GENERALIZED STRESS CONCENTRATION FACTORS FOR EQUILIBRATED FORCES AND STRESSES

REUVEN SEGEV

Abstract. As a sequel to a recent work we consider the generalized stress concentration factor, a purely geometric property of a body that for the various loadings, indicates the ratio between the maximum of the optimal stress and maximum of the loading fields. The optimal stress concentration factor pertains to a stress field that satisfies the principle of virtual work and for which the stress concentration factor is minimal. Unlike the previous work, we require that the external loading be equilibrated and that the stress field be a symmetric tensor field.

Keywords. Continuum mechanics, forces, stresses, stress concentration factor, trace, integrable deformations.

1. Introduction

In a recent article [7] we introduced the notion of a generalized stress concentration factor as a quantitative measure of how bad is the geometry of a body in terms of the ratio between the maximal stresses and the maximum of the applied loads. Specifically, generalized stress concentration factors may be described as follows. Let $F$ be a force on a body $\Omega$ that is given in terms of a body force field $b$ and a surface force field $t$ and let $\sigma$ be any stress field that is in equilibrium with $F$. Then, the stress concentration factor for the pair $F, \sigma$ is given by

$$K_{F,\sigma} = \sup_{x} \{|\sigma(x)|\} / \sup_{x,y} \{|b(x)|, |t(y)|\}, \quad x \in \Omega, \ y \in \partial\Omega.$$  

Here, for $|\sigma(x)|$ we use some norm $|\cdot|$ on the space of stresses at a point—a finite dimensional space. We note that since we do not specify a constitutive relation, for each force $F$ there is a class $\Sigma_{F}$ of stress fields $\sigma$ that are in equilibrium with $F$.

Next, the optimal stress concentration factor for the force $F$ is defined by

$$K_{F} = \inf_{\sigma \in \Sigma_{F}} \{K_{F,\sigma}\},$$

i.e., it is the least stress concentration factor when we allow the stress field to vary over all fields that are in equilibrium with $F$. Finally, the generalized stress concentration factor $K$—a purely geometric property of $\Omega$—is defined by

$$K = \sup_{F} \{K_{F}\} = \sup_{F} \inf_{\sigma \in \Sigma_{F}} \{\sup_{x} \{|\sigma(x)|\}\} / \sup_{x,y} \{|b(x)|, |t(y)|\},$$

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Department of Mechanical Engineering, Ben-Gurion University, Beer Sheva 84105, Israel,
E-mail: rsegev@bgu.ac.il, Fax: 972-8-6472814, Phone: 972-8-6477043.
where $F$ varies over all forces that may be applied to the body. Thus, the generalized stress concentration factor reflects the worst case of loading of the body.

It was shown in [7] that the generalized stress concentration factor is equal to the norm of a mapping associated with the trace operator of Sobolev mappings. Specifically, it was shown that when suprema in the expressions above are replaced by essential suprema, then,

$$K = \sup_{\phi \in W^1_1(\Omega, \mathbb{R}^3)} \frac{\int_{\Omega} |\phi| \, dV + \int_{\partial \Omega} |\hat{\phi}| \, dA}{\int_{\Omega} |\phi| \, dV + \int_{\partial \Omega} |\nabla \phi| \, dV},$$

where $W^1_1(\Omega, \mathbb{R}^3)$ is the Sobolev space of integrable vector fields $\phi$ on $\Omega$ whose gradients $\nabla \phi$ are also integrable, and $\hat{\phi}$ is the trace of $\phi \in W^1_1(\Omega, \mathbb{R}^3)$ on $\partial \Omega$ (whose existence is a basic property of Sobolev spaces).

Consider the Radon measure $\mu$ on $\overline{\Omega}$ defined by $\mu(D) = V(D \cap \Omega) + A(D \cap \partial \Omega)$ ($V$ and $A$ are the volume and area measures, respectively), and let $L^{1,\mu}(\overline{\Omega}, \mathbb{R}^3)$ be the space of fields on $\overline{\Omega}$ that are integrable relative to $\mu$ equipped with the $L^{1,\mu}$-norm so

$$\|w\|_{L^{1,\mu}} = \int_{\Omega} |w| \, dV + \int_{\partial \Omega} |w| \, dA.$$

Then, the trace operator induces an extension mapping $\delta : W^1_1(\Omega, \mathbb{R}^3) \to L^{1,\mu}(\overline{\Omega}, \mathbb{R}^3)$ and the expression for the generalized stress concentration factor above may be written in the form

$$K = \|\delta\|$$

—the basic result of [7].

The treatment in [7] allows stresses and forces that are more general than those treated usually in continuum mechanics. In addition to the usual stress tensor $\sigma_{im}$ the stress object contains a self force field $\sigma_i$. Furthermore, the stress field need not be symmetric and the resultants and total torques due to the forces $F$ need not vanish. The generalized form of the equilibrium equations between the forces and stresses was taken in the form

$$\int_{\Omega} b_i w_i \, dV + \int_{\partial \Omega} t_i w_i \, dA = \int_{\Omega} \sigma_{i} w_i \, dV + \int_{\Omega} \sigma_{ik} w_{i,k} \, dV.$$

Thus, the infimum in the definition of the optimal stress concentration factor may be attained for a stress field that is not admissible physically.

