STRONG CONNECTIVITY AND ITS APPLICATIONS

PETERIS DAUGULIS

Abstract. Directed graphs are widely used in modelling of nonsymmetric relations in various sciences and engineering disciplines. We discuss invariants of strongly connected directed graphs - minimal number of vertices or edges necessary to remove to make remaining graphs not strongly connected. By analogy with undirected graphs these invariants are called strong vertex/edge connectivities. We review first properties of these invariants. Computational results for some publicly available connectome graphs used in neuroscience are described.

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1. Introduction.

1.1. The subject of study. Directed graphs are widely used to model various objects and processes having nonsymmetric features. Directed paths and related notions of strong connectedness and strongly connected components may play important roles in applications. In this paper we discuss invariants of strongly connected graphs which describe stability of strong connectedness with respect to vertex and edge removal operations. Strongly connected graphs having high values of these invariants may be considered to be stably strongly connected with respect to removal of vertices or edges. Therefore strong connectivities may be useful in applications.

As an example of possible application we compute these invariants for connectome graphs which are studied in neuroscience.

1.2. Background.

1.3. Review of related graph theory. In this subsection we list the necessary notions and notations from graph theory, see also \cite{6}.

1.3.1. Directed graphs. We will use notions of reachability between vertices, strong connectedness relation in the set of vertices, strongly connected components (SCC), condensation graph.
The underlying undirected graph of a directed graph $\Gamma$ is denoted by $U(\Gamma)$: nonempty sets of directed edges between any vertices $u$ and $v$ are substituted by undirected edges $u - v$.

For a modern treatment of theory of directed graphs see [1].

1.3.2. Undirected graphs. Let $\Gamma = (V, E)$ be a noncomplete undirected graph. We will denote vertex/edge connectivity of $\Gamma$ by $\zeta_0(\Gamma)/\zeta_1(\Gamma)$ (traditional notations: $\zeta_0 = \kappa$ and $\zeta_1 = \lambda$).

The directed graph obtained from an undirected graph $\Gamma$ by substituting each undirected edge $u - v$ by two directed edges $u \leftrightarrow v$ is denoted as $D(\Gamma)$. Note that $U(D(\Gamma)) = \Gamma$, but $D(U(\Gamma)) \neq \Gamma$, in general.

Given a vertex or edge subset $A$, we denote by $\Gamma - A$ the graph obtained by removing $A$ from $\Gamma$. Given a vertex subset $U$ we denote by $\Gamma[U]$ the $\Gamma$-subgraph induced by $U$. These notations also make sense for directed graphs.

2. Main results.

2.1. Strong connectivity.

Definitions given below can be found in [1].

2.1.1. Definitions.

Definition 2.1. Let $\Gamma = (V, E)$ with $|V| \geq 2$ be a strongly connected directed graph. We call a subset of vertices $U \subseteq V$ a weakening vertex subset provided $\Gamma - U$ is not strongly connected or $\Gamma - U$ has one vertex. We call a subset of edges $D \subseteq E$ a weakening edge subset provided $\Gamma - D$ is not strongly connected.

Definition 2.2. Let $\Gamma = (V, E)$ with $|V| \geq 2$ be a strongly connected directed graph. Minimal cardinality of weakening vertex/edge subsets of $\Gamma$ is called strong vertex/edge connectivity (SVC/SEC) of $\Gamma$ (denoted as $\sigma_0(\Gamma)/\sigma_1(\Gamma)$).

2.1.2. Properties.

Proposition 2.3. $\Gamma = (V, E)$ - a strongly connected directed graph, $|V| \geq 2$.

1. $\sigma_0(\Gamma) \leq \zeta_0(U(\Gamma))$.
2. If $\Gamma = D(\Delta)$ for some undirected graph $\Delta$, then $\sigma_0(\Gamma) = \zeta_0(\Delta)$.

Proof.

1. Removal of a minimal $U(\Gamma)$-disconnecting vertex subset from $\Gamma$ makes the remaining graph disconnected.
2. There are zero or two arrows in both directions between any two vertices of \( \Gamma \). For any \( U \subseteq V \) \( \Gamma - U \) is strongly connected iff \( U(\Gamma) - U \) is connected undirected graph, hence the statement is proved.

**Proposition 2.4.** Let \( a, b \in \mathbb{N} \), \( a \leq b \). There exists a directed graph \( \Gamma_{a,b} \) such that \( \sigma(\Gamma_{a,b}) = a \) and \( \zeta_0(U(\Gamma_{a,b})) = b \).

**Proof.** For any \( b \in \mathbb{N} \) \( \zeta_0(K_{b+1}) = \sigma(D(K_{b+1})) = b \). Thus the case \( a = b \) is proved.

Let \( a < b \). We consider two subcases: \( a \leq \frac{b}{2} \) and \( a > \frac{b}{2} \).

