Integral of Distance Function on Compact Riemannian Manifolds

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Abstract: In this paper we show that, under some curvature assumptions the integral of distance function on a compact Riemannian manifold is bounded below by the product of diameter, volume and a constant only depending on the dimension.

Keywords: Curvature, diameter, volume.

1. introduction

Let $M$ be a compact Riemannian manifold. Let $d(p,q)$ be the distance between points $p$ and $q$. If we fix a point $p \in M$, then we obtain a distance function $d(p,x), x \in M$. It is a continuous function and differentiable almost everywhere. The distance function plays a basic role in Riemannian geometry. In this paper we consider the integral of $d(p,x)$ on $M$. This gives a function

\[ f(p) = \int_M d(p,x)dv, p \in M. \]

Obviously it has an upper bound $d(M)V(M)$. Here $d(M)$ denotes the diameter of $M$ and $V(M)$ is the volume of $M$.

By the mean value theorem, for every $p \in M$ we can find a point $\xi_p \in M$ such that

\[ f(p) = d(p,\xi_p)V(M). \]

So we can ask a natural question: For any compact Riemannian manifold $M$ of dimension $n$, do we have

\[ f(p) \geq c(n)d(M)V(M) \]

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for all \( p \in M \)? The \( c(n) \) is a positive constant only depending on the dimension \( n \). Unfortunately, the answer is negative. In fact we can construct examples such that \( \frac{f(p)}{d(M)V(M)} \) is arbitrarily small for some point \( p \) (example 2.4). But if we add some curvature conditions, the answer is positive. The first result of this paper is

**Theorem 1.1.** Let \( M \) be an \( n \)-dimensional compact Riemannian manifold with nonnegatively Ricci curvature. Then

\[
f(p) > c(n)d(M)V(M)
\]

for all \( p \in M \). The \( c(n) \) can be chosen to equal \( (1 - \frac{1}{n+1})^n \cdot \frac{1}{2^{n+1}(n+1)} \).

A natural problem is to determine the sharp value for \( c(n) \). Examples in section 2 show that \( c(n) \) can achieve \( \frac{1}{2} \).

The Bishop-Gromov’s volume comparison plays an essential role in the proof of the theorem.

In last section we will prove two similar results on complete noncompact Riemannian manifolds.

## 2. examples and properties

Following the triangle inequality, one obviously has

- \( f(p) + f(q) \geq d(p,q)V(M) \),
- \( |f(p) - f(q)| \leq d(p,q)V(M) \).

Let \( p, q \in M \) satisfy \( d(p,q) = d(M) \). One can see that either \( f(p) \) or \( f(q) \) is greater than or equal to \( \frac{1}{2}d(M)V(M) \). So

\[
\max_{y \in M} f(y) \geq \frac{1}{2}d(M)V(M).
\]

We present three examples such that \( f(y) \geq \frac{1}{2}d(M)V(M) \) holds for all \( y \in M \).

**Example 2.1.** Let \( \gamma \) be a closed curve with length \( l \). \( d(\gamma) = \frac{l}{2} \) and \( V(\gamma) = l \). Then \( f = \frac{1}{2}d(\gamma)V(\gamma) \). In fact, if we assume that \( \gamma(t) \) has arc length parameter, then

\[
f = \int_0^{l/2} t\,dt + \int_{l/2}^l (l - t)\,dt = \frac{l^2}{4} = \frac{1}{2}d(\gamma)V(\gamma).
\]
Example 2.2. If $M$ is a compact symmetric space, then for any points $p, q \in M$ there exists an isometric mapping $p$ to $q$. Hence $f$ is a constant. Choose $p, q$ such that $d(p, q) = d(M)$. Then $2f = f(p) + f(q) \geq d(p, q)d(M)$. So we have

$$f \geq \frac{1}{2}d(M)V(M).$$

For a special case when $M$ is sphere space form $S^n_k$ ($k$ is the sectional curvature), let $p \in S^n_k$ and $q$ is the antipodal point of $p$. Then for any $x \in S^n_k$, $d(p, x) + d(q, x) = d(S^n_k)$. Hence we have

