Spin excitations in the integrable open quantum group invariant supersymmetric $t$–$J$ model

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Abstract

The integrable quantum group $spl_q(2,1)$-invariant supersymmetric $t$–$J$ model with open boundaries is studied via an analytic treatment of the Bethe equations. An $su(2)$ feature is seen to hold for states at or close to half-filling. For these states the eigenvalues of the transfer matrix of the $t$–$J$ model satisfy a set of $su(2)$ functional relations. The finite-size corrections to the relevant eigenvalues, and thus the surface effect on the spin excitations, have been calculated analytically by solving the functional relations.

1 Introduction

The integrable supersymmetric $t$–$J$ model of strongly correlated electrons has a long and interesting history (see, e.g., [1, 2, 3, 4, 5] and refs therein). More recently, the construction of the integrable version with open boundary conditions [2, 3] had to await the systematic development of boundary integrability [3, 7]. For open boundaries, the integrable Hamiltonian reads

\[
H = -p \left\{ \sum_{j=1}^{L-1} \sum_{\sigma} \left( c_{j,\sigma}^\dagger c_{j+1,\sigma} + c_{j,\sigma} c_{j+1,\sigma}^\dagger \right) \right\} p 
- 2 \sum_{j=1}^{L-1} \left( S_j^+ S_{j+1}^x + S_j^y S_{j+1}^y + \cos \gamma \left( S_j^z S_{j+1}^z - \frac{n_j n_{j+1}}{4} \right) \right) - \cos \gamma \sum_{j=1}^{L} n_j 
+ i \sin \gamma (n_1 - n_L) - i \sin \gamma \sum_{j=1}^{L-1} (n_j S_j^z - S^z_{j+1} n_{j+1}) + H_+^s + H_+^s, \tag{1.1}
\]

where the boundary fields $H_\pm^s$ are dependent on two arbitrary parameters $\xi_\pm$ [3]. The operators $c_{j,\pm} \ (c_{j,\pm}^\dagger)$ are spin up or down annihilation (creation) operators. The $S_j = (S_j^x, S_j^y, S_j^z)$ are spin operators and $n_j$ the occupation number of electrons at site $j$. The operator $p = \prod_{j=1}^{L} (1 - n_j n_{j+1})$ forbids the double occupancy of electrons at one lattice site.
We begin by recalling the essential ingredients underlying the integrability of the Hamiltonian (1.1). The $R$-matrix (see, e.g., [2, 3]) is given by

$$R(v) = \begin{pmatrix}
a(v) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & b(v) & 0 & c_-(v) & 0 & 0 & 0 & 0 \\
0 & 0 & b(v) & 0 & 0 & c_-(v) & 0 & 0 \\
0 & c_+(v) & 0 & b(v) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a(v) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & b(v) & 0 & c_-(v) \\
0 & 0 & c_+(v) & 0 & 0 & 0 & b(v) & 0 \\
0 & 0 & 0 & 0 & 0 & c_+(v) & 0 & b(v) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & w(v)
\end{pmatrix}, \quad (1.2)$$

where $a(v) = \sin(\gamma - v)$, $b(v) = \sin v$, $c_\pm(v) = e^\pm iv \sin \gamma$ and $w(v) = \sin(\gamma + v)$. Here $v$ is the spectral parameter and $\gamma$ (or $q = e^{-i\gamma}$) is the crossing parameter. This matrix is a trigonometric solution of the Yang-Baxter equation

$$R^{12}(u - v)R^{13}(u)R^{23}(v) = R^{23}(v)R^{13}(u)R^{12}(u - v). \quad (1.3)$$

On the other hand, the boundary $K$-matrices are given in terms of

$$K(v, \xi) = \frac{1}{\sin \xi} \begin{pmatrix} e^{-iv} \sin(\xi + v) & 0 & 0 \\
0 & e^{iv} \sin(\xi - v) & 0 \\
0 & 0 & e^{iv} \sin(\xi - v) \end{pmatrix}, \quad (1.4)$$

with

$$K^-(v) = K(v, \xi_-), \quad (1.5)$$

$$K^+(v) = q^{-1/2} K(-v + \gamma/2, \xi_+) M. \quad (1.6)$$

The crossing matrix $M$ is

$$M = \begin{pmatrix} 1 & 0 & 0 \\
0 & q^2 & 0 \\
0 & 0 & -q^2 \end{pmatrix}. \quad (1.7)$$

The $K$-matrices satisfy the boundary version of the Yang-Baxter equation [3, 4]

$$R_{12}(u - v)K_{1}^{-}(u)R_{21}(u + v)K_{2}^{-}(v) = K_{2}(v)R_{12}(u + v)K_{1}^{-}(u)R_{21}(u - v). \quad (1.8)$$

Following Sklyanin [3], the open boundary condition transfer matrix is defined as [3, 4]

$$T(v) = \sum_{abcd} K_{ba}^\dagger(v)U_{ac}(v)K_{cd}(v)U_{db}(v) \equiv T^{(d)}_{ab(c)}(v), \quad (1.9)$$

with $U_{ac}(v)$ the monodromy matrix defined as the matrix product over the $R$’s,

$$U_{ab(c)}^{(d)}(v) = R_{b_2a_1}(v)R_{b_3a_2}(v)R_{b_4a_3}(v)...R_{b_c}^{d1}(v). \quad (1.10)$$
Here the indices $c$ and $d$ in parentheses are in the quantum space $\mathbb{C}^3 \times \mathbb{C}^3 \times \cdots \times \mathbb{C}^3$ with indices $a$ and $b$ in the horizontal auxiliary space $\mathbb{C}^3$ as usual. The Hamiltonian $H$ is the inverse of $U(v)$, with

$$U_{ab(c)}^{-1}(v) = \tilde{R}_{bc_1}^{ba}(v)\tilde{R}_{bc_2}^{ba}(v)\tilde{R}_{bc_3}^{ba}(v)\cdots\tilde{R}_{bc_L}^{ba}(v),$$

(1.11)

where

$$\tilde{R}_{cd}^{ab}(v) = \frac{R_{dc}^{ab}(-v)}{\sin(\gamma + v)\sin(\gamma - v)}.$$  

(1.12)

The elements of $\tilde{R}$ will be denoted with a tilde, i.e., $\tilde{a}, \tilde{b}, \tilde{c}_\pm$ and $\tilde{w}$. 

