INFINITE ORDER DECOMPOSITIONS OF C*-ALGEBRAS

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Abstract

In the given article infinite order decompositions of C*-algebras are investigated. It is proved that for the infinite order decomposition \( \bigoplus_{\xi,\eta} p_{\xi} A p_{\eta} \) of a C*-algebra \( A \) with respect to an infinite orthogonal set \( \{ p_{i} \} \) of projections of \( A \), if \( p_{\xi} A p_{\xi} \) is a von Neumann algebra for any \( \xi \) then \( \bigoplus_{\xi,\eta} p_{\xi} A p_{\eta} \) is a von Neumann algebra. Also, it is proved that, if a C*-algebra \( A \) with an infinite orthogonal set \( \{ p_{\xi} \} \) of projections in \( A \) such that \( \sup_{\xi} p_{\xi} = 1 \) is not a von Neumann algebra, projections of the set \( \{ p_{\xi} \} \) are pairwise equivalent then \( A \neq \bigoplus_{\xi,\eta} p_{\xi} A p_{\eta} \), and, if the order unit space \( \bigoplus_{\xi,\eta} p_{\xi} A p_{\eta} \) is not weakly closed then \( \bigoplus_{\xi,\eta} p_{\xi} A p_{\eta} \) is not a C*-algebra.

Introduction

In the given article the notion of infinite order decomposition of a C*-algebra with respect to an infinite orthogonal set of projections is investigated. It is known that for any projection \( p \) of a C*-algebra \( A \) the next equality is valid \( A = pA \oplus pA(1 - p) \oplus (1 - p)A(1 - p) \), where \( \oplus \) is a direct sum of spaces. In the given article we investigated an infinite analog of this decomposition, an infinite order decomposition. The notion of infinite order decomposition was introduced in [AFN]. The next theorems belong to [AFN]:

1) if the order unit space \( \bigoplus_{\xi,\eta} p_{\xi} A p_{\eta} \) is monotone complete in \( B(H) \) (i.e. ultra-weakly closed), then \( \bigoplus_{\xi,\eta} p_{\xi} A p_{\eta} \) is a C*-algebra.

2) if \( A \) is monotone complete in \( B(H) \) (i.e. a von Neumann algebra), then \( A = \bigoplus_{\xi,\eta} p_{\xi} A p_{\eta} \).

In the given article we proved that for the infinite order decomposition \( \bigoplus_{\xi,\eta} p_{\xi} A p_{\eta} \) of a C*-algebra \( A \) with respect to an infinite orthogonal set \( \{ p_{i} \} \) of projections of \( A \), if \( p_{\xi} A p_{\xi} \) is a von Neumann algebra for any \( \xi \) then \( \bigoplus_{\xi,\eta} p_{\xi} A p_{\eta} \) is a von Neumann algebra. For this propose it was constructed a multiplication and an involution corresponding to infinite order decompositions. It turns out, the order and the norm defined in the infinite order decomposition of a C*-algebra on a Hilbert space \( H \) coincide with the usual order and the norm in the algebra \( B(H) \). Also, it is proved that, if a C*-algebra \( A \) with an infinite orthogonal set \( \{ p_{\xi} \} \) of projections in \( A \) such that \( \sup_{\xi} p_{\xi} = 1 \) is not a von Neumann algebra, projections of the set \( \{ p_{\xi} \} \) are pairwise equivalent then \( A \neq \bigoplus_{\xi,\eta} p_{\xi} A p_{\eta} \). Moreover if the order unit space \( \bigoplus_{\xi,\eta} p_{\xi} A p_{\eta} \) is not weakly closed then \( \bigoplus_{\xi,\eta} p_{\xi} A p_{\eta} \) is not a C*-algebra.

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1. INFINITE ORDER DECOMPOSITIONS

Let $A$ be a $C^*$-algebra on a Hilbert space $H$, $\{p_\xi\}$ be an infinite orthogonal set of projections of the algebra $A$ with the least upper bound 1 in the algebra $A$. By $\sum_{\xi,\eta} p_\xi p_\eta$ we denote the set

$$\{\{a_{\xi,\eta}\} : a_{\xi,\eta} \in p_\xi p_\eta \text{ for all } \xi, \eta, \text{ and there exists such number } K \in R \text{ that } \| \sum_{k,l=1}^n a_{kl} \| \leq K \text{ for all } n \in N \land \{a_{kl}\}_{k,l=1}^n \subseteq \{a_{\xi,\eta}\}\}$$

and say $\sum_{\xi,\eta} p_\xi p_\eta$ is an infinite order decomposition of the algebra $A$.

Let $A$ be a $C^*$-algebra on a Hilbert space $H$, $\{p_\xi\}$ be an infinite orthogonal set of projections of the algebra $A$ with the least upper bound 1 in the algebra $B(H)$. We define a relation of an order $\leq$ in the vector space $\sum_{\xi,\eta} p_\xi p_\eta$ as follows: for elements $\{a_{\xi,\eta}\}, \{b_{\eta,\xi}\} \in \sum_{\xi,\eta} p_\xi p_\eta$, if for all $n \in N$, $\{p_k\}_{k=1}^n \subseteq \{p_\xi\}$ the inequality $\sum_{k,l=1}^n a_{kl} \leq \sum_{k,l=1}^n b_{kl}$ holds, then we will write $\{a_{\xi,\eta}\} \leq \{b_{\eta,\xi}\}$. Also, the map $\{a_{\xi,\eta}\} \rightarrow \|\{a_{\xi,\eta}\}\|, \{a_{\xi,\eta}\} \in \sum_{\xi,\eta} p_\xi p_\eta$, where $\|\{a_{\xi,\eta}\}\| = \sup\{\|\sum_{k,l=1}^n a_{kl}\| : n \in N, \{a_{kl}\}_{k,l=1}^n \subseteq \{a_{\xi,\eta}\}\}$, is a norm on vector space $\sum_{\xi,\eta} p_\xi p_\eta$.

Example. Let $n$ be an arbitrary infinite cardinal number, $\Xi$ be a set of indexes of the cardinality $n$. Let $\{e_{ij}\}$ be a set of matrix units such that $e_{ij}$ is a $n \times n$-dimensional matrix, i.e. $e_{ij} = (a_{ij})_{\alpha \beta \in \Xi}$, the $(i,j)$-th component of which is 1, i.e. $a_{ij}$ = 1, and the rest components are zeros. Let $\{m_\xi\}_{\xi \in \Xi}$ be a set of $n \times n$-dimensional matrices. By $\sum_{\xi \in \Xi} m_\xi$ we denote the matrix whose components are sums of the corresponding components of matrixes of the set $\{m_\xi\}_{\xi \in \Xi}$. Let

$$M_n(C) = \{\{\lambda_{ij}e_{ij}\} : \text{ for all indexes } i, j \lambda_{ij} \in C,$$

and there exists such number $K \in R$, that for all $n \in N$

$$\text{ and } \{e_{kl}\}_{k,l=1}^n \subseteq \{e_{ij}\} \| \sum_{k,l=1}^n \lambda_{kl} e_{kl} \| \leq K\},$$

where $\| \|$ is a norm of a matrix. It is easy to see that $M_n(C)$ is a vector space.

