SYMMETRY REDUCTION OF EXTERIOR DIFFERENTIAL SYSTEMS AND BÄCKLUND TRANSFORMATIONS FOR PDE IN THE PLANE.

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Abstract. We approach the construction of Bäcklund transformations for Darboux integrable hyperbolic partial differential equations in the plane through the reduction of exterior differential systems.

1. Introduction

Let \( \mathcal{I} \) be an exterior differential system (EDS) on a manifold \( M \) and let \( p : M \to N \) be a smooth submersion. We define the reduced differential system \( \mathcal{I}/p \) on \( N \) by

\[
\mathcal{I}/p = \{ \theta \in \Omega^*(N) | p^*(\theta) \in \mathcal{I} \}.
\]

In the special case where \( G \) is a symmetry group of \( \mathcal{I} \) which acts regularly on \( M \), we shall write \( \mathcal{I}/G \) in place of \( \mathcal{I}/q_{G1} \), where \( q_{G1} : M \to M/G \) is the canonical projection to the space of orbits \( M/G \). Similarly if \( \Gamma \) is a Lie algebra of infinitesimal symmetries of \( \mathcal{I} \) which is regular on \( M \), we shall write \( \mathcal{I}/\Gamma \) in place of \( \mathcal{I}/q_{\Gamma} \), where \( q_{\Gamma} : M \to M/\Gamma \) is the canonical projection to the leaf space \( M/\Gamma \).

Two exterior differential systems (EDS) \( \mathcal{I}_1 \) and \( \mathcal{I}_2 \), defined on manifolds \( M_1 \) and \( M_2 \) are said to be related by a Bäcklund transformation if there exists an EDS \( \mathcal{B} \) on a manifold \( N \) which serves simultaneously as an integrable extensions for both \( \mathcal{I}_1 \) and \( \mathcal{I}_2 \). Precisely there are maps \( \pi_1 : N \to M_1 \) and \( \pi_2 : N \to M_2 \) where

\[
\begin{align*}
\mathcal{B} & \quad \pi_1 \quad \pi_2 \\
\mathcal{I}_1 & \quad \mathcal{I}_2
\end{align*}
\]

and \( \mathcal{B} \) is the Bäcklund transformation.

Bäcklund transformations \([1.2]\) can then be constructed using EDS reduction \([1.1]\) as follows.

**Theorem 1.1.** Let \( \mathcal{I} \) be a Pfaffian system on \( M \) and with symmetry groups \( G_1 \) and \( G_2 \). Let \( H \) be a subgroup of \( G_1 \) and \( G_2 \), and assume that

\[ [i] \] \( M/H, M/G_1 \) and \( M/G_2 \) are smooth manifolds with smooth quotient maps \( q_H : M \to M/H \), \( q_{G1} : M \to M/G_1 \), and \( q_{G2} : M \to M/G_2 \), then

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is a commutative diagram of EDS, where $p_1$ and $p_2$ are the orbit maps defined by

$$p_1(Hx) = G_1x \quad \text{and} \quad p_2(Hx) = G_2x \quad \text{for all } x \in M.$$

[iii] If the action of $G_1$ and $G_2$ are transverse to $I$, then $q_{G_1}$, $q_{G_2}$, $p_1$ and $p_2$ are all integrable extensions, and $I/H$ defines a Bäcklund transformation between $I/G_1$ and $I/G_2$.

In this article we will apply Theorem [1.1] to PDE in the plane. Let $M \subset J^2(\mathbb{R}^2, \mathbb{R})$ be

$$M = \{ p \in J^2(\mathbb{R}^2, \mathbb{R}) \mid F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0 \}$$

where $F$ is a hyperbolic PDE in the plane and let $\mathcal{I}_2$ be the standard rank 3 hyperbolic Pfaffian system determined by the restriction of the contact system on $J^2(\mathbb{R}^2, \mathbb{R})$ [8] to the level set $M$ in equation (1.5). Associated with the hyperbolic system $\mathcal{I}_2$ are a pair of rank 5 Pfaffian systems $\hat{V}$ and $\check{V}$ called the characteristic [8] or singular Pfaffian system [3]. A function $f \in C^\infty(M)$ is called a Darboux invariant if $df \in \hat{V}$ or $df \in \check{V}$, while the rank of the integrable subsystems $\hat{V}^\infty \subset \hat{V}$ and $\check{V}^\infty \subset \check{V}$ determine the number of independent invariant. The differential system $\mathcal{I}_2$ is said to be Darboux integrable (and not Monge integrable) if rank $\hat{V} = \text{rank} \check{V} = 2$. Our first theorem constructs Bäcklund transformations between generic PDE satisfying the Darboux integrability condition and the wave equation given as an EDS by $\mathcal{I}_{s=0}$.

**Theorem 1.2.** Let $\mathcal{I}_2$ be a rank 3 Pfaffian system on a seven manifold $M$ for a second order hyperbolic PDE in the plane which is Darboux integrable with singular Pfaffian systems $\hat{V}$ and $\check{V}$ satisfying $\text{rank} \hat{V}^\infty = 2$, $\text{rank} \check{V}^\infty = 2$. If the Vessiot algebra for $\mathcal{I}_2$ is not $\mathfrak{so}(3)$, then a local Bäcklund transformation $\beta$ with 1-dimensional fibre between $\mathcal{I}_2$ and $\mathcal{I}_{s=0}$ can be constructed through symmetry reduction

$$\mathcal{I}_{s=0} \rightarrow \mathcal{I}_2.$$

The EDS $\mathcal{K}_1 + \mathcal{K}_2$ in this theorem is easily constructed from $\mathcal{I}_2$. In the case where $F = 0$ in equation (1.5) is a Monge-Ampère equation, diagram (1.0) can be refined.
Theorem 1.3. If \( F = 0 \) in equation (1.5) is a Darboux integrable hyperbolic Monge-Ampère equation, then diagram (1.6) can be de-prolonged producing

\[
\begin{align*}
\bar{\mathcal{I}}_1 & \quad \mathcal{B} & \quad \bar{\mathcal{I}}_2 \\
\mathcal{I}_1 & \quad \mathcal{B} & \quad \bar{\mathcal{I}}_2 \\
q_{\mathcal{A}^{X_1}_{2}=0} & \quad \bar{\mathcal{B}} & \quad q_{\mathcal{A}^{X_2}_{2}} \\
\mathcal{I}_1 & \quad \bar{\mathcal{B}} & \quad \bar{\mathcal{I}}_2 \\
p_1 & \quad \bar{p}_1 & \quad \bar{p}_2 \\
\mathcal{B} & \quad \bar{\mathcal{B}} & \quad \mathcal{I}_2 \\
q_{\mathcal{A}^{X_1}_{2}} & \quad q_{\mathcal{A}^{X_2}_{2}} & \quad q_{\mathcal{A}^{X_1}_{2}} \\
\mathcal{K}_1 & \quad \mathcal{K}_2 \\
q_{\mathcal{R}_{\mathcal{G}_1}} & \quad q_{\mathcal{R}_{\mathcal{G}_1}} & \quad q_{\mathcal{R}_{\mathcal{G}_2}} \\
p_1 & \quad \bar{p}_1 & \quad \bar{p}_2 \\
\mathcal{B} & \quad \bar{\mathcal{B}} & \quad \mathcal{I}_2 \\
q_{\mathcal{R}_{\mathcal{G}_1}} & \quad q_{\mathcal{R}_{\mathcal{G}_1}} & \quad q_{\mathcal{R}_{\mathcal{G}_2}} \\
\mathcal{K}_1 & \quad \mathcal{K}_2
\end{align*}
\]  

where \( \bar{\mathcal{B}} \) is a rank 2 Pfaffian system on a 6-manifold and is a Bäcklund transformation with one dimensional fibre between the Monge-Ampère representation \( \bar{\mathcal{I}}_2 \) on a five-dimensional manifold for the PDE \( F = 0 \) and the Monge-Ampère system \( \mathcal{I}_{s=0} \) for the wave equation.

The Bäcklund transformation \( \bar{\mathcal{B}} \) can also be obtained using the following theorem.

Theorem 1.4. Let \( \mathcal{I}_2 \) be a Monge-Ampère system on a five manifold \( M_2 \) which is Darboux integrable (and not Monge integrable) after one prolongation and whose Vessiot algebra \( \text{vess}(\mathcal{I}_2^{1}) \) is not \( \text{so}(3) \), and let \( \mathcal{C}_1 + \mathcal{C}_2 \) be the standard contact structure on \( J^2(R,R) \times J^2(R,R) \). Then there exists two Lie algebras \( \bar{\Gamma}_{G_1} \) of infinitesimal contact transformations on \( J^2(R,R) \times J^2(R,R) \) where \( \bar{\Gamma}_H = \bar{\Gamma}_{G_1} \cap \bar{\Gamma}_{G_2} \) is two dimensional, which produces a (local) Bäcklund transformation \( \bar{\mathcal{B}} \) with one dimensional fibre between \( \mathcal{I}_2 \) and \( \mathcal{I}_{s=0} \) by the reduction diagram

\[
\begin{align*}
\mathcal{I}_{s=0} & = (\mathcal{C}_1 + \mathcal{C}_2) / \bar{\Gamma}_{G_1} \\
\mathcal{I}_2 & = (\mathcal{C}_1 + \mathcal{C}_2) / \bar{\Gamma}_{G_2}.
\end{align*}
\]  

The converse of Theorem 1.3 is proved in [2]. Given a Bäcklund transformation on a 6-dimensional manifold between a Monge-Ampère system \( \mathcal{I}_2 \) which is Darboux integrable (and not Monge integrable) after prolongation, and the Monge-Ampère system for the wave equation, there exists a three dimensional Lie algebras of vector-fields \( \Gamma_{G_2} \), and two-dimensional sub-algebra \( \Gamma_H \subset \Gamma_{G_2} \) such that the Bäcklund transformation \( \mathcal{B} \) can be constructed by symmetry reduction. In particular, the equation

\[
u_{xy} = \frac{\sqrt{1 - u_x^2} \sqrt{1 - u_y^2}}{\sin u},
\]  

has Vessiot algebra \( \text{so}(3) \) which has no two-dimensional subalgebra. Therefore equation (1.9) admits no Bäcklund transformation with one-dimensional fibre with the wave equation. This disagrees with Theorem 1 in [6]. The article [6] also establishes the existence of Bäcklund transformations through
the Cartan-Kähler theorem while the Bäcklund transformations whose existence is given here require only $C^\infty$ group actions.

2. Reduction of Differential Systems

2.1. Preliminaries. An EDS $\mathcal{I} \subset \Omega^*(M)$ on a manifold $M$ is a differentially closed ideal (see [4], and [9]). The EDS $\mathcal{I}$ has constant rank if each of its homogeneous components $I^k \subset \Omega^k(M)$ coincide with the sections $\mathcal{S}(I^k)$ of a constant rank sub-bundle $I^k \subset \Lambda^k(T^*M)$. If $\mathcal{A}$ is a subset of $\Omega^*(M)$, we let $\langle \mathcal{A} \rangle_{\text{alg}}$ and $\langle \mathcal{A} \rangle_{\text{diff}}$ be the algebraic and differential ideals generated by $\mathcal{A}$.

A constant rank Pfaffian differential system $I$ is an EDS for which there exists a sub-bundle $I \subset T^*M$ such that $I = \langle S(I) \rangle_{\text{diff}}$. We also refer to a constant rank sub-bundle $I \subset T^*M$ as a Pfaffian system.

A local first integral of a Pfaffian system $I$ is smooth function $f: U \to \mathbb{R}$, defined on an open set $U$, and such that $df \in I$. For each point $x \in M$ we define

$$I^\infty_x = \{ df_x | f \text{ is a local first integral, defined about } x \}.$$  

We shall always assume that $I^\infty = \bigcup_{x \in M} I^\infty_x$ is a constant rank bundle on $M$. It is easy to verify check that $I^\infty$ is the (unique) maximal, completely integrable, Pfaffian subsystem of $I$. The bundle $I^\infty$ can be computed algorithmically from the derived flag of $I$. The derived system $I' \subset I$ of a Pfaffian system $I$ is defined pointwise by

$$I'(p) = \text{span}\{ \theta_p | \theta \in S(I) \text{ } d\theta \equiv 0 \mod I \}.$$  

The system $I$ is integrable if it satisfies the Frobenius condition if $I' = I$. Letting $I^{(0)} = I$ and assuming $I^{(k)}$ is constant rank we define inductively

$$I^{(k+1)} = (I^{(k)})', \quad k = 0, 1, \ldots, N$$  

where $N$ is the smallest integer where $I^{(N+1)} = I^{(N)}$. Therefore $I^\infty = I^{(N)}$ when $I^{(k)}$ are constant rank. More information about this sequence can be found in [4], and [9].

