A stationary approach for the Kato–Rosenblum theorem in von Neumann algebras

Qihui Li · Rui Wang

Received: 5 September 2022 / Accepted: 28 December 2022 / Published online: 12 January 2023 © Tusi Mathematical Research Group (TMRG) 2023

Abstract
Let \( \mathcal{M} \) be a countable decomposable properly infinite semifinite von Neumann algebra acting on a Hilbert space \( \mathcal{H} \). An analogue of the Kato–Rosenblum theorem in \( \mathcal{M} \) has been proved in Li et al. (J Funct Anal 275(2):259–287, 2018) by showing the existence of generalized wave operators. It is well known that there are two typical approaches to show the existence of wave operators in the scattering theory. One is called time-dependent approach and another is called stationary approach. The main purpose of this article is to exhibit the stationary scattering theory in \( \mathcal{M} \) and then to obtain the Kato–Rosenblum theorem in \( \mathcal{M} \) by the stationary approach instead of a time-dependent approach in Li et al. (2018).

Keywords Stationary approach · Generalized wave operators · von Neumann algebras

Mathematics Subject Classification 47C15 · 47A40 · 47A55

1 Introduction
This paper is a sequel to [7, 8], in which we studied the diagonalizations of self-adjoint operators modulo norm ideal in semifinite von Neumann algebras. (See [3, 9–11] or

Communicated by Esteban Andruchow.

Qihui Li qihui_li@126.com
Rui Wang 18121142030@163.com

1 School of Mathematics, East China University of Science and Technology, Shanghai 200237, People’s Republic of China
2 Shanghai Aerospace Control Technology Institute, Shanghai 200237, People’s Republic of China
[16] for more details about von Neumann algebras.) In particular, we give an analogue of Kato–Rosenblum theorem in a semifinite von Neumann algebra in [8].

Let \( \mathcal{H} \) be a complex separable infinite dimensional Hilbert space. Assume \( H \) and \( H_1 \) are densely defined self-adjoint operators on \( \mathcal{H} \) satisfying that \( H_1 - H \) is in the trace class, then the Kato–Rosenblum theorem asserts that the wave operator \( W_\pm (H_1, H) \) of \( H \) and \( H_1 \) exists and consequently the absolutely continuous parts of \( H \) and \( H_1 \) are unitarily equivalent. Thus, if a self-adjoint operator \( H \) in \( \mathcal{B}(\mathcal{H}) \) has a nonzero absolutely continuous spectrum, then \( H \) can not be a sum of a diagonal operator and a trace class operator. In [8], we introduce the concept of generalized wave operator \( W_\pm \) based on the notion of norm absolutely continuous projections. An analogue of Kato–Rosenblum theorem in a semifinite von Neumann algebra \( \mathcal{M} \) is obtained by showing the existence of the generalized wave operator \( W_\pm \). To be more precise, we proved that for self-adjoint operators \( H \) and \( H_1 \) affiliated with \( \mathcal{M} \) satisfying \( H_1 - H \in \mathcal{M} \cap L^1 (\mathcal{M}, \tau) \), the generalized wave operator \( W_\pm (H_1, H) \) exists and then the norm absolutely continuous part of \( H \) and \( H_1 \) are unitarily equivalent. It implies that a self-adjoint operator \( H \) in \( \mathcal{M} \) can not be a sum of a diagonal operator in \( \mathcal{M} \) (see Definition 1.0.1 in [7] for definition of a diagonal operator in \( \mathcal{M} \)) and an operator in \( \mathcal{M} \cap L^1 (\mathcal{M}, \tau) \) if there is a non-zero norm absolutely continuous projection with respect to \( H \) in \( \mathcal{M} \).

The above statements illustrate that showing the existence of wave operators is the key step to prove Kato–Rosenblum theorem in \( \mathcal{B}(\mathcal{H}) \) or \( \mathcal{M} \). In mathematical scattering theory, wave operator \( W_\pm \) is a fundamental concept and the existence of \( W_\pm \) is one of the main research topics in this area. Actually, there are two typical approaches to show the existence of \( W_\pm \). One is called time-dependent approach which has been used in [4, 13] and another is called stationary approach (see [1, 2, 15] or [17]). The methods which do not make explicit use of the time variable \( t \) are known as the stationary approach. An important merit of the stationary approach is the advanced formula part. We notice that the method in [8] to show the Kato–Rosenblum theorem in \( \mathcal{M} \) is a so-called time-dependent approach. Therefore, it is natural to ask whether there is a stationary approach in \( \mathcal{M} \). Thus, to explore a stationary method in \( \mathcal{M} \) is our main purpose in the current article. To be more precise, we will exhibit conditions under which the generalized stationary wave operators and the generalized wave operators exist and coincide. Consequently, we will be able to show the Kato–Rosenblum theorem in \( \mathcal{M} \) in [8] by a stationary approach.

The notion of the norm absolutely continuous support \( P_{ac}^\infty (H) \) of a self-adjoint operator \( H \) affiliated with \( \mathcal{M} \) plays a very important role in the Kato–Rosenblum theorem in \( \mathcal{M} \). Meanwhile the Kato smoothness given in [5] is a crucial concept in scattering theory. Therefore, it is of interest to give the relation between \( P_{ac}^\infty (H) \) and the Kato smoothness. Actually, we assert that

\[
P_{ac}^\infty (H) = \vee \{ R\left(G^*\right) : G \in \mathcal{M} is H-smooth \}.
\]

Therefore, for a self-adjoint \( H \) in \( \mathcal{M} \), if there is a \( H \)-smooth operator in \( \mathcal{M} \), then \( H \) is not a sum of a diagonal operator in \( \mathcal{M} \) and an operator in \( \mathcal{M} \cap L^1 (\mathcal{M}, \tau) \).

The construction of this paper is as follows. In Sect. 2, we prepare related notation, definitions and lemmas. We list the relation between the resolvent \( R_{H}(z) = \)
(H − z)^{-1} and unitary U_H(t) = \exp(-itH) for a self-adjoint operator H on H. We also recall the definitions of Kato smoothness and generalized wave operators. Some basic properties of generalized wave operators are discussed in this section too. Section 3 is focused on the main results of this paper. We first characterize the norm absolutely continuous support P_{\text{ac}}^\infty (H) of a self-adjoint operator H affiliated with \mathcal{M} in terms of the Kato smoothness. After giving conditions under which the generalized stationary wave operators and the generalized wave operators exist and coincide, we give a stationary proof of the Kato–Rosenblum theorem in \mathcal{M}.

2 Preliminaries and notation

Let \mathcal{H} be a complex Hilbert space and \mathcal{B}(\mathcal{H}) be the set of all bounded linear operators on \mathcal{H}. In this article, we assume that \mathcal{M} \subseteq \mathcal{B}(\mathcal{H}) is a von Neumann algebra and \mathcal{A}(\mathcal{M}) is the set of densely defined, closed operators affiliated with \mathcal{M}.

2.1 The unitary group and resolvent of a self-adjoint operator

Let H be any densely defined self-adjoint operator with domain \mathcal{D}(H) in \mathcal{H} and \{(E_H(\lambda))_{\lambda \in \mathbb{R}}\} be the spectral resolution of the identity for H where E_H(\lambda) = E_H((-\infty, \lambda]). The resolvent R_H(z) = (H − z)^{-1} and unitary U_H(t) = \exp(-itH) for H on \mathcal{H} will be frequently used in the current paper. The following equality gives us the relation between R_H(z) and U_H(t) (see [17, Section 1.4]).

\[
R_H(\lambda \pm i\varepsilon) = \pm i \int_0^\infty \exp(-\varepsilon t \pm i\lambda t) \exp(\pm itH) \, dt \tag{2.1}
\]

From the argument in [17, Section 1.4], we have

\[
\lim_{\varepsilon \to 0} \langle \delta_H(\lambda, \varepsilon) x, y \rangle = \frac{d}{d\lambda} \langle E_H(\lambda) x, y \rangle, \quad \text{a.e. } \lambda \in \mathbb{R} \tag{2.3}
\]

for x or y in \mathcal{H}_{\text{ac}}(H) where \mathcal{H}_{\text{ac}}(H) is the set of all vectors x \in \mathcal{H} such that the mapping \lambda \mapsto \langle E_H(\lambda) x, x \rangle, with \lambda \in \mathbb{R}, is a locally absolutely continuous function on \mathbb{R} (see [6] or [17] for more details). We also have

\[
\frac{d}{d\lambda} \langle E_H(\lambda) E_H(\Lambda) x, y \rangle = \chi_\Lambda(\lambda) \frac{d}{d\lambda} \langle E_H(\lambda) x, y \rangle, \quad \text{a.e. } \lambda \in \mathbb{R}, \tag{2.4}
\]

where \chi_\Lambda(\cdot) is the characteristic function of the Borel set \Lambda by the argument in the proof of Theorem X.4.4 in [6] or Section 1.3 in [17].
2.2 Kato smoothness and generalized wave operators

Kato smoothness plays a very important role in the mathematical scattering theory. It can be equivalently formulated in terms of the corresponding unitary group. We recall it in this section.

