We study the effect of doping away from half-filling in weakly (but finitely) interacting $N$-leg Hubbard ladders using renormalization group and bosonization techniques. For a small on-site repulsion $U$, the $N$-leg Hubbard ladders are equivalent to a $N$-band model, where at half-filling the Fermi velocities are $v_1 = v_N < v_2 = v_{N-1} < \ldots$. We then obtain a hierarchy of energy-scales, where the band pairs $(j, N+1-j)$ are successively frozen out. The low-energy Hamiltonian is then the sum of $N/2$ (or $(N-2)/2$ for $N$ odd) two-leg ladder Hamiltonians without gapless excitations (plus a single chain for $N$ odd with one gapless spin mode) — similar to the $N$-leg Heisenberg spin-ladders. The energy-scales lead to a hierarchy of gaps. Upon doping away from half-filling, the holes enter first the band(s) with the smallest gap: For odd $N$, the holes enter first the nonbonding band $(N+1)/2$ and the phase is a Luttinger liquid, while for even $N$, the holes enter first the band pair $(N/2, N/2+1)$ and the phase is a Luther-Emery liquid, similar to numerical treatments of the $t$-$J$ model, i.e., at and close to half-filling, the phases of the Hubbard ladders for small and large $U$ are the same. For increasing doping, hole-pairs subsequently enter at critical dopings the other band pairs $(j, N+1-j)$ (accompanied by a diverging compressibility): The Fermi surface is successively opened by doping, starting near the wave vector $(\pi/2, \pi/2)$. Explicit calculations are given for the cases $N = 3, 4$.

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I. INTRODUCTION

Interest in the field of ladder systems grew rapidly, when the possibility of superconductivity in doped two-leg ladder materials was proposed [1]. Experimentally, such a behavior has been observed in a two-leg ladder material under high pressure [2]. The $N$-leg ladders have then been studied both as a first step towards the two-dimensional (2D) case [3] and because of their unusual physical properties. For example, spin-ladders exhibit an odd-even effect [4]: while even-leg ladders have a spin-gap, odd-leg ladders have one gapless spin mode. Similarly, the behavior upon doping is very different. The two-leg ladder is a Luther-Emery liquid, but the lightly doped three-leg ladder consists of a Luttinger liquid (LL) plus an insulating spin liquid (ISL) [5]. The latter is due to a truncation of the Fermi surface (FS) in two channels. Such a truncation is argued to take place also in the 2D case, where upon doping some parts of the FS become conducting and other parts remain an ISL [6].

Analytic works for (doped) ladders are usually based on the Hubbard model and restricted to a weak on-site repulsion $U$ (for large $U$, see Ref. [7]). Insulating behavior at half-filling is then due to umklapp processes. The two-leg ladder has been investigated by many authors over the past few years [8-10]. We note that in the lightly doped two-leg ladder paired holes form a dilute gas of hard-core bosons, which are equivalent to spinless fermions [11,12]. Away from half-filling, the phases of the weak interaction limit $U \rightarrow 0$ of the three- and four-leg ladder have been derived in Refs. [13,14] (for possible superconductivity, see Ref. [15]). For the three-leg ladder, a C2S1 phase has been obtained upon doping (a phase with $n$ gapless charge- and $m$ gapless spin modes is denoted by C$n$S$m$). This contrasts numerical findings for the (strongly interacting) $t$-$J$ model, where at low dopings the phase is CIS1 [16]. An idea to explain this discrepancy has been given in a related context, where a one-dimensional electron gas (1DEG) was coupled to an insulating “environment” (another 1DEG) [17].

In this paper, we study the doping $\delta$ away from half-filling of $N$-leg Hubbard ladders in the case of weak but finite repulsive interactions, $0 < U \ll t_\perp, t$ (and $t_\perp$ denote the hopping matrix elements along- and between the chains). Previous works have taken the limit $U \rightarrow 0$ first, keeping the doping $\delta$ finite [18]. Here, we treat the opposite limit, i.e., we keep $U$ finite and then investigate the lightly doped case $\delta \rightarrow 0$, where the umklapp interactions present at half-filling cannot be neglected. It is then advantageous to take the half-filled $N$-leg ladders as a starting point.

For a small $U$, the Hubbard-model reduces to a weakly interacting $N$-band model. After linearization around the Fermi points, each band is characterized by a Fermi velocity $v_j$; at half-filling $v_1 = v_N < v_2 = v_{N-1} < \ldots$. The low-energy physics is then derived by a combination of the renormalization group (RG) [19] and bosonization methods [20]. Integrating the RG equations (RGEs) at half-filling, we find a decoupling into band pairs $(j, N+1-j)$ and a hierarchy of energy-scales $T_j \sim t e^{-\alpha v_j/U}$ ($\alpha$ is a constant of the order of 1), where the band pairs scale towards a two-leg ladder fixed point and become frozen out (this behavior was not noted in previous works [12]; in particular, Ref. [12] concen-
it is advantageous first to diagonalize $H$ is the on-site repulsion. For weak interactions, one band (pair) is doped and then the case when many
in Sec. III, including the doping as a perturbation of the us to study the effect of doping away from half-filling:
The result of this section is a low-energy Hamiltonian allowing and the phases at half-filling are derived. The main re-
given for the cases
For open boundary conditions perpendicular to the legs, the dispersion relations $\epsilon_j$ are given by ($j = 1, \ldots, N$
where $\Psi_{js}$ are the creation- and annihilation operators for the band $j$.
\[
H_0 = \sum_{j,s} d^\dagger_{j\uparrow}(k)\Psi_{js}^{\dagger}(k)\Psi_{js}(k),
\]

where $\Psi_{js}$ and $\Psi_{js}$ are the creation- and annihilation operators for the band $j$.

\begin{equation}
\epsilon_j(k) = -2t \cos(k) - 2t_\perp \cos[\pi j/(N+1)].
\end{equation}

The Fermi momenta in each band $k_{Fj}$ are determined by the chemical potential $\mu = \epsilon_j(k_{Fj})$ and the filling
and the interaction term is
\[
H_{\text{int}} = U \sum_{j,x} d^\dagger_{j\uparrow}(x)d_{j\downarrow}(x)d^\dagger_{j\downarrow}(x)d_{j\uparrow}(x) + \text{H.c.}
\]

The Fermi momenta in each band $k_{Fj}$ are determined by the chemical potential $\mu = \epsilon_j(k_{Fj})$ and the filling
\begin{align}
v_j &= v_j = 2\sqrt{t^2 - \{t_\perp \cos[\pi j/(N+1)]\}^2},
\end{align}

where $j = N+1 - j$. Note that
\begin{align}
v_1 &= v_N < v_2 = v_{N-1} < \ldots .
\end{align}

These particular values of the Fermi velocities will lead to a hierarchy of energy scales (see below).

