Asymptotic behavior of bifurcation curves of one-dimensional nonlocal elliptic equations

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Abstract
We study the one-dimensional nonlocal elliptic equation

\[- \left( \int_0^1 |u(x)|^p dx + b \right)^q u''(x) = \lambda u(x)^p, \ x \in I := (0,1), \ u(x) > 0, \ x \in I,\]
\[u(0) = u(1) = 0,\]

where \(b \geq 0, p \geq 1, q > 1 - \frac{1}{p}\) are given constants and \(\lambda > 0\) is a bifurcation parameter. We establish the global behavior of bifurcation diagrams and precise asymptotic formulas for \(u_\lambda(x)\) as \(\lambda \to \infty\).

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1 Introduction
We consider the following one-dimensional nonlocal elliptic equation

\[
\begin{cases}
- \left( \int_0^1 |u(x)|^p dx + b \right)^q u''(x) = \lambda u(x)^p, \ x \in I := (0,1), \\
u(x) > 0, \ x \in I, \\
u(0) = u(1) = 0,
\end{cases}
\]

(1.1)

where \(b, p, q\) are given constants satisfying

\[b \geq 0, \quad p \geq 1, \quad q > 1 - \frac{1}{p}\]

(1.2)
and \( \lambda > 0 \) is a bifurcation parameter. (1.1) is the model equation of the following nonlocal problem (1.3) considered in [7].

\[
\begin{align*}
-a \left( \int_0^1 |u(x)|^p \, dx \right) u''(x) &= h(x) f(x, u(x)), \ x \in I, \\
u(x) > 0, \ x \in I, \\
u(0) = u(1) = 0,
\end{align*}
\]

(1.3)

where \( a = a(w) \) is a real-valued continuous function. Let \( \|u\|_p := \left( \int_I |u(x)|^p \, dx \right)^{1/p} \). If we put \( a(\|u\|_p) = (\|u\|_p^p + b)^q \), \( h(x) \equiv \lambda \) and \( f(x, u) = u^p \) in (1.3), then we obtain (1.1). Nonlocal elliptic problems as (1.3) have been studied intensively by many authors, since they arise in various physical models. We refer to [4,8,9,11] and the references therein. As for standard and nonlocal bifurcation problems, there are many results for the equations with different types of nonlinearities. We refer to [2, 5] and the references therein. The purpose of this paper is to obtain the precise asymptotic behavior of bifurcation curves \( \lambda = \lambda(\alpha) \) and \( u_\lambda \) as \( \lambda \to \infty \) by focusing on the typical nonlocal problem (1.1). Here, \( \alpha := \alpha_\lambda = \|u_\lambda\|_\infty \) for given \( \lambda > 0 \).

To state our results, we prepare the following notation. For \( p > 1 \), let

\[
\begin{align*}
-W''(x) &= W(x)^p, \ x \in I, \\
W(x) > 0, \ x \in I, \\
W(0) = W(1) = 0,
\end{align*}
\]

(1.4)

We know from [6] that there exists a unique solution \( W_p(x) \) of (1.4).

**Theorem 1.1.** Let \( b = 0 \) in (1.1). Then there exists a unique solution \( u_\lambda \) of (1.1) for any given \( \lambda > 0 \). Furthermore, the following formulas hold.

(i) Assume that \( p > 1 \). Then for a given \( \lambda > 0 \),

\[
\begin{align*}
\lambda &= 2(p + 1)L_{p}^{2-q}M_{p}^{q} \alpha^{p-q-p+1}, \\
u_\lambda(x) &= \lambda^{1/(pq-p+1)} \left\{ (2(p + 1))^{p/(p-1)}L_{p}^{(p+1)/(p-1)}M_{p} \right\}^{-q/(pq-p+1)} W_p(x),
\end{align*}
\]

(1.5) (1.6)

where

\[
\begin{align*}
L_{p} := \int_{0}^{1} \frac{s}{s^{p+1}} \, ds, \ M_{p} := \int_{0}^{1} \frac{s^{p}}{s^{p+1}} \, ds.
\end{align*}
\]

(1.7)

(ii) Assume that \( p = 1 \). Then

\[
\lambda = 2^{q} \pi^{2-q} \alpha^{q}.
\]

(1.8)

We next consider the case \( b > 0 \). To clarify our intention, we start from the simplest case \( p = 2 \) and \( q = 1 \). We have

\[
\begin{align*}
\|W_p\|_p &= (2(p + 1))^{1/(p-1)}M_p^{1/p}L_{p}^{(p+1)/(p(p-1))} \quad (p > 1), \\
\|W_p\|_\infty &= (2(p + 1))^{1/(p-1)}L_{p}^{2/(p-1)}.
\end{align*}
\]

(1.9) (1.10)
We will show (1.9) and (1.10) in the last part of Section 2.

