Locally equivalent quasifree states and index theory

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Abstract
We consider quasifree ground states of Araki’s self-dual canonical anti-commutation relation algebra from the viewpoint of index theory and symmetry protected topological (SPT) phases. We first review how Clifford module indices characterise a topological obstruction to connect pairs of symmetric gapped ground states. This construction is then generalised to give invariants in $KO^*(A_r)$ with $A$ a $C^*$-algebra of allowed deformations. When $A = C^*(X)$, the Roe algebra of a coarse space $X$, and we restrict to gapped ground states that are locally equivalent with respect $X$, a $K$-homology class is also constructed. The coarse assembly map relates these two classes and clarifies the relevance of $K$-homology to free-fermionic SPT phases.

Keywords: index theory, SPT phases, coarse geometry, operator algebras

1. Introduction
Since the influential paper of Kitaev [33], $K$-theory of spaces and $C^*$-algebras has played an important role in studying the phase labels of free-fermionic topological states of matter, see [1, 21, 31, 35, 42, 48] for example. The dual theory, $K$-homology, also features prominently in Kitaev’s paper as a way to characterise gapped local systems. While index pairings with Dirac operators and $K$-homology classes constructed on the (noncommutative) Brillouin torus have been effectively utilised to give numerical topological phase labels [8, 24], the role of $K$-homology as a means to directly characterise local gapped systems appears to be understudied in the mathematical physics literature. The aim of this paper is to provide some first steps in this direction.

For our purposes, it is most convenient to study free-fermionic topological phases via the dynamics induced by gapped Bogoliubov–de Gennes (BdG) Hamiltonians on a Nambu space, a complex Hilbert space with real structure, see [32] for example. Such dynamics give a

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quasifree, gapped and pure ground state of the self-dual algebra of canonical anti-commutation relations (CARs) studied by Araki [3]. Similar to work by Alldridge–Max–Zirnbauer [1], we construct elements in the $K$-theory of a Real $C^*$-algebra $A$ of allowed deformations that characterise pairs of gapped BdG Hamiltonians/quasifree ground states.

To relate these constructions to $K$-homology, we consider the case $A = C^*(X)$, the Roe algebra of a coarse space $X$ constructed from a representation of $C_0(X)$ on the Nambu space and with real structure $\tau$. Taking inspiration from similar constructions in algebraic quantum field theory [14, 38], a notion of local equivalence of gapped quasifree ground states is introduced for BdG Hamiltonians that are compatible with the representation of $C_0(X)$. We show that such locally equivalent ground states give rise to a Fredholm module and $K$-homology class for $C_0(X)$. The coarse assembly map $\rho_X : KO^-(C_0(X))^\tau \to KO_*(C^*(X)^\tau)$ sends this $K$-homology class to the previously constructed $KO_*(C^*(X)^\tau)$-valued indices. The equivariant assembly map can similarly be treated for quasifree gapped ground states with a compact or discrete group symmetry.

The coarse assembly map is an isomorphism for a large class of spaces. Therefore our result helps establish the relevance of $K$-homology as a mathematical characterisation of free-fermionic topological phases as well as its relation with the more well-studied approach via $K$-theory.

### 1.1. Mathematical results

Given a complex Hilbert space $\mathcal{H}$ with real structure $\Gamma$, pure quasifree states of the self-dual CAR algebra $A^\text{qu}_{\mathcal{H}}(\mathcal{H}, \Gamma)$ can be characterised by skew-adjoint unitaries on $\mathcal{H}$ that commute with $\Gamma$. Given a pair $(J_0, J_1)$ of such unitaries whose corresponding quasifree states are equivalent, the space $\text{Ker}(J_0 + J_1)$ is finite-dimensional and has the structure of an ungraded Clifford module. Using the Atiyah–Bott–Shaprio isomorphism [5], the corresponding Clifford module index gives a $K$-theoretic obstruction for the quasifree states to be equivalent when restricted to the even subalgebra of $A^\text{qu}_{\mathcal{H}}(\mathcal{H}, \Gamma)$. In section 3 we review these ideas and their extensions to quasifree pure states that are symmetric with respect to a compact group [11, 41].

Our first task is to extend such Clifford module indices to an index with range $K$-theory of a $C^*$-algebra $A$ with real structure $\tau$. The main technical tool we use to define these indices is the relative Cayley transform considered in [9] for pairs of unitaries acting on a Hilbert $A$-module and whose difference is a compact endomorphism. This construction is reviewed and slightly extended in section 2. We then apply this construction in section 4 to pairs of gapped BdG Hamiltonians $(H_0, H_1)$ acting on $(\mathcal{H}, \Gamma)$ such that $\text{sgn}(H_0) - \text{sgn}(H_1) \in A$.

In the case that $A = C^*(X)$, we show in section 5 that the condition $\text{sgn}(H_0) - \text{sgn}(H_1) \in C^*(X)$ is satisfied when $H_0$ and $H_1$ are quasilocal and the pure quasifree states of $A^\text{qu}_{\mathcal{H}}(\mathcal{H}, \Gamma)$ constructed from $H_0$ and $H_1$ are locally equivalent with respect to the Real representation $C_0(X) \to B(\mathcal{H})$. Because of the close connection between coarse $C^*$-algebras and duality theory, pairs of locally equivalent quasifree states can be used to construct both a $KO_*(C^*(X)^\tau)$-index and a $K$-homology class for $C_0(X)$. Using a description of the assembly map via duality theory and boundary maps in $K$-theory as developed by Roe [45, 46], our main result is that the coarse assembly map relates our constructed $K$-homology and $K$-theory elements. Compact and discrete group symmetries can also be incorporated with minor adjustments.

Because we work in the category of complex $C^*$-algebras with a real structure, the assembly map has a natural description using van Daele $K$-theory [15, 16], which we review in section 2.5. In particular, building from [9, section 5.2], we write down an explicit representative of the boundary map in van Daele $K$-theory composed with the equivalence to $KKR$-theory. Once all the relevant objects are in place, our main result follows relatively easily from this...
general boundary map computation. The boundary map computation can also be applied to systems with a defect that is mathematically encoded by a semi-split short exact sequence (e.g. a codimension one boundary). We lay the mathematical framework to study such systems in section 4.3, though leave a full treatment to another place.

Coarse geometry methods have already been effectively utilised to study free-fermionic topological phases [20, 35, 39, 40]. It would also be interesting to consider analogous methods for more general quasifree dynamics and states such as those defined for Hilbert $C^*$-bimodules and their corresponding Toeplitz and Cuntz–Pinnser algebras [36].

1.2. Applications to topological phases

Gapped BdG Hamiltonians on Nambu space define quasifree ground states of the CAR algebra and provide an effective description of free-fermionic systems. Adopting a framework analogous to the study of symmetry protected topological (SPT) phases of unique gapped ground states, we consider a compact group $G$ and $G$-symmetric ground states which are equivalent but need not be $G$-equivariantly equivalent. When $G$ corresponds to physical (Altland–Zirnbauer) symmetries, the topological obstruction to connect these ground states is given by a Clifford module index. More generally, we can use results from Matsui and Carey–Evans [11, 41] to give a $KO^G_2(\mathbb{R})$-valued obstruction. We extend this work to construct $KO^G_2(\Lambda^2)$-valued indices, which provide a topological obstruction to connect pairs BdG Hamiltonians and ground states with respect to an auxiliary $C^\ast$-algebra $A$ of allowed deformations.

We then consider a coarse space $X$ and pairs of locally equivalent gapped ground states with respect to a representation of $C_0(X)$ on the Nambu space. The coarse index $\mu_X^G : KO^G_2(C_0(X)^T) \to KO^G_2(C^\ast(X)^T)$ then gives a topological obstruction to connect locally equivalent $G$-symmetric gapped ground states via a path of gapped ground states that respects the $G$-symmetry and is local with respect to the representation of $C_0(X)$. For the case of a discrete group $\Gamma$ acting isometrically and cocompactly on $X$, the range of the assembly map is $KO_A(C_0(\Gamma))$, which directly connects to more standard approaches to free-fermionic phases of matter via $K$-theory.

Our result provides a new and potentially useful approach for studying local gapped free-fermionic phases. Coarse geometry methods have been used to consider interacting gapped ground states by Kapustin, Sopenko and Spodyneiko [27, 28], so our framework may also be useful beyond the free-fermionic setting.

1.3. Outline

We collect some basic facts on Fredholm operators and Kasparov theory in section 2. Because gapped quasifree ground states with physical (Altland–Zirnbauer) symmetries can be described via Real mutually anti-commuting skew-adjoint unitaries [32], we also extend some results on the Cayley transform of unitaries [9] to this setting. The Cayley transform provides a way to pass between $KK$-theory and van Daele $K$-theory, which we also introduce as well as its application to boundary maps in Kasparov theory.

Section 3 reviews pure quasifree states of the self-dual CAR algebra $A^\text{car}_{sd}(\mathcal{H}, \Gamma)$ and the construction of Clifford module indices studied in [11, 41] that characterise pairs of symmetric quasifree states. This is extended in section 4 to $KO_A(\Lambda^2)$-valued indices and we compute the image of such indices under the boundary map from a semi-split short exact sequence.

