APPLICATIONS OF WEAK TRANSPORT THEORY

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Abstract. Motivated by applications to geometric inequalities, Gozlan, Roberto, Samson, and Tetali [34] introduced a transport problem for ‘weak’ cost functionals. Basic results of optimal transport theory can be extended to this setup in remarkable generality.

In this article we collect several problems from different areas that can be recast in the framework of weak transport theory, namely: the Schrödinger problem, the Brenier–Strassen theorem, optimal mechanism design, linear transfers, semimartingale transport. Our viewpoint yields a unified approach and often allows to strengthen the original results.

keywords: Schrödinger problem, Brenier–Strassen theorem, linear transfers, semimartingale transport, optimal mechanism design, weak transport problem, duality, cyclical monotonicity.

1. Overview

The optimal transport problem for weak costs was first introduced by Gozlan, Roberto, Samson and Tetali [34] and has immediately generated interest in several groups of researchers, see [45, 46, 29, 32, 1, 2] among others. To present the basic problem we introduce some notation. Throughout \(X\) and \(Y\) denote Polish spaces. Given probability measures \(\mu \in \mathcal{P}(X)\) and \(\nu \in \mathcal{P}(Y)\) we write \(\Pi(\mu, \nu)\) for the set of all couplings on \(X \times Y\) with marginals \(\mu\) and \(\nu\). Given a coupling \(\pi\) on \(X \times Y\) we denote a regular disintegration with respect to the first marginal by \((\pi_x)_{x \in X}\).

We consider cost functionals of the form

\[ C : X \times \mathcal{P}(Y) \rightarrow \mathbb{R} \cup \{+\infty\}, \]

where \(C\) is lower bounded and lower semicontinuous, and \(C(x, \cdot)\) is assumed to be convex on \(\mathcal{P}(Y)\) for every \(x \in X\). The weak transport problem is then to determine

\[ V_C(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \int_X C(x, \pi_x) \mu(dx). \] (OWT)

The classical transport problem is included via \(C(x, \cdot) = \int c(x, y) \, dp(y)\) for a given cost function \(c : X \times Y \rightarrow \mathbb{R} \cup \{+\infty\}\).

Fundamental results in classical optimal transport theory include existence, duality, and cyclical monotonicity of optimizers. Through a series of contributions (see [34, 2, 32, 6]) it has been understood that these results extend in full generality to the weak transport setup. We will recall these results in some detail in Section 2 below.
The purpose of the present article is to advertise the power and flexibility of optimal weak transport theory through the investigation of a broad variety of applications thereof. We will consider the following problems:

- The Schrödinger problem has recently received particular attention since it provides a regularized version of the transport problem that is numerically much more tractable than the classical counterpart. On a technical level, the difference is that quadratic costs are replaced by the entropy wrt a reference measure. Based on monotonicity for weak transport costs we give a short proof of the fundamental characterization of optimizers in the Schrödinger problem. (Section 4.)

- The Brenier–Strassen Theorem of Gozlan and Juillet [32] yields that 1-Lipschitz maps that are gradients to convex functions are the optimizers of transport problems with barycentric costs. This result plays a role in the probabilistic proof of the Caffarelli contraction theorem [28]. We provide a short new derivation which is based on monotonicity for weak transport costs and emphasizes the similarity with the classical Brenier Theorem. (Section 5.)

- A significant problem in the economics literature is to optimize the revenue for a multiple good monopolist. The influential article [25] of Daskalakis, Deckelbaum, and Tzamos suggests a systematic framework to study such problems and provides a dual characterization. We link the multiple good monopolist problem to optimal weak transport and use the weak transport duality theorem to recover and strengthen the results of [25]. (Section 6.)

- Bowles and Ghoussoub [18] have recently introduced the class of linear transfers which includes many specific couplings between probability measures. A main theorem of [18] provides the representation of linear transfers through weak transport under the assumption of compactness of the underlying spaces. We provide a short new derivation based on weak transport duality that allows to drop the compactness condition. In particular, this implies that the representation result of Bowles and Ghoussoub is valid for the important case of Euclidean space. (Section 7.)

- The semimartingale transportation problem was introduced by Tan and Touzi [50] and extends classical optimal transport to the case where mass is transported along the trajectories of a semimartingale. As such, it also contains both martingale optimal transport [30, 13, 14], and the drift control framework of Mikami and Thieullen [41], as particular cases. We show that weak transport theory can be used to strengthen the main duality result of [50]. (Section 8.)

The paper is organized as follows. In Section 2 we give a brief overview of previous works connected to optimal weak transport. In Section 3 we review the basic results of optimal weak transport theory together with the necessary notation. Then we describe the various applications announced above in Sections 4-8.
2. Literature connected to the weak transport problem

The weak transport problem was introduced by Gozlan, Roberto, Samson and Tetali [34], and shortly afterwards by Aliberti, Bouchitte and Champion [2]. The problem has also been designated “general transport problem” and “non-linear transport problem” respectively.

The initial works of Gozlan et al. [34, 33] are mainly motivated by applications to geometric inequalities. Indeed, particular costs of the form \((OWT)\) were already considered by Marton [40, 39] and Talagrand [48, 49]. The theory for problem \((OWT)\) has been further developed in [1, 34, 33, 45, 44, 46, 29, 32, 6, 9, 7]: Basic results of existence and duality are established in the articles [34, 2, 6]. The notion of \(C\)-monotonicity was developed in [4, 32, 6] as an analogue of classical \(c\)-cyclical monotonicity in order to provide a characterization of optimizers to the weak transport problem. A weak transport analogue to the case of quadratic costs in classical optimal transport, is the case of barycentric costs. This case has received particular attention, and we refer to [33, 45, 44, 46, 29, 32, 9, 7, 1].

The weak transport viewpoint is useful for a number of problems loosely related to stochastic optimization: it appears in the recursive formulation of the causal transport problem [5], in [1, 2, 15, 4, 36, 22] it is used to investigate martingale optimal transport problems, in [3] it is applied to prove stability of pricing and hedging in mathematical finance and, as mentioned above, it appears in the recent probabilistic proof to the Caffarelli contraction theorem [28].

3. Fundamental results of weak transport theory

For \(t \geq 1\), \(\mathcal{P}_t(X)\) denotes the set of Borel probability measures with finite \(t\)-th moment for some fixed metric \(d_X\) (compatible with the topology on \(X\)), i.e., a Borel probability measure \(\mu\) is in \(\mathcal{P}_t(X)\) iff for some \(x_0 \in X\) we have

\[
\int_X d_X(x, x_0)^t \mu(dx) < \infty.
\]

The set of continuous functions on \(X\) which are dominated by a multiple of \(1 + d_X(x, x_0)^t\), is denoted by \(\Phi_t(X)\). We equip the set of probability measures \(\mathcal{P}_t(X)\) with the \(t\)-th Wasserstein topology. Specifically, a sequence \((\mu_k)_{k \in \mathbb{N}}\) converges to \(\mu \in \mathcal{P}_t(X)\) if \(\mu_k(f) := \int f d\mu_k\) converges to \(\mu(f)\) for all \(f \in \Phi_t(X)\). The space \(\mathcal{P}(X)\) itself is equipped with the usual weak topology. The same conventions apply to \(Y\) instead of \(X\).

We have already mentioned the basic assumptions on the function \(C\) in the introductory section above. We recall it and make it more precise in the following definition.

**Definition 3.1 (A).** We say \(C : X \times \mathcal{P}_t(Y) \to \mathbb{R} \cup [+\infty]\) satisfies property \([A]\) iff

i) \(C\) is lower-semi continuous wrt the product topology on \(X \times \mathcal{P}_t(Y)\),

ii) \(C\) is bounded from below,
the map \( p \mapsto C(x, p) \) is convex, i.e., for all \( x \in X \) and \( p, q \in \mathcal{P}_t(Y) \) we have
\[
C(x, \lambda p + (1 - \lambda)q) \leq \lambda C(x, p) + (1 - \lambda)C(x, q) \quad \lambda \in [0, 1].
\]

From now on until the end of this section, the cost function \( C \) is assumed to satisfy property (A).

We will need the following existence and continuity result from [6]:

**Theorem 3.2 (Existence and semicontinuity).** The infimum in (OWT) is attained and the value \( V_C(\mu, \nu) \) depends in a lower semicontinuous way on the marginals \((\mu, \nu) \in \mathcal{P}(X) \times \mathcal{P}_t(Y)\).

In optimal transport, the renowned Kantorovich duality states that for lower semi-continuous and lower bounded cost functions \( c : X \times Y \to \mathbb{R} \cup \{+\infty\} \) we have
\[
\inf_{\pi \in \Pi(\mu, \nu)} \int_{X \times Y} c(x, y) \pi(dx, dy) = \sup_{f \in L_1(\mu), \ g \in \Phi_t(Y), \ f + g \leq c} \mu(f) + \nu(g). \quad (3.1)
\]

Here duality takes the following form, which resembles (and generalizes) (3.1), cf. [6, Theorem 3.1]:

**Theorem 3.3 (Kantorovich duality for weak transport).** The weak transport problem (OWT) admits the dual representation
\[
\inf_{\pi \in \Pi(\mu, \nu)} \int_X C(x, \pi_x) \mu(dx) = \sup_{f \in L_1(\mu), \ g \in \Phi_t(Y)} \mu(f) - \nu(g) + \int_X R_C g(x) \mu(dx), \quad (3.2)
\]
where the supremum is taken over functions \( f \in L_1(\mu), \ g \in \Phi_t(Y) \), satisfying \( f(x) + p(g) \leq C(x, p) \) for \( x \in X, \ p \in \mathcal{P}_t(Y) \).

We will often use Theorem 3.3 in the following (equivalent) form:
\[
\inf_{\pi \in \Pi(\mu, \nu)} \int_X C(x, \pi_x) \mu(dx) = -\int_X R_C g(x) \mu(dx), \quad (3.3)
\]
where \( R_C g(x) = \inf_{p \in \mathcal{P}_t(Y)} p(g) + C(x, p) \). Moreover, the right-hand supremum in (3.3) can be restricted to functions in \( g \in \Phi_t(Y) \) which are bounded from below.

