Discriminating properties of the space-time compactification.

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Abstract

Compactification of the 5-dimensional Kaluza-Klein space-time geometry is produced. It is shown, that consideration of the space-time geometry as a physical geometry, i.e. as a geometry described completely by the world function, leads to a discrimination of some values of the particle momentum component along the fifth (charge) coordinate.

1 Introduction

The role of space-time geometry in description of physical phenomena of microcosm has been increased due to appearance of a more general conception of geometry. In the twentieth century the Riemannian geometry was considered to be the most general geometry, suitable for description of the space-time. However, the Riemannian geometry cannot describe such properties of space-time as discreteness, restricted divisibility of geometrical objects and discrete characteristics (mass, charge, angular momentum) of elementary particles. Discrete characteristics of elementary particles are considered to be dynamic properties of elementary particles, although development of the elementary particles theory manifests an increase of the role of the space-time geometry.

In reality, at a use of a true conception of the space-time geometry the elementary particles in themselves, as well as their properties and their dynamics can be described in terms of the proper space-time geometry and only in terms of the space-time geometry. The conventional conception of geometry, which supposes, that any geometry is axiomatizable, and any geometry can be deduced from a system of axioms, is wrong. In any axiomatized geometry the equivalence relation was supposed to be transitive. Only at transitive equivalence relation the set of all geometric
propositions (i.e., geometry) can be deduced from an axiomatics (a finite set of basic geometric propositions).

In the end of the twentieth century one invents a new method of the physical geometry construction [1]. The physical geometry is such a geometry, which is described completely by the world function \( \sigma \). The world function \( \sigma(P, Q) \) is a function of any two points \( P, Q \in \Omega \), where \( \Omega \) is the set of all points (or events), where the geometry is given.

\[
\sigma : \Omega \times \Omega \to \mathbb{R}, \quad \sigma(P, P) = 0, \quad \forall P \in \Omega \quad (1.1)
\]

The world function \( \sigma(P, Q) = \frac{1}{2}\rho^2(P, Q) \), where \( \rho(P, Q) \) is the distance between the points \( P \) and \( Q \).

On the one hand, the proper Euclidean geometry \( \mathcal{G}_E \) is the axiomatizable geometry, which can be deduced from the Euclidean axiomatics [2]. On the other hand, the proper Euclidean geometry \( \mathcal{G}_E \) is a physical geometry. It means, that all propositions \( P_E \) of \( \mathcal{G}_E \) can be expressed in terms of the Euclidean world function in the form \( P_E = P_E(\sigma_E) \). There is a theorem, where this statement has been proved [3, 1]. If now one replaces the Euclidean world function \( \sigma_E \) with the world function \( \sigma \) of some other physical geometry \( \mathcal{G} \) in all propositions \( P_E \):

\[
P_E(\sigma_E) \to P_E(\sigma),
\]

one obtains all propositions \( P_E(\sigma) \) of the physical geometry \( \mathcal{G} \). The procedure of replacement is a deformation of the proper Euclidean geometry, when the Euclidean distance \( \rho_E = \sqrt{2\sigma_E} \) are replaced by the distance \( \rho = \sqrt{2\sigma} \) of the physical geometry \( \mathcal{G} \). Thus, any physical geometry is obtained from the proper Euclidean geometry by means of a deformation.

In general, the physical geometry is not axiomatizable, because the axiomatizability of a geometry is possible only, if the equivalence relation is transitive. Indeed, in the proper Euclidean geometry the vector \( P_0P_1 \) is defined as an ordered set \( P_0P_1 = \{P_0, P_1\} \) of two points \( P_0, P_1 \). The equivalence (equality) of two vectors \( P_0P_1 \) and \( Q_0Q_1 \) is defined by two relations. Two vectors \( P_0P_1 \) and \( Q_0Q_1 \) are equivalent \( (P_0P_1 \equiv Q_0Q_1) \), if

\[
P_0P_1 \equiv Q_0Q_1 : \quad (P_0P_1, Q_0Q_1) = |P_0P_1| \cdot |Q_0Q_1| \wedge |P_0P_1| = |Q_0Q_1| \quad (1.2)
\]

where the scalar product \( (P_0P_1, Q_0Q_1) \) of vectors \( P_0P_1 \) and \( Q_0Q_1 \) is defined by the relation

\[
(P_0P_1, Q_0Q_1) = \sigma(P_0, Q_1) + \sigma(P_1, Q_0) - \sigma(P_0, Q_0) - \sigma(P_1, Q_1), \quad \forall P_0, P_1, Q_0, Q_1 \in \Omega \quad (1.3)
\]

and \( \sigma \) means the world function of the proper Euclidean geometry. The first relation of (1.2) describes parallelism of vectors \( P_0P_1 \) and \( Q_0Q_1 \), whereas the second one describes equality of their lengths. The definition of equivalence of two vectors contains only points \( P_0, P_1, Q_0, Q_1 \), determining the vectors, and world functions between these points. The definition does not refer to a coordinate system and to the dimension of the proper Euclidean geometry \( \mathcal{G}_E \). It is a pure geometric definition, which does not contain a reference to the means of description. In the
proper Euclidean geometry the definition \((1.2)\) of equivalence coincides with the conventional definition of the scalar product on the ground of the linear vector space. The equivalence relation \((1.2)\) is transitive in the proper Euclidean geometry \(G_E\), and this transitivity is a special property of the proper Euclidean geometry.

In the arbitrary physical geometry \(G\) the definition of equivalence has the same form \((1.2)\) with the world function \(\sigma\), describing the geometry \(G\). However, in the general case the equivalence relation \((1.2)\) is not transitive, in general, because in the case of arbitrary world function \(\sigma\) the equivalence of two vectors is multivariant, in general. It means that at the point \(P_0\) may exist many vectors \(P_0P_1, P_0P'_1, P_0P''_1, \ldots\), which are equivalent to the vector \(Q_0Q_1\) at the point \(Q_0\), whereas the vectors \(P_0P_1, P_0P'_1, P_0P''_1, \ldots\) are not equivalent between themselves. In this case it is possible, that

\[
P_0P_1 \equiv Q_0Q_1 \land P_0P'_1 \equiv Q_0Q_1 \land P_0P''_1 \equiv Q_0Q_1
\]

\((1.4)\)

is true. Here the symbol \(\equiv\) means non-equivalency. If relations \((1.4)\) take place, the equivalence relation is intransitive, because for transitive equivalence relation it follows from

\[
P_0P_1 \equiv Q_0Q_1 \land P_0P'_1 \equiv Q_0Q_1
\]

\((1.5)\)

that

\[
P_0P_1 \equiv P_0P'_1
\]

\((1.6)\)

and the relation \((1.2)\) is false. On the other hand, the number of vectors \(P_0P_1, P_0P'_1, P_0P''_1, \ldots\), which are equivalent to the vector \(Q_0Q_1\) at the point \(Q_0\) depends on the number of solutions of two equations \((1.2)\), considered as equations for determination of the point \(P_1\) at fixed points \(P_0, Q_0, Q_1\) (or at fixed point \(P_0\) and vector \(Q_0Q_1\)). The number of these solutions depends on the form of the world function \(\sigma\). We shall differ three different cases:

(1) Single-variance with respect to points \(P_0, Q_0, Q_1\), when there is one and only one solution for \(P_1\) at given points \(P_0, Q_0, Q_1\). In this case the equivalence relation is transitive, if the single-variance takes place for any points \(P_0, Q_0, Q_1\).

(2) Multivariance with respect to points \(P_0, Q_0, Q_1\), when there is more, than one solution at some given points \(P_0, Q_0, Q_1\). In this case the equivalence relation is intransitive.

(3) zero-variance with respect to points \(P_0, Q_0, Q_1\), when there is no solution at some given points \(P_0, Q_0, Q_1\). In this case the equivalence relation may be intransitive and may be transitive.

The second case is strongest in the sense, that appearance of multivariance with respect to some three points \(P_0, Q_0, Q_1\) generates intransitivity of the equivalence relation, and, hence, non-axiomatizability of a physical geometry, because the transitivity of the equivalence relation is a necessary condition of axiomatizability. The second case and the third one are compatible in the sense that the multivariance may take place with respect to some points \(P_0, Q_0, Q_1\), whereas the zero-variance may take place with respect to other points \(P'_0, Q'_0, Q'_1\).
Note that the geometry of Minkowski may be axiomatizable and non-physical, and the geometry of Minkowski may be physical and not axiomatizable. In general, in this case one has two different geometries, having the same world function. We use for them different names. The geometry of Minkowski, which is a physical geometry, will be referred to as the $\sigma$-Minkowskian geometry. The $\sigma$-Minkowskian geometry is not axiomatizable, because it is multivariant with respect to any point $P_0$ and any spacelike vector $Q_0Q_1 = \{Q_0, Q_1\}$. The conventional geometry of Minkowski, which is constructed on the ground of the linear vector space with the scalar product, given on it, is axiomatizable (it is deduced from some axiomatics), but it is not a physical geometry. The geometry of Minkowski cannot be constructed on the basis of the world function only. Construction of the geometry of Minkowski contains a reference to the means of description in the form of the coordinate system. Although the construction of the geometry of Minkowski is invariant with respect to transformation of the coordinate system, it is not invariant with respect to transformation of the coordinate system dimension (see detailed discussion in [4]). The geometry of Minkowski should be qualified as a fortified physical geometry, i.e. a physical geometry with some additional structure, given on the physical geometry. Existence of the additional structure imposes some additional constraints on the geometry.

