GROUP ELEMENTS WHOSE CHARACTER VALUES ARE ROOTS OF UNITY

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Dedicated to Pham Huu Tiep on his 60th birthday

Abstract. We classify all finite groups \( G \) which possesses an element \( x \in G \) such that every irreducible character of \( G \) takes a root of unity value at \( x \).

1. Introduction

Let \( G \) be a finite group. Following [5], an element \( x \in G \) is called a nonvanishing element if \( \chi(x) \neq 0 \) for all irreducible complex characters \( \chi \) of \( G \). This concept has been widely studied in recent years. In this paper, we consider nonvanishing elements of finite groups which satisfy certain minimal condition as follows. Given a nonvanishing element \( x \) of a finite group \( G \), it is not hard to show that \( |C_G(x)| \geq k(G) \) and that the equality holds if and only if \( |\chi(x)| = 1 \) for all irreducible characters \( \chi \) of \( G \) (see Lemma 2.3), where \( C_G(x) \) is the centralizer of \( x \) in \( G \) and \( k(G) \) is the number of conjugacy classes of \( G \). Note that if \( |\chi(x)| = 1 \) for some character \( \chi \) of \( G \), then \( x \) is a root of unity (see, for example, Problem 3.2 in [4]). We will call an element \( x \in G \) a root of unity element if \( |\chi(x)| = 1 \) for all irreducible characters of \( G \). The condition \( |C_G(x)| = k(G) \) alone does not characterizes root of unity elements. For example, if \( G = A_5 \), the alternating group of degree 5, and \( x \in G \) is an element of order 5, then \( |C_G(x)| = 5 = k(G) \) but \( x \) is not a root of unity element. We note that root of unity elements are called totally unitary or TU-elements by S. Ostrovskaya and E. M. Zhmud' and they classify all finite metabelian groups with trivial center that contain a root of unity element in [1] Chapter XXII).

Write \( \text{Irr}(G) \) for the set of irreducible complex characters of \( G \) and \( F(G) \) for the Fitting subgroup of \( G \), that is, the largest normal nilpotent subgroup of \( G \). In our first result, we prove the following.

**Theorem A.** Let \( G \) be a finite group and let \( x \in G \). If \( |\chi(x)| = 1 \) for all irreducible characters \( \chi \) of \( G \), then \( x \in F(G) \) and both \( F(G) \) and \( G/F(G) \) are abelian. In particular, \( G \) is abelian or metabelian.

Thus if a finite group \( G \) has a root of unity element, then it is abelian or metabelian. In particular, such a group is solvable. Theorem A confirms a conjecture proposed in July 31, 2022.

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for root of unity elements. This conjecture states that every nonvanishing element of a finite solvable groups $G$ must lie in the Fitting subgroup $F(G)$.

Write $\text{Irr}(G)$ for the set of all complex irreducible characters of $G$. Our interest in root of unity elements stems from an observation that if $\chi \in \text{Irr}(G)$ and $g \in G$ such that $|\chi(g)| = 1$, then the size of the conjugacy class $g^G$ containing $g$ is always divisible by $\chi(1)$ (see Lemma 2.1). Consequently, if $x$ is a root of unity element of $G$, then $|x^G|$ is divisible by $\chi(1)$ for all $\chi \in \text{Irr}(G)$. This is related to Conjecture C in [6] asserting that if $\chi \in \text{Irr}(G)$ is a primitive character of a finite group $G$, then $\chi(1)$ divides $|g^G|$ for some $g \in G$. Thus the observation above gives us a way to locate the required element $g \in G$. However, not every primitive irreducible character admits a root of unity value. For example, if $G$ is the sporadic simple group O’N, then $G$ has a primitive irreducible character of degree 64790 which does not admit any root of unity value.

In the next result, we classify all finite groups with a root of unity element. Clearly, if $G$ is abelian, then every element of $G$ is a root of unity element. Let $q > 2$ be a prime power. We denote by $\Gamma_q$ the unique doubly transitive Frobenius group with a cyclic complement of order $q - 1$ and degree $q$. So $\Gamma_q \cong AGL_1(\mathbb{F}_q) = \mathbb{F}_q \rtimes \mathbb{F}_q^*$, where $\mathbb{F}_q$ is a finite field with $q$ elements.

**Theorem B.** Let $G$ be a finite group. Then $G$ has a root of unity element $x \in G$ if and only if one of the following holds:

- $G$ is abelian;
- $F(G) = G'Z(G)$ is abelian, $G' \cap Z(G) = 1$; and $G/Z(G) \cong \Gamma_{q_1} \times \Gamma_{q_2} \times \cdots \times \Gamma_{q_m}$, where each $q_i > 2$ is a prime power and $m \geq 1$ is an integer.

