Entanglement and perfect LOCC discrimination of a class of multiqubit states

Somshubhro Bandyopadhyay

DIRO, Université de Montréal, C. P. 6128, Succursale Centre-Ville, Montréal, Québec, Canada H3C 3J7

Abstract

It is shown that while entanglement ensures difficulty in discriminating a set of mutually orthogonal states perfectly by local operations and classical communication (LOCC), entanglement content does not. In particular, for a class of entangled multi-qubit states, the maximum number of perfectly LOCC distinguishable orthogonal states is shown to be independent of the average entanglement of the states, and the spatial configuration with respect to which LOCC operations may be carried out. It is also pointed out that for this class, the make-up of an ensemble, that is, whether it consists only of entangled states or is a mix of both entangled and product states, determines the maximum number of perfectly distinguishable states.

*Present address: Department of Physics, Bose Institute, Kolkata 700009, India; Electronic address: som@bosemain.boseinst.ac.in
Suppose a multipartite quantum state, secretly chosen from a set of mutually orthogonal states, is distributed amongst several observers who are given the task to identify the state without making any error. If the observers are located in the same laboratory, they can perform collective measurements to correctly identify the given state with certainty. However, within the framework of local operations and classical communication (LOCC), wherein they can only perform arbitrary quantum operations on their respective subsystems and communicate by classical channels but are not allowed to exchange quantum states, they may not be able to perfectly distinguish even mutually orthogonal vectors. This shows, at a very basic level, the limitation of LOCC to extract the entire quantum information encoded in a global quantum state. To what extent this global information can be reliably accessed locally then essentially boils down to the problem of faithful discrimination of mutually orthogonal vectors by LOCC [1–6].

A fundamental result is due to Walgate et al who proved that any set of two orthogonal quantum states can always be perfectly discriminated by LOCC [1] irrespective of their entanglement and multipartite structure. However, for sets containing more than two mutually orthogonal states, perfect distinguishability is not always possible and examples can be found in both orthogonal product [8, 9] and entangled ensembles [2–5]. It is important to note that only a set of entangled states can be completely indistinguishable, that is, it is not possible to correctly identify even one state with a non-vanishing probability whereas a set containing at least one product state is always conclusively distinguishable [6, 10], that is, the set contains at least one state (trivially the product state, possibly more) that can be correctly identified with a non-zero probability. Examples of completely indistinguishable sets include the two qubit Bell basis, and more generally, any entangled bipartite basis.

One of the main focuses of LOCC distinguishability of quantum states, regardless of their bipartite or multipartite structure, is to understand the extent to which entanglement is responsible for their indistinguishability. The evidence so far, as noted above, is, at best, mixed although intuitively we expect weakly entangled states should be somewhat more distinguishable than those strongly entangled in the sense weakly entangled states are “less” non-local. A step towards quantifying the relation between entanglement and LOCC distinguishability was taken in a recent work by Hayashi et al [7] who observed that the number of pure states that can be perfectly discriminated by LOCC is bounded above by the total dimension over average entanglement. In particular, if the set of states
\{ |\phi_i\rangle \}_{i=1}^N \text{ is perfectly distinguishable by LOCC, then the average } E(|\phi_i\rangle) \text{ of entanglement “distances” } E(|\phi_i\rangle) \text{ must be less than the total dimension } D/N, \text{ that is, } N \leq D/E(|\phi_i\rangle), \text{ where “entanglement distances” are appropriately defined in terms of an entanglement measure. In its exact form the inequality is hierarchical with respect to different measures of entanglement,}

\[
N \leq \frac{D}{1 + R(|\phi_i\rangle)} \leq \frac{D}{2E_R(|\phi_i\rangle)} \leq \frac{D}{2E_g(|\phi_i\rangle)} \quad (1)
\]

where, corresponding to the state |\phi_i\rangle, R(|\phi_i\rangle) is the global robustness of entanglement [11], \( E_R(|\phi_i\rangle) \) is the relative entropy of entanglement [12], \( E_g(|\phi_i\rangle) \) is an extension of the geometric measure of entanglement [13], and \( \overline{x_i} = \frac{1}{N} \sum_{i=1}^{N} x_i \) denotes the average.

