I. A star orbiting around a supermassive rotating black hole: Free motion and corrections due to star-disc collisions

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II. Relativistic precession of the orbit of a star near a supermassive black hole

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A star orbiting around a supermassive rotating black hole: Free motion and corrections due to star-disc collisions

ABSTRACT
Our aim is to study the evolution of the orbit of a star under the influence of interactions with an accretion disc in an AGN. The model considered consists of a low-mass compact object orbiting a supermassive black hole and colliding periodically with the accretion disc. Approximate calculations based mostly on Newtonian theory of gravity have been carried out by several authors to estimate the effects of circularization of initially eccentric orbits and their dragging to the disc plane. Here, we present the first step to a more adequate general relativistic approach in which the gravitational field of the nucleus is described by the Kerr metric. The star is assumed to move along a geodesic arc between successive interactions with an equatorial accretion disc. We solved relevant formulae for the geodesic motion in terms of elliptic integrals and constructed a fast numerical code which, after specifying details of the star-disc interaction, enables us to follow the trajectory of the star for many revolutions and study the evolution of its eccentricity and inclination with respect to the disc. Lense-Thirring precession of the orbit is potentially a very important effect for observational confirmation of the presence of a rotating black hole in the nucleus. Our approach takes effects of the Lense-Thirring precession into account with no approximation.

The model of star-disc interactions has been suggested to explain the X-ray variability observed in the Seyfert galaxy NGC 6814. We briefly discuss on this subject.

Key words: general relativity — black holes — galaxies: active — galaxies: individual (NGC 6814)

1 INTRODUCTION AND MOTIVATION
General interest in studying star-disc interactions in the nuclei of galaxies has greatly increased in the last years. This is partly due to the fact that
they appear important in explaining the X-ray variability of active galactic nuclei (AGN). Although it is generally believed that many galaxies, and active galaxies in particular, harbour massive black holes in their cores, there is no direct observational confirmation for this paradigm. The origin of this difficulty is apparent: complicated plasma physics of the matter swirling around the black hole makes it difficult to distinguish the effects of general relativity—although they may be essential for the mechanism of energy generation itself. Potentially very important observable is the X-ray data on variability of active galactic nuclei (for the recent review see Wallinder, Kato & Abramowicz 1992). Although our understanding of the origin of X-rays is not satisfactory, it is often accepted that they are generated in the inner regions of the accretion disc where general relativistic effects are significant.

When trying to describe these effects it is useful to separate details of the mechanism generating radiation (described in the local frame co-moving with the matter), i.e. the local physics of the interaction on the one hand, and observable effects as seen by a distant observer on the other. In our previous work (Karas, Vokrouhlický & Polnarev 1992, Paper I) we developed a code which can be used in many astrophysically relevant situations to calculate images of various effects occurring in the close vicinity of the rotating (Kerr) black hole. Our code deals efficiently with problems treated originally by Cunningham & Bardeen (1973). All relativistic effects on photons (like gravitational and Doppler shift of frequency and bending of light rays) were taken into account. As an example, we applied the code to the “hot spot” model of the Seyfert galaxy NGC 6814 which tries to explain its X-ray variability on the scale of approximately 3 hours in terms of a bright orbiting spot (or spots) located on the accretion disc (Abramowicz et al. 1992), and we also discussed the case of a large number of spots with different intrinsic characteristics which may be relevant in explaining the X-ray variability of AGN on still shorter time scales (Abramowicz et al. 1991).

In this paper we present a method for calculating the evolution of the orbital parameters of a compact star orbiting a massive black hole. Such a star may come from a binary or a cluster tidally disrupted by the central black hole (Hills 1988, Novikov, Pethick & Polnarev 1992). It can be deposited in a tightly bound orbit with a close pericentre where relativistic effects are important. The star interacts with the accretion disc only at the moment when it crosses the equatorial plane and this interaction weakly affects its
motion. Cumulative effects of successive tiny interactions circularize the trajectory and change the orbital plane into the plane of the disc. Mutual star-disc interactions have been discussed by Syer, Clarke & Rees (1991) as a possible origin of fueling and variability of AGN. Particularly, they discussed relative time-scales for circularization of the orbit and its “grinding” to the disc plane, and the final radius of the embedded orbit. Naturally, quantitative estimates depend on poorly known details of the interaction (cf. Zurek, Siemiginowska & Colgate 1992). A compact star colliding with an accretion disc is one of viable models for NGC 6814 (Abramowicz 1992, Sikora & Begelman 1992, Rees 1993). All present models possess their advantages and difficulties when compared to observational data and they need to be investigated in greater detail. General relativistic precession of orbital nodes (Lense-Thirring effect)—if detected—would strongly support models of AGN with rotating supermassive black holes. Lense-Thirring precession affects the inclined trajectory of a star, but it has been discussed only in special cases of free orbits with a large radius compared to the gravitational radius of the central black hole (Lense & Thirring 1918) and spherical orbits around an extreme Kerr black hole (Wilkins 1972). We consider the more general case of eccentric orbits which interact with the equatorial disc and we do not assume any particular value of the angular momentum of the central black hole.

Star-disc interactions are a complex problem. Because we assume that the disc produces only a weak perturbation of free motion of the star, we can attack the problem in several steps. The present paper concentrates on an effective method for calculating the free motion of the star between its successive interactions with the disc. Simple examples of how star-disc interactions may change the picture are given in the last Section and they will be discussed in a forthcoming paper in greater detail. Detailed calculation of the Lense-Thirring frequency relevant for this model will be given elsewhere (Karas & Vokrouhlický 1993). Once the motion of the star and its interaction with the disc are specified, we can apply the method of Paper I to compute the shape of the light curve or resultant spectrum. [See also Cunningham & Bardeen (1973), de Felice, Nobili & Calvani (1974), Luminet (1978) and Laor & Netzer (1989)]. Recently, Fabian et al. (1989), Kojima (1991) and Laor (1991) applied analogous approach to study line profiles from accretion discs. For a more complete list of references cf. Paper I.
Let us briefly describe the configuration of the model. We consider a low-mass compact object (white dwarf, neutron star or black hole) orbiting the central massive black hole of the AGN. In our approach, we restrict ourselves to the assumption that the orbiting object moves along a time-like geodesic in the unperturbed background Kerr metric outside the equatorial plane. Thus we implicitly assume that (i) the orbiting object is sufficiently compact and/or far from the central black hole (we neglect any coupling of the star’s higher multipoles to the background curvature), and (ii) its mass is very small compared to the mass of the central black hole (we do not consider perturbations of the background metric). We also neglect the influence of radiating gravitational waves. In several astrophysical situations such assumptions may not be appropriate—see, e.g., Carter & Luminet (1983), Luminet & Marck (1985), and Carter (1992) who studied tidal squeezing of the stars by the nearby black hole. Hartle & Thorne (1985) and Suen (1986) developed a scheme for multipole-tidal interactions of relativistic objects. As for the motion of close and comparable in mass black holes, see, e.g., D’Eath (1975a,b). We do not consider such extreme situations in this paper. Kates (1980) has shown that the star will move closely along a geodesic in the unperturbed background metric for sufficiently long time provided the ratio of its mass and the characteristic reference length of the background metric is a small parameter. We have in mind situations where this parameter is of the order $10^{-5}$ or even less. This is important to note because we will attempt to follow the trajectory of the orbiting object for long periods of time. Next, we assume that the accretion disc is in the equatorial plane of the central black hole. We exclude thick disc models from our present considerations. Each time the object crosses the equatorial plane it interacts with the disc (provided the intersection is between the outer and the inner edges of the disc). In other words, we assume that the trajectory consists of arcs of free geodesic motion above (or below) the disc plane and impulsive changes of the orbital parameters at the moment of passages through the disc. The whole “physics” of the problem is compressed into the prescription governing the changes of the orbital parameters when crossing the equatorial plane. The interaction is assumed very weak which implies that relative changes of energy, angular momentum and other quantities characterizing the orbit of the star are much less than unity in each single event. We should note that the geodesic motion in the Kerr space-time is integrable; thus, we do not expect strong dependence of the shape of the orbit on initial conditions which
is typical for a chaotic motion. An alternative approach which employs a statistical description with appropriately averaged quantities is under preparation. The separation of dynamical and physical aspects, as we introduce it in the present paper, appears very advantageous and it allows us to employ a fast method for computing the evolution of orbital parameters.

2 THE MAPPING – DETAILS OF THE CALCULATION

We consider the geodesic motion of a test particle (representing a star or a low-mass black hole) in the fixed Kerr background metric. We are interested only in short arcs of geodesic motion with the following boundary conditions: the initial point (indexed “i”) lies in the equatorial plane ($\theta = \pi/2$ in Boyer-Lindquist coordinates), the final point (indexed “f”) is the nearest successive intersection of the orbit with the equatorial plane. We employ a ‘mapping’ by which we understand an analytical algorithm to evaluate the final position ($r_f, \phi_f$) from the initial position ($r_i, \phi_i$) with constants of motion assumed to be given. In applications, we also need to know the transformation from initial to final velocities $dr/dt, d\theta/dt, d\phi/dt$ to obtain the full starting information for the physical model of the interaction of the orbiting object with the accretion disc. The whole procedure is trivial in principle because the geodesic motion is separable in the Kerr background metric and the equations of motion can be reduced to a set of ordinary first order differential equations (Carter 1968). Our main task is to handle the problem efficiently.

Carter’s equations involve squares of the velocities. As the star crosses the equatorial plane, the latitudinal velocity changes its sign periodically. In order to treat the case of the radial velocity, we introduce the sign function $\eta \equiv \text{sgn}(dr/dt)$. Thus, our mapping is the analytical transformation

$$(r, \phi; \eta)_i \xrightarrow{\text{M}} (r, \phi; \eta)_f.$$  

We will also be interested in analytical evaluation of the delay in coordinate time which is necessary to pass from the initial to the final configuration in the disc plane: ($t_f - t_i$). In particular, this is important for reconstruction of the AGN photometric curve, provided similar periodic process as described here (star-disc interactions) arise its variability (Karas & Vokrouhlický 1993).
It is worth noting that the code based on this mapping technique is optimized as concerns both the speed and accuracy. The effective step of the method is the whole orbital arc and it cannot be made greater in principle. Moreover, the exact analytical solution of the problem is chosen as a sample function covering one integration step (instead of, for example, polynomials in Runge-Kutta methods). We remark that the name 'mapping’ comes from analogous methods developed in celestial mechanics (e.g. Wisdom 1982, Murray 1986).

We use the standard notation for the Kerr metric (Bardeen 1973). Quantities with the dimension of length in geometrized units are divided by the mass of the central black hole \(M\), and they are thus made dimensionless. Time-like geodesics in the Kerr space-time can be integrated in the form (Carter 1968)

\[
t_f - t_i = \int_{r_i}^{r_f} \frac{r^2(r^2 + a^2)\mathcal{E} + 2ar(a\mathcal{E} - \Phi)}{\Delta R(r)^{1/2}} \, dr + \int_{\theta_i}^{\theta_f} \frac{a^2 \mathcal{E} \cos^2 \theta}{\Theta(\theta)^{1/2}} \, d\theta, \tag{1}
\]

\[
\phi_f - \phi_i = \int_{r_i}^{r_f} \frac{r^2\Phi + 2r(a\mathcal{E} - \Phi)}{\Delta R(r)^{1/2}} \, dr + \int_{\theta_i}^{\theta_f} \frac{\Phi \cot^2 \theta}{\Theta(\theta)^{1/2}} \, d\theta, \tag{2}
\]

\[
\int_{r_i}^{r_f} \frac{dr}{R(r)^{1/2}} = \int_{\theta_i}^{\theta_f} \frac{d\theta}{\Theta(\theta)^{1/2}}. \tag{3}
\]

Here,

\[
R(r) = (\mathcal{E}^2 - 1)r^4 + 2r^3 + [(\mathcal{E}^2 - 1)a^2 - \Phi^2 - Q] r^2 + 2\mathcal{K}r - a^2Q, \tag{4}
\]

\[
\Theta(\theta) = Q - [a^2(1 - \mathcal{E}^2) + \Phi^2 \sin^{-2} \theta] \cos^2 \theta, \quad Q = \mathcal{K} - (\Phi - a\mathcal{E})^2, \tag{5}
\]

and

\[
\Sigma = r^2 + a^2 \cos^2 \theta, \quad \Delta = r^2 + a^2 - 2r, \\
A = (r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta.
\]

The constants of motion, \(\mathcal{E} = -p_t\), \(\Phi = p_\phi\), and \(\mathcal{K}\), can be expressed in terms of components of the four-momentum in the locally non-rotating frame (LNRF):

\[
\mathcal{E} = \left[\left(\frac{\Sigma \Delta}{A}\right)^{1/2} p^r + \frac{2ar}{(\Sigma A)^{1/2}} p^\phi\right]_{r, \theta_i = \frac{\pi}{2}}, \quad \Phi = \left[\left(\frac{A}{\Sigma}\right)^{1/2} p^\phi\right]_{r, \theta_i = \frac{\pi}{2}}, \tag{6}
\]
\[ K = [(\Phi - aE)^2 + \Sigma(p^\theta)^2]_{r_i, \theta_i = \frac{\pi}{2}}. \] (7)

Components of the four-momentum in terms of direction cosines in the local sky of an observer at rest with respect to LNRF are

\[ p^\hat{t} = \gamma, \quad p^\hat{r} = \gamma v \cos \alpha, \]
\[ p^\hat{\theta} = \gamma v \sin \alpha \cos \beta, \quad p^\hat{\phi} = \gamma v \sin \alpha \sin \beta, \] (8)

where \( v \) is the tetrad velocity of the particle in LNRF and the Lorentz factor

\[ \gamma = \frac{1}{\sqrt{1 - v^2}}. \]

Specification of the initial conditions as a result of the star-disc interaction in the local frame co-moving with the matter of the disc requires another boost to the disc co-rotating frame (DCF). In the case of a Keplerian thin disc in the equatorial plane the linear velocity of the DCF with respect to LNRF is

\[ v_{DCF} = \frac{r^2 - 2ar^{1/2} + a^2}{(r^{3/2} + a)\Delta}. \]

It is known that integrals (1)–(3) can be reduced to standard elliptic integrals, but only the simplest cases have been discussed in the literature. The explicit form of relevant formulae depends on the value of the constants of motion and initial conditions. We shall restrict ourselves to the most interesting astrophysical case: stable, energetically bound trajectories which cross the equatorial plane many times repeatedly. Thus, we assume \( 0 \leq E < 1 \). We exclude singular cases of orbits lying exactly in the equatorial plane (\( Q = 0 \); Bardeen, Press & Teukolsky 1972) or those intersecting the rotation axis (\( \Phi = 0 \); Stoghianidis & Tsoubelis 1987); we also exclude the case of the extremely rotating black hole, \( a = 1 \). In the Schwarzschild case (\( a = 0 \)) the geodesic is always planar, while \( a \neq 0 \) leads to the Lense-Thirring precession of the orbit and we have to take into consideration the dragging of the nodes [Wilkins (1972) discussed this effect for spherical orbits in the extreme Kerr case].

