ABSTRACT
Chan, Har-Peled, and Jones [2020] recently developed locality-sensitive ordering (LSO), a new tool that allows one to reduce problems in the Euclidean space $\mathbb{R}^d$ to the 1-dimensional line. They used LSO’s to solve a host of problems. Later, Buchin, Har-Peled, and Oláh [2019,2020] used the LSO of Chan et al. to construct very sparse reliable spanners for the Euclidean space. A highly desirable feature of a reliable spanner is its ability to withstand a massive failure: the network remains functioning even if 90% of the nodes fail. In a follow-up work, Har-Peled, Mendel, and Oláh [2021] constructed reliable spanners for general and topologically structured metrics. Their construction used a different approach, and is based on sparse covers.

In this paper, we develop the theory of LSO’s in non-Euclidean metrics by introducing new types of LSO’s suitable for general and topologically structured metrics. We then construct such LSO’s, as well as constructing considerably improved LSO’s for doubling metrics. Afterwards, we use our new LSO’s to construct reliable spanners with improved stretch and sparsity parameters. Most prominently, we construct $\tilde{O}(n)$-size reliable spanners for trees and planar graphs with the optimal stretch of 2. Along the way to the construction of LSO’s and reliable spanners, we introduce and construct ultrametric covers, and construct 2-hop reliable spanners for the line.

CCS CONCEPTS
• Theory of computation → Computational geometry; Sparingification and spanners.

KEYWORDS
Locality-Sensitive Orderings, Reliable Spanners, Doubling Metric, Ultrametric cover, 2-hop spanners, Minor Free graphs

1 INTRODUCTION
The Algorithmist’s toolkit consists of diverse “tools” frequently utilized for many different problems. In the geometric context, some tools apply to general metric spaces such as metric embeddings [33, 55] and packed decompositions [12, 56, 74], while many tools apply mainly to Euclidean spaces, such as dimension reduction [71], locality-sensitive hashing [70], well-separated pair decomposition (WSPD) [40], and many others. Recently, Chan, Har-Peled, and Jones [42] developed a new and exciting tool for Euclidean spaces called Locality-Sensitive Ordering (LSO).

Definition 1.1 ((τ, ρ)-LSO). Given a metric space $(X, d_X)$, we say that a collection $\Sigma$ of orderings is a $(\tau, \rho)$-LSO (locality-sensitive ordering) if $|\Sigma| \leq \tau$, and for every $x, y \in X$, there is a linear ordering $\sigma \in \Sigma$ such that (w.l.o.g.) $x <_\sigma y$ and the points between $x$ and $y$ w.r.t. $\sigma$ could be partitioned into two consecutive intervals $I_x, I_y$ where $I_x \subseteq B_X(x, \rho \cdot d_X(x, y))$ and $I_y \subseteq B_X(y, \rho \cdot d_X(x, y))$. Parameter $\rho$ is called the stretch parameter.

The main reason that LSO has become an extremely useful tool is that it reduces the problem at hand in the $d$-dimensional Euclidean space to the same problem in a much simpler space: the 1-dimensional line. [42] constructed an $O(1) - \rho^{-d} \log \frac{1}{\epsilon}$-LSO for any given set of points in the $d$-dimensional Euclidean space $\mathbb{R}^d$ (more generally, Chan et al. [42] constructed $O(\epsilon^{-1} \cdot \log n)^{O(d)}$, $\epsilon$-LSO for metric spaces with doubling dimension $d$). They used their LSO to design simple dynamic algorithms for approximate nearest neighbor search, approximate bichromatic closest pair, approximate MST, spanners, and fault-tolerant spanners. Afterwards, Buchin, Har-Peled, and Oláh [36, 37] used the LSO of Chan et al. [42] to construct reliable spanners (see Definition 1.2) for Euclidean spaces following the same methodology: reducing the problem to the construction on the line. In this work, we introduce new notions of LSO and apply them to construct reliable spanners for non-Euclidean metrics.

Given a metric space $(X, d_X)$, a $t$-spanner is a weighted graph $H = (X, E, \omega)$ over $X$ where for every pair of points $x, y \in X$, $d_H(x, y) \leq d_X(x, y) \leq t \cdot d_X(x, y)$, with $d_H$ being the shortest path metric of $H$. The parameter $t$ is called the stretch of the spanner.

A metric space $(X, d)$ has doubling dimension $d$ if every ball of radius $2r$ can be covered by $2^d$ balls of radius $r$.

Often in the literature, the metric space $(X, d_X)$ is the shortest path metric of a graph $G$, and there is a requirement that $H$ will be a subgraph of $G$. We will not have such a requirement in this paper.
A highly desirable property of a $t$-spanner is the ability to withstand extensive vertex failures. Levcopoulos, Narasimhan, and Smid [76] introduced the notion of a fault-tolerant spanner. A subgraph $H = (V, E_H, w)$ is an $f$-vertex-fault-tolerant $t$-spanner of a weighted graph $G = (V, E, w)$, if for every set $F \subseteq V$ of at most $f$ vertices, it holds that $\forall u, v \notin F$, $d_{H,F}(u, v) \leq t \cdot d_{G,F}(u, v)$. A major limitation of fault-tolerant spanners is that the number of failures must be determined in advance; in particular, such spanners cannot withstand a massive failure. One can imagine a scenario where a significant portion (even 90%) of a network fails and ceases to function (due to, e.g., close-down during a pandemic), it is important that the remaining parts of the network (or at least most of it) will remain highly connected and functioning. To this end, Bose et al. [32] introduced the notion of a reliable spanner. Here, given a failure set $B \subseteq X$, the residual spanner $H \setminus B$ is a $t$-spanner for $X \setminus B^*$, where $B^* \supseteq B$ is a set slightly larger than $B$. Buchin et al. [37] relaxed the notion of reliable spanners by allowing the size of $B^*$ to be bounded only in expectation.

**Definition 1.2 (Reliable spanner).** A weighted graph $H$ over point set $X$ is deterministic $\nu$-reliable $t$-spanner of a metric space $(X, d_X)$ if $d_H$ dominates $d_X$, and for every set $B \subseteq X$ of points, called an attack set, there is a set $B^* \supseteq B$, called a fault extension of $B$, s.t.:

1. $|B^*| \leq (1 + \nu)|B|$.
2. For every $x, y \notin B^*$, $d_H(X \setminus B^*) \leq d_X(x, y) \leq t \cdot d_X(x, y)$.

An oblivious $\nu$-reliable $t$-spanner is a distribution $D$ over dominating graphs $H$, such that for every attack set $B \subseteq X$ and $H \in \text{supp}(D)$, there exist a superset $B^*$ of $B$ such that, for every $x,y \notin B^*$, $d_H[X \setminus B^*](x,y) \leq t \cdot d_X(x,y)$, and $\mathbb{E}_{H \sim D}[|B^*|] \leq (1 + \nu)|B|$. We say that the oblivious spanner $D$ has $m$ edges if every graph $H \in \text{supp}(D)$ has at most $m$ edges.

We call the distribution $D$ in Definition 1.2 an oblivious $\nu$-reliable $t$-spanner because the adversary is oblivious to the specific spanner produced by the distribution (it may be aware to the distribution itself).

For constant dimensional Euclidean spaces, Bose et al. [32] constructed a deterministic reliable $(1)$-spanner, such that for every attack $B$, the faulty extension $B^*$ contains at most $O(|B|^2)$ vertices. The construction of reliable spanners where the size of $B^*$ is a linear function of $B$ was left as an open question. For every $\nu, \epsilon \in (0, 1)$, and $n$ points in $d$-dimensional Euclidean space $(\mathbb{R}^d, \|\cdot\|_2)$, Buchin et al. [36] used the LSO of Chan et al. [42] to construct a deterministic $(1 + \epsilon)$-spanner with $n \cdot \nu^{-d} \cdot O(\epsilon^{-7d} \cdot O(\log n))$ edges (see also [31]). Later, for the oblivious case, Buchin et al. [37] applied the same LSO to construct an oblivious $\nu$-reliable $(1 + \epsilon)$-spanner with $n \cdot 2O(\epsilon^{-2d} \cdot O(\nu^{-1} \log (\log n)^2))$ edges.

Very recently, Har-Peled, Mendel, and Oláh [69] constructed reliable spanners for general metric spaces, as well as for topologically structured spaces (e.g. trees and planar graphs). They showed that for every integer $k$, every general $n$-point metric space admits an oblivious $\nu$-reliable $(512 \cdot k)$-spanner with $n^{1+1/k} \cdot O(\nu^{-1} \log 2 \log \nu \log 2)$ edges, where $\nu = \max_{x,y} d_X(x,y)$ is the aspect ratio of the metric space (also known as the spread, which a priori is unbounded). Additionally, they showed that ultrametrics (see Definition 3.1) admit oblivious $\nu$-reliable $(2 + \epsilon)$-spanners with $n \cdot O(\nu^{-1} \log 2)$ edges, tree metrics admit oblivious $\nu$-reliable $(3 + \epsilon)$-spanners with $n \cdot O(\nu^{-1} \log 2 \log n)$ edges, and planar metrics admit oblivious $\nu$-reliable $(3 + \epsilon)$-spanners with $n \cdot O(\nu^{-1} \log 2)$ edges (see Table 2).

The reliable spanner constructions of Har-Peled et al. [69] are based on sparse covers. A $(\tau, \rho)$-sparse cover is a collection $C$ of clusters such that every point belongs to at most $\tau$ clusters, and for every pair $x, y \in X$, there is a cluster $C \in C$ containing both $x, y \in C$ where $\text{diam}(C) \leq \rho \cdot d_X(x, y)$; $\rho$ is called the stretch of the cover $C$. They then treat each cluster in $C$ as a uniform metric, construct a reliable spanner for each cluster, and return the union of all the constructed spanners. Thus the main task becomes constructing a reliable spanner for the uniform metric. Specifically, instead of the oblivious $\nu$-reliable $1$-spanner for the line constructed in [37], Har-Peled et al. [69] constructed an oblivious $\nu$-reliable $2$-spanner for the uniform metric, which is the best stretch possible for subquadratic size spanners (see full version [62]). Indeed, this additional factor 2 appears in the stretch parameter in all the spanners in [69]. Most prominently, for trees they constructed an $(O(\nu^{-1} \log \nu \log n), 2 + \epsilon)$-sparse cover, resulting in a stretch $4 + \epsilon$ spanner, while the natural lower bound is 2 (see full version [62]). A similar phenomenon occurs for planar graphs. An additional drawback in the sparse cover based approach of [69] is its dependency on the aspect ratio $\Phi$ (which a priori can be unbounded). This dependency on the aspect ratio is inherent in their technique and cannot be avoided (see Lemma 20 in [69]).

