Velocity and diffusion constant of an active particle in a one-dimensional force field

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Abstract – We consider a run-and-tumble particle with two velocity states \( \pm v_0 \), in an inhomogeneous force field \( f(x) \) in one dimension. We obtain exact formulae for its velocity \( V \) and diffusion constant \( \sigma \) for arbitrary periodic force \( f(x) \) of period \( L \). They involve the “active potential” which allows to define a global bias. Upon varying parameters, such as an external force \( F \), the dynamics undergoes transitions from non-ergodic trapped states, to various moving states, some with non-analyticities in the \( V \) vs. \( F \) curve. A random landscape in the presence of a bias leads, for large \( L \), to anomalous diffusion \( x \sim t^{\mu} \), \( \mu<1 \), or to a phase with a finite velocity that we calculate.

Persistent random walks, where a walker persists in the same direction for a finite time before changing direction, have been studied extensively [1–5]. The recent years have seen a resurgence of interest in this stochastic process in a different reincarnation, namely the “run-and-tumble particle” (RTP), mostly in the context of active matter [6–14]. While several interesting collective properties of interacting RTPs have been discovered recently, it was realised that even a single RTP exhibits rich and interesting static and dynamic behaviours [15–27]. For example, the stationary state position distribution for an RTP in an external confining potential has been shown to deviate from the equilibrium Gibbs-Boltzmann form [22,23]. Other interesting questions such as the relaxation dynamics towards the stationary state in a confining potential [22], the first-passage properties [15,16,18,19,24,26,28] or the distribution of the current of non-interacting RTPs [29] have been recently studied in the one-dimensional geometry. In this paper, we study a single RTP subjected to an external force periodic in space. We show that, due to the presence of a finite persistence time, the position distribution under the periodic force exhibits a rich and nontrivial behaviour, compared to the ordinary diffusion. In particular, we compute explicitly the velocity and the diffusion constant of the RTP for an arbitrary periodic force \( f(x) \).

The overdamped dynamics of the RTP is described by the stochastic evolution equation

\[
\frac{d\mathbf{x}}{dt} = f(x) + v_0 \sigma(t),
\]

where \( f(x) \) is an external force and \( \sigma(t) = \pm 1 \) represents a telegraphic noise which switches from one state to another at a constant rate \( \gamma \).

In free space on the line in the case where \( f(x) = f \) is uniform, it is well known that the dynamics of the active particle becomes diffusive at large time and can be effectively described on large scale by a Langevin equation

\[
\frac{d\mathbf{x}}{dt} = f + \sqrt{2D_0} \xi(t),
\]

where the effective diffusion constant \( D_0 = v_0^2/(2\gamma) \) and the mean velocity is \( V = f \). This effective description of (1) is valid above a characteristic persistence time \( t^* = O(1/\gamma) \). In fact, the RTP dynamics (1) converges to the Langevin dynamics (2) in the limit where both \( v_0 \rightarrow +\infty \), \( \gamma \rightarrow +\infty \) with fixed \( D_0 \) (see footnote 1).

A natural question is what happens to this effective description when the RTP is subjected to an inhomogeneous force \( f(x) \)? In particular what is the mean velocity \( V \) and the diffusion constant \( D \) for arbitrary \( f(x) \)?

1This property remains true for arbitrary \( f \rightarrow f(x) \).
In the case where \( f(x) = -U'(x) \) with a confining potential \( U(x) \), there exists a stationary solution with zero current \([17,30–33]\). This stationary state was analysed in detail in \([22]\) for potentials of the type \( U(x) = \alpha|x|^p \) and an interesting “shape transition” in the stationary position distribution was found in the \((\alpha,p)\) plane. In that case the RTP motion is bounded which corresponds to \( V = 0 \) and \( D = 0 \). In fact this stationary state is typically non-Boltzmann, which shows that the Langevin equation approximation breaks down.

