Nonlinear automorphism of the conformal algebra in 2D and continuous $\sqrt{TT}$ deformations

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Abstract: The conformal algebra in 2D (Diff($S^1$)$\oplus$Diff($S^1$)) is shown to be preserved under a nonlinear map that mixes both chiral (holomorphic) generators $T$ and $\bar{T}$. It depends on a single real parameter and it can be regarded as a “nonlinear $SO(1,1)$ automorphism.” The map preserves the form of the momentum density and naturally induces a flow on the energy density by a marginal $\sqrt{T\bar{T}}$ deformation. In turn, the general solution of the corresponding flow equation of the deformed action can be analytically solved in closed form, recovering the nonlinear automorphism. The deformed CFT$_2$ can also be described through the original theory on a field-dependent curved metric whose lapse and shift functions are given by the variation of the deformed Hamiltonian with respect to the energy and momentum densities, respectively. The conformal symmetries of the deformed theories can then also be seen to arise from diffeomorphisms that fulfill suitably deformed conformal Killing equations. Besides, Cardy formula is shown to map to itself under the nonlinear automorphism. As a simple example, the deformation of $N$ free bosons is briefly addressed, making contact with recent related results and the dimensional reduction of the ModMax theory. Furthermore, the nonlinear map between the conformal algebra in 2D and its ultra/non-relativistic versions (BMS$_3$$\approx$CCA$_2$$\approx$GCA$_2$), including the corresponding finite $\sqrt{T\bar{T}}$ deformation, are recovered from a limiting case of the nonlinear automorphism. The extension to a three-parameter nonlinear $ISO(1,1)$ automorphism of the conformal algebra, and a discrete nonlinear automorphism of BMS$_3$ are also briefly discussed.
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1 Introduction

Symmetries spanned by infinite-dimensional Lie algebras tightly constrain the possibilities to devise theories or computing amplitudes, among others. Along these lines, consistent nonlinear maps between the generators of infinite-dimensional linear algebras can also be very useful, but since they are just sporadically found, the systematic study of this kind of mappings turns out to be uncharted territory either in physics or mathematics (for a recent related discussion see [1]). A time-honored example of this sort is the Sugawara construction [2], which is known to play a very relevant role in the context of conformal field theories (CFT’s) in two-dimensional spacetimes. In this class of mapping, the generators of the Virasoro algebra are obtained from a precise quadratic combination of those of an affine Kac-Moody algebra (see e.g. [3, 4]). Similar quadratic Sugawara-like relationships, realized in a variety of setups, are also known to exist for the ultra/non-relativistic limit of the conformal algebra in 2D (CCA2≈GCA2≈BMS3)) [5–9], and its supersymmetric extensions [10–15] whose generators also emerge from different quadratic combinations of current algebras.

Another type of nonlinear mappings of the class under discussion is naturally formulated in the context of integrable systems. Indeed, since the Virasoro algebra describes one of the Poisson structures of the KdV hierarchy (see e.g., [16–18]), their generators are also nonlinearly related to the infinite-dimensional Abelian algebra of conserved charges, spanned by the subset of commuting generators of the enveloping algebra. A similar nonlinear relationship of this kind is also known to hold for the generators of the BMS3 algebra [19].

A different kind of nonlinear map that requires going beyond the enveloping algebra has been recently introduced in [20], which relates the generators of the classical (nonanomalous) conformal algebra in 2D (Diff(S1)⊕Diff(S1)) with those of the BMS3 algebra. Under this mapping, the Hamiltonian of a generic CFT2, with chiral (holomorphic) conformal generators given by $T$ and $\bar{T}$, acquires a finite marginal non-analytic deformation determined by $\sqrt{T\bar{T}}$, so that the deformed theory it is no longer a CFT2, but a conformal Carrollian field theory instead, because its energy and momentum densities fulfill the BMS3 algebra1. Remarkably, no limiting process is involved in the nonlinear map between the conformal algebra and its ultra/non-relativistic version. This map, and its corresponding deformation were then shown to be recovered through a class of “infinite boosts” spanned by certain degenerate (non-invertible) linear transformations acting on the coordinates [31]. Further recent interesting results about continuous $\sqrt{T\bar{T}}$ deformations have also been addressed for field theories in 2D along different approaches and points of view in [32], [33], [34] (see also [35], [36]), and for the deformation of two free harmonic oscillators in classical mechanics [37].

