SIGN OF GREEN’S FUNCTION OF PANEITZ OPERATORS
AND THE $Q$ CURVATURE

FENGBO HANG AND PAUL C. YANG

Abstract. In a conformal class of metrics with positive Yamabe invariant, we derive a necessary and sufficient condition for the existence of metrics with positive $Q$ curvature. The condition is conformally invariant. We also prove some inequalities between the Green’s functions of the conformal Laplacian operator and the Paneitz operator.

1. Introduction

Since the fundamental work [CGY] in dimension 4, the Paneitz operator and associated $Q$ curvature in dimension other than 4 (see [B, P]) attracts much attention (see [DHL, GM, HY1, HeR1, HeR2, HuR, QR] etc and the references therein). Let $(M, g)$ be a smooth compact $n$ dimensional Riemannian manifold. For $n \geq 3$, the $Q$ curvature is given by

$$Q = -\frac{1}{2(n-1)}\Delta R - \frac{2}{(n-2)^2} |\text{Rc}|^2 + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2} R^2. \quad (1.1)$$

Here $\Delta$ is the Laplacian, $R$ is the scalar curvature, $\text{Rc}$ is the Ricci tensor. The Paneitz operator is given by

$$P \varphi = \Delta^2 \varphi + \frac{4}{n-2} \text{div} (\text{Rc} (\nabla \varphi, e_i) e_i) - \frac{n^2 - 4n + 8}{2(n-1)(n-2)} \text{div} (R \nabla \varphi) + \frac{n-4}{2} Q \varphi. \quad (1.2)$$

Here $e_1, \cdots, e_n$ is a local orthonormal frame with respect to $g$. For $n \neq 4$, under conformal transformation of the metric, the operator satisfies

$$P_{\rho \frac{\Delta}{\Delta g}} \varphi = \rho^{-\frac{n+4}{n-4}} P_g (\rho \varphi). \quad (1.3)$$

Note this is similar to the conformal Laplacian operator, which appears naturally when considering transformation law of the scalar curvature under conformal change of metric ([LP]). As a consequence we know

$$P_{\rho \frac{\Delta}{\Delta g}} \varphi \cdot \psi d\mu_{\rho \frac{\Delta}{\Delta g}} = P_g (\rho \varphi) \cdot \rho \psi d\mu_g. \quad (1.4)$$

Here $\mu_g$ is the measure associated with metric $g$. Moreover

$$\ker P_g = 0 \iff \ker P_{\rho \frac{\Delta}{\Delta g}} = 0, \quad (1.5)$$

and under this assumption, the Green’s functions $G_P$ satisfy the transformation law

$$G_{P_{\rho \frac{\Delta}{\Delta g}}} (p, q) = \rho (p)^{-1} \rho (q)^{-1} G_{P_g} (p, q). \quad (1.6)$$
For \( u, v \in C^\infty (M) \), we denote the quadratic form associated with \( P \) as
\[
E(u, v) = \int_M P u \cdot v d\mu \quad (1.7)
\]
and
\[
E(u) = E(u, u) \quad (1.8)
\]
By the integration by parts formula in (1.7) we know \( E(u, v) \) also makes sense for \( u, v \in H^2(M) \).

To continue we recall (see [LP]) for \( n \geq 3 \), on a smooth compact Riemannian manifold \( (M^n, g) \), the conformal Laplacian operator is given by
\[
L_g^\ast = -\frac{4(n-1)}{n-2} \Delta \varphi + R \varphi \quad (1.9)
\]
The Yamabe invariant is defined as
\[
Y(g) = \inf \left\{ \frac{\int_M \tilde{R} d\tilde{\mu}}{\mu(M)^{2/n}} : \tilde{g} = \rho^2 g \text{ for some positive smooth function } \rho \right\} \quad (1.10)
\]
\[
= \inf \left\{ \frac{\int_M L_g \varphi \cdot \varphi d\mu}{\| \varphi \|^2_{L^n \frac{n}{n-2}}} : \varphi \text{ is a nonzero smooth function on } M \right\}.
\]
A basic but useful fact about the scalar curvature is
\[
Y(g) > 0 \iff \lambda_1(L_g) > 0 \quad (1.11)
\]
\( \iff g \) is conformal to a metric with scalar curvature \( > 0 \).

Indeed more is true, namely the equivalence still holds if we replace all "\( > \)" by "\( = \)" or "\( < \)". Here \( \lambda_1(L_g) \) is the first eigenvalue of \( L_g \). It is clear \( Y(g) \) is a conformal invariant, on the other hand the sign of \( \lambda_1(L_g) \) is also conformally invariant. The main reason that (1.11) holds is based on the fact the first eigenfunction of a second order symmetric differential operator does not change sign. Unfortunately such kind of property is known to be not valid for higher order operators. The following question keeps puzzling people from the beginning of research on \( Q \) curvature in dimension other than 4, namely: can we find a conformal invariant condition which is equivalent to the existence of positive \( Q \) curvature in the conformal class (in the same spirit as (1.11))? Here we give an answer to this question for conformal class with positive Yamabe invariant.

**Theorem 1.1.** Let \( n > 4 \) and \( (M^n, g) \) be a smooth compact Riemannian manifold with Yamabe invariant \( Y(g) > 0 \), then the following statements are equivalent

1. There exists a positive smooth function \( \rho \) with \( Q_{\rho^2 g} > 0 \).
2. \( \ker P_g = 0 \) and the Green's function \( G_P(p, q) > 0 \) for any \( p, q \in M, p \neq q \).
3. \( \ker P_g = 0 \) and there exists a \( p \in M \) such that \( G_P(p, q) > 0 \) for \( q \in M \setminus \{ p \} \).
Along the way we also find the following comparison inequality between Green’s function of $L$ and $P$.

