Abstract

The "Diophantine" property of the zeros of certain polynomials in the Askey scheme, recently discovered by Calogero and his collaborators, is explained, with suitably chosen parameter values, in terms of the summation theorem of hypergeometric series. Here the Diophantine property refers to integer valued zeros. It turns out that the same procedure can also be applied to polynomials arising from the basic hypergeometric series. We found, with suitably chosen parameters and certain $q$–analogue of the summation theorems, zeros of these polynomials explicitly, which are no longer integer valued. This goes beyond the results obtained by the Authors mentioned above.

Mathematics Subject Classification: 33C20, 33C45.
Key words: Generalized Hypergeometric Series, Basic Hypergeometric Series, Summation Theorems.

1 Introduction

In a series of papers Calogero and his collaborators, see for example [3], [4], and [5], investigated various integrable lattices of the Toda-type, with suitable boundary conditions. These lattices arose as the dressing chains of Adler, Shabat, Yamilov and other. See for example [1], [13] and [16]. It is found that if the small amplitude motion about the equilibrium configuration is assumed to be isochronous, namely, each component is periodic with
the same period, then the characteristic frequencies, must necessarily have integer values. Furthermore, if the assumption of nearest neighbor interaction is made in the lattice models, then the secular equation whose zeros gives the characteristic frequencies reads

\[ \det(x I_N - A_N) = 0, \]

where \( A_N \) is a tri-diagonal matrix of size \( N \). We may take \( N \) to be the number of particles in the many-body problem. See [7] for a detailed treatment. Hence

\[ P_N(x) := \det(xI_N - A_N) \]

maybe interpreted as orthogonal polynomials if the super-diagonal elements of \( A_N \) are real and none of them vanishes. We show that the Diophantine property are the generated when the parameters of the orthogonal polynomials are suitably chosen. The factorization occurs when the polynomials, although are still characteristic polynomials of tri-diagonal matrices, are no longer orthogonal.

The motivation of considering this problem came from reading [6], where a hypergeometric polynomial of degree \( n \) is factored as \( f_m(x)g_{n-m}(x) \), here \( f_m \) has degree \( m \), \( g_{n-m} \) has degree \( n - m \) and the zeros of \( f_m \) are equi-spaced. This holds for all \( m, 1 \leq m \leq n \). This factorization is referred to as having the “Diophantine property” in [6]. Our explanation is that all the Diophantine results in [6] follow from summation theorems for hypergeometric functions and give \( q \)-analogues of all of them. This will be shown in §3. In §4 we provide \( q \)-analogues of all the results of §3, that is all the results in [6]. Section 2 contains the notation, summation theorems, and transformation formulas used in §3 and §4.

It is known that a sequence of monic orthogonal polynomials satisfy a three term recurrence relation

\[ xP_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \beta_n P_{n-1}(x), \quad n > 0, \]

with \( P_0(x) := 1, P_1(x) := x - \alpha_0 \) then \( P_n(x) \) can be represented as a determinant. The monic polynomials have the determinant representation

\[ P_n(x) = \begin{vmatrix}
  x - \alpha_0 & -a_1 & 0 & \cdots & 0 & 0 & 0 \\
  -a_1 & x - \alpha_1 & -a_2 & \cdots & 0 & 0 & 0 \\
  \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\
  0 & 0 & \cdots & -a_{n-2} & x - \alpha_{n-2} & -a_{n-1} \\
  0 & 0 & 0 & \cdots & 0 & -a_{n-1} & x - \alpha_{n-1}
\end{vmatrix}, \]
where \( a^2_n = \beta_n, n > 0 \). It is clear that if \( \beta_k = 0 \) for some \( k < n \) then \( P_n(x) \) factors into a product of two polynomials of the same type, that is a product of two characteristic polynomials of tri-diagonal matrices. All the factorizations in the work of Bruschi, Calogero, and Drogei are of this type. What is surprising is that one of the two characteristic polynomials has equi-spaced zeros.

The zeros of the polynomial \( f_m \) in the factorization of the Askey-Wilson polynomials turned out to be the points \( \left[ a q^k + q^{-k}/a \right]/2, k = 0, 1, 2, \ldots \), where \( a \) is one of the parameters in the Askey-Wilson polynomial. Such points, after \( a \to ia \) are interpolation points in the sense that the values of an entire function \( h \) at these points determine the function uniquely provided that, \( M(h, r) \), the maximum modulus of \( h \) satisfies \( M(h, r) \leq C r^\alpha \exp(b \ln r^2) \) with \(-2b \ln q < 1\), for some \( \alpha \), see [11]. The integers are interpolation points for entire functions \( h \) for which \( M(h, r) \leq C \exp(br) \) with \( b < \pi \).

