On the Poisson Approximation to Photon Distribution for Faint Lasers

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It is proved, that for a certain kind of input distribution, the strongly binomially attenuated photon number distribution can well be approximated by a Poisson distribution. This explains why we can adopt poissonian distribution as the photon number statistics for faint lasers. The error of such an approximation is quantitatively estimated. Numerical tests are carried out, which coincide with our theoretical estimations. This work lays a sound mathematical foundation for the well-known intuitive idea which has been widely used in quantum cryptography.

1. Introduction

The security of Quantum Key Distribution (QKD) is based on the non-cloning principle of an unknown quantum state [1]. In the implementation of QKD based on BB84 protocol [2], one expects that each pulse contains only one photon. If not, the eavesdropper can acquire information using beamsplitter attack [3] without exposing its existence. However, since an ideal single photon state is difficult to prepare, practically, faint laser pulse with ultra-low mean photon number is used as a convenient realization of pseudo-single photon source [4].

By letting a laser source pass through a strong attenuator we get faint laser pulse. For security concern the mean photon number in each faint laser pulse is kept very small (about 0.1). But still, there is a small probability of having more than one photon in each pulse. A precise estimation of this unwelcome probability is crucial for security analysis in QKD systems [3].

In the literature, the photon number in faint laser is treated as Poissonian distributed. It is all right if the input laser before attenuation is Poisson. However, practically we may have input laser whose photon number statistics is not Poisson [5]. If this laser is used as input, the attenuated laser may not be Poisson either. But there is a common belief that no matter what distribution the input laser is, if we attenuate it into a faint laser with sufficient small mean photon number, then Poisson distribution would be a good approximation of photon number distribution in the faint laser pulse. So far, however, this claim has not been mathematical rigorously proved, which is the motivation of this work.

We proved, that for a certain kind of probability distribution, after the binomial decay transformation, which is a mathematical description of laser attenuation [6], the decayed distribution can well be approximated by a Poisson distribution provided that the expectation of the decayed distribution is sufficiently small. It gives a theoretical validation of the above claim, i.e., generally we can use a Poisson distribution to approximate the photon distribution in faint laser and the error of the approximation could be neglected.

2. Preliminary

Consider \( N \) independent particles (photons in laser pulses) passing through an attenuator. Each particle has a probability of \( \eta \) (\( 0 \leq \eta \leq 1 \)) to penetrate the attenuator. We define \( X \) to be the number of particles before decay (Input); and \( X_\eta \) to be the number of particles after decay (Output). \( X \) and \( X_\eta \) are random variables taking values in the natural number system \( \mathbb{N} \), and their

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Eq. (1),
Proof. By the definition of expectation and expectation of the discrete random variables \(X\)
Lemma 1. of this section gives some properties of it.

It is easy to check that
\[
P_\eta(n) \geq 0, \forall n \in \mathbb{N},
\]
and
\[
\sum_{n=0}^{\infty} P_\eta(n) = \sum_{N=0}^{\infty} P(N) = 1.
\]
So \(P_\eta\) is indeed a PMF.

The binomial decay transformation establishes a relation between photon number distribution before and after the attenuation. The remaining of this section gives some properties of it.

Lemma 1. Let \(E(X)\) and \(E(X_\eta)\) denote the expectation of the discrete random variables \(X\) and \(X_\eta\), respectively. Then \(E(X_\eta) = \eta E(X)\).

Proof. By the definition of expectation and Eq. (1),
\[
E(X_\eta) = \sum_{n=0}^{\infty} n P_\eta(n) = \sum_{n=0}^{\infty} \sum_{N=n}^{\infty} \binom{N}{n} \eta^n (1-\eta)^{N-n} P(N)
\]
\[
= \sum_{N=0}^{\infty} P(N) \sum_{n=0}^{N} \binom{N}{n} \eta^n (1-\eta)^{N-n}
\]
\[
= \sum_{N=0}^{\infty} \eta NP(N) = \eta E(X).
\]

Define \(P^\lambda(n)\) the Poisson PMF with parameter \(\lambda\), i.e., \(P^\lambda(n) = e^{-\lambda} \lambda^n / n!\), \(n \in \mathbb{N}\). We will show that the binomial decay transformation preserves the Poisson character.

Lemma 2. Suppose \(P(N) = P^\mu(N)\), then we have \(P_\eta(n) = P^{\eta \mu}(n)\).