First, we simplify the integrals (1)–(3) to a form which can be directly found in standard tables of elliptic integrals (Byrd & Friedman 1971, Gradshteyn & Ryzhik 1980, Gröbner & Hofreiter 1965). For this purpose we need
to find the roots of $R(r)$ and $\Theta(\theta)$. Real roots of are the turning points of the radial and latitudinal motion, respectively. We specify the initial point of the geodesic as $(t_i, r_i, \pi/2, \phi_i)$ and we look for the final position $(t_f, r_f, \pi/2, \phi_f)$ which is in the equatorial plane again. The polynomial $R(r)$ governing the radial motion is of the fourth order with $R(r_i) \geq 0$. In our case, $R(r) < 0$ for $r \to \infty$ and $r = 0$. Thus we can find two real roots, $r_A \in (0, r_i)$, $r_B \in (r_i, \infty)$, and for the remaining two roots we obtain the quadratic equation

$$(\mathcal{E}^2 - 1)r^2 + \left[ (\mathcal{E}^2 - 1)(r_A + r_B) + 2 \right] r - \frac{a^2 Q}{r_A r_B} = 0. \quad (9)$$

Supposing all roots are real we can denote them, in a descending sequence, as $r_1 > r_2 > r_3 > r_4$. We exclude the possibilities of multiple roots because such situations are singular in the sense that they occur for precisely arranged values of $\mathcal{E}$ and $\Phi$. The probability that interaction with the accretion disc will lead to such values is zero (in the measure sense). Thus we have

$$I_r \equiv \int \frac{dr}{R(r)^{1/2}} = \frac{1}{\sqrt{1 - \mathcal{E}^2}} \int \frac{dr}{\sqrt{(r - r_1)(r - r_2)(r - r_3)(r - r_4)}}. \quad (10)$$

In the case of two real and two complex roots ($r_1 > r_2$ and $r_3, r_4$, respectively), we obtain

$$I_r = \frac{1}{\sqrt{1 - \mathcal{E}^2}} \int \frac{dr}{(r_1 - r)(r - r_2)\sqrt{(r - \chi_1)^2 + \chi_2^2}}, \quad (11)$$

where

$$\chi_1 = \Re(e_3), \quad \chi_2 = \Im(e_3).$$

The latitudinal motion is governed by the polynomial $\Theta_{\mu}(\mu) \equiv \sin^2 \theta \; \Theta(\theta)$ [Eq. (5)]. Solving the bi-quadratic equation in $\mu \equiv \cos \theta$,

$$a^2(1 - \mathcal{E}^2)\mu^4 - \left[ Q + a^2(1 - \mathcal{E}^2) + \Phi^2 \right] \mu^2 + Q = 0, \quad (12)$$

we obtain the roots $\mu_+ > \mu_- > 0$; The latitudinal motion is only possible in the region $\mu \in (-\mu_-, \mu_+)$ and

$$I_{\mu} \equiv \int \frac{d\mu}{\Theta_{\mu}(\mu)^{1/2}} = \frac{1}{a\sqrt{1 - \mathcal{E}^2}} \int \frac{d\mu}{\sqrt{(\mu_+^2 - \mu^2)(\mu_-^2 - \mu^2)}}. \quad (13)$$
Analogously, the azimuthal motion can be solved in the form

$$\phi_f - \phi_i = [2(aE - \Phi)A_+ + \Phi B_+] I_+ + [2(aE - \Phi)A_- + \Phi B_-] I_- + \Phi J_\mu, \quad (14)$$

where

$$I_{\pm} = \int \frac{dr}{(r - r_{\pm})R(r)^{1/2}}, \quad J_\mu = \int \frac{d\mu}{(1 - \mu^2)\Theta_\mu(\mu)^{1/2}}, \quad (15)$$

$$A_{\pm} = \pm \frac{r_{\pm}}{r_+ - r_-}, \quad B_{\pm} = \pm \frac{2r_{\pm} - a^2}{r_+ - r_-}, \quad r_{\pm} = 1 \pm \sqrt{1 - a^2}. \quad (16)$$

Finally, for the time coordinate we obtain

$$t_f - t_i = E (J_r + 2K_r) + 2B_+ r_+ E + a(aE - \Phi)A_+ I_+ + 2B_- r_- E + a(aE - \Phi)A_- I_- + 4E I_r + a^2E K_\mu \quad (16)$$

with

$$J_r \equiv \int \frac{r^2 dr}{R^{1/2}}, \quad K_r \equiv \int \frac{r dr}{R^{1/2}}, \quad K_\mu \equiv \int \frac{\mu^2 d\mu}{\Theta_\mu(\mu)^{1/2}}. \quad (17)$$

It is straightforward but moderately tedious to derive the explicit form of the mapping. We give relevant formulae in Appendix.

### 3 THE EVOLUTION OF ORBITS – SIMPLE EXAMPLES

There are several interesting issues closely related to our problem which have not been fully understood as yet. In this Section we present simple examples of secular changes of orbital parameters of a star orbiting a supermassive \((10^6 - 10^9 M_\odot)\) black hole and interacting with the accretion disc. In particular, we study the eccentricity and inclination of the orbit, as they are introduced in Appendix. Differences from previous estimates which have been made within the framework of Newtonian gravity are found to be significant for nearly unstable orbits. These differences can be very subtle and interfere with the details of star-disc interaction and thus at first we adopted an extremely simplified (and perhaps unrealistic) description of the interaction. The Lense-Thirring precession of the orbit was included fully relativistically.
with no approximation. This is important for our particular example we discuss below, NGC 6814. Mittaz & Branduardi-Raymont (1989) and Done et al. (1993) determined the periodicity in modulation of the X-ray emission from NGC 6814 to be \( \approx 12,000 \) seconds. Somewhat uncertain value for the mass of the central black hole was estimated to \( \approx 10^7 M_\odot \) and thus the radius of the corresponding orbit is a few tens of the gravitational radius of the black hole—rather close to the horizon where approximation methods for the Lense-Thirring precession are no longer satisfactory. We should note, for completeness, that if the star-disc interaction is switched off, the eccentricity of the orbit remains constant in time and the value of the precession is exactly that of the Lense-Thirring precession. In the following examples we tune the strength of the star-disc interaction in such a way that the relative change of orbital parameters is of the order \( \approx 10^{-5} \) in each interaction and we follow \( \approx 10^5 \) revolutions. Each of the following Figures shows the sequence of radial coordinates \( r \) for successive intersections of the trajectory with the disc—one point corresponds to one intersection, two intersections correspond to one revolution of the orbiter. (Alternatively, instead of the number of intersections \( N \) we could use coordinate time \( t \) for labelling the \( x \)-axis; Figures remain very similar in shape but the second possibility appears more adequate for plotting computed light curves which should be related to the observer’s time at infinity.) Upper and lower boundaries of the distribution of intersections are current values of the apocentre and the pericentre, respectively. Intersections are (seemingly randomly) scattered in the whole range between these boundaries due to the shift of the pericentre and Lense-Thirring precession (if \( a \neq 0 \)). Frequencies corresponding to both of these effects are quantitatively studied in Karas & Vokrouhlický (1993).

In the first example we assume that, as a result of the interaction, the difference in the azimuthal components of the star and the disc material (evaluated in DCF) is decreased:

\[
\Delta v_{\phi,\text{DCF}} \rightarrow \alpha_* \Delta v_{\phi,\text{DCF}},
\]

where \( \alpha_* \) is a phenomenological parameter \( \lesssim 1 \). Initially retrograde orbits (those with \( I > 90^o \)) decrease the absolute magnitude of their \( \Phi \) component of angular momentum due to interactions with the disc and they either get captured by the black hole or become prograde. Once they are prograde, the star acquires angular momentum from the disc and moves away from
the black hole. Simultaneously, both the eccentricity and the inclination decrease, and the orbital period increases. The effect of energy dissipation due to crashing through the disc is not considered in Eq. (18). The model is of course inadequate when the inclination reaches zero and the star becomes a part of the disc. Figure 1 is an example of such an orbit. As mentioned above, the unperturbed geodesic motion in the Kerr metric is integrable and thus we do not expect any dependence of the characteristic time for circularization or changing of the trajectory to the disc plane on particular values of the initial position or directions of velocity of the star.

Our second example is a modification of the model studied by Syer et al. (1991). It is complementary to the previous one because energy dissipation is now considered while the difference in azimuthal velocities of the star relative to the disc material is ignored. We supposed that the star hits the disc supersonically and pulls some amount of the material with a mass $\Delta m \approx \rho_{\text{disc}} h_{\text{disc}} A_{\text{eff}} \sin^{-1} I$ out of the disc; here, $\rho_{\text{disc}}$ and $h_{\text{disc}}$ are the local density and thickness of the disc, respectively, and $A_{\text{eff}}$ is the effective cross-section for the star-disc interaction. The energy dissipated during the interaction is proportional to the kinetic energy acquired by the disc material, $\Delta E_{\text{diss}} \propto \Delta m (\gamma_{DCF} - 1)$. We assumed that the acceleration of the star which results from this interaction is anti-parallel to the velocity of the star and the corresponding change of the velocity is

$$\Delta \vec{v} = -\frac{\Delta E \vec{v}}{m_\star \gamma^3_{DCF} v^2} \propto -\frac{\rho_{\text{disc}} h_{\text{disc}} A_{\text{eff}} (\gamma_{DCF} - 1) \vec{v}}{m_\star \gamma^3_{DCF} v^2 \sin I}.$$  

(19)

(We should note that the last equation for the star-disc drag becomes inappropriate and must be modified when the motion of the star is subsonic and the disc material is directly accreted onto the star. The motion is highly supersonic with the Mach number of the order $10^2 - 10^3$ under conditions we consider.) Naturally, $\rho_{\text{disc}}$ and $h_{\text{disc}}$ depend on the disc model. Because we do not want to enter into these additional details we assumed that they, as well as $A_{\text{eff}}$, are constant. [We have also carried out computations using the density profiles corresponding to the Novikov & Thorne (1973) relativistic thin disc model which yielded only moderate modifications to the results.]

Figure 2 illustrates two typical cases—both orbits are initially prograde with (a) $I = 35^\circ$ and (b) $I = 80^\circ$. In general, the final radius of the orbit can be either larger (for small values of the initial inclination) or smaller (for large
values) than the initial pericentre. We found that initially retrograde orbits became captured in this model. This feature can be naturally explained as follows. Dissipation of the orbital energy in each intersection with the disc tends to increase the binding energy of the orbiting object. In the case of originally prograde orbits, however, the orbiter obtains sufficient amount of angular momentum, which saves it from being captured by the black hole. Finally, the object settles into a circular orbit in the disc plane. However, an object which started with retrograde orbit does not acquire enough angular momentum during the period of nearly perpendicular intersections with the disc. Due to continuous losses of energy it typically becomes captured by the hole.

To clarify previous results based on the relativistic treatment we compared them with the corresponding Newtonian “elliptic” mapping (see Appendix). To be consistent, we also reduced formula for the star-disc interaction by eliminating the Lorentz factors $\gamma_{DCF}$ in (19) and instead of the Lorentz boost from LNRF to DCF we used the Galilean transformation. Fig. 3a shows the fully relativistic model with the Schwarzschild background metric, while Fig. 3b is the Newtonian analogue. We have chosen formally the same initial eccentricity and inclination in both Figures: $e \simeq 0.83$ and $I = 130^\circ$. The orbit is initially retrograde and in the relativistic case it becomes captured by the central black hole. On the contrary, this does not occur in the Newtonian case and the orbit is circularized to some definite radius. (In a realistic case, however, the orbiter can be tidally disrupted before it is captured but this depends on its internal structure and we do not consider such possibility in the present paper.) Because the interaction of the orbiter with the disc was always chosen to be weak, the time scale for the precession of the pericentre is much shorter than the time scale for the evolution of the other orbital parameters. As a consequence, the points of intersection with the disc fill the interval between the current pericentre and apocentre in the relativistic case (Fig. 3a). There is no precession of the pericentre in the Newtonian case and thus only an ‘adiabatic’ evolution of the orbital parameters is seen (Fig. 3b).

We also show the results of integration with the nearly extreme Kerr black hole. In this example we have chosen $a = 0.9981$ for definiteness (Thorne 1974), the same initial eccentricity and inclination, and the same initial apocentre and pericentre (expressed in the gravitational radii) as in
We observed (Fig. 4) a slightly shorter circularization time than in the corresponding Schwarzschild case, but the qualitative features of the disc-orbit interaction remained unchanged. They may be changed, however, when details of the structure of the accretion disc are taken into account because the structure of the disc and the location of its inner edge depend significantly on $a$. Again, we found that initially retrograde orbits get captured by the black hole.

4 CONCLUSIONS

We assumed that the low-mass compact object interacts with the thin accretion disc twice per each revolution—exactly when it crosses the equatorial plane of the black hole (impulsive approximation). We described the relevant equations and we employed them in the fast numerical code computing the evolution of the trajectory. We found that the effective time of circularization is shorter than the time to change the orbital plane into the plane of the disc. This conclusion is in accordance with previous results of Syer et al. (1991) based on Newtonian gravity. However, we have also observed short periods during the evolution when eccentricity increases. In particular, this increase occurred in model described by Eq. (18) during the transition period where the initially retrograde character of the orbit is changed to a prograde one.

The star-disc interaction was described by a phenomenological parameter characterizing the magnitude of the change of orbital parameters in each collision. This phenomenological description is satisfactory provided the disc remains thin and the orbital parameters are changed only at the moment of transition of the star through the equatorial plane of the central black hole. A description of the interaction which would result from a detailed physical model is not crucial in this case; we want to improve our understanding of the interaction in future work. Effects of the dynamical friction and direct accretion acting on an object moving through the gaseous medium were studied by a number of authors under various conditions (recently by Petrich et al. 1989). In our highly supersonic and turbulent case, the approach outlined by Zurek et al. (1992) appears the most appropriate one.

One can specify parameters of the model for the case of NGC 6814. Each single long-duration observation of EXOSAT or Ginga covers less than 30 revolutions of the orbiter. The characteristic time-scale for the precession
of nodes is much longer than the orbital one. The estimate which adopts the maximum value of $a = 1$ and radius of the orbit $\approx 50$ gravitational radii of the central black hole yields the ratio of the Lense-Thirring to the orbital frequency $\approx 0.005$. This means that the orientation of the orbit is not significantly changed during each observation. However, the interval between EXOSAT observations and Ginga observations was certainly long enough and a resulting change in the orientation suggests a possibility to understand the perfect stability of the orbital period detected by both satellites and at the same time the puzzling change in the light curve profile. We do not want to speculate further on this important subject until the star-disc collisions and dynamical friction acting on the star are better understood (work in preparation).

