ON A FAMILY OF FULLY NONLINEAR INTEGRO-DIFFERENTIAL OPERATORS: FROM FRACTIONAL LAPLACIAN TO NONLOCAL MONGE-AMPÈRE

LUIS A. CAFFARELLI AND MARÍA SORIA-CARRO

Abstract. We introduce a new family of intermediate operators between the fractional Laplacian and the Caffarelli–Silvestre nonlocal Monge-Ampère that are given by infimums of integro-differential operators. Using rearrangement techniques, we obtain representation formulas and give a connection to optimal transport. Finally, we consider a global Poisson problem, prescribing data at infinity, and prove existence, uniqueness, and $C^{1,1}$-regularity of solutions in the full space.

1. Introduction

Integro-differential equations arise in the study of stochastic processes with jumps, such as Lévy processes. A classical elliptic integro-differential operator is the fractional Laplacian,

$$\Delta^s u(x_0) = c_{n,s} \text{PV} \int_{\mathbb{R}^n} (u(x_0 + x) - u(x_0)) \frac{1}{|x|^{n+2s}} \, dx, \quad s \in (0, 1),$$

which can be understood as an infinitesimal generator of a stable Lévy process. These types of processes are very well studied in probability, and their generators may be given by

$$L_K u(x_0) = \int_{\mathbb{R}^n} (u(x_0 + x) - u(x_0) - x \cdot \nabla u(x_0)) K(x) \, dx,$$

where the kernel $K$ is a nonnegative function satisfying some integrability condition.

Over the last few years, there has been significant interest in studying linear and nonlinear integro-differential equations from the analytical point of view. In particular, extremal operators like

$$Fu(x_0) = \inf_{K \in \mathcal{K}} L_K u(x_0)$$

play a fundamental role in the regularity theory. See [6–8, 16] and the references therein. The above equation is an example of a fully nonlinear equation that appears in optimal control problems and stochastic games [12, 15]. The infimum in (1.1) is taken over a family of admissible kernels $\mathcal{K}$ that depends on the applications. In fact, nonlocal Monge-Ampère equations have been developed in the form (1.1), for some choice of $\mathcal{K}$ [4, 9, 11].

The Monge-Ampère equation arises in several problems in analysis and geometry, such as the mass transportation problem and the prescribed Gaussian curvature problem [10]. The classical equation prescribes the determinant of the Hessian of some convex function $u$:

$$\det(D^2 u) = f.$$
In the literature, there are different nonlocal versions of the Monge-Ampère operator that Guille\-n–Schwab [11], Caffarelli–Charro [4], and Caffarelli–Silvestre [9] have considered. See also [13] for a nonlocal linearized Monge-Ampère equation given by Maldonado–Stinga. These definitions are motivated by the following property: if $B$ is a positive definite symmetric matrix, then

$$n \det(B)^{1/n} = \inf_{A \in \mathcal{A}} \text{tr}(A^T B A),$$

where $\mathcal{A} = \{ A \in M_n : A > 0, \det(A) = 1 \}$ and $M_n$ is the set of $n \times n$ matrices. If a convex function $u$ is $C^2$ at a point $x_0$, then by the previous identity with $B = D^2 u(x_0)$, we may write the Monge-Ampère operator as a concave envelope of linear operators. It follows that

$$n \det(D^2 u(x_0))^{1/n} = \inf_{A \in \mathcal{A}} \Delta[u \circ A](A^{-1} x_0).$$

Caffarelli–Charro study a fractional version of $\det(D^2 u)^{1/n}$, replacing the Laplacian by the fractional Laplacian in the previous identity. More precisely,

$$D^s u(x_0) = \inf_{A \in \mathcal{A}} \Delta^s[u \circ A](A^{-1} x_0) = c_{n,s} \inf_{A \in \mathcal{A}} \text{PV} \int_{\mathbb{R}^n} \frac{u(x_0 + x) - u(x_0)}{|A^{-1} x|^n + 2s} \, dx,$$

where $s \in (0,1)$ and $c_{n,s} \approx 1 - s$ as $s \to 1$ (see also [11]). A different approach based on geometric considerations was given by Caffarelli–Silvestre. In fact, the authors consider kernels whose level sets are volume preserving transformations of the fractional Laplacian kernel. Namely,

$$\text{MA}^s u(x_0) = c_{n,s} \inf_{K \in \mathcal{K}^s_n} \int_{\mathbb{R}^n} (u(x_0 + x) - u(x_0) - x \cdot \nabla u(x_0)) K(x) \, dx,$$

where the infimum is taken over the family,

$$\mathcal{K}^s_n = \left\{ K : \mathbb{R}^n \to \mathbb{R}_+ : \left| \{ x \in \mathbb{R}^n : K(x) > r^{-n-2s} \} \right| = |B_r| \quad \text{for all } r > 0 \right\}.$$

Notice that $|A^{-1} x|^{-n-2s} \in \mathcal{K}^s_n$, for any $A \in \mathcal{A}$. Therefore,

$$\text{MA}^s u(x_0) \leq D^s u(x_0) \leq \Delta^s u(x_0).$$

Moreover, both $\text{MA}^s u$ and $D^s u$ converge to $\det(D^2 u)^{1/n}$, up to some constant, as $s \to 1$.

In this paper, we introduce a new family of operators of the form,

$$\inf_{K \in \mathcal{K}^s_k} \int_{\mathbb{R}^n} (u(x_0 + x) - u(x_0) - x \cdot \nabla u(x_0)) K(x) \, dx,$$

for any integer $1 \leq k < n$, which arises from imposing certain geometric conditions on the kernels. Moreover, we will see that $|y|^{-n-2s} \in \mathcal{K}^s_1 \subset \mathcal{K}^s_k \subset \mathcal{K}^s_n$, for $1 < k < n$, and thus, this family will be monotone decreasing, and bounded from above by the fractional Laplacian and by below by the Caffarelli–Silvestre nonlocal Monge–Ampère.

The paper is organized as follows. In Section 2, we construct the family of admissible kernels $\mathcal{K}^s_k$, and give the precise definition of our operators for $C^{1,1}$-functions. We introduce in Section 3 the basic tools from the theory of rearrangements necessary for our goals. In Section 4, we study the infimum in (1.4) and obtain a representation formula, provided some condition on the level sets is satisfied (see Theorem 4.1). We also study the limit as $s \to 1$ and give a connection to optimal transport. The Hölder continuity of $F^s_k u$ is proved in Section 5, following similar geometric techniques from [9]. In Section 6, we consider a global Poisson problem, prescribing data at infinity, and introduce a new definition of our operators for functions that are merely continuous and convex. We show existence of solutions via
Perron’s method and $C^{1,1}$-regularity in the full space by constructing appropriate barriers. Finally, we discuss some future directions in Section 7.

2. CONSTRUCTION OF KERNELS

Let us start with the construction of the family of admissible kernels. Notice that any kernel $K$ in $\mathcal{K}_n^s$, defined in (1.3), will have the same distribution function as the kernel of the fractional Laplacian, since for any $r > 0$,

$$\{x \in \mathbb{R}^n : |x|^{-n-2s} > r^{-n-2s}\} = B_r.$$

Geometrically, this means that the level sets of $K$ are deformations in any direction of $\mathbb{R}^n$ of the level sets of $|x|^{-n-2s}$, preserving the $n$-dimensional volume.

In view of this approach, a natural way of finding an intermediate family of operators between the nonlocal Monge-Ampère and the fractional Laplacian is to consider kernels whose level sets are deformations that preserve the $k$-dimensional Hausdorff measure $\mathcal{H}^k$, with $1 \leq k < n$, of the restrictions of balls in $\mathbb{R}^n$ to hyperplanes generated by $\{e_i\}_{i=1}^k$.

\begin{equation}
\mathcal{H}^k\left(\{y \in \mathbb{R}^k : K(y, z) > r^{-n-2s}\}\right) = \begin{cases} 
\mathcal{H}^k\left(B_{(r^2 - |z|^2)^{1/2}}\right) & \text{if } |z| < r \\
nullo & \text{if } |z| \geq r,
\end{cases}
\end{equation}

where $B_{(r^2 - |z|^2)^{1/2}}$ is the ball in $\mathbb{R}^k$ of radius $(r^2 - |z|^2)^{1/2}$.

In Figure 1 we illustrate condition (2.1) for $k = 2$ and $n = 3$. Note that for $k = n$, we recover the definition of $\mathcal{K}_n^s$. Moreover, $|x|^{-n-2s} \in \mathcal{K}_n^s$ for all $k$.

**Proposition 2.2.** Let $1 \leq k < n$. Then $\mathcal{K}_k^s \subset \mathcal{K}_{k+1}^s \subset \mathcal{K}_n^s$.
We denote by $u$Definition 2.4. Corollary 2.5.

In the classical sense. To obtain a finite number, we need to impose two extra conditions:

Moreover, $z \in R^k$, with $|(t, z)| > r$. Therefore, by Definition 2.1, it follows that $I = 0$. If $|z| < r$, then

If $|z| \geq r$, then for any $t \in R$, we have that $(t, z) \in R^{n-k}$.

Proof. Let $K \in K^s$. Fix any $z \in R^{n-k-1}$ and $r > 0$. Then:

$$H^{k+1}\left(\{y \in R^{k+1}: K(y, z) > r^{n-2s}\}\right) = \int_{R^{k+1}} \chi_{\{y \in R^{k+1}: K(y, z) > r^{n-2s}\}}(y)\,dy$$

$$= \int_{R} \left( \int_{R^{k}} \chi_{\{w, t \in R^{k} \times R: K(w, t, z) > r^{n-2s}\}}(w, t)\,dw \right)\,dt$$

$$= \int_{R} H^{k}\left(\{w \in R^{k}: K(w, t, z) > r^{n-2s}\}\right)\,dt \equiv 1.$$

If $|z| \geq r$, then for any $t \in R$, we have that $(t, z) \in R^{n-k}$, with $|(t, z)| > r$. Therefore, by Definition 2.1, it follows that $I = 0$. If $|z| < r$, then

$$I = \int_{R} H^{k}(B_{(r^2-|z|^2)_{1/2}})\,dt$$

$$= \omega_k \int_{(r^2-|z|^2)_{1/2}} (r^2 - t^2 - |z|^2)^{k/2}\,dt$$

$$= \omega_k (r^2 - |z|^2)^{k/2} \int_{(r^2-|z|^2)_{1/2}} \left(1 - \frac{t}{(r^2 - |z|^2)^{1/2}}\right)^{k/2}\,dt$$

$$= \omega_k (r^2 - |z|^2)^{k+1/2} \int_{-1}^{1} (1 - \sigma^2)^{k/2}\,d\sigma$$

$$= \frac{\pi^{k/2}}{\Gamma\left(\frac{k}{2} + 1\right)} \frac{\pi^{1/2} \Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k+1}{2} + 1\right)} (r^2 - |z|^2)^{k+1/2}$$

$$= \omega_{k+1}(r^2 - |z|^2)^{k+1/2} = H^{k+1}(B_{(r^2-|z|^2)_{1/2}}),$$

where $\omega_l = H^{l}(B_1) = \frac{\pi^{l/2}}{\Gamma\left(\frac{l}{2} + 1\right)}$, and $B_{(r^2-|z|^2)_{1/2}}$ is the ball of radius $(r^2 - |z|^2)^{1/2}$ in $R^{k+1}$.

**Definition 2.3.** A function $u : R^n \rightarrow R$ is said to be $C^{1,1}$ at the point $x_0$, and we write $u \in C^{1,1}(x_0)$, if there is a vector $p \in R^n$, a radius $\rho > 0$, and a constant $C > 0$, such that

$$|u(x_0 + x) - u(x_0) - x \cdot p| \leq C|x|^2,$$

for all $x \in B_\rho$.

We denote by $[u]_{C^{1,1}(x_0)}$, the minimum constant for which this property holds, among all admissible vectors $p$ and radii $\rho$.

**Definition 2.4.** Let $s \in (1/2, 1)$ and $1 \leq k < n$. For any $u \in C^0(R^n) \cap C^{1,1}(x_0)$, we define

$$F^s_k u(x_0) = c_{n,s} \inf_{K \in K^s_k} \int_{R^n} (u(x_0 + x) - u(x_0) - x \cdot \nabla u(x_0)) K(x)\,dx,$$

where $K^s_k$ is the set of kernels satisfying (2.1) and $c_{n,s}$ is the constant in $\Delta^s$.

As an immediate consequence of Proposition 2.2, we obtain that the operators are ordered.

**Corollary 2.5.** Let $s \in (1/2, 1)$ and $1 \leq k < n$. Then for any $u \in C^0(R^n) \cap C^{1,1}(x_0)$,

$$MA^s u(x_0) \leq F^s_k u(x_0) \leq \Delta^s u(x_0).$$

Moreover, $\{F^s_k \}_{k=1}^{n-1}$ is monotone decreasing.

The regularity condition on $u$ in Definition 2.4 allows us to compute $F^s_k u$ at the point $x_0$ in the classical sense. To obtain a finite number, we need to impose two extra conditions:
(P₁) An integrability condition at infinity:
\[ \int_{\mathbb{R}^n} \frac{|u(x)|}{(1 + |x|)^{n+2s}} \, dx < \infty. \]

(P₂) A convexity condition at \( x_0 \):
\[ \tilde{u}(x) \equiv u(x_0 + x) - u(x_0) - x \cdot \nabla u(x_0) \geq 0, \quad \text{for all } x \in \mathbb{R}^n. \]

**Proposition 2.6.** If \( u \in C^0(\mathbb{R}^n) \cap C^{1,1}(x_0) \) and satisfies (P₁) and (P₂), then
\[ 0 \leq F_k^u(x_0) < \infty. \]

**Proof.** Let \( \rho > 0 \) be as in Definition 2.3. Then
\[
0 \leq F_k^u(x_0) \leq \int_{\mathbb{R}^n} (u(x_0 + x) - u(x_0) - x \cdot \nabla u(x_0)) \frac{1}{|x|^{n+2s}} \, dx \\
\leq \int_{B_\rho} \frac{|u|_{C^{1,1}(x_0)} |x|^2}{|x|^{n+2s}} \, dx + \int_{\mathbb{R}^n \setminus B_\rho(x_0)} \frac{|u(x)|}{|x-x_0|^{n+2s}} \, dx \\
+ |u(x_0)| \int_{\mathbb{R}^n \setminus B_\rho} \frac{1}{|x|^{n+2s}} \, dx + |\nabla u(x_0)| \int_{\mathbb{R}^n \setminus B_\rho} \frac{|x|}{|x-x_0|^{n+2s}} \, dx \\
\leq C(s, \rho)(|u(x_0)| + |\nabla u(x_0)| + |u|_{C^{1,1}(x_0)}) \\
+ \frac{1+|x_0|+\rho}{\rho} \int_{\mathbb{R}^n} \frac{|u(x)|}{(1 + |x|)^{n+2s}} \, dx < \infty, \quad \text{since } s \in (1/2, 1). \]

We point out that if \( u \) is not convex at \( x_0 \), then the infimum could be \(-\infty\). We show this result in the next proposition.

**Proposition 2.7.** Let \( u \in C^0(\mathbb{R}^n) \cap C^{1,1}(x_0) \). Assume that \( u \) satisfies (P₁). If there exists \( \bar{x} \in \mathbb{R}^n \) with \( \bar{x} = (\bar{y}, 0) \) and \( \bar{y} \in \mathbb{R}^k \), such that
\[ \tilde{u}(\bar{x}) = u(x_0 + \bar{x}) - u(x_0) - \bar{x} \cdot \nabla u(x_0) < 0, \]
then \( F_k^u(x_0) = -\infty \).