For generic Fermi momenta, the interactions are forward- ($f$) and Cooper- ($c$) scattering within a band or between two different bands. Following Ref. [13], the interacting part of the Hamiltonian can then be written as

\begin{equation}
H_0 = \sum_{j,s} d^\dagger_{j\uparrow}(k)\Psi_{js}^{\dagger}(k)\Psi_{js}(k),
\end{equation}

where $\Psi_{js}$ and $\Psi_{js}$ are the creation- and annihilation operators for the band $j$.

\begin{equation}
\epsilon_j(k) = -2t \cos(k) - 2t_\perp \cos[\pi j/(N+1)].
\end{equation}

The Fermi momenta in each band $k_{Fj}$ are determined by the chemical potential $\mu = \epsilon_j(k_{Fj})$ and the filling

$\Psi_{R/L, js}$ for right- and left movers at the Fermi level in each band.

At half-filling $n = 1$, $\mu = 0$ and the velocities are given by

\begin{align}
v_j &= v_j = 2\sqrt{t^2 - \{t_\perp \cos[\pi j/(N+1)]\}^2},
\end{align}

where $j = N+1 - j$. Note that

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\begin{equation}
\epsilon_j(k) = -2t \cos(k) - 2t_\perp \cos[\pi j/(N+1)].
\end{equation}

The Fermi momenta in each band $k_{Fj}$ are determined by the chemical potential $\mu = \epsilon_j(k_{Fj})$ and the filling $n = 2(\pi N)^{-1}\sum k_{Fj}$. Since we are only interested in the low-energy physics, we linearize the dispersion relation around the Fermi momenta, resulting in Fermi velocities $v_j = 2t \sin(k_{Fj})$. We introduce operators $\Psi_{R/L, js}$ for right- and left movers at the Fermi level in each band.

At half-filling $n = 1$, $\mu = 0$ and the velocities are given by

\begin{align}
v_j &= v_j = 2\sqrt{t^2 - \{t_\perp \cos[\pi j/(N+1)]\}^2},
\end{align}

where $j = N+1 - j$. Note that

\begin{align}
v_1 &= v_N < v_2 = v_{N-1} < \ldots .
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These particular values of the Fermi velocities will lead to a hierarchy of energy scales (see below).

For generic Fermi momenta, the interactions are forward- ($f$) and Cooper- ($c$) scattering within a band or between two different bands. Following Ref. [13], the interacting part of the Hamiltonian can then be written as
FIG. 2. The bands of the four-leg ladder correspond to four different points on a 2D FS (denoted by 1, 2, 3, and 4). The square is the umklapp surface, which is the FS at half-filling (for $t = t_1$). The half-circles indicate the opening of the FS, when the bands two and three are doped, forming a Luther-Emery (LE) liquid, where the pairing takes place between quasi-particles at (close to) the wave vectors $(2\pi/5, 3\pi/5)$ and $(3\pi/5, 2\pi/5)$. At $(4\pi/5, \pi/5)$ and $(\pi/5, 4\pi/5)$, the phase is still an ISL. Similarly as for the two-leg ladder, the order-parameter has a different sign in the two-leg case ($\rho, \sigma$) and two different three-band non-umklapp processes

$$I_{\text{int}} = \sum_{ij} \int dx \left( f_{ij}^{\rho,\sigma} J_{Rij} J_{Lij} - f_{ij}^{\rho,\sigma} J_{Rij} \cdot J_{Lij} \right) + c_{ij}^\rho J_{Rij} J_{Lij} - c_{ij}^\rho J_{Rij} \cdot J_{Lij} \right) + \sum_j \int dx \left( c_{ij}^{\rho,\sigma} J_{Rij} J_{Lij} - c_{ij}^{\rho,\sigma} J_{Rij} \cdot J_{Lij} \right), \quad (7)$$

where the U(1) and SU(2) currents are defined as

$$J_{\text{int}} = \sum_s \Psi_{\text{his}}^s \Psi_{\text{hj}}^s \quad \text{(8)}$$

and

$$J_{\text{int}}^k = \frac{1}{2} \sum_{ss'} \Psi_{\text{his}}^{s,s'} \Psi_{\text{hj}}^{s,s'}, \quad (9)$$

where again $h = R/L$ and the $\tau^k$ are the Pauli matrices ($k = x, y, z$). Due to symmetry, $f_{ij}^{\rho,\sigma} = f_{ji}^{\rho,\sigma}$ and $c_{ij}^{\rho,\sigma} = c_{ji}^{\rho,\sigma}$.

In the half-filled case we must generalize the interacting Hamiltonian. The Fermi momenta of the bands $j$ and $j$ add up to $\pi$, $k_{Fj} + k_{Fj} = \pi$, allowing umklapp interactions to take place between these two bands (for $u_{j,j}^{\rho,\sigma}$ we differ in notation from Ref. [11] by a factor 2),

$$u_{j,j}^{\rho,\sigma} \left( I_{Rij}^\dagger I_{Lij} + \text{H.c.} \right) - u_{j,j}^{\rho,\sigma} \left( I_{Rij}^\dagger \cdot I_{Lij} + \text{H.c.} \right) + u_{j,j}^{\rho,\sigma} \left( I_{Rij}^\dagger I_{Lij} + I_{Rij}^\dagger \cdot I_{Lij} + \text{H.c.} \right). \quad (10)$$

We have used the definitions

$$I_{\text{int}} = \sum_{s,s'} \Psi_{\text{his}}^s \epsilon_{ss'} \Psi_{\text{hj}}^{s'}, \quad (11)$$

and

$$I_{\text{int}}^k = \frac{1}{2} \sum_{s,s'} \Psi_{\text{his}}^s (\epsilon \tau^k)_{ss'} \Psi_{\text{hj}}^{s'}, \quad (12)$$

where $\epsilon = -i\tau^y$. For odd $N$, there exists a single-band umklapp term for the band $r = (N + 1)/2$

$$u_{rr}^\rho \left( I_{Rrr}^\dagger I_{Lrr} + \text{H.c.} \right), \quad (13)$$

In addition to these processes, there exist various interactions involving four different bands or three different bands (only for $N$ odd). For the half-filled three-leg ladder, three different three-band umklapp processes

$$u_{1223}^\rho \left( I_{R12}^\dagger I_{L23} + I_{R23}^\dagger I_{L12} + \text{H.c.} \right) + u_{231}^\rho \left( I_{R23}^\dagger I_{L13} + I_{R13}^\dagger I_{L22} + \text{H.c.} \right) - u_{1223}^\sigma \left( I_{R12}^\dagger \cdot I_{L23} + I_{R23}^\dagger \cdot I_{L12} + \text{H.c.} \right) \quad (14)$$

and two different three-band non-umklapp processes

$$c_{1223}^\rho \left( J_{R12} \cdot J_{L23} + J_{R23} \cdot J_{L12} + \text{H.c.} \right) - c_{1223}^\sigma \left( J_{R12} \cdot J_{L23} + J_{R23} \cdot J_{L12} + \text{H.c.} \right) \quad (15)$$

take place and have to be added to Eq. (10).