To conclude, adopting the model of star-disc interactions as an explanation of the origin of the NGC 6814 light curve, we see one important contribution from general relativistic effects which is due to the pericentre shift and Lense-Thirring precession. These effects drag the point on the orbit where the star crashes through the disc. They also modify the velocity at which the star hits the disc as well as the orientation of the orbit with respect to the observer. This fact has two consequences: (i) additional periodicities corresponding to the precession frequencies are present and can potentially be revealed in the power spectrum of the signal from the source (Karas & Vokrouhlický 1993); (ii) long-term evolution of marginally stable and marginally bound orbits is very different compared to the orbits with identical initial parameters treated in Newtonian theory of gravity. The first consequence above gives us a possibility to detect Lense-Thirring precession induced near the core of an AGN. If the corresponding frequency is not present we will be able to conclude that the central black hole (if any) is non-rotating. This would be extremely important information especially from the point of view of electromagnetic scenarios of AGN which require a rotating black hole. The second consequence is particularly important in describing the capture of the star into a bound orbit around the central black hole. Although many models assume a star located on such an orbit, the very process of the capture is not well understood.
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APPENDIX A: THE MAPPING ALGORITHM

This Appendix outlines the explicit form of the mapping \((r, \phi; \eta)_i \rightarrow (r, \phi; \eta)_f\) from Section 2. Although our approach is straightforward, we believe that it has not yet been employed by other authors. As we found it very advantageous for practical purposes, we describe derivations relevant for this work in some detail. We constructed a numerical code which employs efficient routines for evaluation of elliptic integrals and Jacobian elliptic functions (Press et al. 1986, Press & Teukolsky 1990). The code achieves better precision and about two orders of magnitude higher speed compared to direct numerical integration of the geodesic equation in its equivalent form of first order differential equations. (We used direct integration to check the code.) The two cases, \(0 < a < 1\) and \(a = 0\), are technically somewhat different and we discuss them separately.

A1 The case \(0 < a < 1\)

(i) Evaluate the latitudinal integral between two successive intersections with the equatorial plane:

\[
I_\mu = \frac{2}{a_\mu_+ \sqrt{1 - \xi^2}} K(\mu_-/\mu_+),
\]  

(A1)

where \(K(k)\) denotes the complete elliptic integral of the first kind.

(ii) Distinguish the three cases which may occur:

- Case I—four real roots of \(R(r) = 0\), \(r_4 < r < r_3\).
- Case II—four real roots of \(R(r) = 0\), \(r_2 < r < r_1\).
- Case III—two real and two complex roots of \(R(r) = 0\), \(r_2 < r < r_1\).

Denote \(\eta_i = 1\) (\(\eta_i = -1\)) if \(r\) is increasing (decreasing) at \(r_i\); analogously \(\eta_f\) for \(r_f\).

(iii) Evaluate the increase of \(I_r\) between each two radial turning points:

\[
\delta I_r = \begin{cases} 
\kappa K(k_1) & \text{(Case I and II),} \\
\frac{2}{\sqrt{pq(1-\xi^2)}} K(k_2) & \text{(Case III),}
\end{cases}
\]  

(A2)
where
\[ k_1 = \frac{(r_1 - r_2)(r_3 - r_4)}{(r_1 - r_3)(r_2 - r_4)}, \]
\[ k_2 = \frac{(r_1 - r_2)^2 - (p - q)^2}{4pq}, \]
\[ p^2 = (\chi_1 - r_1)^2 + \chi_2^2, \quad q^2 = (\chi_1 - r_2)^2 + \chi_2^2, \]
\[ \kappa = \frac{2}{\sqrt{(r_1 - r_3)(r_2 - r_4)(1 - E^2)}}. \]

(iv) Denote
\[ f = \begin{cases} 
\kappa F(\varphi, k_1) & \text{(Case I and II)}, \\
\frac{1}{\sqrt{pq(1-E^2)}} F(\varphi, k_2) & \text{(Case III)},
\end{cases} \]
\[ \tilde{I}_r = \begin{cases} 
m \delta I_r - \eta_f f & \text{if } \eta_f \eta_i > 0, \\
(m - \eta_f) \delta I_r + \eta_f f & \text{if } \eta_f \eta_i < 0.
\end{cases} \quad (A3) \]

Here,
\[ \sin^2 \varphi = \begin{cases} 
\frac{(r_1 - r_3)(r_i - r_4)}{(r_3 - r_4)(r_1 - r_i)} & \text{(Case I)}, \\
\frac{(r_1 - r_3)(r_i - r_2)}{(r_1 - r_2)(r_i - r_3)} & \text{(Case II)}, 
\end{cases} \]
\[ \tan^2 \frac{\varphi}{2} = \frac{p(r_i - r_2)}{q(r_1 - r_i)} \quad (\text{Case III}); \]
\[ F(\varphi, k) \] is the incomplete elliptic integral of the first kind and \( m \) is the number of turning points in \( r \) between the two successive intersections. In addition, one has to check whether the trajectory still remains above the horizon if the lower turning point is located below the horizon.

(v) The radial coordinate of the intersection is
\[ r_f = \begin{cases} 
\frac{(r_1 - r_3)r_4 + (r_1 - r_4)r_3 \sigma}{r_1 - r_3 + (r_3 - r_4) \sigma} & \text{(Case I)}, \\
\frac{(r_1 - r_3)r_3 + (r_1 - r_2)r_3 \sigma}{r_1 - r_3 - (r_1 - r_2) \sigma} & \text{(Case II)},
\end{cases} \quad (A4) \]
with
\[ \sigma = \text{sn}^2(u, k_1), \]
\[ u = \frac{1}{2} \sqrt{(r_1 - r_3)(r_2 - r_4)(1 - E^2)(I_\mu - \tilde{I}_r)} , \]

or

\[ r_f = \frac{qr_1 \sigma + pr_2}{p + q\sigma} \quad \text{(Case III)}, \quad \text{(A5)} \]

with

\[ \sigma = \frac{\text{sn}^2(u, k_2)}{1 + \text{cn}(u, k_2)^2}, \]

\[ u = \sqrt{pq(1 - E^2)(I_\mu - \tilde{I}_r)} ; \]

\( \text{sn}(u, k) \) and \( \text{cn}(u, k) \) are Jacobian elliptic functions. At this point we are able to compute the \( r \)-coordinates of the intersections, which is sufficient to determine the evolution of eccentricity and inclination of the orbit and the number of revolutions before the trajectory becomes captured by the black hole or escapes to \( E \geq 1 \) (and then presumably to infinity); \( \phi \) coordinates are also needed if we wish to study the precession. Finally, we need coordinate time to relate the revolutions to time as measured by a distant observer. In the Case III the orbit is in practice captured by the black hole after a few revolutions. Thus we exclude this case from further considerations.

(vi) Evaluate the following quantities.

Case I:

\[ I_\pm = \kappa_\pm [((r_4 - r_1)\Pi(\varphi, n_\pm, k_1) + (r_\pm - r_4)F(\varphi, k_1)] + \tilde{I}_\pm, \quad \text{(A6)} \]

\[ n_\pm = \frac{(r_3 - r_4)(r_\pm - r_1)}{(r_1 - r_3)(r_\pm - r_4)} , \]

\[ J_r + 2K_r = \kappa r_4 \left[ \left( \frac{r_4 \alpha_1^4}{\alpha^4} + 2\frac{\alpha_1^2}{\alpha^2} \right) U + 2\frac{\alpha^2 - \alpha_1^2}{\alpha^4} V_1 + 
2r_4 \alpha_1^2 \frac{\alpha^2 - \alpha_1^2}{\alpha^4} V_1 + r_4 \left( \frac{\alpha^2 - \alpha_1^2}{\alpha^4} \right)^2 V_2 \right] + \tilde{J}_r + 2\tilde{K}_r, \quad \text{(A7)} \]

with

\[ U = F(\varphi, k_1) , \]

\[ V_1 = \Pi (\varphi, -\alpha^2, k_1) , \]
\[ V_2 = \frac{1}{2(\alpha^2 - 1)(k_1^2 - \alpha^2)} \left[ \alpha^2 E(\varphi, k_1) + \left( 2\alpha^2 k_1^2 + 2\alpha^2 - \alpha^4 - 3k_1^2 \right) V_1 + \left( k_1^2 - \alpha^2 \right) U - \text{sn}(U, k_1) \text{cn}(U, k_1) \text{dn}(U, k_1) \frac{\alpha^4}{1 - \alpha^2 \text{sn}^2(U, k_1)} \right], \]

\[ \alpha^2 = \frac{r_1 - r_3}{r_1 - r_3}, \quad \alpha_1^2 = \frac{r_1(r_4 - r_3)}{r_4(r_1 - r_3)}, \]

\[ \sin^2 \varphi = \frac{(r_1 - r_3)(r_i - r_4)}{(r_3 - r_4)(r_1 - r_i)}, \]

\[ \kappa_\pm = \frac{2}{(r_\pm - r_1)(r_\pm - r_4)\sqrt{(r_1 - r_3)(r_2 - r_4)(1 - E^2)}}. \]

Case II:

\[ I_\pm = \kappa_\pm [(r_2 - r_3) \Pi(\varphi, n_\pm, k_1) + (r_\pm - r_2) F(\varphi, k_1)] + \tilde{I}_\pm, \quad \text{(A8)} \]

\[ n_\pm = \frac{(r_2 - r_1)(r_\pm - r_3)}{(r_1 - r_3)(r_\pm - r_2)}, \]

\[ J_r + 2K_r = \kappa r_2 \left[ \left( \frac{r_2\alpha_1^4}{\alpha^4} + 2\frac{\alpha_1^2}{\alpha^2} \right) U + \frac{2\alpha^2 - \alpha_1^2}{\alpha^2} V_1 + 2r_2\alpha_1^2 \frac{\alpha^2 - \alpha_1^2}{\alpha^4} V_1 \right] + \tilde{J}_r + 2\tilde{K}_r, \quad \text{(A9)} \]

with \( U, V_1 \) and \( V_2 \) defined as above, and

\[ \alpha^2 = \frac{r_1 - r_2}{r_1 - r_3}, \quad \alpha_1^2 = \frac{r_3(r_1 - r_2)}{r_2(r_1 - r_3)}, \]

\[ \sin^2 \varphi = \frac{(r_1 - r_3)(r_i - r_2)}{(r_1 - r_2)(r_i - r_3)}, \]

\[ \kappa_\pm = \frac{2}{(r_\pm - r_3)(r_2 - r_\pm)\sqrt{(r_1 - r_3)(r_2 - r_4)(1 - E^2)}}. \]

\( E(\varphi, k) \) and \( \Pi(\varphi, n, k) \) are incomplete elliptic integrals of the second and the third kind, respectively. Integration constants \( \tilde{I}_\pm, \tilde{J}_r \) and \( \tilde{K}_r \) are the values...
of integrals in (15) and (17) evaluated between \( r_i \) and the last turning point. Thus, they depend on the number of turning points in \( r \) \((m)\) and the sign of initial radial velocity \((\eta_i)\) and they can be given in terms analogous to Eq. (A3). We skip explicit expressions because they are rather lengthy.

(vii) Finally,

\[
J_\mu = \frac{2}{a\mu_+\sqrt{1 - \varepsilon^2}} \Pi \left(-\mu_+^2, \mu_-/\mu_+\right), \tag{A10}
\]

\[
K_\mu = \frac{2r_1}{a\sqrt{1 - \varepsilon^2}} \left[K(\mu_-/\mu_+) - E(\mu_-/\mu_+)\right] \tag{A11}
\]

with \( E(k) \) and \( \Pi(n, k) \) being complete elliptic integrals of the second and the third kind. Now we have all necessary quantities required for complete mapping of relevant trajectories in the Kerr metric.

**A2 The Schwarzschild case \((a = 0)\)**

We present the case of the Schwarzschild background metric separately, even though the general formalism developed for the Kerr metric can also be applied. The reason is twofold: (i) the formulae valid for the general Kerr metric often include the angular momentum parameter \(a\) in denominators \([e.g. Eqs. (A10), (A11)]\). These apparent singularities cancel out in the limit \(a \to 0\), but they are the source of difficulties in numerical evaluation; (ii) as the symmetry of the space-time is now higher and geodesics in the Schwarzschild metric remain always planar, we can avoid integration of the latitude \(\theta\), restricting ourselves to the current orbital plane of the test particle spanning one loop of the trajectory above/below the disc (the true orbital plane of the object is changed due to the interactions with the disc). Thus we reduce the order of the mapping by evaluating the integrals in polar coordinates in the orbital plane. The orbital plane differs from the disc plane by the current value of the inclination.

We consider the following coordinate systems: (i) Schwarzschild spherical coordinates \((r, \theta, \phi)\); latitude \(\theta\) is measured from the axis of the disc plane and polar angle \(\phi\) in the disc plane \((\phi = 0\) direction can be chosen arbitrarily), (ii) \((r, \vartheta)\) polar coordinates in the current orbital plane of the orbiting object where the angle \(\vartheta\) is measured from the actual nearest preceding apocentre of the unperturbed trajectory with the current orbital parameters. Let us clarify better the concept of the \(\vartheta\) origin, as it is intimately connected
with our technique. The analytic integration of the geodesic motion in the Schwarzschild space-time is advantageously carried out if the polar angle in the orbital plane is measured from the nearest preceding apocentre. In each step of the mapping procedure we are interested only in one orbital loop above/below the accretion disc; then the interaction with the disc changes the orbital parameters for the successive loop. It is this orbital loop, where the free motion of the test particle in the Schwarzschild background is applied. However, the orbital loop which is under consideration may not necessarily contain the apocentre of the orbit. Thus, apart from the true trajectory of the object we introduce a reference trajectory of the object with the same orbital parameters as for the true one and coinciding with the true trajectory just on the current segment. This fictitious reference orbit defines the \( \theta \) origin—it is measured from the nearest preceding apocentre of the reference trajectory.