**Proof.** Let \( K(x) = |x - \bar{x}|^{-n-2s} \). For any \( r > 0 \) and \( z \in \mathbb{R}^{n-k} \), if \(|z| \leq r\), then
\[ \mathcal{H}^k \left( \{ y \in \mathbb{R}^k : K(y, z) > r^{-n-2s} \} \right) = \mathcal{H}^k \left( \{ y \in \mathbb{R}^k : |y - \bar{y}|^2 + |z|^2 < r^2 \} \right) = \mathcal{H}^k \left( B_{(r^2 - |z|^2)^{1/2}} \right). \]
Also, the measure is clearly zero if \(|z| \geq r\). Therefore, \( K \in K_k^s \). It follows that
\[
F_k^u(x_0) \leq \int_{\mathbb{R}^n} \tilde{u}(x) |x - \bar{x}|^{-n-2s} \, dx \\
= \int_{B_{c}(\bar{x})} \tilde{u}(x) |x - \bar{x}|^{-n-2s} \, dx + \int_{\mathbb{R}^n \setminus B_{c}(\bar{x})} \tilde{u}(x) |x - \bar{x}|^{-n-2s} \, dx = I + \Pi. 
\]
Since \( u \in C^0(\mathbb{R}^n) \cap C^{1,1}(x_0) \), we have that \( \tilde{u} \) is continuous. Hence, given that \( \tilde{u}(\bar{x}) < 0 \), then \( \tilde{u}(x) < 0 \), for all \( x \in B_{c}(\bar{x}) \), for some \( c > 0 \). Moreover, since \( K \notin L^1(B_{c}(\bar{x})) \), it follows that \( I = -\infty \). Arguing similarly as in the proof of Proposition 2.6, we see that \( \Pi < \infty \). Therefore,
\[
F_k^u(x_0) = -\infty. \]

□
Remark 2.8. The operators $F_k^n$ are not rotation invariant. This is because, for simplicity, in the construction of the family of admissible kernels $K_k^n$, we chose the first $k$ vectors from the canonical basis of $\mathbb{R}^n$. In general, we may take any subset of $k$ unitary vectors, $\tau = \{\tau_i\}_{i=1}^k$, and replace the first condition on (2.1) by

$$\mathcal{H}^k(\{y \in \langle \tau \rangle ^\perp : K(y + z\tau) > r^{-n-2s}\}) = \mathcal{H}^k(B_{(r^2-|z|^2)^{1/2}}),$$

for all $z \in \langle \tau \rangle$ and $r > 0$, where $\langle \tau \rangle$ denotes the span of $\{\tau_i\}_{i=1}^k$, and $\langle \tau \rangle ^\perp$ the orthogonal subspace to $\langle \tau \rangle$. Let $SO(n)$ be the group of rotation matrices $n \times n$. Since $\tau_i = Ae_i$, for some $A \in SO(n)$, it follows that any kernel $K_\tau$ satisfying (2.2) can be written as $K_\tau = K \circ A$, where $K$ satisfies (2.1). Therefore to make the operators rotation invariant, one possibility is to take the infimum over all possible rotations. Namely,

$$\inf_{A \in SO(n)} \inf_{K \in \mathcal{K}_k} \int_{\mathbb{R}^n} \tilde{u}(x)K(Ax) \, dx.$$

To focus on the main ideas, we will not explore this operator in this work.

3. Rearrangements and measure preserving transformations

We introduce some definitions and preliminary results regarding rearrangements of non-negative functions. For more detailed information, see for instance [1, 2].

Definition 3.1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative measurable function. We define the decreasing rearrangement of $f$ as the function $f^*$ defined on $[0, \infty)$ given by

$$f^*(t) = \sup \{\lambda > 0 : |\{x \in \mathbb{R}^n : f(x) > \lambda\}| > t\},$$

and the increasing rearrangement of $f$ as the function $f_*$ defined on $[0, \infty)$ given by

$$f_*(t) = \inf \{\lambda > 0 : |\{x \in \mathbb{R}^n : f(x) \leq \lambda\}| > t\}.$$

We use the convention that $\inf \emptyset = \infty$.

Proposition 3.2. Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be nonnegative measurable functions. Then

$$\int_0^\infty f_*(t)g^*(t) \, dt \leq \int_{\mathbb{R}^n} f(x)g(x) \, dx \leq \int_0^\infty f^*(t)g^*(t) \, dt.$$

The upper bound is the classical Hardy–Littlewood inequality. For the proof see [2, Theorem 2.2] or [1, Corollary 2.16]. For the sake of completeness, we give the proof of the lower bound.

Proof. For $j \geq 1$, let $f_j = f|_{B_j}$ and $g_j = g|_{B_j}$, where $B_j$ denotes the ball of radius $j$ centered at 0 in $\mathbb{R}^n$. By [1, Corollary 2.18], it follows that

$$\int_0^{|B_j|} (f_j)_*(t)(g_j)^*(t) \, dt \leq \int_{B_j} f_j(x)g_j(x) \, dx.$$

Since $f, g \geq 0$, we get that

$$\int_{B_j} f_j(x)g_j(x) \, dx \leq \int_{\mathbb{R}^n} f(x)g(x) \, dx.$$

Note that for any $t \in [0, |B_j|]$, we have

$$\{\lambda > 0 : |\{x \in B_j : f_j(x) \leq \lambda\}| > t\} \subset \{\lambda > 0 : |\{x \in \mathbb{R}^n : f(x) \leq \lambda\}| > t\}. $$
Hence, \((f_j)_*(t) \geq f_*(t)\), and
\[
\int_{|B_j|} \sigma(t) \, \mathrm{d}t \geq \int_{|B_j|} f_*(t) \, \mathrm{d}t.
\]
Moreover, \(g_j \nearrow g\) pointwise on \(\mathbb{R}^n\). Then by [1, Proposition 1.39], we have \((g_j)_* \nearrow g_*\) pointwise on \([0, \infty)\), as \(j \to \infty\). By the monotone convergence theorem, we get
\[
\lim_{j \to \infty} \int_{|B_j|} f_*(t) \, \mathrm{d}t = \int_{\mathbb{R}^n} f_*(t) \, \mathrm{d}t.
\]
Combining the previous estimates, we conclude that
\[
\int_0^\infty f_*(t) \, \mathrm{d}t \leq \int_{\mathbb{R}^n} f(x) \, \mathrm{d}x.
\]

**Definition 3.3.** We say that a measurable function \(\psi : \mathbb{R}^l \to \mathbb{R}^m\) is a measure preserving transformation if for any measurable set \(E\) in \(\mathbb{R}^m\), it holds that
\[
\mathcal{H}^l(\psi^{-1}(E)) = \mathcal{H}^m(E).
\]

**Lemma 3.4.** If \(\psi : \mathbb{R}^l \to \mathbb{R}^m\) is a measure preserving, then for any measurable \(f : \mathbb{R}^m \to \mathbb{R}\), and any measurable set \(E\) in \(\mathbb{R}^m\), it follows that
\[
\int_E f(y) \, \mathrm{d}y = \int_{\psi^{-1}(E)} f(\psi(z)) \, \mathrm{d}z.
\]

An important result by Ryff [17] provides a sufficient condition for which we can recover a function given its decreasing/increasing rearrangement, by means of a measure preserving transformation.

**Theorem 3.5** (Ryff’s theorem). Let \(f : \mathbb{R}^n \to \mathbb{R}\) be a nonnegative measurable function. If \(\lim_{t \to \infty} f_*(t) = 0\), then there exists a measure preserving \(\sigma : \text{supp}(f) \to \text{supp}(f_*)\) such that
\[
f = f^* \circ \sigma
\]
almost everywhere on the support of \(f\). Similarly, if \(\lim_{t \to \infty} f_*(t) = \infty\), then \(f = f^* \circ \sigma\).

We will call Ryff’s map, a measure preserving \(\sigma\) satisfying Ryff’s theorem.

**Remark 3.6.** In general, \(\sigma\) is not invertible. Furthermore, there may not exist a measure preserving transformation \(\psi\) such that \(f^* = f \circ \psi\).

As a consequence of Ryff’s theorem, we obtain a representation formula for the admissible kernels. We denote \(\omega_k = \mathcal{H}^k(B_1)\).

**Lemma 3.7.** Let \(K \in K_k^+\). Fix \(z \in \mathbb{R}^{n-k}\) and denote by \(K_z(y) = K(y, z)\). Then
\[
K_z^+(t) = \left((\omega_k^{-1})^{2/k} + |z|^2 \right)^{-\frac{n+2s}{2}}.
\]
In particular, there exists a measure preserving \(\sigma_z : \text{supp}(K_z) \to (0, \infty)\), such that
\[
K(y, z) = K_z^+(\sigma_z(y)), \quad \text{for a.e. } y \in \text{supp}(K_z).
\]

**Proof.** Fix \(z \in \mathbb{R}^{n-k}\). Then
\[
K_z^+(t) = \sup \{ \lambda > 0 : \mathcal{H}^k(\{ y \in \mathbb{R}^k : K(y, z) > \lambda \}) > t \}
\]
\[
= \sup \{ \lambda < |z|^{-n-2s} : \mathcal{H}^k(B(\lambda^{-(n+2s)} - |z|^2)1/2) > t \}
\]
\[
= \sup \{ \lambda < |z|^{-n-2s} : \omega_k(\lambda^{-2/(n+2s)} - |z|^2)k/2 > t \}.
\]
where we denote that rearranges the level sets of \( \tilde{x} \) as attained at some kernel whose level sets depend on the measure preserving transformation \( u \).

\[ \lambda = \sup \{ \lambda < |z|^{-n-2s} : \lambda^{-2/(n+2s)} > (\omega^-_k)^{2/k} + |z|^2 \} \]

\[ = (\omega^-_k)^{2/k} + |z|^2 - \frac{s+2s}{2}. \]

Moreover, \( \lim_{t \to \infty} K^*_k(t) = 0 \). Therefore, the result follows from Theorem 3.5. \( \square \)

In view of Definition 3.1, we introduce the symmetric rearrangement of a function in \( \mathbb{R}^n \) with respect to the first \( k \) variables as follows. Fix \( k \in \mathbb{N} \) with \( 1 \leq k < n \). Given \( x \in \mathbb{R}^n \), we denote \( x = (y, z) \), with \( y \in \mathbb{R}^k \) and \( z \in \mathbb{R}^{n-k} \). Furthermore, for \( z \) fixed, we call \( f_z \) the restriction of \( f \) to \( \mathbb{R}^k \). Namely, \( f_z(y) = f(y, z) \).

**Definition 3.8.** Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a nonnegative measurable function. We define the \( k \)-symmetric decreasing rearrangement of \( f \) as the function \( f^{s,k} : \mathbb{R}^n \to [0, \infty] \) given by

\[ f^{s,k}(x) = f^*_x(\omega_k|y|^k), \]

and the \( k \)-symmetric increasing rearrangement as the function \( f_{s,k} : \mathbb{R}^n \to [0, \infty] \) given by

\[ f_{s,k}(x) = (f_z)_*(\omega_k|y|^k). \]

When \( k = n \), we obtain the usual symmetric rearrangement.

**Remark 3.9.** (1) Notice that \( f^{s,k} \) and \( f_{s,k} \) are radially symmetric and monotone decreasing/increasing, with respect to \( y \). In the literature, this type of symmetrization is also known as the Steiner symmetrization [1, Chapter 6].

(2) By Lemma 3.7, we see that any kernel \( K \in K^*_k \) satisfies

\[ K^{s,k}(x) = |x|^{-n-2s}, \quad \text{for } x \neq 0. \]

### 4. Analysis of \( \mathcal{F}^s_k \)

Our main goal of this section is to study the infimum in the definition of the operator,

\[ \mathcal{F}^s_k u(x_0) = c_{n,s} \inf_{K \in K^*_k} \int_{\mathbb{R}^n} \tilde{u}(x)K(x) \, dx, \]

where \( \tilde{u}(x) = u(x_0 + x) - u(x_0) - x \cdot \nabla u(x_0) \). Throughout the section, we will assume that \( u \in C^0(\mathbb{R}^n) \cap C^{1,1}(x_0) \) and satisfies properties (P1) and (P2), so that \( 0 \leq \mathcal{F}^s_k u(x_0) < \infty \).

#### 4.1. Analysis of the infimum

We will study the following cases:

**Case 1.** For all \( \lambda > 0 \) and \( z \in \mathbb{R}^{n-k} \),

\[ \mathcal{H}^k\{ \{ y \in \mathbb{R}^k : \tilde{u}(y, z) \leq \lambda \} \} < \infty. \]

**Case 2.** There exists some \( \lambda_0 > 0 \) such that for all \( z \in \mathbb{R}^{n-k} \),

\[ \mathcal{H}^k\{ \{ y \in \mathbb{R}^k : \tilde{u}(y, z) \leq \lambda \} \} \begin{cases} < \infty & \text{for } 0 < \lambda < \lambda_0 \\ = \infty & \text{for } \lambda \geq \lambda_0. \end{cases} \]

**Case 3.** For all \( \lambda > 0 \) and \( z \in \mathbb{R}^{n-k} \),

\[ \mathcal{H}^k\{ \{ y \in \mathbb{R}^k : \tilde{u}(y, z) \leq \lambda \} \} = \infty. \]

In the first case, when all of the level sets of \( \tilde{u} \) have finite measure, we show that the infimum is attained at some kernel whose level sets depend on the measure preserving transformation that rearranges the level sets of \( \tilde{u} \). More precisely:
**Theorem 4.1.** Suppose that for all \( \lambda > 0 \) and \( z \in \mathbb{R}^{n-k} \),
\[
\mathcal{H}^k \left( \{ y \in \mathbb{R}^k : \tilde{u}(y, z) \leq \lambda \} \right) < \infty.
\]
Then, for any \( z \in \mathbb{R}^{n-k} \), there exists a measure preserving \( \sigma_z : \mathbb{R}^k \to [0, \infty) \) such that
\[
\mathcal{F}_k u(x_0) = c_{n,s} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \tilde{u}(y, z) \frac{\sigma(z(y))^{2/k} + |z|^2}{n+2s} dydz.
\]
In particular, the infimum is attained.

**Remark 4.2.** Observe that if \( \tilde{u}(\cdot, z) \) is constant in some set of positive measure, then the kernel where the infimum is attained is not unique since the integral is invariant under any measure preserving rearrangement of \( K \) within this set (see [17]).

Before we give the proof of Theorem 4.1, we need a lemma regarding the \( k \)-symmetric increasing rearrangement of \( \tilde{u} \). By Definition 3.8, this is given by the following expression:
\[
\tilde{u}_{s,k}(y, z) = \inf \{ \lambda > 0 : \mathcal{H}^k \left( \{ w \in \mathbb{R}^k : \tilde{u}(w, z) \leq \lambda \} \right) > \omega_k |y|^k \}.
\]

**Lemma 4.3.** Fix \( z \in \mathbb{R}^{n-k} \). If \( \mathcal{H}^k \left( \{ y \in \mathbb{R}^k : \tilde{u}(y, z) \leq \lambda \} \right) < \infty \), for all \( \lambda > 0 \), then
\[
\lim_{|y| \to \infty} \tilde{u}_{s,k}(y, z) = \infty.
\]

**Proof.** Assume there exists \( M > 0 \), independent of \( \lambda \), such that
\[
(4.1) \quad \mathcal{H}^k \left( \{ w \in \mathbb{R}^k : \tilde{u}(w, z) \leq \lambda \} \right) \leq M, \quad \text{for all } \lambda > 0.
\]
Then for any \( y \in \mathbb{R}^k \), with \( \omega_k |y|^k > M \), we have that
\[
\tilde{u}_{s,k}(y, z) = \infty,
\]
since \( \inf \emptyset = \infty \). If (4.1) does not hold, then there must be an increasing sequence \( \{ M_\lambda \}_{\lambda>0} \), with \( M_\lambda \to \infty \), as \( \lambda \to \infty \), such that
\[
\mathcal{H}^k \left( \{ w \in \mathbb{R}^k : \tilde{u}(w, z) \leq \lambda \} \right) = M_\lambda.
\]
Then for any \( M > 0 \), there exists \( \Lambda = \Lambda(M) > 0 \) such that \( M_\Lambda > M \), for all \( \lambda > \Lambda \). Since \( M_\lambda \) is monotone increasing, we can assume without loss of generality that \( M_\Lambda \leq M \). Otherwise, we take \( \Lambda \) to be the minimum for which this property holds. Also, \( \Lambda(M) \) is monotone increasing, and \( \Lambda(M) \to \infty \), as \( M \to \infty \). In particular, it holds that
\[
\inf \{ \lambda > 0 : M_\lambda > M \} \geq \Lambda(M) \to \infty \quad \text{as } M \to \infty.
\]
Then for any \( K > 0 \), there exists \( M > 0 \) such that
\[
\inf \{ \lambda > 0 : M_\lambda > M \} \geq K.
\]
Therefore, for any \( y \in \mathbb{R}^k \), with \( \omega_k |y|^k > M \), we have
\[
\tilde{u}_{s,k}(y, z) = \inf \{ \lambda > 0 : M_\lambda > \omega_k |y|^k \} \geq \inf \{ \lambda > 0 : M_\lambda > M \} \geq K.
\]
We conclude that
\[
\lim_{|y| \to \infty} \tilde{u}_{s,k}(y, z) = \infty.
\]
\[\square\]
Proof of Theorem 4.1. Since $u$ is convex at $x_0$, we have that $\tilde{u}(y, z) \geq 0$. Moreover,

$$\mathcal{F}^k u(x_0) = c_{n,s} \inf_{K \in \mathcal{K}^k} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \tilde{u}(y, z) K(y, z) \ dy \ dz.$$ 

Fix $z \in \mathbb{R}^{n-k}$ and consider the functions $f(y) = \tilde{u}(y, z)$ and $g(y) = K(y, z)$. Since

$$\mathcal{H}^k (\{ y \in \mathbb{R}^k : \tilde{u}(y, z) \leq \lambda \}) < \infty,$$

for any $\lambda > 0$, then by Lemma 4.3, we have

$$\lim_{t \to \infty} f_*(t) = \lim_{|y| \to \infty} f_*(y) = \infty,$$

with $f_*(x) = \tilde{u}_*(y, z)$ and $f_*(x) = f_*(\omega_k|y|^k)$. By Ryff’s theorem (Theorem 3.5), there exists a measure preserving $\sigma : \mathbb{R}^k \to [0, \infty)$, depending on $z$, such that

(4.2) \[ \tilde{u}(y, z) = f_*(\sigma_z(y)), \]

for all $y \in \text{supp} \tilde{u}(-, z) \subseteq \mathbb{R}^k$.