We also recall the definition of an integrable extension [5]. First, if $\mathcal{I}$ and $\mathcal{J}$ are differential systems, we let $\mathcal{I} + \mathcal{J} = \langle \mathcal{I} \cup \mathcal{J} \rangle_{\text{alg}}$ which is also a differential system. Let $p : M \to N$ be a surjective submersion and $\mathcal{I}$ an EDS on $N$. An EDS $\mathcal{E}$ on $M$ is called an integral extension of $\mathcal{I}$ if there exists a subbundle $J \subset \Lambda^1(M)$ of rank $\dim M - \dim N$, such that $J$ is transverse to $p$, that is,

$$\text{(2.2a)} \quad \text{ann}(J) \cap \ker (p_*) = 0,$$

and

$$\text{(2.2b)} \quad \mathcal{E} = \langle p^*(\mathcal{I}) + S(J) \rangle_{\text{alg}}.$$  

A sub-bundle $J$ satisfying these two properties is called an admissible sub-bundle for the extension $\mathcal{E}$. Given any (immersed) integral manifold $s: P \to M$ of $\mathcal{E}$, then by condition $\text{(2.2a)}$ the composition $p \circ s : P \to N$ is an (immersed) integral manifold of $\mathcal{I}$. If $s: P \to N$ is an integral manifold of $\mathcal{I}$, then by condition $\text{(2.2b)}$ the restricted of the EDS $\mathcal{E}$ to $p^{-1}(s(P))$ is an integrable Pfaffian system.
2.2. Reduction, Pullback Bundles, and Semi-basic Forms. Let $p : M \to N$ be a smooth submersion and let $\mathcal{I}$ be an EDS on $M$ and $\mathcal{I}/p$ be the reduction of $\mathcal{I}$ (see (1.1)). In this section we give easily verifiable conditions which guarantee that $\mathcal{I}/p$ is constant rank and provide a simple way to compute local basis of sections for $\mathcal{I}/p$.

Let $\text{Vert}(M) = \ker(p_*) \subset T^*M$ be the vertical distribution for the submersion $p : M \to N$. A vector field $X$ on $M$ taking values in $\text{Vert}(M)$ is called a vertical vector-field. A differential form $\theta \in \Omega^*(M)$ is $p$-semi-basic if $X \lhd \theta = 0$ for all vertical vector-fields $X$, and $\theta$ is called $p$-basic if there exists $\tilde{\theta} \in \Omega^*(N)$ such that $\theta = p^*(\tilde{\theta})$. Likewise, for any sub-bundle $I^k \subset \Lambda^k(T^*M)$ the subset of $p$-semi-basic $k$-forms is defined by

\begin{equation}
I^k_{p,\text{sb}} = \{ \theta \in I^k \mid X \lhd \theta = 0 \text{ for all } X \in \text{Vert}(M) \}.
\end{equation}

Clearly any $p$-semi-basic $k$-form takes values in $I^k_{p,\text{sb}}$.

Now let $\bar{I}^k \subset \Lambda^k(N)$ be a rank $r$ sub-bundle. The pullback bundle $p^*(\bar{I}^k) \subset \Lambda^k(T^*M)$ is the rank $r$ sub-bundle given point-wise by

\begin{equation}
p^*(\bar{I}^k)_x = \{ p^*(\tilde{\theta}_y) \mid \text{for all } \tilde{\theta}_y \in \Lambda^k(T^*_yN), \ y = p(x) \}.
\end{equation}

Note that $p^*(\bar{I}^k) \subset \Lambda^k(T^*M)_{p,\text{sb}}$. Lemma 2.1 below characterizes pullback bundles and is the bundle version of Proposition 6.1.19 in [9].

**Lemma 2.1.** Suppose the fibres of $p : M \to N$ are connected. Given a sub-bundle $I^k \subset \Lambda^k(T^*M)$ there exists a sub-bundle $\bar{I}^k \subset \Lambda^k(T^*N)$ such that $I^k = p^*(\bar{I}^k)$ if and only if $I^k$ is $p$-semi-basic and for all vertical vector fields $X$

\begin{equation}
\Psi^X_t(I^k) = I^k,
\end{equation}

where $\Psi^X_t$ is the flow of $X$.

Condition (2.2) is equivalent to the condition that for all $\theta \in \mathcal{S}(I^k)$

\begin{equation}
X \lhd \theta = 0, \quad L_X \theta \in \mathcal{S}(I^k),
\end{equation}

for all vertical vector-fields $X$. See Proposition 6.1.19 in [9]. Note that $\Lambda^k(T^*M)_{p,\text{sb}}$ satisfies (2.2) (or (2.6)) and that $\Lambda^k(T^*M)_{p,\text{sb}} = p^*(\Lambda^k(T^*N))$.

In analogy with the definition of $\mathcal{I}/p$ in (1.1) the reduction of a bundle $I^k \subset \Lambda^k(T^*M)$ is defined by

\begin{equation}
I^k/p = \{ \theta \in \Lambda^k(T^*N) \mid p^*(\theta) \in I^k \}.
\end{equation}

Lemma 2.1 then has the following corollary.

**Corollary 2.2.** The set $I^k/p$ is a constant rank bundle if and only if $I^k_{p,\text{sb}}$ (equation (2.3)) is constant rank and vertically invariant (equation (2.6)). In which case $I^k_{p,\text{sb}} = p^*(I^k/p)$.

Let $\mathcal{I}$ be a constant rank EDS. The subset $\mathcal{I}_{p,\text{sb}} \subset \mathcal{I}$ of semi-basic forms is

\begin{equation}
\mathcal{I}_{p,\text{sb}} = \{ \theta \in \mathcal{I} \mid X \lhd \theta = 0 \text{ for all vertical vector fields } X \}.
\end{equation}
If $I/p$ is the reduction of $I$ then

$$p^*(I/p) \subset I_{p, sb}.$$  

Note that neither $p^*(I/p)$ or $I_{p, sb}$ are differential ideal. The next theorem is the basic theorem on reduction. It’s proof follows immediately from Lemma 2.1 and Corollary 2.2.

**Theorem 2.3.** Let $p : M \rightarrow N$ be a smooth submersion with connected fibres. If $I$ is a constant rank EDS and if $I^k_{p, sb}$ (see equation (2.3)) are constant rank and satisfy the vertical invariance condition in equation (2.5) (or (2.3)) then

[i] there exists constant rank bundles $\bar{I}^k \subset \Lambda^k(T^*N)$ such that $I^k_{p, sb} = p^*(\bar{I}^k)$,

[ii] $I/p$ is a constant rank EDS (with bundles $\bar{I}^k$), and

[iii] $(I_{p, sb})_{alg} = (p^*(I/p))_{alg}$.

The vertical invariance conditions in Theorem 2.3 can be inferred from the invariance of $I$.

**Lemma 2.4.** If $I$ is constant rank and vertically invariant (equation (2.5)) then

[i] the semi-basic subset $I^k_{p, sb}$ is vertically invariant for each $k$,

[ii] the space of invariants $I^\infty$ is vertically invariant and

[iii] the derived systems $I^{(k)}$ are vertically invariant.

**Proof.** Let $X$ and $Y$ be vertical vector fields on $M$, and let $\Psi^X_t$ be the flow of $X$. Then

$$Y \cdot (\Psi^X_t)^* \theta = \theta((\Psi^X_t)_* Y).$$

Now $\Psi^X_t Y$ is a vertical vector-field (on its domain), and so if $\theta \in I_{p, sb}^k$ then $\theta((\Psi^X_t)_* Y) = 0$ which proves part [i].

Both [ii] and the case if $I^{(1)} = I'$ in part [iii] follow from the vertical invariance of $I$ (equation (2.5)), and the commutativity of the exterior derivative and the pullback. The rest of part [iii] follows by induction.

The following lemma is particularly useful for computing local basis of sections in reduction.

**Lemma 2.5.** Suppose $I^k_{p, sb} = p^*(\bar{I}^k)$, and let $\sigma : U \rightarrow U$ be a local cross-section to $p$. Then

[i] $\sigma^*(I^k_{p, sb}) = \bar{I}^k$, and

[ii] if $\theta \in S(I^k_{p, sb})$ then $\sigma^* \theta \in S(\bar{I}^k)$. 

[iii] Given a local basis of sections $\{\theta^a\}$ for $I^k_{p, sb}|_U$ and a cross-section $\sigma : U \rightarrow U$, then $\sigma^* (\theta^a)$ is a local basis of sections for $\bar{I}^k|_U$.

[iv] About each point $x \in M$ there exists an open set $U$ and a local basis $\{\theta^a\}$ of sections for $I^k_{p, sb}$ where each $\theta^a$ is $p$ basic.

**Remark 2.6.** The point behind Lemma 2.5 is the following. Given generators $I = \langle \alpha^1, \ldots, \alpha^k \rangle_{alg}$ we find generators for the semi-basic forms $I_{p, sb} = span\{\tilde{\alpha}^1, \ldots, \tilde{\alpha}^r \}$ from $I$ using only linear algebra (equation (2.8)). Local generators for $I/p$ are then computed using the pullback $(I/p)|_U = \langle \sigma^* \tilde{\alpha}^1, \ldots, \sigma^* \tilde{\alpha}^r \rangle_{alg}$ by a local section $\sigma : U \rightarrow U$ of $p : M \rightarrow N$. Lemma 2.5 is also useful for producing an adapted set of generators for $I$. In particular if $(I/p)|_U = \langle \tilde{\alpha}^1, \ldots, \tilde{\alpha}^r \rangle_{alg}$ then

$$I|_U = \langle p^* \alpha^1, p^* \alpha^r, \alpha^{r+1}, \ldots, \alpha^k \rangle_{alg},$$

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where the forms $p^*\tilde{\alpha}$ are $p$ basic.

2.3. Reduction by Cauchy Characteristics. Here we briefly review the theory of reduction by Cauchy characteristics again see [4], [9] and relate this to the previous section.

Let $\Delta \subset TM$ be an integrable distribution. Given $x \in M$ we denote by $S_x$ the maximal (connected) integral manifold through $x$. Let $q_{\Delta} : M \to M/\Delta$ be the quotient map by the maximal integral manifolds of $\Delta$ and so $q_{\Delta}(x) = S_x$. We call $\Delta$ regular if $M/\Delta$, endowed with the quotient topology, admits a unique manifold structure such that $q_{\Delta}$ is a smooth submersion. Note that regularity implies $\Delta$ is constant rank.

Let $I$ be a constant rank EDS. The set

\[(2.9) \quad A_I = A(I) = \{X \in TM \mid X \bot I \subset I\}\]

is the space of Cauchy characteristic of $I$ which is an integrable distribution [4]. From here on we will assume that $A(I)$ is regular and denote by $q_A : M \to M/A(I)$ the smooth quotient map to the leaf space.

The main theorem on the reduction by Cauchy Characteristics is the following, see [4], [9].

**Theorem 2.7.** Let $I$ be a constant rank EDS on an $n$-dimensional manifold $M$ and suppose $A(I)$ is regular. If $I_{q_A,\text{sb}}^k \subset I^k$ is constant rank for each $k = 1, \ldots, n$, then the constant rank EDS $\bar{I} = I/q_A$ satisfies

\[I = \langle q_A^*\bar{I} \rangle_{\text{alg}},\]

Equation (2.9) implies that $I$ is vertically invariant (equation (2.5)), and therefore by Lemma 2.4 the hypothesis in Theorem 2.3 are satisfied and $I/q_A$ is a constant rank EDS. The difference between Theorem 2.7 and 2.3, which is an essential point in reduction by Cauchy characteristics, is that $I$ admits a set of $q_A$ semi-basic generators.

Again, local generators for $I/q_A$ can be constructed using Lemma 2.5. For examples of Cauchy characteristic reduction see [4], [9].

**Remark 2.8.** First note that Theorem 2.7 holds for any regular integrable distribution $\Delta \subset A(I)$, where $q_{\Delta} : M \to M/\Delta$ is the smooth quotient map. Let $p : M \to N$ be a smooth submersion with connected fibres and $I$ and EDS on $M$. If the EDS $\langle I_{p,\text{sb}} \rangle_{\text{alg}}$ is vertically invariant (equations (2.5), (2.6)) then

\[\text{Vert}(M) \subset A(\langle I_{p,\text{sb}} \rangle_{\text{alg}})\]

is an integrable sub-distribution of the Cauchy characteristics of $\langle I_{p,\text{sb}} \rangle_{\text{alg}}$. Then

\[I/p = (\langle I_{p,\text{sb}} \rangle_{\text{alg}})/p = (\langle I_{p,\text{sb}} \rangle_{\text{alg}})/\text{Vert}(M).\]

In other words the reduction $I/p$ is Cauchy characteristic reduction of the EDS generated by the semi-basic forms by the subset $\text{Vert}(M)$.  

