ORTHONORMAL STRICHARTZ INEQUALITIES FOR THE
\((k, a)\)-GENERALIZED LAGUERRE OPERATOR AND DUNKL OPERATOR

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Abstract. Let \(\Delta_{k,a}\) and \(\Delta_k\) be the \((k, a)\)-generalized Laguerre operator and the Dunkl Laplacian operator on \(\mathbb{R}^n\), respectively. The aim of this article is twofold. First, we prove a restriction theorem for the Fourier-\(\Delta_{k,a}\) transform. Next, as an application of the restriction problem, we establish Strichartz estimates for orthonormal families of initial data for the Schrödinger propagator \(e^{-it\Delta_{k,a}}\) associated with the operator \(\Delta_{k,a}\). Further, using the classical Strichartz estimates for the free Schrödinger propagator \(e^{-it\Delta_k}\) for orthonormal systems of initial data and the kernel relation between the semigroups \(e^{-it\Delta_{k,a}}\) and \(e^{it\int |\xi|^2 -a\Delta_k}\), we prove Strichartz estimates for orthonormal systems of initial data associated with the Dunkl operator \(\Delta_k\) on \(\mathbb{R}^n\). Finally, we present some applications to our aforementioned results.

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1. Introduction

One of the classical problems in harmonic analysis is the so-called restriction theorem. Historically, the restriction problem originated from studying the boundedness of Fourier transform of an \(L^p\)-function in the Euclidean space \(\mathbb{R}^n\) for some \(n \geq 2\). Later, it was realized that it also arose naturally in other contexts, such as nonlinear PDE and the study of eigenfunctions of the

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Laplacian. However, we recall the most simplest case. For a Schwartz class function \( f \) on \( \mathbb{R}^n \), the Fourier transform and the inverse Fourier transform of \( f \) are defined as

\[
\mathcal{F}(f)(\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} dx, \quad \xi \in \mathbb{R}^n,
\]

and

\[
\mathcal{F}^{-1}(f)(x) = \int_{\mathbb{R}^n} f(\xi) e^{ix\cdot\xi} d\xi, \quad x \in \mathbb{R}^n,
\]

respectively. Then the classical restriction problem is stated as follows.

**Problem 1.1.** Given a surface \( S \) embedded in \( \mathbb{R}^n \) with \( n \geq 2 \), for which exponents \( 1 \leq p \leq 2 \), \( 1 \leq q \leq \infty \), the Fourier transform of a function \( f \in L^p(\mathbb{R}^n) \) belongs to \( L^q(S) \), where \( S \) is endowed with its \( (n-1) \)-dimensional Lebesgue measure \( d\sigma \). i.e.,

\[
\|f\|_{L^q(S)} \leq C\|f\|_{L^p(\mathbb{R}^n)}.
\]

The restriction theorem for the Fourier transform plays an important role in harmonic analysis as well as in the theory of partial differential equations. To understand the general oscillatory integral operators, it plays a crucial role. Moreover, it is also connected to many other important problems in mathematical analysis, geometric measure theory, combinatorics, harmonic analysis, number theory, including several conjectures such as the Bochner-Riesz conjecture, Kakeya conjecture, the estimation of solutions to the wave, Schrödinger, and Helmholtz equations, and the local smoothing conjecture for PDE’s [25].

The case in which the restriction problem often considered in the literature is \( q = 2 \). There are only two types of surfaces for which this problem has been completely settled, namely, smooth compact surfaces with non-zero Gauss curvature and quadratic surfaces. In this direction, an affirmative answer to Fourier restriction problem for compact surfaces with non-zero Gaussian curvature can be seen in the works of Stein and Tomas [23, 26, 27]. On the other hand, for quadratic surfaces, Strichartz [24] gave a complete characterization by splitting the surface into three categories such as paraboloid-like, cone-like, or sphere-like. We refer to the excellent survey of Tao [25] for a detailed study on the historical background of the restriction problem.

The main aim of this paper is to investigate the validity of Problem 1.1 for the Fourier-\( \Delta_{k,a} \) transform and obtain Strichartz type estimates for a system of orthonormal functions for the \((k,a)\)-generalized Laguerre operator \( \Delta_{k,a} \) and Dunkl Laplacian operator \( \Delta_k \) on \( \mathbb{R}^n \).

### 1.1. Restriction problem for the Fourier-\( \Delta_{k,a} \) transform

We refer to Subsection 2.1 for the notational convention used in this subsection.

Let \( f \in L^2_{k,a}(\mathbb{R}^n) \). Suppose \( a > 0 \) and \( k \in \mathcal{K}^+ \) such that \( a + 2\gamma + n - 2 > 0 \). Then the \((k,a)\)-generalized Laguerre transform of \( f \) is defined by

\[
\hat{f}(\ell, m, j) := \int_{\mathbb{R}^n} f(x) \Phi^{(a)}_{\ell,m,j}(x) v_{k,a}(x) dx, \quad \ell, m \in \mathbb{N}, j \in J_m,
\]

where \( \Phi^{(a)}_{\ell,m,j}, v_{k,a} \) and \( J_m \) are all defined in Subsection 2.1. Let \( \mathcal{A} := \{ (\ell, m, j) : \ell, m \in \mathbb{N}, j \in J_m \} \) be a discrete set with respect to the counting measure. If \( f \in L^2_{k,a}(\mathbb{R}^n) \), then \( \hat{f}(\ell, m, j) \in \ell^2(\mathcal{A}) \) and the Plancherel formula is of the form

\[
\|f\|_{L^2_{k,a}(\mathbb{R}^n)} = \sum_{(\ell, m, j) \in \mathcal{A}} |\hat{f}(\ell, m, j)|^2.
\]

The inverse \((k,a)\)-generalized Laguerre transform is given by

\[
f(x) := \sum_{(\ell, m, j) \in \mathcal{A}} \hat{f}(\ell, m, j) \Phi^{(a)}_{\ell,m,j}(x),
\]

for all \( f \in \mathcal{S}(\mathbb{R}^n, v_{k,a}(x) dx) \).

Strichartz [24] first observed that restriction theorems for some quadric surfaces are closely related to space-time decay estimates (also called Strichartz estimates) for some evolution
equations. If we consider the free Schrödinger equation associated with the \((k,a)\)-generalized Laguerre operator (see (2.7)), then the solution is given by \(u(t,x) = e^{-it\Delta_{k,a}}f(x)\) and the \(L^p_{k,a}(\mathbb{R}^n)\) norm is \(\pi\)-periodic in \(t\) variable. Thus by the Strichartz [24], the Strichartz estimate for the Schrödinger propagator \(e^{-it\Delta_{k,a}}\) is also reduced to the restriction theorem on \(\mathbb{T} \times \mathbb{R}^n\), where \(\mathbb{T} = (-\pi, \pi)\). Thus, in order to obtain the Strichartz inequality for the solution \(u(t,x) = e^{-it\Delta_{k,a}}f(x)\), we need to introduce the Fourier-\(\Delta_{k,a}\) transform on \(\mathbb{T} \times \mathbb{R}^n\).

For any \(F \in L^1\left(\mathbb{T}, L^1_{k,a}(\mathbb{R}^n)\right)\), the Fourier-\(\Delta_{k,a}\) transform of \(F\) is given by

\[
\hat{F}(\nu, \ell, m, j) := \frac{1}{2\pi} \int_{\mathbb{R}^n} \int_{\mathbb{T}} F(t,x) \Phi_{\ell,m,j}^{(a)}(x)e^{it\nu} dt v_{k,a}(x) dx, \quad \forall \nu \in \mathbb{Z}, (\ell, m, j) \in \mathcal{A}.
\]

If \(F \in L^2\left(\mathbb{T}, L^2_{k,a}(\mathbb{R}^n)\right)\), then \(\{\hat{F}(\nu, \ell, m, j)\}_{(\nu, \ell, m, j)} \in \ell^2(\mathbb{Z} \times \mathcal{A})\), and the Plancherel formula is of the form

\[
\|F\|_{L^2(\mathbb{T}, L^2_{k,a}(\mathbb{R}^n))} = \sqrt{2\pi} \left\|\{\hat{F}(\nu, \ell, m, j)\}_{(\nu, \ell, m, j)}\right\|_{\ell^2(\mathbb{Z} \times \mathcal{A})}.
\]

Moreover, the inverse Fourier-\(\Delta_{k,a}\) transform is defined as

\[
F(t,x) := \sum_{(\nu, \ell, m, j) \in \mathbb{Z} \times \mathcal{A}} \hat{F}(\nu, \ell, m, j) \Phi_{\ell,m,j}^{(a)}(x) e^{-it\nu}, \quad (t, x) \in \mathbb{T} \times \mathbb{R}^n.
\]

Given a discrete surface \(S\) in \(\mathbb{Z} \times \mathcal{A}\), we define the restriction operator \(\mathcal{R}_S\) as

\[
\mathcal{R}_S F := \{\hat{F}(\nu, \ell, m, j)\}_{(\nu, \ell, m, j) \in S}
\]

and the operator dual to \(\mathcal{R}_S\) (called the extension Fourier-\(\Delta_{k,a}\) operator) as

\[
\mathcal{E}_S(\{\hat{F}(\nu, \ell, m, j)\})(t,x) := \sum_{(\nu, \ell, m, j) \in S} \hat{F}(\nu, \ell, m, j) \Phi_{\ell,m,j}^{(a)}(x) e^{-it\nu}.
\]

Now we consider the following restriction problem:

**Problem 1:** For which exponents of \(p, q, 1 \leq p, q \leq 2\), the sequence of the Fourier-\(\Delta_{k,a}\) transform of a function \(F \in L^q(\mathbb{T}, L^p_{k,a}(\mathbb{R}^n))\) belongs to \(\ell^2(S)\)? i.e.,

\[
\|\mathcal{R}_S F\|_{\ell^2(S)} \leq C\|F\|_{L^q(\mathbb{T}, L^p_{k,a}(\mathbb{R}^n))}.
\]

Let \(p'\) and \(q'\) are the conjugate exponents of \(p\) and \(q\), respectively, then \(\frac{1}{p} + \frac{1}{p'} = 1\) and \(\frac{1}{q} + \frac{1}{q'} = 1\). Thus by the duality argument, Problem 1 is equivalent to the boundedness of the extension operator \(\mathcal{E}_S\) as follows:

**Problem 2:** For which exponents of \(p, q, 1 \leq p, q \leq 2\), the extension Fourier-\(\Delta_{k,a}\) operator is bounded from \(\ell^2(S)\) to \(L^{q'}(\mathbb{T}, L^{p'}_{k,a}(\mathbb{R}^n))\)? i.e.,

\[
\|\mathcal{E}_S(\{\hat{F}(\nu, \ell, m, j)\})\|_{L^{q'}(\mathbb{T}, L^{p'}_{k,a}(\mathbb{R}^n))} \leq C\|\{\hat{F}(\nu, \ell, m, j)\}\|_{\ell^2(S)}.
\]

Since \(\mathcal{E}_S\) is bounded from \(\ell^2(S)\) to \(L^{q'}(\mathbb{T}, L^{p'}_{k,a}(\mathbb{R}^n))\) if and only if \(\mathcal{E}_S(\mathcal{E}_S)^*\) is bounded from \(L^q(\mathbb{T}, L^p_{k,a}(\mathbb{R}^n))\) to \(L^{q'}(\mathbb{T}, L^{p'}_{k,a}(\mathbb{R}^n))\), then Problem 2 also can be reformed as:

**Problem 3:** For which exponents of \(p, q, 1 \leq p, q \leq 2\), the operator \(\mathcal{T}_S\) is bounded from \(L^q(\mathbb{T}, L^p_{k,a}(\mathbb{R}^n))\) to \(L^{q'}(\mathbb{T}, L^{p'}_{k,a}(\mathbb{R}^n))\)? i.e.,

\[
\|\mathcal{T}_S F\|_{L^{q'}(\mathbb{T}, L^{p'}_{k,a}(\mathbb{R}^n))} \leq C\|F\|_{L^q(\mathbb{T}, L^p_{k,a}(\mathbb{R}^n))}.
\]

In this paper, we find exponents of \(p, q, 1 \leq p, q \leq 2\), such that Problem 1 has an affirmative answer when the surface \(S \subset \mathbb{Z} \times \mathcal{A}\) is discrete.
1.2. **Orthonormal Strichartz inequalities for the $\Delta_{k,a}$ operator.** Generalization involving the orthonormal system is strongly motivated by the theory of many body quantum mechanics. In quantum mechanics, a system of $N$ independent fermions in the Euclidean space $\mathbb{R}^n$ can be described by a collection of $N$ orthonormal functions $u_1, \ldots, u_N$ in $L^2(\mathbb{R}^n)$. It is then essential to obtain functional inequalities on these systems whose behavior is optimal in the finite number $N$ of such orthonormal functions. For this particular reason, functional inequalities involving a large number of orthonormal functions are very useful in mathematical analysis of large quantum systems.

