Do $1/r$ potentials require massless particles?

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Abstract

Long-range $1/r$ potentials play a fundamental role in physics. Their ultimate origin is usually traced back to the existence of genuine massless particles as photons or gravitons related to fundamental properties of continuum quantum field theories such as gauge invariance. In this Letter, it is argued that, in principle, an asymptotic, infinitesimally weak $1/r$ potential might also occur in the cutoff version of a simple, one-component spontaneously broken $\Phi^4$ theory, after taking into account the peculiar nature of the zero-momentum limit of the connected scalar propagator. Physical interpretation, phenomenological implications and proposals for a new generation of lattice simulations are also discussed.
1. Long-range $1/r$ potentials play a very important role in physics. Their ultimate origin is usually traced back to the existence of genuine massless particles as photons or gravitons related to fundamental properties of continuum quantum field theories such as gauge invariance. In this Letter, it will be argued that, in principle, an asymptotic, infinitesimally weak $1/r$ behaviour might also occur in the cutoff version of a simple, one-component spontaneously broken $\Phi^4$ theory. To this end, one has to take into account the peculiar nature of the zero 4-momentum limit of the connected scalar propagator $G(p)$.

In fact, from the generally accepted ”triviality” of the theory in four space-time dimensions, one expects a gaussian structure of Green’s functions in the continuum limit. While this implies no observable dynamics at any 4-momentum $p_\mu \neq 0$ and, on the basis of Lorentz invariance, a free-field type form $G^{-1}(p) = (p^2 + m_h^2)$, one cannot exclude a discontinuity in the zero-measure, Lorentz-invariant subset $p_\mu = 0$. This plays a fundamental role in translational invariant vacua characterized by space-time constant expectation values of local operators such as $\langle \Phi \rangle$.

For this reason, if there were arguments for the alternative solution $G^{-1}(p = 0) = 0$, one might ask: could one consider a not entirely trivial continuum limit where, still, $G^{-1}(p) = (p^2 + m_h^2)$ for any $p_\mu \neq 0$ but where there is a discontinuity at $p_\mu = 0$ and $G^{-1}(p = 0) = 0$? In this case, what happens in the presence of an ultraviolet cutoff $\Lambda$ where one expects instead a smooth behaviour? At a certain point, for sufficiently small (“infinitesimal”) momenta, say $|p| \sim m_h^2/\Lambda$, one should necessarily replace the standard massive form $G^{-1}(p) \sim (p^2 + m_h^2) \to m_h^2$ with the different alternative $G^{-1}(p) \to 0$. Therefore, although the continuum theory has only massive, free-field excitations, its cutoff version would exhibit non-trivial qualitative differences, as weak long-range forces, that cannot be considered uninteresting perturbative corrections. This type of qualitative difference is the main point of the present Letter.

In the following, I will first review the basic ingredients of the problem. Some of these preliminary arguments are rather technical and are listed as points 2-5 below. A reader who is only interested in the main conclusions can simply look at the final point 6. Physical interpretation, possible phenomenological implications and proposals for a new generation of lattice simulations will also be discussed.
2. Let us start from lattice simulations. These were performed [1], in the 4D Ising limit of the theory, to objectively test the behaviour of the connected scalar propagator in the broken phase. Differently from the symmetric phase at $\langle \Phi \rangle = 0$, where the simple massive picture works to very high accuracy in the whole range of momenta, the results of the low-temperature phase show unexpected deviations. Namely, when the 4-momentum $p_{\mu} \equiv (p, p_4) \to 0$, the propagator starts to deviate from (the lattice version of) the form $1/(p^2 + \text{const.})$. By expressing the connected Euclidean propagator as

$$G(p) = \frac{1}{p^2 + M^2(p^2)}$$

these deviations can be parameterized by using Stevenson’s sensitive variable [2]