The difference between the space-time geometry of Minkowski and the $\sigma$-Minkowskian space-time geometry appears only at consideration of spacelike vectors, with respect to which the $\sigma$-Minkowskian geometry is multivariant. However, the spacelike vectors do not figure in the particle dynamics, and the difference between the $\sigma$-Minkowskian space-time geometry and the space-time geometry of Minkowski remains to be obscure. If one considers the geometry as a science on mutual disposition of geometrical objects, one should prefer the $\sigma$-Minkowskian geometry as a space-time geometry, because the distance between any pair of points determines mutual disposition of geometrical objects completely. As to axiomatizability of a geometry, this property is important only for deduction of the geometric propositions from the axiomatics. From viewpoint of the geometry as a science on disposition of geometrical object, the axiomatizability is a secondary property of the geometry, and practically all physical geometries are not axiomatizable. The proper Euclidean geometry is a very important exclusion, which admits one to construct physical geometries by means of a deformation of the proper Euclidean geometry.

Deduction of an axiomatizable geometry from axiomatics has two essential defects. Firstly, one needs to formulate geometric propositions and to prove corresponding theorems. The geometric propositions are to be formulated and proved for any new geometry. These procedures are complicated from the technical viewpoint. Besides, only geometries with the transitive equivalence relation can be deduced from axiomatics. Secondly, one needs to invent axioms, and to test their consistency. Inconsistency of a geometry means, that using two different ways of deduction of some statement, one obtains two incompatible statements. Inconsistency of a geometry is a property of the method of the geometry construction, but not a property of the geometry in itself. In the physical geometry, which is constructed on the ground of the deformation principle, the question of its inconsistency is meaningless, because
the problem of geometric propositions formulation is absent at all. All propositions of a physical geometry are taken from the proper Euclidean geometry in the ready-made form. Any logical reasonings and, in particular, proofs of theorems are absent in a physical geometry. Corresponding logical reasonings had been produced in the proper Euclidean geometry.

Finally, the method of a physical geometry construction, based on linear vector space, (for instance, construction of the Riemannian geometry) starts from some $n$-dimensional manifold $M_n$, where the metric tensor $g_{ik}$ is given. The world function $\sigma$ is given by the relation

$$\sigma(x, x') = \frac{1}{2} \left( \int_{L_{xx'}} \sqrt{g_{ik}(x) dx^i dx^k} \right)^2$$ (1.7)

where integration is produced along the geodesic $L_{xx'}$, connecting points $x$ and $x'$. There may be several geodesics, connecting points $x$ and $x'$. In this case the world function $\sigma$ appears to be multivalued. In this case the world function is a derivative quantity, and it may be multivalued. However, in a physical geometry, the world function is a primary quantity, it determines the physical geometry, and it cannot be multivalued.

To make the Riemannian geometry with multivalued world function (1.7) a physical geometry ($\sigma$-Riemannian geometry), one needs to turn the multivalued world function into single-valued one, choosing only one branch of the world function (1.7). Different choice of branches generates different world functions and, hence, different $\sigma$-Riemannian geometries. Thus, the $n$-dimensional manifold $M_n$ with the metric tensor, given on it generates several $\sigma$-Riemannian geometries, if the expression (1.7) appears to be multivalued for some pairs of points $x, x'$.

Construction of $\sigma$-Riemannian geometries by means of transformation of multivalued world function (1.7) into a single-valued world function is accompanied by appearance zero-variance for some points. Of course, this mechanism of construction of a physical geometry with the zero-variance is not unique. However, this mechanism is interesting from the physical viewpoint, because the $\sigma$-Riemannian geometry with the zero-variance is obtained as a result of the compactification of the flat space-time geometry (for instance, compactification of 5-dimensional space-time geometry of Kaluza-Klein [6]). The zero-variance generates some discrimination mechanism, responsible for discrete values of the elementary particle parameters. In particular, compactification of the fifth coordinate in the Kaluza-Klein geometry leads to restrictions on the possible electric charge of the elementary particle.

This paper is devoted to consideration of the procedure of compactification of the Kaluza-Klein geometry, which is accompanied by the construction of a discrimination mechanism, imposing restrictions on the value of the electric charge of the elementary particles. However, at first, we mention about influence of the multi-variance upon the dynamics of elementary particles.
### 2 Influence of the multivariance on the particle dynamics.

In the space-time geometry of Minkowski dynamics of a pointlike particle is described by a timelike world line $L$ of the particle. In the inertial coordinate system $x = \{x^0, x^1, x^2, x^3\}$ the world function $\sigma_M(x, x')$ between two points with coordinates $x$ and $x'$ has the form

$$
\sigma_M(x, x') = \frac{1}{2} g_{ik} (x^i - x'^i) (x^k - x'^k) \quad (2.1)
$$

where the metric tensor has the form $g_{ik} = \text{diag}\{c^2, -1, -1, -1\}$, and $c$ is the speed of the light. The world line $x = x(\tau)$ of a charged particle, moving in the given electromagnetic field $F_{ik}$, is described by the dynamic equation

$$
m \frac{d}{d\tau} \left( c g_{il} \frac{dx^l}{d\tau} \right) = \frac{e}{c} F_{ik}(x) \frac{dx^k}{d\tau}, \quad i = 0, 1, 2, 3 \quad (2.2)
$$

where $m$ is the particle mass, and $e$ is the electric charge of the particle. The constants $m$ and $e$ are non-geometrical characteristics of the pointlike particle.

In general, the mass $m$ and the charge $e$ can be geometrized, i.e. they may be considered as pure geometric characteristics of the pointlike particle. However, it is possible only in the framework of the physical geometry, which is formulated in terms of the world function. The motion of a pointlike particle is described by a world chain, consisting of connected vectors $P_s P_{s+1}$, $s = ...0, 1,...$ The lengths $|P_s P_{s+1}|$ of all vectors are equal

$$
|P_s P_{s+1}| = \mu, \quad s = ...0, 1, ...
$$

Here $\mu$ is the geometrical mass, which is connected with the usual mass $m$ by means of the relation

$$
m = b \mu \quad (2.4)
$$

where $b$ is some universal constant. The geometrical mass $\mu$ is a geometric characteristic of the particle.

The motion (2.2) of a pointlike particle in the electromagnetic field may be described as a free motion of the particle in the 5-dimensional space-time of Kaluza-Klein. The fact, that the motion of a pointlike particle in a physical space-time geometry is free, means that the adjacent vectors in the world chain are equivalent

$$
P_s P_{s+1} \text{equiv} P_{s+1} P_{s+2}, \quad s = ...0, 1,... \quad (2.5)
$$

Let the electromagnetic field is absent. Then dynamic equation (2.2) turn into the dynamic equation

$$
m \frac{d}{d\tau} \left( c g_{il} \frac{dx^l}{d\tau} \right) = 0 \quad (2.6)
$$
Its solution

\[ x^i = x^i(\tau) = X^i + U^i\tau, \quad X^i, U^i = \text{const}, \quad i = 0, 1, 2, 3 \tag{2.7} \]
does not depend on the mass \( m \) and coincides with the solution of equations (2.5), (1.2) in the space-time of Minkowski. However, if the space-time of Minkowski is slightly deformed, the solution may appear to depend on the mass.

Let us consider the space-time geometry \( G_d \), described by the world function

\[ \sigma_d = \sigma_M + d, \quad d = \lambda_0^2 \text{sgn} (\sigma_M), \quad \lambda_0 = \text{const} \tag{2.8} \]

\[ \text{sgn} (x) = \begin{cases} \frac{x}{|x|} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \tag{2.9} \]

where \( \sigma_M \) is the world function of the space-time geometry of Minkowski, and \( \lambda_0 \) is some elementary length of the geometry \( G_d \).

