Searching sequences of resonant orbits between a spacecraft and Jupiter

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Abstracts: This research shows a study of the dynamical behavior of a spacecraft that performs a series of close approaches with the planet Jupiter. The main idea is to find a sequence of resonant orbits that allows the spacecraft to stay in the region of the space near the orbit of Jupiter around the Sun gaining energy from each passage by the planet. The dynamical model considers the existence of only two massive bodies in the systems, which are the Sun and Jupiter. They are assumed to be in circular orbits around their center of mass. Analytical equations are used to obtain the values of the parameters required to get this sequence of close approaches. Those equations are useful, because they show which orbits are physically possible when taking into account that the periapsis distances have to be above the surface of the Sun and that the closest approach distances during the passage by Jupiter have to be above its surface.

Key words: Astrodynamics, Orbital Maneuvers, Close Approach Maneuvers, Space Trajectories.

1. Introduction

The description of the close approach is shown in references [1-3]. There are also several other studies that can be mentioned in this topic, like the study of transfer orbits to find trajectories linking the Lagrangian points and the primaries of the system [4-5] or the existence of atmospheric drag during the maneuver [6]. In terms of practical missions that used this technique, it is possible to mention references [7]-[16], which show several applications of the close approach in orbital maneuvers. Some other studies considered variations of this problem, like the combination of impulsive maneuvers during the close approach [17], noncircular orbits for the primaries [18] or the existence of a cloud of particles resulting from an explosion of a main body [19]. For a maneuver that needs to move a spacecraft that is around Jupiter to send it to another location of the Solar System or beyond, it is possible to mention reference [20]. A three dimensional swing-by is described in reference [21]. In 1968, it was introduced in the literature maneuvers with multiple encounters, like in reference [22]. Consecutive meetings between M3 and the orbit of M2 in the restricted three-body problem for the case \( \mu = 0 \), including elliptical orbits for the primaries, is shown in reference [23].

The present paper is a sequence of those studies, because it extends the previous researches to consider the possibility of more than one close approach with a celestial body. The goal is to search for series of close approaches with Jupiter to find trajectories for missions that want to study the space in the Solar System near the planet by changing its orbit, but without escaping to the interstellar space. The orbital elements, energy, angular momentum and velocity of the spacecraft with respect to the Sun are changed by each of these approaches, and the goal is to study their orbital behavior to know how this trajectory will evolve. The dynamical system given by the “patched-conics” is used and the motion is assumed to be planar. Jupiter is assumed to be in a circular orbit in the present model. Extensions to elliptic orbits were considered, but the results do not have large influences. Figure 1 shows a sketch of the maneuver.
So, this technique uses close approach maneuvers to change the orbit of the spacecraft, to allow it to pass by different parts of the space near the orbit of the planet Jupiter around the Sun. It means that, by changing the orbital elements of the spacecraft, its path in the space covers different areas, so it is possible to make measurements of important scientific data like temperature, radiation level, density of particles and others in different areas without using fuel expenditure to change the orbit of the spacecraft. A complete description of the evolution of the orbits, showing which values of semi-major axis, eccentricities, perigee and apogee distances are covered is also made, in order to known exactly the regions of the space that are included in the trajectories found.

2. Mathematical model

This problem can be studied assuming a system formed by three bodies: The Sun (\(M_1\)), Jupiter (\(M_2\)) and a spacecraft with an infinitesimal mass (\(M_3\)) that remains orbiting the Sun and then makes a close approach with Jupiter. The following variables are used to identify one close approach trajectory: \(r_{ap}\) (the distance from the spacecraft to the center of Jupiter at the moment of the closest approach), \(\vec{v}_1^-\) and \(\vec{v}_1^+\) (velocities of the spacecraft with respect to Jupiter, before and after the maneuver, respectively, in the inertial frame), \(\vec{v}_2\) (velocity of Jupiter with respect to the Sun), \(\delta\) (half of the angle of the curvature due to the close approach) and \(\psi\) (angle of approach). Reference [3] shows more details on that problem.

![Figure 1](https://example.com/figure1.png)

*Figure 1: Overview of the orbital characteristics for some maneuvers.*

The velocity and orbital elements of the spacecraft are changed by the close approach with Jupiter. The orbital elements (\(a\) = semi-major axis, \(e\) = eccentricity), energy (\(E\)) and angular momentum (\(C\)) of the spacecraft before the encounter with Jupiter are obtained from the equations:

\[
a = \frac{r_a + r_p}{2}, \quad e = 1 - \frac{r_p}{r_a}, \quad E = -\frac{\mu}{2a}, \quad C = \sqrt{\mu a(1 - e^2)}
\]

where \(r_p\) is the periapsis of the orbit of the spacecraft around the Sun, \(r_a\) is the apoapsis of that orbit and \(\mu\) is the gravitational parameter of the Sun.