In this paper we consider the RTP dynamics in eq. (1) on an infinite line subjected to an arbitrary force landscape \( f(x) \), periodic in space, of period \( L \), \( f(x) = f(x + L) \) for all \( x \in \mathbb{R} \). In this case, one would anticipate that, for small \( f(x) \), there will not be any stationary position distribution and the particle will keep on moving with time, with a non-zero speed \( V_L \) and a non-zero diffusion constant \( D_L \). One of the principal goals of this paper is to compute \( V_L \) and \( D_L \). But before we do that for the RTP, it is useful to recall what happens for a simple diffusive particle (2) subjected to this periodic force, which has been studied extensively \([34–37]\). In this case, the position distribution \( P(x,t) \) satisfies the Fokker-Planck equation

\[
\partial_t P = -\partial_x J, \quad \text{where} \quad J = -D \partial_x P + f(x)P, \tag{3}
\]

which, for bounded potential, does not have a normalisable steady-state solution. However, its periodised version,

\[
\bar{P}(x,t) = \sum_{n=-\infty}^{\infty} P(x + nL,t), \tag{4}
\]

which satisfies the same Fokker-Planck equation (3), is known to reach a stationary limit \( \bar{P}(x,t) \to P(x) \) as \( t \to +\infty \) \([34–37]\). Indeed \( \bar{P}(x,t) \) corresponds to the position distribution of a diffusive particle on a ring of size \( L \). This stationary periodised solution \( \bar{P}(x) \) can be computed explicitly by setting \( \partial_t \bar{P} = 0 \) in (3), looking for a solution with a non-zero constant current \( J \). The constant \( J \) can be determined from the normalisation condition \( \int_0^L P(x)dx = 1 \). Knowing \( J \), one can then find the velocity \( V_L \) from the general identity \([35]\)

\[
V_L = \lim_{t \to +\infty} \frac{d}{dt} \int dx xP(x,t) = JL, \tag{5}
\]

where \( P(x,t) \) is the non-periodised distribution. Similarly, the diffusion constant \( D_L \), defined as

\[
D_L = \frac{1}{2} \lim_{t \to +\infty} \frac{d}{dt} \left( \overline{x(t)^2} - \overline{x(t)}^2 \right), \tag{6}
\]

where \( \overline{x(t)^2} = \int dx x^2 P(x,t) \), was also computed explicitly \([34–37]\). In addition, if the potential \( U(x) \) is itself periodic\(^2\), \( U(0) = U(L) \), the current vanishes, \( J = 0 \), and the periodised solution converges to \( \bar{P}(x) = e^{-U(x)/D_0}/Z \)

\[
\text{for } x \in [0, L] \text{ where } Z \text{ is a normalisation constant. Thus}
\]

\[
\text{the dimensionless quantity which measures the “tilt” of the potential landscape,}
\]

\[
G_L = \frac{U(0) - U(L)}{D_0}, \tag{7}
\]

can be interpreted as an effective measure of the global bias which determines the sign of the velocity \( V_L \).

In this paper, we carry out a similar procedure for the RTP (1) subjected to this periodic force \( f(x) = f(x + L) \) which we assume to be continuous. However, due to the competition between the periodic force \( f(x) \) and the noise (with a persistent memory) in eq. (1), we show that one obtains a much richer behaviour for the periodised stationary solution leading to different phases and transitions between them. Indeed we find four different phases (denoted by \( A, B, C \) and \( D \)), depending on whether \( f(x) = \pm v_0 \) has real roots or not, leading to an interesting phase diagram shown in fig. 1. In addition, we also compute explicitly for any \( L \), the stationary periodised solution \( \bar{P}(x) \), the velocity \( V_L \) and the diffusion constant \( D_L \). Furthermore, we also compute the mean first passage time to an arbitrary level \( X \).

As mentioned above, the four phases are as follows (see also figs. 1 and 2).

Phase \( A \): \( |f(x)| < v_0 \) for all \( x \). In this case the motion is unbounded and the stationary measure is smooth (if \( f(x) \) is smooth). We obtain a closed formula for \( V_L \) (see eqs. (20) and (21)) and \( D_L \) (see eqs. (24) and (25)).

Phase \( B \): \( f(x) > -v_0 \) and there are roots \( x_i \) (in increasing order) to \( f(x) = v_0 \) (see fig. 2, top panel). Out of these, every alternate ones (denoted by \( x_i^\pm \) in fig. 2, top panel) are attractive fixed points for the RTP dynamics (1) when it is in the \(-v_0 \) state. The motion is a bit more complicated in this case. The position remains unbounded but the stationary periodised solution \( \bar{P}(x) \) has singular points (see eq. (26) and also fig. 3). We also obtain a formula for \( V_L \).
given in eq. (28). A similar phase exists for the symmetric case where \( f(x) < v_0 \) and there are roots to \( f(x) = -v_0 \).

