Averaging out Inhomogeneous Newtonian Cosmologies: II. Newtonian Cosmology and the Navier-Stokes-Poisson Equations

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Abstract
The basic concepts and equations of Newtonian Cosmology are presented in the form necessary for the derivation and analysis of the averaged Navier-Stokes-Poisson equations. A particular attention is paid to the physical and cosmological hypotheses about the structure of Newtonian universes. The system of the Navier-Stokes-Poisson equations governing the cosmological dynamics of Newtonian universes is presented and discussed. A reformulation of the Navier-Stokes-Poisson equations in terms of the fluid kinematic quantities is given and the structure of this system of equations is analyzed.

1 Introduction
In order to derive a system of the averaged Navier-Stokes-Poisson equations and to make use of the system for analysis of inhomogeneous Newtonian universes, it is significant to present the basic concepts and equations of Newtonian Cosmology in the form necessary to approach this problem. In this presentation a particular attention will be paid to the physical and cosmological hypotheses about the structure of Newtonian universes which are spatially infinite configurations of the self-gravitating compressible Newtonian cosmological fluid without viscosity and heat transfer. The system of the Navier-Stokes-Poisson equations governing the cosmological dynamics of Newtonian universes is presented and discussed. A reformulation of the Navier-Stokes-Poisson equations in terms of the fluid kinematic quantities is given and the structure of this system of equations is analyzed.

The structure of this paper is as follows. Chapter 2 describes the full set of the Navier-Stokes-Poisson equations for the self-gravitating compressible Newtonian fluid without viscosity and heat transfer under the gravitational field of its own Newtonian gravitational
potential. The definition of the Newtonian-space and analysis of the coordinate freedom allowed by the Kinematical group of the Navier-Stokes-Poisson equations is carried out in Chapter 3. It is pointed out, in particular, that the Newtonian gravitational potential is not uniquely defined for spatially unbounded fluid configurations. Chapter 4 is devoted to the discussion of the initial boundary value problem for the Navier-Stokes-Poisson equations which is not well-posed and the Heckmann-Schücking boundary conditions for homogeneous and isotropic fluid configurations are formulated. The main hypotheses of Newtonian cosmology are formulated in Chapter 5. In Chapter 6 the Newtonian cosmological principle is formulated and the class of the homogeneous and isotropic Newtonian universes is studied. A definition of inhomogeneous and/or anisotropic Newtonian universes is given in Chapter 7. The nonexistence of the static homogeneous and isotropic Newtonian universe with the vanishing Newtonian cosmological constant is proved in Chapter 8. Chapter 9 makes a brief comparison of Newtonian and general relativistic cosmologies and gives evidence in favor of the remarkable physical similarity of the both cosmological pictures. The full system of the Navier-Stokes-Poisson equations in terms of kinematic quantities is derived in Chapter 10. The next Chapter gives a detailed discussion of this system of equations. It is pointed out, in particular, that in order to become closed it requires an additional evolution equation for the tidal force tensor which is constructed from the gravitational potential and is an analogue of the space-time curvature tensor of general relativity. It means that the cosmological fluid configurations governed by the Navier-Stokes-Poisson equations in terms of kinematic quantities do evolve due to a generalized Newtonian gravitational force. In the last Chapter 12 the Raychaudhuri evolution equation of Newtonian cosmology is shown to be equivalent to the Friedmann equation for the cosmological scale factor for homogeneous and isotropic Newtonian universes.

The formulae and Sections from the paper I [1] will be referred to as (I-XX) and I-X, correspondingly.

Conventions and notations are as follows. All functions \( f = f(x^i, t) \) are defined on 3-dimensional Euclidean space \( E^3 \) in the Cartesian coordinates \( \{x^i\} \) with Latin space indices \( i, j, k, \ldots \) running from 1 to 3, and \( t \) is the time variable. The Levi-Civita symbol \( \varepsilon_{ijk} \) is defined as \( \varepsilon_{123} = +1 \) and \( \varepsilon_{123} = +1 \) and \( \delta^{ij}, \delta_i \) and \( \delta^{ij} \) are the Kronecker symbols. The symmetrization of indices of a tensor \( T^i_{jk} = T^i_{jk}(x^i, t) \) is denoted by round brackets, \( T^i_{(jk)} = \frac{1}{2} \left( T^i_{jk} + T^i_{kj} \right) \), and the antisymmetrization by square brackets, \( T^i_{[jk]} = \frac{1}{2} \left( T^i_{jk} - T^i_{kj} \right) \). A partial derivative of \( T^i_{jk} \) with respect to a spatial coordinate \( x^i \) or time \( t \) is denoted either by comma, or by the standard calculus notation, \( T^i_{jk,t} = \partial T^i_{jk}/\partial t \) and \( T^i_{jk,t} = \partial T^i_{jk}/\partial x^i \), the fluid velocity is \( u^i = u^i(x^j, t) \), and the material (total) derivative is \( T^i_{jk,t} \equiv dT^i_{jk}/dt = \partial T^i_{jk}/\partial t + u^i\partial T^i_{jk}/\partial x^i = T^i_{jk,t} + u^i T^i_{jk,t} \). The Newton gravitational constant, the velocity of light and the Newtonian cosmological constant are \( G, c \) and \( \Lambda \), respectively.
2 The Navier-Stokes-Poisson Equations

Let us consider the self-gravitating compressible Newtonian fluid without viscosity and heat transfer under the gravitational field of its own Newtonian gravitational potential \( \phi = \phi(x^i, t) \). The motion of the fluid is determined by the fluid velocity field \( u^i = u^i(x^j, t) \) satisfying the compressibility condition (I-18),

\[
\frac{\partial u^i}{\partial x^i} \neq 0,
\]

everywhere and by the fluid density field \( \rho = \rho(x^i, t) \) positive everywhere (I-29),

\[
\rho(x^i, t) > 0.
\]

The symmetric fluid stress tensor \( T^{ij} = T^{ij}(x^k, t) \) (I-40) satisfying the Boltzmann postulate (I-75), \( T^{ij} = T^{ji} \), is assumed to be the fluid stress tensor of the perfect fluid, that is, the Newtonian fluid (I-83) with the vanishing viscosity coefficients, \( \lambda(x^i, t) = 0 \) and \( \mu(x^i, t) = 0 \), the vanishing heat flux vector \( q^i(x^j, t) = 0 \) and the constant temperature \( T = \text{const.} \) (I-79) such that

\[
T^{ij} = -\delta^{ij} p.
\]

An equation of state relating the fluid pressure \( p = p(x^i, t) \) and the fluid density \( \rho = \rho(x^i, t) \) is assumed to be defined (I-81) as

The equation of state

\[
p = p(\rho) \quad \text{or} \quad \rho = \rho(p).
\]

The external force \( F^i = F^i(x^j, t) \) for the self-gravitating fluid is defined to be due to the Newtonian gravitational potential \( \phi = \phi(x^i, t) \),

\[
F^i = -\delta^{ij} g_j,
\]

where \( g_i = g_i(x^j, t) \) is the Newtonian gravitational acceleration vector of a fluid particle moving with the velocity \( u^i(x^j, t) \). The acceleration vector has the following property.

The identity for the Newtonian gravitational acceleration \( g_i \),

\[
\varepsilon^{ijk} g_{j,k} = 0,
\]

which means that the Newtonian gravitational field is always locally represented by its scalar potential \( \phi(x^i, t) \)

\[
g_i = \phi_{,i}.
\]

The Newtonian gravitational acceleration vector \( g_i(x^j, t) \) and the Newtonian gravitational potential \( \phi(x^i, t) \) satisfy the Poisson equation with the Newtonian cosmological constant \( \Lambda \)
The Poisson equation for the Newtonian gravitational potential $\phi$

$$\delta^{ij} g_{i,j} = 4\pi G \rho - \Lambda, \quad \text{or} \quad \delta^{ij} \phi_{,ij} = 4\pi G \rho - \Lambda. \quad (8)$$

The conservation of fluid mass (I-30) brings the equation of continuity (I-31) for the fluid density $\rho(x^i, t)$

The equation of continuity for the fluid density $\rho$

$$\rho_t + (\rho u^i)_{,i} = 0, \quad (9)$$

and the conservation of linear momentum (I-37) brings the Cauchy equation of motion (I-41) which for the fluid stress tensor (8) and the external force (5) becomes

The Navier-Stokes equation of motion for the fluid velocity $u^i$

$$u_{i,t} + u_{i,j}u^j = -g_j - \frac{1}{\rho} p_{,j} \quad \text{or} \quad u_{i,t} + u_{i,j}u^j = -\phi_{,i} - \frac{1}{\rho} p_{,j}. \quad (10)$$

Another form of the Navier-Stokes equation (10) can be written by using the total fluid acceleration vector $A^i = A^i(x^j, t)$ (I-45) for a fluid particle,

$$A_i = \frac{du_i}{dt} + g_i \equiv a_i + g_i. \quad (11)$$

The total fluid acceleration vector $A^i(x^j, t)$ can be defined (I-46) by the Navier-Stokes equation of motion (10) as

The total fluid acceleration $A_i$ as determined by the Navier-Stokes equation

$$A_i = -\frac{1}{\rho} p_{,i}. \quad (12)$$

This form of the Navier-Stokes equation means from the physical point of view that the total acceleration $A_i$ of a fluid particle due to the combined effects of the inertial and gravitational forces represented by the fluid particle’s acceleration $a_i$ and the Newtonian gravitational acceleration vector $g_i$ is determined by the gradient of the fluid pressure. Due to the Navier-Stokes equation of motion (12) the total fluid particle acceleration (11) is always directed away from a region with higher pressure towards a region with lower pressure.