### 1.1 Our Contribution

Our major contribution is to the theory of locality-sensitive orderings. Specifically, we significantly improve the parameters of LSO in doubling metrics, and extend the idea of LSO to general metrics, as well as to topologically structured metrics. This is done by introducing left-sided LSO and triangle-LSO (see Table 1). LSO’s are a powerful tool enabling one to reduce many problems to the line. LSO’s already have many applications in computational geometry [42]; we expect that our LSO for doubling metrics, as well as those for general and topologically structured graphs, will find many additional applications in the future. Next, we use these newly introduced LSO’s (or improved in the case of doubling) to construct oblivious reliable spanners. Our constructions have smaller stretch (optimal in the case of topologically structured metrics) and smaller sparsity (see Table 2). Below we describe each type of LSO in detail, and which spanners it was used to construct. Our constructions of LSO for general and doubling metrics are going through the construction of ultrametric covers. An ultrametric cover is a collection of dominating ultrametrics such that the distance between every pair of points is well approximated by some ultrametric in the collection. We construct the first ultrametric cover for doubling metrics with stretch $1 + \epsilon$ (previously only tree covers were known),

\[ \nu \]

With an additional effort, [69] reduced the stretch of the spanner to $3 + \epsilon$. This analysis is tight, and their technique cannot give a reliable spanner with a stretch factor smaller than 3.
and improve the stretch parameter in the ultrametric covers of general metrics (see Table 3). Finally, a crucial ingredient when one constructs reliable spanners using LSO is that the graph does not depend on the metric space. Our spanners for general metrics have significantly smaller stretch compared to \cite{69} (8k compared to 512k^−1), this constant is highly important as it governs the parameter in the power of n. An additional advantage is that we remove the dependency on the aspect ratio (which a priori can be unbounded).

### Table 1: Summary of all known results, on all the different types of locality sensitive orderings (LSO), k ∈ \mathbb{N} is an integer, ε ∈ (0, 1) is an arbitrarily small parameter. ★ denotes the full version of this paper [62].

| LSO type       | Metric Space                  | # of orderings (τ) | Stretch | Ref |
|----------------|-------------------------------|--------------------|---------|-----|
| (Classic) LSO  | Euclidean space R^d           | O(ε−1 · log n)^Ω(d) | τ·log n | \cite{42} |
|                | Doubling dimension d          | O(ε−1 · log n)^Ω(d) | τ·log n | \cite{42} |
| Triangle-LSO   | General metric                | O(ε^{(1−k^−1)})   | 2k + ε  | \cite{5.3} |
|                | Ultrametric                   | 1                  | 1       | Lem. 5.2 |
| Left-sided LSO | Tree                          | log n              | 1       | ★ |
|                | Treewidth k                   | k · log n          | 1       | ★ |
|                | Planar graph                  | 2 · log n          | 1 + ε   | ★ |
|                | Minor Free                    | 2 · log n          | 1 + ε   | ★ |

#### Left-sided LSO.

A (τ, ρ)-left-sided LSO for a metric space (X, d_X) is a collection Σ of linear orderings over subsets of X, called partial orderings, such that every point x belongs to at most τ partial orderings, and for every x, y ∈ X, there is a partial ordering σ ∈ Σ such that for every two points x, y ∈ X satisfying x ≤_σ x' and y ≤_σ y', it holds that d_X(x', y') ≤ ρ · d_X(x, y) (see the full version \cite{62} for formal definition and statements). Note that the stretch guarantee of a (τ, ρ)-left-sided LSO implies that of a (τ, ρ)-LSO (but not the vice versa). However, there could be Ω(n) (partial) orderings in a (τ, ρ)-left-sided LSO. By lifting the restriction on the total number of partial orderings, we can construct a left-sided LSO with an optimal stretch of 1 or a nearly optimal stretch of 1 + ε; see Table 1. This small stretch ultimately leads to the (nearly) optimal stretch for the reliable spanners of tree and planar metrics constructed in this work, which is not attainable in previous work \cite{69}.

We then prove a meta-theorem stating that every metric space admitting a (τ, ρ)-left-sided LSO has an oblivious v-reliable 2p-spanner with n · O(τv−1r^2 log n) edges (see full version \cite{62}). We show that n-vertex trees admit a (log n, 1)-left-sided LSO and conclude that trees have oblivious v-reliable 2-spanners with n · O(v−1 log^3 n) edges. Note that the stretch parameter 2 is optimal.

Later, we show that planar graphs admit a (2 + ε · v^n, 1 + ε)-left-sided LSO for every ε ∈ (0, 1). An oblivious v-reliable (2 + ε · v^n)-spanner with n · O(v^{-1}2 log^2 n) edges follows. The same results also hold for bounded treewidth graphs and graphs excluding a fixed minor.

#### Ultrametric cover.

A (τ, ρ)-tree cover for a metric space (X, d_X) is a set T of τ dominating trees 3 such that the distance between every pair of points is preserved up to a factor ρ in at least one tree (u, v, min_{t ∈ T} d_t(u, v) ≤ ρ · d_X(u, v)). When all trees in the cover are ultrametric, we call it an ultrametric cover (Definition 3.2). The first study on tree covers was for Euclidean spaces by Arya et al. [9] who constructed the so-called Dumbbell trees. For general metrics, Mendel and Naor [79] (implicitly) constructed an ultrametric cover from Ramsey type embeddings. These covers actually have a stronger guarantee, where every vertex v is guaranteed to have an ultrametric in the cover approximating its shortest path tree (υvΣ∀t ⟨u, v⟩ ≤ ρ · d_X(u, v)). There is a long line of work on Ramsey-type embeddings \cite{2, 13, 14, 17, 23, 35, 59, 79, 80}. The state of the art covers follow from Naor and Tao [80], and implies a (2ε · k, O(k · n^{1/k})) ultrametric cover. For doubling metrics, Bartal, Fandina, and Neiman [15] constructed a (1 + ε, ε−2O(d^3)) tree cover. We refer to [15] for further results and background on tree covers.

We observe that every ultrametric admits a (1, 1)-triangle LSO, which implies that given a (τ, ρ)-ultrametric cover, one can construct a (τ, ρ)-triangle LSO (Lemma 5.2). Indeed, the main step in our construction of a triangle LSO for general metrics is a construction of a (O(n^{1/2} · ε−1), 2k + ε)-ultrametric cover (Theorem 3.3). Our construction provides a constant improvement in the stretch parameter (equivalently, a polynomial improvement in the number of ultrametrics in the cover) compared to previous results.
Figure 1: Relationships between different concepts; new concepts introduced in this papers are green-shaded.

Table 2: Comparison between previous and new constructions of reliable spanners. ★ denotes the full version of this paper [62]. All spanners (except [32]) constructed on \( n \)-point metric spaces with reliability parameter \( v \). For doubling metrics, we recover the same strong results previously known only for Euclidean space. Both lower bounds hold for the uniform metric. For all other metric spaces, we improve both stretch and sparsity, and remove the undesirable dependence on the aspect ratio \( \Phi \).

For trees and planar graphs, the stretch was improved from \( 3 + \varepsilon \), to the best possible stretch 2. For general graphs, our spanner has stretch 8\( k \), considerably improving the constant hiding in [69]. This constant governs the parameter in the power of \( n \).

| Family                  | stretch   | guarantee         | size                                      | ref   |
|-------------------------|-----------|-------------------|-------------------------------------------|-------|
| Euclidean (\( \mathbb{R}^d, || \cdot ||_2 \)) | \( O(1) \) | Deterministic     | \( \Omega(n \log n) \)                     | [32]  |
| Doubling dimension \( d \) | \( 1 + \varepsilon \) | Deterministic     | \( n \cdot O(\varepsilon)^{-\delta d} \cdot O(\log n) \) | [36]  |
| General metric          | \( 8k + \varepsilon \) | Oblivious         | \( O(n^{1+\varepsilon/k} \cdot \log^{27} n) \) | [37]  |
| LSO Tree, planar        | \( k \)   | Deterministic     | \( \Omega(n^{1+\varepsilon/k}) \)         | [69]  |
| Ultrametric Tree, planar| \( k < 2 \) | Oblivious         | \( \Omega(n^\delta) \)                     | *     |
| Treewidth \( k \)       | \( 2 \)   | Oblivious         | \( n \cdot O(\varepsilon^{1/(k^2) \log^{27} n}) \) | *     |
| Planar                  | \( 3 + \varepsilon \) | Oblivious         | \( n \cdot O(\varepsilon) \)               | [69]  |
| Minor-free              | \( 2 + \varepsilon \) | Oblivious         | \( n \cdot O(\varepsilon) \)               | *     |

A more structured case is that of a \((\tau, \rho, k, \delta)\)-ultrametric cover, where in addition to being a \((\tau, \rho)\)-ultrametric cover, we require that each ultrametric will be a \( k \)-HST of degree at most \( \delta \) (see Definition 3.1). We show that every \( \Omega(\frac{1}{\varepsilon}) \)-HST of degree bounded by \( \delta \) admits a (classic) \((\frac{3}{\varepsilon}, \varepsilon)\)-LSO (Lemma 4.2). It follows that a \((\tau, \rho, \Omega(\frac{1}{\varepsilon}), \delta)\)-ultrametric cover implies a \((\tau \cdot \frac{\delta}{2}, (1 + \varepsilon)\rho)\)-LSO (Lemma 4.3). The trees in the tree cover for doubling metrics of [15] are far from being ultrametrics and cannot be used in our framework. We then construct an \((\varepsilon^{-O(d)}), 1 + \varepsilon, \frac{1}{\varepsilon}, \varepsilon^{-O(d)}\)-ultrametric cover for spaces with doubling dimension \( d \) (Theorem 3.4), which implies the respective LSO. Interestingly, having such an ultrametric cover is a characterizing property for metric spaces of bounded doubling dimension (Theorem 3.4). See Table 3 for a summary.

