IMPRIMITIVITY THEOREM FOR
GROUPOID REPRESENTATIONS

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Abstract

We define and investigate the concept of the groupoid representation induced by a representation of the isotropy subgroupoid. Groupoids in question are locally compact transitive topological groupoids. We formulate and prove the imprimitivity theorem for such representations which is a generalization of the classical Mackey’s theorem known from the theory of group representations.
1 Introduction

The present paper, devoted to the study of the theory of groupoid representations, is a continuation of my previous work [20] in which one can find a presentation of the groupoid concept and of the groupoid representation concept, important examples as well as relationships between groupoid representations and induced group representations (see also [18], [21], [23], [11]). Groupoids have now found a permanent place in manifold domains of mathematics, such as: algebra, differential geometry, in particular noncommutative geometry, and algebraic topology, and also in numerous applications, notably in physics. It is a natural tool to deal with symmetries of more complex nature than those described by groups (see [23], [11], [2]). Groupoid representations were investigated by many authors and in many ways (see [24], [21], [18], [1], [3]).

In a series of works ([7], [9], [8], [19]) we have developed a model unify-
ing gravity theory with quantum mechanics in which it is a groupoid that describes symmetries of the model, namely the transformation groupoid of the principal bundle of Lorentz frames over the spacetime. To construct the quantum sector of the model we have used a regular representation of a non-commutative convolutive algebra on this groupoid in the bundle of Hilbert spaces. In paper ([6]) we have applied this groupoid representation to investigate spacetime singularities, and in [10] the representation of the fundamental groupoid to the gravitational Aharonov-Bohm effect.

The present work is aimed at introducing the concept of the groupoid representation induced by a representation of the isotropy subgroupoid. We assume that the groupoid in question is a locally trivial topological groupoid and as a topological space it is a locally compact Hausdorff space (Section 4). This concept, framed “in the spirit of Mackey” is a natural generalization of induced representation of locally compact groups, created and investigated by him [15]. Representations, investigated in the present work, are unitary and are realized in Hilbert bundles over the unit spaces of a given groupoid [18]. Section 5 contains the formulation and the proof of the imprimitivity theorem for groupoids which is a generalization of the classical Mackey’s imprimitivity theorem for group representations [14], [15]. The theorem says
that every unitary groupoid representation, for which there exists the imprimitivity system, is a representation induced by a representation of the isotropy subgroupoid.

In Section 6, I investigate induced representations of the transformation groupoid over a homogeneous space of the group $G$ and show that there exists a strict connection between these representations and induced representations of the group $G$ (in the sense of Mackey).

In Section 7, I give a physical interpretation of concepts analyzed in Section 6. I describe the representation that has been used in the mentioned above model unifying gravity and quanta when this model is reduced (as the result of the act of measurement) to the usual quantum mechanics. And then I consider the energy-momentum space for a massive particle (it is a homogeneous space of the Lorentz group) and the transformation groupoid corresponding to this space. I also give a definition (in the sense of Mackey [14], [13]) of a particle as an imprimitivity system for the unitary representation of this groupoid.
2 Preliminaries

Let \( \mathcal{G} \) be a groupoid over a set \( X \) (the base of \( \mathcal{G} \)). We recall (cf. \([4], [18]\) ) that a groupoid \( \mathcal{G} \) is a set with a partially defined multiplication "\( \circ \)" on a subset \( \mathcal{G}^2 \) of \( \mathcal{G} \times \mathcal{G} \), and an inverse map \( g \to g^{-1} \) defined for every \( g \in \mathcal{G} \). The multiplication is associative when defined. One has an embedding \( \epsilon : X \to \mathcal{G} \) called the identity section and two structure maps \( d, r : \mathcal{G} \to X \) such that

\[
\epsilon(d(g)) = g^{-1} \circ g
\]

\[
\epsilon(r(g)) = g \circ g^{-1}
\]

for \( g \in \mathcal{G} \).

Let us introduce the following fibrations in the set \( \mathcal{G} \):

\[
\mathcal{G}_x = \{ g \in \mathcal{G} : d(g) = x \}
\]

\[
\mathcal{G}^x = \{ g \in \mathcal{G} : r(g) = x \}
\]

for \( x \in X \). Let us also denote \( \mathcal{G}_x^y = \mathcal{G}^x \cap \mathcal{G}_y \), and consider the set \( \mathcal{G}_x^x = \mathcal{G}^x \cap \mathcal{G}_x \) for \( x \in X \). It has the group structure and is called the isotropy group of the point \( x \). It is clear that the set \( \Gamma = \bigcup_{x \in X} \mathcal{G}_x^x \) has the structure of a subgroupoid of \( \mathcal{G} \) over the base \( X \) (all the structure maps are the restrictions of the structure maps of \( \mathcal{G} \) to \( \Gamma \)).
We call $\mathcal{G}$ a transitive groupoid, if for each pair of elements $x_1, x_2 \in X$ there exists $g \in \mathcal{G}$ such that $d(g) = x_1$ and $r(g) = x_2$.

A groupoid $\mathcal{G}$ is a topological groupoid if $\mathcal{G}$ and $X$ are topological spaces and all structure maps are continuous (in particular, the embedding $\epsilon$ is a homeomorphism of $X$ onto its image).

In the following we assume that $\mathcal{G}$ (and thus $X$) is a locally compact Hausdorff space.

**Example 1** A pair groupoid. Let $X$ be a locally compact Hausdorff space. Take $\mathcal{G} = X \times X$. We define the set $\mathcal{G}^2$ of composable elements as $\mathcal{G}^2 = \{((x, y), (y, z)) : x, y, z \in X\} \subset \mathcal{G} \times \mathcal{G}$ and a multiplication, for $((x, y), (y, z)) \in \mathcal{G}^2$, by

$$(x, y) \circ (y, z) = (x, z).$$

Moreover, we have: $(x, y)^{-1} = (y, x)$, $d(x, y) = y$, $r(x, y) = x$, $\epsilon(x) = (x, x)$. With such defined structure maps $\mathcal{G}$ is a groupoid, called pair groupoid.

**Example 2** A transformation groupoid. Let $X$ be a locally compact Hausdorff space, and $G$ a locally compact group. Let $G$ act continuously on $X$ to the right, $X \times G \to X$. Denote $(x, g) \mapsto xg$. We introduce the groupoid structure on the set $\mathcal{G} = X \times G$ by defining the following structure maps. The
set of composable elements \( G^2 = \{((xg,h),(x,g)) : x \in X, g, h \in G\} \subset G \times G \), and the multiplication for \(((xg,h),(x,g)) \in G^2\) is given by

\[(xg,h) \circ (x,g) = (x,gh).\]

And also \((x,g)^{-1} = (xg,g^{-1})\), \(d(x,g) = x\), \(r(x,g) = xg\), \(\epsilon(x) = (x,e_G)\). This groupoid is called the transformation groupoid.

Let us recall (cf. [18]) the concept of right Haar System.

