A Proposal for $\aleph_0$ Extended Supersymmetry in Integrable Systems

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ABSTRACT

Utilizing techniques suggested by the recently obtained construction of off-shell spinning particles, we propose the arbitrary $N$-extension of supersymmetry for the KdV system. It is further suggested that the $\aleph_0$ extension for the SKdV system provides a paradigm for all supersymmetric completely integrable systems.

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1 Introduction

The topic of integrable or completely solvable systems is one with a long history having perhaps its best known origin in an observation of John Scott Russell in 1834 [1]. Almost sixty years later the mathematical setting for this class of theories was established [2]. Finally one-hundred and fifty years later, explorations of the connection between supersymmetry and integrable systems began in earnest [3, 4]. The topic of integrable systems has also been found to coalesce with relativistic particle and spinning particles [5] in an unexpected way wherein the KdV and SKdV Lax operators can be found by use of an appropriate set of variables [6]. Due to this last observation, advances in our understanding of the spinning particle quite naturally should have consequences for our understanding of SKdV systems. Along this line of thought, we have recently been able to give, for the first time, an off-shell description of the spinning particle for arbitrary $N$, the degree of the supersymmetry extension. Since $N$ is an arbitrarily large integer, the set of all such integers constitutes a representation of $\aleph_0$, the “smallest” transfinite number.

In the present brief note, we wish to show that the off-shell momentum multiplet of the $\aleph_0$ supersymmetric spinning particle appears to provide the fundamental supersymmetric representation for the construction of the supersymmetric extension of the KdV equation. We show that the well known cases of the $N = 1$ and $N = 2$ theories are “naturally” embedded in a simple algebraic structure. Extending this embedding to the entire structure suggests a form for the supersymmetric extension for arbitrary values of $N$. We discuss the cases of $N = 3, 4$ and compare to the suggestion of Delduc and Ivanov [8] for the SKdV system. We end our letter with a conjecture that the structure we have found is universal for all supersymmetric integrable systems.

2 A Universal Supersymmetry Representation for Integrable Systems

In a related work [9], we have shown that associated with the off-shell spinning particle coordinate there is a “momentum multiplet.” One such multiplet occurs for each coordinate of the spinning particle. After a certain transformation, the component fields $(w_i^j(x,t), \xi^I(x,t), \xi^i(x,t), u(x,t))$ of the multiplet have supersymmetry.

\footnote{In a somewhat jocular way, we may say this was, perhaps the first experimental observation of supersymmetry in Nature.}
variations given by
\[
\delta_Q w^i_j = -i 2 \alpha_1 \left[ (f_{1i})_i^j \xi_1 + (R_1)_k^j \xi_i^k \right],
\]
\[
\delta_Q \xi_1 = \alpha_1 u + d^{-1} \alpha_1 (f_{1i})_j^i \partial_x w^i_j,
\]
\[
\delta_Q \xi_i^k = -\alpha_1 \left[ (L_1)_j^k \partial_x w^i_j + d^{-1} (L_1)_i^k (f_{1i})_j^l \partial_x w^j_l \right],
\]
\[
\delta_Q u = -i 2 \alpha_1 \partial_x \xi_1.
\] (2.1)

here \( d = 2 \), \( w^i_j = (R_1)_k^i \xi_i^k = 0 \). (See [4] for notational conventions as well as the appendix.) For our purposes, it is also useful to introduce the following decomposition
\[
\text{where upon the variations in (2.2) take the forms,}
\]
\[
\delta_Q B^j_i = \epsilon^{a_1} \left[ (f_{1i})_i^j \lambda_{a_1} + (L_1)_k^j \tilde{\lambda}_{a_1}^k \right],
\]
\[
\delta_Q \lambda_{a_1} = i \epsilon^{a_1} \left[ (R_1)_j^i \gamma^a_{a_1} \left[ \partial_x B^j_i + \frac{1}{2} d^{-1} \delta_j^k \epsilon_{abc} F_{bc} \right] \right],
\]
\[
\delta_Q A_a = i \epsilon^{a_1} \left[ (L_1)_k^j \lambda_{a_1}^{a_1} \lambda_{a_1}^k \right].
\] (2.2)