For a self-adjoint operator $H$, an operator $G : \mathcal{H} \to \mathcal{H}$ is called $H$-bounded if $\mathcal{D}(H) \subseteq \mathcal{D}(G)$ and $GR_H(z)$ is bounded for $z$ in the resolvent set $\rho = \rho(H)$.

**Theorem 2.1** (Theorem 4.3.1 in [17] or Theorem 5.1 in [5]) Let $H$ be a densely defined self-adjoint operator in $\mathcal{H}$. Assume that $G : \mathcal{H} \to \mathcal{H}$ is $H$-bounded operator, then the following conditions are equivalent.

1. $\gamma_1^2 = \frac{1}{2\pi} \sup_{x \in \mathcal{D}(H), \|x\|=1} \int_{\mathbb{R}} \| Ge^{\pm \lambda t} H x \|^2 dt < \infty$;
2. $\gamma_2^2 = \frac{1}{(2\pi)^2} \sup_{\|x\|=1, \epsilon > 0} \int_{\mathbb{R}} \left( \| GR_H(\lambda + i\epsilon) x \|^2 + \| GR_H(\lambda - i\epsilon) x \|^2 \right) d\lambda < \infty$;
3. $\gamma_3^2 = \sup_{\|x\|=1, \epsilon > 0} \int_{\mathbb{R}} \| G \delta_H(\lambda, \epsilon) x \|^2 d\lambda < \infty$;
4. $\gamma_4^2 = \sup_{\lambda \in \mathbb{R}, \epsilon > 0} \| G \delta_H(\lambda, \epsilon) G^* \| < \infty$;
5. $\gamma_5^2 = \sup_{\Lambda \subseteq \mathbb{R}} \frac{\| GE_H(\Lambda) G^* \|}{|\Lambda|} < \infty$.

All the constants $\gamma_j = \gamma_j(G)$, $j = 1, \ldots, 5$, are equal to one another.

**Definition 2.2** Let $H$ be a densely defined self-adjoint operator acting on the Hilbert space $\mathcal{H}$. If $G$ is $H$-bounded and one of the inequalities (1)–(5) holds (and then all of them), then operator $G$ is called Kato smooth relative to the operator $H$ ($H$-smooth). The common value of the quantities $\gamma_1, \ldots, \gamma_5$ is denoted by $\gamma_H(G)$.

**Remark 2.3** There are other expressions for the number $\gamma_H(G)$ given in the [17, Section 4.3]. In particular, for each of the sign “$\pm$”

$$\gamma_H^2(G) = \left( \frac{1}{2\pi} \right)^2 \sup_{\|x\|=1, \epsilon > 0} \int_{-\infty}^{\infty} \| GR_H(\lambda \pm i\epsilon) x \|^2 d\lambda. \quad (2.5)$$

Before giving the definition of generalized wave operators, we need to recall the following concepts which appeared first in [8]. For a self-adjoint element $H$ in $\mathcal{A}(\mathcal{M})$, let $\mathcal{P}_\infty^a(H)$ denote the collection of those projections $P$ in $\mathcal{M}$ such that the mapping $\lambda \mapsto PE_H(\lambda) P$ from $\mathbb{R}$ into $\mathcal{M}$ is locally absolutely continuous. Such a $P$ is called a norm absolutely continuous projection with respect to $H$. We set

$$P_\infty^a(H) = \vee \left\{ P : P \in \mathcal{P}_\infty^a(H) \right\}.$$

Such $P_\infty^a(H)$ is called the norm absolutely continuous support of $H$ in $\mathcal{M}$ and denote the range of $P_\infty^a(H)$ by $\mathcal{H}_\infty^a(H)$.

**Remark 2.4** Let $P_\infty^a(H)$ be the projection from $\mathcal{H}$ onto $\mathcal{H}_\infty^a(H)$. In [8], it has been shown that $P_\infty^a(H) \leq P_{\infty}^a(H)$ and $E_H(\Lambda) P_\infty^a(H) = P_\infty^a(H) E_H(\Lambda)$ for any Borel set $\Lambda \subseteq \mathbb{R}$.

The definition of generalized wave operators in $\mathcal{M}$ is given below.
Definition 2.5 [8] Let \( H, H_1 \in \mathcal{A}(\mathcal{M}) \) be a pair of self-adjoint operators and \( J \) be an operator in \( \mathcal{M} \). The generalized wave operator for a pair of self-adjoint operators \( H, H_1 \) and \( J \) is the operator

\[
W_{\pm}(H_1, H; J) = \text{s.o.t-} \lim_{t \to \pm \infty} e^{itH_1} J e^{-itH} P_{\text{ac}}^\infty(H)
\]

provided that the limit exists.

We note that the relation containing the signs “\( \pm \)" is understood as two independent equalities. After slightly modifying the proof of Theorem 5.2.5 in [8], we can conclude that

\[
\varphi(H_1) W_{\pm}(H_1, H; J) = W_{\pm}(H_1, H; J) \varphi(H)
\]

for any Borel function \( \varphi \).

Since different \( J \) might give us different \( W_{\pm} \), we give below a condition on \( J \) such that \( W_{\pm}(H_1, H; J) \) is an isometry on \( P_{\text{ac}}^\infty(H) \). Its proof is similar to Proposition 2.1.3 in [17], so we omit it.

Theorem 2.6 Let \( H, H_1 \in \mathcal{A}(\mathcal{M}) \) be a pair of self-adjoint operators and \( J \) be an operator in \( \mathcal{M} \). If \( W_{\pm}(H_1, H; J) \) exists, then \( W_{\pm}(H_1, H; J) \) is isometric on \( P_{\text{ac}}^\infty(H) \) if the strong operator limit

\[
\text{s.o.t-} \lim_{t \to \pm \infty} (J^* J - I) e^{-itH} P_{\text{ac}}^\infty(H) = 0.
\]

Lemma 2.7 (Lemma 5.2.3 [8]) Suppose \( H \) is a self-adjoint element in \( \mathcal{A}(\mathcal{M}) \). If \( S \in \mathcal{M} \) satisfies that the mapping \( \lambda \mapsto S^* E_H(\lambda) S \) from \( \mathbb{R} \) into \( \mathcal{M} \) is locally absolutely continuous, then \( R(S) \), the range projection of \( S \) in \( \mathcal{M} \), is a subprojection of \( P_{\text{ac}}^\infty(H) \).

Then, we can get the following result.

Proposition 2.8 Let \( H, H_1 \in \mathcal{A}(\mathcal{M}) \) be a pair of self-adjoint operators and \( J \) be an operator in \( \mathcal{M} \). If \( W_{\pm} \triangleq W_{\pm}(H_1, H; J) \) exists and the strong operator limit

\[
\text{s.o.t-} \lim_{t \to \pm \infty} (J^* J - I) e^{-itH} P_{\text{ac}}^\infty(H) = 0,
\]

then \( W_{\pm} W_{\pm}^* \leq P_{\text{ac}}^\infty(H_1) \).

Proof If

\[
\text{s.o.t-} \lim_{t \to \pm \infty} (J^* J - I) e^{-itH} P_{\text{ac}}^\infty(H) = 0,
\]

then \( W_{\pm} W_{\pm} = P_{\text{ac}}^\infty(H) \) by Theorem 2.6. From (2.6), for any \( P \in P_{\text{ac}}^\infty(H) \) and any Borel set \( \Lambda \subseteq \mathbb{R} \),

\[
(W_{\pm} P)^* E_{H_1}(\Lambda) (W_{\pm} P) = P W_{\pm}^* E_{H_1}(\Lambda) W_{\pm} P = P W_{\pm}^* W_{\pm} E_{H}(\Lambda) P = P E_{H}(\Lambda) P.
\]
It implies that the mapping $\lambda \rightarrow (W \pm P)^* E_{H_1}(\lambda) (W \pm P)$ from $\mathbb{R}$ into $\mathcal{M}$ is locally absolutely continuous with respect to $H_1$. Hence the range projection $R(W \pm P) \leq P_{ac}^\infty(H_1)$ by Lemma 2.7. Therefore $R(W) \leq P_{ac}^\infty(H_1)$ by the fact that $W P_{ac}^\infty(H) = W \pm P$. Hence $W \pm W^* \leq P_{ac}^\infty(H_1)$.