The idea of extending functional inequalities involving a single function to a system of orthonormal functions is hardly a new topic. The first initiative work of such generalization goes back to the famous work established by Lieb and Thirring, known as Lieb-Thirring inequality [14, 15] that generalizes the Gagliardo-Nirenberg-Sobolev inequality, which is one of the fundamental tools to prove the stability of matter [14]. In 2013, the classical Strichartz inequality for the Schrödinger propagator $e^{it\Delta}$ has been substantially generalized for a system of orthonormal functions in the works of Frank-Lewin-Lieb-Seiringer [10]. Further, Frank-Sabin [11] improves this result by solving an open problem of [10] about the missing interval and proved the full range orthonormal Strichartz inequality by obtaining a duality principle in terms of Schatten class and generalized the theorems of Stein-Tomas and Strichartz related to the surfaces restrictions of Fourier transforms to systems of orthonormal functions. The duality principle gives an advantage which allows to deduce the restriction bounds for orthonormal systems from Schatten bounds for a single function. Thus the main focus is about how to get the Schatten bounds.

Stein [22] and Strichartz [24] proved the boundedness of $T_S$ by introducing an analytic family of operators $\{T_z\}$ in the sense of Stein defined in a strip $a \leq \text{Re}(z) \leq b$ in the complex plane such that $T_S = T_c$ for some $c \in [a, b]$. Further, they proved that $T_z$ is $L^2 - L^2$ bounded on the line $\text{Re}(z) = b$ and $L^1 - L^\infty$ bounded of on the line $\text{Re}(z) = a$. Using Stein’s complex interpolation theorem [22], they deduced the $L^p - L^{p'}$ boundedness of $T_S$ for some $p \in [1, 2]$. Note that Hölder’s inequality implies that the operator $T_S$ is $L^p - L^{p'}$ bounded if and only if the operator $W_1 T_S W_2$ (composition of the multiplication operator associated with $W_1, T_S$ and the multiplication operator associated with $W_2$) is $L^2 - L^2$ bounded. Indeed, Frank and Sabin [11] proved a more stronger Schatten bound for $W_1 T_S W_2$ which is a more general result than $L^2 - L^2$ boundedness.

Recently, a considerable attention has been devoted to extend Strichartz inequality for a system of orthonormal functions in different framework by several researchers. For example, the restriction theorems and the Strichartz inequality for the system of orthonormal functions for Hermite operator and special Hermite can be found in [5,17,18]. The second author with Feng [9] established the restriction theorem and proved orthonormal Strichartz estimate with respect to the Laguerre operator. Moreover, Nakamura in [19] proved the sharp orthonormal Strichartz inequality on the torus that generalizes Strichartz inequality on torus. We also refer to [6] for the recent developments in the direction of Strichartz estimates for orthonormal families of initial data and weighted oscillatory integral.

Motivated by the recent works and developments in the direction of orthonormal Strichartz inequality, in this paper we establish a restriction theorem for Fourier-$\Delta_{k,a}$ transform and as an application of the restriction problem, we prove Strichartz inequality for a system of orthonormal functions for the Schrödinger propagator $e^{-it\Delta_{k,a}}$ associated with $(k,a)$-generalized Laguerre operator $\Delta_{k,a}$. Finally, we aim to prove Strichartz inequality for orthonormal systems of initial data associated with Dunkl operator $\Delta_k$ on $\mathbb{R}^n$. Here we note that, the results in this paper are more general and Strichartz inequality for a system of orthonormal functions associated with Laplacian, Hermite, and Laguerre operator on $\mathbb{R}^n$ will be a particular case of our results. The main results of this papers are as follows, in particular, we consider the restriction estimate of the form:
Theorem 1.2 (Restriction estimates for orthonormal functions-general case). Suppose $a = 1, 2$ and $k$ is a non-negative multiplicity function such that $a + 2\gamma + n - 2 > 0$. Let $n \geq 1$ and $S = \{(\nu, \ell, m, j) \in \mathbb{Z} \times \mathcal{A} : \nu = 2\ell + \frac{2m}{a}\}$.

If $p, q, n \geq 1$ such that

$$1 \leq p < \frac{4\gamma + 2n + 3a - 4}{4\gamma + 2n + a - 4} \quad \text{and} \quad \frac{1}{q} + \frac{2\gamma + n + a - 2}{pa} = \frac{2\gamma + n + a - 2}{a},$$

for any (possible infinity) orthonormal system $\left\{ \{\hat{F}_i(\nu, \ell, m, j)\}_{(\nu, \ell, m, j) \in \mathbb{Z} \times \mathcal{A}} \right\}_{i \in I}$ in $\ell^2(S)$ and any sequence $\{n_i\}_{i \in I}$ in $\mathbb{C}$, we have

$$\left\| \sum_{i \in I} n_i \left\| E_{S} \left\{ \hat{F}_i(\nu, \ell, m, j) \right\} \right\|_{L^q((-\frac{n}{2}, \frac{n}{2}), L^p_k(\mathbb{R}^n))}^2 \right\| \leq C \left( \sum_{i \in I} |n_i|^{\frac{2p}{p+1}} \right)^{\frac{p+1}{2p}}, \quad (1.2)$$

where $C > 0$ is independent of $\left\{ \{\hat{F}_i(\nu, \ell, m, j)\}_{(\nu, \ell, m, j) \in \mathbb{Z} \times \mathcal{A}} \right\}_{i \in I}$ and $\{n_i\}_{i \in I}$.

As a consequence of the restriction problem, we obtain the following Strichartz inequality for the system of orthonormal functions.

Theorem 1.3 (Strichartz inequality for orthonormal functions-general case). If $p, q, n \geq 1$ such that

$$1 \leq p < \frac{4\gamma + 2n + 3a - 4}{4\gamma + 2n + a - 4} \quad \text{and} \quad \frac{1}{q} + \frac{2\gamma + n + a - 2}{pa} = \frac{2\gamma + n + a - 2}{a},$$

Then for any (possible infinity) system $\left\{ f_i \right\}_{i \in I}$ of orthonormal functions in $L^2_k(\mathbb{R}^n)$ and any coefficients $\{n_i\}_{i \in I}$ in $\mathbb{C}$, we have

$$\left\| \sum_{i \in I} n_i \left\| e^{-it\Delta_k,a} f_i \right\|_{L^q((-\frac{n}{2}, \frac{n}{2}), L^p_k(\mathbb{R}^n))}^2 \right\| \leq C \left( \sum_{i \in I} |n_i|^{\frac{2p}{p+1}} \right)^{\frac{p+1}{2p}},$$

where $C > 0$ is independent of $\{f_i\}_{i \in I}$ and $\{n_i\}_{i \in I}$.

Based on the above result and the kernel relation between the semigroups $e^{-it\Delta_k,a}$ and $e^{it\|x\|^2\Delta_k}$, we obtain the following Strichartz inequality associated with the Dunkl operator for the system of orthonormal functions.

Theorem 1.4 (Strichartz inequality for orthonormal functions associated with Dunkl operator-general case). If $p, q, n \geq 1$ such that

$$1 \leq p < \frac{4\gamma + 2n + 3a - 4}{4\gamma + 2n + a - 4} \quad \text{and} \quad \frac{1}{q} + \frac{2\gamma + n + a - 2}{pa} = \frac{2\gamma + n + a - 2}{a},$$

then for any (possible infinity) system $\left\{ f_i \right\}_{i \in I}$ of orthonormal functions in $L^2_k(\mathbb{R}^n)$ and any coefficients $\{n_i\}_{i \in I}$ in $\mathbb{C}$, we have

$$\left\| \sum_{i \in I} n_i \left\| e^{it\|x\|^2\Delta_k} f_i \right\|_{L^q(\mathbb{R}, L^p_k(\mathbb{R}^n))}^2 \right\| \leq C \left( \sum_{i \in I} |n_i|^{\frac{2p}{p+1}} \right)^{\frac{p+1}{2p}},$$

where $C > 0$ is independent of $\{f_i\}_{i \in I}$ and $\{n_i\}_{i \in I}$.

Apart from the introduction, this paper is organized as follows.

- In Section 2, we recall some basic definitions and important properties of the Dunkl operator and generalized Laguerre semigroup.
- In Section 3, we explain the relation between the Strichartz estimate for the Schrödinger-$\Delta_k,a$ equation and the restriction theorem associated with $\Delta_k,a$ on certain discrete surface.
• In Section 4, we introduce the complex interpolation method and the duality principle in terms of Schatten class and establish the Schatten boundedness for the Fourier-\(\Delta_{k,a}\) extension operator for some \(\lambda_0 > 1\).
• In Section 5, we prove restriction estimates and Strichartz inequalities for systems of orthonormal functions associated with the \((k,a)\)-generalized Laguerre operator \(\Delta_{k,a}\).
• In Section 6, we establish the orthonormal Strichartz inequality for the Dunkl operator \(\Delta_k\) by the relation of the kernel between \(e^{-t\Delta_{k,a}}\) and \(e^{i\Delta_k/\|x\|^{\gamma-n}}\).
• In Section 7, we list some applications to show the results in our paper are more general.

2. Preliminary

In this section, we recall some basic definitions and important properties of the Dunkl operator and generalized Laguerre semigroup to make the paper self contained. A complete account of detailed study on Dunkl operator can be found in [2,20]. However, we mainly adopt the notation and terminology given in [2]. The basic ingredient in the theory of Dunkl operators are root systems and finite reflection groups. We start this section by the definition of root system.

2.1. Dunkl operator. Let \((\cdot,\cdot)\) denotes the standard Euclidean scalar product in \(\mathbb{R}^n\). For \(x \in \mathbb{R}^n\), we denote \(\|x\|\) as \(\|x\| = (x,x)^{1/2}\). For \(\alpha \in \mathbb{R}^n \setminus \{0\}\), we denote \(r_\alpha\) as the reflection with respect to the hyperplane \(\langle \alpha \rangle^\perp\) orthogonal to \(\alpha\) and is defined by

\[
    r_\alpha(x) := x - 2\frac{(\alpha,x)}{\|\alpha\|^2}\alpha, \quad x \in \mathbb{R}^n.\]

A finite set \(\mathcal{R}\) in \(\mathbb{R}^n \setminus \{0\}\) is said to be a root system if the following holds:

\[
\begin{align*}
    & (1) \ r_\alpha(\mathcal{R}) = \mathcal{R} \text{ for all } \alpha \in \mathcal{R}, \\
    & (2) \ \mathcal{R} \cap \mathbb{R}a = \{\pm \alpha\} \text{ for all } \alpha \in \mathcal{R}.
\end{align*}
\]

For a given root system \(\mathcal{R}\), the subgroup \(G \subset O(n,\mathbb{R})\) generated by the reflections \(\{r_\alpha \mid \alpha \in \mathcal{R}\}\) is called the finite Coxeter group associated with \(\mathcal{R}\). The dimension of \(\text{span}\mathcal{R}\) is called the rank of \(\mathcal{R}\). For a detailed on the theory of finite reflection groups, we refer to [13]. Let \(\mathcal{R}^+ := \{\alpha \in \mathcal{R} : \langle \alpha,\beta \rangle > 0\}\) for some \(\beta \in \mathbb{R}^n \setminus \bigcup_{\alpha \in \mathcal{R}} \langle \alpha \rangle^\perp\), be a fix positive root system.

Some typical examples of such system is the Weyl groups such as the symmetric group \(S_n\) for the type \(A_{n-1}\) root system and the hyperoctahedral group for the type \(B_n\) root system. In addition, \(H_3, H_4\) (icosahedral groups) and, \(I_2(n)\) (symmetry group of the regular \(n\)-gon) are also the Coxeter groups.

A multiplicity function for \(G\) is a function \(k : \mathcal{R} \rightarrow \mathbb{C}\) which is constant on \(G\)-orbits. Setting \(k_\alpha := k(\alpha)\) for \(\alpha \in \mathcal{R}\), from the definition of \(G\)-invariant, we have \(k_{ga} = k_\alpha\) for all \(g \in G\). We say \(k\) is non-negative if \(k_\alpha \geq 0\) for all \(\alpha \in \mathcal{R}\). Let us denote \(\gamma\) as \(\gamma = \gamma(k) := \sum_{\alpha \in \mathcal{R}^+} k(\alpha)\).

The \(\mathbb{C}\)-vector space of non-negative multiplicity functions on \(\mathcal{R}\) is denoted by \(\mathcal{K}^+\). For \(\xi \in \mathbb{C}^n\) and \(k \in \mathcal{K}^+\), Dunkl in 1989 introduced a family of first order differential-difference operators \(T_\xi := T_\xi(k)\) by

\[
    T_\xi(k)f(x) := \partial_\xi f(x) + \sum_{\alpha \in \mathcal{R}^+} k_\alpha \langle \alpha,\xi \rangle \frac{f(x) - f(r_\alpha x)}{\langle \alpha, x \rangle}, \quad f \in C^1(\mathbb{R}^n),
\]

where \(\partial_\xi\) denotes the directional derivative corresponding to \(\xi\). The operator \(T_\xi\) defined in (2.1) formally known as Dunkl operator and is one of the most important developments in the theory of special functions associated with root systems [7].

