A Geometric Interpretation of the $p$-adic Littlewood Conjecture

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Abstract

This paper investigates integer multiplication of continued fractions using geometric structures. In particular, this paper shows that integer multiplication of a continued fraction can be represented by replacing one triangulation of an orbifold with another triangulation. This method is used to show that eventually periodic continued fractions have partial quotients which have exponential growth when iteratively multiplied by $n$, for $n$ any fixed, natural number.

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1 Introduction

The main aim of this paper is to find a geometric analogue to the integer multiplication of continued fractions. Our work is based upon the link between continued fractions and geodesics intersecting the Farey tessellation $F$, which was noted by M. Humbert as early as 1916 [8]. This connection was famously used by C. Series in [15], who replaced the usual tessellation of $\mathbb{H}$ by fundamental domains of $SL_2(\mathbb{Z})$ with the Farey complex $F$ 'to clarify the somewhat elusive connection' between the modular surface and continued fractions. Our motivation for this paper stems from a reformulation of the $p$-adic Littlewood Conjecture (pLC), which roughly states that for a fixed prime $p$ and any real number $\alpha$, the partial quotients of the continued fraction expansion $p^p\alpha$ become unbounded as $n$ tends to infinity. However, it is not immediately clear how the continued fraction expansion transforms as we multiply by $p$, and so naturally the question arises:

"How can one multiply a continued fraction by a prime/natural number?"

One could construct such a map between continued fractions by taking a continued fraction $\overline{\alpha}$, recovering the real number $\alpha$, multiplying by a natural number $n$ and then computing the continued fraction expansion of $n\alpha$. However, this algorithm is not very fit for our purposes: it provides little explicit information and when implemented results in errors for $\overline{\alpha} \in \mathbb{R} \setminus \mathbb{Q}$. This is because if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, then $\overline{\alpha}$ will be infinite and so we must first truncate $\overline{\alpha}$ to compute $\alpha$. This truncation leads to computational errors.

Instead, we wish to create a discrete multiplication map $\overline{n} : \overline{\alpha} \mapsto \overline{n\alpha}$, in which we do not have to worry about truncation. Whilst our algorithm is not easily implementable by a computer, it does provide a lot of explicit information and therefore, is useful in theoretical sense. These discrete multiplication maps were shown to exist by J. Vandehey in [10] and an explicit arithmetic construction for $p$ prime, was created by M. Northey in [13]. In this paper, we produce an algorithm to attain such a map, by identifying real numbers with geodesic rays in $\mathbb{H}$ and continued fractions with cutting sequences of these geodesic rays with the Farey complex $F$. In particular, we show that integer multiplication of a continued fraction by $n$, can be understood as a replacement of one triangulation of the orbifold $\Gamma_0(n) \backslash \mathbb{H}$ with another, which we describe in the following theorem.

**Theorem 3.2.** For every continued fraction $\overline{\alpha}$ and any natural number $n$, there are two canonical triangulations $T_{(1,n)}\gamma$ and $T_{(n,n)}\gamma$, and a geodesic ray $\zeta$ on the orbifold $\Gamma_0(n) \backslash \mathbb{H}$, such that the cutting sequence of $\zeta$ with $T_{(1,n)}\gamma$ corresponds to $\overline{\alpha}$ and the cutting sequence of $\zeta$ with $T_{(n,n)}\gamma$ corresponds to $\overline{n\alpha}$. 


We then show that a path on a triangulated orbifold is homotopic to a closed curve if and only if its cutting sequence directly corresponds to an essentially periodic continued fractions (see Definition 3.3(2)).

**Theorem 3.4.** Let $\mathcal{O}$ be a quotient-triangulated orbifold. Then any infinite path $\zeta$ on $\mathcal{O}$ is homotopic to a closed curve if and only if the corresponding cutting sequence is essentially periodic.

By looking at pLC in this geometric setting, we are able to obtain some surprising results pertaining to continued fractions, whilst using relatively simple techniques. The main such results are as follows.

**Theorem 3.6 and Corollary 3.7.** Let $\overline{\alpha}$ be any strictly periodic continued fraction. Then for any natural number $n$, there are infinitely many convergent denominators and infinitely many convergent numerators of $\overline{\alpha}$ which are divisible by $n$.

**Theorem 3.8.** Let $\overline{\beta}$ be an eventually periodic continued fraction. Then for every natural number $n$ there exists natural numbers $a$ and $k$, and an essentially periodic continued fraction $\overline{\alpha}$ such that $mn^k\beta = ma + m\overline{\alpha}$, for $m$ any natural number.

Theorem 3.8 then allows us to relate the growth of eventually periodic continued fractions to the growth of essentially periodic continued fractions. As a result, we get the following proposition.

**Proposition 3.10.** Let $\overline{\alpha}$ be an eventually periodic continued fraction. Then $\overline{\alpha}$ has partial quotients which grow exponentially. In particular, every eventually periodic continued fraction satisfies pLC.

This paper is organised as follows.

Section 2 of this paper is aimed to introduce the premlinary constructions that we will use throughout the paper. In Section 2.1, we provide both the formal statement of both pLC and a reformulation of pLC. In Section 2.2, we recall the classical link between continued fractions and cutting sequences of a geodesic ray $\zeta$ on $\mathbb{H}$ with the Farey complex $\mathcal{F}$, which was first introduced by M. Humbert in [8]. In Section 2.3, we introduce some novel constructions to show how multiplication of a continued fraction can be viewed as taking the cutting sequence of a geodesic ray with respect to a scaled Farey complex. We observe that $\Gamma_0(n)$ induces a common tessellation of the Farey complex and the $\frac{1}{n}$-scaled Farey complex $\frac{1}{n}\mathcal{F}$. We use this information to show, that if a continued fraction $\overline{\alpha}$ has a convergent denominator divisible by some natural number $n$, then $\overline{m\alpha}$ contains a partial quotient of size at least $n$ [Proposition 2.10]. Using the construction of fundamental domains of $\Gamma_0(n)$ introduced by R.S. Kulkarni [9] (which we cover as background in 2.3.2), we describe the $\frac{1}{n}$-scaling of the Farey complex as a change in decoration for a fundamental domain of $\Gamma_0(n)$ and give a theoretical multiplication algorithm for every natural number $n$. 

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In Section 3, we investigate cutting sequences on the orbifolds pertaining to the quotient space $\Phi \backslash \mathbb{H}$ for $\Phi$ a finite subgroup of $PSL_2(\mathbb{Z})$. For all $\Gamma_0(n) \backslash \mathbb{H}$, we show how the discrete multiplication map is induced by a change in quotient triangulation of $\Gamma_0(n) \backslash \mathbb{H}$ [Theorem 3.2]. We then show that an infinite path on a triangulated orbifold is homotopic to a closed curve if and only if the corresponding cutting sequence is essentially periodic (see definition 3.3.(2.)) [Theorem 3.3]. This is used to show that essentially periodic continued fractions are a closed class under multiplication by rational numbers [Corollary 3.5]. We also use Theorem 3.4 to show that for any natural number and any strictly periodic continued fraction, there are infinitely many convergent denominators (and convergent numerators) of the strictly periodic continued fraction which are divisible by this natural number [Theorem 3.6 and Corollary 3.7]. We then show that the integer multiplication of an eventually periodic continued fraction is in some way determined by the natural multiplication of an essentially periodic continued fraction [Theorem 3.8]. We also provide an alternative proof to the statement that eventually periodic continued fractions satisfy pLC, which was first shown in [11]. We improve on this result by showing that eventually periodic continued fractions have partial quotients which grow exponentially, when iteratively multiplied by $n$, for some integer $n$ [Proposition 3.10].

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2 A Geometric Approach to Integer Multiplication of Continued Fractions

The aim of this section is to introduce the main constructions that we will use in Section 3. Sections 2.1 and 2.2 are mostly background, with Section 2.1 introducing both the pLC and reformulation of it as motivation of this paper, as well as some classical results in Diophantine approximation, and Section 2.2 introduces the notion of a cutting sequence and then recalls some classical results of C. Series in [14] and [15]. In Section 2.3, we explain how the continued fraction expansion of $n\alpha$, for a real number $\alpha$ and integer $n$, is equivalent to the cutting sequence of some geodesic ray $\zeta$ with the scaled Farey complex $\frac{1}{n} \mathcal{F}$. We then show how multiplication of a continued fraction by an integer $n$ can be represented by replacing one decoration of a fundamental domain of $\Gamma_0(n)$ with another.
2.1 The $p$-adic Littlewood Conjecture

The $p$-adic Littlewood conjecture (pLC) is a specific case of the mixed Littlewood conjecture, which was first proposed by B. de Mathan and O. Teulié [11] in 2004. The purpose of this conjecture was to gain insight into the Littlewood conjecture, a problem in Diophantine approximation dating back to the 1930’s. However, the $p$-adic Littlewood conjecture has proved very interesting in its own right, with significant progress having been made, but no conclusion. Notably, M. Einsiedler and D. Kleinbock showed in 2005 [5], that the set of counter-examples had Hausdorff dimension zero and D. Badziahin, Y. Bugeaud, M. Einsiedler and D. Kleinbock showed in 2015 [3], that all potential counterexamples must be non-recurrent (in fact, even stronger statements regarding the mixed Littlewood Conjecture were made). Progress has also been made regarding the $t$-adic Littlewood Conjecture, an analogue of pLC over function fields. In particular, the $t$-adic Littlewood conjecture has been shown to be false for $F_3$ by F. Adiceam, E. Nesharim and F. Lunnon in [2], and the paper-folding sequence is given as an explicit counter-example.

In order to explicitly state pLC, we first define the $p$-adic norm and the distance to the nearest integer function. The $p$-adic norm is the function $|\cdot|_p$ given by

$$|x|_p := p^{-\nu_p(x)}$$

where $\nu_p(x) := \max\{n \in \mathbb{N} \cup \{0\} : p^n|x\}$ and the distance to the nearest integer is the function $\|\cdot\|$ given by

$$\|x\| := \min\{|x - n| : n \in \mathbb{Z}\}.$$

Then the statement of the $p$-adic Littlewood conjecture is as follows.

The $p$-adic Littlewood Conjecture. For every $\alpha \in \mathbb{R}$ and $p$ prime, we have:

$$\liminf_{q \to \infty} q \cdot |q|_p \cdot \|q\alpha\| = 0$$

In other words, for every $\alpha \in \mathbb{R}$ and $p$ prime, we can find an infinite subsequence $\{q_k\}_{k \in \mathbb{N}}$ such that:

$$\lim_{k \to \infty} q_k \cdot |q_k|_p \cdot \|q_k\alpha\| = 0$$

In order to remove trivial solutions to pLC, we define the badly approximables as the set of real numbers $\text{Bad} := \{\alpha \in \mathbb{R} : \liminf_{q \to \infty} q \cdot \|q\alpha\| > 0\}$. It follows from the fact $|x|_p \leq 1$ for any $x \in \mathbb{R}$ and the definition of Bad, that if $\alpha \notin \text{Bad}$, then $\alpha$ satisfies pLC. It follows from this, that if $\alpha$ is a counter example to pLC, then necessarily $\alpha \in \text{Bad}$. A useful question which arises from this notion of Bad is:

*For $\alpha \in \text{Bad}$, how can one find a subsequence $\{q_k\}_{k \in \mathbb{N}}$ which minimises the $\|q_k\alpha\|$ term?*

This question has been very well studied from the point of view of the Littlewood conjecture and Diophantine approximation. In particular, it is well known that the convergents of the continued fraction expansion of $\alpha$ gives the best rational approximation of $\alpha$, and thus the convergent denominators of the continued fraction expansion of $\alpha$ minimise the term $\|q_k\alpha\|$. We define continued fractions and convergent denominators below. 
2.1.1 Continued Fractions and Convergents

**Definition 2.1** (Section 10, G.H. Hardy and E.M. Wright [7]). A continued fraction $\overline{\alpha}$ is an expression of the form

$$\overline{\alpha} := a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ddots + \frac{1}{a_r}}}$$

where, $a_0 \in \mathbb{Z}$ and $a_i \in \mathbb{N}$ for $i \geq 1$.

We will usually write continued fractions as a sequence of $a_i$’s, $\overline{\alpha} = [a_0; a_1, \ldots, a_r]$ and refer to the $a_i$’s as partial quotients. The sequence of partial quotients can be either finite or infinite and refer to the corresponding continued fraction as finite or infinite accordingly. Evaluating the continued fraction expression gives a real number $\alpha$, and for any real number $\alpha$ we can find an associated continued fraction expansion. For any $\alpha \in \mathbb{R}$, the continued fraction $\overline{\alpha}$ is finite if and only if $\alpha \in \mathbb{Q}$. For $\alpha \in \mathbb{Q}$, there are two different continued fraction expansions, $[a_0; a_1, \ldots, a_r]$ and $[a_0; a_1, \ldots, a_r - 1, 1]$, where $a_r > 1$. For $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, there is a unique infinite continued fraction expansion.

**Definition 2.2** (Section 10.2, G.H. Hardy and E.M. Wright [7]). Let $\overline{\alpha} = [a_0; a_1, a_2, \ldots]$ be a continued fraction. We define the $k$-th convergent of $\overline{\alpha}$ to be $\frac{p_k}{q_k} : = \left[ a_0; a_1, \ldots, a_k \right]$. We can define this iteratively where:

$$
\begin{align*}
p_{-1} &= 1 \\
q_{-1} &= 0 \\
p_0 &= a_0 \\
q_0 &= 1 \\
p_k &= a_k p_{k-1} + p_{k-2} \\
q_k &= a_k q_{k-1} + q_{k-2}
\end{align*}
$$

We refer to the term $p_k$ as the $k$-th convergent numerator of $\alpha$ and $q_k$ as the $k$-th convergent denominator.

2.1.2 Reformulating pLC in terms of Continued Fractions

Using the notions of continued fractions and convergents of these continued fractions, we can rephrase pLC as a condition on continued fraction expansions. We define the height of $\alpha$, $B(\alpha)$ to be the largest partial quotient in the continued fraction expansion of $\alpha$ (excluding the first partial quotient). In other words,

$$B(\alpha) := \sup \{ a_i : \overline{\alpha} = [a_0; a_1, \ldots], i \in \mathbb{N} \}$$

We then define $\text{Bad}_{CF}$ to be the set of real numbers with bounded partial quotients i.e. $\text{Bad}_{CF} := \{ \alpha \in \mathbb{R} : B(\alpha) < \infty \}$. Using these definitions, we get the following classical lemma.

**Lemma 2.3.** For any $\alpha \in \mathbb{R}$, $\alpha \in \text{Bad}$ if and only if $\alpha \in \text{Bad}_{CF}$. In particular, $\text{Bad} \equiv \text{Bad}_{CF}$. 6
Since the convergent denominators \( \{q_k^{(n)}\}_{k \in \mathbb{N}} \) of \( p^n \alpha \) are "good" approximations for \( \|q_k (p^n \alpha)\| \), that \( \{p^n \cdot q_k^{(n)}\}_{k \in \mathbb{N}} \) are also "good" approximations of \( \|q_k \alpha\| \). Using a more formal version of this reasoning, we can recover the following reformulation of pLC.

