UNSTABLE MODES IN
THREE-DIMENSIONAL SU(2) GAUGE THEORY

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Abstract

We investigate SU(2) gauge theory in a constant chromomagnetic field in three dimensions both in the continuum and on the lattice. Using a variational method to stabilize the unstable modes, we evaluate the vacuum energy density in the one-loop approximation. We compare our theoretical results with the outcomes of the numerical simulations.

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I. INTRODUCTION

It was pointed out by different authors \[1,2\] several years ago that for four-dimensional non Abelian gauge theories without matter fields in the one-loop approximation states with a constant chromomagnetic field lie below the perturbative ground state. This discovery stirred up a lot of interest. It was soon realized, however, by Nielsen and Olesen \[3\] that these states are not stable due to long-range modes, the Nielsen-Olesen unstable modes. It was been, then, conjectured \[4,5\] that the vacuum field configurations which differ from the classical external field only in the unstable mode sector could stabilize the Nielsen-Olesen unstable modes. The conclusion was reached that first the constant chromomagnetic field would form a lattice, which, however, could be unstable under quantum fluctuations. These quantum fluctuations might result in the formation of a quantum liquid, the so-called Copenhagen vacuum \[5\].

On the other hand, the problem of the unstable modes has been reconsidered from a different point of view \[6\]. In Ref. \[6\], henceforth referred as I, it has been showed that, by using variational techniques on a class of approximately gauge-invariant Gaussian wave functionals, the stabilization of the Nielsen-Olesen modes contributes to the energy density with a negative classical term which cancels the classical magnetic energy. Moreover, the stabilization of the Nielsen-Olesen modes induces a further background field which behaves non analytically in the coupling constant and screens almost completely the external chromomagnetic field. As a consequence, even in the strong-field regime, which is the naive perturbative regime, one deals with the non perturbative regime. This means that the calculation of the energy density even in the one-loop approximation is non perturbative \[7\]. In other words, the calculation of the vacuum energy is truly non perturbative and mandates a completely non perturbative approach.

In a previous paper \[8,9\] we investigate four dimensional U(1) and SU(2) lattice gauge theories in an external constant background field. In particular, for the non Abelian case we looked for the effects of the unstable modes on the vacuum energy density. However, the conclusion was reached that the Monte Carlo data did not display effects due to unstable modes owing to finite size of the lattice. Indeed, in four dimensions the unstable modes are long range. Then in lattice simulations one must approximate the continuum and, at the same time, work with a lattice big enough to allow unstable modes. So the external fields which can be put on a lattice with periodic boundary conditions have strength \[8,10\]:

\[
a^2 gB \simeq \frac{2\pi}{L^2} n \tag{1.1}
\]

where \(n\) is an integer, \(L\) is the lattice size, and \(a\) the lattice spacing. On general grounds \[11\], one expects that it exists a critical field \(B_c\) above which the system becomes paramagnetic. As a consequence, we need external magnetic fields lower than the critical field \(B_c\). Using the estimation \[11\] \(B_c \sim T_c^2, T_c/\Lambda_L \simeq 40\) and

\[
\Lambda_L a(\beta) = f(\beta) \simeq \left(\frac{6\pi^2\beta}{11}\right)^{\frac{31}{6}} \exp \left\{ -\frac{3\pi^2\beta}{11} \right\} \tag{1.2}
\]

we obtain for the minimal field \(B_{\text{min}}\) which can be put on a lattice:
For instance, for $\beta = 2.5$ one need $L > 12$ to have $B_{\text{min}}/B_c < 1$. Therefore, we see that in four dimensions working with external magnetic fields lower than the critical fields requires sizeable lattices. We can avoid these problems by considering three-dimensional gauge theories. Indeed, the three-dimensional SU(2) gauge theory is superrenormalizable. Moreover, as we shall discuss later, the unstable modes are not long range.

The purpose of this paper is twofold. First, in Sect. II through Sect. V we consider the SU(2) gauge theory in a constant chromomagnetic field in the fixed-time Schrödinger representation. In Sect. II we show that the quadratic part of the Hamiltonian displays unstable modes. In Sect. III we discuss the stabilization of the unstable modes by using the variational procedure of I. Section IV is devoted to the calculation of the stabilized vacuum energy density in the one-loop approximation. In Sect. V we analyze the Nielsen-Ninomiya Ansatz.

Second, in Sect. VI we reanalyze the problem by employing the lattice formulation of the gauge theories and compare the numerical results with theoretical expectations. Finally, our conclusions are drawn in Sect. VII.

II. SU(2) HAMILTONIAN IN A BACKGROUND FIELD

In this section we consider the (2+1) dimensional pure SU(2) gauge theory in the temporal gauge. We shall follow closely the method of I. In the temporal gauge $A_0^a = 0$ the Hamiltonian is

$$H = \frac{1}{2} \int d^2 x \left\{ (E^a_i(\vec{x}))^2 + (B^a(\vec{x}))^2 \right\},$$

(2.1)

where

$$B^a(\vec{x}) = \frac{1}{2} \epsilon_{ij} F^a_{ij}(\vec{x}),$$

(2.2)

and

$$F^a_{ij}(\vec{x}) = \partial_i A^a_j(\vec{x}) - \partial_j A^a_i(\vec{x}) + g \epsilon^{abc} A^b_i(\vec{x}) A^c_i(\vec{x}).$$

(2.3)

Note that in two spatial dimensions the chromomagnetic field $B^a(\vec{x})$ is a (pseudo)scalar.

In the fixed-time Schrödinger representation the chromoelectric field $E^a_i(\vec{x})$ acts as functional derivative

$$E^a_i(\vec{x}) = +i \frac{\delta}{\delta A^a_i(\vec{x})}$$

(2.4)

on the physical states which are functionals obeying the Gauss’ law:

$$[\partial_i \delta^{ab} + g \epsilon^{abc} A^c_i(\vec{x})] \frac{\delta}{\delta A^a_i(\vec{x})} G(A) = 0.$$  

