LIE ALGEBRAS WITH $S_4$-ACTION AND STRUCTURABLE ALGEBRAS

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Abstract. The normal symmetric triality algebras (STA’s) and the normal Lie related triple algebras (LRTA’s) have been recently introduced by the second author, in connection with the principle of triality. It turns out that the unital normal LRTA’s are precisely the structurable algebras extensively studied by Allison.

It will be shown that the normal STA’s (respectively LRTA’s) are the algebras that coordinatize those Lie algebras whose automorphism group contains a copy of the alternating (resp. symmetric) group of degree 4.

Introduction

Over the years, many different constructions have been given of the exceptional simple Lie algebras in Killing-Cartan’s classification. In 1966, Tits gave a unified construction of these algebras using a couple of ingredients: a unital composition algebra and a simple Jordan algebra of degree 3 [Tit66]. Even though the construction is not symmetric, the outcome (the Magic Square) presents a surprising symmetry.

A symmetric construction was obtained by Vinberg [Vin66] (see also [OV94]) in terms of two unital composition algebras and their Lie algebras of derivations. The two composition algebras play the same role, and hence the symmetry of the construction. Vinberg’s construction was extended by Allison [All78] in terms of structurable algebras [All78], so that Vinberg’s construction becomes the particular case in which the structurable algebra is the tensor product of two composition algebras. Moreover, Allison and Faulkner [AF93] gave a new version of Allison construction which is based on three copies of a structurable algebra and a Lie algebra of Lie related triples. Quite recently, Barton and Sudbery [BS03] (see also the work by Landsberg and Manivel [LM02], [LM04], and the survey by Baez [Bae02]) gave a simple recipe to obtain the Magic Square in terms of two unital composition algebras and their triality Lie algebras, which in perspective is subsumed in Allison-Faulkner’s construction. Also, since simpler formulas for triality are obtained by using the so called symmetric composition algebras instead of the classical unital composition algebras, a version of Barton-Sudbery’s construction in terms of these latter algebras was given in [Eld04], [Eld05]. All
these symmetric constructions of the Magic Square provide $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded models of the Lie algebras involved.

In [Oku05], the second author has studied two classes of algebras: *normal symmetric triality algebras* and *normal Lie related triple algebras*, whose defining conditions reflect respectively the properties of the tensor products of two symmetric composition algebras, needed in the construction of the Magic Square in [Eld04], and the properties of structurable algebras needed in Allison-Faulkner’s construction [AF93]. Algebras in both classes are the building blocks of some $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras.

The aim of this paper is to show that the algebras in these classes are precisely those algebras that *coordinatize* the Lie algebras with an action of either the alternating or the symmetric group of degree 4. This possibility of relating these classes of algebras with an action of these groups, as automorphisms of Lie algebras, was suggested by the $S_4$-action on the exceptional simple real Lie algebras considered by Loke [Lok04].

The paper is organized as follows. The first section deals with Lie algebras with a subgroup of automorphisms isomorphic to the alternating group of degree 4. These Lie algebras are shown to be coordinatized by the normal symmetric triality algebras. Then the second section is devoted to Lie algebras with an action of the larger symmetric group of degree 4, which turn out to be coordinatized by the normal Lie related triple algebras. The unital such algebras are precisely the structurable algebras. Section 3 presents the main examples of algebras in these classes, which include Jordan and structurable algebras, but also Lie algebras, Lie triple systems and tensor products of symmetric composition algebras. Finally, Section 4 shows how, under some restriction on the ground field, Kantor’s 5-graded Lie algebras constructed from structurable algebras are endowed with an action of $S_4$. This gives a folding of the 5-graded Lie algebras into very symmetric $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded algebras, which were considered previously by Allison and Faulkner [AF93].

*Throughout the paper, all the algebras will be considered over a ground field $F$ of characteristic $\neq 2$.*

1. $A_4$-action and normal symmetric triality algebras

Let $A_4$ denote the alternating group of degree 4, and let $\mathfrak{g}$ be a Lie algebra endowed with a group homomorphism

$$A_4 \rightarrow \text{Aut}(\mathfrak{g}).$$

The group $A_4$ is the semidirect product of the normal Klein’s 4-group $V = \langle \tau_1, \tau_2 \rangle$, where $\tau_1 = (12)(34)$ and $\tau_2 = (23)(14)$, and the cyclic group of order 3, $C_3 = \langle \varphi \rangle$, where $\varphi = (123)$ ($1 \mapsto 2 \mapsto 3 \mapsto 1$). Note that $\tau_1 \tau_2 = \tau_2 \tau_1$, $\varphi \tau_1 = \tau_2 \varphi$, and $\varphi \tau_2 = \tau_1 \tau_2 \varphi$. The same notation will be used for the images of these elements in Aut($\mathfrak{g}$).

The action of Klein’s 4-group gives a $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading on $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

(1.1)
where
\[ t = \{ x \in \mathfrak{g} : \tau_1(x) = x, \tau_2(x) = x \} \quad (= \mathfrak{g}_{(0,0)}), \]
\[ \mathfrak{g}_0 = \{ x \in \mathfrak{g} : \tau_1(x) = x, \tau_2(x) = -x \} \quad (= \mathfrak{g}_{(1,0)}), \]
\[ \mathfrak{g}_1 = \{ x \in \mathfrak{g} : \tau_1(x) = -x, \tau_2(x) = x \} \quad (= \mathfrak{g}_{(0,1)}), \]
\[ \mathfrak{g}_2 = \{ x \in \mathfrak{g} : \tau_1(x) = -x, \tau_2(x) = -x \} \quad (= \mathfrak{g}_{(1,1)}). \]

(Here, the subindices 0, 1, 2 must be considered as the elements in \( \mathbb{Z}_3 \).)

Since \( V \) is a normal subgroup of \( A_4 \), \( t \) is invariant under \( \varphi \). Also, for any \( x \in \mathfrak{g}_0 \),
\[ \varphi(x) = \begin{cases} \varphi\tau_1(x) = \tau_2\varphi(x), \\ -\varphi\tau_2(x) = -\tau_1\tau_2\varphi(x) = -\tau_1\varphi\tau_1(x) = -\tau_1\varphi(x), \end{cases} \]
so that \( \varphi(\mathfrak{g}_0) \subseteq \mathfrak{g}_1 \) and, in the same vein, one gets \( \varphi(\mathfrak{g}_i) \subseteq \mathfrak{g}_{i+1} \) for any \( i \) (indices modulo 3).

Let \( \mathfrak{g} = \mathfrak{g}_0 \), and for any \( x \in A \) consider the elements
\[ \iota_0(x) = x \in \mathfrak{g}_0, \quad \iota_1(x) = \varphi\iota_0(x) \in \mathfrak{g}_1, \quad \iota_2(x) = \varphi^2\iota_0(x) \in \mathfrak{g}_2. \]
Thus
\[ \mathfrak{g} = t \oplus (\oplus_{i=0}^2 \iota_i(A)). \]

Since (1.1) is a grading over \( \mathbb{Z}_2 \times \mathbb{Z}_2 \), a bilinear multiplication \( * \) can be defined on \( A \) by means of
\[ [\iota_1(x), \iota_2(y)] = \iota_0(x * y), \] (1.3)
for any \( x, y \in A \). Also, the fact that \( \varphi \) is an automorphism shows that (1.3) is equivalent to
\[ [\iota_i(x), \iota_{i+1}(y)] = \iota_{i+2}(x * y), \] (1.4)
for any \( x, y \in A \) and \( i \in \mathbb{Z}_3 \).

Given any algebra \( (A, *) \), consider its symmetric triality Lie algebra (see [Oku05]):
\[ \text{str}i(A, *) = \{(d_0, d_1, d_2) \in \mathfrak{gl}(A)^3 : d_i(x * y) = d_{i+1}(x) * y + x * d_{i+2}(y) \}
\quad \text{for any } i = 0, 1, 2 \text{ and } x, y \in A \}.

This is a Lie subalgebra of \( \mathfrak{gl}(A)^3 \) (with componentwise Lie bracket), where \( \mathfrak{gl}(A) \) is the Lie algebra of endomorphisms of the vector space \( A \).

Let \( A = \mathfrak{g}_0 \) as above. Three representations of \( t \) on \( A \): \( \rho_i : t \to \mathfrak{gl}(A) \)
\( (i = 0, 1, 2) \), are obtained by means of:
\[ \iota_i(\rho_i(d)(x)) = [d, \iota_i(x)], \]
for any \( d \in t \) and \( x \in A \). Note that
\[ \iota_{i+1}(\rho_{i+1}(\varphi(d))(x)) = \varphi([d, \iota_i(x)]) = \varphi(\iota_i(\rho_i(d)(x))) \]
\[ = \iota_{i+1}(\rho_i(d)(x)), \]
so \( \rho_{i+1}\varphi = \rho_i \), or
\[ \rho_i\varphi^j = \rho_{i-j}, \] (1.5)
for any \( i, j \in \mathbb{Z}_3 \).
Putting together $\rho_0$, $\rho_1$ and $\rho_2$, there appears a Lie algebra homomorphism:

$$\rho : t \longrightarrow \mathfrak{gl}(A)^3$$

$$d \mapsto (\rho_0(d), \rho_1(d), \rho_2(d)).$$

Then:

**Proposition 1.6.** Under the hypotheses above, $\rho(t)$ is contained in $\mathfrak{stri}(A, \ast)$.

**Proof.** For any $d \in t$, $x, y \in A$ and $i \in \mathbb{Z}_3$,

$$\iota_i(\rho_i(d)(x \ast y)) = [d, \iota_i(x \ast y)]$$

$$= [d, [\iota_{i+1}(x), \iota_{i+2}(x)]]$$

$$= [[d, \iota_{i+1}(x)], \iota_{i+2}(y)] + [\iota_{i+1}(x), [d, \iota_{i+2}(y)]]$$

$$= [\iota_{i+1}(\rho_{i+1}(d)(x)), \iota_{i+2}(y)] + [\iota_{i+1}(x), \iota_{i+2}(\rho_{i+2}(d)(y))]$$

$$= \iota_i(\rho_{i+1}(d)(x) \ast y + x \ast \rho_{i+2}(d)(y),$$

which proves the result. 

Therefore, $\rho$ becomes a homomorphism of Lie algebras:

$$\rho : t \longrightarrow \mathfrak{stri}(A, \ast).$$

Note that $\mathfrak{stri}(A, \ast)$ has a natural order 3 automorphism

$$\theta : (d_0, d_1, d_2) \mapsto (d_2, d_0, d_1)$$

(1.7)

and (1.5) is equivalent to:

$$\rho \varphi = \theta \rho.$$  

(1.8)

Consider now the skew-symmetric linear map

$$\delta : A \times A \longrightarrow \mathfrak{stri}(A, \ast)$$

$$(x, y) \mapsto \delta(x, y) = \rho([\iota_0(x), \iota_0(y)]),$$

and denote by $\delta_i(x, y)$ the $i$th component of $\delta(x, y)$ ($\delta_i(x, y) = \rho_i([\iota_0(x), \iota_0(y)])$), so that $\delta(x, y) = (\delta_0(x, y), \delta_1(x, y), \delta_2(x, y))$. Note that $[\mathfrak{g}, \mathfrak{g}] \subseteq t$ ($i = 0, 1, 2$).

