TOWARDS TOPOLOGICAL HOCHSCHILD HOMOLOGY OF JOHNSON-WILSON SPECTRA

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Abstract. We offer a complete description of THH(E(2)) under the assumption that the Johnson-Wilson spectrum E(2) at a chosen odd prime carries an $E_\infty$-structure. We also place THH(E(2)) in a cofiber sequence $E(2) \to THH(E(2)) \to THH(E(2))$ and describe $THH(E(2))$ under the assumption that $E(2)$ is an $E_n$-ring spectrum. We state general results about the $K(i)$-local behaviour of $THH(E(n))$ for all $n$ and $0 \leq i \leq n$. In particular, we compute $K(i)_*THH(E(n))$.

1. Introduction

The first Johnson-Wilson spectrum $E(1)$ at a prime $p$ is the Adams summand of $p$-local periodic complex topological $K$-theory $KU(p)$. It is known that it carries a unique $E_\infty$-structure [MS93,BR05], thus $THH(E(1))$ is a commutative $E(1)$-algebra spectrum. McClure and Staffeldt show that the unit map $E(1) \to THH(E(1))$ is a $K(1)$-local equivalence, hence its cofiber $THH(E(1))$ is a rational spectrum. It is easy to calculate the rational homology of $THH(E(1))$ as

$$H_{\mathbb{Q}}THH(E(1)) \cong \mathbb{Q}[v_1^{\pm 1}] \otimes \mathbb{Q} \Lambda_{\mathbb{Q}}(dv_1)$$

using the B"okstedt spectral sequence with $E_2$-term

$$E_2^{p,q} = HH_{\mathbb{Q}}^{p,q}(\mathbb{Q}[v_1^{\pm 1}]).$$

There is a map

$$\Sigma^{2p-1}E(1) \to THH(E(1)) \to THH(E(1))$$

that factors through $\Sigma^{2p-1}E(1)_\mathbb{Q} \to THH(E(1))$ since $THH(E(1))$ is rational, and that is defined such that the latter map is an equivalence detecting the $H_{\mathbb{Q}}^{p,q}(E(1))$-summand generated by $dv_1$. Since the unit map $E(1) \to THH(E(1))$ splits, this yields a splitting [MS93, Theorem 8.1]

$$THH(E(1)) \simeq E(1) \vee \Sigma^{2p-1}E(1)_\mathbb{Q}$$

as $E(1)$-modules. This computation was also carried out for $KU(p)$ [Aus05], and pushed further to provide formulas for $THH(KU)$ as a commutative $KU$-algebra by Stonek [Sto].

In this paper, we consider the higher Johnson-Wilson spectrum $E(n)$ with coefficient ring

$$E(n) = \mathbb{Z}(p)[v_1, \ldots, v_{n-1}, v_n, v_n^{-1}]$$

for an arbitrary value of $n \geq 1$ and $p$ an odd prime. A main motivation here is to investigate whether the spectrum $THH(E(n))$ also splits into copies of $E(n)$ and its lower chromatic localizations, generalizing McClure and Staffeldt’s intriguing transchromatic result.

As a first step, we compute the Hochschild homology $HH_{\ast}(K(i), E(n))$ of $K(i)_*E(n)$, where $K(i)$ is the $i$th Morava $K$-theory, for $0 \leq i \leq n$, at an odd prime, see Theorem 3.4. We shy away from the prime 2 because Morava $K$-theory is not homotopy commutative at the prime 2. Theorem 3.4 yields a computation of $K(i)_*THH(E(n))$ under the modest assumption that $E(n)$ admits an $E_3$-structure.

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We then focus on $E(2)$, and show in Theorem 5.4 that under the same commutativity assumption $\THH(E(2))$ sits in a cofiber sequence

$$E(2) \to \THH(E(2)) \to \Sigma^{2p-1}L_1E(2) \vee \Sigma^{2p^2-1}E(2)_\mathbb{Q} \vee \Sigma^{2p^2+2p-2}E(2)_\mathbb{Q},$$

where $L_1E(2)$ denotes the Bousfield localization of $E(2)$ with respect to $E(1)$. If the unit $E(2) \to \THH(E(2))$ splits, we then get a decomposition of $\THH(E(2))$ into four summands, a higher analogue of McClure-Staffeldt’s formula for $\THH(E(1))$.

**Remark 1.1.** To study $\THH(E(n))$ by means of the Bökstedt spectral sequence, we need sufficient commutativity of $E(n)$. In this remark, we summarize what is known about multiplicative structures on $E(n)$ and related spectra. Basterra and Mandell showed [BM13] that the Brown-Peterson spectrum $BP$ admits an $E_4$ structure. The Johnson-Wilson spectra $E(n)$ are built out of the $BP(n) = BP/(v_i | i \geq n + 1)$ by inverting $v_n$. In [Law18, Theorem 1.1.2] Tyler Lawson shows that the Brown-Peterson spectrum $BP$ and the spectra $BP(n)$ for $n \geq 4$ at the prime 2 do not possess an $E_{12}$-structure. Andrew Senger [Sen, Theorem 1.2] extends Lawson’s result to odd primes $p$, and shows that $BP$ and the $BP(n)$’s (for $n \geq 4$) do not have an $E_{2(p^2+2)}$-structure.

At the prime 2, Lawson and Naumann [LN12] show that there is an $E_\infty$-model of $BP(2)$ and Hill and Lawson [HL10] prove that $BP(2)$ at the prime 3 possesses a model as an $E_\infty$-ring spectrum. With [MNN15, Theorem A1] this yields $E_\infty$-structures on the corresponding Johnson-Wilson spectra $E(2)$ at these primes.