(2.5)
The effects of an external background field is incorporated by writing
\[ A_i^a(x) = \bar{A}_i^a(x) + \eta_i^a(x) \] (2.6)
where \( \bar{A}_i^a(x) \) is the background field, and \( \eta_i^a(x) \) the fluctuating field. We are interested in a constant abelian chromomagnetic field. Thus we set
\[ \bar{A}_i^a(x) = \delta^{a3} \delta_{i2} x_1 B , \ x = (x_1, x_2) . \] (2.7)
It is straightforward to rewrite the Hamiltonian in terms of the fluctuating fields. We get:
\[ H = V \frac{B^2}{2} + H^{(2)} + H^{(3)} + H^{(4)} , \] (2.8)
where
\[ H^{(2)} = \frac{1}{2} \int d^2 x \left\{ -\frac{\delta^2}{\delta \eta_i^a(x) \delta \eta_i^a(x)} + \eta_i^a(x) D_{ij}^{ab}(\bar{A};x) \eta_j^b(x) - [D_{ij}^{ab}(\bar{A}) \eta_j^b(x)]^2 \right\} , \] (2.9)
\[ H^{(3)} = \frac{g}{2} \epsilon_{jk} \epsilon_{j'k'} \epsilon^{abc} \int d^2 x \ \eta_j^b(x) \eta_j^c(x) D_{ij'}^{ac}(\bar{A}) \eta_{k'}^a(x) \ , \] (2.10)
\[ H^{(4)} = \frac{g^2}{8} \epsilon_{jk} \epsilon_{j'k'} \epsilon^{abc} \epsilon^{a'b'c'} \int d^2 x \ \eta_j^b(x) \eta_j^c(x) \eta_{k'}^{a'}(x) \eta_{k'}^{c'}(x) . \] (2.11)
In Equations (2.8-2.11) \( D_{ij}^{ab}(\bar{A}) \) is the covariant derivative with respect to the background field:
\[ D_{ij}^{ab}(\bar{A}) = \partial_i \delta^{ab} + g \epsilon^{abc} \bar{A}_i^a(x) . \] (2.12)
\( D_{ij}^{ab}(\bar{A}, \bar{x}) \) is the operator
\[ D_{ij}^{ab}(\bar{A}, \bar{x}) = -\delta_{ij} D_{ak}^{bc}(\bar{x}) D_k^{cb}(\bar{x}) + 2g \epsilon_{ij} \epsilon^{3ab} . \] (2.13)
The Gauss’ constraint (2.5) can be rewritten as
\[ \left[ D_{ij}^{ab}(\bar{x}) + g \epsilon^{abc} \eta_i^c(\bar{x}) \right] \frac{\delta G(\bar{x})}{\delta \eta_i^a(\bar{x})} = 0 . \] (2.14)
As is well known [12], the Gauss’ constraint ensures that the physical states are invariant against time-independent gauge transformations. It follows, then, that the physical states are not normalizable. This means that we must fix the residual gauge invariance. Following I, we impose the covariant Coulomb constraint
\[ D_{ij}^{ab}(\bar{x}) \eta_i^b(\bar{x}) = 0 . \] (2.15)
As discussed in I, the functional measure in the scalar product between two physical states gets modified by the Fadeev-Popov determinant associated to the gauge-fixing (2.15).
We are interested in the vacuum energy. The one-loop approximation of the vacuum energy corresponds to consider the quadratic piece of the Hamiltonian. From equation (2.8) we get
\[ H_0 = \frac{V B^2}{2} + H^{(2)}. \] (2.16)

In the same approximation the Gauss’ constraint reduces to
\[ D_{ab}^i(\vec{x}) \frac{\delta \mathcal{G}(\eta)}{\delta \eta^i_b(\vec{x})} = 0 \] (2.17)

Equation (2.17) means that the physical states are functionals of the fields transverse with respect to the operator \( D_{ab}^i(\vec{x}) \).

To diagonalize \( H_0 \), it suffices to solve the eigenvalue equations
\[ D_{ij}^{ab}(\vec{A}; \vec{x}) \phi_j^b(\vec{x}) = \lambda \phi_i^a(\vec{x}) \] (2.18)
with the conditions
\[ D_{ij}^{ab}(\vec{x}) \phi_j^b(\vec{x}) = 0 . \] (2.19)

The method to solve Eqs. (2.18) and (2.19) has been discussed in detail in Appendix B of I. So we merely write down the final results. We find the following transverse eigenvectors:
\[ \phi_i^a(N, \alpha = 1; \vec{x}) = \delta^{a3} \frac{e^{i \vec{k} \cdot \vec{x}}}{\sqrt{2\pi}} e_i(\vec{k}) , \]
\[ N = \vec{k}, \quad \vec{k} \cdot \vec{\epsilon} = 0, \quad \lambda(N, \alpha = 1) = \vec{k}^2 . \] (2.20)

\[ \phi_j^a(N, \alpha = 2; \vec{x}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix} \phi_j^+(N; \vec{x}) + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & +i \\ 0 & 1 \end{pmatrix} \phi_j^-(N; \vec{x}) \]
\[ N = (p, n) \quad \lambda(N, \alpha = 2) = gB(2n + 1), \quad n \geq 1 \] (2.21)

where
\[ \phi_j^+(N, \alpha = 2; \vec{x}) = \frac{\alpha_1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix} F^+(p, n + 1; \vec{x}) + \frac{\alpha_2}{\sqrt{2}} \begin{pmatrix} 1 & +i \\ 0 & 1 \end{pmatrix} F^+(p, n - 1; \vec{x}) \]
\[ \phi_j^-(N, \alpha = 2; \vec{x}) = \left[ \phi_j^+(N, \alpha = 2; \vec{x}) \right]^* . \] (2.22)

In Equation (2.22) \( \alpha_1 = \sqrt{n} a \) and \( \alpha_2 = \sqrt{n + 1} a \); the constant \( a \) is fixed by the normalization condition
\[ \int d^2x \ \phi_i^+(N, \alpha; \vec{x}) \phi_i^-(N', \alpha; \vec{x}) = \delta_{NN'} . \] (2.23)
Moreover

\[
F^+(p, n; \vec{x}) = \frac{e^{+ipx_2}}{\sqrt{2\pi}} N_n \left[ \sqrt{gB} \left( x_1 + \frac{p}{gB} \right) \right] e^{-gB(x_1 + \frac{p}{gB})^2},
\]

\[
N_n = \left( \frac{gB}{\pi} \right)^{\frac{1}{4}} (2^n n!)^{-\frac{1}{2}}. \tag{2.24}
\]

Finally

\[
\phi^a_j(N, \alpha = 3; \vec{x}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \phi^+_j(N; \vec{x}) + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ +i \end{pmatrix} \phi^-_j(N; \vec{x})
\]

\[
N = p, \quad \lambda(p, \alpha = 3) = -gB, \tag{2.25}
\]

with

\[
\phi^+_j(p; \vec{x}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} F^+(p, n = 0; \vec{x})
\]

\[
\phi^-_j(p; \vec{x}) = \left[ \phi^+_j(p; \vec{x}) \right]^*. \tag{2.26}
\]

The presence of the eigenvector \(\phi^a_j(N, \alpha = 3; \vec{x})\) with negative eigenvalue implies that the quadratic Hamiltonian is not positive definite, and whence unbounded from below. According we can decompose the fluctuating field into a stable fluctuation and an unstable one

\[
\eta^a_i(\vec{x}) = \eta^a_{si}(\vec{x}) + \eta^a_{ui}(\vec{x}), \tag{2.27}
\]

where

\[
\eta^a_{si}(\vec{x}) = \sum_{N, \alpha = 1, 2} c(N, \alpha) \phi^a_i(N, \alpha; \vec{x}),
\]

\[
\eta^a_{ui}(\vec{x}) = \sum_N c(N, \alpha = 3) \phi^a_i(N, \alpha = 3; \vec{x}). \tag{2.28}
\]

Whereupon, the quadratic Hamiltonian reads

\[
H^{(2)} = H^{(2)}_s + H^{(2)}_u \tag{2.29}
\]

with \(H^{(2)}_s\) positive definite. From Equation (2.29) it follows that the vacuum functional can be factorized as

\[
\mathcal{G}_{\eta}(\eta) = \mathcal{W}_{\eta}(\eta) \mathcal{Z}_{\eta}(\eta), \tag{2.30}
\]

and the vacuum energy can be written as
\[ E(B) = V \frac{B^2}{2} + E_s + E_u . \] (2.31)

The contribution to the vacuum energy due to the stable modes is readily obtained in the one loop approximation.

