Homotopy reductions of dg morphism spaces between Rouquier complexes

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Abstract
We study morphism complexes between dg lifts of Rouquier complexes. We recover the transitive system of homotopy equivalences between different lifts corresponding to the same braid, as well as the Rouquier formula. We develop a large scale version of Gaussian elimination for complexes and use it to reduce morphism spaces of the form $\text{Hom}^\bullet(1, F_w)$ where $F_w$ is the Rouquier complex associated with a Coxeter word $w$.

1 Introduction
Rouquier complexes were introduced in [24] as a categorification of the braid group $B_W$ associated with a Coxeter system $(W, S)$. Actions of braid groups on categories appeared in representation theory since [5] or [22], a precise definition was made in [7]. They describe higher symmetries of the categories acted upon and usually give rich information about them. A classical example is the braid action on $D^b(O)$, the bounded derived category of Bernstein-Gel’fand-Gel’fand category $O$ associated to a complex semisimple Lie algebra $\mathfrak{g}$, or, as a geometric counterpart, on $D^b(B)$ the bounded derived category of constructible sheaves over the flag variety $B$ associated to an algebraic group $G$. The Coxeter system considered here is the Weyl group $N_G(T)/T$ associated to a maximal torus $T$, with simple reflections identified by a Borel subgroup $B$. Rouquier pointed out the interest of studying the category of self-equivalences induced by these actions and understand tranformations between them. He introduced the 2-braid group $B_W$, that upgrades the braid group to a category which serves as a model to understand these kind of transformations. This is defined in terms of complexes of Soergel bimodules, i.e. some special bimodules over the ring $R = \text{Sym}(\mathfrak{h}^*)$, where $\mathfrak{h}$ is the geometric realization of the Coxeter system $W$. The 2-braid group categorifies the braid group $B_W$, in the sense that it is a strict monoidal category with objects $F_s$ and $F_{s^{-1}}$ for each generator $s$ of $B_W$ such that $F_s F_{s^{-1}} = 1$ and $F_s F_t \cdots = F_t F_s \cdots$ according to braid relations in $B_W$. So one has a natural functor

$$B_W \rightarrow B_W$$

$$b \mapsto F_b$$

(where $B_W$ is considered as a category with only the identity maps) which Rouquier conjectured to be faithful. This was shown in type $A$ in [17], in simply laced finite type in [4], and in all finite types in [13].
This category also plays an important role in algebraic topology. In \cite{KhovanovRozansky1} Khovanov and Rozansky began the construction of link invariants from Rouquier complexes, that was made precise by Khovanov in \cite{Khovanov} as a triply graded link homology. The idea is that the homology of the complex obtained computing Hochschild (co)homology of a Rouquier complex \( F_b \) is an invariant of the link \( \mathcal{L}_b \) obtained by closing the braid \( b \) corresponding to it (Rouquier complexes come with a cohomological and a polynomial degree, ans Hochschild homology provides a third graduation).

In this paper, we will study morphism spaces between Rouquier complexes, mostly over \( \mathbb{Z} \), using the diagrammatic description of Soergel bimodules (which is actually better behaved in more cases) given in \cite{Soergel}. The idea is to consider the dg setting and see the morphism spaces in the usual homotopy category as the cohomology groups of morphism complexes in the dg category. We will first give a description of such spaces in terms of Elias-Williamson diagrams, and show that they are are free dg \( R \)-modules with bases related to light leaves, introduced by Libedinsky in \cite{Libedinsky}. This will allow us to use some homological algebra techniques (based on the so called Gaussian elimination for complexes) to reduce them. In this way, we are able to recover the classical transitive systems of isomorphisms between decompositions of braid elements, as well as the so called Rouquier formula. Then we find a simpler version of \( \text{Hom}^\bullet(1, F_w) \).

The original dg module has a basis labeled by subexpressions of \( w \), and we manage to build a new model labeled by subwords. This allows to simplify the computation of 0-th Hochschild cohomology of links.

The method consists in some large-scale version of Gaussian elimination for complexes, that allows to reduce subquotients to single objects. To each morphism complex we are interested in, we associate a regular CW complex and we show that the chain complex is homotopy equivalent to a single object if the corresponding CW complex is collapsible, in the sense of algebraic topology. Those CW complex and their combinatorics are somehow reminiscent of the associahedron.

The homotopy category of the Hecke category is also used in \cite{Gaitsgory} as a modular replacement for the mixed derived category of sheaves on the (affine) flag variety. In future work we will apply these same techniques to study morphism spaces between objects in this category that correspond to Wakimoto sheaves in the \( A_1 \) case. These lift the lattice part of the affine Hecke algebra and the understanding of their subcategory is a first step towards a Bernstein presentation of the affine Hecke category. The latter plays a central role in modular representation theory of algebraic groups. For instance, in \cite{Tenenberg} character formulas for simple and tilting modules are deduced from a (conjectured) action of this category on the principal block of the category of representations of reductive algebraic groups in positive characteristic. This action was then established by Ciappara in \cite{Ciappara} via Smith-Treumann theory.

2 Soergel calculus with patches

Let \((W,S)\) be a Coxeter system and \(B_W\) the corresponding braid group. First of all, let us fix some notation, following \cite{Humphreys}. We will call Coxeter word (resp. braid word) an element of the free monoid generated by \( S \) (resp. \( S \cup S^{-1} \)).

Given a braid word \( b = \sigma_1 \sigma_2 \cdots \sigma_n \), with \( \sigma_i \in S \cup S^{-1} \), we call subword
any sequence \( \sigma_i, \sigma_i \cdots \sigma_r \), with \( 1 \leq i_1 \leq i_2 \leq \cdots \leq i_r \leq n \). On the other hand, we call subexpression any sequence \( a = (a_1, a_2, \ldots, a_n) \) where \( a_i \in \{ \sigma_i, 1 \} \). Then different subexpressions could correspond to the same subword. We encode (as in [12]) the information of a subexpression \( a \) in a 01-sequence \( e = (e_1, e_2, \ldots, e_n) \in \{0, 1\}^n \), with \( e_i = 1 \) if and only if \( a_i = \sigma_i \). When the word \( b \) is given, we will still refer to such 01-sequences as subexpressions. We denote the subword associated with a subexpression \( e \) by \( b_e \). We use a similar notation for Coxeter subwords and subexpressions. We define the degree of a subexpression to be the difference, with sign, between the numbers of canceled positive letters and canceled negative letters, that is

\[
\text{deg}_b(e) := |\{i \mid e_i = 0, \sigma_i \in S\}| - |\{i \mid e_i = 0, \sigma_i \in S^{-1}\}|
\]

2.1 Review of the diagrammatic Hecke category

We will now briefly recall the construction of [12]. Let \( k \) be a commutative ring, and fix a realization \( h \) of \( W \), in the sense of [12, §3.1]. That is to say, let \( h \) be a finite rank \( k \)-module, over which \( W \) acts linearly via

\[
s(v) = v - \alpha_s(v)\alpha_s^\vee, \quad \forall v \in h, \forall s \in S
\]

for certain distinguished elements \( \alpha_s^\vee \in h \) and \( \alpha_s \in h^* = \text{Hom}(h, k) \), such that

\[
\alpha_s(\alpha_s^\vee) = 2.
\]

We assume in particular the technical condition [12, (3.3)] to make Jones-Wenzl morphisms well defined, and Demazure surjectivity (see [12, Assumption 3.9]), that is: the maps

\[
h \to k \quad \quad k \to h^*
\]

\[
v \mapsto \alpha_s(v) \quad \quad \phi \mapsto \phi(\alpha_s^\vee)
\]

are surjective for all \( s \in S \). So let \( \delta_s \) denote a chosen element in \( h^* \) such that \( \delta_s(\alpha_s^\vee) = 1 \).

We then consider the polynomial ring \( R = \text{Sym}(h^*) \) and the induced action of \( W \) on it. For each \( s \in S \) let \( \partial_s : R \to R \) be the Demazure operator, defined by

\[
\partial_s(f) = \frac{f - s(f)}{\alpha_s}
\]

Then one can define a graded \( k \)-linear monoidal category \( D_{BS}(W, h) \) in the following way. The objects are generated, under direct sum and tensor product, by the family \( \{B_s\}_{s \in S} \). Given a Coxeter word \( w = s_1s_2\cdots s_n \), let \( B_w \) denote the product

\[
B_{s_1} \otimes B_{s_2} \otimes \cdots \otimes B_{s_n}, \quad (1)
\]

called a Bott-Samelson object, and \( 1 \) the monoidal unit.

To describe morphisms, we associate a color to each simple reflection. By \( k \)-linearity it is sufficient to describe the graded module

\[
\text{Hom}(B_w, B_w)
\]

for Coxeter words \( w \) and \( w^\prime \). We consider the planar strip \( \mathbb{R} \times [0, 1] \), and we arrange colored starting points on the bottom line \( \mathbb{R} \times \{0\} \) according to \( w \) and \( w^\prime \).
ending points on the top line $R \times \{1\}$ according to $w_2$. Morphisms are then given by isotopy classes of decorated colored graphs with boundary inside $R \times [0, 1]$ (see [12, Definition 5.1]) relating boundary points, whose vertices (a part from boundary points themselves) are of the following kinds:

- univalent vertices, or dots, that have degree 1;
- trivalent vertices, with all incoming strands of the same color, that have degree $-1$;
- vertices with $2m$ incoming strands of alternate colors corresponding to two different reflections $s$ and $t$ such that $st$ has order $m$ in $W$. These have degree $0$.