$$\zeta(p, m) \equiv (p^2 + m^2)G(p)$$

In terms of this variable, the results can be summarized as follows. One can first define a mass value $m \equiv m_h$ that well describes the higher-momentum part of the propagator data, namely those where one gets a remarkably flat $\zeta_{\text{latt}}(p, m_h) \lesssim 1$. In terms of this mass definition, the resulting $\zeta_{\text{latt}}(p, m_h)$ rapidly increase above unity in the $p_{\mu} \to 0$ limit with a zero-momentum value

$$\zeta_{\text{latt}}(0, m_h) = \frac{m_h^2}{M^2(0)} \bigg|_{\text{latt}}$$

that becomes larger and larger by approaching the continuum limit of the lattice theory (compare Figs. 3, 4 and 5 of Ref.[1]). After the first indications of Ref.[1], Stevenson [2] checked independently the existence of this discrepancy by using different input masses for $\zeta(p, m)$ and plotting the data in various ways. To this end, he used the lattice data of Ref.[3] for the time slices of the connected two-point correlator $C_1(p = 0, t) \sim e^{-E(0)t}$ and generated by Fourier transform equivalent data for the connected scalar propagator $G(p)$. The resulting behaviour of $G(p)$ is in complete agreement with the analogous plots obtained from Ref.[1] (compare Figs.6c, 7, 8 and 9 of Ref.[2]).

The data also indicate that, by approaching the continuum limit, the deviations from $\zeta_{\text{latt}}(p, m_h) \lesssim 1$ become confined to a smaller and smaller region of momenta near $p_{\mu} = 0$. Thus, in the continuum limit where the ultraviolet cutoff $\Lambda \to \infty$, both $M(0)$ and the peculiar infrared region, say $|p| \lesssim \delta$, where the propagator deviates from the


simple massive form, might vanish in units of \( m_h \). In this scenario there would be a hierarchy of scales \( \delta \ll m_h \ll \Lambda \) such that \( \frac{\delta}{m_h} \to 0 \) when \( \frac{m_h}{\Lambda} \to 0 \) (as for instance with the relation \( \delta \sim m_h^2/\Lambda \)). Therefore, if \( m_h \) were taken as the unit mass scale, the deviations from a free-field massive behaviour would simply reduce to the zero-measure set \( p_\mu = 0 \). In this perspective, exact Lorentz covariance would be recovered, since the value \( p_\mu = 0 \) forms a Lorentz-invariant subset. Thus, the whole low-momentum region would represent a typical example of ”reentrant violation of special relativity in the low-energy corner” [4], namely one of those peculiar infrared phenomena of cutoff theories. In the following I will now list three different theoretical arguments that support an unconventional infrared behaviour and point to a similar conclusion.

3. The first theoretical argument is based on the results of Ref.[5]. There, one was studying the effective potential \( V_{\text{eff}}(\phi) \) and the field strength \( Z(\phi) \), as functions of the background field \( \phi \), at various values of the infrared cutoff \( k \). To this end, one starts from a bare action defined at some ultraviolet cutoff \( \Lambda \) and effectively integrates out shells of quantum modes down to an infrared cutoff \( k \). This procedure generates a \( k \)-dependent effective action \( \Gamma_k[\Phi] \) that evolves into the full effective action \( \Gamma[\Phi] \) in the \( k \to 0 \) limit. In this approach, the relevant quantities are the \( k \)-dependent effective potential \( V_k(\phi) \) and field strength \( Z_k(\phi) \), which naturally appear in a derivative expansion of \( \Gamma_k[\Phi] \) around a space-time constant configuration \( \Phi(x) = \phi \).

By integrating numerically the coupled Renormalization-Group (RG) equations for \( V_k(\phi) \) and \( Z_k(\phi) \), one finds the following results. For not too small values of the infrared cutoff \( k \), the effective potential \( V_k(\phi) \) remains a smooth, non-convex function of \( \phi \) as in the loop expansion. In this region of \( k \) one also finds a field strength \( Z_k(\phi) \sim 1 \) for all values of \( \phi \).