**Phase C:** there exist roots to both \( f(x) = +v_0 \), denoted by \( x_i \), and to \( f(x) = -v_0 \), denoted by \( y_i \), increasing order (see fig. 2, bottom panel). In this case the motion is bounded. The stationary periodised measure \( \tilde{P}(x) \) has disjoint supports in a set of intervals with different weights depending on the initial condition. The dynamics is non-ergodic in this case.

**Phase D:** \( f(x) > v_0 \) for all \( x \). The RTP moves to the right in both \( \pm v_0 \) states. A similar situation arises for the symmetric counterpart where \( f(x) < -v_0 \).

Evidently, one can make transitions between these phases by tuning the maximum of the periodic force \( f(x) \). One way to achieve this is to apply an additional constant force \( F \) on top of a periodic force landscape \( f_0(x) = f_0(x + L) \). This amounts to setting \( f(x) = f_0(x) + F \).

Let \( f_0^{\max} \) denote the maximum of \( f_0(x) \), for \( x \in [0, L] \), and, for simplicity, we assume that \( f_0^{\min} = -f_0^{\max} \). From our analysis, an interesting phase diagram emerges in the plane \( (f_0, f_0^{\max}) \) as shown in fig. 1. The motion undergoes transitions along the solid lines \( f_0^{\max} + F = v_0 \) (A to B), \( -f_0^{\max} + F = v_0 \) (B to D) and \( -f_0^{\min} + F = -v_0 \) (C to B). As \( F \) is increased along different lines (dotted lines in fig. 1) the velocity-force characteristics exhibit transitions, with \( V_L = 0 \) in phase C, and non-analyticities in the B phase as new fixed points appear or disappear.

Another interesting question is whether, for the RTP, there exists a single global measure \( G_L \) of the bias as in the diffusive case in eq. (7). For this, it is useful to define an “active external potential”

\[
W(x) = -2\gamma \int_{x_0}^x dy \frac{f(y)}{v_0^2 - f^2(y)},
\]

where \( x_0 \) is an arbitrary position. In the diffusive limit, \( v_0, \gamma \to +\infty \) with fixed \( D_0 = \frac{v_0^2}{2 \gamma} \), \( W(x) \to U(x)/D_0 \) converges to the standard external potential \( U(x) = -\int_{x_0}^x f(x) dx \). We show that in phase A the dimensionless global bias for the RTP, which determines the direction of the velocity, can be expressed in terms of this active potential \( W(x) \)

\[
G_L = W(0) - W(L) = 2\gamma \int_0^L dy \frac{f(y)}{v_0^2 - f^2(y)}.
\]

Indeed we show that the sign of \( V_L \) is the same as the sign of \( G_L \) (and also \( V_L \) vanishes when \( G_L \) vanishes). Clearly, in the diffusive limit, eq. (9) reduces to eq. (7). In addition, we show that in the small bias limit \( (G_L \to 0) \), the velocity satisfies an Einstein-like relation (within linear response in \( G_L \))

\[
V_L \simeq D_L^{\beta} \frac{G_L}{L},
\]

where \( D_L^{\beta} \) denotes the diffusion constant \( D_L \) in the case of zero bias (given below in (24)).

Let us first outline briefly our derivation of the main results. We first define \( P_{\pm}(x, t) \) as the probability densities of the RTP to be in position \( x \) at time \( t \) and in the state \( \sigma(t) = \pm 1 \). They satisfy the pair of Fokker-Planck equations corresponding to eq. (1)

\[
\partial_t P_+ = -\partial_x [f(x) + v_0]P_+ - \gamma P_+ + \gamma P_-, \quad (11)
\]

\[
\partial_t P_- = -\partial_x [f(x) - v_0]P_- + \gamma P_- - \gamma P_+. \quad (12)
\]

The associated periodised distributions, \( \tilde{P}_{\pm}(x, t) = \sum_n \tilde{P}_{\pm}(x + nL, t) \), satisfy the same pair of equations due to the periodicity of \( f(x) \). We also define the total probability \( \tilde{P}(x, t) = \tilde{P}_+(x, t) + \tilde{P}_-(x, t) \), as well as the difference \( \tilde{Q}(x, t) = \tilde{P}_+(x, t) - \tilde{P}_-(x, t) \), which then satisfy the coupled Fokker-Planck equations

\[
\partial_t \tilde{P} = -\partial_x f(x) \tilde{P} + \gamma \tilde{Q}, \quad \partial_t \tilde{Q} = -\partial_x f(x) \tilde{Q} + 2\gamma \tilde{P} - 2\gamma f(x) \tilde{P} = 0. \quad (13)
\]