In the four-leg case, the four-band umklapp interactions read

$$u_{1234}^\rho \left( I_{R12}^\dagger I_{L34} + I_{R34}^\dagger I_{L12} + \text{H.c.} \right) + u_{1324}^\rho \left( I_{R12}^\dagger \cdot I_{L34} + I_{R34}^\dagger \cdot I_{L12} + \text{H.c.} \right)$$

$$+ u_{1234}^\sigma \left( I_{R12}^\dagger I_{L34} + I_{R34}^\dagger I_{L12} + \text{H.c.} \right) - u_{1324}^\sigma \left( I_{R12}^\dagger \cdot I_{L34} + I_{R34}^\dagger \cdot I_{L12} + \text{H.c.} \right) \quad (16)$$

and the four-band non-umklapp interactions take the form

$$c_{1234}^\rho \left( J_{R12} \cdot J_{L34} + J_{R34} \cdot J_{L12} + \text{H.c.} \right) + c_{1324}^\rho \left( J_{R12} \cdot J_{L34} + J_{R34} \cdot J_{L12} + \text{H.c.} \right) - c_{1324}^\sigma \left( J_{R12} \cdot J_{L34} + J_{R34} \cdot J_{L12} + \text{H.c.} \right) \quad (17)$$

In our case $v_1 = v_4$ and $v_2 = v_3$ implying $c_{1234}^\rho = c_{1324}^\rho$, $u_{1234}^\rho = u_{1324}^\rho$. The generalization to $N > 4$ is straightforward such that we do not reproduce it here.

The (initial) values of the couplings are of the order of the on-site repulsion $U$ (for $N = 3$, see Appendix A).
B. Low-energy phases

Using the RG method (and bosonization), we derive the low-energy phases at half-filling and a Hamiltonian allowing us to study the effect of doping, i.e., doping is then included as a (small) perturbation. We have generalized the results given in Refs. [11,13] in order to treat ladders with $N > 2$ legs at half-filling [23]. For weak interactions, $U \ll t$, it is sufficient to take the RGEs in one-loop order, i.e., quadratic in the coupling constants.

Integrating the RGEs, we find, that the Fermi velocities $v_j$ [see Eqs. (3) and (4)] lead to a hierarchy of energy-scales by $T_j \sim t e^{-\alpha v_j/U}$, where the band pairs $(j,j)$ become successively frozen out. This exponential dependence on $v_j/U$ is a consequence of the logarithmic derivatives of the couplings with respect to the energy in the one-loop RGEs. As we show below, the low-energy Hamiltonian is then the sum of $N/2$ ($(N−1)/2$ for $N$ odd) two-leg ladder Hamiltonians corresponding to the band pairs $(j,j)$ (plus the Hamiltonian of a single chain for $N$ odd).

1. Three-leg ladder

It is instructive to start with the RG flow of the three-leg ladder in the limit $v_1 = v_3 \ll v_2$ (i.e., $t_⊥/t \to \sqrt{2}$), where we can neglect in the RGEs all contributions, where contractions over the band 2 take place (see Appendix A). We then obtain the following results. First, the renormalization of the two- and single-band interactions of the bands 1 and 3 is exactly the same as for a two-leg ladder, where the couplings flow towards universal ratios (for a comparison, see Ref. [11])

$$0 < 4g_{13} = 4c_{13}^0 = 4f_{13}^0 = c_1^0 = -c_{11}^0 = -c_{33}^0 = 4u_{1331}^0 = 8u_{1133}^0 = u_{1223}^0,$$  
and

$$c_{11}^0 = c_{33}^0 \approx f_{13}^0 \approx 0,$$

and become of the order of the bandwidth $t$ at the temperature scale $T_1 \sim t e^{-\alpha_1 v_1/U}$ ($\alpha_1$ is a constant of the order of 1). Second, the interactions between the bands 1 and 2 (and 2 and 3) are either not renormalized or flow to 0. Finally, the scaling of the remaining three-band interactions can be calculated as follows (note that $u_{2223}^0$ stays always small). Defining $\mathbf{v} = (c_{1223}^0, c_{1223}^0, u_{1223}^0, u_{1223}^0)$ and

$$M = \begin{pmatrix}
2 & 0.75 & 3 & -0.75 \\
4 & 0 & 4 & -1 \\
3 & 0.75 & 2 & -0.75 \\
-4 & -1 & -4 & 0
\end{pmatrix},$$

we rewrite the RGEs as (the scaling variable $l$ is related to the energy by $T \sim t e^{-\pi l}$, see Appendix A)

$$(v_1 + v_3) \frac{dv}{dl} = g_{13}(l)M \mathbf{v},$$

where, using Eq. (18), we have expressed the interactions of band 1 and 3 in terms of $g_{13}$. The eigenvalues of $M$ are $(7, -1, -1, -1)$ and

$$\frac{1}{v_1 + v_3} \int_{l_0}^{l} g_{13}(l')dl' = \frac{1}{12} \ln \left[ \frac{g_{13}(l)}{g_{13}(l_0)} \right],$$

where $l_0 < l$ denotes the scale at which the interactions of the bands 1 and 3 are sufficiently close to the two-leg ladder ratios. The three-band interactions therefore stay in fixed ratios and

$$\frac{c_{1223}(l)}{g_{13}(l)} = \frac{c_{1223}(l_0)}{g_{13}(l_0)} \left[ \frac{g_{13}(l_0)}{g_{13}(l)} \right]^{5/12} \propto \left[ \frac{U}{g_{13}(l)} \right]^{5/12}$$

such that the three-band interactions are arbitrary small at the scale where $g_{13} \sim t$.

