THE MONOIDS OF THE PATIENCE SORTING ALGORITHM

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Abstract. The left patience sorting (lPS) monoid, also known in the literature as the Bell monoid, and the right patience sorting (rPS) monoid are introduced by defining certain congruences on words. Such congruences are constructed using insertion algorithms based on the concept of decreasing subsequences. Presentations for these monoids are given.

Each finite-rank rPS monoid is shown to have polynomial growth and to satisfy a non-trivial identity (dependent on its rank), while the infinite rank rPS monoid does not satisfy a non-trivial identity. The IPS monoids of finite rank have exponential growth and thus do not satisfy non-trivial identities. The complexity of the insertion algorithms is discussed.

rPS monoids of finite rank are shown to be automatic and to have recursive complete presentations. When the rank is 1 or 2, they are also biautomatic. lPS monoids of finite rank are shown to have finite complete presentations and to be biautomatic.

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Monoids arising from combinatorial objects have been intensively studied in recent years. Important examples include the plactic (see LS81, Lot02, Ful97), the sylvester [HNT03, HNT05], the Chinese [DK94, CEK+01], the hypoplactic [KT97], the Baxter [Gir11, Gir12], the stalactic [HNT08], and the Bell monoids [Rey07]. These monoids are associated to several important combinatorial objects such as Young tableaux (plactic monoid), binary trees (sylvester monoid), Chinese staircases (Chinese monoid) quasi-ribbon tableaux (hypoplactic monoid) or set partitions (Bell monoid), in the sense that their elements can be identified with these objects. Their construction, in general, relies on two distinct approaches.

The first approach uses an insertion algorithm which computes a specific combinatorial object from a over a totally ordered alphabet $\mathcal{A} = \{1 < 2 < \ldots\}$. Then the monoid is constructed by taking the quotient of the free monoid $\mathcal{A}^*$ by the congruence that relates words that yield the same combinatorial object. For example, the plactic monoid can be constructed as the quotient of the free monoid $\mathcal{A}^*$ by the congruence that relates words that give rise to the same semi-standard Young tableau under Schensted’s insertion algorithm (see [Lot02, Chapter 5]). The second method consists of presenting a set of defining relations over $\mathcal{A}^*$, which allows the construction of a congruence (which is proven to coincide with the previous one), and again taking the quotient of $\mathcal{A}^*$ over this congruence.

The richness of these monoids comes from the various perspectives from which we can look at them, giving rise to many interesting questions. For example, in general, to each such monoid we can associate a combinatorial Hopf algebra. Such algebras include for instance the
Poirier–Reutenauer algebra of tableaux $\text{FSym}$ arising from the plactic monoid \cite{PR95,DHT02}, the planar binary tree algebra $\text{PBT}$ of Loday–Ronco obtained from the sylvester monoid \cite{HNT05,LR98}, the Hopf algebra of pairs of twin binary trees $\text{Baxter}$ coming from the Baxter monoid \cite{Gir11,Gir12}, and the Hopf algebras $\text{Sym}$ and $\text{Bell}$ arising from the hypoplactic monoid \cite{KT97,Nov00} and the Bell monoid \cite{Rey07}, respectively. However, the importance of these monoids lies not just in the associated combinatorial algorithms or in the associated Hopf algebras, but also in the fact that they are related to partial orders like the Tamari order and to Robinson–Schensted-like correspondences \cite{Gir12}.

We note that the plactic monoid appears in the literature defined in two different ways. When generalizing Schensted’s insertion algorithm from permutations to words, one can use either row-insertion or column-insertion. Column-insertion is equivalent to row-insertion into a modified ‘Young tableau’ where the entries in each row are required to be strictly increasing and the entries in each column are required to be weakly decreasing. This gives rise to two different anti-isomorphic ‘plactic monoids’, one of which is more commonly used in combinatorics \cite[Chapter 5]{Lot02} and the other in representation theory and the theory of crystals \cite{LLT95,DJM90,Kas95}.

The same issue appears when trying to generalize to words the Patience Sorting (PS) algorithm described for permutations in \cite{BL07}. The PS monoids are the central subject of this paper and arise from the following two possible generalizations: when considering the insertion of a symbol into a PS tableaux we may allow entries to columns to be weakly decreasing or strictly decreasing \cite{AD99}. This choice leads to the rPS monoid in the first case and to the lPS monoid in the second. The lPS monoid is the ‘Bell monoid’ considered in \cite{Rey07}. Contrary to the plactic monoid case, these generalizations give rise to two monoids which, as we will see, are not anti-isomorphic, and, in fact, have very different properties. Section 3 studies the insertion algorithms corresponding to these two types of tableaux and shows how they give rise to congruences.

The plactic monoid of rank $n$ has polynomial growth of degree $(n^2 + n)/2$ \cite{DK94}. It is also known that plactic monoids of rank at most 3 satisfy non-trivial identities but that the infinite-rank plactic monoid does not satisfy a non-trivial identity \cite{KO15,CKK15}. It is an open question whether plactic monoids of finite rank greater than 3 satisfy non-trivial identities. Section 4 shows that the lPS monoids of rank greater than 1 have exponential growth, by proving that free monoids (of rank greater than 1) embed into these monoids, whereas the rPS monoids of finite rank have polynomial growth. As a consequence, we deduce that unlike other monoids associated to combinatorial objects such as the hypoplactic, the sylvester, the Baxter, the stalactic and
others \cite{CM16}, the IPS monoids of rank greater than 1 do not satisfy non-trivial identities. We also prove that finite-rank rPS monoids satisfy identities but that the infinite-rank rPS monoid does not.

Finite-rank plactic, Chinese, and hypoplactic monoids are all presented by finite complete rewriting systems and the finite-rank sylvester monoids are presented by regular complete rewriting systems \cite{CGM15a, CQ08, Kar10, CGM15b}. Section 5 shows that finite-rank lPS monoids are also presented by finite complete rewriting systems. As a corollary, we deduce that these monoids satisfy the homological finiteness condition $FP_\infty$. We also show that rPS monoids are presented by infinite complete rewriting systems.

Many of the monoids mentioned above, such as finite-rank plactic, Chinese, hypoplactic, and sylvester monoids, are biautomatic. Automacity, introduced by Epstein et al. \cite{ECH+92} and later generalized to semigroups and monoids \cite{CRNT01}, is a way of describing and computing with semigroups using finite-state automata. Section 6 proves that finite-rank rPS monoids are automatic and that finite-rank IPS monoids are biautomatic. We deduce as a corollary that the rPS and lPS monoids have word problem solvable in quadratic time.

2. Preliminaries and notation

In this section we introduce the notions that we shall use along the paper. For more details concerning these constructions see for instance \cite{Lot02} and \cite{HNT05} for words and standardization, and \cite[Subsection 1.6]{How95} and \cite{BO93} for presentations and rewriting systems.

2.1. Words and standardization. In what follows, let $\mathcal{A} = \{1 < 2 < 3 < \ldots \}$ be the set of natural numbers viewed as an infinite well-ordered alphabet and let $\mathcal{A}^*$ be the corresponding free monoid. Furthermore, for any $n \in \mathbb{N}$, denote by $\mathcal{A}_n$ the totally ordered subset of $\mathcal{A}$, on the letters $1, \ldots, n$ and $\mathcal{A}_n^*$ the corresponding free monoid.

In general, a word over an arbitrary alphabet is an element of the free monoid over that alphabet with the symbol $\varepsilon$ denoting the empty word. So, if $w = w_1 \cdots w_m$ is an arbitrary word of $\mathcal{A}^*$, with $w_1, \ldots, w_m \in \mathcal{A}$, there are several notions that are directly related with the definition of word, which we define as follows:

- the length of $w$, denoted $|w|$, is the number of symbols from $\mathcal{A}$ in $w$, counting repetitions;
- for any word $u = u_1 \cdots u_k \in \mathcal{A}^*$, with $u_1, \ldots, u_k \in \mathcal{A}$, $u$ is a subsequence of $w$ if there exists a sequence of indexes, $i_1, \ldots, i_k \in \mathbb{N}$, with $1 \leq i_1 < \ldots < i_k \leq m$, such that $u = w_{i_1} \cdots w_{i_k}$;
- for any word $u \in \mathcal{A}^*$, the word $u$ is a factor of $w$ if there exist words $v_1, v_2 \in \mathcal{A}^*$, such that $w = v_1uv_2$;
- for any $a \in \mathcal{A}$, the number of occurrences of $a$ in $w$, is denoted by $|w|_a$.
• the evaluation of $w$, $\text{ev}(w)$, is the infinite sequence of non-negative integers $(|w|_1, |w|_2, |w|_3, \ldots)$ whose $i$-th component is $|w|_i$.

Another related concept that we will frequently use is of standard word. A standard word over a well-ordered alphabet $\Sigma$ is a word $u$ that contains each symbol from the first $|u|$ symbols of $\Sigma$ exactly once. A standard word $u = u_1 \cdots u_k$ in naturally identified with the permutation that maps $i \mapsto u_i$.

Next, we define two different processes for standardizing a word, both allowing us to associate to any given word, a standard word of the same length. Consider the alphabet $C = \{a b : a, b \in A\}$ well-ordered in the following way: for any $a b, c d \in C$,

$$a b \prec c d \iff a < c \lor (a = c \land b < d).$$

Given a symbol $a b \in C$, $a$ will be called the underlying symbol of $a b$ and $b$ the index of $a b$.

For any $w \in A^*$, the left to right standardization of $w$, denoted $\text{std}_L(w)$, which corresponds to the standardization presented in [HNT05], is the word in $C$ obtained by iteratively scanning all the occurrences of 1 and labelling them with $1, 2, \ldots$ from left to right, then scanning all the occurrences of 2 and labelling them with $2, 2, \ldots$ from left to right, and repeating this process for all the symbols occurring in $w$.

The right to left standardization of $w$, denoted $\text{std}_R(w)$, is obtained in the same way, but instead of the labelling being done from left to right, it is done from right to left.

For example, considering the word $w = 1321221 \in A_3^*$, we get

$$w = 1321221$$

$$\text{std}_L(w) = 1_1 3_1 2_1 2_2 3_2 1_3$$

$$\text{std}_R(w) = 1_3 3_1 2_3 2_2 2_1 1_1.$$
where std can be replaced by either stdℓ or stdr.

2.2. **Presentations and rewriting systems.** On a different direction, a **monoid presentation** is a pair \((\Sigma, \mathcal{R})\), where \(\Sigma\) is an alphabet and \(\mathcal{R} \subseteq \Sigma^* \times \Sigma^*\). A monoid \(M\) is said to be defined by a presentation \((\Sigma, \mathcal{R})\) if \(M \cong \Sigma^*/\mathcal{R}^#\), where \(\mathcal{R}^#\) is the smallest congruence containing \(\mathcal{R}\) (see [How95, Proposition 1.5.9] for a combinatorial description of the smallest congruence containing a relation). The presentation is called **finite** if both \(\Sigma\) and \(\mathcal{R}\) are finite and **multihomogeneous** if, for each symbol \(a \in \Sigma\) and every defining relation \((w, w') \in \mathcal{R}\), \(|w|_a = |w'|_a\). A monoid \(M\) is said to be multihomogeneous if there exists a multihomogeneous presentation defining \(M\).

The set of relations \(\mathcal{R}\) can also be referred to as a **rewriting system** whose elements are the **rewriting rules**. This way, the rewriting rules \(r \in \mathcal{R}\) will be written in the form \(r = (r^+, r^-)\).

From this point of view, we can define another binary relation in \(\Sigma^*\), \(\rightarrow_{\mathcal{R}}\), in the following way

\[ u \rightarrow_{\mathcal{R}} v \iff u = w_1 r^+ w_2 \quad \text{and} \quad v = w_1 r^- w_2 \]

for some \((r^+, r^-) \in \mathcal{R}\) and \(w_1, w_2 \in \Sigma^*\). The binary relation \(\rightarrow_{\mathcal{R}}\) is known as a **single-step reduction**. A word \(u \in \Sigma^*\) is said to be **irreducible** if there is no word \(v \in \Sigma^*\) such that \(u \rightarrow_{\mathcal{R}} v\). Define \(\rightarrow_{\mathcal{R}}^*\) to be the transitive and reflexive closure of \(\rightarrow_{\mathcal{R}}\). Note that the relation \(\mathcal{R}^#\) coincides with the reflexive, symmetric, and transitive closure of \(\rightarrow_{\mathcal{R}}\).

We say that the rewriting system \(\mathcal{R}\) is **noetherian** if there is no infinite chain of single step reductions

\[ w_1 \rightarrow_{\mathcal{R}} w_2 \rightarrow_{\mathcal{R}} w_3 \rightarrow_{\mathcal{R}} \cdots. \]

The rewriting system \(\mathcal{R}\) is said to be **confluent** if whenever \(u \rightarrow_{\mathcal{R}}^* v\) and \(u \rightarrow_{\mathcal{R}}^* v'\), there exists \(w \in \Sigma^*\) such that \(v \rightarrow_{\mathcal{R}}^* w\) and \(v' \rightarrow_{\mathcal{R}}^* w\).

In the case of \(\mathcal{R}\) being both noetherian and confluent, we say that \(\mathcal{R}\) is **complete**.

A presentation is said to be noetherian, confluent or complete if the corresponding rewriting system is, respectively, noetherian, confluent or complete.

3. **PS Tableaux, Insertion and PS Monoids**

One of the possible ways of defining the plactic monoid is by considering it as the quotient of a free monoid by the congruence that relates words that yield the same semistandard Young tableau, under Schensted’s insertion algorithm [Sch61].

In this section we shall present the concepts that parallel the notions of semistandard Young tableau and Schensted’s algorithm for words in the case of Patience Sorting. The Patience Sorting algorithm was originally defined to sort a permutation into piles. Two natural generalizations to words (and not only for ‘permutations’) of the Patience
Sorting algorithm appear in [AD99, Subsection 2.4]. Each word gives rise to a PS-tableau, and the set of pairs of words yielding the same PS-tableau is a congruence. This will allow us to define the IPS and the rPS monoids in a similar way to the plactic monoid.

3.1. PS-tableaux and insertion. A composition diagram is a finite collection of boxes arranged in bottom-justified columns, where no order on the length of the columns is imposed. The shape of a composition diagram $B$, denoted by $\text{Sh}(B)$, is the sequence of the heights of the columns of $B$ from left to right. Note that this notion of shape is dual to the usual notion of shape, as given in [Lot02].

An IPS tableau (respectively, an rPS tableau) is a composition diagram with entries from $\mathcal{A}$, so that the sequence of entries of the boxes in each column is strictly (resp., weakly) decreasing from top to bottom, and the sequence of entries of the boxes in the bottom row is weakly (resp., strictly) increasing from left to right.

