Optimal terminal sliding-mode control for second-order motion systems

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Abstract—Terminal sliding mode (TSM) control algorithm, and its non-singular refinement, have been elaborated for two decades and since then belong to a broader class of the finite-time controllers – known to be robust against the matched perturbations. While TSM manifold allows for various forms of the sliding variable, satisfying the $q/p$ ratio for the power of the measurable output state, we demonstrate that $q/p = 0.5$ is optimal for the second-order Newton’s motion dynamics with a bounded control action. The paper analyzes the time optimal sliding surface and proposes an optimal TSM control for the second-order motion systems. Additional insight is given into the finite-time convergence of TSM and robustness against the bounded perturbations. Numerical examples with different type upper bounded perturbations are provided.

I. INTRODUCTION

Terminal sliding mode control belongs to the class of finite time controllers [1], see e.g. [2] for survey, for which a finite-time convergence can be guaranteed. The finite-time convergence is valid for both, reaching phase to the sliding surface and convergence to the state equilibrium, in other words to the origin when the control task is formulated as a zero-reference and initial value problem. In the first case one refers to a finite-time convergence of the sliding variable to zero, while for second case a finite-time convergence of the output tracking error to zero is meant. For more details on reaching and sliding phases and convergence we refer to the seminal literature [3], [4], [5]. Based on [6], [7] the terminal sliding mode control has been proposed in [8] and brought into the non-singular form in [9]. The difference between both will be summarized in the preliminaries provided in Section II. For the second-order Newtonian dynamics of a relative motion, it is possible to bring the system to equilibrium in finite time using a bounded control, i.e. with inherently bounded actuation forces. The well-known approach to this problem leads one to the co-called bang-bang control strategy [10], which is the time-optimal solution for the unperturbed double-integrator control problem. We will briefly sketch it in Section III, for convenience of the reader. In the rest of the paper, we show that based on the time-optimal control of double-integrator, an optimal terminal sliding mode control can be designed for the motion systems with bounded perturbations and control action. The main results are given in Section IV while in Section V three numerical examples of different-type matched perturbations are demonstrated and explained. Finally some conclusions are drawn in Section VI.

This work has received funding from the EU’s H2020-MSCA-RISE research and innovation programme under grant agreement No 734832.

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II. PRELIMINARIES

The original terminal sliding mode control suggests the first-order terminal sliding variable, cf. [9],

$$s = x_2 + \beta x_1^{q/p},$$

where $\beta > 0$ is a design parameter, and $p > q$ are the positive odd constants. Recall that terminal sliding-mode controllers belong to the widely used second-order sliding modes with finite-time convergence, see e.g. [11] for overview. For terminal sliding mode, in the most simple case, the control of a dynamic system $\dot{x}_1 = \dot{x}_2 = u$ is given by

$$u = -\alpha \text{sign}(s),$$

where $\alpha > 0$ is the control parameter. Often the requirement on $p, q$ to be odd is relaxed, and the frequently encountered terminal sliding surface is written as, cf. [11],

$$s = x_2 + \beta \sqrt{|x_1|}\text{sign}(x_1).$$

By taking time derivative of the sliding surface one can show that, cf. [12],

$$\dot{s} = -\alpha \text{sign}(s) + \beta x_2 (2\sqrt{|x_1|})^{-1},$$

One can recognize that a singularity occurs if $x_2 \neq 0$ when $x_1 = 0$. This situation does not occur in an ideal sliding mode, i.e. $s = 0$. However, the terminal sliding mode control cannot guarantee an always bounded control signal before the system is on the sliding manifold $s = 0$. Once on the surface, the sliding variable dynamics reduces to

$$2\dot{s} = -2\alpha \text{sign}(s) - \beta^2 \text{sign}(x_1).$$

Thereupon, one can show that the existence condition $\dot{s} < 0$ of an ideal sliding mode is fulfilled for $\beta^2 < 2\alpha$. For that case, the trajectories of the system reach the surface and remain there for all the time afterwards. This control system behavior is denoted as terminal mode, cf. [13], [12]. When the design parameters are assigned as $\beta^2 > 2\alpha$, the trajectories of the system do not remain on the sliding surface, while the control switches on $s = 0$ according to [2]. That case the control system proceeds with trajectories in the so-called twisting mode, cf. [13], [12].