At large time, assuming a stationary state to exist, we set \( \partial_t \tilde{P} \) to zero in the first equation. This implies that the probability current density \( J(x, t) = f(x) \tilde{P}(x, t) + v_0 \tilde{Q}(x, t) \) converges to a constant \( J = \lim_{t \to +\infty} J(x, t) \) independent of \( x \). Hence, in the stationary state, we have \( f \tilde{P} + v_0 \tilde{Q} = J \), where \( J \) is yet to be determined. Eliminating \( \tilde{Q} \) using eq. (14), and setting \( \partial_x \tilde{Q} = 0 \), one obtains a first-order differential equation for \( \tilde{P} \)

\[
\frac{d}{dx} \left( [v_0^2 - f^2(x)] \tilde{P} + J f(x) \right) + 2\gamma J - 2\gamma f(x) \tilde{P} = 0. \quad (15)
\]

This equation can be explicitly solved for \( \tilde{P}(x) \), using the periodicity condition \( \tilde{P}(x + L) = \tilde{P}(x) \), see below. Knowing \( \tilde{P}(x) \) and \( \tilde{Q}(x) \) from the relation \( f \tilde{P} + v_0 \tilde{Q} = J \), one gets the stationary distribution for each state \( \sigma = \pm 1 \)

\[
\tilde{P}_{\pm}(x) = \frac{\pm J + (v_0 \mp f(x)) \tilde{P}(x)}{2v_0}. \quad (16)
\]

Finally, the unknown constant \( J \) is determined from the normalisation condition \( \int_{x_0}^L \tilde{P}(x) dx = 1 \) and consequently the velocity \( V_L = JL \) is obtained from eq. (5). The computation of the diffusion constant \( D_L \) is a bit more cumbersome, but it can be derived from a generalisation of the method used for the diffusive case [34–36]. The result for the zero-bias case \( G_L = 0 \) for phase A, is simpler and is given explicitly in eq. (24) (see [38] for details).

**Phase A.** In this case \( f(x) < v_0 \) for all \( x \) and the motion of the RTP is unbounded. Assuming a non-zero bias, i.e., \( G(L) \neq 0 \), and following the procedure outlined above, we obtain the stationary distribution

\[
\tilde{P}(x) = \frac{2\gamma J}{v_0^2 - f^2(x)} \left( \int_0^L du \Phi_-(x, u) \right) A_L - \int_0^x du \Phi_-(x, u) - f(x) \frac{v_0^2}{2\gamma}, \quad (17)
\]

where we have defined

\[
\Phi_{\pm}(x, u) = \frac{v_0^2}{v_0^2 - f^2(u)} e^{\pm(W(x) - W(u))}, \quad (18)
\]

\[
A_L = 1 - e^{-W(0) - W(L)} = 1 - e^{-G_L}. \quad (19)
\]

In the limit of zero bias \( G_L \to 0 \), one can show that \( \tilde{P}(x) \to \tilde{A} \Phi_+(0, x) \), for \( x \in [0, L] \) and \( \tilde{A} \) is a normalisation
constant. For arbitrary \( G_L \), by determining \( J \) from the normalisation condition \( f_0^L \mathrm{d}x \tilde{P}(x) = 1 \), we get the velocity \( V_L \) from (5)

\[
\frac{1}{V_L} = \frac{1}{L} \int_{(0, L)^2} \mathrm{d}x \mathrm{d}u \Psi(x, u) \left( \frac{1}{A_L} - \theta(x - u) \right) - \frac{G_L}{2\gamma L},
\]

where we have further defined

\[
\Psi(x, u) = \frac{2\gamma v_0^2 e^{-(W(x) - W(u))}}{(v_0^2 - f^2(x))(v_0^2 - f^2(u))}.
\]

Formula (20) for \( V_L \) is exact for any \( L \) (see footnote \(^3\)).