![FIG. 3. The figure shows (for $t = t_⊥$) the strong coupling value as a function of the initial value $U/t$, where we fix the strong coupling value of $c_{13}^0$ at the bandwidth $t$. The three-band coupling $c_{1223}^0$ then decreases to 0 (the flow of the other three-band couplings is similar), as the initial value is reduced, and the coupling $c_{13}^0$ approaches its asymptotic value, $c_{13}^0 = c_{13}^0/4$. We note that also for the half-filled two-leg ladder [11], the flow to the asymptotic ratios is very slow, i.e., one has to take very small initial values $U/t$.](image)

For other ratios of the velocities $v_1/v_2 < 1$, we have performed a numerical integration of the RGEs: Plotting the strong coupling value as a function of the initial value $U/t$ (we fix the strong coupling value of $c_{13}^0$ at the bandwidth $t$), we find that the couplings given in Eq. (18) always grow up to the bandwidth $t$, approaching their universal ratios, while all the other couplings stay at an order of magnitude smaller, i.e., the asymptotic behavior of the couplings for $U/t \to 0$ is the same as for $v_1 \ll v_2$ (see Fig. 3). We note that in Ref. [11] a flow to strong coupling of the three-band interactions has been found.
for $t_\perp/t < 0.86$; we only obtain that for $v_1 \approx v_2$, i.e., in the strongly anisotropic limit $t_\perp/t < 0.2$.

To conclude our RG analysis of the three-leg ladder: The couplings of the bands 1 and 3 scale towards the two-leg ladder ratios, become of the order of the bandwidth $t$ at the temperature scale $T_1 \sim t e^{-a_1 v_1/U}$, and at (and below) $T_1$, the bands 1 and 3 are decoupled from the band 2. The Hamiltonian becomes therefore $H = H_{13} + H_s$. The first term $H_{13}$ is the Hamiltonian of a two-leg ladder (see Ref. [1]) and the second term $H_s$ is the Hamiltonian of a single “chain”. At the scale $T_1$, the couplings of $H_{13}$ are of the order of $t$ and all the charge and spin-modes of the bands 1 and 3 acquire a gap (this result is obtained by bosonization, see Ref. [1]). The couplings of $H_s$ are still small at $T_1$, i.e., $0 < c^\rho_{22}, c^\delta_{22} \ll t$, such that the charge and spin-modes of band 2 remain gapless.

For a (small but) finite $U$, we then have to investigate the system at energies below $T_1$. For the Hamiltonian $H_s$ it is well-known that it has a charge gap below a temperature $T_2 \sim t e^{-a_2 v_2/U}$ (since $a_2 \approx a_1$ we write in the following $a$ for both of them) and that the spin-excitations remain gapless [24]. Since the two-leg part has no gapless mode, the final phase is COS1 — the same as for the strongly interacting limit, i.e., a three-leg spin-ladder.

2. Four-leg ladder

For the four-leg ladder, we find a similar behavior. The couplings of the bands 1 and 4 grow, become of the order of the bandwidth $t$ at the energy scale $T_1 \sim t e^{-a_1 v_1/U}$, and stay in the same ratio as for a two-leg ladder, while all the other couplings remain small. In the limit $v_1 = v_4 \ll v_2 = v_3$, the RGEs of the four-band couplings become the same as for the three-band couplings and consequently, similarly as above, the four-band interactions stay in fixed ratios and

$$c^\rho_{1234}/c^\rho_{14} \propto (U/c^\rho_{14})^{5/12}.$$  \hspace{1cm} (24)

At the temperature scale $T_1$, the bands 1 and 4 are therefore decoupled from the bands 2 and 3. Again, for a finite $U$, we then have to study the flow of the couplings of band 2 and 3 at energies below $T_1$. We find that they flow at the scale $T_2 \sim t e^{-a_2 v_2/U}$ also to a two-leg ladder fixed point (the flow to the two-leg ladder fixed point takes place for a wide range of initial values, also for non-Hubbard-like). The (bosonized) Hamiltonian is then the sum of two two-leg ladder Hamiltonians, $H = H_{14} + H_{23}$. All charge and spin excitations are therefore gapped (as it is the case in the Heisenberg four-leg spin ladder).

3. The case $N > 4$

Analyzing the RGEs for $N > 4$, we conjecture that this successive freezing out of band pairs $(j,j)$ (plus a single band for $N$ odd) at the characteristic energies $T_j \sim t e^{-a_j v_j/U}$ takes place also for (small) $N > 4$ with $U \ll t/N$. Again, the $T_j$ are a result of the one-loop RGEs: the couplings of the band pair $(j,j)$ scale towards a two-leg ladder fixed point and become of the order of the bandwidth $t$ at $T_j$. Note that we have a hierarchy of energy-scales

$$T_1 > T_2 > \ldots > T_r,$$ \hspace{1cm} (25)

where for $N$ even $r = N/2$ and for $N$ odd $r = (N + 1)/2$. To conclude this section: At low energies the Hubbard-ladder Hamiltonian becomes for $N$ even the sum of $N/2$ two-leg ladder Hamiltonians and for $N$ odd the sum of $(N - 1)/2$ two-leg ladder Hamiltonians plus the Hamiltonian of a single chain

$$H = \sum_j H_j + \delta_{N,\text{odd}} H_s,$$ \hspace{1cm} (26)

where for $N$ odd, the single chain [the band $(N+1)/2]$ has the lowest energy-scale and for $N$ even the two-leg ladder Hamiltonian $H_{N/2,N/2+1}$ [corresponding to the band pair $(N/2, N/2 + 1)$]. The (half-filled) two-leg ladder Hamiltonians have no gapless excitations and the single-chain Hamiltonian present for odd $N$ has only one gapless spin-mode. The phases at half-filling are thus the same as for the Heisenberg spin-ladders [3]. Since the Hubbard model converges for large $U$ onto the $t$-$J$ model (which is at half-filling equivalent to the Heisenberg model) [25], the phases of the half-filled $N$-leg Hubbard ladders are the same for small and large $U$.

We would like to emphasize that we have obtained the final result, Eq. (26), describing decoupled band pairs, from an explicit analysis of the coupled RGEs of the interacting $N$-band problem.

III. HOLE DOPED LADDERS

Since the low-energy cutoff is typically of the order of $t e^{-1/U}$, the effect of the umklapp interactions has to be taken into account for low dopings, $\delta < e^{-1/U}$. It is then advantageous to take the half-filled ladder as a starting point to study the effect of doping. Similarly as done in Refs. [22, 24], we model the doping by adding a chemical potential term $-\mu Q$, where $Q$ is the total charge.

In a single chain, a hole is decomposed in two quasi-particles, one with charge but no spin (holon, here the fermionic “kink”) and the other with spin but no charge (spinon). The quasi-particles of a two-leg ladder are bound hole-pairs (singlet of charge 2). At low doping, these hole-pairs behave as hard-core bosons, which are equivalent to spinless fermions. In Sec. III A, we show that the lightly doped $N$-leg ladders can indeed be described by an effective $N/2$-band $[(N+1)/2]$-band for $N$
odd] model of spinless fermions. We then study the case when only one of these bands is doped. In Sec. III B, we increase the hole doping such that many bands become subsequently doped.

