On the Meaning and Inapplicability of the Zeldovich Relations of Magnetohydrodynamics

ERIC G. BLACKMAN\textsuperscript{1} and GEORGE B. FIELD\textsuperscript{2}

\textsuperscript{1} Department of Physics and Astronomy, University of Rochester, Rochester NY, 14627, USA
\textsuperscript{2} Center for Astrophysics, 60 Garden St., Cambridge MA, 02139, USA

Received; accepted; published online

Abstract. Considering a plasma with an initially weak large scale field subject to nonhelical turbulent stirring, Zeldovich (1957), for two-dimensions, followed by others for three dimensions, have presented formulae of the form $\langle b^2 \rangle = f(R_M)B_0^2$. Such "Zeldovich relations" have sometimes been interpreted to provide steady-state relations between the energy associated with the fluctuating magnetic field and that associated with a large scale or mean field multiplied by a function $f$ that depends on spatial dimension and a magnetic Reynolds number $R_M$. Here we dissect the origin of these relations and pinpoint pitfalls that show why they are inapplicable to realistic, dynamical MHD turbulence and that they disagree with many numerical simulations. For 2-D, we show that when the total magnetic field is determined by a vector potential, the standard Zeldovich relation applies only transiently, characterizing a maximum possible value that the field energy can reach before necessarily decaying in relation to a seed value $B_0$. In 3-D, we show that the standard Zeldovich relations are derived by balancing subdominant terms. In contrast, balancing the dominant terms shows that the fluctuating field can grow to a value independent of $R_M$ and the initially imposed $B_0$, as seen in numerical simulations. We also emphasize that these Zeldovich relations of nonhelical turbulence imply nothing about the amount mean field growth in a helical dynamo. In short, by re-analyzing the origin of the Zeldovich relations, we highlight that they are inapplicable to realistic steady-states of large $R_M$ MHD turbulence.

Key words: galaxies: magnetic fields—ISM: magnetic fields—stars: magnetic fields

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1. Introduction

Stars, galactic interstellar media, the hot plasma of galaxy clusters, and accretion disks all contain magnetohydrodynamic (MHD) turbulence. Over the past 50 years, the ubiquity of MHD turbulence and magnetic fields in astrophysics has stimulated an effort to understand the resulting magnetic and kinetic energy spectra (e.g. Biskamp 1997; 2003; Brandenburg & Subramanian 2005).

Simplified semi-analytic models combined with idealized numerical experiments with restricted boundary conditions have been used in combination, to help bridge the gap between ignorance and realistic astrophysical MHD turbulent settings. One such class of studies focuses on the 3-D amplification of a weak seed magnetic field by a nonhelical turbulent flow in a closed volume or periodic box (e.g. Kazan 1968, Kulvied & Anderson 1992; Maron & Blackman 2002; Haugen, Brandenburg, Dobler 2003; Haugen, Brandenburg, Dobler 2004; Maron, Cowley, McWilliams 2004; Schekochihin et al. 2004). Starting with an initial seed field, the system is forced with an incompressible turbulent flow. The key questions are: how much magnetic energy grows and what determines its saturated value and spectrum? Two initial value problems can be distinguished: (1) An initially weak (magnetic energy $\ll$ turbulent forcing energy) seed magnetic field with a finite power at wave number $k = 0$, subject to forcing, non-helical isotropic turbulence at forcing wave number $k_1 \sim 5k_1$ (e.g. Haugen & Brandenburg 2004) (2) An initially weak seed random magnetic field with no $k = 0$ field, subject to forced, non-helical isotropic turbulence at $k_1 \sim 5$ (e.g. Maron et al. 2004; Schekochihin et al. 2004).

The two initial value problems above address the 3-D small scale dynamo, defined as field amplification at or below the forcing scale. Simulations show that for nonhelical randomly forced flows, no significant field amplification above the scale of the forcing occurs (e.g. Maron & Blackman 2002; Haugen et al. 2003, 2004; Maron et al. 2004). When a $k = 0$
field is imposed and a numerical simulation is performed with periodic box, this mean field cannot change due to the boundary conditions. If the turbulence is instead forced helically at \( k = k_f \), the \( k = 0 \) field still cannot grow, but the field up to \( k = 1 \) can and does grow (e.g. Pouquet, Leorat & Frisch 1976; Brandenburg 2001; Maron & Blackman 2002; Blackman & Field 2002). (The explicit difference between the helical and non-helical forcing can be seen in the set of simulations by Maron & Blackman (2002).) Any significant field amplification on scales above the forcing scale, involves the large scale dynamo mechanism, and requires helical forcing. The nonlinear helical dynamo is the dynamical generalization of the original large scale dynamo (e.g. Moffatt 1978; Parker 1979) proposed to model the large scale fields of rotating stratified systems like the sun or galaxy.

