Research Article

Riddled Attraction Basin and Multistability in Three-Element-Based Memristive Circuit

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By coupling a diode bridge-based second-order memristor and an active voltage-controlled memristor with a capacitor, a three-element-based memristive circuit is synthesized and its system model is then built. The boundedness of the three-element-based memristive circuit is theoretically proved by employing the contraction mapping principle. Besides, the stability distributions of equilibrium points are theoretically and numerically expounded in a 2D parameter plane. The results imply the memristive circuit has a zero unstable saddle focus and a pair of nonzero stable node-foci or unstable saddle-foci depending on the considered parameters. The dynamical behaviors include point attractor, period, chaos, coexisting bifurcation mode, period-doubling bifurcation route, and crisis scenarios, which are explored using some common dynamical methods. Of particular concern, riddled attraction basins and multistability are uncovered under two sets of specified model parameters nearing the tiny neighborhood of crisis scenarios by local attraction basins and phase plane plots. The riddled attraction basins with island-like structure demonstrate that their dynamical behaviors are extremely sensitive to the initial conditions, resulting in the coexistence of limit cycles with period-2 and period-6, as well as the coexistence of period-1 limit cycles and single-scroll chaotic attractors. Moreover, a feasible on-breadboard hardware circuit is manually made and the experimental measurements are executed, upon which phase plane trajectories for some discrete model parameters are captured to further confirm the numerically simulated ones.

1. Introduction

Chaos has attracted appreciable attention due to its potential applications in weather forecasting, aircraft control, and secure communications [1]. Besides, nonlinear electronic circuits have been counted as powerful candidates in supplying experimental and analytical surveys for people to understand chaotic behaviors in these engineering applications. Nonlinear elements including Chua’s diode, PN-junction diode, and memristor, just to refer a few, play a vital role for generating chaos in nonlinear electronic circuits and neuron system [2–8]. To possess the emblematic characteristics of both nonlinearity and nonvolatile memory [2, 3], memristor described by the relation between magnetic flux and electric charge has demonstrated prominent potential in memristor-based applications [4–10]. Particularly, memristor is universally regarded as a basic circuit element that it owns more excellent nonlinearity in constructing chaotic/hyperchaotic electronic circuits [11–27] and systems [28–32] than classical nonlinearities. Namely, a memristor-based circuit is a nonlinear circuit that can display chaos/hyperchaos [11, 12, 15], hidden dynamics [14], and other complex dynamics [16–21]. Thus, the memristor-based circuit is one of the most important application circuits and has received much attention in recent years. Moreover, the memristor-based chaotic circuits have inspired immense research attention in many chaos-based engineering fields including cryptosystem [33], cyber security communication [34], and
signal processing [35], just to mention a few. Thus, it is a quite pregnant work to construct a memristive chaotic circuit with an extremely simple topology and feasible hardware realization to promote the chaos-based engineering applications [27]. However, due to the technical drawbacks and high cost in fabricating nanoscale devices, most of the memristors employed in these memristive circuits are equivalently implemented by operational amplifiers and analog multipliers [16–18] as well as memristive diode bridges cascaded with RC [20, 21], LC [23], or RLC filters [36]. Nowadays, numerous memristive circuits are raised by employing these memristor emulators, from which some unique initial conditions associated phenomena are explored [13, 14].

The most vital experience to construct a memristor-based chaotic circuit is to lead one memristor or more with different nonlinearities into an existing linear or nonlinear electronic circuit [11–27]. Some memristive chaotic/hyperchaotic circuits with relatively simple topologies were derived from Chua’s family electronic circuits by replacing linear resistor or Chua’s diode with different memristors [11–16]. Moreover, by leading memristors into classical linear oscillating circuits or filter circuits, some novel memristive chaotic/hyperchaotic circuits including memristor-based Shinriki’s circuit [20], jerk circuit [21], Wien-bridge circuits [22–24], Sallen–Key filter [25], and bandpass filter [26] have been implemented. Extendedly, the two-memristor-based chaotic circuits developed from Chua’s circuit topology have been proposed [17, 18]. Generally speaking, most of these memristive circuits have the same topologies as the classical oscillating circuits. Herein, a simple three-element-based memristive circuit is synthesized, which only consists of a passive diode bridge-based second-order memristor, an active voltage-controlled memristor, and a coupling capacitor.

