HJB equations with gradient constraint associated with controlled jump-diffusion processes

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Abstract
In this paper, we guarantee the existence and uniqueness (in the almost everywhere sense) of the solution to a Hamilton-Jacobi-Bellman (HJB) equation with gradient constraint and a partial integro-differential operator whose Lévy measure has bounded variation. This type of equation arises in a singular control problem, where the state process is a multidimensional jump-diffusion with jumps of finite variation and infinite activity. We verify, by means of ε-penalized controls, that the value function associated with this problem satisfies the aforementioned HJB equation.

1 Introduction
Our main goal is to study the following HJB equation,

\[
\max \{ \Gamma u - h, |D^1 u| - g \} = 0, \ \text{a.e. in } \mathcal{O}, \quad \text{s.t. } u = 0, \ \text{on } \overline{\mathcal{O}} \setminus \mathcal{O},
\]

where \( \mathcal{O} \) is a convex, open, and bounded set such that \( \mathcal{O} \subset \overline{\mathcal{O}} \subseteq \mathbb{R}^d \) and its boundary \( \partial \mathcal{O} \) is of class \( C^{3,\alpha'} \), with \( \alpha' \in (0,1) \) fixed. The set \( \overline{\mathcal{O}} \) shall be given later on. The partial integro-differential operator \( \Gamma \) is defined by

\[
\Gamma u(x) = \mathcal{L} u(x) - \mathcal{I} u(x),
\]

with

\[
\mathcal{L} u(x) := -\text{tr}[a(x) D^2 u(x)] + \langle b(x), D^1 u(x) \rangle + c(x) u(x),
\]

\[
\mathcal{I} u(x) := \int_{\mathbb{R}^d} [u(x + z) - u(x)] s(x,z) \nu(dz), \quad \text{for } x \in \mathcal{O}.
\]

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Here $| \cdot |$, $\langle \cdot , \cdot \rangle$ and $\text{tr}[\cdot]$ represent the Euclidean norm, the inner product, and the matrix trace, respectively; $D^1 u = (\partial_1 u, \ldots, \partial_d u)$, $D^2 u = (\partial_{ij} u)_{d \times d}$, $h, c : \overline{O} \to \mathbb{R}$, $g : \overline{O}_T \to \mathbb{R}$, $b : \overline{O} \to \mathbb{R}^d$, $a : \overline{O} \to \mathcal{S}(d)$, with $\mathcal{S}(d)$ the set of $d \times d$ symmetric matrices, $\nu$ is a Radon measure on $\mathbb{R}_+^d := \mathbb{R}^d \setminus \{0\}$ satisfying

$$\int_{\mathbb{R}_+^d} ||z| \wedge 1| \nu(dz) \leq C_\nu, \quad (1.4)$$

for some finite positive constant $C_\nu$, and $s : \overline{O} \times \mathbb{R}_+^d \to [0,1]$ is such that

$$\int_{\mathbb{R}_+^d} s(x,z) \mathbb{1}_{\{x+z \notin O\}} \nu(dz) < \infty, \text{ for } x \in O. \quad (1.5)$$

The notations concerning function spaces that we have used in the paper are standard and are discussed in Subsection 1.2.

The HJB equation (1.1) when $\Gamma = \mathcal{L}$ was introduced by Evans in 1979 [9]. Under some regularity assumptions on the coefficients of (1.1), and $\mathcal{L}$ satisfying the elliptic property, he showed that the unique solution to this problem belongs to $W^{1,\infty}(O) \cap W^{2,p}_{\text{loc}}(O)$, for each $p \in [1,\infty)$. Shortly afterwards, Wiegner [32] proved that this solution is in $C^{1,1}(O)$. Later on, Ishii and Koike [18] considered this problem with a gradient constraint more general than Evans proposed in [9]. They verified that the solution to their HJB equation is in $W^{2,\infty}(O)$. Then, Hynd [16] studied the problem with a convex gradient constraint and showed that the solution to this problem is in a viscosity sense and belongs to $C^{1,\alpha}_{\text{loc}}(O) \cap C^{0,1}(O)$, for $\alpha \in (0,1)$.

Recently, Moreno-Franco [27] analysed the HJB equation (1.1) when the domain set is a ball $B_R(0) \subset \mathbb{R}^d$, the coefficients of the partial integro-differential operator $\Gamma$ are constant, $s = g = 1$, $c = q$, with $q$ being a positive constant large enough, and the Lévy measure $\nu$ has a density $\kappa \in C^{0,\alpha'}(\mathbb{R}_+^d)$ with respect to the Lebesgue measure $dz$ such that $\nu(\mathbb{R}_+^d) < \infty$. In this case, assuming that $h \in C^2(B_R(0))$ is non-negative, $\nu$ is such that $\int_{\mathbb{R}_+^d} |z| \nu(dz) < \infty$, and using PDEs and probabilistic methods, the author proved the equation (1.1) has a unique solution in $C^{0,1}(\overline{B_R(0)}) \cap W_{\text{loc}}^{2,p}(B_R(0))$ a.e., for each $p \in (d,\infty)$. It was also shown that there is a relationship between the HJB equation (1.1) on the whole space $\mathbb{R}^d$ and a singular control problem, when the controlled process is a Lévy process, whose components are a $d$-dimensional standard Brownian motion (SBM) with drift and a Poisson compound process.

Notice that the HJB equation (1.1) with the operator $\Gamma$ defined as in (1.2) is more general than in [27]. The Lévy measure $\nu$ can satisfy $\nu(\mathbb{R}_+^d) = \infty$ and it is not required that $\nu$ has a density $\kappa$ with respect to the Lebesgue measure $dz$. This type of HJB equation is also related to a singular control problem when the state process is a jump-diffusion process $X = \{X_t : t \geq 0\}$ (see Eq. (1.10)) with infinitesimal generator of the form

$$\text{tr}[a D^2 u] - \langle b, D^1 u \rangle + \int_{\mathbb{R}_+^d} [u(\cdot+z) - u(\cdot)] s(\cdot,z) \nu(dz), \text{ on } \overline{O}. \quad (1.6)$$

The last term in (1.6) corresponds to the infinitesimal generator of a jump process, whose _jump size and rate_ are given by $z \in \mathbb{R}_+^d$, and $s(x,z)$, respectively. The jump rate of the process $X$ depends on its position at time $t$. For more detail about this problem, see Subsection 1.1.
Assumptions and main results

The following assumptions will henceforth be imposed:

(A1) Assume that \( h, c, a_{ij}, b_i \in C^{1,\alpha'}(\O) \), with \( \alpha' \in (0, 1) \) fixed, \( g \in C^2(\O) \cap C^1(\O) \) and \( \|h\|_{C^{1,\alpha'}(\O)}, \|a_{ij}\|_{C^{1,\alpha'}(\O)}, \|b_i\|_{C^{1,\alpha'}(\O)}, \|c\|_{C^{1,\alpha'}(\O)}, \|g\|_{C^2(\O)} \) and \( \|g\|_{C^1(\O)} \) are bounded by some finite positive constant \( \Lambda \).

(A2) The functions \( h, g \) and \( c \) are such that \( h \geq 0, c > 0 \) on \( \O \), and \( g \geq 0 \) on \( \O_T \).

(A3) The differential part of the operator \( \Gamma \) is strictly elliptic; i.e., there exists a real number \( \theta > 0 \) such that \( \langle a(x)\zeta, \zeta \rangle \geq \theta|\zeta|^2 \), for all \( x \in \O \), \( \zeta \in \mathbb{R}^d \).

(A4) Finally, we assume that \( \nu \) is a Radon measure on \( \mathbb{R}^d_+ \) satisfying (1.4) and \( s \in C^{1,\alpha'}(\O \times \mathbb{R}^d) \) is such that (1.5) holds.

Before introducing the main results of the paper, let us define the support \( \O_I \) of the operator \( I \). Consider the Lévy kernel \( M_s(x, B) = \int_{z \in B} s(x, z)\nu(dz) \), where \( x \in \O \) and \( B \) a Borel measurable set of \( \mathbb{R}^d_+ \). Then,

\[
\O_I := \bigcup_{x \in \O} \{ x + [\mathbb{R}^d \setminus \mathcal{Z}_I(x)] \},
\]  

where \( \mathcal{Z}_I(x) = \{ z' \in \mathbb{R}^d_+ : M_s(x, B_\epsilon(z')) = 0, \text{ for some } \epsilon \in (0, |z'|) \} \); see [12, Definition 2.3.10]. The set \( \mathcal{Z}_I(x) \) is called the zero-jump set. Notice that \( \O \subset \O_I \) and \( \O = \O_I \) if \( s(x, z) = 0 \), for all \( (x, z) \in \O \times \mathbb{R}^d_+ \) such that \( x + z \notin \O \).

Without loss of generality consider \( \O \subset \O_I \subset \mathbb{R}^d_+ \) from now on. Taking the operator \( \Gamma \) as in (1.2) and under Assumptions (A1)–(A4), the main goal obtained in this document is as follows:

**Theorem 1.1.** For each \( p \in (d, \infty) \), there exists a unique non-negative solution \( u \) to the HJB equation (1.1) in the space \( C^{0,1}(\O) \cap W^{2,p}_{\text{loc}}(\O) \).

The solution \( u \) to the HJB equation (1.1) is established in the almost everywhere sense in line with [27]. To prove Theorem 1.1; see Section 3, we will employ a penalization technique, which has been used by [9, 15, 16, 17, 18, 27, 31, 32], when the operator \( \Gamma \) has only the elliptic differential part \( L \) or when the Lévy measure of its integral part \( I \) is finite. Considering the non-linear partial integro-differential Dirichlet (NPIDD) problem

\[
\Gamma u^\varepsilon + \psi_\varepsilon (|D^1 u^\varepsilon|^2 - g^2) = h, \quad \text{in } \O, \quad \text{s.t. } u^\varepsilon = 0, \quad \text{on } \O_I \setminus \O, \tag{1.8}
\]

where the penalizing function \( \psi_\varepsilon : \mathbb{R} \to \mathbb{R} \), with \( \varepsilon \in (0, 1) \), belongs to \( C^\infty(\mathbb{R}) \) and is determined as

\[
\psi_\varepsilon(r) = 0, \quad r \leq 0, \quad \psi_\varepsilon(r) > 0, \quad r > 0, \\
\psi_\varepsilon(r) = \frac{r - \varepsilon}{\varepsilon}, \quad r \geq 2\varepsilon, \quad \psi_\varepsilon'(r) \geq 0, \quad \psi_\varepsilon''(r) \geq 0, \tag{1.9}
\]

we first guarantee the existence and uniqueness of the classical solution \( u^\varepsilon \) to the NPIDD problem (1.8), with \( \Gamma \) as in (1.2). Once this is done, we establish uniform estimates of the sequence \( \{u^\varepsilon\}_{\varepsilon \in (0, 1)} \) that allow us to pass to the limit as \( \varepsilon \to 0 \), in a weak sense in (1.8), which leads to the existence and regularity of the solution to the HJB equation (1.1).

Under Assumptions (A1)–(A4), the other main result obtained in the paper is as follows:
Proposition 1.2. For each $\varepsilon \in (0,1)$, there exists a unique non-negative solution $u^\varepsilon$ to the NPIDD problem (1.8) in the space $C^{3,\alpha'}(\partial \mathcal{O})$.

Although the NPIDD problem (1.8) is a tool to guarantee the existence of the solution to the HJB equation (1.1), this turns out to be a problem of interest itself because, previously to this paper, we find few references related to this class of problems. Say, paper [27] analyses the NPIDD problem (1.8) when the Lévy measure $\nu$ is finite on $\mathbb{R}^d$, and [28] studies a degenerate Neumann problem for quasi-linear elliptic integro-differential operators when the Lévy measure $\nu$ has unbounded variation, i.e., $\int_{\mathbb{R}^d} |z|^2 \wedge 1 \nu(dz) < \infty$, and $s$ satisfies $s(x,z) = 0$, for $(x,z) \in \partial \mathcal{O} \times \mathbb{R}^d$ such that $x + z \notin \partial \mathcal{O}$. This type of problem can also be related to an absolutely continuous optimal control problem when the controlled process is a jump-diffusion with jump measure of finite variation; see Section 4.

To finalize this part, let us make some comments about the assumptions mentioned in the beginning of this subsection. Under (A1), (A3), (A4) and the fact that the boundary $\partial \mathcal{O}$ is of class $C^{3,\alpha'}$, we ensure the existence and uniqueness of the classical solution $u$ to the linear partial integro-differential Dirichlet (LPIDD) problem in (2.22) when $w \in C^{1,\alpha} \left( \partial \mathcal{O} \right)$; see [12, Thm. 3.1.12]. Assumptions (A1), (A2), (A4) and that $\mathcal{O}$ is a bounded convex set are required to show some $a$ priori estimates of the solution $u^\varepsilon$ to the NPIDD problem (1.8), which must be independent of $\varepsilon$; see Lemmas 2.6–2.8. Since $h \geq 0$, $c > 0$ on $\partial \mathcal{O}$ and using Lemma 2.5, it can be verified that $u^\varepsilon$ is the unique non-negative solution to the NPIDD problem (1.8); see Subsection 2.2. Finally, once again making use of $c > 0$ on $\partial \mathcal{O}$, it is proven that the solution to the HJB equation (1.1) is unique; see Subsection 3.2.

The rest of this document is organized as follows: Section 2 is devoted to prove the existence and uniqueness of the solution to the NPIDD problem (1.8). First, some properties of the integral operator $\mathcal{I}$ and some $a$ priori estimates of the solution to the NPIDD problem (1.8) are studied. Afterwards, using Lemmas 2.2, 2.6, 2.8, 2.9, and the Schaefer fixed point Theorem; see [10, Thm. 4, p. 539], it is proven that the classical solution $u^\varepsilon$ to the NPIDD problem (1.8) exists and is unique; see Subsection 2.2. Then, in Section 3, by Lemmas 2.6, 2.8, 3.3, 3.5, using Arzelà-Ascoli Theorem and the reflexivity of $L^p_{\text{loc}}(\mathcal{O})$; see [30, 1, Thm. 7.25, p. 158 and Thm. 2.46, p. 49, respectively], we extract a convergent sub-sequence of $\{u^\varepsilon\}_{\varepsilon \in (0,1)}$, whose limit is the solution to the HJB equation (1.1); see Subsection 3.2. In the following subsection, we present the singular control problem that is related to the HJB equation (1.1). The probabilistic arguments of this part are given in Section 4. Finally, we draw our conclusions and discuss possible extensions of this paper.

1.1 Probabilistic interpretation

Let $W = \{W_t : t \geq 0\}$ and $N$ be a $d$-dimensional SBM and a Poisson random measure on $(\mathcal{S} \times [0, \infty), \mathcal{B}(\mathcal{S}) \times \mathcal{B}([0, \infty)))$, $\eta(d\rho, dz) \times dt$, with $\mathcal{S} = [0, 1] \times \mathbb{R}^d$ and $\eta(d\rho, dz) = d\rho \nu(dz)$, respectively, which are defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that $W$ and $N$ are independent. Let $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ be the filtration generated by $W$ and $N$. We assume furthermore that the filtration $\mathbb{F}$ is completed with the null sets of $\mathbb{P}$. The uncontrolled stochastic process $X = \{X_t : t \geq 0\}$ is governed by the stochastic differential equation (SDE)

$$X_t = \bar{x} - \int_0^t \bar{b}(X_s)ds + \int_0^t \sigma(X_s)dW_s + \int_0^t dJ_s, \ t > 0,$$  

(1.10)
where \( \tilde{x} \in \mathcal{O} \), \( \tilde{b} : \mathbb{R}^d \to \mathbb{R}^d \), and \( \sigma : \mathbb{R}^d \to \mathbb{R}^{d \times d} \). The jump process \( J \) is defined by

\[
J_t = \int_0^t \int_{S_1} z 1_{\{\rho \in [0,s(X_{\cdot},z))\}} \tilde{N}(d\rho,dz,ds) + \int_0^t \int_{S \setminus S_1} z 1_{\{\rho \in [0,s(X_{\cdot},z))\}} \tilde{N}(d\rho,dz,ds), \tag{1.11}
\]

with \( s : \mathbb{R}^d \times \mathbb{R}^d \to [0,1] \), \( S_1 = \{(\rho,z) \in S : |z| \in (0,1)\} \), and \( \tilde{N}(d\rho,dz,dt) = N(d\rho,dz,dt) - \eta(d\rho,dz)dt \) is the compensated Poisson random measure with intensity \( \eta(d\rho,dz)dt \). For each \( \tilde{x} \in \mathcal{O} \), \( \mathbb{P}_{\tilde{x}} \) represents the probability law of \( X \) when it starts at \( \tilde{x} \), and \( \mathbb{E}_x \) is the expected value associated with \( \mathbb{P}_{\tilde{x}} \).

In addition to (A1)–(A4), we need to add other assumption on the whole space \( \mathbb{R}^d \) in such a way that the SDE (1.10) has a unique càdlàg adapted solution \( X \). This assumption will only be used here and in Section 4.

(A5) Assume that there exists a positive constant \( C \) such that

\[
|\sigma(x)|^2 + |\tilde{b}(x)|^2 \leq C[1 + |x|^2],
\]

\[
|\sigma(x) - \sigma(y)| + |\tilde{b}(x) - \tilde{b}(y)| \leq C|x - y|
\]

\[
\int_{\{|z| \in (0,1)\}} |z| |s(x,z) - s(y,z)| \nu(dz) \leq C|x - y|,
\]

for \( x, y \in \mathbb{R}^d \) with \( x \neq y \).

Remark 1.3. Notice that for each \( x, y \in \mathbb{R}^d \) with \( x \neq y \),

\[
\int_{S_1} |z|^2 1_{\{0 \leq \rho \leq s(x,z)\}} \eta(d\rho,dz) = \int_{\{|z| \in (0,1)\}} |z|^2 s(x,z) \nu(dz) \leq C\nu,
\]

\[
\int_{S_1} |z| \left| 1_{\{0 \leq \rho \leq s(x,z)\}} - 1_{\{0 \leq \rho \leq s(y,z)\}} \right| \eta(d\rho,dz) = \int_{\{|z| \in (0,1)\}} |z| |s(x,z) - s(y,z)| \nu(dz) \leq C|x - y|,
\]

since (A4) holds. Then, from (1.12)–(1.13), the SDE (1.10) has a unique càdlàg adapted solution \( X \); see [22].