With decorated we mean that one can add, inside any region, i.e. connected component of the complement of a graph, a box labeled by an arbitrary element of $R$. Composition is given by gluing diagrams vertically, and tensor product by gluing them horizontally.

These diagrams undergo several relations (a part from isotopy equivalence) that we will briefly recall now, recommending the reader to see [12, §5.1] for further detail.

The first polynomial relation is the fact that boxes in the same region multiply, in the sense that one can replace two boxes $f_1$ and $f_2$ with the box $f_1 f_2$. This implies in particular that morphism spaces are enriched in $R$-bimodules: the ring $R$ acts on the left (resp. on the right) by adding a box in the left (resp. right) most region of the diagram.

The other polynomial relations are the following

\[ f = s(f) + \partial_s(f) \quad (3) \]

Here, and below, the dashed circles mean that one can make these replacements anywhere in a diagram.
Then one has the following one-color relations.

\[
\begin{align*}
\text{Diagram} & = \text{Diagram} & (4) \\
\frac{1}{3} \cdot \text{Diagram} & = \text{Diagram} & (5) \\
\text{Diagram} & = 0 & (6)
\end{align*}
\]

The two-color relations are (we give the two versions according to the parity of the order \(m\) of \(st\) in \(W\)):

\[
\begin{align*}
\text{Diagram} & = \text{Diagram} \quad \text{or} \quad \text{Diagram} = \text{Diagram} & (7) \\
\end{align*}
\]

and

\[
\begin{align*}
\text{Diagram} & = \text{Diagram} \quad \text{or} \quad \text{Diagram} = \text{Diagram} & (8)
\end{align*}
\]

where the circles labeled JW are the so-called Jones-Wenzel morphisms, certain \(k\)-linear combinations of diagrams that can be described in terms of the 2-colored Temperley-Lieb algebra. We refer the reader to [12, §5.2] for the details. The only property that we will use is the following. Notice that combining relations (7) and (8), we obtain that the composition of two 2\(m\)-valent vertices gives (the picture is for even \(m\))

\[
\begin{align*}
\text{Diagram} & = \text{Diagram} & (4)
\end{align*}
\]

Then one can show that this is the identity (i.e. the morphism given by parallel vertical strands), modulo morphisms that factor through shorter Coxeter words.
There are also three-color relations that we will not detail here. We will simply say that they give equivalences between certain compositions of 2n-valent vertices, corresponding to loops in the graphs of reduced words of longest elements of all finite rank-3 parabolic subgroups.

**Remark 2.1.1.** (see [23]) Actually, all loops in the graphs of reduced words for any element of \(W\) are generated by the rank-3 ones, so it is natural to expect that no relations with more than three colors are needed.

Finally the diagrammatic Hecke category \(\mathcal{D}(h, k)\), that will also be simply denoted by \(\mathcal{D}\), is defined to be the Karoubi envelope of \(\mathcal{D}_{\text{BS}}(h, k)\).

**Remark 2.1.2.** The Grothendieck ring of this category, which is naturally a \(\mathbb{Z}[v, v^{-1}]\)-algebra is isomorphic to the Hecke algebra \(H_W\) associated with the Coxeter system \((W, S)\). Under certain conditions (namely that \(h\) is reflection faithful and \(\text{char}(k) \neq 2\)) this category is equivalent to the category of Soergel bimodules.

**Remark 2.1.3.** (see [21]) When \(W\) is the Weyl group of a Kac-Moody group \(G\) over \(\mathbb{C}\), and \(h = k \otimes_{\mathbb{Z}} X^*(T)\) (where \(T\) is a maximal torus), and \(k\) is a complete local ring (where we need to assume 2 to be invertible in some cases), then \(\mathcal{D}\) it is equivalent to the category of equivariant parity sheaves on the associated flag variety\(^2\) (see [14]), which is also called the geometric Hecke category. Under this equivalence the objects \(\mathbf{1}\) correspond to the direct images of the constant sheaves along the Bott-Samelson resolutions of Schubert varieties, which explains the terminology. The monoidal structure is provided by convolution.

### 2.2 A dg monoidal category of Rouquier complexes

Let us now consider the homotopy category \(\mathcal{K}^b(\mathcal{D})\) of bounded chain complex with objects in \(\mathcal{D}\). For any simple reflection \(s \in S\) one defines the standard complex \(F_s\) by

\[
\begin{array}{c}
\vdots \\
F_s = \cdots \to 0 \to B_s \to 1(1) \to 0 \to \cdots \\
-1 & 0 & 1 & 2
\end{array}
\]

and the co-standard complex \(F_{s^{-1}}\) by

\[
\begin{array}{c}
\vdots \\
F_{s^{-1}} = \cdots \to 0 \to 1(-1) \to B_s \to 0 \to \cdots , \\
-2 & -1 & 0 & 1
\end{array}
\]

where \((-)\) is the shift in the grading of \(\mathcal{D}\), and the numbers at the bottom of each picture denote the cohomological degrees.

**Remark 2.2.1.** These complexes correspond, in the setting of remark \(2.1.3\) to standard and costandard sheaves over Schubert cells associated with the reflection \(s\).

---

\(^2\)In this case one works with the analytic topology of \(G\). One can also take \(G\) to be defined over any algebraically closed field \(K\) and take \(k\) to be the algebraic closure, or a finite extension, of \(\mathbb{Q}_\ell\), or its ring of integer, or its residue field of characteristic \(\ell\), with \(\ell\) prime to the characteristic of \(K\). In this case one works with étale sheaves.
Given a braid word $\underline{b} = \sigma_1 \sigma_2 \cdots \sigma_n$, with $\sigma_i \in S \cup S^{-1}$, let $F_{\underline{b}}$ denote the tensor product $F_{\sigma_1} \otimes F_{\sigma_2} \otimes \cdots \otimes F_{\sigma_n}.$

**Remark 2.2.2.** When the diagrammatic Hecke category is Krull-Schmidt (see [12, Thm 6.26]), we can reduce $F_{\underline{b}}$ to its minimal complex, which has no contractible summands and can be shown to be canonical (see [11, §6.1]). These minimal complexes are normally called Rouquier complexes. Here we will describe the dg category generated by the $F_{\underline{b}}$ themselves, which are then lifts of Rouquier complexes. We will construct in this setting the classical transitive system of homotopy equivalences between different decompositions of braids. More precisely, if $\underline{b}$ and $\underline{b}'$ are different braid words for $\underline{b} \in \mathcal{B}$ then one has a homotopy equivalence $\gamma_{\underline{b}, \underline{b}'}: F_{\underline{b}} \to F_{\underline{b}'}$ such that all the $\gamma_{\underline{b}, \underline{b}'}$ form a transitive system of maps.

Let $C_{\text{dg}}^b(\mathcal{D})$ denote the dg category of chain complexes with objects in $\mathcal{D}$. Recall that this has the same objects as $K^b(\mathcal{D})$ but morphism spaces are given, for any complexes $A^\bullet$ and $B^\bullet$, by the dg $k$-module $\text{Hom}^\bullet(A^\bullet, B^\bullet)$ whose $p$-th graded piece is

$$\text{Hom}^p(A^\bullet, B^\bullet) = \prod_{q \in \mathbb{Z}} \text{Hom}_{\mathcal{D}}(A^q, B^{q+p}),$$

and the differential is

$$\text{Hom}^p(A^\bullet, B^\bullet) \longrightarrow \text{Hom}^{p+1}(A^\bullet, B^\bullet)$$

$$(f^q)_{q \in \mathbb{Z}} \longmapsto (-1)^i f^q - (\delta_B f^{q+1} + \delta_A f^q).$$

By last section we see that $\text{Hom}^\bullet(A^\bullet, B^\bullet)$ has a structure of dg $R$-bimodule.

Hence one can consider the dg monoidal category

$$\mathcal{R}_{\text{dg}} = \langle F_{\sigma} \mid \sigma \in S \cup S^{-1} \rangle_{\otimes, \oplus, [-]} \subset C_{\text{dg}}^b(\mathcal{D}).$$

whose objects are (direct sums of shifts of) the $F_{\underline{b}}$’s. In this section we will describe this category in terms of the Elias-Williamson presentation and then deduce a diagrammatic model for it. Let us start by describing the general object $F_{\underline{b}}$ for $\underline{b}$ a braid word. Here are some examples.

