An Alternative Construction to the Transitive Closure of a Directed Graph

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November 8, 2021

Abstract

One must add arrows which are forced by transitivity to form the transitive closure of a directed graph. We introduce a construction of a transitive directed graph which is formed by adding vertices instead of arrows and which preserves the transitive relationships formed by distinct vertices in the original directed graph. Our construction does not apply to all directed graphs.

We start by fixing notation and defining compression maps in Section 1. The motivation for this investigation comes from algebra through connections with incidence rings, which are rings of functions defined on sets with relations. None of the algebra applications are covered here, but we mention anyway a good reference for incidence rings is [4]. The class of generalized incidence rings was introduced by G. Abrams in [1]. His construction is based on the balanced relations. In Section 2 we
1 Compression Maps

The directed graphs we consider have a finite number of vertices and no repeated arrows. Loops are allowed (a loop is an arrow from a vertex to itself). The vertex set and the arrow set of a directed graph \( D \) are denoted by \( V(D) \) and \( A(D) \), respectively. An arrow from vertex \( v \) to vertex \( w \) is denoted by \( vw \). The notation \( D^* \) is reserved for the subgraph of \( D \) with vertex set \( V(D^*) = V(D) \) and arrow set \( A(D^*) = \{xy \in A(D) : x, y \in V(D) \text{ and } x \neq y \} \).

We say a directed graph \( D \) is reflexive if \( vv \in A(D) \) for all \( v \in V(D) \) and transitive if \( xy, yz \in A(D) \) implies \( xz \in A(D) \) for all \( x, y, z \in V(D) \). If \( D \) is reflexive and transitive then we say \( D \) is preordered. A transitive triple in \( D \) is an ordered triple of vertices contained in \( \text{Trans}(D) = \{(a, b, c) : a, b, c \in V(D) \text{ and } ab, bc, ac \in A(D)\} \).

**Definition 1.1** Suppose \( D_1 \) and \( D_2 \) are reflexive directed graphs. A compression map is a surjective function \( \theta : V(D_2) \to V(D_1) \) which satisfies 1, 2, and 3 below.

1. \( \theta(x) \theta(y) \in A(D_1) \) for all \( x, y \in D_2 \) such that \( xy \in A(D_2) \).
2. For all \((a_1, a_2, a_3) \in \text{Trans}(D_1)\) there exists \((x_1, x_2, x_3) \in \text{Trans}(D_2)\) such that 
\(\theta(x_i) = a_i\) for \(i = 1, 2, 3\).

3. There is a bijection \(\theta^* : A(D_2^*) \to A(D_1^*)\) given by \(\theta^*(xy) = \theta(x) \theta(y)\) for all \(x, y \in V(D_2)\) with \(xy \in A(D_2^*)\).

A figure showing a reflexive directed graph \(D\) will only display \(D^*\) and will not show the loops. Thus we assume the directed graphs in Figure 1 are both reflexive.

In Example 1.2 we show directed graph (b) is a compression of directed graph (a).

![Figure 1: (a) is transitive and (b) is not transitive.](image)

**Example 1.2** Let \(D_1\) and \(D_2\) be the reflexive directed graphs with \(V(D_1) = \{x, y, z\}\), \(A(D_1^*) = \{xy, yz\}\), \(V(D_2) = \{x, y, z, t\}\), and \(A(D_2^*) = \{xy, tz\}\) where \(x, y, z, t\) are distinct. In Figure 1 we can match up \(D_1\) with (a) and \(D_2\) with (b). A compression map \(\theta : V(D_2) \to V(D_1)\) is given by \(\theta(x) = x, \theta(y) = y, \theta(z) = z,\) and \(\theta(t) = y\). The effect on the directed graphs is to map the two middle vertices of (a) to the middle vertex of (b).

**Lemma 1.3** Let \(D_1\) and \(D_2\) be reflexive directed graphs and let \(\theta : V(D_2) \to V(D_1)\) be a compression. Suppose \(xy, yz \in A(D_2)\) and \((\theta(x), \theta(y), \theta(z)) \in \text{Trans}(D_1)\) for some \(x, y, z \in V(D_2)\). Then \((x, y, z) \in \text{Trans}(D_2)\).

**Proof.** Choose arbitrary \(x, y, z \in V(D_2)\) such that \(xy, yz \in A(D_2)\). If \(x, y, z\) are not distinct then \(xz \in A(D_2)\) follows immediately. If \(x, y, z\) are distinct then set
\[ \theta(x) = a, \ \theta(y) = b, \ \theta(z) = c. \] Assume \((a, b, c) = (\theta(x), \theta(y), \theta(z)) \in \text{Trans}(D_1)\).