The present focus is on the nonhelical dynamo saturation. Two key results emerging from 3-D numerical simulations of the nonhelical dynamo are that (1) no significant field amplification occurs at \( k < k_f \) and (2) the total turbulent magnetic energy integrated for \( k \geq k_f \) grows to within a factor of a few of equipartition with the incompressible turbulent kinetic energy (when the magnetic Prandtl number \( P r_M = \nu/\lambda \) (where \( \nu \) is the viscosity and \( \lambda \) is the magnetic diffusivity) is not too small; Haugen et al. 2004; Boldyrev & Cattaneo 2004. This is independent of whether there is an initial \( k = 0 \), or other large scale (\( 0 < k < k_f \)) field present, as long as that initial large scale field is weak compared to the turbulent forcing (e.g. Haugen and Brandenburg 2004). In fact, the saturated end state of the nonhelical MHD turbulent dynamo spectrum is essentially independent of the shape of the seed spectrum (Maron et al. 2004).

The approximate equipartition and its independence of an initially weak large scale field seem physically reasonable, but contradict a classic relation in MHD referred to as the “Zeldovich relation.” This refers to the Zeldovich (1957, hereafter Z57) result derived for 2-D, and subsequently generalized for 3-D using different approaches (e.g. Steenbeck & Krause 1969; Low 1972; Parker 1973; Krause & Rädler 1980; Zeldovich, Ruzmaikin, Sokoloff 1983 (hereafter Z83)). The Zeldovich relation is given by

\[
\langle b^2 \rangle = f(R_M) \mathcal{B}^2,
\]

where the magnetic field is \( \mathcal{B} = b + \mathcal{B} \), with \( b \) the fluctuating field and \( \mathcal{B} \) the mean or large scale field. Eq. (1) is sometimes used to purport a steady state relation for the fluctuating \( \sim \) total) magnetic energy and the magnetic energy in \( \mathcal{B} \) when the latter provides an initial seed. (We will see later that the distinction between \( \mathcal{B} \) as a large scale field derivable from a vector potential vs. \( \mathcal{B} \) as a mean \( k = 0 \) field has a particular implication for 2-D.)

The function \( f(R_M) \) of the magnetic Reynolds number \( R_M \) is \( \propto R_M^{-1} \) for two dimensions (Z57; Moffatt 1978). In three dimensions, different values have been proposed analytically: As discussed in Krause and Rädler (1980, hereafter KR), for the large \( R_M \) limit, Steenbeck & Krause (1969), Low (1972), Parker (1973) all find \( f(R_M) \sim R_M \), while \( f(R_M) \sim R_M^{2/3} \) for the low \( R_M \) limit. In contrast, in Z83, a more spectrally sensitive result is proposed. There \( f(R_M) \propto ln(R_M) \) for three dimensions and Kolmogorov turbulence, and \( \propto R_M^{5/3} \) in 3-D when the kinetic spectral index \( p \neq 5/3 \).

Two issues immediately arise: First, why do the 3-D analytic calculations differ so strongly in the form of \( f(R_M) \)? Second, having just discussed that modern simulations show that MHD turbulence saturates to a state in which the fluctuating field energy has a value independent of any large scale field, how can a relation like (1) apply regardless of the specific form of \( f(R_M) \)? Where do relations of the form (1) come from and how can we understand why they do not agree with simulated MHD steady states? We address these questions in the present paper.

Note that before the non-linear regime could be studied with numerical simulations, most dynamo theorists studied the kinematic regime. This refers to the regime in which the magnetic field amplified by a turbulent flow does not significantly back-react on this flow, so that the velocity is not subject to strong Lorentz forces. This regime lends itself to analytic studies, but the limitations of the implications must be properly understood. The Zeldovich relations were derived from this perspective, but the specific reasons why different derivations appear to disagree and why none are applicable to the steady-state of fully developed MHD turbulence have not been succinctly elucidated. Doing so is our present goal.

