Josephson effect in graphene SBS junctions

Moitri Maiti and K. Sengupta
TCMP division, Saha Institute of Nuclear Physics, 1/AF Bidhannagar, Kolkata-700064, India.
(Dated: March 23, 2022)

We study Josephson effect in graphene superconductor-barrier-superconductor junctions with short and wide barriers of thickness \( d \) and width \( L \), which can be created by applying a gate voltage \( V_0 \) across the barrier region. We show that Josephson current in such graphene junctions, in complete contrast to their conventional counterparts, is an oscillatory function of both the barrier width \( d \) and the applied gate voltage \( V_0 \). We also demonstrate that in the thin barrier limit, where \( V_0 \to \infty \) and \( d \to 0 \) keeping \( V_0d \) finite, such an oscillatory behavior can be understood in terms of transmission resonance of Dirac-Bogoliubov-de Gennes quasiparticles in superconducting graphene. We discuss experimental relevance of our work.

PACS numbers: 74.50+r, 74.45.+c, 74.78.Na

I. INTRODUCTION

Graphene, a two-dimensional single layer of graphite, has been recently fabricated by Novoselov et al. In graphene, the energy bands touch the Fermi energy at six discrete points at the edges of the hexagonal Brillouin zone. Two of these six Fermi points, referred to as K and K’ points, are inequivalent and the quasiparticle excitations about them obey linear Dirac-like energy dispersion. The presence of such Dirac-like quasiparticles leads to a number of unusual electronic properties in graphene including relativistic quantum hall effect with unusual structure of Hall plateaus, which has been verified in experiments. Further, as suggested in Ref. [5], Dirac quasiparticles in graphene lead to the realization of interesting physical phenomenon such as Klein paradox, Lorenz-boost type phenomenon, and unconventional Kondo effect.

Another interesting consequence of the existence of Dirac-like quasiparticles is superconductivity in graphene. It has been suggested that superconductivity can be induced in a graphene layer in the presence of a superconducting electrode near it via proximity effect or by intercalating it with metallic atoms. Consequently, studies on tunneling conductance on both normal metal-superconductors (NS) and normal metal-barrier-superconductor (NBS) junctions in graphene have been undertaken. It has been shown in Refs. [14] and [15] that the tunneling conductance of such NBS junctions are oscillatory functions of the effective barrier strength and that this oscillatory phenomenon can be understood in terms of transmission resonance phenomenon of Dirac-Bogoliubov-de Gennes (DBdG) quasiparticles in graphene. Josephson effect has also been studied in a superconductor-normal metal-superconducting (SNS) junction in graphene. It has been shown in Ref. [12] that the behavior of such junctions is similar to conventional SNS junctions with disordered normal region. Such Josephson junctions with thin barrier regions have also been experimentally realized recently. However, Josephson effect in graphene superconductor-barrier-superconductor (SBS) junctions has not been studied so far.

In this work, we study Josephson effect in graphene for tunnel SBS junctions. In this study, we shall concentrate on SBS junctions with barrier thickness \( d \ll \xi \) where \( \xi \) is the superconducting coherence length, and width \( L \) which has an applied gate voltage \( V_0 \) across the barrier region. Our central result is that in complete contrast to the conventional Josephson tunnel junctions studied so far, the Josephson current in graphene SBS tunnel junctions is an oscillatory function of both the barrier thickness \( d \) and the applied gate voltage \( V_0 \). We provide an analytical expression for the Josephson current of such a junction. We also compute the critical current of graphene SBS junctions. We find that this critical current is also an oscillatory function of \( V_0 \) and \( d \) and study the amplitude and periodicity of its oscillation.