We also show that if a finite group $G$ has a root of unity element, then it is an $A$-group, that is, $G$ is solvable and all its Sylow subgroups are abelian. For each $i$ with $1 \leq i \leq m$, write $\Gamma_{q_i} = V_i \rtimes A_i$, where $V_i$ is the Frobenius kernel and $A_i$ is the Frobenius complement. Let $U := \prod_{i=1}^m (V_i \setminus \{1\})$ and let $U = \pi^{-1}(U)$ where $\pi : G \rightarrow G/Z(G)$ is the natural homomorphism. Then every element of $U$ is a root of unity element of $G/Z(G)$ by [1] Lemma 3.17 and from the proof of Theorem B, every element of $U$ is a root of unity element of $G$.

Our notation is standard and we follow [4] for the character theory of finite groups.

2. Preliminaries

We collect some properties of root of unity elements in the next lemmas.

**Lemma 2.1.** Let $G$ be a finite group and let $g \in G$. If $\chi \in \text{Irr}(G)$ and $|\chi(g)| = 1$, then $\chi(1)$ divides $|g^G|$. In particular, if $x \in G$ is a root of unity element, then $\chi(1)$ divides $|x^G|$ for all $\chi \in \text{Irr}(G)$.

**Proof.** Assume that $\chi \in \text{Irr}(G)$ and $g \in G$ such that $|\chi(g)| = 1$. Let $K$ be the class sum of the conjugacy class $g^G$, that is, $K = \sum_{y \in g^G} y$. Then

$$
\omega_\chi(K) = \frac{|g^G| |\chi(g)|}{\chi(1)}
$$
is an algebraic integer by [1] Theorem 3.7. Similarly,
\[ \omega_\chi(K) = \frac{|g^G\chi(g)|}{\chi(1)} = \frac{|g^G\overline{\chi(g)}|}{\chi(1)} \]
is also an algebraic integer, where \( \chi \) is the complex conjugate of \( \chi \). Since the products of algebraic integers are algebraic integers, we see that
\[ \omega_\chi(K)\omega_{\overline{\chi}}(K) = \frac{|g^G|^2|\chi(g)|^2}{\chi(1)^2} = \frac{|g^G|^2}{\chi(1)^2} \]
is an algebraic integer. Clearly, \( |g^G|^2/\chi(1)^2 \) is a rational number, so \( |g^G|^2/\chi(1)^2 \) is an integer which implies that \( \chi(1) \) divides \( |g^G| \) as wanted.

If \( x \in G \) is a root of unity element, then for any \( \chi \in \text{Irr}(G) \), we have \( |\chi(x)| = 1 \) and hence \( \chi(1) \) divides \( |x^G| \) as wanted. \( \square \)

**Lemma 2.2.** Let \( G \) be a finite group and let \( x \in G \) be a root of unity element. Then

(a) \( k(G) = |C_G(x)| \).
(b) \( G' \leq \langle x^G \rangle \).
(c) If \( N \trianglelefteq G \), then \( xN \) is a root of unity in \( G/N \).
(d) If \( z \in \mathbb{Z}(G) \), then \( xz \) is also a root of unity element.

**Proof.** From the Second Orthogonal relation, we have
\[ |C_G(x)| = \sum_{\chi \in \text{Irr}(G)} |\chi(x)|^2 = \sum_{\chi \in \text{Irr}(G)} 1 = k(G). \]

Let \( L = \langle x^G \rangle \). For any \( \chi \in \text{Irr}(G/L) \), we see that \( x \in L \subseteq \text{Ker}(\chi) \). Hence \( 1 = |\chi(x)| = \chi(1) \) and thus all characters \( \chi \in \text{Irr}(G/L) \) are linear which implies that \( G/L \) is abelian and so \( G' \leq L \). Since \( \text{Irr}(G/N) \subseteq \text{Irr}(G) \) whenever \( N \trianglelefteq G \), if \( x \) is a root of unity of \( G \) then \( xN \) is a root of unity of \( G/N \).

Finally, let \( z \in \mathbb{Z}(G) \) and \( \chi \in \text{Irr}(G) \). Then \( \chi_{\mathbb{Z}(G)} = \chi(1)\lambda \) for some \( \lambda \in \text{Irr}(\mathbb{Z}(G)) \). We have \( \chi(xz) = \lambda(z)\chi(x) \) and thus if \( x \) is a root of unity element, then so is \( xz \) as \( |\lambda(z)| = 1 \). \( \square \)

The next lemma follows from the proof of Lemma 3.17 in [1] Chapter XXII and the previous lemma. For completeness, we include the proof here.

**Lemma 2.3.** Let \( G \) be a finite group and let \( x \in G \) be a nonvanishing element of \( G \). Then \( |C_G(x)| \geq k(G) \); and the equality holds if and only if \( x \) is a root of unity element.