In section 2, we discuss the Zeldovich relation in 2-D, and explain that it cannot represent a steady state if the total field is derived from a vector potential. We show instead that it represents an absolute maximum that the fluctuating field can attain before necessarily decaying. In section 3, we show that in 3-D, the Zeldovich relations have been derived using arguments ignore key nonlinear growth terms and thus balance subdominant terms. We also show that, although the differences in the 3-D forms of (1) mentioned above can be attributed specifically do different approaches and inclusion of different subdominant terms, all of the analytical approaches drop the same dominant term. As a result, the generalized Zeldovich relation for 3-D MHD turbulence is inapplicable to steady-state large \( R_M \) MHD turbulence. We conclude in section 4.

We emphasize throughout, that the Zeldovich relations were always derived in the context of nonhelical turbulence. They were therefore never meant to apply to a system in which \( \mathcal{B} \) grows, and thus do not present any constraints on helical dynamo theory. This can lead to confusion because the Zeldovich relation sometimes appears in discussions whose main focus is on mean field helical dynamo amplification (e.g. Vainshtein & Cattaneo 1992). Our detailed discussion is therefore focused specifically on understanding the inapplicability of the Zeldovich relation to steady-state nonhelical MHD turbulence.

2. Understanding the 2-D Zeldovich Relation

We first address the 2-D Zeldovich relation derived by Z57. Since no dynamo action can be sustained in 2-D, and the magnetic energy must ultimately decay, one immediately wonders what a relation between \( \mathcal{B}^2 \) and \( \langle b^2 \rangle \) could mean?
In fact ZS7 is primarily a 2-D anti-dynamo theorem paper: we will see that the relation between $B$ and $(b^2)$ represents the maximum that $(b^2)$ could possibly attain before it necessarily decays.

We start with a basic argument showing that in 2-D, when the total magnetic field (mean + fluctuating) is determined by a vector potential, (i.e. $B = \nabla \times A$ and no $k = 0$ component for a periodic box) the total magnetic energy must decay when surface integrals vanish. We write the total the incompressible induction equation for the total magnetic field as (e.g. Moffatt 1978)

$$\partial_t B = \nabla \times (v \times B) + \lambda \nabla^2 B = B \cdot \nabla v - v \cdot \nabla B + \lambda \nabla^2 B,$$

(2)

where $v$ is the velocity flow, assumed to be imposed by external forcing with no mean component, and $\lambda$ is the magnetic diffusivity.

Now following ZS7 and restricting to 2-D ($x, y$) incompressible flow, we can then separate out the equation for the $z$-component

$$\partial_t B_z = -v \cdot \nabla B_z + \lambda \nabla^2 B_z.$$  

(3)

Multiplying by $B_z$ and spatially averaging for incompressible flows, we have

$$\partial_t \langle B_z^2 \rangle = -\partial_i \langle v_i B_z^2 \rangle + \lambda \partial_j \partial_j \langle B_z^2 \rangle - 2\lambda \langle \partial_t B_z^2 \rangle$$

$$= -2\lambda \langle \partial_t B_z \rangle^2,$$

(4)

where the latter equality follows when we assume surface integrals vanish. Eq. (4) shows that the $b_z$ contribution to the magnetic energy will decay.

Because of the first term on the right of (2), the $x, y$ components of the magnetic field need not immediately decay, although they will eventually decay. To see this, note that the vector potential (defined such that $B = \nabla \times A$) satisfies

$$\partial_t A_z = v \times (\nabla \times A) + \lambda \nabla^2 A_z - \lambda \nabla (\nabla \cdot A) - \nabla \phi.$$  

(5)

As above, we can separate out the $z$-component, which determines $B_z$ and $B_y$, to obtain

$$\partial_t A_z = -v \cdot \nabla A_z + \lambda \nabla^2 A_z,$$

(6)

since the $z$-derivatives vanish. Eq. (5) has the same form as (4). Multiplying by $A_z$ and averaging, we again have for incompressible flow

$$\partial_t \langle A_z^2 \rangle = -\partial_i \langle v_i A_z^2 \rangle + \lambda \partial_i \partial_j \langle A_z^2 \rangle - 2\lambda \langle \partial_t A_z^2 \rangle$$

$$= -2\lambda \langle \partial_t A_z \rangle^2,$$

(7)

where the last equality again follows from ignoring the divergence terms for suitable boundary conditions. This shows that $A_z$, like $B_z$ also ultimately decays. The ultimate decay of both $B_z$ and $A_z$ implies that the total magnetic energy also eventually decays. The asymptotic steady-state is one of negligible total magnetic energy, $\langle B^2 \rangle \approx 0$. This is the antidynamo theorem for 2-D (see also Moffatt 1978).