To study the \( L \to \infty \) limit, it is natural to assume that \( f(x) \) satisfies an ergodicity property, namely the existence of translational averages for local observables \( O[f](x) \), denoted as \( \langle O[f](x) \rangle_x = \lim_{L \to \infty} \frac{1}{L} \int_0^L \mathrm{d}x O[f](x) \). In addition (see eq. (9)) we assume that

\[
\lim_{L \to \infty} \frac{G_L}{2\gamma L} = \frac{f_{\text{eff}}}{v_0^2},
\]

where \( f_{\text{eff}} = \frac{s^2 f(x)}{v_0^2 - f^2(x)} \) is an "effective active force" that arises from the global bias \( G_L \). Without loss of generality, we assume \( f_{\text{eff}} > 0 \). Since \( G_L \to \infty \) from eq. (22) it implies \( \lim_{L \to \infty} A_L = 1 \) from (19). Using \( A_L = 1 \), eq. (20) can be re-arranged in a more compact form, leading to \( \lim_{L \to \infty} V_L = \) where

\[
\frac{1}{V} = \int_0^{+\infty} \mathrm{d}z \langle \Psi(x, x + z) \rangle_x = \frac{f_{\text{eff}}}{v_0^2}.
\]

In addition, the diffusion constant \( D_L^{1b} \) for the case of zero bias \( G(L) = 0 \), is obtained as (see [38] for the general case)

\[
\frac{D_0}{D_L^{1b}} = \frac{1}{L^2} \int_0^L \mathrm{d}w \Phi_+(0, u) \int_0^L \mathrm{d}x \Phi_-(0, u').
\]

In the large-\( L \) limit, \( D_L^{1b} \to D^{1b} \) with

\[
\frac{D_0}{D^{1b}} = v_0^2 \left( \frac{e^{-W(x)} e^{W(x)}}{v_0^2 - f^2(x)} \right)_x \times \left( \frac{e^{W(x)} e^{-W(x)}}{v_0^2 - f^2(x)} \right)_x.
\]

This formula is valid provided each translational average in (25) converges. Finally, in the diffusive limit \( v_0, \gamma \to +\infty \) with fixed \( D_0 = \frac{v_0^2}{2\gamma} \), one can check that our formulae (23) and (25) for \( L \to \infty \) reduce to the diffusive results obtained in [36].

**Phase B.** In this phase, there are \( 2n \) roots to the equation \( f(x) = v_0 \) in a period \( L \). Let us denote them by \( x_1^s \) (stable) and \( x_1^u \) (unstable), \( i = 1, \ldots, n \), with \( f'(x_1^s) < 0 \) and \( f'(x_1^u) > 0 \) (we assume for simplicity that \( f(x) \) is differentiable). We choose the period such that the roots are ordered as \( x_1^u < x_2^u < \cdots < x_n^u < x_{n+1}^s = x_1^s + L \), see fig. 2. The \( x_i^u \)s correspond, respectively, to stable and unstable fixed points when the RTP is in the state \( \sigma = -1 \). The motion of the RTP in the state \( \sigma = +1 \) is always to the right. Hence the RTP cannot cross any of stable points

\(^3\)For uniform \( f(x) = f \) it gives \( F = v \), as obvious from (1).

\[\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2a.png}
\caption{Plots of \( f(x) \) in a period \( x \in [0, L] \) showing the stable (s) and unstable (u) fixed points, i.e., the roots of \( f(x) = -\sigma v_0 \), when the RTP is in the state \( \sigma = -1 \) (top line) and \( \sigma = +1 \) state (lowest line). The RTP moves along the arrows and changes state with rate \( \gamma \). Top, phase B: In the \(-\) state the RTP moves left or right towards the fixed points \( x_i^s \) and in the \(+ \) state always to the right, leading to a mean velocity \( V_L > 0 \). Bottom, phase C: The RTP ends up in either intervals \( I_1 = [x_1^1, y_1^1] \) or \( I_2 = [x_2^1, y_2^1] \), which are the supports of the stationary measures (up to periodicity \( L \) ), and \( V_L = 0 \). Starting points in \( [x_1^2 - L, y_1^2] \) and \( [x_1^2, y_2^2] \) end up in \( I_1 \) and \( I_2 \), respectively, with probability one. Starting in \( [y_2^2, x_1^2] \) or \( [y_1^2, x_2^2] \), the RTP ends up randomly in either intervals.}
\end{figure}\]

