Description of $\alpha$-cluster tail in $^8$Be and $^{20}$Ne: Delocalization of $\alpha$ cluster because of quantum penetration

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We analyze the $\alpha$-cluster wave functions in cluster states of $^8$Be and $^{20}$Ne by comparing the exact relative wave function obtained by the generator coordinate method (GCM) with various types of trial functions. For the trial functions, we adopt the fixed range shifted Gaussian of the Brink-Bloch (BB) wave function, the spherical Gaussian with the adjustable range parameter of the spherical Thosaki-Horiuchi-Schuck-Röpke (sTHSR), the deformed Gaussian of the deformed THSR (dTHSR), and a function with the Yukawa tail (YT). The quality of the description of the exact wave function with a trial function is judged by the squared overlap between the trial function and the GCM wave function. The better result is obtained with the sTHSR wave function than the BB wave function, and further improvement can be done with the dTHSR wave function because these wave functions can describe the outer tail better. The YT wave function gives almost the equal quality to or even better quality than the dTHSR wave function indicating that the outer tail of $\alpha$ cluster states is characterized by the Yukawa-like tail rather than the Gaussian tail. In the weakly bound $\alpha$ cluster states with the small $\alpha$ separation energy and the low centrifugal and Coulomb barriers, the outer tail part is the slowly damping function described well by the quantum penetration through the effective barrier. This outer tail characterizes the almost zero-energy free $\alpha$ gas behavior, i.e., the delocalization of cluster.

I. INTRODUCTION

A variety of cluster states have been known in light nuclei, such as $\alpha+\alpha$ and $^{16}$O+$\alpha$ states in $^8$Be and $^{20}$Ne and $3\alpha$ states in $^{12}$C. Cluster motion in these cluster states has been theoretically investigated in details with such microscopic cluster models as the resonating group method (RGM) [1] and the generator coordinate method (GCM) [2, 3] (for example, see Ref. [4] and references therein). In this decade, a new interpretation of the cluster states has been proposed in Refs. [5–10]. That is the dilute cluster gas state where clusters are not localized but they are rather freely moving occupying the lowest orbit in the cluster mean-field potential.

One of the typical examples is the $3\alpha$ cluster state of $^{14}$C($0^+_2$). A new method of cluster model has been constructed for treating the $\alpha$ cluster gas state originally based on the spherical Gaussian. Nowadays, it is called ”Tohsaki-Horiuchi-Schuck-Röpke wave function” (THSR) [5]. The spherical THSR wave function has been extended to the deformed version [6, 7], and it has been shown that, when the $J^\pi$-projection and the orthogonality to the $^{12}$C($0^+_1$) are taken into account, the single deformed THSR wave function is in principle equivalent to the full solution of the $\alpha$ wave function obtained by RGM and GCM calculations. In this paper, we call the spherical and deformed versions of THSR, ”sTHSR” and ”dTHSR”, respectively. Also in the case of $^8$Be($0^+_1$), the exact solution of the $2\alpha$ state obtained by the GCM calculation can be described almost perfectly by the single $2\alpha$ dTHSR wave function.

For the study of $^{20}$Ne, Zhou et al. have introduced the generalized THSR wave function to investigate the cluster structure in $K^\pi = 0^-$ band as well as $K^\pi = 0^+_1$ band and have shown that the single THSR wave functions give the better description than the single Bink-Bloch (BB) wave function [11] having the localized Gaussian form with the fixed range parameter. In particular, for $^{20}$Ne($1^-$), the single dTHSR is almost equivalent to the exact solution of the GCM with 99.98% squared overlap. The fact that the single THSR wave functions give better description than the single BB wave function indicates that the interpretation of the localized $^{16}$O+$\alpha$ cluster for the inversion doublet of $K^\pi = 0^+_1$ and $K^\pi = 0^-_1$ bands in $^{20}$Ne is too simple, but the quantum fluctuation of $\alpha$ cluster position is significant as already known in the success of the GCM calculation with the superposition of many BB wave functions [12]. In the works with the THSR wave functions, the delocalization of clusters in $^{20}$Ne has been stressed and explained as the new concept of the ”nonlocalization” [9, 10].

Based on the success in the description of $2\alpha$, $3\alpha$, and $^{16}$O+$\alpha$ systems with a single THSR wave function, the container picture has been recently proposed to understand the cluster states [13]. In the container picture, clusters are moving in the lowest orbit of the cluster mean-field potential whose spatial size is specified by the Gaussian range parameter of the THSR wave function.

However, one should be careful to discuss physical meaning of cluster wave function, in particular, in the inner region where the cluster wave function is strongly affected by the antisymmetrization effect between clusters. Moreover, the physical or mathematical meaning of the deformation of the Gaussian wave function in the dTHSR is not obvious, because the angular momentum $l$-wave relative wave function projected from the deformed Gaussian shows a behavior quite different from a Gaussian function when the deformation is large.
As already known, in the cluster states such as \(\alpha+\alpha\) and \(^{16}\text{O}+\alpha\) states in \(^8\text{Be}\) and \(^{20}\text{Ne}\), the relative wave function between clusters is characterized by the suppressed inner part, the enhanced surface peak, and the outer tail. The inner suppression and the enhanced amplitude at the surface can be understood by the strong antisymmetrization effect between clusters. It is important that the inner nodal structure and the surface peak structure are dominantly determined by the antisymmetrization, and therefore, it is difficult to discuss the physical meaning of the original \(\alpha\)-cluster wave function before the antisymmetrization. On the other hand, the outer tail part is almost free from the antisymmetrization effect, and it directly shows the \(\alpha\) cluster motion in the physical state.

The outer tail is caused by the quantum penetration and its asymptotic behavior is well defined. Needless to say, the quantum penetration is important in particular in the loosely bound \(\alpha\)-cluster system with the small \(\alpha\) separation energy and the small centrifugal and Coulomb barriers. In such a case, the wave function is slowly damping in the outer region and it has the remarkably long outer tail. As a result, the outer long tail becomes more significant, and hence, the description of the slowly damping tail part is essential for good description of the \(\alpha\)-cluster wave function.

In that sense, it is clear that the BB wave function fails to describe the outer long tail because it has the localized Gaussian form with the fixed range. The Gaussian function may give the better description if the Gaussian range is the adjustable parameter as in the case of the sTHSR wave function. However, as mentioned above, the damping behavior of the tail part in the asymptotic region is well defined by the \(\alpha\) separation energy as well as the centrifugal and Coulomb barriers, and obviously, it should be different from the Gaussian tail. One of the questions is why the projected dTHSR wave function can describe the correct damping behavior of the tail part and succeed to reproduce the exact solution almost perfectly.

In this paper, we investigate the \(\alpha\)-cluster motion in the \(2\alpha\) and \(^{16}\text{O}+\alpha\) cluster states of \(^8\text{Be}\) and \(^{20}\text{Ne}\). The spinless two-body cluster systems of \(^{16}\text{O}+\alpha\) and \(\alpha+\alpha\) can be reduced to the one dimension problem with the relative coordinate \(r\) between clusters. To discuss the physical feature of \(\alpha\) cluster motion we analyze the antisymmetrized relative wave function in one coordinate \(r\) space, and discuss its behavior in three regions of \(r\): the inner part, the surface peak, and the outer tail. In the present work, we consider several kinds of trial functions specified by one or at most two adjustable parameters and examine how accurately the trial function can reproduce the relative wave function of the exact solution obtained by the GCM calculation. Particular attention should be paid on the detailed behavior of the outer tail to discuss the delocalization of the \(\alpha\) cluster in weakly bound cluster states.

We consider the BB, sTHSR, dTHSR, and YT functions as trial functions. The relative wave functions in the BB, sTHSR, dTHSR, and YT functions are characterized by the localized Gaussian with the fixed range, the spherical Gaussian, the deformed Gaussian, and the Yukawa tail function, respectively. Comparing the squared overlap of those trial functions with the GCM wave function, we discuss the quality of the description of the exact solution with those trial functions. We show that the dTHSR is a good trial function which can give almost 100% overlap with the exact solution of weakly bound \(\alpha\)-cluster states such as \(^8\text{Be}(0^+\rangle\) and \(^{20}\text{Ne}(1^-\rangle\) because the projected deformed Gaussian can fit the Yukawa-like tail in the outer region fairly well if the effective barrier height in the outer region is low enough.

We also demonstrate that a kind of wave function with a Yukawa tail gives a good description of those states with almost the same quality as the dTHSR, and it gives even better description for such states as \(^{20}\text{Ne}(0^+)\) and \(^{20}\text{Ne}(2^+_1)\).

This paper is organized as follows. In the next section, the GCM calculation of \(^8\text{Be}\) and \(^{20}\text{Ne}\) is explained. In III, the adopted trial functions are described. The analyses of cluster states in \(^8\text{Be}\) and \(^{20}\text{Ne}\) are given in IV and V, respectively. Finally in VI, the discussion and summary are given. The tail behavior of the relative wave function in the dTHSR wave function in the large deformation limit is explained in the appendix.

II. GCM CALCULATION OF \(^8\text{BE}\) AND \(^{20}\text{NE}\)

The \(\alpha+\alpha\) and \(^{16}\text{O}+\alpha\) cluster models are applied to \(^8\text{Be}\) and \(^{20}\text{Ne}\), respectively. To describe details of the \(\alpha\) cluster motion, we solve the two-cluster problem with the GCM using BB cluster wave functions. In the GCM framework, we can obtain the precise relative wave function between the \(\alpha\) cluster and the other cluster by superposing BB wave functions.

A. Brink-Bloch \(\alpha\)-cluster wave function and GCM

We briefly review the BB wave function and the GCM calculation for a system composed of two spinless clusters \(C_1\) and \(C_2\). The mass numbers of \(C_1\) and \(C_2\) are \(A_1\) and \(A_2\) and the proton numbers are \(Z_1\) and \(Z_2\), respectively. In the present case, \(C_i\) is the \(\alpha\) cluster or \(^{16}\text{O}\). In the GCM calculation of the \(C_1+C_2\) cluster model, the total wave function can be expressed by the linear combination of BB wave functions [11].
A BB wave function of the two-cluster \( C_1 + C_2 \) system with the relative position \( \mathbf{S} \) is expressed as

\[
|\Phi_{\text{BB}}(\mathbf{S})\rangle = \frac{1}{\sqrt{A!}} A \{ \psi(C_1, -\frac{A_2}{A} \mathbf{S}) \psi(C_2, \frac{A_1}{A} \mathbf{S}) \}.
\]  

(1)

Here \( \psi(C_i, \mathbf{S}_i) \) is the wave function of the \( C_i \) cluster localized around \( \mathbf{S}_i \), and it is given by the harmonic oscillator (H.O.) shell model wave function with the shifted center at \( \mathbf{S}_i \). \( A \) is the antisymmetrizer for all nucleons. \( A = A_1 + A_2 \) is the total mass number. The same H.O. width is chosen for \( C_1 \) and \( C_2 \) for simplicity. We set the relative position \( \mathbf{S} \) on the \( z \)-axis \( \mathbf{S} = (0, 0, S) \) in the intrinsic frame and project the BB wave function to the spin-parity \( J^\pi \) eigen state

\[
|\Phi_{\text{BB}}^J(\mathbf{S})\rangle = N_l(S) P^{J\pi}_{MK} |\Phi_{\text{BB}}(\mathbf{S})\rangle,
\]  

(2)

where \( P^{J\pi}_{MK} \) is the spin-parity projection operator, and \( K = M = 0 \) is considered here because the BB wave function of two spinless clusters with \( \mathbf{S} = (0, 0, S) \) is the \( K = 0 \) eigen state. The normalization factor \( N_l(S) \) is chosen to be \( N_l(S) = 1/\sqrt{\langle \Phi_{\text{BB}}(\mathbf{S}) \rangle P^{J\pi}_{MK} |\Phi_{\text{BB}}(\mathbf{S})\rangle} \) to satisfy the normalization \( \langle \Phi_{\text{BB}}^J(\mathbf{S}) | \Phi_{\text{BB}}^J(\mathbf{S}) \rangle = 1 \).