The length \( |\mathbf{P}_0\mathbf{P}_1|_d \) of any vector \( \mathbf{P}_0\mathbf{P}_1 \) has the form

\[ |\mathbf{P}_0\mathbf{P}_1|_d^2 = 2\sigma_d (P_0, P_1) = 2\sigma_M (P_0, P_1) + 2\lambda_0^2 \text{sgn} (\sigma_M (P_0,P_1)) \tag{2.10} \]

In other words, if the distance between points \( P_0, P_1 \) is timelike in \( G_d \), it is also timelike in \( G_M \). If the distance between points \( P_0, P_1 \) is spacelike in \( G_d \), it is also spacelike in \( G_M \). It follows from (2.10) that any timelike (and spacelike) distance is larger, than \( \sqrt{2}\lambda_0 \). It means the in the space-time geometry \( G_d \) there are no close points, and the geometry \( G_d \) should be qualified as a discrete space-time geometry. The geometry \( G_d \) is given on the continuous manifold of Minkowski. It looks rather unexpected, that the discrete geometry may be given on the same point set, on which a continuous geometry can be given. This surprise is explained by the fact, that at the conventional approach, based on the concept of the linear space, the discrete geometry is given on a countable point set, whereas the continuous geometry is given on a continual point set. In reality, the character (discreteness, or continuity) of geometry depends only on the form of the world function. Of course, the continual geometry may be given only on a continual point set. However, as we have seen, the discrete geometry may be given also on a continual point set.

As we have seen, the \( \sigma \)-Minkowskian geometry is multivariant with respect to any point and any spacelike vector. The space-time geometry \( G_d \) is multivariant with respect to timelike vectors also, and this circumstance appears to be important for dynamics of a pointlike particle, because the dynamics deals with timelike vectors. The free motion of a pointlike particle appears to depend on the geometric particle mass \( \mu \) and on the elementary length \( \lambda_0 \), which is responsible for multivariance of \( G_d \) with respect to timelike vectors.

Two adjacent links \( \mathbf{P}_0\mathbf{P}_1 \) and \( \mathbf{P}_1\mathbf{P}_2 \) are equivalent, and, hence, satisfy the relations of the type of (1.2). Let coordinates of the points are

\[ P_0 = \{0, 0, 0, 0\}, \quad P_1 = \{\mu, 0, 0, 0\}, \quad P_2 = \{2\mu + \alpha_0, \alpha_1, \alpha_2, \alpha_3\} \tag{2.11} \]
and coordinates of vectors $\mathbf{P}_0\mathbf{P}_1$, $\mathbf{P}_1\mathbf{P}_2$, $\mathbf{P}_0\mathbf{P}_2$ are

$$
\mathbf{P}_0\mathbf{P}_1 = \{\mu, 0, 0, 0\}, \quad \mathbf{P}_1\mathbf{P}_2 = \{\mu + \alpha_0, \alpha_1, \alpha_2, \alpha_3\}, \quad \mathbf{P}_0\mathbf{P}_2 = \{2\mu + \alpha_0, \alpha_1, \alpha_2, \alpha_3\}
$$

Let us take into account that

$$
|\mathbf{P}_0\mathbf{P}_1|_d^2 = |\mathbf{P}_0\mathbf{P}_1|_M^2 = 2\sigma_M (P_0, P_1) + 2\lambda_0^2 \text{sgn} (\sigma_M (P_0, P_1))
$$

(2.13)

$$(\mathbf{P}_0\mathbf{P}_1, \mathbf{P}_1\mathbf{P}_2)_d = (\mathbf{P}_0\mathbf{P}_1, \mathbf{P}_1\mathbf{P}_2)_M + w (P_0, P_1, P_2)
$$

(2.14)

Here indices "M" and "d" mean that the quantities are calculated in $G_M$ and $G_d$ respectively, and for timelike vectors (2.12)

$$
w (P_0, P_1, P_1, P_2) = d (P_0, P_2) + d (P_1, P_1) - d (P_0, P_1) - d (P_1, P_2) = -\lambda_0^2
$$

(2.15)

The relation $\mathbf{P}_0\mathbf{P}_1 \equiv \mathbf{P}_1\mathbf{P}_2$ has the form of two equations

$$
\mu (\mu + \alpha_0) - \lambda_0^2 = \mu^2 + 2\lambda_0^2
$$

(2.16)

$$
\mu^2 = (\mu + \alpha_0)^2 - \alpha_1^2 - \alpha_2^2 - \alpha_3^2
$$

(2.17)

The quantities $\alpha$ are to be determined from these equations. Solution of equations (2.16), (2.17) has the form

$$
\alpha_0 = \frac{3\lambda_0^2}{\mu}, \quad \alpha_1 = \lambda_0 \sqrt{6 + \frac{9\lambda_0^2}{\mu^2} \sin \theta \cos \varphi},
$$

(2.18)

$$
\alpha_2 = \lambda_0 \sqrt{6 + \frac{9\lambda_0^2}{\mu^2} \sin \theta \cos \varphi}, \quad \alpha_3 = \lambda_0 \sqrt{6 + \frac{9\lambda_0^2}{\mu^2} \cos \theta}
$$

(2.19)

where the quantities $\theta$ and $\varphi$ are arbitrary.

Thus, position of the link $\mathbf{P}_1\mathbf{P}_2$ with respect to the adjacent link $\mathbf{P}_0\mathbf{P}_1$ appears to be indefinite (multivariant). Possible positions of the link $\mathbf{P}_1\mathbf{P}_2$ form generatrices of the cone with the axis $\mathbf{P}_0\mathbf{P}_1$ and the angle $\phi$ at the vertex, which lies at the point $P_1$. The angle $\phi$ is determined by the relation

$$
\tan \phi = \frac{\sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}}{\mu + \alpha_0} = \frac{\lambda_0}{\mu \left(1 + \frac{3\lambda_0^2}{\mu^2}\right)} \sqrt{3 \left(2 + \frac{3\lambda_0^2}{\mu^2}\right)} \approx \frac{\lambda_0\sqrt{6}}{\mu}, \quad \text{if } \lambda_0 \ll \mu
$$

(2.20)

If the elementary length $\lambda_0 \to 0$, the space-time geometry $G_d$ turns into $G_M$, and the cone degenerates into a straight line.

Indefinite (multivariant) position of adjacent links leads to wobbling of the world chain of the pointlike particle. Let us choose elementary length of the space-time geometry $G_d$ in the form

$$
\lambda_0^2 = \frac{\hbar}{2bc}
$$
where $h$ is the quantum constant, $c$ is the speed of the light and the constant $b$ is the universal constant \([2, 4]\), connecting the geometrical mass $\mu$ with the usual mass $m$. Then the statistical description of wobbling world chains is equivalent to the quantum description in terms of the Schrödinger equation \([2]\). The quantum constant $h$ appears in the dynamics of the particle as a parameter of the space-time geometry $G_d$. The conventional quantum principles appear to be needless. Thus, the multivariant space-time geometry admits one to describe quantum effects as geometric effects. Besides, the pointlike particle mass $m$ appears to be geometrized by its connection \([2, 4]\) with the geometrical mass $\mu = |P_0P_1|_d$.

### 3 World function of Kaluza-Klein space-time

The space-time geometry of Kaluza-Klein $G_K$ is given on the 5-dimensional manifold. In the coordinate system with coordinates $x = \{x^0, x^1, x^2, x^3, x^5\}$. Four coordinates $\{x^0, x^1, x^2, x^3\} = \{x^0, \mathbf{x}\}$ describe position of a particle in the space-time of Minkowski, whereas the charge coordinate $x^5$ describes additional characteristic of the particle, which is responsible for interaction of with the electromagnetic field.

Covariant metric tensor $\gamma_{AB}, A, B = 0, 1, 2, 3, 5$ is determined by the relation

$$\gamma_{AB} = \begin{vmatrix} g_{ik} - a_ia_k & a_k & a_k \\ a_i & -1 & 0 \end{vmatrix}, \quad i, k = 0, 1, 2, 3, \quad A, B = 0, 1, 2, 3, 5 \quad (3.1)$$

where $g_{ik}$, $i, k = 0, 1, 2, 3$ is the metric tensor in the conventional 4-dimensional space-time. The quantities $a_k, k = 0, 1, 2, 3$ are connected with electromagnetic potential $A_k$, $k = 0, 1, 2, 3$ by means of the relation

$$a_k = \kappa A_k, \quad k = 0, 1, 2, 3 \quad (3.2)$$

where $\kappa$ is some universal constant. The contravariant metric tensor $\gamma^{AB}, A, B = 0, 1, 2, 3, 5$ has the form

$$\gamma^{AB} = \begin{vmatrix} g^{ik} & g^{ij}a_l \\ g^{kl}a_l & -1 + g^{jl}a_ja_l \end{vmatrix}, \quad i, k = 0, 1, 2, 3, \quad A, B = 0, 1, 2, 3, 5 \quad (3.3)$$

It is supposed that neither electromagnetic potentials $a_k$, nor the metric tensor $g_{ik}$ depend on the charge coordinate $x^5$.