The equations of motion covering a single mapping step are as follows (e.g. Chandrasekhar 1983):

\[
\theta_f - \theta_i \equiv \pi = \int_{r_i}^{r_f} \frac{du}{U(u)^{1/2}}, \tag{A12}
\]

\[
t_f - t_i = \frac{\mathcal{E}}{\mathcal{L}} \int_{\theta_i}^{\theta_f} \frac{d\theta}{u^2(1 - 2u)}, \tag{A13}
\]

where

\[
U(u) = 2u^3 - u^2 + 2\mathcal{L}^{-2}u - (1 - \mathcal{E}^2)\mathcal{L}^{-2},
\]

and \( u = 1/r \). Constants of motion are defined as \( \mathcal{E} = -p_t \) and \( \mathcal{L} = p_{\phi} \) [note, that angular momentum \( \mathcal{L} \) is defined with respect to the fictitious orbital plane, not with respect to the disc plane like \( \Phi \) in the Kerr case]. They are related to the tetrad components of the four-momentum in the locally static frames by means of Eqs. (6)–(7). We define \( I \equiv \frac{\pi}{2} - \beta \) as the inclination of the fictitious orbit with respect to the fixed reference disc plane [it can be equivalently expressed using the LNRF tetrad components in the equatorial plane: \( \tan I = p^\theta/p^\phi \); cf. Eq. (8)].

Again, we will concentrate on orbits characterized by \( \mathcal{E} < 1 \) and \( \mathcal{L} \neq 0 \). The signs of the first and the last term of the polynomial \( U(u) \) guarantee at least one positive root of the equation \( U(u) = 0 \) and, as \( U(u = 0) < 0 \), we conclude that this root corresponds to the apocentre of the orbit. The type of the orbit is determined by the properties of the other two roots of
the equation $U(u) = 0$. The roots cannot be real and negative at the same time (Chandrasekhar 1983). We exclude the possibility of multiple roots of as before. Hence, we are left with the two kinds of orbits characterized by:

- the three positive real roots of $U(u) = 0$ which we arrange according to magnitude: $u_1 < u_2 < u_3$,
- one positive real root ($u_1$) and two complex conjugated roots $(u_c, \overline{u_c})$ of $U(u) = 0$.

The first item still encompasses two types of orbits: (i) those captured by the black hole in the sense that they have no pericentre above the horizon (the apocentre of such orbits is always less than 6 — where the last stable orbit and presumably the inner edge of the accretion disc are located in the Schwarzschild case; hence we do not consider these orbits); (ii) quasi-elliptic orbits bound between the turning points, $u_1$ and $u_2$; we will call them Case I orbits. In terminology used by Chandrasekhar (1983) our Case I orbits correspond to the orbits of the first kind. We call the Case II orbits those of the second item above. They have no pericentre and fall unavoidably to the black hole (they correspond to the orbits with purely imaginary eccentricity in Chandrasekhar’s terminology).

One can easily find a simple rule dividing both cases of the orbits: $\mathcal{L}^2 < 12$ implies the Case II orbit. The Case I orbits are characterized by $\mathcal{L}^2 > 12$ and simultaneously

$$ (1 - 2u_*)(1 + \mathcal{L}^2u_*^2) > \mathcal{E}^2, $$

where $u_* = 1 + |L|^{-1}\sqrt{\mathcal{L}^2 - 12}$. In what follows, we will describe an algorithm for the mapping of these two cases of orbits in detail. We will pay special care to the Case I orbits, as they will be shown to be the most important in astrophysical applications.

**A2.1 Case I orbits**

In close analogy to the Newtonian case, Eq. (A12) is advantageously integrated in terms of the relativistic “true anomaly”

$$ u(\chi) = \mu (1 + e \cos \chi), $$

where

$$ \mu = \frac{u_1 + u_2}{2}, \quad e = \frac{u_2 - u_1}{u_2 + u_1}. $$
The quantity $e$ can be interpreted as the eccentricity of the orbit. We do not write the explicit form of primitive functions obtained by integration (Chandrasekhar 1983) but we give formulae for the mapping which we need in our present work. After some manipulation we arrive at the following form of the mapping:

$$u_f = u_2 - (u_2 - u_1) \left[ \frac{\sqrt{\Phi}\Phi_{\alpha}\sqrt{(1 - k^2\Phi)(1 - k^2\Phi_{\alpha})} + \eta_i\sqrt{\Psi}\Psi_{\alpha}}{1 - k^2\Phi_{\alpha}} \right]^2, \quad (A14)$$

where now

$$\Phi = \frac{u_i - u_1}{u_2 - u_1}, \quad \Psi = \frac{u_2 - u_i}{u_2 - u_1},$$

$$\Phi_{\alpha} = \text{sn}^2\left(\frac{\pi}{2}\omega, k\right), \quad \Psi_{\alpha} = \text{cn}^2\left(\frac{\pi}{2}\omega, k\right),$$

$$\omega = \sqrt{1 - 2u_2 - 4u_1}, \quad k^2 = 2(u_2 - u_1)\omega^{-2},$$

Mapping of the sign function $\eta$ is given as follows:

$$\eta_f = \begin{cases} 
\text{sgn}\left[\sigma(\chi_i) + \pi - \sigma(0)\right] & \text{if } \eta_i = -1, \\
\text{sgn}\left[\sigma(\chi_i) - \pi\right] & \text{if } \eta_i = 1,
\end{cases} \quad (A15)$$

where

$$\chi_i = \arccos(\Phi - \Psi), \quad \chi \in \langle 0, \pi \rangle,$$

$$\sigma(\chi_i) = 2\omega^{-1} F\left(\frac{\pi - \chi_i}{2}, k\right).$$

It is instructive to discuss the Newtonian limit of the mapping formula (A14) in which the terms proportional to some power of $1/c$ are neglected. This limit is now obscured by the fact that we imposed widely used “relativistic convention $c = 1$” while now we want to suppress the terms containing $c$ in the denominator. Careful bookkeeping of $c$ in the preceding equations suggests that the Newtonian limit corresponds to the fictitious procedure: $\omega \to 1, k \to 0$. As a result we obtain

$$u_f = -u_i + u_1^N + u_2^N, \quad (A16)$$

where the Newtonian boundaries are found to be

$$u_{i,2}^N = \mathcal{L}^{-2} \left(1 \pm \sqrt{1 + 2\mathcal{E}^N \mathcal{L}^2}\right), \quad \mathcal{E}^N = \frac{1}{2} \left(\mathcal{E}^2 - 1\right).$$
(we retain the negative value for the Newtonian energy of bound orbits, as seen from the above definition of $E^N$). Eq. (A16) is the correct expression for the "elliptic" mapping. Formula (A16) is surprisingly simple, showing the linearity of the elliptic mapping. Moreover, we also have the simple rule $\eta_f = -\eta_i$. By contrast, the full relativistic mapping (A14) for the orbits of the Case I is highly nonlinear and the mapping in $u$ coordinate is coupled with the mapping of $\eta$ function due to $\eta_i$ in Eq. (A14) and $u$ in Eq. (A15).

Let us turn to Eq. (A13) describing the mapping in the $t$-coordinate. We start by expressing the indefinite integral on the right-hand-side. Changing the $\vartheta$ variable to $\chi$ according to the relation

$$
\frac{d\chi}{d\vartheta} = -\omega \sqrt{1 - k^2 \cos^2(\chi/2)} \equiv -\omega \Delta(\chi, k),
$$

one arrives at the primitive function

$$
T(\chi) = \frac{2\mathcal{E}}{\omega \mathcal{L}} \left\{ \begin{array}{c}
\frac{2}{\mu(1-e)} \Pi \left( \frac{\pi - \chi}{2}, \epsilon^{-1}, k \right) \\
+ \frac{1}{[1 - 2\mu(1-e)]} \Pi \left( \frac{\pi - \chi}{2}, \frac{4\mu e}{2\mu(1-e) - 1}, k \right) \\
+ \frac{1}{2\mu^2(1-e^2)(1+ek^2)} \left[ \Delta(\chi, k) \sin \chi \right] \\
+ [1 + 2\epsilon(1 + k^2) + 3\epsilon^2 k^2] \epsilon^{-1} \Pi \left( \frac{\pi - \chi}{2}, \epsilon^{-1}, k \right) \\
+ E \left( \frac{\pi - \chi}{2}, k \right) - (1 + \epsilon k^2) F \left( \frac{\pi - \chi}{2}, k \right) \right\},
\right.
$$

where

$$
\epsilon = \frac{u_1}{u_2 - u_1}.
$$

Now, the algorithm for evaluating the time step associated with the mapping is as follows:

$\eta_i = -1$: introducing $\zeta_i = \text{sgn}(\sigma(0) - \sigma(\chi_i) - \pi)$ we obtain

$$
t_f - t_i = \begin{cases}
T(\chi_f) - T(\chi_i) & \text{if } \zeta_i \geq 0, \\
2T(0) - T(\chi_f) - T(\chi_i) & \text{if } \zeta_i < 0,
\end{cases}
$$

(A18)
where $\chi_f$ is determined by relations
\[
\begin{bmatrix}
\sin \\
\cos
\end{bmatrix} \chi_f = \begin{bmatrix}
\operatorname{sn} \\
\operatorname{cn}
\end{bmatrix} \left[ \frac{\omega}{2} (\sigma(\chi_i) + \pi), k \right];
\]

\eta_i = 1:\quad \text{introducing } \zeta_i = \operatorname{sgn}(\sigma(\chi_i) - \pi) \text{ we obtain}
\[
t_f - t_i = \begin{cases}
T(\chi_i) - T(\chi_f) & \text{if } \zeta_i \geq 0, \\
T(\chi_i) + T(\chi_f) & \text{if } \zeta_i < 0,
\end{cases}
\] (A19)

and now $\chi_f$ is determined by relations
\[
\begin{bmatrix}
\sin \\
\cos
\end{bmatrix} \chi_f = \begin{bmatrix}
\zeta_i \operatorname{sn} \\
\zeta_i \operatorname{cn}
\end{bmatrix} \left[ \frac{\omega}{2} (\sigma(\chi_i) - \pi), k \right].
\]

The Newtonian limit of Eq. (A18) is
\[
t_f - t_i = \frac{2}{u_1^N u_2^N L} \left[ \frac{1}{\sqrt{1 - e^2}} \arctan \left( \frac{\sqrt{1 - e^2}}{e |\sin \vartheta_i|} \right) + \frac{e \sin \vartheta_i}{1 - e^2 \cos^2 \vartheta_i} \right],
\] (A20)

where
\[
\cos \vartheta_i = \Phi - \Psi, \quad \sin \vartheta_i = \eta_i \sqrt{1 - (\Phi - \Psi)^2}.
\]

and functions $\Phi$ and $\Psi$ are defined as in (A14) and evaluated for $u_1 \equiv u_1^N, u_2 \equiv u_2^N$.

\subsection*{A2.2 Case II orbits}

These orbits have no pericentre. They are captured by the black hole in most cases, even though the interaction with the accretion disc can in principle modify the orbital parameters and change the type of the orbit. Consequently, we do not perform the mapping in full detail—we skip the derivation of the time interval $(t_f - t_i)$ which will not be needed for this type of orbits. We still need to describe the complete algorithm for the mapping in the $u$-coordinate and associated $\eta$-function. We were unsuccessful in finding a compact expression for the mapping $u_i \rightarrow u_f(u_i; u_1; u_2)$ similar to that presented in formula (A14), and thus in the following we give the algorithm of the mapping in several steps (well suited for programming).
(i) Evaluate the following quantities:

\[
\mu = \frac{1}{4}(1 - 2u_1), \quad e = \sqrt{3 - l(1 - l^2)},
\]

\[
\delta = \sqrt{(6\mu - 1)^2 + 4\mu^2e^2}, \quad \gamma_\pm = \delta \pm 6\mu - 1, \quad k_\pm^2 = \frac{\delta^2}{2\delta},
\]

\[
\epsilon = \frac{4u_i + 2u_1 - 1}{e(1 - 2u_1)}, \quad \epsilon_c = \frac{1}{1 + \epsilon^2}, \quad \epsilon_s = \begin{cases} \sqrt{1 - \epsilon_c^2} & \text{if } u_i \geq \mu, \\ -\sqrt{1 - \epsilon_c^2} & \text{if } u_i < \mu, \end{cases}
\]

and the angle \(\omega_i \in (-\pi/2, \pi/2)\) which is defined by

\[
\sin^2 \omega_i = 1 - 2\gamma_+^{-1}\epsilon_c [2e\mu\epsilon_s + (6\mu - 1)\epsilon_c],
\]

where \(\text{sgn}(\omega_i) = \text{sgn}(\epsilon_s - k_-)\).

(ii) Exclude the captured trajectories by evaluating

\[
\vartheta_i = [K(k_+) - F(\omega_i, k_+)]\delta^{-1/2};
\]

the particle will be captured by the black hole before reaching the equatorial plane if \(\vartheta_i \leq \pi\) and \(\eta_i = -1\). Otherwise we continue to the following step.

(iii) Define

\[
\lambda_* = F(\omega, k_+) - \eta_i\pi\delta^{1/2}, \quad \zeta_i = \text{sgn}(K(k_+) + \lambda_*),
\]

and

\[
\begin{align*}
\kappa_s \quad \kappa_c \\
\\zeta_i \text{ sn}(\lambda_*, k_+) \\
\text{ cn}(\lambda_*, k_+)
\end{align*}
\]

(iv) Finally, the mapping which we look for is given by

\[
u_f = \mu \left(1 + e\mathcal{T}_{[\text{sgn}(\kappa_s - s)]}\right), \quad \eta_f = \zeta_i \eta_i, \quad (A21)
\]

where we have used

\[
\mathcal{T}_\pm = \gamma_+^{-1}\kappa_c^{-2} \left[2e\mu \pm \sqrt{4\epsilon^2\mu^2 + \gamma_+\kappa_c^2(\gamma_+\kappa_s^2 - \gamma_-)}\right],
\]

27
These orbits are unstable in the sense that they reach the singularity at \( r = 0 \) and thus in the Newtonian limit there are no analogous orbits corresponding to this case.

Eqs. (A14) and (A21) give the mapping for astrophysically interesting cases of geodesics in the Schwarzschild geometry.

**FIGURE CAPTIONS**

**Figure 1.** The sequence of successive intersections with the disc. We plot radial coordinate \( r \) of the intersection on the ordinate and the number of intersections \( N \) on the abscissa. For \( N \gtrsim 5 \times 10^4 \) the upper and lower boundaries of the distribution of intersections in the Figure get closer to each other, which means that the eccentricity of the orbit decreases; simultaneously as the orbit is circularized it is also ground to the plane of the disc. In this case, \( a = 0 \) and \( \alpha_* = 0.9999 \). The initial pericentre distance is 7, eccentricity 0.7, inclination \( I = 103^\circ \). The arrow indicates the moment when the orbit changes its character from retrograde to prograde \((I = 90^\circ)\).

**Figure 2.** As in Fig. 1 but for the second model of the star-disc interaction [Eq. (40)]. Initial pericentre distance is 30 and eccentricity \( e = 0.83 \). Two initial inclinations are compared—(a) \( I = 35^\circ \), (b) \( I = 80^\circ \). The proportionality constant in (40) is taken \( 10^{-5} \). Originally less inclined orbit (a) settles on the circular orbit in the disc with radius of \( \approx 53 \), while the more inclined orbit (b) grinds to the circular orbit at radius \( \approx 17.7 \). We verified, as an example, that keeping the initial pericentre at 30 the results are not very sensitive to the initial eccentricity provided it is \( \gtrsim 0.75 \).

