Structure of the Coulomb and unitarity corrections to the cross section of $e^+e^-$ pair production in ultra-relativistic nuclear collisions

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Abstract

We analyze the structure of the Coulomb and unitarity corrections to the single $e^+e^-$ pair production as well as the cross section $\sigma_n$ for the multiple pair production in collision of ultra-relativistic nuclei. In the external field approximation we consider the probability of one pair production at fixed impact parameter $\rho$ between colliding nuclei. We obtain the analytical result for this probability at large $\rho$ as compared to the electron Compton wavelength. The energy dependence of this probability as well as that of $\sigma_n$ differ essentially from widely cited but incorrect results. We estimate also the unitary corrections to the total cross section of the process.

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Recently the process of $e^+e^-$ pair production in ultra-relativistic heavy-ion collisions was discussed in numerous papers. This is connected with the beginning of operation of Relativistic Heavy Ion Collider (RHIC) with the Lorentz factor $\gamma = 108$ and charge number of nuclei $Z = 79$. New collider LHC is scheduled to be in operation in the nearest future, with $\gamma = 3000$ and $Z = 82$. The cross section of one pair production in the Born approximation was obtained many years ago [1, 2] and reads

$$\sigma_{\text{Born}} = \frac{28}{27\pi} \frac{\zeta}{m^2} \left[ L^3 - 2.198 L^2 + 3.821 L - 1.632 \right],$$

(1)

where

$$\zeta = (Z_A\alpha)^2(Z_B\alpha)^2, \quad L = \ln(\gamma_A\gamma_B).$$

(2)

Here $\alpha$ is the fine-structure constant, $m$ is the electron mass, $Z_{A,B}$ are the charge numbers of the nuclei $A$ and $B$. The nuclei $A$ and $B$ are assumed to move in the positive and negative directions of the $z$ axis, respectively, and have the Lorentz factors $\gamma_{A,B}$. For the sake of simplicity we consider the process in the frame where both nuclei have the same Lorentz factor $\gamma_A = \gamma_B = \gamma$. In the present paper we consider the case $L \gg 1$. Since cross section (1) is huge, a pair production can be a serious background for many experiments. Besides, it is also important for the problem of beam lifetime and luminosity of colliders (see review [3]). It means that the various corrections to the Born cross section, as well as the cross section $\sigma_n$ for $n$-pair production, are of great importance. At present, there is a lot of controversy in papers devoted to this subject (the corresponding references and critical remarks can be found in [4, 5, 6]). Although some of the corrections were obtained correctly in some special cases, the whole consistent picture of pair production is absent. In the present paper we intend to elucidate some points in this problem.

For $\gamma \gg 1$ it is possible to treat the nuclei as sources of the external field, and calculate the probability of $n$-pair production $P_n(\rho)$ in collision of two nuclei at a fixed impact parameter $\rho$. The corresponding cross section $\sigma_n$ is obtained by the integration over the impact parameter:

$$\sigma_n = \int d^2\rho P_n(\rho).$$

(3)

The quantity which is important in further consideration is an average number of the produced pairs at a given $\rho$:

$$W(\rho) = \sum_{n=1}^{\infty} n P_n(\rho).$$

(4)
The closed expression for $W(\rho)$ was obtained in \([7, 8, 9]\):

\[
W(\rho) = \int \frac{m^2 d^3 p d^3 q}{(2\pi)^6 \epsilon_p \epsilon_q} \left| \int \frac{d^2 k}{(2\pi)^2} \exp[i k \rho] m \mathcal{F}_A(k') \mathcal{F}_B(k) \right|^2,
\]

\[
\mathcal{M} = \bar{u}(p) \left[ \frac{\alpha(k - p_\perp) + \gamma_0 m}{- p_+ + (k - p_\perp)^2 - m^2 \gamma_-} + \frac{- \alpha(k - q_\perp) + \gamma_0 m}{- p_+ - (k - q_\perp)^2 - m^2 \gamma_+} \right] u(-q).
\]

