Renormalization group invariant of lepton Yukawa couplings

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Abstract

By using quark Yukawa matrices only, we can construct renormalization invariants that are exact at the one-loop level in the standard model. One of them $I_q$ is accidentally consistent with unity, even though quark masses are strongly hierarchical. We calculate a lepton version of the invariant $I_l$ for Dirac and Majorana neutrino cases and find that $I_l$ can be also close to unity. For the Dirac neutrino case, if we assume $I_q = I_l$, we can deduce that the hierarchy of neutrino masses is inverted, and the lightest neutrino mass is $3.0 \text{ meV}$ to $8.8 \text{ meV}$.

1 Introduction

In the standard model (SM) of particle physics, fermion masses are determined by their Yukawa couplings to the Higgs field. These couplings are free parameters and cannot be predicted within the SM. There may be some principles that determine the couplings at a high energy scale, such as a grand unified theory or a flavor symmetry. In order to compare the predictions that come from those high energy theory and observed values by experiments, we have to consider renormalization group evolution of the parameters. Renormalization group invariants are very useful because we can infer the high energy physics with low energy inputs. A set of such invariants, exact at the one-loop level, was reported in Ref. [1]. There are two independent invariants that are constructed only by quark Yukawa coupling matrices. One of them, we call $I_q$, is close to unity, and this fact may indicate a rule for the structure of Yukawa couplings.

Today, we know that neutrinos have masses, so it is likely that they are coupled to the Higgs field as other fermions. There are still many mysteries in neutrinos: their type (Dirac or Majorana), the mass hierarchy (normal or inverted), and the absolute masses. We calculate a renormalization group invariant $I_l$ in the lepton sector that is similar to $I_q$ to see implications on these mysteries. If neutrinos are Dirac type, the relation between the neutrino Yukawa coupling matrix and their mass matrix is the same as other fermions. If neutrinos are Majorana type, we need some assumption to relate these matrices. In this letter, we consider simple cases of the type-I seesaw mechanism [2–4]. To calculate $I_l$, we introduce a useful parametrization for neutrino Yukawa couplings.

As a result, we find that $I_l$ can be also $O(1)$ for regardless of the type of neutrinos. For the Dirac neutrino and the inverted hierarchy case, an equality between quark and lepton sector $I_q = I_l$ can hold if the lightest neutrino mass is $3.0 \text{ meV} \leq m_{\text{lightest}} \leq 8.8 \text{ meV}$. For the Majorana neutrino cases, the invariant depends on many unknown parameters. Under some assumptions, the equality can be satisfied for $0.19 \text{ meV} \leq m_{\text{lightest}} \leq 1.5 \text{ meV}$ in the inverted hierarchy case. An interesting condition, $I_l = 1$, can be also realized for Majorana cases.

2 Renormalization group invariants

The Yukawa terms of the Lagrangian that we consider in this letter is

$$-\mathcal{L} = \overline{q}_R Y_u \Phi^c Q_L + \overline{q}_R Y_d \Phi^c Q_L + \overline{\nu}_R Y_{\nu} \Phi^c L_L + \overline{l}_R Y_l \Phi^c L_L + h.c.$$  (1)
There is a combination of the quark Yukawa matrices \( Y_u \) and \( Y_d \) that is exactly invariant under the renormalization group evolution at the one-loop level \([1]\),

\[
I^q \equiv \left( \frac{\text{Det}[Y_u^T Y_u Y_d^T Y_d]}{\text{Tr}[(Y_u^T Y_u Y_d^T Y_d)^{-1}]} \right)^{-2/3} \frac{\text{Tr}[Y_u^T Y_u Y_d^T Y_d]}{\text{Tr}[(Y_u^T Y_u Y_d^T Y_d)^{-1}]} \\
= (m_u m_c m_t m_d m_s m_b)^{-4/3} \frac{\sum m_\alpha^2 m_i^2 |V_{\alpha i}|^2}{\sum m_\alpha^{-2} m_i^{-2} |V_{\alpha i}|^2}.
\]

\( V \) denotes the quark mixing matrix, or the Cabibbo-Kobayashi-Maskawa matrix. The summations are over \( \alpha = u, c, t \) and \( i = d, s, b \). Note that if all quark masses are the same, \( I^q = 1 \). By substituting the observed values (renormalized to \( M_Z \)) \([5, 6]\), \( I^q = 1.35^{+0.54}_{-0.63} = 0.72 \ldots 1.89 \).

