Quasi-quantum secure direct communication scheme using non-maximally entangled states

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Abstract
Quantum communication in general helps deter potential eavesdropping in the course of transmission of bits to enable secure communication between two or more parties. In this paper, we propose a novel quasi-quantum secure direct communication scheme using non-maximally entangled states. The proposed scheme is simple to implement using existing techniques and significantly reduces the number of leaked bits by randomly complementing them. The associated bit error rates are directly controlled by the sender. As a result long sequences or the whole sequence of data can be communicated at once directly, before error checking for a potential eavesdropper. Also a cipher can be used in the protocol for retrieving the bits that are lost due to security. The qubit efficiency of the proposed protocol is found to be 40%.

Keywords Secure communication, Quasi-quantum secure direct communication, Quantum key distribution, Non-maximally entangled states

1. Introduction
Quantum cryptography exploits the fundamental postulates of quantum mechanics to ensure secure communication between two or more parties. Several quantum communication schemes have been proposed since the introduction of the BB84 protocol in 1984 [1]. In the BB84 protocol, the sender makes use of the rectilinear (R) or diagonal (D) basis to encode information in single photons. The quantum no cloning theorem prevents a malicious third party (Eve) from copying the quantum state during the transmission process. The bases used are publicly announced by both the sender (Alice) and the receiver (Bob). They both reject the bits that were generated through

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different bases. The remaining bits that were generated through the same basis are preserved by Bob. He chooses a subset of these bits and sends it to Alice. Alice calculates the error rate which is expected to be within a certain threshold value. If not, the presence of Eve is disclosed [2]. Alternatively the Ekert protocol relies on the non-locality of a shared maximally entangled pair between Alice and Bob. If the measurement is performed by Alice and Bob in a compatible basis, sifted bits are generated after publicly announcing the bases [3].

Quantum key distribution (QKD) schemes, namely, the BB84 and the Ekert protocol are not used to send bits directly. However, they help establish a private key between the sender and the receiver. After establishing the key, bits are sent through a classical channel using an encryption algorithm. On the other hand, quantum secure direct communication (QSDC) schemes are used to communicate directly over a quantum channel without involving key generation [4]. Over the years, QSDC schemes using single photons, entangled states and superdense coding have been proposed [5-9].

While most of the proposed schemes use maximally entangled states like Bell states, we have shown how non-maximally entangled states can help leverage security. Moreover, the transmission of maximally entangled states are more difficult than non-maximally entangled states due to decoherence [10]. Hence, it is worthwhile to seek quantum communication using non-maximally entangled states. The use of non-maximally entangled states for quantum cryptography has been minimal in spite of their generation in laboratories. The simple act of rotating the polarizer for the pump photons before striking the non-linear crystal in standard parametric down conversion experiments can produce such states. In this work, these states are measured by the R (measurement along 0°, 90°) or D (measurement along 45°, 135°) basis similar to the BB84 protocol, and the measurement result is counted as the bit. Utilizing entanglement is possible only when the same basis is used. The results 0° or 45° are counted as 0 bit, and 90° or 135° are counted as 1 bit.

Intentionally skewed probability amplitudes in non-maximally entangled states are used to deliver the intended measurement result to Bob. Incidentally, randomly complemented bits are introduced in the sequence due to the inherent nature of the prepared states. The percentage of bits to be complemented as such is controlled by the sender. The error due to complementing is retrievable as we will see in the ensuing discussion. QSDC schemes do not use classical bits except for error checking. However, our protocol uses a cipher similar to QKD where half of the resulting bits are transmitted in the classical channel. Hence, it can be referred to as a quasi-QSDC scheme. Unlike other QSDC schemes, our protocol does not involve the application of any quantum gates, thereby reducing complexity.

The paper is organized as follows. Section 2 describes the steps involved in the protocol. Section 3 elucidates the security of the given protocol. Section 4 is used for discussing the various cases of quantum states and how they affect security. Section 5 concludes the paper.
2. The protocol

In this section, we introduce the protocol formally. Let N be the sequence of \( n \) number of classical bits to be sent by Alice. The various steps involved in the protocol are as follows:

1. Alice generates a non-maximally entangled state \( \alpha|00> + \beta|11> \), where \( |\alpha|^2 + |\beta|^2 = 1 \). If Alice wants to send a 0 bit in N, she prepares the state such that \( |\alpha| > |\beta| \). To send a 1 bit, she prepares the state such that \( |\alpha| < |\beta| \).
2. She sends one of the qubits in the entangled pair to Bob who is expected to measure it in the same basis as Alice. Both Alice and Bob can measure in the rectilinear (R) or diagonal (D) basis.
3. If Bob’s measurement outcome is 0, he discerns the bit as 0. If he measures 1, it’s a 1 bit.
4. After a predetermined number of iterations or after transmitting all qubits, Bob ends up with a sequence \( N_1 \) having \( n \) bits. Alice and Bob exchange their basis information (whether R or D) through a classical channel. This information can be represented as a sequence with R and D representing 0 and 1, respectively. The sequence can be generated by quantum random number generators which are truly random unlike classical random number generators which are deterministic [11-13].
5. Bob discards the bits that are generated through different bases while noting their positions. Simultaneously Alice generates the sequence of bits, say A that has not reached Bob due to incompatible basis. Let the number of bits in the sequence A be \( d \). Both Alice and Bob have generated a separate sequence of sifted bits (bits generated using the same basis) \( P \) and \( Q \) respectively which will have \( n - d \) number of elements. As one can see in the below security analysis, \( d \approx \frac{n}{2} \).
6. Bob selects a subset of his sifted bits from \( Q \) and sends it along with its respective position to Alice through the classical channel.
7. Alice compares it with the corresponding original bits in \( P \) and checks for any error (if the subset of the elements of \( P \) and \( Q \) are unequal).
8. If the error rate of sifted bits is above a threshold value, Eve’s presence is disclosed and the process is terminated at this point. If the error rate is under a threshold value, Alice applies XOR to each bit in A and the corresponding sifted bit, \( A \oplus P \) and this result is disclosed to Bob through the classical channel. Let this new bit sequence be G.
9. Bob receives G and applies XOR with his sifted bits, \( G \oplus Q = A \oplus P \oplus Q \), which is equivalent to A if \( P \) and \( Q \) are equal. Bob introduces this result in the position of discarded bits with the previously received sifted bits \( Q \). Hence, Bob receives \( n \) bits which contain some randomly complemented bits due to the additional coefficient in the non-maximally entangled states.
10. Alice announces the position of those bits through the classical channel. Bob simply complements the corresponding bits. Thus Bob is left with the sequence N.
### Fig. 1 Quasi-QSDC protocol: dotted line indicates quantum channel while the normal ones indicate classical channel.

#### 3. Security

If Alice and Bob measure using the same basis, their measurement outcomes must be the same due to the correlation in the entangled state. If Eve intercepts Alice’s photons and uses a different basis, her outcome is random. Hence she sends a randomized bit to Bob which might be different from the one Alice measures from her qubit. This reveals the presence of Eve. If the error rate calculated by Alice in Step 8 exceeds a certain percentage, the presence of Eve can be known. The error rate can be calculated as follows:

\[
\text{Error Rate} = \text{Probability of Eve making an error} \times \text{Probability of Bob making an error}
\]

\[
= 50\% \times 50\%
\]

\[
= 25\%
\]

The threshold error rate is therefore 25%. Due to incompatible bases, Bob loses 50% of the bits. This can be retrieved by a predetermined cipher. Here we use the XOR operation. Alice applies XOR with the bits that Bob lost and the sifted bits. Each of these sequences would have about 50% of the total bits sent by Alice. After Alice sends Bob these new bits, Bob applies XOR with the sifted bits and the new bits that Alice sent. Note that the sifted bits are known only to Alice and Bob. Let \( s \) be a sifted bit and \( f \) be a corresponding bit that Bob lost and must receive from Alice. Bob receives \( s \oplus f \) from Alice. With that Bob computes