### 3 The stationary approach and the Kato–Rosenblum theorem in $\mathcal{M}$

#### 3.1 Characterization of norm absolutely continuous projections in $\mathcal{M}$

The cut off function $\omega_n$ is given in [8]. We refer the reader to [8] for its definition. Here, we only recall its useful property.

**Lemma 3.1** (Lemma 4.2.2 in [8]) Suppose $H$ is a self-adjoint element in $\mathcal{A}(\mathcal{M})$. For each $n \in \mathbb{N}$ and cut-off function $\omega_n$, let

$$
\omega_n(H) = \int_{\mathbb{R}} \omega_n(t) \, dE_H(t).
$$

Then $\omega_n(H) \in \mathcal{M}$ and

$$
\omega_n(H) \to I \text{ in strong operator topology, as } n \to \infty.
$$

**Remark 3.2** Let $H$ be a self-adjoint element in $\mathcal{A}(\mathcal{M})$ and $P \in \mathcal{P}_{ac}^\infty(H)$. From Lemma 4.2.3 (vi) in [8],

$$
\int_{\mathbb{R}} \left\| P \omega_n(H) e^{-itH} x \right\|^2 dt \leq \frac{n}{2\pi} \|x\|
$$

for any $x \in \mathcal{H}$ and $n \in \mathbb{N}$. Then,

$$
\sup_{\|x\|=1} \int_{\mathbb{R}} \left\| P \omega_n(H) e^{-itH} x \right\|^2 dt \leq \frac{n}{2\pi}.
$$

By Theorem 2.1 and Definition 2.2, we have $G = P \omega_n(H)$ is $H$-smooth for $n \in \mathbb{N}$.

Next theorem is the main result in this subsection.

**Theorem 3.3** Suppose $H$ is a self-adjoint element in $\mathcal{A}(\mathcal{M})$. Then

$$
P_{ac}^\infty(H) = \cup \{ R(G^*) : G \in \mathcal{M} \text{ is } H \text{-smooth} \}
$$

where $R(G^*)$ is the range projection of $G^*$.

**Proof** Let $P \in \mathcal{P}_{ac}^\infty(H)$. By Remark 3.2, we have $G = P \omega_n(H)$ is $H$-smooth. Hence, from Theorem 2.1,

$$
\sup_{\Lambda \subseteq \mathbb{R}} \frac{\| P \omega_n(H) E_H(\Lambda) \omega_n(H) P \|}{|\Lambda|} = \sup_{\|x\|=1} \frac{1}{2\pi} \int_{\mathbb{R}} \left\| P \omega_n(H) e^{-itH} x \right\|^2 dt \leq \frac{n}{(2\pi)^2}.
$$
Therefore, \( \lambda \to P \omega_n (H) E_H (\lambda) \omega_n (H) P \) is locally absolutely continuous. Then, by Lemma 2.7, we have \( R (\omega_n (H) P) \leq P^{\infty}_{ac} (H) \) for every \( n \in \mathbb{N} \) and \( P \in \mathcal{P}^{\infty}_{ac} (H) \). Hence,

\[
P \leq \lor_n R (\omega_n (H) P) = \lor_n \{ R (G^*) : G = P \omega_n (H) \}
\]

by Lemma 3.1, now we conclude that \( P^{\infty}_{ac} (H) \leq \lor \{ R (G^*) : G \in \mathcal{M} \text{ is } H\text{-smooth} \} \).

On the other hand, if \( G \in \mathcal{M} \) is \( H\)-smooth, then by Theorem 2.1(5) we have \( \lambda \to GE_H (\lambda) G^* \) is locally absolutely continuous. Therefore \( R (G^*) \leq P^{\infty}_{ac} (H) \) by Lemma 2.7. Hence,

\[
\lor \{ R (G^*) : G \text{ is } H\text{-smooth in } \mathcal{M} \} \leq P^{\infty}_{ac} (H).
\]

This completes the proof.

Let \( P^{ac} (H) \) be the projection from \( \mathcal{H} \) onto \( \mathcal{H}^{ac} (H) \). In [8], it has been shown that \( P^{ac} (H) = P^{\infty}_{ac} (H) \) for a densely defined self-adjoint operator \( H \) on \( \mathcal{H} \). Then, we can get the following corollary.

**Corollary 3.4** Let \( H \) be a densely defined self-adjoint operator on \( \mathcal{H} \). Then

\[
P^{ac} (H) = P^{\infty}_{ac} (H) = \lor \{ R (G^*) : G \in \mathcal{B} (\mathcal{H}) \text{ is } H\text{-smooth} \}.
\]

**Corollary 3.5** Suppose \( H \) is a self-adjoint affiliated with \( \mathcal{M} \). Then \( P^{\infty}_{ac} (H) \neq 0 \) if and only if there is at least one \( H\)-smooth operator in \( \mathcal{M} \).

**Proof** If \( P^{\infty}_{ac} (H) \neq 0 \), then there is a projection \( P \in \mathcal{P}^{\infty}_{ac} (H) \). By the argument in the proof of Theorem 3.3, we know that \( P \omega_n (H) \) is \( H\)-smooth operator in \( \mathcal{M} \). The other direction is clear by Theorem 3.3.

### 3.2 The stationary approach in \( \mathcal{M} \)

The stationary approach on a Hilbert space \( \mathcal{H} \) is based on several variations of wave operators, such as weak wave operators and stationary wave operators (see [17]). For the reader who is familiar with the general scattering theory, the following definitions (Definitions 3.6 and 3.7) are natural extensions of the corresponding definitions on \( \mathcal{H} \). For the reader who is not familiar with this area, we refer you to [17] for the general stationary scattering theory and then it is possible for the reader to give these definitions in \( \mathcal{M} \) directly. If the reader are still interested to know the details of these definitions in \( \mathcal{M} \), we refer you to [18].

**Definition 3.6** Let \( H, H_1 \in \mathcal{A} (\mathcal{M}) \) be a pair of self-adjoint operators and \( J \) be an operator in \( \mathcal{M} \). The generalized weak wave operator for a pair of self-adjoint operators \( H, H_1 \) and \( J \) is the operator

\[
\tilde{W}_{\pm} (H_1, H; J) = \text{w.o.t- lim}_{t \to \pm \infty} P^{\infty}_{ac} (H_1) e^{iH_1 J} e^{-iH} P^{\infty}_{ac} (H).
\]
provided that the limit exists.

We have
\[ \tilde{W}_\pm (H, H_1; J^*) = \tilde{W}_\pm^* (H_1, H; J) \]  
(3.2)
it\( \tilde{W}_\pm^n (H_1, H; J) \) exists.

**Definition 3.7** Let \( J \) be an operator in \( \mathcal{M} \), \( H, H_1 \) be self-adjoint operators in \( \mathcal{A}(\mathcal{M}) \). If for any pair of elements \( x \) and \( x_1 \) in \( \mathcal{H} \),
\[ \lim_{\varepsilon \to 0} \varepsilon \pi \langle JR_H (\lambda \pm i \varepsilon) P_{ac}^\infty (H) x, R_{H_1} (\lambda \pm i \varepsilon) P_{ac}^\infty (H_1) x_1 \rangle \]
exists a.e. \( \lambda \in \mathbb{R} \), then the generalized stationary wave operator is defined as
\[ \langle U_\pm (H_1, H; J) x, x_1 \rangle = \int_{-\infty}^{\infty} \lim_{\varepsilon \to 0} \varepsilon \pi \langle JR_H (\lambda \pm i \varepsilon) P_{ac}^\infty (H) x, R_{H_1} (\lambda \pm i \varepsilon) P_{ac}^\infty (H_1) x_1 \rangle d\lambda. \]

From the definition of \( U_\pm (H_1, H; J) \), it is clear that
\[ P_{ac}^\infty (H_1) U_\pm (H_1, H; J) = U_\pm (H_1, H; J) \]  
(3.3)
it\( U_\pm (H_1, H; J) \) exists.

