Schwinger pair production rate and time for some space-dependent fields via worldline instantons formalism

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Abstract. Schwinger pair production rate and time in some space-dependent fields are studied analytically by using the worldline instantons formalism. The instanton path, action, production rate, vacuum-decay time and tunneling time are obtained, which depend on the Keldysh parameter significantly. It is found that under a given field the instanton paths with different Keldysh parameter are exactly in the same plane. Moreover, the normalized instanton actions are all bounded within the region from $\pi$ to $2\pi$. We also find the relationship between vacuum-decay time and tunneling time.

1 Introduction

The instanton was first proposed in Ref. [1], which was obtained as a classical solution to the Euclidean field equations with finite action [2,3]. In general it is difficult to get the analytical solutions, as an alternative method, the Feynman path integral approach is developed [4,5] successfully. Thus, the solutions of paths are also called as worldline instantons. By this approach, many studies on the vacuum pair production have been performed, e.g., a typical case is a constant field in scalar quantum electrodynamics (sQED) [6].

It is well known that the vacuum pair production in a strong external field is not only one of the interesting non-perturbative phenomena in the QED, but also a testable field in a high field science in present and future planned experiments. Since Dirac predicted the existence of positrons [7] theoretically, many works have been performed on the vacuum decay and the electron–positron pair production [8–11], in particular, the vacuum pair production rate was first obtained by Heisenberg and Euler [10] with an effective Lagrangian as $\exp(-\pi E_{cr}/E)$, where $E_{cr} = m^2 c^3/\hbar \approx 1.32 \times 10^{18} \text{ V/m}$ is the critical field strength. And later by the proper-time approach [11], Schwinger studied the vacuum pair creation process in detail and recovered the pair production rate in a constant field as

$$\text{Im} \mathcal{L}[E] = \frac{e^2 E^2}{16\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \exp\left[\frac{-n\pi m^2}{eE}\right]. \quad (1)$$

Therefore, the non-perturbative vacuum pair creation process is now called as Sauter–Schwinger effect or Schwinger pair production.

It should be pointed out that due to the very high critical field strength, it is still a great challenge to achieve this extreme field experimentally for the present laser technology [12–14]. Theoretically, however, in recent years, many topics on Sauter–Schwinger effect have been investigated by different methods that the current widely used includes the worldline instanton technique [15–28], the real-time Dirac–Heisenberg–Wigner (DHW) formalism [29–37], the computational quantum field theory (CQFT) [38–42], the quantum Vlasov equation (QVE) [43–47], the Wentzel–Kramers–Brillouin (WKB) approach [48–54] and so on.

For most of methods among them, the study relies on the numerical calculation strongly while the worldline instanton is the very effective approach that can provide many analytical solutions for various chosen fields. On the other hand, in the research community of pair production, it seems lacking of enough care about the quantum tunneling time with the Schwinger effect in external fields [55]. However, the worldline
instanton approach is a good candidate to investigate the quantum tunneling time because it can get the instanton path which contains the time information in a period motion of an instanton. So it is a convenient way to study the tunneling time [15] and can be made a comparison with the vacuum-decay time.

Motivated by the factors mentioned above, in this paper, we investigate the pair production in some chosen space-dependent inhomogeneous fields via the worldline instanton approach. Furthermore, we would get the analytical solutions of instantons for these fields and also discuss their paths, effective actions and the corresponding tunneling times, as a comparison, the vacuum-decay time, which is the measure of field energy depletion in vacuum, is also examined. It should be pointed out that in our study of the pair production rate for the various fields, due to the special choices about the electric and magnetic field that they are mutual perpendicular, meanwhile, they have the same spatial dependence form; thus, by an appropriate Lorentz transformation (we will further introduce it in detail in Sect. 3), the fields would take the same form as that of only electric fields are presenting.

Even if they are reduced to the case of pure electric field under Lorentz transformation, however, to our knowledge, the case of elliptic cosine field in present study has not been reported before so that this is our new result. By the way, the other four cases of fields (constant, cosine, hyperbolic-secant and Lorentz) would become that has been considered previously in the literature, which are not new. The including of them is necessary in the sense of the convenience for the study and discussion on the later new study about the pair production time; meanwhile, they can provide more examples to illustrate the relation between instanton results like of pair production rate and time.

Our paper is organized as follows. In Sect. 2, we briefly introduce the general form of the one loop worldline instanton formalism for sQED in the electromagnetic fields. In Sect. 3, we consider the model for the space-dependent inhomogeneous crossed electric and magnetic fields. The instanton paths and actions for fields are obtained and discussed. We also discuss the implicit relation of results between only space-dependent and only time-dependent fields. In Sect. 4, we analytically discuss the vacuum-decay and tunneling time for the Schwinger effect and also the relation between the tunneling time and pair production rate. In Sect. 5, we examine the relationship between the vacuum-decay time and the tunneling time. In Sect. 6, we study the variation of the instanton trajectories for studied fields and confirm that all of the instanton paths under a fixed field are in the same planar two-dimensional subspace of the three-dimensional instanton space. Finally, the summarized results and outlook are given briefly in Sect. 7.

### 2 Worldline instanton formalism

Our calculations start with effective action $\Gamma^{\text{Mink}}$ in the Minkowski space; we define it as

$$e^{\Gamma^{\text{Mink}}} : = \langle \mathcal{O}_{\text{out}} \mid \mathcal{O}_{\text{in}} \rangle,$$

where the subscripts in and out represent the initial and final states of vacuum, $\langle \mathcal{O}_{\text{out}} \mid \mathcal{O}_{\text{in}} \rangle$ is the vacuum persistence amplitude, and the pair production probability can be written as

$$P = 1 - |\langle \mathcal{O}_{\text{out}} \mid \mathcal{O}_{\text{in}} \rangle|^2 = 1 - e^{-2\text{Im}\Gamma^{\text{Mink}}} \approx 2\text{Im}\Gamma^{\text{Mink}},$$

where $\text{Im}\Gamma^{\text{Mink}} \ll 1$. The relationship between Minkowski effective action and Euclidean effective action $\Gamma^{\text{Mink}} = i\Gamma^{\text{Eucl}}$ is given in Ref. [16], so that we can obtain

$$\text{Im}\Gamma^{\text{Mink}} = \text{Re}\Gamma^{\text{Eucl}},$$

i.e., we are going to Wick rotate $\text{Im}\Gamma^{\text{Mink}}$ from Minkowski space to Euclidean space in order to simplify the path integral via $x_4 = i\tau$, and we use the $\text{Re}\Gamma^{\text{Eucl}}$ to compute the pair production probability. Now the Minkowski four-potential $A_\mu^M = (\phi, A^M) = (A_0^M, A_1^M, A_2^M, A_3^M)$ turns into the Euclidean four-potential $A_\mu^E = (A_0^E, A_1^E, A_2^E, A_3^E) = (A_4, A)$. The relationship between each component in the Minkowski space and the Euclidean space can be written as

$$A_4^E = \frac{1}{i} A_0(t = -i\tau),$$

$$A_j^E = A(t = -i\tau),$$

where index $j = 1, 2, 3$ mean the $x, y$ and $z$ direction in the position space, and we denote that $x = x_1$, $y = x_2$ and $z = x_3$.