**Theorem 1.2.** Let \( b > 0, \ p = 2, \ q = 1 \) and

\[
\lambda_0 := 2b^{1/2} \|W_2\|_2. \tag{1.11}
\]

(i) If \( 0 < \lambda < \lambda_0 \), then there exists no solution of (1.1).

(ii) If \( \lambda = \lambda_0 \), then (1.1) has a unique solution

\[
u_\lambda(x) = \frac{\lambda_0}{2} \|W_2\|_2^{-2} W_2(x). \tag{1.12}
\]

(iii) If \( \lambda > \lambda_0 \), then there exist exactly two solutions \( u_{1,\lambda}, u_{2,\lambda} \) of (1.1) such that

\[
u_{1,\lambda}(x) = \frac{\lambda \|W_2\|_2^{-1} - \sqrt{\lambda^2 \|W_2\|_2^{-2} - 4b}}{2} \|W_2\|_2^{-1} W_2(x), \tag{1.13}
\]

\[
u_{2,\lambda}(x) = \frac{\lambda \|W_2\|_2^{-1} + \sqrt{\lambda^2 \|W_2\|_2^{-2} - 4b}}{2} \|W_2\|_2^{-1} W_2(x). \tag{1.14}
\]

We see from (1.13) and (1.14) that these two curves start from \((\lambda_0, \alpha_0)\) \((\alpha_0 := \lambda_0 \|W_2\|_2^{-2} \|W_2\|_\infty)\). Further, by Taylor expansion, we see that \(\alpha_1(\lambda) = b \|W_2\|_2^{-1}(1 + o(1))\) and \(\alpha_2(\lambda) = \|W_2\|_\infty \|W_2\|_2^{-2} \lambda(1 + o(1))\) for \(\lambda \gg 1\).

For the case \( p > 1 \) and \( q = 1 \) \((p \neq 2)\), it seems difficult to obtain such exact solutions \(u_\lambda\) as (1.13)–(1.14). Therefore, we try to find the asymptotic shape of solutions \(u_\lambda\) for \(\lambda \gg 1\).

**Theorem 1.3.** Let \( p > 1 \) and \( q = 1 \). Put

\[
\lambda_0 := (b(p - 1))^{1/p} \frac{p}{p - 1} \|W_p\|_p^{p-1}. \tag{1.15}
\]

(i) If \( 0 < \lambda < \lambda_0 \), then there exists no solution of (1.1).

(ii) If \( \lambda = \lambda_0 \), then there exists a unique solution of (1.1).

(iii) If \( \lambda > \lambda_0 \), then there exist exactly two solutions \( u_{1,\lambda} \) and \( u_{2,\lambda} \) of (1.1). Moreover, for \(\lambda \gg 1\),

\[
\left\{
\begin{array}{l}
\lambda := \lambda_1(\alpha) = b \|W_p\|_p^{-1} \alpha^{-\frac{1}{p-1}} \left( 1 + b^{-1} \|W_p\|_p \|W_p\|_\infty \alpha^p(1 + o(1)) \right), \\
u_{1,\lambda}(x) = b^{1/(p-1)} \lambda^{-1/(p-1)} \left( 1 + \frac{1}{p-1} b^{1/(p-1)} \|W_p\|_p \|W_p\|_\infty \alpha^{1-p}(1 + o(1)) \right) W_p(x). \tag{1.16}
\end{array}
\right.
\]

\[
\left\{
\begin{array}{l}
\lambda := \lambda_2(\alpha) = \|W_p\|_p \|W_p\|_\infty \alpha + b \|W_p\|_\infty \alpha^{1-p} + o(\alpha^{1-p}), \\
u_{2,\lambda}(x) = \left( \lambda \|W_p\|_p^{1-p} - b \left( \lambda \|W_p\|_p^{1-p} \right)^{1-p} (1 + o(1)) \right) \|W_p\|_p^{-1} W_p(x). \tag{1.17}
\end{array}
\right.
\]

Finally, we treat (1.1) under the condition (1.2).

