Truncated Gröbner fans and lattice ideals

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Abstract

We outline a generalization of the Gröbner fan of a homogeneous ideal with maximal cells parametrizing truncated Gröbner bases. This “truncated” Gröbner fan is usually much smaller than the full Gröbner fan and offers the natural framework for conversion between truncated Gröbner bases. The generic Gröbner walk generalizes naturally to this setting by using the Buchberger algorithm with truncation on facets.

We specialize to the setting of lattice ideals. Here facets along the generic walk are given by unique (facet) binomials. This along with the representation of binomials as integer vectors give an especially simple version of the generic Gröbner walk.

Computational experience with the special Aardal-Lenstra integer programming knapsack problems is reported. The algorithms of this paper are implemented in the software package GLATWALK, which is available for download at [http://home.imf.au.dk/niels/GLATWALK](http://home.imf.au.dk/niels/GLATWALK).

1 Introduction

The generic Gröbner walk [3] is a version of the classical Gröbner walk algorithm for Gröbner basis conversion in the Gröbner fan of an ideal in a polynomial ring. In the generic walk explicit rational vectors in the Gröbner fan are replaced by computations with infinitesimal numbers which can be handled formally. This leads to an algorithm where input consists only of a source Gröbner basis, a source term order and a target term order.

Truncation of homogeneous ideals have proved very valuable in algebraic computations related to integer programming (see for example [7]). In general there are a lot fewer truncated initial ideals than initial ideals. Similary to initial ideals, truncated initial ideals may be parametrized by the maximal cells in a complete polyhedral fan. We introduce this fan, which is easily constructed from the usual Gröbner fan by inserting the truncation operator at the appropriate places. The truncated Gröbner fan is in general much smaller than the full Gröbner fan and forms the polyhedral setting for a truncated version of the generic Gröbner walk. We prove that the truncated Gröbner fan is
regular along the lines of [6]. This leads to a “truncated” state polytope with vertices enumerating the different reduced truncated Gröbner bases.

In the setting of lattice ideals we give a rather detailed version of the generic Gröbner walk. Algebraic computations with lattice ideals can be greatly simplified representing (saturated) binomials as integer vectors. This along with the fact that the generic walk only traverses facets lead to several simplifications. We report on computational experience in computing saturations of lattice ideals and truncated test sets related to the integer programming problems posed in [2]. Our experiments show that the the generic walk in the truncated Gröbner fan consists of significantly fewer steps, whereas the walk in the full Gröbner fan does not compare well computing directly with the Buchberger algorithm.

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2 Preliminaries

We let \( R = k[x_1, \ldots, x_n] \) denote the ring of polynomials over a field \( k \). We will view \( R \) as the semi-group ring \( k[^n] \).

2.1 Grading on \( R \)

Given \( n \) elements \( a_1, \ldots, a_n \) of an abelian group \((A, +)\) we let \( S_A \) denote the semigroup \( \mathbb{N} a_1 + \cdots + \mathbb{N} a_n \subset A \). For \( v = (v_1, \ldots, v_n) \in \mathbb{N}^n \) we put \( \deg(x^v) = v \cdot a = v_1 a_1 + \cdots + v_n a_n \). These data give a natural \( A \)-grading on \( R \) by defining

\[
R_s = \text{span}_k \{ x^v \mid \deg(x^v) = s \}
\]

for \( s \in S_A \). A non-zero element \( f \in R \) is called homogeneous of degree \( \deg(f) = s \) if \( f \in R_s \). Given an ideal \( J \) in \( R \), we let \( J_s = J \cap R_s \). Recall that an ideal \( J \) is homogeneous ideal if \( J = \oplus J_s \) and that this is equivalent to \( J \) being generated by homogeneous elements. We call \( R \) positively graded if \( R_0 = k \). This is equivalent to \( \dim_k R_s < \infty \) for every \( s \in S_A \).

2.2 Truncating subsets

A subset \( \Omega \subset S_A \) is called truncating if \( s, t \in \Omega \) whenever \( s + t \in \Omega \) for \( s, t \in S_A \). Our standard example of a truncating subset is

\[
\Omega_b = \{ x \in S_A \mid b - x \in S_A \}
\]

for \( b \in S_A \). To a truncating subset \( \Omega \subset S_A \) we associate the monomial ideal (cf. Remark 12.8 in [6])

\[
M_\Omega = \langle x^v \mid \deg(v) \notin \Omega \rangle \subset R.
\]

Given a homogeneous ideal \( J \subset R \) we let

\[
J_\Omega = \bigoplus_{s \in \Omega} J_s.
\]
Lemma 2.1 Let I and J be homogeneous ideals in R. Then
\[ I_\Omega = J_\Omega \text{ if and only if } I + M_\Omega = J + M_\Omega. \]

Proof. If I is a homogeneous ideal, then \( I + M_\Omega \) is a homogeneous ideal and
\[ (I + M_\Omega)_s = \begin{cases} I_s & \text{if } s \in \Omega \\ (M_\Omega)_s & \text{if } s \not\in \Omega. \end{cases} \]

