Nucleon dynamics in effective field theory of nuclear forces

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Low energy nucleon dynamics in the effective field theory (EFT) of nuclear forces is investigated by using the formalism of the generalized quantum dynamics (GQD). This formalism is based on a generalized dynamical equation derived as the most general equation of motion consistent with the current concepts of quantum theory and allows one to extend quantum dynamics to the case where the fundamental interaction in a quantum system is nonlocal in time. The Schrödinger equation follows from this equation in the case where the interaction in a system is instantaneous. We show that the effective theory of nuclear forces gives rise to low energy nucleon dynamics which is governed by the generalized dynamical equation with a nonlocal-in-time interaction operator. This operator is shown to provide a natural parametrization of the interaction of nucleons and can be derived order by order by using methods of subtractive EFT’s. We show that the use of the GQD for describing nucleon dynamics allows one to formulate the EFT of nuclear forces as a completely consistent theory with a well-defined equation of motion which does not require renormalization.

I. INTRODUCTION

Effective field theories (EFT’s) are an important tool for computing physical quantities in theories with disparate energy scales [1-2]. Following the early work of Weinberg and others [3-7], the EFT approach has become very popular in nuclear physics [8,9]. To describe low energy processes involving nucleons and pions, all operators consistent with the symmetries of QCD are included in an effective Lagrangian. A fundamental difficulty is that such a Lagrangian yields graphs which are divergent, and gives rise to singular quantum mechanical potentials. To resolve this problem one has to use some renormalization procedure which regulates the integrals and subtracts the infinities in a systematic way. However, as is well known, there are not any equations for renormalized amplitudes in subtractive EFT’s. For this reason in some EFT’s (see, for example, Refs.[6,10,11]) finite cut-off regularization is used. Nevertheless, even when it is assumed that the cut-off has a physical meaning renormalization is required to render such theories consistent, and certain cut-off-dependent terms have to be absorbed into theories constants before determining them from empirical data.

Another problem of the EFT approach is that one cannot parametrize the interaction of nucleons, by using some Lagrangian or Hamiltonian. In fact, knowing the effective Lagrangian is not sufficient to compute results for physical quantities. In addition, one needs to specify a way to make the physical predictions finite. The counterterms of renormalization involve additional unknown parameters. A renormalization scheme allow one to use experimentally determined parameters instead of the above unknown ones. Thus for computing physical quantities one needs not only knowing the effective Lagrangian of the theory but also knowing some experimentally determined parameters. The nucleon-nucleon (NN) interaction cannot also be parametrized by using the singular potential produced by the above effective Lagrangian: These potentials do not make sense without regularization and renormalization. On the other hand, if the EFT approach, as it is widely believed, is able to provide a fundamental description of the interactions of nucleons at low energies, one can hope that it will give rise to parametrization of these interactions by an interaction operators being so fundamental as the Coulomb potential that parametrizes the interaction of charged particles in low energy QED. In the quantum mechanics of particles interacting via the Coulomb potential, which is an example of an effective theory, one deals with well-defined interaction Hamiltonian and the Schrödinger equation governing the dynamics of the theory. This theory is internally consistent and provides an excellent description of atomic phenomena at low energies. It is natural to expect that the EFT of nuclear forces is also able to provide such a fundamental description of nuclear phenomena at low energies. In particular, one can expect that this theory will allow one not only to calculate scattering amplitudes but also to construct the evolution operator describing the dynamics of nucleon systems. In the present paper we show that recent developments in quantum theory permit such a formulation of the effective theory of nuclear forces.

The above problem of subtractive EFT’s is the same that arises in any quantum field theory with UV divergences: Regularization and renormalization allow one to render the physical predictions finite, however, it is impossible to construct a renormalized Hamiltonian acting on the Fock space, i.e. after renormalization the dynamics of the theory is not governed by the Schrödinger equation. At the same time, in Ref.[12] it has been shown that the Schrödinger equation is not the most general dynamical equation consistent with the current concepts of quantum physics, and a more general equation of motion has been derived as a consequence of the Feynman [13] and canonical approaches to quantum theory. Be-
ing equivalent to the Schrödinger equation in the case of instantaneous interactions, this generalized dynamical equation permits the generalization to the case where the dynamics of a quantum system is generated by a nonlocal-in-time interaction. It has been shown [12] that a generalized quantum dynamics (GQD) developed in this way provides a new insight into the problem of UV divergences.

It should be noted that in the Hamiltonian formalism the interaction generating the dynamics of an isolated system cannot be nonlocal in time in principle. In fact, such a nonlocality of the interaction should be associated with an energy dependence of the interaction Hamiltonian, despite Hamiltonian itself is an operator representing the total energy of the system. This conceptual problem manifests itself in the fact that energy-dependent Hamiltonians are not Hermitean, and, as a result, the evolution of the system is not unitary. Such peculiarities are typical in definitions of renormalized interaction Hamiltonians (see, for example, Refs.[14,15]). The reason for this is quite obvious. The Schrödinger equation is local in time, and the interaction Hamiltonian describes an instantaneous interaction. Hence the introduction of a nonlocal-in-time (energy dependent) interaction Hamiltonian into the Schrödinger equation leads to an internal inconsistency of the theory. A remarkable feature of the generalized dynamical equation is that it provides the extension of the theory to the case of such interactions.

Quantum mechanics is one of the basic ingredients of quantum field theory (EFT). For this reason one may expect that in nonrelativistic limit the Schrödinger equation can be used as an equation of motion in the EFT of nuclear forces. However, this is not true, since within the framework of this theory the Schrödinger equation does not make a since without regularization and renormalization. Meanwhile, as has been shown in Ref.[12], only the generalized dynamical equation must be satisfied in general. This gives us the hope that this equation can provide a perfectly satisfactory description of low energy nucleon dynamics in the effective theory of nuclear forces. In the present paper we show that in leading order of the EFT approach low energy nucleon dynamics is governed by the generalized dynamical equation with a nonlocal-in-time interaction operator. Moreover, this dynamics is just the same as in the case of our model with a separable interaction operator [12,16] that was used as a test model illustrating the possibility of going beyond Hamiltonian dynamics provided by the GQD. As we show in the present paper, in the case of the quantum mechanics of nucleons at low energies, one has to deal with such a non-Hamiltonian dynamics. In Sec.III this is proved precisely in leading order of the EFT approach. We will show that the leading order contact component of the two-nucleon T-matrix obtained by summing bubble diagrams, coincides with the T-matrix of the above model in the particular case where the form factor in the separable interaction operator is of the form $\varphi(p) = 1$. This T-matrix has the properties that are at variance with the ordinary requirements of quantum mechanics, and does not satisfy the Lippmann-Schwinger (LS) equation. At the same time, this T-matrix is the solution of the generalized interaction operator with a nonlocal-in-time interaction operator, and this operator parametrizes the contact component of the NN interaction in leading order.

In Sec.IV we will investigate the features of low energy nucleon dynamics in leading order of the EFT approach. The possibility to extend our results to higher order will be investigated in Sec.V by using the example of the EFT of short-range forces developed by van Kolck [14]. We will show that in next-to-leading order, the dynamics of this theory is also governed by the generalized dynamical equation with a nonlocal-in-time interaction operator. By using this operator as an example, we will demonstrate how one can construct order by order the interaction operator parametrizing the NN interaction in the effective field theory of nuclear forces. The new possibilities that the GQD opens for practical calculations in the EFT approach will be discussed in Sec. VI. The advantages of this formalism become apparent in the case where numerical calculations are needed, i.e. in the case where the long-range components of the NN interactions are taken into account. It will be shown that for solving the evolution problem in this case the generalized dynamical equation can be reduced to an integral equation that does not require regularization and renormalization, and is so convenient for numerical calculations as the LS equation.

II. GENERALIZED QUANTUM DYNAMICS

Let us briefly review the main features of the formalism of the GQD developed in Ref.[12]. As is well known, the basic concept of the canonical formalism of quantum mechanics is that the theory can be formulated in terms of vectors of a Hilbert space and operators acting on this space. This formalism rests on the postulates, which establish the connection between these mathematical objects and observables and prescribe how to compute the probability of an event. In the canonical formalism they are used together with the dynamical postulate according to which the time evolution of a quantum system is governed by the Schrödinger equation. At the same time, in the Feynman formalism [13] quantum theory is formulated in terms of probability amplitudes without resorting to the vectors and operators acting on a Hilbert space. In this approach the following assumption is used as the first basic postulate:

The probability of an event is the absolute square of a complex number called the probability amplitude. The joint probability amplitude of a time-ordered sequence of events is product of the separate probability amplitudes of each of these events. The probability amplitude of an event which can happen in several different ways is a sum of the probability amplitudes for each of these ways.
According to this assumption, the probability amplitude of an event which can happen in several different ways is a sum of contributions from each alternative way. In particular, the amplitude \( < \psi_2 | U(t, t_0) | \psi_1 > \), being the probability of finding the quantum system in the state \( | \psi_2 > \) at time \( t \), if at time \( t_0 \) it was in the state \( | \psi_1 > \), can be represented as a sum of contributions from all alternative ways of realization of the corresponding evolution process. Dividing these alternatives in different classes, we can then analyze such a probability amplitude in different ways. For example, subprocesses with definite instants of the beginning and end of the interaction in the system can be considered as such alternatives. In this way the amplitude \( < \psi_2 | U(t, t_0) | \psi_1 > \) can be written in the form

\[
< \psi_2 | U(t, t_0) | \psi_1 > = < \psi_2 | \psi_1 > + \int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1 < \psi_2 | S(t_2, t_1) | \psi_1 > ,
\]

(1)

where \( < \psi_2 | S(t_2, t_1) | \psi_1 > \) is the probability amplitude that if at time \( t_1 \) the system was in the state \( | \psi_1 > \), then the interaction in the system will begin at time \( t_1 \) and will end at time \( t_2 \), and at this time the system will be in the state \( | \psi_2 > \). Here the interaction picture is used.

