On the force caused by a null Einstein-Maxwell field with the plane symmetry

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Abstract. The article shows that a large flat platform with a constant current, which flows over its surface, accelerates time. It is also shown that if an alternating current flows along the surface of a flat platform while creating a null electromagnetic field then a force repelling from the platform acts on the test particle located near it. This force has no gravitational nature and arises as a result of the curvature of space-time by the electromagnetic field of a flat platform with an alternating current.

1. Problem statement
In this paper, the following task is solved: to find out whether a flat platform, along the surface of which an electric current flows, creates a force whose nature is different from that of gravity. If this force exists, then find the direction of the force relative to the flat platform and the dependence on the current flowing along the surface of the platform.

2. The algorithm for solving the problem
Let the electric current flow along the conductive surface of the platform along the direction of the \( z \)-axis. To find out whether the flat platform creates the current-controlled force, let us apply the following algorithm of actions:
1) Find the solution to the equations of Einstein – Maxwell field describing a flat platform with your current.
2) Since we are talking about the man-made platform, which is unlikely to create a noticeable gravity by its weight, let us move on to the limit of small mass, so that when you turn off the current, the metric becomes flat.
3) Drawing on the behaviour of test particle we find out whether the force acts on it from the side of the platform.

3. The requirements for the metric for the plane with a current
Let us consider that the platform is quite large, and it can be envisioned as a boundless plane within the framework of the problem solved. Therefore, to solve the problem, it is necessary to know the type of the metric for the plane with a current. The metric should depend on two parameters \( I \) and \( \sigma \), where \( I \) is the current flowing along the plane surface band of unit width (linear current density), \( \sigma \) is the surface mass density.
Choose the "Cartesian coordinate system" $\left( t, x, y, z \right)$ so that the coordinate plane $\left( y, z \right)$ coincides with the plane with current, and the direction of the $z$-axis with the direction of the current $I$.

The desired metric must meet the following conditions:

1) In the vacuum case $I = 0$ the metric must turn into Taub’s plane-symmetric vacuum solution.

2) For $I = 0$ and $\sigma = 0$, the metric should become flat:

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2.$$  

3) The current density in this problem has only one non-zero component:

$$J^j = \left( 0; 0; 0; j^3 \right), \text{ where } j^3 = I \delta \left( x \right).$$ (1)

4) In the classical case, the electromagnetic field outside the plane with alternating current will be a plane electromagnetic wave. Therefore, in this problem, we require the field of the plane with alternating current to be null. This means that the invariant

$$J = F^{ij} F_{ij},$$ (2)

where $F_{ik} = \frac{1}{2} \left( F_{ik} + i \tilde{F}_{ik} \right)$, $\tilde{F}_{ik} = \frac{1}{2} \sqrt{-\varepsilon} \delta_{ik} \epsilon_{mn} \epsilon^{mn}$ equals zero.

4. Taub’s Metric

There are various types of writing this metric in the literature [1-4]. In this paper, we will choose the Taub’s metric in the form in which it is presented in [5]:

$$ds^2 = \frac{1}{\sqrt{1 - Mx}} \left( dt^2 - dx^2 \right) - \left( 1 - Mx \right) \left( dy^2 + dz^2 \right),$$ (3)

where $M$ is a constant.

Let us determine the dependence of $M$ on the surface mass density $\sigma$. Considering $Mx$ small and taking into account only the first approximation of this value, we rewrite the metric component $g_{00}$ in the form:

$$g_{00} \approx 1 + \frac{M}{2} x.$$

Whence it follows that the Newtonian gravitational potential of a massive boundless flat platform has the form:

$$\varphi_{\text{grav}} = \frac{M}{4} x,$$

and the intensity of the gravitational field is equal to

$$g_{\text{grav}} = \frac{d \varphi_{\text{grav}}}{dx} = -\frac{1}{4} M.$$

As is known from classical mechanics, the gravitational strength of the field of a boundless flat layer is equal to:

$$g_{\text{grav}} = -2\pi G \sigma,$$

where $G$ is the gravitational constant, $\sigma$ is the surface mass density.