We also recall a further consequence of the proof of Theorem 3.3 in [6] that provides further insight into the dual problem. The convex conjugate of \( \nu \mapsto V_C(\mu, \nu) \) admits a rather concrete representation: For any \( g \in \Phi_t(Y) \) we have
\[
\sup_{\nu \in \mathcal{P}_t(Y)} \nu(g) - V_C(\mu, \nu) = -\int_X R_C g(x) \mu(dx). \quad (3.4)
\]

The notion of \( c \)-cyclical monotonicity constitutes a necessary (and often also sufficient) optimality criterion for transport plans in classical optimal
transport, i.e., for any measurable cost $c : X \times Y \to \mathbb{R} \cup \{-\infty, +\infty\}$ if $\pi^* \in \Pi(\mu, \nu)$ is an optimizer of

$$V_c(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \int_{X \times Y} c(x, y) \pi(dx, dy),$$

where $|V_c(\mu, \nu)| < \infty$, then there exists $\Gamma \subseteq X \times Y$ with $\pi^*(\Gamma) = 1$ such that for all $N \in \mathbb{N}$ and $(x_k, y_k)^N_{k=1}$ in $\Gamma$ we have for any permutation $\sigma$ of $\{1, \ldots, N\}$ that

$$\sum_{k=1}^N c(x_k, y_k) \leq \sum_{k=1}^N c(x_k, y_{\sigma(k)}).$$

The importance of $c$-cyclical monotonicity has been understood at least since the publication of the seminal article [31]. See [52, 17, 10] for minimal conditions that guarantee equivalence of optimality and $c$-cyclical monotonicity. More recently, variants of this ‘monotonicity principle’ have been applied in transport problems for finitely or infinitely many marginals, the martingale version of the transport problem, the Skorokhod embedding problem, and the distribution constrained optimal stopping problem, see [43, 21, 55, 12, 41, 42, 53, 11] among others.

In the context of optimal weak transport the corresponding concept is $C$-monotonicity. Early versions can be found in [4, 6, 32], while the following definition as well as the subsequent result are taken from [9, Section 2].

**Definition 3.4.** A set $\Gamma \subseteq X \times \mathcal{P}_t(Y)$ is called $C$-monotone if for any finite subset of points $(x_k, p_k)^N_{k=1}$ of $\Gamma$ we have

$$\sum_{k=1}^N C(x_k, p_k) \leq \sum_{k=1}^N C(x_k, q_k) \sum_{k=1}^N q_k = \sum_{k=1}^N p_k.$$

A coupling $\pi$ with first marginal $\mu$ is called $C$-monotone if there is a $C$-monotone set $\Gamma$ such that $(\text{id}_X, \delta_{\pi_x})_{\#} \mu$ is concentrated on $\Gamma$.

**Theorem 3.5 (C-monotonicity).** If (OWT) is finitely valued, then any optimizer is $C$-monotone.

The reverse implication holds also true if $C$ is sufficiently regular, see [9, Theorem 2.2].

**Theorem 3.6.** Assume that $\mu \in \mathcal{P}_t(X)$, $\nu \in \mathcal{P}_t(Y)$. If $C$ is continuous and $|C(x, p)| \leq R(d_X(x, x_0)^x + \int_Y d_Y(y, y_0)^y p(dy))$ for some $R$, then any $C$-monotone coupling $\pi \in \Pi(\mu, \nu)$ is optimal for (OWT).

4. Structure of optimizers in the Schrödinger problem

We refer the reader to [38] for a survey on the Schrödinger problem / entropic transport problem (cf. (4.4) below). Starting with the articles [24, 16, 27], the Schrödinger problem has received significant attention as a regularized, numerically tractable version of the classical transport problem.

The main goal of this section is to recover, in Corollary 4.3 below, the characterization of entropic cost optimal transport plans through the product
structure of their density. Given a Polish space \( Z \), the relative entropy of \( \mu \in \mathcal{P}(Z) \) wrt a ‘reference measure’ \( \nu \in \mathcal{P}(Z) \) is defined as
\[
H(\mu|\nu) = \begin{cases} 
\int_X \log \left( \frac{d\mu}{d\nu} \right) \mu(dx) & \mu \ll \nu, \\
+\infty & \text{else}.
\end{cases}
\]

**Theorem 4.1.** Let \( \gamma \) be a probability measure equivalent to the product measure \( \mu \otimes \nu \) for \( \mu \in \mathcal{P}(X) \) and \( \nu \in \mathcal{P}(Y) \). If the coupling \( \pi^* \in \Pi(\mu, \nu) \) is optimal for the problem
\[
\inf_{\pi \in \Pi(\mu, \nu)} \int_X H(\pi (| \gamma)) \mu(dx) < +\infty,
\]
then there is a measurable set \( \Gamma \subseteq X \times \mathcal{P}(Y) \) with \( \mu(\{x \in X : (x, \pi^*_x) \in \Gamma\}) = 1 \) such that for all \( (x, \pi^*_x), (z, \pi^*_z) \in \Gamma \) there is a constant \( a > 0 \) so that
\[
\frac{d\pi^*_x}{d\gamma_x} = a \frac{d\pi^*_z}{d\gamma_z} \quad \nu\text{-a.e.}
\]

**Proof.** The minimization problem (4.1) constitutes a weak transport problem. Therefore, there is by Theorem 3.5 a measurable set \( \Gamma \subseteq X \times \mathcal{P}(Y) \) such that \( \pi^* \) is \( C \)-monotone on \( \Gamma \). Moreover, we may assume w.l.o.g. that \( \gamma_x \sim \nu \) for all \( (x, \pi^*_x) \in \Gamma \) and \( H(\pi^*_x|\gamma_x) < \infty \). Fix \( (x, \pi^*_x), (z, \pi^*_z) \in \Gamma \).

We want to show that \( \pi^*_x \sim \pi^*_z \). By \( C \)-monotonicity, cf. Definition 3.4, we have for all \( q_x, q_z \in \mathcal{P}(Y) \) with \( q_x + q_z = \pi^*_x + \pi^*_z \) that
\[
H(\pi^*_x|\gamma_x) + H(\pi^*_z|\gamma_z) \leq H(q_x|\gamma_x) + H(q_z|\gamma_z).
\]

Then the corresponding first order optimality condition reads as
\[
\pi^*_x \left( \log \left( \frac{d\pi^*_x}{d\gamma_x} \right) \right) + \pi^*_z \left( \log \left( \frac{d\pi^*_z}{d\gamma_z} \right) \right) \leq q_x \left( \log \left( \frac{d\pi^*_x}{d\gamma_x} \right) \right) + q_z \left( \log \left( \frac{d\pi^*_z}{d\gamma_z} \right) \right). \quad (4.3)
\]

This inequality can be easily deduced by differentiation along the segment joining \( \pi^*_x \) with \( q_x \) and \( \pi^*_z \) with \( q_z \), i.e., by computing
\[
\left. \frac{d}{dt} \right|_{t=0} H(q^t_x|\gamma_x) + H(q^t_z|\gamma_z),
\]

where \( q^t = (1-t)\pi^*_x + tq_z \). Assume that there exists a Borel measurable set \( A \subseteq Y \) with \( \pi^*_x(A) > 0 \) but \( \pi^*_z(A) = 0 \). Then on \( A \) we have for \( \nu\text{-a.e.} \) \( y \) that \( \frac{d\pi^*_z}{d\gamma_z}(y) = 0 \). It is straightforward that one can find \( q_x, q_z \) such that \( q_x + q_z = \pi^*_x + \pi^*_z \), \( q_x(A) > 0 \) and \( q_x \left( \log \left( \frac{d\pi^*_z}{d\gamma_z} \right) \right) < +\infty \), so we omit the technical details. As a consequence we find
\[
q_x \left( \log \left( \frac{d\pi^*_x}{d\gamma_x} \right) \right) + q_z \left( \log \left( \frac{d\pi^*_z}{d\gamma_z} \right) \right) = -\infty,
\]

which contradicts the first order optimality criterion since the left-hand side of (4.3) is non-negative. Hence by symmetry \( \pi^*_x \) is equivalent to \( \pi^*_z \) and as a consequence we have \( \pi^*_x \sim \nu \) for all \( (x, \pi^*_x) \in \Gamma \).

Applying Lemma 4.2 below to the pair \( \pi^*_x, \pi^*_z \) yields a contradiction to \( C \)-monotonicity. \( \square \)
Lemma 4.2. Let \( p_i, \gamma_i, i \in \{1, 2\} \) and \( \nu \) be equivalent probability measures on \( Y \). If \( H(p_i|\gamma_i), i = 1, 2, \) is finite and there exists no constant \( \alpha \in \mathbb{R} \) such that
\[
\frac{dp_1}{d\gamma_1} = \alpha \frac{dp_2}{d\gamma_2} \quad \text{\( \nu \)-a.e.},
\]
then there are two probability measures \( q_1, q_2 \in \mathcal{P}(Y) \) with \( q_1 + q_2 = p_1 + p_2 \) and
\[
H(p_1|\gamma_1) + H(p_2|\gamma_2) > H(q_1|\gamma_1) + H(q_2|\gamma_2).
\]

Proof. Since any Polish space is Borel-isomorphic to a measurable subset of \([0, 1]\) we may assume that \( Y = [0, 1] \). By the inverse transform sampling we may assume that \( \nu \) is the uniform distribution on \([0, 1]\). For \( i \in \{1, 2\} \), define the densities
\[
g_i : [0, 1] \to (0, +\infty), \quad g_i = \frac{dp_i}{d\gamma_i}; \quad f_i : [0, 1] \to (0, +\infty), \quad f_i = \frac{d\gamma_i}{d\nu}.
\]
By Lusin’s theorem we find for any \( \epsilon > 0 \) a compact set \( K_\epsilon \subseteq [0, 1] \) with mass \( \nu(K_\epsilon) \geq 1 - \epsilon \) such that \( h := \frac{g_1}{g_2} : [0, 1] \to (0, +\infty) \) is continuous on \( K_\epsilon \). By the assumption that \( g_1 \) is not a multiple of \( g_2 \) we find two disjoint intervals \([a_1, b_1], [a_2, b_2] \subseteq (0, +\infty), b_1 < a_2 \) with
\[
\nu(h^{-1}([a_1, b_1])) > 0 \text{ and } \nu(h^{-1}([a_2, b_2])) > 0.
\]
Choosing \( \epsilon > 0 \) sufficiently small the closed, disjoint sets
\[
A_1 = h^{-1}([a_1, b_1]) \cap K_\epsilon \text{ and } A_2 = h^{-1}([a_2, b_2]) \cap K_\epsilon
\]
have positive mass under \( \nu \). W.l.o.g. we can assume that \( \nu(A_1) = \nu(A_2) > 0 \). The map
\[
T : A_1 \to A_2
\]
\[
x \mapsto F_{v|A_2}^{-1} \circ F_{v|A_1}(x)
\]
provides a measure preserving bijection. Let \( S : [0, 1] \to [0, +\infty) \) be defined as
\[
S(y) = \begin{cases} 
\min \{g_1(y)f_1(y), g_2(y)f_2(y), \hat{g}_1(y)\hat{f}_1(y), \hat{g}_2(y)\hat{f}_2(y)\} & \text{if } y \in A_1, \\
\min \{g_1(z)f_1(z), g_2(z)f_2(z), g_1(y)f_1(y), g_2(y)f_2(y)\} & \text{if } y \in A_2, \\
0 & \text{else},
\end{cases}
\]
where \( \hat{g}_i = g_i \circ T, \hat{f}_i = f_i \circ T \) and \( z = T^{-1}(y) \). Then we define for \( t \in [-1, 1] \) the probability measures \( p'_1 \) and \( p'_2 \) on \([0, 1]\) by
\[
p'_1(dy) = p_1(dy) + tS(y)\nu(dy), \quad p'_2(dy) = p_2(dy) - tS(y)\nu(dy).
\]
The respective densities are then given by
\[
\frac{dp'_1}{d\gamma_1} = g_1 + \frac{tS}{f_1}, \quad \frac{dp'_2}{d\gamma_2} = g_2 - \frac{tS}{f_2}.
\]
Differentiation of
\[ H(p_1' | \gamma_1) + H(p_2' | \gamma_2) \]
\[ = \int_Y \log \left( g_1(y) + t \frac{S}{f_1}(y) \right) p_1'(dy) + \int_Y \log \left( g_2(y) + t \frac{S}{f_2}(y) \right) p_2'(dy) \]
with respect to \( t \) and evaluation at 0 yields
\[ \int_Y S(y) \log \left( \frac{g_1(y)}{g_2(y)} \right) \nu(dy) = \int_Y S(y) \log \left( \frac{g_1(y)g_2(T(y))}{g_2(y)g_1(T(y))} \right) \nu(dy) < 0. \]
By strict convexity of \( t \mapsto H(p_1' | \gamma_1) + H(p_2' | \gamma_2) \) we conclude that there exists a \( t_0 \in [-1,1] \) with
\[ H(p_1' | \gamma_1) + H(p_2' | \gamma_2) > H(p_1' | \gamma_1) + H(p_2' | \gamma_2). \]

Finally we obtain the main result of this section (see [23, Corollary 3.2]).