**Proposition 2.4.** Let \( \alpha \in \text{Bad} \), then \( \alpha \) satisfies pLC if and only if:

\[
\limsup_{i \to \infty} B(p^i \alpha) = \infty
\]

**Proof.** See [13], Appendix.

**Remark.** It is worth noting that our definition of \( B(\alpha) \) excludes the \( a_0 \) term. This is due to the fact that, if we were to include this term, then \( \limsup_{n \to \infty} B(p^n \alpha) = \infty \) for every \( \alpha \in \mathbb{R} \), since \( a_0^{(k)} = [p^k \alpha] \to \infty \), where \( \overline{p^k \alpha} = [a_0^{(k)} ; a_1^{(k)} , \ldots ] \).

Since the latter formulation of pLC is a condition on continued fractions, it would be useful to construct a way of computing \( \overline{p^k \alpha} \) from the continued fraction \( \overline{\alpha} \). For every \( \alpha \in \mathbb{R} \), we can construct a bijective map between \( \overline{\alpha} \) and \( \overline{p\alpha} \) (if \( \alpha \in \mathbb{Q} \), take \( \overline{\alpha} \) with 1 as the final partial quotient) and a bijective map between \( \overline{\alpha} \) and \( p\alpha \). Thus we should be able to construct the bijective map \( \overline{p} : \overline{\alpha} \to \overline{p\alpha} \) to get the following commutative diagram:

\[
\begin{array}{ccc}
\alpha & \xrightarrow{p} & p\alpha \\
\downarrow & & \downarrow \\
\overline{\alpha} & \xrightarrow{\overline{p}} & \overline{p\alpha}
\end{array}
\]

We can view continued fractions as discrete realisations of continuous objects and so, we can think of the \( \overline{p} \) map as a map between discrete structures. Therefore, the reformulation of pLC produces our motivating question:

"Can we construct the \( \overline{p} \) map to directly compute \( \overline{p\alpha} \) from \( \overline{\alpha} \)?"

In our setting we will replace \( p \), prime, with \( n \), a natural number, and \( \overline{p} \) with \( \overline{\alpha} \) analogously.

**2.2 Continued Fractions as Cutting Sequences**

In this section we will introduce the notion of cutting sequences of both geodesic rays and paths with an ideal triangulation in \( \mathbb{H} \), and recall some of the main results of C. Series in [14] and [15].

**2.2.1 Cutting Sequences of Geodesic Rays**

In this paper we will take \( \mathbb{H} \) to be the upper half plane \( \{ z \in \mathbb{C} \cup \{ \infty \} : Im(z) \geq 0 \} \) with boundary \( \partial \mathbb{H} = \mathbb{R} \cup \{ \infty \} \). Geodesic lines are given by Euclidean half-lines of the
form \( \{ a + iy : 0 \leq y \leq \infty \} \) and semicircles centred on \( \partial \mathbb{H} \). We define a hyperbolic \( n \)-gon with vertices \( z_1, z_2, \ldots, z_n \in \mathbb{H} \), to be the closed region bounded by \( l_1, \ldots, l_n \), where \( l_i \) is the geodesic segment between \( z_i \) and \( z_{i+1} \) (taking \( z_{n+1} = z_1 \)). An ideal triangle is a hyperbolic 3-gon, with all vertices lying on \( \partial \mathbb{H} \) and an ideal triangulation of \( \mathbb{H} \) is an infinite collection of ideal triangles \( T \) such that the closure of these triangles cover \( \mathbb{H} \) and for any two triangles \( \tau_1, \tau_2 \in T \), \( \tau_1 \cap \tau_2 = \emptyset \). See Fig. 3 (in Section 2.2.2).

Let \( \zeta \) be an oriented geodesic, which enters a triangle \( \triangle ABC \), labelled clockwise, through the edge \( AB \). We define the triangle to be a left triangle for \( \zeta \) if the geodesic leaves through the edge \( BC \) or a right triangle if the geodesic leaves through the edge \( AC \). If the geodesic, instead leaves through the vertex \( C \), we can view the triangle as either a left triangle or a right triangle. We refer to multiple left triangles in a row as left fans and multiple right triangles in a row as right fans. For either type of fan, all triangles in the fan will have a common vertex. See Fig. 4

![Diagrams of left and right triangles and fans](image)

**Figure 1:** Examples of left and right triangles and fans.

**Definition 2.5.** Let \( T \) be an ideal triangulation of \( \mathbb{H} \), let \( E \) be any edge of \( T \) and let \( \zeta \) be an oriented geodesic ray starting at \( E \) and terminating at some point in \( \partial \mathbb{H} \). The
cutting sequence of $\zeta$ with respect to $T$, denoted $(\zeta, T)$, is the potentially infinite word in the alphabet $F_2 = \langle L, R \rangle$, formed by the following process:

- Start with the (empty) word $L^0$.
- Whenever $\zeta$ cuts $T$ to form a left triangle, add a letter $L$ to the end of the word.
- Whenever $\zeta$ cuts $T$ to form a right triangle, add a letter $R$ to the end of the word.
- Repeat this process iteratively to obtain the cutting sequence $(\zeta, T) := L^{n_0}R^{n_1}L^{n_2} \ldots$, where $n_0 \in \mathbb{N} \cup \{0\}$ and $n_i \in \mathbb{N}$.

We will identify the cutting sequence $(\zeta, T) = L^{n_0}R^{n_1}L^{n_2} \ldots$ with the sequence of indices $\{n_0, n_1, n_2, \ldots\}$, where $n_0 \in \mathbb{N} \cup \{0\}$ and $n_i \in \mathbb{N}$. Since every cutting sequence is of the form $\{\{n_0, n_1, n_2, \ldots\} : n_0 \in \mathbb{N} \cup \{0\}, n_i \in \mathbb{N} \ \forall i \in \mathbb{N}\}$, there is an obvious bijection between cutting sequences of the above form and the continued fraction expansion of some $\alpha \in \mathbb{R}_{>0}$. Explicitly, we can take the following bijection $\eta := \{n_0, n_1, n_2, \ldots\} \mapsto [n_0; n_1, n_2, \ldots]$.

**Remark.** If a cutting sequence of an oriented geodesic ray $\zeta$ is finite, then the geodesic ray terminates at a vertex of the triangulation. The final triangle which $\zeta$ intersects can be thought of either a left triangle or right triangle and thus, can either be added to the final fan or represent a new fan on its own. This is analogous to the fact that the two finite continued fractions expansions $[a_0; a_1, \ldots, a_r]$ and $[a_0; a_1, \ldots, a_r - 1, 1]$ are equivalent.

Occasionally, it may be useful to take the cutting sequence of a geodesic ray $\zeta$ starting at an edge $E$, from an edge $F$ later cutting sequence of $(\zeta, T)$. We will denote this cutting sequence as $(\zeta, T)_F$. We can think of $(\zeta, T)_F$ as a copy of $(\zeta, T)_E$ with a prefix removed. That is, $(\zeta, T)_F$ coincides with $(\zeta, T)$ except for finitely many terms at the start. It is worth noting that due to the convention of always starting with an $L^0$ term, the types of triangle will also coincide for these terms when written as sequences of indices.

Every edge $E$ in $T$ separates $\mathbb{H}$ into two regions, which we will arbitrarily label $E_+$ and $E_-$. Similarly, $E$ will separate any geodesic in $\mathbb{H}$, which it intersects transversely, into two disjoint geodesic rays, one contained entirely in $E_+$ and the other contained entirely in $E_-$. As a result, any geodesic ray $\zeta$ starting at $E$ will be contained entirely in one of these two regions. We will denote the set of all geodesic rays starting at $E$, which are contained entirely in $E_+$ as $Z_{E_+}$ and likewise will denote the set of all geodesic rays starting at $E$, which are contained entirely in $E_-$ as $Z_{E_-}$. Given a particular cutting sequence and triangulation $T$, for every edge $E$ in $T$ we can find two distinct classes of geodesic rays which have this cutting sequence: one in $Z_{E_+}$ and the other in $Z_{E_-}$. These classes are completely determined by the endpoint of one such geodesic ray. That is to say, all geodesic rays starting at $E$ and terminating at a fixed point $\alpha \in \mathbb{R} \cup \{\infty\}$ will have the same cutting sequence relative to $T$. The proof of this statement is an analogue of Lemma 3.1.1 from [15], which we explicitly state in section 2.2.3.
Due to the fact that, for any geodesic ray $\zeta$ and any ideal triangulation $T$, all orientation preserving isomorphisms of $\mathbb{H}$ ($Isom^+(\mathbb{H})$) preserve both the notions of left and right triangles, and how $\zeta$ and $T$ intersect each other, we get the following lemma.

**Lemma 2.6.** Cutting sequences are invariant under $Isom^+(\mathbb{H}) = PSL_2(\mathbb{R})$: If $\varphi \in Isom^+(\mathbb{H})$, then for any geodesic ray $\zeta$ and any triangulation $T$, $(\zeta,T) = \varphi((\zeta,T)) = (\varphi(\zeta),\varphi(T))$.

If we take an orientation-reversing automorphism $\psi \in Isom(\mathbb{H}) \setminus Isom^+(\mathbb{H})$, the notions of left and right triangles swap and as such $(\psi(\zeta),\psi(T)) = (-\zeta,T) = (\zeta^{-1},T)$. Here, we take $(\zeta,T) = \{a_0,a_1,\ldots\}$ and assume $\zeta \in Z_{E_+}$. Then $\zeta^{-1}$ is a geodesic ray in $Z_{E_+}$ with $(\zeta^{-1},T) = \{0,a_0,a_1,\ldots\}$ and $-\zeta$ is a geodesic ray in $Z_{E_-}$ with $(-\zeta,T) = \{0,a_0,a_1,\ldots\}$.

### 2.2.2 Cutting Sequences of Paths

It will often be useful to deal with paths starting from some edge $E$ and terminating at some point in $\partial \mathbb{H}$, instead of geodesic rays. In particular, it will be useful to see how homotopy affects cutting sequences. To do so, we will extend the definition of cutting sequences to include paths which may double back on themselves. We do this by labelling the sides of the triangles in the triangulation and expressing cutting sequences as powers of these labels.

In order to form a labelled triangulation, we first must create two labelling maps $\varphi_1$ and $\varphi_2$. Here, we take an arbitrary triangle $\Delta ABC$, with vertices labelled clockwise. We then define $\varphi_1 : \{L^{-1},L,R\} \rightarrow \Delta ABC$ to be the labelling map such that $L^{-1}$ labels the inside of the edge $AB$, $L$ labels the inside of the edge $BC$ and $R$ labels the inside of the edge $AC$. Similarly, we define $\varphi_2 : \{R^{-1},L,R\} \rightarrow \Delta ABC$ to be the labelling map such that $R^{-1}$ labels the inside of the edge $AB$, $L$ labels the inside of the edge $BC$ and $R$ labels the inside of the edge $AC$. See Fig. 2.

![Fig. 2](image-url)

(a) An example of $\varphi_1$ labelling a triangle $\Delta ABC$.

(b) An example of $\varphi_2$ labelling a triangle $\Delta ABC$.

Figure 2: Examples of $\varphi_1$ and $\varphi_2$ inducing labelling of an arbitrary triangle $\Delta ABC$, with vertices labelled clockwise.
Let $T$ be an ideal triangulation of $\mathbb{H}$, $E$ be an edge of $T$ and $\lambda$ be an oriented path in $\mathbb{H}$, which starts at $E$, terminates at $\alpha \in \partial \mathbb{H}$ and is otherwise disjoint from $\partial \mathbb{H}$. The edge $E$ is an edge of exactly two triangles in $T$: $\tau_+$, which is contained in $E_+$ and $\tau_-$, which is contained in $E_-$. The endpoint $\alpha$ of $\lambda$, will lie in either $E_+$ or $E_-$. Without loss of generality, assume that the end point $\alpha$ lies in $E_+$. Label $\tau_+$ using $\varphi_1$ such that $L^{-1}$ labels the inside of edge $E$. Then the following algorithm produces a labelling for $T$:

Let $\tau$ be a labelled triangle in $T$. Pick an edge $X$ in $\tau$ and let $\tau'$ be the unique other triangle (unlabelled) in $T$ with edge $X$.

- If $X$ has inner label $L$ in $\tau$, label $\tau'$ using $\varphi_1$ such that $L^{-1}$ is the inner labelling of $X$ in $\tau'$.
- If $X$ has inner label $R$ in $\tau$, label $\tau'$ using $\varphi_2$ such that $R^{-1}$ is the inner labelling of $X$ in $\tau'$.
- If $X$ has inner label $L^{-1}$ in $\tau$, label $\tau'$ using $\varphi_1$ such that $L$ is the inner labelling of $X$ in $\tau'$.
- If $X$ has inner label $R^{-1}$ in $\tau$, label $\tau'$ using $\varphi_2$ such that $R$ is the inner labelling of $X$ in $\tau'$.

Repeat ad infinitum. See Fig. 3 for an example of a labelled triangulation.

The generalised cutting sequence of an oriented path $\lambda$ with a labelled triangulation $T$ is defined as follows.

Start with the word $L^0$ in $F_2 = \{L, R\}$. Then every time $\lambda$ passes through an edge of the triangle, append the inner label of that edge (of that triangle) to the word. We define the generalised cutting sequence of $\lambda$ with respect to $T$, which we denote $(\lambda, T)$, to be the word formed by repeating this process iteratively i.e. $(\lambda, T) = L^{m_0} R^{m_1} L^{m_2} \cdots$ with $m_i \in \mathbb{Z}$ $\forall i \in \mathbb{N} \cup \{0\}$. Convention will be to always start with a power of $L$ and to have the word alternate between powers of $L$’s and $R$’s, but the index of these letters may be 0. In particular, a path which passes through an edge and then immediately passes through that edge again would correspond to the term $\cdots L^k L^{k'} R^{k_1} L^{k_2} \cdots$ or $\cdots R^{k_1} L^{k_2} \cdots$, for some $k_1, k_2 \in \mathbb{Z}$. As we did for cutting sequences of geodesic rays, we will identify the generalised cutting sequence of a path $(\lambda, T) = L^{m_0} R^{m_1} L^{m_2} \cdots$ with the sequence of indices $\{m_0, m_1, m_2, \ldots\}$, where $m_i \in \mathbb{Z}$ $\forall i \in \mathbb{N} \cup \{0\}$.

Remark. For $\zeta$ a geodesic ray and an ideal triangulation $T$, the notions of cutting sequence and generalised cutting sequence are equivalent. As a result, we will drop the term "generalised" refer to both as a "cutting sequence".