For overcoming singularity of the sliding surface, the non-singular terminal sliding mode control with

$$s = x_1 + \beta^{-1} x_2^{p/q},$$

has been proposed in [9], while $\beta$, $p$, and $q$ parameters are as before. This control works around the surface singularity problem, while [1] and [6] describe one and the same switching surface, at least in the geometrical sense.
III. TIME OPTIMAL CONTROL OF SECOND-ORDER MOTION SYSTEMS

We consider a generalized second-order motion system
\[ \ddot{x}(t) = m^{-1}u(t) \] of an inertial body \( m \), which is first not affected by any counteracting, correspondingly disturbing, forces. For the sake of simplicity and without loss of generality we focus on the 1DOF motion, so that the scalar motion state variables are \( x_1 = x \) and \( x_2 = \dot{x} \). The control input force, here and for rest of the paper, is constrained as \( u \in [-U, +U] \). Here it is worth emphasizing that the saturated control, correspondingly actuation force, is in focus of our optimal terminal sliding mode control. Consequently \( U \) will appear as a parameter in the following analysis and control synthesis.

Assuming an initial state \([x_1(0), x_2(0)]^T\) and a final state \([x_1(t_f), x_2(t_f)]^T\) the time-optimal control \([14]\) will minimize the cost criterion
\[ J(u) = t_f = \int_0^{t_f} dt, \] thus minimizing the overall convergence time \( t_f \). Following the Pontryagin’s minimum principle and solving the constrained (by control limit \( U \) and initial and final conditions) optimization problem \([8]\) leads to the Hamiltonian function
\[ H = \lambda_0 + \lambda_1(t)x_2(t) + \lambda_2(t)u(t), \] with \( \lambda_0, \ldots, \lambda_2 \) to be the vector of Lagrange multipliers. Minimizing the Hamiltonian function yields the time optimal control of the form, cf. with \([14]\),
\[ u^o(t) = \begin{cases} +U, & \text{for } \lambda_0^2(t) < 0, \\ 0, & \text{for } \lambda_0^2(t) = 0, \\ -U, & \text{for } \lambda_0^2(t) > 0, \end{cases} \] where \( \lambda_0^2(t) \) is the control-related optimal Lagrange multiplier. Substituting this control law into \( x_2 \)-dynamics results in a two-point boundary value problem, see \([14]\) for details, from which the solution of optimal Lagrange multiplier gives
\[ \lambda_0^2(t) = -c_1^2 t + c_2^2. \]

Finally, the optimal constant values \((c_1^2, c_2^2) \neq (0, 0)\) need to be found such that the two-point boundary value problem is solved. With closer look on \([10], [11]\) one can see that: (i) the motion system \([7]\) is always fully accelerated or decelerated, (ii) the calculated multiplier \([11]\) determines the time instant of exactly one switching between both control actions, i.e. from \(+U\) to \(-U\) assuming \( x_1(0) < x_1(t_f) \). Such control behavior is very well known as the so-called “bang-bang” control, while the optimal switching time, resulting from \([11]\), is always depending on the initial and final states.

For the controlled system \([7], [10]\) we want to analyze the decelerating trajectory, i.e. after the time-optimal switching at \( t = \tau \), and that for \( u(t) = -U \) \( \forall \tau \leq t \leq t_f \). For the sake of simplicity and without loss of generality we will assume \( x_1(t_f) = x_2(t_f) = 0 \) and that for the rest of the paper. Consequently, all our analysis and developments, in the following, reduce to a problem of convergence to a stable equilibrium in the origin, while other control tasks with \((x_1(t_f), x_2(t_f)) \neq 0\) can be converted to the one above by an appropriate transformation of coordinates. For the phase-plane trajectories, the system \([7]\) with \( u = -U \) can be rewritten as
\[ \dot{x} = -U x, \] and after integration of the both sides and substitution of \( x_1 \) and \( x_2 \) one obtains the parabolic trajectory
\[ x_1 = -0.5mU^{-1}x_2^2. \] Obviously, the upper branch of \((x_1, x_2)\)-parabola constitutes the motion trajectory until \( t = t_f \), and that with the maximal possible deceleration forced by \( u(t) = -U \). An example of converging state trajectory, achieved with the time-optimal control \([10]\), during decelerating phase is shown in Fig. 1.