The organization of the rest of the paper is as follows. In Sec. II, we obtain an analytical expression for Josephson current for a general SBS junction of thickness \( d \ll \xi \) and applied voltage \( V_0 \) and demonstrate that the Josephson current is an oscillatory function of both \( d \) and \( V_0 \). This is followed by Sec. III, where we discuss the limiting case of thin barrier and demonstrate that the oscillatory behavior of the Josephson current can be understood in terms of transmission resonance of DBdG quasiparticles in graphene. Finally we discuss experimental relevance of our results in Sec. IV.
II. JOSEPHSON CURRENT FOR TUNNEL SBS JUNCTIONS

We consider a SBS junction in a graphene sheet of width $L$ lying in the $xy$ plane with the superconducting regions extending $x = -\infty$ to $x = -d$ and from $x = 0$ to $x = \infty$ to all $0 \leq y \leq L$, as shown in Fig. 1. The superconducting regions $x \geq 0$ and $x \leq -d$ shall be assumed to be kept close to superconducting electrodes so that superconductivity is induced in these regions via proximity effects. Alternatively, one can also possibly use intercalated graphene which may have s-wave superconducting phases. In the rest of this work, we shall assume that these regions are superconducting without worrying about the details of the mechanism used to induce superconductivity. The region B, modeled by a barrier potential $V_0$, extends from $x = -d$ to $x = 0$. Such a local barrier can be implemented by either using the electric field effect or local chemical doping. In the rest of this work, we assume that the barrier region has sharp edges on both sides which requires $d \ll \lambda = 2\pi/k_F$, where $k_F$ and $\lambda$ are the Fermi wavevector and Fermi wavelength for graphene. Such barriers can be realistically created in experiments. The width $L$ of the sample shall be assumed to be large compared to all other length scales in the problem. The SBS junction can then be described by the DBdG equations

$$
(H_a - E_F + U(r) \frac{\Delta(r)}{\Delta^*(r)} E_F - U(r) - H_a) \psi_a = E \psi_a.
$$

Here, $\psi_a = (\psi_{a\alpha}, \psi_{B\alpha}, \psi_{A\bar{\alpha}}, -\psi_{B\bar{\alpha}})$ are the 4 component wavefunctions for the electron and hole spinors, the index $a$ denote $K$ or $K'$ for electron/holes near $K$ and $K'$ points, $\bar{\alpha}$ takes values $K'(K)$ for $a = K(K')$, $E_F$ denote the Fermi energy, $A$ and $B$ denote the two inequivalent sites in the hexagonal lattice of graphene, and the Hamiltonian $H_a$ is given by

$$
H_a = -i\hbar v_F (\sigma_x \partial_x + \text{sgn}(a) \sigma_y \partial_y),
$$

where $\text{sgn}(a)$ takes values $\pm$ for $a = K(K')$.

The pair-potentials $\Delta(r)$ in Eq. 1 connects the electron and the hole spinors of opposite Dirac points. We have modeled the pair-potential as

$$
\Delta(r) = \Delta_0 [\exp(i\phi_2)\theta(x) + \exp(i\phi_1)\theta(x + d)],
$$

where $\Delta_0$ is the amplitude and $\phi_1(2)$ are the phases of the induced superconducting order parameters in regions I (II) as shown in Fig. 1 and $\theta$ is the Heaviside step function. Notice that the mean-field conditions for superconductivity is satisfied as long as $\Delta_0 \ll E_F$ or equivalently $k_F \xi \gg 1$, where $\xi = \hbar v_F / \pi \Delta_0$ is the superconducting coherence length. The potential $U(r)$ gives the relative shift of Fermi energies in the barrier and superconducting regions and is modeled as

$$
U(r) = V_0 \theta(-x)\theta(x + d).
$$

Solving Eq. 1 we obtain the wavefunctions in the superconducting and the barriers regions. In region I, for the DBdG quasiparticles moving along $\pm x$ direction with a transverse momentum $k_y = q = 2\pi n / L$ (for integer $n$) and energy $\epsilon$, the wavefunctions are given by

$$
\psi^\pm_I = \left( u_1^\pm, u_2^\pm, u_3^\pm, u_4^\pm \right) e^{i(\pm k_\pm x + qy) + \epsilon x},
$$

where

$$
\begin{align*}
\frac{u_2^\pm}{u_1^\pm} &= \pm \exp(\pm i\gamma), \quad \frac{u_3^\pm}{u_1^\pm} = \exp[-i(\phi_1 \mp \beta)], \\
\frac{u_4^\pm}{u_1^\pm} &= \pm \exp[\pm i(\mp \phi_1 + \beta + \gamma)],
\end{align*}
$$

