A Note on Approximating 2-Transmitters

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Abstract. A \textit{k-transmitter} in a simple orthogonal polygon \(P\) is a mobile guard that travels back and forth along an orthogonal line segment \(s\) inside \(P\). The \textit{k}-transmitter can see a point \(p\in P\) if there exists a point \(q\in s\) such that the line segment \(pq\) is normal to \(s\) and \(pq\) intersects the boundary of \(P\) in at most \(k\) points. In this paper, we give a 2-approximation algorithm for the problem of guarding a monotone orthogonal polygon with the minimum number of 2-transmitters.

1 Introduction

In the standard version of the art gallery problem, introduced by Klee in 1973 \cite{10}, we are given a simple polygon \(P\) in the plane and the goal is to guard \(P\) by a set of point guards. That is, we need to find a set of point guards such that every point in \(P\) is seen by at least one of the guards, where a guard \(g\) sees a point \(p\) if and only if the segment \(gp\) is contained in \(P\). Chvátal \cite{1} proved that \(\lfloor n/3 \rfloor\) point guards are always sufficient and sometimes necessary to guard a simple polygon with \(n\) vertices. The art gallery problem is known to be NP-hard on arbitrary polygons \cite{8}, orthogonal polygons \cite{11} and even monotone polygons \cite{7}. Eidenbenz \cite{4} proved that the art gallery problem is APX-hard on simple polygons and Ghosh \cite{5} gave an \(O(\log n)\)-approximation algorithm that runs in \(O(n^4)\) time on simple polygons. Krohn and Nilsson \cite{7} gave a constant-factor approximation algorithm on monotone polygons. They also gave a polynomial-time algorithm for the orthogonal art-gallery problem that computes a solution of size \(O(OPT^2)\), where \(OPT\) is the cardinality of an optimal solution.

Many variants of the art gallery problem have been studied. Katz and Morgenstern \cite{6} introduced a variant of this problem in which \textit{k-transmitters} are used to guard orthogonal polygons. A \textit{k}-transmitter \(T\), where \(k \geq 0\), is a point guard that travels back and forth along an orthogonal line segment inside an orthogonal polygon \(P\). A point \(p\) in \(P\) is visible to \(T\), if there is a point \(q\) on \(T\) such that the line segment \(pq\) is normal to \(T\) and it intersects the boundary of \(P\) in at most \(k\) points. In the \textit{Minimum k-Transmitters (MkT)} problem, the objective is to guard \(P\) with the minimum number of \(k\)-transmitters. Katz and Morgenstern introduced the MkT problem for only \(k = 0\) (we remark that 0-transmitters are called \textit{sliding cameras} in \cite{6}). They first considered a restricted version of
Fig. 1: A monotone orthogonal polygon $P$ that can be guarded by a single 2-transmitter $s$ while five 0-transmitters are required to guard $P$ entirely. This example can be extended to show that the exact algorithm of de Berg et al. [2] for the M0T problem does not provide any constant-factor approximation to an exact solution for the M2T problem on $P$.

Our Result. In this paper, we give a polynomial-time 2-approximation algorithm for the M2T problem on simple and monotone orthogonal polygons. Some preliminaries are given in Section 2. We then present our 2-approximation algorithm in Section 3 and conclude the paper in Section 4.

2 Preliminaries

Throughout this paper, let $P$ be a simple and $x$-monotone orthogonal polygon with $n$ vertices. A vertex $u$ of $P$ is called convex (resp., reflex), if the angle at $u$ that is interior to $P$ is $90^\circ$ (resp., $270^\circ$). We denote the leftmost and rightmost vertical edges of $P$ (that are unique) by leftEdge($P$) and rightEdge($P$), respectively. Let $V_P = \{e_1 = \text{leftEdge}(P), e_2, \ldots, e_m = \text{rightEdge}(P)\}$, for some
Let $m > 0$, be the set of vertical edges of $P$ ordered from left to right. Let $P_i^+$ (resp., $P_i^-$), for some $1 \leq i \leq m$, denote the subpolygon of $P$ that lies to the right (resp., to the left) of the vertical line through $e_i$.

Let $s$ be an orthogonal line segment in $P$. We denote the left endpoint and the right endpoint of $s$ by $\text{left}(s)$ and $\text{right}(s)$, respectively. If $s$ is vertical, we define its left and right endpoints to be its upper and lower endpoints, respectively. Moreover, we denote the $k$-transmitter that travels along $s$ by $s^{(k)}$.