2-hop reliable spanners for the path graph. Using different types of LSO, we can reduce the problem of constructing reliable spanners for different complicated metric spaces to that of constructing reliable spanners for the 1-dimensional path graph. Buchin et al. [37] constructed an oblivious \( v \)-reliable 1-spanner...
with \( n \cdot \tilde{O}(v^{-1}(\log \log n)^2) \) edges for the path graph. However, the shortest path between two given vertices in their spanner could contain \( \Omega(\log n) \) edges (called hops). While \((\log n)\)-hop spanners are acceptable when applying them upon a \((\tau, \epsilon)\)-LSO, using a \( h \)-hop spanner of the path graph for \((\tau, \rho)\)-triangle-LSO will result in distortion \( h \cdot \rho \). It is therefore desirable to minimize the number of hops used by the spanner. Having a \( 1 \)-hop spanner will require \( \Omega(n^2) \) edges; we thus settle for the next best thing: a \( 2 \)-hop reliable spanner. Specifically, we construct an oblivious \( v \)-reliable, \( 2 \)-hop \(-\)1-spanner with \( n \cdot \tilde{O}(\log^2 n + v^{-1} \log n \cdot \log \log n) \) edges for the path graph (Lemma 5.7). This spanner is later used in our meta Theorem 5.4 to construct reliable spanners for metric spaces admitting \((\tau, \epsilon)\)-LSO (see the full version [62]).

**Connectivity preservers.** While research on reliable spanners for metric spaces has been fruitful, nothing is known for reliable spanners of graphs, where we require the spanner to be a subgraph of the input graph. In a recent talk, Har-Peled [67] asked a "probably much harder question": whether it is possible to construct a non-trivial subgraph reliable spanner. We show that, even for a much simpler problem where one seeks a subgraph to only preserve connectivity for vertices outside \( B^* \), the faulty extension of \( B \), the subgraph must have \( \Omega(n^2) \) edges in the worst case. Indeed, our lower bound is much more general: it applies to \( g\)-reliable connectivity preservers for some function \( g \). A \( g \)-reliable connectivity preserver of a graph \( G = (V, E) \) is a subgraph \( H \) of \( G \) such that for every attack \( B \subseteq V \), there is a superset \( B^* \supseteq B \) of size at most \( g(|B|) \), such that for every \( u, v \in V \setminus B^* \), if \( u, v \) are connected in \( G \) \( B \), then they are also connected in \( H \setminus B \). Observe that a \( v \)-reliable spanner defined in Definition 1.2 is a \( g \)-reliable (non-subgraph) spanners for the linear function \( g(x) = (1 + v)x \). We showed that there is an \( n \)-vertex graph \( G \) such that every oblivious \( q_n \)-reliable connectivity preserver has \( \Omega(n^{1+1/k}) \) edges for any function \( g = O(x^{k}) \). Taking \( k = 1 \) gives a lower bound \( \Omega(n^2) \) on the number of edges of subgraph \( v \)-reliable spanners. On the positive side, we provide a construction of a deterministic connectivity preserver matching the lower bound (see the full version [62]).

### 2 PRELIMINARIES

Let \((X, d_X)\) be a metric space. The aspect ratio, or spread, denoted by \( \Phi \), is defined as follows: \( \Phi = \max_{x, y} d_X(x, y) / \min_{x, y} d_X(x, y) \). We denote by \( [n] \) the set of integers \([1, 2, \ldots, n]\). For two integers \( a \leq b \), we define \([a : b] = (a, a+1, \ldots, b) \).

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### Table 3: New and previous constructions of tree and ultra-metric covers.

| Space | type       | stretch        | # of trees | ref |
|-------|------------|----------------|------------|-----|
| Euclidean \( \mathbb{R}^d \) | tree       | \( 1 + \epsilon \) | \( O(\frac{d}{\epsilon} \log \frac{2}{\epsilon}) \) | [9] |
| Doubling dimension \( d \) | ultrametric | \( O(d^3) \) | \( O(d \log d) \) | [41] |
| Doubling dimension \( d \) | tree       | \( 1 + \epsilon \) | \( e^{O(d)} \) | [16] |
| Doubling dimension \( d \) | ultrametric | \( 1 + \epsilon \) | \( e^{O(d)} \) | Thm. 3.4 |
| General metric | ultrametric | \( 2e \cdot k \) | \( O(k \cdot n^{1/k}) \) | [79, 80] |
| General metric | ultrametric | \( 2k + \epsilon \) | \( O(n^{2} \cdot e^{-\epsilon}) \) | Thm. 3.3 |

### 1.2 Related Work

The tradeoff between stretch and sparsity (number of edges) of (regular) \( t \)-spanners has been extensively studied \([8, 43, 63, 68, 75, 82]\); see the recent survey of Ahmed et al. [4], and the book [81] and references therein for more details. The bottom line is that \( n \)-point metric spaces admit \( (2k-1) \)-spanners (for every integer \( k \)) with \( O(n^{1+1/k}) \) edges [43], while the metric induced by \( n \) points in \( d \) dimensional Euclidean space admits a \((1 + \epsilon)\)-spanner with \( n \cdot O(\epsilon^{-d}) \) edges [8]. Similarly, \( n \)-point metric spaces with doubling dimension \( d \) admit \((1 + \epsilon)\)-spanners with \( n \cdot e^{-O(d)} \) edges [41].

For vertex-fault-tolerant spanner, it was shown that every \( n \)-point set in \( \mathbb{R}^d \), or more generally in a space of doubling dimension \( d \), admits a \( f \)-vertex-fault-tolerant \((1+\epsilon)\)-spanner with \( e^{-O(d) \cdot f \cdot n} \) edges [76, 77, 84]. For general graphs, after a long line of works \([25, 26, 28, 44, 47, 48]\), it was shown that every \( n \)-vertex graph admits an efficiently constructible \( f \)-vertex-fault-tolerant \((2k-1) \)-spanner with \( O(f^{k-1} \cdot n^{1+1/k}) \) edges, which is optimal assuming the Erdös’ Girth Conjecture [54]. A related notion is that of a vertex-fault-tolerant (VFT) emulator. Unlike spanners, emulators are not required to be subgraphs, and the weight of an emulator edge is determined w.r.t. the faulty set. It was recently shown that vertex-fault-tolerant (VFT) emulators are asymptotically sparser from their spanner counterparts [24].

In addition to vertex-fault-tolerant (VFT) spanners, also edge-fault-tolerant (EFT) spanners were studied, where the guarantee is to withstand up to \( f \)-edge faults (as opposed to \( f \)-vertex faults in VFT). Bodwin, Dinitz, and Robelle [27] constructed an \( f \)-EFT \( 2k \)-1-spanners with \( O(k^2 f \frac{d}{\epsilon} \cdot n^{1+1/k} + k f n) / O(k^2 f \frac{d}{\epsilon} \cdot n^{1+1/k} + k f n) \) edges for odd\( f \) even values of \( k \) respectively. There is also a lower bound of \( \Omega(\frac{f^3}{\epsilon^3} n^{1+1/k}) \) [25].

Abam et al. [1] introduced the notion of region fault-tolerant spanners for the Euclidean plane. They showed that one can construct a \( t \)-spanner with \( O(n \log n) \) edges in such a way that if points belonging to a convex region are deleted, the residual graph is still a spanner for the remaining points.

Spanners with low hop diameter for Euclidean spaces of fixed dimension were studied in the pioneering work of Arya et al. [10]. State of the art is a \((1 + \epsilon)\)-spanner constructible in \( O(\log n) \) time by Solomon [83] that has \( O(\log n \cdot n^2) \) edges and hop diameter \( k \).

In addition to having a small number of edges, it is desirable to have a spanner with a small total edge weight, called a light spanner. Light spanners have been thoroughly studied in the spanner literature \([7, 29, 30, 43, 45, 46, 51, 52, 64, 65, 75]\). Sparse (and light) spanners were constructed efficiently in different computational models such as LOCAL [72], CONGEST [50], streaming \([5, 18, 20, 49, 60, 73]\), massive parallel computation (MPC) [22] and dynamic graph algorithms [19, 21].
Table 4: Construction of reliable spanners for the line. [36] and [37] constructed sparse (both deterministic and oblivious) reliable 1-spanners for points on the line. However, their spanners have $O(\log n)$ hops, which will incur distortion $O(\rho \cdot \log n)$ when applied on a $(\tau, \rho)$-triangle LSO (with $\rho > 1$). We construct a 1-spanner with only 2-hops, which we later use to construct a reliable spanner from a triangle-LSO. In addition, we construct a 2-hop left spanner for the line, which is later used to construct a reliable spanner from a left-sided LSO.

| Type           | guarantee                  | size                        | hops                        | ref         |
|----------------|----------------------------|-----------------------------|-----------------------------|-------------|
| 1 spanner      | Deterministic              | $n \cdot O(\log n \cdot v^{-6})$ | $O(\log n)$                | [36]        |
|                | Oblivious                  | $n \cdot O(v^{-1} \cdot \log v^{-1})$ | $O(\log n)$                | [37]        |
|                | Oblivious                  | $n \cdot O(\log^2 n + v^{-1} \log n \cdot \log n)$ | 2                           | Lemma 5.7   |
| Left spanner   | Oblivious                  | $n \cdot O(v^{-1} \log n)$ | 2                           | Full version [62] |

We use $\tilde{O}$ notation hides poly-logarithmic factors. That is $\tilde{O}(f) = O(f) \cdot \log^{O(1)}(f)$.

