2D Voronoi Coverage Control with Gaussian Density Functions by Line Integration

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Abstract: This paper considers Voronoi coverage control for two-dimensional space with a non-uniform density function. In general, the computation of a mass and a centroid of a Voronoi cell for a non-uniform density function requires spatial discretization since they cannot be represented by a closed form. However the spatial discretization approach may result in exhaustive computation. In this paper, we consider a transformation of a surface integral over a Voronoi cell into a line integral around its boundary by Green’s theorem to alleviate such a computational issue. We show that the proposed method can be implemented only by coordinates of vertices of a Voronoi cell and parameters of a density function when the density function is represented by a sum of Gaussian functions.

Key Words: multi-agent systems, coverage control, centroidal Voronoi partitions.

1. Introduction

Recently, cooperative control of multi-agent systems has attracted growing attention due to its immense potential for various applications [1]–[4]. The consensus control and the coverage control are fundamental problems of cooperative control. In the consensus control, agents make their states converge to a common point [1]. On the other hand, in the coverage control, mobile agents are allocated according to a designated density function [5]–[8]. In particular, the centroidal Voronoi coverage control has been extensively considered in control engineering [9]–[17]. The centroidal Voronoi coverage control was initially introduced by Cortés et al. to optimize a performance function [18]. The authors have shown that the performance function is locally maximized when agents converge to the centroids of the assigned Voronoi cells. Lee et al. have proposed the centroidal Voronoi coverage control with time-varying density functions [19]. Davison et al. have extended the results of the Voronoi coverage control with an Euclidean metric to the case with non-Euclidean metric [20]. Miah et al. have considered Voronoi coverage control with intermittent communications [21].

For the centroidal Voronoi coverage control, each agent has to compute a mass and a centroid of its Voronoi cell. For a uniform density function, the authors of [18] have shown a closed form of a centroid by dividing a Voronoi cell into a union of simplices [22]. However, in general, one cannot obtain a closed form of a mass and a centroid for a non-uniform density function. Therefore spatial discretization is required for the computation of a mass and a centroid. For the two-dimensional case, a double integral is required to compute a surface integral over a Voronoi cell. Moreover each agent has to check whether points are included in its Voronoi cell. Thus the computational load of the spatial discretization method dramatically increases. The authors of [9] have proposed a closed form of a mass and a centroid with a non-uniform density function. However they have approximated a density function by the first-order Taylor series. Thus the method by [9] may not work for the case when the error by the first-order Taylor series cannot be ignored.

On the other hand, integration over polyhedra has been extensively studied in the areas of computational mathematics and computer visions [23]–[25]. There are mainly three different approaches for integration over polygons or polyhedra: integration over divided simplices [26], transformation of a volume integral into a surface integral by Stokes’s theorem [27],[28], and integration based on moment fitting [29].

In this paper, similar to the approach based on the transformation by Stokes’s theorem, we consider a transformation of a surface integral of a mass and a centroid of a Voronoi cell into a line integral around its boundary by Green’s theorem [30]. In this paper, we assume that a density function is represented by a sum of Gaussian functions and each agent knows coordinates of their centers and variances. It is well-known that mixed Gaussian functions give a good approximation of various types of continuous datasets. For example, Hatanaka et al. have proposed a gradient-descent algorithm on $SO(3)$ for monitoring moving objects [31]. They estimate a position of a moving object with the mixed Gaussian functions. In this paper, we also consider parametric representation of the boundary of each Voronoi cell to further reduce the computational load. Then the mass and the centroid are computed only by coordinates of vertices of each Voronoi cell and parameters of the density function, which alleviates the computational issues compared with the spatial discretization method by a double integral over the Voronoi cell. Moreover the proposed method does not depend on the size of a Voronoi cell because a mass and a centroid are computed by a line integral around the boundary of each Voronoi cell with its parametric representation.

We also note that the proposed method does not require the preprocessing procedure of a division of a Voronoi cell into triangular regions as in [22],[26]. This also contributes to the reduction of computational load. In addition to that, the computation of a mass and a centroid by a line integral enables
one to take into consideration of a non-uniform density function, which is not the case for the method proposed in [18]. We show that the proposed method is an extension of the result for a uniform density function in [18] to the case for a non-uniform density function. Finally, in contrast to the method by [9], the proposed method can obtain accurate values of a mass and a centroid because the proposed method uses a density function without the approximation by the Taylor series.

