A CLASS OF MAXIMALLY SINGULAR SETS FOR RATIONAL APPROXIMATION

ANTHONY POËLS

ABSTRACT. We say that a subset of \( \mathbb{P}^n(\mathbb{R}) \) is maximally singular if its contains points with \( \mathbb{Q} \)-linearly independent homogenous coordinates whose uniform exponent of simultaneous rational approximation is equal to 1, the maximal possible value. In this paper, we give a criterion which provides many such sets including Grassmannians. We also recover a result of the author and Roy about a class of quadratic hypersurfaces.

1. Introduction

A basic problem in Diophantine approximation is given a subset \( Z \) of \( \mathbb{P}^n(\mathbb{R}) \) to compute
\[
\lambda(Z) := \sup \{ \lambda(\xi) ; \xi \in Z^\text{li} \} \in [1/n, 1],
\]
where \( Z^\text{li} \) stands for the set of elements of \( Z \) having representatives with \( \mathbb{Q} \)-linearly independent coordinates in \( \mathbb{R}^{n+1} \), and \( \lambda(\xi) \) is the so-called uniform exponent of simultaneous rational approximation to \( \xi \) (see Section 2 for the definition). We say that \( Z \) is maximally singular if \( \lambda(Z) = 1 \).

A famous example is the Veronese curve \( Z = \{(1 : \xi : \xi^2 : \cdots : \xi^n) ; \xi \in \mathbb{R} \} \subseteq \mathbb{P}^n(\mathbb{R}) \) for which the computation of \( \lambda(Z) \), open for \( n \geq 3 \), would have implications on problems of algebraic approximation (see [2]). The case \( n = 2 \) is treated in [4]. More generally, Roy computes in [5] this exponent for any conics in \( \mathbb{P}^2(\mathbb{R}) \). These results are extended by the author and Roy in [3] to the case of any quadratic hypersurface \( Z \) of \( \mathbb{P}^n(\mathbb{R}) \) defined over \( \mathbb{Q} \). We show that, if \( Z^\text{li} \) is not empty, then the exponent \( \lambda(Z) \) is completely determined by \( n \) and the Witt index of the quadratic form defining \( Z \) [3, Theorem 1.1]. In particular, if this index is at least 2, then \( \lambda(Z) = 1 \) and there are uncountably many \( \xi \in Z^\text{li} \) such that \( \lambda(\xi) = 1 \).

Our main result presented in Section 2 gives a geometric condition under which a closed set \( Z \subseteq \mathbb{P}^n(\mathbb{R}) \) (for the topology induced by the projective distance) satisfies \( \lambda(\xi) = 1 \) for an uncountable set of points \( \xi \in Z^\text{li} \), and thus \( \lambda(Z) = 1 \). It allows us to recover the above mentionned result of [3] concerning quadratic hypersurfaces of Witt index \( \geq 2 \) (see Section 6). It also admits the following consequence.

2010 Mathematics Subject Classification. Primary 11J13; Secondary 11J82.

Key words and phrases. Exponents of Diophantine approximation, singular sets, measures of rational approximation, simultaneous approximation, Grassmannians, k-linear maps.
for each non-zero $x$, notation of $[3]$, we set

$\hat{\lambda}(x) = 1$, and so $\hat{\lambda}(Z) = 1$.

Examples of such sets $Z$ are the Grassmannians $G_{k,n}$ of $k$-dimensional subspaces of $\mathbb{R}^n$ as well as the subset of $\mathbb{R}[x_1, \ldots, x_n]_k$ consisting of homogenous polynomials of degree $k$ in $n$ variables which factor as a product of $k$ linear forms over $\mathbb{R}$ (see Section 3 for details). Note that, in general, $\varphi((\mathbb{R}^n)^k)$ need not be a closed subset of $\mathbb{R}^{n+1}$ as an example of Bernau and Wojciechowski [1] shows. So its projectivization may not be closed.