For each reflex vertex $v$ of $P$, extend the edges incident to $v$ inward until they hit the boundary of $P$. Let $C(P)$ be the set of all maximal line segments in $P$ that are obtained in this way. A feasible solution for the M2T problem is a set $M$ of 2-transmitters that guards the entire polygon $P$. A feasible solution $M$ is optimal (or, exact) if $|M| \leq |S'|$, for all feasible solutions $S'$. We say that a feasible solution $M$ for the M2T problem is in standard form if and only if $M \subseteq C(P)$ and every vertical 2-transmitter in $M$ is vertically maximal; that is, it extends as far upwards and downwards as possible.

**Lemma 1.** There exists an optimal solution $OPT^*$ for the M2T problem on $P$ that is in standard form.

**Proof.** Take any optimal solution $OPT$ for the M2T problem on $P$. First, for each line segment $s \in OPT$ that is not aligned with an edge of $P$, move $s$ vertically up or down, or horizontally to the left or right until it hits an edge of $P$. Next, for every line segment $s' \in OPT$ that is not maximal, replace $s'$ with the maximal line segment in $P$ that is aligned with $s'$. Set $OPT^* := OPT$. Clearly, $OPT^*$ is a feasible solution for the M2T problem and every line segment in $OPT^*$ is maximal and aligned with an edge of $P$. So, $OPT^* \subseteq C(P)$. Since $|OPT^*| \leq |OPT|$, we conclude that $OPT^*$ is an optimal solution for the M2T problem that is in standard form. This completes the proof of the lemma. □

For a horizontal line segment $t \in P$ and any $k > 0$, the visibility polygon of a 0-transmitter that travels along $t$ is the same as that of a $k$-transmitter that travels along $t$. We state and prove this observation more formally.

**Lemma 2.** Let $t$ be a horizontal line segment in $P$. Then, $\text{Vis}(t^{(0)}) = \text{Vis}(t^{(k)})$ for any $k > 0$.

**Proof.** It is clear that any point in $P$ that is visible to $t^{(0)}$ is also visible to $t^{(k)}$ and so $\text{Vis}(t^{(0)}) \subseteq \text{Vis}(t^{(k)})$. Now, let $p$ be a point in $P$ that is visible to $t^{(k)}$. Since $t$ is horizontal and $P$ is an $x$-monotone orthogonal polygon, we conclude that the line segment $pq$ does not intersect the boundary of $P$, where $q$ is the projection of $p$ onto $t$. This means that $p$ is also visible to $t^{(0)}$ and therefore, $\text{Vis}(t^{(k)}) \subseteq \text{Vis}(t^{(0)})$. This completes the proof of the lemma. □
3 A 2-Approximation Algorithm

In this section, we give our 2-approximation algorithm for the M2T problem on monotone orthogonal polygons. Recall that in the M2T problem, the objective is to guard the polygon $P$ with minimum number of 2-transmitters, where a 2-transmitter can be either horizontal or vertical. For a point $p \in P$, let $L(p)$ denote the vertical line through $p$. We say that a horizontal 2-transmitter in $P$ is rightward maximal if it extends as far to the right as possible.

The algorithm initially guards a leftmost portion of the polygon $P$ by two 2-transmitters with different orientations, and then will guard the remaining part of $P$ recursively. The order of the two initial 2-transmitters is determined by whether locating first a vertical 2-transmitter and then a horizontal one would guard a larger portion of $P$ than locating first a horizontal 2-transmitter and then a vertical one. In the following, we describe the algorithm more formally.

**Algorithm.** Let $s_v$ be the rightmost maximal vertical 2-transmitter in $P$ such that every point of $P$ that is to the left of $s_v$ is seen by $s_v$; let $p$ be the leftmost point of $P$ that is not seen by $s_v$. Moreover, let $s_h$ be the rightward maximal horizontal 2-transmitter in $P$ such that $\text{left}(s_h)$ lies on $L(p)$. Clearly, $\text{right}(s_h)$ lies on a vertical edge $e_i$ of $P$. Observe that $P_i^-$ is entirely guarded by $s_v$ and $s_h$. Given $P$, we define $vHFinder(P)$ as a method that computes $s_v$ and $s_h$ as described above and returns the triple $(s_v, s_h, e_i)$. Note that $vHFinder(P)$ guards $P_i^-$ by first locating a vertical 2-transmitter and then a horizontal one from left to right. We next consider the other case.