Here $p$ and $\epsilon_p$ ($q$ and $\epsilon_q$) are the momentum and energy of the electron (positron), $u(p)$ and $u(-q)$ are positive- and negative-energy Dirac spinors, $\alpha = \gamma^0 \gamma$, $\gamma_\pm = \gamma^0 \pm \gamma^z$, $\gamma^\mu$ are the Dirac matrices, $p_\pm = \epsilon_p \pm p^z$, $q_\pm = \epsilon_q \pm q^z$, $k$ is a two-dimensional vector lying in the $xy$ plane, $k' = q_\perp + p_\perp - k$, and the function $\mathcal{F}(k)$ is proportional to the electron eikonal scattering amplitude in the Coulomb field.

The function $W(\rho)$ defines the cross section

\[
\sigma_T = \int d^2 \rho W(\rho) = \sum_{n=1}^{+\infty} n \sigma_n,
\]

which is called "the inclusive cross section" in \([10]\). Let us stress that the usual definition of the inclusive cross section is different:

\[
\sigma_{\text{incl}} = \sum_{n=1}^{\infty} \sigma_n.
\]

To obtain $\sigma_T$ it is necessary to perform the regularization of the expression for $W(\rho)$. One of the possible correct regularizations is given in \([3]\) and reads

\[
\mathcal{F}_{A,B}(k) = 2\pi \int d\rho p J_0(\rho k) \left\{ \exp[2i Z_{A,B} \alpha K_0(\rho a_\pm)] - 1 \right\}, \quad a_\pm = (p_\pm + q_\pm)/(2\gamma),
\]

where $J_0$ is the Bessel function and $K_0$ is the modified Bessel function of the third kind.

The Born cross section can be obtained by the replacement

\[
\mathcal{F}_{A,B}(k) \rightarrow \mathcal{F}^0_{A,B}(k) = \frac{4i\pi Z_{A,B} \alpha}{k^2 + a_\pm^2},
\]

where $\mathcal{F}^0(k)$ is the first term in the expansion of $\mathcal{F}(k)$ in $Z\alpha$. After the regularization the cross section $\sigma_T$ can be presented in the form:

\[
\sigma_T = \sigma_{\text{Born}} + \sigma_T^C + \sigma_T^{CC},
\]

where $\sigma_T^C$ is the Coulomb corrections with respect to one of the nuclei, and $\sigma_T^{CC}$ is the Coulomb corrections with respect to both nuclei. In the main logarithmic approximation
they were obtained in [5, 6]

\[
\sigma^C_T = -\frac{28 \zeta}{9\pi m^2} L^2 \left[ f(Z_A\alpha) + f(Z_B\alpha) \right], \\
\sigma^{CC}_T = \frac{56 \zeta}{9\pi m^2} L f(Z_A\alpha)f(Z_B\alpha),
\]

(11)

where

\[
f(x) = x^2 \sum_{n=1}^{\infty} \frac{1}{n(n^2 + x^2)}. \tag{12}
\]

The expression for \(\sigma^C_T\) coincides with that obtained in [4] by means of Weizsäcker-Williams approximation. The accuracy of the expression (10), (11) is determined by the relative order of the omitted terms \(\sim (Z_{A,B}\alpha)^2/L^2\). This accuracy is better than 0.4% for the RHIC and LHC colliders.

In a set of publications [11, 12, 13, 14] it was argued that the factorization of the multiple pair production probability is valid with a good accuracy

\[
P_n(\rho) = \frac{W^n(\rho)}{n!} e^{-W(\rho)}. \tag{13}
\]

The factor \(\exp(-W)\) is nothing but the vacuum-to-vacuum transition probability \(P_0 = 1 - \sum_{n=1}^{\infty} P_n\). Strictly speaking, such a factorization does not take place due to the interference between the diagrams corresponding to the permutation of electron (or positron) lines (see, e.g., [10]). Nevertheless, one can show that this interference gives the contribution which contains at least one power of \(L\) less than that of the amplitude squared. Therefore, in the leading logarithmic approximation we can use the expression (13).