**Theorem 1.4.** Assume (1.2) and \(\lambda \gg 1\). Then there exist exactly two solutions \( u_{1,\lambda} \) and \( u_{2,\lambda} \) of (1.1). Moreover, for \(\lambda \gg 1\),

\[
u_{1,\lambda}(x) = b^{q/(p-1)} \lambda^{-1/(p-1)} \left( 1 + \frac{q}{p-1} b^{(pq-p+1)/(p-1)} \|W_p\|_p^{p/(p-1)} \lambda^{-p/(p-1)} (1 + o(1)) \right) W_p(x), \tag{1.18}
\]

\[
u_{2,\lambda}(x) = \left\{ m \lambda^{1/(pq-p+1)} - \frac{b q m^{1-p}}{pq - p + 1} \lambda^{(1-p)/(pq-p+1)} (1 + o(1)) \right\} \|W_p\|_p^{-1} W_p(x). \tag{1.19}
\]
where \( m := \|W_p\|^{(1-p)/(pq-p+1)}_p \).

The remainder of this paper is organized as follows. In Section 2, we first introduce how to find the solutions of (1.1). Then we prove Theorem 1.1 by time map method. In Section 3, we mainly consider the existence of solutions of (1.1) under the condition (1.2). In Sections 4 and 5, we prove Theorems 1.3 and 1.4 by Taylor expansion and direct calculation.

## 2 Proof of Theorem 1.1

In this section, let \( b = 0 \) in (1.1).

**Lemma 2.1.** For any \( \lambda > 0 \), (1.1) has a unique solution \( u_\lambda \).

**Proof.** We apply the argument [1, Theorem 2] to (1.1). For a given \( \lambda > 0 \), we consider

\[
-\gamma w(x) = \lambda w(x)^p, \quad x \in I,
\]

\[
w(x) > 0, \quad x \in I.
\]

\[
w(0) = w(1) = 0.
\]

Then it is clear that

\[
w_\lambda(x) := \lambda^{1/(1-p)} W_p(x)
\]

is the unique solution of (2.1). For \( t > 0 \), we consider

\[
g(t) := t^{pq/2} - \|w_\lambda\|^{1-p}_{p(p-1)/2} W_p(x).
\]

Then it is known from [1, Theorem 2] that if \( g(t_\lambda) = 0 \), then \( u_\lambda := \gamma w_\lambda (\gamma := t^{1/2}_{\lambda} \|w_\lambda\|^{-1}_{p}) \) satisfies (1.1). Indeed, by (2.1) and (2.3), we have

\[
-\left( \int_0^1 \lambda u_\lambda(x)^p dx \right)^q u_\lambda''(x) = -\|u_\lambda\|^p u_\lambda'(x) = 0
\]

\[
= -\|w_\lambda\|^p \gamma w_\lambda''(x) = t^{pq/2} \gamma \lambda w_\lambda(x) = (t^{1/2}_{\lambda} \|w_\lambda\|^{-1}_{p})^{p-1} \gamma \lambda w_\lambda(x)
\]

\[
= \lambda (\gamma w_\lambda(x))^p = \lambda u_\lambda(x)^p.
\]

On the other hand, assume that \( u_\lambda \) satisfies (1.1). Then we see that \( u_\lambda = \|u_\lambda\|_{p(p-1)/p-1} w_\lambda \). Then we have \( \|u_\lambda\|_{p(p-1)(pq-p+1)/p-1} = \|w_\lambda\|_{p} \). We put \( t^{1/2}_{\lambda} := \|u_\lambda\|_{p} = \|w_\lambda\|_{p(p-1)/pq-p+1} \). Then by (2.3), we have \( g(t_\lambda) = 0 \). Consequently, the solutions \( \{t_{1,\lambda}, t_{2,\lambda}, \ldots, t_{k,\lambda}\} \) of (2.3) correspond to the solutions \( \{u_{1,\lambda}, u_{2,\lambda}, \ldots, u_{k,\lambda}\} \). Therefore, if \( g(t) = 0 \) has a unique soution, then (1.1) also has a unique solution. By (1.2) and (2.3), we see that

\[
g(t) = t^{(p-1)/2} (t^{(pq-p+1)/p-1} - \|w_\lambda\|_{p}) = 0
\]

has a unique positive solution \( t_{\lambda} = \|w_\lambda\|_{p}^{2(1-p)/(pq-p+1)} \). Thus the proof is complete. \( \blacksquare \)

