Non-stationary model of incompressible viscoelastic Kelvin-Voigt fluid of higher order in the Earth’s magnetic field

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Abstract. This work is aimed at studying a higher-order model of magnetohydrodynamics in regards to external effects on a fluid. The first initial boundary value problem for a system of higher-order Kelvin-Voigt equations in the Earth’s magnetic field is considered. The study was carried out in the framework of the theory of Sobolev type semilinear equations. The existence of quasistationary semitrajectories of the indicated problem is proved and its phase space is described.

1. Introduction
We will consider the rheological relation of Newton [1], simulating the dynamics of viscous incompressible fluids

\[ \sigma = 2\nu D - pI \] (1)

Here \( \sigma \) is a stress tensor, \( D \) is a strain rate tensor, \( \nu \in \mathbb{R}_+ \) is a viscosity coefficient, \( I \) is a unit matrix, \( p \) is pressure. We will insert \( \sigma \) from (1) into the equation of a continuous incompressible fluid in the Cauchy form

\[ \frac{dv}{dt} = \nabla \cdot \sigma + f, \nabla \cdot v = 0. \] (2)

We get the famous Navier-Stokes system of equations

\[ v_t = \nu \nabla^2 v - (\nu \cdot \nabla)v - \nabla p + f, \nabla \cdot v = 0. \] (3)

for which initial boundary value problems have been studied by many authors (for example, [2, 3]). But later, researchers appeared who revised relation (1) due to the fact that some effects (for example, the recoil effect and the effect of fading memory) that arise when pumping oil through pipelines did not fit into the framework of the model (3). In particular, V.A. Pavlovskiy [4] proposed a rheological relation

\[ \sigma = 2\nu D + \kappa \frac{D}{\partial t} - pI, \] (4)

in which \( \kappa \) is the coefficient of elasticity. The rheological relation (4) was confirmed experimentally and for negative values \( \kappa \) [5].

If we substitute relation (4) into equation (2), we obtain a system of equations more general than system (3).
\[(1 - \kappa \nabla^2)v_t = \nu \nabla^2 v - (\nu \cdot \nabla)v - \nabla p + f, \nabla \cdot v = 0.\] (5)

At first, it was A.P. Oskolkov who began to study initial boundary value problems for system (5) [6]. Later, relying on the rheological theory [7] - [9] A.P. Oskolkov justifies the relation
\[
\left(1 + \sum_{a=0}^{\Lambda} \kappa_a \frac{\partial^a}{\partial t^a}\right)\sigma = 2\nu \left(1 + \sum_{b=0}^{B} \kappa_b \nu^{-1} \frac{\partial^b}{\partial t^b}\right)D - pI,
\] (6)

which generalizes (4), obtains (depending on the parameters \(A\) and \(B\)) systems of equations modeling the dynamics of fluids by Maxwell, Oldroyd, and Kelvin-Voigt (the latter include equations (5)), and begins a systematic study of initial-boundary value problems for them [10], [11].

An analysis of relation (4), made in [12] from the perspective of a rheological theory [7], [9], showed that in case \(\kappa \in \mathbb{R}\) system (5) exhibits properties characteristic of solids. Therefore, such fluids are proposed to be called media. The first answer on the solvability of the Cauchy – Dirichlet problem for the system of equations (5) in the case of a negative parameter \(\kappa\) was obtained in [13]. By the method of orthogonal projection onto the subspace of solenoidal functions (see [2], [3]), the problem was reduced to the Cauchy problem
\[u(0) = u_0\] (7)

for an operator differential equation of the form
\[Lu = Mu + F(u) + f,\] (8)

where the linear operators \(L, M\) and the nonlinear operator \(F\) act from one Banach space \(U\) to another \(\mathcal{F}\). Then equation (8) is reduced to the equation
\[\dot{u} = Su + N(u) + f,\] (9)

defined on the variety of solutions of equation (8) \(B \subset U\), and the solvability of problem (7), (9) follows from the classical Cauchy theorem (see, for example, [13]). Similarly, using the ideas and methods [14], the thermoconvection problem was studied for system (5) [15].