1 Though the equation of motion (10) is the Euler equation for the perfect fluid (I-80), it will be called here the Navier-Stokes equation of motion in accordance with Eq. (I-83) where the viscosity coefficients are assumed to vanish and the external force is due to the Newtonian gravitational potential $\phi$. 

4
The system of seven first order partial differential equations (6), (8), (9) and (10) has seven unknowns \( u^i(x^j, t) \), \( \rho(x^i, t) \) and \( g_i(x^j, t) \) with a given equation of state (1). It is equivalent to a system of five first and second order equations (8), (9) and (10) for five unknowns \( u^i(x^j, t) \), \( \rho(x^i, t) \) and \( \phi(x^i, t) \). This system is of a mixed hyperbolic-elliptic type and in order to be solved in a physical setting of interest it should be supplemented by the corresponding initial and boundary conditions for the unknowns. As is has been discussed in Section I-13 the problem of solving the equations of classical hydrodynamics is very difficult. The Poisson equation brings additional difficulties when one is looking for a solution for a self-gravitating fluid configuration satisfying the system of the Navier-Stokes-Poisson (4), (8), (9) and (10). A few results have been proved rigorously for compact Friedmann-like solutions and their finite perturbations for the polytropic equation of state, in particular, the linearization stability, a local-in-time existence of the solutions given a set of data not precisely equivalent to initial data, the occurrence of kinematical singularities (see [2]-[5] and reference therein). Conditions for the equilibrium of the self-gravitating rotating systems for particular classes of the equations of state have been also studied (see [6] and references therein).

3 The Newtonian Space-time and the Kinematical Group

The Navier-Stokes-Poisson equations (8), (9) and (10) have been formulated in the framework of the field description of fluid motion when all objects characterizing the fluid and its properties are defined with respect to a position \( \{x^i\} \) at a time \( t \), see Section I-5. Before analyzing and solving the equations one must analyze the mechanics of the fluid motion and reveal the freedom in the choice of possible Eulerian coordinate systems, which is available as far as the equations are formulated in this framework.

As it has been pointed out in Section I-9, the Cauchy equation of motion (I-41) is the Newton second law for a moving fluid particle. From the point of view of Newtonian mechanics the dynamics of the whole fluid configuration under consideration can be represented in the field description by a sequence of its space configurations given as 3-dimensional spaces represented by the full set of all fluid particle’s positions \( \{x^i\} \) which are occupied at each instant of time \( t \). The time \( t \) at positions \( \{x^i\} \) is assumed to be measured by a family of ideal clocks each of which indicates values of time regardless of the previous motion. This is the concept of absolute time. On the basis of this picture of the fluid motion one can naturally define the notion of a Newtonian space-time manifold \( \mathbb{R}^4 \) as follows [7], [10].

The Newtonian space-time A Newtonian space-time is defined as a 4-dimensional differentiable manifold of the following structure:

(a) the real-valued absolute time \( t \) is determined up to a linear transformation \( t \rightarrow \)}
Let us consider now the Cauchy equation of motion (I-41) in the absence of all external and internal forces,

$$\rho \frac{du^i}{dt} = 0 \quad \text{as} \quad F^i = 0, \quad T^{ij} = 0.$$  

(13)

which corresponds the state of free motion of each fluid particle. The following theorem takes place [7].

**Theorem 1 (The inertial coordinates and the Galilean group)** If a fluid is in free motion (13) there always exists a global system of the Eulerian coordinates \(\{x^i, t\}\) called the inertial coordinates which are defined up to the Galilean group of arbitrary linear transformation,

$$x^i \rightarrow A^i_j x^j + B^i + C^i, \quad t \rightarrow at + b,$$

(14)

where \(A^i_j\), \(B^i\) and \(C^i\) are real and constant, and the matrix \(A^i_j\) is nonsingular.

**Proof.** The group of Galilean transformations \((14)\) is found as the general solution of the Cauchy equation for free motion \((13)\) for the Eulerian space coordinates \(\{x^i\}\) along fluid particle paths \((1-2)\), \(x^i = x^i(t)\),

$$\frac{d^2x^i}{dt^2} = 0,$$

(15)

and the law of allowed transformations of the absolute time given above. \textbf{QED}

Evidently the paths of fluid particles moving freely \((13)\) are straight lines in the Newtonian space-time of the fluid.

In the presence of gravitation when a self-gravitating fluid moves in its own Newtonian potential \(\phi = \phi(x^k, t)\) in accordance with the Cauchy equation of motion \((I-41)\) with the external force \((3)\) and the fluid stress tensor \(T^{ij} = 0\),

$$\frac{du^i}{dt} = -\delta^{ij} \frac{\partial \phi}{\partial x^j} \quad \text{as} \quad F^i = -\delta^{ij} g_i, \quad T^{ij} = 0.$$  

(16)

one cannot determine a state of free motion in the sense \((13)\) and \((14)\) and define the class of inertial coordinates \((14)\). It is natural, however, to define another class of privileged coordinates which corresponds to the state of free fall \((16)\) of fluid particles in the gravitational field. It should be pointed out here that the definition of free fall assumes that the inertial and gravitational masses of any fluid particle are the same so that given the same gravitational field any particles move along the same paths if their initial positions and velocities were the same. The following theorem determines the corresponding class of coordinates and its transformation group [3], [10]–[12].
Theorem 2 (The free fall coordinates and the Kinematical group) If a fluid is in free fall motion \((16)\) there always exists a global system of the Eulerian coordinates \(\{x^i, t\}\) called the free fall coordinates which are defined up to the Kinematical group of arbitrary linear transformation,

\[ x^i \rightarrow A^i_j x^j + D^i(t), \quad t \rightarrow at + b, \tag{17} \]

where \(A^i_j\) is the real-valued nonsingular matrix \(A^i_j\) and \(D^i(t)\) is a real-valued arbitrary function of time \(t\). Under the change of coordinates \((17)\) the gravitational potential \(\phi(x^k, t)\) undergoes the transformation

\[ \phi \rightarrow \phi - \delta_{ij} x^i \frac{d^2 D^j}{dt^2}. \tag{18} \]

Proof. The group of kinematical transformations \((17)\) and transformations of the potential \((18)\) are found as the general solution of the Cauchy equation for free fall motion \((16)\) for the Eulerian space coordinates \(\{x^i\}\) of a fluid particle path \((??)\), \(x^i = x^i(t)\),

\[ \frac{d^2 x^i}{dt^2} = -\delta_{ij} \frac{\partial \phi}{\partial x^i}. \tag{19} \]

and the law of allowed transformations of the absolute time given above. QED

It should be pointed out here that the form of the Cauchy equation of motion \((16)\) remains the same in any free fall coordinates \((17)\) and \((18)\).

Thus, in the presence of gravitation the coordinates of a position \(\{x^i\}\) of a free falling fluid particle are defined up to the time-independent rotation \(x^i \rightarrow A^i_j x^j\) at a given instant of time \(t\), and up to the time-dependent translation \(x^i \rightarrow x^i + D^i(t)\) possibly different for different moments of time \(t\). The Newtonian potential is not a coordinate-free function, but rather defined in dependence of a particular choice of the Eulerian coordinates \(\{x^i, t\}\) and it changes due to \((18)\) upon their change. There is, however, a particular class of the self-gravitating fluid configurations called the isolated fluid configurations when a fluid occupies a bounded compact region with a unique common Newtonian potential \([7]\), \([11]\), \([12]\).

Corollary 1 (The isolated fluid configurations) The Newtonian gravitational potential \(\phi \) \((18)\) for the isolated fluid configurations satisfying the global boundary condition at spatial infinity

\[ \phi(x^k, t) \rightarrow 0 \quad \text{as} \quad (\delta_{ij} x^i x^j)^{1/2} \rightarrow \infty \tag{20} \]

is uniquely defined,

\[ \phi \rightarrow \phi. \tag{21} \]

Then the Kinematical group \((17)\) reduces to the Galilean group \((14)\),

\[ D^i(t) = B^i t + C^i. \tag{22} \]
For the case of isolated fluid configurations \( \{20\} \) one is able to distinguish in an invariant manner the inertial and gravitational parts of the total acceleration \( A^i \) \( \{11\} \). Indeed, the left-hand side of the Cauchy equation of motion for the free fall \( \{19\} \) which is an inertial acceleration \( a^i \) and its right-hand side which is a Newtonian gravitational acceleration \( g^i \) are kept invariant independently under the Kinematical group \( \{17\} \) with \( \{22\} \).