It is straightforward to show that the stable mode vacuum functional is:

\[ \mathcal{W}_s(\eta_f) = \exp \left\{ -\frac{\infty}{\Delta} \int [\mathcal{E}] \mathcal{F}_f \left[ \eta_f^i(\vec{s}) (G^i)_j^l(\vec{s}, \vec{t}) \eta_j^l(\vec{t}) \right] \right\} , \] (2.32)

with

\[ (G_s)_{ij}^a(\vec{x}, \vec{y}) = \sum_{N, \alpha = 1, 2} 2\lambda \frac{\sqrt{n}}{\pi} \phi^a_i(N, \alpha; \vec{x}) \phi^b_j(N, \alpha; \vec{y}) . \] (2.33)

Then we obtain

\[ E_s = V \frac{B}{2} \int \frac{d^2 k}{(2\pi)^2} |\vec{k}| + V \frac{gB}{2\pi} \sum_{n=1}^{\infty} \sqrt{gB(2n+1)} . \] (2.34)

In obtaining Equation (2.34) we used

\[ \int dp |F(p, n; \vec{x})|^2 = \frac{gB}{2\pi} . \] (2.35)

On the other hand, we have

\[ H^{(2)}_u = \frac{1}{2} \int d^2 x \left\{ \frac{\delta^2}{\delta \eta_{ui}(\vec{x}) \delta \eta_{ui}(\vec{x})} - gB\eta^a_i(\vec{x})\eta^a_i(\vec{x}) \right\} , \] (2.36)

so that \( H^{(2)}_u \) is unbounded from below. In order to stabilize \( H^{(2)}_u \) we must take into account the quartic coupling which is of order of \( g^2 \). This means that we must take care of the Gauss’ constraint at least at the order \( g^2 \).

The main strategy of these calculation has been discussed in Section III of I. In the next section we shall summarize the main achievements of our analysis.

**III. STABILIZING THE UNSTABLE MODES**

According to the previous section, we would like to stabilize the unstable modes by taking into account the quartic coupling. To do this, we must construct the variational vacuum functional which satisfies the Gauss’ constraint up to order of \( g^2 \). We assume for the vacuum functional the form in Eq. (2.30). The presence in Eq. (2.36) of a negative mass squared term suggests to try with a shifted Gaussian for the functional \( Z_i(\eta) \):

\[ Z_i(\eta) = \exp \left\{ -\frac{\infty}{\Delta} \int [\mathcal{E}] \mathcal{F}_i \left[ \eta_{ij}(\vec{s}) - \eta^l_{ij}(\vec{s}) \right] (G_i^l)_{ij}^l(\vec{s}, \vec{t}) \left[ \eta_{ij}(\vec{t}) - \eta^l_{ij}(\vec{t}) \right] \right\} , \] (3.1)

where
\[(G_u)_{ij}^{ab}(\vec{x}, \vec{y}) = \sum_p 2\rho(p)\phi_i^a(p, \alpha = 3; \vec{x})\phi_j^b(p, \alpha = 3; \vec{y}) , \] (3.2)

\[u_i^a(\vec{x}) = \sum_p b(p)\phi_i^a(p, \alpha = 3; \vec{x}) . \] (3.3)

The vacuum functional \(G_\eta(\eta)\) satisfies the Gauss’ law in the lowest approximation Eq. (2.17). In order to impose the full constraint Eq. (2.14), we write

\[G'_\eta(\eta) = \exp \left[ -\int d^2x \right] G_\eta(\eta) , \] (3.4)

and fix iteratively the functional \(\Gamma_0(\eta)\).

To proceed, we evaluate the expectation value of the Hamiltonian on the vacuum functional (3.4). Fortunately this rather long calculation parallels closely the one done in the case of three spatial dimensions. The reader interested in further details may consult section IV of I.

In the one-loop approximation, the stable mode contribution to the vacuum energy has been already evaluated, Eq. (2.34). For the unstable modes we get in the same approximation

\[E_u = \frac{1}{2} \int d^2x \int d^2y \delta(\vec{x} - \vec{y}) D_{ij}^{ab}(\vec{x}) \left[ (G_u^{-1})_{ij}^{ab}(\vec{x}, \vec{y}) + u_i^a(\vec{x})u_j^b(\vec{y}) \right] \]

\[+ \frac{1}{8} \int d^2x \ (G_u)_{ii}^{aa}(\vec{x}, \vec{x}) + E_u^{(4)} , \] (3.5)

\[E_u^{(4)} = \frac{g^2}{4} \epsilon_{jk} \epsilon_{j'k'} \epsilon^{abc} \epsilon^{ob'c'} \int d^2x \left\{ u_j^b(\vec{x})u_k^c(\vec{x})u_{j'}^{b'}(\vec{x})u_{k'}^{c'}(\vec{x}) \right\} + \left[ (G_u^{-1})_{jj'}^{bb'}(\vec{x}, \vec{x}) u_k^c(\vec{x})u_{k'}^{c'}(\vec{x}) + \text{two permutations} \right] \} . \] (3.6)

In order to minimize \(E_u\) with respect to \(\rho(p)\) and \(b(p)\), we observe that from Eqs. (3.3), (2.23) and (2.26) it follows:

\[u_j^3(\vec{x}) = 0 \]

\[u_j^e(\vec{x}) = \frac{1}{\sqrt{2}}(u_j^1 \pm i u_j^2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \end{pmatrix} g^e(\vec{x}) \] (3.7)

with

\[g^e(\vec{x}) = \sum_p b(p)e^{\pm ipx_2} \frac{gB}{\sqrt{2\pi}} \left( \frac{gB}{\pi} \right)^{\frac{1}{4}} e^{-\frac{gB}{2}(x_1 + \frac{p}{gB})^2} . \] (3.8)

Using Equations (3.2) and (3.7), we recast \(E_u\) into:
\[ E_u = \frac{1}{2} \sum_p \left[ \rho(p) - \frac{gB}{\rho(p)} \right] - gB \int d^2x \ g^+(\vec{x})g^-(\vec{x}) + \frac{g^2}{2} \int d^2x \ \left[ g^+(\vec{x})g^-(\vec{x}) \right]^2 \]
\[ + g^2 \sum_p \frac{1}{2\rho(p)} \int d^2x \ \left\{ 2 \left| F^+(p, n = 0; \vec{x}) \right|^2 g^+(\vec{x})g^-(\vec{x}) \right. \]
\[ + \left. \frac{1}{2} F^+(p, n = 0; \vec{x})g^-(\vec{x}) + \frac{1}{2} F^-(p, n = 0; \vec{x})g^+(\vec{x}) \right\}. \quad (3.9) \]

As discussed in I, the necessary condition so that our configuration contributes to the energy density is:
\[ \int d^2x \ g^+(\vec{x})g^-(\vec{x}) \propto V \equiv L^2. \quad (3.10) \]

Moreover it is easy to show that the minimum of the energy is attained for
\[ g^+(\vec{x})g^-(\vec{x}) = K \quad (3.11) \]
where \( K \) is a positive constant. The condition (3.11) is fulfilled with the choice:
\[ b(p) = \sqrt{2\pi K} \ e^{iLp} \quad (3.12) \]
in the thermodynamic limit \( L \to \infty \). From Equations (3.11) and (3.12) we obtain
\[ E_u = \frac{1}{2} \sum_p \left[ \rho(p) - \frac{gB}{\rho(p)} \right] - VKgB + \frac{g^2}{2} VK^2 \]
\[ + g^2 K \sum_p \frac{1}{\rho(p)} \int d^2x \ \left| F^+(p, n = 0; \vec{x}) \right|^2. \quad (3.13) \]