The one loop Euclidean effective action in an Abelian space-time-dependent background gauge field $A_\mu^E$ for the scalar particle is written by the worldline path integral form [15]

$$\Gamma^E[A] \simeq \sqrt{\frac{2\pi}{m}} \int \mathcal{D}x \frac{1}{\left(\int_0^1 du u^2\right)^{\frac{1}{2}}} \times e^{-\left(m\int_0^1 du u^2 + ie \int_0^1 du A \cdot \dot{x}\right)},$$

where the functional integral $\int \mathcal{D}x$ contains all closed space-time paths $x^\mu(u)$ with period 1. The Eq. (7) satisfies the weak-field condition as

$$m\int_0^1 du u^2 \gg 1.$$
The worldline action (instanton action [15]) $S_0$ which is nonlocal is defined as

$$S_0 = m \sqrt{\int_0^1 du \dot{x}^2 + ie \int_0^1 du A \cdot \dot{x}}. \tag{9}$$

The path $x_\mu(u)$ satisfies nonlinear differential equations system [15]

$$m \frac{\ddot{x}_\mu}{\sqrt{\int_0^1 du \dot{x}^2}} = ieF_{\mu\nu}\dot{x}_\nu, \tag{10}$$

where $m$ is instanton mass, $-e$ is the electron charge, and the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is an antisymmetric tensor. Solutions of Eq. (10) which satisfies the periodicity condition $x_\mu(0) = x_\mu(1)$ are called worldline instantons. We can get

$$\dot{x}^2 = \text{constant} \equiv a^2, \tag{11}$$

and the stationary instanton path $x_\mu(u)$ satisfies Eq. (11) for any $F_{\mu\nu}(x)$. Thus, the Eq. (8) can be rewritten as $ma \gg 1$. Therefore, we just deal with worldline action $S_0$ to calculate one loop Euclidean effective action $\Gamma^E[A]$. Equations (7), (9) and (10) can be written as

$$m\dot{x}_\mu = i e A_\mu, \tag{12}$$

$$S_0 = ma + ie \int_0^1 du A \cdot \dot{x}, \tag{13}$$

$$\Gamma^E[A] \approx \sqrt{\frac{2\pi}{ma}} \int Dxe^{-S_0}. \tag{14}$$

It is not only easy to deal with above three equations, but also to get the basic physical information of the pair production we need. In the next section, we discuss how to deal with space-dependent electromagnetic fields.

### 3 Spatially inhomogeneous electromagnetic fields

In this section, we solve instanton action and effective action analytically in a series of classical space-dependent background electromagnetic fields.

We choose a four-potential $A_\mu$ which has nonzero two components, and it is functions of $x_3$ as

$$A_\mu = A_\mu^E = (A_1^E, A_3^E, 0, 0) = (-iA_0^M(x_3), A_1^M(x_3), 0, 0). \tag{15}$$

From Eq. (15), the electromagnetic fields can be written as

$$E = -\nabla \phi - \frac{\partial A}{\partial t} = E_z(x_3)e_z, \tag{16}$$

$$B = \nabla \times A = B_y(x_3)e_y, \tag{17}$$

where $e_y$ and $e_z$ are the unit vectors, $E_z(x_3)$ and $B_y(x_3)$ are the magnitudes of the electric fields in the $z$ direction and the magnetic fields in the $y$ direction. In our cases, we compute paths $x_\mu$, worldline action $S_0$, and discuss the effective action $\Gamma^E[A]$ for the four-potential $A_\mu$. Finally, we can get the equations of motion

$$m\dot{x}_1 = -i e A_1, \tag{18}$$

$$m\dot{x}_2 = i e A_2, \tag{19}$$

$$m\dot{x}_3 = i e \left(\frac{\partial A_1(x_3)}{\partial x_3} \frac{\partial x_1}{\partial u} + \frac{\partial A_3(x_3)}{\partial x_3} \frac{\partial x_3}{\partial u}\right), \tag{20}$$

$$m\dot{x}_4 = -i e \frac{\partial A_4(x_3)}{\partial x_3} \frac{\partial x_3}{\partial u}. \tag{21}$$

If we assume that $A_1(x_3)$ and $A_4(x_3)$ are odd functions of $x_3$, we get $\dot{x}_3^2 = 1$ and $\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 = a^2$ from above equations, which leads to the result

$$\dot{x}_1 = -i e A_1(x_3), \tag{22}$$

$$\dot{x}_4 = -i e A_4(x_3), \tag{23}$$

$$|\dot{x}_3| = a \sqrt{1 + \left(\frac{e}{m} A_1(x_3)\right)^2 + \left(\frac{e}{m} A_4(x_3)\right)^2}. \tag{24}$$

When $\dot{x}_3 = 0$ in Eq. (24), we can get

$$1 + \left(\frac{e}{m} A_1(x_3^*)\right)^2 + \left(\frac{e}{m} A_4(x_3^*)\right)^2 \equiv 0. \tag{25}$$

At the turning point $x_3^*$, the kinetic energy equals to potential energy of the instanton (see Ref. [56]). It is vital to calculate the path integral in the effective action.

Note that the resulting worldline action is obtained by taking Eq. (15) into Eq. (13) as

$$S_0 = ma + ie \int_0^1 du (A_1 \dot{x}_1 + A_4 \dot{x}_4) = ma + ie \int_0^1 du \left(\frac{im}{ae} \dot{x}_1^2 + \frac{im}{ae} \dot{x}_2^2\right) = ma + \frac{m}{a} \int_0^1 du (\dot{x}_1^2 + \dot{x}_2^2), \tag{26}$$

$$= ma - \frac{m}{a} \int_0^1 du (a^2 - \dot{x}_3^2) = \frac{m}{a} \int_0^1 du \dot{x}_3^2.$$
In general, the expressions of \( A_1(x_3) \) and \( A_4(x_3) \) in the Euclidean space can be written as

\[
A_1(x_3) = \frac{B}{k} f(kx_3), \\
A_4(x_3) = -\frac{iE}{k} f(kx_3),
\]

(27, 28)

where \( i \) is the imaginary unit, \( k \) is the field frequency with space, \( 1/k \) characterize the length scale of the spatial inhomogeneity, \( E \) and \( B \) are peak values of the electric and magnetic fields strength, respectively. Note that \( f \) is the odd function with respect to \( x_3 \).

After substituting Eqs. (27) and (28) into the differential Eqs. (22), (23) and (24), we get

\[
\dot{x}_1 = -\frac{iacB}{mk} f(kx_3), \\
\dot{x}_4 = -\frac{aeE}{mk} f(kx_3), \\
|\dot{x}_3| = a\sqrt{1 - \frac{f^2(kx_3)}{\gamma_{k,e}^2}}.
\]

(29, 30, 31)

For convenience of calculation, we define a modified Keldysh parameter that \( \gamma_{k,e} = mk/e\sqrt{E^2 - B^2} \) and \( \gamma_{\omega,e} = m\omega/e\sqrt{E^2 - B^2} \), which implies that \( \gamma_{k,e}^2 = (\gamma_{k,e} - \gamma_{k,e}^{-2}) \) and \( \gamma_{\omega,e}^2 = (\gamma_{\omega,e} - \gamma_{\omega,e}^{-2}) \), where \( \gamma_{k,e} = mk/eE \) and \( \gamma_{k,b} = mk/eB, \gamma_{\omega,e} = m\omega/eE \) and \( \gamma_{\omega,b} = m\omega/eB \). If \( B = 0, \gamma_{\omega,b} \) determines tunneling or multiphoton process. The instanton action can be written as

\[
S_0 = \frac{m}{a} \int_0^1 du \dot{x}_3^2 = ma \int_0^1 du \left( 1 - \frac{f^2(kx_3)}{\gamma_{k,e}^2} \right)
\]

\[= \frac{4mn^2}{k\gamma_{k,e} b} \int_0^1 dy \sqrt{1 - \frac{y^2}{|f|^2}} = \pi mn \gamma_{k,e} g(\gamma_{k,e}),
\]

(32)

where \( y = \frac{1}{\gamma_{k,e}} f(v) \) with \( v = kx_3 = f^{-1}(\gamma_{k,e} b) \), thus, \( f'(v) \) can be reexpressed as a function of \( y \) again, and the function \( g(\gamma_{k,e}) \) is defined as

\[
g(\gamma_{k,e}) = \frac{2}{\pi} \int_{-1}^1 dy \sqrt{1 - \frac{y^2}{|f|^2}}.
\]

(33)

where integral bounds are related to the turning points [56]

\[
x_3^* = \pm \frac{f^{-1}(\gamma_{k,e})}{k} = x_3^*.
\]

(34)

In the following, we study five types of field; meanwhile, we compare their instanton action in Eqs. (26) and (32) in order to check whether the two results are equivalent.

