Gauge Independence of the Lagrangian Path Integral
in a Higher-Order Formalism

I.A. Batalin
I.E. Tamm Theory Division
P.N. Lebedev Physics Institute
Russian Academy of Sciences
53 Leniniski Prospect
Moscow 117924
Russia

K. Bering
Institute of Theoretical Physics
Uppsala University
P.O. Box 803
S-751 08 Uppsala
Sweden

and

P.H. Damgaard
The Niels Bohr Institute
Blegdamsvej 17
Dk-2100 Copenhagen
Denmark

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Abstract
We propose a Lagrangian path integral based on gauge symmetries generated by a symmetric higher-order \( \Delta \)-operator, and demonstrate that this path integral is independent of the chosen gauge-fixing function. No explicit change of variables in the functional integral is required to show this.
1. Introduction. The usual field-antifield formalism for gauge theories is based on a Grassmann-odd nilpotent operator $\Delta$, which is assumed to be of 2nd order. In covariant form,

$$\Delta = \frac{1}{2}(-1)^{\epsilon_A} \frac{1}{\rho(\Gamma)} \partial_A \rho(\Gamma) E^{AB} (\Gamma) \partial_B .$$

Here $\rho(\Gamma)$ coincides with the measure density in the functional integral, and the non-degenerate $E^{AB}$ satisfies the following symmetry condition:

$$E^{AB} = -(-1)^{(\epsilon_A+1)(\epsilon_B+1)} E^{BA} .$$

The $\Delta$-operator is the basic building block of the whole Lagrangian quantization program in the field-antifield formalism. From it derives the Master Equations as well as the antibracket, which in turn provides an anticanonical framework on the space of fields and antifields. The $\Delta$-operator also generates a global symmetry which is useful for proving independence of the chosen gauge-fixing surface on the field-antifield supermanifold.

In view of this, it is natural to ask if an analogous Lagrangian quantization prescription can be established with the help of a $\Delta$-operator of arbitrary (perhaps even infinite) order. Investigations along these lines were initiated by a study of possible quantum deformations of the field-antifield formalism. More recently, such considerations have been prompted by the observed analogies between the $\Delta$-operator and a certain quantized Hamiltonian BRST operator $\Omega$ in the ghost momentum representation, as well as by a study of the associated algebraic structure. The relevant mathematical foundations date back to work of Koszul.

2. Gauge Independence. Let there be given a functional measure $d\mu \equiv d\Gamma d\lambda \rho(\Gamma)$. While $\Gamma$ represents the usual set of fields and antifields, the $\lambda$’s can be seen either as implementing the gauge (or hypergauge) fixing condition, or as the ghost fields for which the antifields are usual antighosts. We assume that the general odd $\Delta$-operator satisfies two properties: It is nilpotent,

$$\Delta^2 = 0 ,$$

and it is symmetric,

$$\Delta^T = \Delta .$$

Here the transposed operator $\Delta^T$ is defined by

$$\int d\mu \ F \Delta G = (-1)^{\epsilon_F} \int d\mu \left( \Delta^T F \right) G .$$

It is conventional to assume in addition that $\Delta(1) = 0$, but this assumption can easily be discarded, and in fact we do not need it here.

Consider the two Master Equations

$$\Delta e^\lambda_W = 0 , \quad \Delta e^\lambda_X = 0 ,$$

and the path integral

$$Z_X \equiv \int d\mu \ e^\lambda_{[W+X]} .$$

We wish to establish that $Z_X = Z_X'$ for an arbitrary deformation $X \rightarrow X'$ that preserves the master equation for $X$. We consider here $\Delta$-operators and solutions $X$ belonging to the class for which the following represents a maximal deformation:

$$e^\lambda_{X'} = e^{[\Delta, \Psi]} e^\lambda_X .$$
i.e. for an infinitesimal transformation,

\[ \delta e^{\hat{\pi}X} = [\Delta, \Psi] e^{\hat{\pi}X}. \]  

(9)

Proof: We will use 3 ingredients: 1) The Master Equation for \( W \), 2) The Master Equation for \( X \), and 3) The symmetry of \( \Delta \). Then,

\[
Z_{X'} - Z_X = \int d\mu \ e^{i\bar{\hbar}W[\Delta, \Psi]} e^{\hat{\pi}X} = 0 .
\]