Let us represent the cross section \(\sigma_1\) of one pair production as follows

\[
\sigma_1 = \sigma_T + \sigma_{\text{unit}} = \int d^2\rho W(\rho) - \int d^2\rho W(\rho) \left(1 - e^{-W(\rho)}\right) \tag{14}
\]

Thus, the difference between \(\sigma_1\) and \(\sigma_T\) is due to the unitarity correction \(\sigma_{\text{unit}}\). The main contribution to the first term \((\sigma_T)\) comes from \(\rho \gg 1/m\). As for the second term, \(\sigma_{\text{unit}}\), the main contribution to it comes from \(\rho \sim 1/m\). If \(Z_{A,B}\alpha \sim 1\), then the exact \(W(\rho)\) at \(\rho \sim 1/m\) differs essentially from the Born result \(W_0(\rho)\). Since \(\sigma_T\) is studied in detail, here we investigate the unitarity correction and \(\sigma_n\) for \(n \geq 2\). Below we consider two interesting cases:

(i) \(Z_{A,B}\alpha \ll 1, \zeta L \ll 1\);

(ii) \(Z_{A,B}\alpha \ll 1, \zeta L \sim 1\).
Let us consider in detail the case (i), where it is possible to use $W_0$ (obtained in the Born approximation) instead of the exact function $W$, expand the exponent $\exp(-W)$ in the unitarity correction and omit it in $\sigma_n$ for $n \geq 2$. For a few particular values of $\gamma$ the function $W_0(\rho)$ was calculated numerically in \cite{13, 16} around $m\rho \sim 1$. In \cite{3}, for $m\rho \gg 1$, this function was approximated by a simple expression

$$
\tilde{W}_0(\rho) = \frac{14}{9\pi^2} \frac{\zeta}{(m\rho)^2} \left( \ln \frac{0.681\gamma^2}{m\rho} \right)^2, \quad 1 \ll m\rho \ll \gamma^2.
$$

(15)

This expression looks very convenient for fast estimates of various quantities. That is why Eq. (13) is widely cited and used in many papers (see, e.g., Refs. \cite{11, 12, 13, 14, 18, 19, 20, 21}). Now we show that Eq.(13) is incorrect. We find out that there are two scales in dependence of $W_0(\rho)$ on $\rho$: in the region of relatively small impact parameters, $1 \ll m\rho \leq \gamma$, we obtain

$$
W_0(\rho) = \frac{28}{9\pi^2} \frac{\zeta}{(m\rho)^2} \left[ 2 \ln \gamma^2 - 3 \ln (m\rho) \right] \ln (m\rho),
$$

(16)

while in the region of relatively large impact parameters, $\gamma \leq m\rho \ll \gamma^2$, we have

$$
W_0(\rho) = \frac{28}{9\pi^2} \frac{\zeta}{(m\rho)^2} \left( \ln \frac{\gamma^2}{m\rho} \right)^2.
$$

(17)

Note that the function $W_0(\rho)$ given by Eqs. (16) and (17) is the continuous function at $m\rho = \gamma$ together with its first derivative. Certainly, the integration of $W_0(\rho)$ from Eqs. (16) and (17) over $\rho$ gives the main term ($\propto L^3$) in Eq.(1).

The most important distinction between Eq.(13) and our result is a quite different dependence of $W_0(\rho)$ on $\gamma$ for $1 \ll m\rho \ll \gamma$. It follows from Eq.(16) that $W_0(\rho) \propto L$ at $m\rho \sim 1$ while Eq.(13) gives $\tilde{W}_0(\rho) \propto L^2$ at $m\rho \sim 1$. Since $\sigma_{\text{unit}}$ and $\sigma_n$ (for $n \geq 2$) are determined by the region of integration $m\rho \sim 1$, the prediction obtained with the help of Eq. (13) leads to incorrect dependence of these quantities on $\gamma$ (see below).