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2.4. **Group Reduction.** Let $G$ be a Lie group acting on $M$. The action $\mu: G \times M \to M$ is said to be regular if the space of orbits $M/G$ with the quotient topology admits a smooth manifold structure such that the projection map $q_G: M \to M/G$ is a submersion. We let $\Gamma_G$ denote the Lie algebra of infinitesimal generators of $G$ and let $\Gamma_G$ be the distribution spanned by $\Gamma_G$. In particular for regular actions $\text{Vert}(M) = \ker(q_{G,*}) = \Gamma_G$. The action of $G$ is a symmetry group of the EDS $\mathcal{I}$ if $g^*\mathcal{I} = \mathcal{I}$ for all $g \in G$.

We assume the action of $G$ on $M$ is regular and a symmetry group of $\mathcal{I}$ and we shall write $\mathcal{I}/G$ in place of $\mathcal{I}/q_G$. Equation (1.1) gives

\[ I/G = \{ \bar{\theta} \in \Omega^*(M/G) \mid q_G^*\bar{\theta} \in \mathcal{I} \}. \]

The analogue to Corollary 2.3 in [4], which characterizes basic forms, and Lemma 2.1 is the following.

**Lemma 2.9.** A differential form $\theta \in \Omega^k(M)$ is $q_G$ basic if and only if

\[ X \lhd \theta = 0, \quad g^*\theta = \theta, \]

for all $X \in \Gamma$. A rank $r$ sub-bundle $I^k \subset \Lambda^k(M)$ satisfies

\[ X \lhd I^k = 0, \quad g^*I^k = I^k \]

if and only if there exists a rank $r$ sub-bundle $\bar{I}^k \subset \Lambda^k(M/G)$ such that $I^k = q_G^*(\bar{I}^k)$.

The set $\mathcal{I}_{G,\text{sb}} \subset \mathcal{I}$ of $q_G$ semi-basic forms is

\[ \mathcal{I}_{qG,\text{sb}} = \{ \theta \in \mathcal{I} \mid X \lhd \theta = 0 \}. \]

It follows in the same manner as in Lemma 2.5 that the subset $\mathcal{I}_{qG,\text{sb}}$ is $G$-invariant. The analogue of Theorem 2.3 for symmetry groups is the following.

**Theorem 2.10.** Let $G$ be a Lie group acting regularly on $M$ which is a symmetry group of the constant rank EDS $\mathcal{I}$. If the bundles $I^k_{qG,\text{sb}}$ are constant rank, then

[i] there exists constant rank bundles $\bar{I}^k \subset \Lambda^k(T^*(M/G))$ such that $I^k_{qG,\text{sb}} = q_G^*(\bar{I}^k)$,

[ii] $\mathcal{I}/q_G$ is a constant rank EDS (with bundles $\bar{I}^k$), and

[iii] $\langle \mathcal{I}_{qG,\text{sb}} \rangle_{\text{alg}} = \langle q_G^*(\mathcal{I}/q_G) \rangle_{\text{alg}}$.

The vertical invariance condition (2.5) is not necessary in Theorem 2.10 because of the hypothesis that $G$ is a symmetry group of $\mathcal{I}$. Also note, as in Remark 2.8, that the vertical space $\Gamma_G$ is a subset of the Cauchy characteristics for $\langle \mathcal{I}_{qG,\text{sb}} \rangle_{\text{alg}}$. If $G$ has connected orbits then $\mathcal{I}/G$ is Cauchy reduction of $\langle \mathcal{I}_{qG,\text{sb}} \rangle_{\text{alg}}$ by $\Gamma_G$.

A symmetry group $G$ of an EDS $\mathcal{I}$ is said to be **transverse** to $\mathcal{I}$ if

\[ \text{ann}(I^1) \cap \Gamma_G = 0, \]

where $I^1 \subset T^*M$ is the sub-bundle whose sections are the differential 1-forms in $\mathcal{I}$. The actions considered in this article will all satisfy the transversality condition (2.11). For more information on transversality in reduction see [1]. In particular in [1] it is shown that for transverse actions the
hypothesis in Theorem 2.10 are satisfied and \( \mathcal{I}/G \) is a constant rank EDS. For example, by equation (2.11) the rank of the one forms in \( \mathcal{I}/G \) is given by \( \text{rank}(\mathcal{I}/G)^1 = \text{rank} \mathcal{I}^1 - \text{rank} \mathcal{I}_G \).

2.5. The Derived System. We now examine the behavior of the the derived bundles for a Pfaffian system \( I \subset T^*M, \) under reduction.

**Theorem 2.11.** Let \( I \subset T^*M \) be a constant rank sub-bundle and assume that \( \bar{I} = I/p \subset T^*N \) is constant rank. Then for any \( x \in M, \) \( I'_{p,\text{sb}}(x) = p^*(\bar{I}'(y)) \) where \( y = p(x) \), and

\[
\text{rank} \bar{I}'(y) \leq \text{rank} I'(x) \leq \text{rank} \bar{I}'(x) + \text{rank} I - \text{rank} \bar{I}
\]

**Proof.** Let \( x \in M, \) \( y = p(x) \) and let \( U \) be an open set in \( N \) containing \( y \) where \( \bar{\theta}^i, \bar{\eta}^a \) form a local basis of sections for \( I \) such that

\[
\bar{I}|_{\mathcal{T}} = \text{span} \{\bar{\theta}^i, \bar{\eta}^a\}, \quad \bar{I}'(q) = \text{span} \{\bar{\eta}^a\}.
\]

Next let \( U \subset p^{-1}(U) \) be an open subset containing \( x \) where \( p^*\bar{\theta}^i, p^*\bar{\eta}^a \) can be completed to a local basis of sections for \( I \) by \( \theta^r, \)

\[
I|_U = \text{span} \{\theta^r, p^*\bar{\theta}^i, p^*\bar{\eta}^a\}.
\]

If \( \rho \in I'_{p,\text{sb}}(x) \), then there exists \( A_r, B_i, C_a \in C^\infty(U) \) such that

\[
\rho = A_r \theta^r + B_i \bar{\theta}^i + C_a p^*\bar{\eta}^a
\]

and \( A_r(x) = 0 \) by the condition \( \rho \) is \( p \) semi-basic at \( x \). Taking into account that \( A_r(x) = 0 \), the condition \( \rho \in I'(x) \) is

\[
B_i(x) p^*d\bar{\theta}^i_z + C_a(x) p^*d\bar{\eta}^a_z = 0 \mod I.
\]

Equation (2.16) implies there exists \( \alpha_r, \beta_i, \gamma_a \in T^*_ZM \) so that

\[
B_i(x) p^*d\bar{\theta}^i_z + C_a(x) p^*d\bar{\eta}^a_z = \alpha_r \wedge \theta^r_x + \beta_i \wedge p^*\bar{\theta}^i_z + \gamma_a \wedge p^*\bar{\eta}^a_z
\]

where we may assume \( \alpha_r \) contain no \( p^*\bar{\theta}^i_z \) or \( p^*\bar{\eta}^a_z \) terms.

Now \( B_i(x) p^*d\bar{\theta}^i_z + C_a(x) p^*d\bar{\eta}^a_z \) is \( p \) semi-basic, and therefore since \( \theta^r_x \) are now \( p \) semi-basic the term \( \alpha_r \wedge \theta^r_x \) is zero. Furthermore the terms \( \beta_i, \gamma_a \) are \( p \) semi-basic which implies there exists \( \bar{\beta}_i, \bar{\gamma}_a \in T^*_YN \) so that \( \beta^i = p^*\bar{\beta}^i, \gamma^a = p^*\bar{\gamma}^a \). Equation (2.16) then reads

\[
B_i(x) d\bar{\theta}^i_y + C_a(x) d\bar{\eta}^a_y = \bar{\beta}_i \wedge \bar{\theta}^i_y + \bar{\gamma}_a \wedge \bar{\eta}^a_y.
\]

This equation implies \( B_i(x) d\bar{\theta}^i_y \in \bar{I}'|_y \) which by the choice of coframe for \( \bar{I} \) in equation (2.13) gives \( B_i(x) = 0 \). Therefore \( \rho \) in equation (2.14) satisfies \( \rho(x) = C_a(x) p^*\bar{\eta}^a_y \). This implies \( I'_{p,\text{sb}}(x) = p^*\bar{I}'(y) \).

Finally to show the inequality (2.12) note that the index \( r \) in equation (2.14) satisfies \( 1 \leq r \leq \text{rank} I - \text{rank} \bar{I} \). This shows the inequality (2.12) holds.

**Corollary 2.12.** The subset \( I'_{p,\text{sb}} \) is constant rank if and only if \( \bar{I}' \) is constant rank in which case the ranks are equal, \( I'/p = \bar{I}' \), and

\[
0 \leq \text{rank} I' - \text{rank} \bar{I}' \leq \text{rank} I - \text{rank} \bar{I}.
\]
Theorem 2.11 and Corollary 2.12 state that bundle reduction and derivation commute. We also have the following corollary of Theorem 2.11.

**Corollary 2.13.** Let $\mathcal{I} = \langle S(I) \rangle_{aff}$ be the Pfaffian system generated by $I \subset T^*M$, and suppose $\tilde{\mathcal{I}} = \langle S(\tilde{I}) \rangle_{aff}$ is the Pfaffian system generated by $\tilde{I} = I/p$. Then

$$\mathcal{I}'/p \cap \Omega^1(N) = \tilde{\mathcal{I}}' \cap \Omega^1(N), \quad \text{and} \quad \tilde{\mathcal{I}}' \subset \mathcal{I}'/p.$$  

**Remark 2.14.** In Corollary 2.13, by definition $\tilde{\mathcal{I}}'$ is the Pfaffian system generated by $\tilde{I}'$, while $\mathcal{I}'/p$ is not necessarily a Pfaffian system and so there is no reason for these to be equal. Corollary 2.13 does say that these two systems have the same set of one-forms.

Using definition (2.1) we now look at the reduction of $I^\infty$. We begin with the following lemma.

**Lemma 2.15.** Suppose that $I \subset T^*M$ is a constant rank completely integrable sub-bundle, and that $\tilde{I} = I/p$ is constant rank. Then $\tilde{I}$ is completely integrable.

**Proof.** Since $I$ is completely integrable $I' = I$ and so $I'_{p,ab}$ is constant rank and $I'_{p,ab} = p^* \tilde{I}'$. However $I' = I'/p = I/p = I$ and so $\tilde{I}$ is completely integrable.

**Corollary 2.16.** Suppose that $I, I^\infty, \tilde{I} = I/p$, and $I_{p,ab}^\infty$ are constant rank. Then

$$I^\infty/p = \tilde{I}^\infty.$$  

**Proof.** We begin by using Lemma 2.15 to note that $I^\infty/p$ is completely integrable. Now $p^* \tilde{I}^\infty \subset I^\infty$ implies

$$\tilde{I}^\infty \subset I^\infty/p.$$  

However $I^\infty/p \subset I/p$ which by the maximality of $\tilde{I}^\infty$ implies $I^\infty/p \subset \tilde{I}^\infty$. This along with equation (2.18) imply $\tilde{I}^\infty = I^\infty/p$.

The derived series behaves similarly.

**Corollary 2.17.** Let $I^k$ be the $k^{th}$ derived bundle and suppose $I^{k+1}$ is constant rank, $I^k/p$ is constant rank, and $(I^k/p)'$ is constant rank. Then $I^{k+1}/p$ is constant rank and $(I^k/p)' = I^{k+1}/p$. Furthermore, if $I^k/p$ are constant rank for $k = 0 \ldots m$, and $(I/p)^k$ are constant rank for $k = 0 \ldots m$, then $I^k/p$ is constant rank and $(I/p)^k = I^k/p, k = 0 \ldots m$.

### 3. Bäcklund Transformations

We recall the following key result proved in [2], from which Theorem 3.1 follows.

**Theorem 3.1.** Let $G$ and $H$ be Lie groups acting act regularly on $M$ where $H$ is a subgroup of $G$. If $G$ is a symmetry group of the EDS $\mathcal{I}$ which acts transversally to $\mathcal{I}$, then

$$\begin{array}{c}
(M, \mathcal{I}) \xrightarrow{q_H} (M/H, \mathcal{I}/H) \\
\downarrow q_G \downarrow \\
(M/G, \mathcal{I}/G)
\end{array}$$
is a commutative diagram of integrable extensions where

\[ p : M/H \to M/G \]  
(3.1) 

\[ p(Hx) = Gx. \]

Applying Theorem 3.3 with the hypothesis of Theorem 1.1 proves Theorem 1.1 and provides the Bäcklund transformation \( I/H \) in diagram (1.3). We demonstrate Theorem 3.1 and Theorem 1.1 with an example.

