Self-guaranteed measurement-based quantum computation

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In order to guarantee the output of a quantum computation, we usually assume that the component devices are trusted. However, when the total computation process is large, it is not easy to guarantee the whole system when we have scaling effects, unexpected noise, or unaccounted correlations between several subsystems. If we do not trust the measurement basis nor the prepared entangled state, we do need to be worried about such uncertainties. To this end, we propose a “self-guaranteed” protocol for verification of quantum computation under the scheme of measurement-based quantum computation where no prior-trusted devices (measurement basis nor entangled state) are needed. The approach we present enables the implementation of verifiable quantum computation using the measurement-based model in the context of a particular instance of delegated quantum computation where the server prepares the initial computational resource and sends it to the client who drives the computation by single-qubit measurements. Applying self-testing procedures we are able to verify the initial resource as well as the operation of the quantum devices, and hence the computation itself. The overhead of our protocol scales as the size of the initial resource state to the power of 4 times the natural logarithm of the initial state’s size.

I. INTRODUCTION

Quantum computation offers a novel way of processing information and promises solution of some classically intractable problems ranging from factorization of large numbers \cite{Shor97} to simulation of quantum systems \cite{Feynman82}. However, as quantum information processing technologies improve the performance of quantum devices composed of ion traps and superconducting qubits \cite{Knill08, Har95}, a natural question arises; “How can we guarantee the computation outcome of a prepared quantum computation machine?” The solution of this problem is strongly desired in the context of characterization, verification and validation of quantum systems (QCVV), which is actively addressed in recent studies \cite{Mi89, Duan12}. For problems such as factorization, this does not present an issue as verification takes the form of simple multiplication of numbers. However, we cannot deny a possibility that the constructed quantum device suffers from unexpected noise or unaccounted correlations between several subsystems resulting from our insufficient experimental control when implementing the quantum computer. That is, we need to guarantee (verify) the outcome without any noise model. This task is called the verification of quantum computation \cite{YLC98, HN10, S10}.

The concept of verifying a quantum computation is quite different from quantum error correction. In quantum error correction we start with a noise model that can adversely affect the computation and devise quantum codes to counteract this noise provided its strength remains below a certain threshold. In verification of quantum computation we do not make any assumptions about the noise. The prepared states and measurement devices may be behaving ideally or they may be affected by noise. The goal of verification is to ascertain whether the quantum states and measurement devices behave closely according to specifications and how this deviation affects the output of the computation, without assuming any noise model. This is crucial from an experimental point of view as it allows us to test quantum devices and guarantee their reliable operation.

For this purpose, the verification of quantum computation needs to satisfy the following two requirements. One is detectability which means that if the state or the measurement device is far from the ideal one, we reject it with high probability. In this stage, no assumption on the underlying noise model should be made. The other is acceptability which means that the ideal state and the ideal measurement device can pass the test with high probability. Both requirements are needed to characterize performance of test in statistical hypothesis testing \cite{HPW16, He16}.

We need to clarify whether we have already verified the device or not. To address this issue, a device is called trusted when we have already verified it. Otherwise, it is called untrusted. This task may seem daunting at first, particularly when considered in the context of quantum circuit model \cite{S10}. In this model, the computation takes form of a sequence of local and multi-local unitary operations applied to the quantum state resulting in a quantum output that is finally measured out to yield the classical result of the computation. In order to verify the correctness of the output it would appear that one needs to keep track of the entire dynamics, effectively classically simulating the quantum computation. This can of course be achieved only for the smallest of quantum systems due to the exponential increase in the dimensionality of the Hilbert space with increasing system size. Measurement-based model of quantum computation (MBQC), is equivalent to the quantum circuit model but uses non-unitary evolution to drive the computation \cite{KLM01, H07}. In this model, the computation begins
with preparation of an entangled multi-qubit resource state and proceeds by local projective measurements on this state that use up the initial entanglement. In order to implement the desired evolution corresponding to the unitary from the circuit model, the measurements must be performed in an adaptive way where future measurement bases depend on previous measurement outcomes which imposes a temporal ordering on the measurements.

The initial proposal of MBQC in [25] considered a cluster state as the initial state and measurements in the X-Y plane of the qubit’s Bloch sphere at an arbitrary angle along with Z measurements. It has been recently shown that Z measurements are in fact not necessary [24]. We consider measurements of X, Z and X ± Z that are approximately universal when paired with a triangular lattice as the initial resource state [30]. For trusted measurement devices, the computation outcome can be guaranteed only by verifying the initial entangled multi-qubit resource state [31] using stabilizer measurements. However, for untrusted devices this method alone is not sufficient.

Our task is guaranteeing the computation outcome without trusting the measurement devices as well as the initial entangled resource state. To achieve this, we employ self-testing techniques to guarantee prepared states as well as to certify the operation of quantum devices. Self-testing, originally proposed in [22, 33], is a statistical test that compares measured correlations with the ideal ones and based on the closeness of these two cases draws conclusions whether the real devices behave as instructed under a particular definition of equivalence. In any run of the computation we assume that the prepared physical states and devices are untrusted and therefore need testing. Self-testing does not make any artificial assumptions about the Hilbert space structure of the devices or the measurement operators corresponding to classical outcomes observed.

To achieve verification of quantum computation, we need to establish a self-test for a triangular graph state as well as measurements mentioned in the above paragraph. McKague proposed a self-testing procedure for such a graph state in [34] along with measurements in the X-Z plane. However, this method requires many copies of the n-qubit graph state scaling as \( O(n^{32}) \) and therefore is not possible with current or near-future quantum technologies.

In this paper, with feasible experimental realization in mind, we propose a self-testing procedure for a triangular graph state along with measurements of X, Z and X ± Z. One of the main differences between MBQC and the quantum circuit model is the clear split between preparation of the initial entangled resource state and the computation itself. This property suggests a natural approach to guaranteeing the outcome of the computation by splitting the verification process into two parts. Firstly, we have to verify the initial entangled multi-qubit resource state. Secondly, we guarantee the correct operations of the measurement devices that drive the computation. To realize this approach, we begin by introducing a protocol that reduces self-testing for a triangular graph state to a combination of self-tests for a Bell state.

Original proposals of Mayers and Yao [32, 33] have considered self-testing of a Bell state. The method of [34] is based on the Mayers-Yao test while [36] discusses methods based on the Mayers-Yao test as well as the CHSH test. These two approaches require relatively small number of measurement settings. However, direct application of these methods to our protocol results in a huge number of required copies of the graph state. To resolve this issue, we propose a different method for self-testing of a Bell state, which has better precision as previous methods.

II. SELF-TESTING OF MEASUREMENTS BASED ON TWO-QUBIT ENTANGLED STATE

As the first step, we consider a self-testing protocol of local measurements on the untrusted system \( \mathcal{H}_1' \) when the untrusted state \( |\Phi'\rangle \) is prepared on the bipartite system \( \mathcal{H}_1' \otimes \mathcal{H}_2' \). The trusted state corresponding to \( |\Phi'\rangle \) is \( (|0, +\rangle + |1, -\rangle)/\sqrt{2} \). Note that even though the trusted system is a two-qubit state, we do not assume that either of the untrusted systems \( \mathcal{H}_1' \) or \( \mathcal{H}_2' \) are \( \mathbb{C}^2 \). In the rest of our paper we denote untrusted states and operators with primes, such as \( |\psi'\rangle \) and \( X' \), in order to distinguish them from trusted states and devices which have no primes. Our protocol satisfies the following requirements related to our self-testing protocol for three-colorable graph state.

(1-1): Identify measurements of \( X_1, Z_1 \) and \( (X_1 \pm Z_1)/\sqrt{2} \) within a constant error \( \epsilon \). 

(1-2): Measure \( X_1', Z_1' \), \( A(0)' \) and \( A(1)' \) on the system \( \mathcal{H}_1' \), where \( A(i)' := [X_1' + (-1)^iZ_1']/\sqrt{2} \).

(1-3): Measure only \( X_2' \) and \( Z_2' \) on the system \( \mathcal{H}_2' \).

(1-4): Prepare only \( O(\delta^{-4}) \) samples for the required precision level \( \delta \), whose definition will be given latter.

Requirement (1-1) is needed for universal computation based on measurement-based quantum computation [30]. Three-colorable graph states can be partitioned into three subsets of non-adjacent qubits. In the rest of our paper, we refer to one of these subsets as black qubits (B), the second subset is referred to as white qubits (W) and the final subset are red qubits (R). To realize the self-guaranteed MBQC of n-qubit three-colorable graph state with resource size \( O(n^4 \log n) \), we need the requirement (1-4). Indeed, McKague et al. [36] already gave a self-testing protocol for the Bell state. However, their protocol requires resource size that scales as \( O(n^8) \) (Remark 1 of Appendix E).

The self-testing procedure is illustrated in FIG. 1. We prepare \( 8m \) copies of the initial state and split them randomly into 8 groups that are then measured to test the
Based on the above measurements, we check the following 5 inequalities for 8 average values:

\[
\begin{align*}
\text{Av}[X'_1 X'_2] &= 1, \\
\text{Av}[(A(0)|1)(Z'_2 + X'_2)] &\geq \sqrt{2} - \frac{c_1}{\sqrt{m}}, \\
\text{Av}[(1)|1(Z'_2 - X'_2)] &\geq \sqrt{2} - \frac{c_1}{\sqrt{m}}, \\
|\text{Av}[X'_1 X'_2 + Z'_1 Z'_2]| &\leq \frac{c_1}{\sqrt{m}}. 
\end{align*}
\]

Here, for example, the average value in (2) is calculated from the outcomes of the 3rd and 4th groups.

This leads to the following theorem, which is shown in Appendix E.