Consider the following diagrams:

$$B = \begin{array}{c}
5 & 6 \\
4 & 3 & 5 \\
2 & 2 & 2 & 4
\end{array} \quad C = \begin{array}{c}
2 & 5 & 6 \\
2 & 4 & 5 \\
2 & 3 & 4
\end{array}$$

$B$ is an IPS tableau with shape $\text{Sh}(B) = (1, 3, 3, 2)$, whereas $C$ is an rPS tableau with shape $\text{Sh}(C) = (3, 3, 3)$.

Henceforth, we shall often refer to an IPS tableau or to an rPS tableau simply as a PS tableau, not distinguishing the cases when they can be dealt in a similar way.

The Patience Sorting algorithm defined only for permutations by Burstein and Lankham [BL07, Algorithm 1.1], can be generalized to arbitrary words in two different ways, depending on whether or not we allow repeated symbols on the same column [TY11, Subsection 3.2]. We begin by presenting the algorithm that computes from an IPS (resp., rPS) tableau $B$ and a symbol $a \in \mathcal{A}$, an IPS (resp., rPS) tableau $B \leftarrow a$.

**Algorithm 3.1** (Right insertion of a symbol on a PS tableau).

*Input:* An IPS (resp., rPS) tableau $B$ and a symbol $a \in \mathcal{A}$.

*Output:* An IPS (resp., rPS) tableau $B \leftarrow a$.

*Method:*

1. If $a$ is greater than or equal to (resp., greater than) every entry in the bottom row of $B$, create a box with $a$ as an entry at the rightmost end of the bottom row of $B$, and output the resulting tableau. (Note that the insertion can be done to an empty PS tableau, $\emptyset$, which corresponds to the creation of a PS tableau with a single box).

2. Otherwise, let $z$ be the leftmost letter in the bottom row of $B$ that is greater (resp., greater than or equal) than $a$. Create
a box at the top of the column tableau where \( z \) was placed, move each element of the column to the box strictly above, and let \( a \) be the symbol on the bottom box. Output the resulting tableau.

It is straightforward to check that the procedure outputs an lPS (resp., rPS) tableau whenever the algorithm starts with an lPS (resp., rPS) tableau.

If in the previous algorithm, the symbol to be inserted and the symbols in the PS tableaux are all distinct, then Algorithm 3.1 follows the same procedure as Algorithm 1.1 in [BL07]. Moreover, both the lPS version and the rPS version execute the algorithm in the same way.

The iterated insertion of the symbols of a word (read from left to right) into the resulting tableaux using the previous algorithm gives rise to the PS insertion of a word, which is described in detail by the following algorithm:

**Algorithm 3.2** (PS algorithm for words).

**Input:** A word \( w = w_1 \cdots w_k \in A^* \), with \( w_1, \ldots, w_k \in A \).

**Output:** An lPS (resp., rPS) tableau \( \mathcal{R}_\ell(w) \) (resp., \( \mathcal{R}_r(w) \)).

**Method:** Start with the empty tableau \( P_0 = \emptyset \). For each \( i = 1, \ldots, k \), insert \( w_i \) into the lPS (resp., rPS) tableau \( P_{i-1} \) using Algorithm 3.1. Output \( P_k \) for \( \mathcal{R}_\ell(w) \) (resp., \( \mathcal{R}_r(w) \)).

Note that although Algorithms 3.1 and 3.2 consider words on the alphabet \( A \), they also can be used considering words of any totally ordered alphabet. Hence, in any of the results of this paper, the alphabet \( A \) can be replaced by some other totally ordered alphabet.

We use the notation \( \mathcal{R}(w) \) to refer to any of the tableaux \( \mathcal{R}_\ell(w) \) or \( \mathcal{R}_r(w) \), indistinguishably. Denoting by \( \emptyset \) the empty PS tableau, if \( w = w_1 \cdots w_k \), we have

\[
\mathcal{R}(w) = \left( \cdots (\emptyset \leftarrow w_1) \leftarrow w_2) \cdots \right) \leftarrow w_k.
\]

For example, if we are given the word \( w = 254263542 \) its lPS tableau is obtained as pictured in Figure 3.1. As noticed, if all symbols of a word are distinct the resulting tableau is the same, independently of using the lPS or the rPS version of Algorithm 3.2 or [BL07, Algorithm 1.1]. So, if \( \sigma \) is a word where the symbols are all distinct, our notation \( \mathcal{R}(\sigma) \) agrees with the notation used in [BL07] to denote the output of a permutation using Algorithm 1.1, and we have

\[
\mathcal{R}_\ell(\sigma) = \mathcal{R}(\sigma) = \mathcal{R}_r(\sigma)
\]

for any such \( \sigma \).

The following lemma relates the position of repeated symbols in the word to their position in the tableau.

**Lemma 3.3.** Let \( w \) be a word on \( A \) and let \( a_1, a_2 \in A \) be two symbols of \( w \) with \( a_1 \leq a_2 \). Then \( a_1 a_2 \) is a subsequence of \( w \) if, and only if,
Figure 1. Execution of Algorithm 3.2 to the word \( w = 254263542 \), where below each arrow we indicate the symbol to be inserted next, and on its right the result of that insertion into the previous tableau.

The symbol \( a_2 \) is positioned in \( \mathcal{R}(w) \) either in the same column of \( a_1 \), below \( a_1 \), (only in the rPS case) or in a column further to the right of the column containing \( a_1 \).

Proof. Suppose that \( a_1 a_2 \) is a subsequence of \( w \), that is, \( w = w_1 a_1 w_2 a_2 w_3 \). In the computation of \( \mathcal{R}_i(w) \) using Algorithm 3.2, the symbol \( a_2 \) is inserted after \( a_1 \), and thus will be in a column further to the right of the column in which \( a_1 \) is positioned. Indeed, on the step where the symbol \( a_2 \) is inserted, the bottom symbol of the column in which \( a_1 \) is positioned is smaller than \( a_1 \), and thus by Algorithm 3.1, the symbol \( a_2 \) will be inserted to a column further to the right.

In \( \mathcal{R}_r(w) \) the reasoning is similar, with the additional condition that \( a_2 \) can be inserted in the same column below \( a_1 \) if during the execution of Algorithm 3.1, the bottom symbol currently in the column of \( a_1 \) is equal to \( a_1 \).

Conversely, suppose that \( a_1 a_2 \) is not a subsequence of \( w \). This means that \( a_2 a_1 \) is a subsequence of \( w \) and therefore, by the direct part of the statement already proved, applied to the subsequence \( a_2 a_1 \), we reach a contradiction. \( \square \)

Two words \( u, v \), possibly over different totally ordered alphabets, whose IPS tableaux have the same shape (and thus have the same length) are said to have equivalent \( l \)-insertions if for every \( i = 1, \ldots, |u| \) the \( i \)-th symbol of \( u \) and the \( i \)-th symbol of \( v \) are in the same (column-row) position of the respective tableaux \( \mathcal{R}_l(u) \) and \( \mathcal{R}_l(v) \). Similarly, we define words to have equivalent \( r \)-insertions by replacing \( \mathcal{R}_l(u) \) and \( \mathcal{R}_l(v) \) by \( \mathcal{R}_r(u) \) and \( \mathcal{R}_r(v) \), respectively, in the previous definition.
The words \( u = 24131 \) and \( v = 36142 \), whose IPS tableaux are

\[
R_L(u) = \begin{array}{c} 4 \\ 2 \\ 3 \\ 1 \\ 1 \end{array} \quad \text{and} \quad R_L(v) = \begin{array}{c} 6 \\ 3 \\ 4 \\ 1 \\ 2 \end{array}
\]

have equivalent \( l \)-insertions.

**Lemma 3.4.** For any \( w \in \mathcal{A}^* \), the words \( w \) and \( \text{std}_r(w) \) have equivalent \( l \)-insertions, and \( w \) and \( \text{std}_l(w) \) have equivalent \( r \)-insertions.

**Proof.** Let \( w \in \mathcal{A}^* \). The proof follows by induction on the length of \( w \), \( |w| \). The result holds trivially for \( |w| = 1 \).

Suppose by induction hypothesis that the result holds for any word \( w \) with \( |w| = n > 1 \). Let \( a_1, a_2, \ldots, a_k \) and \( a'_1, a'_2, \ldots, a'_k \) be the sequence of entries in the boxes of the bottom row of, respectively \( R_L(w) \) and \( R_L(\text{std}_r(w)) \) (resp., \( R_L(w) \) and \( R_L(\text{std}_l(w)) \)). Consider \( a \in \mathcal{A} \) and let \( a' \) be the rightmost element in the word \( \text{std}_r(wa) \) (resp., \( \text{std}_l(wa) \)).

If \( a_k \leq a \) (resp., \( a_k < a \)), then we are in case (1) of Algorithm 3.1 and therefore a box is added to the rightmost end of the bottom row of \( R_L(w) \) (resp., \( R_L(\text{std}_r(w)) \)), with the symbol \( a \) in it.

The symbol \( a'_k \) is to the left of the symbol \( a' \) in the word \( \text{std}_r(wa) \) (resp., \( \text{std}_l(wa) \)). By the left-to-right standardization (resp., right-to-left standardization) we have \( a'_k < a' \), since \( a_k \leq a \) (resp., \( a_k < a \)).

To compute \( R_L(\text{std}_r(wa)) = R_L(\text{std}_r(w)) \leftarrow a' \) (resp. \( R_L(\text{std}_l(wa)) = R_L(\text{std}_l(w)) \leftarrow a' \)) we also use step (1) of Algorithm 3.1. A box is added to the rightmost end of the bottom row of \( R_L(\text{std}_r(w)) \) (resp., \( R_L(\text{std}_l(w)) \)), with the symbol \( a' \) in it. Therefore, the \( n+1 \)-th symbols of words \( wa \) and \( \text{std}_r(wa) \) (resp., \( \text{std}_l(wa) \)) take the same position of the respective tableaux. Thus, \( wa \) and \( \text{std}_r(wa) \) (resp., \( wa \) and \( \text{std}_l(wa) \)) have equivalent \( l \)-insertions (resp. \( r \)-insertions).

Suppose now that \( a < a_k \) (resp., \( a \leq a_k \)). Let \( a_j \), with \( j \in \{1, 2, \ldots, k\} \) be the leftmost letter in the bottom row of \( R_L(w) \) (resp., \( R_L(\text{std}_r(w)) \)) such that \( a < a_j \) (resp., \( a \leq a_j \)). We have the following situation:

\[
a_1 \leq \ldots \leq a_{j-1} \leq a < a_j \leq \ldots \leq a_k
\]

(resp., \( a_1 < \ldots < a_{j-1} < a \leq a_j < \ldots < a_k \)).

Therefore, case (2) of Algorithm 3.1 is applied and \( a \) bumps all the elements of the column of \( a_j \), being inserted below the \( a_j \) in the bottom row.

By the left-to-right standardization (resp., right-to-left standardization) the inequalities \( a'_{j-1} < a' \) and \( a' < a'_j \) hold, since \( a_{j-1} \leq a \) and \( a < a_j \) (resp., \( a_{j-1} < a \) and \( a \leq a_j \)). Therefore, to compute \( R_L(\text{std}_r(w)) \leftarrow a \) (resp., \( R_L(\text{std}_l(w)) \leftarrow a \)) we apply case (2) of Algorithm 3.1 and so \( a' \) bumps all the elements of the column of \( a'_j \) being inserted in the bottom row below \( a'_j \).
Clearly, the \( n + 1 \)-th symbols of the words \( wa \) and \( std_\ell(wa) \) (resp. \( std_r(wa) \)) take the same position of the respective tableaux. Similarly, the symbols of the columns \( a_j \) and \( a'_j \) will have the same relative positions in the new tableaux, by the induction hypothesis. Therefore, \( wa \) and \( std_\ell(wa) \) (resp., \( wa \) and \( std_r(wa) \)) have equivalent \( l \)-insertions (resp., \( r \)-insertions). The result follows by induction. □

Similarly to the standardization and de-standardization of words presented in Subsection 2.1, we present here corresponding standardization and de-standardization processes of PS tableaux.

Given a PS tableau \( B \), let \( Std_\ell(B) \) be the tableaux obtained from \( B \) reading the entries of \( B \) column by column, from left to right, and on each column from top to bottom, and attaching to each symbol \( a \in \mathcal{A} \) an index \( h \) to the \( h \)-th appearance of \( a \). Similarly, \( Std_r(B) \) is the tableaux obtained from \( B \) reading the entries of \( B \) column by column, from right to left, and on each column from bottom to top, attaching to each symbol \( a \in \mathcal{A} \) an index \( h \) to the \( h \)-th appearance of \( a \).

If the PS tableaux \( B \) has symbols form a standardized word, the de-standardization of \( B \), denoted \( Dstd(B) \), is the tableau produced by erasing the indexes of each of its underlying symbols.

The following result relates both versions of Algorithm 3.2, with Algorithm 1.1 of [BL07].

**Proposition 3.5.** For any word \( w \in \mathcal{A}^* \) and \( x \in \{ \ell, r \} \) we have

1. \( R_x(std_x(w)) = Std_x(R_x(w)) \); and
2. \( Dstd(R_x(std_x(w))) = R_x(w) \).

**Proof.** Let \( w \in \mathcal{A}^* \). By Lemma 3.4 the words \( std_x(w) \) and \( w \) have equivalent \( l \)-insertions, that is, the \( i \)-th symbol of \( std_x(w) \) and the \( i \)-th symbol of \( w \) are, respectively, in the same position of \( R_x(std_x(w)) \) and of \( R_x(w) \). Since elements in corresponding positions of \( w \) and \( std_x(w) \) have the same underlying symbol, and \( Dstd \) just erases the indexes of the symbols in the tableau, we deduce part (2) of the statement.

It remains to show that during the process of standardization of \( R_x(w) \) the indexes will match, so that the symbols in corresponding positions of \( R_x(std_x(w)) \) and \( Std_x(R_x(w)) \) are equal.

Let \( a \in \mathcal{A} \) be a symbol in \( w \) and let \( k = |w|_a \). So \( a_1 \cdots a_k \) is a subsequence of \( std_x(w) \). By Lemma 3.3 equal symbols in \( w \), as we proceed from left to right in \( w \), will be appearing in \( R_x(w) \) as we proceed going column by column, from left to right, and on each column from top to bottom. Thus, by definition of \( Std_x \), the standardization of \( R_x(w) \) will attach indexes in the same way the standardization of \( w \) is processed.

The right-to-left case follows the same reasoning in the first two paragraphs of this proof. In the third paragraph, the main difference is that \( a_k \cdots a_1 \) is the subsequence of \( std_r(w) \) and that also Lemma 3.3...
guarantees that equal symbols in \( w \), as we proceed from right to left in \( w \), will be appearing in \( R_r(w) \) as we proceed going column by column, from right to left, and on each column from bottom to top. This agrees with the standardization \( \text{Std}_r \) of \( R_r(w) \). □

3.2. From PS tableaux to words. All the concepts introduced in the following paragraphs have an lPS and an rPS version depending on whether the considered PS tableau is, respectively, lPS or rPS. So, for the sake of brevity we will only introduce the general notions and when necessary we distinguish them.