This paper is organized as follows. We present preliminary results of Voronoi coverage control in Section 2. We consider the computation of a mass and a centroid by a line integral with Gaussian density functions in Section 3. In Section 4, we show simulation results of the proposed method. We conclude this paper in Section 5.

2. Voronoi Coverage Control

In this section, we summarize the fundamental results on Voronoi coverage control [18]. A partition of \( Q \subset \mathbb{R}^m \) is a collection of the closed connected sets \( W_i = (W_i, W_2, \ldots, W_n) \) such that \( \text{int}(W_i) \cap \text{int}(W_j) = \emptyset \) for \( i \neq j \), and \( \bigcup_{i=1}^n W_i = Q \), where \( \text{int}(A) \) is the interior of a set \( A \). Given a set \( Q \subset \mathbb{R}^n \) and \( n \) distinct points \( p_1, p_2, \ldots, p_n \), the Voronoi partition of \( Q \) generated by \( p_1, p_2, \ldots, p_n \) is the collection of Voronoi cells \( V_i(p) = \{ q \in Q \mid ||q - p_i|| \leq ||q - p_j||, \forall j \in J, j \neq i \} \), where \( p = [p_1, p_2, \ldots, p_n]^T \in \mathbb{R}^n \), \( || \cdot || \) is an Euclidean norm, and \( J = \{1, 2, \ldots, n \} \) [6].

The Delaunay graph for \( n \) distinct points \( p_1, p_2, \ldots, p_n \) is defined as an undirected graph \( G(I, E) \), where \( E = \{(i, j), i \in I \times I \mid V_i(p) \cap V_j(p) \neq \emptyset \} \) is the edge set.

In the rest of this paper, we consider the 2-dimensional case, that is, \( m = 2 \). Let \( Q \subset \mathbb{R}^2 \) be a convex polygonal to be monitored. We consider the following performance function \( J : \mathbb{R}^{2n} \rightarrow R \) for the Voronoi coverage:

\[
J(p) = \int_Q \min_{i \in I} ||q - p_i|| \phi(q) dq = \sum_{i=1}^n \int_{V_i(p)} ||q - p_i|| \phi(q) dq,
\]

where \( \phi : \mathbb{R}^2 \rightarrow \mathbb{R} \) is an integrable density function and \( \phi(q) > 0 \) for all \( q \in Q \). Let \( S = \{p \in \mathbb{R}^n \mid p_j = p_j, i \neq j, i, j \in I\} \) be the set of positions which represents the case when some agents are on the same point. For any \( p \in \mathbb{R}^{2n} \setminus S \), the gradient of the performance function \( J \) is given by

\[
\frac{\partial J}{\partial p_i}(p) = 2M(V_i(p))(p_i - C(V_i(p))),
\]

where \( M(V_i(p)) = \int_{V_i(p)} \phi(q) dq \), \( C(V_i(p)) = \frac{1}{M(V_i(p))} \int_{V_i(p)} q \phi(q) dq \).

We consider a multi-agent system with \( n \) agents whose local communications are given by the Delaunay graph \( G \), that is, if \( (i, j) \in E \), agents \( i \) and \( j \) exchange their positions \( p_i \) and \( p_j \). Each agent updates the position \( p_i \) by the following dynamics:

\[
\dot{p}_i(t) = -\alpha_i(p_i(t) - C(V_i(p_i(t)))) , \quad p_i(0) = p_{i0}, \quad \forall i \in I,
\]

where \( p_{i0} \in Q \) is the initial position of agent \( i \) and \( \alpha_i \in \mathbb{R} \) is a positive constant. Cortés et al. have shown that the position of each agent \( p_i(t) \) converges to the centroid \( C(V_i(p(t))) \) if \( p(0) \in \mathbb{R}^{2n} \setminus S \) and the set of the stationary points of \( J \) is finite [18].

3. Computation of a Mass and a Centroid by Line Integration

To implement the dynamics (5), one should compute a mass and a centroid of a Voronoi cell by (3) and (4). In this section, we consider a transformation of a surface integral of a mass and a centroid over a Voronoi cell into a line integral around its boundary to reduce the load for their computations. To this end, we consider Green’s theorem which gives the relation between a surface integral over a plane and a line integral around its boundary [30].

