Huygens triviality of the time-independent Schrödinger equation. Applications to atomic and high energy physics

Arkady L. Kholodenko, Louis H. Kauffman

1 H.L. Hunter Laboratories, Clemson University, Clemson, SC 29634-0973, USA
2 Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago, 851 South Morgan Street, Chicago, IL, 60607-7045

Abstract

Huygens triviality-a concept invented by Jacques Hadamard-describes an equivalence class connecting those 2nd order partial differential equations which are transformable into the wave equation. In this work it is demonstrated, that the Schrödinger equation with the time-independent Hamiltonian belongs to such an equivalence class. The wave equation is the equation for which Huygens’ principle (HP) holds. The HP was a subject of confusion in both physics and mathematics literature for a long time. Not surprisingly, the role of this principle was obscured from the beginnings of quantum mechanics causing some theoretical and experimental misunderstandings. The purpose of this work is to bring the full clarity into this topic. By doing so, we obtained a large amount of new results related to uses of Lie sphere geometry, of twistors, of Dupin cyclides, of null electromagnetic fields, of AdS-CFT correspondence, of Penrose limits, of geometric algebra, etc. in physical problems ranging from the atomic to high energy physics and cosmology.

Keywords:
Huygens principle
Time-independent Schrödinger equation
Lie sphere geometry
Dupin cyclides

1. Motivation and background

It is well documented [1] that Schrödinger’s equation in its known form emerged only in the 4th installment of Schrödinger’s papers on quantum mechanics—all published in 1926. The Huygens principle was introduced in the 2nd installment. In the 1st installment the Hamilton-Jacobi (H-J) equation was used as point of departure. This equation was then used as an input in his variational derivation of the stationary Schrödinger’s equation. Although the variational way of obtaining this equation was subsequently endorsed by Courant and Hilbert [2], neither Schrödinger himself nor the rest of physics community were using this
(variational) way for obtaining the stationary Schrödinger’s equation. Instead, in the 2nd installment, being guided by results of De Broglie (his PhD thesis was completed in 1924), Schrödinger presented the following arguments. “Hamilton’s variational principle can be shown to correspond to the Fermat principle for wave propagation in configuration (q)-space, and the H-J equation expresses Huygens’ principle for this wave propagation.” In the same paper he also writes “The H-J equation corresponds to Huygens’ principle (in its old simple form, not in the form due to Kirchhoff).” Kirchhoff’s way of dealing with Huygens’s principle is discussed, for example, in the book by Baker and Copson [3]. Thus, for Schrödinger the Huygens principle is synonymous with the H-J equation. Later on, in 1948, Feynman [4] made the following comment about Huygens’ principle. In section 7, page 377, of [4] we read that the equation

\[
\psi(x_{k+1}, t + \epsilon) = \int \exp\left[\frac{i}{\hbar}S(x_{k+1}, x_k)\right]\psi(x_k, t)dx_k/A
\]  

(1.1)

“is easily interpreted physically as the expression of Huygens’ principle.” Here the index \(k = 0, 1, 2, ...,\) represents time ticks, \(\psi(x_k, t)\) is the wave function, \(A\) and \(\hbar\) are known constants, \(S(x_{k+1}, x_k)\) is the classical action between space-time points \(x_{k+1}\) and \(x_k, \epsilon \to 0^+\). Further down he writes: "Actually Huygens’ principle is not correct in optics. It is replaced by Kirchhoff’s modification which requires that both the amplitude and its derivative must be known on the adjacent surface. This is a consequence of the fact that the wave equation in optics is second order in the time. The wave equation of quantum mechanics is first order in time; therefore, Huygens’ principle is correct for matter waves, action replacing time.” From these quotations the question emerges: How the Huygens principle (HP) by Schrödinger is related to that by Feynman? The first attempt to provide mathematically satisfactory answer to this question was made by Gutzwiller [5]. His work is incomplete though as he acknowledges himself. Thus, this paper (and those which will follow in the sequel) is aimed at providing missing details. By doing so, it will become obvious why this topic is still of such profound importance. To demonstrate the importance, it is sufficient to recall some facts from Feynman’s lectures on physics [6] as well as from his book on path integrals [7]. They both begin with the discussion of the two-slit experiment. For the light this experiment was set up originally by Young in 1801 [8]. Its explanation involves uses of HP [4]. For electrons it was performed initially in 1961[9]. After this, use of other heavier particles, including C60 fullerenes, showed the same pattern as that observed by Young for light [10,11]. In view of these experimental results, it is appropriate to make some comments on Gutzwiller’s paper [5]. On page 54 we find the following statement: “The wave equation

\[
\left(\frac{\partial^2}{\partial t^2} \frac{1}{c^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}\right) \Psi = 0
\]  

(1.2)

implies Huygens’s principle which is the relativistic version of Feynman’s path integral\(^1\), valid

\(^1\)Perhaps Gutzviller had in mind that iteration of Eq.(1.1) (which is the epitome of Huygens’ principle for Feynman) is leading to the nonrelativistic path integral satisfying standard Schrödinger’s equation. This
in its usual form for Schrödinger’s equation

\[ i\hbar \frac{\partial \varphi}{\partial t} = -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \varphi + V \varphi = 0, \]

(1.3)

but neither Huygens’ principle nor the path-integral applies to the stationary waves which are the solutions of Eqs (1.4) and (1.5). In enumeration of this paper these equations are respectively given by

\[ \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \psi + \frac{8\pi^2 m}{\hbar^2} (E - V) \psi = 0 \]

(1.4)

and

\[ (\nabla^2 + k^2) \Psi = 0. \]

(1.5)

By making such a claim Gutzwiller contradicts Schrödinger who clearly stated that the H-J equation is mathematically restated Huygens’ principle. Eq.(1.4) is directly obtainable from the H-J equation as shown in Schrödinger’s 1st installment on quantum mechanics [1]. Below, in this paper we shall prove that Schrödinger’s definition of Huygens’ principle is indeed correct in a rigorous mathematical sense. If this is so, then if electrons and photons produce the same interference patterns in the double slit experiment, why then formalism developed in optics [8] cannot be applied to electrons and heavier particles? What makes use of Born’s probabilistic interpretation of the wave function in quantum mechanics superior to that used in optics for description of the two-slit experiment? This issue was carefully investigated by David Bohm in his classical monograph on quantum mechanics [12], pages 97-98. He found that differences do exist but just in few places. The updated comparison was recently made by Sanz and Miret-Artes in their book [13], chapters 4 and 7. From these chapters it follows that all objections made by Bohm in the remaining few places can be removed. The results presented in [13] along with references on which these results are based are incomplete to some extent. In [14] we eliminated this deficiency so that it should be read alongside with reading of this manuscript. Since nowadays the sophisticated quantum mechanical experiments are mainly done optically [15,16], ref. [14] supplies helpful additional guidelines for understanding of these experiments. In view of this, it makes sense to claim that our understanding of all subtleties of quantum mechanics is contingent upon our understanding of optics where Huygens’ principle (HP) is playing a very prominent role. Although the essence of HP is summarized in Definition 3.1. in section 3, details are essential. They are presented in sections 4.5 and Appendix B. In addition, the HP is linked with the conformal invariance of the Huygens-trivial equivalence class of equations. The notion of Huygens’ triviality was formulated by Hadamard (details are presented in sections 4 and 5 and in Appendix B). The equivalence class is made of all 2nd order partial differential

result, apparently, can be generalized to the relativistic case where Huygens’ principle in its conventional form holds. Such path integral is expected to satisfy the wave Eq.(1.2). This program was left unfulfilled by Gutzwiller [5].

2And other heavier particles
equations (PDE’s) which can be transformed in a prescribed way (described in the text) into standard wave equation obeying the HP. This equation is conformally invariant. Its conformal group is $SO(4,2)$. It is the largest symmetry group leaving invariant free Maxwell’s equations or, which is equivalent, the massless Klein-Gordon (or wave) equation [14]. $SO(4,2)$ is the conformal group of Minkowski spacetime [21]. It contains 4 translations, 6 rotations, 1 dilatation and 4 inversions. Hadamard conjectured that Huygens’-trivial equivalence class which he defined is the only one possible. Results of Appendix B demonstrate that this is not the case. Subsequent studies revealed that there are conformally invariant PDE operators which cannot be transformed into D’Alembertian using the transformation rules set up by Hadamard. But they do respect the HP nevertheless! Unexpectedly, their form is determined by the nature of cosmic gravitational plane-wave background. This is explained in the Appendix B.

Our work is made of 7 sections and 4 appendices. The role of appendices is not just auxiliary. Each of them serves as a nucleus for some further work. Therefore, they cannot be omitted upon first reading. In section 2, following the original but not widely known ideas by Schrödinger, we discuss the Schrödinger-style derivation of the uncertainty relations based on relativistic arguments. These arguments are such that they allow to restore the stationary Schrödinger equation. In section 3 we take into account that inclusion of time-dependence, that is replacement of the hyperbolic (wave) equation from which the stationary Shrödinger equation was derived, by the parabolic (truly Schrödinger) equation had occurred only in the 4th installment of Schrödinger’s papers on quantum mechanics [1]. Such an inclusion was associated with a lot of difficulties for Schrödinger. The biggest of this difficulties was the apparent departure from ideas by De Broglie on the physical nature of waves of matter. Clearly, the relativistic considerations leading to Schrödinger’s formulation of the uncertainty principle also need to be sacrificed (e.g. read the additional comments in the Appendix A). As result, the role of HP in the formalism of quantum mechanics had become obscured resulting in some erroneous statements. Subsequent theoretical and experimental works cited in section 3 contain erroneous claims requiring modification of the superposition principle of quantum mechanics-one of the pillars (e.g. the double slit experiment) of quantum mechanics. These circumstances caused us to write a very detailed section 4 describing the mathematical aspects of the HP. Additional information is contained in the Appendix B. In section 4 and Appendix C we find new solutions of the Schrödinger equation-Dupin cyclides. These are having both micro and macro (cosmological) importance. The micro importance ultimately originates from the seminal work of Madelung on hydrodynamical formulation of quantum mechanics which, to our knowledge, was never reproduced in its entirety in English. The cosmological significance of these solutions originates from the remarkable work by Roger Penrose discussed in the Appendix B and also briefly in the Appendix D. Thanks to work by Ward (e.g. read Appendix B), induced by work of Penrose, instead of cosmic microwave background we may think about the cosmic gravitational plane-wave background. It is this background which is the real cause for the Schrödinger equation to exist in its known form. Indeed, Dupin cyclides emerge from analysis of Madelung equations of quantum mechanics, on one hand and, of Huygens- nontrivial (as compared to Huygens-trivial) conformally in-
variant class of PDE’s which should exist in the gravitational plane-wave background, on another. Use of conformal transformations (not to be confused with transformations defined by Hadamard discussed in section 4) relates Hugens-nontrivial and trivial equations to each other.

To strengthen these conclusions, we also developed alternative paths for reaching the same goals. These are coming from the detailed study of group-theoretical properties of the Schrödinger equation initiated some time ago by Niederer [17]. By analogy with Newton’s mechanics, where Newton’s second law is invariant with respect to Galileo-type transformations, one would expect the same for the non relativistic Schrödinger’s equation. This happens not to be the case, however. This fact somewhat complicates the recovery of Newton’s equations from quantum mechanics via Ehrenfest theorem [18]. Studies by many authors had firmly established that the group of symmetries of, say, hydrogen atom is $SO(4,2)$ [19]. This happens to be symmetry of all atoms of periodic system of elements as well as of molecules (at least diatomic) [20,21]. But this symmetry group is the largest symmetry group which leaves invariant free Maxwell’s equations or, which is equivalent, the massless Klein-Gordon equation [14].The de Sitter and anti-de Sitter groups $SO(4,1)$ and $SO(3,2)$ are subgroups of $SO(4,2)$. These subgroups are not isometry groups of the Minkowski spacetime though. Details about these groups and their Lie algebras are summarized in [21]. These groups are also discussed in section 7 and Appendix D. It is fundamentally important that static Einsteinian spacetimes are conformally Minkowskian. Therefore they are invariant under the action of the conformal group $SO(4,2)$ [22]. The group $SO(4,2)$ is a typical representative of the so called dynamical symmetry groups and their spectrum generated algebras [19,23].The relationship between geometry of the underlying spacetimes and quantum mechanics was noticed and developed to a some extent already by Schrödinger. In his book [24] he made an attempt to extend his apparatus of quantum mechanics for flat spacetimes to curved spacetimes. He was interested in finding those spacetimes which permit quantum mechanics to exist. Since the group $SO(4,2)$ supports electromagnetic waves, that is dynamics of massless photons, the same should be true for all massless particles, e.g. neutrino, graviton, gluon, etc. Penrose’s twistor theory was developed initially for description of dynamics of such types of particles [25]. However, it was not used in the atomic and molecular physics because atoms and molecules are massive. This obstacle is possible to by pass as explained in section 6. Thus, in view of this, in this work we initiate use of twistors in the atomic physics.

The Huygens principle is linked with the conformal invariance. Details are provided in section 4 and the Appendix B. The conformal invariance is very nontrivially associated with the Lie sphere geometry. Details on this geometry are given in section 7. By design, this geometry transforms circles into circles and spheres into spheres. Since points are spheres of zero radius and hyperplanes are (hyper)spheres of infinite radius the Dupin cyclides are covering all surfaces made of spheres whose centres are moving along some prescribed curve $c(t)$ parametrized by time $t$ so that the radii of spheres $r(t)$ also change with time. By design, thus made surfaces are invariants of the Lie sphere geometry. The simplest possible examples are: spheres, planes, cones and cylinders. Use of Möbius transformations (this is the subgroup of the Lie sphere transformations group) transforms these simple objects into the whole variety
of Dupin cyclides which stay invariant under the Lie sphere transformations. In the same section 4 we discovered an unusual property of the stationary Schrödinger equation which makes this equation to be treatable gauge-theoretically by employing deep mathematical results of Andreas Floer. This result came as a by product of our efforts to adopt the progressive wave solution method by Friedlander [70] to the stationary Schrödinger equation. His method was applied initially to the conformally invariant D’Alembert wave equation resulting in Dupin cyclides as solutions. In section 4 using mathematically formulated Huygens’ principle we demonstrated that the Dupin cyclides are also valid solutions of the stationary Schrödinger equation. The same result was reobtained in the Appendix C by different methods. In doing so we were motivated by the results obtained in sections 5 and 6. Section 5 begins with removing the mass parameter from the relativistic Klein-Gordon (K-G) equation using Hadamard -type transformations. These are making the K-G equation Huygens-trivial. Next, we notice that every spinor component originating from the Dirac equation is obeying the K-G equation. This means that the massive Dirac equation can also be made Huygens-trivial. Next, in this and in section 6, we discuss all relativistically invariant massive equations with integer or half integer spins and conclude that they are also Huygens-trivial. We continue our study of Huygens triviality in section 6 where we demonstrate Huygens triviality of the stationary Schrödinger equation being influenced by the seminal paper by Vladimir Fock published in 1935 [101],[102]. His goal was not to demonstrate Huygens triviality of the Schrödinger equation (say, for the hydrogen atom). Nevertheless, he came very close to this task. Huygens triviality of the stationary Schrödinger equation enabled us to use twistor methods for solving hydrogen atom problem in section 6 while in section 7 following ideas of Sophus Lee and Felix Klein we demonstrate the isomorphism between the twistor methods and that of the Lie sphere geometry. In the same section we discuss some physical applications such as the interrelation between the Dupin cyclides and torus knots. Such unexpected utility of the Lie sphere geometry we pushed further. In section 7 and the Appendix D, we demonstrated its major role in establishing the AdS-CFT correspondence and in establishing the Penrose limits.

2. Schrödinger-style derivation of uncertainty relations leading to the Schrödinger equation

2.1. Schrödinger-style derivation of uncertainty relations. Role of relativistic arguments

According to Feynman [6], the formalism of quantum mechanics is lying on two pillars: a) Heisenberg’s uncertainty principle and b) the two-slit experiments with photons, electrons, etc. Being guided by these ideas, we found it very illuminating to develop these ideas from scratch based on some considerably lesser known writings by Schrödinger. In an obscure publication [26] he sketched a derivation of the uncertainty relations not found in any other textbooks on quantum mechanics. We begin, however, with equally interesting Schrödinger’s remarks on page 50 of the same reference. "Now the special technique by which classical mechanics dodges the awkward fact of indeterminateness (the fact that equal initial conditions
are followed by different consequences) consists in including initial velocity within the initial conditions...the initial velocity is taken as forming part of the initial condition at any given moment. Velocity, after all, is defined as a differential quotient with respect to time

\[ \frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{x_2 - x_1}{\Delta t}; \Delta t = t_2 - t_1. \]  

(2.1)

This definition refers to two moments of time and not to the state at one moment... It may be that the mathematical apparatus devised by Newton is inadequately adapted to nature; and the modern claim that the concept of velocity becomes meaningless for a precisely defined position in space points strongly in that direction.” These arguments by Schrödinger were taken into consideration seriously only recently, e.g. in 2002 in the book “Quantum Calculus” [27]. Obviously, his arguments serve as precursors for arguments leading to the uncertainty relations. Surprisingly, no such relations can be found in “Quantum Calculus” [27] while in the book by Schrödinger [26], on page 126, we find the following comments: ”...according to the fundamental equation of the Quantum theory

\[ E = h\nu. \]  

(2.2)

To measure frequency we need a certain time. Let us think of the primitive procedure of counting \( n \) vibrations within a definite time \( \Delta t \). Then,

\[ \nu = \frac{n}{\Delta t} \]  

(2.3)

but manifestly with a possible error of \( \Delta \nu = \frac{1}{\Delta t} \) because the process of counting necessarily results in giving the whole number, which is subject to an error of \( \pm 1/2 \). This entails a possible error with respect to energy of \( \Delta E = \frac{h}{\nu} \); hence \( \Delta t \cdot \Delta E = h \)... Now, relativistically, the energy is the fourth component of the energy-momentum vector....Therefore the uncertainty relation can be transferred to the other components as well, for example:

\[ \Delta x \cdot \Delta p_x = h, \]  

(2.4)

where \( p_x \) is the momentum in the \( x \)-direction.” The same result Schrödinger obtains differently on page 129. There, he writes:

”We need but replace the particle by a wave-group and let the wave-length \( \lambda \) and the momentum \( p \) have a relation

\[ p = \frac{h}{\lambda}. \]  

(2.5)

---

3In mechanics (our observation)
4Here \( E \) is energy, \( h \) is Planck’s constant \( h = 2\pi\hbar \) and \( \nu \) is frequency.
5Emphasis is ours
6This is clear from the dispersion relation for the light: \( \omega = ck \) or \( \nu = c/\lambda \). Here \( c \) is speed of light. Multiplying both sides of this dispersion relation by \( h \) we obtain: \( E = cp \), where \( p \) is the same as in Eq.(2.5). Multiplying both sides of Eq.(2.5) by \( c \) and taking into account that \( \nu = c/\lambda \), we obtain: \( cp = h\nu = E \). Therefore, it follows that the De Broglie relation is also valid for light, that is for the massless particle.
In order to build such a group a certain $\lambda$–interval is required. Let $\Delta x$ be the length of the group, then the ratio must be allowed to vary by one unit. Thus:

$$\Delta x \cdot \Delta \left( \frac{1}{\lambda} \right) = 1.$$ 

Multiply by $h$, then $h \cdot \Delta \left( \frac{1}{\lambda} \right)$ is the uncertainty $\Delta p$. And so, $\Delta x \cdot \Delta p = h$.

From these extensive quotations from writings by Schrödinger it follows that:

a) the relationship $\Delta t \cdot \Delta E = h$ implies the relationship $\Delta x \cdot \Delta p = h$;

b) the relationship $\Delta t \cdot \Delta E = h$ is obtained with the assumption that the motion is periodic;

c) the De Broglie wavy relation (2.5) is consistent with the relation (2.4) obtained with the help of relativistic arguments.

It happens, that the stationary Schrödinger’s equation can be obtained based on these three observations only. This can be seen from reading of Schrödinger’s Physical Review paper.[28] In it, Schrödinger acknowledges that his theory came as result of elaboration on works by De Broglie. It is helpful to reproduce Schrödinger’s arguments in order to emphasize some essential elements which were overlooked subsequently in physics literature. Specifically, the connection between the nonrelativistic Schrödinger equation and the relativistic mechanics sketched by Schrödinger has not found its well deserved place in physics literature to our knowledge.

2.2. From uncertainty relations to stationary Schrödinger’s equation

Relations a) and c) of previous subsection are consistent with the relativistically invariant scalar wave equation

$$\Box_4 \phi(x, y, z; t) = 0,$$

(2.6a)

where the D’Alembertian $\Box_4$ is given by

$$\Box_4 = \frac{\partial^2}{\partial t^2} \frac{1}{c^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}.$$  

(2.6b)

Here, as before, $c$ is the speed of light in the vacuum. The simplest solution of (2.6a) is given by the plane wave $\phi(x, y, z; t) = A \exp\{i(\omega t \pm k \cdot x)\}, x = (x, y, z), A = const$ [29]. When it is substituted into Eq.(2.6a), it leads to the dispersion relation

$$\omega^2 - k^2 c^2 = 0$$  

(2.7)

implying $\omega = \pm |k| c$. However, $E = h\omega$ and $p = hk$. Therefore, we obtain: $E = \pm |p| c$. This result was mentioned in the previous subsection. Let furthermore $E' = E + \Delta E$ and $|p'| = |p| + |\Delta p|$, then $\Delta E = |\Delta p| c$. Since $\Delta E \simeq \frac{h}{\Delta x}$, we obtain: $\frac{h}{\Delta x} = |\Delta p| c$, or $|\Delta p| c \Delta t = |\Delta p| |\Delta x| \simeq h$, in accord with Schrödinger [26]. The question arises: Are there solutions of (2.6a) other than plane wave(s) or their linear combinations? This issue was studied in detail.
in [29]. Evidently, the plane waves are valid solutions in the vacuum. But there are other

types, e.g. progressing waves, to be discussed below in section 4.2., etc. If the waves are

propagating in some (inhomogeneous) medium with refractive index \( n(x, y, z) \), the results

from optics require us to replace \( c \) by \( c/n \). In such a case the simple plane wave solution is no

longer suitable and should be replaced by \( \phi(x, y, z; t) = A \exp\{i(2\pi E t)/\hbar\} \psi(x, y, z) \). Clearly,

\[
\frac{2\pi E}{\hbar} = \frac{2\pi \omega}{\hbar} = \omega, \quad \hbar \omega = h\nu.
\]

Substitution of this ansatz into Eq.(2.6a) leads to the equation

\[
\left( \frac{\omega^2 n^2}{c^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \psi(x, y, z) = 0. \tag{2.8}
\]

However, \( 2\pi \nu = \omega \). Therefore, since (in view of Eq.(2.5)) \( \nu^2 = \frac{\hbar}{\lambda} \left( \frac{c}{n} \right)^2 \), we obtain :

\[
\frac{\omega^2 n^2}{c^2} = \frac{4\pi^2}{\lambda^2}.
\]

Using the De Broglie relation \( p = \frac{\hbar}{\lambda} \) and keeping in mind that the total energy \( E \) of the
dynamical system made of a particle moving in the potential \( V \)

\[
E = \frac{p^2}{2m} + V(x, y, z), \tag{2.9a}
\]

we obtain:

\[
p = \sqrt{2m(E - V)}. \tag{2.9b}
\]

With help of this result, Eq.(2.8) acquires the familiar form of stationary Schrödinger’s

Eq.(1.4). In such a form this equation was used by Schrödinger in his first three communications

on quantum mechanics [1] (not to be confused with his paper published in Physical Review [28]).

Only in the 4th paper he found a way of introducing the time dependence correctly. This topic is discussed in the next section.

3. Inclusion of time-dependence into Schrödinger’s equation by Schrödinger and its impact on the latest experimental and theoretical works

Inclusion of time-dependence into Schrödinger’s equation was made only in the 4th installement of Schrödinger’s series of papers on quantum mechanics [1]. There is a good reason why this was done so late. It will be explained at length in the companion paper. In this paper we employ physical arguments, beginning with those by Schrödinger. On page 103 of [1] we find the following comment :

"Thus, when we designated equation (1) or (1') on various occasions as "the wave equation", we were really wrong...". To fix this problem (that is to make things right) Schrödinger suggest to call Eq.(1.4) as the "amplitude equation." He acknowledges, that such an equation is valid only for the conservative (that is time-independent) systems. Any process of measurement involves, however, time-dependent perturbation [31]. Accordingly,
"we must search for the real wave equation" since the amplitude equation is no longer sufficient. Thus, Eq.(1.4) is no longer a real wave equation for Schrödinger. Furthermore, on page 102 he writes: "Equation (11) contains the energy-or frequency-parameter $E$, and is valid, as expressly emphasized in Part II, with a definite $E$-value." Here the italics are Schrödinger's. This statement by Schrödinger is very confusing, nevertheless, since in Eq.(1.4) the parameter $E$ is determined by solving the respective eigenvalue problems. And this is exactly what Schrödinger did repeatedly, starting with his 1st installment on quantum mechanics! Next, he suggests several ways of arriving at the "real wave equation" which we do not want to reproduce here since these results are incorrect (as Schrödinger acknowledges himself). After several wrong attempts, he finally writes: "The dependence of $\psi$ on the time which must exist if (1) to hold, can be expressed by

$$\frac{\partial \psi}{\partial t} = \pm \frac{2\pi i}{\hbar} E \psi$$  \hspace{1cm} (3.1a)$$
as well as by

$$\frac{\partial^2 \psi}{\partial t^2} = -\frac{4\pi^2}{\hbar^2} E^2 \psi.$$ \hspace{1cm} (3.1b)$$

Obviously, "as well as" only means that: a) either his statement is erroneous (since equations (3.1a) and (3.1b) are mutually exclusive because Eq.(3.1a) is leading to the parabolic-type partial differential equation (PDE) while Eq.(3.1b) is leading to the hyperbolic PDE, b) he had in mind to use two times for the same equation, or c) he suggested to replace (in his opinion, definite value) the parameter $E$ (present in the combination $E\psi$) in his "amplitude" equation (1.4) by $\pm i\hbar \frac{\partial \psi}{\partial t}$ in order to arrive at his "true wave equation." Finally, the option c) was selected by Schrödinger so that the obtained equation had become a "true wave equation", known now as Schrödinger's equation. Selection of option c) disconnects this equation from the wave equation (2.8) and makes use of the De Broglie particle-wave arguments much more difficult to implement. On purely mathematical level, selection of the hyperbolic type equation (based on selection of Eq.(3.1b)) requires knowledge of the two initial conditions for the Cauchy problem to be set up correctly, while selection of Eq.(3.1a) requires to use just one initial condition [29] for setting up the Cauchy problem as explained in Appendix A. This difference was noticed by Feynman. In section 1 we brought the following Feynman's quotation [4], page 377: "Actually Huygens’ principle is not correct in optics. It is replaced by Kirchhoff’s modification which requires that both the amplitude and its derivative must be known on the adjacent surface. This is a consequence of the fact that the wave equation in optics is second order in the time. The wave equation of quantum mechanics is first order in time; therefore, Huygens’ principle is correct for matter waves..."
Unfortunately, this statement by Feynman is not correct as we shall explain below. To begin our explanation, we start with some quotations from the book by Feynman and Hibbs on path integrals [7] as well as from Feynman’s lectures on physics [6]. In section 1-1 of [6], on the 1st page, we find the following statement: “There is one lucky break, however—electrons behave just like light. The quantum behavior of atomic objects (electrons, protons, neutrons, and so on) is the same for all, they are all “particle waves”... So, what we learn about the properties of electrons...we will apply also to all “particles” including photons of light.” The interpretation of outcomes of the double slit interference experiment is central for understanding of quantum mechanics as explained in section 1 and in [14]. Because the light is being treated as ”particle waves” by Feynman, we shall begin with the light in the context of the double slit experiment in optics. This experiment was described in detail long before quantum mechanics was born [3,8, 33]. It involves use of Huygens’ principle. Mathematically accurate definition of this principle is given in the next section. For now it is sufficient to think about this principle as follows.

Definition 3.1. The Huygens principle. The new wave front at later time \( t' \) is the envelope of secondary waves emanating from each point of the original wavefront.

Stated in such a form it leaves entirely open the problem: Why the envelope of secondary waves is moving only forward? Later on Fresnel added a superposition principle for the amplitudes of the secondary waves to explain the phenomenon of diffraction. Subsequently, the same problem was looked upon by Kirchhoff, Fraunhofer, Beltrami, Volterra, Hadamard, Arnold, and many others. In optics literature the description of diffraction begins with study of the scalar Helmholtz equation

\[
(\nabla^2 + k^2) \Psi = 0. \tag{3.2}
\]

If we replace \( \frac{c}{n} \) in (2.8) by \( \hat{c} \) and introduce \( k = \omega/\hat{c} \) then, Eq.(2.8) becomes the Helmholtz equation, provided that the potential \( V = 0 \). In such a form Eq.(3.2) is used in scattering theory of quantum mechanics [34]. It is appropriate to notice at this point that it is commonly believed that only Schrödinger’s ansatz, Eq.(3.1a), makes the wave function complex and, with this, it makes Born’s interpretation of quantum mechanics possible. This opinion is erroneous though since the simplest plane wave solution of Eq.(3.2) is known to be [34]

\[
\Psi = \exp(\pm i k \cdot r) \tag{3.3}
\]

leading to the probability current

\[
\mathbf{j} = \frac{\hbar}{2mi} (\Psi^\dagger \nabla \Psi - \Psi \nabla \Psi^\dagger). \tag{3.4}
\]

The question arises: Is it possible to find an analog to this current in optics? From the book by Born and Wolf [8], chapter 8, it follows that the analog of the current in optics can be obtained

\footnote{Here italic emphasis is ours. Historically, the events occurred just in reverse order though, as explained in section 1.}
as result of application of Green’s theorem, also used in quantum mechanics for derivation of the current, Eq.(3.4). In optics $\Psi$ is an amplitude of the wave. Experimentally the intensity is measured. It is expressible through the combination $\Psi^*\Psi$. In optics it is possible to replace the vector wave equations with the scalar ones when diffraction is discussed. Eq.(3.2) is the scalar wave equation for $\Psi$. In optics the intensity $\Psi^*\Psi$ is measured experimentally while in quantum mechanics the very same quantity is the probability. Very detailed analysis of similarities and differences between optics and quantum mechanics was made in our work, ref.[14]. It was stimulated by the footnote on page 387 of [8]. In it, the reference to the 1959 work by Emil Wolf was made. In this work it is demonstrated “that both the (time averaged) energy density and the energy flow in unpolarized quasimonochromatic (optical) wave field may always be derived from one component complex and time-harmonic scalar wave function.” Neither Born nor Wolf attempted to develop these observations subsequently. This was done in [14] where it is demonstrated that Born probabilistic interpretation of quantum mechanics and the optical interpretation (especially for the double-slit experiments) can be made mathematically completely indistinguishable. Initially, the discrepancy of interpretations of quantum mechanics was noticed by De Broglie. On Page 127 of his book [35] we find the following statement: “The probability that the presence of the photon will be made known by photographic action in the apparatus is everywhere proportional to the resultant intensity of the wavetrain.....It is almost certain that the same considerations are valid for the diffraction of material particles.” Thus, results of [14] provide rigorous mathematical support to De Broglie’s intuition.