IV. OPTIMAL TERMINAL SLIDING MODE CONTROL

Now, with the preliminaries given in Section II and time optimal control summarized in Section III we are in the position to formulate the optimal terminal sliding mode control for the perturbed second-order motion systems. The proposed optimal terminal sliding surface is given by
\[ s = x_1 + cmU^{-1}x_2^2 \text{sign}(x_2), \] with \( c > 0 \) to be the single design parameter. Note that the proposed switching, correspondingly sliding, surface \([13]\) has the same form as the non-singular terminal sliding mode algorithm, see \([12]\), that corresponding to the \( q/p = 0.5 \) rate cf. Section II. Important to notice is that both shaping factors \( U \) and \( m \) do not represent any free parameters to be determined, but the inherent physical quantities characterizing the motion system \([1]\). Here one can see that the trajectory \([12]\) (of a time-optimal controlled system) coincides with the sliding surface \([13]\) for \( \alpha = 0.5 \). Further we will show that for that case, the boundary layer of the so-called twisting mode appears, cf. \([13], [12]\). The optimal control value is
\[ u(s) = -U \text{sign}(s), \] in which the control gain factor is the maximal possible one, being inherently limited by the constrained actuation force.

1 We stick on the generalized coordinates \( x \) and forces \( u \) for not explicitly distinguishing between the translational and rotational degrees of freedom (DOFs). As consequence, \( m \) is used for denoting both quantities: inertial mass and moment of inertia respectively.
The sufficient condition for existence of the terminal sliding mode on the surface \( \frac{1}{13} \) can easily be shown by assuming the candidate Lyapunov function
\[
V = 0.5s^2,
\]
for which
\[
0.5 \frac{d}{dt} s^2 < 0
\]
is consequently required. That means for once the trajectory reaches and crosses the surface \( s = 0 \) at \( t = t_r \), it remains on it \( \forall \; t > t_r \). The corresponding motion along \( s = 0 \), denoted as terminal mode, requires \( s \dot{s} < 0 \) to be always fulfilled. For showing it, we consider exemplary \( x_2 > 0 \), thus restricting ourself (yet without loss of generality) to the upper parabolic branch in the second quadrant of the phase-plane. Next we summarize \( \alpha mU^{-1} =: C \), for the sake of clarity, and with these simplifications the terminal sliding surface \( \frac{1}{13} \) reduces to \( s = x_1 + Cx_2^2 \). Taking time derivative and substituting control \( \frac{14}{} \) into dynamics \( \frac{7}{} \) one obtains
\[
\dot{s} = x_2 - 2m^{-1}CUx_2 \text{sign}(s).
\]
Multiplying both sides of \( \frac{17}{} \) with \( s \), the existence condition \( \frac{16}{} \) can be rewritten as
\[
x_2s - 2m^{-1}CUx_2|s| < 0.
\]
Substituting back the system parameters, instead of \( C \), and solving inequality \( \frac{13}{} \) with respect to the single design parameter, which is \( \alpha \), the existence condition of the optimal terminal sliding mode results in
\[
\alpha > 0.5.
\]

When the design parameter is selected as \( 0 < \alpha \leq 0.5 \), the existence condition \( \frac{16}{} \) of sliding mode becomes violated, and the trajectories do not remain on \( s = 0 \) upon crossing. Still the manifold \( \frac{13}{} \) appears as switching surface, and the trajectories reach the origin according to \( \frac{7}{} \). \( \frac{14}{} \). This behavior is known as twisting mode, in which the switching frequency increases towards infinity as the trajectory converges towards origin, while circulating around it.

The phase-portrait with trajectories of system \( \frac{7}{} \), \( \frac{14}{} \) is exemplary shown in Fig. \( \ref{2} \) together with the corresponding surface \( \frac{13}{} \), that for different \( \alpha \)-values. Here the system parameters are assigned as in the numerical examples provided in Section \( \ref{V} \) while the initial values \( (x_1(t_0), x_2(t_0)) = (-1, 0) \) are assumed for all three \( \alpha \) settings. One notices immediately that once the switching surface is reached, the control with \( \alpha = 0.3 \), that violates \( \frac{19}{} \), does not stay sliding on the surface, and the trajectory converges in a twisting mode. Note that here the chattering, due to non-ideal switching and computational issues, does not appear until the states reach the origin, meaning until \( (x_1(t_f), x_2(t_f)) = 0 \).