**Theorem 1** Let \( D \) be the region in the \((x, y)\)-plane bounded by a positively oriented, piecewise-smooth, and simple closed curve \( \partial D \). If \( \Phi(x, y) \) and \( \Psi(x, y) \) are \( C^1 \) functions on an open region that contains \( D \),

\[
\int_{\partial D} (\Phi dx + \Psi dy) = \iint_D \left( \frac{\partial \Psi}{\partial x} - \frac{\partial \Phi}{\partial y} \right) dx dy.
\]

In this paper, we assume that the density function \( \phi \) is given by a sum of Gaussian functions.

**Assumption 1** The density function \( \phi \) is given by

\[
\phi(q) = \sum_{j=1}^M \exp\left(\frac{||q - p_j||^2}{2\sigma_j^2}\right).
\]

where \( p_1, p_2, \ldots, p_M \) are the centers of the \( M \) important areas in the monitoring polyhedron \( Q \) and \( \sigma_j \) is a given variance which represents spatial distribution of the important area \( (M \geq 1) \).

We also assume that each agent knows the coordinates of the centers and the variances of the Gaussian functions. This assumption is slightly different from the problem setting of the standard Voronoi coverage control. However, there are also a number of applications which are compatible with the assumption [10],[31]. For example, in the application to monitoring systems for moving objects, the centers of the important areas and the variances of the Gaussian functions can be considered as the positions of moving objects and the degree of accuracy or uncertainty of the positions, respectively. In [31], the authors have proposed an approach to estimate these values from image data of cameras.

Let \( v_{i\ell} = [X_{i\ell}, Y_{i\ell}]^T \in \mathbb{R}^2 \) be the coordinate of the \( \ell \)-th vertex of the Voronoi cell \( V_i(p) \) \( (\ell = 1, 2, \ldots, N_i) \), where \( N_i \) is the number of vertices of \( V_i(p) \). We assume that the vertices are numbered in the counter clockwise order and \( v_{i\ell} = v_{i\ell} \) for all \( i \in I \) as shown in Fig. 1. For brevity, in what follows, the Voronoi cell \( V_i(p) \) and the coordinate of the \( \ell \)-th vertex of the Voronoi cell \( v_{i\ell} = [X_{i\ell}, Y_{i\ell}]^T \) are denoted by \( V_i \) and \( v_\ell \), respectively.

We here consider the error function \( \text{erf} : \mathbb{R} \rightarrow \mathbb{R} \) defined as follows [32]:

\[
\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt.
\]
where $\mu / n = 6$.

\[ \sum_{i} (\exp M Vi 1) = \int_{\text{area}} \exp \left( \frac{X_{i1} - X_{i2}}{\sqrt{2\sigma_j}} \right) dy. \]  

where $D_1$ is a constant.

Let $p_i = [x_i y_i]_{\in R^2}$ be the coordinate of the center of the $j$-th important area ($j \in \{1, 2, \ldots, M\}$). The following proposition shows that a mass can be computed by coordinates of vertices of a Voronoi cell and parameters of a density function.

**Proposition 1** The mass of the Voronoi cell $V_i$ is given by

\[ M(V_i) = \sum_{j=1}^{M} \sum_{i=1}^{N_j} \frac{\sqrt{2\pi\sigma_j}}{2} (Y_{i1} - Y_{j1}) \]

From (9), for $\mu \in \mathbb{R}$, we have

\[ \int \exp \left( \frac{1}{2} \frac{(y - \mu)^2}{\sigma_j^2} \right) dy = \sqrt{2\pi\sigma_j} \exp \left( \frac{-\mu^2}{2\sigma_j^2} \right) + D_2, \]

where $D_2$ is a constant. Thus, from Theorem 1, we obtain

\[ M(V_i) = \sum_{j=1}^{M} \sum_{i=1}^{N_j} \int_{V_i} \exp \left( \frac{1}{2} \frac{(y - \mu)^2}{\sigma_j^2} \right) dy. \]

\[ = \int_{0}^{1} \left( \exp \left( \frac{-\mu^2}{2\sigma_j^2} \right) + D_2, \right) dx. \]

\[ = \sum_{j=1}^{M} \sum_{i=1}^{N_j} \int_{V_i} \exp \left( \frac{1}{2} \frac{(y - \mu)^2}{\sigma_j^2} \right) dy. \]

\[ = \sum_{j=1}^{M} \sum_{i=1}^{N_j} \int_{V_i} \exp \left( \frac{1}{2} \frac{(y - \mu)^2}{\sigma_j^2} \right) dy. \]

