Singularities of Connection Ricci Flow and Ricci Harmonic Flow

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Abstract

In this paper, we study the singularities of two extended Ricci flow systems — connection Ricci flow and Ricci harmonic flow using newly-defined curvature quantities. Specifically, we give the definition of three types of singularities and their corresponding singularity models, and then prove the convergence. In addition, for Ricci harmonic flow, we use the monotonicity of functional $\nu_\alpha$ to show the connection between finite-time singularity and shrinking Ricci harmonic soliton. At last, we explore the property of ancient solutions for Ricci harmonic flow.

1 Introduction

The Ricci flow theory which was founded by Richard Hamilton in 1980s has a big influence on differential geometry. It is a very good tool to study the structure and characterization of some manifolds and led to the solutions of Poincaré conjecture and geometrization conjecture. When considering the solutions of the Ricci flow equation, it is inevitable to make contact with singularities. That is, the flow stops as a result of the degeneration of some geometric quantities. In this situation, in order to make the flow continue past the singularities, we should adopt the method of geometric surgery introduced by Grigori Perelman. This is the key point in proving the Poincaré conjecture.

On the basis of Ricci flow, in recent years, some extended Ricci flow systems began to be researched by people. The followings are two of them.

The first one is connection Ricci flow (CRF). Its equation is

$$\frac{\partial g}{\partial t} = -2Rc + \frac{1}{2}H,$$

$$\frac{\partial H}{\partial t} = \Delta_{LB}H,$$

(1.1)

where $H$ is a closed 3-form on manifold $M^n$, $H_{ij} = g^{pq}g^{rs}H_{ipr}H_{jqs}$, and $\Delta_{LB} = -dd^* - d^*d$ represents the Laplace-Beltrami operator.

The second one is Ricci harmonic flow. Its equation is

$$\frac{\partial g}{\partial t} = -2Rc + 2\alpha\nabla\phi \otimes \nabla\phi,$$

$$\frac{\partial \phi}{\partial t} = \tau_g \phi,$$

(1.2)
where $\phi(t) : M^m \to N^n$ is a family of smooth maps between Riemannian manifolds, $\tau_g \phi$ is the tension field of $\phi$ with respect to $g$, $\alpha$ is nonnegative and time-dependent in general. But in this paper, we assume that $\alpha \geq 0$ is a time-independent constant.

Connection Ricci flow is a special case of the renormalization group flow in physics, and can also be seen as the Ricci flow in manifolds with non-trivial torsion. [St] and [Li] studied the short-time existence, evolution equations of curvature, derivative estimate, variational structure and compactness theorem from different angles.

Ricci harmonic flow arises from the combination of Ricci flow and harmonic map flow. Main work is done by [Mue] and [List]. In addition to the results above, they also gave the primary property of singularity, no breathers theorem and non-collapsing theorem, and so on.

Considering the above two flows in mathematical sense, if we can prove their long-time convergence under some circumstances, then the limit of connection Ricci flow may be a manifold with constant curvature and torsion, and the limit of Ricci harmonic flow may be a manifold with constant curvature and a harmonic map from it to another manifold. But analogous to Ricci flow, in order to study the convergence of long-time solution, a problem that must be solved is the singularity. Unfortunately, so far, there is not very deep research about the property of singularities of the two flows. Even the natural classification of the singularities has not been obtained. In this paper, we relate the specific properties of the two flows to the work by Hamilton in Ricci flow. Through introducing a new curvature quantity and using it to establish the singularity models, we successfully classify the singularities of connection Ricci flow and Ricci harmonic flow into three relatively simple forms — ancient solutions, eternal solutions and immortal solutions. In addition, for Ricci harmonic flow, we combine the finite-time singularity to soliton which can be seen as the self-similar solution of the flow and also do some research about the ancient solutions. The work in this paper can be helpful in understanding the long-time behavior of the two flows. In the end, we want to point out that our method here may also apply to the singularity problem of some other flows.

1.1 Structure and main results

Now we introduce the organization of the paper and state the main results.

Section 2 is a general review of connection Ricci flow (mainly in [St], [Li]) and Ricci harmonic flow (mainly in [Mue], [List]).

Section 3 is about the singularities of connection Ricci flow. In §3.1, we give the definition of maximum solution regarding the unboundedness of $|Rm|$. In §3.2, we define a new curvature quantity $P$ and use it to classify the singularities into three types. Then we establish three singularity models and prove the convergence theorem.

Theorem 1.1. For any maximal solution to the connection Ricci flow which satisfies the injectivity radius estimate and is of Type I, IIa, IIb, or III, there exists a sequence of dilations of the solution which converges in the $C^\infty_{loc}$ topology to a singularity model of the corresponding type.

Section 4 is about the singularities of Ricci harmonic flow. In §4.1, we also define three types of singularities and three singularity models by a newly-defined curvature quantity $Q$ and get the following conclusion.
Theorem 1.2. For any maximal solution to Ricci harmonic flow which satisfies the injectivity radius estimate and is of Type I, IIa, IIb, or III, there exists a sequence of dilations of the solution which converges in the $C^\infty_{loc}$ topology to a singularity model of the corresponding type.

In §4.2, we prove a theorem concerning the finite-time singularity and shrinking soliton on closed manifolds.

Theorem 1.3. Let $(g(t), \phi(t))_{t \in [0, T)}$, be a maximal solution to the Ricci harmonic flow (1.2) on a closed manifold $M^m$ with singular time $T < \infty$. Let $t_k \to T$ be a sequence of times such that $Q_k = Q(p_k, t_k) \to \infty$. If the rescaled sequence $(M^m, Q_k g(t_k + Q_k^{-1}t), \phi(t_k + Q_k^{-1}t), p_k)$ converges in the $C^\infty$ sense to a closed ancient solution $(M^\infty, g^\infty(t), \phi^\infty(t), p^\infty)$ to the Ricci harmonic flow, then $(g^\infty(t), \phi^\infty(t))$ must be a shrinking Ricci harmonic soliton.

In the third part, we study the ancient solution of Ricci harmonic flow by dealing with the compact case and noncompact case respectively.

Theorem 1.4. Let $(g(t), \phi(t))$ be an ancient solution to Ricci harmonic flow (1.2) on manifold $M^m$, then for any $t$ such that the solution exists, we have $S = R - \alpha|\nabla \phi|^2 \geq 0$. That is, the scalar curvature is always nonnegative.

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2 Preliminaries

In this section, we collect and derive some results about connection Ricci flow and Ricci harmonic flow. Except for Theorem 2.9, all the results have been well known before.

2.1 Connection Ricci flow

Connection Ricci flow is a generalization of Ricci flow to connections with torsion and can be seen as a special case of renormalization group flow which has a physical background. Here we give some main results that will be used in this paper and we suggest the readers to refer [St] and [Li] for more details.

In a Riemannian manifold $(M, g)$, a general connection is defined as

$$\tau(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

In particular, when $\tau = 0$, it is the usual Levi-Civita connection. In [St], Streets let the torsion be geometric (that is, $g_{kl} \tau^k_{ij}$ is a 3-form and $d\tau = 0$) and study this kind of torsions. For a geometric torsion $\tau$, after computation, its curvature tensor is as following

$$R^h_{jkl} = \tilde{R}^h_{jkl} + \frac{1}{2} \left( \frac{\partial \tau^h_{ij}}{\partial x^k} - \frac{\partial \tau^h_{kj}}{\partial x^i} \right) + \frac{1}{4} \left( \tilde{\Gamma}^h_{kp} \tau^p_{ij} + \tau^h_{kp} \tilde{\Gamma}^p_{ij} \right) + \frac{1}{4} \left( \tilde{\Gamma}^h_{lp} \tau^p_{kj} - \frac{1}{2} (\tilde{\Gamma}^h_{kp} \tau^p_{ij} + \tau^h_{kp} \tilde{\Gamma}^p_{ij}) - \frac{1}{4} \tau^h_{kp} \tau^p_{kj} \right) + \frac{1}{2} \left( \tilde{\Gamma}^h_{lp} \tau^p_{kj} - \frac{1}{4} \tau^h_{kp} \tau^p_{kj} \right).$$

(2.1)
where quantities with ∼ are in Levi-Civita connection. From this, the Ricci curvature tensor can be got. It is not symmetric, but has the following symmetric part and skew-symmetric part

\[
\begin{align*}
\text{Re}^\otimes &= \text{Rc} - \frac{1}{4} \mathcal{H}, \\
\text{Re}^\wedge &= -\frac{1}{2} d^* \tau. 
\end{align*}
\] (2.2)

With these preparations, we can consider the connection Ricci flow

\[
\begin{align*}
\frac{\partial}{\partial t} g &= -2 \text{Re}^\otimes, \\
\frac{\partial}{\partial t} \tau &= 2 d \text{Re}^\wedge.
\end{align*}
\]

In another form, it is

\[
\begin{align*}
\frac{\partial}{\partial t} g &= -2 \text{Re} + \frac{1}{2} \mathcal{H}, \\
\frac{\partial}{\partial t} \tau &= \Delta_{\text{LB}} \tau.
\end{align*}
\]

Throughout this paper, we use notation \( H \) instead of \( \tau \). Then the connection Ricci flow equation is exactly (1.1).

[So] and [Li] studied connection Ricci flow equation using different methods. The former viewed the equation as the Ricci flow in general manifolds with torsion while the latter considered the equation in torsion-free manifolds and regarded \( H \) as a separated 3-form. Both methods serve to get some properties about the flow including short-time existence of solution, evolution equations, compactness theorem and some functionals. In this paper, we adopt Li’s method.

The evolution equations under connection Ricci flow are
Proposition 2.1. Under (1.1),
\[
\frac{\partial}{\partial t} R_{ijkl} = \Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}) \\
- (R_{pijkl} R_{pi} + R_{pklt} R_{pj} + R_{ijpl} R_{pk} + R_{ijkl} R_{pk}) \\
- \frac{1}{4} \left[ \nabla_i \nabla_k H_{jl} - \nabla_i \nabla_l H_{jk} - \nabla_j \nabla_k H_{il} + \nabla_j \nabla_l H_{ik} \right] \\
+ \frac{1}{4} [R_{ijkl} H_{pl} + R_{ijpl} H_{lk}],
\]
(2.3)
\[
\frac{\partial}{\partial t} R_{ij} = \Delta R_{ij} + 2R_{piqj} R_{pq} - 2R_{pi} R_{pj} - \frac{1}{4} \left[ R_{piqj} H_{pq} - R_{pi} H_{pq} \right] \\
- \frac{1}{4} \left[ \nabla_i \nabla_j |H|^2 - \nabla_i \nabla_j H_{ip} - \nabla_j \nabla_i H_{jp} + \Delta H_{ij} \right],
\]
(2.4)
\[
\frac{\partial}{\partial t} R = \Delta R + 2|\mathcal{R}|^2 - \frac{1}{2} \langle \mathcal{R} c, H \rangle - \frac{1}{2} \Delta |H|^2 + \frac{1}{2} g^{ik} g^{jl} \nabla_i \nabla_j H_{kl},
\]
(2.5)
\[
\frac{\partial}{\partial t} H_{ij} = 2\langle \Delta_{LB} H_{i,j}, H_{kl} \rangle + 4 \langle R_{ln} - \frac{1}{4} H_{ln}, H_{j,l} H_{j,k} \rangle,
\]
(2.6)
\[
\frac{\partial}{\partial t} |H|^2 = 2\langle \Delta_{LB} H, H \rangle + 6 \langle \mathcal{R} c, H \rangle - \frac{3}{2} |H|^2,
\]
(2.7)
where $B_{ijkl} = R_{piqj} R_{pkql}$.