The set $M_n(C)$, defined above, coincides with the next set:

$$M_n(C) = \{\{\lambda_{ij}e_{ij}\} : \text{ for all indexes } ij \lambda_{ij} \in C,$$

and there exists such number $K \in R$ that for all

$$\{x_i\} \in l_2(\Xi) \text{ the next inequality holds } \sum_{x \in \Xi} \sum_{i \in \Xi} |\lambda_{ij} x_i|^2 \leq K^2 \sum_{i \in \Xi} |x_i|^2,$$

where $l_2(\Xi)$ is a Hilbert space on $C$ with elements $\{x_i\}_{i \in \Xi}$, where $x_i \in C$ for all $i \in \Xi$.

The associative multiplication of elements in $M_n(C)$ can be defined as follows: if $x = \sum_{i,j \in \Xi} \lambda_{ij} e_{ij}$, $y = \sum_{i,j \in \Xi} \mu_{ij} e_{ij}$ are elements of $M_n(C)$ then $xy = \sum_{i,j \in \Xi} \sum_{\Xi} \lambda_{ij} \mu_{ij} e_{ij}$. On this operation $M_n(C)$ is an associative algebra and $M_n(C) = B(l_2(\Xi))$, where $B(l_2(\Xi))$ is the associative algebra of all bounded linear operators on the Hilbert space $l_2(\Xi)$. Then $M_n(C)$ is a von Neumann algebra of infinite $n \times n$-dimensional matrixes on $C$, which is defined by its own infinite order decomposition.
Analogously, if we take the algebra $B(H)$ of all bounded linear operators on an arbitrary Hilbert space $H$ and if \( \{q_i\} \) is an arbitrary maximal orthogonal set of minimal projections of the algebra $B(H)$, then $B(H) = \sum_{i=1}^{\infty} q_i B(H) q_j$ (see [AFN]).

Let $A$ be a $C^*$-algebra, \( \{p_i\} \) be an infinite orthogonal set of projections with the least upper bound 1 in the algebra $A$ and let $\mathcal{A} = \{ (p_i a p_j) : a \in A \}$. Then $A \equiv \mathcal{A}$ (see [AFN]).

**Lemma 1.** Let $A$ be a $C^*$-algebra, \( \{p_\xi\} \) be an infinite orthogonal set of projections of the algebra $A$ with the least upper bound 1 in the algebra $A$. Then, \( \sum_{\xi, \eta}^a p_\xi A p_\eta \) is a vector space with the next componentwise algebraic operations

\[
\lambda \cdot \{a_\xi\} = \{\lambda a_\xi\}, \lambda \in \mathbb{C}
\]

\[
\{a_\xi\} + \{b_\xi\} = \{a_\xi + b_\xi\}, a_\xi, b_\xi \in \sum_{\xi, \eta}^a p_\xi A p_\eta.
\]

And the space $\mathcal{A}$ is a vector subspace of the vector space \( \sum_{\xi, \eta}^a p_\xi A p_\eta \).

**Lemma 2.** Let $A$ be a $C^*$-algebra, \( \{p_\xi\} \) be an infinite orthogonal set of projections of the algebra $A$ with the least upper bound 1 in the algebra $A$. Then, the map \( \{a_\xi\} \to \|\{a_\xi\}\|, \{a_\xi\} \in \sum_{\xi, \eta}^a p_\xi A p_\eta \), where \( \|\{a_\xi\}\| = \sup\{\|\sum_{k=1}^{n} a_{ki}\| : n \in \mathbb{N}, \{a_{ki}\}_{k=1}^{n} \subseteq \{a_\xi\}\} \), is a norm, and \( \sum_{\xi, \eta}^a p_\xi A p_\eta \) is a Banach space with this norm.

**Proof.** It is clear, that for any element \( \{a_\xi\} \in \sum_{\xi, \eta}^a p_\xi A p_\eta \), if \( \|\{a_\xi\}\| = 0 \), then \( a_\xi = 0 \) for all $\xi, \eta$, i.e. \( \{a_\xi\} = 0 \). The rest conditions in the definition of the norm also can be easily checked.

Let \( (a_n) \) be an arbitrary Cauchy sequence in the space \( \sum_{\xi, \eta}^a p_\xi A p_\eta \), i.e. for any positive number $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such, that for all $n_1 \geq n, n_2 \geq n$ the inequality \( \|a_{n_1} - a_{n_2}\| < \varepsilon \) holds. Then the set \( \{\|a_n\|\} \) is bounded by some number $K \in \mathbb{R}_+$ and for any finite set \( \{p_i\}_{k=1}^{n} \subseteq \{p_i\} \) the sequence \( (pa_n p) \) is a Cauchy sequence, where $p = \sum_{k=1}^{n} p_k$. Then, since $A$ is a Banach space, then \( \lim_{n \to \infty} p a_n p \in A \).

Let \( a_\xi = \lim_{n \to \infty} p a_n p \) for all $\xi$ and $\eta$. Then \( \|\sum_{k=1}^{n} a_{ki}\| \leq K \) for all $n \in \mathbb{N}$ and \( \{a_{ki}\}_{k=1}^{n} \subseteq \{a_\xi\} \). Hence \( \{a_\xi\} \in \sum_{\xi, \eta}^a p_\xi A p_\eta \).

The definition of the order in \( \sum_{\xi, \eta}^a p_\xi A p_\eta \) is equivalent to the next condition: for the elements \( \{a_\xi\}, \{b_\xi\} \in \sum_{\xi, \eta}^a p_\xi A p_\eta \), if for all $n \in \mathbb{N}$ and \( \{p_k\}_{k=1}^{n} \subseteq \{p_i\} \) the equality \( \{a_{ki}\}_{k=1}^{n} \subseteq \{b_{ki}\}_{k=1}^{n} \) holds in the algebra $A$, then \( a_\xi \leq b_\xi \).

**Proposition 3.** Let $A$ be a $C^*$-algebra on a Hilbert space $H$, \( \{p_\xi\} \) be an infinite orthogonal set of projections in $A$ with the least upper bound 1 in the algebra $B(H)$. Then the relation \( \leq \), introduced above, is a relation of a partial order, and the space \( \sum_{\xi, \eta}^a p_\xi A p_\eta \) is an order unit space with this order. In this case $A = \{ (p_\xi A p_\eta) : a \in A \}$ is an order unit subspace of the order unit space \( \sum_{\xi, \eta}^a p_\xi A p_\eta \).

