Faster Algorithms for Largest Empty Rectangles and Boxes

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Abstract
We revisit a classical problem in computational geometry: finding the largest-volume axis-aligned empty box (inside a given bounding box) amidst \(n\) given points in \(d\) dimensions. Previously, the best algorithms known have running time \(O(n \log^2 n)\) for \(d = 2\) (Aggarwal and Suri 1987) and near \(n^d\) for \(d \geq 3\). We describe faster algorithms with the following running times (where \(\varepsilon > 0\) is an arbitrarily small constant and \(\tilde{O}\) hides polylogarithmic factors):

- \(O(n^{2+\varepsilon})\) for \(d = 3\), and
- \(\tilde{O}(n^{(5d+2)/6})\) for any constant \(d \geq 4\).

To obtain the higher-dimensional result, we adapt and extend previous techniques for Klee’s measure problem to optimize certain objective functions over the complement of a union of orthants.

Keywords  Largest empty rectangle · Largest empty box · Klee’s measure problem

Mathematics Subject Classification  68Q25 · 68U05

1 Introduction

Two dimensions. In the first part of this paper, we tackle the largest empty rectangle problem: Given a set \(P\) of \(n\) points in the plane and a fixed rectangle \(B_0\), find the largest rectangle \(B \subseteq B_0\) such that \(B\) does not contain any points of \(P\) in its interior.
Here and throughout this paper, a “rectangle” refers to an axis-parallel rectangle; and unless stated otherwise, “largest” refers to maximizing the area.

The problem has been studied since the early years of computational geometry. While similar basic problems such as largest empty circle or largest empty square can be solved efficiently using Voronoi diagrams, the largest empty rectangle problem seems more challenging. The earliest reference on the 2D problem appears to be by Naamad et al. in 1984 [27], who gave a quadratic-time algorithm. In 1986, Chazelle et al. [16] obtained an $O(n \log^3 n)$-time algorithm. Subsequently, in 1987, Aggarwal and Suri [3] presented another algorithm running in $O(n \log^3 n)$ time, followed by a more complicated second algorithm running in $O(n \log^2 n)$ time. The $O(n \log^2 n)$ worst-case bound has not been improved since.¹

A few results on related questions have been given. Dumitrescu and Jiang [20] examined the combinatorial problem of determining the worst-case number of maximum-area empty rectangles and proved an $O(n^{2^\alpha(n)} \log n)$ upper bound; their proof does not appear to have any implication to the algorithmic problem of finding a maximum-area empty rectangle. If the objective is changed to maximizing the perimeter, the problem is a little easier and an optimal $O(n \log n)$-time algorithm can already be found in Aggarwal and Suri’s paper [3]. Another related problem of computing a maximum-area rectangle contained in a polygon has also been explored [17].

We obtain a new randomized algorithm that finds the maximum-area empty rectangle in $n^2 O((\log^* n) \log n)$ expected time. This is not only an improvement of almost a full logarithmic factor over the previous 33-year-old bound, but is also close to optimal, except for the slow-growing iterated-logarithmic-like factor (as $\Omega(n \log n)$ is a lower bound in the algebraic decision tree model). Our solution interestingly uses interval trees to efficiently divide the problem into subproblems of logarithmic size, yielding a recursion with $O(\log^* n)$ depth.

Higher dimensions. The higher-dimensional analog of the problem is largest empty box: Given a set $P$ of $n$ points in $\mathbb{R}^d$ and a fixed box $B_0$, find the largest box $B \subset B_0$ such that $B$ does not contain any points of $P$ in its interior. Here and throughout this paper, a “box” refers to an axis-parallel hyperrectangle; and unless stated otherwise, “largest” refers to maximizing the volume.

Several papers [18–20, 32] have studied related questions in higher dimensions, e.g., proving combinatorial bounds on the number of optimal boxes, or proving extremal bounds on the volume, or designing approximation algorithms. For the original (exact) computational problem, it is not difficult to obtain an algorithm that finds the largest empty box in $\tilde{O}(n^d)$ time (for example, as was done by Backers and Keil [4]).² At the end of their paper, Dumitrescu and Jiang [20] explicitly asked whether a faster algorithm is possible:

¹ Aggarwal and Suri’s first algorithm can be sped up to run in near $O(n \log^2 n)$ time as well, since it relied on a subroutine for finding row minima in Monge staircase matrices, a problem for which improved results were later found [13, 24]; but these results do not appear to lower the cost of Aggarwal and Suri’s second algorithm.

² Throughout the paper, $\tilde{O}$ notation hides polylogarithmic factor.
"Can a maximum empty box in $\mathbb{R}^d$ for some fixed $d \geq 3$ be computed in $O(n^{\gamma_d})$ time for some constant $\gamma_d < d$?"

Dumitrescu and Jiang attempted to give a subcubic algorithm for the 3D problem, but their conditional solution required a sublinear-time dynamic data structure for finding the 2D maximum empty rectangles containing a query point—currently, the existence of such a data structure is not known.

On the lower bound side, Giannopoulos et al. [23] proved that the largest empty box problem is $W[1]$-hard with respect to the dimension. This implies a conditional lower bound of $\Omega(n^{\beta d})$ for some absolute constant $\beta > 0$, assuming a popular conjecture that the $k$-clique problem requires $n^{\Omega(k)}$ time.

We answer the above question affirmatively. For $d = 3$, we give an $O(n^{5/2+\varepsilon})$-time algorithm, where $\varepsilon > 0$ is an arbitrarily small constant. For higher constant $d \geq 4$, we obtain an algorithm with an intriguing time bound that improves over $n^d$ even more dramatically: $\tilde{O}(n^{(5d+2)/6})$. For example, the bound is $O(n^{3.67})$ for $d = 4$, $\tilde{O}(n^{4.5})$ for $d = 5$, and $O(n^{8.67})$ for $d = 10$.

Not too surprisingly, our 3D algorithm achieves subcubic complexity by applying standard range searching data structures (though the application is not immediately obvious). Dynamic data structures are not used.

The techniques for our higher-dimensional algorithm are perhaps more original and significant, with potential impact to other problems. We first transform the largest empty box problem into a problem about a union of $n$ orthants in $D = 2d$ dimensions (the transformation is simple and has been exploited before, such as in [5]). The union of orthants is known to have worst-case combinatorial complexity $O(n^{\lfloor D/2 \rfloor})$ [8]. Interestingly, we show that it is possible to maximize certain types of objective functions over the complement of the union, in time significantly smaller than the worst-case combinatorial complexity.

We accomplish this by adapting known techniques on Klee’s measure problem [9, 11, 12, 28]. Specifically, we build on a remarkable method by Bringmann [9] for computing the volume of a union of $n$ orthants in $D$ dimensions in $n^{D/3+O(1)}$ time (the $O(1)$ term in the exponent was $2/3$ but has been later removed by the author [12]). However, maximizing an objective function over the complement of the union is different from summing or integrating a function, and Bringmann’s method does not immediately generalize to the former (for example, it exploits subtraction). We introduce extra ideas to extend the method, which results in a bigger time bound than $n^{D/3} = n^{2d/3}$ but nevertheless beats $n^{D/2} = n^d$. In particular, we use some simple graph-theoretical arguments, applied to graphs with $O(D)$ vertices.

Paper organization. We present our 2D algorithm in Sect. 2, our 3D algorithm in Sect. 3, and our higher-dimensional algorithms in Sects. 4–6 (all these parts may be read independently).

2 Largest Empty Rectangle in 2D

As in previous work [3, 16], we focus on solving a line-restricted version of the 2D largest empty rectangle problem: given a set $P$ of $n$ points below a fixed horizontal line
Fig. 1 (a, b) Transforming points into horizontal segments. (c) Pseudo-ray $\gamma_s$

$\ell_0$ and a set $Q$ of $n$ points above $\ell_0$, where the $x$-coordinates of all points have been pre-sorted, and given a rectangle $B_0$, find the largest-area rectangle $B \subset B_0$ that intersects $\ell_0$ and is empty of points of $P \cup Q$. By standard divide-and-conquer, an $O(T(n))$-time algorithm for the line-restricted problem immediately yields an $O(T(n) \log n)$-time algorithm for the original largest empty rectangle problem, assuming that $T(n)/n$ is nondecreasing.