As we will see, it will be convenient to pass from diagrams to words representing such diagrams. The first concept that we will need is the generalization of reverse patience word presented in [BL07]. So, given a PS tableau \( B \), the column reading of \( B \), denoted by \( C(B) \), is the word obtained from reading the entries of the PS tableau \( B \), column by column, from the leftmost to the rightmost, starting on the top of each column and ending on its bottom. For example, the column reading of the lPS tableau \( B \) in (1) is \( C(B) = 2 542 632 54 \), whereas the column reading of the rPS tableau \( C \) in (1) is \( C(C) = 222 543 654 \).

The following equality follows from the definition of standardization of words and tableaux: for any \( x \in \{ \ell, r \} \) and any \( x \)PS tableau \( B \),

\[
R_x(C(R_x(w))) = R_x(w).
\]

The relations established between Algorithm 3.2 for words, Algorithm 1.1 of [BL07] for permutations and the standardization processes, allows us to transfer results from the permutation case, such as [BL07, Lemma 2.4], to the general case of words.

**Lemma 3.6.** For any \( x \in \{ \ell, r \} \) and word \( w \in A^* \),

\[
R_x\left(C(R_x(w))\right) = R_x(w).
\]

**Proof.** It follows by Proposition 3.5(2) that

\[
R_x\left(C(R_x(w))\right) = Dstd_x\left(R_x\left(\text{std}_x\left(C(R_x(w))\right)\right)\right).
\]

By (3) and Proposition 3.5(1) we get

\[
R_x\left(C(R_x(w))\right) = Dstd_x\left(R_x\left(C(\text{Std}_x(R_x(w)))\right)\right)
\]

\[
= Dstd_x\left(R_x\left(C(R_x(\text{std}_x(w)))\right)\right).
\]

Now, by [BL07] Lemma 2.4] the result holds for standard words, so

\[
R_x\left(C(R_x(w))\right) = Dstd\left(R_x(\text{std}_x(w))\right).
\]

The result now follows by Proposition 3.5(2). □
Let \( x \in \{ \ell, r \} \). A word on \( A^* \) that is the reading of a column of some \( x \)PS tableau is called an \( x \)PS column word. (In [BL07], the term ‘pile’ is used.) Similarly, a word on \( A^* \) that is the reading of the bottom row of some \( x \)PS tableau is called a \( x \)PS bottom row word. We call a word an \( x \)PS canonical word if it is the column reading of some \( x \)PS tableau. Note that \( r \)PS column words and \( l \)PS bottom row words are weakly increasing, while \( l \)PS column words and \( r \)PS bottom row words are strictly increasing.

Notice that any \( x \)PS canonical word \( c \) has a unique decomposition into \( x \)PS column words \( c_1, \ldots, c_k \), such that \( c = c_1 \cdots c_k \) and such that the subsequence obtained by taking the smallest symbols of each \( c_i \) is a \( x \)PS bottom row word.

**Proposition 3.7.** The mapping \( R_x \) from the set of canonical words to the set of \( x \)PS tableaux is a bijection, and \( C \) is its inverse map.

**Proof.** Let \( c \) be a \( x \)PS canonical word and let \( c = c_1 \cdots c_k \) be its column decomposition. Proceed with Algorithm 3.2 to compute \( R_x(c) \). Note that \( R_x(c_i) \) is a single column, for any \( i = 1, \ldots, k \). Since the subsequence composed by the smallest symbols of the \( c_i \) is a bottom row word, every symbol in \( c_i \) is, in the \( l \)PS case, greater than or equal to, or, in the \( r \)PS case, greater than every symbol in \( R_x(c_1 \cdots c_{i-1}) \), and so is inserted into a new column to the right of \( R_x(c_1 \cdots c_{i-1}) \). Hence \( R_x(c) \) is the juxtaposition of \( R_x(c_1), \ldots, R_x(c_k) \). Thus \( R_x \) is a bijective map on canonical words and \( C(R_x(c)) = c \). \( \square \)

As a consequence of the previous result, we can identify \( x \)PS tableaux with their canonical words. Throughout the remainder of this paper we shall use this identification.

From any word \( w \in A^* \) we can produce a canonical word \( w_c \) by computing the tableau \( R_x(w) \) and identifying it with the column reading \( C(R_x(w)) \). Let \( c_1, \ldots, c_k \) be the unique decomposition of \( w_c \) into \( x \)PS column words, such that \( c = c_1 \cdots c_k \) and the subsequence obtained by taking the smallest symbols of each \( x \)PS column word, is an \( x \)PS bottom row word. We refer to the sequence of column words \( (c_1, \ldots, c_k) \) as the \( x \)PS column configuration of \( w \). (In [BL07], the term ‘pile configuration’ is used.)

By Proposition 3.7 and Lemma 3.6 the mapping from \( A^* \) to the set of \( x \)PS canonical words given by \( w \mapsto C(R_x(w)) \), for any \( w \in A^* \), is a surjection whose restriction to the set of canonical words is bijective.

### 3.3. The left-to-right minimal subsequence

In this subsection we present the construction of sequences of subsequences of a given word, that are built going through the word from left to right. These sequences are the generalization of the computation of the left-to-right minimal subsequence for permutations in [BL07, Definition 2.8]. We shall see that they provide us an alternative way to compute directly
the column configuration of a word. This new approach will allow us to define lPS and rPS left insertion algorithms on next subsection. The algorithm is based on the computation of a strictly (resp., weakly) decreasing subsequence from a given word $w \in A^*$, called its $l$-decreasing (resp., $r$-decreasing) subsequence and denoted $d^l(w)$ (resp., $d^r$):

**Algorithm 3.8.** (Decreasing subsequence)

*Input:* A word $w \in A^*$.

*Output:* Its $l$-decreasing ($r$-decreasing) subsequence $d^l(w)$ (resp. $d^r$).

*Method:*

1. The first symbol of $d^l(w)$ (resp., $d^r(w)$) is the first symbol of $w$;
2. The $i + 1$-th symbol of $d^l(w)$ (resp., $d^r(w)$) is obtained from the $i$-th symbol of $d^l(w)$ (resp., $d^r$), by going to the position that the $i$-th symbol of $d^l(w)$ (resp., $d^r$) occupies in $w$ and finding the first symbol to its right that is less than (resp., less than or equal to) it. It stops when no symbol that is less than (resp., less than or equal to) the $i$-symbol is found and outputs the obtained $l$-decreasing (resp., $r$-decreasing) subsequence.

The algorithm of left-to-right minimal subsequences is then defined in the following way:

**Algorithm 3.9.** (Left-to-right minimal subsequence)

*Input:* A word $w \in A^*$.

*Output:* Its $l$-left-to-right ($r$-left-to-right) minimal subsequence $D^l(w)$ (resp., $D^r(w)$).

*Method:*

1. Let $d^l_1(w)$ (resp., $d^r_1(w)$) be the left-to-right minimal subsequence $d^l(w)$ (resp., $d^r(w)$).
2. For $i \geq 2$, let $w^l_{(i)}$ (resp., $w^r_{(i)}$) be the word obtained from $w$ by deleting the symbols appearing in $d^l_{1}(w), \ldots, d^l_{i-1}(w)$ (resp., $d^r_{1}(w), \ldots, d^r_{i-1}(w)$). Define $d^l_i(w)$ (resp., $d^r_i(w)$) to be $d^l(w^l_{(i)})$ (resp., $d^r(w^r_{(i)})$).
3. Output the sequence of non-empty words $(d^l_1(w), \ldots, d^l_i(w))$ (resp., $(d^r_1(w), \ldots, d^r_i(w))$).

For any $i \in \mathbb{N}$, the word $d^l_i(w)$ (resp., $d^r_i(w)$) computed by the previous algorithm is defined as the $i$-th $l$-decreasing (resp. $r$-decreasing) subsequence of $w$, which is the generalization of the $i$-th left-to-right minimal subsequence for permutations in [BL07].
Lemma 3.10. For any word $w \in A^*$, the words $w$ and $\text{std}_l(w)$ have equivalent $l$-left-to-right minimal subsequences, and $w$ and $\text{std}_r(w)$ have equivalent $r$-left-to-right minimal subsequences.

Proof. We present the full proof for the $l$-case, and whenever the reasoning for the $l$-case and the $r$-case has some significant difference we present the arguments for the $r$-case in brackets.

The proof follows by induction on the length of $w$. If $|w| = 1$, the result holds trivially. Suppose by induction hypothesis that the result holds for a word $w$ with $|w| = n \geq 1$. Let $a \in A$ and let $k$ be the number of non-empty $l$-decreasing subsequences of both $w$ and $\text{std}_l(w)$.

We will prove that $wa$ and $\text{std}_l(wa)$ have equivalent $l$-left-to-right minimal subsequences. Suppose $w = w_1 \cdots w_n$, with $w_1, \ldots, w_n \in A$. Then $\text{std}_l(w) = w'_1 \cdots w'_n$, with $w'_1, \ldots, w'_n \in C$. Denote by $a'$ the rightmost symbol of $\text{std}_l(wa)$.

For $i = 1$, suppose that $w_1$ is the last symbol of $\text{std}_l(w)$ and let $w'_1$ be the corresponding symbol in $\text{std}_l(w)$. As by induction hypothesis $w$ and $\text{std}_l(w)$ have equivalent $l$-left-to-right minimal subsequences, $w'_1$ is the last symbol of $\text{std}_l(\text{std}_l(w))$. 

So, for instance, considering the word $w = 256423542$, we get:

- $d'_l(w) = d'(256423542) = 2$  
  $d'_r(w) = d'(256423542) = 222$
- $d'_l(w) = d'(256423542) = 542$  
  $d'_r(w) = d'(256423542) = 543$
- $d'_l(w) = d'(256423542) = 632$  
  $d'_r(w) = d'(256423542) = 654$
- $d'_l(w) = d'(256423542) = 54$  
  $d'_r(w) = d'(256423542) = \varepsilon$
- $d'_l(w) = d'(256423542) = \varepsilon$

Therefore $\text{D}'(w) = (2, 542, 632, 54)$ and $\text{D}'(w) = (222, 543, 654)$.

Whenever possible, we will use $\text{D}$ instead of $\text{D}'$ (resp., $\text{D}''$) and $\text{D}$ instead of $\text{D}'$ (resp., $\text{D}''$), in order to simplify the notation.

Similarly to the notion of equivalent $l$-insertions previously defined, we say that two words $u, v$, possibly over different totally ordered alphabets, have equivalent $l$-left-to-right minimal subsequences if:

- both words have the same number $k$ of non-empty $l$-decreasing subsequences; and,
- for every $i = 1, \ldots, k$, the positions of the symbols of the subsequence $d'_l(u)$ in $u$ are the same as the positions of the symbols of the subsequence $d'_l(v)$ in $v$.

The notion of equivalent $r$-left-to-right minimal subsequences is defined similarly.

For example, the words $u = 24131$ and $v = 36142$ have equivalent $l$-left-to-right minimal subsequences, since $\text{D}'(u) = (21, 431)$ and $\text{D}'(v) = (31, 642)$.
If \( w_j \leq a \) (resp., \( w_j < a \)) then, by the left-to-right standardization (resp., right-to-left standardization) \( w_j' < a' \). Thus \( \mathcal{D}_l^t(wa) = \mathcal{D}_l^t(w) \) and \( \mathcal{D}_l^t(\text{std}_l(wa)) = \mathcal{D}_l^t(\text{std}_l(w)) \). Increase \( i \) by one, and repeat the previous argument until \( w_j > a \) (resp., \( w_j \geq a \)). If this case never occurs, then both words \( wa \) and \( \text{std}_l(wa) \) have a new \( l \)-decreasing subsequence \( \mathcal{D}_{k+1}^l(wa) = a \) and \( \mathcal{D}_{k+1}^l(\text{std}_l(wk)) = a' \) where its symbols have the same position on \( w \) and \( \text{std}_l(wa) \), respectively.

Suppose we have reached an \( i \in \{1, \ldots, k\} \) such that \( w_j > a \) (resp., \( w_j \geq a \)). Then by the left-to-right standardization (resp., right-to-left standardization) \( w_j' > a' \). So, \( \mathcal{D}_l^t(wa) = \mathcal{D}_l^t(w)a \) and \( \mathcal{D}_l^t(\text{std}_l(wa)) = \mathcal{D}_l^t(\text{std}_l(w))a' \).

After this case \( a \) and \( a' \) have been deleted in the computation of the left-to-right (resp., right-to-left) minimal subsequence, and so the computation of the remaining \( l \)-decreasing subsequences of \( \mathcal{D}_l^t(wa) \) and \( \mathcal{D}_l^t(\text{std}_l(wa)) \) proceeds in the same way as in the computation of \( \mathcal{D}_l^t(w) \) and \( \mathcal{D}_l^t(\text{std}_l(w)) \).

As \( a \) and \( a' \) occupy the same positions in \( wa \) and \( \text{std}_l(wa) \), respectively and by the induction hypothesis \( w \) and \( \text{std}_l(w) \) have equivalent \( l \)-left-to-right minimal subsequences, we can conclude that \( wa \) and \( \text{std}_l(wa) \) have equivalent \( l \)-left-to-right minimal subsequences.

By induction the result follows. \( \square \)

As by the previous result, for any \( w \in \mathcal{A}^* \), the words \( w \) and \( \text{std}_l(w) \) (resp., \( \text{std}_r(w) \)) have equivalent \( (r-) \) \( l \)-left-to-right minimal subsequences and standardization just adds indexes to the symbols of \( w \) and \( \text{std}^{-1} \) just erases them, we can conclude the following:

**Lemma 3.11.** For any \( w \in \mathcal{A}^* \), the de-standardization of the left-to-right minimal subsequence of \( \text{std}(w) \) is the left-to-right minimal subsequence of \( w \).

**Proposition 3.12.** For any \( w \in \mathcal{A}^* \), the left-to-right minimal subsequence of \( w \) produces the column configuration of \( w \).

**Proof.** As the argument applies to both \( l \) and \( r \)-cases, we just provide the proof for the \( l \)-case. Let \( w \in \mathcal{A}^* \) and let \( (c_1, \ldots, c_k) \) be the lPS column configuration of \( w \). By Proposition 3.5, the lPS column configuration of \( w \) can be obtained by computing the IPS column configuration of \( \text{std}(w) \), and then de-standardizing the resulting words (removing its indexes).