**Proof.** Let $\mathcal{M} = \sum_{\xi, \eta}^a p_\xi A p_\eta$. The space $\mathcal{M}$ is a partially ordered vector space, i.e. $\mathcal{M}_+ \cap \mathcal{M}_- = \{0\}$, where $\mathcal{M}_+ = \{ \{a_\xi\} \in \mathcal{M} : \{a_\xi\} \geq 0 \}, \mathcal{M}_- = \{ \{a_\xi\} \in \mathcal{M} : \{a_\xi\} \leq 0 \}$.

By the definition of the order the partially ordered vector space $\mathcal{M}$ is Archimedean. Let \( \{a_\xi\} \in \mathcal{M} \). Since for any finite set \( \{p_k\}_{k=1}^{n} \subseteq \{p_\xi\} \) the inequality \( \|\{a_\xi\}\| p \leq p\{a_\xi\} p \leq \|\{a_\xi\}\| p \) holds, where $p = \sum_{k=1}^{n} p_k$, then by the definition of the order
Indeed, the identification $\mathcal{A} \equiv \mathcal{B}$ gives us $a = \{p_\xi a p_\eta\}$ and $a^* = \{p_\xi^* a^* p_\eta\}$ for all $a \in A$. Then $\mathcal{A}_{aa} = \{p_\xi a p_\eta\} : a \in A_{aa}$. Then $\mathcal{A} = \mathcal{A}_{aa} + i \mathcal{A}_{sa}$. Indeed, $\{p_\xi a p_\eta\}^* = a^* = a = \{p_\xi a p_\eta\}$ for any $a \in A_{aa}$. 

\textbf{Proposition 4.} Let $A$ be a $C^*$-algebra on a Hilbert space $H$, $\{p_i\}$ be an infinite orthogonal set of projections in $A$ with the least upper bound 1 in the algebra $B(H)$. Then the order unit space $A = \{\{p_\xi a p_\eta\} : a \in A\}$ is a $C^*$-algebra, where the operation of multiplication of $A$ defines as follows 

$$\cdot : \{p_\xi a p_\eta\}, \{p_\xi b p_\eta\} \mapsto \{p_\xi a b p_\eta\}, \{p_\xi a p_\eta\}, \{p_\xi b p_\eta\} \in \mathcal{A}.$$ 

\textbf{Proof.} By lemma 4 in [AFN2] the map 

$$\mathcal{I} : a \in A \rightarrow \{p_\xi a p_\eta\} \in \mathcal{A}$$ 

is a one-to-one map. In this case 

$$\mathcal{I}(a)\mathcal{I}(b) = \mathcal{I}(ab)$$ 

by the definition in proposition 4 of the multiplication, and $\mathcal{I}(a) = \{p_\xi a p_\eta\}, \mathcal{I}(b) = \{p_\xi b p_\eta\}, \mathcal{I}(ab) = \{p_\xi a b p_\eta\}$. Hence, the operation, introduced in the formulation of proposition 4 is an associative multiplication and the map $\mathcal{I}$ is an isomorphism of the algebras $A$ and $\mathcal{A}$.

By proposition 3 the isomorphism $\mathcal{I}$ is isometrical. Therefore $\mathcal{A}$ is a $C^*$-algebra with this operation. $\triangleright$

\textbf{Example 1.} We take the algebra $B(H)$ of all bounded linear operators on a Hilbert space $H$. Let $\{q_i\}$ be a maximal orthogonal set of minimal projections of the algebra $B(H)$. Then $\sup_i q_i = 1$ and by lemma 4 in [AFN2] and proposition 4 the space $B(H) = \{\{q_i a q_j\} : a \in B(H)\}$ can be identified with $B(H)$ as $C^*$-algebras in the sense of the map 

$$\mathcal{I} : a \in B(H) \rightarrow \{q_i a q_j\} \in B(H).$$ 

In this case the operation associative multiplication in $B(H)$ is defined as follows 

$$\cdot : \{q_i a q_j\}, \{q_j b q_j\} \mapsto \{q_i a b q_j\}, \{q_i a q_j\}, \{q_j b q_j\} \in B(H).$$ 

Let $a, b \in B(H)$, $q_i a q_j = \lambda_{ij} q_{ij}$, $q_j b q_j = \mu_{ij} q_{ij}$, where $\lambda_{ij}, \mu_{ij} \in \mathbb{C}$, $q_i = q_i^* q_i$, $q_j = q_j^* q_j$, for all indexes $i$ and $j$. Then this multiplication coincides with the next bilinear operation 

$$\cdot : \{q_i a q_j\}, \{q_j b q_j\} \mapsto \{\sum_{\xi} \lambda_{ij} \mu_{ij} q_{ij}\}, \{q_i a q_j\}, \{q_j b q_j\} \in B(H).$$ 

\textbf{Remark 1.} Let $A$ be a $C^*$-algebra on a Hilbert space $H$, $\{p_i\}$ be an infinite orthogonal set of projections in $A$ with the least upper bound 1 in the algebra $B(H)$. Then by proposition 4 $\mathcal{A} = \{\{p_\xi a p_\eta\} : a \in A\}$ is a $C^*$-algebra. In this case the involution on the algebra $\mathcal{A}$ coincides with the next map: 

$$\{p_\xi a p_\eta\}^* = \{p_\xi^* a^* p_\eta\}, \ a \in A.$$ 

Indeed, the identification $\mathcal{A} \equiv \mathcal{B}$ gives us $a = \{p_\xi a p_\eta\}$ and $a^* = \{p_\xi^* a^* p_\eta\}$ for all $a \in A$. Then $\{p_\xi a p_\eta\}^* = a^* = \{p_\xi^* a^* p_\eta\}$ for any $a \in A$. Let $\mathcal{A}_{sa} = \{\{p_\xi a p_\eta\} : a \in A_{sa}\}$. Then $\mathcal{A} = \mathcal{A}_{sa} + i \mathcal{A}_{sa}$. Indeed, $\{p_\xi a p_\eta\}^* = a^* = a = \{p_\xi a p_\eta\}$ for any $a \in A_{sa}$. 

\[\]
Let \( N = \{ \{p_\xi ap_\eta \} : a \in B(H) \} \). By lemma 4 in [AFN2] and by proposition 4 \( N \equiv B(H) \). Therefore we will assume that \( N = B(H) \). Let \( N_{sa} = \{ \{p_\xi ap_\eta \} : a \in B(H), \{p_\xi ap_\eta \}^* = \{p_\xi ap_\eta \} \} \). Then \( N = N_{sa} + iN_{sa} \). Note that \( \{p_\xi ap_\eta \}^* = \{p_\xi ap_\eta \} \) if and only if \( (p_\xi ap_\eta)^* = p_\eta ap_\xi \) for all \( \xi, \eta \).