**Theorem 1.9.** Under the conditions above, for any $a, b, x, y, z \in A$ and $i, j \in \mathbb{Z}_3$:

(i) $[\delta_i(a, b), \delta_j(x, y)] = \delta_j(\delta_{i-j}(a, b)(x), y) + \delta_j(x, \delta_{i-j}(a, b)(y))$,

(ii) $\delta_0(x, y \ast z) + \delta_{-1}(y, z \ast x) + \delta_{-2}(z, x \ast y) = 0$,

(iii) $\delta_0(x, y)(z) + \delta_0(y, z)(x) + \delta_0(z, x)(y) = 0$,

(iv) $\delta_1(x, y) = r_y x - r_x y$ ($l_x : z \mapsto x \ast z$, $r_x : z \mapsto z \ast x$),

(v) $\delta_2(x, y) = l_y r_x - l_x r_y$. 

Proof. For any $d \in \mathfrak{t}$ and $x, y \in A$:

$$[\rho(d), \delta(x, y)] = \rho \left( [d, [\iota_0(x), \iota_0(y)]] \right)$$

$$= \rho \left( \left[ [d, \iota_0(x)], \iota_0(y) \right] + \left[ \iota_0(x), [d, \iota_0(y)] \right] \right)$$

$$= \rho \left( [\iota_0(\rho_0(d)(x)), \iota_0(y)] + [\iota_0(x), \iota_0(\rho_0(d)(y))] \right)$$

$$= \delta(\rho_0(d)(x), y) + \delta(x, \rho_0(d)(y)).$$

Now, for $d = [\iota_i(a), \iota_i(b)]$, using (1.8) one gets

$$\rho(d) = \rho \varphi^i \left( [\iota_0(a), \iota_0(b)] \right)$$

$$= \theta^i [\iota_0(a), \rho \theta^i]$$

$$= \theta^i (\delta(a, b))$$

$$= (\delta_i(a, b), \delta_{1-i}(a, b), \delta_{2-i}(a, b)),$$

and the $j$th component of (1.10) becomes

$$[\delta_{j-i}(a, b), \delta_j(x, y)] = \delta_j (\delta_{-i}(a, b)(x, y) + \delta_j (x, \delta_{-i}(a, b)(y)),$$

which is equivalent to the assertion in item (i).

Now, the Jacobi identity implies:

$$0 = [\iota_0(x), [\iota_1(y), \iota_2(z)]] + [\iota_1(y), [\iota_2(z), \iota_0(x)]] + [\iota_2(z), [\iota_0(x), \iota_1(y)]]$$

$$= [\iota_0(x), \iota_0(y * z)] + [\iota_1(y), \iota_1(z * x)] + [\iota_2(z), \iota_2(x * y)]$$

$$= [\iota_0(x), \iota_0(y * z)] + \varphi \left( [\iota_0(y), \iota_0(z * x)] \right) + \varphi^2 \left( [\iota_0(z), \iota_0(x * y)] \right).$$

Apply $\rho$ and use (1.8) to obtain:

$$\delta(x, y * z) + \theta(\delta(y, z * x)) + \theta^2(\delta(z, x * y)) = 0,$$

whose first component gives (ii).

Also,

$$0 = [\iota_0(x), \iota_0(y), \iota_0(z)] + [\iota_0(y), \iota_0(z), \iota_0(x)] + [\iota_0(z), \iota_0(x), \iota_0(y)]$$

$$= \iota_0 (\delta_0(x, y)(z) + \delta_0(y, z)(x) + \delta_0(z, x)(y)),$$

whence (iii),

$$\iota_1 (\delta_1(x, y)(z)) = [\iota_0(x), \rho \iota_0(y)], \iota_1(z)$$

$$= [\iota_0(x), \iota_1(y)], \iota_0(y) + [\iota_0(x), \iota_0(y), \iota_1(z)]$$

$$= [\iota_2(x * z), \iota_0(y)] + [\iota_0(x), \iota_2(y * z)]$$

$$= \iota_1 \left( (x * z) * y - (y * z) * x \right).$$

hence (iv), and

$$\iota_2 (\delta_2(x, y)(z)) = [\iota_0(x), \rho \iota_0(y)], \iota_2(z)$$

$$= [\iota_0(x), \iota_2(z)], \iota_0(y) + [\iota_0(x), \iota_0(y), \iota_2(z)]$$

$$= - [\iota_1(z * x), \iota_0(y)] - [\iota_0(x), \iota_1(z * y)]$$

$$= \iota_2 \left( y * (z * x) - x * (z * y) \right),$$

which proves (v).
Remark. Note that if $A \ast A = A$, then $\delta_0$ is determined by $\delta_1$ and $\delta_2$ because of item (ii) in Theorem 1.9 (or by “triality”: $\delta_0(x, y)(u \ast v) = \delta_1(x, y)(u) \ast v + u \ast \delta_2(x, y)(v)$). Also, if $\{x \in A : A \ast x = 0\} = 0$, $\delta_0$ is determined by $\delta_1$ and $\delta_2$ by triality, since $\delta_1(x, y)(u \ast v) = \delta_2(x, y)(u) \ast v + u \ast \delta_0(x, y)(v)$, and the same happens if $\{x \in A : x \ast A = 0\} = 0$.

Conditions (i)–(v) in Theorem 1.9 are precisely the conditions (2.15) in [Oku05] defining a normal symmetric triality algebra. Therefore:

**Corollary 1.11.** Under the conditions of Theorem 1.9, the algebra $(A, \ast)$ is a normal symmetric triality algebra.

**Corollary 1.12.** Let $(A, \ast)$ be a normal symmetric triality algebra with respect to the skew-symmetric bilinear map $\delta : A \times A \rightarrow \text{stri}(A, \ast)$. Then $(A, \ast)$ satisfies the degree 5 identity:

$$0 = ((x \ast u) \ast (y \ast z)) \ast v - (((y \ast z) \ast u) \ast x) \ast v$$

$$+ u \ast ((y \ast z) \ast (v \ast x)) - u \ast (x \ast (v \ast (y \ast z)))$$

$$+ (z \ast x) \ast ((u \ast v) \ast y) - y \ast ((u \ast v) \ast (z \ast x))$$

$$+ (z \ast (u \ast v)) \ast (x \ast y) - ((x \ast y) \ast (u \ast v)) \ast z,$$

for any $u, v, x, y, z \in A$.

**Proof.** By ‘triality’, for any $u, v, x, y, z \in A$:

$$\delta_0(x, y \ast z)(u \ast v) = \delta_1(x, y \ast z)(u) \ast v + u \ast \delta_2(x, y \ast z)(v),$$

while item (ii) in Theorem 1.9 gives:

$$\delta_0(x, y \ast z)(u \ast v) = -\delta_2(y, z \ast x)(u \ast v) - \delta_1(z, x \ast y)(u \ast v).$$

Hence,

$$0 = \delta_1(x, y \ast z)(u) \ast v + u \ast \delta_2(x, y \ast z)(v)$$

$$+ \delta_2(y, z \ast x)(u \ast v) + \delta_1(z, x \ast y)(u \ast v).$$

Expanding this last equation, by means of items (iv) and (v) of Theorem 1.9 gives (1.13).

The computations in the proof of Theorem 1.9 can be reversed to get a sort of converse. The straightforward proof is omitted.

**Theorem 1.14.** Let $(A, \ast)$ be a nonzero normal symmetric triality algebra with respect to the skew-symmetric bilinear map $\delta : A \times A \rightarrow \text{stri}(A, \ast)$. Then $t = \sum_{i=0}^{2} \theta^i(\delta(A, A))$ is a Lie subalgebra of $\text{stri}(A, \ast)$, and $\theta^i(\delta(A, A))$ is an ideal of $t$ for any $i = 0, 1, 2$. Moreover, consider three copies $\iota_i(A)$ of $A$ ($i = 0, 1, 2$) and define an anticommutative multiplication on

$$g(A, \ast) = t \oplus (\oplus_{i=0}^{2} \iota_i(A))$$

by means of

- $t$ is a subalgebra of $g(A, \ast)$,
- $[(d_0, d_1, d_2), \iota_i(x)] = \iota_i(d_i(x))$, for any $(d_0, d_1, d_2) \in t$, $x \in A$ and $i \in \mathbb{Z}_3$,
- $[\iota_i(x), \iota_{i+1}(y)] = \iota_{i+2}(x \ast y)$ for any $x, y \in A$ and $i \in \mathbb{Z}_3$. 
Thus, since the any proper ideal invariant under the action of (1.15)

Proof. Let us show first that if \( A \) is simple then \( g(A,*) \) defined in (1.15) is simple as a Lie algebra with \( A_4 \)-action (that is, it does not contain any proper ideal invariant under the action of \( A_4 \)).

Theorem 1.16. Let \((A,*)\) be a normal symmetric triality algebra with \( A \neq 0 \). Then \((A,*)\) is simple if and only if the Lie algebra \( g(A,*) \) defined in (1.15) is simple as a Lie algebra with \( A_4 \)-action (that is, it does not contain any proper ideal invariant under the action of \( A_4 \)).

Proof. Let us show first that if \( g(A,*) \) is simple as a Lie algebra with \( A_4 \)-action, then \( A = A*A \). To see this, note that the subspace

\[
\left( \sum_{i=0}^{2} [\iota_i(A*A), \iota_i(A)] \right) \oplus \left( \oplus_{i=0}^{2} \iota_i(A*A) \right)
\]

is invariant under the action of \( A_4 \), and it is closed under the adjoint action of \( \iota_i(A) \), \( i = 0, 1, 2 \), which generate \( g(A,*) \). Actually,

\[
[[\iota_i(A*A), \iota_j(A)], \iota_{i+1}(A)] \subseteq [\iota_{i+1}(A*A), \iota_i(A)] \subseteq \iota_{i+1}(A*A),
\]

because of the Jacobi identity, and also

\[
[[\iota_i(A*A), \iota_j(A)], \iota_i(A)] \subseteq \iota_i(\delta_0(A*A,A)(A)) \subseteq \iota_i(A*A),
\]

because \( \delta_0(A*A,A) \subseteq \delta_1(A*A,A) + \delta_2(A*A,A) \) by Theorem 1.19. Hence, the subspace above is an ideal of \( g(A,*) \) invariant under the action of \( A_4 \), so it is the whole \( g(A,*) \), and this shows that \( A*A = A \).

Now, since \( \delta_0(A,A) = \delta_0(A*A,A) \subseteq \delta_1(A,A) + \delta_2(A,A) \), it is contained in the Lie multiplication algebra of \( A \) (the Lie subalgebra of \( g(A) \) generated by the left and right multiplications) by Theorem 1.19.