**Proof.** We first claim that if \( x \in G \) is a nonvanishing element, then \( |C_G(x)| \geq k(G) \). Let \( \alpha = \prod_{\chi \in \text{Irr}(G)} \chi(x) \). Let \( n \) be the exponent of \( G \) and let \( \mathbb{Q}_n = \mathbb{Q}(\xi) \), where \( \xi \) is a primitive \( nth \)-root of unity. Let \( \mathcal{G} \) be the Galois group of \( \mathbb{Q}_n \) over \( \mathbb{Q} \). Then \( \mathcal{G} \) acts on \( \text{Irr}(G) \) and we see that \( \chi^\sigma \in \text{Irr}(G) \) if and only if \( \chi \in \text{Irr}(G) \) for all \( \sigma \in \mathcal{G} \). Hence \( \chi^\sigma = \chi \) for all \( \alpha \in \mathcal{G} \). It follows that \( \alpha \in \mathbb{Q} \). Since \( \alpha \) is an algebraic integer, we must have that \( \alpha \in \mathbb{Z} \). As \( x \) is nonvanishing, \( \alpha \neq 0 \) and so \( |\alpha| \geq 1 \).
By the inequality between arithmetic and geometric means, we have that

$$1 \leq |\alpha|^2 = \prod_{\chi \in \text{Irr}(G)} |\chi(x)|^2 \leq \left( \frac{\sum_{\chi \in \text{Irr}(G)} |\chi(x)|^2}{k(G)} \right)^{k(G)} = \left( \frac{|C_G(x)|}{k(G)} \right)^{k(G)}.$$ 

It follows that $|C_G(x)| \geq k(G)$ as wanted.

Next, assume that $x$ is a nonvanishing element of $G$ and $|C_G(x)| = k(G)$. Then $|\alpha| = 1$ from the inequality above. Hence

$$\prod_{\chi \in \text{Irr}(G)} |\chi(x)|^2 = \left( \frac{\sum_{\chi \in \text{Irr}(G)} |\chi(x)|^2}{k(G)} \right)^{k(G)} = 1.$$ 

Therefore, $|\chi(x)| = 1$ for all $\chi \in \text{Irr}(G)$. So $x \in G$ is a root of unity element.

Conversely, if $x$ is a root of unity, then clearly $x$ is nonvanishing and $|C_G(x)| = k(G)$ by Lemma 2.2(a). \hfill \Box

A consequence of the previous lemma is that if $x \in G$ and $k(G) > |C_G(x)|$ or equivalently $|x^G| > |G|/k(G)$, the average of the conjugacy class size of $G$, then $x$ is a vanishing element of $G$, that is, $\chi(x) = 0$ for some $\chi \in \text{Irr}(G)$. Also, if $G$ has a root of unity element $x$, then the commuting probability

$$\text{cp}(G) = \frac{|\{(a, b) \in G \times G : ab = ba\}|}{|G|^2} = \frac{k(G)}{|G|}$$

is equal to $1/|x^G|$.

In the next two lemmas, we quote some results in Chapter XXII of [1].

**Lemma 2.4.** Let $G$ be a finite group and suppose that $x \in G$ is a root of unity element.

(a) If $x \in F(G)$, then $F(G)$ is abelian and $G' \leq F(G)$. In particular, $G$ is abelian or metabelian.

(b) Conversely, if $G$ is metabelian, then $x \in F(G)$.

*Proof.* This is Lemma 1.5 in [1] Chapter XXII. \hfill \Box

The following is the main result of Chapter XXII in [1]. Recall that the socle of a finite group $G$, denoted by $\text{Soc}(G)$, is a product of all minimal normal subgroups of $G$.

**Lemma 2.5.** Let $G$ be a finite metabelian group with trivial center. Then $G$ has a root of unity element if and only if $G \cong \Gamma_{q_1} \times \Gamma_{q_2} \times \cdots \times \Gamma_{q_m}$, where $q_1, q_2, \ldots, q_m$ are prime power $> 2$. Moreover, if $x \in G$ is a root of unity, then $F(G) = \text{Soc}(G) = G' = \langle x^G \rangle$ and $C_G(x) = F(G)$.

*Proof.* The equivalent statements follow from Theorem 1.12 and Corollary 1.11 and the last claim follows from Lemmas 3.8 and 3.14 in [1] Chapter XXII. \hfill \Box

For a finite group $G$, recall that $F(G)$, the Fitting subgroup of $G$, is the largest nilpotent normal subgroup of $G$. The Fitting series of a finite group $G$ is defined by $F_1(G) := F(G)$ and for any integer $i \geq 1$, $F_{i+1}(G)/F_i(G) = F(G/F_i(G))$. Similarly,
the upper central series of \( G \) is defined by \( Z_1(G) := Z(G) \) and for \( i \geq 1 \), we have \( Z_{i+1}(G)/Z_i(G) = Z(G/Z_i(G)) \). The last term of the upper central series of \( G \) is called the hypercenter (or hypercentral) of \( G \) and is denoted by \( Z_\infty(G) \).