**Lemma 5.** Let \( B(H) \) be the algebra of all bounded linear operators on a Hilbert space \( H \). Let \( \{p_\xi \} \) be an infinite orthogonal set of projections of the algebra \( B(H) \) with the least upper bound 1. Then the associative multiplication of the algebra \( N \) (hence of the algebra \( B(H) \)) coincides with the next operation

\[
\{p_\xi ap_\eta \} \star \{p_\xi bp_\eta \} = \left\{ \sum_i p_\xi ap_i p_i bp_\eta \right\}, \{p_\xi ap_\eta \}, \{p_\xi bp_\eta \} \in N
\]

where the sum \( \sum_i \) in the right part of the equality is an ultraweak limit of the net of finite sums of elements in the set \( \{p_\xi ap_i p_i bp_\eta \} \).

**Proof.** Let \( \{p_k \}_{k=1}^n \) be a finite subset of the set \( \{p_\xi \} \). Note that \( \sup_i p_i = 1 \) in the algebra \( B(H) \), i.e., the net of all finite sums of the kind \( \sum_{k=1}^n p_k \) of orthogonal projections of the set \( \{p_k \} \) ultraweakly converges to the identity operator in \( B(H) \). By the ultraweakly continuity of the operator of multiplication \( T(b) = ab, b \in B(H) \), where \( a \in B(H) \), the net of finite sums of elements in the set \( \{p_\xi ap_i p_i bp_\eta \} \) ultraweakly converges in \( B(H) \) and \( \sum_i p_\xi ap_i p_i bp_\eta = p_\xi abp_\eta \) for all \( \xi, \eta \). Hence the operation of multiplication \( \star \) of the algebra \( N \) coincides with the operation, introduced in proposition 4. And the operation of the associative multiplication, introduced in proposition 4 coincides with the multiplication in the algebra \( B(H) \) in the sense \( N \equiv B(H) \). \( \triangleright \)

**Proposition 6.** Let \( A \) be a \( C^* \)-algebra on a Hilbert space \( H \), \( \{p_\xi \} \) be an infinite orthogonal set of projections in \( A \) with the least upper bound 1 in the algebra \( B(H) \).

Then the operation of associative multiplication of the algebra \( A \) coincides with the inducing on \( N \) of the operation, defined in lemma 5.

**Proof.** Let \( \{p_\xi ap_\eta \}, \{p_\xi bp_\eta \} \) be elements of \( A_{sa} \) and \( \{p_k \}_{k=1}^n \) be a finite subset of the set \( \{p_\xi \} \) and \( p = \sum_{k=1}^n p_k \). We have the net of all finite sums of the kind \( \sum_{k=1}^n p_k \) of orthogonal projections of the set \( \{p_\xi \} \) ultraweakly converges to the identity operator in \( B(H) \). Then for all \( \xi, \eta \) the element \( \{p_\xi abp_\eta \} \) is an ultraweak limit in \( B(H) \) of the net \( \{\sum_i p_\xi ap_i p_i bp_\eta \} \) of all finite sums \( \{\sum_{k=1}^n p_\xi ap_i p_i bp_\eta \} \) on all subsets \( \{p_k \}_{k=1}^n \subset \{p_\xi \} \), and the element \( \{p_\xi abp_\eta \} \) belongs to \( A \). Hence the assertion of proposition 6 holds. \( \triangleright \)

**Remark 2.** Let \( A \) be a \( C^* \)-algebra on a Hilbert space \( H \), \( \{p_i \} \) be an infinite orthogonal set of projections in \( A \) with the least upper bound 1 in the algebra \( B(H) \).

Note that then by lemma 4 in [AFN2] the order and the norm in the vector space \( \sum_{i,j} p_i Ap_j \) can be introduced as follows: we write \( \{a_{ij} \} \geq 0 \), if this element is zero or positive element in \( B(H) \) in the sense of the equality \( B(H) = \sum_{\xi, \eta} q_\xi B(H) q_\eta \), where \( \{q_\xi \} \) is an arbitrary maximal orthogonal set of minimal projections of the algebra \( B(H) \): \( \|a_{ij}\| = \|q_\xi\| \) is equal to the norm in \( B(H) \) of this element in the sense of the equality \( B(H) = \sum_{\xi, \eta} q_\xi B(H) q_\eta \) (example 1).

By lemmas 3 and 4 in [AFN2] they coincide with the order and the norm defined in lemma 2 and proposition 3, correspondingly.

**Remark 3.** Suppose that all conditions of remark 2 hold. Then \( B(H) \equiv B(\mathcal{H}) = \sum_{\xi, \eta} q_\xi B(\mathcal{H}) q_\eta \), where \( B(\mathcal{H}) = \{q_\xi a_{ij} : a \in B(H)\} \). Also, we have \( \sum_{ij} p_i Ap_j \) is a Banach space and an order unit space (lemma 2, Proposition 3). Suppose that
Thus, further, when we say that \( p_i = \sup_{q_i} q_i \), for some subset \( \{q_i\} \subset \{q_i\} \), for all \( i \). Note that \( B(H) \equiv \{\{p_i a p_j\} : a \in B(H)\} = \sum_{ij} p_i B(H) p_j \). By propositions 4 and 6 the order unit space \( A = \{\{p_i a p_j\} : a \in A\} \) is closed concerning the associative multiplication of the algebra \( \sum_{ij} p_i B(H) p_j \) (what is the same that \( N = \{\{p_i a p_j\} : a \in B(H)\} \).

At the same time, the order unit space \( \sum_{ij} p_i A p_j \) is the order unit subspace of the algebra \( \sum_{ij} p_i B(H) p_j \).

Since \( B(H) \equiv \sum_{ij} p_i B(H) p_j \), then \( \sum_{ij} p_i B(H) p_j \) is a von Neumann algebra, and without loss of generality, this algebra can be considered as the algebra \( B(H) \).