Then for the value $M$ we find:

$$M = 8\pi G \sigma.$$

The constraint $x < \frac{1}{M}$ in the metric is a consequence of the fact that the plane is infinite. The problem under consideration is local: $Mx \ll 1$. Therefore, this restriction does not affect the final result, and the chosen type of the Taub’s metric is adequate for solving the task.
5. The general form of the metric. The null tetrad and the NP equations

As is known, the plane symmetry is given by three space-like Killing vectors \([1]\):
\[
\xi_1 = \partial_y, \quad \xi_2 = \partial_z, \quad \xi_3 = y\partial_z - z\partial_y,
\]
but for a plane with current, the \(y\) and \(z\) directions are not equal. Therefore, we can assume with greater confidence that there are only two Killing vectors:
\[
\xi_1 = \partial_y, \quad \xi_2 = \partial_z.
\]

In GR, two space-like vectors define solutions with cylindrical symmetry. Therefore, we will choose the desired metric in the Weyl form:
\[
ds^2 = e^{-2U+2K} \left( dt^2 - dx^2 \right) - \left( 1 - Mx \right)^2 e^{-2U} dy^2 - e^{2U} dz^2,
\]
where \(U = U(x,t), \ K = K(x,t)\).

In order to find out the metric needed to solve the problem we apply the Newman-Penrose formalism. Let's choose a null tetrad \((l, n, m, \bar{m})\) in the form:
\[
l' = (1,1,0,0),
\]
\[
n' = \left( n^0, n^1, 0, 0 \right),
\]
\[
m' = \left( 0, 0, m^2, m^3 \right),
\]
\[
\bar{m}' = \left( 0, 0, m^2, -m^3 \right).
\]

Taking into account the fact that the vector \(\bar{m}\) is complex-conjugate to the vector \(m\), it follows from the Weyl metric that the equalities are fulfilled for the components:
\[
n^0 = -n^1, \ m^2 m^3 = \frac{i}{2(1-Mx)}.
\]

For the selected tetrad, the spin coefficients have the form:
\[
\alpha = \beta = \gamma = \tau = \pi = v = k = 0
\]
\[
\rho = \frac{M}{2(1-Mx)}, \ \epsilon = \epsilon^*, \ \bar{\sigma} = \bar{\sigma}^*, \ \mu = \mu^*, \ \lambda = \lambda^*.
\]

From the equations of the Newman-Penrose formalism for the tetrad components and the spin coefficients we have (NP equations):
\[
n^1 = -\frac{2}{M} \left( 1 - Mx \right) \mu,
\]
\[
Dm^2 = \left( \frac{M}{2(1-Mx)} + \bar{\sigma} \right) m^2,
\]
\[
\Delta m^2 = -\left( \mu + \lambda \right) m^2.
\]
\[
\bar{\sigma}^2 - \frac{M \epsilon}{1-Mx} + \Phi_{00} - \frac{M^2}{4(1-Mx)^2} = 0,
\]
\[
-\Delta \epsilon = \Psi_{2},
\]
\[
D\lambda = \frac{\lambda M}{2(1-Mx)} + \mu \bar{\sigma} - 2\epsilon \lambda + \Phi_{20},
\]
\[ D\mu = \frac{\mu M}{(1-Mx)} - 2\mu \varepsilon, \]  
\[ \frac{\mu M}{2(1-Mx)} + \lambda \vec{\sigma} + \Psi_2 = 0, \]
\[ -\Delta \mu = \mu^2 + \lambda^2 + \Phi_{22}, \]
\[ -\Delta \vec{\sigma} = \mu \vec{\sigma} + \frac{\lambda M}{2(1-Mx)} + \Phi_{02}, \]

where \( D = \partial_x + \partial_x \), \( \Delta = n^0 \partial_x + n^1 \partial_x \).

Here \( \Phi_{ab} = \kappa_0^2 \phi_a \phi_b \), where \( \phi_a \) is complex scalar, the electromagnetic field, \( \kappa_0 = 8\pi G \) is the Einstein’s constant.