We begin by reformulating the line-restricted problem as a problem about horizontal line segments. In the subsequent subsections, we will work with this re-formulation. For each point $p \in P$, let $s(p)$ be the longest horizontal line segment inside $B_0$ such that $s(p)$ passes through $p$ and there are no points of $P$ above $s(p)$ (Fig. 1 (a)). We can compute $s(p)$ for all $p \in P$ in $O(n)$ time: this step is equivalent to the construction of the standard Cartesian tree [22, 33], for which there are simple linear-time algorithms (for example, by inserting points from left to right and maintaining a stack, like Graham’s scan, as also re-described in previous papers [3, 16]). Similarly, for each $q \in Q$, let $t(q)$ be the longest horizontal line segment inside $B_0$ such that $t(q)$ passes through $q$ and there are no points of $Q$ below $t(q)$. We can also compute $t(q)$ for all $q \in Q$ in $O(n)$ time.

For a horizontal segment $s$, let $x_s^-$ and $x_s^+$ denote the $x$-coordinates of its left and right endpoints respectively, and let $y_s$ denote its $y$-coordinate. We say that a set $S$ of horizontal segments is laminar if for every $s, s' \in S$, either the two intervals $[x_s^-, x_s^+]$ and $[x_{s'}^-, x_{s'}^+]$ are disjoint, or one interval is contained in the other (in other words, the intervals form a “balanced parentheses” or tree structure). It is easy to see that for the segments defined above, $\{s(p) : p \in P\}$ is laminar and $\{t(q) : q \in Q\}$ is laminar.

The optimal rectangle must have some point $p^* \in P$ on its bottom side and some point $q^* \in Q$ on its top side (except when the optimal rectangle touches the bottom or top side of $B_0$, but the condition can be met by just adding extra points to $P$ and $Q$ on the bottom and top sides of $B_0$ at all the $O(n)$ $x$-coordinates). Chazelle et al. [16] already noted that the case when $[x_{s(p)}^-, x_{s(p)}^+]$ is contained in $[x_{t(q)}^-, x_{t(q)}^+]$ can be handled in $O(n)$ time (in their terminology, this is the case of “three supports in one half, one
in the other”). The key remaining case is when \( x_t^{-}(q^*) < x_s^{-}(p^*) < x_t^{+}(q^*) < x_s^{+}(p^*) \), where the area of the optimal rectangle is \( (x_t^{+}(q^*) - x_s^{+}(p^*)) (y_t(q^*) - y_s(p^*)) \). All other cases are symmetric. The problem is thus reduced to the following (see Fig. 1 (b)):

**Problem 2.1** Given a laminar set \( S \) of \( n \) horizontal segments and a laminar set \( T \) of \( n \) horizontal segments, where all \( x \)-coordinates have been pre-sorted, find a pair \((s, t) \in S \times T\) such that \( x_t^{-} < x_s^{-} < x_t^{+} < x_s^{+} \), maximizing \( (x_t^{+} - x_s^{-})(y_t - y_s) \).

We find it more convenient to work with the corresponding decision problem, as stated below. By the author’s randomized optimization technique \([10]\), an \( O(T(n))\)-time algorithm for Problem 2.2 yields an \( O(T(n))\)-expected-time algorithm for Problem 2.1, assuming that \( T(n)/n \) is nondecreasing:

**Problem 2.2** Given a laminar set \( S \) of \( n \) horizontal segments and a laminar set \( T \) of \( n \) horizontal segments, where all \( x \)-coordinates have been pre-sorted, and given a value \( r > 0 \), decide if there exists a pair \((s, t) \in S \times T\) such that \( x_t^{-} < x_s^{-} < x_t^{+} < x_s^{+} \) and \((x_t^{+} - x_s^{-})(y_t - y_s) > r\), and if so, report one such pair. We call such a pair good.

### 2.1 Preliminaries

To help solve Problem 2.2, we define a curve \( \gamma_s \) for each \( s \in S\):

\[
\gamma_s(x) = \begin{cases} 
\min \left\{ \frac{r}{x - x_s^{-}} + y_s, M_s \right\} & \text{if } x > x_s^{-}, \\
M_s & \text{if } x \leq x_s^{-},
\end{cases}
\]

for some value \( M_s \). (The main part of the curve is a hyperbola \( y = r/(x - x_s^{-}) + y_s \); we place a cap at \( y = M_s \).) We choose distinct large values for the \( M_s \)’s such that \( M_s \) is monotonically increasing in \( x_s^{-} \). The condition \((x_t^{+} - x_s^{-})(y_t - y_s) > r\) is met iff the point \((x_t^{+}, y_t)\) (i.e., the right endpoint of \( t \)) is above the curve \( \gamma_s \), assuming that \( x_t^{+} > x_s^{-} \) and \( M_s \) is sufficiently large. Note that these curves form a family of pseudo-lines: this can be seen from the fact that for any two distinct curves \( \gamma_s \) and \( \gamma_{s'} \) with \( x_s^{-} \geq x_{s'}^{-} \), the difference

\[
\frac{r}{x - x_s^{-}} + y_s - \left( \frac{r}{x - x_{s'}^{-}} + y_{s'} \right) = \frac{r(x_s^{-} - x_{s'}^{-})}{(x - x_s^{-})(x - x_{s'}^{-})} + y_s - y_{s'}
\]

3 The solution is simple: for each \( p \in P \), we find the lowest point \( q_p \in Q \) with \( x \)-coordinate in the interval \([x_s^{-}(p), x_s^{+}(p)]\), and take the maximum of \((x_s^{+}(p)) - x_s^{-}(p)) (y_t(q_p) - y_s(p)) \). All these lowest points \( q_p \) can be found “bottom-up” in the tree formed by the intervals \([x_s^{-}(p), x_s^{+}(p)] : p \in P \), in linear total time.

4 Specifically, this follows immediately from \([10\), Thm. 3.1\], since Problem 2.1 is an example of a “closest-pair”-type problem.

5 In a pseudo-line family, every vertical line intersects each curve exactly once, and every pair of curves intersect at most once. More generally, in a pseudo-segment (resp. pseudo-ray) family, every vertical line intersects each curve at most once, the \( x \)-projection of each curve is an interval (resp. a half-interval), and every pair of curves intersect at most once.
Fig. 2 (a) Proof of Lemma 2.3 (a): inserting $γ_1$ to the lower envelope of pseudo-rays $\{γ_2, \ldots, γ_7\}$. (b) Proof of Lemma 2.3 (b): the $x$-projected intervals and the division into slabs

is monotonically decreasing for $x > x_s^-$ and thus has at most one zero; furthermore, the cap $M_s$ is bigger than the cap $M'_s$.

Define the curve segment $\tilde{γ}_s$ to be the part of $γ_s$ restricted to $x \leq x_s^+$ (see Fig. 1 (c)). These curve segments form a family of pseudo-rays. The lower envelope of $n$ pseudo-rays has at most $2n$ edges, by known combinatorial bounds on order-$2$ Davenport–Schinzel sequences [31]. The following lemma summarizes known subroutines we need on the computation of lower envelopes (proofs are briefly sketched).

Lemma 2.3 Consider $n$ pseudo-lines $γ_1, \ldots, γ_n$, sorted from top to bottom at the vertical line $x = −∞$. Assume that the intersection of any two pseudo-lines can be computed in constant time.

(a) Suppose we define a pseudo-ray $γ'_i$ on $γ_i$ for each $i$, such that the $x$-coordinates of the left endpoints are all $−∞$, and the $x$-coordinates of the right endpoints are monotone (i.e., increasing or decreasing) in $i$. Then the lower envelope of these pseudo-rays $γ'_1, \ldots, γ'_n$ can be computed in $O(n)$ time.

(b) Suppose we define a pseudo-segment $γ''_i$ on $γ_i$ for each $i$, such that $x$-coordinates of the left endpoints are monotone in $i$ and the $x$-coordinates of the right endpoints are monotone in $i$. Then the lower envelope of these pseudo-segments $γ''_1, \ldots, γ''_n$ can be computed in $O(n)$ time.