**Proposition 1.1.** Assume $n > 4$, $(M^n, g)$ is a smooth compact Riemannian manifold with $Y (g) > 0$, $Q \geq 0$ and not identically zero, then $\ker P = 0$ and

$$G_{L}^{\frac{n-4}{2}} \leq c_n G_P. \quad (1.12)$$

Here

$$c_n = 2^{-\frac{n-6}{2}} n^{\frac{n-7}{2}} (n-1)^{-\frac{n-4}{2}} (n-2) (n-4) \omega_n^{\frac{n-2}{2}}, \quad (1.13)$$

$\omega_n$ is the volume of unit ball in $\mathbb{R}^n$. Moreover if $G_{L}^{\frac{n-4}{2}} (p, q) = c_n G_P (p, q)$ for some $p \neq q$, then $(M, g)$ is conformal diffeomorphic to the standard sphere.

In dimension 3 we have

**Theorem 1.2.** Let $(M, g)$ be a smooth compact 3 dimensional Riemannian manifold with Yamabe invariant $Y (g) > 0$, then the following statements are equivalent:

1. there exists a positive smooth function $\rho$ with $Q_{\rho^2} > 0$.
2. $\ker P = 0$ and $G_P (p, q) < 0$ for any $p, q \in M, p \neq q$.
3. $\ker P = 0$ and there exists a $p \in M$ such that $G_P (p, q) < 0$ for $q \in M \setminus \{p\}$.

Similar to Proposition 1.1, we have

**Proposition 1.2.** Let $(M, g)$ be a smooth compact 3 dimensional Riemannian manifold with $Y (g) > 0$, $Q \geq 0$ and not identically zero, then $\ker P = 0$ and

$$G_{L}^{-1} \leq -256\pi^2 G_P. \quad (1.14)$$

If for some $p, q \in M$, $G_{L}^{-1} (p, q) = -256\pi^2 G_P (p, q)$, then $(M, g)$ is conformal diffeomorphic to the standard $S^3$ (note here $p$ can be equal to $q$).

In dimension 4 we have the following (see Corollary 5.1)

**Proposition 1.3.** Assume $(M, g)$ is a smooth compact 4 dimensional Riemannian manifold, $Y (g) > 0$, then for any $p \in M$,

$$\int_M Q d\mu + \frac{1}{2} \int_M \left| R_{L, g}^{2} \right|^2_{G_{L, g}^2} d\mu g_{L, g} = 16\pi^2. \quad (1.15)$$

In particular, $\int_M Q d\mu \leq 16\pi^2$ and equality holds if and only if $(M, g)$ is conformal diffeomorphic to the standard $S^4$.

It is worthwhile to point out that the proof of Theorem B in [G], which gives the inequality in Proposition 1.3, is elementary and does not use the positive mass theorem. Our argument is also elementary and identifies the difference between $\int_M Q d\mu$ and $16\pi^2$.

Theorem 1.1 and 1.2 are motivated by works on the $Q$ curvature in dimension 5 or higher ([GM, HeR1, HeR2, HuR]) and in dimension 3 ([HY1, HY2, HY3]). In [HeR1, HeR2], it was shown in some cases compactness property for solutions of the $Q$ curvature equation can be derived under the assumption that the Green’s function is positive. Recently [GM] showed that the Green’s function is indeed positive when both scalar curvature and $Q$ curvature are positive. Theorem 1.1 says we could relax the assumption to $Y (g) > 0, Q_g > 0$. Whether these two kinds of assumptions are equivalent or not is still unknown. The main approach in [GM] is roughly speaking by applying the maximum principle twice. In [HY3],...
by replacing maximum principle with the weak Harnack inequality it was shown that for metrics with $R > 0$ and $Q > 0$, $P$ is invertible and $G_P(p, q) < 0$ for $p \neq q$. Theorem 1.2 relax the assumption to $Y(g) > 0$ and $Q > 0$. The main new ingredient in our proof of Theorem 1.1 and 1.2 is the formula (2.1), which is closely related to the arguments in [HuR]. In [HY4], we will apply the results on Green’s function to solution of $Q$ curvature equations. In section 2 we will prove the main formula (2.1). In sections 3 and 4 we will prove Theorem 1.1 and 1.2 respectively. In section 5 we will derive the corresponding formula of (2.1) in dimension 4. In particular Proposition 1.3 follows from the formula. In section 6, we will show that the positive mass theorem for Paneitz operator in [GM, HuR] can be deduced from (2.1) too.

2. AN IDENTITY CONNECTING THE GREEN’S FUNCTION OF CONFORMAL LAPLACIAN OPERATOR AND Paneitz OPERATOR

Here we will derive an interesting formula which illustrates the close relation between Green’s function of conformal Laplacian operator and the Paneitz operator. This identity will play a crucial role later.

To motivate the discussion, we note that positive Yamabe invariant implies we have a positive Green’s function for the conformal Laplacian operator. Even though we do not know whether $P$ is invertible or not, we may still try to search for its Green’s function. Note that the possible Green’s function should have same highest order singular term as $G^{n-4}_{L,p}$ (modulus dimension constant), we can use $G^{n-4}_{L,p}$ as a first step approximation. Along this line we compute $P \left( G^{n-4}_{L,p} \right)$ and arrive at the interesting formula (2.1).