From Feynman’s lectures [6], it follows that in all double slit experiments the detectors were used (connected with loudspeakers) to produce "clicks." The number of clicks was counted per unit time as a function of the position ($x$) of the detector. Clearly, the detectors register particle hits. These are not exactly the probability amplitudes but, traditionally treated, they are represented through these amplitudes nevertheless! These amplitudes were experimentally calculated with the help of protocol described in section 1-1 of [6] and, indeed, when compared against the interference patterns for light, the complete agreement was found, e.g. read page 8 of section 1-1[6] and page 5 of [7]. Because of this, further steps were made. On pages 5 and 6 of [7], without saying it explicitly, Feynman formulates the Huygens-Fresnel principle in accord with the results known from optics [8], page 371. Specifically, on pages 5-6 of [7] we find the following description of the 2 hole experiment: “Furthermore, $\phi(x)$ is the sum of two contributions: $\phi_1$ the amplitude of arrival through hole 1, plus $\phi_2$, the amplitude of arrival through hole 2. In other words, there are complex numbers $\phi_1$ and $\phi_2$ such that $P = |\phi|^2$, $\phi = \phi_1 + \phi_2$ and $P_1 = |\phi_1|^2$, $P_2 = |\phi_2|^2$... Here we say only that $\phi_1$, for example, may be evaluated as a solution of a wave equation representing waves spreading from the source to 1, and from 1 to $x$. This reflects the wave properties of electrons (or in the case of light, photons). But the ”motion” of photons is described

\[18\] The italics are ours.

\[19\] This is exactly Huygens’ principle in the form of the ”major premise” by Hadamard described in Appendix B.

\[20\] Here the emphasis is ours.
by Eq.(2.6 a), while electrons-by Schrödinger’s wave equation Eq.(1.3). Hence, the mathematical description is visibly different: Two different equations- the hyperbolic (wave) and the parabolic (Schrödinger) cannot describe the same reality in the same way! The confusion is caused by the fact that the Helmholtz Eq.(3.2) is used both in optics and in quantum mechanics, e.g. read survey on atomic optics [36]. When time-dependence effects can be ignored, then mathematically these equations are indistinguishable! With this observation in our hands we came all the way back to Schrödinger’s dilemma: which one of the equations: Eq.(3.1a) or (3.1b), should be used? Please, recall that the switch from (3.1b) to (3.1a) was caused by Schrödinger’s desire to develop correct description of the time-dependent quantum phenomena. For the time-independent situations, however, it is completely safe to use the wave Eq.(2.8) producing the stationary Schrödinger Eq.(1.4). In this and the following (time-dependent) paper we shall resolve Schrödinger’s dilemma by treating the time-dependent case in such a way that the time-dependence is rigorously eliminated. In such a case, even for the time-dependent situations to be treated in the next paper we can begin with the wave Eq.(2.6a) and, by doing so, we shall bring into correspondence treatments of photons and the rest of "particle waves" (in Feynman’s terminology).

As an input and guidance for what follows, we need now to comment on several recently published papers—all reflecting the existing confusion associated with uses of various definitions of Huygens’ principle in quantum mechanics. The confusion is exacerbated by several additional sources not described above. To illustrate our point, we begin with the Helmholtz Eq.(3.2). The Green’s function \( G(r, r') \) for this equation is obtained from the equation

\[
(\nabla^2 + k^2) G(r, r') = \delta(r - r').
\]  

(3.5)

From quantum mechanics [34], we obtain (for the outgoing standing wave)

\[
G_+(r, r') = \frac{-\exp(ik|r - r'|)}{4\pi |r - r'|}
\]  

(3.6)

(for the incoming wave \( k \) should be replaced by \( -k \) in the exponent). Now, \( G_+(r, r') \) is not at all the Green’s function of the wave equation. Accordingly, one cannot use the Fresnel-Huygens principle in the form hinted by Feynman and implemented in [37]. Specifically, what the authors of [37] are calling as the Huygens-Fresnel principle

\[
G_+(r_x, r_0) = \int d\mathbf{r}_1 G_+(r_x, \mathbf{r}_1)G_+(\mathbf{r}_1, r_0)
\]  

(3.7)

is not all mathematically valid expression! Accordingly, the results of [37] are not valid. Since these results question the double slit experiment lying at the heart of quantum mechanics, we now provide the detailed explanation of what went wrong with the results of [37]. First of all, the propagator \( G_+(r, r') \) does not admit the path integral representation and, accordingly, it does not posses the Markovian property essential for all path integral treatments. This could be seen by direct calculation of Eq.(3.7) in which Eq.(3.6) is used. For the record we quote the result from the book by Ito and McKean [38], paragraph 7.21, containing a remarkable formula.
connecting the Brownian motion propagator with the Newton (or Coulombic) potential

\[ \int_0^\infty dt (2\pi t)^{-\frac{d}{2}} \exp\left\{ -\frac{|b-a|^2}{2t} \right\} = \text{const} |b-a|^{2-d}. \]  

(3.8)

Here \( d \) is the dimensionality of space, \( d \geq 3 \). On the left hand side of this equality we find well known Feynman’s propagator for the free particle of unit mass upon transition \( t \to i\hbar t \). Accordingly, for this propagator one can safely use Eq.(3.7). Unfortunately, Eq.(3.8) cannot be replaced by that involving the Yukawa-type potential, Eq.(3.6). While the attempts to find an identity analogous to Eq.(3.8) for the Yukawa-type potentials, e.g. Eq.(3.6), are still ongoing, it is obvious that, based on current knowledge of this topic, Eq.(3.7) is invalid. Because of this, the main statement of [37] questioning correctness of the superposition principle of quantum mechanics is also incorrect. This is so because of the following. The quantum mechanical superposition principle for two slits A and B in its orthodox form reads: \( \psi_{AB} = \psi_A + \psi_B \). However, the authors of [37], following some earlier works, write instead \( \psi_{AB} = \psi_A + \psi_B + \psi_L \). The third term, supposedly, originates from the entanglement of the Brownian paths originating at the source with two slits, e.g. A and B. That is to say, \( \psi_L \) accounts for the possibility of the Brownian path going through the slits and coming back to the source/the origin. However, according to Huygens’ principle the secondary waves can move only forward! According to Feynman and the cited experiments, both the light and the particle waves produce the same interference patterns. In optics the description of such patterns does involve uses of Huygens’ principle. Therefore, the same should be true for the "particle waves." Accordingly, the equality \( \psi_{AB} = \psi_A + \psi_B + \psi_L \) is incorrect. In many papers attempting to use Feynman’s path integrals for description of the double and triple slit experiments, e.g. read [39] and references therein, the authors used the Markovian analog of Eq.(3.7) (e.g. see Eq.(2.31) on page 37 of the book by Feynman and Hibbs [7]). Use of this analog is illegitimate though because the Brownian propagator in Eq.(3.8) when integrated over time is not the propagator for the wave equation. The parabolic time-dependent Schrödinger equation is surely not the same thing as the hyperbolic wave Eq.(2.6a) since in the first case we need just one initial condition while in the second-two. For the wave equation Huygens’s principle rigorously holds (as explained in the next section and Appendix B). An attempt at rigorous implementation of Huygens’ principle in quantum mechanics formulated in terms of Feynman path integrals was made by Gutzwiller [5] whose sketchy and incomplete results were discussed in the 1st section.

Replacement of the hyperbolic (wave) equation by the parabolic (Schrödinger) could be a source of wrong conclusions about the outcome of three-slit experiments with photons. These were reported in "Science" in 2010 [40] with the purpose of rigorous testing of the superposition principle. These experiments found no deviations from the superposition principle of quantum mechanics. The quality of these experimental studies was further improved in [41] with the same outcome thus confirming the validity of both, Huygens’ and superposition principles. Subsequent analytical and computer studies of the three slit configurations [39], culminating in the latest PRL [37] -all criticized the experimental results of Sinha et al [40]
as well as those by Söllner et al [41]. This critique is invalid however for reasons already explained. We noticed in section 2 that the De Broglie relation \( p = \frac{\hbar}{\lambda} \) is valid also for photons. But the dispersion relation for photons is \( \omega = ck \) while that for the Schrödinger’s equation for the free particle of mass \( m \) is \( \omega = \frac{\hbar k^2}{2m} \) (e.g. read Appendix A). This difference in dispersion relations is equivalent to the replacement of Eq.(3.1b) by (3.1a) as done by Schrödinger. It is fundamental since only under such conditions Heisenberg and Schrödinger pictures of quantum mechanics are in agreement with each other! The paradoxicality of the existing situation can be understood by reading some earlier descriptions of Huygens’ principle, the descriptions of diffraction, etc. [3]. They all involved the wave equation and time-dependence. However, this time-dependence happened to be not essential as could be seen, for example, by reading the authoritative book on optics by Arnold Sommerfeld [42]. Subsequently, the time-dependence was wiped out from books on optics [8,43]. Once it was wiped out, use of the time-independent optics formalism, e.g. describing the double slit diffraction, makes it possible to replace the optical formulas by the quantum mechanical ones. More details are given in [14]. Inclusion of time dependence leads to dramatic effects. Details will be provided in future publications. This can be seeing already from the observation that switching from the hyperbolic (wave) Eq. (2.6a) requiring two functions for the initial conditions (the Cauchy data) to the parabolic (Schrödinger) equation requiring only one function is highly nontrivial, e.g. read subsection 6.2.1. below. Being parabolic, the Schrödinger equation possess the Markovian property while the wave equation is not. To by pass this difficulty, the semigroup analysis of operators was invented [44] In section 1 we quoted Gutzviller [5] who said that “neither Huygens’ principle nor the path-integral applies to the stationary waves which are the solutions of Eq.s (1.4) and (1.5)” . This statement of Gutzviller happens to be incorrect as we shall demonstrate below. Furthermore, in [5] Gutzwiller was using the semigroup analysis [44] for claiming that the time-dependent Schrödinger equation is obtainable from the relativistic (wave-like, a la Dirac, equation) in the limit when \( c \to \infty \). First of all, this limit is unphysical (see, however, Appendix A) and, second of all, we shall demonstrate that Gutzwiller’s claims regarding applicability of Huygens’ principle to Eq.s (1.4) and (1.5) is also incorrect. However, his claims about the Markovian property of path integrals are completely consistent with ours. Eq.(3.7) is surely non Markovian. There is no way of rewriting Eq.(3.7) in the path integral form. Thus, the Markovian property is not coinciding with the Huygens’ principle property! The Hugens principle (and property) in its refined form permits propagation of wave packets only forward (as explained in the following section) while Schrodinger’s equation allows spreading of the wave packets both ways [45]. Interestingly enough, the Huygens principle had been put by Schrödinger into center of his developments of wave mechanics. This can be seen from reading of Part II [1], pages 13-40, of his four foundational installments on quantum mechanics. In view of the variety of opinions in physics literature about the essence of Huygens’ principle (as compared to the variety of opinions about Huygens’ principle prevailing in mathematics literature), in the next section (and, in Appendix B) we are presenting some basic facts about Huygens’ principle as it is

\[ ^{21} \text{Also, read the subsection 6.2.1 and ref.[14] where the Duffin-Kemmer formalism is briefly discussed leading to analogous results.} \]
developed by mathematicians.

4. Basic facts about Huygens’ principle (mathematicians perspective)

4.1. Wavy (that is partial differential equations )-type versus contact-geometric-type aspects of Huygens’ principle

Definition 3.1. provides the essence of Huygens’ principle. In plain words Huygens’ principle can be concisely formulated as follows. Consider a point at the wave front at the moment \( t \) as the source of a new (secondary) wave emanating from this point. The Huygens principle (HP) as formulated by Huygens himself leaves entirely open the problem: Why the envelope of secondary waves is moving only forward? In the previous section we stated that to fix this problem subsequently Fresnel added a superposition principle for amplitudes of the secondary waves. This helped him to explain the phenomenon of diffraction. We had already mentioned that the same problem was looked upon later by Kirchhoff, Beltrami, Volterra, Hadamard, Arnol’ d, and many others. Different authors used the HP with different purposes in mind. This caused differences in interpretation and confusion among users. Basically, the split in interpretations originates from the split of opinions about what is light. If we take the side of proponents of the wavy nature of light, then it is instructive to read carefully the fundamental work by Hadamard [46]. Read also [29], chapter 6. If we take the side of proponents of corpuscular nature of light, then it is instructive to read works by Arnol’d [47-49] in which methods of contact geometry and topology are used. To our knowledge, the detailed connections between these two directions of thought still do not exist. Works by Maslov [50,51] indicate that the connection problem is solved, in principle.

It is appropriate again to bring the quotation by Schrödinger on this topic [26]. On page 154 he writes:22 The light ray, or track of the particle, corresponds to the longitudinal continuity of the propagating process (that is to say, in the direction of spreading); the wave front, on the other hand, to the transversal one, that is to say, perpendicular to the direction of spreading. Both continuities are undoubtedly real. The one has been proved by photographing of particle tracks, and the other by interference experiments. As yet we have not been able to bring the two together into uniform scheme. It is only in extreme cases that the transversal -the spherical-continuity or the longitudinal -the ray-continuity shows itself so predominantly that we believe we can avail ourselves either of the wave scheme or the particle scheme.” Here the italics are Schrödinger’s. To complete this chain of quotations we cite Arnold’s "Lectures on Partial Differential Equations" [48]. At the beginning of Lecture 1 he writes: "In this lecture we shall consider a case in which there is a complete theory, namely the case of one first order equation. From the physical point of view this case (displays) the duality that occurs in describing a phenomenon using waves or particles. The field (of waves) satisfies a certain first-order partial differential equation, the evolution of the particles is described by ordinary differential equations, and there is a method of reducing the partial

\[ \text{This is conclusion of his Nobel Address delivered on December 12th, 1933.} \]
differential equation to a system of ordinary differential equations; in that way one can reduce the study of wave propagation to the study of the evolution of particles.\textsuperscript{23}

In view of particle-wave duality just described it is appropriate to define the dual of Huygens’ principle. In the theory of ordinary differential equations \textsuperscript{52,53} (that is for “rays” in optical terminology) there is so called rectification theorem which, when translated into language of Hamiltonian dynamics, is known as symplectic rectification theorem \textsuperscript{54}. In plain words it is the statement which can formulated as follows.

Definition 4.1. The dual Huygens’ principle. Any point on the dynamical trajectory\textsuperscript{24} can serve as the origin of (new) motion.

That is to say, at any point along the trajectory it is possible to find canonical variables which are making the Hamiltonian of the dynamical system to vanish. At all such points along the trajectory the canonical Hamiltonian equations become trivial. When such dynamical system is quantized, the unitary operator describing system evolution becomes trivial too. Since the dual Huygens’ principle applies to any point \((p,q)\) along the trajectory in phase space, this means that Heisenberg’s uncertainty principle fails. The way out of this difficulty was suggested by Maslov \textsuperscript{50} who suggested to use the so called Lagrangian manifolds \(\mathcal{L}\). These manifolds naturally arise in contact geometry \textsuperscript{47-49}. An introduction to this field can be found in \textsuperscript{55}. In the context of Hamilton-Jacobi formalism the Lagrangian manifolds naturally originate based on familiar equations of the type \(p_i = \frac{\partial S(\{q_j\},t)}{\partial q_i}\). Here \(i, j = 1, \ldots, N\), \(N\) is the dimension of the configurational (tangent) space, \(S\) is the action of the dynamical system. For the prescribed \(p_i'\)'s just stated set of equations defines the Lagrangian manifold \(\mathcal{L}\). Given that the dimension of the symplectic manifold is \(2N\), the dimension of the Lagrangian manifold is \(N\). Since the symplectic manifold is determined locally by the 1-form \(\theta = \sum_{i=1}^{N} p_i dq_i\), the Lagrangian manifold is determined by the requirement: \(d\theta = \sum_{i=1}^{N} dp_i \wedge dq_i = 0\). At the same time, the contact 1-form \(\alpha\) is determined by \(\alpha = dS - \sum_{i=1}^{N} p_i dq\) while the contact plane is determined by the condition \(\alpha = 0\). The Bohr-Sommerfeld quantization conditions for trajectories on \(\mathcal{L}\) are trivial, that is \(\oint p_i dq_i = 0\), since by design \(\{p,q\} = 0\) on \(\mathcal{L}\) where \(\{,\}\) is the Poisson bracket. Existence of the Lagrangian manifolds does not preclude the existence of quantum mechanics, at least in the asymptotic sense \textsuperscript{50}, even though the Groenvold-van Hove theorem \textsuperscript{56,57} causes severe difficulties in systematic development of quantum mechanics.

Resolution, if any, of the above mentioned difficulties caused us to discuss only the wavy (that is, the PDE) side of Huygens’ principle \textsuperscript{29,33}. The dual (particle) side requires uses of methods of contact geometry and topology as we just explained. Much more detailed account of this side of the Huygens principle is given in \textsuperscript{58}, Chr.10. Reference \textsuperscript{59} contains some alternative treatment while very readable but rigorous ref.\textsuperscript{60} claims along with Schrödinger \textsuperscript{1}, part II, that Huygens’ principle and the Hamilton-Jacobi equation are

\textsuperscript{23}Words put in curly brackets as well as italic emphasis are ours
\textsuperscript{24}That is ray trajectory
We begin our discussion of the wavy aspects of the HP by summarizing results by Hadamard [46] In ref.[46] he formulated the following

**Hadamard problem.** How to describe/to classify all second order hyperbolic partial differential equations satisfying HP?

Hadamard discovered that the HP is working only in spaces in which the number of spatial dimensions is odd, e.g. in 3+1 spacetimes of Lorentzian signature. From here, it follows that the behavior of solutions of wave equations in 2+1 and 3+1 dimensional spaces is quite different. A short time point-like perturbation in $\mathbb{R}^3$ will produce sharply defined moving spherical wave so that when it reaches the observer he/she will see (or hear) just a short lasting flash/or splash. At the same time, for waves on the surface of water (this is an example used by Feynman, e.g. read the discussion pertinent to Fig.1-2 in [6] for his preliminary description of the two-slit experiment), that is in $\mathbb{R}^2$, the whole region inside the moving (spreading) circle will be disturbed. They say, that the 3 dimensional event is the result of the action of Huygens’ principle while the 2 dimensional event, say, on the surface of water, is caused by the wave diffusion. The following theorem proven independently by Mathisson[61] and Asgeirsson[62] is playing the central role in studying the HP-type of problems

**Theorem 4.1.** If the hyperbolic equation in space-time of Lorentzian signature is satisfying HP, then it is equivalent to Eq.(2.6a).

Thus, $\Box_4$ is the Huygens operator and a question emerges: How to define the equivalence? Following [63, 64] this equivalence can be defined as follows. Let $L[\phi]$ be a Huygens’ operator (that is the operator which is satisfying HP) and let $\tilde{L}[\phi]$ be another Huygens operator. Then they are equivalent if:

a) $\tilde{L}[\phi]$ can be obtained from $L[\phi]$ by non-singular transformations of the independent variables.

b) $\tilde{L}[\phi] = \lambda^{-1}L[\lambda \phi]$ for some positive, smooth function $\lambda$ in the causal domain $\Omega$ (to be defined momentarily).

c) $\tilde{L}[\phi] = \rho L[\phi]$ for some positive smooth function $\rho$ in $\Omega$.

To define the causal domain $\Omega$ we need to define the distance function \[ \Gamma(t, x; \tau, y) \] given by \[ \Gamma(t, x; \tau, y) = c^2(t - \tau)^2 - \sum_{i=1}^{3} (x_i - y_i)^2. \] Each pair $\langle t, x \rangle \in \mathbb{R}^{1+3}$ is called an event. The Euclidean line segment $r(x, y) = \sqrt{\sum_{i=1}^{3} (x_i - y_i)^2}$ between two events $\langle t, x \rangle$ and $\langle \tau, y \rangle$ can be used to formally define the velocity $v = r(x, y)/(\tau - t), \tau > t$. This definition then allows us to define an open (future)
set $D_+(t, x) \in \mathbb{R}^{1+3}$ of those events that can be reached from $(t, x)$ with the velocity $v \leq c$. Analogously, it is possible to define an open (past) set $D_-(t, x) \in \mathbb{R}^{1+3}$ of events $(\tau, y)$ from which $(t, x)$ can be reached with the velocity $v \leq c$. Thus, $(\tau, y) \in D_+(t, x)$ if $(t, x) \in D_-(\tau, y)$. The boundary of $D_+(t, x)$ is called forward (future) (respectively backward (past)) characteristic cone $C_+(t, x)$ (respectively, $C_-(t, x)$). The lightcone is defined by the requirement $\Gamma(t, x; \tau, y) = 0$. This is an equation of a 3d sphere for a fixed $(\tau - t)^2$. Based on this information, we are now in the position to define

**Definition 4.2. Hadamard criterion.** The operator $L[\phi]$ is of Huygens-type if

$$L[\phi] |_{\Gamma(t,x;\tau,y)=0} = 0. \quad (4.2)$$

Accordingly, the causal domain $\Omega$ is determined by the following requirement [63]: event $(t, x) \in \Omega$ only if $(\tau, y) \in C_\pm(t, x)$.

**Definition 4.3.** A Huygens operator $L[\phi]$ that arises from $\square_4$ via operations a), b) and c) is called trivial Huygens operator.

**Hadamard conjecture.** Every Huygens operator is trivial.

Evidently, the Hadamard conjecture is fully compatible with Theorem 4.1. The question arises: If at any point of (pseudo)Riemannian 3+1 spacetime the metric can be brought into the diagonal form of Lorentzian signature, will the Hadamard conjecture be valid in some open domain of such space? Some studies of this problem were made by Friedlander [65], McLenagan [66], Goldoni [67], Ibragimov [68], Wünsch et al [69]. To discuss these works further from physical standpoint, we need to relate them to Schrödinger’s work [1]. Incidentally, Schrödinger later on in his life studied the problem of wave propagation in curved spacetimes [24]. More details on this topic are given in the Appendix B.

4.2. Connection of mathematical works on Huygens principle with Schrödinger’s foundational papers on quantum mechanics

4.2.1. General background

Already in section 1 we noticed that for Schrödinger (2nd installment in [1]) HP is synonymous with the H-J equation. This point of view is being shared by such famous mathematicians as Gelfand and Fomin [60], pages 208-217. In our opinion, based on [29,33], Gelfand and Fomin results can be presented in such a way that their consistency with results of Hadamard and other mathematicians who used the PDE methods for description of the HP will become obvious. Following [33], we shall assume that at time $t_0$ a light signal had been originated at the point $(x_0, y_0, z_0)$. At the later time $t > t_0$ this signal had penetrated into domain of space enclosed by a surface $V$ given analytically in the form

$$V(x_0, y_0, z_0; x, y, z) = c(t - t_0) \quad (4.3a)$$
The surface $V$ is called wavefront surface. The velocity of the wavefront can be measured with help of the velocity along the trajectories orthogonal to the wavefront (Appendix A). For points $P_1$ and $P_2$ on such trajectory we obtain:

$$V(x_0, y_0, z_0; x_1, y_1, z_1) = c(t_2 - t_1)$$

(4.3b)

or, in the differential form,

$$V_x dx + V_y dy + V_z dz = c dt.$$  

(4.3c)

Mathematically, this result is identical to the condition $\alpha = 0$ for the contact plane mentioned in the previous subsection. From this observation it follows that it is perfectly reasonable to apply methods of contact geometry mentioned in the previous subsection for description of the HP. At the level of PDE this principle can be further elaborated as follows. Since Eq.(4.3a) is describing a wavefront, that is a two-dimensional surface $\Gamma$ in three-dimensional space, it makes sense to introduce coordinates $\xi, \eta$ on this surface so that

$$x = f(\xi, \eta), y = g(\xi, \eta), z = h(\xi, \eta).$$

(4.4a)

Evidently,

$$x(\xi, \eta; 0) = f(\xi, \eta) \equiv x_0, y(\xi, \eta; 0) = g(\xi, \eta) \equiv y_0, z(\xi, \eta; 0) = h(\xi, \eta) \equiv z_0$$

(4.5a)

and

$$\dot{x}(\xi, \eta; 0) = a(\xi, \eta), \dot{y}(\xi, \eta; 0) = b(\xi, \eta), \dot{z}(\xi, \eta; 0) = c(\xi, \eta).$$

(4.5b)

To use these equations, we need to make some detour into theory of PDE following [29]. In particular, equations like Eq. (4.3a) are equations of characteristics. These are equations for some surfaces, e.g. $\Gamma$. These are encountered not only for the first order PDE’s known to standardly trained physics professionals but also to the higher order PDE’s too. Suppose, we are having, say, the 2nd order PDE, e.g like Eq.(2.6a), for which the Cauchy problem (Appendix A) can be set up and is well posed. Then, one can think about extending the Cauchy initial values, e.g. see Eq.s(4.5a),(4.5b), prescribed on $\Gamma$ to solutions of, say, Eq.(2.6a).

For this purpose it is useful to consider some auxiliary problems first. E.g. let us consider a cone, Eq.(4.1), defined on $C_+(t, x)$. Following [29], page 558, we introduce the function

$$\chi = (ct)^2 - x^2 - y^2 - z^2.$$  

By direct calculation we obtain ($c = 1$):

$$\chi_t^2 - \chi_x^2 - \chi_y^2 - \chi_z^2 = 4\chi.$$  

(4.6a)

On the cone $C_+(t, x)$ we have the condition $\chi = 0$ leading to the 1st order PDE

$$\chi_t^2 - \chi_x^2 - \chi_y^2 - \chi_z^2 = 0.$$  

(4.6b)
Just obtained result admits broad generalization. For instance, consider instead \((c=1)\) \(\phi = t - \sqrt{x^2 + y^2 + z^2} = \text{const.} \)

Then, we again obtain

\[
\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi_x}{\partial x^2} - \frac{\partial^2 \phi_y}{\partial y^2} - \frac{\partial^2 \phi_z}{\partial z^2} = 0. \tag{4.7}
\]

Clearly, we can make these results to hold in any number of dimensions. Furthermore, Eqs (4.6b) and (4.7) are looking the same. Now we demonstrate that: a) there are many other functions than \(\chi, \phi, \) satisfying the same 1st order PDE, Eq.(4.7), b) solutions \(\chi, \phi, \) etc. are also solutions of Eq.(2.6a). The demonstration can be achieved with help of the progressive wave solution method developed by Friedlander [70] in 1946. Since his results, to our knowledge, are unknown in physics literature, we would like to make a detour and to discuss his results. This will enable us to develop a variety of physical applications.

4.2.2. Progressive wave solution by Friedlander. Introduction

We have to look for solutions of Eq.(2.6a) in the form

\[
\phi(x, y, z; t) = u(x, y, z)F(ct - f(x, y, z)) \tag{4.8a}
\]

where \(F\) is an arbitrary well behaving function. The ansatz (4.8a) can be further complicated if, instead, we shall look for solutions in the form

\[
\phi(x, y, z; t) = \sum_{m=1}^{N} u_m(x, y, z)F_m(ct - f(x, y, z)), \tag{4.8b}
\]

where

\[
F'_m(\xi) = F_{m-1}(\xi), F_m(\xi) = \int d\xi F_{m-1}(\xi). \tag{4.8c}
\]

In this work we shall use only the ansatz given by Eq.(4.8a). Substitution of this result into Eq.(2.6a) results in the following three coupled PDE’s:

\[
f_x^2 + f_y^2 + f_z^2 = 1, \tag{4.9a}
\]

\[
2(u_xf_x + u_xf_x + u_xf_x) + u\nabla^2f = 0, \tag{4.9b}
\]

\[
\nabla^2u = 0. \tag{4.9c}
\]

Clearly, the simplest case is obtained when \(u = \text{const.}\) In this case we are left only with Eq.(4.9a). Eq.(4.7) will coincide with Eq.(4.9a) if we choose \(\phi(x, y, z; t) = t - f(x, y, z)\). In view of Eq.(2.8) the obtained results can be extended further. For instance, Eq.(2.6a) can be replaced by

\[
\left( \frac{\partial^2}{\partial t^2} - \frac{n^2(x, y, z)}{c^2} \right) \phi(x, y, z; t) = 0. \tag{4.10}
\]
where \( n^2(x, y, z) \) is some dimensionless function of spatial variables. In optics it is associated with the index of refraction [33]. With such a replacement, Eq.(4.9a) is changed into

\[
\tilde{f}_x^2 + \tilde{f}_y^2 + \tilde{f}_z^2 = n^2(x, y, z).
\]

Next, we return back to Eq.(4.3c) which we rewrite as

\[
V_x \dot{x} + V_y \dot{y} + V_z \dot{z} = c.
\]

With a wavefront, Eq.(4.3a), a two parameter \((\xi, \eta)\) family of rays orthogonal to the wavefront is associated. A current point on a ray trajectory is described by \(\{x(\tau), y(\tau), z(\tau)\}\).