However, a tiny momentum scale \( \delta \) exists such that for \( k < \delta \) the effective potential \( V_k(\phi) \) starts to flatten in an inner region of \( |\phi| \) while still matching with an outer, asymptotic shape of the type expected in perturbation theory. The flattening in the inner \( |\phi| \)-region, while reproducing the expected convexity property of the exact effective potential, does not correspond to a smooth behaviour. For such small values of \( k \) there are large departures of \( Z_k(\phi) \) from unity in the inner \( |\phi| \)-region with a strong peaking at the end point \( |\hat{\phi}| = |\hat{\phi}(k)| \) of the flattening region. On the base of the
general convexification property, the \( k \to 0 \) limit of such end point, \( \hat{\phi}(0) \), coincides with one of the minima \( \pm v \) of a suitable semiclassical, non-convex effective potential and is usually taken as the physical realization of the broken phase.

Therefore, the fluctuations with \( |p| \leq \delta \) are non-perturbative for values of the background field in the range \( -\hat{\phi}(|p|) \leq \phi \leq \hat{\phi}(|p|) \). In particular, the very low-frequency modes with \( |p| \to 0 \) behave non-perturbatively for all values of the background in the full range \( -v \leq \phi \leq v \) and thus cannot be represented as standard weakly coupled massive states. Notice that the unexpected effects show up with the emergence of the convexification process. This is induced by the very long-wavelength modes that, so to speak, "live" in the full region \( -v \leq \phi \leq v \).

By itself, the existence of a non-perturbative infrared sector in a region \( 0 \leq |p| \leq \delta \) might not be in contradiction with the assumed exact "triviality" property of the theory if, in the continuum limit, the infrared scale \( \delta \) vanishes in units of the physical parameter \( m_h \) associated with the massive part of the spectrum. Again, this means to establish a hierarchy of scales \( \delta \ll m_h \ll \Lambda \) such that \( \frac{\delta}{m_h} \to 0 \) when \( \frac{m_h}{\Lambda} \to 0 \). Therefore, in units of \( m_h \), the region \( 0 \leq |p| \leq \delta \) would just shrink to the zero-measure set \( p_\mu = 0 \) and one would be left with a massive, free-field theory for all non-zero values of the 4-momentum.

4. As a second theoretical argument, I will compare with Stevenson’s recent analysis [6] of the propagator in the broken-symmetry phase. In his approach, a more faithful representation of the true \( \Phi^4 \) interactions is obtained with the non-local action

\[
\int d^4x \int d^4y \ \Phi^2(x)U(x - y)\Phi^2(y)
\]  

(4)

The kernel \( U(x - y) \) contains, besides the repulsive contact \( \delta \)-function term, say \( U_{\text{core}}(x - y) \), an effective long-range attraction for \( x \neq y \), say \( U_{\text{tail}}(x - y) \). The latter, which is essential for a physical description of spontaneous symmetry breaking as a true condensation process [7], originates from ultraviolet-finite parts of one-loop Feynman graphs and has never been considered in the perturbative RG–approach. Instead, by taking into account both \( U_{\text{core}} \) and \( U_{\text{tail}} \) (and avoiding double counting) one can define a modified RG–expansion [6], as in a theory with two coupling constants. In the end, in the \( \Lambda \to \infty \) limit of the broken phase, the resulting connected Euclidean propagator \( G(p) \) approaches the standard free-field massive form \( G^{-1}(p) = (p^2 + m_h^2) \) except for a
discontinuity at $p_\mu = 0$ where $G^{-1}(p = 0) = 0$. This type of structure, implying the existence of a branch of the spectrum whose energy $E(p) \to 0$ in the $p \to 0$ limit, would indeed support the previous idea that, at least for the continuum theory, all deviations from the massive behaviour are at $p_\mu = 0$.