Exhibiting conditions under which \( U_\pm (H_1, H; J) \) exists and \( U_\pm (H_1, H; J) = W_\pm (H_1, H; J) \) for \( H, H_1 \in \mathcal{A}(\mathcal{M}) \) is the main purpose of this subsection. The next result give us the relation among \( U_\pm (H_1, H; J) \); \( \tilde{W}_\pm (H_1, H; J) \) and \( W_\pm (H_1, H; J) \), it is a natural extension of the similar result in the general scattering theory and the proofs are similar too. For a proof, we refer you to [18].

**Theorem 3.8** [18] Let \( H, H_1 \in \mathcal{A}(\mathcal{M}) \) be a pair of self-adjoint operators and \( J \) be an operator in \( \mathcal{M} \). If \( \mathcal{U}_\pm (H_1, H; J) \), \( \mathcal{U}_\pm (H, H; J^* J) \), \( \mathcal{W}_\pm (H_1, H; J) \) and \( \mathcal{W}_\pm (H, H; J^* J) \) exist as well as
\[ \mathcal{U}_\pm^* (H_1, H; J) \mathcal{U}_\pm (H_1, H; J) = \mathcal{U}_\pm (H, H; J^* J), \]
then \( \mathcal{W}_\pm (H_1, H; J) \) exists and
\[ \mathcal{U}_\pm (H_1, H; J) = \mathcal{W}_\pm (H_1, H; J). \]

Now based on the definition of \( \mathcal{U}_\pm (H_1, H; J) \), we can get the following property.

**Corollary 3.9** Let \( H, H_1 \in \mathcal{A}(\mathcal{M}) \) be a pair of self-adjoint operators and \( J \) be an operator in \( \mathcal{M} \). If \( \mathcal{U}_\pm (H_1, H; J) \) exists, then for any pair of elements \( x \) and \( x_1 \) in \( \mathcal{H} \)
and any Borel sets \( \Lambda, \Lambda_1 \subset \mathbb{R} \),

\[
\mathcal{U}_\pm (H_1, H; J) E_H (\Lambda) x, E_{H_1} (\Lambda_1) x_1) \biggr] = \int_{\Lambda_1 \cap \Lambda} \frac{\varepsilon}{\pi} \left( J R_H (\lambda, \pm i \varepsilon) P^\infty_{ac} (H) x, R_{H_1} (\lambda, \pm i \varepsilon) P^\infty_{ac} (H_1) x_1 \right) d\lambda.
\]

**Proof** Let \( R_H (\lambda, \varepsilon) \) denote \( R_H (\lambda \pm i \varepsilon) \). Set

\[
\alpha (x, x_1; \lambda) = \lim_{\varepsilon \to 0} \frac{\varepsilon}{\pi} \left( J R_H (\lambda, \varepsilon) P^\infty_{ac} (H) x, R_{H_1} (\lambda, \varepsilon) P^\infty_{ac} (H_1) x_1 \right).
\]

Since

\[
P^\infty_{ac} (H) E_H (\Lambda) = E_H (\Lambda) P^\infty_{ac} (H)
\]

and

\[
P^\infty_{ac} (H_1) E_{H_1} (\Lambda_1) = E_{H_1} (\Lambda_1) P^\infty_{ac} (H_1)
\]

by Remark 2.4, we get that

\[
\left| \alpha (E_H (\Lambda) x, E_{H_1} (\Lambda_1) x_1; \lambda) \right|^2 \leq \frac{\varepsilon^2}{\pi^2} \| J \|^2 \lim_{\varepsilon \to 0} \left\| R_H (\lambda, \varepsilon) E_H (\Lambda) P^\infty_{ac} (H) x \right\|^2
\]

\[
= \| J \|^2 \lim_{\varepsilon \to 0} \left( \delta_H (\lambda, \varepsilon) E_H (\Lambda) P^\infty_{ac} (H) x, x \right) \cdot \lim_{\varepsilon \to 0} \left( \delta_{H_1} (\lambda, \varepsilon) E_{H_1} (\Lambda_1) P^\infty_{ac} (H_1) x_1, x_1 \right)
\]

\[
= \| J \|^2 \chi_{\Lambda \cap \Lambda_1} \frac{d \left( E_H (\lambda) P^\infty_{ac} (H) x, x \right)}{d \lambda} \cdot \frac{d \left( E_{H_1} (\lambda) P^\infty_{ac} (H_1) x_1, x_1 \right)}{d \lambda}
\]

by (2.2), (2.3) and (2.4) where \( \chi_{\Lambda \cap \Lambda_1} \) is the characteristic function of \( \Lambda \cap \Lambda_1 \). Therefore,

\[
\chi_{\mathbb{R} \setminus \Lambda \cap \Lambda_1} \alpha (E_H (\Lambda) x, E_{H_1} (\Lambda_1) x_1; \lambda) = 0,
\]

where \( \chi_{\mathbb{R} \setminus \Lambda \cap \Lambda_1} \) is the characteristic function of \( \mathbb{R} \setminus \Lambda \cap \Lambda_1 \). It implies that

\[
\chi_{\Lambda \cap \Lambda_1} \alpha (E_H (\Lambda) x, E_{H_1} (\Lambda_1) x_1; \lambda) = \alpha (E_H (\Lambda) x, E_{H_1} (\Lambda_1) x_1; \lambda).
\]
Hence
\[
\mathcal{X}_{\Lambda \cap \Lambda_1} \alpha_{\pm} (x, x_1; \lambda)
= \mathcal{X}_{\Lambda \cap \Lambda_1} \alpha_{\pm} (E_H (\Lambda) x, E_{H_1} (\Lambda_1) x_1; \lambda)
+ \mathcal{X}_{\Lambda \cap \Lambda_1} \alpha_{\pm} (E_H (\mathbb{R} \setminus \Lambda) x, E_{H_1} (\Lambda_1) x_1; \lambda)
+ \mathcal{X}_{\Lambda \cap \Lambda_1} \alpha_{\pm} (x, E_{H_1} (\mathbb{R} \setminus (\Lambda_1)) x_1; \lambda)
= \alpha_{\pm} (E_H (\Lambda) x, E_{H_1} (\Lambda_1) x_1; \lambda).
\]

It follows that
\[
\langle U_{\pm} (H, H; J) E_H (\Lambda) x, E_{H_1} (\Lambda_1) x_1 \rangle
= \int_{\mathbb{R}} \alpha_{\pm} (E_H (\Lambda) x, E_{H_1} (\Lambda_1) x_1; \lambda)
= \int_{\mathbb{R}} \lim_{\varepsilon \to 0} \frac{\varepsilon}{\pi} \left\{ J R_H (\lambda, \varepsilon) P_{ac}^\infty (H) x, R_{H_1} (\lambda, \varepsilon) P_{ac}^\infty (H_1) x_1 \right\} d\lambda.
\]

The proof is completed.

**Remark 3.10** Let \( H, H_1 \in \mathcal{A} (\mathcal{M}) \) be a pair of self-adjoint operators and \( J \) be an operator in \( \mathcal{M} \) with \( J \mathcal{D} (H) \subseteq \mathcal{D} (H_1) \). Then
\[
H_1 J - J H = (H_1 - z) J - J (H - z)
= (H_1 - z) (J R_H (z) - R_{H_1} (z) J) (H - z). \tag{3.4}
\]

Hence,
\[
J R_H (z) - R_{H_1} (z) J = R_{H_1} (z) (H_1 J - J H) R_H (z). \tag{3.5}
\]

Note
\[
\varepsilon \pi^{-1} \| R_H (\lambda \pm i\varepsilon) x \|^2 = \langle \delta_H (\lambda, \varepsilon) x, x \rangle \tag{3.6}
\]
by (2.2). Let \( R_H (\lambda, \varepsilon) \) denote \( R_H (\lambda \pm i\varepsilon) \). Then from (3.5), we have
\[
\frac{\varepsilon}{\pi} \left\{ J R_H (\lambda, \varepsilon) P_{ac}^\infty (H) x, R_{H_1} (\lambda, \varepsilon) P_{ac}^\infty (H_1) x_1 \right\}
= \frac{\varepsilon}{\pi} \cdot \left\{ (R_{H_1} (\lambda, \varepsilon) J + R_H (\lambda, \varepsilon) (H_1 J - J H) R_H (\lambda, \varepsilon)) P_{ac}^\infty (H) x, R_{H_1} (\lambda, \varepsilon) P_{ac}^\infty (H_1) x_1 \right\}
= \left\{ (J + (H_1 J - J H) R_H (\lambda, \varepsilon)) P_{ac}^\infty (H) x, \delta_H (\lambda, \varepsilon) P_{ac}^\infty (H_1) x_1 \right\}. \tag{3.7}
\]