(10)

3. BRST Symmetry. We can also understand gauge independence of the path integral from the existence of a nilpotent BRST symmetry. We can define two BRST transformations, one associated with \( W \), and one associated with \( X \). They are:

\[
\begin{align*}
\sigma_W F &= \langle \frac{\hbar}{i} \rangle e^{-\frac{i}{\bar{\hbar}} W} \delta e^{\hat{\pi}W} \\
\sigma_X F &= \langle \frac{\hbar}{i} \rangle e^{-\frac{i}{\bar{\hbar}} X} \delta e^{\hat{\pi}W} .
\end{align*}
\]

(11)

The Master Equations for \( W \) and \( X \) are preserved under transformations \( W \rightarrow W + \sigma_W \Psi \) and \( X \rightarrow X + \sigma_X \Psi \). Thus another way of phrasing the gauge independence of the path integral is \( Z_X = Z_{X+\delta X} \) with \( \delta X = \sigma_X \Psi \). This is precisely the content of eqs. (9) and (10). The operators (11) are the natural generalizations of the so-called quantum BRST operator for the case of the conventional 2nd order \( \Delta \)-operator (11).

If we define BRST invariant operators \( G \) by

\[ \sigma_W G = 0 , \]

(12)

we observe that expectation values of such operators do not depend on \( X \):

\[
\langle G \rangle_{X+\delta X} - \langle G \rangle_X = Z^{-1} \int d\mu \ G e^{\hat{\pi}W+\delta X} e^{-\frac{i}{\bar{\hbar}} X} \delta e^{\hat{\pi}W} \\
= (-1)^{\langle G \rangle} Z^{-1} \int d\mu \ \Delta (G e^{\hat{\pi}W}) \Psi e^{\hat{\pi}X} \\
= (-1)^{\langle G \rangle} Z^{-1} \int d\mu \ ([\Delta, G] e^{\hat{\pi}W}) \Psi e^{\hat{\pi}X} \\
= 0 .
\]

(13)

Similarly we can show that \( \langle \sigma_W F \rangle = 0 \) for any \( F \):

\[
\langle \sigma_W F \rangle = Z^{-1} \int d\mu \ e^{\hat{\pi}W+\delta X} \sigma_W F \\
= Z^{-1} \int d\mu \ e^{\hat{\pi}W+\delta X} \langle \frac{\hbar}{i} \rangle e^{-\frac{i}{\bar{\hbar}} W} [\Delta, F] e^{\hat{\pi}W} \\
= Z^{-1} \langle \frac{\hbar}{i} \rangle \int d\mu \ [(\Delta e^{\hat{\pi}X}) Fe^{\hat{\pi}W} - (-1)^{\langle F \rangle} e^{\hat{\pi}X} F \Delta e^{\hat{\pi}W}] \\
= 0 .
\]

(14)

*We can also consider their difference (10), or in fact any linear combination. When both Master Equations are satisfied, we can remove the commutator, but the present form is convenient.*
The formalism is symmetric under exchanges of \( W \) and \( X \). The choice of boundary conditions stipulates which part will play the rôle of action (here taken to be \( W \)), and which part will play the rôle of gauge fixing (here taken to be \( X \)).

4. Generalizations. We note that the requirement of symmetry of \( \Delta \), eq. (4), can be relaxed if we instead impose two conjugate Master Equations on \( W \) and \( X \):

\[
\Delta e^{\pi W} = 0 \quad , \quad \Delta^T e^{\pi X} = 0 .
\]

(15)

Under arbitrary infinitesimal deformations

\[
\delta e^{\pi X} = [\Delta^T, \Psi] e^{\pi X} ,
\]

(16)

one finds again \( Z_X = Z_{X'} \). Similarly the BRST operator \( \sigma_X \) of eq. (11) will have \( \Delta \) replaced by \( \Delta^T \). The rest of the conclusions then remain unaltered.

5. Quantum Deformations of \( \Delta \). Up to this point we have made no further assumptions about \( \Delta \) beyond those stated in eq. (3-4), and indirectly in eq. (8). Gauge independence of the proposed path integral (10) holds in all generality. We shall end by some comments specific to \( \Delta \)-operators obtained by quantum deformations of the classical 2nd order \( \Delta \)-operator (1). Quantum corrections are expected to arise when operator-ordering in the Hamiltonian formalism is properly taken into account.