**Definition 1** A right Haar system for the groupoid \( G \) is a family \( \{\lambda_x\}_{x \in X} \) of regular Borel measures defined on the sets \( G_x \) (which are locally compact Hausdorff spaces) such that the following three conditions are satisfied:

1. the support of each \( \lambda_x \) is the set \( G_x \),

2. (continuity) for any \( f \in C_c(G) \) the function \( f^0 \), where

\[ f^0(x) = \int_{G_x} f d\lambda_x, \]

belongs to \( C_c(X) \),

3. (right invariance) for any \( g \in G \) and any \( f \in C_c(G) \),

\[ \int_{G_{r(g)}} f(h \circ g) d\lambda_{r(g)}(h) = \int_{G_{d(g)}} f(u) d\lambda_{d(g)}(u). \]
One can also consider the family \( \{ \lambda^x \}_{x \in X} \) of left-invariant measures, each \( \lambda^x \) being defined on the set \( G^x \) by the formula \( \lambda^x(E) = \lambda_x(E^{-1}) \) for any Borel subset \( E \) of \( G^x \) (where \( E^{-1} = \{ g \in G : g^{-1} \in E \} \)). Then the invariance condition assumes the form:

\[
\int_{G^d(g)} f(g \circ h) d\lambda^{d(g)}(h) = \int_{G^r(g)} f(u) d\lambda^{r(g)}(u).
\]

Now, let \( \mu \) be a regular Borel measure on \( X \). We can consider the following measures which will be called measures associated with \( \mu \): \( \nu = \int \lambda_x d\mu(x) \) on \( G \), \( \nu^{-1} = \int \lambda_x d\mu(x) \) and \( \nu^2 = \int \lambda_x \times \lambda_x d\mu(x) \) on \( G^2 \).

If \( \nu = \nu^{-1} \) we say that the measure \( \mu \) is a \( G \)-invariant measure on \( X \).

**Definition 2** A topological groupoid \( G \) on \( X \) is called locally trivial if there exist a point \( x \in X \), an open cover \( \{ U_i \} \) of \( X \) and continuous maps \( s_{x,i} : U_i \to G_x \) such that \( r \circ s_i = id_{U_i} \) for all \( i \).

**Proposition 1** Assume that \( G \) is a locally trivial groupoid on \( X \) and \( X \) is second countable space. Let \( \mu \) be a regular Borel measure on \( X \). Then

1. \( G \) is transitive,

2. all isotropy groups of \( G \) are isomorphic with each other,

3. for every \( y \in X \) there exist an open cover \( \{ V_j \} \) of \( X \) and continuous maps \( s_{y,j} : V_j \to G_y \) such that \( r \circ s_j = id_{V_j} \),
4. For every \( x \in X \) there exists a section \( s_x : X \to \mathcal{G}_x \) which is \( \mu \)-measurable, i.e., for every Borel set \( B \) in \( \mathcal{G}_x \), \( s_x^{-1}(B) \) is \( \mu \)-measurable subset of \( X \).

5. If the measure \( \mu \) has the property that \( \mu(\overline{A}) = \mu(A) \) for every \( \mu \)-measurable subset \( A \) of \( X \), then the section \( s_x \) is \( \mu \)-a.e. continuous on \( X \).

Proof:

1. Let \( y_1, y_2 \in X \). Suppose that \( y_1 \in U_1 \) and \( y_2 \in U_2 \). Then \( r(s_1(y_1)) = y_1 \) and \( r(s_2(y_2)) = y_2 \). But \( g = s_2(y_2) \circ s_1(y_1)^{-1} \) has the property \( d(g) = y_1 \) and \( r(g) = y_2 \), and this means that \( \mathcal{G} \) is transitive.

2. For \( x, y \in X \) let \( g_{yx} \) be an element of \( \mathcal{G} \) such that \( d(g_{yx}) = x \) and \( r(g_{yx}) = y \). Then we have an isomorphism of the isotropy groups \( \mathcal{G}_x^x \) and \( \mathcal{G}_y^y \) given by the formula \( \mathcal{G}_x^x \ni \gamma \mapsto g_{yx} \circ \gamma \circ g_{yx}^{-1} \in \mathcal{G}_y^y \).

3. Let \( g_{xy} \) be an element of \( \mathcal{G} \) such that \( d(g_{xy}) = y \) and \( r(g_{xy}) = x \). Then in the fiber \( \mathcal{G}_y \) we can simply define \( s_{y,i}(z) = s_{x,i}(z) \circ g_{x,y} \) for \( z \in U_i \).

4. Since \( X \) is second countable space, we can take a countable cover \( \{U_i\}_{i=1,2,...} \) of the space \( X \). Let us define \( s_x(z) = s_1(z) \) for \( z \in U_1 \),
\( s_x(z) = s_2(z) \) for \( z \in U_2 \setminus U_1, \ldots, s_x(z) = s_n(z) \) for \( z \in U_n \setminus (U_1 \cup U_2 \cup \ldots U_{n-1}) \) etc.

It is easily seen that \( s_x(z) \) is measurable.

5. The set of discontinuity of \( s_x \) is contained in the union of sets \( \partial U_i = \overline{U_i} \setminus U_i, i = 1, 2, \ldots, \) which is of measure zero.

This ends the proof.

From now on we assume that considered groupoids are locally compact and satisfy the assumptions of Proposition 1. It is known that in the case of any locally trivial groupoid there exists a right Haar system (see [18]). Let us choose a collection of sections \( \{s_x\}_{x \in X} \), defined by Proposition 1.3, and denote by \( d\gamma_y \) the right Haar measure on the isotropy group \( \Gamma_y \). (From the assumption that the groupoid \( \mathcal{G} \) is locally compact it follows that all isotropy groups are locally compact and have Haar measures.)

**Definition 3** A right Haar system \( \{\lambda_x\}_{x \in X} \) on the groupoid \( \mathcal{G} \) is called consistent with a Borel regular measure \( \mu \) on the base space \( X \) if, for every \( x \in X \) and any \( f \in C_c(\mathcal{G}_x) \),

\[
\int_{\mathcal{G}_x} f(g)d\lambda_x(g) = \int_X \int_{\Gamma_y} f(\gamma \circ s_x(y))d\gamma_yd\mu(y).
\]
The above formula gives an explicit construction of the right Haar system for many classes of groupoids (see, Section 3 for pair groupoid, and Section 6 for transformation groupoid).

Let us recall the concept of groupoid representation \([18],[21]\). It involves a Hilbert bundle \(H\) over \(X\), \(H = (X, \{H_x\}_{x \in X}, \mu)\) (Dixmier \([3]\) uses for it the name of \(\mu\)-measurable field of Hilbert spaces over \(X\)). Here all Hilbert spaces \(H_x\) are assumed to be separable.

Let \(\mu\) be a \(G\)-invariant measure on \(X\), and \(\nu\) and \(\nu^2\) the associated measures on \(G\).

**Definition 4** A unitary representation of a groupoid \(G\) is the pair \((U, H)\) where \(H\) is a Hilbert bundle over \(X\) and \(U = \{U(g)\}_{g \in G}\) is a family of unitary maps \(U(g) : H_{d(g)} \to H_{r(g)}\) such that:

1. \(U(\epsilon(x)) = id_{H_x}\) for all \(x \in X\),

2. \(U(g) \circ U(h) = U(g \circ h)\) for \(\nu^2\)-a.e. \((g, h) \in G^2\),

3. \(U(g^{-1}) = U(g)^{-1}\) for \(\nu\)-a.e. \(g \in G\),

4. For every \(\phi, \psi \in L^2(X, H, \mu)\),

\[
G \ni g \to (U(g)\phi(d(g)), \psi(r(g)))_{r(g)} \in C
\]
is \( \nu \)-measurable on \( \mathcal{G} \). (Here \( L^2(X, H, \mu) \) denotes the space of square-integrable sections of the bundle \( H \), and \( (\cdot, \cdot)_x \) denotes the scalar product in the Hilbert space \( H_x \).)