The following results are well-known.

**Lemma 2.6.** Let \( G \) be a finite group and let \( N \) be a normal subgroup of \( G \) such that \( N \leq Z(G) \).

1. If \( G/N \) is nilpotent, then \( G \) is nilpotent.
2. \( F(G/N) = F(G)/N \).
3. \( F(G/Z_i(G)) = F(G)/Z_i(G) \) for all \( i \geq 1 \).

**Proof.** The first two claims are well-known. The last claim follows from the second claim and induction. \( \square \)

The next result is Corollary 2.3 in [8].

**Lemma 2.7.** Let \( G \) be a finite solvable group and assume that the Sylow 2-subgroups of \( F_{i+1}(G)/F_i(G) \) are abelian for \( 1 \leq i \leq 9 \). Then every nonvanishing element of \( G \) lies in \( F(G) \).

3. **Solvability of finite groups with a root of unity element**

We first prove Theorem A for finite solvable groups.

**Proposition 3.1.** Let \( G \) be a finite solvable group and suppose that \( x \in G \) is a root of unity element. Then \( G \) is abelian or \( G \) is metabelian, \( x \in F(G) \) and both \( F(G) \) and \( G/F(G) \) are abelian.

**Proof.** If \( G \) is abelian, then we are done. So assume that \( G \) is nonabelian. If \( G \) is metabelian, then the conclusion follows from Lemma 2.4. Thus we only need to show that \( G \) is metabelian. We will prove this by induction on \( |G| \).

Let \( N \) be a minimal normal subgroup of \( G \). Let \( \overline{G} = G/N \) and use the ‘bar’ notation. By Lemma 2.2(c), \( \overline{x} \) is a root of unity element in \( \overline{G} \) and thus by induction, \( \overline{G} \) is abelian or metabelian. If \( \overline{G} \) is abelian, then \( G \) is metabelian. So assume that \( \overline{G} \) is metabelian (but \( G \) is neither metabelian nor abelian). Then \( \overline{x} \in F(\overline{G}) \) and both \( F(\overline{G}) \) and \( \overline{G}/F(\overline{G}) \) are abelian by Lemma 2.4. Further \( N \) is abelian since it is a minimal normal subgroup and \( G \) (and so also \( N \)) is solvable. Thus \( N \leq F(G) \).

Therefore, \( G/F(G) \) is also metabelian. Again by Lemma 2.4, we have that \( F(G/F(G)) = F_2(G)/F_1(G) \) and \( G/F_2(G) \) are abelian. It follows that \( F_3(G) = G \) and \( F_{i+1}(G)/F_i(G) \) is abelian for all \( i \geq 1 \). Now by Lemma 2.7 we have \( x \in F(G) \) as \( x \) is nonvanishing and so \( G \) is metabelian. This contradiction completes the proof. \( \square \)

The following result follows from the proof of Theorem A in [7]. Recall that an element \( g \in G \) is called a vanishing element if \( \chi(g) = 0 \) for some \( \chi \in \text{Irr}(G) \).

**Lemma 3.2.** Let \( G \) be a finite group. Assume that \( G \) has a unique minimal normal subgroup \( N \). If \( N \) is non-abelian and \( G/N \) is solvable, then every element in \( G - N \) is a vanishing element.
Let \( n \geq 2 \) be an integer and let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \) be a partition of \( n \). For \( 1 \leq i \leq k \) and \( 1 \leq j \leq \lambda_i \), we denote by \( h_{i,j}^\lambda \) the hook length of the node \((i, j)\) of the Young diagram of \( \lambda \). Let \( \lambda \) and \( \mu \) be partitions of \( n \). We use the notation \( \chi^\lambda_\mu \) to denote the value of the irreducible character of \( S_n \) labeled by \( \lambda \) evaluated at the conjugacy class with cycle type \( \mu \).

**Lemma 3.3.** Let \( n \geq 6 \) be an integer and let \( x \in A_n \). Then there exists a partition \( \lambda \) of \( n \) which is not self-conjugate such that \( \chi^\lambda(x) \neq \pm 1 \).