**Theorem 1.** Given a significance level \( \alpha \) and an acceptance probability \( \beta \), there exists a pair of positive real numbers \( c_1 \) and \( c_2 \) satisfying the following condition. If the state \((0, +) + (1, -)/\sqrt{2}\) and measurement are prepared with no error, Test (2) of the above \( c_1 \) is passed with probability \( \beta \). Once Test (2) of the above \( c_1 \) is passed, we can guarantee, with significance level \( \alpha \), that there exists an isometry \( U : \mathcal{H}_1 \rightarrow \mathcal{H}_1 \) such that

\[
\begin{align*}
\|UX'_1 U^\dagger - X_1\| &\leq \delta, \\
\|UA(0)|1 U^\dagger - A(0)|1\| &\leq \delta, \\
\|UZ'_1 U^\dagger - Z_1\| &\leq \delta, \\
\|UA(1)|1 U^\dagger - A(1)|1\| &\leq \delta,
\end{align*}
\]

where \( \delta := c_2 m^{-1/4} \), which is called the required precision level.

Note that the significance level \( \alpha \) is the maximum passing probability when one of the conditions in (2-5) does not hold [22]. The acceptance probability \( \beta \) is also called the power of the test in hypothesis testing and is the probability to accept the test in the ideal case. To satisfy the detectability and the acceptability, \( \alpha \) and \( \beta \) are chosen to be constants close to 0 and 1, respectively, which leads to their trade-off relation. In this way, we can show how the measurements forming an approximately universal set for MBQC can be certified using a two-qubit state. Now we proceed to extend this scheme to three-colorable states of arbitrary size.

**III. SELF-TESTING OF A THREE-COLORABLE GRAPH STATE**

Now, we give a self-testing for a three-colorable graph state \( \mathcal{G}' \), composed of the black part (B), the white part (W), and the red part (R), whose total number of qubits is \( n \). Our protocol needs to prepare \( cm \) samples of the state \( \mathcal{G}' \), where \( m \) is \( O(n^4 \log n) \), where the constant \( c \) depends on the structure of the graph \( G \).

To specify it, we introduce three numbers \( l_B, l_W, \) and \( l_R \) for a three-colorable graph \( G \). Consider the set \( S_B := \{1, \ldots, n_B\} \) of black sites, the set \( S_W := \{1, \ldots, n_W\} \) of white sites, and the set \( S_R := \{1, \ldots, n_R\} \) of red sites. We denote the neighborhood of the site \( i \) by \( N_i \subset S_W \cup S_R \). We divide the sites \( S_B \) into \( l_B \) subsets \( S_{B,1}, \ldots, S_{B,l_B} \) such that \( N_i \cap N_j = \emptyset \) for \( i \neq j \in S_{B,k} \) for any \( k = 1, \ldots, l_B \). That is, elements of \( S_{B,k} \) have no common neighbors, which is called the non-conflict condition. We choose the number \( l_B \) as the minimum number satisfying the non-conflict condition. We also define the numbers \( l_W \) and \( l_R \) for the white and red sites in the same way. In FIG. 2 we show that for a triangular graph \( l_B, l_W, l_R \leq 3 \). Based on this structure, testing of measurement devices on each site on \( S_{B,k} \) can be reduced to the two-qubit case as follows:

(3-1): Prepare \( 8m \) states \( |\Phi'\rangle \).

(3-2): Measure \( Z' \) on all sites of \( S_B \setminus S_{B,k} \) for all copies. Then, apply \( Z' \) operators on the remaining sites to correct for the \( Z' \) measurement depending on the outcomes.

(3-3): For all \( i \in S_{B,k} \), choose a site \( j_i \in N_i \). Then, measure \( Z' \) on all sites of \( S_W \setminus \{j_1\} \cap S_{B,k} \) for all copies. Apply \( Z' \) operators on the remaining sites to correct for the \( Z' \) measurements depending on the outcomes.
Steps (4-3), (4-4), and (4-5) perform the stabilizer test given in [31] adapted to a triangular graph state which certifies the graph state $|G \rangle$. For our self-testing, we need to guarantee local measurements of $X_1$, $Z_1$ and $(X_1 \pm Z_1)/\sqrt{2}$ for all sites. Since Test (4) utilizes B-protocol which in turn uses Test (2), Test (4) depends on the parameter $c_1$ of Test (2).

For acceptability, we need to pass Test (2) in all sites, i.e., $n$ qubits. Hence, as shown in Appendix F, to realize an acceptance probability $\beta$ close to 1, we need to choose $c_1$ to be $c_4(\log n)^{1/2}$ with a certain constant $c_4$, which leads to the following theorem.

**Theorem 2.** Given a significance level $\alpha$ and an acceptance probability $\beta$, there exists a pair of positive real numbers $c_2$ and $c_3$ satisfying the following condition.

If the state $|G \rangle$ and our measurements are prepared with no error, Test (4) with $c_1 = c_4(\log n)^{1/2}$ is passed with probability $\beta$. Once Test (4) with $c_1 = c_4(\log n)^{1/2}$ is passed, we can guarantee, with significance level $\alpha$, that there exists an isometry $U_i : \mathcal{H}_i \to \mathcal{H}_i$ such that

\[ ||U_i X_i' U_i^\dagger - X_i||, \ ||U_i Z_i' U_i^\dagger - Z_i|| \leq \delta \quad (7) \]
\[ ||U_i A(0)_{ij} U_i^\dagger - A(0)||, \ ||U_i A(1)_{ij} U_i^\dagger - A(1)|| \leq \delta \quad (8) \]
\[ Tr[\sigma(I - P'_1)], \ Tr[\sigma(I - P'_2)], \ Tr[\sigma(I - P'_3)] \leq \frac{\alpha}{m} \quad (9) \]

where $\delta := c_2(\log n)^{1/4}$, $U := U_1 \otimes \cdots \otimes U_n$, $\sigma$ is the resultant state on the final group, and $P'_1$, $P'_2$, $P'_3$ are POVM elements corresponding to pass of Steps (4-3), (4-4), and (4-5).

Here, the conditions (7)–(8) follow from Theorem 1 and the condition (9) follows from a similar discussion for the stabilizer test given in [31].

**IV. CERTIFICATION OF THE COMPUTATIONAL RESULT**

To guarantee the computational result, we need to guarantee that our computational operation is very close to the true operation based on Theorem 2. When $\{M_i\}_i$ is a POVM realized by an adaptive measurement on each site from $X$, $Z$, $A(0)$, and $A(1)$, as shown in Appendix G, Theorem 2 guarantees that

\[ ||UM_i^\dagger U^\dagger - M_i|| \leq 8n\delta, \quad (10) \]

where $M_i$ is the ideal POVM. This inequality can be shown by a modification of a virtual unitary protocol composed of a collection of unitaries on each site controlled by another trusted system [31 Lemma 3.6]. Thus, as shown in Appendix H, Eqs. (9) combined with the relationship between trace distance and fidelity [37] and the above discussion lead to

\[ ||U\sigma U^\dagger - |G\rangle\langle G||_1 \leq 6n\delta + \frac{3\alpha}{m}. \quad (11) \]
When $M_j'$ is the POVM element of all the outcomes corresponding to the correct computational result, we have

$$\left| \text{Tr}(M_j' \sigma - M_j |G \rangle \langle G |) \right|$$

$$\leq \left| \text{Tr}(UM_j' U^\dagger - M_j) \sigma U^\dagger \right| + \left| \text{Tr}M_j(U \sigma U^\dagger - |G \rangle \langle G |) \right|$$

$$\leq 14n\delta + \frac{3\alpha}{m}. \quad (12)$$

Thus, choosing $m = O(n^\epsilon \log n)$, we can achieve constant upper bound for the probability of accepting an incorrect output of the quantum computation with significance level $\alpha$. Connection between our protocol and interactive proof systems [38, 39] is made explicit in Appendix [1].

V. APPLICATION TO MEASUREMENT-ONLY BLIND QUANTUM COMPUTATION

The above protocol may be applied to the scenario of measurement-only blind quantum computation [20, 31, 40] when the client does not trust the quantum devices performing the measurements. Measurement-only blind quantum computation is a type of delegated quantum computation where the client with limited quantum power instructs a server to prepare a multipartite entangled state which is then sent to the client who performs single-qubit measurements that drive the computation. This protocol is blind by construction, meaning the server cannot find out anything about the computation, and can be verified by stabiliser testing when the client trusts the measurement devices [31]. Measurement-only blind quantum computation was demonstrated experimentally in an optical setup in [11]. Ability to quickly generate and measure quantum states is essential in any verification protocol therefore we believe that this setup shows great promise for implementing the self-guaranteed protocol in the near future.

Now we address the case when the measurement devices are not trusted. We consider the client (Verifier) interacting with two servers, Prover 1 and Prover 2, where Prover 1 prepares the initial state and Prover 2 is used to measure the qubits and therefore test the state and the operation of the quantum devices. There is a possibility that the noise in the initial state is correlated to the noise in the measurement devices. In the language of interactive proof systems this would require Prover 1 and Prover 2 to be independent. This requirement is usually enforced by considering Provers that are permitted to agree on a prior cheat strategy but are not allowed to communicate once the protocol commences.

This protocol requires only independence among two parts, the part of generation of quantum states and the measurement devices. In the language of interactive proof systems this would require Prover 1 and Prover 2 to be independent. This requirement is usually enforced by considering Provers that are permitted to agree on a prior cheat strategy but are not allowed to communicate once the protocol commences.

The assumption of independence between the preparation stage and measurement stage is quite strong. However it is necessary since Prover 1 could quite easily encode the information about the local random unitaries which Prover 2 could later use to his advantage. This also highlights that the scenario considered in this verification scheme is not the usual one of protocols based on interactive proof systems where the Provers are assumed to be non-communicating. Here Prover 1 and Prover 2 engage in one-way quantum communication necessitating our assumption that they are to a large degree honest. On the other hand this assumption is natural in the context of verifying quantum technologies where the Provers are not assumed to be malicious and the only deviation from the verification protocol is caused by unexpected noise. Similar less secure approach has been recently fruitfully used in [12] to efficiently verify adaptive Clifford circuits.