**Corollary 4.3.** Let \( \mu \in \mathcal{P}(X) \) and \( \nu \in \mathcal{P}(Y) \) and \( \gamma \in \mathcal{P}(X \times Y) \) a probability measure equivalent to \( \mu \otimes \nu \). Assume that the value of the corresponding entropic optimal transport problem is finite, i.e.,
\[ \inf_{\pi \in \Pi(\mu,\nu)} H(\pi | \gamma) < \infty. \]
A coupling \( \pi^* \in \Pi(\mu,\nu) \) minimizes \( (4.4) \) if and only \( H(\pi^* | \gamma) < \infty \) and there exist measurable functions \( f \) and \( g \) such that
\[ \frac{d\pi^*}{d\gamma}(x,y) = f(x)g(y) \gamma\text{-a.e.} \]

**Proof.** We use the notation of Theorem 4.1 and rewrite
\[ H(\pi | \gamma) = \int_X H(\pi_x | \gamma_x) \mu(dx) + H(\mu | \gamma_0), \]
where \( \gamma_0 \in \mathcal{P}(X) \) is the X-marginal of \( \gamma \). Lower semicontinuity and strict convexity of the relative entropy yield the existence of a minimizer \( \pi^* \in \Pi(\mu,\nu) \) of \( (4.4) \). The identity \( (4.5) \) shows that \( \pi^* \) also minimizes \( (4.1) \). By Theorem 4.1, calling \( \Gamma \) the set with \( \mu \)-full \( X \)-projection therein, we fix \( (z,\pi^*_z) \in \Gamma \) and define \( g(y) := \frac{d\pi^*_z}{d\gamma_z}(y) \) as well as \( h(x) = \alpha \) for \( \alpha \) as in \( (4.2) \). Thus we have
\[ \frac{d\pi^*_z}{d\gamma_z}(y) = h(x)g(y) \quad (x,\pi^*_z) \in \Gamma, \gamma_x\text{-a.e. } y. \]
Hence,
\[ \frac{d\pi^*}{d\gamma}(x,y) = \frac{d\mu}{d\gamma_0}(x)h(x)g(y) = f(x)g(y) \quad \gamma\text{-a.e.,} \]
where \( f \) is defined appropriately. Conversely, assume that we have two couplings \( \pi, \pi' \) with finite entropy, with marginals \( \mu \) and \( \nu \), and such that their densities w.r.t. \( \gamma \) are of product form. Let
\[ \frac{d\pi}{d\gamma}(x,y) = f(x)g(y) \text{ and } \frac{d\pi'}{d\gamma}(x,y) = f'(x)g'(y) \]
and \( \log(fg) \), \( \log(f'g')f'g' \in L^1(\gamma) \). Since the marginals of \( \pi \) and \( \pi' \) coincide, we have for any \( h \in L^1(\mu) \otimes L^1(\nu) \) that

\[
\int h(x, y)(fg - f'g')\gamma(dx, dy) = 0. \tag{4.6}
\]

We can approximate \( \log(fg) \) by elements in \( L^1(\mu) \otimes L^1(\nu) \) such that on \( [\log(fg) \geq 0] \) we have \( h_n \geq 0 \) and \( h_n \not\rightarrow \log(fg) \) and similar applies on \( [\log(fg) \leq 0] \). Hence

\[
\int \log(fg)d\pi = \lim_n \int h_n fgd\gamma = \lim_n \int h_n f'g'd\gamma = \int \log(fg)d\pi'.
\]

Therefore, \( \log(fg)f'g' \in L^1(\gamma) \) and \( \log(f'g')fg \in L^1(\gamma) \) and

\[
H(\pi|\gamma) - H(\pi|\pi') = \int \log(f'g')d\pi = \int \log(f'g')d\pi' = H(\pi'|\gamma).
\]

Especially, we have shown \( H(\pi|\gamma) = H(\pi'|\gamma) \) since \( H(\pi|\pi') \) is non-negative, which implies \( H(\pi|\pi') \) and \( H(\pi'|\pi) \) vanish, so \( \pi = \pi' \).

5. Brenier-Strassen Theorem

A fundamental result in the theory of optimal transport is Brenier’s theorem \cite{19, 20} which asserts that the optimizer of the Wasserstein-2 distance on \( \mathbb{R}^d \) between \( \mu, \nu \) is given by the gradient of a convex function \( \varphi \). Specifically, the optimal plan \( \pi^* \in \Pi(\mu, \nu) \) is of the form \( (id, \nabla \varphi)(\mu) \) and \( \nabla \varphi(\mu) = \nu \).

We refer to, e.g. \cite[Theorem 2.12]{51} for more details and bibliographical remarks.

Strassen’s theorem \cite{47} asserts that given marginals \( \mu, \nu \in P_1(\mathbb{R}^d) \) there exists a martingale \( (Z_i)_{i=1,2} \) with \( Z_1 \sim \mu, Z_2 \sim \nu \) provided that \( \mu \preceq \nu \). Here \( \preceq \) denotes the usual convex order, i.e. \( \mu \preceq \nu \) means that \( \int \varphi \, d\mu \leq \int \varphi \, d\nu \) for all convex functions \( \varphi \). Note that the condition \( \mu \preceq \nu \) is not only sufficient for the existence of a martingale with these marginals but in fact also necessary by Jensen’s inequality.

An intermediate version between the classical \( W_2 \)-problem and Strassen’s Theorem on the existence of martingales \cite{47} was recently investigated in various forms \cite{32, 34, 33, 11, 46, 6, 7}. The following represents the weak transport analogue of the classical \( W_2 \)-distance:

\[
V_2(\mu, \nu)^2 := \inf_{\eta \leq \nu} \mathcal{W}_2(\mu, \eta)^2 \tag{5.1}
\]

\[
= \inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathbb{R}^d} \left| x - \int_{\mathbb{R}^d} y\pi_s(dy) \right|^2 \mu(dx). \tag{5.2}
\]

Note that the equality of (5.1) and (5.2) is a straightforward consequence of Strassen’s Theorem.

The Brenier-Strassen theorem of Gozlan and Juillet \cite{32} (see also Shu \cite{45} for the one-dimensional case) asserts that \( \eta^* \preceq \nu \) is optimal for (5.1) if and only if there exists a convex function \( \varphi \) with 1-Lipschitz gradient such that \( \nabla \varphi(\mu) = \eta^* \). The proof of the \( if \)-part of the statement can be done by showing dual attainment or more directly using \( C \)-monotonicity whereas the only \( if \)-part was thus far only shown via duality.
The goal of this section is to recover the only if-clause by means of sufficiency of C-monotonicity. We believe that this new proof is appealing in that it mimics the proof of Brenier’s theorem via classical cyclical monotonicity, underlining the similarity of the two results.

To provide an intuition for the proof, we recall the main idea in the classical case: Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be a convex and differentiable function. For any finite number of points $x_1, \ldots, x_n \in \mathbb{R}^d$ it is immediate that

$$\sum_{k=1}^n (x_{k+1} - x_k, \nabla \varphi(x_k)) \leq 0. \quad (5.3)$$

Completing the square yields

$$\sum_{k=1}^n |x_k - \nabla \varphi(x_k)|^2 \leq \sum_{k=1}^n |x_{k+1} - \nabla \varphi(x_k)|^2, \quad (5.4)$$

which shows optimality for any measure supported on finitely many points in the graph of $\nabla \varphi$. Therefore, since cyclical monotonicity is sufficient for optimality, the coupling $(\text{id}, \nabla \varphi)(\mu)$ turns out to be optimal for the quadratic distance transport problem.

In the subsequent proof, we will use as a black box a characterization of convex functions with 1-Lipschitz gradient in the spirit of cyclical monotonicity, see [55]. This permits to draw the desired conclusion for the problem (5.1) in analogy to step from (5.3) to (5.4).

**Theorem 5.1.** Let $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ where $\mathbb{R}^d$ is equipped with the standard euclidean norm $|\cdot|$. Let $\eta^*$ be in convex order dominated by $\nu$. Then the following are equivalent:

1. The values of $\mathcal{W}_2(\mu, \eta^*)$ and $V_2(\mu, \nu)$ coincide, i.e. $\eta^*$ is a solution of the optimization problem (5.1).
2. There is a convex function $\varphi : \mathbb{R}^d \to \mathbb{R}$ such that $\nabla \varphi : \mathbb{R}^d \to \mathbb{R}^d$ is 1-Lipschitz, $\eta^* = \nabla \varphi(\mu)$ and $(\text{id}, \nabla \varphi)_{\#} \mu$ is the unique optimizer of (5.1).

Theorem 5.1 can be found in [32, Theorem 2.1]. A proof of the first implication using different arguments is given in [6, Theorem 1.4].

**Proof of Theorem 5.1**. Here we will only show ‘2 $\implies$ 1’. To this end, we want to verify that

$$\Gamma := \{(x, \delta \varphi(x)) \in \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) : x \in \mathbb{R}^d\}$$

is C-monotone, where $C(x, p) := \left| x - \frac{1}{\int y \mu(dy)} \right|^2$.

