ADMISSIBLE HERMITIAN-YANG-MILLS CONNECTIONS OVER NORMAL VARIETIES

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Abstract. In this paper, we first prove a complete version of the Donaldson-Uhlenbeck-Yau theorem over normal varieties, including normal Kähler varieties and projective normal varieties with multiple polarizations. In particular, this gives the polystability of reflexive sheaves under symmetric and exterior powers and tensor products. As a consequence of the singular Donaldson-Uhlenbeck-Yau theorem, the complete Hitchin-Kobayashi correspondence over normal varieties smooth in codimension two is built by showing that an admissible Hermitian-Yang-Mills connection defines a polystable reflexive sheaf. Furthermore, it is shown that the Hermitian-Yang-Mills connection gives a lower bound for the discriminants of any Kähler resolutions, which gives a Bogomolov-Gieseker inequality over normal varieties and a characterization of the equality using projectively flat connections. We discuss typical cases including normal surfaces and varieties smooth in codimension two where we could simplify the Bogomolov-Gieseker inequality and endow it with topological meanings. We also prove the Bogomolov-Gieseker inequality for semistable reflexive sheaves and characterize the class of semistable sheaves that satisfy the Bogomolov-Gieseker equality. Finally, we give a new criteria for when a normal Kähler variety with trivial first Chern class is a finite quotient of torus.

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1. INTRODUCTION

The celebrated Donaldson-Uhlenbeck-Yau theorem ([13] [44]) confirms the existence of Hermitian-Yang-Mills metrics on stable holomorphic vector bundles over compact Kähler manifolds. Two important corollaries include the polystability of reflexive sheaves under symmetric and exterior powers and tensor products, and the Bogomolov-Gieseker inequality for stable bundles over Kähler manifolds together with a characterization of the equality using projectively flat metrics. There are also further important developments focusing on different aspects: stable Higgs bundles over Kähler manifolds by Simpson ([38]); stable bundles over general compact complex manifold with Gauduchon metrics by Li and Yau ([27]) and over surface by Buchdahl ([3]), with a detailed account by Lubke and Teleman ([29]); stable reflexive sheaves over Kähler manifolds by Bando and Siu ([1]); the most general version over Hermitian manifolds by Lubke and Teleman ([30]). The other direction of the Hitchin-Kobayashi correspondence is much simpler and first proved by Kobayabshi and Lubke ([24]).

The goal of this paper is to prove the singular Donaldson-Uhlenbeck-Yau theorem over normal varieties in complete generality and give various applications in the singular setting. However, the “easy” direction of the Hitchin-Kobayashi correspondence is much more involved in the singular case in general. As a consequence of the singular Donaldson-Uhlenbeck-Yau theorem, we will build the complete Hitchin-Kobayashi correspondence over normal varieties smooth in codimension two.

Now we explain our main motivations for this paper.

First, in recent work [8] where we proved a singular Donaldson-Uhlenbeck-Yau theorem over a class of projective normal varieties, we need to make the assumption that either the projective normal variety has codimension at least three singular set or it comes from the limit of smooth projective varieties with induced Kähler metric from an ambient smooth variety. The techniques developed in [8] has its limits when dealing with normal Kähler
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varieties in general. The first goal of this paper is to remove those assumptions. For this, we will have to deal with more subtle analytic problems and gauge theoretical phenomenon in the critical case due to lack of algebraic geometric tools. There are even simple questions that can be easily solved in the smooth case but not over singular varieties. For example, even in the case of rank 1 reflexive sheaves, the existence of Hermitian-Yang-Mills metrics is nontrivial due to the fact that we are working with singular varieties. Also, a priori, it is not known that an admissible Hermitian metric on a reflexive sheaf does compute the slope of the sheaf, not even to mention the subsheaf. In particular, the easy direction of Hitchin-Kobayashi correspondence that admitting an admissible Hermitian-Yang-Mills metrics imply the sheaf being polystable is highly nontrivial unlike the smooth case, due to lack of the Chern-Weil formula and a sheaf extension result as [1] in the singular setting. We could pass by those subtle problems in [8] by working in the projective case with mild singularities and assuming that the metric on the normal variety is induced from some ambient smooth manifolds. On the other hand, this makes the result not sufficient even when we want to deal some related problems with varieties smooth in codimension two ([9]). To solve the problem in general, we will need to have new ideas and deal with more subtle phenomenon due to the critical singularity assumption.

The second motivation comes from the searching for a Bogomolov-Gieseker inequality over general normal varieties. In the general case, the picture for the Bogomolov-Gieseker inequality is, a priori, not clear due to the fact that the singularities of the varieties could contribute. For example, in the normal surface case, it consists of points, thus it is a difficult problem to weigh the contribution of such singularities on both the algebraic and analytic sides. It seems very mysterious about how the algebraic and analytic sides intertwine in general. It turns out that the correct Bogomolov-Gieseker inequality is surprisingly simple which comes from the basic phenomenon that when we shrink the exceptional divisor on the resolution, there could be loss of Yang-Mills energy. Thus the right picture is given by comparing the quantities on any resolutions and the resulting Hermitian-Yang-Mills connections. This recovers the Bogomolov-Gieseker inequality in the smooth case. At the same time, it is sufficient to give a few interesting and nontrivial applications.

Another related motivation comes from the recent work [6] and [7] which generalizes the classical Donaldson-Uhlenbeck-Yau theorem to complex manifolds with Hermitian metrics of Hodge-Riemann type. More precisely, it was observed in [6] that a new class of Gauduchon metrics can be given by the so-called balanced metrics of Hodge-Riemann type, of which the multipolarizations provide a natural class of examples. Namely, given \((n-1)\) Kähler forms \(\omega_1, \ldots, \omega_{n-1}, \omega_1 \wedge \cdots \wedge \omega_{n-1}\) defines a balanced metric. More importantly, the Hodge-Riemann property still holds for \(\omega_1 \wedge \cdots \wedge \omega_{n-2}\) (see [32]) which gives the Bogomolov-Gieseker inequality in the multipolarization setting (see Section 2.2). The latter had been known when \(\omega_i\) are all Hodge
metrics which follows from the restriction theorem ([26] [23]). For a balanced metric coming from the multipolarization above, \([\omega_1 \wedge \cdots \omega_{n-1}]\) lies in the interior of the so-called cone of movable curves of compact complex manifolds ([2]). Also, the notion of stable sheaves defined via multipolarizations ([18]) already plays an important role in compactifying the moduli space of semistable sheaves over projective manifolds in higher dimensions. More generally the slope stability via movable class has been defined and studied on normal varieties, and it is a very important and useful concept in birational geometry ([17] [4]).

Given the discussions above, we will study the gauge theoretical side for stable reflexive sheaves over normal varieties endowed with multiple Kähler metrics. The key difficulty in this process is to deal with the information near the singularities of the base variety, together with the new features by involving multiple Kähler metrics.

1.1. **Singular Donaldson-Uhlenbeck-Yau theorem.** Suppose \(X\) is a normal variety of dimension \(n\) endowed with \((n-1)\) Kähler forms \(\omega_1, \cdots, \omega_{n-1}\) (see Section 2.1 for definitions). Through the equation \(\omega^{n-1} = \omega_1 \wedge \cdots \omega_{n-1}\), this defines a balanced metric \(\omega\). A stable reflexive sheaf can be defined by passing to a particular resolution \(p: \hat{X} \to X\) so that \(\hat{E} := (p^*E)^{**}\) is stable with respect to \(p^*\omega_1 \wedge \cdots p^*\omega_{n-1}\). The stability is independent of the resolutions since everything is smooth in codimension one (see Section 2.3).

**Theorem 1.1.** Suppose \(E\) is a stable reflexive sheaf over a normal variety \((X, \omega_1 \wedge \cdots \omega_{n-1})\). Then there exists an admissible Hermitian-Yang-Mills metric on \(E\). Moreover, such a metric is unique up to scaling.

**Remark 1.2.** In the Kähler variety case, i.e. \(\omega = \omega_1 = \cdots \omega_{n-1}\), there has been some recent progress on the singular Donaldson-Uhlenbeck-Yau theorem. In the projective case ([8]), if \(\omega\) is the restriction of a Hodge metric from the ambient smooth variety, this has been proved by assuming the base is smooth in codimension two; a singular Donaldson-Uhlenbeck-Yau theorem has also been shown for stable sheaves over projective normal varieties which come from limits of stable sheaves over smooth projective varieties. Assuming a uniform Sobolev constant control for the resolution with perturbed Kähler metrics, the argument in [8] can be used to prove slightly more general results for normal projective varieties and subvarieties of Kähler manifolds smooth in codimension two. However, the control of the Sobolev constants for the perturbed Kähler metrics does not seem to follow from any known results unlike the smooth case in [1]. On the other hand, to use the heat flow to prove the singular Donaldson-Uhlenbeck-Yau theorem, the control of the Sobolev constants is very crucial. Assuming the existence of the uniform Sobolev constant control, the singular Donaldson-Uhlenbeck-Yau theorem could be obtained over normal Kähler varieties using the heat flow method (see [32]).

As a direct corollary, this gives
Corollary 1.3. Suppose $\mathcal{E}_1, \mathcal{E}_2$ are stable reflexive sheaves over $(X, \omega_1 \wedge \cdots \wedge \omega_{n-1})$, then $(\text{Sym}^k \mathcal{E}_1)^{**}, (\wedge^k \mathcal{E}_1)^{**}$ and $(\mathcal{E}_1 \otimes \mathcal{E}_2)^{**}$ are all polystable.

1.2. Hitchin-Kobayashi correspondence over normal varieties smooth in codimension two. By working over normal varieties smooth in codimension two, the complete Hitchin-Kobayashi correspondence could be built by using the singular Donaldson-Uhlenbeck-Yau theorem together with Siu’s sheaf extension result. For this, we have

Corollary 1.4. An admissible Hermitian-Yang-Mills connection over a normal variety $(X, \omega_1 \wedge \cdots \wedge \omega_{n-1})$ smooth in codimension two defines a polystable reflexive sheaf.

Proof. By [39], we know an admissible Hermitian-Yang-Mills connection defines a unique coherent reflexive sheaf since $X$ is smooth in codimension two. Now the polystability follows from Corollary 3.5 which is essentially a consequence of the singular Donaldson-Uhlenbeck-Yau theorem obtained above. □

In particular, combined with the singular Donaldson-Uhlenbeck-Yau theorem above, we have

Theorem 1.5. Over a normal variety $X$ smooth in codimension two endowed with $n-1$ Kähler forms $\omega_1, \cdots, \omega_{n-1}$, there exists a one-to-one correspondence between the space of isomorphism classes of stable reflexive sheaves and the space of gauge equivalent classes of admissible Hermitian-Yang-Mills connections over $(X, \omega_1 \wedge \cdots \wedge \omega_{n-1})$.

Remark 1.6. For the full Hitchin-Kobayashi correspondence over general normal varieties, given the singular Donaldson-Uhlenbeck-Yau theorem above, the essential difficulty lies in proving an admissible Hermitian-Yang-Mills connection defines a coherent reflexive sheaf. More precisely, one needs to construct enough sections of the sheaf across the singular set.

This statement is only on the set level. It seems to the author that an improved map such as an analytic isomorphism as the smooth case is very subtle near singularities of the base. For example, near the singularities, it involves subtle analysis to make sense of a reasonably good moduli space of the analytic solutions. In the normal surface case, as pointed out by Donaldson, this could be related to [14]. We leave this for future work.

1.3. Bogomolov-Gieseker inequality. Now we explain how to build a Bogomolov-Gieseker inequality in the general case. As already mentioned above, this is essentially a consequence of the basic phenomenon that there could be loss of Yang-Mills energy when we shrink the exceptional divisor in the proof of the Donaldson-Uhlenbeck-Yau theorem above.

By the Hodge-Riemann property for multiplicarizations, the Hermitian-Yang-Mills metric $H$ obtained above gives an analytic Bogomolov-Gieseker
inequality. Unless the variety is smooth in codimension two, it is in general expected that it does not compute any corresponding algebraic quantities. However, motivated by the gauge theoretical picture in the proof of the singular Donaldson-Uhlenbeck-Yau theorem, for the algebraic side, fix any $1 \leq i_1 < \cdots i_{n-2} \leq n-2$, we define the discriminant as

$$
\Delta(\mathcal{E})[\omega_{i_1}] \cdots [\omega_{i_{n-2}}] = \inf_{\text{Kähler}} \inf_{\hat{E} \in \mathcal{E}_p} (2rc_2(\hat{E}) - (r-1)c_1(\hat{E})^2)[p^*\omega_{i_1}] \cdots [p^*\omega_{i_{n-2}}]
$$

which is intrinsically associated to $\mathcal{E}$, here $p$ is among all the Kähler resolutions, i.e. $\hat{X}$ admits a Kähler metric and $\mathcal{E}_p$ denotes the space of reflexive sheaves over $\hat{X}$ which are isomorphic to $\mathcal{E}$ away from the exceptional divisor. It turns out the Hermitian-Yang-Mills metric does give a lower bound for the discriminant, thus a version of Bogomolov-Gieseker inequality over normal varieties without requiring the variety being smooth in codimension two. This recovers the classical Bogomolov-Gieseker inequality in the smooth case.

**Corollary 1.7.** Suppose $\mathcal{E}$ is a stable reflexive sheaf over $(X, \omega_1 \wedge \cdots \omega_{n-1})$ and let $H$ be the admissible Hermitian-Yang-Mills metric as above. Then

$$
\Delta(\mathcal{E})[\omega_{i_1}] \cdots [\omega_{i_{n-2}}] \geq \int_X (2rc_2(H) - (r-1)c_1(H)^2) \wedge \omega_{i_1} \wedge \cdots \omega_{i_{n-2}}
$$

for any $1 \leq i_1 < \cdots i_{n-2} \leq n-1$. In particular,

$$
\Delta(\mathcal{E})[\omega_{i_1}] \cdots [\omega_{i_{n-2}}] \geq 0
$$

where $\mathcal{E}$ is projectively flat if the equality holds, i.e. $\mathcal{E}|_{X^{\text{reg}}}$ is defined by a representation $\rho : \pi_1(X^{\text{reg}}) \to \text{PU}(m)$ where $m = \text{rank} \mathcal{E}$.