## 3 Elementary properties of representations of groupoids.

**Definition 5** Unitary representations \((U_1, H_1)\) and \((U_2, H_2)\) of a groupoid \( \mathcal{G} \) are said to be unitarily equivalent if there exists a family \( \{A_x\}_{x \in X} \) of isomorphisms of Hilbert spaces \( A_x : H_{1x} \rightarrow H_{2x}, x \in X \) such that for every \( x, y \in X \) and for \( \nu \)-a.e. \( g \in \mathcal{G}_x^y \) the following diagram commutes

\[
\begin{array}{ccc}
H_{1x} & \xrightarrow{U_1(g)} & H_{1y} \\
\downarrow A_x & & \downarrow A_y \\
H_{2x} & \xrightarrow{U_2(g)} & H_{2y}
\end{array}
\]

**Definition 6** Let \((\mathcal{U}, H)\) be an unitary representation of \( \mathcal{G} \) and let \( H_1 \) be a Hilbert subbundle of \( H \). We say that \( H_1 \) is \( \mathcal{G} \)-invariant if \( U(g)H_{1,x} \subset H_{1,y} \) for every \( x, y \in X \) and for \( \nu \)-a.e. \( g \in \mathcal{G}_x^y \). Then the representation \((\mathcal{U}, H_1)\) is called a subrepresentation of \((\mathcal{U}, H)\). A subrepresentation \((\mathcal{U}, H_1)\) is called proper if \( H_1 \) is proper subbundle of \( H \), i.e. \( H_1 \neq H \) and \( H_1 \) is not null space bundle.
**Definition 7** A unitary representation $(\mathcal{U}, H)$ is called irreducible if it has no proper subrepresentations.

**Example 3** Let $H = X \times H$ be a trivial Hilbert bundle over $X$ with fiber $H$. For $g \in \mathcal{G}_x^n$, $x, y \in X$ define an operator of the representation $U(g) : \{(x) \times H \to \{(y) \times H \text{ by } U(g)(x, h) = (y, h). \text{ Such representation}

(U, H) \text{ of the groupoid } \mathcal{G} \text{ is called trivial representation. A trivial representation is irreducible if and only if it is one-dimensional, i.e., } \dim H = 1.\n
**Example 4** Let $H_x = L^2(\mathcal{G}_x, d\lambda_x)$, for $x \in X$, be a Hilbert space of square $\lambda_x$-integrable functions on $\mathcal{G}_x$, and for $g \in \mathcal{G}_x^n$, $x, y \in X$ and $f \in H_x$ define $U(g) : H_x \to H_y$ by

$$(U(g)f)(g_1) = f(g_1 \circ g),$$

for $g_1 \in \mathcal{G}_y$.

A representation $(\mathcal{U}, H)$ is called regular representation of the groupoid $\mathcal{G}$ [13].

Now let us consider the regular representation of a pair groupoid $\mathcal{G}_0 = X \times X$. Let $\mu$ be a regular Borel measure on $X$. Let us define a right Haar system of measures $\{\mu_x\}_{x \in X}$ on the pair groupoid $\mathcal{G}_0$, $\mu_x$ being given on $\mathcal{G}_{0,x} = X \times \{x\}$ by the formula $\mu_x(f) = \int_{\mathcal{G}_{0,x}} f(y, x) d\mu(y)$. 13
Then we have a simple invariance condition:

\[
\int_{G_{0,x}} f[(y, x) \circ (x, z)]d\mu(y) = \int_{G_{0,z}} f(y, z)d\mu(y).
\]

For each \(x \in X\) the Hilbert space \(L^2(G_{0,x}, d\mu_x)\) is obviously isomorphic to \(L^2(X)\).

**Example 5** The regular representation of a pair groupoid \(G_0\) in the Hilbert bundle \(\{L^2(G_{0,x})\}_{x \in X}\) over \(X\) is given by the following family of operators \(\{U_0(g)\}, \ g \in G_{0,x}, \ x, y \in X\)

\[
[U_0(g)f](z, y) = f(z, x)
\]

where \(z \in X\).

Let us observe that the regular representation of pair groupoid is equivalent to trivial representation in the trivial Hilbert bundle \(X \times L^2(X)\).

Now, we shall introduce the quotient groupoid \(G/\Gamma\) (cf. [12]) and consider its representations.

Let \(\Gamma\) be the isotropy groupoid of a groupoid \(G\). Let us define an equivalence relation \(\sim\) on \(G\), for \(g, h \in G\),

\[
g \sim h \iff \text{there exist } \gamma_1 \in \Gamma \text{ such that } (\gamma_1, g) \in G^2 \text{ and } \gamma_1 \circ g = h.
\]
Let us notice that if $\gamma_2 \in \Gamma_{d(g)}$ then also $g \sim g \circ \gamma_2$. Indeed, $g \circ \gamma_2 = g \circ \gamma_2 \circ g^{-1} \circ g = \gamma_1 \circ g$ where $\gamma_1 = g \circ \gamma_2 \circ g^{-1} \in \Gamma_{r(g)}$.

Denote the equivalence class of $g \in G$ by $[g]$, and the set of such equivalence classes by $G/\Gamma$. Then we can introduce the groupoid structure on $G/\Gamma$. The structure maps $\tilde{d}$ and $\tilde{r}$, the multiplication, the inverse and the identity section $\tilde{\epsilon}$ are given by $\tilde{d}[g] = d(g)$, $\tilde{r}([g]) = r(g)$, $[g] \circ [h] = [g \circ h]$, for $(g, h) \in G^2$, $[g]^{-1} = [g^{-1}]$, $\tilde{\epsilon}(x) = [\epsilon(x)]$, respectively.

It easy to see that the canonical projection $p : G \rightarrow G/\Gamma$ is a homomorphism of (topological) groupoids (in $G/\Gamma$ we choose the quotient topology). Let us denote by $G_0$ the pair groupoid $G_0 = X \times X$ over the base $X$. Recall that in $G_0$ we have $d_0(x, y) = y, r_0(x, y) = x, (x, y) \circ (y, z) = (x, z), (x, y)^{-1} = (y, x)$ and $\epsilon_0(x) = (x, x)$.

We observe that the quotient groupoid $G/\Gamma$ coincides with $G_0$.

**Proposition 2** The map $\Phi : G/\Gamma \rightarrow G_0$, given by $\Phi([g]) = (r(g), d(g))$, is an isomorphism of groupoids over $X$.

**Proof:** It is clear that, for $g, h \in G^2$, one has $\Phi([g] \circ [h]) = (r(g), d(h)) = (r(g), d(g)) \circ (r(h), d(h)) = \Phi([g]) \circ \Phi([h])$. Also $\Phi([g]^{-1}) = (d(g), r(g)) = (\Phi([g]))^{-1}$. Thus $\Phi$ is a groupoid homomorphism. It is clear that $\Phi$ maps $G/\Gamma$ onto $G_0$. Moreover, if $\Phi([g])$ is a unit element in $G_0$, i.e., $\Phi([g]) = (x, x)$
for an element \( x \in X \), then \( d(g) = r(g) = x \), i.e., \( g \in \Gamma \) and \([g]\) in a unit in \( G/\Gamma \). This means that \( \Phi \) is an isomorphism. 

Now, let us assume that a representation \((U, \mathcal{H})\) of the groupoid \( G \) is \( \Gamma \)-invariant, i.e., \( U(\gamma \circ g) = U(g) \) for every \( g \in G \), \( \gamma \in \Gamma \), \((\gamma, g) \in G^2\). Then it is easily seen that one has a unitary representation \((U_0, \mathcal{H}) = \{U_0([g])\}_{[g] \in G/\Gamma}, \mathcal{H}\) of the groupoid \( G/\Gamma \) formed by the family of operators

\[
U_0([g]) = U(g) : H_x \to H_y
\]

for every \( g \) such that \( d(g) = x \), \( r(g) = y \).

