Let $R$ be a Hecke symmetry. There is then a natural quantization $A_n(R)$ of the $n^{th}$ Weyl algebra $A_n$ based on $R$. The aim of this paper is to study some general ring-theoretical aspects of $A_n(R)$ and its relation to formal deformations of $A_n$. We also obtain further information on those quantizations obtained from some well-known Hecke symmetries.

0. Introduction

Let $K$ be a field and fix an $n$-dimensional vector space $V$. If $R : V \otimes V \to V \otimes V$ is a Hecke symmetry for some $q \in K^*$ then, using the relations given in [WZ], there is a natural quantization $A_n(R)$ of the $n^{th}$ Weyl algebra $A_n$ based on $R$. This $A_n(R)$ may be viewed as the algebra of quantized differential operators on the $R$-symmetric algebra as defined in [Gu]. The $R$-symmetric algebra is the quantum coordinate ring of affine $n$-space associated with the quantum function bialgebra $O_R(M(n))$ constructed using the method in [FRT]. The aim of most of this paper is to study some ring-theoretical aspects of $A_n(R)$. Our main result is that $A_n(R)$ is left and right primitive whenever $q$ is not a root of unity and it is not simple if $\dim_K(A_n(R)) = \infty$ and $q \neq \pm 1$. We also show that under some mild assumptions on $R$, the quantum Weyl algebra $A_n(R)$ is an Auslander regular, Cohen-Macaulay, Noetherian domain with Gelfand-Kirillov dimension $2n$. Additionally, we obtain some results on the Krull and global dimensions of those $A_n(R)$ associated with the standard multiparameter and the “Jordan” Hecke symmetries. Finally we show that, suitably interpreted, each $A_n(R)$ is in fact a formal deformation of $A_n$, and, as such, must be isomorphic to $A_n[[t]]$ as a $K[[t]]$-algebra. Although we will not use or discuss this fact, we would like to mention that $A_n(R)$ has been been used to define a quantization of the universal enveloping algebra $U(\mathfrak{gl}(n))$, see [JZ].

1. Quantization of linear spaces and Weyl algebras

In this section we recall the basic methods of quantizing linear spaces and differential operators, (cf [FRT], [Gu], and [WZ]). Fix a field $K$ and let $V$ be an $n$-dimensional vector space with basis $x_1, \ldots, x_n$. If $R : V \otimes V \to V \otimes V$ is a linear transformation then $R_{12} = R \otimes \text{Id}_V$ and $R_{23} = \text{Id}_V \otimes R$ are maps $V \otimes V \otimes V \to V \otimes V \otimes V$.

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Definition 1.1. A linear transformation $R : V \otimes V \to V \otimes V$ is a Hecke symmetry if it satisfies

1. $R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}$ – the braid relation, and
2. $(R - q)(R + q^{-1}) = 0$ for some $q \in K^*$ – the Hecke condition.

If we view the basis $\{x_i \otimes x_j\}$ of $V \otimes V$ as an $n^2 \times 1$ column vector lexicographically ordered then $R : V \otimes V \to V \otimes V$ may be represented as an $n^2 \times n^2$ matrix (also denoted by $R$) whose rows and columns are also lexicographically ordered by pairs $(i, j)$ for $1 \leq i, j \leq n$. Now if $R(x_i \otimes x_j) = \sum R_{ij}^{kl} x_k \otimes x_l$ with $R_{ij}^{kl} \in K$, then the entry in row $(k, l)$ and column $(i, j)$ of $R$ is $R_{ij}^{kl}$. For some purposes, it will be convenient to present some Hecke symmetries as an element of the tensor product $M(n) \otimes M(n)$, which may be identified with $M(n^2)$ via the Kronecker product. Under this identification, the element $e_{ij} \otimes e_{kl}$ of $M(n) \otimes M(n)$ corresponds to the $n^2 \times n^2$ matrix with entry 1 in row $(i, k)$ and column $(j, l)$ and zeroes elsewhere.

If $R(x_i \otimes x_j) = \sum R_{ij}^{kl} x_k \otimes x_l$ then it is easy to check that the braid relation (1.1.1) is equivalent to having

$$\sum_{k,l,s} R_{ij}^{kl} R_{lw}^{sv} R_{ks}^{up} = \sum_{k,l,s} R_{ij}^{kl} R_{lw}^{us} R_{sl}^{pv}$$

(1.2)

for all $(i, j, w)$ and $(u, p, v)$.

The Hecke condition implies that whenever $q^2 \neq -1$ there is a vector space decomposition $V \otimes V \cong W_+ \oplus W_-$ where $W_+$ and $W_-$ are eigenspaces for the eigenvalues $q$ and $-q^{-1}$ of $R$ respectively. Specifically, $W_+ = \text{Im}(R - q)$ and $W_- = \text{Im}(R + q^{-1})$. The basic example of a Hecke symmetry is the permutation operator $\sigma : V \otimes V \to V \otimes V$ defined by $\sigma(x_i \otimes x_j) = x_j \otimes x_i$ for all $i$ and $j$. In this case $q = 1$ and $W_+$ and $W_-$ are the subspaces of $V \otimes V$ consisting of the usual symmetric and skew-symmetric tensors.

Definition 1.3. If $R : V \otimes V \to V \otimes V$ is a Hecke symmetry then the $R$-symmetric algebra, $K_R\langle x_i \rangle$, is the quotient $TV/W_+$.

When $K$ has characteristic zero and $q$ is not a root of unity, a description of $K_R\langle x_i \rangle$ in terms of a subspace of $TV$ can be found in [Gu]. To describe this let $\text{Sym}_R(V^\otimes m) = \{\alpha \in V^\otimes m \mid R_i(\alpha) = q\alpha \text{ for all } i = 1, \ldots, m - 1\}$ where $R_i$ acts as $R$ in tensor factors $i$ and $i + 1$ of $V^\otimes m$ and $\text{Id}_V$ in the others. There exists a projection operator $P_m : V^\otimes m \to$
\( \text{Sym}_R(V^\otimes m) \) for each \( m \) and so \( \bigoplus_{m \geq 0} \text{Sym}_R(V^\otimes m) \) becomes a \( K \)-algebra in which the product of \( \alpha \in \text{Sym}_R(V^\otimes m_1) \) and \( \beta \in \text{Sym}_R(V^\otimes m_2) \) is \( P_{m_1+m_2}(\alpha \otimes \beta) \). This approach of viewing \( K_R\langle x_i \rangle \) as a subspace of the tensor algebra \( TV \) was successfully used to show that, suitably interpreted, \( K_R\langle x_i \rangle \) and the quantum function bialgebra \( O_R(M(n)) \) are formal deformations of their classical counterparts, cf [GGS2].

For the purposes of this paper, it will be most convenient to consider \( K_R\langle x_i \rangle \) in terms of generators and relations. In doing so we have that \( K_R\langle x_i \rangle = K\langle x_i \rangle/I_x \) where \( I_x \) is spanned by quadratic relations of the form \( \sum_{k,l} R_{ij}^{kl} x_k x_l - qx_i x_j \) for all \( i \) and \( j \). In the case \( R = \sigma \), the relations simply become \( x_i x_j = x_j x_i \) for all \( i \) and \( j \) and thus \( K_\sigma\langle x_i \rangle \) is just the polynomial ring \( K[x_i] \). For many choices of \( R \), it is reasonable to think of \( K_R\langle x_i \rangle \) as a graded “deformation” of \( K[x_i] \) which may be viewed as the coordinate ring of a non-commutative version of affine \( n \)-space, see Section 4. However, this is not always the case as there do exist Hecke symmetries for which \( K_R\langle x_i \rangle \) is finite dimensional.

For any \( R \)-symmetric algebra there is a “dual” \( R \)-symmetric algebra \( K_R\langle \partial_i \rangle \) obtained from the Hecke symmetry \( R^* : (V \otimes V)^* \to (V \otimes V)^* \). Now \( K_R\langle \partial_i \rangle = K\langle \partial_i \rangle/I_\theta \) where \( I_\theta \) is spanned by quadratic relations of the form \( \sum_{k,l} R_{ij}^{kl} \partial_k \partial_l - q \partial_i \partial_j \) for all \( i \) and \( j \). It is natural to ask whether there is a way to use \( R \) to give an action of the \( \partial \)'s as “\( R \)-derivatives” of \( K_R\langle x_i \rangle \) analogous to the classical case when \( R = \sigma \) and each \( \partial_i \) is the derivation \( \partial/\partial x_i \) of \( K[x_i] \). The appropriate interaction between the \( x \)'s and \( \partial \)'s is due to Wess and Zumino and the following can essentially be found in [WZ].