According to the above postulate the probability amplitude \( < \psi_2 | S(t_2, t_1) | \psi_1 > \) can itself be represented by the sum of amplitudes for each of the ways in which the subprocess with completely specified instants of the beginning and end of the interaction in a quantum system can happen. However, some supplementary assumptions about the history of the system are needed. In the Feynman approach it is assumed that this history can be represented by some path in space-time. In this case the amplitude \( < \psi_2 | S(t_2, t_1) | \psi_1 > \) can be represented by the sum of contributions from all paths corresponding to processes in which the interaction begins at \( t_1 \) and ends at \( t_2 \). If we assume also that the contribution from a single path is an exponential whose (imaginary) phase is the classical action for this path (the second postulate of Feynman’s theory) and substitute the expression obtained in this manner into Eq.(1), we arrive at Feynman’s sum-over-paths formula for the transitions amplitudes. At the same time, in the formalism of the GQD the history of a quantum system is represented by the version of the time evolution of the system associated with completely specified instants of the beginning and end of the interaction in the system. Such a description of the history of a system is more general and require no supplementary postulates like the second postulate of the Feynman formalism. On the other hand, the probability amplitudes \( < \psi_2 | S(t_2, t_1) | \psi_1 > \), in terms of which the evolution of a system is described within the GQD, are used in the spirit of Feynman’s theory: The probability amplitude of any event is represented as a sum of such amplitudes. In Ref.[12] it has been shown that the use of the operator formalism of the canonical approach allows one to derive a relation for the amplitudes \( < \psi_2 | S(t_2, t_1) | \psi_1 > \) which can be regarded as an equation of motion.

By using the operator formalism, we can represent the probability amplitudes \( < \psi_2 | U(t_2, t_1) | \psi_1 > \) by the matrix elements of the evolution operator, which must be unitary

\[
U^+(t, t_0) U(t, t_0) = U(t, t_0) U^+(t, t_0) = 1,
\]

(2)

and must satisfy the composition law

\[
U(t, t') U(t', t_0) = U(t, t_0), \quad U(t_0, t_0) = 1.
\]

(3)

Meanwhile, \( S(t_2, t_1) \) whose matrix elements are \( < \psi_2 | S(t_2, t_1) | \psi_1 > \) may be only an operator-valued generalized function of \( t_1 \) and \( t_2 \), since only \( U(t, t_0) = 1 + \int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1 S(t_2, t_1) \) must be an operator on the Hilbert space. Nevertheless, it is convenient to call \( S(t_2, t_1) \) an “operator” by using this word in a generalized sense. In the case of an isolated system the operator \( S(t_2, t_1) \) can be represented in the form

\[
S(t_2, t_1) = \exp (iH_0 t) T(t_2 - t_1) \exp (-iH_0 t_1).
\]

(4)

As has been shown in Ref.[12], for the evolution operator \( U(t, t_0) \) given by (4) to be unitary for any times \( t_0 \) and \( t \), the operator \( S(t_2, t_1) \) must satisfy the following equation:

\[
(t_2 - t_1) S(t_2, t_1) = \int_{t_1}^{t_2} dt_4 \int_{t_1}^{t_4} dt_3 \times (t_4 - t_3) S(t_2, t_4) S(t_3, t_1).
\]

(5)

A remarkable feature of this equation is that it works as a recurrence relation, and allows one to obtain \( S(t_2, t_1) \) for any \( t_1 \) and \( t_2 \), if it is known in an infinitesimal neighborhood of the point \( t_2 = t_1 \). Since the operators \( S(t_2, t_1) \) describe the contributions to the evolution operator from the processes in which the interaction in the system begins at \( t_1 \) and ends at \( t_2 \), the above means that in order to construct the evolution operator it is sufficient to know the contributions to this operator from the processes with infinitesimal duration time of interaction. It is natural to associate these processes with the fundamental interaction in the system under study. This make it possible to use the relation (5) as a dynamical equation. One needs only to specify the boundary condition determining the behavior of \( S(t_2, t_1) \) in the limit \( t_2 \to t_1 \) and hence containing the dynamical information about the system. Denoting the contribution to the evolution operator from the processes associated with the fundamental interaction by \( H_{\text{int}}(t_2, t_1) \), such a boundary condition can be written in the form

\[
S(t_2, t_1) \to H_{\text{int}}(t_2, t_1) + o(\tau^2),
\]

(6)

where \( \tau = t_2 - t_1 \). The parameter \( \varepsilon \) is determined by demanding that \( H_{\text{int}}(t_2, t_1) \) must be so close to the solution of Eq.(5) in the limit \( t_2 \to t_1 \) that this equation has a unique solution having the behavior (4) near the
point $t_2 = t_1$. Within the GQD the operator $H_{int}(t_2, t_1)$ plays the same role as the interaction Hamiltonian in the ordinary formulation of quantum theory: It generates the dynamics of a system. Being a generalization of the interaction Hamiltonian, this operator is called the generalized interaction operator.

The operator $H_{int}(t_2, t_1)$ describes fundamental processes, starting from which, one can construct the evolution operator. In this process the system in the state $|\psi_1\rangle$ evolves freely up to some time $t$ when, as a result of the interaction, the state of the system jumps abruptly into the state $|\psi_2\rangle$, and then the system evolves freely again. The contribution from this process into the evolution operator is of the form $\delta(t_2 - t_1) |A(t_1)||\psi_1\rangle$, where the delta-function is needed for this contribution to be nonzero. Thus in this case the interaction operator should be of the form

$$H_{int}(t_2, t_1) = \delta(t_2 - t_1)A(t_1).$$

As has been shown in Ref.[12], the dynamical equation (7) with the boundary condition given by (1) and (7) is equivalent to the Schrödinger equation with the interaction Hamiltonian $H_I(t) = i/2A(t)$. Thus the dynamics governed by Eq.(7) is equivalent to Hamiltonian dynamics in the case where the generalized interaction operator is of the form

$$H_{int}(t_2, t_1) = -2i\delta(t_2 - t_1)H_I(t_1).$$

Correspondingly the interaction operator in the Schrödinger picture

$$H_{int}^{(s)}(t_2 - t_1) = \exp(-iH_0t_2)H_{int}(t_2, t_1)\exp(iH_0t_1)$$

has the form

$$H_{int}^{(s)}(\tau) = -2i\delta(\tau)H_I,$$

where $H_I = \exp(-iH_0\tau)H_I(t)\exp(iH_0\tau)$. In this case the interaction generating the dynamics is instantaneous. On the other hand, from both the physical and mathematical points of view there are no reasons to restrict ourselves to the case where the interaction operator is of the form (8). From the mathematical point of view, the boundary condition (7) with the operator $H_{int}(t_2, t_1)$ given by (8) is not only possible boundary condition for Eq.(8). From the physical point of view, this equation is a unique consequence of the representation (1) and the unitarity condition (2). The representation (1) in turn is a consequence of the first Feynman postulate that, as is well known, is formulated as a result of the analysis of the phenomenon of the quantum interference and hence is one of the most fundamental postulate of quantum theory. Thus Eq.(8) is a unique consequence of the first principles and can be considered as the most general dynamical equation consistent with the current concepts of quantum theory. Note, in this connection, that no new fundamental concepts and postulates are used in the formalism of the GQD. A novelty of this formalism consists in the fact that some basic postulates of the Feynman and canonical approaches to quantum theory are used in combination. This allows one to formulate the theory in terms of the operators $S(t_2, t_1)$. As has been shown in Ref.[12], theory provides a more detailed description of the dynamics of a quantum system then the description directly in terms of the evolution operators, as in the case of the canonical formalism, or in terms of Feynman’s path amplitudes. In the case where the interaction operator is of the form (8), i.e. the interaction is instantaneous, the Schrödinger equation for the evolution operator

$$i\frac{dU(t,t_0)}{dt} = H_I(t)U(t,t_0)$$

and Feynman’s sum-over-paths formula follow from the representation (1) and Eq.(8). At the same time, the dynamical equation permits the generalization to the case where the interaction is nonlocal-in-time, i.e. the time durations of the interaction in the fundamental processes that determine the dynamics of a system are not zero. In this case the dynamics depends not only on the form of the operator $H_{int}^{(s)}(\tau)$ but also on its dependence upon the duration time $\tau$ of the interaction. However, as we have seen, only the behavior $H_{int}^{(s)}(\tau)$ in the limit $\tau \to 0$ is relevant: Knowing the behavior of $H_{int}^{(s)}(\tau)$ in the infinitesimal neighborhood of the point $\tau = 0$ is sufficient to construct the evolution operator by using Eq.(8). Thus within the GQD we deal with a new type of nonlocality. In fact, the ordinary way of nonlocalization of a quantum field theory consists in introducing a nonlocal form factors that depend on parameters determining a scale of nonlocality. As for the operator $H_{int}(t_2, t_1)$, only its values in the infinitesimal neighborhood of the point $t_2 = t_1$ are relevant and hence the scale of its nonlocality in time is infinitesimally small. Thus in this case we deal with some quasilocal operators. This is very important from the point of view of applications to QFT where nonlocalization aimed at resolving the problem of the UV divergences leads to a loss of covariance of the theory. In Ref.[17] it has been shown that after renormalization the dynamics of the three-dimensional theory of a neutral scalar field interacting through a $\phi^4$ coupling is governed by the generalized dynamical equation (8) with a nonlocal-in-time interaction operator.