A. One band (pair) doped

For the following discussion, we use the *bosonized* form of the Hamiltonian \( \{23\} \). Bosonization \( \{19,20\} \) is the method of rewriting fermionic creation (annihilation) operators in terms of field operators \( \Phi_\alpha \) and \( \Pi_\alpha \) satisfying the commutation relation \([\Phi_\alpha(x), \Pi_\alpha(y)] = i\delta(x - y)\) (\( \alpha \) labels the charge and spin modes for the different bands; for the application of bosonization to ladders in the limit \( U \ll t, t_\perp \), see, e.g., Refs. \( \{11,13,20\} \)). It is convenient to introduce the dual field of \( \Phi_\alpha \), \( \partial_x \Phi_\alpha = \Pi_\alpha \). For fermions on a chain or ladder, bosonization applies in the continuum limit.

It is straightforward to generalize the effective Hamiltonian for the half-filled two-leg ladder obtained in Ref. \( \{24\} \) to \( N/2 \) \( [(N - 1)/2 \) for \( N \) odd] two-leg ladder Hamiltonians. We introduce the fields

\[
\Phi_{+j} = \frac{1}{2} (\Phi_{\rho j} + \Phi_{\rho j}^*) \quad \text{and} \quad \Pi_{+j} = \Pi_{\rho j} + \Pi_{\rho j}
\]

combining the charge fields of the band pair \( (j, j) \). Dropping bare energy terms, the Hamiltonian then reads

\[
H = \sum_j \int dx \left\{ \frac{u_{+j}}{2} \left[ \frac{2}{K_{+j}} (\partial_x \Phi_{+j})^2 + \frac{K_{+j}}{2} \Pi_{+j}^2 \right] - g_{+j} \cos(\sqrt{8\pi \Phi_{+j}}) - 2\mu \partial_x \Phi_{+j} \right\},
\]

(28)

plus for odd \( N \) the Hamiltonian of a single chain (dropping the gapless spin-part)

\[
\int dx \left\{ \frac{u_{\rho j}}{2} \left[ \frac{1}{K_{\rho j}} (\partial_x \Phi_{\rho j})^2 + K_{\rho j} \Pi_{\rho j}^2 \right] - g_{\rho j} \cos(\sqrt{8\pi \Phi_{\rho j}}) - \mu \partial_x \Phi_{\rho j} \right\}.
\]

(29)

The coupling \( g_{+j} \) is of the order of \( y_{jj} \), the \( K_\alpha \) are the Luttinger liquid parameters (at half-filling, \( K_{+j} \to 1 \) and \( K_{\rho j} \to 1/2 \) \( \{11,13,20\} \)), and the \( u_\alpha \) are the velocities of the charge modes. The charge (doping) is given by

\[
Q = \sqrt{\frac{2}{\pi}} \int dx \sum_j 2\partial_x \Phi_{+j} + \delta_{N,\text{odd}} \partial_x \Phi_{\rho j}.
\]

(30)

We note that \( \{28\} \) and \( \{24\} \) are the usual Hamiltonians describing a commensurate-incommensurate transition \( \{27\} \). The fields of the band pair \( (j, j) \), \( \Phi_{+j} \) (and of the single chain \( \Phi_{\rho j} \)), become pinned at the energy-scales \( T_j \sim \epsilon \sim \alpha e^{-\alpha e_j}/U \). We therefore define effective gaps for the bands by \( \Delta_j \sim T_j \), resulting in a hierarchy of gaps (see Eq. \( \{25\} \))

\[
\Delta_1 > \Delta_2 > \ldots > \Delta_r.
\]

(31)

The equations of motion following from the Hamiltonians \( \{28\} \) and \( \{24\} \) are a set of sine-Gordon (SG) equations. Here, the physical solutions of the SG equations are the soliton-solutions. Solitons are fermionic particles (kinks). The kinks of the single-chain field \( \Phi_{\rho j} \) have a mass \( M_\rho \sim \Delta_\rho \), and represent the charge part of single holes, while the kinks of the two-leg ladder fields \( \Phi_{+j} \) have a mass \( M_j \sim \Delta_j \), and represent paired holes (singlets). The band structure of these solitons reads

\[
\epsilon_j(k) = \sqrt{M_j^2 + (v_j k)^2}.
\]

(32)

At low doping \( \delta < e^{-t/U} \), the half-filled \( N \)-leg ladders are therefore described by an effective \( N/2 \)-band \( [(N + 1)/2 \) band] model of spinless fermions, where the bottom of the bands are at the energy-scales \( \Delta_j \) (for \( N = 3 \), see Fig. \( 4 \)).

![FIG. 4. The lightly doped three-leg ladder can be described by a two-band model of spinless fermions, where the lower band II contains the charge part of holes and the upper band I paired holes (singlets). At half-filling the phase is C0S1; upon doping, the charge part of the holes enters the band II and the phase becomes C1S1. For dopings \( \delta > \delta_c \), with \( \delta_c = \sqrt{\pi} \), paired holes enter also the band I and the phase becomes C2S1.](image)

At half-filling, \( \mu = 0 \) and minimizing the energy gives \( \Phi_{+j} \approx \Phi_{\rho j} \approx 0 \). When doping the first hole (pair), the chemical potential \( \mu = \mu(\delta) \) jumps from 0 to \( \Delta_r \) and a soliton-solution with charge \( Q > 0 \) minimizes the energy, i.e., the lowest-lying effective band of spinless fermions becomes doped. For odd \( N \), the lowest-lying band is the single-chain band \( (N + 1)/2 \) and the phase upon doping is a dilute gas of spinons and kinks (C1S1), while for even \( N \), the effective band \( N/2 \), corresponding to the two-leg-ladder band-pair \( (N/2, N/2 + 1) \), is doped first and the phase consists of singlets of paired holes (C1S0).
Similarly as for the two-leg ladder, the (singlet) pair-field operator

\[
P_j \propto \Psi_{Rj} \Psi_{Lj} \Psi_{Lj} + \Psi_{Lj} \Psi_{Rj} \Psi_{Rj} \nonumber
\]

\[
\propto \exp \left( -i \sqrt{2 \pi \theta_{pj}} \right) \cos \left( \sqrt{2 \pi \theta_{pj}} \right)
\]  \hspace{1cm} (33)

has a different sign in band \(j\) and \(j\) \[11\],

\[
P_j^\dagger P_j < 0,
\]  \hspace{1cm} (34)

such that the C1SO phase has a \(d\)-wave like symmetry.

It is instructive to rewrite the band operators \(\Psi_{hjs}\) respectively the pair-field operators \(P_j\) in terms of the chain operators \(d_{hjs}\).