Let $N \in \mathbb{N}$, $x_1, \ldots, x_N \in \mathbb{R}^d$ with $y_i = \nabla \varphi(x_i)$. We have to show that if $\sum_{i=1}^N m_i = \sum_{i=1}^N \delta_{y_i}$ then $\sum_{i=1}^N |x_i - y_i|^2 \leq \sum_{i=1}^N |x_i - \int y m_i(dy)|^2$. Clearly we must have $m_i = \sum_{j=1}^N \alpha_{i,j} \delta_{y_j}$ for all $i$, where $(\alpha_{i,j}) \in \mathbb{R}^{N \times N}$ is a bistochastic matrix. We can rewrite $\alpha = (\alpha_{i,j})$ as a convex combination of permutation matrices $(P_{\sigma})_{\sigma \in \Sigma}$, where $\Sigma$ denotes the set of permutations on $\{1, \ldots, N\}$:

$$\alpha = \sum_{\sigma \in \Sigma} \beta_{\sigma} P_{\sigma}, \quad \sum_{\sigma \in \Sigma} \beta_{\sigma} = 1, \quad \beta_{\sigma} \geq 0.$$
Note that the map $F \colon \mathbb{R}^N \to \mathbb{R}$
\[ F((\beta_\sigma)_{\sigma \in \Sigma}) = \frac{1}{2} \sum_{i=1}^{N} \left| x_i - \sum_{\sigma \in \Sigma} \beta_\sigma y_{\sigma(i)} \right|^2 \]
is convex. We seek to show that
\[ \hat{\beta}_\sigma = \begin{cases} 1 & \sigma = id, \\ 0 & \text{else}, \end{cases} \]
is a minimum on the simplex on $\mathbb{R}^N$. The gradient of $F$ at $\hat{\beta}$ has the following form
\[ \nabla F(\hat{\beta}) = \left( \sum_{i=1}^{N} (x_i - y_i) \cdot (y_i - y_{\sigma(i)}) \right)_{\sigma \in \Sigma}. \]
For any permutation $\sigma \in \Sigma$ we compute
\[ \sum_{i=1}^{N} (x_i - y_i) \cdot (y_i - y_{\sigma(i)}) = \sum_{i=1}^{N} x_i \cdot (y_i - y_{\sigma(i)}) - \frac{1}{2} \| y_i - y_{\sigma(i)} \|^2 \]
\[ = \sum_{i=1}^{N} (x_i - x_{\sigma^{-1}(i)}) \cdot y_i - \frac{1}{2} \| y_i - y_{\sigma^{-1}(i)} \|^2 \]
\[ \geq 0. \]
The last inequality above follows from [55, Lemma 4], which states that $\varphi$ being convex and 1-Lipschitz is equivalent to
\[ \varphi(z) - \varphi(x) \geq \nabla \varphi(x) \cdot (z - x) + \frac{1}{2} \| \nabla \varphi(z) - \nabla \varphi(x) \|^2 \quad \forall x, z \in \mathbb{R}^d, \]
so by summing this last inequality we get
\[ 0 = \sum_{i=1}^{N} \varphi(x_{\sigma(i)}) - \varphi(x_i) \]
\[ \leq \sum_{i=1}^{N} (x_i - x_{\sigma(i)}) \cdot y_i - \frac{1}{2} \| y_i - y_{\sigma(i)} \|^2. \]
We conclude that $\nabla F(\hat{\beta})$ is pointwise non-negative proving optimality of $\hat{\beta}$ on the simplex by convexity of $F$.

All in all, the coupling $(id, \nabla \varphi)_{#\mu}$ is concentrated on the $C$-monotone set $\Gamma$. Since both marginals are in $\mathcal{P}_2(\mathbb{R}^d)$ we can apply Theorem 3.6 which yields its optimality for the weak transport problem $V_2(\mu, \nu)$. \qed

An immediate consequence of the Brenier-Strassen Theorem 5.1, coupled with the proof therein, is the following modification of the classical Rockafelll theorem. We denote the subdifferential of a convex function $\varphi$ at $x \in \mathbb{R}^d$ by $\delta \varphi(x)$ and by $\partial \varphi = \{(x, y) \colon x \in \mathbb{R}^d, y \in \delta \varphi(x)\}$ its graph.

**Corollary 5.2** (Rockafellar-Strassen). Let $L \in \mathbb{R}^+$ and $\Gamma \subseteq \mathbb{R}^d \times \mathbb{R}^d$. The following are equivalent:
Theorem 6.1. Then all of the following optimization problems yield the same value:

\[ \sum_{i=1}^{n} (x_{i+1} - x_i) \cdot y_i + \frac{1}{2L} |y_{i+1} - y_i|^2 \leq 0. \]  

(2) there exists a convex \( \varphi \in C^1(\mathbb{R}^d) \) with \( L \)-Lipschitz gradient such that \( \Gamma \subseteq \partial \varphi \).

Proof. The implication \( \mathbb{1} \Rightarrow \mathbb{2} \) can be easily deduced from \cite[Lemma 4]{55}. For \( \mathbb{1} \Rightarrow \mathbb{2} \), since we can always consider \( \Gamma = \{(x,y) : (x,Ly) \in \Gamma \} \), which satisfies \( (5.5) \) for \( L = 1 \), we can assume w.l.o.g. that \( L = 1 \). Let \( \mu \) be a probability measure on \( \mathbb{R}^d \) such that

\[ \text{supp}(\mu) = \text{cl}(\{(x: \exists y, (x,y) \in \Gamma)\}). \]

By \( (5.5) \), if \( (x,y), (x,y') \in \Gamma \) then \( y = y' \). This defines a map \( T: \text{proj}_1 \Gamma \to \text{proj}_2 \Gamma \). Further, for all \( (x,y), (x',y') \in \Gamma \), we have

\[ -|y - y'|^2 \geq (x - x') \cdot (y - y') \geq -|x - x'||y - y'|, \]

which shows that \( T \) is \( 1 \)-Lipschitz. The coupling \( \pi = (id, T)_\# \mu \) is optimal for \( (5.2) \) between its marginals \( \mu \) and \( \eta^* \), by the reasoning in the proof of Theorem 5.1. Hence, by Theorem 5.1 there is a convex function \( \varphi \) with \( 1 \)-Lipschitz gradient such that \( (id, \nabla \varphi)_\# \mu \) is the unique optimizer. Thus, \( \nabla \varphi = T(\mu \text{-a.s.}) \). Due to continuity we conclude \( \nabla \varphi = T \) on \( \text{proj}_1 \Gamma \) and \( \Gamma \subseteq \partial \varphi \). \( \square \)

6. Multiple-Good monopoly problem

The goal of this section is to recover the main result of Daskalakis, Deckelbaum, and Tzamos \cite{25} in Corollary 6.2 below. We will not discuss the economic interpretation and just mention that it can be interpreted as a Kantorovich-Rubinstein Theorem for specific weak transport costs.

We will obtain Corollary 6.2 as a consequence of the more general result Theorem 6.1. We first introduce some notation. Fix \( X := \mathbb{R}^d \) and equip it with the coordinate-wise partial order \( \leq \). Let \( \Phi^{\text{i-c}}(\mathbb{R}^d) \) consist of all \( \leq \)-increasing, convex functions in \( \Phi_1(\mathbb{R}^d) \). For \( \mu, \nu \in \mathcal{P}_1(X) \) we write \( \mu \preceq_{\text{i-c}} \nu \) iff \( \int f \, d\mu \leq \int f \, d\nu \) for all \( f \in \Phi^{\text{i-c}}(X) \). Once again, it follows from the results of Strassen \cite{47} that \( \mu \preceq_{\text{i-c}} \nu \) is tantamount to the existence of a stochastic process \( (Z_i)_{i=1,2} \) satisfying

\[ Z_1 \sim \mu, Z_2 \sim \nu, Z_1, Z_2 \leq \mathbb{E}[Z_2|Z_1]. \]

Theorem 6.1. Let \( \theta \in \Phi^{\text{i-c}}(\mathbb{R}^d) \) be convex and denote

\[ C_{\theta;i-c}(x, p) := \inf_{q \preceq_{\text{i-c}} p} \theta \left( x - \int yq(dy) \right) \text{ and } R_\theta\varphi(x) := \inf_{\varphi \in \mathbb{R} \times \mathbb{R}^{d\times d}} \varphi(z) + \theta(x-y). \]

Then all of the following optimization problems yield the same value:

1. \[ \inf_{\theta \in \Pi(\mu, \nu)} \int_{\mathbb{R}^d} C_{\theta;i-c}(x, x_\pi) \mu(dx), \]
2. \[ \inf_{\theta \in \Pi(\mu, \nu)} \int_{\mathbb{R}^d} \theta(x-y) \pi(dx, dy), \]
3. \[ \inf_{\mu \in \mathcal{P}_1(\mathbb{R}^d)} \sup_{\nu \in \mathcal{P}_1(\mathbb{R}^d)} \inf_{\varphi \in \mathbb{R} \times \mathbb{R}^{d\times d}} \int \theta(x-y) \pi(dx, dy). \]

\[ ^1 \text{Here } (x_{n+1},y_{n+1}) := (x_1, y_1). \]
iv) $\sup_{\varphi \in \Phi^{\pi_i}(\mathbb{R}^d)} -\nu(\varphi) + \int R_{\theta}(x)\mu(dx)$.

Proof. “i) = ii)”: For (i) $\geq$ (ii) we first take $\pi$ to be an (almost) optimizer of (i) and by a measurable selection argument take $q_x$ to be an (almost) optimizer of $C_{\theta,i,e}(x, \pi_x)$. Defining $T(x) := \int yq_x(dy)$ we remark that $T(\mu) \leq_{icx} \nu$, and so (i) $\geq \int \theta(x - T(x))\mu(dx) \geq$ (ii). For the converse, take $\nu \leq_{icx} \nu$ and $P \in \Pi(\mu, \nu)$ which are (almost) optimizers of (ii). By Strassen’s result [47, Theorem 9] there exists a submartingale coupling $\tilde{\pi} \in \Pi(\mu, \nu)$, i.e. a coupling $\pi$ satisfying $\int y \, d\pi_x \geq x$.

Next we define $\pi(dx, dy) = \int P_x(dy)P(dx, d\tilde{x})$ which belongs to $\Pi(\mu, \nu)$. Since by definition $\theta(x - \tilde{x}) \geq C_{\theta,i,e}(x, \tilde{\pi}_x)$, we have by Jensen’s inequality

$$(ii) = \int \theta(x - \tilde{x})P(dx, d\tilde{x}) \geq \int C_{\theta,i,e}(x, \tilde{\pi}_x)P(dx, d\tilde{x})$$

$$\geq \int C_{\theta,i,e}(x, \tilde{\pi}_x)\mu(dx)$$

where we used that $C_{\theta,i,e}(x, \cdot)$ is convex if $\theta$ is convex.

“ii) = iii)”: Fix $\mu \leq_{icx} \tilde{\mu}$ and $\tilde{\nu} \leq_{icx} \nu$. Let $X_1 \leq_{icx} X_2$ and $Y_1 \leq_{icx} Y_2$ be $\mathbb{R}^d$-valued random variables on some probability space whose laws satisfy

$$X_1 \sim \mu, \quad X_2 \sim \tilde{\mu}, \quad Y_1 \sim \tilde{\nu}, \quad Y_2 \sim \nu.$$  

The order $\leq_{icx}$ between random variables has to be understood in the following sense, (where the existence of such random variables is again provided by [47, Theorem 9])

$$X \leq_{icx} Y \iff \mathbb{E}[Y|X] \geq X \quad \text{a.s.}$$

Define the random variable $Z = X_1 + \mathbb{E}[Y_1 - X_2|X_1]$. Then $Z$ is in increasing convex order to $Y_2$, since

$$Z \leq_{icx} Z + \mathbb{E}[X_2 - X_1|X_1] = \mathbb{E}[Y_1|X_1] \leq_{icx} Y_1 \leq_{icx} Y_2.$$  

Applying Jensen’s inequality yields

$$\mathbb{E}[\theta(X_2 - Y_1)] \geq \mathbb{E}[\theta(\mathbb{E}[X_2 - Y_1|X_1])] = \mathbb{E}[\theta(X_1 - Z)] \geq$$(ii).