Let $s'_h$ be the rightward maximal horizontal 2-transmitter in $P$ such that every point of $P$ that is to the left of $L(\text{right}(s'_h))$ is seen by $s'_h$. Suppose that $\text{right}(s'_h)$ lies on some vertical edge $e_{\ell}$ ($1 \leq \ell \leq m$) of $P$. Let $s'_v$ be the rightmost maximal vertical 2-transmitter in $P$ such that every point of $P$ that lies between $L(\text{right}(s'_h))$ and $s'_v$ is guarded by $s'_v$. Moreover, let $p'$ be the leftmost point of $P_j^+$ that is not seen by $s'_v$; clearly, $p'$ lies on a vertical edge $e_j$ ($1 \leq j \leq m$) of $P$. Observe that $s'_h$ and $s'_v$ guard $P_j^-$ entirely. We now define $hVFinder(P)$ as a method that computes $s'_h$ and $s'_v$ as described above and returns the triple $(s'_h, s'_v, e_j)$.

The algorithm is shown in Algorithm [1]. In the first step of the algorithm, we remove from $C(P)$ those line segments whose visibility polygon is a subset of the union of the visibility polygons of all other line segments in $C(P)$. Then, in a while-loop, we iteratively (i) compute the pairs of 2-transmitters $\{s_v, s_h\}$ and $\{s'_h, s'_v\}$ using the methods $vHFinder(P)$ and $hVFinder(P)$, respectively, and then (ii) update $P$ depending on whether $i > j$ (i.e., the 2-transmitters $\{s_v, s_h\}$ guard a larger portion of $P$ than $\{s'_h, s'_v\}$) or $j \geq i$ (i.e., the 2-transmitters $\{s'_h, s'_v\}$ guard a larger portion of $P$ than $\{s_v, s_h\}$). We remark here that by Lemma [1] we can assume that both methods $vHFinder(P)$ and $hVFinder(P)$ select the 2-transmitters from the set $C(P)$. When $P$ is entirely guarded, we return the set $S$ of 2-transmitters.
Algorithm 1 APPROXIMATE2TRANSMITTERS($P$)

1: for each line segment $s \in C(P)$ do
2: if $\text{Vis}(s) \subseteq \bigcup_{s' \in C(P) \setminus \{s\}} \text{Vis}(s')$ then
3: $C(P) \leftarrow C(P) \setminus \{s\}$;
4: $S \leftarrow \emptyset$;
5: while $P \neq \emptyset$ do
6: $(s_{i, j, e_i}) \leftarrow \text{vHFinder}(P)$;\注释：$\{s_v, s_h\} \subseteq C(P)$
7: $(s_{h, s_{i, j}}) \leftarrow \text{hVFinder}(P)$;\注释：$\{s_{h}, s_{i}'\} \subseteq C(P)$
8: if $i > j$ then
9: $S \leftarrow S \cup \{s_v, s_h\}$;
10: $P \leftarrow P_i$;
11: else
12: $S \leftarrow S \cup \{s_{h}, s_{i}'\}$;
13: $P \leftarrow P_j$;
14: return $S$;

Analysis. We first note that by Lemma 1, we can assume that the four 2-transmitters computed by $\text{vHFinder}(P)$ and $\text{hVFinder}(P)$ are always in standard form. That is, we restrict our attention to the line segments in $C(P)$ when computing the set $S$. To see the approximation factor of the algorithm, let $P_1, P_2, \ldots, P_k$ be the partition of $P$ into $k$ subpolygons ordered from left to right such that the subpolygon $P_i$ is guarded in the $i$th iteration of the while-loop. More precisely, $P_i$ is the subpolygon of $P$ that is cut out from $P$ in the $i$th iteration of the while-loop of the algorithm. It is clear that Algorithm 1 locates at most $2k$ 2-transmitters to guard $P$ entirely; that is, $|S| \leq 2k$. In the following, we show that $|\text{OPT}| \geq k$ for any optimal solution $\text{OPT}$ for the M2T problem on $P$.

Lemma 3. Let $\text{OPT}$ be an optimal solution for the M2T problem on $P$. Then, $|\text{OPT}| \geq k$.