**Proposition 2.1.** Assume $n \geq 3$, $n \neq 4$, $(M, g)$ is a $n$ dimensional smooth compact Riemannian manifold with $Y(g) > 0$, $p \in M$, then we have $G^{n-4}_{L,p} \left| \text{Re} \frac{G^{n-4}_{L,p}}{G^{n-4}_{L,p}} \right|^2_g \in L^1(M)$ and

$$P \left( G^{n-4}_{L,p} \right) = c_n \delta_p - \frac{n - 4}{(n - 2)^2} G^{n-4}_{L,p} \left| \text{Re} \frac{G^{n-4}_{L,p}}{G^{n-4}_{L,p}} \right|^2_g$$

in distribution sense. Here

$$c_n = 2^{-\frac{n-6}{n-2}} n^{-\frac{2}{n-2}} (n - 1)^{-\frac{n-4}{n-2}} (n - 2) (n - 4) \omega_n^{-\frac{2}{n-2}},$$

(2.2)

$\omega_n$ is the volume of unit ball in $\mathbb{R}^n$, $G_{L,p}$ is the Green’s function of conformal Laplacian operator $L = -\frac{4(n-1)}{n-2} \Delta + R$ with pole at $p$.

It is worth pointing out that the metric $G^{n-4}_{L,p} g$ on $M \setminus \{p\}$ is exactly the stereographic projection of $(M, g)$ at $p$ ([LP]). To prove the proposition, let us first check what happens under a conformal change of the metric. If $\rho \in C^\infty(M)$ is a positive function, let $\bar{g} = \rho^{\frac{n-4}{n-2}} g$, then using

$$G_{L,p}(q) = \rho(p)^{-1} \rho(q)^{-1} G_{L,p}(q)$$
we see
\[ G_{L,p}^{\frac{n-4}{4}} \left| \frac{\text{Rc}}{G_{L,p}^4} \right|_{g}^2 d\mu = \rho(p)^{-\frac{n-4}{4}} \rho G_{L,p}^{\frac{n-4}{4}} \left| \frac{\text{Rc}}{G_{L,p}^4} \right|_{g}^2 d\mu. \] (2.3)

Hence we only need to check \( G_{L,p}^{\frac{n-4}{4}} \left| \frac{\text{Rc}}{G_{L,p}^4} \right|_{g}^2 \in L^1(M) \) for a conformal metric.

By the existence of conformal normal coordinate ([LP]) we can assume \( \exp_p \) preserve the volume near \( p \). Let \( x_1, \cdots, x_n \) be a normal coordinate at \( p \), denote \( r = |x| \), then (see [LP])
\[ G_{L,p} = \frac{1}{4n(n-1)\omega_n} r^{2-n} \left( 1 + O^{(4)}(r) \right). \] (2.4)

As usual, we say \( f = O^{(m)}(r^\theta) \) to mean \( f \) is \( C^m \) in the punctured neighborhood with \( \partial_{i_1 \cdots i_k} f = O(r^{\theta-k}) \) for \( 0 \leq k \leq m \). By (2.4) and the transformation law
\[ \text{Rc} \left| \frac{G_{L,p}^4}{G_{L,p}^4} \right|_{g}^2 = \text{Rc} - 2D^2 \log G_{L,p} + \frac{4}{n-2} d \log G_{L,p} \otimes d \log G_{L,p} \] (2.5)

\[ - \left( \frac{2}{n-2} \Delta \log G_{L,p} + \frac{4}{n-2} |\nabla \log G_{L,p}|^2 \right) g, \]
careful calculation shows
\[ \left| \text{Rc} \right|_{G_{L,p}^4}^2 = O \left( \frac{1}{r} \right). \] (2.6)

It follows that
\[ G_{L,p}^{\frac{n-4}{4}} \left| \frac{\text{Rc}}{G_{L,p}^4} \right|_{g}^2 = O \left( r^{2-n} \right) \]
hence it belongs to \( L^1(M) \).

To continue, we observe that equation (2.1) is the same as
\[ \int_M G_{L,p}^{\frac{n-4}{4}} P \varphi d\mu = c_\varphi \left( p \right) - \frac{n-4}{(n-2)^2} \int_M G_{L,p}^{\frac{n-4}{4}} \left| \frac{\text{Rc}}{G_{L,p}^4} \right|_{g}^2 \varphi d\mu \] (2.7)
for any \( \varphi \in C^\infty(M) \). A similar check as before shows (2.7) is conformally invariant.

Again we assume \( \exp_p \) preserves the volume near \( p \), then for \( \delta > 0 \) small, it follows from (2.4) that
\[ PG_{L,p}^{\frac{n-4}{4}} = c_\varphi + \text{a L}^1 \text{ function} \] (2.8)
on \( B_\delta(p) \) in distribution sense. On the other hand, on \( M \setminus \{p\} \) using (1.2) and (1.3) we have
\[ P_g \left( G_{L,p}^{\frac{n-4}{4}} \right) = G_{L,p}^{\frac{n-4}{4}} \frac{P}{G_{L,p}^4} 1 \] (2.9)
\[ = -\frac{n-4}{(n-2)^2} G_{L,p} \frac{n-4}{4} \left| \frac{\text{Rc}}{G_{L,p}^4} \right|_{g}^2. \]
Here we have used the fact $R_{G_{L_p}^{-1}g} = 0$. Combine (2.8) and (2.9) we get (2.1).

3. The case dimension $n > 4$

Throughout this section we will assume $(M, g)$ is a smooth compact Riemannian manifold with dimension $n > 4$.

**Lemma 3.1.** Assume $n > 4$, $Y(g) > 0$, $u \in C^\infty(M)$ such that $u \geq 0$ and $Pu \geq 0$.
If for some $p \in M$, $u(p) = 0$, then $u \equiv 0$.

