Binned Hubble parameter measurements and the cosmological deceleration-acceleration transition

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ABSTRACT

Weighted mean and median statistics techniques are used to combine 23 independent lower redshift, $z < 1.04$, Hubble parameter, $H(z)$, measurements and determine binned forms of $H(z)$. When these are combined with 5 higher redshift, $1.3 \leq z \leq 2.3$, $H(z)$ measurements the resulting constraints on cosmological parameters, of three cosmological models, that follow from the weighted-mean binned data are almost identical to those derived from analyses using the 28 independent $H(z)$ measurements. This is consistent with what is expected if the lower redshift measurements errors are Gaussian. Plots of the binned weighted-mean $H(z)/(1 + z)$ versus $z$ data are consistent with the presence of a cosmological deceleration-acceleration transition at redshift $z_{da} = 0.74 \pm 0.05$ [Farooq & Ratra 2013b], which is expected in cosmological models with present-epoch energy budget dominated by dark energy as in the standard spatially-flat $\Lambda$CDM cosmological model.

1. Introduction

In the standard cosmological model\footnote{For recent reviews see, e.g., [Wang (2011), Li et al. (2012b), and Tsujikawa (2013)]. In this paper we assume that general relativity provides an adequate description of gravitation on cosmological scales of interest; for discussions of modified gravity see Capozziello & De Laurentis (2011), Trodden (2012) and references therein.} dark energy dominates the current epoch energy budget, but was less important in the past when non-relativistic (cold dark and baryonic)
matter dominated. The transition from non-relativistic matter dominance to dark energy dominance results in a transition from decelerated to accelerated cosmological expansion. The existence of this transition is a strong prediction of the standard cosmological model and attempts have been made to measure the transition redshift. However, only very recently has this become possible, due to high redshift (i.e., $z$ above the deceleration-acceleration transition) data that recently became available, with the most striking being the measurement of the Hubble parameter $H(z = 2.3) = 224 \pm 8 \text{ km s}^{-1} \text{ Mpc}^{-1}$, well in the matter dominated epoch of the standard ΛCDM model.

From a compilation of 28 independent $H(z)$ measurements over $0.07 \leq z \leq 2.3$ (Farooq & Ratra 2013), hereafter FR, Table 1), the transition redshift was found to be $z_{\text{da}} = 0.74 \pm 0.05$. This was determined from the 6 best-fit transition redshifts measured in three different cosmological models, ΛCDM, XCDM, and φCDM, for two different Hubble constant priors.

The spatially-flat ΛCDM model (Peebles 1984) is the reigning standard cosmological model. In this paper we consider the more general ΛCDM model that allows for non-zero space curvature. In the standard model the cosmological constant, $\Lambda$, contributes around 70% of the present cosmological energy budget, non-relativistic, pressure-less, cold dark matter (CDM) contributes a little more than 20%, and non-relativistic baryonic matter makes up the remaining 5% or so. In the ΛCDM model time-independent dark energy, $\Lambda$, is modeled as a spatially homogeneous fluid with equation of state $p_{\Lambda} = -\rho_{\Lambda}$ where $p_{\Lambda}$ and $\rho_{\Lambda}$ are the fluid pressure and energy density respectively. It has been known for a while now that the spatially-flat ΛCDM model is consistent with most observational data. It is also well known that if, instead of staying constant like $\Lambda$, the dark energy density gradually decreased in time (and correspondingly slowly varied in space), it would alleviate a conceptual coincidence problem associated with the ΛCDM model.

A widely-used parameterization of time-evolving dark energy, XCDM, parameterizes dark energy as a spatially-homogeneous time-varying $X$-fluid with equation of state $p_X = \omega_X \rho_X$. Here, the equation of state parameter $\omega_X$ can take any time-independent value less
than $-1/3$. For computational simplicity we consider only spatially-flat XCDM models. The XCDM parametrization reduces to the flat ΛCDM model for $\omega_X = -1$. For all other values of $\omega_X < -1/3$, the XCDM parametrization is incomplete since it does not describe spatial inhomogeneities (see, e.g., Ratra 1991; Podariu & Ratra 2000).

A simple, consistent, and complete model of slowly-varying dark energy density is the $\phi$CDM model (Peebles & Ratra 1988; Ratra & Peebles 1988). Here dark energy is modeled as a scalar field, $\phi$, with a gradually decreasing (in $\phi$) potential energy density $V(\phi)$. In this paper we assume an inverse-power-law potential energy density $V(\phi) \propto \phi^{-\alpha}$, where $\alpha$ is a nonnegative constant (Peebles & Ratra 1988). When $\alpha = 0$ the $\phi$CDM model reduces to the corresponding ΛCDM case. For computational simplicity we again only consider the spatially-flat cosmological case for $\phi$CDM.

In addition to being affected by the cosmological model used in the analysis, the measured deceleration-acceleration transition redshift $z_{\text{da}}$ depends on the assumed value of the Hubble constant. Consequently, to quantify the effect, we use two Gaussian $H_0$ priors in the analyses. The first prior is $\overline{H}_0 \pm \sigma_{H_0} = 68 \pm 2.8$ km s$^{-1}$ Mpc$^{-1}$. This comes from a median statistics analysis of 553 $H_0$ measurements (Chen & Ratra 2011a) and is consistent with the earlier estimates of Gott et al. (2001) and Chen et al. (2003). The second prior of $\overline{H}_0 \pm \sigma_{H_0} = 73.8 \pm 2.4$ km s$^{-1}$ Mpc$^{-1}$ comes from recent Hubble Space Telescope measurements (Riess et al. 2011).

