ON NON-NEGATIVELY CURVED METRICS
ON OPEN FIVE-DIMENSIONAL MANIFOLDS

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ABSTRACT. Let $V^n$ be an open manifold of non-negative sectional curvature with a soul $\Sigma$ of co-dimension two. The universal cover $\tilde{N}$ of the unit normal bundle $N$ of the soul in such a manifold is isometric to the direct product $M^{n-2}\times \mathbb{R}$. In the study of the metric structure of $V^n$ an important role plays the vector field $X$ which belongs to the projection of the vertical planes distribution of the Riemannian submersion $\pi: V \to \Sigma$ on the factor $M$ in this metric splitting $\tilde{N} = M \times \mathbb{R}$. The case $n = 4$ was considered in [GT] where the authors prove that $X$ is a Killing vector field while the manifold $V^4$ is isometric to the quotient of $M^2 \times (\mathbb{R}^2, g_F) \times \mathbb{R}$ by the flow along the corresponding Killing field. Following an approach of [GT] we consider the next case $n = 5$ and obtain the same result under the assumption that the set of zeros of $X$ is not empty. Under this assumption we prove that both $M^3$ and $\Sigma^3$ admit an open-book decomposition with a bending which is a closed geodesic and pages which are totally geodesic two-spheres, the vector field $X$ is Killing, while the whole manifold $V^5$ is isometric to the quotient of $M^3 \times (\mathbb{R}^2, g_F) \times \mathbb{R}$ by the flow along corresponding Killing field.

1. Introduction

Let $(V^n, g)$ be a complete open Riemannian manifold of non-negative sectional curvature. Remind that as follows from [CG] and [P] an arbitrary complete open manifold $V^n$ of non-negative sectional curvature contains a closed absolutely convex and totally geodesic submanifold $\Sigma$ (called a soul) such that the projection $\pi: V \to \Sigma$ of $V$ onto $\Sigma$ along geodesics normal to $\Sigma$ is well-defined and is a Riemannian submersion (see also [CaS]). The (vertical) fibers $F_P = \pi^{-1}(P), P \in \Sigma$ of $\pi$ define a metric foliation in $V$ and two distributions: a vertical $\mathcal{V}$ distribution of subspaces tangent to fibers and a horizontal distribution $\mathcal{H}$ of subspaces normal to $\mathcal{V}$. For an arbitrary point $P$ on $\Sigma$, an arbitrary geodesic $\gamma(t)$ on $\Sigma$ and arbitrary vector field $V(t)$ which is parallel along $\gamma$ and normal to $\Sigma$ the following

\begin{equation}
\Pi(t,s) = \exp_{\gamma(t)} sV(t)
\end{equation}

are totally geodesic surfaces in $V^n$ of zero curvature, i.e., flats.

When $\dim \Sigma = 1$ or $\text{codim}(\Sigma) = 1$ the manifold $V^n$ is locally isometric to the direct product of $\Sigma$ and Euclidean space of a complementary dimension and of non-negative curvature. Study of the next case $\text{codim}(\Sigma) = 2$ was begun in [M1], where we noted that the manifold $V^4$ or is a direct product when the holonomy of the normal bundle of $\Sigma$ in $V$ is trivial, or the holonomy group acts transitively on normal vectors, every geodesic normal to $\Sigma$ is a ray and (1) holds. The metric structure in this case might be more complicated. In [GT] the authors consider four-dimensional manifolds diffeomorphic to direct products $M^2 \times \mathbb{R}^2$ and prove the following.

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**Theorem A.** [GT]. Every non-negatively curved metric on $M^2 \times R^2$ is isometric to a Riemannian quotient of the form $((M^2, g_0) \times (R^2, g_F) \times R)/R$. Here $R$ acts diagonally on the product by the flow along Killing vector fields on $(M^2, g_0)$ and $(R^2, g_F)$ and by translations on $R$.