It is known [BL07, Lemma 2.9], that the column configuration of \( \text{std}(w) \) corresponds to the left-to-right minimal subsequence of \( \text{std}(w) \).
Thus, the de-standardization of the left-to-right minimal subsequence of \( \text{std}(w) \) is precisely \((c_1, \ldots, c_k)\). The result now follows since, by the previous lemma, the de-standardization of the left-to-right minimal subsequence of \( \text{std}(w) \) is the left-to-right minimal subsequence of \( w \). \( \square \)

For \( x \in \{\ell, r\} \), given a word \( w \in \mathcal{A}^* \), if \( \mathcal{D}_x(w) = (\mathcal{D}_x^1(w), \ldots, \mathcal{D}_x^k(w)) \) and \((c_1, \ldots, c_j)\) is the column configuration of \( w \), then by the previous proposition it follows that \( k = j \) and \( \mathcal{D}_x^i(w) = c_i \), for all \( i \in \{1, \ldots, k\} \), and thus \( \mathcal{C} \left( \mathcal{R}_x(w) \right) = \mathcal{D}_x^1(w) \cdots \mathcal{D}_x^k(w) \).

3.4. **Left-insertion.** This section presents a new algorithm which performs left insertion of symbols on PS tableaux and thus allows us to build PS tableaux from words, proceeding from left to right on the word, which we will prove that has the same output as the Patience Sorting algorithm (Algorithm 3.2). These algorithms will be crucial in proving biautomaticity for the lPS monoid in Section 6.

The procedure makes use of the fact that the computation of the left-to-right minimal subsequence produces the same result as PS algorithm.

The following lemma gives us some of the tools to easily compute decreasing subsequences, by left inserting symbols on a canonical word.

**Lemma 3.13.** Let \( c' \in \mathcal{A}^* \) be a PS column word, and let \( w \in \mathcal{A}^* \) be a canonical word with column configuration \((c_1, \ldots, c_k)\). Then \( \mathcal{D}(c'w) = \mathcal{D}(c'c_1) \) and either:

1. \( c'c_1 \) is a PS canonical word, with \( \mathcal{D}(c'c_1) = c' \) or \( \mathcal{D}(c'c_1) = c'c_1 \), and hence \( c'c_1 \cdots c_k \) is a PS canonical word; or
2. \( c'c_1 \) is not a PS canonical word, and \( \mathcal{D}(c'c_1) \) has the same minimum symbol as \( c_1 \), and the word obtained from \( c'c_1 \) by erasing the symbols from \( \mathcal{D}(c'c_1) \) is the column word \( \mathcal{D}_2(c'c_1) \), that is a prefix of \( c_1 \).

**Proof.** Note that the minimum symbol in \( w \) that is further left (resp., right), is the minimum symbol of \( c_1 \).

Following Algorithm 3.8 to compute \( \mathcal{D}(c'w) \) (resp., \( \mathcal{D}(c'w) \)) we obtain first all symbols from \( c' \), since \( c' \) is an lPS (resp., rPS) column word. If the minimum symbol in \( c' \) is less than or equal (resp., less than) the minimum symbol of \( c_1 \), then \( \mathcal{D}(c'w) = c' = \mathcal{D}(c'c_1) \) (resp., \( \mathcal{D}(c'w) = c' = \mathcal{D}(c'c_1) \)).

Otherwise, the algorithm proceeds passing to symbols from \( w \), it will search for the first (left to right) symbol in \( w \) that is less than (resp., less than or equal to) the minimum symbol in \( c' \). So, the algorithm will find such element in the subsequence \( c_1 \). Indeed, let \( c_1 = a_k \cdots a_1 \), with \( a_i \in \mathcal{A} \), and \( s \in \{1, \ldots, k\} \) be the maximum index such that the minimum of \( c' \) is greater than (resp., greater than or equal to) \( a_s \). (Note that \( a_1 \) is the minimum symbol in \( w \) further left (resp., right).) Then computing \( \mathcal{D}(c'w) \) (resp., \( \mathcal{D}(c'w) \)) we get \( c'a_s \cdots a_1 \) which is equal to
\[ \mathcal{d}'(c_1') \] (resp., \( \mathcal{d}''(c_1') \)). Now, \( \mathcal{d}_k'(c_1') = a_k \cdots a_{k+1} \) is a column word and a prefix of \( c_1 \). Note that when \( s = k \), then \( \mathcal{d}'(c'w) = c_1 = \mathcal{d}''(c_1') \) (resp., \( \mathcal{d}''(c'w) = c_1 = \mathcal{d}''(c_1') \)) and \( \mathcal{d}_k'(c_1') \) (resp., \( \mathcal{d}_k'(c_1') \)) is the empty word.

The following algorithm gives us a left insertion method of a symbol \( a \in \mathcal{A} \) into a PS tableau \( B \), producing a new tableau (more precisely a canonical word) which we denote by \( a \rightarrow B \). The algorithm proceeds from left to right on the columns of \( B \).

**Algorithm 3.14** (Left insertion of a symbol on a PS tableau).

**Input:** A symbol \( a \in \mathcal{A} \) and an IPS (resp., rPS) tableau \( B \).

**Output:** The IPS (resp., rPS) tableau \( a \rightarrow B \).

**Method:** Let \( c_1' \) denote the column \( a \), and let \( (c_1, \ldots, c_k) \) be the IPS (resp., rPS) column configuration of \( \mathcal{C}(B) \). For each \( i = 1, \ldots, k \) proceed as follows:

Step \( i \): Denote by \( d_i \) the IPS (rPS) decreasing subsequence \( \mathcal{d}'(c'_i c_i) \) (resp., \( \mathcal{d}''(c'_i c_i) \)) and let \( c'_{i+1} \) be the prefix of \( c_i \) that is obtained from \( c'_i c_i \) deleting the symbols from \( d_i \).

Let \( d_{k+1} = c'_{k+1} \) and output the word \( d_1 \cdots d_k d_{k+1} = c_1' \cdots c_k' \).

One of the following two cases can occur during the computation of the \( i \)-th step of the previous algorithm:

**Case A:** \( c'_i c_i \) is not a canonical word, so the algorithm computes \( d_i \) and a non-empty column \( c'_{i+1} \), and the minimum symbol of \( d_i \) is the minimum symbol of \( c_i \), by the previous lemma;

**Case B:** \( c'_i c_i \) is a canonical word, and so two cases can occur according to the previous lemma:

(B1) \( d_i = c'_i c_i \) is a single column word, and so \( c'_{i+1} \) is the empty word and the algorithm will compute \( d_j = c_j \), for \( i < j \leq k \), and \( d_{k+1} \) is the empty word; or,

(B2) \( d_i = c'_i \), and so \( c'_{i+1} = c_i \) and the algorithm computes \( d_{j+1} = c_j \), for \( i \leq j \leq k \).

Notice that once the algorithm reaches Case B it maintains in that case, and never returns to Case A in the next iterations. So one of the following situations occurs in the execution of the algorithm:

**Situation (1):** If Case A occurs for all \( i \), then the algorithm will output a word that has column configuration \( (d_1, \ldots, d_k, d_{k+1}) \);

**Situation (2):** If the algorithm first enters on Case B1 in the \( i \)-th iteration, then it will output a word with column configuration \( (d_1, \ldots, d_{i-1}, d_i(= c'_i c_i), d_{i+1}(= c_{i+1}), \ldots, d_k(= c_k)) \);

**Situation (3):** As if the algorithm first enters Case B2 in the \( i \)-th iteration, then it will output a word with column configuration \( (d_1, \ldots, d_{i-1}, d_i(= c'_i), d_{i+1}(= c_i), \ldots, d_{k+1}(= c_k)) \).
Lemma 3.15. Algorithm 3.14 outputs a canonical word. With the notation used in the algorithm, the column configuration of the output is \( (d_1, \ldots, d_k, d_{k+1}) \), with \( d_{k+1} \) possibly empty.

Proof. Observe that on each iteration, the word \( d_i c'_i + 1 \) is a canonical word with column configuration \( (d_i, c'_i + 1) \).

If Situation (2) occurs, then the minimum symbol of \( d_i \) is the minimum symbol of \( c_i \), for \( 1 \leq i \leq k \). Similarly for Situation (1) in which \( c'_{k+1} = d_{k+1} \), and so \( d_k d_{k+1} \) is a canonical word. Therefore, the algorithm outputs a canonical word.

If Situation (3) occurs, then \( c'_i = d_i \) and thus using the observation on the beginning of the proof \( d_i d_i - 1 \) is a canonical word with column configuration \( (d_i - 1, d_i) \).

Since also \( (d_1, \ldots, d_i - 1) \) and \( (d_i + 1, d_{i+1}) \) are column configurations of the corresponding words, we conclude that also in this situation the algorithm outputs a canonical word. \( \square \)

Below, we provide two examples of the execution of the left insertion algorithm for an lPS tableau:

\[
\begin{pmatrix}
4 & 1 & 3 & 5 \\
6 & 8 & 7 & 4 & 8 \\
2 & 3 & 3 & 3 & 5 \\
1 & 1 & 1 & 2 & 3
\end{pmatrix}
\]

\[
= \begin{pmatrix}
4 & 1 \\
2 & 1
\end{pmatrix} \cdot
\begin{pmatrix}
6 & 8 & 7 & 4 & 8 \\
3 & 3 & 3 & 5 \\
1 & 1 & 1 & 2 & 3
\end{pmatrix}
\]

\[
\begin{pmatrix}
6 & 4 & 8 \\
2 & 3 \\
1 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
5 & 1 & 3 & 2 \\
6 & 8 & 7 & 4 & 8 \\
3 & 3 & 3 & 5 \\
1 & 1 & 1 & 2 & 3
\end{pmatrix}
\]

\[\text{Figure 2. Example of Situation (2).}\]

Lemma 3.16. For any word \( w \in A^* \) and \( a \in A \), we have

\[
(a \rightarrow R(w)) = R(aw).
\]

Proof. Maintaining the notation used in Algorithm 3.14 with the input \( a \) and \( B = R(w) \), the procedure outputs a canonical word with column configuration \( (d_1, \ldots, d_k, d_{k+1}) \), by Lemma 3.15.

Next we show that the \( i \)-th decreasing subsequence of \( aw \) is the word \( d_i \), thus proving the result. The initial decreasing subsequence \( d_1(aw) \) of \( aw \) is computed by taking \( d(ac_1 \cdots c_k) \). By Lemma 3.13, \( d_1(aw) = d(ac_1) \), thus equal to \( d_1 \), since \( d_1 = d(c'_1 c_1) \) and \( a = c'_1 \). With the notation used in Algorithm 3.14, \( c'_2 \) is the subsequence of \( ac_1 \) from...
which the symbols of $d_1(aw)$ were erased. Thus to compute $d_2(aw)$ we compute $d(c'_2c_2 \cdots c_k)$. By Lemma 3.13, $c'_2$ is a column word. Hence, the same reasoning can now be applied to compute $d(aw)$, knowing that $d_j(aw) = d_j$, for $1 \leq j < i$, and $d_i(aw) = d(c'_i \cdots c_k) = d(c'_i)$, by Lemma 3.13 for $i \leq k$. Finally, if $c'_{k+1}$ is non empty, these are the non-erased letters after the computation of $d_k(aw)$, and since it is a column word we get $d_{k+1}(aw) = c'_{k+1} = d_{k+1}$. □

For $x \in \{\ell, r\}$, given $w = w_1 \cdots w_n \in A^*$, with $w_1, \ldots, w_n \in A$, the corresponding $x$PS tableau can be computed using right-algorithm by:

$$R_x(w) = \left( ((\emptyset \leftarrow w_1) \leftarrow w_2) \ldots \right) \leftarrow w_n.$$ 

The previous lemma provides us an alternative way:

$$R_x(w) = w_1 \rightarrow \left( \ldots (w_{n-1} \rightarrow (w_n \rightarrow \emptyset)) \right).$$

More formally, the algorithm is defined as follows:

**Algorithm 3.17** (Left PS algorithm for words).

- **Input:** A word $w = w_1 \cdots w_n \in A^*$, with $w_1, \ldots, w_n \in A$.
- **Output:** An lPS (resp., rPS) tableau $R_x(w)$ (resp., $R_y(w)$).
- **Method:** Start with the empty tableau $P_0 = \emptyset$. For each $i = 1, \ldots, k$, insert $w_{n-k+1}$ into the lPS (resp., rPS) tableau $P_{i-1}$ as per Algorithm 3.14. Output $P_k$ for $R_x(w)$ (resp., $R_y(w)$).

### 3.5. Complexity of the algorithms.

Regarding the plactic monoid, it is known that Schensted’s algorithm has time complexity $O(n \log(n))$ on the length $n$ of the input word [Fre75].
In this subsection we study the complexity of the algorithms used in this paper to compute the column configuration of a word, namely the Patience Sorting algorithm (Algorithm 3.2), the left-to-right minimal subsequence algorithm (Algorithm 3.9), and the Left PS algorithm (Algorithm 3.17). (See [Sip96] for background on complexity.)

**Proposition 3.18.** The Patience Sorting algorithm, Algorithm 3.2, has time complexity $O(n \log(n))$, where $n$ is the length of the input word.

**Proof.** Let $w \in \mathcal{A}^*$, with $|w| = n$. Whenever we are inserting a symbol of $w$ using Algorithm 3.1, we can apply the binary search algorithm to the bottom row (which is sorted increasingly) to find the position in which this symbol is going to be inserted. There can be at most $n$ different symbols on the bottom row, and the time complexity of the binary search is thus $O(\log(n))$ (cf. [CLRS09, Ch. 2, Ex. 2.3-5] and [FM71]). We apply this search $|w| = n$ times, so the time complexity for computing $R_\ell(w)$ (resp., $R_r(w)$) using Algorithm 3.2 is $O(n \log(n))$. □

The idea behind the construction of Algorithm 3.9 is the use of the decreasing subsequences algorithm (Algorithm 3.8) in order to iteratively construct the PS column words of the column configuration of $w$.

**Proposition 3.19.** The time complexity of Algorithm 3.9 is $O(n^2)$, where $n$ is the length of the input word.

**Proof.** The decreasing subsequence construction is based on a left to right search on the word comparing symbols to find the next symbol less than (resp., less than or equal on the $r$-case) than the previous one. Given a word $w \in \mathcal{A}^*$ of length $n$, on the first step the algorithm chooses the leftmost symbol of $w$, as for the second step it can do, in the worst-case scenario, $n - 1$ comparisons. In the worst-case scenario, on the $i$-th step the algorithm can perform $n - i + 1$ comparisons. Therefore, the time complexity of the algorithm is

$$O((n - 1) + (n - 2) + \cdots + 1) = O\left(\frac{n(n - 1)}{2}\right) = O(n^2).$$

**Proposition 3.20.** Left PS algorithm, Algorithm 3.17, has time complexity $O(n^2)$, where $n$ is the length of the input word.