Given the above, the commutation relations

$$[T(v), T(u)] = 0$$

(1.13)

are fulfilled. The Hamiltonian $H$ follows from

$$\frac{\partial T(v)}{\partial v} \bigg|_{v=0} = -\frac{1}{4} \sin \gamma H tr K^+(0) + tr \hat{K}^+(0).$$

(1.14)

In the following we consider the quantum group $sp_{n}(2, 1)$-invariant case only, for which $H_\pm = 0$. The transfer matrix $[1, 9]$ has been diagonalised for this and the more general case by means of the algebraic Bethe Ansatz $[2, 3]$. The eigenvalues are given by

$$\lambda(v) = \lambda_A(v) + \lambda_{D_I}(v) + \lambda_{D_{II}}(v),$$

(1.15)

where

$$\lambda_A(v) = \prod_{i=1}^{N} \frac{a(v_i - v)b(v_i + v)}{b(v_i - v)a(v_i + v)} a^L(v)\tilde{a}^L(-v) k_A(v),$$

(1.16)

$$\lambda_{D_I}(v) = k_{D_I}(v)b^L(v)\tilde{b}^L(-v)\frac{b(2v)b(2v + \gamma)}{a(2v)w(2v + \gamma)} \left( \frac{c_ +(2v)}{a(2v)} - 1 \right) \left( 1 - \frac{c_ -(2v + \gamma)}{a(2v + \gamma)} \right)$$

$$\times \prod_{i=1}^{N} \frac{a(v_i - v_i)\tilde{a}(-v_i - v_i)}{b(v_i - v_i)a(v_i + v_i)} \prod_{j=1}^{M} \frac{a(v_j - v_i)b(v_j + v_i + \gamma)}{b(v_j - v_i)a(v_j + v_i + \gamma)},$$

(1.17)

$$\lambda_{D_{II}}(v) = -k_{D_{II}}(v)b^L(v)\tilde{b}^L(-v)\frac{b(2v)b(2v + \gamma)}{a(2v)w(2v + \gamma)} \left( \frac{c_ +(2v)}{a(2v)} - 1 \right)$$

$$\times \left( 1 - \frac{c_ -(2v + \gamma)}{a(2v + \gamma)} \right) \prod_{j=1}^{M} \frac{a(v_j - v_j)b(v_j + v_j + \gamma)}{b(v_j - v_j)a(v_j + v_j + \gamma)}.$$  

(1.18)

The related Bethe equations follow as

$$\left( \frac{a(v_k)\tilde{a}(-v_k)}{b(v_k)\tilde{b}(-v_k)} \right)^L \prod_{i\neq k}^{N} \frac{a(v_i - v_k)b(v_i + v_k)b(v_i + v_k + \gamma)}{a(v_k - v_i)a(v_i + v_k)\tilde{a}(-v_i - v_k - \gamma)}$$

$$\times \prod_{j=1}^{M} \frac{a(v_j + v_k + \gamma)b(v_j - v_k)}{b(v_j + v_k + \gamma)a(v_j - v_k)} = 1, \quad k = 1, \ldots, N,$$

(1.19)

$$\prod_{i=1}^{N} \frac{a(v_i - v_i)\tilde{a}(-v_i - v_i - \gamma)}{b(v_i - v_i)b(-v_i - v_i - \gamma)} = 1, \quad l = 1, \ldots, M.$$  

(1.20)
The local boundary factors are
\[ k_A(v) = k_D(v) = k_{D_{II}}(v) = 1. \] (1.21)

The three possible states \( \uparrow, \downarrow, 0 \) represent either an electron (with spin up or down) or no electron (a hole). These are described by the numbers \( N \) and \( M \) of roots in the nested Bethe equations, where \( M \) is the number of holes and \( N - M \) is the number of down spins. As for the periodic case [8], it follows that the magnetization \( S^x = \frac{1}{2}(n_\uparrow - n_\downarrow) = \frac{1}{2}(L - 2N + M) \) and the number of electrons \( Q = n_\uparrow + n_\downarrow = L - M \) are restricted to \( 0 \leq S^x \leq Q/2 \leq L/2 \). States in the half-filled band have one electron per site \( (M = 0) \).

Our aim is to calculate the massless spin excitations of the states at or close to half-filling. This has been done for the periodic model in the isotropic \( (\gamma \to 0) \) limit via the root density approach to the Bethe equations [9]. In Section 2 we establish an \( su(2) \) structure for the open quantum group invariant model and derive the corresponding functional relations. In Section 3 we derive the spin excitations by solving the functional relations for the finite-size corrections to the transfer matrix eigenspectra. The bulk and surface free energies of the vertex model and related \( t-J \) model are given in Section 4. A discussion and concluding remarks are given in Section 5.

## 2 Functional relations

We begin by considering the eigenvalue expression (1.15) from the functional relation viewpoint. To show the \( su(2) \) structure we use semi-standard Young tableaux as in the study of the six-vertex model with open boundaries [10].

### 2.1 The \( su(2) \) structure

Set \( \mathcal{L} = 2L \) and define
\[
\begin{align*}
\begin{array}{c}
1 \\
2 \\
S
\end{array}^k = & \sin^L(\gamma - k\gamma - v) \sin(2v - 2\gamma + 2k\gamma) \prod_{i=1}^{N} \frac{\sin(v - v_i + \gamma + k\gamma) \sin(v + v_i + k\gamma)}{\sin(v - v_i + k\gamma) \sin(v + v_i - \gamma + k\gamma)}, \quad (2.1) \\
\begin{array}{c}
1 \\
2 \\
S
\end{array}^k = & g_M(v + k\gamma) \sin^L(v + k\gamma) \sin(2v + 2k\gamma) \prod_{i=1}^{N} \frac{\sin(v - v_i - \gamma + k\gamma) \sin(v + v_i + k\gamma - 2\gamma)}{\sin(v - v_i + k\gamma) \sin(v + v_i - \gamma + k\gamma)}, \quad (2.2) \\
\begin{array}{c}
1 \\
2 \\
S
\end{array}^k = & g_M(v + \gamma + k\gamma) \sin^L(v - k\gamma) \sin^L(v + k\gamma + \gamma) \sin(2v + 2k\gamma - 2\gamma) \sin(2v + 2k\gamma + 2\gamma), \quad (2.3)
\end{align*}
\]

where
\[ g_M(v) = \prod_{j=1}^{M} \frac{\sin(v - v_j + \gamma) \sin(v + v_j - \gamma)}{\sin(v - v_j) \sin(v + v_j - 2\gamma)}. \] (2.4)