The last equation tells us that we can assume \( \rho(p) \) independent on \( p \). Thus, by taking into account that (3.14)
\[ \sum_p = V \frac{gB}{2\pi} \quad (3.14) \]
and
\[ \sum_p \left| F^+(p, n = 0; \vec{x}) \right|^2 = \frac{gB}{2\pi}, \quad (3.15) \]
we get finally
\[ \frac{E_u}{V} = \frac{gB}{4\pi} \left[ \rho - \frac{gB}{\rho} \right] + g^2 \frac{gB K}{2\pi \rho} - gBK + \frac{g^2}{2} K^2. \quad (3.16) \]

Varying with respect to \( K \), we get:
\[ K = \frac{gB}{g^2} - \frac{gB}{2\pi} \frac{1}{\rho}. \quad (3.17) \]
Inserting into Eq. (3.16) and neglecting terms of the same order as the two loop contributions, we obtain the simple result:

\[ \frac{E_u}{V} = -\frac{B^2}{2} + \frac{gB}{4\pi} \left[ \rho + \frac{gB}{\rho} \right]. \]  

(3.18)

Now, the minimization with respect to \( \rho \) is straightforward:

\[ \rho^2 = gB. \]  

(3.19)

Whence

\[ \frac{E_u}{V} = -\frac{B^2}{2} + \frac{(gB)^{3/2}}{2\pi} \left[ 1 + \frac{\rho}{\epsilon} \right] \rho. \]  

(3.20)

In conclusion we get for the total vacuum energy in the one-loop approximation the remarkable result:

\[ \frac{E(B)}{V} = \frac{B^2}{2} + \frac{E_s}{2} - \frac{B^2}{2} + \frac{(gB)^{3/2}}{2\pi} \left[ 1 + \frac{\rho}{\epsilon} \right]. \]  

(3.21)

\[ \frac{E_s}{V} = \frac{1}{2} \int \frac{d^2k}{(2\pi)^2} |\vec{k}| \sum_{n=1}^{\infty} \sqrt{gB(2n + 1)}. \]  

(3.22)

The last two terms in Equation (3.21) are the contributions due to the stabilized unstable modes. Note that, as in three spatial dimensions, the unstable modes contribute to the vacuum energy density with a negative classical term which cancels out the classical magnetic energy term.

**IV. THE STABILIZED VACUA**

In the previous section we evaluated the vacuum energy density in the one-loop approximation. If we neglect the unstable mode contribution our result should coincide with the real part of the one-loop effective potential:

\[ V^{(2)}(B) = \frac{B^2}{2} + \frac{1}{2} \int \frac{d^2k}{(2\pi)^2} |\vec{k}| \sum_{n=1}^{\infty} \sqrt{gB(2n + 1)}. \]  

(4.1)

Obviously \( V^{(2)}(B) \) is divergent. However the theory is superrenormalizable. This means that once we subtract the free vacuum energy density we are left with a finite result. As we show below this is the case and the final result

\[ \Delta V^{(2)}(B) \equiv V^{(2)}(B) - V^{(2)}(0) = \frac{B^2}{2} - \frac{(gB)^{3/2}}{2\pi} \left[ 1 - \frac{\sqrt{2} - 1}{4\pi} \zeta \left( \frac{3}{2} \right) \right], \]  

(4.2)

where \( \zeta(z) \) is the Riemann’s Zeta function, is in agreement with the calculation by H. D. Trottier [15].

Indeed, by using the identity:
\[
\sqrt{a} = - \int_{0}^{\infty} \frac{ds}{\sqrt{\pi \sqrt{s}}} \frac{d}{ds} e^{-as},
\tag{4.3}
\]
we have
\[
gB \frac{2\pi}{2\pi} \sum_{n=1}^{\infty} \sqrt{gB(2n+1)} = -gB \frac{2\pi}{2\pi} \sum_{n=1}^{\infty} \int_{0}^{\infty} \frac{ds}{\sqrt{\pi \sqrt{s}}} \frac{d}{ds} e^{-gB(2n+1)s} = -gB \frac{2\pi}{2\pi} \int_{0}^{\infty} \frac{ds}{\sqrt{\pi \sqrt{s}}} \frac{d}{ds} \left[ \frac{1}{e+gBs - e^{-gBs} - e^{-gBs}} \right].
\tag{4.4}
\]

Moreover, we observe that
\[
\int \frac{d^2k}{(2\pi)^2} \left| \vec{k} \right| = -\frac{1}{4\pi} \int_{0}^{\infty} \frac{ds}{\sqrt{\pi \sqrt{s}}} \frac{d}{ds} \left( \frac{1}{s} \right).
\tag{4.5}
\]
Thus, Equations (4.4) and (4.5) ensure that
\[
\frac{3}{2} \int \frac{d^2k}{(2\pi)^2} \left| \vec{k} \right| = V^{(2)}(0).
\tag{4.6}
\]
Whence
\[
\Delta V^{(2)}(B) = \frac{B^2}{2} - \frac{(gB)^{3/2}}{2\pi} - \frac{gB}{2\pi} \int_{0}^{\infty} \frac{ds}{\sqrt{\pi \sqrt{s}}} \frac{d}{ds} \left\{ \frac{1}{e+gBs - e^{-gBs} - e^{-gBs}} \right\}.
\tag{4.7}
\]
A change of the integration variable recasts Eq. (4.7) into (10)
\[
\Delta V^{(2)}(B) = \frac{B^2}{2} - \frac{(gB)^{3/2}}{2\pi} \left[ 1 + \int_{0}^{\infty} \frac{dx}{\sqrt{\pi \sqrt{x}}} \frac{d}{dx} \left\{ \frac{1}{e^x - e^{-x} - \frac{1}{2x}} \right\} \right]
\[
= \frac{B^2}{2} - \frac{(gB)^{3/2}}{2\pi} \left[ 1 - \sqrt{2} \zeta \left( -\frac{1}{2}, \frac{1}{2} \right) \right]
\tag{4.8}
\]
where \(\zeta(z, q)\) is the generalized Riemann’s Zeta function.
Observing that (10)
\[
\sqrt{2} \zeta \left( -\frac{1}{2}, \frac{1}{2} \right) = \frac{\sqrt{2} - 1}{4\pi} \zeta \left( \frac{3}{2} \right),
\tag{4.9}
\]
we obtain the promised result Eq. (4.2).
The one-loop effective potential (1.8) has a negative minimum which, however, lies in a region where the one-loop approximation is not trustworthy.
When we take into account the unstable modes we get (17)
\[
\frac{\Delta E(B)}{V} = +\frac{\sqrt{2} - 1}{8\pi^2} \zeta \left( \frac{3}{2} \right) (gB)^{3/2} + O \left( \frac{1}{\pi^2} \right),
\tag{4.10}
\]
In Equation (4.10) the coefficient in front of \((gB)^{3/2}\) is positive. As a consequence, the minimum of the vacuum energy density is attained for \(B = 0\).
It should be stressed that the calculation of the full contribution to the vacuum energy is non trivial even in the one-loop approximation. As a matter of fact, Equation (4.10) does not include the contributions to the energy arising from the interaction of the stable mode with the induced background field $u_{i}^a(\vec{x})$ which behaves non analytically in the coupling constant. Nevertheless, the cancellation of the classical magnetic energy term due to the stabilization of the unstable modes is sound. So that we can write in general

$$\frac{\Delta E(B)}{V} = a_{3/2}(gB)^{3/2} + O\left(\frac{1}{\epsilon}\right) \ .$$  \hspace{1cm} (4.11)

In order to avoid the unphysical situation where the vacuum energy density decreases without bound for increasing external magnetic field, the constant $a_{3/2}$ should be positive, so that the minimum is again $B = 0$.