**Proof.** We will use the following fact which can follow easily from Murnaghan-Nakayama formula. If \( m \geq 1 \) is an integer and \( \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_r) \), \( \beta = (\beta_1, \beta_2, \ldots, \beta_s) \) are partition of \( m \) with \( h_{2,1}^\gamma < \beta_1 \) and \( \gamma_1 - \gamma_2 \geq \beta_1 \), then

\[
\chi^\gamma_\beta = \chi^{(\gamma_1-\beta_1, \gamma_2, \ldots, \gamma_r)}_{(\beta_2, \beta_3, \ldots, \beta_s)}.
\]

Let \( \alpha \vdash n \) be the cycle partition of \( x \in A_n \). Since \( n \geq 6 \), from the proof of Lemma 1.6 in [9] we may assume that all parts of \( \alpha \) are distinct, except possibly for the part 1, which may have multiplicity 2.

Write \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l) \vdash n \). We consider the following cases.

**Case 1:** \( l \geq 2 \) and \( \alpha_{l-1} = \alpha_l \). In this case, we have \( \alpha_{l-1} = \alpha_l = 1, l \geq 3 \) (as \( n \geq 6 \)) and \( \alpha_{l-2} > 1 \).

Assume first that \( \alpha_{l-2} > 2 \). Since \( n \geq 6 \), the partition \((n - 2, 1, 1)\) of \( n \) is not self-conjugate and we have that

\[
\chi^{(n-2,1,1)}_\alpha = \chi^{(\alpha_{l-2}, 1, 1)}_{(\alpha_{l-2}, 1, 1)} = 0.
\]

The first equality holds by the observation above and the latter equality holds since \( \alpha_{l-2} > 2 \) so

\[
\begin{align*}
  h_{1,1}^{(\alpha_{l-2}, 1, 1)} &= \alpha_{l-2} + 2 > \alpha_{l-2} \\
  h_{1,2}^{(\alpha_{l-2}, 1, 1)} &= \alpha_{l-2} - 1 < \alpha_{l-2} \\
  h_{2,1}^{(\alpha_{l-2}, 1, 1)} &= 2 < \alpha_{l-2}.
\end{align*}
\]

Assume next that \( \alpha_{l-2} = 2 \). Then \( l \geq 4 \) and \( \alpha_{l-3} \geq 3 \). Then the partition \((n - 2, 2)\) of \( n \) is not self-conjugate and by the observation above, we have

\[
\chi^{(n-2, 2)}_\alpha = \chi^{(2, 2)}_{(2, 1, 1)} = 0.
\]

**Case 2:** \( l \geq 2 \) and \( \alpha_{l-1} \neq \alpha_l \).

Assume first that \( l \geq 3 \) or \( \alpha_{l-1} \neq \alpha_l + 1 \). Then

\[
\chi^{(n-\alpha_l, \alpha_l)}_\alpha = \chi^{(\alpha_{l-1}, \alpha_l)}_{(\alpha_{l-1}, \alpha_l)} = 0
\]

as

\[
\begin{align*}
  h_{1,1}^{(\alpha_{l-1}, \alpha_l)} &= \alpha_{l-1} + \alpha_l > \alpha_{l-1} \\
  h_{1,2}^{(\alpha_{l-1}, \alpha_l)} &= \alpha_{l-1} - 1 < \alpha_{l-1} \\
  h_{2,1}^{(\alpha_{l-1}, \alpha_l)} &= \alpha_l < \alpha_{l-1}.
\end{align*}
\]
and \((n - \alpha_l, 1^{\alpha_l})' = (\alpha_l + 1, 1^{n - \alpha_l - 1}) \neq (n - \alpha_l, 1^{\alpha_l})\) as
\[n - \alpha_l \geq \alpha_l - 2 + \alpha_l - 1 \geq 2\alpha_l + 3 > \alpha_l + 1\]
if \(l \geq 3;\) and \(n - \alpha_l \geq \alpha_l - 1 > \alpha_l + 1\) if \(\alpha_l - 1 \neq \alpha_l.\)

Assume next that \(\ell = 2\) and \(\alpha_1 = \alpha_2 + 1.\) Then \(n = 2\alpha_2 + 1.\) As \(n \geq 6,\) we have \(\alpha_2 \geq 3.\) Again \((n - 2, 2)' \neq (n - 2, 2).\)

If \(\alpha_2 = 3,\) then \(n = 7\) and \(\chi_{(4,3)}^{(5,2)} = 0.\) Assume that \(\alpha_2 \geq 4.\) Then
\[\chi_{\alpha}^{(n-2,2)} = \chi_{(\alpha_2 + 1, \alpha_2)}^{(n-2,2)} = \chi_{(\alpha_2)}^{(\alpha_2 - 2,2)} = 0\]
as \(h_{(\alpha_2 - 2,2)}^{(1,1)} = \alpha_2 - 1 < \alpha_2.\)

**Case 3 :** \(l = 1.\) Then \(\alpha = (n).\) Since \(n \geq 6,\) \((n - 2, 2)\) is not self-conjugate and \(\chi_{(n)}^{(n-2,2)} = 0\) as \(h_{(n-2,2)}^{(1,1)} = n - 1 < n.\)

We are now ready to prove Theorem A.