Note that if the space \( \sum_{ij} p_i A p_j \) is closed concerning the associative multiplication of the algebra \( \sum_{ij} p_i B(H) p_j \), then \( \sum_{ij} p_i A p_j \) is a C*-algebra. Also, when we consider the C*-algebra \( A \) with the conditions which are listed above, then we have the algebra \( \sum_{ij} p_i B(H) p_j \) (i.e. actually the algebra \( B(H) \)) and the vector space \( \sum_{ij} p_i A p_j \) as an order unit subspace of the algebra \( \sum_{ij} p_i B(H) p_j \). Then we have

\[
A \subseteq \sum_{ij} p_i A p_j \subseteq \sum_{ij} p_i B(H) p_j.
\]

Thus, further, when we say that \( \sum_{ij} p_i A p_j \) is a C*-algebra we assume that the vector space \( \sum_{ij} p_i A p_j \) is closed concerning the associative multiplication of the algebra \( \sum_{ij} p_i B(H) p_j \).

The involution in the sense of the identification \( \sum_{ij} p_i B(H) p_j \equiv B(H) \) coincides with the next map:

\[
\{a_{ij}\}^* = \{a_{ji}^*\}, \{a_{ij}\} \in \sum_{ij} p_i B(H) p_j.
\]

Indeed, there exists an element \( a \in B(H) \) such that \( a = \{a_{ij}\} = \{p_i a p_j\} \). Then \( a^* = \{p_i a^* p_j\} \) in the sense of \( B(H) \equiv N \). We have \( a_{ij} = p_i a p_j, a_{ji}^* = p_j a^* p_i \) for all \( i, j \). Therefore \( \{p_i a^* p_j\} = \{a_{ji}^*\} \). Hence \( a^* = \{a_{ji}^*\} \). Let \( (\sum_{ij} p_i B(H) p_j)_sa = \{a_{ij}\} : \{a_{ij}\} \in \sum_{ij} p_i B(H) p_j, \{a_{ij}\}^* = \{a_{ij}\}\) . Then

\[
\sum_{ij} p_i B(H) p_j = (\sum_{ij} p_i B(H) p_j)_sa + i(\sum_{ij} p_i B(H) p_j)_sa.
\]

Lemma 7. Let \( A \) be a C*-algebra on a Hilbert space \( H \), \( \{p_i\} \) be an infinite orthogonal set of projections of the algebra \( A \) with least upper bound 1 in \( B(H) \) and \( (\sum_{ij} p_i A p_j)_sa = \{\{a_{ij}\} : \{a_{ij}\} \in \sum_{ij} p_i A p_j, \{a_{ij}\}^* = \{a_{ij}\}\} \). Then

\[
\sum_{ij} p_i A p_j = (\sum_{ij} p_i A p_j)_sa + i(\sum_{ij} p_i A p_j)_sa. \quad (**)
\]

In this case the equality \( \{a_{ij}\}^* = \{a_{ij}\} \) holds for \( \{a_{ij}\} \in \sum_{ij} p_i A p_j \) if and only if \( a_{ij}^* = a_{ji} \) for all \( i, j \).

Proof. Let \( \{a_{ij}\} \in \sum_{ij} p_i A p_j \). We have \( a_{ij} + a_{ji} = a_1 + i a_2 \), where \( a_1, a_2 \in (\sum_{ij} p_i A p_j)_sa \), for all \( i \) and \( j \), since \( a_{ij} + a_{ji} \in A \). Then \( a_{ij} + a_{ji} = p_i a_1 p_j + a_2 \).
Therefore, it can be regarded that $B$ coincides with the operation defined in lemma 8. Let $a_{1j} = p_{1}a_{1j} + p_{j}a_{1j}$, $a_{2} = p_{1}a_{2j} + p_{j}a_{2j}$ for all $i$ and $j$. Let $a_{1j} = p_{1}a_{1j} + p_{j}a_{1j}$, $a_{2j} = p_{1}a_{2j} + p_{j}a_{2j}$ for all $i$ and $j$. Then by the definition of the vector space $\sum_{i,j} p_{i}A_{j}$ we have $\{a_{1j}\}, \{a_{2j}\} \in \sum_{i,j} p_{i}A_{j}$. In this case $\{a_{ij}\}^{*} = \{a_{ij}^{*}\}, k = 1, 2$. Since the element $\{a_{ij}\} \in \sum_{i,j} p_{i}A_{j}$ was chosen arbitrarily we have the equality (**).

The rest part of the assertion of lemma 7 holds by the definition of the self-adjoint elements $\{a_{ij}\}, k = 1, 2$. $\triangleright$

**Lemma 8.** Let $B(H)$ be a *-algebra of all bounded linear operators on a Hilbert space $H$, $\{p_{\xi}\}$ be an infinite orthogonal set of projections of $B(H)$ with the least upper bound 1. Then the associative multiplication of the algebra $\sum_{\xi,\eta} p_{\xi}B(H)p_{\eta}$ (i.e. of the algebra $B(H)$) coincides with the associative multiplication defined as follows:

$\cdot : \{a_{\xi,\eta}\}, \{b_{\xi,\eta}\} \mapsto \{\sum a_{\xi,\eta}b_{\xi,\eta}\}, \{a_{\xi,\eta}\}, \{b_{\xi,\eta}\} \in (\sum_{\xi,\eta} p_{\xi}B(H)p_{\eta})$.

**Proof.** Let $\{a_{\xi,\eta}\}, \{b_{\xi,\eta}\} \in (\sum_{\xi,\eta} p_{\xi}B(H)p_{\eta})$. We have $B(H) = \mathcal{N} = \sum_{\xi,\eta} p_{\xi}B(H)p_{\eta}$. Therefore, it can be regarded that $B(H) = \mathcal{N} = \sum_{\xi,\eta} p_{\xi}B(H)p_{\eta}$. There exists elements $a$, $b$ in the algebra $B(H)$ such that $p_{\xi}ap_{\eta} = a_{\xi,\eta}$, $p_{\xi}bp_{\eta} = b_{\xi,\eta}$ for all $\xi$, $\eta$. Therefore $\{a_{\xi,\eta}\} = \{p_{\xi}ap_{\eta}\}, \{b_{\xi,\eta}\} = \{p_{\xi}bp_{\eta}\}$. Then by lemma 5 we have the associative multiplication of the algebra $\sum_{\xi,\eta} p_{\xi}B(H)p_{\eta}$ (i.e. of the algebra $B(H)$) coincides with the operation defined in lemma 8. $\triangleright$

**Proposition 9.** [AFN] Let $A$ be a von Neumann algebra on a Hilbert space $H$, $\{p_{\xi}\}$ be an infinite orthogonal set of projections of the algebra $A$ with least upper bound 1 in $B(H)$. Then $A = \sum_{\xi} p_{\xi}A_{\xi}$.