The GCM wave function for the \( J^\pi \) state is given by the linear combination of the projected BB wave functions,

\[
|\Phi_{\text{GCM}}\rangle = \sum_k c_k |\Phi_{\text{BB}}^J(\mathbf{S}_k)\rangle.
\]  

(3)

Coefficients \( c_k \) are determined by solving the discretized Hill-Wheeler equation which is equivalent to the diagonalization of the norm and Hamiltonian matrices. Here, the cluster-GCM wave function \( \Phi_{\text{GCM}} \) is normalized as \( \langle \Phi_{\text{GCM}} | \Phi_{\text{GCM}} \rangle = 1 \).

### B. Inter-cluster wave function and antisymmetrization effect

In \(|\Phi_{\text{BB}}(\mathbf{S})\rangle\), the relative wave function between clusters is written by a localized Gaussian wave packet as,

\[
|\Phi_{\text{BB}}(\mathbf{S})\rangle = \frac{1}{\sqrt{A!}} A \{ \Gamma(\mathbf{r}, \mathbf{S}, \gamma) \phi(C_1) \phi(C_2) \phi_{\text{c.m.}} \},
\]  

(4)

\[
\Gamma(\mathbf{S}, \gamma; \mathbf{r}) = \left( \frac{2\gamma}{\pi} \right)^{3/4} e^{-\gamma (r^2 + S^2)},
\]  

(5)

\[
\gamma \equiv \frac{A_1 A_2}{A} \frac{1}{2b^2},
\]  

(6)

\[
\phi_{\text{c.m.}} = \left( \frac{A}{\pi b^2} \right) e^{-\frac{A}{b^2} r_C^2}.
\]  

(7)

\( \mathbf{r} \) is the relative coordinate between mass centers of clusters, \( \phi(C_1) \) and \( \phi(C_2) \) are internal wave functions of clusters, \( r_C \) is the coordinate of the center of the total mass, and \( \phi_{\text{c.m.}} \) is the wave function of the center of total mass motion. \( b \) is the width parameter of the H.O. for two clusters.

With the partial wave expansion of \( \Gamma(\mathbf{S}, \gamma; \mathbf{r}) \), the \( J^\pi \)-projected BB wave function \(|\Phi_{\text{BB}}^{J\pi}(\mathbf{S})\rangle\) for \( \mathbf{S} = (0, 0, S) \) is rewritten,

\[
|\Phi_{\text{BB}}^{J\pi}(\mathbf{S})\rangle = \frac{1}{\sqrt{A!}} A \{ \chi^{\text{BB}}_l(S; r) Y_{00}(\hat{r}) \phi(C_1) \phi(C_2) \phi_{\text{c.m.}} \},
\]  

(8)

\[
\chi^{\text{BB}}_l(S; r) = N_l(S) \sqrt{\frac{2l + 1}{4\pi}} \Gamma_l(S, \gamma; r),
\]  

(9)

\[
\Gamma_l(S, \gamma; r) \equiv 4\pi \left( \frac{2\gamma}{\pi} \right)^{2l} i_l(2\gamma S r) e^{-\gamma (r^2 + S^2)},
\]  

(10)

where \( i_l \) is the modified spherical Bessel function. \( l \) equals to \( J \) because two clusters are spinless. \( \chi^{\text{BB}}_l(S; r) \) is the radial part of the \( l \)-wave relative wave function in \(|\Phi_{\text{BB}}^{J\pi}(\mathbf{S})\rangle\) before the antisymmetrization.

In the GCM wave function, the radial part \( \chi^{\text{GCM}}_l(r) \) of the \( l \)-wave relative wave function is given by the linear combination of \( \chi^{\text{BB}}_l(S; r) \),

\[
|\Phi_{\text{GCM}}\rangle = \sum_k c_k |\Phi_{\text{BB}}^{J\pi}(\mathbf{S}_k)\rangle = \frac{1}{\sqrt{A!}} A \left[ \chi^{\text{GCM}}_l(r) Y_{00}(\hat{r}) \phi(C_1) \phi(C_2) \phi_{\text{c.m.}} \right],
\]  

(12)

\[
\chi^{\text{GCM}}_l(r) = \sum_k c_k \chi^{\text{BB}}_l(S_k; r) = \sum_k c_k \sqrt{\frac{2l + 1}{4\pi}} \Gamma_l(S_k, \gamma; r).
\]  

(13)
It means that the relative wave function $\chi_i(r)$ in the general $C_1+C_2$ cluster wave function

$$|\Phi\rangle = \frac{1}{\sqrt{A!}} A \left[ \chi_i(r) Y_{l0}(r) \phi(C_1) \phi(C_2) \phi_{c.m.} \right], \quad (14)$$

is represented by the expansion of the function $\Gamma_i(S_k, \gamma; r)$ with various $S_k$ values in the GCM framework, and the coefficients are determined so as to minimize the energy of $|\Phi\rangle$.

The cluster wave function $\chi_i(r)$ before the antisymmetrization usually contains Pauli forbidden states of the inter-cluster motion which vanish after the antisymmetrization of nucleons between $C_1$ and $C_2$ clusters. Such forbidden states have no physical meaning in the total $C_1+C_2$ system, and therefore, in discussion of $\alpha$ cluster wave functions in physical states we should extract physical component of the cluster wave function by eliminating unphysical forbidden states. For this aim, we use the antisymmetrized relative wave function $u_1(r)$ defined as follows,

$$\chi_i(r) = \sum_n a_n R_{nl}(b_r; r), \quad (15)$$

$$a_n = \int r^2 dr R_{nl}(b_r; r) \chi_i(r), \quad (16)$$

$$u_1(r) = \sum_n a_n \sqrt{\mu_{nl}} R_{nl}(b_r; r), \quad (17)$$

where $R_{nl}(b_r; r)$ is the radial wave functions of H.O. with the width parameter $b_r = 1/\sqrt{2\gamma} = \sqrt{A/A_1A_2}b$ and $\mu_{nl}$ is the eigen value of the RGM norm kernel [14]. $u_1(r)$ does not contain forbidden states, and it is normalized for the normalized total wave function $\langle \Phi | \Phi \rangle = 1$ as,

$$\int |u_1(r)|^2 r^2 dr = 1. \quad (18)$$

It should be noted that $u_1(r)$ is not the so-called reduced width amplitude as it is defined by the weight $\sqrt{\mu_{nl}}$ instead of the weight $\mu_{nl}$. We regard the function $u_1(r)$ as the physical relative wave function, i.e., the radial part of inter-cluster wave function in the physical component, because it is free from forbidden states and it satisfies the normalization which is essential for the interpretation of the $\alpha$ cluster probability. Moreover, the squared overlap between two normalized wave functions $|\Phi\rangle$ and $|\Phi'\rangle$ for a $C_1+C_2$ cluster system equals to the squared overlap between the antisymmetrized relative wave functions $u_1(r)$ and $u_1'(r)$ for $|\Phi\rangle$ and $|\Phi'\rangle$,

$$|\langle \Phi | \Phi' \rangle|^2 = |\langle u_1(r) | u_1'(r) \rangle|^2. \quad (19)$$

Here we define

$$\langle f(r) | g(r) \rangle \equiv \int f^*(r) g(r) r^2 dr \quad (20)$$

for given functions $f(r)$ and $g(r)$. Then, if we have the exact wave function $|\Phi\rangle$ and an approximated wave function $|\Phi'\rangle$, the accuracy of the approximated wave function can be judged by the squared overlap between the relative wave functions $u_1(r)$ and $u_1'(r)$ for $|\Phi\rangle$ and $|\Phi'\rangle$.

The original function $\chi_i(r)$ before the antisymmetrization and the corresponding antisymmetrized relative wave function $u_1(r)$ have the same asymptotic behavior in the large $r$ region where the antisymmetrization effect between clusters vanishes while they are different in the inner region where clusters largely overlap with each other and the relative wave function $u_1(r)$ is strongly affected by the antisymmetrization effect.

### III. DESCRIPTIONS OF INTER-CLUSTER WAVE FUNCTION

#### A. Description with trial functions and tail behavior

By performing the GCM calculation with enough number basis wave functions given by BB wave functions, we obtained the $C_1+C_2$ cluster wave functions for $J^\pi$ states of $^8$Be and $^{20}$Ne. The obtained GCM wave function $|\Phi_{GCM}\rangle$ is considered to be the exact solution in the full model space of $C_1+C_2$ clusters.

In the GCM calculation, the wave function is expressed by the linear combination of BB wave functions. The superposition of THSR wave functions proposed in Refs. [5, 7, 9, 10] is an alternative choice of basis wave functions.
For a spinless two-body cluster system, which can be reduced to one-dimensional problem, the superposition of THSR wave functions is equivalent to that of BB wave functions because both of them cover the full model space of $C_1+C_2$ cluster states in principle. Namely, the relative wave function is expressed by the linear combination of shifted Gaussian functions with the fixed range in the former case, and it is given by the linear combination of various range Gaussians (the multirange Gaussian) around the origin in the latter case.

If the cluster state can be approximated well by a single basis wave function, one may consider that the basis wave function reflects important character of the inter-cluster motion. In Refs. [6, 8–10], it was shown that the single dTHSR wave function gives pretty good description of $\alpha$ cluster states of $^8$Be and $^{20}$Ne rather than the single BB wave function, and the container picture was proposed that the $\alpha$ cluster is delocalized distributing whole the system.

However, one should take care over the physical meaning of cluster wave function because the original relative wave function $\chi(r)$ before the antisymmetrization contains unphysical forbidden states. Usually, the microscopic wave function of the total system is not sensitive to the inner part of $\chi(r)$ because of the strong antisymmetrization effect. Therefore, it is difficult to judge which trial function is best for the inner region. Physical properties of $\alpha$ cluster states are characterized by the enhanced $\alpha$-cluster probability at the surface and the $\alpha$-cluster tail in the outer region of the physical relative wave function $u_i(r)$ after the antisymmetrization. The outer tail originates in the quantum penetration through the effective barrier and its asymptotic behavior is well defined by the $\alpha$ separation energy $|E_r|$ ($E_r$ is the energy measured from the $\alpha$ threshold energy). In the ideal cluster system without cluster breaking, low-$l$ states with the small separation energy and the low Coulomb barrier should have the enhanced $\alpha$-cluster tail, which can be interpreted as almost zero-energy free $\alpha$ gas. It means that the delocalization of the $\alpha$ cluster in the outer region is an obvious consequence of the quantum penetration in the weakly bound low-$l\alpha$ cluster state in light systems such as $^{20}$Ne and $^8$Be with the low Coulomb barrier.