### 4 The Initial Boundary Value Problem

In general case of the noncompact fluid configurations when the self-gravitating fluid and therefore the gravitational field are extended all over a noncompact 3-dimensional Euclidean space \( E^3 \) such a boundary condition at spatial infinity \( \{20\} \) cannot be imposed. Such cosmological fluid configurations are considered in modelling Newtonian universes represented by Newtonian space-times filled with the matter in the form of self-gravitating cosmological fluid distributions over in noncompact 3-dimensional Euclidean spaces \( E^3 \) evolving during an interval of time possibly infinite. In those cases one is forced to make a choice either to abandon the concept of the inertial Eulerian coordinates \( \{14\} \) or to announce the free fall coordinate systems \( \{17\} \) inertial. The choice is usually made in favour of the latter \[7\], \[11\]-\[13\]. From the physical point of view, such a choice means that all observers belonging the same 3-dimensional Euclidean space \( E^3 \) at \( t = \text{const} \) are inertial though such inertial observers may accelerate relatively to each other when they move with the fluid due its gravitation.

The initial boundary value problem for the system of the Navier-Stokes-Poisson equations \( \{4\}, \{8\}, \{11\} \) and \( \{14\} \) is known not to be well-posed. A 3-dimensional Euclidean space \( t = \text{const} \) is a characteristic surface of the system and it is not permitted to to set initial data on such a surface \[14\]. As a result, the potential is known up to an arbitrary harmonic function \( \psi = \psi(x^i, t) \) satisfying a boundary value problem for the Laplace equation \( \delta_{ij} \psi_{,ij} = 0 \) since it is not defined uniquely \( \{18\} \). Now with the partial time derivatives of the fluid density \( \partial \rho / \partial t \) and the fluid velocity \( \partial u^i / \partial t \) being determined from the equation of continuity \( \{1\} \) and the Navier-Stokes equation of motion \( \{11\} \), the time derivative of the potential \( \partial \phi / \partial t \) must satisfy a boundary value problem for the Poisson equation \( \{8\} \) without \( \Lambda \) term,

\[
\delta_{ij} \frac{\partial^2}{\partial x^i \partial x^j} \frac{\partial \phi}{\partial t} = 4\pi G \frac{\partial \rho}{\partial t}.
\]  

which gives rise again to an arbitrary harmonic function \( \psi_1 = \psi_1(x^i, t) \) satisfying a boundary value problem for the Laplace equation \( \delta_{ij} (\psi_1,t)_{,ij} = 0 \) due to \( \{18\} \). Such a process eventually leads to an infinite number of boundary value problems.

A physical manifestation of the problem with the bad-posedness of the initial value problem for the Navier-Stokes-Poisson equations can be understood if one considers the simplest possible noncompact fluid configuration, that is, a self-gravitating fluid with constant density, \( \rho = \rho(x^i, t) = \text{const} \). On the basis of physical arguments one expects a homogeneous distribution of the self-gravitating fluid throughout the all 3-dimensional
spaces $t = \text{const}$, since a uniform, constant density does not show any particular point in the fluid distribution as preferable. As there is no preferable point in the fluid distribution, it is expected to move as a whole and there should be a particular inertial frame where the fluid is at the state of rest at a time $t = t_0$. A natural initial condition for the fluid velocity therefore is $u^i(x^j, t_0) = 0$. The Navier-Stokes-Poisson equations (8), (9) and (10) in this case read

$$\delta^{ij} \frac{\partial^2 \phi}{\partial x^i \partial x^j} = 4\pi G \rho - \Lambda, \quad \frac{du^i}{dt} = -\frac{\partial \phi}{\partial x^i}, \quad \text{for } \rho(x^i, t) = \text{const}, \quad u^i(x^j, t_0) = 0. \quad (24)$$

The Poisson equation in the system (24) can be solved in the spherical coordinates $(r, \theta, \phi)$ and the resulting solution the Navier-Stokes-Poisson equations

$$u^i(x^j, t) = -\frac{1}{3} (4\pi G \rho - \Lambda) x^i(t - t_0), \quad \phi(x^i) = \frac{1}{6} (4\pi G \rho - \Lambda) r^2, \quad (25)$$

in contradiction with the initial assumptions for the classical Poisson equation (8) when the Newtonian cosmological constant vanishes, $\Lambda = 0$. Indeed, first of all, the solution (25) appears to distinguish the origin of an arbitrarily chosen coordinate system from other points since the gravitational potential vanishes, $\phi(x^i) = 0$, at $r = (\delta_{ij} x^i x^j)^{1/2} = 0$. Secondly, the fluid cannot be at rest at a 3-dimensional space $t = t_0 = \text{const}$, but rather moves with a constant acceleration $a^i(x^k) = -4\pi Gr x^i/3$ away from the point with $r = 0$ where the gravitational potential $\phi(x^i) = 0$.

If now one assumes the condition $4\pi G \rho = \Lambda$, the situation improves, since in this case the solution (25) shows that $u^i(x^j, t) = 0$ and $\phi(x^i) = 0$ everywhere in agreement with the physical analysis made before approaching the equations. It should be emphasized here that the presence of the Newtonian cosmological constant $\Lambda$ in the Poisson equation means a modification of the classical Poisson equation of Newtonian gravity where $\Lambda = 0$. This issue is discussed later in this Section.

Therefore, any way to improve the situation with the initial boundary value problem for the Poisson equation and, as a result, with the Navier-Stokes-Poisson equations, to allow a proper analysis of self-gravitating fluid configurations defined on noncompact 3-dimensional spaces $t = \text{const}$, would assume a modification of the Poisson equation either in its structure, or in the structure of its boundary conditions. The latter also would result in an effective change in structure of the Poisson equation, as any boundary condition on the gravitational potential itself $\phi(x^k, t)$ could not improve the situation as it is clear from the above analysis.

The attempts of the first kind are being made since a long ago, for the unsatisfactory situation with the application of the classical Poisson equation to unbounded systems has been realized very soon after the discovery of this equation. In 1895 Seeliger and Neumann [15], [16] proposed the modified Poisson equation

$$\delta^{ij} \frac{\partial^2 \phi}{\partial x^i \partial x^j} - \chi \phi = 4\pi G \rho, \quad \chi = \text{const}, \quad (26)$$

which assumes a finite range $\chi^{-1/2}$ of this gravitational force as compared with an infinite range of the Newtonian gravitational potential. This equation can be shown to have the
constant solution for the constant fluid density, \( \rho = \rho(x^k, t) = \text{const} \),

\[
  u^i(x^k, t) = 0, \quad \phi(x^k) = \frac{4\pi}{\chi} G \rho = \text{const}, \tag{27}
\]

which in agreement with the expected homogeneous fluid distribution in the state of rest once it was at rest initially. However, one can show that this equation cannot be obtained as a weak field limit of the Einstein equations of general relativity even in the presence of the cosmological constant \( \Lambda = \lambda c^2 \) where \( \lambda \) is the cosmological constant entering the Einstein equations. It is the the Poisson equation (8) that follows as the weak field limit of the Einstein equations. This limiting procedure is singular in the sense that the Einstein equations for perfect fluids with an equation of state (4) have a well-posed initial boundary problem \([17], [18]\), but the Poisson equation has not. It has been assumed therefore that a modification of the Poisson equation must be dictated by a reconsideration of its derivation from the Einstein equations by taking into account the post-Newtonian approximation terms which provide the weak-field correction terms next to the Newtonian approximation. It has been shown \([19]\) that one can construct a generalization of the Navier-Stokes-Poisson equations which does possess a well-defined initial boundary problem, but those equations have very different structure and nonlinear in the gravitational potential itself and in its products with the fluid velocity. Another generalization \([20]\) of the Navier-Stokes-Poisson equations gives a system of equations very similar to the linearized Einstein equations which provide an equation for propagation of the gravitational potential, to permit gravitational waves and a finite speed of gravitational interaction as compared with the infinite speed propagating Newtonian gravitation with no gravitational radiation.

Attempts to modify the boundary conditions for the gravitational potential have proved to be more fruitful because in this approach one can usually assume a condition on the basis of physical arguments \([21]\). First of all, it should be noted that using a simple generalization of the boundary condition (21) such as \( \phi \rightarrow \phi(t)|_\infty \) as \( (\delta_{ij}x^i x^j)^{1/2} \rightarrow \infty \) or a prescription for the value of \( \lim (\delta_{ij}x^i x^j)^{1/2} \rightarrow \infty \), does not work because the potential \( \phi \) diverges at spatial infinity and \( \delta^{kl} \phi_{,kl} \) is determined by the limiting fluid density at infinity through the Poisson equation. Indeed, the structure of the Poisson equation (8) is known to determine the divergence of the acceleration, \( \delta^{ij} g_{i,j} = \delta^{ij} \dot{\phi}_{,ij} \), while the so-called tidal force tensor \( E_{ij} = E_{ij}(x^k, t) \) defined as

**The tidal force tensor \( E_{ij} \) for the Newtonian gravitational potential \( \phi \)**

\[
  E_{ij} = \phi_{,ij} - \frac{1}{3} \delta_{ij} \delta^{kl} \phi_{,kl}, \quad E_{ij} = E_{ji}, \quad \delta^{kl} E_{kl} = 0, \tag{28}
\]

is left undetermined, the tensor being an analogue of the space-time curvature of general relativity. Therefore eight component of the trace-free symmetric tensor \( E_{ij} \) are not determined by the Poisson equation and they are therefore pure kinematic quantities in the Newtonian gravity. The absence of the field equation to govern the tidal force tensor \( E_{ij} \) (28) may be considered as a reflection of the above problems. On the other hand, one can
use the freedom in $E_{ij}$ to get a suitable boundary conditions for particular classes of spatially noncompact self-gravitating fluid configurations. The suitable boundary conditions that allow one to handle spatially homogeneous fluid configurations which necessarily have the uniform fluid density have been formulated by Heckmann and Schücking [11], [12], [21].