In FR we determined the redshift of the deceleration-acceleration transition by finding the mean and standard deviation of the six best-fit $z_{\text{da}}$ values in the 3 models (with 2 different $H_0$ priors). Here we use a different technique to measure $z_{\text{da}}$ and the related uncertainty in each of these six cases. We then determine summary estimates of $z_{\text{da}}$ by considering various weighted mean combinations of these six estimates. The transition redshifts take the forms

$$z_{\text{da}} = \left( \frac{2\Omega_\Lambda}{\Omega_{m0}} \right)^{1/3} - 1,$$

(1)

$$z_{\text{da}} = \left( \frac{\Omega_{m0}}{(\Omega_{m0} - 1)(1 + 3\omega_X)} \right)^{1/3\omega_X} - 1,$$

(2)

for the ΛCDM and XCDM cases where $\Omega_\Lambda$ and $\Omega_{m0}$ are the cosmological constant and non-relativistic matter density parameters. As for $\phi$CDM, from Eqs. (3) of Peebles & Ratra

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5 Other recent measurements are consistent with either the smaller or larger $H_0$ value we consider, see, e.g., Freedman et al. (2012), Sorce et al. (2012), and Tammann & Reindl (2012), although it might now be significant that both BAO (see, e.g., Colless et al. 2012) and Planck CMB anisotropy (Ade et al. 2013) measurements favor the lower $H_0$ value we use. It might also be significant that the lower value of $H_0$ does not require the presence of dark radiation (Calabrese et al. 2012 and references there in).
we first derive

\[
\frac{\dot{a}}{a} = -\frac{4\pi G}{3} \left[ \rho_m + \rho_\phi (1 + 3 \omega_\phi) \right]
\]

\[
= -\frac{1}{2} H_0^2 \left[ \Omega_{m0} (1 + z)^3 + \Omega_\phi (z, \alpha) (1 + 3 \omega_\phi (z)) \right],
\]

(3)

where \( \Omega_\phi (z) \) is the scalar field energy density parameter and

\[
\omega_\phi (z) = \frac{1}{2} \dot{\phi}^2 - V(\phi) - \frac{1}{2} \dot{\phi}^2 + V(\phi).
\]

(4)

The redshift \( z_{\text{da}} \) is determined by requiring that the right hand side of Eq. (3) vanish,

\[
\Omega_{m0} (1 + z_{\text{da}})^3 + \Omega_\phi (z_{\text{da}}, \alpha) \left[ 1 + 3 \omega_\phi (z_{\text{da}}) \right] = 0.
\]

(5)

To determine \( z_{\text{da}} \) we numerically integrate the \( \phi \)CDM model equations of motion, Eqs. (3) of Peebles & Ratra (1988), using the initial conditions described there. These solutions determine the needed functions in Eq. (5), which we then numerically solve for \( z_{\text{da}} (\Omega_{m0}, \alpha) \).

To find the expected values \( \langle z_{\text{da}} \rangle \) and \( \langle z_{\text{da}}^2 \rangle \) we use

\[
\langle z_{\text{da}} \rangle = \frac{\iint z_{\text{da}}(p) \mathcal{L}(p) dp}{\iint \mathcal{L}(p) dp}, \quad \langle z_{\text{da}}^2 \rangle = \frac{\iint z_{\text{da}}^2(p) \mathcal{L}(p) dp}{\iint \mathcal{L}(p) dp}.
\]

(6)

Here \( \mathcal{L}(p) \) is the \( H(z) \) data likelihood function after marginalization over the \( H_0 \) prior in the model under consideration. It depends only on the model parameters \( p = (\Omega_{m0}, \Omega_\Lambda) \) for \( \Lambda \)CDM, = \( (\Omega_{m0}, \omega_X) \) for XCDM, and = \( (\Omega_{m0}, \alpha) \) for \( \phi \)CDM. The standard deviation in \( z_{\text{da}} \) is calculated from the standard formula \( \sigma_{z_{\text{da}}} = \sqrt{\langle z_{\text{da}}^2 \rangle - \langle z_{\text{da}} \rangle^2} \). The results of this computation are summarized in Table 1.

It is reassuring that the results of the penultimate and the last columns of Table 1 are very consistent. FR determined a summary estimate of \( z_{\text{da}} = 0.74 \pm 0.05 \) by computing the mean and standard deviation of the six values in the last column of Table 1. It is of interest to estimate similar summary values for each of the two \( H_0 \) priors. We find that \( z_{\text{da}} = 0.70 \pm 0.05 \) (\( z_{\text{da}} = 0.77 \pm 0.04 \)) for \( H_0 \pm \sigma_{H_0} = 68 \pm 2.8 \) (73.8 \pm 2.4) km s\(^{-1}\) Mpc\(^{-1}\).

Perhaps more realistic summary estimates are determined by the weighted means of the two sets of 3 values in the penultimate column of Table 1: \( z_{\text{da}} = 0.69 \pm 0.06 \) (\( z_{\text{da}} = 0.76 \pm 0.05 \)) for \( H_0 \pm \sigma_{H_0} = 68 \pm 2.8 \) (73.8 \pm 2.4) km s\(^{-1}\) Mpc\(^{-1}\), and \( z_{\text{da}} = 0.74 \pm 0.04 \) is the result if all six values are used.