Interestingly, the novel dynamical behaviors including riddled attraction basin [37] and multistability [13, 14, 17, 20, 21, 30, 31, 38] have been numerically explored in some of these memristive circuits/systems. The riddled attraction basin means that if an attractive basin is riddled, an arbitrarily tiny perturbation in any initial condition can lead the dynamics to change from one to another [39]. Besides, this special phenomenon owning island-like attraction basin has been found in coupled neuron system [40], sea-level rise system [41], coupled chaotic oscillators [42], and integrate-and-fire circuit [43], just to refer a few. Furthermore, the multistability is a striking phenomenon in many memristor-based circuits/systems, which involves the coexistence of more than one long-time behavior for a given set of parameters under different initial conditions [29, 30]. Namely, riddled attraction basin often interplays with multistability naturally. In the previous literature [37], these two novel phenomena are numerically investigated in a 4D memristor-based chaotic system, but the experimental measurements are executed by employing a complex discrete component-based equivalent circuit of the memristor-based system. These gave us the inspiration of that the memristor-based circuit can generate the initial conditions associated behaviors of riddled attraction basin and multistability simultaneously. Besides, hardware level experimental measurements of these novel phenomena can effectively narrow the gap between theoretical investigations and memristor-based engineering applications [29].

Considering that a simple memristive chaotic circuit can serve as an elegant paradigm for better understanding of bifurcation and chaotic dynamics, it is a significant research topic to simplify memristive chaotic circuits by minimizing the number of dynamic elements and physical components [44]. Statistically speaking, most of the aforementioned memristive chaotic circuits can be regarded as two oscillating units coupled directly, linearly, or nonlinearly. Besides, an interesting work addresses the fact that two oscillators can share a common element to construct a chaotic circuit [1]. Inspired by these two considerations, a simple three-element-based memristive circuit including two memristors and one capacitor connected in parallel is presented, which can be regarded as two memristor-capacitor oscillating units with a sharing capacitor. As far as the present authors’ knowledge goes, the three-element-based memristive circuit with such a simple topology has not been reported in any previous literature.

The aim of this work is to reveal the unknown features in the proposed memristive circuit. Toward this purpose, the remainder of this paper is organized as follows. The mathematical model in a dimensionless form is built, and then the equilibrium points and their stabilities are explored, as well as the boundedness is proved in Section 2. MATLAB-based numerical simulations for two considered model parameters are executed in Section 3 by bifurcation plots, dynamical maps, finite-time Lyapunov exponents, and phase plane plots, from which the dynamical behaviors associated with two adjustable model parameters are explored. Besides, riddled attraction basins and multistability are disclosed under two sets of specified model parameters in Section 4. An on-breadboard hardware circuit is manually welded, upon which experimental measurements are executed to verify the numerical simulations in Section 5. The conclusions are drawn in the last section.

### 2. Three-Element-Based Memristive Circuit and Its Boundedness

The proposed three-element-based memristive circuit contains only a capacitor $C_1$, a passive diode bridge-based second-order memristor $M_1$ [23], and an active voltage-controlled memristor $M_2$ [16], as located in the middle of Figure 1. The circuit schematics of the memristors $M_1$ and $M_2$ are illustrated at the left- and right-hand sides, respectively, with dotted rectangles in Figure 1. The reactor $R_3$ is implemented by an NIC [17]. From the viewpoint of circuit topology and compositions, the proposed three-element-based memristive circuit is one of the simplest chaotic circuits in the literature.

For a passive diode bridge-based second-order memristor $M_1$ [23], $v_1$ and $i_1$ are the voltage and current at input terminal, respectively; $v$ and $i$ are the voltage across capacitor $C$ and the current through inductor $L$, respectively. The
The mathematical model of the memristor $M_1$ can be described as

$$I_1 = 2I_S e^{-\rho v_{i_1}} \sinh (\rho v_{i_1}),$$

$$C \frac{dv_{i_1}}{dt} = 2I_S e^{-\rho v_{i_1}} \cosh (\rho v_{i_1}) - 2I_S - i,$$

$$L \frac{di}{dt} = v,$$

where $\rho = 1/(2nV_T)$, $I_S$, $n$, and $V_T$ are three diode model parameters standing for the reverse saturation current, emission coefficient, and thermal voltage, respectively.

The active voltage-controlled memristor $M_2$ [16], as shown in the right of Figure 1, contains $U_1$ to realize a voltage follower for restraining load effect, $U_2$ to implement an integrator with $R_2$ to avoid DC voltage integral drift, two multipliers $U_3$ and $U_4$, and a current inverter. Denoting the integral capacitor voltage as $v_0$, the voltage $v'$ at the output terminal of multiplier $U_4$ can be deduced as

$$v' = g v_0^2 v_1,$$

where $g$ is the total gain of two multipliers $U_3$ and $U_4$. Thus, the current $i_2$ at the input terminal of the memristor $M_2$ can be obtained and simplified as

$$i_2 = i' = \frac{v_1 - v'}{-R_3} = \frac{1}{R_3} (1 - g v_0^2) v_1.$$

By employing Kirchoff’s current law to the inverting input terminal of $U_2$, we can obtain

$$C_0 \frac{dv_0}{dt} + \frac{v_1}{R_1} + \frac{v_0}{R_2} = 0.$$  

Consequently, the mathematical model for the voltage-controlled memristor can be described as

$$i_2 = \frac{1}{R_3} (1 - g v_0^2) v_1,$$

$$C_0 \frac{dv_0}{dt} = \frac{1}{R_1} v_1 - \frac{1}{R_2} v_0.$$

where $v_1$ is the voltage at the input terminal of the memristor $M_2$ and $v_0$ is the memristor inner state variable.