**Proof.** Let $a$ be an element of the vector space $\sum_{\xi,\eta} p_{\xi}A_{\eta}$ and $a = \{a_{\xi,\eta}\}$, where $a_{\xi,\eta} = p_{\xi}ap_{\eta}, a_{\xi,\eta} = p_{\xi}ap_{\eta}$ for all $\xi$, $\eta$. We have $a \in B(H) = \sum_{\xi,\eta} p_{\xi}B(H)p_{\eta}$ and $(\sum_{k=1}^{n} p_{k})a(\sum_{k=1}^{n} p_{k}) \in A$ for any $\{p_{k}\}_{k=1}^{n} \subseteq \{p_{\xi}\}$. Let

$$b_{\eta}^{a} = \sum_{kl=1}^{n} p_{k}^{a}ap_{\eta} = (\sum_{kl=1}^{n} p_{k}^{a})a(\sum_{kl=1}^{n} p_{k}^{a})$$

for all natural numbers $n$ and finite subsets $\{p_{k}^{a}\}_{k=1}^{n} \subseteq \{p_{\xi}\}$. Then by the proof of lemma 3 in [AFN2] the net $(b_{\eta}^{a})$ ultraweakly converges to $a$ in $B(H)$. At the same time $A$ is ultraweakly closed in $B(H)$. Therefore $a \in A$ and $\sum_{\xi,\eta} p_{\xi}A_{\eta} \subseteq A$. $\triangleright$

**Lemma 10.** Let $A$ be a $C^*$-algebra on a Hilbert space $H$, $\{p_{\xi}\}$ be an infinite orthogonal set of projections of $A$ with the least upper bound 1 in the algebra $B(H)$. Then, if projections of the set $\{p_{\xi}\}$ are pairwise equivalent and for every index $\xi$ the component $p_{\xi}A_{\eta}$ is a von Neumann algebra, then the vector space $\sum_{\xi,\eta} p_{\xi}A_{\eta}$ is closed concerning the multiplication of the algebra $\sum_{\xi,\eta} p_{\xi}B(H)p_{\eta}$ and $\sum_{\xi,\eta} p_{\xi}A_{\eta}$ is a $C^*$-algebra.

**Proof.** First, note that $(p_{\xi} + p_{\eta})A(p_{\xi} + p_{\eta})$ is a von Neumann algebra. Indeed, for any net $(a_{\xi})$ in $p_{\xi}A_{\eta}$, weakly converging in $B(H)$ the net $(a_{\xi}x_{\xi}^{*})$ belongs to $p_{\xi}A_{\xi}$, where $x_{\xi}$ is an isometry in $A$ such that $x_{\xi}x_{\xi}^{*} = p_{\xi}, x_{\xi}^{*}x_{\xi} = p_{\eta}$. Then since the net $(a_{\xi}x_{\xi}^{*})$ weakly converges in $B(H)$ then the weak limit $b$ in $B(H)$ of
the net \((aₙ,x_{ξη}ⁿ)\) belongs to \(pξApξ\). Hence \(bx_{ξη} ∈ pξApη\). It is easy to see that \(bx_{ξη}\) is a weak limit in \(B(H)\) of the net \((aₙ)\). Hence \(pξApη\) is weakly closed in \(B(H)\).

Let \(\{aξη\}, \{bξη\} ∈ (\bigoplusξη pξApη\). We have
\[
\sumξη pξApη ⊆ \bigoplusξη pξB(H)pη = B(H).
\]

Therefore there exist elements \(a, b\) in the algebra \(\bigoplusξη pξB(H)pη\) (i.e. in the algebra \(B(H)\)) such that \(pξapη = aξη, pξbpη = bξη\) for all \(ξ, η\). Therefore \(\{aξη\} = \{pξapη\}, \{bξη\} = \{pξbpη\}\). We have
\[
\sum i aξi bηi = pξabpη
\]
calculated in \(\bigoplusξη pξB(H)pη\) belong to \(pξApη\). Since the indexes \(ξ, η\) were chosen arbitrarily and the product \(\{pξapη\}\{pξbpη\} = ab\) belongs to \(\bigoplusξη pξB(H)pη\), then the product of the elements \(a\) and \(b\) belongs to \(\bigoplusξη pξApη\). Therefore the vector space \(\bigoplusξη pξApη\) is closed with respect to the associative multiplication of the algebra \(\bigoplusξη pξB(H)pη\). At the same time, \(\bigoplusξη pξApη\) is a norm closed subspace of the algebra \(\bigoplusξη pξB(H)pη = B(H)\). Hence \(\bigoplusξη pξApη\) is a \(C^*\)-algebra and the multiplication in \(\bigoplusξη pξApη\) can be defined as in the formulation of lemma 8. ▶

Theorem 11. Let \(A\) be a \(C^*\)-algebra on a Hilbert space \(H\), \(\{pξ\}\) be an infinite orthogonal set of projections in \(A\) with the least upper bound 1 in the algebra \(B(H)\). Suppose that projections of the set \(\{pξ\}\) are pairwise equivalent and for any \(ξ pξApξ\) is a von Neumann algebra. Then \(\bigoplusξη pξApη\) is a von Neumann algebra.

Proof. Let \(\{xξη\}\) be such set of isometries in \(A\) that \(pξ = xξηxξη*, pη = xξη*, xξη\) for all \(ξ, η\). Let \(ξ, η\) be arbitrary indexes. We prove that \(pξApη\) is weakly closed. We have \(pξApηApξ ⊆ pξApξ\) and \(pξApη = xξηAxξη\). Let \((aₙ)\) be a net in \(pξApη\), weakly converging to an element \(a\) in \(B(H)\). Then the exists a net \((bₙ)\) in \(pξApη\) such that \(aₙ = xξηbₙξη\) for all \(ξ\). By the weakly continuity of the multiplication separately on multipliers the net \((aₙxₙξη)\) weakly converges to the element \(axξη\) in the algebra \(B(H)\). Since \((aₙxξη*) ⊆ pξApξ\) and \(pξApξ\) is weakly closed in \(B(H)\), then \(axξη* ∈ pξApξ\). Hence there exists an element \(b ∈ A\) such that \(xbξη = xξηbxξηxξη*\). Then \(axξη* xξη = xξηbxξηxξη* xξη = xξηbxξηpη = xξηbxξη ∈ pξApη\). At the same time \(aₙpη = aₙ\) for all \(ξ, η\). Hence, \(apη = a\) in the algebra \(B(H)\). Since \(a = axξηxξη = xξηbxξη ∈ pξApη\), then \(a ∈ pξApη\). Since the net \((aₙ)\) is chosen arbitrarily, then the component \(pξpξApη\) is weakly closed in \(B(H)\). Let \((aₙ)\) be a net in \(\bigoplusξη pξApη\), weakly converging to an element \(a\) in \(B(H)\). Then for all \(ξ, η\) the net \((pξapη)\) weakly converges to \(pξapη\) in \(B(H)\). In this case, by the previous part of the proof \(pξapη ∈ pξApη\) for all \(ξ, η\). Note that \(a ∈ \bigoplusξη pξB(H)pη\). Hence \(a ∈ \bigoplusξη pξApη\). Since the net \((aₙ)\) is chosen arbitrarily, then the vector space \(\bigoplusξη pξApη\) is weakly closed in the algebra \(\bigoplusξη pξB(H)pη = B(H)\). Therefore by lemma 10 \(\bigoplusξη pξApη\) is a von Neumann algebra. ▶