and $\sum_{i=1,4} |u_i|^2 \approx 2\kappa$ is the normalization condition for the wavefunction for $d \ll \kappa^{-1}$, where $\kappa^{-1} = (\hbar v_F)^2 k_s / [E_F \Delta_0 \sin(\beta)]$ is the localization length. Here $k_s = \sqrt{(E_F/hv_F)^2 - q^2}$, $\gamma$, the angle of incidence for the quasiparticles, is given by $\sin(\gamma) = hv_F q / E_F$, and $\beta$ is given by

$$
\beta = \cos^{-1}(\epsilon/\Delta_0) \quad \text{if } |\epsilon| < \Delta_0, \\
= -i \cosh^{-1}(\epsilon/\Delta_0) \quad \text{if } |\epsilon| > \Delta_0.
$$

Note that for $|\epsilon| > \Delta_0$, $\kappa$ becomes imaginary and the quasiparticles can propagate in the bulk of the superconductor. The wavefunctions in region II ($x \geq 0$) can also be obtained in a similar manner

$$
\psi^\pm_II = \left( v_1^\pm, v_2^\pm, v_3^\pm, v_4^\pm \right) e^{i[(\pm k_\pm x + qy) - \epsilon x]},
$$

where $\sum_{i=1,4} |v_i|^2 \approx 2\kappa$ and the coefficients $v_i$ are given

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1}
\caption{A schematic graphene SBS junction with the barrier B sandwiched between two superconductors I and II with pair potentials $\Delta_0 e^{i\phi_1}$ and $\Delta_0 e^{i\phi_2}$. The barrier region is created by an external gate voltage $V_0$.}
\end{figure}
by
\[
\frac{\psi_B^+}{\psi_B^-} = \pm \exp(\pm i\gamma), \quad \frac{\psi_B}{\psi_B^+} = \exp[-i(\phi_2 \pm \beta)],
\]
\[
\frac{\psi_B}{\psi_B^-} = \pm \exp[\pm i(\mp \phi_1 - \beta + \gamma)], \tag{9}
\]

The wavefunctions for electrons and holes moving along \( \pm x \) in the barrier region is given by
\[
\psi_e^{+, -} = (1, \pm e^{+i \vartheta}, 0, 0) \exp[i(\pm k_0 x + q y)] / \sqrt{2d},
\]
\[
\psi_h^{+, -} = (0, 0, 1, \pm e^{+i \vartheta'}) \exp[i(\pm k_0 x + q y)] / \sqrt{2d}.
\tag{10}
\]

Here the angle of incidence of the electron(hole) \( \theta(\theta') \) is given by
\[
\sin[\theta(\theta')] = \left\{ \begin{array}{l}
\frac{\hbar v_F q}{\epsilon + (-\epsilon)(E_F - V_0)}
\end{array} \right.
\]
\[
k_0(k_0') = \left( \frac{\epsilon + (-\epsilon)(E_F - V_0)}{\hbar v_F} \right)^2 - q^2 \tag{11}
\]

To compute the Josephson current in the SBS junction, we now find the energy dispersion of the subgap Andreev bound states which are localized with localization length \( \kappa^{-1} \) at the barrier.\textsuperscript{22,23} The energy dispersion \( \epsilon_n \) (corresponding to the subgap states characterized by the quantum number \( n \)) of these states depends on the phase difference \( \phi = \phi_2 - \phi_1 \) between the superconductors. It is well known that the Josephson current \( I \) across the junction at a temperature \( T_0 \) is given by\textsuperscript{12,22}
\[
I(\phi; T_0) = \frac{4e}{\hbar} \sum_n \sum_{q=-k_F}^{k_F} \frac{\partial \epsilon_n}{\partial \theta} f(\epsilon_n), \tag{12}
\]
where \( f(x) = 1/(e^x/(knT_0) + 1) \) is the Fermi distribution function and \( k_B \) is the Boltzmann constant\textsuperscript{24}.