It is possible to enhance our protocol to the case where the Provers are considered malicious and are actively conspiring against the Verifier. Assume that the Provers share a Bell pair for every qubit that Prover 1 is instructed to prepare. The Verifier then asks Prover 1 to perform a two-qubit Bell measurement on the i-th qubit of a prepared graph state and its corresponding Bell-pair qubit, reporting the outcomes to the Verifier. All the outcomes are denoted by vector $T_i$. The effect of these measurements is to teleport the prepared initial states from Prover 1 to Prover 2 up to a local unitary. The Verifier then proceeds with self-testing protocol of Test (4) taking into account the local rotations $U_T^i (T + T')$. Note that even if Prover 2 has access to the information about $T$ he cannot use it to cheat the Verifier as the vector $T'$ is also uniformly random and unknown to him. By teleporting the copies of initial graph state from Prover 1 to Prover 2, the Verifier can check the computation without making any strong assumptions about the independence of the two Provers. This addition to our protocol introduces only a multiplicative factor and does not affect the scaling of the overhead required by our protocol.
VI. DISCUSSION

The above analysis has been restricted to the case of three-colorable graph states. In fact, the non-conflict condition can be relaxed to the case of graph states which are $k$-colorable as follows. Firstly, we remember that our analysis can be divided into two parts, testing of the measurement basis and testing of the graph state. The first part can be generalized as follows. For each color $i = 1, \ldots, k$, we divide the set of vertices with color $i$ into subsets $S_{i1}, \ldots, S_{iL_i}$ such that there is no common neighborhood for each subset $S_{ij}$. In this case, we can generalized the B-protocol as explained in Appendix [J]. Then, applying this generalization to all colors in the protocol, we can extend the first part. To realize the second part, for each color $i$, we measure non-$i$ color sites with $Z$ and check whether the outcome of measurement $X$ on the sites with color $i$ is the same as the predicted one. We repeat this protocol for all colors. Due to this construction we obtain the same analysis as three-colorable case when the numbers $L_1, \ldots, L_k$ are bounded.

For a computation on an $n$-qubit graph state the resources needed to achieve a constant upper bound for probability of accepting a wrong outcome scale as $O(n^2 \log n)$. This is the same scaling obtained in [8] which raises an interesting open question of minimal overhead required to guarantee an outcome of quantum computation. We have shown how our self-testing protocol can be applied to measurement-only blind quantum computation in the case when also the measurement devices cannot be fully trusted.

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Appendix A: Significance level

In statistical hypothesis testing, the significance level is a key concept for a given test $T$. In general, we say that the statement $S$ holds with the significance level $\alpha$ when the probability of making an incorrect decision is less than $\alpha$ under this claim. More precisely, the probability of the following event is less than $\alpha$: we claim the statement $S$ and the statement $S$ is incorrect.

If the statement $S$ is a property of the true initial state and measurement device, this can be formulated of a simple form as follows. We say that the true initial state and measurement device satisfy the property $S$ with the significance level $\alpha$ (or simply the property $S$ holds with the significance level $\alpha$) when the test $T$ is passed and the following condition holds. When the true initial state and measurement device do not satisfy the property $S$, the probability to pass the test $T$ is smaller than $\alpha$. Usually, there are many probability distributions of the whole system even when the true initial state and measurement device are fixed to not satisfy the property $S$ because we have several possibilities of the true initial state and measurement device. This formulation is the conventional formulation in statistical hypothesis testing. However, in the self-testing, we need to treat the case when the statement $S$ is not a property of the true initial state and measurement device, which requires a more complicated formulation.

Assume that a certain property $S'$ implies that the statement $S$ is correct with probability $\gamma$, which can be regarded as a kind of property of an initial state and a measurement device. Now, we assume that the test $T$ is passed and we claim the statement $S$ as a result. The case of making an incorrect decision is contained in the union of the following two cases. One is the case when the property $S'$ does not hold. The other is the case when the statement $S$ is not correct while the property $S'$ holds. When the property $S'$ holds with the significance level $\gamma$, we can say that the statement $S$ holds with the significance level $\alpha + \gamma$. This is because the probability of making an incorrect decision is less than the sum of the probabilities of the above two cases.

Appendix B: Interval estimation with binomial distribution

We consider how to verify the success probability of a binary system by using sampling. Assume that $n$ binary systems $X_1, \ldots, X_n$ take values in $\{0, 1\}$. We randomly choose $m$ systems and denote the sum by $X$. We assume that the variables $X_1, \ldots, X_n$ independently obey the same distribution $P(1) = p$ and $P(1) = 1 - p$. We randomly choose one variable $X'$ from $n - m$ remaining systems. Then, the variable $X'$ obeys the binomial distribution with average $p$. Now, we consider how to make a statement with respect to the average $p$ from the observed value $X$.

In the following, we denote the binomial distribution of $m$ trials with probability $p$ by $B_p$. Given $p$, we define $x^+(p)$ as $\min\{x|B_p(X \geq x) \leq \alpha\}$, which is often called the percent point with $\alpha$. Then, when the observed $X$ satisfies $X \geq x^+(p_0)$, we can say that the parameter $p$ is larger than $p_0$ with significance level $\alpha$. When $m$ is sufficiently large, $x^+(p_0)$ is approximated to $mp_0 + \sqrt{mp_0(1-p_0)}$. Similarly, we define $x^-(p)$ as $\max\{x|B_p(X \leq x) \leq \alpha\}$. Hence, when the observed $X$ satisfies $x^-(p_1) \geq X \geq x^-(p_0)$, we can say that the parameter $p$ belongs to $(p_0, p_1)$ with significance level $2\alpha$.

For a constant $\alpha$ and a sufficiently large $m$, the value $\sqrt{(p_0 + \frac{\alpha}{\sqrt{m}})\left(1 - (p_0 + \frac{\alpha}{\sqrt{m}})\right)}$ can be approx-
imated to $\sqrt{p_*(1-p_*)}$. We choose $p_0 = p_* - \frac{1}{\sqrt{m}}(\Phi^{-1}(\beta) + \Phi^{-1}(\alpha))\sqrt{p_*(1-p_*)}$ and $p_1 = p_* + \frac{1}{\sqrt{m}}(\Phi^{-1}(\beta) + \Phi^{-1}(\alpha))\sqrt{p_*(1-p_*)}$. We define our test $T(m,p_*,\beta)$ ($T^-(m,p_*,\beta)$) by the condition that the observed $X$ belongs to the interval $[m p_* - \sqrt{m} \Phi^{-1}(\beta) \sqrt{p_*(1-p_*)}, m p_* + \sqrt{m} \Phi^{-1}(\beta) \sqrt{p_*(1-p_*)}]$ (the half interval $[m p_* + \sqrt{m} \Phi^{-1}(\beta) \sqrt{p_*(1-p_*)}, +\infty)$). We have the following two lemmas.

The following lemma guarantees the success probability when the test is passed even when the system is maliciously prepared but the distribution is independent and identical, which relates to the soundness.

**Lemma 1.** When the test $T(m,p_*,\beta)$ ($T^-(m,p_*,\beta)$) is passed, we can say that the parameter $p$ belongs to the interval $[p_* - \frac{1}{\sqrt{m}}(\Phi^{-1}(\beta) + \Phi^{-1}(\alpha))\sqrt{p_*(1-p_*)}, p_* + \frac{1}{\sqrt{m}}(\Phi^{-1}(\beta) + \Phi^{-1}(\alpha))\sqrt{p_*(1-p_*)}]$ with significance level $\alpha$. (the half interval $[p_* - \frac{1}{\sqrt{m}}(\Phi^{-1}(\beta) + \Phi^{-1}(\alpha))\sqrt{p_*(1-p_*)}, +\infty)$ with significance level $\alpha$).

The following lemma guarantees that the test will be passed with high probability when the test is well prepared to be the independent and identical distribution with success probability $p_*$, which relates to the completeness.

**Lemma 2.** Further, when the true parameter $p$ is $p_*$, the test $T(m,p_*,\beta)$ ($T^-(m,p_*,\beta)$) is passed with probability $1 - 2\beta (1-\beta)$.

Appendix C: Interval estimation with hypergeometric distribution

In general, the variables $X_1, \ldots, X_n$ are not necessarily independent and identical. Now, we consider such a general case. Since $X$ is given as the sum of $m$ random samples, the distribution of $X$ is given as $\sum_{k=0}^{n} Q_k(k)p_{HG|n,m,k}$ with the distribution $Q_k$ of the hidden variable $K$ on $\{0, 1, \ldots, 2m + 1\}$, where the hypergeometric distribution $P_{HG|n,m,k}$ is given as

$$P_{HG|n,m,k}(x) := \binom{n}{x} \binom{n-m}{k-x} \binom{m}{k}.$$  

which has been employed in the security analysis on the quantum key distribution, for example [13][15]. Since $X'$ is a random choice from $n - m$ remaining systems, when $K = k$ and $X = x$, the variable $X'$ obeys the distribution $P_{HG|n-m,1,k-x}$, which equals the binary distribution with average $\frac{k-x}{n-m}$. In general, when $X = x$, the success probability of the binary distribution of $X'$ is $\sum_k P_{K|X}(k,x) \frac{k-x}{n-m}$. Define the positive value $c(\alpha, \beta)$ as

$$c(\alpha, \beta)^2 := \left(\frac{1}{2} + \frac{n}{n-m} \Phi^{-1}(\beta) \sqrt{p_*(1-p_*)}\right)^2.$$  

Now, we consider how to make a statement with respect to the success probability of the binary distribution of $X'$ from the observed value $X$. The following lemma guarantees the success probability when the test is passed even when the system is maliciously and the distribution is not independent nor identical, which relates to the soundness in the general case.

**Lemma 3.** Assume that $n = km + o(m)$. When the test $T(m,p_*,\beta)$ is passed, we can say that the success probability of the binary distribution of $X'$ belongs to the interval $[p_* - \frac{c(\alpha, \beta)}{\sqrt{m}}, p_* + \frac{c(\alpha, \beta)}{\sqrt{m}}]$ with significance level $\alpha$.