**Example 6** Let \( H_x = L^2_0(G_x) \) be a Hilbert space of \( \Gamma \)-invariant and \( \mu \)-square integrable functions on \( G_x \), i.e. such that, for Borel-measurable functions \( f \) on \( G_x \), \( f(\gamma \circ g) = f(g) \) for every \( g \in G_x \), \( \gamma \in \Gamma \), \((\gamma, g) \in G^2\), and \( \int_X |f(g)|^2d\mu(r(g)) < \infty \). It is clear that the space \( L^2_0(G_x) \) is isomorphic to \( L^2(X) \). Define for every \( g \in G'_x \), \( x, y \in X \) an operator \( U(g) : H_x \to H_y \) by

\[
(U(g)f)(g_1) = f(g_1 \circ g)
\]

for \( f \in H_x \), \( g_1 \in G_y \). In such a manner we obtain a \( \Gamma \)-invariant unitary representation of the groupoid \( G \) which is called a quasi-regular representation. Let us observe that the corresponding representation \( U_0 \) of the quotient
groupoid $G/\Gamma$ coincides with the regular representation of the pair groupoid $G_0$. 

4 Induced representations of the groupoid $\mathcal{G}$.

In this section, we define the representation of $\mathcal{G}$ induced by a representation of the isotropy subgroupoid $\Gamma$. From now on we assume that on the groupoid $\mathcal{G}$ there exists a right Haar system $\{\lambda_x\}_{x \in X}$ consistent with Borel regular measure $\mu$ on $X$.

First, we have to construct an appropriate Hilbert bundle.

Assume that there is given a unitary representation $(\tau, W)$ of the subgroupoid $\Gamma$. Here $W$ is a Hilbert bundle over $X$. Let $W_x$ denote a fiber over $x \in X$ which is a Hilbert space with the scalar product $\langle \cdot, \cdot \rangle_x$, and let $W = \bigcup_{x \in X} W_x$ denote the total space of the bundle $W$.

Let us define, for every $x \in X$, the space $W_x$ of $W$-valued functions $F$ defined on the set $\mathcal{G}_x$ satisfying the following four conditions:

1. $F(g) \in W_{r(g)}$ for every $g \in \mathcal{G}_x$,

2. for every $\mu$-Borel measurable $r$-section $s_x : X \to \mathcal{G}_x$ (see Proposition 1) the composition $F \circ s_x$ is a $\mu$-measurable section of the bundle $W$,
3. \( F(\gamma \circ g) = \tau(\gamma) F(g) \) for \( g \in \mathcal{G}_x, \, \gamma \in \Gamma_{r(g)} \);

4. \( \int \langle F(s_x(y)), F(s_x(y)) \rangle_y d\mu(y) < \infty \).

If we identify two functions \( F, F' \in \mathcal{W}_x \) satisfying

\[
\int \langle (F - F')(s_x(y)), (F - F')(s_x(y)) \rangle_y d\mu(y) = 0,
\]

we can introduce the scalar product \( \langle \cdot, \cdot \rangle_x \) in the space \( \mathcal{W}_x \)

\[
(F_1, F_2)_x = \int \langle F_1(s_x(y)), F_2(s_x(y)) \rangle_y d\mu(y)
\]

where \( s_x \) is the section determined by Proposition 1, part 3.

The spaces \( \mathcal{W}_x, \, x \in X \), with these scalar products are Hilbert spaces. It is easily seen that they are isomorphic to the Hilbert space \( L^2(X, \mathcal{W}) \) of square-integrables sections of the bundle \( \mathcal{W} \). Now, let us denote \( \mathcal{W} = \{ \mathcal{W}_x \}_{x \in X} \).

It is a Hilbert bundle over \( X \). We define a unitary representation of the groupoid \( \mathcal{G} \) in the Hilbert bundle \( \mathcal{W} \) in the following way

**Definition 8** The representation of the groupoid \( \mathcal{G} \) induced by the representation \((\tau, \mathcal{W})\) of the subgroupoid \( \Gamma \) is the pair \((U^\tau, \mathcal{W})\) where, for \( g \in \mathcal{G}_x^y \), we define \( U^\tau(g) : \mathcal{W}_x \to \mathcal{W}_y \) by

\[
(U^\tau(g_0)F)(g) = F(g \circ g_0).
\]
It is clear that \((U^\tau, \mathcal{W})\) is a unitary groupoid representation.

Sometimes we shall use the notation \(U^\tau = \text{Ind}^G_\Gamma(\tau)\).

5 Systems of imprimitivity.

For a given Hilbert space \(W_0\) we can consider the Hilbert space \(L^2(X, W_0)\) of square integrable \(W_0\) - valued functions on \(X\). In the space \(L^2(X, W_0)\) one has a representation of the commutative algebra \(L^\infty(X)\) given by the multiplication operators by the function: \(L^\infty(X) \ni f \mapsto \pi_0(f) \in B(L^2(X, W_0))\) where, for \(z \in X\),

\[
[\pi_0(f)\psi](z) = f(z)\psi(z).
\]

We shall call \(\pi_0\) the natural representation of \(L^\infty(X)\) in \(L^2(X, W_0)\).

Now, let us consider a representation \(U\) of the groupoid \(\mathcal{G}\) in a Hilbert bundle \(H\) over \(X\). We assume that for \(\mu\) - a.e. \(x \in X\) there exists a Hilbert space \(W_x\) with a scalar product \((\cdot, \cdot)_x\) such that the spaces \(W_x\) are isomorphic with each other. Let us assume that, for \(\mu\) - a.e. \(x\), the fiber \(H_x\) of the bundle \(H\) is isomorphic to \(L^2(X, W_x)\). We shall simply write \(H_x = L^2(X, W_x)\), and \(U(g) : L^2(X, W_x) \to L^2(X, W_y)\) for \(g \in \mathcal{G}^y_x\). It is clear that the collection of the spaces \(\{L^2(X, W_x)\}_{x \in X}\) forms a Hilbert bundle over \(X\) which is
isomorphic to the bundle $H$.

**Definition 9** We say that there exists a system of imprimitivity $(U, \pi)$ for the representation $(U, H)$ of the groupoid $G$ if

1. the representation $(U, H)$ satisfies the above assumption ($H_x = L^2(X, W_x)$ for $\mu$-a.e. $x \in X$),

2. $\pi = (\pi_x)_{x \in X}$ is the family of natural representations of the algebra $L^\infty(X)$ in the Hilbert spaces $L^2(X, W_x)$,

3. for every $f \in L^\infty(X)$, and for $\mu$-a.e. $x, y \in X$, and $\nu$-a.e. $g \in G_y^y$

$$U(g)\pi_x(f)U(g^{-1}) = \pi_y(f).$$

**Example 7** Let $(U, H)$ be the quasi-regular representation of the groupoid $G$, defined in Example 6. Then there exists a system of imprimitivity $(U, \pi)$ for $U$. Indeed, for $f \in L^\infty(X)$, $\psi \in H_y$, $g \in G_y^y$, $h \in G_y$, we have

$$(U(g)\pi_x(f)U(g^{-1})\psi)(h) = (\pi_x(f)U(g^{-1})\psi)(h \circ g) =$$

$$= f(r(h \circ g))(U(g^{-1})\psi)(h \circ g) = f(r(h))\psi(h) = \pi_y(f)\psi(h).$$

The quasi-regular representation can be understood as induced by a trivial one-dimensional representation of the subgroupoid $\Gamma$. 