The very important role in the proof of the Theorem A plays the vector field $X$ which is the projection of the vertical vector field in the universal cover $\tilde{N}$ of the boundary of some metric $s$-tube $N$ of the soul on the "horizontal" factor $M$ in the metric splitting $\tilde{N} = M \times R$, see below. In the four-dimensional case this vector field $X$ restricted to $M^2$ always has zeros since $M^2$ is a two-dimensional sphere. In our case the soul $\Sigma$ of $V$ and $M^3$ are three-dimensional spheres, and hence, $X$ might be nowhere zero as the following simple example shows. Let $h : S^3 \rightarrow S^2$ be the Hopf bundle, i.e., the factoring of a unit sphere $S^3$ in the complex plane $C^2$ by $S^1$ action - multiplication by complex numbers of absolute value 1. Consider $V^5$ which is the the quotient of the direct product of $S^3 \times R^2 \times S^3$, where the $S^1$ acts on $R^2 = C$ by rotations, i.e., again, multiplication in $C^1$ by unit complex numbers. Then for the manifold $V^5 = S^3 \times R^2 \times S^1$ the vector field $X$ is nowhere zero.\(^1\) The objective of this note is to expand an approach from [GT] to the case of non-negatively curved five-dimensional $V^5$ diffeomorphic to a direct product $S^3 \times R^2$ under the following assumption.

**Assumption 1.** The set of zeros of the vector field $X$ is not empty.

Our main result is very similar to the Theorem A above.

**Theorem B.** Let $V^5$ be an open manifold of non-negative sectional curvature and diffeomorphic to $S^3 \times R^2$. Assume that the vector field $X$ has non-empty zero set $Z$. Then $Z$ is a closed geodesic and the manifold $M^3$ admits a singular foliation - "open-book decompositions" by totally geodesic and isometric to each other horizontal two-dimensional spheres $S^2(\psi)$, where the singular set of this decompositions - "bindings", equal the closed geodesic $Z$. The flow along Killing field $X$ acts as "turning pages" in this open-book decomposition, while $V^5$ itself is isometric to a Riemannian quotient of the form $M^3 \times (R^2, g_F) \times R/R$ with $R$ acting diagonally on the product by the flow along Killing vector fields on $M^3$ and $(R^2, g_F)$ and by translations on $R$. The Riemannian submersion $\pi : V^5 \rightarrow \Sigma$ conveys the open-book decomposition of $M^3$ to a similar open-book decomposition of $\Sigma$ with the pages $\Sigma^2(\psi)$ isometric to $S^2(\psi)$.

In the same way as Theorem A in [GT] our Theorem B follows from the fact that the vector field $X$ on $N$ is Killing for every $s$, where $N$ is the boundary of $s$-metric neighborhood of the soul $\Sigma$ in $V$, see Theorem 3 below. Thus, after proving Theorem 3, we complete the proof of Theorem B by referring to the corresponding arguments from [GT], see section 5.

Note that the general case of five-dimensional open manifold $V^5$ with a soul of codimension 2 can be reduced to the one under consideration as follows. First, we note that if the fundamental group of $\Sigma$ (which is isomorphic to that of $V$) is not finite, the universal cover $\tilde{V}$ contains a straight line in the universal cover $\tilde{\Sigma}$ of the soul. Then both $\tilde{V}$ and $\tilde{\Sigma}$ split into direct products, and the case is reduced to the already studied one of open four-dimensional manifolds. When the fundamental group of $\Sigma$ is finite the universal cover $\tilde{\Sigma}$ is diffeomorphic to a sphere $S^3$ due to the non-negativity of the curvature. Next: because an arbitrary vector bundle over simply connected $S^3$ is, obviously, trivial we see that an investigation of the metric structure of an arbitrary $V^5$ with a soul of codimension 2 is reduced to the case when $V^5$ is diffeomorphic to the direct product $S^3 \times R^2$.\(^2\)

Below we assume that the holonomy of the normal bundle is not trivial, for otherwise by a direct product theorem from [M1,4] the manifold $V$ is a metric product.