In order that the dynamical equation (8) with the boundary condition (1) have a unique solution, the operator $H_{int}(t_2, t_1)$ must be sufficiently close to its relevant solution. This means that this operator must satisfy the condition

$$\int(t_2 - t_1)H_{int}(t_2, t_1)_{t_2 \to t_1} \int_{t_1}^{t_2} dt_2 \int_{t_1}^{t_2} dt_3(t_4 - t_3) \times H_{int}(t_2, t_4)H_{int}(t_3, t_1) + o(\tau^{-1}).$$

Note that the value of the parameter $\epsilon$ depends on the form of the operator $H_{int}(t_2, t_1)$. Since $S(t_2, t_1)$ and $H_{int}(t_2, t_1)$ are only operator-valued distributions, the mathematical meaning of the conditions (1) and (10)
needs to be clarified. We will assume that the condition \( |\Psi_1\rangle > |\Psi_2\rangle > 0 \) of the Hilbert space. The condition \( \langle \Psi_1 | > |\Psi_2\rangle > 0 \) has to be considered in the same sense.

If \( H_{\text{int}}(t_2,t_1) \) is specified, Eq. (16) allows one to find the operator \( \tilde{S}(t_2,t_1) \). Formula (16) can then be used to construct the evolution operator \( \hat{U}(t,t_0) \) and accordingly the state vector \( |\psi(t)\rangle = |\psi(t_0)\rangle + \int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1 \tilde{S}(t_2,t_1)|\psi(t_0)\rangle \) (11) at any time \( t \). Thus Eq. (5) can be regarded as an equation of motion for states of a quantum system. By using (11) and (16), the evolution operator can be represented in the form

\[
\langle \Psi_2 | \int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1 \tilde{S}(t_2,t_1)|\Psi_1\rangle \rightarrow t \rightarrow t_0
\]

\[
\langle \Psi_2 | \int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1 H_{\text{int}}(t_2,t_1)|\Psi_1\rangle + o(t^2),
\]

for any vectors \( |\Psi_1\rangle > |\Psi_2\rangle > 0 \) of the Hilbert space. Note in this connection that, while we define the operator \( \tilde{S}(t_2,t_1) \) for which such a Fourier transform exists, the operator \( \tilde{B}(t) \) being an arbitrary operator that has the following behavior in the limit \( \tau \rightarrow 0 \):

\[
\langle \Psi_2 | \tilde{B}(t)|\Psi_1\rangle \rightarrow \tau \rightarrow 0 < n_2|H_{\text{int}}^{(s)}(\tau)|n_1\rangle + o(\tau^2).
\]

Below we will show that in some cases it is convenient knowing the behavior of \( B(z) \) in the limit \( z \rightarrow \infty \) is sufficient to construct the operator \( \hat{T}(\tau) \), for which such a Fourier transform exists. The operators \( \tilde{B}(t) \) is an example of such operators. However, it is not convenient to deal with such interaction operators. In fact, in this case one has to take care of the behavior of \( H_{\text{int}}^{(s)}(\tau) \) not only in the limit \( \tau \rightarrow 0 \) but also in the limit \( \tau \rightarrow \infty \). Nevertheless, formally we can construct the operator \( B(z) \) for any \( z \), by using the operator \( \tilde{B}(t) \) that being dynamically equivalent to the operator \( H_{\text{int}}^{(s)}(\tau) \) satisfies the above requirement.

It should be noted that the T-matrix obtained by solving Eq. (16) satisfies the following equation:

\[
\langle n_2|T(z)|n_1\rangle \rightarrow z \rightarrow \infty < n_2|B(z)|n_1\rangle + o(|z|^{-\beta}),
\]

where \( \beta = 1 + \epsilon \), and

\[
\langle n_2|B(z)|n_1\rangle \rightarrow \tau \rightarrow 0 < n_2|H_{\text{int}}^{(s)}(\tau)|n_1\rangle + o(\tau^2).
\]
where the first term on the right-hand side of this equation describes the instantaneous component of the interaction generating the dynamics of a quantum system, while the term \( H_{\text{non}}(t_2, t_1) \) represents its nonlocal-in-time component.

### III. NUCLEON DYNAMICS IN LEADING ORDER OF THE EFT APPROACH

Let us now show that the leading order nucleon dynamics in the effective theory of nuclear forces is governed by the generalized dynamical equation (13) with a nonlocal-in-time interaction operator. For the sake of simplicity we will consider the nucleons as identical, spinless particles (describe by a field \( \psi \)) with 3-momenta \( Q \) much smaller than their mass \( m \), the mass difference \( \Delta \) to their typical scale of all higher-energy effects. Canonical quantization leads to familiar Feynman rules, and the \( \Psi \) propagator at four-momentum \( p \) is given by

\[
S(p^0, \mathbf{p}) = \frac{i}{p^0 - \mathbf{p}^2/2m + \mathbf{p}^2/8m^3 + \ldots + i\varepsilon},
\]

where the \( C_{2n} \)'s are parameters that depend on the details of the dynamics of range \( \sim 1/M \), where \( M \) characterizes the typical scale of all higher-energy effects. In leading order one has to keep only the first two terms \( p^0/m \) and \( p^0/2m \) in the propagator (20). In this case we deal only with the particles evolving forward in time. Obviously this way is more convenient for constructing the off-shell T-matrix.

Let us consider the two-particle system at energy \( E = \frac{\mathbf{k}^2}{2m} + \frac{\mathbf{k}^2}{2m} + \ldots \) in the center-of-mass frame. The key point of the EFT approach is that the problem can be solved by expanding in the number of derivatives at the vertices or particle lines. In leading order one has to keep only the first term \( C_0 \) in (24), and correspondingly only the first two terms \( p^0/m \) and \( p^0/2m \) in the propagator (20). In this case the two particles evolve according to the familiar nonrelativistic Schrödinger propagator

\[
G_0(z) = \int \frac{d^3k}{(2\pi)^3} \frac{|k > < k|}{z - \frac{\mathbf{k}^2}{2m} + i\varepsilon}.
\]

In this order the Lagrangian can be rewritten in the form

\[
\mathcal{L} = \Psi^+ (i\partial_0 + \frac{1}{2m} \nabla^2 + \frac{1}{8m^3} \nabla^4 + \ldots) \Psi - \frac{1}{2} C_0 \Psi^+ \Psi^+ \Psi^+ \Psi - \frac{1}{8} (C_2 + C_2') \{ \Psi^+ (\nabla \Psi) - \nabla \Psi \}^2 \Psi + \frac{1}{8} (C_2 - C_2') \Psi^+ \Psi \nabla^2 \Psi^+ \Psi + \ldots.
\]

Conservation of particle number reduces the two-nucleon T-matrix to a sum of bubble diagrams. The ultraviolet divergences can all be absorbed in the renormalized parameter \( C^{(R)}_0 \). Summing the bubbles to a geometric series, one gets the T-matrix (14)

\[
< \mathbf{p}_2 | T^{(R)}(z) | \mathbf{p}_1 > = \left[ \frac{1}{C^{(R)}_0} + \frac{imz\sqrt{\varepsilon}}{4\pi} \right]^{-1}.
\]

What is interesting the T-matrix given by (21) is exactly the same as in the model [12,16]. This model provides an example showing that the GQD allows one to extend quantum dynamics to the case where the interaction generating the dynamics of quantum system is nonlocal in time. Thus the model shows that the situation where the dynamics of a quantum system is generated by nonlocal-in-time interaction is possible in principle. From the above it follows that this possibility is realized in low energy nucleon dynamics. The same T-matrix can be obtained via the LS equation with the potential

\[
V(\mathbf{p}_2, \mathbf{p}_1) = \bar{C},
\]

by using some regularization and renormalization procedures. In order to show this fact, let us consider the evolution problem for two nonrelativistic particles in the first excited state, the pion mass \( \mu \), and the range of their interaction. Being straightforward, generalization to spin and nonidentical particles cannot give rise to an essential change of the dynamical situation in a nucleon system. Thus we will consider the dynamics of spinless particles at wave lengths which are large compared to the range of their interaction. Such a dynamics can be described by the EFT of short-range forces developed by van Kolck [14]. In this theory particles are described by a field \( \psi \), and the effective Lagrangian involve arbitrary complicated operators of only \( \psi \) and its derivatives. It is assumed these derivatives are associated with factors of \( 1/m, 1/\Delta, \) or \( 1/\mu \) and, therefore, that the effective Lagrangian can be written as an expansion in \( \partial/(m, \Delta, \mu) \).
where \( f(\tau) \) is some function of the duration time \( \tau \) of interaction, and the form factor \( \varphi(p) \) must have the following asymptotic behavior for \( |p| \to \infty \):
\[
\varphi(p) \sim |p|^{-\alpha}, \quad (|p| \to \infty).
\]

If the interaction is instantaneous, then \( f(\tau) = -2i\lambda \delta(\tau) \), and
\[
\langle p_2|H_{\text{int}}(\tau)|p_1 \rangle = -2i\delta(t_2 - t_1) < p_2|V|p_1 >, \tag{28}
\]
\(< p_2|V|p_1 > \) being the separable potential
\[
< p_2|V|p_1 > = \lambda \varphi^*(p_2)\varphi(p_1). \tag{29}
\]

The solution of the LS equation with this potential is
\[
< p_2|T(z)|p_1 > = \varphi^*(p_2)\varphi(p_1) \left( \frac{1}{\lambda} + \int \frac{d^3k}{(2\pi)^3} \frac{|\varphi(k)|^2}{(z - E_k)} \right)^{-1}. \tag{30}
\]