(where \( B^i_0 = 0 \)), as a starting point. We next separate the gaugini according to the definition
\[
\lambda_{a_1}^i = d^{-1} \left[ (R_1)_k^i \lambda_1 + \tilde{\lambda}_{a_1}^i \right], \quad (L_1)_k^i \tilde{\lambda}_{a_1}^i = 0.
\] (2.3)

where upon the variations in (2.2) take the forms,
\[
\delta_Q B^j_i = \epsilon^{a_1} \left[ (f_{1i})_i^j \lambda_{a_1} + (L_1)_i^k \tilde{\lambda}_{a_1}^j \right],
\]
\[
\delta_Q \lambda_{a_1} = i \epsilon^{a_1} \left[ (R_1)_j^i \gamma^a_{a_1} \left[ \partial_x \lambda_{a_1}^j \right] \right],
\]
\[
\delta_Q \tilde{\lambda}_{a_1}^k = i \epsilon^{a_1} \left[ (R_1)_j^i \left( \partial_x B^j_i \right) - d^{-1} (R_1)_k^l (f_{1i})_l^j (\partial_x B^j_i) \right],
\]
\[
\delta_Q A_a = -i \epsilon^{a_1} \left( \gamma^a_{a_1} \lambda_{a_1}^j \right) \rightarrow \delta_Q \left( \epsilon_{abc} F^{bc} \right) = -i 2 \epsilon^{a_1} \epsilon_{abc} \gamma^b_{a_1} \partial_x \lambda_{a_1}^j
\] (2.4)

after rescaling \( B^j_i \rightarrow d^{-1} B^j_i \). Next we perform a reduction from 3D to 1D defined by
\[
\partial_x \rightarrow (0, \partial_x, 0), \quad A_a(t, x, y) \rightarrow (0, 0, A_y(x)), \quad \epsilon_{abc} F^{bc} \rightarrow 2 (\partial_x A_y, 0, 0),
\] (2.5)

and demand the consistency of the condition \( \delta_Q A_t = \delta_Q A_x = 0 \). These consistency conditions lead to 1D spinors (i.e. solutions take the forms \( \lambda_{a_1}^1 = au_{a_1}(-)s^1 \), \( \lambda_{a_1}^k = bu_{a_1}(+)s^k \) where \( u_{a_1}(\pm) \) are the eigenspinors for \( (\gamma_y)_{a_1}^\beta \) and \( a \) and \( b \) are constants).
that when substituted back into (2.4) yield a set of transformations equivalent to (2.1) with the mappings \( \partial_x A_y \rightarrow u, B^j_i \rightarrow w^j_i, \lambda_{\alpha I} \rightarrow \xi_I \) and \( \hat{\lambda}_{\alpha k}^i \rightarrow \xi_i^k \). So both the \( S_0 \) spinning particle as well as the 3D \( N_0 \) supersymmetric abelian Yang-Mills multiplet lead to the same result.

In order to see how (2.1) is related to the standard known constructions of \( N = 1 \) and \( N = 2 \) SKdV [4], we define (L\(_I^L_i^j \) = \( -I, i\sigma^2 \)), (R\(_I^R_i^j \) = \( I, i\sigma^2 \)) and (f\(_I^F_j^i \) = \( -i \epsilon_{1j} (\sigma^2)^j_i \)) where \( (\sigma^2)^j_i \) denotes the usual second \( 2 \times 2 \) Pauli matrix. From the constraints on \( w^j_i \) and \( \xi_i^k \) it follows that

\[
\begin{align*}
  w^j_i &= w (i\sigma^2)^j_i + \hat{w}^j_i, \\
  \xi_i^k &= \xi_i^1 (\sigma^3)^k_i + \hat{\xi}_i^k.
\end{align*}
\]

In general the off-shell representation in (2.1) is reducible. This feature is seen by writing out the supersymmetry variations of (2.1) in terms of the variables defined in (2.6). A primary irreducible submultiplet is provided by \( (w, \xi_I, u) \) and a secondary irreducible submultiplet is provided by \( (\hat{w}_i^j, \hat{w}^j_i, \xi_I^1, \xi_I^3) \). They have the respective transformations laws,

\[
\begin{align*}
  \delta Q w &= i2 \alpha_1 \epsilon_{1j} \xi_I, \\
  \delta Q \xi_I &= \alpha_1 u + \alpha_1 \epsilon_{1j} \partial_x w, \\
  \delta Q u &= -i2 \alpha_1 \partial_x \xi_I.
\end{align*}
\]

and separately the transformations laws,

\[
\begin{align*}
  \delta Q \hat{w}_i^j &= -i2 \alpha_1 (R^R_i^j)^k \xi_i^k, \\
  \delta Q \xi_i^k &= -\alpha_1 [ (L^L_i^j)^k \partial_x \hat{w}_i^j + d^{-1} (L^L_i^j)^k (f^F_j^i)^l \partial_x \hat{w}_i^j ].
\end{align*}
\]