Assume that \( g(A,*) \) is simple as an algebra with \( A_4 \)-action and let \( 0 \neq I \) be an ideal of \((A,*)\). By the above, \( \delta_0(A,I)(A) \subseteq I \). Then

\[
\left( \sum_{i=0}^{2} [\iota_i(A), \iota_i(I)] \right) \oplus \iota_0(I) \oplus \iota_1(I) \oplus \iota_2(I)
\]

is closed under the adjoint action of \( \iota_i(A) \) for \( i = 0, 1, 2 \) since, for \( i \neq j \neq k \neq i \),

\[
[[\iota_i(I), \iota_j(A)], \iota_k(A)] \subseteq \iota_k(I*A*A) \subseteq \iota_k(I),
\]

\[
[[\iota_i(A), \iota_j(I)], \iota_k(A)] \subseteq [[\iota_k(A), \iota_j(I)] + [\iota_i(A), \iota_k(I)] \subseteq \iota_j(I),
\]

\[
[[\iota_i(A), \iota_j(I)], \iota_i(A)] \subseteq \iota_i(\delta_0(A,I)(A)) \subseteq \iota_i(I).
\]

Thus, since the \( \iota_i(A) \)'s generate \( g(A,*) \), this is a nonzero ideal and hence equals the whole \( g(A,*) \). Therefore, \( I = A \).
Finally, assume that $(A, \ast)$ is simple and let $n = (n \cap t) \oplus \left( \bigoplus_{i=1}^{2} (n \cap \iota_i(A)) \right)$ be an ideal of $g(A, \ast)$ invariant under the action of $A_4$. For any $i$, let $I_i = \{ x \in A : \iota_i(x) \in n \}$. Then for any $i \in \mathbb{Z}_3$, 
\[ [n \cap \iota_i(A), \iota_{i+1}(A)] \subseteq n \cap \iota_{i+2}(A), \]
\[ [\iota_i(A), n \cap \iota_{i+1}(A)] \subseteq n \cap \iota_{i+2}(A), \]
which implies that $I_i \ast A \subseteq I_{i+2}$ and $A \ast I_{i+1} \subseteq I_{i+2}$. But $n$ is invariant under the automorphism $\varphi$, and this shows that $I_1 = I_2 = I_3$ is an ideal of $A$. If this is the whole $A$, $n = g(A, \ast)$, while if this ideal is 0, $n \subseteq t$, which acts faithfully on $\bigoplus_{i=0}^{2} \iota_i(A)$. However, $[n, \iota_i(A)] \subseteq n \cap \iota_i(A) = \iota_i(I_i) = 0$. Hence $n = 0$. 

The restriction $A \ast A \neq 0$ in Theorem 1.16 is necessary, as shown by Example 3.3.a

**Remark.** Let $G$ be a group of automorphisms of a finite dimensional Lie algebra $L$, and assume that $L$ is simple as a Lie algebra with $G$-action. Then, if $I$ is a minimal ideal of $L$, then $\sum_{\sigma \in G} \sigma(I)$ is an ideal, invariant under the action of $G$. Hence $L = \sum_{\sigma \in G} \sigma(I)$, so $L$ is completely reducible as a module over itself (the adjoint module). It follows that $L = \sigma_1(I) \oplus \cdots \oplus \sigma_r(I)$ for some $\sigma_1 = 1, \sigma_2, \ldots, \sigma_r \in G$, and each $\sigma_j(I)$ is simple (by minimality of $I$). Therefore $L$ is a direct sum of simple ideals.

In particular, this applies to the Lie algebras $g(A, \ast)$ for simple $(A, \ast)$ in Theorem 1.16

2. $S_4$-ACTION AND NORMAL LIE RELATED TRIPLE ALGEBRAS

In this section $g$ will be a Lie algebra endowed with a group homomorphism

\[ S_4 \rightarrow \text{Aut}(g), \]

where $S_4$ is the symmetric group of degree 4, which is the semidirect product of $A_4$ and the cyclic subgroup of order 2 generated by the transposition $\tau = (12)$. Besides, $\tau_1 \tau = \tau \tau_1$, $\tau_2 \tau = \tau \tau_2 \tau_1$, and $\tau \varphi = \varphi^2 \tau$. Because of the results in the previous Section,

\[ g \cong t \oplus \left( \bigoplus_{i=0}^{2} \iota_i(A) \right), \]

where $(A, \ast)$ is a normal symmetric triality algebra (normal STA for short).

Since Klein’s 4-group $V$ is a normal subgroup of $S_4$, $t$ is invariant under $\tau$, and hence under $S_4$. For any $x \in g_0 = \iota_0(A) = \{ x \in g : \tau_1(x) = x, \tau_2(x) = -x \}$,

\[ \tau_1 \tau(x) = \tau \tau_1(x) = \tau(x), \]

while

\[ \tau_2 \tau(x) = \tau \tau_2 \tau_1(x) = \tau \tau_2(x) = -\tau(x). \]

Hence $\tau(g_0) \subseteq g_0$. Also, for any $x \in g_1 = \iota_1(A) = \{ x \in g : \tau_1(x) = -x, \tau_2(x) = x \}$,

\[ \tau_1 \tau(x) = \tau \tau_1(x) = -\tau(x), \]

\[ \tau_2 \tau(x) = \tau \tau_2 \tau_1(x) = -\tau \tau_2(x) = -\tau(x). \]
Thus, for any $x, y$, to get $\tau \varphi = \varphi \tau$, next, since
\[ \tau(t_0(x)) = \tau^2(t_0(x)) = -\tau(t_0(x)) = t_0(x), \]
we calculate
\[ \tau(t_1(x)) = \tau(\tau(t_0(x))) = -\varphi^2(t_0(x)) = -t_2(x), \]
\[ \tau(t_2(x)) = \tau(\tau(t_0(x))) = -\varphi(t_0(x)) = -t_1(x). \]

This is shown with the same sort of arguments leading to (1.5).

Proposition 2.1. The map $x \mapsto \bar{x}$ is an involution of $(A, \cdot)$.

Proof. First, we have $\bar{x} = x$, since we calculate
\[ t_0(x) = \tau^2(t_0(x)) = -\tau(t_0(x)) = t_0(x). \]
Next, since $\tau \varphi = \varphi^2 \tau$, for any $x \in A$,
\[ \tau(t_1(x)) = \tau(\tau(t_0(x))) = \varphi^2(t_0(x)) = -\varphi^2(t_0(x)) = -t_2(x), \]
\[ \tau(t_2(x)) = \tau(\tau(t_0(x))) = \varphi(t_0(x)) = -\varphi(t_0(x)) = -t_1(x). \]

Thus, for any $x, y \in A$, apply the automorphism $\tau$ to $[t_0(x), t_1(y)] = t_2(x \cdot y)$ to get $[t_0(\bar{x}), t_2(\bar{y})] = -t_1(x + y)$, or $-t_1(\bar{y} \cdot \bar{x}) = -t_1(x + y)$. Hence $x + y = \bar{y} \cdot \bar{x}$, as required.

Define a new multiplication on $A$ by means of:
\[ x \cdot y = \bar{x} + \bar{y} = \bar{y} \cdot \bar{x}, \]
for any $x, y \in A$. Then $x \mapsto \bar{x}$ is an involution too of $(A, \cdot)$. The second author has shown [Oku05 (1.16)] that
\[ \text{str}(A, \cdot, -) = \text{lrt}(A, \cdot, -), \]
where
\[ \text{lrt}(A, \cdot, -) = \{ (d_0, d_1, d_2) \in \mathfrak{gl}(A)^3 : \bar{d}_i(x \cdot y) = d_{i+1}(x) \cdot y + x \cdot d_{i+2}(y) \]
for any $x, y \in A$ and $i \in \mathbb{Z}_3$, with $\bar{d}(x) = \bar{d}(\bar{x})$ for any $d \in \mathfrak{gl}(A)$ and $x \in A$.

The skew-symmetric bilinear map $\delta$ can be considered now as a map
\[ \delta : A \times A \rightarrow \text{lrt}(A, \cdot, -). \]
As before, $\text{lrt}(A, \cdot, -) = \text{str}(A, *)$ has the natural order 3 automorphism $\theta$ (see [L7]) given by
\[ \theta(d_0, d_1, d_2) = (d_2, d_0, d_1), \]
and also the order 2 automorphism $\xi$ given by
\[ \xi(d_0, d_1, d_2) = (\bar{d}_0, \bar{d}_2, \bar{d}_1), \tag{2.2} \]
which satisfies,
\[ \rho \rho = \xi \rho. \tag{2.3} \]
This is shown with the same sort of arguments leading to (1.3).

Denote by $L_x$ and $R_x$ the left and right multiplications by an element $x$ in $(A, \cdot)$. Hence $L_x(y) = x \cdot y = \bar{x} \cdot \bar{y} = \bar{y} \cdot \bar{x}$, so $L_x = \nu L_x = r_x \nu$, where $\nu(x) = \bar{x}$. Also $R_x = \nu R_x = l_x \nu$.

The maps $\delta_1(x, y), \delta_2(x, y)$ in Theorem [L3] become now:
\[
\begin{cases}
\delta_1(x, y) = r_x l_y - r_y l_x = L_y L_x - L_x L_y, \\
\delta_2(x, y) = l_y r_x - l_x r_y = R_y R_x - R_x R_y,
\end{cases}
\]
Corollary 2.5. Therefore:

Define a skew-symmetric bilinear map \( \delta \) to obtain

\[
\delta_0(x, y)(z) = -\theta_0(\delta_0(x, y)(z))
\]

So \( \delta_0(x, y) = \delta_0(\bar{x}, \bar{y}) \).

Hence, Theorem 1.9 immediately implies the following:

Theorem 2.4. Under the conditions above, for any \( a, b, x, y, z \in A \) and \( i, j \in \mathbb{Z}_3 \):

(i) \( [\delta_i(a, b), \delta_j(x, y)] = \delta_j(\delta_i(a, b)(x) + \delta_i(x, \delta_j(a, b)(y))) \),
(ii) \( \delta_0(x, y \cdot z) + \delta_1(y, z \cdot x) + \delta_2(\bar{z}, x \cdot y) = 0 \),
(iii) \( \delta_0(x, y)(z) + \delta_0(y, z)(x) + \delta_0(z, x)(y) = 0 \),
(iv) \( \delta_1(x, y) = L_y L_x - L_x L_y \),
(v) \( \delta_2(x, y) = R_y R_x - R_x R_y \),
(vi) \( \delta_i(x, y) = \delta_{i-1}(\bar{x}, \bar{y}) \) (or \( \xi(\delta(x, y)) = \delta(\bar{x}, \bar{y}) \)).

Conditions (i)–(vi) above are precisely the conditions (2.34) in [Oku05] defining a normal Lie related triple algebra (or normal LRTA for short). Therefore:

Corollary 2.5. Under the hypothesis above, \( (A, \cdot, \cdot) \) is a normal LRTA.