Then the action

$$\mathcal{A}[x] = \int \left\{ -m_5c\sqrt{\gamma_{AB}\dot{x}^A\dot{x}^B} \right\} d\tau, \quad x = \{x^0(\tau), x^1(\tau), x^2(\tau), x^3(\tau), x^5(\tau)\} \quad (3.4)$$

describes the motion of a charged particle in the gravitational field, described by the metric tensor $g_{ik}$ and in the electromagnetic field $A_k$. Corresponding dynamic equations are obtained as a result of variation of the action \((3.4)\) with respect to $x^A, A = 0, 1, 2, 3, 5$.

$$\frac{dP_A}{d\tau} = -\frac{\partial}{\partial x^A} \left(m_5c\sqrt{\gamma_{AB}\dot{x}^A\dot{x}^B} \right), \quad A = 0, 1, 2, 3, 5 \quad (3.5)$$
where
\[ p_A = -\frac{m_5 c \gamma_{AB} x^B}{\sqrt{\gamma_{CD} x^C x^D}}, \quad A = 0, 1, 2, 3, 5 \] (3.6)

It follows from (3.6)
\[ p_A \gamma^{AB} p_B = (m_5 c)^2 \] (3.7)

As far as \( \gamma_{AB} \) does depend on \( x^5 \), it follows from (3.5), that the canonical momentum component \( p_5 = \text{const} \). It follows from (3.6) and (3.1), that the relation (3.7) may be written in the form
\[ (p_i + p_5 a_i) g^{ik} (p_k + p_5 a_k) = (m_5 c)^2 + p_5^2 \] (3.8)

One can obtain the Hamilton-Jacobi equation for the action (3.4), setting in (3.8) \( p_k = \partial S / \partial x^k, k = 0, 1, 2, 3 \) and \( p_5 = \text{const} \), where \( S \) is the action. One obtains
\[ \left( \frac{\partial S}{\partial x^i} + p_5 a_i \right) g^{ik} \left( \frac{\partial S}{\partial x^k} p_k + p_5 a_k \right) = (m_5 c)^2 + p_5^2 \] (3.9)

Comparing (3.9) with the Hamilton-Jacobi equation
\[ \left( \frac{\partial S}{\partial x^i} + \frac{e}{c} A_i \right) g^{ik} \left( \frac{\partial S}{\partial x^k} p_k + \frac{e}{c} A_k \right) = m^2 c^2 \] (3.10)

describing motion of a pointlike particle of mass \( m \) and of charge \( e \) in 4-dimensional space-time with electromagnetic potential \( A_k, k = 0, 1, 2, 3 \), one concludes that equations (3.9) and (3.10) are equivalent, if
\[ m = \sqrt{m_5^2 + c^{-2} p_5^2}, \quad p_5 = \frac{e}{\kappa c}, \quad a_k = \kappa A_k, \quad k = 0, 1, 2, 3 \] (3.11)

where \( \kappa \) is some universal constant.

The original action (3.4) has the form of the action for a geodesic in 5-dimensional Riemannian space with the metric tensor (3.1). Thus, the motion of a pointlike charged particle in the 4-dimensional Riemannian space-time with the electromagnetic field can be described as a free motion of a particle in the 5-dimensional Riemannian space-time. The electric charge \( e \) of the particle is geometrized in the sense, that it appears to be connected with the component \( p_5 \) of the particle momentum along the fifth (charge) coordinate \( x^5 \).

However, the fifth coordinate \( x^5 \) is unobservable, and one tries to explain this circumstance by the hypothesis, the space-time of Kaluza-Klein is thin in the direction of the fifth coordinate \( x^5 \). One supposes, that the space-time of Kaluza-Klein is compactified in the direction of fifth coordinate \( x^5 \), i.e. the points with coordinates \( \{x^0, x^1, x^2, x^3, x^5\} \) and \( \{x^0, x^1, x^2, x^3, x^5 + 2kL\} \) coincide, where \( L \) is some universal constant and \( k \) is any integer number.
4 Discrimination properties of compactification of Kaluza-Klein geometry

We shall try to analyze influence of compactification on the Kaluza-Klein geometry \( G_K \). For simplicity we shall consider the case, when the gravitational field and the electromagnetic one are absent. Then the metric tensor \((3.1)\) takes the form
\[
\gamma_{AB} = \text{diag}(c^2, -1, -1, -1, -1) \quad \text{and} \quad a_k = 0, \quad k = 0, 1, 2, 3.
\]
Geodesics \( \mathcal{L}_{P_0 P_1} \), passing through points \( P_0 \) and \( P_1 \) with coordinates
\[
P_0 = \{0, 0, 0, 0, 0\}, \quad P_1 = \{y^0, y^1, y^2, y^3, y^5\}, \quad y^0, y^1, y^2, y^3, y^5 \in \mathbb{R}
\]
have the form
\[
x^k = y^k \tau, \quad x^5 = (y^5 + 2nL) \tau, \quad k = 0, 1, 2, 3 \quad (4.2)
\]
where \( \tau \) is a parameter along the geodesic, and \( n \) is an arbitrary integer number.

The compactification may be considered as a conglutination of points with coordinates \( \{x^0, x^1, x^2, x^3, x^5 - L\} \) and \( \{x^0, x^1, x^2, x^3, x^5 + L\} \). As a result one obtains a "cylinder" instead of a plane.

Defining the world function \( \sigma_K (P_0, P_1) \) by means of \((1.7)\) as an integral along the geodesic, connecting points \( P_0 \) and \( P_1 \), one obtains a multivalued world function, because there are many geodesics of different length, connecting the points \( P_0 \) and \( P_1 \). If the space-time geometry is constructed according to conventional method on the basis of the linear vector field, the metric tensor is a primary quantity, whereas the world function is a secondary (derivative) quantity. In this case one may accept situation with multivalued world function, and one may try to interpret this fact in some way.

However, if the space-time geometry is a physical geometry, where the world function is the primary fundamental quantity, one cannot accept a multivalued primary quantity. One needs to use a single-valued world function and to choose only one of many possible variants of the geodesic \((4.2)\). One obtains different space-time geometries for different choice of the geodesic \((4.2)\), determining the world function.

We consider the simplest case, when the world function is defined as integral \((1.7)\) along the "shortest" geodesic, corresponding to the geodesic \((4.2)\). This geodesic makes less, than one convolution around the "cylinder". In this case the world function depends on the standartized value \( x^5_{\text{st}} \) of the coordinate \( x^5 \)
\[
\sigma_K (x, x') = \frac{1}{2} \left( (x^0 - x'^0)^2 - (x - x')^2 - \left( (x^5 - x'^5)_{\text{st}} \right)^2 \right) \quad (4.3)
\]
where \( x = \{x^1, x^2, x^3\} \)
\[
x_{\text{st}} = \begin{cases} 
2L \left\{ \frac{x}{2L} \right\} & \text{if} \quad 2L \left\{ \frac{x}{2L} \right\} \leq L \\
2L \left\{ \frac{x}{2L} \right\} - 2L & \text{if} \quad L < 2L \left\{ \frac{x}{2L} \right\} 
\end{cases}, \quad 2L \left\{ \frac{x}{2L} \right\} \in [0, 2L) \quad (4.4)
\]

Here \( \left\{ x \right\} \) means the fractional part of a decimal number \( x \), and \([x]\) is the integer part of \( x \). In other words, \([x]\) and \( \{x\}\) are defined by relations.
where $Z$ is the set of all integer numbers.

$$\{x\} = x - [x]$$

(4.6)

The choice of the world function $\sigma_K$ in the form (4.3) corresponds to the geodesic (4.2), which makes less, than one convolution around the ”cylinder”.

The world function (4.3) is zero-variant with respect to some vectors. Let us consider the points and corresponding vectors, which may form two adjacent vectors of a world chain.

$$P_0 = \{0, 0, 0, 0, 0\}, \quad P_1 = \{s_0, 0, 0, 0, l\}, \quad P_2 = \{2s_0 + \alpha_0, \alpha_1, \alpha_2, \alpha_3, 2l + \alpha_5\}$$

(4.8)

$$P_0P_1 = s = \{s_0, 0, 0, 0, l\}, \quad P_1P_2 = s + \alpha = \{s_0 + \alpha_0, \alpha_1, \alpha_2, \alpha_3, l + \alpha_5\}$$

(4.9)

$$P_0P_2 = \{2s_0 + \alpha_0, \alpha_1, \alpha_2, \alpha_3, 2l + \alpha_5\}$$

(4.10)

We shall show, that if the fifth coordinate $x^5 = l$ satisfies the relation

$$|l| > \frac{L}{2}$$

(4.11)

then the vector $P_1P_2$, which is equivalent to vector $P_0P_1$ does not exist. It means, that the world chain of a free pointlike particle with the link $P_0P_1$ cannot exist.

The equivalence conditions $P_0P_1 \equiv P_1P_2$ for vectors (4.9) is written in the form

$$|P_0P_1|_K = |P_1P_2|_K$$

(4.12)

$$\sigma_K (P_0P_1) = \sigma_K (P_1P_2)$$

(4.13)

where index ”$K$” means, that the corresponding quantities are taken in the geometry (4.3).