**Example 3.2.** Let \( I = K_1 + K_2 \) be the standard contact system on \( J^3(R, R) \times J^3(R, R) \). In local coordinates \( (x, v, v_x, v_{xx}, y, w, w_y, w_{yy}) \) we have

\[
\mathcal{I} = \left\{ \begin{array}{l}
\theta^v = dv - v_x dx, \quad \theta^{v_x} = dv - v_{xx} dx, \\
\theta^w = dw - w_y dy, \quad \theta^{w_y} = dw - w_{yy} dy
\end{array} \right\} \]  
(3.2)

and let

\[
\Gamma_{G_2} = \text{span}\{ \partial_v - \partial_w, \ pr(v \partial_v + w \partial_w), \ pr(v^2 \partial_v - w^2 \partial_w) \},
\]
(3.3)

\[
\Gamma_{G_1} = \text{span}\{ \partial_v, \partial_w, pr(v \partial_v + w \partial_w) \},
\]
(3.4)

\[
\Gamma_H = \Gamma_{G_1} \cap \Gamma_{G_2} = \text{span}\{ \partial_v - \partial_w, \pr(v \partial_v + w \partial_w) \}.
\]

Let \( M \) be the open set \( v, v_x, w, w_y > 0 \). The distributions \( \Gamma_H \), and \( \Gamma_{G_a} \) are regular on \( M \). The manifolds \( M/\Gamma_{G_a} \) are 7 dimensional while \( M/\Gamma_H \) is 8. With the following choices of coordinates

\[
M/\Gamma_{G_1} = (x, y, z, z_x, z_{xx}, z_{yy}), \quad M/\Gamma_{G_2} = (x, y, u, u_x, u_y, u_{xx}, u_{yy}),
\]
(3.5)

\[
M/\Gamma_H = (x, y, V, W, V_x, W_y, V_{xx}, W_{yy}),
\]

the quotient maps \( q_{\Gamma_{G_1}}, q_{\Gamma_{G_2}} \) and \( q_{\Gamma_H} \) in coordinates \( (3.6) \) are

\[
q_{\Gamma_{G_1}} = (x = x, y, z = \log(w_x/w_y), z_x = D_x(z) = \frac{w_x}{w_y}, z_{xx} = D_x(z_{xx}) = \frac{w_{xx}}{w_y}),
\]
(3.6)

\[
q_{\Gamma_{G_2}} = (x = x, y, u = \log\frac{2w_y v_x}{(v + w)^2}, u_x = D_x(u) = \frac{w_y u_x}{v + w}, u_{xx} = D_x(u_{xx}) = \frac{w_{xx} u}{v + w}),
\]

\[
q_{\Gamma_H} = (x = x, y, V = \log\frac{v_x}{v + w}, W = \log\frac{w_y}{v + w}, W_x = D_x(V), W_y = D_y(W),
\]

\[
V_{xx} = D_x(V_x), W_{yy} = D_y(W_y),
\]

where \( D_x \) and \( D_y \) are the total derivatives. Using the maps in equation (3.6), the projection maps \( p_1 \) and \( p_2 \) which make the commutative diagram (1.10) are easily seen to be given in coordinates by

\[
p_1 = (x = x, y = y, z = V - W, z_x = V_x + e^V, z_y = -W_y - e^W, z_{xx} = D_x(z_x), z_{yy} = D_y(W_y)),
\]
(3.7)

\[
p_2 = (x = x, y = y, u = \log(2) + V + W, u_x = V_x - e^V, u_y = W_y - e^W, u_{xx} = D_x(V_x), u_{yy} = D_y(W_y)).
\]
Since $\Gamma_{G_1}$ and $\Gamma_H$ consist of prolongations of vector-fields, they are infinitesimal symmetries of $I$ and so we can use Lemma 2.5 to compute the reductions that produce diagram (3.3)

$$\frac{\mathcal{I}}{\Gamma_{G_1}} = (du - z_x dx - z_y dy, dz_x - z_{xx} dx, dz_y - z_{yy} dy)_{diff}$$

$$\frac{\mathcal{I}}{\Gamma_{G_2}} = (du - u_x dx - u_y dy, du_x - u_{xx} dx - e^u dy, du_y - e^u dx - u_{yy} dy)_{diff}$$

$$\frac{\mathcal{I}}{\Gamma_H} = (dV - V_x dx + e^W dy, dW + e^V dx - W_y dy, dV_x - V_x dx - e^{V+W} dy, dW_y - e^{V+W} dx - W_y dy)_{diff}.$$

For example the reduction $\frac{\mathcal{I}}{\Gamma_{G_2}}$ can be checked by noting

$$q_{I\Gamma_{G_2}}^r (du - u_x dx - u_y dy) = \frac{-2}{v + w} (\theta^v + \theta^w) + \frac{1}{v_x} \theta^v + \frac{1}{w_y} \theta^w,$$

$$q_{I\Gamma_{G_2}}^r (du - u_x dx - e^u dy) = \frac{2v_x}{(v + w)^2} (\theta^v + \theta^w) - \frac{2v_x^2 + v_{xx} (v + w)}{v_x^2 (v + w)} \theta^v + \frac{1}{v_x} \theta_{xx},$$

or by using a cross-section $\sigma : M/G_2 \to M$ (Lemma 2.4). We also note that

$$p_2^r (du - u_x dx - u_y dy) = \theta^v + \theta^w,$$

$$p_2^r (du - u_x dx - e^u dy) = \theta^v - e^V \theta^v, $$

$$p_2^r (du - u_x dx - u_y dy) = \theta^v - e^V \theta^w,$$

so that

$$\frac{\mathcal{I}}{\Gamma_H} = p_2^r (\frac{\mathcal{I}}{\Gamma_{G_2}}) + \text{span}\{\theta^V\}.$$

The structure equation

$$d\theta^V = -\theta^V_x \wedge dx + e^W \theta^W \wedge dy,$$

demonstrates Theorem 3.1 by directly verifying $\frac{\mathcal{I}}{\Gamma_H}$ is an integral extension of $\frac{\mathcal{I}}{\Gamma_{G_2}}$.

Finally we point out that the reductions in (3.8) are the standard Pfaffian systems for the partial differential equations

$$z_{xy} = 0, \quad u_{xy} = e^u,$$

and the prolongation of

$$V_y = -e^W, W_x = -e^V.$$ 

Note that eliminating $V, W, V_x, W_y$ using equations (3.7) gives

$$z_x - u_x = \sqrt{2} e^{\frac{u}{2}}, \quad z_y + w_y = -\sqrt{2} e^{\frac{u}{2}},$$

which is the classical Bäcklund transformation relating the wave equation and Liouville’s equation $u_{xy} = e^u$. 

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4. Iterated Reduction and De-prolongation

4.1. Iterated Reduction. Let \( \mu : G \times M \to M \) be a regular action of the Lie group \( G \) on \( M \) which preserves the regular integrable distribution \( \Delta \),

\[
(\mu_g)_x \Delta = \Delta_{\mu(g,x)} \quad \text{for all } x \in M, \, g \in G.
\]

These assumption of regularity on the action of \( G \) and the distribution \( \Delta \) give rise to the two quotients

\[
\begin{array}{c}
M \\
\downarrow \text{q}_G \\
M/G
\end{array} \quad \downarrow \text{q}_\Delta \\
\begin{array}{c}
M/\Delta
\end{array}
\]

where the functions \( q_G \) and \( q_\Delta \) are surjective submersions.

We now show, by virtue of the symmetry condition (4.1), that the group \( G \) acts naturally on \( M/\Delta \) in which case we can construct the first iterated reduction \((M/\Delta)/G\). We begin with the following lemma.

Lemma 4.1. Let \( x \in M, \, g \in G \) and let \( S_x \) be the maximal integral manifold of \( \Delta \) through \( x \). Then \( \mu(g, S_x) = S_{\mu(g,x)} \) where \( S_{\mu(g,x)} \) is the maximal integral manifold of \( \Delta \) through \( \mu(g,x) \).

Proof. Condition (4.1) implies, for all \( g \in G \), that the function \( \mu_g : M \to M \) maps integral manifolds of \( \Delta \) to integral manifolds. The image \( \mu(g, S_x) \) is a connected integral manifold through the point \( \mu(g, x) \) and is therefore contained in \( S_{\mu(g,x)} \). Similarly \( \mu(g^{-1}, S_{\mu(g,x)}) \subset S_x \). These two inclusions give

\[
S_{\mu(g,x)} = \mu(g, \mu(g^{-1}, S_{\mu(g,x)})) \subset \mu(g, S_x) \subset S_{\mu(g,x)}
\]

and so \( \mu(g, S_x) = S_{\mu(g,x)} \) as required.

Corollary 4.2. The function \( \tilde{\mu} : G \times M/\Delta \to M/\Delta \)

\[
(\tilde{\mu}(g, S_x) = S_{\mu(g,x)}, \quad g \in G, \, x \in M,
\]

is well-defined and defines an action of \( G \) on \( M/\Delta \). The projection map \( q_\Delta : M \to M/\Delta \) is equivariant with respect to the actions \( \mu \) and \( \tilde{\mu} \) of \( G \), that is,

\[
q_\Delta(\mu(g, x)) = \tilde{\mu}(g, q_\Delta(x)), \quad \text{for all } g \in G, \, x \in M.
\]

Let \( \tilde{q}_G : M/\Delta \to (M/\Delta)/G \) be the quotient map for the action in (4.2).

To define the second iterated reduction, we begin by using the \( G \)-invariance of \( \Delta \) in equation (4.1) to define the bundle reduction \( \tilde{\Delta} = \Delta/G \subset T(M/G) \) by

\[
\tilde{\Delta} = \Delta/G = q_{G*}(\Delta).
\]

If the distribution \( \tilde{\Delta} \) on \( M/G \) is integrable, then the second iterated reduction is \( q_{\tilde{\Delta}} : M/G \to (M/G)/\tilde{\Delta} \).
Theorem 4.3] given below states that if the action of $G$ on $M/\Delta$ is regular, then $\tilde{\Delta} = \Delta/G$ is integrable and regular, and that the two iterated reductions $(M/\Delta)/G$ and $(M/G)/\tilde{\Delta}$ are canonically diffeomorphic. By an abuse of notation we then denote $q_{\tilde{\Delta}} : M/G \to (M/\Delta)/G$ which gives rise to the commutative diagram,

$$
\begin{array}{ccc}
M & \xrightarrow{q_G} & M/G \\
\downarrow q_{\tilde{\Delta}} & & \downarrow q_{\tilde{\Delta}} \\
M/\Delta & \xrightarrow{\pi} & (M/\Delta)/G.
\end{array}
$$

(4.5)

This is the *iterated reduction diagram* for manifolds with $G$ invariant integrable distributions.

We now present two lemmas which are essential for the proof of the Theorem 4.3 (iterated reduction).

**Lemma 4.3.** There exists a unique function $p : M/G \to (M/\Delta)/G$, such that the diagram

$$
\begin{array}{ccc}
M & \xrightarrow{q_G} & M/G \\
\downarrow q_{\Delta} & & \downarrow p \\
M/\Delta & \xrightarrow{\pi} & (M/\Delta)/G
\end{array}
$$

(4.6)

commutes, where $\pi = q_{\tilde{\Delta}} \circ q_{\Delta}$.

**Proof.** Let $\mu(G, x) \in M/G$, and let

$$p(\mu(G, x)) = \tilde{q}_G \circ q_{\Delta}(x).$$

(4.7)

We need to check that $p$ in (4.7) is well-defined by showing by showing $q_{\tilde{\Delta}} \circ q_{\Delta}(x)$ is independent of the point $x$ in the orbit $\mu(G, x)$. Let $x, x' \in M$ be two points which satisfy $q_G(x) = q_G(x')$. If $q_G \circ q_{\Delta}(x) = q_G \circ q_{\Delta}(x')$ then the function $p$ will be well defined as a function on $M/G$.

Since $q_G(x) = q_G(x')$, then $x' = \mu(g, x)$ for some $g \in G$. We then compute,

$$
\tilde{q}_G \circ q_{\Delta}(x') = \tilde{q}_G(\Delta(\mu(g, x)) = \tilde{q}_G \circ \tilde{\mu}(g, q_{\Delta}(x)), \quad \text{by equivariance in equation (4.3)}
$$

$$
= \tilde{q}_G \circ q_{\Delta}(x).
$$

Therefore $p$ in equation (4.7) is well-defined.

We now give the second lemma.

**Lemma 4.4.** Let $p : N \to Q$ be a surjective submersion with connected fibres. Then $\ker(p_*)$ is an integrable and regular distribution, and $N/\ker(p_*)$ is canonically diffeomorphic to $Q$.