Then \(a, b, c\) are distinct since \(ab, bc, ac \in A(D_1)\) by part 3 of Definition 1.1. By part 2 of Definition 1.1 there exist \(x', y', z' \in V(D_2)\) such that \((x', y', z') \in \text{Trans}(D), \ \theta(x') = a, \ \theta(y') = b, \ \text{and} \ \theta(z') = c.\) Moreover \(x', y', z'\) are distinct since \(a, b, c\) are distinct. Then \(x'y', y'z' \in A(D_2)\) so \(\theta^*(xy) = ab = \theta^*(x'y')\) and \(\theta^*(yz) = bc = \theta^*(y'z').\) Therefore \(x' = x, \ y' = y, \ \text{and} \ z' = z\) since \(\theta^*\) is bijective by part 3 of Definition 1.1. We assumed \((x', y', z') \in \text{Trans}(D)\) and proved \(x' = x\) and \(z' = z\) so \(xz \in A(D_2)\). Therefore \((x, y, z) \in \text{Trans}(D_2)\) as desired. □

Lemma 1.3 shows that if a reflexive directed graph has a preordered compression then it must be a preordered directed graph. Example 1.2 shows a preordered directed graph may have a compression which is not preordered. Figure 2 shows there are also directed graphs which are not compressions of preordered directed graphs. Our main result, Theorem 3.2, describes a class of directed graphs which are compressions of directed graphs.

![Figure 2: Directed graphs which are not compressions of transitive directed graphs.](image-url)

\[ \text{Figure 2: Directed graphs which are not compressions of transitive directed graphs.} \]
2 Balanced and Stable Directed Graphs

Definition 2.1 Suppose $D$ is a reflexive directed graph.

1. $D$ is balanced if for all $w, x, y, z \in V(D)$ such that $wx, xy, yz, wz \in A(D)$ there is an arrow from $w$ to $y$ if and only if there is an arrow from $x$ to $z$.

2. $D$ is stable if $D$ is balanced and $ad \in A(D)$ for all distinct $a, b, c, d \in V(D)$ such that $ab, ac, bc, bd, cd \in A(D)$.

A reflexive directed graph is balanced if and only if does not contain an induced subgraph isomorphic to either (a) or (b) in Figure 2 or either of the directed graphs in Figure 3.

![Figure 3: Two directed graphs which are not balanced.](image)

A balanced directed graph is not stable if and only if it does not contain an induced subgraph isomorphic to (c) in Figure 2. Directed graph (d) in Figure 2 is stable but not preordered.

Theorem 2.2 Suppose $\theta : V(D_2) \to V(D_1)$ is a compression map and $D_1$ and $D_2$ are reflexive directed graphs.

1. If $D_1$ is balanced then $D_2$ is balanced.

2. If $D_1$ is stable then $D_2$ is stable.
3. If \(D_1\) is preordered then \(D_2\) is preordered.

**Proof.** (1) Suppose \(D_1\) is balanced and \(wx, xy, yz, wz \in A(D_2)\) for some \(w, x, y, z \in V(D_2)\). Set \(a = \theta(w), b = \theta(x), c = \theta(y),\) and \(d = \theta(z)\). Then \(ab, bc, cd, ad \in A(D_1)\) by property 1 of Definition 1.1.

If \(xz \in A(D_2)\) then property 1 of Definition \ref{def:b} gives \(bd \in A(D_1)\) so \(ac \in A(D_1)\) since \(D_1\) is balanced. Then \((a, b, c) \in Trans(D_1)\) so \((w, x, y) \in Trans(D_2)\) by Lemma \ref{lem:13}. This gives \(wy \in A(D_2)\).

For the other direction assume \(wy \in A(D_2)\). Then property 1 of Definition \ref{def:b} gives \(ac \in A(D_1)\) so \(bd \in A(D_1)\) since \(D_1\) is balanced. Then \((b, c, d) \in Trans(D_1)\) so \((x, y, z) \in Trans(D_2)\) by Lemma \ref{lem:13}. This gives \(xz \in A(D_2)\).

(2) Suppose \(D_1\) is stable and \(wx, wy, xy, xz, yz \in A(D_2)\) for some distinct vertices \(w, x, y, z \in V(D_2)\). Set \(a = \theta(w), b = \theta(x), c = \theta(y),\) and \(d = \theta(z)\). Then \(ab, ac, bc, bd, cd \in A(D_1^*)\) by property 3 of Definition \ref{def:b}. In particular \(a, b, c, d\) are distinct and \(ad \in A(D_1^*)\) since \(D_1\) is stable. Then \((a, c, d) \in Trans(D_1)\) so \((w, y, z) \in Trans(D_2)\) by Lemma \ref{lem:13}. This proves \(wz \in A(D_2^*)\) as desired. Therefore \(D_2\) is stable if \(D_1\) is stable.