**Remark 1.1.** It is important to note that the Wilson polynomials are believed to be the most general orthogonal polynomials of hypergeometric type while the Askey-Wilson polynomials are the most general orthogonal polynomials of basic hypergeometric type. As such we believe that it is unlikely to extend this work to more general polynomials.

## 2 Summation Theorems

Recall that the \( q \)-shifted factorial is

\[
(a; q)_n = \prod_{k=1}^{n} (1 - a q^{k-1}),
\]

and a basic hypergeometric function is

\[
\phi_r \left( \begin{array}{c} a_1, a_2, \ldots, a_{r+1} \\ b_1, b_2, \ldots, b_r \end{array} \right| q, z \right) = \sum_{n=0}^{\infty} \prod_{k=1}^{r+1} \frac{(a_k; q)_n}{(b_k-1; q)_n} z^n,
\]

where \( b_0 := q \).

The Pfaff-Saalschütz theorem is,

\[
_{3}F_{2} \left( \begin{array}{c} -n, A, B \\ C, 1 + A + B - n - C \end{array} \right| 1 \right) = \frac{(C-A)_n (C-B)_n}{(C)_n (C-A-B)_n}.
\]
The summation formula

\[ r+2F_{r+1} \left( \begin{array}{c} A, B, B_1 + m_1, \ldots, B_r + m_r \\ B + 1, B_1, \ldots, B_r \end{array} \bigg| 1 \right) \]

\[ = \frac{\Gamma(B + 1)\Gamma(1 - A)}{\Gamma(1 + B - A)} \prod_{j=1}^{r} \frac{(B_j - B_m)}{(B_j)_{m_j}}, \]

(2.4)

is known as the Karlsson-Minton sum, [9] but it follows from the earlier work of Fields and Wimp [8]. In particular we have

\[ r_{+1}F_r \left( \begin{array}{c} A, B_1 + m_1, \ldots, B_r + m_r \\ B_1, \ldots, B_r \end{array} \bigg| 1 \right) = 0, \]

(2.5)

for \( \text{Re} \ (-a) > m_1 + m_2 + \cdots + m_r \). We will also apply the Whipple transformation [14]

\[ 4F_3 \left( \begin{array}{c} -n, A, B, C \\ D, E, F \end{array} \bigg| 1 \right) = \frac{(E - A)_n(F - A)_n}{(E)_n(F)_n} \]

\[ \times 4F_3 \left( \begin{array}{c} -n, A, D - B, D - C \\ D, A + 1 - n - E, A + 1 - n - F \end{array} \bigg| 1 \right) \]

(2.6)

where \( D + E + F = A + B + C + 1 - n \).

The \( q \)-analogue of the Pfaff-Saalschütz theorem is

\[ 3\phi_2 \left( \begin{array}{c} q^{-n}, A, B \\ C, q^{-n}AB/C \end{array} \bigg| q, q \right) = \frac{(C/A; q)_n(C/B; q)_n}{(C; q)_n(C/AB; q)_n} \]

while the \( q \)-analogue of the Whipple transformation is the Sears transformation [9] (III.15)] is

\[ 4\phi_3 \left( \begin{array}{c} q^{-n}, A, B, C \\ D, E, F \end{array} \bigg| q, q \right) = A^n \frac{(E/A; q)_n(F/A; q)_n}{(E; q)_n(F; q)_n} \]

\[ \times 4\phi_3 \left( \begin{array}{c} q^{-n}, A, D/B, D/C \\ D, q^{-n}A/E, q^{-n}A/F \end{array} \bigg| q, q \right) \]

(2.8)

where \( DEF = q^{1-n}ABC \).