Let $K(y, z) = (\omega^{-1}_k \sigma_z(y))^{2/k} + |z|^2)^{\frac{n+2s}{2}}$. For any $r > |z|$, we have that

$$\mathcal{H}^k (\{ y \in \mathbb{R}^k : K(y, z) > r^{-n-2s} \}) = \mathcal{H}^k (\{ y \in \mathbb{R}^k : (\omega^{-1}_k \sigma_z(y))^{2/k} + |z|^2)^{\frac{n+2s}{2}} > r^{-n-2s} \})$$

$$= \mathcal{H}^k (\{ y \in \mathbb{R}^k : \sigma_z(y) < \omega_k(r^2 - |z|^2/k^2) \})$$

$$= \mathcal{H}^k (\sigma_z^{-1}(0, \omega_k(r^2 - |z|^2/k^2)))$$

$$= \mathcal{H}^k (0, \omega_k(r^2 - |z|^2/k^2))$$

$$= \omega_k(r^2 - |z|^2/k^2) = \mathcal{H}^k(B(r^2 - |z|^2/k^2)),$$

since $\sigma_z$ is measure preserving (see Definition 3.3). Then $K \in \mathcal{K}^k_*$, and thus,

$$\mathcal{F}^k u(x_0) \leq c_{n,s} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \tilde{u}(y, z) \ dydz.$$

To prove the reverse inequality, let $K \in \mathcal{K}^k_*$. Applying Proposition 3.2, we see that

$$\int_{\mathbb{R}^k} \tilde{u}(y, z) K(y, z) \ dy \geq \int_0^\infty f_*(t) g^*(t) \ dt$$

$$= \int_{\mathbb{R}^k} f_*(\sigma_z(y)) g^*(\sigma_z(y)) \ dy$$

$$= \int_{\mathbb{R}^k} \tilde{u}(y, z) g^*(\sigma_z(y)) \ dy,$$

by Lemma 3.4 and (4.2). Moreover, by the definition of rearrangements,

$$g^*(\sigma_z(y)) = \sup \{ \lambda > 0 : \mathcal{H}^k (\{ w \in \mathbb{R}^k : K(w, z) > \lambda \}) > \sigma_z(y) \} = K^*(\tilde{y}, z)$$

with $\omega_k|\tilde{y}|^k = \sigma_z(y)$. By (3.1), we get

$$g^*(\sigma_z(y)) = (|\tilde{y}|^2 + |z|^2)^{-\frac{n+2s}{2}} = (\omega^{-1}_k \sigma_z(y))^{2/k} + |z|^2)^{-\frac{n+2s}{2}}.$$ 

Hence, integrating over all $z \in \mathbb{R}^{n-k}$, and taking the infimum over all kernels $K \in \mathcal{K}^k_*$, we conclude that

$$\mathcal{F}^k u(x) = c_{n,s} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \tilde{u}(y, z) \ dydz.$$

\[ \square \]
Remark 4.4. A natural question that arises from this result is whether there exists a measure preserving \( \varphi_z : \mathbb{R}^k \rightarrow \mathbb{R}^k \) such that

\[
|\varphi_z(y)| = \left( \omega_k^{-1} \sigma_z(y) \right)^{1/k}.
\]

In that case, we would have that the infimum is attained at a kernel \( K \) such that

\[
K(y, z) = |\phi(y, z)|^{-n-2s},
\]

where \( \phi : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a measure preserving with \( \phi(y, z) = (\varphi_z(y), z) \).

Recall that Ryff’s theorem gives a representation of a function \( f \) in terms of its increasing rearrangement \( f_s \), that is, \( f = f_s \circ \sigma \), with \( \sigma : \mathbb{R}^k \rightarrow \mathbb{R} \) measure preserving. If this result were also true for the symmetric increasing rearrangement, given by \( f_{\#}(x) = f_s(\omega_k |x|^k) \), then there would exist a measure preserving \( \varphi : \mathbb{R}^k \rightarrow \mathbb{R}^k \) such that \( f = f_{\#} \circ \psi \). In particular,

\[
f(x) = f_{\#}(\varphi(x)) = f_s(\omega_k |\varphi(x)|^k) = f_s(\sigma(x)).
\]

Hence, it seems reasonable that \( \omega_k |\varphi(x)|^k = \sigma(x) \). As far as we know, this is an open problem.

As an immediate consequence of Theorem 4.1, we obtain the following representation of the function \( F_k^s u \) in terms of the \( k \)-symmetric increasing rearrangement of \( \tilde{u} \).

Corollary 4.5. Under the assumptions of Theorem 4.1, we have

\[
F_k^s u(x_0) = \Delta^s \tilde{u}_{s,k}(0).
\]

Proof. Note that \( \tilde{u}_{s,k}(0) = 0 \), since \( \tilde{u}(0) = 0 \). Therefore, using the same notation as in the proof of Theorem 4.1, we showed that

\[
F_k^s u(x_0) = c_{n,s} \int_{\mathbb{R}^{n-k}} \int_{0}^{\infty} f_s(t) g^s(t) \, dt \, dz
\]

\[
= \omega_k c_{n,s} \int_{\mathbb{R}^{n-k}} \int_{0}^{\infty} f_s(\omega_k r^k) g^s(\omega_k r^k) r^{k-1} \, dr \, dz
\]

\[
= c_{n,s} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} f_s(\omega_k |y|^k) g^s(\omega_k |y|^k) \, dy \, dz
\]

\[
= c_{n,s} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \tilde{u}_{s,k}(y, z) K^{s,k}(y, z) \, dy \, dz
\]

\[
= c_{n,s} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \frac{\tilde{u}_{s,k}(y, z)}{|y|^2 + |z|^2} \, dy \, dz = \Delta^s \tilde{u}_{s,k}(0).
\]

From the previous result and the fact that the family of operators \( \{F_k^s\}_{k=1}^{n-1} \) is monotone decreasing, we see that the fractional Laplacian of the \( k \)-symmetric rearrangements are ordered at the origin.

Corollary 4.6. Suppose we are under the assumption of Theorem 4.1. Then

\[
\Delta^s \tilde{u}_{s,k+1}(0) \leq \Delta^s \tilde{u}_{s,k}(0).
\]

Next we treat the second case.

Theorem 4.7. Suppose that there exists some \( \lambda_0 > 0 \) such that for all \( z \in \mathbb{R}^{n-k} \),

\[
\mathcal{H}^k \left( \{ y \in \mathbb{R}^k : \tilde{u}(y, z) \leq \lambda \} \right) \begin{cases} < \infty & \text{for } 0 < \lambda < \lambda_0 \\ = \infty & \text{for } \lambda \geq \lambda_0. \end{cases}
\]
Then there exists a kernel $K_0 \in K_k^\sigma$, with supp $K_0(\cdot, z) \subseteq \{ y \in \mathbb{R}^k : \hat{u}(y, z) \leq \lambda_0 \}$, such that

$$F_k^* u(x_0) = c_{n,s} \int_{\mathbb{R}^n-k} \int_{\mathbb{R}^k} \hat{u}(y, z) K_0(y, z) dy dz.$$ 

In particular, the infimum is attained.

Proof. Fix $z \in \mathbb{R}^{n-k}$. For $j \geq 1$, define the set

$$A_j(z) = \{ y \in \mathbb{R}^k : \hat{u}(y, z) \leq \lambda_0 - \frac{1}{j} \}.$$

For simplicity, we drop the notation of $z$. We have that $\mathcal{H}^k(A_j) < \infty$, $A_j \subseteq A_{j+1}$, and

$$A_\infty = \bigcup_{j=1}^{\infty} A_j = \{ y \in \mathbb{R}^k : \hat{u}(y, z) < \lambda_0 \}.$$

Observe that if $K \in K_k^\sigma$, then

$$\mathcal{H}^k(\{ y \in \mathbb{R}^k : K(y, z) > 0 \}) = \lim_{r \to 0} \mathcal{H}^k(\{ y \in \mathbb{R}^k : K(y, z) > r \}) = \infty.$$

Hence, we need to distinguish two cases:

Case 2.1. Assume that $\mathcal{H}^k(A_\infty) = \infty$. Let $K \in K_k^\sigma$ and $v_j = \tilde{u} \chi_{A_j}$. By Proposition 3.2,

$$\int_{A_j} \tilde{u}(y, z) K(y, z) dy = \int_{\mathbb{R}^k} v_j(y, z) K(y, z) dy \geq \int_0^\infty (v_j)_*(t) K^*(t) dt.$$

By Lemma 3.4, for any measure preserving $\sigma : \mathbb{R}^k \to [0, \infty)$, it follows that

$$\int_0^\infty (v_j)_*(t) K^*(t) dt = \int_{\mathbb{R}^k} (v_j)_*(\sigma(y)) K^*(\sigma(y)) dy.$$

By Ryff’s theorem (Theorem 3.5), there exists $\sigma_j : A_j \to [0, \mathcal{H}^k(A_j)]$ measure preserving such that $v_j = (v_j)_* \circ \sigma_j$ in $A_j$. Therefore,

$$\int_{A_j} \tilde{u}(y, z) K(y, z) dy \geq \int_{A_j} \tilde{u}(y, z) K^*(\sigma_j(y)) dy.$$

We claim that $\sigma_{j+1}(y) \leq \sigma_j(y)$, for all $y \in A_j$. Indeed, since $A_j \subseteq A_{j+1}$, we have

$$\begin{cases} v_j(y) = v_{j+1}(y), & \text{for all } y \in A_j \\ v_j(y) \leq v_{j+1}(y), & \text{for all } y \in A_{j+1} \setminus A_j. \end{cases}$$

In particular, for all $y \in A_j$,

$$(v_{j+1})_*(\sigma_{j+1}(y)) = (v_j)_*(\sigma_j(y)) \leq (v_{j+1})_*(\sigma_j(y)).$$

Since $(v_{j+1})_*$ is monotone increasing, we must have

$$\sigma_{j+1}(y) \leq \sigma_j(y), \quad \text{for all } y \in A_j.$$

Therefore, there exists $\sigma_{\infty} : A_\infty \to [0, \infty)$ measure preserving such that

$$\sigma_{\infty}(y) = \lim_{j \to \infty} \sigma_j(y).$$

Define the kernel $K_0$ as

$$K_0(y, z) = ((\omega_k^{-1} \sigma_{\infty}(y))^{k/2} + |z|^2)^{-\frac{n+2s}{2}} \chi_{A_{\infty}}(y).$$
Since $\mathcal{H}^k(A_\infty) = \infty$, then $K_0 \in \mathcal{K}_k^+$. Furthermore, note that $\text{supp} K_0(\cdot, z) = \overline{A}_\infty = \{ y \in \mathbb{R}^k : \tilde{u}(y, z) \leq \lambda_0 \}$ and $K_0(y, z) = K_0^*(\sigma_\infty(y))$, for all $y \in A_\infty$. Then by Fatou’s lemma, Lemma 3.7, and (4.3), we get

$$
\int_{\mathbb{R}^k} \tilde{u}(y, z) K_0(y, z) \, dy = \int_{A_\infty} \tilde{u}(y, z) K_0^*(\sigma_\infty(y)) \, dy
$$

$$
\leq \liminf_{j \to \infty} \int_{A_j} \tilde{u}(y, z) K_0^*(\sigma_j(y)) \, dy
$$

$$
= \liminf_{j \to \infty} \int_{A_j} \tilde{u}(y, z) K^*(\sigma_j(y)) \, dy
$$

$$
\leq \int_{\mathbb{R}^k} \tilde{u}(y, z) K(y, z) \, dy,
$$

for any $K \in \mathcal{K}_k^+$. Integrating over $z$ and taking the infimum over all kernels $K$, we conclude the result.

**Case 2.2.** Assume that $\mathcal{H}^k(A_\infty) < \infty$. Set $A = \{ y \in \mathbb{R}^k : \tilde{u}(y, z) = \lambda_0 \}$. Then

$$
\mathcal{H}^k(A) = \infty,
$$

since $\{ y \in \mathbb{R}^k : \tilde{u}(y, z) \leq \lambda_0 \} = A_\infty \cup A$. Fix $\varepsilon > 0$ and define

$$
v_\varepsilon(y, z) = \tilde{u}(y, z) \chi_{A_\infty}(y) + \max\{ \lambda_0, (\lambda_0 + \varepsilon) \phi(y, z) \} \chi_A(y),
$$

with $\phi(y, z) = 1 - e^{-|y|^2 - |z|^2}$. Note that $0 < \phi \leq 1$, $\phi(y, z) \to 1$, as $|(y, z)| \to \infty$, and $\phi(y, z) \approx |y|^2 + |z|^2$, as $|(y, z)| \to 0$. Also, $\{ v_\varepsilon \} \varepsilon > 0$ is a monotone increasing sequence, and

$$
\lim_{\varepsilon \to 0} v_\varepsilon(y, z) = \tilde{u}(y, z) \chi_{A_\infty}(y) + \max\{ \lambda_0, \lim_{\varepsilon \to 0} (\lambda_0 + \varepsilon) \phi(y, z) \} \chi_A(y)
$$

$$
= \tilde{u}(y, z) \chi_{A_\infty}(y) + \max\{ \lambda_0, \lambda_0 \phi(y, z) \} \chi_A(y) = \tilde{u}(y, z) \chi_{A_\infty \cup A}(y).
$$

For any $j \in \mathbb{N}$, with $j > 1/\varepsilon$, consider the set

$$
B_j^\varepsilon(z) = \{ y \in \mathbb{R}^k : v_\varepsilon(y, z) \leq \lambda_0 + \frac{\varepsilon - 1}{j} \}.
$$

Then $B_j^\varepsilon \subseteq B_{j+1}^\varepsilon$ and $B_\infty = \bigcup_{j > 1/\varepsilon} B_j^\varepsilon = \{ y \in \mathbb{R}^k : v_\varepsilon(y, z) < \lambda_0 + \varepsilon \}$. Moreover, we have

$$
\mathcal{H}^k(B_j^\varepsilon) \leq \mathcal{H}^k(A_\infty) + \mathcal{H}^k\{ \{ y \in A : \max\{ \lambda_0, (\lambda_0 + \varepsilon) \phi(y, z) \} \leq \lambda_0 + \varepsilon - \frac{1}{j} \} \}.
$$

Choose $R > 0$ large enough (depending on $\varepsilon$, $j$, $\lambda_0$, and $z$) so that

$$
(\lambda_0 + \varepsilon)e^{-R^2 - |z|^2} < \frac{1}{j}.
$$

Then $(\lambda_0 + \varepsilon) \phi(y, z) > \lambda_0 + \varepsilon - \frac{1}{j} > \lambda_0$, for all $y \in B_R^\varepsilon$, and thus,

$$
\mathcal{H}^k\{ \{ y \in A \cap B_R^\varepsilon : \max\{ \lambda_0, (\lambda_0 + \varepsilon) \phi(y, z) \} \leq \lambda_0 + \varepsilon - \frac{1}{j} \} \} = 0.
$$

By (4.6) and (4.7), we see that

$$
\mathcal{H}^k(B_j^\varepsilon(z)) \leq \mathcal{H}^k(A_\infty) + \mathcal{H}^k(A \cap B_R) < \infty.
$$

Furthermore, $A \subseteq B_\infty$, and thus, by (4.4), we get

$$
\mathcal{H}^k(B_\infty^\varepsilon) \geq \mathcal{H}^k(A) = \infty.
$$

In particular, $v_\varepsilon$ satisfies the assumptions of Case 2.1, so there exists $K_\varepsilon \in \mathcal{K}_k^+$,

$$
K_\varepsilon(y, z) = (\omega_k^{-1} \sigma_\varepsilon(y))^{k/2} + |z|^2)^{-\frac{n+2}{2}} \chi_{B_\infty(y)}(z),
$$
with $\sigma_\varepsilon : B_\infty^c \to [0, \infty)$ measure preserving, depending on $v_\varepsilon$, such that

$$\inf_{K \in K_k} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} v_\varepsilon(y, z) K(y, z) \, dy \, dz = \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} v_\varepsilon(y, z) K_\varepsilon(y, z) \, dy \, dz. \tag{4.9}$$