**Proof of Theorem A.** Let \(x \in G\) be a root of unity element. If \(G\) is solvable, then the theorem follows from Proposition [3.1]. Thus it suffices to show that \(G\) is solvable. Suppose not and let \(G\) be a counterexample to the theorem with \(|G|\) minimal. Then \(x \in G\) is a root of unity but \(G\) is non-solvable. Let \(L = \langle x^G \rangle.\) Then \(G' = L \subseteq G\) by Lemma [2.2] (b).

Let \(N\) be a minimal normal subgroup of \(G.\) By Lemma [2.2] (c), \(G/N\) has a root of unity \(xN.\) Since \(|G/N| < |G|,\) \(G/N\) is solvable. As \(G\) is non-solvable, \(N\) is non-solvable. If \(G\) has two distinct minimal normal subgroups, say \(N_1 \neq N_2,\) then \(N_1 \cap N_2 = 1\) and thus \(G\) embeds into \(G/N_1 \times G/N_2,\) where the latter group is solvable by the argument above. Therefore, \(G\) is solvable, which is a contradiction. It follows that \(G\) has a unique minimal normal subgroup \(N,\) which is non-solvable and \(G/N\) is solvable. Hence \(N = S_1 \times S_2 \cdots \times S_k,\) where each \(S_i \cong S\) for some non-abelian simple group \(S.\) By Lemma [3.2] \(x \in N\) as every element in \(G - N\) is a vanishing element. It follows from Lemma [2.2] (b) that \(G' = \langle x^G \rangle = N,\) and so \(G/N\) is abelian.

Write \(x = (x_1, x_2, \ldots, x_k) \in N,\) where \(x_i \in S_i\) for \(1 \leq i \leq k.\) As \(G\) is nonabelian, \(x\) is nontrivial and so \(o(x),\) the order of \(x,\) is divisible by some prime \(p \geq 2.\) Clearly, \(o(x_i)\) is divisible by \(p\) for some \(i \geq 1.\) Assume that \(S\) has an irreducible character \(\theta\) of \(p\)-defect zero. Then \(\lambda = \theta \times \theta \times \cdots \times \theta \in \text{Irr}(N)\) has \(p\)-defect zero. Clearly, every \(G\)-conjugate of \(\lambda\) also has \(p\)-defect zero and hence if \(\chi \in \text{Irr}(G)\) lying over \(\lambda,\) then \(\chi_N\) is a sum of \(G\)-conjugates of \(\lambda\) so that \(\chi(x) = 0\) since every conjugate of \(\lambda\) vanishes at \(x\) as \(o(x)\) is divisible by \(p.\) Therefore, we can assume that \(S\) has no \(p\)-defect zero character. By \([3,\text{ Corollary 2}],\) the following cases hold.

(i) \(p = 2\) and \(S\) is isomorphic to \(M_{12}, M_{22}, M_{24}, J_2, HS, Suz, Ru, Co_1, Co_3\) or \(A_n\) for some integer \(n \geq 7;\) or

(ii) \(p = 3\) and \(S\) is isomorphic to \(Suz, Co_3\) or \(A_n\) for some integer \(n \geq 7,\)

We make the following observation. Assume that \(S\) has a rational-valued irreducible character \(\theta \in \text{Irr}(S)\) which is extendible to \(\text{Aut}(S).\) Then \(\varphi = \theta \times \theta \times \cdots \times \theta \in \text{Irr}(N)\)
extends to $\chi \in \text{Irr}(G)$ and

$$1 = \lvert \chi(x) \rvert = \lvert \varphi(x) \rvert = \prod_{i=1}^{k} \lvert \theta(x_i) \rvert.$$  

Since $\theta$ is rational, $\theta(x_i)$ is a non-zero integer and thus $\lvert \theta(x_i) \rvert \geq 1$ for all $i$. The previous equation now implies that $\lvert \theta(x_i) \rvert = 1$ for all $i$.

(a) Assume first that $S$ is one of the sporadic simple groups in (i) but not in (ii). Then $x$ and hence $x_i$s must be a 2-element. Using [2], we can find an irreducible rational-valued character $\theta$ which is extendible to $\text{Aut}(S)$ and does not take root of unity on any 2-elements. So this case cannot occur.

Similarly, if $S$ appears in Case (ii), then $x$ and hence $x_i$s are $\{2, 3\}$-elements. Again, by using [2], we can find an irreducible rational-valued character $\theta$ which is extendible to $\text{Aut}(S)$ and does not take root of unity on any $\{2, 3\}$-elements.

(b) Assume that $S \cong A_n$, where $n \geq 7$ is an integer and that $A_n$ has no block of $p$-defect zero for $p = 2, 3$ or both.