The above inequality shows that it is not possible to perfectly distinguish an arbitrary number of mutually orthogonal entangled states. If a set of entangled states is perfectly distinguishable by LOCC then the cardinality of the set must not violate the above inequality. Perhaps more importantly the upper bound which is inversely proportional to the average entanglement of the ensemble indicates the possibility to perfectly distinguish a larger number of weakly entangled states than strongly entangled states. Here one should note that the inequality stretches the upper bound only up to the total dimension as average entanglement approaches zero, as one would expect, but says nothing whether the upper bound is achieved.

In a multipartite setting, besides entanglement, there is another crucial component, viz. the spatial configuration pertaining to a given LOCC protocol, that plays a significant role in distinguishability/indistinguishability of a set of states. Generally speaking, the setting where every party is separated from one another imposes maximum constraint, thereby causing states to be more indistinguishable, whereas in other configurations, most notably those that allow collective operations (like bipartitions), states tend to be more distinguishable. As a result, typically, states that were not perfectly distinguishable before, become perfectly distinguishable when collective operations are allowed. For example, the following three qubit mutually orthogonal states, \( \frac{1}{\sqrt{2}} |000\rangle_{ABC} \pm |111\rangle_{ABC}, \frac{1}{\sqrt{2}} |011\rangle_{ABC} \pm |100\rangle_{ABC} \), are completely indistinguishable in the A-B-C and A-(BC) formations, but are perfectly distinguishable across B-(AC) and C-(AB) bipartitions. These complexities, which are absent in a bipartite situation, greatly enhances the difficulty to obtain non-trivial bounds on the maximum number of perfectly distinguishable states that hold for all spatial configurations. On the other hand, precisely these properties can be cleverly exploited to devise multipar-
tite cryptography primitives like secret sharing and data hiding, and therefore it is very much desirable to search for robust upper bounds, even if they are only applicable for a class of states.

The set of mutually orthogonal multipartite states that we consider in this paper are the canonical $N$-qubit GHZ states. The complete basis can be represented as a collection of $2^{N-1}$ conjugate pairs in which the $i^{th}$ conjugate pair is given by,

$$
|\psi_+^i\rangle = \alpha_i |k_i\rangle + \beta_i |\overline{k}_i\rangle \\
|\psi_-^i\rangle = \beta_i |k_i\rangle - \alpha_i |\overline{k}_i\rangle, \quad i = 1, ..., 2^{N-1}
$$

where, $\alpha_i \geq \beta_i$ are real, and satisfy the normalization $\alpha_i^2 + \beta_i^2 = 1$, $\forall i$, $k$ is a $N$-bit string of $\{0, 1\}$ and $\overline{k}$ is its bitwise orthogonal. We show that for the above class of states, entanglement content is not a key factor, as one would expect from inequality (3), in determining the maximum number of perfectly distinguishable states by LOCC. We find that the maximum number of perfectly distinguishable states by LOCC is $2^{N-1}$ and this does not depend on the average entanglement of the states as long as entanglement is non-zero for every state. This is to say, neither an ensemble of maximally entangled GHZ states nor an ensemble of GHZ states having vanishingly small entanglement would allow more than $2^{N-1}$ states to be perfectly discriminated by LOCC. Moreover this threshold value is maximally robust, in the sense, it holds for all conceivable spatial configurations including every bipartition.

We further show that it is the make up of the ensemble that decides the maximum number of perfectly distinguishable states. That is, the threshold value gradually approaches the total dimension as more and more product states are included in the set. Therefore, if we insist less entanglement leads to more perfectly distinguishable states by LOCC, then it may be achieved, as it does in our example, only by sacrificing entanglement of some states altogether, and not by reducing entanglement of every state to a vanishingly small amount.

To compute entanglement, we choose relative entropy, which, for the above set of states can be exactly obtained and is given by the entanglement entropy. For the $i^{th}$ conjugate pair, entanglement is simply, $E_i^\pm = -\alpha_i^2 \log_2 \alpha_i^2 - \beta_i^2 \log_2 \beta_i^2$ and for any subset of states $S$ of cardinality $|S|$, the average entanglement $E_S = \frac{1}{|S|} \sum_{i, \text{sign}} E_i^{\text{sign}}$ can be smoothly varied between 0 and 1 ebit. For our choice of the measure of entanglement, the relevant inequality that sets an upper bound on the number of perfectly distinguishable states by
LOCC is simply,

\[ N \leq \frac{D}{2E(|\phi_i\rangle)} \]  

(3)

For any pure state \( |\phi_i\rangle \) chosen from the N-qubit GHZ ensemble, \( E(|\phi_i\rangle) \) lies between 0 and 1. Thus, \( 2^{E(|\phi_i\rangle)} \) lies between 1 and 2, and therefore \( 2^{E(|\phi_i\rangle)} \) must also lie between 1 and 2.