Some useful identities are:

\[ (aq^{-n}; q)_n = (q/a; q)_n (-a)^n q^{-\binom{n+1}{2}}, \]

(2.9)

\[ (aq^{-n}; q)_{n-k} = \frac{(q/a; q)_n}{(q/a; q)_k} (-a)^{n-k} q^{\binom{k+1}{2} - \binom{n+1}{2}}, \]

(2.10)

Of course the first is a special case of the second.
3 Complete factorization of the Wilson and Related Polynomials

The Wilson polynomials is

\[
W_n(x; t) = \prod_{j=1}^{3} (t_1 + t_j) \times {}_4F_3 \left( \begin{array}{c} -n, t_1 + t_2 + t_3 + t_4 + n - 1, t_1 + i\sqrt{x}, t_1 - i\sqrt{x} \\ t_1 + t_2, t_1 + t_3, t_1 + t_4 \end{array} \right) ,
\]

where \( t := (t_1, t_2, t_3, t_4) \). It is a fact that the Wilson polynomials is symmetric in the parameters \( t_1, t_2, t_3, t_4 \). The invariance of \( W_n \) under permutations of \( \{t_2, t_3, t_4\} \) is obvious but the invariance under permuting \( t_1 \) and \( t_j \), for \( j = 2, 3, 4 \) is not obvious and is called the Whipple transformation, [14].

If we wish to find a complete factorization of \( W_n \), or equivalently identify all the zeros of \( W_n \), then we must choose the parameters in such a way that the \( {}_4F_3 \) representation can be summed explicitly.

We shall denote the monic Wilson polynomials by \( \{\tilde{W}_n(x; t)\} \), that is

\[
\tilde{W}_n(x; t_1, t_2, t_3, t_4) = \frac{(-1)^n}{(n + t_1 + t_2 + t_3 + t_4 - 1)n} W_n(x; t_1, t_2, t_3, t_4).
\]

Case 1. We reduce the \( {}_4F_3 \) to a \( {}_3F_2 \) and use the Pfaff-Saalschütz theorem. Since we want to keep \( x \) in the factorization we demand that \( n - 1 + \sum_{k=1}^{4} t_k \) be equal to \( t_1 + t_j \) for some \( j \). It is easy to see that this happens if and only if \( t_i + t_j = 1 - n \) for some \( i \neq j, 1 < i, j \leq 4 \). There is no loss of generality in assuming \( t_4 = 1 - n - t_3 \). In this case [23] gives

\[
\tilde{W}_n(x; t_1, t_2, t_3, 1 - n - t_3) = (-1)^n (t_1 + t_3)_n (t_1 + 1 - n - t_3)_n \times {}_3F_2 \left( \begin{array}{c} -n, t_1 + i\sqrt{x}, t_1 - i\sqrt{x} \\ t_1 + t_3, t_1 + 1 - n - t_3 \end{array} \right) ,
\]

\[
= (-1)^n (t_1 + t_3)_n (t_1 + 1 - n - t_3)_n \times \frac{(t_3 + i\sqrt{x})_n (t_3 - i\sqrt{x})_n}{(t_1 + t_3)_n (t_3 - t_1)_n} .
\]

Since \( (-1)^n (t_1 - t_3 + 1 - n)_n = (t_3 - t_1)_n \), it follows that

\[
\tilde{W}_n(x; t_1, t_2, t_3, 1 - n - t_3) = (t_3 + i\sqrt{x})_n (t_3 - i\sqrt{x})_n
\]

\[
= \prod_{k=1}^{n} [x^2 + (t_3 + k - 1)^2] .
\]

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The above factorization is equations (33) and (41a) of [6].

**Case 2.** We identify values of $x$ suitable to apply (3.4). For example, we may choose $i\sqrt{x} = t_4 + j$, and make $t_1 - i\sqrt{x}$, which is $t_1 - t_4 - j$, equal $t_1 + t_3 + k$. Thus we make the parameter identification $x = -(t_4 + j)^2, t_3 = -t_4 - j - k$. Finally we demand that $n + t_3 + t_4 - 1 = s$, which is equivalent to $n - 1 \geq j + k$.

Now we set $m = j + k + 1$, replace $j$ by $j - 1$ and we find that

$$W_n(-(t_4 + j - 1)^2; t_1, t_2, -t_4 - m + 1, t_4) = 0,$$

for $1 \leq j \leq m, 1 \leq m \leq n$. This is (35) in [6]. This is particularly interesting because it seems to give a partial factorization when $m < n$. We shall return to this point at the end of this section.