To discuss the feature of $\alpha$ cluster motion in the physical region, we analyze the antisymmetrized relative wave function $u_i(r)$ and discuss its behavior in three parts, the inner part, the surface peak, and the outer tail. We examine the features of $u_i(r)$ and also those of the original function $\chi_i(r)$ before the antisymmetrization of trial functions. The inner part of $u_i(r)$ is less sensitive to the original relative wave function $\chi_i(r)$ because of the strong antisymmetrization effect, while the outer tail of $u_i(r)$ directly reflects the tail behavior of the original function $\chi_i(r)$. Particular attention should be paid on the tail behavior to discuss the delocalization of $\alpha$ cluster in weakly bound cluster states.

As already mentioned, spinless two-body cluster systems such as $^{16}$O+$\alpha$ and $\alpha+\alpha$ can be reduced to the one dimension problem to describe the relative wave function $\chi_i(r)$. The squared overlap of the total wave function is given by the squared overlap of the antisymmetrized relative wave function $u_i(r)$ as Eq. 19. In the present work, we consider several kinds of trial functions for $\chi_i(a_1, a_2; r)$ specified by one or at most two adjustable parameters $a_1$ and $a_2$ and examine how accurately the trial function can reproduce the exact solution $u_i^{GCM}(r)$. For the criterion of the accuracy, we adopt the squared overlap,

$$N_{\overline{a}}^{over}(a_1, a_2) = |\langle u_i(a_1, a_2; r)|u_i^{GCM}(r)\rangle|^2.$$

If the total wave function given by the trial model function $\chi_i(a_1, a_2; r)$ is 100% equivalent to the exact GCM wave function, the corresponding $u_i(a_1, a_2; r)$ has 100% overlap with the exact solution $u_i^{GCM}(r)$ as $N_{\overline{a}}^{over}(a_1, a_2) = 1$.

Based on the criterion, we determine the optimum parameters $a_1$ and $a_2$ for the trial model wave function solving the mathematical problem to maximize the squared overlap $N_{\overline{a}}^{over}(a_1, a_2)$. With the maximum value $N_{\overline{a}}^{max}$ of the squared overlap, we can discuss the accuracy of the model wave functions.

Since the delocalization of the $\alpha$ cluster in the weakly bound cluster state is characterized by the outer tail, the description of the tail behavior of $u_i^{GCM}(r)$ with the trial model function is crucial in the accurate reproduction of the exact solution. For the quantitative discussion of the tail behavior, we analyze the curvature $C_r(r)$ of $ru_i(r)$ which corresponds to the radial term of the local kinetic energy defined as,

$$C_r(r) \equiv \frac{\hbar^2}{2\mu r u_i(r)} \frac{d^2(r u_i(r))}{dr^2}.$$

We call $C_r(r)$ the radial curvature. In the enough large $r$ region free from the nucleus-nucleus potential, $C_r(r)$ for the exact wave function $ru_i^{GCM}(r)$ approaches the asymptotic solution,

$$C_r(r) = \frac{\hbar^2}{2\mu} \frac{l(l+1)}{r^2} + Z_1Z_2 e^2 \frac{2}{r} - E_r.$$

When we omit the $r$-dependence of the Coulomb potential and consider the $l = 0$ bound state, $C_r(r)$ is approximately constant in the asymptotic region and the tail part of $u_i^{GCM}(r)$ is given by the Yukawa function $e^{-\kappa r}/r$. More generally, the value of the radial curvature $C_r(r)$ in the outer region relates to the local damping factor of the tail, that is, small (large) $C_r(r)$ means the slow (rapid) damping of the $\alpha$-cluster tail.
The delocalization of the \( \alpha \) cluster originates in the quantum penetration though the effectively low barrier and it is characterized by the slowly damping long tail. In good reproduction of the exact solution for the weakly bound cluster state with a trial function, the original trial function \( \chi_l(a_1, a_2; r) \) before the antisymmetrization should be able to describe the correct tail behavior of the exact solution. By analyzing the radial curvature \( C_r(r) \) of \( r\chi_l(r) \) and \( ru_r(r) \) before and after the antisymmetrization, respectively, we show how well trial functions can describe the tail part of the exact \( \alpha \)-cluster wave function.

For the trial functions, we consider a function projected from a shifted spherical Gaussian function and that from a deformed Gaussian around the origin. The former is a model function which contains relative wave functions of the BB wave function and the sTHSR wave function. The latter, the deformed Gaussian, corresponds to the dTHSR wave function. We also consider a trial function with a Yukawa tail (YT). In the later sections, we analyze relative wave functions as well as the radial curvature of trial functions comparing those of the exact solution. We explain the reason why the dTHSR wave function gives good description than the BB wave function for wave functions and radial curvature of trial functions comparing those of the exact solution. We also demonstrate that the YT function can well reproduce the exact solution equivalently to or even better than the dTHSR wave function.

### B. Shifted spherical Gaussian

As a trial function for the inter-cluster wave function \( \chi(r) \) in the \( C_1 + C_2 \) cluster wave function

\[
|\Psi\rangle = \frac{1}{\sqrt{A_1!A_2!}} \Lambda \{ \chi(r) \phi(C_1) \phi(C_2) \},
\]

we adopt the shifted spherical Gaussian (ssG)

\[
\chi(r) = e^{- \frac{(r-S)^2}{\sigma^2}}. \tag{25}
\]

The Gaussian center position \( S \) is chosen to be \( S = (0, 0, S) \). Then using the partial wave expansion of \( \chi(r) \), the \( l \)-wave relative wave function of the \( J^\pi = l^{(-1)^l} \) state projected from \( \Psi \) is written as

\[
\chi_{l}^{\text{ssG}}(S, \sigma; r) \propto i_l \left( \frac{2Sr}{\sigma^2} \right) e^{- \frac{r^2 + \sigma^2}{\sigma^2}}. \tag{26}
\]

For \( \chi_{l}^{\text{ssG}}(S, \sigma; r) \), the antisymmetrized relative wave function \( u_{l}^{\text{ssG}}(S, \sigma; r) \) is defined by the relation 17. The normalization is chosen to be \( \langle u_{l}^{\text{ssG}}(r) | u_{l}^{\text{ssG}}(r) \rangle = 1 \).

The shifted spherical Gaussian is parametrized by \( S \) for the Gaussian center position and \( \sigma \) for the Gaussian range. The parameters \( S \) and \( \sigma \) are optimized so as to maximize the squared overlap \( \langle |u_{l}^{\text{ssG}}(S, \sigma; r) | u_{l}^{\text{GCM}}(r) \rangle^2 \). The wave function is equivalent to the spherical case \( \beta_x = \beta_y = \beta_z \equiv \beta \) of the hybrid THSR wave function proposed in Ref. [10], and present parameters correspond to \( S = S_z \) and \( \sigma \equiv \sqrt{A/A_1A_2\sqrt{b^2 + 2\beta^2}} \) with the parameters \( S_z \) and \( \beta \) defined in Ref. [10].

#### 1. Brink-Bloch wave function

The BB wave function corresponds to the shifted spherical Gaussian with the fixed Gaussian range \( \sigma_{\text{fix}} = 1/\sqrt{7} = \sqrt{A/A_1A_2b} \). \( \chi_{l}^{\text{BB}}(S; r) \) and \( u_{l}^{\text{BB}}(S; r) \) equal to \( \chi_{l}^{\text{ssG}}(S, \sigma_{\text{fix}}; r) \) and \( u_{l}^{\text{ssG}}(S, \sigma_{\text{fix}}; r) \), respectively. The relative wave function in the BB wave function is specified by the parameter \( S \) for the Gaussian center position. The parameter \( S \) for the optimum BB wave function is determined so as to maximize the squared overlap \( \langle u_{l}^{\text{BB}}(S; r) | u_{l}^{\text{GCM}}(r) \rangle^2 \).

\( \sigma_{\text{fix}} \) is comparable to or even smaller than the \( \alpha \) cluster size \( b \). Since the width is fixed to be \( \sigma_{\text{fix}} \), the relative wave function \( \chi_{l}^{\text{BB}}(r) \) of the BB wave function is localized around \( r = S \). The radial curvature \( C_r(r) \) of \( r\chi_{l}^{\text{BB}}(r) \) is roughly estimated to be

\[
C_r(r) \approx \frac{\hbar^2}{\mu} \left( \frac{2(r-S)^2}{\sigma_{\text{fix}}^2} - \frac{3}{\sigma_{\text{fix}}^2} + \frac{2S}{r} \right), \tag{27}
\]

approximating \( \chi_{l}^{\text{BB}}(r) \) by the Gaussian function \( \exp(- \frac{(r-S)^2}{\sigma_{\text{fix}}^2}) \) because it is the function \( l \)-projected from \( \exp(- \frac{(r-S)^2}{\sigma^2}) \). In the tail region \( r > s \), \( C_r(r) \) increases rapidly reflecting the rapid damping tail of \( \chi_{l}^{\text{BB}}(r) \) because of the small range \( \sigma_{\text{fix}} \). It is clear that the BB function is not suitable to describe a slowly damping tail.
2. sTHSR wave function: \( r' \)-weighted spherical Gaussian function

When we take the \( S \to 0 \) limit of the shifted spherical Gaussian, the relative function goes to the \( r' \)-weighted Gaussian around the origin,

\[
\lim_{S \to 0} \chi^G(S, \sigma; r) \propto r' e^{-\frac{r'^2}{2\sigma^2}}. \tag{28}
\]

For even \( l \) states, this is equivalent to the spherical limit \( \beta_\perp \to \beta_z \) (\( \beta_x = \beta_y = \beta_\perp \)) of the deformed THSR wave function used for Be and Ne in Refs. [6, 9]. For odd \( l \) states, it is the spherical case of the zero limit intercluster distance parameter of the hybrid THSR wave function in Ref. [10]. We call this trial function the "spherical THSR" (sTHSR) in this paper,

\[
\chi^\text{sTHSR}(\sigma; r) \propto r' e^{-\frac{r'^2}{2\sigma^2}}. \tag{29}
\]

The sTHSR wave function is parametrized by the Gaussian range \( \sigma \). Strictly speaking, \( \sigma \) should be \( \sigma \geq \sqrt{2A/A_1A_2b} \) in the sTHSR wave function because of the correspondence \( \sigma^2 = 2A(b^2 + 2\beta^2)/A_1A_2 \). In the present work, \( \sigma \) is optimized so as to maximize the squared overlap \( \langle u^\text{sTHSR}_l(\sigma; r)|u^\text{GCM}_l(r)\rangle^2 \) for the normalized wave function as \( \langle u^\text{sTHSR}_l(\sigma; r)|u^\text{sTHSR}_l(\sigma; r)\rangle = 1 \).