Finally in section 5 we consider coarse spaces, pseudolocal gapped BdG Hamiltonians, locally equivalent quasifree states and their topological description via $K$-homology and
We will primarily work in the category of Real $C^\ast$-algebras (with capital R) or $C^\ast$-algebras, which are complex $C^\ast$-algebras with a real structure, an anti-linear order-2 automorphism $a \mapsto a^{\ast 4}$ such that $(a^4)^{\ast 4} = (a^4)^\ast$ for all $a \in A$. We say that $a \in A$ is Real if $a^{\ast 4} = a$. If $A$ has a $\mathbb{Z}_2$-grading $A = A^0 \oplus A^1$ we also assume that $(A^i)^{\ast 4} \subset A^i$, $i \in \{0, 1\}$. We recover the complex theory by ignoring the real structure $\mathfrak{r}_A$. Similarly, restricting to the subalgebra $A^{\ast 4} = \{a \in A : a^{\ast 4} = a\}$ gives a real $C^\ast$-algebra (with lower-case $r$), a $C^\ast$-algebra over the number field $\mathbb{R}$. When the context is unambiguous, we will write $\mathfrak{r}_A$ as $\mathfrak{r}$.

**Example 2.1 (Real Clifford algebras).** Given $r, s \in \mathbb{N}$, the Real $\mathbb{Z}_2$-graded Clifford algebra $\mathbb{C}l_{r,s}$ is the complex $C^\ast$-algebra generated by the elements $\{\gamma_1, \ldots, \gamma_r, \rho_1, \ldots, \rho_s\}$, which are odd, mutually anti-commute and

$$\gamma_j = \gamma_j^\ast = \gamma_j^2, \quad \gamma_j \gamma_k + \gamma_k \gamma_j = 2\delta_{j,k}, \quad \rho_j = \rho_j^\ast = -\rho_j^2, \quad \rho_j \rho_k + \rho_k \rho_j = -2\delta_{j,k}.$$ 

As complex algebras $\mathbb{C}l_{r,s} \cong \mathbb{C}l_{r+s}$. The real Clifford algebra $\mathbb{C}l_{r,s}$ is algebraic span of $\{\gamma_1, \ldots, \gamma_r, \rho_1, \ldots, \rho_s\}$ over $\mathbb{R}$, where $\mathbb{C}l_{r,s}^\ast = \mathbb{C}l_{r,s}$.

We will often make use of the isomorphism $\text{End}(\bigwedge^\ast \mathbb{C}) \cong \mathbb{C}l_{1,1}$ with Real generators $\gamma$ and $\rho$. More generally, $\text{End}(\bigwedge^\ast \mathbb{C}^n) \cong \mathbb{C}l_{n,n}$.

We will occasionally consider ungraded Clifford algebras, though we reserve the notation $\gamma$ and $\rho$ for odd generating elements. In particular, any Clifford algebra appearing in a Kasparov module will always be interpreted as $\mathbb{Z}_2$-graded.

We now briefly review Real Kasparov theory or KKR-theory [29]. Unless otherwise stated, $B$ is a $\sigma$-unital $C^\ast$-algebra and $E_B$ is a countably generated right Hilbert $B$-module, see [37] for the basic theory. We will call such $B$-modules Hilbert $C^\ast$-modules. We denote by $\text{End}_B(E)$ and $\mathbb{K}_B(E)$ the adjointable and compact operators respectively. In the special case where $E = B$ as a vector space with right-action by right-multiplication and $(b_1|b_2)_B = b_1^\ast b_2$, $b_1, b_2 \in B$, we have that $\mathbb{K}_B(B) = B$ and $\text{End}_B(B) = \text{Mult}(B)$, the multiplier algebra of $B$.

A complex Hilbert $C^\ast$-module $E_B$ is a Real Hilbert $C^\ast$-module if there is an antilinear map $\mathfrak{r}_E : E_B \rightarrow E_B$, called the real involution, such that for all $e, e_1, e_2 \in E_B$ and $b \in B$,

$$(e^{\ast \mathfrak{r}})^{\ast \mathfrak{r}} = e, \quad e^{\ast \mathfrak{r}} \cdot b^{\mathfrak{r}} = (e \cdot b)^{\ast \mathfrak{r}}, \quad (e_1^{\ast \mathfrak{r}}|e_2^{\ast \mathfrak{r}})_B = ((e_1|e_2)_B)^{\ast \mathfrak{r}}.$$ 

The real involution on $E_B$ induces a real structure $\tau$ on $\text{End}_B(E)$ via $S^\tau e = (S(e^{\ast \mathfrak{r}}))^{\ast \mathfrak{r}}$ for any $e \in E_B$. Given a separable Real $C^\ast$-algebra $A$, any representation $\pi : A \rightarrow \text{End}_B(E)$ should be compatible with this real structure, $\pi(a^{\ast \mathfrak{r}}) = \pi(a)^{\ast \mathfrak{r}}$ for all $a \in A$.

We will often work with unbounded operators on Hilbert $C^\ast$-modules, see [37, chapter 9] for more details. We recall that a densely defined closed right $B$-linear operator $D : \text{Dom}(D) \subset E_B \rightarrow E_B$ is regular if $D^\ast$ is densely defined and the operator $1 + D^*D : \text{Dom}(D^*D) \rightarrow E_B$ has dense range. Note also that $\text{Dom}(D)$ must be invariant under the right $B$-action in order
to obtain a right $B$-linear operator $D : \text{Dom}(D) \to E_B$. We call $D$ Real and write $D^\sharp = D$ if $(\text{Dom}(D))^\sharp \subset \text{Dom}(D)$ and $(D^e)^\sharp = D e$ for all $e \in \text{Dom}(D)$. We also recall the graded commutator, where for endomorphisms $S, T$ with homogenous parity $[S, T]_\pm = ST - (-1)^{\deg(S)\deg(T)}TS$.

**Definition 2.2.** Let $A$ and $B$ be $\mathbb{Z}_2$-graded Real $C^*$-algebras. A Real Kasparov module $(A, E_B, F)$ consists of

(a) A Real and $\mathbb{Z}_2$-graded Hilbert $C^*$-module $E_B$,

(b) A Real and $\mathbb{Z}_2$-graded $^*$-homomorphism $\pi : A \to \text{End}_B(E)$,

(c) A self-adjoint and odd operator $F = F^\sharp \in \text{End}_B(E)$ such that $[F, \pi(a)]_\pm, \pi(a)(1 - F^2) \in \mathbb{K}_B(E)$ for all $a \in A$.

If $0 = [F, \pi(a)]_\pm = \pi(a)(1 - F^2)$ for all $a \in A$, we say that $(A, E_B, F)$ is degenerate. An unbounded Real Kasparov module is a triple $(A, E_B, F)$ with $A \subset A$ a dense $*$-subalgebra such that conditions (a) and (b) of a Real Kasparov module are satisfied and (c) is replaced by the condition:

(d) There is an unbounded self-adjoint, regular and odd operator $D = D^\sharp$ such that for all $a \in A$, $\pi(a)\text{Dom}(D) \subset \text{Dom}(D)$ and

$$[D, \pi(a)]_\pm \in \text{End}_B(E), \quad \pi(a)(1 + D^2)^{-1/2} \in \mathbb{K}_B(E).$$

When $B = \mathbb{C}$, bounded and unbounded Kasparov modules are also called Fredholm modules and spectral triples respectively.

We will often omit the representation $\pi : A \to \text{End}_B(X)$ if the context is clear. If $(A, E_B, D)$ is an unbounded Real Kasparov module, then the results of [6] can be easily adapted to the Real setting to show that $(A, E_B, D)$ is a Real Kasparov module for $D_0 = D (1 + D^2)^{-1/2}$. Equivalence classes of Real Kasparov modules give an abelian group $\text{KKR}(A, B)$ [29], though this group depends on the choice of real structures for $A$ and $B$. Degenerate Kasparov modules represent the group identity of $\text{KKR}(A, B)$.

If $(A, E_B, F)$ is a Real Kasparov module, then we can ignore the real structures and obtain a complex Kasparov module and class in $\text{KK}(A, B)$. If we restrict the Real Hilbert $C^*$-module $E_B$ to the elements fixed under $\tau_F$, we obtain a real Hilbert $C^*$-module $E_B^{\tau_F}$. Similarly, the Real left action of $A$ becomes a real left action $\pi : A^{\tau_F} \to \text{End}_{B^{\tau_F}}(E^{\tau_F})$. We do not lose any information by restricting Real Kasparov modules to real Hilbert $C^*$-modules and algebras. Similarly, real Kasparov modules can be complexified to obtain Real Kasparov modules and $\text{KKR}(A, B) \cong \text{KKO}(A^{\tau_F}, B^{\tau_F})$.

The Clifford algebras $\mathbb{C}\ell_{s,r}$ also play an important role in the $\text{KKR}$-groups, where we have that $\text{KKR}(A \otimes \mathbb{C}\ell_{s,r}, B) \cong \text{KKR}(A, B \otimes \mathbb{C}\ell_{s,r})$. This isomorphism is obtained by the following composition

$$\text{KKR}(A \otimes \mathbb{C}\ell_{s,r}, B) \xrightarrow{\text{id}_{\mathbb{C}\ell_{s,r}}} \text{KKR}(A \otimes \mathbb{C}\ell_{s,r} \otimes \mathbb{C}\ell_{s,r}, B \otimes \mathbb{C}\ell_{s,r}) \xrightarrow{\gamma} \text{KKR}(A, B \otimes \mathbb{C}\ell_{s,r}),$$

where the first map is the external product with the Kasparov module $\left(\mathbb{C}\ell_{s,r}, \mathbb{C}\ell_{s,r} \otimes \mathbb{C}\ell_{s,r}, 0\right)$ and is an isomorphism of $\text{KKR}$-groups. The second map comes from the identification $\mathbb{C}\ell_{s,r} \otimes \mathbb{C}\ell_{s,r} \cong \text{End}(\bigwedge^n\mathbb{C}^{s+r}) \cong M_{2^{s+r}}(\mathbb{C})$ and the stability of $\text{KKR}$.