More conventionally, cosmological data are used to constrain model parameters values such as \( \Omega_{m0} \) and \( \Omega_\Lambda \) for the \( \Lambda \)CDM model. A number of different data sets have
been used for this purpose. These include Type Ia supernova (SNIa) apparent magnitude versus redshift data (e.g., Ruiz et al. 2012; Chiba et al. 2013; Cardenas & Rivera 2011; Liao et al. 2013; Farooq et al. 2013; Campbell et al. 2013), cosmic microwave background (CMB) anisotropy measurements (Ade et al. 2013, and references therein), baryonic acoustic oscillation (BAO) peak length scale data (Mehta et al. 2012; Anderson et al. 2012; Li et al. 2012a; Scovaricchi et al. 2012; Farooq & Ratra 2013a, and references therein), galaxy cluster gas mass fraction as a function of redshift (e.g., Allen et al. 2008; Samushia & Ratra 2008; Tong & Noh 2011; Lu et al. 2011b; Solano & Nucamendi 2012; Landry et al. 2012), and, of special interest here, measurement of the Hubble parameter as a function of redshift (Jimenez et al. 2003; Samushia & Ratra 2006; Samushia et al. 2007; Sen & Scherrer 2008; Chen & Ratra 2011b; Aviles et al. 2012; Wang et al. 2012b; Campos et al. 2012; Chimento et al. 2013, and references therein). These data, separately and in combination, provide strong evidence for accelerated cosmological expansion at the current epoch. However their error bars are still too large to allow for a discrimination between constant and time-varying dark energy densities.

Of course, both methods are equivalent, since they make use of the same data, but each has its own advantages and disadvantages. In particular, it is of some interest to actually discern the deceleration-acceleration transition in the $H(z)$ data. While the data does indicate the transition, see Fig. 4 of FR, the data points bounce around quite a bit. Given the low reduced $\chi^2$ for the best-fit models (see FR and Table 1 here), all of which show significant evidence for a deceleration-acceleration transition, we investigate different data binning techniques here, to see if binned versions of the $H(z)$ measurements more clearly illustrate the presence of a deceleration-acceleration transition.

Motivated by a similar situation in the early days of CMB anisotropy data constraints, in this paper we use the binning techniques of Podariu et al. (2001) to bin the $H(z)$ data and so construct a smoother representation of the observed $H(z)$ function. While Podariu et al. (2001) considered many more data points then we do here (142 vs. 23), they covered a large range in multipole space with $\ell_{\text{max}} \approx 370 \ell_{\text{min}}$ while we consider a significantly smaller range in redshift space with $z_{\text{max}} \approx 15z_{\text{min}}$. The other striking point is that, unlike in Podariu et al. (2001) for the CMB anisotropy case, we find here that when combining individual $H(z)$

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6Other data, with larger error bars, support these results. See, e.g., Chae et al. (2004), Cao et al. (2012), Chen & Ratra (2012), Jackson (2012), Campaelli et al. (2012), Mania & Ratra (2012), Poitras (2012), and Pan et al. (2013).

7 Compare the analyses of e.g., Ganga et al. (1997) and Ratra et al. (1999), which determine constraints on cosmological model parameters from CMB anisotropy data, to that of Podariu et al. (2001), who bin CMB anisotropy measurements to determine a smoother observed CMB anisotropy angular power spectrum.
measurements the error bars of these measurements are consistent with what is expected for Gaussian errors. This is quiet reassuring. We find that the weighted-mean binned \( H(z) \) equally well constrain cosmological parameters, as well as do the unbinned data. More interestingly, for our purpose here, the binned weighted-mean data clearly indicate the presence of the deceleration-acceleration transition.

Our paper is organized as follows. In the next section we summarize the two techniques we use to bin the \( H(z) \) data and compute binned results for a variety of data points per bin. In Sec. 3 we use the binned \( H(z) \) data to derive constraints on cosmological parameters of the 3 models we consider and show that for the weighted-mean binning these constraints are very close to those that follow from the unbinned data. We conclude in Sec. 4.

2. Binning the data

The 28 individual \( H(z) \) measurements bounce around on the \( H(z)/(1+z) \) plot, Fig. 4 of FR. To try to get a smoother observed \( H(z)/(1+z) \) function we form bins in redshift and then combine the data points in each bin to give a single observed value of \( z, H(z) \), and \( \sigma \) for that bin. The measurements in each bin are combined using two different statistical techniques, weighted mean and median statistics.

Table 2 lists the weighted mean results. These results were computed using the standard formulae (see, e.g., Podariu et al. 2001). That is,

\[
\bar{H}(z) = \frac{\sum_{i=1}^{N} H(z_i)/\sigma_i^2}{\sum_{i=1}^{N} 1/\sigma_i^2},
\]

(7)

where \( N \) is the number of data points in the bin under consideration, \( \bar{H}(z) \) is the weighted mean of the Hubble parameter in that bin, \( H(z_i) \) is the value of the Hubble parameter measured at redshift \( z_i \) and \( \sigma_i \) is the corresponding uncertainty. Weighted mean redshifts, denoted by \( \bar{z} \), were similarly computed,

\[
\bar{z} = \frac{\sum_{i=1}^{N} z_i/\sigma_i^2}{\sum_{i=1}^{N} 1/\sigma_i^2}.
\]

(8)