“i) = iv)”: By duality, see Theorem 3.3 we have

$$i) = \sup_{\varphi \in \Phi^{\pi_i}(\mathbb{R}^d)} -\nu(\varphi) + \int R_{C_{\theta,i,e}}(x, \varphi)\mu(dx).$$

(6.1)

It remains to show that we can additionally restrict the infimum to convex and increasing functions which will be accomplished using the double c-convexification trick. To this end, we note that

$$C_{\theta,i,e}(x, p) = \inf_{x - \int y p(dy) \leq \theta(z)} =: \hat{\theta}(x - \int y p(dy)),$$

where $\hat{\theta}$ is a function on $\mathbb{R}^d$ which is bounded from below and convex. Further note

$$x \mapsto \hat{\theta}(x - y) \text{ is increasing}, \quad y \mapsto \hat{\theta}(x - y) \text{ is decreasing}.$$  

(6.2)
Analogously to Gozlan et al. [33] Proof of Theorem 2.11 (2) we find that
\[
\inf_{y \in \mathbb{R}^d} \varphi(y) + \hat{\theta}(x - y) = R_{C_{1,\text{lex}}} \varphi(x) = R_{C_{1,\text{lex}}} \hat{\varphi}(x) = \inf_{y \in \mathbb{R}^d} \hat{\varphi}(y) + \hat{\theta}(x - y),
\]
where \( \hat{\varphi} \) denotes the convex envelope of \( \varphi \). The inf-convolution \( \psi := R_{C_{1,\text{lex}}} \hat{\varphi} \) is therefore bounded from below, convex and increasing. Note that for any \( p \in \mathcal{P} \) and barycenter \( \int y \rho(dy) = z \) we have
\[
\psi(x) - p(\hat{\varphi}) \leq \psi(x) - \hat{\varphi}(z) \leq \hat{\theta}(x - z),
\]
which allows us to \( \hat{\theta} \)-convexify \( \hat{\varphi} \), i.e.,
\[
\tilde{\varphi}(y) = \sup_x \psi(x) - \hat{\theta}(x - y),
\]
which is in particular an increasing function by (6.2) with
\[
\hat{\varphi} \geq \tilde{\varphi} \geq \psi(y) - \hat{\theta}(0) \geq \min_y \psi(y) - \hat{\theta}(0).
\]
Again, we find that the convex envelope of \( \tilde{\varphi} \)
\[
\tilde{\varphi}(z) = \inf_{p \in \mathcal{P}(Y) : \int y \rho(dy) = z} p(\tilde{\varphi})
\]
is increasing, convex and dominated by \( \tilde{\varphi} \), thus,
\[
\hat{\psi}(x) := R_{C_{1,\text{lex}}} \tilde{\varphi}(x) = R_{C_{1,\text{lex}}} \tilde{\varphi}(x) \geq \psi(x) = R_{C_{1,\text{lex}}} \varphi(x).
\]
Finally, we obtain (since \( \hat{\varphi} \leq \varphi \) and \( \psi \leq \hat{\psi} \)) that
\[
\mu(\psi) - \nu(\varphi) \leq \mu(\hat{\psi}) - \nu(\hat{\varphi}),
\]
which shows that we can replace \( \Phi_{b_1}(\mathbb{R}^d) \) with \( \Phi_{b_1}^{\text{lex}}(\mathbb{R}^d) \) in (6.1).

The proof of Corollary 6.2 resembles the proof of Kantorovich-Rubinstein duality when one already knows that classical Kantorovich duality holds.

**Corollary 6.2.** Setting \( \theta(x) = |x| \) where \( |\cdot| \) is a norm on \( \mathbb{R}^d \), we have
\[
\inf_{\pi \in \Pi(\mu, \nu)} \int \inf_{y \in \mathbb{R}^d} |x - z| \mu(dx) = \sup_{\varphi \in \Phi_{b_1}^{\text{lex}}(\mathbb{R}^d) \text{ and 1-Lipschitz}} \mu(\varphi) - \nu(\varphi).
\]

**Proof.** First we show that given \( \varphi \in \Phi_{b_1}^{\text{lex}}(\mathbb{R}^d) \) the inf-convolution \( \psi := R_{C_{1,\text{lex}}} \varphi \) is 1-Lipschitz: Let \( x, x' \in \mathbb{R}^d \) then
\[
R_{C_{1,\text{lex}}} \varphi(x) - R_{C_{1,\text{lex}}} \varphi(x') \leq \sup_{z \leq y, y \in \mathbb{R}^d} |x - y| - |x' - y| \leq |x - x'|.
\]
By the proof of Theorem 6.1 \( \psi \) is additionally bounded from below, convex and increasing. Using the notation in the proof of Theorem 6.1 the mapping \( \tilde{\varphi} \) is increasing, bounded from below and also 1-Lipschitz. Hence, for \( x, x' \in \mathbb{R}^d \) the increasing, convex and lower bounded function \( \tilde{\varphi} \), see (6.3), is 1-Lipschitz
\[
\tilde{\varphi}(x) - \tilde{\varphi}(x') \leq \sup_{p \in \mathcal{P}(Y)} W_1(p, T_{x'-x}(p)) = |x - x'|,
\]
where \( T_{x'-x} \) is the translation by \( x' - x \). By the conclusion of the proof of Theorem 6.1 we can assume w.l.o.g. that \( \varphi \in \Phi_{b_1}^{\text{lex}}(\mathbb{R}^d) \) is 1-Lipschitz.
It remains to show that $R_{C|x|,\varphi}$ and $\varphi$ coincide. By definition of the inf-convolution we have $\varphi \geq R_{C|x|,\varphi}$. We find by 1-Lipschitz continuity of $\varphi$

$$-\varphi(x) + \varphi(y) + \inf_{z \in Y} |x - z| \geq \inf_{z \in Y} -\varphi(x) + \varphi(z) + |x - z| \geq 0,$$

which shows the reverse inequality. $\square$

7. Backward Transfers

Bowles and Ghoussoub [18] suggest a notion of linear transfer between probability measures which is more encompassing than mass transportation but still admits important traits of the dual theory of mass transport. In particular, they identify many examples that illustrate the scope of their approach.

A main result of Bowles and Ghoussoub yields a representation of linear transfers through weak transport problems, see [18, Theorem 3.1]. In this section, we recover [18, Theorem 3.1] as an application of the weak transport duality theorem. Notably, this approach extends the result of Bowles and Ghoussoub from compact to general Polish spaces without additional effort.

To present the notion of linear transfer, we introduce some notation. The basic object of interest are functionals $T : \mathcal{P}(X) \times \mathcal{P}_t(Y) \to \mathbb{R} \cup \{+\infty\}$. We will use the Legendre transform $T_{\mu}^* : \mathcal{P}_t(Y) \to \mathcal{P}_t(Y)$, which is given by

$$T_{\mu}^*(g) = \sup_{v \in \mathcal{P}_t(Y)} v(g) - T(\mu, v).$$

Since the set $\Phi_t(Y)$ is in separating duality with $M_t(Y)$, the Fenchel duality theorem [54, Theorem 2.3.3] states

$$T_{\mu}(v) = T_{\mu}^{**}(v) = \sup_{g \in \Phi_t(Y)} v(g) - T_{\mu}^*(g),$$

if $T_{\mu}$ is proper convex, bounded from below and lower semicontinuous.

**Definition 7.1.** A proper, convex, bounded from below and lower semicontinuous functional $T : \mathcal{P}(X) \times \mathcal{P}_t(Y) \to \mathbb{R} \cup \{+\infty\}$ is called backward linear transfer if there exists a map $T$ from $\Phi_t(Y)$ to the set of universally measurable functions on $X$ bounded from below by an element in $\Phi_t(X)$, with the following property: for each $\mu \in \mathcal{P}(X)$ with $\inf_{v \in \mathcal{P}_t(Y)} T(\mu, v) < \infty$ the Legendre transform $T_{\mu}^*$ can be represented as

$$T_{\mu}^*(g) = \mu(T(g)) \quad \forall g \in \Phi_t(Y).$$

**Theorem 7.2.** Let $T : \mathcal{P}(X) \times \mathcal{P}_t(Y) \to \mathbb{R} \cup \{+\infty\}$ be such that

$$\forall x \in X, \exists p \in \mathcal{P}_t(Y) : \quad T(\delta_x, p) < \infty.$$  

Then the following are equivalent

(i) $T$ is a backward linear transfer;

(ii) $T$ is a backward linear transfer;

---

2As noted on [18, page 3], the right setting for most applications of linear transfers should be ‘... complete metric spaces, Riemannian manifolds or at least $\mathbb{R}^n$.’ In this respect the extension of [18, Theorem 3.1] beyond the compact setup seems relevant.
(ii) there is a lower semicontinuous cost function \( C: X \times \mathcal{P}_t(Y) \to \mathbb{R} \cup \{ +\infty \} \) which is bounded from below and convex in the second argument such that for all \((\mu, \nu) \in \mathcal{P}(X) \times \mathcal{P}_t(Y)\)

\[
T(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \int_X C(x, \pi_x) \mu(dx).
\]

**Proof.** Evidently, if \( T \) is given as a backward linear transfer, the cost function \( C \) has to satisfy \( C(x, p) = T(\delta_x, p) \) and \( T \) satisfies on \( \Phi_t(Y) \)

\[
T(g)(x) = T^*_{\delta_x} g = \sup_{p \in \mathcal{P}(Y)} p(g) - C(x, p) = -R_C(-g)(x).
\]

Hence by (7.2) and optimal weak transport duality (Theorem 3.3) we have

\[
T(\mu, \nu) = T^*_{\mu}(\nu) = \sup_{g \in \Phi_t(Y)} \nu(g) - \int_X R_C(-g)(x) \mu(dx)
\]

\[
= \sup_{g \in \Phi_t(Y)} \nu(g) - \int_X R_C(g)(x) \mu(dx)
\]

\[
= \inf_{\pi \in \Pi(\mu, \nu)} \int_X C(x, \pi_x) \mu(dx).
\]

Conversely, if (ii) holds, then Theorem 3.3 reveals

\[
T^*_{\mu}(\nu) = \sup_{g \in \Phi_t(Y)} \nu(g) - \mu(T(g)),
\]

for \( T(g)(x) = R_C(-g)(x) \), and by (3.4) we have \( \mu(T(g)) = T^*_{\mu}(g) \). \( \square \)

8. **Semimartingale Transport Duality**

In this part we need to set up some terminology before stating the actual problem. Let

\[
C = C([0, 1]; \mathbb{R}^d)
\]

denote the continuous path space equipped with the supremum norm \( \| \cdot \|_\infty \) and its Borel \( \sigma \)-field. With

\[
W = (W(t))_{t \in [0, 1]}
\]

we denote the canonical (coordinate) process on \( C \), defined by \( W(t)(\omega) = \omega(t) \), so that \( W \) is a standard \( d \)-dimensional Brownian motion under the Wiener measure \( \mathbb{W} \). Let \( \mathcal{F} = (\mathcal{F}_t)_{t \in [0, 1]} \) denote the \( \mathbb{W} \)-complete filtration generated by \( W \). As usual, we denote by \( L^0(\mathbb{W}) \) the space of (real-valued) random variables quotiented with the \( \mathbb{W} \)-a.s. identification, and by \( L^\infty(\mathbb{W}) \) the essentially bounded elements of \( L^0(\mathbb{W}) \). We will likewise identify processes that are \( dt \times d\mathbb{W} \)-almost surely equal. Finally, we denote by \( S^d_+ \) the set of symmetric positive semi-definite matrices of size \( d \times d \). We fix from now on a matrix norm on \( \mathbb{R}^{d \times d} \).