**The Heckmann-Schücking boundary condition for the tidal force tensor $E_{ij}$**

The following boundary condition is assumed to hold for spatially homogeneous and isotropic fluid configurations

$$E_{ij}(x^k, t) \to 0 \quad \text{as} \quad (\delta_{ij}x^ix^j)^{1/2} \to \infty. \quad (29)$$

This boundary condition (29) enables one to describe consistently the unbounded homogeneous and isotropic distributions of the self-gravitating fluid. When taken as cosmological models, such cosmological fluid configurations are Newtonian analogues [1], [11]-[13] of the Friedmann-Lemaitre-Robertson-Walker (FLRW) cosmological models of general relativity [22]-[26]. The conditions, however, can be shown to exclude the spatially homogeneous but anisotropic self-gravitating cosmological fluid configurations. Such Newtonian configurations corresponds to the Kantowski-Sachs and Bianchi spatially homogeneous but anisotropic cosmological models of general relativity [26]-[28]. A generalization of the boundary condition (29) to cover this broader class of fluid configurations is also known [11], [12], [21].

**The generalized Heckmann-Schücking boundary condition for the tidal force tensor $E_{ij}$**

The following boundary condition is assumed to hold for spatially homogeneous anisotropic fluid configurations

$$E_{ij}(x^k, t) \to E(t)_{ij}\big|_{\infty} \quad \text{as} \quad (\delta_{ij}x^ix^j)^{1/2} \to \infty. \quad (30)$$

Here $E(t)_{ij}\big|_{\infty}$ are arbitrary functions of time. The boundary conditions (29) and (30) are considered as physically adequate for this class of cosmological fluid configurations (see [21] for a discussion and further references). In both cases information immediately propagates in from infinity to determine the local physical evolution of a moving fluid. In this approach no new equation for the propagation of the tidal force tensor $E_{ij}$ is explicitly necessary if one remains in the framework of Newtonian gravity, though it leads to impossibility to decide a priori which boundary condition fits a physical setting under consideration. If such a propagation equation for $\partial E_{ij}/\partial t$ is defined in a framework of the generalized Newtonian gravity, then a physical setting would dictate a particular boundary condition at infinity, to determine nonlocally through this condition the local dynamics of a self-gravitating cosmological fluid.
5 The Newtonian Cosmology

Cosmology is a theory of the evolution and structure of our Universe. In dependence on physical and mathematical hypotheses about the structure of the Universe space-time, the content and distribution of cosmological matter and the set of dynamical equations governing the Universe evolution one can formulate different theoretical frameworks for derivation and analysis of cosmological models to compare their results with observations and experimental data (see [7], [13], [29]-[34] for a discussion and references). The following hypotheses are assumed for Newtonian universes, the cosmological models of Newtonian Cosmology.

The Newtonian cosmological space-time hypothesis Newtonian universes are assumed to have the geometry of Newtonian space-time.

A Newtonian universe therefore always possesses an absolute time $t$. A state of a Newtonian universe at every instant of the absolute time $t$ is given by a 3-dimensional Euclidean space $E^3$ and can be therefore described by the Eulerian coordinates $(x^k, t)$ (see Section 3).

The Newtonian cosmological matter hypothesis A Newtonian universe is assumed to be filled with a self-gravitating compressible Newtonian cosmological fluid without viscosity and heat transfer under the gravitational field of its own Newtonian gravitational potential.

Though generally a cosmological fluid can be assumed Stokean, see Section I-13, the Newtonian cosmological fluid is taken as a physically reasonable model for the matter in our Universe and due to its own gravitation as the only reason of the universe evolution. The next hypothesis determines the physical laws of fluid dynamics and its gravitation.

The Newtonian cosmological dynamics hypothesis A Newtonian universe is assumed to be governed by the Navier-Stokes-Poisson equations (4), (8), (9) and (10) supplemented by a set of initial and boundary conditions for a cosmological fluid configuration under study.

Thus, a particular cosmological fluid configuration determines a particular Newtonian universe. A particular cosmological fluid velocity field $u^i(x^j, t)$, the cosmological fluid density distribution $\rho(x^j, t)$ and the Newtonian gravitational potential field $\phi(x^j, t)$ given by a solution to the system of the Navier-Stokes-Poisson equations (4), (8), (9) and (10) for a Newtonian cosmological model determine the evolution of the Newtonian universe as that of the cosmological fluid configuration. The fluid velocity field $u^i(x^j, t)$ determines the equation of motion of cosmological fluid particles $x^i = x^i(\xi^j, t)$ and their trajectories (I-1)-(I-6) through the differential equation (I-10),

$$\frac{dx^i}{dt} = u^i(x^j, t), \quad x^i(0) = \xi^i,$$

(31)
to predict the state of the Newtonian universe for moments of the time interval of the cosmological evolution allowed by the initial and boundary conditions in dependence on
the initial positions \( x^i(0) = \xi^i \) of the cosmological fluid particles. With a particular distribution of the cosmological fluid density \( \rho(x^j, t) \), the motion of the fluid particles taken as matter constituents should be compared with the observed motion of the real cosmological matter to conclude about the physical adequacy of the cosmological model and interpret and/or predict the data.

6 The Newtonian Cosmological Principle

The cosmological matter in our Universe is distributed in a highly inhomogeneous manner on the scales characteristic of typical matter condensations such as stars, galaxies, cluster of galaxies, etc. However, for the largest scales the overall distributions of the matter structures is assumed to be homogeneous and isotropic [13], [29]-[31]. This fundamental assumption about the large-scale structure of the real Universe is called the Cosmological Principle. Let us formulate this Principle in the framework of Newtonian Cosmology [7], [13], [35], [36].

The Newtonian cosmological principle For any observer moving with a cosmological fluid particle the Newtonian universe appears to be the same at different points of space, at different instances and at different directions. Any other observer moving with another cosmological fluid particle at the same time observes the same appearance of the Newtonian universe.

The Newtonian cosmological principle assumes the existence of a privileged family of observers who, at each instant of time, observe the Newtonian universe as homogeneous and isotropic. The distributions of the cosmological fluid over 3-dimensional Euclidean spaces at each instant of time are therefore homogeneous and isotropic. However, that does not mean that the cosmological fluid is static. In general, it does evolve in time since the structure of the Newtonian space-time permits nontrivial time shifts (17) between the 3-dimensinal Euclidean spaces \( t = \text{const} \). A fundamental theorem based on the result of McCrean and Milne [35], [36] establishes the law of the evolution of a Newtonian universe satisfying the Newtonian cosmological principle [7], [13]. Up to a different interpretation of one term, this equation has the same form as the Friedmann equation [22] governing the evolution of the FLRW universes which are homogeneous and isotropic cosmological models in the framework of general relativity [13], [29], [30], [37].

Theorem 3 (The homogeneous and isotropic Newtonian universes) The evolution of a Newtonian universe satisfying the Newtonian cosmological principle is governed by the Friedmann equation

\[
\left( \frac{dR}{dt} \right)^2 = \frac{8\pi G}{3R} \rho_0 - k + \frac{1}{3} \Lambda R^2
\]  

(32)

where \( R = R(t) \) is the so-called scale factor of the cosmological fluid, \( k \) is the constant of integration which represents the total energy of fluid particles, \( \rho_0 = \rho(t_0) \) is the initial value of the cosmological fluid density.
Proof. Let us consider a Newtonian universe filled with a cosmological fluid satisfying the Newtonian cosmological principle an governed by the Navier-Stokes-Poisson equations (4), (8), (9) and (10). An observer moving with a fluid particle and having set up a system of the Eulerian coordinates \((x^k, t)\) with the origin at the fluid particle measures the physical properties of a fluid particle at a position \(\{x^k\}\) as its the velocity \(u^i(x^k, t)\), the density \(\rho(x^k, t)\) and the pressure \(p(x^k, t)\). Another observer residing on another fluid particle with another system of the Eulerian coordinates \((x'^k, t)\) with the origin at that fluid particle measures the physical properties of the same fluid particle at the same position as the first observer, to obtain the velocity \(u'^i(x'^k, t)\), the density \(\rho(x'^k, t)\) and the pressure \(p(x'^k, t)\). Dashes over the density and the pressure have been omitted since these fluid properties are defined independently of any observer. The Newtonian cosmological principle demands now that \(u'^i(x'^k, t)\), \(\rho(x'^k, t)\) and \(p(x'^k, t)\) should be the same functions of \(x'^k\) and \(t\) as \(u^i(x^k, t)\), \(\rho(x^k, t)\) and \(p(x^k, t)\) are functions of \(x^k\) and \(t\),

\[
\begin{align*}
    u^i(x^k, t) &= u^i(x^k, t), & \rho(x^k, t) &= \rho(x^k, t), & p(x^k, t) &= p(x^k, t).
\end{align*}
\]