Proposition 12. Let \(A\) be a monotone complete \(C^*\)-algebra on a Hilbert space \(H\), \(\{pξ\}\) be an infinite orthogonal set of projections in \(A\) with the least upper bound 1 in the algebra \(B(H)\). Then the order unit space \(\bigoplusξη pξApη\) is monotone complete.
Proof. We have the C*-subalgebra $p_\xi A_\xi$ is monotone complete for any index $\xi$. Let $\{p_k\}_{k=1}^n$ be a finite subset of the set $\{p_\xi\}$ and $p = \sum_{k=1}^n p_k$. Then the C*-subalgebra $pA_p$ is also monotone complete.

Let $(a_\alpha)$ be a bounded monotone increasing net in $\bigcap_{pB_{H}} p\xi A_{\xi}$. Since for any finite subset $\{p_k\}_{k=1}^n \subseteq \{p_\xi\}$ the subalgebra $(\sum_{k=1}^n p_k)A(\sum_{k=1}^n p_k)$ is monotone complete then

$$\sup_{\alpha} \{ \sum_{k=1}^n p_k \} a_\alpha (\sum_{k=1}^n p_k) \in \{ \sum_{k=1}^n p_k \} A(\sum_{k=1}^n p_k).$$

Hence, $\{a_{\xi}\} = \{\sup_{\alpha} p_\xi a_\alpha p_\xi\} \cup \{p_\xi(\sup_{\alpha} (p_\xi + p_\eta)a_\alpha (p_\xi + p_\eta))\}_{\xi \neq \eta}$ is an element of the order unit space $\bigcap_{pB_{H}} p\xi A_{\xi}$. It can be checked straightforwardly using the definition of the order in the algebra $\bigcap_{pB_{H}} p\xi A_{\xi}$ that the element $\{a_{\xi}\}$ is the least upper bound of the net $(a_\alpha)$. Since the net $(a_\alpha)$ was chosen arbitrarily then the order unit space $\bigcap_{pB_{H}} p\xi A_{\xi}$ is monotone complete. ▷

Theorem 13. Let $A$ be a monotone complete C*-algebra of bounded linear operators on a Hilbert space $H$. $\{p_\xi\}$ be an infinite orthogonal set of projections in $A$ with the least upper bound 1 in the algebra $B(H)$. Suppose that projections of the set $\{p_\xi\}$ are pairwise equivalent and $A$ is not a von Neumann algebra. Then $A \neq \bigcap_{pB_{H}} p\xi A_{\xi}$ (i.e. $A := \{p_\xi A_{\xi} : a \in A\} \neq \bigcap_{pB_{H}} p\xi A_{\xi}$).

Proof. We have there exists a bounded monotone increasing net $(a_\alpha)$ of elements in $A$, the least upper bound $\sup_{pAp} a_\alpha$ in the algebra $A$ and the least upper bound $\sup_{\sum_{\xiB_{H}} p_\xi B(H)} p_\xi a_\alpha$ in the algebra $\sum_{\xiB_{H}} p_\xi B(H)p_\eta$ of which are different. Otherwise $A$ is a von Neumann algebra.

By the definition of the order in the algebra $\sum_{\xiB_{H}} p_\xi B(H)p_\eta$ there exists a projection $p \in \{p_\xi\}$ such that the least upper bound $\sup_{pAp} p_\xi A_{\xi}$ in the algebra $pAp$ and the least upper bound $\sup_{\sum_{\xiB_{H}} p_\xi B(H)p_\eta} p_\xi A_{\xi}$ in the algebra $\sum_{\xiB_{H}} p_\xi B(H)p_\eta$ are different. Indeed, let $a = \sup_{pAp} a_\alpha$, $b = \sup_{\sum_{\xiB_{H}} p_\xi B(H)p_\eta} a_\alpha$. Since $A \subseteq \sum_{\xiB_{H}} p_\xi B(H)p_\eta$, then $b \leq a$ and $0 \leq a - b$. Hence, if $p_\xi(a - b) = 0$ for all $\xi$, then $p_\xi(a - b) = (a - b)p_\xi = 0$. Therefore by lemma 2 in [AFN2] $a - b = 0$, i.e. $a = b$. Hence $pAp$ is not a von Neumann algebra.

We have there exists an infinite orthogonal set $\{e_i\}$ of projections in $pAp$, the least upper bound $\sup_{pAp} e_i$ in the algebra $pAp$ and the least upper bound $\sup_{pB_{H}B(H)p} e_i$ in the algebra $pB_{H}B(H)p$ of which are different. Otherwise $pAp$ is a von Neumann algebra.

Indeed, any maximal commutative subalgebra $A_o$ of $pAp$ is monotone complete. For any normal positive linear functional $\rho \in B(H)$ and for any infinite orthogonal set $\{q_i\}$ of projections in $A_o$ we have $\rho(\sup_{i} q_i) = \sum_{i} \rho(q_i)$, where $\rho(q_i)$ is the least upper bound of the set $\{q_i\}$ in $A_o$. Hence by the theorem on extension of a $\sigma$-additive measure to a normal linear functional $\rho|_{A_o}$ is a normal functional on $A_o$. Hence $A_o$ is a commutative von Neumann algebra. At the same time the maximal commutative subalgebra $A_o$ of the algebra $\sum_{\xiB_{H}} p_\xi A_{\xi}$ is chosen arbitrarily. Therefore by [GKP] $\sum_{\xiB_{H}} p_\xi A_{\xi}$ is a von Neumann algebra. What is impossible.