**Proof of Theorem 1.1.** (i) Let \( t_{\lambda} = \|w_\lambda\|_{p}^{2(1-p)/(pq-p+1)} \). By Lemma 2.1 and (2.2), we have

\[
u_\lambda(x) = t^{1/2}_{\lambda} \|w_\lambda\|_{p}^{-1} W_p(x) = t^{1/2}_{\lambda} \|W_p\|^{-1}_p W_p(x).
\]
By (2.2) and (2.5), we have
\[ t_\lambda = \|w_\lambda\|^2(1-p)/(pq-p+1) = \lambda^{2/(pq-p+1)}\|W_p\|^2(1-p)/(pq-p+1). \]
(2.7)

We apply the time map argument to (1.4). (cf. [10]). Since (1.4) is autonomous, we have
\[ W_p(x) = W_p(1 - x), \quad x \in [0, 1/2], \]
(2.8)
\[ W'_p(x) > 0, \quad x \in [0, 1/2], \]
(2.9)
\[ \xi := \|W_p\| = \max_{0 \leq x \leq 1} W_p(x) = W_p(1/2). \]
(2.10)

By (1.4), for \(0 \leq x \leq 1\), we have
\[ \{W''_p(x) + W_p(x)^p\}W'_p(x) = 0. \]
(2.11)

By this and (2.10), we have
\[ \frac{1}{2}W'_p(x)^2 + \frac{1}{p+1}W_p(x)^{p+1} = \text{constant} = \frac{1}{p+1}W_p(1/2)^{p+1} = \frac{1}{p+1}\xi^{p+1}. \]
(2.12)

By this and (2.9), for \(0 \leq x \leq 1/2\), we have
\[ W'_p(x) = \sqrt{\frac{2}{p+1}(\xi^{p+1} - W_p(x)^{p+1})}. \]
(2.13)

By this, (2.8) and putting \(\theta := W_p(x)\), we have
\[ \|W_p\|_p^p = 2 \int_0^{1/2} W_p(x)^p dx = 2 \int_0^{1/2} W_p(x)^p \frac{W'_p(x)}{\sqrt{\frac{2}{p+1}(\xi^{p+1} - W_p(x)^{p+1})}} dx \]
(2.14)
\[ = \sqrt{2(p+1)} \int_0^\xi \frac{\theta^p}{\sqrt{\xi^{p+1} - \theta^{p+1}}} d\theta \quad (\theta = \xi s) \]
\[ = \sqrt{2(p+1)}\xi^{(p+1)/2} \int_0^1 \frac{s^p}{\sqrt{1 - s^{p+1}}} ds = \sqrt{2(p+1)}\xi^{(p+1)/2}M_p. \]

By this, (2.6), and (2.7), we have
\[ u_\lambda(x) = \lambda^{1/(pq-p-1)}\{(2(p+1))^{1/2}\xi^{(p+1)/2}M_p\}^{-q/(pq-p-1)}W'_p(x). \]
(2.15)

By putting \(x = 1/2\) in (2.15), we have
\[ \alpha = \lambda^{1/(pq-p-1)}\{(2(p+1))^{1/2}\xi^{(p+1)/2}M_p\}^{-q/(pq-p-1)}\xi. \]
(2.16)

By (2.13), we have
\[ \frac{1}{2} = \int_0^{1/2} dx = \int_0^{1/2} \frac{W'_p(x)}{\sqrt{\frac{2}{p+1}(\xi^{p+1} - W_p(x)^{p+1})}} dx \]
(2.17)
\[ = \sqrt{\frac{p+1}{2}} \int_0^\xi \frac{1}{\sqrt{\xi^{p+1} - \theta^{p+1}}} d\theta \quad (\theta = \xi s) \]
\[ = \sqrt{\frac{p+1}{2}}\xi^{(1-p)/2} \int_0^1 \frac{1}{\sqrt{1 - s^{p+1}}} ds = \sqrt{\frac{p+1}{2}}\xi^{(1-p)/2}L_p. \]
By this, we have
\[ \xi = (2(p + 1))^{1/(p-1)}L_p^{2/(p-1)}. \] (2.18)
By this, (2.15) and (2.16), we obtain (1.5) and (1.6).