It is also interesting to note that the total background field is now given by:

$$\bar{A}^a_i(\vec{x}) = \bar{A}(\vec{x}) + \bar{u}^a(\vec{x}) \ .$$  \hspace{1cm} (4.12)

From Equations (3.7), (3.11), and (3.17) we obtain for the full field strength tensor

$$F^3_{ij} = \frac{g}{2\pi} \sqrt{gB} (\delta_{i1}\delta_{j2} - \delta_{i2}\delta_{j1}) \ .$$  \hspace{1cm} (4.13)

On the other hand the external field strength is

$$F_{ij}^{\text{ext}} \equiv \partial_i A^3_j - \partial_j A^3_i = B (\delta_{i1}\delta_{j2} - \delta_{i2}\delta_{j1}) \ .$$  \hspace{1cm} (4.14)

It follows that

$$\frac{F^3_{12}}{F_{12}^{\text{ext}}} = \frac{1}{2\pi} \left(\frac{B}{g^3}\right)^{-\frac{1}{2}} \ .$$  \hspace{1cm} (4.15)

Equation (4.15) can be compared with the result we should have obtained by neglecting the unstable modes:

$$\frac{F^3_{12}}{F_{12}^{\text{ext}}} = 1 \ .$$  \hspace{1cm} (4.16)

Equation (4.15) tells us that the background field induced by the stabilization of the unstable modes strongly screens the external magnetic field.

In conclusions, the main results of this section are summarized in Eqs. (4.12) and (4.15). Note, however, that our results Eqs. (4.12) and (4.15) are valid only in the thermodynamic limit $L \rightarrow \infty$. In view of the comparison with Monte Carlo simulations on a finite lattice, it is worthwhile to discuss another Ansatz due to H. B. Nielsen and N. Ninomiya [4] which can be realized even on a finite lattice.
V. THE NIELSEN AND NINOMIYA ANSATZ

In this Section we discuss the Ansatz by Nielsen and Ninomiya [4]:

\[
g^+(\vec{x}) = \sum_p b(p) \frac{e^{ipx_2}}{\sqrt{2\pi}} \left( \frac{gB}{\pi} \right)^{1/4} e^{-\frac{gB}{2\pi} (x_1 + \frac{ix_2}{gN})^2}
\]

(5.1)

\[
b(p) = \sqrt{2\pi K_N} \sum_n \delta(p - nc), \quad c = \sqrt{2\pi gB}
\]

(5.2)

with \(K_N\) constant to be fixed by minimizing the vacuum energy density. Using the definition of the third Jacobi \(\theta\) function [16]

\[
\theta_3(z, q) = 1 + 2 \sum_{n \geq 1} q^{n^2} \cos 2nz,
\]

(5.3)

we rewrite Eq.(5.1) as

\[
g^+(\vec{x}) = \sqrt{K_N} \left( \frac{gB}{\pi} \right)^{1/4} e^{-\frac{gB}{2\pi} x_1^2} \theta_3(z, q)
\]

(5.4)

where

\[
q = e^{-\pi},
\]

(5.5)

\[
z = \frac{c}{2} (x_2 + ix_1).
\]

(5.6)

It is easy to show that

\[
\int d^2 x \ g^+(\vec{x})g^-(\vec{x}) = K_N \left( \frac{gB}{\pi} \right)^{1/2} \int dx_1 dx_2 \ e^{-gBx_1^2} |\theta_3(z, q)|^2
\]

\[
= K_N \left( \frac{gB}{\pi} \right)^{1/2} \int dx_1 dx_2
\]

(5.7)

and

\[
\int d^2 x \ [g^+(\vec{x})g^-(\vec{x})]^2 = K_N^2 \frac{gB}{2\pi} [\theta_3(0, q)]^2 \int dx_1 dx_2.
\]

(5.8)

Inserting into Eq.(3.9), we get

\[
E_u = \frac{1}{2} \sum_p \left[ \rho(p) - \frac{gB}{\rho(p)} \right] - gBK_N \left( \frac{gB}{2\pi} \right)^{1/2} V + \frac{g^2}{2} K_N^2 \left( \frac{gB}{2\pi} \right) [\theta_3(0, q)]^2 V
\]

\[
+ g^2 \sum_p \frac{1}{\rho(p)} \int d^2 x \ |F^+(p, n = 0; \vec{x})| g^+(\vec{x})g^-(\vec{x}).
\]

(5.9)
Assuming $\rho(p)$ independent on $p$, and using Eqs. (3.14) and (3.15), we recast Eq. (5.9) into:

$$
\frac{E_U}{V} = \frac{g B}{4\pi} \left[ \rho - \frac{g B}{\rho} \right] - g B \left( \frac{g B}{2\pi} \right)^{1/2} K_N + \frac{g^2}{2} \frac{g B}{2\pi} K_N^2 \left[ \theta_3(0, q) \right]^2 
$$

\begin{equation}
+ g^2 \frac{g B}{2\pi} K_N \left( \frac{g B}{2\pi} \right)^{1/2} \frac{1}{\rho}.
\end{equation}

Varying with respect to $K_N$ we obtain

$$
K_N \left( \frac{g B}{2\pi} \right)^{1/2} = \frac{g B}{g^2 \theta_3(0, q)^2} - \frac{g B}{2\pi} \frac{1}{\theta_3(0, q)^2} \frac{1}{\rho},
\end{equation}

whereupon

$$
E_u = \frac{g B}{4\pi} \left[ \rho + \frac{g B \epsilon}{\rho} \right] - \frac{B^2}{2[\theta_3(0, q)]^2}
\end{equation}

with

$$
\epsilon = \frac{2}{[\theta_3(0, q)]^2} - 1.
\end{equation}

Note that $\theta_3(0, q) = 1 + 2e^{-\pi} + 2e^{-4\pi} + \cdots \approx 1.0864$, so that $\epsilon > 0$. The minimization with respect to $\rho$ is now straightforward. We get:

$$
\rho = \sqrt{g B \epsilon},
\end{equation}

$$
\frac{E_u}{V} = -\frac{1}{2[\theta_3(0, q)]^2} B^2 + \frac{\sqrt{\epsilon}}{2\pi} (g B)^{3/2}.
\end{equation}

Finally, the total vacuum energy density is

$$
\frac{\Delta E(B)}{V} = \frac{1}{2} \left[ 1 - \frac{1}{[\theta_3(0, q)]^2} \right] B^2 - \frac{(g B)^{3/2}}{2\pi} \left[ 1 - \sqrt{\epsilon} - \frac{\sqrt{2}}{4\pi} \zeta(3/2) \right]
\end{equation}

\begin{equation}
+ \mathcal{O} \left( \frac{\epsilon}{\pi} \right).
\end{equation}

We stress once again that the coefficient of $(g B)^{3/2}$ in Eq. (5.16) does not include the contributions due to the interaction between stable modes and the induced background field $u_i^a(\vec{x})$.