**Definition 1.4.** The quantum Weyl algebra associated to \( R \) is the algebra \( A_n(R) \) with generators \( x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \) subject, for all \( i \) and \( j \) to the relations

1. \( \sum_{k,l} R_{ij}^{kl} x_k x_l = qx_i x_j \).
2. \( \sum_{k,l} R_{ij}^{kl} \partial_k \partial_l - q \partial_i \partial_j \).
3. \( \partial_i x_j = \delta_{ij} + q \sum_{k,l} R_{ij}^{kl} x_k \partial_l \).

We will see that these are natural choices for relations defining a quantization of \( A_n \) based on \( R \). Note that relations (1.4.1) generate \( I_x \) and relations (1.4.2) generate \( I_\theta \) and so there are algebra maps \( K_R\langle x_i \rangle \to A_n(R) \) and \( K_R\langle \partial_i \rangle \to A_n(R) \). If we set \( I_{x,\theta} \) to be the subspace
of $K \langle x_i, \partial_l \rangle$ generated by relations (1.4.3) then $A_n(R) = K \langle x_i, \partial_l \rangle/(I_x + I_{\partial} + I_{x,\partial})$.

Our first step in analyzing the structure of $A_n(R)$ is to prove the analog of the well-known fact that the monomials \{\(x_1^{i_1} \cdots x_n^{i_n} \partial_1^{j_1} \cdots \partial_n^{j_n}\)\} form a basis for $A_n$.

**Theorem 1.5.** There is a vector space isomorphism $K_R \langle x_i \rangle \otimes K_R \langle \partial_l \rangle \cong A_n(R)$.

The proof of this proposition will rely on the following important technical lemma which guarantees that relations (1.4.3) introduce no new relations among the $x$’s and $\partial$’s other than those given in (1.4.1) and (1.4.2).

**Lemma 1.6.** Using relations (1.4.3) we have that, for all $u$

\begin{align*}
(1) \quad \partial_u I_x & \subset \sum_w I_x \partial_w. \\
(2) \quad I_{\partial} x_u & \subset \sum_w x_w I_{\partial}.
\end{align*}

**Proof.** To prove (1.6.1) it is only necessary to show that $\partial_u(\sum_{k,l} R_{ij}^{kl} x_k x_l - q x_i x_j) \subset \sum_w I_x \partial_w$.

Now using relations (1.4.3) we get

\[
\partial_u(\sum_{k,l} R_{ij}^{kl} x_k x_l - q x_i x_j) = \sum_{k,l} R_{ij}^{kl}(\delta_{uk} + q \sum_{p,s} R_{ks}^{up} x_p \partial_s)x_l \\
- q(\delta_{ui} + q \sum_{p,s} R_{is}^{up} x_p \partial_s)x_j \\
= \sum_{k,l} R_{ij}^{ul} x_l + q \sum_{k,l,p,s} R_{ij}^{kl} R_{ks}^{up} x_p (\delta_{sl} + q \sum_{v,w} R_{lw}^{sv} x_v \partial_w) \\
- q\delta_{ui}x_j - q^2 \sum_{p,s} R_{is}^{up} x_p (\delta_{sj} + q \sum_{v,w} R_{ij}^{sv} x_v \partial_w). 
\]

(1.7)

In order for (1.7) to lie in $\sum_{w} I_x \partial_w$ we need the sum of all the linear terms in the $x$’s to be identically zero since $I_x$ contains quadratic relations only. The linear terms of (1.7) are

\[
\sum_{k,l} R_{ij}^{ul} x_l + q \sum_{k,l,p,s} R_{ij}^{ks} R_{ks}^{up} x_p - q \delta_{ij} x_j - q^2 \sum_{p} R_{ij}^{up} x_p
\]

which may be written as

\[
\sum_{k,l} R_{ij}^{ul} x_l + q \sum_{k,l,s} R_{ij}^{ks} R_{ks}^{ul} x_l - q \sum_{l} \delta_{ui} \delta_{lj} x_l - q^2 \sum_{l} R_{ij}^{ul} x_l. 
\]

(1.8)
Now having (1.8) equal to zero for all \( u, i, \) and \( j \) is the same as having

\[
R + qR^2 - qR - q^2R = (R - q)(1 + qR) = 0.
\]

Now note that the latter expression is satisfied since \( R \) was assumed to satisfy the Hecke condition \((R - q)(R + q^{-1}) = 0\).

We now show that the sum of the quadratic terms in the \( x \)'s of (1.7) lies in \( \sum_w I_x \partial_w \). These terms are

\[
q^2 \sum_{k,l,p,s,v,w} R_{ij}^{kl} R_{ks}^{lp} R_{lw}^{sv} x_p x_v \partial_w - q^3 \sum_{p,s,v,w} R_{is}^{up} R_{jw}^{sv} x_p x_v \partial_w.
\]

(1.9)

Dividing by \( q^2 \) and using the braid relation (1.2) to rewrite the first term of (1.9) we obtain

\[
\sum_{k,l,p,s,v,w} R_{ik}^{us} R_{jl}^{kl} R_{sl}^{pv} x_p x_v \partial_w - q \sum_{p,s,v,w} R_{is}^{up} R_{jw}^{sv} x_p x_v \partial_w
\]

and by reindexing the second sum this becomes

\[
\sum_{k,l,p,s,v,w} R_{ik}^{us} R_{jl}^{kl} R_{sl}^{pv} x_p x_v \partial_w - q \sum_{k,l,s,w} R_{ik}^{us} R_{jl}^{kl} x_s x_l \partial_w
\]

which is an element of \( \sum_w I_x \partial_w \). Similar computations can be used to prove (1.6.2) and the details are omitted. ■

**Proof of Theorem 1.5.** First note that \( A_n(R) \cong S/(I_x + I_\partial) \) where \( S = K\langle x_i, \partial_i \rangle / I_x, \partial \). Now using relations (1.4.3) which generate \( I_x, \partial \) it is easy to see that every element of \( S \) can be uniquely reduced to the form \( \sum_{I,J} C_{IJ} x_I \partial_J \) where \( C_{IJ} \in K \) and \( x_I \) (resp. \( \partial_J \)) represent the elements \( x_{i_1} \cdots x_{i_v} \) (resp. \( \partial_{j_1} \cdots \partial_{j_v} \)) of \( K\langle x_i, \partial_i \rangle \). It follows from the diamond lemma (cf [B]) that the elements \( \{ x_I \partial_J \} \) of \( S \) are linearly independent. Thus \( \rho : K\langle x_i \rangle \otimes K\langle \partial_i \rangle \to S \) where \( \rho(x_I \otimes \partial_J) = x_I \partial_J \) is a vector space isomorphism. Consequently, \( A_n(R) \cong S/(I_x + I_\partial) \cong K\langle x_i \rangle \otimes K\langle \partial_i \rangle / (I_x \otimes 1 + 1 \otimes I_\partial) \). By Lemma 1.6, any element of the two-sided ideal of \( S \) generated by the relations \( I_x \otimes 1 + 1 \otimes I_\partial \) must lie in \( I_x \otimes K\langle \partial_i \rangle + K\langle x_i \rangle \otimes I_\partial \) and so \( K\langle x_i \rangle \otimes K\langle \partial_i \rangle / (I_x \otimes 1 + 1 \otimes I_\partial) \cong (K\langle x_i \rangle / I_x) \otimes (K\langle \partial_i \rangle / I_\partial) \cong K_R\langle x_i \rangle \otimes K_R\langle \partial_i \rangle \). ■

We will see that the vector space isomorphism \( K_R\langle x_i \rangle \otimes K_R\langle \partial_i \rangle \cong A_n(R) \) is crucial for most of the structure theorems for \( A_n(R) \) that are in Section 3. Immediate consequences of the isomorphism are that the canonical left and right \( A_n(R) \)-modules \( A_n(R)/(\sum_1 A_n(R) \partial_i) \) and \( A_n(R)/(\sum_i x_i A_n(R)) \) may be identified with \( K_R\langle x_i \rangle \) and \( K_R\langle \partial_i \rangle \), respectively.
2. Some Examples

In this section, we give several examples of quantum Weyl algebras using the construction of Section 1 for some well-known Hecke symmetries. We also state some nice ring-theoretic properties of these examples; the precise definitions and proofs of these properties will be given in Section 3.