To obtain these subgap Andreev bound states, we now impose the boundary conditions at the barrier. The wavefunctions in the superconducting and barrier regions can be constructed using Eqs. \textsuperscript{5,8} and \textsuperscript{10} as
\[
\Psi_I = a_1 \psi_I^+ + b_1 \psi_I^- \quad \Psi_{II} = a_2 \psi_{II}^+ + b_2 \psi_{II}^-, \\
\Psi_B = p \psi_B^+ + q \psi_B^- + r \psi_B^+ + s \psi_B^-,
\tag{13}
\]
where \( a_1(b_2) \) and \( b_1(b_2) \) are the amplitudes of right and left moving DBG quasiparticles in region I(II) and \( p(q) \) and \( r(s) \) are the amplitudes of right(left) moving electron and holes respectively in the barrier. These wavefunctions must satisfy the boundary conditions:
\[
\Psi_I|_{x=-d} = \Psi_B|_{x=-d}, \quad \Psi_B|_{x=0} = \Psi_{II}|_{x=0} \tag{14}
\]
Notice that these boundary conditions, in contrast their counterparts in standard SBS interfaces,\textsuperscript{24} do not impose any constraint on derivative of the wavefunctions. Thus the standard delta function potential approximation for short barriers\textsuperscript{22,23} can not be taken the outset, but has to be taken at the end of the calculations.

Substituting Eqs. \textsuperscript{5,8,11} and \textsuperscript{13} in Eq. \textsuperscript{14} we find the equations for boundary conditions at \( x = -d \) to be
\[
a_1 e^{-i k_0 d - \kappa d} + b_1 e^{i k_0 d - \kappa d} = \frac{pe^{i k_0 d} + q e^{i k_0 d}}{\epsilon - i(k - d) - i q}
\]
\[
a_1 e^{i(\gamma - k_0 d) - \kappa d} - b_1 e^{-i(\gamma - k_0 d) - \kappa d} = \frac{pe^{i(k_0 d - d)} - q e^{-i(k_0 d - d)}}{\epsilon - i(k - d) - i q}
\]
\[
a_1 e^{-i(\phi_1 + \beta + k_0 d) - \kappa d} + b_1 e^{-i(\phi_1 - \beta + k_0 d) - \kappa d} = \frac{r e^{-i k_0 d} + s e^{i k_0 d}}{\epsilon - i(k_0 d - d) - i q}
\]
\[
a_1 e^{i(\gamma - \phi_1 + \beta - k_0 d) - \kappa d} - b_1 e^{-i(\gamma - \phi_1 - \beta - k_0 d) - \kappa d} = \frac{r e^{-i(k_0 d - d) - \kappa d} - s e^{i(k_0 d - d)}}{\epsilon - i(k - d) - i q}. \tag{15}
\]

Similar equations at \( x = 0 \) reads
\[
a_2 + b_2 = p + q
\]
\[
a_2 e^{i(\gamma - \phi_2 - \beta) - \kappa d} - b_2 e^{-i(\gamma - \phi_2 + \beta) - \kappa d} = \frac{r e^{-i(k_0 d - d) - \kappa d} - s e^{i(k_0 d - d)}}{\epsilon - i(k - d) - i q}. \tag{16}
\]

Eqs. \textsuperscript{15} and \textsuperscript{16} therefore yields eight linear homogeneous equations for the coefficients \( a_1, b_1, a_2, b_2, p, q, r, \) and \( s \), so that the condition for non-zero solutions of these coefficients can be obtained as
\[
A' \sin(2\beta) + B' \cos(2\beta) + C' = 0 \tag{17}
\]
where \( A', B', \) and \( C' \) are given by
\[
A' = \cos(k_0 d) \cos(\gamma) \cos(\theta') \sin(k_0 d) \sin(\gamma) \sin(\theta) - 1
\]
\[
+ \cos(k_0 d) \cos(\gamma) \cos(\theta) \sin(k_0 d)
\]
\[
+ \frac{1}{2} \cos(k_0 d) \cos(\theta) \sin(2\gamma) \sin(\theta') \sin(k_0 d)
\]
\[
B' = \sin(k_0 d) \sin(k_0 d) \left[-1 + \sin(\theta) \sin(\gamma)
\right.
\]
\[
- \sin(\theta') \sin(\gamma) + \sin(\theta) \sin(\theta') \sin^2(\gamma) \right]
\]
\[
- \cos(k_0 d) \cos(k_0 d) \cos^2(\gamma) \cos(\theta) \sin(\theta')
\]
\[
C' = \cos^2(\gamma) \cos(\theta) \cos(\phi) \sin(k_0 d) \sin(k_0 d)
\]
\[
\times \left[ \sin(\theta) \sin(\theta') - \sin^2(\gamma)
\right.
\]
\[
+ \sin(\gamma) \sin(\theta) - \sin(\theta')) \right] \tag{18}
\]