It is important in this derivation that we took $B = \nabla \times A$ because the first term on the right hand side of (4) would not have emerged as a straightforward total divergence if we were considering a periodic box that included a strictly uniform $k = 0$ periodic field that has no well defined vector potential. (This latter case, was first studied numerically by Moss (1970)).

Now let us take advantage of the positive nature of the last term of Eq. (7) to rewrite that equation

$$\partial_t \langle A_z^2 \rangle = -2\lambda \langle A_z^2 \rangle^2 / \delta^2(t),$$

(8)

where $\delta(t)$ represents the dominant time dependent characteristic variation scale of $A_z$. Allowing for the variation of this scale is important because although the magnetic energy ultimately decays in 2-D, random walk field line stretching can temporarily amplify the field. How much amplification takes place before the field decays?

To answer this, consider that in 2-D, the field initially grows exponentially via the first term on the right hand side of (4), much like in 3-D (Kazantsev 1968; Parker 1979, Kulsrud & Anderson 1992). Early in the kinematic regime,

$$B(t) \simeq B_0 \exp(t/\tau),$$

(9)

where $B_0$ is the root mean square of the initial magnetic field in the $x$-$y$ plane, and $\tau$, for sufficiently steep kinetic energy spectra, is the correlation time of the forcing scale. It should be noted that 2-D forced turbulence exhibits a steeper kinetic energy spectrum than 3-D turbulence because of the tendency for enstrophy to inverse cascade (e.g. Davidson 2004). Approximate inertial range energy spectra $E(k) \propto k^{-3}$ proposed by Batchelor (1969) Kraichnan (1970) seem to be consistent with 2-D simulations (Gotoh 1998; Lindborg & Alvelius 2000) though perhaps for slightly different reasons than originally thought (Davidson 2004). In 3-D, the kinetic energy spectrum $E(k) \propto k^{-5/3}$ is such that the the $\tau$ appearing in (9) would instead be closer to that of the smallest eddy scale than the forcing scale (Kulsrud & Anderson 1992).

As the field lines stretch from the turbulent motions in a confined area, an increase in field energy means that the field must build up on smaller and smaller scales, so $\delta(t)$ decreases. For incompressible flow, the field strength increases linearly with the length of the field line. The length of the field line varies inversely with the scale of the field variation. We can therefore write

$$\delta(t) = L(B_0/B(t))^q = L \exp[-qt/\tau],$$

(10)

where $q$ is an index to allow deviations from a linear relation, $L$ the scale of the initial field $B_0$, and we have used (4). Combining (6) and (10) gives

$$\partial_t \ln \langle A_z^2 \rangle = -2\lambda \exp[2qt/\tau]/L^2$$

(11)

so

$$\ln(\langle A_z^2 \rangle / \langle A_z^2 \rangle_0) = -\left(\tau \lambda / qL^2 \right)(\exp[2qt/\tau] - 1)$$

$$\approx -\left(\frac{1}{qR_{M,L}}\right)(\exp[2qt/\tau] - 1) = -\left(\frac{1}{qR_{M,L}}\right)\left(\frac{B(t)}{B_0}\right)^2,$$

(12)

where we have defined the magnetic Reynolds number $R_{M,L} = L^2 / \chi$, and where the similarity follows for $2qt > \tau$. Thus

$$\langle A_z^2 \rangle \approx \langle A_z^2 \rangle_0 \exp[-(1/qR_{M,L})(B(t)/B_0)^2].$$

(13)

The field grows by stretching until the argument of (13) becomes $\geq 1$. This occurs when

$$B(t) \sim B_0 (qR_{M,L})^{-1/2q}.$$  

(14)

For the case in which $B_0 = B$, we can also write $B(t) \sim (b^2)^{1/2}$ so (14) then gives

$$\langle b^2 \rangle \sim B^2 (qR_{M,L})^{-1/4},$$

(15)
3 UNDERSTANDING THE 3-D ZELDOVICH RELATION

3.1 Deriving the 3-D Zeldovich relation

which for \( q = 1 \) is the 2-D Zeldovich relation.