Finally, we need to pass to the limit. First, we prove that $\{\sigma_\varepsilon\}_{\varepsilon > 0}$ is monotone decreasing. Indeed, let $V_\varepsilon = \{ y \in \mathbb{R}^k : v_\varepsilon(y, z) = \tilde{u}(y, z) \}$. In particular, $A_\infty \subseteq V_\varepsilon \subseteq A_\infty \cup A$. Also, $V_{\varepsilon_2} \subseteq V_{\varepsilon_1}$, for any $\varepsilon_1 \leq \varepsilon_2$. By Ryff’s theorem, recall that

$$v_{\varepsilon_1}(y, z) = (v_{\varepsilon_1})_*(\sigma_{1}(y)) \quad \text{and} \quad v_{\varepsilon_2}(y, z) = (v_{\varepsilon_2})_*(\sigma_{\varepsilon_2}(y)).$$

Since $v_{\varepsilon_2}(y, z) = v_{\varepsilon_1}(y, z)$, for all $y \in V_{\varepsilon_2}$, and $v_{\varepsilon_1}(y, z) \leq v_{\varepsilon_2}(y, z)$, for all $y \in \mathbb{R}^k$, we see that

$$(v_{\varepsilon_2})_*(\sigma_{\varepsilon_2}(y)) = (v_{\varepsilon_1})_*(\sigma_{\varepsilon_1}(y)) \leq (v_{\varepsilon_2})_*(\sigma_{\varepsilon_1}(y)) \quad \text{for all } y \in V_{\varepsilon_2}.$$

Since $(v_{\varepsilon_2})_*$ is monotone increasing, we must have that $\sigma_{\varepsilon_2}(y) \leq \sigma_{\varepsilon_1}(y)$, for all $y \in V_{\varepsilon_2}$. Hence, there exists $\sigma_0 : B_\infty \to [0, \infty)$ measure preserving such that

$$\sigma_0(y) = \lim_{\varepsilon \to 0} \sigma_\varepsilon(y),$$

where $B_\infty = \bigcap_{\varepsilon > 0} B_\infty^c = \{ y \in \mathbb{R}^k : \tilde{u}(y, z) \leq \lambda_0 \} = A_\infty \cup A$. In particular, the sequence of kernels $\{K_\varepsilon\}_{\varepsilon > 0}$ is monotone decreasing. Define

$$K_0(y, z) = \lim_{\varepsilon \to 0} K_\varepsilon(y, z). \tag{4.10}$$

By (4.8) and (4.10), we have

$$K_0(y, z) = \left((\omega_{\varepsilon}^{-1} \sigma_0(y))^{k/2} + |z|^2\right)^{-\frac{n+2s}{2}} \chi_{B_\infty}(y).$$

Moreover, $K_0 \in K_k^s$ since $K_\varepsilon \in K_k^s$, and for any $r > 0$, it follows that

$$\mathcal{H}^k(D_0(r)) = \lim_{\varepsilon \to 0} \mathcal{H}^k(D_\varepsilon(r)), $$

where $D_\varepsilon(r) = \{ y \in \mathbb{R}^k : K_\varepsilon(y, z) > r^{-(n+2s)} \}$. Finally, using (4.5), (4.9), (4.10), and the monotone convergence theorem, we get

$$\int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \tilde{u}(y, z) K_0(y, z) \, dy \, dz = \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \left( v_\varepsilon(y, z) K_\varepsilon(y, z) \right) dy \, dz \leq \lim_{\varepsilon \to 0} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} v_\varepsilon(y, z) K_\varepsilon(y, z) \, dy \, dz \leq \inf_{K \in K_k^s} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \tilde{u}(y, z) \left( K_\varepsilon(y, z) \chi_{A_\infty \cup A}(y) \right) dy \, dz \leq \inf_{K \in K_k^s} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \tilde{u}(y, z) K(y, z) \chi_{A_\infty \cup A}(y) \, dy \, dz.$$
Finally, we deal with the third case, that is, when all of the level sets of \( \tilde{u} \) have infinite measure. In particular, notice that

\[
\tilde{u}_{\alpha_k}(x) = 0, \quad \text{for all } x \in \mathbb{R}^n.
\]

This is the only case where the infimum is not attained. Indeed, we prove in the following theorem that the infimum is equal to zero.

**Theorem 4.8.** Suppose that for all \( \lambda > 0 \) and \( z \in \mathbb{R}^{n-k} \),

\[
\mathcal{H}^k(\{ y \in \mathbb{R}^k : \tilde{u}(y, z) \leq \lambda \}) = \infty.
\]

Then \( \mathcal{F}_k^u(x_0) = 0 \).

**Proof.** From \((P_2)\), we have that \( \mathcal{F}_k^u(x_0) \geq 0 \). To prove the reverse inequality, it is enough to find a sequence of kernels \( \{K_\varepsilon\}_{\varepsilon > 0} \subset \mathcal{K}_k^\varepsilon \) such that

\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \tilde{u}(y, z) K_\varepsilon(y, z) \, dydz = 0.
\]

Fix \( \varepsilon > 0 \) and \( z \in \mathbb{R}^{n-k} \). For any \( j \geq 0 \), we define the set

\[
U_j \equiv U_j(z) = \{ y \in \mathbb{R}^k : \tilde{u}(y, z) < \varepsilon 2^{-j(n+2s)} e^{-|z|^2} \}.
\]

Note that \( U_{j+1} \subseteq U_j \). Also, by assumption, with \( \lambda = \varepsilon 2^{-j(1+2s)} e^{-|z|^2} \), we have that

\[
\mathcal{H}^k(U_j) = \infty, \quad \text{for all } j \geq 0.
\]

We will construct \( K_\varepsilon \in \mathcal{K}_k^\varepsilon \) by describing first where to locate each level set of the form:

\[
A_{-1} \equiv A_{-1}(z) = \{ y \in \mathbb{R}^k : 0 < K_\varepsilon(y, z) \leq 1 \}
\]

\[
A_j \equiv A_j(z) = \{ y \in \mathbb{R}^k : 2^j(n+2s) < K_\varepsilon(y, z) \leq 2^{(j+1)(n+2s)} \}, \quad \text{for } j \geq 0.
\]

Recall that \( K \in \mathcal{K}_k^\varepsilon \) if for all \( r > 0 \), we have

\[
\mathcal{H}^k(\{ y \in \mathbb{R}^k : K(y, z) > r^{-n+2s} \}) = \mathcal{H}^k(\{ y \in \mathbb{R}^k : (|y|^2 + |z|^2)^{-\frac{n+2s}{2}} > r^{-n+2s} \}).
\]

In view of this definition, we define the sets

\[
B_{-1} \equiv B_{-1}(z) = \{ y \in \mathbb{R}^k : 0 < (|y|^2 + |z|^2)^{-\frac{n+2s}{2}} \leq 1 \}
\]

\[
B_j \equiv B_j(z) = \{ y \in \mathbb{R}^k : 2^{j(n+2s)} < (|y|^2 + |z|^2)^{-\frac{n+2s}{2}} \leq 2^{(j+1)(n+2s)} \}, \quad \text{for } j \geq 0.
\]

Note that

\[
\begin{cases}
\mathcal{H}^k(A_{-1}) = \mathcal{H}^k(B_{-1}) = \infty \\
\mathcal{H}^k(A_j) = \mathcal{H}^k(B_j) < \infty, \quad \text{for all } j \geq 0.
\end{cases}
\]

More precisely, for \( j \geq 0 \), if \( |z| < 2^{-j+1} < 2^{-j} \), then

\[
\mathcal{H}^k(A_j) = \mathcal{H}^k(B(2^{-j-1} - |z|^2)^{1/2}) - \mathcal{H}^k(B(2^{-j+1} - |z|^2)^{1/2})
\]

\[
= \omega_k(2^{-2j} - |z|^2)^{k/2} - \omega_k(2^{-2(j+1)} - |z|^2)^{k/2} \leq \omega_k 2^{-kj}.
\]

If \( 2^{-j+1} \leq |z| < 2^{-j} \), then

\[
\mathcal{H}^k(A_j) = \mathcal{H}^k(B(2^{-j-1} - |z|^2)^{1/2}) = \omega_k(2^{-2j} - |z|^2)^{k/2} \leq \omega_k (\frac{3}{4})^{k/2} 2^{-kj}.
\]

If \( |z| \geq 2^{-j} > 2^{-j+1} \), then

\[
\mathcal{H}^k(A_j) = 0.
\]
Therefore, $\mathcal{H}^k(A_j) \leq c2^{-kj}$, where $c > 0$ only depends on $k$. It follows that

$$
(4.12) \quad \mathcal{H}^k \left( \bigcup_{j=0}^{\infty} A_j \right) = \sum_{j=0}^{\infty} \mathcal{H}^k(A_j) \leq c \sum_{j=0}^{\infty} 2^{-jk} < \infty.
$$

For any $i \geq 0$, let $\mathcal{D}_i$ be the collection of all dyadic closed cubes of the form

$$
[m2^{-i}, (m+1)2^{-i}]^k = [m2^{-i}, (m+1)2^{-i}] \times \cdots \times [m2^{-i}, (m+1)2^{-i}].
$$

Note that if $Q \in \mathcal{D}_i$, then $l(Q) = 2^{-i}$, where $l(Q)$ denotes the side length of the cube $Q$. For any $j \geq 0$, since $U_j$ is an open set, by a standard covering argument, we have that there exists a family of dyadic cubes $F_j$ such that

$$
U_j = \bigcup_{Q \in F_j} Q
$$
satisfying the following properties:

1. For any $Q \in F_j$, there exists some $i \geq 0$ such that $Q \in \mathcal{D}_i$.
2. $\text{Int}(Q) \cap \text{Int}(\tilde{Q}) = \emptyset$, for any $Q, \tilde{Q} \in F_j$, with $Q \neq \tilde{Q}$.
3. If $x \in Q \in F_j$, then $Q$ is the maximal dyadic cube contained in $U_j$ that contains $x$.

Analogously, for the sets $B_j$ with $j \geq -1$, there exists a family of dyadic cubes $\tilde{F}_j$ satisfying properties (1) – (3) such that

$$
\text{Int}(B_j) = \bigcup_{Q \in \tilde{F}_j} Q.
$$

Note that $\tilde{F}_j \cap \tilde{F}_{j+1} = \emptyset$ since $B_j \cap B_{j+1} = \emptyset$.

We will construct the sets $A_j$ by properly translating the dyadic cubes partitioning the sets $B_j$ into $U_j$. In particular, we will prove that

$$
\begin{cases}
A_0 = T_0(B_0) \subset U_0 \\
A_j = T_j(B_j) \subset U_j \setminus \bigcup_{i=0}^{j-1} A_i, \quad \text{for all } j \geq 1 \\
A_{-1} = T_{-1}(\tilde{B}_{-1}) \subset U_0 \setminus \bigcup_{i=0}^\infty A_i,
\end{cases}
$$

for some translation mappings $T_j : \tilde{F}_j \rightarrow F_j$ to be determined.

We start with the case $j = 0$. For any $i \geq 0$, denote by

$$
m_i = \mathcal{H}^0(F_0 \cap \mathcal{D}_i) \quad \text{and} \quad n_i = \mathcal{H}^0(\tilde{F}_0 \cap \mathcal{D}_i),
$$

where $\mathcal{H}^0(E)$ is equal to the cardinal of the set $E$. Note that $m_i, n_i \in \mathbb{Z}^+ \cup \{\infty\}$.

We will recursively place $B_0$ into $U_0$. First, fix $i = 0$. If $m_0 \geq n_0$, then for any $\tilde{Q} \in \tilde{F}_0 \cap \mathcal{D}_0$, there exists some $\tau \in \mathbb{R}^k$ and some $Q \in F_0 \cap \mathcal{D}_0$, such that $Q = \tilde{Q} + \tau$. Then define

$$
(4.13) \quad T_0: \quad \tilde{F}_0 \cap \mathcal{D}_0 \rightarrow F_0 \cap \mathcal{D}_0 \\
\tilde{Q} \mapsto Q.
$$

Moreover, we can define $T_0$ one-to-one since $m_0 \geq n_0$, and we can always choose a different $Q$ for each $\tilde{Q}$. Note that there are $p_0$ cubes in $F_0 \cap \mathcal{D}_0$, with $p_0 = m_0 - n_0$, that have not been used. Hence, to all of these cubes, divide each side in half, so that each cube gives rise to $2^k$ cubes with side length $2^{-1}$. Call this collection of new cubes $Q = \{Q_i\}_{i=1}^{2^k p_0} \subset \mathcal{D}_1$, and add them to the family $F_0 \cap \mathcal{D}_1$. Namely, we replace $F_0 \cap \mathcal{D}_1$ by $(F_0 \cap \mathcal{D}_1) \cup Q$.

If $m_0 < n_0$, then take $q_0$ cubes in $\tilde{F}_0 \cap \mathcal{D}_0$, with $q_0 = n_0 - m_0$, and divide each side in half. Call this collection of new cubes $\tilde{Q} = \{\tilde{Q}_i\}_{i=1}^{2^k q_0} \subset \mathcal{D}_1$. Then, we replace $\tilde{F}_0$ by $\tilde{F}_0$, where

$$
\tilde{F}_0 \cap \mathcal{D}_0 = (\tilde{F}_0 \setminus \tilde{Q}) \cap \mathcal{D}_0
$$
\[ \mathcal{F}_0 \cap \mathcal{D}_1 = (\mathcal{F}_0 \cup \mathcal{Q}) \cap \mathcal{D}_1 \]

\[ \mathcal{F}_0 \cap \mathcal{D}_1 = \mathcal{F}_0 \cap \mathcal{D}_i, \quad \text{for all } i \geq 2. \]

If \( \mathcal{H}_0 = \mathcal{H}^0(\mathcal{F}_0 \cap \mathcal{D}_0) \), then \( m_0 = \mathcal{H}_0 \). Hence, by the same argument as in the previous case, we find \( T_0 \) as in (4.13). For \( i \geq 1 \), we can repeat the same process until we run out of cubes from \( \mathcal{F}_0 \) (or the modified family). We know the process will end since \( \mathcal{H}^k(B_0) < \mathcal{H}^k(U_0) \).

When this happens, we will have constructed a one-to-one mapping \( T_0 : \mathcal{F}_0 \rightarrow \mathcal{F}_0 \), since \( \mathcal{F}_0 = \bigcup_{i=0}^{\infty} \mathcal{F}_0 \cap \mathcal{D}_i \) and \( \mathcal{F}_0 = \bigcup_{i=0}^{\infty} \mathcal{F}_0 \cap \mathcal{D}_i \). Then define

\[ A_0 = T_0(B_0) \subset U_0. \]

Iterating this process, we find a sequence of translation mappings \( \{T_j\}_{j=0}^{\infty} \), with \( T_j : \mathcal{F}_j \rightarrow \mathcal{F}_j \), and a sequence of disjoint sets \( \{A_j\}_{j=0}^{\infty} \) such that

\[ A_j = T_j(B_j) \subset U_j \setminus \bigcup_{i=0}^{j-1} A_i. \]

The case \( j = -1 \) is somewhat special since \( \mathcal{H}^k(A_{-1}) = \mathcal{H}^k(B_{-1}) = \infty \). We will see that

\[ A_{-1} = T_{-1}(B_{-1}) \subset U_0 \setminus \bigcup_{i=0}^{\infty} A_i. \]

This is possible because \( \mathcal{H}^k(U_0 \setminus \bigcup_{i=0}^{\infty} A_i) = \infty \) using (4.12). Indeed, we can write

\[ \{ y \in \mathbb{R}^k : 0 < K_\varepsilon(y, z) \leq 1 \} = \bigcup_{j=0}^{\infty} \{ 2^{-j(n+2s)} < K_\varepsilon(y, z) \leq 2^{-j(n+2s)} \}. \]

Now call

\[ C_j = \{ 2^{-j(n+2s)} < (|y|^2 + |z|^2)^{-\frac{n+2s}{2}} \leq 2^{-j(n+2s)} \}, \quad \text{for } j \geq 0. \]

Then \( B_{-1} = \bigcup_{j=0}^{\infty} C_j \), with \( \mathcal{H}^k(C_j) < \infty \), for all \( j \geq 0 \). Hence, instead of partitioning all of \( B_{-1} \) into dyadic cubes, we partition each of its disjoint components \( C_j \). Arguing as before, we place them into \( U_0 \setminus \bigcup_{i=0}^{\infty} A_i \) recursively, according to the following scheme:

\[
\begin{align*}
T^0_1(C_0) &\subset U_0 \setminus \bigcup_{i=0}^{\infty} A_i \\
T^j_{-1}(C_j) &\subset U_0 \setminus \left( \bigcup_{i=0}^{\infty} A_i \cup \bigcup_{i=0}^{j-1} C_i \right), \quad \text{for } j \geq 1,
\end{align*}
\]

where \( T^j_{-1} \) is defined as before. At the end of this process, we find a translation map \( T_{-1} \) with \( T_{-1}(Q) = T^j_{-1}(Q) \), for \( Q \in C_j \). Therefore, we define

\[ A_{-1} = T_{-1}(B_{-1}). \]

Lastly, let \( y \in \mathbb{R}^k = A_{-1} \cup \bigcup_{j=0}^{\infty} A_j \). In particular, there exists some \( j \geq -1 \) such that \( y \in A_j \). Furthermore, recall that \( A_j = T_j(B_j) \), where \( T_j \) is a one-to-one and onto translation map. Hence, there exists a unique \( w \in B_j \) such that \( y = T_j(w) = w + \tau \), for some \( \tau \in \mathbb{R}^k \).