The eigenvalue expression (1.15) now reads
\[
\lambda(v) = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} \frac{\sin^{-1}(2v - 2\gamma)}{\sin^L(\gamma + v) \sin^L(\gamma - v)} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} \frac{1}{\sin^{-1}(2v - 2\gamma) \sin^{-1}(2v - 4\gamma)} \frac{\sin^{-1}(2v - 2\gamma) \sin^{-1}(2v - 4\gamma)}{\sin^L(2\gamma - v)}. \] (2.5)
To proceed further, introduce the auxiliary eigenvalues

\[ T_k^{(1)} = k + \begin{array}{c} 1 \\ 2 \end{array} \quad (2.6) \]

\[ T_0^{(q+1)} = \sum_{q+1} \begin{array}{c} \ldots \\ \varepsilon \end{array} \quad (2.7) \]

where for each Young tableaux it is understood that there are relative shifts in the arguments:

\[
\begin{array}{c}
\varepsilon \\
v + q \\
\ldots \\
v + \gamma \\
v
\end{array}
\quad (2.8)
\]

The zero superscript represents a shift in the right-most box. The number of terms in the sum is \((q + 2)\), the dimension of the irreducible representations of \(su(2)\). Namely they are given by filling the numbers 1 and 2 in the \((q + 1)\)-box Young tableaux according to the rule that the numbers must not decrease moving to the right along the row. We thus get \(q + 2\) numbered Young tableaux. We can show that the auxiliary eigenvalues satisfy

\[ T_0^{(q)} T_1^{(1)} = T_0^{(q+1)} + f_{q-1} T_0^{(q-1)}, \quad (2.9) \]

or pictorially,

\[
\begin{array}{c}
\varepsilon \\
v + q \\
\ldots \\
v + \gamma \\
v
\end{array} \otimes \begin{array}{c} \varepsilon \\
v \\
\end{array} = \begin{array}{c}
\varepsilon \\
v + q \\
\ldots \\
v + q + 1
\end{array} \oplus \begin{array}{c}
\varepsilon \\
v + q + 1 \\
\ldots \\
v + q + 1
\end{array} \quad (2.10)
\]

with

\[ f_k := \begin{array}{c} 1 \\ 2 \end{array} \quad (2.11) \]

The functional relations (2.9) can be further used to show that

\[ T_0^{(q)} T_1^{(q)} = \prod_{k=0}^{q-1} f_k + T_0^{(q+1)} T_1^{(q-1)}, \quad (2.12) \]

which coincides with (2.9) for \(q = 1\). It is also useful to introduce

\[ y_0^q = T_0^{(q+1)} T_1^{(q-1)} / \prod_{k=0}^{q-1} f_k. \quad (2.13) \]

with \(y_0^0 = 0\). Then (2.12) can in turn be used to show that

\[ y_0^q y_1^q = (1 + y_0^{q+1})(1 + y_1^{q-1}). \quad (2.14) \]

The relations (2.9) and (2.12) are known as the \(T\)-system while (2.14) is the \(y\)-system [11, 12, 13, 14].
For half-filling \((M = 0)\) we have \(S_z = \frac{1}{2}L - N\). Now \(g_M(v) = 1\) and the function \(f_k\) is independent of \(M\). It is quite clear from the \(su(2)\) functional relations (2.9) that the term \(f_0\) contributes only to the bulk and surface free energies rather than to the finite-size corrections of higher order. However, the relation \(\partial T(0)/\partial v \simeq \partial T^{(1)}(0)/\partial v\) implies that the finite-size corrections to the eigenvalues of the transfer matrix \(\lambda(v)\) are governed by the auxiliary eigenvalues \(T^{(1)}_0\). Thus the finite-size corrections to the Hamiltonian (1.1) follow from the consideration of \(T^{(1)}_0\). As in the study of the six-vertex model with open boundaries \([10]\), the surface effect on the finite-size corrections can be calculated analytically from the \(T\)-system (2.9) and \(y\)-system (2.14) functional relations.

### 2.2 Zeros and poles

The functional relations have been shown to be very useful in calculating the finite-size corrections to the transfer matrices of exactly solved models \([13, 15, 10]\). To solve the fusion hierarchy (2.9) and (2.14) we need to know the distribution of zeros and poles of the auxiliary eigenvalues \(T^{(q)}\) and \(y^{(q)}\). We consider the model in the strip

\[-\gamma < \text{Re } v < \gamma.\]

Inside this strip the transfer matrix \(T(v)\) in (1.9) is related to the super-symmetric \(t-J\) model (1.1) via (1.14). The clear advantage in working with the transfer matrix formulation is that it allows the application of powerful machinery from complex analysis. In this way we avoid the explicit manipulation of Bethe root densities, etc. The largest, or groundstate, eigenvalue of \(T^{(1)}\) is not expected to possess zeros in the above strip. The zeros contributing to \(T^{(q)}_0\) are of order \(L\) from the bulk. Those contributed by the boundary are only of order 1, which become unimportant in the limit \(L \to \infty\). The bulk zero distribution is

\[\text{zero}[T^{(1)}(v)] = \emptyset,\]

\[\text{zero}[T^{(q)}(v)] = \bigcup_{k=0}^{q-2} \{-k\gamma \}\text{ for } q > 1.\]