### 3.1 Elliptic cosine field

In this subsection, we examine an electric field \( E_z(x_3) = E\c(\kx_3|q) \) and magnetic field \( B_y(x_3) = B\c(\kx_3|q) \) for various values of parameter \( q \) in the Minkowski space as shown in Fig. 1. Corresponding Euclidean space-dependent gauge potential is

\[
A_\mu(x_3) = \left( -\frac{iE}{k} f(kx_3), \frac{B}{k} f(kx_3), 0, 0 \right),
\]

(35)

where \( f(kx_3) = \sn(v|q), f' = \sqrt{1 - \gamma_{k,e}^2 y^2} \sqrt{1 - q^2 \gamma_{k,e}^2 y^2}, \text{sn}(v|q) \) and \( \text{cn}(v|q) \) are the Jacobi elliptic sine and cosine functions [57,58], the two variables \( v \) and \( q \) in the elliptic functions denote the terms of the amplitude and elliptic modulus. Note that \( \text{cn}(\kx_3|0) = \cos(\kx_3) \) and \( \text{cn}(\kx_3|1) = \sech(\kx_3) \), because the \( \text{cn}(\kx_3|q) \) contains an infinite number of functions as shown in Fig. 1. We can obtain the instanton action \( S_0 \) from Eq. (32) directly

\[
S_0 = \frac{4mn^2}{k\gamma_{k,e} b} \int_0^1 dy \sqrt{1 - \frac{y^2}{|f|^2}} = \frac{4mn^2}{\sqrt{E^2 - B^2} \gamma_{k,e} - 1} \Pi \left( \frac{1}{\gamma_{k,e} - 1}, \frac{\arcsin \left( \sqrt{\frac{\gamma_{k,e} - 1}{\gamma_{k,e} + 1}} \right)}{1 - \frac{q^2 \gamma_{k,e}^2}{1 - q^2}} \right)_0^1,
\]

(36)

where \( \Pi(\alpha, \phi, \beta) \) is the incomplete elliptic integral of the third kind in which it has three parameters independently [57,58]

\[
\Pi(\alpha, \phi, \beta) = \int_0^\phi \frac{1}{1 - \sin^2\theta} \sqrt{1 - (\sin\theta\sin\beta)}
\]

(37)

If \( q = 0 \), the instanton action can be written as

\[
S_0 = \frac{4mn^2}{\sqrt{E^2 - B^2} \gamma_{k,e}^2} \left( E \left( \gamma_{k,e} ^2 - 1 \right) - K \left( \gamma_{k,e}^2 \right) \right)
\]

\[= \frac{4mn^2}{\sqrt{E^2 - B^2}} \left( E \left( \gamma_{k,e}^2 - 1 \right) - K \left( \gamma_{k,e}^2 \right) \right) \times \left( \sqrt{\frac{1 - \gamma_{k,e}^2}{\gamma_{k,e}^2}} \right),
\]

(38)

Note that it would be reduced to the result in Eq. (68) when \( q = 0 \), and the approximate result in Eq. (56) when \( q = 1 \), see the latter cases studied in the paper. The general results are shown in Fig. 2, in which the action is normalized in unit of \( mn^2/e\sqrt{E^2 - B^2} \).

We can find that the instantons action increases and the pair production rate decreases with \( \gamma_{k,e} \). The instantons action equals to \( \pi \) and \( 4 \) for \( \gamma_{k,e} = 0 \) and \( \gamma_{k,e} = 1 \) when \( q = 0 \), and \( \pi \) and \( 2\pi \) for \( \gamma_{k,e} = 0 \) and \( \gamma_{k,e} = 1 \) when \( q = 1 \). The instantons action values
Plot of the space-dependent electromagnetic fields $E_z(x_3) = E \text{cn}(kx_3|q)$ and $B_y(x_3) = B \text{cn}(kx_3|q)$ for different values of the parameter $q$. The distributions for blue, yellow, green, magenta, lightblue and orange in the figure correspond to 0, 0.1, 0.5, 0.7, 0.9 and 1 of $q$.

Plot of the stationary worldline instanton action $S_0$ in the $E(x_1, x_3, x_3)$ space for the cases the spatially inhomogeneous electromagnetic fields $E_z(x_3) = E \text{cn}(kx_3|q)$ and $B_y(x_3) = B \text{cn}(kx_3|q)$ of strength $E$ and $B$ with different values of the parameter $q$. The distributions for blue, yellow, green, magenta, lightblue and orange in the figure correspond to 0, 0.1, 0.5, 0.7, 0.9 and 1 of $q$. The $S_0$ have been expressed in units of $nm^2/e \sqrt{(E^2 - B^2)}$ for other fields between $0 \leq q \leq 1$ are within the region from $\pi$ to $2\pi$, as shown in Fig. 2. This is the same result with in Sects. 3.3 and 3.4. We can obtain the same value of instanton action of $S_0$ as $B = 0$ in Ref. [15].

3.2 Constant field

For the simplest constant electromagnetic background fields, we can set $f(kx_3) = kx_3$ and $k = 1$, then we can obtain the Euclidean four-potential for constant electromagnetic background fields

$$A^E_\mu = (-iEx_3, Bx_3, 0, 0). \quad (39)$$

The solutions of Eqs. (29), (30) and (31) are

$$x_1 = i\frac{\gamma_{keb}}{\gamma_{kb}} \cos(2n\pi u), \quad (40)$$

$$x_3 = \gamma_{keb} \sin(2n\pi u), \quad (41)$$
Fig. 3 Parametric plot of the stationary worldline instanton paths in the (Im$x_1$, $x_3$, $x_4$) space for the cases of constant electric and magnetic fields of strength $E$ and $B$. The paths are elliptic curves in (Im$x_1$, $x_3$, $x_4$) space in units of $mB/e(E^2 - B^2)$, $m/e\sqrt{E^2 - B^2}$ and $mE/e(E^2 - B^2)$ for Im$x_1$, $x_3$ and $x_4$.

$$x_4 = \frac{\gamma^{2}_{keb}}{\gamma_{keb}} \cos(2n\pi u),$$

\[ (42) \]

where

$$a = 2n\pi\gamma_{keb}, \quad n \in \mathbb{Z}^+,$$

\[ (43) \]

$n$ is integer number of the closed paths.

The stationary worldline instanton paths are the elliptic curves with different $\gamma_{keb}$, as shown in Fig. 3.

We find the same trajectories for different $\gamma_{keb}$ in (Im$x_1$, $x_3$, $x_4$) space. For $B = 0$, we can obtain the same result with Eq. (26) in Ref. [15].

By taking Eq. (43) into the Eq. (8), we get the weak field condition

$$\frac{2n\pi m^2}{e\sqrt{E^2 - B^2}} \gg 1.$$  

\[ (44) \]

The physical meaning of Eq. (44) is that $\sqrt{E^2 - B^2} \ll \frac{2\pi m^2 c^3}{e\hbar} \sim 10^{16}$ V/cm = $E_{cr}$ or $\sqrt{E^2 - B^2} \ll \frac{2\pi m^2 c^2}{e\hbar} \sim 10^6$ T = $B_{cr}$, which is satisfied for experimentally accessible electromagnetic fields, and the result is the same with Eq.(25) in Refs. [11,15] when $B = 0$.