2. Main result and notation

Let $n$ be an integer $\geq 1$. For each set $Z \subseteq \mathbb{P}^n(\mathbb{R})$ (resp. $S \subseteq \mathbb{R}^{n+1}$) we write $Z(\mathbb{Q}) := Z \cap \mathbb{P}^n(\mathbb{Q})$ (resp. $S(\mathbb{Q}) = S \cap \mathbb{Q}^{n+1}$). We denote by $[\mathbf{x}]$ the class in $\mathbb{P}^n(\mathbb{R})$ of a non-zero point $\mathbf{x}$ of $\mathbb{R}^{n+1}$. Given non-zero points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{n+1}$, we recall that the projective distance between $[\mathbf{x}]$ and $[\mathbf{y}]$ is

$$\text{dist}([\mathbf{x}], [\mathbf{y}]) := \text{dist}(\mathbf{x}, \mathbf{y}) := \frac{\|\mathbf{x} \wedge \mathbf{y}\|}{\|\mathbf{x}\| \|\mathbf{y}\|}.$$ 

Let $\xi \in \mathbb{P}^n(\mathbb{R})$ and let $\xi \in \mathbb{R}^{n+1}$ be a representative of $\xi$ so that $\xi = [\xi]$. Following the notation of [3], we set

$$D_\xi(\mathbf{x}) := \frac{\|\mathbf{x} \wedge \xi\|}{\|\xi\|} = \|\mathbf{x}\| \text{dist}(\xi, [\mathbf{x}])$$

for each non-zero $\mathbf{x} \in Z^{n+1}$. Then, for each $X \geq 1$, we define

$$D_\xi(X) := \min \left\{ D_\xi(\mathbf{x}) ; \mathbf{x} \in Z^{n+1} \setminus \{0\} \text{ and } \|\mathbf{x}\| \leq X \right\}.$$ 

The exponent of uniform approximation $\hat{\lambda}(\xi)$ alluded to in the introduction is defined as the supremum of all $\lambda \in \mathbb{R}$ such that $D_\xi(X) \leq X^{-\lambda}$ for each sufficiently large $X$.

Our main result is the following.

Theorem 2.1. Let $n \geq 2$ and let $Z$ be a closed subset of $\mathbb{P}^n(\mathbb{R})$ (for the topology induced by the projective distance). Suppose that there exists a non-empty set $Z' \subseteq Z(\mathbb{Q})$ with the following property. For each proper projective subspace $H$ of $\mathbb{P}^n(\mathbb{R})$ defined over $\mathbb{Q}$ there is a function $s_H : Z' \rightarrow \mathbb{N}$ such that, for each $x \in Z'$

(i) if $s_H(x) = 0$, then $x \notin H$;
(ii) if $s_H(x) \geq 1$, then there exists a projective line $L$ defined over $\mathbb{Q}$ such that

$$x \in L(\mathbb{Q}) \subseteq Z' \text{ and } \# \{ y \in L(\mathbb{Q}) ; s_H(y) \geq s_H(x) \} < \infty.$$ 

Finally let $\varphi : [1, \infty) \rightarrow (0, 1]$ be a monotonically decreasing function with $\lim_{X \rightarrow \infty} \varphi(X) = 0$ and $\lim_{X \rightarrow \infty} X \varphi(X) = \infty$. Then there are uncountably many $\xi \in Z^{\mathbb{Q}}$ such that $D_\xi(X) \leq \varphi(X)$ for all sufficiently large $X$.
Choosing \( \varphi = \log(3X)/X \) for \( X \geq 1 \), we derive the following consequence.

**Corollary 2.2.** With the hypotheses and notation of the previous theorem, there are uncountably many \( \xi \in Z^\mathbb{Z} \) such that \( \hat{\lambda}(\xi) = 1 \), and so \( \hat{\lambda}(Z) = 1 \).

### 3. Proof of Theorem 2.1

Assume that \( Z \subseteq \mathbb{P}^n(\mathbb{R}) \) and \( Z' \subseteq Z(\mathbb{Q}) \) satisfy the hypotheses of Theorem 2.1.