The reader familiar with the literature of the SKdV equation will immediately recognize (2.7) as a form of the supersymmetric multiplet that is known to occur in SKdV theories. In order to impose the condition that the multiplet of (2.1) should obey the dynamics of the SKdV equations, we must impose two conditions

\[
\begin{align*}
  0 &= (f^F_j^i)^j \partial_t w^j_i + \partial_x^2 w^j_i + 6d^{-1} w^l_i \partial_x k \partial_x w^j_i - 3 \partial_x (uw^j_i), \\
  0 &= \hat{w}^j_i.
\end{align*}
\]

The first supersymmetry variation of these yield,

\[
\begin{align*}
  0 &= \partial_t \xi_I + \partial_x^2 \xi_I - 6\partial_x (w^2 \xi_I) - 3\partial_x (u \xi_I) - 3\epsilon_{1j} \partial_x (w \partial_x \xi_I), \\
  0 &= \xi_i^k.
\end{align*}
\]
Next and finally the second supersymmetry variation yields

\[ 0 = \partial_t u + \partial_x^3 u - 6u\partial_x u - 6\partial_x(uw^2) + 3\partial_x(w\partial_x^2 w) \]
\[ - i6\partial_x(\xi_1\partial_x\xi_1) + i12\epsilon_{1,1}\partial_x(w\xi_1\xi_1), \]
\[ 0 = \partial_x\hat{w}_i^k. \]  

(2.11)

The second equation of (2.11) is already satisfied due to the second equation of (2.9). The first equation of (2.11) is just the supersymmetrically extended version of the KdV equation.

The equations (2.9), (2.10) and (2.11) can be summarized very succinctly in terms of superfields. Let \( A_{IJ} \) be defined by the rhs of the first line of (2.9). Similarly introduce a superfield \( \Omega^{ij} \) whose lowest component corresponds to the second line of (2.9). Introduce a superspace covariant derivative denoted by \( D_I \). The first lines of (2.9), (2.10) and (2.11) then take the respective forms \( A_{IJ} = 0, D_J A_{IJ} = 0 \) and \( D_I D_J A_{IJ} = 0 \). The second lines of (2.9), (2.10) and (2.11) then take the respective forms \( \Omega_i^j = 0, D_J \Omega_i^j = 0 \) and \( D_I D_J \Omega_i^j = 0 \).

### 3 \( \aleph_0 \) Supersymmetry and the Korteweg de Vries Equation

It may seem that so far, all we have done is a recapitulation of the standard and well established SKdV system. In fact, we have much much more. This is implicit in the seemingly strange notation in which we cast our beginning. The point is that the form of the 2 × 2 Pauli matrices is dictated if we view these as a representation of a general algebraic structure that we denote by \( \mathcal{GR}(d, N) \) (dimension \( d \) and rank \( N \)). Any matrix representation consists of \( N \) linearly independent, \( d \times d \), real matrices (denoted by \( L_I \)) that satisfy a general real (\( \equiv \mathcal{GR} \)) Pauli algebra

\[ L_I R_J + L_J R_I = -2\delta_{IJ}I, \quad R_I L_J + R_J L_I = -2\delta_{IJ}I, \]
\[ (3.1) \]

where the \( (R_I) \) matrices are defined by \( (L_I)_{i\hat{k}} + (R_I)_{\hat{k}i} = 0, I = 1, \ldots, N \) and \( i, \hat{k} = 1, \ldots, d \). We emphasize that our L and R matrices are to be manipulated using Van der Waerden techniques. For our later convenience, we define the “complex structure”

\[ ^3 \]It is interesting to note that the first equation of (2.9) corresponds to the choice \( a = 1 \) of reference [4]. This is the only value of this parameter that is consistent with the matrix structure of (2.1)
matrices associated with the $\mathcal{GR}(d, N)$ algebras by $(f_{IJ})_{i}^{j} \equiv \frac{1}{2}(L_{I}R_{J} - L_{J}R_{I})_{i}^{j}$ and $(\tilde{f}_{IJ})_{k}^{i} \equiv \frac{1}{2}(R_{I}L_{J} - R_{J}L_{I})_{k}^{i}$.

An explicit representation of these $\mathcal{GR}(d, N)$ algebras can be found in our works [7, 9]. The first discussion of this type of real Clifford algebra was given by Okubo [10] and this algebraic structure has also been noted in a study of 3D non-linear $\sigma$-models [11].