They commute pairwise and are skew-symmetric with respect to the \(G\)-invariant measure \(h_k(x)dx\), where the weight function \(h_k(x) := \prod_{\alpha \in \mathcal{R}^+} |\langle \alpha, x \rangle|^{2k_\alpha}\) and is homogeneous of degree \(2\gamma\). Thanks to the \(G\)-invariance of the multiplicity function, this definition is independent of the choice of the positive subsystem \(\mathcal{R}^+\). In [8], it is shown that for any \(k \in \mathcal{K}^+\), there is a unique
linear isomorphism $V_k$ (Dunkl’s intertwining operator) on the space $\mathcal{P}(\mathbb{R}^n)$ of polynomials on $\mathbb{R}^n$ such that

1. $V_k(\mathcal{P}_m(\mathbb{R}^n)) = \mathcal{P}_m(\mathbb{R}^n)$ for all $m \in \mathbb{N}$,
2. $V_k|_{\mathcal{P}_0(\mathbb{R}^n)} = id$,
3. $T_{k}(k) = V_{k} \xi_k$.

Here $\mathcal{P}_m(\mathbb{R}^n)$ denotes the space of homogeneous polynomials of degree $m$.

For any finite reflection group $G$ and any $k \in \mathcal{K}^+$, Rösler in [20] proved that there exists a unique positive Radon probability measure $\rho^k_\xi$ on $\mathbb{R}^n$ such that

$$V_k f(x) = \int_{\mathbb{R}^n} f(\xi) d\rho^k_\xi(\xi).$$

The measure $\rho^k_\xi$ depends on $x \in \mathbb{R}^n$ and its support is contained in the ball $B(||x||) := \{\xi \in \mathbb{R}^n : ||\xi|| \leq ||x||\}$. In view of the Laplace type representation (2.2), Dunkl’s intertwining operator $V_k$ can be extended to a larger class of function spaces.

Let $\{\xi_1, \xi_2, \ldots, \xi_n\}$ be an orthonormal basis of $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$. Then the Dunkl Laplacian operator $\Delta_k$ is defined as

$$\Delta_k = \sum_{j=1}^{n} T_{\xi_j}(k)^2.$$ 

The definition of $\Delta_k$ is independent of the choice of an orthonormal basis of $\mathbb{R}^n$. In fact, one can see that the operator $\Delta_k$ also can be expressed as the following:

$$\Delta_m f(x) = \Delta f(x) + \sum_{\alpha \in \mathcal{R}^+} k_\alpha \left\{ \frac{2(\nabla f(x), \alpha)}{\langle \alpha, x \rangle} - ||\alpha||^2 \frac{f(x) - f(r_\alpha x)}{\langle \alpha, x \rangle^2} \right\}, \quad f \in C^1(\mathbb{R}^n),$$

where $\nabla$ and $\Delta$ are the usual gradient and usual Laplacian operator on $\mathbb{R}^n$, respectively. Note that for $k \equiv 0$, the Dunkl Laplacian operator $\Delta_k$ reduces to the Euclidean Laplacian $\Delta$.

2.2. An orthonormal basis in $L^2_{k,a}(\mathbb{R}^n)$. A $k$-harmonic polynomial of degree $m$ is a homogeneous polynomial $p$ on $\mathbb{R}^n$ of degree $m$ such that $\Delta_k p = 0$. Denote by $\mathcal{H}^m_k(\mathbb{R}^n)$ the space of $k$-harmonic polynomials of degree $m$. Spherical harmonics (or just $h$-harmonics) of degree $m$ are then defined as the restrictions of $\mathcal{H}^m_k(\mathbb{R}^n)$ to the unit sphere $\mathbb{S}^{n-1}$. The spaces $\mathcal{H}^m_k(\mathbb{R}^n) |_{\mathbb{S}^{n-1}}$ (for $m \in \mathbb{N}$) are finite dimensional and orthogonal to each other with respect to the measure $h_k(\omega) d\sigma(\omega)$, where $d\sigma$ be the standard measure on the unit sphere $\mathbb{S}^{n-1}$ and $d_k$ the normalizing constant defined by

$$d_k := \left( \int_{\mathbb{S}^{n-1}} h_k(\omega) d\sigma(\omega) \right)^{-1}.$$ 

For $k \equiv 0$, $d_k^{-1}$ is the volume of the unit sphere, namely

$$d_0 = \frac{\Gamma \left( \frac{n}{2} \right)}{2^n \pi^{\frac{n}{2}}}.$$ 

We also have the following spherical harmonics decomposition

$$L^2(\mathbb{S}^{n-1}, h_k(\omega) d\sigma(\omega)) = \sum_{m \in \mathbb{N}} \mathcal{H}^m_k(\mathbb{R}^n) |_{\mathbb{S}^{n-1}}.$$ 

For $a > 0$ and $1 \leq p < \infty$, let $L^p_{k,a}(\mathbb{R}^n)$ be the space of $L^p$-functions on $\mathbb{R}^n$ with respect to the weight

$$v_{k,a}(x) := ||x||^{a-2} h_k(x),$$

which has a degree of homogeneity $a - 2 + 2\gamma$.

Moreover, for the polar coordinates $x = r \omega$ ($r > 0, \omega \in \mathbb{S}^{n-1}$), we have

$$v_{k,a}(x) dx = r^{2\gamma + a - 3} h_k(\omega) dr d\sigma(\omega).$$
According to the spherical harmonic decomposition (2.4), there is a unitary isomorphism (see [2, (3.25)]),
\[ \oplus_{m \in \mathbb{N}} \left( \mathcal{H}^m_k(\mathbb{R}^n) \big|_{S^{n-1}} \right) \cong L^2(\mathbb{R}^+, r^{2\gamma+n+a-3}dr) \cong L^2_{k,a}(\mathbb{R}^n), \]  
(2.5)
where \( \otimes \) stands for the Hilbert completion of the tensor product space of two Hilbert spaces. For any \( \mu > -1 \), and \( \ell, m \in \mathbb{N} \), define the Laguerre polynomial of degree \( \mu \in \mathbb{N} \) on \( \mathbb{R}^+ \) by
\[ L^{(\mu)}_\ell(t) := \sum_{j=0}^{\ell} \frac{\Gamma(\mu + \ell + 1)}{(\ell-j)!\Gamma(\mu+j+1)} (-t)^j j!, \quad t \in \mathbb{R}^+, \]
and set \( \lambda_{k,a,m} := \frac{1}{a}(2m + 2\gamma + n - 2) \).

**Proposition 2.1.** ([2, Proposition 3.15]) For any fixed \( m \in \mathbb{N}, a > 0 \), and a multiplicity function \( k \) satisfying \( \lambda_{k,a,m} > -1 \), set
\[ \psi^{(a)}_{\ell,m}(r) := \left( \frac{2^{\lambda_{k,a,m}} \Gamma(\ell + 1)}{a^{\lambda_{k,a,m}} \Gamma(\lambda_{k,a,m} + \ell + 1)} \right)^{\frac{1}{2}} r^m L^{(\lambda_{k,a,m})}_\ell \left( \frac{2}{a} r^a \right) \exp \left( -\frac{1}{a} r^a \right), \forall \ell \in \mathbb{N}. \]
Then the set \( \{ \psi^{(a)}_{\ell,m} : \ell \in \mathbb{N} \} \) forms an orthonormal basis in \( L^2(\mathbb{R}^+, r^{2\gamma+n+a-3}dr) \).

For every fixed \( m \in \mathbb{N} \), we take an orthonormal basis of \( \mathcal{H}^m_k(\mathbb{R}^n) \big|_{S^{n-1}} \) as
\[ \{ Y^m_j : j \in J_m \}, \]
where \( J_m = \{1, 2, \ldots, \dim(\mathcal{H}^m_k)\} \). Then we obtain an orthonormal basis in \( L^2_{k,a}(\mathbb{R}^n) \) immediately as the following.

**Corollary 2.2.** ([2, Corollary 3.17]) Suppose that \( a > 0 \) and \( k \in \mathcal{K}^+ \) satisfies \( a+2\gamma+n-2 > 0 \). For each \( \ell, m \in \mathbb{N} \) and \( j \in J_m \), we set
\[ \Phi^{(a)}_{\ell,m,j}(x) := Y^m_j \left( \frac{x}{\| x \|} \right) \psi^{(a)}_{\ell,m}(\| x \|). \]
Then the set \( \{ \Phi^{(a)}_{\ell,m,j} : \ell, m \in \mathbb{N}, j \in J_m \} \) forms an orthonormal basis in \( L^2_{k,a}(\mathbb{R}^n) \).

For \( \ell, m \in \mathbb{N} \) and \( p \in \mathcal{H}^m_k(\mathbb{R}^n) \), define the function
\[ \Phi^{(a)}_{\ell,\ell,m,j}(p, x) := p(x) L^{(\lambda_{k,a,m})}_\ell \left( \frac{2}{a} \| x \|^a \right) \exp \left( -\frac{1}{a} \| x \|^a \right). \]
For \( k \in \mathcal{K}^+ \) and \( a > 0 \) such that \( a+2\gamma+n-2 > 0 \), the following vector space
\[ W_{k,a}(\mathbb{R}^n) := \text{span} \left\{ \Phi^{(a)}_{\ell}(p, \cdot) : \ell, m \in \mathbb{N}, p \in \mathcal{H}^m_k(\mathbb{R}^n) \right\} \]
is a dense subspace of \( L^2_{k,a}(\mathbb{R}^n) \).

2.3. \((k,a)\)-generalized Laguerre semigroup. Let \( a > 0 \) to be a deformation parameter and \( k \) be a multiplicity function, then consider the following differential-difference operator
\[ \Delta_{k,a} := \frac{1}{a} \left( \| x \|^a - \| x \|^{2-a} \Delta_k \right), \]
where \( \| x \|^a \) in the right hand side of the formula stands for the multiplication operator by \( \| x \|^a \). Note that when \( a = 2 \) and \( k \equiv 0 \), it reduces to the classical Hermite operator \( H := 2\Delta_{0,2} = \| x \|^2 - \Delta \) on \( L^2(\mathbb{R}^n) \), and when \( a = 1 \) and \( k \equiv 0 \), it reduces to classical Laguerre operator \( L := \Delta_{0,1} = \| x \|^2 - \| x \| \Delta \) on \( L^2(\mathbb{R}^n, \| x \|^{-1}dx) \). For a general multiplicity function \( k \), when \( a = 2 \), it reduces to the Dunkl-Hermite operator \( H_k := 2\Delta_{k,2} = \| x \|^2 - \Delta_k \) on \( L^2(\mathbb{R}^n) \), where \( L_k(\mathbb{R}^n) := L^2(\mathbb{R}^n, h_k(x)dx) \), and when \( a = 1 \), it reduces to Dunkl-Laguerre operator \( L_k := \Delta_{k,1} = \| x \|^2 - \| x \| \Delta_k \) on \( L^2(\mathbb{R}^n, \| x \|^{-1}h_k(x)dx) \).
For $a > 0$ and $k \in \mathbb{K}^+$ such that $a + 2\gamma + n - 2 > 0$, from [2, Corollary 3.22], $\Delta_{k,a}$ is an essentially self-adjoint operator on $L^2_{k,a}(\mathbb{R}^n)$. Moreover, there is no continuous spectrum of $\Delta_{k,a}$ and all the discrete spectra are positive. The orthonormal functions $\{ \Phi_{\ell,m,j}^{(a)} : \ell, m \in \mathbb{N}, j \in J_m \}$ in $L^2_{k,a}(\mathbb{R}^n)$ are eigenfunctions of the operator $\Delta_{k,a}$ corresponding to the eigenvalues $2\ell + \lambda_{k,a,m} + 1$ (see [2, (3.33 a)]), i.e.,

$$\Delta_{k,a} \Phi_{\ell,m,j}^{(a)} = (2\ell + \lambda_{k,a,m} + 1) \Phi_{\ell,m,j}^{(a)}.$$ 

In [2], the authors have studied the so-called $(k,a)$-generalized Laguerre semigroup $I_{k,a}(z)$ with infinitesimal generator $\Delta_{k,a}$, that is

$$I_{k,a}(z) := \exp \left( -z\Delta_{k,a} \right)$$

for $z \in \mathbb{C}$ such that $\text{Re}(z) \geq 0$. Henceforth, we shall denote by $\mathbb{C}^+ = \{ z \in \mathbb{C} : \text{Re}(z) \geq 0 \}$. Suppose $a > 0$ and $k \in \mathbb{K}^+$ satisfying the condition $a + 2\gamma + n - 2 > 0$. Then

1. The map

$$\mathbb{C}^+ \times L^2_{k,a}(\mathbb{R}^n) \longrightarrow L^2_{k,a}(\mathbb{R}^n), \quad (z, f) \longmapsto e^{-z\Delta_{k,a} f}$$

is continuous.

2. For any $p \in \mathcal{H}^m_k(\mathbb{R}^n)$ and $\ell \in \mathbb{N}$, $\Phi_{\ell}^{(a)}(p, \cdot)$ is an eigenfunction of the operator $e^{-z\Delta_{k,a}}$, i.e.,

$$e^{-z\Delta_{k,a}} \Phi_{\ell}^{(a)}(p, x) = e^{-z(\lambda_{k,a,m} + 2\ell + 1)} \Phi_{\ell}^{(a)}(p, x).$$

3. The operator norm $\|e^{-z\Delta_{k,a}}\|_{op}$ is $\exp \left( -\frac{1}{2}(2\gamma + n + a - 2) \text{Re}(z) \right)$.