If the algebra $B$ is trivially graded, $B^1 = \{0\}$, we can also consider Real $K$-theory, where $\text{KKR}(\mathbb{C}\ell_{s,r}, B) \cong \text{KKO}(\mathbb{C}\ell_{s,r}, B) \cong KO_{s+r}(B^1)$. Similarly, the Real $K$-homology groups of a trivially graded algebra $A$ can be expressed as $\text{KKR}(A \otimes \mathbb{C}\ell_{s,r}, \mathbb{C}) \cong KO^{s+r}(A^1)$.
Finally we consider the case of group actions and equivariant Kasparov modules. Fix a compact or discrete group \( G \) and an action \( \beta : G \to \text{Aut}(B) \). We say that \( \beta \) is Real and \( \mathbb{Z}_2 \)-graded if \( \beta_g(b^j) = \beta_j(b)^g \) and \( \beta_g(B^j) \subset B^j \) for all \( g \in G, b \in B \) and \( j \in \{0, 1\} \). A Real Hilbert \( C^* \)-module \( E_B \) is \( G \)-equivariant if there is a homomorphism \( \eta \) from \( G \) into the invertible and bounded (not necessarily adjointable) linear transformations on \( E \) that preserves the \( \mathbb{Z}_2 \)-grading and is such that

\[
\eta_g(e^j) = \eta_g(e)^j, \quad \eta_g(e \cdot b) = \eta_g(e) \cdot \beta_g(b), \quad (\eta_g(e_1) | \eta_g(e_2))_B = \beta_g ((e_1 | e_2)_B)
\]

for all \( e, e_1, e_2 \in E, b \in B \) and \( g \in G \). Such an action then induces a Real and \( \mathbb{Z}_2 \)-graded action \( \tilde{\eta} : G \to \text{Aut}(\text{End}_B(E)) \) where \( \tilde{\eta}(T)e = \eta_e \circ T \circ \eta^{-1}_e(e) \) for any \( T \in \text{End}_B(E), e \in E_B \) and \( g \in G \). If \( A \) is a Real \( C^* \)-algebra with a Real and \( \mathbb{Z}_2 \)-graded group action \( \alpha \), we require that any representation \( \pi : A \to \text{End}_B(E) \) be equivariant with respect to \( \alpha \) and \( \tilde{\eta} \). We say that \( T : \text{Dom}(T) \subset E_B \to E_B \) is \( G \)-invariant if \( \eta_g(\text{Dom}(T)) \subset \text{Dom}(T) \) and \( \tilde{\eta}_g(T) = T \) for all \( g \in G \).

With these preliminaries in place, a \( G \)-equivariant (unbounded) Real Kasparov module is a Real (unbounded) Kasparov module with an equivariant Hilbert \( C^* \)-module and left-action such that the self-adjoint operators \( F \) or \( D \) are \( G \)-invariant.

2.2. Fredholm operators on Hilbert \( C^* \)-modules

We briefly provide some further information on Fredholm theory in the Hilbert \( C^* \)-module setting. A more comprehensive treatment can be found in [18, 26]. We fix a \( \sigma \)-unital \( C^* \)-algebra \( B \) and a Real countably generated Hilbert \( C^* \)-module \( E_B \).

**Definition 2.3.** Let \( S \) be a regular operator on \( E_B \). We say that \( S \) is Fredholm if there is a parametrix \( Q \in \text{End}_B(E) \) such that \( SQ \) and \( QS \) are closable with adjointable closures and \( SQ^* - 1, SQ - 1 \in \mathbb{K}_B(E) \).

If \( T \in \text{End}_B(E) \) (so \( T \) is bounded), then \( T \) is Fredholm if and only if \( q(T) \in \mathbb{K}_B(E) \) is invertible with \( \mathbb{Q}_B(E) = \text{End}_B(E)/\mathbb{K}_B(E) \) the Calkin algebra of the Hilbert \( C^* \)-module \( E_B \) and \( q : \text{End}_B(T) \to \mathbb{Q}_B(E) \) the quotient map.

**Proposition 2.4** ([26, lemma 2.2], [18, proposition 2.14]). If \( S = S^* \) is a skew-adjoint Fredholm operator on a trivially graded Real Hilbert \( C^* \)-module \( E_B \), then the triple

\[
\left( \mathbb{C} \ell_{1,0}, E_B \otimes \bigwedge^r \mathbb{C}, S(1 - S^2)^{-1/2} \otimes \rho \right)
\]

is a Real Kasparov module, where the left \( \mathbb{C} \ell_{1,0} \)-action is generated by \( 1 \otimes \gamma \).

In the case that \( S \) has a compact resolvent, the class \( [S] \in \mathbb{KKR}(\mathbb{C} \ell_{1,0}, B) \) from proposition 2.4 can be directly represented by the unbounded Real Kasparov module,

\[
[S] = \left[ \left( \mathbb{C} \ell_{1,0}, E_B \otimes \bigwedge^r \mathbb{C}, S \otimes \rho \right) \right] \in \mathbb{KKR}(\mathbb{C} \ell_{1,0}, B) \cong KO_1(B^r).
\]
2.3. The Cayley transform of odd self-adjoint unitaries

Let \( A \) be a \( \mathbb{Z}_2 \)-graded and \( \sigma \)-unital \( C^* \)-algebra and \( E_A \) a countably generated and \( \mathbb{Z}_2 \)-graded Real Hilbert \( C^* \)-module over \( A \). We suppose that \( \text{End}_A(E) \) contains as many odd self-adjoint unitaries as we need. We can always ensure this by taking a graded tensor product \( E'_A = E_A \otimes \bigwedge \mathbb{C}^n \), where \( \text{End}_A(E') \simeq \text{End}_A(E) \otimes \mathbb{C} \ell_{2\mathbb{R}} \). Let us then fix a representation of \( \mathbb{C} \ell_{2\mathbb{R}} \) on \( E_A \) with generators \( \{ \gamma_j \}_{j=1}^k \). We are interested in the set

\[
\mathcal{O}_{E_A}^k = \{ V \in \text{End}_A(E) : V \text{ odd, } V^* = V^{-1} = V^t, \quad V_{\gamma j} = -\gamma_j V \text{ for } j = 1, \ldots, k \}. \tag{1}
\]

**Lemma 2.5 (cf [9, lemma 4.5]).** Given \( V_0, V_1 \in \mathcal{O}_{E_A}^k \) with \( \| V_0 - V_1 \|_{\mathcal{O}_4(E)} < 2 \), define the unbounded operator

\[
\mathcal{C}_{V_0}(V_1) = V_0(V_1 + V_0)(V_1 - V_0)^{-1}, \quad \text{Dom}(\mathcal{C}_{V_0}(V_1)) = (V_1 - V_0)E_A.
\]

Then \( \mathcal{C}_{V_0}(V_1) \) is odd, self-adjoint, Real, regular and anti-commutes with \( \{ V_0, \gamma_1, \ldots, \gamma_k \} \) on \( (V_0 - V_1)E_A \), the closure of \( \text{Dom}(\mathcal{C}_{V_0}(V_1)) \) in the module norm of \( E_A \). Furthermore, \( F_{\mathcal{C}_{V_0}(V_1)} = \mathcal{C}_{V_0}(V_1)(1 + \mathcal{C}_{V_0}(V_1)^2)^{-1/2} \) satisfies \( \| 1 - F^2_{\mathcal{C}_{V_0}(V_1)} \|_{\mathcal{O}_4(E)} < 1 \).

**Proof.** We first note that because \( V_0 \) and \( V_1 \) are self-adjoint unitaries, \( V_0(V_1 \pm V_0) = (V_0 \pm V_1)V_1 \). In particular, for any \( e \in E_A \),

\[
V_0(V_1 - V_0)e = -(V_1 - V_0)V_1 e \in (V_1 - V_0)E_A
\]

and so \( V_0 \) preserves the domain of \( \mathcal{C}_{V_0}(V_1) \). Because \( V_0 \) and \( V_1 \) anti-commute with \( \{ \gamma_j \}_{j=1}^k \), we see that \( \{ \gamma_j \}_{j=1}^k \) preserve \( \text{Dom}(\mathcal{C}_{V_0}(V_1)) \) and a simple computation gives that these operators anti-commute with \( \mathcal{C}_{V_0}(V_1) \). We similarly have that on \( \text{Dom}(\mathcal{C}_{V_0}(V_1)) \)

\[
V_0 \mathcal{C}_{V_0}(V_1) = (V_1 + V_0)V_1 V_1(V_1 - V_0)^{-1} = V_0(V_1 + V_0)((V_1 - V_0)V_1)^{-1}
\]

\[
= V_0(V_1 + V_0)(V_0(V_0 - V_1))^{-1} = -V_0(V_1 + V_0)(V_1 - V_0)V_0
\]

\[
= -\mathcal{C}_{V_0}(V_1)V_0.
\]

It is immediate that \( \mathcal{C}_{V_0}(V_1) \) is Real and odd.