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## 2. Rationalized $E(n)$

For $n \geq 1$ the homotopy algebra of $L_E\langle 0 \rangle E(n) = E(n)_\mathbb{Q}$ is $\mathbb{Q}[v_1, \ldots, v_{n-1}, v_n^{1,1}]$ and its algebra of cooperations is

$$\pi_*(E(n)_\mathbb{Q} \wedge E(n)_\mathbb{Q}) \cong \pi_*/E(n)_\mathbb{Q} \otimes_{\mathbb{Q}} \pi_*E(n)_\mathbb{Q} \cong \mathbb{Q}[v_1, \ldots, v_{n-1}, v_n^{1,1}, v_1', \ldots, v_{n-1}', v_n'^{1,1}].$$

This implies the following result.

**Lemma 2.1.** There is a unique $E_\infty$-ring structure on $E(n)_\mathbb{Q}$ for all $n \geq 1$.

**Proof.** The obstruction groups for such an $E_\infty$-ring structure on $E(n)_\mathbb{Q}$ are contained in the Gamma cohomology groups of $\pi_*/E(n)_\mathbb{Q} \wedge E(n)_\mathbb{Q}$ as a $\pi_*E(n)_\mathbb{Q}$-algebra [Rob03, Theorem 5.6]. As we work in characteristic zero, Gamma cohomology agrees with André-Quillen cohomology [RW02, Corollary 6.6]. The algebra $\mathbb{Q}[v_1, \ldots, v_{n-1}, v_n^{1,1}, v_1', \ldots, v_{n-1}', v_n'^{1,1}]$ is smooth over $\mathbb{Q}[v_1, \ldots, v_{n-1}, v_n^{1,1}]$ and therefore André-Quillen cohomology is concentrated in cohomological degree zero where it consists of derivations. The obstructions for existence and uniqueness of an $E_\infty$-ring structure on $E(n)_\mathbb{Q}$ are concentrated in degrees bigger than zero. \qed
As $E_\infty$-ring structures can be rigidified to commutative ring structures (see e.g., [EKMM97, II.3]), we pass to the world of commutative ring spectra from now on.

Topological Hochschild homology of a ring spectrum $A$ can be modelled as the geometric realization of a simplicial spectrum. Using the inclusion of the 1-skeleton, McCrory and Staffeldt [MS93, §3] construct a map

$$\sigma : \Sigma A \to \text{THH}(A).$$

(2.1)

For a commutative ring spectrum $A$ the multiplication maps from $A^{\wedge n+1}$ to $A$ give rise to a map of commutative $A$-algebra spectra from $\text{THH}(A)$ to $A$. Composing this map with the map $A \to \text{THH}(A)$ gives the identity, hence we obtain a splitting of $A$-modules

$$\text{THH}(A) \simeq A \vee \overline{\text{THH}}(A)$$

where $\overline{\text{THH}}(A)$ is the cofiber. The latter spectrum inherits the structure of a non-unital commutative $A$-algebra. In our case this implies the following result.

**Corollary 2.2.** The topological Hochschild homology of $E(n)_Q$ splits, as an $E(n)_Q$-module, as

$$\text{THH}(E(n)_Q) \simeq E(n)_Q \vee \overline{\text{THH}}(E(n)_Q)$$

where $\overline{\text{THH}}(E(n)_Q)$ is the cofiber of the unit map $E(n)_Q \to \text{THH}(E(n)_Q) \simeq \text{THH}(E(n))_Q$. Moreover, the spectrum $\overline{\text{THH}}(E(n)_Q)$ is a non-unital commutative $E(n)_Q$-algebra.

In the sequel, we follow Loday [Lod98, Definition E.1] for the definition of étale algebras. It is straightforward to calculate the topological Hochschild homology of $E(n)_Q$.

**Proposition 2.3.**

$$\pi_* \text{THH}(E(n))_Q \cong Q[v_1, \ldots, v_{n-1}, v_n^\pm 1] \otimes \Lambda_Q(dv_1, \ldots, dv_n)$$

(2.2)

with $|dv_i| = 2p^i - 1$.

**Proof.** The Bökstedt spectral sequence for $\pi_*(\text{THH}(E(n))_Q) \cong HQ_* \text{THH}(E(n))$ is of the form

$$E^2_{s,t} = HH^Q_{s,t}(\pi_* E(n)_Q) \Rightarrow \pi_*(\text{THH}(E(n))_Q).$$

As $Q[v_1, \ldots, v_{n-1}, v_n^\pm 1]$ is étale over $Q[v_1, \ldots, v_{n-1}, v_n]$ and as $Q[v_1, \ldots, v_{n-1}, v_n]$ is smooth, we get

$$HH^Q_{s,t}(\pi_* E(n)_Q) \cong Q[v_1, \ldots, v_{n-1}, v_n^\pm 1] \otimes \Lambda_Q(dv_1, \ldots, dv_n)$$

with $dv_i$ having homological degree one and internal degree $2p^i - 2$. As the Bökstedt spectral sequence is multiplicative and as the algebra generator cannot support any differentials for degree reasons, the spectral sequence collapses at $E^2$. There are no multiplicative extensions and hence we get the result.

**Remark 2.4.** As we work rationally, $\text{THH}(E(n))_Q$ is a commutative $HQ$-algebra spectrum and hence corresponds to a commutative differential graded $Q$-algebra (see [Shi07] or [RS17]).

3. $K(i)_* E(n)$ and $K(i)_* \text{THH}(E(n))$

In the following we assume that $p$ is an odd prime, and that $n$ and $i$ are integers with $1 \leq i \leq n$.