One of the main purposes of our work is showing that the conformal algebra in 2D admits a “nonlinear SO(1,1) automorphism” that maps the algebra to itself, which once

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1These marginal deformations are clearly different from the well-known irrelevant $T\bar{T}$ ones [21–23], whose diverse properties have been studied in e.g., [24–29] (for a review and further references see also [30]).
implemented on generic classical CFT$_2$’s, continuous $\sqrt{TT}$ deformations naturally emerge; as well as exploring some of its dynamical, geometric, thermodynamic and limiting features.

2 Nonlinear $SO(1, 1)$ automorphism

The Poisson brackets of the generators of the conformal algebra in 2D, spanned by two copies of the Witt (centerless Virasoro) algebra, fulfill

$$\{T(\phi), T(\varphi)\} = (2T(\phi) \partial_\phi + \partial_\phi T(\phi)) \delta(\phi - \varphi) ,$$
$$\{\bar{T}(\phi), T(\varphi)\} = - (2\bar{T}(\phi) \partial_\phi + \partial_\phi \bar{T}(\phi)) \delta(\phi - \varphi) ,$$

with $\{T(\phi), \bar{T}(\varphi)\} = 0$. Thus, Poisson brackets of functionals of the form $F = F[T, \bar{T}]$ can be directly computed in terms of the “fundamental” brackets in (2.1). One can then verify that the following nonlinear relations

$$T_\alpha = T \cosh^2 \left( \frac{\alpha}{2} \right) + \bar{T} \sinh^2 \left( \frac{\alpha}{2} \right) - \sqrt{TT} \sinh(\alpha) ,$$
$$\bar{T}_\alpha = \bar{T} \cosh^2 \left( \frac{\alpha}{2} \right) + T \sinh^2 \left( \frac{\alpha}{2} \right) - \sqrt{TT} \sinh(\alpha) ,$$

where $\alpha$ is a dimensionless constant, are such that $T_\alpha$ and $\bar{T}_\alpha$ precisely obey the same conformal algebra of $T$ and $\bar{T}$ in (2.1). In order to carry out an explicit checking, it is useful to note that $\sqrt{T}$ and $\sqrt{\bar{T}}$ behave as $U(1)$ currents, and in terms that currents, the map in (2.2) can be seen as a standard (linear) $SO(1, 1)$ automorphism of their algebra (see section 5).

Therefore, eq. (2.2) provides the searched for nonlinear map of the conformal algebra (2.1) to itself, that can be regarded as a nonlinear $SO(1, 1)$ automorphism.

It is also useful to change the basis of the conformal algebra (2.1) in terms of the energy and momentum densities, given by $H = T + \bar{T}$ and $J = T - \bar{T}$, respectively. In this basis, the nonlinear automorphism (2.2) manifestly preserves the form of the momentum density, since

$$J = T - \bar{T} = T_\alpha - \bar{T}_\alpha ,$$

and naturally induces a flow on the energy density by a continuous marginal $\sqrt{TT}$ deformation, given by $H_\alpha = T_\alpha + \bar{T}_\alpha = \cosh(\alpha) (T + \bar{T}) - 2\sinh(\alpha) \sqrt{TT}$, or equivalently

$$H_\alpha = \cosh(\alpha) H - \sinh(\alpha) \sqrt{H^2 - J^2} .$$

It also worth pointing out that this nonlinear automorphism is the most general one that preserves the momentum density $J$. Indeed, one can prove that for a generic nonlinear relationship of the form $\tilde{H} = \tilde{H}[H, J]$, requiring the deformed generators $J$ and $\tilde{H}$ to fulfill the conformal algebra fixes $\tilde{H} = H_\alpha$ as in (2.4).