For \(N = 3\),

\[
\Psi_{h2s} = \frac{1}{\sqrt{2}}(d_{h1s} - d_{h3s}) \tag{35}
\]

and we find that the holes are situated on the outer legs. Both the phase and the location of the holes is in agreement with previous numerical treatments \[14, 15\].

For \(N = 4\), using \(P_2 P_3^\dagger \approx -1\), we obtain

\[
P_2 \propto 0.22(d_{R1 \perp} d_{L2L} + d_{R2 \perp} d_{L1L} + R \leftrightarrow L) + 0.22(d_{R3 \perp} d_{L4L} + d_{R4 \perp} d_{L3L} + R \leftrightarrow L) - 0.36(d_{R1 \perp} d_{L4L} + d_{R4 \perp} d_{L1L} + R \leftrightarrow L) - 0.14(d_{R2 \perp} d_{L3L} + d_{R3 \perp} d_{L2L} + R \leftrightarrow L),
\]  \hspace{1cm} (36)

such that the singlets are on the top two legs, the bottom two legs, on the legs 1 and 4, and with lowest probability on the legs 2 and 3, similarly as found in Ref. \[11\] for the \(t-J\) model. Finally, also the phases for \(N = 5, 6\) are in agreement with numerical works \[4\]. Since the \(t-J\) model is the large \(U\) limit of the Hubbard model \[27\], we conclude that at and close to half-filling the phases of the \(N\)-leg Hubbard ladders are the same for small and large \(U\). For the two-leg ladder, such a "universal" behavior has already been noted in Ref. \[13\].

### B. Many bands doped

When increasing doping, the chemical potential \(\mu = \mu(\delta)\) increases too and the hole-pairs enter subsequently also the higher-lying (effective) bands of spinless fermions. We thus obtain a series of critical dopings \(\delta_{cj}\), where the first hole-pair enters the band \(j\). For \(N = 3, 4\)

\[
\delta_{c1} \sim \sqrt{e^{-2 \alpha v_1/U} - e^{-2 \alpha v_2/U}} \approx e^{-\alpha v_1/U}. \tag{37}
\]

Since for \(\delta = \delta_{cj}\), \(\partial \mu / \partial \delta = 0\), the compressibility

\[
\kappa \propto \frac{\partial \delta}{\partial \mu} \tag{38}
\]

diverges. The phase transitions at \(\delta_{cj}\) belong to the same universality class as the commensurate-incommensurate transition \[27\]. We note that our qualitative estimate for \(\delta_{cj}\) does not allow for a comparison with numerical works.

The effective band \(j\) corresponds to the band pair \((j, j)\), such that for a given doping, the band pairs \((N/2, N/2 + 1), \ldots, (j, j)\) are conducting, while the band pairs \((j - 1, j + 1), \ldots, (1, N)\) form still an ISL. Next, we interpret this result by mapping each band (pair) on a 2D FS, see Figs. \[1\] and \[2\]. Using the dispersion relation \[1\], the longitudinal Fermi momentum of the band \(j\) is given by

\[
k_{Fj} = \pi - \arccos \left[ \frac{t_{\perp}}{t} \cos \left( \frac{\pi j}{N+1} \right) \right] \tag{39}
\]

and the corresponding transverse Fermi momentum reads

\[
k_{Fyj} = \frac{\pi j}{N+1}. \tag{40}
\]

Upon doping, the holes enter (for \(t_{\perp} \approx t\)) then first near the wave vector \((\pi/2, \pi/2)\) and finally for increasing doping also near \((\pi, 0)\) and \((0, \pi)\). The FS thus becomes truncated, similarly as in the 2D case studied in Ref. \[9\].

For increasing doping, a crossover takes place to the situation away from half-filling, where the umklapp processes can be neglected. It is instructive to study the phases for \(N = 3, 4\), when both effective bands of spinless fermions are doped and to compare with the phase away from half-filling. In the case \(N = 3\) (see Fig. \[3\], the different charges in the effective bands I and II forbid scattering processes of the form

\[
\Psi_{R1}^\dagger \Psi_{R1} \Psi_{L1}^\dagger \Psi_{L1} + I \leftrightarrow II, \tag{41}
\]

where \(\Psi_{R/Lb}^\dagger\) creates a spinless fermion in band \(b = I/II\), since they break \(U(1)\) invariance and only the interactions

\[
\Psi_{R1}^\dagger \Psi_{R1} \Psi_{L1}^\dagger \Psi_{L1} + I \leftrightarrow II \tag{42}
\]

are allowed. For weak interactions, we can bosonize Eq. \[13\] resulting in a density term of the form \(\partial_x \Phi \partial_x \Phi\) (plus the same for the dual fields \(\theta_{1,11}\)) This term does not reduce the number of gapless modes such that the phase for \(\delta > \delta_c\) is C2S1. The same phase has been found by the RG and bosonization methods away from half-filling \[14, 15\]. This contrasts numerical calculations for the strongly interacting \(t-J\) model \[3\], where a spin-gapped phase has been found away from half-filling. Such a phase maybe appears also for weak interactions, if the condensation to the low-energy phase takes place for all bands at the same energy-scale (for another idea, see Ref. \[17\]).

For \(N = 4\), we have two types of paired holes with the same charge in the ladder, localized along the rungs. In contrast to the three-leg case, scattering processes of
the form \( \{41\} \) are allowed. We have shown in Ref. \[28\], that in such a two-band model of spinless fermions with repulsive interactions, slightly away from the band edge, binding takes place and the C2S0 phase close to the band edge becomes a 4-hole C1S0 phase. Away from half-filling, by RG and bosonization, the phase C3S2 has been found \[13\]. Similarly as already noted in Ref. \[17\], we argue that the C3S2 phase is a result of the \( U \to 0 \) limit considered in Ref. \[13\]; and indeed, our phases agree with a numerical treatment of the \( t-J \) four-leg ladder \[21\]. At low doping, the authors obtained a phase of paired holes and at higher doping levels, four-hole clusters (or two-pair clusters). We therefore argue that for increasing doping the phase remains \( U \)-independent in even-leg ladders, but not in odd-leg ladders.

We finally note that including next nearest neighbor hopping terms etc., the Fermi momenta of band pairs add up no more exactly to \( \pi \), i.e., \( v_j = v_j \). However, since the phases are controlled by the relative size of gaps, only a sufficiently large perturbation can qualitatively change the corresponding physics.