**Example 2.2.3.** For $\underline{b} = st^{-1}$, with $s, t \in S$, the complex $F_{\underline{b}}$ is the following (where red correspond to $s$ and blue to $t$):

```
  +1 1
  |   |
B_s B_t
  +1 1
B_s(-1)   B_t(1)
  +1 -1
  1 0 1
```

**Example 2.2.4.** For $\underline{b} = stu^{-1}$, the complex $F_{\underline{b}}$ is the following (s: red; t: blue; u: green):

```
  +1
  |   |   |   |
B_s B_t B_u
  +1 1 1
B_s(-1) B_t(1) B_u(1)
  +1 -1 -1
  1 0 0 1
```
In general, let \( b = \sigma_1 \sigma_2 \cdots \sigma_n \) with \( \sigma_i = s_i^{\epsilon_i} \) for \( \epsilon_i \in \{\pm 1\} \). By the definition of tensor product of complexes, the complex \( F_b \) will have, as in the examples, the form of a cube with vertices indexed by subsets \( I \) of \( \{1, \ldots, n\} \), and their degree will depend on the exponents \( \epsilon_i \) for \( i \not\in I \). This is encoded precisely in a subexpression \( e \) of \( b \) together with its degree. Let \( w \) denote the Coxeter word \( s_1 s_2 \cdots s_n \), obtained by \( b \) by identifying \( s_1^{\pm 1} \) with \( s_1^{-1} \). Then the degree \( q \) part of \( F_b \) is

\[
(F_b)^q = \bigoplus_{e \in \{0,1\}^n, \deg(e) = q} B_{w,e}.
\]  

The differential map will decompose into morphisms \( d_{e'}^e \) for subexpressions \( e \) and \( e' \) of \( b \). We use the notation \( e \xrightarrow{i} e' \) when \( e' \) is obtained from \( e \) by changing the \( i \)-th symbol from 0 to 1. Then \( d_{e'}^e \), by the form of the \( F_{\sigma_i}'s \), will be a dot in those cases when a single change increases the degree by 1. More precisely:

\[
d_{e'}^e = \begin{cases} 
(1)^{\leq i} 
\begin{array}{c}
\cdots \\
B_{w,e'} \\
\cdots \\
\cdots \\
B_{w,e} \\
\end{array}
\end{cases} & \text{if } e \xleftarrow{i} e' \text{ and } \sigma_i \in S \\
(1)^{\leq i} 
\begin{array}{c}
\cdots \\
B_{w,e'} \\
\cdots \\
\cdots \\
B_{w,e} \\
\end{array}
\end{cases} & \text{if } e \xrightarrow{i} e' \text{ and } \sigma_i \in S^{-1} \\
0 & \text{otherwise}
\end{cases}
\]  

(of course the strands should be colored according to the letters in \( w'^e \)). Its sign is determined according to the number \( z_{<i} \) of 0’s in \( e \) before \( i \).

If we consider two braid words \( b_1 \) and \( b_2 \), then the morphism space

\[
\text{Hom}^*(F_{b_1}, F_{b_2})
\]

decomposes into its graded pieces \( \text{Hom}^p(F_{b_1}, F_{b_2}) \) with \( p \in \mathbb{Z} \), and each of these,
according to \((10)\), decomposes in turn into a direct sum of spaces of the form

\[
\text{Hom}_\\mathcal{D}(B^w_{e_1}, B^w_{e_2})
\]

\[
\deg_{B^w_{e_1}}(e_1) = \eta
\]

\[
\deg_{B^w_{e_2}}(e_2) = \eta + p,
\]

where \(w_1\) and \(w_2\) are obtained as above from \(b_1\) and \(b_2\).

The formula for the differential is deduced from \((9)\) and one can find that, over an element \(f\) in the space \((12)\), this is

\[
d(f) = \sum_{e'_2 \in \{0,1\}^n} \left( (d_2 e'_2 \circ f) - (-1)^p \sum_{e'_1 \in \{0,1\}^n} (f \circ (d_1 e'_1)\right),
\]

where the differentials \((d_i)_{e_i}^{e'_i}\) are those of \(F^w_b\) (for \(i = 1, 2\)).

### 2.3 Diagrams with patches

We can now extend the diagrammatic description of \(\mathcal{D}\) to one for \(\mathcal{R}_{dg}\). The discussion in \(\S\) 2.2 implies that we can describe morphisms as follows:

1. Elements of \(\text{Hom}^* (F^w_{b_1}, F^w_{b_2})\) are linear combinations of diagrams obtained with the following procedure:
   - Consider the planar strip \(\mathbb{R} \times [0,1]\) and arrange on the bottom (and on the top) boundary some starting (resp. ending) colored points corresponding to braid letters appearing in the source (resp. target) object;
   - Choose an arbitrary subset of the boundary points and cover them by patches;

\[s \quad t^{-1} \quad u\]

\[
\begin{array}{cccc}
  s & & s & & t^{-1} & & u^{-1} & & t \\
\end{array}
\]

\[+ \quad - \quad +\]

\[+ \quad + \quad - \quad - \quad +\]

**Remark 2.3.1.** This corresponds to choosing a summand of a graded piece of the source and of the target: the subexpressions \(e_1\) and \(e_2\)
of (12) correspond to the arrangements of patches (0’s correspond to patches). In the example we chose $B_s B_t B_t \subset (F_{s+t-1-u} - 1)^0$, corresponding to $e_1 = 10101$ and $B_s B_u \subset (F_{s+t-1-u} - 1)^{-1}$, corresponding to $e_2 = 101$.

- Draw an ordinary Elias-Williamson diagram using points not covered by patches and ignoring signs (but still keeping track of them).

![Diagram]

**Remark 2.3.2.** These diagrams are identified under the same relations as in standard Soergel calculus and patches do not interact with the rest.

2. The cohomological degree of a diagram, that will be denoted by deg, is the following difference

$$+ \# \quad + \# \quad \# \quad \# \quad - \# \quad - \# \quad +$$

So the morphism in our example has degree $-1$.

3. For the differential, consider the following positive *dot-and-patch* diagram and negative *patch-and-dot* diagram:

![Diagram]

The image by the differential map of a diagram will then be the sum, with appropriate signs, of all diagrams obtained from it by one of the following operations (that correspond to formula (13) and the description of the differential (11)):

- i) Attach a (positive) dot-and patch on top of a positive ending point not covered by a patch;
- ii) Attach a (negative) patch-and-dot on top of a negative ending point covered by a patch;
- iii) Attach a (positive) dot-and-patch on bottom of a positive starting point covered by a patch;
- iv) Attach a (negative) patch-and-dot on bottom of a negative starting point not covered by a patch.
Then we eliminate all closed patches as follows:

\[ (14) \]

For example let us see what the image by the differential map of our diagram is:

\[
\begin{pmatrix}
+ & - & + \\
+ & - & + \\
+ & - & + \\
- & + & - \\
+ & - & + \\
+ & - & + \\
+ & - & + \\
- & + & - \\
\end{pmatrix}
= \begin{pmatrix}
+ & - & + \\
- & + & - \\
+ & - & + \\
+ & - & + \\
+ & - & + \\
+ & - & + \\
+ & - & + \\
- & + & - \\
\end{pmatrix}.
\]

which is equal to

\[
\begin{pmatrix}
+ & - & + \\
- & + & - \\
+ & - & + \\
+ & - & + \\
+ & - & + \\
+ & - & + \\
+ & - & + \\
- & + & - \\
\end{pmatrix}
= \begin{pmatrix}
+ & - & + \\
+ & - & + \\
+ & - & + \\
+ & - & + \\
+ & - & + \\
+ & - & + \\
+ & - & + \\
+ & - & + \\
\end{pmatrix}.
\]

The signs are established via the following rule. Consider the \( j \)-th ending point (counting from the left): we call, as before, \( z_{k,j}^{1} \) (where the “1” stands
for “top”) the number of patches (ignoring the signs) over ending points strictly on its left. When attaching a dot-and-patch or a patch-and-dot on top of the \(j\)-th ending point, the sign will be \((-1)^{z^b_j}\). Consider instead the \(i\)-th starting point: we call \(z^b_{\geq i}\) (“b” for “bottom”) the number of patches over starting points beginning from it and to the right. Suppose also that the diagram has \(z^t\) patches on top. Then, when attaching a dot-and-patch or a patch-and-dot on bottom of the \(i\)-th starting point, the sign will be \((-1)^{z^b_{\geq i} + z^t + 1}\).

**Remark 2.3.3.** One can see that this sign rule is obtained from the signs in formulas (11) and (13).

4. Composition of morphisms is given by gluing diagrams vertically with the following additional rules:
   - if patches do not match then the composition is zero;
   - if they match then we apply rule (14).

5. Tensor product of morphism is given as usual by gluing diagrams horizontally, but we have to add a sign in order to satisfy Koszul rule. More precisely the tensor product of the diagrams \(D_1\) and \(D_2\) will have the sign \((-1)^{pq}\) where \(p\) is the number of lower patches present in \(D_1\) and \(q\) is the degree of \(D_2\).

For example, for any \(F_b\) the identity morphism is given by the sum of all possible arrangements of vertical strands and vertical pairs of patches, as follows:

```
  \[ \ldots + \ldots + \ldots + \ldots + \ldots \]
```

2.4 Double leaves with patches

We want now to describe \(R\)-bases for morphism spaces \(\text{Hom}^* (F_{b_1}, F_{b_2})\), seen as left \(R\)-modules. By the results of Libedinsky in [18], described diagramatically in [12], an \(R\)-basis of each space [12] is given by double leaves. We will briefly recall their construction in the case \(b_2\) is the empty word. This actually contains all the information about the general case by adjunction (see later).

First recall the notion of decoration of a subexpression \(f\) of a Coxeter word \(w = s_1 s_2 \cdots s_n\). We consider for each \(i\) the subword \(w_{\leq i} = s_1 s_2 \cdots s_i\) and the corresponding \(f_{\leq i}\). Finally we call \(w_{\leq i}\) the element of \(W\) corresponding to the word \(w_{\leq i}^{f_{\leq i}}\). The decoration of \(f\) goes as follows: for all \(i\) we decorate \(f_i\) with a \(U\) (for “up”) if either \(i = 1\) or \(\ell(w_{\leq i-1} s_i) > \ell(w_{\leq i-1})\) (ignoring the actual value of \(f_i\)), and with a \(D\) (for “down”) otherwise.