**Remark 1.8.** • From our perspective, the most important aspect of this theorem lies in that if the stable reflexive sheaf can be resolved in a nice way, then one can conclude good properties about the original sheaf. Such methods are very often used when one studies stable reflexive sheaves in birational geometry or decomposition of singular spaces. To name a few, for example, [9] [10] [4] [16] [17]. See also Corollary 1.16 for such a nontrivial application.

• We emphasize that we would not expect the equality in the first inequality to hold in general due to the natural gauge theoretical picture. The significance shows up when we study the moduli space of admissible Hermitian-Yang-Mills connections over normal varieties, this will be an important ingredient since it gives us natural bound on the $L^2$ norm of the curvature. Especially in the normal surface case, this could be related to Donaldson theory (see Section 1.3.1).
• When the base is smooth in codimension two, the Bogomolov-Gieseker inequality is known in the case of projective normal varieties, or more generally Kähler normal varieties, and the equality has been characterized with various characteristic classes vanishing conditions [16, 9, 22, 17]. The novelty of our theorem is that it builds the general version over normal varieties without being smooth in codimension three assumption and gives a characterization of the equality in complete generality.

• We also refer the readers to the later recent preprints [33, 20] for related results.

1.4. Simplifications of Bogomolov-Gieseker inequalities. Now we discuss various typical and important cases where for the Bogomolov-Gieseker inequality above, we could simplify or give the formula topological meanings.

1.4.1. Over normal surfaces. We first look at the Bogomolov-Gieseker inequality in the normal surface case which could have potential use for understanding Donaldson theory over normal surfaces.

First, when $\mathcal{E}$ is locally free over a normal Kähler surface $(X, \omega)$, the discriminant can be computed by the natural pull-back for any resolutions. In this case, since $\mathcal{E}$ is locally free, it defines a bundle over $X$, thus there already exist topological quantities corresponding to the ones defined here.

**Corollary 1.9** (When $\mathcal{E}$ is locally free over a normal surface). Suppose $\mathcal{E}$ is a stable locally free sheaf over a normal Kähler surface $(X, \omega)$ and let $H$ be the admissible Hermitian-Yang-Mills metric as above. Then

$$\Delta(\mathcal{E}) = (2rc_2(\mathcal{E}) - (r-1)c_1(\mathcal{E})^2) \cap X$$

$$\geq \int_X (2rc_2(H) - (r-1)c_1(H)^2)$$

$$\geq 0$$

where $\mathcal{E}$ is projectively flat if the last equality holds, i.e. $\mathcal{E}|_{X^{reg}}$ is defined by a representation $\rho : \pi_1(X^{reg}) \to \text{PU}(m)$ where $m = \text{rank} \mathcal{E}$.

**Remark 1.10.**

• An interesting aspect about the first equality lies in that for any resolution $p$, $p^*\mathcal{E}$ will compute the smallest one among the extensions $\mathcal{E}_p$ which is related to a lemma of Du Val [31, Proposition 2.1.12]. It states the natural intersection matrix given by intersecting irreducible components of the exceptional divisor is negative definite.

• Over singular varieties, existence of locally free sheaves is in general a very nontrivial question, but in the normal surface case, abundant locally free sheaves have been constructed in [35]. Combined with the results above, this could be a starting point to study Donaldson theory over normal surfaces, and we leave this for future work.

The second simplification is that the quantity can be computed by using only the minimal resolution for a normal surface.
Corollary 1.11 (On the minimal resolution of a normal surface). Suppose \( \mathcal{E} \) is a stable reflexive sheaf over a normal Kähler surface \((X, \omega)\) and let \( H \) be the admissible Hermitian-Yang-Mills metric as above. Then
\[
\Delta(\mathcal{E}) = \inf_{\hat{\mathcal{E}} \in \mathcal{E}_{\mu_{\text{min}}}} (2rc_2(\hat{\mathcal{E}}) - (r - 1)c_1(\hat{\mathcal{E}})^2)
\geq \int_X (2rc_2(H) - (r - 1)c_1(H)^2)
\geq 0
\]
for any \( 1 \leq i_1 < \cdots < i_{n-2} \leq n - 1 \), where \( \mathcal{E} \) is projectively flat if the last equality holds, i.e. \( \mathcal{E}|_{X_{\text{reg}}} \) is defined by a representation \( \rho : \pi_1(X_{\text{reg}}) \to \text{PU}(m) \) where \( m = \text{rank} \mathcal{E} \).

Remark 1.12. It remains as an interesting question in general to characterize when the Chern-Weil formula holds. We expect the right picture for this has connections with [25]. We leave this for future work.

1.4.2. Smooth in codimension two. When \( X \) is a normal variety with multiple Kähler metrics smooth in codimension two, the quantity can be computed by using any resolutions. Also a Chern-Weil formula holds when the base has isolated singularities.

Corollary 1.13 (When \( X \) is smooth in codimension two). Suppose \( \mathcal{E} \) is a stable reflexive sheaf over \((X, \omega_1 \wedge \cdots \omega_{n-1})\) smooth in codimension two and let \( H \) be the admissible Hermitian-Yang-Mills metric as above.

(1) The following holds
\[
\Delta(\mathcal{E})[\omega_{i_1}] \cdots [\omega_{i_{n-2}}] = (2rc_2(\hat{\mathcal{E}}) - (r - 1)c_1(\hat{\mathcal{E}})^2)[p^*\omega_{i_1}] \cdots [p^*\omega_{i_{n-2}}]
\geq \int_X (2rc_2(H) - (r - 1)c_1(H)^2) \wedge \omega_{i_1} \wedge \cdots \omega_{i_{n-2}}
\geq 0
\]
for any resolution \( p \) and \( \hat{\mathcal{E}} \in \mathcal{E}_p \) and for any \( 1 \leq i_1 < \cdots < i_{n-2} \leq n - 1 \), where \( \mathcal{E}|_{X_{\text{reg}}} \) is projectively flat if the last equality holds. In particular, when \([\omega_1], \cdots, [\omega_{n-1}]\) can be represented by very ample divisors, then
\[
\Delta(\mathcal{E})[\omega_{i_{n-1}}] \cdots [\omega_{i_{n-2}}] = \Delta(\mathcal{E}|_{D_{i_1} \cap \cdots D_{i_{n-2}}})
\geq \int_X (2rc_2(H) - (r - 1)c_1(H)^2) \wedge \omega_{i_1} \wedge \cdots \omega_{i_{n-2}}
\geq 0
\]
where \([D_{i_1}] = [\omega_{i_1}]\) so that \( D_{i_1} \cap \cdots D_{i_{n-2}} \) is a smooth surface in \( X \) and \( \mathcal{E}|_{X_{\text{reg}}} \) is projectively flat if the last equality holds, i.e. \( \mathcal{E}|_{X_{\text{reg}}} \) is defined by a representation \( \rho : \pi_1(X_{\text{reg}}) \to \text{PU}(m) \) where \( m = \text{rank} \mathcal{E} \).
(2) Assume further $X$ has isolated singularities, then
\[ \Delta(\mathcal{E})[\omega_{i_{n-1}}] \cdots [\omega_{i_{n-2}}] \]
\[ = \int_X (2rc_2(H) - (r-1)c_1(H)^2) \wedge \omega_{i_1} \wedge \cdots \omega_{i_{n-2}} \]
\[ \geq 0 \]

where $\mathcal{E}|_{X^{\text{reg}}}$ is projectively flat if the last equality holds, i.e. $\mathcal{E}|_{X^{\text{reg}}}$ is defined by a representation $\rho : \pi_1(X^{\text{reg}}) \to \text{PU}(m)$ where $m = \text{rank } \mathcal{E}$.

1.5. **Bogomolov-Gieseker inequality for semistable sheaves.** Now we study the Bogomolov-Gieseker inequality for semistable reflexive sheaves. For this, we suppose $(X, \omega_1 \wedge \cdots \wedge \omega_{n-1})$ is a normal variety with $n-1$ Kähler metrics. Then we have

**Theorem 1.14.** Given a semistable reflexive sheaf $\mathcal{F}$ over $(X, \omega_1 \wedge \cdots \wedge \omega_{n-1})$, the following holds
\[ \Delta(\mathcal{E}).[\omega_{i_1}] \cdots [\omega_{i_{n-2}}] \geq 0. \]
Suppose for some $\hat{\mathcal{E}} \in \mathcal{E}_p$ and some resolution $p : \hat{X} \to X$,
\[ \Delta(\hat{\mathcal{E}}).[p^*\omega_{i_1}] \cdots [p^*\omega_{i_{n-2}}] = 0, \]
then $\mathcal{F}$ admits a filtration
\[ 0 \subset \mathcal{F}_1 \subset \cdots \mathcal{F}_m = \mathcal{F} \]
so that $\mathcal{F}_i/\mathcal{F}_{i-1}$ is torsion free and $(\mathcal{F}_i/\mathcal{F}_{i-1})|_{X^{\text{reg}}}$ is projectively flat, i.e. $(\mathcal{F}_i/\mathcal{F}_{i-1})|_{X^{\text{reg}}}$ is defined by a representation $\rho : \pi_1(X^{\text{reg}}) \to \text{PU}(m_i)$ where $m_i = \text{rank } (\mathcal{F}_i/\mathcal{F}_{i-1})$.

**Remark 1.15.**
- Given the singular Donaldson-Uhlenbeck-Yau theorem we proved above, the key ingredient in this theorem lies in the fact that a weak Hodge-Riemann property still holds on the resolution for multiple semi-Kähler classes. This follows essentially from [11] which generalizes the classical Hodge-Riemann bilinear relation for a fixed Kähler classes to the case of multiple Kähler classes.
- Assume $X$ is smooth in codimension two, by Corollary 1.13 the condition does not depend on the resolution and thus one even has abundant natural examples from algebraic geometry. As we already see in the projective case with one polarization, this result is already very useful in improving the known algebraic geometric result (see [8]).

1.6. **Quotients of torus.** Now we apply our results to give a new criteria about when a normal complex space with klt singularities is a quotient of a complex torus by a finite group by generalizing the results in [9]. We refer [9] for a detailed account on the importance of such problems.
Corollary 1.16. Let \((X, [\omega])\) be a compact normal Kähler variety with klt singularities satisfying \(c_1(X) = 0 \in H^2(X, \mathbb{R})\). Suppose there exists a reflexive sheaf \(E\) for some resolution \(p: \hat{X} \to X\) with \(E|_{X^{\reg}} \cong \mathcal{T}_X|_{X^{\reg}}\)

\[
\Delta(\hat{E})[p^*\omega]^{n-2} = 0
\]

then \(X\) is a quotient of a complex torus by a finite group acting freely in codimension one.

Proof. This follows from Corollary 1.7 together with the remark in [9] Page 3. More precisely, by [19] Theorem A, we know that the tangent sheaf \(\mathcal{T}_X\) is polystable with slope being zero. By Corollary 1.7, we know \(\mathcal{T}_X|_{X^{\reg}}\) is flat since \(c_1(X) = 0\). Now the conclusion follows from [10] Theorem D. \(\square\)

Remark 1.17. This generalizes the key direction in Theorem A in [9] by removing the assumption of \(X\) being smooth in codimension two. In [9], \(X\) is assumed smooth in codimension two, thus \(E\) can be chosen to be \(\mathcal{T}_X\) and the condition is equivalent to

\[
\int_{\hat{X}} c_2(\hat{X}) \wedge [p^*\omega]^{n-2} = 0.
\]

1.7. Sketch of the proof.

1.7.1. Regarding singular Donaldson-Uhlenbeck-Yau theorem. The strategy for the proof is standard and has been used in [8] in the projective case. It is done by passing to a resolution of singularities and studying the corresponding gauge theoretical limits, which goes back to [12, 44, 1]. The subtlety lies really in how to take care of various technical difficulties in this process with new ideas and prove it in the multi-Kähler setting. Start with any resolution \(p: \hat{X} \to X\) so that \(E := (p^*E)^{**}\) is locally free. Write \(X = X^s \cup X^{\reg}\) as a union of singular and smooth parts. Fix any Kähler metric \(\theta\) on \(\hat{X}\). Then one can show \(E\) is stable with respect to \((p^*\omega_1 + i^{-1}\theta) \wedge \cdots (p^*\omega_{n-1} + i^{-1}\theta)\) for any \(i >> 1\), which essentially follows from the boundedness results in [43]. Thus the Donaldson-Uhlenbeck-Yau theorem for multipolarizations ([6]) gives a family of Hermitian-Yang-Mills metrics \(H_i\) with respect to the perturbed metrics. Let \(\mathcal{E}\) be the underlying smooth bundle for \(E\). By normalizing gauge, we get a sequence of Hermitian-Yang-Mills connections on \(\mathcal{E}\) with a unitary metric \(\mathcal{H}\). By known gauge theoretical results ([7]), after passing to a subsequence, up to gauge transforms, \(A_i\) converges to a limiting Hermitian-Yang-Mills connection \(A_\infty\) over \(X \setminus Z\) where \(Z = X^s \cup \Sigma\) and \(\Sigma\) is a codimension two subvariety of \(X^{\reg}\). The proof is done in the following steps

(1) there exists a nontrivial map \(\Phi: \mathcal{E}_\infty \to \mathcal{E}\) where \(\mathcal{E}_\infty\) the reflexive sheaf defined by \(A_\infty\) over \(X \setminus X^s\); for any global section \(s\) of \(\mathcal{E}_\infty\), \(\log^+ |s|^2 \in W^{1,2} \cap L^\infty\). The non-triviality of this step essentially lies in that we need to take limits of holomorphic sections over noncompact manifolds.
(2) in the rank 1 case, show $A_\infty$ computes $\mu(E)$ and thus conclude $\Phi$ is an isomorphism. Note there is no Chern-Weil formula in the singular setting and the existence of admissible Hermitian-Yang-Mills connections in the rank one case is already highly nontrivial.

(3) prove $\Phi$ is an isomorphism for general rank. This crucially depends on (1) and (2).