Applying Kirchhoff’s current law to node A and allying (1) and (5), the proposed memristive circuit in Figure 1 can be mathematically modeled as

$$C_1 \frac{dv_1}{dt} = \frac{1}{R_3} (1 - g v_0^2) v_1 - 2I_S e^{-\rho v_{i_1}} \sinh (\rho v_{i_1}),$$

$$C_0 \frac{dv_0}{dt} = \frac{1}{R_1} v_1 - \frac{1}{R_2} v_0,$$

$$C_1 \frac{dv}{dt} = 2I_S e^{-\rho v_{i_1}} \cosh (\rho v_{i_1}) - 2I_S - i,$$

$$L \frac{di}{dt} = v.$$

The parameters of the linear circuit elements are determined as $R_1 = 2 \, \text{k} \Omega$, $R_2 = 4 \, \text{k} \Omega$, $R_3 = 1.5 \, \text{k} \Omega$, $C_0 = C_1 = C = 4.7 \, \text{nF}$, $L = 10 \, \text{mH}$, and $g = 0.1 \, \text{V}^2$, and the model parameters of four 1N4148 diodes utilized in the diode bridge are assigned as $I_S = 5.84 \, \text{nA}$, $a = 1.94$, and $V_T = 25 \, \text{mV}$. These circuit parameters are collectively named as the typical circuit parameters in the following sections.

Denote

$$x = \rho v_{i_1}, y = \rho v_0, z = \rho v, w = R_1 \rho i,$$

$$\tau = t/(R_1 C_0), c = \tau (R_1 I_S), d = R_1 / R_2,$$

$$b = g \rho^2, c = 2 \rho R_1 I_S, d = R_1 / R_2,$$

$$f = R_0^2 C_0 / L.$$

Equation (6) can be rewritten as

$$\begin{align*}
\frac{dx}{d\tau} &= a (1 - by^2) x - ce^{-z} \sinh (x), \\
\frac{dy}{d\tau} &= -x - dy, \\
\frac{dz}{d\tau} &= ce^{-z} \cosh (x) - c - w, \\
\frac{dw}{d\tau} &= fz.
\end{align*}$$

Thus, the mathematical model of the proposed memristive circuit in dimensionless form can be built.

Herein, we discuss the boundedness of the solutions of system (8) by using the following theorem, which can be found in [45]. For the reference, it may be stated as follows.
\textbf{Theorem 1} (see 45, Theorem 1.1.1, page 14, Contraction Mapping Principle). Let \((S, \rho)\) be a complete metric space and let \(P: S \rightarrow S\). If there is a constant \(\alpha < 1\) such that for each pair \(\phi_1, \phi_2 \in S\), we have
\[ \rho(P\phi_1, P\phi_2) \leq \alpha \rho(\phi_1, \phi_2), \tag{9} \]

and then there is one and only one point \(\phi \in S\) with \(P\phi = \phi\).

We use the supremum metric denoted by \(\|\|\). Defining \(X = [x, y, z, w]^T\) and taking
\[ A = \begin{bmatrix} a & 0 & 0 & 0 \\ -1 & -d & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & f & 0 \end{bmatrix}, \tag{10a} \]

\[ g(X) = \begin{bmatrix} -abxy^2 - c e^{-z} \sinh(x) \\ 0 \\ c e^{-z} \sinh(x) - c \\ 0 \end{bmatrix}, \tag{10b} \]

system (8) is rewritten by
\[ \dot{X} = AX + g(X). \tag{11} \]

Fixed point method \cite{45} is used to draw the boundedness for system (11). The following theorem gives the boundedness of system (11).

\textbf{Theorem 2.} Let \(g()\) satisfy the Lipschitz condition \(\|f(X) - f(Y)\| \leq L_0 \|X - Y\|\) \((L_0\) is the Lipschitz constant), and let \(L_0 \int_0^t e^{A(t-s)} ds \leq \alpha < 1\) hold. If the initial state \(X(0)\) is bounded, then the solution \(X(t)\) of system (11) is bounded for \(t \geq 0\).