This proves the lemma. \qed

2.3 Initial ideals

Let \( \prec \) denote a total multiplicative ordering on monomials in R (we do not require that the monomial 1 is minimal). If \( \omega \in \mathbb{R}^n \), we let \( \prec_\omega \) denote the multiplicative ordering defined by \( x^u \prec_\omega x^v \) if \( \omega \cdot u \prec_\omega \omega \cdot v \) or \( \omega \cdot u = \omega \cdot v \) and \( x^u \prec x^v \). For \( f \in R \), we let \( \text{supp}(f) \) denote the set of monomials occurring with non-zero coefficient in \( f \). We let \( \text{in}_\prec(f) \) denote the maximal (initial) term in \( \text{supp}(f) \) with respect to \( \prec \). Similarly we let \( \text{in}_\omega(f) \) denote the sum of terms \( a_\omega x^v \) in \( \text{supp}(f) \) with \( \omega \cdot v \) maximal. For a subset \( G \subset R \) we let \( \text{in}_\omega(G) \) and \( \text{in}_\prec(G) \) denote the ideals \( \langle \text{in}_\omega(f) \mid f \in G \setminus \{0\} \rangle \) and \( \langle \text{in}_\prec(f) \mid f \in G \rangle \) respectively. These ideals are homogeneous if \( I \) is homogeneous. A Gröbner basis for \( I \) over \( \prec \) is a finite set \( G := \{f_1, \ldots, f_r\} \subset I \) such that
\[ \text{in}_\prec(I) = \text{in}_\prec(G) = \langle \text{in}_\prec(f_1), \ldots, \text{in}_\prec(f_r) \rangle. \]

The Gröbner basis \( G \) is called minimal if none of \( f_1, \ldots, f_r \) can be left out and reduced if \( \text{in}_\prec(f_i) \) does not divide any of the terms in \( f_j \) for \( i \neq j \) and \( i, j = 1, \ldots, r \). The reduced Gröbner basis of an ideal is unique and consists of homogeneous elements if the ideal is homogeneous. A homogeneous ideal \( J \) in \( R \) always has a reduced Gröbner basis over \( \prec \) if \( R \) is positively graded, since \( \dim_k J_s < \infty \) for \( s \in S_A \). We record the following simple but crucial result ([6], Proposition 1.8) with a complete proof.

Proposition 2.2 Let \( I \subset R \) be any ideal and \( \omega \in \mathbb{R}^n \). Then
\[ \text{in}_\prec(\text{in}_\omega(I)) = \text{in}_\prec(\text{in}_\omega(I)). \]

Proof. Clearly \( \text{in}_\prec(\text{in}_\omega(I)) \subset \text{in}_\prec(\text{in}_\omega(I)). \) The ideal \( \text{in}_\omega(I) \) is homogeneous in the grading given by \( \omega \). So we may decompose an element \( f \in \text{in}_\omega(I) \) as \( f = f_{\lambda_1} + \cdots + f_{\lambda_t} \), where \( f_{\lambda_j} \in \text{in}_\omega(I) \) is homogeneous of \( \omega \)-weight \( \lambda_j \) for \( j = 1, \ldots, t \). Now \( \text{in}_\prec(f) = \text{in}_\prec(f_{\lambda_j}) \) for some \( j \) and we may write
\[ f_{\lambda_j} = a_1 \text{in}_\omega(f_1) + \cdots + a_r \text{in}_\omega(f_r) \]
for suitable \( f_1, \ldots, f_r \in I \), where \( a_1, \ldots, a_r \) are homogeneous elements. Therefore
\[ f_{\lambda_j} = \text{in}_\omega(a_1 f_1 + \cdots + a_r f_r) \]
and \( \text{in}_\prec(f) = \text{in}_\prec(a_1 f_1 + \cdots + a_r f_r) \). This shows that \( \text{in}_\prec(\text{in}_\omega(I)) \supset \text{in}_\prec(\text{in}_\omega(I)). \) \qed

Notice that the multiplicativity of \( \prec \) is not used in the proof of Proposition 2.2. The lifting from \( \text{in}_\omega(I) \) to \( I \) in the proof is a key element in the Gröbner walk algorithm.
2.4 Truncated Gröbner bases

Let $J$ be a homogeneous ideal in $R$ and $\Omega$ a truncating subset. A finite subset $G \subset J$ is called an $\Omega$-Gröbner basis for $J$ over $\prec$ if

$$\text{in}_\prec(J)_\Omega = \text{in}_\prec(G)_\Omega.$$  

If the coefficients of the initial terms in $\text{in}_\prec(g)$ are 1 for $g \in G$ and $\text{in}_\prec(g)$ does not divide any of the terms in $g'$ for $g \neq g' \in G$, then $G$ is called a reduced $\Omega$-Gröbner basis. Reduced $\Omega$-Gröbner bases are unique.