Example 2.1. This example gives the quantum Weyl algebra associated to the “standard” multiparameter $R$-matrix

$$R_{q,p_{ij}} = q \sum_i e_{ii} \otimes e_{ii} + \sum_{i<j} (p_{ij}e_{ji} \otimes e_{ij} + p_{ij}^{-1}e_{ij} \otimes e_{ji}) + (q - q^{-1}) \sum_{i>j} e_{ii} \otimes e_{jj}. \quad (2.2)$$

which depends on $\binom{n}{2} + 1$ non-zero scalars $q$ and $p_{ij}$ for $i < j$. For convenience, we set $p_{ji} = p_{ij}^{-1}$ for $i < j$. When $n = 1$ this is just the $1 \times 1$ scalar $q$ and if $n = 2$ then it becomes the $4 \times 4$ matrix

$$R_{q,p} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q^{-1}p & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

with $p = p_{12}$. Note that if $q = p_{ij} = 1$ for all $i$ and $j$ then $R_{q,p_{ij}}$ is the permutation operator $\sigma$. The quantum groups and quantum linear spaces associated to $R_{q,p_{ij}}$ have been extensively studied in [AST], [GGS2], and [LS]. Some properties of a multiparameter quantum Weyl algebra have been studied in [AD] and [J] but there the algebra differs slightly from $A_n(R_{q,p_{ij}})$; our primary concern here is with the latter since its relations come directly from the general construction. Now, according to Definition 1.4 the relations of $A_n(q, p_{ij})$ are

$$x_ix_j = p_{ij}qx_jx_i, \quad \text{for all } i < j$$

$$\partial_i\partial_j = p_{ij}q^{-1}\partial_j\partial_i, \quad \text{for all } i < j$$

$$\partial_i x_j = p_{ij}^{-1}qx_j\partial_i, \quad \text{for all } i \neq j$$

$$\partial_i x_i = 1 + q^2x_i\partial_i + (q^2 - 1)\sum_{j>i} x_j\partial_j, \quad \text{for all } i. \quad (2.3)$$

Note when $q = 1$ and all $p_{ij} = 1$ these relations reduce to those for $A_n$ and so $A_n(1, 1) = A_n$. The algebra $A_n(q, p_{ij})$ is Auslander regular, Cohen-Macaulay Noetherian domain with GK dimension $2n$. It is simple if and only if $q^2 = 1$ and $\text{char}(K) = 0$. If $q^2 \neq 1$, then $A_n(q, p_{ij})$
has Krull dimension and global dimension 2n and the elements \( x_i \partial_i - \partial_i x_i \) are normal (but not invertible) for all \( i \). If \( q \) is not root of 1, then \( A_n(q, p_{ij}) \) is left and right primitive. If \( q \) and all \( p_{ij} \) are roots of 1, then \( A_n(q, p_{ij}) \) is a PI ring and is not primitive.

If \( n = 1 \), then the resulting quantization of \( A_1 \) is \( A_1(q) \) which has generators \( x (= x_1) \) and \( \partial (= \partial_1) \) and the single relation \( \partial x = 1 + q^2 x \partial \). Many of the above results for \( A_1(q) \) have been previously obtained in [G] and [KS].

**Example 2.4.** For \( n = 2 \), there is another interesting family of quantizations of \( A_2 \) other than \( A_2(q, p) \) of the previous example. Specifically, let

\[
J_{a,b} = \begin{pmatrix}
1 & a & -a & -ab \\
0 & 0 & 1 & b \\
0 & 1 & 0 & -b \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

where \( a \) and \( b \) are two arbitrary scalars in the field \( K \). It follows from [GGS1, 7.1], that \( J_{a,b} \) is a Hecke \( R \)-matrix with \( q = 1 \). The relations of \( A_2(J_{a,b}) \) are

\[
x_1 x_2 = x_2 x_1 + a x_1^2 \\
\partial_2 \partial_1 = \partial_1 \partial_2 + b(\partial_2)^2 \\
\partial_1 x_1 = 1 + x_1 \partial_1 + a x_1 \partial_2 \\
\partial_1 x_2 = -a x_1 \partial_1 - a b x_1 \partial_2 + x_2 \partial_1 + b x_2 \partial_2 \\
\partial_2 x_1 = x_1 \partial_2 \\
\partial_2 x_2 = 1 - b x_1 \partial_2 + x_2 \partial_2
\]

If \( a = b = 0 \) then \( A_2(J_{0,0}) \) reduces to the second Weyl algebra \( A_2 \). Note that in this case \( A_2(J_{a,b}) \cong A_2(J_{\lambda a, \lambda b}) \) for any non zero scalar \( \lambda \in K \). The algebra \( A_2(J_{a,b}) \) is Auslander regular, Cohen-Macaulay Noetherian domain with GKdimension 4. If \( K \) has positive characteristic, then \( A_2(J_{a,b}) \) is a PI ring and has Krull and global dimensions 4. If \( K \) has characteristic zero, then \( A_2(J_{a,b}) \) is always left and right primitive and it is simple if and only if \( a = b \).

**Example 2.6.** Our final example illustrates the fact that \( A_n(R) \) may be finite dimensional. Let \( n = 2 \) and consider the operator

\[
\tau = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]
Note that $\tau$ is a Hecke symmetry with $q = 1$. Now $A_2(\tau)$ has relations
\[
x_1^2 = x_2^2 = \partial_1^2 = \partial_2^2 = 0
\]
\[
x_1 x_2 = x_2 x_1
\]
\[
\partial_1 \partial_2 = \partial_2 \partial_1
\]
\[
\partial_1 x_1 = 1 - x_1 \partial_1
\]
\[
\partial_1 x_2 = x_2 \partial_1
\]
\[
\partial_2 x_1 = x_1 \partial_2
\]
\[
\partial_2 x_2 = 1 - x_2 \partial_2
\]
and it is easy to check directly that $A_2(\tau) \cong M_2(K)$ (or see Corollary 3.3), a simple Artinian $K$-algebra.

3. Some Properties of $A_n(R)$

In this section we will investigate some ring-theoretical properties of $A_n(R)$ for an arbitrary Hecke symmetry $R$. Many of these properties will be proved with the aid of the element $D = \sum_{i=1}^{n} x_i \partial_i$ of $A_n(R)$. As an element of the Weyl algebra $A_n$, the element $D$ is the Euler derivation. Namely it is the derivation of $K[x_i]$ sending a homogeneous element $f_m$ of degree $m$ to $mf_m$. In the quantum case, the action of $D$ on $K_R(x_i)$ will involve the $q$-integers $[m]_{q^2}$. These are defined as follows: set $[m]_{q^2} = \sum_{i=1}^{m} q^{2(i-1)}$ for $m \geq 0$ and $[0]_{q^2} = 0$. Note that if $q^2 = 1$, then $[m]_{q^2} = m$ and if $q^2 \neq 1$, then $[m]_{q^2} = (q^{2m} - 1)/(q^2 - 1)$. The following lemma provides some useful properties about the element $D$.

**Lemma 3.1.** (1) Let $f_m \in K_R(x_i)$ and $g_m \in K_R(\partial_i)$ be arbitrary elements of degree $m$, then
\[
Df_m = [m]_{q^2} f_m + q^{2m} f_m D, \quad \text{and} \quad g_mD = [m]_{q^2} g_m + q^{2m} Dg_m.
\]
(2) The element $E = 1 + (q^2 - 1)D$ is a non-zero normal element in $A_n(R)$. Consequently, if $E$ is not invertible, then $A_n(R)$ is not simple.
(3) Let
\[
D^{[l]} = \sum_{i,j} x_i x_{i_2} \cdots x_{i_l} \partial_{i_1} \cdots \partial_{i_2} \partial_{i_1}.
\]
Then \( D \cdot D^{[l]} = [l] q^{2l} D^{[l]} + q^{2l} D^{[l+1]} \).

(4) The (formal) inverse of \( E \) is \( E^{-1} = \sum_{l \geq 0} (q^{-2} - 1)^l D^{[l]} \). As a consequence, if \( K_R \langle x_i \rangle \) or \( K_R \langle \partial_i \rangle \) is finite dimensional, then \( E \) is invertible.

(5) For every \( l \),

\[
D^{[l]} = \prod_{j=0}^{l-1} q^{-2j} (D - [j] q^2).
\]

In particular, if \( f_m \in K_R \langle x_i \rangle \) and \( g_m \in K_R \langle \partial_i \rangle \) are arbitrary elements of degree \( m \), then

\[
D^{[l]} f_m = \phi(m, l) f_m + \sum_i s_i \partial_i, \quad \text{and} \quad g_m D^{[l]} = \phi(m, l) g_m + \sum_i x_i t_i
\]

for some \( s_i \) and \( t_i \) in \( A_n(R) \), where \( \phi(m, l) = \prod_{j=0}^{l-1} q^{-2j} ([m] q^2 - [j] q^2) \).