(ii) Let \( p = 1 \). Then by (1.1), we have
\[ -u''_\lambda(x) = \frac{\lambda}{\|u_\lambda\|_1^2}u_\lambda(x) = \pi^2 u_\lambda(x). \] (2.19)
It is clear that \( u_\lambda(x) = \alpha \sin \pi x \), and
\[ \|u_\lambda\|_1 = \int_0^1 \alpha \sin \pi x dx = \frac{2\alpha}{\pi}. \] (2.20)
By this and (2.19), we have
\[ \lambda = \pi^2 \|u_\lambda\|^q_1 = 2^q \pi^{2-q} \alpha^q. \] (2.21)
This implies (1.8). Thus the proof of Theorem 1.1 is complete.

By (2.14) and (2.18), for \( p > 1 \), we obtain
\[ \|W_p\|_p = (2(p + 1))^{1/(p-1)} M_p^{1/p} L_p^{(p+1)/(p(p-1))}. \] (2.22)
This implies (1.9). By (2.14) and (2.22), we obtain (1.10).

3 Proof of Theorem 1.2

First, we consider (1.1) under the condition (1.2). The approach to find the solutions of (1.1) is a variant of (2.3)–(2.4). Namely, we seek the solutions of (1.1) of the form
\[ u_\lambda(x) := t\|w_\lambda\|_p^{-1}w_\lambda(x) = t\|W_p\|_p^{-1}W_p(x) \] (3.1)
for some \( t > 0 \). To do this, let \( M(s) := (s + b)^q \). If we have solutions of (1.1) of the form (3.1), then since \( \|u_\lambda\|_p = t \) by (3.1), we have
\[ -M(\|u_\lambda\|_p^p)u''_\lambda(x) = -M(t^p) t \|w_\lambda\|_p^{-1} = -M(t^p) \|w_\lambda\|_p^{-1} \lambda w_\lambda(x)^p = M(t^p) t^{1-p} \|w_\lambda\|_p^{p-1} \lambda u_\lambda(x)^p. \] (3.2)
By this, we look for \( t \) satisfying
\[ M(t^p) t^{1-p} \|w_\lambda\|_p^{p-1} = 1. \] (3.3)
Namely, we solve the equation
\[ g(t) := (t^p + b)^q - \|w_\lambda\|_p^{1-p} t^{p-1} = 0. \tag{3.4} \]
By this, we have
\[ g'(t) = \left( p(t^p + b)^{q-1} t^{p-1} - (p-1)\|w_\lambda\|_p^{1-p} t^{p-2} \right) \tag{3.5} \]
By direct calculation, we see that \( \tilde{g}(t) \) is strictly increasing for \( t > 0 \). Further, by (1.2) and (3.5), we have
\[ g'(t) \leq pqt^p - (p-1)\|w_\lambda\|_p^{1-p} t^{p-2} \to -\infty \quad (t \to 0). \]
Therefore, we see that there exists a unique \( t_0 > 0 \) such that \( g(t_0) = 0 \) and \( g'(t) > 0 \) for \( t > t_0 \). By using (3.4) and (3.5), we find that
\[ g(t_0) = \frac{\|w_\lambda\|_p^{1-p}}{pq t_0} \{ -(pq - p + 1)t_0^p + b(p - 1) \}. \tag{3.6} \]
If \( g(t_0) < 0 \), then there exists exactly \( 0 < t_1 < t_0 < t_2 \) such that \( g(t_1) = g(t_2) = 0 \). If \( g(t_0) > 0 \), then (3.3) has no solutions. This idea will be used in the next sections. Proof of Theorem 1.2 is more simple, since \( t_0 \) is obtained explicitly.