**Proof.** Given a word of length $n$, Algorithm 3.17 executes Algorithm 3.14 $n$ times.

Let $w = w_n \cdots w_1 \in \mathcal{A}^*$. We shall now analyse the time complexity of Algorithm 3.14 when inserting a symbol $w_{s+1}$ into the tableau $\mathcal{T}(w_s \cdots w_1)$. Let $(c_1, \ldots, c_k)$ be the column configuration of $\mathcal{C}(\mathcal{T}(w_s \cdots w_1))$ and let $\text{Sh}(\mathcal{T}(w_s \cdots w_1)) = (j_1, \ldots, j_k)$, where $j_1 + \cdots + j_k = s$. 
To execute Algorithm 3.14 we need only to compute the decreasing subsequences \( d(c_i'c_i) \), \( k_s \) times. (We keep the notation used in the algorithm.) Note that \( c_i' \) and \( c_i \) are column words. As such we can compute \( d(c_i'c_i) \) and \( c_i' + 1 \) scanning through the word \( c_i'c_i \) from left to right only once. Recall that \( c_i' + 1 \) is a prefix of \( c_i \). So reading the word \( c_i'c_i \) we will find first the symbols from \( c_i' \), and if we find either an increase or an equal symbol (an increase on the \( r \) case), it means that we reached the first symbol of \( c_i' + 1 \); keeping in memory the rightmost symbol, say \( a \), of \( c_i' \) we can proceed through the word reading now symbols from \( c_i' + 1 \); the word \( c_i' + 1 \) finishes right before we find the first symbol less (less or equal on the \( r \) case) than \( a \). Having identified \( c_i' + 1 \) we also get \( d(c_i'c_i) \).

Since \( |c_i'| = 1 \) and \( |c_i' + 1| \leq |c_i| = j_i \), we deduce that Algorithm 3.14 needs to do at most \( |c_i'| + |c_i' + 1| + |c_1 \cdots c_{k_s}| \leq 2s + 1 \) comparisons to execute the insertion of symbol \( w_{s + 1} \) into the tableau \( R(w_s \cdots w_1) \). Therefore, it has time complexity \( O(s) \). Hence, Algorithm 3.17 has time complexity \( O(1 + \cdots + n) = O(n^2) \).

3.6. PS congruences. In Subsection 3.1 we defined lPS and rPS tableaux. In each case we can identify words that lead to the same PS tableau, which yields, as will be shown, congruences on \( A^* \). We then give sets of relations that generate each congruence.

Define relations \( \equiv_{lps} \) and \( \equiv_{rps} \) by

\[
\begin{align*}
    u \equiv_{lps} v & \iff R_\ell(u) = R_\ell(v), \\
    u \equiv_{rps} v & \iff R_r(u) = R_r(v),
\end{align*}
\]

for \( u, v \in A^* \). By \( R_\ell(u) = R_\ell(v) \).

By [Rey07], Theorem 3.1, the relation \( \equiv_{lps} \) is a congruence. We present a short proof that both \( \equiv_{lps} \) and \( \equiv_{rps} \) are congruences using Algorithm 3.2 and Lemma 3.16.

**Proposition 3.21.** The relations \( \equiv_{lps} \) and \( \equiv_{rps} \) are congruences on \( A^* \).

**Proof.** Let \( x \in \{\ell, r\} \) and let \( \equiv_{lps} \) denote the corresponding relation \( \equiv_{lps} \) or \( \equiv_{rps} \). Let \( u, v \in A^* \) be such that \( u \equiv_{lps} v \). We will prove that \( ua \equiv_{lps} va \) and that \( au \equiv_{lps} av \) for any \( a \in A \). Then the result follows immediately by an inductive argument on the length of a word \( w = \in A^* \).

Let \( a \in A \). Since \( R_x(u) = R_x(v) \), by Algorithm 3.2 we get

\[
R_x(ua) = (R_x(u) \leftarrow a) = (R_x(v) \leftarrow a) = R_x(va)
\]

and hence \( ua \equiv_{lps} va \).

Now, by Lemma 3.16 we obtain

\[
R_x(au) = (a \rightarrow R_x(u)) = (a \rightarrow R_x(v)) = R_x(av)
\]

and hence \( au \equiv_{lps} av \). \( \square \)
Thus, we define the IPS monoid and the rPS monoid, denoted by $\text{lps}$ and $\text{rps}$, respectively, as the quotients of the free monoid $\mathcal{A}^*$ over the congruences $\equiv_{\text{lps}}$ and $\equiv_{\text{rps}}$, respectively. The IPS monoid is also known as the Bell monoid [Rey07].

So, each PS (lPS or rPS) tableau, and therefore each PS canonical word, is a unique representative of a PS class.

In the following paragraphs we present the analogue of Knuth’s relations for these monoids. Consider the following binary relations on $\mathcal{A}^*$:

$$\mathcal{R}_{\text{lps}} = \{ (yux, yxu) : m \in \mathbb{N}, x, y, u_1, \ldots, u_m \in \mathcal{A}, u = u_m \cdots u_1, x < y \leq u_1 < \cdots < u_m \}$$

and

$$\mathcal{R}_{\text{rps}} = \{ (yux, yxu) : m \in \mathbb{N}, x, y, u_1, \ldots, u_m \in \mathcal{A}, u = u_m \cdots u_1, x \leq y < u_1 \leq \cdots \leq u_m \}.$$

In the remainder of this subsection we prove that $\equiv_{\text{lps}}$ (resp., $\equiv_{\text{rps}}$) is equal to $\mathcal{R}_{\text{lps}}^\#$ (resp., $\mathcal{R}_{\text{rps}}^\#$), the smallest congruence relation containing $\mathcal{R}_{\text{lps}}$ (resp., $\mathcal{R}_{\text{rps}}$). Let us begin with an auxiliary result.

**Lemma 3.22.** For any $w \in \mathcal{A}^*$, if $w_c$ is the lPS (resp., rPS) canonical word associated to $w$, then $(w, w_c) \in \mathcal{R}_{\text{lps}}^\#$ (resp., $(w, w_c) \in \mathcal{R}_{\text{rps}}^\#$).

**Proof.** We give the detailed proof for the $\ell$-case and in parentheses give the main differences for the $r$-case.

We prove the result by induction on the length of $w$. If $w$ is a symbol in $\mathcal{A}$, the result follows immediately since $w = w_c$.

Suppose that the result holds for any word of length $n$. Let $w \in \mathcal{A}^*$ be a word of length $n$ and $a \in \mathcal{A}$. Suppose also that $\mathcal{D}(w) = (c_1, \ldots, c_k)$, and let $a_i$ be the minimum symbol in $c_i$, for $1 \leq i \leq k$. Note that $a_i$ is the rightmost symbol in the word $c_i$, and that $a_1 \leq \cdots \leq a_k$ (resp., $a_1 < \cdots < a_k$).

If $a_k \leq a$, then $\mathcal{C}(\mathcal{R}_\ell(wa)) = c_1 \cdots c_k a = w_c a$, and the result follows trivially. Otherwise, let $j \in \{1, \ldots, k\}$ be the smallest index such that $a_1 \leq \cdots \leq a_{j-1} \leq a < a_j \leq \cdots \leq a_k$ (resp., $a_1 < \cdots < a_{j-1} < a \leq a_j < \cdots < a_k$). Then $\mathcal{D}(wa) = (c_1, \ldots, c_{j-1}, c_j a, c_{j+1}, \ldots, c_k)$.

Now, we have $(a_i c_{i+1} a, a_i a c_{i+1}) \in \mathcal{R}_{\text{lps}}$, for $j \leq i \leq k$, since $a < a_i \leq a_{i+1}$ (resp., $a \leq a_i < a_{i+1}$) and $c_{i+1}$ is an IPS column word. By the induction hypothesis $w \mathcal{R}_{\text{lps}}^\# \mathcal{C}(\mathcal{R}_\ell(w))$. As

$$wa \mathcal{R}_{\text{lps}}^\# \mathcal{C}(\mathcal{R}_\ell(w)) a = c_1 \cdots c_j c_{j+1} \cdots c_{k-1} c_k a$$

$$\mathcal{R}_{\text{lps}}^\# c_1 \cdots c_j c_{j+1} \cdots c_{k-1} a c_k$$

$$\cdots$$

$$\mathcal{R}_{\text{lps}}^\# c_1 \cdots c_j a c_{j+1} \cdots c_k = \mathcal{C}(\mathcal{R}_\ell(wa)),$$

the result follows. \qed
Theorem 3.23. The relations $\equiv_{lps}$ and $\equiv_{rps}$ on $A^*$ are, respectively, the smallest congruences generated by the relations $R_{lps}$ and $R_{rps}$.

Proof. Suppose $(w, w') \in R_{lps}$. Then $w = yux$ and $w' = yxu$, for some $lps$ column word $u = u_m \cdots u_1$. Because $\mathcal{D}(w) = (yx, u) = \mathcal{D}(w')$, it follows that $R_{l}(w) = R_{l}(w')$ and so $w \equiv_{lps} w'$. Therefore, $R_{lps}$ is contained in the congruence $\equiv_{lps}$.

Conversely, let $u, v \in A^*$ be such that $u \equiv_{lps} v$. By definition $R_{l}(u) = R_{l}(v)$ and so $C(R_{l}(u)) = C(R_{l}(v))$. Using Lemma 3.22 we conclude that $(u, v) \in R_{lps}^\#$ by symmetry and transitivity of $R_{lps}^\#$.

The rPS part follows by the same argument using the appropriate rPS definitions.

The PS monoids, $lps$ and $rps$, can now be described as the quotients of the free monoid over the alphabet $A$ by the congruence generated by the relations $R_{lps}$ and $R_{rps}$, respectively. So, we can conclude that $lps$ and $rps$ are defined by the presentations $(A, R_{lps})$ and $(A, R_{rps})$, respectively.

Since for any $a \in A$, $(u, v) \in R_{lps}$ and $(u', v') \in R_{rps}$, we have that $|u|_a = |v|_a$ and $|u'|_a = |v'|_a$, the presentations $(A, R_{lps})$ and $(A, R_{rps})$ are multihomogeneous presentations and therefore $lps$ and $rps$ are multihomogeneous monoids.

Throughout the text we identify words over $A$ with elements of the monoids $lps$ and $rps$ that they represent.

3.7. PS monoids of finite rank. Recall that $A_n = \{1 < \cdots < n\}$. The restriction of the relations $\equiv_{lps}$ (resp., $\equiv_{rps}$) and $R_{lps}$ (resp., $R_{rps}$) to the alphabet $A_n$ yields the IPS (resp., rPS) congruence of rank $n$ denoted by $\equiv_{lps_n}$ (resp., $\equiv_{rps_n}$) and the set of $lps$ (resp., $rps$) relations $R_{lps_n}$ (resp., $R_{rps_n}$).

In a similar way to Theorem 3.23 we conclude that $\equiv_{lps_n}$ (resp., $\equiv_{rps_n}$) is the smallest congruence relation on $A_n^*$ generated by $R_{lps_n}$ (resp., $R_{rps_n}$). We define the IPS (resp., rPS) monoid of rank $n$, denoted by $lps_n$ (resp., $rps_n$), to be the quotient of the free monoid on the alphabet $A_n$ by the congruence $\equiv_{lps_n}$ (resp., $\equiv_{rps_n}$). The pair $(A_n, R_{lps_n})$ is a presentation of the monoid $lps_n$, whereas $(A_n, R_{rps_n})$ is a presentation for $rps_n$.

Since $A_n$ is finite, there are only finitely many words $u = u_m \cdots u_1 \in A_n^*$, with $u_1, \ldots, u_m \in A_n$ such that $u_1 < \cdots < u_m$. As there are also finitely many possibilities for symbols $x, y \in A_n$, the set $R_{lps_n}$ is finite. So the presentation $(A_n, R_{lps_n})$ is finite and we can conclude the following:

Proposition 3.24. For any $n \in \mathbb{N}$, the IPS monoid of rank $n$, $lps_n$, is finitely presented.
Since there are infinitely many decreasing words \( u = u_m \cdots u_1 \in A^*_n \), with \( u_1, \ldots, u_m \in A_n \) such that \( u_1 \leq \cdots \leq u_m \), the set \( \mathcal{R}_{rps_n} \) is infinite and thus the presentation \((A_n, \mathcal{R}_{rps_n})\) is infinite.

**Proposition 3.25.** For any \( n \in \mathbb{N} \) with \( n \geq 2 \), the monoid \( rps_n \) is not finitely presented.

**Proof.** Let \( n \in \mathbb{N} \) be such that \( n \geq 2 \). Suppose, with the aim of obtaining a contradiction, that \( rps_n \) is finitely presented. Then, there exists a finite presentation for \( rps_n \) using the generating set \( A_n \) [Rus95, Proposition 3.1]. Let \((A_n, \mathcal{G})\) be such a presentation. Since \( n \geq 2 \), the relation \( 12^i \equiv_{rps_n} 112^i \) holds, for any \( i \in \mathbb{N} \).

The word \( 12^i1 \) can have factors of three possible types of words: \( 12^k \), \( 2^k \) or \( 2^k1 \) (where \( k \leq i \) can vary). With the generating set \( A_n^* \), none of the given types of words satisfies a non-trivial relation, meaning that if for example \( 12^k = w \) in \( rps_n \), then \( 12^k \) and \( w \) are equal as words.

We conclude that for each \( i \in \mathbb{N} \), \( \mathcal{G} \) must have a relation whose right-hand side or left-hand side is equal to \( 12^i1 \). Thus \( \mathcal{G} \) is infinite, which is a contradiction. \( \square \)

### 4. Growth and identities

#### 4.1. Growth of PS monoids of finite rank

In this subsection we are interested in studying the growth of the finitely ranked IPS and rPS monoids. As we will see the IPS and rPS monoids of finite rank have, in general, distinct type of growths.

Given any monoid \( M \) generated by a finite set \( \Sigma \), the *growth function* of \( M \) with respect to \( \Sigma \), is the function \( \gamma_M : \mathbb{N} \rightarrow \mathbb{N} \) where \( \gamma_M(N) \) is the number of elements of \( M \) that can be expressed as products of length at most \( N \) of generators from \( \Sigma \).

For functions \( f, g : \mathbb{N} \rightarrow \mathbb{N} \), we write \( f \preceq g \) if there exists a positive constant \( c \) such that for all sufficiently large \( N \in \mathbb{N} \), we have \( f(N) \leq g(cN) \). The functions \( f \) and \( g \) are said to be equivalent if \( f \preceq g \) and \( g \preceq f \). We shall identify a function \( f \) with its equivalence class.

For any given finitely generated monoid, the growth functions relative to two distinct finite generating sets are equivalent (see [OK97, Section 9] and [Sap14] for more details on the growth of functions). As such, we consider equivalence classes of functions to study the growth of monoids.

With the above definition, we say that a finitely generated monoid \( M \) has *polynomial growth* if \( \gamma_M \) is bounded above by a function \( N^k \), for some \( k \in \mathbb{N} \). We say that the growth of \( M \) is *exponential*, if \( \gamma_M \) is bounded below by a function \( k^N \) for some \( k \in \mathbb{N} \).