We fix $\epsilon > 0$ and choose $c := \left(\frac{1}{2} + 2\Phi^{-1}(\beta) \sqrt{p_*(1-p_*)}\right)^2$. Due to Lemma 3 to guarantee that the success probability of a binary system belongs to the interval $[p_* - \epsilon, p_* + \epsilon]$ with significance level $\alpha$, we need to prepare $2\frac{\epsilon}{c}$ samples and observe $\frac{n}{\epsilon}$ samples.

Next, we define the test $T(m,0)$ by the condition that the observed $X$ equals 0. We have the following lemma.

**Lemma 4.** When the test $T(m,0)$ is passed, we can say that the success probability of the binary distribution of $X'$ is less than $\frac{1}{m^2}$ with significance level $\alpha$.

We fix $\epsilon > 0$. Due to Lemma 4 to guarantee that the success probability of a binary system is less than $\epsilon$ with significance level $\alpha$, we need to prepare and observe $\frac{1}{\alpha \epsilon}$ samples.

Now, we discuss what kind of test will be used in this paper. In the definition of $c(\alpha, \beta)$, the term $\frac{n}{n-m}$ appears. If $n - m$ does not increase with the order $O(n)$, this term goes to infinity. For example, when $m$ is a half of $n$, this term is 2, which yields a useful application of Lemma 3. Hence, when we need to verify that the binary variable has the success probability close to a certain non-zero value $p_*$, we use the test $T(m,p_*,\beta)$ with a half of observed values, i.e., $m = n/2$. In contrast, when we need to verify that the binary variable has the success probability close to zero, we use all of observed values and employ the test $T(m,0)$.

Appendix D: Proofs of Lemmas 3 and 4

**Proof of Lemma 3**  
Step 1: In this proof, the distribution $Q_k$ does not necessarily have a positive probability at one point. To treat this case, we address the joint distribution $P_{X'K}$ of $X$ and $K$ and the conditional distribution $P_{K|X}$. When we observe $x$ as the outcome of the random variable $X$, the success probability of the binary distribution of $X'$ is $p(x) := \sum_k P_{K|X}(k|x) \frac{k-x}{n-m}$. Using $c_1 := \Phi^{-1}(\beta) \sqrt{p_*(1-p_*)}$ we define the function

$$f(x) := \begin{cases} |p_* - p(x)| & \text{when } |p_* - \frac{x}{m}| \leq \frac{c_1}{\sqrt{m}}, \\ 0 & \text{otherwise.} \end{cases} \quad (D1)$$
The probability of making an incorrect decision is the probability of the event that \( f(X) > c(\alpha, \beta) \). Therefore, it is sufficient to show that this probability is less than \( \alpha \). As shown later, we have

\[
E_X f(X)^2 \leq \frac{1}{m} \left( \frac{1}{2} + \frac{n}{n - m} \right)^2 .
\]  

(D2)

Then, Chebychev inequality guarantees

\[
P_X \left\{ f(X) \geq \frac{c(\alpha, \beta)}{\sqrt{m}} \right\} \leq \alpha,
\]

which is equivalent to the desired statement.

Step 2): In the following, we show (D2). First, we have

\[
X f(X)^2 = \tilde{E}_X \left( E_{K|X} \left( p_* - \frac{K - X}{n - m} \right)^2 \right)
\]

where \( \tilde{E}_X \) expresses the expectation of \( \mathbb{1}_{[p_* - \frac{c}{\sqrt{m}}, p_* + \frac{c}{\sqrt{m}}]}(X)Y \) with respect to \( X \) and \( \mathbb{1}_{[p_* - \frac{c}{\sqrt{m}}, p_* + \frac{c}{\sqrt{m}}]} \) is the indicator function on the set \( [p_* - \frac{c}{\sqrt{m}}, p_* + \frac{c}{\sqrt{m}}] \).

For a given \( k \), we set \( c_2 \) as \( k - p_* = \frac{c_2}{\sqrt{m}} \). When \( \left| \frac{p_* - k}{m} \right| \leq \frac{c_2}{\sqrt{m}} \), we have

\[
\frac{c_2 - c_1}{\sqrt{m}} \leq \frac{X - k}{m - n} \leq \frac{c_2 + c_1}{\sqrt{m}} .
\]  

(D5)

Then, we have

\[
\left| \frac{p_* - k}{n - m} \right| \leq \left| \frac{p_* - k}{m} + \frac{k - X}{n - m} \right| = \left| \frac{p_* - k}{m} \frac{m}{n - m} + \frac{X - k}{m - n} \right| \leq \frac{c_2}{\sqrt{m}} + \frac{m}{n - m} \frac{c_1}{\sqrt{m}} \leq \frac{1}{\sqrt{m}} \left( c_2 + \frac{m}{n - m} c_1 \right) .
\]  

(D6)

When \( c_2 > c_1 \), using (D5), Chebychev inequality guarantees

\[
P_{X|K=k} \left\{ \left| \frac{p_* - X}{m} \leq \frac{c_1}{\sqrt{m}} \right\} \right. \leq \frac{V_k}{m^2} \left( \frac{c_2 - c_1}{\sqrt{m}} \right)^2 \leq \frac{1}{4(c_2 - c_1)^2} ,
\]

where \( V_k \) is the variance of \( X \) and is calculated to be \( \frac{m(n-m)(n-k)}{(n-1)m^2} \leq \frac{m}{4} \).

The combination of (D6) and (D7) yields

\[
\tilde{E}_{X|K=k} \left( p_* - \frac{k - X}{n - m} \right)^2 \leq \min \left( \frac{1}{4(c_2 - c_1)^2}, 1 \right) \frac{1}{m} \left( c_2 + \frac{m}{n - m} c_1 \right)^2 .
\]  

(D8)

Since

\[
\max_{c_2, c_2 \geq c_1} \min \left( \frac{1}{4(c_2 - c_1)^2}, 1 \right) \frac{1}{m} \left( c_2 + \frac{m}{n - m} c_1 \right)^2 = \frac{1}{m} \left( \frac{1}{2} + \frac{n}{n - m} c_1 \right)^2 ,
\]

we have

\[
\tilde{E}_{X|K=k} \left( p_* - \frac{k - X}{n - m} \right)^2 \leq \frac{1}{m} \left( \frac{1}{2} + \frac{n}{n - m} c_1 \right)^2 ,
\]  

(D10)

When \( c_2 \leq c_1 \), we have

\[
\left| \frac{p_* - k}{n - m} \right| \leq \frac{1}{m} \left( c_2 + \frac{m}{n - m} c_1 \right)^2 \leq \frac{1}{m} \left( \frac{1}{2} + \frac{n}{n - m} c_1 \right)^2 ,
\]

(D11)

which implies (D10). The combination of (D4) and (D10) implies (D2).

Proof of Lemma 4 Step 1): We consider the case \( n = m + 1 \). While the true distribution is given as a probabilistic mixture of hypergeometric distributions, it is sufficient to consider the mixture of the cases of \( K \) = 0, 1 because there is no possibility to pass the test when \( K > 1 \). Assume that \( Q_K(1) = p \) and \( Q_K(0) = 1 - p \). The probability to pass is \( \frac{1}{m+1} \) for \( K = 1 \) and \( 1 \) for \( K = 0 \). Hence, in general, the probability to pass is \( 1 - p + \frac{p}{m} \).

When the test \( T(m, 0) \) is passed, the success probability of the binary distribution of \( X' \) is \( \frac{1+m}{n+1-m} \). Therefore, when the probability to pass is greater than \( \alpha \), i.e., \( p \leq \frac{(1+m)(1-\alpha)}{m} \) the success probability of the binary distribution of \( X' \) is less than \( \frac{1}{m+1} \frac{(1+m)(1-\alpha)}{m} = \frac{1-\alpha}{m} \).

Step 2): We consider the general case, i.e., \( n > m + 1 \). Even in this case, if we focus on the observed variables \( X_1, \ldots, X_m \) and the variable \( X' \), the behavior of \( X \) can be written by a mixture of hypergeometric distributions with \( n = m + 1 \). Hence, we obtain the desired statement.
Appendix E: Bell state and Proof of Theorem 1

1. Proof of Theorem 1 using a proposition

In this appendix, we show Theorem 1 of the main body. For this purpose, we focus on the observables

\[ X := |1\rangle\langle 0| + |0\rangle\langle 1|, \quad Z := |0\rangle\langle 0| - |1\rangle\langle 1|. \]  

(E1)

We consider the state \((|0, +\rangle + |1, -\rangle)/\sqrt{2}\) on the composite system \(\mathcal{H}_1 \otimes \mathcal{H}_2\). We also define

\[ A(0) := \frac{X + Z}{\sqrt{2}}, \quad A(1) := \frac{X - Z}{\sqrt{2}}. \]  

(E2)

Here, instead of the ideal systems \(\mathcal{H}_1\) and \(\mathcal{H}_2\), we have the real systems \(\mathcal{H}'_1\) and \(\mathcal{H}'_2\). Also, we assume that we can measure real observables \(X'_1, X'_2, Z'_1, Z'_2, A(0)'_1,\) and \(A(1)'_1\). Here, we choose the real systems \(\mathcal{H}'_1\) and \(\mathcal{H}'_2\) sufficiently large so that our measurements are the projective decompositions of these observables.

In the following, we prepare the real state \(|\psi\rangle\) on the composite system \(\mathcal{H}'_1 \otimes \mathcal{H}'_2\). Then, we have the following proposition.