When $m = n$ we obtain the factorization

$$W_n(x; t_1, t_2, -t_4 - n + 1, t_4) = \prod_{j=1}^{n} [x + (t_4 + j - 1)^2],$$

which is (34) in [6]. The special case $t_4 = (1 - 2n)/4$ is (41a) of [6]. This last factorization also follow from the Pfaff-Saalschütz formulas since the $4F_3$ reduces to a $3F_2$. Indeed in this case we have

$$W_n(x; t_1, t_2, -t_4 - n + 1, t_4) = (t_1 + t_2)_n(t_1 + t_4)_n(t_1 + 1 - n - t_4)_n$$

$$\times 3F_2 \left( \frac{-n, t_1 + i\sqrt{x}, t_1 - i\sqrt{x}}{t_1 + t_4, t_1 + 1 - n - t_4} \Big| 1 \right)$$

$$= (t_1 + t_2)_n(t_1 + t_4)_n(t_1 + 1 - n - t_4)_n \frac{(t_4 + i\sqrt{x})_n(t_4 - i\sqrt{x})_n}{(t_1 + t_4)_n(t_4 - t_1)_n}.$$

We now show how to discover (3.4) from the Whipple transformation. It is clear that $W_n(x; t_1, t_2, 1 - t_4 - m, t_4)$ is a constant multiple of

$$4F_3 \left( \frac{-n, t_1 + t_2 + n - m, t_1 + i\sqrt{x}, t_1 - i\sqrt{x}}{t_1 + t_2, t_1 + 1 - m - t_4, t_1 + t_4} \Big| 1 \right)$$

$$= \frac{(1 - m - t_4 - i\sqrt{x})_n(t_4 - i\sqrt{x})_n}{(t_1 + t_4 + 1 - m)_n(t_1 + t_4)_n}$$

$$\times 4F_3 \left( \frac{-n, t_1 + i\sqrt{x}, m - n, t_2 + i\sqrt{x}}{t_1 + t_2, t_4 + i\sqrt{x} + m - n, i\sqrt{x} + 1 - n - t_4} \Big| 1 \right).$$

In the last step we applied the Whipple transformation (2.6) with the parameter identification

$$A = t_1 + i\sqrt{x}, B = n + t_1 + t_2 - m, C = t_1 - i\sqrt{x},$$

$$D = t_1 + t_2, E = t_1 - t_4 + 1 - m, F = t_1 + t_4.$$
We next apply the Whipple transformation again with the choices

\[ A = t_2 + i\sqrt{x}, \quad B = t_1 + i\sqrt{x}, \quad C = -n, \]
\[ D = t_1 + t_2, \quad E = t_4 + i\sqrt{x} + m - n, \quad F = 1 - n - t_4 + i\sqrt{x}, \]

and \( n \) is now \( n - m \). Therefore the left-hand side of (3.6) is

\[
(1 - m - t_4 - i\sqrt{x})_n (t_4 - i\sqrt{x})_n \quad (t_1 - t_4 + 1 - m)_n (t_1 + t_4)_n \times \frac{(t_4 - t_2 + m - n)_{n-m}(1 - n - t_2 - t_4)_{n-m}}{(t_4 + m - n + i\sqrt{x})_{n-m}(1 - n - t_4 + i\sqrt{x})_{n-m}} \times 4F_3 \left( \frac{m - n, n + t_1 + t_2, t_2 + i\sqrt{x}, t_2 - i\sqrt{x}}{t_1 + t_2, t_2 + 1 - t_4, t_2 + t_4 + m} \right).
\]

(3.7)

It is clear that the \( 4F_3 \) in the above expression is a constant multiple of \( W_{n-m}(x; t_2, t_1, 1 - t_4, m + t_4) \). We now apply the identities

\[
(\alpha + 1)_n = (-1)^n (-\alpha - n)_n, \quad (\alpha)_n = (\alpha)_m (\alpha + m)_{n-m}
\]

(3.8)

to see that

\[
\frac{(1 - m - t_4 - i\sqrt{x})_n (t_4 - i\sqrt{x})_n}{(t_4 + m - n + i\sqrt{x})_{n-m}(1 - n - t_4 + i\sqrt{x})_{n-m}} = \frac{(1 - m - t_4 - i\sqrt{x})_n (t_4 - i\sqrt{x})_n}{(1 - t_4 - i\sqrt{x})_{n-m}(m + t_4 - \sqrt{x})_{n-m}} = \frac{(1 - m - t_4 - i\sqrt{x})_n (t_4 - i\sqrt{x})_n}{(m + t_4 - \sqrt{x})_{n-m}} = (-1)^m (t_4 + i\sqrt{x})_m (t_4 - i\sqrt{x})_m = (-1)^m \prod_{j=0}^{m-1} [x^2 + (t_4 + j)^2].
\]