$$f = \frac{1}{2}d(S^n_k)V(S^n_k).$$

Example 2.3. Let $T^2$ be the flat 2-torus of area 1. Then

$$f = \int_0^\frac{1}{2} r \cdot 2\pi r dr + 4 \int_\frac{1}{2}^\frac{\sqrt{2}}{2} r \cdot \left(\frac{\pi}{2} - 2\arccos\frac{1}{2r}\right) r dr$$

$$= \frac{1}{6}(\sqrt{2} + \ln(\sqrt{2} + 1)) \approx 0.3826.$$

$$f > \frac{1}{2}diam(T^2)V(T^2) = \frac{\sqrt{2}}{4} \approx 0.3535.$$

However the following example shows that $\frac{f(y)}{d(M)V(M)}$ can achieves every value in $(0, 1)$.

Example 2.4. Let $M_1 = \{(x, y, z)|x^2 + y^2 + z^2 = 1, -1 + \varepsilon \leq z \leq 1\}$ and $M_2 = \{(x, y, z)|x^2 + y^2 = 2\varepsilon - \varepsilon^2, -1 + \varepsilon \leq z \leq -1 + \varepsilon\} \cup \{(x, y, z)|x^2 + y^2 \leq 2\varepsilon - \varepsilon^2, z = -1 + \varepsilon - L\}$. Let $M = M_1 \cup M_2$ be $M_1$ glued to $M_2$. We write $C = 2\pi \sqrt{2\varepsilon - \varepsilon^2}$. Let $p = (0, 0, 1)$ and $q = (0, 0, -1 + \varepsilon - L)$. When $C$ is very small,

$$f(p) = \int_{S_1^2} d(p, x) dS + \int_0^L (\pi + t) C dt$$

$$= \frac{1}{2} \pi V(S_1^2) + C(\pi L + \frac{L^2}{2})$$

and

$$d(M)V(M) = (L + \pi)(V(S_1^2) + LC).$$

Set $C = \frac{1}{L^2}$. We can see that $\frac{f(p)}{d(M)V(M)} \to 0$ as $L \to \infty$. We also have $\frac{f(q)}{d(M)V(M)} \to 1$ as $L \to \infty$. 
The following proposition is a consequence of Bishop-Gromov volume comparison.

**Proposition 2.5.** If \( \text{Ric}_M \geq (n-1)k > 0 \), then \( f \leq \frac{1}{2} d(S_k^n) V(S_k^n) \). The equality holds if and only if \( M \) is isometric to \( S_k^n \).

**Proof.** By the Fubini theorem and Bishop-Gromov volume comparison,

\[
\begin{align*}
  f(p) &= \int_M d(p, x) dv = \int_{-\infty}^{+\infty} d\lambda \int_{S_{\lambda}} d(p, x) dv_{\lambda} \\
  &= \int_0^{d(M)} d\lambda \int_{S_{\lambda}} \lambda dv_{\lambda} = \int_0^{d(M)} \lambda V(S_{\lambda}) d\lambda \\
  &\leq \int_0^{d(S_k^n)} \lambda V(S_{k\lambda}) d\lambda \\
  &= \frac{1}{2} d(S_k^n) V(S_k^n).
\end{align*}
\]

The \( S_\lambda \) denotes the sphere center at \( p \) with radius \( \lambda \) and \( V(S_\lambda) \) is the induced volume of \( S_\lambda \). If the equality holds, we have \( d(M) = d(S_k^n) \) and \( V(S_{\lambda}) = V(S_{k\lambda}) \). So \( M \) must be isometric to \( S_k^n \). \( \square \)

### 3. A proof of theorem 1.1

Let \( B_p(r) \) (respectively \( B_o(r) \) ) denote the ball center at \( p \) of radius \( r \) in \( M \) (respectively ball center at origin of radius \( r \) in \( \mathbb{R}^n \)). The \( V_p(r) \) (respectively \( V_o(r) \)) denotes the volume of \( B_p(r) \) (respectively \( B_o(r) \)).