Note that the coefficient of the $(g B)^{3/2}$ term is negative, so that Eq. (5.16) displays a negative minimum. However, the minimum lies in a region where two-loop terms are sizeable.

It is instructive to evaluate the field-strength tensor. The only non-zero component of $F_{\mu\nu}$ is $F_{12}^3$. We obtain
\[ F_{12}^3(\vec{x}) = B - gK_N \left( \frac{gB}{2\pi} \right)^{1/2} e^{-gBx^2} |\theta_3(z, q)|^2, \]  
\hspace{2.5cm} (5.17)

one can show [4] that the chromomagnetic field forms a square lattice in the \( x_1 - x_2 \) plane with lattice constant \( a_N = \sqrt{2\pi/gB} \).

Defining a space averaging
\[
\langle F_{12}^3 \rangle = \frac{\int d^2x \, F_{12}^3(\vec{x})}{\int d^2x},
\]
and using Eq.\((5.17)\), we get
\[
\langle F_{12}^3 \rangle = B - gK_N \left( \frac{gB}{2\pi} \right)^{1/2}.
\]
\hspace{2.5cm} (5.19)

Finally, Equations \((5.11)\) and \((5.14)\) bring on
\[
\frac{\langle F_{12}^3 \rangle}{F_{12}^3} = \left( 1 - \frac{1}{|\theta_3(0, q)|^2} \right) + \frac{1}{2\pi \sqrt{\epsilon \theta_3(0, q)^2}} \frac{1}{\sqrt{B/g^3}},
\]
\hspace{2.5cm} (5.20)

where \( F_{12}^{ex} = B \).

The Nielsen and Ninomiya Ansatz is interesting in view of the comparison with the lattice approach to be discussed in the next section. Indeed the chromomagnetic lattice could be more easily realized on the space-time lattice if the two lattices are commensurate. In the lattice approach the external chromomagnetic field is quantized due to the periodic boundary conditions:
\[
a^2 gB = 2\pi \frac{L^2}{n}, \quad n \text{ integer} \]  
\hspace{2.5cm} (5.21)

where \( a \) is the space-time lattice spacing, and \( L \) the linear lattice size in lattice units. From Equation \((5.21)\) we get
\[
\frac{a_N}{a} = \frac{L}{\sqrt{2n}}.
\]
\hspace{2.5cm} (5.22)

Thus the lattices are commensurate if \( 2n \) is the square of an integer.

**VI. BACKGROUND FIELDS ON THE LATTICE**

From the previous section we learned that the calculation of the vacuum energy is truly non-perturbative even in the one-loop approximation. Thus, we need a non-perturbative approach which can be furnished by the lattice formulation of gauge theories. In Ref. [8] we studied the four-dimensional pure SU(2) lattice gauge theory in an external magnetic field. We looked for the effects due to unstable modes. Due to the limited size of the lattice we used in the Monte Carlo simulations, we failed in observing the unstable mode effects. The reason for such a failure have been already discussed in the Introduction.
The situation should improve considerably working with three-dimensional SU(2) gauge theory. Indeed in $d = 3$ we can perform Monte Carlo simulations on lattices of considerable size. In addition, in two spatial dimensions the unstable modes are not long range. The pure SU(2) gauge theory is implemented on the lattice through the standard Wilson action

$$S_W = \beta \sum_{x,\mu>\nu} U_{\mu\nu}(x) \quad (6.1)$$

where $U_{\mu\nu}(x)$ is the elementary plaquette in the $(\mu, \nu)$-plane at the lattice site $x$.

The external background field on the lattice can be introduced via an external current [8,10]. For the reader convenience, we briefly summarize our method.

In the Euclidean continuum the background action reads

$$S_B = \int d^3 x \ j^a_\mu(x) A^a_\mu(x) \ . \quad (6.2)$$

Using the classical field equations:

$$j^a_\mu(x) = D^{ab}_\nu(\bar{A}) \bar{F}^{b}_{\nu\mu}(x) \ , \quad (6.3)$$

$\bar{F}^{b}_{\nu\mu}$ being the field strength tensor built from the external fields $\bar{A}^a_\mu(x)$, and

$$\bar{F}^{a}_{\nu\mu}(x) = \delta^{a3} F^{\text{ext}}_{\nu\mu}(x) \ , \quad (6.4)$$

we get from Eq. (6.2)

$$S_B = -\frac{1}{2} \int d^3 x \ F^{\text{ext}}_{\nu\mu}(x) \left[ \partial_\mu A^3_\nu(x) - \partial_\nu A^3_\mu(x) \right] \ . \quad (6.5)$$

To discretize the action Eq. (6.5), we must define the Abelian-like piece $\partial_\mu A^3_\nu(x) - \partial_\nu A^3_\mu(x)$ from the plaquette variables. To do this, we use the so-called Abelian projection [18,19].

To implement the Abelian projection, first we fix the gauge by diagonalizing an operator $X(x)$ which transforms according to the adjoint representation of the gauge group:

$$V(x) X(x) V^\dagger(x) = \text{diagonal matrix} \ . \quad (6.6)$$

After that, we rewrite the gauge-fixed links

$$\tilde{U}_\mu(x) = V(x) U_\mu(x) V^\dagger(x + \mu) \quad (6.7)$$

as

$$\tilde{U}_\mu(x) = W_\mu(x) U^A_\mu(x) \ , \quad (6.8)$$

where

$$U^A_\mu(x) = \text{diag} \left[ e^{i\theta^A_\mu(x)} , e^{-i\theta^A_\mu(x)} \right] \quad (6.9)$$

$$\theta^A_\mu(x) = \arg \left\{ \left[ \tilde{U}_\mu(x) \right]_{11} \right\} \ . \quad (6.10)$$
The Abelian-like links (6.9) are called the Abelian projection of \( U_\mu(x) \). The Abelian projected plaquettes \( U^A_{\mu\nu} \) are built from the Abelian projected links in the usual manner. Clearly we have

\[
U^A_{\mu\nu}(x) = \text{diag} \left[ e^{ig^A_{\mu\nu}(x)}, e^{-ig^A_{\mu\nu}(x)} \right].
\]  

(6.11)

It is, thus, natural to define the Abelian field strength tensor as

\[
F^A_{\mu\nu}(x) = \sqrt{\beta} \text{ tr} \left[ \sigma_3 U^A_{\mu\nu}(x) \right] = \sqrt{\beta} \sin \theta^A_{\mu\nu}(x).
\]  

(6.12)

As a consequence, we are led to consider the following background action

\[
S_B = -\sqrt{\beta} \sum_x F_{\mu\nu}^\text{ext} \sin \theta^A_{\mu\nu}(x).
\]  

(6.13)

Taking into account the periodic boundary conditions of the lattice, we write

\[
F_{\mu\nu}^\text{ext}(x) = \sqrt{\beta} \sin \theta_{\mu\nu}^\text{ext}(x),
\]  

(6.14)

\[
\theta_{\mu\nu}^\text{ext}(x) = \frac{2\pi}{L^2} n^\text{ext}_{\mu\nu}(x),
\]  

(6.15)

the \( n^\text{ext}_{\mu\nu}(x) \)'s being integers. In Equation (6.13), \( L \) is the lattice size.