It is possible to obtain the variations of energy and angular momentum from equation [3]. The numerical values are shown in the Tables available in the results part of the paper.

\[
\Delta v = \vec{v}_0 - \vec{v}_1 = 2|\vec{v}_x| \sin \delta
\]

\[
\Delta E = E_+ - E_- = -2v_2 v_x \sin \delta \sin \psi
\]

\[
\Delta C = \frac{\Delta E}{\omega}
\]

where \(\omega\) is the angular velocity between the primaries, \(\delta\) is half of the angle of deflection due to the close approach and \(E_+, E_-\) are the energy before and after the maneuver, respectively. Finally, having determined the variation of energy and angular momentum after the maneuver, it is possible to obtain the semi-major axis and the eccentricity of the orbit after the close approach, by using the equations \(a = -\frac{\mu}{2E}\) and

\[
e = \sqrt{1 - \frac{C^2}{\mu a}}
\]
Next, the distance that the spacecraft has to pass from Jupiter (r_{ap}) to achieve an orbit that has a larger value for the apoapsis (so, achieving the goal of making the spacecraft to pass by different positions in space), but also going to an orbit that is resonant with the orbit of Jupiter, such that a new encounter will occur, is calculated. To solve this problem a list of resonant orbits is made and then it is arranged in order of increasing values of the energy. Then, a search is made to find the value of the r_{ap} for each passage that makes the orbits of the spacecraft to follow this sequence of orbits. Then the analytical solution can be obtained from the theory of hyperbolic orbits [21]:

$$r_{ap} = \frac{\mu}{v_0^2} \left[ \frac{1}{\sin(\delta)} - 1 \right]$$

where $\delta = \text{ArcCos} \left( \frac{1}{\sqrt{2}} \sqrt{\frac{A^2 + 2B(B+C) - A}{A^2 + B^2}} \right)$, $a = \frac{\sin(2\pi + \beta)}{2}$, $B = \cos(2\pi + \beta)$ and $C = -\frac{\Delta E}{2v_2^2v_0^2}$.

3. Tisserand’s Criterion
The Tisserand’s criterion was introduced by astronomer Francois Felix Tisserand. It is an equation in dimensionless coordinates that represents a quantity that should be conserved in a circular restricted three-body system. Thus, two observed orbiting bodies are possibly the same if they nearly satisfy the Tisserand’s Criterion [2]:

$$\frac{1}{a_i} + 2\sqrt{a_i(1-e_i^2)} \cos i_i \approx \frac{1}{a_0} + 2\sqrt{a_0(1-e_0^2)} \cos i_0$$

Where $a$, $e$, and $i$ are the orbital elements before the maneuver and $a_0$, $e_0$ and $i_0$ are the orbital elements after the maneuver. In this study, Tisserand’s Criterion will be used to validate the resonant orbits found. Note that, since only planar orbits are considered, the values for the inclination before and after the close approach is zero (direct orbits) or 180 degrees (retrograde orbits).

3. Numerical Study
The spacecraft starts its motion in a given orbit around the Sun, specified by its periapsis and apoapsis distances. A maximum number of revolutions for Jupiter between two successive close approaches are specified. It generates a large number of potential orbits and the maneuvers required to reach them. The meaning of “potential orbits and maneuvers” is that some of the maneuvers obtained may require a value smaller than the radius of Jupiter as a periapsis distance for one of the passages, or an orbit around the Sun with periapsis below the surface of the Sun. Those maneuvers have to be excluded from a practical list of maneuvers. A number higher than five would generate orbits with too large periods, which has little practical interest and that would be too much influenced by other perturbations, which are not considered here. Other perturbations, in particular the gravitational effects of Saturn, are not added to the dynamical model of the present research, because the idea here is to study possibilities of orbits and not making final decisions on which orbit to use. Those perturbations can be compensated by low cost maneuvers, after the nominal trajectories are found, or added in deeper studies of the orbits selected in this first study. Table 1 shows the orbits, including the information of the number of periods of Jupiter before the next close approach, the equivalent number of orbits of the spacecraft, the period (in years) of the orbit of the spacecraft, the respective semi-major axis (km), and the order of the resonance.

It is then possible to ordinate these orbits in crescent values of the energy and then search for values of r_{ap} for each passage that can make the spacecraft to follow this series. The following assumptions are considered: the passage occurs at the points A_n (Fig. 1); there are no other forces perturbing the spacecraft; the Sun-spacecraft energy and angular momentum are constant after and before the close approach.