After the fundamental work by S.L. Sobolev [16], all equations that are not resolved with respect to the distinguished time derivative (of the form (5), (8)) are now called Sobolev type equations (see, for example, [17–19]).

The most interesting phenomenon in the theory of semilinear (8) and linear Sobolev type equations is the fundamental unsolvability of problem (7) for any initial data \(u \in U_0\) in the case of \(\ker L \neq \{0\}\) [20], [21]. To overcome this difficulty, the phase space method was developed [22], [23], which became a generalization of the reduction of (5) to (8). The essence of this method is to reduce the singular equation (8) to the regular one (9), defined not on the whole space \(U\) (and not even on some dense \(U\) in the linear), but on the set of admissible initial values \(u \in U_0\), understood as the phase space of equation (8). Thus, the problem of solvability of problem (7), (8) reduces to studying the morphology of phase space.

In [14] it is indicated that in [24] a system is presented that describes the fluid flow in the Earth’s magnetic field
\[v_t = \nu \nabla^2 v - (\nu \cdot \nabla)v - \frac{1}{\rho} \nabla p - 2\Omega \times v + \frac{1}{\rho \mu} (\nabla \times b) \times b,\]
\[\nabla \cdot v = 0, \nabla \cdot b = 0,\] (10)
\[b_t = \delta \nabla^2 b + \nabla \times (v \times b).
\]

Here \(b = (b_1(x,t), b_2(x,t)… b_n(x,t))\) is a vector function characterizing magnetic induction; \(\Omega\) is an angular velocity; \(\rho\) is density; \(\mu\) is magnetic permeability; \(\delta\) is magnetic viscosity.

Based on (5) and (10), it is obvious that the system modeling the flow of an incompressible viscoelastic fluid in the Earth’s magnetic field will have the form (at \(\rho = \mu = 1\))
(1 - \kappa \nabla^2) v_i = \nu \nabla^2 v - (\nu \cdot \nabla) v - \nabla p - 2\Omega \times v + (\nabla \times b) \times b, \\
\nabla \cdot v = 0, \nabla \cdot b = 0, \\
b_i = \partial V_2 b + \nabla \times (v \times b). \tag{11}

The first initial boundary value problem for it was investigated by the authors in [25]. In [26], a numerical solution was obtained for this problem. The Cauchy–Dirichlet problems for models of nonzero and higher orders were also studied [27], [28].

In this paper, we will be interested in the solvability of the first initial boundary value problem

\[ v(x,0) = v_0(x), \quad b(x,0) = b_0(x), \quad w_{m,t}(x,0) = w_{m,0}^0(x), \quad \forall x \in D; \]

\[ v(x,t) = 0, \quad b(x,t) = 0, \quad w_{m,t}(x,t) = 0, \quad \forall (x,t) \in \partial D \times R_+, \quad m = 1, M, \quad s = 1, n_m - 1, \tag{12} \]

for a system that is a non-stationary model of a higher order incompressible viscoelastic fluid in the Earth’s magnetic field

\[ (1 - \kappa \nabla^2) v_i = \nu \nabla^2 v - (\nu \cdot \nabla) v + \sum_{m=1}^{M} \sum_{t=0}^{n_m-1} A_{m,t} \nabla^2 w_{m,t} - \frac{1}{\rho} \nabla p - \frac{1}{\rho \mu} (\nabla \times b) \times b + f^1, \]

\[ \nabla \cdot v = 0, \quad b \cdot v = 0, \quad b_i = \partial V_2 b + \nabla \times (v \times b) + f^2, \tag{13} \]

A distinctive feature of this work is the presence of vector functions on the right side of equations (13) \( f^1 = \{f_1^1, ..., f_n^1\}, f^1 = f_1^1(x,t), f^2 = f_2^2(x,t) \). The model of magnetohydrodynamics, taking into account various external effects of zero order, was investigated by the authors in [29]. Here, the authors also obtained a numerical solution to this problem. A nonzero order model was investigated in [30].