The weighted mean standard deviation, denoted by \( \bar{\sigma} \), for each bin was found from

\[
\bar{\sigma} = \left( \sum_{i=1}^{N} 1/\sigma_i^2 \right)^{-1/2}.
\]

(9)

The assumptions underlying use of weighted mean statistics are that the measurement errors
are Gaussian, and there are no systematic errors. Hence, one can compute $\chi^2$, the goodness-of-fit parameter, for each bin,

$$\chi^2 = \frac{1}{N - 1} \sum_{i=1}^{N} \frac{[H(z_i) - \bar{H}(z)]^2}{\sigma_i^2},$$

which has expected value unity and error $1/\sqrt{2(N - 1)}$, so we can use this to determine the number of standard deviations that $\chi$ deviates from unity for each bin,

$$N_\sigma = |\chi - 1|\sqrt{2(N - 1)}.$$  

An unaccounted for systematic error, the presence of significant correlations between the measurements, and breakdown of the Gaussian error assumption for each measurement, are the three factors that can make $N_\sigma$ much greater than unity.

The second technique we use to combine measurements in a bin is median statistics, as developed in Gott et al. (2001). Table 3 lists the median statistics results. The median is the value for which there is a 50% chance of finding a measurement below or above it. It is fair to use median statistics to combine the $H(z)$ data of Table 1 of FR since we assume that all the measurements are independent and there is no over-all systematic error in the $H(z)$ data as a whole (individual measurements can have individual systematic errors, for discussion see Chen & Ratra 2011a). The median will be revealed as a true value as the number of measurements grow to infinity, and this technique reduces the effect of outliers of a set of measurements on the estimate of a true value. If $N$ measurements are considered, the probability of finding the true value between values $N_i$ and $N_{i+1}$ (where $i = 1, 2, ..., N$) is

$$P_i = \frac{2^{-N} N!}{i!(N - 1)!}$$

This process of finding a median value was used for the redshift and the Hubble parameter, and the Hubble parameter probability distribution was used to determine $\sigma$ for each bin.

We would like to have as many measurements as possible in each bin, as well as bins that are as narrow as possible in redshift space. Obviously, since these requirements are contradictory, compromise is necessary. In addition, we require roughly the same number of measurements per bin, so as to have approximately similar errors on the binned measurements. As indicated in Table 2 and Table 3 we consider four different binnings of the 23

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8For other applications of median statistics see, e.g., Sereno (2003), Chen & Ratra (2003), Richards et al. (2009), and Shafieloo et al. (2011).
lower redshift, \( z < 1.04 \), measurements; the five higher redshift measurements are sparsely spread over too large a redshift range to allow for a useful binning.

The last column of Table 2 shows that the first two binnings, with approximately 3 and 5 measurements per bin, do not show any deviation from what is expected from Gaussian errors. On the other hand, the last binning, with about 8 measurements per bin, appears to be not so consistent with the assumption of Gaussian errors. This is likely a consequence of the large width in redshift of these bins, so the measurements at the low \( z \) end and at the high \( z \) end of each bin differ too much to be combined together. Median statistics does not make use of the error bars of the individual measurements. As a result, it is a more conservative technique and when used to combine data in bins it results in larger error bars. A comparison of the results in Tables 2 and 3 clearly illustrates this point. Fortunately the weighted mean results we have found show that the individual lower redshift data points have reasonable error bars and so there is no obvious danger in using the more constraining weighted mean results to draw physical conclusions.

The weighted-mean and median statistics binned results of Tables 2 and 3 are plotted in the top panels of Figs. 1—4 (in purple). These figures also show the 5 higher \( z \) unbinned measurements listed in Table 1 of FR (in cyan). Both sets of observations show 1 and 2 \( \sigma \) error bars. Also shown are the unbinned data (Table 1 of FR) best-fit predictions for \( \Lambda \)CDM (red), XCDM (blue), and \( \phi \)CDM (green) for the two priors, \( \overline{H}_0 \pm \sigma_{H_0} = 68 \pm 2.8 \text{ km s}^{-1} \text{ Mpc}^{-1} \) (dashed lines) and \( \overline{H}_0 \pm \sigma_{H_0} = 7.8 \pm 2.4 \text{ km s}^{-1} \text{ Mpc}^{-1} \) (dotted lines), from FR. Focusing on the weighted-mean panels in each of these plots, and comparing to Fig. 4 of FR, we see that the binned data of Figs. 1—3 clearly demarcates a declaration-acceleration transition.

### 3. Constraints from the binned data

In this section we use the weighted-mean and median statistics binned data to derive constraints on cosmological parameters of \( \Lambda \)CDM, XCDM, and \( \phi \)CDM, and compare these constraints to those that follow from the unbinned data of Table 1 of FR.

In order to derive constraints on the parameters \( \mathbf{p} \) of the dark energy models discussed above, using the binned data from Tables 2 and 3 we follow the procedure of Farooq et al. (2013). The observational data consist of measurements of the Hubble parameter \( H_{\text{obs}}(z_i) \) at redshifts \( z_i \), with the corresponding one standard deviation uncertainties \( \sigma_i \). To constrain parameters of cosmological models, we define the posterior likelihood function \( \mathcal{L}_H(\mathbf{p}) \), that depends only on the model parameters \( \mathbf{p} \), by integrating the product of the \( H_0 \) prior likelihood
function $\propto \exp[-(H_0 - \bar{H}_0)^2/(2\sigma^2_{H_0})]$ and the usual likelihood function $\exp(-\chi^2_H/2)$, as in Eq. (18) of Farooq et al. (2013). Two different Gaussian priors, $\bar{H}_0 \pm \sigma_{H_0} = 68 \pm 2.8$ km s$^{-1}$ Mpc$^{-1}$ (Chen & Ratra 2011a) and $\bar{H}_0 \pm \sigma_{H_0} = 73.8 \pm 2.4$ km s$^{-1}$ Mpc$^{-1}$ (Riess et al. 2011) are used in the marginalization of the likelihood function over the nuisance parameter $H_0$.