By (2.7) and $\Delta_{LB} H = \Delta H + Rm * H$, we get

Theorem 2.2. Let $(M^n, g(x, t), H(x, t))_{t \in [0, T]}$ be a complete solution to connection Ricci flow, and $K_1, K_2$ are arbitrary given nonnegative constants. If
\[
\sup_{M^n \times [0, T]} |Rm(x, t)|_{g(x,t)} \leq K_1, \quad \sup_{M^n} |H(x, 0)|_{g(x,0)}^2 \leq K_2
\]
for all $x \in M^n$ and $t \in [0, T]$, then there exists a constant $C_n$ depending only on $n$ such that
\[
\sup_{M^n \times [0, t]} |H(x, t)|_{g(x,t)}^2 \leq K_2 e^{C_n K_1 t}
\]
for all $x \in M^n$ and $t \in [0, T]$.

The derivative estimates

Theorem 2.3. Let $(M^n, g(x, t), H(x, t))$ be a complete solution to connection Ricci flow, and $K$ is an arbitrary given positive constant. Then for each $\beta > 0$ and each integer $l \geq 1$ there exists a constant $C_l$ depending only on $l, n, \max \{\beta, 1\}$ and $K$ such that if
\[
|Rm(x, t)|_{g(x,t)} \leq K, \quad |H(x, 0)|_{g(x,0)}^2 \leq K
\]
for all $x \in M^n$ and $t \in [0, \frac{\beta}{K}]$, then
\[
|\nabla^{l-1} Rm(x, t)|_{g(x,t)} + |\nabla^l H(x, t)|_{g(x,t)} \leq \frac{C_l}{l^{1/2}}
\]
for all $x \in M^n$ and $t \in (0, \frac{\beta}{K}]$. 

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In addition, [Zheng] gives the locally derivative estimate. Then we show the compactness theorem.

**Theorem 2.4.** Let \((M^n_k, g_k(t), H_k(t), p_k)\) be a sequence of complete pointed solutions to connection Ricci flow for \(t \in (A, \Omega) \ni 0\), such that

1. there is a constant \(C_0 < \infty\) independent of \(k\) such that
   \[
   \sup_{(x,t) \in M^n_k \times (A, \Omega)} |\text{Rm}_k(x, t)|_{g_k(x, t)} \leq C_0, \quad \sup_{x \in M^n_k} |H_k(x, 0)|_{g_k(x, 0)} \leq C_0,
   \]
2. there is a constant \(\iota_0 > 0\) satisfies
   \[
   \text{inj}_{g_k(0)}(p_k) \geq \iota_0 > 0.
   \]

Then there exists a subsequence \(\{j_k\}\) such that \((M^n_{j_k}, g_{j_k}(t), H_{j_k}(t), p_{j_k})\) converges to a complete pointed solution \((M^n_\infty, g_\infty(t), H_\infty(t), p_\infty)\), \(t \in (A, \Omega)\) to connection Ricci flow as \(k \to \infty\).

### 2.2 Ricci harmonic flow

Ricci harmonic flow is a combination of Ricci flow and harmonic map flow, where the latter is an important tool to study harmonic map. [List] and [Mue] did detailed research about this topic. Here we mainly introduce the evolution equations and some results about solitons and maximum solutions. After this, we prove the convergence theorem.

**Proposition 2.5.** Under (1.2),

\[
\frac{\partial}{\partial t} R_{ijkl} = \Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ijkl})
\]
\[
\frac{\partial}{\partial t} R_{ij} = \Delta R_{ij} + 2R_{piqj} R_{pq} - 2R_{pi} R_{pj} - 2\alpha R_{piq} \nabla p \phi \nabla q \phi
\]
\[
\frac{\partial}{\partial t} R = \Delta R + 2|\nabla \phi|^2 - 4\alpha \langle R c, \nabla \phi \otimes \nabla \phi \rangle - 2\alpha |\nabla \phi|^2 - 2\alpha |\nabla \phi|^2
\]
\[
\frac{\partial}{\partial t} (\nabla_i \phi \nabla_j \phi) = \Delta (\nabla_i \phi \nabla_j \phi) - R_{pi} \nabla_j \phi \nabla_i \phi - R_{pj} \nabla_i \phi \nabla_j \phi - 2 \nabla_p \nabla_i \phi \nabla_j \phi
\]
\[
\frac{\partial}{\partial t} |\nabla \phi|^2 = \Delta |\nabla \phi|^2 - 2|\nabla \phi|^2 - 2\alpha |\nabla \phi \otimes \nabla \phi|^2
\]

where \(B_{ijkl} = R_{piqj} R_{pkql}\), and \(N \text{Rm}\) represents the curvature tensor of \(N^n\).
If we set \( S_{ij} = R_{ij} - \alpha \nabla_i \phi \nabla_j \phi \), \( S = R - \alpha |\nabla \phi|^2 \), then

\[
\frac{\partial}{\partial t} S_{ij} = \Delta S_{ij} + 2R_{piqj}S_{pq} - R_{pi}S_{pj} - R_{pj}S_{pi} + 2\alpha \tau g_{ij} \phi \nabla_i \nabla_j \phi, \tag{2.13}
\]

\[
\frac{\partial}{\partial t} S = \Delta S + 2|S_{ij}|^2 + 2\alpha |\tau g\phi|^2. \tag{2.14}
\]

**Definition 2.6.** A solution \((g(t), \phi(t))_{t \in [0, T]}\) of (1.2) is called a soliton if there exists a one-parameter family of diffeomorphisms \( \psi_t : M^m \to M^m \) with \( \psi_0 = \text{id}_{M^m} \) and a scaling function \( c : [0, T) \to \mathbb{R}_+ \) such that

\[
\begin{cases}
g(t) = c(t)\psi_t^* g(0), \\
\phi(t) = \psi_t^* \phi(0).
\end{cases}
\]

The case \( \frac{\partial}{\partial t} \psi_t = \dot{\psi}_t < 0 \), \( \dot{\psi}_t = 0 \) and \( \dot{\psi}_t > 0 \) correspond to shrinking, steady and expanding solitons, respectively. If the diffeomorphisms \( \psi_t \) are generated by a vector field \( X(t) \) that is a gradient of some function \( f(t) \) on \( M^m \), then the soliton is called gradient soliton and \( f \) is called the potential of the soliton.

Given a closed manifold \((M^m, g)\) and a map \( \phi \) from \( M^m \) to a Riemannian manifold \( N^n \), define

\[
\lambda_\alpha(g, \phi) = \inf \left\{ \int_{M^m} (R + |\nabla f|^2 - \alpha |\nabla \phi|^2) e^{-f} dV \left| \int_{M^m} e^{-f} dV = 1 \right. \right\},
\]

for \( \tau > 0 \), again define

\[
\mu_\alpha(g, \phi, \tau) = \inf \left\{ \int_{M^m} \left[ \tau(R + |\nabla f|^2 - \alpha |\nabla \phi|^2) + f - m \right] (4\pi \tau)^{-m/2} e^{-f} dV \left| \int_{M^m} (4\pi \tau)^{-m/2} e^{-f} dV = 1 \right. \right\},
\]

where \( R \) represents the scalar curvature with respect to \( g \), the infimum is attained throughout all functions \( f \in C^\infty(M^m) \).

It is easy to know that \( \lambda_\alpha \) is the first eigenvalue of the operator \( -4 \Delta + R - \alpha |\nabla \phi|^2 \). Similar to Ricci flow, for each \( \tau > 0 \), there exists a smooth minimizer of \( \mu_\alpha(g, \phi, \tau) \). The functional \( \nu_\alpha \) is defined by

\[
\nu_\alpha(g, \phi) = \inf_{\tau > 0} \mu_\alpha(g, \phi, \tau).
\]

By §4.2, we know \( \nu_\alpha(g, \phi) \) may be \(-\infty\).

For the functional \( \nu_\alpha \), we have the following significant monotonicity proposition under Ricci harmonic flow.

**Proposition 2.7.** Let \((g(t), \phi(t))\) be a solution to (1.2) on a closed manifold \( M^m \) with \( \alpha \) a constant. For a constant \( T > 0 \), set \( \tau(t) = T - t \), then \( \mu_\alpha(g(t), \phi(t), \tau(t)) \) is nondecreasing whenever it exists. Moreover, the monotonicity is strict unless \((g(t), \phi(t))\) is a shrinking soliton.

As for the long-time existence of Ricci harmonic flow, there is the following theorem.

**Theorem 2.8.** Let \((g(t), \phi(t))_{t \in [0, T]}\) be a solution to (1.2). Suppose \( T < \infty \) is chosen such that the solution is maximal, i.e. the solution cannot be extended beyond \( T \) in a smooth way. Then the curvature of \((M^m, g(t))\) has to become unbounded for \( t \not\to T \) in the sense that

\[
\limsup_{t \not\to T} \max_{x \in M^m} |Rm(x, t)|^2 = \infty.
\]
In order to deal with the singularity model, we also need the compactness theorem for Ricci harmonic flow. The following theorem is what we will prove in this section.

**Theorem 2.9.** Let \((M^m_k, g_k(t), \phi_k(t), p_k), t \in (A, \Omega) \ni 0\), be a sequence of complete pointed solutions to Ricci harmonic flow such that

(i) there is a constant \(C_0 < \infty\) independent of \(k\) such that

\[
\sup_{(x,t) \in M^m_k \times (A, \Omega)} |Rm_k(x,t)|_{g_k(x,t)} \leq C_0,
\]

(ii) there is a constant \(t_0 > 0\) satisfies

\[
\text{inj}_{g_k(0)}(p_k) \geq t_0 > 0.
\]

Then there exists a subsequence \(\{j_k\}\) such that \((M^m_{j_k}, g_{j_k}(t), \phi_{j_k}(t), p_{j_k})\) converges to a complete pointed solution to Ricci harmonic flow \((M^m_\infty, g_\infty(t), \phi_\infty(t), p_\infty), t \in (a, \omega), \) as \(k \to \infty\).

The proof is a standard procedure given by Richard Hamilton which has already solved the compactness theorem in Ricci flow and connection Ricci flow. Here we only need to prove the following lemma which plays a vital role in the proof of the theorem.