The corresponding instanton action by using Eq. (26) is

$$\mathcal{S}_0 = 2n\pi m\gamma_{keb} \int_{0}^{1} du \cos^2(2n\pi u) = \frac{n\pi m^2}{e\sqrt{E^2 - B^2}}.$$  

\[ (45) \]

Obviously, the instanton action $\mathcal{S}_0$ obtained from Eqs. (32) and (45) has the same results with that in Refs. [11,15], which guarantee the correctness of our definition of the instanton action in Eq. (32). The pair production does not occur when $E \geq B$ or $\gamma_{keb}$ is imaginary number. Therefore, we only consider the pair creation cases when $\gamma_{keb}$ is real number.

### 3.3 Hyperbolic secant field

For the Minkowski space-dependent electric field $E_z(x_3) = E sech^2(kx_3)$ and magnetic field $B_y(x_3) = B sech^2(kx_3)$, the corresponding Euclidean space-dependent gauge potential is

$$A_{\mu}(x_3) = \left(\frac{-iE}{k} \tanh(kx_3), \frac{B}{k} \tanh(kx_3), 0, 0\right).$$

\[ (46) \]

By using Eqs. (22), (23) and (24), we can get the following equations

$$\dot{x}_1 = -\frac{iaeB}{mk} \tanh(kx_3),$$

\[ (47) \]

$$\dot{x}_4 = -\frac{iaeE}{mk} \tanh(kx_3),$$

\[ (48) \]

$$|\dot{x}_3| = a\sqrt{1 + \left(\frac{eB}{mk} \tanh(kx_3)\right)^2 - \left(\frac{aeE}{mk} \tanh(kx_3)\right)^2}.$$  

\[ (49) \]

And the stationary solution for $x_3(u)$ is determined by integrating Eq. (49)

$$x_3 = \frac{1}{k} \arcsinh\left(\frac{\gamma_{keb}}{\sqrt{1 - \gamma^2_{keb}}} \sin\left(\sqrt{1 - \gamma^2_{keb}} k au\right)\right).$$  

\[ (50) \]

We know that $\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 = a^2$ is satisfied. Therefore, the $a$ can be obtained as

$$a = \frac{\gamma_{keb}}{k\sqrt{1 - \gamma^2_{keb}}} 2\pi n, \quad n \in \mathbb{Z}^+.$$

\[ (51) \]

For this inhomogeneous case, we can rewrite Eq. (8) as

$$\frac{m\gamma_{keb}}{k\sqrt{1 - \gamma^2_{keb}}} 2\pi n \gg 1.$$  

\[ (52) \]

It is a weak-field condition, and must satisfies $E > B$ and $\sqrt{E^2 - B^2} \ll E_{cr}$ or $B_{cr}$.

The periodic stationary instanton paths can be obtained from Eqs. (47), (48) and (49)

$$x_1 = \frac{i}{k} \frac{\gamma_{keb}}{\gamma_{keb} \sqrt{1 - \gamma^2_{keb}}} \arcsinh\left[\gamma_{keb} \cos\left(2\pi m n u\right)\right],$$  

\[ (53) \]
The stationary worldline instanton paths are elliptic curves for the different $\gamma_{keb}$ as shown in Fig. 4. As $\gamma_{keb}$ increases, the instanton paths shrinks in size. Note that, when $\gamma_{keb} \to 0$, these loop trajectories of the instantons tend to become the same elliptic curves as shown in Fig. 3. However, the loop paths expand enormously for $\gamma_{keb} = 1$, and no pair creation in this case. Because the spatial width of the electromagnetic field is smaller than the Compton wavelength (see Ref. [15]), thus, there are pair creation only for field is smaller than the Compton wavelength (see Ref. [15]).

The stationary worldline instanton paths are elliptic curves for the different $\gamma_{keb}$ as shown in Fig. 4. As $\gamma_{keb}$ increases, the instanton paths shrinks in size. Note that, when $\gamma_{keb} \to 0$, these loop trajectories of the instantons tend to become the same elliptic curves as shown in Fig. 3. However, the loop paths expand enormously for $\gamma_{keb} = 1$, and no pair creation in this case. Because the spatial width of the electromagnetic field is smaller than the Compton wavelength (see Ref. [15]), thus, there are pair creation only for $\gamma_{keb} < 1$. We can find that all of the elliptical shapes are in the same plane for any value of $\gamma_{keb}$. The stationary instanton paths have the same result with Eq.(61) in Ref. [15] when $B = 0$.

The stationary instanton action $S_0$ is

$$S_0 = m a \int_0^1 du \cos^2 \left( \frac{2\pi nu}{1 + \frac{\gamma_{keb}}{1 - \gamma_{keb}}} \right) \sin^2 \left( \frac{2\pi nu}{1 + \frac{\gamma_{keb}}{1 - \gamma_{keb}}} \right).$$

$$S_0 = \frac{\pi \eta m^2}{e \sqrt{E^2 - B^2}} \frac{2}{1 + \sqrt{1 - \gamma_{keb}}}.$$

This stationary instanton action is plotted in Fig. 5. Note that the instantons action $S_0$ increases with $\gamma_{keb}$, and $S_0 = \pi$ and $2\pi$ for $\gamma_{keb} = 0$ and 1, respectively. We can also obtain the same result for instanton action $S_0$ with in Ref. [15] when $B = 0$.

### 3.4 Cosine field

In the Minkowski space, we consider the electric field $E_x(x_3) = E \cos(kx_3)$ and magnetic field $B_y(x_3) = B \cos(kx_3)$, and corresponding Euclidean space gauge potential is

$$A_\mu(x_3) = \left( -\frac{E}{k} \sin(kx_3), \frac{B}{k} \sin(kx_3), 0, 0 \right).$$

By using Eq. (57), we can obtain the following equations from Eqs. (22), (23) and (24)

$$\dot{x}_1 = -\frac{iae}{m} B \sin(kx_3),$$

$$\dot{x}_4 = -\frac{ae}{m} E \cos(kx_3),$$

$$|\dot{x}_3| = a \sqrt{1 + \left( \frac{e}{m} B \sin(kx_3) \right)^2 - \left( \frac{e}{m} E \sin(kx_3) \right)^2}.$$
the electric fields can be written as blue, pink and green for 0. We get a result from the Eq. (8) which is the real quarter-period of the Jacobi elliptic function [11]. We can find that instanton loop paths tend to become the same elliptical shapes as shown in Fig. 3 when $\gamma_{keb} = 0$. However, the loop paths become infinitely large when $\gamma_{keb} = 1$, and does not occur pair production. The reason of this phenomenon is the same with in Sect. 3.3. Thus, there are pair creation only for $\gamma_{keb} < 1$, and all of the elliptical shapes are in the same plane for any value of $\gamma_{keb}$. The stationary instanton paths have the same result with Eq. (65) in Ref. [15] when $B = 0$.

The stationary instanton action $S_0$ can be written as

$$S_0 = ma \int_0^1 du \, c d^2 \left( \frac{\alpha(u)}{\nu} \right)$$

$$= \frac{4m^2}{k} \frac{\sqrt{1 - \gamma_{keb}^2}}{E^2 - B^2} (E - K)$$

$$\approx \begin{cases} n \frac{m^2 \pi}{e \sqrt{E^2 - B^2}} (1 + \frac{\gamma_{keb}^2}{8} + \frac{3\gamma_{keb}^4}{64} + \ldots), & \gamma_{keb} \ll 1, \\ n \frac{m^2 \pi}{e \sqrt{E^2 - B^2}}, & \gamma_{keb} \to 1, \end{cases}$$

where $E \equiv E(\nu)$ is the complete elliptic integral of the second kind which is the real quarter-period of the Jacobi elliptic function [57,58].