Then, consider a Coxeter word \(w\) and a subexpression \(f\) such that \(w^f\) reduces to the identity element of \(W\), we can construct a morphism \(L_w^f : B_w \to 1\). The construction will depend on some choices. We define \(\lambda_0\) to be \(\text{id}_2\). Then, suppose we have constructed a morphism \(\lambda_i : B_{w_{\leq i}} \to B_{w_{\leq i}}\), where \(w_{\leq i}\) is some reduced
word corresponding in $W$ to the same element as $w_{\leq i}^{f_{\leq i}}$. Then $\lambda_{i+1}$ is defined by one of the following diagram, according to the decorated value of $f_{i+1}$:

\[
\begin{array}{ccc}
\lambda_i & f_{i+1} = U0 \\
\lambda_i & f_{i+1} = U1 \\
\lambda_i & f_{i+1} = D0 \\
\lambda_i & f_{i+1} = D1
\end{array}
\]

In the last two cases $\phi$ is a chosen composition of $2m$-valent vertices corresponding to a sequence of braid relations transforming $w_{\leq i}$ in some reduced word ending with $s_i$. Then the ending word of $\lambda_{i+1}$ is a reduced word $w_{i+1}$ corresponding to the same element as $w_{\leq i+1}$. Hence the induction can continue.

We define $L_{w, f}$ to be $\lambda_n$. By construction it is a morphism in $\text{Hom}_\mathcal{D}(B_{\leq i}, 1)$.

Now take $b$ a braid word and consider $w$ to be its associated Coxeter word as before. Then by the decomposition (12), an $R$-basis of $\text{Hom}^*(F_b, 1)$ is labeled by pairs $(e, f)$ where $e$ is a subexpression of $b$ and $f$ is a subexpression of $b^e$. Notice that, extending $f$ with 0's where $e$ is 0, we can simply encode the above data in a pair $(e, f)$ of subexpressions of $b^e$ with $e \geq f$ and such that $w_f^e$ reduces to the identity element in $W$. These pairs label the basis of light leaves with patches. For the sake of brevity, in the next sections we will refer to the latter simply as light leaves and we will specify without patches, when we need to refer to the classical ones.

Remark 2.4.1. For future use, notice that the decoration of the extended $f$ is compatible with the preceding one: the additional 0's do not change the elements $w_{\leq i}$. We are just “pausing to admire the scenery” a little bit more during our “gentle stroll” ([12 §2.4]).

3 Rouquier complexes

In this section we will use our diagrammatic dg language to state some fundamental properties of the category $\mathcal{R}_{dg}$ and reconstruct some known features of Rouquier complexes. Namely we will build the transitive system of homotopy equivalences mentioned in remark 2.2.2 and we will give a diagrammatic proof of the so called Rouquier formula.

3.1 Bi-adjunction and inverses

An important feature of the object $B_s$ in $\mathcal{D}$, pointed out in [12], is self-biadjunction. This means that we have natural isomorphisms

\[
\begin{align*}
\text{Hom}_\mathcal{D}(B_s, -) & \cong \text{Hom}_\mathcal{D}(-, B_s) \\
\text{Hom}_\mathcal{D}(-B_s, -) & \cong \text{Hom}_\mathcal{D}(-, -B_s)
\end{align*}
\]
given by the following units and counits (for both adjunctions)

\[ 1 \rightarrow B_s B_s \]

\[ B_s B_s \rightarrow 1 \]

**Remark 3.1.1.** In the language of Soergel bimodules, this is a consequence of the fact that \( B_s \) is a Frobenius algebra object in the category of \( R \)-bimodules. In the geometrical setting this is the self-biadjunction of the indecomposable parity sheaf corresponding with the reflection \( s \), which coincides with the simple perverse sheaf \( IC_s \).

Then, either by purely homological algebra arguments or by the geometric interpretation, it is natural to expect a bi-adjunction between standard and co-standard objects \( F_s \) and \( F_{s^{-1}} \). This can actually be formulated at the dg level. Consider the pairs of maps \((\epsilon^+, \eta^+)\) given by

\[ \epsilon^+ : 1 \rightarrow F_s F_{s^{-1}} \]

\[ \eta^+ : F_{s^{-1}} F_s \rightarrow 1 \]

and \((\epsilon^-, \eta^-)\) given by

\[ \epsilon^- : 1 \rightarrow F_{s^{-1}} F_s \]

\[ \eta^- : F_s F_{s^{-1}} \rightarrow 1 \]

It is easy to check that both pairs satisfy the zig-zag equations. This gives, for any \( \sigma \in S \cup S^{-1} \) the following natural isomorphisms of dg functors

\[ \text{Hom}^\bullet(F_{\sigma^{-1}}, -) \cong \text{Hom}^\bullet(-, F_{\sigma^{-1}}) \]

\[ \text{Hom}^\bullet(-, F_{\sigma^{-1}}) \cong \text{Hom}^\bullet(-, -F_{\sigma^{-1}}) \]

Hence any morphism space \( \text{Hom}^\bullet(F_{\alpha}, F_{\beta}) \) is isomorphic to \( \text{Hom}^\bullet(F_{\alpha^{-1}}, F_{\beta^{-1}}) \), as well as \( \text{Hom}^\bullet(F_{\alpha^{-1}}, F_{\beta^{-1}}) \), where the inverse \( \beta^{-1} \) is taken in the free monoid of braid words.

The above units and counits actually allow to show that \( F_s \) and \( F_{s^{-1}} \) are inverse to each other. This was shown in [22] for the corresponding self-equivalences of the derived category \( \mathcal{O} \) and in [24] in terms of Soergel bimodules. The following proposition gives a diagrammatic description of this result.

---

\[ ^3 \text{Here we define an adjunction } E \dashv F \text{ in a dg monoidal category as the data of a closed degree zero unit } 1 \rightarrow FE \text{ and a closed degree zero counit } EF \rightarrow 1 \text{ satisfying the usual zig-zag equations, i.e. such that the compositions} \]

\[ F \rightarrow FEF \rightarrow F \quad E \rightarrow EFE \rightarrow E \]

\[ \text{are the identities of } F \text{ and } E. \]
Proposition 3.1.2. For any \( s \in S \) the following morphisms give mutually inverse homotopy equivalences

\[
F_s F_{s^{-1}} \xrightarrow{-\eta^-} 1[0]
\]

and

\[
F_{s^{-1}} F_s \xrightarrow{-\epsilon^+} 1[0]
\]

Proof. We treat the first one, the second being similar. One can easily compute

\[-\eta^- \epsilon^+ = \text{id}_{1[0]}\]

from the rules of composition of diagrams with patches. On the other hand we have

\[
\text{id}_{F_s F_{s^{-1}}} + \epsilon^+ \eta^- = d \begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix} + \begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}
\]

which one can see by a simple computation, using the equalities

\[
\begin{align*}
\text{id}_{s} \circ \text{id}_{s} &= \text{id}_{s}, \\
\text{id}_{s} \circ \text{id}_{s} &= \text{id}_{s}, \\
\end{align*}
\]

that follow from (3), (4) and (5).

3.2 Braid relation

The homotopy equivalence

\[
\underbrace{F_s F_t F_s \ldots}_{m} \cong \underbrace{F_t F_s F_t \ldots}_{m}
\]

for \( s \) and \( t \) simple reflections with \( st \) of order \( m \), was shown in [24] in the language of Soergel bimodules. In [9] one can find an explicit diagrammatic description in type \( A \).

Here we will reduce the complexes of morphisms in the dg setting to find out some canonical morphisms and show that they give homotopy equivalences.

Then, together with the equivalences of the preceding section, we will build the desired transitive system of homotopy equivalences.

Consider two reduced Coxeter words \( w_1 \) and \( w_2 \), corresponding to the same element \( w \) of \( W \), and call \( N^{w}_{w_1} \subset \text{Hom}^* F_{w_1} F_{w_2} \) the subcomplex spanned by all morphisms factoring through a shorter word: one can in fact see that this property is preserved by the differential. Then we have the following two results.
**Proposition 3.2.1.** We have a short exact sequence of complexes

\[ 0 \rightarrow N_{w^2} \rightarrow \text{Hom}^\bullet(F_{w^1}, F_{w^2}) \rightarrow R[0] \rightarrow 0 \]

*Proof.* In fact any morphism can be written as an \( R \)-linear combination of diagrams from \( N_{w^2} \) and some diagrams not factoring through shorter words. The latter are actually all the same modulo morphisms in \( N_{w^2} \), via cellularity of the category \( \mathcal{D} \) (see [12, §6.4]). This gives the claim. \( \square \)

**Proposition 3.2.2.** The complex \( N_{w^2} \) is nullhomotopic.

For the proof we will need some technical lemmas. Given a braid word \( b \), let \( C_b = \text{Hom}^\bullet(F_b, \mathbb{Z}) \) Recall from §2.4 that the a light leaf with patches is determined by a pair \( (e, f) \) with \( e \geq f \). We consider the decoration of \( f \). We call an independent boundary point a positive starting point that corresponds to some \( U0 \) in \( f \). According to the corresponding value (0 or 1) in \( e \), over independent points we will have either patches or boundary dots. Let \( N_b \) denote the span of light leaves that contain independent boundary points.