Proof. (1) Every polynomial of degree \( m \) is a linear combination of monomials of degree \( m \) and so we may assume \( f_m \) is a monomial. If \( f_m = x_{j_1} \cdots x_{j_m} \) then

\[
D \cdot x_{j_1} \cdots x_{j_m} = \sum_{i=1}^n x_i \partial_i \cdot x_{j_1} \cdots x_{j_m} = \sum_i x_i \{ \delta_{ij_1} + q \sum_{k,l} R_{j_1i}^{lk} x_k \partial_l \} \cdot x_{j_2} \cdots x_{j_m} = \{ x_{j_1} + q^2 x_{j_1} \sum_i x_i \partial_i \} x_{j_2} \cdots x_{j_m} = x_{j_1} \cdots x_{j_m} + q^2 x_{j_1} D x_{j_2} \cdots x_{j_m},
\]

and, by induction on \( m \), we obtain the identity. The second identity is similar to prove.

(2) By (1), it is easy to check that \( E x_i = q^2 x_i E \) and \( \partial_i E = q^2 E \partial_i \) for all \( i \) and thus \( E \) is normal. By Theorem 1.5 \( E \neq 0 \).

(3) This identity follows easily from part (1).

(4) Direct computation shows \( E \cdot E^{-1} = E^{-1} \cdot E = 1 \).

(5) This follows by induction from the identity in part (3). \( \blacksquare \)

In terms of the \( A_n(R) \)-module structure for \( K_R \langle x_i \rangle \), Lemma 4.1.(1) implies that if \( f_m \) is a homogeneous element of degree \( m \) in \( K_R \langle x_i \rangle \) then \( D \cdot f_m = [m] q^2 f_m \). In particular, \( D \cdot (f_m f_{m'}) = [m + m'] q^2 f_m f_{m'} = (D \cdot f_m) f_{m'} + q^{2m} f_m (D \cdot f_{m'}) \) and so \( D \) is a \( \eta \)-derivation where \( \eta \) is the automorphism of \( K_R \langle x_i \rangle \) sending \( f_m \) to \( q^{2m} f_m \).
Theorem 3.2. Suppose that \([m]_{q^2} \neq 0\) for all \(m \geq 1\). Then the canonical left module \(K_R \langle x_i \rangle\) is faithful and simple, and \(\text{End}_{A_n(R)}(K_R \langle x_i \rangle) = K\). The same conclusions hold for the canonical right module \(K_R \langle \partial_i \rangle\), and, as a consequence, \(A_n(R)\) is both left and right primitive.

Proof. Let \(f = \sum_{l \leq m} f_l\) be a non zero element of \(K_R \langle x_i \rangle\) where \(f_l\) is homogeneous of degree \(l\). If \(m = 0\), then \(f\) is a scalar, so the submodule generated by \(f\) is \(K_R \langle x_i \rangle\) itself. If \(m > 0\),

\[
\sum_i x_i \partial_i \cdot f = D \cdot \sum_l f_l = \sum_l [l]_{q^2} f_l \neq 0.
\]

Hence \(\partial_i f \neq 0\) for some \(i\), and the degree of \(\partial_i f\) is strictly less than the degree of \(f\). By induction on \(m\), it follows that the submodule generated by \(f\) is must contain a scalar and thus \(K_R \langle x_i \rangle\) is a simple \(A_n(R)\)-module.

To prove \(K_R \langle x_i \rangle\) is faithful, we need to show the annihilator ideal \(L\) of the module \(K_R \langle x_i \rangle\) is zero. If \(g = \sum C_{iJ} x_i \partial_J\) let \(d(g)\) be the minimum of \(|J|\) appearing in \(g\) where \(|J| = j_1 + \cdots + j_r\) if \(J = (j_1, \ldots, j_r)\). Now pick \(g \in L\) with minimal \(d(g)\), (note that it is necessary for \(d(g) > 0\) since \(g \cdot 1 = 0\)), and consider the element \(g' = gD\). By Lemma 4.1.(1), \(g' = \sum C_{iJ} x_i ([|J|]_{q^2} \partial_J + q^{2|J|} D \partial_J)\). Now since \(d(g') = d(g)\) we must have that \(d(gx_i) < d(g)\) for some \(i\). But \(g \cdot x_i \in L\) and this is a contradiction with minimality of \(d(f)\), so \(L = 0\).

Finally, set \(f = \theta(1)\) where \(\theta \in \text{End}_{A_n(R)}(K_R \langle x_i \rangle)\) is arbitrary. If \(f \notin K\), then \(D \cdot f \neq 0\). But \(D \cdot f = D \cdot \theta(1) = \theta(D \cdot 1) = 0\) and we obtain a contradiction. Therefore \(f \in K\) and \(\text{End}_{A_n(R)}(K_R \langle x_i \rangle) = K\).

Corollary 3.3. Suppose that \([m]_{q^2} \neq 0\) for all \(m\).

(1) \(K_R \langle x_i \rangle\) and \(K_R \langle \partial_i \rangle\) have the same Hilbert series.
(2) If \(\dim(K_R \langle x_i \rangle) = l < \infty\), then \(A_n(R) \cong M_l(K)\), a simple Artinian algebra of rank \(l\).
(3) If \(\dim(K_R \langle x_i \rangle) = \infty\) and \(q^2 \neq 1\) then \(A_n(R)\) is not simple.

Proof. (1) By Theorem 3.2, \(K_R \langle x_i \rangle\) is a simple module and so if \(f_m \in K_R \langle x_i \rangle\) is a nonzero homogeneous element of degree \(m\) there is some element \(g \in A_n(R)\) with \(gf_m = 1\). By counting the degree of \(g\), we may choose \(g \in K_R \langle \partial_i \rangle\) with degree \(m\). Hence the dimension of the \(m\)-homogeneous component of \(K_R \langle \partial_i \rangle\) is at least the dimension of the \(m\)-homogeneous component of \(K_R \langle x_i \rangle\). Similarly, the dimension of the \(m\)-homogeneous component of \(K_R \langle x_i \rangle\)
is at least the dimension of the \(m\)-homogeneous component of \(K_R \langle \partial_i \rangle\) and therefore these dimensions coincide.

(2) By Theorem 3.2, \(A_n(R)\) is primitive and since its faithful simple module \(K_R \langle x_i \rangle\) is finite dimensional, we have \(A_n(R) \cong \text{End}_{E_1}(K_R \langle x_i \rangle)\) where \(E_1 = \text{End}_{A_n(R)}(K_R \langle x_i \rangle) = K\). Hence \(A_n(R) \cong M_l(K)\) where \(l = \dim(K_R \langle x_i \rangle)\).

(3) According to Lemma 3.1.(2), we only need to prove \(E = 1 + (q^2 - 1)D\) is not invertible. It is easy to see that \(\phi(m, l) \neq 0\) for \(m \gg 0\), which implies that \(D^{[l]} \neq 0\) since \(K_R \langle x_i \rangle\) is infinite dimensional (see Lemma 3.1.(5)). Now by Lemma 3.1.(4), \(E^{-1}\) is not in \(A_n(R)\) and hence \(E\) is not invertible. ■

**Remark.** If \(K_R \langle x_i \rangle\) and \(K_R \langle \partial_i \rangle\) are finite dimensional then it is only necessary to require \([m]_{q^2} \neq 0\) for all positive integers \(m \leq d\) where \(d\) is the maximum degree of the elements in \(K_R \langle x_i \rangle\) and \(K_R \langle \partial_i \rangle\) to obtain the conclusions of Theorem 3.2 and Corollary 3.3.(1) and (2).

**Corollary 3.4.** Suppose that \([m]_{q^2} = 0\) for some \(m > 1\). Then \(D^{[m]}\) is central and if \(A_n(R)\) is a domain, then \(A_n(R)\) is not simple.

