The Place of Quasi Topological Structure in the Mathematical Theory of Categorization

Quasi Topological Structure and Soft Sets

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Abstract
This work is a theoretical one bridging two mathematical models namely the Quasi Topological Structure (QTS) and Soft Sets (SST) theories. We prove that the Quasi Topological Structure (QTS) can be viewed as a complexification of Soft Sets, from the point of view of its capacity of analysis. Our strategy is to compare two mathematical structures, namely the structure of Quasi Topological Structure (QTS) and the structure of Soft Sets from the point of view of their mathematical properties. These properties are expressed by means of mathematical notions that translate conceptual features. The notion of conceptual feature is taken in the common sense outside of any scientific domain. However, the mathematical notions are given inside mathematics by their specific conditions. This paper is a theoretical comparison between two new mathematical structures that can become useful in many approaches of Artificial Intelligence (AI). We propose an epistemological analysis in the frame of mathematical foundations and not a tool or a methodology to solving a particular problem.

Introduction
The mathematical theory of categorization encodes in a mathematical way, the idea that given a "space of entities to be categorized" and a set of "criteria of categorization", one must build "clusters" of entities. A cluster is a subset of entities being more or less similar to each other. Soft Sets (6), Fuzzy Sets (7), Rough Sets (8) and Quasi Topological Structure (QTS) (4) are models giving real approximation spaces, while Logic of Determination of Objects (LDO) (5) and Formal Concept Analysis (FCA) (1) are built not only for the aim of categorization but to capture some other features of knowledge as the link between objects and concepts and the typicality of objects. At the same time, all of them can be considered as theories and not just as models.

The degree of generality of the soft sets definition allows an analysis of the other categorization approaches mentioned above in relation with soft sets, i.e. to build a model of the soft set type for each one. In this paper we compare the QTS with soft sets and the LDO with soft sets. This comparison is done from a conceptual point of view through features translated within mathematical properties. It can prove the advantage of modeling a QTS structure by a soft set for reasons of computation and design of the algorithms. The aim of this approach is to study whether the soft set model corresponding to a QTS model or to a LDO model as categorization models provides a more informative or contribution from the point of view of calculus compared to the original corresponding model. This comparison is based on the following assumptions:

1. The set E of parameters in the definition of a soft set (X, E, F) is defined as a general set. Its elements are not subject to any conditions; so, the parameters can be elements of an arbitrary space, for instance, elements of a structured space, or even, structured spaces themselves.
2. A Quasi Topological Structure (QTS) is a particular case of soft sets where the set of parameters are either topological spaces, or approximation spaces in the sense of rough sets, or spaces with other structures as Logic of Determination of Objects (LDO). This comparison is done from a conceptual point of view through features within specific mathematical properties.

This paper is organized in the following parts:
1. Soft Sets;
2. Quasi Topological Structure (QTS);
3. The Quasi Topological Structure (QTS) as a soft set;
4. Logic of Determination of Objects (LDO).
5. The mathematical model of LDO as a soft set.
6. Conclusions

Soft Sets
A soft set (6) is a class of sets built on a space "guided" by parameters belonging to an other set. A soft set is a quadruple (X, E, (A,F)) where:
1. X is an arbitrary space, the space of categorization;
2. E is a set of parameters following which one categorizes the elements of X; A is a subset of parameters, A ⊆ E;
3. F is a function building the categorization: F : A → P(X).
Some specifications must be given:
1. The objects of categorization belong to a space X, distinct from the space E of parameters. This last one is a space of criteria.
2. One does not specify any structure neither for X, nor for E.
3. The only link between X and E is the function F:
   (a) The categorization of the objects in X is given following the pairs (A,F). So it can be built:
      • for each subset of E;
      • only for some subsets of E;
      • for all parameters of E.
   (b) A subset of X is associated to each parameter of A.

**Quasi Topological Structure (QTS)**

We recall the definition from (4).

Definition 1. Let <X, O> be a topological space where X denotes the space and O denote the topology. We say that a set A from this space is structured by a quasi topology or it has a quasi topology structure (QTS) if there exists two open sets O₁ and O₂, and two closed sets F₁ and F₂ such that:

1. \( O₂ \subset O₁ \subset A \subset F₁ \subset F₂ \)
   
   with:

2. O₁ is the biggest open set contained in E, that is, \( O₁ = \text{Int}(A) \)
3. F₁ is the smallest closed set containing E, that is, \( F₁ = \text{Cl}(A) \)
4. O₂ is the biggest open set strictly contained in O₁
5. F₂ is the smallest closed set strictly containing F₁

The set O₂ is said to be the strict interior of E; the set O₁ is the large interior of E. The set F₂ is said to be the large closure of A and the set F₁ the strict closure of A. The internal boundary, the external boundary, the large boundary A are defined by:

6. Int-bound(A) = F₁ - O₂
7. Ext-bound(A) = F₂ - O₁
8. Large-bound(A) = Int-bound(A) ∪ Ext-bound(A)

Definition 1 is presented in an intuitive way in the Figure 1. In Figure 1, we can see that the internal boundary is formed by all elements of \( O₁ - O₂ \) and by all elements of A - O₁, so of elements belonging to \( A \), and of elements of F₁ - A, so of elements do not belonging to A. As for external boundary, it is formed by all elements of A - O₁, so belonging to A and by all elements of F₂ - A, so of elements not in A. In other words, the internal boundary is not completely in A and the external boundary is not completely outside of A. The large boundary is formed by all elements in the large closure not being in the strict interior.