Proof Part (a) follows by a straightforward variant of Graham’s scan [7] (originally for computing planar convex hulls, or by duality, lower envelopes of lines). We insert pseudo-rays in decreasing order of their right endpoints’ $x$-values, while maintaining the portion of the lower envelope to the left of the right endpoint of the current pseudo-ray (see Fig. 2(a)). In each iteration, by the monotonicity assumption, a prefix or suffix of the lower envelope gets deleted (i.e., popped from a stack).

For part (b), the main case is when both the left and right endpoints are monotonically increasing in $i$ (the case when both are monotonically decreasing is symmetric, and the case when they are monotone in different directions easily reduces to two instances of the pseudo-ray case). Greedily construct a minimal set of vertical lines that stab all the pseudo-segments: namely, draw a vertical line at the leftmost right endpoint, remove all pseudo-segments stabbed, and repeat. This process can be done in $O(n)$ time by a linear scan. These vertical lines divide the plane into slabs (see
Fig. 2(b)). In each slab, the pseudo-segments behave like pseudo-rays, so we can compute the lower envelope inside the slab in linear time by applying part (a) twice, for the leftward rays and for the rightward rays (the two envelopes can be merged in linear time). Since each pseudo-segment participates in at most two slabs, the total time is linear. □

As an application of Lemma 2.3 (b), we mention an efficient algorithm for a special case of Problem 2.2, which will be useful later.

**Corollary 2.4** In the case when all segments in \( S \) and \( T \) intersect a fixed vertical line, Problem 2.2 can be solved in \( O(n) \) time.

**Proof** Since \( S \) and \( T \) are laminar, the \( x \)-projected intervals in each set are nested. Let \( s_1, s_2, \ldots \) be the segments in \( S \) with \( [x_{s_1}^-, x_{s_1}^+] \subseteq [x_{s_2}^-, x_{s_2}^+] \) \( \subseteq \ldots \), and let \( t_1, t_2, \ldots \) be the segments in \( T \) with \( [x_{t_1}^-, x_{t_1}^+] \subseteq [x_{t_2}^-, x_{t_2}^+] \) \( \subseteq \ldots \). For each \( s_i \), let \( a(i) \) be the smallest index with \( x_{t_{a(i)}}^- < x_{s_i}^- \), let \( b(i) \) be the smallest index with \( x_{t_{b(i)}}^- < x_{s_i}^- \), and let \( c(i) \) be the largest index with \( x_{t_{c(i)}}^+ < x_{s_i}^+ \). Note that \( a(i) \) is monotonically increasing in \( i \), and \( b(i) \) is monotonically decreasing in \( i \), and \( c(i) \) is monotonically increasing in \( i \). It is straightforward to compute \( a(i), b(i), c(i) \) for all \( i \) by a linear scan.

The problem reduces to finding a pair \((s_i, t_j)\) such that \( \max \{a(i), b(i)\} \leq j \leq c(i) \) and the right endpoint of \( t_j \) is above \( \gamma_{s_i} \). Define the curve segment \( \gamma_{s_i} \) to be the part of \( \gamma_{s_i} \) restricted to \( x \in [\max \{x_{t_{a(i)}}^-, x_{t_{b(i)}}^+, x_{t_{c(i)}}^+\}] \). The problem reduces to finding \( t_j \) whose right endpoint is above some curve segment \( \gamma_{s_i} \), i.e., above the lower envelope of these curve segments. We can compute this lower envelope in \( O(n) \) time by Lemma 2.3 (b) (more precisely, by two invocations of the lemma, as \( \max \{x_{t_{a(i)}}^-, x_{t_{b(i)}}^+, x_{t_{c(i)}}^+\} \) consists of a monotonically increasing and a monotonically decreasing part). The problem can be then be solved by linear scan over the envelope and the endpoints of \( t_j \). □

### 2.2 Algorithm

We are now ready to describe our new algorithm for solving Problem 2.2, using interval trees and an interesting recursion with \( O(\log^* n) \) depth.

**Theorem 2.5** Problem 2.2 can be solved in \( n2^{O(\log^* n)} \) time.

**Proof** As a first step, we build the standard interval tree for the given horizontal segments in \( S \cup T \). This is a perfectly balanced binary tree of with \( O(\log n) \) levels, where each node corresponds to a vertical slab. The root slab is the entire plane, the slab at a node is the disjoint union of the slabs at its two children, and each leaf slab contains no endpoints in its interior. Each segment is stored in the lowest node \( v \) whose slab contains the segment (i.e., the segment is contained in \( v \)’s slab but is not contained in either child’s subslab). Note that each segment is stored only once (unlike in another standard structure known as the segment tree). We can determine the slab containing each segment in \( O(1) \) time, after \( O(n) \)-time preprocessing, by an LCA query [6] (which is easier in the case of a perfectly balanced binary tree).

For each node \( v \), let \( S_v \) (resp. \( T_v \)) be the set of all segments of \( S \) (resp. \( T \)) stored in \( v \). Define the level of a segment to be the level of the node it is stored in.

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Case 1. There exists a good pair \((s^*, t^*)\) where \(s^*\) and \(t^*\) have the same level. Here, \(s^*\) and \(t^*\) must be stored in the same node \(v\) of the interval tree. Thus, a good pair can be found as follows:

1. For each node \(v\), solve the problem for \(S_v\) and \(T_v\) by Corollary 2.4 in \(O(|S_v| + |T_v|)\) time. Note that all segments in \(S_v \cup T_v\) indeed intersect a fixed vertical line (the line that separates \(v\)’s two children’s slabs).

The total running time of this step is \(O(n)\), since each segment is in only one \(S_v\) or \(T_v\).

Case 2. There exists a good pair \((s^*, t^*)\) where \(s^*\) is on a strictly lower level than \(t^*\). To deal with this case, we perform the following steps, for some choice of parameter \(b \geq \log n\):

2a. For each node \(v\), compute the lower envelope of the pseudo-rays \(\{\gamma_s : s \in S_v\}\) by Lemma 2.3(a) in \(O(|S_v|)\) time; let \(E_v\) denote this envelope restricted to \(v\)’s slab. Note that because all segments in \(S_v\) intersect a fixed vertical line and \(S_v\) is laminar, the \(x^+_s\) values are monotonically decreasing in the \(x^-_s\) values for \(s \in S_v\) (and sorting the pseudo-rays \(\gamma_s\) from top to bottom at \(x = -\infty\) is the same as sorting the rays in decreasing order of \(x^-_s\)).

2b. Divide the plane into a set \(\Sigma\) of \(n/b\) vertical slabs each containing \(b\) right endpoints of \(T\).

2c. For each slab \(\sigma \in \Sigma\),

- let \(T_\sigma\) be the set of all segments \(t \in T\) with right endpoints in \(\sigma\), and
- let \(S_\sigma\) be the set of all segments \(s \in S\) such that \(\gamma_s\) appears on \(E_v \cap \sigma\) for some node \(v\).

Divide \(S_\sigma\) (arbitrarily) into blocks of size \(b\) and recursively solve the problem for \(T_\sigma\) and each block of \(S_\sigma\).

Correctness. Consider a good pair \((s^*, t^*)\) with \(s^*\) on a strictly lower level than \(t^*\). Let \(\sigma\) be the slab in \(\Sigma\) containing the right endpoint of \(t^*\), i.e., \(t^* \in T_\sigma\). Let \(v\) be the node \(s^*\) is stored in. Then \(t^*\) intersects the left wall of the slab at \(v\) (since \(t^*\) must be stored in a proper ancestor of \(v\)). Now, the right endpoint of \(t^*\) is below \(\gamma_{s^*}\) and is thus below \(E_v\). Let \(\gamma_s\) be the curve on \(E_v\) that the right endpoint of \(t^*\) is below, with \(s \in S_v\). Then \(\gamma_s\) appears on \(E_v \cap \sigma\), and so \(s \in S_\sigma\). Since the right endpoint of \(t^*\) is below \(\gamma_s\), we have \(x^-_s < x^+_t < x^+_s\), and since \(t^*\) intersects the left wall of \(v\)’s slab, we have \(x^-_t < x^-_s\). So, \((s, t^*)\) is good, and the recursive call for \(T_\sigma\) and some block of \(S_\sigma\) will find a good pair.

