Forcing linearity numbers for coatomic modules

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Abstract
We show that an integer \( n \in \mathbb{N} \cup \{0\} \) is the forcing linearity number of a coatomic module over an arbitrary commutative ring with identity if and only if \( n \in \{0, 1, 2, \infty\} \cup \{q + 2 | q \text{ is a prime power}\} \).

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1. Introduction
Throughout this paper \( R \) shall denote a commutative ring with identity and \( V \) a unital right \( R \)-module. Consider the set \( M_R(V) := \{ f : V \to V | f(vr) = f(v)r \text{ for all } r, v \in V \} \). Under the operations of pointwise addition and composition of functions, \( M_R(V) \) is a near-ring with identity, called the near-ring of homogeneous functions. Note that \( M_R(V) \) contains the endomorphism ring \( \text{End}_R(V) \). The question arises how much linearity is needed on a function \( f \in M_R(V) \) to ensure that \( f \) is linear on all of \( V \), i.e. \( f \in \text{End}_R(V) \). More precisely, we say that a collection \( \{W_i | i \in I\} \) of proper submodules forces linearity on \( V \), if whenever \( f \in M_R(V) \) and \( f \) is linear on each \( W_i, i \in I \), then \( f \in \text{End}_R(V) \). Thus \( M_R(V) = \text{End}_R(V) \) if and only if the empty collection forces linearity on \( V \). The smallest number of modules which force linearity on \( V \) gives rise to the forcing linearity number of \( V \).

Definition 1.1. [3] Let \( V \) be an \( R \)-module. The forcing linearity number \( f \ln(V) \in \mathbb{N} \cup \{0, \infty\} \) of \( V \) is defined as follows:

1. If \( M_R(V) = \text{End}_R(V) \), then \( f \ln(V) = 0 \).
2. If \( M_R(V) \neq \text{End}_R(V) \), and there is some finite collection \( \{W_i | 1 \leq i \leq n\}, n \in \mathbb{N} \), of proper submodules of \( V \) which forces linearity on \( V \), but no collection of fewer than \( n \) proper submodules forces linearity, then we say that \( f \ln(V) = n \).
3. If neither 1. or 2. holds, then we say that \( f \ln(V) = \infty \).

Forcing linearity numbers have been found for several classes of rings and modules, see for example [3], [4], [5] and their references. In section 2 we determine the forcing linearity number of coatomic modules over an arbitrary commutative ring \( R \) with identity. An \( R \)-module \( V \) is called coatomic, if every proper submodule is contained in a maximal submodule of \( V \). For example a finitely generated module or a semisimple module over any ring is coatomic. For a commutative noetherian local ring, the coatomic modules have been characterized in [7].

2. Forcing linearity numbers of coatomic modules
For an \( R \)-module \( V \) and subsets \( S_1, S_2 \) of \( V \) let \( (S_1 : S_2) = \{ r \in R | S_2r \subseteq S_1 \} \). For \( v \in V \) let \( \text{Ann}(v) = \{ r \in R | vr = 0 \} \).
Theorem 2.1. Let $V$ be an $R$–module and let $M,N$ be maximal submodules of $V, M \neq N$. The following are equivalent:

1. The collection $\{M,N\}$ does not force linearity.
2. $\exists w \neq 0 \in V: (M : V) = (N : V) = \text{Ann}(w)$.

Proof. $1 \Rightarrow 2$: Since $\{M,N\}$ does not force linearity on $V$, there exists a function $f \in M_R(V)$ such that $f$ is linear on the submodules $M,N,$ but $f \notin \text{End}_R(V)$. Let $u,v \in V$ be such that $w := f(u+v) - f(u) - f(v) \neq 0$. Since $M \neq N$, and $M,N$ are maximal, we have that $M + N = V$. For every $v \in V - M$, $(M : v) = (M : V)$, therefore $(M : V)$ and $(N : V)$ are maximal ideals. If $(M : V) \neq (N : V)$, then $(M : V) + (N : V) = R$, hence $r + s = 1$ for some $r \in (M : V), s \in (N : V)$. Now $wr = f(ur + vr) - f(ur) - f(vr) = f(ur) + f(vr) - f(ur) - f(vr) = 0$, since $f$ is linear on $M$. Similarly, $ws = 0$, hence $w = w(1) = w(r + s) = 0$, a contradiction. Thus $(M : V) = (N : V)$, and since $(M : V) \subseteq \text{Ann}(w)$ and $(M : V)$ is a maximal ideal, it follows that $(M : V) = \text{Ann}(w)$.