Note that the free monoid of rank 1 has polynomial growth, and for any \( n \in \mathbb{N} \), with \( n \geq 2 \), the free monoid of rank \( n \) has exponential growth.
It is clear that both $lps_1$ and $rps_1$ are free monoids of rank 1, and hence have polynomial growth.

**Proposition 4.1.** For any $n \in \mathbb{N}$, with $n \geq 2$, the free monoid of rank $2^{n-1}$ embeds into $lps_n$.

**Proof.** Let $C_n$ be the subset of $A_n^*$ of IPS column words in which any of its elements has rightmost symbol 1. In $A_n$ there are $n-1$ elements greater than 1. So, as IPS column words are strictly decreasing words, we can conclude that there are $2^{n-1}$ elements in $C_n$.

Let $c_1 \cdots c_k$ be a product of elements of $C_n$. By induction on $i$, it is straightforward to see that the tableau $R_\ell(c_1 \cdots c_i)$ only has symbols 1 (the last symbol of each element of $C_n$) on its bottom row, and thus that $c_{i+1}$ is inserted as a new column at the right of the tableau. Hence the $i$-th column of $R_\ell(c_1 \cdots c_k)$ is simply $c_i$. Since each term of a product of elements of $C_n$ can be recovered from the tableau, it follows that the elements of $C_n$ generate a free submonoid of $lps_n$. □

Using the fact that if $M_1$ is a submonoid $M_2$, then $\gamma_{M_1} \leq \gamma_{M_2}$ [Sap14], we deduce the following:

**Corollary 4.2.** The IPS monoids of rank greater than 1 have exponential growth.

Regarding the monoids $rps_n$, for $n \geq 2$, we will show that they have polynomial growth.

**Theorem 4.3.** For any $n \in \mathbb{N}$, the monoid $rps_n$ has polynomial growth.

**Proof.** Let $n, N \in \mathbb{N}$ and consider the growth function $\gamma_{rps_n}$. The value $\gamma_{rps_n}(N)$ indicates the number of rPS tableaux with at most $N$ entries. Notice that each such rPS tableau has at most $n$ columns and each column has at most height $N$.

A column of height at most $N$ is filled with symbols from the set $A_n$. The number of weakly decreasing words over $A_n$ of length at most $N \in \mathbb{N}$ is uniquely determined by the number of solutions to $m_1 + \cdots + m_n \leq N$, where for any $i \in A_n$, $m_i$ denotes the (non-negative) number of $i$’s occurring in the considered word of length at most $N$. The number of solutions to this inequality is given by $\binom{n+N}{n}$ [Sta11, Subsection 1.2]. Thus there are at most $\binom{n+N}{n}$ different columns of height at most $N$. Thus there are at most

$$n \binom{n+N}{N}$$

rPS tableaux with at most $N$ entries.

Notice that $\binom{n+N}{N} \leq \frac{(n+N)!}{(n)!} = (n+N)(n+N-1) \cdots (N+1) \leq (n+N)^n$, and therefore $\gamma_{rps_n}$ is bounded above by a polynomial function with non-negative coefficients (This is equivalent to have polynomial growth, cf. [dLV11]). □
4.2. Identities on PS monoids. An identity is a formal equality between words of the free monoid over a countable alphabet \( \Sigma \). For any monoid \( M \), we say that \( M \) satisfies the identity \( u = v \) if for any morphism \( f : \Sigma^* \to M \), \( f(u) = f(v) \), that is, for any substitution of symbols in \( u \) and \( v \) by elements of \( M \), the equality holds in \( M \). For instance, the identity \( xy = yx \) is satisfied by any commutative monoid and the bicyclic monoid satisfies Adjan’s identity \( xyyxxyxyyx \) \[\text{Adj66}\]. The lPS and rPS monoids are multihomogeneous, therefore the left side and the right side of any identity satisfied by them has the same length (in fact the same number of each symbol). Thus, we will say that an identity has length \( n \in \mathbb{N} \) if the word on the left side of the identity has length \( n \).

Note first that both \( \text{lps}_1 \) and \( \text{rps}_1 \) are isomorphic to the free monoid of rank 1, which is commutative and therefore satisfies the non-trivial identity \( xy = yx \).

As free monoids of rank greater or equal than 2 do not satisfy non-trivial identities, the following proposition is a consequence of Proposition 4.1.

**Corollary 4.4.** The monoids \( \text{lps} \) and \( \text{lps}_n \) for \( n \geq 2 \) do not satisfy non-trivial identities.

This result is contrary to what happens in other multihomogeneous monoids like the sylvester, the hypoplactic, the Chinese, the Baxter, the stalactic, or the taiga monoid, all of which satisfy non-trivial identities \[\text{CM16}\]. As we will see in the following paragraphs, there is a hierarchy of different identities satisfied by finite-rank rPS monoids, but none satisfies by the infinite-rank rPS monoid. (Compare the situation for plactic monoids: it is known that the infinite-rank plactic monoid does not satisfy a non-trivial identity, but it is not known whether plactic monoids of rank greater than 3 satisfy non-trivial identities \[\text{CKK}^+\].)

Regarding \( \text{rps}_n \), with \( n \geq 2 \), we first prove some auxiliary lemmata.

**Lemma 4.5.** Let \( n \in \mathbb{N} \) and \( w \in \mathcal{A}_n^* \). Suppose that \( w \) has exactly \( k \) different symbols \( i_1 < \cdots < i_k \) (so \( |w| \geq k \)). Then, for \( j \leq k \), the rPS bottom row word of the tableau \( \mathcal{R}_r(w^j) \) has prefix \( i_1i_2\cdots i_j \).

*Proof.* We proceed by induction on \( j \). The result holds trivially for \( j = 1 \) since \( i_1 \) is the smallest symbol in \( w \) and so the leftmost column of \( \mathcal{R}_r(w) \) has \( i_1 \) as the bottom symbol.

Now suppose that the result holds for \( j < k \), and so the bottom row word of the tableau \( \mathcal{R}_r(w^j) \) has prefix \( i_1i_2\cdots i_j \). Denote by \( a_1 \) the symbol in \( w^j \) that takes the position of \( i_j \) in the bottom row of \( \mathcal{R}_r(w^j) \). Since \( w \) has \( k \) different symbols then denote by \( a_2 \) the rightmost symbol \( i_{j+1} \) in \( w^{j+1} \). As result, \( a_1a_2 \) is a subsequence of \( w^{j+1} \). By Lemma 3.3 and since \( a_1 < a_2 \), the symbol \( a_2 \) is positioned in \( \mathcal{R}_r(w^{j+1}) \) in a column further to the right of the column containing \( a_1 \).
Resuming, there is a column, say $c$, further to the right of the column containing $a_1 = i_j$ which has the symbol $i_{j+1}$. Hence, $c$ has a bottom symbol $a$ which satisfies $i_j < a \leq i_{j+1}$. Since there is no other symbol from $w$ between $i_j$ and $i_{j+1}$, we deduce that $a = i_{j+1}$ and that $R_c(w^{j+1})$ has bottom row word with prefix $i_1 i_2 \cdots i_{j+1}$.

**Lemma 4.6.** Let $n \in \mathbb{N}$ and $w \in \mathcal{A}_n^*$. Suppose that $w$ has exactly $k$ different symbols $i_1 < \cdots < i_k$ and that $R_c(w)$ has bottom row word $i_1 \cdots i_k$. Then, for any words $p, q \in \{i_1, \ldots, i_k\}^*$, with $ev(p) = ev(q)$, we get

$$R_p(wp) = R_q(wq).$$

**Proof.** The result follows from the execution of Algorithm 3.2 to compute $R_p(w) \leftarrow p$ and $R_q(w) \leftarrow q$. Each symbol $i_j$ (in $p$ or $q$) will be inserted in the column with bottom symbol $i_j$. Therefore, since $ev(p) = ev(q)$ we get the intended result. □

**Proposition 4.7.** For any $n \in \mathbb{N}$, the monoid $\text{rps}_n$ satisfies the identity:

$$(xy)^{n+1} = (xy)^n y x.$$  

**Proof.** Let $u, v \in \mathcal{A}_n^*$. Denote by $i_1 < \cdots < i_k$ all the symbols from $\mathcal{A}$ in the word $uv$. Note that $ev(uv) = ev(vu)$.

By Lemma 4.5, the tableau $R_c((uv)^k)$ has bottom row word $i_1 \cdots i_k$. Hence, by Lemma 4.6, we obtain $R_c((uv)^{n+1}) = R_c((uv)^n vu)$, since $(uv)^{n+1} = (uv)^k (uv)^{n+1-k}$ and $(uv)^n vu = (uv)^k (uv)^{n-k} vu$, and $ev((uv)^{n+1-k}) = ev((uv)^{n-k} vu)$. □

**Proposition 4.8.** The monoid $\text{rps}_n$ does not satisfy any non-trivial identity of length less than or equal to $n$.

The proof of this result follows closely the proof strategy of [CKK+ Proposition 3.1] and thus we only sketch it.

**Sketch proof.** Suppose, with the aim of obtaining a contradiction, that $\text{rps}_n$ satisfies a non-trivial identity of length less than or equal to $n$. Without loss of generality, assume that it satisfies such an identity over the variable set $\{x, y\}$, say $u(x, y) = v(x, y)$, of length equal to $n$ (that is, with $|u| = |v| = n$), where the $j$-th variable of $u$ is $x$ and the $j$-th variable of $v$ is $y$.

Let $s = n(n-1) \cdots 21 \in \mathcal{A}_n^*$ and let $t = n(n-1) \cdots (j+1)(j-1) \cdots 21 \in \mathcal{A}_n^*$. (So the word $t$ does not contain the generator $j$.) Since $\text{rps}_n$ satisfies the identity $u(x, y) = v(x, y)$, the tableaux $R_c(u(s, t))$ and $R_c(v(s, t))$ are equal, and so the lengths of their bottom rows are equal, which means the lengths of the longest strictly increasing subsequences of $u(s, t)$ and $v(s, t)$ are equal. But $u(s, t)$ contains the strictly increasing subsequence $12 \cdots n$ (since, in particular, $s$ is substituted for the $j$-th variable of $u$), while $v(s, t)$ does not contain a strictly increasing subsequence of length at least $n$, since such a subsequence can include
at most one symbol from each $s$ or $t$ and $t$ was substituted for the $j$-th variable of $v$. This is a contradiction.

Every finite-rank rPS monoid $rps_n$ is a submonoid of the infinite-rank rPS monoid $rps$, so Proposition 4.8 implies the following result:

**Theorem 4.9.** The monoid $rps$ does not satisfy any non-trivial identity.

## 5. Complete presentations

Monoids with finite complete presentations are of great importance, and one of the main reasons is that they provide a solution to the word problem (see [BO93, Subsection 2.2]). The goal of this section is to prove that the presentations $(A_n, R_{lps_n})$ and $(A_n, R_{rps_n})$ are complete.

In [Rey], the author presents two complete rewriting systems, but only on the set of permutations.

In order to prove that the presentations are noetherian we will first introduce an order on $A_n^*$. The length-plus-lexicographic order (also known as shortlex order or radix order) is an ordering on $A_n^*$ where, given $\alpha = \alpha_k \cdots \alpha_1, \beta = \beta_l \cdots \beta_1 \in A_n^*$, with $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_l \in A_n$, then $\alpha \ll \beta$ if

$$k < l \lor \left( k = l \land \exists i \left( \alpha_i < \beta_i \land \forall j < i \left( \alpha_j = \beta_j \right) \right) \right).$$

It is known that $\ll$ is a well order on $A_n^*$ [BO93, Subsection 2.2].

**Lemma 5.1.** For any $n \in \mathbb{N}$, if $(w, w') \in R_{lps_n}$ or $(w, w') \in R_{rps_n}$, then $w' \ll w$. Thus both presentations $(A_n, R_{lps_n})$ and $(A_n, R_{rps_n})$ are noetherian.

**Proof.** To prove that $(A_n, R_{lps_n})$ (resp., $(A_n, R_{rps_n})$) is noetherian we only need to check that $w' \ll w$, whenever $(w, w') \in R_{lps_n}$ (resp., $(w, w') \in R_{rps_n}$), since $\ll$ is admissible, in the sense of being invariant under left and right multiplication, by [BO93, Theorem 2.2.4]. Let $(w, w') \in R_{lps_n}$ (resp., $(w, w') \in R_{rps_n}$). Then, $w = yux, w' = y'u x$ for some $u = u_k \cdots u_1 \in A_n^*, x, y, u_1, \ldots, u_k \in A_n, x < y \leq u_1 < \ldots < u_k$ (resp., $x \leq y < u_1 \leq \ldots \leq u_k$). The words $w$ and $w'$ have the same length and the same first symbol, but $x < u_k$, so $w' \ll w$, as required.

From the previous lemma we deduce that each $lps_n$ (resp., $rps_n$) class has at least one irreducible element with respect to the relation $R_{lps_n}$ (resp., $R_{rps_n}$) (see [BO93, Lemma 2.2.7]).

**Lemma 5.2.** For any $n \in \mathbb{N}$, a word is irreducible with respect to the relation $R_{lps_n}$ if and only if it is an lPS canonical word, and a word is irreducible with respect to the relation $R_{rps_n}$ if and only if it is an rPS canonical word.
Proof. We first of all prove that a canonical word is irreducible. Consider a word \( w \in \mathcal{A}_n^* \) that it is an lPS (resp., rPS) canonical word. Let \((c_1, \ldots, c_k)\) be the IPS (resp., rPS) column configuration of \( w \). For each \( i \in \{1, \ldots, k\} \), let \( a_i \) be the rightmost symbol of \( c_i \). Recall that \( a_1 \leq \cdots \leq a_k \) (resp., \( a_1 < \cdots < a_k \)).

The left-hand side of a rewriting rule in \( R_{lps_n} \) (resp. \( R_{rps_n} \)) has the form \( yu_m \cdots u_1 x \), with \( x, y, u_1, \ldots, u_m \in \mathcal{A}_n \), and \( x < y \leq u_1 < \cdots < u_m \) (resp. \( x \leq y < u_1 \leq \cdots \leq u_m \)). So, if a rewriting rule is applied to \( w \), then necessarily \( y \) is one of the symbols \( a_i \) (the last symbol of a column), because \( y \leq u_m \) (resp., \( y < u_m \)). Note that if \( y \) is the last symbol of the column \( c_j \), then the symbols to its right in the word \( w \) (that is, the symbols in \( c_{j+1} \cdots c_k \)) are all greater than or equal to \( y \). Hence, no symbol \( x \) with \( x < y \) (resp., \( x < y \)), is to right of the symbol \( y \) in \( w \), as it is the case in the word \( yu_m \cdots u_1 x \). Therefore, no rewriting rule can be applied to \( w \).