The Hopf algebroid $(BP_*, BP_* BP)$ represents the groupoid of strict isomorphisms of $p$-typical formal group laws [Lan75] (see also [Rav86, Theorem A2.1.27]). There are isomorphisms of graded $Z(p)$-algebras

$$BP_* \cong \mathbb{Z}(p)[v_1, v_2, \ldots] \quad \text{and} \quad BP_* BP \cong BP_* [t_1, t_2, \ldots],$$

where $|v_i| = |t_i| = 2(p^i - 1)$. By convention $v_0 = p$ and $t_0 = 1$. The $i$th Morava $K$-theory $K(i)$ is complex oriented, and its formal group law $F_i$ (the Honda formal group law) corresponds to the map $BP_* \to K(i)_* = \mathbb{F}_p[v_1^\pm]$ sending $v_i$ to $v_1$ and $v_k$ for $k \neq i$ to zero. The $p$-typical formal
group law $G_n$ over $E(n)_*$ comes from the map $BP_* \to E(n)_*$ that kills all $v_i$ with $i > n$ and inverts $v_n$. Since $E(n)$ is a Landweber exact homology theory, we obtain an isomorphism

$$K(i)_*E(n)_* \cong K(i)_* \otimes_{BP_*} BP_* \otimes_{BP_*} E(n)_*.$$  \hfill (3.1)

Note that $K(i)_*E(n)_*$ is trivial for $i > n$ and that the Bousfield class of $E(n)_*$, $(E(n)_*)$, is $(K(0) \vee \ldots \vee K(n)_*)$.

We first treat the case $i = n$.

**Proposition 3.1.** For all $n \geq 1$ the canonical map $E(n)_* \to \text{THH}(E(n)_*)$ is a $K(n)_*$-local equivalence.

**Proof.** The algebra $K(n)_*E(n)$ is known as $\Sigma(n)$ and it is of the form

$$K(n)_*[t_1, t_2, \ldots]/(v_n t_i^n - v_n^p t_i, i \geq 1),$$

see [Rav86, 6.1.16]. If we set

$$C_s^{(k)} := K(n)_*[t_1, \ldots, t_k]/(v_n t_i^n - v_n^p t_i, 1 \leq i \leq k)$$

then $C_s^{(k)}$ is étale over $K(n)_*$ and $K(n)_*E(n)$ is the directed colimit of the $C_s^{(k)}$'s.

The $K(n)_*$-Bökstedt spectral sequence for $\text{THH}(E(n)_*)$ has as an $E^2$-term

$$\text{HH}^{K(n)_*}_*(K(n)_*E(n)) \cong K(n)_*E(n)$$

concentrated in homological degree zero. Thus $K(n)_*\text{THH}(E(n)_*) \cong K(n)_*E(n)$ and the isomorphism is induced by the map $E(n)_* \to \text{THH}(E(n)_*)$. Therefore, this map is a $K(n)_*$-equivalence and thus $K(n)_*$-locally $\text{THH}(E(n)_*)$ is equivalent to $E(n)_*$. \hfill $\square$

We calculate $K(i)_*E(n)_*$ for $1 \leq i \leq n - 1$ using the following description of morphisms of graded commutative $BP_*$-algebras from $K(i)_*E(n)$ to some graded commutative ring $B_*$. For $n = 2$ we had an argument that was rather involved and Paul Goerss suggested the following simpler proof.

We consider the map $g: BP_*BP \to K(i)_*E(n)_*$ of graded commutative $\mathbb{Z}_p$-algebras given by

$$BP_*BP \to K(i)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} E(n)_* \cong K(i)_*E(n)_*$$

which uses the canonical maps $BP_* \to K(i)_*$ and $BP_* \to E(n)_*$ and the isomorphism from (3.1). By [Rav86, Theorem A2.1.27] this map corresponds to a triple $((\eta_L)_*, F_i, (\eta_R)_*G_n, f)$ where $\eta_L: K(i)_* \to K(i)_*E(n)_*$ is the left unit, $\eta_R: E(n)_* \to K(i)_*E(n)_*$ is the right unit and $(\eta_L)_*F_i$ and $(\eta_R)_*G_n$ are the $p$-typical formal group laws that are given by the corresponding change of coefficients. Here, $f$ is a strict isomorphism between the $p$-typical formal group laws $(\eta_L)_*F_i$ and $(\eta_R)_*G_n$ over $K(i)_*E(n)_*$. By [Rav86, Lemma A2.1.26] such a strict isomorphism is always of the form

$$f(x) = \sum_j (\eta_R)_*G_n t_j x_{p^j}.$$  

The $p$-series of the Honda formal group law $F_i$ is

$$[p]_{F_i}(x) = v_i x_{p^i}$$

and the same is true for $[p]_{(\eta_L)_*F_i}(x)$ because the left unit just embeds $K(i)_*$ into $K(i)_*E(n)_*$. The $p$-series of $(\eta_R)_*G_n$ is

$$[p]_{(\eta_R)_*G_n}(x) = w_1 x_p + (\eta_R)_*G_n \ldots + (\eta_R)_*G_n w_n x_{p^n}$$

for $w_i = \eta_R(v_i)$.

First, we state an elementary lemma about powers of $p$.

**Lemma 3.2.** Let $m \geq 2$, let $r, \ell_1, \ldots, \ell_m$ be natural numbers bigger or equal to 1, and assume that $\ell_j \neq \ell_k$ for $j \neq k$. Then $p^r$ cannot be written as a sum $p^{\ell_1} + \ldots + p^{\ell_m}$. 

Proof. Assume

\[ p^x = p^{x_1} + \ldots + p^{x_m}. \]

Without loss of generality let \( \ell_1 \) be minimal among the \( \ell_j \)'s. Then

\[ p^x = p^{\ell_1}(1 + p^{x_2-\ell_1} + \ldots + p^{x_m-\ell_1}). \]

This is only possible if all the \( \ell_j - \ell_1 \) are equal to zero and if \( m' = m - \ell_1 \). But \( \ell_j - \ell_1 = 0 \) for all \( 2 \leq j \leq m' \) implies that all the \( \ell_j \)'s are equal to \( \ell_1 \) and this contradicts our assumption. \( \square \)

**Proposition 3.3.** For all \( 1 \leq i \leq n \) \( K(i)_*E(n) \) is a colimit of étale \( K(i)_*[w_{i+1}, \ldots, w_n] \)-algebras.

