We derive the BRST cohomology of the $G/G$ topological model for the case of $A_{N-1}^{(1)}$. It is shown that at level $k = \frac{p}{q} - N$ the latter describes the $(p,q) W_N$ minimal model coupled to $W_N$ gravity (plus some extra “topological sectors”).
Recently the space of physical states of the $\frac{G}{G}$ topological models\cite{1,2} for the case of $A^{(1)}_1$ was extracted.\cite{3} By applying the method of ref. \cite{4} the cohomology of the gauge symmetry BRST operator was computed. Using a twisted energy-momentum tensor an intriguing correspondence between the $\frac{SL(2)}{SL(2)}$ model with level $k = \frac{p}{q} - 2$ and $(p, q)$ models coupled to gravity was established. In the present work we extend this analysis to the $A^{(1)}_{N-1}$ algebra and show that the model describes $W_N$ strings.\cite{5} More precisely, the $k = \frac{p}{q} - N$ level model contains $(p, q)$ $W_N$ minimal models\cite{6} coupled to $W_N$ gravity with some extra “topological sectors”. We shall present some evidence which support this result and also work out explicitly the BRST cohomology. Related results were obtained previously within the Hamiltonian reduction approach,\cite{7} which has common features to our approach. In particular it has been shown that the $SL(N, R)$ Kac-Moody algebra at level $k = \frac{p}{q} - N$ gave rise to the $(p, q)$ $W_N$ algebra.\cite{7}

The $\frac{G}{G}$ TQFT is constructed by gauging the anomaly free diagonal $G$ group of the WZW model. The quantum action of the model was shown to be composed of three decoupled parts:\cite{8,9} $S_k(g)$ — a WZW model of level $k$ with $g \in G$, $S_{-(k+2C_G)}(h)$ — a WZW model of level $-(k + 2C_G)$ with $h \in G$, and a dimension $(1, 0)$ system of anticommuting ghosts $\rho$ and $\chi$ in the adjoint representation of the group. The action, thus, reads

$$S_k(g, h, \rho, \chi) = S_k(g) + S_{-(k+2C_G)}(h) - i \int d^2z Tr[\bar{\rho} \partial \chi + \rho \bar{\partial} \chi],$$  \hspace{1cm} (1)$$

where $C_G$ is the second Casimir of the adjoint representation. For fractional level we take for $G$ the non-compact group $SL(N, R)$. In this case parametrizing the gauge fields by orbits of $G_C$, the complexification of $G$, is not permissible* and thus the naive passage from the gauged WZW action to the one given in eqn. (1) may not hold. Nevertheless, one can always take the action of eqn. (1) as the definition of the model.

* We thank E. Witten for paying our attention to this point.
Invariance of each of the three terms under holomorphic \( G \) transformations gives rise to three Kac-Moody currents \( J(z) = g^{-1} \partial g, I(z) = h^{-1} \partial h \) and \( J^{(gh)} = f_{bc}^a \chi^b \rho^c \) of levels \( k, -(k + 2c_G) \) and \( 2c_G \) respectively.

We decompose the \( J^a \) currents (and similarly the \( I^a \) and \( J^{(gh)} \)) to those related to the Cartan sub-algebra generators and to those related to the positive and negative roots. In the present work we are interested in \( A^{(1)}_{N-1} \) and thus there are \( N-1 \) of the former which we denote by \( J^i, i = 1, ..., N-1 \) and \( \frac{N(N-1)}{2} \) of the latter which we denote by \( J^{\pm(ij)} \). Each of the \( J^{\pm(ij)} \) corresponds to a root \( \pm(\alpha_i + \alpha_{i+1} + ... + \alpha_{j-1} + \alpha_j) \) where the simple roots are \( \alpha_1, ..., \alpha_{N-1} \) and with \( 1 \leq i \leq j \leq N - 1 \). The operator product algebra takes now the form

\[
J^i(z)J^j(\omega) = \frac{g^{ij}k}{(z - \omega)^2} + O(z - \omega)
\]

\[
J^i(z)J^{\pm(jk)}(\omega) = \frac{f^{\pm(jk)}_{\pm(lm)}J^{\pm(lm)}(\omega)}{(z - \omega)} + O(z - \omega)
\]

\[
J^{+(ij)}(z)J^{-+(kl)}(\omega) = \delta^{ij} \delta^{kl} \frac{k}{(z - \omega)^2} + \frac{f^{+(ij)-(kl)}_{m}J^{m}(\omega)}{(z - \omega)} + \frac{f^{+(ij)-(kl)}_{\pm(mn)}J^{\pm(mn)}(\omega)}{(z - \omega)} + O(z - \omega),
\]

\[
J^{\pm(ij)}(z)J^{\pm(kl)}(\omega) = \frac{f^{\pm(ij)}_{\pm(mn)}J^{\pm(mn)}(\omega)}{(z - \omega)} + O(z - \omega),
\]

where \( g^{ij} \) is the inverse Cartan matrix, given by \( g^{ij} = \frac{1}{N} \text{Min}(i, j)(N - \text{Max}(i, j)) \), and \( f_{ab}^c \) are the group structure constants.