Using (10), Eq. (14) also implies

\[
\delta_m \simeq \frac{L}{(q R_{M,L})^{1/2}}. 
\]  (16)

This is the minimum scale at which the field energy could peak in the kinematic regime and at which point the vector potential and the total magnetic energy would rapidly decay. If the peak is away from this minimum scale, then the decay rate of \( A_z \) is reduced: Using (16) we can rewrite (13) as

\[
\langle A_z^2 \rangle = \langle A_z^2 \rangle_0 E_{\text{exp}}[-(\delta_m/\delta(t))^2].
\]  (17)

At the time Eq. (15) is satisfied the vector potential rapidly decays from (13) or (14), and so does the total field energy. The Zeldovich relation (15) therefore, simply provides an estimate of the maximum value that the fluctuating field energy could obtain if it were to grow kinematically to this maximum (see also Peckover & Weiss (1978)). There is no steady state for which (15) applies. Moreover, (15) is a kinematic result relevant for \( \langle \nu^2 \rangle > R_{M,L}B_0^2 \). If instead \( \langle \nu^2 \rangle < R_{M,L}B_0^2 \), the dynamical amplification of \( \langle b^2 \rangle \) is ultimately limited by near equipartition with the kinetic energy density well before the value implied by (15) is reached. This latter statement also applies even when \( \vec{B} \) represents a \( k = 0 \) component; while the above proof of the total magnetic energy decay would not be valid in this case, equipartition with the kinetic energy would determine the limiting magnetic energy.

Note also that the mean field enters (15) only if the initial field energy is determined by \( \vec{B} \). The same calculation would go through even if there were no initial mean field, in which case the initial field \( B_0 \) would not be related to \( \vec{B} \).

3. Understanding the 3-D Zeldovich Relation

In 3-D, unlike 2-D, sustained dynamo action is allowed when the system is forced with a velocity flow, whether or not there is a \( k = 0 \) mean field. As discussed in the introduction, simulations of incompressible MHD turbulence show that the magnetic field typically grows to approach and saturate near equipartition with the turbulent kinetic energy in the steady state, independent of any weak initial large scale or \( k = 0 \) mean field. In this section we show why the generalized Zeldovich relation for 3-D does not account for this steady-state correctly. The distinction between large scale 0 < \( k < k_f \) vs. mean \( k = 0 \) is not as essential for 3-D as it was in 2-D in what follows.

3.1.1. The steady-state Z83 approach

We first follow Z83 and employ a spectral formalism where the kinetic energy spectrum \( E(k) \) is defined by

\[
\int E(k) dk = \langle \nu^2 \rangle / 2
\]  (18)

and satisfies

\[
2E(k) = \nu(k)^2/k = \frac{\nu^2(k_f)}{k_f} \left( \frac{k}{k_f} \right)^{-p},
\]  (19)

where \( p \) is the spectral index, \( k_f \) is the wavenumber of the turbulent forcing scale, and \( \nu(k) \) is the turbulent speed at wave number \( k \). Assuming a constant energy transfer rate gives

\[
\nu(k)^2/\tau(k) = \text{constant},
\]  (20)

so the energy transfer time \( \tau(k) \) satisfies

\[
\tau(k) = \tau(k_f) \left( \frac{k}{k_f} \right)^{1-p}.
\]  (21)

The magnetic energy spectrum \( M(k) \), is defined by

\[
\int_{k_1}^{k_3} M(k) dk = E_M
\]  (22)

where \( k_3 \) is the resistive wave number and \( E_M \) is the magnetic energy density.

Using \( \vec{B} = \vec{B} + b \), where \( \vec{B} \) is a fixed large scale field, and subtracting the mean of the magnetic induction equation (22) from Eq. (22) gives the following equation for the fluctuating component of the magnetic field:

\[
\frac{1}{2} \partial_k (b^2) = \langle \nu \times (\nu \times \vec{B}) \rangle + \langle \nu \times (\nu \times b) \rangle + \lambda \langle b \rangle \vec{v} \cdot \nabla \times b.
\]  (23)

We dot this equation with \( b \) and average to find

\[
\nu_T(k) \simeq \frac{1}{\tau(k)k^2}.
\]  (27)

Assuming a steady state, setting Eq. (25) equal to minus Eq. (26) gives

\[
\langle b \rangle \cdot \nu \times (\nu \times \vec{B}) \rangle = 0.
\]  (28)