We are interested in a constant Abelian chromomagnetic field. In this case the lattice action reads:

\[
S = +\beta \sum_{x,\mu,\nu} U_{\mu\nu}(x) - \beta \sum_x \sin \theta^A_{12}(x) \sin \theta^A_{12}(x).
\]  

(6.16)

The quantity of interest is the vacuum energy density at zero temperature in presence of the external magnetic field. In the naive continuum limit one can show easily that the vacuum energy density is given by

\[
E \left( F^\text{ext}_{12} \right) = \beta \left[ P_s \left( F^\text{ext}_{12} \right) - P_t \left( F^\text{ext}_{12} \right) \right]
\]  

(6.17)

with

\[
P_s \left( F^\text{ext}_{12} \right) = 1 - \frac{1}{2} \text{ tr} \left[ U_{12} \left( F^\text{ext}_{12} \right) \right]
\]  

(6.18)

\[
P_t \left( F^\text{ext}_{12} \right) = \sum_{i=1,2} \left[ 1 - \frac{1}{2} \text{ tr} \left[ U_{3i} \left( F^\text{ext}_{12} \right) \right] \right].
\]  

(6.19)

We performed Monte Carlo simulations with the action (6.16) on lattices of size \( L = 20 \), and \( L = 40 \) in the weak coupling region \( \beta \geq 7 \). We measured the quantity

\[
\Delta E \left( F^\text{ext}_{12} \right) \equiv E \left( F^\text{ext}_{12} \right) - E(0) = \beta \left\{ \langle P_s \left( F^\text{ext}_{12} \right) \rangle - \langle P_s(0) \rangle + \langle P_t(0) \rangle - \langle P_t \left( F^\text{ext}_{12} \right) \rangle \right\}.
\]  

(6.20)
Our simulations were done in the unfixed gauge, which corresponds to set \( V(x) = \mathbb{I} \) in Eq. (6.7).

We stress that the background action Eq. (6.13) depends on the gauge-fixing. On the other hand, in the continuum limit the energy density is gauge invariant. This means that one should verify that the lattice definition (6.17) does not depend on the gauge-fixing procedure. In our previous study in four-dimensions we found a very weak dependence on the gauge-fixing. Moreover we are interested in the effects due to the unstable modes. As we have already discussed, the unstable modes modify in a dramatic way the vacuum energy density. So that the effects we are looking for should manifest even without any gauge fixing.

After discarding 500 sweeps, we collect 500 measurements (1 every 5 sweeps). Statistical errors were evaluated by the jackknife algorithm.

In order to reduce the statistical errors the difference \( \langle P_s - P_t \rangle \) was evaluated directly during Monte Carlo runs. Note that the vacuum energy difference can be also measured directly by observing that, within statistical errors, we have

\[
\langle \text{tr } U_{3i} \left( F_{12}^{\text{ext}} \right) \rangle = \langle \text{tr } U_{3i}(0) \rangle = \langle \text{tr } U_{12}(0) \rangle . \tag{6.21}
\]

In Figure 1 we display the adimensional energy density \( \Delta E \left( F_{12}^{\text{ext}} \right) / g^6 \) versus the adimensional external field strenght \( F_{12}^{\text{ext}} / g^3 \) for four different values of \( \beta \).

In general the vacuum energy can be affected by finite lattice effects. To avoid lattice artefacts we used quite large lattices. We found that for \( 20^3 \) lattices there are no sizeable finite size effects up to \( \beta = 10 \) (compare full and open circles in Fig. 1). On the other hand for \( \beta > 10 \) we need larger lattices (see full and open triangles in Fig. 1). As a consequence for \( \beta = 12 \) and 15 we doubled the lattice size, so that we feel that our numerical results are reliable.

In the continuum the adimensional energy density can depend on the unique adimensional combination at our hands, namely \( B / g^3 \). So we can write

\[
\Delta E(B) / g^6 = f \left( \frac{B}{g^3} \right) . \tag{6.22}
\]

In the one-loop approximation we have

\[
f(x) = a_{3/2} x^{3/2} + a_2 x^2 \tag{6.23}
\]

where

\[
a_{3/2} = \frac{1}{2\pi} \left[ 1 - \sqrt{2} - \frac{1}{4\pi} \zeta(3/2) \right] , \quad a_2 = \frac{1}{2} \tag{6.24}
\]

if we neglect the unstable modes. Including the unstable modes we obtained

\[
a_{3/2} = \frac{\sqrt{2} - 1}{8\pi^2} \zeta(3/2) , \quad a_2 = 0 \tag{6.25}
\]

in the approximation of section [IV] and

\[
a_{3/2} = -\frac{1}{2\pi} \left[ 1 - \sqrt{\epsilon} - \sqrt{2} - \frac{1}{4\pi} \zeta(3/2) \right] , \quad a_2 = \frac{1}{2} \left[ 1 - \frac{1}{[\theta_3(0, q)]^2} \right] \tag{6.26}
\]
for the Nielsen and Ninomiya Ansatz. Note that the unstable modes cause a drastic reduction of the classical magnetic energy term.

On the lattice there is another dimensional quantity, namely the lattice spacing. Whereupon Equation (6.22) is a non trivial check for the Monte Carlo outcomes. Any deviations from the scaling law (6.22) can be considered as an indication of the granularity of the lattice.

A glance at Fig. 1 shows that the scaling law (6.22) is satisfied quite well in the weak field strength region $F_{\text{ext}}/g^3 \leq 1$. But, for strong field strength $F_{\text{ext}}/g^3 > 1$ we observe a small deviation from the scaling law (6.22). This found a quite natural explanation. Indeed, in order to approximate the continuum, the magnetic length $1/\sqrt{gF_{\text{ext}}}$ should be greater than the lattice spacing. Thus, at fixed lattice spacing, this condition gets worse by increasing the external field strength.

In Figure 1 we compare our numerical results with the theoretical expectations. The dotted line is the one-loop effective potential (6.24), the dashed line is Eq. (6.25), and the dot-dashed line Eq. (6.26). Note that the theoretical calculations are restricted to the one-loop approximation. However this is not a limitation at all. Indeed the two-loop terms are important for

$$\frac{g^2}{2\pi} \frac{gB}{2\pi} \gtrsim \frac{(gB)^{3/2}}{2\pi},$$

i.e.

$$\frac{B}{g^3} \lesssim \frac{1}{4\pi^2}. \quad (6.27)$$

Unfortunately our Monte Carlo simulations do not allow us to appreciate the vacuum energy differences already for $B/g^3 \lesssim 0.4$. From Figure 1 it is evident that the Monte Carlo data strongly disagree with the one-loop effective potential Eq (6.24). Indeed the vacuum energy density is about an order of magnitude smaller than the classical magnetic energy. We feel that this reduction can be ascribed to the unstable modes. As a matter of fact, our numerical results agree with Eq. (6.25) for $F_{\text{ext}}/g^3 \lesssim 1$. On the other hand for $F_{\text{ext}}/g^3 \gtrsim 1$ the data can fitted by using Eq. (6.23) with $a_2 \sim 10^{-2}$. However, as we have already stressed, the complete cancellation of the classical magnetic energy can be attained for configurations which satisfy Eq. (3.11). This condition can be fulfilled only in the thermodynamic limit. When one deals with a finite lattice, it is natural to expect the presence of a small classical-like term in the vacuum energy density.