We define now \( J^{(tot)} \)

\[
J^{(tot)} = J^a + I^a + J^{(gh)} = J^a + I^a + i f_{bc}^a \chi^b \rho^c
\]

which obeys a Kac-Moody algebra of level

\[
k^{(tot)} = k - (k + 2c_G) + 2c_G = 0.
\]

The total energy-momentum tensor \( T(z) \) is a sum of Sugawara terms of the \( J^a \) and
\( I^a \) currents and the usual contribution of a \((1, 0)\) ghost system, namely\(^8,9\)

\[
T(z) = \frac{1}{k + c_G} : J_a J^a : - \frac{1}{k + c_G} : I_a I^a : + \rho_a \partial \chi^a .
\] (5)

The corresponding Virasoro central charge vanishes

\[
c^{(\text{tot})} = \frac{kd_G}{k + c_G} - \frac{(k + 2c_G)d_G}{-(k + 2c_G) + c_G} - 2d_G = 0
\] (6)

as it is the case for any TCFT. In each of the three sectors of the model one can generalize the symmetry generator \( T(z) \equiv W^{(2)}(z) \) to \( W^{(l)} \) for \( l = 2, \ldots, N \). \( W^{(l)} \) is built as a symmetric normal ordered \( l^{th} \) product of the currents associated with the \( l^{th} \) Casimir operator. For instance the purely \( J \) sector contribution to \( W^{(3)} \)\(^{10} \) will be

\[
W^{(3)}_J = \eta^{(3)} d^{(abc)} : J^a J^b J^c :
\] (7)

where \( d^{(abc)} \) is the symmetric traceless tensor invariant of the underlying Lie algebra, and \( \eta^{(3)} = \sqrt{N} [3(N + k) \sqrt{2(N + k)(N^2 - 4)}]^{-1} \). The generator \( W^{(3)} \) can then be constructed from \( W^{(3)}_J, W^{(3)}_I, W^{(3)}_{J^{(gh)}} \) and similar cubic terms with mixed currents along the lines of ref.\(^{10} \).

It is easy to realize that the basic algebraic structure of a TCFT\(^{11,12} \) is obeyed by the model. This is expressed in terms of two bosonic and two fermionic operators. The former are \( T(z) \) and the \( \text{“} \text{ghost number current} \text{”} \) \( J^\# = \chi^a J^a \). The fermionic currents are a dimension one current which is the BRST current \( J^{(BRST)} \) and a dimension two operator \( G \). These holomorphic symmetry generators are given by

\[
J^{(BRST)} = \chi^a [J^a + I^a + \frac{1}{2} J^{(gh)a}] \\
G = \frac{1}{k + c_G} \rho_a [J^a - I^a]
\] (8)

The TCFT algebraic structure now reads

\[
T(z) = \{Q, G(z)\} \\
J^{(BRST)} = \{Q, J^\#(z)\}
\] (9)
where $Q = \oint J^{(BRST)}(z)$ is the BRST charge. In addition to $T(z)$ and $J^{(BRST)}(z)$, the total current $J^{(tot)}^a$ is also BRST exact,

$$J^{(tot)}^a(z) = \{Q, \rho^a\}.$$  \hspace{1cm} (10)

An essential step for the extraction of the physical states\footnote{The original reference is not provided in the image.} of the theory and for the comparison with $(p, q) W_N$ string models, is the parametrization of the currents in terms of free fields. For the $J^a$ sector we introduce a set of scalars $\phi_i$, $i = 1, \ldots, N-1$ with $\phi_i(z)\phi_j(\omega) = -\delta_{i,j}\log(z-\omega)$ and a commuting $(1, 0)$ system $\beta^{(ij)}, \gamma^{(ij)} (i \leq j)$ for each positive root obeying $\gamma^{(ij)}(z)\beta^{(kl)}(\omega) = \delta^{ik}\delta^{jl}\frac{1}{(z-\omega)}$. The explicit form of the $J^i_n$ and $J^+_{n}^{(ij)}$ currents in terms of these free fields\footnote{The original reference is not provided in the image.} is given by

$$J^+_{n}^{(ij)} = \beta^{(ij)}_n - \sum_m \sum_{k=1}^{i-1} \gamma^{(k,i-1)}_m \beta^{(kj)}_{n-m} :$$

$$J^i_n = \sum_{j=1}^{N-1} g^{ij}[\nu \vec{\alpha}_j \cdot \vec{\phi}_n + 2 \sum_m \gamma^{(jj)}_m \beta^{(jj)}_{n-m} : - \gamma^{(ij)}_m \beta^{(ji)}_{n-m} :] - \sum_m \sum_{k=1}^{j-1} (\gamma^{(k,j-1)}_m \beta^{(kj-1)}_{n-m} : - \gamma^{(kj)}_m \beta^{(jk)}_{n-m} :) - \sum_{m} \sum_{k=j+1}^{N-1} (\gamma^{(j+1,k)}_m \beta^{(j+1,k)}_{n-m} : - \gamma^{(jk)}_m \beta^{(jk)}_{n-m} :)] \hspace{1cm} (11)$$