For maximally entangled GHZ states \( (\alpha_i = \beta_i = 1/\sqrt{2} \forall i) \) for which the entanglement of every state is simply 1 ebit, it is easy to see that the maximum number of perfectly LOCC distinguishable maximally entangled GHZ states (or their local unitary equivalents) is \( 2^{N-1} \) (the bound is tight [7]), whereas, for the computational basis \( (\beta_i = 0 \forall i) \), the inequality suggests an upper bound equal to the total dimension when average entanglement is zero, and indeed, the entire basis is perfectly distinguishable by LOCC. Our goal is to find out the maximum number of perfectly distinguishable GHZ states in the regime \( 0 < E_S \leq 1 \).

We first consider all-entangled ensembles where \( E_i^+ > 0 \forall i \).

**Theorem 1** Let \( S \) be any set of states chosen from the N qubit GHZ basis given by Eq. (2), such that \( E_i \neq 0, \forall i \). If there is a spatially separated configuration, where the set is perfectly LOCC distinguishable, then \( |S| \leq 2^{N-1} \). This upper bound is tight.

Although, the validity of the upper bound for all sets of entangled GHZ states requires that every state in the ensemble be entangled, the upper bound, itself, is independent of the average entanglement \( E_S \), where \( 0 < E_S \leq 1 \). Let us also emphasize that the upper bound holds across all bipartitions, and therefore for any spatially separated configuration. The upper bound is further shown to be tight by showing the existence of an unique set \( S \), \( |S| = 2^{N-1} \) of states, comprising one state from each conjugate pair. This set is perfectly distinguishable by LOCC under the least favorable configuration, namely when every qubit is separated from each other. Remarkably once the threshold value is exceeded, the set ceases to be perfectly distinguishable even when collective operations are allowed except when all qubits are together.

It follows from a result in Ref. [4] that every bipartite entangled basis is completely indistinguishable. Therefore, the entangled GHZ basis is also completely indistinguishable in every spatial configuration as it is completely indistinguishable across every bipartition. It would be interesting to know if there is a non-trivial threshold beyond which any ensemble of GHZ states is completely indistinguishable for a particular spatial configuration, if not for all. The following result, however, negates that possibility.
Theorem 2 There always exist conclusively distinguishable sets $S$, $2^{N-1} \leq |S| < 2^N$. Moreover such sets are conclusively distinguishable even when all qubits are spatially separated.

Our final result shows that the make up of an ensemble is critical in determining the maximum number of LOCC distinguishable GHZ states. When product states are included in the set, the upper bound approaches the total dimension. Consider a hybrid basis consisting of $K, K < 2^{N-1}$ conjugate pairs- Eq. (2), and $2^N - 2K$ product states obtained by assigning $\beta_i = 0$ for some values of $i$. It is obvious that for any set of states $S, 0 \leq E_S \leq 1$.

Theorem 3 Let $S$ be any set of states chosen from such a $N$ qubit hybrid basis. If there is a spatially separated configuration where the set is perfectly LOCC distinguishable, then $|S| \leq 2^N - K$. This upper bound is tight.

Therefore, the only way to decrease average entanglement and increase the number of perfectly distinguishable states at the same time is to get rid of entanglement of some states altogether. For an all-entangled ensemble the upper bound is always $2^{N-1}$, no matter how small or large the average entanglement is. On the other hand, if the ensemble is hybrid, one can perfectly distinguish more states even though its average entanglement could be considerably higher than all-entangled ensembles.

We now prove our results. We begin with two useful lemmas.

Lemma 1 The following set of four mutually orthogonal normalized two-qubit states

\begin{align*}
|\psi^+_1\rangle &= \alpha_1|00\rangle + \beta_1|11\rangle; \alpha_1 \geq \beta_1 > 0 \\
|\psi^-_1\rangle &= \beta_1|00\rangle - \alpha_1|11\rangle \\
|\psi^+_2\rangle &= \alpha_2|01\rangle + \beta_2|10\rangle; \alpha_2 \geq \beta_2 > 0 \\
|\psi^-_2\rangle &= \beta_2|01\rangle - \alpha_1|10\rangle
\end{align*}

is completely indistinguishable, and any subset of three states is not perfectly distinguishable.