**Figure 3.** The graph (a) shows a similar orbit as are those in Fig. 2, but now the initial trajectory is retrograde with an inclination of 130\(^\circ\). As commented in the text, it is captured by the central black hole. The graph (b) shows the ‘Newtonian analogue’ of (a). Only the ‘adiabatic evolution’ of the apocentre (upper curve) and the pericentre (lower curve) is seen and there is no shift of pericentre.

**Figure 4.** Orbits analogous to those in Fig. 2, but with a nearly extreme
Kerr metric \((a = 0.9981)\). The initial pericentre distance is 15 (the same as for the Fig. 2 orbits if expressed in gravitational radii of the central black hole). The initial inclinations are again chosen to be \(I = 35^\circ\) (a) and \(I = 80^\circ\) (b).
Relativistic precession of the orbit of a star near a supermassive black hole

ABSTRACT

We study the gravitomagnetic (Lense-Thirring) precession of the trajectory (approximated by a geodesic) of a star orbiting a supermassive rotating (Kerr) black hole. We do not assume any particular value for the eccentricity or inclination of the orbit or the angular momentum of the black hole. We also discuss the periodicity related to the relativistic shift of the pericenter.

The Seyfert galaxy NGC 6814 is an example of the object for which effects of relativistic precession could be detectable and we discuss the relevant precession frequency for this case. The remarkably stable phase of several patterns and their position in the light curve impose strong restrictions on the model of this object. We conclude that, according to our present knowledge, a star colliding with an accretion disk is somewhat improbable though not completely excluded as a model of NGC 6814. Our arguments are independent of studies of the period stability.

Subject headings: galaxies: active — galaxies: individual (NGC 6814) — general relativity — black holes

1. INTRODUCTION AND MOTIVATION

It is widely accepted that supermassive black holes (SBH) are located in cores of active galaxies and quasars (Begelman, Blandford & Rees 1984; Shlosman, Begelman & Frank 1990). The mass of the SBH is usually estimated to be in the range $M \approx 10^6$–$10^{11} M_\odot$. Anomalous energy output of active galactic nuclei (AGN) may result from an accretion process: the matter is attracted from the surroundings of AGN or it comes from tidally disrupted stars passing too close to the SBH. The accretion disk is formed and the matter eventually falls onto the black hole. Although this scenario can be called standard, the evidence for both accretion disks and SBH in AGN is only indirect (Frank, King & Raine 1985; Blandford, Netzer & Woltjer 1990; Falcke et al. 1993). The best evidence comes from studies of the central surface brightness of
the nuclei, stellar velocity dispersion, spatial distribution of stars and X-ray emission (Young et al. 1978; Sargent et al. 1978; Binney & Tremaine 1987; Dressler & Richstone 1988, 1990; Kormendy 1988a, b; Halpern & Filippenko 1988; Dressler 1989). General relativistic effects may have important consequences for the axial symmetry and stability of accretion disks (Bardeen & Petterson 1975; Abramowicz 1987). However, the presence of black holes in AGN is largely masked by violent plasma processes in the surrounding medium. Electromagnetic models of energy extraction assume that the SBH rotates (Blandford & Znajek 1977; Macdonald & Thorne 1982; Phinney 1983; Kaburaki & Okamoto 1991; Okamoto & Kaburaki 1991 and references cited therein). In this scenario, the energy of an AGN comes, at least partially, from the rotational energy of the SBH. Evolution of the angular momentum of the black hole under such process was studied by Park & Vishniac (1991). It appears that the angular momentum determines the energy output of the AGN in a nontrivial manner, with the maximum at some particular value (Bičák & Janiš 1985). Unfortunately, it is unclear how the value of the angular momentum of the SBH could be determined by an independent observation. The present paper deals with this problem.

It has been proposed that a star could be captured in a bound orbit around the SBH by the tidal distortion and associated dissipation of energy (Fabian, Pringle & Rees 1975; Frank & Rees 1976; Press & Teukolsky 1977; Lee & Ostriker 1986; Rees 1988), tidal disruption of a binary star (Hills 1988) or a cluster (Novikov, Pethick & Polnarev 1992), or by cumulative effects of interactions with an accretion disk (Syer, Clarke & Rees 1991). Various aspects of star-disk collisions were studied by Ostriker (1983), Zentsova (1983), Syer et al. (1991), Zurek, Siemiginowska & Colgate (1992), Sikora & Begelman (1992) and Vokrouhlický & Karas (1993). We assumed that the SBH forms such a binary system with a low mass star in an orbit inclined with respect to the plane of an accretion disk. The disk is presumably thin and its axis is aligned with the rotation axis of the black hole (Bardeen & Petterson 1975; Kumar & Pringle 1985) but the model can easily be generalized to a more complicated geometry. Radiation from the disk is periodically modulated each time the star crosses the disk. We did not attempt to specify any particular mechanism of the modulation. We also ignored secular changes of the orbital parameters of the star due to collisions with the disk and corresponding contributions of the collisions to the precession frequency. We
concentrated on a gravitomagnetically induced precession of the orbit of the
star. This precession is a general relativistic effect; it occurs if the central
black hole rotates and it becomes important when the star revolves close to
it. In particular we attempted to pick up relevant frequencies in the power
spectrum of the simulated signal which are independent of very complex and
poorly understood details of the star-disk interaction. Given the model of
interaction our approach can easily be adapted. We took into account all the
relativistic effects affecting the motion and energy of photons arriving from
the source to a distant observer.

We assume that the star moves along a geodesic around a Kerr black hole.
The gravitomagnetic effect was originally treated by Lense & Thirring (1918)
in the weak-field limit and by Wilkins (1972) in the case of a spherical or-
bit, $r = \text{const}$, around a black hole with the extreme value of the angular
momentum parameter, $a = 1$. According to our knowledge, a generalization
to an eccentric and inclined orbit around a black hole with arbitrary value
of parameter $a \in (0,1)$ has not yet been discussed in the literature. As a
consequence of the gravitomagnetic effect, orbital nodes are dragged in the
sense of rotation of the black hole. This dragging affects the position of the
source with respect to a distant observer. We try to extract the information
which might help us to determine whether the central SBH rotates or not—
provided other details of star-disk collisions are known. The weak-field limit
of the angular velocity of the gravitomagnetic dragging is

\[ \tilde{\Omega}_{LT} \simeq 4 \times 10^5 \frac{M_\odot}{M} \frac{a}{r^3} \text{s}^{-1}, \tag{1} \]

where $r$ and $a$ are the radius of the orbit and the angular momentum pa-
rameter of SBH measured in dimensionless geometrized units, $r = 6.7 \times
10^{-4}(\tilde{r}/1 \text{ cm})$. Several experiments to confirm the dragging effect in the limit
of a weak gravitational field have been proposed but they have not been
carried out as yet (Everitt 1974; Will 1981; Braginskij, Polnarev & Thorne
1984; Ciufolini 1986). The strong field regime of the spin-orbital interaction
is also intensively investigated for a relatively broad class of gravity theories
within the framework of the parametrised post-Keplerian formalism. The
main applications of this approach are directed at the interpretation of the
binary pulsar data (e.g. Damour & Taylor 1992). Though not yet con-
irmed by direct observations, the gravitomagnetic effect is considered as a
firm consequence of general relativity. In the strong-field region close to the black hole the gravitomagnetic precession depends on four parameters—the pericenter distance of the orbit, eccentricity of the orbit, inclination with respect to the equatorial plane of the black hole, and, of course, the angular momentum parameter. The above mentioned processes of tidal interaction can set the star on an initially very eccentric trajectory and we therefore could not restrict ourselves to circular orbits. Star-disk collisions (Syer et al. 1991; Vokrouhlický & Karas 1993), tidal effects (Press, Wiita & Smarr 1975; Lecar, Wheeler & McKee 1976; Boyle & Walker 1986; Zahn 1977 and 1989; Zahn & Bouchet 1989; Tassoul 1988) and gravitational radiation (Peters & Mathews 1963; Zeldovich & Novikov 1971) tend to gradually circularize the elliptic orbit, however. Naturally, these effects modify the precession frequency, but we assumed that the star is a dwarf or a low-mass compact object and ignored the corrections. We also ignore all possible effects of the magnetic field on the space-time geometry. For the discussion of exact solutions of Einstein-Maxwell equations describing a black hole in a magnetic field cf. Ernst (1976), Karas (1991), Manko & Sibgatullin (1992), and references cited therein.

In other words, we ignored all effects which could induce the precession of the star’s orbit except the gravitomagnetic effect. We believe it is an appropriate approach in solving the problem if these effects (like star-disk interactions or gravitational radiation) are also weak. We will discuss other effects elsewhere so that the whole subject is treated in steps.

We determined the value of the precession frequency (or alternatively the nodal shift per one revolution, \( \delta \phi \)) by direct integration of the geodesic equation in terms of elliptic integrals. Gravitational radiation losses can be neglected provided the change of the energy is small; for a circular orbit with energy \( E \) one obtains the dimensionless estimate (e.g. Rees, Ruffini & Wheeler 1974)

\[
| \dot{E} |_{\text{grav}} \approx 6.4 \left( \frac{M}{r} \right)^5 \left( \frac{M_*}{M} \right)^2 \ll 1, \quad (2)
\]

where \( M_* \) is the mass of the star. Suppose we estimate \( M, r \) and an upper limit on the change of orbital period \( \dot{P} \) from observation. Then equation (2) provides us with the upper limit on the mass of the companion star [cf. Sikora & Begelman (1992) and King & Done (1993) for the application to
Analogously, if $R_\star$ is the radius of the star one can obtain a constraint on the pericenter distance $R_p$ which is based on the tidal limit (Carter & Luminet 1983; Luminet & Marck 1985; Rees 1988):

$$R_p \gg R_\star \left(\frac{M_\star}{M}\right)^{1/3}.$$  \hspace{1cm} (3)

This paper is organized as follows. In Section 2 we derive the azimuthal shift of orbital nodes. An unambiguous value can only be given in the case of spherical trajectories. The shift oscillates between the maximum and the minimum values, $\delta\phi_{\text{max}}$ and $\delta\phi_{\text{min}}$, if the orbit is eccentric. We define a suitable probability distribution for $\delta\phi$ and compute, numerically, a mean value $\langle\delta\phi\rangle$ which determines the gravitomagnetic precession averaged over a large number of revolutions of the star on an eccentric trajectory around the SBH. Corresponding formulas are rather complex and, therefore, we also present a table of numerical values in Appendix. In Section 3 we present simple examples to illustrate how the gravitomagnetic frequency could be extracted from observational data. (In the present contribution, we use simulated data from a simple model rather than real data.) Naturally, the frequency cannot be too small so that the data cover at least several periods if the effect is to be detectable. The estimate of the time interval between successive collisions with the disk [see eq. (20) below] leads us to assume that the star crosses the accretion disk near the horizon (typically a few tens of the gravitational radius of the SBH) in the region where the collisions can modulate the disk radiation in the optical/X-ray bands. Possible physical mechanisms for the modulation were discussed by Mushotzky (1982), Guilbert, Fabian & Ross (1982) and Zentsova (1985). The flux of radiation is modulated by the orbital period (short time-scale) and by the perihelion shift and the Lense-Thirring period (longer time-scale). In order to compute observable effects we consider both the time delay and the focusing of photons coming from the disk region to a distant observer. The focusing effect strongly enhances the influence of the Lense-Thirring precession for observers with large inclination (edge on, with respect to the accretion disk). Finally, we apply the model to the case of the Seyfert galaxy NGC 6814. Periodic modulation of the X-ray emission from this galaxy has been confirmed on the time-scale $\approx 12,200$ s (Mittaz & Branduardi-Raymont 1989; Fiore, Massaro & Barone 1992; Done et al. 1992) and possible mechanisms were proposed by several authors (Abramowicz et
al. 1989 and 1992; Done et al. 1991; Honma et al. 1991; Wallinder 1991 and 1992; Abramowicz 1992; Bao 1992a, b; Sikora & Begelman 1992; Rees 1993; Vio et al. 1993; King & Done 1993).

2. PRECESSION FREQUENCY—DETAILS OF THE CALCULATION

2.1 Gravitomagnetic precession

Geodesic motion in the Kerr space-time can be integrated in terms of elliptic integrals (Carter 1968). The appropriate form can be found in Vokrouhlický & Karas (1993). We use the standard notation for the Kerr metric (Bardeen 1973) and we refer the reader not familiar with the Kerr metric to this reference. We assume that the motion of the star is quasi-elliptic with the pericenter outside the black hole horizon, \( r = R_+ \equiv 1 + \sqrt{1 - a^2} \), and with positive energy, \( 0 < \mathcal{E} < 1 \). The locations of the pericenter \( r = R_p \) and the apocenter \( r = R_a \) coincide with the two upper roots of the polynomial

\[
R(r) = (\mathcal{E}^2 - 1)r^4 + 2r^3 + \left[(\mathcal{E}^2 - 1)a^2 - \Phi^2 - Q\right]r^2 + 2Kr - a^2Q, \tag{4}
\]

where \( \mathcal{E} \equiv -p_t, \Phi \equiv p_\phi \) and \( \mathcal{K} \) are the usual constants of motion, \( Q \equiv \mathcal{K} + (\Phi - a\mathcal{E})^2 \). In our case, all roots of \( R(r) \) are real. We denote them by \( R_a \geq R_p \geq R_3 \geq R_4 \); \( R_3 > R_+ \). Other possible combinations—e.g. two complex conjugated roots—are excluded by our previous assumptions on energy and location of the pericenter above the horizon (Stewart & Walker 1973). We further assume that the star periodically crosses the equatorial plane, \( \theta = \pi/2 \); latitudal motion is restricted by \( |\cos \theta| \leq \mu_- \), where \( \mu_- \) is the lower of the two positive roots \( \mu_\pm \) of the polynomial

\[
\Theta(\mu) = a^2 \left(1 - \mathcal{E}^2\right) \mu^4 - \left[Q + a^2 \left(1 - \mathcal{E}^2\right) + \Phi^2\right] \mu^2 + Q; \tag{5}
\]

For \( r \gg R_+ \) one can interpret \( \arccos \mu_- \) as the inclination of stellar orbit.