This stationary instanton action $S_0$ is plotted as a function of the $\gamma_{keb}$ in Fig. 7. Note that the instantons action $S_0$ increases with $\gamma_{keb}$, and $S_0 = \pi$ and for $\gamma_{keb} = 0$ and 1, respectively. This result also is the same with in Ref. [15] when $B = 0$.

Note that, unlike three cases of B-D demonstrated above, in case A and the following case E, it is hard to solve analytically via Eqs. (22), (23) and (24). However, we can find the instanton action directly from Eq. (32) without calculation of instanton paths.

It is easy to verify that the periodic instanton paths are

$$x_1 = \frac{i}{k} \frac{\gamma_{keb}}{\gamma_{kb}} \arcsinh \left( \frac{\gamma_{keb}}{\sqrt{1 - \gamma_{keb}^2}} \frac{cd(\alpha(u)|\nu)}{kau} \right)$$

$$x_3 = \frac{1}{k} \arcsin \left( \frac{\gamma_{keb}}{\sqrt{1 - \gamma_{keb}^2}} \frac{sd(\alpha(u)|\nu)}{\nu} \right),$$

$$x_4 = \frac{1}{k} \frac{\gamma_{keb}}{\gamma_{keb}} \arcsinh \left( \frac{\gamma_{keb}}{\sqrt{1 - \gamma_{keb}^2}} \frac{cd(\alpha(u)|\nu)}{\nu} \right),$$

where $\alpha(u) = 4nKu$, $K \equiv K(\nu)$ and $\nu \equiv -\frac{\gamma_{keb}^2}{1 - \gamma_{keb}^2}$.

![Parametric plot of the stationary worldline instanton paths in the (Im$x_1$, $x_3$, $x_4$) space for the cases of the electric fields $E(x_3) = E\cos(kx_3)$ and the magnetic fields $B(x_3) = B\cos(kx_3)$ of strength $E$ and $B$. The paths are shown for various values of the parameter $\gamma_{keb}$ (red, blue, pink and green for 0.1, 0.5, 0.7 and 0.9), Im$x_1$, $x_3$ and $x_4$ have been expressed in units of $mB/e(E^2 - B^2)$, $m/e\sqrt{E^2 - B^2}$ and $mE/e(E^2 - B^2)$, respectively.](image)
fields \( E \) and \( B \) which (locally) one can form only two scalars. Fact the electromagnetic field is a rank two tensor, from results in this section, we have to say a few words. In the following, for the studied fields and obtained section in the (Im\( x_3 \), \( x_3 \), \( x_3 \)) space for the cases of the electric fields \( E(x_3) = E \cos(kx_3) \) and the magnetic fields \( B(x_3) = B \cos(kx_3) \) of strength \( E \) and \( B \). The \( S_0 \) have been expressed in units of \( \text{nm}^2/e\sqrt{(E^2 - B^2)} \).

### 3.5 Lorentzian field

Now we study the electric field \( E_z(x_3) = E/(1 + (kx_3)^2)^{3/2} \) and magnetic field \( B_y(x_3) = B/(1 + (kx_3)^2)^{3/2} \) in the Minkowski space, and corresponding Euclidean space-dependent gauge potential is

\[
A_\mu(x_3) = \left( -i \frac{E}{k} f(kx_3), \frac{B}{k} f(kx_3), 0, 0 \right),
\]

where \( f(kx_3) = kx_3/(1 + (kx_3)^2) \) and \( f'(kx_3) = (1 - \gamma_{\text{keb}}^2)^{3/2} \) then we can obtain the instanton action \( S_0 \) directly from Eq. (32)

\[
S_0 = \frac{4mn}{k\gamma_{\text{keb}}} \int_0^1 dy \frac{\sqrt{1 - y^2}}{|f'|} = \frac{4nm^2}{\sqrt{E^2 - B^2} \gamma_{\text{keb}}} \left( K \left( \gamma_{\text{keb}}^2 \right) - E \left( \gamma_{\text{keb}}^2 \right) \right).
\]

The instanton action \( S_0 = \pi \) and \( 2\pi \) for \( \gamma_{\text{keb}} = 0 \) and 1, respectively, see Fig. 8. We can find that the instantons action increases and pair creation rate decreases with \( \gamma_{\text{keb}} \). It is similar to results in Sects. 3.3 and 3.4. We can obtain the same result of instanton action \( S_0 \) in Ref. [15] when \( B = 0 \).

Before starting the pair production time examination in the following, for the studied fields and obtained results in this section, we have to say a few words. In fact the electromagnetic field is a rank two tensor, from which (locally) one can form only two scalars

\[
\mathcal{F} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} (B^2 - E^2),
\]

\[
\mathcal{G} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = E \cdot B.
\]

For the fields we have considered in this paper, \( \mathcal{G} = 0 \); only \( \mathcal{F} \) is nonzero. Therefore, in any form of \( f \) we considered, we can find the results, except the case of elliptic cosine field, are related to known results from Ref. [15] by simply replacing

\[
-\mathcal{F} \leftrightarrow E.
\]

Note that the arrow goes both ways: the \( B \) dependence is completely fixed by the \( E \) dependence and can be removed through keeping the same \( \mathcal{F} \).

We can obtain instanton path, action, production rate, production time and tunneling time in time-dependent electromagnetic field by replacing modified Keldysh parameter \( \gamma_{\text{keb}} \) for time-dependent fields with \( \gamma_{\text{keb}} \) after performing above Lorentz boost according to Ref. [15,16]. We can just use this technique to obtain time-dependent result, and thus, we do not need to discuss time-dependent field. The magnitudes of the normalized instantons action are all bounded within the region from 0 to \( \pi \) in the temporally inhomogeneous fields we study [15,16]. For example, the instanton paths, instanton action and pair production rate had been studied by constant, cosine, hyperbolic secant and Lorentzian fields with time-dependent form in Ref. [15,16]. But we have not any analytical expression of pair production time, therefore, we must further investigate it in the next section.

### 4 Pair production time

#### 4.1 Vacuum-decay time

In this section, we discuss the vacuum-decay time (include tunneling process, multiphoton absorption process) for space-dependent electromagnetic fields analytically. Physically, the vacuum-decay time is the time when all the energy of the electromagnetic field is transferred to the particle. Because the vacuum decay
in our cases belongs to the energy transformation process from field to particles, from Eq. (6) in Ref. [59] we can obtain the total pair production time as

$$T_d = U_f \left( \frac{d\langle U_m \rangle}{dt} \right)^{-1},$$

(74)

where $T_d$ denotes the vacuum-decay time, $U_m$ and $U_f$ are the transverse energy density and the field energy density. This energy density of the field is expressed in terms of the invariants $F$ and $G$ as [59,60]

$$U_f = \sqrt{(F^2 + G^2)} f^2 (F, G) + A^2 (F, G),$$

$$L = -\mathcal{F} + \frac{8\alpha^2}{45m^4} F^2 + \frac{14\alpha^2}{45m^4} G^2,$$

(75)

(76)

where $L$ is the effective Heisenberg–Euler Lagrangian, $f = \partial L/\partial F \rightarrow -1$ for the Maxwell’s equation when the lowest-order approximation is taken into account, $A$ is the conformal anomaly induced by external fields, $T_\mu^\nu = 4A$ [59].