Let $G$ be a graph. We denote the vertex set and edge set of $G$ by $V(\mathcal{G})$ and $E(\mathcal{G})$, respectively. When we want to explicitly specify the vertex set $V$ and edge set $E$ of $G$, we write $G = (V, E)$. If $G$ is a weighted graph, we write $G = (V, E, w)$ with $w : E \rightarrow \mathbb{R}_+$ being the weight function on the edges of $G$. For every pair of vertices $u, v \in V$, we denote by $d_H(u,v)$ the shortest path distance between $u$ and $v$ in $G = (V, E, w)$. Given a path $P \subseteq G$, we define the hop length of $P$ to be the number of edges on the path.

A $t$-spanner for a metric space $(X, d_X)$ is a weighted graph $H(V, E, w)$ that has $V = X$, $w(u,v) = d_X(u,v)$ for every edge $(u,v) \in E$ and $d_H(x,y) \leq d_H(x,y) \leq t \cdot d_X(x,y)$ for every pair of points $x, y \in X$. We say that a $t$-spanner $H$ has hop number $h$ if for every pair of vertices $x, y$, there is an $x-y$ path $P$ in $H$ of at most $h$ hops such that $w_H(P) = t \cdot d_X(x,y)$.

The path graph $P_n$ contains $n$ vertices $v_1, v_2, \ldots, v_n$ and there is (unweighted) edge between $v_i$ and $v_j$ if $|i-j| = 1$. A path $v_i, v_{i+1} \ldots v_n$ is monotone if for every $i$, $i_j < i_{j+1}$. Note that if a spanner $H$ contains a monotone path between $v_i, v_j$ then $d_H(v_i, v_j) = d_{P_n}(v_i, v_j) = |i - j|$. We sometimes identify vertices of $P_n$ with numbers in $\{1, 2, \ldots, n\}$, and refer to $\{1, 2, \ldots, n\}$ as the vertex set of $P_n$.

A metric $(X, d_X)$ has doubling dimension $d$ if every ball of radius $\rho$ can be covered by at most $2^d$ balls of radius $\rho/2$. The following lemma gives the standard packing property of doubling metrics (see, e.g., [60]).

**Lemma 2.1 (Packing Property).** Let $(X, d)$ be a metric space with doubling dimension $d$. If $S \subseteq X$ is a subset of points with minimum interpoint distance $\rho$ that is contained in a ball of radius $\rho/2$, then $|S| = \left( \frac{2^d}{\rho} \right)^{O(d)}$.

In the following lemma, we show that when constructing oblivious spanners, it is enough to bound the number of edges in expectation to obtain a worst-case guarantee.

**Lemma 2.2.** Consider an $n$-vertex graph $G = (V, E, w)$ that admits an oblivious $v$-reliable $t$-spanner with $m$ edges in expectation. Then $G$ admits an oblivious $2v$-reliable $t$-spanner with $2m$ edges in the worst case.

**Proof.** Formally, there is a distribution $\mathcal{D}$ over spanners $H$ such that for every attack $B \subseteq V$, $\mathbb{E}[|B^+ \setminus B|] \leq |V|$, and $\mathbb{E}[|H|] \leq m$. Let $\mathcal{D}'$ be the distribution over spanners $H$ obtained by conditioning $\mathcal{D}$ on the event $|H| \leq 2m$. Clearly, all the spanners in supp($\mathcal{D}'$) have at most $2m$ edges. Furthermore, for every attack $B \subseteq V$, it holds that

$$\mathbb{E}_{H \sim \mathcal{D}'}[|B^+ \setminus B|]$$

$$= \mathbb{E}_{H \sim \mathcal{D}'}[|B^+ \setminus B| \mid |H| \leq 2m]$$

$$= \frac{1}{\mathbb{Pr}[|H| \leq 2m]} \cdot \mathbb{E}_{H \sim \mathcal{D}'}[|B^+ \setminus B| \mid |H| \leq 2m] \cdot \mathbb{Pr}[|H| \leq 2m]$$

$$\leq \frac{1}{\mathbb{Pr}[|H| \leq 2m]} \cdot \mathbb{E}_{H \sim \mathcal{D}'}[|B^+ \setminus B|] \leq 2v \cdot |B|,$$

where in the last inequality, we use Markov’s inequality. \hfill $\square$

### 3 ULTRAMETRIC COVERS

**Ultraan.** An ultrametric $(X, d)$ is a metric space satisfying a strong form of the triangle inequality, that is, for all $x, y, z \in X$, $d(x,z) \leq \max \{d(x,y), d(y,z)\}$. A related notion is a $k$-hierarchical well-separated tree ($k$-HST).

**Definition 3.1 ($k$-HST).** A metric $(X, d_X)$ is a $k$-hierarchical well-separated tree ($k$-HST) if there exists a bijection $\varphi$ from $X$ to leaves of a rooted tree $T$ in which:

1. Each node $v \in T$ is associated with a label $\Gamma_v$ such that $\Gamma_v = 0$ if $v$ is a leaf and $\Gamma_v \geq k\Gamma_u$ if $v$ is an internal node and $u$ is any child of $v$.
2. $d_X(x, y) = \Gamma_{\text{lca}(\varphi(x), \varphi(y))}$ where $\text{lca}(u, v)$ is the least common ancestor of any two given nodes $u, v$ in $T$.

It is well known that any ultrametric is a 1-HST, and any $k$-HST is an ultrametric (see [17]).

**Ultraan cover.** Consider a metric space $(X, d_X)$, a distance measure $d_Y$ is said to be dominating if $\forall x, y \in X$, $d_Y(x,y) \leq d_Y(x,y)$. A tree/ultrametric over $X$ is said to be dominating if its metric is dominating. Bartal, Fandina, and Neiman [15] studied tree covers: a metric space $(X, d_X)$ admits a $(\tau, \rho)$-tree cover if there are at most $\tau$ dominating trees $\{T_1, T_2, \ldots, T_\tau\}$ such that $X \subseteq \cup T_i$ for every $i \in [\tau]$ and for every pair of points $x, y \in X$, there is some tree $T_i$ where $d_{T_i}(x, y) \leq \rho \cdot d_X(x, y)$. Bartal et al. [15] observed that the previous constructions of Ramsey trees [2, 79, 80] give an $\tilde{O}(n^{1/k}, 2\varepsilon k)$-tree cover for general metrics, and explicitly constructed an $(\epsilon^{-O(d)}, 1 + \epsilon)$-tree cover for metric spaces with

\footnote{Ramsey trees have additional desired property compared to general tree covers: for every vertex $v$, there is a single tree in the cover satisfying all its pairwise distances, as oppose to union of all the trees in a general tree cover.}
Thus, we obtain a polynomial improvement in the number of ultra-
for every one of the partitions.

Next, in Theorem 3.4 below, we show that every metric space
is a laminar system. The basic idea of doing this is to produce a tree
partition where each vertex is a singleton, and consider
partitions from a pairwise partition cover of the current scale, and
constructing the next level in the hierarchical partition, we take
removal of a metric space
due to the large ratio between consecutive scales, the effects of this
slightly “round” them around the “borders” so that no previously
constructed cluster is included.

Let $X$ be a metric space. We will inductively define a new set of partitions, enforcing it to be a laminar system. The basic idea of doing this is to produce a tree partition where each vertex is a singleton, and consider partitions from a pairwise partition cover of the current scale, and constructing the next level in the hierarchical partition, we take removal of a metric space
Due to the large ratio between consecutive scales, the effects of this
slightly “round” them around the “borders” so that no previously
constructed cluster is included. We will inductively define a new set of partitions, enforcing it to be a laminar system.

3.1 From Pairwise Partition Cover to Ultrametric Cover: Proof of Lemma 3.6

Lemma 3.6 is a reduction from pairwise partition cover scheme to ultrametric cover. In essence, an ultrametric is simply a hierarchical partition. Thus, this reduction takes unrelated partitions in all possible scales, and combines them into hierarchical/laminar partitions. Reductions similar in spirit were constructed in the context of the Steiner tree removal problem [58], stochastic Steiner point removal [53], universal Steiner tree [38], and others. We follow here a bottom-up approach, where the ratio between consecutive scales in a single hierarchical partition (a.k.a. ultrametric) is $O(\frac{\delta}{\log k})$. When constructing the next level in the hierarchical partition, we take partitions from a pairwise partition cover of the current scale, and slightly “round” them around the “borders” so that no previously created cluster will be divided (see Figure 2). The argument is that due to the large ratio between consecutive scales, the effects of this rounding are marginal.

Proof of Lemma 3.6. Assume w.l.o.g. that the minimal pairwise distance in $X$ is 1, while the maximal pairwise distance is $\Phi$. Fix $c \leq 1$ to be determined later. For $i \geq 0$, set $\Delta_i = c \cdot \frac{(2k)^i}{\Phi}$, and let $P_i = \{P_{i,1}^1, \ldots, P_{i,d}^i\}$ be a $(\tau, \rho, \Delta_i)$-padded partition cover (we assume that $P_i$ has exactly $\tau$ partitions; we can enforce this assumption by duplicating partitions if necessary). Fix some $j$, let $P_{i,j+1}$ be the partition where each vertex is a singleton, and consider $(P_{i,j})_{j \geq -1}$. We will inductively define a new set of partitions, enforcing it to be a laminar system. The basic idea of doing this is to produce a tree of partitions where the lower level is a refinement of the higher level, and we do by grouping a cluster at a lower level to one of the clusters at a higher level separating it.
The lowest level $\mathcal{P}^{-1}_j$ where each set in the partition is a singleton, stays as-is. Inductively, for any $i \geq 0$, after constructing $\mathcal{P}^{-i}_j$ from $\mathcal{P}^{i-1}_j$, we will construct $\mathcal{P}^i_j$ from $\mathcal{P}^{i-1}_j$, using $\mathcal{P}^{-i}_j$. Let $\mathcal{P}^{i}_j = \{C_1, \ldots, C_\phi\}$ be the clusters in the partition $\mathcal{P}^{i}_j$. For each $q \in \{1, \phi\}$, let $Y_q = X \setminus \cup_{a \in C_q} C^a$ be the set of unclustered points (w.r.t. level $i$, before iteration $q$). Let $C_q^\prime \in q \setminus Y_q$ be the cluster $C_q$ restricted to vertices in $Y_q$, and let $S = \{C \in \mathcal{P}^{i-1}_j \mid C \cap C_q^\prime \neq \emptyset\}$ be the set of new level-$(i-1)$ clusters with non empty intersection with $C_q'$. We set the new cluster $\tilde{C}_q = S \cup S_q$, and continue iteratively. See Figure 2 for illustration. Clearly, $\mathcal{P}^{i}_{j-1}$ is a refinement of $\mathcal{P}^{i}_j$. We conclude that $\{\mathcal{P}^{i}_{j-1}\}_{j=1}^k$ is a laminar hierarchical set of partitions that refine each other.