The best-fit point (BFP) $p_0$ are those parameter values that maximize the likelihood function $L_H(p)$. To find the 1, 2, and 3 $\sigma$ confidence intervals as two-dimensional parameter sets, we start from the BFP and integrate the volume under $L_H(p)$ until we include 68.27, 95.45, and 99.73 % of the probability.

The lower 3 rows of panels in Figs. 1—4 show the constraints (1, 2, and 3 $\sigma$ contours) from the unbinned $H(z)$ data of Table 1 of FR (in blue dot-dashed contours) and from the binned $H(z)$ data of Tables 2 and 3 here (in red solid contours), for the three dark energy models we consider, and for the two different $H_0$ priors mentioned above. The red filled circles and the blue empty circles are the best fit points for the binned and unbinned data respectively. Some relevant results are listed in Tables 4—7. Comparing the weighted-mean BFP cosmological parameter values listed in these tables, to those listed in the captions of Figs. 1—3 of FR, establishes the very good agreement between the values derived here using the binned data (especially for fewer measurements per bin) and the FR values derived using the unbinned data.

It is clear from the left two columns of the lower three rows of Figs. 1—4 that the weighted-mean binning of the first 23 data points in Table 1 of FR give almost exactly the same constraints on model parameters $p$ for the three cosmological models as do the unbinned data of Table 1 of FR. Since the weighted-mean technique assumes that the error in the measurements has a Gaussian distribution and that the measurements are uncorrelated, this result is consistent with this assumption that the $H(z)$ data in Table 1 of FR have Gaussian errors. Consequently the best way to combine the measurements in a bin is to use the weighted-mean method. It is also useful to note that when there are fewer data points in a narrower bin, the constraints from the binned data matches better with the constraints derived from the unbinned data. This is not unexpected. In the case of median statistics, however, the constraints on model parameters for all three models from the binned data are much less restrictive than those derived from the unbinned data, see the right hand column of panels in the lower three rows of Figs. 1—4. This is because median statistics is a more conservative technique and so, in this case, is not the best way of combining $H(z)$ measurements in bins. It is also interesting to note, from Tables 4—7, that $\chi^2_{\text{min}}$ for the case of median statistics is significantly smaller than $\chi^2_{\text{min}}$ for the weighted mean case. This is a direct consequence of the larger error bars estimated by the more conservative median statistics approach.
4. Conclusion

We have shown that the weighted-mean combinations of the lower redshift $H(z)$ measurements in bins in redshift provide close to identical constraints on cosmological model parameters as do the unbinned $H(z)$ data tabulated in FR. This is consistent with the $H(z)$ measurements errors being Gaussian.

When plotted against $z$, the weighted-mean binned $H(z)/(1+z)$ measurements bounce around much less than the individual measurements considered in FR, and now much more clearly show the presence of a cosmological deceleration-acceleration transition, consistent with the new summary redshift $z_{da} = 0.74 \pm 0.04$ estimated here and consistent with that estimated in FR. This result is also consistent with what is expected in the standard spatially-flat $\Lambda$CDM model and in other cosmological models with present-epoch energy budget dominated by dark energy.

More, and more precise, measurements of $H(z)$ in the redshift range $1 \lesssim z \lesssim 2.5$ will allow for a clearer demarcation of the cosmological deceleration-acceleration transition. We anticipate that such data will soon become available.