Note that in general the coefficients \( A', B', \) and \( C' \) depends on \( \epsilon \) through \( k_0, k_0', \theta \) and \( \theta' \) which makes it impossible to find an analytical solution for Eq. \textsuperscript{17}. However, for subgap states in graphene SBS junctions, \( \epsilon \leq \Delta_0 \ll E_F \). Further, for short tunnel barrier we have \( |V_0 - E_F| \geq E_F \). In this regime, as can be seen from Eqs. \textsuperscript{11} \( A', B', \) and \( C' \) become independent of \( \epsilon \) since \( k_0 \approx k_0' \approx k_1 = \sqrt{[|V_0 - E_F|/\hbar v_F]^2 - q^2} \) and \( \theta \approx \theta' = \theta_1 = \sin^{-1} \left[ \hbar v_F/(E_F - V_0) \right] \) so that the \( \epsilon \) dependence of \( k_0, k_0', \theta \) and \( \theta' \) can be neglected. In this regime one finds that \( A', B', C' \rightarrow A, B, C \) where
The dispersion of the Andreev subgap states and the Josephson current can now be expressed in terms of the applied gate voltage.

\[
A = 0 \\
B = -\sin^2(k_d) \left[ 1 - \sin(\gamma) \sin(\theta_1) \right] \\
C = \sin^2(k_d) \left[ \sin(\gamma) - \sin(\theta_1) \right]^2 \\
+ \cos^2(\gamma) \cos^2(\theta_1) \cos(\phi)
\]  

The dispersion of the Andreev subgap states can now be obtained from Eqs. [17] and [18]. One finds that there are two Andreev subgap states with energies \(\epsilon_\pm = \pm \epsilon\) where

\[
\epsilon = \Delta_0 \sqrt{1/2 - C/2B}
\]  

Using Eq. [12] one can now obtain the expression for the Josephson current

\[
I(\phi, V_0, d, T_0) = I_0 g(\phi, V_0, d, T_0), \\
g(\phi, V_0, d, T_0) = \int_{-\pi/2}^{\pi/2} d\gamma \left[ \frac{\cos^3(\gamma) \cos^2(\theta_1) \sin(\phi)}{B\epsilon/\Delta_0} \right] \\
\times \tanh(\epsilon/2k_d T_0)
\]

where \(I_0 = e\Delta_0 E_F L/2h^2\pi v_F\) and we have replaced \(\Sigma_\delta \rightarrow E_F L/(2\pi h v_F) \int_{-\pi/2}^{\pi/2} d\gamma \cos(\gamma)\) as appropriate for wide junctions [12].

Eqs. [20] and [21] represent the central result of this work. From these equations, we find that both the dispersion of the Andreev subgap states and the Josephson current in graphene SBS junctions, in complete contrast to their conventional counterparts [18,19,22], is oscillatory function of the applied gate voltage \(V_0\) and the barrier thickness \(d\). This statement can be most easily checked by plotting the Josephson current \(I\) as a function of the phase difference \(\phi\) and the applied gate voltage \(V_0\) for a representative barrier thickness \(d = 0.5\lambda\) and temperature \(k_B T_0 = 0.01\Delta_0\), as done in Fig. [2]. In Fig. [3] we plot the critical current of these junctions \(I_c(V_0, d, T_0) = \text{Max}[I(\phi, V_0, d, T_0)]\) as a function of the applied gate voltage \(V_0\) and barrier thickness \(d\) for low temperature \(k_B T_0 = 0.01\Delta_0\). We find that the critical current of these graphene SBS junctions is an oscillatory function of both \(V_0\) and \(d\). This behavior is to be contrasted with those of conventional junctions where the critical current is a monotonically decreasing function of both applied bias voltage \(V_0\) and junction thickness [18,19,22].