These expressions as well as those for the negative root currents are derived by applying the commutation relations for the currents which correspond to the simple roots:

$$J^+_{n}^{(ii)} = \beta^{(ii)}_n - \sum_m \sum_{k=1}^{i-1} \gamma^{(k,i-1)}_m \beta^{(ki)}_{n-m} :$$

$$J^{-}_{n}^{(ii)} = - \sum_m \gamma^{(ii)}_m [\nu \vec{\alpha}_i \cdot \vec{\phi}_{n-m} + (k+i-1)\nu \gamma^{(ii)}_n$$

$$+ \sum_k \sum_{m} \gamma^{(i,k)}_m \beta^{(i+1,k)}_{n-m} : - \sum_k \sum_m \gamma^{(k,i)}_m \beta^{(k+1,1)}_{n-m} :$$

$$- \sum_{l,m} \gamma^{(ii)}_{n-m-l} \sum_k \gamma^{(ik)}_m \beta^{(ik)}_l - \sum_{l=i+1}^{N-1} \gamma^{(i+1,k)}_m \beta^{(i+1,k)}_l \hspace{1cm} (12)$$

These expressions as well as those for the negative root currents are derived by applying the commutation relations for the currents which correspond to the simple roots:
with $\nu^2 = k + N$. The free field parametrization of the $I$ sector is done using a similar set of fields denoted by $\tilde{\phi}, \tilde{\beta}$ and $\tilde{\gamma}$. Our experience with the $A_1^{(1)}$ case taught us that in order to follow the analysis of ref. [4] we would have to perform an involution, namely, $I_n^i \leftrightarrow -I_n^i, I_n^{+(ij)} \leftrightarrow I_n^{-(ij)}, k \rightarrow -k - 2N, \nu \rightarrow i\nu$ and then invoke the free field parametrization. Alternatively one can perform the involution in the $J$ sector and use for the $I$ sector the analogs of eqns. (12), (11).

The correspondence between $SL(2, R)$ $G$ models of level $k = \frac{p}{q} - 2$ and $(p, q)$ minimal models coupled to gravity was demonstrated using a twisted energy-momentum tensor. The generalization of the latter to the $SL(N, R)$ case is given by

$$T(z) \rightarrow \tilde{T}(z) = T(z) + \sum_{i=1}^{N-1} \partial J^{(tot)}_i(z)$$

(13)

Obviously since $T(z)$ and $\partial J^{(tot)}_i(z)$ are BRST exact so is $\tilde{T}(z)$. Using eqn. (2) it is easy to verify that the Virasoro central charge of each sector is shifted to

$$c \rightarrow \tilde{c} = c - 12k \sum_{i,j} g^{ij} = c - d_G c_G k = c - (N - 1)N(N + 1)k,$$

(14)

where $k$ is the level of that sector.

The twisted ghost system will be shown to include the ghosts of a $W_N$ gravity, namely, a sequence of ghosts with dimensions $(i, 1 - i)$ for $i = 2, ..., N$ contributing $\tilde{c}_{Wgh} = -2(N - 1)[(N + 1)^2 + N^2]$ to $\tilde{c}$. The rest of the ghosts are paired with commuting fields of the same conformal structure coming from the $J$ and $I$ sectors. Therefore, the net matter degrees of freedom have the following Virasoro anomaly $c = \tilde{c}_J - \frac{1}{2}[\tilde{c}_{(gh)} - \tilde{c}_{Wgh}] = (N - 1)[(2N^2 + 2N + 1) - N(N + 1)(t + \frac{1}{t})]$ which is exactly that of a $(p, q)$ minimal $W_N$ matter sector provided $t \equiv k + N = \frac{p}{q}$. This can be explicitly verified by analyzing the dimensions and contributions to $\tilde{c}$ of the various free fields in the $J$ sector. Recall that before twisting $(\beta^{(ij)}, \gamma^{(ij)})$ are commuting $(1,0)$ systems and the contribution of the set of all $\phi^i$ to $c$ is $c_\phi = (N - 1) - \frac{12\sum_{k+N} g^{ij}}{k+N} = -(N - 1)[(N^2 + N) - 1]$. Due to the twisting the
$$(\beta^{(ij)}, \gamma^{(ij)})$$ systems acquire dimensions of \((i-j, j-i+1)\) and thus there are \(N-1\) systems of dimension \((0,1)\), \(N-2\) pairs of fields of dimension \((-1,2)\) up to one pair of dimensions \((2-N, N-1)\) where we have used the bosonization of eqns. (12),(11) in this sector. The \(\phi\) central charge is modified to

$$\tilde{c}_\phi = (N-1)[(2N^2 + 2N + 1) - N(N+1)(t + \frac{1}{t})]$$  \(\text{(15)}\)