**Lemma 3.1.** Let \( H \) be a proper projective subspace of \( \mathbb{P}^n(\mathbb{R}) \) defined over \( \mathbb{Q} \) and let \( s_H : Z' \to \mathbb{N} \) be a function satisfying Conditions \((i),(ii)\) of Theorem 2.1 for this choice of \( H \). Then for any non-zero integer point \( x \in \mathbb{Z}^{n+1} \) such that \([x] \in Z' \cap H\) we have \( s_H([x]) \geq 1 \) and there exist infinitely many non-zero integer points \( y \) with \([y] \in Z'\) satisfying

\[
\text{dist}(x, y) \leq \frac{C}{\|y\|} \quad \text{and} \quad s_H([y]) < s_H([x]),
\]

for a constant \( C = C(x, H) > 0 \) independent of \( y \).

**Proof.** Since \([x] \in H\), Condition \((i)\) implies that \( s_H([x]) \geq 1 \). By Condition \((ii)\) of Theorem 2.1 there exists a non-zero integer point \( z \) such that \( \mathbb{P}((x, z)_\mathbb{Q}) \subseteq Z' \) and \( s_H([z + bx]) \leq s_H([x]) - 1 \) for all but finitely many \( b \in \mathbb{Z} \). Putting \( y := z + bx \) we find

\[
\text{dist}(x, y) = \frac{\|x \wedge z\|}{\|x\| \|y\|}
\]

with a numerator that is independent of the choice of \( b \). \( \square \)

**Proof of Theorem 2.1.** This is similar in many aspects to the proof of [3, Proposition 10.4]. Starting with a non-zero integer point \( x_1 \) with \([x_1] \in Z'\), we construct recursively a sequence \((x_i)_{i \geq 1}\) of non-zero points of \( \mathbb{Z}^{n+1} \) which satisfies the following properties. When \( i \geq 1 \) we have

- (a) \([x_{i+1}] \in Z'\);
- (b) \( \|x_{i+1}\| > \|x_i\|\);
- (c) \( s_H([x_{i+1}]) < s_H([x_i]) \) for the subspace \( H = H_i \) of \( \mathbb{P}^n(\mathbb{R}) \) given by

\[
H_i := \mathbb{P}\left((x_j, \ldots, x_i)_\mathbb{R}\right),
\]

where \( j \) is the smallest index \( \geq 1 \) such that \( H_i \) is a proper subspace of \( \mathbb{P}^n(\mathbb{R}) \).

When \( i \geq 2 \), we further ask that

- (d) \( \text{dist}(x_{i+1}, x_i) \leq \frac{C}{\|x_{i+1}\|} \leq \frac{1}{3} \min\left\{ \frac{2\varphi(\|x_{i+1}\|)}{\|x_i\|}, \text{dist}(x_i, x_{i-1}) \right\} \), where \( C = C(x_i, H_i) \)

is the constant given by Lemma 3.1.
Suppose that \( x_1, \ldots, x_i \) are constructed for some \( i \geq 1 \). Then Lemma [3,1] provides a non-zero integer point \( x_{i+1} \) of arbitrarily large norm satisfying Conditions [a] to [c] as well as the left-hand side inequality of [d]. If \( i \geq 2 \) the right-hand side inequality of [d] is also fulfilled for \( \|x_{i+1}\| \) large enough since \( \lim_{X \to \infty} X \varphi(X) = \infty \).

The sequence \( ([x_i])_{i \geq 1} \) converges in \( \mathbb{P}^n(\mathbb{R}) \) to a point \( \xi \) with

\[
\text{dist}(\xi, [x_i]) \leq \sum_{j=1}^{\infty} \text{dist}(x_{j+1}, x_j) \leq \text{dist}(x_{i+1}, x_i) \sum_{j=0}^{\infty} 3^{-j} = \frac{3}{2} \text{dist}(x_{i+1}, x_i)
\]

for each \( i \geq 1 \). Moreover \( \xi \in Z \) since \( Z \) is a closed subset of \( \mathbb{P}^n(\mathbb{R}) \) and \([x_i] \in Z' \subseteq Z\) for each \( i \geq 1 \). When \( i \geq 2 \), Condition [d] combined with [2] yields

\[
D_\xi(x_i) = \|x_i\| \text{dist}(\xi, [x_i]) \leq \frac{3}{2} \|x_i\| \text{dist}(x_{i+1}, x_i) \leq \varphi(\|x_{i+1}\|).
\]