In the present work we restrict the admissible stress fields to symmetric tensor fields and the forces are required to have zero resultants and total torques. These requirements are well known to be equivalent to the requirements that the power produced by the forces and stresses on rigid velocity fields vanishes.

The expression for the generalized stress concentration factor we obtain here for the rigid velocity invariant forces and stresses may be written as

$$K = \|\delta/\omega\|,$$
where \( \mathcal{R} \) denotes the collection of rigid velocity fields, a subspace of the function-spaces we are considering. The extension mapping

\[
\delta|\mathcal{R} : LD(\Omega)|\mathcal{R} \rightarrow L^1(\Omega, \mathbb{R}^3)/\mathcal{R}
\]

between the corresponding quotient spaces is given by \( \delta|\mathcal{R}(w) = [\delta(w)] \). It is well defined for elements of the space \( LD(\Omega) \) containing the vector fields \( w \) of integrable stretchings

\[
\varepsilon(w) = \frac{1}{2}(\nabla w + (\nabla w)^T).
\]

The space \( LD(\Omega) \) and its properties (see [11, 12, 8, 10, 1, 2], and [4] for nonlinear strains) are the main technical tools we use in this work.

For a projection mapping that gives an approximating rigid velocity field to any vector field \( w \) and a corresponding \( w_0 \) that has zero rigid component, this result may be written more specifically as

\[
K = \|\delta_0\| = \sup_{w_0 \in LD(\Omega)_0} \left\{ \int_{\Omega} \sum_i |w_{0i} - r_i| \, dV + \int_{\partial\Omega} \sum_i |w_{0i} - r_i| \, dA \right\} \div \frac{1}{2} \int_{\Omega} \sum_{i,m} |w_{0i,m} + w_{0m,i}| \, dV.
\]

Here \( \delta_0 \) is the extension mapping for vector fields having zero rigid components and \( LD(\Omega)_0 \) is the space of vector fields in \( LD(\Omega) \) having zero rigid components.

Section 2 presents some properties of rigid velocity fields, stretchings and the approximations of velocity fields by rigid ones. Section 3 outlines the definitions and results pertaining to the space \( LD(\Omega) \) and is based on [11]. Finally, Section 4 applies the properties of \( LD \)-fields to the problem under consideration. Some details regarding the notation we use and results on normed spaces and their normed dual spaces are available in [7].

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2. Preliminaries on Stretchings and Rigid Velocities

2.1. Basic definitions. Let \( \Omega \) be an open and bounded 3-dimensional submanifold of \( \mathbb{R}^3 \) with volume \( \|\Omega\| \) having a differentiable boundary and \( w \) a vector field over \( \Omega \). We set \( \varepsilon(w) \) to be the tensor field

\[
\varepsilon(w)_{im} = \frac{1}{2}(w_{i,m} + w_{m,i}),
\]

i.e., the symmetric part of the gradient. As \( w \) is interpreted physically as a velocity field over the body, \( \varepsilon(w) \) is interpreted as the stretching. Alternatively, if \( w \) is interpreted as an infinitesimal displacement field, \( \varepsilon(w) \) is the corresponding linear strain. In the sequel we will refer to \( \varepsilon(w) \) as the stretching associated with \( w \). Here, the partial derivatives are interpreted as the distributional derivatives so one need not care about the regularity of \( w \).

We identify the space of symmetric \( 3 \times 3 \) matrices with \( \mathbb{R}^6 \). For a symmetric tensor field \( \varepsilon \) whose components are integrable functions we use the \( L^1 \)-norm

\[
\|\varepsilon\| = \sum_{i,m} \|\varepsilon_{im}\|_{L^1}.
\]
This norm maybe replaced by other equivalent norms (possibly norms invariant under coordinate transformations). Thus, the space of $L^1$-stretching fields is represented by $L^1(\Omega, \mathbb{R}^6)$ with the $L^1$-norm as above.

A vector field $w$ on $\Omega$ is of integrable stretching if its components are integrable and if each component $\varepsilon(w)_{im} \in L^1(\Omega)$. It can be shown that this definition is coordinate independent. The vector space of velocity fields having integrable stretchings will be denoted by $LD(\Omega)$. This space is normed by

$$
\|w\|_{LD} = \sum_i \|w_i\|_{L^1} + \sum_{i,m} \|\varepsilon(w)_{im}\|_{L^1}.
$$

Clearly, we have a continuous linear inclusion $LD(\Omega, \mathbb{R}^3) \rightarrow L^1(\Omega, \mathbb{R}^3)$. In addition, $w \mapsto \varepsilon(w)$ is given by a continuous linear mapping

$$
\varepsilon : LD(\Omega) \rightarrow L^1(\Omega, \mathbb{R}^6).
$$

2.2. The subspace of rigid velocities. A rigid velocity (or displacement) field is of the form

$$
w(x) = a + \omega \times x, \quad x \in \Omega
$$

where $a$ and $\omega$ are fixed in $\mathbb{R}^3$ and $\omega \times x$ is the vector product. We can replace $\omega \times x$ with $\hat{\omega}(x)$ where $\hat{\omega}$ is the associated skew symmetric matrix so $w(x) = a + \hat{\omega}(x)$.

We will denote the 6-dimensional space of rigid body velocities by $\mathcal{R}^6$. For a rigid motion

$$
\hat{\omega}_{im} = \frac{1}{2}(w_{i,m} - w_{m,i}),
$$

an expression that is extended to the non-rigid situation and defines the vorticity vector field so $w_{i,m} = \varepsilon(w)_{im} + \hat{\omega}_{im}$.