Proof. In the following we fix \( i \) and \( n \). We denote by \( B(i,n)_* \) the graded commutative \( K(i)_-* \)-algebra \( K(i)_*[w_{i+1}, \ldots, w_n] \). For a given \( m \geq 1 \) consider the graded commutative \( BP_* \)-subalgebra \( BP_*[t_1, \ldots, t_m] \) of \( BP_*BP \) and define

\[ B_m = \text{Image}(B(i,n)_*[t_1, \ldots, t_m] \to K(i)_*E(n)). \]

Thus we can express \( B_m \) as \( B(i,n)_*[t_1, \ldots, t_m] / \sim \) where \( \sim \) denotes the quotient that arises from the relations that the \( t_r \)'s and \( w_j \)'s satisfy in \( K(i)_*E(n) \). Note that \( B_{m+1} \) is free as a \( B_m \)-module for all \( m \geq 1 \). Indeed, in each step we adjoin a new polynomial generator \( x \) to a graded commutative ring \( R_1 \) that satisfies relations of the form \( x^p - ux - y \) with a unit \( u \in R_1^\times \) and \( y \in R_1 \).

The strict isomorphism \( f(x) = \sum_j (\eta_R)_*G_{n} t_j x^p \) satisfies

\[ \left[ p \right](\eta_R)_*G_{n}(f(x)) = f([\left[ p \right]|_{\eta_R}, F_{n}(x)) \]

and this yields the equality

\[ w_1(f(x))^p + (\eta_R)_*G_{n} \ldots + (\eta_R)_*G_{n} w_n(f(x))^p = f(v_i x^p) = \sum_j (\eta_R)_*G_{n} t_j (v_i x^p)^p. \]

(3.2)

On the right hand side in \( \sum_j (\eta_R)_*G_{n} t_j x^p x_{p+i} \) the relations for the \( t_r \) are detected by the powers \( x^{p+r} \). Lemma 3.2 ensures that for a given \( x^{p+r} \) we only have to consider the coefficient \( t_j x_{p+i}^p \) with \( i + j = i + r \) coming from the linear term of the \( (\eta_R)_*G_{n} \)-sum \( \sum_j (\eta_R)_*G_{n} t_j x^{p+i} \) and this is \( t_r v_i^p \).

As the right hand side starts with \( x^p \), it is a direct consequence that \( w_1, \ldots, w_{i-1} = 0 \) and from the coefficients of \( x^p \) we obtain that \( w_i = v_i \) in \( K(i)_*E(n) \).

We prove that \( B_1 \) is étale over \( B(i,n)_* \) and that for every \( m, B_m \) is étale over \( B_{m-1} \). It follows that the algebras \( B_m \) are étale over \( B(i,n)_* \).

Thus we have to show that the modules of relative Kähler differentials \( \Omega^1_{B_1/B(i,n)_*} \) and \( \Omega^1_{B_m/B_{m-1}} \) are trivial for all \( m \geq 2 \).

For \( m = 1 \) we compare the coefficients of \( x_{p+i}^{p+1} \) in (3.2). In this case only the linear terms of the \( (\eta_R)_*G_{n} \)-sums contribute something and we obtain

\[ v_i t_{i+1}^p \]

and therefore \( t_1 = v_i^{-p}(v_i t_{i+1}^p + w_{i+1}) \). This gives a flat extension and the Kähler differential on \( t_1 \) is equal to

\[ dt_1 = 0 + v_i^{-p} dw_{i+1} \]

and hence \( B_1 \) is étale over \( B(i,n)_* \).

Consider \( B_m \). Then the first relation for \( t_m \) is given by the relation of the coefficients for \( x_{p+i}^{p+1} \).
We know that the formal group law $G_n(x, y)$ is of the form

$$G_n(x, y) = x + y + \sum_{i,j \geq 1} a_{i,j} x^i y^j$$

where the $a_{i,j} \in E(n)_* = \mathbb{Z}_{(p)}[v_1, \ldots, v_{n-1}, v_n^{\pm 1}]$. Equation (3.2) relates power series with coefficients in $K(i)_*E(n)$, hence the coefficients $\partial_{i,j}$ of $(\eta_R)_* G_n$ are now considered in $K(i)_*E(n)$ and are elements of $\mathbb{F}_p[w_i, \ldots, w_{n-1}, w_n^{\pm 1}]$. On the left hand side of (3.2) we get coefficients that involve some polynomials of $\partial_{i,j}$'s, some $p$th powers of $t_j$'s and some expressions in $w_k$'s. For $m + i \leq n$ we actually get a coefficient $w_{m+i}^{p^{m+i+1}} = w_{i+m}$.

The $\partial_{i,j}$'s are in $B(i, n)_*$, so they don't contribute anything to the relative Kähler differentials. The Kähler differentials on the $t_j^{p^k}$ are trivial because we are over $\mathbb{F}_p$. Hence we can express the Kähler differential $dt_m$ up to a factor of $v_i^{p^m} = w_i^{p^m}$ via Kähler differentials in the $w_k$'s. As $v_i^{p^m}$ is invertible in $B(i, n)_*$, the relative Kähler differentials $\Omega^1_{B_m/B_{m-1}}$ are trivial for all $m \geq 1$. □

**Theorem 3.4.** For all $1 \leq i \leq n$ we have an isomorphism of $K(i)_*E(n)$-algebras

$$\text{HH}_s^{K(i)}(K(i)_*E(n)) \cong K(i)_*E(n) \otimes_{\mathbb{F}_p} \Lambda_{\mathbb{F}_p}(dw_{i+1}, \ldots, dw_n).$$

