of validity.

It is well-known that $D = 3$ theories with 8 scalars and $\mathcal{N} = 4$ ($\mathcal{N} = 2$) supersymmetry have hyper-Kähler (Calabi-Yau) target manifolds. An interesting question that arises then is whether the target manifolds that appear in the $\mathcal{N} = 4$ ($\mathcal{N} = 2$) theories with seven scalars and one vector are in some sense related to these hyper-Kähler (Calabi-Yau) target manifolds. In other words, whether it would be possible to dualise the vector back into a scalar, but without enhancing the supersymmetry back to $\mathcal{N} = 8$. An obvious example is given by the action (3.15), keeping the condition that $B \epsilon = 0$, for the different choices of $B$ as given in Table 2. Of course, the resulting theory can then easily be extended to the original $\mathcal{N} = 8$ theory. A less trivial solution would be to add some ($\mathcal{N} = 8$) supersymmetry breaking potential to the M2-brane action (3.15) and add extra terms to the supersymmetry variations in (3.12), such that the algebra closes and the action is invariant up to terms proportional to $B \epsilon$. It turns out however, that modifications of the M2-brane action and the supersymmetry transformations become very involved and lead inevitably to modifications in the D2-brane side as well, such that it will no longer be clear what the relation is with the obtained action (3.18).

Finally we would like to point out that our results can be easily extended from probes in Minkowski space to probes in any eleven-dimensional supersymmetric background admitting an isometry, leading to for instance probes in spaces describing supersymmetric configurations of D-branes or NS-branes embedded in fluxbrane backgrounds [28, 29].

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A Conventions

We use the convention of [27]. The $x^{10}$ direction is denoted by $z$. We anti-symmetrise with unit coefficient, and always use a mostly plus metric. The covariant derivative on a spinor reads

$$\mathcal{D}_\mu \epsilon = \partial_\mu \epsilon + \frac{1}{4} \hat{\epsilon}_\mu \epsilon.$$  \hfill (A.1)

A.1 $SO(2,1)$ spinors

The $SO(2,1)$ spinors are Majorana (i.e. $\bar{\lambda} = \chi^T C = \alpha^{-1} \chi \gamma_0$, with $C$ charge conjugation and $\alpha$ a unimodular constant). We take the spinors to be Grassmann valued. The Clifford algebra is given by

$$\{\tau_\mu, \tau_\nu\} = 2 \eta_\mu\nu.$$  \hfill (A.2)

with $\tau_2 = -\tau_0 \tau_1$. The gamma-matrices can be represented by Pauli-matrices as

$$\tau_0 = i \sigma_1, \quad \tau_1 = \sigma_2, \quad \tau_2 = \sigma_3.$$  \hfill (A.3)

Hodge duality imposes the following relations between the gamma matrices

$$\tau_{\mu \nu \rho} = -\epsilon_{\mu \nu \rho}, \quad \tau_\mu = \frac{1}{2} \epsilon_{\mu \rho} \tau^{\nu \rho},$$

$$\tau_{\mu \nu} = -\epsilon_{\mu \rho} \tau^{\rho \nu}, \quad 1 = \frac{1}{6} \epsilon_{\mu \rho \sigma} \tau^{\rho \sigma}.$$  \hfill (A.4)

For $\bar{\lambda} \tau^{(n)} \chi = t^{(n)} \bar{\chi} \tau^{(n)} \lambda$, we find the following transposition properties

| $n$ | 0 | 1 | 2 | 3 |
|-----|---|---|---|---|
| $t^{(n)}$ | + | - | - | + |

and the $SO(2,1)$ Fierz identity is given by

$$\chi \bar{\lambda} = -\frac{1}{2} \left( \bar{\lambda} \chi + \bar{\lambda} \tau_\mu \chi \tau^\mu \right).$$  \hfill (A.5)

A.2 $SO(8)$ spinors

The $SO(8)$ spinors are Majorana-Weyl and we take them to be Grassmann even. The chirality matrix $\Gamma_s$ is defined as $\Gamma_s = \prod_{a=1}^8 \Gamma^a$ and $C$ is charge conjugation. For $\bar{\lambda} \Gamma^{(n)} \chi = t^{(n)} \bar{\chi} \Gamma^{(n)} \lambda$, we find the following transposition properties

| $n$ | 0,4,8 | 1,5 | 2,6 | 3,7 |
|-----|------|---|---|---|
| $t^{(n)}$ | + | + | - | - |

In the case that three spinors have the same chirality, we have the following Fierz-identity

$$\eta \bar{\lambda} \chi = \frac{1}{16} \left( 2 \bar{\lambda} \eta \chi - \bar{\lambda} \Gamma_{ab} \eta \Gamma^{ab} \chi + \frac{1}{4!} \bar{\lambda} \Gamma_{abcd} \eta \Gamma^{abcd} \chi \right).$$  \hfill (A.6)