$2 \Rightarrow 1$: Let $v \in V - M$. Then $(M : v) = (M : V) = \text{Ann}(w)$ and $h : V/M \rightarrow Rw, h(vr/M) := wr$ is an isomorphism. Define a function $f : V \rightarrow V$ as follows: For $m \in M, n \in N$ let

$$f(m+n) := \begin{cases} h(n/M) & \text{if } m+n \notin M \cup N \\ 0 & \text{otherwise} \end{cases}$$

Since $M + N = V$, $f$ is defined on $V$. We show that $f$ is well–defined. Suppose $m_1+n_1 = m_2+n_2, m_1,m_2 \in M, n_1,n_2 \in N$. If $m_1+n_1 \in M \cup N$, then $f(m_1+n_1) = f(m_2+n_2) = 0$. If $m_1+n_1 \notin M \cup N$, then $n_1n_2 \in M$, hence $f(m_1+n_1) = h(n_1/M) = h((m_2/M) - f(m_2+n_2)$. Next we show that $f$ is homogeneous. Let $S := V - (M \cup N)$. If $m+n \in S$, then $h(m+n) \neq R$, hence $f(m+n) = h((m+n)/M) = h((m+n)/S)$. If $r \notin S$, then $m+n \notin S$, hence $f(m+n) = h((m+n)/M) = h((m+n)/S) = 0$. Now suppose $m+n \notin S$. Then $m+n \in M \cup N$, hence $(m+n)/M \in S$ for all $r \in R$. Thus $f(m+n) = 0 = f((m+n)/S)$. It now follows that $f \in M_R(V)$.

Since $f/M = f/N = 0$, $f$ is linear on $M$ and $N$. However, for $m \in M - N$ and $n \in N - M$, we have that $m+n \in S$, thus $f(m+n) = h(n/M) \neq 0$, since $h$ is an isomorphism, whereas $f(m) + f(n) = 0$, so $f \notin \text{End}_R(V)$. Therefore the collection $\{M,N\}$ does not force linearity on $V$. $\square$

For an $R$–module $V$ let $\text{Rad}(V)$ denote the Jacobson radical of $V$ and let $J := \text{Rad}(R)$. Recall that an $R$–module $V$ is called local, if $V$ contains a unique maximal submodule.

Theorem 2.2. For a noncyclic coatomic module $V$, the following are equivalent:

1. $f \ln(V) > 2$.
2. $I := (\text{Rad}(V) : V)$ is a maximal ideal and $I = \text{Ann}(w)$ for some $0 \neq w \in V$.

Proof. $1 \Rightarrow 2$: Let $M$ denote the collection of all maximal submodules of $V$. Since $V$ is coatomic, $M \neq 0$. If there exist $M_1,M_2 \in M$ such that $(M_1 : V) \neq (M_2 : V)$, then by Theorem 2.1 the collection $\{M_1,M_2\}$ forces linearity on $V$. Thus $(M_1 : V) = (M_2 : V)$ for all $M_1,M_2 \in M$ and $I = \bigcap\{(M : V) | M \in M\} = (M : V)$ for all $M \in M$, hence $I = (\text{Rad}(V) : V)$ is a maximal ideal. Like in the proof of Theorem 1, we see that $I = \text{Ann}(w)$ for some $w \neq 0$.

$2 \Rightarrow 1$: Suppose that $V$ is a local module with unique maximal submodule $M$. Let $v \in V - M$. If $vR \neq V$, then $vR$ is contained in a maximal submodule, which implies $vR \subseteq M$, a contradiction. Consequently $vR = V$ for all $v \in V - M$, which contradicts our assumption that $V$ is noncyclic. Therefore there exist at least two maximal submodules. Suppose $f \ln(V) \leq 2$. Then there exists a collection of submodules $\{S_1,S_2\}$ which forces linearity on $V$. Since $V$ is coatomic, there exist maximal submodules $M_1,M_2$ such that $S_1 \subseteq M_1, S_2 \subseteq M_2$. Without loss of generality we may assume that $M_1 \neq M_2$ (otherwise we can choose another maximal submodule, since $V$ is not local). Then $\{M_1,M_2\}$ also forces linearity on $V$. We have $(\text{Rad}(V) : V) \subseteq (M_1 : V) \neq R$. By our assumptions $(\text{Rad}(V) : V)$ is a maximal ideal, hence $(\text{Rad}(V) : V) = (M_1 : V) = (M_2 : V)$. Also, $(\text{Rad}(V) : V) = \text{Ann}(w)$ for some $0 \neq w \in V$. Therefore $\{M_1,M_2\}$ does not force linearity by Theorem 1, a contradiction. $\square$