Considering the kernel of the stretching mapping $\varepsilon : LD(\Omega) \rightarrow L^1(\Omega, \mathbb{R}^6)$, a theorem whose classical version is due to Liouville states (see [11, pp. 18–19]) that Kernel $\varepsilon = \mathcal{R}^6$.

2.3. Approximation by rigid velocities. We now wish to consider the approximation of a velocity field by a rigid velocity field. Let $\rho$ be a Radon measure on $\mathcal{M}$ and $1 \leq p < \infty$. For a given $w \in L^{p,\rho}(\mathcal{M}, \mathbb{R}^3)$, we wish to find the rigid velocity $r$ for

$$
\inf_{r' \in \mathcal{R}} \left\{ \|w - r'\|_{L^p,\rho} \right\} = \inf_{r' \in \mathcal{R}} \int \sum_i |w_i - r'_i|^p \, d\rho
$$

is attained. Thus we are looking for vectors $a$ and $b$ that minimize

$$
e = \int \sum_i |w_i - a_i - \varepsilon_{ijk} b_j x_k|^p \, d\rho.
$$

We have

$$
\frac{\partial e}{\partial a_l} = \int \sum_i |w_i - a_i - \varepsilon_{ijk} b_j x_k|^{p-1} \frac{(w_i - a_i - \varepsilon_{ijk} b_j x_k)}{|w_i - a_i - \varepsilon_{ijk} b_j x_k|} (-\delta_{il}) \, d\rho,
$$

$$
\frac{\partial e}{\partial b_l} = \int \sum_i |w_i - a_i - \varepsilon_{ijk} b_j x_k|^{p-1} \frac{(w_i - a_i - \varepsilon_{ijk} b_j x_k)}{|w_i - a_i - \varepsilon_{ijk} b_j x_k|} (-\varepsilon_{ijl} x_k) \, d\rho.
$$
and we obtain the 6 equations for the minimum with the 6 unknowns $a_l, b_m$

\[
0 = \int_{\Omega} |w_l - a_l - \varepsilon_{ijk} b_j x_k|^p - 2 (w_l - a_l - \varepsilon_{ijk} b_j x_k) d\rho,
\]

\[
0 = \int_{\Omega} \sum_i |w_i - a_i - \varepsilon_{ijk} b_j x_k|^p - 2 (w_i - a_i - \varepsilon_{ijk} b_j x_k) \varepsilon_{ilk} x_k d\rho.
\]

Particularly simple are the equations for $p = 2$. In this case we obtain

\[
\int_{\Omega} w d\rho = \int_{\Omega} r d\rho, \quad \text{and} \quad \int_{\Omega} x \times w d\rho = \int_{\Omega} x \times r d\rho.
\]

If we interpret $\rho$ as a mass distribution on $\Omega$, these two conditions simply state that the best rigid velocity approximations should give the same momentum and angular momentum as the original field.

Of particular interest (see [11, p. 120]) is the case where $\rho$ is the volume measure on $\Omega$. Set $\bar{x}$ to be the center of volume of $\Omega$, i.e.,

\[
\bar{x} = \frac{1}{|\Omega|} \int_{\Omega} x dV.
\]

Without loss of generality we will assume that $\bar{x} = 0$ (for else we may replace $x$ by $x - \bar{x}$ in the sequel).

Let $\bar{w}$ be the mean of the field $w$ and $I$ the inertia matrix relative to the center of volume, so

\[
\bar{w} = \frac{1}{|\Omega|} \int_{\Omega} w dV, \quad I_{im} = \int_{\Omega} (x_k x_k \delta_{im} - x_i x_m) dV
\]

and

\[
I(\omega) = \int_{\Omega} x \times (\omega \times x) dV.
\]

The inertia matrix is symmetric and positive definite and so the solution for $r$ gives

\[
r = \bar{w} + \omega \times x
\]

with $\bar{w}$ as above and

\[
\omega = I^{-1} \left( \int_{\Omega} x \times w dV \right).
\]

Thus, $w \mapsto (\bar{w} + \omega \times x)$, with $\bar{w}$ and $\omega$ as above, is well defined for integrable velocity fields and we obtain a mapping

$$\pi_{\mathcal{A}} : L^1(\Omega, \mathbb{R}^3) \to \mathcal{R}.$$ 

It is straightforward to show that $\pi_{\mathcal{A}}$ is indeed a linear projection onto $\mathcal{R}$.

Also of interest below will be the case where $p = 1$ and and the measure $\rho$ is given by

\[
\rho(D) = \mu(D) = V(D \cap \Omega) + A(D \cap \partial \Omega),
\]
as in Section 1. The conditions for best approximations \( r = a + b \times x \) assume the form

\[
\int_{\Omega} (\omega_i - a_i - \varepsilon_{ijk}b_jx_k) \, dV + \int_{\partial\Omega} (\omega_i - a_i - \varepsilon_{ijk}b_jx_k) \, dA = 0,
\]

\[
\int_{\Omega} \sum_i (\omega_i - a_i - \varepsilon_{ijk}b_jx_k) \, \varepsilon_{ijk}x_k \, dV + \int_{\partial\Omega} \sum_i (\omega_i - a_i - \varepsilon_{ijk}b_jx_k) \, \varepsilon_{ijk}x_k \, dA = 0,
\]

where \( z/|z| \) is taken as 0 for \( z = 0 \). (For an analysis of \( L^1 \)-approximations see [9] and reference cited therein.)