Let \( T_z : \mathbb{R}^k \rightarrow \mathbb{R}^k \) be given by \( T_z(y) = w \). Note that \( T_z \) is measure preserving. Then we define the kernel \( K_\varepsilon \) as

\[ K_\varepsilon(y, z) = (|T_z(y)|^2 + |z|^2)^{-\frac{n+2s}{2}}. \]

We have that

\[ \int_{\mathbb{R}^k} \tilde{u}(y, z)K_\varepsilon(y, z) \, dy = \int_{A_{-1}} \tilde{u}(y, z)K_\varepsilon(y, z) \, dy + \sum_{j=0}^{\infty} \int_{A_j} \tilde{u}(y, z)K_\varepsilon(y, z) \, dy \equiv I + II. \]
For I, we use that \( u(y, z) \leq e^{-|z|^2} \), since \( A_{-1} \subset U_0 \). Then by Lemma 3.7 and Lemma 3.4:

\[
I \leq e^{-|z|^2} \int_{|y| \leq 1} K_\varepsilon(y, z) \, dy \\
= e^{-|z|^2} \int_{|\sigma_\varepsilon(y)| \leq 1} |\sigma_\varepsilon(y)|^{-n-2s} \, dy \\
= e^{-|z|^2} \int_{|y| \leq 1} |y|^{-n-2s} \, dy = C e^{-|z|^2},
\]

where \( C > 0 \) depends only on \( n \) and \( s \). For II, we use that \( \tilde{u}(y, z) \leq e^{-j(n+2s)|z|^2} \), since \( A_j \subset U_j \) and \( K_\varepsilon(y, z) \leq 2^{(j+1)(n+2s)} \) in \( A_j \), by definition. Then

\[
II \leq e^{-|z|^2} \sum_{j=0}^\infty 2^{-j(n+2s)} 2^{(j+1)(n+2s)} \mathcal{H}^{k}(A_j) \\
\leq c e^{-|z|^2} \sum_{j=0}^\infty 2^{-kj} \leq C e^{-|z|^2},
\]

where \( C > 0 \) depends only on \( n, s, \) and \( k \).

Integrating over \( z \), we see that

\[
\int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} \tilde{u}(y, z) K_\varepsilon(y, z) \, dy \, dz \leq C \varepsilon \int_{\mathbb{R}^{n-k}} e^{-|z|^2} \, dz \leq C \varepsilon.
\]

Letting \( \varepsilon \to 0 \), we conclude (4.11). \( \square \)

4.2. Limit as \( s \to 1 \). Let \( u \in C^2(\mathbb{R}^n) \). We define \( \text{MA}_k u \) as the Monge-Ampère operator acting on \( u \), with respect to the first \( k \) variables, that is,

\[
\text{MA}_k u(x) = k \left( \det \left( (u_{ij}(x))_{1 \leq i,j \leq k} \right) \right)^{1/k},
\]

with \( D^2 u(x) = (u_{ij}(x))_{1 \leq i,j \leq n} \). We define \( \Delta_{n-k} u \) as the Laplacian of \( u \), with respect to the last \( n-k \) variables, that is,

\[
\Delta_{n-k} u(x) = \sum_{i=k+1}^n u_{ii}(x).
\]

Then under some special conditions, it holds that

\[
(4.14) \quad \lim_{s \to 1} \mathcal{F}_k^s u(x) = \text{MA}_k u(x) + \Delta_{n-k} u(x).
\]

In particular, the family \( \{\mathcal{F}_k^s\}_{k=1}^{n-1} \) can be understood as nonlocal analogs of concave second order elliptic operators, which are decomposed into a Monge-Ampère operator restricted to \( \mathbb{R}^k \) and a Laplacian restricted to \( \mathbb{R}^{n-k} \).

Indeed, by Corollary 4.5, we have \( \mathcal{F}_k^s u(x) = \Delta^s \tilde{u}_{s,k}(0) \). Since the \( k \)-symmetric rearrangement does not depend on \( s \) and \( \Delta \to \Delta \), as \( s \to 1 \), passing to the limit we see that

\[
\lim_{s \to 1} \mathcal{F}_k^s u(x) = \Delta \tilde{u}_{s,k}(0).
\]

Suppose that \( \tilde{u}_{s,k}(y, z) = \tilde{u}(\varphi_s^{-1}(y), z) \), where \( \varphi_z : \mathbb{R}^k \to \mathbb{R}^k \) is an invertible measure preserving transformation, with \( \varphi_z(0) = 0 \), and

\[
\omega_k |\varphi_z(y)|^{1/k} = \sigma_z(y).
\]
Moreover, \( \nabla (4.20) \psi \)

Then there exists a convex function

Theorem 4.10. Let \( f,g \in L^1(\mathbb{R}^k) \). Assume that

\[
\|f\|_{L^1(\mathbb{R}^k)} = \|g\|_{L^1(\mathbb{R}^k)}.
\]

Then there exists a convex function \( \psi : \mathbb{R}^k \to \mathbb{R} \) whose gradient \( \nabla \psi \) pushes forward \( f \) dy to \( g \) dy. Namely, for any measurable function \( h \) in \( \mathbb{R}^k \),

\[
\int_{\mathbb{R}^k} h(y)g(y) \, dy = \int_{\mathbb{R}^k} h(\nabla \psi(y))f(y) \, dy.
\]

Moreover, \( \nabla \psi : \mathbb{R}^k \to \mathbb{R}^k \) is invertible and unique.

In the literature, \( \nabla \psi \) is known as the (optimal) transport map.
Proof of Theorem 4.9. Fix $z \in \mathbb{R}^{n-k}$, $z \neq 0$, and consider $f_z, g_z \in L^1(\mathbb{R}^k)$ given by

$$f_z(y) = (|y|^2 + |z|^2)^{-\frac{n+k}{2}} \quad \text{and} \quad g_z(y) = (\omega_k^{-1} \sigma_z(y))^{2/k} + |z|^2)^{-\frac{n+2k}{2}},$$

where $\sigma_z : \mathbb{R}^k \to [0, \infty)$ is given in Theorem 4.1. Note that

$$\|f\|_{L^1(\mathbb{R}^k)} = \int_{\mathbb{R}^k} (\omega_k^{-1} \sigma_z(y))^{2/k} + |z|^2)^{-\frac{n+2k}{2}} dy = k \omega_k \int_0^\infty (r^2 + |z|^2)^{-\frac{n+2k}{2}} r^{k-1} dr = \int_{\mathbb{R}^k} (|y|^2 + |z|^2)^{-\frac{n+2k}{2}} dy = \|g\|_{L^1(\mathbb{R}^k)},$$

since $\sigma_z$ is measure preserving. By Theorem 4.10, there exists a convex function $\psi_z : \mathbb{R}^k \to \mathbb{R}$ (depending on $z$) whose gradient $\nabla \psi_z$ pushes forward $f_z \, dy$ to $g_z \, dy$. Moreover, $\nabla \psi_z$ is invertible and unique. Call $\varphi_z = (\nabla \psi_z)^{-1}$. Using (4.20), with $h(y) = \tilde{u}(y, z)$, we see that

$$\int_{\mathbb{R}^k} \tilde{u}(y, z) \left( (\omega_k^{-1} \sigma_z(y))^{2/k} + |z|^2)^{-\frac{n+2k}{2}} dy = \int_{\mathbb{R}^k} \tilde{u}(\varphi_z^{-1}(y), z) \left( |y|^2 + |z|^2)^{-\frac{n+2k}{2}} dy. \right.$$}

Integrating over $z \in \mathbb{R}^{n-k}$, we obtain (4.18).

It remains to show that $\varphi_z$ is measure preserving if and only if (4.19) holds. Indeed, for any measurable set $E \subset \mathbb{R}^k$, we have

$$\mathcal{H}^k(\varphi_z^{-1}(E)) = \int_{\varphi_z^{-1}(E)} dy = \int_{\varphi_z^{-1}(E)} \left( |y|^2 + |z|^2)^{-\frac{n+2k}{2}} dy = \int_{\varphi_z^{-1}(E)} \left( |\varphi_z(y)|^2 + |z|^2)^{-\frac{n+2k}{2}} dy = \int_E \left( (\omega_k^{-1} \sigma_z(y))^{2/k} + |z|^2)^{-\frac{n+2k}{2}} dy,$$

where the last equality follows from (4.21) with $h(y) = (|\varphi_z(y)|^2 + |z|^2)^{-\frac{n+2k}{2}} \chi_E(y)$. Therefore,

$$\mathcal{H}^k(\varphi_z^{-1}(E)) = \mathcal{H}^k(E)$$

if and only if $\omega_k|\varphi_z(y)|^k = \sigma_z(y)$, for a.e. $y \in \mathbb{R}^k$. 

\[ \square \]

5. Regularity of $\mathcal{F}_k^*u$

Given $x_0 \in \mathbb{R}^n$, we define the sections

$$D_{x_0} u(t) = \{ x \in \mathbb{R}^n : u(x) - u(x_0) - (x - x_0) \cdot \nabla u(x_0) \leq t \}, \quad \text{for } t > 0.$$

Our main regularity result is the following.

**Theorem 5.1.** Let $s \in (1/2, 1)$ and $1 \leq k < n$. Let $u \in C^{1,1}(\mathbb{R}^n)$ be convex. Fix $x_0 \in \mathbb{R}^n$ and $r_0, \varepsilon > 0$. Suppose that $\Lambda = \sup_{x \in B_{r_0}(x_0)} \text{diam}(D_x u(\varepsilon)) < \infty$ and $M = \sup_{x \in B_{r_0}(x_0)} \mathcal{F}_k^* u(x) < \infty$. Then $\mathcal{F}_k^* u \in C^{0,1-s}(\overline{B_r(x_0)})$ with $r < \min \{ r_0/4, \Lambda, \varepsilon/(8 \Lambda) \}$, and

$$[\mathcal{F}_k^* u]_{C^{0,1-s}(\overline{B_r(x_0)})} \leq C_0 [u]_{C^{1,1}(\mathbb{R}^n)}$$

for some constant $C_0 > 0$ depending only on $n, k, s, \varepsilon, \Lambda,$ and $M$.
This theorem will be a consequence of the next proposition.

**Proposition 5.2.** Fix $x_0 \in \mathbb{R}^n$ and $\varepsilon > 0$. Suppose that $\Lambda = \text{diam}(D_{x_0}u(\varepsilon)) < \infty$ and $[u]_{C^{1,1}(\mathbb{R}^n)} \leq 1$. Then for any $x_1 \in B_r(x_0)$, with $r \leq \varepsilon/(4\Lambda)$, it holds that

$$F_k^u(x_1) - F_k^u(x_0) \leq C\Lambda^{1-s}|x_1 - x_0|^{1-s} + \frac{4A}{\varepsilon}|x_1 - x_0|F_k^u(x_0)$$

for some $C > 0$ depending only on $n$, $k$, and $s$.

First, we prove Theorem 5.1.

**Proof of Theorem 5.1.** Without loss of generality, we may assume that $[u]_{C^{1,1}(\mathbb{R}^n)} \leq 1$. Otherwise, we consider $u/|u|_{C^{1,1}(\mathbb{R}^n)}$. Let $r < \min\{r_0/4, \Lambda, \varepsilon/(8\Lambda)\}$. It is enough to show that

$$[F_k^u]_{C^{0,1-s}(B_r(x_0))} \leq C_0,$$

for some constant $C_0 > 0$ depending only on $n$, $k$, $s$, $\varepsilon$, $\Lambda$, and $M$.

Let $x_1, x_2 \in B_r(x_0)$. Then $x_2 \in B_{2r}(x_1) \subset B_{r_0}(x_0)$, since $4r < r_0$. Moreover, $\text{diam}(D_{x_1}u(\varepsilon)) \leq \Lambda < \infty$. Hence, applying Proposition 5.2 to $u$, replacing $B_r(x_0)$ by $B_{2r}(x_1)$, we get

$$F_k^u(x_2) - F_k^u(x_1) \leq C\Lambda^{1-s}|x_2 - x_1|^{1-s} + \frac{4A}{\varepsilon}|x_2 - x_1|F_k^u(x_1) \leq C_0|x_2 - x_1|^{1-s},$$

where $C_0 = C\Lambda^{1-s} + 4A^{1-s}M/\varepsilon^{2s}$. Since $x_1$ and $x_2$ are arbitrary, we conclude (5.1). □

Before we prove Proposition 5.2, we need several preliminary results.

**Lemma 5.3.** If $f$ is monotone increasing, then

$$\int_0^\infty f(r)\omega(r) \, dr = \int_0^\infty \int_{\mu_f(t)}^\infty \omega(r) \, dr \, dt,$$

with $\mu_f(t) = |\{r > 0 : f(r) \leq t\}|$.

**Proof.** By Fubini’s theorem, we have

$$\int_0^\infty \int_{\mu_f(t)}^\infty \omega(r) \, dr \, dt = \int_0^\infty \omega(r) \int_{\{r > \mu_f(t)\}} dt \, dr.$$

Since $f$ is monotone increasing, then $r > \mu_f(t)$ if and only if $t < f(r)$. Therefore,

$$\int_{\{r > \mu_f(t)\}} dt = \int_0^{f(r)} dt = f(r).$$

□

**Proposition 5.4.** Let $x \in \mathbb{R}^n$. Under the assumptions of Corollary 4.5 it holds that

$$F_k^u(x) = c_{n,s} \int_0^\infty \int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\left(\frac{\mu_x u(t, z)^{1/k}}{|z|}\right) \, d z \, d t,$$

where $\mu_x u(t, z) = \omega_k^{-1} \mathcal{H}^k(\{y \in \mathbb{R}^k : \tilde{u}_x(y, z) \leq t\})$ and

$$W(\rho) = k\omega_k \int_0^\rho \frac{r^{k-1}}{(1 + r^2)^{n+2s/2}} \, dr.$$

**Proof.** By Corollary 4.5, we have that

$$F_k^u(x) = \Delta^s \tilde{u}_{s,k}(0) = c_{n,s} \int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n+2s}} \left( \int_{\mathbb{R}^k} \frac{\tilde{u}_{s,k}(y, z)}{(|z|^{-1}|y|^2 + 1)^{n+2s/2}} \, dy \right) \, d z$$

$$= c_{n,s} \int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} \left( k\omega_k \int_0^\infty v(|z| r, z) \frac{r^{k-1}}{(r^2 + 1)^{n+2s/2}} \, dr \right) \, d z,$$
where \(v(r, z) = \bar{u}_{s,k}(y, z)\) for \(|y| = r\).

Next we apply Lemma 5.3 to \(f(r) = v(|z|r, z)\) and \(\omega(r) = k\omega_k r^{k-1}(r^2 + 1)^{-\frac{n+2s}{2}}\). Note that since \(v\) is the \(k\)-symmetric increasing rearrangement of \(\bar{u}\), we have

\[\mu_f(t) = \frac{1}{|\mathbb{R}|} \int_{\{r > 0 : v(r, z) < t\}} = \frac{\omega_k^{-1/k}}{|\mathbb{R}|} H_k^k \left( \{y \in \mathbb{R}^k : \bar{u}(y, z) < t\} \right)^{1/k} = \frac{1}{|\mathbb{R}|} \mu_{\bar{u}}(t, y)^{1/k}.\]

Therefore,

\[k \omega_k \int_0^\infty \frac{\nu(z r, z) r^{k-1}}{(r^2 + 1)^{\frac{n+2s}{2}}} dr = \int_0^\infty \left( k \omega_k \int_0^\infty \frac{\nu(z r(t), z) r^{k-1}}{|z| (r^2 + 1)^{\frac{n+2s}{2}}} dr \right) dt = \int_0^\infty W \left( \frac{\mu_{\bar{u}}(t, y)^{1/k}}{|z|} \right) dt,\]

where \(W\) is given in (5.2). By Fubini’s theorem, we conclude that

\[\mathcal{F}_k u(x) = c_{n,s} \int_0^\infty \int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W \left( \frac{\mu_{\bar{u}}(t, y)^{1/k}}{|z|} \right) dz dt.\]

\[\square\]

**Lemma 5.5.** Suppose we are under the assumptions of Proposition 5.2. Let \(x_1 \in \overline{B_r(x_0)}\) and \(d = |x_1 - x_0|\). The following holds:

(a) If \(t \in (2\Lambda d, \varepsilon]\), then \(D_{x_0} u(t - 2\Lambda d) \subset D_{x_1} u(t)\).