In particular, we have \( \lim_{i \to \infty} D_\xi(x_i) = 0 \). Let \( i_0, i \) be integers with \( 2 \leq i_0 \leq i \) such that \( V := \mathbb{P}\left(\langle x_{i_0}, \ldots, x_i \rangle_{\mathbb{R}}\right) \) is a strict subspace of \( \mathbb{P}^n(\mathbb{R}) \). By definition of \( H_i \) we have \( V \subseteq H_i \). Moreover \([x_i] \in H_i \), so that \( s_H([x_i]) \geq 1 \), where \( H = H_i \). Since \( s_H \geq 0 \), Condition [c] implies that there exists a smallest integer \( \ell > i \) such that \( H_{\ell} \neq H_i \), and therefore \([x_\ell] \notin H_i \).

In particular \([x_\ell] \notin V\) and this proves that \( (x_i)_{i \geq i_0} \) spans \( \mathbb{R}^{n+1} \) for each \( i_0 \geq 2 \). By [3, Lemma 6.2] we deduce that \( \xi \in Z^{|i} \).

For each \( X \geq \|x_2\| \) we have \( \|x_i\| \leq X < \|x_{i+1}\| \) for some \( i \geq 2 \) and using [3] we obtain

\[
D_\xi(X) \leq D_\xi(x_i) \leq \varphi(\|x_{i+1}\|) \leq \varphi(X).
\]

Therefore the point \( \xi \in Z^{|i} \) has the required property. By varying the sequence \( (x_i)_{i \geq 1} \), we obtain uncountably many such points as in the proof of [3, Proposition 10.4].

4. Proof of Theorem 1.1

Let \( k, n, N \) be positive integers with \( N \geq 2 \) and let \( \varphi : (\mathbb{R}^n)^k \to \mathbb{R}^{N+1} \) be a \( k \)-linear map defined over \( \mathbb{Q} \). We denote by \( Z_\mathbb{R} \) (resp. \( Z_\mathbb{Q} \)) the projectivization of \( \varphi((\mathbb{R}^n)^k) \) (resp. \( \varphi((\mathbb{Q}^n)^k) \)) and by \( S \) the set of points in \((\mathbb{Q}^n)^k\) whose image via \( \varphi \) is non-zero. For each \( \alpha, \beta \in Z_\mathbb{Q} \), we define \( m(\alpha, \beta) \) to be the largest integer \( m \geq 0 \) for which there exist \( (x_1, \ldots, x_k), (y_1, \ldots, y_k) \in S \) satisfying

\[
(4) \quad \alpha = [\varphi(x_1, \ldots, x_k)], \quad \beta = [\varphi(y_1, \ldots, y_k)], \quad \text{and} \quad \# \{i \in [1, k] : x_i = y_i \} = m.
\]

**Lemma 4.1.** Suppose that \( \varphi((\mathbb{Q}^n)^k) \) spans the whole space \( \mathbb{R}^{N+1} \). Let \( H \) be a projective proper subspace of \( \mathbb{P}^N(\mathbb{R}) \). For each \( \alpha \in Z_\mathbb{Q} \) the set \( Z_\mathbb{Q} \setminus H \) is non-empty and, upon defining \( s_H(\alpha) := \min \left\{ k - m(\alpha, \beta) : \beta \in Z_\mathbb{Q} \setminus H \right\} \), we have

\[
(i) \ s_H(\alpha) \geq 0 \text{ with equality if and only if } \alpha \notin H;
\]
(ii) if \( s_H(\alpha) \geq 1 \), then there exists a projective line \( L \subseteq \mathbb{P}^N(\mathbb{R}) \) defined over \( \mathbb{Q} \) such that

\[
\alpha \in L(\mathbb{Q}) \subseteq Z_\mathbb{Q} \quad \text{and} \quad \{ y \in L(\mathbb{Q}) ; s_H(y) \geq s_H(\alpha) \} = \{ \alpha \}.
\]

By Lemma 4.1 the topological closure \( Z = \overline{Z_R} \) of \( Z_R \) satisfies the hypotheses of Theorem 2.1 with \( Z' = Z_\mathbb{Q} \). This, in turn, implies Theorem 1.1.