A.3 $SO(7)$ spinors

These spinors are Majorana and Grassman even. Gamma matrices are $\tilde{\Gamma}^i$ and $\gamma^i$, charge conjugation is $C$. The $SO(7)$ gamma matrices satisfy the following Hodge Duality relations

$$\gamma_{i_1 \ldots i_n} = -\frac{4}{(r-n)!} i \epsilon_{i_1 \ldots i_r} \gamma^{i_{r+1}} \cdots \gamma^{i_n}.$$  \hfill (A.7)

For $\bar{\lambda} \gamma^{(n)} \chi = t^{(n)} \bar{\chi} \gamma^{(n)} \lambda$, we find the following transposition properties

| $n$ | 0.4 | 1.5 | 2.6 | 3.7 |
|-----|------|---|---|---|
| $t^{(n)}$ | + | - | - | + |

and the Fierz identity is given by

$$\lambda \bar{\chi} = \frac{1}{8} \left( \bar{\chi} \lambda + \bar{\chi} \gamma^i \lambda \gamma_i - \frac{1}{2} \bar{\chi} \gamma^{ij} \lambda \gamma_{ij} - \frac{1}{6} \bar{\chi} \gamma^{ijk} \lambda \gamma_{ijk} \right).$$  \hfill (A.8)
References

[1] I. Bakas, Phys. Lett. B343 (1995) 103, hep-th/9410104.
[2] E. Bergshoeff, R. Kallosh, T. Ortín, Phys. Rev. D51 (1995) 3009, hep-th/9410230.
[3] E. Alvarez, L. Alvarez-Gaumé, Y. Lozano, Phys. Lett. B336 (1994) 183, hep-th/9406206.
[4] I. Bakas, K. Sfetsos, Phys. Lett. B349 (1995) 448, hep-th/9502065.
[5] S. Hassan, Nucl. Phys. B460 (1996) 362, hep-th/9504148.
[6] E. Alvarez, L. Alvarez-Gaumé, I. Bakas, Nucl. Phys. B457 (1995) 3, hep-th/9507112.
[7] K. Sfetsos, Nucl. Phys. B463 (1996) 33, hep-th/9510034.
[8] G. Gibbons, D. Wiltshire, Nucl. Phys. B287 (1987) 717, hep-th/0109093.
[9] G. Gibbons, K. Maeda, Nucl. Phys. B298 (1988) 741.
[10] F. Dowker, J. Gauntlett, S. Giddings, G. Horowitz, Phys. Rev. D50 (1994) 2662, hep-th/9312172.
[11] M. Melvin, Phys. Lett. 8 (1964) 65.
[12] F. Dowker, J. Gauntlett, D. Kastor, J. Traschen, Phys. Rev. D49 (1994) 2909, hep-th/9309075.
[13] F. Dowker, J. Gauntlett, G. Gibbons, G. Horowitz, Phys. Rev. D52 (1995) 6929, hep-th/9507143.
[14] F. Dowker, J. Gauntlett, G. Gibbons, G. Horowitz, Phys. Rev. D53 (1996) 7115, hep-th/9512154.
[15] J. Russo, A. Tseytlin, JHEP 9804 (1998) 014, hep-th/9804076.
[16] M. Costa, M. Gutperle, JHEP 0103 (2001) 027, hep-th/0012072.
[17] M. Costa, C. Herdeiro, L. Cornalba, Nucl. Phys. B619 (2001) 155, hep-th/0105023.
[18] R. Emparan, Nucl. Phys. B610 (2001) 169, hep-th/0105062.
[19] D. Brecher, P. Saffin, Nucl. Phys. B613 (2001) 218, hep-th/0106206.
[20] R. Myers, JHEP 9912 (1999) 022, hep-th/9910053.
[21] M. Gutperle, A. Strominger, JHEP 0106 (2001) 035, hep-th/0104136.
[22] A. Uranga, Wrapped fluxbranes, hep-th/0108196.
[23] J. Russo, A. Tseytlin, JHEP 0111 (2001) 065, hep-th/0110107.
[24] J. Figueroa-O’Farrill, J. Simón, JHEP 0112 (2001) 011, hep-th/0110170.
[25] P. Townsend, Phys. Lett. B373 (1996) 68, hep-th/9512062.
[26] P. Townsend, The story of M, hep-th/0205309.
[27] A. Van Proeyen, Tools for supersymmetry, hep-th/9910030.
[28] J. Figueroa-O’Farrill, J. Simón, Supersymmetric Kaluza-Klein reductions of M2 and M5-branes, hep-th/0208107.
[29] J. Figueroa-O’Farrill, J. Simón, Class. Quant. Grav. 19 (2002) 6147, hep-th/0208108.