Due to the plane symmetry, the electromagnetic field is polarized linearly and will be given by the electromagnetic field tensor, whose nonzero covariant components are:

\[ F_{03}, \quad F_{13}. \]

The scalars of the electromagnetic field \( \phi_a \) are equal to:

\[ \phi_0 = F_{01} l^m = (F_{03} + F_{13}) m^3, \]
\[ \phi_1 = \frac{1}{2} F_{y} \left(l^m n^i + \bar{m}^i m^j\right) = 0, \]
\[ \phi_2 = F_{y} \bar{m}^i n^j = (-F_{03} + F_{13}) m^3 n^1. \]

As a result, Maxwell's equations written in terms of these scalars have the form:

\[ D\phi_2 + \lambda \phi_0 - (\rho - 2\varepsilon) \phi_2 = -2\pi j^3, \]
\[ -\Delta \phi_0 - \mu \phi_0 + \vec{\sigma} \phi_2 = -2\pi j^4, \]

where \( j^3 = m^i j^i, \quad j^4 = \bar{m}^i j^i \) are the tetrad components of the 4-vector of the current density (1).

6. The Stokes' Theorem for the plane with a current

Let us find the magnetic field of the boundless plane with a current. Let us take a parallelepiped \( S \times T \), where \( S = \{ x \times y \mid x \in [-a; a], \ y \in [-b; b] \} \) is a rectangle which is perpendicular to the \( z \) -axis, \( T = [t_1, t_2] \) is a time interval. Let an electric current \( \mathcal{J} \) flow through the rectangle \( S \).

First, consider the case when a constant electric current flows through the conductor \( \mathcal{J} \). From the left and right sides of Maxwell's equation

\[ F^{ik}_{\mathcal{J}} = -4\pi j^i, \]

take the integral over the parallelepiped \( S \times T \):

\[ \int F^{ik}_{\mathcal{J}} dS_i = -4\pi \int j^i dS_i, \]

where \( dS_i = \frac{1}{6} e_{ijkl} \sqrt{-g} dS^{kl} \). The integral on the right side of the equality (19)

\[ \int j^i dS_i \equiv \int j^3 dS_3 \]

is equal to the change in the charge that flows through the rectangle \( S \) over a period of time \( \Delta t = t_2 - t_1 \).
\[ \Delta q = \int_j^3 j^3 ds_j = -\frac{1}{6} \int_j^3 e_{jki} \sqrt{-g} dS^{jkl} = \int_j^3 \sqrt{-g} dx dy dt = \oint_j^k \mathcal{J} dt. \]

To calculate the integral \( \int F_{ik}^* dS_i \), we apply the Stokes theorem [6]:

\[ \int F_{ik}^* dS_i = \frac{1}{2} \oint \phi F_{ik}^* df_{ik}^*, \tag{21} \]

where \( df_{ik}^* = -\varepsilon_{iklm} \sqrt{-g} df^{lm} \) is the tensor dual to the bivector of the infinitesimal element of the two-dimensional surface \( df^{lm} = dx^l dx^m dx^k - dx^k dx^m dx^l \).

Given that the two faces of the parallelepiped passing through the points \( x = -a \) and \( x = a \) are parallel to the coordinate plane \( ty \) the integral on the right side (21) will be equal to

\[ -\frac{1}{2} \oint \phi F_{ik}^* df_{ik}^* = \int e_{ikl} \sqrt{-g} df^{lm} = -2 \int F_{13}^* \sqrt{-g} dt dy = -b \int_{t_1}^{t_2} F_{13}^* \sqrt{-g} dt \]

As a result, the equality (20) takes the form:

\[ -4b \int_{t_1}^{t_2} F_{13}^* \sqrt{-g} dt = -4\pi \int_{t_1}^{t_2} \mathcal{J} dt. \]

Whence it follows that the nonzero contravariant component of the electromagnetic field tensor is equal to:

\[ F_{13}^* = \frac{2\pi I}{\sqrt{-g}}, \tag{22} \]

where \( I = \frac{J}{2b} \) is the current flowing through a band of unit width.