**Proof.** We have shown that $K(i)_*E(n)$ is the sequential colimit of the $B_m$'s. As the $K(i)_*$-algebras $B_m$ are étale over $B(i, n)_*$ and as Hochschild homology commutes with localization we can rewrite $\text{HH}_s(B_m)$ as

$$\text{HH}_s(B_m) \cong B_m \otimes_{B(i, n)_*} \text{HH}_s^{K(i)}(B(i,n)_*)$$

$$\cong B_m \otimes_{B(i, n)_*} (B(i,n)_* \otimes_{\mathbb{F}_p} \Lambda_{\mathbb{F}_p}(dw_{i+1}, \ldots, dw_n))$$

$$\cong B_m \otimes_{\mathbb{F}_p} \Lambda_{\mathbb{F}_p}(dw_{i+1}, \ldots, dw_n)$$

using [WG91] and the Hochschild-Kostant-Rosenberg theorem. Hochschild homology commutes with colimits, hence we obtain

$$\text{HH}_s^{K(i)}(K(i)_*E(n)) \cong \text{colim}_m \text{HH}_s^{K(i)}(B_m) \cong K(i)_*E(n) \otimes_{\mathbb{F}_p} \Lambda_{\mathbb{F}_p}(dw_{i+1}, \ldots, dw_n).$$

□

**Theorem 3.5.** Assume that $p$ is an odd prime and that $E(n)$ is an $E_3$-ring spectrum. Then, for all $1 \leq i \leq n$, we have an isomorphism of $K(i)_*E(n)$-algebras

$$K(i)_* \text{THH}(E(n)) \cong K(i)_*E(n) \otimes_{\mathbb{F}_p} \Lambda_{\mathbb{F}_p}(dw_{i+1}, \ldots, dw_n).$$

**Proof.** We use the Bökstedt spectral sequence [Bög], [EKMM97, IX.2.9], with $E^2$-term

$$E^2_{r,s} = (\text{HH}_r^{K(i)}(K(i)_*E(n)))_{s},$$

where $r$ denotes the homological and $s$ the internal degree. By a result of Angeltveit and Rognes [AR05, Prop. 4.3], an $E_3$-structure on $E(n)$ implies that this spectral is one of commutative $K(i)_*E(n)$-algebras. The multiplicative generators $dw_j$ for $i \leq j \leq n$ sit in bidegree $(1, 2p^j - 2)$ and hence they cannot carry any non-trivial differentials. Therefore the spectral sequence collapses at the $E^2$-term. As the abutment is a free graded commutative $K(i)_*E(n)$-algebra, there cannot be any multiplicative extensions. □

**Remark 3.6.** Note if $E(n)$ admits an $E_2$ structure, the Bökstedt spectral sequence is one of $K(i)_*$-algebras by [AR05, Prop. 4.3]. It therefore collapses since all $K(i)_*$-algebra generators lie in columns 0 and 1. This gives the same formula for $K(i)_* \text{THH}(E(n))$ as a $K(i)_*$-module, but not as a $K(i)_*$-algebra, since there is now room for $K(i)_*$-algebra extensions.
4. Blue-shift for THH($E(n)$)

If we assume that $p$ is an odd prime and that $E(n)$ is an $E_\infty$-ring spectrum, then THH($E(n)$) is a commutative $E(n)$-algebra spectrum and the cofiber of the unit map

$$\text{THH}(E(n)) = \text{cofiber}(E(n) \to \text{THH}(E(n)))$$

is a non-unital commutative $E(n)$-algebra spectrum. If $E(n)$ carries an $E_3$-structure, then by [BFV07, §3.3], [BM11] the morphism $E(n) \to \text{THH}(E(n))$ is an $E_2$-map. This implies the following useful fact:

**Lemma 4.1.** If $E(n)$ is an $E_3$-spectrum, then THH($E(n)$) is an $E(n)$-module spectrum and in particular, THH($E(n)$) is $E(n)$-local.

Let $L_n$ denote the localization at $E(n)$, and in particular $L_0$ is the rationalization. Recall that there is a well-known chromatic fracture square

$$
\begin{array}{ccc}
L_nX & \longrightarrow & L_{K(n)}X \\
\downarrow & & \downarrow \\
L_{n-1}X & \longrightarrow & L_{n-1}L_{K(n)}X.
\end{array}
$$

It is shown for instance in [ACB, Example 3.3] and [Bau14, Proposition 2.2] that the homotopy pullback of

$$
\begin{array}{ccc}
L_{K(n)}X & & \\
\downarrow & & \\
L_{n-1}X & \longrightarrow & L_{n-1}L_{K(n)}X.
\end{array}
$$

is an $E(n)$-localization of $X$. The statement in [Bau14, Proposition 2.2] is more general and [ACB] work out far more general local-to-global statements.

We always know from Proposition 3.1 that the unit map is a $K(n)$-local equivalence. The chromatic square for $\overline{\text{THH}}(E(n))$ is:

$$
\begin{array}{ccc}
\overline{\text{THH}}(E(n)) = L_{K(n)} \overline{\text{THH}}(E(n)) & \longrightarrow & L_{K(n)} \overline{\text{THH}}(E(n)) \\
\downarrow & & \downarrow \\
L_{E(n-1)} \overline{\text{THH}}(E(n)) & \longrightarrow & L_{E(n-1)}(L_{K(n)} \overline{\text{THH}}(E(n))).
\end{array}
$$

The $K(n)$-homology of $\overline{\text{THH}}(E(n))$ is zero by Proposition 3.1. It follows that the localization $L_{K(n)} \overline{\text{THH}}(E(n))$ is trivial, and hence $L_{E(n-1)}(L_{K(n)} \overline{\text{THH}}(E(n)))$ is also trivial. Therefore the vertical map on the left hand side is an equivalence and we obtain a nice example of blue-shift:

**Lemma 4.2.** If $E(n)$ is an $E_3$-spectrum, then the cofiber $\overline{\text{THH}}(E(n))$ is $E(n-1)$-local.

5. Topological Hochschild homology of $E(2)$

In this section, we discuss in more detail the topological Hochschild homology of $E(2)$, which we will denote by $E = E(2)$ to simplify the notation. As explained in the proof of Lemma 5.1, the computations of Theorem 3.5 for $E(2)$ can be expressed as follows:

$$
\begin{align*}
K(0)_* \text{THH}(E) & \cong K(0)_* E \otimes \Lambda_\mathbb{Q}(dt_1, dt_2), \\
K(1)_* \text{THH}(E) & \cong K(1)_* E \otimes \Lambda_{\mathbb{F}_p}(dt_1), \\
K(2)_* \text{THH}(E) & \cong K(2)_* E.
\end{align*}
$$

Notice that these computations do not require the assumption that $E$ is an $E_3$-ring spectrum: for the rational case we have a commutative structure anyhow, while in the $K(1)$ and $K(2)$
cases, the $E^2$ page of the Bökstedt spectral sequences is concentrated on columns 0 and 1 (respectively 0).