Squaring this gives

\[
b^2(k) = 2kM(k) = \frac{\nu^2(k)\vec{B}^2}{\nu_T^2(k)k^2} = \frac{E(k)\vec{B}^2}{\nu_T(k)k}.
\]  (29)

Then using (19), (21) and (27) we have

\[
M(k) = \frac{\vec{B}^2}{k_f} \left( \frac{k}{k_f} \right)^{4-3p}.
\]  (30)

Integrating the steady state magnetic energy spectrum from \( k_1 \) to the viscous wave number \( k_v \) then gives

\[
E_{M,\nu} = \int_{k_1}^{k_v} M(k) dk = \vec{B}^2 \int_{k_1}^{k_v} \frac{\sigma^{4-3p} d\sigma}{k_l}.
\]  (31)

The microphysical viscosity \( \nu \) must equal

\[
\nu = \nu_T(k_v) = \frac{1}{\tau(k_v)k_v^2} = \frac{\nu(k_f)}{k_f} \left( \frac{k_v}{k_f} \right)^{p-3}.
\]  (32)
where we have used (19) and (21). This implies
\[ R_{M,l} = P_{TM} \left( \frac{k_l}{k} \right)^p = P_{TM} \left( \frac{k_l}{k} \right)^p, \tag{33} \]
or
\[ k_v = k_1 \left( \frac{R_{M,l}}{P_{TM}} \right)^{1/(3-p)}, \tag{34} \]
where \( R_{M,l} \equiv \frac{v(k_0)}{k_0} \) is the magnetic Reynolds number associated with scale \( k \).

Let us now focus on the case \( P_{TM} = 1 \) to make contact with previous work. In this case, \( E_M = E_{M,v} = \int k_v M(k) \, dk \). Then using (34) in (31) gives, for \( p = 5/3 \) (Kolmogorov spectrum),
\[ \langle b^2 \rangle = 2 \int_{k_1}^{k_0} M(k) \, dk = 2 B^2 \ln(k_d/k_1) = \frac{3}{2} B^2 \ln R_{M,l} \tag{35} \]
For \( p \neq 5/3 \) we instead find
\[ \langle b^2 \rangle = \frac{2 B^2}{5 - 3p} \left( \frac{k_0}{k_1} \right)^{5-3p} - 1 \approx \frac{2 B^2}{5 - 3p} R_{M,l}^{5-3p}, \tag{36} \]
for \( k_d \gg k_1 \). Eqs. (35) and (36) are the generalized Zeldovich relations of Z83.

Note that for \( p = 1 \) the result Eq. (36) would then imply
\[ \langle b^2 \rangle \approx R_{M,l} B^2, \tag{37} \]
as noted by Z83, a formula consistent with the 2-D result (14) for \( L \sim \ell \). However, as we discussed in section 2, there is no true steady state in 2-D and the derivation of (36) relied on the steady state balance between the first two terms on the right of (23). Moreover, as also discussed below Eq. (9), the 2-D turbulent kinetic energy spectrum is more consistent with \( p = 3 \), a value steeper than the Kolmogorov value \( p = 5/3 \), not more shallow.

Before discussing the problems with the 3-D derivation of Z83 and the implications, we summarize a quite different derivation presented in KR that also gives a different result.

### 3.1.2. The time dependent KR approach

KR (p.105; see also Low 1972) also start with (23) with a fixed \( B \), but instead of replacing the nonlinear terms by a diffusion term as in (24), they employ the quasilinear approximation (= second order correlation or first order smoothing approximation (FOSA)) to drop the nonlinear terms altogether. Furthermore, they keep the \( \lambda \) dissipation term and the time evolution term on the left hand side of (26)—both of which were dropped in the approach of the previous section. They then Fourier transform (26) in both space and time to obtain
\[ \tilde{b}_i = \sum_{m} \sum_{LM} B_{M,L}^{ik} \tilde{b}_L, \tag{38} \]
where the tilde indicates Fourier transform. Eq. (38) facilitates a rigorous relation between magnetic and velocity two-point correlators. After some algebra, for isotropic, incompressible turbulence this elegantly leads to (KR)
\[ \langle b^2 \rangle = f(R_M) B^2, \tag{39} \]
where
\[ f(R_M) = \frac{1}{3} \int \int \frac{k^2 Q_{ll}(k, \omega)}{\lambda^2 k^4 + \omega^2} \, dk \, d\omega, \tag{40} \]
and \( Q_{ij}(k, \omega) \delta(k + k', \omega + \omega') \equiv \langle \tilde{v}_i (k, \omega) \tilde{v}_j (k', \omega) \rangle \). Defining \( s = \omega/\lambda \) and taking the \( \lambda \to 0 \) limit, this gives
\[ f(R_M) \approx \frac{\pi}{3\lambda} \int Q_{ll}(k, 0) \, dk \, ds \sim \frac{|P|}{\lambda} = R_M. \tag{41} \]
For \( \lambda \to \infty \), KR instead obtain \( f = R_M^3 \).