Since \( \int_{\{|z| \in (0,1)\}} |z| \nu(dz) < \infty \) and \( \eta(d\rho,dz) = d\rho \nu(dz) \), the infinitesimal generator of \( X \) is given by

\[
\Gamma_1 u(x) = \text{tr}[a(x) D^2 u(x)] - \langle \tilde{b}(x), D^1 u(x) \rangle
\]

\[
+ \int_{\mathcal{S}} [u(x + z 1_{\{\rho \in [0,s(x,z)]\}}) - u(x)] - \langle D^1 u(x), z \rangle \mathbb{1}_{\{\rho \in [0,s(x,z)], |z| \in (0,1)\}} \eta(d\rho,dz)
\]

\[
= \text{tr}[a(x) D^2 u(x)] - \langle \tilde{b}(x), D^1 u(x) \rangle + \int_{\mathbb{R}^d} [u(x + z) - u(x)] s(x,z) \nu(dz), \tag{1.14}
\]

where \( a_{ij} = \frac{1}{2}(\sigma \sigma^T)_{ij} \) and \( \tilde{b} + \int_{\{|z| \in (0,1)\}} zs(\cdot, z) \nu(dz) \). Let \( \mathcal{U} \) be the admissible class of control processes \((n, \zeta)\) that satisfies

\[
\begin{align*}
(n_t, \zeta_t) &\in \mathbb{R}^d \times \mathbb{R}_+, \quad t \geq 0, \quad (n, \zeta) \text{ is adapted to the filtration } \mathcal{F}, \quad \zeta_{t-} = 0 \\
\text{and } \zeta &\text{ is non-decreasing and is right continuous with left hand limits, } t \geq 0, \\
|n_t| &\leq 1 \text{ d}\zeta_t \text{-a.s., } t \geq 0.
\end{align*}
\tag{1.15}
\]
Then, for each \((n, \zeta) \in U\) and \(\tilde{x} \in \mathcal{O}\), the process \(X_{t}^{n,\zeta} = \{X_{t}^{n,\zeta} : t \geq 0\}\) evolves as
\[
X_{t}^{n,\zeta} = \tilde{x} - \int_{0}^{t} \tilde{b}(X_{s}^{n,\zeta})ds + \int_{0}^{t} \sigma(X_{s}^{n,\zeta})dW_{s} + \int_{0}^{t} dJ_{s} - \int_{[0,t]} n_{d}d\zeta_{s}, \ t \geq 0. \tag{1.16}
\]

The process \(n\) provides the direction and \(\zeta\) the intensity of the push applied to the state process \(X_{t}^{n,\zeta}\). Since (1.12)–(1.13) hold, we get that the SDE (1.16) has a unique càdlàg adapted solution \(X_{t}^{n,\zeta}\); see [8]. The jumps of \(X_{t}^{n,\zeta}\) are given by the processes \(J\) and \(\zeta\), i.e., \(\Delta X_{t}^{n,\zeta} = X_{t}^{n,\zeta} - X_{t-}^{n,\zeta} = \Delta J_{t} - n_{t}\Delta \zeta_{t}\), for \(t \geq 0\). The cost function corresponding to \((n, \zeta) \in U\), is defined as
\[
V_{n,\zeta}(\tilde{x}) = \mathbb{E}_{\tilde{x}} \left[ \int_{[0,\tau^{n,\zeta}]} e^{-qs}[h(X_{s}^{n,\zeta})ds + g(X_{s}^{n,\zeta})d\zeta_{s}] \right], \ \tilde{x} \in \mathcal{O}, \tag{1.17}
\]
where \(\tau^{n,\zeta} := \inf\{t > 0 : X_{t}^{n,\zeta} \notin \mathcal{O}\}\), \(q\) is a positive constant and
\[
\int_{[0,t]} e^{-qs} g(X_{s}^{n,\zeta})d\zeta_{s} := \int_{0}^{t} e^{-qs} g(X_{s}^{n,\zeta})d\zeta_{s}^{c} + \sum_{0 \leq s \leq t} e^{-qs} \Delta \zeta_{s} \int_{1}^{t} g(X_{r}^{n,\zeta} + \Delta J_{r} - \lambda n_{r}\Delta \zeta_{s})d\lambda, \ \text{for} \ t > 0, \tag{1.18}
\]
where \(\zeta^{c}\) denotes the continuous part of \(\zeta\) and \(h, g : \mathbb{R}^{d} \rightarrow \mathbb{R}\) are continuous and non-negative. Notice that each control \((n, \zeta) \in U\) generates two types of costs because \((n, \zeta)\) controls the process \(X_{t}^{n,\zeta}\) continuously or by jumps of \(\zeta\) while \(X_{t}^{n,\zeta}\) is inside \(\mathcal{O}\). The term \(\int_{0}^{t} g(X_{s}^{n,\zeta} + \Delta J_{s} - \lambda n_{s}\Delta \zeta_{s})d\lambda\) represents the cost for using the jump \(\Delta \zeta_{s} \neq 0\) with direction \(-n_{s}\) on \(X_{s}^{n,\zeta} + \Delta J_{s}\) at time \(s\). The value function is defined by
\[
V(\tilde{x}) = \inf_{(n, \zeta) \in U} V_{n,\zeta}(\tilde{x}), \ \text{for} \ \tilde{x} \in \mathcal{O}. \tag{1.19}
\]

A heuristic derivation from dynamic programming principle; see [11, Ch. VIII], shows that the HJB equation corresponding to the value function \(V\) is given by
\[
\max\{[g - \Gamma_{1}]u - h, |D^{1}u| - g\} = 0, \ \text{on} \ \mathcal{O}, \ \text{s.t.} \ u = 0, \ \text{in} \ \overline{\mathcal{O}} \setminus \mathcal{O}, \tag{1.20}
\]
where \(\Gamma_{1}\) is as in (1.14). An immediate consequence of Theorem 1.1 is the following corollary.

**Corollary 1.4.** Assume that \(a_{ij}, b_{i}, h, g, s\) satisfy (A1)–(A5). Then, the HJB equation (1.20) has a unique non-negative solution \(u \in C^{0,1}(\overline{\mathcal{O}}) \cap W^{2,p}_{loc}(\mathcal{O})\), for each \(p \in (d, \infty)\).

**Proposition 1.5.** Let \(u\) be the non-negative solution to the HJB equation (1.20). Then, \(V\) defined in (1.19) and \(u\) agree on \(\overline{\mathcal{O}}\).

To give the proof of this proposition, we need to introduce a class of penalized controls which are related to the singular control problem described above and the NPIDD problem (1.8). For more detail, see Section 4.
Comments

Remark 1.6. Previously to the paper by Moreno-Franco [27] and this paper, the singular stochastic control problem described above has been studied extensively in the one-dimensional case when the state process includes the continuous part only; see, e.g., [3, 6, 14, 19, 20]. Several articles focused on the multidimensional case when the state process is a multidimensional SBM [9, 21, 26, 31], a diffusion process [11, 16, 17], or a multidimensional SBM with jumps process, whose Lévy measure $\nu$ satisfies $\int_{\mathbb{R}^d} |z|^p \nu(dz) < \infty$, for all $p \geq 2$ [25]. It should be noted that the results in [3, 6, 14, 19, 20, 21, 25, 26, 31], were given on the whole space $\mathbb{R}^d$, and that in [31], under convexity and polynomial growth assumptions on the function $h$, it is shown that the value function associated with a controlled two-dimensional SBM is in $C^2(\mathbb{R}^2)$.

Remark 1.7. For the one-dimensional case, similar problems to ours can be found in the mathematical finance and risk theory; see, e.g., [7, 33] and [2, 4], respectively. In the risk theory, one wishes to determine an optimal dividend payment strategy for an insurance company (or discovery company) to pay its shareholders, where the insurance company’s surplus is modelled by a spectrally negative (or positive) Lévy process, i.e., a stochastic process which has a càdlàg path, and stationary and independent increments without positive (negative) discontinuity. Using some results of fluctuation theory, it can be shown (in some cases) that the value function associated with this problem is in $C^2(\mathbb{R})$ and satisfies a similar HJB equation as in (1.20) on the whole space $\mathbb{R}$; see, e.g., [2, 4].

Remark 1.8. Some ideas given here and in Section 4 are taken from [34], where the author has shown that the value function associated with a controlled multidimensional diffusion process satisfies the dynamic programming variational inequality in the almost everywhere sense.

1.2 Notation

We introduce the notation and basic definitions of some spaces that are used in this paper. Let $\alpha \in [0, 1]$ and $m \in \{0, \ldots, k\}$, with $k \geq 0$ an integer. The set $C^k(\mathcal{O})$ consists of real-valued functions on $\mathcal{O}$ that are $k$-fold continuously differentiable. We define $C^\infty(\mathcal{O}) = \bigcap_{k=0}^\infty C^k(\mathcal{O})$. The sets $C_c^k(\mathcal{O})$ and $C_c^\infty(\mathcal{O})$ consist of functions in $C^k(\mathcal{O})$ and $C^\infty(\mathcal{O})$, whose support is compact and contained in $\mathcal{O}$, respectively. The set $C^k(\overline{\mathcal{O}})$ is defined as the set of real-valued functions such that $\partial^a f$ is bounded and uniformly continuous on $\mathcal{O}$, for all $a \in \mathcal{D}_m$ and $m \leq k$, where $\mathcal{D}_m$ is the set of all multi-indices of order $m \leq k$. This space is equipped with the following norm $||f||_{C^k(\overline{\mathcal{O}})} = \sum_{m=0}^k \sup_{a \in \mathcal{D}_m} \sup_{x \in \mathcal{O}} \{||\partial^a f(x)||\}$, where $\sum_{a \in \mathcal{D}_m}$ denotes summation over all possible $m$-fold derivatives of $f$. The operator $[\cdot]_{C^{0,\alpha}(\mathcal{O})}$ is given by $[f]_{C^{0,\alpha}(\mathcal{O})} = \sup_{x, y \in \mathcal{O}, x \neq y} \left\{ \frac{|f(x) - f(y)|}{|x - y|^\alpha} \right\}$. We define $C^{k,\alpha}_{loc}(\mathcal{O})$ as the set of functions in $C^k(\mathcal{O})$ such that $[\partial^a f]_{C^{0,\alpha}(\mathcal{K})} < \infty$, for all compact set $\mathcal{K} \subset \mathcal{O}$, $a \in \mathcal{D}_m$ and $m \leq k$. The set $C^{k,\alpha}(\overline{\mathcal{O}})$ denotes the set of all functions in $C^k(\overline{\mathcal{O}})$ such that $[\partial^a f]_{C^{0,\alpha}(\mathcal{O})} < \infty$, for every $a \in \mathcal{D}_m$ and $m \leq k$. This set is equipped with the following norm $||f||_{C^{k,\alpha}(\overline{\mathcal{O}})} = \sum_{m=0}^k \sup_{a \in \mathcal{D}_m} \{||\partial^a f(x)||_{C^{0,\alpha}(\mathcal{O})} + [\partial^a f]_{C^{0,\alpha}(\mathcal{C})} \}$. We understand $C^{k,\alpha}(\mathbb{R}^d)$ as $C^{k,\alpha}(\overline{\mathbb{R}^d})$, in the sense that $[\partial^a f]_{C^{0,\alpha}(\mathbb{R}^d)} < \infty$, for every $a \in \mathcal{D}_m$ and $m \leq k$. As usual, $L^p(\mathcal{O})$ with $1 \leq p < \infty$, denotes the class of real-valued functions on $\mathcal{O}$ with finite norm $||f||_{L^p(\mathcal{O})} := \int_{\mathcal{O}} |f|^p dx < \infty$, where $dx$ denotes the Lebesgue measure. Also, let $L^p_{loc}(\mathcal{O})$ consist of functions whose $L^p$-norm is finite on any compact subset of $\mathcal{O}$. Define the Sobolev space $W^{k,p}(\mathcal{O})$ as the class
Using (A4) and by Dominated Convergence Theorem, it is easy to see that

\[ |f|_{L^p(O)}^p = \sum_{m=0}^{k} \sum_{a \in A_m} \|\partial^a f\|_{L^p(O)}^p. \]

The space \( W^{k,p}_{\text{loc}}(O) \) consists of functions whose \( W^{k,p}_{\text{loc}} \)-norm is finite on any compact subset of \( O \). When \( p = \infty \), the Sobolev and Lipschitz spaces are related. In particular, \( W^{k,\infty}_{\text{loc}}(O) = C^{k-1,1}_{\text{loc}}(\overline{O}) \) and \( W^{k,\infty}(O) = C^{k,1}(\overline{O}) \). Finally, \( C = C(*) \ldots (*) \) and \( K = K(*) \ldots (*) \) represent positive constants that depend only on the quantities appearing in parenthesis.

## 2 Existence and uniqueness of the NPIDD problem

In this section, we are interested in establishing the existence, uniqueness and regularity of the solution to the NPIDD problem (1.8). The arguments used here are based on the Schaefer fixed point Theorem; see [10, Thm. 4, p. 539]. First, we shall analyse some properties of \( I \) \( w \), defined in (1.3), when the function \( w \) is \( C^{0,1} \) and \( C^{1,1} \) on \( \overline{O} \). These results will be helpful to show some properties of the solution to the NPIDD problem (1.8).

**Remark 2.1.** Notice that by definition of \( \overline{O} \); see (1.7), and since \( s \) is a non-negative function on \( \overline{O} \times \mathbb{R}^d \), it follows that \( s(x, z) = 0 \), for each \((x, z) \in \overline{O} \times \mathbb{R}^d \) such that \( x + z \notin \overline{O} \). Then, \( I \) \( w \) can be rewritten as

\[ I \) \( w(x) = \int_{\mathbb{R}^d} [w(x + z) - w(x)] s(x, z) 1_{(x + z \in \overline{O})} \nu(dz), \text{ for } x \in \overline{O}. \]

**Lemma 2.2.**

(i) If \( w \in C^{0,1}(\overline{O}) \), then \( I \) \( w \in C(\overline{O}) \).

(ii) If \( w, v \in C^{0,1}(\overline{O}) \), then \( [w, v]_{\overline{O}} = I[\nu v] - w I v - v I w \) on \( \overline{O} \), where

\[ [w, v]_{\overline{O}} := \int_{\mathbb{R}^d} [w(\cdot + z) - w][v(\cdot + z) - v] s(\cdot, z) \nu(dz), \text{ on } x \in \overline{O}. \]

(iii) If \( w \in C^{1,1}(\overline{O}) \), then \( I \) \( w \in C^1(\overline{O}) \) and \( \partial_i[I w] = I[\partial_i w] + \tilde{I}_i w \) where

\[ \tilde{I}_i w := \int_{\mathbb{R}^d} [w(\cdot + z) - w] \partial_i s(\cdot, z) \nu(dz). \]

**Proof.** Using (A4) and by Dominated Convergence Theorem, it is easy to see that \( I \) \( w \) \( \in C(\overline{O}) \), when \( w \in C^{0,1}(\overline{O}) \). Calculating \( [w(\cdot + z) - w][v(\cdot + z) - v] \), the reader can verify that the statements in (ii) of the lemma above is true. We shall prove the statement given in (iii). Let \( w \) be in \( C^{1,1}(\overline{O}) \), and define \( f_w : \overline{O} \times \mathbb{R}^d \rightarrow \mathbb{R} \) as

\[ f_w(x, z) := \begin{cases} [w(x + z) - w(x)] s(x, z), & \text{if } x + z \in \overline{O}, \\ 0, & \text{otherwise.} \end{cases} \]

Consider \( x, y \in \overline{O} \) such that \( x \neq y \). Then, by Remark 2.1, we see that

\[ |I \) \( w(x) - I \) \( w(y)| \leq \int_{\{z \in (0,1)\}} |f_w(x, z) 1_{x+z \in \overline{O}} - f_w(y, z) 1_{y+z \in \overline{O}}| \nu(dz) \]

\[ + \int_{\{z \geq 1\}} |f_w(x, z) 1_{x+z \in \overline{O}} - f_w(y, z) 1_{y+z \in \overline{O}}| \nu(dz). \]
Meanwhile, from Mean Value Theorem and noting that
\[ s(x, z) \mathbb{1}_{\{y + z \in \mathcal{O}, x + z \notin \mathcal{O}\}} = s(y, z) \mathbb{1}_{\{x + z \in \mathcal{O}, y + z \notin \mathcal{O}\}} = 0, \]
we have
\[
|f_w(x, z) - f_w(y, z)| \leq |x - y| \left[ \mathbb{1}_{\{x + z \in \mathcal{O}, y + z \notin \mathcal{O}\}} \int_0^1 |D^1_x f_w(y + t|x - y|, z)| \, dt 
+ \int_0^1 |D^1_x s(y + t|x - y|, z)| \, dt \left[w(x + z) - w(x)\right] \mathbb{1}_{\{x + z \in \mathcal{O}, y + z \notin \mathcal{O}\}} 
+ |w(y + z) - w(y)| \mathbb{1}_{\{y + z \in \mathcal{O}, x + z \notin \mathcal{O}\}} \right],
\]
(2.1)

where \( D^1_x f_w \) denotes the gradient with respect to \( x \). Observe that
\[
D^1_x f(x, z) = s(x, z) D^1[w(x + z) - w(x)] + [w(x + z) - w(x)] D^1_x s(x, z),
\]
(2.2)

for \((x, z) \in \mathcal{O} \times \mathbb{R}^d \) such that \( x + z \in \mathcal{O} \). If \(|z| < 1\), by (2.1)–(2.2) and using \( w, D^1 w \) are Lipschitz functions on \( \mathcal{O} \), we get
\[
|f_w(x, z) - f_w(y, z)| \leq K_1 |z| |x - y|,
\]
(2.3)

where \( K_1 := \left[ \sum_k ||\partial_k w||^2_{C^{0,1}(\mathcal{O})} \right]^{\frac{1}{2}} + 3K_2[w]_{C^{0,1}(\mathcal{O})} \) and \( K_2 := \left[ \sum_k ||\partial_k s||^2_{C(\mathcal{O} \times \mathbb{R}^d)} \right]^{\frac{1}{2}} \). If \(|z| \geq 1\), by (2.1)–(2.2) and since \( w, D^1 w \) are bounded on \( \mathcal{O} \), it can be verified that
\[
|f_w(x, z) - f_w(y, z)| \leq K_3 |x - y|,
\]
(2.4)

where \( K_3 := 2 \left[ \sum_k ||\partial_k w||^2_{C^1(\mathcal{O})} \right]^{\frac{1}{2}} + 6K_1 ||w||_{C^1(\mathcal{O})} \). Using (2.3)–(2.4), we have that for \((x, z) \in \mathcal{O} \times \mathbb{R}^d \) such that \( x + z \in \mathcal{O}, \frac{1}{\beta} |f_w(x + \beta e_i, z) - f_w(x, z)| \), is bounded by \( K_2 |z| \mathbb{1}_{\{z \in (0,1)\}} + K_3 \mathbb{1}_{\{|z| \geq 1\}} \), which is an integrable function with respect to the Lévy measure \( \nu \). Then, using Dominated Convergence Theorem and (2.2), it follows that \( \partial_i [\mathcal{I} w(x)] = \mathcal{I} \partial_i w(x) + \tilde{\mathcal{I}}_i w(x) \). From here, (A4) and since \( \partial_i w \in C^{0,1}(\mathcal{O}) \), we conclude that \( \mathcal{I} w \in C^{1}(\mathcal{O}) \).