Proposition 1. When

\[ \langle \psi | A(0)'_1 (X'_2 + Z'_2) - A(1)'_1 (X'_2 - Z'_2) | \psi \rangle \geq 2\sqrt{2} - \epsilon_1 \]  

(E3)

\[ \langle \psi | X'_1 Z'_1 | \psi \rangle \geq 1 - \epsilon_2 \]  

(E4)

\[ \langle \psi | Z'_1 X'_2 | \psi \rangle \geq 1 - \epsilon_3 \]  

(E5)

\[ \langle \psi | A(0)'_1 (Z'_2 + X'_2) | \psi \rangle \geq \sqrt{2} - \epsilon_4, \]  

(E6)

\[ \langle \psi | A(1)'_1 (Z'_2 - X'_2) | \psi \rangle \geq \sqrt{2} - \epsilon_4, \]  

(E7)

\[ |\langle \psi | X'_1 Z'_2 + X'_1 Z'_2 | \psi \rangle| \leq \epsilon_5, \]  

(E8)

there exists a local isometry \(U: \mathcal{H}'_1 \rightarrow \mathcal{H}_1\) such that

\[ \|U X'_1 U^\dagger - X_1\| \leq \delta_1 \]  

(E9)

\[ \|U Z'_1 U^\dagger - Z_1\| \leq \delta_1 \]  

(E10)

\[ \left\| U A(0)'_1 U^\dagger - \frac{X_1 + Z_1}{\sqrt{2}} \right\| \leq \delta_2 \]  

(E11)

\[ \left\| U A(1)'_1 U^\dagger - \frac{X_1 - Z_1}{\sqrt{2}} \right\| \leq \delta_2, \]  

(E12)

where \(\delta_1 := \sum_{j=1}^2 \epsilon_j \epsilon_j^\dagger\) and \(\delta_2 := \sum_{j=1}^5 \epsilon_j \epsilon_j^\dagger + \sqrt{2}(\epsilon_2^\dagger + \epsilon_3^\dagger)\), and \(\epsilon_{j}\) and \(\epsilon_{j}\) are constants.

This proposition will be shown in the next subsection. Also, we prepare the following lemma.

Lemma 5. Given an acceptance probability \(\beta\) and a significance level \(\alpha\), there exist positive numbers \(c' > 0\) and \(c'' > c_1 > 0\) satisfying the following. If the state \((|0, +\rangle + |1, -\rangle)/\sqrt{2}\) and measurement are prepared with no error, Test (2) of the above \(c_1\) is passed with probability \(\beta\) (the acceptability). Once all the conditions in Step (2-5) with the above \(c_1\) are satisfied, with the significance level \(\alpha\), we can guarantee the conditions (E3)-(E8) with \(\epsilon_2, \epsilon_3 = \frac{c'}{m}\) and \(\epsilon_4 = \frac{c''}{\sqrt{m}}\), \(\epsilon_5 = \frac{c''}{\sqrt{m}}\).

Proof of Theorem 1: We choose three positive numbers \(c' > 0\) and \(c'' > c_1 > 0\) as in Lemma 5. So, Lemma 5 guarantees the acceptability. Once all of the conditions in Step (2-5) with the above \(c_1\) are satisfied, with the significance level \(\alpha\), we can guarantee the conditions (E3)-(E8) with \(\epsilon_2, \epsilon_3 = \frac{c'}{m}\) and \(\epsilon_1 = \frac{c''}{\sqrt{m}}\), \(\epsilon_4, \epsilon_5 = \frac{c''}{\sqrt{m}}\). Due to Proposition 1 using a suitable isometry \(U\), with the significance level \(\alpha\), we can guarantee the conditions (E9)-(E12) with \(\delta_1, \delta_2 = O(\langle \frac{1}{m} \rangle)^{\dagger}\), which yields the conditions (1) and (0).

Therefore, due to Proposition 1 using a suitable isometry \(U\), with the significance level \(\alpha\), we can guarantee the conditions (E9)-(E12) with \(\delta_1, \delta_2 = O(m^{-\frac{1}{2}})\), which is the desired argument.

Proof of Lemma 5 The observables in the LHS of (E3)-(E5) take a deterministic value in the ideal state. So, the acceptability for Eq. (1) is automatically satisfied. There exists a real number \(c'\) satisfying the following condition. To accept the tests \(A\{X'_1 Z'_2\} = 1\) and \(A\{Z'_1 X'_2\} = 1\) with more than probability \(\beta\), the conditions (E4)-(E5) take a deterministic value in the ideal state. In order that the ideal state pass the tests (2), (3), and (4) with probability \(\beta\), the coefficient \(c_1\) needs to be a constant dependent of \(\beta\). Then, dependently of \(c_1\) and \(\alpha\), there exists a real number \(c'' > c_1\) satisfying the following condition. To accept the tests (1)-(4) with more than probability \(\alpha\), the conditions (E9)-(E12) of \(\epsilon_4, \epsilon_5 = \frac{c''}{\sqrt{m}}\) need to hold. Once these tests \(A\{X'_1 Z'_2\} = 1\) and \(A\{Z'_1 X'_2\} = 1\) are passed, we can guarantee the conditions (E9)-(E12) of \(\epsilon_4, \epsilon_5 = \frac{c''}{\sqrt{m}}\) with significance level \(\alpha\). So, by choosing \(\epsilon_1 = 2\epsilon_4\), (E3) automatically holds.

Remark 1. Here we compare our overhead scaling with that in [30]. Their evaluation can be summarized as follows. Let \(\epsilon\) be the statistical error of observed variables like the quantities given in (E3)-(E8). [30] focuses on the matrix norm of the difference between the ideal observables and the real observables like the quantities appearing in (E9)-(E12) and shows that these quantities are upper bounded by \(O(\epsilon^2)\). However, [30] does not discuss the relation between the error \(\epsilon\) and the number of samples \(m\). Since their statistical errors are in the probabilistic case, the error \(\epsilon\) is given as \(O(1/\sqrt{m})\). Hence, the above matrix norm is bounded by \(O(m^{-1/8})\).

2. Proof of Proposition 1

To show Proposition 1 we need several lemmas.
Lemma 6. When
\[ \langle \psi'| A(0)'_1 (X'_2 + Z'_2) - A(1)'_1 (X'_2 - Z'_2) | \psi' \rangle \geq 2\sqrt{2} - \epsilon_1 \] (E13)
we have
\[ \|(X'_2 Z'_2 + Z'_2 X'_2) | \psi' \rangle \| \leq 2\epsilon'_1, \] (E14)
where \( \epsilon'_1 := 2^{\frac{3}{2}} \epsilon^{'\frac{1}{2}}_1 \).

Lemma 7. When
\[ \langle \psi'| X'_1 Z'_2 | \psi' \rangle \geq 1 - \epsilon_2 \] (E15)
\[ \langle \psi'| Z'_1 X'_2 | \psi' \rangle \geq 1 - \epsilon_3, \] (E16)
we have
\[ \|(X'_1 - Z'_2) | \psi' \rangle \| \leq \epsilon'_2, \] (E17)
\[ \|(Z'_1 - X'_2) | \psi' \rangle \| \leq \epsilon'_3, \] (E18)
where \( \epsilon'_j := 2^{\frac{3}{2}} \epsilon^{'\frac{1}{2}}_j \) for \( j = 2, 3 \).

Proof: Now, we make the spectral decomposition of \( X'_1 Z'_2 \) as \( X'_1 Z'_2 = P - (I - P) \), where \( P \) is a projection. \[ \text{[E15]} \] implies that \( \langle \psi'| (I - P) | \psi' \rangle \leq \frac{\epsilon_2}{2} \). Schwarz inequality yields that
\[ \frac{1}{2} \|(X'_1 - Z'_2) | \psi' \rangle \| = \frac{1}{2} \|(I - X'_1 Z'_2) | \psi' \rangle \|
\[ \leq \sqrt{\frac{\epsilon_2}{2}}. \] (E19)

Similarly, we obtain other inequalities. ■

Lemma 8. When
\[ \langle \psi'| X'_1 Z'_2 | \psi' \rangle \geq 1 - \epsilon_2 \] (E20)
\[ \langle \psi'| Z'_1 X'_2 | \psi' \rangle \geq 1 - \epsilon_3, \] (E21)
\[ \langle \psi'| A(0)'_1 (Z'_2 + X'_2) | \psi' \rangle \geq 2 - \epsilon_4, \] (E22)
\[ \langle \psi'| A(1)'_1 (Z'_2 - X'_2) | \psi' \rangle \geq 2 - \epsilon_4, \] (E23)
\[ \langle \psi'| X'_1 X'_2 + Z'_1 Z'_2 | \psi' \rangle \| \leq \epsilon_5, \] (E24)
we have
\[ \left\| \left( A(0)'_1 - \frac{Z'_2 + X'_2}{\sqrt{2}} \right) | \psi' \rangle \right\| \leq \epsilon'_4 \] (E25)
\[ \left\| \left( A(1)'_1 - \frac{Z'_2 - X'_2}{\sqrt{2}} \right) | \psi' \rangle \right\| \leq \epsilon'_4, \] (E26)
where \( \epsilon'_4 := \sqrt{2} \epsilon_4 + \frac{1}{2} \epsilon_5 + \sqrt{\epsilon_2} + \sqrt{\epsilon_3} \).