Theorem 2.3. Let $V$ be coatomic. Suppose $I := (\text{Rad}(V) : V)$ is a maximal ideal of $R$ and there exists $0 \neq w \in V$ such that $I = \text{Ann}(w)$. Then

$$f \ln_R(V) = f \ln_R(I(V/\text{Rad}(V)))$$

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Proof. We first show that $f \ln_{R/I}(V/Rad(V)) \leq f \ln_{R}(V)$. Let $\{W_{i}| i \in I\}$ be a collection of proper submodules which forces linearity on $V$. Since $V$ is commutative, we may assume that each $W_{i}, i \in I$, is maximal. We show that the collection $\{W_{i}/Rad(V) | i \in I\}$ forces linearity on $V/Rad(V)$. Suppose that this is not the case. Then there exists a homogeneous function $f : V/Rad(V) \to V/Rad(V)$, which is linear on each submodule $W_{i}/Rad(V), i \in I$, but not linear on $V/Rad(V)$. Let $\pi_{M} : V/Rad(V) \to V/M$ denote the projection of $V/Rad(V)$ onto $V/M$ for a maximal submodule $M$. Since $f$ is not linear, there exists a maximal submodule $M$ of $V$ such that $\pi_{M}f : V/Rad(V) \to V/M$ is not linear. Since $I$ is a maximal ideal, $I = \langle M : V \rangle$, hence $w(M : V) = 0$, which implies $V/M \cong wR$. Thus we obtain a homogeneous map $f_{1} : V/Rad(V) \to wR$, which is linear on each submodule $W_{i}/Rad(V), i \in I$. If $g : V \to V$ is defined by $g(v) := f_{1}(v/Rad(V))$, then $g \in M_{R}(V)$ and linear on each $W_{i}, i \in I$, but not linear on $V$, a contradiction to our assumption that $\{W_{i}/Rad(V) | i \in I\}$ forces linearity on $V$. For the reverse inequality suppose first that $f \ln_{R/I}(V/Rad(V)) \leq 1$. Since $V/Rad(V)$ is a vector space over the field $R/I$, it follows from Theorem 3.1 in [3] that $\dim_{R/I}(V/Rad(V)) = 1$. Note that $Rad(V)$ is a superfluous submodule, since $V$ is commutative. It follows that $V$ is cyclic, hence $f \ln_{R/I}(V/Rad(V)) = 0 = f \ln_{R}(V)$. If $\dim_{R/I}(V/Rad(V)) = 2$ or $f \ln_{R/I}(V/Rad(V)) \geq 2$ and $R/I$ is infinite, we have that $f \ln_{R/I}(V/Rad(V)) = 0$ by Theorem 3.1 in [3]. So suppose that $f \ln_{R/I}(V/Rad(V)) \geq 3$ and $|R/I| = q \in \mathbb{N}$. By [3], 3.8 and 3.10, $f \ln_{R/I}(V/Rad(V)) = q + 2$. Choose $\{r_{1}, \ldots, r_{q}\} \subseteq R$ such that $R/I = \langle r_{1}/I, \ldots, r_{q}/I \rangle$. It suffices to give a collection of $q + 2$ proper submodules which forces linearity on $V$. Let $\{b_{i}| i \in I\} \subseteq V$ be such that $\{b_{i}/Rad(V) | i \in I\}$ is a basis of the vector space $V/Rad(V)$. As we have seen above, $|I| \geq 3$, so we can choose pairwise different elements $i_{1}, i_{2}, i_{3} \in I$. Let $(X)$ denote the submodule generated by a subset $X \subseteq V$, and define $S_{1} := (b_{1} + b_{2} + Rad(V), S_{2} := (b_{1} + b_{3} + b_{i}/I \notin \{i_{1}, i_{2}, i_{3}\} + Rad(V), and for $r \in \{r_{1}, \ldots, r_{q}\}$ define $S_{r} := (b_{i_{1}} + r_{b_{2}} + b_{i_{3}} + b_{i}/I \notin \{i_{1}, i_{2}, i_{3}\} + Rad(V)$. Note that all submodules are proper, since $Rad(V)$ is superfluous. Similarly as in Theorems 3.8,3.10 in [3], one can prove that the collection $\{S_{1}, S_{2}\} \cup \{S_{r} | i \in \{1, \ldots, q\}\}$ forces linearity on $V$. \endproof

For $R$ local and $J$-nilpotent, Theorem 2.3 has been proved in [4]. Theorem 5.1. The following example shows that Theorem 2.3 is not true in general, if $I$ is not the annihilator of some $0 \neq w \in V$.