Proof. Since by hypothesis \( Z_\mathbb{Q} \) generates the whole space \( \mathbb{P}^N(\mathbb{R}) \) and since \( H \) is a strict subspace, there exist points \( \beta \in Z_\mathbb{Q} \setminus H \). Assertion (i) is clear because \( m(\alpha, \beta) = k \) if and only if \( \alpha = \beta \).

To prove Assertion (ii) fix \( \alpha \in Z_\mathbb{Q} \) with \( s_H(\alpha) \geq 1 \) and choose \( \beta \in Z_\mathbb{Q} \setminus H \) such that

\[
m := m(\alpha, \beta) = k - s_H(\alpha) \leq k - 1
\]

is maximal. Choose \( (x_1, \ldots, x_k) \) (resp. \( (y_1, \ldots, y_k) \)) in \( S \) satisfying (4) with \( \alpha, \beta \) and \( m \) as above. Without lost of generality, we may assume that \( x_k \neq y_k \). Then set

\[
\alpha = \varphi(x_1, \ldots, x_k), \quad \beta = \varphi(y_1, \ldots, y_k),
\]

By hypothesis, we have \( \alpha, \beta \neq 0 \), \( \alpha = [\alpha] \) and \( \beta = [\beta] \). For any \( \lambda, \mu \in \mathbb{Q} \) the points

\[
(6) \quad \lambda\alpha + \mu\alpha' = \varphi(x_1, \ldots, x_{k-1}, \lambda x_k + \mu y_k) \quad \text{and} \quad \mu\beta + \lambda\beta' = \varphi(y_1, \ldots, y_{k-1}, \lambda x_k + \mu y_k)
\]

belong to \( \varphi((\mathbb{Q}^n)^k) \). Suppose first that \( \alpha' \) is proportional to \( \alpha \). There exist infinitely many \( \mu \in \mathbb{Q}^* \) such that \( \alpha + \mu\alpha' \neq 0 \) and \( \mu\beta + \beta' \neq 0 \). For those \( \mu \), the formulas (6) yield

\[
m(\alpha, [\mu\beta + \beta']) = m([\alpha + \mu\alpha'], [\mu\beta + \beta']) \geq m + 1,
\]

which implies that \( [\mu\beta + \beta'] \in H \) by maximality of \( m \). This is impossible since \( [\beta] = \beta \notin H \). Hence \( \alpha' \) is not proportional to \( \alpha \) and

\[
L := \mathbb{P}(\langle \alpha, \alpha' \rangle_{\mathbb{R}})
\]

is a projective line of \( \mathbb{R}^N(\mathbb{R}) \) defined over \( \mathbb{Q} \), satisfying \( \alpha \in L(\mathbb{Q}) \subseteq Z_\mathbb{Q} \). To conclude, it remains to show that \( L \) satisfies the second condition in (5). We first prove that for any \( \lambda \in \mathbb{Q} \) we have

\[
(7) \quad \beta + \lambda\beta' \neq 0 \quad \text{and} \quad [\beta + \lambda\beta'] \notin H.
\]

It is true if \( \beta' = 0 \). If \( \beta' \neq 0 \), then \( m(\alpha, [\beta']) \geq m + 1 \). This gives \( [\beta'] \in H \) by maximality of \( m \), and (7) follows since \( \beta = [\beta] \notin H \). Combining (6) and (7), we get

\[
m([\lambda\alpha + \alpha'], [\beta + \lambda\beta']) \geq m + 1,
\]

which yields \( s_H([\lambda\alpha + \alpha']) \leq k - m - 1 = s_H(\alpha) - 1 \). \hfill \Box
5. Two examples

In this section we give two examples of sets to which Theorem 1.1 applies.