(3) Part 3 follows immediately from Lemma \ref{lem:13}.

The converse does not hold for every part of Theorem \ref{thm:2.2}. Figure 3 shows two directed graphs which are not balanced. However, they are both compressions of preordered directed graphs. The compressions are defined in a similar fashion as Example \ref{ex:1.2} by constructing a directed graph which splits the middle vertex in two.
3 Clasps and Soloists

The next definition helps us identify vertices where the transitive relation fails.

**Definition 3.1** Suppose $D$ is a reflexive directed graph and $x \in V(D)$.

1. We say $x$ is a clasp if there exist $w, y \in V(D) \setminus \{x\}$ such that $wx, xy \in A(D)$ and there is no arrow from $w$ to $y$.

2. We say $x$ is a locked clasp if there exist $u, v, w, y \in V(D) \setminus \{x\}$ such that $(u, x, y), (u, x, v), (w, x, v) \in \text{Trans} (D)$ and there is no arrow from $w$ to $y$.

3. An unlocked clasp is a clasp which is not locked.

Directed graph (d) in Figure 2 contains a locked clasp determined by the vertex in the lower right corner. We are now able to state our main Theorem.

**Theorem 3.2** Let $D$ be a stable directed graph. Then $D$ is the compression of a preordered directed graph if and only if every clasp in $D$ is unlocked.

**Remark 3.3** Suppose $D$ is a reflexive directed graph. Theorem 3.2 shows $D$ is the compression of a preordered directed graph if $D$ does not contain an induced subgraph isomorphic to one of the directed graphs in Figure 2 or in Figure 3. This is reminiscent of Kuratowki’s characterization of planar graphs (see [2]). Theorem 3.2 is not a complete classification since both directed graphs in Figure 3 are compressions of preordered directed graphs. But both directed graphs in Figure 3 contain directed cycles so we can give a complete classification if $D^*$ is acyclic.
Corollary 3.4 Suppose $D$ is a reflexive directed graph such that $D^*$ is acyclic. Then $D$ is the compression of a preordered directed graph if and only if $D$ does not contain an induced subgraph isomorphic to one of the directed graphs in Figure 2.

Definition 3.5 Suppose $D$ is a reflexive directed graph. If $r, s \in V(D)$ satisfy $rs, sr \in A(D)$ then $r$ and $s$ are said to be paired in $V(D)$. We say an element $s \in V(D)$ is a soloist in $D$ if $r$ and $s$ are not paired for all $r \in V(D) \setminus \{s\}$.

Lemma 3.6 Suppose $D$ is a reflexive directed graph.

1. If $x \in V(D)$ is a clasp then $x$ is a soloist.

2. Suppose $D_2$ is a stable directed graph and $\theta : V(D_2) \rightarrow V(D)$ is a compression map. There is a locked clasp in $D_2$ if and only if there is a locked clasp in $D$.

3. If $D$ contains a locked clasp then $D$ is not the compression of a preordered directed graph.

Proof. (1) Since $x$ is a clasp there exist $w, y \in V(D)$ such that $w, y \in V(D) \setminus \{x\}$, $wx, xy \in A(D)$, and there is no arrow from $w$ to $y$. Suppose there exists $z \in V(D) \setminus \{x\}$ such that $x$ is paired with $z$. Applying the balance property to $w, x, z, x$ and to $x, z, x, y$ yields $wz, zy \in A(D)$. This gives $w \neq y, y \neq z,$ and $w \neq z$ since there is no arrow from $w$ to $y$. Applying the stable property to $w, x, z, y$ gives $wy \in A(D)$, which is a contradiction.

(2) If $x \in V(D_2)$ is a locked clasp then there exist $u, v, w, y \in V(D_2) \setminus \{x\}$ such that $(u, x, y), (u, x, v), (w, x, v) \in \text{Trans}(D_2)$ and there is no arrow from $w$ to $y$ in $D_2$. Properties (1) and (3) of Definition 1.1 give $\theta(u), \theta(v), \theta(w), \theta(y) \in V(D) \setminus \{\theta(x)\}$
such that $(\theta(u), \theta(x), \theta(y)), (\theta(w), \theta(x), \theta(v)) \in \text{Trans}(D)$, and there is no arrow from $\theta(w)$ to $\theta(y)$ in $D$. Therefore $\theta(x)$ is a locked clasp in $V(D)$.