The radial curvature \( C_r(r) \) of \( r\chi^\text{sTHSR}_l(\sigma; r) \) is trivial because the \( r' \)-weighted Gaussian is the lowest solution for the \( l \) state in the H.O. potential,

\[
C_r(r) = \frac{\hbar^2}{\mu} \left( \frac{2r^2}{\sigma^4} - \frac{2l + 3}{\sigma^2} \right) + \frac{\hbar^2}{2\mu} \frac{l(l+1)}{r^2} = \frac{\hbar^2}{2\mu} \frac{l(l+1)}{r^2} - \hbar \omega \left( l + \frac{3}{2} \right) + \frac{1}{2} \mu \omega^2 r^2, \tag{30}
\]

\[
\omega = \frac{2\hbar}{\mu \sigma^2}. \tag{31}
\]

In the outer region where the \( 1/r^2 \) term is negligible and the \( r^2 \) term is dominant, \( C_r(r) \) increases quadratically and crosses the \( C_r(r) = 0 \) line around \( r = \sqrt{l + \frac{3}{2}} \sigma \). Because the width \( \sigma \) is the adjustable parameter, the sTHSR can be a better function to describe the outer tail of the relative wave function than the BB wave function with the fixed range \( \sigma_{\text{fix}} \). However, since its curvature \( C_r(r) \) contains the quadratic term, it is difficult to perfectly reproduce a Yukawa-like long tail. Namely, the damping behavior of Gaussian tail is inconsistent with the Yukawa-like tail. It is the mathematically consequence of Gaussian function. It indicates that some improvement is necessary in the tail part of trial functions for better agreement to the exact solution.

C. Deformed Gaussian function: deformed THSR wave function

Another extension of the Gaussian function is the axial symmetric deformed Gaussian (dG) function around the origin,

\[
\chi^\text{dG}(\sigma_\perp, \sigma_z; r) \propto \exp\left( -\frac{x^2}{\sigma_\perp^2} - \frac{y^2}{\sigma_\perp^2} - \frac{z^2}{\sigma_z^2} \right) \tag{32}
\]

\[
= \exp\left( -\frac{r^2}{\sigma_\perp^2} + \frac{r^2}{\Delta} \cos \theta^2 \right) \tag{33}
\]

\[
\frac{1}{\Delta} = \frac{1}{\sigma_\perp^2} - \frac{1}{\sigma_z^2}. \tag{34}
\]

The relative wave function \( \chi_l(r) \) of the even \( l \) wave is given as

\[
\chi^\text{dG}_l(\sigma_\perp, \sigma_z; r) \propto \int Y_{l0}(\hat{r}) \chi^\text{dG}(r) d\Omega
= \sqrt{(2l+1)\pi} \int_0^\pi \exp\left( -\frac{r^2}{\sigma_\perp^2} + \frac{r^2}{\Delta} \cos \theta^2 \right) P_l(\cos \theta) \sin \theta d\theta
= 2\sqrt{(2l+1)\pi} \exp\left( -\frac{r^2}{\sigma_\perp^2} \right) \int_0^1 P_l(t) \exp\left( \frac{t^2}{\Delta} \right) dt, \tag{35}
\]
where $P_l(t)$ is the Legendre polynomial. For odd $l$ state, we adopt the axial symmetric deformed Gaussian function infinitesimally shifted to $z$ direction from the origin,

$$
\chi^G_{\text{odd}}(\sigma_\perp, \sigma_z; r) \propto z \exp\left(-\frac{x^2}{\sigma^2_\perp} - \frac{y^2}{\sigma^2_\perp} - \frac{z^2}{\sigma^2_z}\right)
$$

$$
= r \cos \theta \exp\left(-\frac{r^2}{\Delta} + \frac{r^2 \cos \theta^2}{\Delta}\right).
$$

The relative wave function $\chi_l(r)$ for the odd $l$ wave is given as

$$
\chi^G_l(\sigma_\perp, \sigma_z; r) \propto \int Y_{l0}(\hat{r}) \chi^G_{\text{odd}}(r) d\Omega
$$

$$
= 2\sqrt{(2l+1)\pi} \exp\left(-\frac{r^2}{\sigma^2_\perp}\right) \int_0^1 P_l(t) t \exp\left(-\frac{t^2}{\Delta}\right) dt,
$$

where $P_l(t)$ is the Legendre polynomial. $\chi^G_l(\sigma_\perp, \sigma_z; r)$ is parametrized by the range parameters $\sigma_\perp$ and $\sigma_z$ which are optimized so as to maximize the squared overlap $|\langle u^G_l(\sigma_\perp, \sigma_z; r)|u^G_{\text{CM}}(r)\rangle|^2$ for the normalized wave function as $|\langle u^V_{\text{THSR}}(\sigma_\perp, \sigma_z; r)|u^G_{\text{THSR}}(\sigma_\perp, \sigma_z; r)\rangle| = 1$. The present deformed Gaussian wave function for even $l$ states corresponds to the zero limit of the intercluster distance parameter of the hybrid THSR proposed in Ref. [10]. The parameters $\beta_{2l}$ of the deformed THSR and the hybrid THSR relate to the present parameters $\sigma_\perp, \sigma_z$ as $\sigma^2_\perp = 2A(b^2 + 2\beta^2)A_1A_2$ and $\sigma^2_z = 2A(b^2 + 2\beta^2)/A_1A_2$. Because of these relations, the parameters $\sigma_\perp$ and $\sigma_z$ should be $\sigma_\perp \geq \sqrt{2A/A_1A_2b}$ and $\sigma_z \geq \sqrt{2A/A_1A_2b}$ in the deformed THSR. When we impose this condition, $\sigma_\perp, \sigma_z \geq \sqrt{2A/A_1A_2b}$, we call the deformed Gaussian wave function ”deformed THSR” (dTHSR).

In the spherical limit $\sigma_\perp \to \sigma_z$, $\chi^G_l(r)$ goes to the $S \to 0$ limit $\chi^G_l(r)$, and its radial curvature $C_r(r)$ becomes the quadratic form given in Eq. 30. In the other hand, in the case of the largely deformed Gaussian with $\sigma_z \gg \sigma_\perp$, the curvature $C_r(r)$ of $r\chi^G_l(r)$ can be small in the outer region. As explained in the appendix, for $\chi^G_l(r)$ with $\sigma_z \gg \sigma_\perp$, $C_r(r) \to 0$ in the $\sigma_\perp \ll r < \sigma_z$ region. It means that the deformed Gaussian should be a better trial function that can efficiently describe the slow damping behavior of long tail than the spherical Gaussian having the Gaussian tail.

D. Yukawa tail function

To describe the slow damping behavior of the $\alpha$-cluster tail in the outer region, we consider another trial function with a Yukawa tail in the outer region instead of Gaussian functions. To avoid the singularity of the Yukawa function in the small $r$ region, we introduce a Yukawa tail (YT) function by smearing the inner part of the Yukawa function and continuously connecting to the $r^l$ function in the $r \to 0$ limit which is the correct asymptotic behavior at $r \to 0$ of the regular wave function in the finite potential well as follows,

$$
\chi^Y_l(a_R, a_Y; r) \propto \left(\frac{r}{R(r)}\right)^l F^\text{yukawa} \left(\frac{R(r)}{a_Y}\right)
$$

$$
F^\text{yukawa}(x) = \frac{e^{-x}}{x}
$$

$$
R(r) = \frac{r}{\sqrt{1 - \exp(-r^2/a_R^2)}}
$$

Here $R(r)$ is the scaling function that approaches $R(r) = a_R$ in $r \to 0$ and it goes to $R(r) = r$ in the large $r$ region. Therefore, the function $\chi^Y_l(a_Y, a_R; r)$ has the Yukawa tail in the outer region. The damping behavior of the tail is characterized by the parameter $a_Y$. The parameter $a_R$ corresponds to the range for smearing the Yukawa function in the inner region. To describe the Yukawa tail in the outer region, the smearing range $a_R$ should be the same order of or smaller than the surface peak position of the $\alpha$-cluster wave function. For the optimum YT function, these two parameters $a_Y$ and $a_R$ are optimized so as to maximize the squared overlap $|\langle u^Y_l(a_Y, a_R; r)|u^G_{\text{CM}}(r)\rangle|^2$ for the normalized wave function as $|\langle u^Y_l(a_Y, a_R; r)|u^G_{\text{THSR}}(\sigma_\perp, \sigma_z; r)\rangle| = 1$.

It is clear that the curvature $C_r(r)$ of $r\chi^Y_l(\sigma, a; r)$ goes to constant in the outer tail part,

$$
C_r(r) = \frac{\hbar^2}{2\mu a_Y^2}
$$

(43)
Instead of Yukawa function, we can also consider an alternative function having the tail of modified spherical Hankel function \( H_l(r) \) as,

\[
\chi^{HT}_l(a_R, a_Y; r) \propto \left( \frac{r}{R(r)} \right)^l H_l \left( \frac{R(r)}{a_Y} \right),
\]

which has the curvature in the outer tail,

\[
C_r(r) = \frac{\hbar^2}{2\mu} \frac{l(l+1)}{r^2} + \frac{\hbar^2}{2\mu a_Y^2}.
\]

We checked this trial function and found that it gives almost the same quality as the YT function in fitting the exact solution \( u^GCM_l(r) \). In this paper, we show only the results of the YT function.

IV. RESULTS OF \(^8\)BE

A. GCM calculation of \(^8\)Be

For \(^8\)Be, we perform the GCM calculation of the 2\(\alpha\) cluster model. We use the Volkov No.1 with \( m = 0.56 \) and the width parameter \( b = 1.36 \) fm the same as the 2\(\alpha\) calculation in Ref. [6]. \( S_k \) is 0.5, 1.25, 2.0, \ldots, 20.0 fm are chosen for the basis BB wave functions in the GCM calculation. In the present calculation, two-body Coulomb force is approximated by the seven-range Gaussian. The calculated energy is chosen for the basis BB wave functions in the GCM calculation. In the present calculation, two-body Coulomb force is approximated by the seven-range Gaussian. The calculated energy is \( E_r = -0.32 \) MeV which slightly underestimates the experimental energy \( E_r = 0.092 \) MeV.

B. Squared overlap of trial functions with the GCM wave function of \(^8\)Be

We consider how well trial functions can describe the exact solution obtained by the GCM calculation. Trial functions are specified by one of two parameters. We vary the parameter(s) and search for the optimum parameter(s) which gives the maximum value of the squared overlap \( \mathcal{N}^{\text{over}} = |\langle u_l(r) | u^GCM_l(r) \rangle|^2 \) with the GCM wave function. For trial functions, we adopt the BB, sTHSR, dTHSR, and YT wave functions. The maximum values \( \mathcal{N}^{\text{over}} \) of the squared overlap and the optimized parameters are shown in table I. The squared overlap of \( R_{nl}(b_r; r) \) for the relative wave function of the SU(3) shell model (SM) limit is also shown.

The results for the BB, sTHSR, dTHSR wave functions are in principle the same as those discussed by Funaki et al. in Refs. [6, 8]. The description of the BB wave function is worse compared with the sTHSR wave function although it is much better than the SM wave function. The reason is that the BB wave function can describe the enhanced surface peak of the antisymmetrized relative wave function \( u_l^GCM(r) \) in \(^8\)Be\(\left( 0^+ \right) \) better than the SM one but it fails to describe the long outer tail. The sTHSR wave function can describe the outer tail part better than the BB wave function but it is not sufficient for the perfect description because a Gaussian tail is different from the correct asymptotic behavior of the relative wave function. On the other hand, the dTHSR wave function describes the GCM wave function almost perfectly as 99.99% squared overlap. This is because the outer tail behavior is described fairly well by the dTHSR as shown in Ref. [8].