\(^1\)For corresponding $M^3$ and $S^3$ the one-form given by the scalar product with $X$ is a (nowhere degenerated) contact form $\alpha$ with $\alpha \wedge d\alpha$ - the volume form.

\(^2\)The case when five-dimensional $V^5$ has a soul of codimension 3 we considered in [M5].
2. Vector field $X$ and its zeros

Fix some positive $s_0$ smaller than a focal radius of $\Sigma$ in $V$. For some $s < s_0$ denote by $N\Sigma(s)$, or simply by $N$, the boundary of an $s$-neighborhood of $\Sigma$. Due to our choice it is a smooth manifold. It consists of all points $Q(P,V) = \exp_{P}(sV)$, where $P$ is a point on $\Sigma$ and $V$ is a unit vector normal to $\Sigma$ at $P$.

**Lemma 1.** $N(s)$ has non-negative curvature if $s$ is sufficiently small.

**Proof.** This follows from the Gauss equations and the fact that $N(s)$ bounds a convex subset in a manifold $V$ of non-negative curvature. The last is obviously true when the holonomy of the (trivial) normal bundle $\nu\Sigma$ of the soul is trivial, i.e., all parallel translations along closed curves in $\Sigma$ acts identically on vectors normal to $\Sigma$ because then $V$ is isometric to the direct metric product $\Sigma \times (\mathbb{R}^2,h)$, see [M1]. If the holonomy is not trivial, then all normal vectors are so called ray directions, and $N(s)$ coincides with the boundary $\partial C_s$ of an absolutely convex set constructed in [CG], see again [M1]. The Lemma 1 is proved.

**Lemma 2.** The universal cover $\tilde{N}$ of $N(s)$ is isometric to the direct product $(M,g) \times \mathbb{R}$, where $M$ is diffeomorph to $S^3$. The composition of a covering map and a submersion $\pi$ provides a diffeomorphism between an arbitrary factor $M$ and the soul $\Sigma$ which we denote by $\pi^M: M \to \Sigma$.\(^3\)

**Proof.** This follows from the fact that $N(s)$ is diffeomorphic to the trivial circle bundle over three-dimensional sphere $\Sigma$, i.e., has an infinite cycle fundamental group generated by a homotopy class of a fiber. Then by standard arguments the universal cover $\tilde{N}(s)$ admits a straight line, and hence by Toponogov splitting theorem is isometric to the direct product $(M,g) \times \mathbb{R}$.

Denote by $E$ the unit vector field in $N(s)$ tangent to the projections of straight lines (i.e., $R$-factor) from $\tilde{N}$ to $N$. By $W$ we denote the (vertical) vector field on $N$ which is the speed of the natural $S^1$-action on $N$ given by rotations in a positive direction of a normal vectors to $\Sigma$ as follows: for $Q = Q(P,V)$ denote by $Q_\phi = Q_\phi(P,V) = Q(P,V_\phi)$, where $V_\phi$ is $V$ rotated by the angle $\phi$ in the bundle of unit normals to $\Sigma$ in $V$ (which is correctly defined since the bundle is topologically trivial). Finely, denote by $X$ the vector field on $N$ which is the component of $W$ normal to $E$.\(^4\)

$$X = W - (W,E)E. \quad (2)$$

Note, that $N$ naturally inherits from $V$ a horizontal distribution $\mathcal{H}$, while the vector field $W$ belongs to the vertical distribution. If by $M$ we denote an image of some $(M,g)$-factor in the direct metric product $\tilde{N}$ under the projection $pr: \tilde{N} \to N$, then (the restriction of) $E$ on $M$ would be the unit vector field of normals to $M$, $X$ is a vector field tangent to $M$, while another vector field $Y$ tangent to $M$ would be a horizontal if and only if it is normal to $X$. In particular, the tangent subspace $T_QM$ is horizontal $H_Q$ if and only if $X(Q) = 0$.