In the case \( \alpha \leq \frac{1}{2} \), the potential (25) does not make sense without renormalization, since it gives rise to UV divergences. In fact, in this case the integral in (30) is not convergent, and one has to use some regularization procedure. Using the dimensional regularization we can write the following expression for the regulated T-matrix:
\[
< p_2|T(z)|p_1 > = \varphi^*(p_2)\varphi(p_1) \left( \frac{1}{\lambda} + \int \frac{d^{3-\varepsilon}k}{(2\pi)^3} \frac{|\varphi(k)|^2}{(z - E_k)} \right)^{-1}. \tag{32}
\]

This expression can be rewritten in the form
\[
< p_2|T(z)|p_1 > = \varphi^*(p_2)\varphi(p_1) \left( \frac{1}{C_0^{(\varepsilon)}} - z \int \frac{d^{3-\varepsilon}k}{(2\pi)^3} \frac{|\varphi(k)|^2}{(z - E_k)} \right)^{-1}, \tag{31}
\]

where \( (C_0^{(\varepsilon)})^{-1} = \lambda^{-1} - \int \frac{d^3k}{(2\pi)^3} \frac{|\varphi(k)|^2}{E_k} \). Let us use the scattering length \( a_0 \) as an additional experimental information that is needed for renormalization. The parameter \( C_0^{(\varepsilon)} \) in the expression (32) for the T-matrix is related to the scattering length by
\[
a_0 = \frac{m}{4\pi} C_0^{(\varepsilon)} |\varphi(0)|^{-2}.
\]

The above means that we have to fix the value \( C_0^{(\varepsilon)} \) which we will denote by \( C_0^{(R)} \). Now we can let \( \varepsilon \to 0 \) in Eq.(32), and get
\[
< p_2|T(z)|p_1 > = \varphi^*(p_2)\varphi(p_1) \left( \frac{1}{C_0^{(R)}} - z \int \frac{d^3k}{(2\pi)^3} \frac{|\varphi(k)|^2}{(z - E_k)} \right)^{-1}. \tag{33}
\]

It is easy to see that this renormalized T-matrix is not a solution of the LS equation with some potential because of the slow rate of decay of the form factor \( \varphi(p) \) as \( p \to \infty \). Note in this connection that the renormalized strength of the potential \( \lambda_\varepsilon = (C_0^{(\varepsilon)} - \int \frac{d^3k}{(2\pi)^3} \frac{|\varphi(k)|^2}{E_k} )^{-1} \) tends to zero as \( \varepsilon \to 0 \). This means that after renormalization the interaction Hamiltonian is zero. This is a manifestation of the well-known fact that one cannot construct a renormalized Hamiltonian, and there are not any equations for renormalized amplitudes within Hamiltonian formalism. At the same time, one can easily verify that the T-matrix given by (33) satisfies the generalized dynamical equation (3). Moreover, as stated above, it coincides with the T-matrix of the model [12,16]. In this model it is assumed that the generalized interaction operator is of the form (26). In this case, \( < p_2|\tilde{S}(t_2,t_1)|p_1 > \) can be represented in the form
\[
< p_2|\tilde{S}(t_2,t_1)|p_1 > = \varphi^*(p_2)\varphi(p_1)\tilde{s}(t_2,t_1). \tag{34}
\]

Correspondingly, the T-matrix should be of the form
\[
< p_2|T(z)|p_1 > = \varphi^*(p_2)\varphi(p_1)t(z), \tag{35}
\]

From (16) it follows that the function \( t(z) \) must satisfies the equation
\[
\frac{dt(z)}{dz} = -i^2(z) \int \frac{d^3k}{(2\pi)^3} \frac{|\varphi(k)|^2}{(z - E_k)^2}. \tag{36}
\]

This equation can be rewritten in the form
\[
\frac{dt(z)}{dz} = \int \frac{d^3k}{(2\pi)^3} \frac{|\varphi(k)|^2}{(z - E_k)^2}. \tag{37}
\]

with \( t(z) = (t(z))^{-1} \). All solutions of Eq.(36) satisfying the condition
\[
(t^*(z) = t(z), \quad z \in (-\infty,0), \tag{38}
\]

are physically realizable [12]. Each of them corresponds to the definite function \( f(\tau) \) in the generalized interaction operator (26). As we have noted, only the asymptotic behavior of the function \( f(\tau) \) as \( \tau \to 0 \) is relevant, and formally one can use any function that has this asymptotic behavior. Knowing the asymptotic behavior of \( H_{\text{int}}^{(s)}(\tau) \) in the limit \( \tau \to 0 \) allows one to determine the asymptotic behavior of the function \( t(z) \) in the limit \( z \to \infty \) (the one-to-one correspondence between these behaviors follows from the Fourier transform). This behavior in turn can be used as a boundary condition for the differential equation (36). The T-matrix obtained in this way can then be used for constructing the evolution operator. Thus knowing the asymptotic behavior of \( H_{\text{int}}^{(s)}(\tau) \) in the limit \( \tau \to 0 \) uniquely determines the dynamics of the system. On the other hand, as it follows from Eq.(37), the asymptotic behavior of \( t(z) \) is determined by the large-momentum behavior of the form factor \( \varphi(p) \). In the case \( \alpha \geq \frac{1}{2} \), at it is easily seen from Eq.(37), all solutions of
where the function $f'(\tau)$ has no such a singularity at the point $\tau = 0$ as the delta-function. Thus, in the case $\alpha > \frac{1}{2}$, the interaction operator should be of the form

$$< p_2 | H_{\text{int}}^{(s)}(\tau) | p_1 > = -2i\delta(\tau) < p_2 | V | p_1 >, \tag{41}$$

$$< p_2 | V | p_1 >$$

being the separable potential

$$< p_2 | V | p_1 > = \lambda \varphi^*(p_2) \varphi(p_1), \tag{42}$$

and hence the dynamics generated by this operator is equivalent to the dynamics governed by the Schrödinger equation with the separable potential $\lambda \varphi^*(p_2) \varphi(p_1)$. Solving Eq.\eqref{eq:36} with the boundary condition \eqref{eq:39}, for the T-matrix, we easily get the expression \eqref{eq:30}, i.e. in the case $\alpha > \frac{1}{2}$, the interaction in the system may be only instantaneous.

Let us now consider the case $-\frac{1}{2} < \alpha < \frac{1}{2}$ (the restriction $\alpha > -\frac{1}{2}$ is necessary for the integral in \eqref{eq:30} to be convergent) where the ordinary separable-potential model does not make a sense without renormalization. In this case solutions of Eq.\eqref{eq:30} tends to zero as $|z| \to \infty$:

$$t(z) \to b_1(-z)^{\alpha - \frac{1}{2}} + b_2(-z)^{2\alpha - 1} + o(|z|^{2\alpha - 1}),$$

where $b_1 = -4\pi \cos(\alpha \pi) m^{\alpha - \frac{1}{2}}$, and $b_2$ is some arbitrary constant. Thus the T-matrix has the following asymptotic behavior for $|z| \to \infty$:

$$< p_2 | T(z) | p_1 > \to \varphi^*(p_2) \varphi(p_1)$$

$$\times \left( b_1(-z)^{\alpha - \frac{1}{2}} + b_2(-z)^{2\alpha - 1} + o(|z|^{2\alpha - 1}) \right), \tag{43}$$

By using asymptotic methods (see, for example, Ref.\cite{18}) it is easy to show that this large $z$ behavior of the function $t(z)$ corresponds to the following behavior of $T(\tau)$ related to the T-matrix by Eq.\eqref{eq:13} in the limit $\tau \to 0$:

$$< p_2 | \tilde{T}(\tau) | p_1 > \to \varphi^*(p_2) \varphi(p_1)$$

$$\times \left( a_1 \tau^{\alpha - \frac{1}{2}} + a_2 \tau^{2\alpha - 2\alpha} + o(\tau^{2\alpha - 2\alpha}) \right), \tag{44}$$

where $a_1 = -ib_1 \Gamma^{-1}(\frac{1}{2} - \alpha) \exp[i(-\alpha \pi + \frac{1}{2} \pi)]$, and $a_2 = b_2 \Gamma^{-1}(1 - 2\alpha) \exp[-i\alpha \pi]$, $\Gamma(z)$ being the gamma-function. On the other hand, according to Eqs.\eqref{eq:4} and \eqref{eq:6} the asymptotic behavior of $< p_2 | T(\tau) | p_1 >$ is determined by the interaction operator $H_{\text{int}}^{(s)}(\tau)$

$$< p_2 | \tilde{T}(\tau) | p_1 > \to < p_2 | H_{\text{int}}^{(s)}(\tau) | p_1 > + o(\tau), \tag{45}$$

where the parameter $\varepsilon$ is determined by demanding that $\exp(iH_0t_2) H_{\text{int}}^{(s)}(\tau) \exp(-iH_0t_1)$ must be so close to the relevant solution of Eq.\eqref{eq:30} in the limit $\tau \to 0$ that this solution is unique having the asymptotic behavior \eqref{eq:45}. It is easy to see that in the separable case all solutions of Eq.\eqref{eq:30} for given $\alpha$ have the same leading term in the expansion \eqref{eq:43}, since the parameter $b_1$ is uniquely determined by the value of the parameter $\alpha$. Only the second term containing free parameter $b_2$ distinguishes the different solutions of this equation. On the other hand, the first two term in the expansion \eqref{eq:44} are uniquely determined by the first two term of the expansion \eqref{eq:43}. This means that the parameter $\varepsilon$ in \eqref{eq:45} is equal to $2\alpha$, and hence the generalized interaction operator $H_{\text{int}}^{(s)}(\tau)$ should be of the form

$$< p_2 | H_{\text{int}}^{(s)}(\tau) | p_1 > = \varphi^*(p_2) \varphi(p_1) \left( a_1 \tau^{\alpha - \frac{1}{2}} + a_2 \tau^{2\alpha - 2\alpha} \right) \tag{46}$$

Of course, the interaction operator \eqref{eq:46} may be supplemented by any term being of order $O(\tau^{-2\alpha})$. However, this will not change the solution of Eq.\eqref{eq:5}, i.e. in the is case we will get the generalized interaction operator being dynamically equivalent to the interaction operator \eqref{eq:46}. As we have noted, the operator $H_{\text{int}}^{(s)}(\tau)$ describes the contributions to the evolution operator from the processes with infinitesimal duration time $\tau$ of interaction, and only its behavior in the limit $\tau \to 0$ is relevant for us.