**Lemma 3.11** Let \( H, H_1 \in \mathcal{A} (\mathcal{M}) \) be a pair of self-adjoint operators and \( J \) be an operator in \( \mathcal{M} \) with \( J \mathcal{D} (H) \subseteq \mathcal{D} (H_1) \). Suppose there is an \( H \)-bounded operator \( G \)
and an $H_1$-bounded operator $G_1$ in $A(M)$ satisfying $H_1 J - J H = G_1^* G$. If

$$\lim_{\varepsilon \to 0} GR_H (\lambda \pm i \varepsilon) P_{ac}^\infty (H) x$$

exists a.e. $\lambda \in \mathbb{R}$ for every $x \in \mathcal{H}$ and

$$\lim_{\varepsilon \to 0} \left[ G_1 \delta H_1 (\lambda, \varepsilon) P_{ac}^\infty (H_1) x_1, y \right]$$

exists a.e. $\lambda \in \mathbb{R}$ for any $x_1$ and $y \in \mathcal{H}$, then $U \pm (H_1, H; J) \in \mathcal{D}(H, H; J^* J)$ exist and

$$U_\pm^* (H_1, H; J) U_\pm (H_1, H; J) = U_\pm (H, H; J^* J).$$

**Proof** Let $R_H (\lambda, \varepsilon)$ denote $R_H (\lambda \pm i \varepsilon)$. Since $H_1 J - J H = G_1^* G$, by (3.7)

$$\frac{\varepsilon}{\pi} \left\{ J R_H (\lambda, \varepsilon) P_{ac}^\infty (H) x, R_H (\lambda, \varepsilon) P_{ac}^\infty (H_1) x_1 \right\}$$

$$= \left\{ (J + (H_1 J - J H) R_H (\lambda, \varepsilon)) P_{ac}^\infty (H) x, \delta H_1 (\lambda, \varepsilon) P_{ac}^\infty (H_1) x_1 \right\}$$

$$= \left\{ (J + G_1^* GR_H (\lambda, \varepsilon)) P_{ac}^\infty (H) x, \delta H_1 (\lambda, \varepsilon) P_{ac}^\infty (H_1) x_1 \right\}$$

$$= \left\{ J P_{ac}^\infty (H) x, \delta H_1 (\lambda, \varepsilon) P_{ac}^\infty (H_1) x_1 \right\}$$

$$+ \left\{ (G R_H (\lambda, \varepsilon)) P_{ac}^\infty (H) x, G_1 \delta H_1 (\lambda, \varepsilon) P_{ac}^\infty (H_1) x_1 \right\}.$$

From (2.3),

$$\lim_{\varepsilon \to 0} \left\{ J P_{ac}^\infty (H) x, \delta H_1 (\lambda, \varepsilon) P_{ac}^\infty (H_1) x_1 \right\}$$

exists a.e. $\lambda \in \mathbb{R}$.

Since

$$\lim_{\varepsilon \to 0} GR_H (\lambda, \varepsilon) P_{ac}^\infty (H) x$$

and

$$\lim_{\varepsilon \to 0} \left[ G_1 \delta H_1 (\lambda, \varepsilon) P_{ac}^\infty (H_1) x_1, y \right]$$

exist a.e. $\lambda \in \mathbb{R}$ for every $x, x_1$ and $y \in \mathcal{H}$, we can easily check that

$$\lim_{\varepsilon \to 0} \left\{ (G R_H (\lambda, \varepsilon)) P_{ac}^\infty (H) x, G_1 \delta H_1 (\lambda, \varepsilon) P_{ac}^\infty (H_1) x_1 \right\}$$

exists a.e. $\lambda \in \mathbb{R}$ in (3.10). Hence $U \pm (H_1, H; J)$ is well-defined.

For any Borel set $\Lambda$,

$$\left\{ E_{H_1} (\Lambda) \cdot U \pm (H_1, H; J) x, x_1 \right\}$$

$$= \int_{\Lambda} \lim_{\varepsilon \to 0} \left\{ \delta H_1 (\lambda, \varepsilon) U \pm (H_1, H; J) x, x_1 \right\} d\lambda$$

(3.11)
by (2.3). By Corollary 3.9, we also have
\[
\langle E_{H_1}(\Lambda) \cdot U_\pm (H_1, H; J) x, x_1 \rangle
\]
\[
= \int_{\Lambda} \lim_{\varepsilon \to 0} \frac{\varepsilon}{\pi} \langle J R_H(\lambda, \varepsilon) P_{ac}^\infty (H) x, R_{H_1}(\lambda, \varepsilon) P_{ac}^\infty (H_1) x_1 \rangle d\lambda.
\] (3.12)

Comparing two conditions in (3.11) and (3.12), we have
\[
\lim_{\varepsilon \to 0} \langle \delta_{H_1}(\lambda, \varepsilon) U_\pm (H_1, H; J) x, x_1 \rangle
\]
\[
= \lim_{\varepsilon \to 0} \frac{\varepsilon}{\pi} \langle J R_H(\lambda, \varepsilon) P_{ac}^\infty (H) x, R_{H_1}(\lambda, \varepsilon) P_{ac}^\infty (H_1) x_1 \rangle \; \text{a.e.} \; \lambda \in \mathbb{R}.
\] (3.13)

Therefore, this equality holds only when
\[
\lim_{\varepsilon \to 0} \frac{\varepsilon}{\pi} \langle J R_H(\lambda, \varepsilon) P_{ac}^\infty (H) x, R_{H_1}(\lambda, \varepsilon) P_{ac}^\infty (H_1) x_1 \rangle
\]
essists a.e. \( \lambda \in \mathbb{R} \). By (3.7) and the fact that \( H_1 J - J H = G_1^* G \), we have
\[
\frac{\varepsilon}{\pi} \langle J R_H(\lambda, \varepsilon) P_{ac}^\infty (H) x, R_{H_1}(\lambda, \varepsilon) P_{ac}^\infty (H_1) x_1 \rangle
\]
\[
= \langle J P_{ac}^\infty (H) x, \delta_{H_1}(\lambda, \varepsilon) P_{ac}^\infty (H_1) x_1 \rangle
\]
\[
+ \langle (G R_H(\lambda, \varepsilon)) P_{ac}^\infty (H) x, G_1 \delta_{H_1}(\lambda, \varepsilon) P_{ac}^\infty (H_1) x_1 \rangle.
\]

Then, by (3.8) and (3.9), we can conclude that
\[
\lim_{\varepsilon \to 0} \frac{\varepsilon}{\pi} \langle J R_H(\lambda, \varepsilon) P_{ac}^\infty (H) x, R_{H_1}(\lambda, \varepsilon) P_{ac}^\infty (H_1) x_1 \rangle
\] (3.14)
essists a.e. \( \lambda \in \mathbb{R} \).
In (3.14), replace $x_1$ by $\mathcal{U}_\pm (H_1, H; J) y$, so from (2.2), (3.7) and (3.13) we have
\[
\lim_{\varepsilon \to 0} \frac{\varepsilon}{\pi} \left[ J R_H (\lambda, \varepsilon) P_{ac}^\infty (H) x, R_{H_1} (\lambda, \varepsilon) \mathcal{U}_\pm (H_1, H; J) P_{ac}^\infty (H) y \right]
\]
\[
= \lim_{\varepsilon \to 0} \frac{\varepsilon}{\pi} \left[ (J + (H_1 I - J H) R_H (\lambda, \varepsilon)) P_{ac}^\infty (H) x, \delta_{H_1} (\lambda, \varepsilon) \mathcal{U}_\pm (H_1, H; J) P_{ac}^\infty (H) y \right]
\]
\[
= \lim_{\varepsilon \to 0} \frac{\varepsilon}{\pi} \left[ (J + (H_1 I - J H) R_H (\lambda, \varepsilon)) P_{ac}^\infty (H) x, R_{H_1} (\lambda, -\varepsilon) J R_H (\lambda, \varepsilon) P_{ac}^\infty (H) y \right]
\]
\[
= \lim_{\varepsilon \to 0} \frac{\varepsilon}{\pi} \left[ (J + (H_1 I - J H) R_H (\lambda, \varepsilon)) P_{ac}^\infty (H) x, R_H (\lambda, -\varepsilon) J^* J R_H (\lambda, \varepsilon) P_{ac}^\infty (H) x, P_{ac}^\infty (H) y \right] \text{ a.e. } \lambda \in \mathbb{R}.
\]