2.1 À priori estimates of the solution to the NPIDD problem

To apply the Schaefer fixed point Theorem in our problem, we need to show an à priori estimate of the classical solution \( u^\varepsilon \) to the NPIDD problem (1.8) on the space \( (C^{1,\alpha'}(\mathcal{O}), ||\cdot||_{C^{1,\alpha'}(\mathcal{O})}) \); see Lemma 2.9.

Remark 2.3. Notice that if the solution \( u^\varepsilon \) is at least \( C^2 \) on \( \mathcal{O} \), then using [12, Thm. 3.1.22] and the Sobolev embedding Theorem [1, Thm. 4.12, p. 85], it can be verified that for each \( \varepsilon \in (0,1) \) fixed,
\[
||u^\varepsilon||_{C^{1,\alpha'}(\mathcal{O})} \leq C \left[ ||h||_{L^{p'}(\mathcal{O})} + ||\psi_{\varepsilon}(\cdot) ||_{L^{p'}(\mathcal{O})} \right],
\]
(2.5)

for some \( C = C(\Lambda, \nu, s, \alpha') \), where \( p' \in (d, \infty) \) is such that \( \alpha' = 1 - \frac{d}{p'} \). We see that the second term in the RHS of (2.5) depends on \( \psi_{\varepsilon}(\cdot) ||D^1 u^\varepsilon||^2 \). If \( ||D^1 u^\varepsilon|| \leq C' \) for some
constant $C'$ independent of $\epsilon$; see Lemma 2.8, then, from (A.1), (1.9) and (2.5), it follows that $\|u^\varepsilon\|_{C^{1,\alpha'}(\overline{\Omega})} \leq [\Lambda + \frac{1}{\varepsilon}][C']^2 + \Lambda^2 + 1]C \int_{\Omega} dx$. Although this estimation depends on $1/\varepsilon$, it is sufficient to use the Schaefer fixed point Theorem in our problem, since $\varepsilon$ is fixed; see Subsection 2.2. Later on, in Section 3, we will give a local estimation for $\psi_\varepsilon(|D^1 u^\varepsilon|^2 - g^2)$ which is independent of $\varepsilon$; see Lemma 3.3.

Before continuing, we need to introduce the concepts of sub-solution and super-solution for the NPIDD problem (1.8).

**Definition 2.4.** 1. A function $f$ in $C^2(\overline{\Omega}) \cap C^{0,1}(\overline{\Omega})$ is a sub-solution of (1.8) if
\[ \Gamma f + \psi_\varepsilon(|D^1 f|^2 - g^2) \leq h, \text{ in } \Omega, \text{ s.t. } f = 0, \text{ on } \partial \Omega \setminus \Omega. \]

2. A function $f$ in $C^2(\overline{\Omega}) \cap C^{0,1}(\overline{\Omega})$ is a super-solution of (1.8) if
\[ \Gamma f + \psi_\varepsilon(|D^1 f|^2 - g^2) \geq h, \text{ in } \Omega, \text{ s.t. } f = 0, \text{ on } \partial \Omega \setminus \Omega. \]

An immediate consequence of this definition is the following result, which is used to prove Lemma 2.6.

**Lemma 2.5.** If $\varphi$ and $\eta$ are a sub-solution and a super-solution of (1.8), respectively, then $\varphi - \eta \leq 0$ on $\overline{\Omega}$.

**Proof.** From Definition 2.4, we get
\[ \Gamma[\varphi - \eta] + \psi_\varepsilon(|D^1 \varphi|^2 - g^2) - \psi_\varepsilon(|D^1 \eta|^2 - g^2) \leq 0, \text{ in } \Omega, \text{ s.t. } \varphi - \eta = 0, \text{ on } \partial \Omega \setminus \Omega. \] (2.6)

Let $x^* \in \overline{\Omega}$ be a maximum point of $\varphi - \eta$. If $x^* \in \partial \Omega$, trivially, we have $\varphi - \eta \leq 0$ on $\overline{\Omega}$. Suppose that $x^* \in \Omega$. This means that
\[ D^1[\varphi - \eta](x^*) = 0, \text{ tr}[a(x^*)D^2[\varphi - \eta](x^*)] \leq 0, \]
\[ [\varphi - \eta](x^* + z) - [\varphi - \eta](x^*) \leq 0, \text{ for } z \in \mathbb{R}^d_+ \text{ with } x^* + z \in \overline{\Omega}. \] (2.7)

Since $[\varphi - \eta](x^* + z) = 0$ when $x^* + z \in \overline{\Omega} \setminus \Omega$, it follows that $[\varphi - \eta](x^*) \geq 0$. Meanwhile, applying (2.7) in (2.6), it yields $c(x^*)[\varphi - \eta](x^*) \leq 0$. Then, $[\varphi - \eta](x^*) \leq 0$, since $c > 0$ on $\overline{\Omega}$. Therefore, $[\varphi - \eta](x) \leq [\varphi - \eta](x^*) = 0$ for all $x \in \overline{\Omega}$. 

**Lemma 2.6.** If $u^\varepsilon \in C^2(\overline{\Omega}) \cap C^{0,1}(\overline{\Omega})$ is a solution to the NPIDD problem (1.8), there exists a positive constant $C_1$ independent of $\varepsilon$, such that $0 \leq u^\varepsilon \leq C_1$ on $\overline{\Omega}$ and $|D^1 u^\varepsilon| \leq d^{\frac{1}{2}}C_1$ on $\partial \Omega$.

From now on, for simplicity of notation, we replace $u^\varepsilon$ by $u$ in the proofs of the results.

**Proof of Lemma 2.6.** For $h$ which is a $C^{1,\alpha'}$-function on $\overline{\Omega}$, let $v \in C^{2,\alpha'}(\overline{\Omega})$ be the unique solution to the LPIDD problem
\[ \Gamma v = h, \text{ in } \Omega, \text{ s.t. } v = 0, \text{ on } \partial \Omega \setminus \Omega. \]

Then, $\|v\|_{C^{2,\alpha'}(\overline{\Omega})} \leq K_4\|h\|_{C^{0,\alpha'}(\overline{\Omega})} \leq K_4\Lambda =: C_1$, where $K_4 = K_4(d, \Lambda, \nu, s, \alpha')$; see [12, Thm. 3.1.12]. Then $v$ is a super-solution of (1.8). Meanwhile, we know that $h \geq 0$, this implies that the zero function is a sub-solution of (1.8). Therefore, using Lemma 2.5, it follows
that $0 \leq u \leq K_4 A$ on $\overline{\mathcal{O}}$. Take a point $x$ in $\partial \mathcal{O}$ and a unit vector $n_x$ outside $\mathcal{O}$ such that it is not tangent to $\mathcal{O}$. Defining $v = -n_x$, we have that $\langle v, D^1 v(x) \rangle = \lim_{\theta \to 0} \frac{v(x+\theta e)}{\theta} \geq \lim_{\theta \to 0} \frac{u(x+\theta e)}{\theta} = \langle v, D^1 u(x) \rangle$. Since $v = -n_x$, it yields $\langle n_x, D^1 v(x) \rangle \leq \langle n_x, D^1 u(x) \rangle \leq 0$.

Then, $|\langle n_x, D^1 u(x) \rangle| \leq \left( \sum_i |\partial_i v|^2 \right)^{\frac{1}{2}} \leq d^\frac{1}{2} C_1$. Suppose that $|D^1 u(x)| \neq 0$ and that the vector $D^1 u(x)$ is outside $\mathcal{O}$. Taking $n_x = \frac{D^1 u(x)}{|D^1 u(x)|}$, it follows that $|D^1 u(x)| \leq d^\frac{1}{2} C_1$. If the vector $v = D^1 u(x)$ is inside $\mathcal{O}$, proceeding as before, we have $0 \leq \langle v, D^1 u(x) \rangle \leq \langle v, D^1 v(x) \rangle$. Therefore, $|D^1 u(x)| \leq d^\frac{1}{2} C_1$. In the case that $|D^1 u(x)| = 0$, the inequality is trivially true. Consequently, we have finished the proof.

Before checking $|D^1 u^\varepsilon| \leq C'$ on $\overline{\mathcal{O}}$, for some constant $C' > 0$ independent of $\varepsilon$, we need to define an auxiliary function $\varphi$, which satisfies (2.8) on $\mathcal{O}$. In particular, (2.8) is true, when $\varphi$ is evaluated at its maximum $x^* \in \mathcal{O}$, which helps us to prove Lemma 2.8.

**Lemma 2.7.** Let $u^\varepsilon \in C^0(\overline{\mathcal{O}}) \cap C^2(\overline{\mathcal{O}}_T)$ be a solution to the NPIDD problem (1.8). Define the auxiliary function $\varphi : \overline{\mathcal{O}}_T \rightarrow \mathbb{R}$ as $\varphi := |D^1 u^\varepsilon|^2 - \lambda M_\varepsilon u^\varepsilon$, on $\overline{\mathcal{O}}_T$, where $M_\varepsilon := \sup_{x \in \partial \mathcal{O}} |D^1 u^\varepsilon(x)|$ and $\lambda > 0$. Then, $\varphi \in C^0(\overline{\mathcal{O}}_T)$ and there exists a positive constant $C_2$ independent of $\varepsilon$ such that

$$- \text{tr}[a D^2 \varphi] - \mathcal{L}\varphi \leq C_2 |D^1 u^\varepsilon|^2 + C_2 [1 + M_\varepsilon [1 + \lambda]] |D^1 u^\varepsilon| + \lambda C_2 M_\varepsilon$$

$$- \psi'_\varepsilon(\cdot)[2(D^1 u^\varepsilon, D^1 |D^1 u^\varepsilon|^2) - C_2 |D^1 u^\varepsilon| - \lambda M_\varepsilon |D^1 u^\varepsilon|^2 - g^2]], \quad \text{on} \mathcal{O}, \quad (2.8)$$

where $\psi'_\varepsilon(\cdot)$ denotes $\psi'_\varepsilon(|D^1 u|^2 - g^2)$.

**Proof.** Notice that $\varphi \in C^0(\overline{\mathcal{O}}_T \setminus \partial \mathcal{O}) \cap C^1(\overline{\mathcal{O}}_T)$, $\partial_i \varphi = 2(D^1 \partial_i u, D^1 u) - \lambda M_\varepsilon \partial_i u$ and $\partial_{ij} \varphi = 2[(D^1 \partial_{ij} u, D^1 u) + (D^1 \partial_i u, D^1 \partial_j u)] - \lambda M_\varepsilon \partial_{ij} u$ on $\overline{\mathcal{O}}$. Then, from here and using $u = \partial_i u = \partial_{ij} u = 0$ on $\partial \mathcal{O}$, it is easy to see that $\partial_{ij} \varphi = 0$ on $\partial \mathcal{O}$ and thus $\varphi \in C^2(\overline{\mathcal{O}}_T)$. Observe that

$$- \text{tr}[a D^2 \varphi] - \mathcal{L}\varphi = -2 \sum_k \langle a D^1 \partial_k u, D^1 \partial_k u \rangle - 2 \sum_k \text{tr}[a D^2 \partial_k u \partial_k u]$$

$$- \mathcal{L}[D^1 u]^2 + \lambda M_\varepsilon [\text{tr}[a D^2 u] + \mathcal{L} u], \quad (2.9)$$

From (1.2) and (1.8), it follows that

$$\lambda M_\varepsilon [\text{tr}[a D^2 u] + \mathcal{L} u] = \lambda M_\varepsilon \psi'_\varepsilon(\cdot) + \lambda M_\varepsilon [\tilde{D}_1 u - h], \quad (10.10)$$

where $\tilde{D}_1 u := \langle b, D^1 u \rangle + cu$. Differentiating (1.8) and by Lemma 2.2.iii, we see that

$$- \text{tr}[a D^2 \partial_k u] - \mathcal{L} \partial_k u = \text{tr}[\partial_k a] D^2 u + \mathcal{L} \partial_k u$$

$$+ \partial_k [h - \langle b, D^1 u \rangle - cu] - \psi'_\varepsilon(\cdot) \partial_k [D^1 u]^2 - g^2]. \quad (11.11)$$

By Lemma 2.2.ii, it yields

$$- 2 \partial_k u \mathcal{L} \partial_k u = [\partial_k u, \partial_k u]_{\mathcal{I}} - \mathcal{L} [\partial_k u]^2. \quad (12.12)$$
From here, multiplying (2.11) by $2\partial_k u$ and taking summation over all $k$’s,

$$-2 \sum_k \text{tr}[a D^2 \partial_k u][\partial_k u - I]|D^1 u|^2$$

$$= \tilde{D}_2 u + 2\langle D^1 u, D^1 h \rangle - 2\psi_\varepsilon'(\cdot)<D^1 u, D^1[u|D^1 u|^2 - g^2]> - \sum_k [\partial_k u, \partial_k u]_I, \quad (2.13)$$

with $\tilde{D}_2 u := 2[\sum_k \partial_k u[\text{tr}[[\partial_k a] D^2 u] + \tilde{I}_k u] - \langle D^1 u, D^1[(b, D^1 u) + cu])\rangle$. Applying (2.10) and (2.13) in (2.9), we get

$$-\text{tr}[a D^2 \varphi] - I\varphi = -2\psi_\varepsilon'(\cdot)<D^1 u, D^1[u|D^1 u|^2 - g^2]> + \lambda M_\varepsilon \psi_\varepsilon(\cdot)$$

$$- 2 \sum_k \langle a D^1 \partial_k u, D^1 \partial_k u \rangle - \lambda M_\varepsilon h + 2\langle D^1 u, D^1 h \rangle$$

$$- \sum_k [\partial_k u, \partial_k u]_I + \tilde{D}_2 u + \lambda M_\varepsilon \tilde{D}_1 u. \quad (2.14)$$

Notice that

$$\lambda M_\varepsilon \psi_\varepsilon(\cdot) \leq \lambda M_\varepsilon \psi_\varepsilon'(\cdot)|D^1 u|^2 - g^2], \quad (2.15)$$

since $\psi_\varepsilon$ is a convex function. From (A1), (A3) and since $h \geq 0$ on $\overline{O}$, it follows that

$$-2 \sum_k \langle a D^1 \partial_k u, D^1 \partial_k u \rangle - \lambda M_\varepsilon h + 2\langle D^1 h, D^1 u \rangle \leq -2\theta|D^2 u|^2 + 2d\Lambda|D^1 u|. \quad (2.16)$$

By (A4), Mean Value Theorem and the estimate $0 \leq u \leq C_1$, with $C_1$ as in Lemma 2.6, we get

$$2 \sum_k \partial_k u \tilde{I}_k u \leq 2dK_2|D^1 u| \left[ \int_{|z| \in (0,1)} \left[ \int_0^1 |D^1 u(\cdot + tz)|dt \right]|z| \nu(dz) \right.$$  

$$+ \int_{|z| \geq 1} [u(\cdot + z) + u] \nu(dz) \right] \leq 2dK_2C_\nu[M_\varepsilon + 2C_1]|D^1 u|,$$

where $C_\nu, K_2$ are as in (1.4), (2.3), respectively. Then, using the inequality above and since $[\partial_k u, \partial_k u]_I \geq 0$, we have

$$- \sum_k [\partial_k u, \partial_k u]_I + \tilde{D}_2 u + \lambda M_\varepsilon \tilde{D}_1 u \leq 4d^3 \Lambda|D^2 u| |D^1 u| + 2d^2 \Lambda|D^1 u|^2$$

$$+ 2dC_1[\Lambda + 2K_2C_\nu]|D^1 u| + dM_\varepsilon[\lambda \Lambda + 2K_2C_\nu]|D^1 u| + \lambda M_\varepsilon C_1. \quad (2.17)$$

Therefore, applying (2.15)–(2.17) in (2.14) and noting that $-\theta|D^2 u|^2 + 2d^3 \Lambda|D^2 u| |D^1 u| \leq \frac{d^3 \Lambda^2}{\theta}|D^1 u|^2$, we obtain the inequality (2.8), where $C_2 = C_2(d, \Lambda, \nu, s, \alpha')$.  

**Lemma 2.8.** If $u^\varepsilon \in C^3(\overline{O}) \cap C^2(\overline{O}_T)$ is a solution to the NPIDD problem (1.8), then there exists a positive constant $C_3$ independent of $\varepsilon$, such that $|D^1 u^\varepsilon| \leq C_3$ on $\overline{O}$.  