The solution of Eq.\eqref{eq:37} with the initial condition $\tilde{t}(z = 0) = \tilde{t}_0$ is

$$\tilde{t}(z) = \tilde{t}_0 - z \int \frac{d^3k}{(2\pi)^3} \frac{|\varphi(k)|^2}{(z - E_k)E_k}. \tag{47}$$

From this we easily get the expression \eqref{eq:33} for the T-matrix. This solution of Eq.\eqref{eq:16} has the behavior \eqref{eq:43}, provided $\tilde{t}_0 = -b_2 b_1^{-2\alpha}$. Taking this fact into account, one can rewrite this solution in terms of the parameters $b_1$ and $b_2$

$$< p_2 | T(z) | p_1 > = N(z) \varphi^*(p_2) \varphi(p_1), \tag{48}$$

where

$$N(z) = \left( b_2 \frac{b_1^{-\alpha}}{b_1} - \frac{(-z)^{\frac{1}{2} - a}}{b_1} - M(z) \right)^{-1}, \tag{49}$$

with

$$M(z) = \int \frac{d^3k}{(2\pi)^3} \frac{|\varphi(k)|^2 - |k|^{-2\alpha}}{z - E_k}. \tag{50}$$

By using Eqs.\eqref{eq:12} and \eqref{eq:47}, we can construct the evolution operator

$$< p_2 | U(t, t_0) | p_1 > = < p_2 | p_1 > + \frac{i}{2\pi} \int_{-\infty}^{\infty} dx \exp(-izt) \frac{N(z) \varphi^*(p_2) \varphi(p_1)}{(z - E_{p_2})(z - E_{p_1})}. \tag{51}$$
where $z = x + iy$, $y > 0$. The evolution operator (51) satisfies the composition law (3) and is unitary provided the parameter $b_2$ is real. Thus, in the case $-\frac{1}{2} < \alpha < \frac{1}{2}$, the dynamics is well defined, and all the requirements of quantum theory are satisfied.

The model under consideration is an transparent illustration of the possibility of the extension of quantum dynamics provided by the GQD. From the point of view of Eq. (33), the restriction to Hamiltonian dynamics is equivalent to the restriction to the solutions that tend to some nonzero constant in the limit $|z| \to \infty$. In this case the interaction in the system is instantaneous, and the dynamics is Hamiltonian. At the same time, this equation has solutions that tend to zero as $|z| \to \infty$. This takes place in the case $-\frac{1}{2} < \alpha \leq \frac{1}{2}$. In this case the interaction is nonlocal in time and is parameterized by the interaction operator (46). In this case the dynamics is non-Hamiltonian. There are no reasons to restrict ourselves to the solutions of Eq. (33) which tend to nonzero constants, because all solutions of this equation satisfying the condition (38) are physically realizable, and such an extension of quantum dynamics is possible in principle. The remarkable fact is that this possibility is realized in low energy nucleon dynamics, and, as we have noted, the leading order contribution to the nucleon-nucleon T-matrix is described by our model. In fact, in the case $\psi(p) = 1$, for the T-matrix given by (47), we have

$$\langle p_2 | T(z) | p_1 \rangle = -\frac{b_1^2}{b_2 - i b_1 \sqrt{z}} = -\left[ \frac{1}{C_0^{(R)}} + \frac{i m \pi \sqrt{z}}{4 \pi} \right]^{-1},$$

where $b_1 = -\frac{4\pi}{m\sqrt{m}}$, and $C_0^{(R)} = \frac{b_1^2}{b_2}$. In this case the generalized interaction operator is of the form

$$\langle p_2 | H^{(t)}(\tau) | p_1 \rangle = \frac{4\sqrt{\pi} \exp(i \pi \frac{1}{4})}{m \pi \sqrt{\tau}} + \frac{16\pi^2}{m^3 C_0^{(R)}}.$$

This operator parametrizes the leading order contact component of the NN interaction and generates the two-nucleon dynamics that is described by the evolution operator

$$\langle p_2 | U(t, t_0) | p_1 \rangle = \langle p_2 | p_1 \rangle + i \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \exp(-izt)N(z) \left( z - E_{p_2} \right) \left( z - E_{p_1} \right).$$

It should be noted that the interaction operator (53) contains the renormalize parameter $C_0^{(R)}$, while in the ordinary methods based on the use of a renormalization procedure this parameter appears in renormalized amplitudes at the final stage. The generalized dynamical equation (3) with the interaction operator (53) is well defined and allows one to describe, in a consistent way, the dynamics generated by the leading order component of the NN interaction.

IV. NON-HAMILTONIAN CHARACTER OF LOW ENERGY NUCLEON DYNAMICS

Let us now discuss advantages of the GQD and new possibilities that it opens for describing low energy nucleon dynamics. As we have seen, the T-matrix (33) obtained by solving the dynamical equation (16) with the interaction operator (46), can be also obtained starting with the singular potential

$$V(p', p) = \lambda \varphi^*(p') \varphi(p), \quad \varphi(p) \sim |p|^{-\alpha}, \quad \alpha \leq \frac{1}{2}.$$ 

However, in this way one cannot determine a potential that could parametrize the interaction in the system. The above singular potential does not make sense without renormalization and is only of formal importance for the problem under consideration. In fact, as we have shown, the strength of potential, which in the regulated LS equation is adjusted to give the correct scattering length, becomes zero after removing regularization. All the information contained in this potential is that the T-matrix is of the form (35) with the same form factor. This information is not sufficient to construct the T-matrix. In addition, one needs to use some experimental data, for example, the scattering lengths. However, one cannot construct an interaction Hamiltonian that could contain all this information. The T-matrix (33) obtained in this way does not satisfy the LS equation. Moreover, as we will show below, it has the properties that are at variance with the Hamiltonian formalism. Thus in this case we have only a calculation rule that allows one to compute the T-matrix: One cannot derive any renormalized equation containing a well-defined potential that parametrizes the interaction in the system. This problem is the price for trying to describe the dynamics of the system after renormalization in terms of the Hamiltonian formalism, despite this dynamics is non-Hamiltonian. From the more general point of view provided by the GQD we see that the T-matrix (33) satisfies the generalized dynamical equation (16) with the nonlocal-in-time interaction operator (46), and this operator describes the fundamental interaction in the system. Knowing this operator is sufficient to construct the T-matrix and hence the evolution operator, and in this case we deal with the well-defined equation that does not require renormalization. As we have seen, an example of such a dynamics is the leading order nucleon dynamics where, as a consequence of the symmetries of the underlying theory, the form factor must satisfy the condition $\varphi(p) = 1$.

In order to clarify the character of the leading order nucleon dynamics let us examine the properties of the evolution operator of the more general theory with the interaction operator (46). In the Schrödinger picture, this operator $\langle p_2 | V(t) | p_1 \rangle = \langle p_2 | U_0(t, t_0) | p_1 \rangle$ can be rewritten in the form

$$\langle p_2 | V(t) | p_1 \rangle = \langle p_2 | p_1 \rangle > i \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \exp(-izt) \langle p_2 | T(z) | p_1 \rangle \left( z - E_{p_2} \right) \left( z - E_{p_1} \right).$$


where \( < p_2 | T(z) | p_1 > \) is given by (18). Since this T-matrix satisfies Eqs.(16) and (18), the evolution operator (51) is unitary, and satisfies the composition law (3). Correspondingly the operators \( V(t) \) constitute a one-parameter group of unitary operators, with the group property

\[
V(t_1 + t_2) = V(t_1)V(t_2), \quad V(0) = 1. \quad (56)
\]

Assume that this group has a self-adjoint infinitesimal generator \( H \) which in the Hamiltonian formalism is identified with the total Hamiltonian. Then for \( |\psi > \in \mathcal{D}(H) \) we have

\[
\frac{V(t)|\psi> - |\psi>}{t} \rightarrow_{t \rightarrow 0} -iH|\psi>. \quad (57)
\]

From this and (55) it follows that

\[
H = H_0 + H_1,
\]

with

\[
< p_2 | H_1 | p_1 > = \frac{i}{2\pi} \int_{-\infty}^{\infty} dz \frac{< p_2 | T(z) | p_1 >}{(z - E_{p_2})(z - E_{p_1})}, \quad (58)
\]

where \( z = x + iy \), and \( y > 0 \). Since \( < p_2 | T(z) | p_1 > \) is an analytic function of \( z \) and, in the case \( \alpha \leq \frac{1}{2} \), tends to zero as \( |z| \rightarrow \infty \), from Eq.(58) it follows that \( < p_2 | H_1 | p_1 > = 0 \) for any \( p_2 \) and \( p_1 \), and hence \( H = H_0 \). This means that, if the infinitesimal generator of the group of the operators \( V(t) \) exists, then it coincides with the free Hamiltonian, and the evolution operator is of the form \( V(t) = \exp(-iH_0t) \). Thus, since this, obviously, is not true, the group of the operators \( V(t) \) has no infinitesimal generator, and hence the dynamics is not governed by the Schrödinger equation.