One additional algebraic structure that we find useful to introduce is what we call $UGR$ defined by

$$UGR = \sum_{N} \oplus \mathcal{GR}(N). \quad (3.2)$$

It is a simple matter to show that the transformations in (2.1) close uniformly on all of the fields

$$[ \delta_{Q}(\alpha_{1}), \delta_{Q}(\alpha_{2}) ] = i 4 \alpha_{1}^{I} \alpha_{2}^{I} \partial_{\tau}, \quad (3.3)$$

without the use of equations of motion. This is a consequence of (3.1). However, the more useful observation is that the transformations in (2.1) are “$UGR$-covariant.” By this we mean that for each value of $N$ there exist $d \times d$ matrices contained in $UGR$ for which the algebra in (3.3) can be shown to close. Furthermore, the equations of (2.9) are also $UGR$-covariant. Thus, we may say that (2.1) together with (2.9) defines an $\aleph_{0}$ supersymmetric extension of the known SKdV equations!

4 Proposed Extensions for $N = 3, 4$ SKdV

In order to demonstrate the significance of the statements at the end of the last section, we believe it is useful to explicitly show what the $UGR$-covariant formalism suggests as $N = 3, 4$ SKdV theories. We begin with the $N = 3$ theory ($d = 4$) where the $G\mathcal{R}$-Van der Waerden (1,1) tensors are denoted by

$$(I)_{ik}, \quad (L_{I})_{ik}, \quad (E_{I})_{ik}, \quad (L_{I}E_{J})_{ik}, \quad (4.1)$$

and have multiplicities $1 + 3 + 3 + 9$. The (2,0) $G\mathcal{R}$-Van der Waerden tensors have the same multiplicities and are denoted by

$$(I)_{ij}, \quad (f_{I})_{ij}, \quad (F_{I})_{ij}, \quad (f_{I}F_{J})_{ij}, \quad (4.2)$$
where \((E_1)_{ik}(E_4)_{jk} = \delta_{ij} \delta_{ik} + \epsilon_{ijk} (F_K)_{ij}\). The \(N = 3\) quantities \(w^i_j\) and \(\xi^i_j\) take the forms

\[
\begin{align*}
w^i_j &= w^1 (f_1)_{ij}^* + \tilde{w}^{I} (F_1)_{ij}^* + \tilde{w}^{IJ} (f_1 F_4)_{ij}^* , \\
\equiv w^1 (f_1)_{ij}^* + \tilde{w}^{i j} , \\
\xi^i_j &= \xi \hat{\delta}^i_j + \hat{\xi}^I (E_1)_{ik}^* + \hat{\xi}^{IJ} (L_4 E_4)_{ij}^* .
\end{align*}
\]

(4.3)

For the \(N = 4\) theory \((d = 4)\) we must introduce two types of indices \(I = 1, 2, 3, 4\) and \(\tilde{I} = 1, 2, 3, 4\). The \(\mathcal{GR}-\text{Van der Waerden} \) tensors are denoted by

\[
(L_1)_{ik} , \quad (E_1)_{ik} , \quad (f_1 F_4)_{ik} ,
\]

(4.4)

and have multiplicities \(4 + 3 + 9\). The \(\mathcal{GR}-\text{Van der Waerden} \) tensors have the multiplicities (the \((0,2)\) tensors have the same decomposition) \(1 + 3 + 3 + 9\) and are denoted by

\[
(I)_{ij} , \quad (f_1)_{ij} , \quad (F_4)_{ij} , \quad (f_1 F_4)_{ij} .
\]

(4.5)

The tensorial quantities in (4.5) satisfy antisymmetry and self-duality conditions

\[
(f_{11}) = -(f_{11}) , \quad (f_{11} F_4) = -(f_{11} F_4) ,
\]

\[
(f_{11}) = \frac{1}{2} \epsilon_{ijkl} (f_{kl}) , \quad (f_{11} F_4) = \frac{1}{2} \epsilon_{ijkl} (f_{kl} F_4) .
\]

(4.6)

The \(N = 4\) quantities \(w^i_j\) and \(\xi^i_j\) take the forms

\[
\begin{align*}
w^i_j &= w^{1J} (f_{11})_{ij}^* + \tilde{w}^{\tilde{K}} (F_4)_{ij}^* + \tilde{w}^{1\tilde{K}} (f_{11} F_4)_{ij}^* , \\
\equiv w^{1J} (f_{11})_{ij}^* + \tilde{w}^{i j} , \\
\xi^i_j &= \hat{\xi}^I (E_1)_{ik}^* + \hat{\xi}^{1\tilde{K}} (f_{11} E_4)_{ij}^* .
\end{align*}
\]

(4.7)