A non-equatorial geodesic around a rotating black hole is not planar; intersections with the \( \theta = \pi/2 \) plane are dragged in the sense of rotation. At first, we apply the mapping

\[
[r, \phi, t, \text{sign}(\dot{r})]_n \rightarrow [r, \phi, t, \text{sign}(\dot{r})]_{n+1} \tag{6}
\]

35
which expresses coordinates of the \((n+1)\)-st intersection in terms of coordinates of the previous, \(n\)-th intersection with the equatorial plane. The azimuthal shift of the orbital node \(\delta \phi\) per one revolution is then defined by

\[
\delta \phi = \phi_{n+2} - \phi_n - 2\pi
\]

(\(\phi\)-coordinate is not restricted to \([0, 2\pi]\) in this convention). One can find

\[
r_{n+1} = \frac{(R_a - R_3)R_p + (R_a - R_p)R_3 \sigma}{R_a - R_3 - (R_a - R_p) \sigma},
\]

\[
\phi_{n+1} - \phi_n = [2(aE - \Phi)A_+ + \Phi B_+] I_+ + [2(aE - \Phi)A_- + \Phi B_-] I_- + \Phi J, \tag{9}
\]

\[
t_{n+1} - t_n = \mathcal{E}(J + 2K_r) + 2[B_+ R_+ \mathcal{E} + a(aE - \Phi)A_+] I_+ \\
+ 2[B_- R_- \mathcal{E} + a(aE - \Phi)A_-] I_- + 4\mathcal{E} I_r + a^2 \mathcal{E} K, \tag{10}
\]

We denoted

\[
\sigma = \text{sn}^2(u, k_1),
\]

\[
u = \frac{1}{2} \sqrt{(R_a - R_3)(R_p - R_4)(1 - \mathcal{E}^2) \left(I_\mu - \bar{I}_r\right)},
\]

\[
k_1 = \frac{(R_a - R_p)(R_3 - R_4)}{(R_a - R_3)(R_p - R_4)},
\]

\[
A_\pm = \pm \frac{R_\pm}{R_+ - R_-}, \quad B_\pm = \pm \frac{2R_\pm - a^2}{R_+ - R_-},
\]

\[
R_\pm = 1 \pm \sqrt{1 - a^2},
\]

\[
I_\mu = \frac{2}{a\mu_+ \sqrt{1 - \mathcal{E}^2}} K(\mu_-/\mu_+),
\]

\[
I_\pm = \kappa_\pm [(R_p - R_3)\Pi(\varphi, n_\pm, k_1) + (R_\pm - R_p)F(\varphi, k_1)] + \bar{I}_\pm, \tag{14}
\]

\[
n_\pm = \frac{(R_p - R_a)(R_\pm - R_3)}{(R_a - R_3)(R_\pm - R_p)};
\]

\(F(\varphi, k)\) is the incomplete elliptic integral of the first kind, \(K(k) \equiv F(\pi/2, k)\) is the corresponding complete integral and \(\text{sn}(u, k)\) is the Jacobian elliptic
function \((e.g. \text{Byrd} \& \text{Friedman} 1971; \text{Gradshteyn} \& \text{Ryzhik} 1980)\). Analogously, \(\Pi(\phi, n, k)\) is the elliptic integral of the third kind. In equations (8)–(10),

\[
J_r + 2K_r = \kappa R_p \left[ \left( \frac{R_p \alpha_1^4}{\alpha^4} + 2 \frac{\alpha_1^2}{\alpha^2} \right) U + 2 \frac{\alpha^2 - \alpha_1^2}{\alpha^2} V_1 \right. \\
+ 2R_p \frac{\alpha^2 - \alpha_1^2}{\alpha^4} V_1 + R_p \frac{(\alpha^2 - \alpha_1^2)^2}{\alpha^4} V_2 \bigg] + \bar{J}_r + 2\bar{K}_r,
\]

(15)

with \(U, V_1\) and \(V_2\) defined by

\[
U = F(\varphi, k_1), \\
V_1 = \Pi(\varphi, -\alpha^2, k_1),
\]

\[
V_2 = \frac{1}{2(\alpha^2 - 1)(k_1^2 - \alpha^2)} \left[ \alpha^2 E(\varphi, k_1) + \left(2\alpha^2 k_1^2 + 2\alpha^2 - \alpha^4 - 3k_1^2\right) V_1 \right. \\
+ \left( k_1^2 - \alpha^2 \right) U - \text{sn}(U, k_1) \text{cn}(U, k_1) \text{dn}(U, k_1) \frac{\alpha^4}{1 - \alpha^2 \text{sn}^2(U, k_1)} \bigg],
\]

and

\[
\alpha^2 = \frac{R_a - R_p}{R_a - R_3}, \quad \alpha_1^2 = \frac{R_3(R_a - R_p)}{R_p(R_a - R_3)}, \\
\sin^2 \varphi = \frac{(R_a - R_3)(r_n - R_p)}{(R_a - R_p)(r_n - R_3)}, \\
\kappa = \frac{2}{\sqrt{(R_1 - R_3)(R_2 - R_4)(1 - \mathcal{E}^2)}}, \\
\kappa_\pm = \frac{\kappa}{(R_\pm - R_3)(R_p - R_\pm)}.
\]

\(E(\varphi, k)\) denotes the incomplete elliptic integral of the second kind, \(\text{cn}(u, k)\) and \(\text{dn}(u, k)\) are the Jacobian elliptic functions,

\[
J_\mu = \frac{2}{a\mu_+ \sqrt{1 - \mathcal{E}^2}} \Pi\left(-\mu_-^2, \mu_-/\mu_+\right), \quad \text{(16)}
\]

\[
K_\mu = \frac{2R_a}{a \sqrt{1 - \mathcal{E}^2}} \left[K(\mu_-/\mu_+) - E(\mu_-/\mu_+)\right] \quad \text{(17)}
\]
and $\Pi(n, k) \equiv \Pi(\pi/2, n, k)$. $\bar{I}_r$, $\bar{I}_\pm$, $\bar{J}_r$ and $\bar{K}_r$, in equations (12), (14) and (15) are the integration constants. They depend on the number of turning points in $r$-coordinate between successive intersections with the equatorial plane ($m$) and on the sign of the radial velocity $[\text{sign}(\dot{r})]$ at the intersections. For example,

$$\bar{I}_r = \kappa [m\bar{K}(k_1) - \text{sign}(\dot{r}_{n+1}) F(\varphi, k_1)].$$

(18)

Further details of the derivation are given in Appendix of Vokrouhlický & Karas (1993) where the orbits under consideration are called the “Case II orbits.”

The above expressions can be considerably simplified in the special case of spherical orbits with $r$-coordinate constant. (It can be seen that $r = \text{const}$ orbits do satisfy the geodesic equation due to integrability and separability of the geodesic motion in the Kerr spacetime; Chandrasekhar 1983). The gravitomagnetic precession of the spherical orbits around an extreme ($a = 1$) Kerr black hole was studied by Wilkins (1972). One can generalize his results to the case of spherical orbits $r \equiv R_s = \text{const}$ with an arbitrary value of the angular momentum parameter, $|a| < 1$. Taking into account equation (9), we obtain

$$(\delta \phi)_{r=R_s} = \frac{4}{a\mu_+ \sqrt{1 - \mathcal{E}^2}} \left\{ \Phi \Pi \left( -\mu_-^2, -\mu_+/\mu_+ \right) + a \left[ (\mathcal{E}(r^2 + a^2) - a\Phi \right] \Delta^{-1} - \mathcal{E} \right\} K(\mu_-/\mu_+) - 2\pi, \quad (19)$$

where $\Delta = r^2 - 2r + a^2$. The corresponding orbital period is

$$P_{r=R_s} = 2 \left[ \frac{r^2 (r^2 + a^2) \mathcal{E} + 2ar (a\mathcal{E} - \Phi)}{\Delta} I_\mu + a^2 \mathcal{E} K_\mu \right], \quad (20)$$

and the precession frequency is

$$(\Omega_{LT})_{r=R_s} = \left( \frac{\delta \phi}{P} \right)_{r=R_s}. \quad (21)$$

Setting $a = 0$ in equations (19)–(20) one arrives at $\delta \phi = 0$ and $P = 2\pi R_s^{3/2}$ which corresponds to Keplerian motion with vanishing nodal shift in the Schwarzschild metric, as expected. On the other hand, if $a \neq 0$ the dominant term in the asymptotic expansion of equation (19) for $r \rightarrow \infty$ coincides
with the well-known result of Lense & Thirring (1918): \( \delta \phi = 2\pi a R_s^{-3/2} \). The dependence of period \( P \) on the inclination \( \mu \) is only weak and equation (20) can be approximated by its value for equatorial orbits:

\[
P \approx 2\pi \left( r^{3/2} \pm a \right),
\]

(22)

where +/- signs correspond to direct/retrograde orbits with respect to the rotating SBH. Analogously, the nodal shift of spherical polar orbits (\( \Phi = 0 \)) can be obtained from equation (19) with \( \mu = 1 \). Polar orbits were studied by Stoghianidis & Tsoubelis (1987).

2.2. Motion of the pericenter

In the present paper we are mainly interested in the frequency of the gravitomagnetic precession. In the data, however, there will also be present a periodicity associated with another relativistic effect—the pericenter shift. We thus need to estimate the corresponding precession frequency \( \Omega_p \). In the Schwarzschild case

\[
\Omega_p = \frac{\delta \phi}{P},
\]

(23)

where

\[
\delta \phi = \frac{4}{\omega} K(k) - 2\pi,
\]

(24)

\[
P = \sqrt{\frac{u_2 + u_1}{u_3 - u_1} (1 - 2u_1) (1 - 2u_2)} \left\{ \frac{4}{u_1} \Pi \left( \alpha^2, k' \right) + \frac{8}{1 - 2u_1} \right. \\
\times \left. \Pi \left( \frac{2u_2 - u_1}{1 - 2u_1}, k' \right) + \frac{1}{u_1^2 \alpha^2} \left( k'^2 - \alpha^2 \right) \left[ \alpha^2 E(k') + \left( k'^2 - \alpha^2 \right) K(k') \right] \right. \\
+ \left. \left( 2\alpha^2 k'^2 + 2\alpha^2 - \alpha^4 - 3k'^2 \right) \Pi \left( \alpha^2, k' \right) \right\}.
\]

(25)

Here,

\[
\omega = \sqrt{1 - 2u_2 - 4u_1}, \quad k^2 = 2(u_2 - u_1)\omega^{-2},
\]

\[
\alpha^2 = 1 - \frac{u_2}{u_1}, \quad k'^2 = \frac{u_2 - u_1}{u_3 - u_1},
\]

where \( u_1 \equiv 1/R_a, u_2 \equiv 1/R_{p} \) and \( u_3 = \frac{1}{2} - u_1 - u_2 \) are roots of the polynomial in \( u = 1/r \):

\[
U(u) = 2u^3 - u^2 + 2\mathcal{L}^{-2}u - (1 - \mathcal{E}^2)\mathcal{L}^{-2}
\]

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The angular momentum \( L \) is defined with respect to the axis perpendicular to the orbital plane of the star, not with respect to the axis of the disk like \( \Phi \) in the Kerr case. This definition is naturally more advantageous in the Schwarzschild case because the orbit is planar.

An analogous effect of the pericenter shift can be expected in the Kerr case, too. Now, however, there is no unambiguous value of \( \Omega_P \) because the orbit does not remain planar and the pericenter shift is mixed up with the gravito-magnetic precession. We calculate \( \Omega_P \) numerically by applying the mapping algorithm (6) for the \( r \)-coordinate over a large number of revolutions of the star around the SBH.

### 2.3. The mean value of the precession frequencies

A specific value of the nodal shift \( \delta \phi \) and corresponding precession frequency can only be associated with spherical orbits. In this case \( \delta \phi \) is specified by equation (19) with three parameters—\( a, \mu_- \) and \( R_s \). The case of a quasi-elliptic orbit with nonzero eccentricity requires knowledge of four parameters. The appropriate parameters are \( a, \mu_- \), pericenter distance \( R_p \), and eccentricity \( e \equiv (R_a - R_p)/(R_a + R_p) \). The nodal shift oscillates between the maximum and the minimum values, \( \delta \phi_{\text{max}} \) and \( \delta \phi_{\text{min}} \). Figures 1 and 2 show the graphs of \( \delta \phi(r) \) for some typical values of the parameters (here \( r \) denotes the radius of the intersection with the equatorial plane). For practical purposes one needs the mean value of the shift which characterizes an average taken over a large number of revolutions. We define

\[
\langle \delta \phi \rangle = \int_{R_p}^{R_a} \delta \phi(x) P(x) \delta \phi dx,
\]

where \( P(r) \) is the probability distribution for the radial coordinate of the intersections of the specified trajectory with the disk. \( P \) is normalized to unity, \( \int P(x) dx = 1 \). We applied the following numerical procedure for evaluating equation (26): (i) at first, for a given set of orbital parameters \( (E, \Phi, Q) \), we constructed the probability distribution by following the course of an ensemble of orbits with identical orbital parameters and randomly chosen initial conditions; (ii) then we applied equation (7) and performed the integration in equation (26). The resulting values of the mean nodal shift are tabulated.
in the Appendix. One can see that the dispersion is often quite small and the mean value is a satisfactory approximation to the current value of $\delta \phi$, except for very close orbits.

Analogously to $\langle \delta \phi \rangle$, we introduce the mean orbital period $\langle P \rangle$ of an eccentric orbit and we calculate $\langle \Omega P \rangle$ by applying the mapping algorithm for the $r$-coordinate over a large number of revolutions of the star around the SBH.

3. NUMERICAL MODELS

3.1. Simple examples

In this Section we will show how the relativistic precession becomes evident in a simulated signal from the source. We take into account general relativistic effects on photons coming from the source to the observer with no approximation. The method for integrating light trajectories in the Kerr metric has been discussed by many authors (Cunningham & Bardeen 1973; de Felice, Nobili & Calvani 1974; Cunningham 1975 and 1976; Asaoka 1989). We employ an efficient approach described by Karas, Vokrouhlický & Polnarev 1992. We do not include the noise component which is of course present in a real signal. The reason is that the form of the noise depends on its particular model and it does not affect the periodicities we are interested in. The way to account for the noise in the synthetic light curve is straightforward, in principle, if one adopts a specific model of its origin. Such a model can be based on mechanisms proposed by Moskalik & Sikora 1986, Chagelishvili, Lominadze & Rogava 1989, Abramowicz et al. 1991, or Baring 1992.

The position of a distant observer is characterized by his inclination $\theta_o$ with respect to the symmetry axis. We assume that the star-disk collisions modulate the disk radiation at the moment when the star crashes through the disk from the far side to the near side with respect to the observer, i.e. once per revolution. The light travel time to the observer depends on the location of the intersection and it is thus periodically affected by the precession. Frequencies corresponding to the periodic modulation of the signal are evident in the Fourier transform of arrival times.