Let \(F(x(\tau), y(\tau), z(\tau))\) be some yet arbitrary function on this trajectory. Then,

\[
\frac{dF}{d\tau} = F_x \dot{x} + F_y \dot{y} + F_z \dot{z}.
\]

Without loss of generality, we can let \(F(x(\tau), y(\tau), z(\tau)) = \tilde{f}(x(\tau), y(\tau), z(\tau))\). Furthermore, let

\[
\frac{dx}{dt} = \frac{v}{\sqrt{\tilde{f}_x^2 + \tilde{f}_y^2 + \tilde{f}_z^2}}, \quad \frac{dy}{dt} = \frac{\tilde{f}_y}{\sqrt{\tilde{f}_x^2 + \tilde{f}_y^2 + \tilde{f}_z^2}}, \quad \frac{dz}{dt} = \frac{\tilde{f}_z}{\sqrt{\tilde{f}_x^2 + \tilde{f}_y^2 + \tilde{f}_z^2}},
\]

where \(v\) is the absolute value of the velocity. With such identifications we now replace \(V(x, y, z)\) in Eq.(4.12) by \(\tilde{f}(x, y, z)\) and use the r.h.s. of Eq.s(4.14) in Eq.(4.12) to arrive at

\[
\sqrt{\tilde{f}_x^2 + \tilde{f}_y^2 + \tilde{f}_z^2} = \frac{c}{v}.
\]

But \(\frac{c}{v} = n\). Therefore, by squaring we reobtain back Eq.(4.11). In addition, however, using Eq.s (4.14) we obtain

\[
\tilde{f}_x = \frac{1}{v} \frac{dx}{dt} = \frac{c}{v^2} \dot{x}.
\]

Accordingly,

\[
\tilde{f}_x^2 + \tilde{f}_y^2 + \tilde{f}_z^2 = \left(\frac{c}{v}\right)^2 = \left(\frac{c}{v^2}\right)^2 [\dot{x}^2 + \dot{y}^2 + \dot{z}^2] \quad \text{or} \quad \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = v^2
\]

By combining Eq.s (4.5b) and (4.16b) we obtain:

\[
a^2 + b^2 + c^2 = v^2(x_0, y_0, z_0)
\]

This result is holding on \(\Gamma\) along with Eq.s(4.5a). Therefore \(V(x_0, y_0, z_0; x_0, y_0, z_0) = 0 = \tilde{f}(\xi, \eta)\). Let \(t_0 = 0\) in Eq.(4.3a) and rewrite Eq.(4.3a) as

\[
\tilde{f}(\xi, \eta; x, y, z) - ct = 0
\]

(4.3.d)
so that at \( t = 0 \) the characteristic surface \( \Gamma \) is \( \tilde{f}(\xi, \eta) = 0 \). During the time evolution each point \((\xi, \eta)\) of \( \Gamma \) creates its own wavefront according to Eq.(4.3b). Thus, a two-parameter set of wavefronts is obtained. The essence of Huygens’ principle lies in demonstration that the envelope of all these wavefronts is again a wavefront \( \psi(x, y, z) − ct = 0 \) such that for \( t = 0 \) we reobtain \( \psi(x, y, z) \big|_{t=0} = \tilde{f}(\xi, \eta) \). The procedure for finding an envelope is reduced to eliminating the parameters \((\xi, \eta)\) from three equations

\[
\begin{align*}
\tilde{f}_\xi(\xi, \eta; x, y, z) &= 0, \\
\tilde{f}_\eta(\xi, \eta; x, y, z) &= 0, \\
\tilde{f}(\xi, \eta; x, y, z) − ct &= 0.
\end{align*}
\]

Suppose that

\[
\xi = A(x, y, z), \eta = B(x, y, z)
\]

are calculated from Eqs.(4.18a) and (4.18b), respectively, then \( \tilde{f}(A(x, y, z), B(x, y, z); x, y, z) \) is a solution of Eq.(4.11). Indeed,

\[
\tilde{f}_x = \tilde{f}_\xi A_x + \tilde{f}_\eta B_x + \tilde{f}_x = \tilde{f}_x
\]

in view of Eqs.(4.18a),(4.18b). Anlogously, \( \tilde{f}_y = \tilde{f}_y \) and \( \tilde{f}_z = \tilde{f}_z \) implying that Eq.(4.11) holds. Thus, the surface (the characteristic) \( \Gamma : 0 = \tilde{f}(\xi, \eta) \) at time \( t \) is converted into surface described by Eq.(4.3d). Evidently, since \( \tilde{f}(\xi, \eta; x, y, z) = \tilde{f}(A(x, y, z), B(x, y, z); x, y, z) \) it is obeying the same H-J equation (4.11) in view of Eq.(4.20) (and those for \( y \) and \( z \) components).

As we know already (e.g. read section 1), the essence of Huygens’ principle for Schrödinger is the H-J Eq.(4.11). With explanations just made superimposed with those in [60], it should be clear to our readers that this is indeed the case.

4.2.3. Progressive wave solution method by Friedlander. From mechanics of Bohm to gauge-theoretic mechanics of Floer

Results of previous subsection were obtained under the assumption that \( u(x, y, z) \) in Eq.(4.8a) is a constant. If it is not a constant, situation becomes much more complicated mathematically [65],[70]. To decide what to do with these complications physically requires some work. First, we notice that the H-J equation was used in the 1st paper by Schrödinger on quantum mechanics [1], pages 1-12, as an input for obtaining the stationary Schrödinger equation. Although his method of deriving this equation was endorsed by Courant and Hilbert [2], pages 445-450, to our knowledge, it was left without attention in physics literature. For the sake of results we shall develop momentarily, we are going to reproduce the 1st Shrödinger method now. For this purpose, we should notice that: a) the ansatz, Eq.(4.8a), was made without account of dimensionality arguments; b) when the dimensionality arguments are taken into account, the H-J equation, e.g. Eq.(4.11), becomes a simple statement.
about the classical momentum of the particle:\footnote{Following Schrödinger, and for the sake of argument, we are discussing only the one and two-body problems.}

\[
\left( \frac{\partial S}{\partial x} \right)^2 + \left( \frac{\partial S}{\partial y} \right)^2 + \left( \frac{\partial S}{\partial z} \right)^2 = 2m(E - V). \tag{4.21}
\]

Schrödinger makes the following reversible substitution \( S \rightleftharpoons h \ln \psi \) into Eq.(4.21) resulting in his Eq.(1") (ours Eq.(1.4)). For our readers convenience we rewrite it here again:

\[
\left( \frac{\partial \psi}{\partial x} \right)^2 + \left( \frac{\partial \psi}{\partial y} \right)^2 + \left( \frac{\partial \psi}{\partial z} \right)^2 = \frac{2m}{\hbar^2} (E - V) \psi^2. \tag{4.22}
\]

Because of noticed reversibility, this is still the classical H-J equation, even though it has \( \hbar \) in it. It is exactly the same as Eq.(4.21). Next, using Eq.(4.22), instead of Eq.(4.21), Schrödinger considers the following optimization problem: Find the minimum of the functional

\[
J[\psi] = \frac{1}{2} \int d^3x \left[ (\nabla \psi)^2 - \frac{2m}{\hbar^2} (E - V) \psi^2 \right] \tag{4.23}
\]

under the subsidiary condition

\[
\int d^3x \psi^2 = 1. \tag{4.24}
\]

The result of such a minimization is the stationary Schrödinger Eq.(1.4). By design, any solution of Eq.(1.4) should be a minimum of \( J[\psi] \). If \( \psi \) coming as solution of Eq.(1.4) is such that \( J[\psi] = 0 \), then it is describing the classical trajectory according to Eq.(4.22) since the substitution \( S \rightleftharpoons h \ln \psi \) is reversible. But, in view of results of previous subsection, this is indeed the case! E.g. solutions of Eq.(4.11) are solutions of Eq.(4.10) if in the ansatz Eq.(4.8a) we replace \( c \) by \( \omega \) and take into account Eq.s (2.8), (2.9). This observation allows us to reinterpret variational results of the 1st Schrödinger paper in terms of the gauge-theoretic formalism developed by Floer. A quick introduction to Floer’s theory is provided in [71] while the detailed account can be found, for instance, in [72]. Even though Floer theory studies a multitude of closed orbits on symplectic manifolds classically, the extremely sophisticated computational methods developed by Floer exactly parallel those used in nonperturbative (instanton) treatments of the Yang-Mills theory. These results establish a connection between the non relativistic Schrödinger equation (with time-independent Hamiltonian) and the gauge-theoretic Yang-Mills-type theory. We shall say more on this subject in subsection 7.4.

If \( u(x,y,z) \) in Eq.(4.8a) is not a constant then, at the very least, we end up with the formalism of quantum mechanics developed by David Bohm [73]. See also Appendix C. It also can be considered as classical because quantum mechanical corrections in Bohmian mechanics are treated by methods of classical mechanics (in fact, of hydrodynamics). Bohm formalism is based on representation given by Eq.(4.8a) with an extra restriction \( F(ct - f(x,y,z)) \rightarrow \exp \{ \frac{1}{\hbar} (Et - S(x,y,z)) \} \). It remains to be investigated how results of Bohm formalism might change if Eq.(4.8b) is used instead. This task is left for further study.
4.2.4. Progressive wave solution method by Friedlander. Dupin cyclides, the Lie sphere geometry and the conformal group SO(4,2)

From what was discussed thus far it follows that both the hyperbolic (wave-like) and the parabolic (Schrödinger-like) equations can be treated with help of the progressive wave solution method. If this is so then, according to Friedlander [70], the most general solution in both cases should be expressible in terms of cyclides of Dupin. For the hyperbolic PDE’s this was demonstrated by Friedlander [65,70]. In this subsection, and also in subsection 7.5.3. and Appendix C, we shall demonstrate that this is also true for the stationary Schrödinger equation. By doing so, we shall put the obtained results in a much broader context. This context will allow us to find a place for (thus far) very exotic Dupin cyclides in atomic, high energy physics and cosmology.

To begin, we notice that 1946 result by Friedlander [70] was reconsidered in 2005 by Sym [74]. Not only he reobtained it in a much shorter and simpler form in ref.s [74,75] but, in addition, he was able to find some (not serious) mistakes in the original work. Both Friedlander and Sym were solving the system of Eq.s (4.9a-c). Sym very cleverly used symmetry to solve these equations. His solution strategy can be summarized as follows. First, solve Eq.(4.9a) by cleverly using the symmetry built into the Dupin cyclides. Second, to take the full advantage of this symmetry, rewrite Eq.s (4.9a-c) in curvilinear (actually geodesic) coordinates reflecting this symmetry. The choice of coordinates in Eq.s (4.9b,c) is determined by the fact that Eq.(4.9a) should admit a natural solution respecting the symmetry of Dupin cyclides. Third, use this solution in Eq.(4.9b) and insure that Eq.(4.9c) is solved in such coordinates as well.

Being armed with these results, we apply them to the Helmholtz Eq.(3.2) which is obtainable from both the D’Alembert and the free particle Schrödinger equations. Therefore, the results of separation of variables in Eq.(4.9c) can be used for solving the Helmholtz equation as well. The next step is made by checking what kinds of potentials in the stationary Schrodinger’s equation allow the full separation of variables with help of these geodesic coordinates. This problem is non trivial. It was studied, for example, in [76,77]. We shall not go into full details, however, since we shall develop much broader vision of the role of Dupin cyclides in such kind of problems which, in addition, will make such calculation much simpler and much more physically appealing.

For this purpose we need to explain: a) What are the Dupin cyclides? b) How their presence/absence affects the results of Schrödinger and Bohm? c) What happens to the Huygens principle if Dupin cyclides are taken into account? The answer to b) is given in part in the already cited [76,77]. But it is also given in Apendices B and C from the entirely different standpoint. The positive answer to c) follows in part from works by Sym [74,75] just mentioned and also follows from results of the Appendix B. Therefore, we need only to provide the answer to a). The answers to b) and c) are provided in just mentioned appendices and in section 7. The answer to a) is provided immediately below.

We begin with ref.[78]. It is associated with the notion of canal surface. Such a surface can be designed as follows. Choose some sphere $S_{c,r}(t)$ whose radius $r(t)$ is changing in time.
The center of $S_{c,r}(t)$ is moving along some curve (the trajectory) $c(t)$ parametrized by time $t$ so that such a motion is described analytically as

$$F(x, t) = \|x - c(t)\|^2 - r^2(t) = 0. \quad (4.25)$$

**Definition 4.4. Canal surface.** The canal surface $\Sigma_{c,r} \subset \mathbb{R}^3$ is an envelope of a 1-parameter family $S_{c,r}(t)$ of spheres centered at the spine curve $c(t)$.

The envelope is being defined as joint solution of two equations

$$F(x, t) = 0, \quad (4.26a)$$

$$\frac{\partial F(x, t)}{\partial t} = <x - c(t), \dot{c}(t)> + r(t) \cdot \dot{r}(t) = 0. \quad (4.26b)$$

Here $\langle, \rangle$ is the scalar product in $\mathbb{R}^3$. The canal surface $\Sigma_{c,r}$ is fully determined by a curve in 4 dimensional Minkowski space $\mathbb{R}^{3,1}$. Clearly, every point $(c(t), r^2(t))$ of $\mathbb{R}^{3,1}$ corresponds to a sphere $S_{c,r}(t)$. Dupin cyclides are made of canal surfaces but not another way around.

This follows from the ingenious observation by J.C. Maxwell who constructed some of the first Dupin cyclides following the original work by Chales Dupin done in 1803. In 1868 Maxwell noticed that, even though every point $(c(t), r^2(t))$ belongs to a sphere $S_{c,r}(t)$, there could be two different sets: a) $c_1$ and $r_1$, b) and $c_2$ and $r_2$ producing the same canal surface.

**Definition 4.5 a).** Dupin cyclides are the only canal surfaces which can be designed in two different ways.

Consider the simplest example—the torus

a) the spine curve $c_1(t) = a(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}, 0)$, the radius function $r_1(t) = c$;

b) the spine curve $c_2(t) = a(0, 0, \frac{2t}{1+t^2})$, the radius function $r_2(t) = c - a\frac{1+t^2}{1-t^2}$.

Using of a) and b) in Eq.s (4.26a,b) and eliminating $t$ leads to the following quadric (describing the torus—the canal surface)

$$(x^2 + y^2 + z^2 + a^2 - c^2)^2 - 4a^2(x^2 + y^2) = 0. \quad (4.27)$$

From this elementary example the following general definition follows

**Definition 4.5 b).** Dupin cyclides are the surfaces whose lines of curvatures are circles. Elementary examples of (degenerate) Dupin cyclides include spheres, planes (that is spheres of infinite radius), toruses and cones of revolution.

The above results admit generalization to higher dimensions [79]. More on this will be said in section 7. In the meantime, we would like to notice the following. An affine isometry $f$ of $\mathbb{R}^n$ is described as follows. Let $x \in \mathbb{R}^n$, then $f(x) = Qx + b$ where $Q$ is $n \times n$ orthogonal matrix (that is $QQ^T = 1$) whose entries belong to $\mathbb{R}$, $b \in \mathbb{R}^n$. 26
Definition 4.6. If \( f(c) = c \) (respectively \( f(\Sigma_{c,r}) = \Sigma_{c,r} \)), then \( f \) is called symmetry of \( c \) (respectively of \( \Sigma_{c,r} \)).

To get a feeling of this symmetry it is sufficient to consider the space \( \mathbb{R}^2 \). Since the canal surfaces are made of circles moving along the spine curve, the symmetry transformations should convert circles into circles. This is possible with help of the Möbius (or conformal) transformations \( f(z) = \frac{az + b}{cz + d}, \ z \in \mathbb{C}, \ \mathbb{C} = \mathbb{R}^2 \cup \{\infty\} \). It is clear then, that the transformation \( f(\Sigma_{c,r}) = \Sigma_{c,r} \) is a conformal transformation. It is possible to extend these ideas to higher dimensions naively. This will yield the multidimensional Möbius transformations. These are not quite yet the transformations having the Dupin cyclides unchanged. They form a subgroup of the Lie sphere geometry group [80] which by design leaves the Dupin cyclides unchanged. In two dimensions the Möbius group is the isometry group of the Poincare' hyperbolic upper half plane. The interrelationship between the hyperbolic and the Minkowski spaces is nicely described in the book by Ratcliffe [81]. From it we find that the Möbius group is also group of isometries of the Minkowski spacetime. In sections 6,7 we shall demonstrate that the group of isometries of the compactified Minkowski spacetime is the conformal group \( \text{SO}(4,2) \). It is described, for example, in [19], pages 345-348. The significance of this group was noticed already in section 1. In section 7 we shall demonstrate that this group coincides with the Lie sphere group. This observation is of fundamental physical significance to be explained in section 7 and the Appendix B. In the meantime we would like to explain the differences between the Möbius and the Lie sphere groups. Since the Möbius group is the subgroup of the Lie sphere group, the additional symmetry elements originate from the fact that a) spheres could have an extra label- orientation. This can be seen already in dimension two. A circle can have the radius -vector with the base sitting on the circle and directed toward the center of circle. Another option is for the same vector to play the role of the outward normal to the circle. Besides, one should pay attention to the fact that the circles can touch each other and each circle could have an infinite family of circles sitting inside or outside of the given one and touching it at the same point. To these objects one should add points-circles of zero radius and, also -lines, the circles of infinite radius (in two dimensions) or the planes in dimensions higher than two. According to Klein’s Erlangen program, the following definition is the most appropriate.

**Definition 4.7.** The essence of Lie sphere geometry lies in the study of properties of transformations mapping oriented spheres (including points and planes) to oriented spheres while preserving the oriented contact of sphere pairs.

**Corollary 4.8.** Dupin cyclides are invariants of the Lie sphere geometry.

**Corollary 4.9.** The conformal group \( \text{SO}(4,2) \) was used in atomic [19] and in high energy physics [83] without any reference to the Lie sphere geometry symmetry group. It was discussed already in section 1. Use of conformal group caused introduction of the unphysical two-times formalism in [83]. Use of Lie sphere geometry removes this deficiency. In the rest of this paper we demonstrate how the conformal group \( \text{SO}(4,2) \) is identified with the Lie sphere geometry group. This identification opens the door for new results in conformal
dynamics, conformal quantum mechanics, conformal quantum field theory, conformal gravity, etc. The AdS-CFT correspondence to be discussed in section 7 and Appendix D is part of this “conformal program”. It is important to remember that the source-free Maxwellian electrodynamics is invariant with respect to SO(4,2) group as well while the results of conformal quantum mechanics recently were utilized with great success for recovery of the Regge-like spectrum of hadrons.

5. Closer look at the Hadamard conjecture. Recovery of some known equations for massive/massless particles with or without spins

In this paper we are not going to discuss the developments associated with counterexamples to the Hadamard conjecture. All these counterexamples are discussing the validity of the Huygens’ principle in spacetimes of dimensionality higher than four [68]. In the light of results of section 1 use of spacetimes of dimensionality higher than four for multielectron atoms and for molecules is also not necessary, apparently. Fortunately, in four dimensional spacetimes no counterexample to the Hadamard conjecture was found. Results of Appendix B should be considered as a cosmologically inspired ramification of the already known results. Because of this, it is convenient to restate the Theorem 4.1. as follows

Fundamental Principle 5.1. Quantum mechanical behavior of all elementary particles (massive and massless, of integer and half integer spin) is inseparably linked with the Lorentzian signature of ambient spacetimes.

Conjecture 5.2. Although not immediately obvious, the Fundamental Principle is synonymous with the central role of the Lie sphere geometry acting in conformally flat spacetimes of Lorentzian signature. Its influence on physics is ranging from conformal mechanics to conformal wave mechanics and conformal gravity. Group-theoretical classification of physically sensible spacetimes is contingent upon their ability to sustain quantum mechanics.

5.1. Hadamard triviality and mass generation. Panoramic view

We continue this section with demonstration of Huygens’ triviality of telegrapher’s and the Klein-Gordon equations. Using Eq.(2.6) and the equivalence condition $\tilde{L}[\phi] = \lambda^{-1}L[\lambda \phi]$ with $\lambda = e^{at}$, where $a$ is some constant, we obtain ($c = 1$):

$$ e^{-at}\left\{\left[\frac{\partial^2}{\partial t^2} e^{at} \phi \right] - \left[\nabla^2 e^{at} \phi \right]\right\} = \frac{\partial^2}{\partial t^2} \phi + 2a \frac{\partial}{\partial t} \phi - \nabla^2 \phi + a^2 \phi = 0. \quad (5.1) $$

26Associated with accounting for Penrose limits of physical spacetimes.

27Further details are presented in section 7 and Appendix B.
Next, let $\lambda_1 = e^{ibx}$ and $\lambda_2 = e^{-icx}$. Substitute these factors into Eq.(5.1) and apply again Hadamard’s equivalence rules. After a short calculation we arrive at

$$\frac{\partial^2}{\partial t^2} \phi + 2a \frac{\partial}{\partial t} \phi - \nabla^2 \phi + 2ib \frac{\partial}{\partial x} \phi - 2ic \frac{\partial}{\partial x} \phi + (a^2 - b^2 - c^2) \phi = 0. \quad (5.2a)$$

If now $b = c$ and $a^2 = 2b^2$, we obtain telegrapher’s equation

$$\frac{\partial^2}{\partial t^2} \phi + 2a \frac{\partial}{\partial t} \phi - \nabla^2 \phi = 0. \quad (5.2b)$$

One dimensional version of this equation is discussed at length in the book [86]. In one and two dimensions this equation admits the path integral treatment. In the meantime, the Klein-Gordon (K-G) equation is obtained now if we make a replacement $\phi = e^{at} \psi$ in Eq.(5.2b) with subsequent replacement of $a$ by $ia$. After this, we obtain:

$$\frac{\partial^2}{\partial t^2} \psi - \nabla^2 \psi + a^2 \psi = 0. \quad (5.3)$$

Thus, we just demonstrated that the K-G equation is Huygens-equivalent to the D’Alembert Eq.(2.6a). According to [87], page 99, every spinor component of the Dirac equation with nonzero mass is satisfying the K-G equation. This fact establishes the Huygens equivalence between the Dirac and D’Alembert equations. Apparently, the particles with higher spin, e.g. spin-2 gravitons, etc. also belong to the same Huygens equivalence class [87 – 89].

**Corollary 5.3.** Huygens equivalence between the D’Alembert and all relativistic equations of integer and half integer spin explains why the double slit experiments made with photons and massive particles produce the same fringe patterns. More details on this is given in ref.[14].

Thanks to the seminal work by Mark Kac [90] the propagator for telegrapher’s and Dirac’s equations can be presented in the path integral form at least in 1+1 dimensions [91]. In view of Eq.s (5.2b) and (5.3) the same is true for the K-G equation as explained in detail in [92].

It should be noted that in all these cases the associated path integrals do not involve the Gaussian-type random processes. They are designed with help of the Poissonian-type random processes. Excellent description of these types of processes in conjunction with the telegrapher’s equation is given in [86]. The noticed connection with random walks is helpful but not crucial for the tasks to be completed in the rest of this paper. Furthermore, the above path integrals can be designed rigorously only in 1+1 and 2+1 dimensions. In [93] it was rigorously demonstrated that these results cannot be extended to higher dimensions.

To stay focused, we are not going to discuss any further the connection between the random walks of various kinds and PDE’s. Instead, we notice the following.

**First.** The diffusion/Schrödinger is not the wave-type equation studied by Hadamard. The diffusion/Schrödinger equation describes the dispersive waves discussed in the Appendix A. Therefore, contrary to Feynman’s claims made in [4], it apparently does **not** obey the
Huygens principle. However, the results of sections 4 and 6 indicate that this apparent deficiency of the Schrödinger equation can be repaired, and quite rigorously, for as long as the quantum Hamiltonian of this equation is manifestly time-independent. What remains to be proven is Huygens triviality of Schrödinger's equation with time-independent Hamiltonian.

**Second.** Initially, Hadamard obtained his results in 3+1 dimensions as explained in previous section. Subsequently, he developed the method of descent allowing use of 3+1 dimensional results as an input for obtaining 2+1 dimensional solutions. Still later, these results were extended by others to 1+1 dimensions, again with use of the method of descent,[94], pages 315, 316.

**Third.** Clearly, by going down from 3+1 to 1+1 dimensions some information is lost. Otherwise the distinction between, say, 2 and 3 dimensional wave propagation is going to disappear. But it is not! Therefore, the attempt to use the method of descent in reverse cannot help us in extending 2+1 dimensional results obtainable with help of, say, the Poissonian statistics to 3+1 dimensions. Other methods, e.g. those using Grassmann variables should be used instead [95]. Different method, also using Grassmann variables but employing differential-geometric considerations for 3 dimensional paths evolving in 3+1 dimensional spacetime (leading to 3+1 Dirac propagator) was developed in [96]. In connection with [96], the following observation is appropriate.

**Forth.** The Standard Model of particle physics uses the widely accepted Higgs mechanism responsible for the mass generation. At the same time, the twistor formalism describing all massless particles had been extended recently by accounting for the rigidity of the worldlines of the massless particles [97]. Such differential-geometric mass generation method is analogous to that proposed in [96]. The latest results in this direction can be found in [98].

**Fifth.** After Eq.(5.3) we stated that the K-G, the Dirac and other basic equations for massive particles can be made Huygens-trivial using Hadamard’s transformation rules. Since establishing of this equivalence is reversible process, this means that it is possible to avoid use of the Higgs mechanism. The two-time formalism developed by Itzhak Bars (summarized in [83]) leads to the same conclusions. Because the De Broglie relation is valid for massless photons as well as for the massive particles as demonstrated in the footnote 6 and because the stationary Schrödinger equation can be restored from the relativistically obtained uncertainty relations (as demonstrated by Schrödinger (e.g. read section 2)) the notion of mass should be linked with the De Broglie relation. Its actual value is controlled by the rigidity of world lines [97],[98],[30].

In view of these remarks, our first task is to address and to solve the problem formulated in the first remark. In section 3 we provided the following Schrödinger’s comments: "Thus, when we designated equation (1) or (1′)[29] on various occasions as "the wave equation", we were really wrong..." Well, actually, Schrödinger was not at all wrong! Eq.(1.4) was used systematically by Schrödinger himself in Parts I-III of [1] to solve the Hydrogen atom, the harmonic oscillator, the rigid and non rigid rotators and the Stark effect quantum mechanically. He also developed the perturbation theory based on the exact results which

---

28The latest results on this topic are mentioned in the "Note added in proof"
29Our Eq.(4.22).
he obtained. All these results are correct! In section 3 we explained what made Schrödinger unhappy with his "amplitude equation", that is with our Eq.(1.4). In section 4 using methods of characteristics and progressive waves we demonstrated that Eq.(1.3) is indeed the correct wave equation in the sense of De Broglie [35] obeying the Huygens' principle in the sense of Hadamard. Furthermore, in section 6 we demonstrate that for the time-independent Hamiltonians Schrödinger's "real wave equation" can be embedded into the Hadamard scheme of calculations relating a given 2nd order PDE with the wave Eq.(2.6a).

In conclusion, we would like to mention that by using the two-time formalism Itzhak Bars established the equivalence (in the sense of gauge equivalence(duality) defined in his paper) between the D'Alembert Eq.(2.6a), on one hand, and the Schrödinger equations for the hydrogen atom and the harmonic oscillator, on another [99]. In [99] Bars acknowledged that he was not able yet to provide a complete classification of all gauge-equivalent(dual) quantum mechanical systems originating from the same (gauge-invariant) model. The attempt to do so was made later, in [100]. However, making choices between the two times in such a formalism still remained mysterious. Below, we shall obtain the same results using entirely different methods enabling us to avoid the two-times formalism altogether.

6. Huygens triviality of the stationary Schrödinger equation

6.1. Some comments about 1935 work by V. Fock on hydrogen atom

On February 8, 1935, Vladimir Fock presented his seminal lecture entitled "On theory of the hydrogen atom" at the theory seminar of the Leningrad State University. English translation of his talk can be found in [101]. It is based on Fock's article [102] published in German.

To begin our comments on his paper we rewrite Eq.(1.4) in the following standard form

$$-\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi = E\psi. \quad (6.1a)$$

Without loss of generality we shall consider only potentials for which the above equation is exactly solvable. At the classical level use of canonical transformations makes all such exactly solvable problems equivalent since they all can be brought into standard action-angle form known for the harmonic oscillator Accordingly, it is sufficient to consider only the hydrogen atom problem for which in known system of units the potential $V(r)$ is defined as $V(r) = -\frac{Ze^2}{r} \equiv -\frac{k}{r}$. For this (Coulombic) potential Fock replaces the partial differential Eq.(6.1a) by the equivalent integral equation

$$\left(\frac{P^2}{2m} - E\right)\psi(p) = \frac{k}{2\pi^2\hbar} \int \frac{d^3p'\psi(p')}{|p - p'|^2}. \quad (6.1b)$$

Citing Fock, the rationale for using the integral equation method instead of solving the partial differential equation is caused by the following observations.
It has long been known that the energy levels of the hydrogen atom are degenerate with respect to the azimuthal quantum number $l$. But any degeneracy of eigenvalues is linked to the transformation group of the relevant equation: e.g. the degeneracy with respect to the magnetic quantum number $m$ is attributed to the usual rotational group. Here Fock refers to the degeneracy of the following type. For a given $l$ (associated with the rigid rotator energy $l(l+1)$) one has $2l+1$ wavefunctions labeled by the magnetic number $m$:

$$-l \leq m \leq l.$$  

He calls such a degeneracy "accidental" and in his paper he finds the symmetry group causing this accidental degeneracy. He demonstrated that the group causing accidental degeneracy is four-dimensional rotational group $SO(4)$.

Instead of copying Fock’s arguments, we shall arrive at the same results much more economically. For this purpose we introduce the notations:

$$H = \frac{p^2}{2m} - \frac{k}{r}; \quad (6.2a)$$

$$L = \mathbf{r} \times \mathbf{p}; \quad (6.2b)$$

$$A = \mathbf{p} \times L - mk\hat{\mathbf{r}}, \quad \hat{\mathbf{r}} = \frac{\mathbf{r}}{r}. \quad (6.2c)$$

It can be easily demonstrated that in addition to the Hamiltonian $H$ which is constant of motion, both the angular momentum $L$ and the Laplace-Runge-Lentz vector $A$ are also constants of motion [101]. These objects can be looked upon either at the classical level with subsequent quantization or at the quantum level. The last route was chosen initially by Pauli [103]. The results in both classical and quantum cases depend upon the value of the energy constant $E = H: E > 0, E < 0$ and $E = 0$. In this work the case $E = 0$ will not be considered since it is not related to the tasks we would like to accomplish. Traditional analysis involving bound orbits/states typically begins with the case $E < 0$. In this case it is more convenient to replace the vector $A$ with the rescaled vector $D$ defined as ([104], page 421):

$$D = \frac{A}{\sqrt{-2mE}}. \quad (6.3a)$$

In the case of $E > 0$, the vector $D$ is defined accordingly as

$$D = \frac{A}{\sqrt{2mE}}. \quad (6.3b)$$

In terms of such notations the Poisson brackets $\{,\}$ commutation relations are readily obtained with the result [104]

$$\{L_i, L_j\} = \varepsilon_{ijk} L_k, \quad (6.4)$$

$$\{D_i, L_j\} = \varepsilon_{ijk} D_k,$$

$$\{D_i, D_j\} = \varepsilon \varepsilon_{ijk} L_k.$$

Here $\varepsilon = 1(E < 0), \varepsilon = -1(E > 0)$. Using this Poisson algebra, the respective Lie algebra is obtained via standard Dirac quantization prescription. As result, the so(4) Lie algebra (for
\( E < 0 \) and \( \text{so}(3,1) \) Lie algebra (for \( E > 0 \)) is obtained. The first one is the Lie algebra of the \( \text{SO}(4) \) rotation Lie group while the second is the Lie algebra of the \( \text{SO}(3,1) \) Lorentz group. In his talk Fock did mention the Lorentz group but provided no computational details.