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\]

(1\( \sigma \) errors are used in this letter except Eq. (9)). It is surprising that this invariant is close to and consistent with unity, even though masses of quarks are very hierarchical (say, \( m_t/m_u \sim 10^5 \)). By keeping the leading term, \( I^q \approx 1 \) means

\[
\frac{m_u m_t m_d m_b}{m_c^2 m_s^2} \approx 1.
\]

This equation constrains the relative quarks masses of the second generation between the masses of the first and the third generations. A mass spectrum that realizes Eq. (4) was studied in Ref. \([7]\). In this letter, we calculate the invariant in the lepton sector,

\[
I^l \equiv \left( \frac{\text{Det}[Y_\nu^T Y_\nu Y_\nu^T Y_\nu]}{\text{Tr}[(Y_\nu^T Y_\nu Y_\nu^T Y_\nu)^{-1}]} \right)^{-2/3} \frac{\text{Tr}[Y_\nu^T Y_\nu Y_\nu^T Y_\nu]}{\text{Tr}[(Y_\nu^T Y_\nu Y_\nu^T Y_\nu)^{-1}]}.
\]

The relation between neutrino Yukawa couplings and the observed neutrino mass matrix depends on the type of neutrinos: Dirac or Majorana.

### 3 Dirac neutrinos

If neutrinos are Dirac-type, the Yukawa matrix \( Y_\nu \) is directly converted to neutrino mass matrix \( m_\nu \) as

\[
Y_\nu = \frac{1}{v} m_\nu = \frac{1}{v} U_R m U^\dagger, \quad m_{ij} = m_i \delta_{ij} \quad (i, j = 1, 2, 3)
\]

\( U \) and \( U_R \) are unitary matrices that diagonalize \( m_\nu \), and \( v \) is the vacuum expectation value of the Higgs doublet \( \Phi \) (invariants do not depend on \( v \), though). In the basis where the charged lepton mass matrix is diagonal \( (m_{\alpha \beta} \equiv Y_{\alpha \beta} v = m_\alpha \delta_{\alpha \beta}) \), \( U \) corresponds to the neutrino mixing matrix. The invariant \( I^l \) can be explicitly written similarly to \( I^q \),

\[
I^l = (m_1 m_2 m_3 m_e m_\mu m_\tau)^{-4/3} \frac{\sum m_\alpha^2 m_i^2 |U_{\alpha i}|^2}{\sum m_\alpha^{-2} m_i^{-2} |U_{\alpha i}|^2}.
\]
Figure 1: Neutrino mass dependence of the renormalization group invariant $I_l$ for Dirac neutrino case. Upper and lower curves show the IH and NH cases with 1σ deviation roughly estimated from Eqs. (13), (14). Straight line shows the lower limit of $I_q$.

The summations are over $\alpha = e, \mu, \tau, i = 1, 2, 3$. Among the parameters in Eq. (8), absolute values of the neutrino masses cannot be determined by oscillation experiments. The strongest limit on the neutrino mass scale is set by the observation of cosmic microwave background [8],

$$\sum m_i < 0.23 \text{ eV (95% limit)} \Rightarrow m_i \lesssim 0.08 \text{ eV},$$

so we consider the region $m_i \leq 0.1 \text{ eV}$ in this letter. The dependence of $I_l$ on the lightest neutrino mass is shown in Fig. 1. We have used charged lepton masses (renormalized to $M_Z$ [5]) and the results of neutrino oscillation experiments [9]. To better understand Eq. (8), we discuss three limiting cases below.