The article, in addition to the Introduction and the List of references, contains two chapters. In the first Chapter, using the theory of \( p \)-sectorial operators, we consider the necessary and sufficient condition for the existence of a solution to the abstract Cauchy problem for a semilinear nonstationary Sobolev type equation. In the second Chapter, problem (12), (13) is considered as a concrete interpretation of an abstract problem.

2. Abstract problem

We will consider a Cauchy problem

\[ u(0) = u_0 \]

for a semilinear autonomous Sobolev type equation

\[ \dot{L}u = Mu + F(u) + f. \tag{14} \]

Here operator \( L \in \mathcal{L}(U, F), \; M \in \mathcal{C}(U : F) \) at that \( \ker L \neq \{0\} \).

We will denote by \( U_M \) the lineal \( \text{dom } M \) equipped with the norm of the graph, i.e.

\[ U_M = \{ u \in \text{dom } M : \| u \| = \| Mu \|_F + \| u \|_U \}. \]

Let operator \( F \in \mathcal{C}^{\infty}(U_M : F) \).

We call a local solution (hereinafter simply a solution) of problem (14), a vector function (15) \( u \in \mathcal{C}^{\infty}((0,T); U_M) \) that satisfies equation (15) and is such that \( u(t) \to u_0 \) for \( t \to 0^+ \)

Let the operator \( M \) be strongly \( (L, p) \)-sectorial, then the solution to problem (14), (15) can be nonunique [32]. Therefore, in the future, we will see only such solutions to problem (14), (15) that are
quasistationary semitrajectories. It is known from [33] that solutions to problem (14), (15) do not exist for all \( u_0 \in U_M \).

**Definition 1.** Let space \( U \) be split into a direct sum \( U = U_0 \oplus U_1 \) so that \( \ker L \subset U_0 \). The solution \( u = v + w \) where \( v(t) \in U_0 \), and for all \( t \in (0, T) \), of equation (15) is called a quasistationary semitrajectory, if \( L v = 0 \).

**Definition 2.** The set \( B_1 \subset U_M \times \mathbb{R}_+ \) will be called with the extended phase space of equation (15) if for any point \( u_0 \in B \) there exists a unique solution to problem (14), (15), moreover \( (u(t), t) \in B_1 \).

Based on the condition of strong \((L, p)\)-sectoriality of the operator \( M \), the spaces \( U \) and \( F \) will be split into direct sums \( U = U^0 \oplus U^1, F = F^0 \oplus F^1 \). Here \( U^0 \) and \( F^0 \) are the kernels, and \( U^1 \) and \( F^1 \) are the images of analytic resolving semigroups \( U^t \) and \( F^t \) of a linear homogeneous equation \( L \hat{u} = Mu \).

The above-mentioned semigroups are:

\[
U^t = \frac{1}{2\pi i} \int_{\Gamma} R^L_\mu(M) e^{\mu t} d\mu, \quad F^t = \frac{1}{2\pi i} \int_{\Gamma} L^R_\mu(M) e^{\mu t} d\mu,
\]

where \( \Gamma \subset \rho^L(M) \) is such contour that \( \arg \mu \to \pm \theta \) at \( |\mu| \to +\infty \), at that

\[
\rho^L = \{ \mu \in C : (\mu L - M)^{-1} \in L(U, F) \} - L \text{ is the resolvent set of operator } M, \quad R^L_\mu(M) = (\mu L - M)^{-1}, \quad L^R_\mu(M) = L(\mu L - M)^{-1} \}
\]

By virtue of the results of [33], we reduce problem (14), (15) to the equivalent system:

\[
\begin{align*}
R \hat{u}^0 &= u^0 + G(u) + g(t), \quad u^0(0) = u_{00}, \\
\hat{u}^1 &= Su^1 + H(u) + h(t), \quad u^1(0) = u_{01}.
\end{align*}
\]

Here \( u^k \in U^k, k = 0, 1, \quad u = u^0 + u^1 \), operators \( R = M_{0}^{-1}L_{0}, \quad S = L_1^{-1}M_{1}, \quad G = M_{0}^{-1}(I - Q)F, \quad H = L_1^{-1}QF, g = M_{0}^{-1}(I - Q)f, h = L_1^{-1}Qf, (Q \in L(F) \equiv L(F,F)) \) is a projector that splits space as required. We call system (5) the *normal form* of problem (14), (15).