If we redefine the rate of vacuum decay (combine with Eqs. (2) and (4) in Ref. [59]) by replacing the worldline instantons action $S_0$ with the action under constant electric field as

$$\frac{d\langle U_m \rangle}{dt} = \frac{eE}{2\pi^2} \int_m^{\infty} d\epsilon \int_{E/B}^{\infty} e^{-\beta(\gamma_{c\text{eb}})\epsilon^2} \left( 1 + \frac{\sqrt{\pi} e^S_0}{2\sqrt{S_0}} \text{erfc} \left( \sqrt{S_0} \right) \right),$$

(77)

where $\beta(\gamma_{c\text{eb}}) = S_0/m^2$, $\epsilon = \sqrt{m^2 + p^2}$, $\gamma_{c\text{eb}} = \sqrt{m^2 + p^2}$. Therefore, the vacuum-decay time can be obtained as

$$T_d = U_f \left( \frac{eE}{4\pi^2} \frac{m e^{-S_0}}{\beta(\gamma_{c\text{eb}})} \left( 1 + \frac{\sqrt{\pi} e^{S_0}}{2\sqrt{S_0}} \text{erfc} \left( \sqrt{S_0} \right) \right) \right)^{-1}.$$

(78)

If we consider constant electric field, the vacuum-decay time becomes that

$$T_d = U_f \left( \omega_0 E^2 e^{-\omega_0/4} \left( 1 + h \left( \sqrt{\frac{\pi E_0}{E}} \right) \right) \right)^{-1},$$

(79)

where $\omega_0 = \alpha c/\pi^2$, $\omega = \alpha m c^2/\pi^2 h = 5.740 \times 10^{17} s^{-1}$ and $h(z) = \sqrt{\pi} e^{z^2} \text{erfc}(z)/2z$. If we ignore the second and third terms of $\mathcal{L}$ in Eq. (76), it is the same result with Eq. (6) in Ref. [59] when $B/E = 0$ (see Fig. 9). The orange line in Fig. 9 (left) represents the minimum value ($\omega_0^{-1}/4$) of the vacuum-decay time for constant electric field. We can find that the time increases with the decrease of $\gamma_{c\text{eb}}$ for the same parameter $B/E$, but does not depend on the $x_3$.

From Eq. (78), we can find that the vacuum-decay time depends on $x_3$ and electromagnetic field modes. To facilitate the discussion of the effect of coordinate $x_3$ and electromagnetic field modes, we give four examples of different electromagnetic fields with parameters $k = 0.3$ and $E/B = 0.01$ for the Maxwell’s equation as shown in Fig. 10. We can find that the time increases with decrease of parameter $\gamma_{c\text{eb}}$ when $x_3$ is fixed. Meanwhile, if we consider the overall effect in the distributions of vacuum-decay time for sech$^2$ ($kx_3$), cos ($kx_3$), $1/(1 + (kx_3)^2)^{3/2}$ and cn ($kx_3|0.7$), the distributions of these times are similar to those shapes of the applied electromagnetic field modes for the same parameter $E/E_{cr}$.

Certainly, we can apply Eq. (78) not only to higher-order Heisenberg–Euler Lagrangian, but also to multiphoton and tunneling processes (or mixing process).
4.2 Tunneling time

In this subsection, we discuss the expression of the tunneling time of the Bohm viewpoint by using the instanton action. The time taken by the particle from $x_3^+$ in the barrier region is quantum tunneling time or tunneling time. From Ref. [55], the tunneling time can be written by using Eq. (B7) in Appendix B

$$T_t = \int_{u(x^-)}^{u(x^+)} \dot{x}_4 du = \int_{x_3^+}^{x_3^-} \dot{x}_4 dx_3, \quad (80)$$

where $T_t$ denotes the tunneling time. According to Eqs. (30) and (31), we can obtain that

$$T_t = \int_{x_3^+}^{x_3^-} -\frac{ae}{f} \frac{f(kx_3)}{\gamma_{ke}^2} dx_3 = -\frac{2}{\gamma_{ke}} \int_{x_3^+}^{x_3^-} \frac{f(kx_3)}{\sqrt{1 - \frac{f^2(kx_3)}{\gamma_{ke}^2}}} dx_3. \quad (81)$$

Thus, the tunneling times corresponding to the space-dependent electromagnetic fields researched in Sect. 3 are

$$T_t = \begin{cases} \frac{2\gamma_{ke}}{\gamma_{ke}} \arcsin \left[ \gamma_{ke} \right], & \text{constant} \\ \frac{2}{\gamma_{ke} \sqrt{1 - \gamma_{ke}^2}} \arcsinh \left[ \frac{2\gamma_{ke}}{\gamma_{ke} \sqrt{1 - \gamma_{ke}^2}} \right], & \text{sech}^2(kx_3) \\ \frac{2}{\gamma_{ke} \sqrt{1 - \gamma_{ke}^2}} \arcsinh \left[ \frac{2\gamma_{ke}}{\gamma_{ke} \sqrt{1 - \gamma_{ke}^2}} \right], & \cos(kx_3) \\ \frac{2\gamma_{ke}^2}{\gamma_{ke} \sqrt{1 - \gamma_{ke}^2}} \left(1 - \gamma_{ke}^2 \right)^{1/2}, & \text{sech}^2(kx_3) \\ \frac{2}{\gamma_{ke} \sqrt{1 - \gamma_{ke}^2}} \arcsinh \left[ \frac{2\gamma_{ke}}{\gamma_{ke} \sqrt{1 - \gamma_{ke}^2}} \right], & \text{cos}^2(kx_3) \\ \frac{2}{\gamma_{ke} \sqrt{1 - \gamma_{ke}^2}} \arcsinh \left[ \frac{2\gamma_{ke}}{\gamma_{ke} \sqrt{1 - \gamma_{ke}^2}} \right], & \text{cos}^2(kx_3) \end{cases} \quad (82)$$

respectively. On the other hand, we know that there is a simple relationship between the tunneling time $T_t$ and the $x_4$ as

$$T_t = 2x_4(0) = \left( x_4^{\text{max}} - x_4^{\text{min}} \right). \quad (83)$$

For example, it is valid for cases of constant, sech$^2(kx_3)$ and cos$(kx_3)$ fields by comparing Eqs. (42), (55) and (67) with (82) where the instanton number $n = 1$ is given. Obviously, the relationship Eq. (83) is valid generally for any field that has instanton solutions, which is proven in Appendix B by using different methods. Physically, the tunneling time is just one that an instanton "particle" takes when it travels the half loop path from $x_3^+$ to $x_3^-$, see Figs. 3, 4 and 6.

We can find another nature of tunneling time for the worldline instanton approach that according to Eq. (82), the tunneling time increases with parameter $\gamma_{ke}^2$ when $\gamma_{ke}$ and $k$ are given, while the pair production rate decreases. Thus, the larger the tunneling time, the smaller the pair creation. Interestingly, the tunneling time no longer depends on the $x_3$, which is different from the vacuum-decay time.

4.3 Tunneling time and production rate

In order to prove this relationship between tunneling time and particle number, we find out the analytical solution of the production rate and prove whether this relationship is correct. By the way, if it is not easy to observe the pair production by using our studied electromagnetic field, then we can instead of it observe it in the pure electric field since the Lorentz transformation.
enables two cases to be equivalent with each other as mentioned above at the end of Sect. 3.