20
We are now in a position to state our main theorem (the imprimitivity theorem for groupoids):

**Theorem 1** If, for a representation \((U, H)\), there exists a system of imprimitivity \((U, \pi)\) then the representation \(U\) is equivalent to the representation \(U^\tau\) induced by some representation \((\tau, W)\) of the subgroupoid \(\Gamma\).

Let us observe that, for \(\gamma \in \Gamma_x = \mathcal{G}_x^x\), condition 3 of the Definition 9 of the imprimitivity system reduces to the following one

\[U(\gamma)\pi_x(f)U(\gamma^{-1}) = \pi_x(f).\]

Let us denote by \(\mathcal{M}_{0,x}\) the following subalgebra in \(B(H_x) = B(L^2(X, W_x))\):

\[\mathcal{M}_{0,x} = \{U(\gamma) : \gamma \in \Gamma_x\}.\]

Then we have, for \(\mu\) - a.e. \(x \in X\), \(\pi_x(L^\infty(X)) \subset \mathcal{M}_{0,x}'\), where \(\mathcal{M}_{0,x}'\) denotes the commutant of the algebra \(\mathcal{M}_{0,x}\) in \(B(H_x)\).

**Definition 10** The system of imprimitivity \((U, \pi)\) is irreducible if, for \(\mu\) - a.e. \(x \in X\), \(\pi_x(L^\infty(X)) = \mathcal{M}_{0,x}'\).

**Theorem 2** If for a representation \((U, H)\) there exists an irreducible system of imprimitivity \((U, \pi)\) then the representation \(U\) is equivalent to the representation \(U^\tau\) induced by some irreducible representation \((\tau, W)\) of the subgroupoid \(\Gamma\).
First, let us notice that from the fact that all operators \( U(\gamma), \gamma \in \Gamma_x \), commute with all \( \pi_x(f), f \in L^\infty(X) \), it follows that \( U(\gamma) \) are decomposable (see [5], part II, 2.5). This means that, for \( \mu - \text{a.e.} \ y \in X \), there exists an operator \( U(\gamma)_y \in B(W_x) \) such that, for \( \psi \in L^2(X,W_x) \), \( (U(\gamma)\psi)(y) = U(\gamma)_y(\psi(y)) \). It is easily seen that all \( U(\gamma)_y \) are unitary (cf. [5], II.2, Ex.2).

We can prove even more.

**Lemma 1** If for a representation \((U,H)\) there exists a system of imprimitivity, then

1. there exists a unitary representation \((\tau_x,W_x)\) of the group \(\Gamma_x\) such that \(\tau_x(\gamma) = U(\gamma)\) for every \(\gamma \in \Gamma_x\) and \(\mu - \text{a.e.} \ x \in X\). (In particular it means that the function \(X \ni y \to U(\gamma)_y \in B(H_x)\) is a constant field of operators),

2. we can define a representation \((\tau,W)\) of the subgroupoid \(\Gamma\) such that, for \(\gamma \in \Gamma_x\), \(\tau(\gamma) = \tau_x(\gamma)\),

3. if the system of imprimitivity \((U,\pi)\) is irreducible then \((\tau,W)\) is an irreducible representation of \(\Gamma\), i.e., for \(\mu - \text{a.e.} \ x \in X\), the representations \((\tau_x,W_x)\) of the groups \(\Gamma_x\) are irreducible.
**Proof:** Notice that the Hilbert space $L^2(X,W_x)$ is isomorphic to the tensor product of Hilbert spaces $L^2(X) \otimes W_x$. A decomposable operator in such a space has the form $[A(\psi \otimes h)](y) = \psi(y) \otimes A_y h$. We have to show that it is of the form $id_{L^2} \otimes A_0$, where $A_0 \in B(W_x)$. Let $\{\psi_i\}_{i=1,2,...}$ be an orthonormal basis of the space $L^2(X)$. Consider the unitary operators $U_{ij}$ in the space $L^2(X)$ defined by $U_{ij}\psi_j = \psi_i$. If $A$ is decomposable then $A$ commutes with all operators of the form $U_{ij} \otimes id_{H_x}$. Then it follows that $A = id_{L^2} \otimes A_0$ by Lemma 2 ([3], section I.2.3). For the operators $U(\gamma), \gamma \in \Gamma_x$ let us denote by $\tau_x(\gamma)$ the operators $W_x \to W_x$ such that $U(\gamma) = id_{L^2} \otimes \tau_x(\gamma)$. It is clear that all $\tau_x(\gamma)$ are unitary in $W_x$, and $\tau_x(\gamma_1 \circ \gamma_2) = \tau_x(\gamma_1) \circ \tau_x(\gamma_2)$ for $\gamma_1, \gamma_2 \in \Gamma$. Thus $\tau_x$ is a unitary representation of the group $\Gamma_x$ in the Hilbert space $W_x$. This ends the proof of part 1.

Now the assertion 2 of the Lemma is obvious.

To obtain part 3 it is sufficient to see that the condition of irreducibility of the imprimitivity system implies that only operators of the form $\lambda id_{W_x}$, $(\lambda \in \mathbb{C})$ commute with all $\tau_x(\gamma), \gamma \in \Gamma_x$. But by Schur’s lemma it follows that the representation $\tau_x$ of $\Gamma_x$ is irreducible. ⋄

The next lemma gives us more properties of the representation $(\tau, W)$ of the groupoid $\Gamma$ as well as of the representation $(U, H)$ that has a system of
Lemma 2

1. The representations $\tau_x, x \in X$ are equivalent to each other, as representations of isomorphic groups $\Gamma_x$.

2. The operators $U(g) : H_x \to H_y$, where $H_x = L^2(X, W_x)$, $H_y = L^2(Y, W_y)$ for $g \in G^y_x$, are decomposable, i.e., there exist unitary operators $U^0(g) : W_x \to W_y$ such that for $\psi \in L^2(X, W_x)$ and, for $z \in X$,

$$(U(g)\psi)(z) = (U^0(g))(\psi(z)).$$

Moreover, the operator $U^0(g) : W_x \to W_y$ does not depend of $z \in X$.

Proof: First we shall prove part 2. First of all, let us notice that all spaces $W_x$, for $\mu$-a.e. $x \in X$, are isomorphic to each other as Hilbert spaces. Denote by $i^y_x : W_x \to W_y$ the isomorphism and define the unitary map $R^y_x : L^2(X, W_x) \to L^2(X, W_y)$ by $(R^y_x \psi)(z) = i^y_x(\psi(z))$, $\psi \in L^2(X, W_x), z \in X$.

Consider the composition of unitary maps $U(g) \circ (R^y_x)^{-1} : L^2(X, W_y) \to L^2(X, W_y)$ where $g \in G^y_x$. By using the property of the imprimitivity system for $U(g)$, we obtain

$$U(g) \circ (R^y_x)^{-1} \circ \pi_y(f) = \pi_y(f) \circ U(g) \circ (R^y_x)^{-1}.$$
for $f \in L^\infty(X)$.

This means that the operator $U(g) \circ (R^y_x)^{-1}$ is decomposable in $L^2(X, W_y)$. But $(R^y_x)$ is a decomposable map by definition, therefore $U(g)$ is decomposable as the composition of decomposable maps. As in the proof of Lemma 1 we conclude that $U^0(g)$ does not depend of $z \in X$ and is unitary.