Hence, applying the definition of $\mathcal{U}_\pm (H_1, H; J)$, we have
\[
\mathcal{U}_\pm (H_1, H; J) x, \mathcal{U}_\pm (H_1, H; J) y
\]
\[
= \int_{-\infty}^{\infty} \lim_{\varepsilon \to 0} \frac{\varepsilon}{\pi} \left[ R_H (\lambda, -\varepsilon) J^* J R_H (\lambda, \varepsilon) P_{ac}^\infty (H) x, P_{ac}^\infty (H) y \right]
\]
\[
= \{ \mathcal{U}_\pm (H, H; J^* J) x, y \}.
\]

It implies that
\[
\mathcal{U}_\pm^* (H_1, H; J) \mathcal{U}_\pm (H_1, H; J) = \mathcal{U}_\pm (H, H; J^* J).
\]

**Remark 3.12** According to Theorem 3.8, if $\tilde{W}_\pm (H_1, H; J)$ and $\tilde{W}_\pm (H, H; J^* J)$ both exist, then the condition of Lemma 3.11 guarantee that $\mathcal{U}_\pm (H_1, H; J) = W_\pm (H_1, H; J)$ in $\mathcal{M}$.

### 3.3 The Kato–Rosenblum theorem in $\mathcal{M}$

In the rest of this paper, we assume that $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ is a countable decomposable properly infinite semifinite von Neumann algebra with a faithful normal tracial weight $\tau$. The results below involve noncommutative $L^p$-spaces associated to $\mathcal{M}$, so we refer the reader to [12] for more details about it.

**Remark 3.13** For a separable Hilbert space $\mathcal{H}$, we denote by $H_\pm^2 (\mathcal{H})$ the class of functions with values on $\mathcal{H}$, holomorphic on the upper (lower) half-plane and such...
that

$$\sup_{\varepsilon > 0} \int_{\mathbb{R}} \| u (\lambda \pm i \varepsilon) \|^2 d\lambda < +\infty.$$  

Then by the result in Sect. I of Chapter V in [14], we know that the radial limit exists almost everywhere, i.e., \( \lim_{\varepsilon \to 0} u (\lambda \pm i \varepsilon) \) exists a.e. \( \lambda \in \mathbb{R} \).

**Lemma 3.14** Let \( H \in \mathcal{A}(\mathcal{M}) \) be a self-adjoint operator and \( A \in L^2(\mathcal{M}, \tau) \cap \mathcal{M} \). Then

$$s.o.t.- \lim_{\varepsilon \to 0} A R_H (\lambda \pm i \varepsilon) P_{ac}^\infty (H)$$

and

$$s.o.t.- \lim_{\varepsilon \to 0} A \delta_H (\lambda, \varepsilon) P_{ac}^\infty (H)$$

exist in the strong operator topology a.e. \( \lambda \in \mathbb{R} \).

**Proof** By Remark 3.2, and Theorem 2.1, we get

$$\sup_{\| x \| = 1} \frac{1}{2\pi} \int_{\mathbb{R}} \left\| P \omega_n (H) e^{-i t H} x \right\|^2 dt = \sup_{\Lambda \subseteq \mathbb{R}} \left\| P \omega_n (H) E_H (\Lambda) \omega_n (H) P \right\| \leq \frac{n}{(2\pi)^2}.$$  

Hence by (2.5), we have

$$\sup_{\| x \| = 1} \frac{1}{2\pi} \int_{\mathbb{R}} \left\| P \omega_n (H) e^{-i t H} x \right\|^2 dt$$

$$= \frac{1}{(2\pi)^2} \sup_{\| x \| = 1, \varepsilon > 0} \int_{\mathbb{R}} \left( \| P \omega_n (H) H (\lambda \pm i \varepsilon) x \|^2 \right) d\lambda \leq \frac{n}{(2\pi)^2}.$$  

From Lemma 2.1.1 in [8], there is a sequence \( \{ x_m \}_{m \in \mathbb{N}} \) of \( H \) such that

$$\| A \|^2 = \tau (A^* A) = \sum \langle A^* A x_m, x_m \rangle$$

and

$$\vee \{ A' x_m : A' \in \mathcal{M}' \text{ and } m \in \mathbb{N} \}$$

is dense where \( \mathcal{M}' \) is the commutant of \( \mathcal{M} \). Then, for these \( \{ x_m \}_{m \in \mathbb{N}} \), we have

$$\int_{\mathbb{R}} \left( \| P \omega_n (H) R_H (\lambda \pm i \varepsilon) A x_m \|^2 \right) d\lambda \leq n \| A x_m \|^2 \leq n \| A \|^2.$$  

© Birkhäuser
for $P \in \mathcal{P}_{a.c}^\infty (H)$. We further note that for every $A \in \mathcal{L}^2 (\mathcal{M}, \tau) \cap \mathcal{M}$,

$$\int_{\mathbb{R}} \| P \omega_n (H) R_H (\lambda \pm i \varepsilon) A x_m \|_2^2 \, d\lambda = \int_{\mathbb{R}} \sum_m \| P \omega_n (H) R_H (\lambda \pm i \varepsilon) A x_m \|_2^2 \, d\lambda$$

$$= \sum_m \int_{\mathbb{R}} \| P \omega_n (H) R_H (\lambda \pm i \varepsilon) A x_m \|_2^2 \, d\lambda$$

$$\leq n \sum_m \| A x_m \|^2 \leq n \| A \|^2_2 .$$

Combining it with the equality

$$\| X \|^2_2 = \tau (X^* X) = \tau (XX^*) = \| X^* \|^2_2$$

for every $X \in \mathcal{M}$,

we get the following inequality

$$\int_{\mathbb{R}} \| A R_H (\lambda \pm i \varepsilon) \omega_n (H) P x_m \|_2^2 \, d\lambda \leq \int_{\mathbb{R}} \| A R_H (\lambda \pm i \varepsilon) \omega_n (H) P \|_2^2 \, d\lambda$$

$$= \int_{\mathbb{R}} \| P \omega_n (H) R_H (\lambda \mp i \varepsilon) A \|_2^2 \, d\lambda$$

$$\leq n \| A \|^2_2 .$$

It implies that the vector-valued function $A R_H (\lambda \pm i \varepsilon) \omega_n (H) P x_m$ belongs to the Hardy classes $H^2_\pm (\mathcal{H})$ in the upper and lower half planes. By Remark 3.13, the radial limit values of functions in $H^2_\pm (\mathcal{H})$ exist a.e. $\lambda \in \mathbb{R}$, therefore

$$\lim_{\varepsilon \to 0} A R_H (\lambda \pm i \varepsilon) \omega_n (H) P x_m$$

exists a.e. $\lambda \in \mathbb{R}$ for every $x_m$.

Since the linear span of the set $\{ A' x_m : A' \in \mathcal{M}' \text{ and } m \in \mathbb{N} \}$ is dense in $\mathcal{H}$, we have

$$\lim_{\varepsilon \to 0} A R_H (\lambda \pm i \varepsilon) \omega_n (H) P A' x_m = A' \lim_{\varepsilon \to 0} A R_H (\lambda \pm i \varepsilon) \omega_n (H) P x_m$$

and then this indicates that

$$\text{s.o.t.-} \lim_{\varepsilon \to 0} A R_H (\lambda \pm i \varepsilon) \omega_n (H) P$$

exists in the strong operator topology. From the fact that $\omega_n (H) \to I$ ($n \to \infty$) in Lemma 3.1, we can conclude that

$$\text{s.o.t.-} \lim_{\varepsilon \to 0} A R_H (\lambda \pm i \varepsilon) P$$

exists for $A \in \mathcal{L}^2 (\mathcal{M}, \tau) \cap \mathcal{M}$ and $P \in \mathcal{P}_{a.c}^\infty (H)$.