At this stage, there are some fundamental differences between (4.3) and (4.7) that are very important. In the \(N = 3\) case, we note that the number of components of \(\tilde{w}^i_j\) is equal to the sum of the numbers of \(\hat{\xi}^1\) and \(\hat{\xi}^{1J}\). This is a signal that an off-shell \(N = 3\) theory occurs if we only retain \(w^1\) and \(\xi^i_j\). The primary \(N = 3\) irreducible off-shell submultiplet consists of \((w_1, \xi_1, \xi, u)\). This is very different from the \(N = 4\) case. There we see that the number of components of \(\tilde{w}^i_j\) is equal to the number of components of \(\xi^i_j\). In the \(N = 4\) case an off-shell formulation occurs if we retain only \(w^{1J}\). The primary \(N = 4\) irreducible off-shell multiplet consists of \((w_{11}, \xi_1, u)\). The primary \(N = 3\) and \(N = 4\) submultiplets have exactly the same number of fields and with their respective transformation laws derived from (2.1) as

\[
\begin{align*}
\delta_Q w_1 &= i 2 \alpha_1 \xi - i 2 \epsilon_{1JK} \alpha_j \xi_K , \quad \delta_Q \xi &= - \alpha_1 \partial_x w_1 , \\
\delta_Q \xi_I &= \alpha_I u - \epsilon_{1JK} \alpha_j \partial_x w_K , \quad \delta_Q u &= - i 2 \alpha_1 \partial_x \xi_I ,
\end{align*}
\]

(4.8)
for $N = 3$ theory and as

$$
\delta_Q w_{1J} = i2\alpha[\xi_I] + i2\epsilon_{1JKL}\alpha_K\xi_L ,
$$

$$
\delta_Q \xi_I = \alpha_I u - \alpha_J\partial_x w_{1J} ,
$$

$$
\delta_Q u = -i2\alpha_I\partial_x\xi_I .
$$

(4.9)

for $N = 4$ theory.

Now we are exactly in the same position as with the $N = 2$ theory. We begin again with (2.9). Applying one supersymmetry variation leads to the $N = 3, 4$ analog of (2.10). Applying a second supersymmetry variation leads to the $N = 3, 4$ analog of (2.11). However, it is amusing to note that our proposal for the $N = 4$ SKdV equation is almost identical in form to the $N = 2$ theory,

$$
0 = \partial_t u + \partial^3_x u - 6u\partial_x u - 3\partial_x( uw_{1J}w_{1J} ) + \frac{3}{2}\partial_x( w_{1J}\partial^2_x w_{1J} )
$$

$$
- i 6\partial_x( \xi_I\partial_x\xi_I ) + i 12\partial_x( w_{1J}\xi_I\xi_J ) .
$$

(4.10)

We caution the reader that the cases of $N \leq 4$ are the exception rather than the rule. In each of these cases, we are able to formulate the theory solely in terms of a primary submultiplet. We are able to set the secondary submultiplet to zero as a constraint (as opposed to an equation of motion) without disturbing the off-shell supersymmetry of the primary submultiplet. For general values of $N$ this is not the case and the treatment of the secondary submultiplet must handled carefully. The simplest way to proceed is to impose the first equation in (2.9) without taking the “trace” with the $f$-tensor and not use the second equation. Under this circumstance we are guaranteed to find a manifestly off-shell supersymmetric system that includes the KdV equation for all values of $N$.

The suggestion that the KdV equation admits $N = 3, 4$ supersymmetric extensions was first made in reference [8] based on the use of harmonic superspace and superconformal algebras. It is therefore useful to make some comparisons. Foremost, since the off-shell structure of our formulation in (2.1) is determined from representations of the $\mathcal{GR}(d, N)$ algebras, we begin with a finite set of auxiliary fields compared to the infinite set required by harmonic superspace. For the $N = 3$ case we seem to be in general agreement with regard to the on-shell theory. In particular, our $\mathcal{UGR}$-covariant formalism picks the $a = 1$ theory upon reduction to $N = 2$. For the $N = 4$ case we again agree with the previous results in terms of spectrum. However, after reduction to $N = 2$ we find only the $a = 1$ theory whereas the most recent results of Delduc et. al. seem to suggest that the $a = 2, 4$ cases are preferred. The source of this disagreement is at present unclear.
5 \textbf{UGR}-covariant Lax Operator

As discussed previously by Ramos and Roca, there is a relation between spinning particles and the Lax operator. In the following we first review this observation briefly and make some modifications that will be convenient later in this section.