**Lemma 2.10.** Let \((M^m, g)\) be a Riemannian manifold, \(K\) a compact subset of \(M^m\), and \((g_k(t), \phi_k(t))\) a collection of solutions to Ricci harmonic flow defined on neighborhoods of \(K \times [\gamma, \delta]\) with \([\gamma, \delta]\) containing \(0\). Suppose that for each \(r\),

(a) \(C_0^{-1} g \leq g_k(0) \leq C_0 g, \) on \(K\), for all \(k\),

(b) \(|\nabla^r g_k(0)| + |\nabla^r \phi_k(0)| \leq C_r, \) on \(K\), for all \(k\), \(r = 1, 2, \ldots\),

(c) \(|\nabla^r \phi_k| \leq C'_r, |\nabla^r Rm_k| \leq C'_r, \) on \(K \times [\gamma, \delta]\), for all \(k\), \(r = 0, 1, 2, \ldots\),

for some positive constants \(C'_r, C_r, C'_r\) independent of \(k\), where \(Rm_k\) are the curvature tensor of the metrics \(g_k(t)\), \(\nabla_k\) denote covariant derivative with respect to \(g_k(t)\), \(|\cdot|\) are the length of a tensor with respect to \(g_k(t)\), and \(|\cdot|\) is the length with respect to \(g\). Then the metrics \(g_k(t)\) satisfy

\[
\hat{C}_0^{-1} g \leq g_k(t) \leq \hat{C}_0 g, \text{ on } K \times [\gamma, \delta]
\]

and

\[
\left| \frac{\partial^s}{\partial t^s} \nabla^r g_k \right| + \left| \frac{\partial^s}{\partial t^s} \nabla^r \phi_k \right| \leq \hat{C}_{r,s}, \text{ on } K \times [\gamma, \delta], \text{ for all } k, r, s = 1, 2, \ldots,
\]

for all \(k\), where \(\hat{C}_{r,s}\) are positive constants independent of \(k\).

**Proof.** First, the equation

\[
\frac{\partial}{\partial t} g_k = -2Rc_k + 2\alpha \nabla \phi_k \otimes \nabla \phi_k
\]

and the assumption (c) give that

\[
\hat{C}_0^{-1} g \leq g_k(t) \leq \hat{C}_0 g, \quad (2.15)
\]
on $K \times [\alpha, \beta]$ for some positive constant $\tilde{C}_0$ independent of $k$.

For the second part of the conclusion. When $s = 0$, we divide the proof into two parts. That is, we will prove the following

$$|\nabla^r g_k| \leq \tilde{C}'_{r, 0}, \quad |\nabla^r \phi_k| \leq \tilde{C}''_{r, 0}.$$ 

Here we show the case of $r = 1, 2$, and the higher covariant derivative cases can be derived by the same argument.

Taking the difference of the connection $\Gamma_k$ of $g_k$ and the connection $\Gamma$ of $g$ with $\Gamma$ being fixed in time, we get

$$\frac{\partial}{\partial t}((\Gamma_k)^c_{ab} - \Gamma^c_{ab}) = \frac{\partial}{\partial t} \left( \frac{1}{2} (g_k)^cd \left[ (\nabla_k)_a(g_k)_{bd} + (\nabla_k)_b(g_k)_{ad} - (\nabla_k)_d(g_k)_{ab} \right] \right) = \frac{1}{2} (g_k)^cd \left[ (\nabla_k)_a(-2(R_{ck})_{bd} + 2\alpha(\nabla_k)_b\phi_k \cdot (\nabla_k)_d\phi_k) + (\nabla_k)_b(-2(R_{ck})_{ad} + 2\alpha(\nabla_k)_a\phi_k \cdot (\nabla_k)_d\phi_k) - (\nabla_k)_d(-2(R_{ck})_{ab} + 2\alpha(\nabla_k)_a\phi_k \cdot (\nabla_k)_b\phi_k) \right],$$

and by assumption (c) and (2.15),

$$\left| \frac{\partial}{\partial t}((\Gamma_k) - \Gamma) \right| \leq C,$$ for all $k$.

For time $t = 0$, at a normal coordinate of the metric $g$ at a fixed point, note that

$$(\Gamma_k)^c_{ab} - \Gamma^c_{ab} = \frac{1}{2} (g_k)^cd \left( \nabla_a(g_k)_{bd} + \nabla_b(g_k)_{ad} - \nabla_d(g_k)_{ab} \right),$$

then by assumption (b) and (2.15)

$$|\Gamma_k(0) - \Gamma| \leq C,$$ for all $k$.

Integrating over time we deduce that

$$|\Gamma_k - \Gamma| \leq C,$$ on $K \times [\gamma, \delta]$, for all $k$.

Again using assumption (c), (2.15) and (2.17), we have

$$\left| \frac{\partial}{\partial t}((\nabla g_k)) \right| = \left| -2\nabla((R_{ck} - \alpha \nabla_k \phi_k \otimes \nabla_k \phi_k)) \right| = \left| -2\nabla_k((R_{ck} - \alpha \nabla_k \phi_k \otimes \nabla_k \phi_k) + (\Gamma_k - \Gamma)_* (R_{ck} - \alpha \nabla_k \phi_k \otimes \nabla_k \phi_k)) \right| \leq C,$$ for all $k$.

Thus by combining with assumption (b) we get bounds

$$|\nabla g_k| \leq \tilde{C}'_{1, 0},$$ on $K \times [\gamma, \delta],$ (2.18)
where \( \tilde{C}_{1,0} \) is a positive constant independent of \( k \).

Similarly, assumption (c), (2.15) and (2.17) also indicate

\[
\left| \frac{\partial}{\partial t} (\nabla \phi_k) \right| = |\nabla (\tau_{g_k} \phi_k)|
\]
\[
= |\nabla_k (\tau_{g_k} \phi_k) + (\Gamma_k - \Gamma) * (\tau_{g_k} \phi_k)|
\]
\[
\leq C, \quad \text{for all } k.
\]

This, combined with assumption (b), gives the bounds

\[
|\nabla \phi_k| \leq \tilde{C}_{1,0}', \quad \text{on } K \times \left[ \gamma, \delta \right],
\]

where \( \tilde{C}_{1,0}' \) is a positive constant independent of \( k \).

The case of \( r = 1 \) is finished.

Next we begin to deal with the case of \( r = 2 \). In order to bound \( \nabla^2 g_k \), again regarding \( \nabla \) as fixed in time, we know

\[
\frac{\partial}{\partial t} (\nabla^2 g_k) = -2 \nabla^2 (R_{c_k} - \alpha \nabla_k \phi_k \otimes \nabla_k \phi_k).
\]

Write

\[
\nabla^2 (R_{c_k} - \alpha \nabla_k \phi_k \otimes \nabla_k \phi_k)
\]
\[
= \left[ (\nabla - \nabla_k) \nabla + \nabla_k (\nabla - \nabla_k) \right] (R_{c_k} - \alpha \nabla_k \phi_k \otimes \nabla_k \phi_k)
\]
\[
= (\Gamma - \Gamma_k) * (\nabla (R_{c_k} - \alpha \nabla_k \phi_k \otimes \nabla_k \phi_k))
\]
\[
+ \nabla_k (\Gamma - \Gamma_k) * (R_{c_k} - \alpha \nabla_k \phi_k \otimes \nabla_k \phi_k)
\]
\[
+ \nabla_k \Gamma (R_{c_k} - \alpha \nabla_k \phi_k \otimes \nabla_k \phi_k)
\]
\[
+ \nabla_k \Gamma (R_{c_k} - \alpha \nabla_k \phi_k \otimes \nabla_k \phi_k),
\]

where we have used (2.16). Then using assumption (c), (2.15), (2.17) and (2.18), we have

\[
\left| \frac{\partial}{\partial t} \nabla^2 g_k \right| \leq C + C \cdot |\nabla_k \nabla g_k|
\]
\[
= C + C \cdot |\nabla^2 g_k + (\Gamma_k - \Gamma) * \nabla g_k|
\]
\[
\leq C + C |\nabla^2 g_k|.
\]

Thus by combining with assumption (b) we get bounds

\[
|\nabla^2 g_k| \leq \tilde{C}_{2,0}', \quad \text{on } K \times \left[ \gamma, \delta \right],
\]

where \( \tilde{C}_{2,0}' \) is a positive constant independent of \( k \).

Similarly, assumption (c), (2.15) and (2.17) also indicate

\[
\left| \frac{\partial}{\partial t} (\nabla^2 \phi_k) \right| = |\nabla^2 (\tau_{g_k} \phi_k)|
\]
\[
\leq C + C |\nabla^2 g_k|
\]
\[
\leq C, \quad \text{for all } k.
\]
This, combined with assumption (b), gives the bounds
\[ |\nabla^2 \phi_k| \leq \tilde{C}_{2,0}'' \quad \text{on } K \times [\gamma, \delta], \]
where \( \tilde{C}_{2,0}'' \) is a positive constant independent of \( k \). The case of \( r = 2 \) is also proved.

In the end, we discuss the case \( s \geq 1 \) in the second part of the conclusion. Since
\[
\frac{\partial^s}{\partial s^s} \nabla^r g_k = \nabla^s \frac{\partial^{s-1}}{\partial t^{s-1}} \left( -2Rc_k + 2\alpha \nabla_k \phi_k \otimes \nabla_k \phi_k \right),
\]
\[
\frac{\partial^s}{\partial s^s} \nabla^r \phi_k = \nabla^s \frac{\partial^{s-1}}{\partial t^{s-1}} \left( \tau_{g_k} \phi_k \right),
\]
using the evolution equations for curvature, by induction, we can see that the above two quantities are bounded by a sum of terms which are products of \( |\nabla^r \nabla_k Rm_k|, |\nabla^r \nabla_k^2 Rc_k|, |\nabla^r \nabla_k^3 R_k| \) and \( |\nabla^r \nabla_k^4 \phi_k| \). Hence we get
\[
\left| \frac{\partial^s}{\partial s^s} \nabla^r g_k \right| \leq \tilde{C}_{r,s}'', \quad \left| \frac{\partial^s}{\partial s^s} \nabla^r \phi_k \right| \leq \tilde{C}_{r,s}'', \quad \text{on } K \times [\gamma, \delta].
\]

\[ \square \]

3 Singularities of connection Ricci flow

3.1 Maximal solution of connection Ricci flow

Before studying the singularities of connection Ricci flow, first giving the definition of maximal solution.

**Definition 3.1.** Suppose \( (g(t), H(t))_{t \in [0, T]} \) is a solution to the connection Ricci flow (1.1) on \( M^n \), where either \( M^n \) is compact or at each time \( t \) the metric \( g(\cdot, t) \) is complete and has bounded curvature. We say that \( (g(t), H(t)) \) is a maximal solution if either \( T = \infty \) or \( T < \infty \) and
\[
\lim_{t \to T} \sup_{x \in M^n} |Rm(x, t)| = \infty.
\]

**Remark 3.2.** In fact, on \( M^n \), both the curvature and the 3-form evolve under connection Ricci flow. Hence the flow will end if any one of them goes to infinity. However, according to Theorem 2.2, if \( |H|^2 \) goes to infinity, then it can be indicated that \( |Rm| \) must also go to infinity. So the definition of maximal solution can still be given by the unboundedness of \( |Rm| \).

3.2 Three types of singularities and singularity models of connection Ricci flow

By §3.1, if connection Ricci flow has finite-time singularities, then
\[
\lim_{t \to T} \sup_{x \in M^n} |Rm(x, t)| = \infty,
\]
where $T$ is the maximum time.