We will say that the cutting sequence $(\lambda, T)$ is reduced if the word contains no term of the form $g x^0 g^{-1}$, where $g \in \{L, R\}$ and $x \in \{L, R\}$. We can reduce the cutting sequence by reducing the corresponding word and will denote the class of all equivalent cutting sequences up to reduction as $[\lambda, T]$. It follows quite simply that cutting sequences will be reduced if and only if the corresponding path $\lambda$ does not pass through any edge of
Figure 3: An example of a labelled triangulation.

T more than once. We can view reduction of the cutting sequence as a homotopy of the path \( \lambda \), which preserves \( \partial \mathbb{H} \). As a result, the classes of equivalent cutting sequences \([ \lambda, T ]\) are exactly the classes of homotopic paths \([ \lambda ]^E_\alpha\) with the same starting edge \( E \) and endpoint \( \alpha \).

We can analogously define the map between cutting sequences and continued fraction expansions \( \eta : \{ m_0, m_1, m_2, \ldots \} \mapsto [m_0; m_1, m_2, \ldots] \). Since we take alternating letters, continued fractions reduce in the same way that the corresponding words do. For example, if we had a non-reduced word \( g_1 L^{k_1} R^0 L^{-k_1} g_2 \cdots = g_1 L^0 g_2 \cdots \) (for \( g_1, g_2 \in \langle L, R \rangle \)), then the corresponding continued fraction expansion would be \([ g_1, k_1, 0, -k_1, g_2, \ldots ]\), which is equivalent under concatenating terms to \([ g_1, k_1 + (-k_1), g_2, \ldots ]\). The above construction ensures that the reduction of a cutting sequence directly corresponds to the reduction of the equivalent continued fraction. See Fig. 4 for an example of two equivalent cutting sequences.

2.2.3 The Farey Complex \( \mathcal{F} \)

The Farey complex \( \mathcal{F} \) is an ideal triangulation of the upper-half plane \( \mathbb{H} \). The vertices are the set \( \mathbb{Q} \cup \{ \infty \} \). Two vertices \( A \) and \( B \) have a geodesic edge between them if once written in reduced form, \( A = \frac{p}{q} \) and \( B = \frac{r}{s} \), we have \( | ps - qr | = 1 \). We say two vertices are neighbours, if they have an edge between them. In this definition, we treat \( \infty \) as \( \frac{1}{0} \). An equivalent way of interpreting \( \mathcal{F} \) is by taking the image of the line between 0 and \( \infty \).
Figure 4: An example of two different paths intersecting a labelled triangulation to form the same cutting sequence up to reduction.
under all possible elements of $\text{SL}_2(\mathbb{Z})$. See Fig. 5 for a truncated picture of the Farey Complex.

Given two vertices $A = \frac{p}{r}$ and $B = \frac{q}{s}$ in $\mathbb{Q} \cup \{\infty\}$ in reduced form, we can define \textit{Farey addition} $\oplus$ and \textit{Farey subtraction} $\ominus$, as follows:

\[
A \oplus B := \frac{r+p}{q+s} = \frac{s+q}{r+q} =: B \oplus A \\
A \ominus B := \frac{r-s}{q-s} = \frac{s-r}{r-q} =: B \ominus A
\]

Simple arithmetic can show that any two neighbours $A$ and $B$ in $\mathcal{F}$ have exactly two neighbours in common, $A \oplus B$ and $A \ominus B$. It is a well known fact, that given any two neighbours in $\mathcal{F}$, you can generate the whole of $\mathcal{F}$ by iteratively using Farey addition and subtraction of these two points.

The following theorem, highlights the importance of the Farey Complex with regards to continued fractions.

\textbf{Theorem 2.7} (Theorem A, C. Series [15]). \textit{Let $\zeta$ be a geodesic in $\mathbb{H}$ with endpoints $\alpha_1 > 0$ and $\alpha_2 < 0$, and let $I_+$ be the geodesic line between 0 and $\infty$, $I_-$ be the region \{z : Re(z) > 0\} and $I_-$ be the region \{z : Re(z) < 0\}. Then, for $\zeta^+ = \zeta \cap Z_{I_+}$ and $\zeta^- = \zeta \cap Z_{I_-}$ (with implicit orientation), $\eta((\zeta^+, \mathcal{F}))$ is the continued fraction expansion of $\alpha_1$ and $\eta((\zeta^-, \mathcal{F}))$ is the continued fraction expansion of $\frac{-1}{\alpha_2}$.}

An immediate consequence of this theorem is that if we take $Z_\alpha$ to be the set of all geodesic rays starting at the $y$-axis $I$ and terminating at a fixed point $\alpha \in \mathbb{R} \setminus \{0\}$, then $(\zeta, \mathcal{F}) = (\zeta', \mathcal{F})$ for all $\zeta, \zeta' \in Z_\alpha$. This result can be extended to Lemma 3.3.1 from [15]:

\textbf{Lemma 2.8} (Lemma 3.1.1, C. Series [15]). \textit{Let $\zeta_{\alpha,1}$ and $\zeta_{\alpha,2}$ be two geodesic rays with the same endpoint $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ but potentially different start points. Then the cutting sequences of $\zeta_{\alpha,1}$ and $\zeta_{\alpha,2}$ with respect to $\mathcal{F}$, eventually coincide. In particular, there exists an edge $E$ which both $\zeta_{\alpha,1}$ and $\zeta_{\alpha,2}$ intersect, such that $(\zeta_{\alpha,1}, \mathcal{F})_E = (\zeta_{\alpha,2}, \mathcal{F})_E$.}

Let $\zeta_\alpha$ be a geodesic ray starting at $I$ and terminating at some point $\alpha > 0$. For each fan in the cutting sequence of $\zeta_\alpha$ with $\mathcal{F}$, there is a common vertex of all the triangles in this fan. Using Theorem 2.7 and taking truncations of $\overline{\alpha}$, it is easy to show that these vertices are exactly the convergents of $\overline{\alpha}$. Thus, when a cutting sequence of $\zeta_\alpha$ with $\mathcal{F}$ changes fan, the vertices of the edge at which it changes fan are both convergents of $\overline{\alpha}$. Because $\zeta_\alpha$ passes through pairs of edges of a triangle, we can similarly recover the set of convergents by taking all the vertices which belong to two or more edges with which the geodesic ray intersects, as well as taking the point at $\infty$. In other words, if two edges of the cutting sequence have a common vertex in $\mathcal{F}$, this vertex is a convergent of $\overline{\alpha}$. Similarly, if we know a vertex is a convergent, then it is either the point at $\infty$ or the endpoint of at least two edges of $\mathcal{F}$ in our cutting sequence (for $\alpha > 1$, $\infty$ will also be the endpoint of at least two edges). See Fig. 5.
2.3 Constructing the Discrete Multiplicative Map $\overline{n}$

Let $n^* := \begin{pmatrix} \sqrt{n} & 0 \\ 0 & \frac{1}{\sqrt{n}} \end{pmatrix} \in PSL_2(\mathbb{R})$ and define $\frac{1}{n^*} := (n^*)^{-1}$ for $n \in \mathbb{N}$. These two maps scale both $\mathbb{H}$ and $\mathcal{F}$ by a factor of $n$ and $\frac{1}{n}$, respectively. In particular, they multiply the real axis by $n$ and $\frac{1}{n}$. These maps do not preserve $\mathcal{F}$ and we will refer to the images of $\mathcal{F}$ under these maps as $n\mathcal{F}$ and $\frac{1}{n}\mathcal{F}$ respectively. It is worth noting that both of these maps preserve the line $I$ between 0 and $\infty$, which is our conventional starting edge for our geodesic rays in $\mathcal{F}$. The initial direction of departure is also preserved since $n^*$ and $\frac{1}{n^*}$ preserve the orientation of the $\mathbb{H}$. It follows that for all geodesic rays $\zeta$ starting at $I$, $(\zeta, \frac{1}{n}\mathcal{F}) = (n^*(\zeta), \mathcal{F})$. As a result, we can view the map multiplying continued fractions by integer $\overline{n} : \overline{\alpha} \rightarrow \overline{n\alpha}$ in terms of a map between $\mathcal{F}$ and $\frac{1}{n}\mathcal{F}$ which preserves $\zeta_\alpha$, a geodesic ray starting at $I$ and terminating at $\alpha$. Explicitly, we can express $\overline{n}$ as the map between the pairs $(\zeta_\alpha, \mathcal{F}) \rightarrow (\zeta_\alpha, \frac{1}{n}\mathcal{F})$.

Since the $\overline{n}$ map in this context is dependent upon the $\frac{1}{n}$ map, we have only described the $\overline{n}$ map via continuous action on $\mathbb{H}$. Instead we want to describe $\overline{n}$ on discrete structures. To find such a discrete map we will claim that for any natural number $n$, there exists a polygon $P_n$ with side pairings and two decorated copies of $P_n$, $T_{\{1,n\}}$ and
such that $T_{(1,n)}$ tessellates $\mathcal{F}$ and $T_{(n,n)}$ tessellates $\frac{1}{n}\mathcal{F}$, under the group action induced by the side pairings of $P_n$. We will take $P_n$ containing the $y$-axis $I$ and will take this edge to be our starting edge, unless otherwise stated. Then, we express our geodesic ray $\zeta_n$ as a collection of ordered sub-paths $\bigcup_{i=1}^{\infty} \zeta_{i,n}$ intersecting the tessellation induced by $P_n$, such that each sub-path $\zeta_{i,n}$ is entirely contained in some image of $P_n$ in this tessellation. Then, the ordered product of the cutting sequences derived by the sub-paths $\bigcup_{i=1}^{\infty} \zeta_{i,n}$ is equivalent the cutting sequences of $\zeta_n$. In particular, replacing $T_{(1,n)}$ with $T_{(n,n)}$ encodes the multiplication discrete map $\overline{n} : \overline{a} \rightarrow \overline{n\alpha}$.

### 2.3.1 Common Structure of $\mathcal{F}$ and $\frac{1}{n}\mathcal{F}$

For a tiling $T$, any element in $Isom^+(T)$ preserves both $T$ and the orientation, so any subgroup of $Isom^+(T)$ also preserves $T$. We can say more, any subgroup $G < Isom^+(T)$ produces a fundamental domain $F$, which when imbedded with the correct structure, tessellates $T$. This extra structure is $F_T := T \cap F$ embedded into $F$ as a decoration. We refer to $F \cup F_T$ as a decorated tile of $G$.

Since $Isom^+(\mathcal{F}) = SL_2(\mathbb{Z})$ is the maximal orientation-preserving group which preserves $\mathcal{F}$, one can show that $Isom^+(\frac{1}{n}\mathcal{F}) = \left\{ n^{-1} \circ A \circ n : A \in SL_2(\mathbb{Z}) \right\}$ is the maximal orientation preserving group which preserves $\frac{1}{n}\mathcal{F}$. We can view this form as a composition of maps: first the map scaling $\frac{1}{n}\mathcal{F}$ to $\mathcal{F}$, followed by an isomorphism of $\mathcal{F}$ and finally the map scaling $\mathcal{F}$ back to $\frac{1}{n}\mathcal{F}$. We can also write $Isom^+(\frac{1}{n}\mathcal{F}) = \left\{ \left( \begin{array}{cc} a & b \\ \frac{nc}{d} & \frac{d}{d} \end{array} \right) \in SL_2(\mathbb{R}) : \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{Z}) \right\}$ and $Isom^+(\frac{1}{n}\mathcal{F})$ takes on a natural group structure induced by $Isom^+(\mathcal{F})$.

We can recover a common subgroup of the group of isomorphisms for $\mathcal{F}$ and $\frac{1}{n}\mathcal{F}$ by taking the intersection of $Isom^+(\mathcal{F})$ and $Isom^+(\frac{1}{n}\mathcal{F})$. $Isom^+(\mathcal{F}) \cap Isom^+(\frac{1}{n}\mathcal{F}) = \left\{ \left( \begin{array}{cc} a & b \\ d & d \end{array} \right) \in SL_2(\mathbb{Z}) : c \equiv 0 (\text{mod } n) \right\}$. $\Gamma_0(n)$ is a subgroup of both $Isom^+(\mathcal{F})$ and $Isom^+(\frac{1}{n}\mathcal{F})$ by construction and therefore preserves the structure of both $\mathcal{F}$ and $\frac{1}{n}\mathcal{F}$. As a result, any fundamental domain $D$ of $\Gamma_0(n)$ with decoration $D \cap \mathcal{F}$ or $D \cap \frac{1}{n}\mathcal{F}$ will tessellate $\mathcal{F}$ or $\frac{1}{n}\mathcal{F}$, respectively.

Finding a fundamental domain $D$ for $\Gamma_0(n)$ has been relatively well studied by mathematicians in [9] and [10]. We can recover the $\overline{n}$ map by replacing the decoration $D \cap \mathcal{F}$ of $D$ with the decoration $D \cap \frac{1}{n}\mathcal{F}$. Since $\Gamma_0(n_1) < \Gamma_0(n_2)$ if and only if $n_2 \mid n_1$, given a fundamental domain of $\Gamma_0(n)$, we can also embed structure to tessellate $\frac{1}{d}\mathcal{F}$ for all $d \mid n$ and as a result can recover the $\overline{d}$ map.

It will often be useful to tell when two vertices are neighbours in both $\mathcal{F}$ and $\frac{1}{n}\mathcal{F}$. The following lemma provides an if and only if condition.

**Lemma 2.9.** Two points $A$ and $B$ are neighbours in both $\mathcal{F}$ and $\frac{1}{n}\mathcal{F}$ if and only if they have reduced form $\frac{a}{cn_1}$ and $\frac{b}{dn_2}$ with $n = n_1n_2$ and $|adn_2 - bcn_1| = 1$.

**Proof.** ($\Rightarrow$): If $A$ and $B$ are neighbours in $\mathcal{F}$ and $\frac{1}{n}\mathcal{F}$, then it follows that $n \cdot A$ and $n \cdot B$ are also neighbours in $\mathcal{F}$. Let $A = \frac{a'}{d'}$ and $B = \frac{b'}{d'}$ be in reduced form, then since
A and B are neighbours in $F$, $|a'd' - b'c'| = 1$. Let $g_1 := \gcd(c', n)$ and $h_1 := \gcd(d', n)$, then we can write $c' = c g_1$ and $d' = d h_1$ for some $c, d \in \mathbb{N}$ and take $g_2 = \frac{n}{g_1}$ and $h_2 = \frac{n}{h_1}$. We compute $\gcd(c, g_2) = \gcd(c, a') = 1$ and $\gcd(d, h_2) = \gcd(d, b') = 1$, and therefore, the points $n \cdot A = \frac{a}{c} n$ and $n \cdot B = \frac{b}{d} n$ are in reduced form. Since $n \cdot A$ and $n \cdot B$ are neighbours in $F$, it follows that $|a'g_2d - b'h_2c| = 1$. We require that $\gcd(g_2, h_2) = 1$, since otherwise $|a'g_2d - b'h_2c| \equiv 0 \mod \gcd(g_2, h_2)$, which for $\gcd(g_2, h_2) \neq 1$ would lead to a contradiction to $A$ and $B$ being neighbours in $F$. It follows from the fact that $A$ and $B$ are neighbours in $F$ and from writing $c' = c g_1$ and $d' = d h_1$, that $\gcd(g_1, h_1) = 1$.