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Proof. Let \( \varphi \) and \( M_\varepsilon \) be as in Lemma 2.7, with \( \lambda \geq 1 \) a constant that shall be selected later on. Observe that if \( M_\varepsilon \leq 1 \), we obtain a bound for \( M_\varepsilon \) that is independent of \( \varepsilon \). We assume henceforth that \( M_\varepsilon > 1 \). Let \( x^* \in \overline{\mathcal{O}} \) be a point where \( \varphi \) attains its maximum on \( \mathcal{O} \). Then,

\[
|D^1 u(x)|^2 \leq |D^1 u(x^*)|^2 + \lambda M_\varepsilon [u(x) - u(x^*)] \leq |D^1 u(x^*)|^2 + \lambda M_\varepsilon C_1,
\]

for all \( x \in \overline{\mathcal{O}} \). The last inequality in (2.18) is obtained from Lemma 2.6. If \( x^* \in \partial \mathcal{O} \), by Lemma 2.6, it is easy to deduce \( \varphi(x^*) = |D^1 u(x^*)|^2 \leq dC_1^2 \). Then, from (2.18), \( |D^1 u|^2 \leq dC_1^2 + \lambda M_\varepsilon C_1 \) in \( \overline{\mathcal{O}} \). Notice that for all \( \varepsilon \), there exists \( x_0 \in \overline{\mathcal{O}} \) such that \( [M_\varepsilon - \varepsilon]^2 \leq |D^1 u(x_0)|^2 \). Then,

\[
[M_\varepsilon - \varepsilon]^2 \leq dC_1^2 + \lambda M_\varepsilon C_1.
\]

Letting \( \varepsilon \to 0 \) in (2.19), it follows that \( |D^1 u| \leq M_\varepsilon \leq \frac{dC_1^2}{M_\varepsilon} + \lambda C_1 \leq dC_1^2 + \lambda C_1 \), since \( M_\varepsilon > 1 \). Let \( x^* \) be in \( \mathcal{O} \). We have \( \text{tr}[a(x^*)]D^2 \varphi(x^*) \leq 0 \), \( \varphi(x^*) \geq \varphi(x^* + z) \) for \( x^* + z \in \overline{\mathcal{O}} \), and \( \partial_i \varphi(x^*) = \partial_i |D^1 u(x^*)|^2 - \lambda M_\varepsilon \partial_i u(x^*) = 0 \). Then, \( 0 \leq -\text{tr}[aD^2 \varphi] - \mathcal{I} \varphi \) and \( 2(D^1 u, D^1 u|D^1 u|^2) = 2\lambda M_\varepsilon |D^1 u|^2 \) at \( x^* \). From here, using Lemma 2.7 and since \( \psi_\varepsilon(\cdot) \geq 0 \), it follows that

\[
0 \leq C_2|D^1 u|^2 + C_2[1 + M_\varepsilon[1 + \lambda]]|D^1 u| + \lambda C_2 M_\varepsilon - \psi_\varepsilon(\cdot)[\lambda M_\varepsilon|D^1 u|^2 - C_2|D^1 u| + \lambda M_\varepsilon g^2], \text{ at } x^*,
\]

where \( C_2 \) is as in Lemma 2.7. If \( \psi_\varepsilon(\cdot) < 1 \), by definition of \( \psi_\varepsilon \), given in (1.9), we obtain that \( \psi_\varepsilon(\cdot) \leq 1 \). It follows that \( |D^1 u(x^*)|^2 \leq 2 + \Lambda^2 \). Then, by (2.18) and arguing as in (2.19), we obtain \( M_\varepsilon \leq 2 + \Lambda^2 + \lambda C_1 \). If \( \psi_\varepsilon(\cdot) \geq 1 \), then, multiplying by \( \frac{1}{M_\varepsilon \psi_\varepsilon(\cdot)} \) in (2.20), it can be verified that

\[
0 \leq [C_2 - \lambda]|D^1 u|^2 + C_2[3 + \lambda]|D^1 u| + \lambda C_2, \text{ at } x^*.
\]

Notice that this inequality is satisfied for any \( \lambda \geq 1 \) fixed, where the maximum point of \( \varphi \), \( x^* \in \mathcal{O} \), depends on \( \lambda \). Then, taking \( \lambda \geq \max\{1, C_2\} \) fixed, from (2.21), it follows that \( |D^1 u(x^*)| < K_5 \), for some \( K_5 = K_5(d, \Lambda, \nu, s, \alpha', \lambda) \). Using (2.18) and an argument similar to (2.19), we conclude that there exists \( C_3 = C_3(d, \Lambda, \nu, s, \alpha', \lambda) \) such that \( |D^1 u| \leq M_\varepsilon \leq C_3 \) on \( \overline{\mathcal{O}} \).

By the previous results seen here and using (2.5), we obtain the following estimate of \( u_\varepsilon \) on the space \( (C^{1,\alpha'}(\overline{\mathcal{O}}), \| \cdot \|_{C^{1,\alpha'}(\overline{\mathcal{O}})}) \).

**Lemma 2.9.** If \( u_\varepsilon \in C^3(\overline{\mathcal{O}}) \cap C^2(\overline{\mathcal{O}}_F) \) is a solution to the NPIDD problem (1.8), there exists \( C_4 = C_4(\varepsilon, \Lambda, \nu, s, \alpha') \) such that \( \|u_\varepsilon\|_{C^{1,\alpha'}(\overline{\mathcal{O}})} \leq C_4 \).

**Proof.** By Remark 2.3 and Lemma 2.8, it yields \( \|u\|_{C^{1,\alpha'}(\overline{\mathcal{O}})} \leq C \int_\mathcal{O} dx[\Lambda + K_6[C_3^2 + \Lambda^2 + 1]] \) for some \( C = C(\Lambda, \nu, s, \alpha') \) and \( K_6 = K_6(\varepsilon, \Lambda) \). Taking \( C_4 := C \int_\mathcal{O} dx[\Lambda + K_6[C_3^2 + \Lambda^2 + 1]] \), it follows that \( \|u\|_{C^{1,\alpha'}(\overline{\mathcal{O}})} \leq C_4 \) on \( \overline{\mathcal{O}} \).

\[\boxed{}\]
2.2 Proof of Proposition 1.2

In this subsection, we present the proof of the existence, uniqueness and regularity of the solution \( u^\varepsilon \) to the NPIDD problem (1.8).

**Proof of Proposition 1.2.** Let \( \varepsilon \in (0, 1) \) be fixed. Observe that the following LPIDD problem

\[
\Gamma u = h - \psi_\varepsilon(|D^1 w|^2 - g^2), \quad \text{in } \mathcal{O}, \quad \text{s.t. } u = 0, \quad \text{on } \partial \mathcal{O} \setminus \mathcal{O},
\]

has a unique solution \( u \in C^{2,\alpha'}(\mathcal{O}) \), for each \( w \in C^{1,\alpha'}(\mathcal{O}) \), since \( h - \psi_\varepsilon(|D^1 w|^2 - g^2) \in C^{0,\alpha'}(\mathcal{O}) \), and (A1), (A3) and (A4) hold; see [12, Thm. 3.1.12]. Defining the map \( T : C^{1,\alpha'}(\mathcal{O}) \to C^{2,\alpha'}(\mathcal{O}) \) as \( T[w] = u \), for each \( w \in C^{1,\alpha'}(\mathcal{O}) \), where \( u \) is the solution to the LPIDD problem (2.22), we see that \( T \) is well defined. Notice that \( T \) is continuous and maps bounded sets in \( C^{1,\alpha'}(\mathcal{O}) \) into bounded sets in \( C^{2,\alpha'}(\mathcal{O}) \) which are pre-compact in the Hölder space \( (C^{1,\alpha'}(\mathcal{O})), || \cdot ||_{C^{1,\alpha'}(\mathcal{O})} \); see [5, Thm. 16.2.2]. To use the Schaefer fixed point Theorem; see [10, Thm. 4, p. 539], we need to verify that the set \( \tilde{A} := \{ w \in C^{1,\alpha'}(\mathcal{O}) : \rho T[w] = w, \text{ for some } \rho \in [0, 1] \} \), is bounded uniformly, i.e. \( ||w||_{C^{1,\alpha'}(\mathcal{O})} \leq C \) for all \( w \in \tilde{A} \), where \( C \) is some positive constant which is independent of \( w \) and \( \rho \). Let \( w \) be in \( \tilde{A} \). Notice that if \( \rho = 0 \), then it follows immediately that \( w = 0 \). So, assume \( w \in C^{1,\alpha'}(\mathcal{O}) \) such that \( T[w] = \frac{w}{\rho} \), for some \( \rho \in (0, 1] \); or, in other words, \( w \in C^{2,\alpha'}(\mathcal{O}) \), since the map \( T \) is defined from \( C^{1,\alpha'}(\mathcal{O}) \) to \( C^{2,\alpha'}(\mathcal{O}) \), and

\[
\Gamma w = \rho[h - \psi_\varepsilon(|D^1 w|^2 - g^2)], \quad \text{in } \mathcal{O}, \quad \text{s.t. } w = 0, \quad \text{on } \partial \mathcal{O},
\]

Taking \( f = \rho[h - \psi_\varepsilon(|D^1 w|^2 - g^2)] + Tw \), from (A1), (A4), Lemma 2.2 and since \( \rho[h - \psi_\varepsilon(|D^1 w|^2 - g^2)] \in C^{1,\alpha'}(\mathcal{O}) \), we have that \( f \in C^{1,\alpha'}(\mathcal{O}) \). Then, the linear Dirichlet problem

\[
\mathcal{L} \tilde{v} = f, \quad \text{in } \mathcal{O}, \quad \text{s.t. } \tilde{v} = 0, \quad \text{on } \partial \mathcal{O},
\]

has a unique solution \( \tilde{v} \in C^{3,\alpha'}(\mathcal{O}) \), since (A3) holds and the boundary \( \partial \mathcal{O} \) is of class \( C^{3,\alpha'} \); see [13, Thms. 6.14, 9.19, pp. 107, 244, respectively]. Recall that the elliptic differential operator \( \mathcal{L} \) is defined in (1.3). From (1.2) and (2.23)–(2.24), it follows that

\[
\mathcal{L}w = \mathcal{L}\tilde{v}, \quad \text{in } \mathcal{O}, \quad \text{s.t. } w = \tilde{v}, \quad \text{on } \partial \mathcal{O}.
\]

From here and using [13, Thm. 6.14 p. 107], \( w = \tilde{v} \in \mathcal{O} \), and hence \( w \in C^{3,\alpha'}(\mathcal{O}) \). Therefore \( \tilde{A} \subset C^{3,\alpha'}(\mathcal{O}) \). Now, applying similar arguments, seen in proofs of Lemmas 2.6, 2.8 and 2.9, to (2.23), it can be verified that \( 0 \leq w \leq C_1, |D^1 w| \leq C_3 \), on \( \overline{\mathcal{O}} \), and \( ||w||_{C^{1,\alpha'}(\mathcal{O})} \leq C_4 \), where \( C_1, C_3, C_4 \) are positive constants as in Lemmas 2.6, 2.8 and 2.9, respectively. Notice that these constants are independent of \( \rho \) and \( w \). This means that \( \tilde{A} \) is bounded uniformly on \( (C^{1,\alpha'}(\mathcal{O})), || \cdot ||_{C^{1,\alpha'}(\mathcal{O})} \). Since \( T \) is a continuous and compact mapping from the Banach space \( (C^{1,\alpha'}(\mathcal{O})), || \cdot ||_{C^{1,\alpha'}(\mathcal{O})} \) to itself and the set \( \tilde{A} \) is bounded uniformly, we conclude, by the Schaefer fixed point Theorem, there exists a fixed point \( u^\varepsilon \in C^{1,\alpha'}(\mathcal{O}) \) to the problem \( T[u^\varepsilon] = u^\varepsilon \) which satisfies the NPIDD problem (1.8). In addition, we have \( u^\varepsilon = T[w] \in C^{2,\alpha'}(\mathcal{O}) \) and by similar arguments seen previously, it can be shown that \( u^\varepsilon \) is non-negative and belongs to \( C^{3,\alpha'}(\mathcal{O}) \). The uniqueness of the solution \( u^\varepsilon \) to the problem (1.8), is obtained from Lemma 2.5. With this remark we finish the proof. \( \blacksquare \)
3 Existence and uniqueness of the HJB equation

Since \( u^\varepsilon \) satisfies Lemmas 2.6 and 2.8, for each \( \varepsilon \in (0,1) \), and the constants that appear in these Lemmas are independent of \( \varepsilon \), we only need to show that \( \psi_\varepsilon(|D^1 u^\varepsilon|^2 - g^2) \) is locally bounded by a positive constant independent of \( \varepsilon \); see Lemma 3.3. This estimate implies that \( u^\varepsilon \) is locally bounded uniformly in \( \varepsilon \), i.e., \( ||u^\varepsilon||_{W^{2,p}(B_{\beta r})} \leq C \) where \( B_{\beta r} \subset O \) is an open ball with radius \( \beta r \), where \( \beta \in (0,1) \) and \( r > 0 \); see Lemmas 3.4 and 3.5. From here, we extract a convergent sub-sequence \( \{u^\varepsilon_n\}_{n \geq 1} \) of \( \{u^\varepsilon\}_{\varepsilon \in (0,1)} \), whose limit function is the solution to the HJB equation (1.1); see Subsection 3.2.

3.1 Some local properties of the solution to the NPIDD problem

Before showing that \( \psi_\varepsilon(|D^1 u^\varepsilon|^2 - g^2) \leq C \) is locally bounded by some constant \( C \) independent of \( \varepsilon \), we need to define an auxiliary function \( \phi_\varepsilon \), which satisfies (3.1) on \( B_{\beta' r} \), with \( \beta' \) as in Remark 3.1. In particular, (3.1) is true, when \( \phi \) is evaluated at its maximum \( x^* \in B_{\beta' r} \subset O \), which helps us to prove Lemma 3.3.

**Remark 3.1.** Let \( \overline{O}_r \) be the interior set of \( \overline{O}_r \). In Lemmas 3.2–3.5 and their proofs, we consider cut-off functions \( \xi \in C_0^\infty(\overline{O}_r) \) which satisfy \( 0 \leq \xi \leq 1 \), \( \xi = 1 \) on the open ball \( B_{\beta r} \subset B_{\beta' r} \subset O \) and \( \xi = 0 \) on \( \overline{O}_r \setminus B_{\beta' r} \), with \( r > 0 \), \( \beta' = \frac{\beta + 1}{2} \) and \( \beta \in (0,1) \). It is also assumed that \( ||\xi||_{C^2(\overline{O}_r)} \leq K_7 \), where \( K_7 > 0 \) is a constant independent of \( \varepsilon \).

**Lemma 3.2.** If \( \phi := \xi \psi_\varepsilon(|D^1 u^\varepsilon|^2 - g^2) \) on \( \overline{O}_r \), then

\[
\text{tr}[a D^2 \phi] \geq \psi_\varepsilon'(\cdot)[\theta \xi |D^2 u^\varepsilon|^2 - C_5 |D^2 u^\varepsilon| - C_5] - \xi \psi_\varepsilon'(\cdot) |D^1 u^\varepsilon|^2 + 2\psi_\varepsilon'(\cdot)\langle D^1 \phi, D^1 u^\varepsilon \rangle, \quad \text{on } B_{\beta' r},
\]

for some positive constant \( C_5 \) independent of \( \varepsilon \), where \( \theta \) is given in (A3).