If $x_1 \in V(D)$ is a locked clasp then there exist $u_1, v_1, w_1, y_1 \in V(D) \setminus \{x_1\}$ such that $(u_1, x_1, y_1), (u_1, x_1, v_1), (w_1, x_1, v_1) \in \text{Trans}(D)$ and there is no arrow from $w_1$ to $y_1$ in $D$. By part 2 of Definition 1.1 there exist $a, b, c, d, e, f, u_2, x_2, y_2 \in V(D_2)$ such that $(a, b, c), (d, e, f), (u_2, x_2, y_2) \in \text{Trans}(D_2)$, $\theta(a) = u_1$, $\theta(b) = x_1$, $\theta(c) = v_1$, $\theta(d) = w_1$, $\theta(e) = x_2$, $\theta(f) = y_1$, $\theta(u_2) = u_1$, $\theta(x_2) = x_1$, and $\theta(y_2) = y_1$. We have $\theta^*(u_2x_2) = \theta^*(ab)$ and $\theta^*(bc) = \theta^*(ef)$ so $u_2 = a$, $x_2 = b$, $e = b$, and $c = f$ by part 3 of Definition 1.1. Setting $w_2 = a$ and $v_2 = c$ gives $u_2, v_2, w_2, y_2 \in V(D) \setminus \{x_2\}$ such that $(u_2, x_2, y_2), (u_2, x_2, v_2), (w_2, x_2, v_2) \in \text{Trans}(D)$, $u_2x_2, x_2y_2 \in A(D_2)$, and there is no arrow from $u_2$ to $y_2$ in $D_2$. Therefore $x_2$ is a locked clasp in $D_2$.

(3) Part 3 follows immediately from part 2. ■

**Lemma 3.7** Suppose $D$ is a stable directed graph and $s \in V(D)$ is a soloist.

1. Suppose $(a, b, s) \in \text{Trans}(D)$ for some $a, b \in V(D) \setminus \{s\}$ with $a \neq b$.

   (a) If $sc \in A(D)$ for some vertex $c$ then $ac \in A(D)$ if and only if $bc \in A(D)$.

   (b) If $xa \in A(D)$ for some vertex $x$ then $xs \in A(D)$ if and only if $xb \in A(D)$.

2. Suppose $(a, s, c) \in \text{Trans}(D)$ for some $a, c \in V(D) \setminus \{s\}$.

   (a) If $xa \in A(D)$ for some vertex $x$ then $xc \in A(D)$ if and only if $xs \in A(D)$.

   (b) If $cd \in A(D)$ for some vertex $d$ then $ad \in A(D)$ if and only if $sd \in A(D)$.
3. Suppose \((s, b, c) \in \text{Trans}(V(D))\) for some \(b, c \in V(D) \setminus \{s\}\) with \(b \neq c\).

(a) If \(cd \in A(D)\) for some vertex \(d\) then \(bd \in A(D)\) if and only if \(sd \in A(D)\).

(b) If \(as \in A(D)\) for some vertex \(a\) then \(ab \in A(D)\) if and only if \(ac \in A(D)\).

**Proof.** The proofs of parts 1, 2, and 3 are similar. Note that in part 2 we have \(a \neq c\) since \(s\) is a soloist. We prove part 1. Assume \((a, b, s) \in \text{Trans}(D)\) for some \(a, b \in V(D) \setminus \{s\}\) with \(a \neq b\).

(a) Suppose \(sc \in A(D)\) for some \(c \in V(D) \setminus \{s\}\). If \(ac \in A(D)\) then applying the balance property to \(a, b, s, c\) gives \(bc \in A(D)\). On the other hand if \(bc \in A(D)\) then \(a \neq c\) and \(b \neq c\) since \(s\) is a soloist. Applying the stable property to \(a, b, s, c\) gives \(ac \in A(D)\).

(b) Suppose \(xa \in A(D)\) for some \(x \in V(D) \setminus \{a, b\}\). If \(xs \in A(D)\) then applying the balance property to \(x, a, b, s\) gives \(xb \in A(D)\). On the other hand if \(xb \in A(D)\) then \(x \neq s\) since \(s\) is a soloist. Applying the stable property to \(x, a, b, s\) gives \(xs \in A(D)\). ■

The proof of Theorem \ref{3.2} is a constructive algorithm described in section \ref{4}. In each iteration of the algorithm we construct a preordered directed graph with one more vertex and define a compression. The algorithm stops when we arrive at a preordered directed graph and the desired compression is obtained by composition.