2.4. Distortions. Let \( W \) be a vector space of velocities on \( \Omega \) containing the rigid velocities \( \mathcal{R} \) and let \( w_1 \) and \( w_2 \) be two velocity fields in \( W \). We will say that the two have the same distortion if \( w_2 = w_1 + r \) for some rigid motion \( r \in \mathcal{R} \). This clearly generates an equivalence relation on \( W \) and the corresponding quotient space \( W/\mathcal{R} \) will be referred to as the space of distortions. If \( \chi \) is an element of \( W/\mathcal{R} \) then \( \varepsilon(w) \) is the same for all members of \( w \in \chi \). The natural projection

\[ \pi: W \to W/\mathcal{R} \]

associates with each element \( w \in W \) its equivalence class \( [w] = \{ w + r | r \in \mathcal{R} \} \).

If \( w \) is a normed space, then, the induced norm on \( W/\mathcal{R} \) is given by (see Appendix A)

\[ ||[w]|| = \inf_{w' \in [w]} ||w'|| = \inf_{r \in \mathcal{R}} ||w - r||. \]

Thus, the evaluation of the norm of a distortion, is given by the best approximation by a rigid velocity as described above.

Let \( W \) be a vector space of velocities contained in \( L^1(\Omega, \mathbb{R}^3) \), then, \( \pi_{\mathcal{R}} \) defined above induces an additional projection

\[ \pi_0(w) = w - \pi_{\mathcal{R}}(w). \]

The image of \( \pi_0 \) is the kernel \( W_0 \) of \( \pi_{\mathcal{R}} \) and it is the subspace of \( W \) containing velocity fields having zero approximating rigid velocities. Clearly, we have a bijection \( \beta: W/\mathcal{R} \to W_0 \). On \( W_0 \) we have two equivalent norms: the norm it has as a subspace of \( W \) and the norm that makes the bijection \( \beta: W/\mathcal{R} \to W_0 \) an isometry.

With the projections \( \pi_0 \) and \( \pi_{\mathcal{R}} \), \( W \) has a Whitney sum structure \( W = W_0 \oplus \mathcal{R} \).

2.5. Equilibrated forces. Let \( W \) be a vector space of velocities (we assume that it contains the rigid velocities). A force \( F \in W^* \) is equilibrated if \( F(r) = 0 \) for all \( r \in \mathcal{R} \). This is of course equivalent to \( F(w) = F(w + r) \) for all \( r \in \mathcal{R} \) so \( F \) induces a unique element of \( (W/\mathcal{R})^* \). Conversely, any element of \( G \in (W/\mathcal{R})^* \) induces an equilibrated force \( F \) by \( F(w) = G([w]) \), where \([w]\) is the equivalence class of \( w \). In other words, as the quotient projection is surjective, the dual mapping \( \pi^*: (W/\mathcal{R})^* \to W^* \) is injective and its image—the collection of equilibrated forces—is orthogonal to the kernel of \( \pi \). Furthermore, as in Appendix A \( \pi^* \) is norm preserving. Thus, we may identify the collection of equilibrated forces in \( W^* \) with \( (W/\mathcal{R})^* \).

If \( i_{\mathcal{R}}: \mathcal{R} \to W \) is the inclusion of the rigid velocities, then,

\[ i_{\mathcal{R}}^*: W^* \to \mathcal{R}^*, \]

is a continuous and surjective mapping. The image \( i_{\mathcal{R}}^*(F) \) will be referred to as the total of the force. In particular, its component dual to \( \bar{w} \) will be referred to as
the force resultant and the component dual to \( \omega \) will be referred to as the resultant torque. Thus, in particular, the resultant force and torque vanish for an equilibrated force. This structure may be illustrated by the sequences

\[
\begin{align*}
0 & \longrightarrow R \xrightarrow{\iota_R} W \xrightarrow{\pi} W/\mathcal{R} \longrightarrow 0 \\
0 & \leftarrow \iota_{R^*} \xleftarrow{} W^* \xleftarrow{\pi^*} (W/\mathcal{R})^* \leftarrow 0.
\end{align*}
\]

Using the projection \( \pi_\mathcal{R} \) and the Whitney sum structure it induces we have a Whitney sum structure \( W^* = W_0^* \oplus \mathcal{R}^* \) and it is noted that the norm on \( W_0^* \) is implied by the choice of norm on \( W_0 \).

3. Fields of Integrable Stretchings

In this section we list the basic properties of vector fields of integrable stretching (or deformation) as in [11] (see also [12, 8, 10, 1, 2] and [4] for nonlinear strains). The presentation below is adapted to the application we consider and is not necessarily the most general.

If both \( w \) and \( \varepsilon(w) \) are in \( L^p \) for \( 1 < p < \infty \), the Korn inequality (see [3]) implies that \( w \in W_1^1(\Omega) \). This would imply in particular that \( w \) has a trace on the boundary of \( \Omega \). However, as shown by Ornstein [5], \( w \) need not necessarily be in \( W_1^1(\Omega, \mathbb{R}^3) \) for the critical value \( p = 1 \). Nevertheless, the theory of integrable stretchings shows that the trace is well defined even for \( p = 1 \).