We consider

\[
g: [0, 1] \times \mathbb{R}^d \times \mathbb{R}^d \times S^d_+ \to \mathbb{R} \cup \{ \infty \},
\]
and assume

**Assumption 8.1.**

1. \( g \) is jointly measurable and lower-bounded.
2. For each \( t \in [0, 1] \) the function
   \[
   \mathbb{R}^d \times \mathbb{R}^d \times S^d_+ \ni (q, a) \mapsto g(t, x, q, a),
   \]
   is jointly lower semicontinuous. Furthermore
   \[
   (q, a) \mapsto g(t, x, q, a)
   \]
   is convex for each fixed \((t, x)\).
3. Either \( g \) is finite and coercive in the sense that
   \[
   \lim_{|q| + |a| \to \infty} \inf_{t, x} g(t, x, q, a) = +\infty,
   \]
   or
   \[
   \text{dom}(g(t, x, \cdot, \cdot)) := \{(q, a) \in \mathbb{R}^d \times S^d_+ : g(t, x, q, a) < \infty\}
   \]
   is a compact convex set which does not depend of \((t, x)\).

For \( Q \in \mathcal{P}(\mathcal{C}) \) we denote by

\[
M^\text{ac}_0(Q)
\]
the space of continuous \( \mathbb{R}^d \)-valued \( Q \)-martingales, which are started at zero, whose quadratic variation matrix is absolutely continuous and integrable: Namely \( M \in M^\text{ac}_0(Q) \) iff it is a \( Q \)-martingale started at zero, \( \frac{d\langle M \rangle}{dt} \) exists \( Q \)-a.s. and

\[
\mathbb{E}^Q[|\langle M \rangle(1)|] < \infty.
\]
This last condition is equivalent to asking

\[
\mathbb{E}^Q \left[ \int_0^1 \left| \frac{d\langle M \rangle(t)}{dt} \right| dt \right] < \infty.
\]

On the other hand we write \( \mathcal{L}^1(Q) \) for the set of progressively measurable \( \mathbb{R}^d \)-valued processes which are integrable with respect to \( dt \times dQ \).

We can now introduce the set of semimartingale laws relevant to our work:

\[
S := \{ Q \in \mathcal{P}(\mathcal{C}) : \begin{array}{l}
W(\cdot) = \int_0^\cdot q^Q(s)ds + M^Q(\cdot) \text{ under } Q, \\
\text{for some } M^Q \in M^\text{ac}_0(Q), \ q^Q \in \mathcal{L}^1(Q) \}
\end{array}
\]

We remark that for \( Q \in S \) the process \( q^Q \) above is uniquely determined. Likewise, the \( S^d_+ \)-valued process

\[
a^Q := \frac{d\langle M^Q \rangle}{dt}
\]

is uniquely determined.

Let

\[
a^q : S \to \mathbb{R} \cup \{+\infty\},
\]
be given by
\[
\alpha^\varepsilon(Q) := \mathbb{E}^Q \left[ \int_0^1 g(t, W(t), q^Q(t), a^Q(t)) \, dt \right].
\] (8.2)

Note that \(\alpha^\varepsilon(Q)\) is well-defined and takes values in \(\mathbb{R} \cup \{+\infty\}\), as \(g\) is bounded from below. As a final bit of notation, we introduce \(\Pi(\mu, \nu)\) for the set of those \(Q \in S\) with initial and final marginals equal to \(\mu\) and \(\nu\) respectively. We further write \(\Pi(\mu, \cdot)\) when only the initial marginal is prescribed.

For \(\mu, \nu \in \mathcal{P}(\mathbb{R}^d)\) we define
\[
V(\mu, \nu) := \inf_{Q \in \Pi(\mu, \nu)} \alpha^\varepsilon(Q).
\]

This is a stochastic mass transport problem (or optimal transport of semimartingales) as introduced by Tan and Touzi [50]. Specification of \(\alpha^\varepsilon\) allow to cover classical optimal transport, martingale transport, and some instances of the Schrödinger problem in this framework. We now prove a duality result, originally obtained by the aforementioned authors, by means of optimal weak transport:

**Theorem 8.2.** Under the standing assumptions, we have
\[
V(\mu, \nu) = \sup_{\phi \in C^b_{\mathcal{P}}(\mathbb{R}^d)} \mu(\bar{\Psi}) - \nu(\phi),
\]
where \(\bar{\Psi}(x) := \inf_{Q \in \Pi(\delta_x, \cdot)} \mathbb{E}^Q \left[ \int_0^1 g(t, W(t), q^Q(t), a^Q(t)) \, dt + \Psi(X_1) \right].\)

**Proof.** Define \(C(x, p) := V(\delta_x, p)\).

Remark that if \(Q_1 \in \Pi(\delta_x, p_1)\) and \(Q_2 \in \Pi(\delta_x, p_2)\), then for \(\alpha \in [0, 1]\) we have \(\alpha Q_1 + (1 - \alpha) Q_2 \in \Pi(\delta_x, \alpha p_1 + (1 - \alpha) p_2)\). As we will see in Theorem 8.3 the function \(\alpha^\varepsilon(\cdot)\) is convex. From these facts it follows that \(C(x, \cdot)\) is convex for each \(x\) fixed. Further, Theorem 8.3 also shows that \(\alpha^\varepsilon(\cdot)\) is coercive and lower semicontinuous, which then implies that \(C(x, \cdot)\) is lower semicontinuous for each \(x\) fixed.

A standard measurable selection argument shows that
\[
V(\mu, \nu) = \inf_{\pi \in Cpl(\mu, \nu)} \int V(\delta_x, \pi^\nu) \mu(dx) = \inf_{\pi \in Cpl(\mu, \nu)} \int C(x, \pi^\nu) \mu(dx),
\]
where we wrote \(Cpl(\mu, \nu)\) for the set of measures in \(\mathbb{R}^d \times \mathbb{R}^d\) with the given marginals. Applying the duality (3.3) we deduce
\[
V(\mu, \nu) = \sup_{\psi} \left\{ \int R_C \psi(x) \mu(dx) - \nu(\psi) \right\},
\]

3Equivalently, one may minimize the functional \(\mathbb{E}[\int_0^1 g(t, X(t), q(t), \sigma(t)\sigma'(t)) \, dt]\) over all semimartingales \(dX(t) = q(t) \, dt + \sigma(t) \, dB(t)\) on some stochastic basis, such that \(X(0) \sim \mu, X(1) \sim \nu.\)
where
\[ R_C\psi(x) := \inf_{p}\{p(\psi) + C(x, p)\} \]
\[ = \inf_{Q \in \Pi(\delta x, \cdot)} E_Q \left[ \int_0^1 g(t, W(t), q(t), a(t)) dt + \Psi(X_1) \right] \]
\[ = \tilde{\psi}(x). \]
\[ \square \]

For the previous result we employed:

**Theorem 8.3.** The functional \( \alpha^x \) is convex, lower semicontinuous with respect to weak convergence, and coercive in the sense that \( \{Q : \alpha^x(Q) \leq c\} \) is weakly compact for each \( c \in \mathbb{R} \).

In order to prove this we need the following auxiliary result first:

**Lemma 8.4.** Suppose \((q_n)_n\) is a sequence of \( L^1([0, 1], dt; \mathbb{R}^d)\)-valued random variables possibly defined in different probability spaces, and call \( A_n(t) := \int_0^t q_n(s)ds \). In the same space where \( q_n \) is defined we are given a further \( C \)-valued random variable \( M_n \) such that \( M_n(0) = 0 \), \( M_n \) is a martingale wrt. its completed filtration, and such that \( a_n(t) := \frac{M_n(t)}{dt} \) exists a.s. Finally assume the existence of \( c > 0 \) such that, for all \( n \),
\[ E\left[ \int_0^1 g(t, M_n(t) + A_n(t), q_n(t), a_n(t)) dt \right] \leq c. \tag{8.3} \]

Then there exist an \( L^1([0, 1], dt; \mathbb{R}^d)\)-valued random variable \( A \), a \( C \)-valued random variable \( M \), and subsequences \( A_{n_k} \) and \( M_{n_k} \) such that

1. \( A_{n_k} \) converges in law in \( C \) to \( A \),
2. \( M_{n_k} \) converges in law in \( C \) to \( M \),
3. \( A(t) = \int_0^t q(s)ds \), some \( L^1([0, 1], dt; \mathbb{R}^d)\)-valued random variable \( q \),
4. \( \langle M \rangle(t) = \int_0^t a(t)dt \), some \( S_d^+ \)-valued process \( a \) with \( E\left[ \int_0^1 |a(t)| dt \right] < \infty \)
5. the following inequality holds:
\[ E\left[ \int_0^1 g(t, M_n(t) + A_n(t), q_n(t), a_n(t)) dt \right] \leq \lim inf_{k \to \infty} E\left[ \int_0^1 g(t, M_{n_k}(t) + A_{n_k}(t), q_{n_k}(t), a_{n_k}(t)) dt \right] \tag{8.4} \]

In particular, the laws of \((A_n)\) and \((M_n)\) form tight sequences.

The following proofs follow very closely the arguments in [8]. As a small technical improvement over [50], we observe that the coercivity condition \((3.1)\) assumed here, is weaker than the analogue in the cited paper.