It should be also noted that in the case \( \alpha \leq \frac{1}{2} \), \( \tilde{S}(t_2, t_1) \) is not an operator on the Hilbert space. In fact, the wave function

\[
\psi(p) \equiv < p | \psi > = < p | \tilde{S}(t_2, t_1) | \psi_1 > = \varphi^*(p) \tilde{s}(t_2, t_1) \int \frac{d^3k}{(2\pi)^3} \varphi(k) < k | \psi_1 > \quad (59)
\]

is not square integrable for any nonzero \( |\psi_1 > \), because of the slow rate of decay of the form factor \( \varphi(p) \) as \( |p| \rightarrow \infty \). Correspondingly, in the case \( \alpha \leq \frac{1}{2} \), the T-matrix given by (48) does not represent an operator on the Hilbert space. However, as we have stated, in general \( \tilde{S}(t_2, t_1) \) may be only an operator-valued generalized function such that the evolution operator is an operator on the Hilbert space. Correspondingly the T-matrix need not be an operator on the Hilbert space. It is enough that the evolution operator given by (51) is such an operator. The T-matrix and \( \tilde{S}(t_2, t_1) \) satisfy these requirements not only for \( \alpha > \frac{1}{2} \) but also for \( -\frac{1}{2} < \alpha \leq \frac{1}{2} \), since the evolution operator (51) is an operator on the Hilbert space. At the same time, in the case \( \alpha \leq \frac{1}{2} \) we go beyond Hamiltonian dynamics.

The above means that low energy nucleon dynamics in the effective theory of nuclear forces cannot be governed by the Schrödinger equation and hence is non-Hamiltonian. Correspondingly the interaction of nucleons cannot be parametrized by an interaction Hamiltonian defined on the Hilbert space. At the same time, as has been shown on the example of the effective theory with the Lagrangian (23), this dynamics is governed by the generalized dynamical equation (3) with a nonlocal-in-time interaction operator. In the above theory describing the leading order nucleon dynamics this operator is of the form (53), and two nucleon dynamics is described by the evolution operator (54). This evolution operator is unitary and satisfies the composition law (3). Thus the use of Eq.(3) as an equation of motion in the effective theory of nuclear forces allows one not only to calculate scattering amplitudes but also to construct the evolution operator describing low energy nucleon dynamics.

V. NEXT-TO-LEADING ORDER

We have shown that in leading order the dynamics of the EFT under consideration is governed by the generalized dynamical equation with a nonlocal-in-time interaction operator. Let us now show that this is true also in next-to-leading order. For this let us come back to the theory with the Lagrangian (19) in which it is assumed that particles evolve forward in time, and particle number is conserved. Summing the graphs of the perturbation series yield the following on-shell T-matrix (14):

\[
T_{\alpha\beta}(k, p' \cdot p) = T_{\alpha\beta}^{(0)}(k) - 2C_{2}^{(R)}k_{p'} \cdot \hat{p}\]

\[
+ O\left( \frac{4\pi}{mM} (Q/M)^4 \right),
\]

where \( T_{\alpha\beta}^{(0)}(k) \) is the S-wave amplitude and is given by

\[
T_{\alpha\beta}^{(0)}(k) = \left[ \frac{1}{C_0^{(R)}} - \frac{2}{C_0^{(R)}} k^2 + \frac{imk}{4\pi} \left( 1 + \frac{k^2}{2m^2} \right) \right]^{-1}
\]

\[
\times [1 + O((Q/M)^4)],
\]

k = |p_1| = |p_2|, \( \hat{p}_{p'}/k, C_0^{(R)}, C_{2}^{(R)} \) and \( \tilde{C}_{2}^{(R)} \) are renormalized parameters. In this way one can also construct the off-shell T-matrix. At the same time, as it has been shown in leading order, the T-matrix can be obtained without summing the diagrams and resorting to regularization and renormalization procedures, provided that the UV behavior of the T-matrix elements as functions of momenta is known. As we have seen, the requirement that the T-matrix being of the form \( < p_2 | T(z) | p_1 > \) satisfies the generalized dynamical equation (16) yields the formula (52) where only the parameter \( C_0^{(R)} \) is free. In next-to-leading order the relation between the Lagrangian (19) and the form of the T-matrix is not so straightforward, because in this case one has to take into account the fact that this Lagrangian
is only of the formal importance, and should be supplemented by some counterterms. Nevertheless, let us assume that they do not effect on the form of the T-matrix. In this case from the analysis of the perturbation series of the theory it follows that the two-particle T-matrix (in this paper we focus on the S-wave channel) should be of the form

\[ \langle p_2 | T(z) | p_1 \rangle = t_1(z) + C_1 t_2(z) (p_1^2 + p_2^2) + \ldots \]

where \( C_1 = C_2^{(R)} / C_0^{(R)} \). Substituting (62) into (16), we get

\[ \frac{dt_1(z)}{dz} = - \int \frac{d^3k}{(2\pi)^3} \left[ \frac{1}{z - \frac{k^2}{m}} \right] \left[ t_2(z) \left( 1 + \frac{2k^4}{4m^3(z - \frac{k^2}{m})} \right) + 2C_1 t_1(z) t_2(z) k^2 \right] + O\left( \frac{4\pi}{mM} (Q/M)^4 \right). \] (63)

Since the problem with UV divergences must not arise in treating the generalized dynamical equation, we have

\[ C_1 = - \frac{1}{4m^2} + \delta C_1 \] (64)

with \( \delta C_1 = O(\frac{1}{M^2}) \).

Correspondingly, the function \( \hat{t}(z) \equiv (t(z))^{-1} \) satisfies the following equation:

\[ \frac{d\hat{t}(z)}{dz} = \int \frac{d^3k}{(2\pi)^3} \left( 1 + \frac{2z^2 k^2}{2m^2(z - \frac{k^2}{m})} \right). \] (67)

Its solution with the initial condition \( \hat{t}(z = 0) = \hat{t}_0 \) is

\[ \hat{t}(z) = \hat{t}_0 + \int_{\hat{z}}^z ds \int \frac{d^3k}{(2\pi)^3} \left( 1 + \frac{2sk^2}{2m^2(s - \frac{k^2}{m})} \right) \left( \frac{z + \frac{2m}{s}}{2m(z - \frac{k^2}{m})} \right) = \hat{t}_0 - \frac{im}{4\pi} \sqrt{1 + z^2} \left( 1 + \frac{z}{8m} \right). \] (68)

Denoting \( \hat{t}_0 = \frac{1}{C_0^{(R)}} \), for the function \( t(z) \), we get

\[ t(z) = - \left[ \frac{1}{C_0^{(R)}} + \frac{im}{4\pi} \sqrt{1 + z^2} \left( 1 + \frac{z}{8m} \right) \right]^{-1} \left[ 1 + O((Q/M)^4) \right]. \] (69)

Substituting (65) with \( t(z) \) given by (69) into (63) yields

\[ \langle p_2 | T(z) | p_1 \rangle = - \left( 1 + \frac{C_2^{(R)}}{C_0^{(R)}} \right) \left( \frac{1}{C_0^{(R)}} + \frac{im}{4\pi} \sqrt{\frac{k^2}{m} - \frac{k^4}{4m^3} \left( 1 + \frac{k^2}{8m^2} + \frac{k^4}{32m^4} \right) \right)^{-1} \left( \frac{1}{C_0^{(R)}} + \frac{im}{4\pi} \sqrt{\frac{k^2}{m} - \frac{k^4}{4m^3} \left( 1 + \frac{k^2}{8m^2} + \frac{k^4}{32m^4} \right) \right) \] (70)
Thus the requirement that the T-matrix of the form (62) satisfies the generalized dynamical equation (16) yields the formula (70), and in this way we get the same two-particle on-shell T-matrix that in Ref. [12] has been obtained by summing the bubble graphs of the EFT.

In order to obtain the form of the generalized interaction operator that leads to the T-matrix (70), one has to examine the large \( z \) behavior of the function \( t(z) \) in the limit \( |z| \to \infty \). From (69) it follows that

\[
t(z) \to \frac{b_1}{|z|} + \frac{b_2}{|z|} + o(|z|^{-1}),
\]

where \( s = z (1 + \frac{1}{m \sqrt{m}}) \), \( b_1 = -\frac{4\pi}{m \sqrt{m}} \), and \( b_2 = \frac{\pi}{C_0^{(R)}} \).

Of course, the asymptotic behavior of \( t(z) \) can be represented in the form of the expansion in \( z^{-\frac{3}{2}} \):

\[
t(z) \to \frac{b'_1}{|z|} + \frac{8mb'_1}{(-z)^{\frac{3}{2}}} + \frac{b'_2}{(-z)^3} + o(|z|^{-3}),
\]

where \( b'_1 = \frac{32\pi}{\sqrt{m}} \) and \( b'_2 = \frac{(\omega \pi)^{\frac{4}{3}}}{C_0^{(R)}} \). However, for the problem under consideration the values of \( z \) much larger than \( Q \) but much smaller than \( m \) should be considered as infinitely large, and the behavior of the function in this region is described by the expansion (72). The interaction operator \( H^{(s)}_{int}(\tau) \) that gives rise to this behavior of \( t(z) \) is

\[
< p_2 | H^{(s)}_{int}(\tau) | p_1 > = \left( 1 + \frac{C_0^{(R)}}{C_0^{(R)}} \right) \left( \frac{4\sqrt{\pi} \exp(i\frac{3\pi}{4})}{m^2 \sqrt{\tau - \frac{i}{8m}}} + \frac{16\pi^2}{m^2 C_0^{(R)}} \right). \tag{74}
\]

This operator parametrizes the contact term of the NN interaction in the S-wave channel up to next-to-leading order. The generalized dynamical equation with this interaction operator uniquely determines the dynamics of the system. The equation of motion (38) with this interaction operator is well-defined and allows one to obtain the T-matrix and the evolution operator without resorting to regularization and renormalization. It is easy to show that the evolution operator constructed in this way (the evolution operator (12) with the T-matrix given by (70)) is unitary and satisfies the composition law (38) up to next-to-leading order.