The first part of the lemma was proved in [3, 4]. The proof of the second part follows from [3, 6].
Lemma 2 The following set of four mutually orthogonal two qubit states

\[
|\psi^+\rangle = \alpha_1|00\rangle + \beta_1|11\rangle; \alpha_1 \geq \beta_1 > 0 \\
|\psi^-\rangle = \beta_1|00\rangle - \alpha_1|11\rangle \\
|\phi\rangle = |01\rangle \\
|\chi\rangle = |10\rangle
\]

and its following subsets, \{\{|\psi^\pm\rangle, |\phi\rangle\}, \{|\psi^\pm\rangle, |\chi\rangle\}\} are not perfectly distinguishable by LOCC.

For the proof of lemma 2, see [3].

Proof of Theorem 1. To prove that a set of states is not perfectly distinguishable for all spatial configurations, we must show that the states are not perfectly distinguishable across any bipartition. We first prove our result choosing an arbitrary bipartition. Then we will show how the proof can be worked out similarly for any other bipartition.

Denote a bipartition Alice – Bob as \((m, Q)\) where \(m\) qubits belong to Alice, and the rest to Bob; \(1 \leq m \leq N/2\) for even \(N\) and \(1 \leq m \leq (N - 1)/2\) for odd \(N\); the index \(Q\) represents the specific set of \(m\) qubits that belong to Alice (note that there are \((\binom{N}{m})\) ways to choose the specific \(m\) qubits). Rewrite a conjugate pair, say, \(|\psi^\pm_i\rangle\) (Eq. (2)), to explicitly reflect the bipartite form:

\[
|\psi^+_i(m, Q)\rangle_{AB} = \alpha_i|m\rangle_A|(N - m)\rangle_B + \beta_i|m\rangle_A|(N - m)\rangle_B \\
|\psi^-_i(m, Q)\rangle_{AB} = \beta_i|m\rangle_A|(N - m)\rangle_B - \alpha_i|m\rangle_A|(N - m)\rangle_B
\]

(6)

where \(m\) is a \(m\)-bit string, and \((N - m)\) is a \((N - m)\)-bit string of \(\{0, 1\}\); \(m, (N - m)\) are their bit-wise orthogonals. Corresponding to the above pair, across the same bipartition \((m, Q)\), there also exists another unique conjugate pair, say, \(|\psi^\pm_j\rangle\),

\[
|\psi^+_j(m, Q)\rangle_{AB} = \alpha_j|m\rangle_A|(N - m)\rangle_B + \beta_j|m\rangle_A|(N - m)\rangle_B \\
|\psi^-_j(m, Q)\rangle_{AB} = \beta_j|m\rangle_A|(N - m)\rangle_B - \alpha_j|m\rangle_A|(N - m)\rangle_B
\]

(7)

Changing to the notation \(|m\rangle_A = |0\rangle_A; |N - m\rangle_B = |0\rangle_B; |\overline{m}\rangle_A = |1\rangle_A; |N - m\rangle_B = |1\rangle_B\)
the states can be written in a compact way,

\[ |\psi^+ (m, Q)\rangle_{AB} = \alpha_i |0\rangle_A |0\rangle_B + \beta_i |1\rangle_A |1\rangle_B \]
\[ |\psi^- (m, Q)\rangle_{AB} = \beta_i |0\rangle_A |0\rangle_B - \alpha_i |1\rangle_A |1\rangle_B \]
\[ |\psi^+_j (m, Q)\rangle_{AB} = \alpha_j |0\rangle_A |1\rangle_B + \beta_j |1\rangle_A |0\rangle_B \]
\[ |\psi^-_j (m, Q)\rangle_{AB} = \beta_j |0\rangle_A |1\rangle_B - \alpha_j |1\rangle_A |0\rangle_B \] (8)

It now follows from lemma 1 that the above set is completely indistinguishable, and any subset of three states chosen from the above set is not perfectly distinguishable. Let’s emphasize that, for the above pairs of states their indistinguishability holds strictly across the bipartition \( (m, Q) \). It can be shown that across any other bipartition, the above conjugate pair is perfectly distinguishable.