And, as in Section 2, everything can be reversed to get:

Theorem 2.6. Let \( (A, \cdot, \cdot) \) be a nonzero normal LRTA with respect to the skew-symmetric bilinear map \( \delta : A \times A \rightarrow \mathrm{Irr}(A, \cdot, \cdot) \). Then \( t = \sum_{i=0}^{2} \theta^i(\delta(A, A)) \) is a Lie subalgebra of \( \mathrm{Irr}(A, \cdot, \cdot) \), and \( \theta^i(\delta(A, A)) \) is an ideal of \( t \) for any \( i = 0, 1, 2 \). Moreover, consider three copies \( t_i(A) \) of \( A \) \( (i = 0, 1, 2) \) and define an anticommutative multiplication on

\[
g(A, \cdot, \cdot) = t \oplus (\oplus_{i=0}^{2} t_i(A))
\]

by means of

- \( t \) is a subalgebra of \( g(A, \cdot, \cdot) \),
- \( [t_0, t_1, t_2] = 0 \), for any \( (d_0, d_1, d_2) \in t \), \( x, y \in A \) and \( i \in \mathbb{Z}_3 \),
- \( [t_i(x), t_{i+1}(y)] = t_{i+2}(\overline{x \cdot y}) \) for any \( x, y \in A \) and \( i \in \mathbb{Z}_3 \),
- \( [t_i(x), t_j(y)] = \theta^i(\delta(x, y)) \), for any \( x, y \in A \) and \( i \in \mathbb{Z}_3 \).

Then \( g(A, \cdot, \cdot) \) is a Lie algebra and the symmetric group \( S_4 \) embeds as a subgroup of \( \text{Aut}(g(A, \cdot, \cdot)) \) by means of

- The restrictions of the elements of the 4-group \( V \) to \( t \) are trivial: \( \tau_1|_t = \text{id} = \tau_2|_t \). Moreover, \( \varphi|_t = \theta|_t \) and \( \tau|_t = \xi|_t \). \( (\theta \text{ and } \xi \text{ as in } \text{(1.17)} \text{ and } \text{(2.2)}) \)
- \( \varphi(t_i(x)) = t_{i+1}(x) \) for any \( x \in A \) and \( i \in \mathbb{Z}_3 \).
- For any \( x \in A \), \( \tau_1(t_0(x)) = t_0(x) \), \( \tau_1(t_i(x)) = -t_i(x) \) for \( i = 1, 2 \), while \( \tau_2(t_1(x)) = t_1(x) \), \( \tau_2(t_i(x)) = -t_i(x) \) for \( i = 0, 2 \).
• For any $x \in A$, $\tau(t_0(x)) = -t_0(\bar{x})$, $\tau(t_1(x)) = -t_1(\bar{x})$, and $\tau(t_2(x)) = -t_1(\bar{x})$.

Given an algebra with involution $(A, \cdot, \bar{\cdot})$, the Steinberg unitary Lie algebra $\mathfrak{stu}_3(A, \cdot, \bar{\cdot})$ is defined as the Lie algebra generated by the symbols $u_{ij}(a)$, $1 \leq i \neq j \leq 3$, $a \in A$, subject to the relations:

$$u_{ij}(a) = u_{ji}(-\bar{a}),$$

$a \mapsto u_{ij}(a)$ is linear,

$$[u_{ij}(a), u_{jk}(b)] = u_{ik}(ab)$$

for distinct $i, j, k$.

Then [AF93, Lemma 1.1],

$$\mathfrak{stu}_3(A, \cdot, \bar{\cdot}) = s \oplus u_{12}(A) \oplus u_{23}(A) \oplus u_{31}(A),$$

with $s = \sum_{i<j}[u_{ij}(A), u_{ij}(A)]$. This is a $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading of $\mathfrak{stu}_3(A, \cdot, \bar{\cdot})$.

**Proposition 2.7.** Let $(A, \cdot, \bar{\cdot})$ be a nonzero normal LRTA, then there is a surjective homomorphism of Lie algebras

$$\phi : \mathfrak{stu}_3(A, \cdot, \bar{\cdot}) \rightarrow \mathfrak{g}(A, \cdot, \bar{\cdot})$$

such that, for any $a \in A$, $\phi(u_{12}(a)) = -t_0(a)$, $\phi(u_{23}(a)) = -t_1(a)$, and $\phi(u_{31}(a)) = -t_2(a)$.

**Proof.** It is enough to realize that $[u_{12}(a), u_{23}(b)] = u_{13}(a \cdot b) = -u_{31}(a \cdot b)$, while $[t_0(a), t_1(b)] = t_2(a \cdot b)$, and cyclically. \hfill $\square$

Also, the following result is proved by a straightforward computation:

**Proposition 2.8.** Let $(A, \cdot, \bar{\cdot})$ be an algebra with involution. Then $\mathfrak{stu}_3(A, \cdot, \bar{\cdot})$ is endowed with an action of $S_4$ by automorphisms by means of:

$$\tau_1 : u_{12}(a) \mapsto u_{12}(a)$$

$$u_{23}(a) \mapsto -u_{23}(a)$$

$$u_{31}(a) \mapsto -u_{31}(a)$$

$$\varphi : u_{12}(a) \mapsto u_{23}(a)$$

$$u_{23}(a) \mapsto u_{31}(a)$$

$$u_{32}(a) \mapsto u_{12}(a)$$

With these last two results, the result in [Oku05, Theorem 2.6] follows easily:

**Theorem 2.9.** The unital normal LRTA’s are precisely the structurable algebras.

**Proof.** By Proposition 2.7, any unital normal LRTA is 3-faithful, and hence it is a structurable algebra [AF93, Theorem 5.5]. Conversely, if $(A, \cdot, \bar{\cdot})$ is a structurable algebra, then $\mathfrak{g} = \mathfrak{stu}_3(A, \cdot, \bar{\cdot})$ is endowed with an action of $S_4$ by automorphisms (Proposition 2.8), which shows that $(A, \cdot, \bar{\cdot})$ is a normal LRTA because of Theorem 2.4. \hfill $\square$

With the same arguments as for Theorem 1.16 one gets too:
Theorem 2.10. Let \((A,\cdot,\bar{-})\) be a normal Lie related triple algebra with \(A \cdot A \neq 0\). Then \((A,\cdot,\bar{-})\) is simple if and only if \(g(A,\cdot,\bar{-})\) is simple as a Lie algebra with \(S_4\)-action.

3. Examples

Example 3.1. (Structurable algebras)

Theorem 2.9 shows that structurable algebras are precisely the unital normal LRTA’s, thus providing many examples of these latter algebras. Given any structurable algebra \((A,\cdot,\bar{-})\), the associated Lie algebra with \(S_4\)-action in Theorem 2.6 is

\[ g(A,\cdot,\bar{-}) = t \oplus \left( \bigoplus_{i=0}^{2} \theta^i(\delta(A, A)) \right), \]

where \(t = \sum_{i=0}^{2} \theta^i(\delta(A, A))\). But Theorem 2.4 shows that

\[
\begin{align*}
\delta_0(x, y) &= \delta_0(x, y \cdot 1) = -\delta_1(y, x) - \delta_2(1, x \cdot y) \\
&= -(L_xL_y - LyL_x) - (R_yx - x\cdot y) \\
&= R_{(x-y\cdot y)} + LyL_x - L_xL_y.
\end{align*}
\]

Therefore, \(t\) is precisely the subspace \(T_t\) of inner triples in \(\text{AF93}\) Equation (I).

By identifying \(u_{12}(x)\) with \(-i_0(x)\), \(u_{23}(x)\) with \(-i_1(x)\), and \(u_{31}(x)\) with \(-i_2(x)\) as in Proposition 2.7, it follows that \(g(A,\cdot,\bar{-})\) is precisely the Lie algebra \(K(A,\cdot,\gamma,\theta)\) constructed in \(\text{AF93}\) Section 4, with \(\gamma = (1,1,1)\). (Note that other choices of \(\gamma\) prevent this algebra from having the symmetry induced by the action of \(S_4\).)

Example 3.2. (Jordan algebras)

Unital Jordan algebras are examples of structurable algebras (where the involution is the identity map). Nonunital Jordan algebras are examples too of normal LRTA’s.

Given a Jordan algebra \(J\), with multiplication \(\cdot\) and involution \(\bar{-} = \text{id}\), the left and right multiplications coincide: \(L_x = R_x\) for any \(x\), and \(J\) is a normal LRTA \(\text{Oku05}\) Example 2.3 with

\[
\delta_i(x, y) = -[L_x, L_y]
\]

for any \(x, y \in J\) and \(i \in \mathbb{Z}_3\). Since \([L_x, L_y]\) is a derivation of \(J\), conditions (i)–(vi) in Theorem 1.9 are clearly satisfied. Hence,

\[
t = \sum_{i=0}^{2} \theta^i(\delta(J, J)) = \text{span} \left\{ ([L_x, L_y], [L_x, L_y], [L_x, L_y]) : x, y \in J \right\},
\]

which is isomorphic to \(\text{ind}er(J) = \text{span} \{ [L_x, L_y] : x, y \in J \}\), the Lie algebra of inner derivations (see \(\text{Jac68}\)). Thus \(g(J,\cdot,\bar{-})\) is isomorphic to the Lie algebra

\[
g = \text{ind}er(J) \oplus (\mathfrak{s} \oplus_F J)
\]

considered in \(\text{Tit62}\), where \(\mathfrak{s}\) is the three-dimensional simple Lie algebra with a basis \(\{e_0, e_1, e_2\}\) such that \([e_i, e_{i+1}] = e_{i+2}\), indices modulo 3. (Over the reals this is just \(\mathfrak{su}_2\).)
The Lie bracket in $g$ is given, for any $d \in \text{ind}(J)$, $s, t \in \mathfrak{s}$ and $x, y \in J$, by
\[
\begin{align*}
[d, s \otimes x] &= s \otimes d(x), \\
[s \otimes x, t \otimes y] &= [s, t] \otimes x \cdot y - \frac{1}{2} \kappa(s, t)[L_x, L_y],
\end{align*}
\]
where $\kappa$ denotes the Killing form of $\mathfrak{s}$. The isomorphism just sends $\iota_i(x)$ to $e_i \otimes x$ for any $x$ and $i$.