Let us consider the derivation of Eqs. (13) and (17). The main contribution to the integrals in (3) at $\rho \gg 1/m$ comes from the region of integration

$$
|k|, |p_\perp + q_\perp| \ll m, \quad |p_z|, |q_z| \ll m\gamma, \quad |p_\perp - q_\perp| \sim m.
$$

Passing to the variables $P = p_\perp + q_\perp$, $r = (p_\perp - q_\perp)/2$, $E = p_z + q_z$, and $x = (p_z - q_z)/(p_z + q_z)$, we take the integrals over $r$ and $x$ (see \cite{3}). Then, we obtain:

$$
\begin{align*}
W_0(\rho) &= \frac{112}{9\pi} \frac{\zeta}{m^2} \int_{m}^{m\gamma} \frac{dE}{E} \int d^2P L^{ij} L^{ij*}, \\
L^{ij} &= \int \frac{d^2k\, k^i (P - k)^j \exp(ik\rho)}{(2\pi)^2 [k^2 + E^2/\gamma^2][(P - k)^2 + m^4/(E^2\gamma^2)]}.
\end{align*}
$$

(18)
After the integration over \( k \) we come to

\[
W_0(\rho) = \frac{56}{9\pi^2m^2} \int m \gamma dE \left[ \int \frac{dP}{\gamma/E} \vartheta(\gamma/E - \rho) + \int_{\text{max}\{1/\rho, E/\gamma\}}^{m} \frac{dP}{\pi^2} \vartheta(\gamma E/m^2 - \rho) \right],
\]

(19)

where \( \vartheta(x) \) is the step function. Straightforward integration leads to Eqs. (16), (17). From our derivation it is clear that two large logarithms at \( m\rho \gg 1 \) come from the integration over \( E \) and \( P \), while at \( m\rho \sim 1 \) the logarithm from the integration over \( P \) is absent and the only logarithm \( L \) arises from the integration over \( E \).

The result (16), (17) can also be obtained within the standard Weizsäcker-Williams approximation. In this approximation the colliding nuclei with impact parameters \( \rho_1 \) and \( \rho_2 \) emit \( dn_1 \) and \( dn_2 \) equivalent photons with energies \( \omega_1 \) and \( \omega_2 \):

\[
dn_i = \frac{Z_i^2}{\pi^2} \frac{d\omega_i}{\omega_i} \frac{d^2\rho_i}{\rho_i^2}, \quad i = 1, 2, \quad \omega_i \ll m\gamma, \quad \frac{1}{m} \ll \rho_i \ll \frac{\gamma}{\omega_i}.
\]

(20)

Then these photons collide and produce a pair with the invariant mass squared \( s = 4\omega_1\omega_2 \).

Therefore, the function \( W_0(\rho) \) is

\[
W_0(\rho) = \int d\omega_1 d\omega_2 \delta(\rho_1 + \rho_2 - \rho) \sigma_{\gamma\gamma}(s)
\]

(21)

where \( \sigma_{\gamma\gamma} \) is the cross section of the process \( \gamma\gamma \rightarrow e^+e^- \). Then we perform the integration over \( \omega_2 \) using the relation

\[
\int_{4m^2}^{\infty} \frac{ds}{s} \sigma_{\gamma\gamma}(s) = \frac{14\pi}{9} \frac{\alpha^2}{m^2}.
\]

(22)

The main contribution to this integral is given by the region near the lower integration limit \( s = 4m^2 \). That is why we can extend the upper limit up to infinity and substitute \( \omega_2 = m^2/\omega_1 \) in the step function. After that we have

\[
W_0(\rho) = \frac{14}{9\pi^3m^2} \int_{m/\gamma}^{m\gamma} d\omega_1 \int \frac{d^2\rho_1}{\rho_1^2(\rho - \rho_1)^2} \vartheta\left(\frac{\gamma}{\omega_1} - \rho_1\right) \vartheta\left(\frac{\gamma\omega_1}{m^2} - |\rho - \rho_1|\right).
\]

(23)

The main contribution to this integral is given by two regions: \( 1/m \ll \rho_1 \ll \rho \) and \( 1/m \ll |\rho - \rho_1| \ll \rho \). Then we have

\[
W_0(\rho) = \frac{28}{9\pi^2(m\rho)^2} \int_{m/\gamma}^{m\gamma} d\omega_1 \int \frac{d\rho_1}{\rho_1} \times \left[ \vartheta(\gamma/\omega_1 - \rho_1) \vartheta(\gamma\omega_1/m^2 - \rho) + \vartheta(\gamma/\omega_1 - \rho) \vartheta(\gamma\omega_1/m^2 - \rho_1) \right].
\]

(24)

The further integration leads again to (16), (17).