We suppose, that the vector $P_0P_1$ is timelike in the sense, that

$$s_0^2 > L^2$$

(4.14)

As far as

$$\sigma_K (P_0P_1) = \sigma_K (P_0, P_2) - \sigma_K (P_0, P_1) - \sigma_K (P_1, P_2)$$

(4.15)

the developed form of equations (4.12) and (4.13) has the form

$$s_0^2 - l^2 = (s_0 + \alpha_0)^2 - \alpha_1^2 - \alpha_2^2 - \alpha_3^2 - ((l + \alpha_5)_{st})^2$$

(4.16)
\[(2s_0 + \alpha_0)^2 - \alpha_1^2 - \alpha_2^2 - \alpha_3^2 - (2l + \alpha_5)_{st}^2 = 4\left(s_0^2 - l^2\right)\]  (4.17)

Eliminating \(\alpha_1^2 + \alpha_2^2 + \alpha_3^2\) from equations (4.17) and (4.16), one obtains

\[2s_0\alpha_0 - (2l + \alpha_5)_{st}^2 + ((l + \alpha_5)_{st})^2 + 3l^2 = 0\]  (4.18)

\[\alpha_0 = -\frac{(2l + \alpha_5)_{st}^2 + (l + \alpha_5)_{st}^2 + 3l^2}{2s_0}\]  (4.19)

Substituting (4.19) in (4.18), one obtains

\[\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = (2l + \alpha_5)_{st}^2 - 2(l + \alpha_5)_{st}^2 - 2l^2 + \left(-\frac{(2l + \alpha_5)_{st}^2 + (l + \alpha_5)_{st}^2 + 3l^2}{2s_0}\right)^2\]  (4.20)

Let us set

\[\beta = \beta_{st} = (l + \alpha_5)_{st}\]  (4.21)

Then

\[(2l + \alpha_5)_{st} = (l + \beta)_{st} = \begin{cases} 
    l + \beta & \text{if } -L < l + \beta \leq L \\
    -2L + l + \beta & \text{if } L < l + \beta \leq 2L \\
    2L + l + \beta & \text{if } -2L < l + \beta \leq -L 
\end{cases}\]  (4.22)

\[\gamma = \begin{cases} 
    0 & \text{if } -L < l + \beta \leq L \\
    -2L & \text{if } L < l + \beta \leq 2L \\
    2L & \text{if } -2L < l + \beta \leq -L 
\end{cases}\]  (4.23)

Note, that we are interested in the quantity \(\beta\), because it is the fifth coordinate of the vector \(P_1P_2\), which is determined to within \(2kL\), where \(k\) is an arbitrary integer number

\[P_1P_2 = \{s_0 + \alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta\} = \{s_0 + \alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta + 2nL\}\]  (4.25)

Substituting (4.21) and (4.23) in (4.20), one obtains after transformations

\[\alpha^2 = (l + \gamma)(l + 2\beta + \gamma) - 2l^2 + \frac{(l + \gamma)(l + 2\beta + \gamma) - 3l^2}{4s_0^2} - \beta^2\]  (4.26)

where

\[\alpha^2 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2\]  (4.27)

Let us consider the case, when

\[\gamma = 0\]  (4.28)

Substituting (4.28) in (4.26), one obtains

\[\alpha^2 = -(l - \beta_{st})^2\left(1 - \frac{l^2}{s_0^2}\right)\]  (4.29)
If vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{P}_1\mathbf{P}_2$ are timelike in the sense (4.14), rhs of (4.29) is nonpositive, and the unique solution of (4.29) is $\beta = l$. It follows from (4.23), (4.19) and (4.28), that

$$\alpha^2 = 0, \quad \beta = \beta_{\text{st}} = l, \quad \alpha_0 = 0, \quad -\frac{L}{2} < l \leq \frac{L}{2} \quad (4.30)$$

In other words, the vector $\mathbf{P}_1\mathbf{P}_2$ has coordinates

$$\mathbf{P}_1\mathbf{P}_2 = \{s_0, 0, 0, 0, l\} \quad (4.31)$$

which coincide with coordinates of vector $\mathbf{P}_0\mathbf{P}_1$, if

$$-\frac{L}{2} < l \leq \frac{L}{2} \quad (4.32)$$

Let us consider the case, when

$$\gamma = -2L, \quad L < l + \beta \leq 2L \quad (4.33)$$

In this case the equation (4.26) takes the form

$$\alpha^2 = -4L (l + \beta - L) - (l - \beta)^2 + \frac{((l - 2L)(l - 2L + 2\beta) - 3l^2)^2}{4s_0^2} \quad (4.34)$$

There is the unique solution of (4.34), which satisfies the condition (4.33). It has the form

$$\alpha = 0, \quad \beta = l = \frac{L}{2} \quad (4.35)$$

In the case, when

$$\gamma = 2L, \quad -2L < l + \beta \leq -L \quad (4.36)$$

the equation (4.26) takes the form

$$\alpha^2 = 4L (L + l + \beta) - (l - \beta)^2 + \frac{((l + 2L)(l + 2\beta + 2L) - 3l^2)^2}{4s_0^2} \quad (4.37)$$

The unique solution of equation (4.37), which satisfies (4.36), has the form

$$\alpha^2 = 0, \quad \beta = l, \quad l = -\frac{L}{2} \quad (4.38)$$

Thus, one obtains, that at the point $P_1$ there is only one vector $\mathbf{P}_1\mathbf{P}_2 = \{s_0, 0, 0, 0, l\}$, which is equivalent to the vector $\mathbf{P}_0\mathbf{P}_1 = \{s_0, 0, 0, 0, l\}$ at the point $P_0$. This equivalence takes place only, if $l$ satisfies the relation

$$|l| \leq \frac{L}{2} \quad (4.39)$$

If the relation (4.39) is not satisfied, at the point $P_1$ there is no vector $\mathbf{P}_1\mathbf{P}_2$, which is equivalent to the vector $\mathbf{P}_0\mathbf{P}_1 = \{s_0, 0, 0, 0, l\}$ at the point $P_0$. 

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If the Kaluza-Klein geometry is not compactified the vectors \( \mathbf{P}_0 \mathbf{P}_1 = \{s_0, 0, 0, 0, l\} \) and \( \mathbf{P}_1 \mathbf{P}_2 = \{s_0, 0, 0, 0, l\} \) are equivalent at any value of the charge \( l \) (at \( l^2 < s_0^2 \)). Thus, the compactification discriminates large values of the charge coordinate \( x^5 = l \). At the conventional approach to the Kaluza-Klein geometry, based on the linear vector space, the compactification does not discriminate any values of the charge coordinate \( x^5 \).

The charge coordinate \( x^5 = l \) is the charge component of the vector \( \mathbf{P}_0 \mathbf{P}_1 = \{s_0, 0, 0, 0, l\} \), which is a momentum vector. The discrimination of values of the quantity \( l \) is a discrimination of the charge component \( p_5 \) of the momentum vector \( \mathbf{P}_0 \mathbf{P}_1 \), i.e. it is a discrimination of the particle charge.

### 5 The case of distorted Kaluza-Klein space-time

Let us consider compactification of the distorted Kaluza-Klein space-time. The world function has the form

\[
\sigma_{dK}(x, x') = \sigma_K(x, x') + \lambda_0^2 \text{sgn} (\sigma_K(x, x'))
\]

(5.1)

where \( \sigma_K \) is determined by the relation (4.3), (4.4). We consider two timelike vectors \( \mathbf{P}_0 \mathbf{P}_1 \) and \( \mathbf{P}_1 \mathbf{P}_2 \) of the world chain. The vectors are determined by the relation (4.8) - (4.10). These vectors are supposed to be equivalent and to satisfy the relations of the type (4.12), (4.13). Now they have the form

\[
|\mathbf{P}_0 \mathbf{P}_1|_{dK}^2 = |\mathbf{P}_1 \mathbf{P}_2|_{dK}^2
\]

(5.2)

\[
(\mathbf{P}_0 \mathbf{P}_1. \mathbf{P}_1 \mathbf{P}_2)_{dK} = |\mathbf{P}_0 \mathbf{P}_1|_{dK}^2
\]

(5.3)

As far as vectors \( \mathbf{P}_0 \mathbf{P}_1 \) and \( \mathbf{P}_1 \mathbf{P}_2 \) are timelike, the equations may be rewritten in the developed form

\[
s_0^2 - l^2 = (s_0 + \alpha_0)^2 - \alpha_1^2 - \alpha_2^2 - \alpha_3^2 - (l + \alpha_5)^2_{st}
\]

(5.4)

\[
((2s_0 + \alpha_0)^2 - \alpha_1^2 - \alpha_2^2 - \alpha_3^2 - (2l + \alpha_5)^2_{st}) + 2\lambda_0^2 = 4 \left( (s_0^2 - l^2) + 2\lambda_0^2 \right)
\]

(5.5)

Eliminating \( \alpha_1^2 + \alpha_2^2 + \alpha_3^2 \) from (5.5) and (5.4), one obtains

\[
2s_0\alpha_0 - (2l + \alpha_5)_{st}^2 + (l + \alpha_5)_{st}^2 - 6\lambda_0^2 + 3l^2 = 0
\]

(5.6)

or

\[
\alpha_0 = \frac{(2l + \alpha_5)_{st}^2 - (l + \alpha_5)_{st}^2 - 3l^2 + 6\lambda_0^2}{2s_0}
\]

(5.7)

Substituting (5.7) in (5.4), one obtains

\[
\alpha^2 = (2l + \alpha_5)_{st}^2 - 2(l + \alpha_5)_{st}^2 - 2l^2 + 6\lambda_0^2 + \left( \frac{(2l + \alpha_5)_{st}^2 - (l + \alpha_5)_{st}^2 - 3l^2 + 6\lambda_0^2}{2s_0} \right)^2
\]

(5.8)
where \( \alpha \) is determined by the relation (4.27).