The sets $G_{k,n}$. Let $k, n$ be two integers with $1 \leq k < n$ and $n \geq 3$. We define the Grassmannian $G_{k,n}$ as the projectivization of the set

$$\{ x_1 \wedge \cdots \wedge x_k \mid x_1, \ldots, x_k \in \mathbb{R}^n \}$$

inside $\mathbb{P}(\wedge^k \mathbb{R}^n)$. By identifying $\wedge^k \mathbb{R}^n$ to $\mathbb{R}^{N}$ via an ordering of the Plücker coordinates, where $N = \binom{n}{k} \geq 3$, the set $G_{k,n}$ is identified to a subset $\mathbb{P}(\mathbb{R}^N)$. It is Zariski closed, thus closed. By construction it is the projectivization of the image of the $k$-linear map $\varphi : (\mathbb{R}^n)^k \rightarrow \mathbb{R}^N$ defined by

$$\varphi(x_1, \ldots, x_k) = x_1 \wedge \cdots \wedge x_k, \quad x_1, \ldots, x_k \in \mathbb{R}^n.$$

It is easily seen that $\varphi((\mathbb{Q})^n)$ spans $\mathbb{R}^N$, so that the conditions of Lemma 4.1 are fulfilled and we get $\hat{\lambda}(G_{k,n}) = 1$.

The sets $H_{n,k}$. Let $k, n$ be positive integers with $n \geq 2$ and $n + k \geq 4$. The vector space $\mathbb{R}[x_1, \ldots, x_n]_k$ of homogenous polynomials of degree $k$ in $n$ variables admits for basis the set of monomials

$$x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad \alpha_1 + \cdots + \alpha_n = k$$

which identifies it to $\mathbb{R}^N$, where $N = \binom{n+k-1}{k} \geq 3$. We denote by $H_{n,k} \subseteq \mathbb{P}^N(\mathbb{R})$ the projectivization of the set

$$\left\{ \prod_{j=1}^k L_j \mid L_1, \ldots, L_k \in \mathbb{R}[x_1, \ldots, x_n]_1 \right\} = \psi((\mathbb{R}[x_1, \ldots, x_n]_1)^k),$$

where the $k$-linear map $\psi : (\mathbb{R}[x_1, \ldots, x_n]_1)^k \rightarrow \mathbb{R}[x_1, \ldots, x_n]_k \cong \mathbb{R}^N$ is defined by

$$\psi(L_1, \ldots, L_k) = L_1 \cdots L_k$$

for any $L_1, \ldots, L_k \in \mathbb{R}[x_1, \ldots, x_n]_1$. It can be shown that $H_{n,k}$ is closed. Since $\psi((\mathbb{Q}[x_1, \ldots, x_n]_1)^k)$ spans $\mathbb{R}^N$, we can once again apply Lemma 4.1 to get $\hat{\lambda}(H_{n,k}) = 1$.

6. Quadratic hypersurfaces of Witt index $> 1$

Let $n \geq 1$ be an integer. A quadratic hypersurface of $\mathbb{P}^n(\mathbb{R})$ defined over $\mathbb{Q}$ is a non-empty subset which is the set of zeros in $\mathbb{P}^n(\mathbb{R})$ of an irreductible homogeneous polynomial $q$ of $\mathbb{Q}[t_0, \ldots, t_n]$ of degree $2$. The Witt index (over $\mathbb{Q}$) $m$ of $q$ is the largest integer $m \geq 0$ such that $\mathbb{Q}^{n+1}$ contains an orthogonal sum (with respect to the symmetric bilinear form associated to $q$) of $m$ hyperbolic planes for $q$.