**Proof of Lemma 8.4.**

We first check tightness. If the final part of Assumption \((8.1)\) holds, this is trivial. Otherwise, by \((8.1)\), for each \( r > 0 \) we may find \( N > 0 \) such that \( g(t, x, q, a) \geq r|q| + r|a| \) whenever \( |q| \vee |a| \geq N \). Moreover, there
exists \( b \geq 0 \) such that \( g(t, x, q, a) \geq -b \). In particular, for all \((t, q)\) we have \(|q| + |a| \leq 2N + \frac{1}{r}(g(t, x, q, a) + b)\). Hence, for \( 0 \leq s < t \leq 1 \),

\[
|A_n(t) - A_n(s)| \leq \int_s^t |q_n(u)|\,du \\
\leq \frac{1}{r} \int_s^t (g(u, W(u), q_n(u), a_n(u)) + b)\,du + 2N(t - s) \\
\leq \frac{1}{r} \int_0^1 (g(u, W(u), q_n(u), a_n(u))\,du + \frac{b}{r} + N(t - s).
\]

Hence, for any \( \delta_n \downarrow 0 \), \((8.3)\) yields

\[
\lim \sup_{n \to \infty} \sup_{\tau} \mathbb{E}[|A_n(\tau + \delta_n) - A_n(\tau)|] \leq \lim \sup_{n \to \infty} \left( \frac{c + b}{r} + N\delta_n \right) = \frac{c + b}{r},
\]

where the \( \sup \) is over all stopping times with values in \([0, 1 - \delta_n]\). As \( r > 0 \) was arbitrary, this shows that

\[
\lim \sup_{n \to \infty} \sup_{\tau} \mathbb{E}[|A_n(\tau + \delta_n) - A_n(\tau)|] = 0,
\]

and from Aldous’ criterion for tightness [37, Theorem 16.11] we conclude that \((A_n)\) is tight. The Cauchy-Schwartz inequality and similar calculations allow to conclude that

\[
\mathbb{E}[|M_n(\tau + \delta) - M_n(\tau)|] \leq \sqrt{N\delta + \frac{c + b}{r}},
\]

so as before \((M_n)\) is a tight sequence. Furthermore, \((M_n)\) is in fact precompact in the 1-Wasserstein space \( \mathcal{W}_1(C) \) of measures on \( C \) which integrate the supremum of the norm of the path of the canonical process. This follows by showing that

\[
\lim \sup_{K \to \infty} \mathbb{E} \left[ \sup_{n \leq 1} |M_n(t)| \mathbf{1}_{\sup_{n \leq 1} |M_n(t)| \geq K} \right] = 0,
\]

which is a consequence of Cauchy-Schwartz, Doob’s inequality, and Assumption [8.13].

Passing to a subsequence and applying Skorokhod’s representation, let us now assume that there exists continuous process \( A \) and \( M \) such that \( A_n \to A \) and \( M_n \to M \) almost surely (in \( C \)), with all processes defined on some common probability space \((\Omega, \mathcal{F}, P)\). The process \( M \) is in fact a martingale thanks to \( \mathcal{W}_1(C)\)-precompactness. From Assumption [8.13] and a standard argument as in the de la Vallée Poisson Theorem, we conclude that \([q_n : n \in \mathbb{N}] \subseteq L^1 : = L^1([0, 1] \times \Omega, dt \otimes dP)\) is uniformly integrable and thus weakly precompact. Similarly, using further that \([a_n : n \in \mathbb{N}]\) is weakly precompact in the Bochner space \( L^1_{d} : = L^1([0, 1] \times \Omega, dt \otimes dP; \mathbb{R}^{d \times d})\) of matrix-valued integrable processes if and only if \([a_n : n \in \mathbb{N}]\) is uniformly integrable (cf. [26]), we deduce that \([a_n : n \in \mathbb{N}]\) is weakly precompact. By passing to a further subsequence, we may now assume that \( q_n \to q \) weakly in \( L^1 \), that \( a_n \to a \) weakly in \( L^1_{d} \), and that \( a \) is almost surely \( S^d_{+}\)-valued. Because \( g \) is bounded from below and lower semicontinuous in its last three variables, the map \((\bar{X}, \bar{q}, \bar{a}) \mapsto \mathbb{E} \int_0^1 g(t, \bar{X}(t), \bar{q}(t), \bar{a}(t))dt\) is lower semicontinuous in
the norm topology of $C \times L^1 \times L^1_d$, by Fatou’s lemma. Because it is also convex in the last two, this map is therefore weakly lower semicontinuous when $L^1 \times L^1_d$ is given the weak topology. This yields [8.4]. By dominated convergence, it holds for each bounded random variable $Z$ that

$$
\mathbb{E}[ZA(t)] = \lim_{n \to \infty} \mathbb{E}[ZA_n(t)] = \lim_{n \to \infty} \mathbb{E} \left[ Z \int_0^t q_n(s)ds \right] = \mathbb{E} \left[ Z \int_0^t q(s)ds \right].
$$

Hence $A(t) = \int_0^t q(s)ds$ a.s. for each $t$, and by continuity we have $A(\cdot) = \int_0^\cdot q(s)ds$ a.s. A similar argument shows that for each $1 \leq i, j \leq d$ and $0 \leq s \leq t \leq 1$ and $Z$ measurable up to time $s$, we have

$$
0 = \lim_{n \to \infty} \mathbb{E} \left[ Z \left( M_n^i M_n^j(t) - M_n^i M_n^j(s) - \int_s^t a_n^i(r)dr \right) \right] = \mathbb{E} \left[ Z \left( M^i M^j(t) - M^i M^j(s) - \int_s^t a^j(r)dr \right) \right],
$$

from which $\langle M \rangle(\cdot) = \int_0^\cdot a(r)dr$. □

**Proof of Theorem 8.3**

**Convexity:** Let $\lambda \in [0, 1]$, and fix $Q_0, Q_1 \in S$. We work on the extended probability space $C \times \{0, 1\}$, and we write $(W, X)$ to denote the identity map on this space. We define a measure $M$ on $C \times \{0, 1\}$ by requiring that the second marginal of $M$ be $\lambda \delta_0 + (1 - \lambda) \delta_1$, and the conditional law of $W$ given $X$ be $Q_X$. In particular, the first marginal of $M$ is precisely $Q := \lambda Q_0 + (1 - \lambda) Q_1$. Abbreviate $q_i := q^{Q_i}$ and $a_i := a^{Q_i}$. It easily follows that the process

$$
W(t) - \int_0^t q_X(s)ds
$$

defines an $M$-martingale with respect to the filtration $\mathcal{F} = (\mathcal{F}_t)_{t \in [0, 1]}$ defined by $\mathcal{F}_t = \mathcal{F}_i \otimes C(X)$ on the product space. Furthermore, the quadratic variation of $W(\cdot) - \int_0^\cdot q_X(s)ds$ has a density explicitly given by $t \mapsto a_X(t)$. Now define the processes $q = (q(t))_{t \in [0, 1]}$ and $a = (a(t))_{t \in [0, 1]}$ respectively as the optional projections of the processes $(q_X(t))_{t \in [0, 1]}$ and $(a_X(t))_{t \in [0, 1]}$ on the filtration $\mathcal{F}$ generated by $W$. In particular,

$$
q(t) = \mathbb{E}^M[q_X(t) | (W_s)_{s \leq t}] = \mathbb{E}^M[1_{X=0} q_0(t) + 1_{X=1} q_1(t) | (W_s)_{s \leq t}],
$$

$$
a(t) = \mathbb{E}^M[a_X(t) | (W_s)_{s \leq t}] = \mathbb{E}^M[1_{X=0} a_0(t) + 1_{X=1} a_1(t) | (W_s)_{s \leq t}],
$$

A few computations reveal that $W(\cdot) - \int_0^\cdot q(t)dt$ is still an $M$-martingale. On the other hand, since $W(\cdot) - \int_0^\cdot q_X(s)ds$ has $a_X$ as the density of its quadratic variation under $M$, so does $W$ itself. But then for all $i \leq i, j \leq d$, if $R$ is
bounded and $\mathcal{F}_t$-measurable and $0 \leq h \leq 1 - t$, we have
\[0 = \mathbb{E}^M \left( W^i(t + h) - W^i(t) - \int_t^{t+h} a^i(s) \, ds \right) R\]
\[= \mathbb{E}^M \left( W^iW^j(t + h) - W^iW^j(t) - \int_t^{t+h} a^i(s) \, ds \right) R\]
\[= \mathbb{E}^Q \left( W^iW^j(t + h) - W^iW^j(t) - \int_t^{t+h} a^i(s) \, ds \right) R,
\]
recalling that $Q$ is the first marginal of $M$. Since the $M$ martingale $W(\cdot) - \int_0^t q(s) \, ds$ is $\mathcal{F}_t$-adapted, it follows that it is a $Q$-martingale (when seen as living in the filtered probability space $(C, \mathcal{F}, Q)$) and the above display shows that the density of its quadratic variation is precisely $\lambda a$. In summary, we conclude that $Q \in S$, and that $q = q^0$ as well as $a = a^0$. Finally, using Jensen’s inequality, we compute
\[
\lambda a^\delta(Q_0) + (1 - \lambda) a^\delta(Q_1)
\]
\[= \lambda \mathbb{E}^Q_{0} \left[ \int_0^1 g(t, W(t), q_0(t), a_0(t)) \, dt \right] + (1 - \lambda) \mathbb{E}^Q_{1} \left[ \int_0^1 g(t, W(t), q_1(t), a_1(t)) \, dt \right]
\]
\[= \mathbb{E}^M \left[ \int_0^1 g(t, W(t), Q_X(t), a_X(t)) \, dt \right]
\]
\[\geq \mathbb{E}^M \left[ \int_0^1 g(t, W(t), q(t), a(t)) \, dt \right]
\]
\[= \mathbb{E}^Q \left[ \int_0^1 g(t, W(t), q(t), a(t)) \, dt \right]
\]
\[= a^\delta(Q).
\]
**Inf-compactness:** Let $c \in \mathbb{R}$ and $\Lambda_c := \{Q : a^\delta(Q) \leq c\}$. It is convenient in this step and the next to define
\[W^Q(t) := W(t) - \int_0^t q^Q(s) \, ds, \quad t \in [0, 1],
\]
for $Q \in S$, noting that $W^Q$ is a $Q$-martingale with volatility $a^Q$. Letting $A^Q(t) := \int_0^t q^Q(s) \, ds$, it follows from Lemma 8.4 that $\{Q \circ (A^Q)^{-1} : Q \in \Lambda_c \} \subseteq \mathbb{P}(C)$ is tight. On the other hand, $\{Q \circ (W^Q)^{-1} : Q \in \Lambda_c \}$ is tight as well by the same argument. Since each marginal is tight, we deduce that $\{Q \circ (W^Q, A^Q)^{-1} : Q \in \Lambda_c \} \subseteq \mathbb{P}(C \times C)$ is tight. Finally, by continuous mapping, the set $\{Q \circ (W^Q + A^Q)^{-1} : Q \in \Lambda_c \} = \Lambda_c$ is tight.