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\[ = \frac{1}{M(V_i)} \sum_{j=1}^{M} \sum_{i=1}^{N_j} \exp \left( \frac{-\mu^2}{2\sigma_j^2} \right) dy. \]

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\[ = \frac{1}{M(V_i)} \sum_{j=1}^{M} \sum_{i=1}^{N_j} \exp \left( \frac{-\mu^2}{2\sigma_j^2} \right) dy. \]

where $\partial V_i$ is the boundary of the Voronoi cell $V_i$.

Let $L_{ij}$ be the segment that connects the two vertices $v_i$ and $v_{i+1}$. Note that the Voronoi cell $V_i$ is a polyhedron with $V_i$ vertices which are numbered in the counter clockwise order and $v_{V_i+1} = v_1$ (Fig. 1). Thus, from (11), we have

\[ M(V_i) = \frac{1}{M(V_i)} \sum_{j=1}^{M} \sum_{i=1}^{N_j} \exp \left( \frac{-\mu^2}{2\sigma_j^2} \right) dy. \]

\[ = \frac{1}{M(V_i)} \sum_{j=1}^{M} \sum_{i=1}^{N_j} \exp \left( \frac{-\mu^2}{2\sigma_j^2} \right) dy. \]

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\[ = \frac{1}{M(V_i)} \sum_{j=1}^{M} \sum_{i=1}^{N_j} \exp \left( \frac{-\mu^2}{2\sigma_j^2} \right) dy. \]
\[
\exp \left( -\frac{1}{2} \left( \frac{y - y_j}{\sigma_j} \right)^2 \right) dxdy. \tag{15}
\]

We here consider the following functions:

\[
\Phi_j(x, y) = -\frac{\sqrt{\pi} \sigma_j}{2} x \exp \left( -\frac{1}{2} \left( \frac{x - x_j}{\sigma_j} \right)^2 \right) \text{erf} \left( \frac{y - y_j}{\sqrt{2} \sigma_j} \right),
\]

\[
\Psi_j(x, y) = 0.
\]

The derivative of the error function is given as follows [32]:

\[
\frac{d}{dx} \text{erf}(x) = \frac{2}{\sqrt{\pi}} \exp(-x^2). \tag{16}
\]

Hence, we obtain

\[
\frac{\partial \Phi_j}{\partial y} = -x \exp \left( -\frac{1}{2} \left( \frac{x - x_j}{\sigma_j} \right)^2 \right) \exp \left( -\frac{1}{2} \left( \frac{y - y_j}{\sigma_j} \right)^2 \right),
\]

Thus, from (6) and (15), we have

\[
M(V)C_\ell(V) = \sum_{j=1}^{M} \int_{V_j} \left( \Phi_j \frac{d\Phi_j}{dy} - \Psi_j \frac{d\Psi_j}{dy} \right) dxdy
\]

\[
= \sum_{j=1}^{M} \int_{V_j} \left( \Phi_j \frac{d\Phi_j}{dy} \right) dxdy
\]

\[
= -\frac{\sqrt{\pi}}{2} \sum_{j=1}^{M} \int_{V_j} \sigma_j \exp \left( -\frac{1}{2} \left( \frac{x - x_j}{\sigma_j} \right)^2 \right) \text{erf} \left( \frac{y - y_j}{\sqrt{2} \sigma_j} \right) dxdy.
\]

Then, from the parametric representation of a Voronoi cell (12), we have

\[
M(V)C_\ell(V) = \sum_{j=1}^{M} \sum_{i=1}^{N_j} \frac{\sqrt{\pi} \sigma_j}{2} \int_{0}^{1} \left( (X_{\ell+1} - X_\ell) x + X_\ell \right)
\]

\[
\exp \left( -\frac{1}{2} \left( \frac{(X_{\ell+1} - X_\ell) x + X_\ell - x_i}{\sigma_j} \right)^2 \right)
\]

\[
\text{erf} \left( \frac{(Y_{\ell+1} - Y_\ell) s + Y_\ell - y_i}{\sqrt{2} \sigma_j} \right) \right) dx ds.
\]

Since the density function \( \phi \) is given by (7), we have \( M(V) > 0 \) for all \( i \in I \). Thus the centroid \( C_\ell(V) \) is given by (13).