For (1), we crucially use the fact that $Z$ has codimension two as in [5]. Fix a smooth metric $H'$ on $E$. The idea is to take limit of the sections given by the identity map $id \in \text{Hom}(E, E)$. We can take a precompact exhaustion $X^\epsilon$ of $X \setminus Z$ where $X \setminus X^\epsilon \to Z$ as $\epsilon \to 0$. Normalize the identity map to have $L^2$ norm equal to one over a fixed region $X^\epsilon$ with respect to the metric $H^*_1 \otimes H'$ on $\text{Hom}(E, E)$ and the metric on $X$ given by $\omega_1 \wedge \cdots \omega_{n-1}$. This gives us a sequence of holomorphic sections. By standard elliptic theory, passing to a subsequence, one can take a limit of this sequence. But the problem is that we are working with noncompact base, thus the limit might be trivial. However, we can prove a useful property using the structure of $Z$: for any $z \in X^\epsilon$, there exists a holomorphic curve $D \subset X^{reg}$ passing $z$ and $\partial D \subset X^\epsilon$. This enables us to restrict our sections to $D$, and apply maximum principle to get control over $X^\epsilon$. In particular, we get a nontrivial limit over $X^\epsilon$. For the regularity statement about sections $s$ of $E_\infty$, one can first show $\log^+ |s|^2 \in W^{1,2}_{\text{loc}}$ by an adapation of Bando and Siu’s argument ([1, Theorem 2]) using one dimensional slices instead of two dimensional slices. Then it follows from [25] and [37] that there exists a global Sobolev inequality for $W^{1,2}(X^{reg})$ functions, thus one can apply Moser iteration to get the $L^\infty$ bound (see Section 2.5).

For (2), if we assume rank $E = 1$, then the Remmert-Stein extension theorem implies $E_\infty$ can be extended to be a reflexive sheaf over $X$ since we have a nontrivial map $\Phi : E_\infty \to E$. If we can show $\mu(E_\infty) = \mu(E)$ which will be a very crucial fact needed in (3) as well, the map has to be an isomorphism. This relies on a key observation that such a Chern-Weil formula still exists by using the fact that $A_\infty$ comes from the limit of smooth ones on the resolutions and the fact that $X^s \subset X$ is a complex subvariety of codimension at least two.

For (3), by Siu’s theorem ([40]), one can show actually the saturation $G$ of the image of $\Phi$ defines a coherent analytic subsheaf of $E$. If rank $G < \text{rank } E$, then $\mu(G) < \mu(E)$ since $E$ is stable. By applying the Weizenböck formula to sections of $\text{Hom}(\wedge^{\text{rank } G} E_\infty, \det G)$, this will give a contradiction. Here we need the crucial fact that $\det G$ admits a Hermitian-Yang-Mills metric which does compute the slope of $\det G$ by step (2). In particular, $\Phi$ has full rank. Now by (2), we know $\det(E_\infty) \cong \det(E)$ which will force $\Phi$ to be an isomorphism. The conclusion follows.
1.7.2. Bogomolov-Gieseker inequality. As we mentioned in the introduction, the Bogomolov-Gieseker inequality follows from the gauge theoretical picture when we shrink the exceptional divisor naturally. We first deal with $p : \hat{X} \to X$ so that $\hat{E} = (p^*E)^{**}$ is locally free. On the resolution, the quantities in the Bogomolov-Gieseker inequality can be directly computed by the perturbed Hermitian-Yang-Mills metrics. Now the limit of this equation will give us what we need. This is due to the fact that by the Hodge-Riemann property for multipolarizations, the integrand given by the Hermitian-Yang-Mills metrics on the resolution defines a sequence of Radon measures with uniformly bounded mass on $\hat{X}$, thus the inequality follows from Fatou’s lemma. For general resolutions $p : \hat{X} \to X$, this follows from the same argument by our main theorem applied to stable reflexive sheaves over $\hat{X}$ and that the Chern-Weil formula still holds for such with admissible Hermitian-Yang-Mills metrics.

Now the various cases of simplifications follow from some topological arguments as well as some more subtle analytic properties.

The Bogomolov-Gieseker inequality for semistable sheaves follows essentially from results on Hodge-Riemann properties for multiple Kähler metrics [11] together with our main results for stable reflexive sheaves.

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2. Preliminary results

2.1. Varieties with multiple Kähler metrics. In this section, we will recall the notion of Kähler metrics on normal varieties from [45].

Let $X$ be a normal variety. We will always write $X = X^s \cup X^{reg}$ where $X^s$ denotes the singular part of $X$ having codimension at least two; $X^{reg}$ denotes the smooth part of $X$. A local function $f$ on $X$ is called to be smooth if for any local embedding of $X$, i.e. $X \cap U \to U \subset \mathbb{C}^N$, it can be extended to be a smooth function. Now a local smooth strongly plurisubharmonic function on $X$ means a smooth function which can be extended to be a smooth strongly plurisubharmonic function for some embedding.

Definition 2.1 (Definition-Lemma [45]). A Kähler metric $\omega$ on $X$ is defined by a cover $\{(U_i, \rho_i)\}$, where $U_i \subset \mathbb{C}^N$ is open subset, $\rho_i$ is a smooth strongly plurisubharmonic function on $U_i$, i.e. in a neighborhood of $U_i$ in $\mathbb{C}^N$, and $\omega|_{X^{reg} \cap U_i} = \sqrt{-1} \partial \bar{\partial} \rho_i|_{X^{reg} \cap U_i}$. By [45, Page 23], this gives a Čech class $[\omega] \in H^2(X, \mathbb{R})$, which is usually referred as a Kähler class on $X$.

Remark 2.2. We will also need to pull back a Kähler class through resolutions $p : \hat{X} \to X$. For this, we will keep using the fact that $p^*[\omega]$ is the same thing as locally pulling back the smooth defining functions $\rho_i$ as $\rho_i \circ p$ in the
definition above, i.e. \( p^*\omega \) as a smooth de-Rham class is locally given by \( \sqrt{-1} \partial \bar{\partial} (\rho \circ p) \) (see [15, Proposition 3.5]). Abusing notation, we will not make a difference between the Čech class and the corresponding de Rham class while the context should make it clear.

Now take any smooth resolution of \( p : \tilde{X} \to X \). As a consequence of the definition, we have the following properties for Kähler metrics on varieties.

1. For each \( i \), \( p^*\omega_i \) defines a smooth de-Rham class in \( \tilde{X} \) which follows from that we can pull back the smooth plurisubharmonic functions to define the pull-back of the corresponding Kähler class;
2. For dimensional reasons, the Čech class \( p^*[\omega_1] \wedge \cdots p^*[\omega_{n-1}]|_{p^{-1}(Z)} = 0 \) if \( Z \) has codimension at least two;
3. Similarly \( p^*[\omega_1] \wedge \cdots p^*[\omega_{n-2}]|_{p^{-1}(Z)} = 0 \) if \( Z \) has codimension at least three.

We need the following well-known fact, for which we include a short explanation

\[ \text{Lemma 2.3. Let } Z \subset X \text{ be a subvariety containing } X^s. \text{ Suppose } a \in H^*(\tilde{X}, \mathbb{R}) \text{ with } a|_{p^{-1}(Z)} = 0. \text{ Denote by } [\Omega] \text{ the corresponding de-Rham class for } a, \text{ then} \]

\[ \Omega = \Omega^0 + d\Phi \]

where \( \Omega^0 \) is a smooth closed form with compact support in \( \tilde{X} \setminus p^{-1}(Z) \).

\[ \text{Proof. Indeed, choose an open neighborhood } U \text{ of } p^{-1}(Z) \text{ which deformation retracts onto } p^{-1}(Z), \text{ then } H^*(U, \mathbb{R}) \cong H^*(p^{-1}(Z), \mathbb{R}) \text{ through the restriction map, which can be seen by using the singular Cohomology. Here we used the functorial properties of the isomorphism between the Čech cohomology and singular Cohomology over } \mathbb{R}. \text{ In particular, we know as an element in the Čech cohomology, } a|_U \text{ is also trivial, thus the corresponding de-Rham cohomology class } [\Omega|_U] \text{ is also trivial, i.e. } \Omega|_U = d\Phi' \text{ over } U. \text{ Choose a cut-off function } \rho \text{ which is 1 near } p^{-1}(Z). \text{ Then } \Phi = \rho\Phi' \text{ does the job.} \]

Given this, if \( Z \) has codimension at least two, we can always write

\[ p^*\omega_1 \wedge \cdots p^*\omega_{n-1} = \Omega^0_{n-1} + d\Phi_{n-1} \]

where \( \Omega^0_{n-1} \) is compact supported in \( \tilde{X} \setminus p^{-1}(Z) \) and if \( Z \) has codimension at least three then

\[ p^*\omega_{i_1} \wedge \cdots p^*\omega_{i_{n-2}} = \Omega^0_{n-2} + d\Phi_{n-2} \]

where \( \Omega^0_{n-2} \) is compact supported in \( \tilde{X} \setminus p^{-1}(Z) \).

We will also need the following observation

\[ \text{Proposition 2.4. Given two Kähler metrics } \omega_1, \omega_2, \text{ there exists a constant } C > 0 \text{ so that} \]

\[ C^{-1}\omega_1 \leq \omega_2 \leq C\omega_1. \]

\[ \text{Later in various places, we will use such a property without mentioning.} \]
Proof. It suffices to build such a bound near each point $z \in X^s$. Cover $X$ with open sets $(U_i, \rho_i)$ where $U_i \subset \mathbb{C}^N$ and $\rho_i$ is a smooth strongly PSH function on $U_i$ and $\omega_1 = i\partial \bar{\partial} \rho_i$. Assume $z \in U_i$ for some fixed $i$. By assumption, we can always extend the local defining function $\rho'_i$ for $\omega_2$ to be a smooth function $U_i$. Thus we know
\[ \sqrt{-1} \partial \bar{\partial} \rho'_i \leq C_i \sqrt{-1} \partial \bar{\partial} \rho_i. \]
By compactness, we can cover $X$ with finitely many such open sets and choose $C$ to be the largest $C_i$. This gives $\omega_2 \leq C \omega_1$. The other inequality follows from symmetry where we might need to change $C$ a little. □

Corollary 2.5. Given $(n-1)$ Kähler metrics $\omega_{i_1}, \cdots \omega_{i_{n-1}}$ on $X$, then for some constant $C > 0$
\[ C^{-1} \omega_{i_1}^{n-1} \leq \omega_1 \wedge \cdots \wedge \omega_{n-1} \leq C \omega_{i_1}^{n-1} \]
i.e. for any $(1,0)$ form $\theta$,
\[ C^{-1} \sqrt{-1} \theta \wedge \overline{\theta} \wedge \omega_{i_1}^{n-1} \leq \sqrt{-1} \theta \wedge \overline{\theta} \wedge \omega_1 \wedge \cdots \wedge \omega_{n-1} \leq C \sqrt{-1} \theta \wedge \overline{\theta} \wedge \omega_{i_1}^{n-1} \]

2.2. Donaldson-Uhlenbeck-Yau theorem for balanced metrics of Hodge-Riemann type. In this section, we recall some results from [6].

Let $\mathcal{F}$ be a holomorphic vector bundle over a smooth compact complex manifold $Y$ with $n-1$ Kähler forms $\omega_1, \cdots, \omega_{n-1}$. Then by [42], we know $\omega_1, \cdots, \omega_{n-1}$ define a balanced metric $\omega$ through the following
\[ \omega^{n-1} = \omega_1 \wedge \cdots \wedge \omega_{n-1} \]
i.e. $\omega$ is a Hermitian metric with $d\omega^{n-1} = 0$. The Donaldson-Uhlenbeck-Yau theorem over complex manifolds with Gauduchon metrics gives ([27])

Theorem 2.6. Suppose $\mathcal{F}$ is stable over $(X, \omega_1 \wedge \cdots \wedge \omega_{n-1})$. There exists a Hermitian-Yang-Mills metric $H$ on $\mathcal{F}$, i.e.
\[ \sqrt{-1} \Lambda_\omega F_H = \lambda \text{id} \]
where $F_H$ is the Chern curvature of the metric $H$.

The Hermitian-Yang-Mills equation implies
\[ (\frac{\sqrt{-1}}{2\pi} F_H - \frac{1}{r} \text{tr}(\frac{\sqrt{-1}}{2\pi} F_H \text{id}) \wedge \omega_1 \wedge \cdots \wedge \omega_{n-1} = 0 \]
where $r = \text{rank}(\mathcal{F})$. By Timorin’s results which generalize the classical Hodge-Riemann property for one Kähler form to multiple Kähler forms (see [42]), we know for any $1 \leq i_1 < \cdots < i_{n-2} \leq n-1$ the following holds point-wisely
\[ -\text{tr}(\frac{\sqrt{-1}}{2\pi} F_H - \frac{1}{r} \text{tr}(\frac{\sqrt{-1}}{2\pi} F_H \text{id}) \wedge (\frac{\sqrt{-1}}{2\pi} F_H - \frac{1}{r} \text{tr}(\frac{\sqrt{-1}}{2\pi} F_H \text{id})) \wedge \omega_{i_1} \wedge \cdots \wedge \omega_{i_{n-2}} \geq 0, \]
where the equality holds if and only if
\[ \frac{\sqrt{-1}}{2\pi} F_H = \frac{1}{r} \text{tr}(\frac{\sqrt{-1}}{2\pi} F_H \text{id}). \]
As a corollary, this generalizes the classical Bogomolov-Gieseker inequality to the multi-polarization setting

**Corollary 2.7 (Bogomolov-Gieseker inequality).** Suppose $\mathcal{F}$ is stable over $(X, \omega_1 \wedge \cdots \omega_{n-1})$. Then for any $1 \leq i_1 < \cdots < i_{n-2} \leq n-1$,

$$
(2rc_2(\mathcal{F}) - (r - 1)c_1(\mathcal{F}))^2 \cdot [\omega_{i_1}] \cdots [\omega_{i_{n-2}}] \geq 0
$$

where the equality holds if and only if $\mathcal{F}$ is projectively flat.

### 2.3. Chern classes and stability.

In this section, we will give definitions of stable reflexive sheaves in the case of normal varieties.