3 Continuous $\sqrt{TT}$ deformations

As a direct consequence of the nonlinear automorphism (2.4), if the energy density of a CFT$_2$ is replaced by the “spectrally flowed” one in (2.4), i.e., $H \rightarrow H_\alpha$, the corresponding
deformed action \( I_\alpha \) necessarily retains the conformal symmetry. The deformation of a generic CFT\(_2\) action can then be readily implemented in Hamiltonian form, which in the conformal gauge explicitly reads

\[
I_\alpha = \int d^2 x \, L_\alpha = \int d^2 x \left( \Pi \dot{\Phi} - H_\alpha \right),
\]

(3.1)

where \( \Phi \) and \( \Pi \) collectively denote the fields and their momenta.

It is also worth noting that, since the marginally deformed energy density \( H_\alpha \) in (2.4) identically fulfills

\[
\frac{\partial H_\alpha}{\partial \alpha} = \sqrt{H_\alpha^2 - J^2},
\]

(3.2)

the deformed Lagrangian satisfies the following flow equation

\[
\frac{\partial L_\alpha}{\partial \alpha} = \sqrt{H_\alpha^2 - J^2} = 2 \sqrt{T_\alpha T_\alpha} = \sqrt{\det (T^{(\alpha)\mu\nu})},
\]

(3.3)

where \( T^{(\alpha)\mu\nu} \) is the stress-energy tensor of the deformed CFT\(_2\). Thus, eq. (3.3) precisely agrees with the flow that has been recently considered in [32], [33] and [34] (see also [35], [36]), for some concrete examples in the Lagrangian formalism. Indeed, in [32] it was shown that a dimensional reduction of the ModMax theory [38] (see also [39, 40]) to two spacetime dimensions corresponds to the continuous \( \sqrt{T T} \) deformation of \( N = 2 \) free bosons, and then extended the deformation to arbitrary \( N \). The deformation of free bosons was also addressed in [33] from a more general approach, where it was argued that the deformation should also work well for a wider class of examples, as confirmed in [36]. The same result for free bosons was also obtained in [34] from a perturbative approach. Later on, in [35] it was shown that the flow equation (3.2) in Lagrangian form can be reduced to a set of first order non-linear PDE’s by method of characteristics, and applied them to recover the deformation of free bosons.

We should highlight that one of the advantages of the Hamiltonian approach is that the general solution of the Lagrangian flow equation (3.3) can be analytically solved in closed form, recovering the nonlinear automorphism of the conformal algebra. Indeed, solving (3.3) in Hamiltonian form amounts to find the general solution of (3.2), which is given by

\[
H_\alpha = |J| \cosh (\alpha - f (H, J)),
\]

(3.4)

with \( f \) an arbitrary function of \( H \) and \( J \), whose precise form becomes fixed as

\[
f (H, J) = \cosh^{-1} \left( \frac{H}{|J|} \right),
\]

(3.5)

once the initial condition \( H_0 = H \) is imposed. Therefore, the energy flow \( H_\alpha \) in (2.4) is recovered once (3.5) is replaced into (3.4).

As a simple explicit example, the continuous \( \sqrt{T T} \) deformation of \( N \) free bosons with flat target metric

\[
I_0 [\Phi^I] = -\frac{1}{2} \int d^2 x \sqrt{-g} \delta_{1K} \partial^I \Phi^J \partial^\mu \Phi^K,
\]