Proof: Since
\[ \|(X'_1 Z'_2 - I) | \psi' \rangle \| \leq 2\sqrt{\epsilon_2}, \] (E27)
\[ \|(Z'_1 X'_2 - I) | \psi' \rangle \| \leq 2\sqrt{\epsilon_3}, \] (E28)
we have
\[ \| (\psi'| Z'_2 X'_2 + X'_2 Z'_2 | \psi' \rangle - \langle \psi'| Z'_2 Z'_2 + X'_2 X'_2 | \psi' \rangle \|
\[ \leq \| (\psi'| Z'_2 X'_2 + X'_2 Z'_2 | \psi' \rangle - \langle \psi'| Z'_2 Z'_2 | \psi' \rangle \|
\[ + \| (\psi'| X'_2 Z'_2 + Z'_2 X'_2 | \psi' \rangle - \langle \psi'| I - Z'_1 Z'_2 | \psi' \rangle \|
\[ \leq 2\sqrt{\epsilon_2} + 2\sqrt{\epsilon_3}. \] (E29)

So, we obtain \[ \text{[E25]} \] as follows.
\[ \left\| \left( A(0)'_1 - \frac{Z'_2 + X'_2}{\sqrt{2}} \right) | \psi' \rangle \right\|^2
\[ = \langle \psi'| A(0)'_1 - \sqrt{2} A(0)'_1 (Z'_2 + X'_2) + \frac{Z'_2^2 + X'_2^2 + Z'_2 X'_2 + X'_2 Z'_2}{2} | \psi' \rangle
\[ = 2 - \langle \psi'| \sqrt{2} A(0)'_1 (Z'_2 + X'_2) + \frac{Z'_2 X'_2 + X'_2 Z'_2}{2} | \psi' \rangle
\[ \leq 2 - \sqrt{2} \langle \psi'| A(0)'_1 (Z'_2 + X'_2) + \frac{Z'_2 X'_2 + X'_2 Z'_2}{2} | \psi' \rangle
\[ + \frac{1}{2} (\langle \psi'| Z'_2 Z'_2 + X'_2 X'_2 | \psi' \rangle + \sqrt{\epsilon_2} + \sqrt{\epsilon_3} \leq 2 - 2 + \frac{\sqrt{2} \epsilon_4 + \frac{1}{2} \epsilon_5 + \sqrt{\epsilon_2} + \sqrt{\epsilon_3} = \sqrt{2} \epsilon_4 + \frac{1}{2} \epsilon_5 + \sqrt{\epsilon_2} + \sqrt{\epsilon_3}. \] (E30)

Similarly, we have
\[ \left\| \left( A(1)'_1 - \frac{Z'_2 + X'_2}{\sqrt{2}} \right) | \psi' \rangle \right\|^2
\[ \leq 2 - \sqrt{2} \langle \psi'| A(1)'_1 (Z'_2 - X'_2) | \psi' \rangle
\[ - \frac{1}{2} (\langle \psi'| Z'_2 Z'_2 + X'_2 X'_2 | \psi' \rangle + \sqrt{\epsilon_2} + \sqrt{\epsilon_3} \leq \sqrt{2} \epsilon_4 + \frac{1}{2} \epsilon_5 + \sqrt{\epsilon_2} + \sqrt{\epsilon_3}, \] (E31)
which shows \[ \text{[E20]} \]. ■

Lemma 9. When
\[ \langle \psi'| X'_1 Z'_2 | \psi' \rangle \| \leq 2\epsilon'_1, \] (E32)
\[ \|(X'_1 - Z'_2) | \psi' \rangle \| \leq \epsilon'_2, \] (E33)
\[ \|(Z'_1 - X'_2) | \psi' \rangle \| \leq \epsilon'_3, \] (E34)
\[ \left\| \left( A(0)'_1 - \frac{Z'_2 + X'_2}{\sqrt{2}} \right) | \psi' \rangle \right\| \leq \epsilon'_4, \] (E35)
\[ \left\| \left( A(1)'_1 - \frac{Z'_2 - X'_2}{\sqrt{2}} \right) | \psi' \rangle \right\| \leq \epsilon'_4, \] (E36)
there exist local isometries \( U_j : \mathcal{H}_j' \rightarrow \mathcal{H}_j \) for \( j = 1, 2 \).
such that

\[ \|U|\psi\rangle - |junk\rangle|\psi\rangle\| \leq \delta'_1, \]  

\[ \|UX'_1|\psi\rangle - X'_1|junk\rangle|\psi\rangle\| \leq \delta'_1, \]  

\[ \|UZ'_1|\psi\rangle - Z'_1|junk\rangle|\psi\rangle\| \leq \delta'_1, \]  

\[ \|UX'_2|\psi\rangle - X'_2|junk\rangle|\psi\rangle\| \leq \delta'_1, \]  

\[ \|UZ'_2|\psi\rangle - Z'_2|junk\rangle|\psi\rangle\| \leq \delta'_1, \]

\[
\left\|UA(0)^1|\psi\rangle - \frac{X_1 + Z_1}{\sqrt{2}}|junk\rangle|\psi\rangle\right\| \leq \delta_2, \tag{E37}
\]

\[
\left\|UA(1)^1|\psi\rangle - \frac{X_1 + Z_1}{\sqrt{2}}|junk\rangle|\psi\rangle\right\| \leq \delta_2, \tag{E38}
\]

where \( \delta'_1 := \sum_{j=1}^3 c'_j c'_j \) and \( \delta_2 := \sqrt{2} \delta'_1 + c'_4 \) and \( U := U_2U_1. \)

**Proof:** We set the initial state on \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) to be \( |0, +\rangle. \)

Define \( U_1 := \{(0,0) + X'_1|1\rangle(1)\}H_1\{(0,0) + Z'_1|1\rangle(1)\}H_1 \) and \( U_2 := \{(0,0) + Z'_2|1\rangle(1)\}H_2\{(0,0) + X'_2|1\rangle(1)\}H_2, \) where \( H := |+\rangle(0) + |-\rangle(1) = |0\rangle + |1\rangle |-\rangle. \)

Hence, we have

\[
U|\psi\rangle = \frac{1}{4}((I + Z'_1)(I + X'_2)|\psi\rangle|0+\rangle + Z'_2(I + Z'_1)(I - X'_2)|\psi\rangle|0-\rangle + X'_1(I - Z'_1)(I + X'_2)|\psi\rangle|1+\rangle + X'_1Z'_2(I - Z'_1)(I - X'_2)|\psi\rangle|1-\rangle. \tag{E39}
\]

When \( |junk\rangle := \frac{\sqrt{2}}{4}(I + Z'_1)(I + X'_2)|\psi\rangle, \) we have

\[
\|U|\psi\rangle - |junk\rangle|\psi\rangle\||\psi\rangle \leq \frac{1}{2}(2c'_3 + 2c'_4 + 4c'_3 + 2c'_4 + 4c'_2). \tag{E40}
\]

Thus,

\[
\|U|\psi\rangle - |junk\rangle|\psi\rangle\||\psi\rangle \leq \frac{1}{2}(2c'_3 + c'_1 + 2c'_2). \tag{E41}
\]

So, we obtain \( \text{E37}. \) Inequalities \( \text{E38} - \text{E41} \) can be shown by using the anti-commutation relation and exchanging \( X_1, Z_1, Z_1 \) and \( X_2, X_2, Z_2. \) The coefficients \( c_j \) for \( \delta \) are given by counting the number of these operations.
Now, we show (E42). We have
\[
\begin{align*}
&\left\| UA(0\rangle_1^1 |\psi\rangle - \frac{X_1 + Z_1}{\sqrt{2}} |\text{junk}\rangle |\psi\rangle \right\| \\
&= \left\| UA(0\rangle_1^1 |\psi\rangle - \frac{Z_2 + X_2}{\sqrt{2}} |\text{junk}\rangle |\psi\rangle \right\| \\
&\leq \left\| UA(0\rangle_1^1 |\psi\rangle - U\left(\frac{Z_2^* + X_2^*}{\sqrt{2}}\right) |\text{junk}\rangle |\psi\rangle \right\| \\
&\leq \left\| \left( \frac{A(0\rangle_1^1)^* - \frac{Z_2^* + X_2^*}{\sqrt{2}} \right) |\psi\rangle \right\| \\
&+ \frac{1}{\sqrt{2}} \left\| U\left(\frac{Z_2^*}{\sqrt{2}}\right) |\psi\rangle - \frac{Z_2}{\sqrt{2}} |\text{junk}\rangle |\psi\rangle \right\| \\
&+ \frac{1}{\sqrt{2}} \left\| U\left(\frac{X_2^*}{\sqrt{2}}\right) |\psi\rangle - \frac{X_2}{\sqrt{2}} |\text{junk}\rangle |\psi\rangle \right\| \\
&\leq \epsilon_4' + \frac{3\delta_1'}{2}.
\end{align*}
\]
We obtain (E42). In the same way, we can show (E43).

**Lemma 10.** The local isometries \(U_j : \mathcal{H}_j \rightarrow \mathcal{H}_j\) for \(j = 1, 2\) satisfy
\[
\begin{align*}
&\left\| U |\psi\rangle - |\text{junk}\rangle |\psi\rangle \right\| \leq \delta_1', \\
&\left\| UX_1 |\psi\rangle - X_1 |\text{junk}\rangle |\psi\rangle \right\| \leq \delta_1', \\
&\left\| UZ_1 |\psi\rangle - Z_1 |\text{junk}\rangle |\psi\rangle \right\| \leq \delta_1', \\
&\left\| UX_2 |\psi\rangle - X_2 |\text{junk}\rangle |\psi\rangle \right\| \leq \delta_1', \\
&\left\| UZ_2 |\psi\rangle - Z_2 |\text{junk}\rangle |\psi\rangle \right\| \leq \delta_1', \\
&\left\| UA(0\rangle_1^1 |\psi\rangle - \frac{X_1 + Z_1}{\sqrt{2}} |\text{junk}\rangle |\psi\rangle \right\| \leq \delta_2', \\
&\left\| UA(1\rangle_1^1 |\psi\rangle - \frac{X_1 - Z_1}{\sqrt{2}} |\text{junk}\rangle |\psi\rangle \right\| \leq \delta_2',
\end{align*}
\]
for \(U := U_2U_1\), we have
\[
\begin{align*}
&\left\| U_1X_1 |U_1\rangle^1 |\psi\rangle - X_1 |\psi\rangle \right\| \leq 2\sqrt{2}\delta_1', \\
&\left\| U_1Z_1 |U_1\rangle^1 |\psi\rangle - Z_1 |\psi\rangle \right\| \leq 2\sqrt{2}\delta_1', \\
&\left\| UA(0\rangle_1^1 |U_1\rangle^1 - \frac{X_1 + Z_1}{\sqrt{2}} |\psi\rangle \right\| \leq \sqrt{2}(\delta_1' + \delta_2'), \\
&\left\| UA(1\rangle_1^1 |U_1\rangle^1 - \frac{X_1 - Z_1}{\sqrt{2}} |\psi\rangle \right\| \leq \sqrt{2}(\delta_1' + \delta_2').
\end{align*}
\]