This result is identical to the net matter contribution to \(c\) given above and hence the \(\phi^i\) fields are in fact those of the \(W_N\) model. A further indication of this equivalence is the dimensions of the \(\phi\) fields which correspond to the maximal weights \(\lambda^j = \sum_k g^{jk}[(r_k - 1) - t(s_k - 1)]\) where \(\lambda^j\) are defined below in eqn. (20). After the twisting the dimensions are \(\Delta_{r_1,\ldots,r_{N-1},s_1,\ldots,s_{N-1}} = \frac{12 \sum_{i,j} g^{ij}(ps_i - qr_i)(ps_j - qr_j) - N(N^2 - 1)(p-q)^2}{24pq}\)  \(\text{(16)}\)
as in the \(W_N\) minimal models.\(^6\) If one parametrizes the \(I\) sector in the same way as the \(J\) sector, then clearly the modified dimensions of the \((\tilde{\beta}^{(ij)}, \tilde{\gamma}^{(ij)})\) fields are the same as those without tilde. From the point of view of their contribution to \(\tilde{c}\) the \(\tilde{\phi}^j\) fields are then identical to those of \(W_N\) gravity. This is achieved by replacing \(t\) with \(-t\) in \(\tilde{c}_\phi\) defined above. Using an inverse parametrization in this sector, the \(W_N\) gravity modes and the fields which pair with the redundant ghosts are not any more the \(\tilde{\phi}^j\) and \((\tilde{\beta}^{(ij)}, \tilde{\gamma}^{(ij)})\) fields respectively but rather some combination of them. With respect to the untwisted \(T\) the ghosts \((\rho^a, \chi^a)\) are all of dimension \((1,0)\). The ghost part of \(\tilde{T}(z)\) has the form

$$\tilde{T}^{(gh)}(z) = g_{ab} : \rho^a \partial \chi^b : + \sum_{1 \leq i \leq j \leq N-1} (j - i + 1) \partial[ : \rho^{- (ij)}(z) \chi^{+ (ij)}(z) : - : \rho^{+ (ij)}(z) \chi^{- (ij)}(z) : ].$$  \(\text{(17)}\)

It is thus obvious that the members of the Cartan sub-algebra \(\rho^i, \chi^i\) remain \((1,0)\) fields. On the other hand the pair \((\chi^{+ (ij)}, \rho^{- (ij)})\) carries now dimensions \((i-j-\)
$1, 2 + j - i$) and $(\chi^{-\langle ij \rangle}, \rho^{+\langle ij \rangle})$ carry dimensions $(j - i + 1, i - j)$. Altogether one finds for the $(\chi, \rho)$ ghosts $N - 1$ pairs of fields of dimension $(0, 1)$ coming from the Cartan sub-algebra, $N - 1$ pairs of dimension $(-1, 2)$, $N - 2$ pairs of dimension $(-2, 3)$ up to a pair of dimension $(1 - N, N)$, and similarly $N - 1$ pairs of dimension $(1, 0)$ up to one pair of dimension $(N - 1, 2 - N)$. It is now clear that when the dust settles the $G_n$ model of $SL(N, R)$ at level $k = \frac{p}{q} - N$ has the field content of a minimal $W_N (p, q)$ model coupled to $W_N$ gravity plus pairs of “topological sectors” namely pairs of commuting and anti-commuting $(i, 1 - i)$ ghost systems for $i = 1, \ldots, N - 1$.

Next we proceed to extract the space of physical states of the model. The physical states are in the cohomology of $Q$, the BRST charge, $|\text{phys}>, \in H^*(Q)$. Since both $J^{(\text{tot})} a_n$ and $L_n$ are $Q$ exact (eqn.(9),(10)), it is clear that

$$L_0|\text{phys}>, = 0 \quad J^{(\text{tot})} 0_i|\text{phys}>, = 0 \quad (i = 1, \ldots, N - 1)$$

and therefore we can work on the space annihilated by $L_0$ and $J^{(\text{tot})} 0_i$. Note that since $J^{(\text{tot})} 0_i$ is $Q$ exact the physical states before and after twisting are the same. We can, therefore, concentrate on the untwisted $G_n$ theory. The BRST charge can be decomposed into