4. If $\text{Re}(z) > 0$, then $e^{-z\Delta_{k,a}}$ is a Hilbert Schmidt operator.

5. If $\text{Re}(z) = 0$, then $e^{-z\Delta_{k,a}}$ is an unitary operator.

For $\text{Re}(z) \geq 0$, $I_{k,a}(z)$ is an integral operator for all $a > 0$ and has the following expression

$$e^{-z\Delta_{k,a}} f(x) = c_{k,a} \int_{\mathbb{R}^n} \Delta_{k,a}(x, y; z)f(y)v_{k,a}(y)dy,$$  

where the constant

$$c_{k,a} = a ^{-2\gamma + n - 2} \frac{G \left( \frac{2\gamma + n + a - 2}{a} \right)}{d_k},$$

and $d_k$ is defined in (2.3). Moreover, a series expansion for the kernel $\Delta_{k,a}$ can be found in [2, Theorem 4.20]. Since the series is expressed compactly for $a = 1, 2$, throughout the paper we will also assume that $a = 1, 2$.

Recall the expression of the kernel $\Lambda_{k,a}(x, y; z)$ in the expression (2.6) given by $\Lambda_{k,a}(x, y; z) := V_k^\omega h_a(r, s; z; (\omega, \cdot))(\eta)$ (using the polar coordinate $x = rw, y = so$), where for all $\zeta \in [-1, 1]$ with parameters $r, s > 0$ and $z \in \mathbb{C}^+ \backslash i\pi \mathbb{Z}$:

$$h_a(r, s; z; \zeta) := \frac{\exp \left( -\frac{1}{a} (ra^a + sa^a) \coth z \right)}{\sinh z^{2\gamma + n - 1}} \frac{\Gamma \left( \frac{1}{2}(\gamma + n - a) \right) \Gamma \left( \frac{1}{2}(\gamma + n + a + 2 - 2) \right)}{\Gamma \left( \frac{1}{2}(\gamma + n + a - 2) \right)} \left( 1 + \zeta \right)^{1/2}, \quad a = 1,$$

$$a = 2.$$

Here $I_k(\omega) = \left( \frac{\omega}{2} \right)^{-\lambda} I_k(\omega)$ is the (normalized) modified Bessel function of the first kind, $V_k$ is the Dunkl intertwining operator, and the superscript in $V_k^\omega$ denotes the corresponding variable. We point out that, when $k \equiv 0$, $\Lambda_{k,a}(x, y; z) = h_a(r, s; z; (\omega, \eta))$.

For $a = 1, 2$, it follows that the kernel $\Lambda_{k,a}(x, y; i\mu)$ of $e^{-i\mu\Delta_{k,a}}$ satisfies

1. $\Lambda_{k,a}(x, y; -i\mu) = \Lambda_{k,a}(x, y; i\mu)$;

2. $\Lambda_{k,a}(x, y; i(\mu + \pi)) = e^{-i\pi \frac{n + 2\gamma + a - 2}{a}} \Lambda_{k,a} \left( -i \frac{2}{a} x, y; i\mu \right)$.
for all $\mu \in \mathbb{R} \setminus \pi \mathbb{Z}$, which follows that the $L^p_{k,a}(\mathbb{R}^n)$ norm of $e^{-i\mu \Delta_{k,a} f}$ is $\pi$-periodic in $t$ and thus determined by its values for $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Moreover, the kernel $\Lambda_{k,a}(x, y; z)$ of $e^{-z \Delta_{k,a}}$ satisfies the following upper bound estimates (see [2, Proposition 4.26]).

**Proposition 2.3.** For $a = 1, 2$, the function $\Lambda_{k,a}(x, y; z)$ satisfies the following inequalities:

1. For $\text{Re}(z) > 0$, there exists a constant $C > 0$ depending on $z$ such that
   $$|\Lambda_{k,a}(x, y; z)| \leq \frac{1}{|\sin z|} \exp \left(-C(||x||^a + ||y||^a)\right).$$

2. For $z = \epsilon + i\mu$ such that $\epsilon \geq 0$ and $\mu \in \mathbb{R} \setminus \pi \mathbb{Z}$, we have
   $$|\Lambda_{k,a}(x, y; z)| \leq \frac{1}{|\sin \mu|} \exp \left(-\frac{2\gamma + n + a - 2}{\epsilon^2}\right).$$

Now, we consider the Cauchy problem for the free Schrödinger equation associated with the $(k, a)$-generalized Laguerre operator $\Delta_{k,a}$, namely

$$\begin{cases}
i \partial_t u(t, x) - \Delta_{k,a} u(t, x) = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}^n, \\
u(0, x) = f(x) \in L^p_{k,a}(\mathbb{R}^n).&
\end{cases}$$

(2.7)

Then $u(t, x) = e^{-it \Delta_{k,a}} f(x)$ is the solution of the above system. As the solution of the problem (2.7) is $\pi$-periodic in $t$, we introduce the mixed normed space $L^q \left((-\frac{\pi}{2}, \frac{\pi}{2}), L^p_{k,a}(\mathbb{R}^n)\right)$ which is the set of measure functions $h$ on $(-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}^n$ such that

$$\|h\|_{L^q \left((-\frac{\pi}{2}, \frac{\pi}{2}), L^p_{k,a}(\mathbb{R}^n)\right)} := \left\|\|h(t, \cdot)\|_{L^p_{k,a}(\mathbb{R}^n)}\right\|_{L^q(-\frac{\pi}{2}, \frac{\pi}{2})}.$$

A pair $(p, q)$ is called admissible if $\left(\frac{1}{p}, \frac{1}{q}\right)$ belongs to the trapezoid

$$\frac{1}{2} \left(\frac{2\gamma + n - 2}{2\gamma + n + a - 2}\right) < \frac{1}{p} \leq \frac{1}{2} \text{ and } \frac{1}{2} \leq \frac{1}{q} \leq 1$$

or

$$0 \leq \frac{1}{q} < \frac{1}{2} \text{ and } \frac{1}{q} \geq \left(\frac{2\gamma + n + a - 2}{a}\right) \left(\frac{1}{2} - \frac{1}{p}\right).$$

The Strichartz estimates for the solution of (2.7) is proved by [3] and may be stated as:

**Theorem 2.4** (Strichartz inequality for a single function). Suppose $a = 1, 2$ and $k$ is a non-negative multiplicity function such that

$$a + 2\gamma + n - 2 > 0.$$ 

Let $(p, q)$ be an admissible pair and $u = e^{-it \Delta_{k,a}} f$ be the solution to the homogeneous problem (2.7) with $f \in L^2_{k,a}(\mathbb{R}^n)$. Then we have the estimate

$$\|e^{-it \Delta_{k,a}} f\|_{L^q \left((-\frac{\pi}{2}, \frac{\pi}{2}), L^p_{k,a}(\mathbb{R}^n)\right)} \leq C \|f\|_{L^2_{k,a}(\mathbb{R}^n)}.$$ 

Next we turn our attention to the free Schrödinger equation with respect to the differential-difference part of $\Delta_{k,a}$

$$\begin{cases}
i \partial_t w(t, x) + \frac{1}{2}||x||^{2-a} \Delta_{k,a} w(t, x) = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}^n, \\
w(0, x) = f(x),&
\end{cases}$$

Then $e^{it \frac{||x||^{2-a}}{2} \Delta_{k,a} f}(x)$ is the solution of the above Schrödinger equation. Let $\Gamma_{k,a}(x, y; it)$ be the kernel of $e^{it \frac{||x||^{2-a}}{2} \Delta_{k,a}}$. Using the change of variable $s = \tan(t)$ with $t \in (-\pi/2, \pi/2)$, we get

$$\Lambda_{k,a}(x, y; i \tan^{-1} s) = c_{k,a}^{-1} \left(1 + s^2\right)^\frac{2\gamma + n + a - 2}{2a} \exp \left(-\frac{is||x||^a}{a}\right) \Gamma_{k,a} \left((1 + s^2)\frac{a}{2} x, y; is\right),$$

where $c_{k,a}$ is the constant derived from the Strichartz estimates.
\[ e^{-i\tan^{-1}(s)\Delta_{k,a}} f(x) = (1 + s^2)\frac{2s^{2n+1}}{2n} e^{-is\frac{x_1^2}{a}} e^{i\frac{s}{a}x_2^2} \Delta_k f \left( (1 + s^2)\frac{1}{2} x \right), \]

for any \( f \in L^2_{k,a}(\mathbb{R}^n) \).

**Theorem 2.5** ([3]). Suppose \( a = 1, 2 \) and \( k \) is a non-negative multiplicity function such that \( a + 2\gamma + n - 2 > 0 \).

If \( 1 \leq p, q \leq \infty \) satisfy
\[
\left( \frac{1}{2} - \frac{1}{p} \right) \frac{2\gamma + n + a - 2}{a} - \frac{1}{q} = 0,
\]
then for all \( f \in L^2_{k,a}(\mathbb{R}^n) \) we have
\[
\|e^{\frac{i}{a}s^2x^2}\Delta_k f\|_{L^q((0,\infty),L^p_{k,a}(\mathbb{R}^n))} = \|e^{-it\Delta_k,a} f\|_{L^q((-\infty,0),L^p_{k,a}(\mathbb{R}^n))},
\]
\[
\|e^{\frac{i}{a}s^2x^2}\Delta_k f\|_{L^q((-\infty,0),L^p_{k,a}(\mathbb{R}^n))} = \|e^{-it\Delta_k,a} f\|_{L^q((0,\infty),L^p_{k,a}(\mathbb{R}^n))}.
\]
Moreover, we have
\[
\|e^{\frac{i}{a}s^2x^2}\Delta_k f\|_{L^q(\mathbb{R},L^p_{k,a}(\mathbb{R}^n))} \leq C\|f\|_{L^2_{k,a}(\mathbb{R}^n)}.
\]

**2.4. Schatten class.** If \( \mathcal{H} \) is a complex, separable Hilbert space, a linear compact operator \( A : \mathcal{H} \to \mathcal{H} \) belongs to the \( r \)-Schatten-von Neumann class \( \mathcal{G}^r(\mathcal{H}) \) if
\[
\sum_{n=1}^{\infty} (s_n(A))^r < \infty,
\]
where \( s_n(A) \) denote the singular values of \( A \), i.e. the eigenvalues of \( |A| = \sqrt{A^*A} \) with multiplicities counted. For \( 1 \leq r < \infty \), the Schatten space \( \mathcal{G}^r(\mathcal{H}) \) is defined as the space of all compact operators \( A \) on \( \mathcal{H} \) such that \( \sum_{n=1}^{\infty} (s_n(A))^r < \infty \).

For \( 1 \leq r < \infty \), the class \( \mathcal{G}^r(\mathcal{H}) \) is a Banach space endowed with the norm
\[
\|A\|_{\mathcal{G}^r} = \left( \sum_{n=1}^{\infty} (s_n(A))^r \right)^{\frac{1}{r}}.
\]
For \( 0 < r < 1 \), the \( \| \cdot \|_{\mathcal{G}^r} \) as above only defines a quasi-norm with respect to which \( \mathcal{G}^r(\mathcal{H}) \) is complete. An operator belongs to the class \( \mathcal{G}^1(\mathcal{H}) \) is known as **Trace class** operator. Also, an operator belongs to \( \mathcal{G}^2(\mathcal{H}) \) is known as **Hilbert-Schmidt** operator.

**2.5. An analytic family of operators.** Let us first recall that a family of operators \( \{ T_z \} \) on \( \mathbb{T} \times \mathbb{R}^n \) defined in a strip \( \alpha \leq \text{Re}(z) \leq \beta \) in the complex plane is analytic in the sense of Stein if it has the following properties:

1. For each \( z : \alpha \leq \text{Re}(z) \leq \beta \), \( T_z \) is a linear transformation of simple functions on \( \mathbb{T} \times \mathbb{R}^n \) (i.e., functions that take on only a finite number of nonzero values on sets of finite measure on \( \mathbb{T} \times \mathbb{R}^n \) to measurable functions on \( \mathbb{T} \times \mathbb{R}^n \)).
2. For all simple functions \( F, G \) on \( \mathbb{T} \times \mathbb{R}^n \), the map \( z \to \langle G, T_z F \rangle \) is analytic in \( a < \text{Re}(z) < b \) and continuous in \( a \leq \text{Re}(z) \leq b \).
3. Moreover, \( \sup_{\alpha \leq \lambda \leq \beta} |\langle G, T_{\lambda + it} F \rangle| \leq C(s) \) for some \( C(s) \) with at most a (double) exponential growth in \( s \).
3. Restriction theorem for the \((k, a)\)-generalized Laguerre operator

For \(a = 1, 2\), let \(S\) be the discrete surface \(S = \{(\nu, \ell, m, j) \in \mathbb{Z} \times A : \nu = 2\ell + \frac{2m}{a}\}\) with respect to the counting measure. Choose

\[
\hat{F}(\nu, \ell, m, j) = \begin{cases} 
\hat{f}(\ell, m, j), & \text{if } \nu = 2\ell + \frac{2m}{a}, \\
0, & \text{otherwise},
\end{cases}
\]

for some measurable function \(f\) on \(\mathbb{R}^n\). Then for any \(f \in L^2_{k,a}(\mathbb{R}^n)\), by the Plancherel formula, we have

\[
\|F\|_{L^2(\mathbb{T}, L^2_{k,a}(\mathbb{R}^n))} = \sqrt{2\pi} \|\{\hat{F}(\nu, \ell, m, j)\}\|_{\mathcal{E}(S)} = \sqrt{2\pi} \|\{\hat{f}(\ell, m, j)\}\|_{\mathcal{E}(A)} = \sqrt{2\pi} \|f\|_{L^2_{k,a}(\mathbb{R}^n)}.
\]