However, unlike Ricci flow, for connection Ricci flow, we cannot know how $|Rm|$ goes to infinity near maximum time. To deal with this problem, we define a new curvature quantity.

**Definition 3.3.** For connection Ricci flow, we call $P = |Rm| + |\nabla H| + |H|^2$ the absolute compound curvature (AC curvature).

Under this definition, if connection Ricci flow has finite-time singularities, then

$$\lim_{t \to T} \sup_{x \in M^n} P(x, t) = \infty.$$  

**Remark 3.3.** By (2.1), the three terms in the AC curvature defined above are exactly terms in the curvature on manifold which has $H$ as its torsion. So in the view of geometry, the AC curvature stands for the real curvature bound here.

**Proposition 3.4.** If $0 \leq t < T < \infty$ is the maximal interval of existence of the solution $(M^n, g(t), H(t))$ to connection Ricci flow, there exists a constant $c_0 > 0$ depending only on $n$ such that

$$\sup_{x \in M^n} P(x, t) \geq \frac{c_0}{T-t}. \quad (3.1)$$

**Proof.** By Proposition 2.1,

\[
\frac{\partial}{\partial t} Rm = \Delta Rm + Rm \ast Rm + Rm \ast H \ast H + H \ast \nabla^2 H + \nabla H \ast \nabla H,
\]
\[
\frac{\partial}{\partial t} \nabla H = \Delta(\nabla H) + Rm \ast \nabla H + \nabla Rm \ast H + H \ast H \ast \nabla H,
\]
\[
\frac{\partial}{\partial t} |H|^2 \leq \Delta |H|^2 - 2|\nabla H|^2 + C|Rm||H|^2 + C|H|^4,
\]

so

\[
\frac{\partial}{\partial t} |Rm|^2 \leq \Delta |Rm|^2 - 2|\nabla Rm|^2 + C|Rm|^2|H|^2 + C|Rm||H||\nabla^2 H|
\]
\[+ C|Rm||\nabla H|^2 + C|Rm|^3, \]
\[
\frac{\partial}{\partial t} |H|^2 \leq \Delta |H|^2 - 2|\nabla^2 H|^2 + C|Rm||\nabla H|^2 + C|\nabla Rm||H||\nabla H|
\]
\[+ C|H|^2|\nabla H|^2, \]
\[
\frac{\partial}{\partial t} |H|^4 \leq \Delta |H|^4 + C|Rm||H|^4 + C|H|^6.
\]

Then

\[
\frac{\partial}{\partial t}(|Rm|^2 + |\nabla H|^2 + |H|^4)
\]
\[\leq \Delta(|Rm|^2 + |\nabla H|^2 + |H|^4) - 2|\nabla Rm|^2 - 2|\nabla^2 H|^2
\]
\[+ C|\nabla Rm||H||\nabla H| + C|Rm||H||\nabla^2 H| + C|Rm||\nabla H|^2
\]
\[+ C|Rm|^2|H|^2 + C|Rm||H|^4 + C|H|^2|\nabla H|^2 + C|Rm|^3 + C|H|^6. \]

Using Cauchy inequality,

\[
|\nabla Rm|^2 + C_1|H|^2|\nabla H|^2 \geq C|\nabla Rm||H||\nabla H|,
\]
\[
|\nabla^2 H|^2 + C_2|Rm|^2|H|^2 \geq C|Rm||H||\nabla^2 H|. \]
This gives

\[
\frac{\partial}{\partial t}(|Rm|^2 + |\nabla H|^2 + |H|^4) \\
\leq \Delta(|Rm|^2 + |\nabla H|^2 + |H|^4) + C|Rm||\nabla H|^2 + C|Rm|^3|H|^2 \\
+ C|Rm||H|^4 + C|H|^2|\nabla H|^2 + C|Rm|^3 + C|H|^6 \\
\leq \Delta(|Rm|^2 + |\nabla H|^2 + |H|^4) + C(|Rm| + |\nabla H| + |H|^2)^3 \\
= \Delta(|Rm|^2 + |\nabla H|^2 + |H|^4) + C(|Rm| + |\nabla H| + |H|^2)^{3/2} \\
\leq \Delta(|Rm|^2 + |\nabla H|^2 + |H|^4) + C(|Rm|^2 + |\nabla H| + |H|^4)^{3/2}.
\]

\(C, C_1, C_2\) above are all constants depending only on \(n\). Whenever the supremum is finite, we set

\[K(t) = \sup_{x \in M^n} (|Rm(x, t)|^2 + |\nabla H(x, t)|^2 + |H(x, t)|^4).\]

By the maximum principle, we have

\[\frac{dK}{dt} \leq CK^{\frac{3}{2}},\]

which implies

\[\frac{d}{dt} K^{-\frac{1}{2}} \geq -\frac{C}{2}.\]

Integrating this inequality from \(t\) to \(\tau \in (t, T)\) and using the fact that

\[\lim\inf_{\tau \to T} K(\tau)^{-\frac{1}{2}} = 0,\]

we obtain

\[K(t)^{-\frac{1}{2}} \leq \frac{C}{2}(T - t).\]

Hence

\[\sup_{x \in M^n} (|Rm(x, t)|^2 + |\nabla H(x, t)|^2 + |H(x, t)|^4)^{1/2} \geq \frac{2}{C(T - t)}.\]

Then as \(P \geq (|Rm|^2 + |\nabla H|^2 + |H|^4)^{1/2}\), we finally get

\[\sup_{x \in M^n} P(x, t) \geq \frac{c_0}{T - t}.\]

According to this result, we can classify the singular solutions of connection Ricci flow.

**Definition 3.5.** Define the following three types of singularities of connection Ricci flow (\(T\) is the maximum time),

**Type I singularity:**

\[T < \infty, \quad \sup_{M^n \times [0, T)} P \cdot (T - t) < \infty,\]

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**Type IIa singularity:**

\[ T < \infty, \quad \sup_{M^n \times [0,T)} P \cdot (T - t) = \infty, \]

**Type IIb singularity:**

\[ T = \infty, \quad \sup_{M^n \times [0,\infty)} P \cdot t = \infty, \]

**Type III singularity:**

\[ T = \infty, \quad \sup_{M^n \times [0,\infty)} P \cdot t < \infty. \]

Similar to Ricci flow, we can then give the definition of three singularity models.

**Definition 3.6.** A solution \((M^n, g(t), H(t))\) to the connection Ricci flow (1.1), where either \(M^n\) is compact, or at each time \(t\), the metric \(g(\cdot, t)\) is complete and has bounded AC curvature, is called a singularity model if it is not flat in the sense of AC curvature \((P \not\equiv 0)\) and of one of the following three types:

**Type I singularity model:** The solution exists for \(t \in (-\infty, \omega)\) for some constant \(\omega\) with \(0 < \omega < \infty\) and for any \(x \in M^n\), any \(t \in (-\infty, \omega)\),

\[ P(x, t) \leq \frac{\omega}{\omega - t}, \]

with equality at \(t = 0\) and a point \(y \in M^n\).

**Type II singularity model:** The solution exists for \(t \in (-\infty, +\infty)\), and for any \(x \in M^n\), any \(t \in (-\infty, \omega)\),

\[ P(x, t) \leq 1, \]

with equality at \(t = 0\) and a point \(y \in M^n\).

**Type III singularity model:** The solution exists for \(t \in (-a, +\infty)\), for some constant \(a\) with \(0 < a < \infty\) and for any \(x \in M^n\), any \(t \in (-\infty, \omega)\),

\[ P(x, t) \leq \frac{a}{a + t}, \]

with equality at \(t = 0\) and a point \(y \in M^n\).

In order to use compactness theorem (Theorem 2.4), we then define the injectivity radius estimate.

**Definition 3.7.** A solution \((M^n, g(t), H(t))\) to the connection Ricci flow on the time interval \([0, T)\) is said to satisfy an injectivity radius estimate if there exists a constant \(c_I > 0\) such that

\[ \text{inj}(x, t)^2 \geq \frac{c_I}{\sup_{M^n} P(\cdot, t)} \]

for all \((x, t) \in M^n \times [0, T)\).

Following is the main result of this section.
**Theorem 3.8.** For any maximal solution to the connection Ricci flow which satisfies the injectivity radius estimate and is of Type I, IIa, IIb, or III, there exists a sequence of dilations of the solution which converges in the $C^\infty_{\text{loc}}$ topology to a singularity model of the corresponding type.

We will prove this theorem in next subsection.

### 3.3 Convergence of the dilated solutions to connection Ricci flow

Consider the solution $(M^n, g(t), H(t))$ to the connection Ricci flow (1.1), where either $M^n$ is compact, or at each time $t$, the metric $g(\cdot, t)$ is complete and has bounded AC curvature. To dilate, we choose $(x_i, t_i)$ such that

$$\sup_{M^n} P(\cdot, t_i) \geq c_1 \sup_{M^n \times [s_i, t_i]} P,$$

and

$$P(x_i, t_i) \geq c_2 \sup_{M^n} P(\cdot, t_i),$$

in which $c_1, c_2 \in (0, 1]$, and $s_i$ satisfies $(t_i - s_i) \sup_{M^n} P(\cdot, t_i) \to \infty$ (usually we set $s_i = 0$).

Once we have chosen a sequence $\{(x_i, t_i)\}$ such that $t_i \nearrow T \in (0, \infty]$, consider the solutions $(M^n, g_i(t), H_i(t))$ defined below

$$g_i(t) = P(x_i, t_i) \cdot g\left(t_i + \frac{t}{P(x_i, t_i)}\right),$$

$$H_i(t) = P(x_i, t_i) \cdot H\left(t_i + \frac{t}{P(x_i, t_i)}\right),$$

in the time interval

$$-t_i P(x_i, t_i) \leq t < (T - t_i) P(x_i, t_i).$$

(The right endpoint is $\infty$ if $T = \infty$.) It can be easily verified that after this dilation, they are still solutions to connection Ricci flow. And the AC curvature of the new metric $g_i$ and 3-form $H_i$ has norm 1 at the point $x_i$ and the new time 0:

$$P[g_i, H_i](x_i, 0)|_{g_i} = 1.$$ (3.6)

This guarantees that the limit of the pointed solutions $(M^n, g_i(t), H_i(t), x_i)$, if it exists, will not be flat in the sense of AC curvature.

Assuming (3.2) and (3.3), one has the uniform AC curvature bound

$$P[g_i, H_i](x, t) \leq \frac{1}{c_1 c_2}$$

for all $x \in M^n$ and $t \in [- (t_i - s_i) P(x_i, t_i), 0]$. Combined with the above injectivity radius estimate, by Theorem 2.4, there exists a subsequence of the pointed sequence $(M^n, g_i(t), H_i(t), x_i)$ converges to a complete pointed solution $(M^n_\infty, g_\infty(t), H_\infty(t), x_\infty)$ to connection Ricci flow on an ancient time interval $-\infty < t < \tau \leq \infty$. This solution satisfies

$$P[g_\infty, H_\infty](x, t)|_{g_\infty} \leq \frac{1}{c_1 c_2}.$$
for all \((x, t) \in M^n_\infty \times (-\infty, 0]\).