Note that the vector field $W$ in $N$ is never tangent to any of the $M$-factor, or (equivalently) never orthogonal to $E$. Indeed, if so then some homotopically non-trivial closed geodesic $\Gamma(s)$ in $N$ which is the images of a straight line in the universal cover $\tilde{N}$ is never horizontal at some point, and therefore, horizontal everywhere, which obviously can not be homotopically non-trivial in $N$. To see this denote by $\tilde{\Gamma}(s)$ its image under $\pi$ in $\Sigma$, and by $V(s)$ the normal vector field of vertical geodesics connecting $\tilde{\Gamma}(s)$ and $\Gamma(s)$. Since $\Sigma$ is simply connected their exists a disk $D$ in $\Sigma$ with a boundary $\Gamma$ and extension of the vector field $V$ over $D$ (because the restriction of a normal bundle to $D$ is trivial). The vertical lift of $D$ along this extension will provide us a disk in $N$ with a boundary $\Gamma$ implying that $\Gamma$ is contractible in $N$. The obtained contradiction proves that $E$ is never horizontal, or that the map $\pi^M: M \to \Sigma$ from any of the image $M$ of a factor in the direct product $N = (M^3,g) \times \mathbb{R}$ into the soul $\Sigma$ is a diffeomorphism. The Lemma 2 is proved.\(^5\)

\(^3\)Note, that this statement and forthcoming (3) both are true for an arbitrary $V^n$ with simply connected soul of codimension two.

\(^4\)Note, that our $X$ is different from similar $X$ of [GT].

\(^5\)This also fills the gap in the arguments from Lemma 2.1 in [GT].
By definition the differential of the diffeomorphism $\pi^M : M \to \Sigma$ is an isometry on the subspace of horizontal vectors, i.e., on the subspace in $T_Q M$ normal to $X$ (or on the whole $T_Q M$ if $X(Q) = 0$), while

$$\|d\pi^M_Q(X)\| = \cos(\alpha(Q))\|X\|,$$

where $\alpha(Q)$ denotes the angle at the point $Q \in M$ between vectors $E$ and $W$. The map $\pi^N : N \to \Sigma$ is the composition of the projection in the universal cover to the horizontal factor and then $\pi^M$.

When $X$ is identically zero the submanifold $M$ is horizontal in $N$, isometric to $\Sigma$ by (3), the holonomy of the normal bundle $\nu \Sigma$ is trivial, and, again, $V$ is isometric to the direct product $\Sigma \times (R^2, h)$ of the soul $\Sigma$ and some non-negatively curved plane $(R^2, h)$.

Next we prove that if $X$ is not identically zero, or has no zeros at all, then $X$ vanish along some closed geodesic.

**Theorem 1.** If the set of zeros $Z$ of the vector field $X$ in $M$ is a proper subset (i.e., is not $M$ itself or empty) then $Z$ is a closed geodesic. Every minimal geodesic connecting two points from $Z$ is itself a subset of $Z$.

**Proof.** If $Z$ is a proper subset of $M$ then for some $P \in Z$ there exists a sequence of points $Q_i \to P$ such that $X(Q_i) \neq 0$. As in [GT], see Lemma 2.1; we note that every geodesic $L(P, Q; t)$ in $M$ connecting a point $P$ where $X$ vanish with an arbitrary point $Q$ with non-vanishing $X(Q)$ is orthogonal to $X(Q)$,

$$QP \perp X(Q),$$

where $\bar{P}Q$ denotes the vector of direction of $L(P, Q; t)$ at the point $Q$. Hence, $Z$ belongs to the exponential image $\Pi(Q, X(Q))$ of a plane in $T_Q M$ of all vectors normal to $X(Q)$:

$$Z \subset \Pi(Q, X(Q)).$$

The surface $\Pi(Q, X(Q))$ near $Q$ is "almost a plane" - smooth with a second form vanishing at $Q$. Fix for a moment some $Q = Q_1$ close enough to $P$. Then in a small closed ball $B$ around $P$ with radius $dist(P, Q)$ zeros of $X$ belong to this "almost a plane" $\Pi(Q, X(Q))$. Thus there exists the farthest point $Q' \in B$ to $\Pi$ where $X(Q') \neq 0$. Then by (5) the part of the set $Z$ inside the ball $B$ belongs to the intersection of two "almost planes" $\Pi(Q, X(Q))$ and $\Pi(Q', X(Q'))$ which both are "almost orthogonal". This intersection, as easy to see, is a smooth curve with geodesic curvature of the order $dist(P, Q)$. Because the point $Q = Q_1$ can be chosen arbitrary close to $P$ we conclude that the set $Z$ is inside some finite collection of intervals of geodesics. Next, we verify that $Z$ is connected. Indeed, if not we may find two different points $P_1$ and $P_2$ from its different components and such that the minimal geodesic $L = L(P_1, P_2; t)$ connecting these points does not intersect $Z$. In some small ball $B$ around the middle point $Q$ of this geodesic the vector field $X$ will be non-zero with $X(Q)$ normal to $L$. Consider the "almost plane" $\Pi$ in $B$ going through the point $Q$ and normal to $L$. From (4) we see that the vector field $X$ in this plane not only is almost tangent to $\Pi$, but also almost tangent to small circles in $\Pi$ around $Q$. Which implies that the projection of $X$ on $\Pi$ has index $\pm 1$ at the center $Q$ of these circles, i.e., equals zero at $Q$. The obtained contradiction proves that $Z$ is a closed geodesic.\(^6\) Clearly, if some minimal geodesic $L$ connects two zeros $P$ and $Q$ from $Z$, but does not belong to $Z$ we may repeat arguments above to show that there exists one more point in the interior of $L$ where $X$ vanish. Which completes the proof of our theorem.

Note, that in our arguments we used only condition (4). Hence we have the following.

\(^6\)By the arguments above we immediately deduce that $Z$ is a connected geodesic. The fact that this geodesic can not be infinite follows from the compactness of $M$ and that it can not accumulate to something other than itself.
Corollary 1. If some smooth (not identically zero) vector field $X$ in some compact three-dimensional manifold $M$ satisfies (4), then its set of zeros $Z$ is a closed geodesic.

Next we consider the metric structures of $M$ when $Z$ is a closed geodesic.\(^7\)

3. $Z = S^1$

Assume that the set of zeros $Z$ of the vector field $X$ is some closed geodesic $Z = Z(t), 0 \leq t \leq 1$. Then $Z(t)$ is horizontal, its projection by the submersion $\pi$ to the soul $\Sigma$ is again a closed geodesic $\tilde{Z}(t)$ of the same length, for every $t$ the geodesic $l_t(s)$ connecting $\tilde{Z}(t)$ and $Z(t)$ is normal to $\Sigma$ with a direction $V(t)$ parallel along $\tilde{Z}(t)$. Now take an arbitrary point $Q$ in $M$ and connect it with all the points $Z(t)$ by minimal geodesics $L_t(s)$. All this geodesics are horizontal, and, if $\tilde{Q} = \pi(Q)$, their projections $L_t(s)$ are minimal geodesics connecting $\tilde{Q}$ with $\tilde{Z}(t)$. Also, if $V(t,s)$ denotes the (unit) vector of the direction of the (vertical) geodesic connecting $L_t(s)$ with $L_t(s)$, then $V(t,s)$ is parallel along $L_t(s)$. In particular, it follows that the parallel translation along a closed path from $\tilde{Q}$ to $\tilde{Q}$ consisting of two $L_t(s)$ and $L_t(s)$ and a part $L(t), t' \leq t \leq t''$ acts trivially on $V$ - the direction of $QQ$. Which by the prism construction from [M1-3] implies that the O’Neill’s fundamental tensor $\alpha$ vanishes at $Q$ for horizontal vectors tangent to the family of geodesics $L_t(s)$. As we already saw, this family belongs to the plane $\Pi(Q,X(Q))$ of all geodesics, issuing from $Q$ in directions normal to $X(Q)$. Thus we have

(6) \[ A_Y Z(Q) = 0 \]

for all $Y,Z$. Again, by the same prism construction we have

(7) \[ R(\tilde{Y}(t), \tilde{Z}(t))V(t) = 0 \]

along $\tilde{Z}(t)$, where $\tilde{Z}(t)$ is the unit tangent to $\tilde{Z}(t)$, $\tilde{Y}(t)$ any tangent to $\Sigma$, and $V(t)$ is the direction of the vertical geodesic $\tilde{Z}(t)Z(t)$.