Table 2 shows several different options for the first orbit. The possible orbits, after the first one, are shown in Table 3, that list the correct sequence of maneuvers that keeps the spacecraft increasing energy,
semi-major axis and apoapsis distance. This Table shows the number of the maneuver, the orbital period (days), the distance of the closest approach obtained from the algorithm (in units of radius of Jupiter), semi-major axis (km), eccentricity, energy (km²/s²), periapsis distance (km), apoapsis distance (km), half of the deflection angle (degree), angle of approach (degree), order of the resonance and the time elapsed since the start of the maneuvers (years). The \( r_{ap} \), the distance of the close approach between the spacecraft and Jupiter, is the key factor of the sequence, because it is assumed to be the only variable available to control the motion of the spacecraft. The radius of Jupiter is called \( R_J \).

Now, it is necessary to study the second restriction and verify if the values of \( r_{ap} \) are possible to reach, which means that they have to be larger than the radius of Jupiter. Figures 2 to 5 show the results. Note that after maneuver 8, the distance of the closest approach is always below the surface of Jupiter. The times that the spacecraft remains in each orbit are visible in the plots, because the dots represent the instant of each maneuver.

| Revolutions of Jupiter | Revolutions of the spacecraft | Period of the spacecraft (years) | Semi-major axis of the spacecraft \( x 10^8 \) (km) | Order of resonance |
|------------------------|-------------------------------|---------------------------------|---------------------------------|-------------------|
| 1                      |                               | 11.859                          | 7.785                           | 1:1               |
|                        | 2                             | 5.9295                          | 4.904                           | 2:1               |
| 2                      | 1                             | 23.718                          | 12.358                          | 1:2               |
|                        | 3                             | 7.90599                         | 5.941                           | 3:2               |
|                        | 5                             | 4.7436                          | 4.226                           | 5:2               |
| 3                      | 1                             | 35.577                          | 16.193                          | 1:3               |
|                        | 2                             | 17.7885                         | 10.201                          | 2:3               |
|                        | 4                             | 8.89424                         | 6.426                           | 4:3               |
|                        | 5                             | 7.11539                         | 5.538                           | 5:3               |
|                        | 7                             | 5.08242                         | 4.425                           | 7:3               |
|                        | 8                             | 4.44712                         | 4.048                           | 8:3               |
| 4                      | 1                             | 47.436                          | 19.615                          | 1:4               |
|                        | 3                             | 15.812                          | 9.431                           | 3:4               |
|                        | 5                             | 9.48719                         | 6.709                           | 5:4               |
|                        | 7                             | 6.77657                         | 5.361                           | 7:4               |
|                        | 9                             | 5.27066                         | 4.534                           | 9:4               |
| 5                      | 1                             | 59.295                          | 22.76                           | 1:5               |
|                        | 2                             | 29.6475                         | 14.341                          | 2:5               |
|                        | 3                             | 19.765                          | 10.94                           | 3:5               |

The initial orbit for the spacecraft has periapsis of 155,000,000 km, which is near the orbit of the Earth around the Sun, and apoapsis of 1,011,830,000 km, that is higher than the orbit of Jupiter around the Sun. In that sequence it was not found the problem of reaching values for the periapsis that are inside the Sun. Some of the orbits (the first 4) have periapsis below the initial value, so those orbits intercept the orbit of the Earth around the Sun. It causes a potential risk of collision, but with very small probability. This possibility is neglected in the present study. In fact, these periapsis distances are interesting, because they generate trajectories that cross the path of the Earth and that can be used to recover the spacecraft, if desired. Note that the values for the distances of the closest approach decreases, to compensate the increase of the velocity of approach in the equation of the gain of energy (Eq. 2) and this fact limit the number of revolutions before a value below the surface of Jupiter is found for \( r_{ap} \). This situation is a characteristic of the sequence of orbits shown here, where the values of important elements like the angle of approach and the velocity of approach are given in advance. Of course situations that generate an escape orbit in the first approach can be found for Jupiter, but this is not the goal of this research.

The energy, angular momentum, semi-major axis and apoapsis distances increase after each close approach. This characteristic is forced by the goal of finding series of orbits. The energy goes from
-113.98 km$^2$/s$^2$ before the first passage until -65.19 km$^2$/s$^2$ after the last one, in crescent steps. This variation in energy causes the semi-major axis to go from 583.41x10$^6$ km to 1020.15x10$^6$ km, which corresponds to a variation of the apoapsis distances from 1011.83 x10$^6$ km (above the orbit of Jupiter) to 1679.94 x10$^6$ km. The eccentricity has a decreasing sequence and the time span for this sequence is years.