**Lemma 3.2.3.** The complex \( N_b \) (where the differential map is just induced by the one of \( C_b \)) is null-homotopic.

*Proof.* We proceed by induction on the cohomological length of the \( N_b \). When this is 0, there is nothing to prove. Otherwise, consider, for each light leaf its full version, obtained by replacing every independent patch by the corresponding dot. Now consider the subcomplex spanned by the light leaves whose full version has maximal degree: that is a subcomplex because the differential map operates there just over independent points. For the same reason this subcomplex is isomorphic to a direct sum of complexes of the form \( (R \rightarrow R)^{\otimes 3} \), hence null-homotopic. For example, the subcomplex spanned by the light leaves whose full version is

![Diagram](image)

is the following

![Diagram](image)

which is isomorphic to \( (R \rightarrow R)^{\otimes 3} \).

We can then conclude by induction and the following lemma. \( \square \)
Lemma 3.2.4. Let \( \mathcal{C} \) be any \( k \)-linear category and let

\[
0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0
\]  

be an exact sequence in the category of chain complexes \( \mathcal{C}(\mathcal{C}) \). If the complex \( A \) (resp. \( C \)) is null-homotopic, then the map \( B \rightarrow C \) (resp. \( A \rightarrow B \)) is a homotopy equivalence.

Proof. We prove the homotopy equivalence between \( B \) and \( C \) (the case for \( A \) and \( B \) is analogous): consider the triangulated structure on the homotopy category \( \mathcal{K}(\mathcal{C}) \). The short exact sequence (15) induces a distinguished triangle

\[
A \longrightarrow B \longrightarrow C \quad \text{or} \quad 0
\]

Now, by the contractibility of \( A \), we have the following morphism of distinguished triangles

\[
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
0 & \longrightarrow & C
\end{array}
\]

which implies that the middle vertical arrow is an isomorphism in the homotopy category, i.e. a homotopy equivalence of complexes.

Proof of 3.2.2. By adjunction we can consider the complex \( C_{w}^{w_{1}-1} \) and notice that \( N_{w}^{w_{1}} \) corresponds precisely to \( N_{w_{1}}^{w_{1}-1} \) from above. In fact all light leaves corresponding by adjunction to morphisms that factor through shorter words have some independent boundary point in some (positive) starting point of \( w_{1} \).

As a byproduct of these techniques, we can now prove in our language the so-called Rouquier formula (conjectured in [25], and proved in [19], and in [20])

Corollary 3.2.5. Let \( w \) and \( v \) be elements of \( W \), and let \( \underline{w} \) and \( \underline{v} \) be reduced words corresponding to them. Then

\[
\text{Hom}^{\bullet}(F_{\underline{w}}, F_{\underline{v}^{-1}}) \simeq \begin{cases} R[0] & \text{if } w = v^{-1} \\ 0 & \text{otherwise} \end{cases}
\]

Proof. Applying bi-adjunction we can study \( \text{Hom}^{\bullet}(F_{\underline{w}}, R) \) and suppose that \( \ell(w) \geq \ell(v) \). Now, if \( wv \) is not the identity, then every subexpression corresponding to the identity must contain a 0 and there has to be at least one in \( w \). Then consider the first 0 from the left. It has to be a \( U_{0} \) because \( w \) is reduced. Hence the complex is null-homotopic by 3.2.3.

If instead \( w = v^{-1} \), then, by what precedes, the only subexpression with no \( U_{0} \) must be \( U_{1}, U_{1}, \ldots, U_{1}, D_{1}, \ldots, D_{1}, D_{1} \). So again by 3.2.3 we obtain a single copy of \( R \) in degree 0 as claimed.

By lemma 3.2.3 we can actually say that \( \text{Hom}^{\bullet}(F_{\underline{w}}, R) \) is zero whenever \( \underline{w} \) does not admits subexpressions without \( U_{0} \)'s.

By combining proposition 3.2.1 and lemma 1.1.2 we get
Corollary 3.2.6. The second map of the short exact sequence of proposition 3.2.1 is a homotopy equivalence
\[
\text{Hom}^* (F_{w_1}, F_{w_2}) \xrightarrow{\sim} R[0].
\]
Then a morphism \( F_{w_1} \to F_{w_2} \) is non-zero in the homotopy category if and only if it does not belong to \( N_{w_2}^{w_1} \).

Proof. The first statement follows from Proposition 3.2.1 and Lemma 3.2.4.

For the second part of the statement, take the cohomology at 0 of the equivalence: the left hand side becomes just the space of morphisms in the homotopy category.

Hence we can prove the following

Proposition 3.2.7. For any \( s, t \in S \) such that \( st \) has order \( m \) in \( W \). Let \( w_1 = st \ldots \) and \( w_2 = ts \ldots \). Then there is a homotopy equivalence
\[
\gamma_{s,t} : F_{w_1} \to F_{w_2}
\]

Proof. Take \( \gamma_{s,t} \) to be the image of \( 1 \in R \) via the inverse homotopy equivalence of corollary 3.2.6 for the \( w_1 \) and \( w_2 \) of the statement, and \( \gamma_{t,s} \) to be the same thing but exchanging \( w_1 \) and \( w_2 \). Then we have
\[
\gamma_{s,t} \in + N_{w_1}^{w_2}
\]
\[
\gamma_{t,s} \in + N_{w_1}^{w_2}
\]

We claim that \( \gamma_{s,t} \) and \( \gamma_{t,s} \) give inverse homotopy equivalences. In fact, for instance, the composition \( \gamma_{t,s} \gamma_{s,t} \) (the other one is analogous) will belong to
\[
+ N_{w_1}^{w_2}
\]
and the composition of two \( 2m \)-valent vertices gives (see §2.1) the diagram with parallel strands (the identity of \( B_{w_1} \)) modulo some terms coming from the Jones-Wenzl morphism that belong to \( N_{w_1}^{w_2} \). This is also true for the identity morphism of \( F_{w_1} \). Hence the difference belongs to \( N_{w_1}^{w_2} \), which implies that the two morphisms are homotopy equivalent.

By adjunction we have a similar homotopy equivalence when we replace \( s \) and \( t \) with \( s^{-1} \) and \( t^{-1} \).

Now consider two braid words \( b_1 \) and \( b_2 \), such that \( b_2 \) is obtained from \( b_1 \) by applying one of the relations of \( B_W \). By appropriately tensoring with identity to the left and to the right the morphisms from proposition 3.1.2 or 3.2.7 we get a homotopy equivalence \( \gamma_{b_1, b_2} \) corresponding to the relation used.
If $b_2$ is obtained from $b_1$ via a sequence of relations then we call $\gamma_{b_1, b_2}$ the corresponding composition of morphisms as above.

A priori, the $\gamma_{b_1, b_2}$ could depend on the choice of relations, so to show that the above morphisms are well-defined, we have to prove that, given a loop of relations transforming a braid word $b$ into itself, the corresponding composition $\gamma$ is the identity. We have a homotopy equivalence $\text{Hom}^\bullet(F_b, F_{b_2}) \simeq \text{Hom}^\bullet(1[0], 1[0]) \simeq R[0]$ such that $1 \in R$ corresponds to the identity of $F_b$.

Hence the morphism $\gamma$ will be homotopy equivalent to $f \cdot \text{id}$ for some $f \in R$. But in all the equalities that we use to reduce $\gamma$ to $f \cdot \text{id}$ the coefficients are always 1’s. This implies that $f$ is 1.

As in [24] then, for a given element $b \in B_W$ one can take the limit $F_b$ in $K^b(\mathcal{D})$ of all isomorphisms $\gamma_{b, b'}$ for $b$ and $b'$ among all possible braid word corresponding to $b$, that by the above discussion form a transitive system. Then $b \mapsto F_b$ gives a functor $B_W \to K^b(\mathcal{D})$ (considering $B_W$ as a category with only identity morphisms). If we define $F_{b_1} F_{b_2}$ to be $F_{b_1 b_2}$ then the objects $F_b$ form a strict monoidal category $B_W$ called the 2-braid group.

4 Morphisms between Rouquier complexes

We start by introducing the homological algebra techniques that we will use to reduce the dg complexes of morphisms in $\mathcal{R}_{dg}$ in order to study morphisms in $K^b(\mathcal{D})$.