For simplicity we shall consider the case, when

\[ s_0^2 \gg l^2, L^2 \]  

(5.9)

Let us use designations (4.21) - (4.24). Substituting (4.21) and (4.23) in (5.8) and taking into account (5.9), one obtains after transformations

\[ \alpha^2 = (l + \gamma)(l + 2\beta + \gamma) - \beta^2 - 2l^2 + 6\lambda_0^2 \]  

(5.10)

Let us consider the case, when \( \gamma = 0 \) and according to (4.21), (4.24)

\[-L < \beta \leq L \]  

(5.11)

Then the equation (5.10) is transformed to the form

\[ \alpha^2 + (\beta - l)^2 = 6\lambda_0^2 \]  

(5.12)

The relation (5.12) describes a sphere of the radius

\[ r_1 = \sqrt{6}\lambda_0 \]  

(5.13)

with the center \( \{\alpha, \beta_c\} = \{\alpha, l\} \) in the 4-dimensional space of coordinates \( \{\alpha, \beta\} = \{\alpha_1, \alpha_2, \alpha_3, \beta\} \).

Solution of equations (5.13), (5.7) has the form

\[ \begin{align*}
\alpha_1 &= r_1 \cos \theta_1, \\
\alpha_2 &= r_1 \sin \theta_1 \cos \theta_2, \\
\alpha_3 &= r_1 \sin \theta_1 \sin \theta_2 \cos \theta_3, \\
\beta &= l + r_1 \sin \theta_1 \sin \theta_2 \sin \theta_3, \\
\alpha_0 &= 0
\end{align*} \]  

(5.14)

(5.15)

where \( r_1 \) is defined by relation (5.13), and \( \theta_1, \theta_2, \theta_3 \) are arbitrary numbers. Although the solution (5.14), (5.15) is multivalued, but it is placed at the distance of the order of \( \lambda_0 \) from the single-valued solution (4.30). At \( \lambda_0 = 0 \) the radius \( r_1 \) of the sphere (5.12) vanishes and the solution (5.14), (5.15) coincides with (4.30), if one takes into account (5.9).

The sphere (5.12), describing the set of solutions, has points, which lie inside the region (5.11), if the following conditions are fulfilled

\[ (-L - l < \beta_c + r_1) \land (\beta_c - r_1 \leq L - l) \]  

(5.16)

where \( \beta_c = l \) is the coordinate \( \beta \) of the center of sphere. It follows from (5.16) and (5.13) that

\[ l > -\frac{L + r_1}{2}, \quad l \leq \frac{L + r_1}{2} \]  

(5.17)

All points of the sphere (5.12) lie inside the region (5.11), if

\[ (-L - l < \beta_c - r_1) \land (\beta_c + r_1 \leq L - l) \]  

(5.18)
It means that
\[ l > -\frac{L - r_1}{2}, \quad l \leq \frac{L - r_1}{2} \] (5.19)

In the case, when the condition (5.17) is fulfilled, but the condition (5.19) is not fulfilled, only those points \( \{\alpha, \beta\} \) of the sphere (5.12) describe solutions, which satisfy the condition
\[ \frac{L - r_1}{2} \leq l \leq \frac{L + r_1}{2}, \quad -\frac{L + r_1}{2} < l < -\frac{L - r_1}{2} \] (5.20)
and parameters \( \theta_1, \theta_2, \theta_3 \), determining solution (5.14), (5.15) satisfy the condition
\[ -\frac{L}{2} - l < \frac{r_1}{2} \sin \theta_1 \sin \theta_2 \sin \theta_3 \leq \frac{L}{2} - l \] (5.21)

This relation is a corollary of (5.11) and (5.15).

For instance, if
\[ l = \frac{L}{2} + \frac{r_1}{2} \delta, \quad -1 < \delta \leq 1 \] (5.22)
then it follows from (5.11)
\[ \beta < \frac{L}{2} - \frac{r_1}{2} \delta \] (5.23)

It follows from the relation (5.15), that
\[ \beta \in \left(\frac{L}{2} - \left(1 - \frac{\delta}{2}\right) r_1, \frac{L}{2} + \left(1 + \frac{\delta}{2}\right) r_1\right) \] (5.24)

In particular,
\[ \beta \in \begin{cases} \left(\frac{L}{2} - r_1, \frac{L}{2} - \frac{r_1}{2}\right) & \text{if } l = \frac{L}{2} + \frac{r_1}{2}, \delta = 1 \\ \left(\frac{L}{2} - r_1, \frac{L}{2} + \frac{r_1}{2}\right) & \text{if } l = \frac{L}{2}, \delta = 0 \\ \left(\frac{L}{2} - \frac{3}{2} r_1, \frac{L}{2} + \frac{r_1}{2}\right) & \text{if } l = \frac{L}{2} - \frac{r_1}{2}, \delta = -1 \end{cases} \] (5.25)

One can see from (5.25), that the value \( x^5 = \beta \) of the charge coordinate \( x^5 \) of vector \( P_1 P_2 \) is always less, than \( \frac{L}{2} + \frac{r_1}{2} \). Besides, the value \( x^5 = \beta \) of the charge coordinate \( x^5 \) of vector \( P_1 P_2 \) is less, than \( L/2 \), if the charge coordinate \( x^5 = l \) of vector \( P_0 P_1 \) is larger, than \( L/2 \). In other words, the charge coordinate \( x^5 \) "reflects itself" from the value \( x^5 = L/2 \).

Let us consider the case, when
\[ \gamma = -2L, \quad L < l + \beta \leq 2L \] (5.26)

In this case the equation (5.10) takes the form
\[ \alpha^2 + (\beta - l + 2L)^2 = r_2^2, \quad r_2 = \sqrt{8L^2 - 8lL + 6\lambda_0^2} \] (5.27)
or
\[ l \leq L + \frac{1}{8L} r_1^2 \] (5.28)
Besides, it is necessary, that the following restrictions were fulfilled

\[(L - l < \beta_c + r_2) \land (\beta_c - r_2 \leq 2L - l)\], \quad \beta_c = l - 2L \tag{5.29}\]

By means of (5.27) the first relation (5.29) is reduced to the form

\[L^2 - 4LL + 4l^2 - 6\lambda_0^2 < 0\tag{5.30}\]

This inequality is fulfilled, if

\[l \in \left(\frac{1}{2}L - \frac{1}{2}\sqrt{6\lambda_0}, \frac{1}{2}L + \frac{1}{2}\sqrt{6\lambda_0}\right) = \left(\frac{L - r_1}{2}, \frac{L + r_1}{2}\right)\tag{5.31}\]

where \(r_1\) is determined by the relation (5.13).

The second inequality (5.29) may be reduced to the form

\[-8L^2 + 8Ll - 4l^2 + 6\lambda_0^2 \leq 0\tag{5.32}\]

The inequality is satisfied identically, if

\[\frac{3}{2}\lambda_0^2 = \frac{r_1^2}{4} < L^2\tag{5.33}\]

In this case the relations (5.28) and (5.31) are the only constraints imposed on \(l\).

If the inequality (5.33) is violated, the relation (5.32) is fulfilled in the case, when

\[l \notin \left(L - \sqrt{\frac{r_1^2}{4} - L^2}, L + \sqrt{\frac{r_1^2}{4} - L^2}\right)\tag{5.34}\]

Let us consider the case (5.33), when the inequality (5.32) is satisfied identically. In this case the charge coordinate \(x^5 = \beta\) of the vector \(P_1P_2\) satisfies to inequalities (5.26) and (5.30). We write them in the form

\[L - l < \beta \leq \beta_c + r_2, \quad \beta_c - r_2 \leq \beta \leq \beta_c + r_2\tag{5.35}\]

\[L - l < \beta \leq 2L - l\tag{5.36}\]

As far as the relation (5.31) coincides with the first relation (5.20), we represent \(l\) in the form (5.22). Let \(r_1^2 \ll L^2\). By means of (5.30), and the second relation (5.29) the inequalities (5.35), (5.36) are reduced to the form

\[\frac{1}{2}L - \frac{1}{2}\delta r_1 < \beta \leq \frac{L}{2} - \frac{\delta r_1}{2} + \frac{1}{4L} (1 - \delta^2) r_1^2 + o\left(\frac{r_1^2}{L^2}\right), \quad \delta^2 \leq 1\tag{5.37}\]

It means that the range of possible values \(\beta\) is very narrow (of the order of \((1 - \delta^2) \frac{r_1^2}{4L}\)) At \(\lambda_0 \to 0\) the value of \(\beta\) tends to \(L/2\).