To prove part 1 let us first observe that the isotropy groups $\Gamma_x$ are isomorphic to each other $x \in X$. Indeed, taking an element $g \in G^y_x$ we define the isomorphism $i : \Gamma_x \to \Gamma_y$ by the formula $i(\gamma) = g \circ \gamma \circ g^{-1}$ for $\gamma \in \Gamma_x$. Now, we have $U(i(\gamma)) = id_{L^2} \otimes \tau_y(i(\gamma))$ as in the proof of Lemma 1. On the other hand, $U(i(\gamma)) = U(g) \circ U(\gamma) \circ U(g^{-1}) = (id_{L^2} \otimes U^0(g)) \circ (id_{L^2} \otimes \tau_x(\gamma)) \circ (id_{L^2} \otimes U^0(g)^{-1}) = id_{L^2} \otimes (U^0(g) \circ \tau_x(\gamma) \circ U^0(g)^{-1})$. Therefore, we have $\tau_y(i(\gamma)) = U^0(g) \circ \tau_x(\gamma) \circ U^0(g)^{-1}$, but this means that the representations $\tau_y$ and $\tau_x$ are equivalent.

Now, we are in a position to give proofs of Theorems 1 and 2.

**Proof.** Let us consider the spaces $\{W_x\}_{x \in X}$, introduced in Section 1, connected to the representation $\tau$ of Lemma 1 and the corresponding induced representation $U^\tau$. We shall show that the representation $(U, H)$ is equivalent to $(U^\tau, W)$. We define a family of isomorphisms of Hilbert spaces $J_x : H_x \to
\(\mathcal{W}_x\) for \(\mu\) - a.e. \(x \in X\). Since \(H_x = L^2(X, W_x)\), for \(\psi \in H_x, g \in \mathcal{G}_x,\) and \(r(g) = y\), we put \(F(g) = (J_x \psi)(g) = (U(g)(\psi))(y)\). The definition is correct since by Lemma 2 we have \((U(g)\psi)(y) = U^0(g)(\psi(y))\), and \(U^0(g)\) does not depend on \(y \in X\). Since \(U(g)\psi \in L^2(X, W_y)\), therefore \([U(g)(\psi)](y) \in W_y\). Also it is clear that \(F(\gamma \circ g) = \tau(\gamma)(F(g))\) for \(\gamma \in \Gamma_y\). To see the square-integrability let us write

\[
\int \langle F(s_x(y)), F(s_x(y)) \rangle_y d\mu(y) =
\]

\[
= \int \langle U^0(s_x(y))(\psi)(y), U^0(s_x(y))(\psi)(y) \rangle_y d\mu(y) = \int \langle \psi(y), \psi(y) \rangle_y d\mu(y) =
\]

\[
= \| \psi \|_{H_x} < \infty.
\]

This also shows that \(J_x\) are unitary maps and are injective. To see that \(J_x\) map onto \(\mathcal{W}_x\), we can give the formula for \(J_x^{-1}: (J_x^{-1}F)(y) = (U^0(g))^{-1}(F(g))\) where \(F \in \mathcal{W}_x\) and \(g \in \mathcal{G}_x^y\). Then the right-hand side does not change if we take other element \(g_1 \in \mathcal{G}_x^y\). Indeed, since \(g_1 = \gamma \circ g\), for an element \(\gamma \in \Gamma_y\), therefore we have \((U^0(\gamma \circ g))^{-1}(F(\gamma \circ g)) = ((U^0(g))^{-1}(\tau(\gamma))^{-1}(\tau(\gamma))(F(g)) = (U^0(g))^{-1}(F(g))\). This shows that \(J_x\), \(x \in X\), are isomorphisms of Hilbert spaces. Now we can see that \(J_x\) are intertwining maps for the representations \(U\) and \(U^\tau\), i.e., that the following
diagram commutes

\[
\begin{array}{c}
H_x \xrightarrow{U(g)} H_z \\
\downarrow J_x & \downarrow J_z \\
W_x \xrightarrow{U^*(g)} W_z
\end{array}
\]

for \( \mu \text{-a.e. } x, z \in X \) and \( \nu \text{- a.e. } g \in \mathcal{G}^z_x \). Let \( \psi \in H_x \). Then, for \( h \in \mathcal{G}^y_z \), we have \( [(J_z U(g)) (\psi)](h) = [(U(h)(U(g))(\psi)](y) = U(h \circ g)(\psi(y)) = U^0(h \circ g)(\psi(y)) \). On the other hand, \( U^*(g) J_x(\psi)(h) = [J_x(\psi)](h \circ g) = [U(h \circ g)(\psi)](y) \). This ends the proof of Theorem 1.

The theorem 2 is now a simple consequence of Theorem 1 and Lemma 1, part 3.

6 Representations of the transformation groupoid \( \mathcal{G} = X \times G, X = K \backslash G \)

As an introduction to this section we recall the concept of induced representation in Mackey sense (cf [15], [13], [22]) of a Lie group \( G \) by a unitary representation \((L,V)\) of its closed subgroup \( K \) defined in a Hilbert space \( V \).

We assume, for simplicity, that \( X = K \backslash G \) has a \( G \)-invariant measure \( \mu \). We consider \( \mathcal{H}_L \), a Hilbert space consisting of measurable functions \( \phi \) on \( G \)
with values in $V$, such that

$$
\phi(hg) = L(h)\phi(g), \ h \in K,
$$

and

$$
\int_X ||\phi([g])||_V^2 d\mu([g]) < \infty
$$

where $[g]$ denotes the image of $g$ in $X$ under the projection $G \to K\backslash G = X$.

We introduce the inner product

$$(\phi_1, \phi_2)_{\mathcal{H}_L} = \int_X (\phi_1(x), \phi_2(x))_V d\mu(x).$$

Then we define the representation $U^L$ of $G$ on $\mathcal{H}_L$ given by the formula

$$(U^L(g)f)(g_0) = f(g_0g), \ g_0, g \in G, \ f \in \mathcal{H}_L.$$

It is easily seen that $U^L$ is unitary. The representation $(U^L, \mathcal{H}_L)$ is called induced by the representation $L$ of $K$.

Let $G$ be a noncompact Lie group and $K$ its compact subgroup. We assume that $G$ is unimodular. Then the homogeneous space $X = K\backslash G$ is a $G$-manifold with right action of the group $G$: $X \times G \ni (x, g) \mapsto xg \in X$.

As above, we assume that there exists a $G$-invariant measure $\mu$ on the space $X$, i.e., for $f \in C_c(G)$ we have

$$
\int f(xg)d\mu(x) = \int f(x)d\mu(x).
$$
We shall consider the structure of transformation groupoid on $G = X \times G$ (cf. Example 2) and construct a right Haar system on $G$ consistent with the measure $\mu$.

Let us denote

$$G_x = \{(x, g) \in G : g \in G\},$$

$$G^y = \{(yg^{-1}, g) \in G : g \in G\}.$$

Let us also denote the isotropy group $G^x_x$ by $\Gamma_x$, $\Gamma_x = \{(x, k) : k \in K_x\}$, where $K_x$ is a subgroup of $G$ of the form $K_x = g_0^{-1}Kg_0$ where $g_0 \in G$ is an element of the coset $x \ (x = [g_0])$. Indeed, for $k_x \in K_x$ we have $xk_x = [g_0]g_0^{-1}kg_0 = [kg_0] = x$.