Now the production rate can be written as (for the convenience of discussion, we only write the analytical solutions of three kinds of fields)

\[
\frac{\text{Im} \Gamma_{\text{semi}}}{(V_2 T)^{\text{Mink}}} = \begin{cases} 
\exp \left[ -\frac{\pi m^2}{e\sqrt{E^2 - B^2}} \right], \\
(1-\gamma_{kebp})^{3/4} \exp \left[ -\frac{4m^2}{e\sqrt{E^2 - B^2}} \sqrt{1-\gamma_{kebp}} \right] \left[ E \left( \frac{-\gamma_{kebp}^2}{1-\gamma_{kebp}} \right) - K \left( \frac{-\gamma_{kebp}^2}{1-\gamma_{kebp}} \right) \right], \\
\frac{(1-\gamma_{kebp})^{5/4}}{1} \exp \left[ -\frac{m^2 \pi}{e\sqrt{E^2-B^2}} \right] \left( \frac{2}{1+\sqrt{1-\gamma_{kebp}}} \right), \\
\frac{(1-\gamma_{kebp})^{3/4}}{1} \exp \left[ -\frac{2m^2 \pi}{e\sqrt{E^2-B^2}} \right] \left( \frac{2}{1+\sqrt{1-\gamma_{kebp}}} \right), \\
\frac{1}{d(\gamma_{kebp}^2 g(\gamma_{kebp}))} \left( \frac{2m^2 \pi}{e\sqrt{E^2-B^2}} \right), \\
\text{constant} \\
\cos (kx_3) \\
\text{sech}^2 (kx_3) \\
\text{cn} (kx_3 | q) \\
\end{cases}
\]

For these three fields, the tunneling times and production rates are shown in Fig. 11 on the left and right side, respectively. We can find that the tunneling time is from small to large for \(\cos (kx_3)\), \(\text{sech}^2 (kx_3)\) and \(\frac{1}{1+(kx_3)^2} \text{sech}^2 (kx_3)\), respectively and vice versa for the production rate from high to low for these fields. This result is consistent with the result in the Sect. 4.2. The particle number decreases with the increase of tunneling time, and increases with the decrease of tunneling time.

5 Relationship between the decay time and the tunneling time

In this subsection, we will study the relationship between the decay time and the tunneling time for external field though we do not have any mathematical expression to connect. Note that the decay time depends on space coordinate, but the tunneling time does not. Thus, we can find a ratio about the average decay time for space coordinate and the tunneling time, and then can discuss this relationship. If we define the average decay time as

\[
\bar{T}_d := \frac{\int_a^b T_d dx_3}{\int_a^b dx_3},
\]

where we choose \(-2\pi/k\) and \(2\pi/k\) for \(a\) and \(b\) in our work.

The ratio \(\bar{T}_d/T_t\) is plotted for different fields in Fig. 12. Note that the ratios for \(\text{sech}^2 (kx_3)\) and \(1/(1+(kx_3)^2)^{3/2}\) are very similar because the distributions of the two fields are approximately the same. We can find that there are different distributions for different fields, but on the whole, they have two common features. First, this ratio increases linearly with the increase of the \(E/E_{cr}\) (note that \(E/E_{cr} \approx 0.03\) ). The order of the ratio of various fields from small to large is Lorentzian, secant, elliptic cosine and cosine. Second, the decay time for \(E/E_{cr} \approx 0.05\) is always greater than the tunneling time. The physical meaning is that the process of transferring the energy of the external field to the electron is not over yet, but the tunneling process is over.

6 Discussion

In this section, we would like to discuss whether all of the instanton paths for electromagnetic fields as the Eqs. (16) and (17) are within the same plane in the \((\text{Im} x_1, x_3, x_4)\). To do so, we compare the instanton paths of different electromagnetic fields with different parameter \(\gamma_{kebp}\), as shown in Fig. 13. We can find that the paths of the instantons are in the same plane for all electromagnetic fields. As the parameter \(\gamma_{kebp} \to 0\), all paths tend to become the same elliptical shape, and the size of the instantons paths expands with \(\gamma_{kebp}\). For the same \(\gamma_{kebp}\), the size of instanton path that caused by constant electromagnetic fields is the smallest, \(\cos (kx_3)\) is in the middle, and \(\text{sech}^2 (kx_3)\) is the largest.

In the previous section, we discussed that for the same electromagnetic field, the paths of instantons are in the same plane. Therefore, all of the paths of instantons in our work are in the same plane indeed. For the same \(\gamma_{kebp}\), the instanton action \(\mathcal{S}_0\) is the smallest for the constant electromagnetic field. However, the instanton actions are the middle and the largest for \(\cos (kx_3)\) and \(\text{sech}^2 (kx_3)\). According to Eq. (3), for the same parameter \(\gamma_{kebp}\), the pair production probabilities are the largest, middle and the smallest for constant,
Fig. 11 Plots of the tunneling time $T_t$ (left) and the production rate $\text{Im} \Gamma_{\text{semi}}/(V_2 T)^{\text{Mink}}$ (right) for fields $\cos(kx_3)$ (green), $\text{sech}^2(kx_3)$ (magenta) and $1/(1+(kx_3)^2)^{3/2}$ (blue), where $(V_2 T)^{\text{Mink}}$ is the volume in the Minkowski space. The $S_0$ part in the production rate has been expresses in units of $m^2/e\sqrt{(E^2 - B^2)}$.

Fig. 12 Ratio plot of the average vacuum-decay time $T_d$ and the tunneling time $T_t$ for fields $\cos(kx_3)$ (red), $\text{sech}^2(kx_3)$ (cyan), $1/(1+(kx_3)^2)^{3/2}$ (blue) and $\text{cn}(kx_3|0.7)$ (green) with $k = 0.03$ and $B/E = 0.5$

where the function $g(\gamma_{keb})$ is defined as

$$g(\gamma_{keb}) = \frac{2}{\pi} \int_{-1}^{1} dy \sqrt{\frac{1 - y^2}{|f'(y)|}},$$

(87)

where $y = \frac{1}{\gamma_{keb}}f(v)$, $f'(v)$ is to be reexpressed as a function of $y$. For the electromagnetic fields we discussed, the path of the instanton expands with $\gamma_{keb}$. At the same time, the instanton action and the tunneling time increase with $\gamma_{keb}$, the pair production rate decreases with $\gamma_{keb}$ decrease. Most importantly, the magnitudes of the instantons produced by various external electromagnetic fields are in the region between $\pi$ and $2\pi$ in normalized unit according to the analytical solutions of instantons action. Furthermore, we investigated how to obtain all results for time-depended field through space-depended results, and those results can be found by replacing modified Keldysh parameter $\gamma_{\omega eb}$ with $i\gamma_{keb}$ after perform above Lorentz boost. The magnitudes of the normalized instantons action are all bounded within the region from 0 to $\pi$ in the temporally inhomogeneous fields we study.

At last, for the pair production time, we have identified two definitions and obtained the analytical expression of them. The distributions of the vacuum-decay times are similar to those shapes of the applied electromagnetic field modes for the same parameter $E/E_{cr}$. We find a nature of tunneling time for the worldline instanton approach that the tunneling time increases with the increase of parameter $\gamma_{keb}$ while $\gamma_{ke}$ and $k$ are fixed. Physically, the time it takes for an instanton to travel half loop path from $x_3^-$ to $x_3^+$ is the tunneling time. The larger the tunneling time, the smaller the pair creation. Interestingly, the tunneling time no longer depends on the $x_3$. Finally, we obtain the relationship between tunneling time and $x_3$

$$T_t = 2x_4(0) = (x_4^{\text{max}} - x_4^{\text{min}}).$$

(88)

Meanwhile, we studied the relationship between the vacuum-decay time and the tunneling time. Ratio of

7 Conclusion

In conclusion, we have investigated the Schwinger effect in different space-dependent inhomogeneous electromagnetic fields with variation of parameter $\gamma_{keb}$ via the worldline instanton approach. It is found that all results would recover to that in Ref. [15] if $B = 0$. Also we found that the range of loop paths of all instantons expand with $\gamma_{keb}$. All of the paths of instantons in our paper are in the same plane with various electromagnetic fields for different parameter $\gamma_{keb}$.