It is found that the present YT wave function gives almost equal quality to the dTHSR in describing the GCM wave function. It indicates that the Yukawa-like tail is essential to reproduce the exact solution.

The main reason for the failure of the BB wave function in describing the GCM wave function is that the long tail is missing because the range \( \sigma \) of the shifted spherical Gaussian is fixed to be the small value \( \sigma_{\text{fix}} = 1.36 \) fm in the BB wave function. In Fig. 1, we show the squared overlap \( \mathcal{N}^{\text{over}} = |\langle u_l(r) | u^GCM_l(r) \rangle|^2 \) for the ssG wave function \( u^{\text{ssG}}_l(S, \sigma; r) \) as functions of the parameters \( S \) and \( \sigma \). The values on the \( \sigma = 1.36 \) fm line correspond to \( \mathcal{N}^{\text{over}} \) for the BB wave function with a given \( S \) value, while those on the \( S = 0 \) line are \( \mathcal{N}^{\text{over}} \) for the sTHSR wave function with a given \( \sigma \) value. There exists a wide plateau with \( \mathcal{N}^{\text{over}} \geq 95\% \) in the region \( \sigma = 4 \) to 5 fm and \( S = 0 \) to 4 fm. Even if the parameter \( S \) is fixed to \( S = 4 \) fm, the ssG function can have about 95% overlap with the GCM wave function by adjusting \( \sigma \). The optimized \( \sigma \) for \( S = 4 \) fm is \( \sigma = 3.5 \) fm which is much larger than the \( \sigma_{\text{fix}} \) in the BB wave function and can describe the outer tail of the GCM wave function reasonably. The maximum \( \mathcal{N}^{\text{over}} \) is given at \( S = 0 \) which is contained in the model space of the sTHSR function. This behavior is consistent with the argument for the delocalization of \( \alpha \) cluster in Refs. [8, 10]. It should be noted that the inner part of \( \chi_l(r) \) is less significant because the physical wave function is not so sensitive to the inner part of the original function \( \chi_l(r) \) before the antisymmetrization, while the tail part of \( \chi_l(r) \) is important relatively. In the optimized ssG wave function, the \( S = 0 \) is chosen for the
best fit of the slow damping feature of the outer long tail within the ssG model space. Nevertheless, the wide plateau from $S = 0$ fm to $S \sim 4$ fm at $\sigma = 4 \sim 5$ fm indicates less importance of the inner part but the particular importance of the long outer tail which should be regarded as the delocalization of cluster because of the quantum penetration.

![Graph](image)

**FIG. 1:** Squared overlap $\Lambda^\text{over} = |\langle u_l(r)|u_l^\text{GCM}(r)\rangle|^2$ for the ssG wave function $u_l^\text{ssG}(S, \sigma; r)$ as functions of the distance parameter $S$ and the range parameter $\sigma$. The data on the horizontal axis at $\sigma = 1.36$ fm correspond to $\Lambda^\text{over}$ for the BB wave function with a given $S$ value, while those on the vertical axis at $S = 0$ are $\Lambda^\text{over}$ for the sTHSR function with a given $\sigma$ value. Solid lines indicate contour for 90%, 80%, 70%, · · ·. Between the solid lines, thin dotted lines are drawn at 2.5% intervals.

**TABLE I:** Maximum values $\Lambda^\text{over}_{\text{max}}$ (%) of the squared overlap $\Lambda^\text{over} = |\langle u_l(r)|u_l^\text{GCM}(r)\rangle|^2$ and the optimized parameters (fm) for the trial functions, BB, sTHSR, dTHSR, and YT wave functions for $^8\text{Be}(0^+_1)$. The squared overlap $\Lambda^\text{over}$ of $R_{20}(b_r; r)$ for the SU(3) shell model limit with $u_l^\text{GCM}(r)$ is also shown.

|          | BB   | sTHSR | dTHSR | YT   | SM   |
|----------|------|-------|-------|------|------|
| $\Lambda^\text{over}_{\text{max}}(S)$ | 77.27(4.01) | 97.29(4.77) | 99.99(2.88,11.06) | 99.98(2.02,3.27) | 21.79 |
| $\Lambda^\text{over}_{\text{max}}(\sigma)$ |          |       |       |      |      |
| $\Lambda^\text{over}_{\text{max}}(\sigma_\perp, \sigma_z)$ |          |       |       |      |      |
| $\Lambda^\text{over}_{\text{max}}(a_R, a_Y)$ |          |       |       |      |      |

**C. Analysis of $\alpha$-$\alpha$ intercluster wave functions**

We analyze the antisymmetrized relative wave functions $u_l(r)$ as well as the non-antisymmetrized ones $\chi_l(r)$ of the trial functions comparing them with the exact solution $u_l^\text{GCM}(r)$ obtained by the GCM calculation.

The antisymmetrized relative wave functions $ru_l(r)$ of the optimized trial wave functions are shown in Fig. 2. The relative wave function for the SU(3) SM limit given by the function $rR_{20}(b_r; r)$ is also shown in the figure for comparison. $ru_l^\text{GCM}(r)$ of the GCM wave function is characterized by three parts, the inner part, the surface peak, and the outer tail. The inner part has the oscillating nodal structure because of the antisymmetrization effect. Compared with the shell model limit case, the inner part is suppressed, while the surface peak is enhanced and shifted toward outward in $ru_l^\text{GCM}(r)$. Moreover, $ru_l^\text{GCM}(r)$ has the long tail in the outer region because of the quantum penetration. The tail part is damping very slowly because of the small separation energy $|E_r| = 0.32$ MeV in $^8\text{Be}(0^+_1)$. As a result, the tail component contributes to the significant fraction of the total probability $|\langle u_l^\text{GCM}(r)|u_l^\text{GCM}(r)\rangle| = 1$.

The original relative wave functions $r\chi_l(r)$ before the antisymmetrization of the optimized trial functions are shown compared with the antisymmetrized relative wave functions $ru_l(r)$ after the antisymmetrization in Fig. 3. In the figure, the scaled functions $r\chi_l^\text{sc}(r) \equiv r\chi_l(r)/\sqrt{\langle \chi_l(r)|\chi_l(r)\rangle}$ and $ru_l^\text{sc}(r) \equiv ru_l(r)/\sqrt{\langle \chi_l(r)|\chi_l(r)\rangle}$ are shown as well as the normalized wave function $ru_l(r)$. The structures of the inner nodal oscillation and the surface peak in $ru_l(r)$ are not so sensitive to $r\chi_l(r)$ because of the antisymmetrization effect. On the other hand, in the region $r > 4$ fm, the $ru_l^\text{sc}(r)$ almost agrees to $r\chi_l^\text{sc}(r)$ because the antisymmetrization effect vanishes in the outer region. This means that the shape of the tail part of $ru_l(r)$ is directly determined by the shape the original function $r\chi_l(r)$.

In the comparison of the BB wave function with the GCM wave function, it is found that the surface peak structure as well as the suppressed inner part are described reasonably but the outer long tail is missing in the BB wave function.
because of the rapid damping of the Gaussian with the fixed range $\sigma_{\text{fix}}$. The description of the outer tail is drastically improved by the sTHSR wave function in which the Gaussian range $\sigma$ is the adjustable parameter. However, the Gaussian shape of the sTHSR wave function is insufficient to describe the detailed tail behavior. The dTHSR and YT wave functions show almost the perfect agreement to $\nu_u^{\text{GCM}}(r)$. The success of the YT wave function indicates that the Yukawa-type tail of the YT function is suitable to reproduce the outer long tail rather than the Gaussian tail.

As shown in Table I, the squared overlap $\Lambda_{\text{over}}$ of the trial functions with the GCM wave function is much small as about 20% in the case of SU(3) shell model. It remarkably increases in the BB wave function as $\Lambda_{\text{max}} \sim 80\%$. It means that 80% of the wave function can be reproduced by describing the inner suppression and the enhanced surface peak while the description of the outer tail behavior is essential for the remaining 20% accuracy.

The relative wave function $\chi(r)$ before the antisymmetrization contains unphysical forbidden states, and therefore, one should care about that the inner part of $\chi(r)$ has no or less physical meaning. Indeed, the physical relative wave function $u(r)$ after the antisymmetrization does not depend so much on the detailed behavior of the inner part of $\chi(r)$. In Fig. 4, we demonstrate that almost the same $u(r)$ can be obtained from various $\chi(r)$ having different inner structures. In the figure, we show $\chi(r)$ for the dTHSR and YT wave functions. Although both of them give almost the same $u(r)$ functions almost equivalent to the exact solution $u^{\text{GCM}}(r)$, some difference is shown in $\chi(r)$ in the $r < 2$ fm region. We also show a modified relative wave function by subtracting $R_{00}(b_i;0)$ for the forbidden state from $\chi^{\text{dTHSR}}(r)$ by hand as $\chi^{\text{dTHSR}}(r) = \chi^{\text{dTHSR}}(r) - (\chi^{\text{dTHSR}}(0)/R_{00}(b_i;0))R_{00}(b_i;r)$ (labeled dTHSR* in the figure). The modified function gives the physical relative wave function $u(r)$ completely same as that of $\chi^{\text{dTHSR}}(r)$. We also show the $\chi^{\text{GCM}}(r)$ before the antisymmetrization of the GCM wave function. It is found that the inner part of $\chi(r)$ does not affect the physical state.

Thus, in the inner region it is in principle difficult to discuss the localization or delocalization of cluster because of the insensitivity to the original trial function. On the other hand, the cluster structure is characterized by the enhanced surface peak and the outer tail. The localization and delocalization of cluster can be definitely distinguished by the damping behavior of the outer tail, which is caused by the quantum penetration. Because of the small $\alpha$ separation energy of $^{8}\text{Be}(0^+_1)$, $u^{\text{GCM}}(r)$ has the slowly damping tail indicating the delocalization of cluster.

To discuss the damping behavior of the outer tail more quantitatively, we analyze the radial curvature $C_r(r)$ of $r\nu_u(r)$ and $r\chi(r)$ defined in Eq. 22. In Fig. 5, the $r$ dependence of $C_r(r)$ of $r\nu_u(r)$ for the optimized trial function is compared with that for the GCM wave function. $C_r(r)$ of $r\chi(r)$ is also shown in Fig. 6.