The zeros and poles of \(y^{(q)}\) are determined by (2.13), which gives

\[\begin{align*}
&\text{(I) } q = 1 : \text{zero}[t^{(1)}(v)] = \{0\}, \\
&\qquad \text{pole}[t^{(1)}(v)] = \{-\gamma\}, \quad \text{for } q = 1, \\
&\text{(II) } q \geq 2 : \text{zero}[t^{(q)}(v)] = \emptyset, \\
&\qquad \text{pole}[t^{(q)}(v)] = \{-q\gamma\}, \quad \text{for } q \geq 2.
\end{align*}\]

for the bulk contribution only. Here the boundary contribution to the zeros and poles, of order greatly less than \(L\), are not listed and contribute less than those of order \(L\) when the system size \(L\) becomes large. Only these zeros or poles of order \(L\) are especially important in the thermodynamic limit \(L \to \infty\) \([10]\).
2.3 The functional relations for finite-size corrections

The finite-size corrections to the eigenvalues $T^{(1)}(v)$ can be obtained by solving the functional relations (2.9) and (2.14) in the strip (2.15). Denote the finite-size corrections of $T^{(1)}(v)$ by $T^{(1)}_{\text{finite}}(v)$ and write

$$T^{(1)}(v) = T^{(1)}_{\text{finite}}(v) T^{(1)}_{\text{free}}(v). \quad (2.20)$$

The bulk and the surface free energy contributions together satisfy

$$T^{(1)}_{\text{free}}(v) T^{(1)}_{\text{free}}(v + \gamma) = f_0. \quad (2.21)$$

Inserting (2.20) into (2.12) or (2.9) we find that

$$T^{(1)}_{\text{finite}}(v) T^{(1)}_{\text{finite}}(v + \gamma) = 1 + y^{(1)}(v). \quad (2.22)$$

The finite-size corrections to $T^{(1)}(v)$ are thus represented by the $y$-system component $y^{(1)}(v)$. In the following we give an analytical treatment of (2.22) and (2.14). We will see that the finite-size corrections in the scaling limit are dependent only on the braid asymptotics and the bulk behavior of the functional relations.

The Bethe equations (1.19)-(1.20) render $T^{(1)}(v)$ analytic. Since all functions involved in the eigenvalues are $\pi$-periodic, the analyticity domains for $T^{(1)}(v)$ are not unique. It is thus useful to introduce functions of a real variable by restricting the eigenvalue functions to certain lines in the complex plane,

$$T(x) := T^{(1)}_{\text{finite}} \left( \frac{i}{\pi} x \gamma + \frac{1}{2} \gamma \right), \quad (2.23)$$

$$\alpha^{(q)}(x) := y^{(q)} \left( \frac{i}{\pi} x \gamma + \frac{1 - q}{2} \gamma \right), \quad (2.24)$$

$$A^{(q)}(x) := 1 + \alpha^{(q)}(x). \quad (2.25)$$

For the groundstate the functions $A^{(1)}(x)$ and $T(x)$ are analytic, non-zero (for those of order $L$) in $-\pi < \text{Im } x < \pi$ and possess constant asymptotics for $\text{Re } x \to \pm \infty$ (the ANZC property), which can be seen directly from the eigenvalues.

Eqn (2.22) can be solved using the new functions and applying Fourier transforms,

$$F_{\mathcal{T}}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \left[ \ln T(x) \right]' e^{-ikx},$$

$$\left[ \ln T(x) \right]' = \int_{-\infty}^{\infty} dk \ F_{\mathcal{T}}(k) e^{ikx}, \quad (2.26)$$

$$A^{(q)}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \left[ \ln F^{(q)}_{A}(x) \right]' e^{-ikx},$$

$$\left[ \ln F^{(q)}_{A}(x) \right]' = \int_{-\infty}^{\infty} dk \ A^{(q)}(k) e^{ikx}. \quad (2.27)$$

We obtain

$$\ln \mathcal{T} = k * \ln A^{(1)}, \quad (2.28)$$
where the kernel \(k(x)\) is
\[
k(x) := \frac{1}{2\pi \cosh(2x)}.
\] (2.29)

Here the convolution \(f \ast g\) of two functions \(f\) and \(g\) is defined by
\[
(f \ast g) := \int_{-\infty}^{\infty} f(x-y)g(y) \, dy = \int_{-\infty}^{\infty} g(x-y)f(y) \, dy.
\] (2.30)

There is an integration constant \(C\) in (2.28) which we drop because it does not contribute to the \(1/L\) corrections. In case of the low-lying excitations we have to take care of zeros in the analyticity strips so that the simple ANZC properties hold. The result (2.28) is still correct if we change the integration path \(L\) so that \(T(x)\) is ANZC and the Cauchy theorem can be applied as discussed elsewhere [13, 10].

The function \(A^{(1)}\) is determined by the \(y\)-system (2.14). According to Section 2.2 the analyticity strip (2.15) for \(y^{(1)}(v)\) contains a zero of order \(\mathcal{L}\) at \(u = 0\) and a pole of order \(\mathcal{L}\) at \(u = \pm \gamma\). All other functions \(y^{(q)}\) are analytic and non-zero in their analyticity strips \(-\gamma < u < \gamma\). Taking care of these properties, applying Fourier transforms to the logarithmic derivative of the equations (2.14) with the new functions (2.23)-(2.25) and then integrating the equations back we obtain the nonlinear integral equations
\[
\ln \alpha^{(q)} = \ln \epsilon^{(q)} + k \ast \ln A^{(q-1)} + k \ast \ln A^{(q+1)} + D^{(q)},
\] (2.31)

where
\[
\epsilon^{(q)}(x) := \begin{cases} 1, & q \neq 1 \\
\tanh\mathcal{L}\left(\frac{1}{2}x\right), & q = 1. \end{cases}
\] (2.32)

Here \(D^{(q)}\) are integration constants. For the same reason we have to take care of the ANZC property in the analyticity strips in (2.31).