\textbf{Proof.} For a given \(X(0)\), applying the variation of parameters formula to obtain
\[ X(t) = X(0)e^{At} + \int_0^t e^{A(t-s)} g(X(s))ds. \tag{12} \]

We use the supremum metric and define the complete metric space \((S, \|\|)\) by
\[ S = \{\phi: [0, \infty) \rightarrow R^n | \phi \in C, \phi(0) = X(0), \phi \text{ bounded}\}, \tag{13} \]

where \(\phi \in C\) represents \(\phi\) to be continuous.

For \(\phi \in S\), define \((P\phi)(0) = X(0)\). And for \(t \geq 0\) yield
\[ (P\phi)(t) = X(0)e^{At} + \int_0^t e^{A(t-s)} g(\phi(s))ds. \tag{14} \]

Then, \(P\phi\) is continuous and bounded, and it is now clear that \(P: S \rightarrow S\). As \(g()\) satisfy the Lipschitz condition \(\|f(X) - f(Y)\| \leq L_0 \|X - Y\|\) and \(L_0 \int_0^t e^{A(t-s)} ds \leq \alpha < 1\), if \(\phi, \eta \in S\), then
\[ ||(P\phi)(t) - (P\eta)(t)|| \leq \int_0^t e^{A(t-s)} |g(\phi(s)) - g(\eta(s))|ds \leq L_0 \int_0^t e^{A(t-s)} \|\phi - \eta\|ds \leq \alpha \|\phi - \eta\|. \tag{15} \]

Therefore, by the contraction mapping principle, there is a unique fixed point \(\phi\) residing in \(S\), and that is the solution of system (11). Hence, the solution of three-element-based memristive circuit (8) is bounded.

The normalized typical model parameters in (8) are calculated from the typical circuit parameters by \cite{7} as
\[ a = 4/3, b = 9.409 	imes 10^{-4}, c = 2.4082 	imes 10^{-4}, d = 0.5, f = 1.88. \tag{16} \]

In the following investigations, the model parameters \(d\) and \(e\) are selected as two adjustable parameters with the other model parameters in (16) fixed.

Making the left-hand side of (8) equal to 0, the equilibrium point of (8) can be determined, which is given as
\[ P = (\overline{x}, \overline{y}, \overline{z}, \overline{w}) = (\overline{x}, -\overline{x}/d, 0, c \cosh(\overline{x}) - c), \tag{17} \]

where \(\overline{x}\) is the solution of the following transcendental equation:
\[ a[1 - b(\overline{x}/d)^3] \overline{x} - c \sinh(\overline{x}) = 0 \tag{18} \]

According to (8) and (17), the Jacobian matrix at the equilibrium point \(P\) is obtained and digested as
\[
\begin{bmatrix}
1 - b(\overline{x}/d)^3 & -c \cosh(\overline{x}) & 2ab\overline{x}^2/d & c \sinh(\overline{x}) \\
-1 & -d & 0 & 0 \\
c \sinh(\overline{x}) & 0 & -c \cosh(\overline{x}) & -1 \\
0 & 0 & f & 0
\end{bmatrix}
\tag{19}
\]

The characteristic equation of (19) is then obtained as
\[ P(\lambda) = \lambda^4 + m_1\lambda^3 + m_2\lambda^2 + m_3\lambda + m_4 = 0, \tag{20} \]

where
\[ m_1 = 2c \cosh(\overline{x}) + a \left( \frac{6\overline{x}^2}{d^2} - 1 \right) + d, \]
\[ m_2 = \left( \frac{c^2 + f}{d} + c \cosh(\overline{x}) \right) \left( 2d - a + \frac{6ab\overline{x}^2}{d^2} \right) + a \left( \frac{3b\overline{x}^2}{d} - d \right), \]
\[ m_3 = c \left( f + \frac{3ab\overline{x}^2}{d} - ad \right) \cosh(\overline{x}) - f(d + a) + c^2d, \]
\[ m_4 = f \left( a \left( \frac{3b\overline{x}^2}{d} - d \right) + cd \cosh(\overline{x}) \right). \tag{21} \]
The model parameters $d$ and $f$ are selected as two adjustable parameters within the regions of $0.1 \leq d \leq 0.9$ and $0.1 \leq f \leq 6$ to disclose the dynamical behaviors in (8). By numerical simulations, the solutions to $\vec{x}$ for the equilibrium points in the considered model parameter ranges are solved and plotted in Figure 2(a), which implies that (8) has one zero equilibrium point (marked as $P_0$) and two nonzero equilibrium points (marked as $P_\pm$) being symmetric about the origin. The distributions for $P_\pm$ are changed with the variation of the model parameter $d$, but independent of the model parameter $f$. Besides, four eigenvalues for $P_0$ and $P_\pm$ by applying different model parameters $d$ and $f$ are numerically obtained from (20), from which the stabilities can be determined.