**Proof.** By using Lemma 3.1.(1) and (4), it is easy to check \(D^{[m]}\) is central. Now since \(A_n(R)\) is a domain, \(D^{[m]} = 0\) if and only if \(D = [j]_{q^2}\) for some \(j < m\). But \(A_n(R) \cong K_R \langle x_i \rangle \otimes K_R \langle \partial_i \rangle\) and so \(D\) is not equal to any scalar element in \(K\). Thus \(D^{[m]}\) is not zero. Moreover, it is not invertible since \(D^{[m]}x_i = 0\) in \(K_R \langle x_i \rangle\). Therefore \(A_n(R)\) is not simple. ■

As a consequence of Corollary 3.3 and Corollary 3.4, if \(A_n(R)\) is a simple domain, then \(q^2 = 1\) and \(\text{char}(K) = 0\). The converse, however, is not true and we do not know the sufficient conditions on \(R\) for \(A_n(R)\) to be simple. However, for those examples with \(q^2 = 1\) discussed in the last section we are are able to distinguish which are simple.

**Theorem 3.5.** If \(\text{char}(K) = 0\) then \(A_n(\pm 1, p_{ij})\) is simple.

**Proof.** We may assume \(q = 1\) since \(A_n(1, p_{ij}) = A_n(-1, -p_{ij})\). Now according to Theorem 1.5, it is easy to see that \(\{x_1^{i_1} \cdots x_n^{i_n} \partial_1^{j_1} \cdots \partial_n^{j_n}\}\) forms a basis for \(A_n(1, p_{ij})\). Now \(A_n(1, p_{ij})\) is a \(\mathbb{Z}^n\)-graded algebra with \(\deg(x_1^{i_1} \cdots x_n^{i_n} \partial_1^{j_1} \cdots \partial_n^{j_n}) = (i_1 - j_1, \ldots, i_n - j_n)\). We first show that \(A_n(1, p_{ij})\) is \(\mathbb{Z}^n\)-graded simple — that is, any ideal which contains a non-zero homogeneous element also contains a scalar and thus must be the entire ring. Now if \(w = \sum C_{I,J} x_1^{i_1} \cdots x_n^{i_n} \partial_1^{j_1} \cdots \partial_n^{j_n}\) is homogeneous of degree \((e_1, \ldots, e_n)\) then direct calculation
shows that for all $m$

$$x_m w - \left( \prod_{s \neq m} p_{ms}^{e_s} \right) w x_m = \sum j_m C'_{I,J} x_1^{i_1} \cdots x_n^{i_n} \partial_1^{j_1} \cdots \partial_m^{j_m-1} \partial_n^{j_n}$$

and

$$\partial_m w - \left( \prod_{s \neq m} p_{ms}^{e_s} \right) w \partial_m = \sum i_m C''_{I,J} x_1^{i_1} \cdots x_m^{i_m-1} \cdots x_n^{i_n} \partial_1^{j_1} \cdots \partial_n^{j_n}$$

where $C'_{I,J}$ and $C''_{I,J}$ are non-zero elements of $K$. By induction, it follows that this ideal contains a scalar.

Now let $L$ be an arbitrary non-zero ideal of $A_n(1,p_{ij})$ and suppose $f \in L$ with $f = f_1 + \cdots + f_i$ where $f_i$ is homogeneous. Since $A_n(1,p_{ij})$ is graded simple, we may assume $f_1 = 1$. For each homogeneous $a \in A_n(1,p_{ij})$ the element $af - fa \in L$. By choosing $f$ with minimal $l$, we get that each $f_i$ is central. It follows that $f_i$ must be invertible since $f_i$ is homogeneous and $A$ is graded simple. But the only invertible elements of $A_n(1,p_{ij})$ lie in $K$ and so $f$ is a scalar and thus $L = A_n(1,p_{ij})$. ■

Apart from the algebra $A_2(\tau) \cong M_2(K)$ of Example 2.6 which obviously is simple, the other example we have considered with $q^2 = 1$ is the algebra $A_2(J_{a,b})$ of Example 2.4. In contrast with $A_n(1,p_{ij})$ and $A_2(\tau)$, the algebra $A_2(J_{a,b})$ is not always simple.

**Theorem 3.6.** If char$(K) = 0$ then $A_2(J_{a,b})$ is simple if and only if $a = b$.

**Proof.** It is easy to see from the defining relations (2.5) for $A_2(J_{a,b})$ that $\{x_1^m | m \geq 0\}$ is an Ore set and consequently the localization $A_2(J_{a,b})[x_1^{-1}]$ is well-defined. The element $x_1^{-1} + (a-b)\partial_2$ is normal in $A_2(J_{a,b})$ and is not invertible unless $a = b$. Thus when $a \neq b$, the localization $A_2(J_{a,b})[x_1^{-1}]$ not simple and hence neither is $A_2(J_{a,b})$.

To examine the case when $a = b$ we may, as remarked earlier, assume that $a = b = 1$. To show $A_2(J_{1,1})$ is simple we need the following lemma which can be found in [W].

**Lemma 3.7.** Let $T$ be a simple ring, let $\eta$ be an automorphism of $T$ and let $\delta$ be a $\eta$-derivation. If for every $m \geq 0$, $D_m = \sum_{i=0}^{m} \eta^i \delta \eta^{-i}$ is not an $\eta$-inner derivation, then the Ore extension $T[x; \eta, \delta]$ is simple. ■

To apply this to our situation note that $A_2(J_{1,1})[x_1^{-1}]$ may be written as an iterated Ore extension

$$A_2(J_{1,1})[x_1^{-1}] = K_R(x_1, x_2, x_1^{-1})[\partial_2, \eta_2, \delta_2][\partial_1, \eta_1, \delta_1]$$
where \( K_R(x_1, x_2, x_2^{-1}) \cong K(x_1, x_2, x_2^{-1})/(x_1 x_2 - x_2 x_1 - x_1^2) \) and

\[
\begin{array}{c|c|c}
\eta_2 : & x_1 & x_1 \\
x_2 & x_2 - x_1 & \\
\eta_1 : & x_1 & x_1 \\
x_2 & x_2 - x_1 & \\
\delta_2 : & x_1 & 0 \\
x_2 & x_2 - x_1 & 1 \\
\delta_1 : & x_1 & 1 + x_1 \partial_2 \\
x_2 & (x_2 - x_1) \partial_2 & \\
\end{array}
\]

Now it is easy to see that \( \eta_2 \delta_2 = \delta_2 \eta_2 \) and the corresponding \( D_m \) in Lemma 3.7 is just \((m + 1) \delta_2 \). Now we claim that \( \delta_2 \) is not an \( \eta_2 \)-inner derivation. Suppose \( \eta_2 \) is such a derivation, that is, assume that there is some homogeneous \( a \in K_R(x_1, x_2, x_1^{-1}) \) with \( \delta_2(f) = \eta_2(f) a - a f \) for every \( f \in K_R(x_1, x_2, x_1^{-1}) \). Now since \( 0 = \delta_2(x_1) = \eta_2(x_1) a - ax_1 = x_1 a - ax_1 \) it follows that \( a \) is a polynomial in \( x_1 \) and \( x_1^{-1} \). Since \( \delta_2 \) has degree \(-1\) the element \( a \) must have the form \( \lambda x_1^{-1} \) for some \( \lambda \in K \). But \( 1 = \delta_2(x_2) = \eta_2(x_2) \cdot \lambda x_1^{-1} - \lambda x_1^{-1} \cdot x_2 = 0 \) which gives a contradiction and so \( \delta_2 \) cannot be an \( \eta_2 \)-inner derivation. Now by Lemma 3.7, \( S = K_R(x_1, x_2, x_2^{-1})[\partial_2, \eta_2, \delta_2] \) is simple since \( K_R(x_1, x_2, x_2^{-1}) \cong A_1[\partial^{-1}] \) is simple. Now we can consider \( S[\partial_1, \eta_1, \delta_1] (= A_2(J_{1,1})[x_1^{-1}]) \). In this case, the \( D_m \) in Lemma 3.7 satisfies \( D_m(x_1) = (m + 1)(1 + x_1 \partial_2) \) and \( D_m(\partial_2) = - (m + 1) \partial_2^2 \). Now the unique homogeneous solution to the equation \( x_1 a - ax_1 = (m + 1)(1 + x_1 \partial_2) \) is \( (m + 1)(x_1^{-2} + x_1^{-1} \partial_2) x_2 \). For this choice of \( a \), we have that \( D_m(\partial_2) \neq \eta_1(\partial_2) a - a \partial_2 \) and therefore \( D_m \) is not an \( \eta_1 \)-inner derivation. Once again, by Lemma 3.7 it follows that \( A_2(J_{1,1})[x_1^{-1}] \) is simple. Finally we need to establish that \( A_2(J_{1,1}) \) is simple. Let \( I \) be a non-zero ideal of \( A_2(J_{1,1}) \). Then \( I[x_1^{-1}] \) is a non zero ideal of \( A_2(J_{1,1})[x_1^{-1}] \) and so \( I[x_1^{-1}] = A_2(J_{1,1})[x_1^{-1}] \) since \( A_2(J_{1,1})[x_1^{-1}] \) is simple. Hence \( I \) contains \( x_1^m \) for some \( m > 0 \). But from the defining relations for \( A_2(J_{1,1}) \) we have that \( \partial_1 x_1^m - x_1^m \partial_1 = mx_1^{m-1} + mx_1^m \partial_2 \). It follows that \( x_1^{m-1} \in I \) since \( x_1^m \in I \). By induction on \( m \), the ideal \( I \) of \( A_2(J_{1,1}) \) must contain a scalar and therefore \( I = A_2(J_{1,1}) \) and \( A_2(J_{1,1}) \) is simple.