**Lemma 3.1.** For any \( p \in M \), we have \( \max_{x \in M} d(p, x) \geq \frac{1}{2} d(M) \).

**Proof.** If on the contrary, \( \max_{x \in M} d(p, x) < \frac{1}{2} d(M) \). Choosing \( q_1, q_2 \in M \) such that \( d(q_1, q_2) = d(M) \), thus

\[
  d(M) \leq d(p, q_1) + d(p, q_2) < \frac{1}{2} d(M) + \frac{1}{2} d(M) = d(M).
\]

This is a contradiction. \( \square \)

Because the Ricci curvature of \( M \) is nonnegative. The Bishop-Gromov’s volume comparison implies that

\[
\frac{V_p(r)}{V_o(r)} \geq \frac{V_p(R)}{V_o(R)}
\]
Integral of Distance Function on Compact Riemannian Manifolds

for $r \leq R$. Hence

$$V_p(r) \geq \frac{V_o(r)}{V_o(R)} V_p(R) = \frac{r^n}{R^n} V_p(R).$$

Let $R \to d = d(M)$. We get

$$V_p(r) \geq \frac{r^n}{d^n} V(M)$$

for all $r \leq d$.

By the lemma, for any $p \in M$, we can choose $q \in M$ such that $d(p, q) \geq \frac{1}{2} d(M)$. Thus

$$f(p) = \int_{B_p(\frac{1}{2}d - r)} d(p, x) dv + \int_{M \setminus B_p(\frac{1}{2}d - r)} d(p, x) dv$$

$$> \int_{M \setminus B_p(\frac{1}{2}d - r)} d(p, x) dv$$

$$> (\frac{1}{2}d - r)V(M \setminus B_p(\frac{1}{2}d - r))$$

$$> (\frac{1}{2}d - r)V_q(r)$$

$$\geq V(M)(\frac{1}{2}d - r)^\frac{r^n}{d^n}.$$

Let $g(r) = (\frac{1}{2}d - r)^\frac{r^n}{d^n}, 0 \leq r \leq \frac{1}{2}d$. When

$$g'(r) = \frac{1}{d^n}\left[\frac{nd}{2} r^{n-1} - (n+1)r^n\right] = 0,$$

$$r = \frac{n}{2(n+1)} d, g(r)$$

achieves the maximal value $(1 - \frac{1}{n+1})^n \cdot \frac{1}{2^{n+1}(n+1)} d$. Hence we get

$$f(p) > (1 - \frac{1}{n+1})^n \cdot \frac{1}{2^{n+1}(n+1)} d(M)V(M).$$

4. noncompact analogues of theorem 1.1

In this section we consider the noncompact version of theorem 1.1. Let $M$ be a complete noncompact Riemannian manifold of dimension $n$. For a point
\( p \in M \) and \( d > 0 \), we write
\[
f(p, d) = \int_{B_p(d)} d(p, x)dv\]

Then we have

**Theorem 4.1.** If \( M \) is a Cartan-Hadamard manifold, then
\[
f(p, d) > \frac{n}{n+1} \cdot \frac{1}{\sqrt{n+1}} dV_p(d),
\]
for any \( p \in M \) and all \( d > 0 \).

Note that \( \frac{n}{n+1} \cdot \frac{1}{\sqrt{n+1}} \) tends to 1 as \( n \) goes to \(+\infty\). On the other hand, we always have \( f(p, d) < dV_p(d) \). So theorem 4.1 is more or less surprise.