**Proposition 2.3** Let $G$ be the reduced Gröbner basis for $J$ over $\prec$. Then $G$ consists of homogeneous elements and $G_\Omega = \{g \in G \mid \deg(g) \in \Omega\}$ is the reduced $\Omega$-Gröbner basis for $J$ over $\prec$.

**Proof.** If $g$ is an element of the reduced Gröbner basis of $J$ and $g = g_1 + \cdots + g_r$ is written as a sum of homogeneous elements, then $\text{in}_\prec(g) = \text{in}_\prec(g_j)$ for some $j = 1, \ldots, r$. Therefore $g$ has to be homogeneous. The monomial ideal $\text{in}_\prec(J)$ is spanned as a vector space by $\{\text{in}_\prec(f) \mid f \in J_s, \text{for some } s \in S\}$. This shows that $\text{in}_\prec(J)_\Omega$ is the $k$-span of $\text{in}_\prec(f)$ for $f \in J_\Omega$. Suppose that $f \in J_\Omega$. Since $G$ is a Gröbner basis for $J$ we may find $g \in G$, such that $\text{in}_\prec(g)$ divides $\text{in}_\prec(f)$. This shows that $\deg(g) \in \Omega$, since $\Omega$ is a truncating subset. Therefore $\text{in}_\prec(f) \in \langle \text{in}_\prec(g) \mid g \in G_\Omega \rangle$. □

**Corollary 2.4** If $G_\Omega$ is the reduced $\Omega$-Gröbner basis for $I$ over $\prec_\omega$ then $\{\text{in}_\omega(g) \mid g \in G_\Omega\}$ is the reduced $\Omega$-Gröbner basis for $\text{in}_\omega(I)$ over $\prec$.

**Proof.** Let $G$ be the reduced Gröbner basis for $I$ over $\prec_\omega$. Then we know from Proposition 2.2 that $\{\text{in}_\omega(g) \mid g \in G\}$ is the reduced Gröbner basis of $\text{in}_\omega(I)$. Now Proposition 2.3 gives that

$$\{\text{in}_\omega(g) \mid g \in G, \deg(\text{in}_\omega(g)) \in \Omega\} = \{\text{in}_\omega(g) \mid g \in G_\Omega\}$$

is the reduced $\Omega$-Gröbner basis of $I$. □

Proposition 2.3 reveals that the $\Omega$-truncated Gröbner basis can be obtained from the reduced Gröbner basis by picking out the elements with degree in $\Omega$. Truncated Gröbner bases can be computed from a homogeneous generating set using Buchberger’s algorithm discarding $S$-polynomials with degree outside $\Omega$. This follows from the fact that the division algorithm preserves the degree of a homogeneous polynomial. Very often only Gröbner bases up to a certain degree are needed.

3 The truncated Gröbner fan

In this section we assume that $R$ is positively graded. Analogously to ([6], Proposition 2.3) we define

$$C_\Omega[\omega] = \{v \in \mathbb{R}^n \mid \text{in}_v(I)_\Omega = \text{in}_\omega(I)_\Omega\}$$

for a homogeneous ideal $I \subset R$. We call the closure of $C_\Omega[\omega]$ in $\mathbb{R}^n$ a truncated Gröbner cone.

**Theorem 3.1** The collection

$$\mathcal{F}_\Omega(I) = \{C_\Omega[\omega] \mid \omega \in \mathbb{R}^n\}$$

of truncated Gröbner cones form a complete fan in $\mathbb{R}^n$. 
Proof. A monomial ideal $M$ satisfies $\text{in}_u(J) + M = \text{in}_u(J + M)$ for every ideal $J \subset R$ and $u \in \mathbb{R}^n$. Now Lemma 2.1 shows that $\text{in}_v(I) + M = \text{in}_v(I + M)$ if and only if $\text{in}_v(I + M) = \text{in}_v(I + M)$. This proves that $\mathcal{F}_\Omega(I)$ is the usual Gröbner fan of the homogeneous ideal $I + M$. Now the conclusion follows from (6, Proposition 2.4).

The truncated Gröbner fan is available from the usual Gröbner fan by eliminating inequalities given by polynomials of degree outside $\Omega$. We give an example illustrating this.

Example 3.2 Consider the (toric) ideal

$$I_A = \langle a^2c - b^2e, a^2d - be^2, ce - bd \rangle \subset k[a, b, c, d, e].$$

This ideal is homogeneous in the grading given by the columns of

$$A = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
0 & 1 & 2 & 1 & 0 \\
0 & 0 & 1 & 2 & 1
\end{pmatrix}. $$

For example, the degree of the variable $c$ is $(1, 2, 1)$. The Gröbner fan $\mathcal{F}(I_A)$ of $I_A$ is the normal fan of an octagon in $\mathbb{R}^5$ (cf. Example 1.1 in [4]). It is pictured in $\mathbb{R}^2$ below with reduced Gröbner bases labeling the maximal cells.

Putting $\Omega = \{ v \in \mathbb{Z}^3 \mid v \cdot (1, 1, 1) < 6 \}$ we get the truncated fan $\mathcal{F}_\Omega(I_A)$ with the reduced $\Omega$-Gröbner
bases labeling the maximal cells.