Because $Q$ (and $\tilde{Q}$ correspondingly) was arbitrary, the tensor $\alpha$ vanishes identically in $M$ on vectors normal to the vector field $X$.

Theorem 2. Distribution in $M \setminus Z$ of the two-planes normal to the vector field $X$ is integrable. It is tangent to the family of totally geodesic spheres with a common intersection set - the closed geodesic $Z$.

Proof. Indeed, from (6) immediately follows that the Lie bracket of arbitrary fields $Y,Z$ orthogonal to $X$ is also orthogonal to $X$:

(8) \[ ([Y,Z],X) = ([Y,Z],W-(W,E)E) = ([Y,Z],W) = (A_Y Z - A_Z Y) = 0. \]

\(^7\)When $X$ has no zeros we may introduce the following $A$-"contact" structure on $M$. Denote by $\alpha$ the 1-form on $M^3$ given by the scalar product with a vector field $X$. Because the vector field $E$ on $N$ is parallel, from $\nabla^N E \equiv 0$ (here $\nabla^N$ is the covariant derivative in $N$ in a metric induced by $N \subset V$) we see that

(9) \[ (\nabla^N_Y X, Z) = (\nabla_Y W - (\nabla_Y W), E)E, Z) = (\nabla_Y W, Z) \]

for arbitrary $Y,Z$ tangent to $M$. Therefore, because $M$ is totally geodesic in $N$ we have by direct calculations that

\[ \alpha \wedge d\alpha = a(P)d\text{vol}^M, \]

where $d\text{vol}^M$ is the volume form of $M$ and the function $a(P)$ is given by

\[ a(P) = (A_Y Z, X) \]

or by $(A'_Y, Z', X')$ where $(X', Y', Z')$ an arbitrary orthonormal (positively orientated) basis in $T_P M$. The horizontal distribution on $M$ is not involutive outside zeros of $\alpha$. It would be interesting to find examples with $\alpha$ vanishing somewhere and nowhere zero $\alpha$. 

Thus, the vector field $X$ is, actually, the field of normals to some family of hyper-surfaces in $M$. But, as we already know, every geodesic $L_t$ connecting $Q$ with a point $Z(t)$ of $Z$ belongs to such a surface. From which, obviously, follows that this family of surfaces coincide with the family of our planes $\Pi(Q, X(Q))$. Because at $Q$ the second form of this surface vanish, and $Q$ is arbitrary our surfaces have vanishing second forms or are totally geodesic. For definiteness, from now on we call by $\Pi(Q)$ the union of all geodesics $L_t$ connecting $Q$ with $Z$. It is a totally geodesic surface which boundary is the closed geodesic $Z(t)$. Therefore, the vector $Y(t)$ tangent to $\Pi$ and normal to this boundary at the point $Z(t)$ is parallel along $Z$. Because the tangent vector $Z(t)$ to this geodesic is also (auto-)parallel, we see that the holonomy around $Z$ is trivial, i.e., parallel translation along $Z(t)$ is the identity operator. If we choose some parallel vector field $Y^*(y)$ along $Z$ normal to $Z(t)$, we can define the angle function $\psi$ for vectors $Y(t)$ normal to $Z(t)$ as the the angle between $Y(t)$ and $Y^*(t)$. Corresponding $\Pi(Q)$ we denote also by $\Pi(\psi)$. To complete the proof of the theorem we note that for (a half-sphere) $\Pi(\psi)$ there exists another one $\Pi(\psi + \pi)$ which normal to their common boundary $Z(t)$ equals $-Y(t)$. Their union in a neighborhood of $Z(t)$ is again an exponential image of planes tangent to $Y(t)$ and $Z(t)$, and therefore, is a smooth surface: a sphere which we denote by $S^2(Q)$, or by $S^2(\psi)$ (then $S^2(\psi) = S^2(\psi + \pi)$). Theorem 2 is proved.