**Lemma 5.1.** For $i = 1, 2$, there exist classes $\lambda_i \in \text{THH}_{2p^i-1}(E)$ with the following properties. Under the Hurewicz homomorphism

(a) the class $\lambda_i$ maps to $dt_i \in K(0)_{2p^i-1} \text{THH}(E)$, for $i = 1, 2$;
(b) the class $\lambda_i$ maps to $dt_1 \in K(1)_{2p^i-1} \text{THH}(E)$.

**Proof.** We use McClure-Staffeldt's computation of $\text{THH}_*(BP)$ in [MS93, Remark 4.3], which has been validated by the proof [BM13] that $BP$ admits an $E_4$ structure. We briefly recall the computation. The integral, rational and mod $p$ homology of $BP$ are given as

$$Hz_*BP \cong \mathbb{Z}_{(p)}[t_i \mid i \geq 1], \quad K(0)_*BP \cong \mathbb{Q}[t_i \mid i \geq 1] \quad \text{and} \quad HF_p_*BP \cong \mathbb{Z}[\xi_i \mid i \geq 1],$$

where the class $t_i \in Hz_{2p^i-1}BP$ maps to $\xi_i$ under mod $(p)$ reduction [Rav86, Proof of Theorem 5.2.8] and to the class with same name $t_i$ under rationalization. The associated Bökstedt spectral sequences collapse, providing isomorphisms

$$Hz_*\text{THH}(BP) \cong Hz_*BP \otimes \Lambda_{\mathbb{Z}_{(p)}}(dt_i \mid i \geq 1),$$

$$K(0)_*\text{THH}(BP) \cong K(0)_*BP \otimes \Lambda_{\mathbb{Q}}(dt_i \mid i \geq 1) \quad \text{and} \quad HF_p_*\text{THH}(BP) \cong HF_p_*BP \otimes \Lambda_{\mathbb{F}_p}(d\xi_i \mid i \geq 1),$$

with $dx = \sigma_*(x)$, where $\sigma: \Sigma BP \to \text{THH}(BP)$ is the map given in (2.1). There is an isomorphism

$$\text{THH}_*(BP) \cong BP_* \otimes \Lambda_{\mathbb{Z}_{(p)}}(\lambda_i \mid i \geq 1),$$

and the Hurewicz homomorphism

$$\text{THH}_*(BP) \to Hz_*\text{THH}(BP)$$

is an inclusion mapping $\lambda_i$ to $dt_i$. In particular, the classes $dt_i$ (integral and rational) and $d\xi_i$ are spherical: they are the image of $\lambda_i$ under the Hurewicz homomorphism mapping from $\text{THH}_*(BP)$. For $i \geq 1$, let us define

$$\lambda_i \in \text{THH}_{2p^i-1}(E)$$

as the image of the class with same name under the natural map

$$\text{THH}_*(BP) \to \text{THH}_*(E).$$

In the rational case, we have

$$\eta_R(v_i) \equiv \alpha_i t_i$$

modulo decomposables in $K(0)_*BP$, where $\alpha_i \in \mathbb{Q}$ is a unit. We deduce that

$$K(0)_*E \cong \mathbb{Q}[t_1, t_2][\eta_R(v_2)^{-1}]$$

and the Bökstedt spectral sequence recovers

$$K(0)_*\text{THH}(E) \cong K(0)_*E \otimes \Lambda_{\mathbb{Q}}(dt_1, dt_2).$$

By naturality, comparing with the case of $BP$, we deduce that the Hurewicz homomorphism $\text{THH}_*(E) \to K(0)_*\text{THH}(E)$ maps $\lambda_i$ to $dt_i$.

For $K(1)_*$-homology, we argue similarly, using the commutative square

$$\begin{array}{ccc}
\text{THH}_*(BP) & \longrightarrow & K(1)_*\text{THH}(BP) \\
\downarrow & & \downarrow \\
\text{THH}_*(E) & \longrightarrow & K(1)_*\text{THH}(E).
\end{array}$$

We have $K(1)_*BP \cong K(1)_*[t_i \mid i \geq 1]$, and the Bökstedt spectral sequence yields

$$K(1)_*\text{THH}(BP) \cong K(1)_*BP \otimes \Lambda_{\mathbb{F}_p}(dt_1 \mid i \geq 1).$$
Comparing the Bökstedt spectral sequences for $HZ_* \text{THH}(BP)$ and $K(1)_* \text{THH}(BP)$, we deduce that the class $\lambda_1 \in \text{THH}_*(BP)$ maps to $dt_1 \in K(1)_* \text{THH}(BP)$. Recall that

$$K(1)_*E = K(1)_*[t_i \mid i \geq 1][\eta_R(v_2)^{-1}]/(\eta_R(v_j) \mid j \geq 3)$$

is a colimit of étale algebras over $K(1)_*[w_2, w_2^{-1}]$, where

$$w_2 = \eta_R(v_2) = u_1^p t_1 - v_1^p t_1.$$  

In particular $dw_2 = v_1^p dt_1$, and the Bökstedt spectral sequence provides the formula given above for $K(1)_* \text{THH}(E)$. Now obviously $dt_1 \in K(1)_* \text{THH}(BP)$ maps to $dt_1 \in K(1)_* \text{THH}(E)$. This implies assertion (b) of the lemma. \hfill $\Box$