To establish the reachability condition, see \( \ref{15} \), \( \ref{5} \), one consider the candidate Lyapunov function \( \frac{15}{} \) for which the following conditions should be satisfied: (i) \( V(s) \) is positive definite \( \forall s \), (ii) \( \dot{V} < 0 \) for \( s \neq 0 \). For showing the finite-time convergence and, therefore, global stability the so-called called \( \eta \)-reachability \( \ref{15} \) condition
\[
s \dot{s} < -\eta|s|
\]
can be evaluated, in which \( \eta > 0 \) is a small design constant which ensures the reachability condition is satisfied. One slightly modifies the above condition (ii) to be
\[
\dot{V} \leq -\gamma \sqrt{V}, \quad \gamma = \sqrt{2\eta}.
\]
Integrating both sides of inequality \( \ref{21} \) one obtains, cf. \( \ref{5} \),
\[
\sqrt{V(t)} \leq -0.5\gamma t + \sqrt{V(0)}.
\]
Since reaching the sliding surface at finite-time \( t_r \), means \( V(t_r) = 0 \), one can easily obtain from \( \ref{22} \) the following
\[
t_r \leq \frac{\gamma^{-1}}{2} \sqrt{V(0)} = \gamma^{-1}2\sqrt{V(0)}|s(0)|.
\]
One can recognize that a bounded finite time can be guaranteed by \( \ref{23} \) and it depends on the initial value of the sliding manifold. In order to evaluate the \( \eta \)-reachability condition for the control system \( \ref{7} \), \( \ref{13} \), \( \ref{14} \), we rewrite \( \ref{23} \) as
\[
\text{sign}(s)\dot{s} < -\eta.
\]
Taking the time derivative of \( \frac{13}{} \) and substituting the control system dynamics \( \frac{7}{} \), \( \frac{14}{} \), one obtains
\[
\dot{s} = x_2 - 2\alpha|x_2|\text{sign}(s).
\]
Combining \( \ref{24} \) and \( \ref{25} \) and separating the variables we achieve the \( \eta \)-reachability condition as
\[
x_2\text{sign}(s) < -\eta + 2\alpha|x_2|.
\]
One can see that it is always possible to find a small positive constant \( \eta \) such that the condition \( \ref{26} \) is fulfilled \( \forall x_2 \).

Once the existence and reachability conditions, \( \ref{19} \) and \( \ref{26} \) respectively, are satisfied it is indicative to analyze the control system robustness against the perturbations. The external and internal perturbations\(^2\) matched by the control will directly affect the right-hand side of \( \frac{7}{} \) and, in worst
case, produce a counteracting disturbing force during the reaching phase, for which \(|u(t)| = U\) at \(0 < t < t_r\). On the other hand, the sudden and rather ‘fast’ perturbations, when occurring during the sliding mode, will diverge the trajectories from the sliding manifold, thus anew giving rise to the basic reachability problem. Obviously, to guarantee that the optimal terminal sliding mode control can compensate for the unknown perturbations \(\xi(t)\), the single boundedness condition

\[ |\xi| < U, \]  

is required for all time intervals \(T < t_r\), that to allow trajectories for reaching the sliding surface.

In the following, we will demonstrate different examples of an optimal terminal sliding mode control applied to the perturbed motion system. For that purpose, the system plant is extended by the matched upper bounded perturbation that results in

\[ \dot{x}(t) = m^{-1}(u(t) + \xi(t)). \]  

V. NUMERICAL EXAMPLES

The assumed numerical parameters, here and for the rest of the paper, are \(m = 0.1, U = 1, \alpha = 0.6\). All the controlled motion trajectories start at the initial state \((x_1(t_0), x_2(t_0)) = (-1,0)\), while the control set point is placed in the origin, meaning \((x_1(t_f), x_2(t_f))_{ref} = 0\). The implemented and executed simulations are with 1kHz sampling rate and the (most simple) first-order Euler numerical solver.