Analysis. The total number of edges in all envelopes \(E_v\) is at most \(2 \sum_v |S_v| \leq 2n\). Since the envelopes \(E_v\) have disjoint \(x\)-projections for nodes \(v\) at the same level, and since there are \(O(\log n)\) levels, the \(O(n/b)\) dividing vertical lines of \(\Sigma\) intersect at most \(O((n/b) \log n)\) edges among all the envelopes. Thus, \(\sum_{\sigma \in \Sigma} |S_\sigma| \leq 2n + O((n/b) \log n) = O(n)\) if \(b \geq \log n\), and so the total number of recursive calls in step 2c over all \(\sigma \in \Sigma\) is \(O(n/b)\).

Case 3. There exists a good pair \((s^*, t^*)\) where \(s^*\) is on a strictly higher level than \(t^*\). This remaining case is symmetric to Case 2 (by switching \(S\) and \(T\) and negating \(y\)-coordinates).

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By running the algorithms for all three cases, a good pair is guaranteed to be found if one exists. The running time satisfies the recurrence $T(n) \leq O(n/b)T(b) + O(n)$. Setting $b = \log n$ gives $T(n) \leq n2^{O(\log^* n)}$.

By the observations from the beginning of this section, we can now solve Problem 2.1 and the line-restricted problem in $n2^{O(\log^* n)}$ expected time, and the original largest empty rectangle problem in $n2^{O(\log^* n)} \log n$ expected time.

**Corollary 2.6** Given $n$ points in $\mathbb{R}^2$ and a rectangle $B_0$, we can compute the maximum-area empty rectangle inside $B_0$ in $n2^{O(\log^* n)} \log n$ expected time.

### 3 Largest Empty Box in 3D

In this section, we describe a subcubic algorithm for the largest empty box problem in 3D. The key is the following result on an “asymmetric” case with a left point set and right point set of different sizes:

**Theorem 3.1** Given a set $P$ of $n$ points in $(-\infty, 0) \times \mathbb{R}^2$, and a set $Q$ of $m$ points in $(0, \infty) \times \mathbb{R}^2$, we can compute the maximum-volume box that contains the origin and is empty of points in $P \cup Q$ in $\tilde{O}(n^2 + m^{1+\varepsilon})$ time for an arbitrarily small constant $\varepsilon > 0$.

**Proof** Map a box $b = (-x_1, x'_1) \times (-x_2, x'_2) \times (-x_3, x'_3) \subset \mathbb{R}^3$ to a point $b^* = (x_1, x'_1, x_2, x'_2, x_3, x'_3)$ in 6D. Map a point $p = (p_1, p_2, p_3) \in \mathbb{R}^3$ to an orthant $p^* = (-p_1, \infty) \times (p_1, \infty) \times (-p_2, \infty) \times (p_2, \infty) \times (-p_3, \infty) \times (p_3, \infty)$ in 6D. Then the point $p$ is in the box $b$ iff the point $b^*$ is in the orthant $p^*$.

Say $B_0 = (-c_1, c'_1) \times (-c_2, c_2) \times (-c_3, c_3)$. Define $\mathcal{X} = [0, c_1] \times [0, c'_1] \times [0, c_2] \times [0, c'_2] \times [0, c_3] \times [0, c'_3]$. Our goal is to find a point $b^* = (x_1, x'_1, x_2, x'_2, x_3, x'_3) \in \mathcal{X}$ maximizing the function $H(b^*) := (x_1 + x'_1)(x_2 + x'_2)(x_3 + x'_3)$, such that $b^*$ is in the complement of both $U(P) = \bigcup_{p \in P} p^*$ and $U(Q) = \bigcup_{q \in Q} q^*$. Let $Z$ be the set of all vertices of $U(P) \cap \mathcal{X}$ and $A$ be the set of all vertices of $U(Q) \cap \mathcal{X}$. The constraint can then be restated as follows: $b^*$ is dominated by some vertex in $Z$ and by some vertex in $A$.

Since all points $(p_1, p_2, p_3) \in P$ have $p_1 < 0$, $U(P) \cap \mathcal{X}$ corresponds to a union of $n$ orthants in 5D (the second dimension is irrelevant); by known results on the union complexity of orthants [8], the set $Z$ has $O(n^2)$ size and can be constructed in $\tilde{O}(n^2)$ time. Similarly, since all points $(q_1, q_2, q_3) \in Q$ have $q_1 > 0$, $U(Q) \cap \mathcal{X}$ corresponds to a union of $n$ orthants in 5D (the first dimension is irrelevant); the set $A$ has $O(m^2)$ size and can be constructed in $\tilde{O}(m^2)$ time.

For each $z = (z_1, z'_1, z_2, z'_2, z_3, z'_3) \in Z$ and each $a = (a_1, a'_1, a_2, a'_2, a_3, a'_3) \in A$, the maximum of $h(b^*)$ over all $b^* \in \mathcal{X}$ that are dominated by both $z$ and $a$ is given by the function

---

6 We say that a point $(p_1, \ldots, p_d)$ dominates a point $(q_1, \ldots, q_d)$ if $p_i \geq q_i$ for all $i = 1, \ldots, d$.\[\text{ Springer}\]
Thus, our problem is reduced to computing the maximum of \( f(z, a) \) over all \( z \in Z \) and \( a \in A \). This new problem can be solved by standard range searching technique. First consider the decision problem of testing whether the maximum exceeds a given fixed value \( r \). We build a data structure for \( A \) such that for each query point \( z \in Z \), we can quickly decide whether there exists an \( a \in A \) with \( f(z, a) \geq r \). The structure will be a two-level structure. The primary structure is a standard data structure for orthogonal range searching, namely, the range tree [7], which generates a collection of \( \tilde{O}(n) \)-so-called canonical subsets \( A_i \) of total size \( \tilde{O}(n) \), so that given any query box, the subset of all points inside the box can be expressed as a union of \( \tilde{O}(1) \) canonical subsets. For each canonical subset \( A_i \), we will build some secondary structure, as explained below.

For each \( z \in Z \), in one case, we identify all \( a \in A \) with \( z_1 \leq a_1, z_1' \geq a_1', z_2 \leq a_2, z_2' \geq a_2', z_3 \leq a_3, \) and \( z_3' \geq a_3' \), by orthogonal range searching. Express the answer as a union of \( \tilde{O}(1) \) canonical subsets \( A_i \). For each such canonical subset \( A_i \), we want to decide whether there exists an \( a \in A_i \) with \((z_1 + a_1')(z_2 + a_2')(z_3 + a_3') \geq r \). This can be done by point location in a lower envelope of surfaces of the form \( z_3 = r / ((z_1 + a_1')(z_2 + a_2')) - a_3' \) in 3D. By known results on lower envelopes of surfaces [30], after building a secondary structure for each \( A_i \) in \( O(|A_i|^{2+\varepsilon}) \) time, such a query can be answered in \( O(\log m) \) time.

In another case, where \( z_1 \leq a_1, z_1' \leq a_1', z_2 \leq a_2, z_2' \geq a_2', z_3 \leq a_3, \) and \( z_3' \geq a_3' \), we want to instead decide whether there is an \( a \in A_i \) with \((z_1 + z_1')(z_2 + a_2')(z_3 + a_3') \geq r \). With a change of variable \( z_3'' = z_1 + z_1' \), this can also be done by point location in a lower envelope of surfaces in 3D, this time of the form \( z_3 = r / (z_1''(z_2 + a_2')) - a_3' \). All other cases can be handled similarly, with the inequality changed, e.g., to \((z_1 + z_1')(z_2 + z_2')(z_3 + a_3') \geq r \), or \((z_1 + z_1')(z_2 + a_2')(a_3 + a_3') \geq r \), or \((z_1 + a_1') \geq 2 \geq a_2'(a_3 + a_3') \geq r \), or \((z_1 + a_1')(a_2 + a_2')(a_3 + a_3') \geq r \), or \((z_1 + a_1')(a_2 + a_2')(a_3 + a_3') \geq r \), or \((a_1 + a_1')(a_2 + a_2')(a_3 + a_3') \geq r \) (remaining cases are symmetric.) In each of these cases, it is easy to see (after a change of variables) that the subproblem also reduces to point location in a lower envelope, in fact, in 2D or 1D.