(b) If \(t \in (\varepsilon, \infty)\), then \(D_{x_0} u(t - 2\Lambda d/\varepsilon) \subset D_{x_1} u(t)\).

**Proof.** First we prove (a). Fix \(t \in (2\Lambda d, \varepsilon]\) and let \(x \in D_{x_0} u(t - 2\Lambda d)\). Then

\[u(x) - u(x_0) - (x - x_0) \cdot \nabla u(x_0) \leq t - 2\Lambda d.\]

Using (5.3), convexity, and \(|u|_{C^{1,1}(\mathbb{R}^n)} \leq 1\), we see that

\[u(x) - u(x_1) - (x - x_1) \cdot \nabla u(x_1) = u(x) - u(x_0) - (x - x_0) \cdot \nabla u(x_0) - \left( (u(x_1) - u(x_0) - (x - x_0) \cdot \nabla u(x_0)) \right) + (x - x_1) \cdot (\nabla u(x_0) - \nabla u(x_1)) \leq t - 2\Lambda d + |x - x_1|d.\]

Moreover, \(x \in D_{x_0} u(\varepsilon)\), since \(t \leq \varepsilon\), and thus,

\[|x - x_1| \leq |x - x_0| + |x_0 - x_1| \leq \Lambda + d \leq 2\Lambda.\]

Therefore, \(x \in D_{x_1} u(t)\).

Next we prove (b). Fix \(t \in (\varepsilon, \infty)\) and let \(x \in D_{x_0} u(t - 2\Lambda d/\varepsilon)\). By the previous computation, we have that

\[u(x) - u(x_1) - (x - x_1) \cdot \nabla u(x_1) \leq t - 2\Lambda d/\varepsilon + (|x - x_0| + \Lambda)d.\]

To control the distance from \(x\) to \(x_0\), we need to estimate the diameter of \(D_{x_0} u(t)\). Take \(y \in D_{x_0} u(t) \setminus D_{x_0} u(\varepsilon)\) and let \(z\) be in the intersection between \(\partial D_{x_0} u(\varepsilon)\) and the line segment joining \(x_0\) and \(y\). Then there is some \(\lambda > 1\) such that \(y - x_0 = \lambda(z - x_0)\). By convexity of \(u\),

\[u(z) \leq \frac{\lambda - 1}{\lambda} u(x_0) + \frac{1}{\lambda} u(y).\]

Therefore,

\[\lambda \varepsilon = \lambda (u(z) - u(x_0) - (z - x_0) \cdot \nabla u(x_0)) \leq (\lambda - 1) u(x_0) + u(y) - \lambda u(x_0) - (y - x_0) \cdot \nabla u(x_0) = u(y) - u(x_0) - (y - x_0) \cdot \nabla u(x_0) \leq t,\]

\[\square\]
so $\lambda \leq t/\varepsilon$. By convexity, we have that $D_{x_0}u(t) \subset x_0 + \frac{t}{\varepsilon}(D_{x_0}u(\varepsilon) - x_0)$. It follows that

\[
\text{diam } D_{x_0}u(t) \leq t/\varepsilon \text{ diam } D_{x_0}u(\varepsilon) = \Lambda t/\varepsilon.
\]

Hence, $|x - x_0| \leq \Lambda t/\varepsilon$, and by (5.4), we get

\[
u(x) - u(x_1) - (x - x_1) \cdot \nabla u(x_1) \leq t - 2\Lambda d t/\varepsilon + (\Lambda t/\varepsilon + \Lambda)d \leq t,
\]

which means that $x \in D_{x_1}u(t)$.

We are ready to give the proof of Proposition 5.2.

Proof of Proposition 5.2. Let $x_1 \in B_r(x_0)$, with $r \leq \varepsilon/(4\Lambda)$, and call $d = |x_0 - x_1|$. We will estimate $F^k_ku(x_1)$ using Proposition 5.4:

\[
F^k_ku(x_1) = c_{n,s} \int_0^{\infty} \int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\left(\frac{\mu_{\lambda}(t,z)}{|z|}^{1/k}\right) dz dt.
\]

In view of Lemma 5.5, we divide the integral with respect to $t$ in three parts:

1. $t \in (0, 2\Lambda d]$,
2. $t \in (2\Lambda d, \varepsilon)$,
3. $t \in (\varepsilon, \infty)$.

Let us start with I. Since $u \in C^{1,1}(\mathbb{R}^n)$ with $[u]_{C^{1,1}(\mathbb{R}^n)} \leq 1$, then

\[
\mu_{\lambda}(t,z) \geq (t - |z|^2)^{k/2}.
\]

Hence, using that $W(\rho)$ is monotone decreasing, we get

\[
W\left(\frac{\mu_{\lambda}(t,z)}{|z|}^{1/k}\right) \leq W\left(\left(\frac{t}{|z|^2} - 1\right)^{1/k}\right).
\]

Therefore,

\[
\int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\left(\frac{\mu_{\lambda}(t,z)}{|z|}^{1/k}\right) dz \leq \int_{\{|z|<t^{1/2}\}} \frac{1}{|z|^{n-k+2s}} W\left(\left(\frac{t}{|z|^2} - 1\right)^{1/k}\right) dz
\]

\[
+ W(0) \int_{\{|z|>t^{1/2}\}} \frac{1}{|z|^{n-k+2s}} dz = I_1 + I_2.
\]

Note that $W(0) = C(n, k, s) < \infty$. Then

\[
I_2 \lesssim \int_{t^{1/2}}^{\infty} \frac{1}{\rho^{n-k+2s}} \rho^{n-k-1} d\rho \approx t^{-s}.
\]

For $I_1$, we make the change of variables, $w = z/t^{1/2}$. We see that

\[
I_1 = \int_{\{|w|<1\}} \frac{1}{t^{n-k+2s} w^{n-k+2s}} W\left(\left(\frac{1}{|w|^2} - 1\right)^{1/k}\right) \frac{d^n w}{d^n w} \approx \frac{1}{t^s} \int_0^1 \frac{1}{\rho^{1+2s}} W\left(\left(\frac{1}{\rho^2} - 1\right)^{1/k}\right) d\rho.
\]

Note that if $0 < \rho \leq 1/2$, then $\left(\frac{1}{\rho^2} - 1\right)^{1/k} \geq \frac{1}{\sqrt{2}\rho}$. Hence,

\[
W\left(\left(\frac{1}{\rho^2} - 1\right)^{1/k}\right) \leq W\left(\frac{1}{\sqrt{2}\rho}\right) = \int_{\sqrt{2}\rho}^{\infty} \frac{r^{k-1}}{(1 + r^2)^{n+2s}} dr \lesssim \rho^{n-k+2s}.
\]

Therefore,

\[
I_1 \lesssim t^{-s} \int_0^{1/2} \frac{1}{\rho^{1+2s}} \rho^{n-k+2s} d\rho + t^{-s} W(0) \int_1^{1/2} \frac{1}{\rho^{1+2s}} d\rho \approx t^{-s},
\]

since $n - k > 0$. We conclude that

\[
I = c_{n,s} \int_0^{2\Lambda d} \int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\left(\frac{\mu_{\lambda}(t,z)}{|z|}^{1/k}\right) dz dt
\]
\[
\leq \int_{0}^{2\Lambda d} t^{-s} \, dt \approx (2\Lambda d)^{1-s} = (2\Lambda)^{1-s}|x_1 - x_0|^{1-s}.
\]

Next we estimate the integral for \( t \in (2\Lambda d, \varepsilon] \). To this end, we use Lemma 5.5, part (a):
\[
D_{\mu_0}u(t - 2\Lambda d) \subset D_{x_1}u(t).
\]
In particular, for any \( z \in \mathbb{R}^{n-k} \) fixed, we have
\[
\{ y \in \mathbb{R}^k : \tilde{u}_{x_0}(y, z) \leq t - 2\Lambda d \} \subset \{ y \in \mathbb{R}^k : \tilde{u}_{x_1}(y, z) \leq t \}.
\]
Hence, \( \mu_{x_0}(t - 2\Lambda d, z) \leq \mu_{x_1}(t, z) \), which yields
\[
\begin{align*}
\II &= c_{n,s} \int_{2\Lambda d}^{\varepsilon} \int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\left( \frac{\mu_{x_1}u(t, z)^{1/k}}{|z|} \right) \, dz \, dt \\
&\leq c_{n,s} \int_{0}^{\varepsilon - 2\Lambda d} \int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\left( \frac{\mu_{x_1}u(t, z)^{1/k}}{|z|} \right) \, dz \, dt.
\end{align*}
\]
Finally, we estimate the integral for \( t \in [\varepsilon, \infty) \). By Lemma 5.5, part (b):
\[
D_{\mu_0}u(t - 2\Lambda dt/\varepsilon) \subset D_{x_1}u(t).
\]
Hence, \( \mu_{x_0}(t - 2\Lambda dt/\varepsilon, z) \leq \mu_{x_1}(t, z) \), and
\[
\begin{align*}
\III &= c_{n,s} \int_{\varepsilon}^{\infty} \int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\left( \frac{\mu_{x_1}u(t, z)^{1/k}}{|z|} \right) \, dz \, dt \\
&\leq \int_{\varepsilon}^{\infty} \int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\left( \frac{\mu_{x_0}u(t - 2\Lambda dt/\varepsilon, z)^{1/k}}{|z|} \right) \, dz \, dt \\
&= \frac{1}{2 - 2\Lambda d/\varepsilon} \int_{\varepsilon - 2\Lambda d}^{\infty} \int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\left( \frac{\mu_{x_0}u(t, z)^{1/k}}{|z|} \right) \, dz \, dt.
\end{align*}
\]
Note that
\[
\II + \III \leq \frac{c_{n,s}}{2 - 2\Lambda d/\varepsilon} \int_{\varepsilon - 2\Lambda d}^{\infty} \int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\left( \frac{\mu_{x_0}u(t, z)^{1/k}}{|z|} \right) \, dz \, dt = \frac{\varepsilon}{2 - 2\Lambda d} \mathcal{F}_k^s u(x_0).
\]
Therefore, we conclude that
\[
\mathcal{F}_k^s u(x_1) - \mathcal{F}_k^s u(x_0) \leq CA^{1-s}|x_1 - x_0|^{1-s} + \left( \frac{\varepsilon}{2 - 2\Lambda d} - 1 \right) \mathcal{F}_k^s u(x_0) \\
\leq CA^{1-s}|x_1 - x_0|^{1-s} + \frac{4\Lambda}{\varepsilon}|x_1 - x_0|\mathcal{F}_k^s u(x_0)
\]
since \( d < r \leq \varepsilon/(4\Lambda) \), and thus, \( \varepsilon - 2\Lambda d \geq \varepsilon/2 \).

\[ \square \]

6. A GLOBAL POISSON PROBLEM

We consider the following Poisson problem in the full space:
\[
\begin{cases}
\mathcal{F}_k^s u = u - \varphi & \text{in } \mathbb{R}^n \\
(u - \varphi)(x) \to 0 & \text{as } |x| \to \infty,
\end{cases}
\] (6.1)
\[
\text{where } \varphi : \mathbb{R}^n \to \mathbb{R} \text{ is nonnegative, smooth, and strictly convex. Furthermore, we ask that } \varphi \text{ behaves asymptotically at infinity as a cone } \phi, \text{ that is,}
\]
\[
\lim_{|x| \to \infty} (\varphi - \phi)(x) = 0.
\] (6.2)

Similar problems have been studied for nonlocal Monge-Ampère operators in [4, 9].

We will prove the following theorem.
Theorem 6.1. There exists a unique solution $u$ to (6.1) such that $u \in C^{1,1}(\mathbb{R}^n)$ with

$$[u]_{C^{1,1}(\mathbb{R}^n)} \leq [\varphi]_{C^{1,1}(\mathbb{R}^n)}.$$

To define the notion of solution, we introduce a natural pointwise definition of $F_k^* u$ for functions $u$ that are merely continuous.

Definition 6.2. Let $u \in C^0(\mathbb{R}^n)$.

(a) We say that a linear function $l(y) = y \cdot p + b$, with $p \in \mathbb{R}^n$, and $b \in \mathbb{R}$, is a supporting plane of $u$ at a point $x$ if $l(x) = u(x)$ and $l(y) \leq u(y)$, for all $y \in \mathbb{R}^n$.

(b) We define the subdifferential of $u$ at a point $x$ as the set $\partial u(x)$ of all vectors $p \in \mathbb{R}^n$ such that $l(y) = y \cdot p + b$ is a supporting plane of $u$ at $x$, for some $b \in \mathbb{R}$.

Definition 6.3. Let $u \in C^0(\mathbb{R}^n)$ be a convex function. For $x_0 \in \mathbb{R}^n$, we define

$$F_k^* u(x_0) = c_{n,s} \sup_{p \in \partial u(x_0)} \inf_{K \in K_k} \int_{\mathbb{R}^n} (u(x_0 + x) - u(x_0) - x \cdot p)K(x) dx.$$

Remark 6.4. Note that if $u \in C^{1,1}(x_0)$, then $\partial u(x_0) = \{\nabla u(x_0)\}$, and the previous definition coincides with Definition 2.4.

The following properties of $F_k^* u$ will be useful for our purposes. The proof is analogous to the one in [9], so we omit it here.

Lemma 6.5. Let $u, v \in C^0(\mathbb{R}^n)$ be convex functions. The following holds:

(a) (Homogeneity). For any $\lambda > 0$,

$$F_k^* (\lambda u) = \lambda F_k^* u.$$

(b) (Monotonicity). Assume that $u(x_0) = v(x_0)$ and $u(x) \geq v(x)$ for all $x \in \mathbb{R}^n$. Then

$$F_k^* u(x_0) \geq F_k^* v(x_0).$$

(c) (Concavity). For any $x \in \mathbb{R}^n$,

$$F_k^* \left( \frac{u + v}{2} \right)(x) \geq \frac{F_k^* u(x) + F_k^* v(x)}{2}.$$

(d) (Lower semicontinuity). Assume that $u \in C^{1,1}(\mathbb{R}^n)$. Then

$$F_k^* u(x_0) \leq \liminf_{x \to x_0} F_k^* u(x).$$

Definition 6.6. Let $u \in C^0(\mathbb{R}^n)$ be a convex function. We say that $u$ is a subsolution to $F_k^* u = u - \varphi$ in $\mathbb{R}^n$ if

$$F_k^* u(x_0) \geq u(x_0) - \varphi(x_0), \quad \text{for all } x_0 \in \mathbb{R}^n.$$

Similarly, $u$ is a supersolution if

$$F_k^* u(x_0) \leq u(x_0) - \varphi(x_0), \quad \text{for all } x_0 \in \mathbb{R}^n.$$

We say that $u$ is a solution if it is both a subsolution and a supersolution.

Lemma 6.7. If $u$ and $v$ are subsolutions, then $\max\{u, v\}$ is a subsolution.

Proof. Let $w = \max\{u, v\}$. Then $w$ is continuous and convex. Fix $x_0 \in \mathbb{R}^n$. Without loss of generality, we may assume that $u(x_0) \geq v(x_0)$. Then $w(x_0) = u(x_0)$ and $w(x) \geq u(x)$, for any $x \in \mathbb{R}^n$. By monotonicity (see Lemma 6.5), we have

$$F_k^* w(x_0) \geq F_k^* u(x_0) \geq u(x_0) - \varphi(x_0) = w(x_0) - \varphi(x_0).$$

Hence, $w$ is a subsolution. □
We will show existence and uniqueness of solutions to \((6.1)\) using Perron’s method. The key ingredients are the comparison principle, and the existence of a subsolution (lower barrier) and a supersolution (upper barrier). We state this in the following proposition. We omit the proof since it is similar to that in [9].

**Proposition 6.8.** Consider the equation \(F_k^\ast u = u - \varphi\) in \(\mathbb{R}^n\). The following holds:
(a) (Comparison principle). Let \(u\) and \(v\) be a subsolution and supersolution, respectively. Assume that \(u \leq v\) in \(\mathbb{R}^n \setminus \Omega\), for some bounded domain \(\Omega \subset \mathbb{R}^n\). Then \(u \leq v\) in \(\mathbb{R}^n\).
(b) (Lower-barrier). The function \(\varphi\) is a subsolution.
(c) (Upper-barrier). The function \(\varphi + w\) is a supersolution, where \(w = (I - \Delta^s)^{-1} \Delta^s \varphi\). In particular, \(w(x) \leq C(1 + |x|)^{1-2s}, \) for some \(C > 0\).