5. Finally, as a third theoretical argument, I emphasize that the possibility $G^{-1}(p = 0) = 0$ is also in agreement with the analogous indication of Ref. [8] that, in the broken-symmetry phase, $G^{-1}(p = 0)$ is a \textit{two-valued} function that, in addition to the standard value $G_a^{-1}(p = 0) = m^2_h$, includes the solution $G_b^{-1}(p = 0) = 0$ as in a massless theory. To this end, it becomes crucial to take the $\phi \to \pm v$ limit of the broken phase by first including the one-particle reducible tadpole graphs where zero-momentum propagator lines are attached to the one-point function $\Gamma_1(p = 0) = V'_\text{NC}(\phi)$, the first derivative of the standard non-convex effective potential $V_{\text{NC}}(\phi)$ of the loop expansion. By implicitly assuming the regularity of the zero-momentum propagator, these graphs are usually ignored at $\phi = \pm v$ where $V'_\text{NC}(\pm v) = 0$. Thus, $G^{-1}(p)$ is identified with the 1PI two-point function $\Gamma_2(p)$, whose zero-momentum value $\Gamma_2(p = 0)$ is nothing but $V''_{\text{NC}}(\pm v)$, a positive-definite quantity. On the other hand, by allowing for a singular $G(p = 0)$, one is faced with a completely different diagrammatic expansion and thus the simple picture of the broken phase as a pure massive theory, based on the chain

$$
G^{-1}(p = 0) = \Gamma_2(p = 0) = V''_{\text{NC}}(\pm v) > 0
$$

breaks down.

It is interesting that, as in point 3 above, one can find a relation with the convexity property of the exact effective potential. In fact, the existence of the two solutions for $G^{-1}(p = 0)$ at $\phi = \pm v$ can also be derived by evaluating in the saddle point approximation the generating functional $W[J]$ for a constant source and taking the double limit where $J \to \pm 0$ and the space-time volume $\Omega \to \infty$ [8].

As such, the two solutions admit a geometrical interpretation in terms of left and right second derivatives of the exact, Legendre transformed effective potential $V_{\text{LTI}}(\phi)$. This is convex downward and is not an infinitely differentiable function when $\Omega \to \infty$ [9]. These non-trivial differences should induce to check those physical aspects of the spontaneously broken phase, such as the mass spectrum, that depend crucially on
the identification \( V_{\text{eff}}(\phi) \equiv V_{\text{NC}}(\phi) \). In particular, the \( k \)-dependent effective potential \( V_k(\phi) \), obtained by integrating out shells of quantum modes down to some infrared cutoff \( k \) and mentioned in Sect.3, is clearly approaching convexity in the \( k \to 0 \) limit. Therefore, this well defined theoretical construction supports the identification of \( V_{\text{LT}}(\phi) \) as the true effective potential in the infinite-volume limit of the theory.

6. It is conceivable that the subtleties of \( G(p) \) at \( p_\mu = 0 \) might have been missed in most conventional approximation schemes. At the same time, the possibility of an infrared sector which is richer than expected has far reaching phenomenological implications. To see this, let us first summarize the results of sects. 2-5 as follows: by assuming a continuum limit where Lorentz invariance and ”triviality” hold exactly, a possible deviation from the entirely trivial, massive free-field limit, with \( G^{-1}(p) = (p^2 + m_h^2) \) identically, can only have the form of a discontinuity at \( p_\mu = 0 \) where \( G^{-1}(p) = 0 \).