The lensing effect enhances the radiation when the source is behind the black hole and thus contributes to the periodic modulation of the signal. As an
alternative to previously described Fourier transform of arrival times, one can detect relevant frequencies in the power spectrum of the photometric radiative flux. However, in this alternative approach the input signal for the Fourier transform depends on the model of the star-disk interaction (see below). We assume, for simplicity, that the shape of the observed signal from successive collisions has always an identical profile: a sharp onset and then an exponential decrease of the local luminosity in the static frame or, alternatively, in the disk co-rotating frame. In the latter case both the lensing and the Doppler effect contribute to the signal strength. Figure 3 is an example of typical light curves. We assumed a spherical orbit, \( r = 6 \), around a nearly-extreme Kerr black hole. It is not clear whether such a close orbit is astrophysically realistic because the process of capture is not well understood (Rees 1993). A more plausible configuration is treated in the next paragraph.

The basic frequency in the power spectrum of the observed signal corresponds to the orbital motion. One can detect two fundamental lines in the power spectrum. The first line at higher frequency \( \Omega_o = 2\pi/P \), see equation (20) \( \) corresponds to the orbital motion and the second one at lower frequency \( \Omega_{LT} \), see equation (21) \( \) corresponds to the gravitomagnetic precession. Figure 4 shows the normalized power spectrum of the light curve from the previous Figure, strictly speaking the coefficient of spectral correlation as defined by Ferraz-Mello (1981). The power spectrum contains also linear combinations of fundamental frequencies.

The shape of the light curve naturally depends on the particular model and position of the observer, however, we expect that identical periodicities caused by the precession of the orbit will be present in the signal independently of the details. The power spectrum constructed from arrival times has a simple form. In Figure 5 we applied the Fourier transform on time intervals elapsed between successive flares in the light curve. Now, the shape of the spectrum is determined by orbital parameters of the star and angular momentum of the SBH but it depends only very weakly on other details of the model—decay time of the flare, viewing angle of the observer, etc. This advantageous property is easy to understand because the arrival times of successive flares are directly related to the precession of the orbit and we thus avoid the model-dependent features in the light curve.
As mentioned above, several effects tend to circularize the orbit of the star and for this reason spherical orbits are of special interest. However, the star can initially be captured in an eccentric orbit. Therefore, we also considered the case of quasi-elliptic orbits ($e \neq 0$). The pericenter shift reveals itself as another line in the power spectrum. In Figure 6 one can detect lines corresponding to the orbital motion ($\Omega_o$), the gravitomagnetic precession ($\Omega_{LT}$), and the pericenter shift ($\Omega_P$). The gravitomagnetic precession disappears in the Schwarzschild case and the power spectrum of an eccentric orbit is then rather similar to the case of a spherical orbit near a Kerr black hole. To distinguish between the both extreme cases one needs an independent estimate of the radius and eccentricity of the orbit (Figure 7).

3.2. The case of NGC 6814

At present there is no generally accepted explanation of the periodicity observed in the X-ray flux from the Seyfert galaxy NGC 6814. Among the most promising models are those which relate the periodicity to the orbital motion of some object (unspecified for the time being) around the SBH. These models can in a natural way deal with the enormous stability of the period over several years. (For a discussion of different mechanisms which have been proposed in the literature see, e.g., Abramowicz 1992.) The model of a star colliding with an accretion disk has been considered as a viable model (Syer et al. 1991; Sikora & Begelman 1992; Rees 1993). The basic periodicity at approximately 12,200 s is interpreted as the orbital period of the star. It is a straightforward conclusion that if the SBH rotates then another periodicity associated with the gravitomagnetic effect should be present in the signal. This conclusion is independent of the unknown details of the star-disk interaction and it only assumes a nonzero viewing angle. Data which are currently available do not cover sufficiently long interval of time to reveal additional periodicities besides the basic one. For this reason we were unable to give a definitive conclusion about the general relativistic precession in NGC 6814. This could be changed if data from ROSAT also show up the periodic component. However, already now one can see that the model of star-disk interactions faces serious difficulties (see below). Considering the time scale of the periodicity of NGC 6814 and the estimate of the black hole mass one can write

$$\tilde{P} = 4.038 \times 10^{-4} M_6 \langle P \rangle,$$
\[ \tilde{P}_{LT} = \tilde{P} \frac{2\pi}{\langle \delta \phi \rangle}, \]

where \( \tilde{P} \) and \( \tilde{P}_{LT} \) are measured in units of 12,200 s and \( M_6 \equiv M/(10^6 M_\odot) \). Assuming \( e \approx 0 \), \( \tilde{P} \approx 1 \) and \( M_6 \approx 1 \), one can estimate the radius of the orbit to be

\[ R_s = (394 \mp a)^{2/3} \approx 53.7. \]

The periods associated with the gravitomagnetic precession and the pericenter shift are thus substantially longer than the orbital one, although one should remember that the estimate of the mass of the SBH in NGC 6814 and consequently the above value of \( R_s \) are rather uncertain (Bao 1992b); \( \langle \delta \phi \rangle \) can be estimated from the tables which are given in Appendix.

We constructed the predicted light curve in the way described in Fig. 3 and compared main frequencies in the power spectrum with those in observational data. The eccentricity was assumed to be limited to a very small value by the effects of orbital circularization; the low eccentricity is also required by the phase stability of the light curve, as discussed below. As a consequence, the dependence of \( \Omega_P \) on \( e \) can be ignored. Also the inclination of the orbit is a free parameter but the dependence of all relevant frequencies on \( \mu_- \) is very weak. One can see that the period related to the pericenter shift is in the range \( 16 \tilde{P} \lesssim \tilde{P}_P \lesssim 20 \tilde{P} \) (the lower limit corresponds to \( a = 0 \), the upper limit corresponds to \( a \to 1 \)). Time scales related to both the orbital motion and the pericenter shift are significantly shorter than the one related to the gravitomagnetic precession, \( \tilde{P}_{LT} \gtrsim 215 \tilde{P} \) (indicated value is the lower limit corresponding to \( a \to 1 \)).

We considered the scenario in which the X-ray flares are associated with bright spots in the disk corona occurring in the place where the star crashes through the disk. After their creation the spots subsequently rotate with the disk matter, radiate with decreasing luminosity in their local frame, and eventually disappear after several revolutions. As the spots rotate, their radiation is periodically enhanced by the lensing effect which causes secondary maxima in the light curve. Secondary maxima can also be related to the event of the star crashing through the disk in the direction away from the observer. By comparing the EXOSAT (Mittaz & Branduardi-Raymont 1989; Fiore et al. 1992) and Ginga (Done et al. 1992) folded light curves, Abramowicz et al. (1993) demonstrated that the phase of important peaks in the light

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curve remains remarkably stable. This fact imposes strong restrictions on the model and if it is confirmed one can conclude that the constant phase of individual patterns in the light curve can only be understood if the black hole does not rotate and the orbit of the star is nearly circular. The change in the shape of the light curve is then related to instabilities in the accretion process which affect the structure of the disk. A significant change in the accretion rate is evident from the change of the X-ray flux of NGC 6814 during the interval between the EXOSAT and Ginga observations.

The peaks in the light curve are not regularly spaced. Within the framework of the discussed model it is very difficult to understand this non-regular (but constant in time) distribution of phases of the peaks in the light curve which has been reported at different ranges of energy. We remind that one of the peaks in the light curve is, according to the above model, associated with the creation of a bright spot at the moment of the star-disk collision. Subsequent peaks are gravitationally lensed images of the orbiting spot. These two mechanisms—star-disk interaction and lensing—are necessary because we deal with non-regular spacing of the peaks. The peaks originating by different mechanisms should very probably be distinguishable by comparing light curves at different energies. The distinction is not apparent in data, nevertheless, the model can definitively be ruled out only if the physics of X-ray flares in the disk corona is well understood; this subject is beyond the scope of the present paper. Further restrictions on the model come from the limits on \( \dot{P} \) derived from the studies of the effects of gravitational radiation and star-disk collisions (King & Done 1993). Alternative explanations of the flares are based either on occultations of the central disk regions by matter ejected in the star-disk collisions or interaction of ejected matter with a jet. Within such models we need more data to restrict the angular momentum of the SBH and the eccentricity of the orbit, but these models face even more severe problems with the relatively narrow width of the profile of the peaks in the light curve and they require a precisely tuned inclination of the stellar orbit.

4. CONCLUSIONS

We considered the general relativistic precession of a star orbiting a supermassive black hole and colliding with an accretion disk in the nucleus of an
AGN. We derived formulae for the azimuthal shift due to the gravitomagnetic precession and perihelion shift during the free (geodesic) part of motion between subsequent interactions with the accretion disk. No restrictions on the orbital parameters and the black hole angular momentum were imposed except that they describe a stable bound trajectory around the Kerr black hole. Within the framework of this model our results restrict possible values of the angular momentum of the central black hole. To illustrate the precession effect, we adopted a simplified model of the star-disk interaction and we determined relevant frequencies in the power spectrum of the observed signal. Both types of the relativistic precession which are relevant for our problem—the precession related to the pericenter shift and the gravitomagnetic (Lense-Thirring) precession—expose themselves clearly in the power spectrum. We found the analysis of arrival times conceptually more trivial and at the same time more advantageous than the analysis of the complete photometric curve.

Our conjecture is that the typical character of the power spectrum will remain conserved at some level in astrophysically more realistic models of the interaction. Such an assumption is well-founded provided the orbital parameters of the star are not changed significantly during several periods associated with the precession motion ($\Omega_o \gg \Omega_{LT}, \Omega_P$). More realistic models must include the effects of tidal interactions, gravitational radiation and star-disk collisions on the orbital parameters (the work in preparation).

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APPENDIX

We tabulate the azimuthal shift of orbital nodes due to the gravitomagnetic precession. The three tables correspond to $a = 0.33$ (Table 1), $a = 0.67$
(Table 2), and \( a = 1 \) (Table 3). In the tables, \( R_p \) denotes the pericenter distance in units of the gravitational radius \( R_+ \), \( e \) is the eccentricity of the orbit, and \( \mu_- \) is the parameter characterizing inclination in the asymptotic region, \( r \gg R_+ \) [see equation (3)]. The columns denoted by “+” and “−” correspond to direct (\( \Phi > 0 \)) and retrograde (\( \Phi < 0 \)) orbits, respectively. (The difference naturally disappears for polar orbits which are characterized by \( \mu_- = 1 \) and \( \Phi = 0 \).) The nodal shift is given in the form

\[
\langle \delta \phi \rangle + \frac{(\delta \phi_{\text{max}} - \langle \delta \phi \rangle)}{(\delta \phi_{\text{max}} - \delta \phi_{\text{min}})} \left[ 10^{-4} \text{rad} \right],
\]

where \( \langle \delta \phi \rangle \) is the mean value of the shift and \( \delta \phi_{\text{max}} \), \( \delta \phi_{\text{min}} \) are the maximum and the minimum values of the shift (for details see the text.) In particular, for spherical orbits (\( r = \text{const}, e = 0 \)) we obtain \( \delta \phi_{\text{max}} = \langle \delta \phi \rangle = \delta \phi_{\text{min}} \); In this case, \( \langle \delta \phi \rangle \) is given by equation (19). An ellipsis “…” in Table 3 excludes those combinations of parameters which do not correspond to a time-like geodesic.

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FIGURE CAPTIONS

FIG. 1.—The nodal shift $\delta \phi (r)$ (in radians) per revolution for different eccentricities of the orbit as a function of the radial distance of the intersection with the equatorial plane. Solid lines correspond to direct orbits, dashed lines correspond to retrograde orbits. Numbers given with each curve indicate the eccentricity. The inclination parameter is $\arccos \mu_- = 45^\circ$, $a = 0.9981$. Two values of $\delta \phi$ for given $r$ correspond to $\dot{r} > 0$ and $\dot{r} < 0$, respectively. The case of spherical orbits is denoted by a star, “∗”.

FIG. 2.—As in Fig. 1 but for different inclinations and constant eccentricity. Now, $e = 0.5$ and the numbers indicate the value of $\arccos \mu_-$. In particular, $\arccos \mu_- = 90^\circ$ corresponds to equatorial orbits. Naturally, the curves of direct and retrograde polar orbits ($\arccos \mu_- = 0$) coincide.

FIG. 3.—The numerical simulation of the observed light curve. Radiative flux is given in arbitrary units on the ordinate. The flux is periodically affected by the lensing effect (main peaks) when the source of radiation is behind the black hole. In this example we consider a source of radiation which is located in the equatorial plane in the place where the star intersected the disk. The position of the intersection is affected by the gravitomagnetic precession provided the central supermassive black hole rotates. As a consequence, the places of intersection are dragged in the azimuthal direction; the Figure covers 17 periods. The luminosity of the source is isotropic in the locally static frame and decreases exponentially with e-fold time $\tau$. The position of the observer at infinity is characterized by the viewing angle $\theta_o$; $\theta_o = 0$ is the rotation axis of the black hole. The angular momentum parameter is chosen to be $a = 0.9981$. The star moves in an inclined spherical orbit: $r = 6$, $\mu_- = 0.5$. The three cases correspond to (a) $\theta = 80^\circ$, $\tau = 0.2P$, (b) $\theta = 40^\circ$, $\tau = 0.2P$, (c) $\theta = 80^\circ$, $\tau = 1.5P$ where $P_{r=6}$ is the orbital period given by equation (22). Further details of the model are described in the text.

FIG. 4.—Normalized power spectrum of the light curve from Fig. 3. The coefficient of spectral correlation is on ordinate and the spectral window on abscissa.
FIG. 5.—Normalized power spectrum of the arrival times for the signal from Fig. 3. The axes are as in Fig. 4. One can verify that the typical shape of the spectrum is quite insensitive to the details of configuration in a broad range of parameters.

FIG. 6.—As in Fig. 5. The orbit of the star is now eccentric: $e = 0.1$, $R_p = 6$, $\mu_- = 0.5$. Parameters of the three cases (a)–(c) as before.