This shows that

\[
\frac{W_n(x; t_1, t_2, 1 - t_4 - m, t_4)}{(t_1 + t_2)_n (t_1 + t_2, 1 - t_4 - m)_n (t_1 + t_4)_n} = \frac{(-1)^m (t_4 + i\sqrt{x})_m (t_4 - i\sqrt{x})_m}{(t_1 + t_2)_{n-m} (t_1 + t_4)_{n-m} (t_1 - t_4 + 1 - m)_{n-m}} \times W_{n-m}(x; t_2, t_1, 1 - t_4, t_4 + m).
\]
4 The Askey-Wilson Polynomials

The Askey-Wilson polynomials are defined through the representation

\[
p_n(x; t) = t_1^{-n} \prod_{j=1}^{3} (t_1 t_j; q)_n \times _4 \phi_3 \left( q^{-n}, t_1 t_2 t_3 t_4 q^{n-1}, t_1 e^{i \theta}, t_1 e^{-i \theta} \left| q, q \right. \right).
\]

As in Case 1 of §3 we reduce the \( _4 \phi_3 \) to a \( _3 \phi_2 \). There is no loss of generality in assuming that \( q^{n-1} t_1 t_2 t_3 t_4 = t_1 t_2 \), that is \( t_4 = q^{n-1} / t_3 \). Applying the summation formula (2.7) we get

\[
p_n(\cos \theta; t_1, t_2, t_3, q^{n-1} / t_3) = (t_3 e^{i \theta}; q)_n (t_3 e^{-i \theta}; q)_n \frac{(t_1 t_2; q)_n q^{1-n} t_1 / t_3; q)_n}{t_1^n (q/t_1 t_3; q)_n}
\]

It is clear that

\[
(t_3 e^{i \theta}; q)_n (t_3 e^{-i \theta}; q)_n = \prod_{k=1}^{n} [1 - 2t_3 x q^{k-1} + t_3^2 q^{2k-2}],
\]

from which we can find the zeros explicitly.

As in of Case 2 §3 we choose \( t_3 = q^{1-m} / t_4 \) then apply the Sears transformation (2.8) with the parameter identification

\[
A = t_1 e^{i \theta}, \quad B = q^{n-m} t_1 t_2, \quad C = t_1 e^{-i \theta},
\]

\[
D = t_1 t_2, \quad E = q^{1-m} t_1 / t_4, \quad F = t_1 t_4.
\]

The result is

\[
p_n(\cos \theta; t) = e^{i m \theta} (t_1 t_2; q)_n (q^{1-m} e^{i \theta} / t_4; q)_n (t_4 e^{-i \theta}; q)_n \times _4 \phi_3 \left( q^{-n}, t_1 e^{i \theta}, t^{m-n} t_4 e^{i \theta}, t^{1-n} e^{i \theta} / t_4 \left| q, q \right. \right).
\]

We next apply the Sears transformation again with the choices

\[
A = t_2 e^{i \theta}, \quad B = t_1 e^{i \theta}, \quad C = q^{-n},
\]

\[
D = t_1 t_2, \quad E = t_4 q^{m-n} e^{i \theta}, \quad F = q^{1-n} e^{i \theta} / t_4,
\]
and the terminating parameter $n$ is replaced by $n - m$. This leads to

$$p_n(x; t) = \frac{t_2^{2(n-m)}(t_1t_2; q)_n(q^{1-m}e^{-i\theta}/t_4; q)_n(t_4e^{-i\theta}; q)_n}{(t_1t_2; q)_{n-m}(t_2q/t_4; q)_{n-m}(q^m t_2 t_4; q)_{n-m}} \times \frac{(q^{n-m}t_4/t_2; q)_{n-m}(q^{1-n}/t_2t_4; q)_{n-m}e^{-i(2n-m)\theta}}{(q^{m-n}t_4q^{m}; q)_{n-m}(q^{1-n}e^{i\theta}/t_4; q)_{n-m}} \times p_{n-m}(x; t_2, t_1, q/t_4, q^m t_4),$$

with $x = \cos \theta$. Applying equations (2.9)–(2.10) we finally establish the factorization

$$p_n(x; t) = (t_4e^{i\theta}; q)_m(t_4e^{-i\theta}; q)_m p_{n-m}(x; t_2, t_1, q/t_4, q^m t_4)$$

(4.4) \times (-1)^m t_4^{n-2m} t_2^{m-n} \frac{(q^m t_1 t_2; q)_{n-m}}{(q^m t_1 t_4; q)_{n-m}},

again with $x = \cos \theta$. Note that

$$ (t_4e^{i\theta}; q)_m(t_4e^{-i\theta}; q)_m = \prod_{k=1}^{n} [1 - 2xt_4 q^{k-1} + t_4^2 q^{2k-2}].$$

(4.5)

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