From the same argument, we can also show that the y-coordinate of the centroid \( C_\ell(V) \) is given by (14).

We here consider the special case when \( M = 1 \), \( p_0 = [x_0, y_0]^T = 0 \) with \( \sigma_j \to +\infty \). From the definition of a density function (7), this is approximately equivalent to the case for a uniform density function, that is, \( \phi(q) \approx 1 \) for all \( q \in Q \). Hereafter, for simplicity of notation, the variance of the density function \( \sigma^2 \) is denoted by \( \sigma \).

First we consider the limit of the mass \( M(V) \). Since

\[
\lim_{\sigma \to +\infty} \text{erf} \left( \frac{\xi}{\sqrt{2} \sigma} \right) = 0 \quad \text{for} \quad \xi \in \mathbb{R},
\]

from L’Hospital’s rule and (16), we have

\[
= \lim_{\sigma \to +\infty} \frac{\sqrt{\pi} \sigma}{2} \text{erf} \left( \frac{\xi}{\sqrt{2} \sigma} \right) = \lim_{\sigma \to +\infty} \frac{\sqrt{\pi} \sigma}{2} \text{erf} \left( \frac{\xi}{\sqrt{2} \sigma} \right) = \lim_{\sigma \to +\infty} \frac{\sqrt{\pi} \sigma}{2} \frac{d}{d\sigma} \text{erf} \left( \frac{\xi}{\sqrt{2} \sigma} \right)
\]

\[
= \lim_{\sigma \to +\infty} \frac{\sqrt{\pi} \sigma}{2} \frac{d}{d\sigma} \text{erf} \left( \frac{\xi}{\sqrt{2} \sigma} \right).
\]

Thus, from (10) and the bounded convergence theorem [33], we have

\[
\lim_{\sigma \to +\infty} M(V) = \sum_{\ell=1}^{N} \left( \int_0^1 \left( \frac{(X_{\ell+1} - X_\ell) s + X_\ell}{\sqrt{2} \sigma} \right)^2 ds \right)
\]

\[
\left[ \frac{(Y_{\ell+1} - Y_\ell) s + Y_\ell}{\sqrt{2} \sigma} \right] ds.
\]

From (17) and (18), we obtain

\[
\lim_{\sigma \to +\infty} M(V) = \sum_{\ell=1}^{N} \left( \int_0^1 \left( \frac{(X_{\ell+1} - X_\ell) s + X_\ell}{\sqrt{2} \sigma} \right)^2 ds \right)
\]

\[
\left[ \frac{(Y_{\ell+1} - Y_\ell) s + Y_\ell}{\sqrt{2} \sigma} \right] ds.
\]

This coincides with the closed form of a mass for \( \phi(q) \equiv 1 \) [18].

Next we consider the limit of the centroid \( C(V) \). From (13), (17) and the bounded convergence theorem, we have

\[
\lim_{\sigma \to +\infty} M(V)C_\ell(V) = \sum_{\ell=1}^{N} \left( \int_0^1 \left( \frac{(X_{\ell+1} - X_\ell) s + X_\ell}{\sqrt{2} \sigma} \right)^2 ds \right)
\]

\[
\left[ \frac{(Y_{\ell+1} - Y_\ell) s + Y_\ell}{\sqrt{2} \sigma} \right] ds.
\]
We compare the performance of the proposed method with the one of the spatial discretization method. In the spatial discretization method, the mass $M(V_i(p))$ is computed by summing up the values of the density function on the grid points over the Voronoi region $V_i(p)$. The centroid $C(V_i(p))$ is also computed in the similar manner. The grid interval used in the spatial discretization method is $\Delta = 0.01$. Figures 4 and 5 show the trajectories of agents for $0 \leq t \leq 30$ and the Voronoi partition at $t = 30$ by the spatial discretization method (Fig. 4) and the proposed method with a line integral (Fig. 5). In these figures, trajectories of agents are represented by dotted lines and a final position of each agent is represented by $\circ$. Figure 6 shows the evolution of the performance functions in (1) by the spatial discretization method $J_{SD}$ and the proposed method with a line integral $J_{LI}$. These figures show that the result by the proposed method is almost the same as the one by the spatial discretization method. On the other hand, the execution times required to run the simulation for $0 \leq t \leq 30$ by the spatial discretization method and the proposed method are $831.67 \text{s}$ and $68.802 \text{s}$, respectively.