**Proof.** The distribution $\ker(p_*)$ is integrable and so we need to show it is regular.

We begin with a standard construction (see for example [10], page 142). Define a partition on $N$ by $x \sim x'$ if $p(x) = p(x')$. The equivalence class of $x \in N$ is then

$$
[x] = p^{-1}(p(x)).$$

(4.8)
The function $p$ is a submersion and therefore an open map, and the canonical bijection $\tau: Q \rightarrow N/\sim$ given by
\[
\tau(y) = p^{-1}(y)
\]
is a homeomorphism where $N/\sim$ has the quotient topology. The differentiable structure on $Q$ induces a differentiable structure on $N/\sim$ compatible with the quotient topology in which case the map $\tau$ is a diffeomorphism. With this differentiable structure on $N/\sim$, the function $\hat{q}: N \rightarrow N/\sim$ is then a surjective submersion.

We now show that $\ker(p_\ast)$ is regular by first showing that for each $x \in N$, $[x] = p^{-1}(p(x)) = S_x$, where $S_x$ is the maximal connected integral manifold of $\ker(p_\ast)$ through $x$. Therefore the elements of $N/\sim$ and $N/\ker(p_\ast)$ are identical, the functions $q_{\ker(p_\ast)}$, and $\hat{q}$ are identical and so $q_{(\ker(p_\ast))}$ is a surjective submersion.

Let $x \in N$. Then $p^{-1}(p(x))$ is a connected (by hypothesis) integral manifold of $\ker(p_\ast)$ and therefore $[x] = p^{-1}(p(x)) \subseteq S_x$. To show the reverse inclusion let $x' \in S_x$, and since $S_x$ is connected, let $\gamma: [0,1] \rightarrow S_x$ be a smooth curve in $S_x$ joining $x$ to $x'$. Since $\gamma(t) \in S_x$, and $\gamma(t)$ is smooth as a curve into $N$ we have the tangent vector $\gamma'(t)$ satisfying $\gamma'(t) \in \ker(p_\ast)$ for all $t \in (0,1)$. Consequently, the tangent vector to the curve $p \circ \gamma: [0,1] \rightarrow Q$ satisfies $\frac{d}{dt}(q \circ \gamma) = 0$, for all $t \in (0,1)$. Therefore $p \circ \gamma$ is constant and $p(x') = p \circ \gamma(1) = p \circ \gamma(0) = p(x)$. That is $S_x \subseteq p^{-1}(p(x))$, which then shows $S_x = p^{-1}(p(x))$. Therefore the elements of $N/\sim$ and $N/\ker(p_\ast)$ are identical which finishes the proof.

We now come to the main theorem on iterated reduction.

**Theorem 4.5.** Let $G$ be a Lie group preserving the regular integrable distribution $\Delta$ on the manifold $M$, and suppose that $G$ acts regularly on $M$ and on $M/G$. Then [i] the distribution $\bar{\Delta} = \Delta/G$ on $M/G$ is integrable and regular, and [ii] there exists a canonical diffeomorphism of $\tau: (M/\Delta)/G \rightarrow (M/G)/\bar{\Delta}$, giving rise to the commutative diagram
\[
\begin{array}{ccc}
M/G & \xrightarrow{q_{\Delta}} & (M/\Delta)/G \\
\downarrow p & & \downarrow \tau^{-1} \\
(M/\Delta)/G & \xleftarrow{\tau} & (M/G)/\bar{\Delta}
\end{array}
\]

**Proof.** The hypothesis that $G$ acts regularly on $M/\Delta$ implies that $\pi = \hat{q}_G \circ q_{\Delta}$ in diagram (4.6) is a surjective submersion. By applying a generalization of Theorem 3.1 (to $\pi$ and $q_G$ in diagram (4.6)) we have $p$ is a surjective submersion (see Theorem 3.1 in [2]).

We now show $\ker(p_\ast)$ is integrable and regular and that $\ker(p_\ast) = \bar{\Delta}$. Let $\tilde{x} \in (M/\Delta)/G$, then $p^{-1}(\tilde{x}) = q_G(S_x)$ where $x \in M$ and $S_x$ is the maximal integral manifold of $\Delta$ through $x \in M$. In particular $p^{-1}(\tilde{x})$ is connected. By Lemma 4.3 $\ker(p_\ast)$ is integrable and regular. Now from the commutative diagram (4.10) we have
\[
\ker(p_\ast) = q_G_{\ast}(\ker(\pi_\ast)) = q_G_{\ast}((\Delta + \Gamma_G) = q_{G\ast}(\Delta) = \Delta/G.
\]
Therefore \( \bar{\Delta} = \Delta/G \) is an integrable regular distribution. This proves part [i].

We now apply Lemma 4.4 with \( N = M/G, Q = (M/\Delta)/G \) and \( \ker(p_*) = \bar{\Delta} \). The canonical diffeomorphism \( \tau: (M/\Delta)/G \to (M/G)/\bar{\Delta} \) in equation (4.9) is then

\[
\tau(\mu(G,S_x)) = \bar{S}_{\mu(G,x)},
\]

where \( \bar{S}_{\mu(G,x)} \) is the maximal integral manifold of \( \bar{\Delta} \) through \( \mu(G,x) \in M/G \). This proves part [ii] of the theorem.

Theorem 4.5 provides the identification of \( (M/\Delta)/G \) and \( (M/G)/\bar{\Delta} \). In terms of this identification the function \( p \) in equation (4.6) becomes \( q_{\bar{\Delta}} \) so that diagram (4.6) becomes diagram (4.5).

**Remark 4.6.** Let \( p_1: M \to (M/\Delta)/G \) be the surjective submersion \( p_1 = \bar{q}_G \circ q_{\Delta} \). As described in the proof of Lemma 4.4, the manifold \( (M/\Delta)/G \) is canonically diffeomorphic to the quotient space \( M/\sim_1 \), where the \( \sim_1 \) equivalence class \([x]_1 \) of \( x \in M \) is given by

\[
[x]_1 = p_1^{-1}(p_1(x)) = \mu(G,S_x)
\]

and \( S_x \) is the maximal integral manifold of \( \Delta \) through \( x \).

Similarly, let \( p_2: M \to (M/G)/\bar{\Delta} \) be the surjective submersion \( p_2(x) = q_{\Delta} \circ q_G(x) \). Then \( (M/G)/\bar{\Delta} \) is canonically diffeomorphic to the quotient space \( M/\sim_2 \), where the \( \sim_2 \) equivalence class, \([x]_2 \) of \( x \in M \) is given by

\[
[x]_2 = p_2^{-1}(p_2(x)) = q_{\Delta}^{-1}(\bar{S}_{q_G(x)})
\]

and \( \bar{S}_{q_G(x)} \) is the maximal integral manifold of \( \bar{\Delta} \) through \( q_G(x) \).

Theorem 4.5 implies that the equivalence classes (4.12) and (4.13) are identical. By applying the projection \( q_G \) to these equivalence classes we find

\[
q_G([x]_1) = q_G(\mu(G,S_x)) = q_G(S_x), \quad \text{for all } x \in M.
\]

and

\[
q_G([x]_2) = \bar{S}_{q_G(x)}, \quad \text{for all } x \in M.
\]

In other words, by equations (4.12) and (4.13), the equivalence classes (4.12) and (4.13) are the same if and only if \( q_G \) maps maximal integral manifolds of \( \Delta \) to maximal integral manifolds of the quotient \( \bar{\Delta} \),

\[
q_G(S_x) = \bar{S}_{q_G(x)} \quad \text{for all } x \in M.
\]

Condition (4.16) is precisely the condition we need in order to construct the commutative diagram (4.5) and is satisfied given the regularity hypothesis stated in Theorem 4.5.
4.2. De-prolongation. Recall from section 2.3 that the Cauchy characteristics $A(I) \subset TM$ of an EDS $I$ is an integrable distribution which we will assume to be regular where $q_{A\Gamma}: M \to M/A\Gamma$ is the quotient map and $\bar{I} = I/q_{A\Gamma}$ the reduced EDS (see Theorem 2.7). Combining iterated reduction and Cauchy reduction produces the following theorem.

**Theorem 4.7.** Let $G$ be a symmetry group of $I$ acting regularly on $M$. Then

[i] the Cauchy characteristics $A(I)$ are $G$-invariant and,

[ii] $G$ is a symmetry group of $I/q_{A\Gamma}$, where the action of $G$ on $M/A\Gamma$ is defined by equation (4.18) (with $\Delta = A(I)$).

[iii] If the action of $G$ on $M$ and $M/A\Gamma$ is regular then the following diagram commutes,

$$
\begin{array}{ccc}
(M, I) & \xrightarrow{q_G} & (M/G, I/G) \\
\downarrow q_{A\Gamma} & & \downarrow q_{A\Gamma} \\
(M/A\Gamma, I/q_{A\Gamma}) & \xrightarrow{\bar{q}_G} & ((M/A\Gamma)/G, (I/q_{A\Gamma})/G)
\end{array}
$$

(4.17)

where $q_G$ is the quotient by the action of $G$ on $M/A\Gamma$ from part [ii], and $A\Gamma = A(I)/G$.

**Proof.** We begin by proving $A\Gamma$ is $G$ invariant. If $X \in A(I)$ then

$$g_*X \cdot I = X \cdot I (g^*I) = X \cdot I \subset I.
$$

Therefore $g_*X$ is a Cauchy characteristic and $A(I)$ is $G$-invariant. This proves part [i].

We now check that the action of $G$ on $M/A\Gamma$ is a symmetry of $I/q_{A\Gamma}$. Let $\theta \in I/q_{A\Gamma}$, and $g \in G$. We compute $q_{A\Gamma}^* \circ \bar{\mu}_g(\bar{\theta})$ using the equivariance of $q_{A\Gamma}$ (equation (4.3)) to obtain

$$q_{A\Gamma}^* \circ \bar{\mu}_g(\bar{\theta}) = \mu_g^* \circ q_{A\Gamma}^*(\bar{\theta}).
$$

(4.18)

By hypothesis $q_{A\Gamma}^*(\bar{\theta}) \in I$ and $G$ is a symmetry group of $I$, therefore $\mu_g^* \circ q_{A\Gamma}^*(\bar{\theta}) \in I$. Together with equation (4.18), this shows $\bar{\mu}_g(\bar{\theta}) \in I/q_{A\Gamma}$, and thus $G$ is a symmetry group of $I/q_{A\Gamma}$. This proves part [ii].

We now prove part [iii]. Part [i] shows that $A(I)$ is $G$-invariant and, if we assume that the action of $G$ is regular on both $M$ and $M/A(I)$, then Theorem 4.5 can be applied to construct the commutative diagram (4.5), again with $\Delta = A(I)$ and $A\Gamma = A(I)/G$. Combining this with a simple generalization of Theorem 4.1 (see Theorem 3.1 in [2]) we obtain the commutative diagram of differential systems in (4.17).

The issue we now address is to what extent the reduced distribution $A\Gamma = A(I)/G$ in (4.17) are precisely the Cauchy characteristics of $I/G$. Denote the Cauchy characteristics of the reduced system $I/G$ by $A_{I/G} = A(I/G)$. If diagram (4.5) holds and if the condition $A(I)/G = A(I/G)$ is satisfied, then we say that characteristic reduction and symmetry reduction commute. We investigate this condition starting with the following lemma.

**Lemma 4.8.** Let $A\Gamma \cap \Gamma_G$ be constant rank. Then the distribution $A\Gamma = A(I)/G = q_{G^*}(A\Gamma)$ has constant rank, is integrable, and $A\Gamma/G \subset A(I/G)$.
Clearly, by equations (4.22) and (4.23),

\[ \text{local basis} \]

We claim that the tangent vector, \( q \), which satisfies (4.20), \( \Gamma \) for reverse inclusion. We utilize the transversality hypothesis by working with a set of local generators \( \Gamma \). Let \( \bar{A}_I \) by Theorem 2.9 in [7]. We proceed with the second part of the lemma.