Now we prove that a non-canonical word is reducible. Let \( n \in \mathbb{N} \). Consider a word \( w \in \mathcal{A}_n^* \) that is not an IPS (resp. rPS) canonical word. Let \( c_1, \ldots, c_k \) be IPS (resp. rPS) column words in \( \mathcal{A}_n^* \) such that \( w = c_1 \cdots c_k \), and for any \( i \in \{1, \ldots, k-1\} \), \( c_i c_{i+1} \) is not an IPS (resp. rPS) column word, and for some \( j \in \{1, \ldots, k-1\} \) the rightmost symbol in \( c_j \) is greater (resp. greater or equal) than the rightmost symbol in \( c_{j+1} \). Note that this decomposition of \( w \) exists since \( w \) is not an IPS (resp. rPS) canonical word.

Denote by \( y \) the rightmost symbol in \( c_j \), by \( x \) the leftmost symbol in \( c_{j+1} \) that is smaller (resp. less or equal) than \( y \), and let \( u_m, \ldots, u_1 \in \mathcal{A}_n \) and \( z \in \mathcal{A}_n^* \) be such that \( c_j c_{j+1} = u_m \cdots u_1 x z \). (Note that \( u_m \cdots u_1 \) is non-empty since \( c_j c_{j+1} \) is not an IPS (resp. rPS) column word.) Then \( yu_m \cdots u_1 x \), with \( x < y \leq u_1 < \cdots < u_m \) (resp. \( x < y \leq u_1 < \cdots < u_m \)), is a factor of \( w \), and hence \( w \) is reducible with respect to the relation \( R_{lps_n} \) (resp. \( R_{rps_n} \)). \( \square \)

**Lemma 5.3.** Let \( n \in \mathbb{N} \). Each \( lps_n \) class and each \( rps_n \) class has a unique irreducible element.

**Proof.** As the arguments follow in the same way, we just provide the proof for the \( lps_n \) case. If \( w, w' \in \mathcal{A}_n^* \) are irreducible words that are in the same \( lps_n \) class, then by the previous lemma \( w \) and \( w' \) are IPS canonical words. Since they are in the same \( lps_n \) class, that is, \( w \equiv_{lps_n} w' \) (equivalently, \( R_l(w) = R_l(w') \)), we get \( w = \mathcal{E}(R_l(w)) = \mathcal{E}(R_l(w')) = w' \), by Proposition 3.7. \( \square \)

Combining the previous results we extract the following:

**Lemma 5.4.** For any \( n \in \mathbb{N} \), the IPS and the rPS canonical words are the smallest words, with respect to the length-plus-lexicographic order, of its \( lps_n \) and \( rps_n \) classes, respectively.
Theorem 5.5. For every $n \in \mathbb{N}$, the pair $(A_n, R_{lps_n})$ is a finite and complete presentation for the lPS monoid of rank $n$, $lps_n$, whereas $(A_n, R_{rps_n})$ is a complete presentation for the rPS monoid of rank $n$, $rps_n$.

Proof. By Lemma 5.1 the presentations $(A_n, R_{lps_n})$ and $(A_n, R_{rps_n})$ are noetherian. By Lemma 5.3 the presentations are confluent. Thus both presentations are complete. \qed

Note that, despite not being finite, we can still algorithmically decide whether a word is a left-hand side of a rule in $R_{rps_n}$ and compute the corresponding right-hand side. Thus, we are able to compute normal forms and therefore solve the word problem for $rps_n$ (for more information regarding these constructions, see, for example, [Kob95]).

Corollary 5.6. For any $n \in \mathbb{N}$, the monoids $lps_n$ and $rps_n$ have solvable word problem.

Combining Theorem 5.5 with [SOK94, Theorem 5.3], we obtain the following corollary:

Corollary 5.7. For any $n \in \mathbb{N}$, the $lps_n$ monoid has finite derivation type.

By [Coh97] also conclude the following:

Corollary 5.8. For any $n \in \mathbb{N}$, the $lps_n$ monoid satisfies the homological conditions left- and right-$FP_\infty$.

6. Automaticity and Biautomaticity

6.1. Preliminaries. This subsection recall some basic definitions and necessary results from the theory of automatic and biautomatic monoids. For further information on automatic semigroups, see [CRRT01] and [CGM15b]. We assume familiarity with the theory of finite automata and regular languages, and also with the theory of transducers and rational relations (see, for example, [Ber79]).

An automatic structure for a monoid uses automata in order to compare pairs of words over an alphabet $\Sigma$. As these pairs of words might not have the same length we have to use a padding symbol $\$\$ to lengthen the shorter word to be the same length as the longer. More precisely, let $\delta_R : \Sigma^* \times \Sigma^* \rightarrow ((\Sigma \cup \{\$\}) \times (\Sigma \cup \{\$\}))^*$ be defined by:

$$(u_1 \cdots u_m, v_1 \cdots v_n) \mapsto \begin{cases} (u_1, v_1) \cdots (u_m, v_n) & \text{if } m = n, \\ (u_1, v_1) \cdots (u_n, v_n)(u_{n+1}, \$) \cdots (u_m, \$) & \text{if } m > n, \\ (u_1, v_1) \cdots (u_m, v_m)(\$, v_{m+1}) \cdots (\$, v_n) & \text{if } m < n. \end{cases}$$
and let $\delta_L : \Sigma^* \times \Sigma^* \to \left(\left(\{\$\} \cup \Sigma\right) \times \left(\{\$\} \cup \Sigma\right)\right)^*$ be defined by

\[
(u_1 \cdots u_m, v_1 \cdots v_n) \mapsto \\
\begin{cases} 
(u_1, v_1) \cdots (u_m, v_n) & \text{if } m = n, \\
(u_1, \$) \cdots (u_{m-n}, \$) (u_{m-n+1}, v_1) \cdots (u_m, v_n) & \text{if } m > n, \\
(\$, v_1) \cdots (\$, v_{n-m}) (u_1, v_{n-m+1}) \cdots (u_m, v_n) & \text{if } m < n.
\end{cases}
\]

where $u_i, v_i \in \Sigma$.

Let $M$ be a monoid, $\Sigma$ a finite generating set and let $L \subseteq \Sigma^*$ a regular language such that every element of $M$ has at least one representative in $L$. Define the relations

\[
L_a = \{(u, v) : u, v \in L, ua =_M v\}, \\
aL = \{(u, v) : u, v \in L, au =_M v\}.
\]

We say that $(\Sigma, L)$ is an automatic structure for $M$ if $(L_a) \delta_R$ is a regular language over $(\Sigma \cup \{\$\}) \times (\Sigma \cup \{\$\})$ for all $a \in \Sigma \cup \{\varepsilon\}$. A monoid is automatic if it admits an automatic structure with respect to some generating set.

We say that the pair $(\Sigma, L)$ is a biautomatic structure for $M$ if $(L_a) \delta_R$, $(aL) \delta_R$, $(L_a) \delta_L$, and $(aL) \delta_L$ are all regular languages over $(\Sigma \cup \{\$\}) \times (\Sigma \cup \{\$\})$ for all $a \in \Sigma \cup \{\varepsilon\}$. A monoid $M$ is biautomatic if it admits a biautomatic structure with respect to some generating set. Clearly, biautomaticity implies automaticity.

In order to prove that a relation $R\delta_R$ or $R\delta_L$ is regular, the following result, which is a combination of [FS93, Corollary 2.5] and [HT05, Proposition 4], will be useful:

**Proposition 6.1.** If $R \subseteq \Sigma^* \times \Sigma^*$ is rational relation and there is a constant $k$ such that $|u| - |v| \leq k$ for all $(u, v) \in R$, then $(R) \delta_R$ and $(R) \delta_L$ are regular.

### 6.2. Proving automaticity for the rPS monoids of finite rank.

Our goal in this subsection is to prove that the monoid $\text{rps}_n$ is automatic for any $n \in \mathbb{N}$. The first step will be to construct a regular language $L \subseteq \mathcal{A}_n^*$ which maps surjectively (and in fact bijectively) onto $\text{rps}_n$. We then prove that $(L_a) \delta_R$ is a regular language for every $a \in \mathcal{A}_n \cup \{\varepsilon\}$ and thus show that $(\mathcal{A}_n^*, L)$ is an automatic structure for $\text{rps}_n$. However, as we will see, $(\mathcal{A}_n^*, L)$ is only a biautomatic structure for this monoid when $n = 1$. We finish the subsection constructing a biautomatic structure for $\text{rps}_2$. 
Consider the following languages over $A_n$:

$$L^{(1)} = \{n\}^*\{n-1\}^*\cdots\{2\}^*\{1\}^+$$

$$L^{(2)} = \{n\}^*\{n-1\}^*\cdots\{3\}^*\{2\}^+$$

$$\vdots$$

$$L^{(n-1)} = \{n\}^*\{n-1\}^+$$

$$L^{(n)} = \{n\}^+;$$

it is immediate that these are regular languages.

For any $B \subseteq A_n$, with $B = \{a_1 < \ldots < a_k\} \neq \emptyset$, define

$$L^B = \prod_{a_i \in B} L^{(a_i)} = L^{(a_1)} \cdots L^{(a_k)};$$

this is a finite product of regular languages and thus is itself regular. Define $L^\emptyset = \{\varepsilon\}$.

Furthermore, since finite unions of regular languages are also regular, the language

$$L = \bigcup_{B \subseteq A_n} L^B$$

is also regular.

**Proposition 6.2.** For any $n \in \mathbb{N}$, the language $L \subseteq A_n^*$ maps bijectively onto $\text{rps}_n$.

**Proof.** First note that $L^{(a_i)}$ is the set of (non-empty) weakly decreasing words ending in $a_i$, or equivalently the set of readings of rPS columns whose bottom symbol is $a_i$. Thus $L^B$ (for $B \subseteq A_n$) is the set of readings of rPS tableaux whose bottom row consists of the symbols in $B$ (in increasing order). Since the sets $L^B$ are pairwise disjoint, it follows that $L$ is in one-to-one correspondence with the set of rPS tableaux. Hence $L$ maps bijectively onto $\text{rps}_n$. $\square$

It remains to prove that for any $i \in A_n \cup \{\varepsilon\}$, the language $(L_i)\delta_R$ is regular. We will first prove that the relation $L_i$ is rational. The first step is to construct some auxiliary relations. For any $i \in A_n$, let

$$L^{(j,\varepsilon)} = \{(u, u) : u \in L^{(j)}\},$$

and for any $i, j \in A_n$ such that $i \leq j$, let

$$L^{(j,i)} = L^{(j,\varepsilon)} \{(\varepsilon, i)\} = \{(u, u \cdot i) : u \in L^{(j)}\}.$$  

It is immediate from the definitions that for any $j \in A_n$, the relation $L^{(j,\varepsilon)}$ is rational, and that for any $i, j \in A_n$ with $i \leq j$, the relation $L^{(j,i)}$ is rational. Clearly, $(u, v) \in L^{(j,i)}$ if and only if $u$ is the reading of an rPS column whose bottom symbol is $j$, and $v$ is the reading of the column resulting from right-inserting a symbol $i \leq j$ (which will be inserted at the bottom of this column).
Let \( j \in \mathcal{A}_n \) and let \( B \subseteq \mathcal{A}_n \). If \( j \) is greater than every symbol in \( B \) (which, in particular, holds if \( B = \emptyset \)), then define

\[
L^{(B,j)} = \left( \prod_{a_i \in B} L^{(a_i, \varepsilon)} \right) \{(\varepsilon, j)\}.
\]

In this case, \((u, v) \in L^{(B,j)}_j\) if and only if \( u \) is the column reading of an rPS tableau whose bottommost row contains the symbols in \( B \) (in increasing order) and \( v \) is the column reading of the tableau that arises from right-inserting the symbol \( j \) (which is greater than every symbol in \( B \) and is thus appended to the right side of the bottommost row).

Otherwise \( j \) is less than or equal to some symbol in \( B \). Suppose \( B = \{a_1 < \ldots < a_k\} \) and let \( a_m = \min\{a_i \in B : a_i \geq j\} \). Partition \( B \) as \( B = C \cup \{a_m\} \cup E \), where \( j > a \) for all \( a \in C \) and \( j < a \) for all \( a \in E \). Now define

\[
L^{(B,j)} = \left( \prod_{a_i \in C} L^{(a_i, \varepsilon)} \right) L^{(a_m, j)} \left( \prod_{a_i \in E} L^{(a_i, \varepsilon)} \right).
\]

In this case, \((u, v) \in L^{(B,j)}_j\) if and only if \( u \) is the column reading of an rPS tableau whose bottommost row contains the symbols in \( B \) (in increasing order) and \( v \) is the column reading of the tableau that arises from right-inserting the symbol \( j \) (which is inserted at the bottom of the column ending in \( a_m \)).

Combining the last two paragraphs, and noting that the sets \( L^{(B,j)} \) are disjoint, we see that

\[
L_j = \bigcup_{B \subseteq \mathcal{A}_n} L^{(B,j)}.
\]

Since each \( L^{(B,j)} \) is a concatenation of rational relations, it too is a rational relation. Hence \( L_j \) is a rational relation.

Finally, note that

\[
L_\varepsilon = \{(u, u) : u \in L\},
\]

by Proposition 6.2 and is thus a rational relation.

For any \( j \in \mathcal{A}_n \cup \{\varepsilon\} \), if \((u, v) \in L_j\) then \(|u|−|v| \leq 1\), since insertion into a tableau increases the length of the column reading by 1. Hence \( L_j \delta_R \) is a regular language by Proposition 6.1. Thus,

**Theorem 6.3.** For any \( n \in \mathbb{N} \), \((\mathcal{A}_n, L)\) is an automatic structure for \( \text{rps}_n \).

For \( n = 1 \), the languages

\[
(1L)\delta_R = \{(1, 1)\}^*\{(\$, 1)\},
\]

\[
(1L)\delta_L = \{(\$, 1)\}\{(1, 1)\}^*,
\]

\[
(1L)\delta_L = \{(\$, 1)\}\{(1, 1)\}^*.
\]
are all regular, and so \((A_1, L)\) is a biautomatic structure for \(\text{rps}_1\) and thus \(\text{rps}_1\) is biautomatic. However, \((A_n, L)\) is not in general a biautomatic structure for \(\text{rps}_n\):

**Proposition 6.4.** For any \(n \in \mathbb{N}\) with \(n \geq 2\), the language \((1L)\delta_R\) is not regular.

**Proof.** Let \(n \in \mathbb{N}\) with \(n \geq 2\). Suppose, with the aim of obtaining a contradiction, that

\[(1L)\delta_R = \{(u, v)\delta_R : u, w \in L \land 1u \equiv_{\text{rps}_n} v\}\]

is a regular language over \((A_n \cup \{\$\}) \times (A_n \cup \{\$\})\). Let \(N\) be the number of states in a finite automaton \(A\) recognizing \((1L)\delta_R\) and fix \(\alpha > N\).