3.1. Definition. We recall that \( LD(\Omega) \) is the vector space of fields with integrable stretchings. With the norm

\[
\|w\|_{LD} = \sum_i \|w_i\|_{L^1} + \sum_{i,m} \|\varepsilon(w)_{im}\|_{L^1}
\]

\( LD(\Omega) \) is a Banach space.

3.2. Approximation. \( C^\infty(\overline{\Omega}, \mathbb{R}^3) \) is dense in \( LD(\Omega) \).

3.3. Traces. The trace operator can be extended from \( W_1^1(\Omega, \mathbb{R}^3) \) onto \( LD(\Omega, \mathbb{R}^3) \).

Thus, there is a unique continuous linear mapping

\[
\gamma: LD(\Omega) \longrightarrow L^1(\partial\Omega, \mathbb{R}^3)
\]

such that \( \gamma(w) = w|_{\partial\Omega} \), for every field \( w \) of bounded stretching that is a restriction to \( \Omega \) of a continuous field on the closure \( \overline{\Omega} \). Thus, the norm of the trace mapping is given by

\[
\|\gamma\| = \sup_{w \in LD(\Omega)} \frac{\|\gamma(w)\|_{L^1}}{\|w\|_{LD}}.
\]

As a result of the approximation of fields of bounded stretchings by smooth vector fields on \( \overline{\Omega} \), \( \|\gamma\| \) may be evaluated using smooth vector fields in the expression above, i.e.,

\[
\|\gamma\| = \sup_{w \in C^\infty(\overline{\Omega}, \mathbb{R}^3)} \frac{\|w|_{\partial\Omega}\|_{L^1}}{\|w\|_{LD}}.
\]

3.4. Extensions. There is a continuous linear extension operator

\[
E: LD(\Omega) \longrightarrow LD(\mathbb{R}^3)
\]

such that \( E(w)(x) = w(x) \) for almost all \( x \in \Omega \).
3.5. **Regularity.** If \( w \) is any distribution on \( \Omega \) whose corresponding stretching is \( L^1 \), then \( w \in L^1(\Omega, \mathbb{R}^3) \).

3.6. **Distortions of integrable stretching.** On the space of \( LD \)-distortions, \( LD(\Omega)/\mathcal{R} \), we have a natural norm

\[
\|\chi\| = \inf_{w \in \chi} \|w\|_{LD}.
\]

This norm is equivalent to

\[
\|\varepsilon(\chi)\| = \sum_{i,m} \|\varepsilon(w)_{im}\|_{L^1}
\]

where \( w \) is any member of \( \chi \). Clearly, the value of this expression is the same for all members \( w \in \chi \) and we can use any other equivalent norm on the space of symmetric tensor fields.

Using the projection \( \pi_{\mathcal{R}} \) as above we denote by \( LD(\Omega)_0 \) the kernel of \( \pi_{\mathcal{R}} \) and by \( \pi_0 \) the projection onto \( LD(\Omega)_0 \) so

\[
(\pi_0, \pi_{\mathcal{R}}) : LD(\Omega) \to LD(\Omega)_0 \oplus \mathcal{R}.
\]

Then, there is a constant \( C \) depending only on \( \Omega \) such that

\[
\|\pi_0(w)\|_{L^1} = \|w - \pi_{\mathcal{R}}(w)\|_{L^1} \leq C \|\varepsilon(w)\|_{L^1}.
\]

3.7. **Equivalent norms.** Let \( p \) be a continuous seminorm on \( LD(\Omega) \) which is a norm on \( \mathcal{R} \). Then,

\[
p(w) + \|\varepsilon(w)\|_{L^1}
\]

is a norm on \( LD(\Omega) \) which is equivalent to the original norm in 3.1.

4. **Application to Equilibrated Forces and Stresses**

4.1. **LD-velocity fields and forces.** The central object we consider is \( LD(\Omega) \) whose elements are referred to as \( LD \)-velocity fields. Elements of the dual space \( LD(\Omega)^* \) will be referred to as \( LD \)-forces. Our objective is to represent \( LD \)-forces by stresses and by pairs containing body forces and surface forces.

Rather than the original norm of 3.1, it will be convenient to use an equivalent norm as asserted by 3.7 as follows. Let

\[
\pi_{\mathcal{R}} : LD(\Omega) \to \mathcal{R}
\]

be the continuous linear projection defined in Paragraph 2.3 and let \( q : \mathcal{R} \to \mathbb{R} \), be a norm on the finite dimensional \( \mathcal{R} \). Then,

\[
p = q \circ \pi_{\mathcal{R}} : LD(\Omega) \to \mathbb{R}
\]

is a continuous seminorm that is a norm on \( \mathcal{R} \subset LD(\Omega) \). It follows from 3.7 that

\[
\|w\|_{LD}' = q(\pi_{\mathcal{R}}(w)) + \|\varepsilon(w)\|_{L^1}
\]

is a norm on \( LD(\Omega) \) which is equivalent to the original norm defined in 3.1.
4.2. **LD-distortions**. With the norm $\|\cdot\|_{LD}$, the induced norm on $LD(\Omega)\mathscr{R}$ is given by

$$\|w\|_{LD}^\prime = \inf_{r \in \mathbb{R}} \|w + r\|_{LD},$$

so using $\pi_{\mathscr{R}}(r) = r$, $\varepsilon(r) = 0$ and choosing $r = -\pi_{\mathscr{R}}(w)$, we have