Configuration we described in the last theorem is well-known and is called an open book decomposition.

**Corollary 2.** If the set of zeros of $X$ is a closed geodesic $Z(t)$, then $M$ admits an open book decomposition with a bending $Z$ and pages $\Pi(\psi)$ which are totally geodesic half-spheres.

Next we look more closely on the family of diffeomorphisms between pages $\Pi(\psi)$ of our open book decomposition given by shifts in directions normal to them. Let $f_\theta : \Pi(\psi) \to \Pi(\psi + \theta)$ denotes the map sending the point $Q$ in $\Pi(\psi)$ into the intersection of $\Pi(\psi + \theta)$ with an integral curve of the field of normals to pages issuing from $Q$. If we denote

$$\frac{\partial f_\theta(Q)}{\partial \theta} = X^*(f_\theta(Q)),$$

then the field $X^*$ is proportional to $X$, i.e., $X^*(Q) = k'(Q)X(Q)$ for some positive function on $M \setminus Z$. By $k$ we denote its norm: $k(Q) = \|X^*(Q)\|$. Because all pages $\Pi(\psi)$ are totally geodesic all maps $f_\theta$ are isometries. Therefore, we call the family of these isometries: "turning pages". If, in addition, $k$ is constant along trajectories of $X^*$ (or $X$, which is the same) then $f_\theta : M \to M$ is a family of isometries of the entire $M$,

and the vector field $X^*$ is a Killing vector field. Note also the following trivial statement.

**Lemma 3.** All trajectories of the vector field $X^*$ in $M$ are closed circles around $Z$.

**Proof.** Indeed, take some geodesic $L_t(s)$ in $S^2(\psi)$ connecting some $Q$ with the point $Z(t)$ which is nearest to it, and consider the orbit of this geodesic under our family of "rotations": $\Phi(s, \theta) = f_\theta(L_t(s))$. We choose natural parameter $s$ on $L_t(s)$ in such a way that $Z(t) = L_t(0)$. Because for every $\theta$ the curve $f_\theta(L_t(s))$ lies in the totally geodesic $\Pi(\psi + \theta)$ and $f_\theta$ is an isometry, this $f_\theta(L_t(s))$ is again the geodesic in $M$. Therefore, $\Phi(s, \theta)$ is a part of the "plane" $\Pi(Z(t), Z(t))$ of all geodesics issuing from $Z(t)$ in directions normal to $Z(t)$. For a fixed $s$ the line $\Phi(s, \theta)$ is a closed circle in this "plane".

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8"reading the book"

9It is interesting to note also the following property of these circles. As we will show, the vector field $X^*$ is Killing and constant along its trajectories, i.e., circles $f_\theta(Q)$. Therefore, the norm $k$ of $X^*$ attains its maximum on some set $Z^*$ which is invariant under rotations $f_\theta$. We claim that $Z^*$ is a collection of closed geodesics in $M$. Indeed, from $Y(X^*, X^*) \equiv 0$ for every $Y$ tangent to $S^2(\psi)$ at some point $Q$ of $Z^*$ it follows that $\nabla_{X^*} X^* \equiv 0$, or that the geodesic curvature of the orbit $f_\theta(Q)$ equals zero. Every closed geodesic from $Z^*$ is linked with $Z$.  