6.2. Huygens’ triviality of the stationary Schrödinger equation in the light of Fock’s work

6.2.1. General consideration

Just obtained results, when superimposed with results by Fock, are sufficient for group-theoretical proof of Huygens’ triviality of the stationary Schrödinger equation. Nevertheless, it is instructive to arrive at the final destination via extremely informative detour. This detour is possible to perform by using, for example, fundamental work by Gelfand, Milnos and Shapiro [105]. Alternatively, the same results can be obtained using the Duffin-Kemmer formalism [14]. The authors of [105] discussed carefully the problem of classification of all relativistically invariant equations. That is of all Lorentz-invariant equations. Clearly, the protocol of study of these equations is the same as that for study of rotationally invariant equations. Therefore, we shall study both problems simultaneously. The study begins with the equation of the type

\[
\sum_{i=0}^{n} g_{ij} \frac{\partial^2 \psi}{\partial x_i \partial x_j} + m^2 \psi = 0,
\]

where \( n + 1 \) is the dimensionality of spacetime, \( m \) is the mass parameter, and the metric tensor \( g_{ij} \) is given by

\[
g_{ij} = \delta_{ij} \text{ (rotational group)}; \\
g_{ij} = 1(i, j = 0), g_{ij} = -1(i, j \neq 0, i = j), g_{ij} = 0, i, j \neq 0, i \neq j. \text{ (Lorentz group)}
\]

Typically, \( n = 3 \) but other dimensionalities could be considered as well. With such defined metric tensor, the 2nd order PDE, Eq.(6.5a), is convenient to rewrite in the form of the system of the 1st order PDE’s. Incidentally, in such a case, one can think about the method of characteristics for such PDE’s and the H-J equations, etc.[29]. We shall not touch this topic in this section though since the relevant information was already provided in section 4. Instead, following [105], we introduce an auxiliary functions \( \psi_i \) via equation

\[
m\psi_i = \frac{\partial \psi}{\partial x_i}, i = 0, ..., n.
\]

A simple calculation produces (for the Lorentzian case, \( n = 3 \))

\[
\sum_{i=1,2,3} \frac{\partial \psi_i}{\partial x_i} - \frac{\partial \psi_0}{\partial x_0} - m\psi = 0; \frac{\partial \psi}{\partial x_i} = m\psi_i,
\]

and (for the Euclidean case, \( n = 3 \))

\[
\sum_{i=1,2,3} \frac{\partial \psi_i}{\partial x_i} + \frac{\partial \psi_0}{\partial x_0} + m\psi = 0; \frac{\partial \psi}{\partial x_i} = m\psi_i.
\]
At this point it is convenient to introduce the extended wave vector \( \Phi = \{ \psi, \psi_0, \psi_1, \psi_2, \psi_3 \}^T \) (where \( T \) means ”transpose”) and to rewrite the system of the above equations into the form

\[
\sum_{k=0}^{n} L_k \frac{\partial}{\partial x_k} \Phi \pm im \Phi = 0.
\] (6.8)

Here \( L_k \) are the \( 5 \times 5 \) matrices. E.g. in the Lorentzian case

\[
L_0 = \begin{pmatrix}
0 & i & 0 & 0 & 0 \\
-i & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\] (6.9)

etc. Let \( G \) be the matrix of either Euclidean rotations or of Lorentz transformations acting on coordinates \( x_i \) as \( x' = Gx \). Let \( T_G \) be some unitary operator such that

\[
\Phi'(x') = T_G \Phi(x) \quad \text{and} \quad \Phi(x) = T^{-1}_G \Phi'(x')
\] (6.10)

Accordingly, Eq.(6.8) can be rewritten now as

\[
\sum_{k,l} L_k T^{-1}_G \frac{\partial}{\partial x'_l} \Phi'(x') G_{lk} \pm im T^{-1}_G \Phi'(x') = 0 \] (6.11a)

or, equivalently, as

\[
\sum_{k,l} T_G L_k T^{-1}_G G_{lk} \left( \frac{\partial}{\partial x'_l} \Phi'(x') \right) \pm im \Phi'(x') = 0. \] (6.11b)

The condition of invariance follows immediately from Eq.(6.11b)

\[
\sum_k T_G L_k T^{-1}_G G_{lk} = L_l \] (6.12)

Since this condition is unchanged in the massless case, it is sufficient to consider the massless case only. Clearly, the simplest equation invariant with respect to either the Euclidean rotation or Lorentz group transformations is the four-dimensional Laplacian in Euclidean case or D’Alembertian in Lorentzian case respectively. In his paper [102] Fock obtained indeed the four-dimensional Laplacian equation as an equation whose solutions are those of the integral Eq.(6.1b). To obtain solutions in the Lorentzian case requires us only to perform the Wick rotation, that is to make a replacement: \( x_0 \rightarrow \pm ix_0 \). This innocently looking operation routinely used in physics literature required 126 pages of proof in mathematics literature [106 – 108]. Thanks to its existence, we are spared from the necessity to repeat needed proofs. The same results could be obtained group-theoretically via development of the unified treatment of SO(4) and S(3,1) Lie groups [109]. A transparent and motivating
example of such an interrelation is given in the pedagogically written article by John Milnor [110]. Nevertheless, below we shall provide yet another derivation. It involves the twistor formalism. Use of twistor formalism enables us to present Fock’s results in a different light. To accomplish this task requires several steps which we would like to describe now. We begin with the following.

6.2.2. Grassmannians and Plücker embedding

First of all, we would like to take a careful look at the Hamiltonian $H$ for the hydrogen atom at the classical level

$$H = \frac{\mathbf{p}^2}{2m} - \frac{k}{r} = E. \quad (6.13)$$

In particular, let initially $E < 0$. This condition connects the momenta and coordinates. In particular, it should also hold for $r \to 0$. In such a case, to maintain the equality, the momenta should become infinite. If initially we had $p = \{p_x, p_y, p_z\} \in \mathbb{R}^3$, now we must add a point at infinity $p_\infty$ to keep the relation, Eq(6.13), unchanged. The addition of $p_\infty$ leads to the compactification of $\mathbb{R}^3$ thus converting it to $S^3 = \mathbb{R}^3 \cup \{\infty\}$. The $SO(4)$ group is the group of isometries of $S^3[111]$. Next, instead of rescaling the Runge-Lentz vector $A$ in Eq.(6.3) we can rescale the angular momentum $L$. This is effectively done in Fock’s paper as we would like to explain now. For this purpose we follow [55], pages 361-364.

Consider 2 points $\mathbf{x}$ and $\mathbf{y}$ on the line $\mathcal{L}$ in 3d space. Since the coordinates $\mathbf{x} = \{x_1, x_2, x_3\}$ and $\mathbf{y} = \{y_1, y_2, y_3\}$ are taken with respect to some fixed origin, these are vectors. We can construct from these vectors two other vectors: $\mathbf{Z} = \mathbf{y} - \mathbf{x}$ and $\mathbf{L} = \mathbf{x} \times \mathbf{y}$. The new element is coming from the following step. We enlarge the embedding into Euclidean space by increasing its dimensionality, that is by replacing vectors $\mathbf{x}$ and $\mathbf{y}$ by $\mathbf{x}_p = \{x_0, \mathbf{x}\}$ and $\mathbf{y}_p = \{y_0, \mathbf{y}\}$. With such an enlargement the vectors $\mathbf{x}_p, \mathbf{y}_p$ can be looked upon either as vectors in $\mathbb{R}^4$ or as points in the projective space $\mathbb{P}^3$. Now we take into account that in 3 dimensions $a \times b \equiv a \wedge b$ [55]. We would like to apply this correspondence to the vectors $\mathbf{x}_p$ and $\mathbf{y}_p$. Specifically, we consider the exterior product $(x_0, x_1, x_2, x_3) \wedge (y_0, y_1, y_2, y_3) \equiv (x_0, \mathbf{x}) \wedge (y_0, \mathbf{y}) \equiv (l_{01}, l_{02}, l_{03}, l_{23}, l_{31}, l_{12})$. For the sake of illustration, we can choose $x_0 = y_0 = 1$, then $l_{01} = y_1 - x_1 = Z_1, ..., l_{23} = x_2y_3 - y_2x_3 = L_1, ..., l_{12} = x_1y_2 - y_1x_2 = L_3$. Clearly, $L_1, L_2, L_3$ can be looked upon as components of the angular momenta $\mathbf{L}$. In such a case, following Fock [101,102] we can make an identification $\mathbf{y}_p = \{y_0, \mathbf{y}\} \to \{p_0, \mathbf{p}\}$ with $p_0 = \sqrt{-2mE}, E < 0$. Evidently, in such a case we no longer can choose $x_0 = y_0 = 1$. This is not essential, however. In particular, this could be seen if we notice that $Z_1, Z_2, Z_3$ are proportional to the components of the Laplace-Runge-Lentz vector. If this is so, then we require that

$$Z \cdot \mathbf{L} = 0. \quad (6.14a)$$

However, for the Kepler problem we have:

$$(\mathbf{p} \times \mathbf{L} - mk \hat{r}) \cdot \mathbf{L} = (\mathbf{p} \times \mathbf{L}) \cdot \mathbf{L} - mk \hat{r} \times \mathbf{p} = 0. \quad (6.14b)$$
At the same time, \( p \times L = p \times (r \times p) = r(p \cdot p) - p(p \cdot r) \). Therefore, \( p \times L - mk \hat{r} = ar + bp \), where \( a \) and \( b \) are known constants. By the appropriate rescaling, this linear combination can be made the same as \( Z_1, Z_2, Z_3 \). There is another meaning of Eq. (6.14a) however. It is purely mathematical. And, as such, it is totally independent of its relevance to the Kepler problem. It is associated with the concept of Grassmannian manifold and of Plücker’s embedding of this manifold into the projective space.

To shorten our discussion, without loss of generality we can always replace the real space \( \mathbb{R}^n \) by the complex space \( \mathbb{C}^n \). Let \( \{e_1, \ldots, e_n\} \) be some vector basis in \( \mathbb{C}^n \). Introduction of such a basis requires introduction of the scalar product. This can be done by analogy with the scalar products of quantum mechanics. With help of this basis we introduce the exterior products, e.g., \( e \cdot \wedge e \) (where \( e \) is a vector). It is convenient to keep track of these products by defining the subsets \( J_p = \{i_1, \ldots, i_p\}, p \leq n \). Let the set of basis vectors \( \{e_{i_1}, \ldots, e_{i_p}\} \) be associated with such a subset. There are exactly \( n!/(n-p)! \) ways to select such a set from the set \( \{e_1, \ldots, e_n\} \) and to make the exterior products \( e_{i_1} \wedge \cdots \wedge e_{i_p} \) out of selected vectors. The Grassmannian \( Gr_p(C^n) \) is the manifold made out of all \( p \)-dimensional subspaces of \( C^n \).

Suppose that in \( C^n \) we changed the basis from \( \{e_1, \ldots, e_n\} \) to \( \{e'_1, \ldots, e'_n\} \). Changes of the basis in \( C^n \) lead to changes in the basis for subsets, e.g.,

\[
A_{i_1} \cdot \wedge \cdots \cdot A_{i_p} e_{i_1} \cdots \cdot e_{i_p} = A_{i_1} \cdot \wedge \cdots \cdot A_{i_p} e_{i_1} \wedge \cdots \wedge e_{i_p} = \det(A) e_{i_1} \wedge \cdots \wedge e_{i_p}.
\]

This relation can be looked upon as the equivalence relation defining a point in the complex projective space \( \mathbb{P}^N \). Thus, we just (Plücker) embedded the Grassmannian \( Gr_p(C^n) \) into the complex projective space \( \mathbb{P}^N \) of dimension \( N = (n!/(n-p)!) - 1 \). Since \( n!/(n-p)! = n!/(n-p)! \) we also have \( Gr_p(C^n) = Gr_{n-p}(C^n) \) causing the exterior products \( e_{i_1} \wedge \cdots \wedge e_{i_p} \) and \( e_{i_{p+1}} \wedge \cdots \wedge e_{i_{n-p}} \) to be related to each other in the way known from the Hodge theory of differential forms. Specifically, if we introduce the notations: \( e_{i_1} \wedge \cdots \wedge e_{i_p} \in \Lambda^p \mathcal{E} \) and \( e_{i_{p+1}} \wedge \cdots \wedge e_{i_{n-p}} \in \Lambda^{n-p} \mathcal{E} \), this definition is implying that in both cases we use the basis \( \{e_1, \ldots, e_n\} \), so that \( e_i \in \mathcal{E} \forall i \). If this is so, we can then construct the products of the type \( \Lambda^p \mathcal{E} \wedge (\Lambda^{n-p} \mathcal{E}) \). Construction of such products is subject to the Hodge-type constraints (\( \Lambda^p \mathcal{E} \wedge (\Lambda^{n-p} \mathcal{E}) \sim (\Lambda^{p+1} \mathcal{E}) \wedge (\Lambda^{n-p-1} \mathcal{E}) \sim \Lambda^n \mathcal{E} \)). The sign \( \sim \) means (Hodge-type) "equivalence". Evidently, it is permissible to have as well the following equivalences (\( \Lambda^p \mathcal{E} \) \( \wedge (\Lambda^{n-p} \mathcal{E}) \sim (\Lambda^{p+2} \mathcal{E}) \wedge (\Lambda^{n-p-2} \mathcal{E}) \sim \Lambda^n \mathcal{E}, \) and so on. Just described results lead us to the following

**Definition 6.1.** Let \( \{e_1, \ldots, e_n\} \) be the basis for \( \mathbb{C}^n \), then we define the set \( e_{J_p} = e_{i_1} \wedge \cdots \wedge e_{i_p} \), with \( 1 \leq i_1 < i_2 < \cdots < i_p \leq n \). If \( x \in \Lambda^n \mathcal{E} \), then \( x \) is totally decomposable if

\[
x = \sum_{J_p \in \{1 \ldots n\}} a_{J_p} e_{J_p} \equiv \sum_{1 \leq i_1 < i_2 < \cdots < i_p \leq n} a_{i_1 i_2 \ldots i_p} (e_{i_1} \wedge \cdots \wedge e_{i_p}).
\]

The homogenous coordinates \( a_{J_p} \) are called Plücker coordinates on \( \mathbb{P}^N \).
From this definition, it follows at once that
\[ x \wedge x = 0. \]  
(6.17)

This is the condition for the Plücker embedding. We want to demonstrate now that Eq. (6.14a) is the condition for the Plücker embedding. For this purpose, consider now the Grassmannian \( \text{Gr}_2(C^4) \). For it, we have \( \{e_0, \ldots, e_3\} \) as the basis for \( E \). Next, we construct \( \wedge^2 E \) built as follows:
\[ \wedge^2 E \sim \{e_0 \wedge e_1, e_0 \wedge e_2, e_0 \wedge e_3, e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\} \]  
(6.18a)

Since \( x^p = \sum_{i,j=0}^{3} x_{ij} e_i e_j \) and \( y^p = \sum_{i,j=0}^{3} y_{ij} e_i e_j \), the condition for the Plücker embedding acquires the following form:
\[ x^p \wedge y^p = 0 = (l_{01}l_{23} + l_{02}l_{31} + l_{03}l_{12}) e_0 \wedge e_1 \wedge e_2 \wedge e_3. \]  
(6.18b)

From here we obtain:
\[ l_{01}l_{23} + l_{02}l_{31} + l_{03}l_{12} = 0, \]  
(6.18c)

where the determinants \( l_{ij} \) are defined by
\[ l_{ij} = \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}, i,j = 0, 1, 2, 3. \]  
(6.18d)

Since, \( x_0y_1 - x_1y_0 = Z_1 = l_{01} \), while \( x_1y_2 - y_1x_2 = L_3 \) and so on, the Plücker embedding condition, Eq. (6.18), is exactly the same as the mechanical condition, Eq. (6.14a).

6.2.3. From Plücker embedding to conformal compactification of Minkowski space via twistor formalism

The results obtained in previous subsection enable us to complete our proof of Huygens triviality. In the previous subsection we discussed the rationale for compactification of the momentum space \( p = \{p_x, p_y, p_z\} \in \mathbb{R}^3 \). Naturally, we ended up with \( S^3 \). The 3-sphere \( S^3 \) is living in \( \mathbb{R}^4 \). There is some advantage in compactification of \( \mathbb{R}^4 \) as well leading to \( S^4 \). The argument goes as follows. In the previous subsection we introduced vectors \( x^p, y^p \) living in \( \mathbb{R}^4 \). We noticed that they can be looked upon either as vectors (in \( \mathbb{R}^4 \)) or as points (in \( \mathbb{P}^3 \)). By complexification we end up with these vectors living either in \( \mathbb{C}^4 \) or \( \mathbb{CP}^3 \). This allows us to develop the compactification of the Minkowski space (whose isometry group is \( \text{SO}(3,1) \)) and the Euclidean space (whose isometry group \( \text{SO}(4) \)) using the same formalism. Furthermore, using the compactification procedure leads us directly to the formalism of twistors and twistor spaces.

Definition 6.2. The twistor space \( T \) is \( \mathbb{C}^4 \) with coordinates \( Z = \{Z^0, Z^1, Z^2, Z^3\} \). The projective twistor space is \( \mathbb{P}T = \mathbb{CP}^3 \) with homogenous coordinates \( \{Z^0 : Z^1 : Z^2 : Z^3\} \).
Next, we notice that if for the description of the complex plane \( C \) we have to use the complex numbers, e.g. \( z = x + iy \), then for description of \( C^2 \) we have to use the quaternions, e.g. \( q = t + ix + jy + kx \). Both complex numbers and quaternions admit matrix presentation \([55]\). For instance,

\[
z = x + iy \in C = \mathbb{R}^2 \iff A = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}.
\]

(6.19a)

The scalar product \( x \cdot y \) in \( \mathbb{R}^2 \) can be presented either as \( x \cdot y = x_1x_2 + y_1y_2 \) or as \( x \cdot y = \frac{1}{2}(z_1 \bar{z}_2 + \bar{z}_1z_2) \). The last presentation can be equivalently rewritten as

\[
x \cdot y = \frac{1}{2}tr(A_1A_2^T).
\]

(6.19b)

The squared norm \(|z|^2 = x^2 + y^2\) of the complex number can be alternatively represented as

\[
|z|^2 = \det A.
\]

(6.19c)

To extend these results to \( C^2 \) we notice that \( C^2 \cong \mathbb{R}^4 \). The quaternion \( q = t + ix + jy + kx \) is encoded by the \( \{t, x, y, z\} \in \mathbb{R}^4 \). In view of the correspondence \( C^2 \cong \mathbb{R}^4 \) we introduce 2 complex numbers \( z_1 = t + ix \) and \( z_2 = y + iz \) and then, by analogy with Eq.\((6.19a)\), we can write

\[
q = t + ix + jy + kx 
\]

\[
\in H = \mathbb{R}^4 \cong A = \begin{pmatrix} z_1 & -z_2 \\ z_2 & \bar{z}_1 \end{pmatrix}.
\]

(6.20a)

Accordingly, the squared norm \(|q|^2 = t^2 + x^2 + y^2 + z^2\) is given by

\[
|q|^2 = \det A.
\]

(6.20b)

Consider now a spacetime of Euclidean signature \((+, +, +, +)\) in which the vector \( x \) is given componentwise as

\[
(x^0, x^1, x^2, x^3) \in \mathbb{R}^4 \cong (x^{AB}) = \begin{pmatrix} x^0 + ix^1 & x^2 + ix^3 \\ -x^2 + ix^3 & x^0 - ix^1 \end{pmatrix}, \ A, B = 0, 1.
\]

(6.21)

The square of the Euclidean norm for \( x \) is given by \( \det(x^{AB}) = |z_1|^2 + |z_2|^2 \). Here \( z_1 = x^0 + ix^1, z_2 = -x^2 + ix^3 \). Because of this, we also have \( (x^0, x^1, x^2, x^3) \cong (z_1, z_2) \in C^2 \). Furthermore, we also obtain: \( C^2 \cong \mathbb{R}^4 = H \). Accordingly, \( C^4 \cong \mathbb{H}^2 \). Next, we recall that the 3-sphere \( S^3 \) can be analytically represented as \( |z_1|^2 + |z_2|^2 = 1 \). The Hopf map \( S^3 \to S^2 \) can be constructed now as follows. Begin with \( \{z_1, z_2\} \in C^2 \) and consider the ratio \( z_1/z_2 \) (or \( z_2/z_1 \)). Since \( z = |z| \exp(i\varphi) \), it is clear that \( z_1/z_2 \sim z \sim C \sim S^2 \). Alternatively, \( \{z_1 : z_2\} = \mathbb{CP}^1 = S^2 \). Consider now two quaternions \( q_1 = Z^0 + Z^1j \) and \( q_2 = Z^2 + Z^3j \) and consider the quaternionic projective space with homogenous coordinates \( \{q_1 : q_2\} \in \mathbb{HP}^1 \). If in the case of complex numbers we had the correspondence \( z_1/z_2 \sim S^2 \sim C \) equivalent to the Hopf map \( S^3 \to S^2 \), in the quaternionic case we analogously have another Hopf map: \( \mathbb{CP}^3 \to \mathbb{S}^4 = \mathbb{HP}^1 \). To demonstrate that this is indeed the case, we recall (from the Definition 6.2.) that \( \mathbb{PT} = \mathbb{CP}^3 \). At the same time, \( \{Z^0, Z^1, Z^2, Z^3\} \in T \) and, using this fact,
we construct a quaternion $q$ of the type $q = \frac{Z_0 + Z_1 j}{Z^2 + Z_3 j}$. In complete analogy with the complex numbers, where we have: $z_1/z_2 = z \to \frac{q_1}{q_2} = q$, now we have:

$$S^4 = \mathbb{H} \cup \{\infty\} \ni q = z_1 + z_2 j = \frac{Z_0 + Z_1 j}{Z^2 + Z_3 j}. \quad (6.22)$$

Recall [55] that the projective space $\mathbb{CP}^n$ can be defined as the quotient

$$\left\{ \sum_{j=1}^{n+1} |\xi_j|^2 = 1 \right\} / (\xi_i \rightarrow e^{i\varphi} \xi_i), \; \xi_i \in \mathbb{C}. \quad (6.23)$$

Therefore, each point in $\mathbb{CP}^n$ should be identified with the circle $S^1$ on $S^{2n+1}$. From here, we obtain the familiar Hopf fibration (for $n = 1$): $S^3/S^1 \simeq S^2$. This logic fails for $n = 3$ where by analogy we formally should expect to have $S^7/S^1$. This quotient is not the Hopf map though. For $n = 3$ the Hopf map is given by the quotient $S^7/S^3 \simeq S^4$. From here we are obtaining the already mentioned correspondence:

$$S^3 \rightarrow z_1/z_2 \sim z \sim \mathbb{CP}^1 \sim S^2 \sim \mathbb{R}^2 \cup \{\infty\} \iff S^7 \rightarrow q_1/q_2 \sim \mathbb{HP}^1 \sim S^4 \sim \mathbb{R}^4 \cup \{\infty\}. \quad (6.24)$$

In view of these results and, taking into account that $q = z_1 + z_2 j; \; z_1, z_2 \in \mathbb{C}$, we can formally write: $q = f(z, w), \; z, w \in \mathbb{C}$. In addition, we have as well: $\{f + gj : h + kj\} \in \mathbb{HP}^1$ and think about the correspondence $f \sim Z^0, g \sim Z^1, h \sim Z^2, k \sim Z^3$ as defining some analytic functions of $z$ and $w$. In terms of such notations Eq.(6.22) can be rewritten as

$$f + gj - (h + kj)(z_1 + z_2 j) = 0 \quad (6.25)$$

Using the multiplication table for quaternions [112], p.39, it is possible to rewrite this result in more suggestive form as

$$(h, k) \left( \begin{array}{cc} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{array} \right) = (f, g). \quad (6.26)$$

By comparing Eq.s(6.21) and (6.26) we identify $z_1$ with $x_0 + ix_1$ and $z_2$ with $x^2 + ix^3$. This identification can be extended by identifying $Z^2$ and $Z^3$ with $h$ and $k$ and $f$ and $g$ with $Z^0$ and $Z^1$. In literature on twistors [113] the spinor language is used. To comply with the standard twistor notations we should make an identification : $Z^0 = \omega^0, Z^1 = \omega^1$ on the one hand and, $Z^2 = \pi_0, Z^3 = \pi_1$ on the other. In terms of such notations Eq.(6.26) acquires the following form [114]:

$$\omega^A = x^{AB} \pi_B, \; A, B = 0, 1. \quad (6.27)$$

Here $\omega$ and $\pi$ are the two-component spinors.

**Definition 6.3.** Eq.(6.27) is known in twistor literature as *incidence relation*. It connects points in $\mathbb{R}^4$ (or Euclidean) space with points in $\mathbb{PT}$.  

39
By changing the matrix $x^{AB}$ describing the Euclidean space to that describing the Minkowski space

$$(x^0, x^1, x^2, x^3) \leftrightarrow (x^{AB}) = \left( \begin{array}{cc} -i (x^0 - x^1) & x^2 + ix^3 \\ -x^2 + ix^3 & -i (x^0 + x^1) \end{array} \right), \ A, B = 0, 1,$$

(6.28)

leads to the appropriately changed determinant: $\det(x^{AB}) = -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2$

and to the incidence relation for the Minkowski space $\mathfrak{M}$.

From previous discussion it is clear that the incidence relation, Eq.(6.27), could be used for both the Euclidean and the Minkowski spaces if we use the complexification: $(x^0, x^1, x^2, x^3) \leftrightarrow (z_1, z_2) \in \mathbb{C}^2$ leading to the matrix $x^{AB}$ defined by

$$x^{AB} = \left( \begin{array}{cc} \bar{z} & w \\ \bar{w} & z \end{array} \right)$$

(6.29)

in which $\bar{z}$ and $\bar{w}$ should not be treated as complex conjugates of $z$ and $w$ (unless otherwise specified). In terms of such notations the incidence relation, Eq.(6.27), can be rewritten as

$$\bar{z}Z^2 + wZ^3 = Z^0,$$

$$\bar{w}Z^2 + zZ^3 = Z^1.$$

(6.30a)

Geometrically, this is the system of equations for two hyperplanes. In the language of algebraic/projective geometry they can be conveniently rewritten as

$$Z^\alpha A_\alpha = 0 \text{ and } Z^\alpha B_\alpha = 0.$$  

(6.30b)

The question of interest is: Under what conditions these two planes intersect? Why should we be interested in this question? Because we would like to connect just described twistor formalism with that presented in previous subsection. To do so, following [55] we consider $2 \times 4$ matrix $\mathbf{M}$ of the type

$$\mathbf{M} = \left( \begin{array}{cccc} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{array} \right)$$

(6.31)

Its 1st and 2nd rows are made of vectors $\mathbf{x}^p$ and $\mathbf{y}^p$ respectively. All Plücker coordinates are obtainable now as determinants of columns $i$ and $j$ of $\mathbf{M}$. In view of Eq.(6.16), the matrix $\mathbf{M}$ can be used in the following matrix equation

$$x_i^p = \sum_{j=0}^{3} M_{ij} e_j, \ i = 1, 2.$$  

(6.32)

In the system of linear equations, Eq.(6.30a), one can identify $Z^0, Z^1, Z^2, Z^3$ with the vectors $e_j, j = 0 \div 3$, in Eq.(6.32). Then, the matrix $\mathbf{M}$ acquires the following form

$$\mathbf{M} = \left( \begin{array}{cccc} \bar{z} & w & 0 & 1 \\ \bar{w} & z & 1 & 0 \end{array} \right).$$

(6.33)
Since the Plücker coordinates for this matrix are:
\[ l_{01} = \tilde{z}z - \tilde{w}w, l_{02} = \tilde{z}, l_{03} = -\tilde{w}, l_{12} = w, l_{13} = -z, l_{23} = -1, \]
the Plücker embedding condition, Eq.(6.18), acquires the following form
\[ (-1)(\tilde{z}z - \tilde{w}w) + \tilde{z}z - \tilde{w}w = 0 \quad (6.34) \]
and, is trivially satisfied.