**Proof.** Let \( \phi \) be as in Lemma 3.2. First and second derivatives of \( \phi \) on \( B_{\beta' r} \subset O \), are given by

\[
\begin{align*}
\partial_\phi &= \psi_\varepsilon(\cdot)\partial_\xi + \psi_\varepsilon(\cdot)[2\langle D^1 u, D^1 \partial_\xi u \rangle - \partial_\xi [g^2]], \\
\partial_{ij} \phi &= \psi_\varepsilon(\cdot)\partial_\xi \partial_j + \psi_\varepsilon(\cdot)[\partial_\xi \partial_j [D^1 u^2] - g^2] + \partial_\xi \partial_j |D^1 u|^2 - g^2] + \xi \psi_\varepsilon'(\cdot)[2\langle D^1 \partial_i u, D^1 \partial_j u \rangle + \langle D^1 u, D^1 \partial_j u \rangle - \partial_{ij} [g^2]] + 4\xi \psi_\varepsilon''(\cdot)\bar{\eta}_i \bar{\eta}_j,
\end{align*}
\]

where \( \bar{\eta} = (\bar{\eta}_1, \ldots, \bar{\eta}_d) \) with \( \bar{\eta}_i := \langle D^1 u, D^1 \partial_i u \rangle - \frac{\partial_i [g^2]}{2} \). Then,

\[
\begin{align*}
\text{tr}[a D^2 \phi] &= 2\psi_\varepsilon(\cdot) \sum_k \partial_k u \text{tr}[a D^2 \partial_k u] + 4\xi \psi_\varepsilon''(\cdot)\langle a \bar{\eta}, \bar{\eta} \rangle + \psi_\varepsilon(\cdot) \text{tr}[a D^2 \xi] \\
&\quad + \xi \psi_\varepsilon'(\cdot) \left[ 2 \sum_{ij} a_{ij} \langle D^1 \partial_i u, D^1 \partial_j u \rangle - \text{tr}[a D^2 [g^2]] \right] \\
&\quad + 2\psi_\varepsilon'(\cdot) \left[ 2 \sum_{ij} a_{ij} \partial_i \xi \langle D^1 u, D^1 \partial_j u \rangle - \langle a D^1 \xi, D^1 [g^2] \rangle \right].
\end{align*}
\]
Using (2.11), (2.12) and (3.2), it can be verified
\[ 2\xi \psi'_\epsilon(\cdot) \sum_k \partial_k u \text{tr}[a D^2 \partial_k u] = \psi'_\epsilon(\cdot) [\xi \tilde{D} u + 2(D^1 \varphi, D^1 u) - 2\psi_\epsilon(\cdot)(D^1 \varphi, D^1 u)] \]
\[ + \xi \psi'_\epsilon(\cdot) \left[ \sum_k [\partial_k u, \partial_k u]_\epsilon - I \left| D^1 u \right|^2 \right], \quad (3.4) \]
with \( \tilde{D} u = -2 \sum_k \partial_k u \text{tr}[[\partial_k a] D^2 u + \tilde{I}_k u] - 2(D^1 u, D^1 [h - \langle b, D^1 u \rangle - cu]) \). Observe that
\[ \xi \tilde{D} u \geq -2d\Lambda C_3 [d[d + 1]|D^2 u| + 2dC_3 + C_1 + 1] - 2dK_2 C_\nu C_3 [C_3 + 2C_1], \quad (3.5) \]
where \( \Lambda, C_\nu, K_2, C_1, C_3, K_7 \) are as in (A1), (A4), (2.3), Lemmas 2.6, 2.8, and Remark 3.1, respectively. By (1.3) and (1.8), it yields \( \psi_\epsilon(\cdot) = h + \text{tr}[a D^2 u] - \langle b, D^1 u \rangle - cu + I u \) on \( O \). From here, we see that
\[ \psi_\epsilon(\cdot) \leq d^2 \Lambda |D^2 u| + \Lambda [1 + dC_3] + C_\nu [C_3 + 2C_1]. \quad (3.6) \]
Then,
\[ -2\psi_\epsilon(\cdot)(D^1 \varphi, D^1 u) \geq -2dC_3 K_7 [d^2 \Lambda |D^2 u| + \Lambda [1 + dC_3] + C_\nu [C_3 + 2C_1]]. \quad (3.7) \]
Since \( [\partial_k u, \partial_k u]_\epsilon \geq 0 \), we get
\[ \xi \psi'_\epsilon(\cdot) \left[ \sum_k [\partial_k u, \partial_k u]_\epsilon - I \left| D^1 u \right|^2 \right] \geq -\xi \psi'_\epsilon(\cdot) I \left| D^1 u \right|^2, \quad (3.8) \]
Meanwhile, by (A1) and (A3), it follows that
\[ 4\xi \psi''_\epsilon(\cdot) \langle a\bar{\eta}, \bar{\eta} \rangle \geq 4\xi \psi''_\epsilon(\cdot) |\eta|^2 \geq 0, \]
\[ \xi \psi'_\epsilon(\cdot) \left[ 2 \sum_k \langle a D^1 \partial_k u, D^1 \partial_k u \rangle - \text{tr}[a D^2 |g^2|] \right] \geq \psi'_\epsilon(\cdot) |\xi \theta| D^2 u^2 - 4d^2 \Lambda^3]. \quad (3.9) \]
Since \( \psi_\epsilon(r) \leq \psi'_\epsilon(r) r \), for all \( r \in \mathbb{R} \), and from Lemmas 2.6, 2.8 and Remark 3.1, we get
\[ \psi_\epsilon(\cdot) \text{tr}[a D^2 \xi] \geq -\psi_\epsilon(\cdot) d^2 \Lambda K_7 \geq -\psi'_\epsilon(\cdot) d^2 K_7 \Lambda C_3^2, \]
\[ 2\psi'_\epsilon(\cdot) \left[ \sum_{ij} a_{ij} \partial_i \xi \langle D^1 u, D^1 \partial_j u \rangle - \langle a D^1 \xi, D^1 |g^2| \rangle \right] \geq -4\psi'_\epsilon(\cdot) d^3 \Lambda K_7 [C_3 |D^2 u| + \Lambda^2]. \quad (3.10) \]
Applying (3.4), (3.5), (3.7)–(3.10) in (3.3), we obtain the inequality given in (3.1), with
\[ C_5 = C_5(d, \Lambda, \nu, s, \alpha', K_7). \]

**Lemma 3.3.** Let \( \phi \) be as in Lemma 3.2. There exists a positive constant \( C_6 \) independent of \( \epsilon \) such that \( \phi \leq C_6 \) on \( B_{\beta r} \subset O \), for \( r > 0 \) and \( \beta' \) as in Remark 3.1.

**Proof.** Taking \( x^* \in \overline{B}_{\beta r} \) as a point where \( \phi \) attains its maximum on \( B_{\beta r} \), it suffices to bound \( \phi(x^*) \) by a constant independent of \( \epsilon \). If \( x^* \in \partial B_{\beta r} \), then \( \phi(x) \leq \phi(x^*) = 0 \). Let \( x^* \) be in \( B_{\beta r} \). Observe, if \( |D^1 u(x^*)|^2 - g(x^*)^2 < 2\epsilon \), from (1.9), we see that \( \phi(x) \leq \phi(x^*) = 0 \).
\( \xi(x^*)\psi_\varepsilon(\|D^1 u(x^*)\|^2 - g(x^*)^2) \leq 1 \) on \( B_{3r} \). Therefore, we obtain the result of Lemma 3.2. Assume that \( \|D^1 u(x^*)\|^2 - g(x^*)^2 \geq 2\varepsilon \). Since \( x^* \in B_{3r} \), we know that

\[
D^1 \phi(x^*) = 0, \quad \text{tr}[a(x^*) D^2 \phi(x^*)] \leq 0,
\]

\[
\phi(x^* + z) - \phi(x^*) \leq 0, \quad \text{for z with } x^* + z \in \overline{O}_I.
\]

Then, evaluating \( x^* \) in (3.1) and by \( \psi_\varepsilon'(\cdot) = 1/\varepsilon \) at \( x^* \); see (1.9), we get

\[
0 \geq \frac{1}{\varepsilon}[\theta \xi |D^2 u|^2 - C_5 |D^2 |u|^2 - C_5] - \frac{\xi}{\varepsilon} I |D^1 u|^2, \quad \text{at } x^*,
\]

where \( C_5 \) as in Lemma 3.2. Meanwhile, note that

\[
- \frac{\xi}{\varepsilon} I |D^1 u|^2 = - \frac{1}{\varepsilon} [\xi I_B |D^1 u|^2 + \xi I_{B^c} |D^1 u|^2], \quad \text{at } x^*,
\]

where \( I_C |D^1 u|^2 := \int_C |D^1 u(\cdot + z)|^2 - |D^1 u|^2 s(\cdot, z)\nu(dz) \), with \( C \subseteq \mathbb{R}^d_+ \), and \( B := \{ z \in \mathbb{R}^d : |D^1 u(\cdot + z)|^2 - [g(\cdot + z)]^2 \leq |D^1 u|^2 - g^2 \}, \quad \text{at } x^* \}. \) By Remark 2.1 and Lemma 2.2.i, the operator \( I_C \) is well defined. Now, from (A1), (A4) and Mean Value Theorem, it yields

\[
- \frac{\xi}{\varepsilon} I_B |D^1 u|^2 \geq \frac{1}{\varepsilon} \left[ -I_{B^c} [\xi |D^1 u|^2] + |D^1 u|^2 I_{B^c} \xi + \sum_i [\xi, [\partial_i u]|I_B^c \right], \quad \text{at } x^*,
\]

with \( \Lambda, C_\nu \) as in (A1), (1.4), respectively. By Lemma 2.2.ii, it follows that

\[
- \frac{\xi}{\varepsilon} I_{B^c} |D^1 u|^2 = \frac{1}{\varepsilon} \left[ -I_{B^c} [\xi |D^1 u|^2] + |D^1 u|^2 I_{B^c} \xi + \sum_i [\xi, [\partial_i u]|I_{B^c} \right], \quad \text{at } x^*,
\]

where \( [\xi, [\partial_i u]|I_{B^c} := \int_{B^c} [\xi(\cdot + z) - \xi][[\partial_i u(\cdot + z)]^2 - [\partial_i u]^2 s(\cdot, z)\nu(dz) \). Proceeding in a similar way that in (3.14), and using Lemma 2.8 and Remark 3.1, it is easy to verify that

\[
|D^1 u|^2 I_{B^c} \xi + \sum_i [\xi, [\partial_i u]|I_{B^c} \geq -3K_T C_\nu C_3^2 [2 + d^2], \quad \text{at } x^*.
\]

If \( z \in B^c \), then, from (1.9) and (3.11), it can be verified that

\[
\frac{1}{\varepsilon} [\xi(\cdot + z)|D^1 u(\cdot + z)|^2 - \xi |D^2 u|^2 \leq \frac{1}{\varepsilon} [\xi(\cdot + z)g(\cdot + z)]^2 - \xi g^2 - [\xi(\cdot + z) - \xi, \quad \text{at } x^*.
\]

Then, proceeding as before,

\[
- \frac{1}{\varepsilon} I_{B^c} [\xi |D^1 u|^2 \geq - \frac{\Lambda^2 K_T C_\nu}{\varepsilon} [2 + 3d^2] - K_T C_\nu [2 + d^2], \quad \text{at } x^*.
\]

Applying (3.13)–(3.17) in (3.12), we obtain

\[
0 \geq \theta \xi |D^2 u|^2 - K_8 |D^2 u| - K_8 [2 + \varepsilon] \quad \text{at } x^*,
\]

for some \( K_8 = K_8(d, \Lambda, \nu, s, \alpha') \). Then, \( |D^2 u(x^*)| - K_9 |D^2 u(x^*)| - K_{10} \leq 0 \), where
\[ K_9 := \frac{K_8 + K_7 + 40K_5 \xi(x^*)|2+\varepsilon|}{2K_9(x^*)^{1/2}} \quad \text{and} \quad K_{10} := \frac{K_8 - [K_7 + 40K_5 \xi(x^*)|2+\varepsilon|/2]}{2K_9(x^*)^{1/2}}. \] Notice that \( K_{10} < 0 < K_9. \) This implies that \(|D^2 u(x^*)| \leq \frac{K_8 + [K_7 + 12\theta K_9]^{1/2}}{2\theta}.\) From here and (3.6),

\[
\phi(x) = \xi(x)\psi_\varepsilon(\|D^1 u\varepsilon(x^*)\|^2 - g(x)^2) 
\leq \xi(x^*)[d^2 \Lambda |D^2 u(x^*)| + K_{11}] \leq \frac{d^2 \Lambda [K_8 + [K_7 + 12\theta K_9]^{1/2}]}{2\theta} + K_{11},
\]

with \( K_{11} := \Lambda[1 + dC_3] + C_\varepsilon[C_3 + 2C_1]. \) We conclude that \( \phi \leq C_6 \) on \( B_{\beta r} \) with some constant \( C_6 = C_6(d, \Lambda, \nu, s, \alpha', \beta). \)

From Lemmas 2.6, 2.8, 3.3, the following estimate is obtained in \( L^p_{\text{loc}}(O). \)

**Lemma 3.4.** Let \( p \in (1, \infty). \) There exists a positive constant \( C_7 \) independent of \( \varepsilon \) such that \( \|D^2 u\|_{L^p(B_{\beta r})} \leq C_7, \) for \( \beta \in (0, 1] \) and \( r > 0. \)

**Proof.** Taking \( w = \xi u, \) we obtain \( \|D^2 w\|_{L^p(B_{\beta r})} \leq \|D^2 w\|_{L^p(B_{\beta r})}, \) with \( B_{\beta r} \subset O, \) \( p \in (1, \infty), \) \( r > 0 \) and \( \beta' \) as in Remark 3.1. By calculating the first and second derivatives of \( w \) in \( B_{\beta r}, \) \( \partial_i w = u\partial_i \xi + \xi \partial_i u, \) \( \partial_{ij} w = \partial_{ji} u\partial_j \xi + \partial_i u\partial_j \xi + \partial_j u\partial_i \xi + \partial_{i}w\partial_j \xi + \xi \partial_{ij} u, \) and from (1.8), we get

\[
Lw = f, \quad \text{in} \ B_{\beta r}, \quad \text{s.t.} \ \ w = 0, \ \text{on} \ \partial B_{\beta r},
\]

where \( f := \xi[h + Lu - \psi_\varepsilon(\|D^1 u\varepsilon\|^2 - g^2)] - u[\text{tr}[a D^2 \xi] - (D^1 \xi, b)] \leq 2(a D^1 \xi, D^1 u). \) We know that for the linear Dirichlet problem (3.18) (see [23, Lemma 3.1]), \( \|D^2 w\|_{L^p(B_{\beta r})} \leq K_{12}[\|f\|_{L^p(B_{\beta r})}] \) for some \( K_{12} = K_{12}(d, \Lambda, p, \beta', r). \) Estimating the terms of \( f \) with the norm \( \|\cdot\|_{L^p(B_{\beta r})} \) and using (A1)–(A4) and Lemmas 2.6, 2.8, 3.3, it follows that there exists \( C_7 = C_7(d, \Lambda, \nu, s, \alpha', p, \beta, r) \) such that \( \|D^2 u\|_{L^p(B_{\beta r})} \leq \|D^2 w\|_{L^p(B_{\beta r})} \leq C_7. \)

By Lemmas 2.6, 2.8, 3.3 and 3.4, the following result can be easily verified, and the proof is omitted.

**Lemma 3.5.** Let \( p \in (1, \infty). \) There exists a positive constant \( C_8 \) independent of \( \varepsilon \) such that \( \|u\varepsilon\|_{W^{2,p}(B_{\beta r})} \leq C_8, \) for \( \beta \in (0, 1] \) and \( r > 0. \)

### 3.2 Proof of Theorem 1.1

This subsection is devoted to proving Theorem 1.1. Let \( p \in (1, \infty) \) be fixed, by Lemmas 2.6, 2.8, and 3.3–3.5, we obtain that for each open ball \( B_{\beta r} \subset O, \) \( \beta \in (0, 1] \) and \( r > 0, \) there exist positive constants \( C_9, C_{10} \) independent of \( \varepsilon \) such that

\[
\|u\varepsilon\|_{W^{1,\infty}(O)} < C_9 \quad \text{and} \quad \|u\varepsilon\|_{W^{2,p}(B_{\beta r})} < C_{10}.
\]

Taking \( p \in (d, \infty) \) fixed, from (3.19) and the Sobolev embedding Theorem, we have that for each open ball \( B_{\beta r} \subset O, \) there exists a positive constant \( C_{11} \) independent of \( \varepsilon \) such that

\[
\|u\varepsilon\|_{C^{1,\alpha}(B_{\beta r})} \leq C_{11}, \quad \text{with} \ \alpha = 1 - \frac{d}{p}.
\]
Using Arzelà-Ascoli Theorem, the reflexivity of $L^p_{\text{loc}}(\mathcal{O})$; see [30, 1, Thm. 7.25, p. 158 and Thm. 2.46, p. 49, respectively], and (3.19)–(3.20), we get that there exists a subsequence $\{u^\varepsilon\}_{\varepsilon \geq 1}$ of $\{u^\varepsilon\}_{\varepsilon \in (0,1)}$, and $\tilde{u} \in C^{0,1}(\overline{\mathcal{O}}) \cap W^{2,p}_{\text{loc}}(\mathcal{O})$ such that $u^\varepsilon \overset{\varepsilon \to 0}{\longrightarrow} \tilde{u}$ in $C(\overline{\mathcal{O}})$, $\partial_i u^\varepsilon \overset{\varepsilon \to 0}{\longrightarrow} \partial_i \tilde{u}$ in $C_{\text{loc}}(\mathcal{O})$, $\partial_{ij} u^\varepsilon \overset{\varepsilon \to 0}{\longrightarrow} \partial_{ij} \tilde{u}$, weakly in $L^p_{\text{loc}}(\mathcal{O})$. Now, define
\[
\begin{align*}
\hspace{1cm} u(x) := \begin{cases} 
\tilde{u}(x), & \text{if } x \in \mathcal{O}, \\
0, & \text{if } x \in \overline{\mathcal{O}} \setminus \mathcal{O}.
\end{cases}
\end{align*}
\]
It can be verified that $u$ is a continuous function on $\overline{\mathcal{O}}_I$, which satisfies $u \in C^{0,1}(\overline{\mathcal{O}}) \cap W^{2,p}_{\text{loc}}(\mathcal{O})$ and
\[
\begin{align*}
\hspace{1cm} u^\varepsilon & \overset{\varepsilon \to 0}{\longrightarrow} u, \text{ in } C(\overline{\mathcal{O}}_I), \\
\partial_i u^\varepsilon & \overset{\varepsilon \to 0}{\longrightarrow} \partial_i u, \text{ in } C_{\text{loc}}(\mathcal{O}), \\
\partial_{ij} u^\varepsilon & \overset{\varepsilon \to 0}{\longrightarrow} \partial_{ij} u, \text{ weakly in } L^p_{\text{loc}}(\mathcal{O}).
\end{align*}
\]
(3.21)

Since (1.8) and (3.21) hold, we only need to verify (3.22). Hence, we can conclude that the limit function $u$ is the solution to the HJB equation (1.1).

**Lemma 3.6.** Let $\{u^\varepsilon\}_{\varepsilon \geq 0}$ and $u$ be the subsequence and the limit function that satisfy (3.21). Then,
\[
\int_{B_r} \varsigma \mathcal{I} u^\varepsilon \, dx \overset{\varepsilon \to 0}{\longrightarrow} \int_{B_r} \varsigma \mathcal{I} u \, dx, \text{ for any } \varsigma \in C^\infty_c(\mathcal{O}) \text{ with } \text{supp}[\varsigma] \subset B_r \subset \mathcal{O}.
\]
(3.22)

**Proof.** Let $\varsigma \in C^\infty_c(\mathcal{O})$ and $\text{supp}[\varsigma] \subset B_r \subset \mathcal{O}$, with $0 < r_0 < \text{dist} (\text{supp}[\varsigma], \partial B_r) \land 1$. Then,
\[
\begin{align*}
\hspace{1cm} & \left| \int_{B_r} \varsigma(x) \mathcal{I}[u^\varepsilon - u](x) \, dx \right| \\
\leq & \int_{\text{supp}[\varsigma]} |\varsigma(x)| \int_{\{ |z| < r_0 \}} \int_0^1 |D^1(u^\varepsilon - u)(x + tz)| \, dt \, |z| s(x, z) \nu(\,dz\,) \, dx \\
& + \frac{1}{r_0} \int_{\text{supp}[\varsigma]} |\varsigma(x)| \int_{\{|z| \geq 1\}} |[u^\varepsilon - u](x + z) - [u^\varepsilon - u](x)| \, |z| s(x, z) \nu(\,dz\,) \, dx \\
& + \int_{\text{supp}[\varsigma]} |\varsigma(x)| \int_{\{|z| \geq 1\}}|[u^\varepsilon - u](x + z) - [u^\varepsilon - u](x)| \, |z| s(x, z) \nu(\,dz\,) \, dx \\
\leq & C_\nu ||\varsigma||_{L^1(B_r)} \left[ ||D^1[u^\varepsilon - u]||_{C(B_r)} + 2 \left( \frac{1}{r_0} + 1 \right) ||u^\varepsilon - u||_{C(\overline{\mathcal{O}})} \right].
\end{align*}
\]
(3.23)

From (3.21) and letting $\varepsilon \to 0$ in (3.23), it follows that (3.22). With that, we finish the proof.