In the next part of this section we return to study some general properties of \( A_n(R) \) constructed Hecke symmetries \( R \) which are “skew invertible”, by which we mean that there is a matrix \( P \) with \( \sum_{i,j} P_{gj}^{fi} R_{jk}^{il} = \delta_{fj} \delta_{gi} = \sum_{i,j} R_{gj}^{fi} P_{jl}^{ik} \) for all \( f, g, k \) and \( l \). This property on \( R \) insures that in \( A_n(R) \) every \( x_i \partial_j \) can be written as a linear combination of elements of the form \( \partial_k x_i \). All of the choices of \( R \) used in this paper satisfy this property.

Let \( B \) be a ring and \( M \) be a \( B \)-module. The Krull, global and Gelfand-Kirillov dimensions of \( M \) will be denoted by \( Kdim(M) \) and \( GKdim(M) \) and \( gl.dim(B) \), respectively. If \( B \) is
a Noetherian ring with finite GK and global dimensions then, $B$ is **Auslander regular** if, for every Noetherian $B$-module $M$ and every submodule $N \subseteq \text{Ext}^j_B(M, B)$, one has $\text{Ext}_B^i(N, B) = 0$ for all $i < j$. The ring $B$ is **Cohen-Macaulay** if $j(M) + \text{GKdim}(M) = \text{GKdim}(B)$ holds for every Noetherian $B$-module $M$ where $j(M) = \min\{j|\text{Ext}^j_B(M, B) \neq 0\}$. The next lemma contains some facts about these properties which will later be applied to $A_n(R)$.

**Lemma 3.8.** (1) Let $B = \bigcup_{i \geq 0} B_i$ be a filtered algebra with $B_0 = K$. If the associated graded ring $gr(B)$ is an Auslander regular, Cohen-Macaulay, and Noetherian domain with GK dimension $d$, then $B$ also has these properties.

(2) Let $B = \bigoplus_{i \geq 0} B_i$ be a graded algebra with $B_0 = K$ and suppose that $\eta$ is a graded algebra automorphism and that $\delta$ is a $\eta$-derivation. If $B$ is an Auslander regular, Cohen-Macaulay, and Noetherian domain, then so is the Ore extension $B[x; \sigma, \delta]$.

**Proof.** See [SZ, 4.4] for (1) and the Lemma of [LS] for (2). $\blacksquare$

A ring $B$ is an **iterated Ore extension starting with $K$** if, for each $i = 1, \ldots, l$, there is a subring $B_i$ of $B$ with $B_0 = K$ and $B_l = B$ such that $B_i$ is an Ore extension of $B_{i-1}$.

**Lemma 3.9.** Suppose $R$ is a skew-invertible Hecke $R$-matrix such that $K_R\langle x_i \rangle$ and $K_R\langle \partial_i \rangle$ are iterated Ore extensions starting with $K$. If the relations between $x_1, \ldots, x_n, \partial_1, \ldots, \partial_l$ do not involve $\partial_{l+1}, \ldots, \partial_n$ then $A_n(R)$ is an iterated Ore extensions starting with $K$.

**Proof.** Let $B_l$ be the subring of $A_n(R)$ generated by $x_1, \ldots, x_n, \partial_1, \ldots, \partial_l$. Clearly, $B_0 = K_R\langle x_i \rangle$ and $B_n = A_n(R)$. We must show that $B_l$ is an Ore extension of $B_{l-1}$. First note that Theorem 1.5 and the fact that $K_R\langle x_i \rangle$ and $K_R\langle \partial_i \rangle$ are iterated Ore extensions starting with $K$ imply that the set of monomials

$$\{x_i^{i_1} \cdots x_n^{i_n} \partial_1^{j_1} \cdots \partial_l^{j_l} | i_s, j_t \geq 0\}$$

is linearly independent. Now since the relations between $x_1, \ldots, x_n$ and $\partial_1, \ldots, \partial_l$ do not involve $\partial_{l+1}, \ldots, \partial_n$, the above set must span $B_l$. Thus $B_l = B_{l-1}[\partial_l; \eta, \delta]$ for some ring endomorphism $\eta$ of $B_{l-1}$ and $\eta$-derivation $\delta$. Since $R$ is skew-invertible, the endomorphism $\eta$ must actually be an automorphism. $\blacksquare$
Remarks: (1) The conclusion of this lemma clearly holds when $K_R\langle x_i \rangle$ and $K_R\langle \partial_i \rangle$ are iterated Ore extensions and the relations between $x_1, \ldots, x_n, \partial_l, \ldots \partial_n$ do not involve $\partial_1, \ldots, \partial_{l-1}$. Note that this is the case for the quantizations $A_n(q, p_{ij})$ and $A_2(J_{a,b})$.

(2) In the same way, it is easy to prove that $gr(A_n(R))$ is an iterated Ore extension starting with $K$.

As a consequence of Lemmas 3.8 and 3.9, we have the following:

**Corollary 3.10.** Let $R$ satisfy the same conditions as in Lemma 3.9.

(1) $A_n(R)$ and $gr(A_n(R))$ are Auslander regular, Cohen-Macaulay, and Noetherian domains.

(2) The global and GK dimensions of $gr(A_n(R))$ are both $2n$.

(3) The Krull and global dimensions of $A_n(R)$ are both $\leq 2n$ and its GK-dimension is $2n$.

**Proof.** (1) This follows immediately from Lemma 3.8 and Lemma 3.9.

(2) Since any connected graded ring has the simple module $K$, we obtain from [MR, 7.9.18] that the global dimension of the graded Ore extension $gr(A_n(R))$ is $2n$. By Theorem 1.5, the GK dimensions of $gr(A_n(R))$ and $A_n(R)$ are both $2n$.

(3) The bounds on the Krull and global dimensions follow from [MR, 6.5.4 and 7.5.3].

Next we apply the previous results to further study some of the examples described in Section 2.

**Theorem 3.11.** (1) The algebra $A_n(q, p_{ij})$ is an Auslander regular, Cohen-Macaulay, Noetherian domain with GK dimension $2n$.

(2) $A_n(q, p_{ij})$ is a PI ring if and only if $[m]q^2 = 0$ and $p_{ij}^m = 1$ for some $m$ and all $i$ and $j$.

(3) If $q^2 \neq 1$ or $\text{char}(K) \neq 0$, then $A_n(q, p_{ij})$ has Krull dimension and global dimension $2n$.

**Proof.** (1) This follows from Corollary 3.10 since it is well known that for this example both $K_R\langle x_i \rangle$ and $K_R\langle \partial_i \rangle$ are iterated Ore extensions starting with $K$, and the relations between $x_1, \ldots, x_n, \partial_{l+1}, \ldots, \partial_n$ are independent of $\partial_1, \ldots, \partial_l$ for all $l$.

(2) If $A_n(q, p_{ij})$ is a PI ring, then the subring generated by $x_i$ and $x_j$ is also a PI ring and hence some power of $p_{ij}q$ is one since $x_ix_j = p_{ij}q x_j x_i$. Similarly, we get that some power of $p_{ij}^{-1}q$ is 1 by considering the subalgebra generated by $\partial_i$ and $\partial_j$. Consequently, there exist an integer $m$ such that $p_{ij}^m = 1$ and $q^{2m} = 1$ for all $i$ and $j$. Now if $q^2 \neq 1$, then we also have $[m]q^2 = 0$. If, however, $q = \pm 1$ then we must have that $\text{char}(K) = p > 0$ since otherwise
gl.dim(\(A_n(q, p_{ij})\)) would be simple by Theorem 3.5. Now \(m' = pm\) satisfies \([m']_q^2 = 0\) and \(p_{ij}^{m'} = 1\) for all \(i\) and \(j\). Conversely, if \([m]_q^2 = 0\) and \(p_{ij}^m = 1\) for some \(m\) then \(x_i^m\) and \(\partial_i^m\) are central for all \(i\), and thus \(A_n(q, p_{ij})\) is a PI ring.