The KR approach produces a different (and spectra independent) result for \( f(R_M) \) from the Z83 approach of the previous subsection because KR keeps the time derivative term in (23) and the resistive term, and drops the nonlinear terms. In contrast, Z83 drops the time derivative term and the resistive term and replaces the triple correlations by a turbulent diffusivity. If KR were to include the nonlinear terms by a turbulent diffusivity, the effect would be to replace \( \lambda \) in this subsection by a turbulent diffusion coefficient. The strong \( R_M \) dependence in (41) would disappear, and the result would be spectrally dependent.

### 3.2. Invalidating the 3-D Zeldovich relations

Despite their differences, both approaches to deriving 3-D Zeldovich relations in the previous two subsections are missing the same key ingredient which ultimate invalidates them for realistic steady states. Both approaches do not include the growth of the small scale field from the third term in (23).

Fundamental to KR approach, was ignoring any contribution from the third term in (23) via the second order correlation approximation or FOSA. But this excludes all the nonlinear terms that make a nonlinear turbulent spectrum. These terms are not ignorable and as soon as \( b \) grows at all beyond the fixed \( B \), they become dominant.

While the Z83 approach of section 3.1.1 does not ignore the third term in (23), it treats this term as a diffusion term, and incorrectly assumes that a steady-state arises between the first two terms on the right of (23). This is the central problem with the derivation of section 3.1.1 and invalidates (36) for realistic steady MHD turbulence. As discussed earlier, for steady state MHD turbulence initiated with a weak field, the small scale field can build up even in the absence of a large scale field. The saturated magnetic energy thus has nothing to do with a mean field, and Eq. (36) is ruled out for the steady state. This discrepancy between (36) and numerical simulations of incompressible MHD turbulence (e.g. Maron & Blackman 2002; Haugen et al. 2003,2004; Maron et al. 2004) arises because the third term of (23) is a actually a source of both growth and decay (e.g. Kulsrud & Anderson 1992; Haugen & Brandenburg 2004). The third term of (36) and should therefore be replaced by the sum of a growth term and a decay term. That is,
\[ \langle b \cdot \nabla \times (v \times b) \rangle_{(k)} = \Gamma b^2(k) - \nu_T(k) k^2 v^2(k), \tag{42} \]
where \( \Gamma \) and \( \nu_T(k) \) indicate a growth rate. For a given \( k \gg k_\lambda \), the terms on the right can largely balance independent of any term with the mean field or magnetic diffusivity. Accordingly, the energies of \( v \) and \( b \) could equilibrate without reference to the second term in (23) which contains \( B \). If the only
initial seed field were a weak mean field, then for a very short
time, the second term of (24) would be the dominant term.
For a $k = 0$ mean field in a box with periodic boundaries, no
change in the mean field is possible. Thus the second term of
(24) would swiftly become subdominant for an initially weak
mean field, in contrast to that implied by the crucial step Eq.
(26), which appears in Z83.

If instead of being a weak seed, the initial $\mathcal{B}$ were im-
posed to be a fraction of order unity of equipartition with the
turbulent kinetic energy, then the first term on the right of
(24) can be always competitive with the remaining terms on
the right of (24). In this case, the small scale dynamo action
is modestly suppressed by the strong mean field (Figs. 2 and
6 of Haugen & Brandenburg 2004; also Brandenburg 1993).