IV. CONCLUSIONS

We have studied the doping away from half-filling in \( N \)-leg Hubbard ladders in the case of a small on-site repulsion \( U \), where the ladders are equivalent to a \( N \)-band model. At half-filling, the \( N \) Fermi velocities are such that \( v_1 = v_N < v_2 = v_{N-1} < \ldots \). Using RG and bosonization techniques, we have obtained a hierarchy of energy-scales \( \Delta_j \sim e^{-\alpha v_j/U} \), where the couplings of the band pair \( (j, j) \) and the single band \( (N + 1)/2 \) for \( N \) odd are of the order of the bandwidth \( t \), decoupled from the other bands, and at a two-leg ladder fixed point. The low-energy Hamiltonian is then the sum of \( N/2 \) \( [(N-1)/2 \) for \( N \) odd] two-leg ladder Hamiltonians, corresponding to the band pairs \( (j, j) \), without gapless excitations (plus for odd \( N \) the Hamiltonian of a single chain with one gapless spin-mode). We thus obtain the same phases as for the Heisenberg spin-ladders.

Upon doping, the holes enter first the band (pair) with the smallest gap \( \Delta_j \). In odd-leg ladders, this is the nonbonding band \( (N + 1)/2 \) and the phase is a dilute gas of holons (here, the fermionic kinks) and spinors (C1S1), while in even-leg ladders, the band pair \( (N/2, N/2 + 1) \) is doped first and the phase consists of singlets of paired holes (C1S0), reflecting the odd-even effect at half-filling (i.e., of Heisenberg spin-ladders). The same phases have been found in numerical works for the \( t-J \) model \[13\], \[14\], \[22\], i.e., at and close to half-filling, the phases of the \( N \)-leg Hubbard ladders are the same for small and large \( U \).

For a finite \( U \) and at low doping, the ladders can be described by an effective \( N/2 \)-band model \( [(N + 1)/2 \)-band model for \( N \) odd] of spinless fermions, where the (effective) band \( j \) corresponds to the band pair \( (j, j) \) and the bottoms of these bands are at the energy-scales \( \Delta_j \). Upon doping, the hole-pairs then subsequently enter these bands and the FS is successively opened, first near the wave vector \( \pi/2, \pi/2 \) and finally also near \( (\pi, 0) \) and \( (0, \pi) \), similarly as in the 2D case \[8\]. In addition, we obtain peaks in the compressibility at critical dopings \( \delta_{j,j} \), where the first hole-pair enters the band pair \( (j, j) \).

For increasing doping, the \( U \)-independence of the phase seems to break down, at least for odd-leg ladders. However, the \( U \)-independence and the similarity to the 2D case at and close to half-filling support the hope that also in 2D, some of the weak-coupling RG results can be extended to large \( U \) \[8\].

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APPENDIX A: RENORMALIZATION GROUP

The renormalization group (RG) method is a controlled way of subsequently eliminating (integrating out) high-energy modes in a given Hamiltonian. While the noninteracting part \( H_0 \) is at a RG fixed point, couplings of the interacting part \( H_{\text{int}} \) may grow or decrease under a RG transformation (i.e., when lowering the energy). A subsequent (perturbative) elimination of high-energy modes in \( H_{\text{int}} \) then results in RGEs, which give the change of the coupling when lowering the energy, see Ref. \[13\].

Below, we give the RGEs for the couplings of the three-leg ladder. Fully taking into account the symmetries [i.e., \( SU(2), (U(1), \text{and } v_1 = v_3) \), there are 21 independent RGEs. Using operator product expansion for the products of the various currents, the derivation is straightforward \[24\]. Similar RGEs are obtained in the case \( N > 3 \) \[24\].

For the half-filled three-leg ladder \( v_1 = v_3 \), such that \( f_{12}^\rho = f_{23}^\rho, c_{12}^\rho = c_{23}^\rho, \) and \( c_{11}^\rho = c_{33}^\rho \). The initial values of the two and single band couplings are given by

\[ f_{13}^\rho = c_{11}^\rho = c_{33}^\rho = \frac{3U}{16}, \quad f_{12} = c_{12}^\rho = \frac{U}{8}, \quad c_{22} = \frac{U}{4}, \]

\[ \omega_{1331} = \frac{3U}{8}, \quad \omega_{1133} = \frac{3U}{32}, \quad \omega_{1331} = 0, \quad \omega_{22} = \frac{U}{8}. \] \hspace{1cm} \text{(A1)}

and \( f_{ij}^\rho = 4 f_{ij}^\rho, c_{ij}^\rho = 4 c_{ij}^\rho \). The initial values of the three-band couplings read

\[ c_{1223} = \frac{U}{8}, \quad c_{2223} = \frac{U}{2}, \]

\[ \omega_{1223} = \frac{U}{4}, \quad \omega_{2213} = \frac{U}{8}, \quad \omega_{1223} = 0. \] \hspace{1cm} \text{(A2)}
The RGES take the following form, where the energy-scale is related to $l$ by $T \sim t e^{-\pi l}$. Starting with the above initial values the RGES are then numerically integrated up to the scale $l_c$, where the first coupling is of the order of the bandwidth $t$. For one-loop RGES, $l_c = \alpha t/U$, where $\alpha \sim 1$ (this can easily be checked by multiplying the RGES with $t/U$). We use the abbreviations

$$A_\pm = \pm \left[ (c_{123}^\rho)^2 + \frac{3}{16} (c_{1223}^\rho)^2 \right]$$

$$+ (u_{123}^\rho)^2 + \frac{3}{16} (u_{112}^\rho)^2$$

$$B_\pm = \pm 4 \left( (c_{123}^\rho c_{1223}^\rho - u_{123}^\rho u_{1223}^\rho) - (c_{1223}^\rho)^2 - (u_{1223}^\rho)^2 \right)$$

$$C_{\pm \sigma} = \frac{c_{13}^\sigma + f_{13}^\sigma}{2 v_1} \pm \frac{f_{12}^\sigma}{v_2} + \frac{c_{22}^\sigma}{2 v_2}$$

$$D = c_{1233}^\rho u_{1223}^\rho - \frac{3}{16} c_{1223}^\rho u_{1223}^\rho,$$

where $v_{12} = v_1 + v_2$.