As we argued above, the function \( W_0(\rho) \) at \( m\rho \ll \gamma \) and \( L \gg 1 \) has the form

\[
W_0(\rho) = \zeta LF(m\rho)
\]

(25)
FIG. 1: The function $F(x)$ from Eq. (25) vs. $x = m\rho$ (solid curve) and its asymptotic form $F_{\text{asymp}}(x)$, Eq. (26) (dashed curve).

with the universal function $F(x)$ independent of $Z_{A,B}$ and $\gamma$.

We extract this function (see Fig. 1) from the numerical results in [15] for the case $\gamma = 3400$. It follows from (16) that the asymptotic form of the function $F(x)$ at $x \gg 1$ reads

$$F_{\text{asymp}}(x) = \frac{56 \ln x}{9\pi^2 x^2}.$$ (26)

Using this function we obtain in the region (i) the unitarity correction to the one-pair production cross section

$$\sigma_{\text{unit}} = -2C_2 \frac{\zeta L^2}{m^2}$$ (27)

and the cross sections for the n-pair production ($n \geq 2$)

$$\sigma_n = C_n \frac{\zeta^n L^n}{m^2},$$ (28)

$$C_n = \frac{2\pi}{n!} \int_0^{\infty} F^n(x) xdx,$$ (29)

$$C_2 = 1.33, \quad C_3 = 0.264, \quad C_4 = 0.066, \quad C_5 = 0.0176.$$ (30)

Our result $\sigma_n \propto L^n$ for $n \geq 2$ differs considerable from the incorrect results $\sigma_n \propto L^{2n}$ of [11] and $\sigma_n \propto L^{3n}$ of [21].

Let us pass to the consideration of the case (ii). If $\zeta L \sim 1$, but $\zeta (Z_{A,B} \alpha)^2 L \ll 1$, then we can neglect the Coulomb corrections in $W(\rho)$ but should keep the exponent in Eq. (13). It gives the result similar to Eq. (28) for $\sigma_n$ with the replacement

$$C_n \to \tilde{C}_n(\gamma, Z_{A,B}) = \frac{2\pi}{n!} \int_0^{\infty} F^n(x) \exp[-\zeta L F(x)] xdx.$$ (31)
For unitarity correction we have

$$\sigma_{\text{unit}} = -2\pi \frac{\zeta}{m^2} L \int_0^{\infty} F(x) \{1 - \exp[-\zeta \, L \, F(x)]\} \, dx . \quad (32)$$

If $\zeta (Z_{A,B} \alpha)^2 L \sim 1$, then we should use in Eq. (13) the function $W(\rho)$ calculated exactly with respect to the parameters $Z_{A,B} \alpha$. Note that $W(\rho)$, for $L \gg 1$ and $m\rho \sim 1$, has the form similar to Eq. (25)

$$W(\rho) = \zeta \, L \, \tilde{F}(m\rho, Z_{A} \alpha, Z_{B} \alpha) . \quad (33)$$

The function $\tilde{F}(m\rho, Z_{A} \alpha, Z_{B} \alpha) \rightarrow F(m\rho)$ for $Z_{A,B} \alpha \rightarrow 0$, but the difference $\tilde{F} - F$ can be neglected in the exponential factor $\exp(-W)$ only if this difference is small as compared to $1/(\zeta L)$ rather than to $F$.

![Graph of W(x)/ζ vs. x](image)

**FIG. 2:** The exact function $W(x)$ (solid curve), Born approximation $W_0(x)$ (dashed curve), and the asymptotic form of $W_0(x)$ from (16), (17) (dotted curve) in units of $\zeta$.