Now consider the case, an alternating current with a linear density flows along the plane \( I = I(t) \).

It follows from the energy-momentum tensor for the electromagnetic field

\[ T^{ik} = -\varepsilon_{ilm} F^{ij} F^{lm} + \frac{1}{4} \varepsilon^{ik} F_{lm} F^{lm} \]

that the Poynting vector for the desired metric has the form:

\[ \mathbf{S} = (T^{01}, T^{02}, T^{03}) = \left( \frac{1}{2m^3 (m^3)^3} F^{03}, F^{13} ; 0 ; 0 \right). \]

We assume that \( \mathbf{S} \) is directed from the plane. This is possible when the values \( F^{03} \) and \( F^{13} \) have the same signs.

In accordance with requirement 4, the invariant (2) is zero:

\[ J = F_{13} F^{13} + F^{13} F^{13} = 16 \phi_0 \phi_2 = 0 \]

from where, for the specified direction of the Poynting vector, we have:

\[ \phi_0 = 0 \iff F^{03} = F^{13}. \tag{23} \]

Outside the plane, we get from Maxwell's equations:

\[ \partial_t \left( \sqrt{-g} F^{13} \right) + \partial_x \left( \sqrt{-g} F^{13} \right) = 0, \tag{24} \]

\[ \partial_t F_{13} + \partial_x F_{13} = 0. \tag{25} \]

From equation (24) follows
\[ \partial_t \left( \sqrt{g_{22}/g_{33}} F_{13} \right) + \partial_x \left( \sqrt{g_{22}/g_{33}} F_{13} \right) = 0. \]

Comparing it with (25), we find: \( g_{22} = g_{33} \). Then it follows from (4) that for a plane with an alternating current creating a null electromagnetic field around itself, the equation holds:

\[ g_{22} = g_{33} = \frac{1}{1 - Mx}. \]  

While integrating the equation (25), the nonzero contravariant components of the electromagnetic field tensor are obtained in the form:

\[ F^{13} = F^{03} = \frac{2\pi I(u)}{\sqrt{-g}}, \]

where \( u = t - x \).

7. A plane with a direct current. The acceleration of time

The solution for a plane with a direct current is known and presented in [7], but in this solution it is not clear how the linear current density \( I \) and the surface mass density \( \sigma \) are included in the metric. Their location in the metric is the key point in solving the problem. In order to find out this, we will carry out the process of integrating the NP equations.

For the field of a plane with a direct current, the nonzero component of the tensor \( F^{ij} \) is \( F^{13} \). The field will be non-null, since for the invariant (2) we have

\[ J = 16\phi_0\phi_2 \neq 0, \]

where \( \phi_0 = F_{13}m^3 \), \( \phi_2 = -F_{13}n^1m^3 \). These scalars are related by the equality

\[ \phi_2 = n^1\phi_0 = \frac{2}{M} \left( 1 - Mx \right) \mu \phi_0. \]

From (22) we have

\[ I = \frac{1}{2\pi} F^{13} \sqrt{-g} = -\frac{1}{\pi} \left( 1 - Mx \right) m^3 \phi_0 = \text{Const}. \]

As a consequence of (28), from the equations (18) and (19) the equations follow:

\[ \dot{\lambda} = -2(1 - Mx) \mu \sigma, \]

\[ \partial_x \phi_0 = \partial \phi_0 + \frac{M}{2(1 - Mx)} \phi_0, \]