Proof. From Eq. (1),
\[
P_\eta(n) = \sum_{N=n}^{\infty} \binom{N}{n} \eta^n (1-\eta)^{N-n} P(N)
\]
\[
= \frac{(\eta \mu)^n}{n!} \sum_{N=n}^{\infty} \frac{e^{-\mu}}{N!} [\mu (1-\eta)]^{N-n}
\]
\[
= \frac{(\eta \mu)^n}{n!} e^{-\eta \mu}.
\]

3. Poisson Approximation

From Lemma 2 and Lemma 1 we know that if \(P(N)\), the photon number distribution in the laser before attenuation, is Poisson with parameter \(\mu\), then the decayed distribution, \(P_\eta(n)\), is also a Poisson whose parameter is \(\eta \mu\), with \(\eta\) being the attenuating coefficient. However, practically \(P(N)\) may not be a Poisson distribution [5]. If so, \(P_\eta(n)\) would not be a Poisson distribution. Nevertheless, in QKD the faint laser is treated as Poisson distributed. The reason of doing this is based on the common belief that any input distribution would reduce to Poisson distribution provided that the attenuation is strong enough. Next we justify it quantitatively.

Proposition. In the case of faint laser, the decayed distribution can be approximated as a Poisson distribution,
\[
P_\eta(n) \approx P^\lambda(n), \ n \in \mathbb{N},
\]
where
\[
\lambda = E(X_\eta) = \eta E(X) \ll 1.
\]
More concretely, if we expand \(P^\lambda(n)\) into Taylor series of \(\lambda\):
\[
P^\lambda(0) = 1 - \lambda + \frac{\lambda^2}{2} - O(\lambda^3),
\]
\[
P^\lambda(1) = \lambda - \lambda^2 + O(\lambda^3),
\]
\[
P^\lambda(2) = \frac{\lambda^2}{2} - O(\lambda^3),
\]
\[
P^\lambda(n) = O(\lambda^3), n \geq 3.
\]
Then we have
\[
P_\eta(0) = 1 - \lambda + \frac{\lambda^2}{2} + C(X) \lambda^2 - D_0(X) \lambda^3,
\]
(3a)
\[ P_0(1) = \lambda - \lambda^2 - 2C(X)\lambda^2 + D_1(X)\lambda^3, \quad (3b) \]
\[ P_0(2) = \frac{\lambda^2}{2} + C(X)\lambda^2 - D_2(X)\lambda^3, \quad (3c) \]
\[ \sum_{n=3}^{\infty} P_n(n) = [D_0(X) + D_2(X) - D_1(X)]\lambda^3, \quad (3d) \]
where
\[ C(X) = \frac{\text{Var}(X) - E(X)}{2E(X)^2} \]

and
\[ 0 \leq D_i(X) \leq D(X) = \frac{M(X)}{E(X)^3}, \quad (i = 0, 1, 2). \]
\[ M(X) = E[X(X-1)(X-2)] \text{ is the 3rd factorial moment of } X. \]

**Proof.** The generating function of \( X \) is
\[ G(z) = \sum_{N=0}^{\infty} P(N)z^N, \quad z \in \mathbb{R}. \]

Taking the \( n \)-th order derivatives of \( G(z) \) with respect to \( z \) yields,
\[ G^{(n)}(z) = \sum_{N=n}^{\infty} N(N-1)\cdots(N-n+1)P(N)z^{N-n}. \]

Let \( z = 1 \), one has,
\[ G^{(n)}(1) = E \left[ \sum \frac{X!}{(X-n)!} \right]. \]

For \( n = 0, 1, 2, 3 \) we have
\[ G(1) = 1, \]
\[ G'(1) = E(X), \]
\[ G''(1) = \text{Var}(X) + [E(X)]^2 - E(X), \]
\[ G'''(1) = E[X(X-1)(X-2)] = M(X). \]

From Eq. (1),
\[ P_0(n) = \sum_{N=n}^{\infty} \binom{N}{n} \eta^n (1 - \eta)^{N-n} P(N) = \frac{\eta^n}{n!} \sum_{N=n}^{\infty} N(N-1)\cdots(N-n+1)(1-\eta)^{N-n} P(N) = P_0(n) \]
Expanding \( G^{(n)}(1 - \eta) \) into Taylor series, in the case of \( n = 0 \), one has,
\[ P_0(0) = G(1) - \eta G'(1) + \frac{\eta^2}{2} G''(1) - \frac{\eta^3}{6} G'''(1 - \theta_0 \eta) \]
\[ = 1 - \lambda + \frac{\lambda^2}{2} \text{Var}(X) - E(X) \lambda^2 - \frac{G'''(1 - \theta_0 \eta)}{6E(X)^3} \lambda^3 \]
where \( \theta_0 \in [0,1] \) and
\[ 0 \leq D_0(X) \leq \frac{G'''(1 - \theta_0 \eta)}{6E(X)^3} \leq \frac{M(X)}{E(X)^3} = D(X). \]