### Table 2 – Options for initial conditions for the initial orbit (maneuver 0).

| Man. | period (day) | ap$_o$ (10$^6$ km) | a (10$^6$ km) | e | Energy | r$_o$ (10$^6$ km) | r$_a$ (10$^6$ km) | $\delta$ (deg) | $\Psi$ (deg) | resonance | Time (years) | Tisserand’s Criterion |
|------|--------------|---------------------|--------------|---|--------|------------------|------------------|--------------|-------------|------------|-------------|-----------------------|
| 0    | 2266.8       | -                   | 505.58       | 0.693 | -131.5 | 155.0          | 856.16          | 27.34        | 372.61     | -           | 0           | 1.3101                 |
| 0    | 2533.6       | -                   | 544.49       | 0.715 | -122.1 | 155.0          | 933.99          | 36.38        | 372.65     | -           | 0           | 1.2931                 |
| 0    | 2810.0       | -                   | 583.41       | 0.734 | -113.98 | 155.0          | 1011.83         | 40.42        | 373.52     | -           | 0           | 1.2550                 |

### Table 3 - Sequence of orbits performing close approaches with Jupiter.

| Man. | period (day) | r$_p$ (R$_J$) | a (10$^6$ km) | e | Energy | r$_o$ (10$^6$ km) | r$_a$ (10$^6$ km) | $\delta$ (deg) | $\Psi$ (deg) | resonance | Time (years) | Tisserand’s Criterion |
|------|--------------|--------------|--------------|---|--------|------------------|------------------|--------------|-------------|------------|-------------|-----------------------|
| 0    | 2810.00      | -            | 583.41       | 0.734 | -113.98 | 155.0          | 1011.83         | 40.42        | 373.52     | -           | 0           | 1.2550                 |
| 1    | 1856.39      | 11.49        | 442.54       | 0.867 | -150.27 | 58.75        | 826.33          | 33.31        | 335.39     | 7.3         | 35.60       | 1.2551                 |
| 2    | 2165.79      | 17.41        | 490.44       | 0.814 | -135.59 | 91.3          | 889.57          | 43.73        | 356.00     | 9.4         | 83.00       | 1.2545                 |
| 3    | 2475.19      | 9.47         | 536.10       | 0.771 | -124.04 | 122.93       | 949.27          | 48.99        | 358.74     | 2.1         | 142.31      | 1.2544                 |
| 4    | 2598.94      | 8.92         | 553.82       | 0.765 | -120.07 | 135.08       | 972.57          | 52.70        | 357.55     | 7.4         | 177.89      | 1.2548                 |
| 5    | 2887.71      | 5.44         | 594.12       | 0.727 | -111.93 | 162.07       | 1026.18         | 57.55        | 357.60     | 5.3         | 201.60      | 1.2549                 |
| 6    | 3248.68      | 3.92         | 642.66       | 0.700 | -103.48 | 192.97       | 1092.34         | 61.14        | 358.81     | 3.2         | 237.18      | 1.2544                 |
| 7    | 3465.19      | 3.00         | 670.90       | 0.687 | -99.12  | 209.99       | 1131.81         | 65.93        | 356.40     | 4.3         | 264.16      | 1.2547                 |
| 8    | 4331.57      | 2.01         | 778.52       | 0.656 | -85.42  | 267.77       | 1289.27         | 73.25        | 356.27     | 5.4         | 296.48      | 1.2547                 |
| 9    | 5775.33      | 0.93         | 943.10       | 0.644 | -70.51  | 335.64       | 1580.56         | 78.28        | 358.70     | 1.1         | 343.91      | 1.2547                 |
| 10   | 6497.36      | 0.45         | 1020.15      | 0.647 | -65.19  | 360.36       | 1679.94         | 80.65        | 358.93     | 3.4         | 379.49      | 1.2544                 |

Figure 2: Energy of the spacecraft as a function of time.

Figure 3: Semi-major axis of the spacecraft as a function of time.

Figure 4: Apoapsis distance of the spacecraft as a function of time.

Figure 5: Periapsis distance of the spacecraft as a function of time.

4. Conclusion

This study was made to show the evolution of the trajectories, as well as the amplitudes of the variations of the keplerian elements, velocity, energy and angular momentum in a trajectory of a spacecraft that perform a series of close approaches with the planet Jupiter. A set of analytical equations is used to allow
the calculation of the distance of the closest approach that generates a specified orbit. Then, a series of resonant orbits with Jupiter that is increasing apoapsis distance to cover a large area of the space near the orbit of Jupiter is found. Using these equations it is possible to establish a sequence of close approaches that meets the goals. The results showed that it is possible to find useful sequences of close approaches to study the space near the orbit of Jupiter by using these natural changes of orbits to pass by different positions in the space without the expenses of applying a control to the spacecraft.

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