From the equations (7)-(16) we get:

\[ \partial_x \varepsilon = \frac{M^2}{4(1 - Mx)^2} - \dot{\sigma}^2, \]

\[ \partial_x \sigma = \frac{M}{(1 - Mx)} \dot{\sigma} - \Phi_{00}, \]

\[ \Phi_{00} = \frac{M^2}{4(1 - Mx)^2} - \dot{\sigma}^2 - \frac{M}{(1 - Mx)} \varepsilon, \]

\[ \partial_x \mu = \frac{M}{(1 - Mx)} \mu - 2 \mu \varepsilon, \]

\[ \partial_x m^3 = -\dot{\sigma} m^3 + \frac{M}{2(1 - Mx)} m^3. \]
Given the equality \( (29) \), integrating these equations yields:

\[
\phi_0 = i \frac{2\pi I}{\sqrt{2} \sqrt{1-Mx \left( C \left( 1-Mx \right)+1 \right)}},
\]

\[
\varepsilon = -\frac{3M}{4\left(1-Mx\right)} + \frac{M}{\left(1-Mx\right)\left(C\left(1-Mx\right)+1\right)},
\]

\[
\sigma = \frac{M}{\left(1-Mx\right)} - \frac{M}{\left(1-Mx\right)\left(C\left(1-Mx\right)+1\right)},
\]

\[
\mu = \frac{M}{4\sqrt{1-Mx \left( C \left( 1-Mx \right)+1 \right)^2}},
\]

\[
m^3 = i \frac{\left(C\left(1-Mx\right)+1\right)}{\sqrt{2} \sqrt{1-Mx}}.
\]

As a result, while using the equalities \((6)\) and \((7)\), we get the metric in the form:

\[
ds^2 = \left[C\left(1-Mx\right)+1\right]^2 \left(dt^2 - dx^2\right) - \left(1-Mx\right)\left[C\left(1-Mx\right)+1\right]^2 dy^2 - \frac{\left(1-Mx\right)}{\left[C\left(1-Mx\right)+1\right]^2} dz^2,
\]

\[(30)\]

where \( C = \frac{\pi I^2}{4G\sigma^2}, \) \( M = 8\pi G\sigma \), \( I \) is the linear current density.

The metric \((30)\) has the following properties:

1) When the current is switched off \((I = 0)\), the metric turns into the Taub’s vacuum metric \((3)\).

2) When \( I = 0 \) and \( M = 0 \), the metric becomes flat.

3) It follows from \((22)\) that the magnetic field is given by the non-zero covariant component of the tensor of the electromagnetic field:

\[
F_{13} = \frac{2\pi I}{\left(C\left(1-Mx\right)+1\right)^2}.
\]

In the case when \( I \) and \( M \) are small, the magnetic field strength is equal to:

\[
\vec{H} = 2\pi \vec{j},
\]

where \( \vec{j} \) – unit vector along the \( z \) -axis. From where, in SI units, we get the magnetic field induction in the form:

\[
\vec{B} = \frac{\mu_0 I}{2} \vec{j},
\]

where \( \mu_0 \) is the magnetic constant. This expression coincides with the classical value of the magnetic field induction, which is created by a boundless plane with a direct current.

From the metric it can be seen that in the case of a light platform \((Mx \ll 1)\) a flat platform with a direct current does not create any tangible force directed along or against the \( x \) -axis and having a non-gravitational nature. In this case, the electric current causes the deformation of space-time. As a result, when a direct current flowing along the surface of the plane is turned on the acceleration of time occurs:
\[ \Delta t_0 = \Delta t \left( 1 + \frac{\pi I^2}{4G\sigma^2} \right). \]

\( \Delta t_0 \) is proper time.

**8. The metric of a plane with an alternating current**

Let us consider the case when an alternating current flows through the plane. It follows from (23):

\[ \phi_0 = 0. \]

For convenience, let us use the coordinate system \((u, r, y, z)\), where \( u = t - r \). In these coordinates

\[ n^0 = \frac{4}{M} (1 - Mx) \mu, \quad n^1 = -\frac{2}{M} (1 - Mx) \mu. \]

At points outside the \(z\)-axis, the current is zero: \( j^i = 0\). Therefore, it follows from Maxwell’s equations (19) that space-time is shift-free: \( \sigma = 0 \). As a result, equations (12), (14) and (18) follow

\[ \lambda = 0, \; \Psi_2 = \frac{M}{2(1 - Mx)} \mu, \; \varepsilon = \frac{M}{4(1 - Mx)}. \]