An immediate consequence of the comparison principle is the uniqueness of solutions.

**Lemma 6.9** (Uniqueness). There exists at most one solution to \((6.1)\).

**Proof.** Suppose by means of contradiction that there exist two functions \(u, v \in C^0(\mathbb{R}^n)\) with \(u \neq v\), satisfying \((6.1)\). Then \(|u(x) - v(x)| \to 0\), as \(|x| \to \infty\). Hence, for any \(\varepsilon > 0\), there exists a compact set \(\Omega_\varepsilon \subset \mathbb{R}^n\), depending on \(\varepsilon\), such that
\[
v(x) - \varepsilon \leq u(x) \leq v(x) + \varepsilon \quad \text{for all} \quad x \in \mathbb{R}^n \setminus \Omega_\varepsilon.
\]
Moreover, for any \(x_0 \in \mathbb{R}^n\), the function \(v + \varepsilon\) satisfies
\[
F_k^\ast(v + \varepsilon)(x_0) = v(x_0) - \varphi(x_0) < (v(x_0) + \varepsilon) - \varphi(x_0).
\]
Therefore, \(v\) is a supersolution and by the comparison principle, it follows that \(u \leq v + \varepsilon\) in \(\mathbb{R}^n\). Similarly, we see that \(v - \varepsilon\) is a subsolution and \(u \geq v - \varepsilon\) in \(\mathbb{R}^n\). Hence,
\[
\|u - v\|_{L^\infty(\mathbb{R}^n)} \leq \varepsilon,
\]
and letting \(\varepsilon \to 0\), we get \(u = v\) in \(\mathbb{R}^n\), which is a contradiction. \(\square\)

To prove existence of a solution, we define
\[
(6.3) \quad u(x) = \sup_{v \in \mathcal{S}} v(x),
\]
where \(\mathcal{S}\) is the set of admissible subsolutions given by
\[
\mathcal{S} = \{ v \in C^{0,1}(\mathbb{R}^n) : v \text{ subsolution, } \varphi \leq v \leq \varphi + w, \text{ and } |v|_{C^0(\mathbb{R}^n)} \leq |\varphi|_{C^{0,1}(\mathbb{R}^n)} \}.
\]
Note that \(\mathcal{S} \neq \emptyset\) since \(\varphi \in \mathcal{S}\), and the supremum is finite since \(v \leq \varphi + w\), for any \(v \in \mathcal{S}\).

Moreover, \(u\) is convex and Lipschitz, with
\[
[u]_{C^{0,1}(\mathbb{R}^n)} \leq [\varphi]_{C^{0,1}(\mathbb{R}^n)}.
\]
From \(\varphi \leq u \leq \varphi + w\), and the upper bound for \(w\) in Proposition 6.8, it follows that
\[
0 \leq (u - \varphi)(x) \leq w(x) \leq C(1 + |x|)^{1-2s} \to 0,
\]
as \(|x| \to \infty\) since \(1 - 2s < 0\).

**Proposition 6.10.** The function \(u\) given in \((6.3)\) is \(C^{1,1}(\mathbb{R}^n)\) with
\[
[u]_{C^{1,1}(\mathbb{R}^n)} \leq [\varphi]_{C^{1,1}(\mathbb{R}^n)}.
\]
Proof. We will show that for any $x_0, x_1 \in \mathbb{R}^n$,
$$0 \leq u(x_0 + x_1) - u(x_0 - x_1) - 2u(x_0) \leq |\varphi|_{C^{1,1}(\mathbb{R}^n)}|x_1|^2.$$ 
Indeed, the lower bound follows from convexity of $u$. Hence, we only need to prove the upper bound. Call $M = |\varphi|_{C^{1,1}(\mathbb{R}^n)}$. Then
\begin{equation}
\varphi(x_0 + x_1) - \varphi(x_0 - x_1) - M|x_1|^2 \leq 2\varphi(x_0).
\end{equation}
Letting $v \in S$ and fix $x_1 \in \mathbb{R}^n$. Define
$$\hat{v}(x_0) = \frac{1}{2}(v(x_0 + x_1) + v(x_0 - x_1) - M|x_1|^2), \quad \text{for } x_0 \in \mathbb{R}^n.$$ 
We claim that $\hat{v}$ is a subsolution to $F_k^s u = u - \varphi$ in $\mathbb{R}^n$. Indeed, since $F_k^s$ is homogeneous of degree 1, concave, and translation invariant (see Lemma 6.5), we have
$$\mathcal{F}_k^s \hat{v}(x_0) = \mathcal{F}_k^s \left( \frac{1}{2}v(x_0 + x_1) + \frac{1}{2}v(x_0 - x_1) \right)$$
$$\geq \frac{1}{2} \mathcal{F}_k v(x_0 + x_1) + \frac{1}{2} \mathcal{F}_k v(x_0 - x_1)$$
$$\geq \frac{1}{2} \left( v(x_0 + x_1) - \varphi(x_0 + x_1) + v(x_0 - x_1) - \varphi(x_0 - x_1) \right)$$
$$= \frac{1}{2} \left( v(x_0 + x_1) - v(x_0 - x_1) - M|x_1|^2 \right) - \frac{1}{2} \left( \varphi(x_0 + x_1) + \varphi(x_0 - x_1) - M|x_1|^2 \right)$$
$$\geq \hat{v}(x_0) - \varphi(x_0).$$
Moreover, using that $v \leq \varphi + w$, we get
$$\hat{v}(x_0) \leq \frac{1}{2} (\varphi(x_0 + x_1) + \varphi(x_0 - x_1) - M|x_1|^2) + \frac{1}{2} (w(x_0 + x_1) + w(x_0 - x_1)).$$
By (6.4) and the upper bound of $w$ in Proposition 6.8, part (c), we see that
$$\hat{v}(x_0) - \varphi(x_0) \leq \frac{C}{2} (1 + |x_0 + x_1|^{1-2s}) + \frac{C}{2} (1 + |x_0 - x_1|^{1-2s}) \to 0,$$
as $|x_0| \to \infty$ and $x_1$ fixed, since $1 - 2s < 0$. Then for all $\varepsilon > 0$, there is some compact set $\Omega_\varepsilon$, depending on $\varepsilon$ and $x_1$, such that
$$\hat{v}(x_0) - \varepsilon \leq \varphi(x_0), \quad \text{for all } x_0 \in \mathbb{R}^n \setminus \Omega_\varepsilon.$$ 
Consider $\hat{v}_\varepsilon = \max\{\hat{v} - \varepsilon, \varphi\}$. Then $\hat{v}_\varepsilon$ is a subsolution, since the maximum of subsolutions is a subsolution (see Lemma 6.7). Also, $\hat{v}_\varepsilon = \varphi \leq \varphi + w$ in $\mathbb{R}^n \setminus \Omega_\varepsilon$, and $\varphi + w$ is a supersolution by Proposition 6.8, part (c). Applying the comparison principle, we get $\varphi \leq \hat{v}_\varepsilon \leq \varphi + w$. Moreover, $[\hat{v}_\varepsilon]_{C^{0,1}(\mathbb{R}^n)} \leq [\varphi]_{C^{0,1}(\mathbb{R}^n)}$. Therefore, $\hat{v}_\varepsilon \in S$.

Since $u(x_0) = \sup_{v \in S} v(x_0)$, it follows that $u(x_0) \geq \hat{v}_\varepsilon(x_0) \geq \hat{v}(x_0) - \varepsilon$. Letting $\varepsilon \to 0$, we conclude that for any $v \in S$ and $x_0, x_1 \in \mathbb{R}^n$,
\begin{equation}
u(x_0) \geq \frac{1}{2} \left( v(x_0 + x_1) + v(x_0 - x_1) - M|x_1|^2 \right).
\end{equation}
Finally, by definition of supremum, for any $\delta > 0$, and $x_0, x_1 \in \mathbb{R}^n$, there exist $v_1, v_2 \in S$ such that $u(x_0 + x_1) - \delta < v_1(x_0 + x_1)$ and $u(x_0 - x_1) - \delta < v_2(x_0 - x_1)$. Let $v = \max\{v_1, v_2\}$. Then using (6.5) for this $v$, we get
$$u(x_0) \geq \frac{1}{2} (u(x_0 + x_1) - \delta + u(x_0 - x_1) - \delta - M|x_1|^2).$$
Letting $\delta \to 0$, we conclude that
$$u(x_0 + x_1) - u(x_0 - x_1) - 2u(x_0) \leq |\varphi|_{C^{1,1}(\mathbb{R}^n)}|x_1|^2.$$
\[\square\]
Lemma 6.11. For any \( x_0 \in \mathbb{R}^n \) and \( \varepsilon > 0 \), the set
\[
D_{x_0} u(\varepsilon) = \{ x \in \mathbb{R}^n : u(x) - u(x_0) - (x - x_0) \cdot \nabla u(x_0) \leq \varepsilon \}
\]
is compact.

Proof. Let \( x_0 \in \mathbb{R}^n \) and \( \varepsilon > 0 \). Without loss of generality, we may assume that \( x_0 = 0 \). Let \( l \) be the supporting plane of \( u \) at 0, that is, \( l(x) = u(0) + x \cdot \nabla u(0) \). Clearly, \( D_{x_0} u(\varepsilon) \) is closed. Hence, we only need to show that it is bounded. Recall that
\[
\phi(x) < \varphi(x) \leq u(x), \quad \text{for all} \ x \in \mathbb{R}^n,
\]
where \( \phi \) is a cone. Note that the strict inequality in (6.6) follows from the strict convexity of \( \varphi \). Moreover, by (6.1) and (6.2), we have
\[
\lim_{|x| \to \infty} (u - \phi)(x) = 0.
\]
Therefore, \( D_{x_0} u(\varepsilon) \subset \{ \phi < l + \varepsilon \} \). We claim that
\[
\lim_{|x| \to \infty} (\phi - l)(x) = \infty.
\]
If this condition holds, then for all \( M > 0 \), there exists \( R > 0 \), such that
\[
\phi(x) - l(x) > M, \quad \text{for all} \ |x| > R.
\]
Choosing \( M = \varepsilon \), we see that \( \{ \phi < l + \varepsilon \} \subset B_R \), for some \( R \) depending on \( \varepsilon \). Hence, the set \( D_{x_0} u(\varepsilon) \) is bounded.

To prove the claim, we distinguish two cases. If \( u(0) = 0 \), then \( u \) attains an absolute minimum at 0, so \( \nabla u(0) = 0 \). In particular, \( l(x) = 0 \), for all \( x \in \mathbb{R}^n \), and thus, (6.7) is clearly satisfied. Hence, it remains to show the claim when
\[
u(0) > 0.
\]
We will prove it by contradiction. If (6.7) is not true, then there exists a sequence of points \( \{ x_j \}_{j=1}^\infty \subset \mathbb{R}^n \) such that \( |x_j| \to \infty \), as \( j \to \infty \), and
\[
\lim_{j \to \infty} (\phi - l)(x_j) < \infty.
\]
Using that \( \phi \) is continuous and homogeneous of degree 1, and letting \( j \to \infty \), we get
\[
\frac{\phi(x_j)}{|x_j|} - \frac{l(x_j)}{|x_j|} = \frac{\phi(x_j)}{|x_j|} - \frac{u(0)}{|x_j|} - \frac{x_j}{|x_j|} \cdot \nabla u(0) \to \phi(e) - D_e u(0) = 0,
\]
where \( x_j/|x_j| \to e \), up to a subsequence. Therefore, \( \phi(e) = D_e u(0) \). For any \( \lambda > 0 \), we have
\[
l(\lambda e) = u(0) + \lambda e \cdot \nabla u(0) = u(0) + \lambda \phi(e) = u(0) + \phi(\lambda e).
\]
Since \( l \) is a supporting plane of \( u \), we know that \( u(x) \geq l(x) \), for all \( x \in \mathbb{R}^n \), and thus,
\[
u(\lambda e) \geq l(\lambda e) = \phi(\lambda e) + u(0).
\]
Letting \( \lambda \to \infty \), we see that
\[
0 = \lim_{\lambda \to \infty} (u - \phi)(\lambda e) \geq u(0) > 0,
\]
which is a contradiction. \( \square \)

Proposition 6.12 \((u \text{ is a subsolution})\). The function \( u \) given in (6.3) satisfies
\[
\mathcal{F}_k^n u(x_0) \geq u(x_0) - \varphi(x_0), \quad \text{for all} \ x_0 \in \mathbb{R}^n.
\]
Proof. By Proposition 6.10, we know that \( u \in C^{1,1}(\mathbb{R}^n) \). Without loss of generality, we may assume that \( [u]_{C^{1,1}(\mathbb{R}^n)} = 1 \). Otherwise, consider \( u/[u]_{C^{1,1}(\mathbb{R}^n)} \).

Let \( x_0 \in \mathbb{R}^n \). Then the quadratic polynomial
\[
P(x) = u(x_0) + \nabla u(x_0) \cdot (x - x_0) + |x - x_0|^2
\]
touches \( u \) from above at \( x_0 \). Moreover, we may assume that \( P \) touches \( u \) strictly from above at \( x_0 \). If not, we replace \( P \) by \( P + \varepsilon |x - x_0|^2 \) with \( \varepsilon > 0 \) small.

Fix \( \delta > 0 \). Then there exists \( h > 0 \), with \( h \to 0 \) as \( \delta \to 0 \), such that
\[
P(x) - u(x) \geq h > 0, \quad \text{for all } x \in \mathbb{R}^n \setminus B_{\delta}(x_0).
\]

Since \( u(x) = \sup_{v \in \mathcal{S}} v(x) \) and \( v \in \mathcal{S} \) is uniformly continuous, there is a monotone sequence \( \{v_j\}_{j=1}^{\infty} \subset \mathcal{S} \) such that \( v_j \to u \) uniformly in compact subsets of \( \mathbb{R}^n \). In particular, there exists \( j_0 \geq 1 \), depending on \( h \), such that for all \( j > j_0 \),
\[
(6.8) \quad u(x) - h < v_j(x), \quad \text{for all } x \in \overline{B_{\delta}(x_0)}.
\]

Call \( v = v_j \) for some \( j > j_0 \). It follows that
\[
\begin{cases}
P - v \geq h & \text{in } \mathbb{R}^n \setminus B_{\delta}(x_0) \\
P - v < P - u + h & \text{in } B_{\delta}(x_0).
\end{cases}
\]

Let \( d = \inf_{\mathbb{R}^n} (P - v) \). Then \( d = P(x_1) - v(x_1) \), for some \( x_1 \in \overline{B_{\delta}(x_0)} \), with \( 0 \leq d < h \), and
\[
\begin{cases}
P(x_1) - d = v(x_1) \\
P(x) - d \geq v(x), \quad \text{for all } x \in \mathbb{R}^n.
\end{cases}
\]

Hence, \( P - d \) is a quadratic polynomial that touches \( v \) from above at \( x_1 \). In particular, since \( v \) is convex, then \( v \) has a unique supporting plane \( l \) at \( x_1 \), so \( \partial v(x_1) = \{\nabla l\} \).