4 Truncated state polytopes

The truncated Gröbner fan of $I$ is the usual Gröbner fan of $I + M\Omega$. From this it follows that the truncated Gröbner fan of a homogeneous ideal $I$ is the normal fan of a natural Minkowski summand in a state polytope for $I$. Emphasizing the simple Lemma 4.1 below, we briefly sketch a proof of this along the lines of [6]. Let $I$ be a homogeneous ideal in $R$ and $a \in A$.

**Lemma 4.1** Let $\prec$ and $\prec'$ be two total multiplicative orderings on monomials in $R$. Let $x^{v_1}, \ldots, x^{v_s}$ denote the monomials in $\text{in}_{\prec}(I)_a$ and $x^{u_1}, \ldots, x^{u_t}$ the monomials in $\text{in}_{\prec'}(I)_a$. Then after permuting $u_1, \ldots, u_t$ and $v_1, \ldots, v_s$ we may assume that $x^{v_1} \prec x^{v_2} \prec \cdots \prec x^{v_s}$ and

- $x^{u_1} \prec x^{u_2}$
- $x^{u_2} \prec x^{u_3}$
- $\vdots$
- $x^{u_t} \prec x^{v_s}$.

**Proof.** We may find a vector space basis $f_1, \ldots, f_s$ of $I_a$, such that $\text{in}_{\prec}(f_1) = x^{v_1}, \ldots, \text{in}_{\prec}(f_s) = x^{v_s}$ and $x^{v_1} \prec x^{v_2} \prec \cdots \prec x^{v_s}$. Now put $f'_1 = f_1$. Move on to $f_2$. If $\text{in}_{\prec}(f_2) = \lambda f'_1$ for $\lambda \in k$, put $f'_2 = f_2 - \lambda f'_1$. Then clearly $\text{in}_{\prec}(f'_2) \prec \text{in}_{\prec}(f_2)$. In general if $\text{in}_{\prec}(f'_j) \prec \text{in}_{\prec}(f_j)$ for $j < m$ and $\text{in}_{\prec}(f'_m) \in W = \text{span}_k \{ \text{in}_{\prec}(f'_1), \ldots, \text{in}_{\prec}(f'_{m-1}) \}$, then $f'_m = f_m - \lambda_1 f'_1 - \cdots - \lambda_{m-1} f'_{m-1}$ satisfies $\text{in}_{\prec}(f'_m) \not\in W$ for suitable $\lambda_1, \ldots, \lambda_{m-1} \in k$. Furthermore $\text{in}_{\prec}(f'_m) \prec \text{in}_{\prec}(f_m)$. In this way we get the monomials $x^{u_1} = \text{in}_{\prec}(f'_1), \ldots, x^{u_t} = \text{in}_{\prec}(f'_s)$ of $\text{in}_{\prec}(I)_a$ written up in the desired way. \qed

Let

$$
\Sigma \text{in}_{\prec}(I)_a = \sum_{v \in \text{in}_{\prec}(I)_a} v
$$

where $a \in A$ and $\prec$ is a multiplicative total ordering.
Corollary 4.2 Let $\prec_1$ and $\prec_2$ be two total multiplicative orderings. If $\Sigma \in_{\prec_1} (I)_a = \Sigma \in_{\prec_2} (I)_a$, then $\in_{\prec_1} (I)_a = \in_{\prec_2} (I)_a$.

Proof. This is an easy consequence of Lemma 4.1

Definition 4.3 A state polytope in degree $a$ for $I$ is defined as

$$\text{State}_a(I) = \text{conv}\{\Sigma \in_{\prec} (I)_a \mid \prec \text{ total multiplicative ordering}\}.$$ 

For a finite subset $S \subset S_A$ we let

$$\text{State}_S(I) = \sum_{a \in S} \text{State}_a(I).$$

Corollary 4.4 For $\omega \in \mathbb{R}^n$ we have

$$\text{face}_\omega(\text{State}_a(I)) = \text{State}_a(\in_\omega(I)).$$

Proof. First assume that $\in_\omega(I)$ is a monomial ideal and $\text{face}_\omega(\text{State}_a(I))$ is a vertex $\{\Sigma \in_{\prec} (I)_a\}$ for some multiplicative monomial ordering $\prec$ by picking a generic $\omega$. Apply Lemma 4.1 to $\in_{\omega(I)}$ and $\in_{\omega(I)}$. In this setting we then have $u_1 = v_1, \ldots, u_s = v_s$ since $\omega \cdot (v_1 + \cdots + v_s) \geq \omega \cdot (u_1 + \cdots + u_s)$. Therefore $\{\Sigma \in_{\prec} (I)_a\} = \{\Sigma \in_\omega(I)_a\}$ when $\omega$ is generic. General $\omega$ are reduced to generic $\omega$ as in the last part of the proof of Lemma 2.6 in [6].