Further, we will study such quasistationary semitrajectories of equation (15) for which \( R \hat{u}^0 \equiv 0 \). For this, we will assume that the operator \( R \) is bi-splitting [33] (its kernel \( \ker R \) and image \( \im R \) are complemented in space \( U \)). Let \( U^{00} = \ker R \), denoting by \( U^{01} = U^0 - U^{00} \) some addition to the subspace \( U^{00} \). Then the first equation of system (16) takes the form

\[
R \hat{u}^{01} = u^{00} + u^{01} + G(u) + g(t),
\]

where \( u = u^{00} + u^{01} + u^1 \).

**Theorem 1.** Let the operator \( M \) be strongly \((L, p)\)-sectorial, the operator \( R \) is bi-splitting, and there exists a quasistationary semitrajectory of equation (15). Then it satisfies the relations

\[
0 = u^{00} + u^{01} + G(u) + g(t), \quad u^{01} = \text{const.}
\]

We now turn to the consideration of sufficient conditions. It is known from [33] that if the operator \( M \) is strongly \((L, p)\)-sectorial, then the operator \( S \) is sectorial. It means that the operator \( M \) generates in \( U^1 \) an analytic semigroup, which we denote by \( \{U^1_{t} : t \geq 0\} \), since in reality the operator \( U^1_{t} \) is the restriction of the operator \( U^t \) to \( U^1 \). From the splitting \( U = U^0 \oplus U^1 \), it follows that there is a projector \( P \in L(U|M) \) that corresponds to this splitting. But \( P \in L(U|M) \) and, therefore, \( U_M \) splits into a direct sum \( U = U^0_M \oplus U^1_M \) so that the investment \( U^k_M \subset U^k, k = 0, 1 \) is dense and continuous.
Theorem 2. Suppose that the operator $M$ is strongly $(L, p)$-sectorial, the operator $R$ is bi-splitting, and the operator $F$ belongs to $C^\infty(U_M; \mathcal{F})$. Besides, let also

(A) in some neighborhood $O_{u_0} \subset U_M$ of the point $u_0$ the relation is satisfied

$$0 = u_0^{\alpha_0} + (I - P_k)(G(u^{00} + u_0^{01} + u^1) + g(r));$$

(B) projector $P_R$ belongs to $L(U_M)$ and operator $I + P_R G_{u_0} : U_M^{00} \to U_M^{00}$ is a toplinear isomorphism $(U_M^{00} = U_M \cap U^{00})$;

(C) for analytical groups $\{U_1^t : t \geq 0\}$ the relation is satisfied

$$\int_0^\tau \left\|U_1^t\right\|_{L(U_1^{1}; U_0^{1})} dt < \infty, \tau \in \mathbb{R}_+.$$  \hspace{1cm} (20)

Then there is a unique solution to problem (14), (15) which is a quasistationary semitrajectory of equation (15).

Remark 1. For ordinary analytic semigroups that have an estimate $\left\|U_1^t\right\|_{L(U_1^{1}; U_0^{1})} \leq \text{const} \ t$, condition (20) is not satisfied. Since in the future we are going to use Theorem 2 exactly for this case, we will make some necessary explanations. We will denote some interpolation space constructed by the operator $S$ by $U_1^t = [U_1^t; U_0^t]_{\alpha}, \alpha \in [0,1]$. The condition $F \in C^\infty(U_M; \mathcal{F})$ in Theorem 2 is supplemented with the condition $H \in C^\infty(U_M; U_1^t), \alpha \in [0,1]$, and condition (9) is replaced by

$$\int_0^\tau \left\|U_1^t\right\|_{L(U_1^{1}; U_0^{1})} dt < \infty, \tau \in \mathbb{R}_+.$$  \hspace{1cm} (21)

Then the statement of Theorem 2 does not change (for a discussion of these questions, see [33]).