4.1 Gaussian reductions

First we recall the so-called Gaussian elimination for complexes, which was introduced in [2] for computing Khovanov homology. It is summarized by the following remark (see also [8, §5.4])

**Remark 4.1.1.** Let $\mathcal{C}$ be a $k$-linear category, and suppose we have a complex of the following form:

\[
\begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix}
\begin{pmatrix}
\eta \\
\theta
\end{pmatrix}
\]

\[
\begin{pmatrix}
\epsilon \\
\zeta
\end{pmatrix}
\]

\[
A \xrightarrow{\begin{pmatrix}
\epsilon \\
\zeta
\end{pmatrix}} B \oplus E \xrightarrow{\begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix}} C \oplus E' \xrightarrow{\begin{pmatrix}
\eta \\
\theta
\end{pmatrix}} D,
\]

(16)

where $\delta$ is an isomorphism. Then observe that the following map is an isomorphism of complexes

\[
\begin{pmatrix}
\epsilon \\
\zeta
\end{pmatrix}
\]

\[
\begin{pmatrix}
\alpha - \beta \delta^{-1} \\
\gamma \\
\delta
\end{pmatrix}
\]

\[
\begin{pmatrix}
\eta \\
\theta
\end{pmatrix}
\]

\[
\begin{pmatrix}
\epsilon \\
\zeta
\end{pmatrix}
\]

\[
\begin{pmatrix}
\alpha - \beta \delta^{-1} \gamma \\
\delta
\end{pmatrix}
\]

\[
\begin{pmatrix}
\eta \\
\theta
\end{pmatrix}
\]

\[
A \xrightarrow{\begin{pmatrix}
\epsilon \\
\zeta
\end{pmatrix}} B \oplus E \xrightarrow{\begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix}} C \oplus E' \xrightarrow{\begin{pmatrix}
\eta \\
\theta
\end{pmatrix}} D
\]

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In other words, the null-homotopic complex $E \to E'$ is a summand of the original complex (16), and a complement is given by

$$A \xrightarrow{\epsilon} B \xrightarrow{\alpha-\beta \delta^{-1} \gamma} C \xrightarrow{\eta} D$$

which is then homotopy equivalent to (16). For future use, we also explicitly write down the inverse homotopy equivalences:

$$A \oplus B \cong C \oplus E \oplus E' \cong D$$

where

$$\pi = (\text{id}_B, 0) , \quad \iota = \begin{pmatrix} \text{id}_B \\ -\delta^{-1} \gamma \end{pmatrix} , \quad \pi' = (\text{id}_C, -\beta \delta^{-1}) , \quad \iota' = \begin{pmatrix} \text{id}_C \\ 0 \end{pmatrix}.$$

When working in Krull-Schmidt categories, given a complex $C$, one can (repeatedly) use Gaussian elimination to compute its aforementioned minimal complex $C_{\text{min}}$, that has no contractible summands and is homotopy equivalent to $C$. Here we will use Gaussian elimination in a category which is not necessarily Krull-Schmidt, in order to simplify the computation of cohomology.

To that purpose we now describe a large scale version of Gaussian elimination for complexes, consisting in many applications of remark 4.1.1 in a convenient order.

We consider complexes of the form

$$\bigoplus_{p=0}^{i_0} A_p^0 \longrightarrow \bigoplus_{p=0}^{i_1} A_p^1 \longrightarrow \cdots \longrightarrow \bigoplus_{p=0}^{i_n} A_p^n.$$  (18)

with the $A_p^h$ in $\mathcal{C}$. The arrows in (18) are given by matrices of the form $(\phi_{qp}^h)$, where

$$\phi_{qp}^h \in \text{Hom}_\mathcal{C}(A_p^h, A_q^{h+1})$$

We say that such a complex is right quasi-collapsible if all non-zero entries of these matrices are isomorphisms, and one can construct a regular CW complex with the following properties:

- the $h$-cells are labeled by numbers $p$ from 0 to $i_h$ (corresponding to the objects $A_p^h$);
- the $h$-cell $p$ is face of the $(h+1)$-cell $q$ if and only if $\phi_{qp}^h$ is non-zero, hence an isomorphism.

Then this CW complex is called right adjacency diagram of the chain complex (18).

Consider now the 1-skeleton of a right adjacency diagram. Take an edge (i.e. 1-cell) labeled $q$ and choose an orientation, suppose its vertices are the 0-cells labeled $p_0$ and $p_1$, in this order. Let $\tilde{q}$ denote the oriented edge. We have isomorphisms $\phi_{0p}^q$ and $\phi_{1p}^q$, and we can define the isomorphism associated to the oriented edge $\tilde{q}$ to be

$$\Phi(\tilde{q}) = -(\phi_{qp}^0)^{-1}\phi_{qp}^1.$$
(the minus sign is for technical reason that will become clear later). Notice that if \( -\tilde{q} \) is the same edge with opposite orientation, then \( \Phi(-\tilde{q}) = (\Phi(\tilde{q}))^{-1} \). Given \( r \geq 1 \) and oriented edges \( \tilde{q}_1, \tilde{q}_2, \ldots, \tilde{q}_r \) forming an oriented path \( P \) along the 1-skeleton of a right adjacency diagram, we call isomorphism associated to \( P \) the composition

\[
\Phi(P) := \Phi(\tilde{q}_r) \cdots \Phi(\tilde{q}_2) \Phi(\tilde{q}_1).
\]

Clearly \( P' \) is another path starting from the endpoint of \( P \) and \( P'' \) is the concatenation, then

\[
\Phi(P'') = \Phi(P') \Phi(P).
\]

Dually, we can define left quasi-collapsible complexes and adjacency diagrams: the face relationship then goes the other way around (lower dimensional cells correspond to higher cohomological degrees). We also define isomorphisms associated to paths similarly (just reversing the arrows).

Recall that, in a regular CW complex, a face of a maximal cell is called free when it is not the face of any other cell. Recall also that a collapse or elementary contraction is a deformation of a regular CW complex consisting in removing a free face of a cell and the cell itself. Then a complex is said to be collapsible if and only if there is a sequence of collapses that reduces it to a single 0-cell. (this notion appears to be first introduced in [26] for simplicial complexes)

We can now give the result that incarnates large scale Gaussian elimination.

**Lemma 4.1.2.** A right quasi-collapsible complex of the form (18), whose adjacency diagram is collapsible, is homotopy equivalent to one of its degree zero objects. More precisely, if the right adjacency diagram can be collapsed to the 0-cell \( p_0 \) then the following are inverse homotopy equivalences

\[
\begin{array}{ccc}
\bigoplus_{p=0}^{i_0} A^0_p & \longrightarrow & \bigoplus_{p=0}^{i_1} A^1_p & \longrightarrow & \cdots & \longrightarrow & \bigoplus_{p=0}^{i_n} A^n_p \\
\downarrow \pi_{p_0} & & & & & & \downarrow \pi_{p_0} \\
A^0_{p_0} & & & & & & A^0_{p_0}
\end{array}
\]

where \( \pi_{p_0} \) is the projection to the factor \( A^0_{p_0} \) and \( P^p_{p_0} \) is any path from the 0-cell \( p_0 \) to \( p \).

**Proof.** Notice that the collapse of an \( (h+1) \)-cell \( q \) and a free face \( p \) in the adjacency diagram corresponds to the elimination of the isomorphism \( \phi^h_{qp} \). The condition on \( p \) to be free corresponds to the condition \( \phi^h_{qp} = 0 \) for any \( q' \neq q \). So that, in the notation of the remark [11], we have \( \beta = 0 \), which implies that no changes are needed on the remaining morphisms after the elimination. Hence we can proceed by induction.

To find an explicit homotopy equivalence, suppose, without loss of generality, that the sequence of collapses removes successively the points \( i_0, i_0 - 1, \ldots, 1 \) (each one together with some edge) and that \( p_0 = 0 \). Notice that the composition of the cohomological degree zero part of the homotopy equivalences in (17) is not affected by collapses of bigger dimension, hence it is the one in figure [1]. One can then check that the composition of these morphisms gives, in each component, the isomorphism associated to the path obtained during the collapses.

What remains to be proved is that the morphism does not depend on the choice of the path. In other words, we need to show that, for any vertex \( p \) of the
\[ A_0^0 \oplus A_1^0 \oplus \cdots \oplus A_{i_0-1}^0 \oplus A_{i_0}^0 \]

\[ \begin{pmatrix}
\text{id} & 0 & \cdots & 0 & 0 \\
0 & \text{id} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \text{id} & 0 \\
\end{pmatrix} \]

\[ A_0^0 \oplus A_1^0 \oplus \cdots \oplus A_{i_0-1}^0 \]

\[ \begin{pmatrix}
\text{id} & 0 & \cdots & 0 & 0 \\
0 & \text{id} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \text{id} & \Phi_{i_0-1} \\
\end{pmatrix} \]

\[ \begin{pmatrix}
\text{id} & 0 \\
0 & \text{id} \\
\text{id} & 0 \\
0 & \text{id} \\
\end{pmatrix} \]

\[ \begin{pmatrix}
\text{id} & 0 \\
0 & \text{id} \\
\text{id} & 0 \\
0 & \text{id} \\
\end{pmatrix} \]

\[ A_0^0 \oplus A_1^0 \]

\[ \begin{pmatrix}
\text{id} & 0 \\
0 & \text{id} \\
\text{id} & 0 \\
\end{pmatrix} \]

\[ A_0^0 \]

Figure 1: The degree 0 parts of the homotopy equivalences from Gaussian elimination.

diagram, if we choose two paths connecting it to \( p_0 \), the isomorphism associated
is the same. It is sufficient to prove that the isomorphism associated to a loop is
the identity. Consider a loop \( P \). By contractibility this loop is the boundary of
the union of a finite number of faces and, by splitting the isomorphism, we can
restrict to the case where \( P \) is the boundary of a single 2-cell, as in the example
below.

