Logarithmic Exotic Conformal Galilean Algebras

Malte Henkel\textsuperscript{a}, Ali Hosseiny\textsuperscript{b,d}, and Shahin Rouhani\textsuperscript{c,d}

\textsuperscript{a} Groupe de Physique Statistique, Institut Jean Lamour (CNRS UMR 7198), Université de Lorraine Nancy, B.P. 70239, F–54506 Vandœuvre-lès-Nancy Cedex, France.

\textsuperscript{b} Department of Physics, Shahid Beheshti University G.C., Evin, Tehran 19839, Iran.

\textsuperscript{c} Department of Physics, Sharif University of Technology P.O. Box 11165-9161, Tehran, Iran.

\textsuperscript{d} School of Particles and Accelerators, Institute for Research in Fundamental Sciences (IPM) P.O. Box 19395-5531, Tehran, Iran.

e-mail: malte.henkel@univ-lorraine.fr, al_hosseiny@sbu.ac.ir and rouhani@ipm.ir

Abstract

Logarithmic representations of the conformal Galilean algebra (CGA) and the Exotic Conformal Galilean algebra (ECGA) are constructed. This can be achieved by non-decomposable representations of the scaling dimensions or the rapidity indices, specific to conformal galilean algebras. Logarithmic representations of the non-exotic CGA lead to the expected constraints on scaling dimensions and rapidities and also on the logarithmic contributions in the co-variant two-point functions. On the other hand, the ECGA admits several distinct situations which are distinguished by different sets of constraints and distinct scaling forms of the two-point functions. Two distinct realisations for the spatial rotations are identified as well. The first example of a reducible, but non-decomposable representation, without logarithmic terms in the two-point function is given.
1 Introduction

Logarithmic conformal field theories (LCFT) arose, by noticing that the independent solutions of the null vector equation governing the behaviour of the four point function, could coincide in certain cases; giving rise to new independent solutions involving logarithms [1,2]. Previously this possibility was ignored because unitarity ruled it out; however, applications for such non-unitary theories could be found within condensed-matter or statistical physics (for reviews of LCFT and applications see [3–7]). On another front, recent developments has attracted interest towards non-relativistic conformal field theories (NRCFT) [8–18]. These are theories based on attempted extensions of the Galilean symmetries, the motivation being that they may apply to low-energy and/or time-dependent systems in condensed-matter or statistical physics. The best-known special cases of such symmetry algebras are the Schrödinger algebra and the conformal Galilei algebra (CGA), both to be defined below. The natural question arises as to whether logarithmic correlators may be found for such NRCFTs [19–21], for a recent review see [22]. The answer is affirmative. Furthermore, applications including the one-dimensional contact process (Regge on field-theory) and the one-dimensional Kardar-Parisi-Zhang equation have been suggested [23,24]. In this paper, we present new logarithmic correlators for the Exotic Galilean Algebra (ecga) [25,26], which is CGA in 2+1 dimensions, but with an ‘exotic’ central charge.

Naturally, non-relativistic conformal symmetries are based on Galilean symmetry. A Galilean transformation \((x \mapsto x', t \mapsto t')\) acts on a point \(x\) in \(d\)-dimensional Euclidean space, at a given time \(t\), according to:

\[
x' = R x + b t + a, \quad t' = t + c
\]

where \(R \in \text{SO}(d)\) is a \(d \times d\) rotation matrix, \(b\) and \(a\) are \(d\)-dimensional vectors and \(c\) is a constant. However, we shall look at larger symmetries. For instance the symmetry group (called the Schrödinger group) of the free Schrödinger equation is larger:

\[
x' = \frac{R x + b t + a}{ft + g}, \quad t' = \frac{dt + c}{ft + g}, \quad dg - fc = 1
\]

The Lie algebra (Schrödinger algebra) spanned by the infinitesimal generators of the transformations \([1,2]\) is given below for \(1+1\) dimensions. Being non-semi-simple, this Lie algebra admits a non-trivial central charge, related to projective transformations of the solutions of the Schrödinger equation. It is related to the (non-relativistic) ‘mass’ \(M\) of the system. This can be generalised straightforwardly to what we shall call \(l\)-Galilei algebra\(^1\) by admitting a more complex transformation \([28,29]\):

\[
x' = \frac{R x + b_i t^{2l} + \ldots + b_1 t + b_0}{(ft + g)^{2l}}, \quad t' = \frac{dt + c}{ft + g}, \quad dg - fc = 1
\]

where the \(b_i, i = 0,1,\ldots, 2l\) are \(d\)-dimensional vectors. The transformations \([1.3]\) form a closed set and their infinitesimal generators span a closed Lie algebra only for \(l \in \frac{1}{2}Z\) half-integer or integer. The Schrödinger group and its Lie algebra are recovered for \(l = \frac{1}{2}\;\text{the case}\)

\(^1\)In some papers these are referred to as \(\text{spin} - l\) Galilei algebras [27], however the index \(l\) has nothing to do with spin.
\( l = 1 \) gives the *conformal Galilei group* and its Lie algebra, the *Conformal Galilean Algebra (CGA)* \(^{30}\). These transformations, and more generally those of (1.3), have in common the existence of a well-defined *dynamical exponent* \( z \) such that under a dilatation
\[
x \to \lambda x, \quad t \to \lambda^z t
\]
such that \( z = 1/l \) for the \( l \)-Galilei transformations (1.3). The two important special cases \( l = \frac{1}{2}, 1 \) also arise from two distinct more general approaches:

1. One may try to generalise (1.3) further by extending the projective conformal (or Möbius) transformations in the time \( t \) to arbitrary conformal transformations. Taking the projective terms describing the transformation of the wave functions into account, the only cases with local generators which close as a Lie algebra are, besides evidently conformal transformations in space-time, the cases \( l = \frac{1}{2}, 1 \) of the Schrödinger algebra and the CGA \(^{12}\).

2. When considering the non-relativistic limit of space-time conformal transformations and assuming the existence of a dynamical exponent \( z \), restriction to time-like and light-like geodesics reproduces exactly the Schrödinger algebra and the CGA, for \( z = 2 \) and \( z = 1 \), respectively \(^{16}\).

Physical applications either refer to strongly anisotropic systems at equilibrium, where the ‘time’ \( t \) is just a name for a peculiar spatial direction with strongly modified interactions such that \( z = \theta \) is better referred to as an ‘anisotropy exponent’ (paradigmatic examples are uniaxial Lifshitz points in lattice spin models with competing interactions); or else to real dynamics, at or far away from equilibrium. In the first case, co-varient \( n \)-point functions (such as we shall calculate later on) will represent physical correlators; the second case, causality constraints\(^2\) imply that \( n \)-point functions are to be interpreted as response functions with respect to some external perturbation. See \(^{15}\) for an introduction and overview on recent results. For brevity, we shall refer throughout to the two-point functions to be computed as ‘correlators’.

Finally returning to the Lie algebra of the symmetry transformations (1.3), in \( 1 + 1 \) dimensions it spanned by the generators:
\[
H = -\partial_t, \quad P^n = -t^n \partial_x; \quad n = 0, \ldots, 2l, \\
D = -(t\partial_t + lx\partial_x), \quad C = -(2ltx\partial_x + t^2\partial_t).
\]
(1.5)
with the following non-vanishing commutators
\[
[D, H] = H, \quad [D, C] = -C, \quad [C, H] = 2D, \\
[D, P^n] = (l - n)P^n, \quad [H, P^n] = -nP^{n-1}, \quad [C, P^n] = (2l - n)P^{n+1}.
\]
(1.6)
Known physical realisations of these algebras are known for \( l = 1/2 \) as the Schrödinger algebra\(^3\), for \( l = 1 \) as the CGA and for \( l = 2 \) and \( l = 3 \) in the Lifshitz points of first and

\(^2\)An algebraic method of derivation uses an embedding into a parabolic sub-algebra of the conformal algebra in \( d + 2 \) dimensions, see \(^{31}\) for details.

\(^3\)Especially in the phase-ordering kinetics far from equilibrium for spin systems quenched to temperatures \( T < T_c \) below the critical temperature \( T_c > 0 \), when for a non-conserved order parameter one has naturally \( z = 2 \) \(^{18, 32, 33}\).
second order in the ANNNS model, which adds uniaxial competing interactions to the so-called spherical model. It remains an open problem to find physical realisations for generic values of \( l \).

CGA is special because it can be obtained from the relativistic conformal algebra through contraction. When contracting, in some sense we are investigating the symmetry for low velocities. In other words we allow:

\[
\begin{align*}
  x &\to x, \\
  t &\to t, \\
  c &\to \infty.
\end{align*}
\]

(1.7)

In \( 1 + 1 \) dimensions, CGA is even more special since it has an infinite-dimensional extension (which is called ‘full CGA/altern-Virasoro algebra’ in the literature, contains a Virasoro sub-algebra and admits two independent central charges [38]) which in turn can be obtained fully from contraction [39,40]. This infinite-dimensional extension of the CGA is almost solvable [41], a property which helps to investigate logarithmic representations and holographic realisation easily [19,20].

Here, we study some properties of the finite-dimensional CGA (and leave aside its infinite-dimensional extensions). In \( 2 + 1 \) dimensions, CGA admits a non-trivial central extension (the so-called “exotic” central charge [25,26]) which forbids Galilean boosts to commute, reminiscent of non-commutative theories. Its physical significance has been of interest [42,43]. The central charge can also be obtained by contraction and two-point function is realised using auxiliary coordinates [27]. In this paper we consider this exotic algebra \( \text{ecga} \) and show that logarithmic representations exist. A new feature arises in the CGA and the \( \text{ecga} \) in that the rapidities allow for extra types of logarithmic representations, which we shall construct. We work out two-point functions for realisations in which the rapidity index is included. The ‘exotic’ extension of the CGA in \( 2 + 1 \) dimensions leads to several unexpected results on the form of the two-point functions; notably, we discuss the consequences of two distinct realisations of rotation-invariance (which from a purely algebraic point of view are indistinguishable). We hope these results to be useful in future attempts in identifying specific models with conformal galilean symmetries.

This paper is organised as follows: In section 2 we give a very brief presentation of LCFT, and recall the derivation of the two-point functions in logarithmic representations of the LCFT, the Schrödinger algebra and the CGA, using the elegant formalism of nilpotent variables. In section 3 we give a short introduction to the exotic CGA, and derive the two-point functions, both for scalar and logarithmic representations. Some conclusions are presented in section 4, with a table summarising our findings in a compact manner. Several appendices treat technical aspects of the calculations, either in the ECAG or on rotation-invariance.

\(^4\)See [12,28]. When considering the uniaxial Lifshitz points in systems with competing interactions such as the ANNNO(\( n \)) model, field-theoretic two-loop calculations have shown that the anisotropy exponent \( \theta - 1 = O(\varepsilon^2) \) in \( d = 4.5 - \varepsilon \) dimensions or \( \theta - 1 = O(1/n) \), which known, non-vanishing coefficients which are of the order \( \approx 10^{-3} - 10^{-2} \) [34,37]. The ANNNS model corresponds to \( n \to \infty \).
2 Logarithmic CFT: background

2.1 Basic formalism

Logarithmic conformal field theories (LCFTs) arise when indecomposable but reducible representations of the Virasoro algebra are taken [1,2] (for reviews see [3–6]). When the action of the scaling operator on the Verma module is not diagonal it gives rise to staggered modules [44, 45]. In the simplest case, the highest weight primary operator and its logarithmic partner form a rank-2 Jordan cell:

\[ L_0 \phi_h(Z)|0\rangle = h\phi_h(Z)|0\rangle, \quad L_0 \psi_h(Z)|0\rangle = h\psi_h(Z)|0\rangle + \phi_h(Z)|0\rangle. \] (2.1)

There is a simple method for dealing with case by introducing nilpotent variables \( \theta_i \) which satisfy the following relations:

\[ \theta_i^2 = 0, \quad \theta_i \theta_j = \theta_j \theta_i. \] (2.2)

These nilpotent variables also admit complex conjugation which go into the anti-holomorphic part of the primary operators:

\[ \bar{\theta}_i^2 = 0, \quad \bar{\theta}_i \bar{\theta}_j = \bar{\theta}_j \bar{\theta}_i. \] (2.3)

Now we can define our super-fields as

\[ \Phi(z, \theta) = \phi(z) + \theta \psi(z), \] (2.4)

and thereby equation (2.1) is written compactly as [46]:

\[ L_0|h + \theta\rangle = (h + \theta)|h + \theta\rangle, \] (2.5)

where the state \( |h + \theta\rangle \) is:

\[ |h + \theta\rangle = |h, 0\rangle + \theta|h, 1\rangle. \] (2.6)