(3.6)
can then be readily implemented just by replacing the energy density $H = \frac{1}{2} (\Pi' I + \Phi'^I \Phi'_I)$ by $H_\alpha$ in (2.4), where $I = \frac{\delta L}{\delta \dot{\Phi}^I}$ and $J = \Pi_I \Phi'^I$, so that the deformed Hamiltonian action in the conformal gauge (3.1) in this case reads

$$I_\alpha [\Phi^I, \Pi_J] = \int dx^2 \left( \Pi_I \dot{\Phi}^I - H_\alpha \right), \quad (3.7)$$

with

$$H_\alpha = \frac{1}{2} (\Pi' I + \Phi'^I \Phi'_I) \cosh (\alpha) - \sinh (\alpha) \sqrt{\frac{1}{4} (\Pi' I + \Phi'^I \Phi'_I)^2 - (\Pi_I \Phi'^I)^2}. \quad (3.8)$$

Note that, as it also occurs for the ModMax theory [38] (see also [41]), the equivalence of the deformed Lagrangian action for the free bosons obtained in [32], [33], [34] and also in [35] with the Hamiltonian action given by (3.7) with (3.8), requires careful implementation of a suitable Legendre transformation. In the case of a single free boson ($N = 1$) the deformation is trivial because the spectrally flowed energy simplifies as $H_\alpha = \frac{1}{2} (\Pi^2 e^{-\alpha} + e^{\alpha} \Phi'^2)$, so that the momentum is obtained from its own field equation $\Pi = e^{\alpha} \dot{\Phi}$. Thus, substituting in (3.7), the deformed action is just a rescaling of (3.6), given by $I_\alpha = e^{\alpha} I_0$.

A manifestly covariant form of the continuously $\sqrt{T \bar{T}}$-deformed action is briefly discussed in section 5.

4 Geometric realization

Following the lines of [20] one can show that the continuous $\sqrt{T \bar{T}}$ deformation can be geometrically implemented in terms of the original (undeformed) CFT$_2$ on a precise field-dependent Riemannian metric. It is then useful and instructive to consider the undeformed theory in a generic (non-conformal) gauge, so that in a local patch, the two-dimensional background metric belongs to the conformal class of the following one

$$ds^2 = -N^2 dt^2 + \left( d\phi + N^\phi dt \right)^2, \quad (4.1)$$

where $N$, $N^\phi$ are the lapse and shift functions, respectively. Since the total Hamiltonian of the undeformed CFT$_2$ is now given by

$$\mathcal{H}_0 = \int d\phi \left( \mu T + \bar{\mu} \bar{T} \right) = \int d\phi \left( NH + N^\phi J \right), \quad (4.2)$$

with $\mu = N + N^\phi$ and $\bar{\mu} = N - N^\phi$, that of deformed theory is obtained through the replacement $H \rightarrow H_\alpha$, which reads

$$\mathcal{H}_\alpha = \int d\phi \left( \mu T_\alpha + \bar{\mu} \bar{T}_\alpha \right) = \int d\phi \left( NH_\alpha + N^\phi J \right). \quad (4.3)$$

The conservation laws then acquire the following form

$$\dot{T}_\alpha = \{ T_\alpha, \mathcal{H}_\alpha \} = 2T_\alpha \mu' + \mu T'_\alpha, \quad (4.4)$$

$$\dot{\bar{T}}_\alpha = \{ \bar{T}_\alpha, \mathcal{H}_\alpha \} = - \left( 2\bar{T}_\alpha \bar{\mu}' + \bar{\mu} \bar{T}'_\alpha \right), \quad (4.5)$$

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and in absence of global obstructions, the canonical generators can be written as
\[ Q_\alpha [\eta, \bar{\eta}] = \int d\phi (\eta T_\alpha + \bar{\eta} \bar{T}_\alpha) , \]  

being conserved \( \dot{Q}_\alpha = 0 \) provided the parameters fulfill
\[ \dot{\eta} = \mu \eta' - \eta \mu', \quad \dot{\bar{\eta}} = \bar{\eta} \bar{\mu}' - \bar{\mu} \bar{\eta}' . \]  