We fix $(X, \omega_1 \wedge \cdots, \omega_{n-1})$ to be a normal Kähler variety with $n-1$ Kähler metrics. Denote $\omega$ to be the balanced metric defined by

$$
\omega^{n-1} = \omega_1 \wedge \cdots \omega_{n-1}.
$$

Let $\mathcal{E}$ be a reflexive sheaf over $X$. We will recall the notion of Chern classes we want to deal with. We fix $p : \hat{X} \to X$ to be any resolution. Recall in the introduction, we let $E_p$ be the space of reflexive sheaves on $\hat{X}$ which are isomorphic to $\mathcal{E}$ away from the exceptional divisor. Pick $\hat{E} \in E_p$. Then the Chern numbers we need can be defined as

$$
(2.5) \quad \deg(\mathcal{E}) = c_1(\hat{E})p^* [\omega_1] \cdots p^* [\omega_{n-1}]
$$

and the slope of $\mathcal{E}$ is defined as

$$
(2.6) \quad \mu(\mathcal{E}) := \frac{\deg \mathcal{E}}{\text{rank} \mathcal{E}}.
$$

Given a connection $A$ on $\mathcal{E}$ defined away from the singular set, if everything is smooth, then the Chern numbers above can be computed by using Chern-Weil theory. In the following, we will still denote $c_i(A)$ or $c_i(\mathcal{H})$ as the forms corresponding to $c_i(\mathcal{E})$, if $A$ is the Chern connection given by $\mathcal{H}$ on $\mathcal{E}$.

As a direct corollary of Equation (2.1), we have

**Proposition 2.8.** $\deg(\mathcal{E})$ is independent of the choice of the resolutions $p$ and $E_p$.

**Proof.** Suppose $\hat{E} \in E_{p_1}$ and $\hat{E}' \in E_{p_2}$ are two such extensions for two different resolutions $p_1 : \hat{X}^1 \to X$ and $p_2 : \hat{X}^2 \to X$. Let $p : \hat{X} \to X$ be a resolution so that the following diagram commutes

$$
\begin{array}{c}
\hat{X} \\
q_1 \downarrow \quad q_2 \\
\hat{X}_1 \\
p_1 \downarrow \quad p_2 \\
X
\end{array}
$$
for some map $q_1, q_2$. Here $\hat{X}$ for example can be taken to be the resolution of the fiber product of $\hat{X}_1$ and $\hat{X}_2$ over $X$, which again is a resolution of $X$. Take $\hat{E} \in E_p$. By equation (2.1) applied to $p_1$, we can write
\[ p_1^*\omega_1 \wedge \cdots p_1^*\omega_{n-1} = \Omega_{n-1}^0 + d\Phi_{n-1} \]
over $\hat{X}_1$ where $\Omega_{n-1}^0$ is a compact supported form over $X \setminus Z$ and $Z = \text{Sing}(\hat{E}) \cup \text{X}^*$. Then we know
\[
\text{deg}(\hat{E}) = \int_{X_1 \setminus p_1^{-1}(Z)} c_1(\hat{E}_1) \wedge \Omega_{n-1}^0.
\]
On the other hand, by definition, we know $c_1(\hat{E}_1)|_{X \setminus Z} = c_1(\hat{E})|_{X \setminus Z}$, thus
\[
\int_{X_1 \setminus p_1^{-1}(Z)} c_1(\hat{E}_1) \wedge \Omega_{n-1}^0 = \int_{X_1 \setminus p_1^{-1}(Z)} c_1(\hat{E}) \wedge \Omega_{n-1}^0.
\]
Since $[\Omega_{n-1}^0]$ can be naturally viewed as the same de Rham class over $\hat{X}$ as $[(\pi)^*\omega_i_1] \wedge \cdots [(\pi)^*\omega_i_{n-2}]$, we have
\[
\int_{X_1 \setminus p_1^{-1}(Z)} c_1(\hat{E}) \wedge \Omega_{n-1}^0 = c_1(\hat{E}).[\pi^*\omega_1].\cdots[\pi^*\omega_{n-1}] = \text{deg}(\hat{E})
\]
which implies $\text{deg}(\hat{E}_1) = \text{deg}(\hat{E})$. Similarly $\text{deg}(\hat{E}_2) = \text{deg}(\hat{E})$. The conclusion follows. \qed

In particular, the following slope stability is well-defined

**Definition 2.9.** $\mathcal{E}$ is called slope stable (resp. semistable) if for any $\mathcal{F} \subset \mathcal{E}$, $\mu(\mathcal{F}) < \mu(\mathcal{E})$ (resp. $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$). $\mathcal{E}$ is called polystable if it is a direct sum of stable ones.

The following fact will be used in later sections which we include a proof for completeness

**Lemma 2.10.** Suppose $\mathcal{E}_1$ and $\mathcal{E}_2$ are two stable reflexive sheaves with the same slope. Then any nontrivial map between $\mathcal{E}_1$ and $\mathcal{E}_2$ is an isomorphism. In particular, a stable reflexive sheaf is simple.

**Proof.** Suppose $\phi$ is such a map. By stability, we must have $\text{Ker} \phi = 0$, thus $\phi$ is injective. Then we have
\[
0 \to \det(\mathcal{E}_1) \xrightarrow{\det(\phi)} \det(\mathcal{E}_2) \to \tau_D \to 0
\]
for some $\tau_D$ which is a torsion sheaf supported on a subvariety $D \subset X$. By definition, we have
\[
\text{deg}(\mathcal{E}_2) - \text{deg}(\mathcal{E}_1) = c_1(\tau_D).[\omega_1] \cdots [\omega_{n-1}].
\]
Now any pure codimension one component of $D$ will contribute strictly positively to the equality above and give a contradiction, thus $\text{codim}_C D \geq 2$. The conclusion follows. \qed
2.4. **Discriminant.** We need another important concept in our setting, i.e. the so-called *discriminant*. As we mentioned in the introduction, due to the fact that $X$ is a normal variety where in general the singularities could contribute in an essential way to higher Chern classes, unlike the slope, it could not be defined by using a single resolution. Instead, motivated by the gauge theoretical picture in the proof of the Donaldson-Uhlenbeck-Yau theorem in our setting, we include all the resolutions. For this, as in the introduction, given a sheaf $\mathcal{H}$, we denote

$$\Delta(\mathcal{H}) = 2rc_2(\mathcal{H}) - (r - 1)c_1(\mathcal{H})^2$$

when the chern classes are well-defined in the ordinary sense. Now for any $1 \leq i_1 < \cdots < i_{n-2} \leq n - 1$, the discriminant is defined as

$$\Delta(\mathcal{E})[\omega_{i_1}] \cdots [\omega_{i_{n-2}}] = \inf_{\text{Kähler}} \inf_{\mathcal{E} \in \mathcal{E}_p} \Delta(\hat{\mathcal{E}})p^*[\omega_{i_1}] \cdots p^*[\omega_{i_{n-2}}].$$

(2.7)

Now we discuss typical cases where this could be simplified and computed.

2.4.1. **Locally free sheaves over projective normal surface.** When $\mathcal{E}$ is locally free, $\mathcal{E}$ defines a topological bundle over $X$ which is a CW complex. Thus, the Chern classes of $\mathcal{E}$ are naturally defined in the singular cohomology as well as the corresponding Čech cohomology. We will abuse notation for not making a difference between the two cohomologies. In particular, the term we defined above can be computed using the cup products in singular cohomology.

**Lemma 2.11.** Assume $\mathcal{E}$ is locally free over a normal surface. Then $\Delta(\mathcal{E})$ coincides with the topological definition.

The proof of this follows directly from the following inequality that has independent interest

**Proposition 2.12.** For any $\hat{\mathcal{E}} \in \mathcal{E}_p$ and any resolution $p : \hat{X} \to X$, the following holds

$$\Delta(\mathcal{E}) \cap X \leq \Delta(\hat{\mathcal{E}}) \cap \hat{X}$$

**Proof.** Passing to a further resolution, we can assume that for any $x \in \hat{X}$, the exceptional divisors are simple normal crossings. We first observe that

$$c(\hat{\mathcal{E}}) = (1 + p^*x_1 + e_1) \cdots (1 + p^*x_m + e_m)$$

where

$$c(\mathcal{E}) = (1 + x_1) \cdots (1 + x_m)$$

and $e_i$ can be represented by the exceptional divisors. In particular, by computation, we have

$$(\Delta(p^*\mathcal{E}) - \Delta(\hat{\mathcal{E}})) \cap \hat{X} = \sum_{i<j}(e_i - e_j)^2 \leq 0$$

where
for the first equality, we used the observation
\[ e_t \cdot p^* x_s = 0 \]
since \( p^* x_s \mid X_s = 0 \) due to dimensional reasons.
for the second inequality, this follows from a lemma of Du Val ([31, Proposition 2.1.12]) which states the natural intersection matrix given by intersecting irreducible components of the exceptional divisor is negative definite.

The conclusion follows. \( \square \)

2.4.2. On the minimal resolution of a normal surface. Now let \( p^{\text{min}} : \hat{X} \to X \) be the minimal resolution, i.e. for any other resolution \( p' : \hat{X}' \to X \), it factors through a unique map \( q : \hat{X}' \to \hat{X} \) so that the following diagram commutes

\[ \begin{array}{ccc} 
\hat{X}' & \xrightarrow{q} & \hat{X} \\
\downarrow{p'} & & \downarrow{p^{\text{min}}} \\
X & & X 
\end{array} \]
i.e. \( p' = p^{\text{min}} \circ q \).

**Proposition 2.13.** The discriminant can be computed on the minimal resolution, i.e.
\[ \Delta(E) = \inf_{\hat{E} \in \mathcal{E}_{p^{\text{min}}}} \Delta(\hat{E}). \]

**Proof.** As above, fix any resolution \( p' : \hat{X}' \to X \), it factors as \( p' = p^{\text{min}} \circ q \). Given any \( \hat{E}' \in \mathcal{E}_{p'} \), then \( (q_*\hat{E}')** \in \mathcal{E}_{p^{\text{min}}} \). It suffices to prove that
\[ \Delta((q_*\hat{E}')**) \leq \Delta(\hat{E}'). \]
Indeed, we know
\[ \Delta((q_*\hat{E}')**) = \Delta(q^*((q_*\hat{E}')**)) \leq \Delta(\hat{E}') \]
where the first equality follows from definition and the second equality follows from Proposition 2.12. \( \square \)

2.4.3. When \( X \) is smooth in codimension two.

**Proposition 2.14.** Assume \( X \) is smooth in codimension two, the discriminant \( \Delta(\mathcal{E})[\omega_{i_1}] \cdots [\omega_{i_{n-2}}] \) can be computed by using any resolutions. Furthermore, if \([\omega_1], \cdots [\omega_{n-1}]\) can be represented by very ample Cartier divisors, then
\[ \Delta(\mathcal{E})[\omega_{i_1}] \cdots [\omega_{i_{n-2}}] = (2rc_2(\mathcal{E}|_{D_{i_1} \cap \cdots \cap D_{i_{n-2}}}) - (r - 1)c_1(\mathcal{E}|_{D_{i_1} \cap \cdots \cap D_{i_{n-2}}})^2) \]
where for each \( i \), \([D_i] = [\omega_i]\) and \( D_{i_1} \cap \cdots \cap D_{i_{n-2}} \) is a smooth surface in \( X \).
Proof. Suppose \( \hat{\mathcal{E}} \in \mathcal{E}_p \) and \( \hat{\mathcal{E}}' \in \mathcal{E}_{p'} \) for two different resolutions \( p : \hat{X} \to X \) and \( p' : \hat{X}' \to X \). Similar to Proposition 2.8, it suffices to prove the case when we have the following commutative diagram

\[
\begin{array}{ccc}
\hat{X} & \xrightarrow{q} & \hat{X}' \\
\downarrow{p'} & & \downarrow{p} \\
X & & 
\end{array}
\]

By Equation (2.2) applied to \( p \), we can write

\[
p^*\omega_1 \wedge \cdots \cdot p^*\omega_{n-2} = \Omega^0_{n-2} + d\Phi_{n-2}
\]

over \( \hat{X}' \) where \( \Omega^0_{n-2} \) is a smooth \( 2n-4 \) forms with compact support in \( X^{\text{reg}} \), and \( \Phi_{n-2} \) is a smooth \( 2n-4 \) form over \( \hat{X}' \). Then

\[
\Delta(\hat{\mathcal{E}})[\omega_1]\cdots[\omega_{n-2}] = \int_{X \setminus Z} \Delta(\hat{\mathcal{E}}) \wedge \Omega^0_{n-2}
\]

\[
= \int_{X \setminus Z} \Delta(\hat{\mathcal{E}}') \wedge \Omega^0_{n-2}
\]

\[
= \Delta(\hat{\mathcal{E}}')[\omega_1]\cdots[\omega_{n-2}]
\]

where the first equality follows from definition, the second inequality follows from \( \Delta(\hat{\mathcal{E}}) = \Delta(\hat{\mathcal{E}}') \) over \( X \setminus Z \), and the last equality follows from \( [(p')^*\omega_1]\cdots[(p')^*\omega_{n-2}] \) and \( [\Omega^0_{n-2}] \) can be naturally viewed as the same de Rham class over \( \hat{X} \). For the second part, it suffices to notice that viewed as linear functionals on \( H^1_{dR}(\hat{X}, \mathbb{R}) \), \( D_1 \cap \cdots D_{n-2} \) and \( \Omega^0_{n-2} \) define the same element. \( \square \)

2.5. Admissible Hermitian-Yang-Mills metrics and regularity results. We use the following notion of admissible Hermitian-Yang-Mills metric (\cite{1})

**Definition 2.15.** An admissible Hermitian-Yang-Mills metric is defined as a smooth Hermitian metric \( H \) on a holomorphic bundle \( F \) over \( X \setminus Z \) where \( X_s \subset Z \) and \( Z \setminus X_s \) is a subvariety of \( X^{\text{reg}} \) of codimension at least two, and the metric \( H \) satisfies the following

- \( \int_{X \setminus Z} |F_H|^2 < \infty \);
- \( \sqrt{-1} \Lambda_{\omega} F_H = \lambda \text{id} \) where \( \lambda \) is usually referred as the Einstein constant where \( F_H \) denotes the curvature. The associated Chern connection \( A \) is usually referred as an (admissible) Hermitian-Yang-Mills connection.