We next argue by induction that $\mathcal{P}^{i}_j$ has diameter $\Delta_i(1 + e)$. Consider $\tilde{C}_q \in \mathcal{P}^{i}_j$, it consists of $C_q' \subseteq C_q \in \mathcal{P}^{i-1}_j$ and of clusters in $\mathcal{P}^{i-1}_j$ intersecting $C_q'$. As the diameter of $C_q'$ is bounded by diam($C_q') \leq \Delta_i$, and by the induction hypothesis, the diameter of each cluster $C \in \mathcal{P}^{i-1}_j$ is bounded by $(1 + e)\Delta_{i-1}$, we conclude that the diameter of $\tilde{C}_q$ is bounded by

$$
\Delta_i + 2 \cdot (1 + e)\Delta_{i-1} = \Delta_i + 2(1 + e)\Delta_i \leq (1 + e)\Delta_i,
$$

since $\rho \geq 1$ and $e < 1$.

Next we argue that $\mathcal{P}^{i}_j = \{\mathcal{P}^{i}_{j-1} \cup \mathcal{P}^{i}_j\}$ is a $(\tau, (1 + e)\rho, 0, (1 + e)\Delta)$-pairwise partition cover. Observe that it contains $\tau$ partitions, and we have shown that all the clusters have diameter at most $(1 + e)\Delta_i$. Thus, it remains to prove that for every pair $x, y$ at distance $d_X(u, v) \in \left[\frac{(1 + e)\Delta_i + (1 + e)\Delta_i}{2(1 + e)\rho}, \frac{(1 + e)\Delta_i}{1 + \rho}\right]$ contained in some cluster. As $d_X(u, v) \in \left[\frac{(1 + e)\Delta_i}{2}, \frac{1 + e}{2}\right]$, there is some index $j$ such that $B_X(u, e\Delta_i), B_X(v, e\Delta_i) \subseteq C_j \in \mathcal{P}^{i}_j$. That is, the balls of radius $e\Delta_i$ around $u, v$ are contained in a cluster of $\mathcal{P}^{i}_j$. We argue that $u, v \in C_j \in \mathcal{P}^{i}_j$. Let $C_{\hat{u}}, C_{\hat{v}} \in \mathcal{P}^{i-1}_j$ be the clusters containing $u, v$ respectively at $(i-1)$-th level. Note that they both have diameter at most $(1 + e)\Delta_{i-1} = \frac{(1 + e)(1 + e)}{2(1 + e)\rho} < e\Delta_i$. Hence $C_{\hat{u}}, C_{\hat{v}} \subseteq B_X(u, e\Delta_i), B_X(v, e\Delta_i) \subseteq C_j \in \mathcal{P}^{i}_j$, and similarly $\tilde{C}_q \subseteq C_j \in \mathcal{P}^{i}_j$. By the partitioning algorithm, it follows that $C_{\hat{u}}, C_{\hat{v}} \subseteq C_j \in \mathcal{P}^{i}_j$ (as $C_{\hat{u}}, C_{\hat{v}}$ do not intersect any other clusters), and in particular $u, v \in C_j \in \mathcal{P}^{i}_j$ as required.

Finally, we construct an ultrametric cover. Fix an index $j \in [1, r]$; we construct a $(\frac{2\rho}{\tau})$-HST $U_j$ as follows. Leaves of $U_j$ bijectively correspond to points in $X$ and have label 0. For each $i \in [0, l]$ where $l = \log_{\log_e \sqrt{\rho}}$, internal nodes at level $i$ bijectively correspond to the clusters $\mathcal{P}^i_j$ (leaves of $U_j$ is at level $-1$), and have label $(1 + e)\Delta_i$. There is an edge from each node corresponding to a cluster $C_{i-1} \in \mathcal{P}^i_j$ to the node corresponding the unique cluster $\hat{C}_{i-1} \in \mathcal{P}^i_j$, containing $\hat{C}_{i-1}$. The root of $U_j$ is the unique single cluster in $\mathcal{P}^l_j$. Clearly, the ultrametric cover $\{U_j\}_{j=1}^r$ is dominating, and every ultrametric is a $(\frac{2\rho}{\tau})$-HST.

To bound the stretch, we will construct such an ultrametric cover with $c = (1 + e)$ for every $l \in [0, \log_{\log_e \sqrt{\rho}}]$. The final ultrametric cover will be a union of these $O(\log \log_e \sqrt{\rho})$ ultrametric covers. Clearly, their cardinality is bounded by $\tau \cdot O(\log \log_e \sqrt{\rho}) = O(\frac{\rho}{\tau} \log \frac{\rho}{\tau})$.

Consider a pair $x, y \in X$. Let $l \in [0, \log_{\log_e \sqrt{\rho}}]$, and $i \geq 0$ be the unique indices such that $(1 + e)^{-i} \left(\frac{4\rho}{\tau}\right)^i \leq (1 + e)^i \cdot d_X(x, y) \leq (1 + e)^i \cdot (\frac{4\rho}{\tau})^i$. For $c = (1 + e)^i$, there is some index $j$, and a cluster $C_j \in \mathcal{P}^i_j$ such that $x, y \in C_j \in \mathcal{P}^i_j$. Thus the in the corresponding ultrametric, $x, y$ both descend to an internal node with label $(1 + e)^i \cdot (\frac{4\rho}{\tau})^i \leq (1 + e)^i \cdot d_X(x, y)$, the stretch guarantee follows.

In summary, we have constructed an $O(\frac{\rho}{\tau} \log \frac{\rho}{\tau})$, $(\rho(1 + 7e))$-ultrametric cover, consisting of $(\frac{4\rho}{\tau})$-HST’s.

\[\Box\]

### 3.2 Pairwise Partition Cover for General Metrics: Proof of Lemma 3.7

Fix parameter $\delta \in (0, 1]$. We begin by creating a distribution over partitions, such that for every pair of points $u, v$ at distance $\Delta$, there is a non trivial probability that the same balls around $u, v$ contained in a single cluster. Later, Lemma 3.7 will follow by taking the union of many independently sampled such partitions.

**Lemma 3.7.** For every $n$-point metric space $(X, d_X)$, integer $k \geq 1$, $\delta \in (0, 1]$, and $\Delta > 0$ there is a distribution over $\Delta$-bounded partitions such that for every pair of points $u, v$ where $d_X(u, v) \leq \Delta$, with probability at least $\frac{n^2 \Delta}{\delta}$, the balls $B_X(u, \frac{\delta \Delta}{4k(2k+\delta)}) \cup B_X(v, \frac{\delta \Delta}{4k(2k+\delta)})$ contained in a single cluster.

For the case where $\delta = 0$, is a distribution formerly constructed by the first author [57]. Our proof here follows the steps of [57] (which is based on the partition in [39]).

**Proof of Lemma 3.8.** Pick u.a.r. a radius $r \in \{\frac{1}{2}, \frac{1}{3}, \ldots, \frac{k}{k}\}$, and a random permutation $\pi = \{d_1, d_2, \ldots, d_k\}$ over the points. Then set $C_\pi = B_X(u_\pi, r \cdot \frac{\delta}{\Delta}) \cup \bigcup_{j=1}^{k} B_X(v_{\pi}, r \cdot \frac{\delta}{\Delta})$. As a result we obtain a $\Delta$ bounded partition $\{C_\pi\}_{\pi=1}^{k}$.

For a pair $u, v$ where $d_X(u, v) \leq \Delta$, let $T = B_X(u, \frac{\delta \Delta}{4k(2k+\delta)}) \cup B_X(v, \frac{\delta \Delta}{4k(2k+\delta)})$. Note that every $d_X(x, y) \leq d_X(u, v) + \frac{\delta \Delta}{4k(2k+\delta)} \leq \frac{\delta \Delta}{4k(2k+\delta)}$. Hence all the points in $T$ will join the cluster of $x \in T$. Denote by $\Psi$ the event that all the vertices in $T$ are contained in a single cluster. Using the law of total probability, we conclude

$$
\Pr[\Psi] = \frac{1}{k} \sum_{i=1}^{k} \Pr[\Psi \mid r = \frac{\delta}{\Delta}] \geq \frac{1}{k} \sum_{i=1}^{k} \frac{|A_{k-1}|}{|A_k|} \geq \frac{|A_{k-1}|}{|A_k|} \geq n^{-\frac{t}{r}} .
$$

---

This is the full version, see also the conference version [56].
where the second inequality follows by the inequality of arithmetic and geometric means. □

We now ready to prove Lemma 3.7 (restated for convenience).

**Lemma 3.7.** Every $n$-point metric space $(X, d_X)$ admits an $(O(n^{1/3} \log n), 2k + \delta, \frac{\delta}{8k(2k+\Delta)})$-pairwise partition cover scheme for any $\delta \in [0, 1]$ and integer $k \geq 1$.