**Proof:** We have
\[
\begin{align*}
&U_1X_1 |U_1\rangle^1 |\psi\rangle |\text{junk}\rangle \\
&= U_2U_1X_1 |U_1\rangle^1 |U_2\rangle^1 |\text{junk}\rangle |\psi\rangle \\
&\quad = U_2U_1X_1 |U_1\rangle^1 |U_2\rangle^1 U_2U_1 |\psi\rangle' \\
&\quad + U_2U_1X_1 |U_1\rangle^1 U_2 |\text{junk}\rangle |\psi\rangle' - U_2U_1 |\psi\rangle' \\
&= U_2U_1X_1 |\psi\rangle' + U_2U_1X_1 |U_2\rangle^1 |\text{junk}\rangle |\psi\rangle - U_2U_1 |\psi\rangle' \\
&\quad = X_1 |\psi\rangle' |\text{junk}\rangle + (U_2U_1X_1 |\psi\rangle' - X_1 |\text{junk}\rangle |\psi\rangle) \\
&\quad + U_2U_1X_1 |U_1\rangle^1 |\text{junk}\rangle |\psi\rangle - U_2U_1 |\psi\rangle',
\end{align*}
\]
Hence, we obtain
\[
\left\| U_1X_1 |U_1\rangle^1 |\psi\rangle - X_1 |\psi\rangle \right\| \leq 2\delta_1',
\]
which implies that
\[
\left\| U_1X_1 |U_1\rangle^1 - X_1 \right\| \leq 2\sqrt{2}\delta_1'.
\]
So, we obtain (E59). Similarly, we obtain other inequalities.

**Proof of Proposition [1]Chose \(\delta_1 = 2\sqrt{2}\delta_1'\) and \(\delta_2 = \sqrt{2}(\delta_1' + \delta_2') = \sqrt{2}(1 + \sqrt{2})\delta_1' + \epsilon_4'). Then, combining these lemmas, we obtain Proposition [1].**

**Appendix F: Proof of Theorem 2**

To show Theorem 2 of the main body, we prepare Lemma 11 as follows.

**Lemma 11.** Given an acceptance probability \(\beta\) and a significance level \(\alpha\), there exist positive numbers \(\epsilon'' > \epsilon_4 > 0\) satisfying the following. Here, we use the same \(\epsilon'\) as the proof of Theorem 1. If the state \(|\psi\rangle\) and our measurements are prepared with no error, Test (4) with \(c_1 = c_4(\log^2 n)^{1/2}\) is passed with probability \(\beta\) (the acceptability). Once all of the conditions in Step (2-5) with \(c_1 = c_4(\log^2 n)^{1/2}\) are satisfied in all sites, with the significance level \(\alpha\), we can guarantee the conditions (E3)-(E8) in all sites with \(\epsilon_2, \epsilon_3 \leq \epsilon' / m\) and \(\epsilon_1 = \frac{2\epsilon''(\log n)^{1/2}}{\sqrt{m}}, \epsilon_4, \epsilon_5 = \frac{\epsilon''(\log n)^{1/2}}{\sqrt{m}}\).

We choose three positive numbers \(\epsilon' > 0\) and \(\epsilon'' > \epsilon_4 > 0\) as in Lemma 11. So, Lemma 11 guarantees the acceptability. Once all of the conditions in Step (2-5) with \(c_1 = c_4(\log^2 n)^{1/2}\) are satisfied in all sites, with the significance level \(\alpha\), we can guarantee the conditions (E3)-(E8) with \(\epsilon_2, \epsilon_3 \leq \epsilon' / m\) and \(\epsilon_1 = \frac{2\epsilon''(\log n)^{1/2}}{\sqrt{m}}, \epsilon_4, \epsilon_5 = \frac{\epsilon''(\log n)^{1/2}}{\sqrt{m}}\). Due to Proposition [1] using a suitable isometry \(U\), with the significance level \(\alpha\), we can guarantee the conditions (E9)-(E12) with \(\delta_1, \delta_2 = O((\log n)^{1/4})\), which yields the conditions (7) and (8).

Eqs. [9] can be shown as follows. When \(Tr\sigma(I-P_2) \geq \alpha / m\), we accept the stabilizer test with respect to \(P_2\) with probability smaller than \(\alpha\). So, we can guarantee that \(Tr\sigma(I-P_2) \leq \alpha / m\) with significance level \(\alpha\).

**Proof of Lemma 11** To accept the tests \(Av[X_1^1 Z_2] = 1\) and \(Av[Z_1^1 X_2] = 1\) in all sites with more than probability \(\alpha\), the conditions (E4)-(E5) of \(\epsilon_2, \epsilon_3 \leq \epsilon' / m\) need to hold in all sites. More precisely, the summand of \(\epsilon_2\) and \(\epsilon_3\) with respect to all sites need to be \(\epsilon' / m\), which is a stronger condition than the above condition. Once these tests \(Av[X_1^1 Z_2] = 1\) and \(Av[Z_1^1 X_2] = 1\) are passed in all sites, we can guarantee the conditions (E4)-(E5) of \(\epsilon_2, \epsilon_3 \leq \epsilon' / m\) in all sites with significance level \(\alpha\).
The observables in the LHS of $[E4] - [E5]$ and the stabilizer test take a deterministic value in the ideal state. On the other hand, the observables in the LHS of the remaining cases $[E3], [E6] - [E8]$ behave probabilistically even in the ideal state. Hence, for the acceptability, we need to care about the accepting probability only for $[E3], [E6] - [E8]$ in all sites because $[E3]$ follows from $[E3]$ and $[E8]$. In order that the ideal state accepts all of the tests $[1] - [4]$ in all sites, i.e., totally $3n$ tests, with probability $\beta$, the coefficient $c_1$ needs to increase with respect to $n$. For example, when we choose $c_1$ to be $c_1(\log n)^{1/2}$ with a certain constant $c_4$, the ideal state accepts these tests with probability $\beta$ in all sites due to the following reason. To satisfy the above condition, we consider the case when the ideal state accepts each test of each site with probability $1 - \frac{1-\beta}{4n}$, which implies that the ideal state accepts all of $4n$ tests with more than probability $\beta$ because the test $|Av[X'\cdots X' + Z'\cdots Z']| \leq \frac{1-\beta}{\sqrt{m}}$ is regarded as two tests. For example, we focus on the test $Av[A(0)[(Z_1' + X_2')]] \geq \sqrt{2} - \frac{c_4(\log n)^{1/2}}{\sqrt{m}}$. Due to the central limit theorem, the accepting probability with the ideal state is approximated to $\int_{\sqrt{2} - \frac{c_4(\log n)^{1/2}}{\sqrt{m}}}^{\sqrt{2}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$, where $v$ is the variance of $A(0)[(Z_1' + X_2')]$. By solving the condition $\int_{\sqrt{2} - \frac{c_4(\log n)^{1/2}}{\sqrt{m}}}^{\sqrt{2}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = \frac{1-\beta}{4n}$, we obtain $c_1 = c_1(\log n)^{1/2}$ with a certain constant $c_4$. The remaining tests can be treated in the same way. So, we can conclude that the above choice of $c_1$ guarantees the condition for the acceptance probability $\beta$.

Then, dependently of $c_4$ and $\alpha$, there exists a real number $c''' > c_4$ satisfying the following condition. To accept the tests $Av[A(0)[(Z_1' + X_2')]] \geq \sqrt{2} - \frac{c_4(\log n)^{1/2}}{\sqrt{m}}$, $Av[A(1)[(Z_1' - X_2')]] \geq \sqrt{2} - \frac{c_4(\log n)^{1/2}}{\sqrt{m}}$, and $|Av[X_1'X_2' + Z_1'Z_2']| \leq \frac{c_4(\log n)^{1/2}}{\sqrt{m}}$, $Av[A(1)[Z_1'X_2' - Z_2'X_1']] = 1$, and $Av[Z_1'X_2'] = 1$ in all sites with more than probability $\alpha(< \beta)$, we need to accept the tests $Av[A(0)[(Z_1' + X_2')]] \geq \sqrt{2} - \frac{c_4(\log n)^{1/2}}{\sqrt{m}}$, $Av[A(1)[(Z_1' - X_2')]] \geq \sqrt{2} - \frac{c_4(\log n)^{1/2}}{\sqrt{m}}$, and $|Av[X_1'X_2' + Z_1'Z_2']| \leq \frac{c_4(\log n)^{1/2}}{\sqrt{m}}$. $Av[A(1)[Z_1'X_2' - Z_2'X_1']] = 1$, and $Av[Z_1'X_2'] = 1$ in each sites with more than probability $\alpha(< \beta)$, which implies that the conditions $[E6] - [E8]$ of $c_4, c_5 = c_3(\log m)^{1/2}$ hold in each site with a certain constant $c_1$. Since $c''''(\log n)^{1/2} > c_4(\log n)^{1/2} + c_1$, the above condition implies the conditions $[E6] - [E8]$ of $c_4, c_5 = c_3(\log m)^{1/2}$ in all sites with significance level $\alpha$. So, by choosing $\epsilon_1 = 2\epsilon_4$, $[E3]$ automatically holds in all sites.

Appendix G: Error of POVM element: Proof of Inequality (10)

Similarly to [9], we introduce $n$ ideal trusted systems spanned by $[0], [1]$ while each untrusted system is spanned by $[1], [-1]$. Let $U_j$ be a unitary on the trusted system. Let $V_j$ be a unitary controlling the $j$-th untrusted system by the trusted system, defined as follows. The operators on the untrusted system are restricted to $I$ and $s$ operators $\{D_{(i)}\}_{i=1}^{s}$ such that their eigenvalues are $1$ and $-1$. So, $U(D_{(i)}U^\dagger - D_{(i)}) \leq \delta$. In the main text, $s = 4$ and $\{D_{(i)}\}_{i=1}^{s} = \{X, Z, A(0), A(1)\}$. Then, we assume $V_j$ has the form $\sum_{k\in F_2^*} \langle k|D_j(k)$, where $D_j(k)$ is one of $I$ and $\{D_{(i)}\}_{i=1}^{s}$. According to FIG. 7 of [9], we define $W_j := U_jV_jW_{j-1}$, and $W_0 = U_0$ and $U := U_1 \cdots U_n$.