The second and third conditions (33) mean that the density and pressure should be independent of a position of a fluid particle,

\[
\begin{align*}
    \rho &= \rho(t), & p &= p(t),
\end{align*}
\]

and they are functions of the absolute time \(t\) only. If the relative velocity of the fluid particle associated with the first observer with respect to the fluid particle associated with the second observer is \(u^i(x'^k - x^k, t)\), then, by taking into account the first condition (33), the velocity \(u'^i(x'^k, t)\) of the fluid particle at the position \(\{x'^k\}\) can be represented as

\[
\begin{align*}
    u^i(x'^k, t) &= u^i(x^k, t) + u^i(x'^k - x^k, t) \quad \text{or} \quad u^i(x'^k - x^k, t) = u^i(x'^k, t) - u^i(x^k, t). 
\end{align*}
\]

The system of functional equations (35) has a solution

\[
\begin{align*}
    u^i(x^k, t) &= H^i_j(t) x^j.
\end{align*}
\]

Now by the Newtonian cosmological principle the motion of the cosmological fluid is isotropic, that is, it must not depend on any direction, \(H^i_j(t) = H(t) \delta^i_j\), and the cosmological fluid velocity (36) takes the form

\[
\begin{align*}
    u^i(t) &= H(t) x^i.
\end{align*}
\]

If \(\{\xi^i\}\) is an initial position of a cosmological particle at \(t = t_0\), \(x^i(t_0) = \xi^i\), the solution to this initial value problem gives the equations of motion of a cosmological fluid particle, see Eq. (31), in the following form

\[
\begin{align*}
    x^i(t) &= R(t) \xi^i \quad \text{where} \quad \frac{1}{R} \frac{dR}{dt} = H(t), \quad R(t_0) = 1.
\end{align*}
\]
This shows that the only motions of the cosmological fluid compatible with the Newtonian cosmological principle, that is, with a homogeneous and isotropic distribution of the fluid, are those of the uniform expansion or contraction determined by one time-dependent scalar scale factor \( R(t) \) which is always positive, \( R(t) \geq 0 \). This competes the analysis of the cosmological fluid kinematics due to Newtonian cosmological principle.

To study the dynamics of the cosmological fluid satisfying the Newtonian cosmological principle, one should analyze the system of the Navier-Stokes-Poisson equations for the fluid moving in accordance with Eqs. (31) and (38). The equation of continuity takes the form

\[
\frac{\partial \rho}{\partial t} + 3H(t)\rho(t) = \frac{d\rho}{dt} + 3H(t)\rho(t) = 0, \tag{39}
\]

which has a general solution

\[
\rho(t) = \frac{\rho(t_0)}{R^3(t)}. \tag{40}
\]

The solution admits an obvious interpretation, since if all linear dimensions in an evolving homogeneous and isotropic cosmological fluid are scaled up by the factor \( R(t) \), then all volumes are scaled up by the factor \( R^3(t) \) with the correspondent increase or decrease in the cosmological fluid density.

The Poisson equation for the gravitational potential \( \phi = \phi(x^k, t) \) of a homogeneous and isotropic distribution of the cosmological fluid with the density \( \rho = \rho(t) \) and with the Heckmann-Schücking boundary condition reads

\[
\delta_{ij} \frac{\partial^2}{\partial x^i \partial x^j} \phi(x^k, t) = 4\pi G \rho(t) - \Lambda, \quad E_{ij}(x^k, t) \to 0 \quad \text{as} \quad (\delta_{ij} x^i x^j)^{1/2} \to \infty. \tag{41}
\]

This equation has a solution

\[
\phi(x^k, t) = \frac{2}{3} \pi G \rho(t) \delta_{ij} x^i x^j - \frac{1}{2} \Lambda \delta_{ij} x^i x^j, \quad \text{and} \quad \delta_{ij} x^i x^j = \frac{4}{3} \pi G \rho(t) \delta_{ij} - \frac{1}{3} \Lambda \delta_{ij}. \tag{42}
\]

The second relation in (42) shows that the Heckmann-Schücking boundary condition is satisfied by the solution and, in fact, the tidal force tensor \( E_{ij} \) vanishes everywhere

\[
E_{ij} = 0. \tag{43}
\]

The physical meaning of the solution (42) is that the gravitational potential \( \phi(x^k, t) \) is spherically symmetric and constant over surfaces \( \delta_{ij} x^i x^j = \text{const} \) for each moment \( t = \text{const} \). Now the Navier-Stokes equation takes the form

\[
\frac{\partial u^i}{\partial t} + u^j \frac{\partial u^i}{\partial x^j} + \frac{1}{\rho} \delta_{ij} \frac{\partial p}{\partial x^j} = -\delta_{ij} \frac{\partial \phi}{\partial x^j} \quad \rightarrow \quad \left( \frac{dH}{dt} + H^2 \right) x^i = -\frac{4}{3} \pi G \rho(t) x^i + \frac{1}{3} \Lambda x^i. \tag{44}
\]

By expressing the function \( H(t) \) and the fluid density \( \rho(t) \) in (44) through the scale factor \( R(t) \) by (38) and (40) one gets the equation for \( R(t) \),

\[
R^2 \frac{d^2 R}{dt^2} + \frac{4}{3} \pi G \rho(t_0) - \frac{1}{3} \Lambda R^3 = 0. \tag{45}
\]
The first integral of this equation (45) can be easily found by integration, to bring the Friedmann equation (32) for the scale factor $R(t)$. Here $-k$ is the integration constant which has meaning of the total energy of a cosmological fluid particle with a coordinate $R(t)$ moving along its path (38). Indeed the integration of (43) gives

$$-k = \left(\frac{dR}{dt}\right)^2 - \frac{8\pi G}{3R} \rho_0 - \frac{1}{3} \Lambda R^2 = K(t) + V(t)$$

(46)

where $K(t)$ and $V(t)$ are the kinetic and potential energies of the fluid particle. QED

7 **Inhomogeneous and Anisotropic Newtonian Universes**

On the basis of the notion of homogeneous and isotropic Newtonian universe one can define a notion of an inhomogeneous and/or anisotropic Newtonian universe.

**Inhomogeneous and/or anisotropic Newtonian universes** A *Newtonian universe is inhomogeneous and/or anisotropic if it does not satisfy the Newtonian cosmological principle, for, at least, an interval of the cosmological evolution time.*

As it has been pointed out the Newtonian cosmological principle assumes the existence of a privileged family of observers who, at each instant of time, observe the Newtonian universe as homogeneous and isotropic. If a Newtonian universe is inhomogeneous and/or anisotropic, a free falling observer comoving with a fluid particle will be measuring deviations in the velocities, density an pressure for a fluid particle under observation as it moves, and different values of these quantities for different fluid particles moving around the observer. From the geometrical point of view that means that distributions of the evolving cosmological fluid over 3-dimensional Euclidean spaces at each instant of time are different and therefore the fluid is inhomogeneous and/or anisotropic during its evolution.

8 **The Static Newtonian Universe**

The equation (32) may be integrated for different values of $\Lambda$ and $k$ to bring a number of Newtonian cosmological models similar to the homogeneous and isotropic cosmological models of general relativity [7], [13], [37]. One of the most important results of the analysis is that if the Newtonian cosmological constant vanishes, $\Lambda = 0$, then no static Newtonian universes, that is, either spatially compact or noncompact self-gravitating distributions of cosmological fluid for which scale factor $R(t)$ (37) and (38) is constant,

$$R(t) = \text{const} \rightarrow x^i(t) = \text{const} \xi^i, \quad u^i(t) = 0,$$

(47)

can exist [7], [13], [37], see also Section 4. Therefore, Newtonian cosmology, as well as general relativistic cosmology, also predicts that our Universe is evolving in time and there
was a time when the evolution began. If the Newtonian cosmological constant does not vanish, \( \Lambda \neq 0 \), then there exists a critical value of the Newtonian cosmological constant \( \Lambda = \Lambda_c \) for which the Newtonian universe is static and the cosmological fluid configuration in this case neither expands, nor contracts. This Newtonian universe is an analogue of Einstein’s static cosmological model of general relativity.

**Corollary 2 (No static Newtonian universe with \( \Lambda = 0 \))** No static Newtonian universe exists if the cosmological constant vanishes, \( \Lambda = 0 \). A static Newtonian universe exists for \( k > 0 \) if the Newtonian cosmological constant \( \Lambda \) and the scale factor \( R(t) \) have the critical values, \( \Lambda_c = k^3/16\pi^2G^2\rho_0^2 \) and \( R_c = 4\pi G \rho_0/k \).

**Proof.** Let us assume that for a homogeneous and isotropic Newtonian universe the Newtonian cosmological constant vanishes, \( \Lambda = 0 \). When the equation of motion (45) for the scale factor \( R(t) \) of a fluid particle reads

\[
R^2\frac{d^2R}{dt^2} + \frac{4}{3} \pi G \rho(t_0) = 0.
\]  

(48)

Under the condition of staticity \( (17) \) Eq. (48) gives

\[
\rho(t_0) = 0
\]  

(49)

that in contradiction with the definition of a Newtonian cosmology as a configuration of a self-gravitating cosmological fluid with everywhere nonvanishing positive fluid density \( (2) \) and the positivity of the mass \( (1-30) \) of any fluid region \( \Sigma(t) \). Therefore, such a static fluid configuration is impossible.