We now show that, the entire basis can be grouped into \( 2^{N-2} \) unique (unique with respect to the bipartition being considered) blocks, each block consisting of two conjugate pairs having exactly the same LOCC distinguishability properties like those in Eq. (8). To see how it’s done, consider another conjugate pair \( |\psi^\pm_k\rangle \),

\[ |\psi^+_k (m, Q)\rangle_{AB} = \alpha_k |m\rangle_A |(N-m)'\rangle_B + \beta_k |\overline{m}\rangle_A |(N-m)'\rangle_B \]
\[ |\psi^-_k (m, Q)\rangle_{AB} = \beta_k |m\rangle_A |(N-m)'\rangle_B - \alpha_k |\overline{m}\rangle_A |(N-m)'\rangle_B \] (9)

The conjugate pair \( |\psi^\pm_k (m, Q)\rangle \) is different from those in Eqs. (6) and (7) only in the bit values of B. As before, there must also exist another conjugate pair, \( |\psi^\pm_l (m, Q)\rangle \),

\[ |\psi^+_l (m, Q)\rangle_{AB} = \alpha_l |m\rangle_A |(N-m)'\rangle_B + \beta_l |\overline{m}\rangle_A |(N-m)'\rangle_B \]
\[ |\psi^-_l (m, Q)\rangle_{AB} = \beta_l |m\rangle_A |(N-m)'\rangle_B - \alpha_l |\overline{m}\rangle_A |(N-m)'\rangle_B \] (10)

Proceeding in the same way \( 2^{N-m-1} \) such distinct blocks can be constructed for the same bit values of A, that is for the same ordered pair \( (\overline{m}, \overline{m}) \). This is because, for a fixed \emph{m-bit} string of A, there are \( 2^{N-m} \) possible strings corresponding to B. Out of those \( 2^{N-m} \) strings, each block uses two \( (N-m)$-bit$ \) strings. Therefore, the number of distinct blocks is \( 2^{N-m}/2 = 2^{N-m-1} \). Now there are \( 2^m \) distinct \emph{m-bit} strings that are possible for A, however, each block uses two of them, say \( m' \) and \( \overline{m'} \). Therefore, the number of distinct blocks corresponding to different bit values of A but the same bit values of B is \( 2^{m-1} \).
Therefore, altogether there are $2^{m-1}2^{N-m-1} = 2^{N-2}$ distinct blocks, each block consists of two conjugate pairs and has the same LOCC distinguishability property as the block of states specified by Eqs. (6,7). Now, any set consisting of more than $2^{N-1}$ states must have at least three states from one block. Because these three states are not perfectly distinguishable, hence the entire set is also not perfectly distinguishable.

To show that the proof indeed holds for all bipartitions, first observe that for another bipartition of the type $(m, Q')$ (that is a different set of $m$ qubits are selected), only the constituent states of each block change. The indistinguishability property of the states in each block, which is the key feature of the entire proof, remains unaffected. This means, the entire basis can again be grouped into $2^{N-2}$ blocks where any three states from each block are not perfectly distinguishable, and therefore, it is not possible to pick more than $2^{N-1}$ states and still be able to distinguish the states perfectly. For any other bipartition with a different $m$ value, what changes are the length of the $m - bit$ string which corresponds to the number of qubits on Alices’s side, and the constituent states of each block; the crux of the argument does not change. This concludes the proof. □

**Proof of Theorem 2.** Let’s recall that a set of states is conclusively distinguishable, if at least one state can be correctly identified with a nonzero probability, no matter how small. Consider a set of GHZ states containing a state whose conjugate partner is not included in the set. This particular state can always be correctly identified just by doing measurement in the computational basis which can indeed be carried out even when all qubits are separated from each other. It is easy to see that such set of states, $S$ can always be constructed for all values of cardinality $|S| \leq 2^N - 1$. □

**Proof of Theorem 3.** Consider the hybrid basis consisting of $K$ conjugate pairs, and $2^N - 2K$ product states. Construct a set $S$ of cardinality $|S| = 2^N - K$ in the following way: include all the product states, and one state from each conjugate pair. This set is again perfectly distinguishable by LOCC when all qubits are spatially separated, and therefore for all spatial configurations. To show that any set of cardinality greater than $2^N - K$ is not perfectly distinguishable, first note that a conjugate pair $|\psi^\pm_k\rangle$ is reduced to a product pair by setting $\beta_k = 0$. Recall that when the entire basis was entangled, then across every bipartition we were able to group the states into $2^{N-2}$ blocks, each block containing two conjugate pairs. For the hybrid case, some of the blocks now contain either one conjugate pairs and two product states (the same product states whose superposition would have given
rise to the corresponding conjugate pair in the all-entangled case) or four product states (these product states are those corresponding to the two conjugate pairs in the all-entangled case).