**Proof.** By (2.1) we have

$$
\int_M G_{L,p}^{\frac{n-4}{4}} Pu \, d\mu = -\frac{n-4}{(n-2)^2} \int_M G_{L,p}^{\frac{n-4}{4}} \left| \frac{Rc}{G_{L,p}^{-1}g} \right|_g^2 \, d\mu.
$$

Hence $Pu = 0$ and $\left| \frac{Rc}{G_{L,p}^{-1}g} \right|_g^2 u = 0$.

If $u$ is not identically zero, then by unique continuation property we know $\{u \neq 0\}$ is dense, hence $\frac{Rc}{G_{L,p}^{-1}g} = 0$. Since $\left( M \setminus \{p\}, G_{L,p}^{-1}g \right)$ is asymptotically flat, it follows from relative volume comparison theorem that $\left( M \setminus \{p\}, G_{L,p}^{-1}g \right)$ is isometric to the standard $\mathbb{R}^n$. In particular $(M, g)$ must be locally conformally flat and simply connected compact manifold, hence it is conformal to the standard $S^n$ by [K]. But in this case we have $\ker P = 0$, hence $u = 0$, a contradiction. 

**Remark 3.1.** Indeed the same argument gives us the following statement: If $n > 4$, $Y(g) > 0$, $u \in L^1(M)$ such that $u \geq 0$ and $Pu \geq 0$ in distribution sense, for some $p \in M$, $u$ is smooth near $p$ and $u(p) = 0$, then $u \equiv 0$.

A straightforward consequence of Lemma 3.1 is the following useful fact.

**Proposition 3.1.** Assume $n > 4$, $Y(g) > 0$, $Q \geq 0$. If $u \in C^\infty(M)$ such that $Pu \geq 0$ and $u$ is not identically constant, then $u > 0$.

**Proof.** If the conclusion of the proposition is false, then $u(p) = \min_M u \leq 0$ for some $p$. Let $\lambda = -u(p) \geq 0$, then $u + \lambda \geq 0$, $u(p) + \lambda = 0$ and

$$
P(u + \lambda) = Pu + \frac{n-4}{2} \lambda Q \geq 0.
$$

It follows from the Lemma 3.1 that $u + \lambda \equiv 0$. This contradicts with the fact $u$ is not a constant function.

Proposition 3.1 helps us determine the null space of $P$ without information on the first eigenvalue.

**Corollary 3.1.** Assume $n > 4$, $Y(g) > 0$, $Q \geq 0$, then

$$
\ker P \subset \{ \text{constant functions} \}.
$$

If in addition, $Q$ is not identically zero, then $\ker P = 0$ i.e. $0$ is not an eigenvalue of $P$.

**Proof.** Assume $Pu = 0$. If $u$ is not a constant function, then it follows from Proposition 3.1 that $u > 0$ and $-u > 0$, a contradiction.
Now we ready to prove half of Theorem 1.1.

**Lemma 3.2.** Assume \( n > 4 \), \( Y (g) > 0 \), \( Q \geq 0 \) and not identically zero, then \( \ker P = 0 \), moreover the Green’s function \( G_{p,p} (q) = G_P (p,q) > 0 \) for \( p \neq q \).

**Proof.** By Corollary 3.1, we know \( \ker P = 0 \). Hence for any \( f \in C^\infty (M) \), there exists a unique \( u \in C^\infty (M) \) with \( Pu = f \), moreover

\[
 u(p) = \int_M G_{p,p} (q) f(q) \, d\mu(q).
\]

If \( f \geq 0 \), it follows from the Proposition 3.1 that \( u \geq 0 \). Hence \( G_{p,p} \geq 0 \). If \( G_{p,p} (q) = 0 \) for some \( q \), since \( PG_{p,p} = \delta_p \geq 0 \) in distribution sense, we know from the Remark 3.1 that \( G_{p,p} \equiv 0 \), a contradiction. Hence \( G_{p,p} (q) > 0 \) for \( p \neq q \). \( \square \)

Next let us give the full argument of Theorem 1.1.

**Proof of Theorem 1.1.** (1)⇒(2): This follows from Lemma 3.2, (1.5) and (1.6).

(2)⇒(1): This follows from the classical Krein-Rutman theorem ([L]). Since our case is relatively simple, we provide the argument here. Define the integral operator \( T \) as

\[
 Tf(p) = \int_M G_P (p,q) f(q) \, d\mu(q).
\]

Then we have

\[
 T' = \frac{1}{\alpha_1} \sup_{f \in L^2(M) \setminus \{0\}} \frac{\int_M T f \cdot f \, d\mu}{\|f\|_{L^2}^2} > 0.
\]

\( \alpha_1 \) is an eigenvalue of \( T \). We note all eigenfunctions of \( \alpha_1 \) does not change sign. Indeed say \( T \varphi = \alpha_1 \varphi, \int_M \varphi^2 d\mu = 1 \), we have

\[
 \int_M (\varphi_+^2 + \varphi_-^2) \, d\mu = 1.
\]

Here \( \varphi_+ = \max \{ \varphi, 0 \}, \varphi_- = \max \{ -\varphi, 0 \} \). Without losing of generality, we assume \( \varphi_+ \) is not identically zero. Then

\[
 \alpha_1 = \int_M T \varphi \cdot \varphi \, d\mu = \int_M T \varphi_+ \cdot \varphi_+ \, d\mu + \int_M T \varphi_- \cdot \varphi_- \, d\mu - 2 \int_M T \varphi_+ \cdot \varphi_- \, d\mu \leq \alpha_1 - 2 \int_M T \varphi_+ \cdot \varphi_- \, d\mu.
\]