(i) Degenerate masses

The simplest case is that neutrino masses are almost the same, $m_1 \simeq m_2 \simeq m_3$. Since the charged lepton masses are strongly hierarchical $m_e \ll m_\mu \ll m_\tau$, we can approximate

$$I_l \simeq (m_1^2 m_e m_\mu m_\tau)^{-4/3} \frac{m_\mu^2 m_\tau^2 \sum_i |U_{\tau i}|^2}{m_1^2 m_e^2 \sum_i |U_{e i}|^2}$$

$$= \left(\frac{m_e m_\tau}{m_\mu^2}\right)^{2/3} \left(\equiv I^l_d\right)$$

$$= 0.18648 \pm 0.00001.$$
(ii) Normal Hierarchy (NH)  
We take $m_1 ≪ m_2 < m_3$. By subtracting the most dominant term of $I'$, 
\[
I' \simeq \left( \frac{m_e m_\tau m_1 m_3}{m_\mu^2 m_2^2} \right)^{2/3} \frac{|U_{e3}|^2}{|U_{e1}|^2} = I'_d \left( \frac{m_1 m_3}{m_2^2} \right)^{2/3} \frac{s_{23}}{c_{13}}. \tag{13}
\]
We have used conventional parametrization of $U$ given in Ref. [9]. The experimental error of $I'$ is roughly estimated from this equation to be $+5.1\%$. 

(iii) Inverted Hierarchy (IH)  
We take $m_1 \simeq m_2 \gg m_3$. The dominant term is 
\[
I' \simeq \left( \frac{m_e m_\tau m_3}{m_\mu^2 m_2} \right)^{2/3} \frac{|U_{\tau 1}|^2 + |U_{\tau 2}|^2}{|U_{e1}|^2} \approx I'_d \left( \frac{m_3}{m_2} \right)^{2/3} \frac{s_{23}}{s_{13}} \tag{14}
\]
Since $s_{13} \ll 1$, $I'$ tends to be larger for the IH case than that for the NH case. The experimental error of $I'$ is roughly estimated from this equation to be $+6.3\%$. 

From Fig. 1, we find that $I'$ is $O(0.1) \sim O(1)$ for both mass hierarchy cases. In the IH case, $I' = I''$ can be satisfied for narrow and hierarchical region of the lightest neutrino mass 3.0 meV $\leq m_3 \leq 8.8$ meV $\ll m_1, m_2$ (see Table 1). This mass range gives 
\[
0.101 \text{ eV} \leq \sum m_i \leq 0.109 \text{ eV}. \tag{15}
\]
Such a total neutrino mass is within the reach of future cosmological and astrophysical surveys [10]. Unlike $I''$, an interesting condition $I' = 1$ is highly excluded for both NH and IH cases. 

4 Majorana neutrinos  
If neutrinos are Majorana type, we need some assumption to relate their mass and Yukawa matrix. Here we assume that the Majorana masses are generated through the type-I seesaw mechanism [2–4]. We add a Majorana mass term of three right-handed neutrinos to the Lagrangian, 
\[
\mathcal{L} \rightarrow \mathcal{L} - \frac{1}{2} \left( \bar{\nu}_R M \nu_R + \bar{\nu}_R^T M \nu_R \right), \quad M_{IJ} = M_I \delta_{IJ} (I, J = 1, 2, 3). \tag{16}
\]
We take the basis of $\nu_R$ such that $M$ is diagonal. We expect that the Majorana masses of $\nu_R$ are much larger than the Dirac mass term. Then we obtain the neutrino mass matrix 
\[
m_\nu = -v^2 Y_\nu^T M^{-1} Y_\nu. \tag{17}
\]
In this case, we can parametrize $Y_\nu$ as [11] 
\[
Y_\nu^\dagger Y_\nu = \frac{1}{v^2} U \sqrt{m R^\dagger M R} \sqrt{m U^\dagger}. \tag{18}
\]
$R$ is an arbitrary complex matrix that satisfies $R^T R = 1$. This matrix cannot be determined by low energy experiments. Determinant of Eq. (18) does not depend on $R$, 
\[
\text{Det}[Y_\nu^\dagger Y_\nu] = \text{Det}[\left( \frac{1}{v^2} M R^\dagger M R \right)] = \frac{1}{v^6} m_1 m_2 m_3 M_1 M_2 M_3. \tag{19}
\]
Here we have used $\text{Det}[R^\dagger]\text{Det}[R]= (\pm 1)^2 = 1$. By these parametrization, the invariant is