As explained before, in the outer region where the nucleus-nucleus interaction vanishes, the radial curvature $C_r(r)$ of the exact solution is well defined by the centrifugal barrier, the Coulomb barrier, and the constant value $-E_\alpha$ of the $\alpha$ separation energy as given in Eq. 23. For the GCM wave function of $^{8}\text{Be}(0^+_1)$, $C_r(r)$ is small and almost flat in the the $r > 5$ fm region because of the small separation energy $|E_\alpha|$ and the small Coulomb barrier and no centrifugal barrier. For trial functions, $C_r(r)$ of $r\nu_u(r)$ in $r > 5$ fm region directly reflects $C_r(r)$ of the original $r\chi(r)$ which is simply given by the form of the model function. The BB wave function has the steep $r$ dependence of $C_r(r)$ and can not describe the flat behavior of correct $C_r(r)$ of the GCM wave function. The sTHSR wave function gives better results than the BB wave function. However, the $C_r(r)$ of the Gaussian tail in the sTHSR is given by the Harmonic oscillator potential as shown in Eq. 30, it is different from the flat behavior of the correct $C_r(r)$. On the other hand, the dTHSR and the YT wave functions can reproduce well the $r$ dependence of the correct $C_r(r)$. These results indicate that the slow damping of the outer tail of the GCM wave function is approximately described by the Yukawa tail rather than Gaussian tail. The optimized dTHSR wave function mathematically has the slowly damping tail similar to the Yukawa tail.

The $r$ dependence of the radial curvature of $C_r(r)$ in the outer region is trivial for the BB, sTHSR, and YT wave functions as explained in the previous section, while that for the dTHSR wave function is not trivial. We show, in Fig. 7, $C_r(r)$ of $r\chi(r)$ for trial functions with various parameter sets. In the BB wave function, the function $C_r(r)$ is shifted with the change of the parameter $S$ while keeping the shape almost unchanged. In the case of the sTHSR wave function, $C_r(r)$ is given by the H.O. potential with the frequency $\omega = 2\hbar/\mu\sigma^2$ as shown in Eq. 30. The size of H.O., i.e., the slope of $C_r(r)$ depends on $\sigma$, and $C_r(r)$ crosses the $C_r(r) = 0$ line around $r = \sqrt{1 + 3/2\sigma}$. Certainly, it is not be able to adjust the slope and the crossing point independently. As seen in Fig. 5, the optimized parameter $\sigma = 4.77$ fm shows the reasonable agreement to the correct $C_r(r)$ in the $r = 5$ fm region. However, $C_r(r)$ of the sTHSR is the increasing function and fails to describe the flat behavior even though the agreement is improved by the dTHSR with the large range parameter $\sigma$ than the BB wave function. $C_r(r)$ for the YT function is constant to be $h^2/2\mu a_Y^2$ in the region roughly larger than $\sim 2a_R$. With the range parameter $a_Y$ of the Yukawa tail, the constant value of the flat region of $C_r(r)$ can be freely adjusted. Indeed, with the optimized parameter $a_Y = 3.27$ fm, $C_r(r)$ of the GCM wave function in the tail region is reproduced well.

The $r$ dependence of $C_r(r)$ for the dTHSR is not trivial. In Fig. 7, we show $C_r(r)$ of the dTHSR of the prolately deformed case $\sigma_2 > \sigma_1$ with the fixed $\sigma_1$. In the deformed case, the slope of $C_r(r)$ changes around $r \sim 2\sigma_1$, and it is more gentle in the outer region than the inner region. As $\sigma_2$ increases, the gradient of $C_r(r)$ in the outer region
be noted that the having the large deformation, the becomes small and close to zero and it is in good agreement to the correct tail behavior of the GCM wave function having the small radial curvature because of the small α separation energy as well as the small Coulomb barrier and no centrifugal barrier. Such the slowly damping tail can be described well by the dTHSR wave function with the large deformation.

In the analysis of the relative wave function of ¹⁰Be, we can reach the following conclusions. The relative wave function between two α clusters in ¹⁰Be system is characterized by three parts, the oscillating inner part, the enhanced surface peak, and the outer tail. The inner part is suppressed while the surface peak is relatively enhanced. The nodal structure in the inner region and the enhanced peak structure at the surface in the physical wave function ru(σ) are not so sensitive to the original trial functions χi(r) because of the strong antisymmetrization effect between clusters. The outer tail is caused by the quantum penetration and its asymptotic behavior is well defined. Since ¹⁰Be(0⁺) is the weakly bound cluster state having the small α separation energy and the low Coulomb barrier and no centrifugal barrier, its wave function is slowly damping in the outer region, and the outer tail becomes the remarkably long tail.

To get a good approximation of the exact solution (GCM wave function) for such the weakly bound α-cluster state, it is essential to fit the outer tail part, in particular, its slow damping behavior. Since the inner and peak parts are mainly determined by the antisymmetrization effect and therefore it is relatively less important in the fitting. The outer tail part is the slowly damping function characterized by the obvious effective barrier given by the Coulomb force and the separation energy free from the nuclear interaction, and therefore, it is interpreted as almost zero-energy free α gas. This "free α-cluster gas part" in the outer region can be understood as the delocalization of cluster. It should be pointed out that the origin of the delocalization, i.e., the "free α-cluster gas part" in the outer region, is the quantum penetration and it is the natural consequence of the weak binding of the α cluster. In the asymptotic region where the nucleus-nucleus interaction vanishes, the damping behavior of the outer tail in this free gas region is well defined by the α separation energy. In the ideal case that the separation energy is small and the Coulomb and centrifugal barriers are not high, the dTHSR wave function can describe the exact wave function fairly well because it has the suitable form to fit the long tail in the outer free gas region. However, it should be stressed that the success of the dTHSR is the mathematical result of the fact that the dTHSR gives the form different from the Gaussian tail but rather similar to the Yukawa-type tail.

V. RESULTS OF ²⁰Ne

A. GCM calculation of ²⁰Ne

For ²⁰Ne, we perform the GCM calculation of the ¹⁶O+α cluster model. We use Volkov No.1 with m = 0.60 and the width parameter b = 1.46 fm the same as the ¹⁶O+α calculations in Refs. [9, 10]. Sₖ = 0.6, 1.2, ⋅⋅⋅, 12.0 fm are
FIG. 3: Relative wave functions $r\chi_l(r)$ before the antisymmetrization and $ru_l(r)$ after the antisymmetrization of the optimized trial functions for $^8$Be($0^+$). The scaled functions $r\chi^\text{sc}_l(r) \equiv r\chi_l(r)/\sqrt{\langle \chi_l(r)|\chi_l(r) \rangle}$ and $ru^\text{sc}_l(r) \equiv ru_l(r)/\sqrt{\langle \chi_l(r)|\chi_l(r) \rangle}$ are shown as well as $ru_l(r)$.

FIG. 4: Relative functions $\chi^\text{sc}_l(r)$ of the GCM wave function, and the optimized dTHSR and YT wave functions for $^8$Be($0^+$). The function $\chi_l(r) - (ch_i(0)/R_{00}(b_r;0))R_{00}(b_r;r)$ (dTHSR') modified from $\chi_l(r)$ of the dTHSR function is also shown.

chosen for the basis BB wave functions in the GCM calculation. In the present calculation, two-body Coulomb force is approximated by the seven-range Gaussian. For the $^{16}$O cluster wave function, we use the $4\alpha$ BB cluster wave function with an enough small inter-cluster distance which is equivalent to the $p$-shell closed H.O. configuration.

The energy of $0^+$, $2^+$, $4^+$, $6^+$, and $8^+$ states in the $K^\pi = 0^+$ band and that of $1^-$ and $3^-$ states in the $K^\pi = 0^-$ band are shown in table II. The calculated $6^+$, $8^+$, and $3^-$ states are obtained within the bound state approximation in the present GCM basis $S_k \leq 12$ fm though they are resonances above the $\alpha$-decay threshold energy. The present GCM calculation is in principle consistent with the "Brink GCM" calculation in Refs. [9, 10].
FIG. 5: Radial curvature $C_r(r)$ of the antisymmetrized relative wave function $ru(r)$ of the optimized trial functions for $^8\text{Be}(0^+_1)$ compared with that of the GCM wave function.

FIG. 6: Radial curvature $C_r(r)$ of the original relative wave function $r\chi_l(r)$ of the optimized trial functions for $^8\text{Be}(0^+_1)$.

B. Squared overlap of trial functions with the GCM wave function of $^{20}\text{Ne}$

In a similar way to the analysis of $^8\text{Be}$, we see how well trial functions can describe the exact solution of $^{16}\text{O}+\alpha$ cluster states obtained by the GCM calculation. For trial functions, we adopt the BB, sTHSR, dTHSR, and YT wave functions. In addition we use the deformed Gaussian "dG" wave function which has the same form of the dTHSR wave function but no restriction of $\sigma_{\perp, z} \geq \sqrt{A/A_1 A_2 b}$ differently from the dTHSR wave function.

The maximum values $\Lambda_{\text{max}}$ of the squared overlap $\Lambda_{\text{max}}^\text{over} = |\langle u_l(r)|u_l^{\text{GCM}}(r)\rangle|^2$ and the optimized parameters are shown in table III. The squared overlap of $u_l(r) = R_{nl}(b;r)$ in the SU(3) shell model limit with $u_l^{\text{GCM}}(r)$ is also shown. Here the node number $n$ is $2n + l = 8$ and $2n + l = 9$ for even and odd $l$ states, respectively.

The results for the BB and dTHSR wave functions are in principle the same as those discussed by Zhou et al.
FIG. 7: Radial curvature $C_r(r)$ of $r\chi_l(r)$ of the BB, sTHSR, dTHSR, and YT functions with various parameters. (a) $C_r(r)$ of the BB wave function with the parameters $S=3, 4, 5$ fm. (b) $C_r(r)$ of the sTHSR wave function with the parameters $\sigma=3, 5, 7, 9$ fm. (c) $C_r(r)$ of the dTHSR wave function with the parameters $(\sigma_x, \sigma_z)=(3, 3), (3, 5), (3, 7),$ and $(3, 11)$ fm. (d) $C_r(r)$ of the YT wave function with the parameters $(a_R, a_Y)=(2, 1), (2, 2),$ and $(2, 4)$ fm.

TABLE II: Energy of $^{20}$Ne calculated with the cluster GCM using Volkov No.1 $m=0.60$ and $b=1.46$ fm. The energy $E_r$ (MeV) is measured from the $\alpha$ threshold energy. The experimental energy of the states in $K^\pi=0^+_1$ and $K^\pi=0^-_1$ bands is also listed.

|         | $E_r$ (MeV) | GCM  | Exp. |
|---------|-------------|-------|------|
| $^{20}$Ne$(0^+_1)$ | $-5.92$ | $-4.73$ |
| $^{20}$Ne$(2^+_1)$ | $-4.71$ | $-3.1$ |
| $^{20}$Ne$(4^+_1)$ | $-1.90$ | $-0.48$ |
| $^{20}$Ne$(6^+_1)$ | $2.55$ | $4.05$ |
| $^{20}$Ne$(8^+_1)$ | $8.94$ | $7.22$ |
| $^{20}$Ne$(1^-_1)$ | $-1.24$ | $1.06$ |
| $^{20}$Ne$(3^-_1)$ | $1.07$ | $2.43$ |

in Refs. [9, 10]. The $\mathcal{N}_{\text{over}}$ of the BB wave function is much larger than $\mathcal{N}_{\text{over}}$ of the SM wave function because it describes the enhanced surface peak better than the SM. However the description of the GCM wave function is as much as $\mathcal{N}_{\text{over}}=90 \sim 95\%$ and it is not satisfactory because of the missing of the outer tail part in the BB wave function. The description is improved by the sTHSR wave function, in particular, for the weakly bound states such as $^{20}$Ne$(1^-_1)$ and $^{20}$Ne$(3^-_1)$. Further improvement is given by the dTHSR wave function; $\mathcal{N}_{\text{over}} > 99\%$ is obtained by the dTHSR for $^{20}$Ne$(0^+_1)$, $^{20}$Ne$(1^-_1)$, and $^{20}$Ne$(3^-_1)$ as already shown in Ref. [10]. However, the description is not perfect for $J^\pi=2^+, 4^+, 6^+$, and $8^+$ states and maximum overlap is as much as $\mathcal{N}_{\text{over}} < 99\%$.