By the observation above, if $\lambda$ is a partition of $n$ which is not self-conjugate, then $\chi^\lambda$, the irreducible character of $S_n$ labeled by $\lambda$, remains irreducible upon reduction to $A_n$ and thus $\lvert \chi^\lambda(x_i) \rvert = 1$. Note that $\chi^\lambda$ is rational-valued. Now Lemma 3.3 provides a contradiction.

Therefore, $G$ must be solvable as wanted. The proof is now complete. \qed

4. Finite metabelian groups with a root of unity element

In this section, we will characterizing finite metabelian groups with a root of unity element. Such a group with trivial center was classified by S. Ostrovskaya and E. M. Zhmud’. Recall that if $q > 2$ is a prime power, then $\Gamma_q$ is a doubly transitive Frobenius group with a cyclic complement of order $q - 1$ and degree $q$. Note that $\Gamma_q$ has a root of unity element and every root of unity element of $\Gamma_q$ lies in $F(\Gamma_q)$, which is an elementary abelian $p$-group, where $q$ is a power of a prime $p$. Moreover, if $\Gamma = \Gamma_{q_1} \times \Gamma_{q_2} \times \cdots \times \Gamma_{q_m}$, where each $q_i > 2$ are prime powers, then $\Gamma$ has a root of unity element and furthermore, all Sylow subgroups of $\Gamma$ are abelian.

**Lemma 4.1.** Let $G$ be a finite group and let $x \in G$ be a root of unity element. Let $K = Z_\infty(G)$ be the hypercenter of $G$. Assume that $G$ is nonabelian. Then

1. $G/K \cong \Gamma_{q_1} \times \Gamma_{q_2} \times \cdots \times \Gamma_{q_m}$, where each $q_i > 2$ is a prime power and $m \geq 1$ is an integer. Moreover, $F(G) = G'K$ is abelian, and $C_{G/K}(xK) = F(G/K) = F(G)/K$.
2. $F(G) = C_G(x)$ and $k(G) = \lvert F(G) \rvert = \lvert C_G(x) \rvert$.
3. If $N = Z_i(G)$ for some $i \geq 1$ or $N \leq Z(G)$, then $C_{G/N}(xN) = F(G/N) = F(G)/N$ and $k(G/N) = \lvert F(G) : N \rvert$.

**Proof.** By Theorem A, $x \in F := F(G)$, $F$ is abelian and $G' \leq F$. Since $K \leq G$ is nilpotent, we have $Z(G) \leq K \leq F$. Now the center of $G/K$ is trivial by the definition
of \( K \). Moreover \( F(G/K) = F/K \) by Lemma 2.6(3). Since \( G/K \) has a root of unity element \( xK \), part (1) follows from Lemma 2.5.

Since \( x \in F \) and \( F \) is abelian, we have \( K \leq x \leq C_F(x) \). Let \( \overline{G} = G/K \). From part (1), we have \( C_{\overline{G}}(x) = F(\overline{G}) = \overline{F} \). Hence

\[
\overline{F} \leq C_{\overline{G}}(x) \leq C_{\overline{G}}(x) = \overline{F}.
\]

Thus \( \overline{F} = C_{\overline{G}}(x) \) and hence \( F = C_G(x) \). As \( k(G) = |C_G(x)| \) by Lemma 2.2(a), part (2) follows.

Finally, let \( N = Z_i(G) \) for some \( i \geq 1 \) or \( N \leq Z(G) \). Then \( G/N \) has a root of unity and it is not nilpotent. By Lemma 2.6 \( F(G/N) = F/N \). Now part (3) follows by applying part (2) to \( G/N \).

Following P. Hall, a finite solvable group \( G \) is called an \( A \)-group if every Sylow subgroup of \( G \) is abelian. The next lemma shows that any finite group with a root of unity element is an \( A \)-group.

**Proposition 4.2.** If a finite group \( G \) has a root of unity element, then \( G \) is an \( A \)-group.

**Proof.** Let \( G \) be a finite group with a root of unity element \( x \in G \). Clearly, if \( G \) is abelian, then \( G \) is an \( A \)-group. So, we can assume that \( G \) is nonabelian and hence by Theorem A, \( x \in F := F(G) \) and both \( F \) and \( G/F \) are abelian so \( G \) is solvable. We proceed by induction on \(|G|\) that all Sylow subgroups of \( G \) are abelian.