Hence \( \int_M T \varphi_+ \cdot \varphi_- \, d\mu = 0 \). Since \( T \varphi_+ > 0 \), we see \( \varphi_- = 0 \). Hence \( \varphi \geq 0 \). Because \( T \varphi = \alpha_1 \varphi \) we see \( \varphi \in C^\infty (M) \) and \( \varphi > 0 \). It follows that \( \alpha_1 \) must be a simple eigenvalue and \( P \varphi = \alpha_1^{-1} \varphi \), hence

\[
 Q_\varphi P_{\varphi}^{\frac{1}{n-4}} \varphi = \frac{2}{n-4} P_{\varphi}^{\frac{1}{n-4}} \varphi = \frac{2}{n-4} \varphi^{\frac{n+4}{n-4}} P_{\varphi} \varphi = \frac{2}{n-4} \alpha_1^{-1} \varphi^{\frac{n+4}{n-4}} > 0.
\]

(2)⇒(3): Assume \( p_0 \in M \) such that \( G_{p,p_0} > 0 \). For \( p \in M \), define

\[
 \Theta(p) = \min_{q \in M \setminus \{p\}} G_P (p,q) \tag{3.1}
\]

Then we have \( \Theta(p_0) > 0 \). We note that \( \Theta(p) \neq 0 \) for any \( p \in M \). Otherwise, say \( \Theta(p) = 0 \), then \( G_{p,p} \geq 0 \) and \( G_{p,p} (q) = 0 \) for some \( q \neq p \). It follows from Remark

\[
 \min_{q \in M \setminus \{p\}} G_{p,q} (q) > 0.
\]
3.1 that $G_{P,p} = \text{const}$, a contradiction. Since $M$ is connected we see $\Theta (p) > 0$ for all $p$. In another word, $G_P (p,q) > 0$ for $p \neq q$.

**Remark 3.2.** In the proof of (2)$\Rightarrow$(1), a similar argument tells us if $\beta$ is an eigenvalue of $T$, $\beta \neq \alpha_1$, then $|\beta| < \alpha_1$. It follows that when $G_P$ is positive, the smallest positive eigenvalue of $P$ must be simple and its eigenfunction must be either strictly positive or strictly negative. Moreover if $\lambda$ is a negative eigenvalue of $P$, then $|\lambda|$ is strictly bigger than the smallest positive eigenvalue.

**Proof of Proposition 1.1.** By Lemma 3.2 we know $\ker P = 0$ and $G_P > 0$. From (2.1) we know

$$P \left( G_{L,p}^{\frac{n-4}{4}} - c_n G_{P,p} \right) = -\frac{n-4}{(n-2)^{n-4}} G_{L,p}^{\frac{n-4}{4}} \left| \text{Rc}_{G_{L,p}^{\frac{4}{n}}} \right|_g^2 \leq 0.$$ 

Hence $G_{L,p}^{\frac{n-4}{4}} \leq c_n G_{P,p}$. If for some $q \neq p$, $G_{L,p}^{\frac{n-4}{4}} (q) = c_n G_{P,p} (q)$, then $\text{Rc}_{G_{L,p}^{\frac{4}{n}}} = 0$, hence $(M,g)$ is conformal diffeomorphic to the standard $S^n$ by the argument in the proof of Lemma 3.1.

### 4. 3 Dimensional Case

Throughout this section we assume $(M,g)$ is a smooth compact Riemannian manifold of dimension 3.

If $Y (g) > 0$, then for $p \in M$, (2.1) becomes

$$P \left( G_{L,p}^{-1} \right) = -256 \pi^2 \delta_p + G_{L,p}^{-1} \left| \text{Rc}_{G_{L,p}^{\frac{4}{n}}} \right|_g^2 .$$

(4.1)

Note here $G_{L,p}^{-1} \in H^2 (M)$.

**Lemma 4.1.** Assume $Y (g) > 0$, $u \in H^2 (M)$ such that $u \geq 0$, $Pu \leq 0$ in distribution sense. If for some $p \in M$, $u (p) = 0$, then either $u \equiv 0$ or $(M,g)$ is conformal diffeomorphic to the standard $S^3$ and $u$ is a constant multiple of $G_{P,p}$.

**Proof.** Using the fact $G_{L,p}^{-1} \in H^2 (M)$, it follows from (4.1) that

$$\int_M G_{L,p}^{-1} Pu \, \mu - \int_M G_{L,p}^{-1} \left| \text{Rc}_{G_{L,p}^{\frac{4}{n}}} \right|_g^2 \, u \, \mu = 0 .$$

Note here

$$\int_M G_{L,p}^{-1} Pu \, \mu = E \left( G_{L,p}^{-1}, u \right) .$$

Hence $\int_M G_{L,p}^{-1} Pu \, \mu = 0$ and $\int_M G_{L,p}^{-1} \left| \text{Rc}_{G_{L,p}^{\frac{4}{n}}} \right|_g^2 \, u \, \mu = 0$. Hence $\left| \text{Rc}_{G_{L,p}^{-1}} \right|_g^2 \, u = 0$. Since $Pu$ must be a measure, we see $Pu = \text{const} \cdot \delta_p$. In particular $u$ is smooth on $M \setminus \{ p \}$. If $u$ is not identically zero, it follows from unique continuation property that the set $\{ u \neq 0 \}$ is dense, and hence $\text{Rc}_{G_{L,p}^{\frac{4}{n}}} = 0$. Same argument as in the proof of Lemma 3.1 tells us $(M,g)$ must be conformal diffeomorphic to the standard $S^3$, and hence $u = \text{const} \cdot G_{P,p}$.

**Proposition 4.1.** Assume $Y (g) > 0$, $Q \geq 0$. If $u \in C^\infty (M)$ such that $Pu \leq 0$ and $u$ is not identically constant, then $u > 0$. 