$$\|w\|_{LD}^\prime = \inf_{r \in \mathbb{R}} \{q(\pi_{\mathscr{R}}(w + r)) + \|\varepsilon(w + r)\|_{L^1}\}$$

$$= \inf_{r \in \mathbb{R}} \{q(\pi_{\mathscr{R}}(w) + r) + \|\varepsilon(w)\|_{L^1}\}$$

$$= \|\varepsilon(w)\|_{L^1}.$$

Let $\pi_0: LD(\Omega) \rightarrow LD(\Omega)_0$ be the projection onto $LD(\Omega)_0 \subset LD(\Omega)$, the kernel of $\pi_{\mathscr{R}}$. Then,

$$\|\pi_0(w)\|_{LD}^\prime = \|w - \pi_{\mathscr{R}}(w)\|_{LD}^\prime$$

$$= q(\pi_{\mathscr{R}}(w - \pi_{\mathscr{R}}(w))) + \|\varepsilon(w - \pi_{\mathscr{R}}(w))\|_{L^1}$$

$$= \|\varepsilon(w)\|_{L^1}.$$

We conclude that with our choice of norm $\|\cdot\|_{LD}^\prime$ on $LD(\Omega)$, the two norms in 3.6 are not only equivalent but are actually equal. Thus, this choice makes $LD(\Omega)_0$ isometrically isomorphic to $LD(\Omega)\mathscr{R}$.

4.3. **Equilibrated LD-forces and their representations by stresses**. Summarizing the results of the previous sections we can draw the commutative diagram

$$LD(\Omega) \xrightarrow{\varepsilon} L^1(\Omega, \mathbb{R}^6)$$

$$LD(\Omega)\mathscr{R} \xrightarrow{\varepsilon/\mathscr{R}} L^1(\Omega, \mathbb{R}^6).$$

Here, Liouville’s rigidity theorem implies that the kernels of $\varepsilon$ and $\pi$ are identical, the rigid velocity fields, and $\varepsilon/\mathscr{R}$ given by $\varepsilon/\mathscr{R}(\chi) = \varepsilon(w)$, for some $w \in \chi$, is an isometric injection.

This allows us to represent $LD$-forces—elements of $LD(\Omega)^*$—using the dual diagram.

$$LD(\Omega)^* \xrightarrow{\varepsilon^*} L^\infty(\Omega, \mathbb{R}^6)$$

$$(LD(\Omega)\mathscr{R})^* \xrightarrow{(\varepsilon/\mathscr{R})^*} L^\infty(\Omega, \mathbb{R}^6).$$

Now, $(\varepsilon/\mathscr{R})^*$ is surjective and as in [1] the Hahn-Banach Theorem implies that any $T \in (LD(\Omega)\mathscr{R})^*$ may be represented in the form

$$T = (\varepsilon/\mathscr{R})^*(\sigma)$$

for some essentially bounded symmetric stress tensor field $\sigma \in L^\infty(\Omega, \mathbb{R}^6)$. Furthermore, the dual norm of $T$ is given by

$$\|T\| = \inf_{T = (\varepsilon/\mathscr{R})^*(\sigma)} \|\sigma\|_{L^\infty} = \inf_{T = (\varepsilon/\mathscr{R})^*(\sigma)} \left\{ \operatorname{ess} \sup_{i,m,x \in \Omega} |\sigma_{im}(x)| \right\}.$$
$\pi^*$ is a norm-preserving injection, any equilibrated $LD$-force $S \in LD(\Omega)^*$ may be represented in the form

$$S = \varepsilon^*(\sigma)$$

for some stress field $\sigma$ and

$$\|S\| = \inf_{S = \varepsilon^*(\sigma)} \|\sigma\|_{L^\infty} = \inf_{S = \varepsilon^*(\sigma)} \left\{ \text{ess sup}_{i,m,x \in \Omega} |\sigma_{im}(x)| \right\}.$$ 

4.4. $\mu$-integrable distortions and equilibrated forces on bodies. Following [7] we use $L^{1,\mu}(\Omega, \mathbb{R}^3)$ to denote the space of integrable vector fields on $\Omega$ whose restrictions to $\partial \Omega$ are integrable relative to the area measure on $\partial \Omega$. On this space we use the norm

$$\|w\|_{L^{1,\mu}} = \int \Omega |w| \, dV + \int_{\partial \Omega} |w| \, dA = \|w\|_{L^1(\Omega, \mathbb{R}^3)} + \|w\|_{L^1(\partial \Omega, \mathbb{R}^3)}.$$ 

Alternatively, the $L^{1,\mu}$-norm may be regarded as the $L^1$-norm relative to the Radon measure $\mu$, defined above and hence the notation.

Forces, being elements of the dual space $L^{1,\mu}(\Omega, \mathbb{R}^3)^*$, may be identified with elements of $L^{\infty,\mu}(\Omega, \mathbb{R}^3)$. A force $F$ on a body, given in terms of a body force $b$ and a surface force $t$, may be identified with a continuous linear functional relative to the $L^{1,\mu}$-norm if the body force components $b_i$ and surface force components $t_i$ (alternatively, $|b|$ and $|t|$) are essentially bounded relative to the volume and area measures, respectively. In this case, the representation is of the form

$$F(w) = \int \Omega b_i w_i \, dV + \int_{\partial \Omega} t_i w_i \, dA.$$ 

Moreover, the dual norm of a force is the $L^{\infty,\mu}$-norm, given as

$$\|F\|_{L^{\infty,\mu}} = \|F\|_{L^{1,\mu}}^* = \text{ess sup}_{x \in \Omega, y \in \partial \Omega} \{|b(x)|, |t(y)|\}$$

as anticipated.