For each \(\beta \in \mathbb{N}\), let \(u_\beta\) and \(v_\beta\) be representatives of the elements of \(\text{rps}_n\) whose column readings are \(2^{\beta}1^\beta\) and \(1^{\beta+1}2^\beta\), respectively. Since these words are both in \(L\), we conclude by Proposition 6.2 that \(u_\beta = 2^{\beta}1^\beta\) and \(v_\beta = 1^{\beta+1}2^\beta\). It is straightforward to see that

\[(u_\beta, v_\beta)\delta_R = (2^{\beta}1^\beta, 1^{\beta+1}2^\beta)\delta_R \in (1L)\delta_R.\]

Since \(\alpha > N\), when \(A\) reads \((u_\alpha, v_\alpha)\delta_R\), it enters the same state after reading two different symbols 2 of \(u_\alpha\). Let \(u'\) and \(u''\) be the prefixes of \(u_\alpha\) up to and including these symbols 2; note that \(|u''| > 0\). Then \(u' = 2^\eta\) and \(u'' = 2^\eta 2^\gamma\) for some \(\eta, \gamma \in \mathbb{N}\) with \(\gamma > 0\). Let \(v'\) and \(v''\) be the prefixes of \(v_\alpha\) of lengths \(\eta\) and \(\gamma\), respectively; thus \(v' = 1^\eta\), \(v'' = 1^\eta 1^\gamma\). Pumping \((u'', v'')\delta_R\) shows that

\[(2^{\eta+2\gamma}2^{\alpha-\eta-\gamma}1^\alpha, 1^\eta 2^\gamma 1^{\alpha+1-\eta-\gamma}2^\alpha)\delta_R \in (1L)\delta_R.\]

Hence \(1 \cdot 2^{\alpha+\gamma+1}1^\alpha \equiv_{\text{rps}_n} 1^{\alpha+\gamma+1}2^\alpha\). As \(\gamma > 0\), the number of symbols 1 on the left side of the equality is smaller than the number of symbols 1 on the right side of the equality. This is a contradiction since \(\text{rps}_n\) is multihomogeneous. \(\square\)

We can also show that \(\text{rps}_2\) is biautomatic as follows: let

\[J = \{2^i 12^k 1^{j-1} : i, k \in \mathbb{N}_0, j \in \mathbb{N}\} \cup \{2^i : i \in \mathbb{N}_0\}\]

\[= \{2\}^*\{1\}\{2\}^*\{1\}^* \cup \{2\}^*;\]

note that \(J\) is regular.

The map \(\varphi : J \to \text{rps}_2\), defined by

\[
\varphi : x \mapsto \begin{cases} 
2^{i}1^{j}2^{k} & \text{if } x = 2^{i}12^{k}1^{j-1} \\
2^{i} & \text{if } x = 2^{i}
\end{cases}
\]
is a bijection between \( J \) and \( \text{rps}_2 \) (viewed as column readings of tableaux). Thus, \( J \) is a regular language that maps bijectively onto \( \text{rps}_2 \). Since

\[
J_1 = \{ (2, 2) \}^* \{ (1, 1) \} \{ (2, 2) \}^* \{ (1, 1) \}^* \{ (\varepsilon, 1) \} \cup \{ (2, 2) \}^* \{ (\varepsilon, 1) \} \\
J_2 = \{ (2, 2) \}^* \{ (1, 1) \} \{ (2, 2) \}^* \{ (\varepsilon, 2) \} \{ (1, 1) \}^* \cup \{ (2, 2) \}^* \{ (\varepsilon, 2) \}
\]

1. \( J = \{ (\varepsilon, 1) \} \{ (2, 2) \}^* \{ (1, 1) \} \{ (2, 2) \}^* \{ (\varepsilon, 1) \} \cup \{ (\varepsilon, 2) \} \{ (2, 2) \}^* \)

we deduce that \((i, J)\delta_R, (i, J)\delta_L, (J_i)\delta_R\) and \((J_i)\delta_L\) for \( i \in \{ 1, 2 \} \) are regular languages. Therefore we have the following result:

**Proposition 6.5.** The pair \((A_2, J)\) is a biautomatic structure for \( \text{rps}_2 \).

The idea of the biautomatic structure for \( \text{rps}_2 \) does not seem to generalize to higher ranks, so the following question remains open:

**Question 6.6.** For each \( n \geq 3 \), is \( \text{rps}_n \) biautomatic?

### 6.3. Proving biautomaticity for the IPS monoids of finite rank.

#### 6.3.1. The language of representatives.

In order to prove that finite-rank IPS monoids are biautomatic, we will first work with a language \( K \) defined over a set of generators different from \( \mathcal{A}_n \), and then later switch back to \( \mathcal{A}_n \).

Define the alphabet

\[
\mathcal{E}_n = \{ e_\alpha : \alpha = \alpha_n \cdots \alpha_1 \in \mathcal{A}_n^+, \, \alpha_1, \ldots, \alpha_n \in \mathcal{A}_n, \, \alpha_1 < \ldots < \alpha_n \}.
\]

The idea is that each IPS column word \( \alpha \) of \( \text{rps}_n \) is represented by a single IPS column \( e_\alpha \). Note that since \( \mathcal{A}_n \) is finite and the subscript of each letter \( e_\alpha \) is a strictly decreasing word, \( \mathcal{E}_n \) is finite.

For any \( e_\alpha \in \mathcal{E}_n \), let \( \alpha_1 \) denote the rightmost (that is, smallest) symbol in \( \alpha \). Define the following language of representatives over \( \mathcal{E}_n \),

\[
K = \{ e_{\alpha_1} e_{\alpha_2} \cdots e_{\alpha_k} : k \in \mathbb{N}_0, e_{\alpha_i} \in \mathcal{E}_n, \, \alpha_i^i \leq \alpha_1^{i+1} \text{ for all } i \}
\]

Notice that \( e_{\alpha_1} e_{\alpha_2} \cdots e_{\alpha_k} \in K \) if and only if \( \alpha_1 \alpha_2 \cdots \alpha_k \) is the IPS column decomposition of the corresponding IPS tableau, or in other words, \( \alpha_1 \alpha_2 \cdots \alpha_k = \mathcal{C}(\mathcal{R}_e(\alpha_1 \alpha_2 \cdots \alpha^k)) \). Thus \( K \) maps bijectively to \( \text{rps}_n \).

Observe also that in order to check that \( \alpha_i^i \leq \alpha_1^{i+1} \), the automaton needs only to store the previously read symbol. So \( K \) is a regular language.

#### 6.3.2. Right multiplication by transducer.

We will now prove that, for any \( \gamma \in \mathcal{A}_n \), the relation

\[
K_{e_\gamma} = \{ (e_{\alpha_1} \cdots e_{\alpha_k}, e_{\beta_1} \cdots e_{\beta_l}) \in K \times K : e_{\alpha_1} \cdots e_{\alpha_k} e_\gamma = e_{\beta_1} \cdots e_{\beta_l} \}
\]

is a rational relation. In order to do so, we will prove that \( K_{e_\gamma} \) can be recognized by a transducer that reads right-to-left; this will be enough
to prove that $K_{e_{\gamma}}$ is rational since the class of rational relations is closed under reversal [Ber79] pp. 65–66).

We will also make use of non-determinism, in that our transducer will be able to guess the next symbol to be read. This symbol is stored in the state of the transducer so that later it can be compared with next symbol when it is read. Then, if the guess was correct, the transducer continues, otherwise it enters a failure state.

Thus the idea is that the transducer will read a pair of words

$$(e_{\alpha^1}e_{\alpha^2}\cdots e_{\alpha^k}, e_{\beta^1}e_{\beta^2}\cdots e_{\beta^l}) \in K \times K$$

from right to left, with the aim of checking whether the pair is in $K_{e_{\gamma}}$ or not. We imagine the transducer as reading symbols from the top tape and outputting symbols on the bottom tape. Essentially, the transducer will perform the insertion algorithm using the alphabet $E_n$ as a column representation of the tableau.

The transducer stores either the symbol $e_{\gamma}$ or the symbol $\infty$. It non-deterministically guesses the next symbol to be read (that is, one symbol to the left of the current symbol). Initially it stores the symbol $e_{\gamma}$; the idea is that when the stored symbol is $e_{\gamma}$, the transducer is searching for the correct column in which to insert $e_{\gamma}$. Following Algorithm 3.1 the transducer knows that it has found the correct column, $\alpha^i$ (represented by $e_{\alpha^i}$) in which to insert $e_{\gamma}$ if the last symbol of $\alpha^i$ is strictly greater than $\gamma$ and the last symbol of $\alpha^{i-1}$ (which it knows since it has non-deterministically guessed $e_{\alpha^{i-1}}$) is less or equal than $\gamma$. When the stored symbol is $\infty$, the transducer has completed the insertion and simply reads the remainder symbols (which were not read yet) from the input tape and writes them on the output tape.

Initially the transducer stores $e_{\gamma}$ and non-deterministically knows $e_{\alpha^k}$. If the last symbol of $\alpha^k$ is less than or equal to $\gamma$ the transducer outputs $e_{\gamma}$ before reading any symbol and stores $\infty$.

When reading a symbol $e_{\alpha^i}$, the transducer non-deterministically knows $e_{\alpha^{i-1}}$, or non-deterministically guesses that it has reached $e_{\alpha^i}$. In the later case, the transducer reads $e_{\alpha^i}$, outputs $e_{\beta}$, with $\beta = \alpha^i\gamma$, and stores $\infty$. Otherwise, there are two possibilities. If, when reading $e_{\alpha^i}$, the last symbol of $\alpha^{i-1}$ is less or equal than $\gamma$, then the transducer outputs $e_{\beta}$, with $\beta = \alpha^i\gamma$, and stores $\infty$. Otherwise, it keeps the stored symbol $e_{\gamma}$ and proceeds to read the symbol $e_{\alpha^{i-1}}$.

This heuristic description of the transition function corresponds to a more formal description using a finite lookup table. The transducer performs right-insertion of a symbol $e_{\gamma}$ in terms of representatives in the language $K$. Thus $K_{e_{\gamma}}$ is a rational relation.

6.3.3. Left multiplication by transducer. We want to prove that for any $\gamma \in A_n$, the relation

$$e_{\gamma}K = \{(e_{\alpha^1}\cdots e_{\alpha^k}, e_{\beta^1}\cdots e_{\beta^l}) \in K \times K : e_{\gamma}e_{\alpha^1}\cdots e_{\alpha^k} \equiv_{lps_n} e_{\beta^1}\cdots e_{\beta^l}\}$$
is a rational relation.

The strategy will be analogous to the one used in the right multiplication by transducer, but this time the transducer will read pairs of words from left to right. Similarly, we will make use of the left insertion algorithm (see subsection 3.4) but using the symbols $E_n$. The transducer will read a pair of words 

$$(e_{\alpha_1} e_{\alpha_2} \cdots e_{\alpha_k}, e_{\beta_1} e_{\beta_2} \cdots e_{\beta_l}) \in K \times K$$

from left to right with the purpose of checking whether the pair is in $e_\gamma K$.

The transducer stores in its state a symbol $e_\eta$ from $E_n$ or a symbol $\infty$. Initially, the transducer stores $e_\gamma$. The idea is that when the stored symbol is some $e_\eta$, the transducer will insert $\eta$ into the column represented by the symbol it is reading from the input tape. When the stored symbol is $\infty$, the transducer has completed the algorithm and simply reads the remaining symbols from the input tape and writes them in the output tape.

If the transducer is storing $e_\eta$ and reads $e_{\alpha^i}$, there are two possibilities:

- If the last symbol of $\eta$ is greater than the first symbol of $\alpha^i$, it outputs $e_{\eta\alpha^i}$ and stores $\infty$.
- Otherwise, it factors $\alpha^i$ as $\alpha^i = \mu \nu$, where the length of $\mu$ is minimal such that last symbol of $\mu$ is greater than or equal to the last symbol of $\eta$. It outputs $e_{\eta\nu}$ and stores $e_\mu$.

If the transducer reaches the end of its input (which it can know by non-deterministically looking ahead) while storing $e_\eta$, it outputs $e_\eta$.

Again, this heuristic description of the transition function corresponds to a more formal description using a finite lookup table. The transducer performs left-insertion of a symbol $e_\gamma$ in terms of representatives in the language $K$. Thus $e_\gamma K$ is a rational relation.

6.3.4. Deducing biautomaticity. Therefore, for any $\gamma \in A_n$, both $e_\gamma K$ and $K e_\gamma$ are rational relations. Define $Q \subseteq E_n^* \times A_n^*$ by

$$Q = \{(e_{\alpha_1} e_{\alpha_2} \cdots e_{\alpha^k}, \alpha^1 \alpha^2 \cdots \alpha^k) : e_{\alpha^i} \in E_n\}.$$

It is easy to see that $Q$ is a rational relation. (In fact, $Q$ is the homomorphism extending $e_a \mapsto \alpha$.) Let

$$L = (K)Q = K \circ Q = \{v \in A_n^* : \exists u \in K (u, v) \in Q\}.$$

Since $K$ maps bijectively onto $lps_n$, so does $L$. Moreover, $L$ is a regular language, because $K$ is a regular language and a $Q$ is a rational relation and the class of regular languages is closed under applying rational
relations. For any $\gamma \in A_n$,

\[(u, v) \in L_\gamma \iff u \in L \land v \in L \land u \gamma \equiv_{lps_n} v\]

\[\iff \exists u', v' \in K \ (u', u) \in Q \land (v', v) \in Q \land u' e \gamma \equiv_{lps_n} v'\]

\[\iff \exists u', v' \in K \ (u, u', v') \in Q^{-1} \land (v', v) \in K e \gamma \land (v, v) \in Q\]

\[\iff (u, v) \in Q^{-1} \circ K e \gamma \circ Q\]

Therefore, as $Q^{-1}, K e \gamma$, and $Q$ are rational relations, $L_\gamma$ is a rational relation.

Similarly, from the fact that $e \gamma K$ is a rational relation, we deduce that $\gamma L = Q^{-1} \circ e \gamma K \circ Q$ is a rational relation.

Finally, since $L$ maps bijectively onto $L$,

\[L_\varepsilon = \varepsilon L = \{(u, u) : u \in L\},\]

which is a rational relation.

For any $\gamma \in A_n$, if $(u, v) \in L_\gamma$ or $(u, v) \in \gamma L$, then $|v| \leq |u| + 1$ since $u \gamma \equiv_{lps_n} v$ and $lps_n$ is multihomogeneous. By Proposition 6.1, $(L_\gamma)\delta_R$, $(L_\gamma)\delta_L$, $(\gamma L)\delta_R$ and $(\gamma L)\delta_L$ are all regular. This proves the following result:

**Theorem 6.7.** For any $n \in \mathbb{N}$, $(A_n, L)$ is a biautomatic structure for $lps_n$.

### 6.4. Consequences of automaticity

The automaticity of the rPS and IPS monoids together with \[\text{Corollary 3.7}\] imply the following result:

**Proposition 6.8.** For any $n \in \mathbb{N}$, the monoids $rps_n$ and $lps_n$ have word problem solvable in quadratic time.

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