On the other hand, the YT function gives fairly good results for all states with more than 99% accuracy except for $^{20}$Ne$(6^+_1)$. This indicates that the YT function is the better trial function to fit the GCM wave function. It is
TABLE III: Maximum values \( N_{\text{max}}^{\text{over}} \) (%) of the squared overlap \( N_{\text{over}} = |\langle u(r) | u_{\text{GCM}}^l(r) \rangle|^2 \) and optimized parameters (fm) for the trial functions of BB, sTHSR, dTHSR, dG, and YT wave functions for \( ^{20}\text{Ne} \). The squared overlap \( N_{\text{over}} \) for \( R_{\text{opt}}(l;r) \) of the SU(3) shell model (SM) wave function with \( u_{\text{GCM}}^l(r) \) is also shown.

|       | BB     | sTHSR | dTHSR | dG     | YT     | SM     |
|-------|--------|--------|--------|--------|--------|--------|
| \( ^{20}\text{Ne}(0^+_1) \) | 95.30(3.24) | 98.47(2.39) | 99.29(1.56, 3.00) | 99.67(0.81, 3.02) | 99.94(0.68, 0.83) | 41.04   |
| \( ^{20}\text{Ne}(2^+_1) \) | 93.32(3.13) | 97.46(2.04) | 98.80(1.15, 2.75) | 98.84(1.02, 2.77) | 99.46(1.40, 0.75) | 43.61   |
| \( ^{20}\text{Ne}(4^+_1) \) | 93.16(2.87) | 96.15(1.75) | 97.84(1.15, 2.39) | 99.25(0.77, 2.71) | 99.79(1.41, 0.76) | 50.08   |
| \( ^{20}\text{Ne}(6^+_1) \) | 93.72(2.41) | 95.33(1.50) | 96.69(1.15, 1.93) | 97.23(1.00, 2.07) | 98.34(2.04, 0.53) | 61.04   |
| \( ^{20}\text{Ne}(8^+_1) \) | 95.65(1.73) | 96.15(1.30) | 98.45(1.15, 1.87) | 98.45(1.15, 1.87) | 99.57(2.03, 0.55) | 76.21   |
| \( ^{20}\text{Ne}(1^+_1) \) | 91.75(4.08) | 99.53(2.97) | 99.98(4.31, 1.91) | 99.98(4.31, 1.91) | 99.99(2.60, 1.19) | 19.99   |
| \( ^{20}\text{Ne}(3^+_1) \) | 89.87(4.03) | 97.94(2.53) | 99.85(4.32, 1.15) | 99.99(4.47, 0.87) | 99.98(2.09, 1.39) | 20.43   |

C. Analysis of \( ^{16}\text{O}-\alpha \) intercluster wave functions

In a similar way to the previous analysis of \( ^8\text{Be} \), we analyze the antisymmetrized relative wave function \( u_l(r) \) as well as the non-antisymmetrized one \( \chi_l(r) \) before the antisymmetrization. \( ru_l(r) \) of the optimized trial functions is compared with the exact solution \( ru_l^{\text{GCM}}(r) \) in Figs. 8 and 9, and the relative wave function \( r\chi_l(r) \) and \( ru_l(r) \) before and after the antisymmetrization of the trial functions for \( ^{20}\text{Ne}(0^+_1) \) are shown in Fig. 10. The inner oscillating part and the enhanced surface peak structures of \( ru_l(r) \) are not as sensitive to the details of the original trial function \( r\chi_l(r) \) at least in low \( l \) states because of the strong antisymmetrization effect. To describe well the exact solution with the trial function it is essential to fit well the outer tail with \( r\chi_l(r) \). Compared with \( ru_l(r) \) of the SM wave function, the BB wave function describes well the suppressed inner nodal region and the enhanced surface peak but it fails to fit the outer tail because of the fixed Gaussian range. The sTHSR describes the outer tail somewhat better than the BB wave function, however, it is not so good in particular for positive-parity states in the \( K^\pi = 0^+_1 \) band. The dTHSR wave function gives better results in the description of the outer tail than the sTHSR, especially fairly good description of the outer tail in \( ^{20}\text{Ne}(1^-_1) \). However, it is still insufficient for positive-parity states. The best fit of the outer tail part of \( ru_l^{\text{GCM}}(r) \) is given by the YT function.

To discuss the damping behavior of the outer tail more quantitatively, we analyze the radial curvature \( C_r(r) \) of \( ru_l(r) \) and \( r\chi_l(r) \) defined in Eq. 22. In Figs. 11 and 12, the \( r \) dependence of \( C_r(r) \) of \( ru_l(r) \) for the optimized trial functions is compared with that for the GCM wave function. \( C_r(r) \) of \( r\chi_l(r) \) for the trial functions of \( ^{20}\text{Ne}(0^+_1) \) and \( ^{20}\text{Ne}(1^-_1) \) is also shown in Figs. 13 and 14.

\( C_r(r) \) of \( ru_l^{\text{GCM}}(r) \) for the GCM wave function in the tail region shows rather flat behavior with a finite offset. For \( ^{20}\text{Ne}(0^+_1) \), \( C_r(r) = 5 \sim 10 \text{ MeV} \) in the outer tail region and it is much larger than \( C_r(r) \) of \( ^8\text{B}(0^+_1) \) because of the larger \( \sigma \) separation energy (\( -E_c = 5.92 \text{ MeV} \)) and the larger Coulomb barrier in \( ^{20}\text{Ne}(0^+_1) \). Also for \( ^{20}\text{Ne}(4^+_1) \) and \( ^{20}\text{Ne}(8^+_1) \), \( C_r(r) \) in the tail region is as large as \( C_r = 5 \sim 10 \text{ MeV} \) because of the Coulomb and centrifugal barriers. It is obvious that the BB and sTHSR wave functions fail to describe such the behavior of \( C_r(r) \). The dTHSR gives the better result but it is difficult to fit the plateau of \( C_r(r) \) in the outer tail region except for the case that \( C_r(r) \) is small enough.

As for \( ^{20}\text{Ne}(1^-_1) \) and \( ^{20}\text{Ne}(3^-_1) \) in the \( K^\pi = 0^-_1 \) band, \( C_r(r) \) of \( ru_l^{\text{GCM}}(r) \) of the GCM wave function in the tail region is relatively smaller than the \( K^\pi = 0^-_1 \) band states as \( C_r(r) < 5 \text{ fm} \) because of the small separation energy and the rather low centrifugal barrier. Moreover, since the surface peak is shifted outward because of the stronger antisymmetrization effect in the \( K^\pi = 0^-_1 \) band states, the position crossing the \( C_r(r) = 0 \) line shifts to the larger \( r \) region compared with the \( K^\pi = 0^-_1 \) band states. In such the case, the sTHSR and dTHSR wave functions can give rather gentle slope of \( C_r(r) \), and therefore, it is easier to describe the \( K^\pi = 0^-_1 \) band states than the \( K^\pi = 0^-_1 \) band states. However, \( C_r(r) \) of the dTHSR is a gradually increasing function in the outer region and it gives a slightly
steeper slope of $C_\alpha(r)$ than that of the exact solution, in particular, of $^{20}\text{Ne}(3^-_1)$. On the other hand, the YT function can fit the flat region of $C_\alpha(r)$ in the outer region, and therefore it gives better results in reproducing $u_\alpha(r)$ and $C_\alpha(r)$ even for the $K^\pi = 0^-_1$ band states than the dTHSR as well as for the $K^\pi = 0^+_1$ band states. This indicates that the Yukawa-like tail is essential in the $\alpha$-cluster states of $^{20}\text{Ne}$.

The radial curvature $C_\alpha(r)$ can be regarded as the effective potential with the offset $-E_r$ as $C_\alpha(r) = V_{\text{eff}}(r) - E_r$ in which the $\alpha$ cluster is confined and moving in the relative wave function. As seen in Figs. 13 and 14, the effective potential for the physical wave function $ru_\alpha(r)$ at the surface region is dominantly described by the antisymmetrization effect, and it is quite different from that for the original relative wave function $r\chi_\alpha(r)$ before the antisymmetrization. On the other hand, in the outer region, the effective potential for $ru_\alpha(r)$ is consistent with that for $r\chi_\alpha(r)$. As clearly seen, the effective potential shown by $C_\alpha(r)$ for the exact solution $ru_\alpha^{\text{GCM}}(r)$ can not be described by the Harmonic oscillator potential, but it shows a rather normal shape of the Coulomb and centrifugal barriers with a certain finite range attraction. Therefore, it is not surprising that the YT function with the Yukawa tail gives the best fit rather than the sTHSR with the Gaussian tail. As explained before, the dTHSR wave function is not necessarily successful to describe the Yukawa-like tail except for the small $C_\alpha(r)$ case.

These results indicate the following facts. The fixed-range Gaussian tail in the BB wave function is not suitable to reproduce the outer tail of cluster wave functions in $^{20}\text{Ne}$ system. The Gaussian tail with the adjustable range is better than the fixed-range Gaussian as argued in Refs. [9, 10]. However, the Yukawa-type tail can describe the outer tail well rather than Gaussian tail. The dTHSR wave function does not work so well as the YT function except for $^{20}\text{Ne}(1^-_1)$, because the cluster states of $^{20}\text{Ne}$ are not weakly bound states but they are rather well "bound" states because of the larger separation energy and/or higher centrifugal and Coulomb barriers than $^8\text{Be}(0^+_1)$.

VI. DISCUSSION AND SUMMARY

We analyze the $\alpha$-cluster wave functions in the cluster states of $^8\text{Be}$ and $^{20}\text{Ne}$ by comparing various types of trial functions such as the BB, sTHSR, dTHSR, and YT functions with the exact wave function obtained by the GCM calculation. The relative wave functions in the BB, sTHSR, dTHSR, and YT functions are given by the localized Gaussian with the fixed range, the spherical Gaussian, the deformed Gaussian, and the Yukawa-tail function, respectively. By investigating the squared overlap of the trial functions with the GCM wave function, we study how well the trial functions can describe the exact cluster wave function. Compared with the SU(3) shell-model limit wave function, the description of the suppressed inner part and the enhanced surface peak of the physical relative wave function $ru_\alpha(r)$ is improved with the BB wave function. The better result is obtained with the sTHSR wave function than the BB wave function, and further improvement is given with the dTHSR wave function because these wave functions can describe better the outer tail part. The YT function gives almost equal quality to the dTHSR wave function for $^8\text{Be}(0^+_1)$ and $^{20}\text{Ne}(1^-_1)$, and even better description for such states as $^{20}\text{Ne}(0^+_2)$, $^{20}(2^+_1)$, and $^{20}(3^-_1)$. This result indicates that the outer tail of $\alpha$ cluster states is characterized by the Yukawa-like tail rather than the Gaussian tail.