Let \( \tau \geq 0 \) be such that \( l + \tau \) is the supporting plane of \( u \) at some point \( x_2 \). Note that \( x_2 \) approaches \( x_0 \) as \( h \) goes to 0, and thus, there exists some \( r = r(h) > 0 \) such that \( r \to 0 \), as \( h \to 0 \), and \( x_2 \in B_r(x_0) \). Furthermore, since \( l(x_1) + d = v(x_1) + d = P(x_1) \geq u(x_1) \), then \( \tau \leq d < h \) (see Figure 2).

\[\text{Figure 2. Geometry involved in the proof of Proposition 6.12.}\]
Fix $\varepsilon > 0$. By Lemma 6.11, we have that $D_{x_0} u(\varepsilon)$ is bounded, so $\Lambda = \text{diam} \ D_{x_0} u(\varepsilon) < \infty$. Choose $\delta$ sufficiently small, so that $r < \varepsilon/(4\Lambda)$. Then by Proposition 5.2, it holds that

$$F_k^s u(x_2) \leq F_k^s u(x_0) + C \Lambda^{1-s} |x_2 - x_0|^{1-s} + \frac{4\Lambda}{\varepsilon} F_k^s u(x_0) |x_2 - x_0| \leq F_k^s u(x_0) + C(r),$$

where $C(r) \to 0$, as $r \to 0$. Next we will show that

$$F_k^s u(x_1) - C \tau^{1-s} \leq F_k^s u(x_2)$$

for some constant $C > 0$ depending only on $n$, $k$, and $s$. Since $\partial v(x_1) = \{\nabla l\}$, then $v \in C^{1,1}(x_1)$, and using Proposition 5.4, we get

$$F_k^s v(x_1) = c_{n,s} \int_0^\infty \int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\left(\frac{\mu_x v(t,z)}{|z|}\right) \, dz \, dt,$$

where $\partial_x v(t,z) = \omega_{n-1} H^k(\{y \in \mathbb{R}^k : \tilde{v}_x(y,z) \leq t\})$, and $W$ is the monotone decreasing function given in (5.2). Observe that since $v \leq u$, $l$ is the supporting plane of $v$ at $x_1$, and $l + \tau$ is the supporting plane of $u$ at $x_2$, then for any $t > 0$, it follows that

$$D_{x_2} u(t) = \{u - (l + \tau) \leq t\} \subseteq \{v - l \leq t + \tau\} = D_{x_1} v(t + \tau).$$

In particular, $\mu_x u(t,z) \leq \mu_x v(t + \tau, z)$, for any $z \in \mathbb{R}^{n-k}$. Therefore,

$$W(\mu_x u(t,z)) \geq W(\mu_x v(t + \tau, z)),$$

which yields

$$F_k^s u(x_2) \geq c_{n,s} \int_0^\infty \int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\left(\frac{\mu_x v(t,z)}{|z|}\right) \, dz \, dt = F_k^s v(x_1) - c_{n,s} \int_0^\tau \int_{\mathbb{R}^{n-k}} \frac{1}{|z|^{n-k+2s}} W\left(\frac{\mu_x v(t,z)}{|z|}\right) \, dz \, dt \geq F_k^s v(x_1) - C \tau^{1-s},$$

where the last inequality follows from the fact that $\mu_x v(t,z) \geq C(t - |z|^2)_{1+}$ and $W$ is monotone decreasing.

Combining (6.9) and (6.10), using that $v$ is a subsolution, and (6.8), we get

$$F_k^s u(x_0) + C(r) \geq F_k^s v(x_1) - C \tau^{1-s} \geq v(x_1) - \varphi(x_1) - C \tau^{1-s} > u(x_1) - h - \varphi(x_1) - C \tau^{1-s}.$$

Letting $\delta \to 0$, it follows that $h \to 0$, $C(r) \to 0$, $\tau \to 0$, and $x_1 \to x_0$. By continuity of $u$ and $\varphi$, we conclude the result. \hfill $\Box$

**Proposition 6.13** (u is a supersolution). The function $u$ given in (6.3) satisfies

$$F_k^s u(x_0) \leq u(x_0) - \varphi(x_0), \quad \text{for all} \ x_0 \in \mathbb{R}^n.$$  

**Proof.** Assume the statement is false. Then there exists some $x_0 \in \mathbb{R}^n$ such that

$$F_k^s u(x_0) > u(x_0) - \varphi(x_0).$$

Without loss of generality, we may assume that $u(x_0) = 0$ and $\nabla u(x_0) = 0$. Otherwise, consider $v(x) = u(x) - u(x_0) - (x - x_0) \cdot \nabla u(x_0)$. Then there exists some $\delta > 0$ such that

$$F_k^s u(x_0) \geq -\varphi(x_0) + \delta.$$

Fix $\varepsilon > 0$ and let $u^\varepsilon(x) = \max\{u(x), \varepsilon\}$. We will show that for $\varepsilon$ sufficiently small, $u^\varepsilon$ is an admissible subsolution, and thus, reaching a contradiction with $u$ being the largest subsolution. Indeed, $u^\varepsilon$ is convex and $u^\varepsilon \in C^{0,1}(\mathbb{R}^n)$ with $|u^\varepsilon|_{C^{0,1}(\mathbb{R}^n)} \leq |\varphi|_{C^{0,1}(\mathbb{R}^n)}$. Moreover,
note that \( u^\varepsilon(x) = u(x) \), for \( x \) large. Hence, once we show that \( u^\varepsilon \) is a subsolution, it will follow from the comparison principle that \( \varphi \leq u^\varepsilon \leq \varphi + w \).

If \( x \in \{u_\varepsilon = u\} \), then \( u_\varepsilon(x) = u(x) \) and \( u_\varepsilon \geq u \) in \( \mathbb{R}^n \). By monotonicity (Lemma 6.5),

\[
F^s_k u^\varepsilon(x) \geq F^s_k u(x) \geq u(x) - \varphi(x) = u^\varepsilon(x) - \varphi(x),
\]

since \( u \) is a subsolution, by Proposition 6.12.

If \( x \in \{u^\varepsilon > u\} \), then \( u^\varepsilon(x) = \varepsilon \) and \( \partial u^\varepsilon(x) = \{0\} \). In particular,

\[
(6.12) \quad F^s_k u^\varepsilon(x) = F^s_k u^\varepsilon(x_0).
\]

Moreover, for any \( t > 0 \), we have \( D_{x_0} u^\varepsilon(t) = \{u^\varepsilon - \varepsilon \leq t\} = \{u \leq t + \varepsilon\} = D_{x_0} u(t + \varepsilon) \).

Therefore, in view of Proposition 5.4, we get

\[
(6.13) \quad F^s_k u^\varepsilon(x_0) = F^s_k u(x_0) - \int_0^\varepsilon \int_{\mathbb{R}^{n-k+2\varepsilon}} \frac{1}{|z|^{n-k+2\varepsilon}} W\left(\frac{\mu_{x_0}(t,z)^{1/k}}{|z|}\right) dz dt \geq F^s_k u(x_0) - C\varepsilon^{1-s}
\]

since \( u \in C^{1,1}(\mathbb{R}^n) \) and \( \mu_{x_0}(t,z) \geq (t - |z|^2)^{k/2} \).

Combining (6.11), (6.12), and (6.13), we see that

\[
F^s_k u^\varepsilon(x) = F^s_k u^\varepsilon(x_0) \geq F^s_k u(x_0) - C\varepsilon^{1-s} \geq -\varphi(x_0) + \delta - C\varepsilon^{1-s}
\]

\[
= u^\varepsilon(x) - \varphi(x) + (\varphi(x) - \varphi(x_0) + \delta - C\varepsilon^{1-s} - \varepsilon),
\]

since \( u^\varepsilon(x) = \varepsilon \). We need the term inside the parenthesis to be nonnegative. Hence, it remains to control \( \varphi(x) - \varphi(x_0) \). Since \( \varphi \) is smooth,

\[
|\varphi(x) - \varphi(x_0)| \leq |\varphi|_{C^{0,1}(\mathbb{R}^n)} |x - x_0|.
\]

We distinguish two cases. If \( \{u = 0\} = \{x_0\} \), then \( |x - x_0| \leq d_\varepsilon \to 0 \), as \( \varepsilon \to 0 \). Hence, choosing \( \varepsilon \) sufficiently small, we see that

\[
\varphi(x) - \varphi(x_0) + \delta - C\varepsilon^{1-s} - \varepsilon \geq \delta - |\varphi|_{C^{0,1}(\mathbb{R}^n)} d_\varepsilon - C\varepsilon^{1-s} - \varepsilon \geq 0.
\]

Therefore, \( u^\varepsilon \in S \), which contradicts \( u^\varepsilon(x_0) > u(x_0) = \sup_{v \in S} v(x_0) \geq u^\varepsilon(x_0) \).

Suppose now that \( \{u = 0\} \) contains more than one point. By compactness of \( \{u = 0\} \) and continuity of \( \varphi \), there exists some \( x_1 \in \{u = 0\} \) where \( \varphi \) attains its maximum. Then

\[
F^s_k u(x_1) = F^s_k u(x_0) \geq u(x_0) - \varphi(x_0) + \delta \geq u(x_1) - \varphi(x_1) + \delta.
\]

Moreover, by convexity of \( \{u = 0\} \) (since \( u \geq \varphi \geq 0 \)) and \( \varphi \), we must have that \( x_1 \in \partial\{u = 0\} \).

Hence, there exists \( \{x_j\}_{j=2}^\infty \subset \{u > 0\} \) such that \( x_j \to x_1 \) and \( u \) is strictly convex at \( x_j \).

Namely, there is a supporting plane that touches \( u \) only at \( x_j \).

By continuity of \( u \), there exists some \( j_0 \geq 2 \) such that

\[
u(x_1) > u(x_j) - \delta/4, \quad \text{for all } j > j_0.
\]

By continuity of \( \varphi \), there exists some \( j_1 \geq 2 \) such that

\[
\varphi(x_1) < \varphi(x_j) + \delta/4, \quad \text{for all } j > j_1.
\]

By lower semicontinuity of \( F^s_k u \), up to a subsequence, there exists some \( j_2 \geq 2 \) such that

\[
F^s_k u(x_j) > F^s_k u(x_1) - \delta/4, \quad \text{for all } j > j_2.
\]

Let \( J > \max\{j_0, j_1, j_2\} \). Then

\[
F^s_k u(x_j) > F^s_k u(x_1) - \delta/4 \geq u(x_1) - \varphi(x_1) + 3\delta/4 > u(x_j) - \varphi(x_j) + \delta/4,
\]

and we can repeat the previous argument, replacing \( x_0 \) by \( x_j \). We conclude that

\[
F^s_k u(x_0) \leq u(x_0) - \varphi(x_0), \quad \text{for all } x_0 \in \mathbb{R}^n.
\]
7. Future directions

As mentioned in the introduction, the main idea to define a nonlocal analog to the Monge-Ampère operator is to write it as a concave envelope of linear operators. More precisely,

\[ n \det(D^2u(x))^{1/n} = \inf_{M \in \mathcal{M}} \text{tr}(MD^2u(x)), \]

where \( \mathcal{M} = \{ M \in \mathcal{S}^n : M > 0, \det(M) = 1 \} \) and \( \mathcal{S}^n \) is the set of \( n \times n \) symmetric matrices. Note that this identity is equivalent to the one given in (1.2) taking \( M = AA^T \) and \( B = D^2u(x) \), since \( \text{tr}(A^TBA) = \text{tr}(AA^TB) \). In fact, this extremal property does not only hold for \( n \det(B)^{1/n} \) with \( B \in \mathcal{S}^n \) and \( B > 0 \). If \( \lambda = (\lambda_1, \ldots, \lambda_n) \), where \( \lambda_i \) are the eigenvalues of \( B \), then the function \( f \) defined in \( \Gamma = \{ \lambda \in \mathbb{R}^n : \lambda_i > 0, \text{ for all } i = 1, \ldots, n \} \), given by

\[ f(\lambda) = n\left( \prod_{i=1}^{n} \lambda_i \right)^{1/n} = n \det(B)^{1/n} \]

is differentiable, concave, and homogeneous of degree 1. In general, if \( f \) satisfies these conditions in an open convex set \( \Gamma \) in \( \mathbb{R}^n \), then

\[ f(\mu) = \inf_{\mu \in \Gamma} \{ f(\mu) + \nabla f(\mu) \cdot (\lambda - \mu) \} = \inf_{\mu \in \Gamma} \nabla f(\mu) \cdot \lambda, \]

where the second identity follows by Euler’s theorem. Therefore,

\[ f(\lambda) = \inf_{M \in \mathcal{M}_f} \text{tr}(MB), \]

where \( \mathcal{M}_f = \{ M \in \mathcal{S}^n : \lambda(M) \in \nabla f(\Gamma) \} \), \( \nabla f(\Gamma) = \{ \nabla f(\mu) : \mu \in \Gamma \} \), and \( \lambda(M) \) are the eigenvalues of \( M \).

For instance, the \( k \)-Hessian functions introduced by Caffarelli–Nirenberg–Spruck in [5] satisfy these conditions and, in fact, fractional analogs have been recently studied by Wu [18]. It would be interesting to explore fractional analogs to a wider class of fully nonlinear concave operators, as the ones mentioned above.

We remark that the 1-Hessian is equal to the Laplacian, and the \( n \)-Hessian is equal to the Monge-Ampère operator. Moreover, for \( 1 < k < n \), we obtain an intermediate discrete family between these operators. In view of this observation, a natural question of finding a continuous family connecting the Laplacian with the Monge-Ampère operator arises. Here we suggest possible families that connect smoothly these two operators, passing through the \( k \)-Hessians, in some sense. Indeed, let \( \alpha \in (0, 1]^n \) and denote \( |\alpha| = \alpha_1 + \cdots + \alpha_n \). For \( \lambda \in \mathbb{R}_+^n \), we consider the functions,

\[ f_\alpha(\lambda) = \left( \sum_{\sigma \in S} \lambda_{\sigma(1)}^{\alpha_1} \cdots \lambda_{\sigma(n)}^{\alpha_n} \right)^{1/|\alpha|}, \]

where \( S \) is the set of all cyclic permutations of \( \{1, \ldots, n\} \). Observe that for any \( 1 \leq k \leq n \), if \( \alpha = \sum_{i \in \mathcal{I}} \epsilon_i \), with \( |\mathcal{I}| = k \), then \( f_\alpha \) is precisely the \( k \)-Hessian function. Consider any smooth simple curve \( \gamma : [0, 1] \to (0, 1]^n \) such that

1. \( \gamma(0) = e_i \), for some \( 1 \leq i \leq n \),
2. \( \gamma(t_k) = \sum_{i \in \mathcal{I}_k} \epsilon_i \), with \( |\mathcal{I}_k| = k \), and \( 0 < t_k < t_{k+1} < 1 \), for all \( 1 < k < n \), and
3. \( \gamma(1) = (1, \ldots, 1) \).

Then the family \( \{ f_\alpha \}_{\alpha \in \text{Im}(\gamma)} \) is as we described. In particular, fractional analogs of these functions would give a continuous family from the fractional Laplacian to the nonlocal Monge-Ampère. We will study this problem in a forthcoming paper.
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REFERENCES

[1] A. Baernstein II, Symmetrization in Analysis, Cambridge University Press (2019).
[2] C. Bennett and M. Sharpley, Interpolation of Operators, Pure and Applied Mathematics 129, Academic Press (1988).
[3] Y. Brenier, Polar factorization and monotone rearrangement of vector-valued functions, Comm. Pure Appl. Math. 44 (1991), no. 4, 375–417.
[4] L. Caffarelli and F. Charro, On a fractional Monge-Ampère operator, Ann. PDE 1, Art. 4, 47 pp. (2015).
[5] L. Caffarelli, L. Nirenberg, J. Spruck, The Dirichlet problem for nonlinear second-order elliptic equations. III. Functions of the eigenvalues of the Hessian, Acta Math. 155 (1985), no. 3-4, 261–301.
[6] L. Caffarelli and L. Silvestre, The Evans-Krylov theorem for nonlocal fully nonlinear equations, Ann. of Math. 174 (2011), no. 2, 1163–1187.
[7] L. Caffarelli and L. Silvestre, Regularity theory for fully nonlinear integro-differential equations, Comm. Pure Appl. Math. 62 (2009), no. 5, 597–638.
[8] L. Caffarelli and L. Silvestre, Regularity results for nonlocal equations by approximation, Arch. Ration. Mech. Anal. 200 (2011), no. 1, 59–88.
[9] L. Caffarelli and L. Silvestre, A nonlocal Monge-Ampère equation, Comm. Anal. Geom. 24 (2016), 307–335.
[10] G. De Philippis and A. Figalli, The Monge-Ampère equation and its link to optimal transportation, Bull. Amer. Math. Soc. 51 (2014), no. 4, 527–580.
[11] N. Guillen and R. W. Schwab, Aleksandrov-Bakelman-Pucci type estimates for integro-differential equations, Arch. Ration. Mech. Anal. 206 (2012), no. 1, 111–157.
[12] N. V. Krylov, Controlled diffusion processes, Springer-Verlag (1980).
[13] D. Maldonado and P. R. Stinga, Harnack inequality for the fractional nonlocal linearized Monge-Ampère equation, Calc. Var. Partial Differential Equations 56 (2017), no. 4, 45 pp.
[14] R. J. McCann, Existence and uniqueness of monotone measure-preserving maps. Duke Math. J. 80 (1995), no. 2, 309–323.
[15] M. Nisio, Stochastic differential games and viscosity solutions of Isaacs equations, Nagoya Math. J. 110 (1988), 163–184.
[16] X. Ros-Oton and J. Serra, Boundary regularity for fully nonlinear integro-differential equations, Duke Math. J. 165 (2016), no. 11, 2079–2154.
[17] J. V. Ryff, Measure preserving transformations and rearrangements, J. Math. Anal. Appl. 31 (1970), 449–458.
[18] Y. Wu, Regularity of fractional analogue of k-Hessian operators and a non-local one-phase free boundary problem, Ph.D. Thesis, The University of Texas at Austin (2019).