The most general quantum deformation of the 2nd order \( \Delta \) can be written in terms of its homogeneous components as

\[
\Delta = \sum_{n=0}^{\infty} (\hbar i)^n \Delta_n
\]

\[
\Delta_n = \sum_{m=0}^{n+2} \Delta_{n,m} \quad , \quad \Delta_{n,m} = \Delta^{A_{m-1} \cdots A_1}(\Gamma) \partial_{A_1} \cdots \partial_{A_m} .
\]

(17)

in an ordering with all (left) derivatives standing to the right. \((\hbar i)^n \Delta_{n,m}\) is the contribution to \( \Delta \) of order \( n \) in \( \hbar \) and differential order \( m \). So the classical part is

\[
\Delta_0 = \Delta_{0,2} + \Delta_{0,1} + \Delta_{0,0} .
\]

(18)

Each new order of \( \hbar \) gives rise to one extra order of differentiation.

The Master Equation has an expansion in terms of higher antibrackets \( \Phi^k \):

\[
0 = (\hbar i)^2 e^{-\pi W} \Delta e^{\pi W} = (\hbar i)^2 \sum_{k=0}^{\infty} \frac{1}{k!} \Phi^k (\pi W, \ldots , \pi W) = \sum_{n=0}^{\infty} \sum_{k=0}^{n+2} (\hbar i)^{n+2-k} \frac{1}{k!} \Phi^k \Delta_n (W, \ldots , W) = \sum_{k=2}^{\infty} \frac{1}{k!} \Phi^k \Delta_{k-2} (W, \ldots , W) + \frac{4}{\hbar} \sum_{k=1}^{\infty} \frac{1}{k!} \Phi^k \Delta_{k-1} (W, \ldots , W) + \sum_{\ell=2}^{\infty} \frac{(\hbar i)^\ell}{\ell !} \sum_{k=0}^{\infty} \frac{1}{k!} \Phi^k \Delta_{k+\ell-2} (W, \ldots , W) .
\]

(19)

\(\dagger\) The original \( \Delta \)-operator (1) is of this form, with \( \Delta_{0,0} = 0 \).
Here we have introduced the generalized antibracket\footnote{7}
\[
\Phi^k_\Delta (A_1, \ldots, A_k) = [\cdots [\Delta, A_1], \cdots, A_k] 1, \tag{20}
\]
in terms of $k$ nested commutators.

Assuming a semiclassical expansion of $W$, $W = \sum_{n=0}^{\infty} \hbar^n W_n$, one sees that the classical master equation
\[
\Phi^2_{\Delta_0} (W_0, W_0) = 0 \tag{21}
\]
will be replaced, in general, by a (possibly infinite\footnote{All sums over the order of the bracket $k$ truncate at order $N$ if the $\Delta$-operator is of order $N$.}) sum,
\[
\sum_{k=2}^{\infty} \frac{1}{k!} \Phi^k_{\Delta_{k-2}} (W_0, \ldots, W_0) = 0. \tag{22}
\]

A similar analysis for the operator $\sigma_W$ yields
\[
\sigma_W F = \sum_{k=1}^{\infty} \frac{1}{k!} \Phi^{k+1}_{\Delta_{k-1}} (W, \ldots, W, F) + \sum_{\ell=1}^{\infty} (\frac{\hbar}{\ell})^\ell \sum_{k=0}^{\infty} \frac{1}{k!} \Phi^{k+1}_{\Delta_{k+\ell-1}} (W, \ldots, W, F). \tag{23}
\]

The classical transformation
\[
\sigma^{cl}_W (F) = \Phi^2_{\Delta_0} (W_0, F) \tag{24}
\]
will therefore in general be replaced by
\[
\sum_{k=1}^{\infty} \frac{1}{k!} \Phi^{k+1}_{\Delta_{k-1}} (W_0, \ldots, W_0, F) \tag{25}
\]

Conversely, if one assumes that quantum deformations of $\Delta$ must be of such a form as to preserve the original classical Master Equation (22), then this places restrictions on the expansion (17). The condition is that $\Delta_n$ must be a differential operator of order $\leq n + 1$ for $n \geq 1$. Or, equivalently, $\Delta_{n,n+2} = 0$ for $n \geq 1$.

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