The following result is part of [3, Theorem 1.1].
Theorem 6.1 (Poëls-Roy, 2019). Let \( n \geq 3 \) be an integer and let \( Z \) be a quadratic hypersurface of \( \mathbb{P}^n(\mathbb{R}) \) defined over \( \mathbb{Q} \), and let \( m \) be the Witt index (over \( \mathbb{Q} \)) of the quadratic form on \( \mathbb{Q}^{n+1} \) defining \( Z \). If \( m \geq 2 \), then there are uncountably many \( \xi \in Z^m \) such that \( \hat{\lambda}(\xi) = 1 \), and thus \( \hat{\lambda}(Z) = 1 \).

In this section we use our Theorem 2.1 to give an alternative shorter proof of this statement. We choose \( Z' = Z(\mathbb{Q}) \) and for each proper projective subspace \( H \subseteq \mathbb{P}^n(\mathbb{R}) \) defined over \( \mathbb{Q} \) and each \( \alpha \in Z(\mathbb{Q}) \) we set

\[
(8) \quad s_H(\alpha) = \begin{cases} 
0 & \text{if } \alpha \notin H, \\
1 & \text{if } \alpha \in H \text{ and } \langle \alpha \rangle^\perp \neq H, \\
2 & \text{if } \alpha \in H \text{ and } \langle \alpha \rangle^\perp = H,
\end{cases}
\]

where for any \( \beta \in \mathbb{P}^n(\mathbb{R}) \) the set \( \langle \beta \rangle^\perp \) denotes the orthogonal in \( \mathbb{P}^n(\mathbb{R}) \) of \( \beta \) (with respect to the symmetric bilinear form associated to \( q \)). This choice of \( s_H \) satisfies Condition [(i)] of Theorem 2.1. If \( s_H(\alpha) \geq 1 \) we show that there exists a projective line \( L \) defined over \( \mathbb{Q} \) with the following properties stronger than Condition [(ii)] of Theorem 2.1:

(I) \( \alpha \in L(\mathbb{Q}) \subseteq Z(\mathbb{Q}) \);

(II) \( s_H(\beta) < s_H(\alpha) \) for any \( \beta \in L(\mathbb{Q}) \setminus \{\alpha\} \).

If \( s_H(\alpha) = 2 \), we choose a projective line \( L \) defined over \( \mathbb{Q} \) satisfying [(I)] (this exists since the Witt index \( m \) of \( q \) is \( \geq 2 \)). We claim that \( L \) also satisfies [(II)]. Indeed, since \( H^\perp = \{\alpha\} \), we have \( \langle \beta \rangle^\perp \neq \langle \alpha \rangle^\perp = H \) for any \( \beta \neq \alpha \), and so [(II)] follows.

If \( s_H(\alpha) = 1 \), the situation is more complicated. Arguing as in the proof of [3, Lemma 10.3], we note that there exists a zero \( \beta \) of \( q \) in \( Z(\mathbb{Q}) \cap \langle \alpha \rangle^\perp \setminus H \). Since \( \alpha \in H \) and \( \beta \notin H \), any point \( \gamma \neq \alpha \) in the projective line \( L \subseteq Z \) generated by \( \alpha \) and \( \beta \) does not belong to \( H \) and thus satisfies \( s_H(\gamma) = 0 < s_H(\alpha) \).

References

[1] S. Bernau and P. Wojciechowski. Images of bilinear mappings into \( \mathbb{R}^3 \). Proc. Amer. Math. Soc., 124(12):3605–3612, 1996.
[2] H. Davenport and W. Schmidt. Approximation to real numbers by algebraic integers. Acta Arith., 15(4):393–416, 1969.
[3] A. Poëls and D. Roy. Rational approximation to real points on quadratic hypersurfaces. \texttt{arXiv:1909.01499 [math.NT]}, 2019.
[4] D. Roy. Approximation to real numbers by cubic algebraic integers I. Proc. Lond. Math. Soc., 88(1):42–62, 2004.
[5] D. Roy. Rational approximation to real points on conics. Ann. Inst. Fourier, 63(6):2331–2348, 2013.

Département de Mathématiques, Université d'Ottawa, 150 Louis Pasteur, Ottawa, Ontario K1N 6N5, Canada

E-mail address: anthony.poels@uottawa.ca