It is well known that if $F$ is equilibrated, i.e., $F \in \pi_0^*(G)$, for some $G \in (L^{1,\mu}(\Omega, \mathbb{R}^3)/\mathcal{K})^*$, then,

$$\int \Omega b \, dV + \int_{\partial \Omega} t \, dA = 0,$n and $$\int \Omega x \times b \, dV + \int_{\partial \Omega} x \times t \, dA = 0.$n

4.5. $LD$-forces represented by body forces and surface forces. Using the trace operator $\gamma$, for each $w \in LD(\Omega)$ we may define

$$\delta(w) : \overline{\Omega} \rightarrow \mathbb{R}^3$$

by $\delta(w)(x) = w(x)$ for $x \in \Omega$ and $\delta(w)(y) = \gamma(w)(y)$ for $y \in \partial \Omega$. The trace theorem and the original definition in 3.1 of the norm on $LD(\Omega)$ imply that we defined a linear and continuous mapping

$$\delta : LD(\Omega) \rightarrow L^{1,\mu}(\overline{\Omega}, \mathbb{R}^3).$$

By the linearity of the extension mapping and using $\delta(r) = r$ for $r \in \mathcal{K}$, we set

$$\delta|\mathcal{K} : LD(\Omega)|\mathcal{K} \rightarrow L^{1,\mu}(\overline{\Omega}, \mathbb{R}^3)|\mathcal{K}$$
Thus, we obtain

\[
\begin{array}{ccc}
L^1(\Omega, \mathbb{R}^3) & \xleftarrow{\delta} & LD(\Omega) \\
\pi & & \downarrow \pi \\
L^1(\Omega, \mathbb{R}^3)/\mathcal{R} & \xleftarrow{\delta/\mathcal{R}} & LD(\Omega)/\mathcal{R}.
\end{array}
\]

The dual commutative diagram is

\[
L^{\infty}(\Omega, \mathbb{R}^3) \xrightarrow{\pi^*} LD(\Omega)^* \\
\delta/\mathcal{R} \downarrow \pi^* \\
(L^1(\Omega, \mathbb{R}^3)/\mathcal{R})^* \xrightarrow{\delta/\mathcal{R}^*} (LD(\Omega)/\mathcal{R})^*.
\]

In particular, the image under \(\delta^*\) of an equilibrated force \(F \in L^{\infty}(\Omega, \mathbb{R}^3)\) is an equilibrated \(LD\)-force.

As the norm of a mapping and its dual are equal, we have

\[
\|\delta/\mathcal{R}\| = \|(\delta/\mathcal{R})^*\| = \sup_{G \in (L^1(\Omega, \mathbb{R}^3)/\mathcal{R})^*} \frac{\|\delta/\mathcal{R}^*(G)\|}{\|G\|} = \sup_{G \in (L^1(\Omega, \mathbb{R}^3)/\mathcal{R})^*} \frac{\inf_{\delta/\mathcal{R}^*(G) = (\varepsilon/\mathcal{R})^*(\sigma)} \|\sigma\|}{\|G\|}.
\]

Using the fact that the two mappings \(\pi^*\) are isometric injections onto the respective subspaces of equilibrated forces, we may replace \(G\) above by an equilibrated force \(F \in L^{\infty}(\Omega, \mathbb{R}^3)\), and \((\delta/\mathcal{R})^*(G) = (\varepsilon/\mathcal{R})^*(\sigma)\) is replaced by \(\delta^*(F) = \varepsilon^*(\sigma)\). Thus, we obtain

\[
\|\delta/\mathcal{R}\| = \sup_{F} \inf_{\delta^*(F) = \varepsilon^*(\sigma)} \left\{ \text{ess sup}_{i,m,x} \left\{ \|\sigma_{im}(x)\| \right\} \right.,
\]

over all equilibrated forces in \(L^{\infty}(\Omega, \mathbb{R}^3)\). Explicitly, the condition \(\delta^*(F) = \varepsilon^*(\sigma)\) is

\[
\int_{\Omega} b \cdot w \, dV + \int_{\partial \Omega} t \cdot w \, dA = \int_{\Omega} \sigma \cdot \varepsilon(w) \, dV
\]

as anticipated and we conclude that

\[
K = \|\delta/\mathcal{R}\|.
\]

Remark 4.1. If we want to regard \(\delta/\mathcal{R}\) as a mapping between function spaces we should use the decompositions of the respective spaces into Whitney sums. We already noted that \(LD(\Omega)/\mathcal{R}\) is isometrically isomorphic to \(LD(\Omega)_0\) — the space of \(LD\)-vector fields having zero rigid components. Now \(L^{1,\mu}(\Omega, \mathbb{R}^3)_0\) is bijective to \(L^{1,\mu}(\Omega, \mathbb{R}^3)/\mathcal{R}\) but as a subspace of \(L^{1,\mu}(\Omega, \mathbb{R}^3)\) it has a different norm (see Paragraph 2.4). Since we are interested in the quotient norm in order to use the essential supremum for the dual norm, we will endow \(L^{1,\mu}(\Omega, \mathbb{R}^3)_0\) with the quotient norm \(\|u_0\| = \inf_{\tau \in \mathcal{R}} \|u_0 - \tau\|_{L^{1,\mu}}\) — which brings us back to the problem of best