Now let $G$ be a universal Gröbner basis for $I$ consisting of homogeneous elements. Put $S = \{\deg(g) \mid g \in G\}$. Given a truncating subset $\Omega \subset S_A$ we let

$$\text{State}_\Omega(I) = \text{State}_{\Omega \setminus S}(I)$$

denote a truncated state polytope. Notice that $\text{State}_\Omega(I)$ is a Minkowski summand in a state polytope $\text{State}_S(I)$ of $I$. Using Corollaries 4.2 and 4.4 the same arguments as in the last part of the proof of Theorem 2.5 in [6] show that the normal fan of $\text{State}_\Omega(I)$ is $\mathcal{F}_\Omega(I)$. It follows that $\mathcal{F}_\Omega(I)$ is the normal fan of a Minkowski summand of a state polytope and that $\mathcal{F}_\Omega(I)$ is a coarsening of the usual Gröbner fan.

5 Walking in the truncated Gröbner fan

The Gröbner walk can be carried out in the truncated Gröbner fan converting one truncated Gröbner basis to another. We sketch the appropriate generalization of Proposition 3.2 in [3]. The term orders of ([3], Proposition 3.2) are represented by weight vectors below. If a weight vector $\omega$ represents the term order $\prec$, then we refer to $\prec_\eta$ as $\omega$ modified by $\eta$.

Proposition 5.1 Let $I$ be a homogeneous ideal in $R$ and $\Omega$ a truncating subset. Let $C_{\Omega}[\omega_1]$ and $C_{\Omega}[\omega_2]$ be maximal cells in the truncated Gröbner fan $\mathcal{F}_\Omega(I)$ of $I$. Suppose that $G$ is the reduced $\Omega$-Gröbner basis for $I$ over $\omega_1$. If $\omega \in \overline{C_{\Omega}[\omega_1]} \cap C_{\Omega}[\omega_2]$, then

(i) The reduced $\Omega$-Gröbner basis for $\in_\omega(I)$ over $\omega_1$ is $G_{\omega} = \{\in_\omega(g) \mid g \in G\}$. 

(ii) If \( H \) is the reduced \( \Omega \)-Gröbner basis for \( \text{in}_\omega(I) \) over \( \omega_2 \), then
\[
\{ f - f^G | f \in H \}
\]
is a minimal \( \Omega \)-Gröbner basis for \( I \) over \( \omega_2 \) modified by \( \omega \).

(iii) The reduced \( \Omega \)-Gröbner basis for \( I \) over \( \omega_2 \) modified by \( \omega \) coincides with the reduced \( \Omega \)-Gröbner basis for \( I \) over \( \omega_2 \).

Proof. The items (i) and (iii) follow as in Proposition 3.2 of [3] taking Proposition 2.3 and Corollary 2.4 into account. For the proof of (iii) observe that if \( H' = \{ f_1, \ldots, f_s \} \) is the reduced Gröbner basis of \( \text{in}_\omega(I) \) over \( \omega_2 \), then \( G'' := \{ f_1 - f_1^G, \ldots, f_s - f_s^G \} \) is a minimal Gröbner basis for \( I \) over \( \omega_2 \) modified by \( \omega \) by Proposition 3.2(ii) in [3]. Here \( G' \) is the reduced Gröbner basis for \( I \) over \( \omega_1 \). Notice that \( G'' \) consists of homogeneous elements and that
\[
G''_\Omega = \{ g \in G'' | \deg(g) \in \Omega \}
\]
is a minimal \( \Omega \)-Gröbner basis for \( I \) over \( \omega_2 \) modified by \( \omega \). If \( \deg(f_i - f_i^G) \in \Omega \), then \( \deg(f_i) \in \Omega \). In this case \( f_i^G = f_i^G \). Using Proposition 2.3 this finishes the proof of (ii). \( \square \)

Now the generic Gröbner walk ([3], §4) carries over verbatim to the truncated setting using the Buchberger algorithm with truncation in step (iv).

6 Lattice ideals

In the rest of this paper we will remain exclusively in the setting of lattice ideals. Recall the decomposition of an integral vector \( v \in \mathbb{Z}^n \) into \( v = v^+ - v^- \), where \( v^+, v^- \in \mathbb{N}^n \) are vectors with disjoint support. For \( u, v \in \mathbb{N}^n \) we let \( u \preceq v \) denote the partial order given by \( v - u \in \mathbb{N}^n \). For a subset \( B \subseteq \mathbb{Z}^n \) we associate the ideal
\[
I_B = \langle x^{v^+} - x^{v^-} | v \in B \rangle \subset R.
\]
In the case where \( B = \mathcal{L} \) is a lattice we call \( I_\mathcal{L} \) the lattice ideal associated to \( \mathcal{L} \). Recall that lattice ideals are saturated i.e. if \( f \in I_\mathcal{L} \) is divisible by a variable \( x_i \), then \( f/x_i \in I_\mathcal{L} \). This means that we apply the homogeneous Buchberger algorithm with sat-reduction as explained in [5].