Starting from this observation, in the presence of a finite ultraviolet cutoff \( \Lambda \), where one expects instead a smooth behaviour, one can try to construct a not entirely trivial \( G(p) \) as a smooth interpolation between these two distinct propagator forms of the continuum theory, say

\[
G^{-1}(p) = (p^2 + m_h^2) f(p^2/\delta^2) \tag{6}
\]

The function \( f \) refers to some infrared momentum scale \( \delta \neq 0 \) (with \( \delta/m_h \to 0 \)) in such a way that

\[
\lim_{\delta \to 0} f(p^2/\delta^2) = 1 \quad (p_\mu \neq 0) \tag{7}
\]

with the only exception

\[
\lim_{p_\mu \to 0} f(p^2/\delta^2) = 0 \tag{8}
\]

(think for instance of \( f(x) = \tanh(x) \), \( f(x) = 1 - \exp(-x) \), \( f(x) = x/(1 + x)\),...). In the following I will adopt Eq.(6). However, as one can easily check, there would be no significative change by employing the alternative form \( G^{-1}(p) = p^2 + m_h^2 f(p^2/\delta^2) \). In fact, analogous results would persist in any cutoff version where the function \( M^2(p^2) \) of Eq.(4) vanishes for \( p_\mu \to 0 \). Notice that, by adopting Eq.(6), one simply finds \( f(p^2/\delta^2) = \zeta^{-1}(p, m_h) \) in terms of Stevenson’s \( \zeta \)-function [2].
To understand what kind of instantaneous potential $V(r)$ in coordinate space is associated with such propagator for the scalar field, one has to consider the standard Fourier transform of the zero-energy propagator $G(p, p_4 = 0)$

$$D(r) = \int \frac{d^3p}{(2\pi)^3} \frac{e^{ip\cdot r}}{(p^2 + m^2) f(p^2/\delta^2)}$$

that in the case of the one-photon exchange, $G(p, p_4 = 0) = 1/p^2$, gives a $1/r$ potential.

Now, a straightforward replacement $f(p^2/\delta^2) = 1$ would produce the well known Yukawa potential $e^{-m_h r}/r$. However, if we consider the finite-cutoff theory, we have to take into account the region $p^2 \ll \delta^2$ where the relevant limiting relation is rather

$$\lim_{p \to 0} f(p^2/\delta^2) = 0$$

For this reason, since the dominant contribution for $r \to \infty$ comes from $p = 0$, where the denominator in (9) vanishes, there would be long-range forces that have never been considered. In this case, by expanding around $p = 0$ and replacing

$$f(p^2/\delta^2) \sim \frac{p^2}{\delta^2} f'(0)$$

one obtains the two leading behaviours

$$\lim_{p \to 0} (p^2 + m^2_h) f(p^2/\delta^2) \sim p^2 m^2_h \delta^2 f'(0)$$

and

$$\lim_{p^2 \to \infty} (p^2 + m^2_h) f(p^2/\delta^2) \sim p^2$$

Therefore, on the basis of the Riemann-Lebesgue theorem on Fourier transforms (see the Appendix), whatever the detailed form of $f(x)$ at intermediate $x$, the leading contribution at asymptotically large $r$ will be $1/r$. One thus gets

$$\lim_{r \to \infty} D(r) = D_\infty(r) = \frac{\delta^2}{f'(0)m^2_h} \frac{1}{4\pi r}$$

all dependence on the interpolating function being contained in the factor $f'(0) = \mathcal{O}(1)$.

To put some numbers (in units $\hbar = c = 1$), let us consider for definiteness the scenario $\delta \sim m^2_h/\Lambda$. This is motivated by a description of spontaneous symmetry breaking as a true condensation process and by the identification of $\delta$ as the momentum
scale below which collective oscillations of the condensate starts to propagate \[12\]. Thus, if \( m_h \) were around the Fermi scale and \( \Lambda \) around the Planck scale, \( \delta \) would be around \( 10^{-5} \) eV. For this particular case, let us compute the asymptotic potential between two fermions \( i \) and \( j \) of masses \( m_i \) and \( m_j \) that in the Standard Model couple to the singlet Higgs boson with strength \( y_i = m_i/v \) and \( y_j = m_j/v \). Besides the short-distance Yukawa potential governed by the Fermi constant \( G_F \equiv 1/v^2 \)

\[
V_{\text{yukawa}}(r) = -\frac{G_F m_i m_j}{4\pi r} e^{-m_h r}
\]  