For the \( \mu \) function, which completely determines the type of metric, the system of equations follows from the NP equations:

\[ \partial_x \mu = -\frac{M}{(1 - Mx)} \mu, \]

\[ \partial_x \phi_2 = 0, \]

\[ -\Delta \mu = \mu^2 + \Phi_{22}, \]

where \( \Delta = n^0 \partial_u + n^1 \partial_\lambda \). While integrating these equations, we find

\[ \mu = \frac{M}{4\sqrt{1 - Mx}} e^{\frac{8\sigma^2}{\sigma} \int dt}, \; \phi_2 = i\sqrt{2\pi} e^{\frac{8\sigma^2}{\sigma} \int dt}. \]

As a result, taking into account (26), we get the metric in the form

\[ ds^2 = \frac{1}{\sqrt{1 - Mx}} e^{\frac{8\sigma^2}{\sigma} \int dt} \left( dt^2 - dx^2 \right) - (1 - Mx) \left( dy^2 + dz^2 \right). \] (31)

The solution (31) is a particular solution of the general solution obtained by Misra and Radhakrishna for the null electromagnetic fields with the cylindrical symmetry [8].

The metric has the following properties:

1) Just as in the case when a direct current flows through the plane, when the current is turned off \((I = 0)\), the metric turns into the Taub’s static metric (3), and in the case when there is no plane with a current \((I = 0, \; M = 0)\), the metric becomes flat.

2) It follows from (27) that the magnetic field of the plane with an alternating current is given by non-zero covariant components of the electromagnetic field tensor:

\[ F_{13} = -F_{03} = 2\pi I (t - x). \]

3) In the tetrad (5), the only non-zero Weyl scalar has the form:
\[
\Psi_2 = \frac{M^2}{8(1-Mx)^{\frac{3}{2}}} e^{-\frac{g^2x^2}{\sigma}} \int_0^\infty \frac{r^2}{d\tau^2} .
\]

This means that the null vector \( l \) as well as the null vector \( n \) form a shear-free null geodesic congruence and the solution is of Petrov type D. In the literature, space-time of type D is usually interpreted as the gravitational field of insular massive bodies, and the scalar \( \Psi_2 \) is associated with their mass, which also confirms the correctness of the conclusion about the connection of the constant \( M \) with the surface mass density.

9. The acceleration of the test particle

Consider the case of a weak gravitational field (\( Mx \ll 1, \left\langle \frac{I^2}{\sigma} \right\rangle \ll 1 \)), where \( \left\langle \cdot \right\rangle \) is the averaging over time). In this case, the \( g_{00} \) component of the metric tensor is equal to

\[
g_{00} \approx 1 - \frac{4\pi^2}{\sigma} \int_0^\infty \frac{I^2}{d\tau}.
\]

In such a field, the acceleration of the test sample particle along the \( x \)-axis will be equal to

\[
\frac{d^2x}{dt^2} \approx -I_{00}^{-1} \approx -\frac{1}{2} \frac{\partial g_{00}}{\partial x} \approx \frac{4\pi^2}{\sigma} I^2.
\]

The acceleration of the particle is directed from the plane, i.e. the particle experiences the force which repels it from the plane:

\[
F \sim \frac{I^2}{\sigma} .
\]

For the current which changes according to the harmonic law

\[
I = I_0 \cos \omega t,
\]

the time-averaged acceleration of the particle has the form:

\[
\frac{d^2x}{dt^2} \approx \frac{2\pi^2 I_0^2}{\sigma}.
\]

The cause of the force (55) is the curvature of space-time by the null Einstein-Maxwell field.

10. Conclusion

Thus, we can draw the following conclusion: if an alternating current flows along the surface of a large flat platform, creating a null electromagnetic field, then a test particle located near this platform, in addition to the gravitational attraction of the platform, will be repelled by the force from the platform, the value of which is directly proportional to the square of the linear current density and inversely proportional to the surface mass density:

\[
F \sim \frac{I^2}{\sigma} .
\]

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