The entire two-level data structure has \( O(|A|^{2+\varepsilon}) \) preprocessing time and \( \tilde{O}(1) \) query time. The total time for \(|Z| \) queries is \( \tilde{O}(|A|^{2+\varepsilon} + |Z|) = \tilde{O}(m^{4+O(\varepsilon)} + n^2) \).

The original problem of computing \( \max_{z \in Z, a \in A} f(z, a) \) can be reduced to the decision problem, for example, by the author’s randomized optimization technique [10], Thm. 3.1, or deterministically, by parametric search [26] (since the preprocessing algorithm can be further parallelized).

**Corollary 3.2** Given \( n \) points in \( \mathbb{R}^3 \) and a box \( B_0 \), we can compute the maximum-volume empty box inside \( B_0 \) in \( O(n^{5/2+\varepsilon}) \) time for an arbitrarily small constant \( \varepsilon > 0 \).

**Proof** By divide-and-conquer, it suffices to solve the plane-restricted version where the box is constrained to intersect a given axis-parallel plane. By another application of divide-and-conquer, the problem can be further reduced to the line-restricted version.
when the box is constrained to intersect a given axis-parallel line \( \ell_0 \). The running time increases by at most two logarithmic factors. Without loss of generality, assume that \( \ell_0 \) is the first coordinate axis.

Divide space into \( n/m \) slabs, by planes orthogonal to the first coordinate axis, where each slab contains \( m \) points. Order the slabs from left to right. Let \( Q_i \) be the subset of all input points in the \( i \)-th slab \( \sigma_i \).

Consider the case when the optimal box has its right side in \( \sigma_i \) but is not contained in \( \sigma_i \). This case reduces to an instance of the above lemma for the two point sets \( Q_1 \cup \cdots \cup Q_{i-1} \) and \( Q_i \) (after translation to make the left wall of \( \sigma_i \) pass through the origin). The total cost over all \( n/m \) slabs \( \sigma_i \) is thus \( \tilde{O}(n/m)(n^2 + m^{4+\varepsilon}) \). Setting \( m = \sqrt{n} \) gives a time bound of \( O(n^{5/2+\varepsilon}) \).

The remaining case when the optimal box is contained in \( \sigma_i \) for some \( i \) can be handled by recursion. The total time is

\[
T(n) = \sqrt{n} T(\sqrt{n}) + O(n^{5/2+\varepsilon}),
\]

yielding \( T(n) = O(n^{5/2+\varepsilon}) \). (Alternatively, instead of recursion, we could switch to some known cubic algorithm.) \( \square \)

In higher dimensions \( d \geq 4 \), a similar approach can yield an algorithm with running time of the form \( n^{d-1+O(1/d)} \), but the approach in Sect. 5 is better. On the other hand, for \( d = 3 \), the algorithm in Sect. 5 gives time bound \( \tilde{O}(n^{(5d+2)/6}) = \tilde{O}(n^{17/6}) \), which is worse than \( n^{5/2} \).

## 4 Largest Empty Anchored Box in Higher Dimensions (Warm-Up)

To prepare for our solution to the largest empty box problem in higher constant dimensions, we first investigate a simpler variant, the largest empty anchored box problem: given a set \( P \) of \( n \) points in \([0, \infty)^d\) and a fixed box \( B_0 \), find the largest-volume anchored box in \( B_0 \) that does not contain any points of \( P \) in its interior, where an anchored box has the form \( B = (0, x_1) \times \cdots \times (0, x_d) \) (having the origin as one of its vertices).

Let \( \bigcup S \) denote the union of a set \( S \) of objects. By mapping a box \( B = (0, x_1) \times \cdots \times (0, x_d) \) to the point \((x_1, \ldots, x_d)\), and mapping each input point \((p_1, \ldots, p_d)\) to the orthant \((p_1, \infty) \times \cdots \times (p_d, \infty)\), the largest empty anchored box problem reduces to

**Problem 4.1** Define the function \( H_{\text{vol}}(x_1, \ldots, x_d) = x_1 x_2 \cdots x_d \). Given a set \( S \) of \( n \) orthants in \( \mathbb{R}^d \) and a box \( B_0 \), find the maximum of \( H_{\text{vol}} \) over \( B_0 - \bigcup S \).

(In the application to largest empty anchored box, the orthants all contain \((\infty, \ldots, \infty)\), but our algorithm does not require all orthants to be of the same type.)

By known results [8], the union of \( n \) orthants in \( \mathbb{R}^d \) has worst-case combinatorial complexity \( O(n^{d/2}) \) and can be constructed in \( \tilde{O}(n^{d/2}) \) time. We will show that Problem 4.1 can be solved faster than explicitly constructing the union.
4.1 Preliminaries

A key tool we need is a spatial partitioning scheme due to Overmars and Yap [28] (originally developed for solving Klee’s measure problem in $\tilde{O}(n^{d/2})$ time). The version stated below is taken from [12, Lem. 4.6]; see that paper for a short proof. (The partitioning scheme is also related to “orthogonal BSP trees” [15, 21].)

**Lemma 4.2** Given a set of $n$ axis-parallel flats (of possibly different dimensions) in $\mathbb{R}^d$, and given a parameter $r$, we can divide $\mathbb{R}^d$ into $O(r^d)$ cells (bounded and unbounded boxes) so that each cell intersects $O(n/r^j)$ $(d-j)$-flats. The construction of the cells, along with the conflict lists (lists of all flats intersecting each cell), can be done in $\tilde{O}(n + r^d + K)$ time, where $K$ is the total size of the conflict lists.

Call a function $H : \mathbb{R}^d \rightarrow \mathbb{R}$ simple if it has the form

$$H(x_1, \ldots, x_d) = h_1(x_1)h_2(x_2)\cdots h_d(x_d),$$

where each $h_i$ is a univariate step function. The complexity of $H$ refers to the total complexity (number of steps) in these step functions. As an illustration of the usefulness of Lemma 4.2, we first show how to maximize simple functions over the complement of a union of orthants in $\tilde{O}(n^{d/2})$ time:

**Lemma 4.3** Let $H$ be a simple function with $O(n)$ complexity. Given a set $S$ of $n$ orthants in $\mathbb{R}^d$ and a box $B_0$, we can compute the maximum of $H$ in $B_0 - \bigcup S$ in $\tilde{O}(n^{d/2})$ time for any constant $d \geq 2$.

**Proof** Apply Lemma 4.2 to the $\binom{d}{2}n=O(n)$ $(d-2)$-flats that pass through the $(d-2)$-faces of the given orthants. This yields a partition of $B_0$ into cells.

Consider a cell $\Delta$. The number of $(d-2)$-flats intersecting $\Delta$ is bounded by $O(n/r^2)$, which can be made 0 by setting $r := \Theta(\sqrt{n})$. Consequently, only $(d-1)$-faces of the given orthants may intersect $\Delta$, i.e., all orthants are 1-sided inside $\Delta$. The complement of the union of 1-sided orthants is a box $B$ (we can use orthogonal range searching or intersection data structure to identify the 1-sided orthants intersecting $\Delta$ and compute this box $B$ in $O(1)$ time [1, 7]). For a simple function $H(x_1, \ldots, x_d) = h_1(x_1)\cdots h_d(x_d)$, we can maximize $H$ over the box $\Delta \cap B$ by maximizing $h_i(x_i)$ over an interval (the $x_i$-projection of $\Delta \cap B$) for each $i \in \{1, \ldots, d\}$ separately. This corresponds to a 1D range maximum query for each $i$, which can be done straightforwardly in $O(\log n)$ time (or more carefully in $O(1)$ time [6]). As the number of cells is $O(r^d) = O(n^{d/2})$, the total running time is $\tilde{O}(n^{d/2})$.  