The function $W(\rho)$ was calculated numerically in [22] for the particular case $\gamma = 100$, $Z = 79$. In Fig. 2 it is shown together with $W_0(x)$, calculated also in [22], and our asymptotics (16), (17). There is a good agreement of our analytical results with the exact numerical one already at $m\rho > 2$. It is also seen a noticeable difference between $W(\rho)$ and $W_0(\rho)$ in the region $m\rho \sim 1$, that is essential in calculations of the unitarity correction and $\sigma_n$ for $n \geq 2$. Emphasize that the Coulomb corrections $\sigma^C_T \propto L^2$ in (14) arise due to the difference between $W(\rho)$ and $W_0(\rho)$ in the region $m\rho \gg 1$.

Using the numerical results for $W(\rho)$ and $W_0(\rho)$ from [22] we find that the exact value of $\sigma_{\text{unit}}/\sigma_{\text{Born}}$ is equal to $-4.1\%$, while the result without Coulomb effects gives $-6.4\%$. With the help of Eqs.(32) and (27) we obtain $-9.3\%$ and $-12\%$, respectively. Thus, in the
example considered ($\gamma = 100, \ Z = 79$) the account of the Coulomb effects as well as the exponential factor is very important.

It is interesting to estimate the unitarity correction for the LHC case ($\gamma = 3000, \ Z = 82$). Using the recent numerical results of K. Hencken (private communication) for $W(\rho)$ at $\gamma = 3400, \ Z = 82$, we find that the exact value of $\sigma_{\text{unit}}/\sigma_{\text{Born}}$ for this case is equal to $-3.2\%$. This ration remains approximately the same for the LHC case.

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[1] L.D. Landau, E.M. Lifshitz. Phys. Z. Sowjet. 6, 244 (1934).
[2] G. Racah, Nuovo Cim. 14, 93 (1937).
[3] C. Bertulani, G. Baur. Phys. Rep. 163, 299 (1988).
[4] D.Yu. Ivanov, A. Schiller, V.G. Serbo, Phys. Lett. B 454, 155 (1999).
[5] R.N. Lee, A.I. Milstein, Phys. Rev. A. 61, 032103 (2000).
[6] R.N. Lee, A.I. Milstein, Phys. Rev. A 64, 032106 (2001).
[7] B. Segev, J.C. Wells, Phys. Rev. A 57, 1849 (1998); physics/9805013.
[8] A.J. Baltz, L. McLerran, Phys. Rev. C 58, 1679 (1998).
[9] U. Eichmann, J. Reinhardt, S. Schramm, and W. Greiner, Phys.Rev. A 59, 1223 (1999).
[10] A.J. Baltz, F. Gelis, L. McLerran, and A. Peshier, nucl-th/0101024.
[11] G. Baur. Phys. Rev. D 41, 3535 (1990)
[12] M.J. Rhoades-Brown, J. Weneser. Phys. Rev. A 44, 330 (1991).
[13] C. Best, W. Greiner, and G. Soff. Phys. Rev A 46, 261 (1992)
[14] K. Hencken, D. Trautmann, G. Baur. Phys. Rev. A 51, 998 (1995).
[15] K. Hencken, D. Trautmann, G.Baur. Phys. Rev. A 51, 1874 (1995).
[16] M.C. Güçlü et al. Phys. Rev. A 51, 1836 (1995).
[17] G. Baur. Phys. Rev. A 42, 5736 (1990).
[18] A. Alscher, K. Hencken, D. Trautmann, G.Baur. Phys. Rev. A 55, 396 (1997).
[19] G.Baur, K. Hencken, D. Trautmann. J. Phys. G: Nucl. Part. Phys. 24, 1657 (1998).
[20] M.C. Güçlü, J. Li, A.S. Umar, D.J. Ernest, M.R. Stayer. Ann. of Phys. 272, 7 (1999).

[21] M.C. Güçlü. Nucl. Phys. A 668, 149 (2000).

[22] K. Hencken, D. Trautmann, G.Baur. Phys. Rev. C 59, 841 (1999).