VI. INTEGRAL EQUATIONS

As we have shown up to next-to-leading order of the EFT with the Lagrangian (19), the dynamics of nucleons at low energies is governed by the generalized dynamical equation (38) with the nonlocal-in-time interaction operator (74), provided the parameters of the theory satisfy the condition (64). Within the GQD this operator is well defined. Of course, the dynamical situation in the theory with such an interaction differs from that in the theory with ordinary potentials. The operator \( \tilde{S}(t_2, t_1) \) is only an operator-valued generalized function on the Hilbert space. However, as we have noted this is not at variance with the general requirements of quantum theory, since the evolution operator given by (51) is an operator on this space. The generalized dynamical equation (38) is not equivalent to the Schrödinger equation in this case. Nevertheless, it is a well-defined equation, and allows one to obtain the T-matrix without resorting to regularization and renormalization procedures. This is very important for practical calculations, since for solving realistic problems one has to deal not only with the contact component of the NN interaction but also with its long-range one. In this case the interaction operator is of the form

\[
< p_2 | H^{(s)}_{int}(\tau) | p_1 > = -2i\delta(\tau) V(p_2, p_1) + < p_2 | H^{(non)}_{int}(\tau) | p_1 >, \tag{75}
\]

where \( < p_2 | H^{(non)}_{int}(\tau) | p_1 > \) represents the contact nonlocal-in-time component and \( V(p_2, p_1) \) is a potential describing the long-range component of the NN interaction. In general it consists of the meson-exchange potentials and the Coulomb potential in the proton-proton channel. In the leading order the nonlocal component is given by (53) and for the interaction operator, we can write

\[
< p_2 | H^{(s)}_{int}(\tau) | p_1 > = \frac{4\sqrt{\pi} \exp(i\frac{3\pi}{4}) + 16\pi^2}{m^2 \sqrt{\tau - \frac{i}{8m}}} + \frac{16\pi^2}{m^2 C_0^{(R)}} \frac{2i\delta(\tau) V(p_2, p_1)}. \tag{76}
\]

In the case of such an interaction operator, the solution of the dynamical equation (16) can be represented (see Appendix A) in the form

\[
< p_2 | T(z) | p_1 > = t_0(z) + t_1(z; p_1) + t_1(z; p_2) + t_2(z; p_2, p_1), \tag{77}
\]

where \( t_2(z; p_2, p_1) \) is a solution of the equation

\[
t_2(z; p_2, p_1) = V(p_2, p_1) + \int \frac{d^3q}{(2\pi)^3} \frac{K(z; p_2, q)}{z - E_q} t_2(z; q, p_1), \tag{78}
\]

with

\[
K(z; p_2, q) = N(z) \int \frac{d^3k}{(2\pi)^3} \frac{V(p_2, k)}{z - E_k} + V(p_2, q). \tag{79}
\]
and the functions $t_0(z)$ and $t_1(z; p)$ are defined as

$$ t_1(z; p) = N(z) \int \frac{d^3k}{(2\pi)^3} \frac{t_2(z; p, q)}{z - E_k}, \quad (80) $$

$$ t_0(z) = N(z) \left( 1 + \int \frac{d^3q}{(2\pi)^3} \frac{t_1(z; q)}{z - E_q} \right), \quad (81) $$

where

$$ N(z) = -\left( \frac{1}{C_0^{(R)}} + \frac{im^2\sqrt{z}}{4\pi} \right)^{-1}. $$

Equations with the next-to-leading order corrections can be derived in the same way.

In Weinberg’s power counting the one-pion-exchange potential is of leading order. Hence in this order the NN interaction operator can be expressed as

$$ \langle p_2|H^{(s)}_{\text{int}}(\tau)|p_1 \rangle = \frac{4\sqrt{\pi}\exp(i\frac{3\pi}{4})}{m^2\sqrt{\tau}} + \frac{16\pi^2}{m^3C_0^{(R)}} - 2i\delta(\tau)V_\pi(p_2, p_1), \quad (82) $$

where $V_\pi(p_2, p_1)$ is the conventional one-pion-exchange potential. Substituting this potential into Eq. (82) and solving it numerically, one can easily obtain the T-matrix and hence the evolution operator. Note that conventional way of solving the above problem is the formal use of the potential

$$ V_0(p_2, p_1) = \tilde{C} + V_\pi(p_2, p_1) \quad (83) $$

(see, for example, Refs. [7,11]). We say “formal” since the use of such a potential leads to UV divergences, and the Schrödinger and LS equations require regularization and renormalization. On the other hand, as we have shown, the contact interaction, which is in (83) formally represented by the term $\tilde{C}$, is parametrized by the operator (53) (the first two terms in the operator (82)). In this case we deal with the well defined interaction operators and Eq. (82) which does not require regularization and renormalization. By using Eq. (57), one can obtain the T-matrix so easily as in the case of the pure one-pion-exchange potential.

In order to take into account electromagnetic corrections, one has to include the Coulomb potential into the NN interaction operator

$$ \langle p_2|H^{(s)}_{\text{int}}(\tau)|p_1 \rangle = \frac{4\sqrt{\pi}\exp(i\frac{3\pi}{4})}{m^2\sqrt{\tau}} + \frac{16\pi^2}{m^3C_0^{(R)}} - 2i\delta(\tau)V_\pi(p_2, p_1) - 2i\delta(\tau)\frac{e_{12}\alpha}{q^2}, \quad (84) $$

where $e_{12} = 1$ for the proton-proton channel and zero otherwise. Eq. (75) with the interaction operator (84) can be easily solved numerically. Thus this equation allows one to investigate electromagnetic corrections without resorting to renormalization.

VII. SUMMARY AND DISCUSSION

We have shown that the formalism of the GQD allows one to formulate the effective field theory of nuclear forces as an internally consistent theory with equations that do not require regularization and renormalization. It has been shown that the effective NN interaction is nonlocal in time, and low energy nucleon dynamics is governed by the generalized dynamical equation (83) with a nonlocal-in-time interaction operator. In leading order of the EFT approach this operator is given by (83). It should be noted that the generalized interaction operator $H^{\text{int}}_{\text{int}}(t_2, t_1)$ is not an effective interaction Hamiltonian (only in the local case they are related by Eq. (8)), and, in contrast with such Hamiltonians, the use of the generalized interaction operators permits a natural parametrization of the NN interaction: The generalized interaction operator (83) generates the unitary evolution of a nucleon system. In Sec. III we have demonstrated the advantages of the GQD in describing the leading order nucleon dynamics in comparison with the ordinary methods based on the use of the singular potential $\langle p_2|V|p_1 \rangle = C_0$. This potential does not make sense without renormalization and hence does not contain all the needed dynamical information. In addition, one needs to use some empirical data as renormalization constants. In contrast with this singular potential, the generalized interaction operator (83) contains all the needed dynamical information. For example, it contains the observable, renormalized parameter $C_0^{(R)}$. The dynamical equation with the interaction operator (83) is well defined and allows one to construct the T-matrix and the evolution operator without resorting to regularization and renormalization procedures. The dynamical information contains not only in the form of the operator (83) that, as has been shown, is a consequence of the symmetries of QCD but also in its dependence on the time duration $\tau$ of the interaction in a system.

As has been shown, only values of $H^{(s)}_{\text{int}}(\tau)$ in the infinitesimal neighborhood of the point $\tau = 0$, i.e. at scales of the underlying theory are relevant. As already stated, the operator $H^{(s)}_{\text{int}}(\tau)$ describes the contribution to the evolution operator from the processes with infinitesimal time duration of interaction. The above means that

$$ \langle \psi_2|H^{(s)}_{\text{int}}(\tau)|\psi_1 \rangle \rightarrow \tau_{t_2 \rightarrow t_1} \langle \psi_2|\tilde{S}(t_2, t_1)|\psi_1 \rangle \quad (85) $$

where $\langle \psi_2|\tilde{S}(t_2, t_1)|\psi_1 \rangle$ describe the contributions to the evolution operator from the processes in which the quark and gluon degrees of freedom can come to play. Here one of the advantages of the formulation of the theory in terms of the operators $\tilde{S}(t_2, t_1)$ becomes apparent: The relevant amplitudes $\langle \psi_2|\tilde{S}(t_2, t_1)|\psi_1 \rangle$ of the underlying theory can be directly used as the matrix elements of the generalized interaction operator $H^{\text{int}}_{\text{int}}(t_2, t_1)$ generating low energy dynamics. These degrees of freedom manifest themselves through the $\tau$ dependence of
the operator \( \mathbb{L} \). Thus the parameter \( C_0^{(R)} \) in Eq. (53) parametrizes the leading order effects of the underlying physics on low energy dynamics, and hence can be computed in terms of parameters in QCD. At the same time, this parameter can be determined from low energy experiments.