Since $P_{a.c}^\infty (H) = \vee \{ P : P \in \mathcal{P}_{a.c}^\infty (H) \}$, we conclude that

$$\text{s.o.t.-} \lim_{\varepsilon \to 0} A R_H (\lambda \pm i \varepsilon) P_{a.c}^\infty (H)$$

exists for $A \in \mathcal{L}^2 (\mathcal{M}, \tau) \cap \mathcal{M}$. 

\[\Box\]
Note that $\delta_H(\lambda, \varepsilon) = \frac{1}{2\pi i} \left[ R_H(\lambda + i\varepsilon) - R_H(\lambda - i\varepsilon) \right]$, so we can conclude that

$$
\lim_{\varepsilon \to 0} A\delta_H(\lambda, \varepsilon) P_{ac}^\infty(H)x = \frac{1}{2\pi i} \left( \lim_{\varepsilon \to 0} AR_H(\lambda + i\varepsilon) P_{ac}^\infty(H)x - \lim_{\varepsilon \to 0} AR_H(\lambda - i\varepsilon) P_{ac}^\infty(H)x \right)
$$

eexists for $A \in L^2(M, \tau) \cap M$. The proof is completed.

**Remark 3.15** By Lemma 2.1.6 in [7], we know that $\mathcal{L}^p(M, \tau) \cap M$ is a two-sided ideal of $M$ for $1 \leq p < \infty$.

**Theorem 3.16** Let $H, H_1 \in \mathcal{A}(M)$ be a pair of self-adjoint operators and $J$ be an operator in $M$ with $J\mathcal{D}(H) \subseteq \mathcal{D}(H_1)$. Assume $H_1J - JH \in \mathcal{L}^1(M, \tau) \cap M$. Then $U_\pm(H_1; J)$ and $U_\pm(H, H; J^*J)$ both exist and

$$
U_\pm^*(H_1, H; J)U_\pm(H_1, H; J) = U_\pm(H, H; J^*J).
$$

**Proof** Let $G = |H_1J - JH|^{1/2} \in L^2(M, \tau) \cap M$ and $G_1^* = V|H_1J - JH|^{1/2} \in L^2(M, \tau) \cap M$ for some partial isometry $V$ in $M$. Then the proof is completed by Lemmas 3.14 and 3.11.

According to Remark 3.12, the next task is to show that $\tilde{W}_\pm(H_1; H; J)$ and $\tilde{W}_\pm(H, H; J^*J)$ both exist.

**Lemma 3.17** (Lemma 2.5.1 in [8]) Let $H, H_1 \in \mathcal{A}(M)$ be a pair of self-adjoint operators, $J$ be an operator in $M$ with $J\mathcal{D}(H) \subseteq \mathcal{D}(H_1)$ and $H_1J - JH \in \mathcal{M}$. Let $W_J(t) = e^{itH_1}Je^{-itH}$, for $t \in \mathbb{R}$. Then, for all $x \in \mathcal{H}$ and $s, w \in \mathbb{R}$, the mapping $t \mapsto e^{itH_1}(H_1J - JH)e^{-itH}x$ from $[s, w]$ into $\mathcal{H}$ is Bochner integrable with

$$
(W_J(w) - W_J(s))x = i \int_s^w e^{itH_1}(H_1J - JH)e^{-itH}xdt.
$$

**Lemma 3.18** Let $H \in \mathcal{A}(M)$ be a self-adjoint operator and $G$ be an operator in $M$. Then, there is a linear manifold $\mathcal{D}$ in $\mathcal{H}_{ac}^\infty(H)$ with $\overline{\mathcal{D}} = \mathcal{H}_{ac}^\infty(H)$ such that

$$
\int_{-\infty}^{\infty} \|Ge^{-itH}y\|^2 dt < \infty, \quad y \in \mathcal{D}.
$$

**Proof** For any $x \in \mathcal{H}_{ac}^\infty(H)$, by (2.3), we have

$$
\lim_{\varepsilon \to 0} \langle G\delta_H(\lambda, \varepsilon)x, h \rangle = \frac{d}{d\lambda}\langle GE_H(\lambda)x, h \rangle \quad \text{a.e. } \lambda \in \mathbb{R} \quad \text{for any } h \in \mathcal{H}.
$$

Let $F_x(\lambda) \in \mathcal{H}$ be the weak limit of $G\delta_H(\lambda, \varepsilon)x$, i.e. $\lim_{\varepsilon \to 0} \langle G\delta_H(\lambda, \varepsilon)x, h \rangle = \langle F_x(\lambda), h \rangle$ a.e. $\lambda \in \mathbb{R}$ for every $h \in \mathcal{H}$. We set

$$
X_{N,n}(x) = \{ \lambda : |\lambda| \leq n, \|F_x(\lambda)\| \leq N \}$$

© Birkhäuser
and $\mathcal{D}$ to be the set of linear combinations of all elements of the form $y = E \left( X_{N,n} \right) x$ for $x \in \mathcal{H}^\infty_{ac} (H)$ and $n, N \in \mathbb{N}$. Since for $x \in \mathcal{H}^\infty_{ac} (H)$ and $n, N \in \mathbb{N}$,

$$E \left( X_{N,n} \right) x = E \left( X_{N,n} \right) P^\infty_{ac} (H) x = P^\infty_{ac} (H) E \left( X_{N,n} \right) x$$

by Remark 2.4, we have $\mathcal{D} \subset \mathcal{H}^\infty_{ac} (H)$. Note

$$\lim_{N \to \infty} \left| (-n, n) \setminus X_{N,n} \right| = 0,$$

then $x$ can be approximated by the elements $E \left( X_{N,n} \right) x$ for $x \in \mathcal{H}^\infty_{ac} (H)$. Hence $\mathcal{D} = \mathcal{H}^\infty_{ac} (H)$.

Let $\{e_j\}_{j \in \mathbb{Z}}$ be an orthonormal basis in $\mathcal{H}$. Since for $x, y \in \mathcal{H}$,

$$\langle e^{-itH} x, y \rangle = \int_{-\infty}^{\infty} e^{-i\lambda t} d \langle E_H (\lambda) x, y \rangle,$$

we have for $y = E \left( X_{N,n} \right) x$

$$s \left( Ge^{-itH} y, e_j \right) = \int_{\mathbb{R}} e^{-i\lambda t} d \left[ GE_H (\lambda) y, e_j \right] = \int_{X_{N,n}(x)} e^{-i\lambda t} \frac{d \langle GE_H (\lambda) y, e_j \rangle}{d \lambda} d \lambda = \int_{X_{N,n}(x)} e^{-i\lambda t} \langle F_y (\lambda), e_j \rangle d \lambda.$$

Then, by the Parseval equality, for each $j \in \mathbb{Z}$

$$\int_{\mathbb{R}} \left| \langle Ge^{-itH} y, e_j \rangle \right|^2 dt = 2\pi \int_{X_{N,n}(x)} \left| \langle F_y (\lambda), e_j \rangle \right|^2 d \lambda.$$

Hence,

$$\int_{\mathbb{R}} \left\| Ge^{-itH} y \right\|^2 dt = 2\pi \int_{X_{N,n}(x)} \left\| F_y (\lambda) \right\|^2 d \lambda \leq 4\pi N^2 n.$$

Therefore we have

$$\int_{-\infty}^{\infty} \left\| Ge^{-itH} y \right\|^2 dt < \infty, \text{ for } y \in \mathcal{D}.$$