This method allows a quick calculation of the two-point function. Concentrating on the holomorphic part of quasi-primary operators we obtain [46]:

\[ G(z_1, \theta_1; z_2, \theta_2) = \langle \Phi_1(z_1, \theta_1) \Phi_2(z_2, \theta_2) \rangle = g(\theta_1, \theta_2)(z_1 - z_2)^{-h_1 + \theta_1 + h_2 + \theta_2}\delta_{h_1, h_2}. \] (2.7)

where \( g(\theta_1, \theta_2) \) is given by

\[ g(\theta_1, \theta_2) = a(\theta_1 + \theta_2) + b\theta_1 \theta_2. \] (2.8)

and \( a, b \) are normalisation constants. Now, expanding both sides of (2.7) in powers of \( \theta_1, 2 \), one obtains the well-known logarithmic CFT two-point functions (with \( z := z_1 - z_2 \))

\[ \langle \phi(z_1)\phi(z_2) \rangle = 0, \]
\[ \langle \phi(z_1)\psi(z_2) \rangle = a z^{-2h_1} \delta_{h_1, h_2}, \]
\[ \langle \psi(z_1)\psi(z_2) \rangle = z^{-2h_1} (b - 2a \ln z) \delta_{h_1, h_2}. \] (2.9)

This offers a simple and fast way of obtaining logarithmic correlators in other algebras as well, as we shall demonstrate below. Of course, we merely discussed here the most simple scenario for the appearance of logarithmic representations and shall leave to future work the description of more complex situations.
2.2 Logarithmic representations of the Schrödinger algebra

The Schrödinger algebra is the symmetry of the free Schrödinger equation. It is naturally tied in with Galilean symmetry. It is the smallest \((l = 1/2)\) element of the \(l − \text{Galilei}\) algebras, plus a central extension:

\[
[P_i^0, P_j^1] = \mathcal{M}\delta_{ij} \tag{2.10}
\]

where the scalar \(\mathcal{M}\) is the non-relativistic mass and \(i, j = 1, \ldots, d\). The Schrödinger algebra \(\mathfrak{sch}(d)\) has a well-known infinite-dimensional extension (with a Virasoro sub-algebra) which is now usually called the ‘Schrödinger-Virasoro algebra’ \((\mathfrak{sv})\) [10]. In \(1 + 1\) dimensions, the algebra \(\mathfrak{sv}\) is represented by differential operators as (with \(n \in \mathbb{Z}\) and \(m \in \mathbb{Z} + \frac{1}{2}\)):

\[
X_n = -t^{n+1}\partial_t - \frac{1}{2}(n+1)t^n x\partial_x - \frac{1}{4}n(n+1)\mathcal{M}t^{n-1}x^2 - (n+1)ht^n, \\
Y^m = -t^{m+1/2}\partial_x - (m+\frac{1}{2})t^{m-1/2}\mathcal{M}x, \\
M^n = -\mathcal{M}t^n. \tag{2.11}
\]

These generators make up the dynamical symmetry algebra of the free Schrödinger equation \(\mathcal{S}\phi = 0\), with the Schrödinger operator \(\mathcal{S} := 2\mathcal{M}\partial_t - \partial_x^2 = 2\mathcal{M}0X^{-1} - (Y^{-\frac{1}{2}})^2\). All generators \((2.11)\) of \(\mathfrak{sch}(1) := \langle X^0, X^\pm 1, Y^\pm \frac{1}{2}, M^0 \rangle\) commute with \(\mathcal{S}\), with the two exceptions

\[
[S, X^0] = -\mathcal{S}, \quad [S, X^1] = -2t\mathcal{S} - 2\mathcal{M} \left( h - \frac{1}{2} \right) \tag{2.12}
\]

such that solutions of \(\mathcal{S}\phi = 0\) which have \(h = \frac{1}{2}\) will be mapped onto another solution (and an obvious generalisation to \(d \geq 1\) dimensions).

\(^5\) Representations of this algebra \(\mathfrak{sch}(1)\) may be constructed by invoking scaling states:

\[
X^0|h\rangle = 0. \tag{2.13}
\]

Now, similar to CFT, a rank 2 logarithmic representation may be found where two states exist, \(|h, 1\rangle\) and \(|h, 2\rangle\) such that the action of \(X^0\) on them is non-diagonizable

\[
X^0|h, 1\rangle = 0, \quad X^0|h, 2\rangle = |h, 1\rangle. \tag{2.14}
\]

We follow the formalism of the previous sub-section\(^6\). The two-point function is well known\(^7\):

\[
\langle \Phi_1(x_1, t_1, \theta_1), \Phi_2^*(x_2, t_2, \theta_2) \rangle = \delta_{h_1, h_2} \delta_{\mathcal{M}_1, \mathcal{M}_2} t^{-2h_1} \exp \left[ -\frac{\mathcal{M}_1 x^2}{2t} \right] \times \left( b(\theta_1 + \theta_2) + \theta_1 \theta_2 (c - 2b \ln t) \right) \tag{2.15}
\]

\(^5\) There is an unitary bound \(h \geq d/2\) for the Schrödinger algebra \(\mathfrak{sch}(d)\) in \(d\) dimensions [17].

\(^6\) At first sight, one might believe that because of the commutator \((2.10)\), with \(\mathcal{M} \neq 0\), invariance under space-translations and Galilei-transformations could not be required simultaneously. However, invariance under \(M^0\) gives the Bargman super-selection rule \(M^{[2]} = M_1 + M_2 = 0\). Hence the action of the commutator \((2.10)\) vanishes on any \(n\)-point function.

\(^7\) Here, the ‘complex conjugate’ \(\Phi^*\) is obtained from \(\Phi\) by changing the sign of the mass: \(\mathcal{M} \mapsto -\mathcal{M}\) [18].
where \( t := t_1 - t_2, x := x_1 - x_2 \) and \( b, c \) are normalisation constants. Expanding, one has
\[
\langle \phi_1(t_1, x_1) \phi_2^*(t_2, x_2) \rangle = 0, \\
\langle \phi(t_1, x_1) \psi_2^*(t_2, x_2) \rangle = b t^{-2h_1} \exp \left[ -\frac{\mathcal{M}_1 x^2}{2t} \right] \delta_{h_1, h_2} \delta_{\mathcal{M}_1, \mathcal{M}_2}, \tag{2.16} \\
\langle \psi_1(t_1, x_1) \psi_2^*(t_2, x_2) \rangle = t^{-2h_1} (c - 2b \ln t) \exp \left[ -\frac{\mathcal{M}_1 x^2}{2t} \right] \delta_{h_1, h_2} \delta_{\mathcal{M}_1, \mathcal{M}_2}.
\]

### 2.3 Logarithmic CGA in 1 + 1 dimensions

Galilean conformal algebra in 1 + 1 and 2 + 1 dimensions is special. In 1 + 1 dimensions, it is unique because it can be obtained directly from contracting 2-dimensional CFTs. Following this contraction many aspects of the fields can be extracted from CFT\(_2\). In 2 + 1 dimensions, it admits an ‘exotic’ central charge \[25\] [26].

For the moment, and to remain faithful to the method of previous sections, consider 1+1 dimensions:

\[
P = -\partial_x, \qquad K = -t \partial_x - \gamma, \qquad F = -t^2 \partial_x - 2t \gamma, \tag{2.17} \\
H = -\partial_t, \qquad D = -(t \partial_t + x \partial_x) - \Delta, \qquad C = -(2tx \partial_x + t^2 \partial_t) - 2t \Delta,
\]

in which \( \Delta \) is eigenvalue of dilation \( D \) and \( \gamma \) is eigenvalue of \( K \) which is called *rapidity*. These can be further embedded into an infinite-dimensional set of generators (where \( X^{-1,0,1} = H, D, C \) and \( Y^{-1,0,1} = P, K, F \)) which generate the infinite-dimensional Lie algebra called ‘Full CGA/altern-Virasoro algebra’ in the literature \((n \in \mathbb{Z})\):

\[
X^n = -\left[ t^{n+1} \partial_t + (n+1)t^n x \partial_x + (n+1)(t^n \Delta + nt^{n-1} \gamma x) \right] \\
Y^n = -\left[ t^{n+1} \partial_x + (n+1)t^n \gamma \right] \tag{2.18}
\]

with the commutators

\[
[X^m, X^n] = (m-n)X^{m+n}, \quad [X^m, Y^n] = (m-n)Y^{m+n}, \quad [Y^m, Y^n] = 0. \tag{2.19}
\]

Co-variant two-point functions are \[12\] [39] [48] (with \( x := x_1 - x_2 \) and \( t := t_1 - t_2 \))

\[
\langle \phi_1(t_1, x_1) \phi_2(t_2, x_2) \rangle_{\text{CGA}} = a \delta_{\Delta_1, \Delta_2} \delta_{\gamma_1, \gamma_2} t^{-2\Delta_1} \exp \left[ -\frac{2\gamma_1 x}{t} \right] \tag{2.20}
\]

As mentioned above, the interesting point regarding CGA in \( d = 1 + 1 \) dimensions is that we can obtain them from two-dimensional conformal symmetry by contraction. Briefly, \( d = 2 \) conformal symmetry consists of two commuting Virasoro algebras, with generators:

\[
L_n = -z^{n+1} \partial_z, \quad L_n = -\bar{z}^{n+1} \partial_{\bar{z}}. \tag{2.21}
\]

Under the contraction limit \((1.7)\), one observes that:

\[
X^n = L^n + \bar{L}^n \\
Y^n = \frac{1}{c}(L^n - \bar{L}^n) \tag{2.22}
\]
generate the algebra given by (2.19). The central charges of the two chiral copies of Virasoro algebra, namely $C$ and $\bar{C}$ combine to give the two independent central charges in the CGA, making it non-unitary [41], are a contracted limit of CFT$_2$, it might be possible that its representations are contracted limit of CFT$_2$ representations [41]. To observe this consider primary states in CFT$_2$:

$$X^0|h,\bar{h}\rangle = (L_0 + \bar{L}_0)|h,\bar{h}\rangle = (h + \bar{h})|h,\bar{h}\rangle,$$

$$Y^0|h,\bar{h}\rangle = \frac{L_0 - \bar{L}_0}{c}|h,\bar{h}\rangle = \frac{h - \bar{h}}{c}|h,\bar{h}\rangle.$$  

(2.23)

We observe that the scaling states of CFT$_2$ are scaling states of CGA, too. Now, they are identified by their scaling weight and rapidity. In other words

$$|h,\bar{h}\rangle \rightarrow |\Delta,\gamma\rangle,$$

(2.24)

in which

$$\Delta := h + \bar{h},$$

$$\gamma := \frac{h - \bar{h}}{c}.$$  

(2.25)

Now, to build a logarithmic representation of the full CGA, we expect logarithmic partners to appear in (2.23). The standard way to introduce them is to formally replace the real numbers/vectors $\Delta, \gamma$ by 2 $\times$ 2 matrices (we carry this out for any spatial dimension; the case $d = 2$ will be needed in section 3 below for the ECGA)

$$\Delta \mapsto \hat{\Delta} := \begin{pmatrix} \Delta & \Delta' \\ 0 & \Delta \end{pmatrix}, \quad \gamma \mapsto \hat{\gamma} := \begin{pmatrix} \gamma & \gamma' \\ \gamma'' & \gamma \end{pmatrix}$$

(2.26)

where we already used that one of the two matrices can without restriction of the generality be assumed to have a (non-diagonalisable) Jordan form.\footnote{This discussion is quite analogous to the one which applies to the logarithmic representations of the ‘ageing’ sub-algebra of the Schrödinger algebra (without time-translations) [23]; the two scaling dimensions $x, \xi$ used therein and their matrix generalisations play the same rôles as $\Delta, \gamma$ in the CGA studied here.}

In order to find the most general admissible form, we write the above representations (2.18) of the CGA as follows (and include all terms which describe the transformation of the scaling operators), with $n \in \mathbb{Z}$

$$X^n = -t^{n+1} \partial_t - (n+1)t^n x \cdot \partial_x - (n+1)t^n \begin{pmatrix} \Delta & \Delta' \\ 0 & \Delta \end{pmatrix} - n(n+1)t^{n-1} \begin{pmatrix} \gamma & \gamma' \\ \gamma'' & \gamma \end{pmatrix} \cdot x,$$

$$Y^n = -t^{n+1} \partial_x - (n+1)t^n \begin{pmatrix} \gamma & \gamma' \\ \gamma'' & \gamma \end{pmatrix},$$

$$R = -\epsilon_{ij} x_i \partial_j - \epsilon_{k\ell} \gamma_k \frac{\partial}{\partial \gamma_{\ell}} - \epsilon_{k\ell} \gamma'_k \frac{\partial}{\partial \gamma'_{\ell}} - \epsilon_{k\ell} \gamma''_k \frac{\partial}{\partial \gamma''_{\ell}}$$

(2.27)

(with $\partial_j := \partial/\partial x_j$) such that the non-vanishing commutators become

$$[X^n, X^m] = (n - m)X^{n+m} + (n + 1)(m + 1)(m - n)t^{n+m-1} \Delta' \gamma'' \cdot x \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$[X^n, Y^m] = (n - m)Y^{n+m} + (n + 1)(m + 1)t^{n+m} \Delta' \gamma'' \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

(2.28)