To prove that \( \mathcal{C}_{V_0}(V_1) \) is self-adjoint and regular, one considers the bounded transform \( F_{\mathcal{C}_{V_0}(V_1)} = \mathcal{C}_{V_0}(V_1)(1 + \mathcal{C}_{V_0}(V_1)^2)^{-1/2} \). To make sense of this operator, we first compute using the normality of \( V_0 V_1 \)

\[
(1 + \mathcal{C}_{V_0}(V_1)^2)^{-1/2} = (1 + V_0(V_1V_0 + 1)(V_1V_0 - 1))^{-1/2}
\]

\[
= (1 - (2 + V_1V_0 + V_0V_1)(-2 + V_1V_0 + V_0V_1))^{-1/2}
\]

\[
= ((2 - V_1V_0 - V_0V_1 + 2 + V_1V_0 + V_0V_1)
\]

\[
\times (2 - V_0V_1 - V_1V_0)^{-1})^{-1/2}
\]

\[
= (4(2 - V_0V_1 - V_1V_0))^{-1/2}
\]

\[
= \frac{1}{2}(2 - V_0V_1 - V_1V_0)^{1/2}.
\]

Therefore we can write \( F_{\mathcal{C}_{V_0}(V_1)} = \frac{1}{2}V_0(V_1V_0 + 1)(V_1V_0 - 1)^{-1}(2 - V_0V_1 - V_1V_0)^{1/2} \). It is shown in [9, lemma 4.5] that \( F_{\mathcal{C}_{V_0}(V_1)} \) is self-adjoint and has norm bounded by 1. Then using
that $V_0$ commutes with $V_0V_1 + V_1V_0$ and the normality of $V_1V_0$,

$$F_{C_{V_0}(V_1)}^2 = \frac{1}{4} \left( V_0(V_1 V_0 + 1)(V_1 V_0 - 1)^{-1}(2 - V_0 V_1 - V_1 V_0)^{1/2} \right)^2$$

$$= -\frac{1}{4} (V_1 V_0 + 1)(V_0 V_1 + 1)(V_1 V_0 - 1)^{-1}(1 - V_1 V_0)^{-1}$$

$$\times (2 - V_0 V_1 - V_1 V_0)$$

$$= \frac{1}{4} (V_1 V_0 + 1)(V_0 V_1 + 1) = \frac{1}{4} (2 + V_0 V_1 + V_1 V_0). \quad (2)$$

We therefore have that

$$1 - F_{C_{V_0}(V_1)}^2 = \frac{1}{2} - \frac{1}{4} (V_0 V_1 + V_1 V_0) = \frac{1}{4} (2 - V_0 V_1 - V_1 V_0) = \frac{1}{4} (V_0 - V_1)^2. \quad (3)$$

In particular, $1 - F_{C_{V_0}(V_1)}$ is positive and $(1 - F_{C_{V_0}(V_1)})^{1/2}$ has dense range $(V_1 - V_0)E_A$. Applying [37, theorem 10.4], $C_{V_0}(V_1)$ is self-adjoint and regular.

Finally using equation (3),

$$\|1 - F_{C_{V_0}(V_1)}^2\|_{\mathcal{A}(E)} = \frac{1}{4} \| (V_0 - V_1)^2 \|_{\mathcal{A}(E)} < 1. \quad \square$$

The operator $C_{V_0}(V_1)$ maps $(V_1 - V_0)E_A$ to $(V_1 + V_0)E_A$ in analogy to the standard Cayley transform for unitary operators on Hilbert spaces. Because the operator $V_0 - V_1$ need not be dense in $E_A$ nor have closed range, the operator $\tilde{C}_{V_0}(V_1)$ is a densely-defined unbounded operator on the submodule $(V_0 - V_1)E_A \subset E_A$. One may consider $C_{V_0}(V_1)$ as densely defined right $A$-linear map $(V_0 - V_1)E_A \to E_A$ such that it is self-adjoint and regular on $(V_0 - V_1)E_A$. Because the operators $\{\gamma_j\}_{j=1}^k$ anti-commute with $V_0$ and $V_1$, they restrict to mutually anti-commuting odd self-adjoint unitaries acting on $(V_0 - V_1)E_A$.

**Proposition 2.6.** Let $V_0, V_1 \in \mathcal{O}_A^k$ with $\|V_0 - V_1\|_{\mathcal{Q}(E)} < 2$. Then the triple

$$\left( C_{\bar{\ell}_k+1,0}, E_A, F_{C_{V_0}(V_1)} = C_{V_0}(V_1)(1 + C_{V_0}(V_1)^2)^{-1/2} \right)$$

is a Real Kasparov module with left Clifford generators $\{V_0, \gamma_1, \ldots, \gamma_k\}$. If $V_0 - V_1 \in \mathbb{K}_A(E)$, then the class in $\text{KKR}(C_{\bar{\ell}_k+1,0}, A)$ of this Kasparov module can be represented by the unbounded Real Kasparov module

$$\left( C_{\bar{\ell}_k+1,0}, (V_0 - V_1)E_A, C_{V_0}(V_1) \right)$$

with Clifford generators $\{V_0, \gamma_1, \ldots, \gamma_k\}$.

**Proof.** By lemma 2.5, the estimate $\|V_0 - V_1\|_{\mathcal{Q}(E)} < 2$ implies that $\|1 - F_{C_{V_0}(V_1)}^2\|_{\mathcal{Q}(E)} < 1$ and so $F_{C_{V_0}(V_1)}$ is invertible in the Calkin algebra and, hence, Fredholm. Because $C_{V_0}(V_1)$ anti-commutes with $\{V_0, \gamma_1, \ldots, \gamma_k\}$, so does $F_{C_{V_0}(V_1)}$. Thus the triple $\left( C_{\bar{\ell}_k+1,0}, E_A, F_{C_{V_0}(V_1)} \right)$ is a Real Kasparov module.
Similar to equations (2) and (3), we compute that
\[ 1 + C_{\mathcal{V}_0}(V_i)^2 = 1 + (2 + V_0 V_i + V_i V_0)(2 - V_0 V_i - V_i V_0)^{-1} = 4(2 - V_0 V_i - V_i V_0)^{-1} = 4(V_i - V_0)^{-2}. \]

Therefore \((1 + C_{\mathcal{V}_0}(V_i)^2)^{-1/2} = 1/2|V_0 - V_i|\), which will be compact if \(V_0 - V_i \in K_A(E)\). This result combined with lemma 2.5 shows that \((\mathcal{C}_{\mathcal{F}^{+1+0}}(V_i)\mathcal{V}_0, C_{\mathcal{V}_0}(V_1))^*\) is an unbounded Kasparov module and is an unbounded lift of \((\mathcal{C}_{\mathcal{F}^{+1+0}}, E_A, F_{\mathcal{V}_0}(V_1))^*\).

\[ \square \]

2.4. The Cayley transform of skew-adjoint ungraded unitaries

Fix a \(\sigma\)-unital, ungraded and Real \(C^*\)-algebra \(B\) and an ungraded and countably generated Real Hilbert \(C^*\)-module \(E_B\). We also suppose that there exist operators \(\{\kappa_j\}_{j=1}^n \subset \text{End}_B(E)\) such that for all \(j, k \in \{1, \ldots, n\}\),
\[ \kappa_j^* = -\kappa_j, \quad \kappa_j^{\ell} = \kappa_j, \quad \kappa_j^* \kappa_j + \kappa_j \kappa_j = -2\delta_{jk}. \]

Such an assumption can always be satisfied by taking an ungraded (Real) representation of \(C_{\mathcal{F}^{+1+0}}\) on \(C^\omega\) and considering \(E_B = E_B \otimes C^\omega\). We then define the set
\[ U_{E_B}^0 = \{ J \in \text{End}_B(E) : J = J^*, J^2 = -1, \kappa_j J = -J \kappa_j \text{ for all } j = 1, \ldots, n \}. \]

**Lemma 2.7.** Let \(J_0, J_1 \in U_{E_B}^0\) be such that \(\|J_0 - J_1\|_{\text{Q}_B(E)} < 2\). Define the operator
\[ C_{\mathcal{B}}(J_1) = J_0(J_1 + J_0)(J_1 - J_0)^{-1}, \quad \text{Dom}(C_{\mathcal{B}}(J_1)) = (J_1 - J_0)E_B. \]

Then \(C_{\mathcal{B}}(J_1)\) is an unbounded, Real, regular and skew-adjoint operator on \((J_1 - J_0)E_B\) that anti-commutes with \(\{J_0, \kappa_1, \ldots, \kappa_n\}\).

**Proof.** Given \(\{J_0, J_1, \kappa_1, \ldots, \kappa_n\}\) acting on \(E_B\) we can consider \(\{J_0 \otimes \rho, J_1 \otimes \rho, \kappa_1 \otimes \rho, \ldots, \kappa_n \otimes \rho\}\) acting on \((E \otimes C_{\mathcal{F}^{0+1}})B \otimes C_{\mathcal{F}^{0+1}}\) and with \(\rho\) the skew-adjoint generator. All operators are now odd self-adjoint unitaries and so we can apply lemma 2.5. Expressing these results in terms of operators on \(E_B\), we get the desired results, e.g. \(C_{\mathcal{B}}(J_1) \otimes \rho = C_{\mathcal{B}}(J_1 \otimes \rho)\), so the self-adjointness and regularity of \(C_{\mathcal{B}}(J_1) \otimes \rho\) gives the skew-adjointness and regularity of \(C_{\mathcal{B}}(J_1)\).