We finish this section with an example which covers the steps and constructions given in the proof of Theorem \ref{3.2}. The directed graphs in Figure \ref{4} are stable. We may identify (i) as a compression of (ii) by mapping 2 and \(t_1\) to 2. We may also identify (ii) as a compression of (iii) by mapping 4 and \(t_2\) to 4.
Example 3.8 Let $D$ be reflexive directed graph (i) shown in Figure 4. The clasps are 2 and 4 and we set $x_1 = 2$.

![Figure 4: A construction using the proof of Theorem 3.2](image)

Step 1 $Y_1 = \{4, 6\}$ and $A_1$ is empty.

Step 2 Use construction A since $A_1$ is empty. Let $D_2$ be the reflexive directed graph with $V(D_2) = V(D_1) \cup \{t_1\}$ and $A(D_2^*) = \sigma_1 \cup \tau_1$ where $B_1 = \{4, 6\}$, $\sigma_1 = A(D_1^*) \setminus \{24, 26\}$, and $\tau_1 = \{t_14, t_16\}$. This gives (ii) in Figure 4.

Step 3 Define $\theta_1 : V(D_2) \rightarrow V(D_1)$ by $\theta_1(t_1) = 2$ and $\theta_1(u) = u$ for all $u \in V(D_1)$.

Step 4 We go back to step 1 with $x_2 = 4$ since 4 is the only clasp in $D_2$.

Step 1 We have $Y_2 = \{6, 7\}$ and $A_2 = \{t_1, 3\}$.

Step 2 Use construction B with $a_2 = 3$, $b_2 = t_1$, and $y_2 = 6$. Let $D_3$ be the reflexive directed graph with $V(D_3) = V(D_2) \cup \{t_2\}$ and $A(D_3^*) = \sigma_2 \cup \tau_2$ where $\sigma_2 = A(D_2^*) \setminus \{34, 47\}$, $\tau_2 = \{3t_2, t_27\}$. This gives (iii) in Figure 4.
Step 3 Define $\theta_2 : V(D_3) \to V(D_2)$ by $\theta_2(t_2) = 4$ and $\theta_2(u) = u$ for all $u \in V(D_2)$.

Step 4 $(X_3, \rho_3)$ is preordered so the algorithm stops. The compression map is $\theta_1 \circ \theta_2$.

4 Proof of Theorem \textbf{3.2}

If $D$ is the compression of a preordered directed graph then every clasp in $D$ is unlocked by part 3 of Lemma \textbf{3.6}. We assume $D$ is stable and every clasp in $D$ is unlocked and prove $D$ is the compression of a preorder. We set $D_1 = D$, and describe an algorithm to construct stable directed graphs $D_1, \ldots, D_m$ such that $D_i$ is a compression of $D_{i+1}$ for each $i < m$. In the last iteration $D_m$ is preordered and the desired compression map is obtained by composition.

Assume $D_1$ is not preordered and fix a clasp $x_1 \in V(D_1)$. In the first iteration of our algorithm we have $i = 1$.

Step 1

The sets $Y_i$ and $A_i$ are defined below.

- $Y_i = \{y \in V(D_i) \setminus \{x_i\} : wx_i, x_iy \in A(D) \text{ and } wy \notin A(D_i) \text{ for some } w \in V(D_i)\}$
- $A_i = \{a \in V(D_i) \setminus \{x_i\} : (a, x_i, y) \in \text{Trans}(D_i) \text{ for some } y \in Y_i\}$.

Note that $Y_i$ is nonempty since $x_i$ is a clasp.
Step 2

We fix $t_i \notin V(D_i)$ and construct a reflexive directed graph $D_{i+1}$ such that $V(D_{i+1}) = V(D_i) \cup \{t_i\}$, and $A(D_{i+1}^*) = \sigma_i \cup \tau_i$ where $\sigma_i$ and $\tau_i$ are defined using construction A or construction B. In both constructions $\tau_i = A(D_{i+1}^*) \setminus \sigma_i$ and $|\tau_i| = |A(D_i^*) \setminus \sigma_i|$ so $|A(D_i^*)| = |A(D_{i+1}^*)|$. The arrows in $\tau_i$ will all contain $t_i$. If an arrow does not contain $x_i$ then it will be in $\sigma_i$. Moreover $\sigma_i$ consists of the arrows belonging to both $D_{i+1}$ and $D_i$. Depending on the construction, an arrow may be contained in $\sigma_i$ even if it contains $x_i$.

Use construction B if there exist $a_i, b_i \in A_i$ such that $(b_i, x_i, y_i) \in \text{Trans}(D_i)$ and $a_i y_i \notin A(D_i)$ for some $y_i \in Y_i$. Otherwise use construction A.