FIG. 7.—Calculated light curves for the simple model described in § III a. The case (a) is for the Kerr black hole with $a = 0.9981$ and a circular trajectory of the star at $R_s = 40$ (the value consistent with current estimates for NGC 6814). The long-term modulation of the light curve (three main peaks) is due to the gravitomagnetic precession (angular frequency $\Omega_{LT}$). The case (b) is for a nonrotating black hole and an eccentric trajectory ($e = 0.25$, $R_p = 40$). Now the modulation is due to the pericenter shift and it has a different period which is given by $\langle \Omega_P \rangle$. 
| $R_p$ | $\epsilon$ | 0.  | .2  | .4  | .6  | .8  |
|-------|------------|-----|-----|-----|-----|-----|
|       |            | +   |   -|    |    |    |
| 0.    | 1311 ± 0   | 1441 ± 0 | 1005 ± 69 | 1110 ± 126 | 810 ± 89 | 900 ± 156 | 677 ± 92 | 754 ± 155 |
| .25   | 1313 ± 0   | 1439 ± 0 | 1006 ± 70 | 1109 ± 125 | 812 ± 90 | 898 ± 154 | 678 ± 93 | 754 ± 154 |
| 5     | .5 1321 ± 0 | 1434 ± 0 | 1012 ± 72 | 1104 ± 121 | 816 ± 92 | 894 ± 150 | 682 ± 95 | 749 ± 150 |
| .75   | 1336 ± 0   | 1422 ± 0 | 1024 ± 77 | 1093 ± 114 | 826 ± 98 | 885 ± 142 | 690 ± 101 | 741 ± 142 |
| 1.    | 1381 ± 0   | 1429 ± 0 | 1059 ± 94 | 856 ± 118 | 1059 ± 120 | 715 ± 120 | 614 ± 114 | 614 ± 118 |
| 0.    | 466 ± 0    | 510 ± 0 | 356 ± 10 | 389 ± 15 | 285 ± 13 | 310 ± 20 | 236 ± 14 | 256 ± 20 |
| .25   | 467 ± 0    | 509 ± 0 | 357 ± 10 | 388 ± 15 | 286 ± 14 | 310 ± 20 | 236 ± 14 | 256 ± 20 |
| 10    | .5 469 ± 0 | 507 ± 0 | 359 ± 10 | 387 ± 15 | 287 ± 14 | 309 ± 19 | 238 ± 15 | 255 ± 20 |
| .75   | 474 ± 0    | 503 ± 0 | 362 ± 11 | 383 ± 14 | 290 ± 14 | 306 ± 19 | 240 ± 15 | 253 ± 19 |
| 1.    | 488 ± 0    | 373 ± 13 | 298 ± 16 | 246 ± 17 | 209 ± 17 | 209 ± 17 | 209 ± 17 | 209 ± 17 |
| 0.    | 166 ± 0    | 179 ± 0 | 127 ± 2 | 136 ± 2 | 101 ± 2 | 108 ± 3 | 84 ± 2 | 89 ± 3 |
| .25   | 167 ± 0    | 179 ± 0 | 127 ± 2 | 136 ± 2 | 102 ± 2 | 108 ± 3 | 84 ± 2 | 89 ± 3 |
| 20    | .5 167 ± 0 | 178 ± 2 | 128 ± 2 | 135 ± 2 | 102 ± 2 | 108 ± 3 | 84 ± 2 | 88 ± 3 |
| .75   | 169 ± 0   | 177 ± 0 | 129 ± 2 | 135 ± 2 | 103 ± 2 | 107 ± 3 | 84 ± 3 | 88 ± 3 |
| 1.    | 173 ± 0    | 132 ± 2 | 105 ± 3 | 86 ± 3 | 73 ± 3 | 73 ± 3 | 73 ± 3 | 73 ± 3 |
| 0.    | 120 ± 0    | 128 ± 0 | 91 ± 1 | 97 ± 1 | 73 ± 1 | 77 ± 2 | 60 ± 1 | 63 ± 2 |
| .25   | 120 ± 0    | 128 ± 0 | 91 ± 1 | 97 ± 1 | 73 ± 1 | 77 ± 2 | 60 ± 1 | 63 ± 2 |
| 25    | .5 120 ± 0 | 127 ± 0 | 92 ± 1 | 97 ± 1 | 73 ± 1 | 77 ± 2 | 60 ± 1 | 63 ± 2 |
| .75   | 121 ± 0   | 126 ± 0 | 92 ± 1 | 96 ± 1 | 73 ± 1 | 76 ± 2 | 60 ± 1 | 63 ± 2 |
| 1.    | 124 ± 0    | 94 ± 1 | 75 ± 1 | 62 ± 2 | 52 ± 1 | 52 ± 1 | 52 ± 1 | 52 ± 1 |
| 0.    | 72 ± 0    | 77 ± 0 | 55 ± 0 | 58 ± 1 | 44 ± 1 | 46 ± 1 | 36 ± 1 | 38 ± 1 |
| .25   | 73 ± 0    | 77 ± 0 | 55 ± 0 | 58 ± 1 | 44 ± 1 | 46 ± 1 | 36 ± 1 | 38 ± 1 |
| 35    | .5 73 ± 0 | 76 ± 0 | 56 ± 0 | 58 ± 0 | 44 ± 1 | 46 ± 1 | 36 ± 1 | 38 ± 1 |
| .75   | 73 ± 0    | 76 ± 0 | 56 ± 0 | 58 ± 0 | 44 ± 1 | 46 ± 1 | 36 ± 1 | 38 ± 1 |
| 1.    | 75 ± 0    | 57 ± 0 | 45 ± 1 | 37 ± 1 | 31 ± 1 | 31 ± 1 | 31 ± 1 | 31 ± 1 |
| 0.    | 59 ± 0    | 63 ± 0 | 45 ± 0 | 48 ± 0 | 36 ± 0 | 38 ± 0 | 30 ± 0 | 31 ± 0 |
| .25   | 59 ± 0    | 63 ± 0 | 45 ± 0 | 48 ± 0 | 36 ± 0 | 38 ± 0 | 30 ± 0 | 31 ± 0 |
| 40    | .5 60 ± 0 | 62 ± 0 | 45 ± 0 | 47 ± 0 | 36 ± 0 | 38 ± 0 | 30 ± 0 | 31 ± 0 |
| .75   | 60 ± 0    | 62 ± 0 | 46 ± 0 | 47 ± 0 | 36 ± 0 | 38 ± 0 | 30 ± 0 | 31 ± 0 |
| 1.    | 61 ± 0    | 46 ± 0 | 37 ± 0 | 30 ± 0 | 25 ± 0 | 25 ± 0 | 25 ± 0 | 25 ± 0 |
| 0.    | 50 ± 0    | 52 ± 0 | 38 ± 0 | 40 ± 0 | 30 ± 0 | 32 ± 0 | 25 ± 0 | 26 ± 0 |
| .25   | 50 ± 0    | 52 ± 0 | 38 ± 0 | 40 ± 0 | 30 ± 0 | 32 ± 0 | 25 ± 0 | 26 ± 0 |
| 45    | .5 50 ± 0 | 52 ± 0 | 38 ± 0 | 40 ± 0 | 30 ± 0 | 32 ± 0 | 25 ± 0 | 26 ± 0 |
| .75   | 50 ± 0    | 52 ± 0 | 38 ± 0 | 40 ± 0 | 30 ± 0 | 32 ± 0 | 25 ± 0 | 26 ± 0 |
| 1.    | 51 ± 0    | 39 ± 0 | 31 ± 0 | 25 ± 0 | 21 ± 0 | 18 ± 0 | 18 ± 0 | 18 ± 0 |
| 0.    | 43 ± 0    | 45 ± 0 | 33 ± 0 | 34 ± 0 | 26 ± 0 | 27 ± 0 | 21 ± 0 | 22 ± 0 |
| .25   | 43 ± 0    | 45 ± 0 | 33 ± 0 | 34 ± 0 | 26 ± 0 | 27 ± 0 | 21 ± 0 | 22 ± 0 |
| 50    | .5 43 ± 0 | 45 ± 0 | 33 ± 0 | 34 ± 0 | 26 ± 0 | 27 ± 0 | 21 ± 0 | 22 ± 0 |
| .75   | 43 ± 0    | 44 ± 0 | 33 ± 0 | 34 ± 0 | 26 ± 0 | 27 ± 0 | 21 ± 0 | 22 ± 0 |
| 1.    | 44 ± 0    | 33 ± 0 | 26 ± 0 | 22 ± 0 | 18 ± 0 | 18 ± 0 | 18 ± 0 | 18 ± 0 |
| $R_0$ | $e$ | $0$ | $.2$ | $.4$ | $.6$ | $.8$ |
|-------|-----|-----|-----|-----|-----|-----|
|       |     |     |     |     |     |     |
| 0.    | 0.  | 288.3 ± 348.0 | 221.2 ± 128.0 | 272.3 ± 462.0 | 1787 ± 171.0 | 2229.5 ± 526.0 | 1496 ± 181.0 | 1883 ± 555.0 |
| 0.25  | 1035 ± 1246.0 | 794 ± 21.0 | 950 ± 51.0 | 638 ± 29.0 | 760 ± 65.0 | 529 ± 31.0 | 628 ± 65.0 | 450 ± 31.0 |
| 0.5   | 1050 ± 1233.0 | 806 ± 22.0 | 941 ± 49.0 | 647 ± 30.0 | 1526 ± 62.0 | 536 ± 33.0 | 621 ± 62.0 | 456 ± 32.0 |
| 1.    | 1146 ± 650.0 | 658 ± 9.0 | 451 ± 19.0 | 500 ± 14.0 | 361 ± 12.0 | 298 ± 12.0 | 246 ± 11.0 | 280 ± 19.0 |
|       |     |     |     |     |     |     |     |     |
| Rp  | e → | 0.   | .2  | .4  | .6  | .8  | . | 
|-----|-----|------|-----|-----|-----|-----|---| 
|     |     |      |     |     |     |     | | 
| 0   | ±   | 8538 | ±   | 6548 | ±   | 5322 | ±   | 4496 | ±   | 3909 | ± 2075 | 
| .25 |   | 8623 | ±   | 6609 | ±   | 5370 | ±   | 5796 | ±   | 4394 | ± 2075 | 
| 5   |   | 8882 | ±   | 6799 | ±   | 5520 | ±   | 6862 | ±   | 4523 | ± 2075 | 
| .75 |   | 9341 | ±   | 7151 | ±   | 5808 | ±   | 7407 | ±   | 4065 | ± 2075 | 
| 1   |   | ...  | ±   | 8143 | ±   | 6732 | ±   | 5752 | ±   | 5035 | ± 2075 | 
|     |     | 3066 | ±   | 2366 | ±   | 2262 | ±   | 1601 | ±   | 1373 | ± 526 | 
| .25 |   | 3089 | ±   | 2382 | ±   | 2361 | ±   | 1839 | ±   | 1498 | ± 526 | 
| 5   |   | 3161 | ±   | 2434 | ±   | 2163 | ±   | 1407 | ±   | 1358 | ± 526 | 
| .75 |   | 3300 | ±   | 2535 | ±   | 2044 | ±   | 1460 | ±   | 1260 | ± 526 | 
| 1   |   | 3695 | ±   | 2834 | ±   | 2284 | ±   | 1630 | ±   | 1093 | ± 526 | 
|     |     | 1708 | ±   | 1317 | ±   | 1062 | ±   | 754  | ±   | 566  | ± 526 | 
| .25 |   | 1718 | ±   | 1325 | ±   | 1068 | ±   | 738  | ±   | 566  | ± 526 | 
| 5   |   | 1752 | ±   | 1345 | ±   | 1086 | ±   | 903  | ±   | 696  | ± 526 | 
| .75 |   | 1818 | ±   | 1396 | ±   | 1122 | ±   | 932  | ±   | 696  | ± 526 | 
| 1   |   | 2018 | ±   | 1542 | ±   | 1235 | ±   | 1023 | ±   | 869  | ± 526 | 
|     |     | 1129 | ±   | 870  | ±   | 700  | ±   | 581  | ±   | 500  | ± 526 | 
| .25 |   | 1135 | ±   | 874  | ±   | 703  | ±   | 584  | ±   | 500  | ± 526 | 
| 5   |   | 1154 | ±   | 888  | ±   | 714  | ±   | 592  | ±   | 500  | ± 526 | 
| .75 |   | 1193 | ±   | 915  | ±   | 734  | ±   | 609  | ±   | 500  | ± 526 | 
| 1   |   | 1312 | ±   | 1001 | ±   | 800  | ±   | 660  | ±   | 500  | ± 526 | 
|     |     | 629  | ±   | 506  | ±   | 410  | ±   | 300  | ±   | 235  | ± 526 | 
| .25 |   | 632  | ±   | 509  | ±   | 413  | ±   | 303  | ±   | 235  | ± 526 | 
| 5   |   | 641  | ±   | 512  | ±   | 416  | ±   | 306  | ±   | 236  | ± 526 | 
| .75 |   | 659  | ±   | 507  | ±   | 410  | ±   | 309  | ±   | 236  | ± 526 | 
| 1   |   | 715  | ±   | 545  | ±   | 434  | ±   | 357  | ±   | 302  | ± 526 | 
|     |     | 503  | ±   | 403  | ±   | 302  | ±   | 235  | ±   | 260  | ± 526 | 
| .25 |   | 505  | ±   | 406  | ±   | 306  | ±   | 238  | ±   | 260  | ± 526 | 
| 5   |   | 512  | ±   | 409  | ±   | 310  | ±   | 241  | ±   | 260  | ± 526 | 
| .75 |   | 525  | ±   | 403  | ±   | 315  | ±   | 244  | ±   | 260  | ± 526 | 
| 1   |   | 567  | ±   | 432  | ±   | 344  | ±   | 300  | ±   | 230  | ± 526 | 
|     |     | 415  | ±   | 319  | ±   | 256  | ±   | 211  | ±   | 179  | ± 526 | 
| .25 |   | 416  | ±   | 320  | ±   | 257  | ±   | 212  | ±   | 179  | ± 526 | 
| 5   |   | 422  | ±   | 324  | ±   | 259  | ±   | 214  | ±   | 181  | ± 526 | 
| .75 |   | 432  | ±   | 331  | ±   | 264  | ±   | 218  | ±   | 184  | ± 526 | 
| 1   |   | 464  | ±   | 354  | ±   | 282  | ±   | 232  | ±   | 195  | ± 526 | 
|     |     | 350  | ±   | 269  | ±   | 215  | ±   | 178  | ±   | 150  | ± 526 | 
| .25 |   | 351  | ±   | 270  | ±   | 216  | ±   | 178  | ±   | 150  | ± 526 | 
| 5   |   | 355  | ±   | 272  | ±   | 218  | ±   | 180  | ±   | 150  | ± 526 | 
| .75 |   | 363  | ±   | 278  | ±   | 222  | ±   | 184  | ±   | 150  | ± 526 | 
| 1   |   | 389  | ±   | 297  | ±   | 236  | ±   | 194  | ±   | 163  | ± 526 | 
|     |     | 300  | ±   | 230  | ±   | 218  | ±   | 178  | ±   | 150  | ± 526 | 
| .25 |   | 301  | ±   | 231  | ±   | 218  | ±   | 178  | ±   | 150  | ± 526 | 
| 5   |   | 305  | ±   | 234  | ±   | 216  | ±   | 178  | ±   | 150  | ± 526 | 
| .75 |   | 311  | ±   | 239  | ±   | 210  | ±   | 175  | ±   | 150  | ± 526 | 
| 1   |   | 332  | ±   | 253  | ±   | 201  | ±   | 165  | ±   | 150  | ± 526 |