### 2.4 The functional relations in the limit \(L \to \infty\)

The finite-size corrections can be extracted from the nonlinear integral equations (2.31) and (2.28). The system size \(L\) enters the nonlinear equations (2.31) through (2.32). The function \(\epsilon^{(1)}\) has three asymptotic regimes with transitions in scaling regimes when \(x\) is of the order of \(-\ln L\) or \(\ln L\). We suppose that \(\alpha^{(q)}\) and \(A^{(q)}\) scale similarly. Thus in the following scaling limits,
\[
e_{\pm}^{(q)}(x) := \lim_{L \to \infty} \epsilon^{(q)}\left(\pm (x + \ln L)\right),
\]
\[
a_{\pm}^{(q)}(x) := \lim_{L \to \infty} \alpha^{(q)}\left(\pm (x + \ln L)\right), (2.33)
\]
\[
A_{\pm}^{(q)}(x) := \lim_{L \to \infty} A^{(q)}\left(\pm (x + \ln L)\right) = 1 + a_{\pm}^{(q)}(x),
\]
eqn (2.31) takes the form
\[
\ell \alpha^{(q)} = \ell \epsilon^{(q)} + k \ast \ell A^{(q-1)} + k \ast \ell A^{(q+1)} + D^{(q)},
\] (2.34)
where we use the abbreviations

\[ \ell a(q)(x) := \ln a(q)(x), \quad \ell A(q)(x) := \ln A(q)(x), \]

\[ \ell e(q)(x) := \begin{cases} 0, & q \neq 1, \\ -2e^{-x}, & q \neq 1, \end{cases} \]  \hspace{1cm} (2.35) \]

and suppress the ± subscripts. The transfer matrix \( T(x) \) in the \( L \to \infty \) limit now becomes

\[
\ln T(x) = (k \ln A^{(1)})(x) \\
= \frac{1}{2\pi} \int_{-\ln N}^{\infty} \left( \frac{\ln A^{(1)}(y + \ln L)}{\cosh(x - y + \ln L)} + \frac{\ln A^{(1)}(-y - \ln L)}{\cosh(x + y + \ln L)} \right) dy + o\left(\frac{1}{L}\right) \\
= \frac{e^x}{L\pi} \int_{-\infty}^{\infty} e^{-y\ell A^{(1)}_1(y)} dy + \frac{e^{-x}}{L\pi} \int_{-\infty}^{\infty} e^{-y\ell A^{(1)}_2(y)} dy + o\left(\frac{1}{L}\right) \\
= \frac{2}{L\pi} \int_{-\infty}^{\infty} -e^{-y\ell A^{(1)}_1(y)} dy + o\left(\frac{1}{L}\right), \]  \hspace{1cm} (2.36) \]

The above equation converges and can actually be evaluated explicitly with the help of the dilogarithmic function

\[ L(x) = -\int_{0}^{x} dy \frac{\ln(1 - y)}{y} + \frac{1}{2} \ln x \ln(1 - x). \]  \hspace{1cm} (2.37) \]

Multiplying the derivative of (2.34) with \( \ell A^q \), and (2.34) itself with \( (\ell A^q)' \), taking the difference, summing over \( q \) and using (2.35), we are able to obtain

\[ 2 \int_{-\infty}^{\infty} e^{-y\ell A^{(1)}_1(y)} dy = -\sum_{q} L\left(\frac{1}{A^{(q)}}\right) \left|_{-\infty}^{\infty} \right. + \frac{1}{2} \sum_{q} D^{(q)}\ell A^{(q)} \left|_{-\infty}^{\infty} \right., \]  \hspace{1cm} (2.38) \]

where the constants \( D^{(q)} \) are given in terms of the \( x \to \infty \) asymptotics by

\[ D^{(q)} = \ell a^{(q)} - \frac{1}{2} \ell A^{(q-1)} - \frac{1}{2} \ell A^{(q+1)}. \]  \hspace{1cm} (2.39) \]

The result (2.38) shows that the finite-size corrections in the scaling limit depend only on the braid asymptotics and the bulk behaviour of the functional relations.

### 2.5 Asymptotics and bulk behavior

The nonlinear integral equations (2.34) can be easily solved in the limit \( x \to \pm \infty \) with

\[ \gamma = \frac{\pi}{h}, \quad h = 3, 4, \ldots. \]  \hspace{1cm} (2.40) \]

In many cases different \( h \) correspond to different models. The equation (2.38) shows that these asymptotic solutions are enough to obtain the finite-size corrections of the transfer matrix \( T^{(1)}(v) \).
It is obvious to see that the $x \to \infty$ asymptotics corresponds to the braid limit $u \to \pm \infty$. In this limit (2.14) reduces to
\[(y^{(q)}_{\infty})^2 = (1 + y^{(q-1)}_{\infty})(1 + y^{(q+1)}_{\infty}).\] (2.41)

This equation in turn means
\[2\ell a^{(q)} = \ell A^{(q-1)} + \ell A^{(q+1)} + D^{(q)},\] (2.42)
in terms of the functions $a^q$. The constants $D^q$ can be either zero or non-zero as the different branches can be taken for the logarithmic functions in the nonlinear integral equations.

To solve for $y^{(q)}_{\infty}$ we write $y^{(1)}_{\infty}$ as
\[y^{(1)}_{\infty} = \frac{\sin 3\theta}{\sin \theta},\] (2.43)
where the parameter $\theta$ is to be determined. The recursion relation (2.41) implies
\[y^{(q)}_{\infty} = \sin q\theta \sin(q + 2)\theta / \sin^2 \theta,\]
\[y^{(q)}_{\infty} + 1 = \sin^2(q + 1)\theta / \sin^2 \theta,\] (2.44)
for all $q = 1, 2 \cdots$. This solution has to be consistent with the braid limit of $T^{(q)}(v)$. To fix the constant parameter $\theta$ let us consider the groundstate, for which $N = \frac{1}{2}(L + M)$ and
\[\lim_{\text{Im} v \to \pm \infty} T^{(1)}(v)/\phi(v) = 2 \cos \left(\frac{\pi}{h}\right).\] (2.45)

Recalling (2.43) and using the relations
\[y^{(1)}_{\infty} = \lim_{\text{Im} u \to \pm \infty} y^{(2)}_{0}/f_0 = \lim_{\text{Im} u \to \pm \infty} T^{(1)}_0 T^{(1)}_1/f_0 - 1 = 4 \cos^2\left(\frac{\pi}{h}\right) - 1\] (2.46)
we have
\[\theta = \gamma = \frac{\pi}{h}.\] (2.47)

Moreover, the special values of $\theta$ lead to the closure condition
\[y^{(h-2)}_{\infty} = 0.\] (2.48)
For the sector $S^z = \frac{1}{2}(L + M) - N$ we have to modify $\theta$ to be
\[\theta = m\gamma = \frac{m\pi}{h}\] (2.49)
where \( m = 2S^z + 1 = 1, 3, \ldots \leq h - 1 \).