We observe that since $\gcd(g_1, h_1) = 1$, $g_1 = \gcd(g_1, n) = \gcd(g_1, h_1 h_2) = \gcd(g_1, h_2)$. Similarly, we observe that $h_2 = \gcd(h_2, n) = \gcd(h_2, g_1 h_2) = \gcd(h_2, g_1)$ and so $g_1 = h_2$.

By a similar procedure we find that $g_2 = h_1$ and the result follows by relabelling $a = a'$, $c = c'$, $n_1 = g_1$ and $n_2 = h_1$.

($\Leftarrow$): Let $A = \frac{a}{nc_1}$ and $B = \frac{b}{dn_2}$, with $n = n_1 n_2$ and $|adn_2 - bcn_1| = 1$. Since $|adn_2 - bcn_1| = 1$ we see that $A$ and $B$ are neighbours in $F$. Also $n \cdot A = \frac{an_2}{c}$ and $n \cdot B = \frac{bn_1}{d}$ in reduced form and $|an_2d - bn_1c| = |adn_2 - bcn_1| = 1$. Therefore, $n \cdot A$ and $n \cdot B$ are neighbours in $F$. By rescaling we now see that $A$ and $B$ are neighbours in $\frac{1}{n} F$ as required.

The condition that $A$ and $B$ have reduced form $\frac{a}{nc}$ and $\frac{b}{dn}$ with $n = n_1 n_2$ and $|adn_2 - bcn_1| = 1$, translates to saying that if $A$ and $B$ are neighbours of this form in either $F$ or $\frac{1}{n} F$, then necessarily they are neighbours in both $F$ and $\frac{1}{n} F$. A direct result of this, is that two points of the form $\frac{a}{nc}$ and $\frac{b}{dn}$ are neighbours in $F$ if and only if they are neighbours in $\frac{1}{n} F$. This implies that for any two neighbours $\frac{a}{nc}$ and $\frac{b}{dn}$ in both $F$ and $\frac{1}{n} F$, there is an element of the form $(\frac{a}{nc}, \frac{b}{dn}) \in \Gamma_0(n) \subset SL_2(\mathbb{Z})$ which maps $0$ to $\frac{a}{nc}$ and $0$ to $\frac{b}{dn}$. The reverse is also true, any element of $\Gamma_0(n)$ maps the vertices 0 and $\infty$, to a pair of neighbours in both $F$ and $\frac{1}{n} F$.

For any point of the form $A = \frac{a}{nc}$ and any two consecutive neighbours $B_1 = \frac{b_1}{d_1}$ and $B_2 = \frac{b_2}{d_2}$ of $A$ in $F \cap \frac{1}{n} F$, we can always find a map $\varphi \in \Gamma_0(n)$, such that $\varphi(0) = A$, $\varphi(\infty) = B_1$ and $\varphi(\frac{1}{n}) = B_2$ (up to relabelling). Since $\Gamma_0(n)$ preserves both $F$ and $\frac{1}{n} F$, the number of neighbours that $A$ has between $B_1$ and $B_2$ in $F$ (or equivalently in $\frac{1}{n} F$) will be equivalent to the number of neighbours that $0$ has between $\infty$ and $\frac{1}{n}$ in $F$ (or in $\frac{1}{n} F$). Similarly, for any point of the form $B = \frac{b}{dn}$ ($\gcd(n, d) = 1$) and any two consecutive neighbours $A_1 = \frac{a_1}{nc_1}$ and $A_2 = \frac{a_2}{nc_2}$ of $B$ in $F \cap \frac{1}{n} F$, the number of neighbours that $B$ has between $A_1$ and $A_2$ in $F$ (or in $\frac{1}{n} F$) will be equivalent to the number of neighbours that $\infty$ has between 0 and 1 in $F$ (or in $\frac{1}{n} F$). We summarise this in the following table:
| Points of the form | Number of neighbours in $F$ between consecutive neighbours in $F \cap \frac{1}{n} F$ | Number of neighbours in $\frac{1}{n} F$ between consecutive neighbours in $F \cap \frac{1}{n} F$ |
|------------------|---------------------------------|---------------------------------|
| $\frac{a}{nc}$  | 0                               | $n - 1$                         |
| $\frac{b}{d}$, $\gcd(n,d) = 1$ | $n - 1$                        | 0                               |

This information is used to prove the following result.

**Proposition 2.10.** If a continued fraction $\alpha$ has a convergent denominator $q_k$, such that $n \mid q_k$ for $n \in \mathbb{N}$ and $n < q_k$, then $B(n \alpha) \geq n$. Further, if $\frac{pk}{q_k} = \frac{pk}{nq_k}$ is a convergent of $\alpha$, $\frac{pk}{q_k}$ is a convergent of $\alpha \frac{1}{n} \alpha$.

**Proof.** Let $A = \frac{pk}{q_k}$ be a convergent of $\alpha$ with geodesic representative $\zeta_0$ in $\mathbb{H}$, such that $n \mid q_k$ and $n \in \mathbb{N}$. Then $A$ is a common vertex of a fan in the cutting sequence of $\zeta_0$ with $F$, and so, at least two edges of the cutting sequence have $A$ as an endpoint. Let $B = \frac{s}{u}$ and $C = \frac{r}{s}$ be the other two endpoints of two such edges, with $\triangle ABC \in F$. Since $A$ is a neighbour of both $B$ and $C$ in $F$, $\gcd(q_k, s) = \gcd(n, s) = \gcd(q_k, u) = \gcd(n, u) = 1$. From Lemma 2.9, the edges $AB$ and $AC$ are in $F \cap \frac{1}{n} F$. By the above paragraph, there exists a map in $T_0(n)$ which takes $\infty$ to $A$, 0 to $B$ and 1 to $C$ (up to relabelling $B$ and $C$). Since $AB$ and $AC$ are both in $F \cap \frac{1}{n} F$, there are $n - 1$ edges between $AB$ and $AC$ in $\frac{1}{n} F$, all of which $\zeta_0$ passes through. All these edges share $A$ as an endpoint and so, they are all edges in the same fan. It follows from this, that the fan $\zeta_0$ forms with $\frac{1}{n} F$ containing both $AB$ and $AC$, contains at least $n$ triangles. Therefore, the cutting sequence $(\zeta_0, \frac{1}{n} F)$ contains a partial quotient with value at least $n$.

Since $n < q_k$, there exists a $q_k' > 1$ such that $q_k = nq_k'$. By the above argument $A = \frac{pk}{q_k}$ is a common vertex of a fan in the cutting sequence $(\zeta_0, \frac{1}{n} F)$. When we rescale using the $n$-scaling map, $\frac{pk}{q_k}$ in $\frac{1}{n} F$ maps to $\frac{pk}{q_k}$ in $F$, which is a common vertex in the cutting sequence $(\zeta_0, \frac{1}{n} F)$. Therefore, $\frac{pk}{q_k}$ is a convergent of $\frac{1}{n} \alpha$. Since $q_k' > 1$, it follows that $\frac{pk}{q_k}$ is not the common vertex of the first fan but necessarily of some fan after. As a result, the partial quotient of $\frac{1}{n} \alpha$ with value at least $n$ is not the first partial quotient. By definition $B(n \alpha) \geq a_i^{(n)}$ for all $i \in \mathbb{N}$ and since there exists an $a_i^{(n)} \geq n$, it follows that $B(n \alpha) \geq n$. \hfill $\square$

We can improve on this result by taking all such triangles in the fan with the common vertex $\frac{pk}{q_k}$ and subdividing each of these $n$ times, when taking $\frac{1}{n} F$. There are $a_k$ such triangles, where $a_k$ is the $k$-th partial quotient. Note that there may be extra terms in this fan (added either side). As a result, we get the following corollary.

**Corollary 2.11.** If a continued fraction $\alpha$ has a convergent denominator $q_k$, such that $n \mid q_k$ for $n \in \mathbb{N}$ and $n < q_k$, then $B(n \alpha) \geq n a_k$. 

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2.3.2 Fundamental domains of $\Gamma_0(n)$

All results and constructions in this sub-section are contained in [9].

Fundamental domains of $\Gamma_0(n)$ have been well studied with relation to modular forms. Notably, R.S. Kulkarni gives an explicit construction of a fundamental domain (with side pairings) using Farey symbols in [9]. This is the construction which we will use and as such we recall important results for ease.

A Farey Sequence is a sequence of vertices in $F$, $\{\infty, x_0, \ldots, x_r, \infty\}$, such that each consecutive pair of vertices $x_i$ and $x_{i+1}$ are neighbours in $F$ and there is some $i \in \{0, \ldots, r\}$ with $x_i = 0$. Given a Farey sequence, we construct a Farey symbol $\sigma$ by identifying each pair of consecutive vertices $x_i, x_{i+1}$ with one of the following intervals:

1. A free interval with label $a$ such that there is another pair of consecutive vertices $x_j, x_{j+1}$, which form a free interval and have the same label,

\[
x_i \overset{a}{\longrightarrow} x_{i+1} \quad \text{and} \quad x_j \overset{a}{\longrightarrow} x_{j+1}
\]

2. An even interval,

\[
x_i \overset{\bullet}{\longrightarrow} x_{i+1}
\]

3. An odd interval.

\[
x_i \overset{\circ}{\longrightarrow} x_{i+1}
\]

An example of a Farey symbol is:

\[
\left\{ \infty \overset{0}{\longrightarrow} \begin{array}{c} 1 \\ \hline \end{array} \overset{1}{\longrightarrow} \begin{array}{c} 2 \\ \hline \end{array} \overset{1}{\longrightarrow} \begin{array}{c} \infty \\ \hline \end{array} \right\}
\]

Remark. This Farey symbol corresponds to a fundamental domain for $\Gamma_0(7)$.

For each Farey symbol $\sigma$, we can then construct a special polygon $P_{\sigma}$ with edge identifications induced by the interval type, as seen below. We will see shortly that every special polygon is a fundamental domain for some finite index subgroup of $PSL_2(\mathbb{Z})$ and every finite index subgroup of $PSL_2(\mathbb{Z})$ admits a special polygon as a fundamental domain. We construct $P_{\sigma}$ as follows:

1. Given two free intervals $x_i \overset{a}{\longrightarrow} x_{i+1}$ and $x_j \overset{a}{\longrightarrow} x_{j+1}$, we construct a geodesic edge between $x_i$ and $x_{i+1}$ and another between $x_j$ and $x_{j+1}$. These edges are identified by the map taking the vertex $x_i$ to $x_{j+1}$ and $x_{i+1}$ to $x_j$. We refer to such pairs of edges as free sides of $P_{\sigma}$.
2. Given an even interval \( x_i \overset{\theta}{\rightarrow} x_{i+1} \), we construct a geodesic edge between \( x_i \) and \( x_{i+1} \). The edge maps to itself by mapping \( x_i \to x_{i+1} \) and vice versa (by the elliptic involution about the midpoint of the edge between \( x_i \) and \( x_{i+1} \)). We refer to such an edge as an even side of \( P_\sigma \).

3. Given an odd interval \( x_i \overset{\theta}{\rightarrow} x_{i+1} \), we take the unique triangle between \( x_i, x_{i+1} \) and \( x_i \oplus x_{i+1} \). We take \( y_i \) to be the centre of this triangle and construct the geodesic edges between \( x_i \) and \( y_i \) and between \( y_i \) and \( x_{i+1} \). See Fig. 6a and 6b. These edges are identified by the map which preserves \( y_i \) and maps \( x_i \) to \( x_{i+1} \) to \( x_i \oplus x_{i+1} \) and \( x_i \oplus x_{i+1} \) to \( x_i \). We refer to such edges as an odd sides of \( P_\sigma \).

To construct our side pairings we note that if we have a pair of vertices in \( \mathcal{F} \), \( x_i = \frac{a_i}{b_i} \) and \( x_{i+1} = \frac{a_{i+1}}{b_{i+1}} \) and we want a map in \( SL_2(\mathbb{Z}) \) taking \( x_i \) to \( x_j = \frac{a_j}{b_j} \) and \( x_{i+1} \) to \( x_{j+1} = \frac{a_{j+1}}{b_{j+1}} \), then this map will be of the form:

\[
\varphi := \begin{pmatrix} a_j b_i + a_{j+1} b_{i+1} & -a_j a_{i+1} \cr b_i b_j + b_{i+1} b_{j+1} & -a_i b_j - a_{i+1} b_{j+1} \end{pmatrix}
\]  

(1)

For any Farey symbol \( \sigma \), there is a collection of maps (defined above) corresponding to these edge identifications. We define \( \Phi_\sigma \) to be group generated by the edge identifications of \( \sigma \). By the Poincaré Polyhedron Theorem, \( P_\sigma \) is a fundamental domain for \( \Phi_\sigma \). Theorem (6.1) in [2] states that the edge identifications for any Farey symbol \( \sigma \), form an independent set of generators for \( \Phi_\sigma \). Moreover, the following Theorem explains the importance of special polygons.

**Theorem 2.12** (Theorem (3.2) and (3.3), R.S. Kulkarni, [2]). **Every special polygon is a fundamental domain for a finite index subgroup of \( PSL_2(\mathbb{Z}) \), which is generated by the side pairings.** Every finite index subgroup of \( PSL_2(\mathbb{Z}) \) admits a special polygon as a fundamental domain.

Assuming we have a Farey symbol \( \sigma \) with \( \Phi_\sigma = \Gamma_0(n) \) for some \( n \) an positive integer, every edge identification \( \varphi \) (as in (1)) must satisfy \( b_j b_j + b_{j+1} b_{j+1} \equiv 0 \mod n \). Therefore, for two pairs of neighbours in \( \mathcal{F} \), \( x_i \) and \( x_{i+1} \), and \( x_j \) and \( x_{j+1} \), there is a transformation in \( \Gamma_0(n) \) which maps the edge between \( x_i \) and \( x_{i+1} \), and \( x_j \) and \( x_{j+1} \) if and only if \( b_j b_j + b_{j+1} b_{j+1} \equiv 0 \mod n \), where \( b_k \) is the denominator of \( x_k \) in reduced form. In the case that \( b_i \neq b_{i+1} \) and \( b_{i+1} \neq b_j \), these edges form a pair of free sides. For an odd edge \( b_j = b_i \) and \( b_{j+1} = b_i \) and for an even edge \( b_j = b_i \) and \( b_{j+1} = b_i \), therefore, two neighbours in \( \mathcal{F} \), \( x_i \) and \( x_{i+1} \), form an odd edge if and only if \( b_i^2 + b_i b_{i+1} + b_{i+1}^2 \equiv 0 \mod n \). Similarly, two neighbours in \( \mathcal{F} \), \( x_i \) and \( x_{i+1} \), form an even edge if and only if \( b_i^2 + b_{i+1}^2 \equiv 0 \mod n \).