**Remark 5.2.** Note that the above proof does not require the map $BP \to E(n)$ to be an $E_3$-map.

The class $\lambda_1 \in \text{THH}_{2p-1}(E)$ of Lemma 5.1 corresponds to a map $\lambda_1 : S^{2p-1} \to \text{THH}(E)$. Smashing with $E$, using the $E$-module structure of $\text{THH}(E)$ (assuming an $E_3$ structure on $E$), and composing with the cofiber $\text{THH}(E) \to \overline{\text{THH}}(E)$ of the unit, we obtain a map

$$j_1 : \Sigma^{2p-1}E \cong E \wedge S^{2p-1} \to E \wedge \overline{\text{THH}}(E) \to \overline{\text{THH}}(E) \to \overline{\text{THH}}(E).$$

In the same fashion, we obtain a map $j_2 : \Sigma^{2p^2-1}E \to \overline{\text{THH}}(E)$ corresponding to the class $\lambda_2$.

**Lemma 5.3.** The map $j_1$ factors through a map

$$\tilde{j}_1 : \Sigma^{2p-1}L_1E \to \overline{\text{THH}}(E)$$

that is a $K(1)_*$-isomorphism, and whose cofiber $C(\tilde{j}_1)$ is a rational spectrum.

**Proof.** Recall from Lemma 4.2 that the cofiber $\overline{\text{THH}}(E)$ of the unit map is $E(1)$-local. In particular, the map $j_1$ factors through a map

$$\tilde{j}_1 : \Sigma^{2p-1}L_1E \to \overline{\text{THH}}(E).$$

The localization map $E \to L_1E$ is a $K(1)_*$-isomorphism, and therefore so are the induced maps $\ell : \text{THH}(E) \to \text{THH}(L_1E)$ and $\tilde{\ell} : \overline{\text{THH}}(E) \to \overline{\text{THH}}(L_1E)$, by convergence of the $K(1)$-based Bökstedt spectral sequence. Hence, to prove the claim, it suffices to show that the composition

$$(5.4) \Sigma^{2p-1}L_1E \xrightarrow{\tilde{j}_1} \overline{\text{THH}}(E) \xrightarrow{\ell} \overline{\text{THH}}(L_1E)$$

is a $K(1)_*$-isomorphism. The $K(1)_*$-based Bökstedt spectral sequence for $L_1E$ is identical to the one of $E$, computed above as

$$E_{*,*}^2 = K(1)_*E \otimes \Lambda_{E_*}(dt_1) \Rightarrow K(1)_*\text{THH}(E),$$

where $K(1)_*E$ is in filtration degree zero and $K(1)_*E\{dt_1\}$ is in filtration degree 1, and where all differentials are zero. By definition of the map $j_1$, if $1 \in K(1)_0E$ is the unit, then $j_1_*(\Sigma^{2p-11})$ is represented modulo lower filtration by the permanent cycle $dt_1$ in $E_{1,*}^2$. Since this is a spectral sequence of $K(1)_*E$-modules, the composition $(5.4)$ induces a map in $K(1)$ homology that is represented modulo lower filtration by the isomorphism $\Sigma^{2p-1}K(1)_*E \to E_{1,*}^2 = K(1)_*E\{dt_1\}$ sending a class $\Sigma^{2p-1}w$ to $wdt_1$. It is therefore a $K(1)_*$-isomorphism, proving the claim.

Now we consider the cofiber $C(\tilde{j}_1)$ of $\tilde{j}_1$, sitting in an exact triangle

$$(5.5) \Sigma^{2p-1}L_1E \xrightarrow{\tilde{j}_1} \overline{\text{THH}}(E) \xrightarrow{\ell} C(\tilde{j}_1) \xrightarrow{\delta} \Sigma^{2p}L_1E.$$  

Since $\tilde{j}_1$ is a $K(1)_*$-isomorphism, we know that $K(1)_*C(\tilde{j}_1) = 0$, and since $\overline{\text{THH}}(E)$ and thus $C(\tilde{j}_1)$ are $E(1)$-local, we deduce (as in Lemma 4.2) that $C(\tilde{j}_1)$ is $E(0)$-local (i.e., rational). \hfill $\Box$

We now define a map $\lambda_{12} : L_0S^{2p^2-2p-2} \to C(\tilde{j}_1)$ as a composition over the cofibers

$$L_0S^{2p^2-2p-2} \to L_0\overline{\text{THH}}(E) \to L_0\overline{\text{THH}}(E) \to C(\tilde{j}_1),$$

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where the first map above realizes the class $dt_1 dt_2 \in K(0)_* \text{THH}(E)$. Smashing $\lambda_2$ with $E$ and using the module structure we obtain a map

$$j_{12} : \Sigma^{2p^2 - 2p - 2} L_0 E \to C(j_1).$$

Similarly, $\lambda_2$ induces a map

$$j_2 : \Sigma^{2p^2 - 1} L_0 E \to C(j_1).$$

**Theorem 5.4.** Let $p$ be an odd prime such that $E = E(2)$, the second Johnson-Wilson spectrum at $p$, is an $E_3$-ring spectrum. Then the map $j_2 \lor j_{12}$ lifts to a map

$$\bar{j}_2 \lor j_{12} : \Sigma^{2p^2 - 1} L_0 E \lor \Sigma^{2p^2 - 2p - 2} L_0 E \to \text{THH}(E)$$

and the sum $\beta$ of $\bar{j}_1$, $\bar{j}_2$ and $j_{12}$ is a weak equivalence of $E$-modules

$$\beta : \Sigma^{2p-1} L_1 E \lor \Sigma^{2p^2 - 1} L_0 E \lor \Sigma^{2p^2 + 2p - 2} L_0 E \to \text{THH}(E).$$