Proof. If the conclusion of the proposition is false, then \( u(p) = \min_M u \leq 0 \) for some \( p \). Let \( \lambda = -u(p) \geq 0 \), then \( u + \lambda \geq 0 \), \( u(p) + \lambda = 0 \) and

\[
P(u + \lambda) = Pu - \lambda Q \leq 0.
\]

It follows from the Lemma 4.1 that \( u + \lambda \equiv 0 \). This contradicts with the fact \( u \) is not a constant function. \( \square \)

Corollary 4.1. Assume \( Y(q) > 0 \), \( Q \geq 0 \), then \( \ker P \subset \{ \text{constant functions} \} \). If in addition, \( Q \) is not identically zero, then \( \ker P = 0 \) i.e. 0 is not an eigenvalue of \( P \).

Proof. Assume \( Pu = 0 \). If \( u \) is not a constant function, then it follows from Proposition 4.1 that \( u > 0 \) and \( -u > 0 \), a contradiction. \( \square \)

Lemma 4.2. Assume \( Y(q) > 0 \), \( Q \geq 0 \) and not identically zero, then \( \ker P = 0 \), and the Green’s function \( G_{P,p}(q) = G_P(p,q) < 0 \) for \( p \neq q \). Moreover if for some \( p \in M \), \( G_{P,p}(p) = 0 \), then \( (M,g) \) is conformal diffeomorphic to the standard \( S^3 \).

Proof. By Corollary 4.1, we know \( \ker P = 0 \). Hence for any \( f \in C^\infty(M) \), there exists a unique \( u \in C^\infty(M) \) with \( Pu = f \), moreover

\[
u(p) = \int_M G_{P,p}(q) f(q) \, dq.
\]

If \( f \leq 0 \), it follows from the Proposition 4.1 that \( u \geq 0 \). Hence \( G_{P,p} \leq 0 \). If \( G_{P,p}(q) = 0 \) for some \( q \), since \( PG_{P,p} = \delta_p \geq 0 \), it follows from Lemma 4.1 that \( (M,g) \) must be conformal diffeomorphic to the standard \( S^3 \) and \( G_{P,p} \) is a constant multiple of \( G_{P,q} \), this implies \( p = q \). Hence \( G_{P,p} < 0 \) on \( M \setminus \{ p \} \). \( \square \)

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. (1) \( \Rightarrow \) (2): This follows from Lemma 4.2 and (1.5), (1.6).

(2) \( \Rightarrow \) (1): This follows from Krein-Rutman theorem, or one may use the argument in the proof of Theorem 1.1. We also remark it follows that the largest negative eigenvalue of \( P \) must be simple and its eigenfunction must be strictly positive or strictly negative. Moreover if \( \lambda \) is a positive eigenvalue of \( P \), then \( \lambda \) is strictly bigger than the absolute value of the largest negative eigenvalue.

(3) \( \Rightarrow \) (2): We can assume \( (M,g) \) is not conformal diffeomorphic to the standard \( S^3 \). For any \( p \in M \), we let

\[
\Theta(p) = \max_M G_{P,p}.
\]

Then it follows from Lemma 4.1 that \( \Theta(p) \neq 0 \) for any \( p \in M \). Since \( \Theta(p_0) < 0 \) for some \( p_0 \in M \), we see \( \Theta(p) < 0 \) for all \( p \in M \). In another word, \( G_P < 0 \). \( \square \)

With all the above analysis, we can easily deduce Proposition 1.2.

Proof of Proposition 1.2. Under the assumption of Proposition 1.2, it follows from Lemma 4.2 that \( \ker P = 0 \) and \( G_P(p,q) < 0 \) for \( p \neq q \). From (4.1) we see

\[
P \left( G_{L,p}^{-1} + 256 \pi^2 G_{P,p} \right) = G_{L,p}^{-1} \left| R_{G_{L,p}^{-1}} \right|_g \geq 0.
\]

Hence \( G_{L,p}^{-1} + 256 \pi^2 G_{P,p} \leq 0 \). If it achieves 0 somewhere, then \( R_{G_{L,p}^{-1}} = 0 \) and hence \((M,g)\) is conformal diffeomorphic to the standard \( S^3 \). \( \square \)
At last we want to point out based on Proposition 1.2, using the arguments in [HY3] we have the following statement: Let
\[ M = \left\{ g : g \text{ is a smooth metric with } Y(g) > 0 \text{ and there exists a positive smooth function } \rho \text{ such that } Q_{\rho^2 g} > 0 \right\} \]
be endowed with \( C^\infty \) topology. Then

(1) For every \( g \in M \), there exists \( \rho \in C^\infty (M), \rho > 0 \) such that \( Q_{\rho^2 g} = 1 \).

Moreover as long as \((M, g)\) is not conformal diffeomorphic to the standard \( S^3 \), the set
\[ \{ \rho \in C^\infty (M) : \rho > 0, Q_{\rho^2 g} = 1 \} \]
is compact in \( C^\infty \) topology.

(2) Let \( \mathcal{N} \) be a path connected component of \( M \). If there is a metric in \( \mathcal{N} \) satisfying condition NN, then every metric in \( \mathcal{N} \) satisfies condition NN. Hence as long as the metric is not conformal to the standard \( S^3 \), it satisfies condition \( P \). As a consequence, for any metric in \( \mathcal{N} \),
\[ \inf \left\{ E(u) \left\| u^{-1} \right\|^2_{L^2(M)} : u \in H^2(M), u > 0 \right\} > -\infty \]
and is always achieved.

We omit the details here.