**Limits of Type I Singularities.** Given a Type I singular solution \((M^n, g(t), H(t))\) on \([0, T)\), define
\[
\Omega \triangleq \sup_{M^n \times [0,T)} P(x, t) \cdot (T - t) < \infty,
\]
and
\[
\omega \triangleq \limsup_{t \to T} P(\cdot, t) \cdot (T - t).
\]
By Proposition 3.4, we have \(\omega \in [c_0, \Omega] \subset (0, \infty)\).

Taking a sequence of points and times \((x_i, t_i)\) with \(t_i \nearrow T\) such that
\[
P(x_i, t_i) \cdot (T - t_i) \xrightarrow{i \to \infty} \omega_i \to \omega. \tag{3.8}
\]
Consider the dilated solutions \((g_i(t), H_i(t))\) to connection Ricci flow defined by (3.4). By the definition of \(\omega\), there is for every \(\epsilon > 0\) a time \(t_\epsilon \in [0, T)\) such that
\[
P(x, t) \cdot (T - t) \leq \omega + \epsilon
\]
for all \(x \in M^n\) and \(t \in [t_\epsilon, T)\). The AC curvature norm of \(g_i\) and \(H_i\) then satisfies
\[
P_i(x, t) = \frac{1}{P(x_i, t_i)} P\left(x, t + \frac{t}{P(x_i, t_i)}\right)
= \frac{P(x, t_i + \frac{1}{P(x_i, t_i)} \cdot (T - t_i - \frac{t}{P(x_i, t_i)})}{P(x_i, t_i) \cdot (T - t_i) - t}
\leq \frac{\omega + \epsilon}{\omega_i - t}. \tag{3.9}
\]
if \(t_\epsilon \leq t_i + t \cdot P(x_i, t_i)^{-1} < T\), hence if \(t \in [-P(x_i, t_i) \cdot (t_i - t_\epsilon), \omega_i)\).

Note that \(\omega_i \to \omega\) and that \(\lim_{i \to \infty} P(x_i, t_i) \cdot (t_i - t_\epsilon) = \infty\) for any \(\epsilon > 0\). So if a pointed limit solution \((M^n_\infty, g_\infty(t), H_\infty(t), x_\infty)\) of a subsequence \(\{M^n, g_i(t), H_i(t)\}\) exists, we can let \(i \to \infty\) and then \(\epsilon \to 0\) in estimate (3.9) to conclude that
\[
P_\infty(x, t) \leq \frac{\omega}{\omega - t}
\]
for all \(x \in M^n_\infty\) and \(t \in (-\infty, \omega)\). Because \(P_i(x_i, 0) = 1\) for all \(i\), the limit satisfies \(P_\infty(x_\infty, 0) = 1\).

**Limits of Type IIa Singularities.** Given a Type IIa singular solution \((M^n, g(t), H(t))\) to connection Ricci flow on \([0, T)\), first let \(T_i\) and \(c'_i > 0\) such that \(T_i \nearrow T, c'_i \nearrow 1\) as \(i \to \infty\). Then take a sequence \(\{(x_i, t_i)\}\), such that \(t_i \nearrow T_i\), and
\[
P(x_i, t_i) \cdot (T_i - t_i) \geq c'_i \sup_{M^n \times [0, T_i]} P(x, t) \cdot (T_i - t). \tag{3.10}
\]
Consider the dilated solutions \((g_i(t), H_i(t))\) to connection Ricci flow defined by (3.4).
(3.10) implies that for \( t \in [-t_i P(x_i, t_i), (T_i - t_i) P(x_i, t_i)] \) we have

\[
P_i(x, t) \left( T_i - t_i - \frac{t}{P(x_i, t_i)} \right)
= \frac{P(x, t) - t}{P(x_i, t_i)(T_i - t_i)} (T_i - t_i)
\leq \frac{1}{C_i} (T_i - t_i),
\]

or equivalently,

\[
P_i(x, t) \leq \frac{1}{C_i} \left( \frac{T_i - t_i}{P(x_i, t_i)} \right) P(x_i, t_i) - t.
\]

Since \( T_i \to T \) and

\[
\lim_{i \to \infty} \sup_{M^n \times [0, T_i]} P(x, t) \cdot (T_i - t) = \infty
\]

by the condition of Type IIa singularities, so we have

\[
P_i(x_i, t_i) \cdot (T_i - t_i) \to \infty
\]

for the selected points and times \((x_i, t_i)\). Hence the pointed limit \((M^n_\infty, g_\infty(t), H_\infty(t), x_\infty)\), if it exists, is defined for all \( t \in (-\infty, \infty) \) and satisfies the uniform AC curvature bound

\[
\sup_{M^n_\infty \times (-\infty, \infty)} P_i \leq 1
\]

with equality at \((x_\infty, 0)\) because \( P_i(x_i, 0) = 1 \).

**Limits of Type IIb Singularities.** The condition for this type of singularities is

\[
\sup_{M^n \times [0, \infty)} t \cdot P(x, t) = \infty.
\]

(3.11)

Similar to the Type IIa case, let \( T_j \to \infty \) and choose \((x_i, t_i)\) such that

\[
\frac{t_i (T_i - t_i) \cdot P(x_i, t_i)}{\sup_{M^n \times [0, T_i]} t (T_i - t) \cdot P(x, t)} \triangleq 1 - \delta_i \to 1.
\]

(3.12)

Define

\[
\alpha_i \triangleq t_i \cdot P(x_i, t_i),
\]

\[
\omega'_i \triangleq (T_i - t_i) \cdot P(x_i, t_i).
\]

(3.11) and (3.12) guarantee that \( \alpha_i \to \infty \) and \( \omega'_i \to \infty \). This is because

\[
\frac{1}{\alpha_i^{-1} + \omega_i'} = \frac{\alpha_i \omega'_i}{\alpha_i + \omega_i'} = \frac{t_i \cdot P(x_i, t_i) \cdot (T_i - t_i)}{T_i}
= \frac{1}{T_i} \sup_{M^n \times [0, T_i]} t \cdot P(x, t) \cdot (T_i - t)
\geq \frac{1}{2} \sup_{M^n \times [0, T_i]} t \cdot P(x, t) \to \infty,
\]

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\[
\alpha_i^{-1} \to 0 \quad \text{and} \quad \omega_i^{-1} \to 0.
\]

Meanwhile, for all \(x \in M^n\) and \(t \in [-\alpha_i, \omega_i')\), we have
\[
P_i(x, t) = \frac{1}{P(x_i, t_i)} P \left( x, t_i + \frac{t}{P(x_i, t_i)} \right) = \frac{(t_i + \frac{t}{P(x_i, t_i)})(T_i - t_i - \frac{t}{P(x_i, t_i)})P(x, t_i + \frac{t}{P(x_i, t_i)})}{t_i(T_i - t_i)P(x_i, t_i)} \times \frac{t_i \cdot p(x_i, t_i)(T_i - t_i - \frac{t}{P(x_i, t_i)})}{(t_i \cdot p(x_i, t_i) + t)(T_i - t_i - \frac{t}{P(x_i, t_i)})} \leq \frac{1}{1 - \delta_i} \times \frac{\alpha_i}{\alpha_i + t} \omega_i' - t.
\]

Since \(\delta_i \to 0\), \(\alpha_i \to \infty\) and \(\omega_i' \to \infty\), we conclude that the pointed limit solution \((M^n_\infty, g_\infty(t), H_\infty(t), x_\infty)\), if it exists, is defined for all \(t \in (-\infty, \infty)\) and satisfies the AC curvature bound
\[
\sup_{M^n_\infty \times (-\infty, \infty)} P_\infty \leq 1 = P_\infty(x_\infty, 0).
\]

**Limits of Type III Singularities.** If the solution \((M^n, g(t), H(t))\) to connection Ricci flow is a Type III singularity, then it exists for \(t \in [0, \infty)\) and satisfies
\[
\sup_{M^n \times [0, \infty)} t \cdot P(\cdot, t) < \infty.
\]

Define
\[
a \equiv \limsup_{t \to \infty} (t \cdot \sup_{M^n} P(\cdot, t)) \in [0, \infty).
\]

First we need to show that \(a\) is strictly positive. By the conclusion in Riemann geometry, for a fixed curve \(\gamma\) joining two points \(p_1, p_2 \in M^n\), under connection Ricci flow, the length of the curve evolves like
\[
\frac{d}{dt} L_t(\gamma) = \frac{1}{2} \int_\gamma \frac{\partial g}{\partial t}(\dot{\gamma}, \dot{\gamma}) ds = -\int_\gamma \left( \text{Rc} - \frac{1}{4} \mathcal{H} \right)(\dot{\gamma}, \dot{\gamma}) ds.
\]

Next we argue by contradiction. Assume \(a = 0\), then we have

**Claim.** If there is an \(\epsilon = \epsilon(n) > 0\) small enough such that
\[
a = \limsup_{t \to \infty} (t \cdot \sup_{M^n} P(\cdot, t)) \leq \epsilon, \quad (3.13)
\]

then there exist \(C < \infty\), \(\delta > 0\) and \(T_\epsilon < \infty\) depending only on \(n\) such that
\[
\text{diam}(M^n, g(t)) \leq C t^{\frac{2}{3} - \delta} \quad (3.14)
\]

for all \(t \geq T_\epsilon\).

To prove this claim, let \(\gamma : [a, b] \to M^n\) be a fixed path, then
\[
\left| \frac{dL}{dt} \right| \leq \int_\gamma \left| \text{Rc} - \frac{1}{4} \mathcal{H} \right|_{g(t)} ds.
\]
If (3.13) holds for some $\epsilon > 0$, there exists a time $T_\epsilon < \infty$ such that for all $t > T_\epsilon$, we have

$$t \cdot \sup_{M^n} P(\cdot, t) \leq 2\epsilon.$$ 

In particular,

$$\sup_{M^n} \left| \text{Rc} - \frac{1}{4} H(\cdot, t) \right| \leq \frac{C\epsilon}{t}$$

for all $t \geq T_\epsilon$, where $C$ depends only on $n$. Hence

$$\frac{dL}{dt}(\tau) \leq \frac{|dL}{dt}(\tau)}{\tau} \leq \frac{C\epsilon}{\tau} L(\tau)$$

for $\tau \geq T_\epsilon$. Integrating this inequality from time $T_\epsilon$ to time $t > T_\epsilon$ implies

$$L(t) \leq L(T_\epsilon) \left(\frac{t}{T_\epsilon}\right)^{C\epsilon} = L(T_\epsilon) T_\epsilon^{-C\epsilon} \cdot t^{C\epsilon}.$$ 

In particular, if we choose $0 < \epsilon < \frac{1}{2C}$, then any two points in $(M^n, g(t))$ can be joined by a path of length

$$L(t) \leq \text{diam}(M^n, g(T_\epsilon)) \cdot T_\epsilon^{-C\epsilon} \cdot t^{\frac{C}{2} - \delta},$$

where $\delta \triangleq \frac{1}{2} - C\epsilon$. This implies (3.14) and proves the claim.