Thus, in the case (5.11) the possible values of \(\beta\) are located on the 4-dimensional sphere surface of the radius \(r_1 = \sqrt{6\lambda_0}\). They occupy the whole sphere surface, if

\[|l| < \frac{L - r_1}{2}\tag{5.38}\]
and they occupy only part of the sphere surface, if

\[ \frac{L - r_1}{2} < |l| < \frac{L + r_1}{2} \]  
(5.39)

In both cases the area of the region on the sphere surface, where the solutions are placed, is not greater than the whole area of the sphere surface, which is equal to \(2\pi^2 r_1^3\).

In the case (5.26) the possible values of \(\beta\) are located on the surface of a 4-dimensional sphere of the radius \(r_2 \approx 2L\). They occupy the surface of a thin truncated spherical segment, which has the height

\[ h = \frac{1}{4L} (1 - \delta^2) r_1^2 \]  
(5.40)

where \(\delta\) is determined by relations (5.22):

\[ l = \frac{L}{2} + \frac{r_1}{2} \delta, \quad -1 < \delta \leq 1 \]  
(5.41)

The area \(S_{\text{seg}}\) of the spherical part of the segment surface is equal to

\[ S_{\text{seg}} = \pi r_2^3 \Delta \phi_2 \]  
(5.42)

where the angle \(\Delta \phi_2\) is defined by the relation

\[ \Delta \phi_2 = \frac{h}{r_2} \approx \frac{r_1^2}{8L^2} (1 - \delta^2) \]  
(5.43)

It follows from (5.42) and (5.43), that the area of the segment surface

\[ S_{\text{seg}} = \pi L r_1^2 (1 - \delta^2) \gg 2\pi^2 r_1^3, \]  
(5.44)

if \(\delta^2\) is not too close to 1 and \(r_1 \ll r_2 \approx 2L\). Besides, if \(l = \frac{L}{2} + \frac{1}{2} r_1 \delta\), then

\[ \beta \in \left( \frac{1}{2} L - \frac{1}{2} r_1 \delta, \frac{L}{2} - \frac{r_1}{2} \delta + \frac{1}{4L} (1 - \delta^2) r_1^2 \right), \quad |\delta| \leq 1 \]  
(5.45)

The difference between \(l\) and \(\beta\)

\[ l - \beta \in \left( r_1 \delta, r_1 \delta - \frac{1}{4L} (1 - \delta^2) r_1^2 \right) \]  
(5.46)

The point \(x^5 = L/2\) is placed between the values \(l\) and \(\beta\). If \(\delta \to 0\), the difference \(|l - \beta| \to r_1^2/4L \ll r_1\).
6 World chain in the distorted Kaluza-Klein spacetime

Let us consider the world chain, consisting of equivalent connected vectors \( \mathbf{P}_0 \mathbf{P}_1, \mathbf{P}_1 \mathbf{P}_2, \ldots, \mathbf{P}_k \mathbf{P}_{k+1}, \ldots \) in the distorted Kaluza-Klein geometry (5.1). Vector \( \mathbf{P}_k \mathbf{P}_{k+1} \) has the form

\[
\mathbf{P}_k \mathbf{P}_{k+1} = \{ s_0, \alpha_k, l_k \}
\]

(6.1)

We are not interested in values of coordinates \( s_0 \gg L \), and \( \alpha_k \). These coordinates may be transformed to the case \( \alpha_k = 0 \) by means of the Lorentz transformation in the space of coordinates \( \{ s_0, \alpha_k \} \). We are interested only in evolution of the charge coordinate \( x^5 = l_k \).

We consider the case, when \( r_1 = \sqrt{6 \lambda_0} \ll L \). As we have seen in the previous section, if \( |l_k| < L/2 - r_1/2 \), the next value \( l_{k+1} \) of the charge coordinate appears to be multivariate. It belongs to the interval

\[
l_{k+1} \in \{ l_k - r_1, l_k + r_1 \}
\]

(6.2)

The 4-vector \( \{ \alpha_{k+1}, l_{k+1} \} \) lies on surface of the 4-dimensional sphere \( S_k \) of radius \( r_1 \) with the center at the point \( \{0, l_k\} \). Exact value of \( l_{k+1} \) is not determined, and we are forced to use a statistical description of the quantity \( l_{k+1} \).

It is reasonable to suppose that all solutions are equiprobable. It means, that the probability of the fact, that the quantity \( l_{k+1} \) appears on some domain of the surface of the sphere \( S_k \), is proportional to the area of this domain. Then the point \( \{ \alpha_{k+1}, l_{k+1} \} \) walks in the 4-dimensional space of coordinates \( \{ x, x^5 \} \), making each next step in the random direction. The length of any step is equal to \( r_1 \). These solutions we shall qualify as standard solutions. Such a random walk continues until the value of charge coordinate \( x^5 = l \notin (\frac{L-r_1}{2}, \frac{L+r_1}{2}) \), or \( l \notin (\frac{-L-r_1}{2}, \frac{-L+r_1}{2}) \). If \( l_k \) appears inside the interval \( (\frac{L-r_1}{2}, \frac{L+r_1}{2}) \), the additional solutions appear. The point \( \{ \alpha_{k+1}, l_{k+1} \} \), describing these additional solutions, lies on the surface of 4-dimensional sphere of radius \( r_2 = \sqrt{8L^2 - 8kL + 6\lambda_0^2} \approx 2L \). If \( l_k = L/2 + r_1 \delta/2 \), \( |\delta| < 1 \), the area of the corresponding domain on the surface of the 4-dimensional sphere is given by the relation (5.44). It is much more, than the area of the whole surface of the sphere \( S_k \), corresponding to standard solutions. It means that the probability of additional solutions is much more, than the probability of standard solutions.

Additional solutions lie very close to the boundary \( x^5 = L/2 \). According to (5.41), (5.45), if \( l_k = L/2 + r_1 \delta/2 \),

\[
l_{k+1} \in \left( \frac{1}{2} L - \frac{1}{2} r_1 \delta, \frac{1}{2} L - \frac{r_1}{2} \delta + \frac{1}{4L} \left( 1 - \delta^2 \right) r_1^2 \right), \quad |\delta| \leq 1
\]

(6.3)

If \( l_k \notin \left( \frac{L}{2} - \frac{r_1}{2}, \frac{L}{2} + \frac{r_1}{2} \right) \), the additional solutions are absent. In other words, for additional solutions the boundary \( x^5 = L/2 \) is placed between \( l_k \) and \( l_{k+1} \). Each next additional solution is on the other side of the boundary \( x^5 = L/2 \). The sequence
where \( q \) with respect to reflection of the charge coordinate \( x \) to the value \( x^5 = L/2 \). It is possible also, that the value of the charge coordinate \( x^5 = l_k \) of the vector \( \mathbf{P}_k \mathbf{P}_{k+1} \) tends to the value \( x^5 = -L/2 \). Situation is symmetric with respect to reflection of the charge coordinate \( x^5 \), and boundaries \( x^5 = L/2 \) and \( x^5 = -L/2 \) have equal status.

The random walk of charge coordinate \( l_k \) was modelled. The segment \((0, \frac{L}{2} + r_1)\) was divided into six equal cells. Let \( q = \{q_1, q_2, q_3, q_4, q_5, q_6\} \) be the mean number of points in cells. The first five cells lie in interval \((0, (L - r_1)/2)\) and are associated with standard solutions, whereas the sixth cell corresponds to the interval \(((L - r_1)/2, L/2 + r_1)\), and it is associated with additional solutions. The particles walk randomly from a cell to adjacent cells. The mean number \( q^{(k+1)} \) of particles at the step \( k + 1 \) is described by the equation

\[
q^{(k+1)} = T q^{(k)}
\]

where \( q^{(k)} \) and \( q^{(k+1)} \) are columns and \( T \) is the matrix, describing the transition probabilities

\[
q^{(k)} = \begin{pmatrix}
q_1^{(k)} \\
q_2^{(k)} \\
q_3^{(k)} \\
q_4^{(k)} \\
q_5^{(k)} \\
q_6^{(k)}
\end{pmatrix}, \quad T = \begin{pmatrix}
\frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\
0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\
0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\
0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3}
\end{pmatrix}
\]

\[
(6.4)
\]

\[
(6.5)
\]

The transition probability from the internal cell to adjacent cell is equal to \( 1/3 \). The probability of the transition absence is also \( 1/3 \). The transition probabilities from the boundary cells are different. At the end, corresponding to the point \( x^5 = 0 \), the transition probability to the adjacent internal cell is equal to \( 2/3 \) (reflection from the end), whereas the probability of the transition absence is equal to \( 1/3 \). In the end cell, corresponding to the point \( x^5 = L/2 \), the probability of the transition absence is equal to \( n/(n + 1) = 6/7 \), and the probability of transition is equal to \( 1/(n + 1) = 1/7 \), where \( n \) is the number of cells. The large probability of the transition absence is conditioned by the fact, that all additional solutions are placed at the end cell, and the ratio \( p_b/p \) is proportional to \( r_2/r_1 \), where \( p_b \) is the probability of the transition absence in the boundary cell, and \( p \) is the probability of transition from the boundary cell. The quantity \( r_2/r_1 \) proportional to \( L/r_1 \). As far as the length of a cell is equal approximately to \( r_1 = L/n \), then quantity \( p_b/p \) is proportional to \( n = L/r_1 \).