4.2 Algorithm

To improve over $n^{d/2}$, we adapt an approach by Bringmann [9] (originally for solving Klee’s measure problem for orthants in $n^{d/3+O(1)}$ time). The approach involves first

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7 A weaker time bound was stated in [12, Lem. 4.6], but the output-sensitive time bound follows directly from the same construction.
solving the 2-sided special case, and then applying Overmars and Yap’s partitioning scheme. A 2-sided orthant is the set of all points \((x_1, \ldots, x_d) \in \mathbb{R}^d\) satisfying a condition of the form \([x_i \leq a] \land [x_j \leq b]\) for some \(i, j \in \{1, \ldots, d\}\), where each occurrence of “?” is either \(\leq\) or \(\geq\). We will adapt the author’s subsequent re-interpretation [12, Sect. 4.1] of Bringman’s technique, described in terms of monotone step functions.

**Theorem 4.4** In the case when all the input orthants are 2-sided, Problem 4.1 can be solved in \(\widetilde{O}(n^{[d/2]})\) time for any constant \(d \geq 4\).

**Proof** The boundary of the union of 2-sided orthants of the form \([x_i \leq a] \land [x_j \leq b]\) with a fixed \(i, j\) and a fixed choice for the two “?”s is a staircase, i.e., the graph of a univariate monotone (increasing or decreasing) step function (see Fig. 3). There are \(O(d^2)\) choices in total for \(i, j\) and the two “?”s. Thus, the complement of the union of 2-sided orthants can be expressed as the set of all points \((x_1, \ldots, x_d) \in \mathbb{R}^d\) satisfying an expression \(E(x_1, \ldots, x_d)\) which is a conjunction of \(O(d^2)\) predicates each of the form \([x_i \leq f(x_j)]\), where \(i, j \in \{1, \ldots, d\}\), “?” is \(\leq\) or \(\geq\), and \(f\) is a monotone step function. The total complexity of these step functions is \(O(n)\). Conversely, any such expression can be mapped back to the complement of a union of \(O(n)\) 2-sided orthants.

We first observe a few simple rules for rewriting expressions:

1. \([x_i \leq f(x_j)] \land [x_i \leq g(x_j)]\) can be rewritten as \([x_i \leq \min\{f, g\}(x_j)]\) if \(f\) and \(g\) are both increasing or both decreasing. Note that the lower envelope \(\min\{f, g\}\) is still a monotone step function with \(O(n)\) complexity. A similar rule applies for \(\geq\).
2. \([x_i \leq f(x_j)]\) can be rewritten as \([x_j \geq f^{-1}(x_i)]\) if \(f\) is increasing (the inequality is flipped if \(f\) is decreasing). Note that the inverse \(f^{-1}\) is still a monotone step function.
3. More generally, \([f(x_i) \leq g(x_j)]\) can be rewritten as \([x_j \geq (g^{-1} \circ f)(x_i)]\) if \(f\) is increasing (the inequality is flipped if \(f\) is decreasing). Note that the composition \(g^{-1} \circ f\) is still a monotone step function with \(O(n)\) complexity.
4. \([x_i \leq f(x_j)] \land [x_i \leq g(x_k)]\) can be rewritten as the disjunction of \([x_i \leq f(x_j)] \land [f(x_j) \leq g(x_k)]\) and \([x_i \leq g(x_k)] \land [g(x_k) \leq f(x_j)]\). A similar rule applies for \(\geq\).

The plan is to decrease the dimension by repeatedly eliminating variables.

We maintain a simple function \(H\). Initially, \(H(x_1, \ldots, x_d) = \sigma(x_1) \cdots \sigma(x_d)\), where \(\sigma(x)\) denotes the successor of \(x\) among the \(O(n)\) input coordinate values (\(\sigma\) is a step function). We call an index \(i\) free if the variable \(x_i\) appears exactly once in \(H\) and is “unaltered”, i.e., \(h_i(x_i) = \sigma(x_i)\). All indices are initially free.

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Fig. 3 The union of one type of 2-sided orthants
In each iteration, we pick a free index $i$. Whenever $x_i$ appears more than twice in $E$, we can apply rule 4 (in combination with rules 1–3) to obtain a disjunction of two subexpressions, where in each subexpression, the number of occurrences of $x_i$ is decreased. By repeating this process $O(1)$ times (recall that $d$ is a constant), we obtain a disjunction of $O(1)$ subexpressions, where in each subexpression, only at most two occurrences of $x_i$ remain—in at most one predicate of the form $[x_i \leq f(x_j)]$, and at most one predicate of the form $[x_i \geq g(x_k)]$.

We branch off to maximize $H$ over each of these subexpressions separately. In such a subexpression, to eliminate the variable $x_i$ while maximizing $H$, we replace the two predicates $[x_i \leq f(x_j)]$ and $[x_i \geq g(x_k)]$ with $[f(x_j) \geq g(x_k)]$, and replace $x_i$ with $f(x_j)$ in $H$ (i.e., reset $h_j(x_j)$ to $h_j(x_j)\sigma(f(x_j))$, which is still a step function with $O(n)$ complexity). Now, $i$ and $j$ are not free.

We stop a branch when there are no free indices left. At the end, we get a large but $O(1)$ number of subproblems, where in each subproblem, at least $\lfloor d/2 \rfloor$ variables have been eliminated, i.e., the dimension is decreased to $d' \leq \lfloor d/2 \rfloor$. We solve each subproblem by Lemma 4.3 in $\tilde{O}(n^{d/2})$ time.

We now combine Theorem 4.4 and Lemma 4.2 to solve Problem 4.1:

**Corollary 4.5** Problem 4.1 can be solved in $\tilde{O}(n^{d/3+\lfloor d/2 \rfloor/6})$ time for any constant $d \geq 4$.

**Proof** Apply Lemma 4.2 to the $O(n)$ $(d-3)$- and $(d-2)$-flats through the $(d-3)$-faces and $(d-2)$-faces of the given orthants. This yields a partition of $B_0$ into cells.

Consider a cell $\Delta$. The number of $(d-3)$-flats intersecting $\Delta$ is $O(n/r^3)$, which can be made 0 by setting $r := \Theta(n^{1/3})$. The number of $(d-2)$-flats intersecting $\Delta$ is $O(n/r^2) = O(n^{1/3})$. So, inside the cell $\Delta$, all orthants are 2-sided or 1-sided, with $O(n^{1/3})$ 2-sided orthants. The union of 1-sided orthants simplifies to the complement of a box (we can use orthogonal range searching or intersection data structures [1, 7] to identify the 1-sided orthants intersecting $\Delta$ and compute this box). We can thus apply Theorem 4.4 to maximize $H$ over the cell $\Delta$ in $\tilde{O}(n^{1/3}\lfloor d/2 \rfloor/2)$ time. As there are $O(r^d) = O(n^{d/3})$ cells, the total running time is $\tilde{O}(n^{d/3+\lfloor d/2 \rfloor/6})$. \hfill $\Box$

**Corollary 4.6** Given $n$ points in $\mathbb{R}^d$ and a box $B_0$, we can compute the maximum-volume empty anchored box inside $B_0$ in $\tilde{O}(n^{d/3+\lfloor d/2 \rfloor/6}) \leq \tilde{O}(n^{5d/12})$ time for any constant $d \geq 4$.

### 5 Largest Empty Box in Higher Dimensions

We now adapt the approach from Sect. 4 to solve the original largest empty box problem in higher constant dimensions. By $d$ levels of divide-and-conquer, it suffices to solve the point-restricted version of the problem: given a set $P$ of $n$ points in $\mathbb{R}^d$, a fixed box $B_0$, and a fixed point $o$, find the largest-volume box $B \subset B_0$ that contains $o$ and is empty of points of $P$. An $O(T(n))$-time algorithm for the point-restricted problem immediately yields an $O(T(n) \log^d n)$-time algorithm for the original problem (in fact, the polylogarithmic factor disappears if $T(n)/n^{1+b}$ is increasing for some constant $\delta > 0$). Without loss of generality, assume that $o$ is the origin.
By mapping a box \( B = (-x_1, x_1') \times \cdots \times (-x_d, x_d') \) (of volume \((x_1 + x_1') \cdots (x_d + x_d')\)) to the point \((x_1, x_1', \ldots, x_d, x_d')\) in \(2d\) dimensions, and mapping each input point \( p = (p_1, \ldots, p_d)\) to the orthant \((-p_1, \infty) \times (p_1, \infty) \times \cdots \times (-p_d, \infty) \times (p_d, \infty)\) in \(2d\) dimensions (and changing \( B_0 \) appropriately), the problem reduces to the following variant of Problem 4.1, after doubling the dimension:

**Problem 5.1** Define \( H_{\text{new-vol}}(x_1, \ldots, x_d) = (x_1 + x_2)(x_3 + x_4) \cdots (x_{d-1} + x_d) \) for an even \(d\). Given a set \( S\) of \( n\) orthants in \(\mathbb{R}^d\) and a box \( B_0 \), find the maximum of \( H_{\text{new-vol}} \) over \( B_0 - \bigcup S \).