We proceed to show the existence and uniqueness of the solution to the HJB equation (1.1).

**Proof of Theorem 1.1.** Existence. Let $p \in (d, \infty)$ be fixed, $\{u^\varepsilon\}_{\varepsilon \geq 0}$ and $u$ be the subsequence and the limit function, respectively, that satisfy (3.21) and (3.22). Recall that $u \in C^{0,1}(\overline{\mathcal{O}}) \cap W^{2,p}_{\text{loc}}(\mathcal{O})$ and $u^\varepsilon \in C^{3,\alpha}(\overline{\mathcal{O}})$ is the unique solution to the NPIDD problem (1.8) when $\varepsilon = \varepsilon_\kappa$. Then, (1.8), (3.21) and (3.22) imply $\int_{B_r} \varsigma \mathcal{I} u \, dx \leq \int_{B_r} \varsigma h \, dx$ for each
non-negative function $\zeta$ in $C^\infty_c(B_r)$, where $\text{supp}[\zeta] \subset B_r \subset \mathcal{O}$. From here, it follows that $\Gamma u \leq h$ a.e. in $\mathcal{O}$. Meanwhile, since $\psi_\varepsilon([|D^1 u|] + g^2)$ is locally bounded (uniformly in $\varepsilon$); see Lemma 3.3, it follows that for each $x \in \mathcal{O}$, there exists an $\varepsilon'$ such that for all $\varepsilon \leq \varepsilon'$, $|D^1 u(x)| \leq g(x)$. From here and that $\lim_{\varepsilon \to 0} |D^1 u(x)| = 0$, it yields $|D^1 u| \leq g$ in $\mathcal{O}$. Suppose that $|D^1 u(x^*)| < g(x^*)$, for some $x^* \in \mathcal{O}$. Then, by the continuity of $D^1 u$, there exists a small open ball $B_r \subset \mathcal{O}$ such that $x^* \in B_r$ and $|D^1 u(x)| < g(x)$ in $B_r$. Since $\lim_{\varepsilon \to 0} |D^1 u(x)| = 0$, we obtain that there exists $\varepsilon_{\kappa_0}$ such that for each $\varepsilon \leq \varepsilon_{\kappa_0}$, $|D^1 u(x)| < g(x)$ in $B_r$. Then, from (1.8) and the definition of $\psi_\varepsilon$, it follows that for each $\varepsilon \leq \varepsilon_{\kappa_0}$, $\Gamma u = h$ in $B_r$. Then, $\int_{B_r} \Gamma u \zeta \, dx = \int_{B_r} h \zeta \, dx$, for any non-negative function $\zeta \in C^\infty_c(B_r)$, with $\text{supp}[\zeta] \subset B_r \subset \mathcal{O}$. From here and using (3.21)-(3.22), we get $\int_{B_r} \zeta \Gamma u \, dx = \int_{B_r} \zeta h \, dx$. Therefore, $\Gamma u = h$, a.e. in $B_r$. By the arguments seen previously, we conclude that $u$ is a solution to the HJB equation (1.1) a.e. in $\mathcal{O}$.

Proof of Theorem 1.1. Uniqueness. Let $p \in (d, \infty)$ be fixed. Suppose there exist $u_1, u_2 \in C^{0,1} \cap W^{2,p}_{\text{loc}}(\mathcal{O})$, two solutions to the HJB equation (1.1). Let $x^* \in \overline{\mathcal{O}}$ be the point where $u_1 - u_2$ attains its maximum. If $x^* \in \partial \mathcal{O}$, it is easy to see $[u_1 - u_2](x) \leq [u_1 - u_2](x^*) = 0$ for all $x \in \overline{\mathcal{O}}$. Let us assume that $x^* \in \mathcal{O}$. In this case one wishes to prove that $[u_1 - u_2](x^*) = 0$, which we demonstrate by contradiction. Suppose $[u_1 - u_2](x^*) > 0$ and take $f := [1 - \rho] u_1 - u_2$ on $\mathcal{O}$ such that $f(x^*) > 0$, for some $\rho > 0$ small enough. Using $f = 0$ on $\partial \mathcal{O} \setminus \mathcal{O}$, it follows that $f(x^*) > 0$, where $x^* \subset \mathcal{O}$ is the point where $f$ attains its maximum. Besides, we have $D^1 f(x^*) = [1 - \rho] D^1 u_1(x^*) - D^1 u_2(x^*) = 0$ and $f(x^*) \leq f(x^*)$, for $z \in \mathbb{R}^d$ with $x^* + z \in \overline{\mathcal{O}}$. Then, $\mathcal{I} f(x^*) \leq 0$. Since $D^1 f(x^*) = 0$, $|D^1 u_2(x^*)| \leq |g(x^*)|$ and $1 - \rho < 1$, we get $|D^1 u_2(x^*)| = [1 - \rho] |D^1 u_1(x^*)| < g(x^*)$. This implies that there exists $\mathcal{V}_{x^*}$ a neighborhood of $x^*$ such that $\Gamma u_2 = h$ and $\Gamma u_1 \leq h$ in $\mathcal{V}_{x^*}$. Then, $\mathcal{I} f \leq -\rho h$ in $\mathcal{V}_{x^*}$, and hence $\text{tr}[a D^2 f] \geq \langle b, D^1 f \rangle + cf - \mathcal{I} f + \rho h$, in $\mathcal{V}_{x^*}$. By using Bony’s maximum principle (see [24]), it yields $0 \geq \liminf_{x \to x^*} \text{tr}[a(x) D^2 f(x)] \geq c(x^*) f(x^*) - \mathcal{I} f(x^*) + \rho h(x^*)$, which is a contradiction since $c(x^*) f(x^*) > 0$, $-\mathcal{I} f(x^*) \geq 0$ and $\rho h(x^*) \geq 0$. The application of Bony’s maximum principle is permitted here because $u_1, u_2 \in W^{2,p}_{\text{loc}}(\mathcal{O})$ and $d < p < \infty$. Therefore, it yields $[u_1 - u_2](x) \leq [u_1 - u_2](x^*) \leq 0$ for all $x \in \overline{\mathcal{O}}$. Taking $u_2 - u_1$ and proceeding in the same way as before, it follows that $u_2 - u_1 \leq 0$ in $\mathcal{O}$, and hence we conclude that the solution $u$ to the HJB equation (1.1) is unique.

4 Penalized control problem and proof of Proposition 1.5

This section is devoted to verifying that the value function $V$ and $u$ agree on $\mathcal{O}$, which are the value function defined in (1.19) and the solution to the HJB equation (1.20), respectively. For this purpose, we introduce a class of penalized controls that belong to $\mathcal{U}$. Recall that $\mathcal{U}$ is the set of admissible controls $(n, \zeta)$ that satisfy (1.15). Take the penalized controls set $\mathcal{U}_\varepsilon$ by $\mathcal{U}_\varepsilon := \{(n, \zeta) \in \mathcal{U} : \zeta_\varepsilon \text{ is absolutely continuous, } 0 \leq \zeta_\varepsilon \leq 2C_3/\varepsilon\}$, with $\varepsilon \in (0, 1)$ fixed.
where $C_3$ is a positive constant as in Lemma 2.8, which is independent of $\varepsilon$. Then, for each $(n, \zeta) \in \mathcal{U}^{\varepsilon}$ and $\tilde{x} \in \mathcal{O}$, the process $X^{n,\zeta} = \{X^{n,\zeta}_t : t \geq 0\}$ evolves in the following way

$$X^{n,\zeta}_t = \tilde{x} - \int_0^t [\tilde{b}(X^{n,\zeta}_s) + n_s \tilde{\zeta}_s]ds + \int_0^t \sigma(X^{n,\zeta}_s)dW_s + \int_0^t dJ_s, \quad \text{with } t \geq 0,$$

where $W$ is a $d$-dimensional SBM as in Subsection 1.1 and $J$ is the jump process given by (1.11). Notice that $\Delta X^{n,\zeta} = \Delta J$. Recall that here $\tilde{b} : \mathbb{R}^d \rightarrow \mathbb{R}^d$, $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$, and $s : \mathcal{O} \times \mathbb{R}^d \rightarrow [0, 1]$ satisfy (A1)–(A5). Then, the SDE (4.1) has a unique càdlàg adapted solution $X^{n,\zeta}$; see [8]. The penalized cost related to this class of controls is defined by

$$V_{n,\zeta}(\tilde{x}) = \mathbb{E}_\tilde{x} \left[ \int_{t_{\varepsilon}}^{\tau^{n,\zeta}} e^{-qs}[h(X^{n,\zeta}_s) + l_\varepsilon(X^{n,\zeta}_s, \tilde{\zeta}_s n_s)]ds \right], \quad \text{for } (n, \zeta) \in \mathcal{U}^{\varepsilon},$$

where $\tau^{n,\zeta} := \inf\{t > 0 : X^{n,\zeta}_t \notin \mathcal{O}\}$, $h : \mathbb{R}^d \rightarrow \mathbb{R}$ is continuous and non-negative, and $l_\varepsilon(x, y) := \sup_{\gamma \in \mathbb{R}^d}\{(\gamma, y) - H_\varepsilon(x, \gamma)\}$ is the Legendre transform of $H_\varepsilon(x, \gamma) := \psi_\varepsilon(|\gamma|^2 - g(x)^2)$, where $g : \mathbb{R}^d \rightarrow \mathbb{R}$ is continuous and non-negative. Notice that, for each $x \in \mathbb{R}^d$ fixed, $H_\varepsilon(x, \gamma)$ is a $C^2$ and convex function with respect to the variable $\gamma \in \mathbb{R}^d$, since $\psi_\varepsilon \in C^\infty(\mathbb{R})$ is convex function; see (1.9). The value function for this problem is given by

$$V_\varepsilon(\tilde{x}) := \inf_{(n, \zeta) \in \mathcal{U}^{\varepsilon}} V_{n,\zeta}(\tilde{x}).$$

A heuristic derivation from dynamic programming principle (see [11, Ch. VIII]) shows that the NPIDD problem corresponding to the value function $V_\varepsilon$ is of the form

$$[q - \Gamma_1]u_\varepsilon + \sup_{y \in \mathbb{R}^d}\{(\mathbf{D}^1 u_\varepsilon, y) - l_\varepsilon(\cdot, y)\} = h, \quad \text{in } \mathcal{O}, \quad \text{s.t. } u_\varepsilon = 0, \quad \text{on } \overline{\mathcal{O}_T} \setminus \mathcal{O},$$

where $\Gamma_1$ is as in (1.14). Since $H_\varepsilon(x, \gamma)$ is $C^2$ with respect to the variable $\gamma$, it follows that $H_\varepsilon(x, \gamma) = \sup_{y \in \mathbb{R}^d}\{(\gamma, y) - l_\varepsilon(x, y)\}$. Then, the NPIDD problem (4.3) can be written as

$$[q - \Gamma_1]u_\varepsilon + \psi_\varepsilon(|\mathbf{D}^1 u_\varepsilon|^2 - g^2) = h, \quad \text{in } \mathcal{O}, \quad \text{s.t. } u_\varepsilon = 0, \quad \text{on } \overline{\mathcal{O}_T} \setminus \mathcal{O}.$$

Assuming from now on that $a_{ij} = \frac{1}{2}(\sigma\sigma^T)_{ij}$, $b_i$, $h$, $g$, $s$ satisfy (A1)–(A5), an immediate consequence of Proposition 1.2 is the following corollary.

**Corollary 4.1.** The NPIDD problem (4.4) has a unique non-negative solution $u_\varepsilon$ in $C^{3,\alpha'}(\overline{\mathcal{O}})$, for each $\varepsilon \in (0, 1)$.

**Remark 4.2.** Without loss of generality we assume that $\psi_\varepsilon$ is non-decreasing as $\varepsilon \downarrow 0$; see [34].

**Corollary 4.3.** Let $u_\varepsilon$ be the unique non-negative solution to the NPIDD problem, for each $\varepsilon \in (0, 1)$. Then, $u_\varepsilon$ is non-increasing as $\varepsilon \downarrow 0$.

**Proof.** Let $u^{\varepsilon_1}$, $u^{\varepsilon_2}$ be the unique solutions to the NPIDD problem (1.8) when $\varepsilon = \varepsilon_1, \varepsilon_2$, respectively, with $\varepsilon_2 \leq \varepsilon_1$. Since $\psi_{\varepsilon_2} \geq \psi_{\varepsilon_1}$ and $u^{\varepsilon_2}$ is the unique solution to (1.8) when $\varepsilon = \varepsilon_2$, we see that

$$[q - \Gamma_1]u^{\varepsilon_2} + \psi_{\varepsilon_1}(|\mathbf{D}^1 u^{\varepsilon_2}|^2 - g^2) \leq h, \quad \text{in } \mathcal{O}, \quad \text{s.t. } u^{\varepsilon_2} = 0, \quad \text{on } \overline{\mathcal{O}_T} \setminus \mathcal{O}.$$

From Lemma 2.5, it follows that $u^{\varepsilon_2} \leq u^{\varepsilon_1}$ on $\overline{\mathcal{O}}$. Therefore, $u_\varepsilon$ is non-increasing as $\varepsilon \downarrow 0$. □
Now we construct our optimal control candidate \((n^{\xi,*}, \zeta^{\xi,*})\) for the problem (4.2). Consider the following SDE
\[
X^{\xi,*}_{t,\xi} = \bar{x} + \int_0^t \sigma(X^{\xi,*}_s) dW_s + \int_0^t \sigma(J_s) dJ_s \\
- \int_0^t \left[ \tilde{b}(X^{\xi,*}_s) + 2\psi_\epsilon'(|D^1 u^\epsilon(X^{\xi,*}_s)|^2 - g(X^{\xi,*}_s)^2) D^1 u^\epsilon(X^{\xi,*}_s) \right] ds,
\]
with \(\bar{x} \in \mathcal{O}, \ t \geq 0\) and \(\tau^*_\epsilon = \inf\{t > 0 : X^{\xi,*}_t \notin \mathcal{O}\}\). Observe that \(\psi_\epsilon'(|D^1 u^\epsilon|^2 - g^2) D^1 u^\epsilon\) satisfies (1.12), since it is a bounded Lipschitz continuous function on \(\overline{\mathcal{O}}\). Then, the SDE (4.5) has a unique càdlàg adapted solution \(X^{\xi,*};\) see [22]. Defining the control process \((n^{\xi,*}, \zeta^{\xi,*})\) by
\[
n^{\xi,*}_t = \begin{cases} \\
\frac{D^1 u^\epsilon(X^{\xi,*}_t)}{|D^1 u^\epsilon(X^{\xi,*}_t)|}, & \text{if } |D^1 u^\epsilon(X^{\xi,*}_t)| \neq 0, \\
\gamma_0, & \text{if } |D^1 u^\epsilon(X^{\xi,*}_t)| = 0,
\end{cases}
\]
with \(\gamma_0 \in \mathbb{R}^d\) a unit vector fixed, and \(\zeta^{\xi,*}_t = \int_0^t \zeta^{\xi,*}_s ds\), with
\[
\dot{\zeta}^{\xi,*}_t = 2\psi_\epsilon'(|D^1 u^\epsilon(X^{\xi,*}_t)|^2 - g(X^{\xi,*}_t)^2) D^1 u^\epsilon(X^{\xi,*}_t),
\]
we see that for \(t \in [0, \tau^*_\epsilon]\), \(n^{\xi,*}_t \zeta^{\xi,*}_t = 2\psi_\epsilon'(|D^1 u^\epsilon(X^{\xi,*}_t)|^2 - g(X^{\xi,*}_t)^2) D^1 u^\epsilon(X^{\xi,*}_t)\), \(\Delta \zeta^{\xi,*}_t = 0\), \(|n^{\xi,*}_t| = 1\) and, by (1.9) and Lemma 2.8, \(\dot{\zeta}^{\xi,*}_t \leq 2\chi a_\epsilon\). On the event \(\{\tau^*_\epsilon = \infty\}\), the control process \((n^{\xi,*}, \zeta^{\xi,*})\) belongs to \(U^\epsilon\). On the event \(\{\tau^*_\epsilon < \infty\}\), since \(u^\epsilon \in C^{3,a_\epsilon}(\overline{\mathcal{O}})\), \(u^\epsilon = 0\) on \(\overline{\mathcal{O}} \setminus \mathcal{O}\) and \(X^{\xi,*}_{\tau^*_\epsilon} \in \overline{\mathcal{O}} \setminus \mathcal{O}\) and \(\zeta^{\xi,*}_{\tau^*_\epsilon} \equiv 0\) and \(n^{\xi,*}_{\tau^*_\epsilon} \equiv \gamma_0\), for \(t > \tau^*_\epsilon\). In this way, we have that \((n^{\xi,*}, \zeta^{\xi,*}) \in U^\epsilon\).