Moreover, the instanton action can be written generally as

$$S_0 = \frac{\pi m n}{k\gamma_{keb}} g(\gamma_{keb}),$$

(86)
Fig. 13  Parametric plot of the stationary worldline instanton paths in the $(\text{Im}x_1, x_3, x_4)$ space for the cases of the constant (blue), sech$^2(kx_3)$ (pink) and cos$(kx_3)$ (green) electric fields of strength $E$ and $B$. The paths are shown for various values of the parameter $\gamma_{kEB}$ (top right, top left, bottom right and bottom left for 0.1, 0.5, 0.7 and 0.9). $\text{Im}x_1$, $x_3$ and $x_4$ have been expressed in units of $mB/e(E^2 - B^2)$, $m/e\sqrt{E^2 - B^2}$ and $mE/e(E^2 - B^2)$

the average vacuum-decay time and the tunneling time increases linearly with the increase of the $E/E_{cr}$ (note that $E/E_{cr} \geq 0.03$) when $k = 0.03$ and $B/E = 0.5$.

These results suggest that the Schwinger effect in the space- and time-dependent inhomogeneous electromagnetic fields is vital when we consider real laser pulses analytically. Our results may not only help the analytical interpretation of pair production, but also has great prospects for the study on the essence of the pair production time.

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Author contributions

All the authors contributed equally to the paper in the preparation of the manuscript and writing. All the authors have read and approved the final manuscript.

Data Availability Statement  This manuscript has no associated data or the data will not be deposited. [Authors comment: This article is a theoretical paper. All the useful formulas and data are already in the text of the article.]
Appendix A: Tunneling time in the QED

For the tunneling process, the particle energy is smaller than the potential energy of the particle so that the particle momentum is the imaginary number. It is convenient to calculate the tunneling time easily by using momentum of the instanton after perform Wick-rotation in the space-time $x_{\mu} = (t, -x)$, where $x = |x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$, because the instanton momentum is the real value in the tunneling process. Thus, the relativistic momentum can be expressed with the instanton momentum via the Einstein’s mass-energy relation

$$p = \sqrt{\mathcal{E}^2/c^2 - m_0^2c^2} = i\sqrt{m_0^2c^2 - \mathcal{E}^2/c^2} = iq,$$  
(A1)

$$\mathcal{E} = mc^2 = m_0c^2/\sqrt{1 - v^2/c^2} = \gamma m_0c^2,$$  
(A2)

where $p, \mathcal{E}, m_0$ and $m$ are the particle momentum, energy, rest mass and the relativistic mass of motion, $c$ is the speed of the light, $v$ denotes the magnitude of particle velocity $v = (v_1, v_2, v_3)$, and $\gamma$ is the relativistic factor. We define the instanton momentum $q$ in order to calculate the tunneling time, and the tunneling time can be written as [55]

$$T_t = \int_{x_-}^{x_+} \frac{dx}{v},$$  
(A3)

where $T_t$ and $x_{\pm}$ are the tunneling time and the classical turning points where $q(x_{\pm}) = 0$ [61].

Now we calculate the tunneling time under constant electric field $E$ in order to compare with worldline instanton result conveniently. Thus, we can obtain the tunneling time in this case

$$T_t = \int_{x_-}^{x_+} \frac{dx}{\gamma/\sqrt{1 - (W - eEx)^2}},$$  
(A4)

here, $W$ denotes the energy of the instanton. If we define $u = (W - eEx)/m_0c^2$ in order to performing integral transformation, $\gamma = (W - eEx)/m_0c^2 = u$ because of the $\mathcal{E} = \gamma m_0c^2 = W - eEx$ under the constant electric field. Therefore, the tunneling time can be written as

$$T_t = 2 \int_0^1 \left( \frac{(1 - u E/m_0c^2)^{-1}}{E/c\sqrt{1 - u^2}} \right) du = 2m_0c \int_0^1 \frac{udu}{E/c\sqrt{1 - u^2}}$$
$$= \frac{2m_0c}{E/c}. $$  
(A5)

We can find that the tunneling time $T_t = 2x_4(u = 0)$ when $B = 0$, $n = 1$ and $\hbar = c = 1$ in Eq. (42). Therefore, it can be written as

$$T_t = (x_4^{max} - x_4^{min}).$$  
(A6)

Appendix B: Tunneling time via the Hamilton–Jacobi equation

We derive the same result Eq. (A6) by using the Hamilton-Jacobi equation (HJE). From Eq. (16.11) in Ref. [62], the HJE can be written as

$$\left( \nabla S - \frac{e}{c} A \right)^2 - \frac{1}{c^2} \left( \frac{\partial S}{\partial t} + eA_t \right)^2 + m_0^2c^2 = 0,$$  
(B1)

where $A_\mu = (A_t, A)$, $S$ is the action. The first and second terms represent particle energy and momentum according to the Einstein’s mass-energy relation $p^2 - \mathcal{E}^2/c^2 + m_0^2c^2 = 0$. Thus, the tunneling time can be expressed by the instanton momentum $q$

$$T_t = \int_{x_-}^{x_+} \frac{dx}{v} = 2 \int_0^1 \frac{\mathcal{E}^2/c^2 dx}{|p|}$$
$$= 2 \int_0^{x_+} \frac{\mathcal{E}/c^2 dx}{q}$$
$$= 2 \int_0^{x_+} \frac{(\partial S/c^2 + eA_t) dx}{c^2 \left( \nabla S - \frac{e}{c} A \right)}.$$  
(B2)

After change the Einstein’s mass-energy relation to $\mathcal{E}^2/m_0^2c^4 - p^2/m_0^2c^2 = \mathcal{E}^2/m_0^2c^4 + q^2/m_0^2c^2 = 1$ form and two side multiply constant $a^2$ from Eq. (11), we can obtain

$$\frac{\mathcal{E}^2a^2}{m_0^2c^4} + \frac{q^2a^2}{m_0^2c^2} = a^2 \equiv \dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 + \dot{x}_4^2 = \dot{x}_4^2 + \dot{x}_2^2.$$  
(B3)

We can obtain the energy $\mathcal{E}$ and momentum $p$ by comparing with each term in above equation

$$\mathcal{E}^2 = \left( \frac{\partial S}{\partial t} + eA_t \right)^2 = \frac{m_0^2c^4}{a^2} \dot{x}_4^2.$$  
(B4)

$$p^2 = \left( \nabla S - \frac{e}{c} A \right)^2 = -\frac{m_0^2c^2}{a^2} x_4^2 \equiv -q^2.$$  
(B5)

After we take above equations into Eq. (B2), the tunneling time can be rewritten as
\[ T_I = 2 \int_0^{x_+} \frac{\dot{x}_4}{x} dx = 2 \int_{1/4}^{0} \frac{\dot{x}_4}{x} \frac{dx}{du} = 2 \int_{1/4}^{0} \dot{x}_4 du = 2 \left( x_4(0) - x_4(1/4) \right) = 2x_4(0) = \left( x_4^{\text{max}} - x_4^{\text{min}} \right), \] (B6)

where \( x_4^{\text{min}} \) and \( x_4^{\text{max}} \) are maximum and minimum values for \( x_4 \). It is the same result Eq. (A6), but Eq. (B6) does not depend on the external electromagnetic field.

Most importantly, we can use this result for the space-time-dependent inhomogeneous electromagnetic field

\[ T_I = \int_{u(x_-)}^{u(x_+)} \dot{x}_4 du. \] (B7)

Equation (B7) is general form for any field. We can prove Eq. (B7) via another convenient way: the tunneling time can be found by using instanton velocity

\[ T_I = \int_{x_-}^{x_+} \frac{dx}{\dot{u}} = \int_{x_-}^{x_+} \frac{dx}{\dot{x}_4} = \int_{u(x_-)}^{u(x_+)} \dot{x}_4 du. \] (B8)

This illustrates that the three results in the appendix are equivalent.

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