If $J$ is any unital Jordan algebra, its Tits-Kantor-Koecher Lie algebra $\mathcal{TKK}(J)$ (see [Jac68, Ch. VIII]) is isomorphic to the Lie algebra in (5.3), but with $\mathfrak{s}$ replaced by $\mathfrak{sl}_2$. If $\sqrt{-1} \in F$, then $\mathfrak{s}$ is isomorphic to $\mathfrak{sl}_2$ and, hence, $\mathcal{TKK}(J)$ becomes a Lie algebra with $S_4$-action. \hfill \square

\textbf{Example 3.4.} (Lie algebras)

Given any Lie algebra $g$, the direct sum of four copies of $g$: $g^4$, is endowed with a natural action of $S_4$:
\[
S_4 \rightarrow \text{Aut}(g^4)
\]
\[
\sigma \mapsto \left((x_1, x_2, x_3, x_4) \mapsto (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, x_{\sigma^{-1}(3)}, x_{\sigma^{-1}(4)})\right).
\]
The Lie algebra $g^4$ may be identified naturally with $g \otimes_F F^4$, and $F^4$ contains the basis $\{1, e_0, e_1, e_2\}$, where $1 = (1, 1, 1, 1)$, $e_0 = (1, 1, -1, -1)$, $e_1 = (1, -1, -1, 1)$ and $e_2 = (1, -1, 1, -1)$. Then, the components in $\{1,2\}$ become:
\[
\begin{align*}
t &= \{X \in g^4 : \tau_1(X) = X, \tau_2(X) = X\} = g \otimes 1, \\
g_0 &= \{X \in g^4 : \tau_1(X) = X, \tau_2(X) = -X\} = g \otimes e_0, \\
g_1 &= \{X \in g^4 : \tau_1(X) = -X, \tau_2(X) = X\} = g \otimes e_1, \\
g_2 &= \{X \in g^4 : \tau_1(X) = -X, \tau_2(X) = -X\} = g \otimes e_2.
\end{align*}
\]
It follows that the corresponding normal LRTA can be identified with $g$, with $x \cdot y = [x, y]$, $x^\circ = -x$ and $\delta(x, y) = (\text{ad}_{[x,y]}, \text{ad}_{[x,y]}, \text{ad}_{[x,y]})$ for any $x, y \in g$, in accordance to [Oku03, Example 2.3]. Therefore, any Lie algebra is a normal LRTA. \hfill \square

\textbf{Example 3.5.} (Lie triple systems)

Any Lie triple system is the odd part of a $\mathbb{Z}_2$-graded Lie algebra $g = g_0 \oplus g_1$, with the triple product being given by $[x y z] = [[x, y], z]$ for any $x, y, z \in g_1$.

Given a $\mathbb{Z}_2$-graded Lie algebra $g = g_0 \oplus g_1$, let us denote by $\nu$ the grading automorphism: $\nu(x_0 + x_1) = x_0 - x_1$. Then $S_4$ acts as automorphisms of $g^3$ as follows:
\[
\begin{align*}
\tau_1(x, y, z) &= (x, \nu(y), \nu(z)), \\
\tau_2(x, y, z) &= (\nu(x), y, \nu(z)), \\
\varphi(x, y, z) &= (z, x, y), \\
\tau(x, y, z) &= (x, z, y).
\end{align*}
\]
The subspaces in (1.2) are
\[ t = \mathfrak{g}_0^3, \]
\[ \mathfrak{g}_0 = \{(x_1, 0, 0) : x_1 \in \mathfrak{g}_1\}, \]
\[ \mathfrak{g}_1 = \{(0, x_1, 0) : x_1 \in \mathfrak{g}_1\}, \]
\[ \mathfrak{g}_2 = \{(0, 0, x_1) : x_1 \in \mathfrak{g}_1\}. \]
Hence the associated normal LRTA is \( A \simeq \mathfrak{g}_1 \), with \( x \cdot y = 0, \bar{x} = -x \) and \( \delta(x, y) = (\text{ad}_{[x,y]}, 0, 0) \) for any \( x, y \in \mathfrak{g}_1 \).

If \( \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) is a simple Lie algebra, then \([\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_0\) and \( x_0 \mapsto \text{ad}_{x_0} \mid_{\mathfrak{g}_1} \) is one-to-one. Hence, with \( A = \mathfrak{g}_1 \) as before, \( \mathfrak{g}(A, \cdot, \cdot) \) coincides with \( \mathfrak{g}^3 \), and this has no proper ideals invariant under the action of \( S_4 \). Hence \( \mathfrak{g}(A, \cdot, \cdot) \) is simple as a Lie algebra with \( S_4 \)-action, even though \( (A, \cdot) \) is a trivial algebra.

**Example 3.6.** (Tensor products of symmetric composition algebras)

A symmetric composition algebra is a triple \((S, \ast, q)\), where \( q : S \to F \) is a regular quadratic form satisfying for any \( x, y, z \in S \):
\[
q(x \ast y) = q(x)q(y),
\]
\[
q(x \ast y, z) = q(x, y \ast z),
\]
where \( q(x, y) = q(x + y) - q(x) - q(y) \) is the polar of \( q \) (see [KMRT98, Ch. VIII])

The classification of the symmetric composition algebras was obtained in [EM93, Eld97].

Given any unital composition algebra (or Hurwitz algebra) \( C \) with norm \( q \) and standard involution \( x \mapsto \bar{x} \), the new algebra defined on \( C \) but with multiplication
\[ x \ast y = \bar{x}y, \]
is a symmetric composition algebra, called the associated para-Hurwitz algebra. In dimensions 1, 2 or 4, any symmetric composition algebra is a para-Hurwitz algebra (with a few exceptions in dimension 2), while in dimension 8, apart from the para-Hurwitz algebras, there is a new family of symmetric composition algebras termed Okubo algebras.

If \((S, \ast, q)\) is any symmetric composition algebra, consider the corresponding orthogonal Lie algebra \( \mathfrak{o}(S, q) = \{d \in \mathfrak{gl}(S) : q(d(x), y) + q(x, d(y)) = 0 \ \forall x, y \in S\} \), and the subalgebra of \( \mathfrak{o}(S, q)^3 \) defined by
\[
\text{tri}(S, \ast, q) = \{d_0, d_1, d_2 \in \mathfrak{o}(S, q)^3 : d_0(x \ast y) = d_1(x) \ast y + x \ast d_2(y) \ \forall x, y \in S\}
\]
\[
= \{(d_0, d_1, d_2) \in \mathfrak{o}(S, q)^3 : \langle d_0(x), y, z \rangle + \langle x, d_1(y), z \rangle + \langle x, y, d_2(z) \rangle = 0 \ \forall x, y, z \in S\},
\]
where \( \langle x, y, z \rangle = q(x, y \ast z) \). It turns out that the map,
\[
\theta : \text{tri}(S, \ast, q) \to \text{tri}(S, \ast, q)
\]
\[
(d_0, d_1, d_2) \mapsto (d_2, d_0, d_1)
\]
is an automorphism of \((S, *, q)\) of order 3. Its fixed subalgebra is (isomorphic to) the derivation algebra of \((S, *)\) which, if the dimension is 8 and the characteristic of the ground field is \(\neq 2, 3\), is a simple Lie algebra of type \(G_2\) in the para-Hurwitz case and a simple Lie algebra of type \(A_2\) (a form of \(\mathfrak{sl}_3\)) in the Okubo case.

A straightforward computation (see [EO01] for a more general setting) shows that for any \(x, y \in S\), the triple
\[
t_{x,y} = \left(\sigma_{x,y}, \frac{1}{2}q(x,y)id - r_xl_y, \frac{1}{2}q(x,y)id - l_xr_y\right)
\]
is in \(\text{tri}(S, *, q)\), where \(\sigma_{x,y}(z) = q(x,z)y - q(y,z)x\), \(r_x(z) = z \ast x\), and \(l_x(z) = x \ast z\) for any \(x, y, z \in S\).

Given two symmetric composition algebras \((S, *, q)\) and \((S', *, q')\), their tensor product \(A = S \otimes_F S'\) is a normal STA with multiplication
\[
(a \otimes x) \ast (b \otimes y) = (a \ast b) \otimes (x \ast y),
\]
and skew-symmetric bilinear map \(\delta : A \times A \to \text{stri}(A, *)\) given by
\[
\delta(a \otimes x, b \otimes y) = q'(x,y)t_{a,b} + q(a,b)t_{x,y} \in \text{tri}(S, *, q) \oplus \text{tri}(S', *, q') \subseteq \text{stri}(A, *)
\]
for any \(a, b \in S\) and \(x, y \in S'\) (see [Eld04]).

This example can be extended by considering the so called generalized symmetric composition algebras [Oku05, Examples 2.4 and 2.5].

Note also that the tensor product of two unital composition algebras is a structurable algebra, and hence a normal LRTA. \(\square\)

**Example 3.7.** Any Okubo algebra \((S, *, q)\) over \(F\) either contains a nonzero idempotent, or there exists a cubic field extension \(L/F\) such that \(L \otimes_F S\) contains a nonzero idempotent. In the first case, there is an order 3 automorphism \(\phi\) of \(S\) such that \(S\) becomes a para-Hurwitz algebra with the new multiplication given by \(x \ast y = \phi(x) \ast \phi^2(y)\), for any \(x, y\), and the same norm \(q\) (see [KMRT98, §3.4]). This is used in [Eld04] to relate the models of the exceptional Lie algebras in [Eld05] constructed in terms of Okubo algebras, to those given by para-Hurwitz algebras.

This can be extended as follows. Let \((A, *)\) be a normal STA and consider an order 3 automorphism \(\phi \in \text{Aut}(A,*)\) such that \(\phi \delta_0(x, y)\phi^{-1} = \delta_0(\phi(x), \phi(y))\) for any \(x, y \in A\).

Note that since \(l_{\phi(x)} = \phi l_x\phi^{-1}\) and \(r_{\phi(x)} = \phi r_x\phi^{-1}\), because \(\phi\) is an automorphism, then \(\phi \delta_i(x,y)\phi^{-1} = \delta_i(\phi(x), \phi(y))\) for \(i = 1, 2\), because of Theorem 3.9. Also, if \((d_0, d_1, d_2) \in \text{stri}(A,*),\) then \((\phi d_0 \phi^{-1}, \phi d_1 \phi^{-1}, \phi d_2 \phi^{-1}) \in \text{stri}(A,*)\) too, and hence, for any \(x, y \in A\):

\[
\begin{cases}
(\delta_0(\phi(x), \phi(y)), \delta_1(\phi(x), \phi(y)), \delta_1(\phi(x), \phi(y))) \in \text{stri}(A,*), \\
(\phi \delta_0(x,y)\phi^{-1}, \phi \delta_1(x,y)\phi^{-1}, \phi \delta_2(x,y)\phi^{-1}) \in \text{stri}(A,*)
\end{cases}
\]

so its difference also belongs to \(\text{stri}(A,*):\)

\[
\left(\phi \delta_0(x,y)\phi^{-1} - \delta_0(\phi(x), \phi(y)), 0, 0 \right) \in \text{stri}(A,*)
\]
But if either $A = A \ast A$, or $\{ x \in A : x \ast A = 0 \} = 0$, or $\{ x \in A : A \ast x = 0 \} = 0$, then $(d_0, 0, 0) \in \text{stri}(A, \ast)$ implies that $d_0 = 0$. Hence the condition $\phi \delta_0(x, y) \phi^{-1} = \delta_0(\phi(x), \phi(y))$ is superfluous in these cases.