Next, we analyze the temperature dependence of the amplitude of oscillations of \(I_c\). To find the amplitude of oscillation, we have computed \(I_c\) as a function of \(V_0\) for a representative value of \(d = 0.3\lambda\), noted the maximum \((I_{c,\text{max}})\) and minimum \((I_{c,\text{min}})\) values of \(I_c\), and calculated the amplitude \(I_{c,\text{max}} - I_{c,\text{min}}\). The procedure is repeated for several temperatures \(T_0\) and the result is plotted in Fig. [4]. which shows that the amplitude of oscillations decreases monotonically as a function of temperature.

Finally, we discuss the period of oscillation of the critical current. To obtain the period, we obtain the critical current \(I_c\) as a function of barrier width \(d\) for the fixed applied gate voltage \(V_0\) and note down \(d_{\text{period}}\). We then compute \(\lambda_{\text{period}} = V_0 d_{\text{period}}/h v_F\) and plot \(\chi_{\text{period}}\) as a function of \(V_0\) for \(k_B T_0 = 0.01\Delta_0\) as shown in Fig. [5]. We find that \(\chi_{\text{period}}\) decreases with \(V_0\) and approaches an universal value \(\pi\) for large \(V_0 \geq 20 E_F\). This property, as we shall see in the next section, can be understood by analysis of graphene SBS junctions in the thin barrier limit \((V_0 \rightarrow \infty\) and \(d \rightarrow 0\) such that \(\chi = V_0 d/h v_F\) remains finite [14]), and is a direct consequence of transmission...
III. THIN BARRIER LIMIT

In the limit of thin barrier, where \( V_0 \rightarrow \infty \) and \( d \rightarrow 0 \) such that \( \chi = \frac{V_0 d}{\hbar v_F} \) remains finite, \( \theta_1 \rightarrow 0 \) and \( k_1 d \rightarrow \chi \). From Eqs. (19) and (20) we find that in this limit, the dispersion of the Andreev bound states becomes

\[
\epsilon_{\pm}^{tb}(q, \phi; \chi) = \pm \Delta_0 \sqrt{1 - T(\gamma, \chi) \sin^2(\phi/2)}, \quad (22)
\]

\[
T(\gamma, \chi) = \frac{\cos^2(\gamma)}{1 - \cos^2(\chi) \sin^2(\gamma)}. \quad (23)
\]

where the superscript ‘tb’ denote thin barrier limit. The Josephson current \( I \) can be obtained substituting Eq. (23) in Eq. (12). In the limit of wide junctions, one gets

\[
I^{tb}(\phi, \chi, T_0) = I_0 g^{tb}(\phi, \chi, T_0),
\]

\[
g^{tb}(\phi, \chi, T_0) = \int_{-\pi/2}^{\pi/2} d\gamma \left[ \frac{T(\gamma, \chi) \cos(\gamma) \sin(\phi)}{\sqrt{1 - T(\gamma, \chi) \sin^2(\phi/2)}} \right] \times \tanh(\epsilon_+ / 2k_B T_0). \quad (24)
\]

Eqs. (23) and (24) represent the key results of this section. From these equations, we find that the Josephson current in graphene SBS junctions is a \( \pi \) periodic oscillatory function of the effective barrier strength \( \chi \) in the thin barrier limit. Further we observe that the transmission probability of the DBdG quasiparticles in a thin SBS junction is given by \( T(\gamma, \chi) \) which is also the transmission probability of a Dirac quasiparticle through a square potential barrier as noted in Ref. 5. Note that the transmission becomes unity for normal incidence (\( \gamma = 0 \)) and when \( \chi = n\pi \). The former condition is a manifestation of the Klein paradox for DBdG quasiparticles. However, this property is not reflected in the Josephson current which receives contribution from quasiparticles approaching the junction at all angles of incidence. The latter condition (\( \chi = n\pi \)) represents transmission resonance condition of the DBdG quasiparticles. Thus the barrier becomes completely transparent to the approaching quasiparticles when \( \chi = n\pi \) and in this limit the Josephson current reduces to its value for conventional tunnel junctions in the Kulik-Omelyanchuk limit: \( I^{tb}(\phi, n\pi, T_0) = 4I_0 \sin(\phi/2) \text{Sgn}(\cos(\phi/2)) \tanh(\Delta_0 |\cos(\phi/2)| / 2k_B T_0) \).