**Proof.** Fix $\Delta$. Sample $s = n^{\frac{1}{3}} \cdot 2 \ln n$ i.i.d. partitions using Lemma 3.8. Consider a pair of points $u, v$ such that $d_X(u, v) \leq \frac{\delta}{2k+\Delta}$. Then in each sampled partition, the probability that balls $B_X(u, \frac{\delta}{4k(2k+\Delta)})$, $B_X(v, \frac{\delta}{4k(2k+\Delta)})$ contained in a single cluster is at least $p = n^{-\frac{k}{3}}$. The probability that $u, v$ are not satisfied by any partition, is at most $(1 - p)^s \leq e^{-\delta^2 s} = e^{-\frac{\delta^2}{8k(2k+\Delta)}} n^{-2}$. As there are at most $\binom{n}{2} \leq \frac{n^2}{2}$ pairs at distance at most $\frac{\delta}{2k+\Delta}$, by union bound, with probability at least $\frac{1}{2}$, every pair is satisfied by some partition. It follows that the union of $s$ random partitions is, with a probability at least $\frac{1}{2}$, an $(O(n^{1/3} \log n), 2k + \delta, \frac{\delta}{8k(2k+\Delta)})$-pairwise partition cover as required. □

### 3.3 Ultrametric Cover for Doubling Spaces

In this section, we will construct a pairwise partition cover for doubling spaces, and then use it to construct ultrametric covers, and thus proving Theorem 3.4. We begin with the following combinatorial lemma.

**Lemma 3.9.** Consider a graph $G = (V, E_b \cup E_r)$ with disjoint sets of blue edges $E_b$ and red edges $E_r$, such that the maximal blue degree is $\delta_b \geq 1$, and the maximal red degree is $\delta_r \geq 1$. Then there is a set of at most $s = O(\delta_b \delta_r)$ matching $M = \{M_1, M_2, \ldots, M_s\}$ of $G$ such that (a) $E_b \subseteq \bigcup_{i=1}^{s} M_i$, and (b) for every matching $M \in M$, there is no red edge whose both endpoints are matched by $M$.

**Proof.** We construct $M$ greedily. Initially, $M = \emptyset$. Let $E'_b$ be the set of blue edges of $G$ that are not added to any matching in $M$. Let $M \subseteq E'_b$ be a maximal matching such that there is no red edge whose endpoints are both matched by $M$ (such maximal matching could be found greedily in linear time); we add $M$ to $M$ and repeat. We argue by contradiction that the greedy algorithm adds at most $4\delta_b \delta_r$ matching to $M$. Consider a vertex $v$ such that after $\delta_b (2\delta_r + 2)$ maximal matchings added to $M$, there remains at least one blue edge incident to $v$ that is not covered by any matching in $M$. Since there is at most $\delta_b$ blue edges incident to $v$, there must be a set $M_v \subseteq M$ of at least $\delta_b (2\delta_r + 1)$ matchings where $v$ is not matched by any of the matchings in $M_v$. By the maximality, in each matching $M \in M_v$, either:

(a) A red neighbor of $v$ is matched by $M$.

(b) For every blue neighbor $u$ of $v$, either $u$ is matched, or a red neighbor of $u$ is matched by $M$, which prevents $u$ from being matched.

Since $v$ has at most $\delta_b$ red neighbors, and each of them can be matched at most $\delta_b$ times, case (a) happens at most $\delta_b \delta_r$ times. The blue neighbors of $v$ could be matched at most $\delta_b - 1$ times, while their red neighbors could be matched at most $\delta_b \delta_r$ times. Thus, case (b) happens at most $\delta_b - 1 + \delta_b \delta_r < \delta_b (\delta_r + 1) - 1$ times. We conclude that $|M_v| \leq \delta_b \delta_r + \delta_b (\delta_r + 1) - 1 = \delta_b (2\delta_r + 1) - 1$, a contradiction. □

**Lemma 3.10.** Every metric space $(X, d_X)$ with doubling dimension $d$ admits an $(\epsilon^{-O(d)}, 1 + \epsilon, \epsilon)$-pairwise partition cover scheme for any $\epsilon \in (0, 1/16)$.

**Proof.** Let $\Delta > 0$ be any given real number. We show that $(X, d_X)$ admits an $(\epsilon^{-O(d)}, 1 + 8\epsilon, \epsilon (1 + 8\epsilon) \Delta)$-pairwise partition cover $\mathcal{P}$, the lemma then follows by rescaling $\epsilon$ and $\Delta$.

Let $N$ be an $(\epsilon \Delta)$-net of $(X, d_X)$. We construct a graph $G$ with $N$ as the vertex set; there is a blue edge $(u, v) \in E_b$ in $G$ iff $d_X(u, v) \in \left(1 - 4\epsilon\Delta, 1 + 2\epsilon \Delta\right]$, and there is a red edge $(u, v) \in E_r$ iff $d_X(u, v) \leq 4\epsilon \Delta$. As $\epsilon < \frac{1}{16}$, the set of blue and red edges are disjoint. By the packing property of doubling metrics (Lemma 2.1), every vertex in $G$ has blue degree $\epsilon^{-O(d)}$ and
red degree $2^{O(d)}$. Let $M$ be the set of matching of $G(N, E_b \cup E_r)$ guaranteed by Lemma 3.9: $|M| = O(e^{-O(d)}2^{O(d)}) = e^{-O(d)}$.

For each matching $M \in M$, we construct a partition $\mathcal{P}_M$ as follows: for every edge $\{u, v\} \in M$, we add $B_X(u, 2\varepsilon \Delta) \cup B_X(v, 2\varepsilon \Delta)$ as a cluster to $\mathcal{P}_M$. Denote by $N_M$ the set of net points that remain unclustered. For every net point $x \in N_M$, we initiate a new cluster $C_x$ containing $x$ only. Then, every remaining unclustered point $z \in X$ joins the cluster of its closest net point $x_z$ (from either $N_M$ or $N \setminus N_M$). See Figure 3 for an illustration.

We observe that for any two edges $(u, v)$ and $(u', v')$ in matching $M$, $B_X(u, 2\varepsilon \Delta) \cap B_X(u', 2\varepsilon \Delta) = \emptyset$ since otherwise, there is a red edge between $u$ and $u'$, contradicting item (b) in Lemma 3.9. Thus, $\mathcal{P}_M$ is indeed a partition of $X$. We next bound the diameter of each cluster in $\mathcal{P}_M$. Clearly every cluster $C_x$ for $x \in N_M$ has diameter at most $2\varepsilon \Delta$. On the other hand, by the construction and the triangle inequality, the diameter of every cluster resulting from the matching is bounded by $2 \cdot (\varepsilon \Delta + 2\varepsilon \Delta) + (1 + 2\varepsilon)\Delta = (1 + 8\varepsilon)\Delta$. Thus $\mathcal{P}_M$ is $(1 + 8\varepsilon)\Delta$-bounded.

Let $\mathcal{P} = \{\mathcal{P}_M\}_{M \in M}$. It remains to show that for every $x, y \in X$ such that $d_X(x, y) \in [\frac{1}{(1 + 8\varepsilon)\Delta}, \frac{1}{(1 + 8\varepsilon)\Delta} - \frac{1}{(1 + 8\varepsilon)\Delta}]$, there is a cluster $C$ in a partition $\mathcal{P}$ containing both $B_X(x, \frac{\varepsilon}{2} \cdot (1 + 8\varepsilon)\Delta)$ and $B_X(y, \frac{\varepsilon}{2} \cdot (1 + 8\varepsilon)\Delta)$. Note that as $\varepsilon \leq \frac{1}{10k}$, $\frac{\varepsilon}{2} \cdot (1 + 8\varepsilon)\Delta \leq \delta \varepsilon$. Let $x', y' \in N$ be net points such that $d_X(x', y') \leq \delta \varepsilon$. Then by the triangle inequality $|d_X(x, y') - d_X(x, y)| \leq 2\varepsilon \Delta$, implying that $d_X(x', y') \in [1 - 4\varepsilon \Delta, 1 + 2\varepsilon \Delta]$. Hence, $G$ contains a blue edge between $x', y'$. It follows that there is a matching $M$ containing the edge $(x', y')$, and a partitions $\mathcal{P}_M$ containing the cluster $C = B_X(x', 2\varepsilon \Delta) \cup B_X(y', 2\varepsilon \Delta)$. In particular, $B_X(x, \varepsilon \Delta) \cup B_X(y, \varepsilon \Delta) \subseteq C$ as required.

We are finally ready to prove Theorem 3.4 that we restate below for convenience.

**Theorem 3.4 (Ultrametric Cover For Doubling Metrics).** Every metric space $(X, d_X)$ with doubling dimension $d$ admits an $(e^{-O(d)}, 1 + \varepsilon, \varepsilon, e^{-O(d)})$-ultrametric cover for any parameter $\varepsilon \in (0, \frac{1}{2})$.

Conversely, if a metric space $(X, d_X)$ admits a $(\tau, \rho, k, \delta)$-ultrametric cover for $k \geq 2\rho$, then it has doubling dimension $d \leq \log(\tau \rho)$.

**Proof.** We begin with the first assertion (doubling metrics admit ultrametric covers). After appropriate rescaling, by Lemma 3.6 and Lemma 3.10, we obtain an $(e^{-O(d)}, 1 + \varepsilon)$-ultrametric cover where every ultrametric in the cover is a $\frac{1}{2}$-HST. It remains to show that every ultrametric in the cover returned by Lemma 3.6 w.r.t. the pairwise partition cover scheme constructed in Lemma 3.10 has bounded degree.

We will use the terminology of Lemmas 3.6 and 3.10. Consider some ultrametric $U_j$ and some cluster $C$ at level $i$ with label $(1 + \varepsilon)\Delta_i$. The clusters at level $i - 1$ correspond to points in $(\varepsilon \Delta_i, -1)$-net. The cluster $C$ has diameter $(1 + \varepsilon)\Delta_i$, and hence it has at most $e^{-O(d)}(\varepsilon \Delta_i - 1)$-net points. In particular $C$ can contain at most $e^{-O(d)}$ level-$(i - 1)$ clusters. The bound on the degree follows.