Note that because \(\kappa_j\) anti-commute with \(J_0\) and \(J_1\) for all \(j = 1, \ldots, n\), the operators \(\kappa_j\) also restrict to skew-adjoint unitaries on the submodule \((J_0 - J_1)E_B\). An adaptation of proposition 2.6 to the ungraded and skew-adjoint setting gives the following.

**Proposition 2.8.** Let \(J_0, J_1 \in U_{E_B}^0\) be such that \(\|J_0 - J_1\|_{\text{Q}_B(E)} < 2\). Then the triple
\[ \left( \mathcal{C}_{\mathcal{F}^{+2+0}}, E_B \otimes \bigwedge C_{\mathcal{B}}(J_1)(1 - C_{\mathcal{B}}(J_1)^2)^{-1/2} \otimes \rho \right) \]
is a Real Kasparov module, where the $\mathcal{C}^\infty_{n+2,0}$-action has generators $\{1 \otimes \gamma, J_0 \otimes \rho, \kappa_1 \otimes \rho, \ldots, \kappa_n \otimes \rho\}$. For any choice of basepoint $e$.

As it will be useful for several of our results below, we give a brief overview of van Daele K-theory and boundary maps.

Let $A$ be a complex $C^*$-algebra. We say that $A$ has a balanced $\mathbb{Z}_2$-grading if $A$ contains an odd self-adjoint unitary. That is, there is an odd element $e$ satisfying $e = e^* = e^{-1}$.

In particular, $A$ is unital. If $A$ has a real structure, we also require $e^{\tau_A} = e$.

For simplicity, we will assume that any $\mathbb{Z}_2$-graded and unital $C^*$-algebra $A$ is balanced graded, taking the tensor product $A = A' \otimes \mathcal{C}^\infty_{1,1}$ if necessary. We can extend the grading and real structure of $A$ to $M_2(A)$ entrywise.

Let $V(A) = \bigsqcup \pi_0(\text{OSU}(M_2(A)))$ be the disjoint union of homotopy classes of odd self-adjoint unitaries in $M_2(A)$, which is an abelian semigroup under direct summation, $[x] + [y] = [x \oplus y]$. The Grothendieck group obtained from this semigroup will be denoted $GV(A)$.

The semigroup homomorphism $d : V(A) \to \mathbb{N}$ taking the value $k$ on $M_2(A)$ induces a group homomorphism $d : GV(A) \to \mathbb{Z}$.

**Definition 2.9.** Let $A$ be a complex $C^*$-algebra. We say that $A$ has a balanced $\mathbb{Z}_2$-grading if $A$ contains an odd self-adjoint unitary. That is, there is an odd element $e$ satisfying $e = e^* = e^{-1}$.

In particular, $A$ is unital. If $A$ has a real structure, we also require $e^{\tau_A} = e$.

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Let $V(A) = \bigsqcup \pi_0(\text{OSU}(M_2(A)))$ be the disjoint union of homotopy classes of odd self-adjoint unitaries in $M_2(A)$, which is an abelian semigroup under direct summation, $[x] + [y] = [x \oplus y]$. The Grothendieck group obtained from this semigroup will be denoted $GV(A)$.

The semigroup homomorphism $d : V(A) \to \mathbb{N}$ taking the value $k$ on $M_2(A)$ induces a group homomorphism $d : GV(A) \to \mathbb{Z}$.

**Definition 2.10.** If $A$ is unital and has a balanced $\mathbb{Z}_2$-grading, then the van Daele K-theory group of $A$ is $DK(A) = \text{Ker}(d : GV(A) \to \mathbb{Z})$.

If $A$ is not unital then we set $DK(A) = \text{Ker}(q_r : DK(A^-) \to DK(\mathbb{C}))$ where $q : A^- \to \mathbb{C}$ quotients the minimal unitisation $A^-$ by the ideal $A$.

Elements of $DK(A)$ are formal differences of odd self-adjoint unitaries, denoted by $[x] - [y]$.

It will also be useful to consider van Daele K-theory relative to a choice of basepoint [15]. For a balanced graded algebra $A$ and $e \in A$ an odd self-adjoint unitary, we let $V(A) = \bigsqcup \pi_0(\text{OSU}(M_2(A)))$ where we embed $M_2(A)$ into $M_{k+1}(A)$ via $x \mapsto x \oplus e$. van Daele’s $K$-theory group with a basepoint is defined as the Grothendieck group $DK_e(A) = GV_e(A)$.

The group $DK_e(A)$ does not depend on the choice of $e$ up to isomorphism [15, proposition 2.12]. For any choice of basepoint $e$, $DK_e(A) \cong DK(A)$ [9, section 2.1.1].
Examples 2.11.

(a) If $A$ is a unital and trivially graded algebra, then odd self-adjoint unitaries of $A \otimes \mathbb{C} \ell_{1,1}$ are of the form

$$U = \frac{1}{2}(u + u^*) \otimes \gamma + \frac{1}{2}(u - u^*) \otimes \rho = \begin{pmatrix} 0 & u^* \\ u & 0 \end{pmatrix}$$

where $u \in A$ is unitary and we make have made the identification $\{\gamma, \rho\} \sim \{(\sigma_1, -i\sigma_2)\}$. If $U^* = U$, then $u^* = u$ and the map $U \mapsto u$ furnishes an isomorphism $DK_{1\otimes \gamma}(A \otimes \mathbb{C} \ell_{1,1}) \cong KO(A^e)$. 

(b) If $A$ is trivially graded and unital, then odd self-adjoint unitaries in $A \otimes \mathbb{C} \ell_{1,0}$ take the form $U = x \otimes \gamma$. Hence $x = x^* = x^e$ is an even unitary in $A$ and the map $DK_{1\otimes \gamma}(A) \ni [x \otimes \gamma] \mapsto \left[\frac{1}{2}\right] \in KO(A^e)$ is an isomorphism.

For a balanced graded algebra $A$ with a closed, two-sided and graded ideal $I$ we define the relative van Daele group

$$DK(A, A/I) := \{[x] - [y] : x, y \in OSU(M_n(A)), x - y \in M_n(I)\}.$$ 

Here $[\cdot]$ denotes homotopy classes in $OSU(M_n(A))$. As expected, there is an excision isomorphism $DK(I) = DK(A, A/I) [9, proposition 2.4]$. 

The excision isomorphism gives us a basepointed description of the van Daele $K$-theory for non-unital $C^*$-algebras $A$ such that $\text{Multi}(A)$ is balanced graded. For such algebras we fix an odd self-adjoint unitary $e \in \text{Multi}(A)$ and let $A^{-e}$ be the subalgebra of $\text{Multi}(A)$ generated by $A$ and $e$. We may then consider $DK(A) = DK(A^{-e}, A^{-e}/A)$, see [9, section 2.1] for the full details.

Remark 2.12. Let $A$ be balanced graded and fix the $\mathbb{C} \ell_{k,0}$-generators $\{\gamma_1, \ldots, \gamma_k\} \subset A$. Recalling $\mathcal{O}_A^k$ from equation (1) on page 7, we may also wish to consider homotopy classes of odd self-adjoint unitaries in the set

$$\mathcal{O}_A^k = \{V = V^* = V^e = V^{-1} \in A : \text{V odd}, V\gamma_j = -\gamma_j V \text{ for } j = 1, \ldots, k\}.$$

Extending to matrices, we can define another semigroup of homotopy classes of odd self-adjoint unitaries in $\oplus_n M_n(A)$ that anti-commute with $\{\gamma_j^{\otimes n}\}_{j=1}^k$. However, noting that $\mathbb{C} \ell_{k,0} \otimes \mathbb{C} \ell_{j,0} \cong \text{End}(\wedge^j \mathbb{C} \ell_k)$, and that the representations of $\mathbb{C} \ell_{k,0}$ and $\mathbb{C} \ell_{j,0}$ on $\wedge^j \mathbb{C} \ell_k$ graded-commute, homotopy classes of odd self-adjoint unitaries in $\oplus_n M_n(A)$ that anti-commute (graded-commute) with $\{\gamma_j^{\otimes n}\}_{j=1}^k$ are equivalent to homotopy classes of odd self-adjoint unitaries in $\oplus_n M_n(A \otimes \mathbb{C} \ell_{0,k})$. Hence from the perspective of $K$-theory, it suffices to consider $DK(A \otimes \mathbb{C} \ell_{0,k})$.

We can use the Cayley transform of odd self-adjoint unitaries from section 2.3 to relate van Daele $K$-theory to $KKR$-theory.

Theorem 2.13 ([9, theorem 4.15]). Let $A$ be a Real $C^*$-algebra such that $\text{Multi}(A)$ is balanced graded. Suppose that $V_0, V_1 \in \text{Multi}(A)$ are odd self-adjoint unitaries anti-commuting with the $\mathbb{C} \ell_{l,0}$-generators $\{\gamma_1, \ldots, \gamma_k\} \subset \text{Multi}(A)$ and such that $V_0 - V_1 \in A$. Then the unbounded Real Kasparov module from proposition 2.6,

$$(V_0, V_1) \mapsto (\mathbb{C} \ell_{k+1,0}, \mathcal{A}_A, \mathcal{A}_{V_0}(V_1))$$

gives an isomorphism $DK(\text{Multi}(A) \otimes \mathbb{C} \ell_{0,k}, \text{Multi}(A) \otimes \mathbb{C} \ell_{0,k}/A \otimes \mathbb{C} \ell_{0,k}) \cong KKR(\mathbb{C} \ell_{k+1,0}, A)$. 