Construction A.

- $B_i = \{b \in V(D_i) \setminus \{x_i\} : (x_i, b, y) \in \text{Trans}(D_i) \text{ or } (x_i, y, b) \in \text{Trans}(D_i) \text{ for some } y \in Y_i\}$

- $\sigma_i = A(D_i^*) \setminus (\{ax_i : a \in A_i \setminus \{x_i\}\} \cup \{x_i b : b \in B_i\})$

- $\tau_i = \{at_i : a \in A_i\} \cup \{t_i b : b \in B_i\}$

If $y \in Y_i$ then $y \in B_i$ since $(x_i, y, y) \in \text{Trans}(D_i)$. Therefore $Y_i \subseteq B_i$.

Construction B.

- $T_i = \{cz : c \in V(D_i), z \in V(D_i) \setminus \{x_i\}, (c, x_i, z) \in \text{Trans}(V(D_i)) \text{, and } cy_i \notin A(D_i)\}$

- $\sigma_i = A(D_i^*) \setminus \{cx_i, x_i, z : c, z \in V(D_i) \text{ and } cz \in T_i\}$
\[ \tau_i = \{ ct_i, t_iz : c, z \in V(D_i) \text{ and } cz \in T_i \} \]

There exists \( z \in Y_i \) such that \( (a_i, x_i, z) \in \text{Trans}(D_i) \) since \( a_i \in A_i \). Moreover \( a_iy_i \notin A(D_i) \), \( z \in x_i \), and \( x_i \notin Y_i \) so \( a_i z \in T_i \). Therefore \( T_i \) is nonempty.

**Step 3**

Define \( \theta_i : V(D_{i+1}) \to V(D_i) \) so that \( \theta_i(t_i) = x_i \) and \( \theta_i(u) = u \) for all \( u \in V(D_i) \).

Before moving on we prove \( \theta_i \) is a compression. Routine calculations show part (1) of Definition 1.1 hold and there is a well-defined map \( \theta_i^* : A(D_{i+1}) \to A(D_i) \) given by \( \theta_i^*(uv) = \theta_i(u) \theta_i(v) \) for all \( u, v \in V(D_i) \) such that \( uv \in A(D_{i+1}) \). It is easy to see \( \theta_i^*(\sigma_i) \cup \theta_i^*(\tau_i) = A(D_{i+1}) \) hence \( \theta_i^* \) is surjective. We have already shown \( |A(D_{i+1})| = |A(D_i)| \) so \( \theta_i^* \) is a bijection.

The only condition left is part 2 of Definition 1.1. Suppose \( d_1, d_2, d_3 \in V(D_i) \) and \( (d_1, d_2, d_3) \in \text{Trans}(D_i) \). We must show \( (d_1, d_2, d_3) \) is the image of a transitive triple in \( D_{i+1} \). This is easy if \( d_1, d_2, d_3 \) are not distinct since every arrow in \( D_i \) is the image of an arrow in \( D_{i+1} \). Assume \( d_1, d_2, d_3 \) are distinct.

We check every possible case and make repeated use of the fact \( uv \in \sigma_i \) if and only if \( uv \in A(D_i) \) for all \( u, v \in V(D_i) \setminus \{x_i\} \). In cases 2, 3, and 4 we have \( x_i = d_r \) for some \( r \in \{1, 2, 3\} \). We will show either \( (d_1, d_2, d_3) \in \text{Trans}(D_i) \) so that \( \theta_i(d_j) = d_j \) for \( j = 1, 2, 3 \) or the desired transitive triple is obtained by replacing \( d_r \) with \( t_i \) so that \( \theta_i(d_j) = d_j \) for \( j \neq r \), and \( \theta_i(t_i) = d_r \).

We split cases between construction B and construction A when necessary. Note that \( x_i \) is a soloist by part 1 of Lemma 3.6.

**Case 1** \( d_1, d_2, d_3 \in V(D_i) \setminus \{x_i\} \)
We have $d_1d_2, d_2d_3, d_1d_3 \in \sigma_i$ thus $(d_1, d_2, d_3) \in \text{Trans}(D_i)$.

**Case 2** If $d_1, d_2 \in V(D_i) \setminus \{x_i\}$ and $d_3 = x_i$ then $d_1d_2 \in \sigma_i$ since $d_1d_2 \in A(D^*_i)$.