$$Q = \chi^0_0 J^{(\text{tot})i} 0_i + M^i \rho^0_0 + \hat{Q}$$

$$M^i = -\frac{1}{2} f_{bc}^{ij} \sum_n : \chi^{-n}_n \chi^{-1}_n :$$

where the $i-$ sum is over the Cartan subalgebra, so that on the sub-space of states annihilated by $\rho^0_0$, $Q = \hat{Q}$. We build the states of $H^*(\hat{Q})$ on a highest weight vacuum $|J, I>$ obeying the following relations

$$J^{(\text{tot})} 0_i|\lambda J, \lambda I> = 0 \quad J^{(\text{tot})} 0_i|\lambda J, \lambda I> = 0 \quad J^{(\text{tot})} 0_i|\lambda J, \lambda I> = 0$$

$$J^{(\text{tot})} 0_i|\lambda J, \lambda I> = 0 \quad J^{(\text{tot})} 0_i|\lambda J, \lambda I> = 0$$

The weights in this parametrization are $\sum \alpha_i \lambda^{\hat{J}}_I$ and $\sum \alpha_i \lambda^{\hat{J}}_J$, respectively. Using the free field parametrization of the currents (11), it is straightforward to check
that \( \beta_{0}^{(ij)}|\lambda_{J}, \lambda_{I} >= 0 \). In order to apply the spectral sequence decomposition \(^{3}\) of the BRST charge, we have to assign a degree to the various fields. We use an assignment that generalizes the one used for the \( A_{1}^{(1)} \) case \(^{3}\) and which produces as the lowest degree term in \( Q \) a term with zero degree.

This assignment reads

\[
\begin{align*}
\text{deg}(\chi^{a}) &= \text{deg}(\gamma^{(ij)}) = \text{deg}(\tilde{\gamma}^{(ij)}) = \text{deg}(\phi^{i+}) = 1 \\
\text{deg}(\rho^{a}) &= \text{deg}(\beta^{(ij)}) = \text{deg}(\tilde{\beta}^{(ij)}) = \text{deg}(\phi^{i-}) = -1
\end{align*}
\]

where \( \phi_{n}^{i\pm} = \frac{1}{\sqrt{2}}(\phi_{n}^{i} \pm i\tilde{\phi}_{n}^{i}) \). The decomposition of \( \hat{Q} \) into terms of different degrees is given by

\[
\hat{Q} = Q^{(0)} + Q^{(1)} + \ldots + Q^{(N+1)}
\]

\[
Q^{(0)} = \sum_{i,j} \sum_{1 \leq i \leq j \leq N-1} (\chi_{-n}^{(ij)} \beta_{n}^{(ij)} + \chi_{-n}^{+(ij)} \tilde{\beta}_{n}^{(ij)}) + \sqrt{2}\nu \sum_{n \neq 0} \sum_{i=1}^{N-1} \chi_{-n}^{i}(\tilde{\alpha}_{i} \cdot \tilde{\phi}_{n}^{-})
\]

where \( \text{deg}(Q^{(n)}) = n \). We do not write down expressions for \( Q^{(n \neq 0)} \) since they would not be needed for the extraction of physical states. Let us now examine the cohomology \( H^{*}(Q^{(0)}) \). The conformal dimension operator \( L_{0} \) when acting on a given state has two parts: one that determines the level of the vacuum state and the other that determines the dimension of the excitations

\[
L_{0} = \hat{L}_{0} + \frac{1}{2(k+N)} \left[ \sum_{i,j} g_{ij} (\lambda_{J}^{i} \lambda_{J}^{j} - \lambda_{I}^{i} \lambda_{I}^{j}) - 2 \sum_{i} (\lambda_{J}^{i} - \lambda_{I}^{i}) \right]
\]

\[
\hat{L}_{0} = \sum_{n} \sum_{1 \leq i \leq j \leq N-1} n \left[ \beta_{-n}^{(ij)} \gamma_{n}^{(ij)} : + : \tilde{\beta}_{-n}^{(ij)} \tilde{\gamma}_{n}^{(ij)} : + : \rho_{-n}^{+(ij)} \chi_{n}^{-(ij)} : + : \rho_{-n}^{-(ij)} \chi_{n}^{+(ij)} : \right]
\]

\[
+ \sum_{n \neq 0} \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} g_{ij} \left[ \rho_{-n}^{i} \lambda_{n}^{j} : + : \phi_{-n}^{i+} \phi_{n}^{i-} : \right]
\]