The above objective function \( H_{\text{new-vol}} \) is a bit more complicated than the one from Sect. 4, and so further ideas are needed, as we will next describe.

### 5.1 Preliminaries

For a multigraph \( G \) with vertex set \( \{1, \ldots, d\} \) (without self-loops), define a \( G \)-function \( H: \mathbb{R}^d \to \mathbb{R} \) to be a function of the form

\[
H(x_1, \ldots, x_d) = \prod_{i=1}^d h_i(x_i) \cdot \prod_{e=ij \in G} (h'_e(x_i) + h''_e(x_j)),
\]

where \( h_i, h'_e, \) and \( h''_e \) are univariate step functions. The complexity of \( H \) refers to the total complexity of these step functions. A pseudo-forest is a graph where each component is either a tree, or a tree plus an edge—in the latter case, the component is called a 1-tree (and we allow the extra edge to be a duplicate of an edge in the tree).

**Lemma 5.2** Let \( H \) be a \( G \)-function with \( O(n) \) complexity. Given a box \( B_0 \), we can compute the maximum of \( H \) over \( B_0 \) in \( \tilde{O}(n) \) time if \( G \) is a forest, or \( \tilde{O}(n^2) \) time if \( G \) is a pseudo-forest, for any constant \(d\).

**Proof** For the forest case: Pick a leaf \( i \). Then \( H \) is of the form \( h(x_i) \cdot (h'(x_i) + h''(x_j)) \cdots \), where \( h, h', h'' \) are step functions and \( x_j \) does not appear in “...”. Define \( F(\xi) := \max \{h(x) \cdot (h'(x) + \xi) \} \). Then \( F \) is the upper envelope of \( O(n) \) linear functions in the single variable \( \xi \), and can be constructed in \( \tilde{O}(n) \) time by the dual of a planar convex hull algorithm [7]. We can eliminate the variable \( x_i \) by replacing the \( h(x_i) \cdot (h'(x_i) + h''(x_j)) \) factor with \( F(h''(x_j)) \) (which is a step function in \( x_j \) with \( O(n) \) complexity). As a result, \( H \) becomes a \((G - \{i\})\)-function in \(d - 1\) variables. After \( d \) iterations, the problem becomes trivial.

For the pseudo-forest case: We may assume the graph is connected, since we can maximize the parts of \( H \) corresponding to different components separately. Pick a vertex \( i \) that belongs to the unique cycle of \( G \) (if exists). Then \( G - \{i\} \) is a forest. By trying out all \( O(n) \) different settings of \( x_i \) (breakpoints of the step functions), the problem reduces to \( O(n) \) instances of the forest case. \(\square\)

**Lemma 5.3** Let \( H \) be a \( G \)-function with \( O(n) \) complexity, where \( G \) is a pseudo-forest. Given a set \( S \) of \( n \) boxes in \(\mathbb{R}^d\) and a box \( B_0 \), we can compute the maximum of \( H \) over \( B_0 - \bigcup S \) in \( \tilde{O}(n^{d/2+1}) \) time for any constant \(d\).
**Proof** Apply Lemma 4.2 to the $O(n)$ $(d - 2)$-flats through the boundaries of the orthants, together with the $O(n)$ $(d - 1)$-flats $x_j = a$ for all breakpoints $a$ of the step functions appearing in $H$. This yields a partition of $B_0$ into cells.

Consider a cell $\Delta$. The number of $(d - 2)$-flats intersecting $\Delta$ is $O(n/r^2)$, which can be made 0 by setting $r := \Theta(\sqrt{n})$. So, inside the cell $\Delta$, we see only 1-sided orthants, and their union simplifies to the complement of a box. In addition, the number of $(d - 1)$-flats intersecting $\Delta$ is $O(n/r) = O(\sqrt{n})$; in other words, the breakpoints of the step functions in $H$ relevant to the cell $\Delta$ is $O(\sqrt{n})$. We can thus apply Lemma 5.2 to maximize $H$ over the cell $\Delta$ in $\tilde{O}(n^{d/2})$ time. As the number of cells is $O(n^d) = O(n^{d/2})$, the total running time is $\tilde{O}(n^{d/2}(\sqrt{n})^2)$.

\[ \square \]

### 5.2 Algorithm

We now modify the proof of Theorem 4.4 to solve Problem 5.1 for the 2-sided orthant case:

**Theorem 5.4** In the case when all input orthants are 2-sided, Problem 5.1 can be solved in $\tilde{O}(n^{d/4+1})$ time for any constant even $d$.

**Proof** We maintain a $G$-function $H$. Initially,

\[ H(x_1, \ldots, x_d) = (\sigma(x_1) + \sigma(x_2))(\sigma(x_3) + \sigma(x_4)) \cdots (\sigma(x_{d-1}) + \sigma(x_d)), \]

with $G$ being a matching with $d/2$ edges, where $\sigma(x)$ denotes the successor of $x$ among all $O(n)$ input coordinate values. We call an index $i$ free if $x_i$ appears exactly once in $H$ and is “unaltered” (i.e., $H$ is of the form $(\sigma(x_i) + h(x_i)) \cdots$ where $x_i$ does not appear in “…”). All indices are initially free. We maintain the following invariants: at any time, (i) $G$ is a pseudo-forest with at most $d/2$ edges, and (ii) for each component $T$ of $G$ which is a tree (not a 1-tree), $T$ has at least two free leaves.

In each iteration, we pick a free leaf $i$ in some component $T$ of $G$ which is a tree. As before, we rewrite the expression $E$ as a disjunction of $O(1)$ subexpressions, where in each subexpression, only two occurrences of $x_i$ remain—in a predicate of the form $[x_i \leq f(x_j)]$, and another predicate of the form $[x_i \geq g(x_k)]$.

We branch off to maximize $H$ for each of these subexpressions separately. In such a subexpression, to eliminate the variable $x_i$ while maximizing $H$, we replace the two predicates $[x_i \leq f(x_j)]$ and $[x_i \geq g(x_k)]$ with $[f(x_j) \geq g(x_k)]$, and replace $x_i$ with $f(x_j)$ in $H$ (since $x_i$ is free). Now, $i$ and $j$ are not free. Also, in the graph $G$, the unique edge $i \ell$ incident to $i$ is replaced by $j \ell$ (unless $j = \ell$). If $j$ is in the same component $T$ as $i$, then $T$ becomes a 1-tree; otherwise, two components are merged and the new component is either a tree with at least two free leaves, or a 1-tree (see Fig. 4). So, the invariants are maintained.

We stop a branch when there are no free indices left. At the end, we get $O(1)$ subproblems, where in each subproblem, all components are 1-trees, and so the number of nodes is exactly equal to the number of edges, implying that the dimension is $d' \leq d/2$. Now we can apply Lemma 5.3 to solve these subproblems in $\tilde{O}(n^{d/4+1})$ time.

\[ \square \]
Corollary 5.5  Problem 5.1 can be solved in $\tilde{O}(n^{(5d+4)/12})$ time for any constant even $d$.

Proof  Following the proof of Corollary 4.5 but using Theorem 5.4 instead of Theorem 4.4 gives running time $\tilde{O}(n^{d/3}(n^{1/3}d/4+1))$.

Applying the above corollary in $2d$ dimensions, we finally obtain:

Corollary 5.6  Given $n$ points in $\mathbb{R}^d$ and a box $B_0$, we can compute the maximum-volume empty box inside $B_0$ in $\tilde{O}(n^{(5d+2)/6})$ time for any constant $d$.