This discussion is quite analogous to the one which applies to the logarithmic representations of the ‘ageing’ sub-algebra of the Schrödinger algebra (without time-translations) [23]; the two scaling dimensions $x, \xi$ used therein and their matrix generalisations play the same rôles as $\Delta, \gamma$ in the CGA studied here.
and \([Y^a_i, R] = -\epsilon_{ij} Y^a_j\). In order to recover the commutators (2.19) of the CGA, we must have

\[
\Delta' \gamma'' = 0
\]

(2.29)

Hence, either \(
\Delta' = 0
\)

such that the matrix \(\hat{\Delta} = \Delta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\)

and \(\hat{\gamma}\) is either diagonalisable (which would give a pair of non-logarithmic representations) or else it has a Jordan form where one can always arrange for \(\gamma'' = 0\). Alternatively, we have directly \(\gamma'' = 0\). Therefore,

**one can always set \(\gamma'' = 0\) in (2.26).**

In summary without loss of generality, we can repeat eq. (2.14) by admitting as the most general case states:

\[
\begin{align*}
X^0|\Delta, \gamma; 1\rangle &= 0, & X^0|\Delta, \gamma; 2\rangle &= \Delta'|\Delta, \gamma; 1\rangle \\
Y^0|\Delta, \gamma; 1\rangle &= 0, & Y^0|\Delta, \gamma; 2\rangle &= \gamma'|\Delta, \gamma; 1\rangle
\end{align*}
\]

(2.30)

In the language of nilpotent variables, we define an eigenstate \(|\tilde{\Delta}, \tilde{\gamma}\rangle\) where

\[
\tilde{\Delta} = \Delta + \Delta' \theta, \quad \tilde{\gamma} = \gamma + \gamma' \theta.
\]

(2.31)

Equation (2.30) then reads as:

\[
\begin{align*}
X^0|\tilde{\Delta}, \tilde{\gamma}; 2\rangle &= \Delta'|\tilde{\Delta}, \tilde{\gamma}; 1\rangle, & Y^0|\tilde{\Delta}, \tilde{\gamma}; 2\rangle &= \gamma'|\tilde{\Delta}, \tilde{\gamma}; 1\rangle.
\end{align*}
\]

(2.32)

Now we follow on to calculate two-point functions:

\[
F(x_1, t_1, \theta_1; x_2, t_2, \theta_2) = \langle \tilde{\Delta}_1, \tilde{\gamma}_1|\phi_1(x_1, t_1)\phi_2(x_2, t_2)|\tilde{\Delta}_2, \tilde{\gamma}_2 \rangle
\]

(2.33)

Before going further let’s redefine our parameters such that for the general variable \(w\) we set:

\[
w = w_1 - w_2, \quad w^+ = w_1 + w_2.
\]

(2.34)

For instance:

\[
\begin{align*}
t &= t_1 - t_2, & t^+ &= t_1 + t_2 \\
\theta &= \theta_1 - \theta_2, & \theta^+ &= \theta_1 + \theta_2 \\
\Delta &= \Delta_1 - \Delta_2, & \Delta^+ &= \Delta_1 + \Delta_2
\end{align*}
\]

(2.35)

Two-point functions should be invariant under the Ward identities arising out of CGA elements \(X^{-1}, X^0, X^1, Y^{-1}, Y^0, Y^1\). First, since \(F\) must be invariant under space- and time-translation, it must be a function merely of \(t\) and \(x\) an not of \(t^+\) and \(x^+\). Invariance under \(Y^0\) is expressed as

\[
(t_1 \partial_{x_1} + \gamma_1 + \gamma'_1 \theta_1 + t_2 \partial_{x_2} + \gamma_2 + \gamma'_2 \theta_2) F = 0,
\]

(2.36)

\footnote{Implicitly, \(\Delta'\) and \(\gamma'\) are assumed to have the same value for both scaling operators.}
which reduces to:
\[(t\partial_x + \gamma^+ + \gamma'_1\theta_1 + \gamma'_2\theta_2)F = 0\]  
(2.37)

restricting \(F\) to:
\[F = e^{-(\gamma^+ + \gamma'_1\theta_1 + \gamma'_2\theta_2)\tilde{T}} [g_0(t) + g_1(t)\theta_1 + g_2(t)\theta_2 + g_3(t)\theta_1\theta_2].\]  
(2.38)

Now, invariance under \(Y^1\) gives:
\[t^+(t\partial_x + \gamma^+ + \gamma'_1\theta_1 + \gamma'_2\theta_2) + t(\gamma + \gamma'_1\theta_1 - \gamma'_2\theta_2)F = 0.\]  
(2.39)

So, we find that
\[\gamma = 0, \quad g_0(t) = 0, \quad \gamma'_1g_2 = \gamma'_2g_1,\]  
(2.40)

reducing \(F\) to:
\[F = e^{-(\gamma^+ + \gamma'_1\theta_1 + \gamma'_2\theta_2)\tilde{T}} \{w(t)\gamma'_1\theta_1 + w(t)\gamma'_2\theta_2 + g_3(t)\theta_1\theta_2\} \delta_{\gamma_1,\gamma_2}.\]  
(2.41)

in which \(w(t) = g_1(t)/\gamma'_1\). Now let’s look at \(X^0\) which appears as
\[(t\partial_t + x\partial_x + \Delta^+ + \Delta'_1\theta_1 + \Delta'_2\theta_2)F = 0,\]  
(2.42)

Inserting \(F\) from (2.41) in the above equation leads to:
\[(t\partial_t + \Delta^+)g_1(t) = 0,\]
\[(t\partial_t + \Delta^+)g_3(t) + \Delta'_1\gamma'_2w(t) + \Delta'_2\gamma'_1w(t) = 0,\]  
(2.43)

which results in
\[w(t) = at^{-\Delta^+},\]
\[g_3(t) = t^{-\Delta^+} (b - a(\Delta'_1\gamma'_2 + \Delta'_2\gamma'_1) \ln |t|).\]  
(2.44)

Action of \(X^1\) gives nothing new but super-selection rules:
\[\Delta = 0, \quad \Delta'_1 = \Delta'_2, \quad \gamma'_1 = \gamma'_2,\]  
(2.45)

This constraint will appear in \(G_{12}\) and \(G_{21}\). Since under exchange 1 ↔ 2 we have \(G_{12} ↔ G_{21}\) we can renormalize \(\phi\) in a way to arrange for a perfect symmetry under exchange of the scaling operators, such that \(\langle \phi_1\psi_2 \rangle = G_{12} \overset{!}{=} G_{21} = \langle \psi_1\phi_2 \rangle\). This leads to the stronger constraints:
\[\Delta'_1 = \Delta'_2, \quad \gamma'_1 = \gamma'_2.\]  
(2.46)

So, the final result is
\[F = e^{-(2\gamma_1 + \gamma^+\theta_1 + \gamma^+\theta_2)\tilde{T}} \left[ at^{-2\Delta_1\theta^+} + t^{-2\Delta_1} (b - 2a\Delta' \theta^+ \ln |t|) \theta^+ \theta^+ \right] \delta_{\gamma_1,\gamma_2} \delta_{\Delta_1,\Delta_2} \delta_{\Delta'_1,\Delta'_2} \delta_{\gamma'_1,\gamma'_2}.\]  
(2.47)
Expanding both sides of (2.33) in terms of nilpotent variables, we find the two-point functions for logarithmic primaries of CGA

\[ \langle \phi_1 \phi_2 \rangle = 0, \]
\[ \langle \phi_1 \psi_2 \rangle = a e^{-2\gamma_1 x} t^{-2\Delta_1} \delta_{\Delta_1, \Delta_2} \delta_{\gamma_1, \gamma_2} \delta_{\gamma_1', \gamma_2'}, \]
\[ \langle \psi_1 \psi_2 \rangle = e^{-2\gamma_1 x} t^{-2\Delta_1} \left[ -2a \Delta_1' \ln |t| - 2a \gamma_1' x + b \right] \delta_{\Delta_1, \Delta_2} \delta_{\Delta_1', \Delta_2} \delta_{\gamma_1, \gamma_2} \delta_{\gamma_1', \gamma_2'}. \tag{2.48} \]

One needs to notice that since \( \delta \phi \phi \) term is equal to zero, then we can re-scale \( \phi_1 \) so that \( \Delta_1' = \Delta_2' \) and thereby \( \gamma_1' = \gamma_2' \). These results can be obtained as well by contraction from a LCFT_2 where both chiral components have logarithmic partners.

Since we wrote the generators in (2.27) in an arbitrary number of space dimensions \( d \), it is now straightforward to write down the extension of (2.48) to \( d + 1 \) dimensions

\[ \langle \phi_1 \phi_2 \rangle(t, x) = 0, \]
\[ \langle \phi_1 \psi_2 \rangle(t, x) = a |t|^{-2\Delta_1} e^{-2\gamma_1 x/t} \delta_{\Delta_1, \Delta_2} \delta_{\gamma_1, \gamma_2} \delta_{\gamma_1', \gamma_2'}, \]
\[ \langle \psi_1 \psi_2 \rangle(t, x) = |t|^{-2\Delta_1} e^{-2\gamma_1 x/t} \left[ b - 2a \frac{x}{t} \cdot \gamma_1' - 2a \Delta_1' \ln |t| \right] \delta_{\Delta_1, \Delta_2} \delta_{\gamma_1, \gamma_2} \delta_{\gamma_1', \gamma_2'}, \tag{2.49} \]

with a manifest invariance under the spatial rotations (2.27)\(^{10}\). We also list explicitly the several super-selection rules, as they apply to the non-vanishing elements of the matrices \( \hat{\Delta} \) and \( \hat{\gamma} \).

### 3 Exotic CGA

The exotic CGA (ecga) is a centrally extended CGA in \( 2 + 1 \) dimensions. The generators \( P, K, F \) now become 2-dimensional vectors \( P, K, F \) (or equivalently \( Y^n \) is replaced by \( Y^n \)) such that the immediate extension of the commutators (2.19) is centrally extended by the nontrivial commutators \( [25, 26] \):

\[ [K_i, K_j] = \Xi \epsilon_{ij}, \quad [P_i, F_j] = -2\Xi \epsilon_{ij}; \quad i, j = 1, 2, \tag{3.1} \]

\( \epsilon_{ij} \) are the elements of the totally antisymmetry matrix \( \tilde{\epsilon} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) and and \( \epsilon_{12} = 1 \).

For realising the central charge, following (2.27) one may invoke two operators \( \chi_i \) such that

\[ [\chi_i, \chi_j] = \Xi \epsilon_{ij}, \quad [\chi_i, \Xi] = 0. \tag{3.2} \]

Since \( \Xi \) is central, one may represent it by its eigenvalue \( \Xi = \xi \). The ecga generators read:

\[ H = -\partial_t, \quad D = -x_i \partial_t - t \partial_i, \quad C = -2tx_i \partial_t - t^2 \partial_t - 2x_i \chi_i, \]
\[ P_i = -\partial_i, \quad K_i = -t \partial_i - \chi_i, \quad F_i = -t^2 \partial_i - 2t \chi_i - 2x_j \epsilon_{ij} \xi, \]
\[ J = -\epsilon_{ij} x_i \partial_j - \frac{1}{2\xi} \chi_i \chi_i, \tag{3.3} \]

\(^{10}\)In principle, the constants \( a, b \) can depend on the vectors \( \gamma_1 \) and \( \gamma_1' \). Rotation-invariance then implies \( a = a(\gamma_1^2, \gamma_1'^2, \gamma_1 \cdot \gamma_1') \) and analogously for \( b \).
Here, the generator $J$ of rotations was explicitly included as well. Its commutators with the other generators of the ECGA read

\[
\begin{align*}
[J, H] &= [J, D] = [J, C] = 0 \\
[J, P] &= \tilde{\epsilon} P \quad , \quad [J, K] = \tilde{\epsilon} K \quad , \quad [J, F] = \tilde{\epsilon} F
\end{align*}
\]  

(3.4)

Martelli and Tachikawa [27] obtain the above algebra by making a contraction of the $2+1$ dimensional conformal algebra where spin has been taken into account. In other words, they start by: (with $\mu$ and $\nu = 0, 1, 2$)

\[
\begin{align*}
\tilde{M}_{\mu\nu} &= -x_\mu \partial_\nu + x_\nu \partial_\mu - \Sigma_{\mu\nu}, \\
\tilde{P}_\mu &= -\partial_\mu, \\
\tilde{K}_\mu &= -x^\nu x_\nu \partial_\mu + 2x_\mu x^\nu \partial_\nu + 2x^\nu \Sigma_{\mu\nu} \\
\tilde{D} &= -x^\mu \partial_\mu,
\end{align*}
\]

(3.5)

where $\Sigma_{\mu\nu}$ is the spin. Now under the contraction limit (1.7) and redefining operators as:

\[
\begin{align*}
P_i &= \tilde{P}_i \frac{c}{c}, \\
H &= \tilde{P}_0, \\
K_i &= \tilde{M}_{0i} \frac{c}{c}, \\
D &= \tilde{D}, \\
F_i &= \tilde{K}_i \frac{c}{c}, \\
C &= -\tilde{K}_0, \\
\chi_i &= \frac{\Sigma_{0i}}{c}, \\
\xi &= \frac{\Sigma_{21}}{c^2}, \\
J &= \tilde{M}_{12} + \frac{1}{2c^2}(\Sigma_{0i} \Sigma_{0i}) + \Sigma_{12}
\end{align*}
\]

(3.6)

they end up with (3.3). Clearly, the operators $\chi_i$ and the central charge $\xi$ are remnants of the spin components.