For any \( \Gamma_0(n) \), we can choose a special polygon \( P_\sigma \) (as fundamental domain) such that the \( y \)-axis \( I \) and \( I + 1 \) are paired sides of \( P_\sigma \). In particular, we can take \( x_0 = 0 \) and \( x_r = 1 \) in the corresponding Farey sequence. This is due to the fact that \( \{ \frac{1}{1}, \frac{1}{2} \} \in \Gamma_0(n) \), for all \( n \in \mathbb{N}_{>1} \).
For $p$ prime, we can find a Farey symbol $\sigma$ (for which $P_\sigma$ is a fundamental domain of $\Gamma_0(p)$), in which the vertices are symmetric in the line $x = \frac{1}{2}$ to $\infty$. In other words, the underlying Farey sequence will be of the form $\{\infty, 0, x_1, x_2, \ldots, x_n, 1, \infty\}$, where $x_i = 1 - x_i$. The term $\frac{1}{2}$ will be in every Farey symbol of $\Gamma_0(p)$ for $p \geq 5$. This is due to the fact that the line $0$ to $\frac{1}{2}$ separates $\mathbb{H}$ into two regions: one containing the vertex $1$ and the other containing all other neighbours of $0$. Therefore, to get a Farey symbol containing the terms $0$ and $1$, the underlying Farey sequence must either only contain the vertices $\infty$, $0$ and $1$ or the sequence must contain the vertex $\frac{1}{2}$. If we have either an odd or even interval, the interval identifications are symmetric in the line $\frac{1}{2}$ to $\infty$. However, for free intervals we have antisymmetry, i.e. the free interval labelled $a$ will be replaced with the label $a'$ in this symmetry. Due to the symmetry of the vertices in the line $x = \frac{1}{2}$ to $\infty$ and the pseudo-symmetry of the interval identifications, we will shorten the sequence up to the term $\frac{1}{2}$ (for $p \geq 5$, since for $p = 2, 3$ we only use the vertices $\infty$, $0$ and $1$). Similarly, due to identification of the lines $x = 0$ to $\infty$ and $x = 1$ to $\infty$, we will not include these terms in our sequence, with this identification being implicit. For example:

$$\{0 \quad x_1 \quad x_2 \quad x_3 \quad \frac{1}{2} \quad \text{refl.}\} = \{0 \quad x_1 \quad x_2 \quad x_3 \quad \frac{1}{2} \quad x_3' \quad x_2' \quad x_1' \quad 1\}$$

We can explicitly state how many odd, even and free intervals there will be in each fundamental domain of $\Gamma_0(n)$. This can be derived from the properties of the quotient space $\Gamma_0(n)\backslash \mathbb{H}$. For a prime $p \geq 5$ with $p \equiv 1 \mod 3$ there are exactly two odd intervals (either side of $\frac{1}{2}$), otherwise there are no odd intervals. Similarly, if $p \equiv 1 \mod 4$ there are exactly two even intervals (either side of $\frac{1}{2}$), otherwise there are no even intervals. If $p \equiv 1 \mod 3$, the Farey symbol has $\frac{p+2}{3}$ terms, otherwise the Farey symbol has $\frac{b+4}{3}$ terms.

Given $\Phi$ a subgroup of $PSL_2(\mathbb{Z})$, we can use the Riemann-Hurwitz formula to relate some geometric invariants the quotient space $\Phi\backslash \mathbb{H}$, as follows:

$$d = 3e_2 + 4e_3 + 12g + 6t - 12$$

where:

- $d$ is the index of $[PSL_2(\mathbb{Z}) : \Phi]$
- $e_2$ is the number orbifold points in $\Phi\backslash \mathbb{H}$ with cone angle $\pi$ (or equivalently the number of even intervals in a corresponding special polygon)
- $e_3$ is the number orbifold points in $\Phi\backslash \mathbb{H}$ with cone angle $\frac{2\pi}{3}$ (or equivalently the number of odd intervals in a corresponding special polygon)
- $g$ is the genus of $\Phi\backslash \mathbb{H}$
- $t$ is the number of cusps for $\Phi\backslash \mathbb{H}$
For $\Phi = \Gamma_0(n)$:

$$d = n \prod_{q \mid n} \left(1 + \frac{1}{q}\right),$$

$$t = \sum_{a \mid n} \varphi \left(\gcd \left(a, \frac{n}{a}\right)\right),$$

where $q$ is a prime number, $a \in \mathbb{N}$ and $\varphi$ is the Euler totient function.

Calculating this information for $\Gamma_0(p)$, we observe that the quotient space $\Gamma_0(p) \backslash \mathbb{H}$ will have 2 punctures, $e_2$ even intervals, $e_3$ odd intervals and genus $g$. The above relation then reduces to:

$$p + 1 = 3e_2 + 4e_3 + 12g$$

### 2.3.3 Decorated tiles of $\Gamma_0(n)$

**Definition 2.13.** We define $P_n$ to be a special polygon, with the y-axis $I$ and $I + 1$ as paired sides, which is a fundamental domain for $\Gamma_0(n)$.

If we take $P_n$ to be a fundamental domain for $\Gamma_0(n)$, we can construct $T_{\{1,n\}} := P_n \cup (P_n \cap \mathcal{F})$ and $T_{\{n,n\}} := P_n \cup (P_n \cap \frac{1}{d} \mathcal{F})$ to be two decorated tiles of $\Gamma_0(n)$ such that $\Gamma_0(n) \cdot T_{\{1,n\}} = \mathcal{F}$ and $\Gamma_0(n) \cdot T_{\{n,n\}} = \frac{1}{n} \mathcal{F}$. See Fig. 6 for images of $T_{\{1,n\}}$ and $T_{\{n,n\}}$ for $n = 7, 11$. We can similarly define $T_{\{d,n\}} := P_n \cup (P_n \cap \{\frac{1}{d} \mathcal{F}\})$ for every $d \mid n$ and the decorated tile $T_{\{d,n\}}$ together with the side pairings induced by $\Gamma_0(n)$ encodes sufficient data to recover $\frac{1}{d} \mathcal{F}$, for all $d \mid n$.

For any geodesic ray $\zeta_\alpha$, we can decompose $\zeta_\alpha$ into an ordered collection of sub-paths $\bigcup_{i=1}^\infty \zeta_{i,\alpha}$, such that each $\zeta_{i,\alpha}$ is entirely contained in an image of $P_n$ under its tessellation by $\Gamma_0(n)$. We will abuse notation and think of each $\zeta_{i,\alpha}$ as a sub-path in $P_n$. Then the cutting sequence of $\zeta_\alpha$ with $\frac{1}{d} \mathcal{F}$ for $d \mid n$, is equivalent to ordered product of the cutting sequences for each $\zeta_{i,\alpha}$ with $T_{\{d,n\}}$. Explicitly, $\prod_{i=1}^\infty (\zeta_{i,\alpha}, T_{\{d,n\}}) = (\zeta_{1,\alpha}, T_{\{d,n\}}) \cdot (\zeta_{2,\alpha}, T_{\{d,n\}}) \cdot \cdots = (\zeta_\alpha, \mathcal{F})$, where $(\zeta_{1,\alpha}, T_{\{d,n\}}) \cdot (\zeta_{2,\alpha}, T_{\{d,n\}}) = \{L^{R_{n_0}} R^{n_1} \cdots \}$. $\{L^{R_{n_0}} R^{n_1} \cdots \} = \{L^{R_{n_0}} R^{n_1} \cdots L^{R_{n_0}} R^{n_1} \cdots \}$.

**Remark.** On a technical note, we take $(\zeta_{i,\alpha}, T_{\{d,n\}})$ to be the generalised cutting sequence with the canonically induced labelled triangulation on $\mathcal{F}$. This is simply so that we do not need to form a "full" left or right triangle in $T_{\{d,n\}}$, and so the first term in $(\zeta_{i,\alpha}, T_{\{d,n\}})$ and the last term in $(\zeta_{i-1,\alpha}, T_{\{d,n\}})$ are both well-defined and without error.

**Algorithm for Integer Multiplication of a Continued Fraction by $n$.**

By using the above notions of sub-path, we obtain the following algorithm for multiplying a continued fraction $\alpha$ by $n$ some integer:

1. Construct the fundamental domain $P_n$ of $\Gamma_0(n)$.

2. Construct the decorated tiles of $T_{\{1,n\}}$ and $T_{\{n,n\}}$. 


3. Use the continued fraction $\pi$ to algorithmically recover a curve $\zeta_\alpha$ as a sequence of sub-paths $\bigcup_{i=1}^{\infty} \zeta_{i,\alpha}$ intersecting $T_{\{1,n\}}$.

4. Take the cutting sequence of each $\zeta_{i,\alpha}$ with respect to $T_{\{n,n\}}$.

5. Compute $\prod_{i=1}^{\infty} (\zeta_{i,\alpha}, T_{\{n,n\}}) = (\zeta_{1,\alpha}, T_{\{n,n\}}) \cdot (\zeta_{2,\alpha}, T_{\{n,n\}}) \cdot \ldots = (\zeta_\alpha, F)$.

Figure 6: Images of $P_7$ and $P_{11}$, with embedded structure of $F$ for (a) and (c), and embedded structure of $\frac{1}{2}F$ and $\frac{1}{11}F$ on (b) and (d) respectively. Side pairings are indicated by bold, short dashed and long dashed lines. For (a) and (c), the dashed lines are part of $F$.

**Remark.** The algorithm that we obtain is not particularly useful for explicit computation (such an algorithm can be derived from taking all possible sub-paths), however it does have some useful theoretic properties, some of which we will see later in this paper and some of which we leave for future work. A more explicit algorithm can be found in [13].
### 3 Cutting sequences on $\Gamma_0(n)\setminus \mathbb{H}$

In this section, we define cutting sequences on triangulated orbifolds and show the bijective relation between positive essentially periodic continued fractions (see definition 3.3.2) and closed curves on these triangulated orbifolds. Using this result, we show that the bounds of essentially periodic continued fractions grow exponentially when iteratively multiplied by a fixed integer. We then relate this result to the convergents of essentially periodic continued fractions and show that every eventually periodic continued fraction multiplies like an essentially periodic continued fraction.

#### 3.1 Cutting Sequences on $\Gamma_0(n)\setminus \mathbb{H}$

In Section 2.2, we defined cutting sequences of geodesic rays with respect to ideal triangulations of $\mathbb{H}$. The concept of a geodesic ray intersecting a triangle to form a left or right triangle is independent of metric and thus, for any triangulated surface we can define a cutting sequence for a geodesic ray relative to this triangulation.

However, when we take $\Phi_\sigma \setminus \mathbb{H}$, for $\sigma$ some Farey symbol (or equivalently, take the corresponding special polygon $P_\sigma$ and identify sides, which we denote $P_\sigma / \sim$), we are not necessarily left with a surface. Instead, we obtain a two-dimensional orbifold. Here we define a **two-dimensional orbifold** to be a surface $S$ (possibly with boundary), with a set of marked points $M$ and a potentially empty set of orbifold points $Q$. In the case that $Q$ is empty, the orbifold will be a surface. When we supply the orbifold with a metric, each element of $M$ will correspond to a **cusp** with angle 0 and each orbifold point will correspond to a **cone point** with angle $\frac{2\pi}{k}$ for some $k \in \mathbb{N} > 1$. We will only consider orbifolds with empty boundary, at least one cusp (element of $M$) and a potentially empty set of orbifold points $Q$. When we take the corresponding special polygon $P_\sigma$ and identify sides via the side pairings, we see that elements of $E_2$ correspond to the central points the even edges, elements of $E_3$ correspond to the interior vertices of $P_\sigma$ formed by the odd edges and elements of $M$ correspond to vertices on the boundary of $\mathbb{H}$ quotient the side pairings.

**Remark.** Following the definitions in Section 2.3.2, the number of cusps is exactly given by $|M| = t$ and the number of orbifold points with cone angles $\pi$ and $\frac{2\pi}{3}$ are $|E_2| = e_2$ and $|E_3| = e_3$, respectively.

We define an **arc** $\gamma$ on an orbifold $O$ to be a geodesic path, which is disjoint from $M \cup Q$ except from its endpoints, with the following properties:

- The endpoints of $\gamma$ are contained in $M \cup Q$ and at least one endpoint is in $M$,
- The only self-intersections of $\gamma$ occur at the endpoints of $\gamma$, if at all,
- If $\gamma$ bounds a monogon i.e. both endpoints of $\gamma$ are at the same point in $M$, then this monogon either contains one element of $M$, one element of $E_3$ or two elements of $E_2$.
If $\gamma$ has one endpoint in $E_3$, then we will say that $\gamma$ is a structural arc. We will say that a pair of arcs $\gamma, \gamma'$ are compatible, if $\gamma \cap \gamma' \subset M$ (i.e. $\gamma$ and $\gamma'$ only intersect at endpoints which are also marked points). Similarly, we define a quotient triangulation $T$ of an orbifold $O$ to be a maximal collection of pairwise compatible arcs on $O$. There are six possible types of triangles that can arise from a quotient triangulation, which we list in Table 1.

| Type | Name |
|------|------|
| (I)  | Standard triangle |
| (II) | Self-folded triangle |
| (IIIa)| Quotient-2 triangle (a) |
| (IIIb)| Quotient-2 triangle (b) |
| (IIIc)*| Quotient-2 triangle (c) |
| (IV) | Quotient-3 triangle |

Table 1: A table of the six possible types of triangles that can appear in a quotient triangulation and their lifts in $\mathbb{H}$. Elements of $P$ are indicated by $\bullet$, elements of $E_2$ are indicated by $\circ$ and elements of $E_3$ are indicated by $\Box$. Dashed lines indicate structural arcs and their lifts.

Remark. * The quotient-2 triangle (IIIc) occurs as a triangulation for exactly one orbifold. This orbifold has three elements in $E_2$ and a single cusp, and the triangle is formed by taking an arc between each point in $E_2$ and the cusp. Only one subgroup $\Phi$
Lemma 3.1. Let \( \Phi \) be a finite subgroup of \( PSL_2(\mathbb{Z}) \) (excluding \( \Gamma_3 \)), let \( P_\Phi \) be a special polygon which is a fundamental domain for \( \Phi \), and let \( T \) be an ideal triangulation of \( \mathbb{H} \), which is invariant under \( \Phi \). Then the projection of \( T \) decomposes \( \Phi \backslash \mathbb{H} \) into triangles of type (I)-(IIIb) or into monogons containing a single element of \( E_3 \). In particular, the projection of \( \Phi \) induces a unique quotient triangulation \( T_\Phi \) of \( \Phi \backslash \mathbb{H} \).