Proposition 2 (9 Lemma 6) with modification. We have

\[\|UW_jU^\dagger - W_j\| \leq s\delta.\] (G1)

Proof: We have

\[UW_jU^\dagger - W_j = UU_jU_j^\dagger V_jW_{j-1}U^\dagger - U_jV_jW_{j-1}^\dagger U^\dagger = (UU_jU_j^\dagger - U_j)W_{j-1}^\dagger U^\dagger + U_jV_j(UW_{j-1}^\dagger - W_{j-1}).\] (G2)

By induction, it is enough to show

\[\|UW_jU^\dagger - U_j\| \leq s\delta.\] (G3)

We have

\[UW_jU^\dagger - U_j = U \sum_{k\in F_2^*} \langle k|D_j(k)U_j^\dagger \sum_{k\in F_2^*} \langle k|D_j(k)

\[= \sum_{i=1}^{s} \langle k|D_{(i)}U_j^\dagger - D_{(i)}\rangle\] (G4)

For $i$, we have

\[\| \sum_{k\in F_2^*} \langle k|D_{(k)}(U_{(i)}U_j^\dagger - D_{(i)})\| \leq \sum_{k\in F_2^*} \langle k|D_{(k)}(U_{(i)}U_j^\dagger - D_{(i)})\| \leq \delta.\] (G5)

So, we have (G3).
set the initial state $|\pm\rangle^\otimes n$ on the trusted system. Choose $U_j$ as the application of the Hadamard operator $H$ on the $j$-th trusted system. Then, we define

$$V_j := \sum_{k_1, \ldots, k_{j-1}} |k_1, \ldots, k_{j-1}, 0\rangle\langle k_1, \ldots, k_{j-1}, 0|$$
$$+ D_j(k_1, \ldots, k_{j-1})|k_1, \ldots, k_{j-1}, 1\rangle\langle k_1, \ldots, k_{j-1}, 1|.$$ 

Then, we define the TP-CP map $\Lambda$ from the untrusted $n$-qubit system to the trusted $n$-qubit system as

$$\Lambda(\rho) := \text{Tr}_{UT} U T W_{n}\rho \otimes |\pm\rangle^\otimes n W_n^\dagger,$$  

where $\text{Tr}_{UT}$ expresses the partial trace with respect to the untrusted system. Due to the construction, $\Lambda(\rho)$ is the same as the output distribution when the above adaptive measurement is applied.

**Proposition 3** ([9] Corollary 2 with modification). For any state $\rho$, we have

$$\|UA'(U^\dagger \rho U)^\dagger - \Lambda(\rho)\|_1 \leq 2s\eta.$$  

Hence, when $M_i$ is a POVM element of an adaptive measurement, we have

$$\|UM_iU^\dagger - M_i\| \leq \max\rho \text{Tr}(UM_iU^\dagger - M_i)\rho \leq 2s\eta,$$  

which implies inequality [10] of the main text by substituting 4 for $s$.

**Proof:** We have

$$U W_j U^\dagger (\rho \otimes |\pm\rangle^\otimes n) U W_j^\dagger U^\dagger - W_j (\rho \otimes |\pm\rangle^\otimes n) W_j^\dagger$$

$$= (U W_j U^\dagger - W_j) (\rho \otimes |\pm\rangle^\otimes n) U W_j^\dagger U^\dagger + W_j (\rho \otimes |\pm\rangle^\otimes n) (U W_j^\dagger U^\dagger - W_j^\dagger).$$  

Also, we have

$$\| (U W_j U^\dagger - W_j) (\rho \otimes |\pm\rangle^\otimes n) U W_j^\dagger U^\dagger \|_1$$

$$\leq \|U W_j U^\dagger - W_j\| \| (\rho \otimes |\pm\rangle^\otimes n) U W_j^\dagger U^\dagger \|_1$$

$$\leq \|U W_j U^\dagger - W_j\| \leq 2\eta,$$  

$$\|W_j (\rho \otimes |\pm\rangle^\otimes n) (U W_j^\dagger U^\dagger - W_j^\dagger)\|_1$$

$$\leq \|W_j (\rho \otimes |\pm\rangle^\otimes n)\| \| (U W_j^\dagger U^\dagger - W_j^\dagger)\|$$

$$\leq \|U W_j^\dagger U^\dagger - W_j^\dagger\| \leq 2\eta.$$  

Combining (G9), (G10), and (G11), we have

$$\|UA'(U^\dagger \rho U)^\dagger - \Lambda(\rho)\|_1$$

$$\leq \|U W_j U^\dagger (\rho \otimes |\pm\rangle^\otimes n) U W_j^\dagger U^\dagger - W_j (\rho \otimes |\pm\rangle^\otimes n) W_j^\dagger U^\dagger\|_1$$

$$\leq 2s\eta,$$  

where the first inequality follows from the information processing inequality with respect to the trace of the untrusted system. Hence, we obtain (G7).

---

**Appendix H: Error in the initial state: Proof of inequality [11]**

In this section, we show a slightly stronger inequality than inequality [11] of main text:

$$\|U \sigma U^\dagger - |G\rangle\langle G\|_1^2 \leq 6\eta + 3\alpha/m$$  

by assuming Inequalities (3)–(5) in Theorem 2.

Now, we have the relation

$$\|U \sigma U^\dagger - |G\rangle\langle G\|_1^2$$  

$$(a) \leq 1 - \text{Tr} (G |U \sigma U^\dagger G) = \text{Tr} (I - |G\rangle\langle G|) U \sigma U^\dagger$$

$$\leq \text{Tr} (I - P_1) U \sigma U^\dagger + \text{Tr} (I - P_2) U \sigma U^\dagger$$

$$+ \text{Tr} (I - P_3) U \sigma U^\dagger,$$  

where (a) follows from the relation between the trace norm and the fidelity [37, (6.106)] and (b) follows from the inequality $I - |G\rangle\langle G| \leq (I - P_1) + (I - P_2) + (I - P_3)$.

We can apply (G8) with $s = 2$ to $P_1$ because $P_1$ is a POVM element of an adaptive measurement based on $X$ and $Z$. So, we have

$$\|U P_i U^\dagger - |G\rangle\langle G\|_1^2$$

$$= \|\text{Tr} (U P_i U^\dagger - |P_i\rangle\langle P_i| U \sigma U^\dagger)\|_1^2$$

$$\leq \|U P_i U^\dagger - |P_i\rangle\langle P_i|\|_2 \leq 2\eta.$$  

Thus, Inequality (9) in Theorem 2 implies

$$\|\text{Tr} (U (I - P_1) U^\dagger - (I - P_1) U \sigma U^\dagger)\|_1^2$$

$$\leq \|U P_i U^\dagger - |P_i\rangle\langle P_i|\|_2 \leq 2\eta.$$  

The combination of (H2) and (H4) yields (H1).

**Appendix I: Interactive proof system**

Although not explicitly stated in the main part, our protocol is an instance of an interactive proof system [38, 39] for any language in BQP with a quantum prover (Bob) and a nearly-classical verifier (Alice) equipped with a random number generator. More formally, for every language $L \in \text{BQP}$ and input $x$ there exists a poly(|x|)-time verifier $V$ interacting with a poly(|x|) number of quantum provers such that if $x \in L$, there exists a set of honest provers for which $V$ accepts with probability at least $c = 2/3$ (completeness). If $x \notin L$, then for any set of provers, $V$ accepts with probability at most $p_{\text{incorrect}} \leq s = 1/3$ (soundness), where $p_{\text{incorrect}}$ is given by Eq. (12) of the main part.
We measure $Z'$ on all sites of $S_i \setminus S_{i,j}$ for all copies. Then, we apply $Z'$ operators on the remaining sites to correct applied $Z'$ operators depending on the outcomes.

(3-4): Due to the above steps, the resultant state should be $\otimes_{a \in S_{i,j}} |\Phi'_a\rangle_{ab}$. Then, we apply the self-testing procedure to all of $\{|\Phi'_a\rangle_{ab}\}_{a \in S_{i,j}}$.

The above protocol verifies the measurement device on sites with $i$-th color. Then, applying this generalization to all colors in the protocol, we can extend the first part.

To realize the second part, for each color $i$, we measure non-$i$ color sites with $Z$ basis and check whether the outcome of measurement $X$ on the sites with color $i$ is the same as the predicted one. Then, we denote the projection corresponding to the passing event for this test by $P_i$. Hence, we have $\prod_{i=1}^k P_i = |G\rangle\langle G|$ because only the state $|G\rangle$ can pass all of these tests. Thus, applying this test for all colors, we can test whether the state is the desired graph state.

Then, choosing $c_3$ to be $k + 8(\sum_{i=1}^k t_i)$, we propose our self-testing protocol as follows,

(4-1): We prepare $c_3m + 1$ $n$-qubit states $|G'\rangle$.

(4-2): We randomly divide the $c_3m + 1$ copies into $c_3 + 1$ groups. The first $c_3$ groups are composed of $m$ copies and the final group is composed of a single copy.

(4-3): For the first $k$ groups, we apply the following test. For the $i$-th group, we measure $Z'$ on the sites with non-$i$ color and $X'$ on the sites with $i$-th color, and check that the outcome of $X'$ measurements is the same as predicted from the outcomes of $Z'$ measurements.

(4-4): We run the $i$-protocol with $S_{i,j}$ for the $k + 8(j - 1 + \sum_{i=1}^{j-1} l_i) + 1$-th $k + 8(j + \sum_{i=1}^{j-1} l_i)$-th groups. Then, we check $8$ conditions in Step (2-5). We repeat this protocol for $j = 1, \ldots, l_i$ and $i = 1, \ldots, k$.

When we employ the above protocol for $k$-colorable case, the difference from the 3-colorable case is only the number of samples. We have the same analysis for the certification of computation result as the 3-colorable case.

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