Let $\{x_{\xi}\}$ be such set of isometries in $A$ that $p_\xi = x_{\xiB_{x_{\xiB_{\xi}}}B_{\xi}}$, $p_\eta = x_{\etaB_{x_{\etaB_{\eta}}}B_{\eta}}$ for all $\xi$, $\eta$, and let $p_1 = p$. Let $\{x_{\xi}\}$ be the subset of the set $\{x_{\xi}\}$ such that $p_1 = x_{1B_{1}B_{1}}$, $\eta$, for all $\xi$. Without loss of generality we regard that the set of indexes $i$ for $\{e_i\}$ is a subset of the set of indexes $\xi$ for $\{p_\xi\}$. Let $\{e_i x_{1i}\}$ be a set of
all components of some infinite dimensional matrix \( \{a_{\xi n}\} \), the components, which are not present, are zeros and \( \{x_i^* e_i\} \) be also an analogous matrix, which coincides with \( \{a_{\xi n}^*\} \). We have \( \sum_i e_i x_i^* e_i^* = \sum_i e_i p_i e_i^* = \sum_i e_i e_i^* = \sum_i e_i \leq \sup_p p \sum_i e_i \). Therefore \( \{a_{\xi n}\} \in \sum_{\xi.n} p \xi A p_n \). Therefore if \( \{a_{\xi n}\} \in A \) (i.e. in \( A := \{\{p \xi A p_n\} : a \in A\} \)) then the product \( \{a_{\xi n}\} \cdot \{a_{\xi n}^*\} \) in \( \sum_p p_i B(H) p_j \) belongs to \( \sum_{\xi.n} p \xi A p_n \). In this case we have the infinite dimensional matrix \( \{a_{\xi n}\} \cdot \{a_{\xi n}^*\} \) contains the component \( \sum_i e_i x_i x_i^* e_i^* \) such that \( \sum_i e_i x_i x_i^* e_i^* = \sum_i e_i x_i \cdot x_i^* e_i^* = p_1(\sum_i e_i x_i x_i^* e_i^*) \). Consequently, \( p_1(\{a_{\xi n}\} \cdot \{a_{\xi n}^*\}) = \sum_i e_i x_i x_i^* e_i^* \). Hence \( \sum_i e_i x_i x_i^* e_i^* = p_1(\sum_{\xi.n} p \xi A p_n) p_1 = p_1 A p_1 \). Since \( \sum_i e_i x_i x_i^* e_i^* = \sum_i e_i p_1 e_i^* = \sum_i e_i e_i^* = \sum_i e_i \) then \( \sum_i e_i \in p_1 A p_1, i.e. \sup_p B(H) e_i \in \sum_i e_i \). The last statement is a contradiction. Therefore \( \{a_{\xi n}\} \notin A \). Hence \( A \neq \sum_{\xi,n} p \xi A p_n \) (i.e. \( A := \{\{p \xi A p_n\} : a \in A\} \neq \sum_{\xi.n} p \xi A p_n \) ).

The next assertion follows by theorem 13 and it’s proof.

*Corollary 14.* Let \( A \) be a C\(^*\)-algebra on a Hilbert space \( H \), \( \{p \xi\} \) be an infinite orthogonal set of projections in \( A \) with the least upper bound 1 in the algebra \( B(H) \). Suppose that the order unit space \( \sum_{\xi.n} p \xi A p_n \) is monotone complete and there exists a bounded monotone increasing net \((a_n)\) of elements in \( \sum_{\xi.n} p \xi A p_n \), the least upper bound \( \sum_{\xi.n} p \xi A p_n a_n \) in the algebra \( \sum_{\xi.n} p \xi A p_n \) and the least upper bound \( \sup_{\xi,n} p \xi B(H) p_n a_n \) in the algebra \( \sum_{\xi.n} p \xi B(H) p_n \) are different. Then the vector space \( \sum_{\xi.n} p \xi A p_n \) is not closed concerning the multiplication of the algebra \( \sum_{\xi.n} p \xi B(H) p_n \).

2. Application

Let \( n \) be an infinite cardinal number, \( \Xi \) a set of indexes of cardinality \( n \). Let \( \{e_{ij}\} \) be a set of matrix units such that \( e_{ij} \) is a \( n \times n \)-dimensional matrix, i.e. \( e_{ij} = (a_{i,j})_{i,j=1}^n, \) whose \( (i,j)\)-s component is 1, i.e. \( a_{ij} = 1, \) and the rest components are zero. Let \( X \) be a hyperstonean compact, \( C(X) \) the commutative algebra of all complex-valued continuous functions on the compact \( X \) and

\[
\mathcal{M} = \{ \sum_{i,j \in \Xi} \lambda_{ij}(x) e_{ij} : (\forall j) \lambda_{ij}(x) \in C(X) \}
\]

\[
(\exists K \in R)(\forall m \in N)(\forall \{e_{kl}\}_{k,l=1}^m \subseteq \{e_{ij}\}) || \sum_{k,l=1}^m \lambda_{kl}(x) e_{kl} || \leq K,
\]

where \( || \sum_{k,l=1}^m \lambda_{kl}(x) e_{kl} || \leq K \) means \( (\forall x \in X) || \sum_{k,l=1}^m \lambda_{kl}(x) e_{kl} || \leq K \). The set \( \mathcal{M} \) is a vector space with pointwise algebraic operations. The map \( || . || : \mathcal{M} \rightarrow R_+ \) defined as

\[
||a|| = \sup_{\{e_{kl}\}_{k,l=1}^m} \| \sum_{k,l=1}^n \lambda_{kl}(x) e_{kl} ||,
\]

is a norm on the vector space \( \mathcal{M} \), where \( a \in \mathcal{M} \) and \( a = \sum_{i,j \in \Xi} \lambda_{ij}(x) e_{ij} \).

*Theorem 15.* \( \mathcal{M} \) is a von Neumann algebra of type I\(_n\), and \( \mathcal{M} = C(X) \otimes M_n(C) \).

*Proof.* It is easy to see that the set \( \mathcal{M} \) is a vector space with the componentwise algebraic operations. It is known that the vector space \( C(X, M_n(C)) \) of continuous matrix-valued maps on the compact \( X \) is a C\(^*\)-algebra. Let \( A = C(X, M_n(C)) \)
and $e_i$ be a constant $e_i$-valued map on $X$, i.e. $e_i$ is a projection of the algebra $A$. Then $\{e_i\}$ is an orthogonal set of projections with $\sup_i e_i = 1$ in the algebra $A$. Then $\sum_{i,j} e_i A e_j = M$. We have a $C^*$-algebra $A$ can be embedded in $B(H)$ for some Hilbert space $H$. Then $\sum_{i,j} e_i A e_j$ can be embedded in $B(H)$. For any $i$ $e_i A e_i = C(X) e_i$, i.e. the component $e_i A e_i$ is weakly closed in $B(H)$. Hence, by theorem 11 the image of vector space $M$ in $B(H)$ is a von Neumann algebra. Hence, $M$ is a von Neumann algebra. Note, that the set $\{e_i\}$ is a maximal orthogonal set of abelian projections with central support 1. Hence, $M$ is a von Neumann algebra of type $I_n$. Moreover the center $Z(M)$ of the algebra $M$ is isomorphic to $C(X)$ and $M = C(X) \otimes M_n(C)$. \begin{flushright} ▷ \end{flushright}

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