The action for the ordinary massless relativistic particle is well known to be described by an action that contains an einbein ($e$), momentum ($P$) and coordinate ($X$),

$$\mathcal{L} = -\frac{1}{2} e^{-1} P^2 + P \partial_r X ,$$

whose equations of motion follow from the calculus of variations as

$$P^2 = 0 , \quad P = e (\partial_r X) , \quad \partial_r P = 0 .$$

Now motivated by the work of Ramos and Roca, let us perform the change of variable described by

$$\tilde{X} \equiv e^{\frac{1}{2}} X , \quad \tilde{P} \equiv e^{-\frac{1}{2}} P + \frac{1}{2} e^{\frac{1}{2}} (\partial_r \ln e) X ,$$

and concentrate on the latter two equations in (5.2). These become

$$\tilde{P} = (\partial_r \tilde{X}) , \quad \partial_r \tilde{P} = -\mathcal{U}[e] \tilde{X} ,$$

where $\mathcal{U}[e]$ is defined by

$$\mathcal{U}[e] \equiv -\frac{1}{2} \left[ (\partial_r^2 \ln e) + \frac{1}{2} (\partial_r \ln e)^2 \right] .$$

This quantity has a number of interesting properties including

$$\mathcal{U}[J^{-1}] = \left( \frac{f''}{f'} \right) - \frac{3}{2} \left( \frac{f''}{f'} \right)^2 \equiv S(f) ,$$

where $J \equiv \partial_r f$ is the Jacobian of the coordinate transformation $\tau \to f(\tau) \equiv \exp(K) \tau$, $K = K^r \partial_r$ and $S(f)$ is the Schwartzian derivative. Also under a scale transformation of the einbein $e \to e \exp \lambda(\tau)$ (with $\lambda(\tau)$ an arbitrary function) we see

$$\mathcal{U}[e \exp \lambda(\tau)] = \mathcal{U}[e] - \frac{1}{2} \left( \nabla^2 \lambda \right) - \frac{1}{4} \left( \nabla \lambda \right)^2 ,$$

where

$$\nabla \lambda \equiv \partial_r \lambda , \quad \nabla^2 \lambda \equiv (\partial_r + (\partial_r \ln e)) \partial_r \lambda .$$

Since $\mathcal{U}(1) = 0$, it follows that we can also write $\mathcal{U}(e) = -\frac{1}{2} \left( \nabla^2 \ln e \right) + \frac{1}{4} \left( \nabla \ln e \right)^2$.

Clearly, the equations of (5.4) for $\tilde{X}$ and $\tilde{P}$ are derivable from the Hamiltonian

$$\mathcal{H} = \frac{1}{2} [ \tilde{P}^2 + \mathcal{U}[e] \tilde{X}^2 ] .$$
by use of a standard Poisson bracket. Finally the second order operator form of (5.4) becomes

\[
\left[ \partial_{\tau}^2 + \mathcal{U}[e] \right] \tilde{X} = 0.
\]  

(5.9)

and this is the Lax operator (after we switch \( \tau \rightarrow x \)). In [6] this argument has been extended to the case of the \( N = 1 \) spinning particle and the \( N = 1 \) SKdV system. We should be able to find a \( \mathcal{UGR} \)-covariant formulation by embedding the component results into superfield equations involving the spinning particle superfields in a manner that is independent of \( N \) and make the switch \( \tau \rightarrow x \) at the end.

We can easily embed these results into superfield equations. For example, (5.3) can be seen to occur as components of the equations

\[
\tilde{\Pi}_1 = E^{-\frac{1}{2}} \Pi_1 + i \frac{1}{2} E^{\frac{1}{2}} \left( D_1 \ln E \right) X , \quad \tilde{X} = E^{\frac{1}{2}} X .
\]  

(5.10)

Here \( X \) and \( \Pi_1 \) denote superfields whose component formulation is described in reference [9] as well as in the appendix. The spinorial derivative \( D_1 \) due to (3.1) satisfies

\[
\left[ D_1 , D_J \right] = -i 4 \delta_{1J} \partial_{\tau} ,
\]  

(5.11)

and \( E \) denotes the superdeterminant of the 1D supergravity vielbein that satisfies,

\[
\left[ E_1 , E_J \right] = -i 4 \delta_{1J} E_{\tau} , \quad \left[ E_1 , E_{\tau} \right] = 0 .
\]  

(5.12)

Similarly, the results in (5.4) can be embedded into the following superfield equations

\[
\tilde{\Pi}_1 = i \frac{1}{2} D_1 \tilde{X} , \quad \partial_{\tau} \tilde{\Pi}_1 = -i \frac{1}{2} \mathcal{U}_1 [E] \tilde{X} ,
\]

\[
\mathcal{U}_1 [E] \equiv - \frac{1}{4} \left[ (\partial_{\tau} D_1 \ln E) + \frac{1}{2} (D_1 \ln E) (\partial_{\tau} \ln E) \right] .
\]  

(5.13)