The transformation laws of \( T_\alpha \) and \( \bar{T}_\alpha \) then follow from \( \delta T_\alpha = \{ T_\alpha, Q_\alpha [\eta] \} \) and \( \delta \bar{T}_\alpha = \{ \bar{T}_\alpha, Q_\alpha [\bar{\eta}] \} \), which in terms of null coordinates, \( x = t + \phi \) and \( \bar{x} = t - \phi \), read
\[ \delta T_\alpha = 2T_\alpha \partial \eta + \partial T_\alpha \eta , \quad \delta \bar{T}_\alpha = 2\bar{T}_\alpha \partial \bar{\eta} + \partial \bar{T}_\alpha \bar{\eta} . \]  

Note that in the conformal gauge \(( N = 1, N^\phi = 0 )\) the independent components of the stress energy tensor and the parameters become (anti-)chiral, i.e., \( T_\alpha = T_\alpha^\prime, \quad \bar{T}_\alpha = -\bar{T}_\alpha^\prime; \) \( \dot{\eta} = \eta', \quad \dot{\bar{\eta}} = -\bar{\eta}' \).

A geometric description of the continuous \( \sqrt{T\bar{T}} \) deformation then follows from the fact that the deformed total Hamiltonian \((4.3)\) is a homogeneous functional of degree one in \( T \) and \( \bar{T} \) (as well as in \( J, H \)) so that it fulfills
\[ \mathcal{H}_\alpha = \int d\phi \left( \frac{\delta \mathcal{H}_\alpha}{\delta T} T + \frac{\delta \mathcal{H}_\alpha}{\delta \bar{T}} \bar{T} \right) = \int d\phi \left( \frac{\delta \mathcal{H}_\alpha}{\delta H} H + \frac{\delta \mathcal{H}_\alpha}{\delta J} J \right) . \]  

Therefore, comparing \( \mathcal{H}_\alpha \) in \((4.9)\) and \((4.2)\), it is apparent that the deformed theory becomes equivalently described by the original CFT\(_2\) on a field-dependent curved metric, whose lapse and shift functions are respectively given by \( N_\alpha = \frac{\delta \mathcal{H}_\alpha}{\delta H} \) and \( N^\phi_\alpha = \frac{\delta \mathcal{H}_\alpha}{\delta J} \); i.e.,
\[ ds^2_{(\alpha)} = -\left( \frac{\delta \mathcal{H}_\alpha}{\delta H} \right)^2 dt^2 + \left[ d\phi + \left( \frac{\delta \mathcal{H}_\alpha}{\delta J} \right) dt \right]^2 , \]  

which in terms of \( T_\alpha \) and \( \bar{T}_\alpha \) reads
\[ ds^2_{(\alpha)} = -\left[ \cosh (\alpha) + \frac{T_\alpha + \bar{T}_\alpha}{2\sqrt{T_\alpha \bar{T}_\alpha}} \sinh (\alpha) \right]^{-2} N^2 dt^2 + \left[ d\phi + \left( N^\phi + \frac{(T_\alpha - \bar{T}_\alpha) \tanh (\alpha)}{(T_\alpha + \bar{T}_\alpha) \tanh (\alpha) + 2\sqrt{T_\alpha \bar{T}_\alpha}} N \right) dt \right]^2 , \]  

where \( N \) and \( N^\phi \) stand for the (field-independent) lapse and shift functions of the original undeformed background metric in \((4.1)\).\(^2\)

Note that once the theory is deformed, the manifest field dependence of the lapse and shift functions \( N_\alpha \) and \( N^\phi_\alpha \) of the deformed metric \((4.11)\) yields a local obstruction that prevents to gauge them away.

\(^2\)As in [20], the Ricci scalar of the field-dependent metric \( g^{(\alpha)}_{\mu\nu} \) is not diffeomorphic to that of the undeformed one \( g_{\mu\nu} (R^{(\alpha)} \neq R) \). Following another approach, and in the context of \( TT\)-like deformations, different field-dependent modifications of the background metric have also been proposed in [32]. In contradistinction, note that in the geometric interpretation of the \( TT \) deformations, the corresponding background metrics turn out to be related through field-dependent diffeomorphisms [42–45].