**Lower semicontinuity:** Suppose $\{Q_n : n \in \mathbb{N}\} \subseteq \Lambda_c$ with $Q_n \rightharpoonup Q$ weakly for some $Q \in \mathbb{P}(C)$. We must show that $Q$ belongs to $\Lambda_c$. Define the continuous process
\[A_n(t) := \int_0^t q^{Q_n}(s) \, ds = W(t) - W^{Q_n}(t),
\]
Recalling that \( Q_n \circ (W, W^{Q_n}, A^n)^{-1} : n \in \mathbb{N} \) is tight. Relabelling a subsequence, suppose that \( Q_n \circ (W, W^{Q_n}, A^n)^{-1} \) converges weakly to the law of some \( C^3 \)-valued random variable \((X, B, A)\). Using Lemma 8.4, we may assume also that \( A(t) = \int_0^t q(s)ds \) and \( a(t) := \frac{d(B(t))}{dt} \) satisfying
\[
\mathbb{E} \int_0^1 g(t, X(t), q(t), a(t))dt \leq \liminf \mathbb{E} Q_n \int_0^1 g(t, W(t), q^{Q_n}(t), a^{Q_n}(t))dt \leq c.
\]
Clearly \( W^{Q_n} \) is a martingale in the filtration of \((W, W^{Q_n}, A^n)\), and hence \( B \) is a martingale in the filtration of \((X, B, A)\). Finally, notice that
\[
X(t) = B(t) + A(t) = B(t) + \int_0^t q(s)ds,
\]
as the same relation holds in the pre-limit. A standard argument shows that \( X - \int_0^t \tilde{q}(s)ds \) is a martingale with respect to the filtration of \( X \), where \( \tilde{q} \) is the optional projection of \( q \) onto such filtration. Writing
\[
X(t) = B(t) + \int_0^t [g(s) - \tilde{q}(s)]ds + \int_0^t \tilde{q}(s)ds =: \tilde{B}(t) + \int_0^t \tilde{q}(s)ds,
\]
we deduce that \( \tilde{B} \) is an \( X \)-adapted martingale with density of quadratic variation \( a \). By convexity of \( g(t, x, \cdot, \cdot) \), we have
\[
\mathbb{E} \int_0^1 g(t, X(t), \tilde{q}(t), a(t))dt \leq \mathbb{E} \int_0^1 g(t, X(t), q(t), a(t))dt \leq c.
\]
Recalling that \( Q \) denoted the law of \( X \), we conclude that \( Q \in \Lambda_c \).

\[\Box\]

REFERENCES

[1] A. Alfonsi, J. Corbetta, and B. Jourdain. Sampling of probability measures in the convex order and approximation of Martingale Optimal Transport problems. ArXiv e-prints, Sept. 2017.
[2] J.-J. Alibert, G. Bouchitte, and T. Champion. A new class of cost for optimal transport planning. hal-preprint, 2018.
[3] J. Backhoff-Veraguas, D. Bartl, M. Beiglböck, and M. Eder. Adapted Wasserstein Distances and Stability in Mathematical Finance. arXiv e-prints, 2019.
[4] J. Backhoff-Veraguas, M. Beiglböck, M. Huesmann, and S. Källblad. Martingale Benamou–Brenier: a probabilistic perspective. Ann. Probab., to appear, 2020.
[5] J. Backhoff-Veraguas, M. Beiglböck, Y. Lin, and A. Zalashko. Causal transport in discrete time and applications. SIAM Journal on Optimization, 27(4):2528–2562, 2017.
[6] J. Backhoff-Veraguas, M. Beiglböck, and G. Pammer. Existence, duality, and cyclical monotonicity for weak transport costs. Calculus of Variations and Partial Differential Equations, 58(6):203, 2019.
[7] J. Backhoff-Veraguas, M. Beiglböck, and G. Pammer. Weak monotone rearrangement on the line. Electronic Communications in Probability, 25, 2020.
[8] J. Backhoff-Veraguas, D. Lacker, and L. Tangpi. Non-exponential Sanov and Schilder theorems on Wiener space: BSDEs, Schrödinger problems and Control. Forthcoming at Annals of Applied Probability, 2018.
[9] J. Backhoff-Veraguas and G. Pammer. Stability of martingale optimal transport and weak optimal transport. arXiv e-prints, 2019.
[10] M. Beiglböck. Cyclical monotonicity and the ergodic theorem. *Ergodic Theory Dynam. Systems*, 35(3):710–713, 2015.
[11] M. Beiglböck, A. Cox, and M. Huesmann. Optimal transport and Skorokhod embedding. *Invent. Math.*, 208(2):327–400, 2017.
[12] M. Beiglböck and C. Griessler. A land of monotone plenty. *Annali della SNS, to appear*, Apr. 2016.
[13] M. Beiglböck, P. Henry-Labordère, and F. Penkner. Model-independent bounds for option prices—a mass transport approach. *Finance Stoch.*, 17(3):477–501, 2013.
[14] M. Beiglböck, N. Juillet, et al. On a problem of optimal transport under marginal martingale constraints. *The Annals of Probability*, 44(1):42–106, 2016.
[15] M. Beiglböck and N. Juillet. Shadow couplings. *ArXiv e-prints*, Sept. 2016.
[16] J.-D. Benamou, G. Carlier, M. Cuturi, L. Nenna, and G. Peyré. Iterative Bregman projections for regularized transportation problems. *SIAM J. Sci. Comput.*, 37(2):A1111–A1138, 2015.
[17] S. Bianchini and L. Caravenna. On optimality of $c$-cyclically monotone transference plans. *C. R. Math. Acad. Sci. Paris*, 348(11-12):613–618, 2010.
[18] M. Bowles and N. Ghoussoub. Mather measures and ergodic properties of Kantorovich operators associated to general mass transfers. 2019.
[19] Y. Brenier. Décomposition polaire et réarrangement monotone des champs de vecteurs. *C. R. Acad. Sci. Paris Sér. I Math.*, 305(19):805–808, 1987.
[20] Y. Brenier. Polar factorization and monotone rearrangement of vector-valued functions. *Comm. Pure Appl. Math.*, 44(4):375–417, 1991.
[21] M. Colombo, L. De Pascale, and S. Di Marino. Multimarginal optimal transport maps for one-dimensional repulsive costs. *Canad. J. Math.*, 67(2):350–368, 2015.
[22] A. M. Cox and M. Vidmar. The structure of non-linear martingale optimal transport problems. *arXiv preprint arXiv:1903.06606*, 2019.
[23] I. Csiszár. I-divergence geometry of probability distributions and minimization problems. *The Annals of Probability*, pages 146–158, 1975.
[24] M. Cuturi. Sinkhorn distances: Lightspeed computation of optimal transport. In *Advances in neural information processing systems*, pages 2292–2300, 2013.
[25] C. Daskalakis, A. Deckelbaum, and C. Tzamos. Strong Duality for a Multiple-Good Monopolist. *Econometrica*, 85(3):735–767, 2017.
[26] J. Diestel. Uniform integrability: an introduction. 1991.
[27] S. Eckstein and M. Kupper. Computation of optimal transport and related hedging problems via penalization and neural networks. *Applied Mathematics & Optimization*, pages 1–29, 2019.
[28] M. Fathi, N. Gozlan, and M. Prodhomme. A proof of the caffarelli contraction theorem via entropic regularization. *arXiv preprint arXiv:1904.06053*, 2019.
[29] M. Fathi and Y. Shu. Curvature and transport inequalities for Markov chains in discrete spaces. *Bernoulli*, 24(1):672–698, 2018.
[30] A. Galichon, P. Henry-Labordère, and N. Touzi. A stochastic control approach to no-arbitrage bounds given marginals, with an application to lookback options. *Ann. Appl. Probab.*, 24(4):312–336, 2014.
[31] W. Gangbo and R. McCann. The geometry of optimal transportation. *Acta Math.*, 177(2):113–161, 1996.
[32] N. Gozlan and N. Juillet. On a mixture of breiner and strassen theorems. *arXiv preprint arXiv:1808.02681*, 2018.
[33] N. Gozlan, C. Roberto, P.-M. Samson, Y. Shu, and P. Tetali. Characterization of a class of weak transport-entropy inequalities on the line. *Ann. Inst. Henri Poincaré Probab. Stat.*, 54(3):1667–1693, 2018.
[34] N. Gozlan, C. Roberto, P.-M. Samson, and P. Tetali. Kantorovich duality for general transport costs and applications. *J. Funct. Anal.*, 273(11):3327–3405, 2017.
[35] C. Griessler. $c$-cyclical monotonicity as a sufficient criterion for optimality in the multi-marginal Monge-Kantorovich problem. *ArXiv e-prints*, Jan. 2016.
[36] J. Guyon, R. Menegaux, and M. Nutz. Bounds for vix futures given s&p 500 smiles. *Finance and Stochastics*, 21(3):593–630, 2017.

[37] O. Kallenberg. *Foundations of modern probability*. Springer Science & Business Media, 2006.

[38] C. Léonard. A survey of the Schrödinger problem and some of its connections with optimal transport. *Discrete Contin. Dyn. Syst.*, 34(4):1533–1574, 2014.

[39] K. Marton. A measure concentration inequality for contracting markov chains. *Geometric & Functional Analysis GAFA*, 6(3):556–571, 1996.

[40] K. Marton et al. Bounding $d$-distance by informational divergence: A method to prove measure concentration. *The Annals of Probability*, 24(2):857–866, 1996.

[41] T. Mikami and M. Thieullen. Duality theorem for the stochastic optimal control problem. *Stochastic Process. Appl.*, 116(12):1815–1835, 2006.

[42] M. Nutz and F. Stebegg. Canonical Supermartingale Couplings. *Ann. Probab., to appear*, Sept. 2018.

[43] B. Pass. On the local structure of optimal measures in the multi-marginal optimal transportation problem. *Calc. Var. Partial Differential Equations*, 43(3-4):529–536, 2012.

[44] P.-M. Samson. Transport-entropy inequalities on locally acting groups of permutations. *Electron. J. Probab.*, 22:Paper No. 62, 33, 2017.

[45] Y. Shu. From hopf-lax formula to optimal weak transfer plan. *arXiv preprint arXiv:1609.03405*, 2016.

[46] Y. Shu. Hamilton-Jacobi equations on graph and applications. *Potential Anal.*, 48(2):125–157, 2018.

[47] V. Strassen. The existence of probability measures with given marginals. *Ann. Math. Statist.*, 36:423–439, 1965.

[48] M. Talagrand. Concentration of measure and isoperimetric inequalities in product spaces. *Publications Mathématiques de l’Institut des Hautes Etudes Scientifiques*, 81(1):73–205, 1995.

[49] M. Talagrand. New concentration inequalities in product spaces. *Inventiones mathematicae*, 126(3):505–563, 1996.

[50] X. Tan and N. Touzi. Optimal transportation under controlled stochastic dynamics. *Ann. Probab.*, 41(5):3201–3240, 2013.

[51] C. Villani. *Topics in optimal transportation*, volume 58 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2003.

[52] C. Villani. *Optimal Transport. Old and New*, volume 338 of *Grundlehren der mathematischen Wissenschaften*. Springer, 2009.

[53] D. Zaev. On the Monge-Kantorovich problem with additional linear constraints. *Mathematical Notes*, 98(5-6):725–741, 2015.

[54] C. Zalinescu. *Convex analysis in general vector spaces*. World scientific, 2002.

[55] X. Zhou. On the fenchel duality between strong convexity and lipschitz continuous gradient. *arXiv preprint arXiv:1803.06573*, 2018.