This expression is smooth around the unstable points \( x_i^a \) but has singularities near the stable points \( x_i^s \) assuming that \( f'(x) \) is continuous. From (16) we also obtain the singularity associated to each state as \( \tilde{P}_\pm(x) \sim (x_i^\pm + \gamma) \phi_{i+1}^{\pm} \) with \( \phi_{i+1}^\pm = \phi_{i+1}^\pm + 1 \) and \( \phi_0^\pm = \phi_1^\pm \). The velocity is then obtained from normalisation \( \int_{x_1^s + L}^{x_1^s + L} \mathrm{d}x \tilde{P}(x) = 1 \) and \( J = V_L/L \) leading to the result for
Velocity and diffusion constant of an active particle in a one-dimensional force field

\[ \tilde{P}(x) = \sum_{i \in S} A_i \frac{B_1(|x_j - y_{i,0}|)}{V_{0}^2} e^{-2g_j u_i - \frac{f_0(\gamma)}{\sigma^2 F_{\text{tr}}}}, \quad (30) \]

where \( u_i \) can be chosen as the midpoint \( u_i = \frac{x_i + y_{i,0}}{2} \). The stationary measure has a support made of a collection of disjoint intervals and is zero elsewhere, which correspond to “downwards travels” of \( f(x) \), as represented in the bottom panel of fig. 2. The coefficients \( A_i, i \in S \), are however determined by the initial condition, together with the normalisation condition. Hence if there is more than one element in \( S \) the system is non-ergodic.

**Phase D.** In this case there are no roots to \( f(x) = \pm v_0 \). However there is a global bias and the velocity \( V_s \neq 0 \). It turns out that both the stationary \( \tilde{P}(x) \) and \( V_s \) are given by exactly the same formula as in phase A, namely by eqs. (17) and (20), respectively.

In the limit \( \gamma \to 0^+ \), transitions being rare, the velocity simplifies (in all phases) as \( V_s = \frac{1}{2}(V_+ + V_-) \), where \( V_+ = L/\int_0^L \frac{dx}{f(x)} \) is the velocity of an RTP frozen in state \( \sigma = \pm 1 \), with \( V_- = 0 \) if a root to \( f(x) = -v_0 \sigma \) exists [38].

**Transitions and velocity force characteristics.** As mentioned earlier, dynamical transitions can occur between these phases as some external parameters are varied, such that \( f(x) \) crosses the levels \( \pm v_0 \), see, e.g., fig. 1. Let us give a concrete example of this transition for the model \( f(x) = f_0(x) + F \) with \( f_0(x) = 4|x - 2| - 1 \) for \( x \in [0,1] \) and we set \( v_0 = 1 \) as well as \( L = 1 \). Clearly, in this case, \( f_{\text{max}}^{\text{root}} = v_0 = 1 \). If we now vary \( F \), we move along the horizontal line \( f_{\text{max}}^{\text{root}} = v_0 \) in the phase diagram in fig. 1. For any \( F > 0 \), the system is in phase B, a special case of this was discussed before for \( F = 1 \). However, exactly at \( F = 0 \) the system is in phase C. This critical point in this example is exactly at \( F = F_c = 0 \). As \( F \to 0 \), the velocity \( V_s \) vanishes as a power law \( V_s \sim (F - F_c)^{\beta} \) where the exponent \( \beta = \beta(\gamma) \) depends continuously on \( \gamma \). For example, we find \( (\beta(4) = 2 \) and \( \beta(2) = 1 \) [38].

Similarly, by varying \( f_{\text{max}}^{\text{root}} \) in fig. 1 one can induce a transition from phase A to phase C along the vertical line at \( F = 0 \). Here we provide a concrete example of this transition by considering the attractive logarithmic potential, \( f(x) = f_0(x) = -\frac{\alpha}{x^2 + \alpha^2} \), on the interval \([-L/2, +L/2] \), of period \( L \gg a \) (see footnote 5).

Note that in this case the global bias in eq. (9) vanishes, \( G_s = W(-L/2) - W(L/2) = 0 \), due to \( f(x) \) being an odd function. To proceed, we look for the possible real roots of \( f(x) = \pm v_0 \). It is easy to verify that the four roots are given by \( a \pm \sqrt{r^2 - 1} \) where \( r = \alpha/(2v_0) \). Clearly, if for \( L \gg a \) we can neglect the small jump in \( f(x) \) at \( x = \pm L/2 \).