**Case** \( a \leq \frac{b}{2} \).

\( \Gamma_{a,b} = (V_{a,b}, E_{a,b}) \), where \( V_{a,b} = \{ U, V, W_a, W'_{b-a} \} \), \( |U| = |V| = b + 1 \), \( |W_a| = a \), \( |W'_{b-a}| = b - a \). \( E_{a,b} \) contains 1) edges from every element of \( U \) to every element of \( W_a \), 2) edges from every element of \( W_a \) to every element of \( V \), 3) edges from every element of \( V \) to every element of \( W'_{b-a} \), 4) edges from every element of \( W'_{b-a} \) to every element of \( U \).

In this case \( a = \min(a, b - a) \), \( W_a \) is a minimal weakening vertex subset for \( \Gamma_{a,b} \).

**Case** \( a > \frac{b}{2} \).

\( \Gamma_{a,b} = (V_{a,b}, E_{a,b}) \), where \( V_{a,b} = \{ U, V, W_a, W'_{b-a} \} \), \( |U| = |V| = b + 1 \), \( |W_a| = a \), \( |W'_{b-a}| = b - a \). \( E_{a,b} \) contains 1) edges in both directions between every element of \( U \) and every element of \( W_a \), 2) edges in both directions between every element of \( W_a \) and every element of \( V \), 3) edges from every element of \( U \) to every element of \( W'_{b-a} \), 4) edges from every element of \( W'_{b-a} \) and every element of \( V \).

In this case \( W_a \) is a minimal weakening vertex subset for \( \Gamma_{a,b} \).

In both cases \( W_a \cup W'_{b-a} \) is a minimal disconnecting vertex subset of \( U(\Gamma_{a,b}) \) and \( \zeta_0(\Gamma_{a,b}) = |W_a \cup W'_{b-a}| = b \).

**Proposition 2.5.** SVC of a directed graph on \( n \) vertices can be computed in \( O(n^3) \) time.

**Proof.** Assume again \( \Gamma = (V, E) \) is an edge-weighted graph with weight of each edge equal to 1. For any two vertices \( u, v \) define \( MF(u, v) \) equal to the maximal flow from \( u \) to \( v \) in \( \Gamma \). \( MF(u, v) \) is equal to the cardinality of a maximal system of disjoint \((u, v)\)-paths. Define \( \sigma(u, v) := \min(MF(u, v), MF(v, u)) \). The minimal number of vertices necessary to remove to make \( u \) and \( v \) in different SCC (denoted by \( \sigma(u, v) \))
is equal to \( \min(MF(u,v), MF(v,u)) \), it can be computed in \( O(n^3) \) time, see [5]. By statement 1 of [2,3] we have that \( \sigma(\Gamma) = \min_{u \in V, v \in V} \sigma(u,v) \). Since there are \( \frac{n(n-1)}{2} \) vertex pairs, it follows that \( \sigma(\Gamma) \) can be computed in \( O(n^3) \cdot n^2 = O(n^5) \) time.

Example 2.6. \( \Gamma_{1,3} \) is shown in Fig.1. \( W_1 = \{1\} \), \( W'_2 = \{2,3\} \). A minimal weakening vertex subset of \( \Gamma_{1,3} \) is \( W_1 \). A minimal disconnecting vertex subset of \( U(\Gamma_{1,3}) \) is \( W_1 \cup W'_2 \).

Example 2.7. \( \Gamma_{2,3} \) is shown in Fig.2. Pairs of oriented edges with common vertices are shown as undirected edges. \( W_2 = \{1,2\} \), \( W'_1 = \{3\} \). A minimal weakening vertex subset of \( \Gamma_{2,3} \) is \( W_2 \). A minimal disconnecting vertex subset of \( U(\Gamma_{2,3}) \) is \( W_2 \cup W'_1 \).

Remark 2.8. Computation of SC can be iterated to get more information about graph structure as follows. Suppose we are given a strongly connected graph \( \Gamma \).

1. Find all (or at least one) minimal weakening set, find all resulting SCC \( \Gamma_{1,1}, \Gamma_{1,2}, \ldots, \Gamma_{1,n_1} \) and the corresponding condensation graph \( \Delta_1 \) with \( n_1 \) vertices.
2. Repeat Step 1 with each nontrivial SCC. Get a family of acyclic condensation graphs and a family of SCC.
2.2. An application. It makes sense to study strong connectivity in graph models where directed paths and reachability play important roles. A very simple example of such an application would be a city graph model where vertices are street intersections and edges are (possibly one-way) streets. A meaningful question: how many one-way streets or intersections can be closed so that the remaining street network is still strongly connected.

In this subsection we describe some computational results related to strong connectivity of connectome graphs considered in neuroscience.