Using the claim, let $\Omega' \triangleq \sup_{M^n \times [0, \infty)} (t \cdot \sup_{M^n} P(\cdot, t)) < \infty$, then for any $t \in (0, \infty)$,

$$\sup_{M^n} P(\cdot, t) \cdot \text{diam}(M^n, g(t))^2 \leq \Omega' C^2 t^{-2\delta}.$$ 

Since $\Omega' C^2 t^{-2\delta} \to 0$ as $t \to \infty$, this contradicts the injectivity radius estimate! Hence our original assumption is false, that is $a > 0$.

By the definition of $a$, there exist sequences $(x_i, t_i)$ with $t_i \to \infty$ such that

$$a_i \triangleq t_i \cdot P(x_i, t_i) \to a.$$ 

Choose any such sequence. Also by the definition of $a$, there is for any $\xi > 0$ a time $T_\xi \in [0, \infty)$ such that

$$t \cdot P(x, t) \leq a + \xi$$

for all $x \in M^n$ and $t \in [T_\xi, \infty)$. The dilated solutions $(g_i(t), H_i(t))$ exist on the time intervals $[-a_i, \infty)$ and satisfy

$$P_i(x, t) = \frac{1}{P(x_i, t_i)} P\left(x, t_i + \frac{t}{P(x_i, t_i)}\right)$$

$$= \frac{P(x, t_i + \frac{t}{P(x_i, t_i)} \cdot (t_i + \frac{t}{P(x, t_i)})}{P(x_i, t_i) \cdot t_i + t}$$

$$\leq \frac{a + \xi}{a_i + t},$$

if $t_i + t \cdot P(x_i, t_i)^{-1} \geq T_\xi$, that is if $t \geq P(x_i, t_i) \cdot (T_\xi - t_i)$. Note that for any fixed $\xi > 0$, we have

$$P(x_i, t_i) \cdot (T_\xi - t_i) \to -a.$$
and
\[ \frac{a + \xi}{a_i + t} \rightarrow \frac{a + \xi}{a + t} \]
for \( t > -a \). Hence the pointed limit solution \((M^n_{\infty}, g_{\infty}(t), H_{\infty}(t), x_\infty))\), if it exists, is defined for \( t \in (-a, \infty) \) and satisfies
\[
\sup_{M^n_{\infty} \times (-a, \infty)} P_{\infty} \leq \frac{a}{a + t} = P_{\infty}(x_\infty, 0).
\]

To sum up, we finish the proof of Theorem 3.8.

4 Singularities of Ricci harmonic flow

4.1 Singularity models and convergence of dilated solutions

By Theorem 2.8, if Ricci harmonic flow has finite-time singularities, then
\[
\lim_{t \to T} \sup_{x \in M^m} |Rm(x, t)| = \infty,
\]
where \( T \) is the maximum time.

Similar to the case of connection Ricci flow, define an AC curvature
\[
Q = |Rm| + |
abla^2 \phi| + |\nabla \phi|^2.
\]

Similarly, if Ricci harmonic flow has finite-time singularities, then
\[
\lim_{t \to T} \sup_{x \in M^m} Q(x, t) = \infty.
\]

Besides, here we also have

**Proposition 4.1.** If \( 0 \leq t < T < \infty \) is the maximal interval of existence of the solution \((M^m, g(t), \phi(t))\) to Ricci harmonic flow, there exists a constant \( c_0 > 0 \) depending only on \( m, \alpha \) and the curvature of manifold \( N^n \) such that
\[
\sup_{x \in M^m} Q(x, t) \geq \frac{c_0}{T - t}.
\]

**Proof.** By Proposition 2.5,
\[
\frac{\partial}{\partial t} |Rm|^2 \leq \Delta |Rm|^2 - 2|\nabla Rm|^2 + C |Rm|^2 |\nabla \phi|^2 + C |Rm||\nabla^2 \phi|^2 + C |Rm||\nabla \phi|^4 + C |Rm|^3,
\]
\[
\frac{\partial}{\partial t} |\nabla^2 \phi|^2 \leq \Delta |\nabla^2 \phi|^2 - 2|\nabla^3 \phi|^2 + C |Rm||\nabla^2 \phi|^2 + C |\nabla^3 \phi|^2 |\nabla \phi|^2 + C |\nabla^2 \phi||\nabla \phi|^4,
\]
\[
\frac{\partial}{\partial t} |\nabla \phi|^4 \leq \Delta |\nabla \phi|^4 + C |\nabla \phi|^6.
\]
After computation, we can also get

\[
\frac{\partial}{\partial t} (\|Rm\|^2 + |\nabla^2 \phi|^2 + |\nabla \phi|^4) \\
\leq \Delta (\|Rm\|^2 + |\nabla^2 \phi|^2 + |\nabla \phi|^4) + C (\|Rm\|^2 + |\nabla^2 \phi|^2 + |\nabla \phi|^4)^{3/2}
\]

for a constant \( C > 0 \) depending only on \( m, \alpha \) and the curvature of manifold \( N^n \). The following is like that in the proof of Proposition 3.4. \( \square \)

Replace the \( P \) in the case of connection Ricci flow by \( Q \), we can then obtain the classification of singular solutions, singularity models and convergence of dilated solutions of Ricci harmonic flow.

**Definition 4.2.** Define the following three types of singularities of Ricci harmonic flow (\( T \) is the maximum time),

**Type I** singularity:
\[
T < \infty, \quad \sup_{M^m \times [0,T]} Q \cdot (T - t) < \infty,
\]

**Type IIa** singularity:
\[
T < \infty, \quad \sup_{M^m \times [0,T]} Q \cdot (T - t) = \infty,
\]

**Type IIb** singularity:
\[
T = \infty, \quad \sup_{M^m \times [0,\infty)} Q \cdot t = \infty,
\]

**Type III** singularity:
\[
T = \infty, \quad \sup_{M^m \times [0,\infty)} Q \cdot t < \infty.
\]

Then the definition of the corresponding three types of singularity models

**Definition 4.3.** A solution \((M^m, g(t), \phi(t))\) to the Ricci harmonic flow (1.2), where either \( M^m \) is compact, or at each time \( t \), the metric \( g(\cdot, t) \) is complete and has bounded AC curvature, is called a **singularity model** if it is not flat in the sense of AC curvature (\( Q \neq 0 \)) and of one of the following three types:

**Type I** singularity model: The solution exists for \( t \in (-\infty, \omega) \) for some constant \( \omega \) with \( 0 < \omega < \infty \) and for any \( x \in M^m \), any \( t \in (-\infty, \omega) \),
\[
Q(x, t) \leq \frac{\omega}{\omega - t},
\]
with equality at \( t = 0 \) and a point \( y \in M^m \).

**Type II** singularity model: The solution exists for \( t \in (-\infty, +\infty) \), and for any \( x \in M^m \), any \( t \in (-\infty, \omega) \),
\[
Q(x, t) \leq 1,
\]
with equality at $t = 0$ and a point $y \in M^m$.

**Type III singularity model:** The solution exists for $t \in (-a, +\infty)$, for some constant $a$ with $0 < a < \infty$ and for any $x \in M^m$, any $t \in (-\infty, \omega)$,

$$Q(x, t) \leq \frac{a}{a + t},$$

with equality at $t = 0$ and a point $y \in M^m$.

The injectivity radius estimate for Ricci harmonic flow,

**Definition 4.4.** A solution $(M^m, g(t), \phi(t))$ to Ricci harmonic flow on the time interval $[0, T]$ is said to satisfy an injectivity radius estimate if there exists a constant $c_I > 0$ such that

$$\text{inj}(x, t)^2 \geq \frac{c_I}{\sup_{M^m} Q(\cdot, t)},$$

for all $(x, t) \in M^m \times [0, T]$.

Then we give the convergence of dilated solutions.

**Theorem 4.5.** For any maximal solution to Ricci harmonic flow which satisfies the injectivity radius estimate and is of Type I, IIa, IIb, or III, there exists a sequence of dilations of the solution which converges in the $C^\infty$ topology to a singularity model of the corresponding type.

For Ricci harmonic flow, the dilated solutions are defined as

$$g_i(t) = Q(x_i, t_i) \cdot g(t_i + \frac{t}{Q(x_i, t_i)}),$$

$$\phi_i(t) = \phi(t_i + \frac{t}{Q(x_i, t_i)}),$$

(4.1)

And note that we have established the compactness theorem (Theorem 2.9) for Ricci harmonic flow. The rest of the proof is the same as that in connection Ricci flow.

### 4.2 Compact blow-up limits of finite-time singularities are shrinking solitons

Analogous to the work for Ricci flow in [Zhang], we can connect the finite-time singularities of Ricci harmonic flow with shrinking Ricci harmonic solitons.

**Theorem 4.6.** Let $(g(t), \phi(t))_{t \in [0, T]}$, be a maximal solution to the Ricci harmonic flow (1.2) on a closed manifold $M^m$ with singular time $T < \infty$. Let $t_k \rightarrow T$ be a sequence of times such that $Q_k = Q(p_k, t_k) \rightarrow \infty$. If the rescaled sequence $(M^m, Q_k g(t_k + Q_k^{-1} t), \phi(t_k + Q_k^{-1} t), p_k)$ converges in the $C^\infty$ sense to a closed ancient solution $(M^\infty, g_\infty(t), \phi_\infty(t), p_\infty)$ to the Ricci harmonic flow, then $(g_\infty(t), \phi_\infty(t))$ must be a shrinking Ricci harmonic soliton.

First we derive some estimates about $\lambda_\alpha$ and $\mu_\alpha$ functionals.
Lemma 4.7. We have the upper bound

\[ \mu_\alpha(g, \phi, \tau) \leq \tau \lambda_\alpha(g, \phi) + \text{Vol}(g) - \frac{m}{2} \ln(4\pi\tau) - m, \]  

(4.2)

and the lower bound for \( \tau > \frac{m}{8} \),

\[ \mu_\alpha(g, \phi, \tau) \geq \tau \lambda_\alpha - \frac{m}{2} \ln(4\pi\tau) - m - \frac{m}{8} (\lambda_\alpha - \inf \{ R - \alpha |\nabla \phi|^2 \} ) - m \ln C_s, \]  

(4.3)

where \( C_s \) denotes the Sobolev constant for \( g \) such that \( \| \psi \|_{L_{\text{meas}}^2(g)} \leq C_s \| \psi \|_{H^{1,2}(g)} \) for all \( \psi \in C^\infty(M^m) \).

Proof. By definition, set \( u = (4\pi\tau)^{-m/4} e^{-f/2} \), then \( \int_{M^m} u^2 dV = 1 \) and so

\[
\int_{M^m} [\tau (R + |\nabla f|^2 - \alpha |\nabla \phi|^2) + f - m](4\pi\tau)^{-m/2} e^{-f} dV
\]

\[
= \tau \int_{M^m} (Ru^2 + 4|\nabla u|^2 - \alpha |\nabla \phi|^2 u^2) dV - \int_{M^m} u^2 \ln u^2 dV - \frac{m}{2} \ln(4\pi\tau) - m
\]

\[
\leq \tau \int_{M^m} (Ru^2 + 4|\nabla u|^2 - \alpha |\nabla \phi|^2 u^2) dV + \text{Vol}(g) - \frac{m}{2} \ln(4\pi\tau) - m,
\]

where we used that \( -t \ln t \leq 1 \) for all \( \tau > 0 \). The upper bound follows by choosing \( f \) such that \( u \) is the eigenfunction of the first eigenvalue of \( -4\Delta + R - \alpha |\nabla \phi|^2 \).