A. Motion system with nonlinear friction

The motion system dynamics is first considered affected by the nonlinear Coulomb friction \(f(x_2, t)\) which moreover includes the continuous presliding transitions, see [16] for details. It is worth emphasizing that the modeled Coulomb friction force is not discontinuous at zero velocity crossing, while a saturated friction force at steady-state results yet in \(F_c \text{sign}(\dot{x})\), where \(F_c > 0\) is the Coulomb friction constant. The friction force, which appears as an internal perturbation \(\xi = -f\), is assumed to be bounded by the half of the control amplitude, i.e. \(F_c = 0.5\). The controlled state trajectory is shown in Fig. 3 (a), while the region around origin is additionally zoomed-in for the sake of visualization. One can see that after convergence in the sliding mode, a stable low-amplitude \((\times 10^{-5})\) limit cycle around zero appears due to the by-effects caused by nonlinear presliding [16] friction. This is not surprising since \(F_c = 0.5U\), and no explicit presliding friction compensation is performed by the terminal sliding mode control which can only switch between \(u = \pm U\). The convergence of both state variables \((x_2)\) is down-scaled by factor 0.1 for the sake of better visualization is shown in Fig. 3 (b) over the progress of frictional perturbation. It can also be side-noted that feasibility of the Coulomb friction force compensation by means of a classical (first-order) sliding mode control with discontinuous control action was demonstrated in combination with a time-optimal bang-bang strategy in an experimental study [17].

As next, we assume an internal/external perturbation to be a periodic function of time, modeled by \(\xi(t) = 0.5\sin(20t)\). Note that such disturbing harmonic forces (or torques) may appear in various types of the motors, more generally actu-
double-integrator dynamics, for which only one dedicated control scheme uses the time-optimal trajectory of the systems with matched bounded perturbations. The proposed sliding mode control aimed for the second-order motion. It can be stressed that from all three types of the perturbation appears each time the binary perturbation changes the sign, and piecewise in the twisting-like mode. The latter transiently converges to zero, piecewise in the sliding-mode segments. The motion trajectory is highly perturbed. Yet, both states and, alike, in (b) the states convergence and perturbation are plotted together as the time series. One can recognize that the controlled state trajectory is shown in Fig. 5 (a), switching provides the maximal possible acceleration and deceleration in presence of the inherent control bounds. The proposed optimal terminal sliding mode has the single free design parameter α which allows for both, terminal and twisting control modes. The time-optimal twisting mode on the boundary layer, i.e. for α = 0.5, is in particular interesting for those motion control applications where a frequent sliding-mode switching of the control signal may be undesirable, like in the hydraulic control valves [19]. Here we abstain from discussion on pros and cons of the discontinuous and continuous sliding-mode controllers, see [20] for details. Rather we focus on the analysis and prove of existence and reachability conditions of the proposed optimal terminal sliding mode control. Three illustrative examples of the bounded internal and external perturbations demonstrate the control efficiency in a numerical setup.

VI. CONCLUSIONS

This paper proposes and analyzes the optimal terminal sliding mode control aimed for the second-order motion systems with matched bounded perturbations. The proposed control scheme uses the time-optimal trajectory of the double-integrator dynamics, for which only one dedicated

\[ x(t) = \begin{cases} \frac{d^2 x}{dt^2}, & \text{if } |\dot{x}(t)| < \alpha \frac{dx}{dt} \\ \frac{dx}{dt}, & \text{if } |\dot{x}(t)| \geq \alpha \frac{dx}{dt} \end{cases} \]

and, alike, in (b) the states convergence and perturbation are plotted together as the time series. One can recognize that the motion trajectory is highly perturbed. Yet, both states converge to zero, piecewise in the sliding-mode segments and piecewise in the twisting-like mode. The latter transiently appears each time the binary perturbation changes the sign. It can be stressed that from all three types of the perturbation examples, that one constitute a ‘worst’ case, which mostly challenges the designed control system.

\[ x(t) = \begin{cases} \frac{d^2 x}{dt^2}, & \text{if } |\dot{x}(t)| < \alpha \frac{dx}{dt} \\ \frac{dx}{dt}, & \text{if } |\dot{x}(t)| \geq \alpha \frac{dx}{dt} \end{cases} \]

Fig. 5. Phase portrait of state trajectory (a) and convergence of output tracking variable \( x_1(t) \) and down-scaled \( x_2(t) \)-state over perturbation (b).