The operators $\chi_i$ and the central charge $\xi$ can be realised explicitly via an auxiliary space with coordinates $\nu_1, \nu_2$:

\[
\chi_i = \partial_{\nu_i} - \frac{1}{2} \epsilon_{ij} \nu_j \xi
\]

(3.7)

Alternatively, one may use instead of $J$ a more natural-looking generator of infinitesimal rotations, including its action on the auxiliary space

\[
R := -\epsilon_{ij} x_i \partial_{x_j} - \epsilon_{ij} \gamma_i \partial_{\nu_j} - \epsilon_{ij} \nu_i \partial_{\nu_j}
\]

(3.8)

which obeys the same algebraic properties as the generator $J$.

In the above realisation, one expects the operators $D$ and $\Xi$ to have simultaneous eigenvectors since they commute, which is the primary state we use to construct the correlators. In order to include the rapidities as well, and to simplify the computation of two-point functions, we include all those terms which describe the transformation of the scaling operators into the generators. Then the action of all generators on two-point functions simply vanishes. The important Bargman super-selection rule of the central charge $\xi^{[2]} = \xi_1 + \xi_2 = 0$ follows. This is completely analogous to the treatment of the central charge in the Schrödinger algebra in section 2.2. In this new representation, the generators of the ECGA read (those given...
As a first step towards the logarithmic two-point functions from the non-logarithmic two-point functions, we will discuss central extensions for the existence of non-trivial non-conformal wave equations.

We observe that rotation invariance under the action of the generator $R$ leads to a different result than requiring co-variance under the rotation generator $J$ from (3.9), as used in [27]. In what follows, we distinguish these two cases and speak of ‘$J$-invariance’ if $J$ is used along with the generators of (3.9) and of ‘$R$-invariance’ when $R$ is used.

Quite analogously to Schrödinger-invariance treated above, the generators (3.9) are dynamical symmetries of the wave equation $S \phi = 0$, where the Schrödinger operator is

$$S := -\xi \partial_t + \epsilon_{ij}(\chi_i + \gamma_i)\partial_j = -\xi \partial_t + (\chi + \gamma) \hat{\epsilon} \partial_x$$

(3.10)

The only generators of the ECGA (3.9) which do not commute with $S$ are $D$ and $C$:

$$[S,D] = -S, \quad [S,C] = -2tS - 2\xi(\Delta - 1)$$

(3.11)

Rotation-invariance holds as well: $[S,J] = [S,R] = 0$. Hence one has a dynamical symmetry on the space of solutions of the equation $S \phi = 0$ where $\Delta = \Delta_\phi = 1$, consistent with the known unitary bound $\Delta \geq 1$ [27]. This illustrates again the importance of these non-trivial central extensions for the existence of non-trivial non-conformal wave equations.

### 3.1 Non-logarithmic two-point functions

As a first step towards the logarithmic two-point functions from the ECGA, we begin with the non-logarithmic case. This was done first by Martelli and Tachikawa [27], but only for vanishing rapidities $\gamma_i = 0$. It is one of the aims of this section to allow for $\gamma_i \neq 0$ and to analyse systematically the possible constraints. The two-point function is defined as

$$F := F(t_1, t_2; x_1, x_2; \nu_1, \nu_2) = \langle \phi_1(t_1, x_1, \nu_1)\phi_2(t_2, x_2, \nu_2) \rangle.$$  

(3.12)

We shall use throughout the variables as defined in (2.35) and apply the ECGA-Ward identities derived from (3.9) to $F$. Space- and time-translation-invariance restrict $F$ to be a function of $t = t_1 - t_2$ and $x = x_1 - x_2$. The invariance under the central charge $\Xi$ gives the important Bargman super-selection rule

$$\xi_1 + \xi_2 = 0$$

(3.13)

Invariance under the dilatations $D$ and the two generalised Galilei-transformations $K$ gives

$$(-t\partial_t - x \cdot \partial - \Delta_1 - \Delta_2) F = 0$$

(3.14)

$$(-t\partial - \gamma_1 - \gamma_2 - \chi_1 - \chi_2) F = 0$$

(3.15)
The first of those is solved by the ansatz
\[ f \] to use these identities first to simplify the remaining conditions and especially to derive
Rather than using these to parametrise immediately an explicit scaling form, we prefer
identity of the two generators \( F \), which gives \((-t^2 \partial_t - 2t (\chi_1 + \gamma_1) - 2\xi_1 \hat{e} \cdot x) F = 0 \) and
where the eqs. \( (3.13,3.14,3.15) \) have been used. Using again \( (3.15) \), this is further simplified to
\[ \left(-t (\chi_1 - \chi_2 + \gamma_1 - \gamma_2) - 2\xi_1 \hat{e} \right) F = 0 \] \( (3.16) \)
Similarly, invariance under \( C \) gives \((-t^2 \partial_t - 2t x \cdot \partial - 2\Delta_1 t - 2(\chi_1 + \gamma_1) \cdot x) F = 0 \), where
eqs. \( (3.13,3.15) \) have been used. Applying \( (3.14) \), again, leads to
\[ \left(-t x \cdot \partial - (\Delta_1 - \Delta_2) t - 2 (\chi_1 + \gamma_1) \cdot x \right) F = 0 \] \( (3.17) \)
Eqs. \( (3.14,3.15,3.16,3.17) \) contain the complete available information on the shape of the
two-point function \( F \), up to rotation-invariance, to be discussed below.
Multiplying \( (3.16) \) with \( x \), one has
\[ x \cdot (\chi_1 + \gamma_1) F = x \cdot (\chi_2 + \gamma_2) F \] \( (3.18) \)
such that comparison of eqs. \( (3.15,3.17) \) leads to \((\Delta_1 - \Delta_2) t F = 0 \). Hence, the constraint
\( \Delta_1 = \Delta_2 \) follows. \(^{11} \)
The remaining three independent equations can be further simplified via the ansatz
\[ F = t^{-2\Delta_1} e^{-(\gamma_1 + \gamma_2) u} f (u, \nu_1, \nu_2) \), \( u := x/t \] \( (3.19) \)
which leads to the following two conditions
\[ \left(\partial_u + \chi_1 + \chi_2 \right) f = 0 \), \( (\chi_1 - \chi_2 + \gamma_1 - \gamma_2 + 2\xi_1 \hat{e} u) f = 0 \] \( (3.20) \)
Only now, we use the explicit form \( (3.17) \). Further, we introduce the new variables \( \nu^\pm := \frac{1}{2} (\nu_1 \pm \nu_2) \) and write \( f = f(u, \nu_1, \nu_2) \) such that
\[ \left(\partial_u + \partial_{\nu^+} - \xi_1 \hat{e} \nu^- \right) f = 0 \), \( \left(\partial_{\nu^-} - \xi_1 \hat{e} \nu^+ + 2\xi_1 \hat{e} u + \gamma_1 - \gamma_2 \right) f = 0 \] \( (3.21) \)
The first of those is solved by the ansatz \( f = \exp [\xi_1 u \hat{e} \nu^-] \phi(w, \nu^-) \), with \( w := u - \nu^+ \). The last function \( \phi \) can be found from the equation \( (\partial_{\nu^-} + \xi_1 \hat{e} w + \gamma_1 - \gamma_2) \phi = 0 \). Hence\[ \phi (w, \nu^-) = \phi_0 (w) \exp \left[-\xi_1 \nu^- \cdot \hat{e} w - \nu^- \cdot (\gamma_1 - \gamma_2) \right] \] \( (3.22) \)
where \( \phi_0 \) is an arbitrary differentiable function, which besides on \( w \), can in principle also depend on the parameter vectors \( \gamma_{1,2} \).
Summarising our results, we can write the explicit two-point function, with \( u = x/t \)
\[ \langle \phi_1 \phi_2 \rangle = f_0 \left( u - \frac{\nu_1 + \nu_2}{2} \right) |t|^{-2\Delta_1} e^{-(\gamma_1 + \gamma_2) u - \frac{1}{2} (\gamma_1 - \gamma_2) (\nu_1 - \nu_2) \hat{e} \xi_1 u \hat{e} (\nu_1 - \nu_2) + \frac{1}{2} \xi_1 \nu_1 \cdot \nu_2 \delta_{\Delta_1,\Delta_2} \delta_{\xi_1,\xi_2,0}} \] \( (3.23) \)
\(^{11} \)For the non-exotic CGA, \( (3.10) \) or \( (3.18) \) would further imply \( \gamma_1 = \gamma_2 \).
where the symmetry under exchange of position 1 ↔ 2 is taken into account. One also uses the notation of the skew product \( a \wedge b := a \tilde{c} b = \epsilon_{ij} a_i b_j \). If the rotation-invariance is taken into account as well, the undetermined function written above as \( f_0 = f_0(w) \) becomes
\[
\begin{align*}
J\text{-invariance} &: \quad f_0 = f_0 \left( \gamma_1^2, \gamma_2^2, \gamma_1 \cdot \gamma_2, w + \tilde{c} (\gamma_1 - \gamma_2) \left( 2\xi_1 \right)^{-1} \right) \\
R\text{-invariance} &: \quad f_0 = f_0 \left( w^2, \gamma_1^2, \gamma_2^2, w \cdot \gamma_1, w \cdot \gamma_2 \right)
\end{align*}
\]
as is shown in appendix B.

Remarkably, the ECGA-covariant two-point function no longer needs to obey the constraint \( \gamma_1 = \gamma_2 \) of the rapidities which we had obtained in (2.49) for the non-exotic case.

### 3.2 Logarithmic Two-Point Functions

We are finally prepared for the computation of the co-variant two-point functions in the logarithmic representation of the ECGA, which in the most simple case can be obtained formally from the representation (3.9) by making the replacements
\[
\Delta \mapsto \hat{\Delta} := \begin{pmatrix} \Delta & \Delta' \\ 0 & \Delta \end{pmatrix}, \quad \gamma \mapsto \hat{\gamma} := \begin{pmatrix} \gamma & \gamma' \\ 0 & \gamma \end{pmatrix}
\]
see section 2.3. We seek the two-point functions
\[
F = \langle \phi_1 \phi_2 \rangle, \quad G_{12} = \langle \phi_1 \psi_2 \rangle, \quad G_{21} = \langle \psi_1 \phi_2 \rangle, \quad H = \langle \psi_1 \psi_2 \rangle
\]
where the arguments are implicit. Surprisingly, it turns out that the non-modified contribution \( F = \langle \phi_1 \phi_2 \rangle \) does not necessarily vanish, in contrast with the non-exotic CGA (2.49).

Throughout, temporal and spatial translations-invariance and invariance under the central charge \( \Xi \) shall be used, such that all two-point functions depend on \( t \) and \( x \) and the Bargman super-selection rule (3.13) is valid.

We now state the explicit result and refer to appendix A for the details of the calculation. Two distinct cases must be recognised, namely

1. **Case 1**: \( \Delta'_1 \neq 0 \) or \( \Delta'_2 \neq 0 \) and \( F = 0 \). This is the most direct extension of the logarithmic representations of the non-exotic CGA.

2. **Case 2**: \( \Delta'_1 = \Delta'_2 = 0 \) and \( F \neq 0 \). Here, only the rapidity matrices \( \hat{\gamma}_i \) will take a Jordan form, while \( \hat{\Delta} = \Delta \tilde{I} \) is diagonal.

In what follows, we shall use the notations from section 3.1.

In **Case 1**, we have \( F = 0 \) and \( G_{12} = G(t, x) = G(-t, -x) = G_{21} =: G \) such that
\[
\begin{align*}
G &= |t|^{-2 \Delta_1} e^{-(\gamma_1 + \gamma_2) u - \frac{1}{2} (\gamma_1 - \gamma_2) \cdot (\nu_1 - \nu_2)} e^{\xi_1 u \wedge (\nu_1 - \nu_2) + \frac{1}{2} \xi_1 \nu_1 \wedge \nu_2} g_0(w) \\
H &= |t|^{-2 \Delta_1} e^{-(\gamma_1 + \gamma_2) u - \frac{1}{2} (\gamma_1 - \gamma_2) \cdot (\nu_1 - \nu_2)} e^{\xi_1 u \wedge (\nu_1 - \nu_2) + \frac{1}{2} \xi_1 \nu_1 \wedge \nu_2} h(u, \nu_1, \nu_2) \\
h &= h_0(w) - g_0(w) \left( 2 \Delta'_1 \ln |t| + u \cdot (\gamma'_1 + \gamma'_2) + \frac{1}{2} (\nu_1 - \nu_2) \cdot (\gamma'_1 - \gamma'_2) \right)
\end{align*}
\]
together with the abbreviations \( u = x/t \) and \( w := u - \frac{1}{2} (\nu_1 + \nu_2) \) and the constraints \( \Delta_1 = \Delta_2, \Delta'_1 = \Delta'_2 \) and \( \xi_1 + \xi_2 = 0 \). The undetermined functions \( g_0(w) \) and \( h_0(w) \) still are subject to rotation-invariance, see below.