As for the lower bound, let \( \bar{f} \) be the minimizer of \( \mu_\alpha(g, \phi, \tau) \) for fixed \( \tau > 0 \) and set \( \bar{u} = (4\pi\tau)^{-m/4} e^{-\bar{f}/2} \). We estimate the term \( -\int_{M^m} \bar{u}^2 \ln \bar{u}^2 dV \) by

\[
-\int_{M^m} \bar{u}^2 \ln \bar{u}^2 dV = -\frac{m}{2} \int_{M^m} \bar{u}^2 \ln \frac{\bar{u}^2}{\bar{u}^2} dV \geq -m \ln \| \bar{u} \|_{L_{\text{meas}}^2(g)} \]

\[
\geq -\frac{m}{2} \ln \left( 1 + \int_{M^m} |\nabla \bar{u}|^2 dV \right) - m \ln C_s,
\]

where we used the Jensen and Sobolev inequality in the first and the second inequality. Then we have

\[
\mu_\alpha(g, \phi, \tau)
\]

\[
= \tau \int_{M^m} (R\bar{u}^2 + 4|\nabla \bar{u}|^2 - \alpha |\nabla \phi|^2 \bar{u}^2) dV - \int_{M^m} \bar{u}^2 \ln \bar{u}^2 dV - \frac{m}{2} \ln(4\pi\tau) - m
\]

\[
\geq \tau \int_{M^m} (R\bar{u}^2 + 4|\nabla \bar{u}|^2 - \alpha |\nabla \phi|^2 \bar{u}^2) dV - \frac{m}{2} \ln \left( 1 + \int_{M^m} |\nabla \bar{u}|^2 dV \right)
\]

\[
- \frac{m}{2} \ln(4\pi\tau) - m - m \ln C_s,
\]

\[
\geq \left( \tau - \frac{m}{8} \right) \int_{M^m} (R\bar{u}^2 + 4|\nabla \bar{u}|^2 - \alpha |\nabla \phi|^2 \bar{u}^2) dV + \frac{m}{8} \int_{M^m} (R - \alpha |\nabla \phi|^2) \bar{u}^2 dV
\]

\[
- \frac{m}{2} \ln(4\pi\tau) - m - m \ln C_s,
\]

which proves the lower bound if we set \( \tau > \frac{m}{8} \). \( \square \)
Corollary 4.8.

$$\nu_\alpha(g, \phi) = -\infty \iff \lambda_\alpha(g, \phi) \leq 0, \quad \nu_\alpha(g, \phi) \neq -\infty \iff \lambda_\alpha(g, \phi) > 0.$$  

Proof. From the conclusion of [Ro] which indicates that the functional $$\mu_\alpha(g, \phi, \tau)$$ in Ricci harmonic flow is always attainable by some smooth function, it can be easily seen that $$\mu_\alpha(g, \phi, \tau)$$ is continuous for $$\tau > 0$$.

When $$\lambda_\alpha(g, \phi) \leq 0$$, since

$$\lim_{\tau \to \infty} \left[ \tau \lambda_\alpha(g, \phi) - \frac{m}{2} \ln(4\pi \tau) \right] = -\infty,$$

from the first inequality of above lemma, $$\nu_\alpha(g, \phi) = -\infty$$.

When $$\lambda_\alpha(g, \phi) > 0$$, since

$$\lim_{\tau \to \infty} \left[ \left( \tau - \frac{m}{8} \right) \lambda_\alpha(g, \phi) - \frac{m}{2} \ln(4\pi \tau) \right] = +\infty,$$

from the second inequality of above lemma, $$\mu_\alpha(g, \phi, \tau) > -\infty$$ as $$\tau \to \infty$$. Like the functional $$\mu(g, \tau)$$ in Ricci flow, $$\mu_\alpha(g, \phi, \tau) > -\infty$$ as $$\tau \to 0^+$$. This gives $$\nu_\alpha(g, \phi) \neq -\infty$$. Corollary is proved. \qed

It follows that $$\nu_\alpha$$ functional is valuable only when $$\lambda_\alpha > 0$$. The assumption of our main theorem implies the positivity of $$\lambda_\alpha$$ along the Ricci harmonic flow. This fact will be proved later.

Corollary 4.9. If $$\lambda_\alpha(g, \phi) \leq 0$$, then $$\mu_\alpha(g, \phi, \tau) \leq \ln \text{Vol}(g) - \frac{m}{4} \ln(4\pi \tau) - m + 1$$.

Proof. First note that $$\mu_\alpha(ag, \phi, a\tau) = \mu_\alpha(g, \phi, \tau)$$ for any $$a > 0$$ by a direct computation. Set $$V = \text{Vol}(g)^{-2/m}$$, then by the above lemma,

$$
\mu_\alpha(g, \phi, \tau) = \mu_\alpha(Vg, \phi, V\tau) \\
\leq V\tau \lambda_\alpha(Vg, \phi) - \frac{m}{2} \ln(4\pi V\tau) - m + \text{Vol}(Vg) \\
\leq \ln \text{Vol}(g) - \frac{m}{2} \ln(4\pi \tau) - m + 1.
$$

\qed

Lemma 4.10. Let $$(g(t), \phi(t))$$, $$t \in [0, T)$$, be a solution to the Ricci harmonic flow on a closed manifold $$M^m$$. If $$\lambda_\alpha(g(t), \phi(t)) \leq 0$$ for all $$t$$, then there exist constants $$c_1, c_2 > 0$$ depending only on $$m, \alpha$$ and $$(g(0), \phi(0))$$, such that for all $$t \geq 0$$ we have $$\text{Vol}(g(t)) \geq c_1 e^{-c_2 t}$$.

Proof. By Proposition 2.7 and Lemma 4.7, we have

$$
\mu_\alpha(g(t), \phi(t), \frac{m}{8}) \geq \mu_\alpha(g(0), \phi(0), \frac{m}{8} + t) \\
\geq \lambda_\alpha(g(0), \phi(0)) t - \frac{m}{2} \ln \left( 4\pi \left( t + \frac{m}{8} \right) \right) \\
+ \frac{m}{8} \inf \{ R(\cdot, 0) - \alpha |\nabla \phi|^2(\cdot, 0) \} - m - m \ln C_s(g(0)) \\
\geq \left( \lambda_\alpha(g(0), \phi(0)) - 4 \right) t - \frac{m}{2} \ln \left( \frac{m}{2} \pi \right) \\
+ \frac{m}{8} \inf \{ R(\cdot, 0) - \alpha |\nabla \phi|^2(\cdot, 0) \} - m - m \ln C_s(g(0)),
$$
where $C_s(g(0))$ denotes the Sobolev constant of $(M^m, g(0))$. Setting
\[ c_1 = \exp \left( \frac{m}{8} \inf \{ R(\cdot, 0) - \alpha |\nabla \phi|^2(\cdot, 0) \} - m \ln C_s(g(0)) - 1 \right), \quad c_2 = -\lambda_\alpha(g(0), \phi(0)) + 4, \]
and substituting $\tau = \frac{m}{8}$ into Corollary 4.9, we obtain the estimate
\[ \text{Vol}(g(t)) \geq \exp \left( \mu_\alpha \left( g(t), \phi(t), \frac{m}{8} \right) \right) + \frac{m}{2} \ln \left( \frac{m}{2} \pi \right) + m - 1 \geq c_1 \exp(-c_2 t). \]

**Corollary 4.11.** Let $(g(t), \phi(t))$, $t \in [0, T)$, be a maximal solution to the Ricci harmonic flow on a closed manifold $M^m$ with $T < \infty$. If $\lambda_\alpha(g(t), \phi(t)) \leq 0$ for all $t$, then any blow-up limit is noncompact.

**Proof.** Suppose we have a blow-up sequence $(M^m, Q_k g(t_k + Q_k^{-1} t), \phi(t_k + Q_k^{-1} t), p_k)$ of Ricci harmonic flow solutions with $Q_k \to \infty$. By assumption and above lemma, we have that the rescaled volume at time zero equals $Q_k^{m/2} \text{Vol}(g(t_k)) \to \infty$. So the limit has infinite volume and consequently cannot be compact.

Now we are ready to give a **Proof of Theorem 4.6.** By above corollary, we may assume that $\lambda_\alpha(g(0), \phi(0)) > 0$. So Proposition 2.7 uses and there is a limit $\sigma = \lim_{t \to T^-} \nu_\alpha(g(t), \phi(t))$. Then for any $t \in (-\infty, 0]$, by the smooth convergence,
\[ \nu_\alpha(g_\infty(t), \phi_\infty(t)) = \lim_{k \to \infty} \nu_\alpha(Q_k g(t_k + Q_k^{-1} t), \phi(t_k + Q_k^{-1} t)) = \lim_{k \to \infty} \nu_\alpha(g(t_k + Q_k^{-1} t), \phi(t_k + Q_k^{-1} t)) = \lim_{t \to T^-} \nu_\alpha(g(t), \phi(t)) = \sigma. \]
That is, the $\nu_\alpha$ functional is constant on the limit flow. Then Corollary 4.8 and Proposition 2.7 imply that $(g_\infty(t), \phi_\infty(t))$ must be a shrinking Ricci harmonic soliton.

**4.3 Ancient solutions of Ricci harmonic flow have nonnegative scalar curvature**

In this part, we study the ancient solutions of Ricci harmonic flow in the case of compact and noncompact manifolds. Specifically, the Type I singularity model given by §4.2 is just an ancient solution.

First is the compact case.

**Theorem 4.12.** Let $(g(t), \phi(t))$ be an ancient solution to Ricci harmonic flow (1.2) on a compact manifold $M^m$, then for any $t$ such that the solution exists, we have $S = R - \alpha |\nabla \phi|^2 \geq 0$. That is, the scalar curvature is always nonnegative.

**Proof.** The main idea of the proof is using maximum principle to (2.14)
\[ \frac{\partial}{\partial t} S = \Delta S + 2|S_{ij}|^2 + 2\alpha |\tau_\phi|^2. \]
Define \( S_{\text{min}}(t) = \min_{M^m} S(\cdot, t) \), then

\[
\frac{d}{dt} S_{\text{min}} \geq \frac{2}{m} S_{\text{min}}^2 \geq 0.
\]

Suppose there exists a time \( t_0 \in (-\infty, \Omega) \) and a point \( x_0 \in M^m \), such that \( S(x_0, t_0) < 0 \), then \( S_{\text{min}}(t_0) < 0 \). Pick some time \( t_1 < t_0 \), integrating (4.4) from \( t_1 \) to \( t_0 \) yields

\[
\frac{1}{S_{\text{min}}(t_1)} \geq \frac{2}{m} (t_0 - t_1) + \frac{1}{S_{\text{min}}(t_0)}.
\]

Since \((g(t), \phi(t))\) is an ancient solution, \( t_1 \) can be chosen in the whole interval \((-\infty, t_0)\). Let \( t_1 = t_0 + \frac{m}{2 S_{\text{min}}(t_0)} \), then we have

\[
\frac{1}{S_{\text{min}}(t_1)} \geq 0 \Rightarrow S_{\text{min}}(t_1) \geq 0 \geq S_{\text{min}}(t_0).
\]

But by (4.4), \( S_{\text{min}} \) increases, which is a contradiction! So the assumption is not true, and the theorem is proved.