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Fig. 1.— Top left (right) panel shows the $H(z)/(1+z)$ data, binned with 3 or 4 measurements per bin, as well as 5 higher $z$ measurements, and the FR best-fit model predictions, dashed (dotted) for lower (higher) $H_0$ prior. The 2nd through 4th rows show the $H(z)$ constraints for $\Lambda$CDM, XCDM, and $\phi$CDM. Red (blue dot-dashed) contours are 1, 2, and 3 $\sigma$ confidence interval results from 3 or 4 measurements per bin (unbinned FR Table 1) data. In these three rows, the first two plots include red weighted-mean constraints while the second two include red median statistics ones. The filled red (empty blue) circle is the corresponding best-fit point. Dashed diagonal lines show spatially-flat models, and dotted lines indicate zero-acceleration models. For quantitative details see Table 4.
Fig. 2.— Top left (right) panel shows the $H(z)/(1+z)$ data, binned with 4 or 5 measurements per bin, as well as 5 higher $z$ measurements, and the FR best-fit model predictions, dashed (dotted) for lower (higher) $H_0$ prior. The 2nd through 4th rows show the $H(z)$ constraints for $\Lambda$CDM, XCDM, and $\phi$CDM. Red (blue dot-dashed) contours are 1, 2, and 3 $\sigma$ confidence interval results from 4 or 5 measurements per bin (unbinned FR Table 1) data. In these three rows, the first two plots include red weighted-mean constraints while the second two include red median statistics ones. The filled red (empty blue) circle is the corresponding best-fit point. Dashed diagonal lines show spatially-flat models, and dotted lines indicate zero-acceleration models. For quantitative details see Table 5.
Fig. 3.— Top left (right) panel shows the $H(z)/(1+z)$ data, binned with 5 or 6 measurements per bin, as well as 5 higher $z$ measurements, and the FR best-fit model predictions, dashed (dotted) for lower (higher) $H_0$ prior. The 2nd through 4th rows show the $H(z)$ constraints for $\Lambda$CDM, XCDM, and $\phi$CDM. Red (blue dot-dashed) contours are 1, 2, and $3 \sigma$ confidence interval results from 5 or 6 measurements per bin (unbinned FR Table 1) data. In these three rows, the first two plots include red weighted-mean constraints while the second two include red median statistics ones. The filled red (empty blue) circle is the corresponding best-fit point. Dashed diagonal lines show spatially-flat models, and dotted lines indicate zero-acceleration models. For quantitative details see Table 6.
Fig. 4.— Top left (right) panel shows the $H(z)/(1+z)$ data, binned with 7 or 9 measurements per bin, as well as 5 higher $z$ measurements, and the FR best-fit model predictions, dashed (dotted) for lower (higher) $H_0$ prior. The 2nd through 4th rows show the $H(z)$ constraints for $\Lambda$CDM, XCDM, and $\phi$CDM. Red (blue dot-dashed) contours are 1, 2, and 3 $\sigma$ confidence interval results from 7 or 9 measurements per bin (unbinned FR Table 1) data. In these three rows, the first two plots include red weighted-mean constraints while the second two include red median statistics ones. The filled red (empty blue) circle is the corresponding best-fit point. Dashed diagonal lines show spatially-flat models, and dotted lines indicate zero-acceleration models. For quantitative details see Table 7.
Table 1. Deceleration-Acceleration Transition Redshifts\textsuperscript{a}

| $h$ Prior\textsuperscript{b} | Best-Fit Values | $\chi^2_{\text{min}}$ | $z_{da} \pm \sigma_{z_{da}}$\textsuperscript{c} | $z_{da}$\textsuperscript{d} |
|-----------------------------|------------------|------------------------|---------------------------------|------------------|
| $\Lambda$CDM               |                  |                        |                                 |                  |
| $0.68 \pm 0.028$            | $(\Omega_m, \Omega_\Lambda) = (0.29, 0.72)$ | 18.2                   | $0.690 \pm 0.096$              | 0.706            |
| $0.738 \pm 0.024$           | $(\Omega_m, \Omega_\Lambda) = (0.32, 0.91)$ | 19.3                   | $0.781 \pm 0.067$              | 0.785            |
| XCDM                        |                  |                        |                                 |                  |
| $0.68 \pm 0.028$            | $(\Omega_m, \omega_X) = (0.29, -1.04)$ | 18.2                   | $0.677 \pm 0.097$              | 0.695            |
| $0.738 \pm 0.024$           | $(\Omega_m, \omega_X) = (0.26, -1.30)$ | 18.2                   | $0.696 \pm 0.082$              | 0.718            |
| $\phi$CDM                   |                  |                        |                                 |                  |
| $0.68 \pm 0.028$            | $(\Omega_m, \alpha) = (0.29, 0.00)$ | 18.2                   | $0.724 \pm 0.148$              | 0.698            |
| $0.738 \pm 0.024$           | $(\Omega_m, \alpha) = (0.25, 0.00)$ | 20.7                   | $0.850 \pm 0.116$              | 0.817            |

\textsuperscript{a}Estimated using the unbinned data in Table 1 of FR.

\textsuperscript{b}Hubble constant in units of 100 km s\textsuperscript{-1} Mpc\textsuperscript{-1}.

\textsuperscript{c}Computed using Eqs. (1-6).

\textsuperscript{d}The deceleration-acceleration transition redshift in the model with the best-fit values of the cosmological parameters, as computed in FR.
| Bin | N  | $z^a$  | $H(z)$ (km s$^{-1}$ Mpc$^{-1}$) | $H(z)$ (1 $\sigma$ range) (km s$^{-1}$ Mpc$^{-1}$) | $H(z)$ (2 $\sigma$ range) (km s$^{-1}$ Mpc$^{-1}$) | $N_\sigma$ |
|-----|----|-------|-------------------------------|---------------------------------|---------------------------------|---------|
| 3 or 4 measurements per bin |
| 1  | 3  | 0.096 | 69.0                          | 59.4–78.5                       | 49.9–88.0                       | 2.00    |
| 2  | 4  | 0.185 | 76.0                          | 73.1–78.9                       | 70.2–81.8                       | 1.73    |
| 3  | 3  | 0.338 | 76.6                          | 71.5–81.8                       | 66.4–86.9                       | 1.89    |
| 4  | 3  | 0.417 | 84.4                          | 78.1–90.7                       | 71.8–97.0                       | 1.55    |
| 5  | 3  | 0.598 | 90.9                          | 85.4–96.4                       | 79.9–102                        | 0.73    |
| 6  | 3  | 0.720 | 96.6                          | 91.8–101                        | 87.0–106                        | 1.17    |
| 7  | 4  | 0.929 | 129                           | 118–140                         | 107–151                         | 0.13    |
| 4 or 5 measurements per bin |
| 1  | 4  | 0.139 | 77.2                          | 71.1–83.3                       | 64.9–89.5                       | 1.41    |
| 2  | 5  | 0.191 | 75.2                          | 72.1–78.2                       | 69.1–81.2                       | 2.71    |
| 3  | 5  | 0.380 | 79.9                          | 75.8–84.1                       | 71.6–88.3                       | 1.81    |
| 4  | 5  | 0.668 | 94.1                          | 90.5–97.7                       | 86.8–101                        | 0.91    |
| 5  | 4  | 0.929 | 129                           | 118–140                         | 107–151                         | 0.13    |
| 5 or 6 measurements per bin |
| 1  | 5  | 0.167 | 75.7                          | 72.3–79.0                       | 69.0–82.3                       | 1.86    |
| 2  | 6  | 0.271 | 76.2                          | 72.7–79.7                       | 69.3–83.1                       | 2.89    |
| 3  | 6  | 0.569 | 89.4                          | 85.5–93.2                       | 81.7–97.0                       | 1.70    |
| 4  | 6  | 0.787 | 106                           | 101–112                         | 95.8–117                        | 2.50    |
| 7 or 9 measurements per bin |
| 1  | 7  | 0.177 | 75.4                          | 72.7–78.2                       | 69.9–81.0                       | 2.66    |
| 2  | 9  | 0.448 | 83.6                          | 80.3–86.8                       | 77.1–90.0                       | 1.29    |
| 3  | 7  | 0.754 | 102                           | 97.6–106                        | 93.2–111                        | 2.99    |