We have the following essentially due to Bando and Siu (see \cite{1} Theorem 2 and \cite{7} Proposition 46)

**Theorem 2.16.** An admissible Hermitian-Yang-Mills connection \( A \) defines a reflexive sheaf \( \mathcal{F} \) over \( X^{\text{reg}} \). Furthermore, the admissible Hermitian-Yang-Mills metric could be extended and defined wherever \( \mathcal{F} \) is locally free.
Now an adaption of the slicing argument in [1] using one dimensional slices instead of two dimensional slices gives the following regularity result. We include the proof for completeness here.

**Proposition 2.17.** For any local section $s$ of $F$, $\log^+ |s|^2 \in W^{1,2}_{loc}$.

**Proof.** We will use $Q_1 \lesssim (\geq) Q_2$ to denote $Q_1 \lesssim (\geq) CQ_2$ for some constant $C$. By Corollary 2.5, we can assume the metric is induced from the flat metric on $\mathbb{C}^N$. The statement is local. Fix $z \in Z$. We can assume $z \in X \subset U \subset \mathbb{C}^N$ with $\omega$ being the restriction of the standard flat metric. By shrinking $U$, we can assume $U = B^l \times B^l$ and $z = 0$

$$p : X \cap U \to B^l$$

and $p^{-1}(t)$ is a smooth curve while $p^{-1}(0)$ is a smooth curve with a point singularity at the origin. By the assumption on $Z$, we know $p^{-1}(t) \cap Z = \emptyset$ for generic $t$. Let $u_t = \log^+ |s|^2|_{p^{-1}(t)}$ which is smooth for generic $t$. We have

$$\Delta_t u_t \gtrsim -|F_H|.$$ 

Let $\chi$ be a cut-off function on $B^l$ supported near 0. Then by doing integration by parts and applying Cauchy-Schwartz inequality, we have

$$\int_{p^{-1}(t)} |\nabla'(\chi u_t)|^2 \lesssim \delta \int_{p^{-1}(t)} (\chi u_t)^2 + \delta^{-1} \int_{p^{-1}(t)} \chi^2 |F_H|^2 + \int_{p^{-1}(t)} |\nabla' \chi|^2 u_t^2.$$ 

for any fixed $0 < \delta << 1$. With the induced metric on $p^{-1}(t)$, by [37, Theorem 18.6], there exists a Poincaré inequality for $p^{-1}(t)$ and the Sobolev constant is independent of $t$ since $p^{-1}(t)$ is a minimal submanifold of $U$. So we have

$$\int_{p^{-1}(t)} |\nabla'(\chi u_t)|^2 \lesssim \delta^{-1} \int_{p^{-1}(t)} \chi^2 |F_H|^2 + \int_{p^{-1}(t)} |\nabla' \chi|^2 u_t^2$$

which further implies

$$\int_{p^{-1}(K)} |\nabla'(\chi u_t)|^2 \lesssim \delta^{-1} \int_{p^{-1}(K)} \chi^2 |F_H|^2 + \int_{p^{-1}(K)} |\nabla' \chi|^2 u_t^2$$

for some compact set $0 \in K \subset p(X \cap U)$. Here $\nabla'$ denotes the derivative in the fiber direction. Now choose $(n-1)$ more such projections and add them together. This gives the bound we need. \hfill \Box

**Corollary 2.18.** For any global section $s$ of $F$, $\log^+ |s|^2 \in L^\infty$.

**Proof.** By [37], we know $X$ admits a Sobolev inequality for compact supported functions over $X^{reg}$ if $X$ is endowed with a fixed Kähler metric i.e.

$$\|f\|_{L^2_{n+m}} \leq C(\|f\|_{L^2(X)} + \|\nabla f\|_{L^2(X)})$$

for any $f \in C^1_c(X^{reg})$. More precisely, locally we can assume the Kähler metric is induced from the flat metric on $\mathbb{C}^N$. In particular, $X$ is a minimal submanifold of $\mathbb{C}^N$ away from the singular set, thus by [37, Theorem 18.6], there exists a Sobolev inequality for $X$. Now the approximation result in [28]...
Section 4 implies a Sobolev inequality for functions in $W^{1,2}(X^{reg})$ where as pointed out in [28, Section 0], the essential argument does not require the variety to be projective. By Corollary 2.18, we also have a Sobolev inequality for $(X, \omega_1 \wedge \cdots \wedge \omega_{n-1})$. Given this, we can apply the Moser iteration to $\log^+ |s|^2$ which satisfies

$$\Delta \log^+ |s|^2 \geq \lambda_\infty$$

where $\lambda_\infty$ is the Einstein constant of $A_\infty$, thus $\log^+ |s|^2 \in L^\infty$. This concludes the proof.

Remark 2.19. We also refer the readers to [21] for the Sobolev inequalities built for a more general class of Kähler spaces.

Corollary 2.20. The admissible Hermitian-Yang-Mills connection on $F$ is unique if it exists. Furthermore, if $F$ can be extended to be a stable reflexive sheaf over $X$, the Hermitian-Yang-Mills metric is unique up to scaling.

Proof. Otherwise, suppose we have two such metrics $H_1$ and $H_2$ with Einstein constants $\lambda_1 \geq \lambda_2$. Consider the identity map $id \in \text{Hom}(F, F)$ with the endowed metric $H_1^* \otimes H_2$. Then straightforward computation shows

$$\Delta |id|^2 = 2 |\nabla id|^2 - (\lambda_2 - \lambda_1) |id|^2.$$ 

Since $id \in L^\infty$ by Corollary 2.18, we can do integration by parts to conclude $\nabla id = 0$, i.e. $id$ is parallel and $\lambda_2 = \lambda_1$. The uniqueness follows. If $F$ comes from a stable reflexive sheaf over $X$, write $H_1(\ldots) = H_2(g\ldots)$ where $g$ is Hermitian with respect to $H_2$. The fact that $id$ is parallel implies $g$ is holomorphic, thus a multiple of the identity map by Lemma 2.10.

2.6. Hermitian-Yang-Mills metrics using perturbations. The following has been observed in the Kähler case (see [9] [46]) by using the boundedness result in [43]. Similar argument also works in our setting. Take a resolution $p : \hat{X} \to X$ and let $\hat{\mathcal{E}} := (p^* \mathcal{E})^\otimes_\mathbb{C}$.

Proposition 2.21. Suppose $\mathcal{E}$ is a stable reflexive sheaf over $(X, \omega_1 \wedge \cdots \wedge \omega_{n-1})$, then $\hat{\mathcal{E}}$ is stable over $(\hat{X}, (p^* \omega_1 + i^{-1}\theta) \wedge \cdots (p^* \omega_{n-1} + i^{-1}\theta))$ for $i >> 1$.

Proof. By passing to a further resolution ([34, Theorem 3.5]), it suffices to prove it when $\hat{\mathcal{E}}$ is locally free. By definition, $\hat{\mathcal{E}}$ is stable with respect to $p^* \omega_1 \wedge \cdots p^* \omega_{n-1}$. Otherwise, the push-forward of the destabilizing subsheaf will destabilize $\mathcal{E}$ as well. Now we argue by contradiction for the main statement. By passing to a subsequence, we have a sequence of quotient maps $q_i : \hat{\mathcal{E}} \to \mathcal{F}_i$ and $\mu_i(\mathcal{F}_i) \leq \mu_i(\hat{\mathcal{E}})$ where $\mu_i$ denotes the slope of a sheaf with respect to the metric $(p^* \omega_1 + i^{-1}\theta) \wedge \cdots (p^* \omega_{n-1} + i^{-1}\theta)$. By choosing a metric on $\hat{\mathcal{E}}$, an easy computation shows

$$c_1(\mathcal{F}_i)[\theta][p^* \omega_{k_1}] \cdots [p^* \omega_{k_{n-t-1}}] \geq C$$
for some $C$ independent of $l, k_1, \cdots k_{l-1}$. In particular, this implies

$$\mu(F_i) + O(\frac{1}{i}) \leq \mu_i(\mathcal{E}).$$

By the boundedness result in [43, Corollary 6.3] applied to the degree defined by $\omega_1 \wedge \cdots \omega_{n-1}$, we can assume

$$p_\ast q_i : \mathcal{E} \to \text{Im}(p_\ast q_i)$$

lies in the same component of the corresponding Douady space and has a limit over $X$. Thus we get a quotient map $q_\infty : \mathcal{E} \to \mathcal{F}_\infty$ where $\mu(\mathcal{F}_\infty) \leq \mu(\mathcal{E})$. This contradicts the stability of $\mathcal{E}$. \hfill \Box \\

Now we will fix a resolution $p : \hat{X} \to X$ so that $\hat{\mathcal{E}} := (p^\ast \mathcal{E})^{**}$ is locally free. From the above, for any $i >> 1$, by the Donaldson-Uhlenbeck-Yau theorem for multipolarizations (see Theorem 2.6), there exists a Hermitian-Yang-Mills metric $H_i$ on $\hat{\mathcal{E}}$ over $(\hat{X}, (p^\ast \omega_1 + i^{-1} \theta) \wedge \cdots (p^\ast \omega_{n-1} + i^{-1} \theta))$. In particular,

$$\sqrt{-1} F_{H_i} - \frac{\text{tr}(\sqrt{-1} F_{H_i})}{\text{rank} \mathcal{E}} \text{id} \wedge (p^\ast \omega_1 + i^{-1} \theta) \wedge \cdots (p^\ast \omega_{n-1} + i^{-1} \theta) = 0$$

i.e.

$$\sqrt{-1} F_{H_i} - \frac{\text{tr}(\sqrt{-1} F_{H_i})}{\text{rank} \mathcal{E}} \text{id}$$

is primitive with respect to $(p^\ast \omega_{k_1} + i^{-1} \theta) \wedge \cdots (p^\ast \omega_{k_{n-2}} + i^{-1} \theta)$ for any $1 \leq k_1 < \cdots k_{n-2} \leq n - 1$. Now the Hodge-Riemann property applied to $(p^\ast \omega_1 + i^{-1} \theta) \wedge \cdots (p^\ast \omega_{n-2} + i^{-1} \theta)$ (see Equation (2.4)) implies the following Bogomolov-Gieseker inequality

$$\int_X (2rc_2(A_i) - (r-1)c_1^2(A_i)) \wedge (p^\ast \omega_1 + i^{-1} \theta) \wedge \cdots (p^\ast \omega_{n-2} + i^{-1} \theta) \geq 0$$

where the inequality holds if and only if $A_i$ is projectively flat.

Let $\hat{\mathcal{E}}$ be the underlying smooth bundle of $\hat{\mathcal{E}}$. By doing complex gauge transforms ([44, Section 5]), we have a sequence of Hermitian-Yang-Mills connections $A_i$ on a fixed unitary bundle $(\hat{\mathcal{E}}, H)$. Then we can take a Uhlenbeck limit using gauge theory (see [7]). Namely, by passing to a subsequence, up to gauge transforms, we can assert $A_i$ converges locally smoothly to a limiting connection $A_\infty$ defined on $E|_{X \setminus \Sigma}$ where $\Sigma$ is the so-called bubbling set defined as

$$\Sigma = \{ z \in X^{\text{reg}} : \lim_{r \to 0^+} \liminf_{i \to \infty} r^{4n-2} \int_{B_r(z)} |F_{A_i}|^2 \text{dvol}_i > 0 \}.$$

Here the base is given by $X \setminus \Sigma$ together with a sequence of metrics that converge to the metric given by the multiple Kähler forms, thus Uhlenbeck compactness applies.

By [8, Proposition 46] (see also [41, Theorem 4.3.3] for the Kähler case), we know
Lemma 2.22. Σ ⊂ X^{reg} is a subvariety of codimension at least two.

Remark 2.23. As we will see later, Σ is actually empty.

We need the following key observation regarding the holomorphic vector bundle E∞ over X^{reg} defined by A∞.

Proposition 2.24. Assume E∞ can be extended to be a reflexive sheaf over X. Then μ(E∞) = μ(E). In particular, the admissible Hermitian-Yang-Mills metric computes the slope of E∞.

Proof. Denote 𝔭E∞ = (p∗E∞)**. By Equation 2.1, we can write

\[ p^*(\omega_1 \wedge \cdots \omega_{n-1}) = \Omega_{n-1}^0 + d\Phi_{n-1} \]

where Ω_{n-1}^0 is compact supported in X \setminus p^{-1}(X^s), which is possible because X^s ⊂ X is a subvariety of codimension at least two. Given this, we have the following

\[
\mu(E∞) = \int_X c_1(\mathcal{E}_∞) \wedge p^*(\omega_1 \wedge \cdots \omega_{n-1}) \\
= \int_X c_1(\mathcal{E}_∞) \wedge (\Omega_{n-1}^0 + d\Phi_{n-1}) \\
= \int_X c_1(\mathcal{E}_∞) \wedge \Omega_{n-1}^0 \\
= \int_X c_1(A∞) \wedge \Omega_{n-1}^0 \\
= \lim \int_X c_1(A_i) \wedge \Omega_{n-1}^0 \\
= \lim \int_X c_1(A_i) \wedge p^*(\omega_1 \wedge \cdots \omega_{n-1}) \\
= \lim \int_X c_1(A_i) \wedge (p^*(\omega_1) + i^{-1}\theta) \wedge \cdots (p^*(\omega_{n-1}) + i^{-1}\theta) \\
= \lim \mu_i(\mathcal{E})
\]

All the equalities are straightforward except the fourth and fifth one: the fourth one follows from the fact that c_1(A∞) is smooth over X^{reg} by Theorem 2.16 applied to det(E∞) and Ω_{n-1}^0 is compact supported away from the exceptional divisors; the fifth one follows from the strong L^1 convergence of FAi over the support of Ω_{n-1}^0. Combined with that the Einstein constant of A_i converges to the Einstein constant of A∞, we know the admissible Hermitian-Yang-Mills metric computes the slope of E∞. □

2.7. Nontrivial limits of holomorphic sections away from the singular set. In this section, we will prove a technical result needed later. By Corollary 2.5, we can assume X is endowed with a Kähler metric.
The convergence of the Hermitian-Yang-Mills connections in the previous sections can be now described by saying we have a sequence of connections, up to gauge transforms, converging locally smoothly over $X \setminus Z$ where $Z = X^* \cup \Sigma$ as defined before. Here we fix the original metric on $X$ given by $\omega_1 \wedge \cdots \wedge \omega_{n-1}$ to look at the convergence. Namely, $A_i$ is a sequence of unitary connections on a unitary bundle $(\mathcal{E}, H)$ satisfying

- $F^0_{A_i} = 0$;
- up to gauge transforms, $A_i$ converges to $A_\infty$ locally smoothly over $X \setminus Z$.