Proof. 1. Firstly, we will show that if a triangle \( \delta \) in the ideal triangulation \( T \) of \( \mathbb{H} \) does not contain a lift of an orbifold point in \( \Phi \backslash \mathbb{H} \), then \( \delta \) projects to a triangle of type (I) or (II) in \( \Phi \backslash \mathbb{H} \).

Let \( \delta \) be a triangle in the ideal triangulation \( T \) of \( \mathbb{H} \), which does not contain a lift of an orbifold point in \( \Phi \backslash \mathbb{H} \). Then, the projection of \( \delta \) in \( \Phi \backslash \mathbb{H} \) will not contain any elements of \( Q = E_2 \cup E_3 \). Since \( T \) is invariant under \( \Phi \), geodesics in \( \mathbb{H} \) will project to geodesic arcs in \( \Phi \backslash \mathbb{H} \) and these geodesic arcs will be pairwise disjoint except for at \( P \). As a result, the projection of \( \delta \) will be triangles of type (I) or (II).

2. We now show that if \( \delta \) contains the lift of an orbifold point in \( E_2 \), then \( \delta \) projects to a triangle of type (IIIa)-(IIIb) in \( \Phi \backslash \mathbb{H} \).

Claim: If \( P_\Phi \) contains an even edge \( e \), then any triangulation \( T \) preserved by \( \Phi \) must contain an edge that runs through \( m_e \), where \( m_e \) is the fixed point of \( \varphi_e \), the side pairing induced by the even edge \( e \).

Proof of Claim: First, we assume the opposite, that \( m_e \) is not intersected by any edge of \( T \). Then, since \( e_2 \) lies in the interior of \( \mathbb{H} \), \( m_e \) must lie in the interior of \( \delta \), some triangle in \( T \). Two vertices of \( \delta \) will lie on one side \( e_+ \) of the even edge \( e \) and one vertex of \( \delta \)
will lie on the other side $e_-$. Since $\varphi_e$ is an elliptic involution of order 2 with fixed point $m_e$, the image of $\varphi_e(\delta)$ will contain $m_e$ and have 2 vertices in $e_-$ and one vertex in $e_+$. Since $\varphi_e$ is an element of $\Phi$, it follows that $\varphi_e(\delta)$ must be a triangle in $T$ (since, $T$ is invariant under $\Phi$). Both triangles $\delta$ and $\varphi_e(\delta)$ contain the point $m_e$, however $\delta \neq \varphi_e(\delta)$, since the number of endpoints in $e_+$ and $e_-$ are different for $\delta$ and $\varphi_e(\delta)$. This implies $\delta$ and $\varphi_e(\delta)$ have non-trivial intersection and do not intersect along a common edge (since then $m_e$ would lie on this edge). Therefore, $T$ can not be an ideal triangulation and this is a contradiction to our initial assumptions.

QED.

It follows from the above claim, that if a triangle $\delta$ in $T$ contains the point $m_e$, then this point lies on one of the edges of $\delta$. Such a triangle can either have one, two or three edges which each contain the lift of a point in $E_2$. These triangles will project to a quotient triangle in $\Phi_\setminus \mathbb{H}$ of type (IIIa), (IIIb) or (IIIc) (which occurs only for $\Phi = \Gamma_3$), respectively.

3. Finally, we show that if $\delta$ contains the lift of an orbifold point in $E_3$, then $\delta$ projects to a monogon containing a single element of $E_3$. As seen above, we can then construct a unique structural arc between this element of $E_3$ and the element of $M$ which lies on the boundary of this monogon.

**Claim:** No edge in $T$ projects to a structural arc in $\Phi_\setminus \mathbb{H}$.

**Proof of claim.** Assume that $E$ is an edge of an ideal triangulation $T$ in $\mathbb{H}$, which projects to an structural arc in $\Phi_\setminus \mathbb{H}$. Then, $E$ must intersect the centre $c_e$ of a triangle formed be an odd edge $e$ in $P_\Phi$. We define $\varphi_e$ to be the side pairing induced by this odd edge. Then, $\varphi_e$ is an elliptic involution of order 3 with fixed point $c_e$. In particular, the images of $E$ under $Id$, $\varphi_e$ and $\varphi_e^{-1}$ are three geodesics which all intersect at $c_e$. Since $T$ is invariant under $\Phi$, all of the images of $E$ under $\Phi$ (and therefore under $Id$, $\varphi_e$ and $\varphi_e^{-1}$) must be edges in $T$. See Fig. 7(a). However, we have multiple geodesics intersecting inside $\mathbb{H}$ and therefore, $T$ can not be an ideal triangulation and this is a contradiction to our initial assumptions. 

QED.

Following this claim, the point $c_e$ must lie in the interior of some triangle $\delta$ in $T$. The elliptic involution about $c_e$ will split $\mathbb{H}$ into three different regions, each containing a vertex of $\delta$. Therefore, the projection of $\delta$ on to $\Phi_\setminus \mathbb{H}$ will be a monogon containing a single orbifold point with cone angle $\frac{2\pi}{3}$. 

Let $\lambda$ be a path on $\Phi_\setminus \mathbb{H}$ and $\delta$ be a triangle in a quotient triangulation $T_\Phi$, which $\lambda$ passes through. Then we will say that $\lambda$ cuts $\delta$ to form a left (or right) triangle if, once having removed the structural arcs, the lift of $\lambda$ cuts the lift of $\delta$ to form a left (or right) triangle. We then derive the cutting sequence of $\lambda$ with $T_\Phi$ in the usual sense, which we denote $(\lambda, T_\Phi)$. Here the space $\Phi_\setminus \mathbb{H}$ is implied by the quotient triangulation. Obviously, the cutting sequence $(\lambda, T_\Phi)$ will be equivalent to the cutting sequence $(\lambda, \overline{T})$, where $\overline{\lambda}$
A geodesic line (bold) passing through the point \( c_e \) (left), and its images under \( Id, \varphi_e \) and \( \varphi_e^{-1} \) (right).

A pair of geodesic rays (left), which form a triangle under the action of \( Id, \varphi_e \) and \( \varphi_e^{-1} \) (right).

Figure 7: Examples of edges and their images under the actions of \( Id, \varphi_e \) and \( \varphi_e^{-1} \), where \( \varphi_e \) is an elliptic involution of order 3 with fixed point \( c_e \). The corresponding odd edge \( e \) is also shown for structure.

is the lift of \( \lambda \) in \( \mathbb{H} \) and \( \overline{T} \) is the lift of \( T_\Phi \) with structural arcs removed (which will be an ideal triangulation).

Note that since we remove the structural arcs when defining the left and right triangles, we do not necessarily need a quotient triangulation to take a cutting sequence and a quotient triangulation with structural arcs removed will be sufficient. However, since there is a unique way to construct these structural arcs, we will equate these two objects anyway.

Since \( \Gamma_0(n) \setminus \mathbb{H} \) is equivalent to \( P_n/\sim \) (a special polygon for \( \Gamma_0(n) \) quotient the side identifications), it follows that the projection of \( \frac{1}{d}F \) onto \( \Gamma_0(n) \setminus \mathbb{H} \) for \( d \mid n \), is equivalent to \( T_{(d,n)/\sim} \), the copy of \( P_n \) with decoration induced by \( \frac{1}{d}F \), quotient the side identifications. Using this information and Lemma 3.1, we get the following theorem.

**Theorem 3.2.** For every geodesic ray \( \tilde{\zeta} \) in \( \mathbb{H} \) starting at the y-axis \( I \) with endpoint \( \alpha > 0 \), there is a canonical projection \( \zeta \) onto \( \Gamma_0(n) \setminus \mathbb{H} \) such that \( (\tilde{\zeta}, \frac{1}{d}F) = (\zeta, T_{(d,n)/\sim}) \), for all \( d \mid n \).

**Proof.** By Lemma 3.1 we see that the projection of \( \frac{1}{d}F \) on \( \Gamma_0(n) \setminus \mathbb{H} \) decomposes \( \Gamma_0(n) \setminus \mathbb{H} \) into triangles of type (I)-(IIIb) or into monogons containing a single element of \( E_3 \), for
all \( d \mid n \). Since the y-axis \( I \) is an edge in \( \frac{1}{d} \mathcal{F} \) for all \( d \in \mathbb{N} \), the projection \( \zeta \) of \( \tilde{\zeta} \) in \( \Gamma_0(n) \backslash \mathbb{H} \) is unique and has a well defined starting edge and direction of departure, for all \( T_{(d,n)} \sim \). It follows that, since the cutting sequence in \( \Gamma_0(n) \backslash \mathbb{H} \) of \( \zeta \) with \( T_{(d,n)} \sim \) is independent of structural arcs, the cutting sequence in \( \Gamma_0(n) \backslash \mathbb{H} \) of \( \zeta \) with \( T_{(d,n)} \sim \) is both well-defined and equivalent to the cutting sequence in \( \mathbb{H} \) of \( \zeta \) with \( \frac{1}{d} \mathcal{F} \) for all \( d \mid n \).

3.2 Closed Curves as Cutting Sequences

For an arbitrary infinite sequence \( \{a_i\}_{i \in \mathbb{N}} \), we will say the sequence is periodic if there exists an \( s \in \mathbb{N} \) such that \( a_i = a_{s+i} \) for all \( i \in \mathbb{N} \). We will write this sequence as \( \{a_i\}_{i \in \mathbb{N}} = a_1, a_2, \ldots, a_s, a_1, a_2, \ldots, a_s, \ldots \) and refer to \( s \) as a period of the sequence. We use this to define the following types of continued fraction.

Definition 3.3. 1. A strictly periodic continued fraction, is any continued fractions with partial quotient expansion of the form \( [a_0; a_1, \ldots, a_{s-1}] \) or \( [0; a_1, \ldots, a_s] \). We refer to the set of all strictly periodic continued fractions as \( \text{SP} \).

2. An essentially periodic continued fraction, is any continued fractions with partial quotient expansion of the form \( [a_0; a_1, \ldots, a_s] \) with \( a_0 \leq a_s \) or \( [0; a_1, a_2, \ldots, a_{s+1}] \) with \( a_1 \leq a_s \). We refer to the set of all essentially periodic continued fraction as \( \text{ESP} \).

3. An eventually periodic continued fraction, is any continued fraction with partial quotient expansion of the form \( [b_0; \ldots, b_r, a_1, \ldots, a_s] \) where \( r \in \mathbb{N} \cup \{0\} \). We refer to the set of all eventually periodic continued fraction as \( \text{EVP} \).

Remark. An immediate consequence of these definitions is that \( \text{SP} \subset \text{ESP} \subset \text{EVP} \). We refer to the set of all positive strictly periodic continued fractions as \( \text{SP}^+ \). Similarly, we refer to the sets of positive essentially periodic continued fractions and positive eventually periodic continued fractions as \( \text{ESP}^+ \) and \( \text{EVP}^+ \), respectively.

We can equivalently define essentially periodic continued fractions to be continued fractions of the form \( [a_0; a_1, \ldots, a_s] \) or \( [0; a_1, \ldots, a_s] \), where \( a_0 \in \mathbb{Z} \) and \( a_i \in \mathbb{N} \cup \{0\} \). We will simplify this alternative definition to the case where \( a_i \in \mathbb{N} \) for \( 0 < i < s \) but \( a_n \in \mathbb{N} \cup \{0\} \), since we can always find such a presentation. In the case that \( a_s \neq 0 \), we just get a strictly periodic continued fraction. In the case that \( a_s = 0 \), we concatenate terms either side of the 0 term. Explicitly, we have the following:

\[
[a_0; a_1, \ldots, a_{s-1}, 0] = [a_0; a_1, \ldots, a_{s-1}, 0, a_0, a_1, \ldots, a_{s-1}, 0, a_0, \ldots] \\
= [a_0; a_1, \ldots, a_{s-1} + a_0, a_1, \ldots, a_{s-1} + a_0, \ldots] \\
= [a_0; a_1, \ldots, a_{s-1} + a_0]
\]

\[
[0; a_1, \ldots, a_{s-1}, 0] = [0; a_1, \ldots, a_{s-1}, 0, a_1, \ldots, a_{s-1}, 0, a_1, \ldots] \\
= [0; a_1, \ldots, a_{s-1} + a_1, a_2, \ldots, a_{s-1} + a_1, \ldots] \\
= [0; a_1, a_2, \ldots, a_{s-1} + a_1]
\]
In terms of fans, a zero term corresponds to having an empty fan. Therefore, since the fans either side are of the same type, they both collapse into one bigger fan. It is occasionally useful to have a place holder fan of size zero, particularly when dealing with closed curves on our orbifold.

**Remark.** We can similarly define the notion of being strictly, essentially or eventually periodic for all sequences of numbers and in particular for cutting sequences.

For all eventually periodic continued fractions, we can write the periodic part with even period. To do this we simply take two copies of the period and take this to be our new period i.e. \([b_0; b_1, \ldots, b_r, a_1^1, \ldots, a_j^1, \ldots, a_{2s}^1] = [b_0; b_1, \ldots, b_r, a_1, \ldots, a_s, a_1, \ldots, a_s] = [b_0; b_1, \ldots, b_r, a_1^1, \ldots, a_{2s}^1]\), where \(a_j^1 = a_i\) for \(j \equiv i \mod s\). For the rest of the paper, we will write all eventually periodic continued fractions with an even period. Taking the period to be even ensures that when we take the associated cutting sequence, the initial term and the final term of the period will correspond to different letters i.e. \(L^{a_0} \cdots R^{a_2}\).

This ensures that the parity of the cutting sequence will be nice, i.e. when we take multiple copies of the period, every term in the sequence will alternate. The following example emphasises why we take even periods:

\[
\begin{align*}
[2;1,1] &= \eta(L^2 R L R^2 L R) \neq \eta(L^2 R L) \\
[2;1,1,0] &= \eta(L^2 R L R^0) = \eta(L^2 R L R^0 L^2 R L^0) = \eta(L^2 R L L^2 R L) = \eta(L^2 R L)
\end{align*}
\]

We can also ensure that the finite prefix of an eventually periodic continued fraction has an even number of terms by shifting the period by a single term, if necessary. In other words, if we had the continued fraction expansion \([b_0; b_1, \ldots, b_{2s}, a_1^1, \ldots, a_{2s}^1]\) then this is equivalent to \([b_0; b_1, \ldots, b_{2s}, a_1, a_2, \ldots, a_{2s}, a_1^1]\).

Whilst \(ESP^+\) is perhaps an unnatural object in a typical number theory setting, it is a very natural object with regards to the geometric approach. This is emphasised in the following theorem.

**Theorem 3.4.** Let \(O\) be an orbifold with quotient triangulation \(T\). Then a path \(\zeta\) relative to a starting edge \(E\) in \(T\) (excluding structural arcs) is homotopic to a closed curve on \(O\) if and only if \(\eta((\zeta, T)) \in ESP^+\).