(15)

(that dominates for \( r \lesssim 1/m_h \)) they would feel the asymptotic potential associated with Eq.(14). This can be conveniently expressed as

\[
\lim_{r \to \infty} V(r) = V_\infty(r) = -\frac{G_\infty m_i m_j}{4\pi r}
\]  

with the effective coupling

\[
G_\infty = \frac{\delta^2 f'(0) m_h^2}{G_F} \sim 10^{-33} G_F
\]  

(17)

Strictly speaking, this asymptotic potential represents a cutoff artifact since the continuum theory has only massive, free-field excitations, with the only exception of a discontinuity at \( p_\mu = 0 \) where \( G^{-1}(p) = 0 \). At least, this seems the only possible remnant of symmetry breaking allowed by exact Lorentz invariance and ”triviality”. However in the cutoff theory, where one expects a smooth behaviour, the deviation from the massive form will necessarily extend, from the zero-measure set \( p_\mu = 0 \), to a tiny momentum region \( \delta \). It is this momentum region, that vanishes in the continuum theory but remains finite in the cutoff theory, to produce the long-range \( 1/r \) potential of strength \( \delta^2/m_h^2 \). Since in this momentum region the propagator looks like in a massless theory, the answer to the question posed in the title of this Letter depends on the personal taste, even though there are no genuine massless particles (i.e. with propagator \( 1/p^2 \) in the whole range of momenta). Notice that, in the context of a condensate physical picture, the idea of long-range \( 1/r \) interactions in \( \Phi^4 \) theory below the condensation temperature was also considered in Ref.\[13\] by following a different approach.

A possible physical interpretation of the phenomenon is the following. By representing the broken-symmetry phase as a physical condensate, one is naturally lead
to consider superfluid $^4$He, the physical system that is usually considered as a non-relativistic realization of $\Phi^4$. One can thus try to understand the double-valued nature of the zero-momentum connected propagator in the broken phase in analogy with Landau’s original idea [14] of two different branches in the energy spectrum of $^4$He, namely gapless density oscillations (phonons) and massive vortical excitations (rotons) [15]. Experiments however have shown that these two different branches actually merge into a single energy spectrum, a sort of ”hybrid” that smoothly interpolates between the two different functional forms. In our case, the interpolating propagator produces similar effects.

One may object that Eq.(11) might be too simple. In principle, the function $f(p^2/\delta^2)$, for $p\rightarrow 0$, might vanish as $(p^2/\delta^2)^{1+\eta}$, where $\eta$ plays the role of an anomalous dimension and might be needed for a proper matching of the inverse propagator in the infrared region. In this case, the asymptotic $1/r$ potential in coordinate space would exhibit corrections proportional to $(r\delta)^\eta$.

Another possible objection is that a scale $\delta \sim 10^{-5}$ eV is probably ruled out by experiments. Thus, in the scenario $\delta \sim m_h^2/\Lambda$, to get a sufficiently small strength, one should take a $\Lambda$ which is larger than the Planck scale or a $m_h$ well below 300 GeV or both. However, comparison with experiments represents a separate issue. If some assumption behind the above numerical analysis is in conflict with phenomenology, still, the basic ambiguity of $G(p)$ at $p\rightarrow 0$ remains a peculiarity of the broken-symmetry vacuum and represents a challenge for any consistent cutoff version of the theory.