Now let us suppose that $U_k$ and $F_k$ are Banach spaces, the operators $A_k$ belong to $L(U_k, F_k)$, and the operators $B_k : \text{dom} B_k \to$ are linear and closed with domains of definitions $\text{dom} B_k$ dense in $U_k$, $k = 1, 2$. We construct spaces $U = U_1 \times U_2, F = F_1 \times F_2$ and operators $L = A_1 \otimes A_2, M = B_1 \otimes B_2$. By construction, the operator $L$ belongs to $L(L(U, F))$ and the operator $M : \text{dom} M \to F$ is linear, closed and tightly defined, $\text{dom} M = \text{dom} B_1 \times \text{dom} B_2$.

Theorem 3. Let the operators $B_k$ be strongly $(A_k, p_k)$-sectorial, $k = 1, 2$. Then the operator $M$ is strongly $(L, p)$-sectorial, $p = \max(p_1, p_2)$.

3. Conclusion

Following [32] we reduce the problem (12)–(13) to (14)–(15). With this purpose we will see (13) as follows

$$\begin{align*}
(1 - \kappa \nabla^2) v_i &= v \nabla^2 v - (\nabla \cdot v) v + \sum_{m=1}^M \sum_{s=0}^{n_m - 1} A_{m,s} \nabla^2 w_{m,s} - 2 \Omega \times v + (\nabla \times b) \times b + f^1, \\
\nabla \cdot v &= 0, \nabla \cdot b = 0, \ b_i = \delta \nabla^2 b + \nabla \times (v \times b) + f^2, \\
\frac{\partial w_{m,0}}{\partial t} &= v + \alpha_m w_{m,0}, \quad \alpha_m \in \mathbb{R}_+, \quad m = 1, M, \\
\frac{\partial w_{m,s}}{\partial t} &= s w_{m,s-1} + \alpha_m w_{m,s}, \quad s = 1, n_m - 1.
\end{align*}$$
Further, similarly to [32], we introduce the spaces $U$ and $F$, the operators $L$ and $M$, generated by problem (14) – (15). Strong $(L,l)$-sectoriality of the operator $M$ under the condition that $\kappa^{-1} \notin \sigma(A)$, $A = \nabla^2 E_n$ ($E_n$ is the identity matrix of $n$ order) is established on the basis of Theorem 3. The form of the nonlinear operator $F$ will be the same as in [30]. Belonging of $F$ to the class $C^\infty(U;F)$ is proved by calculating the Frechet derivatives of this operator [30]. As a result of checking the conditions of theorems 1 and 2, we establish.

**Theorem 4.** Let $\kappa^{-1} \notin \sigma(A) \cup \sigma(A_\sigma)$. Then, for any $u_0$, such that $u_0 \in M$, and some $T \in \mathbb{R}_+$, there exists a unique solution $u = (u_x,0,u_p,u_b)$ to problem (1), (2), which is a quasistationary semitrajectory, at that $u(t) \in M$ for all $t \in (0,T)$.

Here $A_\sigma$ is the restriction of the operator $A$ to the corresponding subspace of solenoidal vectors $u_x,0,u_p,u_b$ - elements of the corresponding subspaces of a Banach space $U$. $M$ is the expanded phase space of problem (12) – (13), having the form

$$M = \{u \in \mathcal{U}_M : u_x = 0, b_x = 0, u_p = \Pi (u_x, \nabla) u_\sigma + \sum_{m=1}^{M} \sum_{k=0}^{M} A_{m,q} \nabla^2 w_{m,q} - 2\Omega \times u_\sigma + (\nabla \times b_\sigma) \times b_\sigma + f^1(t)\}.$$

**Remark 2.** In recent decades, the theory of equations has received a powerful stimulus in its development, as evidenced by the monographs that have appeared [34–37]. This work is adjacent to the scientific direction developed by Professor G. A. Sviridyuk and his disciples [38–41].

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7
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