Let \( f \) denote the face enclosed by \( P \). Let \( \tilde{q} \) and \( \tilde{q}' \) be consecutive oriented edges
of \( P \) connecting, respectively, \( p \) to \( p' \) and \( p' \) to \( p'' \). Then

\[ \Phi(\tilde{q}')\Phi(\tilde{q}) = (\phi_{q'q''}^0)^{-1}\phi_{q'q''}^0(\phi_{qp'}^0)^{-1}\phi_{qp}^0 \]

and, by the fact that composition of successive differential maps is 0, we have

\[ \phi_{fq}^1\phi_{qp}^0 + \phi_{fq'}^1\phi_{q'p}^0 = 0 \]

which implies

\[ -\phi_{qp}^0(\phi_{q'p}^0)^{-1} = (\phi_{fq}^1)^{-1}(\phi_{fq'}^1). \quad (19) \]
Hence one can successively use (19) in the expression of the isomorphism associated to $P$ so that the $\phi^1$’s cancel each other, and one gets the identity.

Of course one can reverse all the arrows in the lemma and obtain an analogous statement for left adjacency diagrams.

We finish this section with another homological algebra lemma which will allow us to reduce collapsible complexes inside bigger ones.

Lemma 4.1.3. Let $C$ and $C'$ be two complexes with filtrations $C_\bullet$ and $C'_\bullet$ and let $f : C \to C'$ be a map of complexes compatible with the filtrations. If the induced maps on subquotients are homotopy equivalences then also $f$ itself is a homotopy equivalence.

Proof. It is sufficient to prove that, given a diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & X \\
\downarrow & & \downarrow \\
0 & \longrightarrow & X'
\end{array}
\]

\[
\begin{array}{ccc}
Y & \longrightarrow & Z \\
\downarrow & & \downarrow \\
Y' & \longrightarrow & Z'
\end{array}
\]

where the rows are short exact sequences and the left and right vertical arrows are homotopy equivalences, then the middle one is also a homotopy equivalence. But for this it is sufficient to pass to the homotopy category where the above diagram becomes a morphism between distinguished triangles, where the left and right vertical arrows are isomorphisms, which implies that the middle one is an isomorphism too.

We can now start applying these techniques to our case.

4.2 A small example: type $A_1$

Let us start with type $A_1$. Here we have objects $F_n$ and $F_{n-1}$, and all (direct sums of) tensor products between them. In this case it is actually easy to reduce directly the objects $F^n_\alpha$, for any integer $n$, also with $k = \mathbb{Z}$ (see [10, Exercise 19.29]), but we will instead reduce morphism complexes to illustrate how these techniques work.

By bi-adjunction and proposition 3.1.2 in order to study morphism spaces up to homotopy equivalence, we can restrict ourselves to the following

\[\text{Hom}^\bullet(F^n_\alpha, R),\]

with $n \in \mathbb{Z}$.

Let us introduce some notation specific to this case. Diagrams representing morphisms of the above form are those with only one color and no ending points. We will call a bridge a sub-diagram consisting of one single connected component of the strands, without dots, and possibly some patches under it. For example, the following diagram contains three bridges, one patch outside bridges and two boundary dots.
Hence, one can see that the light leaves, labeled \((e,f)\), which form an \(R\)-basis for the dg-module \(A = \text{Hom}^\bullet(F^n_s, R)\) are precisely the arrangements of bridges, patches and boundary dots, with a total of \(n\) starting points on the bottom (patches outside bridges and boundary dots correspond to \(f_i = U0\) and \(e_i = 0\) or \(1\) respectively, and bridges correspond to subsequences of \(f\) of the form \(U1, D0, \ldots, D0, D1\), where the \(D0\)'s correspond to either patches or pillars according to the value of \(e_i\)).

We will start from the case \(n > 0\). Recall that, in this case, the differential map is given by the replacement of a patch with a boundary dot, with an appropriate sign

**Theorem 4.2.1.** For \(n \geq 2\), the complex \(\text{Hom}^\bullet(F^n_s, R)\) is homotopy-equivalent to

\[
R(-n + 2) \xrightarrow{\alpha} R(-n + 4) \xrightarrow{0} R(-n + 6) \xrightarrow{\alpha} \cdots \xrightarrow{\alpha} R(n - 2)
\]

where the last object is in degree 0 and the morphisms are alternatively \(\alpha\) and 0. For \(n = 1\), the complex is null-homotopic.

**Proof.** By \(\text{3.2.3}\) we can get rid of all light leaves with independent points and the quotient, that will be denoted by \(A\), is spanned by those light leaves that are arrangements of bridges. Let us then introduce a short notation for these light leaves. We will represent them by sequences of 0’s and 1’s in parentheses: every \(U1\) gives an open parenthesis, every \(D1\) a closed parenthesis and every \(D0\) gives 0 or 1 according to the value \(e_i\), as in the following example.

\[
e \quad f \quad \begin{array}{ccccccc} U1 & D0 & D0 & D0 & D0 & D0 & D1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ ( & 0 & 1 & 1 & 0 & 1 & 1 ) \end{array}
\]

For example the following arrangement of bridges is represented by the sequence \((0)(10)(001)(11)\).

\[
\text{(20)}
\]

This is for example the complex \(A\), when \(n = 4\),

\[
\begin{array}{c}
\text{\(R \cdot (00)\)} \\
\xleftarrow{\alpha} \\
\xrightarrow{-\alpha} \\
\text{\(R \cdot (01)\)} \\
\xleftarrow{1} \\
\xrightarrow{\text{\(R \cdot (10)\)}} \\
\xrightarrow{\text{\(R \cdot (11)\)}} \\
\end{array}
\]

Given any such light leaf, we define its weight to be the sum of the number of bridges (i.e. the number of pairs of parentheses) and the number of patches (i.e. the number of 0’s). For example the weight of the above light leaf \((20)\) is 7.

We can then filter \(A\) according to the weight, because the differential map will decrease it. So we will have

\[
0 = C_{\leq 0} \leftrightarrow C_{\leq 1} \leftrightarrow \cdots \leftrightarrow C_{\leq n-1} = A
\]
where $C_{\leq i}$ is the sub-complex spanned by all the light leaves with weight not greater than $i$. We want to apply lemma 4.1.3. First let us consider each sub-quotient $C_i = C_{\leq i} / C_{\leq i-1}$ and notice that it is a quasi-collapsible complex. In fact, when we replace a patch by a boundary dot under a bridge we get

\[ \cdots = \ \delta s \cdots = \ \delta s \cdots = \ \delta s \cdots = \ \cdots \]

in particular the coefficients of light leaves with the same weight is an invertible element of $R$. Furthermore, by the above, we see that the light leaf $L$ has an arrow towards $L'$ if and only if the sequence of $L$ can be obtained by replacing one occurrence of the sub-sequence “)(" in the sequence of $L'$ by one of the two sub-sequences “01” or “10”. This allows to construct a CW complex with the required adjacency properties, which is the right adjacency diagram of $C_i$.

For example this is the right adjacency diagram corresponding to $C_3$, in the case $n = 7$.

\begin{center}
\begin{tikzpicture}
\end{tikzpicture}
\end{center}

Now, one can show that all the adjacency diagrams that we obtain are collapsible, hence lemma 4.1.2 implies that each of the corresponding $C_i$’s reduces to the single copy of $R$, in degree $i$, represented by the sequence $L_i = (11..i00..0)$, with $i$ zeroes and $n - i - 1$ ones.

Now let $A'$ be the complex of the statement and consider the map $A' \to A$ defined as follows. The generator 1 of the copy of $R$ in degree $i$ + 1 is sent to the combination of all light leaves $L$ consisting of a single bridge of weight $i$, each with coefficient $(-1)^t$, where $t$ is the minimal number of transpositions that are necessary to turn the sequence of $L$ into that of $L_i$. It is easy to see that this map is a morphism of complexes.

Consider the following filtration of $A'$

\[ 0 = C'_{\leq 0} \hookrightarrow C'_{\leq 1} \hookrightarrow \cdots \hookrightarrow C'_{\leq n-1} = A' \]
where $C_{≤i}'$ is the subcomplex consisting of the last $i$ terms. Then the above map is compatible with the filtrations and, on subquotients, it induces the homotopy equivalences of lemma 4.1.2 (in fact $t$ is precisely the length of a path from $L$ to $L_i$). Then lemma 4.1.3 concludes the proof.

The argument for the case $n ≤ 0$ is very similar (the following theorem will actually follow by the main result of the next section). We have

**Theorem 4.2.2.** For $n = −m ≤ 0$, the complex $\text{Hom}^\bullet(F_s^{-m}, R)$ is homotopy-equivalent to

$$R(-m) \rightarrow \cdots \rightarrow R(m-4) \rightarrow R(m-2) \rightarrow R(m)$$

where the first object is in degree 0 and the morphisms are alternatively $\alpha$ or 0 (starting from the last).

**Proof.** The light leaves are the same as in the previous case, what changes is their degree and the differential map. The following is, for example, the complex for $m = 2$.

![Complex diagram]

This time we will call *weight* the number of connected components of a diagram, excluding the patches. So let $A$ be the starting complex, and let $A_{≤i}$ the subcomplex spanned by the light leaves of weight less than or equal to $i$. They are indeed sub-complexes because the differential map, in this case, can only decrease the number of connected components.

Now, define $A_i = A_{≤i}/A_{≤i-1}$. It is easy to see that $A_i$ is a collapsible complex, and we can construct a homotopy equivalence between $A$ and the claimed complex in a similar way as before.