If one more state is added, then the set now includes either a block of four states containing one conjugate pair and two product states (because all product states, and one state from each conjugate pair have already been included) which is not perfectly distinguishable (by lemma 2), or a block of three states containing one conjugate pair and one state from another conjugate pair which are not perfectly distinguishable (by lemma 1). Note that we just described the worst case scenario, and other sets containing $2^N - K$ states would invariably contain a block(s) which is(are) not perfectly distinguishable either by lemma 1 or lemma 2. This holds for all bipartitions (see the proof of Theorem 1), and therefore for all spatial configurations. □

To conclude, it is shown that for GHZ states, entanglement content is not a contributing factor in determining the maximum number of states that are perfectly distinguishable by LOCC, although, entanglement certainly is. This goes against our intuition and inequality (3), in the sense both suggest a larger number of weakly entangled states can be perfectly distinguished than strongly entangled ones. One surprising feature, in the context of the result obtained is the spatial configuration independence of the threshold value. We have also shown a hybrid ensemble comprising both entangled GHZ states and the product states is less indistinguishable in the sense, the upper bound on the maximum number of perfectly distinguishable states is always greater than $2^{N-1}$. In particular the upper bound approaches the total dimension as more and more product states are included.

The open question is whether entanglement independence of the upper bound is a generic feature. Supporting evidence might be obtained by looking into the non-maximal canonical basis in $d \otimes d$. This basis, where every state has Schmidt rank $d$ is a direct generalization of the $2 \otimes 2$ non-maximal canonical basis considered in lemma 1. Taking cue from the distinguishability of maximally entangled canonical basis in $d \otimes d$ and the result presented here, we could expect that no more than $d$ states can be perfectly distinguished irrespective of the average entanglement. However, if we mix entangled states of different Schmidt ranks the upper bound is likely to increase, but the exact functional form is not immediately clear.
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[1] J. Walgate, A. J. Short, L. Hardy and V. Vedral, Phys. rev. Lett 85, 4972 (2000).
[2] S. Ghosh, G. Kar, A. Roy, A. Sen(De), and U. Sen, Phys. Rev. Lett. 87, 277902 (2001).
[3] S. Ghosh, G. Kar, A. Roy, D. Sarkar, A. Sen(De), and U. Sen, Phys. Rev. A 65, 062307 (2002).
[4] M. Horodecki, A. Sen(De), U. Sen, and K. Horodecki, Phys. rev. Lett. 90, 047902 (2003).
[5] H. Fan, Phys. Rev. Lett. 92, 177905 (2004); M. Nathanson, J. Math. Phys. 46, 062103 (2005);
J. Walgate and L. Hardy, Phys. Rev. Lett. 89, 147901 (2002); R. Duan, Y. Feng, Z. Ji, M. Ying, Phys. Rev. Lett. 98, 230502 (2007).
[6] S. Bandyopadhyay and J. Walgate, J. Phys. A: Math. Theor. 42, 072002 (2009).
[7] M. Hayashi, D. Markham, M. Murao, M. Owari, and S. Virmani, Phys. Rev. Lett. 96, 040501 (2006).
[8] C. H. Bennett, D. P. DiVincenzo, C. A. Fuchs, T. Mor, E. Rains, P. W. Shor, J. A. Smolin, W. K. Wootters, Phys. Rev. A 59, 1070 (1999).
[9] C.H. Bennett, D.P. DiVincenzo, T. Mor, P.W. Shor, J.A. Smolin, B.M. Terhal, Phys. Rev. Lett. 82, 5385 (1999).
[10] A. Chefles, Phys. Rev. A 69, 050307 (2004).
[11] G. Vidal and R. Tarrach, Phys. Rev. A 59, 141 (1999); A. W. harrow and M. A. Nielsen, Phys.
Rev A 68, 012308 (2003); M. Steiner, Phys. Rev. A 67, 054305 (2003).
[12] V. Vedral and M. B. Plenio, Phys. Rev. A 57, 1619 (1998).
[13] A. Shimony, Ann. NY. Acad. Sci 755, 675 (1995); H. Barnum and N. Linden, J. Phys. A 34, 6787 (2001); T.-C. Wei and P. M. Goldbart, Phys. Rev. A 68, 042307 (2003).
[14] D. Markham and B. C. Sanders, Phys. Rev. A 78, 042309 (2008)
[15] C. H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. Smolin, and W. K. Wootters, Phys.
Rev. Lett. 70, 1895 (1993).