In Case 2, we find the constraints \( \Delta_1 = \Delta_2 \) and \( \xi_1 + \xi_2 = 0 \) and

\[
\begin{align*}
F &= |t|^{-2\Delta_1} e^{-\frac{1}{2} (\gamma_1 - \gamma_2)} \left( u - \frac{1}{2} (\nu_1 - \nu_2) \right) e^\xi u^{\wedge} (\nu_1 - \nu_2) + \frac{1}{2} \xi_{1} u^{\wedge} \nu_{2} f_0(w) \\
G_{12} &= |t|^{-2\Delta_1} e^{-\frac{1}{2} (\gamma_1 - \gamma_2)} \left( u - \frac{1}{2} (\nu_1 - \nu_2) \right) e^\xi u^{\wedge} (\nu_1 - \nu_2) + \frac{1}{2} \xi_{1} u^{\wedge} \nu_{2} g_{12}(u, \nu_1, \nu_2) \\
G_{21} &= |t|^{-2\Delta_1} e^{-\frac{1}{2} (\gamma_1 - \gamma_2)} \left( u - \frac{1}{2} (\nu_1 - \nu_2) \right) e^\xi u^{\wedge} (\nu_1 - \nu_2) + \frac{1}{2} \xi_{1} u^{\wedge} \nu_{2} g_{21}(u, \nu_1, \nu_2) \\
H &= |t|^{-2\Delta_1} e^{-\frac{1}{2} (\gamma_1 - \gamma_2)} \left( u - \frac{1}{2} (\nu_1 - \nu_2) \right) e^\xi u^{\wedge} (\nu_1 - \nu_2) + \frac{1}{2} \xi_{1} u^{\wedge} \nu_{2} h(u, \nu_1, \nu_2)
\end{align*}
\]

where

\[
\begin{align*}
g_{12} &= g_0(w) - f_0(w) \left( u - \frac{1}{2} (\nu_1 - \nu_2) \right) \cdot \gamma'_2 \\
g_{21} &= g_0(w) - f_0(w) \left( u + \frac{1}{2} (\nu_1 - \nu_2) \right) \cdot \gamma'_1 \\
h &= h_0(w) - g_0(w) \left( u \cdot (\gamma'_1 + \gamma'_2) + \frac{1}{2} (\nu_1 - \nu_2) \cdot (\gamma'_1 - \gamma'_2) \right) \\
&+ \frac{1}{2} f_0(w) \left( u + \frac{1}{2} (\nu_1 - \nu_2) \right) \cdot \gamma'_1 \left( u - \frac{1}{2} (\nu_1 - \nu_2) \right) \cdot \gamma'_2
\end{align*}
\]

Finally, in both cases, rotation-invariance must be taken into account, see appendix B for the details. If we use \( R \)-invariance, in both cases the functions \( f_0(w), g_0(w) \) and \( h_0(w) \) are short-hand notations for undetermined functions of 9 rotation-invariant combinations of \( w, \gamma_{1,2} \) and \( \gamma'_{1,2} \), for example

\[
f_0 = f_0 \left( w^2, \gamma_1^2, \gamma_2^2, \gamma_1'{}^2, \gamma_2'{}^2, w \cdot \gamma_1, w \cdot \gamma_2, w \cdot \gamma'_1, w \cdot \gamma'_2 \right)
\]

and analogously for \( g_0 \) and \( h_0 \). We point out that in both cases, there is no constraint neither on the \( \gamma_i \), nor the \( \gamma'_i \). Alternatively, if we use \( J \)-invariance we find that the \( \gamma \)-matrices are diagonal, viz. \( \gamma'_1 = \gamma'_2 = 0 \). Then only case 1 retains a logarithmic structure and we have

\[
\begin{align*}
g_0 &= g_0 \left( \gamma_1^2, \gamma_2^2, \gamma_1 \cdot \gamma_2, w + \hat{e} (\gamma_1 - \gamma_2) (2\xi_1)^{-1} \right) \\
h_0 &= h_0 \left( \gamma_1^2, \gamma_2^2, \gamma_1 \cdot \gamma_2, w + \hat{e} (\gamma_1 - \gamma_2) (2\xi_1)^{-1} \right)
\end{align*}
\]

So, the task is done and two-point functions of logarithmic representations of the ECGA have been worked out.

## 4 Conclusions

The exotic Galilean algebra corresponds to \( d = 2 \) and \( l = 1 \) case of \( l \)-Galilei algebras. This algebra arises as the singular limit of the conformal algebra when the speed of light tends to infinity. In other words it should describe the low velocity limit of conformal systems.
Table 1: Summary on the constraints obeyed by co-variant two-point functions in several variants of conformal galilean algebras. The first column indicates the non-exotic algebra CGA or the exotic ecga, where a prefix ‘L-’ indicates a logarithmic representation. The equation labels refer to the explicit form of the two-point function, as derived in the text. The various constraints on either scaling dimensions $\Delta$, rapidities $\gamma$ or the Bargman super-selection rule on the ‘masses’ $\xi$ are listed. The last three lines refer to the logarithmic representations of the exotic ecga. Therein, the extra labels refer to the two distinct choices of the rotation generator: either $R$-invariance with the two distinct case 1 (R1) and case 2 (R2), or else $J$-invariance (J).

However the low energy limit is often described by the Schrödinger algebra which is the $l = \frac{1}{2}$ case of $l$-Galilei algebras. This is rather paradoxical and the physical candidates for the realisation of CGA have proved hard to find.

In this work, we analysed the generic form of co-variant two-point functions, for scalar and logarithmic representations of conformal galilean algebra. The transformation of quasi-primary scaling operators under these algebras can be characterised in terms of a scaling dimension $\Delta$, a rapidity vector $\gamma$ and in the case of the exotic ecga also in terms of a ‘mass’ $\xi$. If one considers logarithmic representations, the scaling dimension and the rapidities can acquire a matrix form. Restricting to the most simple case of two-component logarithmic representations, these matrices have been shown to be simultaneously of a Jordan form

$$\Delta \mapsto \hat{\Delta} = \begin{pmatrix} \Delta & \Delta' \\ 0 & \Delta \end{pmatrix}, \quad \gamma \mapsto \hat{\gamma} = \begin{pmatrix} \gamma & \gamma' \\ 0 & \gamma \end{pmatrix}$$ (4.1)

Qualitatively very different results were obtained for the non-exotic CGA and the exotic ecga, as summarised in table 1

1. When considering the CGA, the extension to logarithmic representation essentially produced the expected generalisations of the constraints on both the scaling dimension $\Delta$ and the rapidity $\gamma$ also to the non-diagonal elements of the corresponding matrices, according to (4.1). In addition, the various two-point functions simply take up the same kind of logarithmic prefactors, see eq. (2.49), as one would have expected from logarithmic conformal or even logarithmic Schrödinger-invariance, see eqs. (2.9) (2.16).

2. Therefore, by comparing the results (2.9) of logarithmic conformal invariance and (2.16) of logarithmic Schrödinger-invariance, one might have believed that going over
to the ECGA would merely lead to the naturally expected Bargman super-selection rule \(\xi_1 + \xi_2 = 0\), which would be the analogue of non-relativistic mass conservation in Schrödinger-invariant systems. Remarkably, it turned out that the form of the co-variant two-point functions in the exotic ECGA is considerably richer.

3. For scalar representations, the presence of extra internal dimensions needed to realise the extra exotic structure releases the constraint on the rapidities \(\gamma_i\) of the two scaling operators. A finer difference arise through the possibility of two distinct choices for the generator of rotation, labelled \(J\) and \(R\), and referred to as ‘\(J\)-invariance’ and ‘\(R\)-invariance’. The precise shape of a last undetermined scaling function \(f_0\) depends on whether \(J\)-invariance or \(R\)-invariance is assumed, see eq. (3.24).

4. Stronger qualitative differences appear in the logarithmic representations of the ECGA. For \(R\)-invariance, two distinct cases emerge. The first one, labelled R1 in table [1], keeps the conventional result that the two-point function \(F = \langle \phi \phi \rangle = 0\) of the partner vanishes. But if the matrices \(\Delta\) are diagonal, a new case arises, labelled R2 in table [1] where \(F \neq 0\) and new additional terms in the remaining two-point functions \(\langle \phi \psi \rangle\) and \(\langle \psi \psi \rangle\) arise. In both cases, the remaining scaling functions are of the generic form (3.30). On the other hand, for \(J\)-invariance, labelled J in table [1], the rapidity matrices \(\hat{\gamma}\) are forced to be diagonal, such that the logarithmic terms reduce to those known from the non-exotic CGA. Here, only case 1 retains a logarithmic structure and the form of the remaining scaling functions is given in (3.31).

5. What has been referred to in the literature as “logarithmic” conformal field theory, uses in fact reducible but non-decomposable representations of the conformal algebra. In all cases known so far, the correlators also acquired a logarithmic term as well as power-law-dependence on distance, which motivated the name ‘logarithmic’. Here, a first example has been found (case R2 of the L-ECGA in table [1]) where a reducible but non-decomposable representation does not lead to explicit logarithms in the two-point functions.

The present study looked at co-variant two-point functions of conformal galilean algebras from an abstract point of view. We hope that the results presented here will turn out to be helpful in identifying specific physical model with one of them as a dynamical symmetry. We hope to come back to this in future work.

Acknowledgments

We are grateful to S. Moghimi-Araghi and A. Naseh for discussions. Two of us (MH and SR) would like to thank the organisers of the ADCFTA at the IHP Paris 2011, for warm hospitality, where this work was initiated. MH thanks the organisers of the Symposium ‘Models from statistical mechanics in applied sciences’ at Warwick University 2013 for warm hospitality and acknowledges partial support from the Collège Doctoral franco-allemand Nancy-Leipzig-Coventry (Systèmes complexes à l’équilibre et hors équilibre) of UFA-DFH. SR would like to thank warm hospitality at Van der Waals-Zeeman Instituut, University of...
Appendix A. Calculational details

The results (3.27) and (3.28,3.29) of the main text, respectively, are derived.

Starting from the definitions (3.26), the function $F$ was already found in section 3.1. As we shall see that $F \neq 0$ may occur, we revert to the standard formulation of LCFT. Temporal and spatial translation-invariance and the Bargman super-selection rule $\xi_1 + \xi_2 = 0$ are obvious.

We begin with the two-point function $G_{12} = \langle \phi_1 \psi_2 \rangle$. Co-variance under the generators $X_0, Y_0, Y_1$ and $X_1$, via calculations totally analogous to the ones presented in section 3.1, lead to the conditions

\[
\begin{align*}
(-t\partial_t - x \cdot \partial - \Delta_1 - \Delta_2) G_{12} - \Delta_1' F &= 0 \\
(-t\partial_t - \gamma_1 - \gamma_2 - \chi_1 - \chi_2) G_{12} - \gamma_2' F &= 0 \\
(-t (\gamma_1 - \gamma_2 + \chi_1 - \chi_2) - 2\xi_1 \tilde{r}) G_{12} + t\gamma_2' F &= 0 \\
(-t \cdot \partial - t(\Delta_1 - \Delta_2) - 2r \cdot (X_1 + \gamma_1)) G_{12} + t\Delta_2' F &= 0
\end{align*}
\]

(A.1)

If $F \neq 0$, then from section 3.1 we have $\Delta_1 = \Delta_2$. Otherwise, if $F = 0$, the above conditions are then identical to those treated in section 3.1, so that again $\Delta_1 = \Delta_2$ follows. Hence, we always have the constraint $\Delta_1 = \Delta_2$. Next, multiply the 3rd eq. (A.1) with $x$. On the other hand, simplify the 4th eq. (A.1) by using again $Y_0$-covariance. This gives the two simultaneous conditions

\[
\begin{align*}
t x \cdot (X_1 - \chi_2 + \gamma_1 - \gamma_2) G_{12} - t x \cdot \gamma_2' F &= 0 \quad \text{(A.2)} \\
x \cdot (X_1 - \chi_2 + \gamma_1 - \gamma_2) G_{12} - x \cdot \gamma_2' F - t\Delta_2' F &= 0
\end{align*}
\]

which requires that

\[
t^2 \Delta_2' F = 0 \quad \text{(A.3)}
\]

Similarly, if we consider the other mixed two-point function $G_{21} = \langle \psi_1 \phi_2 \rangle$, we find $t^2 \Delta_1' F = 0$. Therefore, the following cases must be distinguished:  

1. **Case 1**: Let $\Delta_1' \neq 0$ or $\Delta_2' \neq 0$. Then $F = 0$.