2.2.1. Connectome graphs. Connectome graphs are discrete mathematical models used for modelling nervous systems on different scales, see [7], [10]. On the microscale level these graphs are special cases of cell graphs which are studied for many types of human body tissues. In such graphs vertices correspond to cells and edges correspond to physical cell contacts or substance transfer (volume transmission, see [2]) links. See [3] for an example of cell graph application in tumour tissue modelling. On mesoscale and macroscale levels connectome graphs are essentially quotient graphs of microscale cell graphs. In this paper we do not deal with connectome scale, nature of graph edges and other modelling issues, we are interested only in applications of graph-theoretic concepts and algorithms. Connectome graph edges may be directed, undirected and weighted (labelled). We consider only directed graph structure of connectomes, each edge is assigned the constant weight 1. We use data and references available at [8].

We assume that direction of connectome edges, directed paths and reachability have considerable biological meaning. For example, direction of edges may be correlated with flow of signals or substances.

2.2.2. Cat. Graph (C) description - macroscale connectome graph (cortico-cortical connections, see [9]), strongly connected graph with 65 vertices and 1139 edges, \( \zeta_i(U(C)) = 3 \), diameter 3, minimal degree 3, maximal degree 45.

\[ \sigma_0(C) = \sigma_1(C) = 1. \] There is a unique weakening vertex for \( C \). After removing it \( C \) splits into one trivial SCC and a SCC having 63 vertices. There is a unique weakening edge pair for \( C \). After removing it \( C \) splits into two SCC having 64 and 1 vertices.

We do 6 iterations. At each step the graph splits into one trivial and one large strongly connected component. The list of SC of large components starting from \( C \) is \([1, 2, 3, 3, 3, 3, 2]\). The list of vertex connectivities of underlying graphs of large components is \([3, 3, 7, 7, 7, 6]\).
2.2.3. Rat. There are 3 macroscale connectomes \( \{R_1, R_2, R_3\} \) for *Rattus norvegicus* available at [8]. Every graph has one nontrivial SCC having 502 or 493 vertices. Graphs have similar properties in all cases. \( \zeta_i(\mathcal{U}(R_n)) = 7 \).

\( \sigma_0(R_n) = \sigma_1(R_n) = 2 \). Each graph has a unique weakening vertex pair. After removing a weakening vertex pair \( R_n \) splits into one or several trivial SCC and one large SCC.

2.2.4. Fly. A mesoscale fly connectome graph (see [11]) description - 1781 vertices and 9735 edges, 996 strongly connected components - one with 785 vertices, one with 2 vertices, the other components trivial, minimal degree 1, maximal degree 927.

Let \( F_1 \) - SCC of the described graph with 785 vertices. \( \zeta_i(\mathcal{U}(F_1)) = 1 \). \( \sigma_0(F_1) = \sigma_1(F_1) = 1, 173 \) weakening vertices. After removing a weakening vertex \( F_1 \) splits into several trivial and 1 large SCC. \( F_1 \) has 245 weakening edges.

2.3. Conclusion. We discuss study of invariants directed graphs - strong connectivities. Vertex connectivity can be computed in polynomial-time using network algorithms.

We present some computation results involving graph models in biology - connectome graphs of nerve tissues.

Some of our observations concerning connectome graphs:

1) after removing a minimal weakening vertex subset the considered connectome graphs split into one nontrivial and one or several trivial SCC;
2) strong connectivities are close to undirected connectivities;
3) often there is a unique weakening vertex or edge subset.

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REFERENCES

[1] Bang-Jensen, J., Gutin G. Digraphs: Theory, Algorithms and Applications. Springer.2010
[2] Bentley, B., Branicky, R., Barnes, C. L. et al. The multilayer connectome of Caenorhabditis elegans. www.arxiv.org, 2016, [arXiv:1608.08793]
[3] Bilgin, C., Demir, C., Nagi, C., Yener, B. Cell-graph mining for breast tissue modeling and classification. Conf Proc IEEE Eng Med Biol Soc, 2007, 5311-4.
[4] Bosma, W., Cannon, J., and Playoust, C. The Magma algebra system. I. The user language. J. Symbolic Comput., 24, 2997, pp. 235-265.
[5] Cormen, T., Leiserson C., Rivest R., Clifford S. Introduction to Algorithms, Second Edition. MIT Press and McGraw-Hill, 2001, pp. 643668.
[6] Diestel, R. Graph Theory. *Graduate Texts in Mathematics*, Vol.173, Springer-Verlag, Heidelberg, 2010.

[7] Kaiser, M. A Tutorial in Connectome Analysis: Topological and Spatial Features of Brain Networks. *Neuroimage*, 57, 2011, pp.892-907.

[8] http://www.openconnectomeproject.org. 2016.

[9] Reus M.A., Heuvel M.P. Rich Club Organization and Intermodule Communication in the Cat Connectome. The Journal of Neuroscience, 7 August 2013, 33(32), pp. 12929-12939; doi: 10.1523/JNEUROSCI.1448-13.2013.

[10] Sporns, O. The human connectome: Origins and challenges. *Neuroimage* 80, 2013, pp.53-61.

[11] Takemura S., Bharioke A., Lu Z. et al. A visual motion detection circuit suggested by Drosophila connectomics. Nature, 500, 8 August, 2013, pp.175181, doi:10.1038/nature12450.