Next, we would like to explain how just obtained result is connected with the intersection of two hyperplanes, Eq.s(6.30b), in \( \text{PT} = \mathbb{CP}^3 \). Following [115], page 83, consider a hyperplane through the origin \( O \) in \( \mathbb{CP}^3 \). It is described by the system of two equations
\[
\begin{align*}
l_0 z_0 + l_1 z_1 + l_2 z_2 + l_3 z_3 &= 0, \\
m_0 z_0 + m_1 z_1 + m_2 z_2 + m_3 z_3 &= 0.
\end{align*}
\]
(6.35)
The description of this plane remains unchanged if instead of \( l_0, l_1, l_2, l_3 \) we would use \( l_0 + \lambda m_0, l_1 + \lambda m_1, \ldots \), where \( \lambda \) is an arbitrary parameter. This observation makes the system of Eq.s(6.30a) equivalent to the system of Eq.s(6.35). A symmetrical set of coordinates is obtained by defining the six expressions
\[ l_{ij} = l_i - l_j, i, j = 0, 1, 2, 3. \]
Use of these expressions in Eq.s(6.35) allows us to eliminate successively \( z_0, z_1, z_2, z_3 \) resulting in the following system of equations
\[
\begin{align*}
l_{01} z_1 + l_{02} z_2 + l_{03} z_3 &= 0, \\
l_{10} z_0 + l_{12} z_2 + l_{13} z_3 &= 0, \\
l_{20} z_0 + l_{21} z_1 + l_{23} z_3 &= 0, \\
l_{30} z_0 + l_{31} z_1 + l_{32} z_2 &= 0.
\end{align*}
\]
(6.36)
Elimination of \( z_0, z_1 \) and \( z_2 \) from these equations results in the Plücker relation, Eq.(6.18c), and the reminder equation is the equation for a complex projective line in \( \mathbb{CP}^3 \). So that, indeed, two planes, Eq.s(6.30b), are intersecting in a line. More details can be found in [116], pages 141-144. In view of Eq.s(6.15)-(6.17) six Plücker coordinates \( l_{01}, l_{02}, l_{03}, l_{23}, l_{31}, l_{12} \) represent a point in \( \mathbb{CP}^5 \) so that Eq.(6.18c) represents a complex quadric \( Q_4 \) in this projective space \( \mathbb{CP}^5 \)[116]. Following [115] it is useful to rewrite this quadric in terms of new variables
\[
\begin{align*}
l_{03} &= u_0 + u_3, l_{13} = u_1 + u_4, l_{23} = u_2 + u_5, \\
l_{12} &= u_0 - u_3, l_{20} = u_1 - u_4, l_{01} = u_2 - u_5.
\end{align*}
\]
(6.37)
Use of these variables converts Eq(6.18c) into
\[
\begin{align*}
u_0^2 + u_1^2 + u_2^2 - u_3^2 - u_4^2 - u_5^2 = 0.
\end{align*}
\]
(6.38)
Since \( u_i \)'s are complex we can adopt them for physically relevant situations. We begin with

\[ ^{30} \text{In the context of the Lie sphere geometry this equation is discussed in connection with Eq.(7.15) of section 7.} \]
**Definition 6.4.** The *null cone* \( \Gamma \) of the origin is defined (according to Eq.(4.1)) as

\[
0 = \Gamma(t, x; \tau, y) = c^2(t - \tau)^2 - \sum_{i=1}^{3}(x_i - y_i)^2 = T^2 - X^2 - Y^2 - Z^2. \tag{6.39a}
\]

The *generators* of \( \Gamma \)- the *null rays*— are subject to the constraint

\[
T : X : Y : Z = \text{const.} \tag{6.39b}
\]

The Lorentz transformation \( L \) sends the set of generators of \( \Gamma \) into another set of generators of \( \Gamma \) preserving \( \Gamma \). Following Penrose [117] we define the group \( C(2) \) as the group of all conformal maps of the compactified complex plane, that is of \( S^2 \), to itself. The connected component of the identity in \( C(2) \) consists of the orientation-preserving conformal maps \( \varsigma \to f(\varsigma) \) of \( S^2 \), given by

\[
\varsigma \to f(\varsigma) = \frac{\alpha \varsigma + \beta}{\gamma \varsigma + \delta}. \tag{6.40}
\]

Details are given in [105]. From this reference we find that the homomorphism of the Lorentz group \( O(3,1) \) into \( C(2) \) is 2 ÷ 1. The quadratic form \( \Gamma \) defined by Eq.(6.39a) is obtainable from the quadric \( Q_4 \) defined in Eq.(6.38) if we require \( u_0^2 + u_1^2 = 0 \). The geometrical and topological meaning of these two extra terms is associated with conformal symmetry typical for the Lie sphere geometry to be described below. In the meantime, at this moment, we restrict ourselves by the concepts which were already in use in mathematical physics literature. For this purpose we recall some results from section 1. In it we mentioned that locally the static Einsteinian space-times are Minkowskian. Let \( \mathcal{M}^3 \) be some 3-manifold so that topologically all static Einsteinian spacetimes are of the form \( \mathcal{M}^3 \times \mathbb{R} \) [118], where \( \mathbb{R} \) represents time. The positivity of mass theorem in general relativity, used in [119], superimposed with use of the Yamabe theorem [120], allows us to replace \( \mathcal{M}^3 \times \mathbb{R} \) by \( S^3 \times \mathbb{R} \). The compactification requirement causes us to replace this manifold by \( S^3 \times S^1 \). But analytically \( S^1 \) is \( u_0^2 + u_1^2 = \text{const.} \). Accordingly, Eq.(6.38) does contain information about compactification. We already know that the twistor space is \( \mathbb{C}^4 \) which is the complexification of \( \mathbb{R}^4 \) whose compactification is \( S^4 \). Thus, we end up with the compactified Minkowski space \( \mathcal{M} \simeq S^3 \times S^1 \) and the compactified Euclidean \( \mathcal{E} \) space \( S^4 \). Both are described by the Kleinian quadric \( Q_4 \) [116] defined by Eq.(6.38). The sphere \( S^4 \) is described by

\[
\mathcal{E} : u_0^2 + u_1^2 = u_2^2 + u_3^2 + u_4^2 - u_5^2 = 0 \tag{6.41a}
\]

while the (compactified) Minkowski space by

\[
\mathcal{M} : u_0^2 + u_1^2 = u_2^2 + u_3^2 + u_4^2 - u_5^2 = 0 \tag{6.41b}
\]

\[\text{Eq.}(6.41b) \text{ is discussed in terms of the formalism of the Lie sphere geometry in section 7, in connection with Eq.s (7.10) and (7.20).}\]
**Definition 6.5.** The *null cone* of the origin of the compactified Minkowski space $\mathfrak{M}$ is described by Eq.(6.41b). The symmetry group leaving this null cone invariant is the conformal group $SO(4,2)$. This is the largest symmetry group of the hydrogen atom.

We have reached these conclusions using arguments entirely different from those developed in the group-theoretic [19] and high energy physics[83] literature. Use of the concepts of Lie sphere geometry to be discussed in section 7 provides us with solid theoretical framework for dealing with just discussed problems. At the same time, the obtained results are sufficient for finishing our study of Huygens triviality of the stationary Schrödinger equation.

### 6.3. Harmonic analysis and Huygens’ triviality

In Fock’s paper [102] the stereographic projection was used to relate $p = \{p_x, p_y, p_z\} \in \mathbb{R}^3$ and $\bar{p} = \{p_0, p_x, p_y, p_z\} \in \mathbb{R}^4$. The momentum space $p$ was one point ($p_\infty$) compactified to 3-sphere $S^3$ living in $\mathbb{R}^4$. If $N$ is the North pole of the 3-sphere, $N:=(0,0,0,1)$, we define $U_N := S^3 \setminus N$.

**Definition 6.6.** The *stereographic projection* $\pi$ is the map $\pi : U_N \to \mathbb{R}^3$ defined by

$$\pi : x_k = \frac{\xi_k}{1 - \xi_4}, \quad k = 1, 2, 3. \quad (6.42a)$$

Its inverse $\pi^{-1}$ is defined by

$$\pi^{-1} : \xi_k = \frac{2x_k}{1 + \|x\|^2}, \quad \xi_4 = \frac{\|x\|^2 - 1}{\|x\|^2 + 1}, \quad k = 1, 2, 3. \quad (6.42b)$$

The map $\pi$ can be extended to $\bar{\pi}$ defined as follows. By relating the North pole $N$ for the 3-sphere to the compactification point $p_\infty = \{\infty\}$ all $S^3$ is covered. Since $\xi_4 \neq 1$, just defined extended stereographic projection $\bar{\pi}$ provides a bijective correspondence between points on $S^3$ and points of the compactified momentum space $p \cup \{\infty\}$. Using such stereographic projection Fock transformed the integral equation Eq.(6.1b), defined on $\mathbb{R}^3$, into the integral equation on $S^3$. As result, he demonstrated that solutions of the integral equation inside $S^3$ are harmonic functions. That is they are solutions of the 4-dimensional Laplacian. The D’Alembertian, Eq.(2.6b), is obtainable from the Laplacian via formal replacement $x_0 \to \pm ix_0$ discussed in subsecton 6.2.1. Although Fock did not discuss the $E > 0$ case in his paper, he did mention about the relevance of the Lobachevsky (that is hyperbolic) space for the description of $E > 0$ case. At present, it is possible to find in literature solutions describing $E > 0$ case. The detailed calculations are given, for example, in [121]. Although these calculations formally solve the $E > 0$ problem, they are not revealing its mathematical essence. This essence was described already in Fock’s paper [102] but was left undeveloped. Only recently Fock’s remarks were put into plausible mathematical form. In 2008 in the paper...
by Frenkel and Libine [122] in Advances in Mathematics entitled "Quaternionic analysis, representation theory and physics" Fock's results for both $E < 0$ and $E > 0$ were rederived. As result, it is sufficient to use only results by Frenkel and Libine for proving Huygens' triviality of the Schrödinger equation (with time-independent Hamiltonian).

At this point it is logical to present a condensed summary of Fock's results along with their subsequent improvements. Fock noticed that Eq.(6.1b) when stereographically lifted to the 3-sphere is looking very much the same as the Poisson formula for the harmonic functions inside the circle. It should be noticed, though, that Fock was not referring to the Poisson formula explicitly. The description of this formula employs the standard complex analysis [123]. Its derivation begins with the use of the Cauchy-Riemann equations. Recall, that these are defined as follows. Let $u(x, y)$ and $v(x, y)$ be some analytic functions satisfying the Cauchy-Riemann equations

$$
\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}.
$$

Since

$$
\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} \quad \text{and} \quad \frac{\partial^2 v}{\partial x \partial y} = -\frac{\partial^2 v}{\partial y \partial x},
$$

both $u$ and $v$ are harmonic functions. That is they satisfy the Laplace equation: $\triangle u = \triangle v = 0$, $\triangle = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2}$. The crucial theorem leading to the Poisson formula exploits this fact.

**Theorem 6.7.** Let $u(x, y)$ be the harmonic function in some simply connected domain $G$ of the complex plane $C$. Then, there is a regular in $G$ function $f(z)$ such that $u(x, y) = \text{Re} f(x + iy)$. Function $f(z)$ is determined with help of $u(x, y)$ with accuracy up to purely imaginary constant.

The Poisson formula can then be defined as solution of the Dirichlet problem: Find a function harmonic in $G$ and coinciding with some prescribed function $g(z)$ at the boundary of $G$. Since any connected domain can be converted into a disc of, say, radius $R$, the solution of the Dirichlet problem can be represented via the Poisson integral formula as follows

$$
u(x, y) = \frac{1}{2\pi} \int_0^{2\pi} \frac{u(Re^{i\varphi})}{R^2 - 2Rr \cos(\varphi - \psi) + r^2} \, d\varphi. \quad (6.43a)$$

Let now $R= 1$ in Eq.(6.43), then this formula acquires look very similar to formula Eq.(9.19) of Fock's paper [102]. Nevertheless, Fock's Eq.(9.19) cannot be qualified as the Poisson formula. The Poisson formula is just a modification of the Cauchy integral formula [123]

$$
f(z) = \frac{1}{2\pi} \int \frac{f(w) \, dw}{w - z}. \quad (6.43b)$$

In previous subsections we demonstrated how quaternions replace complex numbers when $C \rightarrow C^2 \sim R^4$. The question arises: Can the rest of results of complex analysis be rewritten in the quaternionic language in $R^4$? The development of the theory of quaternionic functions
was initiated by Rudolf Fueter. In 1935, the year of publication of Fock’s paper, he produced an exact quaternionic counterpart of the Cauchy integral formula [122],[124]. However, it took another 73 years before this formula was converted into quaternionic analog of the Poisson formula. This happened only in 2008. Frenkel and Libine [122] used the quaternionic version of the Poisson formula for demonstration that it correctly reproduces both the discrete and continuous spectrum of the hydrogen atom. The way of derivation of these spectra is opposite to that used by Fock. Specifically, Fock used the stereographic projection to bring the 3 dimensional integral equation, Eq.(6.1b), into 4-dimensional Poisson -like form. Furthermore, the wavefunction satisfying the Laplace Eq.(9.12) of Fock’s paper is not the same as that obtainable from his 4-dimensional Poisson-like integral equation. For \( E < 0 \) the integral Eq.(6.1b), when converted into the 4-dimensional form, acquires the following look

\[
\frac{1}{2\pi^2} \int_{S^3} \frac{\Psi(X')}{\|X - X'\|^2} d\Omega' = p_0 \Psi(X)
\] (6.44)

to be compared with Eq.(6.43b). Here \( d\Omega' \) is known volume element of the 3-sphere. The \( X \) coordinates are \( X=\{\xi_1,\ldots,\xi_4\} \). These are given by Eq.s(6.42b). The relation between \( \Psi(X) \) in Eq.(6.44) and \( \psi(p) \) in Eq.(6.1b) is given by [125], page 83,

\[
\Psi(X) = \left[ \frac{p^2 + p_0^2}{2p_0^2} \right] \psi(p), \quad (6.45a)
\]
or

\[
\Psi(X) = \frac{p_0^2}{X_k^2} \psi(p). \quad (6.45b)
\]

It would appear that the analogous relation for \( E > 0 \) formally would solve the Hyugens triviality problem in view of the transformation rules defined in section 4. This is not the case, however. To explain the existing problem it is useful to briefly discuss the case \( E < 0 \) first, since it is easier. In his paper Fock introduces the stereographic projection by defining coordinates \( \xi, \eta, \zeta \) and \( \chi \) such that \( \xi^2 + \eta^2 + \zeta^2 + \chi^2 = 1, \xi = \frac{2p_0 x}{r_0^2 + p^2}, \text{etc} \). In addition, he introduces in a rather arbitrary fashion still another set of coordinates

\[
x_1 = r\xi; x_2 = r\eta; x_3 = r\zeta; x_4 = r\chi \quad (6.46a)
\]
along with the 4-dimensional Laplace equation

\[
\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} + \frac{\partial^2 u}{\partial x_4^2} = 0. \quad (6.46b)
\]

Since the harmonic function \( u(x_1, x_2, x_3, x_4) \) satisfying this equation is not a solution of the integral Eq.(6.44) (surprisingly), clearly Eq.(6.44) cannot be qualified as the Poisson formula adopted to four dimensions. At the same time, if \( r \) in Eq.(7.46a) is the radius of the 3-sphere (eventually \( r = 1 \)) then, following Fock, it is possible to represent the harmonic function as

\[
u = r^{n-1} \Psi_n(X).
\]
Following Cordani [125], page 90, we would like to rewrite Fock’s results a bit differently. Thus, let \( h_l(X) \) be a homogenous harmonic polynomial of degree \( l \) in \( \mathbb{R}^{n+1} \). That is \( \Delta_{\mathbb{R}^{n+1}} h_l(X) = 0 \) and \( \sum_{i=1}^{n+1} X_i \frac{\partial}{\partial X_i} h_l(X) = l h_l(X) \). Then, if \( \varsigma \in S^n \),

\[
\Delta_{\mathbb{R}^{n+1}} h_l(X) = \Delta_{\mathbb{R}^{n+1}}(r^l Y_l(\varsigma)) = \left( \frac{\partial^2}{\partial r^2} + \frac{n}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{S^n} \right)(r^l Y_l(\varsigma)) = 0 \quad (6.47a)
\]

leading to

\[
[(l(l + n - 1) + \Delta_{S^n}) Y_l(\varsigma) = 0. \quad (6.47b)
\]

Let \( \vec{X} = r \vec{n} \), where \( \vec{n} \) is the variable unit vector on \( \mathbb{R}^{n+1} \). In 4 dimensions \( \mathbb{n} = \{ \xi, \eta, \zeta, \chi \} \), e.g. see Eq.(6.46a). The normal derivative of \( h_l(X) \) on the unit sphere \( S^n \) is given by \( \frac{\partial}{\partial n}(r^l Y_l(\varsigma)) = l Y_l(\varsigma) \). Application of Green’s third identity [121], page 333, and [125], page 87, permits us then to rewrite the l.h.s. of Eq.(6.44) with help of just defined results in the following form

\[
\frac{1}{2\pi^2} \int_{S^3} \frac{Y_l(X')}{||X - X'||^2} dY' = \frac{1}{1 + l} Y_l(X), \quad l = 0, 1, 2, \ldots \quad (6.48)
\]

Comparison with the r.h.s. of Eq.(6.44) yields the discrete portion of the spectrum, \( p_0 = \sqrt{-2mE} = (1 + l)^{-1} \), for the hydrogen atom. To get the continuum portion of the hydrogen atom spectrum requires us to make some redefinitions of the already obtained results. It is convenient to represent the continuum and discrete results side-by-side. Specifically,

Discrete spectrum :

\[
\text{unit sphere} \quad S^n = \{ x \in \mathbb{R}^{n+1} \mid ||x||^2_L = 1 \}; \quad (6.49a)
\]

Continuous spectrum :

\[
\text{unit hyperboloid of two sheets} \quad F^n = \{ x \in \mathbb{R}^{n+1} \mid ||x||^2_L = -1 \}. \quad (6.49b)
\]

Here \( E \) stands for scalar product in space of Euclidean signature while \( L \) stands for scalar product in the space of Lorentzian signature. In particular, in the last case we have

\[
< x, y>_L = x_1 y_1 + \cdots + x_n y_n - x_{n+1} y_{n+1}. \quad < x, x>_L = ||x||^2_L
\]

so that the result \( ||x||^2_L = -1 \) should be looked upon either as an option \( H^n_+ = \{ x \in \mathbb{R}^{n+1} \mid ||x||^2_L = -1, x_{n+1} > 0 \} \) or as an option \( H^n_- = \{ x \in \mathbb{R}^{n+1} \mid ||x||^2_L = -1, x_{n+1} < 0 \} \). Therefore, \( F^n = H^n_+ \cup H^n_- \).

Use of stereographic projection is defined by [81] by analogy with Eqs. (6.42a), (6.42b) leads to the following identifications:

Unit hyperboloid of two sheets :

\[
F^n, \pi : x_k = \frac{\xi_k}{1 + \xi_{n+1}}, \quad k = 1, 2, \ldots n;
\]

\[
\pi^{-1} : \xi_k = \frac{2x_k}{1 - ||x||^2}, \xi_{n+1} = \frac{||x||^2 + 1}{||x||^2 - 1}, \quad k = 1, \ldots, n.
\]
Accordingly, for Eq. (6.44) we obtain:

\[
\frac{1}{2\pi^2} \int_{S^3} \frac{\Psi(X')}{\|X - X'\|^2} d\Omega' = p_0 \Psi(X) \quad \text{(3-sphere);} \tag{6.50a}
\]

\[
-\varepsilon(\xi_4) \frac{1}{2\pi^2} \int_{F^3} \frac{\Psi(X')}{\|X - X'\|^2} d\Omega' = p_0 \Psi(X) \quad \text{(hyperboloid of two sheets).} \tag{6.50b}
\]

Here \( \varepsilon(\xi_4) = 1 \) for \( \xi_4 \geq 1 \) and \( \varepsilon(\xi_4) = -1 \) for \( \xi_4 < 1 \).

For \( E > 0 \) the Laplace Eq. (6.46b) is replaced by the D'Alembert Eq. (2.6a). Accordingly, Eq. (6.47a) is now being replaced by

\[
\square_{R^{n+1}} [r^{\lambda} H_{N,\alpha,\beta}(\varsigma)] = 0 \tag{6.51}
\]

with \( \lambda = -\frac{1}{2}(n - 1) + iN \). \( N \) is being a real number. Since the solution of Eq. (6.51) presented in [121] contains many gaps in logic it is not going to be discussed any further in this paper. However one should keep in mind that it is this solution which is being used in solving the integral Eq. (6.50b). Thus, the harmonic function \( u = r^{n-1} \Psi_n(X) \) solving the Laplace Eq. (6.46b) for \( E < 0 \) is being replaced by \( u = r^N H_{N,\alpha,\beta}(\varsigma) \), where the function \( u \) is solution of the D’Alembert Eq. (2.6a) (or Eq. (6.51)) with the boundary conditions on \( F^3 \). As in the case of \( E < 0 \), the spherical harmonic \( \Psi(X) = H_{N,\alpha,\beta}(\varsigma) \) solves the integral Eq. (6.50b) for \( E > 0 \). However, the solution protocol for \( E > 0 \) is very different. For us it is essential that the function \( u \) is the solution of D’Alembert equation in the Minkowski spacetime having the hyperboloid of two sheets \( F^3 \) as boundary. Eqs (6.45a) and (6.45b) are being replaced now by

\[
\Psi(X) = \left[ \frac{p_0^2 - p^2}{2p_0^2} \right]^2 \psi(p), \tag{6.52a}
\]

or

\[
\Psi(X) = \frac{p_k^2}{X_k} \psi(p). \tag{6.52b}
\]

Suppose that solution \( H_{N,\alpha,\beta}(\varsigma) \) of Eq. (6.51) of the integral Eq. (6.50b) is found, then the solution of the D’Alembert Eq. (6.51) is given by \( r^N H_{N,\alpha,\beta}(\varsigma) \). In the case \( E < 0 \) the stereographic coordinates \( X = \{\xi_1, \ldots, \xi_4\} = n = \{\xi, \eta, \zeta, \chi\} \) had been artificially extended to \( \vec{X} = r\vec{n} \) so that if \( \Psi(X) \) defined on the 3-sphere is the solution of the integral equation, Eq. (6.50), \( \Psi(\vec{X}) \) is the solution of the Laplace Eq. (6.46b). Analogous replacement in \( E > 0 \) case produces solution of the D’Alembert equation from solution of the integral Eq. (6.50b). When just described extension of stereographic coordinates is combined with Eqs (6.52) such hyperbolic analog of \( \Psi(\vec{X}) \) solves the problem of Hadamard triviality. This follows in view of the fact that the Schrödinger operator \( \hat{L}[\phi], \) Eq. (6.1b), is obtainable from the D’Alembert operator \( \hat{L}[\phi] \) via sequence of non-singular transformations of independent variables (stereographic projection and extension). Thus, the Schrödinger operator for the time-independent Schrödinger equation is Hyugens’ trivial. At the physical level of rigor the same result was obtained by I. Bars [99] using two time formalism.
Frenkel and Libine solution of hydrogen atom model resulting in spectrum for bound \( E < 0 \) and scattering \( E > 0 \) states \([122]\) improve and considerably simplify just described results since the quaternionic Poisson formula is relating the harmonic functions inside \( S^3 \) and \( F^3 \) with solutions of the stationary Schrödinger Eq.(6.1) for both \( E < 0 \) and \( E > 0 \) as required.

7. Physical uses of Lie sphere geometry.

7.1. Bird's view of the two-times formalism of I.Bars

In \([83]\) the program of two-times formalism was outlined while in \([99]\) detailed calculations illustrating general principles are presented. In short, the main idea is to replace 3+1 Minkowski spacetime by more general (fundamental) 4+2 spacetime having 4 space and 2 time variables so that the signature of this a spacetime is \((+,+,+,+,-,-)\). Evidently, the symmetry of such spacetime is either \(O(4,2)\) or \(SO(4,2)\). In such an extended spacetime there could possibly exist one fundamental model casting countable number of "shadows" (projections) into much more familiar Minkowski spacetime. Each "shadow" is perceived as known distinct particle physics model. Under an umbrella of 2-times formalism, it appears, that apparently different physical models actually are having the same origin in 4+2 spacetime. Mathematically speaking, the idea is to distribute all particles (and their bound states) of high energy physics into equivalence classes in accordance with the basic (fundamental) models living in 4+2 dimensional spacetime. Since the already made calculations connect massless relativistic particles with the massive ones (even accounting for the spin), the concept of a mass of particle and significance of the Higgs mechanism for mass generation loose their central importance in this formalism. Furthermore, the same formalism connects, for example, the relativistic particles (massive or not) with the non-relativistic massive particles, the extended (bound) systems, such as hydrogen atom, with the massless relativistic particles, etc.

In section 6 we demonstrated how the nonrelativistic Schrödinger equation for hydrogen atom is connected with the D’Alembert (in 3+1 dimensions) or the Laplace (in 4 dimensions), that is with equations describing the massless and spinless relativistic particle. In section 5 we demonstrated how relativistic equations for massive particles can be reduced to those whose masses are zero. We were guided by the Hadamard ideas of Huygens triviality casting all Huygens-trivial equations (e.g. read section 4) into the same equivalence class. In another paper \([126]\) Bars connected the hydrogen atom with the massless relativistic free particle and with the massive 3-dimensional harmonic oscillator. Thus, he effectively demonstrated Huygens triviality of the hydrogen-atom and harmonic oscillator. His work develops results of \([127]\) published previously in which Bars (with collaborators) raised a question: Which of 2 times in the 2 time formalism is the familiar time coordinate? Unfortunately for his project, no clear-cut answer to this question was given either in \([127]\) or in the basic reference \([83]\) summarizing all accomplishments of the 2-times formalism. Accordingly, the 2-times formalism did not gain much popularity among the high energy physicists. Nevertheless, in our opinion, upon development, e.g. based on results of this work, there could be a chance to make formalism developed by Bars more viable.
We initiate this process of further development with the discussion of the relationship between the dynamics of hydrogen atom and harmonic oscillator. Being driven by the same problem of regularization of singularities (e.g. read again comments to Eq.(6.13) for the Kepler problem), Kustaanhemo and Stiefel (K-S) using the Hopf mapping (section 6) transformed the 3 dimensional classical dynamics of the Kepler problem into the 4-dimensional problem of classical dynamics of the harmonic oscillator on $S^3$ [128]. Their work was extended by many authors both at the classical and quantum levels. At the quantum level readable account of uses of the K-S transformation converting hydrogen atom problem into harmonic oscillator problem is given in [129]. An independent and much simpler treatment of the same conversion problem was given in [130] by Chen. By utilizing results of [130], Chen demonstrated in [131] the isomorphism between the largest symmetry group SO(4,2) of the hydrogen atom and the group SU(2,2) used in physics of twistors [25]. Later on he connected group-theoretically hydrogenic, oscillator and free-particle massless relativistic systems in [132]. In his derivations Chen used only one-time formalism. His results do not use twistors or compactification of the Minkowski space. Thus, they cannot be immediately linked with discussions involving the Lie sphere geometry-a tool essential for generalization of these results. At the same time, the formalism by Bars, perhaps if further developed, can accommodate the concepts of Lie sphere geometry.

7.2. Lie sphere geometry. Fundamentals

The examples discussed in previous subsection indicate that 2-times formalism invented and developed by Bars in some instances can be entirely replaced by more familiar 1 time formalism. The examples of such a replacement can hardly be generalized though. At the same time, the diversity of results obtained with help of the 2 times formalism is appealing. Thus, in this subsection we would like to suggest a reliable direction enabling us to replace questionable 2-times formalism by mathematically well developed formalism of the Lie sphere and the Möbius geometries. In physics literature the formalism based of utilizing results of Möbius geometry is known as "geometric algebra".

We begin with the description of Lie geometry since the Möbius geometry (and, therefore, geometric algebra) is more restrictive. The Lie geometry is the geometry of generalized oriented hyperspheres living in the compactified Euclidean space $S^n = \mathbb{R}^n \cup \{\infty\}$. The elements of this geometry are:

a) Oriented hyperspheres. These are familiar from section 4 spheres $S_{c,r}$ of finite radius $r > 0$ with center $c \in \mathbb{R}^n$ so that

$$S_{c,r} = \{ x \in \mathbb{R}^n \mid \| x - c \|^2 = r \}.$$  \hspace{1cm} (7.1)

The hypersphere $S_{c,r}$ divides $\mathbb{R}^n$ into 2 parts. If we denote one of these parts as "positive", then another part is "negative". By introducing such a distinction we are introducing the
oriented hyperspheres. In such a case a given $S_{c,r}$ is replaced by $S_{c,r}^\pm$. These results generalize 3 dimensional result, Eq.(4.25), needed for definition of canal surfaces and Dupin cyclides.

In higher dimensions the cyclides of Dupin are being replaced by the Dupin hypersurfaces. Such hypersurfaces can live in Euclidean and hyperbolic spaces [133].

**Definition 7.1.** Hypersurfaces with constant principal curvatures are **Dupin hypersurfaces**.

To distinguish the oriented hyperspheres analytically it is convenient to introduce *signed radius* which is telling us whether we should consider the inward or outward field of unit normals. According to convention, the positive radii $r > 0$ are assigned to hyperspheres with the inward field of unit normals while the negative radii $r < 0$ are assigned to hyperspheres with the outward field of unit normals. By doing so we just had introduced a bijection between the hypersurfaces of non-vanishing radius and tuples

$$(c, r), c \in \mathbb{R}^n, \ r \in \mathbb{R}^*.$$  \hspace{1cm} (7.2)

Here $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$.

b) **Oriented hyperplanes.** A hyperplane $P$ in $\mathbb{R}^n$ is characterized by the equation

$$P = \{x \in \mathbb{R}^n \mid \langle n, x \rangle = d\}$$  \hspace{1cm} (7.3)

with a unit normal $n \in S^{n-1}$ and $d = \mathbb{R}$. Evidently, the tuples $(n, d)$ and $(-n, -d)$ represent the same hyperplane. As a hypersphere, it divides $\mathbb{R}^n$ into two half spaces. By declaring one of the two half spaces to be positive, we are getting the notion of oriented hyperplane. Thus, by analogy with hyperspheres, we obtain a splitting: $P \to P^\pm$ for any $P$.

c) **Points.** Points are hyperspheres of zero radius.

d) **Infinity.** Infinity point is making $\mathbb{R}^n$ compactified, that is $S^n = \mathbb{R}^n \cup \{\infty\}$.

e) **Contact elements.** A set of all hyperspheres through $x \in \mathbb{R}^n$ which are in oriented contact with $P$ and with one another thus all sharing normal vector $n$ at $x$.

**Remark 7.2.** The connection/replacement with/of 2 times formalism by Bars follows now from the observation that all just described elements are modelled as points, respectively lines, in the projective space $P(\mathbb{R}^{n+1,2})$ with the space of homogenous coordinates $\mathbb{R}^{n+1,2}$. Details follow below.

From now on, for simplicity, we shall only discuss the case: $n = 3$. Following Suris [80] we equip the space $\mathbb{R}^{4,2}$ with 6 linearly independent unit vectors $e_1, \ldots, e_6$ whose scalar product is defined as

$$\langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } i = j \in \{1, \ldots, 4\}, \\ -1 & \text{if } i = j \in \{5, 6\}, \\ 0 & \text{if } i \neq j. \end{cases}$$
To exemplify the properties of Lie sphere geometry in the most efficient way, it is convenient to make the following redefinitions:

\[ e_0 = \frac{1}{2}(e_5 - e_4), e_\infty = \frac{1}{2}(e_5 + e_4) \quad \text{implying} \]
\[ < e_0, e_0 > = < e_\infty, e_\infty > = 0; < e_0, e_\infty > = -\frac{1}{2}. \quad (7.4a) \]

In terms of just made definitions we redefine the objects of Lie sphere geometry as follows:

**Oriented sphere** \( \hat{s} \) with center \( c \in \mathbb{R}^3 \) and signed radius \( r \in \mathbb{R} \):

\[ \hat{s} = c + e_0 + (|c|^2 - r^2)e_\infty + re_6. \quad (7.5) \]

**Oriented plane** \( \hat{P} \) defined by \( < n, x > = d \) with \( n \in S^2 \) and \( d \in \mathbb{R} \):

\[ \hat{P} = n + 0 \cdot e_0 + 2d e_\infty + 0 \cdot e_6. \quad (7.6) \]

**Point** \( \hat{x} \in \mathbb{R}^3 \):

\[ \hat{x} = x + e_0 + |x|^2 e_\infty + 0 \cdot e_6. \quad (7.7) \]

**Infinity**:

\[ \hat{\infty} = e_\infty. \quad (7.8) \]

**Contact element**:

\[ \text{span}(\hat{x}, \hat{P}). \quad (7.9) \]

**Remark 7.3.** In the projective space \( P(\mathbb{R}^{n+1,2}) \) the first four types of elements are represented by the **points**. These are the equivalence classes of the above vectors with respect to the equivalence relation \( \xi \sim \eta \iff \xi = \lambda \eta, \lambda \in \mathbb{R}^*, \xi, \eta \in \mathbb{R}^{n+1,2} \).