**Proof.** (\(\Rightarrow\)): Let \(\zeta\) be a closed curve which passes through the edge \(E\) in \(O\). Let \(\zeta_i\) be the representation of \(\zeta\), which starts and ends at \(E\) and follows \(\zeta\) exactly once. Similarly, define \(\zeta_i\) for \(i \in \mathbb{N}\), to be the representation of \(\zeta\) that starts and ends at \(E\) and goes round \(\zeta\) exactly \(i\) times. By using a zero term as place holder (if required), we can always write \((\zeta_i, S)\) in the form \(\{n_0, n_1, \ldots, n_{2m-1}\} = L^{n_0} R^{n_1} \cdots R^{n_{2m-1}}\), where \(n_0, n_{2m-1} \in \mathbb{N} \cup \{0\}\) and \(n_i \in \mathbb{N}\) for \(i \in \{1, \ldots, 2m - 2\}\). As a result, we can express \((\zeta_i, T)\) in alternating fans, as follows:

\[
(\zeta_i, T) = \{n_0, n_1, \ldots, n_{2m-1}, \ldots, n_0, n_1, \ldots, n_{2m-1}\}
= L^{n_0} R^{n_1} \cdots R^{n_{2m-1}} \cdots L^{n_0} R^{n_1} \cdots R^{n_{2m-1}}
\]
Taking the limit as \( i \) tends to infinity, we observe that \((\zeta, T) = \lim_{i \to \infty}(\zeta_i, T) = (n_{0}, n_{1}, \ldots, n_{2m-1}) = L^{n_{0}}R^{n_{1}}\ldots R^{n_{2m-1}}\). We investigate the four cases, which arise from this, based on whether \( n_{0}, n_{2m-1} \) are zero or non-zero:

- If \( n_{0} \neq 0 \) and \( n_{2m-1} \neq 0 \), then the resulting word is reduced and therefore \( \eta((\zeta, T)) \in SP^{+} \subset ESP^{+} \).

- If \( n_{0} = n_{2m-1} = 0 \), then \((\zeta, S) = L^{0}R^{n_{1}}\ldots R^{0} = R^{n_{1}}\ldots R^{n_{2m-2}} \). This is now in reduced form and therefore \( \eta((\zeta, T)) \in SP^{+} \subset ESP^{+} \).

- If \( n_{0} = 0 \) and \( n_{2m-1} \neq 0 \), then \((\zeta, T) = L^{0}R^{n_{1}}\ldots R^{n_{2m-1}} = R^{n_{1}}\ldots R^{n_{2m-1}} = R^{n_{1}}L^{n_{2}}\ldots R^{n_{1}+n_{2m-2}} \). This is now in reduced form and therefore \( \eta((\zeta, T)) \in ESP^{+} \).

- If \( n_{0} \neq 0 \) and \( n_{2m-1} = 0 \), then \((\zeta, T) = L^{n_{0}}R^{n_{1}}\ldots R^{0} = L^{n_{0}}R^{n_{1}}\ldots L^{n_{2m-2}} = L^{n_{0}}L^{n_{2}}\ldots L^{n_{0}+n_{2m-2}} \). This is now in reduced form and therefore \( \eta((\zeta, T)) \in ESP^{+} \).

Since all possible forms of \((\zeta, T)\) satisfy \( \eta((\zeta, T)) \in ESP^{+} \), the result follows.

\((\Leftarrow):\) Since \( O \) is finitely triangulated with quotient triangulation \( T \), we can define a cutting sequence on it. We denote the set of non-structural arcs in \( T \) as \( \mathcal{E} \). Every edge \( E \in \mathcal{E} \) will be an edge of at most two triangles in the triangulation \( T \), which we arbitrarily label \( \tau^{+}_{E} \) and \( \tau^{-}_{E} \). For each of these triangles, we can approach \( E \) from at most two directions, that is, either via a left triangle or a right triangle. Note that for \( E \) an edge from an element of \( M \) to an element of \( E_{2} \), this edge will have a single direction of approach. We define \( \overline{E} \) to be the set of all possible directions of approach for all edges in \( O \). We can then think of the abstract set of all theoretical directions of approach as the Cartesian product \( \mathcal{E} \times \{+, -\} \times \{L, R\} \), where the set \( \{+, -\} \) represents the choice of approaching an edge \( E \) via \( \tau^{+}_{E} \) or \( \tau^{-}_{E} \) and the set \( \{L, R\} \) represents the choice of approaching an edge \( E \) via left or right triangles. Since \( \mathcal{E}, \{+, -\} \) and \( \{L, R\} \) are all finite sets, \( \overline{E} \subset \mathcal{E} \times \{+, -\} \times \{L, R\} \) is also a finite set. See Fig. 8 for a pictorial representation of direction of approach.

Let \( \overline{\alpha} \) be a positive essentially periodic continued fraction with one of two forms:

- (i) \( \overline{\alpha} = [a_{0}; a_{1}, \ldots, a_{2s-1}] \) with \( a_{s-1}, a_{2s-1} \in \mathbb{N} \cup \{0\} \) and \( a_{i} \in \mathbb{N} \) otherwise.
- (ii) \( \overline{\alpha} = [0; a_{1}, \ldots, a_{2s}] \) with \( a_{s}, a_{2s} \in \mathbb{N} \cup \{0\} \) and \( a_{i} \in \mathbb{N} \) otherwise.

For case (i), we can write \( \eta^{-1}(\overline{\alpha}) = L^{a_{0}}R^{a_{1}}\ldots R^{a_{2s-1}} \). Similarly, for case (ii), we can write \( \eta^{-1}(\overline{\alpha}) = R^{a_{0}}L^{a_{2}}\ldots L^{a_{s}} \). We define the object \( \overline{\alpha}_{1} \) as follows:

\[ \overline{\alpha}_{1} = \begin{cases} [a_{0}; a_{1}, \ldots, a_{2s-1}] & \text{for case (i)} \\ [0; a_{1}, a_{2}, \ldots, a_{2s}] & \text{for case (ii)} \end{cases} \]

For both of the above values of \( \overline{\alpha}_{1} \), \( \eta^{-1}(\overline{\alpha}_{1}) \) starts and ends with different letters, since \( \overline{\alpha}_{1} \) is of even length.
Figure 8: An image depicting the four possible directions of approach to the blue edge $E$.

Given a starting edge $E_0$ with chosen direction of departure, we can find a homotopy class of geodesic rays $[\zeta]$, such that $\eta((\zeta, T)) = \bar{\alpha}$, for all $\zeta \in [\zeta]$. We define $\zeta_1$ to be a geodesic path (unique up to homotopy), which starts at $E_0$ (with the same direction of departure as $\zeta$), terminates at some edge $E_1$ and satisfies $\eta((\zeta_1, T)) = \bar{\alpha}_1$. We similarly define $\zeta_i$ to be a geodesic path (up to homotopy) in $O$ relative to $E_{i-1}$, with direction of departure opposite to the direction of approach for $\zeta_{i-1}$, such that $(\zeta_i, T) = \bar{\alpha}_1$. Due to the fact we took $\alpha_1$ with even length, $\zeta_i$ approaches $E_i$ via the same type of triangles for every $i \in \mathbb{N}$, i.e. if $\zeta_1$ approaches $E_1$ via left triangles, then every $\zeta_i$ approaches $E_i$ via left triangles. Note that this fan may in fact be empty but this does not change the fact that the cutting sequence consists solely of alternating terms. Since the cutting sequences $(\zeta, T)$ and $(\bigcup_{i=1}^{\infty} \zeta_i, T)$ (with canonical ordering) are equivalent and $\zeta$ and $\zeta_1$ have the same starting edge, we can homotope the collection of $\zeta_i$, $\bigcup_{i=1}^{\infty} \zeta_i$, to look like $\zeta$. Then, the set of edges (with the direction of approach) which are the endpoints for each $\zeta_i$, $\bigcup_{i=1}^{\infty} E_i$ is a subset of $\bar{E}$. Since $i$ runs from 1 to $\infty$ and $\bar{E}$ is finite, it follows from the pigeon-hole principle that there exist $j, k \in \mathbb{N}$ such that $E_j = E_{j+k}$, with the same direction of approach and the same type of approach via left/right triangles.

As seen above, for each $E_i$, we can uniquely define up to homotopy the path $\zeta_i$. By the same argument, for each $E_i$ we can uniquely define up to homotopy the path $\zeta_{i-1}$. Hence, if $E_{j+k} = E_j$, then $\zeta_{j-1}$ and $\zeta_{j+k-1}$ are homotopic and $E_{j-1} = E_{j+k-1}$. Using iteration, we recover that for such a $k$, $E_0 = E_k$. For each $l, l' \in \mathbb{N}$ with $l \equiv l'$ mod
k, the paths \( \zeta \) and \( \zeta' \) are homotopic. We can homotope all such paths \( \zeta \) and \( \zeta' \) to be concurrent and since \( E_0 = E_k \), we recover a closed curve. Since \( \zeta \) is homotopic to \( \bigcup_{i=1}^{\infty} \zeta_i \), \( \zeta \) is therefore homotopic to a closed curve.

It is worth noting that this technique works for any type of periodicity, that is, if \( \overline{\alpha} \) has a periodic tail, the representation \( \zeta_\alpha \) of \( \overline{\alpha} \), relative to some quotient triangulation of \( \mathcal{O} \), is homotopic to a closed curve. If \( \overline{\alpha} \) is essentially periodic, \( \zeta_\alpha \) is homotopic to a closed curve, as above. If \( \overline{\alpha} \) is eventually periodic, \( \zeta_\alpha \) can be decomposed into a finite path which joins onto an infinite path homotopic to a closed curve. As a result, we can show that \( ESP^+ \) and \( EVP^+ \) are both closed classes under multiplication by any rational number. Below we provide the statement for \( ESP^+ \), but the proof that \( EVP^+ \) is closed under rational multiplication follows trivially.

**Corollary 3.5** (Theorem 3.4). The \( \overline{\pi} \) map, maps \( ESP^+ \) to \( ESP^+ \). In particular, for all \( q \in \mathbb{Q}_{>0} \) and \( \overline{\alpha} \in ESP^+ \), we have \( \overline{q\alpha} \in ESP^+ \).

**Proof.** Let \( \zeta_\alpha \) be a geodesic ray, starting at the y-axis and terminating at some point \( \alpha > 0 \) with \( \overline{\alpha} \in ESP^+ \). Then \( \overline{\zeta_\alpha} \) is homotopic to a closed curve in \( T_{(1,n)}/\sim \). The map \( \overline{\pi} : (\zeta_\alpha, T_{(1,n)}/\sim) \to (\overline{\zeta_\alpha}, T_{(n,n)}/\sim) \) maps \( \overline{\zeta_\alpha} \) to itself. As such, \( \zeta_\alpha \) is a closed curve in \( T_{(n,n)}/\sim \) and \( \overline{\alpha} = (\zeta_\alpha, T_{(n,n)}/\sim) \) defines a positive essentially periodic continued fraction by Theorem 3.4.

We can similarly define the map \( \overline{\pi}^{-1} : (\zeta_\alpha, T_{(n,n)}/\sim) \to (\zeta_\alpha, T_{(1,n)}/\sim) \) to recover \( \overline{\pi} = (\zeta_\alpha, T_{(1,n)}/\sim) \) from \( \overline{\alpha} = (\zeta_\alpha, T_{(n,n)}/\sim) \). Let \( q = \frac{\alpha}{s} \). Since \( \overline{q\alpha} \in ESP^+ \) and \( \overline{\pi} \in ESP^+ \), \( \overline{q\alpha} = \overline{\frac{\alpha}{s}} \in ESP^+ \). □

### 3.2.1 Convergents of Essentially Periodic Continued Fractions

The property of \( \zeta_\alpha \) being homotopic to a closed curve on some orbifold \( \mathcal{O} \) can also be used to prove some interesting facts regarding the convergent denominators/numerators of both strictly periodic and essentially periodic continued fractions. Here we take our orbifold to be \( \Gamma_0(n) \backslash \mathbb{H} \) for some \( n \in \mathbb{N} \). Using the link between convergents and common vertices of the fans in \( (\zeta_\alpha, \mathcal{F}) \), and Theorem 3.4, we show that for any natural number \( n \) and \( \overline{\alpha} \) any strictly periodic continued fraction, \( \alpha \) has infinitely many convergent denominators and numerators divisible by \( n \). We also show for any natural number \( n \) and any essentially periodic continued fraction \( \overline{\alpha} \), that:

- For \( \alpha > 1 \), there are infinitely many convergent denominators of \( \overline{\alpha} \) which are divisible by \( n \).

- For \( \alpha < 1 \), there are infinitely many convergent numerators of \( \overline{\alpha} \) which are divisible by \( n \).

**Theorem 3.6.** Let \( \overline{\alpha} \in SP^+ \), then for every \( n \in \mathbb{N} \) there are infinitely many convergent denominators \( q_k \) of \( \overline{\alpha} \) such that \( n \mid q_k \).

Let \( \overline{\alpha} \in ESP^+ \) with \( \alpha > 1 \), then for every \( n \in \mathbb{N} \) there are infinitely many convergent denominators \( q_k \) of \( \overline{\alpha} \) such that \( n \mid q_k \).
Proof. Let $\zeta_\alpha$ be a geodesic ray, starting at the $y$-axis $I$ and terminating at some point $\alpha \in \mathbb{R}_{>0}$, with $\bar{\alpha} \in ESP^+$. We can take $\bar{\zeta}_\alpha$ to be the image of $\zeta_\alpha$ in the orbifold $\Gamma_0(n) \setminus \mathbb{H}$, with $(\bar{\zeta}_\alpha, \overline{T(1,n)\gamma}) = (\zeta_\alpha, F)$. By Theorem 3.4 it follows that $\bar{\zeta}_\alpha$ is homotopic to a closed curve in $\Gamma_0(n) \setminus \mathbb{H}$. We can take $E$ to be the image of $I$ in $\Gamma_0(n) \setminus \mathbb{H}$ and as a result $\bar{\zeta}_\alpha$ will start at the edge $E$. Since $\bar{\zeta}_\alpha$ is homotopic to a closed curve, it follows that $\bar{\zeta}_\alpha$ intersects this edge $E$ infinitely often.

Since $P_n$ is a fundamental domain for $\Gamma_0(n)$ and $I$ maps to $E$, the edge $E$ lifts to the set of all edges $\Gamma_0(n)(I) := \{ \rho(I) : \rho \in \Gamma_0(n) \}$ in $\mathbb{H}$. From section 2.3.1 we can explicitly describe $\Gamma_0(n)(I)$ as the set of edges with vertices $\frac{a}{nc}$ and $\frac{b}{d}$, with $(\frac{a}{nc}, \frac{b}{d}) \in \Gamma_0(n)$. For $\rho = (\frac{a}{nc}, \frac{b}{d}) \in \Gamma_0(n)$, $\rho(\infty) = \frac{a}{nc}$ and $\rho(0) = \frac{b}{d}$.