In analogy to the \( SL(2) G \) model \(^{3}\) one can easily verify that \( \hat{L}_{0} \), the contribution
to $L_0$ of the excitations, is $Q(0)$ exact

$$\hat{L}_0 = \{Q(0), \hat{G}(0)\}$$

$$\hat{G}_0 = -\sum_n \sum_{1 \leq i \leq j \leq N-1} n[\rho^{+\langle ij \rangle}_n \gamma^{\langle ij \rangle}_n + \rho^{-\langle ij \rangle}_n \tilde{\gamma}^{\langle ij \rangle}_n] + \frac{1}{\sqrt{2\nu}} \sum_{n \neq 0} \sum_{i,j,k=1}^{N-1} g_{ij} \rho^i_n \epsilon^j_k \phi^k_{-n}.$$ (24)

where $\epsilon^j_k$ are a set of numbers obeying $\sum_{i=1}^{N-1} \alpha_i \epsilon^j_i = \delta^j_i$. Hence, $\hat{L}_0$ annihilates the states in the cohomology of $Q(0)$ on the Fock space and there are no excitations in $H^*(Q^0)$. Since both $L_0$ and $\hat{L}_0$ annihilate physical states so does $L_0 - \hat{L}_0$ and therefore there is a restriction on the vacuum $|\vec{\lambda}_J, \vec{\lambda}_I \rangle$. The states built on this vacuum may contain only the zero modes $\tilde{\beta}^{\langle ij \rangle}_0$, $\gamma^{\langle ij \rangle}_0$, $\chi^{-\langle ij \rangle}_0$, $\chi^{+\langle ij \rangle}_0$. The computation of $\frac{\text{Ker}(Q(0))}{\text{Image}(Q(0))}$ for the $A_{1}$^{(1)} is written down in ref. [3]. An analogous computation for $A_{N-1}$^{(1)} gives

$$H^{rel}(Q(0)) = \{ \prod_{1 \leq i \leq j \leq N-1} \chi^{+\langle ij \rangle}_0 |\vec{\lambda}_J, \vec{\lambda}_I \rangle; \quad \lambda^i_J + \lambda^i_I = -i(N - i) \}. \quad (25)$$

The condition $\lambda^i_J + \lambda^i_I = -i(N - i)$ is analogous to $I = -(J + 1)$ for the $SL(2)$ case. The absolute cohomology (without the restriction $\rho^i_0 = 0$ ) is

$$H^{abs}(Q(0)) \simeq H^{rel}(Q(0)) \oplus \sum_{\{k_1, \ldots, k_l\}} \chi^{k_1}_0 \ldots \chi^{k_l}_0 H^{rel}(Q(0)) \quad (26)$$

where the sum is over $\{k_1, \ldots, k_l\}$ which are all possible subsets of the set 1, ..., $N-1$. Thus, each state in the relative cohomology gives rise to $2^{N-1}$ states in the absolute cohomology. Next we want to examine whether $H^*(Q(0)) \simeq H^*(Q)$. In ref. [4] it was shown that the latter holds when the degree is bounded on the Fock space built on $|\vec{\lambda}_J, \vec{\lambda}_I \rangle$. Since the physical states are annihilated by $L_0$, it is clear that for a given $\vec{\lambda}_J, \vec{\lambda}_I$ the degree carried by the excitations is bounded. Hence, one has to consider possible contributions to the total degree from zero modes. Obviously, there can be only one zero mode for each $\chi$ that corresponds to a root. If we denote by $N_{\langle ij \rangle}$ and $\tilde{N}_{\langle ij \rangle}$ the number of $\gamma^{\langle ij \rangle}_0$ and $\tilde{\beta}^{\langle ij \rangle}_0$ respectively, the condition
\[ \sum_i J_i^0 = 0 \] bounds \[ \sum_{1 \leq i \leq j \leq N-1} (N_{ij} + \tilde{N}_{ij}) \] from both sides. Since \( N_{ij} \) and \( \tilde{N}_{ij} \) are non-negative it implies that all of them are bounded separately and thus also their contribution to the total degree which is \[ \sum_{1 \leq i \leq j \leq N-1} (N_{ij} - \tilde{N}_{ij}) \]. We conclude that using the lemmas of ref. [4] the isomorphism \( H^*(Q^{(0)}) \simeq H^*(Q) \) holds, and thus in the cohomology of \( Q \) there is a single state built on the vacuum \( |\bar{\lambda}_J, \bar{\lambda}_I > \) and it carries ghost number \( G = \frac{N(N-1)}{2} \).

To derive the space of physical states one has to translate the cohomology on the Fock spaces into the space of irreducible representations of \( A^{(1)}_{N-1} \) in the matter sector. In analogy to the totally reducible representations of \( SL(2, R) \) \[ [3] \], there are similar ones for \( SL(N, R) \) provided \( k + N = \frac{p}{q} \) and for particular highest weights. A brief discussion of the latter is presented in the appendix.