As for the noncompact case, refer to the work for Ricci flow in [Chen], similar conclusion can still be obtained.

Before giving the theorem, we first introduce the following lemma.

**Lemma 4.13.** Let \((g(x, t), \phi(x, t))\) be a solution to Ricci harmonic flow (1.2) on \( M^m \), and denote by \( d_t(x, x_0) \) the distance between \( x \) and \( x_0 \) with respect to the metric \( g(t) \). Suppose \( \text{Rc}(\cdot, t) \leq (m - 1)K \) on \( B_{r_0}(x_0, r_0) \) for some \( x_0 \in M^m \) and some positive constant \( K \) and \( r_0 \). Then at \( t = t_0 \) and outside \( B_{r_0}(x_0, r_0) \), the distance function \( d(x, t) = d_t(x, x_0) \) satisfies the differential inequality

\[
\left( \frac{\partial}{\partial t} - \Delta \right) d \geq -(m - 1) \left( \frac{2}{3} K r_0 + r_0^{-1} \right).
\]

**Proof.** Let \( \gamma : [0, d(x, x_0)] \rightarrow M^m \) be a shortest normal geodesic from \( x_0 \) to \( x \) with respect to the metric \( g(t_0) \). We may assume that \( x \) and \( x_0 \) are not conjugate to each other in the metric \( g(t_0) \), otherwise we can understand the differential inequality in the barrier sense. Let \( X = \gamma(0) \), and let \( \{X, e_1, \ldots, e_{m-1}\} \) be an orthonormal basis of \( T_{x_0}M \). Extend this basis parallel along \( \gamma \) to form a parallel orthonormal basis \( \{X(s), e_1(s), \ldots, e_{m-1}(s)\} \) along \( \gamma \).

Let \( X_i(s) \), \( i = 1, \ldots, m - 1 \) be the Jacobian fields along \( \gamma \) such that \( X_1(0) = 0 \) and \( X_i(d_t(x, t_0)) = e_i(d(x, t_0)) \) for \( i = 1, \ldots, m - 1 \). Then it is well-known that

\[
\Delta d_{t_0}(x, x_0) = \sum_{i=1}^{m-1} \int_0^{d(x, t_0)} (|\dot{X}_i|^2 - R(X, X_i, X_i))ds.
\]

Define vector fields \( Y_i \), \( i = 1, \ldots, m - 1 \), along \( \gamma \) as follows:

\[
Y_i(s) = \begin{cases} \frac{\partial}{\partial s} e_i(s), & \text{if } s \in [0, r_0], \\ e_i(s), & \text{if } s \in [r_0, d(x, t_0)]. \end{cases}
\]
They have the same value as the corresponding Jacobian fields \( X_i(s) \) at the two end points of \( \gamma \). Then by standard index comparison theorem we have

\[
\Delta d_{t_0}(x,x_0) = \sum_{i=1}^{m-1} \int_0^{d(x,t_0)} (|\dot{X}_i|^2 - R(X,X,X,X_i))ds
\]

\[
\leq \sum_{i=1}^{m-1} \int_0^{d(x,t_0)} (|\dot{Y}_i|^2 - R(X,Y,Y,Y_i))ds
\]

\[
= \int_0^{r_0} \frac{1}{r_0}(m - 1 - s^2Rc(X,X))ds + \int_{r_0}^{d(x,t_0)} (-Rc(X,X))ds
\]

\[
= -\int_\gamma Rc(X,X) + \int_0^{r_0} \left( \frac{m - 1}{r_0^2} + \left( 1 - \frac{s^2}{r_0^2} \right)Rc(X,X) \right)ds
\]

\[
\leq -\int_\gamma Rc(X,X) + (m - 1)\left( \frac{2}{3}Kr_0 + r_0^{-1} \right).
\]

On the other hand,

\[
\frac{\partial}{\partial t} d_t(x,x_0) = \frac{\partial}{\partial t} \int_0^{d(x,t_0)} \sqrt{g_{ij}X_iX_j}ds
\]

\[
= -\int_\gamma (Rc - \alpha\nabla\phi \otimes \nabla\phi)(X,X)ds.
\]

Make the subtraction and consider that \( (\nabla\phi \otimes \nabla\phi)(X,X) = |\nabla_X\phi|^2 \geq 0 \), then the lemma is proved.

Now we state and prove the result for noncompact case.

**Theorem 4.14.** Let \((g(t),\phi(t))\) be a complete ancient solution to Ricci harmonic flow \((1.2)\) on a noncompact manifold \(M^m\), then for any \(t\) such that the solution exists, we have \(S = R - \alpha|\nabla\phi|^2 \geq 0\). That is, the scalar curvature is always nonnegative.

**Proof.** Suppose \((g(t),\phi(t))\) is defined for \(t \in (-\infty,T]\) for some \(T > 0\). We divide the arguments into two steps.

Step 1: Consider any complete solution \((g(t),\phi(t))\) defined on \([0,T]\). For any fixed point \(x_0 \in M^m\), pick \(r_0 > 0\) sufficiently small so that

\[
|Rc(\cdot,t)| \leq (m - 1)r_0^{-2} \quad \text{on } B_t(x_0,r_0)
\]

for all \(t \in [0,T]\). Then for any positive number \(A > 2\), pick \(K_A > 0\) such that \(S \geq -K_A\) on \(B_0(x_0,Ar_0)\) at \(t = 0\). We claim that there exists a universal constant \(C > 0\) (depending on the dimension \(m\)) such that

\[
S(\cdot,t) \geq \min \left\{ -\frac{m}{t + K_A^{-1}}, -\frac{C}{Ar_0^2} \right\} \quad \text{on } B_t \left( x_0, \frac{3A}{4}r_0 \right)
\]

for each \(t \in [0,T]\).
Take a smooth nonnegative decreasing function \( \bar{\psi} \) on \( \mathbb{R} \) such that \( \bar{\psi} = 1 \) on \( (-\infty, \frac{7}{8}] \), and \( \bar{\psi} = 0 \) on \([1, \infty)\). Also, let it satisfy \( |\bar{\psi}'| \leq C_1 \bar{\psi}^{1/2}, |\bar{\psi}''| \leq C_2 \). If we set \( \psi = \bar{\psi}^p \), then we can find suitable \( p > 0 \), such that

\[
|\psi'| \leq C_1 \psi^{\frac{p}{2}}, \quad \left| \frac{2\psi'^2}{\psi} - \psi'' \right| \leq C_2 \psi^{\frac{p}{2}}.
\]

It can be verified that \( \psi \) is still a smooth nonnegative decreasing function on \( \mathbb{R} \) which has value 1 on \( (-\infty, \frac{7}{8}] \) and value 0 on \([1, \infty)\).

Consider the function \( u(x, t) = \psi \left( \frac{d_t(x_0, x)}{A r_0} \right) \cdot S(x, t) \).

Then we have

\[
\left( \frac{\partial}{\partial t} - \Delta \right) u = \frac{\psi' S}{A r_0} \left( \frac{\partial}{\partial t} - \Delta \right) d_t(x_0, x) - \frac{\psi'' S}{(A r_0)^2} + 2\psi(|S_{ij}|^2 + \alpha |\tau g|^2) - 2\nabla \psi \cdot \nabla S \quad (4.7)
\]

at smooth points of the distance function \( d_t(x_0, \cdot) \).

Let \( u_{\min}(t) = \min_{x \in M} u(\cdot, t) \). Whenever \( u_{\min}(t_0) \leq 0 \), assume \( u_{\min}(t_0) \) is achieved at some point \( \bar{x} \), then at \((\bar{x}, t_0)\),

\[
\nabla (\psi S) = 0, \Delta (\psi S) \geq 0, \psi' S \geq 0.
\]

When \( \bar{x} \) is outside \( B_{r_0}(x_0, r_0) \), by (4.5) and Lemma 4.13,

\[
\left( \frac{\partial}{\partial t} - \Delta \right) d_t(x_0, x) \geq -\frac{5(m - 1)}{3r_0}.
\]

Taking these inequalities into (4.7), we get

\[
\left. \frac{d}{dt} \right|_{t=t_0} u_{\min} \geq -\frac{5(m - 1)}{3A r_0} \psi' S + \frac{2}{m} \psi S^2 + \frac{1}{(A r_0)^2} \left( \frac{2\psi'^2}{\psi} - \psi'' \right) S
\]

\[
\geq \frac{2}{m} \psi S^2 + \frac{C'}{A r_0^2} \psi^{\frac{p}{2}} S
\]

as long as \( u_{\min}(t_0) \leq 0 \). Here we use the definition

\[
\left. \frac{d}{dt} \right|_{t=t_0} u_{\min} = \liminf_{h \to 0^+} \frac{u_{\min}(t_0 + h) - u_{\min}(t_0)}{h},
\]

and the property of \( \psi \). Then by Cauchy inequality,

\[
\frac{1}{A r_0^2} \psi^{\frac{p}{2}} S \geq -\frac{C_3^2}{(A r_0)^2} - \delta S^2, \quad \delta \text{ is a small positive constant.}
\]

At last we come to

\[
\left. \frac{d}{dt} \right|_{t=t_0} u_{\min} \geq \frac{1}{m} u_{\min}^2(t_0) + \left( \frac{1}{2m} u_{\min}^2(t_0) - \frac{C_2^2}{(A r_0)^2} \right). \quad (4.8)
\]
When $\bar{x} \in B_{t_0}(x_0, r_0)$, since $\psi' = 0$, the above inequality still holds.

Integrating (4.8) yields

$$u_{\min}(t) \geq \min \left\{ -\frac{m}{t + K_A^{-1}}, -\frac{C}{Ar_0^2} \right\} \text{ on } B_t \left( x_0, \frac{3A}{4}r_0 \right),$$

where $C$ is a positive constant depending only on $m$. So Claim (4.6) follows.

Step 2: Now if our solution $(g(t), \phi(t))$ is ancient, we can replace $t$ by $t - \beta$ in (4.6) and get

$$S(\cdot, t) \geq \min \left\{ -\frac{m}{t - \beta + K_A^{-1}}, -\frac{C}{Ar_0^2} \right\} \text{ on } B_t \left( x_0, \frac{3A}{4}r_0 \right).$$

Letting $A \to \infty$ and then $\beta \to -\infty$, we complete the proof of the theorem.

\[ \square \]

**Remark 4.15.** By the proof of Theorem 4.12 and Theorem 4.14, provided $(g(t), \phi(t))$ exists in a time interval which goes to $-\infty$, the conclusions hold. Therefore, for eternal solutions of Ricci harmonic flow, we still have the scalar curvature is nonnegative.

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