Next, we prove the second assertion. Consider a metric space $(X, d_X)$ admitting a $(\tau, \rho, k, \delta)$-ultrametric cover with $k \geq 2\rho$. Let $C_X(x, r)$ be some ball of radius $r$. In each ultrametric $U_i$ in the cover, let $L_i$ be the node closest to root that is an ancestor of $x$ and has label at most $\rho \cdot r$. Let $\{L_{i,1}, L_{i,2}, \ldots\}$ be the set of at most $\delta$ children of $L_i$ in $U_i$. For each $L_{i,j}$, we pick an arbitrary leaf $u_{i,j} \in X$ descendent of $L_{i,j}$. We argue that

$$B_X(x, r) \subseteq \bigcup_{i, j} B_X(u_{i,j}, \frac{r}{2}),$$

as the number of balls in the union is at most $\tau \delta$, the theorem will follow. Consider a vertex $y \in B_X(x, r)$. There is necessarily an ultrametric $U_i$ such that $d_{U_i}(x, y) \leq r \cdot \rho$. In particular, in $U_i$, $x, y$ are both descendents of a node with label at most $\rho \cdot r$. Recall that $L_i$ is such a node with maximal label. Let $L_{i,j}$ be the child of $L_i$ such that $y$ is descendent of $L_{i,j}$. As $U_i$ is a $\kappa$-HST, the label of $L_{i,j}$ is bounded by $\frac{\rho \cdot r}{\kappa} \leq \frac{r}{2}$ since $k \geq 2\rho$. In particular, $y \in B_X(u_{i,j}, \frac{r}{2})$.

\[ \square \]

## 4 Locality Sensitive Ordering

**Locality-Sensitive Ordering.** Chan et al. [42] introduced and studied the notion of locality-sensitive ordering (Definition 1.1). In the same paper, Chan et al. [42] showed that the Euclidean metric of dimension $d$ has an $(O(e^{-d} \log \frac{1}{d}), e)$ -LSO. They also presented various applications of the LSO to solve fundamental geometry problems in Euclidean spaces.

**Lemma 4.1.** Given a set of $n$ elements $[n] = \{1, \ldots, n\}$, there exists a set $S$ of $[\frac{n}{2}]$ orderings such that for any two elements $i \neq j \in [n]$, there exists an ordering $\sigma$ in which $i$ and $j$ are adjacent.

\[ \square \]
We show that metrics admitting an ultrametric cover of bounded degree have an LSO with a small number of orders. Our proof relies on the following lemma.

**Lemma 4.2.** Every \(\alpha\)-HST \((U, d_U)\) of degree \(\delta\) admits a \(\left(\frac{\delta}{2}, \frac{1}{\alpha}\right)\)-ULSO.

**Proof.** For simplicity, we will assume that the number of children in each node is exactly \(\delta\). This could be achieved by adding dummy nodes. By Lemma 4.1, every set of \(\delta\) vertices can be ordered into \(\left\lceil \frac{\delta}{2} \right\rceil\) orderings such that every two vertices are adjacent in at least one of them. Denote these orderings by \(\sigma_1, \ldots, \sigma_{\left\lceil \frac{\delta}{2} \right\rceil}\). We construct the set of orderings for \((U, d_U)\) inductively.

Let \(A\) be the root of the HST with children \(A_1, \ldots, A_\alpha\). By the induction hypothesis, each \(A_i\) admits a set \(\sigma^i_1, \ldots, \sigma^i_{\left\lceil \frac{\delta}{2} \right\rceil}\) orderings.

We construct \(\left\lceil \delta/2 \right\rceil\) orderings as follows: for each \(j \in \left[\left\lceil \delta/2 \right\rceil\right]\), order the vertices inside each \(A_i\) w.r.t. \(\sigma^i_j\) and order the sets in between w.r.t. \(\sigma_j\). The resulting ordering is denoted by \(\sigma_j\). This finishes the construction.

Next, we argue that this is a \(\left(\frac{\delta}{2}, \frac{1}{\alpha}\right)\)-ULSO. Clearly, we used exactly \(\left\lceil \frac{\delta}{2} \right\rceil\) orderings. Let \(\Delta\) be the label of the root. Consider a pair of leaves \(x, y\). If \(d_U(x, y) < \Delta\), then there is some \(i\) such that \(x, y \in A_i\). By the induction hypothesis, there is some ordering \(\sigma^i_j\) of \(A_i\) such that \((\text{w.l.o.g.}) x \prec_{\sigma^i_j} y\), and the points between \(x\) and \(y\) w.r.t. \(\sigma^i_j\) could be partitioned into two consecutive intervals \(I_x, I_y\) such that \(I_x \subseteq B_U(x, \overline{d_U(x, y)}/\alpha)\) and \(I_y \subseteq B_U(y, \overline{d_U(x, y)}/\alpha)\). The base case is trivial since every leaf has label 0. Note that \(\sigma^i_j\) is a sub-ordering of \(\sigma_j\). In particular, the lemma holds.

The next case is when \(d_U(x, y) > \Delta\). Then there is \(i 
eq i'\) such that \(x \in A_i\) and \(y \in A_{i'}\). There is some ordering \(\sigma_{ij}\) such that \(A_i\) and \(A_{i'}\) are consecutive. In particular all the vertices between \(x\) to \(y\) in \(\sigma_{ij}\) can be partitioned to two sets, the first belonging to \(A_i\), and the second to \(A_{i'}\). The lemma follows as all the vertices in \(A_i (A_{i'})\) are at distance at most \(\frac{\delta}{2} = \frac{d_U(x, y)}{\alpha}\) from \(x (y)\).

**Lemma 4.3.** If a metric \((X, d_X)\) admits a \((\tau, \rho, k, \delta)\)-ultrametric cover, then it has a \((\tau, \frac{\delta}{2}, \frac{\rho}{k})\)-LSO.

**Proof.** Let \(\mathcal{U}\) be an ultrametric cover for \((X, d_X)\). For each ultrametric \((U, d_U)\), let \(\Sigma_U\) be the set of orderings obtained by applying Lemma 4.2. Let \(\Sigma = \cup_{U \in \mathcal{U}} \Sigma_U\). We show that \(\Sigma\) is the LSO claimed by the lemma. Clearly, it contains at most \(\tau \cdot \left\lceil \frac{\delta}{2} \right\rceil\) orderings.

Consider two points \(x \neq y \in X\). Let \(U\) be an ultrametric in \(\mathcal{U}\) such that \(d_U(x, y) \leq \rho \cdot d_X(x, y)\). By Lemma 4.2, there is an ordering \(\rho \in \Sigma_U\) such that \((\text{w.l.o.g.}) x \prec_{\rho} y\) and points between \(x\) and \(y\) w.r.t. \(\rho\) can be partitioned into two consecutive intervals \(I_x\) and \(I_y\) where \(I_x \subseteq B_U(x, \overline{d_U(x, y)}/k)\) and \(I_y \subseteq B_U(y, \overline{d_U(x, y)}/k)\). Since \(d_U(x, y) \leq \rho \cdot d_X(x, y)\), we conclude that \(I_x \subseteq B_X(x, \frac{\rho}{k} \cdot d_X(x, y))\) and \(I_y \subseteq B_X(y, \frac{\rho}{k} \cdot d_X(x, y))\) as desired.

Corollary 4.4 (LSO for Doubling Metrics). For every \(\epsilon\) sufficiently smaller than 1, every metric space \((X, d_X)\) of doubling dimension \(d\) admits an \((1-\epsilon, d)\)-LSO.

**Reliable (1+\(\epsilon\))-Spanners from LSO.** Buchin et al. [36, 37] constructed reliable \((1+\epsilon)\)-spanners for Euclidean metrics using \((\tau, \epsilon)\)-LSO by Chan et al. [42]. Specifically, their spanner for the deterministic case has \(n \cdot O(\epsilon^{-2d} \log^{1/2}(\frac{1}{\epsilon}) \cdot \frac{1}{\epsilon} \cdot \log n \log n)^6\) edges, while for the oblivious case, they constructed a spanner with an almost linear number of edges: \(n \cdot O(\epsilon^{-2d} \log^{1/2}(\frac{1}{\epsilon}) \cdot \frac{1}{\epsilon} \cdot \log^{-1}(\log n)^2 \log n \log n\). Their key idea is to reduce the problem to the construction of reliable \((1+\epsilon)\)-spanners for the (unweighted) path graph \(\mathcal{P}_n\) with \(n\) vertices. We observe that their construction of the reliable \((1+\epsilon)\)-spanners did not use any propery of the metric space other than the existence of an LSO.

Theorem 4.5 ([36, 37], implicit). Suppose that for any \(\epsilon \in (0, 1)\), an n-point metric space \((X, d_X)\) admits a \((\tau (\epsilon, \epsilon), \epsilon)\)-LSO for some function \(\tau : (0, 1) \to \mathbb{N}\). Then for every \(\nu \in (0, 1)\) and \(\epsilon \in (0, 1)\),

1. \((X, d_X)\) admits a deterministic \(\nu\)-reliable \((1+\epsilon)\)-spanner with \(n \cdot O\left(\left(\frac{\epsilon}{2} \right)^2 \log n \log n\right)^\nu\) edges for some universal constant \(c\).
2. \((X, d_X)\) admits an oblivious \(\nu\)-reliable \((1+\epsilon)\)-spanner with \(n \cdot O\left(\left(\frac{\epsilon}{2} \right)^2 \log n \log n\right)^\nu\) edges for some universal constant \(c\).

By Theorem 4.5 and Corollary 4.4, we have:

**Corollary 4.6.** Consider a metric space \((X, d_X)\) with doubling dimension \(d\). Then for every \(\nu \in (0, 1)\) and \(\epsilon \in (0, 1)\), \((X, d_X)\) admits a deterministic \(\nu\)-reliable \((1+\epsilon)\)-spanner with \(n \cdot \epsilon^{-O(d)} \cdot \log^{-1}(\log n)^2 \cdot \log \left(\frac{\epsilon}{2} \log n\right)^\nu\) edges, and an oblivious \(\nu\)-reliable \((1+\epsilon)\)-spanner with \(n \cdot \epsilon^{-O(d)} \cdot \log^{-1}(\log n)^2 \cdot \log \left(\frac{\epsilon}{2} \log n\right)^\nu\) edges.

**5 Triangle Locality-Sensitive Ordering**

A triangle locality-sensitive ordering (triangle-LSO) is defined as follows.

**Definition 5.1** \((\tau, \rho)\)-LSO. Given a metric space \((X, d_X)\), we say that a collection \(\Sigma\) of orderings is a \((\tau, \rho)\)-triangle-LSO if \(|X| \leq \tau\), and for every \(x, y \in X\), there is an ordering \(\sigma \in \Sigma\) such that \((\text{w.l.o.g.}) x \prec_{\sigma} y\), and for every \(a, b \in X\) such that \(x \preceq a \preceq b \preceq y\) it holds that \(d_X(a, b) \leq \rho \cdot d_X(x, y)\).