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4f′(x) is discontinuous but the integral (26) is well defined.
$r < 1$, there is no real root – this corresponds to phase A. In contrast when $r > 1$ there are four real roots – this corresponds to phase C. Thus, by tuning $r$ across the critical value $r = 1$, the system can go from phase A to C.

- For $r > 1$, in phase C, following our general discussion before (see also fig. 2 bottom panel), there is only one region of space $[x_a, y_a]$ with $x_a = a(-r + \sqrt{r^2-1})$ and $y_a = a(r - \sqrt{r^2-1})$, where the particle gets trapped in the stationary state, irrespective of the initial condition (since $L \gg 2y_a$). Thus in this phase, both $V_L$ and $D_L$ vanish and the particle position is always localised (bound) at long times.

- In contrast, for $r < 1$, i.e., in phase A, the particle position at long times may or may not be localised in the limit $L \to \infty$. This can be clearly seen by examining the zero-bias ($G_1 = 0$) stationary distribution $\hat{P}(x) = \hat{A} \Phi_+(0,x)$ where $\Phi_+$ is given in (18). It is easy to check that, for large $1 \ll |x| < L/2$, $\hat{P}(x) \propto |x|^{-g}$ with the exponent $g = 2\gamma_\alpha/\omega^0$. If $g > 1$, the stationary distribution $\hat{P}(x)$ becomes independent of $L$ in the large-$L$ limit, since it can be normalised on the interval $(\infty, +\infty)$. Thus the particle position is bound in the large-$L$ limit. This can also be seen from the asymptotic behaviour of $D_L$ for large $L$, where eq. (24) predicts $D_L \sim d_r L^{-g}$, with $d_r$ some $r$-dependent constant. Thus for $g > 1$, the diffusion constant vanishes asymptotically for large $L$, confirming the bound state. On the other hand, if $g < 1$, there is no stationary distribution in the large-$L$ limit and the diffusion constant, for large-$L$, approaches a constant $D_L = D_0(1 - g^2)$. This leads to the phase diagram in the $(r, g)$ plane as shown in fig. 4. For $g > 1$, as $r \to 1^-$ from below (from phase A), $D_L$ has an essential singularity [38], i.e., $D_L \sim e^{-\pi g/\sqrt{1-r}}$ as $r \to 1^-$. **Mean first passage time (MFPT).** One can calculate the MFPT at a fixed level $x$, $T_+(x)$, for an RPT starting from $x$ in the state ±. In phase A, for an infinite line (not assuming periodicity) assuming $W(-\infty) = +\infty$, it reads

$$T_+(x) = \int_{x}^{\infty} \frac{2\gamma dy}{v_0 - f(y)} \int_{-\infty}^{y} \frac{e^{W(y)-W(z)}dz}{v_0 + f(z)} + T_-(X).$$

(31)

Here $T_-(X) = \frac{L}{\gamma} + 2 \int_{-\infty}^{X} \frac{e^{W(X)-W(z)}dz}{v_0 + f(z)}$ is the mean first return time to level $X$, which, for an RPT started in the state is non-zero, while $T_+(X) = 0$. In fact the difference is given for general $x$ by

$$T_+(x) - T_-(x) = 2 \int_{-\infty}^{x} \frac{dy e^{W(x)-W(z)}}{v_0 + f(z)} + \frac{1}{\gamma}.$$  

(32)

Note that in the diffusive limit $T_-(X) \to 0$, $T_+(x) - T_-(x) \to 0$ and one recovers the formula given in [37]. These are thus purely active quantities. We have checked [38] that the velocity in (23) can also be obtained from the limit $\lim_{X \to +\infty} T_\pm (0)/X = 1/V$.