There is another subtlety in the derivation of (36). Expanding
out the term containing $\mathcal{B}$ in (24) gives
\[
\langle b \cdot \nabla \times (v \times \mathcal{B}) \rangle = \langle b_q \partial_n v_q \rangle \mathcal{B}_n - \langle b_n v_q \rangle \partial_q \mathcal{B}_n, \quad (43)
\]
If correlations involving arbitrary products of small scale ve-
locity and magnetic fluctuations were isotropic (that is, if the
velocity and magnetic fluctuations were jointly distributed
isotropically), both of the terms on the right would vanish:
the first because the correlation is a vector, and the second
because isotropy would make that term proportional to $\delta_{qs}$.
The value of the terms on the right hand side of (43) for non-
helical flows and a weak mean field is therefore less than a
simple order of magnitude estimate of those terms would
indicate. Accordingly, the second term of (24) will be smaller
than (25) would indicate. Symbolically, Eq. (24) should be
replaced by
\[
|\langle b \cdot \nabla \times (v \times \mathcal{B}) \rangle| \langle a \rangle \simeq |\langle b(k)k v(k) \rangle|_{a \mathcal{B}}, \quad (44)
\]
where the subscript $a$ indicates that only an anisotropic part
of the correlation contributes. This issue does not arise for the
second term on the right of (24) (which was taken to balance
the left side of (14) in (28)) which is given by
\[
\langle b \cdot \nabla \times (v \times b) \rangle = \langle b_q b_n \partial_n v_q \rangle - \langle b_n v_q \partial_q b_n \rangle. \quad (45)
\]
Although the second term on the right of (45) vanishes for
incompressible turbulence with the joint isotropy assumption,
as can be seen by pulling the divergence out of the average,
the first term on the right of (45) has a surviving isotropic
contribution. Thus, the fact that (43) might be even smaller
than estimated in (25), further highlights that the terms on the
right of (24) could largely maintain the steady state of
(24) by themselves.

4. Conclusion

We have re-examined the meaning and derivation of the Zel-
dovich relations (Z57, Z83, ZR) of MHD for 2-D and 3-D.
These relations (e.g. Eqs. 15, 36, 37, 39) have been purported
to relate the fluctuating magnetic field energy to a large scale
or mean magnetic field energy in steady state forced MHD
turbulence with an imposed mean field. We have identified
why these relations do not apply for realistic steady states
and highlight some common misconceptions about them.

First, we have shown that these relations focus on the re-
response of the total magnetic energy to non-helical turbulence
driven by kinetic energy in the presence of an initially weak
large scale field. Under the circumstances of their standard
derivations, the large scale field serves only as a seed field,
and does not grow. The relations say nothing about how large
a mean field can grow with respect to a fluctuating field in hel-
ical MHD turbulence and therefore place no constraints on
mean field helical MHD dynamo theory.

Second, we point out that the presence of a mean field
as a seed field in 3-D is the only reason that the small scale
field that results from MHD turbulence would depend on it.
Were the initial field a small scale seed field with no net
mean, the small scale field would still grow, as seen in non-
helically driven MHD turbulence simulations (e.g. Maron &
Blackman 2002; Haugen et al. 2003, 2004; Maron et al. 2004)
Schekochihin et al. 2004), and the saturation value could not
possibly depend on the mean field. For 3-D, we have indeed
shown that standard derivations of the Zeldovich relation do
not include (KR) or replace key nonlinear terms in the mag-
netic energy equation by a turbulent diffusion term (Z83),
ignoring an equally important growth term. The Zeldovich
relations that emerge therefore result from an inappropriate
balance of non-dominant terms. Accordingly, as seen in 3-D
simulations, the ultimate steady-state saturation of the small
scale field is determined by equipartition with the turbulent
kinetic energy, independent of the mean field or magnetic
Reynolds number for large $R_M$ non-helical systems.

Finally, for 2-D, we have shown that when the magnetic
field has no $k = 0$ component (but can still have a large scale
$k < k_f$ component) the net magnetic energy must decay for
a forced 2-D MHD turbulence system. Then the Zeldovich
relation (15) or (37) cannot represent a true steady state.
We have shown that instead, this relation represents the maxi-
mum that the fluctuating field could obtain if the initial seed
field were a large scale seed field (and if $R_M \mathcal{B}^2$ were less
than the turbulent kinetic energy—otherwise the latter deter-
mines a quasi-steady saturated value) after which the mag-
netic energy would decay. In this respect, the import of the
Zeldovich relation of Z57, was mainly to illustrate quantita-
tively, the now well known result that 2-D MHD turbulence
does not sustain a steady-state dynamo without a fixed seed.

Acknowledgements. EGB acknowledges support from NSF grant
AST-0406799 and NASA grant ATP04-0000-0016, and the Isaac
Newton Institute, Cambridge, and A. Brandenburg & M. Proctor
for comments and discussions.

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