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4. $X^*$ is Killing

As we saw above, $A_Z$ vanishes along $Z$, see (6,7). Also the holonomy of the normal bundle is trivial along the projection $\bar{Z}(t)$ of $Z$ under submersion $\pi$ which is the closed geodesic in the soul $\Sigma$. Therefore, applying the simplified version of arguments\(^\text{10}\) from the proof of the Theorem A from [M5] (see section 5 there) we get

\begin{equation}
\nabla_W W \equiv 0 \quad \text{and} \quad R[W(t), Z(t)] \equiv 0
\end{equation}

for a unit vertical field $W(t)$ along $Z$.

Take another vector $V(\phi)$ normal to $\Sigma$ at $\bar{P} = \bar{Z}(0)$ with an angle $\phi$ to $V$. Its parallel transport along $\bar{Z}(t)$ is again $V(\phi)$. Denote by $V(\phi, t)$ the corresponding parallel vector field along $\bar{Z}(t)$. The vertical lifts of $\bar{Z}(t)$ into $N$ along this vector field are again closed geodesics which we denote by $Z(\phi, t)$. Easy to see that (6,7) are satisfied along them\(^\text{11}\), which in turn implies (10) along $Z(\phi, t)$, or that the vertical fibers of the submersion $\pi : Z(\phi, t) \to \bar{Z}(t)$ have zero geodesic curvature, or are geodesic lines in $N$. Hence, they coincide with projections of straight lines, i.e., $R$-factors under universal cover $M \times R \to N$. We formulate the obtained result as follows.

**Lemma 4.** The set of zeros of the vector field $X$ in $N$ is a tori which is the image of the direct product of $Z \subset M$ with a straight-line factor $R$ in the universal cover $\bar{N} = M \times R$ under covering map $\bar{N} \to N$. For an arbitrary choice of $M$ in $N$ the $\pi$-projection of the set of zeros of $X$ in $M$ is the same closed geodesic $\bar{Z}$ in $\Sigma$.

The obtained claim means that every vertical fibre in $N$ stays in the set of zeros of the vector field $X$ if it contains some of the point where $X$ vanish. Now we can repeat arguments from [GT] and prove the following statement.

**Theorem 3.** The vector field $X^*$ is Killing, if it has non-empty zero set.

**Proof.** Indeed, the Lemma 4’s claim enable us to repeat arguments from [GT]: for every point $Q$ of $M$ denote by $(f_i(Q), t)$ the points of the fiber of the submersion $\pi : N \to \Sigma$ issuing from $Q$. These are trajectories of the vector field $W$ in $N$. As we saw, the distance between $f_i(Q)$ and $f_i(F)$ is constant for every $F$ from the zero set $Z$ since the geodesic connecting them in $M \times \{t\}$ is horizontal. By the Lemma 4 we see $f_i$ is the identity map on $Z$. Therefore, $f_i(Q)$ are circles $S^1_{Q}$ around $Z$, they coincide with the circles $f_0(Q)$ which are orbits of the vector field $X^*$ above. To show that $X^*$ is Killing consider the cylinder $C_Q = \{(f_0(Q), t)\}$, (see Lemma 2.1 in [GT]). The restriction of $\pi$ on $C_Q$ is a Riemannian submersion of a flat cylinder onto some circle in $\Sigma$, or by [GG] has fibers tangent to some Killing field on $C_Q$. This proves that $X^*$ has constant norm along $C_Q$ and is a Killing vector field.

5. Proof of the Theorem B

From Theorem 3 it follows that the restriction of the Riemannian submersion $\pi : V \to \Sigma$ on $N$, which is the boundary of some s-metric neighborhood of the soul, can be described as the factoring by the action along trajectories of the Killing vector field $X^*$. From this fact the Theorem B follows in the same way as Theorem A; see section 3 in [GT] for the meticulous analysis of the cooperation between Killing vector fields $X^*$ on different s-metric neighborhoods of the soul which ensures the claim of both Theorems A and B.

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\(^{10}\)when $\text{codim}\Sigma = 2$ instead of $\text{codim}\Sigma = 3$.

\(^{11}\)for the proof note, that (7) implies (6) through the prism construction, see the Lemma 2 in [M5]
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