The evolution operator governing the two-nucleon dynamics in leading order is of the form (54). In Sec.IV we have shown that the group of these operators has no infinitesimal generator and hence the dynamics is non-Hamiltonian. This means that within the EFT approach there are no potential satisfying the requirements of ordinary quantum mechanics that could govern low energy nucleon dynamics. For example, the T-matrix given by (52) does not satisfy the LS equation and has properties that are at variance with the Hamiltonian formalism. On the other hand, as we have seen, the T-matrix satisfies the generalized dynamical equation (16) with the interaction operator given by (53). The essential lesson we have learned from the above analysis is that many problems of the EFT of nuclear forces arise because of ignoring the fact that low energy dynamics produced by this theory is non-Hamiltonian. For example, the above statement that there are not any equations for renormalized amplitudes in a subtractive EFT should mean that there are no such equations within Hamiltonian formalism. From a more general point of view provided by the GQD we see that such equations exist, and the theory can be formulated in a completely consistent way. An EFT describes low energy physics in terms of a few parameters, and these low energy parameters can be computed in terms of a more fundamental high energy theory. It is remarkable that the interaction operator \( \mathbb{L} \) parametrizing the leading order contact component of the NN interaction is uniquely determined by the symmetries of QCD and contains one of these parameters \( C_0^{(R)} \). Other renormalized parameters will be contained in higher order corrections to the generalized interaction operator, and this corrections can be obtained order by order within the EFT approach. This has been demonstrated in next-to-leading order by using the example of the EFT of short-range forces developed by van Kolck [4].

We have shown that up to next-to-leading order the dynamics of the theory is governed by the generalized dynamical equation with the nonlocal-in-time interaction operator (74). These results have been obtained in the particular case where the parameters of the theory satisfy the condition (64). This limitation is a consequence of the fact that we used the assumption that the two-particle T-matrix of the theory is of the form (62), while this assumption does not fulfilled in general. The generalization of these results to the case where the condition (64) is not satisfied can only lead to another form of the generalized interaction operator. The general case will be discussed in detail in another paper. At the same time, the model we have considered in Sec.V provide the transparent illustration of the fact that in any order of the EFT approach low energy dynamics of nucleons is governed by the generalized dynamical equation \( \mathbb{L} \) with a nonlocal-in-time interaction operator. As has been proved, the assumption that the relevant solution of Eq.(16) should be of the form (62) uniquely determines this solution and leads to the condition (64). In this way we arrive at exactly the same on-shell T-matrix that has been obtained in Ref.[4] by summing the bubble diagrams up to next-to-leading order. The corresponding generalized interaction operator is of the form (74). The generalized dynamical equation \( \mathbb{L} \) with this interaction operator is well defined and allows one to describe the dynamics of the system in a consistent way.

The above method for constructing the effective interaction operator is inapplicable in general, because the parameters of the theory need not satisfy the condition (64). In general, in order to construct the generalized interaction operator parametrizing the interaction in the EFT of nuclear forces, one has to analyze the contributions from the off-shell amplitudes to \( T(z) \) in the limit \( |z| \to \infty \). In this limit the main contribution to the T-matrix comes from the bubble diagrams including potential pions that produce the pion-exchange potential. By summing the bubble diagrams in the large \( z \) limit, one can determine the contact component of the interaction operator parametrizing the NN interaction. Thus the NN interaction operator can be represented as the sum (75) of the nonlocal contact and long-range components that can be obtained separately. The contact component is represented by a nonlocal-in-time interaction operator, while the long-range one is represented by the pion-exchange potential. In general, the long-range component should be supplemented by the Coulomb potential. It is extremely important, that starting with this NN interaction operator, we can construct the T-matrix and the evolution operator without resorting to summing all relevant diagrams and using regularization and renormalization procedures. The generalized dynamical equation \( \mathbb{L} \) with this interaction operator is well defined and can be reduced to integral equations which do not require renormalization. This has been demonstrated in leading order of the EFT approach. The operator (76) parametrizes the interaction that includes not only the leading order contact component but also the long-range one being described by the potential \( V(p_2,p_1) \). The generalized dynamical equation with the interaction operator (76) can be reduced to the integral equation (78). This equation is so convenient for numerical calculations as the LS equation which is its particular case where the parameter \( C_0^{(R)} \) tends to zero and hence one can neglect the contact component. In this case the theory is reduced to the ordinary theory of the NN interaction based on the use of the pion-exchange potential.

Finally, we have shown that the use of the formalism of the GQD allows one to formulate the effective field theory of nuclear forces as a completely consistent theory based on the well-defined equation of motion that does not require regularization and renormalization. Being formulated in this way, the effective nuclear theory
permits a natural parametrization of the interaction of nucleons by the generalized interaction operator that be derived order by order by using the methods of subtractive EFT’s. One can hope that this operator will be able to play the same role in nuclear physics as the Coulomb potential in quantum mechanics of atomic phenomena. A remarkable feature of such a formulation of the theory is that in this case all advantages of the operator formalism of quantum mechanics can be used. The equation of motion (5), for example, allows one to construct not only the S-matrix but also the evolution operator describing the dynamics of nucleon systems. This is very important, because the S-matrix is not everything. For example, at finite temperature there is no S-matrix because particles cannot get out to infinite distances from a collision without bumping into things. In conclusion, it should be emphasized that the above is not a new approach to the EFT of nuclear forces. We have only shown that the EFT approach gives rise to low energy nucleon dynamics which can be described, in a natural way, only by using the generalized dynamical equation (5), and its use as the equation of motion describing low energy nucleon dynamics can allow one to formulate the effective field theory of nuclear forces as a perfectly satisfactory theory like the quantum mechanics of atomic phenomena.

APPENDIX A

Let us consider the solution of Eq.(16) in the case where the interaction operator is of the form (76). From (17) and (18) it follows that this solution can be represented in the form

\[ < p_2 | T(z) | p_1 > = \lim_{u \to -\infty} < p_2 | T_u(z) | p_1 >, \]

(A1)

where the operator \( T_u(z) \) is the solution of the equation

\[ T_u(z) = B(u) + (u - z)B(u)G_0(u)G_0(z)T_u(z). \]

(A2)

Here the operator \( B(z) \) is given by

\[ < p_2 | B(z) | p_1 > = f_1(z) + V(p_2, p_1), \]

(A3)

with

\[ f_1(z) = -\frac{4\pi}{m^2 \sqrt{-z}} - \frac{16\pi^2}{m^2 C_0(R) z}. \]

The solution of Eq.(A2) can be represented in the form

\[ < p_2 | T_u(z) | p_1 > = t_0^{(u)}(z, z; p_1) + t_1^{(u)}(z; p_2) + t_2^{(u)}(z; p_1, p_2). \]

(A4)

Substituting this representation in Eq.(A2) yields the following equations for \( t_0^{(u)}(z), t_1^{(u)}(z; p), t_1^{(u)}(z; p_2) \) and \( t_2^{(u)}(z; p_1, p_2): \)

\[ t_0^{(u)}(z; p) = (u - z) \int \frac{d^3k}{(2\pi)^3} \frac{\left( t_0^{(u)}(z) + t_1^{(u)}(z; k) \right)}{(z - E_k)(u - E_k)} \times V(k, p); \] (A6)

\[ t_0^{(u)}(z; p) = (u - z) f_1(u) \]

\[ \times \int \frac{d^3k}{(2\pi)^3} \frac{\left( t_0^{(u)}(z; p) + t_2^{(u)}(z; k, p) \right)}{(z - E_k)(u - E_k)}; \] (A7)

\[ t_2^{(u)}(z; p_2, p_1) = V_z(p_2, p_1) + (u - z) \int \frac{d^3k}{(2\pi)^3} \]

\[ \frac{\left( t_1^{(u)}(z; p) + t_2^{(u)}(z; k, p) \right)}{(z - E_k)(u - E_k)} V(k, p_1). \]

(A8)

It is not difficult to verify that

\[ t_1^{(u)}(z; p) \equiv t_1^{(u)}(z; p). \]

By solving the above set of equations, one can obtain the functions \( t_0^{(u)}(z), t_1^{(u)}(z, p) \) and \( t_2^{(u)}(z, p_2, p_1) \) that in turn can be used for constructing the T-matrix. In fact, from (A1) and (A2) it follows that the T-matrix can be represented in the form [7] where the functions \( t_0(z), t_1(z, p) \) and \( t_2(z, p_2, p_1) \) are given by

\[ t_0(z) = \lim_{u \to -\infty} t_0^{(u)}(z, z; p_1), \]

\[ t_1(z, p) = \lim_{u \to -\infty} t_1^{(u)}(z, p), \]

\[ t_2(z, p_2, p_1) = \lim_{u \to -\infty} t_2^{(u)}(z, p_2, p_1). \]

Taking into account that

\[ \int \frac{d^3k}{(2\pi)^3} \frac{u - z}{(z - E_k)(u - E_k)} = \frac{m \sqrt{m}}{4\pi} \left( -\frac{4\pi}{m^2 \sqrt{-u}} - \frac{16\pi^2}{m^2 C_0(R) u} \right), \]

Eq.(A3) can be rewritten in the form

\[ t_0^{(u)}(z) = -\frac{4\pi}{m^2 \sqrt{-u}} + t_0^{(u)}(z) \]

\[ \times \left( -\frac{4\pi}{m^2 \sqrt{-u}} - \frac{16\pi^2}{m^2 C_0(R) u} \right) \frac{m \sqrt{m}}{4\pi} \left( -\frac{4\pi}{m^2 \sqrt{-u}} - \frac{16\pi^2}{m^2 C_0(R) u} \right) \]

\[ -\frac{4\pi}{m^2 \sqrt{-u}} \int d^3k \frac{t_1(z, k)}{z - E_k} + o(|u|^{-1/2}). \]

Letting \( u \to -\infty \) in this equation and assuming that \( V(p_2, p_1) \) satisfies the ordinary requirements of quantum mechanics, one can easily get Eq.(81). In the same way, from (A6) and (A8) one can derive Eqs.(78) and (80).

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