| Alice | Bob |
|-------|-----|
| Needs to send sequence \( N \) with \( n \) bits | |
| Prepares \( \alpha|00> + \beta|11> \): \( n \) times | |
| Measurement done in \( R \) or \( D \) | Measurement done in \( R \) or \( D \) |
| Choice of bases | Choice of bases |
| Sifted bits: \( P \), and remaining bits: \( A \) | Sifted bits: \( Q \), and discards others |
| Checks \( Q \) against \( P \) for error | Randomly chooses a subset in \( Q \) |
| Prepares \( A \oplus P \) | Computes \( (A \oplus P) \oplus Q \) |
| Position of bits to be complemented | |
| Receives sequence \( N \) | |
Applying XOR twice with a given bit would result in the same bit. Hence Bob receives the previously discarded bits. He has now received all $n$ bits, where some are randomly complemented. The increased security introduced by such randomly complemented bits due to non-maximally entangled states can be significant. This is made possible by the small probabilities introduced via non-maximally entangled states. If the bit needed to be sent is 0, this error is introduced by the coefficient $|\beta|$. Thus the bits deciphered by Eve when she uses the compatible basis will also be wrong by a probability of $|\beta|^2$. Hence, the combined bit error rate for Eve (and Bob) would be $[50 + 50*(|\beta|^2)]$ % (note that the error additional to 50% is recoverable through complementing) before and during error checking. In short, the leaked bits due to compatible basis are not reliable too for Eve. This allows us to send long sequences of data before error checking as Eve’s presence will introduce her to redundant information from randomly complemented bits (even when she uses the same bases as Alice). The value of $|\beta|$ can be increased by Alice as the sequence of bits to be transmitted at a time gets longer. This is in contrast to the block transmission technique employed in most QSDC schemes. Hence, the proposed protocol offers a solution for communicating through an eavesdropping strategy. If the transmission is free from eavesdropping, Alice can announce the position of bits to be complemented at the end of the protocol. She can rather avoid the announcement if she finds Bob suspicious too. Alice can keep a track on the result of Bob’s qubit due to the correlation. As a result, qubits are not lost in the process of increasing security through the use of non-maximally entangled states.

Additionally, classical bits are used in error checking and for the cipher apart from qubits. It’s therefore important to quantify the efficiency of the scheme. The theoretical qubit efficiency of a protocol is defined as

$$\eta = \frac{c}{q + b},$$

where $c$ is the number of bits received by Bob, $q$ is the number of qubits transmitted by Alice and $b$ is the number of classical bits exchanged between Alice and Bob [14]. Here $c = 1$, $q = 1$ and $b = 1 + 0.5 = 1.5$, where $b$ includes the basis exchanged and the position of bits to be complemented. Thus $\eta = 0.4$ and the efficiency of the proposed protocol is 40%.

### 4. Role of various quantum states in security

The coefficients $|\alpha|$ and $|\beta|$ can also follow specific values that are set by Alice unlike the one mentioned in Step 1 of the protocol. For example, she may set $|\alpha| < |\beta|$ to send a 0 bit. The number of bits to be complemented gets higher as a result. The bit error rate will approach 100% as $|\beta|^2$ gets closer to 1. The following graph shows the bit error rate for corresponding values of $|\beta|$ before error checking.
The case of maximally entangled states where $|\alpha|^2 = |\beta|^2 = 0.5$ or $|\alpha| = |\beta| = 0.7071$ is also possible. The bit error rate in this case would be 75%. Since the coefficients are independent of the message bits, the maximally entangled case corresponds to perfect secrecy. Values for $|\beta|^2$ close to 1 are not useful as Eve can complement all the sifted bits if she wants. At the same time, very small values are not useful as only a few bits would be complemented. Alice can also control the bit error rate if she becomes suspicious during the protocol as she can control the coefficients of the non-maximally entangled states in real time. This can be achieved by rotating the pump polarization with respect to the vertical or horizontal by an angle $\theta$. A non-maximally entangled state can be expressed as $(\varepsilon|00\rangle + |11\rangle)/\sqrt{\varepsilon^2 + 1}$ where $0 < \varepsilon < 1$ and the degree of entanglement, $\varepsilon = tan \theta$ [15]. Hence Alice can control $\alpha$ and $\beta$ by controlling $\Theta$. A central satellite source can also help share an entangled pair (preferably maximal) between Alice and Bob [16].

5. Conclusion

In the proposed quasi-QSDC scheme, different bases are used along with non-maximally entangled states to achieve secure communication. This is also favorable for transmitting qubits as non-maximally entangled states are more robust than maximally entangled states. One half of the sequence of bits is sent by generating sifted bits and the other half uses a cipher with the sifted bits. The position of bits to be complemented is announced by Alice at the end. The scheme also highlights how security can be increased through non-maximally entangled states without losing any qubits, but at the cost of post processing the bits on the receiver side. The resulting efficiency of the protocol is 40%. The case of maximally entangled states can serve as the middle ground for the bit error rates. For practical applications, the sender can use the range of bit error rates lesser
than 75% to reduce post processing and achieve faster results. Alice can constantly make a choice between faster post processing and increased security.

6. References

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