A generalization of the procedure of ref. [17] enables us to write down the irreducible representations as the cohomology of a BRST like operator \( Q_J \) \[ [18] \] acting on a union of the Fock spaces, at zero degree. The relevant cohomology with ghost number \( G = n \) is given by \[ [3] \]

\[ H^{(n)}[H^{(0)}(\mathcal{F}_{\bar{\lambda}_J}, Q_J) \times \mathcal{F}_{\bar{\lambda}_I} \times \mathcal{F}^G, Q] \tag{27} \]

where \( \mathcal{F}_{\bar{\lambda}_J} \) is the relevant union of Fock spaces and where we shift the definition of the ghost number by \( \frac{N(N-1)}{2} \). Due to the fact that \( Q_J \) commutes with all the currents, and that both cohomologies are non-zero at a single degree, this cohomology is isomorphic to \[ [3] \]

\[ H^{(n)}[H^{(0)}_{rel}(\mathcal{F}_{\bar{\lambda}_J} \times \mathcal{F}_{\bar{\lambda}_I} \times \mathcal{F}^G, Q), Q_J]. \tag{28} \]

The cohomology in the brackets is exactly the one computed above. Hence, for a given \( \bar{\lambda}_J \), there is a physical state iff it is built on a vacuum \( \bar{\lambda}_J \) contained in \( \mathcal{F}_{\bar{\lambda}_J} \) obeying \( \bar{\lambda}_J^i + \bar{\lambda}_J^i = -i(N - i) \) and carrying a ghost number equal to the degree of \( \bar{\lambda}_J \) in the \( Q_J \) complex. Denoting \( J_i = \frac{1}{2} \sum_{j=1}^{N-1} g_{ij} \bar{\lambda}_J^j \), the condition on the highest
weight state reads

\[ 2(J_i)_{r_i,s_i} + 1 = r_i - (s_i - 1)(k + N) \]  

(29)

where \( \sum_{i=1}^{N-1} r_i < p, \sum_{i=1}^{N-1} s_i < q + N - 1 \). In the \( A_1^{(1)} \) model it was found\(^3\) that there is a single state at each ghost number \( G = -2l \) built on \( |J_{r,s}, -J_{r+2lp,s} - 1 \rangle \) and a single state for \( G = 1 - 2l \) built on \( |J_{r,s}, -J_{-r+2lp,s} - 1 \rangle \). In the present case of \( A_{N-1}^{(1)} \) there is an \( N - 2 \) dimensional lattice of states for each ghost number and \( J \). This follows from the \( N - 1 \) dimensional lattice of Fock spaces which are derived by Weyl reflections as well as shifts by linear combination of roots.\(^{18}\)

In the present work the space of physical states of the \( G \) model based on \( A_{N-1}^{(1)} \) was derived by computing the cohomology of the gauge BRST charge on the space of Kac-Moody irreducible representations in the “matter sector”. We showed that after twisting, apart from some ”topological sectors”, the conformal properties of the fields of the model at level \( k + N = \frac{p}{q} \) are the same as those of \( (p, q) W_N \) strings. In the \( A_1^{(1)} \) case\(^3\) with \( k + 2 = \frac{p}{q} \) the partition function on the torus was shown to be identical to that of the \( (p, q) \) minimal model coupled to gravity. In the present case one can repeat this calculation by inserting the values of the \( \hat{L}_0 \) and \( \hat{J}^{(tot)}_0 \) of the physical states into an index of the form \( \text{Tr}[(-)^G q^{\hat{L}_0 + \pi \theta^i \hat{J}^{(tot)}_0}] \), where \( \theta^i \) are associated with the moduli of flat gauge connection on the torus. For a particular value of the moduli this reduces to \( \text{Tr}[(-)^G q^{\hat{L}_0}] \), where \( \hat{L}_0 \) is the twisted one given in eqn. (13), and should reproduce the torus partition function of the \( (p, q) W_N \) strings. An interesting question is the relation between the \( A_{N-1}^{(1)} \) \( G \) models and the twisted Kazama-Suzuki models.\(^{14}\) The explicit formulation of the Kazama-Suzuki models as gauged WZW models was discussed in ref. \[15\]. Moreover, the \( G \) models can be embedded into topological matter theories obtained by twisting hermitian symmetric \( N=2 \) supersymmetric coset models.\(^{16}\) The Kazama Suzuki models based on the coset \( \frac{SU(N)}{SU(N-1) \times U(1)} \) fall into this class of models. In addition, it has been shown that these models have \( W_N \) algebra as their chiral algebra.\(^{20}\) It is interesting to note that one can consider a whole series of \( N = 2 \) supersymmetric coset models by taking for \( G = SU(N) \) the following subgroups
\[ H = U(1)^{N-1}, SU(2) \times U(1)^{N-2}, ..., SU(N-1) \times U(1) \] with fermions residing in the coset. Upon twisting one obtains a corresponding series of topological models. Formally the \( \frac{G}{2} \) model will be the next one in this series. The cohomologies associated with these models and their relations to the cohomology of the \( \frac{G}{2} \) model is under current investigation. Some related work for the case of \( H = U(1)^{N-1} \) can be found in ref. [19]. Another issue that has to be resolved is the precise relation between the cohomology of the Kac-Moody BRST operator and that of the Virasoro and \( W_N \) cohomologies. Resolving that question may open the way to the use of the Kac-Moody BRST charge (linear in the currents) to deduce \( W_N \) cohomologies.