Define
\[
\text{bin}(w) = x^{v^+} - x^{v^-}
\]
for \( w \in \mathbb{Z}^n \). The (saturated) \( S \)-polynomial of \( \text{bin}(u) \) and \( \text{bin}(v) \) is then given by \( \text{bin}(u - v) \). Similarly if \( v^+ \leq u^+ \) we may reduce \( \text{bin}(u) \) by \( \text{bin}(v) \) giving \( \text{bin}(u - v) \). We have silently assumed that the initial term of \( \text{bin}(w) \) is \( x^{v^+} \) for the term ordering in question.

Usually a generating set \( B \) for \( \mathcal{L} \) as an abelian group is given. Computing the lattice ideal \( I_\mathcal{L} \supset I_B \) can be done using that
\[
I_\mathcal{L} = I_B : (x_1 \cdots x_n)^\infty,
\]
where \( I : f^\infty \) denotes the ideal given by
\[
\{ r \in R | rf^m \in I, \text{ for } m > 0 \}
\]
for an ideal \( I \subset R \) and an element \( f \in R \) ([6], Lemma 12.2). If \( B \) contains a positive vector, then \( I_B = I_\mathcal{L} \) ([6], Lemma 12.4). If \( \mathcal{L} \cap \mathbb{N}^n = \{0\}, I_\mathcal{L} \) may be computed from \( I_B \) using Gröbner basis computations for different reverse lexicographic term orderings ([6], Lemma 12.1).
6.1 The generic Gröbner walk for lattice ideals

We now specialize the generic Gröbner walk to the setting of lattice ideals representing binomials by integer vectors as bin\((v)\). In our implementation of the generic walk we walk from \(\prec_{c_1}\) to \(\prec_{c_2}\), where \(\prec\) is the reverse lexicographic order given by \(x_1 \prec \cdots \prec x_n\) and \(c_1, c_2\) are integer vectors.

In the algorithm outlined below we walk between two arbitrary multiplicative orderings \(\prec_1, \prec_2\) and move inside Gröbner cones given by minimal Gröbner bases (cf. [3], Proposition 2.3). Autoreduction is replaced by a simplified lifting step.

In the notation of ([3], §2.3) we have

\[
\delta_{\prec}(\text{bin}(w)) = w
\]

assuming that \(x^w^- \prec x^w^+\). The facet preorder \(\prec\) is now given on binomials bin\((u)\) and bin\((v)\) as in ([3], §4, (3)) by

\[
\text{bin}(u) \prec \text{bin}(v) \iff Tuv_1 \prec_1 Tvu_1,
\]

where \(T\) is a matrix defining the target term order \(\prec_2\). We get as in ([3], §4) that bin\((u) \prec \text{bin}(v)\) and bin\((v) \prec \text{bin}(u)\) imply that \(u\) is a multiple of \(v\). If \(G := \{\text{bin}(v_1), \ldots, \text{bin}(v_r)\}\) is a Gröbner basis then \(v_1, \ldots, v_r\) lie in a common half space and the facet preorder induces a total ordering on \(G\). In particular one gets that the facet in the generic Gröbner walk is given by a unique facet binomial. This means that Gröbner basis computations on facets proceed as in ([4], Algorithm 3.1). To give some more details we introduce the notation

\[
\text{mon}(w) = x^w^+
\]

for \(w \in \mathbb{Z}^n\). Suppose that \(G\) above is a minimal Gröbner basis and that bin\((v_1)\) is minimal in the facet preorder. Then step (c) of ([4], Algorithm 3.1) is to compute a Gröbner basis of the ideal

\[
\langle \text{bin}(-v_1), \text{mon}(v_2), \ldots, \text{mon}(v_r) \rangle.
\]  

(1)

Working with minimal Gröbner bases it may happen that \(\text{mon}(-v_1)\) is divisible by a monomial \(\text{mon}(v_j), j = 2, \ldots, r\). In this case the reduced Gröbner basis of (1) is

\[
\{\text{mon}(v_1), \ldots, \text{mon}(v_r)\}
\]

and \(\text{mon}(v_1)\) lifts to \(\text{bin}(w)\), where \(\text{bin}(-w)\) is the reduction of \(\text{bin}(-v_1)\) modulo

\[
\{\text{bin}(v_2), \ldots, \text{bin}(v_r)\}.
\]

In this way the usual autoreduction of the Gröbner walk is built into the lifting. Notice that \(\text{mon}(v_j)\) lifts to \(\text{bin}(v_j)\) for \(j = 2, \ldots, r\). These observations account for step (ii) in \texttt{facet\_buchberger} below. On the other hand, if \(\text{mon}(-v_1)\) is not divisible by any of the monomials, then bin\((v_1)\) is a (real) facet binomial of a facet in the Gröbner cone corresponding to the reduced Gröbner basis. In this case we end up with a minimal Gröbner basis

\[
\{\text{bin}(-v_1), \text{mon}(w_1), \ldots, \text{mon}(w_s)\}
\]

of the ideal in (1). This lifts to the minimal Gröbner basis

\[
\{\text{bin}(-v_1), \text{bin}(w_1), \ldots, \text{bin}(w_s)\}.
\]
The details of the algorithm are given below. The variable \textit{facet\_list} contains a list of binomials ordered in ascending order according to the facet preorder (these are potential facet binomials). The variable \textit{G} contains the current minimal Gröbner basis. The procedure \textbf{initialize\_facet\_list} initializes \textit{G} and \textit{facet\_list} given \textit{B}. The procedure \textbf{insert} inserts a given binomial into \textit{G} and updates \textit{facet\_list}.