Because $\bar{\zeta}_\alpha$ intersects $E$ infinitely often, $\zeta_\alpha$ intersects $\Gamma_0(n)(I)$ infinitely often. If $\zeta_\alpha$ intersects $\rho(I)$, for some $\rho \in \Gamma_0(n)$, then due to the fact that every point $\rho(\infty)$ behaves locally like $\infty$ in $F$ and $\frac{1}{n}F$ (see section 2.3.1), $\rho(\infty)$ will be a convergent of $\bar{\alpha}$ if and only if $\zeta_\alpha$ approaches the edge $\rho(I)$ via a left triangle or departs the edge $\rho(I)$ via a left triangle.

We consider four different cases for the possible values of $(\zeta_\alpha, F)$ for $\bar{\alpha} \in ESP^+$:

(i) $\bar{\alpha} \in SP^+$ $\alpha > 1$: then $(\zeta_\alpha, F) = L^{a_0}R^{a_1}\cdots R^{a_{2s-1}}$ and $a_i \in \mathbb{N}$.

(ii) $\bar{\alpha} \in SP^+$ $\alpha < 1$: then $(\zeta_\alpha, F) = R^{a_1}\cdots R^{a_2}$ and $a_i \in \mathbb{N}$.

(iii) $\bar{\alpha} \in ESP^+$ $\alpha > 1$: then $(\zeta_\alpha, F) = L^{a_0}R^{a_1}\cdots L^{a_{2s}} = L^{a_0}R^{a_1}\cdots L^{a_{2m-\alpha_0}}R^0$ with $0 < a_0 \leq a_{2s}$ and $a_i \in \mathbb{N}$.

(iv) $\bar{\alpha} \in ESP^+$ $\alpha < 1$: then $(\zeta_\alpha, F) = R^{a_1}L^{a_2}\cdots R^{a_{2s-1}} = R^{a_1}L^{a_2}\cdots R^{a_{2s-1}}L^0$ with $0 < a_1 \leq a_{2s-1}$ and $a_i \in \mathbb{N}$.

From the construction in Theorem 3.4, we can guarantee that we approach the edge $E$ via the last term in the period and depart $E$ via the first term in our period. Therefore, we can guarantee that $\zeta_\alpha$ approaches or departs infinitely many edges $\rho(I)$ via the last term in our period or the first term in our period respectively. For (i)-(iii) we can always write the period to start or end with non-empty left fan and as such, there exist infinitely many elements $\rho \in \Gamma_0(n)$ such that $\rho(\infty)$ will be a convergent of $\bar{\alpha}$. For case (iv), we can not guarantee such a result. Since $\alpha \neq \infty$, $\zeta_\alpha$ can only intersect finitely many edges $\rho(I)$, where $\rho(\infty) = \infty$. In particular, for cases (i)-(iii), $\bar{\alpha}$ has infinitely many convergents of the form $\frac{a}{nc}$ for $a, c, \in \mathbb{N}$.

Unfortunately, this result does not hold in general for $\bar{\alpha} \in ESP^+$ with $0 < \alpha < 1$. When we write the corresponding cutting sequence, the period both starts and ends with a non-empty right fan and as such we can not guarantee a convergent of the form $\frac{a}{nc}$. An example of such a result is that for $\bar{\alpha} = [0; 1, 1, 2]$, 5 does not divide any of the convergent denominators. Since, for $0 < \alpha < 1$ with $\bar{\alpha} \in ESP^+$, $\frac{1}{\alpha} > 1$ and $\frac{1}{\alpha} \in ESP^+$, we get the following corollary.

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Corollary 3.7. Let $\alpha \in \text{SP}^+$, then for every $n \in \mathbb{N}$ there are infinitely many convergent numerators $p_k$ of $\alpha$ such that $n \mid p_k$. 

Let $\alpha \in \text{ESP}^+$ with $0 < \alpha < 1$, then for every $n \in \mathbb{N}$ there are infinitely many convergent numerators $p_k$ of $\alpha$ such that $n \mid p_k$.

3.3 Eventually Periodic Continued Fractions in Relation to Essentially Periodic Continued Fractions

Theorem 3.8. Let $\beta$ be in $\text{EVPP}^+ \setminus \text{ESP}^+$. Then for every $n \in \mathbb{N}$ there exists an $a, k \in \mathbb{N}$ and $\alpha \in \text{ESP}^+$ such that $mn^k \beta = ma + \overline{n \alpha}$. In particular, $n^k \beta$ will be of the form $[a_0 \mid a_1, \ldots, a_s]$ with $a_0 > a_s$.

Outline of proof: For any $\beta \in \text{EVPP}^+ \setminus \text{ESP}^+$, we take the corresponding geodesic ray $\zeta_\beta$, starting at the $y$-axis $I$ and terminating at $\beta$ and we take an edge $E'$ in $F$, which splits $\zeta_\beta$ into a periodic part $\xi_\beta$ and a non-periodic part $\lambda_\beta$. Note that we can view periodicity of $\xi_\beta$ as a geometric property in the sense that when we take the quotient orbifold $\Gamma_0(m) \setminus \mathbb{H}$, $\xi_\beta$ is homotopic to a closed curve. This is true for all $m \in \mathbb{N}$. In particular, any sub-ray of $\zeta_\beta$, which starts after $E'$ will be "geometrically periodic". Since $\text{ESP}^+$ we show that there exists $a, k \in \mathbb{N}$ such that the line $E$ from $\frac{a}{n^k}$ to $\infty$ intersects $\xi_\beta$. If we take $\xi_\beta \in \mathbb{N}$ by $n^k$, we see that $n^k \xi_\beta$ is still "geometrically periodic" and starts at the line from $a$ to $\infty$. Since $n^k \xi_\beta \in \mathbb{N}$ it follows that $\eta((n^k \xi_\beta, E)) = a + \eta((n^k \xi_\beta, F))$ and $\eta((n^k \xi_\beta, E)) = ma + \eta((n^k \xi_\beta, E))$ for any $m \in \mathbb{N}$. See Fig. 9.

Proof. We begin by looking at geodesic rays on $\mathbb{H}$. Let $\zeta_\beta$ be a geodesic ray, starting at the $y$-axis $I$ and terminating at some point $\beta > 0$, such that $\beta \in \text{EVPP}^+ \setminus \text{ESP}^+$. We can write $(\zeta_\beta, F)$ to be in the form $L_{a_0} \cdots R_{a_0} \cdots L_{a_0} \cdots R_{a_0} \cdots$ with $s_0 \in \mathbb{N} \cup \{0\}$ and $a_0, a_i \in \mathbb{N}$. We take $\lambda_\beta$ to be the finite sub-path of $\zeta_\beta$, which starts at $I$, terminates at some edge $E'$ and has cutting sequence $(\lambda_\beta, F) = L_{a_0} \cdots R_{a_0} \cdots$. Similarly, we take $\xi_\beta$ to be the infinite sub-path of $\zeta_\beta$, which starts at $E$, terminates at $\beta$ and has cutting sequence $(\xi_\beta, F) = L_{a_0} \cdots R_{a_0} \cdots$. Let $(x, y)$ be the Cartesian co-ordinates of $\zeta_\beta \cap L'$. Necessarily $x < \beta$ since the geodesic ray approaches $\beta$ from the left and if we were to assume $x \geq \beta$ then the unique geodesic which passes through $(x, y)$ and $\beta$ could not also pass through the line from $0$ to $\infty$. Thus, the interval $[x, \beta)$ is non-empty. By continuity, we can find values $a, k \in \mathbb{N}$ such that $\frac{a}{n^k} \in [x, \beta)$ and the line $E$ from $\frac{a}{n^k}$ to $\infty$, must intersect $\xi_\beta$ (and by extension $\zeta_\beta$). Since $E$ is in $1_{mn^k} F$ for all $m \in \mathbb{N}$, it follows by rescaling that $mn^k \zeta_\beta$ passes through the line from $a$ to $\infty$ for all $m \in \mathbb{N}$.

When we take $\xi_\beta$ to be the projection of $\zeta_\beta$ in $\Gamma_0(mn^k) \setminus \mathbb{H}$, we see that $\xi_\beta$ is homotopic to a closed curve in $\Gamma_0(mn^k) \setminus \mathbb{H}$, since $\eta((\xi_\beta, F(1_{mn^k} / \zeta))) = \eta((\xi_\beta, F)) \in \text{ESP}^+$. Since
Figure 9: Diagrams to illustrate $\lambda_\beta$, $\xi_\beta$ and $\xi_{\beta,E}$, as sub-paths of $\zeta_\beta$. 

(a) A diagram showing $\lambda_\beta$ and $\xi_\beta$ as sub-paths of $\zeta_\beta$.

(b) A diagram showing $\xi_{\beta,E}$ as a sub-path of $\zeta_\beta$. 

Figure 9: Diagrams to illustrate $\lambda_\beta$, $\xi_\beta$ and $\xi_{\beta,E}$, as sub-paths of $\zeta_\beta$. 

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\(\xi_\beta\) intersects the edge \(E\) from \(\ell\) to \(\infty\) and \(E\) is an edge of \(\frac{1}{n^k}\mathcal{F}\). \(\xi_\beta\) intersects \(\overline{E}\), the image of \(E\) in \(\Gamma_0(mn^k)\mathcal{H}\), infinitely often. If we remove the prefix of \(\xi_\beta\) such that it starts from the edge \(E\), which we denote \(\xi_{\beta,E}\), then the projection \(\xi_{\beta,E}\) in \(\Gamma_0(mn^k)\mathcal{H}\) will also be homotopic to a closed curve (since this is equivalent to just moving the base point of the curve). Then, by Theorem 3.4, \((\xi_{\beta,E}, T_{(n^k,mn^k)}(-))\) and \((\xi_{\beta,E}, T_{(mn^k,mnk)}(-))\) are both essentially periodic. For \(m \in \mathbb{N}\), there is a map \(\varphi := \begin{pmatrix} 1 & -\frac{m^k}{mn^k} \\ 0 & 1 \end{pmatrix}\) \(\in Isom^+(\frac{1}{mn^k}\mathcal{F}) \subset Isom^+(\frac{1}{n^k}\mathcal{F})\), which maps \(E\) to \(I\). For \(\xi_{\alpha} := \varphi(\xi_{\beta,E})\), since \(\varphi \in Isom^+(\frac{1}{mn^k}\mathcal{F})\) for \(m \in \mathbb{N}\), we have \((\xi_{\beta,E}, \frac{1}{mn^k}\mathcal{F})_E = \varphi((\xi_{\beta,E}, \frac{1}{mn^k}\mathcal{F})_E) = (\varphi(\xi_{\beta,E}), \varphi(\frac{1}{mn^k}))_E = (\xi_{\alpha}, \frac{1}{mn^k}\mathcal{F})_I\).

If we take \(\alpha\) to be the endpoint of \(\xi_{\alpha}\), we can see that \(\alpha + \frac{1}{n^k} = \beta\) and therefore, \(mn^k\alpha + ma = mn^k\beta\). Finally, since \(mn^k\beta\) intersects the line from \(ma\) to \(\infty\), the terms before this line can only affect the first term i.e. \(\eta((\xi_{\beta}, \frac{1}{mn^k}\mathcal{F})) = ma + \eta((\xi_{\beta}, \frac{1}{mn^k}\mathcal{F}))_E = ma + \eta((\xi_{\alpha}, \frac{1}{mn^k}\mathcal{F}))\). Here we can take \(\alpha' = n^k\alpha\) which will be in \(ESP^+\) by Corollary 3.5. The result follows by relabelling.

### 3.4 Bounds of Eventually Periodic Continued Fractions Grow at least Exponentially

In this section we give an alternative proof to the statement that every element of \(EVP^+\) satisfies pLC \([\mathbb{I}]\). We also show that for \(\overline{\alpha} \in ESP^+\) \(\lim_{m \to \infty} B(m\alpha) = \infty\). Finally, we show that for every \(\overline{\alpha} \in EVP^+\) there exists \(a, k \in \mathbb{N}\) such that \(n^i \alpha \leq B(n^{i+k}\alpha)\) for every \(i \in \mathbb{N}\). In other words, \(B(n^i\alpha)\) grows at least exponentially (after some point) for any \(\overline{\alpha} \in EVP^+\).

**Proposition 3.9.** If \(\alpha \in \mathbb{R}_{>0}\) with \(\overline{\alpha} \in ESP^+\), then \(\lim_{m \to \infty} B(m\alpha) = \infty\).

**Proof.** By corollary 3.5 for every \(m \in \mathbb{N}\) we have \(\overline{ma} \in ESP^+\). For \(k \in \mathbb{N}\) big enough, \(k\alpha > 1\) and for all \(m \geq k\), \(\overline{ma}\) is of the form \([a_0^{(m)}; a_1^{(m)}, \ldots, a_{r(m)}^{(m)}]\) with \(0 < a_0^{(m)} \leq a_{r(m)}^{(m)}\).

Here \(r(m)\) is the length of the period for \(ma\). For each \(m \in \mathbb{N}\), \(B(m\alpha) \geq a_{r(m)}^{(m)}\) by the definition of the function \(B(x)\), \(a_{r(m)}^{(m)} \geq a_0^{(m)}\) by the definition of essentially periodic continued fractions and \(a_0^{(m)} = [ma]\) by the construction of continued fractions. In particular, \(B(m\alpha) \geq a_{r(m)}^{(m)} \geq a_0^{(m)} = [ma]\). Since \([ma] \to \infty\) for every \(\alpha \in \mathbb{R}_{>0}\), it follows that \(B(m\alpha) \to \infty\). \(\square\)

**Proposition 3.10.** If \(\alpha \in \mathbb{R}_{>0}\) with \(\overline{\alpha} \in EVP^+\), then \(\lim_{i \to \infty} B(n^i\alpha) = \infty\). In particular, there exists \(a, k \in \mathbb{N}\) such that \(n^i \alpha \leq B(n^{i+k}\alpha)\) for every \(i \in \mathbb{N}\). Every element in \(EVP^+\) satisfies pLC.

**Proof.** By Theorem 3.8 for \(\beta \in EVP^+\) we can find some \(k' \in \mathbb{N} \cup \{0\}\), such that \(n^{k'}\beta\) behaves like some element \(n^{k'}\alpha\) of \(ESP^+\). In particular, by Proposition 3.9 for every \(\beta \in EVP^+\), \(\lim_{i \to \infty} B(n^{k'+i}\beta) = \infty\) and by taking \(n = p\) prime, pLC follows. We can take
$k \in \mathbb{N}$, with $k \geq k'$, such that $\lfloor n^k \alpha \rfloor > 1$. We know that $B(n^{k+i}\alpha) = B(n^{k+i}\beta)$, since $n^k \beta$ and $n^k \alpha$ only differ by their first term (by Theorem 3.8). Therefore, $B(n^{i+k}\beta) = B(n^{i+k}\alpha) \geq \lfloor n^{i+k}\alpha \rfloor \geq \lfloor n^i \lfloor n^k \alpha \rfloor \rfloor = n^i \lfloor n^k \alpha \rfloor = n^i a_0^{(k)}$.

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