Analogously, Eq. (3b) and Eq. (3c) can be derived. Finally, apply the relation
\[ \sum_{n=3}^{\infty} P_n(n) = 1 - [P_0(1) + P_0(1) + P_0(2)], \]
one yields Eq. (3d). \( \square \)

**Remark 1.** If \( C(X) \) and \( D(X) \) is not too big, \( \lambda^2 C(X) \) and \( \lambda^3 D(X) \) can be ignored when \( \lambda \ll 1 \). By comparing Eqs. (3a)-(3d) with Eqs. (1a)-(1d), we can see that \( P_0(n) \) is well approximated by the Poisson distribution \( P^\lambda(n) \).

**Remark 2.** The approximation of Poisson distribution is an asymptotic result in the limit \( \lambda \to 0 \). If \( \lambda \) is large, this approximation will be broken, as can be found from the numerical Example 1 in the following.

**Remark 3.** The approximation error \( \Delta(n) = P_0(n) - P^\lambda(n) \) can be written as
\[ \Delta(0) = \lambda^2 C(X) + O(\lambda^3), \quad (4a) \]
\[ \Delta(1) = -2\lambda^2 C(X) + O(\lambda^3), \quad (4b) \]
\[ \Delta(2) = \lambda^2 C(X) + O(\lambda^3), \quad (4c) \]
\[ \Delta(n) \leq O(\lambda^3), \quad n \geq 3. \quad (4d) \]

Here \( C(X) = [\text{Var}(X) - E(X)]/[2E(X)^2] \) is determined by \( P(N) \), the input distribution only. If
\[ Var(X) = E(X), \text{ then } C(X) = 0 \text{ and the error decreases to } O(\lambda^3). \] On the other hand, for some singular input distribution, \( C(X) \) is so big that \( P_\eta(n) \) can no longer be approximated by Poisson. In Example 3 we give a typical example that Poisson approximation fails.

In QKD we use faint laser to simulate the single photon source. For security analysis, it is important to estimate \( P_\eta(n > 1|n > 0) \), the probability that a pulse contains more than one photon. According to our estimation,

\[
P_\eta(n > 1|n > 0) = \frac{1 - P_\eta(0) - P_\eta(1)}{1 - P_\eta(0)} \approx \frac{\lambda^2 + C(X)\lambda^2}{\lambda - \frac{\lambda^2}{2} - C(X)\lambda^2}.
\]

Here we have neglected the \( \lambda^3 \) and higher order terms. We further simplify it by removing the \( \lambda^2 \) terms in the denominator, which gives

\[
P_\eta(n > 1|n > 0) \approx \left[ \frac{1}{2} + C(X) \right] \lambda. \tag{5}
\]

If the input distribution is Poisson, then \( C(X) = 0 \). After we attenuate it to faint laser that contains an average of 0.1 photon in each pulse, which means \( \lambda = 0.1 \), we would have \( P_\eta(n > 1|n > 0) = 0.05 \), i.e., each pulse has about 5% chance to contain more than one photon.

From Eq. (5) we can see that, the risk of a QKD system rises as \( C(X) \) grows. In Example 3 we use an ill-shaped input distribution whose \( C(X) = 9.11 \). After we attenuate it to \( \lambda = 0.1 \), then \( P_\eta(n > 1|n > 0) \approx 1 \). So for this big a \( C(X) \), the QKD system would be totally unsecured since almost every pulse contains at least two photons.

On the other hand, if the input distribution satisfies \( Var(X) < E(X) \) then \( C(X) < 0 \). One can expects the attenuated faint laser be more secure than a Poisson laser because we get a smaller \( P_\eta(n > 1|n > 0) \). The last row of table II gives an example of negative \( C(X) \). However, the possible smallest \( C(X) \) is \( -\frac{1}{2}E(X)^{-1} \). For practical input laser, \( E(X) \) is so big that \( -\frac{1}{2}E(X)^{-1} \) can be ignored. This means, for attenuated faint laser, a Poisson distribution is almost as good as one can expects.