Notice that we do not really compute the \( S\)-polynomials in (iv.b) below. We optimize the algorithm by replacing the \( S\)-polynomial \( S(\text{bin}(w), \text{mon}(v)) \) with the initial term \( x^{(w-v)^+} \) of the saturated \( S\)-polynomial \( \text{sat}(S(\text{bin}(w), \text{bin}(v))) \).

**Algorithm 6.1 (Generic Gröbner walk for lattice ideals)**

**INPUT:** Integer vectors \( c_1, c_2 \). Integer vectors \( B = \{v_1, \ldots, v_r\} \) such that \( \{\text{bin}(v_1), \ldots, \text{bin}(v_r)\} \) is a minimal Gröbner basis for \( I_{\varphi} \) over \( \prec_1 \).

**OUTPUT:** Integer vectors \( G = \{w_1, \ldots, w_s\} \) such that \( \{\text{bin}(w_1), \ldots, \text{bin}(w_s)\} \) is a minimal Gröbner basis over \( \prec_2 \).

(i) \textbf{initialize\_facet\_list}:

(ii) while (\textit{facet\_list} \neq \emptyset) do

(a) \textit{facet\_bin} := first element in \textit{facet\_list}

(b) \textbf{facet\_buchberger};

\textbf{facet\_buchberger}:

(i) Delete \textit{facet\_bin} from \textit{G} and put \textit{bin} := \(-\textit{facet\_bin}\);

(ii) if \( w^+ \leq \textit{bin}^+ \) for some \( w \in \textit{G} \)

reduce \textit{bin} by \textit{G};

\textbf{insert}(\(-\textit{bin}\));

return;

(iii) \textit{Spairs} := \emptyset;

(iv) for \( v \) in \textit{G} do

(a) if \( (\textit{bin}^+ \land v^+) = 0 \)

continue;

(b) \textit{Spairs} := \textit{Spairs} \cup \{v - \textit{bin}\}

(v) Delete \( v \in \textit{G} \) if \( \textit{bin}^+ \leq v^+ \);

(vi) while (\textit{Spairs} \neq \emptyset) do

(a) Select \( s \) in \textit{Spairs} and put \textit{Spairs} := \textit{Spairs} \setminus \{s\}. 
(b) Reduce $s$ by $\text{bin}$;
(c) if $(v^+ \leq s^+)$ for some $v \in G$
    continue;
(d) Delete $v \in G$ if $s^+ \leq v^+$.
(e) $\text{insert}(s)$
(f) $\text{Spairs} := \text{Spairs} \cup \{s - \text{bin}\}$

(vii) $\text{insert}(\text{bin})$;

Truncation blends in easily with Algorithm 6.1. Suppose that $\Omega$ denotes the truncating subset. First binomials in $B$ with degrees outside $\Omega$ are discarded. With every addition of an $S$-binomial in step iv(b) of $\text{facet}_{\text{buchberger}}$, a test for degree membership of the truncating subset $\Omega$ is done. If the test fails for the $S$-binomial it is not added to $\text{Spairs}$.

**Example 6.2** We give a very simple example illustrating Algorithm 6.1. Consider the ideal

$$I = \langle x - t^2, y - t^3 \rangle \subset k[t,x,y].$$

Clearly $G := \{ x - t^2, y - t^3 \}$ is a Gröbner basis for $I$ over the weight vector $\sigma = (-1, 0, 0)$, where $t$ corresponds to $(1,0,0)$ etc. We wish to walk to the weight vector $\tau = (1,0,0)$ breaking ties with the reverse lexicographic order given by $t < x < y$. Let $<$ denote the corresponding facet preorder. Then $x - t^2 < y - t^3$ and we begin by “computing” a Gröbner basis for $\langle t^2 - x, y \rangle$ giving $G = \{ t^2 - x, y - t^3 \}$ after lifting. In the following step the facet binomial is $y - t^3$, which gets replaced by the reduction $y - tx$ in step (ii) of $\text{facet}_{\text{buchberger}}$. This accounts for the next facet binomial. We then compute a Gröbner basis of $\langle tx - y, t^2 \rangle$ giving $\{ tx - y, t^2, ty, y^2 \}$. This lifts to $\{ tx - y, t^2 - x, ty - x^2, y^2 - x^3 \}$, which is the reduced Gröbner basis for $I$ over $(1,0,0)$, since $ty - x^2$ and $y^2 - x^3$ are not candidates for facet binomials as the vectors $(1, -2, 1)$ and $(0, -3, 2)$ both are outside $C_{<\sigma,<\tau}$ (cf. §4 of [2]).