Notice first that if \( 1 < N \leq G \), then \( G/N \) has a root of unity element \( xN \) and so by induction, every Sylow subgroup of \( G/N \) is abelian. Now let \( N \) be a minimal normal subgroup of \( G \). Since \( G \) is solvable, \( N \) is an elementary abelian \( p \)-group for some prime \( p \). If \( Q \) is a Sylow \( r \)-subgroup of \( G \) for some prime \( r \neq p \), then \( QN/N \cong Q/Q \cap N \cong Q \) is abelian. Thus it remains to show that every Sylow \( p \)-subgroup of \( G \) is abelian. Let \( P \in \text{Syl}_p(G) \). Then \( N \leq P \) and \( P/N \) is abelian as \( P/N \in \text{Syl}_p(G/N) \). Hence \( P' \leq N \).

Now if \( G \) has another minimal normal subgroup, say \( M \neq N \), then \( M' \leq N \cap M = 1 \) and hence \( P \) is abelian as wanted if \( M \) is also a \( p \)-group. If instead \( M \) is not a \( p \)-group then we can conclude as in the \( r \neq p \) case that \( P \) is abelian. Therefore, we may assume that \( N \) is the unique minimal normal subgroup of \( G \).

Since \( F \) is abelian and \( N \leq F \) is the unique minimal normal subgroup of \( G \), \( F \) must be a \( p \)-group and so \( F = O_p(G) \). Let \( K = Z_\infty(G) \). By Lemma 4.4 \( K \leq F \leq G \) and \( G/K \cong \Gamma_{q_1} \times \Gamma_{q_2} \times \cdots \times \Gamma_{q_m} \) for some integer \( m \geq 1 \) and each \( q_i > 2 \) is a prime power. Write \( \Gamma_{q_i} = V_i \rtimes A_i \), where \( V_i \) is the Frobenius kernel which is an elementary abelian group of order \( q_i \) and \( A_i \) is cyclic of order \( q_i - 1 \). Now \( F/K = F(G/K) = V_1 \times V_2 \times \cdots \times V_m \) is a direct product of the Frobenius kernels of the groups \( \Gamma_{q_i} \). It follows that each \( q_i \) is a power of \( p \). Hence \( p \nmid |A_i| \) for all \( i \) and so \( |G:F| = \prod_{i=1}^m |A_i| = \prod_{i=1}^m (q_i - 1) \) is coprime to \( p \). Therefore, \( F \) is a Sylow \( p \)-subgroup of \( G \) which is abelian and we are done.

**Corollary 4.3.** Let \( G \) be a finite group and let \( x \in G \) be a root of unity. Then \( Z(G) = Z_\infty(G) \), that is, \( G/Z(G) \) has a trivial center, and \( G' \cap Z(G) = 1 \).

**Proof.** By Proposition 4.2 \( G \) is a finite solvable \( A \)-group. The result now follows from (3.8) and Theorem 4.1 in [10].
Proof of Theorem B. Let $G$ be a finite group, let $Z := Z(G)$ and $F := F(G)$. Suppose first that $x \in G$ is a root of unity element. If $G$ is abelian, then we are done. Assume that $G$ is non-abelian. By Theorem A, $x \in F$, $F$ and $G/F$ are abelian and $G$ is metabelian. By Corollary 4.3, $Z(\infty)(G) = Z(G)$, $G/Z$ has trivial center and $G' \cap Z(G) = 1$. Now, the conclusion follows from Lemma 4.1 (1).

For the converse, assume that $G$ is nonabelian. So $F = G' \times Z$. Write $G = G/Z$ and use the ‘bar’ notation. By Lemma 2.5, $G$ has a root of unity element $x$ for some $x \in G$. Note that the hypothesis above implies that $F = G' \times Z$.

We claim that $x$ is also a root of unity element of $G$, that is, $|\chi(x)| = 1$ for all $\chi \in \text{Irr}(G)$. As $|\chi(x)| = |\chi(\bar{x})| = 1$ for every $\chi \in \text{Irr}(G/Z)$, it suffices to show that if $1 \neq \lambda \in \text{Irr}(Z)$ and $\chi \in \text{Irr}(G)$ lying over $\lambda$, then $|\chi(x)| = 1$.

Let $1 \neq \lambda \in \text{Irr}(Z)$. Since $F = Z \times G'$, $\theta = \lambda \times 1_{G'} \in \text{Irr}(F)$ is an extension of $\lambda$ and $G' \leq \text{Ker}(\theta)$. So $\theta$ can be considered an irreducible character of $F/G'$ and thus $\theta$ extends to $\phi \in \text{Irr}(G/G')$. Thus $\lambda$ extends to $\phi \in \text{Irr}(G)$. By Gallagher’s theorem, every $\chi \in \text{Irr}(G)$ lying above $\lambda$ has the form $\phi \mu$ for some irreducible character $\mu \in \text{Irr}(G)$. Since $\phi$ is linear, we have $|\phi(x)| = 1$. We also have $|\mu(x)| = |\mu(\bar{x})| = 1$ as $\mu \in \text{Irr}(G)$ and $\bar{x}$ is a root of unity element of $G$.

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