**Theorem 3.19** Let the operators $H, H_1 \in A (\mathcal{M})$ be a pair of self-adjoint operators and $J$ be an operator in $\mathcal{M}$ with $\mathcal{D} (H) \subseteq \mathcal{D} (H_1)$. If $H_1 J - J H \in \mathcal{M}$, then the generalized weak wave operator $\tilde{W}_{\pm} (H_1, H; J)$ exists.
Proof Let \( W_J(t) = e^{itH_1} J e^{-itH} \) for \( t \in \mathbb{R} \) and \( H_1 - JH = G^* G \) for \( G_1 \) and \( G \) in \( \mathcal{M} \). Then, by Lemmas 3.17 and 3.18, there are linear spaces \( \mathcal{D} \subseteq \mathcal{H}_{ac}^\infty (H) \) and \( \mathcal{D}_1 \subseteq \mathcal{H}_{ac}^\infty (H_1) \) with \( \mathcal{D} = \mathcal{H}_{ac}^\infty (H) \) and \( \mathcal{D}_1 = \mathcal{H}_{ac}^\infty (H_1) \) such that for \( x \in \mathcal{D} \) and \( y \in \mathcal{D}_1 \)

\[
\| (W_J(t) - W_J(s)) x , y \| = \left| \int_s^w \left(e^{itH_1} (H_1 J - JH) e^{-itH} x , y \right) dt \right| \\
\leq \int_s^w \left| \left(G e^{-itH} x , G_1 e^{-itH_1} y \right) \right| d\lambda \\
\leq \left( \int_s^w \| G e^{-itH} x \|^2 dt \cdot \int_s^t \| G_1 e^{-itH_1} y \|^2 dt \right)^{1/2}
\]

and

\[
\int_s^w \| G e^{-itH} x \|^2 dt \to 0, \quad \int_s^w \| G_1 e^{-itH_1} y \|^2 dt \to 0
\]

as \( s, w \to \pm \infty \). Hence

\[
\lim_{t \to \pm \infty} \langle W_J(t) x , y \rangle = \lim_{t \to \pm \infty} \left( W_J(t) P_{ac}^\infty (H) x , P_{ac}^\infty (H_1) y \right)
\]

exists for \( x \in \mathcal{D} \) and \( y \in \mathcal{D}_1 \). Since \( \mathcal{D} = \mathcal{H}_{ac}^\infty (H) \) and \( \mathcal{D}_1 = \mathcal{H}_{ac}^\infty (H_1) \), we have

\[
\lim_{t \to \pm \infty} \left( P_{ac}^\infty (H_1) W_J(t) P_{ac}^\infty (H) x , y \right)
\]

exists for any \( x, y \in \mathcal{H} \). Therefore, \( \tilde{W}_{\pm} (H_1, H; J) \) exists.

Corollary 3.20 Let the operators \( H, H_1 \in \mathcal{A}(\mathcal{M}) \) be a pair of self-adjoint operators and \( J \) be an operator in \( \mathcal{M} \) with \( J' \mathcal{D} (H) \subseteq \mathcal{D} (H_1) \). If \( H_1 J - JH \in \mathcal{M} \), then the generalized weak wave operator \( \tilde{W}_{\pm} = \tilde{W}_{\pm} (H, H; J^* J) \) exists.

Proof Since

\[
H J^* J - J^* JH = J^* (H_1 J - JH) - \left(J^* H_1 - H J^* \right) J \in \mathcal{M},
\]

we have \( \tilde{W}_{\pm} = \tilde{W}_{\pm} (H, H; J^* J) \) exists by Theorem 3.19.

Next result is the Kato–Rosenblum Theorem in a semifinite von Neumann algebra \( \mathcal{M} \) which was first proved in [8] by a time-dependent approach. One of the main purpose of this article is to obtain this result by applying the stationary scattering theory. Now, we are ready to show it here.

© Birkhäuser
Theorem 3.21 (Theorem 5.2.5 in [8]) Let $H, H_1 \in \mathcal{A}(\mathcal{M})$ be a pair of self-adjoint operators. Assume $H_1 - H \in L^1(\mathcal{M}, \tau) \cap \mathcal{M}$, then

$$W_\pm \triangleq W_\pm(H_1, H) \text{ exists in } \mathcal{M}.$$  
Moreover, $W_\pm^* W_\pm = P_{ac}^\infty(H)$, $W_\pm W_\pm^* = P_{ac}^\infty(H_1)$ and $W_\pm H W_\pm^* = H_1 P_{ac}^\infty(H_1)$.

**Proof** Combining Theorem 3.19, Corollary 3.20 and Theorems 3.16 and 3.8, we know that $W_\pm(H_1, H)$ and $W_\pm(H, H_1)$ both exist. By Theorem 2.6, we have

$$W_\pm^* W_\pm = W_\pm^*(H_1, H) W_\pm(H_1, H) = P_{ac}^\infty(H)$$
and

$$W_\pm^*(H, H_1) W_\pm(H, H_1) = P_{ac}^\infty(H_1).$$

Since $H_1 - H \in \mathcal{M}$, we have $\tilde{W}_\pm(H_1, H)$ and $\tilde{W}_\pm(H, H_1)$ both exist and

$$W_\pm^* = W_\pm^*(H_1, H) = \tilde{W}_\pm^*(H_1, H) = \tilde{W}_\pm(H, H_1) = W_\pm(H, H_1);$$
$$W_\pm^*(H, H_1) = \tilde{W}_\pm^*(H, H_1) = \tilde{W}_\pm(H_1, H) = W_\pm(H_1, H) = W_\pm$$

by equality (3.2). Thus,

$$W_\pm^* W_\pm = P_{ac}^\infty(H), \ W_\pm W_\pm^* = P_{ac}^\infty(H_1).$$

Meanwhile, by (2.6),

$$W_\pm H W_\pm^* = H_1 W_\pm W_\pm^* = H_1 P_{ac}^\infty(H_1).$$

So the proof is completed.

**Remark 3.22** For a self-adjoint $H \in \mathcal{M}$, if there is a $H$-smooth operator in $\mathcal{M}$, then $H$ is not a sum of a diagonal operator in $\mathcal{M}$ and an operator in $\mathcal{M} \cap L^1(\mathcal{M}, \tau)$ by Theorems 3.21 and 3.3.

**Acknowledgements** The authors wish to express their thanks to the referees for their helpful suggestions. Q. Li was partly supported by NSFC (Grant no. 11671133, 11871021).

**References**

1. Birman, M.Sh., Entina, S.B.: The stationary approach in abstract scattering theory. Izv. Akad. Nauk SSSR Ser. Mat. 31(2), 401–430 (1967)
2. Deich, V.G.: The stationary local method in scattering theory for a pair of spaces. Probl. Math. Fiz. 6, 76–90 (1973)
3. Kadison, R., Ringrose, J.: Fundamentals of the Theory of Operator Algebras. Vol I. Elementary Theory and Vol. II. Advanced Theory. Corrected Reprint of the 1986 Original. Graduate Studies in Mathematics, vols. 15 and 16. American Mathematical Society, Providence (1997)
4. Kato, T.: Perturbation of continuous spectra by trace class operators. Proc. Jpn. Acad. 33, 260–264 (1957)
5. Kato, T.: Wave operators and similarity for some non-selfadjoint operators. Math. Ann. 162, 258–279 (1966)
6. Kato, T.: Perturbation Theory for Linear Operators. Reprint of the 1980 edition. Classics in Mathematics. Springer-Verlag, Berlin, 1995. xxii+619 pp
7. Li, Q., Shen, J., Shi, R.: A generalization of Voiculescu’s theorem for normal operators to semifinite von Neumann algebras. Adv. Math. 375, 107347 (2020)
8. Li, Q., Shen, J., Shi, R., Wang, L.: Perturbations of self-adjoint operators in semifinite von Neumann algebras: Kato–Rosenblum theorem. J. Funct. Anal. 275(2), 259–287 (2018)
9. Murray, F.J., von Neumann, J.: On rings of operators. Ann. Math. 37, 116–229 (1936)
10. Murray, F.J., von Neumann, J.: On rings of operators, II. Trans. Am. Math. Soc. 41, 208–248 (1937)
11. Murray, F.J., von Neumann, J.: On rings of operators, IV. Ann. Math. 44, 716–808 (1943)
12. Pisier, G., Xu, Q.: Non-commutative $L^p$-Spaces. Handbook of the Geometry of Banach Spaces, vol. 2. North-Holland, Amsterdam, pp. 1459–1517 (2003)
13. Rosenblum, M.: Perturbation of the continuous spectrum and unitary equivalence. Pac. J. Math. 7, 997–1010 (1957)
14. Sz.-Nagy, B., Foias, C.: Harmonic Analysis of Operators on Hilbert Space. Translated from the French and revised North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York; Akadémiai Kiadó, Budapest 1970 xiii+389 pp
15. Tiedra de Aldecoa, R.: Stationary scattering theory for unitary operators with an application to quantum walks. J. Funct. Anal. 279(7), 108704 (2020)
16. von Neumann, J.: On rings of operators, III. Ann. Math. 41, 94–161 (1940)
17. Yafaev, D.R.: Mathematical Scattering Theory. General Theory. Translated from the Russian by J. R. Schulenberger. Translations of Mathematical Monographs, vol. 105. American Mathematical Society, Providence (1992). ISBN:0-8218-4558-6
18. Zhan, X., Ruan, Y., Huang, H., Li, Q.: Generalized wave operator in von Neumann algebras. Chin. Q. J. Math. 37(1), 52–60 (2022)