7 Computational experience

In [1] a collection of integer knapsacks are constructed related to the classical Frobenius problem of finding the largest number, which is not a sum of given relatively prime natural numbers. Feasibility for these knapsacks turn out to be very hard for traditional branch and bound software like CPLEX, but easy for lattice reduction methods as shown in [1].

In [2] these knapsacks are equipped with a feasible right hand side and a specific cost vector $c$. These examples form the point of departure in this section, where we specifically document performance for computing (truncated) test sets using the package GLATWALK\textsuperscript{1}. It turns out that test sets in the feasibility case is by far the hardest computations. Test sets with respect to the cost vector in [2] finish in negligible timings ($< 0.05$ seconds) using both the generic walk and the Buchberger algorithm with truncation.

Each of the examples are of the form: maximize $cx$, where

$$Ax = b,$$

\[ (*) \]

\[ ^1 \text{home.imf.au.dk/niels/GLATWALK} \]
$x \in \mathbb{N}^n$ and $A$ is a $1 \times n$-matrix $(a_1 \cdots a_n)$. The cost vector $c$ and the matrices $A$ may be found in [2]. The first step is finding a feasible solution to ($*$). As in [3] this results in the knapsack: minimize $t$ subject to $Ax + t = b$, where $t \in \mathbb{N}$ and $x \in \mathbb{N}^n$. This leads to the problem of finding a (truncated) Gröbner basis of

$$\langle x_1 - t^{a_1}, \ldots, x_n - t^{a_n} \rangle$$

with respect to the vector $\tau = (1,0,\ldots,0)$, where $t$ is the “first” variable. We may compute this Gröbner basis directly using the Buchberger algorithm or walk from the vector $\sigma = (-1,0,\ldots,0)$. The performance of the functions walk and gbasis of GLATWALK for computing a full Gröbner basis of ($**$) over $\tau$ is reported in [3]. The second step is the computation of the toric ideal $I_A$ (associated with the integer matrix $A$) and its Gröbner basis over the vector $-c$. The function saturate of GLATWALK performs the saturation necessary in computing lattice ideals. Below\(^2\) we have computed the ideals $I_A$ using saturate after LLL-reducing Ker$(A)$ with the function LLL. In many of the examples, LLL-reduction offers great savings in the computation of the saturation. The third column shows the timing of gbasis in computing a full Gröbner basis over $-c$ for $I_A$. The fourth column is the timing of walk in walking from $-e_1$ to $-c$. The fifth and sixth columns show sizes of the full and truncated reduced Gröbner bases of $I_A$ over $-c$.

| EXAMPLE | saturate | gbasis | walk | size | tr size |
|---------|----------|--------|------|------|---------|
| cuww1   | 0.1      | 3.1    | 1.8  | 2618 | 7       |
| cuww2   | 0.0      | 0.3    | 1.3  | 898  | 16      |
| cuww3   | 0.1      | 0.4    | 10.1 | 963  | 16      |
| cuww4   | 0.0      | 3.3    | 59.6 | 3143 | 5       |
| cuww5   | 0.0      | 0.0    | 102.8| 267  | 32      |
| prob1   | 0.0      | 0.0    | 2.7  | 180  | 75      |
| prob2   | 0.0      | 0.0    | 0.5  | 280  | 45      |
| prob3   | 0.5      | 0.0    | 2.5  | 163  | 94      |
| prob4   | 0.2      | 0.1    | 122.3| 475  | 83      |
| prob5   | 0.6      | 0.0    | 0.0  | 68   | 56      |
| prob6   | 0.4      | 9.2    | 39.5 | 4541 | 94      |
| prob7   | 0.4      | 1.8    | 72.8 | 2036 | 79      |
| prob8   | 0.9      | 0.0    | 2.4  | 227  | 103     |
| prob9   | 0.0      | 0.0    | 0.5  | 108  | 108     |
| prob10  | 1.4      | 0.1    | 517.5| 536  | 119     |

Both walk and gbasis finish in negligible timings ($< 0.05$ seconds) in computing the truncated Gröbner bases. However in computing the full Gröbner bases in the above table, walk does not

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\(^2\)All timings are in seconds. The computations were carried out on an ACER notebook 1.6 GHz Pentium mobile with 1MB L2 cache.
compare well with \texttt{gbasis}. Typically to compute a target Gröbner basis with less than 1,000 binomials, the generic walk traverses cones associated with reduced Gröbner bases of more than 20,000 binomials along a straight line intersecting many cones in the Gröbner fan.

Most of the examples above indicate that the truncated Gröbner fan is much smaller than the full Gröbner fan. The straight line path in the truncated Gröbner fan traverses significantly fewer cones. It is open for further research exactly when the walk is a substantial improvement (as in many of the feasibility examples reported in [3]). Perhaps a combination of direct Gröbner basis computations for suitably chosen (easier) weight vectors tending to the target vector followed by a walk to the target order may lead to improvements.

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