For the zero equilibrium point $P_0$, (20) always owns a pair of conjugate roots with negative real parts and two real roots with opposite signs, which manifests that $P_0$ is an unstable saddle focus. For the nonzero equilibrium points $P_\pm$, the stabilities for them are the same. Thus, only the corresponding stability distributions for the equilibrium point $P$ are shown in Figure 2(b) in the considered model parameter ranges, where the yellow and green regions represent stable node-foci (marked as SNF), and the red region stands for the unstable saddle-foci (marked as USF), respectively. The yellow and green regions behave with the same stability, but (20) has two pairs of conjugate roots with negative real parts in the yellow region, whereas it has a pair of conjugate roots with negative real parts and two real roots with negative sign in the green region. The unstable saddle-foci in the red region has a pair of conjugate roots with positive real parts and two real roots with negative signs. Obviously, the sign for the real parts of the conjugate roots transmits from negative to positive when the two parameters changed from green region to red region, leading to the occurrence of Hopf bifurcation [27]. That is to say, system (8) loses its stability at the boundary between the green and red regions, leading to the occurrence of state transitions. Since the stabilities of the nonzero equilibrium points are associated with the two model parameters, system (8) can display rich dynamical behaviors when varying these model parameters.

3. Numerically Uncovered Dynamical Behaviors

As aforementioned, the model parameters $d$ and $f$ are selected as two adjustable parameters to numerically explore the dynamical behaviors in (8). In addition, MATLAB ODE23 algorithm with the time step 0.1 s and time interval [4.5 ks, 5 ks] is utilized for drawing bifurcation plots, and ODE23-based Wolf’s Jacobi method [46] with the time step 1 s and time span 30 ks is applied for plotting the dynamical maps and Lyapunov exponents.

To fully demonstrate the dynamical behaviors, two-dimensional (2D) bifurcation plots depicted by the periodicities of the variable $x$ and dynamical maps described by the largest Lyapunov exponent in the $d$-$f$ parameter plane are drawn in Figures 3(a) and 3(b), respectively, where the initial conditions $(0.1, 0.1, 10^{-3}, 10^{-5})$ are utilized.

In Figure 3(a), different colors labeled by different marks on the colorbar located at the right-hand side represent the corresponding dynamical behaviors, among which the color regions labeled by SS and DS denote the chaotic behaviors with single-scroll and double-scroll attractors, respectively. Besides, the color region labeled by P00 denotes the point attractor [47] and the other color regions labeled by P01 ∼ P09 represent the periodic behaviors with different periodicities from period-1 to period-9. Whereas in Figure 3(b), the different colors on the colorbar located at the right-hand side represent the corresponding values of the largest Lyapunov exponent, among which the red and deep yellow colors with the positive largest Lyapunov exponent denote the chaotic behaviors, and the other colors with the values of zero and negative ones represent the periodic behavior and point attractor, respectively.

It is clearly uncovered that the dynamical behaviors exhibited by the 2D bifurcation plots in Figure 3(a) match perfectly with those depicted by the dynamical maps in Figure 3(b), verifying the occurrence of complex dynamics for the three-element-based memristive circuit with the evolvement in the $d$-$f$ parameter plane.

To further brighten the dynamical behaviors in (8), one-dimensional (1D) bifurcation plots and finite-time
Lyapunov exponents for the parameter $d$ in the region $[0.1, 0.9]$ with the fixed parameter $f = 1.88$ and another parameter $f$ in the region $[0.1, 6]$ with the fixed parameter $d = 0.5$ are drawn in Figures 4(a) and 4(b), respectively. Remark that the numerical algorithms for drawing the 1D bifurcation plots and finite-time Lyapunov exponents in Figure 4 are the same as those employed in Figure 3. Herein, to clearly demonstrate the Lyapunov exponents, only the first three Lyapunov exponents are given. The trajectories colored in red are those starting from the initial conditions $(0.1, 0.1, 10^{-9})$ and the trajectories colored in blue are those starting from the initial conditions $(-0.1, 0.1, 10^{-9})$ in the bifurcation plots, whereas the initial conditions $(0.1, 0.1, 10^{-8}, 10^{-7})$ are employed in the calculation process of finite-time Lyapunov exponents.

In Figure 4(a), we consider the parameter $d$ as the bifurcation control parameter. The bifurcation diagram is plotted by checking the local maxima of variable $x$. The orbit begins with a stable point. After Hopf bifurcation happens at Hopf bifurcation point (HBP) [27] with $d = 0.3487$, the orbit loses its stability and runs in a limit cycle with period-1. Afterwards, the orbit evolves to single-scroll chaos following the period-doubling bifurcation route. It is noted that when considering two sets of initial conditions, two symmetric coexisting attractors can be revealed as $d < 0.444$, indicating the occurrence of coexisting bifurcation mode. Thus, dynamical behaviors including stable point, period, chaos, coexisting bifurcation mode, and period-doubling bifurcation route are explored as the parameter $d$ changes.