2. **Case 2**: Let $\Delta_1' = \Delta_2' = 0$. Then $F \neq 0$ is possible.

Before we enter into this distinction, we write down the conditions for the last two-point function $H = \langle \psi_1 \phi_2 \rangle$. Proceeding as in section 3.1, we find

\[
\begin{align*}
(-t\partial_t - x \cdot \partial - \Delta_1 - \Delta_2) H - \Delta_1' G_{12} - \Delta_2' G_{21} &= 0 \\
(-t\partial_t - \gamma_1 - \gamma_2 - \chi_1 - \chi_2) H - \gamma_1' G_{12} - \gamma_2' G_{21} &= 0 \\
(-t (\gamma_1 - \gamma_2 + \chi_1 - \chi_2) - 2\xi_1 \tilde{r}) H - t (\gamma_1' G_{12} - \gamma_2' G_{21}) &= 0 \\
(-t \cdot \partial - t(\Delta_1 - \Delta_2) - 2r \cdot (X_1 + \gamma_1)) H \\
- t (\Delta_1' G_{12} - \Delta_2' G_{21}) - 2\gamma_1' \cdot x G_{12} &= 0
\end{align*}
\]

\[\text{13} \text{We leave out here distributional contributions } F \sim \delta(t), \delta'(t) \text{.} \]
Again, we multiply the 3\textsuperscript{rd} of these by $x$ and re-use $Y_0$-covariance on the 4\textsuperscript{th}, along with the known constraint $\Delta_1 = \Delta_2$. This gives simultaneously

\begin{align}
 x \cdot (\chi_1 - \chi_2 + \gamma_1 - \gamma_2) H + x \cdot (\gamma'_1 G_{12} - \gamma'_2 G_{21}) &= 0 \quad (A.5) \\
 x \cdot (\chi_1 - \chi_2 + \gamma_1 - \gamma_2) H + x \cdot (\gamma'_1 G_{12} - \gamma'_2 G_{21}) + t (\Delta'_1 G_{12} - \Delta'_2 G_{21}) &= 0
\end{align}

which implies the constraint

\begin{equation}
 t (\Delta'_1 G_{12} - \Delta'_2 G_{21}) = 0 \quad (A.6)
\end{equation}

In case 2, we have $\Delta'_1 = \Delta'_2 = 0$ and this constraint is already satisfied. In case 1, $F = 0$ and the form of $G_{12}$ and $G_{21}$ can be read off from the non-logarithmic representation of section 3.1. Since under the exchange $1 \leftrightarrow 2$ one has $G_{12} \leftrightarrow G_{21}$, one can always arrange their amplitudes such that

\begin{equation}
 \Delta'_1 = \Delta'_2 \quad (A.7)
\end{equation}

\begin{equation}
 G(t, x) = G_{12} = G_{21} = G(-t, -x)
\end{equation}

Now, we can consider the two cases separately and work out the two-point functions explicitly.

**Case 1.** With the two constraints $\Delta_1 = \Delta_2$ and $\Delta'_1 = \Delta'_2$, $H$ is to be found from the three independent equations

\begin{align}
 (t \partial_t + x \cdot \partial + 2\Delta_1) H + 2\Delta'_1 G &= 0 \\
 (t \partial_t + \gamma_1 + \gamma_2 + \chi_1 + \chi_2) H + (\gamma'_1 + \gamma'_2) G &= 0 \quad (A.8) \\
 ((\gamma_1 - \gamma_2 + \chi_1 - \chi_2) + 2\xi \hat{e} u) H + (\gamma'_1 - \gamma'_2) G &= 0
\end{align}

We also require the explicit form of the operators $\chi_i$

\begin{equation}
 (\chi_1 + \chi_2) f = (\partial_{\nu^+} - \xi_1 \hat{e} \nu^-) f, \quad (\chi_1 - \chi_2) f = (\partial_{\nu^-} - \xi_1 \hat{e} \nu^+) f \quad (A.9)
\end{equation}

with the variables $\nu^\pm := \frac{1}{2} (\nu_1 \pm \nu_2)$. Since $F = 0$, the mixed correlator $G$ can be read off from (A.1). The result has already been obtained in section 3.1 and reads

\begin{equation}
 G = |t|^{-2\Delta_1} e^{-(\gamma_1 + \gamma_2) u} g(u, \nu^+, \nu^-) = g_0(w) |t|^{-2\Delta_1} e^{-(\gamma_1 + \gamma_2) u - (\gamma_1 - \gamma_2) \nu^-} e^{\xi (u + w) \hat{e} \nu^-} \quad (A.10)
\end{equation}

where $w := u - \nu^+$ and $g_0(w)$ is an undetermined function. Analogously, we write

\begin{equation}
 H = |t|^{-2\Delta_1} e^{-(\gamma_1 + \gamma_2) u} h(t, u, \nu^+, \nu^-) \quad (A.11)
\end{equation}

and proceed to extract $h$ from the three conditions (A.8). The first one reduces to $t \partial_t h + 2\Delta'_1 g = 0$ and has the solution

\begin{equation}
 h(t, u, \nu^+, \nu^-) = -2\Delta'_1 \ln |t| g(u, \nu^+, \nu^-) + h_1(u, \nu^+, \nu^-) \quad (A.12)
\end{equation}

Next, the second condition (A.8) becomes

\begin{equation}
 (\partial_u + \partial_{\nu^+} - \xi_1 \hat{e} \nu^-) h_1 + (\gamma'_1 + \gamma'_2) g = 0 \quad (A.13)
\end{equation}
This is solved by the transformation \( h_1 = e^{\xi_1 u \cdot \nu^-} \) such that the resulting equation for \( h_1 \) is readily integrated, with the result

\[
h_1 = -u \cdot (\gamma'_1 + \gamma''_1) g_0(w) e^{-(\gamma_1 - \gamma_2) \cdot \nu^-} e^{\xi_1 u \cdot \nu^-} + e^{\xi_1 u \cdot \nu^-} h_2(w, \nu^-) \tag{A.14}
\]

The last condition \((A.8)\) has the form

\[
(\partial \nu^- + \xi_1 \bar{e} (u + w) + \gamma_1 - \gamma_2) h_1 + (\gamma'_1 - \gamma''_1) g = 0 \tag{A.15}
\]

Inserting \( h_1 \) from \((A.14)\), and with the transformation \( h_2 = e^{-\nu^- \cdot \bar{e} w} e^{-(\gamma'_1 - \gamma''_1) h_2(w, \nu^-)} \), this reduces to \( \partial \nu^- h_2 + (\gamma'_1 - \gamma''_1) g_0(w) \). Hence \( h_2 = -\nu^- \cdot (\gamma'_1 - \gamma''_1) g_0(w) + h_0(w) \) such that finally

\[
h_1 = -u \cdot (\gamma'_1 + \gamma''_1) g_0(w) e^{-(\gamma_1 - \gamma_2) \cdot \nu^-} e^{\xi_1 (u + w) \cdot \nu^-} + h_0(w) e^{-\gamma_1 \cdot \nu^-} e^{\xi_1 (u + w) \cdot \nu^-} \tag{A.16}
\]

and where \( h_0(w) \) remains undetermined. Summarising, we have found that \( F = 0 \) and

\[
G = |t|^{-2\Delta_1} e^{-(\gamma_1 + \gamma_2) \cdot u - (\gamma_1 - \gamma_2) \cdot \nu^-} e^{\xi_1 (u + w) \cdot \nu^-} g_0(w) \\
H = |t|^{-2\Delta_1} e^{-(\gamma_1 + \gamma_2) \cdot u - (\gamma_1 - \gamma_2) \cdot \nu^-} e^{\xi_1 (u + w) \cdot \nu^-} h(u, w, \nu^-) \\
h = h_0(w) - g_0(w) (2\Delta'_1 \ln |t| + u \cdot (\gamma'_1 + \gamma''_1) + \nu^- \cdot (\gamma'_1 - \gamma''_1)) \tag{A.17}
\]

We have the constraints \( \Delta_1 = \Delta_2, \Delta'_1 = \Delta'_2 \) and \( \xi_1 + \xi_2 = 0 \). At this stage, the functions \( g_0(w) \) and \( h_0(w) \) remain undetermined.

The further consequences of rotation-invariance are discussed in appendix B.

**Case 2** Since now \( \Delta'_1 = \Delta'_2 = 0 \), the first mixed correlator \( G_{12} \) is obtained from the first three equations \((A.1)\). Similarly, the other mixed correlator \( G_{21} \) is found from the equations

\[
(t \partial_t + x \cdot \partial + 2\Delta_1) G_{21} = 0 \\
(t \partial + \gamma_1 + \gamma_2 + \chi_1 + \chi_2) G_{21} + \gamma'_1 F = 0 \\
((\gamma_1 - \gamma_2 + \chi_1 - \chi_2) + 2\xi_1 \bar{e} u) G_{12} + \gamma'_1 F = 0
\]

and the last one is determined from

\[
(t \partial_t + x \cdot \partial + 2\Delta_1) H = 0 \\
(t \partial + \gamma_1 + \gamma_2 + \chi_1 + \chi_2) H + \gamma'_1 G_{12} + \gamma''_2 G_{21} = 0 \\
((\gamma_1 - \gamma_2 + \chi_1 - \chi_2) + 2\xi_1 \bar{e} u) H + \gamma'_1 G_{12} - \gamma''_2 G_{21} = 0 \tag{A.18}
\]

and all subject to the constraints \( \Delta_1 = \Delta_2 \) and \( \xi_1 + \xi_2 = 0 \). Since \( F \) was already found in section 3.1, we can also adapt eq. \((A.8)\) from the Case 1 treated above and write directly down the mixed correlators, with the result

\[
F = |t|^{-2\Delta_1} e^{-(\gamma_1 + \gamma_2) \cdot u - (\gamma_1 - \gamma_2) \cdot \nu^-} e^{\xi_1 (u + w) \cdot \nu^-} f_0(w) \\
G_{12} = |t|^{-2\Delta_1} e^{-(\gamma_1 + \gamma_2) \cdot u - (\gamma_1 - \gamma_2) \cdot \nu^-} e^{\xi_1 (u + w) \cdot \nu^-} [g_0(w) - f_0(w) (u - \nu^-) \cdot \gamma''_1] \tag{A.19} \\
G_{21} = |t|^{-2\Delta_1} e^{-(\gamma_1 + \gamma_2) \cdot u - (\gamma_1 - \gamma_2) \cdot \nu^-} e^{\xi_1 (u + w) \cdot \nu^-} [g_0(w) - f_0(w) (u + \nu^-) \cdot \gamma'_1]
\]
Herein, we have taken into account that under the exchange 1 $\leftrightarrow$ 2 of the sites, one has the permutation $G_{12} \leftrightarrow G_{21}$. Then $f_0$ and $g_0$ remain undetermined. The correlator $H$ is written in the form

$$H = |t|^{-2\Delta_i} e^{-(\gamma_1+\gamma_2)u_-(\gamma_1-\gamma_2)\nu^-} e^{\xi_1(\nu_1+\nu^-)h(u, \nu, \nu^-)}$$

(A.20)

Then the first of eqs. (A.18) is automatically satisfied, whereas the second and third lead to the system

$$\partial_u h + (\gamma'_1 + \gamma'_2) g_0 - (\gamma'_1 (u - \nu^-) \cdot \gamma'_2 + \gamma'_2 (u + \nu^-) \cdot \gamma'_1) f_0 = 0$$
$$\partial_{\nu^-} h - (\gamma'_1 - \gamma'_2) g_0 - (\gamma'_1 (u - \nu^-) \cdot \gamma'_2 - \gamma'_2 (u + \nu^-) \cdot \gamma'_1) f_0 = 0$$

(A.21)

These are decoupled by defining $y^\pm := \frac{1}{2} (u \pm \nu^-)$. Considering $h = h(y^+, y^-, w)$, one has

$$\partial_{y^+} h = -2\gamma'_1 g_0 + 2\gamma'_1 y^- \cdot \gamma'_2 f_0$$
$$\partial_{y^-} h = -2\gamma'_2 g_0 + 2\gamma'_2 y^+ \cdot \gamma'_1 f_0$$

(A.22)

such that finally, with $h_0(w)$ an undetermined function

$$h = h_0(w) - 2g_0(w) y^+ \cdot \gamma'_1 - 2g_0(w) y^- \cdot \gamma'_2 + 2f_0(w) y^+ \cdot \gamma'_1 y^- \cdot \gamma'_2$$
$$= h_0(w) - g_0(w) (u + \nu^-) \cdot \gamma'_1 - g_0(w) (u - \nu^-) \cdot \gamma'_2$$

$$+ \frac{1}{2} f_0(w) (u + \nu^-) \cdot \gamma'_1 (u - \nu^-) \cdot \gamma'_2$$

(A.23)

The discussion of rotation-invariance is analogous to Case 1 and carried out in appendix B. Combining the results eqs. (A.19) (A.20) (A.23) and reverting to the original coordinates, one arrives at the final form (3.28) (3.29) stated in the main text.