For any $\epsilon > 0$, define $Z^\epsilon$ to be a closed $\epsilon$-neighborhood of $Z$ in $X$. Furthermore, $Z^\epsilon \subset \text{Interior}(Z^\epsilon')$ if $\epsilon < \epsilon'$ and $Z^\epsilon$ converges to $Z$ as $\epsilon \to 0$.

Suppose we have a sequence of holomorphic sections $\{s_i \in H^0(X \setminus Z, \mathcal{E})\}$ with $\|s_i\|_{L^2(X, \omega)} = 1$ where $X_\epsilon = X \setminus Z^\epsilon$. Standard elliptic theory for holomorphic sections guarantees the existence of a limit over $X^\epsilon$. The goal is to show the limit is actually nontrivial using similar argument in [5].

2.7.1. Curve property.

**Definition 2.25.** A closed subset $S \subset X$ admits a good cover if $X$ can be covered by finitely many open sets $U_k \subset \mathbb{C}^N$ where $\omega|_{U_k \cap X} = \sqrt{-1} \partial \bar{\partial} \rho|_{X \cap U_k}$ and $\rho$ is a strongly smooth pluri-subharmonic function on $\overline{U_k}$ such that

- $U_k = B^l_{\delta_2} \times B_{\delta_3}^l$ for some $\delta_2^k, \delta_3^k > 0$, where $B^l_{\delta_2}$ denotes the ball $\{|z| < \delta_2^k\}$ in $\mathbb{C}^l$ and $B_{\delta_3}^l$ denote the ball $\{|z| < \delta_3^k\}$ in $\mathbb{C}^l$. Here $l + l' = N$.

- $\overline{U_k} \cap S \subset V_k = B_{\delta_1}^l \times B_{\delta_3}^{l'}$ for some $0 < \delta_1^k < \delta_2^k$ where $\overline{U_k} = B_{\delta_2}^l \times B_{\delta_3}^l$.

- for any $z \in B_{\delta_1}^l \cap \rho_k(X \cap \overline{U_k})$, $\rho_k^{-1}(z) \cap X$ is a smooth surface away from finitely many point. Here $\rho_k : B_{\delta_1}^l \times B_{\delta_3}^{l'} \to \overline{B_{\delta_2}^l}$ denotes the natural projection.

**Lemma 2.26.** $Z \subset X$ admits a good cover.

**Proof.** For any $z \in X$, since $X_y$ is a codimension 2 complex subvariety of $X$ and $\Sigma$ is a subvariety of $X^{reg}$ of codimension at least two, locally near $z$, we can cover $X$ with an open set $U \subset \mathbb{C}^N$ by assuming $z = 0$ so that for some orthogonal projection $\rho : U \subset \mathbb{C}^N = \mathbb{C}^l \times \mathbb{C}^{l'} \to \mathbb{C}^l$, $\rho^{-1}(y) \cap X \cap B_{\delta_2}^l$ is a smooth surface away from $\rho^{-1}(y) \cap X_y \cap B_{\delta_2}^l$ which consists of finitely many points for any $y \in \rho(X \cap U)$ for some $\delta_3 > 0$; furthermore, $\rho^{-1}(y) \cap \Sigma \cap B_{\delta_2}^l$ consists of points which accumulate at most near $X_y \cap B_{\delta_2}^l$. Then near $z$, we can easily construct a neighborhood $U_p = B_{\delta_2}^l \times B_{\delta_3}^l$ for some $\delta_2, \delta_3 > 0$ and $\overline{U_p} \cap S \subset V_p$ where $V_p = B_{\delta_1}^l \times B_{\delta_3}^{l'}$ for some $0 < \delta_1 < \delta_2$. Now we get such an open cover of $X$ given by $\cup_p U_p$. Since $X$ is compact, we can take a finite subcover $\cup_k U_{p_k}$ which gives a good cover for $Z \subset X$. \hfill $\square$

**Corollary 2.27.** For $0 < \epsilon << 1$, $Z^\epsilon \subset X$ admits a good cover.
Proof. Otherwise, suppose the statement is false for a sequence \( \epsilon_i \to 0^+ \). By definition, \( Z^{\epsilon_i} \) converges to \( Z \). Let \( \cup_i U_k \) be a good cover for \( Z \). We want to show it is a good cover for \( Z^{\epsilon_i} \) for \( i \) large, thus get a contradiction. We only need to verify that \( \overline{U_k} \cap Z^{\epsilon_i} \subset V_k \) for \( i \) large. Otherwise, by passing to a subsequence and using the finiteness of \( \{ U_k \}_k \), we can assume for some fixed \( k \), there always exists \( z_i \in (\overline{U_k} \cap Z^{\epsilon_i}) \setminus V_k \) for each \( i \) and \( z_i \) converges to \( z \in \overline{U_k} \cap Z \). Then \( z \in V_k \) and thus \( z_i \in V_k \) for \( i \) large. Contradiction. \( \square \\

Lemma 2.28. For any \( \epsilon > 0 \), let \( \cup_i U_k \) be a good cover of \( Z^\epsilon \subset X \). Then there exists a constant \( C = C(\epsilon) > 0 \) so that for any \( x \in X \setminus Z^\epsilon \), there exists a curve \( D_z \subset U_k \cap X \) for some \( k \) such that \( \partial D_z \subset X \setminus Z^\epsilon \cap \partial(X \setminus Z^\epsilon) \geq C \) and \( d(D_z, X) \geq C \). Here \( d \) denotes the distance between two sets.

Proof. For each \( k \), \( \overline{U_k} \cap Z \subset \overline{U_k} \cap Z^\epsilon \subset \overline{U_k} \cap Z^\epsilon \subset V_k \) where \( \overline{U_k} = \overline{B_{\delta_k}^i} \times \overline{B_{\delta_k}^i} \) and \( V_k = B_{\delta_k}^i \times B_{\delta_k}^i \). Consider the projection \( \rho_k : \overline{U_k} \to \overline{B_{\delta_k}^i} \). By assumption, we have

\[
\overline{(B_{\delta_k}^i \times B_{\delta_k}^i)} \cap Z = \overline{U_k} \cap Z \subset B_{\delta_k}^i \times B_{\delta_k}^i
\]

which implies \( \rho_k^{-1}(y) \cap Z \cap B_{\delta_k}^i \) is a compact subset of \( B_{\delta_k}^i \) for any \( y \in \rho_k(X \cap \overline{U_k}) \). Also we know \( \rho_k^{-1}(y) \cap X \cap B_{\delta_k}^i \) is a complex subvariety of \( B_{\delta_k}^i \), thus consists of finitely many points. Since \( ((B_{\delta_k}^i \times B_{\delta_k}^i) \cap Z \cap X) \cap \rho_k^{-1}(y) \) is a subvariety of \( B_{\delta_k}^i \), we know it consists of finitely many points which accumulate at most near \( X \cap B_{\delta_k}^i \). Now for any \( z \in X \setminus Z^\epsilon \), suppose \( z \in U_k \), then \( \rho_k^{-1}(\rho_k(z)) \cap Z \cap B_{\delta_k}^i \) consists of points which accumulate mostly near \( X \cap B_{\delta_k}^i \). As a result, one can easily find a curve \( D_z \subset U_k \) containing \( z \) such that \( D_z \subset Z = \emptyset \) and \( \partial D_z \subset (U_k \setminus \overline{V_k}) \cap X \subset U_k \cap X \setminus Z^\epsilon \). By perturbing the curve \( D_z \), we can find an open neighborhood \( V_z \) of \( z \) so that for each \( z' \in V_z \) there exists such a curve \( D_{z'} \) so that \( D_{z'} \subset Z = \emptyset \) and \( \partial D_{z'} \subset (U_k \setminus \overline{V_k}) \cap X \subset U_k \cap X \setminus Z^\epsilon \). Furthermore, \( \inf_{x \in V_z} d(D_{z'}, Z) 0 \) and \( \inf_{x \in V_z} d(\partial D_{z'}, \partial(X \setminus Z^\epsilon)) > 0 \). As a result, we get an open cover \( \bigcup_{z \in X \setminus Z^\epsilon} V_z \) of \( X \setminus Z^\epsilon \). Since \( X \setminus Z^\epsilon \) is compact, we can find a finite subcover \( \cup_i V_{z_i} \). Let \( C(\epsilon) = \min \{ \inf_{x \in V_{z_i}} d(D_z, Z), \inf_{x \in V_{z_i}} d(\partial D_z, \partial(X \setminus Z^\epsilon)) \} \). This finishes the proof. \( \square \\

2.7.2. Nontrivial limits.

Proposition 2.29. For any fixed \( 0 < \epsilon << 1 \), \( s_i \) converges to a nontrivial holomorphic section \( s_\infty \in H^0(X \setminus Z, E_\infty) \).

Proof. We consider any \( 0 < \epsilon << 1 \) so that \( X^\epsilon = X \setminus Z^\epsilon \) satisfies the conditions needed in Lemma 2.28. By induction, it suffices to show that

\[
\|s_i\|_{L^\infty(X^\epsilon)} \leq C := C(\epsilon, \|s_\infty\|_{L^2(X^\epsilon)}).
\]
which implies the strong convergence of \( s_i \) over \( X^\epsilon \) for any \( 0 < \epsilon << 1 \). More precisely, given the estimates, we first have strong convergence over \( X^\epsilon \), then over \( X^{\hat{\epsilon}} \), we have a uniform \( L^\infty \) estimate, thus the inductive estimates

\[
\|s_i\|_{L^\infty(X^{\hat{\epsilon}})} \leq C := C\left(\frac{\epsilon}{2}, \|s_i\|_{L^2(X^{\hat{\epsilon}})}\right)
\]

give strong convergence over \( X^{\hat{\epsilon}} \). Continuing this way, we obtain the convergence by passing to a subsequence. Now for any point \( z \in X^{\hat{\epsilon}} \), we want to prove

\[
|s_i(z)| \leq C.
\]

Let \( D \) be the holomorphic curve obtained in Lemma 2.28. Let \( t = s_i|_D \), we know

\[
\Delta_D \log(|t|^2 + 1) \geq -|F_{A_{\infty}}|_D \geq -C_\epsilon.
\]

Here since the disk \( D \) has a definite distance of \( Z \) by assumption which might depend on \( \epsilon \), we have \( |F_{A_{\infty}}|_D \leq C_\epsilon \) for some \( C_\epsilon \). Now near the point \( z \), \( X \cap U \subset U_k \subset \mathbb{C}^n \), \( \omega = i\partial\bar{\partial}\rho_k|_{X \cap U} \) for some smooth strongly plurisubharmonic function on \( U_k \). By definition, \( \Delta_D \rho_k = 1 \) thus we have

\[
\Delta_D \left( \log(|t|^2 + 1) + C_\epsilon \rho_k \right) \geq 0
\]

which by Maximum principle implies

\[
\log(|t|^2 + 1) + C_\epsilon \rho_k \leq C_\epsilon \sup_{\partial D} \rho_k + \sup_{\partial D} \log(|t|^2 + 1) \leq C(\epsilon, \|s_i\|_{L^2(X^\epsilon)}).
\]

For the last inequality, the bound of the first term is trivial, and we only explain the second one. The interior estimate over \( X^\epsilon \) for holomorphic sections implies \( s_i \) is uniformly bounded within any precompact subset of \( X^\epsilon \). In particular, it is uniformly bounded over \( \partial D \) which lies in a fixed precompact subset of \( X^\epsilon \), since \( \partial D \subset X^\epsilon \) has a definite distance to \( \partial X^\epsilon \). This finishes the proof. \( \square \)

3. **Proof of Singular Donaldson-Uhlenbeck-Yau Theorem**

We first prove the rank 1 case. Recall we have a limiting Hermitian-Yang-Mills connection \( A_\infty \) coming from the perturbed Hermitian-Yang-Mills connections and \( A_\infty \) defines a holomorphic vector bundle \( E_\infty \) over \( X \setminus Z \) where \( Z = X_\delta \cup \Sigma \) as before.

By Proposition 2.29, we can conclude the existence of a nontrivial map \( \Phi : E_\infty \to E \) coming from the normalized identity map \( \text{id} : E \to E \). More precisely, to get the limit, we consider \( \text{Hom}(E,E) \) with the metrics \( H_i^* \otimes H' \) where \( H_i \) are the sequence of Hermitian-Yang-Mills metrics and \( H' \) is any fixed smooth metric on \( E \) away from \( X_\delta \). Now we apply Proposition 2.29 to the sequence of holomorphic sections given by the identity map and get \( \Phi : E_\infty \to E \).
Lemma 3.1. Given a rank 1 reflexive sheaf $E$ over $X$, there exists an admissible Hermitian-Yang-Mills metric $H$ on $E$ with
\[ \mu(E) = \frac{\int_X c_1(H) \wedge \omega_1 \wedge \cdots \wedge \omega_{n-1}}{\text{rank } E}. \]

Proof. Since there exists a nontrivial map between $E$ and $E_\infty$ over $X_{\text{reg}}$, $E_\infty$ can be extended to be a reflexive rank 1 sheaf over $X$. Indeed, by the Remmert-Stein extension theorem, $(E \otimes E_\infty^*)^*$ can be extended to be a rank 1 reflexive sheaf from which the extension of $E_\infty$ follows trivially. It has to be an isomorphism since $E$ and $E_\infty$ have the same slope by Proposition 2.24 and Lemma 2.10.