**Remark 7.4.** A contact element in \( \mathbb{R}^n \) is an **isotropic line** in \( P(\mathbb{R}^{n+1,2}) \). This line is defined in Eq. (7.11) below.

To be specific, we need to introduce the **Lie quadric**. This is accomplished as follows. First, we define the set of isotropic vectors in \( \mathbb{R}^{n+1,2} \) via

\[ L^{n+1,2} = \{ \hat{v} \in \mathbb{R}^{n+1,2} | < \hat{v}, \hat{v} > = 0 \}, \]

then the Lie quadric \( Q^{n+1,2} \) is defined as \( Q^{n+1,2} = P(L^{n+1,2}) \).

**Remark 7.5.** For \( n = 3 \) the Lie quadric just defined coincides with that defined by Eq. (6.41b) representing properties of the compactified Minkowski space \( \mathfrak{M} \).

It was obtained via twistor formalism in previous section thus hinting at the connection between the twistor and the Lie sphere formalisms. That this is indeed the case will be explained below, in subsection 7.4.

Going back to contact elements, choose now two hyperspheres (or spheres if \( n = 3 \)) \( \hat{s}_1, \hat{s}_2 \in \hat{v} \subset \mathbb{R}^{n+1,2} \) such that \( < \hat{s}_1, \hat{s}_2 > = 0 \). Then, these spheres are in oriented contact with...
each other. An elementary proof can be found in [134], page 15. Using this fact, we can define a line \( L \) in projective space \( P(L^{n+1,2}) \). For this purpose let \( \alpha_1 \) and \( \alpha_2 \) be some real numbers. Using these numbers we define a linear combination \( \hat{s} = \alpha_1 \hat{s}_1 + \alpha_2 \hat{s}_2 \), a projective line. Using it, we obtain:

\[
< \hat{s}, \hat{s} > = \alpha_1^2 < \hat{s}_1, \hat{s}_1 > + \alpha_2^2 < \hat{s}_2, \hat{s}_2 > + 2 \alpha_1 \alpha_2 < \hat{s}_1, \hat{s}_2 > = 0
\]  

(7.11)
since both \( \hat{s}_1 \) and \( \hat{s}_2 \) are isotropic vectors. If \( \hat{s}_1 \) and \( \hat{s}_2 \) in \( \mathbb{R}^{4,2} \) represent two spheres in oriented contact, then the line \( L \) in \( P(L^{4,2}) \) through the corresponding points is isotropic as just demonstrated. It lies entirely on the Lie quadric \( P(L^{4,2}) \). There are no projective subspaces of higher dimensions completely contained in \( P(L^{4,2}) \). This is proven in [134], page 17. In just described formalism planes are spheres of infinite radii and points are spheres of zero radii. The Definition 4.7. can be restated now as follows

**Definition 7.6.** The essence of Lie geometry lies in study of projective transformations of \( P(L^{4,2}) \) (for \( n = 3 \)) leaving the Lie quadric \( Q^{4,2} \) invariant. The group of such transformations is factor group \( O(n+1,2)/\{1, -1\} = PO(n+1,2) \).

This quotient is just the higher dimensional extension of the earlier discussed quotient \( O(3,1)/\{1, -1\} = PO(3,1) \) describing a homomorphism of embedding of the Lorentz group \( O(3,1) \) into \( C(2) \) group introduced in connection with Eq.(6.40). These transformations are preserving the isotropy property described by Eq.(7.10). Furthermore, the (non) vanishing of \( e_0 \) or of \( e_6 \) component of a point in \( P(L^{4,2}) \) is not invariant under a general Lie sphere transformation leading to absence of distinction in this geometry between the oriented spheres, oriented planes and points.

Presented background is sufficient for reading of [135], chapter 15, where many additional facts about the Lie sphere geometry are nicely explained.

### 7.3. Möbius geometry and geometric algebra. Fundamentals

In this subsection we connect the Lie sphere geometry with the Möbius geometry known in physics literature as "geometric algebra" [136]. To demonstrate interconnection between Möbius geometry and geometric algebra we have to provide some basics on Möbius geometry first. This is easy to do since Möbius geometry is just part of the Lie sphere geometry. Möbius geometry studies subgroups of Lie sphere geometry preserving subsets of \( P(L^{4,2}) \) with vanishing \( e_6 \). Thus, it deals with non-oriented spheres, non-oriented planes, points \( x \in \mathbb{R}^3 \), the infinity point \( \infty \) compactifying \( \mathbb{R}^3 \) to \( S^3 \). The elements \( x \in S^3 \) are in one-to-one correspondence with the points on the projectivized light cone \( P(L^{4,1}) \) where

\[
L^{n+1,1} = \{ \hat{\nu} \in \mathbb{R}^{n+1,1} | < \hat{\nu}, \hat{\nu} > = 0 \}
\]  

(7.12)
to be compared with Eq.(7.10). Points \( x \in \mathbb{R}^3 \) correspond to points of \( P(L^{4,1}) \) with non-vanishing \( e_6 \) while the point \( \infty \) corresponds to the only one point of \( P(L^{4,1}) \) with vanishing \( e_0 \) component.
Remark 7.7. In view of the Remark 7.2., by comparing Eq.s (7.10) and (7.12) it is clear that results of Bars can be redone only with help of the Lie sphere geometry. Nevertheless, Möbius geometry is used in geometric algebra which found its way into physics for some time [136]. This fact deserves some further discussion. In particular, we begin with the following definition

Definition 7.8. The essence of Möbius geometry lies in the study of properties of nonoriented (hyper)spheres invariant with respect to projective transformations \( P(L^{4,1}) \) mapping points to points. The group of such transformations is isomorphic to \( PO(n + 1, 1) = O(n + 1, 1)/\{1, -1\} \approx O^+(n + 1, 1) \). It is the group of Lorentz transformations of \( R^{n+1,1} \) preserving the time-like direction. Every conformal diffeomorphism of \( S^n = R^n \cup \{\infty\} \) is induced by the restriction of Möbius transformation to \( P(L^{4,1}) \). Further details can be found in [137].

Remark 7.9. Although the Dupin cyclides are described by the Lie sphere geometry in 3 dimensions [138], additional studies, e.g. see for example [139], demonstrated that in 3 dimensions use of the Möbius geometry and, hence, the geometric algebra is sufficient in the sense described in the next subsection. This algebra describes all conformal motions. In [140] it is demonstrated how the Dupin cyclides emerge as sets of orbits of conformal motions.

7.4. Crown achievement of Sophus Lie- discovery of the isomorphism between the Lie sphere and Plücker line geometries.

The essence of 2-times formalism invented by I. Bars and described in his book [83] is summarized in Fig.7.7. of this book. In it, the 2-times formalism is placed at the center of this figure while the twistor formalism is presented in the upper left corner. It is depicted as some kind of corollary of the 2-times formalism. We have already explained that the 2-times formalism is nothing else but the Lie sphere geometry formalism. Now, following ideas of Sophus Lie [141], we are going to demonstrate that the Lie sphere formalism is isomorphic to the Plücker line formalism. Plücker formalism was discussed in section 6 in connection with twistors. That such an isomorphism is possible, is hinted in the Remark 7.5. Now we provide the details. By describing the Lie sphere-Plücker line correspondence we are establishing the isomorphism between the Penrose twistors and the 2-times formalisms. Since the twistor formalism is associated with the most of the exactly integrable systems originating from all kinds of reductions of the Abelian and non Abelian-Yang Mills gauge fields [142, 143], it is hoped, that in view of this isomorphism it is sufficient to study properties of these gauge fields in order to address all problems of high energy physics, including those pertinent to the Standard Model and gravity. Furthermore, the unexpected connection between the gauge-theoretic (Floer) and Schrödinger’s formalisms noticed in subsection 4.2.3. now acquires an independent support.

We develop Plücker’s line geometry by analogy with the Möbius and Lie sphere geometries. For this purpose, we recall first Eq.s (6.16)-(6.17) describing representative elements of the exterior algebra as well as Plücker embedding of this algebra into complex projective space.
We are adopting general formalism to four dimensional complex (twistor) space \( C^4 \). By treating this space as vector space we have \( \{e_0, ..., e_3\} \) as the basis for \( E \in C^4 \). Accordingly, we also can construct a complex six dimensional space \( \wedge^2 E \) of bivectors:

\[
\wedge^2 E \sim \{e_0 \wedge e_1, e_0 \wedge e_2, e_0 \wedge e_3, e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\} \tag{7.13}
\]

Notice, that in the case of the Lie sphere geometry we used also six dimensional vector space \( \{e_1, ..., e_6\} \). Therefore, by analogy with the Lie sphere case, we define the scalar product according to the rules [80]:

\[
\langle e_0 \wedge e_1, e_2 \wedge e_3 \rangle = -\langle e_0 \wedge e_2, e_1 \wedge e_3 \rangle = \langle e_0 \wedge e_3, e_1 \wedge e_2 \rangle = 1 \tag{7.14}
\]

Notice that thus far we nowhere used the fact that the underlying vector space is complex. Thus, following [80] we shall initially treat it as real. Then, based on the multiplication table given by Eq.(7.14), we conclude that the signature of such a space is (3,3). Accordingly, \( \wedge^2 E \simeq R^{3,3} \). In accord with Eq.(7.10) we introduce the set of isotropic vectors

\[
L^{3,3} = \{\hat{g} \in \wedge^2 E \simeq R^{3,3} | \langle \hat{g}, \hat{g} \rangle = 0 \}. \tag{7.15}
\]

The Plücker quadric \( Q^{3,3} \) is now defined by \( P(L^{3,3}) \). It was obtained in section 6, Eq.(6.38). Use of transformations given by Eq.(6.37) brings us back to the well known Plücker embedding result, Eq.(6.18c). Notice that the standard Plücker relation, Eq.(6.18c), does not require use of complex variables. This happens to be essential as we shall explain momentarily. For this we recall that Lie discovered line-spherical geometry correspondence in 1870 following ideas of Poncelet, Plücker and Darboux. All these ideas, including contributions of Sophus Lie, are presented in the book by Klein [144] while the detailed account of the Lie Sphere geometry, Dupin cyclides, etc. is presented in the book by Blaschke [145]. In particular, in addition to the sketchy presentation of the line-sphere geometry correspondence given in [141], Felix Klein provides accessible and comprehensive discussion on the line-sphere geometry correspondence in his book [144], pages 262-274. Since books by Blaschke and Klein are both in German, for readers convenience, we provide only summary of what was accomplished which we are seamlessly connecting with the previous material. We refer to just mentioned books for further details.

Consider a single sphere \( S^2 \) in \( R^3 \). Analytically it can be described as follows

\[
x^2 + y^2 + z^2 - 2ax - 2by - 2cz + D = 0. \tag{7.16}
\]

In addition to parameters \( a, b, c, D \), it is necessary to introduce the radius \( r \) of the sphere via

\[
r^2 = a^2 + b^2 + c^2 - D. \tag{7.17}
\]

These parameters are to be considered as coordinates in the space of spheres. In addition, it is convenient to introduce the homogenous coordinates

\[
a = \frac{\xi}{\nu}, \quad b = \frac{\eta}{\nu}, \quad c = \frac{\zeta}{\nu}, \quad r = \frac{\lambda}{\nu}, \quad D = \frac{\mu}{\nu} \tag{7.18}
\]
enabling us to insert the sphere $S^2$ into projective space. To do so, we insert Eq.(7.18) into Eq.(7.17) with the result:

$$\xi^2 + \eta^2 + \zeta^2 - \lambda^2 - \nu\mu = 0. \quad (7.19)$$

Planes (spheres of infinite radius) and points (spheres of zero radius) are included in this parametrization. For points we have to put $\lambda = 0$ in Eq.(7.19) while for planes we require $\nu = 0$. To represent Eq.(7.19) in terms of the Lie sphere quadric $Q^{4,2}$ (defined after Eq.(7.10)), it is sufficient to represent the combination $\nu\mu$ as $\nu\mu = \alpha^2 - \beta^2$. Then, Eq.(7.19) is replaced by

$$\xi^2 + \eta^2 + \zeta^2 - \lambda^2 - \alpha^2 = 0 \quad (7.20)$$

which is the same as previously obtained Eq.(6.41b). Questions remains: a) Is it possible to convert the quadric $Q^{4,2}$ into $Q^{3,3}$ by some kind of nonsingular and real transformation? b) If this is not possible, can the same result be achieved by using complex transformations? In other words: Is it possible to prove that the projective groups $PO(4,2)$ and $PO(3,3)$ are isomorphic? The answer is "NO" in the domain of real numbers and "YES" in the domain of complex numbers. Explicitly, consider Plücker’s result, Eq.(6.18c), and perform the following transformation (attributed to Lie)

$$l_{01} = \xi + i\eta, \quad l_{23} = \xi - i\eta,$$

$$l_{02} = \zeta + \lambda, \quad l_{31} = \zeta - \lambda,$$

$$l_{03} = \mu, \quad l_{12} = -\nu. \quad (7.21)$$

Substitution of this result into Eq.(6.18c) brings us back to Eq.(7.19). Although algebraically this substitution looks very simple, geometrically, it is at the heart of the line-spherical geometry correspondence. All missing details of this correspondence are given in the book by Klein [144]. Thus, all results of 2-times formalism developed by Bars should be obtainable from the twistor formalism.

7.5. Some physical implications of the Lie sphere geometry

7.5.1. Lie sphere vs Möbius geometry. Important facts

To study physical implications, we need to extend results describing the line-Lie sphere geometry correspondence presented in Klein’s book [144]. This is needed because the Dupin cyclides were not discussed by Klein in the context of this correspondence even though they were discussed in his book. This is excusable in view of Definitions 4.4 and 4.5. However, in describing the Lie sphere geometry we introduced isotropic vectors via Eq.(7.10) and described the oriented contact between two spheres in Eq.(7.11). A simple calculation done in [134], page 15, indicates that the oriented contact condition $\langle \hat{s}_1, \hat{s}_2 \rangle = 0$ is equivalent to

$$|c_1(t) - c_2(t)|^2 = (r_1(t) - r_2(t))^2. \quad (7.22)$$
We would like to test this result using the simplest Dupin cyclide described after Definition 4.5. Using these results, we obtain: 
\[(r_1(t) - r_2(t))^2 = \left(a \frac{1+t^2}{1-t^2}\right)^2.\] At the same time
\[|c_1(t) - c_2(t)|^2 = a^2\left[(\frac{1-t^2}{1+t^2})^2 + (\frac{2t}{1+t^2})^2 + (\frac{2t}{1-t^2})^2\right] = \left(a \frac{1+t^2}{1-t^2}\right)^2.\] Thus, we just demonstrated that whenever there is an oriented contact between two spheres, there is a Dupin cyclide associated with such a contact. The contact between two spheres takes place along some curve on the surface of the Dupin cyclide. Definition 4.5 indicates that the oriented contact between spheres is the result of coincidence of envelopes originating from two different canal surfaces. According to Maxwell, only Dupin cyclides are being formed in this way. Evidently, the Lie sphere geometry permits both \(\langle \hat{s}_1, \hat{s}_1 \rangle = 0\) (or \(\langle \hat{s}_2, \hat{s}_2 \rangle = 0\)) and \(\langle \hat{s}_1, \hat{s}_2 \rangle = 0\) implying that this geometry permits the existence of both the canal surfaces and the Dupin cyclides. At the same time, the spine curve \(c(t)\) could be closed or not. Accordingly, the most primitive canal surfaces are cylinders, cones of revolution and tori. It happens that use of Möbius geometry and, in particular, the inversion transformation, is sufficient for generation of all known Dupin cyclides [139, 140, 146] starting with cylinders, cones of revolution and tori. For the case of tori of revolution, a very accessible proof is given in [147], Proposition 20.36. This fact allows trivially reobtain the results by Friedlander [65, 70] and Sym [74] starting with cylinders, cones of revolution and tori as solutions of the D’Alembert Eq.(2.6).

The Dupin cyclides as much as D’Alembertian, Eq.(2.6), are invariants of the Lie sphere transformations, e.g. read again the discussion around Eq.(7.11), while the cyclides deform under Möbius transformations. Complicated Dupin cyclides are formed from the simplest ones via application of succession of Möbius transformations. Since the line-Lie sphere geometry correspondence does not survive under use of Möbius transformations, it appears, that Möbius transformations are playing an auxiliary role. This conclusion is reinforced by the observation that in Möbius geometry the condition \(\langle \hat{s}_1, \hat{s}_2 \rangle = 0\) is not the contact condition given Eq.(7.22). In Möbius geometry we have instead [80]:
\[|c_1 - c_2|^2 = r_1^2 + r_2^2\] (7.23)
which is the condition of two spheres to intersect orthogonally. Notice that the above condition does not require use of the \(t\) parameter. Surprisingly, these conclusions are wrong! To explain why, following [140], we need to describe the conformal motion first.

7.5.2. Fundamentals of conformal kinematics

As in physics, everything begins with mechanics and mechanics begins with kinematics. The Euclidean kinematics is composed of rotations around some axis followed by translation along the same axis. It is a special kind of conformal kinematics. The kinematics of Möbius geometry is based on the known fact that it is an isometry of hyperbolic space [81]. This means that in such geometry circles are geodesics. They are analogs of straight lines in Euclidean geometry. Using this observation, we cover a plane \(\mathbb{R}^2\) (or sphere \(S^2\)) with a network of mutually orthogonal circles (instead of Euclidean plane being covered by a lattice made of

\[\text{More details are given below.}\]
mutually orthogonal straight lines). On thus formed Euclidean (or non Euclidean) lattice we can move along geodesics, that is along the segments of straight lines or along the segments of circles. In both cases the result of elementary motion along the segment followed by motion along the orthogonal segment can be interpreted as rotation. In going from two to three dimensions, in Euclidean geometry, the planar square lattice is replaced by three dimensional lattice. In hyperbolic space we have to cover \( \mathbb{R}^3 \) by the network of mutually orthogonal spheres. This explains at once the significance of Eq.(7.22) in Möbius geometry. By avoiding technicalities clearly explained in [140] we only notice that any three dimensional conformal motion can be made of two commuting elementary motions. These are analogs of Euclidean translation along the axis and rotation around the same axis (screw motion). Consider now the most elementary Dupin cyclide—the torus. For the torus the meridians and parallels are mutually orthogonal. Both are circles. But circles are crosssections of spheres! Accordingly, if we embed such a torus into \( \mathbb{R}^3 \) (or, better, in \( S^3 \)) foliated by mutually orthogonal spheres in such a way that meridians and parallels of the torus are the crosssections of the associated spheres, then we can think about the conformal trajectories originating on the surface of such a torus. In view of results of previous subsection, any nontrivial cyclide of Dupin is obtainable from torus via operation of Möbius inversion. This means the following:

a) The network of mutually orthogonal circles on the torus will transfer to the network of mutually orthogonal circles on the cyclide.

b) The set of all orbits originating at the torus will form two dimensional surface—Dupin cyclide. If we look only at some orbits, they are developing at the surface of Dupin cyclide.

c) When an initial point \( x(0) \) is given, it determines both the orbit and the type of cyclide.

**Remark 7.10.** The above statement a) can be formulated as a theorem. Originally it was known as conjecture (attributed to Ulrich Pinchall): Cyclides of Dupin are the only surfaces in Euclidean space on which two families of orthogonal circles lie. Although we explained why this is so above, the full proof was given by Thomas Ivey [148] in 1995. This proof explains why many researchers, e.g. read [140] or [149], use Möbius geometry for generating Dupin cyclides. Surely, the same results are achievable with the help of Lie sphere geometric methods as explained in [150],[151].

Just obtained results can be used for reobtaining in the most economical (physical) way both—the result by by Friedlander [70] discussed in section 4, and that for knots made of null fields discussed both in electrodynamics [152] and in quantum mechanics [14]. Details are given in the next subsection and Appendices B and C.

### 7.5.3. Knots and Dupin cyclides in quantum mechanics and electrodynamics

We provided an evidence for existence of Dupin cyclides in quantum mechanics and in electrodynamics in section 4 and, from different perspective, in Appendix C. We would like now to reobtain results by Friedlander using physical arguments. Following [14], we begin
with the observation that the set of source-free Maxwell equations in the vacuum can be compactly rewritten as
\[ i \frac{\partial}{\partial t} F = c \nabla \times F , \nabla \cdot F = 0. \] (7.24)

Here \( c \) is the speed of light, \( F = E + iH \) is the Riemann-Silberstein vector involving both the electric \( E \) and magnetic \( H \) fields. By representing this vector as
\[ F(r,t) = F_+ + F_-, F_{\pm}(r,t) = \int d\omega e^{\pm i\omega t} F_{\pm}(r) \] (7.25)
the above set of Maxwell’s equations is converted into
\[ \nabla \times F_\omega = k F_\omega, \] (7.26a)
\[ \nabla \cdot F_\omega = 0, \] (7.26b)
where \( \omega \) is \( +\omega \) or \( -\omega \) and \( k = \omega/c \). In plasma physics Eq.(7.26a) is known as "force-free equation" while in hydrodynamics it is known as "Beltrami equation". Eq.(7.26a) was discussed in detail in our book [55] in connection with problems emerging in contact geometry.

By applying the curl operator to both sides of Eq.(7.26a) and taking into account Eq.(7.26b) we obtain:
\[ \nabla^2 F_\omega + k^2 F_\omega = 0. \] (7.27)
This vector version of the Helmholtz equation should be compared with its more familiar version, Eq.(3.2), discussed in section 3. This (scalar) version is used both in quantum mechanics and in electrodynamics. The way to relate these two equations to each other is nontrivial [14]. In part, it is based on the useful identity
\[ (\nabla^2 + k^2)(r \cdot F_\omega) = 2 \nabla F_\omega + r \cdot (\nabla^2 + k^2) F_\omega. \] (7.28)

In [14] it is shown how the vector, Eq.(7.27), can be restored from the scalar Eq.(3.2) in which \( \Psi = (r \cdot F_\omega) \). By applying operation \( \text{div} \) to both sides of Eq.(7.26a) and by assuming that \( k = \text{const} = \kappa(x,y,z) \) we obtain, \( \text{div}(\kappa F_\omega) = F_\omega \cdot \nabla \kappa = 0 \). Let \( r(t) = \{x(t), y(t), z(t)\} \) be some trajectory on the surface \( \text{const} = \kappa(x,y,z) \). Since \( \frac{d}{dt} \kappa(x(t), y(t), z(t)) = v_x \kappa_x + v_y \kappa_y + v_z \kappa_z = v \cdot \nabla \kappa = 0 \), by identifying \( v \equiv F_\omega \) we conclude that the "velocity" is always tangential to the surface \( \text{const} = \kappa(x,y,z) \). Since the vector field \( v \) is assumed to be nowhere vanishing, the surface \( \text{const} = \kappa(x,y,z) \) can only be a torus \( T^2 \). In the case if \( \kappa \) is rational number the field lines \( v \) on \( T^2 \) are closed implying that these lines are forming torus knots. Using results of previous subsection and applying Möbius transformation we end up with torus-like knots wound around cyclides of Dupin (that is around the distorted tori). Depending on the type of Möbius transformation, different "wrapped" Dupin cyclides will be formed. But this is exactly the result of the paper by Friedlander [70]!

**Remark 7.11.** Use of conformal transformation for generation of knots/links was suggested initially by Bateman and Cunningham in 1910. Their results were recently rediscovered and utilized for knot/link generation in [152]. See also [153]. No mention of Dupin cyclides, etc. was made in these references.
Remark 7.12 Results of refs [152] and [153] demonstrate how the torus knots/links can be generated from null and not null electromagnetic fields (discussed in some detail in the Appendix B). Being technically permissible, such torus knot generation is not allowed for the non null fields on physical grounds.

7.6. Place of Lie sphere geometry in development of foundations of AdS-CFT correspondence

We conclude our paper with a brief discussion of the foundations of AdS-CFT correspondence. An excellent physical introduction to this subject is given in the recent book by Nastase [154]. Some mathematical aspects of this correspondence are discussed in our work [155]. Use of these references allow us to reduce our discussion to the absolute minimum. In physical and mathematical literature on AdS-CFT correspondence no mention of relevance of the Lie sphere geometry exist to our knowledge. Thus, what follows below is the fist attempt at elimination of this deficiency.

We begin with familiar examples, e.g. discussion of the simplest model of hyperbolic space $\mathbb{H}^2$. It is known in literature as the Poincare' disc model $\mathbb{D}^2$. Geodesics in this model are made of the horocycles. These are circular segments whose both ends lie at the boundary $S^1_\infty$ of the disc $\mathbb{D}^2$. The boundary $S^1_\infty$ is considered to be "the spatial infinity". By the appropriate choice of constants $a, b, c, d$ in the Möbius transformation $f(z)$ given by $f(z) = \frac{az + b}{cz + d}, z \in \mathbb{C}, \mathbb{C} = \mathbb{R}^2 \cup \{\infty\}$, the Poincare' disc model can be transformed into Poincare' half plane model of the hyperbolic space $\mathbb{H}^2$. Clearly, use of the inverse conformal transformation converts the half plane model back into the disc model. Let $z$ be some point inside such disc model. Successive applications of the Möbius transformation will propel this point to the spatial infinity $S^1_\infty$ after infinite number of iterations. For detailed examples, please, read [156]. This two-dimensional model is generalizable to higher dimensions where it is known as the hyperbolic ball model [81]. Thus, when interested in higher dimensions, we need to replace a combination $(\mathbb{H}^2, S^1_\infty)$ by $(\mathbb{H}^{n+1}, S^n_\infty)$. In all described models of hyperbolic geometry the boundary at infinity is playing an important role. This role can be seen already in two dimensional model $(\mathbb{H}^2, S^1_\infty)$. In it, the deformations of $S^1_\infty$ lead to the Virasoro algebra -central object of the conformal field theory [155,157], the Teichmüller spaces, etc. Thus, already at the level of two dimensions we are dealing with a kind of AdS-CFT correspondence. By analogy with two dimensions, it is expected that the hyperbolic-like interior- actually, the anti–de Sitter spacetime (AdS) (in higher dimensions)- is affected by (linked with) the conformal field theory (CFT) residing at the boundary. The higher dimensional AdS-CFT correspondence in higher dimensions cannot be rigorously developed based on the hyperbolic ball model though. This is because of the Mostow rigidity theorem. The Mostow rigidity makes deformations of conformal 3-sphere $S^3_\infty$ impossible to perform. Nevertheless, the mathematics of this model is serving as a guideline for more realistic models as demonstrated by Frances [158]. The bottom line is the following. The hyperbolic space $\mathbb{H}^n$ is replaced by the Anti–de Sitter space (AdS)-the Einstein space of constant negative curvature. This space is obtainable from the vacuum Einstein equations with added (negative) cosmological constant [155]. The hyperbolic ball at infinity $S^n_\infty$ is now replaced by the space
of conformally flat solutions Ein$^{n,1}$ of Einstein’s equations. Being conformally flat these are
conformally equivalent to Minkowski spacetime, that is to say their conformal symmetry
group is SO(4,2). In the Appendix D we rigorously demonstrate that

$$Ein^{n,1} = \partial^\infty AdS^{n+2} = Q^{n+1,2}. \tag{7.29}$$

Here $\partial^\infty AdS^{n+2}$ is the boundary of the Anti-de Sitter space while $Q^{n+1,2}$ is the Lie sphere quadric introduced after Eq.(7.10).

**Acknowledgement** Authors are deeply grateful to Professor (Dr.) David Delphenich for supplying us with the original of Madelung’s paper, ref. [177], as well as with the English-German translation of this paper. Authors are also very thankful to the unknown referee for his thoughtful remarks which lead to considerable improvement of our presentation.

**Note added in proof.** The interplay between the differential and the Lie Sphere geometries, the rigidity and quantization (and hence mass generation) in the context of the transversal knots (discussed in our earlier work, Ann.Phys.371 (2016) 77) is developed in great detail in two latest works by E.Musso and L.Nicolodi (Comm.Anal.Geom. 25 (2017) 209 and Nonlin.Anal. 143 (2016) 224). The connection between knots and the Regge-mass spectrum is discussed in our earlier work (Int. J.Mod. Phys. A 30 (2015), id. 1550189-212).

**Appendix A**

**Comparison between the Cauchy problems for parabolic and hyperbolic equations**

1. **Phase velocity.** Without loss of generality, we consider partial differential equations (PDE) involving just two variables $t$ and $x$. Suppose that solution $u(t, x)$ can be represented in the traveling wave form [45] as follows: $u(t, x) = f(x - ct).$ Here $c$ is the velocity of the profile $f$. More generally, in the case when $x = \{x_1, ..., x_n\}$ we get $u(t, x) = f(k \cdot x - \omega t).$ It is called *plane wave* with wavefront normal to $k$, speed $c = \frac{\omega}{|k|}$ (phase velocity), and profile $f$.

2. **Exponential solutions for simplest PDE’s and group velocity.** When studying linear partial differential equations, it is convenient to specify the profile function $f$ in order to study the complex-valued plane wave solutions of the form

$$u(t, x) = \exp\{i(k \cdot x - \omega t)\}. \tag{A.1}$$

Using such an ansatz in the diffusion equation leads to

$$u_t - \triangle u = (-i\omega + |k|^2) = 0 \tag{A.2}$$
resulting in the dispersion relation \( i|k|^2 = \omega \). Anticipating generalizations, this result can be rewritten as \( \omega = \omega(|k|^2) \). Therefore Eq.(A.1) can be presented in the form
\[
    u(t, x) = \exp\{i(k \cdot x - \omega(|k|^2)t)\}. \tag{A.3}
\]
This representation allows us to introduce the group velocity \( c_g \) as
\[
    c_g = \nabla_k \omega(|k|^2). \tag{A.4}
\]
In the case of simplest Schrödinger’s equation \((m = 1, \hbar = 1)\) \( iu_t + \Delta u = 0 \), the dispersion relation is given by
\[
    |k|^2 = \omega. \tag{A.5}
\]
Therefore, the group velocity is \( c_g = 2k \). Upon restoring the usual system of units, we obtain:
\[
    E = \hbar \omega = \frac{(\hbar k)^2}{2m} \text{ implying } \omega(k^2) = \frac{\hbar k^2}{2m}. \nonumber
\]
Accordingly, \( \nabla_k \omega(|k|^2) = c_g = \frac{\hbar k}{m} \). Using the De Broglie relation: \( p = \hbar k \), we conclude that the group velocity \( c_g \) coincides with the particle velocity in quantum mechanics.