Example 2.4. Let $R := F[[x]]$ denote the ring of formal power series over a field $F$ and let $V := R \times R$. Since $R$ is local with radical $J = \langle x \rangle$, $Rad(V) = VJ = \langle x \rangle \times \langle x \rangle$ and $I = \langle Rad(V) : V = \langle x \rangle \rangle$ is maximal. By [3], Corollary 2.4, $f \ln_{R}(V) = 1$. However, $f \ln_{R/I}(V/Rad(V)) = f \ln_{F}(F^{2}) = \infty$, by [3], Theorem 3.1.

Theorem 2.5. Let $n \in \mathbb{N} \cup \{0, \infty\}$. Then $n$ is the forcing linearity number of a coatomic module over a commutative ring if and only if $n \in \{0, 1, 2, \infty\} \cup \{q + 2| q \text{ is a prime power}\}$. \endproof

Proof. It is well-known that there exist coatomic modules $V$ over a commutative ring $R$ such that $f \ln_{R}(V) \in \{0, 1, 2, \infty\}$, see for example [5]. If $V$ is a cyclic module, then $M_{R}(V) = End_{R}(V)$, hence $f \ln_{R}(V) = 0$. Now suppose $f \ln_{R}(V) > 2$. By Theorem 2.2, $I = \langle Rad(V) : V \rangle$ is a maximal ideal and $I = Ann(w)$ for some $0 \neq w \in V$. By Theorem 2.3, $f \ln_{R}(V) = f \ln_{R/I}(V/Rad(V))$ and as we have remarked previously, $f \ln_{R/I}(V/Rad(V)) \in \{\infty\} \cup \{q + 2| q \text{ is a prime power}\}$. \endproof

It is not known to the author, whether Theorem 2.5 is true for every module over a commutative ring.

There is a class of rings which have the property that every right module is coatomic, or which is easily seen to be equivalent, every nonzero right module has a maximal submodule.

Definition 2.6. A ring $R$ is called a right max–ring, if every right $R$–module is coatomic. See [6].

Theorem 2.7. [2] For a commutative ring $R$, the following are equivalent:

1. $R$ is a max–ring.

2. $J$ is $T$–nilpotent and $R/J$ is von Neumann regular.

Theorem 2.8. Let $V$ be a module over a commutative max–ring $R$. If $R$ is not local, then $f \ln_{R}(V) \leq 2$. \endproof

Proof. Suppose that $R$ is not local, but $f \ln_{R}(V) > 2$. Since $R$ is a max-ring, it follows from Theorem 2.7 and from [1], Proposition 18.3 that $Rad(V) = VJ$. By Theorem 2.2, $\langle Rad(V) : V \rangle = \langle VJ : V \rangle$ is a maximal ideal. We have $J \subseteq \langle VJ : V \rangle$. Suppose that there exists an element $r \in \langle VJ : V \rangle$. Then $r \notin M$ for some maximal ideal $M$ of $R$. Let $R_{M}, V_{M}$ denote the localisations of $R, V$ at $M$. By [1], Proposition 18.3, $Rad(V_{M}) = V_{M}J_{M}$. Since $R$ is a max-ring $J$ is $T$-nilpotent, thus $J_{M}$ is $T$-nilpotent. It follows from Theorem 2.5 that $R_{M}$ is a max-ring, hence $Rad(V_{M}) = V_{M}J_{M} \neq V_{M}$. So let $w/1 \in V_{M} - Rad(V_{M})$. From $r \in \langle VJ : V \rangle$, $w/1 \cdot r/1 = wr/1 \in V_{M}J_{M}$. Since $r \notin M$, $r/1$ is invertible in $R_{M}$, hence $w/1 \in V_{M}J_{M} = Rad(V_{M})$, a contradiction. It now follows that $J = \langle VJ : V \rangle$ is a maximal ideal of $R$, which contradicts our assumption that $R$ is not local.
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