In order to have a further check, we looked at the Abelian and non Abelian chromomagnetic field strength. Indeed, a further clear signature of the unstable modes resides in a drastic reduction of the non Abelian chromomagnetic field with respect to the Abelian one. In Figure 2 we display the ratio $\langle F_{\text{A}}^{12} \rangle / F_{\text{ext}}^{12}$ versus $F_{\text{ext}}^{12}/g^3$. From Figure 2 we see that the system feels a rather strong Abelian chromomagnetic field. Note, however, that the Abelian chromomagnetic field is strongly affected by lattice artifacts. In Ref. [21] we found a similar situation even in the U(1) lattice gauge theory. The situation is quite different for non Abelian chromomagnetic field. As Figure 3 shows, the ratio $\langle F_{\text{3}}^{12} \rangle / F_{\text{ext}}^{12}$ depends only on $F_{\text{12}}/g^3$ (the small deviations in the strong field strength region have been already discussed). Moreover the non Abelian chromomagnetic field strength is reduced by one order
of magnitude with respect to the Abelian one. In Figure 3 we also compare the numerical data with Eqs. (1.13) and (5.20). A few comments are in order. Without unstable modes and in the one-loop approximation we should have obtained

$$\langle F_{12}^3 \rangle / F_{12}^{\text{ext}} = 1.$$  \hspace{1cm} (6.28)

Equation (6.28) is the dotted line in Fig. 3. We see that Eq. (6.28) is in total disagreement with the numerical results. In addition, if we take into account that a small quadratic term in the energy density adds a constant term of the same order in Eq. (4.15), then we can conclude that the data corroborate Eq. (4.13) at least in the region $F_{12}^{\text{ext}} / g^3 \gtrsim 0.5$. As concerns the weak field strength region, on the one hand we expect that in Eq. (4.13) there are sizeable effects due to higher order contributions. On the other hand, the numerical results are not reliable because the magnetic length became comparable to the lattice size. Thus we are confident that Figs. 1 and 3 together give evidence of the unstable modes on the lattice.

Let us conclude this section by comparing our approach to the one recently proposed by H. D. Trottier and R. M. Voloshyn [22]. These authors consider three-dimensional lattice gauge theories in a background field. The background field is induced by means of an external current Eq. (6.2). Thus in the continuum the background action they used coincides with Eq. (1.3). For the U(1) gauge theory the action of Ref. [22] agrees with our lattice background action Eq. (6.13). However the authors of Ref. [22] do not enforce periodic boundary conditions, so that the external background field strength is not restricted by the constraint Eq. (6.15) [23]. As concerns the SU(2) gauge theory, Trottier and Voloshyn adopt a different discretization of the Abelian field strength tensor:

$$F_{A \mu \nu}(x) = \sqrt{\beta} \, \text{tr} \left\{ \sigma_3 / 2i \, (U_{\mu \nu} - [U_{\mu}, U_{\nu}]) \right\}. \hspace{1cm} (6.29)$$

It should be stressed that, whatever discretization one chooses the total action is no longer SU(2)-invariant, but only U(1)-invariant. This means there is always the freedom of reducing the original local SU(2) invariance to a local U(1) invariance with a suitable gauge fixing. In particular, the authors of Ref. [22] used the discretization Eq. (6.29) without gauge-fixing. As concerns the numerical results, the authors of Ref. [22] found that the vacuum energy density is more than an order of magnitude smaller than the classical magnetic energy over the whole range of the applied external magnetic field, in qualitative agreement with our results.

However, for $F_{12}^{\text{ext}} / g^3 \lesssim 1$ the vacuum energy density of Ref. [22] is negative. This result looks puzzling to us because we do not expect that in the weak external field strength region the vacuum energy density displays a dramatic dependence on the discretization of the Abelian field strength tensor. To check this point, we performed Monte Carlo simulations for $F_{12}^{\text{ext}} / g^3 \leq 1$ by using Eq. (6.29) in the background action. It turns out that the vacuum energy density difference is positive even though it is about a factor two smaller than our previous results. This disagreement can be ascribed to the different discretization adopted for the Abelian field strength. It turns out, however, that the discretization Eq. (6.29) leads to a vacuum energy density which is close to Eq. (1.25). Thus the method of Ref. [22] corroborate the evidence of unstable modes on the lattice.
Contrarily to our findings the authors of Ref. [22] found that the vacuum energy density is negative for $F_{12}^\text{ext}/g^3 \lesssim 1$. We do not believe that this disagreement can be ascribed to the different boundary conditions. We do not yet understand the reasons of this discrepancy. We hope to clarify this point in future studies.

VII. CONCLUSIONS

Let us conclude by summarizing the main achievements of this paper. We investigated the three-dimensional SU(2) gauge theory in a constant chromomagnetic field, both in the continuum and on the lattice. As in the four-dimensional case, we found the presence of unstable modes. The stabilization procedure of the unstable modes introduces a new background field with non analytic behaviour in the coupling constant. The most striking consequences of the induced background field are the cancellation (or a drastic reduction) of the classical magnetic energy, and an almost complete screening of the applied magnetic field. Moreover, we found that the non analytic behaviour of the induced background field makes the calculation of the ground-state energy highly non trivial. We studied, then, the problem with the non perturbative techniques offered us by the lattice approach to the gauge theories. We feel that we have convincingly put out the evidence of the unstable modes.

We would like to end by briefly discussing some consequences of our finding. As is known since long time, in the non Abelian gauge theories in the perturbative one-loop approximation, states with a constant chromomagnetic field lie below the perturbative ground state. It follows that these states could be a better approximation to the true ground state than the perturbative one. In particular, it was believed [5] that from these states one could set up an approximate ground state which confines color charges. However, the presence of unstable modes leads to a muddled state. In the case of (2+1)-dimensions, we argued (see the discussion after Eq. (4.10)) that, after the stabilization of the unstable modes, the states with a constant chromomagnetic background field are not energetically favoured with respect to the the perturbative ground state. This means that these states are not relevant to the confinement at least in three space-time dimensions. However, if we believe that the confinement mechanism does not depend on the space-time dimensions, then we are led to the same conclusions even in the more interesting four-dimensional case.
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the periodic lattice, the Monte Carlo simulations gave results which agree with Ref. [22],
namely the vacuum energy density was close to the classical value $\frac{1}{2} (F_{12}^{\text{ext}})^2$. 
FIGURES

FIG. 1. Vacuum energy density versus the applied magnetic field. Full and open symbols refer to $L=20$ and $L=40$ respectively. Squares correspond to $\beta = 7$, circles to $\beta = 10$, triangles to $\beta = 12$, and diamonds to $\beta = 15$. The curves are discussed in the text.

FIG. 2. The Abelian chromomagnetic field strength versus the applied magnetic field (symbols as in Fig. 1).

FIG. 3. The full chromomagnetic field strength versus the applied magnetic field (symbols as in Fig. 1). The dashed line is Eq. (4.15), the dot-dashed line Eq. (5.21), and the dotted line is Eq. (6.28).
Figure 1
Figure 2
Figure 3