**Check case 2 for construction A.** If $x_iy \in A(D_i)$ then $d_1y \in A(D_i)$ if and only if $d_2y \in A(D_i)$ by part 1(a) of Lemma 3.7. This gives $d_1 \in A_i$ if and only if $d_2 \in A_i$ so $d_1t_i \in \tau_i$ if and only if $d_2t_i \in \tau_i$ and either $(d_1, d_2, t_i) \in \text{Trans}(D_i)$ or $(d_1, d_2, x_i) \in \text{Trans}(D_i)$.

**Check case 2 for construction B.** We have $d_1z \in A(D_i)$ if and only if $d_2z \in A(D_i)$ for all $z \in V(D_i) \setminus \{x_i\}$ such that $x_iz \in A(D_i)$ by part 1(a) of Lemma 3.7. This gives $d_1z \in T_i$ if and only if $d_2z \in T_i$ for all $z \in V(D_i) \setminus \{x_i\}$. Therefore $d_1t_i \in \tau_i$ if and only if $d_2t_i \in \tau_i$ and either $(d_1, d_2, t_i) \in \text{Trans}(D_i)$ or $(d_1, d_2, x_i) \in \text{Trans}(D_i)$.

**Case 3** If $d_1, d_3 \in V(D_i) \setminus \{x_i\}$ and $d_2 = x_i$ then $d_1d_3 \in \sigma_i$ since $d_1d_3 \in A(D^*_i)$.

**Check case 3 for construction A.** If $d_3 \in B_i$ then there exists $y \in Y_i$ such that $(x_i, d_3, y) \in \text{Trans}(D_i)$ or $(x_i, y, d_3) \in \text{Trans}(D_i)$. This gives $d_1y \in A(D_i)$ by applying either part 2(b) or part 3(b) of Lemma 3.7. Therefore $d_1 \in A_i$.

On the other hand if $d_1 \in A_i$ then $(d_1, x_i, z) \in \text{Trans}(D_i)$ for some $z \in Y_i$. Since $z \in Y_i$ there exists $w \in V(D_i)$ such that $wx_i \in A(D_i)$ and $wz \notin A(D_i)$. If $wd_3 \in A(D_i)$ then $(d_1, x_i, z), (d_1, x_i, d_3), (w, x_i, d_3) \in \text{Trans}(D_i)$ and $x_i$ is a locked clasp, which is a contradiction. We are left with $wd_3 \notin A(D_i), d_3 \in Y_i$, and $d_3 \in B_i$.

We have shown $d_1 \in A_i$ if and only if $d_2 \in B_i$ so $d_1t_i \in \tau_i$ if and only if $t_id_3 \in \tau_i$ and either $(d_1, x_i, d_3) \in \text{Trans}(D_i)$ or $(d_1, t_i, d_3) \in \text{Trans}(D_i)$.
Check case 3 for construction B. If $d_3y_i \notin A(D_i)$ then $d_1d_3 \in T_i$ and $(d_1, t_i, d_3) \in \text{Trans}(D_i)$.

If $d_1y_i \in A(D_i)$ then $d_1x_i \in \sigma_i$ and we must show $x_id_3 \in \sigma_i$. Note that $(d_1, x_i, d_3), (d_1, x_i, y_i), (b_i, x_i, y_i) \in \text{Trans}(D_i)$ so $b_id_3 \in A(D_i)$ since $x_i$ is an unlocked clasp. If $x_id_3 \notin \sigma_i$ then $cd_3 \in T_i$ for some $c \in V(D_i)$ such that $cy_i \notin A(D_i)$. This gives $(b_i, x_i, y_i), (b_i, x_i, d_3), (c, x_i, d_3) \in \text{Trans}(D_i)$ with $cy_i \notin A(D_i)$ and $x_i$ is a locked clasp. This is a contradiction. We are left with $d_1x_i, x_id_3 \in \sigma_i$ and $(d_1, x_i, d_3) \in \text{Trans}(D_i)$.

Case 4 If $d_2, d_3 \in V(D_i) \setminus \{x_i\}$, $d_1 = x_i$ then $d_2d_3 \in \sigma_i$ since $d_2d_3 \in A(D_i^*)$.

Check case 4 for construction A. Suppose $d_j \in B_i$ for $j = 2$ or $j = 3$ and let $k \in \{2, 3\}$ be such that $k \neq j$. If $d_j \in B_i$ then $(x_i, d_j, y) \in \text{Trans}(D_i)$ or $(x_i, y, d_j) \in \text{Trans}(D_i)$ for some $y \in Y_i$. There exists $w \in V(D_i) \setminus \{x_i\}$ such that $wx_i \in A(D_i)$ and $wy \notin A(D_i)$ since $y \in Y_i$. Then $wd_2, wd_3 \notin A(D_i)$ by two applications of part 3(b) of Lemma 3.7. Thus $d_2, d_3 \in Y_i$ and $(t_i, d_2, d_3) \in \text{Trans}(D_i)$.