This yields the critical Josephson current \( I^{tb}_c(\chi = n\pi) = 4I_0 \) for \( k_B T_0 \ll \Delta_0 \). Note, however, that in contrast to conventional junctions \( T(\gamma, \chi) \) cannot be made arbitrarily small for all \( \gamma \) by increasing \( \chi \). Hence \( I^{tb}_c \) never reaches the Ambegaokar-Baratoff limit of conventional tunnel junctions. Instead, \( I^{tb}_c(\chi) \) becomes a \( \pi \) periodic oscillatory function of \( \chi \). The amplitude of these oscillations decreases monotonically with temperature as discussed in Sec. III.

Finally, we compute the product \( I^{tb}_c R_N \) which is routinely used to characterize Josephson junctions, where \( R_N \) is the normal state resistance of the junction. For graphene SBS junctions \( R_N \) corresponds to the resistance of a Dirac quasiparticle as it moves across a normal metal-barrier-normal metal junction. For short and wide junctions discussed here, it is given by \( R_N = R_0 / s_1(\chi) \) where \( R_0 = \pi^2 v_F \hbar^2 / (e^2 E_F L) \) and \( s_1(\chi) \) is given by

\[
s_1(\chi) = \int_{-\pi/2}^{\pi/2} d\gamma T(\gamma, \chi) \cos(\gamma). \quad (25)
\]

Note that \( s_1(\chi) \) and hence \( R_N \) is an oscillatory function of \( \chi \) with minimum 0.5\( R_0 \) at \( \chi = n\pi \) and maximum
FIG. 6: Plot of $I_{c}^{th}R_{N}$ as a function of $\chi$. $I_{c}^{th}R_{N}$ is an oscillatory bounded function of $\chi$ and never reaches its value $(\pi\Delta_{0}/2e)$ for conventional junctions in the Ambegaokar-Baratoff limit.

$0.75R_{0}$ at $\chi = (n + 1/2)\pi$. The product $I_{c}^{th}R_{N}$, for thin SBS junctions is given by

$$I_{c}^{th}R_{N} = (\pi\Delta_{0}/2e)g_{\text{max}}(\chi, T)/s_{1}(\chi),$$

where $g_{\text{max}}(\chi)$ is the maximum value of $g^{th}(\phi, \chi)$. Note that $I_{c}^{th}R_{N}$ is independent of $E_{F}$ and hence survives in the limit $E_{F} \rightarrow 0^{\pm}$. For $k_{B}T_{0} \ll \Delta_{0}$, $g_{\text{max}}(n\pi) = 4$ and $s_{1}(n\pi) = 2$, so that $I_{c}^{th}R_{N}|_{\chi=n\pi} = \pi\Delta_{0}/e$ which coincides with Kulik-OMelyanchuk limit for conventional tunnel junctions. However, in contrast to the conventional junction, $I_{c}^{th}R_{N}$ for graphene SBS junctions do not monotonically decrease to the Ambegaokar-Baratoff limit of $\pi\Delta_{0}/2e \simeq 1.57\Delta_{0}/e$ as $\chi$ is increased, but demonstrates $\pi$ periodic oscillatory behavior and remains bounded between the values $\pi\Delta_{0}/e$ at $\chi = n\pi$ and $2.27\Delta_{0}/e$ at $\chi = (n + 1/2)\pi$, as shown in Fig. 6.