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Recalling remark 2.12, because $V_0$ and $V_1$ anti-commute with $\{\gamma_j\}_{j=1}^k$, they give a class in the degree shifted $DK(Mult(A) \otimes \mathbb{C}^{\ell_0})$ and because $V_0 - V_1 \in A$ we can take the relative class $[V_0] - [V_1] \in DK(Mult(A) \otimes \mathbb{C}^{\ell_0})$. Recalling the excision isomorphism, the cited result in [9] then finishes the proof.

Theorem 2.13 also shows that if $A$ is trivially graded, then $DK(A \otimes \mathbb{C}^{\ell_0}) \cong KO_1 + s_{-\nu}(A^5)$, see also [46].

Finally let us consider boundary maps associated to the $\mathbb{Z}_2$-graded short exact sequence of Real $C^*$-algebras,

$$0 \rightarrow B \rightarrow E \xrightarrow{\partial} A \rightarrow 0.$$  

The corresponding boundary map $\partial : DK(A) \rightarrow DK(B \otimes \mathbb{C}^{\ell_1,0})$ was studied by van Daele [16]. We assume that $E$ is balanced graded and consider the composition

$$DK(A \otimes \mathbb{C}^{\ell_0}) \xrightarrow{\partial} DK(B \otimes \mathbb{C}^{\ell_1,0}) \xrightarrow{\mathcal{C}_B} KKR(\mathbb{C}^{\ell_0}, B),$$

where $\mathcal{C}_B$ is the Cayley isomorphism from theorem 2.13 and we have made the identification $KKR(\mathbb{C}^{\ell_1,0}, B \otimes \mathbb{C}^{\ell_1,0}) \cong KKR(\mathbb{C}^{\ell_0}, B)$.

Let us fix a basepoint odd self-adjoint unitary $e \in E$ and mutually anti-commuting odd self-adjoint unitaries $\{\gamma_j\}_{j=1}^k \subset E$ that anti-commute with $e$. These elements descend to anti-commuting odd self-adjoint unitaries in $Mult(A)$ via the quotient map.

**Proposition 2.14 (cf [9, section 5.2]).** Let $V \in M_n(A^{-\theta(\nu)})$ be an odd self-adjoint unitary that anti-commutes with $\{q(\gamma_j^{\nu})\}_{j=1}^k$ and $V - q(e^{\nu}) \in M_n(A)$. Suppose that $\hat{V} \in M_n(E)$ is a Real, odd and self-adjoint lift of $V$ that anti-commutes with $\{\gamma_j^{\nu,1}\}_{j=1}^k$. Then the map $\mathcal{C}_B \circ \partial([V] - [e]) \in KKR(\mathbb{C}^{\ell_0,0}, B)$ can be represented by the bounded Real Kasparov module

$$\mathcal{C}_B \circ \partial([V] - [e]) \in KKR(\mathbb{C}^{\ell_0,0}, B)$$

with Clifford generators $\{\gamma_j^{\nu,1}\}_{j=1}^k$.

In the above result we have implicitly used that $E \subset Mult(B)$ as $B$ is an ideal in $E$.

**Proof.** By [9, proposition 5.7], the composition $\mathcal{C}_B \circ \partial$ is represented by the unbounded Kasparov module

$$\left( \mathcal{C}_B \circ \partial([V] - [e]) \right).$$

Because the operators $\{\gamma_j^{\nu,1}\}_{j=1}^k$ anti-commute with $\hat{V}$, they give a well-defined representation of $\mathbb{C}^{\ell_0,0}$ on $\cos(\frac{\pi}{2} V) B_B^{\nu,1} \otimes \mathbb{C}^{\ell_0,0}$ and anti-commute with $\tan(\frac{\pi}{2} V)$. Therefore, up to the isomorphism $KKR(\mathbb{C}_B \otimes \mathbb{C}^{\ell_0,0}) \cong KKR(\mathbb{C}^{\ell_0,0}, B)$, $\mathcal{C}_B \circ \partial$ can be represented by the unbounded Kasparov module

$$\left( \mathcal{C}_B \circ \partial \left( \cos(\frac{\pi}{2} V) B_B^{\nu,1} \otimes \mathbb{C}^{\ell_0,0}, \tan(\frac{\pi}{2} V) \right) \right).$$
with Clifford generators $\{\gamma_1, \ldots, \gamma_k\}$. We take the bounded transform to get the Kasparov module

$$\left( \mathbb{C} \ell_{k, 0}, B^m_{\ell} : \sin \left( \frac{\pi}{2} V \right) \right).$$

Finally, the straight-line operator homotopy $F_t = (1 - t) \sin(\frac{\pi}{2} V) + tV$ for $t \in [0, 1]$ does not change the KKR-class and gives the result.

By the equivalence of skew-adjoint unitaries $J$ in a trivially graded $C^*$-algebra $A$ with odd self-adjoint unitaries $J \otimes \rho = A \otimes \mathbb{C} \ell_{0, 1}$, we can also apply the results of this section to the ungraded skew-adjoint setting.

### 3. Quasifree ground states from the viewpoint of SPT phases

#### 3.1. Definition and properties

Fermionic quasifree ground states can be naturally studied using Araki's self-dual CAR algebra. Fix a separable complex Hilbert space $\mathcal{H}$ and a real involution, a self-adjoint anti-unitary $\Gamma$. Equivalently, $\mathcal{H}$ is a Real Hilbert $\mathbb{C}$-module with real involution $v^* = \Gamma v$. The self-dual CAR algebra $A^w_{\mathfrak{d}}(\mathcal{H}, \Gamma)$ is the $C^*$-algebra generated by $1$ and $c(v)$ for $v \in \mathcal{H}$ such that $v \mapsto c(v)$ is linear and with relations

$$c(v^*) = c(\Gamma v), \quad \{c(v^*), c(w)\} = \langle v, w \rangle_{\mathcal{H}}, \quad v, w \in \mathcal{H}.$$

The self-dual CAR algebra is $\mathbb{Z}_2$-graded by the parity automorphism $\Theta$, where $\Theta(c(v)) = -c(\Theta(v))$ for all $v \in \mathcal{H}$. One recovers the more familiar CAR algebra by means of a basis projection, an orthogonal projection $P$ on $\mathcal{H}$ such that $P + \Gamma PT = 1_{\mathcal{H}}$. Given a basis projection, there is a graded isomorphism $A^w_{\mathfrak{d}}(PH) \cong A^w_{\mathfrak{d}}(\mathcal{H}, \Gamma)$ which on generators is given by

$$a^*(Pv) \mapsto c(Pv), \quad a(Pv) \mapsto c(\Gamma Pv), \quad v \in \mathcal{H}.$$

Basis projections also are used to construct pure quasifree states on $A^w_{\mathfrak{d}}(\mathcal{H}, \Gamma)$. We summarise some of the key results of [3].

**Theorem 3.1 ([3, theorem 1]).** Let $P$ be a basis projection on $(\mathcal{H}, \Gamma)$.

(a) There is a quasifree, pure and $\Theta$-invariant state $\omega_P$ on $A^w_{\mathfrak{d}}(\mathcal{H}, \Gamma)$ such that

$$\omega_P \left( c(v^*)c(w) \right) = \langle v, Pw \rangle_{\mathcal{H}}, \quad v, w \in \mathcal{H},$$

and is extended to $A^w_{\mathfrak{d}}(\mathcal{H}, \Gamma)$ by the formulas

$$\omega_P(c(v_1) \cdots c(v_{2n+1})) = 0,$$

$$\omega_P(c(v_1) \cdots c(v_{2n})) = (-1)^n \sum_{\sigma} (-1)^{\sigma} \times \prod_{j=1}^{n} \omega_P \left( c(v_{\sigma(j)})c(v_{\sigma(j+n)}) \right),$$

where $n \in \mathbb{N}$, $v_j \in \mathcal{H}$ for all $j$ and the sum is over permutations $\sigma$ such that

$$\sigma(1) < \sigma(2) < \cdots < \sigma(n), \quad \sigma(j) < \sigma(j + n), \quad j = 1, \ldots, n.$$

(b) Let $P_0$ and $P_1$ be basis projections on $(\mathcal{H}, \Gamma)$. Then $\omega_{P_0}$ and $\omega_{P_1}$ are unitarily equivalent if and only if $P_0 - P_1$ is in the ideal of Hilbert–Schmidt operators.
A simple method to construct quasifree states is to consider the unitary dynamics on \((\mathcal{H}, \Gamma)\) generated by a self-adjoint operator \(H = H^*\) such that \(\Gamma(\text{Dom}(H)) \subset \text{Dom}(H)\) and \(\Gamma H = -H\Gamma\). We will call such operators BdG Hamiltonians. We will furthermore restrict to gapped BdG Hamiltonians by assuming that \(0 \notin \sigma(H)\).