(3.1)

Thus for \(F \in L^2(\mathbb{T}, L^2_{k,a}(\mathbb{R}^n))\), we get

\[
\mathcal{E}_S(\{\hat{F}(\nu, \ell, m, j)\})(t, x) = \sum_{(\nu, \ell, m, j) \in S} \hat{F}(\nu, \ell, m, j) \Phi_{\nu,\ell,m,j}(x)e^{-it\nu} \\
= \sum_{(\ell, m, j) \in A} \hat{f}(\ell, m, j) \Phi_{\ell,m,j}(x)e^{-it(2\ell + \frac{2m}{a})} \\
= \sum_{(\ell, m, j) \in A} \langle f, \Phi_{\ell,m,j}^{(a)} \rangle \Phi_{\ell,m,j}(x)e^{-it(2\ell + \frac{2m}{a})} \\
= e^{it(\frac{1}{2}(2\gamma+n-2))} \sum_{(\ell, m, j) \in A} e^{-it(2\ell+\frac{1}{2}(2m+2\gamma+n-2))} \langle f, \Phi_{\ell,m,j}^{(a)} \rangle \Phi_{\ell,m,j}(x) \\
= e^{it(\frac{1}{2}(2\gamma+n-2))} e^{-it\Delta_{k,a}} f(x).
\]

(3.2)

Now once we assume that Problem 2 holds, i.e., \(\mathcal{E}_S\) is bounded from \(\ell^2(S)\) to \(L^p(\mathbb{T}, L^p_{k,a}(\mathbb{R}^n))\) for some \(1 \leq p, q \leq 2\), then from from (3.1) and (3.2), the Strichartz inequality follows as

\[
\|e^{-it\Delta_{k,a}} f\|_{L^p(\mathbb{T}, L^p_{k,a}(\mathbb{R}^n))} = \|\mathcal{E}_S(\{\hat{F}(\nu, \ell, m, j)\})\|_{L^p(\mathbb{T}, L^p_{k,a}(\mathbb{R}^n))} \\
\leq C \|\{\hat{F}(\nu, \ell, m, j)\}\|_{\mathcal{E}(S)} \\
= C\sqrt{2\pi} \|f\|_{L^2_{k,a}(\mathbb{R}^n)}.
\]

Thus from the above, we can conclude that the Strichartz inequality for the solution of (2.7) associated with the \((k, a)\)-generalized Laguerre operator holds if only if the restriction theorem holds on the particular surface \(S = \{(\nu, \ell, m, j) \in \mathbb{Z} \times A : \nu = 2\ell + \frac{2m}{a}\}\).

By Theorem 2.4, we have the following restriction estimates.

**Theorem 3.1** (Restriction theorem for a single function). Let \(S = \{(\nu, \ell, m, j) \in \mathbb{Z} \times A : \nu = 2\ell + \frac{2m}{a}\}\) be the discrete surface. Then under the same hypotheses as in Theorem 2.4, we have

\[
\|\mathcal{E}_S(\{\hat{F}(\nu, \ell, m, j)\})\|_{L^q((-\frac{T}{2}, \frac{T}{2}), L^p_{k,a}(\mathbb{R}^n))} \leq C \|\{\hat{F}(\nu, \ell, m, j)\}\|_{\mathcal{E}(S)},
\]

with \(C > 0\) is a constant independent of \(\{\hat{F}(\nu, \ell, m, j)\}_{(\nu,\ell,m,j) \in S}\).

4. The Schatten boundedness of \(T_S\)

4.1. The complex interpolation method and the duality principle. In order to generalize the restriction and Strichartz estimates involving a single function to the system of orthonormal functions, we need to introduce the complex interpolation method and the duality principle lemma in our context. We refer to Proposition 1 and Lemma 3 of [11] with appropriate modifications to obtain the following two results:
Proposition 4.1. Let \( \{T_z\} \) be an analytic family of operators on \( \mathbb{T} \times \mathbb{R}^n \) in the sense of Stein defined on the strip \( -\lambda_0 \leq \text{Re}(z) \leq 0 \) for some \( \lambda_0 > 1 \). Assume that the following bounds
\[
\begin{align*}
\|T_is\|_{L^2(T,L_{k,a}^2(\mathbb{R}^n))} & \rightarrow L^2(T,L_{k,a}^2(\mathbb{R}^n)) \leq M_0 e^{a|s|}, \\
\|T_{-\lambda_0}+is\|_{L^1(T,L_{k,a}^2(\mathbb{R}^n))} & \rightarrow L^\infty(T,L_{k,a}^\infty(\mathbb{R}^n)) \leq M_1 e^{b|s|},
\end{align*}
\]
for all \( s \in \mathbb{R} \), for some \( a, b, M_0, M_1 \geq 0 \). Then, for all \( W_1, W_2 \in L^{2\lambda_0}(\mathbb{T} \times \mathbb{R}^n) \), the operator \( W_1 T_{-1} W_2 \) belongs to \( G^{2\lambda_0}(L^2(T,L_{k,a}^2(\mathbb{R}^n))) \) and we have the estimate
\[
\|W_1 T_{-1} W_2\|_{G^{2\lambda_0}(L^2(T,L_{k,a}^2(\mathbb{R}^n)))} \leq M_0^{1-\frac{a}{\lambda_0}} M_1^{\frac{a}{\lambda_0}} \|W_1\|_{L^{2\lambda_0}(T,L_{k,a}^2(\mathbb{R}^n))} \|W_2\|_{L^{2\lambda_0}(T,L_{k,a}^2(\mathbb{R}^n))}.
\]

Lemma 4.2 (Duality principle). Let \( \lambda \geq 1 \) and \( S \subset \mathbb{Z} \times \mathcal{A} \) be a discrete surface. Assume that \( A \) is a bounded linear operator from \( \ell^p(S) \) to \( L^q(\mathbb{T}, L^p_{k,a}(\mathbb{R}^n)) \) for some \( p, q \geq 1 \). Then the following statements are equivalent.

1. There is a constant \( C > 0 \) such that for all \( W \in L^{\frac{2q}{p+q}}(\mathbb{T}, L^{\frac{2p}{p+q}}_{k,a}(\mathbb{R}^n)) \)
\[
\|W A A^* \Phi\|_{G^\lambda(L^2(T,L_{k,a}^2(\mathbb{R}^n)))} \leq C \|W\|^2_{L^{\frac{2q}{p+q}}(\mathbb{T}, L^{\frac{2p}{p+q}}_{k,a}(\mathbb{R}^n))},
\]
where the function \( W \) is interpreted as an operator which acts by multiplication.

2. There is a constant \( C' > 0 \) such that for any orthonormal system \( \{f_i\}_{i \in I} \) in \( L^2_{k,a}(\mathbb{R}^n) \) and any sequence \( \{n_i\}_{i \in I} \) in \( \mathbb{C} \)
\[
\left\| \sum_{i \in I} n_i |A f_i|^2 \right\|_{L^{\frac{2q}{p+q}}(\mathbb{T}, L^{\frac{2p}{p+q}}_{k,a}(\mathbb{R}^n))} \leq C' \left( \sum_{i \in I} |n_i|^{p'} \right)^{\frac{q}{p'}}.
\]

Remark 4.3. Proposition 4.1 and Lemma 4.2 is also valid in the domain \( (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}^n \).

4.2. On general discrete surface. In this subsection we establish the Schatten boundedness of \( T_S \) on general discrete surface. Let \( S = \{ (\nu, \ell, m, j) \in \mathbb{Z} \times \mathcal{A} : R(\ell, m, j, \nu) = 0 \} \) be a discrete surface, where \( R(\nu, \ell, m, j) \) is a polynomial of degree one, with respect to the counting measure.

For some \( \lambda_0 > 1 \) and \( -\lambda_0 \leq \text{Re}(z) \leq 0 \), consider the analytic family of generalized functions
\[
G_z(\nu, \ell, m, j) = \psi(z) R(\nu, \ell, m, j)^z,
\]
where \( \psi(z) \) is an appropriate analytic function with a simple zero at \( z = -1 \) with exponential growth at infinity when \( \text{Re}(z) = 0 \) and
\[
R(\nu, \ell, m, j)^z = \begin{cases} 
R(\nu, \ell, m, j)^z & \text{for } R(\nu, \ell, m, j) > 0, \\
0 & \text{for } R(\nu, \ell, m, j) \leq 0.
\end{cases}
\]
Restricting the Schwartz class function \( \Phi \) on \( \mathbb{Z} \times \mathcal{A} \), we have
\[
\langle G_z, \Phi \rangle := \psi(z) \sum_{r \in \mathbb{Z}} \sum_{(\nu, \ell, m, j) : R(\nu, \ell, m, j) = r} \Phi(\nu, \ell, m, j),
\]
where \( r^z \) is defined as in (4.5). Using one dimensional analysis of \( r^z \) (see [12]), we have
\[
\lim_{z \to -1} \langle G_z, \Phi \rangle = \sum_{(\nu, \ell, m, j) \in S} \Phi(\nu, \ell, m, j),
\]
and this ensures that \( G_{-1} = \delta_S \).
For $-\lambda_0 \leq \text{Re}(z) \leq 0$, define the analytic family of operators $T_z$ acting on Schwartz class functions on $\mathbb{T} \times \mathbb{R}^n$ by

$$T_z F(t, x) = \sum_{(\nu, \ell, m, j) \in \mathbb{Z} \times A} \hat{F}(\nu, \ell, m, j) G_z(\nu, \ell, m, j) \Phi_{\ell,m,j}(x) e^{-it\nu}.$$ 

Then $\{T_z\}$ is an analytic in the sense of Stein defined on the strip $-\lambda_0 \leq \text{Re}(z) \leq 0$ for some $\lambda_0 > 1$. Moreover, we have the identity $T_{\text{S}} = T_{-1}$ and

$$T_z F(t, x) = \sum_{(\nu, \ell, m, j) \in \mathbb{Z} \times A} \hat{F}(\nu, \ell, m, j) G_z(\nu, \ell, m, j) \Phi_{\ell,m,j}(x) e^{-it\nu}$$

$$= \frac{1}{2\pi} \sum_{(\nu, \ell, m, j) \in \mathbb{Z} \times A} \left[ \int_{\mathbb{R}^n} \int_{\mathbb{T}} F(\tau, y) \Phi_{\ell,m,j}(y) e^{i\nu \tau} d\tau v_{k,a}(y) dy \right] G_z(\nu, \ell, m, j) \Phi_{\ell,m,j}(x) e^{-it\nu}$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}^n} \int_{\mathbb{T}} \left[ \sum_{(\nu, \ell, m, j) \in \mathbb{Z} \times A} \Phi_{\ell,m,j}(y) \Phi_{\ell,m,j}(x) G_z(\nu, \ell, m, j) e^{-i\nu(t-\tau)} \right] F(\tau, y) d\tau v_{k,a}(y) dy$$

$$= \int_{\mathbb{R}^n} \left( K_z(\cdot, x, y) * F(\cdot, y) \right)(t) v_{k,a}(y) dy,$$

(4.6)

where

$$K_z(t, x, y) = \frac{1}{2\pi} \sum_{(\nu, \ell, m, j) \in \mathbb{Z} \times A} \Phi_{\ell,m,j}(y) \Phi_{\ell,m,j}(x) G_z(\nu, \ell, m, j) e^{-i\nu t}.$$ 

When $\text{Re}(z) = 0$, we have

$$||T_{is}||_{L^2(\mathbb{T}, L^2_{k,a}(\mathbb{R}^n)) \rightarrow L^2(\mathbb{T}, L^2_{k,a}(\mathbb{R}^n))} = ||G_{is}||_{\mathcal{L}(L^2(\mathbb{T}, L^2_{k,a}(\mathbb{R}^n)))} \leq ||\psi(i\lambda)||.$$ 

(4.7)

Again an application of Hölder and Young inequalities in (4.6) gives

$$||T_z||_{L^1(\mathbb{T}, L^1_{k,a}(\mathbb{R}^n)) \rightarrow L^{\infty}(\mathbb{T}, L^\infty_{k,a}(\mathbb{R}^n))} \leq \sup_{x,y \in \mathbb{R}^n, t \in \mathbb{T}} |K_z(t, x, y)|.$$ 

(4.8)

Using Proposition 4.1, we obtain the following Schatten boundedness of the form (4.3).