![Figure 1. If attenuation is not strong enough, the decayed distribution may not be approximated by Poisson distribution. In this example the input random variable takes integer values from 0 to 1000, \( \eta \) is set to 0.1, and the expectation after decay is \( \lambda = 48.85 \).](image)

4. Numerical Examples

Following we give some numerical simulations and the results coincide with our theoretical estimations quite well.

**Example 1.** First we show that if the mean value of the decayed distribution \( \lambda = \eta E(X) \) is not sufficiently small, then in general, approximation using Poisson distribution fails. We choose a random variable \( X \) whose PMF takes the shape in Figure 1 and set \( \eta = 0.1 \). In this case \( \lambda \approx 48.85 \), and it can be observed that \( P_\eta(n) \) is far away from Poisson distribution.

**Example 2.** Using the same input random variable \( X \) as Example 1 but we take \( \eta = 0.001 \), which means that our new attenuator is 100 times stronger than the old one. Now we have \( \lambda = 0.4885 \) and the binomial decayed PMF \( P_\eta(n) \) is close to a Poisson distribution, as indicated in Figure 2. We further decrease \( \eta \) to 0.0002, then \( \lambda = 0.0977 \) and Figure 3 shows a perfect match between the decayed distribution and Poisson distribution, which claims strong support for the
common belief that photon number in faint laser pulse can be treated as Poisson distributed.

Example 3. For some ill-shaped input distributions, $C(X)$ could be very large. If this happens, even if $\lambda$ is small, Poisson approximation could still fail. As an example, we construct a singular input distribution of which $P(X=1)=0.95$ and $P(X=1001)=0.05$. In this case, $C(X)=9.11$. If we attenuate it to $\lambda=0.1$, as Figure 4 shows, $P^{\lambda}(n)$ fails to converge to $P^{\eta}(n)$.

To quantitatively test the approximation error, Eqs. (4a)-(4d), we choose different input distributions and computer their attenuated distribution. We adjust the attenuating coefficient $\eta$ to keep $\lambda=0.1$. The numerical result, which is listed in Table 1 supports our theoretical estimation very well.

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REFERENCES

1. W. K. Wootters and W. H. Zurek, Nature 299 (1982) 802.
2. C. H. Bennett and G. Brassard, Proc. of IEEE Int. Conf. on Computers, Systems, and Signal Processing, Piscataway, NJ: IEEE, (1984) 175.
3. N. Lütkenhaus, Phys. Rev. A 61 (2000) 052304.
4. N. Gisin, G. Riordy, W. Tittel and H. Zbinden, Rev. Mod. Phys. 74 (2002) 145.
5. B. E. A. Saleh, Phy. Rev. Lett. 58 (1987) 2656.
6. D. Marcuse, IEEE J. Quantum Electron. QE-20 (1984) 1139.
7. C. W. Gardiner, Handbook of Stochastic Methods, Springer, Berlin, 1983.
8. R. Durrett, Probability: Theory and Examples, Wadsworth and Brooks: Pacific Grove, CA, 1991.
Table 1
Numerical validation of the approximate error, Eqs. (4a)-(4d). It can be observed that $\Delta(0) \approx \Delta(2) \approx \lambda^2 C(X)$, which goes on well with Eq. (4a) and Eq. (4d); And $\Delta(1) \approx -2\lambda^2 C(X)$, which also agrees with Eq. (4c).

| $\lambda^2 C(X)$ | $\Delta(0)$ | $\Delta(1)$ | $\Delta(2)$ | $\Delta(3)$ | $\Delta(4)$ |
|-----------------|-------------|-------------|-------------|-------------|-------------|
| 0.0045          | 0.0039      | -0.0073     | 0.0028      | 0.0005      | 0.0000      |
| 0.0030          | 0.0027      | -0.0050     | 0.0020      | 0.0003      | 0.0000      |
| 0.0018          | 0.0017      | -0.0032     | 0.0013      | 0.0002      | 0.0000      |
| 0.0011          | 0.0010      | -0.0019     | 0.0008      | 0.0001      | 0.0000      |
| 0.0005          | 0.0005      | -0.0010     | 0.0004      | 0.0000      | 0.0000      |
| -0.00016        | -0.00014    | 0.00027     | -0.00011    | -0.00001    | -0.00000    |

Figure 4. A typical example when error is large even if $\lambda$ is small. The input distribution is constructed as $P(X = 1) = 0.95$ and $P(X = 1001) = 0.05$. We decay it to $\lambda = 0.1$ and it turns out that $P_\lambda(n)$ fails to converge to $P_\eta(n)$. 