$$I^\dagger = (m_1 m_2 m_3 M_1 M_2 M_3 m_\mu^2 m_\tau^2 m_\nu^2 m_\mu^2)^{-2/3} \frac{\text{Tr} \left[ R^\dagger M R \sqrt{m U^T m_u U \sqrt{m}} \right]}{\text{Tr} \left[ (R^\dagger M R \sqrt{m U^T m_u U \sqrt{m}})^{-1} \right]}.$$  

(21)

This is a general expression of $I^\dagger$. To be specific, we assume $M_1 = M_2 = M_3$. In this case, $I^\dagger$ does not depend on $M$ explicitly, but depend on $R^\dagger R$. This matrix is Hermitian and orthogonal. We introduce a useful parametrization for such a matrix.

$$R^\dagger R = \sqrt{1 + a^2} I - (\sqrt{1 + a^2} - 1) \frac{\bar{a} a^T}{a^2} + i A,$$

(22)

$$A \equiv \begin{pmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{pmatrix}, \quad \bar{a}^T \equiv (a_1, a_2, a_3), \quad a \equiv |\bar{a}|.$$  

(23)

If $R$ is real, $R^\dagger R = R^T R = 1$, which corresponds to $\bar{a} = 0$. In principle, $a_i$ can take any real value, and $I^\dagger$ can be infinitely large. Consider the case $a_2 = a_3 = 0, a_1 = a \gg 1$. In this limit, we obtain

$$R^\dagger R \simeq a \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & i \\ 0 & -i & 1 \end{pmatrix}.$$  

(25)

By substituting it into Eq. (21), the invariant is

$$I^\dagger = (m_1 m_2 m_3 m_\mu^2 m_\tau^2 m_\nu^2 m_\mu^2 m_\tau^2 m_\nu^2 m_\mu^2)^{-2/3} \frac{\sum_{i,j=2,3} \sqrt{m_i m_j} (R^\dagger R)_{ij} (U^T m_u U)_{ji}}{\sum_{i,j=2,3} \sqrt{m_i^{-1} m_j^{-1}} (R^\dagger R)_{ij} (U^T (m_l^2)^{-1} U)_{ji}}$$

(26)

$$\propto m_1^{-2/3}.$$  

(27)

$I^\dagger$ diverges in the limit $m_1 \to 0$. The limit $a \to \infty$ is, however, highly unnatural, since it means that the absolute values of the components of $Y_\nu^T Y_\nu$ are much larger than that of $Y_\tau^T Y_\tau$ (compare Eq. (17) and Eq. (18)). To be concrete, we consider the region $|a_i| \leq 5$.

The dependence of $I^\dagger$ on the lightest neutrino mass is shown in Figs. 2 and 3. For each NH and IH cases, we have shown three curves: (i) $a_i = 0$, which corresponds to the case of real $R$, (ii) $I^\dagger(0.1 \text{ eV})$ is largest and (iii) smallest. We have used charged lepton masses renormalized to $M_Z$ and neutrino oscillation parameters [9]. $I^\dagger$ is defined at the energy scale larger than $M$, but we can expect that the renormalization group effect between $M_Z$ and $M$ is negligible. The reasons are as follows. First, $I^\dagger$ include charged lepton masses in the form of ratios, which are almost constant in renormalization group evolution. Second, $I^\dagger$ does not depend on the evolution of the overall scale of neutrino masses, and the renormalization effect on mixing angles is negligible [13] since the Yukawa coupling of $\tau$ lepton is small in the SM (the effect is estimated to be smaller than $1^\circ$ [14]). For simplicity, we have assumed two Majorana phases are zero. Note that if $R$ is real, $I^\dagger$ does not depend on the Majorana phases.

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1. Another possible parametrization is $R^\dagger R = \exp(iB)$, where $B$ is a real antisymmetric matrix [12]. This is concise, but values of each components are not explicit.