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It is also worth highlighting that the field-dependent Riemannian metric $g^{(α)}_{μν}$ and the conserved stress-energy tensor $T^{(α)}_{νμ}$, become inextricably intertwined. Thus, since both structures are field dependent, their functional variation

$$
\delta ξ g^{(α)}_{μν} = \frac{δ g^{(α)}_{μν}}{δ T^α_ν} \delta ξ T^α_ν + \frac{δ g^{(α)}_{μν}}{δ T^α_ν} \delta ξ \bar{T}^α_ν , \ \ δ ξ T^{(α)}_{νμ} = \frac{δ T^{(α)}_{νμ}}{δ T^α_ν} \delta ξ T^α_ν + \frac{δ T^{(α)}_{νμ}}{δ T^α_ν} \delta ξ \bar{T}^α_ν , \ (4.12)
$$

has to be taken into account when acting on them with diffeomorphisms $ξ = ξ^μ \partial_μ$. Hence, the symmetries of the deformed theory can be seen to geometrically arise from diffeomorphisms preserving both relevant structures up to local scalings, which yields to deformed conformal Killing equations, given by

$$
\nabla_μ^{(α)} ξ^ν + \nabla_ν^{(α)} ξ^μ - λ g^{(α)}_{μν} = δ ξ g^{(α)}_{μν} , \\
\mathcal{L}_ξ T^{(α)}_{νμ} = δ ξ T^{(α)}_{νμ} , \quad (4.13)
$$

where $\nabla_μ^{(α)}$ is the covariant derivative with respect to the metric $g^{(α)}_{μν}$, and $\mathcal{L}_ξ$ stands for the Lie derivative.

Noteworthy, the deformed conformal Killing equations (4.13) can be exactly solved from scratch for the deformed metric and stress-energy tensor, $g^{(α)}_{μν}$ and $T^{(α)}_{νμ}$, whose solution is given by diffeomorphisms of the form

$$
ξ^μ = N^{-1} \left( η_H, N η_J - N^φ η_H \right) , \quad (4.14)
$$

with parameters $η = η_H + η_J$ and $\bar{η} = η_H - η_J$, that fulfill the same eq. (4.7) above, while the transformation laws of $T^α_ν$ and $\bar{T}^α_ν$ are found to be given by the expected ones as in (4.8). It is then amusing to verify that the standard conformal Killing vectors emerge from solving (4.13) on the field-dependent deformed metric $g^{(α)}_{μν}$ in (4.11), due to the fact that the conformal Killing equation has been suitably deformed. Therefore, the transformation law of the fields in the deformed theory agrees with those of the original undeformed (primary) fields. Indeed, collectively denoting the fields by $Φ$, and expressing them in covariant form, the conformal symmetries of the deformed theory become just spanned by $δ_Φ = \mathcal{L}_ξ Φ$.

5 Overview and ending remarks

Since the nonlinear automorphism (2.2) preserves the conformal symmetries, its corresponding continuous $\sqrt{T \bar{T}}$ deformation also yields to a CFT$_2$, and hence the deformation keeps the integrability properties of the undeformed theory. Furthermore, as pointed out in section 2, $\sqrt{T}$ and $\sqrt{\bar{T}}$ behave as currents, because their corresponding Poisson brackets fulfill

$$
\begin{align}
\{ \sqrt{T}(φ), T(φ) \} &= \partial_φ \left( \sqrt{T}(φ) δ(φ - φ) \right) , \quad (5.1) \\
\{ \sqrt{T}(φ), \sqrt{\bar{T}}(φ) \} &= \frac{1}{2} \partial_φ δ(φ - φ) , \quad (5.2)
\end{align}
$$
and similarly for $\sqrt{T}$ with a corresponding minus sign on the r.h.s. Thus, $\sqrt{T}$ and $\sqrt{\bar{T}}$ can be naturally assembled as the components of a vector

$$j^I = \left( \begin{array}{c} \sqrt{T} \\ \sqrt{\bar{T}} \end{array} \right),$$