Proof. By Lemma 4.8 we have \( A(I(G)/\mathbb{Z}) = A(I(G)/\mathbb{Z}) \subset A(I(G), \text{alg}) \), and so we just need to show the reverse inclusion. We utilize the transversality hypothesis by working with a set of local generators \( \Gamma \) and \( I(G) \) chosen such that (see [2], [1]),

\[ I|U = (\theta^i, \theta^G, \tau_G^\alpha)_{\text{alg}}, \quad \text{and} \quad (I(G)|q_G(U)) = (\tilde{\theta}^i, \tilde{\tau}^\alpha)_{\text{alg}}, \]

where

\[ q_G^*(\tilde{\theta}^i) = \theta^G, \quad q_G^*(\tilde{\tau}^\alpha) = \tau_G^\alpha. \]

Let \( y \in q_G(U) \) and suppose \( Y \in A(I(G)/\mathbb{Z}) \) is a Cauchy characteristic at \( y \). Pick \( x \in U \) with \( q_G(x) = y \) and \( X \in T_xU \) with \( q_G^*(X) = Y \). The transversality hypothesis implies there exists a local basis \( \{ Z_b \} \) for \( \Gamma_G \) at \( x \) such that

\[ \theta^a(Z_b) = \delta^a_b. \]

We claim that the tangent vector,

\[ Z = X - \theta^a(X)Z_a, \]

which satisfies \( q_G^*(Z) = Y \), also satisfies \( Z \in A(I_x) \). Once this is shown, the theorem is proved.

We check the characteristic condition for \( Z \) on the algebraic generators of \( I|U \) in equation (4.20). Clearly, by equations (4.22) and (4.23), \( \theta^a(Z) = 0 \). Now if \( \omega_G \) is a \( G \)-basic form in \( I \), then \( \omega_G = q_G^*\omega \),

\[ \bar{A}_I \text{ is constant rank and integrable follows from Section 2.5 and is also given by Theorem 2.9 in [7]. We proceed with the second part of the lemma.} \]
Proof. The proofs of (i), (ii), (iii) are the same as in Theorem 4.9. Part [iv] is a direct consequence of Theorem 4.9 applied to $\mathcal{I}'$ and so there is nothing to prove.

To prove part [v] let $Y \in A(\mathcal{I}/G)$. We show that $Y \in A((\mathcal{I}/G)')$ by checking the Cauchy characteristic condition on generators for $(\mathcal{I}/G)'$. We begin with the 1-forms in $\mathcal{I}'/G$ and $(\mathcal{I}/G)'$ which by equation (4.21) in Corollary 2.13 are the same. Therefore the Cauchy characteristic condition on the one-forms for these two systems is identical. Since $(\mathcal{I}/G)'$ is Pfaffian we only need to check that if $\tau \in (\mathcal{I}/G)' \cap \Omega^2(M/G)$ then $Y \lhd \tau \in (\mathcal{I}/G)'$. 

Remark 4.10. Theorem 4.9 shows for an action of $G$ transverse to $\mathcal{I}$, that symmetry reduction and reduction by Cauchy characteristics commute. In which case we may replace $q_{\Lambda_X}$ by $q_{\Lambda_{X/G}}$ in diagram (4.17) since $A(\mathcal{I}/G) = A(\mathcal{I})/G$. 

Finally, we apply iterated reduction to the situation which will allow us to pass from diagram (1.6) to diagram (1.7). Let $\mathcal{I}$ be a Pfaffian system on $M$, with symmetry group $G$ acting transversally to the derived system $\mathcal{I}'$. Then $\mathcal{I}/G$ is a Pfaffian system (1). However, even with the hypothesis that $G$ acts transversally to $\mathcal{I}'$, it is not necessarily true that $(\mathcal{I}/G)' = \mathcal{I}'/G$. One easy way to see why these may not be equal is that $(\mathcal{I}/G)'$ is always a Pfaffian system, while $\mathcal{I}'/G$ is a Pfaffian system if and only if $G$ is transverse to $\mathcal{I}$.

Let $A(\mathcal{I}')$ be the Cauchy characteristics of $\mathcal{I}'$. The analogue to Theorem 4.7 is the following.

Theorem 4.11. Let $G$ be a symmetry group of the Pfaffian system $\mathcal{I}$.

[i] The Cauchy characteristics $A(\mathcal{I}')$ of $\mathcal{I}'$ are $G$ invariant and,

[ii] $G$ is a symmetry group of $\mathcal{I}/q_{\Lambda_{X'}}$, where the action on $M/A(\mathcal{I}')$ is defined by equation (4.2) (with $\Delta = A(\mathcal{I})'$).

[iii] If $G$ acts regularly on $M$ and $M/A_{\mathcal{I}}$, then the following diagram commutes

\[
\begin{array}{ccc}
(M, \mathcal{I}) & \xrightarrow{q_G} & (M/G, \mathcal{I}/G) \\
\downarrow q_{\Lambda_X} & & \downarrow q_{\Lambda_{X'}} \\
(M/A_{\mathcal{I}}, \mathcal{I}/q_{\Lambda_X}) & \xrightarrow{q_{G}} & ((M/A_{\mathcal{I}})/G, (\mathcal{I}/q_{\Lambda_X})/G)
\end{array}
\]

where $q_G$ is the quotient by the action of $G$ on $M/A_{\mathcal{I}}$.

[iv] If $G$ is transverse to $\mathcal{I}'$ then $A(\mathcal{I}')/G = A(\mathcal{I}'/G)$ and

[v] $A(\mathcal{I}'/G) \subset A((\mathcal{I}/G)')$, where $A((\mathcal{I}/G)')$ are the Cauchy characteristics of the Pfaffian system $(\mathcal{I}/G)'$.

Proof. The proofs of [i],[ii],[iii] are the same as in Theorem 4.7. Part [iv] is a direct consequence of Theorem 4.9 applied to $\mathcal{I}'$ and so there is nothing to prove.

To prove part [v] let $Y \in A(\mathcal{I}'/G)$. We show that $Y \in A((\mathcal{I}/G)')$ by checking the Cauchy characteristic condition on generators for $(\mathcal{I}/G)'$. We begin with the 1-forms in $\mathcal{I}'/G$ and $(\mathcal{I}/G)'$ which by equation (2.17) in Corollary 2.13 are the same. Therefore the Cauchy characteristic condition on the one-forms for these two systems is identical. Since $(\mathcal{I}/G)'$ is Pfaffian we only need to check that if $\tau \in (\mathcal{I}/G)' \cap \Omega^2(M/G)$ then $Y \lhd \tau \in (\mathcal{I}/G)'$. 

\[
Z \lhd \omega_G = X \lhd q_G \bar{\omega}
\]

using $\omega_G$ is semi-basic.

(4.24) \[ q_G (\omega_G) = \omega_G = q_G (q_G (X \lhd \bar{\omega})) \]

using $q_G (X) = Y$.

By hypothesis, $Y$ is a Cauchy characteristic and so $Y \lhd \bar{\omega} \in (\mathcal{I}/G)_Y$. The last line in equation (4.24) then implies $Z \lhd \omega_G \in \mathcal{I}$. Applying this result to the generators $\theta^i_G$ and $\tau^a_G$ in $\mathcal{I}|_{\mathcal{I}'}$ in equation (4.20) implies that $Z \in A(\mathcal{I})_x$ and is a Cauchy characteristic of $\mathcal{I}$. 

\[
\omega \in \mathcal{I}/G
\]

\[
\bar{\omega} \in \mathcal{I}/G
\]

where $\bar{\omega}$ is the Cauchy characteristics of the Pfaffian system $\mathcal{I}$.
By Lemma 2.13 \((I/G)’ \subset I'/G\), and so \(\bar{\tau} \in (I'/G)\). Since \(Y\) is a characteristic for \(I'/G\),

\[ Y \cup \bar{\tau} \in (I'/G) \cap \Omega^1(M/G). \]

By equation (2.17) \((I'/G) \cap \Omega^1(M/G) = (I/G)' \cap \Omega^1(M/G)\), and so \(Y \cup \bar{\tau} \in (I/G)’\). Therefore \(Y \in A((I/G)’)\).

\[
\text{Remark 4.12. In Theorem 4.11 the set } A(I')/G \text{ is not necessarily the full space of Cauchy characteristics } A((I/G)'). \text{ However, suppose that the conditions in Theorem 4.11 hold and } \]

\[
\begin{align*}
[i] & \text{ pr}(I/q_{\Gamma_{G_a}}) = I \quad \text{and} \\
[ii] & A(I')/G = A((I/G)')
\end{align*}
\]

where \(pr\) is prolongation. Then \(I/q_{\Gamma_{G_a}}\) is the \textbf{de-prolongation of } \(I\), the quotient by \(A(I')/G\) in diagram (4.25) can be replaced with the quotient by \(A((I/G)')\), and the diagram (4.25) is the \textbf{de-prolongation diagram}. With these additional hypothesis, \textbf{de-prolongation and group reduction commute}.

\[
\text{Example 4.13. As a precursor to the proofs of Theorems 1.3 and 1.4 and to demonstrate de-prolongation, we revisit Example 3.2. In this case we have in equation (3.8) the rank 3 Pfaffian systems } I/\Gamma_{G_a} \text{ on the 7 manifolds } M/\Gamma_{G_a} \text{ and the rank 4 Pfaffian system } B = I/\Gamma_H \text{ on the 8 manifold } M/\Gamma_H. \text{ Now from equation (3.2) we have}
\]

\[A(I') = \text{span}\{ \partial_{xxx}, \partial_{yyy} \}
\]

and \(M/A(I') = J^2(R, R) \times J^2(R, R)\), and \(I/A(I') = C_1 + C_2\) (the standard contact structures). The reductions \(A(I')/\Gamma_{G_a}\) and \(A(I')/\Gamma_H\) are all two dimensional and satisfy

\[
\begin{align*}
A(I')/\Gamma_{G_1} &= \text{span}\{ \partial_{zz}, \partial_{zy} \} = A_{I_{z=0}} \\
A(I')/\Gamma_{G_2} &= \text{span}\{ \partial_{uu}, \partial_{uy} \} = A_{I_2} \\
A(I')/\Gamma_H &= \text{span}\{ \partial_{vv}, \partial_{Wy} \} = A_{B'}
\end{align*}
\]

\text{Therefore conditions } [i], [ii] \text{ in Remark 4.12 hold and we can form the de-prolongation diagram (4.25) for each of } (I, \Gamma_{G_1}), (I, \Gamma_{G_2}), \text{ and } (I, \Gamma_H).

Let \(\hat{\Gamma}_{G_a}\) and \(\hat{\Gamma}_H\) be the projection (or reduction by \(A(I')\) as in Corollary 4.2) of the Lie algebras \(\Gamma_{G_a}\) and \(\Gamma_H\) to \(M/A(I') = J^2(R, R) \times J^2(R, R)\). Combing the application of Theorem 4.11 three
times, we get the commutative diagram

\[
\begin{array}{ccc}
q_{\Gamma_{G_1}} & \xrightarrow{q_{\Gamma_H}} & q_{\Gamma_{G_2}} \\
\downarrow & & \downarrow \\
\bar{p}_1 & & \bar{p}_2 \\
\end{array}
\]

where \( q_{\Gamma_{G_1}}, q_{\Gamma_{G_2}}, q_{\Gamma_H} \) are each the bottom arrow in the corresponding diagram (3.25).

Diagram (4.13) produces the Bäcklund transformation in equation (1.7) where \( \tilde{p}_1 \) and \( \tilde{p}_2 \) are the de-prolongation of the maps \( p_1 \) and \( p_2 \) in equation (3.7), and are given in coordinates by

\[
\begin{align*}
\tilde{p}_1 &= (x = x, y = y, z = V - W, z_x = V_x + e^V, z_y = -W_y - e^W) \\
\tilde{p}_2 &= (x = x, y = y, u = \log(2) + V + W, u_x = V_x - e^V, u_y = W_y - e^W)
\end{align*}
\]

5. Bäcklund Transformations for Darboux Integrable PDE in the Plane

5.1. The Vessiot Algebra and Quotient Representation. Let \( F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0 \) be a hyperbolic PDE in the plane and let \( \mathcal{I} \) be the standard rank 3 Pfaffian system determined by the restriction of the contact system on \( J^2(\mathbb{R}^2, \mathbb{R}) \) [8] to the level set \( M = \{ p \in J^2(\mathbb{R}^2, \mathbb{R}) \mid F(p) = 0 \} \). The following theorem is proved in [8].

**Theorem 5.1.** Let \( \mathcal{I} \) be the standard rank 3 Pfaffian system for a hyperbolic PDE in the plane on the seven dimensional manifold \( M \). Then about each point \( x \in M \) there exists an open set \( U \) and a coframe \( \{\theta^i, \omega^a, \pi^a\}_{0 \leq i \leq 2, 1 \leq a \leq 2} \) on \( U \) where \( \mathcal{I}|_U = \langle \theta^0, \theta^1, \theta^2 \rangle_{\text{diff}} \) and

\[
\begin{align*}
d\theta^0 &= \theta^1 \wedge \omega^1 + \theta^2 \wedge \omega^2 & \text{mod } \theta^0 \\
d\theta^1 &= \omega^1 \wedge \pi^1 + \mu_1 \theta^2 \wedge \pi^2 & \text{mod } \theta^0, \theta^1 \\
d\theta^2 &= \omega^2 \wedge \pi^2 + \mu_2 \theta^1 \wedge \pi^1 & \text{mod } \theta^0, \theta^2.
\end{align*}
\]

The functions \( \mu_1, \mu_2 \) are the Monge-Ampère invariants.