A BdG Hamiltonian defines a quasifree dynamics \(\beta : \mathbb{R} \to \text{Aut}(\mathcal{A}_\text{sd}(\mathcal{H}, \Gamma))\) given on generators by \(\beta_t(\varepsilon(v)) = e^{itH} \varepsilon(v)\) for all \(v \in \mathcal{H}\). The ground state of this action is then completely described by the basis projection \(P = \chi_{(0,\infty)}(H)\), \(P + \Gamma P \Gamma = 1_{\mathcal{H}}\).

**Proposition 3.2** ([19, proposition 6.37], [10, proposition 5.3]). Let \(\beta : \mathbb{R} \to \text{Aut}(\mathcal{A}_\text{sd}(\mathcal{H}, \Gamma))\) be a quasifree dynamics with BdG Hamiltonian \(H\). If \(0 \notin \sigma(H)\), then the quasifree state \(\omega_P\) associated to the basis projection \(P = \chi_{(0,\infty)}(H)\) is the unique ground state for the dynamics \(\beta\). Furthermore, this ground state is gapped in the sense that the generator of the dynamics on the Gelfand-Naimark-Segal (GNS) Hilbert space has a spectral gap above 0.

**Example 3.3 (BdG Hamiltonians from superconductors).** The canonical example we will consider is the Nambu space \(\mathcal{H} = \mathcal{V} \oplus \mathcal{V}^*\) with \(\mathcal{V}\) the Hilbert space of electrons and \(\mathcal{V}^*\) the space of holes related by the (anti-linear) Riesz map \(R : \mathcal{V} \to \mathcal{V}^*\). In particular, \(\mathcal{H}\) has the natural real involution \(\Gamma = \begin{pmatrix} 0 & R^{-1} \\ R & 0 \end{pmatrix}\). We will also use the isomorphism \(\mathcal{V} \oplus \mathcal{V}^* \simeq \mathcal{V} \otimes \mathbb{C}^2\) with real involution \(\Gamma = \mathbb{C}(1 \otimes \sigma_1)\) and \(\mathbb{C}\) complex conjugation. Typical examples of \(\mathcal{V}\) for discrete models are \(\ell^2(L^2)\) with \(L\) a countable set. For continuous models we will consider \(\mathcal{V} = L^2(\mathbb{R}^d, \mathbb{C}^n)\) or \(L^2(M, V)\) with \(V \to M\) a complex Hermitian vector bundle over a complete Riemannian manifold.

Generically, we will consider BdG Hamiltonians on \(\mathcal{V} \otimes \mathbb{C}^2\) of the form

\[
\begin{pmatrix}
\hbar & \delta \\
\delta^* & -\hbar
\end{pmatrix}, \quad \Theta = \mathbb{C} \Lambda \Xi, \quad \delta^* = -\delta.
\]

In discrete systems, e.g. \(\mathcal{H} = \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^{2n}\), we typically have that \(\hbar = (\rho(S_1, \ldots, S_d) - \mu) \otimes 1_n\), with \(\rho(S_1, \ldots, S_d)\) is a self-adjoint finite polynomial of the shift operators (e.g. the discrete Laplacian) and \(\mu \in \mathbb{R}\) the Fermi energy. The pairing potential \(\delta\) is then determined by the type of superconductor under consideration (s-wave, d-wave \((\rho \pm ip)\)-wave etc). Concrete examples for \(d = 2\) can be found in [17, section 2.3].

For continuous models such as \(\mathcal{H} = L^2(\mathbb{R}^d, \mathbb{C}^{2n})\), the BdG Hamiltonians we consider take the same form as equation (6), but now where \(\hbar = (\sum (\frac{-1}{2} A_j^2) - \mu) \otimes 1_n\) is a (possibly magnetic) Laplacian. The coupling term \(\delta^*\) is often a first-order differential operator and depends on the example under consideration, see [47, section 2] for example. For the purposes of this paper, the specific form of \(H\) is not so important provided it is sufficiently local, elliptic and \(0 \notin \sigma(H)\).

Let us also note that we may also consider weakly disordered BdG Hamiltonians, i.e. the disordered Hamiltonian still anti-commutes with \(\Gamma\) and has a spectral gap at 0.

Given a unitary or anti-unitary operator \(W\) on \(\mathcal{H}\) that commutes with \(\Gamma\), we can define a linear or anti-linear automorphism \(\beta_W\) of \(\mathcal{A}_\text{sd}(\mathcal{H}, \Gamma)\) such that \(\beta_W(\varepsilon(v)) = \varepsilon(Wv)\). Note also that any such quasifree automorphism will commute with the parity automorphism \(\Theta = \beta_{-1}\).

**Lemma 3.4.** Fix a basis projection \(P + \Gamma P \Gamma = 1\) on \((\mathcal{H}, \Gamma)\).

(a) If \(W\) is a unitary operator on \(\mathcal{H}\) such that \([W, \Gamma] = 0 = [W, P]\), then \(\omega_P(\beta_W(a)) = \omega_P(a)\) for all \(a \in \mathcal{A}_\text{sd}(\mathcal{H}, \Gamma)\).
(b) If $T$ is an anti-unitary operator on $\mathcal{H}$ such that $[T, \Gamma] = [T, P] = 0$, then the quasifree state $\omega_P$ is such that $\omega_P(\beta_f(a^\ast)) = \omega_P(a)$ for all $a \in \mathcal{A}_\text{ad}^\text{sr}(\mathcal{H}, \Gamma)$.

**Proof.** Because $\omega_P$ is quasifree, we only need to check invariance on operators of the form $c(u)^\ast c(v)$ for arbitrary $u, v \in \mathcal{H}$. For the linear case,

$$
\omega_P \left( \beta_W(c(u)^\ast c(v)) \right) = \omega_P(c(Wu)^\ast c(Wv)) = \langle Wu, PWv \rangle_{\mathcal{H}} = \langle u, Pv \rangle_{\mathcal{H}} = \omega_P(c(u)^\ast c(v)).
$$

For the anti-linear case, we first note that because $(\mathcal{H}, \Gamma)$ is a Real Hilbert space, $(\langle u, v \rangle) = (\Gamma u, \Gamma v)$. We now compute

$$
\omega_P \circ \beta_T \left( ((c(u)^\ast c(v))^\ast) \right) = \omega_P \left( \beta_T((c(v)^\ast c(u))) \right) = \omega_P(c(\Gamma TT v)^\ast c(Tu))
$$

$$
= \langle (\Gamma TT v), PTu \rangle_{\mathcal{H}} = \langle Tv, PTu \rangle_{\mathcal{H}} = \langle v, Pu \rangle_{\mathcal{H}} = \langle u, Pv \rangle_{\mathcal{H}} = \omega_P(c(u)^\ast c(v)).
$$

Note that the assumptions $[T, \Gamma] = [W, \Gamma] = 0$ imply that $T$ and $W$ are (linear) orthogonal operators on the real Hilbert space $\mathcal{H}_R = \{ v \in \mathcal{H} : \Gamma v = v \}$.

### 3.2. Parity symmetry and $\mathbb{Z}_2$-indices

Let us recall the basic classifying principle of SPT phases of gapped ground states with an on-site symmetry.

Fix a reference ground state $\omega_0$ and on-site symmetry $G$. We assume that a gapped ground state $\omega$ is connected to $\omega_0$ but need not be $G$-equivariantly connected.

Usually $\omega_0$ is taken to be a product state. Here we will consider a similar notion.

Let $\omega_0$ and $\omega_1$ be quasifree pure gapped ground states of $\mathcal{A}_\text{ad}^\text{sr}(\mathcal{H}, \Gamma)$. We assume that $\omega_1$ is unitarily equivalent to $\omega_0$, but their restrictions to the $\Theta$-invariant (even) subalgebra $\mathcal{A}_\text{ad}^\text{sr}(\mathcal{H}, \Gamma)^0$ need not be unitarily equivalent.

Hence, let $H_0$ and $H_1$ be BdG Hamiltonians with $0 \notin \sigma(H_0) \cup \sigma(H_1)$ and such that the corresponding quasifree ground states $\omega_0$ and $\omega_1$ are unitarily equivalent. By theorem 3.1, the basis projections $P_0 = \chi_{(0, \infty)}(H_0)$ and $P_1 = \chi_{(0, \infty)}(H_1)$ are such that $P_0 - P_1$ is Hilbert–Schmidt. Let us therefore consider the Real skew-adjoint unitaries $J_0 = i(2P_0 - 1)$, $J_1 = i(2P_1 - 1)$, where $\Gamma J_0 = J_0$ and $\Gamma J_1 = J_1$. In the sequel we will often write $J_k = i \text{sgn}(H_k) = iH_k|H_k|^{-1}$, $k \in \{0, 1\}$, which is defined via the Borel functional calculus.

**Lemma 3.5 ([7, proposition 4.4]).** The sum $J_0 + J_1$ is a Real and Fredholm operator on $\mathcal{H}$.

Let us therefore consider the finite-dimensional space $\text{Ker}(J_0 + J_1)$. Noting that $J_1(J_0 + J_1) = (J_0 + J_1)J_0$, we see that both $J_0$ and $J_1$ act on $\text{Ker}(J_0 + J_1)$. Choosing $J_0$ or $J_1$ as the generator, we therefore obtain an ungraded $\mathbb{C}\ell_{0,1}$-action on $\text{Ker}(J_0 + J_1)$. Note that we cannot take a $\mathbb{C}\ell_{0,2}$-action as $J_0$ and $J_1$ neither commute nor anti-commute in general. Hence
Ker($J_0 + J_1$) can be regarded as an ungraded $\mathbb{C}l_{0,1}$-module and we can ask whether it extends to a $\mathbb{C}l_{0,2}$-module. As the following result shows, this extension only occurs when $\omega_0$ and $\omega_1$ are equivalent on the even subalgebra.