Note that every \((\tau, \rho)\)-triangle LSO is a \((\tau, \rho)\)-LSO; however, a \((\tau, \rho)\)-LSO is a \((\tau, 2\rho + 1)\)-triangle LSO by triangle inequality. Hence for stretch parameter \(\rho > 1\), triangle-LSO is preferable to the classic LSO. Similar to Lemma 4.3, we show that a metric space admitting an ultrametric cover has a triangle-LSO with a small number of orderings. The proof of the following lemma is differed to the full version [62].

**Lemma 5.2.** If a metric \((X, d_X)\) admits a \((\tau, \rho)\)-ultrametric cover \(\mathcal{U}\), then it has a \((\tau, \rho)\)-triangle-LSO.
Using Theorem 3.3 and Lemma 5.2, we conclude.

**Corollary 5.3.** For every $k \in \mathbb{N}$, and $e \in (0, 1)$, every $n$-point metric space admits an $O(n^{k} \cdot \log n \cdot \frac{2}{e} \cdot \log \frac{1}{e})$-triangle-LSO.

**Reliable Spanners from triangle-LSO.** We show that if a metric space admits a $(\tau, \rho)$-triangle-LSO, it has an oblivious $2\rho$-spanners with about $\mathcal{O}(n \cdot \tau^{1/2} \cdot \log n)$ edges. We use this result to construct reliable $(8k-2)(1+e)$-spanners for general metrics (and reliable $(2\rho)$-spanners for ultrametrics).

**Theorem 5.4.** Suppose that a metric space $(X, d)$ admits a $(\tau, \rho)$-Triangle-LSO. Then for every $v \in (0, 1)$, $X$ admits an oblivious $v$-reliable, $(2\rho)$-spanner with $\mathcal{O}(n \cdot \log^{2} n + \rho^{-1} \tau \log n \cdot \log \log n)$ edges.

The proof of Theorem 5.4 is deferred to the end of the section. Using Corollary 5.3 with parameters $2k$ and $\frac{\rho}{2}$, and Theorem 5.4 we conclude:

**Theorem 5.5 (Oblivious Reliable Spanner for General Metric).** For every $n$-point metric space $(X, d)$ and parameters $v \in (0, 1)$, $e \in (0, \frac{1}{2})$, $k \in \mathbb{N}$, $(X, d)$ admits an oblivious $v$-reliable, $8k + e$-spanner for $X$ with $n^{1+k} \cdot \nu^{-1} \cdot \log^{2} n \cdot \log \log n \cdot O(\log \frac{1}{e})^{2} = n^{1+k} \cdot \nu^{-1} \cdot \mathcal{O}(\log^{3} n \cdot \frac{1}{e})$ edges.

By Lemma 5.2 and Theorem 5.4, we obtain:

**Corollary 5.6.** For every parameter $v \in (0, 1)$, every $n$-point ultrametric space $(X, d)$ admits an oblivious $v$-reliable, 2-spanner with $n \cdot \mathcal{O}(\log^{2} n + \nu^{-1} \log n \cdot \log \log n)$ edges.

The stretch parameter in Corollary 5.6 is tight (see the full version [62]).

For the rest of this section, we show how to construct reliable spanners for metric spaces admitting a triangle LSO. Following the approach of Buchin et al. [36, 37], we reduce the problem to the construction of reliable spanners for the (unweighted) path graph $P_n$. However, in our setting, we face a very different challenge: when $\rho > 1$, the stretch of the reliable spanner for $(X, d_X)$ grows linearly w.r.t. the number of hops of the reliable spanner for the path graph. In the Euclidean setting studied by Buchin et al. [37], the stretch parameter is $\rho = \epsilon < 1$, and as a result, the stretch is not significantly affected by the number of hops of the spanner for the path graph.

Buchin et al. [37] constructed an oblivious $\nu$-reliable 1-spanner for the path graph $P_n$ with $O(n \cdot \nu^{-1} \log \nu^{-1})$ edges. However, their spanner has hop diameter $2 \log n$; if we use their reliable spanner for the path graph, we will end up in a spanner for $(X, d_X)$ with stretch $2\rho \log n$. To have a stretch $2\rho$, we construct a reliable spanner for the path graph $P_n$ with hop diameter 2. As a consequence, the sparsity of our spanner has some additional logarithmic factors. Note that a 2-hop spanner for the path graph $P_n$ even without any reliability guarantee must contain $\Omega(n \log n)$ edges (see Exercise 12.10 [81]). Our result is summarized in the following lemma whose proof is deferred to the full version [62].

**Lemma 5.7 (2-hop-spanner).** For every $v \in (0, 1)$, the path graph $P_n$ admits an oblivious $v$-reliable, 2-hop 1-spanner $H$ with $n \cdot \mathcal{O}(\log^{2} n + v^{-1} \log n \cdot \log \log n)$ edges.

Using Lemma 5.7, we can construct a reliable spanner for metric spaces admitting a $(\tau, \rho)$-triangle LSO as claimed by Theorem 5.4.

**Proof of Theorem 5.4.** Let $\Sigma$ be a $(\tau, \rho)$-triangle LSO as assumed by the theorem. Let $\nu = \frac{1}{2v}$. For every ordering $\sigma \in \Sigma$, we form an unweighted path graph $P_{\sigma}$ with vertex set $X$ and the order of vertices along the path is $\sigma$. We construct a $v$-reliable 2-hop spanner $H_{\sigma}(X, E_{\sigma}, w_{\sigma})$ for $P_{\sigma}$ with $n \cdot \mathcal{O}(\log^{2} n + \nu^{-1} \log n \cdot \log \log n)$ edges by Lemma 5.7. Note that for every edge $(u, v) \in E_{\sigma}$ of $H_{\sigma}$, $w_{\sigma}(u, v)$ is the distance between $u$ and $v$ in the (unweighted) path graph $P_{\sigma}$.

We form a new weight function $w_{\chi}$ that assigns each edge $(u, v) \in E_{\sigma}$ a weight $w_{\chi}(u, v) = d_X(u, v)$. The reliable spanner for $(X, d_X)$ is $H = \bigcup_{\sigma \in \Sigma} H_{\sigma}(X, E_{\sigma}, w_{\chi})$. We observe that the total number of edges in $H$ is bounded by

$$\sum_{\sigma \in \Sigma} n \cdot \mathcal{O}(\log^{2} n + \nu^{-1} \log n \cdot \log \log n) = \nu \cdot \mathcal{O}(\log^{2} n + \nu^{-1} \tau \log n \cdot \log \log n).$$

Let $B \subseteq X$ be an oblivious attack. Let $B^*_{\sigma}$ be the faulty extension of $B$ in $H_{\sigma}$, and $B^* = \bigcup_{\sigma \in \Sigma} B^*_{\sigma}$ be the faulty extension of $B$ in $H$. We observe that

$$|B^*| \leq |B| + \sum_{\sigma} |B^*_{\sigma} \setminus B| \leq |B| + \tau \nu \cdot |B| \leq (1 + \nu) \cdot |B|.$$  

It remains to show the stretch guarantee of $H$. For every pair of points $x, y \notin B^*$, let $\sigma \in \Sigma$ be the ordering that satisfies the ordering property for $x$ and $y$: for every $a, b \in X$ such that $x \leq_{\Sigma} a \leq_{\Sigma} b \leq_{\Sigma} y$, $d_X(a, b) \leq \rho \cdot d_X(x, y)$. (Here we assume w.l.o.g. that $x \leq_{\Sigma} y$). Since $x, y \notin B^*$, $x, y \notin B^*_{\sigma}$. Since $H_{\sigma}(X, E_{\sigma}, w_{\sigma})$ is a 2-hop 1-spanner for $P_{\sigma}$, there must be $z \notin B$ such that $x \leq_{\Sigma} z \leq_{\Sigma} y$ and $(x, z), (z, y) \in E_{\sigma}$. We conclude that $d_H(x, y) \leq w_{\chi}(x, z) + w_{\chi}(z, y) = d_X(x, z) + d_X(z, y) \leq 2\rho \cdot d_X(x, y)$; the theorem follows. \hfill $\Box$

## 6 CONCLUSIONS

In this paper, we have presented different types of locality-sensitive orderings and used them to construct reliable spanners. For the construction of the LSO’s, we introduced and constructed ultrametric covers. Finally, in order to use the LSO’s to construct reliable spanners, we construct 2-hop spanners and left spanners for the path graph. Several open questions naturally arise from our work:

1. Can we construct a $v$-reliable 2-hop 1-spanner for the path graph $P_n$ with $O(n \log n)$ edges for constant $v$? Note that a 2-hop spanner for the path graph $P_n$ even without any reliability guarantee must contain $\Omega(n \log n)$ edges (see Exercise 12.10 [81]).

2. A major open question is the construction of deterministic reliable spanners general metric spaces. [69] constructed deterministic reliable $O(t^2)$-spanners for general metrics with $O(n^{1+\frac{1}{t}})$ edges, and deterministic reliable $O(t)$-spanners for...
trees and planar graphs with $\tilde{O}(n^{1+\frac{1}{m}})$ edges, while showing an $\Omega(n^{1+\frac{1}{m}})$ lower bound on the number of edges in a deterministic reliable $t$-spanners for the uniform metric. Using the new LSO’s constructed in this paper, it could be possible to improve the stretch parameters by a constant factor and remove the dependency on aspect ratio from the sparsity. However, the lower bound for uniform metrics applies to trees and planar graphs as well; thus using $\tilde{O}(n^{1+\frac{1}{m}})$ edges to obtain stretch $t$ is necessary. On the other hand, general metrics are far from understood. Closing the gap between the current $O(t^3)$ upper bound to the $t$ lower bound is a fascinating open question.

(3) Can we construct a reliable spanner of stretch 2 (as opposed to stretch $2 + e$ presented in this paper) for planar metrics with a nearly linear number of edges?

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