The singular or characteristic Pfaffian system for \( \mathcal{I} \) are given by

\[
\hat{\mathcal{I}} = \{\theta^i, \omega^1, \pi_1\}_{0 \leq i \leq 2}, \quad \hat{\mathcal{V}} = \{\theta^i, \omega^2, \pi_2\}_{0 \leq i \leq 2}.
\]

For the next lemma we note that the Cauchy characteristics for \( \mathcal{I}' = \theta^0_{\text{diff}} \) are given in the dual frame by

\[
A(\mathcal{I}') = \text{span}\{\partial_{\pi_1}, \partial_{\pi_2}\}.
\]
We assume $A(I')$ to be regular and $q_{A_{x'}} : M \to M/A_{x'}$ to be the smooth quotient map.

The vanishing of the Monge-Ampère invariants permits the following reduction lemma.

**Lemma 5.2.** Suppose $I$ is the standard rank 3 Pfaffian system for a hyperbolic PDE in the plane, and that the invariant conditions $\mu_1 = \mu_2 = 0$ in equation (5.1) are satisfied. Then $\bar{I} = I/q_{A_{x'}}$ is a hyperbolic Monge-Ampère system on the five dimensional manifold $M/A_{x'}$, and $\bar{I} = I/q_{A_{x'}}$ is the de-prolongation of $I$.

**Proof.** We follow Section 4.2. The structure equations (5.1) with $\mu_1 = \mu_2 = 0$ determine the $q_{A_{x'}}$ semi-basic forms as

$$(5.4) \quad \langle I_{q_{A_{x'}}}, sb \rangle = \langle \theta_0, \omega^1 \wedge \theta_1, \omega^2 \wedge \theta_2 \rangle_{alg}.$$ 

The quotient $I/q_{A_{x'}} = I_{q_{A_{x'}}, sb}/q_{A_{x'}}$, and by equation (5.4) is a hyperbolic Monge-Ampère EDS. Finally prolongation using $\omega^1 \wedge \omega^2$ as independence condition proves the final claim. \qed

**Remark 5.3.** The PDE $F = 0$ satisfies the invariant conditions $\mu_1 = \mu_2 = 0$, if and only if $F = 0$ is a Monge-Ampère equation, see [8].

The EDS $I$ is said to be Darboux integrable (and not Monge integrable) if $\hat{V}^\infty$ and $\tilde{V}^\infty$ are rank two, in which case we may assume equations (5.1) and (5.2) hold and

$$(5.5) \quad \hat{V}^\infty = \{\omega^1, \pi_1\} \quad \tilde{V}^\infty = \{\omega^2, \pi_2\}.$$ 

We now recall a pivotal result from [3] on the geometric properties of Darboux integrable systems applied to the case of hyperbolic PDE in the plane.

**Theorem 5.4.** (The quotient representation of a Darboux integrable system) Let $I$ be Darboux integrable with singular Pfaffian systems (5.2) satisfying (5.5), and let $x \in M$. Let $M_1$ be the five dimensional maximal integral manifold of $\hat{V}^\infty$ through $x$ and let $K_1$ be rank 3 Pfaffian system which is the restriction of $\hat{V}$ to $M_1$. Similarly let $M_2$ be the five dimensional maximal integral manifold of $\tilde{V}^\infty$ though $x$ and $K_2$ the rank 3 Pfaffian system which is the restriction of $\tilde{V}$ to $M_2$. Then

[i] there exists a pair of isomorphic three dimensional Lie algebra of vector-fields $\Gamma_a$ on $M_a$ each of which is point-wise linearly independent, transverse to $K_a$ and symmetries of $K_a$;

[ii] there exists open sets $U \subset M, U_1 \subset M_1$ and $U_2 \subset M_2$ containing $x$ such that

$$(5.6) \quad I|_U = (K_1 + K_2)|_{U_1 \times U_2/\Gamma_{diag}}.$$ 

The algebra $\Gamma_a$ is called the Vessiot algebra of $I$ which is determined by a sequence of coframe adaptations, is a fundamental invariant of Darboux integrable systems whose construction is given in [3] (see also [2]).

Before proceeding to the proofs of Theorems [1.2, 1.3] and [1.4] we need to examine the derived sequence of $K_a$. Using the inclusion map $i_1 : M_1 \to U$ we have $K^1_1 = i_1^*(\hat{V})$ which from (5.2) is to $M_1$

$K^1_1 = \text{span}\{ \theta^0, \theta^1, \theta^2 \}$. 

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The structure equations for the Pfaffian system $K_1$ are easily obtained by restricting equations (5.1) to $M_1$. On $M_1$ we have $\omega^1 = 0$ and $\pi_1 = 0$ on $M_1$ and the structure equations from (5.1) pullback to (by an abuse of notation),

$$
\begin{align*}
    d\theta^0 &= \theta^2 \wedge \omega^2 \mod \theta^0 \\
    d\theta^1 &= \mu_1 \theta^2 \wedge \pi^2 \mod \theta^0, \theta^1 \\
    d\theta^2 &= -\pi_2 \wedge \omega^2 \mod \theta^0, \theta^1.
\end{align*}
$$

(5.7)

Analogous equations hold for $K_2$ on $M_2$. These equations easily imply the following.

**Corollary 5.5.** If the Monge-Ampère invariants $\mu_1, \mu_2$ are nowhere vanishing then by equation (5.7) the derived series for $K_1$ and $K_2$ are

$$(K_1)' = \{ \theta^0, \theta^1 \}, \; (K_1)'' = \{ 0 \}; \quad (K_2)' = \{ \theta^0, \theta^2 \}, \; (K_2)'' = \{ 0 \}.
$$

If the Monge-Ampère invariants satisfy $\mu_1 = \mu_2 = 0$ then

$$(K_1)' = \{ \theta^0, \theta^1 \}, \; (K_1)'' = \{ \theta^1 \}, \; (K_1)''' = \{ 0 \}; \quad (K_2)' = \{ \theta^0, \theta^2 \}, \; (K_2)' = \{ \theta^2 \}, \; (K_2)''' = \{ 0 \}.
$$

5.2. Bäcklund Transformations for PDE in the plane: Proof of Theorem 1.2

**Proof.** (Proof of Theorem 1.2) Let $\mathcal{I}_2$ be the standard rank three Pfaffian system on a seven manifold $M$ representing a Darboux integrable hyperbolic PDE in the plane satisfying (5.5).

We then construct diagram (1.6) as a commutative diagram of EDS by identifying the EDS $\mathcal{I}$ in Theorem 1.1 along with the three Lie algebras $\Gamma_H$, $\Gamma_G_1$ and $\Gamma_G_2$ with $\Gamma_H = \Gamma_G_1 \cap \Gamma_G_2$. Applying Theorem 1.1 produces diagram (1.6).

Our starting point is Theorem 5.4. Let $x \in M$, $U, U_1, U_2$, $\Gamma_1$ and $\Gamma_2$ be as in Theorem 5.4. Since $\Gamma_1$ and $\Gamma_2$ are isomorphic, choose basis so that $\Gamma_1 = \text{span}\{X_1\}_{1 \leq i \leq 3}$, and $\Gamma_2 = \text{span}\{Y_1\}_{1 \leq i \leq 3}$ have the same structure constants. The infinitesimal action of $\Gamma_{G_2} = \Gamma_{\text{diag}}$ is

$$
\Gamma_{G_2} = \Gamma_{\text{diag}} = \text{span}\{X_1 + Y_1\}_{1 \leq i \leq 3},
$$

and equation (5.6) produces the right hand side of (1.6) where $\mathcal{I} = \mathcal{K}_1 + \mathcal{K}_2$, is on $M_1 \times M_2$ and $\Gamma_{G_2} = \Gamma_{\text{diag}} : U_1 \times U_2 \to U$ as described in Theorem 5.3. We now identify $\Gamma_H$ and $\Gamma_G_1$.

A cursory glance at the classification of real 3-dimensional Lie algebras shows that except for the case $\Gamma_1 = \Gamma_2 = \text{so}(3)$, every such algebra admits a 2-dimensional subalgebra $\mathfrak{h}$. Therefore a basis for the infinitesimal generators $\Gamma_a$ may be chosen so that $X_1, X_2$ and $Y_1, Y_2$ are the infinitesimal generators for the action of $\mathfrak{h}$ so that the structure equations satisfy,

$$
[X_1, X_2] = \epsilon X_1 \quad \text{and} \quad [Y_1, Y_2] = \epsilon Y_1, \quad \epsilon = 0, 1.
$$

(5.9)

In a neighbourhood of $x$, the sub-algebras $\{X_1, X_2\}$ and $\{Y_1, Y_2\}$ may be chosen transverse to the two-dimensional derived systems $\mathcal{K}_1'$ and $\mathcal{K}_2'$. Relabel the corresponding open sets, $U_1, U_2, U$.

Using the notation in equation (5.9), let $\Gamma_H \subset \Gamma_G_2$ be the two dimensional subalgebra $\Gamma_H = \{X_1 + Y_1, X_2 + Y_2\}$. Let $q_{\Gamma_H} : U_1 \times U_2 \to (U_1 \times U_2)/\Gamma_H$ be the quotient map, and $p_2$ the induced orbit map from Theorem 5.1 with the subalgebra $\Gamma_H \subset \Gamma_{\text{diag}}$. Letting $B = (\mathcal{K}_1 + \mathcal{K}_2)/\Gamma_H$, we have
completed the right triangle in diagram in (1.6). Theorem 3.1 also shows that each of the elements in the right triangle of (1.6) is an integrable extension.

We now identify \( \Gamma_{G_1} \). Using the basis of infinitesimal generators \( \Gamma_a \) adapted to the two dimensional sub-algebras in (5.9) let

\[
(5.10) \quad \Gamma_{G_1} = \{X_1, Y_1, X_2 + Y_2\}.
\]

Note that \( \Gamma_H = \Gamma_{G_2} \cap \Gamma_{G_1} \). Let \( q_{\Gamma_{G_1}} : U_1 \times U_2 \rightarrow (U_1 \times U_2)/\Gamma_{G_1} \) and let \( p_1 \) the induced orbit map from Theorem 3.1 with the subalgebra \( \Gamma_H \subset \Gamma_{G_1} \).

The fact that \( \Gamma \Gamma_{G_1} \) part \( \Pi \) of Theorem 5.1 gives the quotient algebras (5.11) as a commutative diagram of integral extensions. Theorem 6.1 in [2] (see also Lemma 6.7 in [2]). Let \( \hat{\Gamma}_1 = \{X_1, \hat{\Gamma}_2 = \{Y_1\} \), then the reference above,

\[
(5.11) \quad \mathcal{I}_1 = (\mathcal{K}_1 + \mathcal{K}_2)/\Gamma_{G_1} = (\mathcal{K}_1/\hat{\Gamma}_1 + \mathcal{K}_2/\hat{\Gamma}_2)/(\Gamma_{G_1}/(\hat{\Gamma}_1 \times \hat{\Gamma}_2))
\]

Now by the choice of \( \{X_1, X_2\} \) and \( \{Y_1, Y_2\} \) being transverse to \( \mathcal{K}_a \), the quotient system \( \mathcal{K}_1/\hat{\Gamma}_1 + \mathcal{K}_2/\hat{\Gamma}_2 \) is a Pfaffian system (see [1]) where each \( \mathcal{K}_a/\hat{\Gamma}_a \) has derived sequence rank \( [2, 1, 0] \). Furthermore the quotient algebras \( \{X_1, X_2\}/\{X_1\} \) and \( \{Y_1, Y_2\}/\{Y_1\} \) are transverse to \( (\mathcal{K}_a/\hat{\Gamma}_a) \). The action of one-dimensional Lie algebra \( \Gamma_{G_1}/(\hat{\Gamma}_1 \times \hat{\Gamma}_2) = \{X_2 + Y_2\} \) is therefore transverse to \( (\mathcal{K}_1/\hat{\Gamma}_1 + \mathcal{K}_2/\hat{\Gamma}_2) \).

Again using [2] we conclude from these remarks and equation (5.11) that \( (\mathcal{K}_1 + \mathcal{K}_2)/\Gamma_{G_1} \) is a rank 3 hyperbolic Pfaffian system.