**Lemma 4.4** (Verification Lemma for penalized control problem). *Let \(\epsilon \in (0,1)\) be fixed. Then,*

(i) *For each \((n, \zeta) \in U^\epsilon, \ u^\epsilon \leq \mathcal{V}_{n,\zeta}^\epsilon\) on \(\overline{\mathcal{O}}\).*

(ii) *Let \(X^{\xi,*}, (n^{\xi,*}, \zeta^{\xi,*})\) be the solution process to the SDE (4.5) and the control process given by (4.6)-(4.7), respectively. Then, \(u^\epsilon = \mathcal{V}_{n^{\xi,*}, \zeta^{\xi,*}} = V^\epsilon\) on \(\overline{\mathcal{O}}\).*

From now on, for simplicity of notation, we replace \(X^{n,\zeta}\) by \(X\) in the proofs of the results.

**Proof of Lemma 4.4.** *Let \(\epsilon \in (0,1)\) be fixed, \(X = \{X_t : t \geq 0\}\) be the process which evolves as in (4.1), with \((n, \zeta) \in U^\epsilon\) and \(\bar{x} \in \mathcal{O}\) an initial state. Notice that \(u^\epsilon\) is in \(C^3(\overline{\mathcal{O}})\). Then, integration by parts and Itô’s formula imply (see [29, Cor. 2 and Thm. 33, pp. 68 and 81, respectively])
\[
\begin{align*}
\dot{u}^\epsilon(\bar{x}) &= e^{-q[t_\Lambda]} u^\epsilon(X_{t_\Lambda}) - \int_0^{t_\Lambda} e^{-q_s} (D^1 u^\epsilon(X_s), dJ_s) \\
&+ \int_0^{t_\Lambda} e^{-q_s} [q u^\epsilon(X_s) + \langle D^1 u^\epsilon(X_s), \tilde{b}(X_s) \rangle - \text{tr}[a(X_s) D^2 u^\epsilon(X_s)] ds] \\
&- \int_0^{t_\Lambda} e^{-q_s} \langle D^1 u^\epsilon(X_s), \sigma(X_s) dW_s \rangle + \int_0^{t_\Lambda} e^{-q_s} (D^1 u^\epsilon(X_s), n_s) d\zeta_s \\
&- \sum_{0 \leq s \leq t_\Lambda} e^{-q_s} [u^\epsilon(X_{s+} + \Delta X_s) - u^\epsilon(X_{s-}) - \langle D^1 u^\epsilon(X_{s-}), \Delta X_s \rangle], \quad (4.8)
\end{align*}
\]
where \( a_{ij} = \frac{1}{2}(\sigma \sigma^T)_{ij} \) and \( \tau = \inf\{ t > 0 : X_t \notin \mathcal{O} \} \). Meanwhile, since \( \Delta \zeta \equiv 0 \), it can be verified

\[
- \sum_{0 \leq s \leq t \wedge \tau} e^{-q t}[u^\varepsilon(X_s^- + \Delta X_s) - u^\varepsilon(X_s^-) - \langle D^1 u^\varepsilon(X_s^-), \Delta X_s \rangle]
\]

\[
= - \int_0^{t \wedge \tau} \int_S e^{-q t}[u^\varepsilon(X_s^- + z) - u^\varepsilon(X_s^-) - \langle D^1 u^\varepsilon(X_s^-), z \rangle] \mathbb{1}_{\{\rho \in [0,s,z]\}} N(d\rho, dz, ds)
\]

\[
= - \tilde{M}_{t \wedge \tau}^\varepsilon + \int_0^{t \wedge \tau} e^{-q t} \langle D^1 u^\varepsilon(X_s^-), dJ_s \rangle - \int_0^{t \wedge \tau} e^{-q t} \mathcal{L} u^\varepsilon(X_s) ds
\]

\[
+ \int_0^{t \wedge \tau} \int_{\{|z| \in (0,1)\}} e^{-q t} \langle D^1 u^\varepsilon(X_s), z \rangle s(X_s, z) \nu(dz) ds,
\]

where

\[
\tilde{M}_{t \wedge \tau}^\varepsilon := \int_0^{t \wedge \tau} \int_S e^{-q t}[u^\varepsilon(X_s^- + z) - u^\varepsilon(X_s^-)] \mathbb{1}_{\{\rho \in [0,s,z]\}} \tilde{N}(d\rho, dz, ds).
\]

Recall that \( \mathcal{S} = [0,1] \times \mathbb{R}^d \) and \( \tilde{N}(d\rho, dz, dt) = N(d\rho, dz, dt) - \eta(d\rho, dz) dt \) is the compensated Poisson random measure with intensity \( \eta(d\rho, dz) dt = d\rho \nu(dz) dt \). Then, from (4.8), (4.9) and noting that \( d\xi_s = \dot{\xi} ds \) and

\[
\int_0^{t \wedge \tau} e^{-q t} \langle D^1 u^\varepsilon(X_s), b(X_s) \rangle ds = \int_0^{t \wedge \tau} e^{-q t} \left[ \langle D^1 u^\varepsilon(X_s), \tilde{b}(X_s) \rangle + \int_{\{|z| \in (0,1)\}} \langle D^1 u^\varepsilon(X_s), z \rangle s(X_s, z) \nu(dz) \right] ds,
\]

it follows that

\[
u^\varepsilon(\tilde{x}) = e^{-q [t \wedge \tau]} u^\varepsilon(X_{t \wedge \tau}) - \tilde{M}_{t \wedge \tau}^\varepsilon + \int_0^{t \wedge \tau} e^{-q t}[q - \Gamma_1] u^\varepsilon(X_s) + \langle D^1 u^\varepsilon(X_s), \dot{\xi} n_s \rangle ds,
\]

with \( \Gamma_1 \) as in (1.14),

\[
\tilde{M}_{t \wedge \tau}^\varepsilon = \tilde{M}_{t \wedge \tau}^\varepsilon + \tilde{M}_{t \wedge \tau}^\varepsilon \quad \text{and} \quad \tilde{M}_{t \wedge \tau}^\varepsilon := \int_0^{t \wedge \tau} e^{-q t} \langle D^1 u^\varepsilon(X_s), \sigma(X_s) dW_s \rangle.
\]

Observe that

\[
\mathbb{E}[\tilde{f}(X_{t \wedge \tau}^- \cdot ) ] N(d\rho, dz, ds) = \int_0^t \int_S \mathbb{E}[\tilde{f}(X_{t \wedge \tau}^- \cdot ) ] N(d\rho, dz, ds).
\]

From here and taking \( \tilde{f}(X_{\tau}^-, z) = e^{-q t}[u^\varepsilon(X_{\tau}^- + z) - u^\varepsilon(X_{\tau}^-)] \mathbb{1}_{\{\rho \in [0,s(z)]\}} \), it can be verified that \( \tilde{M}^\varepsilon = \{ \tilde{M}^\varepsilon_{t \wedge \tau} : t \geq 0 \} \) is a martingale. Moreover, \( \tilde{M}^\varepsilon \) is square integrable, since

\[
\mathbb{E}[\tilde{M}_{t \wedge \tau}^\varepsilon|^2] \leq t \left[ C_3^2 \int_{\{|z| \in (0,1)\}} |z|^2 \nu(dz) + 4C_1^2 \int_{\{|z| \geq 1\}} \nu(dz) \right] < \infty, \quad \text{for } t \geq 0 \text{ fixed},
\]
where $C_1$ and $C_3$ are as in Lemmas 2.6 and 2.8, respectively. Meanwhile, Itô’s isometry and the continuity of $\sigma$ and $u^\varepsilon$ on $\overline{\mathcal{O}}$, imply that

$$
\mathbb{E}_\tilde{x}[\mathcal{M}_{t\wedge \tau}^\varepsilon] \leq 2 \sum_{i,j} \mathbb{E}_\tilde{x} \left[ \int_0^{t\wedge \tau} e^{-qs} \partial_t u^\varepsilon(X_{t\wedge \tau}) \sigma_{ij}(X_{t\wedge \tau})^2 ds \right] \leq 2 t C_3^2 \sum_{i,j} \sup_{x \in \overline{\mathcal{O}}} \{ \sigma_{ij}(x)^2 \} < \infty,
$$

for $t \geq 0$ fixed. This implies that $\mathcal{M}^\varepsilon = \{ \mathcal{M}_{t\wedge \tau}^\varepsilon : t \geq 0 \}$ is a square integrable martingale. Therefore, the process $\mathcal{M}^\varepsilon = \{ \mathcal{M}_{t\wedge \tau}^\varepsilon : t \geq 0 \}$ is also a square integrable martingale, with $\mathcal{M}_0^\varepsilon = 0$. Notice that, by Doob’s stopping theorem, $\mathbb{E}_\tilde{x}[\mathcal{M}_{t\wedge \tau}^\varepsilon] = \mathbb{E}_\tilde{x}[\mathcal{M}_0^\varepsilon] = 0$. Taking the expected value in (4.11), it follows that

$$
u^\varepsilon(\tilde{x}) = \mathbb{E}_\tilde{x} \left[ e^{-q[t\wedge \tau]} u^\varepsilon(X_{t\wedge \tau}) + \int_0^{t\wedge \tau} e^{-qs} \left[ \nu - \Gamma_1 \right] u^\varepsilon(X_s) + \langle D^1 u^\varepsilon(X_s), \zeta, n_s \rangle \right] ds. \quad (4.13)
$$

From (4.4) and inequality $\langle \gamma, y \rangle \leq \psi_\varepsilon(\langle \gamma \rangle^2 - g(x)^2) + l_\varepsilon(x, y)$, we have

$$
u^\varepsilon(\tilde{x}) \leq \mathbb{E}_\tilde{x} \left[ e^{-q[t\wedge \tau]} u^\varepsilon(X_{t\wedge \tau}) + \int_0^{t\wedge \tau} e^{-qs} [h(X_s) + l_\varepsilon(X_s, \zeta, n_s)] ds \right], \quad (4.14)
$$

since $h + l_\varepsilon \geq 0$. Observe that

$$
\mathbb{E}_\tilde{x}[e^{-q[t\wedge \tau]} u^\varepsilon(X_{t\wedge \tau})] = \mathbb{E}_\tilde{x}[e^{-q[t\wedge \tau]} u^\varepsilon(X_{t\wedge \tau}) \mathbb{I}_{\{\tau < \infty\}}] + \mathbb{E}_\tilde{x}[e^{-q[t\wedge \tau]} u^\varepsilon(X_t) \mathbb{I}_{\{\tau = \infty\}}].
$$

On the event $\{ \tau < \infty \}$, we have $\lim_{t \to \infty} e^{-q[t\wedge \tau]} u^\varepsilon(X_{t\wedge \tau}) = e^{-q[t\wedge \tau]} u^\varepsilon(X_t) = 0$, since $u^\varepsilon = 0$ on $\overline{\mathcal{S}}_I \setminus \mathcal{O}$, and by Lemma 2.6, $0 \leq e^{-q[t\wedge \tau]} u^\varepsilon(X_{t\wedge \tau}) \leq C_1$ for all $t \geq 0$. Then, by Dominated Convergence Theorem, we see $\mathbb{E}_\tilde{x}[e^{-q[t\wedge \tau]} u^\varepsilon(X_{t\wedge \tau}) \mathbb{I}_{\{\tau < \infty\}}] \to 0$. Now, on $\{ \tau = \infty \}$, we observe that $e^{-q[t\wedge \tau]} \to 0$ and $X_t \in \mathcal{O}$, for all $t > 0$. Since $u^\varepsilon$ is a bounded continuous function on $\overline{\mathcal{O}}$, we have that $\mathbb{E}_\tilde{x}[e^{-q[t\wedge \tau]} u^\varepsilon(X_t) \mathbb{I}_{\{\tau = \infty\}}] \leq C_1 e^{-q[t\wedge \tau]} \to 0$. Then,

$$
\mathbb{E}_\tilde{x}[e^{-q[t\wedge \tau]} u^\varepsilon(X_{t\wedge \tau})] \to 0. \quad (4.15)
$$

Therefore, from here and letting $t \to \infty$ in (4.14), it yields $u^\varepsilon \leq \mathcal{V}_{n, \zeta}$ on $\overline{\mathcal{O}}$. Let $X^{n, \varepsilon}$ be the solution process to the SDE (4.5), with control $(n^{\varepsilon, \zeta}, e^{\varepsilon, \zeta})$ given in (4.6)–(4.7). Proceeding in a similar way that in (4.13) and noting that the supremum of $l_\varepsilon(x, n)$ is attained if $\gamma$ is related to $\eta$ by $\eta = 2 u_\varepsilon(\langle \gamma \rangle^2 - g(x)^2)\gamma$, i.e.,

$$
l_\varepsilon(x, 2u_\varepsilon(\langle \gamma \rangle^2 - g(x)^2)\gamma) = 2u_\varepsilon(\langle \gamma \rangle^2 - g(x)^2)|\gamma|^2 - \psi_\varepsilon(\langle \gamma \rangle^2 - g(x)^2),
$$

it follows that

$$
u^\varepsilon(\tilde{x}) = \mathbb{E}_\tilde{x} \left[ e^{-q[t\wedge \tau^\varepsilon]} u^\varepsilon(X^{n, \varepsilon}_{t\wedge \tau^\varepsilon}) + \int_0^{t\wedge \tau^\varepsilon} e^{-qs} [h(X^{n, \varepsilon}_s) + l_\varepsilon(X^{n, \varepsilon}_s, n^{\varepsilon, \zeta}_s \zeta_s)] ds \right], \quad (4.16)
$$

with $\tau^\varepsilon = \inf \{ t : X^{n, \varepsilon}_t \notin \mathcal{O} \}$. Notice that

$$
\int_0^{t\wedge \tau^\varepsilon} e^{-qs} [h(X^{n, \varepsilon}_s) + l_\varepsilon(X^{n, \varepsilon}_s, n^{\varepsilon, \zeta}_s \zeta_s)] ds \uparrow \int_0^{\tau^\varepsilon} e^{-qs} [h(X^{n, \varepsilon}_s) + l_\varepsilon(X^{n, \varepsilon}_s, n^{\varepsilon, \zeta}_s \zeta_s)] ds,
$$

as $t \to \infty$, since $h + l_\varepsilon \geq 0$. Then, by Monotone Convergence Theorem,

$$
\mathbb{E}_\tilde{x} \left[ \int_0^{t\wedge \tau^\varepsilon} e^{-qs} [h(X^{n, \varepsilon}_s) + l_\varepsilon(X^{n, \varepsilon}_s, n^{\varepsilon, \zeta}_s \zeta_s)] ds \right] \to \mathcal{V}_{n, \varepsilon, \zeta, \varepsilon}(\tilde{x}). \quad (4.17)
$$

Letting $t \to \infty$ in (4.16) and using (4.15), (4.17), we conclude $u^\varepsilon = \mathcal{V}_{n, \varepsilon, \zeta, \varepsilon} = V^\varepsilon$ on $\overline{\mathcal{O}}$. ■
To finalize, we present the proof of the main result given in Subsection 1.1.

**Proof of Proposition 1.5.** By Subsection 3.2 and Corollaries 4.1, 4.3, we have that there exists a non-increasing sub-sequence \( \{u^{\varepsilon_k}\}_{k \geq 0} \) of \( \{u^{\varepsilon}\}_{\varepsilon \in (0,1)} \) such that for each \( \kappa \geq 0 \), \( u^{\varepsilon_k} \) is the unique non-negative solution to the NPIDD problem (4.1), with \( \varepsilon = \varepsilon_\kappa \), and

\[
u^{\varepsilon_k}_{\varepsilon_\kappa \to 0} \to u \quad \text{in} \quad C(\overline{O_T}), \quad \partial_t u^{\varepsilon_k}_{\varepsilon_\kappa \to 0} \to \partial_t u \quad \text{in} \quad C_{\text{loc}}(O), \quad \partial_{ij} u^{\varepsilon_k}_{\varepsilon_\kappa \to 0} \to \partial_{ij} u \quad \text{weakly in} \quad L^p_{\text{loc}}(O),
\]

where \( p \in (d, \infty) \) is fixed and \( u \) is the unique non-negative solution to the HJB equation (1.20). Also, from Lemma 4.4, we know that \( u^{\varepsilon_k} = V_{n^{\varepsilon_k,\ast},\zeta^{\varepsilon_k,\ast}} = V^{\varepsilon_k} \) on \( \overline{O} \), with \( (n^{\varepsilon_k,\ast},\zeta^{\varepsilon_k,\ast}) \) as in (4.6)–(4.7). Notice that \( l_\varepsilon(x,\beta \gamma) \geq \langle \beta \gamma, g(x) \rangle - \psi_\varepsilon(|g(x)|^2 - |g(x)|^2) = \beta g(x) \), with \( \beta \in \mathbb{R} \) and \( \gamma \in \mathbb{R}^d \) a unit vector. Then, from here and considering \( X^{\varepsilon_k,\ast} \) as in (4.5), it follows that

\[
V(\bar{x}) \leq V_{n^{\varepsilon_k,\ast},\zeta^{\varepsilon_k,\ast}}(\bar{x}) = \mathbb{E}\bar{\pi} \left[ \int_0^{\tau_{\varepsilon_k}^*} e^{-qt}[h(X_t^{\varepsilon_k,\ast}) + \zeta_t^{\varepsilon_k,\ast} g(X_t^{\varepsilon_k,\ast})]dt \right] \\
\leq \mathbb{E}\bar{\pi} \left[ \int_0^{\tau_{\varepsilon_k}^*} e^{-qt}[h(X_t^{\varepsilon_k,\ast}) + l_\varepsilon(X_t^{\varepsilon_k,\ast},\zeta_t^{\varepsilon_k,\ast}n_t^{\varepsilon_k,\ast})]dt \right] = u^{\varepsilon_k}(\bar{x}), \quad (4.18)
\]