**Lemma 3** *Let $s_0$ be a Borel section of the principal bundle $G \to K \backslash G = X$, i.e., $[s_0(x)] = Ks_0(x) = x$. Then

1. For every $x \in X$ there exists a section $s_x : X \to G_x$ with respect to the map $r$, i.e., $r(s_x(y)) = y$,

2. Every element $g \in G_x$ can be represented as $g = k \circ s_x(y)$ where $k \in \Gamma_y = G^y_y$.***

**Proof:**
1. Let \( o \in X \) denotes the origin point, i.e., \( o = [k], k \in K \). Then we have 
\[ kso(x) = x \] or, equivalently, \( os_0(x) = x \). Analogously, \( os_0(y) = y \) for \( y \in X \). Thus we can define the section \( s_x : X \to G_x \) by the formula 
\[ s_x(y) = (x, s_0(x)^{-1}s_0(y)) \]. It is clear that \( r(s_x(y)) = y \).

2. Let us observe that the product \( k \circ s_x(y) \), where \( k \in G_y \), is of the form 
\[ (y, s_0(y)^{-1}ks_0(y)) \circ (x, s_0(x)^{-1}s_0(y)) = (x, s_0(x)^{-1}ks_0(y)). \]

We have to show that if \( g = (x, g) \) with \( g \) such that \( xg = y \), then there exists \( k \in K \) such that \( g = s_0(x)^{-1}ks_0(y) \) or, equivalently, 
\[ s_0(x)gs_0(y)^{-1} = k. \] Now, \( os_0(x)gs_0(y)^{-1} = xgs_0(y)^{-1} = ys_0(y)^{-1} \). But the isotropy group of the origin point \( o \) is equal to \( K \), what means \( s_0(x)gs_0(y)^{-1} \in K \).

Now, for a function \( f \in C_c(G_x) \), let us define \( f_x(y, k) = f(x, s_0(x)^{-1}ks_0(y)) \), and 
\[ \int_{G_x} f(g) d\lambda_x(g) = \int_X \int_K f_x(y, k) dk d\mu(y). \]

**Proposition 3** The collection \( \{\lambda_x\}_{x \in X} \) is a right Haar system on the groupoid \( G \) consistent with the measure \( \mu \) on \( X \).
Proof: We have to show the right-invariance of the system $\{\lambda_x\}$. Let $z = r(g_0)$, $g_0 = (x, g_0)$ such that $xg_0 = z$ and let us compute

$$\int_{g_z} f(g \cdot g_0) d\lambda_z(g) = \int_X \int_K f([z, s_0(z)^{-1}k s_0(y)] \circ (x, g_0)) dk d\mu(y) =$$

$$\int \int f(x, g_0 s_0(z)^{-1}k s_0(y)) dk d\mu(y).$$

Let us observe that $os_0(x)g_0s_0(z)^{-1} = zs_0(z)^{-1} = o$, thus $s_0(x)g_0s_0(z)^{-1} \in K$ and $s_0(x)g_0s_0(z)^{-1}k = k_1 \in K$. But this means that $g_0s_0(z)^{-1}k = s_0(x)^{-1}k_1$. Thus, continuing the computation, we have

$$\int \int f(x, g_0 s_0(z)^{-1}k s_0(y)) dk d\mu(y) = \int \int f(x, s_0(x)^{-1}k_1 s_0(y)) dk_1 d\mu(y) =$$

$$\int_{g_z} f(g) d\lambda_z(g). \quad \diamond$$

Now, we shall consider representations of the isotropy subgroupoid $\Gamma$. As we have seen, $\Gamma = \bigcup_{x \in X} \{x\} \times K_x$ with $K_x = g^{-1}Kg$ and $g \in G$ such that its coset in $X$ is equal to $x$ ($[g] = x$). We can use $g = s_0(x)$.

Let $(\tau, W)$ be a unitary representation of the groupoid $\Gamma$ in a Hilbert bundle $W = \{W_x\}_{x \in X}$.

**Definition 11** A representation $(\tau, W)$ is called $X$-consistent if there exist a unitary representation $(\tau_0, W_0)$ of the group $K$ and a family of Hilbert space
isomorphisms

\[ A_x : W_0 \rightarrow W_x, \ x \in X \]

such that, for \( \gamma \in \Gamma_x \) of the form \( \gamma = (x, s_0(x)^{-1}k s_0(x)) \),

\[ \tau(\gamma) = A_x \tau_0(k) A_x^{-1}. \]

**Proposition 4** Let \((U, W)\) be a unitary representation of the groupoid \( \mathcal{G} \).

Then the restriction \((\tau, W)\) to the subgroupoid \( \Gamma \) of the representation \((U, W)\), given by the formula \( \tau(\gamma) = U(\gamma) \) for \( \gamma \in \Gamma \), is a \( X \)-consistent representation of \( \Gamma \).

**Proof.** We can write \( \gamma = (x, s_0(x)^{-1}k s_0(x)) = (o, s_0(o)) \circ (o, k) \circ (o, s_0(x))^{-1}. \) Then \( U(\gamma) = U((o, s_0(o)))U((o, k))U((o, s_0(x)))^{-1}. \) Let us denote \( A_x = U((o, s_0(o))) \) and \( \tau_0(k) = U((o, k)) \), \( W_0 = W_o. \) Then it is clear that \( \{A_x\}_{x \in X} \) and \( \tau_0 \) satisfies the conditions of \( X \)-consistent representation.

\( \diamond \)

In the sequel we shall consider the representation of the groupoid \( \mathcal{G} = X \times G \) induced by \( X \) - consistent representation \((\tau, W)\) of the subgroupoid \( \Gamma \), and we shall establish its connection with the induced representation in the Mackey sense of the group \( G \). We use the notation of section 3. Now
condition 3 of the definition of the space $W_x$ assumes the form

$$F(\gamma \circ (x, g)) = \tau(\gamma)F(x, g)$$

where $x, y \in X$, $y = xg$, $g \in G$, $\gamma \in \Gamma_y = \{y\} \times K_y$. Thus we have $\gamma = (y, s_0(y)^{-1}ks_0(y))$ for an element $k \in K$. Then, by the definition of $X$-consistent representation, we can write

$$F(\gamma \circ (x, g)) = (A_y\tau_0(k)A_y^{-1})F(x, g).$$

Let introduce a function $\phi : G \to W_0$ defined by the formula $\phi(ks_0(y)) = A_y^{-1}(F(x, s_0(x)^{-1}ks_0(y)))$. Then the function $\phi$ has the property $\phi(kg) = \tau_0(k)\phi(g)$.

It is sufficient to check the above formula for $g = s_0(y)$. If $\gamma = (y, s_0(y)^{-1}ks_0(y))$ then we have $\phi(kg) = A_y^{-1}(F(\gamma \circ (x, s_0(x)^{-1}s_0(y)))) = A_y^{-1}(A_y\tau_0(k)A_y^{-1})F(x, s_0(y)) = \tau_0(k)\phi(g)$.

We shall use the notation $(L, W_0)$ for the unitary representation of the group $K$ in the space $W_0$, $L = \tau_0$. Thus we have $\phi(kg) = L(k)\phi(g)$ and we can consider the Hilbert space $\mathcal{H}_L$ introduced above as well as the representation $(U^L, \mathcal{H}_L)$ of the group $G$ induced in the sense of Mackey by $L$ from the subgroup $K$.

The following theorem establishes a connection of the induced represen-
tation \((U^r, \mathcal{W})\) of the groupoid \(G\) with the representation \((U^L, \mathcal{H}_L)\) of the group \(G\).

Denote by \(R_g, g \in G\), the following operator acting in the space \(\mathcal{W}_x, x \in X, y = xg,\)

\[
(R_g F)(x, h) = (A_{xh}A_{xh}^{-1})(F(x, hg)).
\]

Then we have the family of unitary \(G\)-representations \((R, \mathcal{W}_x), x \in X\). (The unitarity follows from the fact that the measure \(\mu\) is \(G\)-invariant and the operators \(A_{xh}, A_{xhg}\) are Hilbert space isomorphisms.)