The two-band forward-scattering interactions are renormalized according

$$\frac{d\rho_{13}}{dl} = \frac{1}{2 v_1} \left[ (c_{13}^\rho)^2 + \frac{3}{16} (c_{13}^\rho)^2 + (u_{1331}^\rho)^2 \right]$$

$$+ \frac{1}{2 v_1} \left[ 16 (u_{1133}^\rho)^2 + \frac{3}{16} (u_{1131}^\rho)^2 \right] + \frac{A_+}{2 v_2}$$

$$\frac{d\rho_{12}}{dl} = \frac{1}{2 v_2} \left[ (c_{13}^\sigma)^2 + \frac{3}{16} (c_{13}^\sigma)^2 + 4 (u_{2213}^\sigma)^2 \right]$$

$$+ \frac{1}{2 v_1} \left[ 2 u_{1331}^\sigma (u_{1331}^\sigma) - \frac{1}{2} (u_{1331}^\sigma)^2 \right]$$

$$+ \frac{B_+}{2 v_2},$$

the single-band interactions according

$$\frac{d\rho_{11}}{dl} = - \sum_{k \neq 1} \frac{1}{2 v_k} \left[ (c_{1k}^\rho)^2 + \frac{3}{16} (c_{1k}^\rho)^2 \right]$$

$$+ \frac{1}{2 v_1} \left[ (u_{1331}^\rho)^2 + \frac{3}{16} (u_{1331}^\rho)^2 \right]$$

$$\frac{d\rho_{22}}{dl} = - \sum_{k \neq 2} \frac{1}{2 v_k} \left[ (c_{2k}^\sigma)^2 + \frac{3}{16} (c_{2k}^\sigma)^2 \right]$$

$$+ \frac{8}{v_2} (u_{22}^\sigma)^2 + \frac{A_+}{v_1}$$

$$\frac{d\sigma_{11}}{dl} = - \sum_{k \neq 1} \frac{1}{2 v_k} \left[ \frac{1}{2} (c_{1k}^\sigma)^2 + 2 c_{1k}^\sigma c_{1k}^\rho \right]$$

$$- \frac{1}{2 v_1} \left[ 2 u_{1331}^\rho (u_{1331}^\rho) + \frac{1}{2} (u_{1331}^\rho)^2 \right]$$

$$\frac{d\sigma_{22}}{dl} = - \sum_{k \neq 2} \frac{1}{2 v_k} \left[ \frac{1}{2} (c_{2k}^\sigma)^2 + 2 c_{2k}^\sigma c_{2k}^\rho \right]$$

$$\frac{d\sigma_{12}}{dl} = - \sum_{k \neq 1} \frac{1}{2 v_k} \left[ \frac{1}{2} (c_{1k}^\sigma)^2 + 2 c_{1k}^\sigma c_{1k}^\rho \right]$$

the two-band back-scattering interactions according

$$\frac{d\rho_{13}}{dl} = - \sum_{k} \frac{1}{2 v_k} \left( c_{1k}^\rho c_{1k}^{\rho \sigma} + \frac{3}{16} c_{1k}^\rho c_{1k}^{\sigma \rho} \right)$$

$$+ \frac{1}{v_1} \left( c_{13}^\rho f_{13}^\rho + \frac{3}{16} c_{13}^\rho f_{13}^\sigma + 4 u_{1133}^\sigma u_{1331}^\sigma \right) + \frac{A_+}{2 v_2}$$

$$\frac{d\rho_{12}}{dl} = - \sum_{k} \frac{1}{2 v_k} \left( c_{1k}^\rho c_{1k}^{\rho \sigma} + \frac{3}{16} c_{1k}^\rho c_{1k}^{\sigma \rho} \right)$$

$$+ \frac{2}{v_{12}} \left( c_{12}^\rho f_{12}^\rho + \frac{3}{16} c_{12}^\rho f_{12}^\sigma \right) + \frac{4}{v_{12}} u_{2213}^\sigma u_{1331}^\sigma$$

$$\frac{d\rho_{12}}{dl} = - \sum_{k} \frac{1}{2 v_k} \left( c_{1k}^\rho c_{1k}^{\rho \sigma} + \frac{3}{16} c_{1k}^\rho c_{1k}^{\sigma \rho} \right)$$

$$+ \frac{1}{v_1} \left( c_{13}^\rho f_{13}^\rho + c_{13}^\rho f_{13}^\sigma + \frac{1}{2} c_{13}^\sigma c_{13}^\rho \right) + \frac{B_+}{4 v_2}$$

$$+ \frac{1}{v_{12}} \left( c_{12}^\rho f_{12}^\rho + c_{12}^\rho f_{12}^\sigma - \frac{1}{2} c_{12}^\sigma f_{12}^\rho \right)$$

$$+ \frac{4}{v_{12}} u_{2213}^\rho u_{1223}^\rho,$$

and for the three-band interactions, we find

$$\frac{d\rho_{13}}{dl} = \frac{u_{1331}^\rho u_{1223}^\rho + 4 u_{1133}^\sigma u_{1223}^\sigma - 3 u_{1131}^\rho u_{1223}^\rho}{2 v_1}$$

$$+ \frac{2 u_{22}^\sigma u_{1223}^\rho + c_{1223}^\rho C_{\rho}}{v_2} + \frac{3 c_{1223}^\rho C_{\rho}}{16}$$

$$\frac{d\sigma_{13}}{dl} = - \frac{u_{1331}^\rho u_{1223}^\rho}{2 v_1} \left( u_{1331}^\rho + 4 u_{1133}^\sigma - \frac{1}{2} u_{1331}^\rho \right)$$

$$+ \frac{u_{1331}^\rho u_{1223}^\rho - 2 u_{22}^\sigma u_{1223}^\rho}{v_2}$$

$$+ \frac{c_{1223}^\rho C_{\rho} + c_{1223}^\rho C_{\sigma} - \frac{1}{2} c_{1223}^\rho C_{\rho}}{2 v_1}.$$
and the three-band umklapp interactions according

\[
\frac{du^{\rho}_{1223}}{dt} = \frac{u^{\rho}_{1331}c_{1223}^{\rho} + 4u^{\rho}_{1133}c_{1223}^{\rho} + 3u^{\sigma}_{1331}c_{1223}^{\rho}}{2v_1} + \frac{4u^{\rho}_{2213}c_{12}^{\rho}}{v_2} + \frac{2u^{\rho}_{222}c_{1223}^{\rho}}{v_2} + u^{\rho}_{1223}C_{+}^{\rho} - \frac{3u^{\sigma}_{1223}C_{+}^{\sigma}}{16} + \frac{4u^{\rho}_{2213}f^{\rho}_{12}}{v_2}f_{12} \]

\[
\frac{du^{\sigma}_{1223}}{dt} = -\frac{c_{1223}^{\sigma}}{2v_1} \left( u^{\sigma}_{1331} + 4u^{\rho}_{1133} - \frac{1}{2} u^{\sigma}_{1331} \right) - \frac{u^{\sigma}_{1331}c_{1223}^{\sigma}}{2v_1} - \frac{4u^{\rho}_{2213}c_{12}^{\sigma}}{v_2} - \frac{2u^{\rho}_{222}c_{1223}^{\sigma}}{v_2} - u^{\sigma}_{1223}C_{+}^{\sigma} + u^{\sigma}_{1223}C_{+}^{\rho} - \frac{u^{\rho}_{1223}}{2}C_{+}^{\tau}.
\]

(A9)

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