**Lemma 4.4.** Let $n \geq 1$ and $S \subset \mathbb{Z} \times A$ be a discrete surface. Suppose that for each $x, y \in \mathbb{R}$ and $t \in \mathbb{T}$, $|K_z(t, x, y)|$ is uniformly bounded by a constant $C(s)$ with at most exponential growth in $s$ at when $z = -\lambda_0 + is$ for some $\lambda_0 > 1$. Then for all $W_1, W_2 \in L^{2\lambda_0}(\mathbb{T}, L^2_{k,a}(\mathbb{R}^n))$, the operator $W_1T_zW_2 = W_1T_{-1}W_2$ belongs to $G^{2\lambda_0}(L^2(\mathbb{T}, L^2_{k,a}(\mathbb{R}^n)))$ and we have the estimate

$$||W_1T_zW_2||_{g^{2\lambda_0}(L^2(\mathbb{T}, L^2_{k,a}(\mathbb{R}^n)))} \leq C ||W_1||_{L^{2\lambda_0}(\mathbb{T}, L^2_{k,a}(\mathbb{R}^n))} ||W_2||_{L^{2\lambda_0}(\mathbb{T}, L^2_{k,a}(\mathbb{R}^n))}.$$ 

**Remark 4.5.** Lemma 4.4 also holds true in the domain $(-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}^n$.

4.3. **On the particular surface** $S = \{(\nu, \ell, m, j) \in \mathbb{Z} \times A : \nu = 2\ell + \frac{2m}{a}\}$. In this subsection we work on this particular discrete surface $S = \{(\nu, \ell, m, j) \in \mathbb{Z} \times A : \nu = 2\ell + \frac{2m}{a}\}$ with respect to the counting measure.

Setting $\psi(z) = \frac{1}{1(z+1)}$ and $R(\nu, \ell, m, j) = \nu - 2\ell - \frac{2m}{a}$ in (4.4), then we have

$$G_z(\nu, \ell, m, j) = \frac{1}{\Gamma(z+1)} \left( \nu - (2\ell + \frac{2m}{a}) \right)^z_+$$

and

$$\lim_{z \to -1} \langle G_z, \Phi \rangle = \lim_{z \to -1} \frac{1}{\Gamma(z+1)} \sum_{r \in \mathbb{Z}} \sum_{\{\nu, \ell, m, j, \nu - (2\ell + \frac{2m}{a}) = r\}} \Phi(\nu, \ell, m, j).$$
Thus $G_{-1} = \delta_S$ and
\[
T_z F(t, x) = \int_{\mathbb{R}^n} (K_z(\cdot, x, y) * F(\cdot, y))(t) \psi_ka(y) dy,
\]
where
\[
K_z(t, x, y) = \frac{1}{2\pi} \sum_{(\nu, \ell, m, j) \in S} \Phi_{\ell, m, j}(y) \Phi_{\ell, m, j}(x) G_z(\nu, \ell, m, j) e^{-i\nu t}
\]
\[
= \frac{1}{2\pi \Gamma(z + 1)} \sum_{(\ell, m, j, n) \in A, \ell, m, j \in \mathbb{Z} \times A} \Phi_{\ell, m, j}(y) \Phi_{\ell, m, j}(x) \left( \nu - (2\ell + 2m + 2n) \right) z e^{-i\nu t}
\]
\[
= \frac{e^{it(1 + z/2)(\gamma + n - 2)}}{2\pi \Gamma(z + 1)} \sum_{(\ell, m, j, n) \in A, \ell, m, j \in \mathbb{Z} \times A} e^{-it(2\ell + 2m + 2\gamma + n - 2)} \Phi_{\ell, m, j}(y) \Phi_{\ell, m, j}(x) \sum_{r=0}^{\infty} r_+^z e^{-irt}
\]
\[
= \frac{e^{it(1 + z/2)(\gamma + n - 2)}}{2\pi \Gamma(z + 1)} \Delta_{k,a}(x, y; t) \sum_{r=0}^{\infty} r_+^z e^{-irt},
\]
where the last equality comes from the spectral decomposition of $e^{-i\Delta_{k,a}}$. In order to obtain a uniformly estimate of the kernel $K_z$, we need to estimate $\sum_{r=0}^{\infty} r_+^z e^{-irt}$.

**Lemma 4.6** ([17]). Let $-\lambda_0 \leq \text{Re}(z) < 0$ for some $\lambda_0 > 1$. Then the series $\sum_{r=0}^{\infty} r_+^z e^{-irt}$ is the Fourier series of an integrable function on $[-\pi, \pi]$ which is of class $C^\infty$ on $[-\pi, \pi] \setminus \{0\}$. Near the origin this function has the same singularity as the function whose values are $\Gamma(z + 1)(it)^{-z-1}$, i.e.,
\[
\sum_{r=0}^{\infty} r_+^z e^{-irt} \sim \Gamma(z + 1)(it)^{-z-1} + b(t),
\]
where $b \in C^\infty[-\pi, \pi]$. Here the notation $x \sim y$ stands for, there exists constants $C_1, C_2 > 0$ such that $C_1 x \leq y \leq C_2 x$.

When $\text{Re}(z) = 0$, as (4.7), we have
\[
\|T_{is}\|_{L^2((-\frac{\pi}{2}, \frac{\pi}{2}), L^2_{k,a}(\mathbb{R}^n)) \to L^2((-\frac{\pi}{2}, \frac{\pi}{2}), L^2_{k,a}(\mathbb{R}^n))} = \|G_{is}\|_{\ell^\infty(Z \times A)} 
\leq |\psi(is)| = \frac{1}{\Gamma(1 + is)} \leq C e^{\pi|s|/2}.
\]
(4.12)

When $z = -\lambda_0 + is$, (4.8) gives $T_z$ is bounded from $L^1\left((-\frac{\pi}{2}, \frac{\pi}{2}), L^1_{k,a}(\mathbb{R}^n)\right)$ to $L^\infty\left((-\frac{\pi}{2}, \frac{\pi}{2}), L^\infty_{k,a}(\mathbb{R}^n)\right)$ if $|K_z(t, x, y)|$ is uniformly bounded for each $x, y \in \mathbb{R}^n$ and $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$. When $a = 1, 2$, from (4.10), Proposition 4.6, and Proposition 2.3, we get
\[
|K_z(t, x, y)| \sim \frac{1}{|t|^{\text{Re}(z) + \frac{2\gamma + n + a - 2}{a}}}, \quad \forall x, y \in \mathbb{R}^n, t \in (-\frac{\pi}{2}, \frac{\pi}{2}).
\]
(4.13)

Therefore, for every $x, y \in \mathbb{R}$ and $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $|K_z(t, x, y)|$ is uniformly bounded if $\text{Re}(z) = -1 - \frac{2\gamma + n + a - 2}{a}$. Thus choosing $\lambda_0 = 1 + \frac{2\gamma + n + a - 2}{a}$, $T_z$ is bounded from $L^1\left((-\frac{\pi}{2}, \frac{\pi}{2}), L^1_{k,a}(\mathbb{R}^n)\right)$ to $L^\infty\left((-\frac{\pi}{2}, \frac{\pi}{2}), L^\infty_{k,a}(\mathbb{R}^n)\right)$.

Now the Schatten boundedness comes out from Lemma 4.4,
Lemma 4.7. Suppose $a = 1, 2$ and $k$ is a non-negative multiplicity function such that

$$a + 2\gamma + n - 2 > 0.$$ 

Let $n \geq 1$ and $S = \{(n, \ell, m, j) \in \mathbb{Z} \times \mathbb{A} : \nu = 2\ell + \frac{2mn}{a}\}$ be a discrete surface on $\mathbb{Z} \times \mathbb{A}$ and $\lambda_0 = 1 + \frac{2\gamma + n + a - 2}{a}$. Then for all $W_1, W_2 \in L^{2\lambda_0} \left(\left(-\frac{\pi}{2}, \frac{\pi}{2}\right), L^{2\lambda_0}_{k,a}(\mathbb{R}^n)\right)$, the operator $W_1 T_S W_2 = W_1 T_{-1} W_2$ belongs to $G^{2\lambda_0} \left(L^2\left((-\frac{\pi}{2}, \frac{\pi}{2}), L^2_{k,a}(\mathbb{R}^n)\right)\right)$ and we have the estimate

$$\|W_1 T_S W_2\|_{G^{2\lambda_0} \left(L^2\left((-\frac{\pi}{2}, \frac{\pi}{2}), L^2_{k,a}(\mathbb{R}^n)\right)\right)} \leq C \|W_1\|_{L^{2\lambda_0} \left(\left(-\frac{\pi}{2}, \frac{\pi}{2}\right), L^{2\lambda_0}_{k,a}(\mathbb{R}^n)\right)} \|W_2\|_{L^{2\lambda_0} \left(\left(-\frac{\pi}{2}, \frac{\pi}{2}\right), L^{2\lambda_0}_{k,a}(\mathbb{R}^n)\right)}.$$

5. Restriction theorems and Strichartz inequalities for orthonormal functions

Let $S = \{(n, \ell, m, j) \in \mathbb{Z} \times \mathbb{A} : \nu = 2\ell + \frac{2mn}{a}\}$ be a discrete surface on $\mathbb{Z} \times \mathbb{A}$ and $\lambda_0 = 1 + \frac{2\gamma + n + a - 2}{a}$. From Theorem 4.7, we have

$$\|W_1 E_S (E_S)^* W_2\|_{G^{2\lambda_0} \left(L^2\left((-\frac{\pi}{2}, \frac{\pi}{2}), L^2_{k,a}(\mathbb{R}^n)\right)\right)} \leq C \|W_1\|_{L^{2\lambda_0} \left(\left(-\frac{\pi}{2}, \frac{\pi}{2}\right), L^{2\lambda_0}_{k,a}(\mathbb{R}^n)\right)} \|W_2\|_{L^{2\lambda_0} \left(\left(-\frac{\pi}{2}, \frac{\pi}{2}\right), L^{2\lambda_0}_{k,a}(\mathbb{R}^n)\right)}.$$ 

(5.1)

Taking $A = E_S$, by Theorem 3.1, Lemma 4.2, and (5.1), we have the restriction theorem for the system of orthonormal functions.

Theorem 5.1 (Restriction estimates for orthonormal functions-diagonal case). Suppose $a = 1, 2$ and $k$ is a non-negative multiplicity function such that

$$a + 2\gamma + n - 2 > 0.$$ 

Let $n \geq 1$ and $S = \{(n, \ell, m, j) \in \mathbb{Z} \times \mathbb{A} : \nu = 2\ell + \frac{2mn}{a}\}$. For any (possible infinity) orthonormal system $\{\hat{F}(n, \ell, m, j)\}_{\ell \in \mathbb{Z}}$ in $\ell^2(S)$ and any sequence $\{n_i\}_{i \in I}$ in $\mathbb{C}$

$$\left\| \sum_{i \in I} n_i |E_S \{\hat{F}(n, \ell, m, j)\}|^2 \right\|_{L^{1+\frac{2\gamma + n + a - 2}{a}} \left((-\frac{\pi}{2}, \frac{\pi}{2}), L^{1+\frac{2\gamma + n + a - 2}{a}}_{k,a}(\mathbb{R}^n)\right)} \leq C \|\{n_i\}_{i \in I}\|_{\ell^{\frac{2(2\gamma + n + 2a - 2)}{a}}}.$$

where $C > 0$ is independent of $\{\hat{F}(n, \ell, m, j)\}_{(\mu, \nu) \in \mathbb{N}^n \times \mathbb{Z}}$ and $\{n_i\}_{i \in I}$.

By (3.2), we generalize the Strichartz inequality involving systems of orthonormal functions.

Theorem 5.2 (Strichartz inequalities for orthonormal functions-diagonal case). Suppose $a = 1, 2, n \geq 1$ and $k$ is a non-negative multiplicity function such that

$$a + 2\gamma + n - 2 > 0.$$ 

For any (possible infinity) orthonormal system $\{f_i\}_{i \in I}$ in $L^2_{k,a}(\mathbb{R}^n)$ and any sequence $\{n_i\}_{i \in I}$ in $\mathbb{C}$, we have

$$\left\| \sum_{i \in I} n_i |e^{-it\Delta_{k,a}} f_i|^2 \right\|_{L^{1+\frac{2\gamma + n + a - 2}{a}} \left((-\frac{\pi}{2}, \frac{\pi}{2}), L^{1+\frac{2\gamma + n + a - 2}{a}}_{k,a}(\mathbb{R}^n)\right)} \leq C \|\{n_i\}_{i \in I}\|_{\ell^{\frac{2(2\gamma + n + 2a - 2)}{a}}}.$$

where $C > 0$ is independent of $\{f_i\}_{i \in I}$ and $\{n_i\}_{i \in I}$.

To obtain the Strichartz inequality for the general case, we have to prove the following Schatten boundedness.

Proposition 5.3 (General Schatten bound for the extension operator). Suppose $a = 1, 2$ and $k$ is a non-negative multiplicity function such that

$$a + 2\gamma + n - 2 > 0.$$ 

Let $n \geq 1$ and $S = \{(n, \ell, m, j) \in \mathbb{Z} \times \mathbb{A} : \nu = 2\ell + \frac{2mn}{a}\}$. Then for any $p, q \geq 1$ satisfying

$$\frac{1}{q} + \frac{2\gamma + n + a - 2}{pa} = \frac{1}{2}, \quad \frac{4\gamma + 2n + 3a - 4}{a} < p \leq \frac{2(2\gamma + n + 2a - 2)}{a}.$$

we have
\[ \|W_1 T_2 W_2\|_{L^p((-\frac{\pi}{2}, \frac{\pi}{2}), L^q_{x,a}(\mathbb{R}^n))} \leq C \|W_1\|_{L^q((-\frac{\pi}{2}, \frac{\pi}{2}), L^p_{x,a}(\mathbb{R}^n))} \|W_2\|_{L^q((-\frac{\pi}{2}, \frac{\pi}{2}), L^p_{x,a}(\mathbb{R}^n))}, \]
for all \( W_1, W_2 \in L^q((-\frac{\pi}{2}, \frac{\pi}{2}), L^p_{x,a}(\mathbb{R}^n)) \), where \( C > 0 \) is independent of \( W_1, W_2 \).