5. 4 Dimension Case Revisited

Throughout this section we will assume \((M, g)\) is a smooth compact Riemannian manifold of dimension 4. In this dimension the \( Q \) curvature is written as
\[ Q = -\frac{1}{6} \Delta R - \frac{1}{2} |Rc|^2 + \frac{1}{6} R^2. \quad (5.1) \]
The Paneitz operator can be written as
\[ P \varphi = \Delta^2 \varphi + 2 \text{ div } (Rc(\nabla \varphi, e_i) e_i) - \frac{2}{3} \text{ div } (R \nabla \varphi). \quad (5.2) \]
Here \( e_1, e_2, e_3, e_4 \) is a local orthonormal frame with respect to \( g \). \( P \) satisfies
\[ P e^{2w} g \varphi = e^{-4w} P_g \varphi \quad (5.3) \]
for any smooth function \( w \). The \( Q \) curvature transforms as
\[ Q e^{2w} g = e^{-4w} (P_g w + Q_g). \quad (5.4) \]
In the spirit of Proposition 2.1, we have

**Proposition 5.1.** Assume \((M, g)\) is a 4 dimensional smooth compact Riemannian manifold with \( Y(g) > 0 \), \( p \in M \), then we have \( \left| Rc_{G^2_{L,p}} g \right|^2 \in L^1(M) \) and
\[ P (\log G_{L,p}) = 16\pi^2 \delta_p - \frac{1}{2} \left| Rc_{G^2_{L,p}} g \right|^2 - Q \quad (5.5) \]
in distribution sense. Here \( G_{L,p} \) is the Green’s function of conformal Laplacian operator \( L = -6\Delta + R \) with pole at \( p \).
Proof. If \( \rho \) is a positive smooth function on \( M \), \( \bar{g} = \rho^2 g \), then

\[
\left| Rc_{G^2_{L,p}} g \right|_{\bar{g}}^2 d\bar{\mu} = \left| Rc_{G^2_{L,p}} g \right|_g^2 d\mu. \tag{5.6}
\]

Hence to show \( \left| Rc_{G^2_{L,p}} g \right|_g^2 \in L^1(M) \), in view of the existence of conformal normal coordinate, we can assume \( \exp_p \) preserves volume near \( p \). Let \( x_1, x_2, x_3, x_4 \) be normal coordinate at \( p \), \( r = |x| \), then (see [LP])

\[
G_{L,p} = \frac{1}{24\pi^2} \frac{1}{r^2} \left( 1 + O^{(4)}(r^2) \right). \tag{5.7}
\]

Using

\[
Rc_{G^2_{L,p}} g = Rc - 2D^2 \log G_{L,p} + 2d \log G_{L,p} \otimes d \log G_{L,p} - \left( \Delta \log G_{L,p} + 2 |\nabla \log G_{L,p}|^2 \right) g, \tag{5.8}
\]

we see \( \left| Rc_{G^2_{L,p}} g \right|_g = O(1) \), hence \( \left| Rc_{G^2_{L,p}} g \right|_g^2 \in L^1(M) \).

On the other hand, (5.5) means

\[
\int_M \log G_{L,p} \cdot P \varphi d\mu = 16\pi^2 \varphi(p) - \frac{1}{2} \int_M \left| Rc_{G^2_{L,p}} g \right|_g^2 \varphi d\mu - \int_M Q \varphi d\mu. \tag{5.9}
\]

Careful check shows (5.9) is conformally invariant. Hence we can assume \( \exp_p \) preserves volume near \( p \). It follows from (5.7) that on \( B_\delta(p) \) for \( \delta > 0 \) small,

\[
P(\log G_{L,p}) = 16\pi^2 \delta_p + a L^1 \text{ function} \tag{5.10}
\]

in distribution sense. On \( M \setminus \{p\} \), we have

\[
P(\log G_{L,p}) = G^4_{L,p} Q_i^2 g - Q = \frac{1}{2} \left| Rc_{G^2_{L,p}} g \right|_g^2 - Q. \tag{5.11}
\]

(5.5) follows. \( \square \)

By integrating (5.5) on \( M \) and observing that

\[
\left| Rc_{G^2_{L,p}} g \right|_g^2 d\mu_g = \left| Rc_{G^2_{L,p}} g \right|_{G^2_{L,p}}^2 d\mu_{G^2_{L,p}}
\]

we immediately get

**Corollary 5.1.** Assume \( Y(g) > 0 \), then for any \( p \in M \),

\[
\int_M Q d\mu + \frac{1}{2} \int_M \left| Rc_{G^2_{L,p}} g \right|_{G^2_{L,p}}^2 d\mu_{G^2_{L,p}} = 16\pi^2. \tag{5.12}
\]

In particular, \( \int_M Q d\mu \leq 16\pi^2 \) and equality holds if and only if \((M, g)\) is conformal diffeomorphic to the standard \( S^4 \).
6. Positive mass theorem for Paneitz operator revisited

Throughout this section we will assume \((M, g)\) is a smooth compact Riemannian manifold with dimension \(n > 4\).

In [HuR], for locally conformally flat manifold with \(Y (g) > 0\) and positive Green’s function \(G_P\), a positive mass theorem for Paneitz operator was proved by a nice calculation. Note that this result plays similar role for \(Q\) curvature equation as the classical positive mass theorem for the Yamabe problem ([LP]). It was observed that similar calculation works for \(n = 5, 6, 7\) in [GM] and for \(n = 3\) in [HY3]. Since the case \(n = 3\) can be covered by Lemma 4.1, we concentrate on the case \(n > 4\). The main aim of this section is to show the positive mass theorem for Paneitz operator follows from the formula (2.1).

**Lemma 6.1.** Assume \(n > 4\), \(Y (g) > 0\), \(\ker P = 0\). Let \(x_1, \ldots, x_n\) be a coordinate near \(p\) with \(x_i (p) = 0\), \(r = |x|\). If either \(M\) is conformally flat near \(p\) or \(n = 5, 6, 7\), then

\[
c_n G_{P,p} - G_{L,p}^\frac{n-4}{n} = \text{const} + O^{(4)} (r) .
\]

(6.1)

Here \(c_n\) is the constant given by (1.13).