IV. EXPERIMENTS

As a test of our predictions, we suggest measuring DC Josephson current in these junctions as a function of the applied voltage $V_{0}$. Such experiments for conventional Josephson junctions are well-known. Further SNS junctions in graphene has also been recently been experimentally created. For experiments with graphene junctions which we suggest, the local barrier can be fabricated by applying an additional gate voltage in the normal region of the junctions studied in Ref. [17]. In graphene, typical Fermi energy can reach $E_{F} \leq 80$ meV with Fermi-wavelength $\lambda = 2\pi/k_{F} \geq 100$ nm. Effective barrier strengths of 500-1000 meV and barrier widths of $d \approx 20-50$ nm can be achieved in realistic experiments. These junctions therefore meet our theoretical criteria: $d \ll \lambda$ and $|V_{0} - E_{F}| \geq E_{F}$. To observe the oscillatory behavior of the Josephson current, it would be necessary to change $V_{0}$ in small steps $\delta V_{0}$. For barriers with fixed $d/\lambda = 0.3$ and $V_{0}/E_{F} = 10$, this would require changing $V_{0}$ in steps of approximately 30 meV which is experimentally feasible. The Joule heating in such junctions, proportional to $I_{c}^{2}R_{N}$, should also show measurable oscillatory behavior as a function of $V_{0}$.

In conclusion, we have shown that the Josephson current in graphene SBS junction shows novel oscillatory behavior as a function of the applied bias voltage $V_{0}$ and the barrier thickness $d$. In the thin barrier limit, such a behavior is the manifestation of transmission resonance of DBdG quasiparticles in superconducting graphene. We have suggested experiments to test our predictions.

KS thanks V. M. Yakovenko for discussions.

1. K.S. Novoselov et al., Science 306, 666 (2004).
2. T. Ando, J. Phys. Soc. Jpn. 74 777 (2005).
3. V.P. Gusynin and S.G. Sharapov, Phys. Rev. Lett. 95, 146801 (2005); N.M.R. Peres, F. Guinea, and A. Castro Neto, Phys. Rev. B 73, 125411 (2006).
4. K.S. Novoselov et al., Nature 438, 197 (2005); Y. Zhang et al., Nature 438, 201 (2005).
5. M.I. Katsnelson et al., Nature Phys. 2, 620 (2006).
6. O. Klein, Z. Phys. 53, 157 (1929).
7. V. Lukose, R. Shankar and G. Baskaran, Phys. Rev. Lett., 98 116802 (2007).
8. K. Sengupta and G. Baskaran, [arXiv:0705.0257] (unpublished).
9. M. Hentschel and F. Guinea, [arXiv:0705.0522] (unpublished).
10. C.W.J. Beenakker, Phys. Rev. Lett. 97, 067007 (2006).
11. A.F. Volkov et al., Physica C 242, 261 (1995).
12. M. Titov and C.W.J. Beenakker, Phys. Rev. B 74, 041401(R) (2006).
13. B. Uchoa and A. Castro Neto, Phys. Rev. Lett. 98, 146801 (2007).
14. S. Bhattacharjee and K. Sengupta, Phys. Rev. Lett. 97, 217001 (2006).
15. S. Bhattacharjee, M. Maiti and K. Sengupta, [arXiv:0704.2760] (unpublished).
16. A.G. Moghaddam and M. Zareyan, cond-mat/0611577 (unpublished).
17. H. Heersche et al., Nature 446, 56 (2006).
18. K.K. Likharev, Rev. Mod. Phys. 51, 101 (1979).
19. A.A. Golubov, M. Y. Kupryanov, and E. Il’ichev, Rev. Mod. Phys. 76, 411 (2004).
20. I.O. Kulik and A. Omelyanchuk, JETP Lett. 21, 96(1975); ibid Sov. Phys. JETP 41, 1071 (1975).
21. V. Ambegaokar and S. Baratoff, Phys. Rev. Lett. 10, 486, (1963).
22. A.M. Zagoskin Quantum Theory of Many Body Systems,
23 H.-J. Kown, K. Sengupta and V.M. Yakovenko, Eur. Phys. J. B 37, 349 (2004).
24 For short junctions ($d \ll \xi$), the main contribution to the Josephson current comes from the subgap states$^{12}$.
25 P.W. Anderson and J. Rowel, Phys. Rev. Lett. 10, 230 (1963).