We use Theorem 6.1 part [I] in [2] to compute number of Darboux invariants, which states \( \mathcal{I}/\Gamma_{G_1} \) has \( M_1/\pi_1(\Gamma_{G_1}) \) and \( M_2/\pi_2(\Gamma_{G_1}) \) invariants where \( \pi_a : M_1 \times M_2 \rightarrow M_a \), \( a=1,2 \). We have from equation (5.10),

\[
\pi_1(\Gamma_{G_1}) = \text{span}\{X_1, X_2\}, \quad \pi_2(\Gamma_{G_1}) = \text{span}\{Y_1, Y_2\}.
\]

Since \( M_1 \) and \( M_2 \) are 5 dimensional, we conclude (equation 6.10 in [2]) that

\[
\text{rank } \hat{V}_0^\infty = \dim M_2 - \text{rank } \pi_2(\Gamma_{G_1}) = 3, \quad \text{rank } \hat{V}_0^\infty = \dim M_1 - \text{rank } \pi_1(\Gamma_{G_1}) = 3.
\]

If we now show that \( \mathcal{I}_1 \) is the EDS for a PDE in the plane, the local equivalence with \( \mathcal{I}_{s=0} \) will follow from a well-known theorem of Lie’s which states: a hyperbolic PDE in the plane where each characteristic system admits three Darboux invariants is locally equivalent to \( \mathcal{I}_{s=0} \).

By the transversality of the one-dimensional Lie algebra \( \Gamma_{G_1}/(\hat{\Gamma}_1 \times \hat{\Gamma}_2) = \{X_2 + Y_2\} \) to the rank two distribution \( \mathcal{K}_1/\hat{\Gamma}_1 + \mathcal{K}_2/\hat{\Gamma}_2 \) we have rank \( I_1' = 1 \). Let \( \theta \) be a local generator for \( I' \). Then \( I_1 \) represents a PDE in the plane if and only if \( \theta \) has rank 2 (see (11)). Let \( \theta_{sb} \in \mathcal{K}_1/\hat{\Gamma}_1 + \mathcal{K}_2/\hat{\Gamma}_2 \), be \( \Gamma_{G_1}/(\hat{\Gamma}_1 \times \hat{\Gamma}_2) = \{X_2 + Y_2\} \) semi-basic. Then \( \theta \) has rank 2 if and only if \( \theta_{sb} \) is rank 2 because \( q_{\Gamma_{G_1}} \theta = \lambda \theta_{sb} \).
By the transversality of the action of \( \{X_1, X_2\}/\tilde{\Gamma}_1 \) to \( K_1/\tilde{\Gamma}_1 \) on \( M_1/\tilde{\Gamma}_1 \) and \( \{Y_1, Y_2\}/\tilde{\Gamma}_2 \) to \( K_2/\tilde{\Gamma}_2 \) on \( M_2/\tilde{\Gamma}_2 \) we may choose \( \theta^1 \in K_1/\tilde{\Gamma}_1 \) and \( \eta^1 \in K_2/\tilde{\Gamma}_2 \) satisfying \( \theta^1(X_2) = 1, \eta^1(Y_2) = 1 \). Then

\[
\theta_{ab} = \theta^1 - \eta^1
\]
is semi-basic. The derived system for \( K_a/\tilde{\Gamma}_a \) are \([2,1,0]\) so we may choose one-forms \( \omega^1, \theta^2 \) on \( M_1/\tilde{\Gamma}_1 \) and \( \sigma^1, \eta^2 \) on \( M_2/\tilde{\Gamma}_2 \) so that \( K_1/\tilde{\Gamma}_1 = \langle \theta^1, \theta^2 \rangle_{\text{diff}}, K_2/\tilde{\Gamma}_2 = \langle \eta^1, \eta^2 \rangle_{\text{diff}} \) and

\[
d\theta^1 = \sigma^1 \wedge \theta^2 \mod \theta^1, \quad d\eta^1 = \tau^1 \wedge \eta^2 \mod \eta^1.
\]

Therefore \( \theta_{ab}^1 \wedge (d\theta_{ab}^1)^2 \neq 0 \) and so both \( \theta_{ab} \) and \( \theta \) have rank 2.

**Example 5.6.** The non-linear partial differential equation \( 3u_{xx}(u_{yy})^3 + 1 = 0 \) is Darboux integrable. Applying the algorithm in [2] produces the following quotient representation. Let \( K_1 + K_2 \) be the sum of two copies of the standard Pfaffian system for the Cartan-Hilbert equation \( u' = (u'')^2 \), on \( \mathbb{R}^5 \times \mathbb{R}^5 \) with coordinates \( (s, u, v, s, t, y, z, z, z_t) \),

\[
(K_1 + K_2 = \langle du - (sv)_2 ) ds, dv - vs ds, dv - vs ds, dy - (zv)_2 dt, dz - zv dt, dz_t - zv dt \rangle_{\text{diff}}.
\]

The Lie algebra \( \Gamma_{G_2} \) in Theorem [2:3] is

\[
\Gamma_{G_2} = \text{span}\{\partial_v - \partial_z, s\partial_v + \partial_v + t\partial_z + \partial_z, \partial_u - \partial_y\}.
\]

Now let

\[
\Gamma_{G_1} = \text{span}\{\partial_v, \partial_z, s\partial_v + \partial_v + t\partial_z + \partial_z\}
\]

\[
\Gamma_H = \Gamma_{G_1} \cap \Gamma_{G_2} = \text{span}\{\partial_v - \partial_z, s\partial_v + \partial_v + t\partial_z + \partial_z\}.
\]

Coordinates can be chosen on the 8-manifold \( M/\Gamma_H \) so that the quotient map \( q_H : M \to M/\Gamma_H \) is

\[
q_H = (x_1 = t, x_2 = s, x_3 = z_t, x_4 = v + s, x_5 = z - v, x_6 = u, x_7 = y, x_8 = z + v - (s + t) v).
\]

Coordinates on \( M/\Gamma_1 \) can be chose so the the map \( p_1 \) is

\[
p_1 = \left( X' = -x_2, Y' = x_1 x_3, U' = x_5 - x_1 x_3 + x_2 x_4, P' = \frac{1}{2x_4}, Q' = \frac{1}{2x_3}, R' = -\frac{1}{2x_2(x_4)^2}, T' = \frac{1}{2x_1(x_3)^2} \right).
\]

Finally coordinates on \( M/\Gamma_2 \) can be chosen so that \( p_2 \) has the form

\[
p_2 = \left( X = -2x_3 + x_4, Y = x_5 - \frac{1}{2}(x_3 - x_4)(x_1 + x_2), S = \frac{1}{2}(x_2 - x_1), T = \frac{2}{x_1 + x_2}, U = 2x_3 - x_4 (x_1 + x_2) \right), \quad U = 2x_3 + x_4 (x_1 + x_2),
\]

\[
P = x_8 - x_1 x_6 + \frac{1}{6}(x_1 + x_2)((2x_1 - x_2)x_3 - (x_1 - 2x_2)x_4), \quad Q = \frac{1}{2}x_1 x_2 x_3 x_4, \quad T = \frac{2}{x_1 + x_2}.\]
The choice of variables (5.15) and (5.16) give the Bäcklund transformation in diagram (1.6) as

\[
\begin{align*}
&dx_5 - x_3 dx_1 + x_4 dx_2 \\
&dx_6 - x_2^2 dx_2, \quad dx_7 - x_3^2 dx_1 \\
&dx_8 - x_5 dx_1 + x_4(x_1 + x_2) dx_2
\end{align*}
\]

(5.17)

\[
\begin{align*}
&P_1 \quad dU' - P'dX' - Q'dY' \\
&P_2 \quad dP' - R'dX', \quad dQ' - T'dY'
\end{align*}
\]

\[
\begin{align*}
&P_1 \quad dU - PdX - QdY \\
&P_2 \quad dP + \frac{1}{3d^2}dX - SdY, \quad dQ - SdX -TdY
\end{align*}
\]

Where the standard representation the for PDE \( U'_{X'Y'} = 0 \) and \( 3U_{XX}U_{YY} + 1 = 0 \) as EDS appear.

5.3. The Monge-Ampère Case: Proof of Theorems 1.3 and 1.4. Let \( \tilde{I}_2 \) be a rank 3 Pfaffian system on a 7 manifold \( M \) for a Darboux integrable hyperbolic Monge-Ampère equation satisfying (5.5). Let \( \tilde{I}_2 = I_2/q_{A_{x_{2}^2}} \) be the corresponding Monge-Ampère system on the five-manifold \( M/A_{x_{2}^2} \) (Lemma 5.2), where \( q_{A_{x_{2}^2}} : M \to M/A_{x_{2}^2} \) is the smooth quotient map. Likewise let \( A(I'_{s=0}) \) be the two dimensional space of Cauchy characteristics for the derived system \( I'_{s=0} \) with smooth quotient map \( q_{A_{x_{2}^2}_{s=0}} : M_{s=0} \to M_{s=0}/A_{x_{2}^2}_{s=0} \). The EDS \( \tilde{I}_{s=0} = I_{s=0}/q_{A_{x_{2}^2}_{s=0}} \) is the Monge-Ampère form of the wave equation.

Proof. (Theorem 1.3) Let \( x \in M \) and recall that \( M_1 \) is the maximal integral manifold of \( \hat{V}^\infty \) through \( x \) and \( K_1 \) is the restriction of \( \hat{V} \) to \( M_1 \) with the analogous definition of \( M_2 \) and \( K_2 \) (Theorem 5.4). By the proof of Theorem 1.2 in Section 5.2 there exists open sets \( U \subset M, U_1 \subset M_1, U_2 \subset M_2 \) where \( x \in U \) and diagram (1.6) holds. In order to produce diagram (1.7) we need to de-prolong diagram (1.6) by applying Theorem 4.11.

By Corollary 5.3 we have that \( K_1' = \{ \theta^0, \theta^1 \} \), and from equation (5.7) with \( \mu_1 = 0 \), that span\{\( \partial_{\pi_2} \)\} is the Cauchy characteristic of \( \hat{K}_1' \). A similar argument holds for \( K_2 \) on \( U_2 \) and so the Cauchy characteristics for \( K_1' + K_2' \) on \( U_1 \times U_2 \) are

\[
A(K_1' + K_2')_{U_1 \times U_2} = \text{span}\{\partial_{\pi_2} + 0, 0 + \partial_{\pi_1}\}.
\]

By part [ii] in Theorem 4.11 let \( \tilde{G}_u \) and \( \tilde{G}_H \) be the projected Lie algebras of symmetries of \( (K_1 + K_2)/q_{A_{K_1} + K_2} \) from \( G_u \) and \( G_H \) on \( M_1 \times M_2 \). In the proof of Theorem 1.2 in Section 5.2 it was noted that all the algebras \( \tilde{G}_u \) and \( \tilde{G}_H \) are transverse to \( K_1' + K_2' \). This combined with the fact \( A(K_1' + K_2') \subset \text{ann}(K_1' + K_2') \) implies \( A(K_1' + K_2')/\Gamma_H \) and \( A(K_1' + K_2')/\Gamma_G \) are rank 2. Therefore part [v] of Theorem 4.11 and dimensional considerations imply

\[
A(\mathcal{B}') = A(K_1' + K_2')/\Gamma_H, \quad A(\mathcal{I}_2') = A(K_1' + K_2')/\Gamma_G, \quad A(\mathcal{I}_s'=0) = A(K_1' + K_2')/\Gamma_G.
\]
Diagram (4.25) (or Remark 4.12) along with equation (5.15) then give the de-prolongation diagrams,

\[
\begin{array}{ccc}
\mathcal{K}_1 + \mathcal{K}_2 & \xrightarrow{q_{\mathcal{K}_1}} & \mathcal{B} \\
\text{with a similar diagram holding for } \mathcal{K}_1 \text{ and } \mathcal{I}_{s=0}. \text{ Now applying Theorem 1.1 with } \tilde{\Gamma}_H = \tilde{\Gamma}_{G_1} \cap \tilde{\Gamma}_{G_2} \text{ produces the diagram}
\end{array}
\]

(5.20)

\[
\begin{array}{ccc}
\tilde{\Gamma}_H & \xrightarrow{q_{\mathcal{K}_1'}} & \mathcal{B} \\
\text{Combining the bottom edges of diagrams (5.19) together with (5.20) produces the bottom part of diagram (1.7).} \tag{5.19}
\end{array}
\]

Proof. (Theorem 1.4). Theorem 1.4 will follow immediately from diagram (5.20) in the previous proof once we identify \( K_1 \) and \( K_2 \). By Corollary 5.5. \( K_a \) have derived flag \([3, 2, 1, 0]\). Therefore Enge’s Theorem 4.1 applies and \( K_a \) may be identified locally with the canonical contact system on \( J^3(\mathbb{R}, \mathbb{R}) \). Finally \( C_a = K_a/q_{\mathcal{K}_a} \) are the de-prolongation of \( K_a \), and may be identified locally with the canonical contact system on \( J^2(\mathbb{R}, \mathbb{R}) \). This finishes the proof of Theorem 1.4 and produces diagram (1.5).

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