**Proof.** First we observe that if \(\rho\) is a positive smooth function on \(M\), \(\bar{g} = \rho^{-\frac{4}{n-4}} g\), then

\[
c_n G_{P,p} - G_{L,p}^\frac{n-4}{n} = \rho (p)^{-1} \rho^{-1} \left( c_n G_{P,p} - G_{L,p}^\frac{n-4}{n} \right) .
\]

(6.2)

Hence we only need to verify (6.1) for a conformal metric.

For the case \(M\) is conformally flat near \(p\), by a conformal change of metric, we can assume \(g\) is Euclidean near \(p\). Then under the normal coordinate at \(p\) we have

\[
G_{P,p} = \frac{1}{2n (n-2) (n-4) \omega_n} \left( r^{4-n} + A + O^{(4)} (r) \right) .
\]

(6.3)

Here \(\omega_n\) is the volume of unit ball in \(\mathbb{R}^n\) and \(A\) is a constant. People usually call \(A\) as the mass of Paneitz operator. The Green’s function of conformal Laplacian

\[
G_{L,p} = \frac{1}{4n (n-1) \omega_n} \left( r^{2-n} + O^{(4)} (r^{-1}) \right) .
\]

(6.4)

It is worth pointing out one has better estimate for the Green’s function than the one in (6.3) and (6.4), but the formula we wrote above also works for \(n = 5, 6, 7\) without locally conformally flat assumption. More precisely, for \(n = 5, 6, 7\), under the conformal normal coordinate, (6.3) and (6.4) remain true (see [GM, LP]). It follows that

\[
c_n G_{P,p} - G_{L,p}^\frac{n-4}{n} = 4n (n-1) \omega_n \frac{n-4}{n} A + O^{(4)} (r) .
\]

(6.5)

To continue, we note that under the assumption of Lemma 6.1, by (2.1) we have

\[
P \left( c_n G_{P,p} - G_{L,p}^\frac{n-4}{n} \right) = \frac{n-4}{(n-2)} G_{L,p}^\frac{n-4}{n} \left[ R_{G_{L,p}} g \right]_g^2 .
\]

(6.6)

hence

\[
G_{L,p}^\frac{n-4}{n} \left[ R_{G_{L,p}} g \right]_g^2 = O (r^{-3})
\]

(6.7)
and

\[(4n(n-1)\omega_n)^{-\frac{n-4}{n-2}}A = \frac{n-4}{(n-2)^2} \int_M G_{p,p} G_{L,p}^{\frac{n-4}{2}} \left( \frac{1}{G_{L,p}} \right)^2 d\mu. \quad (6.8)\]

If in addition we know the Green’s function $G_{p,p} > 0$, then it follows from (6.8) that $A \geq 0$, moreover $A = 0$ if and only if $(M,g)$ is conformal equivalent to the standard $S^n$. This proves the positive mass theorem for Paneitz operator.

References

[B] T. Branson. Differential operators canonically associated to a conformal structure. Math. Scand. 57 (1985), no. 2, 293–345.

[CGY] S.-Y. A. Chang, M. J. Gursky and P. C. Yang. An equation of Monge-Ampere type in conformal geometry, and four-manifolds of positive Ricci curvature. Ann. of Math. (2) 155 (2002), 709–787.

[DHL] Z. Djadli, E. Hebey and M. Ledoux. Paneitz-type operators and applications. Duke Math. Jour. 104 (2000), 129–169.

[G] M. J. Gursky. The principal eigenvalue of a conformally invariant differential operator. Comm. Math. Phys. 207 (1999), no. 1, 131–143.

[GM] M. J. Gursky and A. Malchiodi. A strong maximum principle for the Paneitz operator and a nonlocal flow for the $Q$ curvature. Preprint (2014).

[HY1] F. B. Hang and P. Yang. The Sobolev inequality for Paneitz operator on three manifolds. Calculus of Variations and PDE. 21 (2004), 57–83.

[HY2] F. B. Hang and P. Yang. Paneitz operator for metrics near $S^3$. Preprint (2014).

[HY3] F. B. Hang and P. Yang. $Q$ curvature on a class of 3 manifolds. Comm Pure Appl Math, to appear.

[HY4] F. B. Hang and P. Yang. $Q$ curvature on a class of manifolds with dimension at least 5. Preprint (2014).

[HeR1] E. Hebey and F. Robert. Compactness and global estimates for the geometric Paneitz equation in high dimensions. Electron Res Ann Amer Math Soc. 10 (2004), 135–141.

[HeR2] E. Hebey and F. Robert. Asymptotic analysis for fourth order Paneitz equations with critical growth. Adv Calc Var. 4 (2011), no. 3, 229–275.

[HuR] E. Humbert and S. Raulot. Positive mass theorem for the Paneitz-Branson operator. Calculus of Variations and PDE. 36 (2009), 525–531.

[K] N. H. Kuiper. On conformally-flat spaces in the large. Ann of Math. 50 (1949), 916–924.

[L] P. Lax. Functional analysis. John Wiley & Sons, Inc. 2002.

[LP] J. M. Lee and T. H. Parker. The Yamabe problem. Bull AMS. 17 (1987), no. 1, 37–91.

[P] S. Paneitz. A quartic conformally covariant differential operator for arbitrary pseudo-Riemannian manifolds. Preprint (1983).

[QR] J. Qing and D. Raske. On positive solutions to semilinear conformally invariant equations on locally conformally flat manifolds. Int Math Res Not. Art. id 94172 (2006).