Let us now consider the Friedmann equation \( (32) \) for the scale factor \( R(t) \). For a static Newtonian universe \( (17) \) the equation takes the form

\[
\frac{8\pi G}{3R}\rho_0 - k + \frac{1}{3} \Lambda R^2 = 0.
\]  

(50)

For \( k > 0 \) when the potential energy of a fluid particle is larger than its kinetic energy \( (10) \), the critical value of the cosmological constant, \( \Lambda_c = k^3/16\pi^2G^2\rho_0^2 \), ensures that the function in the right-hand side of Eq. \( (20) \) always positive for all \( R(t) \geq 0 \) except it vanishes at a double root at \( R(t) = \text{const} = 4\pi G \rho_0/k = R_c \). One can show that for the cases \( k = 0 \) and \( k < 0 \) this function never vanishes for \( R(t) \geq 0 \). **QED**

9 **The Newtonian Cosmology versus the General Relativistic Cosmology**

The general relativistic Friedmann equation for the scale factor \( R_{GR}(t) \) has the same form \( (12) \) with three differences:

[a] the pressure is absent in the case of general relativity;
[b] the Newtonian scale factor $R(t)$ has no physical dimension while general relativistic scale factor $R_{GR}(t)$ has that of time;

[c] the constant $k$ which is an integration constant representing the total energy $-k$ \cite{comment1} in Newtonian cosmology, in general relativity represents the curvature of 3-dimensional space orthogonal to the world lines of cosmological fluid particles, $k$ having one of the value $+1, 0$ or $-1$ for three different types of 3-dimensional space geometry.

In all other respects the identity of the Newtonian and general-relativistic Friedmann equation \cite{eq1} is complete \cite{comment2}, \cite{comment3}, \cite{comment4}, \cite{comment5}. There are three physical consequences of this remarkable similarity of fundamental significance for the construction of a realistic homogeneous and isotropic cosmological model of our Universe in the frameworks of Newtonian and general relativistic cosmologies.

\{a\} The history of an homogeneous and isotropic Universe as described in the frameworks of Newtonian and general relativistic cosmologies is identical for both theories since the Friedmann equation \cite{eq1} for the variation of the scale factor $dR/dt$ has the same structure and there is the same set of cosmological models in both cosmological settings.

\{b\} The difference between the Newtonian and general relativistic cosmologies is governed by the [pressure]/[density] and [gravitational potential]/[rest mass] ratios. Therefore for the current state of our Universe where both ratios are always small, general relativity cannot offer anything radically new. When the propagation of light over the distances comparable with the radius of the curvature of Universe is to be taken into consideration, general relativistic cosmology will be more physically adequate compared with a Newtonian cosmology consideration. However, one cannot expect any radical differences in both approaches for the scales characteristic, for example, for galaxies in analysis of galaxy formation, galaxy structure, etc.

\{c\} The ability of general relativity to deal with the cosmological settings when the [pressure]/[density] and [gravitational potential]/[rest mass] ratios are not appreciable is not much greater as compared with the Newtonian cosmology, though the results more easily and directly interpreted in the framework of general relativity. On the other hand, the mathematical difficulties one encounters while solving cosmological problems in general-relativistic setting frequently obscure the physical interpretation of obtained results.

10 The Navier-Stokes-Poisson Equations in Kinematic Quantities

For analysis of the dynamics of inhomogeneous Newtonian universes, that is, the evolution of self-gravitating cosmological fluids, it is very useful to reformulate \cite{comment6} the system of the Navier-Stokes-Poisson equations \cite{eq2}, \cite{eq3}, \cite{eq4} and \cite{eq5} in terms of the kinematic quantities, see Section I-10.
10.1 The Evolution Equations

The first issue in the reformulation of the system is to derive the evolution equations for the fluid expansion scalar \( \theta = \theta(x^i, t) \) (I-59), the fluid shear tensor \( \sigma_{ij} = \sigma_{ij}(x^k, t) \) (I-58), and the fluid vorticity tensor \( \omega_{ij} = \omega_{ij}(x^k, t) \) (I-49) or the fluid vorticity vector \( \omega^i = \omega^i(x^j, t) \) (I-50) which must replace the Navier-Stokes equation of motion (10) describing the evolution of the fluid velocity vector \( u^i = u^i(x^j, t) \).

**Theorem 4 (The evolution equations for the kinematic quantities)** The Navier-Stokes equation (17) for the velocity vector \( u^i \) and the Poisson equation (8) lead to the following system of evolution equations for the fluid expansion scalar \( \theta(x^i, t) \), the fluid shear tensor \( \sigma_{ij}(x^k, t) \) and fluid vorticity vector \( \omega^i(x^j, t) \):

The Raychaudhuri evolution equation for the expansion scalar \( \theta \)

\[
\frac{d\theta}{dt} + \frac{1}{3} \theta^2 + 2(\sigma^2 - \omega^2) + 4\pi G \rho - \Lambda - \delta^{ij} A_{ij} = 0, \tag{51}
\]

The propagation equation for the shear tensor \( \sigma_{ij} \)

\[
\frac{d\sigma_{ij}}{dt} + \delta^{kl} \sigma_{ik} \sigma_{lj} + \frac{2}{3} \theta \sigma_{ij} - \frac{1}{3} \delta_{ij} (2\sigma^2 + \omega^2 - \delta^{kl} A_{k,l}) + \omega_i \omega_j + E_{ij} - A_{(i,j)} = 0, \tag{52}
\]

The propagation equation for the vorticity vector \( \omega^i \)

\[
\frac{d\omega^i}{dt} + \frac{2}{3} \theta \omega^i - \delta^{ij} \sigma_{jk} \omega^k - \frac{1}{2} \epsilon^{ijk} A_{j,k} = 0. \tag{53}
\]

**Proof.** To derive the evolution equations for the kinematic quantities, one needs to consider the tensor \( u_{i,j} \) which describes the spatial change of the fluid particle’s velocity by taking a spatial derivative of the velocity vector \( u_i = \delta_{ij} u^j \). Due to the Cauchy decomposition theorem, see Section I-10, the tensor \( u_{i,j} \) can be always represented (I-46) in terms of the fluid expansion scalar \( \theta \) (I-59), the fluid shear tensor \( \sigma_{ij} \) (I-58) and fluid vorticity tensor \( \omega_{ij} \) (I-49) as

The kinematic decomposition of the tensor \( u_{i,j} \)

\[
u_{i,j} = \sigma_{ij} + \frac{1}{3} \delta_{ij} \theta + \omega_{ij}. \tag{54}\]
The first identity for the tensor $u_{i,j}$

$$u_{i,jt} = u_{i,tj}, \quad (55)$$

The second identity for the tensor $u_{i,j}$

$$u_{i,jk} = u_{i,kj}, \quad (56)$$

to a spatial derivative of the Navier-Stokes equation (10) written with using the total acceleration vector $A^i$ (11) for a fluid particle

$$u_{i,tj} + \delta^{kl}u_{i,kj}u_l + \delta^{kl}u_{i,k}u_{l,j} + g_{i,j} - A_{i,j} = 0, \quad (57)$$

one gets the equation

$$u_{i,jt} + \delta^{kl}u_{i,jk}u_l + g_{i,j} - A_{i,j} = 0 \quad \rightarrow \quad \frac{d}{dt} u_{i,j} + \delta^{kl}u_{i,k}u_{l,j} + \phi_{i,j} - A_{i,j} = 0. \quad (58)$$

Making use of the definition of the tidal force tensor $E_{ij}$ (28), the Poisson equation (8) and substituting the decomposition rule (54) into the spatial derivative of the Navier-Stokes equation (58) with taking consequently its trace, the symmetric trace-free part and the antisymmetric part using the definition for the vorticity vector $\omega^i$ (1-50) brings the the evolution equations for the fluid expansion scalar $\theta$ (51), the fluid shear tensor $\sigma_{ij}$ (52), and fluid vorticity vector $\omega^i$ (53).

QED

It should be pointed out here that Eq. (58) is the propagation equation for the tensor $u_{i,j}$ along the fluid particle trajectories. In absence of the fluid pressure when the total fluid acceleration vector $A^i = 0$ (12), if one considers two neighboring free falling fluid particles moving along their trajectories the equation (58) describes a relative change in the positions of both particles due to action of gravitation through the second spatial derivative of potential $\phi_{i,j}$. This is the Newtonian analogue of the geodesic deviation equation in general relativity [38] and the quantity $\phi_{i,j}$ or the tidal force tensor $E_{ij}$ (28), is the Newtonian analogue of the space-time curvature tensor of General Relativity [3].

10.2 The Constraint Equations

The identities (55) and (56) for the tensor $u_{i,j}$ are Newtonian analogues of the so-called Ricci identities in general relativity, which relate the second derivatives of an arbitrary vector field with the space-time curvature tensor [3].

Due to the second identity (56) for the tensor $u_{i,j}$ there is also another set of equations which put constraints on the kinematic quantities [3].