where \( \tau_{\varepsilon_k}^* = \inf\{t > 0 : X_t^{\varepsilon_k,\ast} \notin O\} \). Recall that \( V_{n^{\varepsilon_k,\ast},\zeta^{\varepsilon_k,\ast}} \) is the cost function given in (1.17) corresponding to the control \( (n^{\varepsilon_k,\ast},\zeta^{\varepsilon_k,\ast}) \), and note that, for this control, the second term in the RHS of (1.18) is zero, since \( \zeta^{\varepsilon_k,\ast} \) has the continuous part only. Letting \( \varepsilon_\kappa \to 0 \) in (4.18), it yields \( V \leq u \) on \( \overline{O} \). Let \( X \) be the process that evolves as in (1.16) and \( \tau = \inf\{t > 0 : X_t \notin O\} \), with \( (n, \zeta) \in \mathcal{U} \). Define \( \tau_m = \inf\{t > 0 : X_t \notin O_m\} \) and \( O_m := \{x \in O : \text{dist}(x, \partial O) > 1/m\} \), where \( m \) is a positive integer large enough. Replacing \( \varepsilon, \tau \) by \( \varepsilon_\kappa, \tau_m \) in (4.8), respectively, using integration by parts and Itô’s formula for \( e^{-q(t \wedge \tau_m)} u^{\varepsilon_k}(X_t \wedge \tau_m) \), it can be verified that (4.8) holds for this case. Notice that

\[
\int_0^{t \wedge \tau_m} e^{-qt}(D^1 u^{\varepsilon_k}(X_{t-}), n_s) d\zeta_s = \int_0^{t \wedge \tau_m} e^{-qt}(D^1 u^{\varepsilon_k}(X_{t-}), n_s) d\zeta^c_s + \sum_{0 \leq s \leq t \wedge \tau_m} e^{-qt}(D^1 u^{\varepsilon_k}(X_{s-}), n_s) \Delta \zeta_s, \quad (4.19)
\]

where \( \zeta^c \) denotes the continuous part of \( \zeta \). Meanwhile, since \( \Delta X_t = \Delta J_t - n_t \Delta \zeta_t \), it can be
\[
\sum_{0 \leq t \leq \tau_m} e^{-qt} [u^\varepsilon(X_{s-} + \Delta J_s - n_s \Delta \zeta_s) - u^\varepsilon(X_{s-} + \Delta J_s)]
\]

\[
= -\int_0^{t \wedge \tau_m} e^{-qt} [u^\varepsilon(X_{s-} + z) - u^\varepsilon(X_{s-})] - (D^1 u^\varepsilon(X_{s-}), [\Delta J_s - n_s \Delta \zeta_s])] N(d\rho, dz, ds)
\]

\[
= -\sum_{0 \leq s \leq t \wedge \tau_m} e^{-qt} [D[u^\varepsilon]_s + (D^1 u^\varepsilon(X_{s-}), n_s) \Delta \zeta_s] \mathbb{1}_{\{\Delta \zeta_s \neq 0\}}
\]

\[
= -\sum_{0 \leq s \leq t \wedge \tau_m} e^{-qt} [D[u^\varepsilon]_s + (D^1 u^\varepsilon(X_{s-}), n_s) \Delta \zeta_s] \mathbb{1}_{\{\Delta \zeta_s \neq 0\}},
\]

(4.20)

with \(\tilde{M}^{\varepsilon} \) as in (4.10) and \(D[u^\varepsilon]_s := [u^\varepsilon(X_{s-} + \Delta J_s - n_s \Delta \zeta_s) - u^\varepsilon(X_{s-} + \Delta J_s)] \mathbb{1}_{\{\Delta \zeta_s \neq 0\}}\). Applying (4.19)–(4.20) in (4.8), it is easy to verify that

\[
u^\varepsilon(\tilde{x}) = e^{-q[t \wedge \tau_m]} u^\varepsilon(X_{t \wedge \tau_m}) - \sum_{0 \leq t \leq \tau_m} e^{-qt} D[u^\varepsilon]_s,
\]

(4.21)

where \(q > 0\), \(\Gamma_1\) is as in (1.14), and \(\mathcal{M}^{\varepsilon}\) is the square integrable martingale given by (4.12). From (4.4), \([q - \Gamma_1]u^\varepsilon(X_s) \leq h(X_s)\), for all \(s \in [0, t \wedge \tau_m)\). Then, taking expected value in (4.21),

\[
\nu^\varepsilon(\bar{x}) \leq \mathbb{E}_\bar{x} \left[ e^{-q[t \wedge \tau_m]} u^\varepsilon(X_{t \wedge \tau_m}) + \int_0^{t \wedge \tau_m} e^{-qt} [q - \Gamma_1] u^\varepsilon(X_s)ds \right. 
\]

\[
- \sum_{0 \leq t \leq \tau_m} e^{-qt} D[u^\varepsilon]_s.
\]

(4.22)

Define \(g_1(t \wedge \tau_m, X_{t \wedge \tau_m}) = \sum_{0 \leq s \leq t \wedge \tau_m} e^{-qt} D[u]_s\). Then, letting \(\varepsilon \to 0\) in (4.22), by Dominated Convergence Theorem, and using \(u^\varepsilon\), \(|D^1 u^\varepsilon|\) are uniformly bounded by \(C_1, C_3\) on \(\overline{\Omega}_x\) and \(\overline{\Omega}_m\), respectively, \(u^\varepsilon(X_s) \to u(X_s), \quad |D^1 u^\varepsilon(X_s)| \to |D^1 u(X_s)|, D[u^\varepsilon]_s \to D[u]_s\), for \(s \leq t \wedge \tau_m\), and \(|D^1 u| \leq g\) on \(\Omega\), it follows that

\[
u(\bar{x}) \leq \mathbb{E}_\bar{x} \left[ e^{-q[t \wedge \tau_m]} u(X_{t \wedge \tau_m}) + \int_0^{t \wedge \tau_m} e^{-qt} [h(X_s)ds + g(X_s)d\zeta] - \sum_{0 \leq t \leq \tau_m} \mathbb{E}_\bar{x} [g_1(t \wedge \tau_m, X_{t \wedge \tau_m})].
\]

(4.23)

By similar arguments used in (4.15) and (4.17), and noting that \(\tau = m \to \infty, \mathbb{P}_\bar{x}\)-a.s.,
it can be verified that
\[
\lim_{t \to \infty} \lim_{m \to \infty} \mathbb{E}_x \left[ e^{-q[t \wedge \tau_m]} u(X_{t \wedge \tau_m}) + \int_0^{t \wedge \tau_m} e^{-q s} [h(X_s) ds + g(X_s) d\zeta_s] \right] = \mathbb{E}_x \left[ \int_0^\tau e^{-q s} [h(X_s) ds + g(X_s) d\zeta_s] \right]. \tag{4.24}
\]

On the event \(\{\tau = \infty\}\), we have that for each \(s > 0\) such that \(\Delta \zeta_s \neq 0\), \(X_{s-} + \Delta J_s - n_s \Delta \zeta_s \in \mathcal{O}\). Meanwhile, \(X_{s-} + \Delta J_s \in \mathcal{O}\) or \(X_{s-} + \Delta J_s \in \mathcal{O}_T \setminus \mathcal{O}\). If \(X_{s-} + \Delta J_s \in \mathcal{O}\), by Mean Value Theorem,
\[
-D[u] \leq \Delta \zeta_s \int_0^1 |D^1 u(X_{s-} + \Delta J_s - \lambda n_s \Delta \zeta_s)| d\lambda \leq \Delta \zeta_s \int_0^1 g(X_{s-} + \Delta J_s - \lambda n_s \Delta \zeta_s) d\lambda,
\tag{4.25}
\]
since \(|D^1 u| \leq g\) in \(\mathcal{O}\). If \(X_{s-} + \Delta J_s \in \mathcal{O}_T \setminus \mathcal{O}\), we have that the line segment between \(X_{s-} + \Delta J_s\) and \(X_{s-} + \Delta J_s - n_s \Delta \zeta_s\), that is described by \(X_{s-} + \Delta J_s - \lambda n_s \Delta \zeta_s\), with \(\lambda \in [0, 1]\), intersects \(\partial \mathcal{O}\) in a unique point \(X^*_s := X_{s-} + \Delta J_s - \lambda^* n_s \Delta \zeta_s\), for some \(\lambda^* \in (0, 1)\), since \(\mathcal{O}\) is convex. Then, noting that \(X_{s-} + \Delta J_s - n_s \Delta \zeta_s - X^*_s = -[1 - \lambda^*] n_s \Delta \zeta_s\), and using again Mean Value Theorem and the fact that \(u(X^*_s) = 0\),
\[
-u(X_{s-} + \Delta J_s - n_s \Delta \zeta_s) = -[u(X_{s-} + \Delta J_s - n_s \Delta \zeta_s) - u(X^*_s)] \\
\leq [1 - \lambda^*] \Delta \zeta_s \int_0^1 g(X^*_s - \lambda [1 - \lambda^*] n_s \Delta \zeta_s) d\lambda. \tag{4.26}
\]

Meanwhile, observe that
\[
\int_{\lambda^*}^1 g(X_{s-} + \Delta J_s - \lambda n_s \Delta \zeta_s) d\lambda = [1 - \lambda^*] \int_0^1 g(X^*_s - \lambda [1 - \lambda^*] n_s \Delta \zeta_s) d\lambda.
\]
Then, from here, it is easy to verify
\[
[1 - \lambda^*] \int_0^1 g(X^*_s - \lambda [1 - \lambda^*] n_s \Delta \zeta_s) d\lambda \leq \int_0^1 g(X_{s-} + \Delta J_s - \lambda n_s \Delta \zeta_s) d\lambda. \tag{4.27}
\]
Therefore, by (4.26)–(4.27) and that \(u(X_{s-} + \Delta J_s) = 0\), it yields (4.25). From here and by Monotone Convergence Theorem, we have
\[
\lim_{t \to \infty} \lim_{m \to \infty} \mathbb{E}_x \left[ -g_1(t \wedge \tau_m; X_{t \wedge \tau_m}) \mathbb{1}_{\{\tau = \infty\}} \right] \\
\leq \lim_{t \to \infty} \lim_{m \to \infty} \mathbb{E}_x \left[ \mathbb{1}_{\{\tau = \infty\}} \sum_{s \leq t \wedge \tau_m} e^{-q s} \Delta \zeta_s \int_0^1 g(X_{s-} + \Delta J_s - \lambda n_s \Delta \zeta_s) d\lambda \right] \\
\leq \mathbb{E}_x \left[ \mathbb{1}_{\{\tau = \infty\}} \sum_{s \geq 0} e^{-q s} \Delta \zeta_s \int_0^1 g(X_{s-} + \Delta J_s - \lambda n_s \Delta \zeta_s) d\lambda \right]. \tag{4.28}
\]

Now, on \(\{\tau < \infty\}\), for \(s \leq \tau\) we can use the same arguments as for the case \(\{\tau = \infty\}\); for \(s = \tau\) we have \(X_{\tau-} + \Delta J_{\tau-} - n_\tau \Delta \zeta_{\tau} \in \mathcal{O}_\tau \setminus \mathcal{O}\) and either \(X_{\tau-} + \Delta J_{\tau} \in \mathcal{O}\) or \(X_{\tau-} + \Delta J_{\tau} \in \mathcal{O}_\tau \setminus \mathcal{O}\),
then similar arguments as before apply. Then, by using the Monotone Convergence Theorem, we have

$$
\lim_{t \to \infty} \lim_{m \to \infty} \mathbb{E}_\bar{x}[-g_1(t \wedge \tau_m, X_{t \wedge \tau_m}) \mathbb{1}_{\{\tau_\infty\}}] \\
\leq \mathbb{E}_\bar{x} \left[ \mathbb{1}_{\{\tau_\infty\}} \sum_{0 \leq s \leq \tau} e^{-qs} \Delta \zeta_s \int_0^1 g(X_{s-} + \Delta J_s - \lambda n \Delta \zeta_s) d\lambda \right].
$$

(4.29)

Therefore, letting $m \to \infty$ and $t \to \infty$ in (4.23), and using (1.18), (4.24), (4.28) and (4.29), it yields $u \leq V_{n,\epsilon}$ on $\mathcal{O}$. From here and (1.19), $u \leq V$ on $\mathcal{O}$. By the arguments seen previously, we conclude that $u = V$ on $\mathcal{O}$.

4.1 About penalized optimal controls

As discussed previously, the value function $V$, given in (1.19), satisfies the HJB (1.20). This means that the domain $\mathcal{O}$ is divided into two parts. The first part, defined as $\mathcal{E} \subseteq \mathcal{O}$, is where $V$ satisfies the elliptic integro-differential equation $[q - \Gamma_1]V = h$, which suggests that the optimal control corresponding to this problem will not be exercised on $\mathcal{E}$. Otherwise the ‘optimal control’ will exercise a force and a direction at $x \in \mathcal{O} \setminus \mathcal{E}$ in such a way that the process $X_{t \wedge n,\epsilon}$ will be pushed back to some point $y \in \partial \mathcal{E}$.

To construct an optimal strategy to the problem (1.19), it is necessary to verify that $\partial \mathcal{E}$ is at least of class $C^1$, which is not easy to get; this is currently the topic of a work in progress by the authors. In the literature, we can find problems of this type that have been successfully solved in some cases; see, e.g., [2, 3, 4, 6, 14, 19, 21, 31].

Another way to address the problem (1.19) is by means of $\epsilon$-penalized optimal controls which have been constructed in (4.6)–(4.7). From Lemma 4.4 and proof of Lemma 1.5, we know that $V_{\epsilon,*,\epsilon} \downarrow V$ as $\epsilon \downarrow 0$ on $\mathcal{O}$, and $V \leq V_{n,\epsilon,*,\epsilon} \leq V_{\epsilon,*,\epsilon}$ on $\mathcal{O}$, where $(n_{\epsilon,*,\epsilon}^*, \zeta_{\epsilon,*,\epsilon}^*)$ as in (4.6)–(4.7), and $V_{n,\epsilon,*,\epsilon} \leq V_{\epsilon,*,\epsilon}$ on $\mathcal{O}$, are given by (1.17), (1.19) and (4.2), respectively. Taking $\epsilon$ small enough, we have that the control $(n_{\epsilon,*,\epsilon}^*, \zeta_{\epsilon,*,\epsilon}^*)$ is exercised as follows: if the controlled process $X_{t \wedge n,\epsilon} \satisfies |D^1 V_{\epsilon,*,\epsilon}(X_{t \wedge n,\epsilon}^*)| \leq g(X_{t \wedge n,\epsilon}^*)$ with $t \in [0, \tau_{\epsilon,*,\epsilon}^*]$ and $\tau_{\epsilon,*,\epsilon}^* = \inf\{t > 0 : X_{t \wedge n,\epsilon}^* \notin \mathcal{O}\}$, then $\zeta_{\epsilon,*,\epsilon}^* \equiv 0$ and $X_{t \wedge n,\epsilon}^*$ will stay in $\mathcal{E}$. If $0 < |D^1 V_{\epsilon,*,\epsilon}(X_{t \wedge n,\epsilon}^*)|^2 - g(X_{t \wedge n,\epsilon}^*)^2 < 2\epsilon_\kappa$, with $t \in [0, \tau_{\epsilon,*,\epsilon}^*]$, the process $X_{t \wedge n,\epsilon}^*$ will be crossing $\partial \mathcal{E}$ persistently. Otherwise, $(n_{\epsilon,*,\epsilon}^*, \zeta_{\epsilon,*,\epsilon}^*)$ will exercise a force $\frac{2\epsilon}{\lambda} D^1 V_{\epsilon,*,\epsilon}(X_{t \wedge n,\epsilon}^*)$ and a direction $-\frac{D^1 V_{\epsilon,*,\epsilon}(X_{t \wedge n,\epsilon}^*)}{|D^1 V_{\epsilon,*,\epsilon}(X_{t \wedge n,\epsilon}^*)|}$ at $X_{t \wedge n,\epsilon}^*$ in such a way that it will be pushed back to $\partial \mathcal{E}$.

To finalize this section, Zhu [34] also solved a similar problem by means of $\epsilon$-penalized optimal controls when the state process is a multidimensional diffusion process.

5 Conclusions and some further work

In this paper we have guaranteed, under Assumptions (A1)–(A4), the existence and uniqueness for the strong (in the a.e. sense) and classical solutions to the HJB and NPIDD equations presented in (1.1) and (1.8), respectively. It should be noted that one of main contributions of this work is Assumption (A4), which permits the Lévy measure $\nu$ to be infinite on $\mathbb{R}^d_+$. This assumption also played an important role in the proofs of Lemmas 2.7 and 3.3.

Another main result achieved in this paper is the establishment of the strong relationship between the value functions $V, V_\epsilon$ given in (1.19), (4.2), and the solutions $u, u_\epsilon$ to the equations (1.20), (4.4), respectively. Although, the optimal control process for the singular
stochastic problem (1.19) was not given, and this is still an open problem, we constructed a family of ε-optimal absolutely continuous control processes \( \{(n^{\varepsilon, \kappa}, \zeta^{\varepsilon, \kappa, \kappa})\}_{\kappa \geq 1} \); see (4.6)–(4.7), such that the limit of their value functions \( V^{\varepsilon, \kappa} \) (as \( \varepsilon \to 0 \)) agrees with the value function \( V \).

There are some extensions to be considered and directions for future research:

(i) One of the natural extensions of this work would be to study the HJB and NPIDD equations (1.1) and (1.8), respectively, when the integral operator \( Iw \) has the form

\[
\int_{\mathbb{R}^d} \left[ \left| w(\cdot + z) - w - (D^1 w, z) \mathbb{1}_{\{|z| \leq 1\}} \right| \nu(dz) \right] \left| z \right|^2 \nu(dz) < \infty.
\]

In this case, the main difficulty lies in obtaining results similar to Lemmas 2.7 and 3.3 because we must have an à priori estimate of

\[
\int_{\{|z| \leq 1\}} \left[ \int_0^1 \left| D^2 u^{\varepsilon, t}(\cdot + tz) \right| dt \right] |z|^2 \nu(dz)
\]

independent of \( \varepsilon \).

(ii) Another extension is to generalize the gradient constraint that appears in (1.1), i.e., to study the HJB equation presented in works as [16] or [17], when the operator is a partial integro-differential operator as in (1.3).

(iii) In parallel to this research, the stochastic control problems in different branches of applied probability (insurances, inventories, etc.), which are closely related to these HJB equations, may be analyzed.

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