Proof. By Lemma 1 we assume that $\text{OPT}$ is in standard form; that is, $\text{OPT} \subseteq C(P)$ and every vertical 2-transmitter in $\text{OPT}$ is vertically maximal. Consider the partition $T = \{P_1, P_2, \ldots, P_k\}$ of $P$ induced by the recursive steps of the algorithm, and let $s$ be a horizontal line segment in $P$. We say that $s$ originates from $P_j$, for some $1 \leq j \leq k$, if $\text{left}(s)$ lies inside $P_j$. Suppose for a contradiction that $|\text{OPT}| < k$. Then, there must be a subpolygon $P_j \in T$ such that neither a vertical 2-transmitter of $\text{OPT}$ lies in $P_j$ nor a horizontal 2-transmitter of $\text{OPT}$ originates from $P_j$. We then must have one of the followings (w.l.o.g., we assume that Algorithm 1 located the pair $\{s_v, s_h\}$ in $P_j$):

- There exists at least one horizontal 2-transmitter in $\text{OPT}$ that intersects $\text{leftEdge}(P_j)$ (and, therefore its left endpoint lies to the left of $\text{leftEdge}(P_j)$). Let $s_{i}'$ be the rightward maximal horizontal 2-transmitter among all such 2-transmitters. Clearly, $s_{i}'$ does not see $P_j$ entirely because then $\text{hVFinder}(P)$
would have selected the portion of $s^*_h$ that lies in $P_i$ along with the vertical line segment $s^*_h$ and so $P_i$ would have been extended further to the right. Now, let $P'_i := P_i \setminus \text{Vis}(s^*_h)$. Since $s^*_h$ is rightward maximal and there is no horizontal 2-transmitter of $OPT$ that is originated from $P_i$, we conclude that no horizontal 2-transmitter in $OPT$ sees a point in $P'_i$. Therefore, there must a vertical 2-transmitter $s^*_v$ that guards $P'_i$ and that $s^*_v$ lies to the left of $\text{leftEdge}(P_i)$ or to the right of $\text{rightEdge}(P_i)$ (recall that there is no vertical 2-transmitter of $OPT$ inside $P_i$). (i) If $s^*_v$ lies to the right of $\text{rightEdge}(P_i)$, then our algorithm would have added $s^*_v$ and the portion of $s^*_h$ that lies in $P_i$ into $S$ and so $P_i$ would have been extended further to the right — a contradiction. (ii) If $s^*_v$ lies to the left of $\text{leftEdge}(P_i)$, then we observe that $s^*_v$ and $s_h$ (i.e., the horizontal 2-transmitter located in $P_i$ by our algorithm) would together guard $P_i$ entirely. This means that $\text{Vis}(s_v) \subseteq (\text{Vis}(s^*_v) \cup \text{Vis}(s_h))$ and so $s_v$ should have been removed from $C(P)$ in the first step of the algorithm — a contradiction.

• There is no horizontal 2-transmitter of $OPT$ intersecting $\text{leftEdge}(P_i)$. This means that no point inside $P_i$ is seen by a horizontal 2-transmitter in $P_i$. Moreover, since no vertical 2-transmitter of $OPT$ lies in $P_i$, we conclude that $P_i$ is guarded by a set $M \subseteq OPT$ of only-vertical 2-transmitters that lie to the left of $\text{leftEdge}(P_i)$ or to the right of $\text{rightEdge}(P_i)$. That is, $P_i \subseteq \bigcup_{s_j \in M} \text{Vis}(s_j)$. But, this means that $\text{Vis}(s_v) \subseteq \bigcup_{s_j \in M} \text{Vis}(s_j)$, which is a contradiction because then $s_v$ should have been removed from $C(P)$ in the first step of the algorithm.

By the two cases above, we conclude that $|OPT| \geq k$. This completes the proof of the lemma.

Each call to methods $vH\text{Finder}(P)$ and $hV\text{Finder}(P)$ is completed in polynomial time. Moreover, the while-loop of Algorithm 1 terminates after at most $m$ iterations (recall that $m$ is the number of the vertical edges of $P$) because at least one new vertical edge of $P$ is guarded at each iteration. Therefore, Algorithm 1 runs in polynomial time. Therefore, by Lemma 3 and the fact that $|S| \leq 2k$, we have the main result of this paper:

**Theorem 1.** There exists a polynomial-time 2-approximation algorithm for the $M2T$ problem on monotone orthogonal polygons.

## 4 Conclusion

In this paper, we gave a polynomial-time 2-approximation algorithm for the $M2T$ problem on monotone orthogonal polygons. The complexity of the problem remains open on simple orthogonal polygons. Similar to Katz and Morgenstern [6], it might be interesting to first consider the problem with only-vertical 2-transmitters.
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