(3) We will prove the results on the Krull and global dimensions with the aid of a suitable regular sequence. Recall that elements \(a_1, \ldots, a_l\) of a ring \(T\) form a regular sequence if \(a_{i+1}\) is a regular normal element in \(T_i = T/(a_1, \ldots, a_i)\) for each \(i = 0, \ldots l - 1\).

First assume that \(q^2 \neq 1\). From the defining relations of \(A_n(q, p_{ij})\) it is easy to check that \(1 + (q^2 - 1)x_n\partial_n, x_{n-1}, \partial_{n-1}, \ldots, x_1, \partial_1, x_n - 1\) is a regular sequence of \(A_n(q, p_{ij})\). Now from [MR, 6.3.9] and [MR, 7.3.5] we obtain \(Kdim(A_n(q, p_{ij})) \geq 2n\) and \(gl.dim(A_n(q, p_{ij})) \geq 2n\). Together with Corollary 3.10.(3), this implies that the Krull and global dimensions of \(A_n(q, p_{ij})\) are both \(2n\).

If \(\text{char}(K) = p > 0\) then we may assume \(q^2 = 1\). Now since \(A_n(q, p_{ij}) = A_n(-q, -p_{ij})\) we may further assume that \(q = 1\). In this case it is again routine to check that \(x_1^p, \partial_1^p, \ldots, x_n^p, \partial_n^p\) forms a sequence of regular elements in \(A_n(1, p_{ij})\). Moreover, the algebra \(A_n(1, p_{ij})/(x_i^p, \partial_i^p)\) is finite dimensional and so by [MR, 6.3.9] and Corollary 3.10.(3) we have \(Kdim(A_n(1, p_{ij})) = 2n\). We can find the global dimension from the Cohen-Macaulay property. Since \(A_n(1, p_{ij})/(x_i^p, \partial_i^p)\) is finite dimensional, \(GKdim(A_n(1, p_{ij})/(x_i^p, \partial_i^p)) = 0\), and \(j(A_n(1, p_{ij})/(x_i^p, \partial_i^p)) = j(A_n(1, p_{ij})) = 2n\). Thus \(gl.dim(A_n(1, p_{ij})) \geq j(A_n(1, p_{ij})/(x_i^p, \partial_i^p)) = 2n\) and then Corollary 3.10.(3) forces \(gl.dim(A_n(1, p_{ij})) = 2n\).

\(\square\)

Remark 3.12 (1) In a similar fashion, we can prove that the Krull and global dimensions are \(2n\) for the quantum Weyl algebra \(A_n^{\Lambda}A\) studied in [AD] and [J].

(2) If \(\text{char}(K) = 0\) then S.P. Smith has shown that \(Kdim(A_n(1, p_{ij})) = n\). We conjecture that \(gl.dim(A_n(1, p_{ij})) = n\) as well.

We now turn to further study the algebra \(A_2(J_{a,b})\). First we start with the following elementary result.

**Proposition 3.13.** The algebra \(A_2(J_{a,b})\) is an Auslander regular, Cohen-Macaulay, Noetherian domain with GK dimension 4.

**Proof.** First note that \(J_{a,b}\) is skew-invertible and both \(K_{J_{a,b}}(x_1, x_2)\) and \(K_{J_{a,b}}(\partial_1, \partial_2)\) are iterated Ore extensions starting with \(K\). Now the results follow from Corollary 3.10 since the relations between \(x_1, x_2\) and \(\partial_2\) do not involve \(\partial_1\).

\(\square\)
Lemma 3.13. Let $K$ be a field of characteristic $p > 0$ and let $R$ be an affine PI $K$-algebra which is a finite module over its Noetherian center $C$. If $\sigma$ is an automorphism of $R$ of finite order and $\delta$ is a $\sigma$-derivation, then the Ore extension $R[x; \sigma, \delta]$ is also a PI ring.

Proof. By the Artin-Tate Lemma [MR, 13.9.10], $C$ is affine. For every central element $c$, $\sigma(c) \in C$ and so $\sigma$ restricts to an automorphism of $C$. Since $\sigma$ has finite order, $C$ must be integral over the fixed subring $C^\sigma = \{c \in C \mid \sigma(c) = c\}$ and thus $C$ is a finitely generated $C^\sigma$-module since $C$ is affine. Hence $C^\sigma$ is also affine by the Artin-Tate Lemma [MR, 13.9.10]. Now for every $r \in C^\sigma$, we have

$$xr = rx + \delta(r)$$

where $\delta(r) \in R$. Consider the subring $C'$ generated by the set $\{r^p\}$ for all $r \in C^\sigma$. If the algebra $C^\sigma$ is generated by $\{c_1, \cdots, c_t\}$, then $C'$ is generated by $\{c_1^p, \cdots, c_t^p\}$. Consequently, $C'$ is affine and $C^\sigma$ is finitely generated over $C'$. For every $r = c^p \in C'$, it follows that

$$xr = xc^p = c^p x + pc^{p-1}\delta(c) = c^p x = rx$$

and hence (3.14) holds for all $r \in C'$. This implies that the subalgebra of $R[x; \sigma, \delta]$ generated by $C'$ and $x$ is isomorphic to the commutative algebra $C'[x]$. Since $R$ is a finite $C'$-module, the Ore extension $R[x; \sigma, \delta]$ must be a finite $C'[x]$-module. From [MR, 13.4.9], we conclude that $R[x; \sigma, \delta]$ is a PI algebra. ■

Theorem 3.15. The algebra $A_2(J_{a,b})$ is a PI ring if and only if $\text{char}(K) = p > 0$. In this case, $A_2(J_{a,b})$ is a finite module over its Noetherian center.

Proof. If $\text{char}(K) = 0$, Theorem 3.2 implies that $A_2(J_{a,b})$ is not a PI ring. Now assume $\text{char}(K) = p > 0$. By Lemma 3.13, $K_R(x_1, x_2) = K[x][x_2; 1, \delta]$ is a PI ring and hence $K_R(x_1, x_2)$ is finite module over its Noetherian center, see [St, 2.12]. Similar to the proof of Theorem 3.6, it is easy to see that

$$A_2(J_{a,b}) = K_R(x_1, x_2)[\partial_2; \sigma_2, \delta_2][\partial_1; \sigma_1, \delta_1]$$

where

$$\sigma_2 : x_1 \rightarrow x_1, \ x_2 \rightarrow x_2 - bx_1, \ \text{and}$$

$$\sigma_1 : x_1 \rightarrow x_1, \ x_2 \rightarrow x_2 - ax_1, \ \partial_2 \rightarrow \partial_2.$$
It is easy to check that $\sigma_1$ and $\sigma_2$ have finite order $p$. Then the assertions of the theorem follow by applying Lemma 3.13 and [St, 2.12]. Note that these rings satisfy the conditions in [St, 2.12] by Lemma 3.9, Corollary 3.10, and Quillen’s Theorem (see [MR, 12.6.13]).

**Corollary 3.15.** If $\text{char}(K) = p > 0$ then the Krull and global dimensions of $A_2(J_{a,b})$ are both 4.

**Proof.** Since $A_2(J_{a,b})$ is a semiprime Noetherian PI ring, it follows from [MR, 6.4.8 and 10.10.6] that $\text{Kdim}(A_2(J_{a,b})) = 4$. From [RSS], we get $\text{Kdim}(A_2(J_{a,b})) \leq \text{gl.dim}(A_2(J_{a,b}))$. But by Corollary 3.10.(3) $\text{gl.dim}(A_2(J_{a,b})) \leq 4$ and finally we obtain $\text{gl.dim}(A_2(J_{a,b})) = 4$. ■

### 4. Deformations and cohomology of Weyl algebras

In this section we discuss the formal deformation theory of the Weyl algebra $A_n$ and its relation to the quantum Weyl algebras $A_n(R)$. In particular, we will see that, suitably interpreted, $A_n(R)$ is a deformation of the classical Weyl algebra $A_n$. Throughout this section assume $\text{char}(K) = 0$. We first state a result from [Sr] concerning the Hochschild cohomology groups $H^m(A_n, A_n)$. This will be useful in determining which types of deformations of $A_n$ that are possible.