Define a new multiplication on $A$ by means of

$$x \ast y = \phi(x) \ast \phi^2(y)$$

for any $x, y \in A$ as before. Then it is easily checked that $(A, \ast)$ is again a normal STA with $\delta \cdot : A \times A \to \text{stri}(A, \ast)$ given by

$$\delta^i(x, y) = \phi^{-i} \delta_i(x, y) \phi^i$$

for any $x, y \in A$ and $i \in \mathbb{Z}_3$, because

$$\delta^i(x, y)(u \ast v) = \phi^{-i} \delta_i(x, y) \phi^i(\phi(u) \ast \phi^2(v))$$

$$= \phi^{-i} \delta_i(x, y)(\phi^{i+1}(u) \ast \phi^{i+2}(v))$$

$$= \phi^{-i} \left( \delta_{i+1}(x, y)(\phi^{i+1}(u)) \ast \phi^{i+2}(v) \right)$$

$$+ \phi^{i+1}(u) \ast \delta_{i+2}(x, y)(\phi^{i+2}(v))$$

$$= \phi(\delta_{i+2}(x, y)(u)) \ast \phi^2(v) + \phi(u) \ast \phi^2(\delta_{i+2}(x, y)(v))$$

$$= \delta^i_{i+2}(x, y)(u) \ast v + u \ast \delta^i_{i+2}(x, y)(v),$$

for any $x, y, u, v \in A$ and $i \in \mathbb{Z}_3$.

Under these conditions, the linear map:

$$\Phi : \mathfrak{g}(A, \ast) \longrightarrow \mathfrak{g}(A, \ast),$$

such that

$$\begin{cases}
\Phi(\iota^i_*)(x) = \iota_i(\phi^i(x)), \\
\Phi(d_0, d_1, d_2) = (d_0, \phi d_1 \phi^2, \phi^2 d_2 \phi),
\end{cases}$$

is an isomorphism of Lie algebras.

If now $(A, \cdot, -)$ is a normal LRTA and $\phi$ is an order 3 automorphism of $(A, \cdot)$ such that $\bar{\phi} = \phi^2$, then a new multiplication can be defined by

$$x \cdot y = \phi(x) \cdot \phi^2(y),$$

for any $x, y \in A$, which satisfies

$$x \cdot y = \phi(x) \cdot \phi^2(y) = \phi^2(x) \cdot \phi(x)$$

$$= \phi^2(y) \cdot \phi(x) = \phi(y) \cdot \phi^2(x)$$

$$= \bar{y} \cdot \bar{x}.$$}

Hence $\bar{-}$ is an involution too of $(A, \cdot)$. Now, assuming that $\phi \delta_0(x, y) \phi^{-1} = \delta_0(\phi(x), \phi(y))$ for any $x, y \in A$, $(A, \cdot, \bar{-})$ is another normal LRTA relative to the skew-symmetric bilinear map $\delta^\cdot : A \times A \to \text{ltt}(A, \cdot, \bar{-})$ given by:

$$\delta^\cdot_i(x, y) = \phi^{-i} \delta_i(x, y) \phi^i,$$

for any $x, y \in A$ and $i \in \mathbb{Z}_3$.

This allows the construction of new normal STA’s or LRTA’s out of old ones. For instance, let $\mathfrak{s}$ be the three-dimensional simple Lie algebra that appears in Example 3.2. This is a normal LRTA, since it is a Lie algebra.
Consider the order 3 automorphism given by \( \phi(e_i) = e_{i+1} \), for any \( i \in \mathbb{Z}_3 \). This automorphism induces the new multiplication \( \bullet \) given by
\[
s \bullet t = [\phi(s), \phi^2(t)]
\]
which gives:
\[
e_i \bullet e_{i+1} = -e_{i+2}, \quad e_{i+1} \bullet e_i = 0, \quad e_i \bullet e_i = e_i,
\]
for any \( i \in \mathbb{Z}_3 \). This normal LRTA does not fit into any of the previous examples.

\[\square\]

4. \( S_4 \)-action on Kantor’s construction

Given any structurable algebra \((A, \cdot)\) (multiplication denoted by juxtaposition), Kantor’s construction gives a 5-graded Lie algebra \( K(A, \cdot, d) \) (see \[All79\], based on \[Kan72\], \[Kan73\]), where \( d \) is a Lie subalgebra of the Lie algebra of derivations which contains the inner derivations. On the other hand, Allison and Faulkner defined a \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-graded Lie algebra \( K(A, \cdot, \gamma, v) \) in \[AT93\] Section 4, where \( \gamma \in F^3 \), and \( v \) is a Lie subalgebra of \( \text{lrt}(A, \cdot) \) containing the subspace of inner triples (see Example 3.1). A precedent of this construction appears in \[Vin66\].

The aim of this section is to show that the Lie algebra \( K(A, \cdot, \bar{d}) \) is isomorphic to the Lie algebra \( K(A, \cdot, \gamma, v) \), for some suitable \( \gamma \) and \( v \), and that, under some restrictions on the ground field, it is endowed with an action of the symmetric group \( S_4 \).

Let \((A, \cdot)\) be a structurable algebra over a ground field of characteristic \( \neq 2, 3 \). This assumption on the field will be kept throughout the section. Let \( \text{indert}(A, \cdot) \) (the inner derivation algebra) be the linear span of the derivations \( \{D_{x,y} : x, y \in A\} \), where \[All78\] Eq. (15):
\[
D_{x,y}(z) = \frac{1}{3} \left( [x, y] + [\bar{x}, \bar{y}], z \right) + (z, y, x) - (z, \bar{x}, \bar{y}).
\]
(Here \((x, y, z) = (xy)z - x(yz)\) is the associator.) The subspace \( \text{indert}(A, \cdot) \) is an ideal of the Lie algebra of derivations \( \text{dert}(A, \cdot) \). Also, consider the Lie algebra of inner related triples
\[
\text{inlrt}(A, \cdot) = \sum_{i=0}^{2} \text{span} \left\{ \theta^i(\delta(x, y)) : x, y \in A \right\},
\]
where
\[
\delta(x, y) = \left( R_{(xy)\bar{z}} + L_y L_x - L_x L_y, L_y L_x - L_x L_y, R_y R_x - R_x R_y \right).
\]
(See Example 3.1 and note that \( \text{inlrt}(A, \cdot) \) was denoted by \( \mathcal{T}_I \) in \[AF93\].)

The following notation will be used. Given the structurable algebra \((A, \cdot)\), let \( S \) be the set of skew-symmetric elements: \( S = \{ x \in A : \bar{x} = -x \} \). Then \( \mathcal{T}_S \) will denote the vector space
\[
\mathcal{T}_S = \left\{ (L_{s_1} - R_{s_2}, L_{s_2} - R_{s_0}, L_{s_0} - R_{s_1}) : s_0, s_1, s_2 \in S, \ s_0 + s_1 + s_2 = 0 \right\}.
\]
Also, for any subspace \( s \subseteq \mathfrak{gl}(A) \), denote by \( s^{<3>} \) the ‘diagonal subspace’ \( \{(s, s, s) : s \in s \} \leq \mathfrak{gl}(A)^3 \).
Lemma 4.1. Let \((A,\cdot)\) be a structurable algebra. Then:

(a) For any \(x,y \in A\), \(\sum_{i=0}^{2} \delta_i(x,y) = -3D_{x,y}\).
(b) \(\text{frt}(A,\cdot) = \text{der}(A,\cdot)^{<3>} \oplus T_S\), while \(\text{infrt}(A,\cdot) = \text{inder}(A,\cdot)^{<3>} \oplus T_S\).
(c) The map

\[
\{d : \text{inder}(A,\cdot) \leq d \leq \text{der}(A,\cdot)\} \to \{1 : \text{infrt}(A,\cdot) \leq 1 \leq \text{frt}(A,\cdot)\}
\]

is a bijection.

Proof. By [AF93, Eq. (A1)],

\[(x,\tilde{y},z) - (y,\tilde{x},z) = (z,\tilde{x},y) - (z,\tilde{y},x),\]

which is equivalent to

\[L(x\tilde{y} - y\tilde{x}) + R(\tilde{x}y - \tilde{y}x) = L_xL_y - L_yL_x - R_xR_y + R_yR_x.\]

Hence

\[L(\tilde{y}x + [x,\tilde{y}]) + L(x\tilde{y} - y\tilde{x}) = -R_xR_y - R_yR_x - (L_xL_y - L_yL_x) + (L_xL_y - L_yL_x).\]

Thus,

\[3D_{x,y} = L(\tilde{y}x + [x,\tilde{y}]) - R(\tilde{y}x + [x,\tilde{y}]) + 3R(\tilde{x}y - \tilde{y}x) + 3(R_xR_y - R_yR_x)\]

\[= -2\left(R(\tilde{x}y - \tilde{y}x) + R(\tilde{x}y - \tilde{y}x)\right) + 3R(\tilde{x}y - \tilde{y}x) + 3(R_xR_y - R_yR_x)\]

\[+ (L_xL_y - L_yL_x) + (L_xL_y - L_yL_x)\]

\[= (R(\tilde{x}y - \tilde{y}x) - 2R(\tilde{x}y - \tilde{y}x)) - (R_xR_y - R_yR_x) + 2(R_xR_y - R_yR_x) + (L_xL_y - L_yL_x)\]

\[= (R(\tilde{x}y - \tilde{y}x) - 2R(\tilde{x}y - \tilde{y}x)) - (R_xR_y - R_yR_x) + 2(R_xR_y - R_yR_x) + (L_xL_y - L_yL_x)\]

But \(D_{x,y} = D_{x,y} [AL73,\text{Lemma 6}]\), so

\[9D_{x,y} = 6D_{x,y} + 3D_{x,y}\]

\[= -3R(\tilde{x}y - \tilde{y}x) + 3(R_xR_y - R_yR_x)\]

\[+ 3(L_xL_y - L_yL_x) + 3(L_xL_y - L_yL_x),\]

which proves (a).

Now, (b) follows from [AF93, Corollary 3.5] and its proof, using (a).

For (c) consider the linear map

\[\phi : \text{frt}(A,\cdot) \longrightarrow S \times S\]

\[(d_0, d_1, d_2) \mapsto (d_1(1), d_2(1)).\]

Any \((d_0, d_1, d_2) \in \ker \phi\) satisfies \(\tilde{d_0}(1) = 0\), so \(d_0(1) = 0\), and

\[\tilde{d_i}(x) = \begin{cases} \tilde{d_i}(x) = d_{i+1}(x) \\
\tilde{d_i}(1x) = d_{i+2}(x) \end{cases}\]

for any \(x \in A\), so \(\tilde{d_i} = d_{i+1} = d_{i+2}\) for any \(i\). This implies that \(d_i = d_{i+1} = d_{i+2}\) for any \(i\), and hence \(\ker \phi = \text{der}(A,\cdot)^{<3>}\). It follows that \(\text{frt}(A,\cdot) = \ker \phi \oplus T_S\), \(\text{infrt}(A,\cdot) = \ker \phi|_{\text{infrt}(A,\cdot)} \oplus T_S\), and for any
subalgebra \( I \) of \( lrt(A,^-) \) containing \( \text{infrt}(A,^-) \), \( I = \ker \phi_{|I} \oplus T_S \). Finally, the map \( I \mapsto \ker \phi_{|I} \) is the inverse of the map in (e).