$^a$Weighted mean of $z$ values of measurements in the bin.
Table 3. Median Statistics Results For 23 Lower Redshift Measurements

| Bin | N  | $z^b$ | $H(z)$ (km s$^{-1}$ Mpc$^{-1}$) | $H(z)$ (1 $\sigma$ range) (km s$^{-1}$ Mpc$^{-1}$) | $H(z)$ (2 $\sigma$ range) (km s$^{-1}$ Mpc$^{-1}$) |
|-----|----|-------|-------------------------------|---------------------------------|---------------------------------|
|     |    |       |                               |                                 |                                 |
| 3 or 4 measurements per bin |     |       |                               |                                 |                                 |
| 1   | 3  | 0.100 | 69.0                          | 49.4—88.6                      | 29.8—108                       |
| 2   | 4  | 0.189 | 75.0                          | 68.5—81.5                      | 62.0—88.0                      |
| 3   | 3  | 0.280 | 77.0                          | 63.0—91.0                      | 49.0—105                       |
| 4   | 3  | 0.400 | 83.0                          | 69.0—97.0                      | 55.0—111                       |
| 5   | 3  | 0.593 | 97.0                          | 84.0—110                       | 71.0—123                       |
| 6   | 3  | 0.730 | 97.3                          | 89.3—105                       | 81.3—113                       |
| 7   | 4  | 0.890 | 121                           | 99.5—143                       | 78.0—164                       |
| 4 or 5 measurements per bin |     |       |                               |                                 |                                 |
| 1   | 4  | 0.110 | 69.0                          | 53.2—84.8                      | 37.4—101                       |
| 2   | 5  | 0.200 | 75.0                          | 61.0—89.0                      | 47.0—103                       |
| 3   | 5  | 0.400 | 83.0                          | 69.0—97.0                      | 55.0—111                       |
| 4   | 5  | 0.680 | 97.3                          | 89.3—105                       | 81.3—113                       |
| 5   | 4  | 0.890 | 121                           | 99.5—143                       | 78.0—164                       |
| 5 or 6 measurements per bin |     |       |                               |                                 |                                 |
| 1   | 5  | 0.120 | 69.0                          | 57.0—81.0                      | 45.0—93.0                      |
| 2   | 6  | 0.275 | 76.7                          | 62.7—90.7                      | 48.7—105                       |
| 3   | 6  | 0.537 | 93.5                          | 83.0—104                       | 72.5—115                       |
| 4   | 6  | 0.878 | 111                           | 92.5—130                       | 74.0—148                       |
| 7 or 9 measurements per bin |     |       |                               |                                 |                                 |
| 1   | 7  | 0.170 | 72.9                          | 60.9—84.9                      | 48.9—96.9                      |
| 2   | 9  | 0.400 | 87.9                          | 73.9—102                       | 59.9—116                       |
| 3   | 7  | 0.875 | 105                           | 88.0—122                       | 71.0—139                       |

$^b$Median of $z$ values of measurements in the bin.
Table 4: Best-Fit Points And Minimum $\chi^2$s For 3 Or 4 Measurements Per Bin

| Model | $h$ Prior  | Weighted Mean | Median  |
|-------|------------|---------------|---------|
|       |            | $\Omega_{m0}$ | $\chi^2_{\text{min}}$ | $\Omega_{m0}$ | $\chi^2_{\text{min}}$ |
| ΛCDM  | 0.68 ± 0.028 | 0.29 | 13.0 | 0.24 | 8.75 |
|       | 0.738 ± 0.024 | 0.32 | 14.1 | 0.26 | 9.37 |
| XCDM  | 0.68 ± 0.028 | 0.29 | 13.0 | 0.28 | 8.85 |
|       | 0.738 ± 0.024 | 0.26 | 13.0 | 0.25 | 9.18 |
| φCDM  | 0.68 ± 0.028 | 0.29 | 13.0 | 0.27 | 8.82 |
|       | 0.738 ± 0.024 | 0.25 | 15.4 | 0.24 | 9.43 |