**Proof.** The composition $\delta \circ (j_2 \lor j_{12})$ is trivial, so that $j_2 \lor j_{12}$ lifts to a map $\bar{j}_2 \lor j_{12}$:

Indeed, $\Sigma^{2p} L_1 E$ fits in the chromatic fracture pullback diagram

$$\Sigma^{2p} L_1 E \longrightarrow \Sigma^{2p} L_{K(1)} E$$

$$\downarrow \quad \downarrow$$

$$\Sigma^{2p} L_0 E \longrightarrow \Sigma^{2p} L_0 (L_{K(1)} E).$$

The composition of $\delta \circ (j_2 \lor j_{12})$ with the left vertical map to $\Sigma^{2p} L_0 E$ is trivial, since it factors over the composition

$$L_0 \text{THH}(E) \to L_0 C(j_1) \to \Sigma^{2p} L_0 E$$

of two consecutive maps in the $(E(0)$-localized) cofiber sequence (5.5). The composition of $\delta \circ (j_2 \lor j_{12})$ with the top map to $\Sigma^{2p} L_{K(1)} E$ is trivial as well; indeed, there is no non-trivial map from a $K(1)$-acyclic to a $K(1)$-local spectrum. This finishes the proof that $\delta \circ (j_2 \lor j_{12})$ is trivial and that the lift exists. We now define $\beta$ as the sum

$$\beta = \bar{j}_1 \lor \bar{j}_2 \lor j_{12} : \Sigma^{2p-1} L_1 E \lor \Sigma^{2p^2 - 1} L_0 E \lor \Sigma^{2p^2 + 2p - 2} L_0 E \to \text{THH}(E)$$

Finally, we claim that $\beta$ is a $K(0)_*\text{-isomorphism}$: this is analogous to the proof above that $\bar{j}_1$ is a $K(1)_*\text{-isomorphism}$, working this time with the $K(0)$-based Bökstedt spectral sequence. Since $\beta$ is a $K(0)_*\text{-}$ and a $K(1)_*\text{-}$isomorphism of $E(1)$-local spectra, it is a weak equivalence. \qed

Assume now that in addition to $E$ being an $E_3$-ring spectrum, the unit map $E \to \text{THH}(E)$ splits in the homotopy category (this holds for example if $E$ is an $E_\infty$-ring spectrum). We then have a weak equivalence of $E$-modules $E \lor \text{THH}(E) \to \text{THH}(E)$. On the other hand, summing $\beta$ with the identity of $E$ gives a weak equivalence

$$\text{id} \lor \beta : E \lor \Sigma^{2p-1} L_1 E \lor \Sigma^{2p^2 - 1} L_0 E \lor \Sigma^{2p^2 + 2p - 2} L_0 E \to E \lor \text{THH}(E).$$

This implies the following corollary of Theorem 5.4.

**Corollary 5.5.** Assume that $p$ is an odd prime, and that the second Johnson-Wilson spectrum $E = E(2)$ admits an $E_3$-structure. If the unit map $E \to \text{THH}(E)$ splits in the homotopy category, then the maps above provide a weak equivalence of $E$-modules

$$E \lor \Sigma^{2p-1} L_1 E \lor \Sigma^{2p^2 - 1} L_0 E \lor \Sigma^{2p^2 + 2p - 2} L_0 E \to \text{THH}(E).$$
Remark 5.6. Corollary 5.5 implies that
- the $2^0$ summand of $K(2)_* E$ in $K(2)_* \text{THH}(E)$ indexed by 1,
- the $2^1$ summands of $K(1)_* E$ in $K(1)_* \text{THH}(E)$ indexed by 1 and $dt_1$,
- the $2^2$ summands of $K(0)_* E$ in $K(0)_* \text{THH}(E)$ indexed by 1, $dt_1$, $dt_2$ and $dt_1 dt_2$
assemble, in $\text{THH}(E)$, into
- the $2^0$ summand $E$ indexed by 1 and detected by $K(0)_*$, $K(1)_*$ and $K(2)_*$,
- the $2^1 - 2^0$ summand $L_1E$ indexed by $dt_1$ and detected by $K(0)_*$ and $K(1)_*$, and
- the $2^2 - 2^1$ summands $L_0E$ indexed by $dt_2$ and $dt_1 dt_2$ and detected by $K(0)_*$.

Notice that Bruner and Rognes [BR] obtain very similar computations for $K(i)_* \text{THH}(tmf)$ for $i = 0, 1, 2$, where $tmf$ denotes the connective spectrum of topological modular form.

We can picture the summands of $\text{THH}(E)$ in a 2-dimensional cube of local pieces (up to suspensions, where $E = L_2 E$):

\[
\begin{array}{ccc}
1 & dt_1 \\
& E & L_1 E \\
& dt_2 & L_0 E & L_0 E
\end{array}
\]

We conjecture that this picture extends to describe a decomposition of $\text{THH}(E(n))$ into $2^n$ summands, with summands placed in an $n$-dimensional cube, where the $i$th edge has two coordinates 1 and $dt_i$. We formulate this as follows.

**Conjecture 5.7.** If $p$ is an odd prime such that $E(n)$ is a sufficiently commutative $S$-algebra, then $\text{THH}(E(n))$ decomposes as a sum of $2^n$ factors, namely $2^{n-i-1}$ suspended copies of $L_i E(n)$ for each $0 \leq i \leq n-1$, plus one copy of $E(n)$. More precisely, the $L_i E(n)$ summands are indexed by the $2^{n-i-1}$ monomial generators
\[\omega \in \Lambda_0(dt_1, \ldots, dt_{n-i-1}) \{dt_{n-i}\} \subset K(0)_* \text{THH}(E(n)),\]
and the summand corresponding to such a monomial $\omega$ is $\Sigma^{|\omega|} L_i E(n)$.

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