(5.3)

so that

$$\{ j^I (\phi), j^J (\varphi) \} = -\frac{1}{2} \eta^{IJ} \partial_\phi (\phi - \varphi),$$

(5.4)

with $\eta^{IJ} = \text{diag}(-1,1)$. Note that in terms of the currents $j^I$, the map (2.2) is equivalently expressed as

$$j^I (\alpha) = \Lambda^I_K \left( \frac{\alpha}{2} \right) j^K,$$

(5.5)

with

$$\Lambda^I_K (\alpha) = \left( \begin{array}{cc} \cosh (\alpha) & -\sinh (\alpha) \\ -\sinh (\alpha) & \cosh (\alpha) \end{array} \right),$$

(5.6)

and therefore, the nonlinear automorphism in (2.2) can be seen to emerge from a standard (linear) $SO(1,1)$ one, through a sort of “inverse Sugawara construction”. The currents $j^I$ are also useful in order to construct manifestly invariant objects under the nonlinear automorphism (2.2), as it is the case of the momentum density, since it can be expressed as

$$J = T - \bar{T} = -\eta_{IJ} j^I j^J.$$  

(5.7)

In this sense, it is also worth pointing out that for negative values of $T$ and $\bar{T}$, consistency of the map requires using the negative branch of the square root in (2.2). Hence for $T = -T$ and $T = -\bar{T}$, with $T, \bar{T} > 0$, the corresponding currents assemble as components of a contravariant vector, i.e.,

$$\mathcal{J}^I = \left( \sqrt{-T} \ \sqrt{-\bar{T}} \right),$$

(5.8)

because they transform according to

$$\mathcal{J}^I_{\alpha} = \mathcal{J}^K \Lambda^I_K \left( \frac{\alpha}{2} \right).$$

(5.9)

Besides, the centrally extended conformal algebra, described by two copies of the Virasoro algebra, can also be seen to admit a nontrivial automorphism that is necessarily nonlocal. Nonetheless, the nonlinear automorphism (2.2) still holds when only the zero modes are involved in the map. Consequently, Cardy formula can be seen to be invariant under the nonlinear automorphism. Indeed, once expressed in terms of the (negative) left and right ground state energies, denoted by $T$ and $\bar{T}$, respectively, the asymptotic growth of the number of states can be written as

$$S = 4\pi \left( \sqrt{-T \bar{T}} + \sqrt{-\bar{T} T} \right) = 4\pi \mathcal{J}_I j^I = 4\pi \mathcal{J}^a_I j^I_{\alpha} = 4\pi \left( \sqrt{-T_{\alpha} T_{\alpha}} + \sqrt{-\bar{T}_{\alpha} T_{\alpha}} \right),$$

(5.10)

being manifestly invariant under the nonlinear $SO(1,1)$ automorphism.
Furthermore, the nonlinear map between the conformal algebra in 2D and its ultra/non-relativistic version recently found in [20], including the corresponding finite $\sqrt{TT}$ deformation from a CFT$_2$ to a field theory invariant under BMS$_3$, can be recovered from the nonlinear automorphism in a limiting case. This can be seen as follows: rescaling $H_\alpha$ in (2.4) as

$$P_\alpha = \frac{H_\alpha}{\cosh (\alpha)} = H - \tanh (\alpha) \sqrt{H^2 - J^2}, \tag{5.11}$$

one can perform an In"on"u-Wigner contraction of conformal algebra spanned by $J$ and $P_\alpha$, so that in the limit $\alpha \to \pm \infty$, both branches of the supertranslation generators of the map

$$P_{(\pm)} = H \mp \sqrt{H^2 - J^2}, \tag{5.12}$$

clearly fulfill BMS$_3$ algebra; in full agreement with the result in [20] that was directly obtained from (5.12) without any sort of limiting process. Note that since the energy density rescales as in (5.11), consistency with time evolution implies that the time coordinate rescales according to $\tilde{t} = t \cosh (\alpha)$, so that in terms of $\tilde{t}$ the deformed action reads

$$I_\alpha = I_0 - 2 \tanh (\alpha) \int d\tilde{t} d\phi \sqrt{\tilde{T} \tilde{T}}. \tag{5.13}$$