In the limit \( x \to -\infty \), \( y^{(q)} \) can be considered as the bulk behaviour in large \( L \). According to Section 2.2, the analyticity strip for \( y^{(1)}(v) \) contains a zero of order \( N \) at \( v = 0 \) and poles of order \( N \) at \( v = -\pm \gamma \). All other functions \( y^{(q)} \) are analytic and non-zero in their analyticity strips in \(-\gamma < v < \gamma\). For large \( N \) we find that the leading bulk behaviour to \( y^{(q)} \) is

\[
y^{(q)}_{\text{bulk}}(v) = \begin{cases} 
\text{constant}, & q \neq 1, \\
\text{constant} \left[ \tan \left( \frac{1}{2} hv \right) \right]^\ell, & q = 1.
\end{cases}
\]

The constants are fixed by the functional equations (2.14) and can be calculated similarly to the asymptotics of \( y^{(q)} \). As for the ABF model \cite{13}, it is easy to see that the limit

\[
\lim_{x \to -\infty} \lim_{N \to \infty} y^{(1)}(v) \sim \lim_{x \to -\infty} \exp (-2e^{-x}) = 0 .
\]

Therefore the functional equations (2.14) are modified and we find the constants for \( 2 \leq q \leq h - 3 \), with

\[
y^{(q)}_{\text{bulk}} = \sin(q - 1) \tau \sin(q + 1) \tau / \sin^2 \tau,
\]
\[
y^{(q)}_{\text{bulk}} + 1 = \sin^2 q \tau / \sin^2 \tau ,
\]

where

\[
\tau = \frac{m'' \pi}{h - 1} ,
\]

which is consistent with the closure condition (2.48).

The transfer matrix eigenspectra has only one “quantum number” \( S^z \). There should be one free parameter between \( m' \) and \( m'' \). For the largest (groundstate) eigenvalue the appropriate choices are \( m = m'' = 1 \). The open boundary \( t-J \) system under consideration possesses \( sp_{l_q}(2, 1) \) invariance. In the case of a fixed \( M \) the Bethe equations and transfer matrix eigenvalues \( T^{(1)} \) are similar to those of the six-vertex model with open \( su_q(2) \)-invariant boundaries. Thus we suppose that \( m'' = 1 \) as in the study of the XXZ-chain \cite{16, 17} and the related six-vertex model \cite{10}. The low-lying excited states are given by \( m > 1 \).

The solution \( y^{(1)}_{\text{bulk}}(v) \) is given by

\[
y^{(1)}_{\text{bulk}}(v) y^{(1)}_{\text{bulk}}(v + \gamma) = (1 + y^{(2)}_{\text{bulk}}) = 4 \cos^2 \tau .
\]

Thus we find lastly that

\[
y^{(1)}_{\text{bulk}}(v) = \pm 2 \cos \tau \left[ \tan \left( \frac{1}{2} hv \right) \right]^\ell .
\]
3 Finite-size corrections

The finite-size corrections are only dependent on the braid and bulk limits. In these limits
the functional relations are truncated and the summation in (2.38) can be replaced with

\[
2 \int_{-\infty}^{\infty} e^{-y} e^{yA^{(1)}(y)} dy = - \sum_{q=1}^{h-3} L \left( \frac{1}{A^{(q)}} \right) \bigg|_{-\infty}^{\infty} + \frac{1}{2} \sum_{q=1}^{h-3} D^{(q)} e^{A^{(q)}} \bigg|_{-\infty}^{\infty} .
\]  

(3.1)

Recall that the constants \(D^{(q)}\) are dependent on the branches of the dilogarithmic func-
tions in the nonlinear integral equations. The appropriate choice yie lds the correct finite-
size corrections. Simply taking \(D^{(q)} = 0\) is consistent with the asymptotics solutions given
in Section 2.5. To take nonzero \(D^{(q)}\) we need to single out the appropriate branches from
the asymptotic solutions of the equations, as has been shown for ABF models [13].

A relevant dilogarithm identity has been established by Kirillov [18]. Cons ider the
functions

\[y^{(q)}(j, r) := \frac{\sin(q + 2)\varphi \sin(q\varphi)}{\sin^{2}(\varphi)}, \quad 1 \leq b \leq n - 1, \quad 1 \leq q \leq r,\]  

(3.2)

with

\[\varphi = \frac{(1 + j)\pi}{2 + r}, \quad 0 \leq j \leq r .\]  

(3.3)

It is obvious that they represent the solutions of the asymptotic equations (2.34) with
\(r = h - 2\) or the solutions of (2.52) for the bulk behaviour with \(r = h - 3\). Then we have
the dilogarithm identity

\[s(j, r) := \sum_{q=1}^{r} L \left( \frac{1}{1 + y^{(q)}(j, r)} \right) = \frac{\pi^{2}}{6} \left[ \frac{3r}{2 + r} - \frac{6j(j + 2)}{2 + r} + 6j \right].\]  

(3.4)

Now in terms of the dilogarithm function the finite-size corrections (2.36) are expressed
as

\[\ln T(x) = \frac{\cosh x}{L\pi} \left[ \sum_{q=1}^{h-3} L \left( \frac{1}{A^{(q)}} \right) \bigg|_{-\infty}^{\infty} + \frac{1}{2} \sum_{q=1}^{h-3} D^{(q)} e^{A^{(q)}} \bigg|_{-\infty}^{\infty} \right] + o \left( \frac{1}{L} \right).
\]  

(3.5)

Note that the nonlinear integral equations (2.34) including the closure condition (2.48)
and their solutions presented in Section 2.5 are the same as those of the ABF models [13].
Therefore we can calculate the finite-size corrections in the same way. Similarly to [13],
with \(D^{(q)} \neq 0\), it can thus be shown that in terms of the functions \(s(j, r)\) the finite-size corrections
(3.3) for the quantum-invariant open boundary system can be written

\[\ln T(x) = \frac{\pi \cosh x}{6L} \left[ s(0, h - 3) + s(0, 1) - s(m - 1, h - 2) - 6(1 - m)(2 - m) \right] + o \left( \frac{1}{L} \right).
\]  

(3.6)
Inserting (3.4) into this equation we have the finite-size corrections in the more recognizable form [19]

\[ \ln \mathcal{T}(x) = \frac{\pi}{6L} \left( c - 24\Delta_m \right) \cosh x + o \left( \frac{1}{L} \right), \] (3.7)

where the central charge and conformal weights are given by

\[ c = 1 - \frac{6}{h(h-1)}, \]
\[ \Delta_m = \frac{[h - (h-1)m]^2 - 1}{4h(h-1)}, \] (3.8)

with \( m = 1, 3, \cdots \leq h - 1 \).