**Proof of Theorem 1.2.** Since \( p = 2, q = 1 \), by (3.4), we have
\[ g(t) = t^2 + b - \lambda \|W_2\|_2^{-1} t = 0. \tag{3.7} \]
Then
\[ t_{1,\lambda} = \frac{\lambda \|W_2\|_2^{-1} - \sqrt{\lambda^2 \|W_2\|_2^{-2} - 4b}}{2}, \tag{3.8} \]
\[ t_{2,\lambda} = \frac{\lambda \|W_2\|_2^{-1} + \sqrt{\lambda^2 \|W_2\|_2^{-2} - 4b}}{2}. \tag{3.9} \]
By this and (3.1), we obtain (i), (ii) and (iii). Thus the proof is complete. \( \blacksquare \)

### 4 Proof of Theorem 1.3

In this section, let \( p > 1, q = 1 \). Since \( \|w_\lambda\|_p = \lambda^{-1/(p-1)} \|W_p\|_p \) by (2.2), we have from (3.5) that
\[ g'(t) = t^{p-2} \{ pt - \lambda \|W_p\|_p^{1-p} (p-1) \}. \tag{4.1} \]
We put
\[ t_0 := \frac{p-1}{p} \lambda \|W_p\|_p^{1-p}. \tag{4.2} \]
Then \( g'(t_0) = 0 \). By this and (3.4), we have
\[
g(t_0) = \left( \frac{p-1}{p} \lambda \|W_p\|_p^{1-p} \right)^p + b - \lambda \|W_p\|_p^{1-p} \left( \frac{p-1}{p} \lambda \|W_p\|_p^{1-p} \right)^{p-1} \tag{4.3}
\]
\[
= - \frac{1}{p-1} \left( \frac{p-1}{p} \lambda \|W_p\|_p^{1-p} \right)^p + b.
\]
We put
\[
\lambda_0 := \left( \frac{b(p-1)}{\lambda} \right)^{1/p} \frac{p}{p-1} \|W_p\|_p^{p-1}.
\tag{4.4}
\]
By this, we see that (i)–(iii) are valid, since if \( \lambda_0 \) satisfies (4.4), then \( g(t_0) = 0 \) by (4.3).
Further, \( g(0) = b > 0 \) and \( g(t) > 0 \) when \( t \gg 1 \).