When the parameter $f$ is increased from 0.1, the orbit of system (8) starts from period and goes into chaos via crisis scenario at $f = 0.2159$. When $f$ increases to 0.3846, the orbit enters into period again. Thereafter, the orbit enters into chaos again via crisis scenario at $f = 0.4886$. The reverse period-doubling bifurcation scenarios occur at $f = 4.7408$, 4.8730, and 5.6980, respectively, and the orbit enters into period, as illustrated in Figure 4(b). The bifurcation diagrams show some narrow periodic windows in the chaos regions with the zero largest Lyapunov exponent. Moreover, the coexisting bifurcation mode occurs in the parameter regions of $0.1 \leq f \leq 1.177$ and $2.834 \leq f \leq 6$. Consequently, rich dynamical behaviors including period, chaos, coexisting bifurcation mode, period-doubling bifurcation route, and crisis scenarios are explored with the variation of the parameter $f$.

As shown in Figure 4, the 1D bifurcation plots coincide well with finite-time Lyapunov exponents, which further indicate the occurrence of period, chaos, period-doubling bifurcation route, and coexisting bifurcation mode with the variations of the adjustable parameters and initial conditions.

With respect to some discrete $d$ and $f$, the phase plane plots of the model (8) in the $x$-$y$ plane are numerically simulated to further confirm the emerging dynamical behaviors, as shown in Figures 5(a)–5(d), respectively, where the initial conditions of the trajectories colored in red and blue are the same as those utilized for the 1D bifurcation plots in Figure 4.

When $d = 0.1$ and $f = 1.88$, two coexisting point attractors are shown in Figure 5(a), with $LE_1 = 0.0011$, $LE_2 = -0.0046$, $LE_3 = -0.0515$, and $LE_4 = -0.1044$; when $d = 0.35$ and $f = 1.88$, two coexisting limit cycles with period-2 are displayed in Figure 5(b), with $LE_1 = 0.0011$, $LE_2 = -0.0283$, $LE_3 = -0.0394$, and $LE_4 = -8.8207$; when $d = 0.5$ and $f = 1.88$, a double-scroll chaotic attractor is demonstrated in Figure 5(c), with $LE_1 = 0.1339$, $LE_2 = 0.0001$, $LE_3 = -0.4610$, and $LE_4 = -0.6482$; when $d = 0.5$ and $f = 3$, two coexisting single-scroll chaotic attractors are illustrated in Figure 5(d), with $LE_1 = 0.1660$, $LE_2 = 0.0006$, $LE_3 = -0.4701$, and $LE_4 = -11.1751$.

4. Riddled Attraction Basin and Multistability

It is interesting that the occurrence of crisis scenarios leads to the appearance of imperfect bifurcation routes in the 1D bifurcation plots, as shown in Figure 4. The dynamical
behavior switches from one to another suddenly when the crisis scenario happens [16], which may lead to the occurrence of multistability [48]. Additionally, the generated multistability mechanism in our proposed memristive circuit must be different from the ones in the chaotic systems owning the special structure of symmetry or with offset boosting [49, 50]. For clarity, two narrow parameter ranges are chosen as representative examples to further disclose the dynamical behaviors near the tiny neighborhood of the parameters with crisis scenarios happening. 1D bifurcation plots in the two considered narrow parameter ranges are illustrated in Figure 6, which indicates that the three-element-based circuit can really possess the initial conditions-associated periodic and chaotic behaviors.

With the generality and comparison, four sets of typical model parameters $d=0.35$ and $f=1.88$, $d=0.5$ and $f=3$, $(0.1, 0.1, 10^{-9}, 10^{-9})$, $(-0.1, 0.1, 10^{-9}, 10^{-9})$. Figure 5: Phase plane plots in the $x$–$y$ phase plane for different $d$ and $f$. (a) Coexisting point attractors for $d=0.1$ and $f=1.88$. (b) Coexisting limit cycles with period-2 for $d=0.35$ and $f=1.88$. (c) Double-scroll chaotic attractors for $d=0.5$ and $f=1.88$. (d) Coexisting single-scroll chaotic attractors for $d=0.5$ and $f=3$. Figure 4: Numerically simulated 1D bifurcation plots of the maxima of the variable $x$ and finite-time Lyapunov exponents with respect to $d$ and $f$, respectively. (a) The parameter $d$ varying with $f=1.88$. (b) The parameter $f$ varying with $d=0.5$. Complexity 7
$d = 0.3593$ and $f = 1.88$, and $d = 0.5$ and $f = 0.2155$ are selected to further explore if the initial conditions associated dynamical behavior is near the crisis scenario or not. The local attraction basins defined as the region of the initial conditions [51] $(x_0, y_0, 10^{-9}, 10^{-9})$ for the four sets of model parameters are plotted in Figures 7(a)–7(d), respectively. The adjustable initial conditions $x_0$ and $y_0$ are all scanned in the region of $[-15, 15]$. Remark that the local attraction basins in Figure 7 are drawn with the identical time step and time interval utilized in the bifurcation plots in Figure 6, and different attraction regions are padded by different colors to distinguish the attractor types. The color regions labeled by L-SS and R-SS denote the chaotic attractors with single-scroll located at the top left and bottom right corners in the $x$-$y$ plane, respectively. Besides, the color regions labeled by L-P1, L-P2, and L-P6, as well as R-P1, R-P2, and R-P6 denote the limit cycles with different periodicities of period-1, period-2, and period-6 located at the top left and bottom right corners in the $x$-$y$ phase plane, respectively.