### 4 Free energies

Consider now the bulk and surface free energies. Let

\[ \lambda_{\text{free}}(v) = T_b(v)T_s(v) = \frac{T^{(1)}_{\text{free}}((v))}{\sin(2v - 2\gamma) \sin^L(\gamma + v) \sin^L(\gamma - v)}, \] (4.9)

where \( T_b \) and \( T_s \) are the bulk and surface contributions. They are determined by (2.21). Factors of order \( L \) in \( f_0 \) contribute to \( T_b \) and otherwise to \( T_s \). From (2.5) and (2.21) we should have

\[ T_b(v)T_b(v + \gamma) = \frac{\sin^L(\gamma + v) \sin^L(\gamma - v)}{\sin^2(2\gamma) \sin[\gamma + (\gamma + v)] \sin[\gamma - (\gamma + v)]} \] (4.10)

for the bulk and

\[ T_s(v)T_s(v + \gamma) = \frac{\sin(2v + 2\gamma) \sin(2v - 2\gamma)}{\sin^2(2\gamma) \sin[\gamma + (\gamma - 2v)] \sin[\gamma - (\gamma - 2v)]} \] (4.11)

for the surface. Solving these equations we find

\[ \log T_b(v) = 2L \int_{-\infty}^{\infty} dk \frac{\cosh(2\gamma k - \pi k) \sinh(vk) \sinh(\gamma k) \cosh(vk)}{k \cosh(\gamma k) \sinh(\pi k)}, \] (4.12)

\[ \log T_s(v) = \int_{-\infty}^{\infty} dk \frac{\cosh(4\gamma k - \pi k) \sinh(2vk) \sinh(2\gamma k - 2vk)}{k \cosh(2\gamma k) \sinh(\pi k)} - \int_{-\infty}^{\infty} dk \frac{\cosh(2\gamma k - \pi k) \sinh(2vk) \sinh(\gamma k - 2vk)}{k \cosh(\gamma k) \sinh(\pi k)}. \] (4.13)

#### 4.1 \( t-J \) model

The eigenvalues \( E \) of the quantum invariant \( t-J \) Hamiltonian (1.1) follow via the relation (1.14), with

\[ E_m = -\frac{4}{\sin \gamma} \frac{\partial \lambda(v)}{\partial v} \bigg|_{v=0}. \] (4.14)
From the above, we have that the groundstate energy is given by 
\[ E_0 = 2L e_b + 2e_s + e_s^+ + e_s^-, \]
where
\[
\begin{align*}
e_b &= -\frac{2}{L} \left. \frac{\partial \log T_b(v)}{\partial v} \right|_{v=0} \\
&= -\frac{4}{\sin \gamma} \int_{-\infty}^{\infty} \frac{dk}{\sinh(\gamma k) \sinh(\pi k)} \cosh(2\gamma k - \pi k) \sinh(\gamma k),
\end{align*}
\]
and
\[
\begin{align*}
e_s &= -\frac{2}{\sin \gamma} \left. \frac{\partial \log T_s(v)}{\partial v} \right|_{v=0} \\
&= -\frac{4}{\sin \gamma} \int_{-\infty}^{\infty} \frac{dk}{\sinh(\gamma k) \sinh(\pi k)} \cosh(4\gamma k - \pi k) \sinh(2\gamma k) \\
&+ \frac{4}{\sin \gamma} \int_{-\infty}^{\infty} \frac{dk}{\sinh(\gamma k) \sinh(\pi k)} \cosh(2\gamma k - \pi k) \sinh(\gamma k). \tag{4.16}
\end{align*}
\]

Here the local surface energies \( e_s^\pm = 0 \) for the quantum invariant case. At half-filling, the results for \( e_b \) and \( e_s \) are in agreement with those of the XXZ chain [16], as expected. The spin excitations are given by
\[
E_m = E_0 - \frac{4}{\sin \gamma} \left. \frac{\partial \log T_{\text{finite}}(v)}{\partial v} \right|_{v=0} \\
= E_0 - \frac{v_s \pi}{24L} (c - 24\Delta_m) + o\left(\frac{1}{L}\right), \tag{4.17}
\]
where the sound velocity is
\[
v_s = \begin{cases} 
\frac{\pi}{3} & \gamma \neq 0, \\
\frac{\gamma \sin \gamma}{\pi} & \gamma = 0.
\end{cases} \tag{4.18}
\]

5 Conclusion and discussion

We have exploited the \( su(2) \) structure of the transfer matrix functional relations to calculate the massless spin excitations of the integrable quantum group invariant \( t - J \) model (1.1) at close to half-filling. We took the special value \( \gamma = \pi/h \) of the anisotropy parameter in order to close the functional relations. Results for the isotropic model are recovered in the limit \( h \to \infty \). Explicitly, in this limit the scaling dimensions of the spin excitations, or spinons, follow from (3.8) as
\[
X_{S_z} = 2\Delta_{S_z} = 2(S^z)^2, \tag{5.1}
\]
where \( S^z = 0, 1, \ldots \).
5.1 General parameter \(0 \leq \gamma \leq \pi/2\)

The excitation spectrum can be obtained for general anisotropy parameter \(\gamma\) via a related analytic nonlinear integral equation approach [20, 21, 22, 23, 24]. For the quantum invariant model at or close to half-filling we expect

\[
\log \lambda(v) = \log T_b + \log T_s + \frac{\pi}{6L} \left( c - 24\Delta_m \right) \sin\left( \frac{\pi v}{\gamma} \right) + o\left( \frac{1}{L} \right),
\]

with the bulk and surface free energies given by (4.12) and (4.13). The central charge and conformal dimensions are

\[
c = 1 - \frac{6\gamma^2}{\pi(\pi - \gamma)},
\]

\[
\Delta = \frac{[\gamma m - (m - 1)\pi]^2 - \gamma^2}{4\pi(\pi - \gamma)},
\]

where \(m = 2S^z + 1 = 1, 3, \cdots\). It follows that the finite-size corrections to the anisotropic \(t - J\) model are

\[
E_m = 2L e_b + 2e_s - \frac{\pi v_s}{6L} \left( c - 24\Delta_m \right) + o\left( \frac{1}{L} \right)
\]

where the bulk and surface terms are given by (4.15) and (4.16).