**Theorem 4.1.** [Sr] The Hochschild cohomology groups $H^m(A_n, A_n) = 0$ for all $m > 0$. ■

We now turn to the deformation theory of the Weyl algebras. First we briefly recall the basic definitions and notions of algebraic deformation theory, cf [Ge].

**Definition 4.2.** A $K[[t]]$-algebra $A_t$ is a formal deformation of a $K$-algebra $A$ if it is a flat, $t$-adically complete $K[[t]]$-module equipped with a $K$-algebra isomorphism $A_t \otimes K[[t]] K \cong A$.

Since $K$ is a field, the flatness hypothesis of the definition can simply be replaced with $t$-torsion free; (the former concept is required when considering algebras over a commutative ring). When $A_t$ is a deformation of $A$ we may identify $A_t$ with $A[[t]]$ as $K[[t]]$-modules and, in doing so, the deformed multiplication $\mu_t : A[[t]] \otimes K[[t]] A[[t]] \to A[[t]]$ necessarily has the form $\mu_t = \mu + \mu_1 t + \mu_2 t^2 + \cdots$ where $\mu$ is the multiplication in $A$ and each $\mu_i \in \text{Hom}_K(A \otimes A, A)$ is extended to be $K[[t]]$-linear. The trivial deformation has $\mu_i = 0$ for all $i \geq 1$ and so is the algebra $A[[t]]$. Deformations $A_t$ and $A'_t$ are equivalent if there is a $K[[t]]$-algebra isomorphism $\phi_t : A_t \to A'_t$ of the form $\phi_t = \text{Id}_A + \phi_1 t + \phi_2 t^2 + \cdots$ where each $\phi_i \in \text{Hom}(A, A)$ is extended
to be $K[[t]]$-linear. An algebra is rigid if every deformation is equivalent to the trivial deformation.

As shown in [Ge], the associativity condition $\mu_t((\mu_t(a, b), c) = \mu_t((a, \mu_t(b, c))$ imposes restrictions on the $\mu_i$ which are intimately associated with the Hochschild cohomology groups $H^*(A, A)$ in low dimensions. We shall not need or describe this connection in detail, but we note in particular that the first non-zero $\mu_r$ is a Hochschild 2-cocycle and can be changed by any coboundary by passing to an equivalent deformation. Thus, if $\mu_r$ is itself a coboundary, then there is an equivalent deformation with $\mu_i = 0$ for all $i \leq r$. Consequently, $A$ is rigid whenever $H^2(A, A) = 0$; (the converse is false if $K$ has positive characteristic and it is unknown if $K$ has characteristic 0).

In order to study the relation of the quantizations $A_n(R)$ to deformations of $A_n$ we must first make some modifications in their definition since, as defined in Section 1, $A_n(R)$ is a $K$-algebra while deformations are $K[[t]]$-algebras. Throughout the remainder of this section let $V_t = V[[t]] = V \otimes K[[t]]$ and view all tensor products over $K[[t]]$ (i.e. $\otimes = \otimes_{K[[t]]}$). Let $\mathcal{R} : V_t \otimes V_t \to V_t \otimes V_t$ be a linear transformation of the form $\mathcal{R} = \sigma + t\mathcal{R}_1 + t^2\mathcal{R}_2 + \cdots$ where $\sigma$ is again the operator sending $a \otimes b$ to $b \otimes a$ for $a$ and $b$ in $V_t$. In this context, we say $\mathcal{R}$ is a formal Hecke symmetry if it satisfies the braid relation, $\mathcal{R}_{12}\mathcal{R}_{23}\mathcal{R}_{12} = \mathcal{R}_{23}\mathcal{R}_{12}\mathcal{R}_{23}$ as linear maps $V_t \otimes V_t \otimes V_t \to V_t \otimes V_t \otimes V_t$, and the formal Hecke condition $(\mathcal{R} - q(t))(\mathcal{R} + q(t)^{-1}) = 0$ for some $q(t) \in K[[t]]$ with $q(0) = 1$. Now if $\mathcal{R}(x_i \otimes x_j) = \sum_{k,l} \mathcal{R}^{kl}_{ij} x_k \otimes x_l$ with $\mathcal{R}^{kl}_{ij} \in K[[t]]$, we define $A_n(\mathcal{R})$ to be the quotient of $K\langle x_1, \cdots x_n, \partial_1, \cdots \partial_n \rangle[[t]]$ subject to the relations

(a) $\sum_{k,l} \mathcal{R}^{kl}_{ij} x_k x_l = q(t)x_i x_j$. \\
(b) $\sum_{k,l} \mathcal{R}^{ij}_{lk} \partial_k \partial_l = q(t)\partial_i \partial_j$. \\
(c) $\partial_i x_j = \delta_{ij} + q(t) \sum_{k,l} \mathcal{R}^{ik}_{jl} x_k \partial_l$.

These relations are of course obtained from those for $A_n(R)$ by replacing $R^{kl}_{ij}$ with $\mathcal{R}^{kl}_{ij}$. A natural question is to determine whether $A_n(\mathcal{R})$ is a deformation of $A_n$. By definition, $A_n(\mathcal{R})$ is $t$-adically complete and, moreover, $A_n(\mathcal{R}) \otimes_{K[[t]]} K \cong A_n$ since the defining relations of $A_n(\mathcal{R})$ reduce to those for $A_n$ when $t = 0$. Thus $A_n(\mathcal{R})$ will be a deformation of $A_n$ if and only if it is $t$-torsion free. Now since $\mathcal{R}$ is a formal Hecke symmetry, the results of [GGS2] imply that $K_\mathcal{R}\langle x_i \rangle$ and $K_\mathcal{R}\langle \partial_i \rangle$ are deformations of the polynomial rings $K[x_i]$
and $K[\partial_i]$ and consequently there are $K[[t]]$-module isomorphisms $K\mathcal{R}\langle x_i \rangle \cong K[x_i][[t]]$ and $K\mathcal{R}\langle \partial_i \rangle \cong K[\partial_i][[t]]$. Now reasoning in the same way as in Theorem 1.5, there is a $K[[t]]$-module isomorphism $A_n(\mathcal{R}) \cong K\mathcal{R}\langle x_i \rangle \hat{\otimes} K\mathcal{R}\langle \partial_i \rangle$ where $\hat{\otimes}$ indicates the completion of the algebraic tensor product $\otimes_{K[[t]]}$ with respect to the $t$-adic topology. Combining these facts gives a $K[[t]]$-module isomorphism $A_n(\mathcal{R}) \cong (K[x_i] \otimes K[\partial_i])[t]$ and the latter module is isomorphic to $A_n[[t]]$. Thus we have the following:

**Theorem 4.4.** If $\mathcal{R}$ is a formal Hecke symmetry then $A_n(\mathcal{R})$ is a deformation of $A_n$. ■

Now the Weyl algebra $A_n$ is rigid since, according to Theorem 4.1, $H^2(A_n, A_n) = 0$. Thus for any formal Hecke symmetry there is an algebra isomorphism $A_n(\mathcal{R}) \cong A_n[[t]]$. As an illustration of this fact, consider the quantization multiparameter quantization $A_n(R_{q,p,ij})$ of Example 2.1. Now if we replace $q$ with $e^t$ and $p_{ij}$ with $e^{c_{ij}t}$ for $c_{ij} \in K$ in the definition of $R_{q,p,ij}$ then we obtain a formal Hecke symmetry $\mathcal{R}_{q,p,ij}$. Consequently, $A_n(\mathcal{R}_{q,p,ij})$ is a deformation of $A_n$ and $A_n(\mathcal{R}_{q,p,ij}) \cong A_n[[t]]$. For the classical case when all $p_{ij} = 1$ this isomorphism has been explicitly constructed in [Og]. It is important to note that, even if a Hecke symmetry $R$ is obtainable as a specialization of a formal Hecke symmetry $\mathcal{R}$, the rigidity of $A_n$ does not guarantee the existence of a $K$-algebra isomorphism $A_n(R) \cong A_n$. This may be seen even in the “simplest” quantum Weyl algebra, $A_1(q)$. If $q \neq 1$ then $A_1(q)$ is not simple and thus is not isomorphic to $A_1$. To further illustrate this point, note that in $A_1[[t]]$ the elements $x$ and $\partial_t = \frac{1}{x} \cdot \frac{e^{tx}\partial - 1}{e^t - 1}$ satisfy $\partial_t x = 1 + q^2 x \partial_t$ with $q = e^{t/2}$ and thus provide an explicit isomorphism $A_1[[t]] \cong A_1(e^{t/2})$. Even if $K = \mathbb{C}$ this analytic isomorphism has zero radius of convergence.
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