The notion of Quasi Topology Structure (QTS) was founded by Jean-Pierre Desclés (4). It was defined as a cognitive model with applications in several fields of science beginning with linguistics, especially formal semantics, and continuing with anthropology, sociology and laws in the area of humanities. In computer science this notion can be applied at least in the field of digital image processing. Till now, the QTS was instantiated and studied within two types of approximation spaces: topological spaces (4) and rough sets spaces.

The notions of “openness” and “closeness” expressed in a topological space by the own conditions will be expressed in an approximation space by conditions related to the nature of this type of space, that is “relations”. In the rough sets approach, we endow the approximation space with two relations and introduce the notion of double rough set.

The set theory conditions for a structure to be a quasi-topology are directly outcome from real examples. These conditions are established in (4). That is the reason for an analysis of quasi topology structure (QTS) as mathematical model, in other words, to build the mathematical model associated to the quasi topology structure in different mathematical spaces taken as mathematical structures. The model of quasi topology structure must be a mathematical model encoding the idea that a set, as member of an abstract space, can have a “strict interior” and a “large interior” a “strict closure” and a “large closure”. It follows, obviously, that this set can be provided with an “internal boundary” and an “external boundary”. These ideas come from the linguistic expressions of the space, the linguistic expressions of the time, even from basic notion in law as “legal” opposite to “illegal” or from the social notion of “inhabitant of a city”.

The strict interior, the strict exterior, the internal boundary and the external boundary are subsets of the given set whose elements have a slightly different status from the other elements although they belong to the given set. These are the cognitive elements of the structure of quasi topology. From the technical point of view, it is a question of “approximate categorization”, the term “approximate” being taken in its large meaning.

- The “conceptual metaphor” which relates “interior” of the topology to “lower approximation” of rough sets, to the
“heart” of locology (9), to some degree of membership of fuzzy sets (7) have not the same “mathematical meaning”. For the “interior” versus “closure” in topology the basic privileged idea is that one of continuity versus separation. For “lower approximation” versus “upper approximation” there is an algebraic relation defining them. For the locology, it is also an algebraic relation but with some additional conditions.

- The informative power of QTS in some applications is greater than this one of approaches mentioned above. This fact was proved using some examples from social sciences and humanities, particularly, from linguistics.

We can define a quasi topological space as follows:

**Definition 2.** A quasi topological space \(<X, O_1, O_2, >\) is a triple formed by a space \(X\) and two topologies \(O_1, O_2\) with the properties from Definition 1.

**The Quasi Topology Structure (QTS) as soft set**

The soft set model is a good framework that capture on one hand the notion of QTS and from the other hand each of models mentioned above. From the mathematical point of view it is a generalization of a lot of other mathematical structures. Keeping in mind the idea that the transformation of a cognitive model into a mathematical model has the following characteristic: the more cognitive traits are added, the higher the degree of particularity of the mathematical model. As a consequence, one could say that the more general the mathematical model is, the further away it is from the immediate reality. Less general it is, the closer it is to the applications. We can easily see that the QTS can be defined not only based on the notion of topology but taking into account other mathematical notions (general relations, particular relations, membership degrees ...). In this case, the QTS becomes defined outside topology. The reason of this choice depends on the type of application. The soft sets theory seems to be a good framework in order to unified that. The idea of interpreting the QTS in a soft set approach is the following: Definition 3. A QTS is a soft set is a quadruple \((X, E, (A,F))\) where:

1. \(X\) is an arbitrary space, the space of categorization;
2. The set \(E\) of parameters is a set of parameters \(E = \{O^1, O^2\}\) where \(O^1\), expressed the “interior type” and \(O^2\), expressed the “strict interior type”;
3. The function \(F, F: A \rightarrow P(X)\) is defined by \(f_1: O^1 \rightarrow \{\text{all interiors}\}\) and \(f_2: O^2 \rightarrow \{\text{all strict interiors}\}\)

**Logic of Determination of Objects (LDO)**

The LDO (5) as a non classical logic is issued from the idea that all logic as system is composed by two parts:

- A *structure* of its notions;
- A deductive subsystem.