5.2 Local surface energies

The above results are for vanishing boundary fields \(H^\pm_s\). The quantum group \(spl_q(2,1)\) invariance is broken for \(H^\pm_s \neq 0\). However, the model remains integrable if \[3\]

\[
H_s^- = i\sin \gamma (\cot \xi_- - 1)(S^z_1 - n^h_1/2)
\]

\[
H_s^+ = -i\sin \gamma (\cot \xi_+ - 1)(S^z_L - n^h_L/2)
\]

where \(H_s^\pm = 0\) is recovered in the limit \(\xi^\pm \to \infty\). For finite \(\xi^\pm\) the eigenvalues and Bethe equations are given by (1.13)-(1.20) with \[3\]

\[
k_A(v) = \frac{q \sin(\xi_+ - v) \sin(\xi_- + v)}{\sin \xi_+ \sin \xi_-},
\]

\[
k_{D_2}(v) = \frac{\sin(\xi_+ + v - \gamma) \sin(\xi_- - v + \gamma)}{\sin \xi_+ \sin \xi_-},
\]

\[
k_{D_{II}}(v) = \frac{\sin(\xi_- - v + \gamma) \sin(\xi_+ + v - \gamma)}{\sin \xi_+ \sin \xi_-}.
\]

At or close to half-filling the free energy can be calculated in a similar manner to that presented here for the quantum invariant case. As seen in Section 2.3, the total ground-state energy of the \(t-J\) model satisfies (2.21). The local surface free energy, on the other hand, satisfies

\[
T^\pm_s(v)T^\pm_s(v + \gamma) = \frac{q \sin(\xi_+ + v) \sin(\xi_- - v)}{\sin(\xi_-) \sin(\xi_+)}.
\]
Here we omit the factor \( q \) in the following as it shifts \( \log T_s^\pm(v) \) with no contribution to the surface energy of the quantum chain. We find

\[
\log T_s^\pm(v) = \int_{-\infty}^{\infty} dk \frac{e^{k(\gamma-\xi^\pm)} \cosh(2\xi^\pm k - \pi k) \sinh(vk) \sinh(\xi^\pm k - vk)}{k \cosh(\gamma k) \sinh(\pi k)}
\]

(5.12)

Hence the local surface energy of the quantum chain is given by

\[
e_s^\pm = -\frac{4}{tr K^\pm(0) \sin \gamma} \left. \frac{\partial \log T_s^\pm(v)}{\partial v} \right|_{v=0}
\]

\[
= -\frac{4}{tr K^\pm(0) \sin \gamma} \int_{-\infty}^{\infty} dk \frac{e^{k(\gamma-\xi^\pm)} \cosh(2\xi^\pm k - \pi k) \sinh(\xi^\pm k)}{\cosh(\gamma k) \sinh(\pi k)}
\]

(5.13)

5.3 Hole excitations

We have considered spin excitations, the so-called spinons, at or close to the half-filled band. The spinon excitations are given by the dimensions (3.8) and (5.4), with (5.1) for the isotropic model, where \( S_z = \frac{1}{2}(L - 2N + M) \). One also needs to consider the holon part of the spectrum. According to the results obtained for the excitation spectra of the periodic \( t-J \) model, the holons should contribute another independent conformal theory [9].

In contrast to the spinon part, the calculation of the free energies from \( \lambda(v) \) is no longer as per Section 4. Let us write \( \lambda(v) = \lambda_{\text{free}}(v) \lambda_{\text{finite1}}(v) \lambda_{\text{finite2}}(v) \), then \( \lambda_{\text{finite1}}(v) = T_{\text{finite}}^{(1)}(v) \). Define \( \lambda_{\text{free,finite2}}(v) = \lambda_{\text{free}}(v) \lambda_{\text{finite2}}(v) \). Both the bulk and surface free energies should follow from

\[
\lambda_{\text{free,finite2}}(v) \lambda_{\text{free,finite2}}(v + \gamma) = d(v) g_M(v + \gamma),
\]

which is clearly dependent on \( M \), where

\[
d(v) = \frac{\sin^L(\gamma + v) \sin^L(\gamma - v) \sin(2\gamma + 2v)}{\sin^L(2\gamma + v) \sin^L(-v) \sin(2v)}.
\]

(5.15)

This shows that the bulk and surface free energies follow from two parts, namely \( g_M(v) \) and \( d(u) \). The energies \( e_b \) and \( e_s \) given in (4.13) and (4.16) are the contribution from \( d(v) \).

The above relation also encodes the finite-size corrections to the eigenvalues, and thus the holon part of the spectra, through the contribution \( \lambda_{\text{finite2}}(v) \). It is obvious that we need to analyse the Bethe equation (1.19)-(1.20) to solve the inversion relation (5.14). It is possible that we will need to use a very different method to obtain the holon excitations. Whether or not the central charge associated with the holon part of the spectrum is also less than one for the anisotropic quantum invariant \( t-J \) model remains to be explored.

Note Added. After completing this work we received a preprint by Asakawa and Suzuki in which the finite-size corrections are calculated for the open isotropic \( t-J \) model via the root density method [23]. They find the central charge \( c = 1 \) for both spinons and holons in agreement with the periodic case [3]. The conformal weights of the spinons agree with our result (5.1). In addition to treating the corresponding vertex model our results for the spinon conformal spectra generalise those of Asakawa and Suzuki to the anisotropic quantum invariant \( t-J \) model.
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