### Appendix B. On rotation-invariance in the ECGA

Having discussed in the main text the shape of two-point functions c-o-variant under the ECGA, we now consider the additional consequences when rotation-invariance is taken into account as well.

#### B.1 Rotation-invariance for vanishing rapidities

We shall compare the consequences of using two distinct representations for the infinitesimal generator of rotations, namely (also recall that $a \wedge b = \epsilon_{ij} a_i b_j$)

$$J := -\epsilon_{ij} x_i \frac{\partial}{\partial x_j} - \frac{1}{2\xi} \chi_i \chi_i = -x \wedge \partial_x - \frac{1}{2\xi} \chi^2$$
$$R := -\epsilon_{ij} x_i \frac{\partial}{\partial x_j} - \epsilon_{ij} \nu_i \frac{\partial}{\partial \nu_j} = -x \wedge \partial_x - \nu \wedge \partial_\nu$$

(B.1)

Martelli and Tachikawa [27] advocated in favour of the generator $J$, because it naturally appears in the contraction procedure they used in deriving the ECGA. Here, we wish to compare with the results found for the naturally-looking generator $R$, also mentioned as a possible alternative in [27].
From a purely algebraic point of view, there is no criterion which would lead one to prefer one of these two choices. Both obey the same commutation relations with the other generators of the ECGA and both commute with the Schrödinger operator $S$ defined in section 3.

Here, we shall show that physically distinct results are found, depending on the use of $J$ (which we shall refer to as ‘$J$-invariance’) or $R$ (which we shall refer to as ‘$R$-invariance’). Namely, the rapidity-less ECGA-covariant two-point function $F = \langle \phi_1 \phi_2 \rangle$ has the form

$$F = f_0(u - \nu^+ e^{-\xi_1 \nu^- (2u - \nu^+)} \delta_{\Delta_1, \Delta_2} \delta_{\xi_1, \xi_2, 0}$$

(B.2)

where $\nu^\pm = \frac{1}{2} (\nu_1 \pm \nu_2)$ and still contains an undetermined differentiable function $f_0 = f_0(w)$ of the single variable $w = u - \nu^+$. Any explicit dependence on the single variables $\nu^\pm$ of the two-point function $F$ is already contained in (B.2). The additional requirement of rotation-invariance leads to a clear distinction

$$\begin{cases} f_0 \text{ is arbitrary} & ; J\text{-invariance} \\ f_0 = f_0(w_1^2 + w_2^2) & ; \text{R}\text{-invariance} \end{cases}$$

(B.3)

Proof: Begin by writing down the two-particle form of the generators $J$ and $R$ (where in view of the coming application to $F$, the Bargman super-selection rule $\xi_2 = -\xi_1$ has already been used)

$$\begin{align*}
J &= -u \wedge \partial_u - \frac{1}{2\xi_1} (\partial_{\nu^+} \cdot \partial_{\nu^-} + \xi_1^2 \nu^+ \cdot \nu^-) - \frac{1}{2} (\nu^+ \wedge \partial_{\nu^+} + \nu^- \wedge \partial_{\nu^-}) \\
R &= -u \wedge \partial_u - \nu_1 \wedge \partial_{\nu_1} - \nu_2 \wedge \partial_{\nu_2} = -u \wedge \partial_u - \nu^+ \wedge \partial_{\nu^+} - \nu^- \wedge \partial_{\nu^-} 
\end{align*}$$

(B.4)

In working out the condition of rotation-invariance $JF \equiv 0$ or $RF \equiv 0$, respectively, we shall need the following auxiliary formulæ, with $\tilde{\epsilon} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $a, b \in \mathbb{R}^2$ such that $a \wedge b = a \cdot (\tilde{\epsilon} b)$

$$\begin{align*}
\partial_a e^{-a^\wedge b} &= - (\tilde{\epsilon} b) e^{-a^\wedge b} \\
(\tilde{\epsilon} a) \cdot b &= -a \wedge b \\
(\tilde{\epsilon} a) \cdot (\tilde{\epsilon} b) &= a \cdot b \\
a \wedge (\tilde{\epsilon} b) &= -a \cdot b 
\end{align*}$$

(B.5)

Then, application of the two distinct rotation generators to the two-point function (B.2) leads straightforwardly in the case of $J$-invariance simply to the identity $JF = 0$, whereas in the case of $R$-invariance we obtain $w \wedge \partial_w f_0(w) = 0$. Hence, given the form (B.2), $J$-invariance is automatic and does not impose any further condition on the function $f_0$. On the other hand, in the case of $R$-invariance, $f_0$ can only be a function of the scalar $|w|^2 = w_1^2 + w_2^2$. This proves the assertion (B.3).

As we shall see below, this distinction between the generators $J$ and $R$ will lead to further consequences in the case of non-vanishing rapidities.
B.2 Rotation-invariance for non-vanishing rapidities: non-logarithmic case

In the non-logarithmic case, the ECGA-covariant two-point function reads

\[ F = f_0(u - \nu^+) t^{-2 \Delta_1} e^{-2 \gamma - u - 2 \gamma - \nu} e^{-\xi_1 \nu^+ \wedge (2 u - \nu^+)} \delta_{\Delta_1, \Delta_2} \delta_{\xi_1 + \xi_2, 0} \]  
(B.6)

where in addition to the previous conventions, we also defined \( \gamma_{\pm} := \frac{1}{2} (\gamma_1 \pm \gamma_2) \). Therefore, the two rotation-generators must be generalised to include rapidity terms and read for a single particle

\[ J = -x \wedge \partial_x - \gamma \wedge \partial_\gamma - \frac{1}{2} \chi \cdot \chi, \quad R = -x \wedge \partial_x - \gamma \wedge \partial_\gamma - \nu \wedge \partial_\nu \]  
(B.7)

and for two particles

\[ J = -u \wedge \partial_u - \frac{1}{2} \xi_1 (\partial_{\nu^+} \cdot \partial_{\nu^-} + \xi_1^2 \nu^+ \cdot \nu^-) - \frac{1}{2} (\nu^+ \wedge \partial_{\nu^+} + \nu^- \wedge \partial_{\nu^-}) \]
\[ - \gamma_+ \wedge \partial_{\gamma_+} - \gamma_- \wedge \partial_{\gamma_-} \]
\[ R = -u \wedge \partial_u - \gamma_+ \wedge \partial_{\gamma_+} - \gamma_- \wedge \partial_{\gamma_-} - \nu^+ \wedge \partial_{\nu^+} - \nu^- \wedge \partial_{\nu^-} \]  
(B.8)

Again, the undetermined function \( f_0 = f_0(\gamma_+, \gamma_-, w) \) depends on the single variable \( w = u - \nu^+ \). However, since the rotations can also transform the rapidities \( \gamma_{\pm} \), \( f_0 \) can in addition also depend explicitly on them. Hence, \( f_0 = f_0(\gamma_+, \gamma_-, w) \) will be a function of 6 variables, subject to a single condition coming form rotation-invariance.

Using the same auxiliary identities (B.5) as before, straightforward but slightly tedious calculations lead to

\[ J\text{-invariance} : \quad \gamma_+ \wedge \frac{\partial f_0}{\partial \gamma_+} + \gamma_- \wedge \frac{\partial f_0}{\partial \gamma_-} + \frac{\gamma_-}{\xi_1} \cdot \frac{\partial f_0}{\partial w} = 0 \]
\[ R\text{-invariance} : \quad \gamma_+ \wedge \frac{\partial f_0}{\partial \gamma_+} + \gamma_- \wedge \frac{\partial f_0}{\partial \gamma_-} + w \wedge \frac{\partial f_0}{\partial w} = 0 \]  
(B.9)

In order to find the general solutions of these, in the case of \( J\text{-invariance} \), one introduces a new variable \( \nu := \gamma_- - \xi_1 \hat{\nu} w \) and takes \( f_0 \) as a function \( f_0(\gamma_+, \gamma_-, \nu) \). Then the first of eqs. (B.9) reduces to \( (\gamma_+ \wedge \partial_{\gamma_+} + \gamma_- \wedge \partial_{\gamma_-}) f_0 = 0 \). Three obvious and independent solutions of this are \( \gamma_+^2, \gamma_-^2 \) and \( \gamma_+ \gamma_- \), from which the general solution can be constructed.

On the other hand, in the case of \( R\text{-invariance} \) one easily lists 5 independent solutions so that finally

\[ J\text{-invariance} : \quad f_0 = f_0(\gamma_+^2, \gamma_-^2, \gamma_+ \cdot \gamma_-, w + \hat{\chi} \gamma_- \xi_1^{-1}) \]
\[ R\text{-invariance} : \quad f_0 = f_0(w^2, \gamma_+^2, \gamma_-^2, w \cdot \gamma_+, w \cdot \gamma_-) \]  
(B.10)

Reverting to the original variables \( \gamma_{1,2} \) gives the expressions in the main text or appendix A.

A comparison of the two distinct eqs. (B.9) shows the origin of these two distinct forms of the function \( f_0 \); while in the case of \( R\text{-invariance} \), the habitual form of the rotation generator guarantees that formal scalar products of the vectors \( w, \gamma_+ \) and \( \gamma_- \) are always rotation-invariant, this holds no longer true in the case of \( J\text{-invariance} \), where only scalar...
products between the rapidity vectors $\gamma_\pm$ have this property. Model-specific calculations will permit to distinguish between these possibilities.

We also observe that in the case of $J$-invariance, taking the non-exotic limit $\xi_1 \to 0$ enforces $\gamma_- \to 0$, in order to maintain a finite value in the last argument of the function $f_0$. In this way, one can understand how the constraint $\gamma_1 = \gamma_2$ in non-exotic CGA is recovered. No such limit argument can be made in the case of $R$-invariance.

### B.3 Rotation-invariance for non-vanishing rapidities: logarithmic case

One must must take further into account that the $\gamma$’s become Jordan matrices, such that the rotation generators have to be generalised to the forms

$$J = -x \wedge \partial_x - \gamma \wedge \partial_\gamma - \gamma' \wedge \partial_{\gamma'} - \frac{1}{2x} X \cdot X, \quad R = -x \wedge \partial_x - \gamma \wedge \partial_\gamma - \gamma' \wedge \partial_{\gamma'} - \nu \wedge \partial_\nu \quad (B.11)$$

for a single particle and with analogous extensions in the two-particle case. In addition to the two-point function $F = \langle \phi_1 \phi_2 \rangle$ already analysed in the non-logarithmic representations, one now has to consider the additional two-point functions $G = \langle \phi_1 \psi_2 \rangle$ and $H = \langle \psi_1 \psi_2 \rangle$. Furthermore, we have already seen in the main text that two distinct cases have to be distinguished, depending on whether the matrices $\hat{A}_{1,2}$ have Jordan form (case 1) or are diagonal (case 2).

**A)** If we consider $R$-invariance, begin with case 1 (where $F = 0$ and require co-variance $RG = 0 \equiv RH$, one has

$$w \wedge \frac{\partial g_0}{\partial w} + \gamma_1 \wedge \frac{\partial g_0}{\partial \gamma_1} + \gamma_2 \wedge \frac{\partial g_0}{\partial \gamma_2} + \gamma'_1 \wedge \frac{\partial g_0}{\partial \gamma'_1} + \gamma'_2 \wedge \frac{\partial g_0}{\partial \gamma'_2} = 0 \quad (B.12)$$

and analogously for $h_0$, since all individual terms in (A.17) are explicitly rotation-invariant. In principle, the functions $g_0, h_0$ depend on the 10 variables $w, \gamma_{1,2}, \gamma'_{1,2}$. Since rotation-invariance imposes a single extra condition, there remains a function of 9 rotation-invariant variables, for example

$$g_0 = g_0 \left( w^2, \gamma_1^2, \gamma_2^2, \gamma_1'^2, \gamma_2'^2, w \cdot \gamma_1, w \cdot \gamma_2, w \cdot \gamma'_1, w \cdot \gamma'_2 \right)$$

$$h_0 = h_0 \left( w^2, \gamma_1^2, \gamma_2^2, \gamma_1'^2, \gamma_2'^2, w \cdot \gamma_1, w \cdot \gamma_2, w \cdot \gamma'_1, w \cdot \gamma'_2 \right) \quad (B.13)$$

For case 2, an analogous argument applies and one recovers [B.13] along with an analogous form for $f_0$.