**Theorem 3**

1. For every \(x \in X\) the \(G\)-representation \((R, \mathcal{W}_x)\) is unitarily equivalent to the induced representation \((U^L, \mathcal{H}_L)\).

2. All representations \((R, \mathcal{W}_x), x \in X\), are unitarily equivalent to each other. The equivalence is given by the operators \(I^y_x : \mathcal{W}_x \rightarrow \mathcal{W}_y,\)

\[
(I^y_x F)(y, s_0(y)^{-1}k s_0(z)) = (A_y A_x^{-1})(F(x, s_0(x)^{-1}k s_0(z))),
\]

\(x, y \in X.\)

**Proof.**
1. We define the linear map $J_x: \mathcal{W}_x \to \mathcal{H}_L$ by $(J_x F)(g) = \phi(ks_0(y)) = A_y^{-1}(F(x, s_0(x)^{-1}k) s_0(y))$ where $g = k s_0(y)$. $J_x$ is a linear isomorphism since $A_y$ is an isomorphism and it is easily seen that $J_x$ preserves scalar products of $\mathcal{W}_x$ and $\mathcal{H}_L$ and so it is a Hilbert space isomorphism. To see that it defines an equivalence of representations, we have to show that, for $g \in G$, the following diagram is commutative

\[
\begin{array}{ccc}
\mathcal{W}_x & \xrightarrow{R_g} & \mathcal{W}_x \\
\downarrow J_x & & \downarrow J_x \\
\mathcal{H}_L & \xrightarrow{U^L(g)} & \mathcal{H}_L
\end{array}
\]

Let us compute $(U^L(g)J_x)(F)(h)$. It is sufficient to take $h = s_0(y)$ and to notice that each $g \in G$ can be written in the form $g = s_0(y)^{-1}ks_0(z)$, for $z \in X, z = yg$ and an element $k \in K$.

\[
(U^L(s_0(y)^{-1}ks_0(z))J_x)(F)(s_0(y)) = (J_x F)(s_0(y)) = A_y^{-1}((R_g F)(x, s_0(x)^{-1}s_0(y))) = L(k)A_z^{-1}(F(x, s_0(x)^{-1}s_0(z))).
\]

On the other hand

\[
(J_x R_g)(F)(s_0(y)) = A_y^{-1}((R_g F)(x, s_0(x)^{-1}s_0(y))) = A_y^{-1}(A_y A_z^{-1}(F(x, s_0(x)^{-1}k) s_0(z))) = A_z^{-1}(A_z \gamma_0(k) A_z^{-1}(F(x, s_0(x)^{-1}s_0(z)))) = L(k)A_z^{-1}(F(x, s_0(x)^{-1}s_0(z))).
\]
2. Now it is a simple observation that $I_x^y = J_y^{-1} J_x$.

7 A physical picture. A concept of particle
in the representation theory framework

In papers ([7], [9], [8], [19]) we have studied a model unifying general relativity and quantum mechanics based on noncommutative geometry. The principal structure of the model is provided by a transformation groupoid $G = E \times G$ where $G$ is the Lorentz group and $E$ is the principal $G$-bundle over the spacetime $M$ (the total space of the bundle is formed by all Lorentz frames at all points of $M$). We have defined a right action of $G$ on $E$, and the multiplication of elements of the groupoid is introduced as follows

$$(pg, g_1) \circ (p, g) = (p, gg_1),$$

$p \in E, g, g_1 \in G$.

The model is reduced to the usual quantum mechanics when an act of measurement is performed. Then we choose a frame $p \in E$ which represents a reference frame in which the measurement is done. In the sequel we consider the situation when we want to observe a particle from a different reference
frame situated at a fixed point \( x \in M \). In such a case, our groupoid reduces to the groupoid \( G = E_x \times G \) where \( E_x \) is the fiber of the bundle \( E \) over \( x \). The groupoid \( G \) is transitive and its isotropy subgroupoid is trivial \( \Gamma = E_x \times \{ e \} \), where \( e \in G \) is the neutral element of the group \( G \).

A representation \( (\tau, W) \) of \( \Gamma \) is realized in a trivial Hilbert bundle \( W = \{ W_p \}_{p \in E_x} \) with \( W_p = \mathbb{C} \) and \( \tau(\gamma) = id_{W_p} \) for \( \gamma = (p, e) \). Therefore, the induced representation \( (U^\tau, W) \) of \( G \) is simply a regular representation (cf \(...\)). Indeed, for \( p \in E \), we have \( G_p = \{(p, g), g \in G\} \), \( \mathcal{W}_p = L^2(G_p) \cong L^2(G) \) and, for \( F \in \mathcal{W}_p \), \( h \in G \),

\[
(U^\tau(p, g)F)(ph, h) = F((pg, h) \circ (p, g)) = F(p, gh).
\]

Let us notice that this regular representation can serve to define random operators on the groupoid and then to define the von Neumann algebra of the groupoid (cf. \[8\], \[19\], \[9\]).

Now we pass to quantum mechanical momentum representation of a particle with the mass \( m \). Having fixed (by an act of measurement) \( p \in E \), we have reduced our initial space to \( \{ p \} \times G \cong G \). But we want to consider the energy-momentum space \( H \) of the particle, \( H = \{(p_0, p_1, p_2, p_3) \in \mathbb{R}^4 : p_0^2 - p_1^2 - p_2^2 - p_3^2 = m\} \). We have an action of the group \( G = SL_2(\mathbb{C}) \) on the
hyperboloid $H$ (see [22]).

To describe the action we identify $H$ with the set $\mathcal{H}$ of hermitian $2 \times 2$-matrices with determinant equal to $m$,

$$(p_0, p_1, p_2, p_3) \mapsto \begin{pmatrix} p_0 - p_3 & p_2 - ip_1 \\ p_2 + ip_1 & p_0 + p_3 \end{pmatrix}$$

and we let to act $g \in G$ on $\mathcal{H}$ to the right in the following way, $\mathcal{H} \ni A \mapsto g^* Ag \in \mathcal{H}$. (It is clear that $\det(g^* Ag) = \det A = m$). Next, we see that the isotropy group of the element $(p_0, 0, 0, 0)$, $p_0 = \sqrt{m}$ is equal to $K = SU(2)$. Thus we deduce that the homogoneus space $K \backslash G$ is diffeomorphic to $H$. We can take the phase space of a particle of the mass $m$ as the space $\mathcal{G} = K \backslash G \times G = H \times G$ and consider the algebraic structure of transformation groupoid on it.

Let $(\mathcal{U}, \mathcal{W})$ be a unitary representation of the groupoid $\mathcal{G}$ in a Hilbert bundle $\mathcal{W}$. Assume that there exists an imprimitivity system $(\mathcal{U}, \pi)$ for $(\mathcal{U}, \mathcal{W})$. We say that a particle of mass $m$ is represented by the pair $(\mathcal{U}, \pi)$. We say that it is an elementary particle if the imprimitivity system $(\mathcal{U}, \pi)$ is irreducible [13], [14]. Equivalently (on the strength of Theorem 1), we can say that the particle is an induced representation $(\mathcal{U}^\tau, \mathcal{W})$ where $\tau$ is a
unitary representation of the isotropy subgroupoid $\Gamma$. In the same way, we can say that the particle is elementary if the inducing representation $\tau$ is irreducible and, in turn, this means that the representation $(L, W_0)$, $L = \tau_0$, of the group $K = SU(2)$ is irreducible. Then the representation $(L, W_0)$ is called the spin of the particle.

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