Thus, before taking the limit, and for a generic gauge choice, the continuously $\sqrt{TT}$-deformed deformed action can be written in a manifestly covariant way as

$$I_\alpha = I_0 - \tanh (\alpha) \int d^2 \tilde{x} \sqrt{\det T_{\mu\nu}}, \tag{5.14}$$

where $T_{\mu\nu}$ stands for stress-energy tensor of the undeformed CFT$_2$, and it is implicitly assumed that $I_0$ is given in Hamiltonian form.

Hence, in the limit $\alpha \to \pm \infty$, the finite $\sqrt{TT}$ deformation in [20], given by

$$\tilde{I} = I_0 \pm \int d^2 \tilde{x} \sqrt{\det T_{\mu\nu}}, \tag{5.15}$$

is recovered. From this procedure once concludes that that the finite $\sqrt{TT}$ deformation necessarily yields to an ultra-relativistic theory, since the speed of light can be identified as $c = 1/\cosh (\alpha)$.

An interesting remark that concerns the pure BMS$_3$ algebra is in order. Note that each branch of the nonlinear map from the conformal algebra in 2D to BMS$_3$ in (5.12) reproduces precisely the same algebra. Thus, since both branches are related as

$$P_{(+)} = \frac{J^2}{P_{(-)}}, \tag{5.16}$$

this last equation provides a nontrivial discrete nonlinear automorphism of the BMS$_3$ algebra. Indeed, one can prove that this is the most general nonlinear automorphism of the form $\tilde{P} = \tilde{P} \{P, J\}$ that preserves superrotations $J$ (up to a trivial constant rescaling of supertranslations). The discrete automorphism (5.16) appears to be deeply related to the
inequivalent kinds of Carrollian limits discussed in [46], dubbed as those of electric and magnetic type (see also [47]). Indeed, for a single scalar field, the corresponding deformation of the action associated to the nonlinear discrete automorphism of BMS$_3$ in (5.16) maps the Carrollian electric-like action to the magnetic-like one and vice-versa. In this sense, (5.16) could be regarded as a duality relation between Carrollian theories of electric and magnetic type.

As an ending remark, we point out that the nonlinear $SO(1,1)$ automorphism of the conformal algebra in 2D can be extended to a three-parameter $ISO(1,1)$ one, given by

$$T_{\alpha,\eta} = \left[ \sqrt{T} \cosh \left( \frac{\alpha}{2} \right) - \sqrt{\bar{T}} \sinh \left( \frac{\alpha}{2} \right) + \eta \right]^2,$$

$$T_{\alpha,\bar{\eta}} = \left[ \sqrt{T} \cosh \left( \frac{\alpha}{2} \right) - \sqrt{\bar{T}} \sinh \left( \frac{\alpha}{2} \right) + \bar{\eta} \right]^2,$$

with $\alpha$, $\eta$ and $\bar{\eta}$ constants. Note that in terms of the currents (5.3) the extended automorphism reads

$$j^I_{\langle \alpha, \eta, \bar{\eta} \rangle} = \Lambda^I_J \left( \frac{\alpha}{2} \right) j^J + \eta^I,$$

with

$$\eta^I = \begin{pmatrix} \eta \\ \bar{\eta} \end{pmatrix},$$

which clearly corresponds to a standard linear automorphism. Three-parameter deformations of CFT$_2$'s can then be readily performed by virtue of the extended nonlinear automorphism. Nonetheless, the momentum density is no longer preserved, and the energy density ceases to be homogeneous of degree one, which precludes a direct geometric interpretation of the 3-parameter deformation of the action, as that performed in section 4. Further details about the extended automorphism, as well as its application in the context of celestial holography (see e.g., [48–50]), are expected to be discussed elsewhere.

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