We have shown $d_2 \in B_i$ or $d_3 \in B_i$ imply $(t_i, d_2, d_3) \in \text{Trans}(D_i)$. On the other hand if $d_2 \notin B_i$ and $d_3 \notin B_i$ then $x_id_2, x_id_3 \in \sigma_i$ and $(x_i, d_2, d_3) \in \text{Trans}(D_i)$.

Check case 4 for construction B. We have $cd_2 \in A(D_i)$ if and only if $cd_3 \in A(D_i)$ for all $c \in V(D_i)$ such that $cx_i \in A(D_i)$ and $cy_i \notin A(D_i)$ by part 3(b) of Lemma 3.7. This gives $cd_2 \in T_i$ if and only if $cd_3 \in T_i$ for all $c \in V(D_i) \setminus \{x_i\}$. Therefore $t_id_2 \in \tau_i$ if and only if $t_id_3 \in \tau_i$ and either $(x_i, d_2, d_3) \in \text{Trans}(D_i)$ or $(t_i, d_2, d_3) \in \text{Trans}(D_i)$. 

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Step 4

If $D_{i+1}$ is preordered then the algorithm stops and the compression from $D_{i+1}$ to $D$ is determined by composition. Otherwise fix a clasp $x_{i+1} \in V(D_i)$ and go back to step 1 with $i$ replaced by $i + 1$.

To study the algorithm we consider a given iteration $i$. Then $A(D_{i+1})$ is stable by Theorem 2.2 and $D_{i+1}$ contains no unlocked clasps by part 2 of Lemma 3.6. This means we may repeat the algorithm as often as necessary. We must prove the algorithm stops eventually.

In each iteration of the algorithm we are adding a new vertex but not adding any arrows other than loops. The only way this can continue indefinitely is if our algorithm forces vertices to not form arrows with any other elements. We assume every vertex of $D_i$ forms an arrow with some other vertex of $D_i$ and show every vertex of $D_{i+1}$ forms an arrow with some other vertex of $D_{i+1}$.

Suppose $x, y \in V(D_i)$ satisfy $x \neq y$ and $xy \in A(D_i)$. If $x \neq x_i$ and $y \neq x_i$ then $xy \in \sigma_i$ so $xy \in A(D_{i+1})$. Note that $t_i$ forms an arrow with some other vertex of $D_{i+1}$ by construction. We are left with proving $x_i$ forms an arrow with some other vertex of $D_{i+1}$.

Assume there is not an arrow formed by $x_i$ and any other vertex of $D_{i+1}$ after using construction A. There exist $b, z \in V(D_i)$ such that $bx_i \in A(D_i)$, $x_i z \in A(D_i)$, and $bz \notin A(D_i)$ since $x_i$ is a clasp. Then $bt_i \in A(D_{i+1})$ and $t_i z \in A(D_{i+1})$ by assumption so $b \in A_i$ and $z \in B_i$. Since $b \in A_i$ there exists $y \in V(D_i)$ such that $(b, x_i, y) \in \text{Trans}(D_i)$. There must also exist $a \in V(D_i)$ such that $ax_i \in A(D_i)$ and $ay \notin A(D_i)$ since $y \in Y_i$. This gives $at_i \in A(D_{i+1})$ by assumption so $a \in A_i$.  

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Thus $a, b \in A_i$ satisfy the conditions in step 2 for construction B. This contradicts our assumption that we used construction A, so there is an arrow formed by $x_i$ and another vertex of $D_{i+1}$.

In construction B we have $b_iy_i \in A(D_i)$ so $b_iz \notin T_i$ for all $z \in V(D_i) \setminus \{x_i\}$. This gives $b_ix_i \in \sigma_i$ and $b_ix_i \in A(D_{i+1})$ so there is an arrow formed by $x_i$ with another vertex of $D_{i+1}$.

**Acknowledgements** The author wishes to thank Martin Erickson of Truman State University and Steven Szydlik of the University of Wisconsin Oshkosh for encouragement and suggesting improvements to the writing in earlier drafts.

**References**

[1] G. Abrams, Groups Gradings and Recovery Results for Generalized Incidence Rings, *J. Alg.* **164** (1994) 859-875.

[2] K. Kuratowski, Sur le probléme des courbes gauches en topologie, *Fund. Math.* **15** (1930) 271–283.

[3] K. Price, Good Gradings of Generalized Incidence Rings, submitted

[4] E. Spiegel and C. O’Donnell, ”Incidence Algebras,” Marcel Dekker, New York, NY (1997).