![Fig. 4: Dynamical phase diagram of the attractive logarithmic potential model, in terms of $r$ (activity parameter) and $g$ (potential strength), exhibiting two binding transitions of different nature. In the A bound phase the stationary measure $\hat{P}(x)$ becomes bimodal for $g < 8r^2$.](image)

All our results extend to inhomogeneous transition rates $\gamma \to \gamma(x)$ and velocity $v_0 \to v_0(x)$, see [38] for details. For example, in the absence of an external force $f(x) = 0$, the velocity vanishes and the diffusion constant is given by

$$D_L = \left( \frac{L^2}{\int_0^L dx \frac{2\gamma(x)}{v_0(x)}} \right) \left( \frac{\int_0^L dx \frac{1}{v_0(x)}}{v_0^2} \right).$$

(33)

**Random landscape: velocity.** Consider now $f(x)$ a random force where each realisation is periodic $f(x + L) = f(x)$, but the probability distribution of $f(x)$ is independent of $x$. We restrict to the phase A in the large-$L$ limit. Let us define $\int f(x) = v_0^0 f(x)/v_0(x)$, with $f(x) = f_{\text{eff}}$, the effective bias defined in (22), which we choose to be non-negative (overbars denote averages over the random force). We assume that the translational average $\langle \ldots \rangle_z$ coincides with the disorder average. This implies, from (23),

$$V^{-1} = \int_0^{+\infty} dz K(z) = \frac{f_{\text{eff}}}{v_0^0},$$

(34)

in terms of the two-point correlator

$$K(z) = \frac{2\gamma_\alpha^2 e^{-W(0)-W(\text{z})}}{(v_0^0 - f^2(z))}.$$  

(35)

There are thus two possible phases separated by a threshold force $f_c$: i) if $f_{\text{eff}} \ll \infty$, $x$ does not have zero velocity $V > 0$ since the bias is positive, and ii) $f_{\text{eff}} \to +\infty$, for which the velocity vanishes. The first case occurs for large enough $f_{\text{eff}} > f_c$ since $W(0) - W(z) = f_{\text{eff}} z/D_0$.

**Random landscape: anomalous diffusion.** The existence of a $V = 0$ phase is a signature of anomalous diffusion. By tuning the random force we first consider the case $f_{\text{eff}} = 0$. Consider the case where $f(x)$ is short range correlated. Then $W(x)$ performs an unbiased random walk as a function of $x$. From (24), a good estimate, which is also a lower bound $\log \frac{D_L}{D_0} \geq -[\max_{x \in [0, L]} W(x) - \min_{x \in [0, L]} W(x)] + c$, with $c = \log \min_{x \in [0, L]} (1 - \frac{f(x)}{v_0^0})$. If $W(x)$ has bounded
moments, it behaves, under rescaling, as a Brownian motion, growing typically as $W(x) \sim \pm \sqrt{x}$, with $\sigma^2 = \int_0^\infty dxf(x)$. This lower bound then leads to the estimate $\frac{1}{\nu} \log D_{\nu} \approx -2\gamma \sigma \omega$, where the PDF of $\omega > 0$ is known. The diffusion time $T_{\nu}$ on scale $L$ is thus $\log T_{\nu} = \log(L^2/D_{\nu}) \sim 2\gamma \sigma \sqrt{L}$. This is similar to the Sinai problem [40] for a passive particle, as noted in [27].

Let us return to the case of non-zero bias, $f_{\text{eff}} > 0$, not studied in [27]. As discussed above, $V = 0$ for $f_{\text{eff}} < f_c$. The Brownian approximation for $W(x)$ allows to characterise the anomalous behaviour. Discarding the pre-exponential factors in (35) one obtains $V \sim f_{\text{eff}} - f_c$ for $f_{\text{eff}} > f_c$, with $f_c = \frac{\sigma^2}{\omega}$. In the zero velocity phase, anomalous diffusion $x \sim t^\mu$ is expected, as in the Sinai problem [40]. Qualitatively, from eq. (20), $\frac{1}{V_L} \sim f_{\text{eff}} e^{U(0)} - e^{W(x)} \sim e^{\Delta m} \sim L^{\mu - 1}$, where $\Delta_m$ is the maximum drawdown of the Brownian motion with $\Delta m = \mu \log L$ where $\mu = \frac{\sigma^2}{2\omega} = f_{\text{eff}}/f_c$ [41].

In conclusion, we have obtained analytical expressions for the stationary measure, the velocity and the diffusion constant for a single RTP in an arbitrary 1D periodic force field with period $L$. We obtained exact results both for finite $L$ and in the large-$L$ limit. We showed that, even for a single particle, the dynamics exhibits interesting phase transitions, with power law or exponential singularities in these observables. We also investigated [38] how the Fick’s law gets modified for non-interacting RTP’s subjected to a concentration gradient. It would be interesting to explore how these results are modified for interacting RTPs.

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