6 Largest Empty Anchored Box in Higher Dimensions (Further Improved)

In this section, we return to the largest empty anchored box problem and describe a further improvement to the result in Sect. 4, by incorporating the graph-theoretic approach from Sect. 5.

For a multigraph $G$ with vertex set $\{1, \ldots, d\}$ (without self-loops), define a generalized $G$-function $H : \mathbb{R}^d \to \mathbb{R}$ to be a function of the form

$$H(x_1, \ldots, x_d) = \prod_{i=1}^{d} h_i(x_i) \cdot \prod_{e=ij \in G} h'_e(x_i, x_j),$$

where each $h_i$ is a univariate step function and each $h'_e$ is a bivariate step function. Here, in a bivariate step function $h'_e$, the domain is divided into grid cells by horizontal and vertical lines, and $h'_e$ is constant in each grid cell; the complexity of $h'_e$ refers to the number of horizontal and vertical lines. The complexity of $H$ is the total complexity of the univariate and bivariate step functions.

Lemma 6.1  Let $H$ be a generalized $G$-function with $O(n)$ complexity, where $G$ is a matching. Given a box $B_0$, we can compute the maximum of $H$ over $B_0$ in $O(n^2)$ time for any constant $d$.

Proof  Trivial.  \qed
Lemma 6.2 Let \( H \) be a generalized \( G \)-function with \( O(n) \) complexity, where \( G \) is a matching. Given a set \( S \) of \( n \) boxes in \( \mathbb{R}^d \) and a box \( B_0 \), we can compute the maximum of \( H \) in \( B_0 - \bigcup S \) in \( O(n^{d/2+1}) \) time for any constant \( d \).

**Proof** Similar to the proof of Lemma 5.3, but using Lemma 6.1 instead of Lemma 5.2 as subroutine. \( \square \)

We now improve Lemma 4.3 for 2-sided orthants:

**Lemma 6.3** Let \( H \) be a simple function with \( O(n) \) complexity. Given a set \( S \) of \( n \) 2-sided orthants in \( \mathbb{R}^d \) and a box \( B_0 \), we can compute the maximum of \( H \) in \( B_0 - \bigcup S \) in \( \tilde{O}(n^{d/3+1}) \) time for any constant \( d \).

**Proof** We maintain a generalized \( G \)-function \( H \). Initially, \( G \) consists of \( d \) isolated vertices. We repeatedly find variables \( x_i \) to eliminate:

**Case 1** There is an isolated index \( i \) in \( G \). As before, we rewrite the expression \( E \) as a disjunction of \( O(1) \) subexpressions, where in each subexpression, only two occurrences of \( x_i \) remain—in a predicate of the form \([x_i \leq f(x_j)]\), and another predicate of the form \([x_i \geq g(x_k)]\). We branch off to maximize \( H \) for each of these subexpressions separately. In such a subexpression, to eliminate the variable \( x_i \) while maximizing \( H \), we replace the two predicates \([x_i \leq f(x_j)]\) and \([x_i \geq g(x_k)]\) with \([f(x_j) \geq g(x_k)]\), and replace \( h_i(x) := \min_{x: g(x_k) \leq x \leq f(x_j)} h_i(x) \) in \( H \). We remove \( i \) from \( G \), and add edge \( jk \) to \( G \).

**Case 2** There is an index \( i \) of degree at least 2 in \( G \). We try out all \( O(n) \) different settings of \( x_i \) (breakpoints of the step functions), and obtain \( O(n) \) instances in which \( i \) is removed from \( G \).

We stop a branch when neither case is applicable, i.e., all indices in \( G \) have degree 1, i.e., \( G \) is a matching. Here, we can apply Lemma 6.2 to solve the problem.

Consider one branch. Suppose Case 1 is applied \( s \) times and Case 2 is applied \( t \) times. At the end, the number of vertices in \( G \) is \( d' := d - s - t \), and the number of edges in \( G \) is at most \( s - 2t \) (since Case 1 adds one edge and Case 2 removes at least two edges). Since \( G \) is a matching at the end, the number of vertices is twice the number of edges. Thus, \( d - s - t \leq 2(s - 2t) \), i.e., \( s \geq d/3 + t \), i.e., \( d' \leq 2d/3 - 2t \). Since \( O(n') \) instances are generated and Lemma 6.2 has cost \( \tilde{O}(n^{d/2+1}) \), the total cost is \( \tilde{O}(n'^n d'^{d/2+1}) \leq \tilde{O}(n^{d/3+1}) \).

The above lemma improves the time bound in Theorem 4.4 to \( O(n^{(d/2)/3+1}) \). This in turn improves the time bound in Corollary 4.5 to \( O(n^{d/3}(n^{1/3} d^{d/2}/3+1)) = O(n^{(d+1)/3 + (d/2)/9}) \).

**Corollary 6.4** Given \( n \) points in \( \mathbb{R}^d \) and a box \( B_0 \), we can compute the maximum-volume empty anchored box inside \( B_0 \) in \( \tilde{O}(n^{(d+1)/3 + (d/2)/9}) \) time for any constant \( d \geq 3 \).
7 Remarks

On the 2D algorithm. The \(2^{O(\log^* n)}\) factor can be analyzed more precisely (an upper bound of \(3^{\log^* n}\) can be shown with minor changes to the algorithm). A question remains whether the extra factor could be further lowered to inverse-Ackermann, or eliminated completely.

The previous algorithm by Aggarwal and Suri [3] used matrix searching techniques, namely, for finding row minima in certain types of partial Monge matrices. We are able to bypass such subroutines because we have focused our effort on solving the decision problem (due to the author’s randomized optimization technique [10]). Generally, the row minima problem is equivalent to the computation of lower envelopes of pseudo-rays and pseudo-segments, not necessarily of constant complexity [13]. However, to solve the decision problem, we only need lower envelopes of pseudo-rays and pseudo-segments of constant complexity (formed by hyperbolas), for which there are simpler direct methods, as we have noted in Lemma 2.3. (Incidentally, the proof we gave for reducing Lemma 2.3 (b) to (a) is essentially equivalent to Aggarwal and Klawe’s reduction of row minima in double-staircase to staircase matrices [2]; a similar idea has also been used in dynamic data structures with “FIFO updates” [14].)

On the other hand, it should be possible to modify our approach to get improved deterministic algorithms for 2D largest empty rectangle, by solving the optimization problem directly and using known matrix searching subroutines [24], though details are more involved and the running time seems slightly worse than in our randomized algorithm.

It is theoretically possible to devise an optimal algorithm for Problem 2.1 without knowing the true complexity of the algorithm, since by a constant number of rounds of recursion in our method, the problem is reduced to subproblems of very small size (say, \(\log \log \log \log n\)), for which we can afford to explicitly build an optimal decision tree (this type of trick appeared before in the literature [25, 29]).

On the higher-dimensional algorithms. Our approach in higher dimensions works for maximizing the perimeter (sum of edge lengths) of the box as well. In fact, the algorithm for the simpler, largest empty anchored box problem should suffice here after doubling the dimension, since the required objective function here is \(H_{\text{perim}}(x_1, \ldots, x_d) = x_1 + \cdots + x_d\), which is “similar” to \(H_{\text{vol}}(x_1, \ldots, x_d) = x_1 \cdots x_d\).

For the largest empty anchored box problem, we have improved the \(\tilde{O}(n^{\lfloor d/2 \rfloor})\) time bound to \(\tilde{O}(n^{5d/12})\) in Sect. 4, and then to \(\tilde{O}(n^{(7d+6)/18})\) in Sect. 6 using graph-theoretic ideas. Still further improvements of the exponent is likely possible, by working with \(G\)-functions for hypergraphs \(G\), not just graphs, though improvement on the fraction 7/18 appears very tiny and requires \(d\) to be a very large constant, and the algorithm becomes more complicated. For the largest empty box problem, we currently don’t know how to improve the fraction 5/6, even using hypergraphs. It remains a fascinating question what the best fraction \(\beta\) is for which the problem could be solved in \(O(n^{\beta d + o(d)})\) time.

On the conditional lower bound side, another relevant question is whether Problems 4.1 or 5.1 remain \(W[1]\)-hard with respect to the parameter \(d\) in the special case of 2-sided orthants.
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