For comparison, two sets of the model parameters $d = 0.35$ and $f = 1.88$, $d = 0.5$ and $f = 3$ away from the crisis scenarios utilized in Figures 5(b) and 5(d) are employed to sketch the attraction basins in Figures 7(a) and 7(b), with which only the coexistence of limit cycles with period-2 and single-scroll chaotic attractors are generated in the proposed memristive circuit, respectively. The riddled attraction basins merely appear within some initial condition ranges, which divides the attraction basins into unriddled zone and small riddled ones [43]. Whereas for the model parameters $d = 0.3593$ and $f = 1.88$ that are near the crisis scenario, it is interesting that the riddled attraction basin with the island-like structure is formed in the whole initial conditions plane, as shown in Figure 7(c). For the model parameters $d = 0.5$ and $f = 0.2155$, the attraction basin is divided into finite zones, revealing the dynamical behaviors of coexistence of limit cycles with period-1 and single-scroll chaotic attractors. Obviously, the riddled attraction basins and multistability easily appeared under the set of model parameters near the tiny neighborhood of the parameters with crisis scenarios, which can help us to find or to avoid them in the engineering applications of the proposed memristive circuit. To the best knowledge of the authors, the behavior of riddled attraction basin has been reported in many dynamical systems, but rarely in memristor-based physical circuit.

By numerical simulations, the coexistence of limit cycles with period-2 and period-6 are demonstrated corresponding to the dynamical behaviors revealed in the riddled attraction basins, as shown in Figure 8(a), where the initial conditions $(\pm 3, \mp 4.01, 10^{-9}, 10^{-9})$ for period-2 and $(\pm 3, \mp 4, 10^{-9}, 10^{-9})$ for period-6 located in different adjacent islands are employed. Note that the initial conditions are symmetrical about the origin and arbitrarily tiny perturbation in the initial conditions, which lead to the fact that the periodic behaviors change from period-2 to period-6, or vice versa. For $d = 0.5$ and $f = 0.2155$, the attraction basin of multistability behaviors is depicted in Figure 7(d). The coexistence of limit cycles with period-1 and single-scroll chaotic attractors are demonstrated, as shown in Figure 8(b), where the initial conditions $(\pm 5, \mp 5, 10^{-9}, 10^{-9})$ for limit cycles with period-1 and $(\pm 5, \mp 2.5, 10^{-9}, 10^{-9})$ for the single-scroll chaotic attractors are utilized. The numerical simulations show that the riddled attraction basin and multistability emerge in such a simple memristive circuit under specified model parameters near the crisis scenarios happening.

5. Hardware Experiments

A hardware experimental circuit is manually welded on a breadboard using commercially available discrete components, upon which the circuit running trajectories can be captured to verify the MATLAB numerical simulations. The hardware experimental circuit employs seven potentiometers (two of the seven potentiometers are utilized to adjust the gain of multiplier $U_0$), a manually winding inductor, three monolithic capacitors, four N4148 diodes as well as three operational amplifiers TL082CP, and two multipliers AD633 with bipolar $\pm 15$ V supply, as shown in Figure 9. The typical circuit parameters are employed in hardware measurements. All the resistor values in our hardware experiments are measured by Tonghui TH2816A Precision LCR Meter. Additionally, the running trajectories are captured by a four-channel digital oscilloscope in XY mode.