Consider now the wave equation using the ansatz (A.1). We obtain instead:
\[
    u_{tt} - \Delta u = (\omega^2 + |k|^2) = 0, \tag{A.6}
\]
resulting in
\[
    |k|^2 = \omega^2. \tag{A.7}
\]
From here the phase velocity \( c \) is obtained as \( c = \frac{\omega}{|k|} = \text{const} \) and the modulus of group velocity \( c_g \) coincides with the phase velocity.

3. **Cauchy problems.** The Cauchy problem for the 1 dimensional diffusion equation is formulated as follows. For the diffusion equation
\[
    u_t - Du_{xx} = 0 \tag{A.8}
\]
defined on the whole line \( -\infty < x < \infty \), whose solution at the time \( t = 0 \) is \( u(0, x) = \phi(x) \), find a solution \( u(t, x) \) for \( t > 0 \) and \( -\infty < x < \infty \). The solution of this problem is given by
\[
    u(t, x) = \int_{-\infty}^{\infty} dy S(x - y, t)\phi(y) = \int_{-\infty}^{\infty} dz S(z, t)\phi(x - z) \tag{A.9a}
\]
Here \( z = x - y \), and
\[
    S(z, t) = \frac{1}{\sqrt{4\pi Dt}} \exp(-z^2/4Dt). \tag{A.9b}
\]
The last two equations can be combined into
\[
    u(t, x) = \int_{-\infty}^{\infty} dp \exp\{-\frac{p^2}{4}\}\phi(x - p\sqrt{Dt}). \tag{A.10}
\]
The analogous problem for the Schrödinger’s equation is obtained now by replacing \( t \) by \( it \) in the above result. This fact is helpful for the following reason. Suppose that the initial condition \( u(0,x) = \phi(x) \) is zero everywhere, except on some small interval \((a,b)\). Because the exponential function is never zero, the integral, Eq.(A.9a), is not going to be zero, even for \( t \to 0^+ \) and \( x \) arbitrary far from \((a,b)\). This can be restated as

**Infinite propagation speed** [45]. The initial condition \( u(0,x) = \phi(x) \) affects the solution \( u(t,x) \) for all \( x \) no matter how small \( t \) is. Thus, heat propagates with infinite speed.

Evidently, the replacement of \( t \) by \( it \) in the above results is leading to the Schrödinger equation. But use of the same arguments as in the diffusion case are not changing this conclusion. This fact is one of the sources of Einstein’s spooky action at the distance.

Diffraction of light—main evidence of quantum mechanical behavior—uses time-independent Helmholtz Eq.(3.2) of the main text. This equation emerges in both quantum mechanics and in optics. However, if one is willing to study the Cauchy problem for the wave equation \( u_{tt} - u_{xx} = 0 \) defined on \(-\infty < x < \infty\), it should be formulated as follows. In addition to the initial data \( u(0,x) = \phi_1(x) \) one has to supply the initial velocity \( u_t(0,x) = \phi_2(x) \). Thus, while for the well posedness of the Cauchy problem for the diffusion equation one needs just one initial condition, one needs two initial conditions for the wave equation. At the mathematical level no more comments are required. At the physical level, more comments are needed. These are originating from the fact that, say, optical waves are generated by atoms [16]. The two-level atomic systems—primary sources of photons require for their description the time-dependent Schrödinger equation which is identical in its structure with the Pauli equation for spin 1/2 particle in varying magnetic field [16,31,55]. For such two-level system two initial conditions are required. In addition, technologically it is important to generate single polarized photons and to use the 50:50 beam splitter to cause such a single photon to be self-entangled [160]. Thus, even though a dispersion relation Eq.(A.7) originating from the wave Eq.(A.6) can be used for description of a single photon [33] in quantum optics polarization of photons is exploited essentially [16] causing use of the Pauli-type Schrödinger equation for their description. In such a case quantum phenomenon of entanglement [160] coexist with the classical fact of finite speed propagation of wave signals.

**Appendix B**

**Unimaginable universality of Hadamard premises**

B.1. Brief review of Hadamard’s premises

There are 3 ”premises” formulated by Hadamard. E.g. read [162], pages 445-450. These are:

---

33E.g. via the de Broglie relation
a) **Major premise.** Suppose we are observing events within the time interval $0 < t < t_0$. In order to find a state at the moment $t = t_0$, we need to know a state at the time $t = 0$, then find a state at the time $t = t'$ and, using this information, find a state at $t = t_0$;

b) **Minor premise.** Suppose within a short period of time $\varepsilon \geq t \geq 0$ there emerges some light disturbance localized in the vicinity of point $O$ then, at time $t = t'$ this disturbance will be concentrated in the very thin spherical layer enclosing a sphere of radius $\omega t'$ centered at $O$;

c) **Corollary.** In order to evaluate the action of the initial light perturbation, located at the point $O$ at $t = 0$, it is permissible to replace this perturbation by a set of perturbations emerged at time $t = t'$ and distributed on the surface of the sphere centered at $O$ and having radius $\omega t'$.

According to Hadamard, different writers identified Huygens’ principle either with a) (e.g. Feynman, as described in section 1) or with b) (e.g. read our section 4) or with c). In his early work on hyperbolic equations and Huygens’ principle Hadamard considered only such wavefronts which did not contain any singularities known as caustics. The major premise does not exclude existence of caustics. Therefore, what is considered by mathematicians as “Huygens’ principle”, corresponds to Hadamard’s minor premise and to corollary. In such a form it is discussed in the main text, e.g. read Definition 4.2. Alternatively, read [68], page 138. In short, Hadamard restricted himself by studying of the wake-free waves. Situations when wakes are present (leading to caustics) is considered in [50]. Much more comprehensive and detailed treatment (but not mathematically well supported) is given in [163]. The same but mathematically supported is given in [59]. Huygens triviality is defined in Definition 4.3. Following this definition, the Hadamard conjecture is formulated as follows:

**Hadamard conjecture.** Every Huygens operator is trivial.

For purposes of quantum mechanics it is almost always sufficient to use the results of Theorem 4.1. In this appendix we would like to explain that in the broader (but physically also very relevant) context Theorem 4.1. sometimes fails. This is so if, following [68], we reformulate Theorem 4.1. as

**Theorem 4.1.1.** If equation of hyperbolic type in conformally flat Minkowski (3+1) spacetime satisfies Huygens principle, then it is equivalent of the wave Eq.(2.6a).

The new element here is mention of conformal flatness. The wave Eq.(2.6a) is surely conformally invariant, e.g. read [82]. Based on this fact, a broader question can be posed:

**Hadamard-like conjecture.** Is it true that the Huygens principle holds if the underlying equation is conformally invariant? That is to say: Is the conformal invariance and the Huygens principle are equivalent statements (or interdependent concepts)?

Stated still a bit differently, the above conjecture can be formulated as follows.
Hadamard-like conjecture. Second version. Is it always true that the Huygens principle is obeyed whenever it is possible to relate the conformally invariant equation of the 2nd order to the wave Eq.(2.6a)?

B.2. Conformal invariance versus the Huygens principle

In any given dimension \(d\), say, in \(d = 3 + 1\), all conformal groups are classified in flat and curved spaces [164]. Many important additional details are given in [165] and [166]. Accordingly, all conformally invariant equations can be explicitly written for all Einstein spaces of general relativity. If this is so, will equations with different conformal groups lead to the same Eq.(2.6a)?

It is instructive to provide some details to answer this question. Following [166] we introduce the second order differential (Beltrami) operator \(\Delta_2\) as follows

\[
\Delta_2 u = g^{ij}(\frac{\partial^2 u}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial u}{\partial x^k}).
\]  

(B.1)

With help of this operator any 2nd order linear differential equation can be written as

\[
\Delta_2 u + a^i(x) \frac{\partial u}{\partial x^i} + c(x)u = 0.
\]  

(B.2)

In the same metric conformally invariant 2nd order equation can be written (in 3+1 dimensions) as

\[
\Delta_2 u + \frac{1}{6} Ru = 0,
\]  

(B.3)

where \(R\) is the scalar curvature. At the same time, using [165],[166] it is possible to prove the following:

**Theorem B.1.** Any spacetime \(\mathcal{M}_{n+1}, n \geq 3\), with a given metric \(g\) possesses nontrivial group of conformal motions if and only if this spacetime is conformally equivalent to the Lorentzian spacetime. The most general Lorentzian spacetime \(V_{3+1}\) is described by the metric

\[
ds^2 = (dt)^2 - (dx_1)^2 - \sum_{i,j=2}^4 a_{ij}(x_1-t)dx^i dx^j.
\]  

(B.4)

Here \(a_{ij}\) is the positively definite matrix.

**Corollary B.2.** Any Lorentzian spacetime with nontrivial conformal group is conformally equivalent to the space for which the Ricci tensor \(R_{ij} = 0\).

**Corollary B.3.** Every 3+1 Lorentzian spacetime \(V_{3+1}\) with nontrivial conformal group admits the 2nd order partial differential equation

\[
u_{tt} - u_{xx} - f(x-t)u_{yy} - 2\varphi(x-t)u_{yz} - u_{zz} = 0
\]  

(B.5)
...satisfying Huygens’ principle.

**Remark B.4.** It is clear that for a special choice of functions \( f \) and \( \varphi \) Eq.\((B.5)\) will coincide with the wave Eq.\((2.6a)\). However for other choices these two equations are not always conformally equivalent since they might involve different groups of conformal motions. This will be further explained below.

**Remark B.5.** We brought to our reader’s attention Eq.\((B.5)\) because of the remarkable paper by Ward [167] allowing us to connect the results of this appendix with those in the main text.

In section 4 we introduced and discussed the progressive wave solutions of the wave equation as well as the Dupin cyclides. In section 7 we reobtained Friedlander’s results [70] in much simpler way by using methods of conformal and contact geometries. Now, using these methods, we want to rederive Ward’s results having additional purposes in mind.

In his paper Ward was not discussing the Huygens principle or its connection with the conformal invariance. Ward’s purpose was to demonstrate that if instead of the wave Eq.\((2.6a)\) one considers the 2nd order wave-like equation with metric taken from the plane-wave gravitational background, one still can apply the progressive wave solution method by Friedlander to this equation and to reobtain all types of Dupin cyclides discussed by Friedlander. Based on results of this appendix and those in sections 4 and 7 of the main text, Ward’s goals can be restated as follows:

**Question B.6.** If the wave Eq.\((2.6a)\) admits progressing waves solutions resulting in Dupin cyclides, is the same is true for the Eq.\((B.5)\)?

In our discussion of Eq.\((B.5)\) we have not touched the subject of plane gravitational waves while Ward was not looking any further at the origins of Eq.\((B.5)\) which he interpreted in terms of the plane gravitational pp waves background. We would like now to complement Ward’s results with missing details. By doing so we simplify Ward’s results as well and explain why plane gravitational waves are fundamentally relevant for our paper.

**B.3. Conformal groups and Lie sphere groups, Rainich, Misner and Wheeler geometrodynamics of coupled Einstein-Maxwell fields, the AdS-CFT correspondence and Penrose limit**

We begin our discussion with results summarized in [168]. From chapter 3 of this book it follows that the flat Minkowski metric

\[
ds^2 = -(dt)^2 + (dx)^2 + (dy)^2 + (dz)^2
\]

\(\text{(B.6a)}\)

can be converted into

\[
ds^2 = -2dudv + 2d\zeta d\bar{\zeta}
\]

\(\text{(B.6b)}\)
with help of substitutions \( u = \frac{1}{\sqrt{2}}(t - z), v = \frac{1}{\sqrt{2}}(t + z), \zeta = \frac{1}{\sqrt{2}}(x + iy) \). From chapter 17 of the same book we obtain the metric for the pp-gavitational waves (obtained for the first time by Brinkman in 1925 and interpreted in terms of gravitational waves by Peres in 1959):

\[
ds^2 = -2dudv - 2H(\zeta, \bar{\zeta}, u)du^2 + 2d\zeta d\bar{\zeta}. \tag{B.7}\]

The physical meaning of the perturbational term \( H(\zeta, \bar{\zeta}, u) \) is explained in detail in [169],[169]. From these references it follows that solutions of Einstein’s field equations leading to the metric of the type given by Eq.(B.7) originate from the exact solution of the coupled Einstein-Maxwell fields. E.g.read pp 324-325 of [168] and ch.r 24 of [169]. Analysis of solutions for these fields made in [170]-[171] indicates that coupled Einstein-Maxwell fields can be of two types: a) non null and b) null. The null electromagnetic fields are such for which \( H^2 - E^2 = 0 \) and \( E \cdot H = 0 \). We used null electromagnetic fields in [71,96] in connection with formation of torus-like knots while we explained their relevance to the Schrödinger fields in [14] and in section 7.5.3. Although the content of just cited references reproduces many results of Ward’s paper [167], we still have to add several important pieces of information. This is so, because we have not made yet a connection between metrics Eq.(B.7) and (B.4).

Toward this goal, we begin with the observation that the exact solution of the coupled Einstein-Maxwell fields was obtained by Rainich [172] in 1925 and rediscovered by Misner and Wheeler at the end of 1950ies [173]. They reformulated results by Rainich in the context of geometrodynamics. In short, this means that the exact solution of Einstein’s equations for the coupled gravity-Maxwell fields is obtainable in terms of the metric from which it is possible to restore both the gravitational and electromagnetic fields which are non null. The solution for the null fields was obtained much later [170],[171].

We are interested in reinterpreting these null field results in terms of the Lie sphere geometry discussed in section 7 and Appendix D. To do so, initially we are following the paper by Kuiper [174]. From this paper it follows that the (pseudo) Riemannian space \( \mathcal{M}_d \) of dimensionality \( d \) is flat if some region about any point \( x \in \mathcal{M}_d \) if it can be covered by a metric -preferred coordinate system such that

\[
ds^2 = g_{ij}(x)dx^i dx^j. \tag{B.8}\]

Above, \( g_{ij} = e_i \delta_{ij}, e_i = +1 \) or \( -1 \) for all points of \( \mathcal{M}_d \).

**Definition B.7.** A (pseudo) Riemannian space \( \mathcal{M}_d \) is called conformally flat if some region about any point \( x \in \mathcal{M}_d \) can be mapped conformally into a flat space. A conformally -flat space is obtainable from the flat space via change of metric, that is

\[
\tilde{g}_{ij}(x) = \omega^2(x)g_{ij}(x), \tag{B.9}\]

where \( g_{ij}(x) \) is the metric tensor defined in Eq.(B.8) while \( \omega^2(x) \) is some positive function of \( x \).

When \( \omega^2(x) \) is constant, then the above transformation is called ”similarity” or ”homothety”. When \( \omega^2(x) = 1 \), then the homothety is an ”isometry”. Obviously, these definitions are not restricted to the diagonal metric tensor defined by Eq.(B.8).
Theorem B.8. (Liouville theorem) For spaces of dimensionality $>2$ under the conformal transformations described by Eq. (B.9) the (hyper) spheres are carried to (hyper) spheres.

Corollary B.9. From the same reference [174] and from section 7, Eq.(7.10), it follows that the conformal transformations are induced by the projective transformations leaving the Lie quadric $Q_{n+1,2}$ invariant. From here the connection with the Lie sphere geometry follows. More on Liouville theorem can be found in [175].

Definition B.10. Killing vectors $X$ are defined as solutions of the Killing equations

$$\mathcal{L}_X g_{ij} = 0,$$

(B.10)

where $\mathcal{L}_X$ is the Lie derivative in the direction of $X$. These equations are describing the isometric-type of motion on $\mathcal{M}_d$. Accordingly, the conformal Killing vectors $X$ are defined as solutions of the conformal Killing equations

$$\mathcal{L}_X g_{ij}(x) = 2\psi(x) g_{ij}(x).$$

(B.11)

These are describing the conformal-type of motion on $\mathcal{M}_d$. Examples of solutions of Eq.s (B.11) are demonstrated in [166].

Definition B.11. The conformal Lie group $\hat{\mathcal{C}}(\mathcal{M}_d)$ of a connected $d$-dimensional (pseudo) Riemannian manifold $\mathcal{M}_d$ is conformally trivial if by using transformations defined by Eq.(B.9) it is possible to find a conformally-equivalent space $\tilde{\mathcal{M}}_d$ in which Eq.(B.11) is replaced by Eq.(B.10).

Definition B.12. If use of Eq.(B.9) cannot bring Eq.(B.11) into Eq.(B.10), such conformal group is nontrivial.

In [166] the following remarkable theorem is proved

Theorem B.13. (pseudo) Riemannian space $\mathcal{M}_d$ has nontrivial conformal group only if it is conformally equivalent to the conformally flat space

Remark B.14. Following [174] for conformally flat spaces about any point $x \in \mathcal{M}_d$ there is a region in which (for $d \geq 3$) the $\frac{1}{2}(d + 2)(d + 1)$-parameter group of infinitesimal conformal transformations may exist. For $d = 4$ we obtain 15 parameters group of conformal motions. This is $SO(4,2)$ group describing conformal symmetries of hydrogen atom described in section 1. Based on Theorem B.8., this is the Lie sphere geometry group of motions.

Remark B.15. Following [176] the result $\frac{1}{2}(d + 2)(d + 1)$ provides maximal number of conformal generators for conformally flat spacetimes. However further studies demonstrated that Theorem B.13. can be extended. For (pseudo) Riemannian spaces which are not conformally flat the number of conformal generators is strictly less than the maximal number. It can be demonstrated [177], [178] that for the metric of the type given by Eq.(B.7) the maximal number is 7.

Remark B.16. Such not conformally flat metrics are of no physical interest, however, for variety of reasons. First, we immediately lose the connection with the Lie sphere geometry.
in view of Theorem B.8. Second, once this connection is lost, we also lose the connection with the AdS-CFT correspondence (section 7 and Appendix D).

**Remark B.17.** In [166] it was proven that every Lorentzian spacetime with nontrivial conformal group is conformally equivalent to the Ricci flat space, e.g. read the Corollary B.2. In [177] it is argued (by invoking the content of Brinkmann’s theorem) that the only Ricci flat but nonflat 4-manifolds (that is 4-manifolds whose scalar curvature is not identically zero) admitting nonhomothetic conformal vector fields are manifolds described in terms of the pp waves metric given by Eq.(B.7). In the same reference it is being argued that for nonconformally flat spacetimes the dimension of the conformal group is at most 7.

Based on the last two remarks, it is physically meaningful to consider the conformally flat pp waves. In such a case we obtain the pp wave metric with circle-preserving 15 components conformal group which is Ricci flat. The condition of Ricci flatness is known to be ([169], page 386):

\[
\frac{\partial^2}{\partial \zeta \partial \bar{\zeta}} H(\zeta, \bar{\zeta}, u) = 0.
\]

The pp metric, Eq.(B.7), with \( H(\zeta, \bar{\zeta}, u) \) determined from Eq.(B.12) is describing the coupled Einstein-Maxwell null fields. Eq.(24.43) of [169], page 384, explains why the condition, Eq.(B.12), is equivalent to the condition of Ricci flatness. The \( u \)-dependence of \( H(\zeta, \bar{\zeta}, u) \) can be determined from the condition on the conformal Weil tensor to be zero. Such a condition selects spaces which are conformally flat. Examples of such selection are discussed in [166]. Based on just presented results, it should be clear that, although technically permissible, parts of the results of [152],[153] describing non null electromagnetic knots are without physical justification.

The seminal work of Penrose [179], pages 271-275, entitled: "Any space-time has a plane wave as a limit" explains the universal significance of the metric given by Eq.(B7) which is also known in physics literature as Penrose limit metrics. The significance of Theorem B.1. and appreciation of its role in relating the conformal transformations to Huygens’ principle apparently, totally escaped the attention of physics community. In Appendix D we provide solid evidence that the conformally flat space-times supporting the Lie sphere geometry are indeed the boundaries of space-times in which the AdS-CFT correspondence holds.

**Appendix C**

**Dupin cyclides from Madelung’s hydrodynamical reformulation of the Schrödinger equation**

Using progressive waves method by Friedlander [70] we provided enough evidence in sections 4 and 7 that Dupin cyclides can exist as solutions of the Shrödinger equation. In this appendix we shall approach the same problem from yet another direction which is very illuminating in its own right. In our study we were motivated by two papers by Ferapontov
Ferapontov was interested in establishing the connection between the dynamics of Hamiltonian systems of hydrodynamical type and its realization in terms of differential geometry of hypersurfaces embedded in Minkowski-type spacetimes [180]. Using general results developed in [180], in [181] Ferapontov illustrated general results of [180] using hydrodynamical Hamiltonian systems which do not possess Riemann invariants. As result, he demonstrated the duality between such Hamiltonian systems and Dupin hypersurfaces. These are the hypersurfaces with constant principal curvatures [133]. In 3 dimensions mathematicians call them as ”Dupin surfaces”.

Almost immediately after Schrödinger published the installment of his foundational papers on quantum mechanics in Annalen der Physik in 1926 [1], on 25th of October of 1926 Madelung submitted his paper entitled ”Quantentheorie in hydrodynamischer form” to the Zeitschrift für Physik [182] where it was published in 1927. Although equations (3) and (4) of his paper contain typos, the key new equations (3’),(3’’)) and (4’) are correct. It is worth to reproduce these equations in this appendix. To avoid any ambiguities, we shall use exactly the same symbols as Madelung was using. He begins with the stationary Schrödinger equation

$$\Delta \psi_0 + \frac{8\pi^2 m}{\hbar^2} (W - U) \psi_0 = 0$$ \hspace{1cm} (C.1)

in which $W$ is an energy. Next, he writes $\psi = \psi_0 e^{i \frac{2\pi W}{\hbar} t}$ and uses this result in the time-dependent version of the Schrödinger equation

$$\Delta \psi - \frac{8\pi^2 m}{\hbar^2} U \psi - i \frac{4\pi m}{\hbar} \frac{\partial \psi}{\partial t} = 0.$$ \hspace{1cm} (C.2)

Next, he was looking for a solution in the form $\psi = \alpha e^{i \beta}$, where he is saying that, in view of Eq.(C.1), it is sufficient to consider only $\beta$ to be linearly-dependent upon $t$ while if we use Eq.(C.2), then it makes sense to consider both $\alpha$ and $\beta$ to be time-dependent. Surely, the consistency between equations (C.1) and (C.2) requires $\beta$ to be linearly dependent upon $t$. Substituting $\psi = \alpha e^{i \beta}$ into Eq.(C.2) and separating the real part from imaginary in resulting equation Madelung had obtained the following two equations

$$\Delta \alpha - \alpha (\text{grad} \beta)^2 + \frac{8\pi^2 m}{\hbar^2} U \alpha + 4\pi m \frac{\partial \beta}{\partial t} = 0,$$ \hspace{1cm} (C.3)

$$\alpha \Delta \beta + 2(\text{grad} \alpha \cdot \text{grad} \beta) - 4\pi m \frac{\partial \alpha}{\partial t} = 0.$$ \hspace{1cm} (C.4)

By introducing new notation $\varphi = -\beta \frac{\hbar}{4\pi m}$ Eq.(C.4) is converted into the continuity equation (this is Madelung’s Eq.(4'))

$$\text{div}(\alpha^2 \text{grad} \varphi) + \frac{\partial \alpha^2}{\partial t} = 0.$$ \hspace{1cm} (C.5)

---

34Riemann invariants are to be discussed in a separate publication.
35We reproduce these equations without obvious typos which apparently were overlooked by Madelung when he was proofreading the galleys of his paper. His new key equations (3’),(3’’)) and (4’) are correct though.
Now, following Madelung, we introduce the velocity $\vec{u} = \text{grad}\varphi \equiv \vec{u}$. In terms of such defined velocity Eq. (C.3) acquires the following form (this is Madelung’s Eq. (3’))

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}(\text{grad}\varphi)^2 + \frac{U}{m} - \frac{\Delta \alpha}{\alpha} \frac{h^2}{8\pi^2 m^2} = 0. \quad (C.6a)$$

This equation can be rewritten in recognizable hydrodynamical form by taking into account that $\nabla \times \vec{u} = 0$ and by applying the gradient operator to Eq. (C.6a). In the end, we obtain: (Madelung’s Eq. (3’’))

$$\frac{\partial \vec{u}}{\partial t} + \frac{1}{2} \text{grad}(\vec{u})^2 = \frac{d}{dt} \frac{\vec{u}}{m} - \frac{h^2}{8\pi^2 m^2} \text{grad}\Delta \alpha. \quad (C.6b)$$

**Remark C.1.** Eq. (C.6b) along with the continuity Eq. (C.5) describes the irrotational fluid moving under the action of conservative forces. In our book [55], page 67, we noticed (and described in various places of the book) the following chain of correspondences:

- classical mechanics $\rightleftharpoons$ thermodynamics $\rightleftharpoons$ electrodynamics
- geometrical optics $\rightleftharpoons$ hydrodynamics $\rightleftharpoons$ magnetohydrodynamics
- superconductivity $\rightleftharpoons$ non Abelian gauge Yang-Mills theories.

These correspondences are all derivable from the formalism of contact geometry and topology discussed in our book. Evidently, just described result by Madelung is fully consistent with the gauge-theoretic Floer-type description of Schrödinger’s quantum mechanics outlined in main text, subsection 4.2.3.

Now we recall that, according to Madelung, if we are only interested in the description of the stationary Schrödinger Eq. (C.1) the $t$—dependence of $\alpha$ can be omitted. In such a case using Eq. (C.3) we obtain:

$$\Delta \alpha = 0 \quad (C.7)$$

and

$$(\text{grad}\beta)^2 = \frac{8\pi^2 m}{h^2}(W - U). \quad (C.8)$$

Finally, Eq. (C.4) now acquires the following form:

$$\alpha \Delta \beta + 2(\text{grad}\alpha \cdot \text{grad}\beta) = 0. \quad (C.9)$$

A quick look at Eq.s (4.9 a-c) (producing Dupin’s cyclides solutions) and comparing these with just obtained Eq.s (C.7)-(C.9) indicates that these two sets of equations are the same when $\frac{8\pi^2 m}{h^2}(W - U) = 1$. But, as we know, in general $\frac{8\pi^2 m}{h^2}(W - U) \neq 1$.

Following ingenious ideas by Luneburg [33], which we are about to describe, just noticed difficulty can be resolved. By doing so, we are also going to rederive the results of section 6 via entirely different set of arguments.

We begin with Eq. (C.8) which we formally rewrite as

$$\beta_x^2 + \beta_y^2 + \beta_z^2 = n^2(x, y, z). \quad (C.10)$$
Consider now a set of wavefronts: $\beta(x, y, z) = \text{const}$. An orthogonal trajectory (a ray) through this wavefront at any point $x, y, z$ is normal to the wavefront through this point. In analogy with Eq.(4.13), we introduce the parameter $\tau$ along the ray trajectory so that this ray can be described in terms of the set $\{x(\tau), y(\tau), z(\tau)\}$. This allows us to introduce the equations for these rays as follows:

$$\frac{dx}{d\tau} = \lambda \beta_x, \quad \frac{dy}{d\tau} = \lambda \beta_y, \quad \frac{dz}{d\tau} = \lambda \beta_z. \quad (C.11)$$

Here $\lambda = \lambda(x, y, z; \tau) > 0$ is parameter describing various choices of parametrization of the ray trajectory. This freedom of reparametrization is the key element in achieving our goal-to bring Eq(C.10) into the form of Eq.(4.9a). Toward this goal, using Eq.(C.11) we consider the following chain of equalities

$$\frac{d}{d\tau} \left( \frac{1}{\lambda} \frac{dx}{d\tau} \right) = \beta_{xx} \frac{dx}{d\tau} + \beta_{xy} \frac{dy}{d\tau} + \beta_{xz} \frac{dz}{d\tau} = \lambda (\beta_{xx} \beta_x + \beta_{xy} \beta_y + \beta_{xz} \beta_z) = \frac{\lambda}{2} \frac{d}{dx} (\beta_x^2 + \beta_y^2 + \beta_z^2) \quad (C.12a)$$

Clearly, proceeding analogously, we obtain as well:

$$\frac{d}{d\tau} \left( \frac{1}{\lambda} \frac{dy}{d\tau} \right) = \frac{\lambda}{2} \frac{d}{dy} n^2, \quad (C.12b)$$

$$\frac{d}{d\tau} \left( \frac{1}{\lambda} \frac{dz}{d\tau} \right) = \frac{\lambda}{2} \frac{d}{dz} n^2. \quad (C.12c)$$

In equations (C.11) let $\lambda = \frac{1}{n}$ then, in view of Eq.(C.10), we obtain

$$\left( \frac{dx}{d\tau} \right)^2 + \left( \frac{dy}{d\tau} \right)^2 + \left( \frac{dz}{d\tau} \right)^2 = 1. \quad (C.13)$$

That is with such choice of the parameter $\lambda$ the yet arbitrary parameter $\tau$ becomes a parameter describing natural parametrization along the ray. We shall qualify it as "time". One still can do a better job, though, by noticing that, say,

$$\frac{1}{\lambda} \frac{d}{d\tau} \left( \frac{1}{\lambda} \frac{dx}{d\tau} \right) = \frac{d^2\mathbf{r}}{d\sigma^2} = \frac{1}{2} \frac{d}{dx} n^2. \quad (C.14)$$

Here $d\sigma = d(\lambda\tau)$. Evidently, Eq.s (C.12) become the Newtonian equations of motion

$$\frac{d^2\mathbf{r}}{d\sigma^2} = \frac{1}{2} \mathbf{n} \cdot \nabla n^2. \quad (C.16)$$

Here $\mathbf{r} = \{x, y, z\}$. If this is so, then in view of Eq.s(C.10) and (C.11) we easily obtain\(^{36}\)

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = n^2 \quad (C.18)$$

\(^{36}\)Since now $\dot{x} = \frac{dx}{d\sigma}$ and $\sigma = \lambda\tau$ and we can use Eq.s(C.10) and (C.11)
This time, however, we can bring Eq.s(C.13) and (C.18) in correspondence with each other by properly selecting "time". Introduce as well $\tilde{\beta}_x = \lambda \beta_x$, $\tilde{\beta}_y = \lambda \beta_y$, $\tilde{\beta}_z = \lambda \beta_z$. Using this definition and selecting $\lambda = \frac{1}{n}$ we convert Eq.(C10) into

$$\tilde{\beta}_x^2 + \tilde{\beta}_y^2 + \tilde{\beta}_z^2 = 1$$

(C.19)
easily recognizable as Eq.(4.9a) of the main text. Now we multiply Eq.(C.9) by $\lambda$ to obtain instead

$$\alpha \lambda \Delta \beta + 2 (\text{grad} \alpha \cdot \text{grad} \beta) = 0.$$  (C.20)

Notice also that $\Delta \beta = \nabla_x \beta_x + \nabla_y \beta_y + \nabla_z \beta_z$. But $\tilde{\beta}_x / \lambda = \beta_x, \text{etc}$. Since $\lambda = \frac{1}{n}$ then $\Delta \beta = \nabla_x n \tilde{\beta}_x + \nabla_y n \tilde{\beta}_y + \nabla_z n \tilde{\beta}_z = n \Delta \tilde{\beta} + \tilde{\beta}_x \nabla_x n + \tilde{\beta}_y \nabla_y n + \tilde{\beta}_z \nabla_z n$. In view of Eq.(C.20) and taking again into account that $\lambda = \frac{1}{n}$, it only remains to demonstrate that $\lambda^{-1} \left( \tilde{\beta}_x \nabla_x n + \tilde{\beta}_y \nabla_y n + \tilde{\beta}_z \nabla_z n \right) = 0$. Recall Eq.(C.11) and present $\tilde{\beta}_x \nabla_x n + \tilde{\beta}_y \nabla_y n + \tilde{\beta}_z \nabla_z n$ as $\frac{dx}{dt} \nabla_x n + \frac{dy}{dt} \nabla_y n + \frac{dz}{dt} \nabla_z n$. Use equations of motion, Eq.s(C.12), in order to write $\nabla_x n = \frac{dx}{dt} (n \frac{dx}{dt})$, etc. Finally, write $n \frac{dx}{dt} \frac{d}{dt} (n \frac{dx}{dt}) = \frac{1}{2} \frac{d}{dt} (n \frac{dx}{dt})^2$, etc. Using this result along with Eq.s(C.10),(C.11) we finally must prove that $\frac{d}{dt} n^2 = 0$. But $n^2 = \frac{8\pi^2 m}{c^4}(W - U)$ and for the time-independent $U$ the desired result follows. Alternatively, by comparing Eq.s(C.13) and (C.18) and by noticing that Eq.(C.18) is converted to (C.13) for $\lambda = \frac{1}{n}$ we obtain: $n = 1$ for $\lambda = \frac{1}{n}$. Thus, we just demonstrated that the Dupin cyclides generating set of equations, Eq.s (4.9), of the main text can be made to coincide with the the set of equations obtained by Madelung [177].