To evaluate the execution time in detail, we conduct a simulation for the different numbers of agents ($n = 5, 10, 20$). Table 1 shows the averages of the execution times of 100 trials by the spatial discretization method and the proposed method. In this simulation, the execution time is defined as

\begin{equation}
\frac{1}{N} \sum_{i=1}^{N} (X_{t+1} - X_t)Y_t + X_t(Y_{t+1} - Y_t) + X_tY_t
\end{equation}

Similarly, we also have

\begin{equation}
\lim_{\alpha \to 0^+} M(V_i)C_i(V_i) = \frac{1}{6} \sum_{i=1}^{N} (X_t + X_{t+1})(X_tY_{t+1} - X_{t+1}Y_t).
\end{equation}

These results coincide with the closed form of a centroid for $\phi(q) \equiv 1$ [18].

From the above argument, we see that the proposed method of the computation of a mass and a centroid by a line integral is an extension of the existing method with a uniform density function to the case with a non-uniform density function. Note that the computation of (10), (13), and (14) does not depend on the size of a Voronoi cell because surface integrals of each point in $Q$ depend on the size of a Voronoi cell because surface integrals of the coordinates of the vertices $v_i = [x_i, y_i]^T$ of each Voronoi cell and the parameters of the density function (the variance $\sigma_i$ and the coordinate of the center $p_i = [x_i, y_i]^T$). Thus the proposed method does not require the procedure to check whether each point in $Q$ is included in a Voronoi cell nor the computation of a double integral over a Voronoi cell, which are computationally demanding for agents with limited resources.

4. Simulation

We consider a multi-agent system with 7 agents ($n = 7$) which move within the square region $Q = \{ [x, y]^T \in \mathbb{R}^2 \mid 0 \leq x, y \leq 1 \}$. The density function $\phi$ is given by (7) with $M = 2$, $p_1 = [0.1, 0.8]^T$, $p_2 = [0.6, 0.6]^T$, $\sigma_1 = 0.2$, $\sigma_2 = 0.1$. The contour of this density function on $Q$ is shown in Fig. 2. The initial positions of the agents are $[0.24, 0.06]^T$, $[0.21, 0.11]^T$, $[0.22, 0.21]^T$, $[0.11, 0.05]^T$, $[0.22, 0.26]^T$, $[0.05, 0.2]^T$, $[0.15, 0.23]^T$. The initial positions of the agents and the corresponding Voronoi partition are shown in Fig. 3. In Fig. 3, the marks $\Box$ and $\circ$ represent the positions of an agent and a centroid of each Voronoi cell, respectively. Each agent updates its position by the dynamics (5) with $\alpha_i = 0.9$ for all $i \in \{1, 2, \ldots, 7\}$. The simulation is performed by MATLAB R2016a on a PC with Intel Core i7-2600, 3.4 GHz.

![Fig. 2 Contour of the density function $\phi$.](image1)

![Fig. 3 Initial positions of 7 agents and the corresponding Voronoi partition. $\Box$ and $\circ$ are the positions of agents and centroids.](image2)

![Fig. 4 Trajectories of agents and Voronoi partition at $t = 30$ by the spatial discretization method. $\Box$ are the positions of agents at $t = 0$, $+$ and $\circ$ are the positions of agents and centroids at $t = 30$.](image3)
5. Conclusion

This paper has considered 2D Voronoi coverage control with the Gaussian density functions. We have considered a transformation of a surface integral of a mass and a centroid over a Voronoi cell into a line integral around its boundary with parametric representations. The proposed method can be implemented only by coordinates of vertices of a Voronoi cell and parameters of a density function. Therefore the proposed method can reduce the computational load due to the preprocessing procedure to obtain a region of a Voronoi cell and the computation of a double integral.

In this paper, we have assumed that a density function is given by a sum of Gaussian functions. To derive the results for a more general type of density functions is our subsequent work.

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