Table 5: Best-Fit Points And Minimum $\chi^2$s For 4 Or 5 Measurements Per Bin

| Model | $h$ Prior  | Weighted Mean | Median  |
|-------|------------|---------------|---------|
|       |            | $\Omega_{m0}$ | $\chi^2_{\text{min}}$ | $\Omega_{m0}$ | $\chi^2_{\text{min}}$ |
| ΛCDM  | 0.68 ± 0.028 | 0.29 | 12.9 | 0.17 | 7.62 |
|       | 0.738 ± 0.024 | 0.32 | 13.7 | 0.20 | 8.04 |
| XCDM  | 0.68 ± 0.028 | 0.29 | 12.9 | 0.24 | 7.75 |
|       | 0.738 ± 0.024 | 0.26 | 12.5 | 0.23 | 8.17 |
| φCDM  | 0.68 ± 0.028 | 0.29 | 13.0 | 0.22 | 7.70 |
|       | 0.738 ± 0.024 | 0.25 | 15.2 | 0.23 | 8.13 |
Table 6: Best-Fit Points And Minimum \( \chi^2 \)s For 5 Or 6 Measurements Per Bin

| Model | \( h \) Prior | Weighted Mean BFP | \( \chi^2_{\text{min}} \) | Median BFP | \( \chi^2_{\text{min}} \) |
|-------|---------------|-------------------|----------------|------------|----------------|
| \( \Lambda \text{CDM} \) | 0.68 ± 0.028 | \( \Omega_{m0} = 0.28 \) | 10.2 | \( \Omega_{m0} = 0.18 \) | 7.65 |
|       |               | \( \Omega_{\Lambda} = 0.70 \) |               | \( \Omega_{\Lambda} = 0.45 \) |               |
|       | 0.738 ± 0.024 | \( \Omega_{m0} = 0.31 \) | 11.2 | \( \Omega_{m0} = 0.20 \) | 8.13 |
|       |               | \( \Omega_{\Lambda} = 0.89 \) |               | \( \Omega_{\Lambda} = 0.66 \) |               |
| \( X\text{CDM} \) | 0.68 ± 0.028 | \( \Omega_{m0} = 0.29 \) | 10.2 | \( \Omega_{m0} = 0.24 \) | 7.77 |
|       |               | \( \omega_X = -1.01 \) |               | \( \omega_X = -0.68 \) |               |
|       | 0.738 ± 0.024 | \( \Omega_{m0} = 0.26 \) | 10.2 | \( \Omega_{m0} = 0.23 \) | 8.25 |
|       |               | \( \omega_X = -1.28 \) |               | \( \omega_X = -0.89 \) |               |
| \( \phi \text{CDM} \) | 0.68 ± 0.028 | \( \Omega_{m0} = 0.29 \) | 10.2 | \( \Omega_{m0} = 0.22 \) | 7.72 |
|       |               | \( \alpha = 0.00 \) |               | \( \alpha = 1.73 \) |               |
|       | 0.738 ± 0.024 | \( \Omega_{m0} = 0.25 \) | 12.4 | \( \Omega_{m0} = 0.23 \) | 8.21 |
|       |               | \( \alpha = 0.00 \) |               | \( \alpha = 0.30 \) |               |

Table 7: Best-Fit Points And Minimum \( \chi^2 \)s For 7 Or 9 Measurements Per Bin

| Model | \( h \) Prior | Weighted Mean BFP | \( \chi^2_{\text{min}} \) | Median BFP | \( \chi^2_{\text{min}} \) |
|-------|---------------|-------------------|----------------|------------|----------------|
| \( \Lambda \text{CDM} \) | 0.68 ± 0.028 | \( \Omega_{m0} = 0.29 \) | 9.7 | \( \Omega_{m0} = 0.17 \) | 7.76 |
|       |               | \( \Omega_{\Lambda} = 0.73 \) |               | \( \Omega_{\Lambda} = 0.43 \) |               |
|       | 0.738 ± 0.024 | \( \Omega_{m0} = 0.31 \) | 10.4 | \( \Omega_{m0} = 0.19 \) | 7.88 |
|       |               | \( \Omega_{\Lambda} = 0.90 \) |               | \( \Omega_{\Lambda} = 0.64 \) |               |
| \( X\text{CDM} \) | 0.68 ± 0.028 | \( \Omega_{m0} = 0.29 \) | 9.7 | \( \Omega_{m0} = 0.24 \) | 7.82 |
|       |               | \( \omega_X = -1.04 \) |               | \( \omega_X = -0.69 \) |               |
|       | 0.738 ± 0.024 | \( \Omega_{m0} = 0.26 \) | 9.5 | \( \Omega_{m0} = 0.23 \) | 7.97 |
|       |               | \( \omega_X = -1.28 \) |               | \( \omega_X = -0.90 \) |               |
| \( \phi \text{CDM} \) | 0.68 ± 0.028 | \( \Omega_{m0} = 0.28 \) | 9.7 | \( \Omega_{m0} = 0.22 \) | 7.80 |
|       |               | \( \alpha = 0.00 \) |               | \( \alpha = 1.69 \) |               |
|       | 0.738 ± 0.024 | \( \Omega_{m0} = 0.25 \) | 11.8 | \( \Omega_{m0} = 0.22 \) | 7.93 |
|       |               | \( \alpha = 0.00 \) |               | \( \alpha = 0.48 \) |               |