In particular, for general rank, this gives

Corollary 3.2. The induced admissible Hermitian-Yang-Mills connection on $\det(E_\infty)$ defines a sheaf isomorphic to $\det(E)$. Furthermore, $A_\infty$ computes the slope of $\det(E)$.

The proof can be concluded by showing

Proposition 3.3. $\Phi : E_\infty \to E$ is an isomorphism.

Proof. Let $G$ denote the saturated subsheaf of $E$ given by the image of $\Phi$ in $E$. Following the argument in [44, Section 7], by Siu’s theorem ([40]), $G$ can be extended to be a reflexive sheaf over $X$. Indeed, it suffices to prove it locally. We can assume $G$ is a saturated subsheaf of $O \oplus N$ over some open subset $U$ away from $\Sigma$. Then $G \subset O \oplus N$ defines a map $f$ from the base to $Gr(N, N - \text{rank } G)$ away from some codimension two subvariety $Z'$. By Siu’s theorem ([40, Page 441]), we know the graph $\Gamma_f \subset (U \setminus \Sigma \cup X^s) \times Gr(N, N - \text{rank } G)$ can be extended to be a subvariety of $U \times Gr(N, N - \text{rank } G)$ by taking its closure $\overline{\Gamma_f}$. Let $\pi_1 : \overline{\Gamma_f} \to U$ and $\pi_2 : \overline{\Gamma_f} \to Gr(N, N - \text{rank } G)$ denote the natural projections. Let $\overline{F}$ denote the tautological locally free sheaf over $Gr(N, N - \text{rank } G)$. Then $((\pi_1)_* \pi_2^* \overline{F})^*$ gives an extension for $G$. Assume $\text{rank } G < \text{rank } E$. Since $E$ is stable, $\mu(G) < \mu(E)$. By considering
\[ F := \text{Hom}(\bigwedge^{\text{rank } G} E_\infty, \det G) \]
we can reduce to the following setting by Corollary 3.1 a sheaf $F$ defined over $X_{\text{reg}}$ admits an admissible Hermitian-Yang-Mills metric with negative Einstein constant and a nonzero global section $s := \bigwedge^{\text{rank } G} \Phi$. By Corollary 2.18 we know $s \in L^\infty$. Then we can use
\[ \Delta|s|^2 = 2|\nabla s|^2 - (\mu(G) - \mu(E))|s|^2. \]
By doing a cut-off argument, we know $\int_X \Delta|s|^2 = 0$. This is a contradiction since the integral of the right hand side is strictly positive. In particular, $\Phi$ has full rank at a generic point. By Corollary 3.2 $\det \Phi$ is nowhere vanishing, thus $\Phi$ is an isomorphism.

□
In particular, this gives

**Corollary 3.4.** $\Sigma = \emptyset$.

**Proof.** Indeed, by Proposition 3.3, we know away from $X^* \cup \text{Sing}(E)$, the sequence of Hermitian-Yang-Mills metrics $H_i$ are actually locally uniformly equivalent if we normalize properly. In particular, fixing a background metric, we have a local uniform $C^0$ bound away from $X^* \cup \text{Sing}(E)$. Then by the elliptic estimates for Hermitian-Yang-Mills metric ([1]), we have local uniform higher order estimates for the sequence of metrics as well. In particular, $H_i$ converges smoothly away from $X^* \cup \text{Sing}(E)$. The conclusion follows. \hfill \qed

As a direct corollary, we have the following

**Corollary 3.5.** If a reflexive sheaf $F$ over $(X, \omega_1 \wedge \cdots \omega_{n-1})$ admits an admissible Hermitian-Yang-Mills metric $H$, it is polystable.

**Proof.** We first show that $H$ computes the slope of $F$. Indeed, by the singular Donaldson-Uhlenbeck-Yau theorem for the line bundle case, there exists a Hermitian-Yang-Mills metric $H_0$ on $\det(F)$ which computes the slope of $\det(F)$. Now by Corollary 2.20, this defines the same Hermitian-Yang-Mills connection as $H$, in particular $H$ computes the slope of $F$. Let $F_1 \subset F$ be a stable subsheaf of $F$ with $\mu(F_1) \geq \mu(F)$. By the Singular Donaldson-Uhlenbeck-Yau theorem, there exists an admissible Hermitian-Yang-Mills metric $H_1$ on $F_1$. Similar as the proof of Corollary 2.20, we can conclude that the map $F_1 \to F$ is parallel with respect to the connection defined by $H_1$ and $H$ on $\text{Hom}(F_1, F)$ and $\mu(F_1) = \mu(F)$. In particular, we know $F = F_1 \oplus (F/F)^\ast$.

The conclusion now follows from induction. \hfill \qed

In particular, Corollary 1.3 follows directly from this.

4. **Bogomolov-Gieseker inequality**

4.1. **General version.** We separate the discussions into two cases.

4.1.1. **Locally free resolutions.** Fix any Kähler resolution $p : \hat{X} \to X$ so that $\hat{E} = (p^*E)^\ast$ is locally free. Then

$$
\int_{\hat{X}} (2rc_2(\hat{E}) - (r - 1)c_1(\hat{E})^2) \wedge p^*\omega_{i_1} \wedge \cdots p^*\omega_{i_{n-2}}
$$

$$= \lim_i \int_{\hat{X}} (2rc_2(A_i) - (r - 1)c_1(A_i)^2) \wedge (p^*\omega_{i_1} + i^{-1}\theta) \wedge \cdots (p^*\omega_{i_{n-2}} + i^{-1}\theta)
$$

$$= \lim_i \int_{\hat{X}} (2rc_2(A_{\infty}) - (r - 1)c_1(A_{\infty})^2) \wedge (p^*\omega_{i_1} + i^{-1}\theta) \wedge \cdots (p^*\omega_{i_{n-2}} + i^{-1}\theta)
$$

$$\geq \int_{X} (2rc_2(A_{\infty}) - (r - 1)c_1(A_{\infty})^2) \wedge \omega_{i_1} \wedge \cdots \omega_{i_{n-2}}.
$$
The first equality is trivial while the second follows from the Chern-Weil theory in the smooth case. For the last inequality, it follows from the curvature of the induced connection on $\text{Hom}(\hat{E}, \hat{E})$ being primitive, thus

$$(2rc_2(A_i) - (r - 1)c_1(A_i)^2) \wedge (p^*\omega_1 + i^{-1}\theta) \wedge \cdots (p^*\omega_{i_{n-2}} + i^{-1}\theta)$$

defines a sequence of Radon measures on $\hat{X}$ (see Equation 2.4). Furthermore, it has uniformly bounded mass. Then the inequality follows from Fatou’s lemma. Since the limiting Hermitian-Yang-Mills connection is independent of the resolutions by Corollary 2.20, the conclusion follows. About the case of equality, it follows from exactly the same argument as [8, Corollary 1.3].

4.1.2. General case. Fix a general Kähler resolutions $p : \hat{X} \to X$. Our main theorem implies there exists a family of admissible Hermitian-Yang-Mills connections on $(p^*\mathcal{E})^\ast$ on $\hat{X}$ with respect to the perturbed metrics. Since over $\hat{X}$ the quantities

$$(2rc_2(A_i) - (r - 1)c_1(A_i)^2)$$

given by the admissible Hermitian-Yang-Mills connections defines a sequence of closed currents (see [7, Proposition 46]), we know the quantities in the Bogomolov-Gieseker inequality can still be computed by the admissible Hermitian-Yang-Mills connections on $(p^*\mathcal{E})^\ast$ over $\hat{X}$. Furthermore, exactly the same argument as the smooth case implies the family of admissible Hermitian-Yang-Mills connections converges to the admissible Hermitian-Yang-Mills connections on $\mathcal{E}$ we obtained. So the same argument as above gives the statement about general resolutions.

4.2. Various simplifications. Now we discuss various cases where the Bogomolov-Gieseker inequality could be simplified and one can extract more from the gauge theoretical picture when shrinking the exceptional divisor.

4.2.1. Over a normal surface. We first give a proof of Corollary 1.9. The equality

$$\Delta(\mathcal{E}) = (2rc_2(\mathcal{E}) - (r - 1)c_1(\mathcal{E})^2) \cap X$$

follows from Proposition 2.12. Corollary 1.11 follows from Corollary 1.7 and Proposition 2.13.

4.2.2. When $X$ is smooth in codimension two. The equality

$$(2rc_2(\mathcal{E}) - (r - 1)c_1(\mathcal{E})^2).[\omega_1] \cdots [\omega_{i_{n-2}}]$$

follows from Proposition 2.14. It remains to show that the Chern-Weil formula holds when $X$ has isolated singularities, i.e.

$$(2rc_2(\hat{E}) - (r - 1)c_1(\hat{E})^2).[p^*\omega_1] \cdots [p^*\omega_{i_{n-2}}]$$

follows from Proposition 2.14. It remains to show that the Chern-Weil formula holds when $X$ has isolated singularities, i.e.

$$\int_X (2rc_2(H) - (r - 1)c_1(H)^2) \wedge \omega_1 \wedge \cdots \omega_{i_{n-2}}$$
for any resolution \( p : \hat{X} \to X \). For this, by definition, we can choose an embedding \( U \to \mathbb{C}^N \) near \( x \) so that \( \omega_i = \sqrt{-1} \partial \bar{\partial} (\rho_i | U) \) for any \( i \) where \( \rho_i \) is a smooth function near \( x \) in \( \mathbb{C}^n \). In particular,
\[
\omega_i \wedge \cdots \omega_{i_{n-2}} | U = d(\psi | U)
\]
for some \( \psi \) smooth on \( \mathbb{C}^N \), thus by cutting-off near the isolated singularities, we can globally write
\[
\omega_i \wedge \cdots \omega_{i_{n-2}} = d(\psi_0 + \Omega_0)
\]
where \( \psi_0 \) is a restriction of form from \( \mathbb{C}^n \) that is smooth supported near \( x \). By [8, Lemma 2.5], we know
\[
\int_X (2rc_2(H) - (r - 1)c_1(H)^2) \wedge \omega_i \wedge \cdots \omega_{i_{n-2}} = \int_X \Omega_0.
\]
The conclusion follows.

5. Bogomolov-Gieseker inequality for semistable sheaves

Now we go back to the general set-up and assume \( E \) is a semistable torsion free sheaf over a normal variety \((X, \omega_1 \wedge \cdots \omega_{n-1})\) endowed with \(({\mathcal{O}}_X)\) Kähler metrics. The goal is to prove Theorem 1.7. For this, we first prove the inequality for semistable sheaves

**Proposition 5.1** (Gieseker-Bogomolov inequality for semistable sheaves). Given any semistable reflexive sheaf \( E \) over \((X, \omega_1 \wedge \cdots \omega_{n-1})\), the Gieseker-Bogomolov inequality holds i.e.
\[
\Delta(E)[\omega_1] \cdots [\omega_{n-1}] \geq 0
\]

**Proof.** Fix \( \hat{E} \in \mathbf{E}_p \) for some resolution \( p : \hat{X} \to X \). Let
\[
0 \subset E_1 \subset \cdots E_m = E
\]
be a Jordan-Hölder filtration for \( \mathcal{E} \). Then this naturally gives a filtration for
\[
0 \subset \hat{E}_1 \subset \cdots \hat{E}_m = \hat{E}
\]
so that \( \hat{E}_k/\hat{E}_{k-1} \) is torsion free for any \( k \) and \( \hat{E}_k|_{\mathcal{O}_{X_{\text{reg}}}} \cong \mathcal{E}_k|_{\mathcal{O}_{X_{\text{reg}}}} \). By Proposition 2.21, we know for any \( k \), \( \hat{E}_k/\hat{E}_{k-1} \) is stable with respect to \( [p^*\omega_1 + i^{-1}\theta] \wedge \cdots [p^*\omega_{n-1} + i^{-1}\theta] \). Thus
\[
\Delta(\hat{E}_k/\hat{E}_{k-1})[p^*\omega_1 + i^{-1}\theta] \wedge \cdots [p^*\omega_{n-1} + i^{-1}\theta] \geq 0.
\]
Denote the Chern classes of \( \hat{E}_k/\hat{E}_{k-1} \) (resp. \( \mathcal{E} \)) by \( \gamma_k \) (resp. \( \gamma \)), and the rank of \( \hat{E}_k/\hat{E}_{k-1} \) (resp. \( \mathcal{E} \)) by \( r_k \) (resp. \( r \)). Then \( r = \sum \gamma_k \), and \( \gamma = \sum r_k \). By definition, we have
\[
\sum_k \frac{\Delta(\hat{E}_k/\hat{E}_{k-1})}{r_k} \frac{\Delta(\hat{E})}{r} = \sum_{s<t} \frac{1}{r<r} (\frac{\gamma_s}{r_s} - \frac{\gamma_t}{r_t})^2.
\]
By assumption, we know
\[
(\frac{\gamma_s}{r_s} - \frac{\gamma_t}{r_t})[p^*\omega_1] \cdots [p^*\omega_{n-1}] = 0.
\]
To finish the proof, it suffices to prove that

\[(\frac{\gamma_s}{r_s} - \frac{\gamma_t}{r_t})^2[p^*\omega_{i_1}] \cdots [p^*\omega_{i_{n-1}}] \leq 0\]

In the smooth case, this follows from [11] which generalize the classical Hodge-Riemann theorem over Kähler manifolds to the case of multiple Kähler metrics ([42]). Let \(c_{st}^i\) so that

\[(\frac{\gamma_s}{r_s} - \frac{\gamma_t}{r_t} - c_{st}^i[p^*\omega_{i_{n-1}}])(\frac{\gamma_s}{r_s} - \frac{\gamma_t}{r_t} - c_{st}^i[p^*\omega_{i_{n-1}}])\]

which implies

\[(\frac{\gamma_s}{r_s} - \frac{\gamma_t}{r_t} - c_{st}^i[p^*\omega_{i_{n-1}}])\]

\[= 0\]

Since \(c_{st}^i \to 0\) as \(i \to \infty\), by taking the limit of the equation above, we have

\[(\frac{\gamma_s}{r_s} - \frac{\gamma_t}{r_t})\]

\[= 0\]

The conclusion follows. \(\square\)

Assume now

\[\Delta(\hat{\mathcal{E}})[p^*\omega_{i_1}] \cdots [p^*\omega_{i_{n-2}}] = 0\]

for some \(\hat{\mathcal{E}} \in \mathcal{E}_\mathfrak{p}\) and some resolution \(p : \hat{X} \to X\). To finish the proof of Theorem [1.4], it remains to show

**Proposition 5.2.** \(\mathcal{E}\) admits a filtration

\[0 \subseteq \mathcal{E}_1 \subseteq \cdots \mathcal{E}_m = \mathcal{E}\]

so that \(\mathcal{E}_i/\mathcal{E}_{i-1}\) is torsion free and \((\mathcal{E}_i/\mathcal{E}_{i-1})|_{X^{\text{reg}}}\) is projectively flat.