Not surprisingly we find

\[
\mathcal{U}_1 [E \exp \Lambda] = \mathcal{U}_1 [E] - \frac{1}{4} \left[ (\nabla_{\tau} D_1 \Lambda) + (\nabla_1 \partial_{\tau} \Lambda) + (D_1 \Lambda) (\partial_{\tau} \Lambda) \right] ,
\]

\[
(\nabla_{\tau} D_1 \Lambda) \equiv \left[ \partial_{\tau} + (\partial_{\tau} \ln E) \right] (D_1 \Lambda) , \quad (\nabla_1 \partial_{\tau} \Lambda) \equiv \left[ D_1 + (D_1 \ln E) \right] (\partial_{\tau} \Lambda) ,
\]

\[
S(K) \equiv \mathcal{U}_1 [J^{-1}] , \quad J \equiv (1 \cdot e^K) , \quad K \equiv K^1 D_1 + K^\tau \partial_{\tau} ,
\]  

(5.14)

where on the last line above we have expressed the super-Schwartzian in terms of the super-Jacobian of the coordinate transformation induced by the exponentiation of the super-vector field \( K \) (i.e. the transformation \( (\zeta^1, \tau) \rightarrow e^K (\zeta^1, \tau) \)).

Combining the first two equations of (5.16) we obtain

\[
\left\{ \partial_{\tau} D_1 + \mathcal{U}_1 [E] \right\} \tilde{X} = 0 ,
\]  

(5.15)
as the \textit{UGR}-covariant generalization of (5.9). For the case of $N = 1$, the operator in this equation is precisely the super-Lax operator of reference \cite{6}. Since this last equation is \textit{UGR}-covariant, we propose that its interpretation as the super-Lax operator should extend to all $N$.

\section{Conclusion}

One of the interesting points regarding supersymmetric systems is the proposal of De Crombrugghe and Rittenberg that states that all supersymmetric systems with $N > 4$ supersymmetry must necessarily be integrable systems. With this as a background it is not surprising that our proposal for the $\aleph_0$ supersymmetric extension of the KdV equation should be made. However, we emphasize that we have \underline{not} given a proof that the system of equations in (2.9) (or (6.1) below) are completely integrable.

We believe that our results are robust. In fact it is tempting to conjecture that the multiplet of (2.1) is universal for all supersymmetric integrable systems in the sense that it provides the basic supersymmetry representations for these theories. We should mention that there are lots of embeddings of the equations of integrable systems into $\aleph_0$-extended systems. What seems fairly unique about (2.1) are the close relations to both spinning particle and 3D $\aleph_0$-extended supersymmetric Yang-Mills theory.

It is not such a great leap to propose that other integrable systems are amenable to a similar treatment. For example, we propose that the Kadomtsev-Petviashvili (KP) system \cite{10} works much the same way and for exactly the same multiplet. We begin by now assuming that each field in the multiplet of (2.1) depends on bosonic variables $(x, y, t)$ but we use exactly the same set of transformation laws. The only difference is to replace the first equation in (2.9) by

$$0 = \left( f_{1j} \right) - i \left[ \partial_y^2 w_i^j + \partial_x \left[ \partial_t w_i^j + \partial_x^2 w_i^j + 6d^{-1} w_k^l w_l^k \partial_x w_i^j - 3 \partial_x (uw_i^j) \right] \right], \quad (6.1)$$

while modifying (dropping where appropriate) the second equation of (2.9). Once again if one studies the case of $N = 2$ utilizing the parametrization in section 2, then (6.1) is found to contain the $N = 1$ theory \cite{10} as well as the proposal for the $N = 2$ \cite{12} theory.

Our present results suggest a number of interesting departures for the future. Foremost, there is the issue of the rigorous proof of integrability for the $\aleph_0$ supersymmetric models. Should this prove to be the case, then an interesting situation
develops. The $\aleph_0$-extended supersymmetric integrable systems are embedded in 3D $\aleph_0$-extended supersymmetric Yang-Mills theories. In particular 3D $\aleph_0$ supersymmetric Chern-Simons theories (possibly coupled to matter) might then provide a universal starting point. Such an approach would begin with $\aleph_0$ supersymmetric non-Abelian multiplets similar to (2.2) coupled to $\aleph_0$ supersymmetric scalar multiplets in such a way that the spin-1 field strength is algebraically related to currents constructed from the matter scalar multiplets. This constitutes an equation of motion for a Chern-Simons matter-coupled system. With this possibility realized, we might be able to construct an elementary proof that the supersymmetric version of the Atiyah conjecture is false. The key point is that 4D self-dual supersymmetric Yang-Mills theories when reduced to 3D can never produce via simple mechanisms any theories that possess more than $N = 8$ supersymmetry! Thus, it appears that the role of 4D self-dual theory as originally envisioned by Atiyah might be taken over instead by 3D Chern-Simons and supersymmetric Chern-Simons theory as the universal generators of all integrable systems.