**B)** A very different result is found for $J$-invariance. If one considers case 1 first, one has again $F = \langle \phi_1 \phi_2 \rangle = 0$, whereas $G = \langle \phi_1 \psi_2 \rangle$ has the same form as the two-point function $F$ treated above in the non-logarithmic case. It remains to consider the two-point function

$$H = \langle \psi_1 \psi_2 \rangle = t^{-2\Delta_1} e^{-2\gamma_+ \cdot u - 2\gamma_- \cdot \nu} h \quad (B.14)$$

where the scaling function $h$ can be written as

$$h = h_0(u - \nu^+) - g_0(u - \nu^+) \left( 2\Delta_1 \ln |t| + u \cdot (\gamma'_1 + \gamma'_2) + \nu^- \cdot (\gamma'_1 - \gamma'_2) \right) \quad (B.15)$$
In complete analogy to the previous sub-section, $J$-invariance implies the conditions

$$Dh = 0, \ Dg_0 = 0$$  \hspace{1cm} (B.16)

with the differential operator

$$D := γ_+ \wedge \frac{∂}{∂γ_+} + γ_- \wedge \frac{∂}{∂γ_-} + γ'_1 \wedge \frac{∂}{∂γ'_1} + γ'_2 \wedge \frac{∂}{∂γ'_2} + \frac{γ_2}{ξ_1} \cdot \frac{∂}{∂w}$$  \hspace{1cm} (B.17)

This gives the following equation for $h_0$:

$$Dh_0 - g_0D \left( 2Δ'_1 \ln |t| + u \cdot (γ'_1 + γ'_2) + υ^− \cdot (γ'_1 - γ'_2) \right) = 0$$  \hspace{1cm} (B.18)

Working out the differential operator and taking the condition $w = u - υ^+$ into account, leads to

$$Dh_0 - \left( (γ'_1 + γ'_2) \cdot \left( \frac{1}{ξ_1} γ_- + u \right) + (γ'_1 - γ'_2) \wedge υ^- \right) g_0 = 0$$  \hspace{1cm} (B.19)

However, since $h_0$ depends only on $w$ and not on $u$ or $υ^±$ separately, this condition is only compatible with our previous results if

$$γ'_1 = γ'_2 = 0$$  \hspace{1cm} (B.20)

Then only the matrix $\hat{Δ}$ of the conformal weight can have a Jordan form and

$$g_0 = g_0 \left( γ^2_+, γ^2_-, γ_+ \cdot γ_-, w + \hat{c} γ_-ξ_1^{-1} \right)$$
$$h_0 = h_0 \left( γ^2_+, γ^2_-, γ_+ \cdot γ_-, w + \hat{c} γ_-ξ_1^{-1} \right)$$  \hspace{1cm} (B.21)

Reverting to the original $γ_{1,2}$ gives the expressions in the main text or in appendix A.

Similar arguments apply to case 2: since now $F = ⟨φ_1φ_2⟩ ≠ 0$, consideration of $G_{12}$ leads to $γ'_2 = 0$ and the other mixed two-point function $G_{21}$ gives $γ'_1 = 0$. Then no logarithmic structure remains. For $J$-invariance, there is but a single case, see table[1]

References

[1] H. Saleur, “Polymers and percolation in two dimensions and twisted $N = 2$ supersymmetry,” Nucl. Phys. B382, 486 (1992) [arXiv:9111007[hep-th]].

[2] V. Gurarie, “Logarithmic operators in conformal field theory,” Nucl. Phys. B410, 535 (1993) [arXiv:9303160[hep-th]].

[3] M.R. Gaberdiel, H.G. Kausch, “A rational logarithmic conformal field theory,” Phys. Lett. B386, 131 (1996) [arXiv:9606050[hep-th]].

[4] M.R. Gaberdiel, H.G. Kausch, “A local logarithmic conformal field theory,” Nucl. Phys. B538, 631 (1999) [arXiv:9807091[hep-th]].

[5] M. Flohr, “Bits and pieces in logarithmic conformal field theory,” Int. J. Mod. Phys. A18, 4497 (2003) [arXiv:0111228[hep-th]].
[6] see a recent special issue in J. Phys. A Math. Theor. (at press, 2013) on LCFT

[7] P. Mathieu and D. Ridout, “From Percolation to Logarithmic Conformal Field Theory,” Phys. Lett. B657, 120 (2007) [arXiv:0708.0802]; and “Logarithmic $\mathcal{M}(2,p)$ minimal models, their logarithmic coupling and duality”, Nucl. Phys. B801, 268 (2008) [arXiv:0711.3541].

[8] U. Niederer, “The maximal kinematical invariance group of the free Schrödinger equation,” Helv. Phys. Acta 45, 802 (1972).

[9] C. R. Hagen, “Scale and conformal transformations in galilean-covariant field theory,” Phys. Rev. D5, 377 (1972).

[10] M. Henkel, “Schrödinger invariance in strongly anisotropic critical systems,” J. Stat. Phys. 75, 1023 (1994) [arXiv:9310081[hep-th]].

[11] C. Duval, G. W. Gibbons and P. Horvathy, “Celestial Mechanics, Conformal Structures, and Gravitational Waves,” Phys. Rev. D43, 3907 (1991) [arXiv:hep-th/0512188].

[12] M. Henkel, “Phenomenology of local scale-invariance: from conformal invariance to dynamical scaling”, Nucl. Phys. B641, 405 (2002) [hep-th/0205256].

[13] D.T. Son, “Toward an AdS/cold atoms correspondence: a geometric realization of the Schrödinger symmetry,” Phys. Rev. D78, 046003 (2008) [arXiv:0804.3972 [hep-th]].

[14] K. Balasubramanian and J. McGreevy, “Gravity duals for non-relativistic CFTs,” Phys. Rev. Lett. 101, 061601 (2008) [arXiv:0804.4053 [hep-th]].

[15] M. Alishahiha, R. Fareghbal, A. E. Mosaffa and S. Rouhani, “Asymptotic symmetry of geometries with Schrödinger isometry,” [arXiv:0902.3916 [hep-th]].

[16] C. Duval and P.A. Horváthy, “Non-relativistic conformal symmetries and Newton-Cartan structures”, J. Phys. A: Math. Theor. 42, 465206 (2009). [arXiv:0904.0531]

[17] N. Aizawa and P. S. Isaac, “On irreducible representations of the exotic conformal Galilei algebra,” J. Phys. A: Math. Theor 44, 035401 (2011) [arXiv:1010.4075 [math-ph]].

[18] M. Henkel and M. Pleimling, Nonequilibrium phase transitions, vol 2: Ageing and dynamical scaling far from equilibrium, Springer (Heidelberg 2010)

[19] A. Hosseiny and S. Rouhani, “Logarithmic Correlators in Non-relativistic Conformal Field Theory,” [arXiv:1001.1036 [hep-th]].

[20] A. Hosseiny and A. Naseh, “On holographic realization of logarithmic Galilean conformal algebra”, J. Math. Phys. 52 092501 (2011). [arXiv:1101.2126]
[21] K. Hotta, T. Kubota and T. Nishinaka, “Galilean Conformal Algebra in Two Dimensions and Cosmological Topologically Massive Gravity”, Nucl. Phys. B838, 358 (2010) [arXiv:1003.1203 [hep-th]].

[22] M. Henkel and S. Rouhani, “Logarithmic correlators or responses in non-relativistic analogues of conformal invariance”, J. Phys A Math. Theor, at press (2013) [arXiv:1302.7136 [hep-th]].

[23] M. Henkel, “On logarithmic extensions of local scale-invariance,” Nucl. Phys. B869[FS], 282 (2013) [arXiv:1009.4139 [hep-th]].

[24] M. Henkel, J. D. Noh and M. Pleimling, “Phenomenology of ageing in the Kardar-Parisi-Zhang equation,” Phys. Rev. E85, 030102(R) (2012) [arXiv:1109.5022 [cond-mat.stat-mech]].

[25] J. Lukierski, P. C. Stichel and W. J. Zakrzewski, “Exotic Galilean conformal symmetry and its dynamical realisations,” Phys. Lett. A357, 1 (2006) [arXiv:0511259[hep-th]].

[26] J. Lukierski, P. C. Stichel and W. J. Zakrzewski, “Acceleration-Extended Galilean Symmetries with Central Charges and their Dynamical Realizations,” Phys. Lett. B650, 203 (2007) [arXiv:0511259[hep-th]].

[27] D. Martelli and Y. Tachikawa, “Comments on Galilean conformal field theories and their geometric realization,” J. High. Phys. 1005, 091 (2010) [arXiv:0903.5184 [hep-th]].

[28] M. Henkel “Local scale invariance and strongly anisotropic equilibrium critical systems,” Phys. Rev. Lett. 78, 1940 (1997) [arXiv:cond-mat/9610174v2 [cond-mat.stat-mech]].

[29] J. Negro, M.A. del Olmo and A. Rodríguez-Marco, “Nonrelativistic conformal groups”, J. Math. Phys. 38, 3786 and 3810 (1997).

[30] P. Havas, J. Plebanski “Conformal extensions of the Galilei group and their relation to the Schrödinger group,” J. Math. Phys. 19, 482 (1978)

[31] M. Henkel and J. Unterberger, “Schrödinger invariance and space-time symmetries”, Nucl. Phys. B660, 407 (2003) [hep-th/0302187]; M. Henkel, “Causality from dynamical symmetry: an example from local scale-invariance”, proceedings of the 7th AGMP conference 24-26 oct. 2011 at Mulhouse (France) (at press (2013)) [arXiv:1205.5901]

[32] A.J. Bray, “Theory of phase-ordering”, Adv. Phys., 43,357 (1994); A.J. Bray and A.D. Rutenberg, “Growth laws for phase-ordering”, Phys. Rev. E49, R27 (1994).
[33] M. Henkel and M. Pleimling, “Local scale-invariance as dynamical space-time symmetry in phase-ordering kinetics”, Phys. Rev. E68, 065101(R) (2003) [cond-mat/0302482];

M. Henkel, A. Picone, and M. Pleimling, “Two-time autocorrelation function in phase-ordering kinetics from local scale-invariance”, Europhys. Lett. 68, 191 (2004) [cond-mat/0404464];

E. Lorenz and W. Janke, “Numerical tests of local scale-invariance in ageing q-state Potts models”, Europhys. Lett. 77, 10003 (2007).

[34] H.W. Diehl and M. Shpot, “Critical behavior at m-axial Lifshitz points: Field-theory analysis and ε-expansion results”, Phys. Rev. B62, 12338 (2000) [cond-mat/0006007].

[35] M. Shpot and H.W. Diehl, “Two-loop renormalization-group analysis of critical behaviour at m-axial Lifshitz points”, Nucl. Phys. B612, 340 (2001) [cond-mat/0106105].

[36] M.A. Shpot, Yu.M. Pis’mak, and H.W. Diehl, “Large-n expansion for m-axial Lifshitz points”, J. Phys. Condens. Matter, 17, S1947 (2005) [cond-mat/0412405].

[37] M.A. Shpot, H.W. Diehl, Yu.M. Pis’mak, “Compatibility of 1/n and epsilon expansions for critical exponents at m-axial Lifshitz points”, J. Phys. A Math. Theor. 41, 135003 (2008) [arXiv:0802.2434]

[38] V. Ovsienko and C. Roger, “Generalisations of Virasoro group and Virasoro algebra through extensions by modules of tensor-densities on S¹”, Indag. Mathem. 9, 277 (1998).

[39] M. Henkel, R. Schott, S. Stoimenov and J. Unterberger, “The Poincaré algebra in the context of ageing systems: Lie structure, representations, Appell systems and coherent states,”, Confluentes Math. 4, 1250006 (2012) [arXiv:0601028 [math-ph]].

[40] A. Hosseiny and S. Rouhani, “Affine Extension of Galilean Conformal Algebra in 2+1 Dimensions,” J. Math. Phys. 51, 052307 (2010) [arXiv:0909.1203 [hep-th]].

[41] A. Bagchi, R. Gopakumar, I. Mandal and A. Miwa, “CGA in 2d,” JHEP 1008, 004 (2010) [arXiv:0912.1090 [hep-th]].

[42] C. Duval and P. A. Horvathy, “The exotic Galilei group and the ’Peierls substitution’,” Phys. Lett. B479, 284 (2000) [arXiv:0002233[hep-th]].

[43] R. Cherniha and M. Henkel, “The exotic conformal Galilei algebra and non-linear partial differential equations”, J. Math. Anal. Appl. 369, 120 (2010). [arXiv:0910.4822]

[44] K. Kytölä and D. Ridout, “On staggered indecomposable Virasoro modules ,” J. Math. Phys. 50, 123503 (2007) [arXiv:0905.0108].

[45] P. Mathieu and D. Ridout, “From Percolation to Logarithmic Conformal Field Theory,” Phys. Lett. B657, 120 (2007) [arXiv:0708.0802 [hep-th]].
[46] S. Moghimi-Araghi, S. Rouhani and M. Saadat, “Logarithmic conformal field theory through nilpotent conformal dimensions,” Nucl. Phys. B599, 531 (2001) [arXiv:0008165[hep-th]].

[47] K.M. Lee, S. Lee and S. Lee, “Nonrelativistic Superconformal M2-Brane Theory”, JHEP 0909:030 (2009) [arXiv:0902.3857]

[48] A. Bagchi and I. Mandal, “On Representations and Correlation Functions of Galilean Conformal Algebras,” Phys. Lett. B675, 393 (2009) [arXiv:0903.4524 [hep-th]].