In order to prove Proposition 5.3, we need to obtain the following revised Hardy-Littlewood-Sobolev inequality.

**Lemma 5.4.** For \( 0 \leq \lambda < 1 \) and \( 1 < p, q < \infty \) such that \( \frac{1}{p} + \frac{1}{q} + \lambda = 2 \), we have
\[ \left| \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{g(t) h(\tau)}{\sin(t-\tau)^{\lambda}} \, dt \, d\tau \right| \leq C \|g\|_{L^p(-\frac{\pi}{2}, \frac{\pi}{2})} \|h\|_{L^q(-\frac{\pi}{2}, \frac{\pi}{2})}. \]

**Proof.** Without loss of generality, we assume \( g, h \geq 0 \). We divide \((-\frac{\pi}{2}, \frac{\pi}{2})^2\) into three disjoint subsets:
\[ B_1 = \left\{ (t, \tau) \in (-\frac{\pi}{2}, \frac{\pi}{2})^2 : |t-\tau| \leq \frac{\pi}{2} \right\}, \]
\[ B_2 = \left\{ (t, \tau) \in (-\frac{\pi}{2}, \frac{\pi}{2})^2 : \frac{\pi}{2} < t-\tau < \pi \right\}, \]
and
\[ B_3 = \left\{ (t, \tau) \in (-\frac{\pi}{2}, \frac{\pi}{2})^2 : \pi < t-\tau < \pi \right\}. \]

Then the integral is split into three parts
\[ I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{g(t) h(\tau)}{|\sin(t-\tau)|^{\lambda}} \, dt \, d\tau : = I_1 + I_2 + I_3, \]
where
\[ I_j = \int_{B_j} \frac{g(t) h(\tau)}{|\sin(t-\tau)|^{\lambda}} \, dt \, d\tau, \quad \forall j = 1, 2, 3. \]

For any \( t \in [0, \frac{\pi}{2}] \), we have \( \frac{3}{2}t \leq \sin t \leq t \). Thus when \( (t, \tau) \in B_1 \), we get \( \frac{3}{2}|t-\tau| \leq |\sin(t-\tau)| \leq |t-\tau| \). Then
\[ I_1 = \int_{B_1} \frac{g(t) h(\tau)}{|\sin(t-\tau)|^{\lambda}} \, dt \, d\tau \leq \left( \frac{\pi}{2} \right)^{\lambda} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{g(t) h(\tau)}{|t-\tau|^{\lambda}} \, dt \, d\tau \leq C \|g\|_{L^p(-\frac{\pi}{2}, \frac{\pi}{2})} \|h\|_{L^q(-\frac{\pi}{2}, \frac{\pi}{2})}. \]

When \( (t, \tau) \in B_2 \), we have \( 0 < \pi + \tau - t < \frac{\pi}{2} \) and \( (t, \pi + \tau) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \times (\frac{\pi}{2}, \frac{3\pi}{2}) \). Then \( \frac{3}{2}(\pi + \tau - t) \leq \sin(\pi + \tau - t) \leq (\pi + \tau - t) \), i.e., \( \frac{3}{2}(\pi + \tau - t) \leq \sin(t-\tau) \leq (\pi + \tau - t) \). Thus it follows that
\[ I_2 = \int_{B_2} \frac{g(t) h(\tau)}{|\sin(t-\tau)|^{\lambda}} \, dt \, d\tau \leq \left( \frac{\pi}{2} \right)^{\lambda} \int_{B_2} \frac{g(t) h(\tau)}{|\pi + \tau - t|^{\lambda}} \, dt \, d\tau \leq \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g(t) dt \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{h(s-\pi)}{|s-t|^{\lambda}} ds \leq C \|g\|_{L^p(-\frac{\pi}{2}, \frac{\pi}{2})} \|h\|_{L^q(-\frac{\pi}{2}, \frac{\pi}{2})} = C \|g\|_{L^p(-\frac{\pi}{2}, \frac{\pi}{2})} \|h\|_{L^q(-\frac{\pi}{2}, \frac{\pi}{2})}. \]

When \( (t, \tau) \in B_3 \), analogous to the subset \( B_2 \), we also have
\[ I_3 \leq C \|g\|_{L^p(-\frac{\pi}{2}, \frac{\pi}{2})} \|h\|_{L^q(-\frac{\pi}{2}, \frac{\pi}{2})}. \]
Therefore, from (5.2), (5.3), and (5.4), we have
\[ I = I_1 + I_2 + I_3 \leq C \|g\|_{L^p\left(\left(-\frac{x}{T},\frac{x}{T}\right)\right)} \|h\|_{L^q\left(\left(-\frac{x}{T},\frac{x}{T}\right)\right)}. \]

Now we are in a position to prove Proposition 5.3.

**Proof of Proposition 5.3.** For \( 0 < \lambda < \lambda_0 = 1 + \frac{2\gamma + n + a - 2}{a} \), the operator \( T_{-\lambda+is} \) is an integral operator with kernel \( K_{-\lambda+is}(t - \tau, x, y) \) defined in (4.10), Applying Lemma (5.4) along with (4.12) and (4.13), we have
\[
\|W_1^{\lambda - is}T_{-\lambda+is}W_2^{\lambda - is}\|_{L^2((-\frac{x}{T},\frac{x}{T}), L^2_{k,a}(\mathbb{R}^n))}^2 \\
= \int_{-\frac{x}{T}}^{\frac{x}{T}} \int_{-\frac{x}{T}}^{\frac{x}{T}} \int_{\mathbb{R}^n} |W_1(t, x)(2\lambda)K_{-\lambda+is}(t - \tau, x, y)|^2 |W_2(\tau, y)(2\lambda)|^2 v_{k,a}(x)dx \ v_{k,a}(y)dydtd\tau \\
\leq C \|W_1\|_{L^{2\lambda}_{\mathbb{R}^n}} \|W_2\|_{L^{2\lambda}_{\mathbb{R}^n}} \ \|W_1\|_{L^{2\lambda}_{\mathbb{R}^n}} \|W_2\|_{L^{2\lambda}_{\mathbb{R}^n}} \\
= \|W_1\|_{L^{2\lambda}_{\mathbb{R}^n}} \|W_2\|_{L^{2\lambda}_{\mathbb{R}^n}} \\
\leq C\|W_1\|_{L^{2\lambda}_{\mathbb{R}^n}} \|W_2\|_{L^{2\lambda}_{\mathbb{R}^n}} \\
\leq C\|W_1\|_{L^{2\lambda}_{\mathbb{R}^n}} \|W_2\|_{L^{2\lambda}_{\mathbb{R}^n}} \ \|W_1\|_{L^{2\lambda}_{\mathbb{R}^n}} \|W_2\|_{L^{2\lambda}_{\mathbb{R}^n}}.
\]

provided \( 0 \leq 2\lambda_0 - 2\lambda < 1 \), i.e., \( 2\lambda_0 - 1 < 2\lambda \leq 2\lambda_0 \). By Theorem 2.9 of [21] and the identity \( T_S = T_{-1} \), we have
\[
\|W_1T_5W_2\|_{L^{2\lambda}_{\mathbb{R}^n}} = \|W_1T_5W_2\|_{L^{2\lambda}_{\mathbb{R}^n}} \\
\leq C\|W_1\|_{L^{2\lambda}_{\mathbb{R}^n}} \|W_2\|_{L^{2\lambda}_{\mathbb{R}^n}}. \tag{5.5}
\]

Again by Lemma 4.2, we have
\[
\left\| \sum_{i \in I} n_i |E_i\{\{F_i(\nu, \ell, m, j)\}\}|^2 \right\|_{L^{2\lambda}_{\mathbb{R}^n}} \leq C \left( \sum_{i \in I} |n_i|^{(2\lambda)'} \right)^{(2\lambda)'}, \tag{5.6}
\]
for any orthonormal system \( \{\{F_i(\nu, \ell, m, j)\}\}_{i \in I} \) in \( \ell^2(S) \) and \( \{n_i\}_{i \in I} \) in \( 
\mathbb{C} \), which is equivalent to
\[
\left\| \sum_{i \in I} n_i |e^{-it\Delta k,a}f_i|^2 \right\|_{L^{2\lambda}_{\mathbb{R}^n}} \leq C \left( \sum_{i \in I} |n_i|^{(2\lambda)'} \right)^{(2\lambda)'}, \tag{5.7}
\]
for any orthonormal system \( \{f_i\}_{i \in I} \) in \( L^{2\lambda}_{k,a}(\mathbb{R}^n) \) and \( \{n_i\}_{i \in I} \) in \( 
\mathbb{C} \). This completes the proof of the proposition. \( \square \)

Now we are ready to establish the restriction theorem for systems of orthonormal functions.

**Theorem 5.5** (Restriction estimates for orthonormal functions-general case). Suppose \( a = 1, 2 \) and \( k \) is a non-negative multiplicity function such that \( a + 2\gamma + n - 2 > 0 \). Let \( n \geq 1 \) and \( S = \{(\nu, \ell, m, j) \in \mathbb{Z} \times A : \nu = 2\ell + \frac{2m}{a} \} \). If \( p, q, n \geq 1 \) such that
\[
1 < p < \frac{4\gamma + 2n + 3a - 4}{4\gamma + 2n + a - 4} \quad \text{and} \quad 1 < \frac{2\gamma + n + a - 2}{pa} = \frac{2\gamma + n + a - 2}{a},\n\]
the restriction estimate
\[
\|F\|_{L^p(\mathbb{R}^n)} \leq C\|\phi_{\chi_S}(F)\|_{L^q(\mathbb{R}^n)},
\]
holds for the functions \( F \).
for any (possible infinity) orthonormal system \( \{ \{ \hat{F}_i(\nu, \ell, m, j) \} \}_{(\nu, \ell, m, j) \in \mathbb{Z} \times A} \) in \( \ell^2(S) \) and any sequence \( \{ n_i \}_{i \in I} \) in \( \mathbb{C} \), we have

\[
\left\| \sum_{i \in I} n_i \mathcal{E}_\Sigma \{ \hat{F}_i(\nu, \ell, m, j) \} \right\|^2_{L^n((-\frac{n}{2}, \frac{n}{2}), L^p_{k,a}(\mathbb{R}^n))} \leq C \left( \sum_{i} |n_i|^{\frac{p+1}{2p}} \right)^2,
\]

(5.8)

where \( C > 0 \) is independent of \( \{ \{ \hat{F}_i(\nu, \ell, m, j) \} \}_{(\nu, \ell, m, j) \in \mathbb{Z} \times A} \) and \( \{ n_i \}_{i \in I} \).

**Proof.** Using the fact that the operator \( e^{-it\Delta_{k,a}} \) is unitary, we have

\[
\left\| \sum_{i \in I} n_i |e^{-it\Delta_{k,a} f_i}|^2 \right\|^2_{L^n((-\frac{n}{2}, \frac{n}{2}), L^p_{k,a}(\mathbb{R}^n))} \leq \sum_{i \in I} |n_i|^2,
\]

for any (possible infinity) system \( \{ f_i \}_{i \in I} \) of orthonormal functions in \( L^2(\mathbb{R}^n_+, dw) \) and any coefficients \( \{ n_i \}_{i \in I} \) in \( \mathbb{C} \). It is equivalent to

\[
\left\| \sum_{i \in I} n_i \mathcal{E}_\Sigma \{ \hat{F}_i(\nu, \ell, m, j) \} \right\|^2_{L^n((-\frac{n}{2}, \frac{n}{2}), L^p_{k,a}(\mathbb{R}^n))} \leq \sum_{i \in I} |n_i|^2,
\]

for any (possible infinity) orthonormal system \( \{ \{ \hat{F}_i(\nu, \ell, m, j) \} \}_{(\nu, \ell, m, j) \in \mathbb{Z} \times A} \) in \( \ell^2(S) \) and any sequence \( \{ n_i \}_{i \in I} \) in \( \mathbb{C} \).