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Appendix A: Explicit $GR(4, 4)$ and $GR(4, 3)$ Representations

As examples of the explicit form of the $GR$ matrix representations used in the text, we present here explicit results. In the $GR(4, 3)$ case we define

$$(L_1)_{ik} \equiv (ia\hat{\sigma}^2 \otimes \hat{\sigma}^1, iL \otimes \hat{\sigma}^2, i\hat{\sigma}^2 \otimes \hat{\sigma}^3)_{ik} = -(R_1)_{ki} \quad ,$$

$$(E_1)_{ik} \equiv (i\hat{\sigma}^1 \otimes \hat{\sigma}^2, i\hat{\sigma}^2 \otimes L, i\sigma^2 \otimes \sigma^2)_{ik} \quad ,$$

$$(f_1)_{ij} \equiv (i\sigma^2 \otimes \hat{\sigma}^1, iL \otimes \sigma^2, i\hat{\sigma}^3 \otimes \hat{\sigma}^3)_{ij} \quad ,$$

$$(F_1)_{ij} \equiv (i\sigma^1 \otimes \hat{\sigma}^2, i\hat{\sigma}^2 \otimes L, i\sigma^3 \otimes \sigma^2)_{ij} \quad .$$

Here the matrices $f_1$ are related to the usual $f_{13}$-matrices via the equation

$$f_{1j} = \epsilon_{1jk}f_k \quad .$$

We note that explicit expressions for $(L_1E_1)_{ik}$ and $(f_1F_1)_{ij}$ follow from the matrix multiplications

$$(L_1E_1)_{ik} = (L_1)_{ij}(1)_{ji}(E_1)_{ik} \quad , \quad (f_1F_1)_{ij} = (f_1)_{il}(F_1)_{lj} \quad ,$$

respectively.
In the case of $\mathcal{GR}(4, 4)$ we define,

$$(L_i)_{ik} \equiv (I \otimes I, i\sigma^2 \otimes \sigma^1, iI \otimes \sigma^2, i\sigma^3 \otimes \sigma^3)_{ik} = -(R_i)_{ki},$$

$$(E)_{ik} \equiv (i\sigma^1 \otimes \sigma^2, i\sigma^2 \otimes I, i\sigma^5 \otimes \sigma^2)_{ik}.$$  \hspace{1cm} (A.4)

The explicit forms of the matrices $(f_{1I})_k^l$ and $(\tilde{f}_{1I})_k^l$ follow from the definitions below equation (3.1). For the matrices denoted by $E_i^I$ and $F_i^I$, we simply use exactly the same matrices as for the case of $N = 3$.

**Appendix B: $\mathcal{GR}(d, N)$ Off-Shell Spinning Particle Supermultiplets**

In this appendix, we simply include the component level description of the multiplets required to describe the off-shell spinning particle. First there is a supermultiplet that contains the coordinate $X$. The complete multiplet and transformation laws are given by,

$$\delta_Q X = i\alpha^I \Psi_I,$$

$$\delta_Q \Psi_I = -2 \left[ \alpha_1 (\partial_\tau X) + d^{-1} \alpha^J (f_{1I})_i^j \mathcal{F}_j^i \right],$$

$$\delta_Q \mathcal{F}_i^j = i\alpha^I (f_{1K})_i^j (\partial_\tau \Psi_K) + i\alpha^K (L_K)_i^k \Lambda_k^j,$$

$$\delta_Q \Lambda_k^j = 2\alpha^K \partial_\tau \left[ (R_K)_k^l \mathcal{F}_l^j + d^{-1} (R_I)_k^j (f_{1K})_k^l \mathcal{F}_l^k \right],$$

where the algebraic restrictions $\mathcal{F}_i^i = (L_1)_i^j \Lambda_j^i = 0$ are imposed.

Next there is a second supermultiplet that contains the canonically conjugate momentum $P$. The complete multiplet and transformation laws take the forms,

$$\delta_Q \pi_i = \alpha_1 P + d^{-1} \alpha^K (f_{1K})_i^j \mathcal{G}_j^i,$$

$$\delta_Q \mu_i^k = -\alpha_K (L_K)_i^k \mathcal{G}_i^k + d^{-1} \alpha_K (L_1)_i^k \mathcal{G}_i^k,$$

$$\delta_Q P = -i 2\alpha_1 \partial_\tau \pi_i,$$

$$\delta_Q \mathcal{G}_i^j = -i 2 \left[ \alpha_1 (f_{1I})_i^j \partial_\tau \pi_1 + \alpha_K (R_K)_i^j \partial_\tau \mu_i^k \right],$$

where the algebraic restrictions $\mathcal{G}_i^i = (R_1)_i^j \mu_j^i = 0$ are imposed.
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