Upon completion of this work we have received a work[21] where the cohomology of the \( A_1^{(1)} \) \( \frac{G}{2} \) case is discussed.

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APPENDIX

Totally irreducible representations of \( SL(N,R) \)

According to the Kac-Kazhdan formula[22] a highest weight representation \( \Lambda \) has a singular vector of degree \( n\bar{\alpha} \) for \( n \) a positive integer and \( \bar{\alpha} \) a positive root of level \( k \) \( SL(N,R) \), iff

\[
\Phi_{\bar{\alpha},n}(\Lambda) \equiv (\Lambda + \bar{\mu}, \bar{\alpha}) - \frac{n}{2}(\bar{\alpha}, \bar{\alpha}) = 0
\]  

(A.1)

where \( \bar{\mu} \) is the sum of the fundamental weights which obey \( (\Lambda_i, \alpha_j) = \delta_{ij} \) with \( i, j = 0, ..., N - 1 \). Let us now parametrize the highest weight representations as
follows

$$\vec{\Lambda} = (k - 2J_1 - 2J_2 - \ldots - 2J_{N-1})\vec{\Lambda}_0 + \sum_{i=1}^{N-1} 2J_i \vec{\Lambda}_i$$ (A.2)

The positive roots of the $A_{N-1}^{(1)}$ algebra are

(i) $\alpha = n'(\alpha_0 + \alpha_1 + \ldots + \alpha_{N-1}) + (\alpha_i + \alpha_{i+1} + \ldots + \alpha_j)$ $n' \geq 0, \; 1 \leq i \leq j \leq N - 1$

(ii) $\alpha = n'(\alpha_0 + \alpha_1 + \ldots + \alpha_{N-1})$ $n' \geq 1$

(iii) $\alpha = n'(\alpha_0 + \alpha_1 + \ldots + \alpha_{N-1}) - (\alpha_i + \alpha_{i+1} + \ldots + \alpha_j)$ $n' \geq 1, \; 1 \leq i \leq j \leq N - 1$ (A.3)

The simple roots (for $N > 2$) obey the scalar product

$$(\alpha_i, \alpha_j) = \begin{cases} 2 & i = j \\ -1 & i - j = \pm 1 \; (mod \; N) \\ 0 & \text{otherwise} \end{cases}$$ (A.4)

Substituting eqn.(A.2) and eqn.(A.3) for $\vec{\Lambda}$ and $\vec{\alpha}$ respectively, the condition for singular vectors eqn.(A.1) takes the form

(i) $\Phi_{\vec{\alpha},n}(\vec{\Lambda}) = (k + N)n' - n + (2J_1 + 2J_{i+1} + \ldots + 2J_j) + (j - i + 1) = 0$

(ii) $\Phi_{\vec{\alpha},n}(\vec{\Lambda}) = (k + N)n' = 0$

(iii) $\Phi_{\vec{\alpha},n}(\vec{\Lambda}) = (k + N)n' - n - (2J_i + 2J_{i+1} + \ldots + 2J_j) - (j - i + 1) = 0$ (A.5)

Equation (ii) has obviously a solution only for $k = -N$. Let us define the totally reducible representation as the one that corresponds to the maximal number of $n, n'$ which solve equations (i) and (iii). Infinitely many solutions for $n, n'$ exist provided $k + N = \frac{p}{q}$ where $p, q$ are two integers with no common divisor. In that case for every $n, n'$ solution, $n + p, n' + q$ also solves the equations. In fact $n + lp, n' + lq$ solves equation (i) and $-(n + l'p), -(n' + l'q)$ solves (iii) when the shifted numbers are still in the allowed domains. The maximal number of solutions
occurs therefore only for $0 \leq n' < q$ and $0 < n < p$. Denoting $n, n'$ for $i = j$ by $r_i, s_i$ one gets

$$2J_i + 1 = r_i - (k + N)(s_i - 1)$$ (A.6)

which has the same form as the condition for the singular state for the $A_1^{(1)}$ case. Imposing the previous constraints for all $\vec{\alpha}$ one finds that $r_i, s_i$ are positive integers that obey

$$\sum_{i=1}^{N-1} (s_i - 1) < q \quad \sum_{i=1}^{N-1} r_i < p$$ (A.7)

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