### 4.3 From subexpressions to subwords

We want now to use the same ideas to give a partial simplification of complexes of morphisms of the form

$$\text{Hom}^\bullet(\mathbb{1}, F_w)$$

for a Coxeter word $w$.

**Remark 4.3.1.** In many cases we can actually restrict to the case where $w$ is a reduced word. In fact let us first suppose that $k$ is a characteristic 0 field. Following [10 §6] one can define a t-structure $(K^{≤0}, K^{≥0})$ on $K^b(D)$ using the decomposition properties of objects in $\mathcal{D}$, that are encoded in Soergel’s conjecture (which was actually shown in [11] using this t-structure). Some interesting properties of this t-structure for our purpose are the following:

- One has that $F_s \otimes -$ (resp. $F_{s-1} \otimes -$) preserves $K^{≥0}$ (resp. $K^{≤0}$);
• If \( w = s_1 s_2 \cdots s_n \) and \( v = s_1 \ast s_2 \ast \cdots \ast s_n \), where \( \ast \) denotes the Demazure (or greedy) product, then one has, in \( K_b(\mathcal{D}) \), that \( \tau_{\leq 0}(F_w) \cong F_v(m) \), for some \( m \in \mathbb{Z} \), where \( (1) \) is the Tate twist \( (1)[-1] \).

Then, by the general axioms of a t-structure, because \( 1 \in K_{\leq 0} \), one can deduce that

\[
\text{Hom}_{K_b(\mathcal{D})}(1, F_w) = \text{Hom}_{K_b(\mathcal{D})}(1, F_v(m))
\]

One can define a t-structure also when \( k \) is either an extension of \( \mathbb{Q}_\ell \), its ring of integers, or its residue field. This is done in \( [1] \) in order to define a modular analog of the equivariant mixed derived category. The heart of the t-structure is then an (alternative) modular version of equivariant mixed perverse sheaves, originally related to the eigenvalues of the Frobenius morphism, and the combination of shifts \( (1)[−1] \) corresponds to the original Tate twist, which explains the terminology.

We want to describe a slightly simpler version of \( \text{Hom}^*(1, F_w) \) which is homotopy equivalent to it. The idea is that, by eliminating some light leaves, we can identify different starting subexpressions representing the same subwords, somehow forgetting about patches.

Let \( \bar{b} = w^{-1} \) (in the free monoid of braid words): by adjunction we can consider

\[
C_{\bar{b}} = \text{Hom}^*(F_{w^{-1}}, 1).
\]

Let \( v \) be the Coxeter word associated to \( b \), so \( v \) is just the mirror image of \( w \).

Let \( I_0 \) be the set of pairs \( (u_0, f_0) \) with \( u_0 \) a subword of \( v \) and \( f_0 \) a subexpression of \( u_0 \) such that the corresponding light leaf without patches with starting word \( u_0 \) does not have empty arcs, i.e. subdiagrams consisting in consecutive starting points of the same color connected by a strand (which corresponds to one of the subsequences \( U1D1, U1D0, D0D1 \) or \( D0D0 \) in \( f_0 \)). We will define a new complex with a basis labeled by \( I_0 \) instead of pairs \( (e, f) \).

More precisely we will have to choose a representative for each subword, and the complex will depend on this choice: actually the complexes obtained are all identical up to sign. Let \( E \) be the set of pairs \( (e, f) \) such that \( e f \) reduces to the identity and that the light leaf labeled \( (e, f) \) does not have empty arcs with patches i.e. subdiagrams of the form

![Diagram](https://via.placeholder.com/150)

which correspond to subsequences of \( e \) of the form \( 10\ldots01 \) such that the \( f_i \) corresponding to the first 1 is either \( U1 \) or \( D0 \), and the one corresponding to the last one is either \( D1 \) or \( D0 \) (and the number of patches is arbitrary, including 0).

\footnote{One can define it inductively as follows:}

\[
w \ast s = \begin{cases} 
ws & \text{if } \ell(ws) > \ell(w) \\
w & \text{otherwise}
\end{cases}
\]

One can then show that it actually gives a well-defined associative product on \( W \).
Given a pair of subexpressions \((e, f)\) of \(v\), denote \(\mathbf{f}|e\) the subexpression induced by \(f\) on \(v^e\), obtained by forgetting the 0’s of \(f\) corresponding to 0’s of \(e\). Then consider the map

\[ E \xrightarrow{\rho} I_0 \]

\[(e, f) \mapsto (\mathbf{v}^e, \mathbf{f}|e)\]

and choose a complete set \(I\) of representatives for its fibers.

Then let \(C_b(I)\) be a dg module that is a free \(R\)-module with basis \(\{L'_e, f|e\} (e, f) \in I\), where each \(L'_e, f|e\) is in degree \(\deg_b (e, f)\), and the differential map is defined as follows. First we compute the image by the differential map in \(C_b\) of \(L'_e, f|e\). This will be a linear combination of the form

\[ \sum_{(e', f') \in I} \lambda_{(e', f')} L'_{e', f'} \]

with coefficients \(\lambda_{(e', f')}\) in \(R\). Now, given two subexpressions \(e'\) and \(e''\) for the same subword of \(v\), we call \(n^{e''}_{e'}\) the minimal number of transpositions on symbols 0 and 1 that are necessary to get \(e''\) from \(e'\). Then the differential in \(C_b(I)\) is defined as

\[ d(L'_e, f|e) = \sum_{(e', f') \in I} \left( \sum_{(e'', f'') \in E \atop \rho(e', f') = \rho(e', f'')} (-1)^{n^{e''}_{e'}\lambda_{e'', f''}} \right) L'_{e', f'} \]

Then we can state the following result:

**Theorem 4.3.2.** For each \((e, f) \in E\) let \((e', f')\) be the only element of \(I\) such that \((e, f) \sim (e', f')\). Then the map \(C_b \to C_b(I)\) given by

\[ L_e, f \mapsto \begin{cases} (-1)^{n^{e'}_{e''}} L'_{e', f'} & \text{if } (e, f) \in E \\ 0 & \text{otherwise} \end{cases} \]

is a homotopy equivalence.

**Proof.** Given any light leaf \((e, f)\) we first consider its simplified starting word, i.e. the word \(\mathbf{u}_e\) obtained from \(\mathbf{u}_e = \mathbf{v}^e\) by reducing to one single letter those sequences of repeated letters connected by empty arcs. Then we call simplified version of \((e, f)\) the light leaf without patches with starting word \(\mathbf{u}_e\) obtained by eliminating every empty arc: this will correspond to a subexpression \(\mathbf{f}_0\) of \(\mathbf{u}_e\). Finally we call standard simplified version of \((e, f)\) the light leaf \((e', f')\) \(\in I\) corresponding to \((\mathbf{u}_e, \mathbf{f}_0)\) (the only difference is that \((e', f')\) has some patches whose positions are prescribed by the chosen representative).

For example, the standard simplified version of the following light leaf
that has starting word and simplified starting word respectively

\[ u = s_1 s_1 s_2 s_1 s_3 s_4 s_5 s_4 s_3 = \mathbf{u}^{(10111111111)}, \]

\[ u_0 = s_1 s_2 s_1 s_3 s_4 s_5 s_3, \]

is the following

![Diagram](image)

if the chosen subexpression \( e' \) is

\[(00111011111).\]

Now let \( A_{<i} \) be the sub-complex of \( C_b \) spanned by the light leaves whose simplified starting word \( u \) has length not greater then \( i \). In parallel, let \( A'_{<i} \) be the sub-complex of \( C_b(I) \) spanned by the pairs \( (e,f) \) with the length of \( v \) not greater than \( i \). Then notice that \( A_i := A_{<i}/A_{<i-1} \) splits into summands \( A_{\mathbf{u}, f_0} \), according to the simplified light leaves without patches \( (u_0, f_0) \), for \( u_0 \) subword of \( u \) of length \( i \), and \( f_0 \) giving a light leaf without patches that has no empty arcs. This because the differential map reduces the length of the starting word, so either it maps to a simpler word with same simplification, or to a word with shorter simplification.

Each of these \( A_{\mathbf{u}, f_0} \)'s is a left quasi-collapsible complex: in fact the differential acts by \( \pm 1 \) eliminating some pillars of the empty arcs, and one can again construct a CW complex with the desired properties.

For example, let us consider \( v = ststst \), the subword \( u_0 = st \) and the subexpression \( f_0 = 00 \) of \( u_0 \). The (left) adjacency diagram of \( A_{\mathbf{u}, f_0} \) is

![Diagram](image)

In general, one can see that the adjacency diagram is collapsible to any of its 0-cells, for any choice of \( u, u_0 \) and \( f_0 \). Furthermore the isomorphism associated to
a single edge is $+1$ if there is an odd number of patches between the two pillars over which the differential is acting, and $-1$ otherwise. Then the isomorphism associated to a path from $e$ to $e'$ is precisely $(-1)^{2e}$.

Hence corollary 4.1.2 implies that $A_{(\omega, f_0)}$ is homotopy-equivalent to a single copy of $R$, namely the one labeled by the light leaf $(e, f) \in I$ corresponding to $(\omega, f_0)$.

One can then see that the map in the statement is compatible with the filtrations and that, over subquotients, it gives the homotopy equivalences from lemma 4.1.2. Hence we conclude by lemma 4.1.3.

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