**Proof.** Since the sectional curvature of \( M \) is nonpositive and \( M \) has no cut point. By the Bishop-Gromov’s volume comparison (c.f. [1] page 169), one has
\[
\frac{V_p(r)}{V_o(r)} \leq \frac{V_p(d)}{V_o(d)}
\]
for \( r \leq d \). Hence
\[
V_p(r) \leq \frac{V_o(r)}{V_o(d)} V_p(d) = \frac{r^n}{d^n} V_p(d).
\]

We estimate the lower bound of \( f \).
\[
f(p, d) = \int_{B_p(r)} d(p, x)dv + \int_{B_p(d) \setminus B_p(r)} d(p, x)dv
\geq \int_{B_p(d) \setminus B_p(r)} d(p, x)dv
\geq rV(B_p(d) \setminus B_p(r)) = r(V_p(d) - V_p(r))
\geq V_p(d)(1 - \frac{r^n}{d^n}).
\]

Let \( g(r) = r(1 - \frac{r^n}{d^n}) \), \( 0 \leq r \leq d \). When
\[
g'(r) = 1 - (n+1)\frac{r^n}{d^n} = 0,
\]
\[r = \frac{d}{\sqrt{n+1}}, \text{ } g(r) \text{ achieves the maximal value} \frac{n}{n+1} \cdot \frac{1}{\sqrt{n+1}} \cdot d.\] So we have
\[
f(p, d) > \frac{n}{n+1} \cdot \frac{1}{\sqrt{n+1}} dV_p(d).
\]
Theorem 4.2. If the Ricci curvature of \( M \) is nonnegative, then

\[
f(p, d) > c(n) d V_p(d),
\]

for any \( p \in M \) and all \( d > 0 \). The concrete value of \( c(n) \) is given in the following proof.

The constant \( c(n) \) is different to the one in Theorem 1.1.

Proof. Let \( 0 \leq t \leq \frac{d}{2} \), \( q \) is a point satisfying \( d(p, q) = d - t \). Then

\[
f(p, d) = \int_{B_p(d-2t)} d(p, x) dv + \int_{B_p(d) \setminus B_p(d-2t)} d(p, x) dv
\]
\[
> \int_{B_p(d) \setminus B_p(d-2t)} d(p, x) dv
\]
\[
> (d - 2t) V(B_p(d) \setminus B_p(d - 2t))
\]
\[
> (d - 2t) V_q(t)
\]
\[
\geq (d - 2t) \frac{t^n}{(2d - t)^n} V_q(2d - t)
\]
\[
> (d - 2t) \frac{t^n}{(2d - t)^n} V_p(d).
\]

Since \( B_q(t) \subset B_p(d) \setminus B_p(d - 2t) \), the third “\( > \)” holds. The last “\( > \)” follows from \( B_q(2d - t) \supset B_p(d) \). Let \( g(t) = (d - 2t) \frac{t^n}{(2d - t)^n}, 0 \leq t \leq \frac{d}{2} \). When

\[
g'(t) = -2(\frac{t}{2d - t})^n + (d - 2t)n(\frac{t}{2d - t})^{n-1} \frac{2d}{(2d - t)^2} = 0,
\]

namely,

\[
t^2 - 2d(n + 1)t + nd^2 = 0,
\]

\( t = (n + 1 - \sqrt{n^2 + n + 1})d \). \( g(t) \) achieves the maximal value

\[
\frac{3}{2\sqrt{n^2 + n + 1} + 2n + 1} \left( \frac{n}{n + 2 + 2\sqrt{n^2 + n + 1}} \right)^n d.
\]

Choose

\[
c(n) = \frac{3}{2\sqrt{n^2 + n + 1} + 2n + 1} \left( \frac{n}{n + 2 + 2\sqrt{n^2 + n + 1}} \right)^n.
\]
We obtain
\[ f(p, d) \geq c(n)dV_p(d). \]
\[ \square \]

Recall a well-known theorem of Calabi and Yau [2]: Let \( M \) be a complete noncompact Riemannian manifold with nonnegative Ricci curvature. For any \( p \in M \), we have \( V_p(r) \geq c(n, p)r \). Consequently \( M \) has infinite volume. Combining this result with Theorem 4.2, we obtain

**Corollary 4.3.** Let \( M \) be a complete noncompact Riemannian manifold with nonnegative Ricci curvature. Then

\[
\lim_{r \to \infty} \int_{B_p(r)} \frac{d(p, x)}{r} dv = +\infty,
\]

for all \( p \in M \).

**References**

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