For any subalgebra \( \mathfrak{d} \) with \( \text{inder}(A,^-) \leq \mathfrak{d} \leq \mathfrak{der}(A,^-) \), denote by \( I(A,^-), \mathfrak{d} \) the subalgebra \( \mathfrak{d}^{<3>} \oplus T_S \) of \( \text{ftrt}(A,^-) \).

Let us recall Kantor's construction from [All79, Section 3].

Let \( \mathfrak{d} \) be a subalgebra of \( \mathfrak{der}(A,^-) \) containing \( \text{inder}(A,^-) \), then

\[
\mathcal{K}(A,^-, \mathfrak{d}) = \tilde{\mathfrak{n}} \oplus (T_A \oplus \mathfrak{d}) \oplus \mathfrak{n},
\]

where \( \mathfrak{n} = A \times S \), \( \tilde{\mathfrak{n}} \) is another copy of \( \mathfrak{n} \), and \( T_x = V_{x,1} \), where \( V_{x,y}(z) = (xy)z + (zy)x - (zx)y \) for any \( x, y, z \in A \). Then \( T_A \oplus \mathfrak{d} \) is a Lie subalgebra of \( \mathfrak{gl}(A) \) and the bracket of any two of its elements in \( \mathcal{K}(A,^-, \mathfrak{d}) \) coincides with its bracket in \( \mathfrak{gl}(A) \). Moreover,

\[
\begin{align*}
[f,(x,s)] &= (f(x), f^\delta(s)), \\
[f,(x,s)]^\gamma &= (f^\epsilon(x), f^\delta(x))^\gamma, \\
[(x,r),(y,s)] &= (x, yx - yx), \\
[(x,r),(y,s)]^\gamma &= (x, yx - yx)^\gamma, \\
[(x,r),(y,s)]^- &= -(sx, 0)^\gamma + V_{x,y} + L_xL_y + (ry, 0),
\end{align*}
\]

for any \( x, y \in A, r, s \in S, f \in T_A \oplus \mathfrak{d} \), where \( f^\epsilon = f - T_{(f(1)+f(1))} \) and \( f^\delta = f + R_{\frac{(f(1))}{2}} \), so that \( d^\epsilon = d^\delta = d \) for any \( d \in \mathfrak{der}(A,^-) \), while \( T_x^\epsilon = -T_x^\delta \) and \( T_x^\delta = T_x + R_{\frac{2x}{x}} = L_x + R_x \) for any \( x \in A \).

The Lie algebra \( \mathcal{K} = \mathcal{K}(A,^-, \mathfrak{d}) \) is 5-graded:

\[
\mathcal{K} = \mathcal{K}_{-2} \oplus \mathcal{K}_{-1} \oplus \mathcal{K}_0 \oplus \mathcal{K}_1 \oplus \mathcal{K}_2,
\]

where \( \mathcal{K}_{-2} = (0 \times S)^\gamma, \mathcal{K}_{-1} = (A \times 0)^\gamma, \mathcal{K}_0 = T_A \oplus \mathfrak{d}, \mathcal{K}_1 = A \times 0 \) and \( \mathcal{K}_2 = 0 \times S \). Allison [All79, Eq. (15)] considered the order two automorphism \( \chi \) such that

\[
\chi(\tilde{\mathfrak{n}} + f + \mathfrak{n}) = \tilde{\mathfrak{n}} + f^\epsilon + \mathfrak{n},
\]

for any \( m, n \in A \times S \) and \( f \in T_A \oplus \mathfrak{d} \). Also, for any \( 0 \neq \alpha \in F, \sigma_\alpha(x_j) = \alpha^j x_j \) \( (x_j \in \mathcal{K}_j) \) defines an automorphism too.

Fix \( 0 \neq \alpha \in F \) and consider the two commuting order two automorphisms \( \tau_1, \tau_2 \) of \( \mathcal{K}(A,^-, \mathfrak{d}) \) given by

\[
\tau_1 = \sigma_{-1}, \quad \tau_2 = \sigma_{\alpha \chi}.
\]

These two automorphisms induce a \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-grading of \( \mathcal{K} = \mathcal{K}(A,^-, \mathfrak{d}) \) as in (12), where

\[
\begin{align*}
\mathcal{K}_{(0,0)} &= T_S \oplus \mathfrak{d} \oplus \{\alpha(0, s) + \alpha^{-1}(0, s)^\gamma : s \in S\}, \\
\mathcal{K}_{(1,0)} &= T_H \oplus \{\alpha(0, s) - \alpha^{-1}(0, s)^\gamma : s \in S\}, \\
\mathcal{K}_{(0,1)} &= \{(x, 0) + \alpha^{-1}(x, 0)^\gamma : x \in A\}, \\
\mathcal{K}_{(1,1)} &= \{(\alpha(x, 0) - (x, 0)^\gamma : x \in A\}.
\end{align*}
\]

Note that \( \mathcal{K}_{(1,0)}, \mathcal{K}_{(0,1)} \) and \( \mathcal{K}_{(1,1)} \) are vector spaces isomorphic to \( A \).
For ease of notation, write
\[
\begin{align*}
e_1(x) &= (x, 0) + \alpha^{-1}(x, 0)^{-} \in \mathcal{K}_{(0, 1)}, \\
e_2(x) &= \alpha(\bar{x}, 0) - (\bar{x}, 0)^{-} \in \mathcal{K}_{(1, 0)} \\
e_0(x) &= \frac{1}{2} \left(T_{(x+\bar{x})} + \alpha(0, x - \bar{x}) - \alpha^{-1}(0, x - \bar{x})^{-}\right) \in \mathcal{K}_{(1, 1)}.
\end{align*}
\]
Then the bracket in \(\mathcal{K}\) gives, for any \(x, y \in A\):
\[
[e_1(x), e_2(y)] = \left[ (x, 0) + \alpha^{-1}(x, 0)^{-}, \alpha(y, 0) - (\bar{y}, 0)^{-} \right] \\
= \alpha(0, xy - \bar{y}\bar{x}) - \alpha^{-1}(0, xy - \bar{y}\bar{x})^{-} - V_{x, \bar{y}} - V_{\bar{y}, x} \\
= \alpha(0, xy - \bar{y}y, 0) - \alpha^{-1}(0, xy - \bar{y}y)^{-} - T_{(xy+\bar{y}y)} \\
= -2\epsilon_0(\bar{xy}),
\]
as \(V_{x, \bar{y}} + V_{\bar{y}, x})(z) = (x\bar{y} + y\bar{x})z = T_{(xy+\bar{y}y)}(z).
Also, for any \(a \in H (= \{ x \in A : \bar{x} = x \})\), \(x \in A\) and \(s \in S\), \([T_a, (x, 0)^{-}] = (ax, 0)^{-}\), while \([T_a, (x, 0)^{-}] = -(ax, 0)^{-}\), and
\[
[\alpha(0, s) - \alpha^{-1}(0, s)^{-}, (x, 0)] = \alpha^{-1}[(x, 0), (0, s)] = -\alpha^{-1}(sx, 0)^{-},
\]
while
\[
[\alpha(0, s) - \alpha^{-1}(0, s)^{-}, (x, 0)^{-}] = \alpha([(0, s), (x, 0)^{-}] = \alpha(sx, 0).
\]
Therefore,
\[
[e_0(x), e_1(y)] = \frac{1}{2} \left[T_{(x+\bar{x})} + \alpha(0, x - \bar{x}) - \alpha^{-1}(0, x - \bar{x})^{-}, (y, 0) + \alpha^{-1}(y, 0)^{-}\right] \\
= \frac{1}{2} \left( (x + \bar{x})y, 0) - \alpha^{-1}((x + \bar{x})y, 0)^{-} \\
- \alpha^{-1}((x - \bar{x})y, 0)^{-} + ((x - \bar{x})y, 0) \right) \\
= (xy, 0) - \alpha^{-1}(xy, 0)^{-} \\
= \alpha^{-1}\epsilon_2(\bar{xy}),
\]
and
\[
[e_2(x), e_0(y)] = -\frac{1}{2} \left[T_{(y+\bar{y})} + \alpha(0, y - \bar{y}) - \alpha^{-1}(0, y - \bar{y})^{-}, \alpha(\bar{x}, 0) - (\bar{x}, 0)^{-}\right] \\
= -\frac{1}{2} \left( \alpha((y + \bar{y})\bar{x}, 0) + ((y + \bar{y})\bar{x}, 0)^{-} \\
- ((y - \bar{y})\bar{x}, 0)^{-} - \alpha((y - \bar{y})\bar{x}, 0) \right) \\
= -\alpha(\bar{y}\bar{x}, 0) + (\bar{y}\bar{x}, 0)^{-} \\
= -\alpha\epsilon_1(\bar{xy}).
\]
That is,
\[
[e_0(x), e_1(y)] = \alpha^{-1}\epsilon_2(\bar{xy}), \\
[e_1(x), e_2(y)] = -2\epsilon_0(\bar{xy}), \\
[e_2(x), e_0(y)] = -\alpha\epsilon_1(\bar{xy}).
\]
Lemma 4.3. The linear map
\[\psi : \mathcal{K}_{(0,0)} \longrightarrow \mathfrak{trt}(A,^\perp)\]
\[p \mapsto (\delta_0(p), \delta_1(p), \delta_2(p)),\]
where \(\delta_i(p)\) is determined by
\[\left[p, \epsilon_i(x)\right] = \epsilon_i(\delta_i(p)(x)),\]
is a one-to-one Lie algebra homomorphism with image \(\mathfrak{l}(A,^\perp, \partial)\).

Proof. First of all, \(\psi\) is well defined since the Jacobi identity shows that, for any \(p \in \mathcal{K}_{(0,0)}, x, y \in A\) and \(i \in \mathbb{Z}_3, \)
\[\left[p, \left[\epsilon_i+1(x), \epsilon_i+2(y)\right]\right]
= \left[\left[p, \epsilon_i+1(x)\right], \epsilon_i+2(y)\right] + \left[p, \left[\epsilon_i+1(x), \epsilon_i+2(y)\right]\right]
= \left[\epsilon_i+1(\delta_i+1(p)(x)), \epsilon_i+2(y)\right] + \left[\epsilon_i+1(x), \epsilon_i+2(\delta_i+2(p)(y))\right]
= \xi_i \epsilon_i(\delta_i+1(p)(x)y + x \delta_i+2(p)(y)),\]
where \(\xi_0 = -2, \xi_1 = -\alpha, \xi_2 = \alpha^{-1}\) by (4.3), but also
\[\left[p, \left[\epsilon_i+1(x), \epsilon_i+2(y)\right]\right] = \left[p, \xi_i \epsilon_i(x\overline{y})\right] = \xi_i \epsilon_i(\delta_i(p)(x\overline{y})),\]
so
\[\overline{\delta_i(p)}(xy) = \delta_i+1(p)(x)y + x \delta_i+2(p)(y).\]
The fact that \(\psi\) is a Lie algebra homomorphism is clear.