---

1Note that we cannot use

\[
\|\delta\| = \|\delta^*\| = \sup_{F \in L^{\infty}(\Omega, \mathbb{R}^3)} \|\delta^*(F)\| = \sup_{F \in L^{\infty}(\Omega, \mathbb{R}^3)} \frac{\inf_{\delta^*(F) = \varepsilon^*(\sigma)} \|\sigma\|}{\|F\|_{L^{\infty}(\Omega, \mathbb{R}^3)}}
\]

because \(\varepsilon^*\) is not surjective so there might be no \(\sigma\) satisfying the condition \(\delta^*(F) = \varepsilon^*(\sigma)\).
approximation by rigid velocity as described in the end of paragraph 2.3. Thus, $\delta / R$ becomes identical to the restriction

$$\delta_0 = \delta\big|_{LD(\Omega)_0} : LD(\Omega)_0 \rightarrow L^{1,\#}(\Omega, \mathbb{R}^3)_0$$

of $\delta$ to vector fields having zero rigid components. Its norm is given by

$$\|\delta / R\| = \|\delta_0\| = \sup_{w_0 \in LD(\Omega)_0} \left\{ \frac{1}{2} \int_\Omega \sum_{i,m} |w_{0i,m} + w_{0m,i}| \, dV \right\}.$$

Again, one may use smooth vector fields to evaluate the supremum as these are dense in $LD(\Omega)$.

**Appendix A. Elementary Properties of Quotient Spaces**

We describe below some elementary properties of quotient spaces of normed spaces (e.g., [9, p. 227]).

**A.1. The quotient norm.** Let $W$ be a normed vector space with a norm $\|\cdot\|$ and $\mathcal{R}$ a closed subspace of $W$ (e.g., a finite dimensional subspace). Then, the quotient norm $\|\cdot\|_0$ is defined on $W / \mathcal{R}$ by

$$\|w_0\|_0 = \inf_{w \in w_0} \|w\|.$$

Denoting by $\pi : W \rightarrow W / \mathcal{R}$ the natural linear projection $\pi(w) = [w]$, we clearly have

$$\|\pi(w)\|_0 = \|\pi(w + r)\|_0 = \inf_{r \in \mathcal{R}} \|w + r\|,$$

for any $r \in \mathcal{R}$. The quotient norm makes the projection mapping $\pi$ continuous and the topology it generates on the quotient space is equivalent to quotient topology.

**A.2. Dual spaces.** We note that as the projection $\pi$ is surjective, its dual mapping

$$\pi^* : (W / \mathcal{R})^* \rightarrow W^*$$

is injective. Clearly, it is linear and continuous relative to the dual norms. If $\phi \in \text{Image } \pi^*$ so $\phi = \pi^*(\phi_0)$, $\phi_0 \in (W / \mathcal{R})^*$, then, for each $r \in \mathcal{R}$,

$$\phi(r) = \pi^*(\phi_0)(r)$$

$$= \phi_0(\pi(r))$$

$$= \phi_0(0)$$

$$= 0.$$

On the other hand, if for $\phi \in W^*$, $\phi(r) = 0$ for all $r \in \mathcal{R}$, then, we may define $\phi_0 \in (W / \mathcal{R})^*$ by $\phi_0(w_0) = \phi(w)$, for some $w \in W$ such that $\pi(w) = w_0$. The choice of $w \in w_0$ is immaterial because $\phi(w + r) = \phi(w) + \phi(r) = \phi(w)$, for any $r \in \mathcal{R}$. We conclude that

$$\text{Image } \pi^* = \mathcal{R}^\perp = \{ \phi \in W^* | \phi(r) = 0 \text{ for all } r \in \mathcal{R} \}.$$
Next we consider the dual norm of elements of the dual to the quotient space. For \( \phi_0 \in (W/\mathcal{R})^* \), we have
\[
\|\phi_0\| = \sup_{w_0 \in W/\mathcal{R}} \frac{|\phi_0(w_0)|}{\|w_0\|_0}.
\]
Thus,
\[
\|\phi_0\| = \sup_{w_0 \in W/\mathcal{R}} \left\{ \sup_{r \in \mathcal{R}} \frac{|\pi^*(\phi_0)(w + r)|}{\|w + r\|} \right\} \quad \text{for some } w \in w_0
\]
\[
= \sup_{w_0 \in W/\mathcal{R}} \left\{ \sup_{w' \in w_0} \frac{|\pi^*(\phi_0)(w')|}{\|w'\|} \right\} \quad \text{for some } w \in w_0
\]
\[
= \sup_{w' \in W} \frac{|\pi^*(\phi_0)(w')|}{\|w'\|} = \|\pi^*(\phi_0)\|.
\]

We conclude that \( \pi^* \) is norm preserving.

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Department of Mechanical Engineering, Ben-Gurion University, P.O.Box 653, Beer-Sheva 84105 Israel.

E-mail address: rsegev@bgu.ac.il