2. Without this condition, a general orthogonal Hermitian matrix can be written as

$$\pm \left[ \sqrt{1 + a^2} I - (\sqrt{1 + a^2} \pm 1) \bar{a} a^T / a^2 + i A \right].$$  

(24)
Figure 2: Majorana neutrino, NH case. Blue solid curve is the case of real $R$, or $\vec{a} = 0$. The upper and the lower dashed curves are taking $\vec{a}^T = (5.00, 2.89, 0.572)$ and $\vec{a}^T = (-1.98, 3.80, -5.00)$. Lower limit of the invariant $I^q$ is also shown by straight line.

As we can see from Figs. 2 and 3, the invariant $I^l$ largely depends on the imaginary part of $R^\dagger R$. If $R$ is real, the equality $I^l = I^q$ is satisfied only in the IH case with small $m_3$,

$$0.19 \text{ meV} \leq m_3 \leq 1.5 \text{ meV} \quad (28)$$

(see Table 1). Majorana neutrino masses can be searched by not only cosmological observations but also neutrinoless double beta decay experiments that are sensitive to $m_{\beta\beta} \equiv |\sum U_{ei}^2 m_i|$. By Eq. (28), we obtain

$$0.098 \text{ eV} \leq \sum m_i \leq 0.100 \text{ eV}, \quad (29)$$

$$0.018 \text{ eV} \leq m_{\beta\beta} \leq 0.048 \text{ eV} \quad (30)$$

(Majorana phases are considered for $m_{\beta\beta}$). These regions can be searched in a foreseeable future [10][16]. If $R$ is complex, $I^l$ can be much larger, and an interesting condition $I^l = 1$ can be satisfied for both NH and IH cases.

5 Summary

We can construct renormalization group invariants with Yukawa coupling matrices only. As pointed out in Ref. [1], an invariant $I^q$ in the quark sector is close to one, even though quark masses are highly hierarchical. This fact seems to imply a rule for the structure of Yukawa matrices. In this letter, we have calculated an invariant $I^l$ in the lepton sector, which is defined similarly to $I^q$. To calculate $I^l$ for Majorana neutrino cases, we introduced a useful parametrization of neutrino Yukawa couplings. We found that $I^l$ can be $O(1)$ for both neutrino types. A
hypothesized equality $I^l = I^q$ tends to favor inverted and strongly hierarchical neutrino masses, and such a mass spectrum can be explored by several experiments. As the quark sector invariant, $I^l = 1$ is also consistent for Majorana neutrino cases, but not for Dirac neutrino cases.

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**Table 1:** Summary of our numerical result. For Majorana cases, $R$ is assumed to be real. $I^l_{\text{peak}}$ denote the extremum of $I^l$ (when the lightest neutrino mass is $m_{\text{lightest}} = m_{\text{peak}}$). $m_{\mu = I^q}$ is the lightest neutrino mass that satisfies the condition $I^l = I^q$ within one standard deviation.

|                | Dirac | Majorana |
|----------------|-------|-----------|
| $I^l_{\text{peak}}$ |       |           |
| NH             | 0.260$^{+0.013}_{-0.025}$ | 0.756$^{+0.041}_{-0.061}$ | 0.191$^{+0.019}_{-0.018}$ | 0.754$^{+0.041}_{-0.061}$ |
| IH             | 0.756$^{+0.041}_{-0.061}$ |           | 0.191$^{+0.019}_{-0.018}$ | 0.754$^{+0.041}_{-0.061}$ |
| $m_{\text{peak}}$/meV | 7.06  | 5.30      | 7.59         | 0.568         |
| $m_{\mu = I^q}$/meV |       | 3.0...8.8 |       |       |

Figure 3: Majorana neutrino, IH case. The upper and the lower dashed curves are taking $\vec{a}^T = (5.00, 2.60, -0.235)$ and $\vec{a}^T = (-3.58, 5.00, -5.00)$. Straight horizontal lines show the central value and lower limit of $I^q$. Other notations are the same as in Fig.
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