We now prove (1.17). We assume that \( \lambda \gg 1 \). Then there exists \( 0 < t_1 < t_0 < t_2 \) which satisfy \( g(t_1) = g(t_2) = 0 \). Since \( t_0 \to \infty \) as \( \lambda \to \infty \), we see that \( t_2 \to \infty \) as \( \lambda \to \infty \). Then by (3.4), we have
\[
t_2 = \lambda \|W_p\|_p^{-1-p} + R, \tag{4.5}
\]
where \( R \) is the remainder term, and \( R = o(\lambda) \). By (3.4), (4.5) and Taylor expansion, we have
\[
(\lambda \|W_p\|_p^{1-p})^p \left( 1 + \frac{R}{\lambda \|W_p\|_p^{1-p}} \right)^p + b - (\lambda \|W_p\|_p^{1-p})^p \left( 1 + \frac{R}{\lambda \|W_p\|_p^{1-p}} \right)^{p-1} \tag{4.6}
\]
\[
= (\lambda \|W_p\|_p^{1-p})^p \left( 1 + \frac{pR}{\lambda \|W_p\|_p^{1-p}} (1 + o(1)) \right) + b
\]
\[
- (\lambda \|W_p\|_p^{1-p})^p \left( 1 + \frac{R(p-1)}{\lambda \|W_p\|_p^{1-p}} (1 + o(1)) \right) = 0.
\]
This implies that
\[
R = -b \left( \lambda \|W_p\|_p^{1-p} \right)^{1-p} (1 + o(1)). \tag{4.7}
\]
By this, (3.1) and (4.5), for \( \lambda \gg 1 \), we have
\[
u_{2,\lambda}(x) = \left\{ \lambda \|W_p\|_p^{1-p} - b (\lambda \|W_p\|_p^{1-p})^{1-p} (1 + o(1)) \right\} \|W_p\|_p^{-1} W_p(x). \tag{4.8}
\]
By putting \( x = 1/2 \) in (4.8), we have
\[
\alpha = \left\{ \lambda \|W_p\|_p^{1-p} - b (\lambda \|W_p\|_p^{1-p})^{1-p} (1 + o(1)) \right\} \|W_p\|_p^{-1} \|W_p\|_\infty. \tag{4.9}
\]
By this, we obtain
\[
\lambda = \|W_p\|_p^p \|W_p\|_\infty^{-1} \alpha + b \|W_p\|_\infty^{p-1} \alpha^{1-p} + o(\alpha^{1-p}). \tag{4.10}
\]
By this and (4.8), we obtain (1.17). We next prove (1.16). To do this, we consider the asymptotic behavior of \( t_1 \) as \( \lambda \to \infty \). If there exists a constant \( C > 0 \) such that \( C < t_1 < C^{-1} \). Then by (3.3), for \( \lambda \gg 1 \), we have
\[
g(t_1) = t_1^p + b - \lambda \|W_p\|_p^{1-p}t_1^{p-1} < 0. \tag{4.11}
\]
This is a contradiction, since \( g(t_1) = 0 \). If \( t_1 \to \infty \) as \( \lambda \to \infty \), then by (4.2) and (4.11), \( t_1 > t_0 \) for \( \lambda \gg 1 \). This is a contradiction. Therefore, \( t_1 \to 0 \) as \( \lambda \to \infty \). By (3.4), we have
\[
t_1^{p-1}(\lambda \|W_p\|_p^{1-p} - t_1) = b. \tag{4.12}
\]
By this and Taylor expansion, we have
\[
t_1 = \left( \frac{b}{\lambda \|W_p\|_p^{1-p} - t_1} \right)^{1/(p-1)} \tag{4.13}
\]
\[
= \|W_p\|_p b^{1/(p-1)} \lambda^{-1/(p-1)} \left( \frac{1}{1 - t_1 \lambda^{-1} \|W_p\|_p^{p-1}} \right)^{1/(p-1)}
\]
\[
= \|W_p\|_p b^{1/(p-1)} \lambda^{-1/(p-1)} \left( 1 + \frac{1}{p-1} \frac{t_1}{\lambda \|W_p\|_p^{1-p}(1 + o(1))} \right). \tag{4.14}
\]
By this and (3.1), we have
\[
u_{1,\lambda}(x) = b^{1/(p-1)} \lambda^{-1/(p-1)} \left( 1 + \frac{1}{p-1} \frac{b^{1/(p-1)} \|W_p\|_p^p}{\lambda^{p/(p-1)}} (1 + o(1)) \right) W_p(x). \tag{4.15}
\]
By this, we obtain
\[
\lambda_1 = b \|W_p\|_p^{p-1} \lambda^{-1/(p-1)} \left( 1 + \frac{1}{p-1} \frac{b^{1/(p-1)} \|W_p\|_p^p}{\lambda^{p/(p-1)}} (1 + o(1)) \right) \xi. \tag{4.16}
\]
By this and (1.10), we have
\[
\lambda_1 = b \|W_p\|_p^{p-1} \alpha^{-(p-1)} \left( 1 + b^{-1} \|W_p\|_p^p \|W_p\|_\infty^p \alpha^p (1 + o(1)) \right). \tag{4.17}
\]
By this and (4.14), we obtain (1.16). Thus the proof is complete. \( \blacksquare \)

## 5  Proof of Theorem 1.4

In this section, we assume (1.2) and \( \lambda \gg 1 \). We put \( k := ((p-1)\|W_p\|_p^{1-p}/pq)^{1/(pq-p+1)} \). By (3.5), we have
\[
t_0 = k \lambda^{1/(pq-p+1)}(1 + o(1)). \tag{5.1}
\]
By this, (1.2) and (3.4), we see that \( g(t_0) < 0 \). Then there exists \( 0 < t_1 < t_0 < t_2 \) such that \( g(t_1) = g(t_2) = 0 \). By (5.1), we see that \( t_2 \to \infty \) as \( \lambda \to \infty \). We first prove (1.19). We recall that \( m := \|W_p\|_p^{(1-p)/(pq-p+1)} \). By (3.4), we have
\[
t_2 = m \lambda^{1/(pq-p+1)} + r, \tag{5.2}
\]
where \( r \) is the remainder term satisfying \( r = o(\lambda^{1/(pq-p+1)}) \). It is clear that (3.4) is equivalent to

\[
(t_2^p + b)^q = \lambda \| W_p \|_p^{1-p} t_2^{p-1}. \tag{5.3}
\]