Starting from uniform distribution of particles over cells, one obtains after nu-
Numerical calculations by means of the formulae (6.4), (6.5)

\[
q^{(0)} = \begin{pmatrix}
1/6 \\
1/6 \\
1/6 \\
1/6 \\
1/6 \\
1/6
\end{pmatrix}, \quad q^{(33)} = \begin{pmatrix}
0.07620 \\
0.15171 \\
0.14980 \\
0.14710 \\
0.14423 \\
0.33096
\end{pmatrix}
\]

(6.6)

\[
q^{(60)} = \begin{pmatrix}
0.07353 \\
0.14698 \\
0.14675 \\
0.14643 \\
0.14609 \\
0.34021
\end{pmatrix}, \quad q^{(78)} = \begin{pmatrix}
0.07326 \\
0.14649 \\
0.14644 \\
0.14636 \\
0.14628 \\
0.34116
\end{pmatrix}, \quad q^{(79)} = \begin{pmatrix}
0.07325 \\
0.14648 \\
0.14643 \\
0.14636 \\
0.14628 \\
0.34118
\end{pmatrix}
\]

(6.7)

One see from relations (6.6), (6.7), that the mean number of particles in the end cell, corresponding to the boundary point \(x^5 = L/2\) is larger, than in other cells, because of the large probability of the transition absence.

If the number \(n\) of cells increase, the probability of transition from the boundary cell, which is equal to \(1/(n+1)\), decreases, and the mean number of particles in the boundary cell increases. For instance, let the probability of the transition absence be equal to 12/13, that corresponds to the number of cells \(n = 12\). The corresponding probability of transition from the boundary cell is equal to 1/13. Considering the random walk with the number of cells \(n = 6\) and the probability of the transition absence equal to 12/13, one obtains

\[
q^{(66)} = \begin{pmatrix}
0.05772 \\
0.11525 \\
0.11469 \\
0.11387 \\
0.11294 \\
0.48555
\end{pmatrix}
\]

(6.8)

Comparison of (6.7) and (6.8) shows, that in the established state of the random walk process the mean numbers of particles in internal cells are practically equal. Let \(p\) and \(p_b\) be the probability of the transition from the boundary cell and the probability of transition absence respectively. Using the fact, that in the established state the mean number of of particles in internal cells is the same, one can write

\[
p_b q_b = p q, \quad q_b + n q = 1 \quad (6.9)
\]

where \(q_b\) and \(q\) are the probabilities of finding the particle in the boundary cell and the probability of finding the particle in internal cell respectively. The quantity \(n\) is the number of internal cells. Solving equations (6.9) with respect to \(q\) and \(q_b\), one obtains

\[
q_b = \frac{p}{p_b n + p}, \quad q = \frac{p_b}{p + np_b}
\]

(6.10)
Let us consider interval \((0, \frac{L}{2} + r_1)\) of the axis \(x^5\). Let the boundary cell be the interval \((\frac{L}{2} - r_1, \frac{L}{2} + r_1)\). The internal cells belong to the interval \((0, \frac{L}{2} - r_1)\) of the axis \(x^5\). The internal cells are of equal length. The length of the internal cell is chosen in such a way, that three probabilities: (1) of transition of \(l_k\) into the left adjacent cell, (2) of transition of \(l_k\) into the right adjacent cell and (3) of the transition absence, were equal between themselves. These conditions are realized, if the length \(l_c\) of the cell is equal \(l_c = \kappa r_1\), \(\kappa \approx \frac{1}{3}\). Then the number \(n\) of internal cells has the form

\[
n = \frac{(L/2 - r_1)}{2\kappa r_1} \approx \frac{3L}{4r_1} \gg 1
\]

(6.11)

It means that the probability of transition from the internal cell is equal to \(p = 1/3\).

Let us consider the case, when \(l_k \in (\frac{L}{2} - r_1, \frac{L}{2} + r_1)\), i.e. \(l_k\) is inside the boundary cell. In this case all additional solutions \(l_{k+1} \in \left(\frac{L-r_1}{2}, \frac{L+r_1}{2}\right)\). Only standard solutions \(l_{k+1}\) may appear outside the boundary cell. According to (5.22), (5.24)

\[
l_{k+1} \in \left(\frac{L}{2} - \left(1 - \frac{\delta}{2}\right)r_1, \frac{L}{2} + \left(1 + \frac{\delta}{2}\right)r_1\right)
\]

(6.12)

if

\[
l_k = \frac{L}{2} + \frac{r_1}{2}\delta, \quad -1 < \delta \leq 1
\]

(6.13)

In other words, if

\[
l_k = \frac{L}{2} - \frac{r_1}{2}\varepsilon, \quad 1 > \varepsilon \geq 0
\]

(6.14)

\[
l_{k+1} \in \left(\frac{L}{2} - r_1 - \frac{\varepsilon}{2}r_1, \frac{L}{2} + r_1 - \frac{\varepsilon}{2}r_1\right)
\]

(6.15)

The area of the region on the sphere surface, corresponding to standard solutions, is not more, than \(2\pi^2 r_1^3\), whereas according to (5.44) the area on the sphere surface, corresponding to additional solutions is equal to \(\pi L r_1^2 \left(1 - \delta^2\right)\). If \(\delta^2\) is not too close to 1, this area is much more, than \(2\pi^2 r_1^3\). Taking the mean value of \(\delta^2 = 1/2\), one obtains that the probability \(p_b\) of transition from the boundary cell may be evaluated as follows

\[
p_b \leq \frac{2\pi^2 r_1^3}{\pi L r_1^2 \left(1 - 1/2\right)} = \frac{4\pi r_1}{L} \ll 1
\]

(6.16)

Using the relation (6.10), one obtains for the probability \(q_b\) of the fact, that \(l_{k+1} \in (\frac{L}{2} - r_1, \frac{L}{2} + r_1)\) the following estimation

\[
q_b = \frac{1/3}{\frac{4\pi r_1}{L} + 1/3} = 0.03416, \quad q = \frac{p_b}{p + np_b} = \frac{1.287\, 8}{L r_1}
\]

(6.17)

Thus, at the established state the probability of finding the charge coordinate in the boundary cell is much larger, than the probability of finding it in any internal cell. However, the probability of finding the charge coordinate in all internal cells is equal to \(q = 1 - 0.03416 = 0.965\, 84\). It is larger, than the probability 0.03416 of finding the charge coordinate in the boundary cell. At first sight, it is rather strange.
Apparently, it is connected with this fact, that at small $r_1$ the transition to established state need large time (many steps). For instance, in the case $r_1 = \sqrt{6}\lambda_0 = 0$ the established state cannot be approached. We have seen, that in the geometry (4.3) any walk of the charge coordinate $x^5 = l_k$ is absent completely, and distribution of $l_k$ at the initial moment does not change. The established state in the sense of relation (6.9) cannot be approached.

7 Concluding remarks

Thus, a compactification of the Kaluza-Klein space-time leads to discrimination of some values $x^5$ of the charge momentum component (charge). The discrimination is a corollary of the fact, that not all links of the world chain of a particle are possible in the compactificated geometry. In the conventional approach to the Kaluza-Klein geometry, when the space-time geometry is constructed as a Riemannian geometry (but not as a physical geometry), the discrimination of the momentum component $p_5$ values is absent. The constraints on the value of $p_5$ are imposed artificially as postulates.

One should expect, that discrimination of values of $p_5$ may be conditioned not only by the space-time geometry. The component $p_5$ of the momentum is the electric charge of the particle. Fluctuations of the charge $p_5$ generate fluctuations of the electric current, connected with the motion of the particle. Fluctuations of the current generate electromagnetic radiation, which influence on the particle motion via the Kaluza-Klein space-time geometry. This influence continues, until the particle charge takes the state with minimal fluctuation. The state with minimal fluctuations is approached, if the coordinate $l_k$ of the world chain vector $P_k P_{k+1}$ takes the values $x^5 = l_k = \pm L/2$. At these states the probability of transition to another value of $l_k$ is minimal.

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