20
Theorem 5 (The constraints on the kinematic quantities) The second identity for the tensor \( u_{i,j} \) (56) puts the following constraints on the fluid expansion scalar \( \theta(x^i, t) \), the fluid shear tensor \( \sigma_{ij}(x^k, t) \) and fluid vorticity vector \( \omega^i(x^j, t) \):

The first constraint equation

\[
\delta^{jk}(\sigma_{ij,k} - \omega_{ij,k}) - \frac{2}{3}\theta_j = 0,
\]

(59)

The second constraint equation

\[
\omega^i_{;i} = 0,
\]

(60)

The third constraint equation

\[
\delta^{j(i}(\sigma_{jk,l} + \omega_{jk,l})\varepsilon^{m)kl} = 0.
\]

(61)

Proof. The first constraint (59) follows from taking a trace of the second identity (56) for the tensor \( u_{i,j} \) with respect to indices \( i \) and \( j \). The second constraint (60) follows from the fact that the total antisymmetrization of (56) in its indices gives

\[
u_{[i,jk]} = 0
\]

(62)

which provides (60) after total contraction with the Levi-Civita symbol \( \varepsilon^{ijk} \). The third constraint (61) follows after the contraction of the second identity for the tensor \( u_{i,j} \) (56) with \( \varepsilon^{mjk} \) and symmetrization of the two remaining free indices

\[
\delta^{j(i}n\varepsilon^{m)jk}(u_{i,jk} - u_{i,kj}) = 0.
\]

(63)

QED

It should be noted here that the trace of the constraint (61) vanishes identically due to the constraint (60).

10.3 The Integrability Conditions

There are three more constraints on the kinematic quantities following from the first and the second identities (55) and (56) for the tensor \( u_{i,j} \) as their integrability conditions.

Theorem 6 (The integrability conditions for the kinematic quantities) There is the following set of the integrability conditions of the first identity (55) and the second identity (56) for the tensor \( u_{i,j} \). No more integrability conditions exist.
The first integrability condition

\[ E^j_{i,j} = \frac{8\pi G}{3} \rho, \quad (64) \]

The second integrability condition

\[ E^{(i}_{k,j} \varepsilon^{jk)} = 0, \quad (65) \]

The third integrability condition

\[ \sigma^{[i,j]}_{[k,l]} + \frac{2}{3} \delta^{[i}_{[k} \theta^{j]}_{l]} = 0. \quad (66) \]

**Proof.** The derivation of the integrability conditions is straightforward. The first and the second integrability conditions follow from taking a spatial derivative of the first identity \((64)\) and the requirement \(u_{i,t[jk]} = 0\) which brings an integrability condition

\[ u_{i,jtk} - u_{i,ktj} - u_{i,t[jk]} = 0 \quad \rightarrow \quad u_{i,jtk} - u_{i,ktj} = 0. \quad (67) \]

Taking the trace of \((67)\) with respect to indices \(i\) and \(j\) with using the evolution equations and the Poisson equation \((8)\) gives the first integrability condition \((64)\), while the antisymmetrization of \((67)\) with respect to indices \(i\) and \(j\) gives the second integrability condition \((65)\). No more constraints on the kinematic quantities can be found from the integrability condition \((67)\). The third integrability conditions follows from taking a spatial derivative of the second identity \((56)\) and the requirement \(u_{i,j[kl]} = 0\) which brings an integrability condition

\[ u_{i,kjl} - u_{i,ljk} - u_{i,j[kl]} = 0 \quad \rightarrow \quad u_{i,kjl} - u_{i,ljk} = 0. \quad (68) \]

Taking the antisymmetrization of \((68)\) with respect to indices \(i\) and \(j\) gives the third integrability condition \((66)\). No more constraints on the kinematic quantities can be found from the integrability condition \((68)\).

It is easy to show that calculations of the integrability conditions for \((67)\) and \((68)\) does not bring any new constraints since they are satisfied identically. \( \Box \)

It should be noted here that the trace of the integrability condition \((65)\) vanishes identically because the tensor \(E_{ij}\) is symmetric.

The integrability conditions \((64), (65)\) and \((66)\) are analogues of the Bianchi identities for the space-time curvature tensor of general relativity \([33]\) that can be seen from the structure of the integrability conditions \((67)\) and \((68)\) which are just a cyclic combination for the third derivatives of the fluid velocity \(u^t\). Therefore taking further integrability conditions results in expressions identically vanishing in a 3-dimensional space for they involve the antisymmetrization of more than three indices.
11 The Structure of the System of Equations

While the Poisson equation (8) keeps the same form in this reformulation of the Navier-Stokes-Poisson equations in terms of the kinematic quantities, the equation of continuity (9) takes the following form.

**The equation of continuity as the evolution equation for the fluid density \( \rho \)

\[
\frac{d\rho}{dt} + \rho \theta = 0. \tag{69}
\]

Thus, the system of the Navier-Stokes-Poisson equations for the fluid expansion scalar \( \theta(x^i, t) \), the fluid shear tensor \( \sigma_{ij}(x^k, t) \) and fluid vorticity vector \( \omega^i(x^j, t) \) includes the evolution equations (51)-(53), the constraints (59)-(61), the integrability conditions (64)-(66), the equation of continuity (69), the Poisson equation (8) and (4), to replace the Navier-Stokes-Poisson equations (4), (6), (8), (9) and (10). Upon having determined the kinematic quantities by solving the above system, one must solve the nonhomogeneous equations (54) for the fluid velocity \( u^i(x^j, t) \) together with the identities (55) and (56). Finally, to determine the equation of motion of fluid particles \( x^i = x^i(\xi^j, t) \), or \( \xi^j = \xi^j(x^i, t) \), (I-1)-(I-4) from the fluid velocity \( u^i(x^j, t) \) the initial value problem (31) has to be solved.

This system of equations for the kinematic quantities and the Navier-Stokes-Poisson equations are usually considered to be equivalent (33). However, this has not been shown as yet, strictly speaking. As it has been noted above, the Navier-Stokes-Poisson equations is a system of a mixed hyperbolic-elliptic type for seven first order nonlinear partial differential equations for seven unknowns \( \rho, u^i \) and \( g_i \), or it can be taken as a system of five first and second order equations (8), (9) and (10) for five unknowns \( \rho, u^i \) and \( \phi \) to be supplemented by the corresponding initial and boundary conditions for the unknowns and an equation of state (4). It allows application of some techniques for analyzing the questions of existence and uniqueness, at least, for a class of boundary and/or initial conditions, see Section 2. Once reformulated in terms of kinematic quantities, the system of nonlinear equations for a self-gravitating fluid changes its structure and it is does not fit into the standard classification of the partial differential equations. Four main issues appear to be important in consideration of the Navier-Stokes-Poisson equations written in terms of kinematic quantities:

(A) The system is not closed since the evolution equation for the shear tensor (52) contains the tidal force tensor \( E_{ij} \) (28) which does not have any evolution equation for itself. This is a reflection of the noncloseness of the Navier-Stokes-Poisson equations (4), (8), (9) and (10) with respect to boundary conditions and its initial boundary value problem, see Section 4. Upon reformulation in terms of kinematic quantities this problem becomes, however, critical for the Navier-Stokes-Poisson equations. Indeed, the system (8), (9) and (10) can be supplemented by some boundary conditions, for example, the Heckmann-Schücking boundary conditions (29) or (30), to allow its solution without adding any other equations and an explicit change in the structure of the system of equations (4), (8), (9) and (10).
On the contrary, the system \((4), (8), (12), (51)-(56), (59)-(61), (64)-(66)\) and \((69)\) must be explicitly supplemented by an evolution equation for the tidal force tensor \(E_{ij}\) \((28)\) in order to allow its solution. Thus, the Navier-Stokes-Poisson equations in terms of kinematic quantities must admit a generalization of Newtonian gravity in terms of an evolution equation for the \(E_{ij}\) in addition to the Poisson equation \((8)\) which then acquires a status of a constraint equation. As it has been pointed out in Section 4, there are a number of approaches known to modify the Newtonian gravity satisfying the Poisson equation, see \([19], [20], [21]\) and references therein. One possibility is to accept the so-called local tidal approximation \([20]\).

The evolution equations for the tidal force tensor \(E_{ij}\)

\[
\frac{dE_{ij}}{dt} + \theta E_{ij} - \delta_{ij}\delta^{kn}\delta^{lm}\sigma_{kl}E_{nm} - 3\delta^{kl}\sigma_{k(i}E_{j)l} - \delta^{kl}\omega_{k(i}E_{j)l} + 4\pi G\rho\sigma_{ij} = 0
\]

\((70)\)

which is a generalization of the Zel’dovich approximation \([39], [40]\) by taking into account the second spatial derivatives in the equation of motion for a fluid particle \([1]\). With such an evolution equation \((70)\) for the tidal force tensor \(E_{ij}\) \((28)\) added, the system of the Navier-Stokes-Poisson equations in kinematic quantities \((4), (8), (12), (51)-(56), (59)-(61), (64)-(66), (69)\) becomes closed. Then one can look for a solution of the system with a suitable set of initial and boundary conditions characteristic for the problem under study. One can assume another evolution equation for the tidal force tensor \(E_{ij}\) which may be dictated by a particular Newtonian cosmological model.