In conclusion, for its conceptual relevance and the potential phenomenological implications, it seems worth to further sharpen our understanding of the low-momentum region of spontaneously broken $\Phi^4$ theories. In particular, with a new generation of lattice simulations one should study the $p\rightarrow 0$ limit of the connected propagator on much larger lattices and try to determine the interpolating function $f(p^2/\delta^2)$ in Eq.(6). By reaching the critical region $|p| \lesssim \delta$, the deviations from the pure massive behaviour $f \sim 1$ (that remain below 30% on the lattices used so far [11,2]) should become macroscopic. In the scenario $\delta \sim m_h^2/\Lambda$, by using the relation [17] $\Lambda \sim \frac{4893(3)}{a}$ to relate the ultraviolet cutoff of a $\Phi^4_4$ theory to the lattice spacing $a$, and setting $\delta = \frac{2\pi}{L_{\min}}$, this
means a minimal lattice size
\[ \frac{L_{\text{min}}}{a} \sim \frac{30.74}{(m_h a)^2} \]  
(18)

For mass values in the scaling region, one gets \( \frac{L_{\text{min}}}{a} \sim 123, 192, 342, 769, 3074 \) for \( m_h a = 0.5, 0.4, 0.3, 0.2, 0.1 \) respectively. Four-dimensional lattices with \( \frac{L}{a} = \mathcal{O}(100) \) should be attainable with the present supercomputers.

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**Appendix**

Let us consider a function $F(q^2)$ that exhibits the two asymptotic trends ($\alpha > 0$, $A > 0$)

\[
\lim_{q\to0} \frac{F(q^2)}{q^2} = \alpha + O(q^2), \quad \lim_{q\to\infty} \frac{F(q^2)}{q^2} = A + O(1/q^2)
\]

(19)

so that

\[
\int_0^\infty \frac{q \ dq}{|F(q^2)|} = +\infty
\]

(20)

I will assume the general requirements needed for the existence of the Fourier transform

\[
I(r) = \int \frac{d^3q}{(2\pi)^3} \frac{e^{iqr}}{F(q^2)} = \frac{1}{2\pi^2r} \int_0^\infty \frac{q \ dq}{F(q^2)} \sin(qr)
\]

(21)
Our aim is to determine the leading behaviour of $I(r)$ for $r \to \infty$. To this end, one can introduce a momentum scale $\delta$ and decompose $I(r)$ as

$$I(r) = I_1(r) + I_2(r) + I_3(r)$$

where

$$I_1(r) = \frac{1}{2\pi^2r} \int_0^\infty \frac{1}{q} \left( \frac{q^2}{F(q^2)} - \frac{q^2 + \delta^2}{Aq^2 + \alpha\delta^2} \right) \sin(qr) \, dq$$

$$I_2(r) = \frac{1}{2\pi^2r} \int_0^\infty \frac{1}{q} \left( \frac{q^2 + \delta^2}{Aq^2 + \alpha\delta^2} - \frac{1}{\alpha} \right) \sin(qr) \, dq$$

$$I_3(r) = \frac{1}{2\pi^2\alpha r} \int_0^\infty \frac{\sin(qr)}{q} \, dq = \frac{1}{4\pi\alpha r}$$

By introducing the function

$$g(q) = \frac{1}{q} \left( \frac{q^2}{F(q^2)} - \frac{q^2 + \delta^2}{Aq^2 + \alpha\delta^2} \right)$$

one gets

$$\int_0^\infty dq \, |g(q)| < +\infty$$

(i.e. $g(q) \in L^{(1)}$). For this reason by defining

$$\hat{g}(r) = \int_0^\infty g(q) \sin(qr) \, dq$$

one finds

$$\lim_{r \to \infty} \hat{g}(r) = 0$$

for the Riemann-Lebesgue theorem [10]. Thus $I_1(r) = \hat{g}(r)/(2\pi^2r)$ vanishes faster than $1/r$ when $r \to \infty$. On the other hand, one also finds

$$I_2(r) = \frac{1}{2\pi^2r} \frac{\alpha - A}{\alpha} \int_0^\infty \frac{q}{Aq^2 + \alpha\delta^2} \sin(qr) \, dq = \frac{\alpha - A e^{-\sqrt{\frac{2}{\alpha}} \delta r}}{A\alpha} \frac{\delta r}{4\pi r}$$

Therefore, in the $r \to \infty$ limit, the leading behaviour of $I(r)$ is given by $I_3(r)$. 

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