**Informal description of LDO**

The Logic of Determination of Objects (LDO) is a non-classical logic which the basic notion are concepts and objects. Its *conceptual structure* is represented by their definitions and the relations between them. It contains a theory of typicality of objects.

The LDO is due to Jean-Pierre Desclés and described as a logical model in (5). LDO is defined within the framework of Combinatory Logic (2) with functional types. LDO is inspired by the semantics of natural languages. It solves some problems that classical logic cannot describe and solve:

- It supply a solution for the mismatch between logic categories and linguistic categories (adjectives, intransitive verbs often represented by unary predicates);
- It consider the determination as a logic operator in order to represent linguistic expression as a book, a red book, a book which is on the table;
- It reconsider the duality extension – intension via its theory of typicality; the extension and the intension of a concept are no longer in duality.

In a first time, the relation between concepts and objects was described by:

- The concepts are organized in a network by the relation of inheritance. The objects are also organized in a network of more or less determinate objects by the operation of determination. One pass from an object more or less determinate to another more determinate than the first one by a determination. Objects which can determinate no more are called completely (fully) determinate objects. In the logical model (quotation) the concepts and objects have different types.

  - the basic (primitive) notion used both in logical model and mathematical model is this one of property. A property can be taken as membership of a concept intension \(\text{Int}\), or it can generate a determination of an object. In this way a concept is conceived as \(f = (\text{property} (f), \text{essence} (\text{Ess} f), \text{intension} (\text{Int} f), \text{typical object completely indeterminate} (\tau f), \text{relation} \text{of inheritance} (\rightarrow \tau))\).
  - The concepts are organized in a network by the the determination \(\rightarrow\):)

**Mathematical structural models of LDO**

In Figure 2, it is represented a network of the couple \((F, O)\). The first one is the subspace of concepts, the second one the subspace of objects associated with a concept \(f\). One can remark that the essence \((\text{Ess} f)\) is contained in the intension \((\text{Int} f)\). For some concepts in intension \((\text{Int})\) one needs their negations. The set \(N\text{Int}\) is the set of these negations in order to describe the typicality-atypicality. The relation on inheritance in the subspace of concepts \(F\) is represented by an arrow. The relation of determination is represented by a dotted arrow in the subspace \(O\).

**The mathematical model of LDO as soft set.**

The first characteristic of LDO as logic is that it enlarge the structure of basic logical elements of reasoning by
introducing an axiomatisation of two key notions of our reasoning, namely that of concept and that of object. Its second characteristic related to the structure of QTS is the categorization of objects in typical objects and atypical objects.

The network in Figure 2 will be denoted by Net. It show that to each subclass of typical objects associated to a concept $\hat{f}$ we can associate a subnetwork $\text{Net}_{i\tau}$. Equally, to each class of atypical objects associated to a concept $\hat{f}$ we can associate a subnetwork $\text{Net}_{i\alpha}$.

The mathematical model of LDO as a soft set $X$ is:

$$X = (\text{Net}, \text{Net}_{i\tau}, \{\text{Net}_{i\tau}, \text{Net}_{i\alpha}\}, G)$$

where

- Net is the whole network;
- $\text{Net}_{i\tau}$ and $\text{Net}_{i\alpha}$ are typical networks, atypical networks respectively; they model the set of parameters;
- $G$ is transition function between The space ans the parameters defined as: $G : \text{Net} \rightarrow (\mathcal{P}(\text{N})), \text{defined by } G(\text{Net}_{i\tau}) = \{o_{\tau} / o \text{ typical object}\}$ and $G(\text{Net}_{i\alpha}) = \{o_{\alpha} / o \text{ atypical object}\}$.

**Conclusions**

In our opinion, the way to build efficient computational tools solving real problems passes by the "best understanding" of the reality. The process of understanding is a "generalized compilation" which is a construction of a chain of successive representations that the last one is the algorithm or the system of algorithms able to solve the initial problem. The mathematical modeling is inside this chain. All passes through the mathematical modeling. It is the mathematical modeling whose depends the successful of the algorithm. The implementation of the algorithm is quasi independent of the mathematical model. The mathematical model is a key point in automatic system construction. The more it is adapted to the reality, the more the success of the solution of the problem is acquired. By the other hand the mathematical model can be constructed inside several mathematical theory depending on problem. So, it derives the importance of comparing mathematical theories from the point of view of their "cognitive power". In this work, we compare the soft set theory with the quasi topological structure theory and with the theory of LDO. This study is a theoretical study linking a set theory model, namely soft set theory model to two theoretical models of categorization, QTS and LDO.

This study leads to the following conclusions:

1. The QTS can be expressed inside the soft set theory.
2. Taking into account the fact that LDO is a logic not only a categorization model, a link between the soft set model and the structural model of LDO is given as an example.
3. The purpose of these comparisons, inside mathematical theory is to have a choice in the construction of automatic categorization procedures.

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