By this, (5.2) and Taylor expansion, we have

\[
\text{r.h.s. of (5.3)} = \lambda \| W_p \|_p^{1-p} \left( m\lambda^{1/(pq-p+1)} + r \right)^{p-1}
\]

\[
= \lambda \| W_p \|_p^{1-p} m^{p-1} \lambda^{(p-1)/(pq-p+1)} \left( 1 + (p-1) \frac{r}{m\lambda^{1/(pq-p+1)}} (1 + o(1)) \right). \tag{5.4}
\]

By (5.2) and Taylor expansion, we have

\[
\text{l.h.s. of (5.3)} = t_2^{pq} \left( 1 + \frac{b}{t_2} \right)^q = t_2^{pq} \left( 1 + \frac{bq}{t_2^p} (1 + o(1)) \right)
\]

\[
= (m\lambda^{1/(pq-p+1)} + r)^{pq} \left\{ 1 + bq(m\lambda^{1/(pq-p+1)} + r)^{-p} (1 + o(1)) \right\}
\]

\[
= (m\lambda^{1/(pq-p+1)})^{pq} \left( 1 + \frac{r}{m\lambda^{1/(pq-p+1)}}(1 + o(1)) \right)^{pq}
\]

\[
\times \left\{ 1 + bq(m\lambda^{1/(pq-p+1)})^{-p} (1 + o(1)) \right\}.
\]

By the definition of \( m \), we see that the leading terms of (5.4) and (5.5) coincide each other.

By (5.4) and (5.5), we have

\[
pq \frac{r}{m\lambda^{1/(pq-p+1)}} + \frac{bq}{m\lambda^{(1-p)/(pq-p+1)}(1 + o(1))} = (p-1) \frac{r}{m\lambda^{1/(pq-p+1)}}.
\]

This implies that

\[
r = - \frac{bqm^{1-p}}{pq - p + 1} \lambda^{(1-p)/(pq-p+1)}(1 + o(1)). \tag{5.7}
\]

By this and (5.2), for \( \lambda \gg 1 \), we have

\[
t_2 = \left\{ m\lambda^{1/(pq-p+1)} - \frac{bqm^{1-p}}{pq - p + 1} \lambda^{(1-p)/(pq-p+1)}(1 + o(1)) \right\}.
\]

By this and (3.1), we have

\[
u_{2,\lambda}(x) = \left\{ m\lambda^{1/(pq-p+1)} - \frac{bqm^{1-p}}{pq - p + 1} \lambda^{(1-p)/(pq-p+1)}(1 + o(1)) \right\} \| W_p \|_p^{p-1} W_p(x). \tag{5.9}
\]

This implies (1.19). We next show (1.18). We consider the asymptotic behavior of \( t_1 \) as \( \lambda \to \infty \). By the same argument as that in Section 4, we find that \( t_1 \to 0 \) as \( \lambda \to \infty \). By (5.3), we have

\[
\lambda \| W_p \|_p^{1-p} t_1^{p-1} = b^q (1 + o(1)). \tag{5.10}
\]

This implies that

\[
t_1 = b^{n/(p-1)} \| W_p \|_p \lambda^{-(1-p)/(p-1)} (1 + \eta), \tag{5.11}
\]
where $\eta$ is the remainder term. Then by Taylor expansion and (5.11), we have

$$\text{l.h.s. of (5.3)} = (b + t_1^p)^q = b^q(1 + b^{-1}t_1^p)^q = b^q(1 + b^{-1}qt_1^p + o(t^p)), \quad (5.12)$$

$$\text{r.h.s. of (5.3)} = b^q(1 + (p-1)\eta + o(\eta)).$$

By this, we have

$$\eta = \frac{q}{p-1}b^{-1}t_1^p = \frac{q}{p-1}b^{(pq-p+1)/(p-1)}\|W_p\|_{p\lambda-p/(p-1)}(1 + o(1)). \quad (5.13)$$

By this and (5.11), we have

$$u_{1,\lambda}(x) = b^{q/(p-1)}\lambda^{-1/(p-1)} \left\{ 1 + \frac{q}{p-1}b^{(pq-p+1)/(p-1)}\|W_p\|_{p\lambda-p/(p-1)}(1 + o(1)) \right\} W_p(x). \quad (5.14)$$

This implies (1.18). Thus the proof is complete.

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