By Lemma 4.2, equivalently, we have the operator

\[
W \in L^2((-\frac{n}{2}, \frac{n}{2}), L^\infty_{k,a}(\mathbb{R}^n)) \mapsto WT_\Sigma W \in \mathcal{G}^\infty \left( L^2((-\frac{n}{2}, \frac{n}{2}), L^2_{k,a}(\mathbb{R}^n)) \right)
\]

is bounded. By Lemma 4.7, the operator

\[
W \in L^{\frac{2(2+n+a-2)}{2\gamma + n + a - 2}}((-\frac{n}{2}, \frac{n}{2}), L^{\frac{2(2+n+a-2)}{a}}_{k,a}(\mathbb{R}^n)) \mapsto WT_\Sigma W \in \mathcal{G}^{2(1+n+a-2)} \left( L^2((-\frac{n}{2}, \frac{n}{2}), L^2_{k,a}(\mathbb{R}^n)) \right)
\]

is also bounded. Applying the complex interpolation method in the Chapter 4 of [4], the operator

\[
W \in L^p((-\frac{n}{2}, \frac{n}{2}), L^p_{k,a}(\mathbb{R}^n)) \mapsto WT_\Sigma W \in \mathcal{G}^p \left( L^2((-\frac{n}{2}, \frac{n}{2}), L^2_{k,a}(\mathbb{R}^n, dw)) \right)
\]

is bounded for any \( p, q \geq 1 \) satisfying

\[
\frac{1}{q} + \frac{2\gamma + n + a - 2}{pa} = \frac{1}{2}, \quad \frac{2(2\gamma + n + 2a - 2)}{a} \leq p \leq \infty.
\]

Combined with Lemma 5.3, we have the operator

\[
W \in L^p((-\frac{n}{2}, \frac{n}{2}), L^p_{k,a}(\mathbb{R}^n)) \mapsto WT_\Sigma W \in \mathcal{G}^p \left( L^2((-\frac{n}{2}, \frac{n}{2}), L^2_{k,a}(\mathbb{R}^n, dw)) \right)
\]

is still bounded for any \( p, q \geq 1 \) satisfying

\[
\frac{1}{q} + \frac{2\gamma + n + a - 2}{pa} = \frac{1}{2}, \quad \frac{4\gamma + 2n + 3a - 4}{a} \leq p \leq \infty.
\]

Again by Theorem 3.1 and Lemma 4.2, we obtain our desired estimates. \( \square \)

As a consequences of the above result, we obtain the following Strichartz inequalities for the system of orthonormal functions.
Theorem 5.6 (Strichartz inequalities for orthonormal functions-general case). If $p,q,n \geq 1$ such that
\[
1 \leq p < \frac{4\gamma + 2n + 3a - 4}{4\gamma + 2n + a - 4} \quad \text{and} \quad \frac{1}{q} + \frac{2\gamma + n + a - 2}{pa} = \frac{2\gamma + n + a - 2}{a},
\]
then for any (possible infinity) system \( \{f_i\}_{i \in I} \) of orthonormal functions in \( L^2_{k,a}(\mathbb{R}^n) \) and any coefficients \( \{n_i\}_{i \in I} \in \mathbb{C} \), we have
\[
\left\| \sum_{i \in I} n_i |e^{-it\Delta_{k,a}}f_i|^2 \right\|_{L^q((-\frac{\pi}{2\gamma}, \frac{\pi}{2\gamma}), L^p_{k,a}(\mathbb{R}^n))} \leq C \left( \sum_{i \in I} |n_i|^{\frac{2p}{2p+1}} \right)^{\frac{2+1}{2p}},
\]
where \( C > 0 \) is independent of \( \{f_i\}_{i \in I} \) and \( \{n_i\}_{i \in I} \).

6. Orthornormal Strichartz inequalities for the Dunkl operator

In this section we prove Strichartz estimate for system of orthonormal functions associated with Dunkl operator \( \Delta_k \) using the relation between the semigroups corresponding to \( (k,a) \)-generalized Laguerre operator \( \Delta_{k,a} \) and the Dunkl operator \( \Delta_k \).

Corresponding to the Dunkl operator \( \Delta_k \), we have the solution \( e^{i\frac{t}{a}||x||^2-a\Delta_k}f \) to the free Schrödinger equation
\[
\begin{cases}
  i\partial_t w(t,x) + \frac{1}{a}||x||^{2-a}\Delta_k w(t,x) = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\
  w(0,x) = f(x), \quad x \in \mathbb{R}^n.
\end{cases}
\]

For \( a = 1,2 \) and \( k \) is a non-negative multiplicity function such that \( a + 2\gamma + n - 2 > 0 \), from Theorem 2.5, the classical estimates in this case may be stated as
\[
\left\| e^{i\frac{t}{a}||x||^{2-a}\Delta_k}f \right\|_{L^q(\mathbb{R}, L^p_{k,a}(\mathbb{R}^n))} \leq C \|f\|_{L^2_{k,a}(\mathbb{R}^n)}
\]
under the same conditions on \( p \) and \( q \) is that \( 1 \leq p, q \leq \infty \) satisfies
\[
\left( \frac{1}{2} - \frac{1}{p} \right) \frac{2\gamma + n + a - 2}{a} - \frac{1}{q} = 0.
\]

As a consequences of the above result, we obtain the following Strichartz inequality associated with Dunkl operator for the system of orthonormal functions.

Theorem 6.1 (Strichartz inequalities for orthonormal functions associated with Dunkl operator-general case). If \( p,q,n \geq 1 \) such that
\[
1 \leq p < \frac{4\gamma + 2n + 3a - 4}{4\gamma + 2n + a - 4} \quad \text{and} \quad \frac{1}{q} + \frac{2\gamma + n + a - 2}{pa} = \frac{2\gamma + n + a - 2}{a},
\]
then for any (possible infinity) system \( \{f_i\}_{i \in I} \) of orthonormal functions in \( L^2_{k,a}(\mathbb{R}^n) \) and any coefficients \( \{n_i\}_{i \in I} \in \mathbb{C} \), we have
\[
\left\| \sum_{i \in I} n_i |e^{\frac{t}{a}||x||^{2-a}\Delta_k}f_i|^2 \right\|_{L^q(\mathbb{R}, L^p_{k,a}(\mathbb{R}^n))} \leq C \left( \sum_{i \in I} |n_i|^{\frac{2p}{2p+1}} \right)^{\frac{2+1}{2p}},
\]
where \( C > 0 \) is independent of \( \{f_i\}_{i \in I} \) and \( \{n_i\}_{i \in I} \).

Proof. Let \( \Gamma_{k,a}(x,y;it) \) and \( \Lambda_{k,a}(x,y;it) \) be the kernel of \( e^{i\frac{t}{a}||x||^2-a\Delta_k} \) and \( e^{-it\Delta_{k,a}} \), respectively. Then we know that, using the change of variable \( s = \tan(t) \) with \( t \in (-\pi/2, \pi/2) \), we get
\[
\Lambda_{k,a}(x,y;it) = c_{k,a}^1 \left( 1 + s^2 \right)^{\frac{2\gamma + n + a - 2}{2a}} \exp \left( -is\frac{||x||^2}{a} \right) \Gamma_{k,a} \left( \left( 1 + s^2 \right)^{\frac{1}{2}} x, y; is \right)
\]
for any \( s > 0 \). From this, and using the scaling condition \( \frac{1}{q} + \frac{2\gamma + n + a - 2}{pa} = \frac{2\gamma + n + a - 2}{a} \), it follows (as in Theorem 2.5 for a single function) that
\[
\left\| \sum_{i \in I} e^{it\Delta_k f_i} \right\|_{L^q((0, +\infty), L^p_{k,a}(\mathbb{R}^n))} = \left\| \sum_{i \in I} e^{-it\Delta_k f_i} \right\|_{L^q((0, +\infty), L^p_{k,a}(\mathbb{R}^n))},
\]
and
\[
\left\| \sum_{i \in I} e^{it\Delta_k f_i} \right\|_{L^q((-\infty, 0), L^p_{k,a}(\mathbb{R}^n))} = \left\| \sum_{i \in I} e^{-it\Delta_k f_i} \right\|_{L^q((-\infty, 0), L^p_{k,a}(\mathbb{R}^n))},
\]
for any system of orthonormal functions \( \{f_i\}_{i \in I} \) in \( L^2_{k,a}(\mathbb{R}^n) \) and any coefficients \( \{n_i\}_{i \in I} \) in \( \mathbb{C} \). Hence, we immediately obtain our desired results from Theorem 5.6. \( \square \)

7. Final remarks

(1) When \( a = 2 \) and \( k \in \mathcal{K}^+ \), then Theorem 6.1 becomes: if
\[
p, q, n \geq 1, \quad 1 \leq p < \frac{2\gamma + n + 1}{2\gamma + n - 1} \quad \text{and} \quad \frac{2}{q} + \frac{2\gamma + n}{p} = 2\gamma + n,
\]
for any (possible infinity) system \( \{f_i\}_{i \in I} \) of orthonormal functions in \( L^2_k(\mathbb{R}^n) \) and any coefficients \( \{n_i\}_{i \in I} \) in \( \mathbb{C} \), we have
\[
\left\| \sum_{i \in I} n_i |e^{it\Delta_k f_i}|^2 \right\|_{L^q(\mathbb{R}, L^p(\mathbb{R}^n))} \leq C \left( \sum_{i \in I} |n_i|^{\frac{2p}{2p - n}} \right)^{\frac{p+1}{p}},
\]
which generalizes the known Strichartz estimates for the Schrödinger-Dunkl equation in [16] involving systems of orthonormal functions.

Moreover, when \( a = 2 \) and \( k \equiv 0 \), then Theorem 6.1 reduces: if
\[
p, q, n \geq 1, \quad 1 \leq p < \frac{n + 1}{n - 1} \quad \text{and} \quad \frac{2}{q} + \frac{n}{p} = n,
\]
for any (possible infinity) system \( \{f_i\}_{i \in I} \) of orthonormal functions in \( L^2(\mathbb{R}^n) \) and any coefficients \( \{n_i\}_{i \in I} \) in \( \mathbb{C} \), we have
\[
\left\| \sum_{i \in I} n_i |e^{it\Delta f_i}|^2 \right\|_{L^q(\mathbb{R}, L^p(\mathbb{R}^n))} \leq C \left( \sum_{i \in I} |n_i|^{\frac{2p}{2p - n}} \right)^{\frac{p+1}{p}},
\]
which is nothing but the classical orthonormal Strichartz inequality associated with Laplacian \( \Delta \) on \( \mathbb{R}^n \) proved in [10, 11].

(2) When \( a = 2 \) and \( k \in \mathcal{K}^+ \), then Theorem 5.6 becomes the orthonormal Strichartz inequality for the Schrödinger equation associated with the Dunkl-Hermite operator \( H_k \) which states: if
\[
p, q, n \geq 1, \quad 1 \leq p < \frac{2\gamma + n + 1}{2\gamma + n - 1} \quad \text{and} \quad \frac{2}{q} + \frac{2\gamma + n}{p} = 2\gamma + n,
\]
for any (possible infinity) system \( \{f_i\}_{i \in I} \) of orthonormal functions in \( L^2_k(\mathbb{R}^n) \) and any coefficients \( \{n_i\}_{i \in I} \) in \( \mathbb{C} \), we have
\[
\left\| \sum_{i \in I} n_i |e^{-itH_k f_i}|^2 \right\|_{L^q((-\frac{a}{2}, \frac{a}{2}), L^p_k(\mathbb{R}^n))} \leq C \left( \sum_{i \in I} |n_i|^{\frac{2p}{2p - n}} \right)^{\frac{p+1}{p}},
\]
which generalizes the known Strichartz estimates for the Dunkl-Hermite operator in Theorem 4.5 of [16].
Moreover, when $a = 2$ and $k \equiv 0$, then Theorem 5.6 reduces to: for

$$1 \leq p < \frac{n+1}{n-1} \quad \text{and} \quad \frac{2}{q} + \frac{n}{p} = n$$

and any (possible infinity) system $\{f_i\}_{i \in I}$ of orthonormal functions in $L^2(\mathbb{R}^n)$ and any coefficients $\{n_i\}_{i \in I}$ in $\mathbb{C}$, we have

$$\left\| \sum_{i \in I} n_i |e^{-itH}f_i|^2 \right\|_{L^q((-\pi,\pi),L^p(\mathbb{R}^n))} \leq C \left( \sum_{i \in I} |n_i|^{2p} \right)^{\frac{p+1}{2p}},$$

which is nothing but the classic Strichartz inequality for systems of orthonormal functions associated with the Hermite operator $H$ on $\mathbb{R}^n$ proved in [5,17].

(3) When $a = 1$ and $k \in K^+$, then Theorem 5.6 generalizes the Strichartz inequality for the Schrödinger propagator associated with the Dunkl-Laguerre operator $L_k$ on $\mathbb{R}^n$ (see Theorem A in [1]) involving systems of orthonormal functions and it states that: If $p, q, n \geq 1$ such that

$$1 \leq p < \frac{4\gamma + 2n - 1}{4\gamma + 2n - 3} \quad \text{and} \quad \frac{1}{q} + \frac{2\gamma + n - 1}{p} = 2\gamma + n - 1,$$

we have

$$\left\| \sum_{i \in I} n_i |e^{-itL_k}f_i|^2 \right\|_{L^q((-\pi,\pi),L^p_{k,1}(\mathbb{R}^n))} \leq C \left( \sum_{i \in I} |n_i|^{2p} \right)^{\frac{p+1}{2p}},$$

for any (possible infinity) system $\{f_i\}_{i \in I}$ of orthonormal functions in $L^2_{k,1}(\mathbb{R}^n)$ and any coefficients $\{n_i\}_{i \in I}$ in $\mathbb{C}$. Besides, it reduces the orthonormal Strichartz inequality for the propagator $e^{it\|x\|\Delta_k}$ which is a generalization of Theorem D in [1]. Moreover, when $k \equiv 0$, we can also obtain the orthonormal Strichartz inequality associated with the classical Laguerre operator $L$. We shall not list any more. We just intend to show the generality of our approach.

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