Appendix D

Various models of hyperbolic and Anti- de Sitter spaces

D.1. Hyperboloid model of hyperbolic space

Following Danciger [159], consider $\mathbb{R}^{n,1}$ denoting $\mathbb{R}^{n+1}$ equipped with $(n, 1)$ Minkowskitype metric tensor $g$:

$$g = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}. \quad (D.1)$$

Using $g$ we define the hyperboloid of two sheets $x^T g x = -1$. Sheets are determined by the sign of the first(or last) coordinate $x_1$, e.g. read [105] or [159] for more details. Traditionally, the hyperbolic space $\mathbb{H}^n$ is defined by the sheet for which $x_1 > 0$. Alternatively, it can be also defined as

$$\mathbb{H}^n = \{ x \in \mathbb{R}^{n+1} : x^T g x = -1 \}/\{ \pm I \} \quad (D.2)$$

Taking a quotient $\{ \pm I \}$ identifies two sheets. The hyperboloid $x^T g x = -1$ inherits the Riemannian metric of constant curvature $-1$ from that defined by Eq.(D.1). Isometries of $\mathbb{H}^n$ are defined by

$$\text{Isom}(\mathbb{H}^n) = \text{PO}(n.1) := \{ A \in GL(n + 1, \mathbb{R}) : A^T g A = g \}/\{ \pm I \}. \quad (D.3)$$
This result defines the Möbius-type transformation in accord with Definition 7.8. The orientation-preserving isometries are isometries lying in the identity component of $PO(n,1)$: $\text{Isom}(\mathbb{H}^n) = PO_0(n,1)$. For $n$ even (and this is our case) we have $PO_0(n,1) \simeq SO_0(n,1)$. If we had chosen to think about $\mathbb{H}^n$ as positive sheet of the hyperboloid $x^Tgx = -1$, then we should think about $PO(n,1)$ as the subgroup of $O(n,1)$ that preserves the positive sheet.

D.2. The projective model of hyperbolic space and Möbius geometry

This model is easily obtainable from the hyperboloid model. Instead of Eq.(D.2) we have now

$$\mathbb{H}^n = \{ x \in \mathbb{R}^{n+1} : x^Tgx > 0 \}/\sim \quad (D.4)$$

where the symbol $\sim$ denotes equivalence. That is $x \sim x'$ whenever there is some nonzero $\lambda \in \mathbb{R}^*$ such that $x = \lambda x'$. Thus, every hyperbolic structure is also a projective structure defined on some domain of $\mathbb{RP}^n$ determined by the equivalence relation. This fact then allows us to define the boundary at infinity as

$$\partial^\infty \mathbb{H}^n = \{ x \neq 0, x \in \mathbb{R}^{n+1} : x^Tgx = 0 \}/\sim. \quad (D.5)$$

We had encountered this equation already as Eq.(7.12). It can be interpreted either as the projectivized light cone or as the condition for two spheres to intersect transversally. $\partial^\infty \mathbb{H}^n$ has invariant flat conformal structure in the sense of Eq.(D.3). Since a geodesic in $\mathbb{H}^n$ is determined by two distinct points on $\partial^\infty \mathbb{H}^n$ this means that in this formalism every geodesic in $\mathbb{H}^n$ is in one-to-one correspondence with the orthogonal intersection of two spheres.

D.3. Hyperboloid model of Anti-de Sitter space

Let $\mathbb{R}^{n+1,2}$ denote $\mathbb{R}^{n+3}$ equipped with $(n + 1, 2)$ Minkowski-like metric tensor $g$

$$g = \begin{pmatrix} I_{n+1} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (D.6)$$

Using $g$, we define the hyperboloid $x^Tgx = -1$. Instead of Eq.(D2) defining the hyperbolic space $\mathbb{H}^n$ we are having now the following definition of the Anti-de Sitter space (AdS)[159]

$$AdS^{n+2} = \{ x \in \mathbb{R}^{n+3} : x^Tgx = -1 \}/\{\pm I\} \quad (D.7)$$

The hyperboloid $x^Tgx = -1$ inherits a Lorentzian metric of constant curvature $-1$ from the form defined by Eq.(D.6). Isometries of $AdS^{n+2}$ are defined as

$$\text{Isom}(AdS^{n+2}) = PO(n+1,2) := \{ A \in GL(n+3,\mathbb{R}) : A^TgA = g \}/\{\pm I\}. \quad (D.8)$$

D.4. The projective model of the Anti-de Sitter space and Lie sphere geometry
The projective model of AdS is easily obtainable from the hyperboloid AdS model. Instead of Eq. (D.7) we have now
\[
AdS^{n+2} = \{ x \in \mathbb{R}^{n+3} : x^T g x > 0 \} / \sim \tag{D.9}
\]
The boundary of AdS space is defined now by analogy with Eq. (D.5) that is
\[
\partial^\infty AdS^{n+2} = \{ x \neq 0, x \in \mathbb{R}^{n+3} : x^T g x = 0 \} / \sim \tag{D.10}
\]
In view of Remark 7.4. this result is easily recognizable as the Lie quadric \( Q^{n+1,2} \). It can be interpreted either as the projectivized light cone or as a condition for two spheres to touch each other. Accordingly, the geodesics in Anti-de Sitter space can be of three types:

a) A space-like geodesics is determined by two distinct points lying on \( \partial^\infty AdS^{n+2} \), that is on \( Q^{n+1,2} \). In terms of the Lie sphere geometry such a geodesics corresponds to two spheres touching each other.

b) A light-like geodesics has its both ends lying at the same point of the quadric \( Q^{n+1,2} \). This situation is characterizes the projectivized light cone. In the language of the Lie sphere geometry such situation corresponds to spheres participating in the formation of canal surfaces other than Dupin cyclides as explained in section 7.5.1.

c) A time-like periodic geodesics which does not touch the boundary \( \partial^\infty AdS^{n+2} \).

The action of \( PO(n+1,2) \) preserves \( \partial^\infty AdS^{n+2} \) in accord with Definition 7.6. and Corollary B.9. Furthermore, according to [161], page 184, Einstein’s space \( \text{Ein}^{n,1} = \partial^\infty AdS^{n+2} \). Thus, Einstein’s space is the same as the the space where Lie sphere geometry acts and, using the discovered by Sophus Lie correspondence between the Lie sphere and Plücker line geometries described in section 7.4., it is possible to map such defined Einstein’s space into the space of twistors. The connection with Einstein spaces (and with spaces where Lie sphere geometry acts) and, therefore, with Penrose boundary can be easily explained now. For this, following [161] and using Eq. (7.10) (or Eq. (6.41a)) of the main text, we relate the quadratic form
\[
\left< v, v \right> = v_1^2 + \cdots + v_{n+1}^2 - v_{n+2}^2 - v_{n+3}^2 \tag{D.11a}
\]
i to the nullcone as:
\[
v_1^2 + \cdots + v_{n+1}^2 = v_{n+2}^2 + v_{n+3}^2. \tag{D.11b}
\]
Since each of \( v’ \)’s is never zero, it is possible to divide both sides of Eq. (D.11b) by the positive number \( \sqrt{v_{n+2}^2 + v_{n+3}^2} \) so that this equation can be rewritten as
\[
v_1^2 + \cdots + v_{n+1}^2 = 1 = v_{n+2}^2 + v_{n+3}^2. \tag{D.12}
\]
It describes the product \( S^n \times S^1 \). For \( n = 3 \) we obtain: \( S^3 \times S^1 \). This is an example of static Einstein spacetime with compactified time axis [22]. In fact, \( S^3 \times S^1 = \text{Ein}^{n,1} \)
while $\text{Ein}^{n,1} = \hat{\text{Ein}}^{n,1}/\{\pm 1\}$. Kühnel and Rademacher had classified all (pseudo) Riemannian Einstein spacetimes having local or global conformal conformal groups [183]. These are surely including the pp wave type Ricci-flat spaces which, based on results of Appendix B, can be identified with Penrose limits.

References

[1] E.Schrödinger, Collected Papers on Wave Mechanics, Chelsea Publ.Co, New York, 1978.
[2] R.Courant and D.Hilbert, Methods of Mathematical Physics, Vol.1, Interscience Publishers, Inc., New York, 1953.
[3] B.Baker and E.Copson, The Mathematical Theory of Huygens’ Principle, Clarendon Press, Oxford, 1939.
[4] R.Feynman, Rev. Mod.Phys. 20 (1948) 367.
[5] M.Gutzwiller, Huygens’ principle and the path integral, in Path Summation: Achievements and Goals (Trieste, 1987), pp. 47–73, L.Schulman editor, World Sci. Publishing, Singapore, 1988.
[6] R.Feynman, R.Leighton and M.Sands, The Feynman Lectures on Physics, Vol.3, Addison-Wesley Publishing Co., New York, 1989.
[7] R.Feynman and A.Hibbs, Quantum Mechanics and Path Integrals, McGraw-Hill Co, New York, 1965.
[8] M.Born and E.Wolf, Principles of Optics, Cambridge U. Press, Cambridge, UK, 1980.
[9] C. Jönsson, Zeitschrift für Physik, 161 (1961) 454.
[10] M.Arndt, O.Nairz, J.Vos-Andreae, C.Keller, G.Zouw and A.Zellinger, Nature, 401 (1999) 680.
[11] S.Eibenberger, S.Gerlich, M.Arndt, M.Mayor and J.Tüxen, Phys. Chem. Chem. Phys. 15 (2013) 14696.
[12] D.Bohm, Quantum Theory, Dover Publications Inc., New York, 1989.
[13] A. Sanz, and S. Miret-Artés, A Trajectory Description of Quantum Processes I. Fundamentals, Springer-Verlag, Berlin, 2012.
[14] A.Kholodenko, arXiv:1703.04674.
[15] U. Leonhardt, Measuring the Quantum State of Light, Cambridge U. Press, Cambridge, UK, 1997.
[16] M. Fox, Quantum Optics: An Introduction, Oxford U. Press, Oxford, UK, 2006.
[17] U.Niederer, Helv. Phys. Acta 45 (1972) 802.
[18] R.Littlejohn, Phys. Rep. 138 (1986) 193.
[19] B.Wybourne, Classical Groups for Physicists, Wiley-Interscience Publ., New York, 1974.
[20] M.Kibler, Found.Chem. 9 (2007) 221.
[21] R. Campoamor-Stursberg, J.of Phys.: Conference Series 538 (2014) 012004.
[22] A.Keane and R.Barrett, Class.Quantum Grav. 17 (2000) 201.
[23] A. Bohm, Y. Ne’eman and A.Barut, Dynamical Groups and Spectrum Generating Algebras, Vol’s 1 & 2, World Scientific, Singapore, 1988.
[24] E.Schrödinger, Expanding Universes, Cambridge University Press, Cambridge, 1956.
[25] S.Huggett and K.Tod, An Introduction to Twistor Theory Cambridge University Press, Cambridge, 1994.
[26] E.Schrödinger, Science and the Human Temperament, George Allen & Unwin LTD, London, 1935.
[27] V. Kac and P.Cheung, Quantum Calculus, Springer-Verlag, Berlin, 2002.
[28] E.Schrödinger, Phys.Rev.28 (1926) 1049.
[29] R.Courant and D.Hilbert, Methods of Mathematical Physics, Vol.2, Interscience Publishers, Inc., New York, 1962.
[30] F.Cardin, Elementary Symplectic Topology and Mechanics, Springer-Verlag, Heidelberg, 2015.
[31] D.Tannor, Introduction to Quantum Mechanics. A Time-Dependent Perspective, University Science Books, Sausalito, Ca., 2007.
[32] M.Gosson and B.Hiley, Found.Phys.41 (2011) 1415.
[33] R.Luneburg, Mathematical Theory of Optics, U.of California Press, Berkeley, 1966.
[34] A.Davydov, Quantum Mechanics, Pergamon Press, Oxford, 1976.
[35] L.De Broglie, An Introduction to the Study of Wave Mechanics, Methuen & Co. LTD., 36 Essex Street, W.C., 1930.
[36] C.Adams, M.Siegel and J. Mlynek, Phys. Reports 240 (1994) 143.
[37] R.Sawant, J.Samuel,A.Sinha,S.Sinha and U.Sinha, PRL 113 (2014) 120406.
[38] K.Ito and H. McKea Jr., Diffusion Processes and Their Sample Paths, Springer-Verlag, Berlin, 1965.
[39] H.De Raedt, K.Michelsen and K.Hess, Phys.Rev.A 85 (2012) 012101.
[40] U.Sinha,C.Couteau, T.Jennewein,R.Laflamme and G.Weihs, Science 329 (2010) 418.
[41] I. Söllner, B.Gschösser, P. Mai, B. Pressl, Z. Vörös and G. Weihs, Found. Phys. 42 (2012) 742.
[42] A. Sommerfeld, Optics, Academic Press, NY, 1964.
[43] L.Landau and E.Lifshitz, Classical Theory of Fields, Butterworth-Heinemann, Oxford, 2000.
[44] A.Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, Berlin, 1982.
[45] L.Evans, Partial Differential Equations,
AMS Publishers, Providence, RI, 2010.

[46] J. Hadamard, Lectures on Cauchy’s Problem in Linear Partial Differential Equations, Nauka Co, Moscow, 1978 (Russian translation from French original).

[47] V. Arnol’d, Contact Geometry and Wave Propagation, Enseign. Math. 36 (1990) 215.

[48] V. Arnol’d, Lectures on Partial Differential Equations, Springer-Verlag, Berlin, 2004.

[49] V. Arnol’d, Mathematical Methods of Classical Mechanics, Springer-Verlag, Heidelberg, 1978.

[50] V. Maslov and M. Fedoryuk, Semiclassical Approximation in Quantum Mechanics, D. Reidel Publishing Co., Dordrecht-Boston, Mass., 1981.

[51] M. Fedoryuk, Partial Differential equations V, Springer-Verlag, Berlin, 1999.

[52] V. Arnol’d, Ordinary Differential Equations, Springer-Verlag, Berlin, 1992.

[53] C. Chicone, Ordinary Differential equations with Applications, Springer-Verlag, Berlin, 1999.

[54] A. Fasano and S. Marmi, Analytical Mechanics, Oxford U. Press, Oxford, 2006.

[55] A. Khoshledenko, Applications of Contact Geometry and Topology in Physics, World Scientific, Singapore, 2013.

[56] M. Gotay, J. Math. Phys. 40 (1999) 2017.

[57] V. Guillemin and S. Sternberg, Variations on the Theme by Kepler, AMS Publishers, Providence, RI 1990.

[58] M. Giaqinta and S. Hildebrandt, Calculus of variations II, Springer-Verlag, Berlin, 1996.

[59] S. Benenti, Hamiltonian Structures and Generating Families, Springer-Verlag, Berlin, 2011.

[60] I. Gelfand and S. Fomin, Calculus of Variations, Prentice-Hall Inc., New Jersey, 1963.

[61] M. Mathisson, Acta Math. 71 (1939) 249.

[62] L. Asgeirsson, Comm. Pure Appl. Math. 9 (1956) 307.

[63] P. Günther, Math. Intelligencer 13 (1991) 56.

[64] P. Günther, Huygens’ Principle and Hyperbolic Equations, Academic Press, Inc., Boston, MA, 1988.

[65] F. Friedlander, The Wave Equations on a Curved Space-Time, Cambridge U. Press, Cambridge, 1975.

[66] R. McLenaghan and J. Carminati, Ann. Inst. Henri Poincare 44 (1986) 115.

[67] R. Goldoni, J. Math. Phys. 18 (1977) 2125.

[68] N. Ibragimov, A. Oganesyan, Russian Math. Surveys 46 (1991) 137.

[69] M. Belger, R. Schimming and V. Wunsch, J. for Analysis and its
Applications 16 (1997) 9.

[70] F. Friedlander, Proc. Camb. Phil. Soc. 43 (1946) 360.
[71] A. Kholodenko, Anal. Math. Phys. 6 (2016) 163.
[72] M. Audin and M. Damian, Morse Theory and Floer Homology, Springer-Verlag, Berlin, 2014.
[73] D. Bohm and B. Hiley, Undivided Universe, Routledge, New York and London, 1993.
[74] A. Sym, J. Nonlinear Math. Phys. 12, Supplement 1 (2005) 648.
[75] R. Pruss and A. Sym, Phys. Lett. A 336 (2005) 459.
[76] A. Sym and A. Szereszewski, SIGMA 7 (2011) 095.
[77] P. Broadbridge, C. Chau and W. Miller Jr. SIGMA 8 (2012) 089.
[78] J. Alcázar, H. Dahl and G. Muntingh, arXiv:1611.06768.
[79] T. Cecil and P. Ryan, Geometry of Hypersurfaces, Springer-Verlag, Berlin, 2015.
[80] Y. Suris, in Sophus Lie and Felix Klein. The Erlangen Program and its Impact in Mathematics and Physics, EMS 23 (2015).
[81] J. Ratcliffe, Foundations of Hyperbolic Manifolds, Springer-Verlag, Berlin, 1994.
[82] C. Codirla and H. Osborn, Ann. Phys. 260 (1997) 91.
[83] I. Bars and J. Terning, Extra Dimensions in Space and Time, Springer Science + Business Media, LLC 2010.
[84] P. Arvidsson, R. Marnelius, arXiv:hep-th/0612060.
[85] S. Brodsky, G. de Teramond, H. Dosch and J. Elich, Phys. Reports 584 (2015) 1.
[86] A. Kolesnik and N. Ratanov, Telegraph Processes and Option Pricing, Springer-Verlag, Berlin, 2013.
[87] W. Greiner, Relativistic Quantum Mechanics. Wave Equations, Springer-Verlag, Berlin, 2000.
[88] V. Berestetskii, E. Lifshitz and L. Pitaevskii, Relativistic Quantum Theory, Pergamon Press, Oxford, UK, 1971.
[89] N. Bogoliubov and D. Shirkov, Introduction to the Theory of Quantized Fields, John Wiley & Sons, New York, 1976.
[90] M. Kac, Rocky Mountain J. of Math. 4 (1974) 497.
[91] B. Gaveau, T. Jacobson, M. Kac and L. Schulman, PRL 53 (1984) 419.
[92] C. Dewitt-Morette and P. Cartier, Functional Integration: Action and Symmetries, Cambridge U. Press, Cambridge, 2006.
[93] A. Kolesnik and M. Pinsky, J. Stat. Phys. 142 (2011) 828.
[94] E. Zauderer, Partial Differential Equations of Applied Mathematics, Wiley-Interscience, New York, 1989.
[95] B. Gaveau and L. Schulman, Il Nuovo Cimento 11 (1989) 31.
[96] A. Kholodenko, Ann. Phys. 201 (1990) 186.
[97] S. Deguchi and T. Suzuki, Phys. Lett. B 731 (2014) 337.
[98] S. Deguchi and S. Okano, Phys. Rev. D 93 (2016) 045016.
[99] I. Bars, Phys. Rev. D 58 (1998) 066006.
[100] I. Araya and I. Bars, Phys. Rev. D 89 (2014) 066011.
[101] S. Singer, Linearity, Symmetry and Prediction in the Hydrogen Atom, Springer-Verlag, Berlin, 2005.
[102] V. Fock, Z. Phys. 98 (1935) 145.
[103] L. Landau and E. Lifshitz, Quantum Mechanics, Elsevier Publ. Co., Amsterdam, 1981.
[104] H. Goldstein, Classical Mechanics, Addison-Wesley Publ. Co., Reading, MA, 1980.
[105] I. Gelfand, R. Milnos and Z. Shapiro, Representations of the Rotation and Lorentz Groups and Their Applications, Martino Fine Books, P.O. Box 913, Eastford, CT, 2012. (Translation from the original 1958 Russian edition).
[106] J. Bros and G. Viano, Forum Math. 8 (1996) 621.
[107] J. Bros and G. Viano, Forum Math. 8 (1996) 659.
[108] J. Bros and G. Viano, Forum Math. 9 (1997) 165.
[109] D. Basu and S. Srinivasan, Czech. J. Phys. B 27 (1997) 635.
[110] J. Milnor, The American Mathematical Monthly 90 (1983), 353.
[111] T. de Laat, Regularization and quantization of the Kepler problem, PhD Thesis, Department of Mathematics, Radboud University, Nijmegen, Netherlands, 2010.
[112] A. Fomenko, Symplectic Geometry, Gordon and Breach Co., New York, 1988.
[113] M. Dunajski, Solitons, Instantons and Twistors, Oxford U. Press, Oxford, UK, 2010.
[114] R. Penrose and W. Rindler, Spinors and Space-Time, Cambridge U. Press, Cambridge, UK, 1984.
[115] D. Sommerville, An Introduction to the Geometry of N Dimensions, Dower Publications, Inc., New York, 1958.
[116] H. Pottmann and J. Wallner, Computational Line Geometry, Springer-Verlag, Berlin, 2010.
[117] R. Penrose, Relativistic symmetry groups, in Group Theory and Nonlinear Problems, A. Barut Editor, pp. 1-58, D. Reidel Publ. Co., Boston, 1974.
[118] M. Kriele, Spacetime, Springer-Verlag, Berlin, 1999.
[119] A. Kholodenko, Int’l J. Mod. Phys. A 30 (2015) 1550189.
[120] A. Kholodenko, E. Ballard, Physica A 380 (2007) 115.
[121] M. Bander and C. Itzykson, Rev. Mod. Phys. 38 (1966) 346.
[122] I. Frenkel and M. Libine, Adv. Math. 218 (2008) 1806.
[123] A. Hurwitz and R. Courant, Complex Function Theory (in German)
Interscience Publishers, Inc., New York, 1944.

[124] R. Fueter, Comment. Math. Helv. 8 (1) (1935) 371.
[125] B.Cordani, The Kepler Problem, Birkhäuser, Basel, 2003.
[126] I.Bars, Phys.Rev. D 58 (1998) 066006.
[127] I.Bars, C. Deliduman and O.Andreev, Phys. Rev. D 58 (1998) 066004.
[128] P.Kustaanheimo and E.Stiefel, J.Fur.Reine und Angevante Math. 218 (1965), 204.
[129] F.Cornish, J.Phys.A 17 (1984) 323.
[130] A.Chen, Phys.Rev.A 22 (1980) 333.
[131] A.Chen, Phys.Rev.A 23 (1981) 1655.
[132] A.Chen,Phys.Rev.A 26 (1982) 669.
[133] G.Thorbergsson, Bull. London Math.Soc. 15 (1983) 493.
[134] E.Huhnen-Venedey, PhD Thesis, department of Mathematics, Technical University,Berlin, 2007.
[135] E.Musso, G.Jensen and L.Nicolodi, Surfaces in Classical Geometries, Springer -Verlag, Berlin, 2016.
[136] C.Doran and A.Lasenby, Geometric Algebra, Cambridge U.Press, Cambridge, 2004.
[137] A.Bobenko and Y.Suris, Discrete Differential Geometry, AMS Publishers, Providence, RI, 2008.
[138] A.Bobenko and E.Huhnen-Venedey, Geom. Dedicata 159 (2012) 207.
[139] L.Drouton, L.Fuchs, L.Garnier and R.Langevin, Adv. in Appl.Clifford Algebra 24 (2014) 515.
[140] L.Dorst, Math.Comput. Sci 10 (2016) 97.
[141] S.Helgason, in The Sophus Lie memorial conference (Oslo 1992), pages 3-21, Scand.U.Press, Oslo, 1994.
[142] L.Mason and N.Woodhouse, Integrability, Self-Duality, and Twistor Theory, Clarendon Press, Oxford, 1996.
[143] R.Ward, R.Wells,Jr. , Twistor Geometry and Field Theory, Cambridge University Press, Cambridge, 1990.
[144] F.Klein, Vorlesungen Uber Höhere Geometrie, Spinger-Verlag, Berlin, 1926.
[145] W. Blaschke, Vorlesungen Uber Differential-Geometrie III, Springer-Verlag , Berlin, 1929.
[146] M.Schrott and B.Odenhal, J. for Geometry and Graphics, 10 (2006)73.
[147] A.Gray, Modern Differential Geometry of Curves and Surfaces with Mathematica, Taylor&Francis, Roca Baton, FL 2006.
[148] T. Ivey, AMS Proceedings 123 (1995) 865.
[149] P. Colapino, Articulating Space: Geometric Algebra for Parametric Design-Symmetry Kinematics and Curvature, PhD Thesis, U.of California, Santa Barbara, 2016.
[150] A.Bobenko and E.Huhnen -Venedey, Geom.Dedicata 159 (2012) 207.
[151] M. Lavicka and J. Vrsek, J. for Geometry and Graphics 13 (2009) 145.
[152] M. Arrayas, D. Bouwmeester and J. Trueba, Phys. Reports 667 (2017), 1.
[153] C. Hoyos, N. Sicar and J. Sonnenschein, J. Phys. A 48 (2015) 255204.
[154] H. Nastase, Introduction to the AdS-CFT Correspondence, Cambridge U. Press, Cambridge, UK, 2015.
[155] A. Kholodenko, J. Geom. Phys. 35 (2000), 193.
[156] A. Kholodenko, J. Geom. Phys. 38 (2001), 81.
[157] A. Kholodenko, J. Geom. Phys. 43 (2005), 45.
[158] C. Frances, Comm. Math. Helv. 80 (2005), 883.
[159] J. Danciger, Geometric Transitions: From Hyperbolic to AdS Geometry, PhD Thesis, Department of Mathematics, Stanford University, 2011.
[160] S. Takeuchi, Japanese J. of Appl. Physics 53 (2014) 030101.
[161] T. Barbot, V. Charette, T. Drumm, W. Goldman and K. Melnick, in Recent Developments in Pseudo-Riemannian Geometry, D. Alekseevsky and H. Baum Editors, pp 179-229, ESI Lect. Math. Phys., Eur. Math. Soc., Zürich, 2008.
[162] V. Mazya and T. Shaposhnikova, Jacques Hadamard, A Universal Mathematician, AMS Publishers, Providence, RI, 1998.
[163] Y. Kravtsov, Geometric Optics in Engineering Physics, Alpha Sci. International Ltd., Harrow, UK, 2005.
[164] A. Petrov, Einstein Spaces, Pergamon Press, New York, 1969.
[165] A. Petrov, New Methods in General Theory of Relativity, Nauka, Moscow, 1966 (in Russian).
[166] N. Ibragimov, Transformation Groups Applied to Mathematical Physics, D. Reidel Publ. Co., Boston, MA, 1985.
[167] R. Ward, Class. Quant. Grav. 4 (1987) 775.
[168] J. Griffits and J. Podolsky, Exact Space-Times in Einstein’s General Relativity, Cambridge U. Press, Cambridge UK, 2009.
[169] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers and E. Herlt, Exact Solutions of Einstein’s Field Equations, Cambridge U. Press, Cambridge, UK, 2003.
[170] R. Geroch, Ann. Phys. 36 (1966) 147.
[171] C. Torre, Class. Quantum Grav. 31 (2014) 045022.
[172] G. Rainich, AMS Transactions 27 (1925) 106.
[173] Ch. Misner and J. Wheeler, Ann. Phys. 2 (1957) 525.
[174] N. Kuiper, Ann. Math. 30 (1949) 916.
[175] W. Kühnel and H-B. Rademacher, J. Math. Pures Appl. 88 (2007) 251.
[176] A. Aminova, Russ. Math. Surveys 50 (1995) 69.
[177] A. Keane and B. Tupper, Class. Quantum Grav. 21 (2004) 2037.
[178] W. Kühnel and H-B. Rademacher, Geom. Dedicata 109 (2004) 175.
[179] M. Cahen and M. Flato, Differential Geometry and Relativity,
[180] E. Ferapontov, Soviet Jorn.Math. 55 (1991) 1970.
[181] E. Ferapontov, Differential Geom. and Applications 5 (1995) 121.
[182] E. Madelung, Zeit.f.Phys.40 (1927) 322.
[183] W. Kühnel and H-B. Rademacher, Result.Math. 56 (2009) 421.