(B) The system is overdetermined since it has been derived by differentiation of the Navier-Stokes-Poisson equations \((4), (8), (9)\) and \((10)\). It now contains higher derivatives of the fluid velocity through the expansion scalar \(\theta = \theta(x^k, t)\) \((I-59)\), the shear tensor \(\sigma_{ij} = \sigma_{ij}(x^k, t)\) \((I-58)\) and the vorticity vector \(\omega^i = \omega^i(x^j, t)\) \((I-50)\) and these kinematic became now additional unknowns to the velocity field \(u^i = u^i(x^j, t)\), the fluid density scalar \(\rho = \rho(x^i, t)\) and the gravitational potential scalar \(\phi = \phi(x^i, t)\). There are 14 unknowns, namely, 3 components of the fluid velocity \(u^i\), 1 function of the density \(\rho\), 1 function of the gravitational potential \(\phi\), 1 function of the expansion \(\theta\), 5 components of the trace-free symmetric shear tensor \(\sigma_{ij}\) and 3 components of the fluid vorticity vector \(\omega^i\). All together they should satisfy 69 equations, namely, 9 evolution equations \((51)-(53)\), the equation of continuity \((69)\), the equation of state \((4)\), the Poisson equation \((8)\), 5 evolution equations \((70)\), 9 constraints including 3 constraints \((59)\), 1 constraint \((60)\), 5 constraints \((61)\), 16 integrability conditions including 3 integrability conditions \((64)\), 5 integrability conditions \((65)\) and \((66)\) and 9 integrability conditions \((67)\) and \((68)\) which satisfy 18 integrability conditions \((55)\) and \((56)\).

(C) Even after the reformulation in terms of the kinematic quantities of the fluid expansion scalar \(\theta = \theta(x^i, t)\) \((I-59)\), the fluid shear tensor \(\sigma_{ij} = \sigma_{ij}(x^k, t)\) \((I-58)\), and the fluid vorticity tensor \(\omega_{ij} = \omega_{ij}(x^k, t)\) \((I-49)\) or the fluid vorticity vector \(\omega^i = \omega^i(x^j, t)\) \((I-50)\) the system of the Navier-Stokes-Poisson equations \((4), (8), (12), (51)-(56), (59)-(61), (64)-(66), (69), (28)\) and \((70)\) do contain the fluid velocity \(u^i(x^j, t)\) explicitly. Indeed, the material derivatives in the evolution equations \((51)-(53)\) have the inertial term \(u_{i,j}u^j\)
involving $u^i(x^j, t)$. That means from the mathematical point of view that the information stored in the spatial derivatives of the fluid velocity $u_{i,j}(x^k, t)$ through the decomposition (54) is not enough to fully specify the fluid velocity field $u^i(x^j, t)$ in terms of the kinematic quantities. Therefore, if only the kinematic quantities $\theta(x^i, t)$, $\sigma_{ij}(x^k, t)$ and $\omega^i(x^j, t)$ are considered to be the unknowns, the system of the Navier-Stokes-Poisson equations is actually a system of integro-differential equations.

(D) The space of solutions of the system (4), (8), (12), (51)-(56), (59)-(61), (64)-(66), (69), (28) and (70) is bigger than that of the Navier-Stokes-Poisson equations (4), (8), (9) and (10). This is reflected in the presence, first of all, of an evolution equation for the tidal force tensor (70) which does not take place in the original system. Secondly, there are constraints and integrability conditions which include even higher derivatives of the tensor $u_{i,j}$. Since the decomposition law (54) is a Pfaffian system when it is considered as a system of nonhomogeneous partial differential equations for the 3 unknown components of velocity $u_i$, due to the Frobenius theorem (42), (43) the integrability conditions (55) and (56) are necessary and sufficient for the existence of a solution locally. The problem, however, is that the identities (73) and (75) taken as a system of nonhomogeneous partial differential equations for the tensor $u_{i,j}$ form a system of partial differential equations which is not Pfaffian. In such a case, their integrability conditions (64)-(66) are only sufficient conditions for integrability of (55) and (56) in general (see, for instance, (43) and references therein). One may expect they become also necessary for some cosmological fluid configurations. When and under which assumptions about the physics of a self-gravitating fluid, and finally, for which class(es) of Newtonian cosmologies this can occur definitely deserves a careful study. It should be pointed out here that this problem of the integrability of the Navier-Stokes-Poisson equations in kinematic quantities is affected also by the structure of the evolution equations for the tidal force tensor $E_{ij}$ (70), or another equation of this kind, which has been added to close the system.

(E) The dynamics of the self-gravitating fluid governed by the system of Navier-Stokes-Poisson equations in terms of kinematic quantities (4), (8), (12), (51)-(56), (59)-(61), (64)-(66), (69), (28) and (70) is not, strictly speaking, due to Newtonian gravity of the cosmological fluid, but rather due to its generalization in accordance with the evolution equation (70) for the gravitational potential through the tidal force tensor $E_{ij}$ (28). The universes corresponding to such dynamically evolving cosmological fluid configurations under this modified Newtonian gravitational force will be still called Newtonian, since if the Newtonian cosmological principle is satisfied, the tidal force tensor should vanish asymptotically (29) and (30) on the largest cosmological scales and the Newtonian universes homogeneous and isotropic (32) on such large scales are governed then by the Newtonian gravitational potential satisfying the Poisson equation (8). If a Newtonian universe is inhomogeneous and/or anisotropic the full system of the Navier-Stokes-Poisson equations in terms of kinematic quantities must be taken into consideration.

Though the system of the Navier-Stokes-Poisson equations in terms of kinematic quantities look a bit complicated as compared with the original Navier-Stokes-Poisson equations, the former has proved to be mathematically efficient and physically adequate in cosmological studies, in particular, where some simplifying assumptions can be adopted.
(see [38] and recent applications [14]-[18] and references therein).

12 The Raychaudhuri Equation in a Homogeneous and Isotropic Newtonian Universe

The evolution equation for the expansion scalar (51) is known to have the very similar form to the Raychaudhuri equation for the expansion scalar of the 4-velocity of the cosmological fluid in general relativity [49], [50] with the only different term with pressure involved in the general relativistic case. This equation is very significant in both Newtonian and general relativistic cosmologies since the expansion scalar of the evolving cosmological fluid serves as its fundamental parameter. For an homogeneous and isotropic Newtonian universe it is the only dynamical fluid characteristic. One can show in this case that the Raychaudhuri equation (51) is equivalent for the Friedmann equation (32), and the expansion scalar is then directly expressible in terms of scale factor [11], [12], [37].

Theorem 7 (The Homogeneous and Isotropic Raychaudhuri Equation) The evolution of a Newtonian universe satisfying the Newtonian cosmological principle is governed by the Raychaudhuri equation

$$\frac{d\theta}{dt} + \frac{1}{3} \theta^2 + 4\pi G \rho - \Lambda = 0 \quad (71)$$

for the cosmological fluid expansion scalar $\theta = \theta(t)$, while the shear tensor $\sigma_{ij}(x^k, t)$, vorticity vector $\omega^i(x^j, t)$ and the tidal force tensor $E_{ij}(x^k, t)$ vanish

$$\sigma_{ij} = 0, \quad \omega^i = 0, \quad E_{ij} = 0. \quad (72)$$

Then the Raychaudhuri equation is equivalent to the Friedmann equation (32) for the fluid scale factor $R = R(t)$ with

$$\theta = \frac{3}{R} \frac{dR}{dt}. \quad (73)$$

Proof. Let us consider a Newtonian universe filled with a cosmological fluid satisfying the Newtonian cosmological principle and governed by the system of Navier-Stokes-Poisson equations in terms of kinematic quantities (4), (8), (12), (51)-(56), (59)-(61), (64)-(66), (69), (28) and (70). The same analysis as in the case of an homogeneous and isotropic Newtonian universe satisfying the Newtonian cosmological principle, Section 6, shows that the fluid density $\rho(x^k, t)$ and fluid pressure $p(x^k, t)$ are functions of absolute time $t$ only and the velocity of the cosmological fluid $u^i(x^k, t)$ has the following form,

$$\rho = \rho(t), \quad p = p(t), \quad u^i(t) = H(t)x^i. \quad (74)$$

Calculation of the tensor $u_{i,j}$ and the kinematic quantities for the fluid gives

$$u_{i,j} = H(t)\delta_{ij}, \quad \theta = H(t), \quad \sigma_{ij} = 0, \quad \omega^i = 0. \quad (75)$$
The kinematic quantities (74) together with a solution to the equation of continuity (10) satisfy the equations (1), (53)-(56), (59)-(61), (66) and (69) from the system of Navier-Stokes-Poisson equations in terms of kinematic quantities. The evolution equation (72) for the shear tensor results in everywhere vanishing tidal force tensor $E_{ij}$ for the fluid (43) which satisfies the evolution equation for the tidal force tensor (70). The vanishing tensor $E_{ij}$ (43) is consistent with the solution (12) of the Poisson equation (8). Then the Raychaudhuri evolution equation for the expansion scalar $\theta = H(t)$ is the only remaining equation which takes the form (71). By taking into account the definition of the scale factor $R(t)$ of the cosmological fluid (38), see also the representation of the expansion scalar (I-61), one can easily derive the evolution equation (45) for the scale factor from the Raychaudhuri equation (71). The first integral of (45) leads the Friedmann equation (32) for the scale factor $R(t)$.

This result shows, in particular, that for the case of homogeneous and isotropic Newtonian universes the system of Navier-Stokes-Poisson equations in terms of kinematic quantities (4), (8), (12), (51)-(56), (59)-(61), (64)-(66), (69), (28) and (70) is consistent.

**Acknowledgments**

I would like to thank Alan Coley for his hospitality in Dalhousie University. The work has been supported in part by the Swiss National Science Foundation, Grant 7BYPJ065731.

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