Besides, \(\psi(0) = 0^{<3>}, \) and for any \(s \in S\) and \(x \in A, \)
\[\left[T_s, \epsilon_1(x)\right] = \left[T_s, (x, 0) + \alpha^{-1}(x, 0)^\perp\right] = (T_s(x), 0) + \alpha^{-1}(T_s(x), 0)^\perp
= \epsilon_1(T_s(x)),\]
\[\left[T_s, \epsilon_2(x)\right] = \left[T_s, \alpha(\bar{x}, 0) - (\bar{x}, 0)^\perp\right] = \alpha(T_s(\bar{x}), 0) - (T_s(\bar{x}), 0)^\perp
= \alpha(T_s(x), 0) - (T_s(x), 0) = \epsilon_2(T_s(x)).\]

(Note that \(T_s = L_s + 2R_s, \) so \(\overline{T_s} = -(R_s + 2L_s).\)

From Lemma 4.4, we know that \((-R_s, L_s + R_s, -L_s)\) and \((-L_s, -R_s, L_s + R_s)\) belong to \(\mathfrak{trt}(A,^\perp)\), and so does their difference \((L_s - R_s, T_s, \overline{T_s})\). Also, since any element \((d_0, d_1, d_2)\) in \(\mathfrak{trt}(A,^\perp)\) is determined by the pair \((d_1, d_2)\), it follows that
\[\psi(T_s) = (L_s - R_s, T_s, \overline{T_s}).\]

Moreover,
\[\left[\alpha(0, s) + \alpha^{-1}(0, s)^\perp, \epsilon_1(x)\right] = \left[\alpha(0, s) + \alpha^{-1}(0, s)^\perp, (x, 0) + \alpha^{-1}(x, 0)^\perp\right]
= \left[[0, s], (x, 0)^\perp\right] + \alpha^{-1}\left[[0, s], (0, s)^\perp\right]
= (sx, 0) + \alpha^{-1}(sx, 0)^\perp = \epsilon_1(sx),\]
\[\left[\alpha(0, s) + \alpha^{-1}(0, s)^\perp, \epsilon_2(x)\right] = \left[\alpha(0, s) + \alpha^{-1}(0, s)^\perp, \alpha(\bar{x}, 0) - (\bar{x}, 0)^\perp\right]
= -\alpha\left[[0, s], (\bar{x}, 0)^\perp\right] - \left[[\bar{x}, 0], (0, s)^\perp\right]
= -\alpha(s\bar{x}, 0) + (s\bar{x}, 0)^\perp = \epsilon_2(xs),\]
whence we conclude that
\[\psi(\alpha(0, s) + \alpha^{-1}(0, s)^\perp) = -(L_s + R_s), L_s, R_s).\]
Therefore, for any $s_0, s_1, s_2 \in S$ with $s_0 + s_1 + s_2 = 0$, 
\[
(L_{s_1} - R_{s_2}, L_{s_2} - R_{s_0}, L_{s_0} - R_{s_1})
\]
\[
= \frac{1}{2} \left( \psi(T_{s_1+s_2}) - \psi(\alpha(0, s_1 - s_2) + \alpha^{-1}(0, s_1 - s_2)^{-1}) \right),
\]
so that
\[
\psi \left( T_S \oplus \{ \alpha(0, s) + \alpha^{-1}(0, s)^{-1} : s \in S \} \right) = T_S,
\]
and hence $\psi$ gives an isomorphism onto $\mathfrak{l}(A, -) = \mathfrak{d}^{<3\gamma}_3 \oplus T_S$. □

Let us recall now Allison and Faulkner’s construction of the Lie algebra $\mathcal{K}(A, -, \gamma, \mathfrak{v})$, for any structurable algebra $(A, \gamma)$, $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in F^3$ and subalgebra $\mathfrak{v}$ of $\mathfrak{gl}(A, -)$ containing $\mathfrak{l}(A, -)$. As a vector space, $\mathcal{K}(A, -, \gamma, \mathfrak{v}) = \mathfrak{v} \oplus A[12] \oplus A[23] \oplus A[31]$, where $A[ij]$ is a copy of $A$ with $a[ij] = -\gamma_i \gamma_j^{-1} \tilde{a}[ji]$ for any $1 \leq i \neq j \leq 3$, and the multiplication is obtained by extending the bracket in $\mathfrak{v}$ by setting for any $a, b \in A$
\[
[a[ij], b[jk]] = ab[ik],
\]
\[
[T, a[ij]] = T_k(a)[ij], \text{ for } T \in \mathfrak{v}
\]
\[
[a[ij], b[ij]] = \gamma_i \gamma_j^{-1} T,
\]
where $(i, j, k)$ is a cyclic permutation of $(1, 2, 3)$ and the $T$ in the last row is $T = (T_1, T_2, T_3)$ with
\[
T_i = L_\delta L_a - L_a L_\delta,
\]
\[
T_j = R_\delta R_a - R_a R_\delta,
\]
\[
T_k = R_{(\delta b - ba)} + L_\delta L_a - L_a L_\delta,
\]
(Compare with Example 3.1)

**Proposition 4.4.** Let $(A, -)$ be a structurable algebra and let $\mathfrak{d}$ be a subalgebra of $\mathfrak{gl}(A, -)$ containing the inner derivations. Then $\mathcal{K}(A, -, \mathfrak{d})$ is isomorphic to $\mathcal{K}(A, -, \gamma, \mathfrak{l}(A, -, \mathfrak{d}))$, with $\gamma = (1, -1, 2\alpha)$.

**Proof.** Consider the following elements in $\mathcal{K}(A, -, \mathfrak{d})$:
\[
\tilde{\epsilon}_0(x) = \epsilon_0(x), \quad \tilde{\epsilon}_1(x) = \frac{1}{2} \epsilon_1(x), \quad \tilde{\epsilon}_2(x) = -\epsilon_2(x),
\]
for any $x \in A$. Then (4.12) becomes
\[
[\tilde{\epsilon}_0(x), \tilde{\epsilon}_1(y)] = -\frac{1}{2\alpha} \tilde{\epsilon}_2(xy),
\]
\[
[\tilde{\epsilon}_1(x), \tilde{\epsilon}_2(y)] = \tilde{\epsilon}_0(xy),
\]
\[
[\tilde{\epsilon}_2(x), \tilde{\epsilon}_0(y)] = 2\alpha \tilde{\epsilon}_1(xy).
\]
Note that for $\gamma = (1, -1, 2\alpha), -\frac{1}{2\alpha} = -\gamma_1 \gamma_3^{-1}, 1 = -\gamma_2 \gamma_1^{-1}$ and $2\alpha = -\gamma_3 \gamma_2^{-1}$. Now, the isomorphism $\psi : \mathcal{K}(0, 0) \rightarrow \mathfrak{l}(A, -, \mathfrak{d})$, can be extended to an isomorphism $\Psi : \mathcal{K}(A, -, \mathfrak{d}) \rightarrow \mathcal{K}(A, -, \gamma, \mathfrak{l}(A, -, \mathfrak{d}))$ by means of
\[
\Psi(\tilde{\epsilon}_0(x)) = x[12], \quad \Psi(\tilde{\epsilon}_1(x)) = x[23], \quad \Psi(\tilde{\epsilon}_2(x)) = x[31],
\]
while, for any $p \in \mathcal{K}(0, 0), \Psi(p) = (d_1, d_2, d_0)$ if $\psi(p) = (d_0, d_1, d_2)$. The only difficulty in proving that $\Psi$ is an isomorphism lies in proving that $\Psi[\tilde{\epsilon}_i(x), \tilde{\epsilon}_j(y)] = [\Psi(\tilde{\epsilon}_i(x)), \Psi(\tilde{\epsilon}_j(y))]$ for any $i \in \mathbb{Z}_3$, and $x, y \in A$. But the
action of both sides on $\Psi(\bar{e}_{i+1}(A) \oplus \bar{e}_{i+2}(A))$ coincide, while any element in $\mathfrak{r}(A, -)$ is determined by its action on two of the direct summands in $A[12] \oplus A[23] \oplus A[31] = \Psi(\bar{e}_0(A) \oplus \bar{e}_1(A) \oplus \bar{e}_2(A))$. \hfill \Box

**Corollary 4.5.** Let $(A, -)$ be a structurable algebra over a field $F$ satisfying that $-1 \in F^2$, and let $\mathfrak{d}$ be a subalgebra of $\mathfrak{d}(A, -)$ containing the inner derivations. Then, $\text{Aut}(\mathcal{K}(A, -, \mathfrak{d}))$ contains a subgroup isomorphic to the symmetric group of degree 4.

**Proof.** Take $\alpha = 2$ above and define

$$
i_0(x) = \sqrt{-1}\bar{e}_0(x), \quad \iota_1(x) = \sqrt{-1}\bar{e}_1(x), \quad \iota_2(x) = \frac{1}{2} \bar{e}_2(x),$$

for any $x \in A$. Then (4.2) becomes

$$(\iota_i(x), \iota_{i+1}(y) = \iota_{i+2}(xy)$$

for any $x, y \in A$ and $i \in \mathbb{Z}_3$. With the arguments of the last proof, it is readily seen that $\mathcal{K}(A, -, \mathfrak{d})$ is isomorphic to $\mathcal{K}(A, -, \gamma, I(A, -, \mathfrak{d}))$, where $\gamma = (-1, -1, -1)$. Now, besides the automorphisms $\tau_1, \tau_2$ used to obtain the grading over $\mathbb{Z}_2 \times \mathbb{Z}_2$, there appears the order 3 automorphism $\phi$ such that

$$\begin{aligned}
\phi(\iota_i(x)) &= \iota_{i+1}(x), \\
\phi(\psi^{-1}(d_0, d_1, d_2)) &= \psi^{-1}(d_2, d_0, d_1)
\end{aligned}$$

and the order two automorphism $\tau$ such that

$$\begin{aligned}
\tau(\iota_0(x)) &= -\iota_0(x), \\
\tau(\iota_1(x)) &= -\iota_2(x), \\
\tau(\iota_2(x)) &= -\iota_1(x), \\
\tau(\psi^{-1}(d_0, d_1, d_2)) &= \psi^{-1}(\bar{d}_0, \bar{d}_2, \bar{d}_1),
\end{aligned}$$

with $x \in A$, $i \in \mathbb{Z}_3$ and $(d_0, d_1, d_2) \in I(A, -, \mathfrak{d})$. These automorphisms generate a subgroup of the automorphism group isomorphic to $S_4$. \hfill \Box

It must be remarked that Allison proved in [All91] Theorem 2.2 and Section 4] that if $-\gamma_1\gamma_2^{-1} \in (F^\times)^2$, then $\mathcal{K}(A, -, \gamma, I(A, -, \mathfrak{d}))$ is isomorphic to $\mathcal{K}(A, -, \mathfrak{d})$. In particular, $\mathcal{K}(A, -, \mathfrak{d})$ is isomorphic to $\mathcal{K}(A, -, (1, -1, 1), I(A, -, \mathfrak{d}))$.

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