**Proof.** Let \(\mathcal{E}_1\) (resp. \(\hat{\mathcal{E}}_1\)) be a stable subsheaf of \(\mathcal{E}\) (resp. \(\hat{\mathcal{E}}\)) having the same slope as \(\mathcal{E}\) (resp. \(\hat{\mathcal{E}}\)) so that \(\mathcal{E}/\mathcal{E}_1\) (resp. \(\hat{\mathcal{E}}/\hat{\mathcal{E}}_1\)) is torsion free stable with the same slope as \(\mathcal{E}\) (resp. \(\hat{\mathcal{E}}\)) and

\[\hat{\mathcal{E}}_1|_{X^{\text{reg}}} \cong \mathcal{E}_1|_{X^{\text{reg}}}\]

By induction, we need to show

1. \(\hat{\mathcal{E}}_1\) is projectively flat away from the exceptional divisor.
2. \(\hat{\mathcal{E}}/\hat{\mathcal{E}}_1\) is reflexive away from the exceptional divisor and

\[\Delta((\hat{\mathcal{E}}/\hat{\mathcal{E}}_1)^*)[p^*\omega_{i_1}] \cdots [p^*\omega_{i_{n-2}}] = 0\]

We first prove (1). Since \(\hat{\mathcal{E}}_1\) and \(\hat{\mathcal{E}}/\hat{\mathcal{E}}_1\) are stable, by Corollary [1.7] we know

\[\Delta(\hat{\mathcal{E}}_1)[p^*\omega_{i_1}] \cdots [p^*\omega_{i_{n-2}}] \geq 0\]

and

\[\Delta((\hat{\mathcal{E}}/\hat{\mathcal{E}}_1)^*)[p^*\omega_{i_1}] \cdots [p^*\omega_{i_{n-2}}] \geq 0\]

By Corollary [1.7] it suffices to show

\[\Delta(\hat{\mathcal{E}}_1)[p^*\omega_{i_1}] \cdots [p^*\omega_{i_{n-2}}] = 0\]
As the proof of Proposition 5.1, we know
\[
\tag{5.2} \frac{\Delta(\hat{E}_1)}{r_1} + \frac{\Delta(\hat{E}/\hat{E}_1)}{r_1} - \frac{\Delta(\hat{E})}{r} [p^*\omega_{i_1}] \cdots [p^*\omega_{i_{n-1}}] \leq 0
\]
which combined with the inequality above implies
\[
\Delta(\hat{E}_1) = 0
\]
and
\[
\frac{\Delta(\hat{E}/\hat{E}_1)}{r_1} [p^*\omega_{i_1}] \cdots [p^*\omega_{i_{n-1}}] = 0
\]
if we can show
\[
\tag{5.3} \Delta(\hat{E}/\hat{E}_1)[p^*\omega_{i_1}] \cdots [p^*\omega_{i_{n-2}}] \geq \Delta((\hat{E}/\hat{E}_1)^{**}).[p^*\omega_{i_1}] \cdots [p^*\omega_{i_{n-2}}].
\]
Assume this, by Corollary 1.7, we know \(\hat{E}_1\) is projectively flat over \(X^{reg}\).

Now we prove Equation 5.3. Indeed, we have
\[
0 \to \hat{E}/\hat{E}_1 \to (\hat{E}/\hat{E}_1)^{**} \to \tau \to 0
\]
where \(\text{supp}(\tau)\) has codimension at least two. Let \(\Sigma_k\) denotes the irreducible pure codimension two components of \(\text{supp}(\tau)\), and to each \(\Sigma_k\), one can associate it with an analytic multiplicity \(m_k = h^0(\tau|_{\Delta}, \Sigma_k)\) where \(\Delta\) is a transverse slice of \(\Sigma_k\) at a generic point. Then
\[
\text{ch}_2(\tau) = \text{PD}(\sum_k m_k \Sigma_k)
\]
by [36, Proposition 3.1]. We then know from definition that
\[
\Delta((\hat{E}/\hat{E}_1)^{**}).[p^*\omega_{i_1}] \cdots [p^*\omega_{i_{n-2}}] = \sum_k m_k \int_{\Sigma_k} p^*\omega_{i_1} \wedge \cdots \omega_{i_{n-2}}
\geq 0
\]
where the equality holds if and only if \(\Sigma_k \cap X^{reg} = \emptyset\) for any \(k\).

For (2), from the discussion above, we know \(\text{supp}(\tau)\) has codimension at least three over \(X^{reg}\). Furthermore, we have
\[
\Delta((\hat{E}/\hat{E}_1)^{**}).[p^*\omega_{i_1}] \cdots [p^*\omega_{i_{n-2}}] = 0.
\]
By induction, \((\hat{E}/\hat{E}_1)^{**}\) is locally free over \(X^{reg}\). It remains to show \(\hat{E}/\hat{E}_1\) is reflexive over \(X^{reg}\). This follows from the argument in [36, Proposition 2.3]. The key point is that over \(X^{reg}\), the short exact sequence
\[
0 \to \hat{E}_1 \to \hat{E} \to \hat{E}/\hat{E}_1 \to 0
\]
locally splits away from a codimension three set over \(X^{reg}\) since \(\hat{E}_1\) and \((\hat{E}/\hat{E}_1)^{**}\) are locally free over \(X^{reg}\). Thus this will force \(\hat{E}/\hat{E}_1\) to be reflexive over \(X^{reg}\), i.e. locally free over \(X^{reg}\) in this case. \(\square\)
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**References**

1. Shigetoshi Bando and Yum-Tong Siu, *Stable sheaves and Einstein-Hermitian metrics*, Geometry and analysis on complex manifolds, World Sci. Publ., River Edge, NJ, 1994, pp. 39–50. MR 1463962
2. Sébastien Boucksom, Jean-Pierre Demailly, Mihai Paun, and Thomas Peternell, *The pseudo-effective cone of a compact kähler manifold and varieties of negative kodaira dimension*, Journal of Algebraic Geometry 22 (2013), no. 2, 201–248.
3. NP Buchdahl, *Hermitian-einstein connections and stable vector bundles over compact complex surfaces*, Mathematische Annalen 280 (1988), 625–648.
4. Frédéric Campana and Mihai Păun, *Foliations with positive slopes and birational stability of orbifold cotangent bundles*, Publications mathématiques de l’IHÉS 129 (2019), no. 1, 1–49.
5. Xuemiao Chen and Song Sun, *Analytic tangent cones of admissible Hermitian-Yang-Mills connections*, Geometry & Topology 25 (2021), no. 4, 2061–2108.
6. Xuemiao Chen and Richard A. Wentworth, *The nonabelian Hodge correspondence for balanced Hermitian metrics of Hodge-Riemann type*, Accepted by Mathematical Research Letter, arxiv:2106.09133.
7. , *Compactness for Ω-Yang-Mills equations*, Calc. Var. Part. Differ. Equ 61 2, 58 (2022).
8. , *A Donaldson-Uhlenbeck-Yau theorem for normal varieties and semistable bundles on degenerating families*, Mathematische Annalen 388 (2024), no. 2, 1903–1935.
9. Benoît Claudon, Patrick Graf, and Henri Guenancia, *Numerical characterization of complex torus quotients*, arXiv preprint arXiv:2109.06738 (2021).
10. Benoît Claudon, Patrick Graf, Henri Guenancia, and Philipp Naumann, *Kähler spaces with zero first chern class: Bochner principle, albanese map and fundamental groups*, Journal für die reine und angewandte Mathematik (Crelles Journal) (2022).
11. Tien-Cuong Dinh and Việt-Anh Nguyén, *The mixed Hodge-Riemann bilinear relations for compact Kähler manifolds*, Geom. Funct. Anal. 16 (2006), no. 4, 838–849. MR 2255382
12. Simon Donaldson, *Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles*, Proc. London Math. Soc. 50 (1985), 1–26.
13. , *Infinite determinants, stable bundles, and curvature*, Duke Math. J. 54 (1987), 231–247.
14. Simon Kirwan Donaldson, *Floer homology groups in yang-mills theory*, vol. 147, Cambridge University Press, 2002.
15. Patrick Graf and Tim Kirschner, *Finite quotients of three-dimensional complex tori*, Annales de l’Institut Fourier, vol. 70, 2020, pp. 881–914.
16. Daniel Greb, Stefan Kebekus, and Thomas Peternell, *Étale fundamental groups of Kawamata log terminal spaces, flat sheaves, and quotients of abelian varieties*, Duke Math. J. 165 (2016), no. 10, 1965–2004. MR 3522654
17. , *Movable curves and semistable sheaves*, Int. Math. Res. Not. IMRN (2016), no. 2, 536–570. MR 3493425
18. Daniel Greb and Matei Toma, *Compact moduli spaces for slope-semistable sheaves*, Algebr. Geom. 4 (2017), no. 1, 40–78. MR 3592465
19. Henri Guenancia, *Semistability of the tangent sheaf of singular varieties*, Algebraic Geometry 3 (2016), no. 5, 508–542.
20. Henri Guenancia and Mihai Păun, *Bogomolov-Gieseker inequality for log terminal Kähler threefolds*, arXiv preprint arXiv:2405.10003 (2024).
21. Bin Guo, Duong H Phong, Jian Song, and Jacob Sturm, *Sobolev inequalities on Kähler spaces*, arXiv preprint arXiv:2311.00221 (2023).
22. Andreas Höring and Thomas Peternell, *Algebraic integrability of foliations with numerically trivial canonical bundle*, Invent. Math. **216** (2019), no. 2, 395–419. MR 3953506
23. Daniel Huybrechts and Manfred Lehn, *The geometry of moduli spaces of sheaves*, second ed., Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2010. MR 2665168
24. Shoshichi Kobayashi, *Differential geometry of complex vector bundles*, Princeton University Press, 1987.
25. Adrian Langer, *Chern classes of reflexive sheaves on normal surfaces*, Mathematische Zeitschrift **235** (2000), no. 3, 591–614.
26. *Semistable sheaves in positive characteristic*, Annals of mathematics (2004), 251–276.
27. Jun Li and Shing-Tung Yau, *Hermitian-Yang-Mills connection on non-Kähler manifolds*, Mathematical aspects of string theory (San Diego, Calif., 1986), Adv. Ser. Math. Phys., vol. 1, World Sci. Publishing, Singapore, 1987, pp. 560–573. MR 915839
28. Peter Li and Gang Tian, *On the heat kernel of the bergmann metric on algebraic varieties*, Journal of the American Mathematical Society (1995), 857–877.
29. Martin Lübke and Andrei Teleman, *The Kobayashi-Hitchin correspondence*, World Scientific Publishing Co., Inc., River Edge, NJ, 1995. MR 1370660
30. Martin Lübke and Andrei Teleman, *The universal kobayashi-hitchin correspondence on hermitian manifolds*, vol. 13, American Mathematical Soc., 2006.
31. András Némethi, *Normal surface singularities*, vol. 74, Springer Nature, 2022.
32. Wenhao Ou, *Admissible metrics on compact kähler varieties*, arXiv preprint arXiv:2201.04821 (2022).
33. *Orbifold modifications of complex analytic varieties*, arXiv preprint arXiv:2401.07273 (2024).
34. Hugo Rossi, *Picard variety of an isolated singular point*, Rice Institute Pamphlet-Rice University Studies **54** (1968), no. 4.
35. Stefan Schröer and Gabriele Vezzosi, *Existence of vector bundles and global resolutions for singular surfaces*, Compositio Mathematica **140** (2004), no. 3, 717–728.
36. Benjamin Sibley and Richard A. Wentworth, *Analytic cycles, Bott-Chern forms, and singular sets for the Yang-Mills flow on Kähler manifolds*, Adv. Math. **279** (2015), 501–531. MR 3345190
37. Leon Simon, *Lectures on geometric measure theory*, Proceedings of the Centre for Mathematical Analysis, Australian National University, vol. 3, Australian National University, Centre for Mathematical Analysis, Canberra, 1983. MR 756417
38. Carlos Tschudi Simpson, *Systems of Hodge bundles and uniformization*, ProQuest LLC, Ann Arbor, MI, 1987, Thesis (Ph.D.)–Harvard University. MR 2636035
39. Yum-Tong Siu, *A harkogs type extension theorem for coherent analytic sheaves*, Annals of Mathematics **93** (1971), no. 1, 166–188.
40. *Extension of meromorphic extension theorem on kahler manifolds*, Annals of Mathematics **102** (1975), no. 3, 421–462.
41. Gang Tian, *Gauge theory and calibrated geometry. I*, Ann. of Math. (2) **151** (2000), no. 1, 193–268. MR 1745014
42. V. A. Timorin, *Mixed Hodge-Riemann bilinear relations in a linear context*, Funktsional. Anal. i Prilozhen. **32** (1998), no. 4, 63–68. 96. MR 1678857
43. Matei Toma, *Bounded sets of sheaves on Kähler manifolds, ii*, arXiv preprint arXiv:1906.05853 (2019).
44. Karen Uhlenbeck and Shing-Tung Yau, *On the existence of Hermitian-Yang-Mills connections in stable vector bundles*, Comm. Pure Appl. Math. **39** (1986), no. 2, S257–S293.

45. Jean Varouchas, *Kähler spaces and proper open morphisms*, Mathematische Annalen **283** (1989), no. 1, 13–52.

46. Xiaojun Wu, *The Bogomolov’s inequality on a singular complex space*, arXiv:2106.14650 (2021).

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