By keeping the typical circuit parameters and tuning the potentiometer $R_2$ and the inductor $L$ to meet the model

\[x_{\text{max}}(0.1, 0.1, 10^{-9}, 10^{-9})
(−0.1, 0.1, 10^{-9}, 10^{-9})
0.19 0.21 0.23 0.250.17\]

Figure 6: Numerically simulated 1D bifurcation plots of the maxima of the variable $x$ in two narrow parameter ranges of $d$ and $f$, respectively. (a) The parameter $d$ varying with $f = 1.88$. (b) The parameter $f$ varying with $d = 0.5$. Complexity
parameters utilized in the MATLAB numerical simulations, the experimentally captured trajectories in the $v_1-v_0$ plane are shown in Figure 10 and Figure 11, respectively. The values of $R_2$ and $L$ can be calculated as $R_2 = 20 \, \text{k}\Omega$, $5.71 \, \text{k}\Omega$, and $4 \, \text{k}\Omega$ with $L = 10 \, \text{mH}$ as well as $R_2 = 4 \, \text{k}\Omega$ with $L = 6.27 \, \text{mH}$ by (7) corresponding to Figures 5(a)–5(d), respectively. In Figures 11(a) and 11(b), the circuit parameters are calculated as $R_2 = 5.57 \, \text{k}\Omega$ with $L = 10 \, \text{mH}$ and $R_2 = 4 \, \text{k}\Omega$ with $L = 87.24 \, \text{mH}$ corresponding to Figures 8(a) and 8(b). It is remarked that, to capture the coexisting attractors in the hardware experiments, the desired initial conditions are difficult to assign in the hardware experiments, which are randomly sensed through turning on the hardware circuit power supplies repeatedly.
Figure 9: The hardware experimental circuit of the three-element-based circuit. (a) An overviewed graph. (b) An enlargement of on-breadboard circuit.

Figure 10: Experimentally captured trajectories in the $v_1$–$v_0$ plane for different $R_2$ and $L$. (a) Coexisting point attractors for $R_2 = 20\, \text{k}\Omega$ and $L = 10\, \text{mH}$. (b) Coexisting limit cycles with period-2 for $R_2 = 5.71\, \text{k}\Omega$ and $L = 10\, \text{mH}$. (c) Double-scroll chaotic attractors for $R_2 = 4\, \text{k}\Omega$ and $L = 10\, \text{mH}$. (d) Coexisting single-scroll chaotic attractors for $R_2 = 4\, \text{k}\Omega$ and $L = 6.27\, \text{mH}$. 

Complexity
This randomly sensed way is different from the accurately assigned way in digitally FPGA-based hardware experiments [52]. Without considering the minor deviations by the calculation error, parameter error of the circuit components, and nonideality of the active device [53], the experimentally measured results shown in Figures 10 and 11 are well consistent with the MATLAB numerical simulations in Figures 5 and 8, respectively, which validate that the three-element-based memristive circuit can indeed generate these complex dynamical behaviors. Note that the trajectories of the coexisting attractors for different initial conditions are dividedly captured by the digital oscilloscope and are tackled by Adobe Photoshop to overlap them in one figure.

6. Conclusion

This paper mainly presents a comprehensive investigation of the dynamical behaviors in theoretical, numerical, and experimental surveys in a three-element-based memristive circuit. The memristive circuit having fewer circuit elements with a simply connected topology contains only two memristors and one capacitor, which can be regarded as two memristor-capacitor oscillating units with a shared capacitor. The boundedness of the three-element-based memristive circuit is theoretically proved by employing the contraction mapping principle, which implies that the trajectories of the memristive circuit are all confined in a limited phase space. The memristive circuit has three equilibrium points including one zero equilibrium point and two nonzero ones. The zero equilibrium point is always an unstable saddle focus, but the stability is stable node-foci or unstable saddle-foci for the two nonzero equilibrium points with respect to two considered model parameters, which lead to the occurrence of rich dynamical behaviors in the proposed memristive circuit. Dynamical behaviors associated with the two adjustable model parameters including point attractor, limit cycles with different periodicities, single-scroll and double-scroll chaotic attractors, coexisting bifurcation mode, period-doubling bifurcation route, and crisis scenarios are numerically simulated by MATLAB-based programs for 2D/1D bifurcation plots, dynamical maps, finite-time Lyapunov exponents, and phase plane plots. Particularly, under two sets of specified model parameters near crisis scenarios, the phenomena of riddled attraction basins and multistability are numerically discovered by local attraction basins. This is the first time to demonstrate the riddled attraction basins and multistability in such a simple memristive circuit. Moreover, an on-board circuit is manually made by the off-the-shelf components and hardware experiments are executed, from which the memristive circuit running trajectories under some discrete model parameters are captured to further validate the numerically simulated ones. Besides, the analog implementation of memristive circuit can effectively promote the integrated circuit design. These revealed results demonstrate that the presented three-element-based memristive circuit has a great potential for the application in real chaos-based engineering. Moreover, this memristive circuit is one of the simplest chaotic circuits in the literature, which also can provide powerful experimental and analytical platforms for people to understand the dynamical behaviors in electronic engineering and neurology. Moreover, the work done lets us conjecture that there are still some unknown features of this three-element-based memristive circuit, which are to be uncovered in our future research.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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