The relative wave functions in the $\alpha$-cluster states of $^8\text{Be}$ and $^{20}\text{Ne}$ are characterized by three parts, the oscillating inner part, the enhanced surface peak, and the outer tail. In the $\alpha$-cluster states, the inner part is suppressed while the surface peak is relatively enhanced because of the antisymmetrization effect between clusters. The nodal structure in the inner region and the enhanced peak structure at the surface in the physical wave function $ru_\alpha(r)$ are not so sensitive to the original tria l functions $r\chi_\alpha(r)$ before the antisymmetrization because of the strong antisymmetrization effect between clusters. The outer tail is caused by the quantum penetration and its asymptotic behavior is well defined. In the weakly bound $\alpha$ cluster states with the small $\alpha$ separation energy and the low angular momentum $l$, the wave function is slowly damping in the outer region and it has the remarkably long outer tail.

To get a good approximation of the exact solution (GCM wave function) for the weakly bound $\alpha$-cluster states, it is essential to fit the outer tail part, in particular, its slow damping behavior. On the other hand, the inner and peak parts are relatively less important in the fitting because they are mainly determined by the antisymmetrization effect. The outer tail part is the slowly damping function which characterizes the almost zero-energy free $\alpha$ gas behavior. This "free $\alpha$-cluster gas part" in the outer region is understood as the delocalization of cluster. It should be pointed out that the origin of this delocalization, i.e., the "free $\alpha$-cluster gas part" in the outer region, is the quantum penetration and it is the obvious consequence of the weakly bound system. In the asymptotic region where the nucleus-nucleus interaction vanishes, the damping of the outer long tail in the free gas region is well defined by the $\alpha$ separation energy. In the ideal case that the separation energy is small and the Coulomb and centrifugal barriers are not high, the dTHSR wave function can describe the exact wave function fairly well because it has the suitable form to fit the long tail in the outer free gas region. However, the success of the dTHSR is the mathematical result of the fact that the dTHSR can describe the Yukawa-like tail rather than the Gaussian tail.

Compared with the outer tail region free from the antisymmetrization, it is difficult to discuss physical meaning of the
inner region and even of the surface peak region, because the physical wave function in these regions is not so sensitive to the original trial function before the antisymmetrization but it is strongly affected by the antisymmetrization effect. Instead, in the outer region, the alpha cluster wave function is well defined, and the features of the original model wave functions before the antisymmetrization are reflected more directly in the physical wave function after the antisymmetrization. The physical meaning of the delocalization of clusters is clearly given in this region by the outer long tail caused by the quantum penetration, which can be regarded as the almost zero-energy "free α gas".

It should be stressed that the dTHSR with the large deformation has the damping behavior of the tail part quite different from the Gaussian tail of the sTHSR. The effective potential evaluated by the radial curvature $C_r(r)$ for the physical relative wave function of the exact solution does not show the feature of the Harmonic oscillator potential, but it shows a normal shape of the Coulomb and centrifugal barriers with a certain finite range attractive potential. Therefore, it is not surprising that the YT function with the Yukawa tail gives the best fit among the present trial functions rather than the sTHSR having the Gaussian form. The dTHSR wave function can successful describe the Yukawa-like tail in the exact solution in the case that $C_r(r)$ is small in the outer tail region. If we apply the "container picture" proposed in Ref. [13], it is better to consider a constant barrier with a small hight rather than Harmonic oscillator potential as the confining potential to understand the slowly damping Yukawa-like tail of the cluster wave function in the physical region.

FIG. 8: Relative wave functions $r_u(r)$ of the optimized trial wave functions for $K = 0^+_1$ band states of $^{20}$Ne compared with that of the GCM wave function. $r_u(r) = R_{nl}(b_r; r)$ for the SU(3) shell model limit is also shown.
FIG. 9: Same as Fig. 8 but for the \( K = 0^- \) band states of \( ^{20}\text{Ne} \).

In Refs. [13], the container picture has been proposed to understand nuclear clustering such as the \( 3\alpha \) state in \(^{12}\text{C} \) system. It has been shown that the RGM wave function for \(^{12}\text{C}(0_2^+)\), which corresponds to the exact solution of the \( 0_2^+ \) state within the \( 3\alpha \) cluster model space, can be described fairly well by a single deformed THSR wave function [8, 13]. The result of the deformed THSR is much better than the spherical THSR function. It should be noted that the deformation of the optimized THSR wave function for \(^{12}\text{C}(0_2^+)\) is large with about 1:3 ratio. One should be careful again that the tail behavior of the \( J^\pi \)-projected wave function of the largely deformed Gaussian is different from the Gaussian tail. If we omit the angular momentum coupling and consider only the \(^{8}\text{Be}(0^+)+\alpha\) component, the radial wave function of the \( \alpha \) cluster in the outer tail region is given by the form similar to the \( 2\alpha \) system discussed in the present work. The largely deformed THSR wave function may not give the Gaussian tail but it might describe the slow damping behavior of the outer tail, in which the \( \alpha \) clusters behave as almost zero-energy "free \( \alpha \) gas". In this outer tail region, the dynamics is governed by the quantum penetration though the small Coulomb barrier free from potential, and it is determined dominantly by the \( \alpha \) separation energy. One should also take care about the "physical region" in the \( 3\alpha \) system. Since the inner part is strongly affected by the antisymmetrization between clusters and also by the orthogonality to the lowest state \(^{12}\text{C}(0_1^+)\), it is not easy to clearly mention the physical meaning of the inner part. Instead, the delocalization of clusters can be defined in the outer region free from the effects of the antisymmetrization and orthogonalization. As the separation energy becomes small, the outer tail part of the wave function becomes more and more important. In such a case, the delocalization could be characterized by the slowly damping long tail of the almost zero-energy "free \( \alpha \) gas" as mentioned before. It may be useful to investigate the details of the outer tail of the \( 3\alpha \) state, in particular, its damping behavior in order to understand the proper shape of the confining effective potential in the container picture.

In general, the delocalization occurs in weakly bound cluster states definitely at least in the outer region. Even though the Gaussian tail of the single sTHSR function is not sufficient to perfectly reproduce the details of the outer tail, in particular, the damping behavior, it is usually better in description of the long tail than the fixed-range Gaussian of the single BB wave function. Further drastic improvement in the description of the outer tail can be obtained by the dTHSR function. It should be also pointed that the dTHSR wave function is not only mathematically useful but also it is a powerful tool to investigate the \( \alpha \) cluster states because the total microscopic wave function of the system is given in the quite simply form and it is easy to practically calculate the energy expectation value of the total wave function. In contrast to the dTHSR function, the present analysis of the YT relative wave function.
is just a mathematical game, and it is not practical to use the YT function in the actual microscopic calculation of many-body systems.

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Appendix A: \( \chi^dG(r) \) in large deformation limit

In the case of largely deformed Gaussian with \( \sigma_z \gg \sigma_\perp \), the curvature \( C(r) \) of \( \chi^dG(r) \) can become small in the outer region as explained below. \( \chi^dG(r) \) in Eq. 35 for even \( l \) is rewritten as

\[
\chi^dG(r) \propto \int_0^\pi P_l(\cos \theta) \exp \left( -\frac{r^2}{\sigma_z^2} + \frac{r^2 \cos \theta^2}{\Delta} \right) \sin \theta d\theta
\]

\[
= \exp \left( -\frac{r^2}{\sigma_z^2} \right) \int_0^\pi P_l(\cos \theta) \exp \left( -\frac{r^2 \sin \theta^2}{\Delta} \right) \sin \theta d\theta. \quad (A1)
\]

FIG. 10: Relative wave functions \( r\chi_l(r) \) before the antisymmetrization and \( ru_l(r) \) after the antisymmetrization of the optimized trial functions for \( ^{20}\text{Ne}(0^+_1) \). The scaled functions \( r\chi^{sc}_l(r) \equiv r\chi_l(r)/\sqrt{\langle \chi_l(r)|\chi_l(r) \rangle} \) and \( ru^{sc}_l(r) \equiv ru_l(r)/\sqrt{\langle \chi_l(r)|\chi_l(r) \rangle} \) are shown as well as \( ru_l(r) \).
In the $\sigma_z \gg \sigma_\perp$ case, in the asymptotic region of $r \gg \sigma_\perp$, only small $\theta$ region contributes to the integral, and then $\chi_l^{dG}(r)$ can be approximately estimated as

$$\chi_l^{dG}(r) \approx \exp \left( -\frac{r^2}{\sigma_z^2} \right) 2 \int_0^\infty \rho_l(1) \exp \left( -\frac{r^2}{2\Delta} \right) \theta d\theta$$

$$= \frac{\Delta}{r^2} \exp \left( -\frac{r^2}{\sigma_z^2} \right). \quad (A2)$$

In the $r \lesssim \sigma_z$ or $r \approx \sigma_z$ region, the $\exp \left( -\frac{r^2}{\sigma_z^2} \right)$ term changes gradually and $\chi_l^{dG}(r)$ approximately has the $\frac{1}{r^2}$ behavior, and therefore, the curvature $C_r(r)$ of $r\chi_l^{dG}(r)$ is roughly estimated as $C_r(r) \approx \frac{k^2}{\mu r^2}$. Namely, for $r\chi_l^{dG}(r)$ with $\sigma_z \gg \sigma_\perp$, the curvature $C_r(r)$ goes to 0 in the $\sigma_\perp \ll r \lesssim \sigma_z$ region. It means that the deformed Gaussian should be a better trial function that can efficiently describe the slow damping behavior of the outer tail than the spherical Gaussian having more rapid damping of the Gaussian tail.

The $\frac{1}{r^2}$ behavior in the $r \lesssim \sigma_z$ region of $\chi_l^{dG}(r)$ with $\sigma_z \gg \sigma_\perp$ can be more intuitively understood by a cylinder
FIG. 12: Same as Fig. 11 but for the $K = 0_1^-$ band states of $^{20}$Ne.

picture. The angle average of a cylinder with a diameter $\sigma_\perp$ at $r \gg \sigma_\perp$ is approximated as $\pi \sigma_\perp^2 / 4\pi r^2$.

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FIG. 13: Radial curvature $C_r(r)$ of $r\chi_l(r)$ of the optimized trial functions for $^{20}\text{Ne}(0^+)^+$.

FIG. 14: Radial curvature $C_r(r)$ of $r\chi_l(r)$ of the optimized trial functions for $^{20}\text{Ne}(1^-)^+$. 

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(a) $^{20}\text{Ne}(0^+)\text{ BB}$

(b) $^{20}\text{Ne}(0^+)\text{ sTHSR}$

(c) $^{20}\text{Ne}(0^+)\text{ dTHSR}$

(d) $^{20}\text{Ne}(0^+)\text{ MY}$

(a) $^{20}\text{Ne}(1^-)^+\text{ BB}$

(b) $^{20}\text{Ne}(1^-)^+\text{ sTHSR}$

(c) $^{20}\text{Ne}(1^-)^+\text{ dTHSR}$

(d) $^{20}\text{Ne}(1^-)^+\text{ MY}$