**Proposition 3.6 ([4, theorem 4]).** The states $\omega_0$ and $\omega_1$ restricted to $A_{\text{sat}}^\text{even}(\mathcal{H}, \Gamma)^0$ are equivalent if and only if $\frac{1}{2} \dim \text{Ker}(J_0 + J_1)$ is even.

If $H_0$ and $H_1$ are bounded, the index $\frac{1}{2} \dim \text{Ker}(J_0 + J_1)$ also computes the $\mathbb{Z}_2$-valued spectral flow of a skew-adjoint and Real Fredholm path $\{iH_t\}_{t \in [0,1]}$ connecting $iH_0$ and $iH_1$ [13, proposition 6.2]. If $H_0$ or $H_1$ is unbounded, the same is true using the unbounded version of the $\mathbb{Z}_2$-valued spectral flow [7, section 6].

### 3.3. Altland–Zirnbauer symmetries

Here we consider the physical symmetries of free-fermionic systems as considered by Altland and Zirnbauer [1, 2, 32]. Such symmetries may be unitary or anti-unitary operators on $\mathcal{H}$ and commute with the BdG Hamiltonians. The quasifree ground state $\omega_F$ will then be invariant under such symmetries by lemma 3.4. In particular, we will take advantage of the following result first noted by Kennedy and Zirnbauer and then further developed by Alldridge, Max, and Zirnbauer.

**Proposition 3.7 ([32, section 2], [1, sections 3.1–3.2]).** Let $H$ be a BdG Hamiltonian on $(\mathcal{H}, \Gamma)$ with $0 \notin \sigma(H)$ and $J = iH[H]^{-1}$. If $H$ has Altland–Zirnbauer symmetries, then there are real mutually anti-commuting skew-adjoint unitaries $\{\kappa_j\}_{j=1}^n \subset B(\mathcal{H})$ such that $\kappa_jJ = -J\kappa_j$ for all $j \in \{1, \ldots, n\}$ and where the integer $n \in \{0, \ldots, 7\}$ depends on the symmetry.

Proposition 3.7 as stated hides many details, but we remark that the skew-adjoint unitaries $\{\kappa_j\}_{j=1}^n$ are concretely constructed from the physical symmetry operators that commute with $H$. For example, if $H$ is time-reversal symmetric via a self-adjoint anti-unitary $T$ such that $[T, H] = [T, \Gamma] = 0$ and $T^2 = -1$, we take $\kappa_T = \Gamma T$.

We consider gapped BdG Hamiltonians $H_0$ and $H_1$ which have the same Altland–Zirnbauer symmetry type and whose gapped ground states $\omega_0$ and $\omega_1$ are unitarily equivalent if we ignore symmetries. That is, $J_0 - J_1$ is Hilbert–Schmidt and there are operators $\{\kappa_j\}_{j=1}^n$ such that $J_0\kappa_j = -\kappa_j J_1$ for all $i \in \{0, 1\}, j \in \{1, \ldots, n\}$.

Therefore we again consider the finite-dimensional space Ker($J_0 + J_1$), where $\kappa_j \cdot \text{Ker}(J_0 + J_1) \subset \text{Ker}(J_0 + J_1)$. Hence, Ker($J_0 + J_1$) is an ungraded $\mathbb{C}l_{0,n+1}$-module with generators $\{J_0, \kappa_1, \ldots, \kappa_n\}$. Letting $\mathcal{M}_n$ denote the Grothendieck group of equivalence classes of ungraded $\mathbb{C}l_{0,n}$-modules, we can consider the class $[\text{Ker}(J_0 + J_1)] \in \mathcal{M}_{n+1}/\mathcal{M}_{n+2} \cong KO_{n+2}(\mathbb{R})$ via the Atiyah–Bott–Shapiro isomorphism [5, theorem 11.5]. The vanishing of this class implies that the $KO_{n+2}(\mathbb{R})$-valued spectral flow between the (bounded or unbounded) skew-adjoint endpoints $iH_0$ and $iH_1$ vanishes [7, sections 5–6]. A non-trivial Clifford index guarantees that the ground state gap will close on any Fredholm path connecting $\omega_0$ and $\omega_1$. We therefore say that $[\text{Ker}(J_0 + J_1)] \in KO_{n+1}(\mathbb{R})$ is a topological obstruction to connect the two ground states $\omega_0$ and $\omega_1$ in a way that respects the Altland–Zirnbauer symmetries of $H_0$ and $H_1$.

**Remark 3.8 (Reinterpretation via Cayley map).** Let $J_0$ and $J_1$ be Real skew-adjoint unitaries on $(\mathcal{H}, \Gamma)$ that anti-commute with the ungraded $\mathbb{C}l_{0,n}$-generators $\{\kappa_j\}_{j=1}^n$. Suppose further that and $J_0 - J_1$ is Hilbert–Schmidt. In particular, $\|J_0 - J_1\|_{\mathcal{M}(\mathcal{H})} = 0$ and so the operators $J_0, J_1 \in \text{Mult}(\mathcal{K}(\mathcal{H}))$ fall into the framework of section 2.4. Hence there is an unbounded
Real Kasparov module
\[
\left(\mathbb{C}[n+2,0, J_0 = J_1] \otimes \bigotimes_{\ell} \mathbb{C}, J_0(J_1 + J_0)(J_1 - J_0) \otimes \rho\right), \ K = \mathbb{K}(\mathcal{H})
\]
with left Clifford generators \(\{1 \otimes \gamma, J_0 \otimes \rho, \kappa_1 \otimes \rho, \ldots, \kappa_n \otimes \rho\}\). We therefore obtain a class in \(KKR(\mathbb{C}[n+2,0, \mathbb{K}(\mathcal{H})]) \cong KO_{n+2}(\mathbb{K}(\mathcal{H})) \cong KO_{n+2}(\mathbb{R})\) which can be determined by the ungraded Clifford module index \([\text{Ker}(J_0 + J_1)] \in KO_{n+2}(\mathbb{R})\).

### 3.4. Compact group symmetries

Let \(G\) be a compact group and \(\nu : G \to \mathbb{Z}_2\) a homomorphism. A representation \(W\) of \(G\) is unitary/anti-unitary with respect to \(\nu\) if \(W_g\) is unitary (respectively anti-unitary) for all \(g \in G\) such that \(\nu(g) = 0\) (respectively \(\nu(g) = 1\)). We fix a unitary/anti-unitary representation \(W\) and assume that \(W_g \Gamma = \Gamma W_g\) for all \(g \in G\). This then gives a linear/anti-linear action \(\beta\) on \(A^{\text{car}}(\mathcal{H}, \Gamma)\) relative to \(\nu : G \to \mathbb{Z}_2\) such that \(\beta_g(c(v)) = c(W_g v)\) for any \(g \in G\) and \(v \in \mathcal{H}\). Now take a gapped BdG Hamiltonian such that \([H, W_g]\) is well-defined and \([H, W_g]\) = 0 for all \(g \in G\). Then for \(P = (\chi_{\nu \neq 0,\gamma}(H)\) the basis projection on \((\mathcal{H}, \Gamma)\), \([W_g, P] = 0\) for any \(g \in G\) and the ground state \(\omega_P\) is invariant under \(\beta\) by lemma 3.4. For \(J = i \text{sgn}(H) = i[H][H]^{-1}\), a simple computation shows that
\[
W_g J W_g^* = (-1)^{\nu(g)} J \quad \text{for all } g \in G.
\]

Following the viewpoint of SPT phases, we are also interested in ground states on the \(G\)-invariant subalgebra
\[
A^{\text{car}}_G(\mathcal{H}, \Gamma) = \{a \in A^{\text{car}}(\mathcal{H}, \Gamma) : \beta_g(a) = a^{\nu(g)} \quad \text{for all } g \in G\},
\]
\[
a^{\nu(g)} = \begin{cases} a, & \nu(g) = 0 \\ a^*, & \nu(g) = 1 \end{cases}.
\]

Given a state \(\omega\) on \(A^{\text{car}}(\mathcal{H}, \Gamma)\), we let \(\omega^G\) denote its restriction to \(A^{\text{car}}_G(\mathcal{H}, \Gamma)\).

Let us now take two \(G\)-symmetric gapped BdG Hamiltonians \(H_0\) and \(H_1\) such that \(\omega_0\) and \(\omega_1\) are equivalent without symmetry. Then the space \(\text{Ker}(J_0 + J_1)\) is finite dimensional and \(W_g \cdot \text{Ker}(J_0 + J_1) \subset \text{Ker}(J_0 + J_1)\) for all \(g \in G\). Therefore \(\text{Ker}(J_0 + J_1)\) determines an element of \(R(G)\), the representation ring of \(G\).

**Definition 3.9 (cf [11, section 5]).** We denote by \(R(G)_k\) the Grothendieck group of ungraded and Real \((\mathbb{C}\ell_{0,k-1}, G)\)-bimodules modulo those extendable to ungraded and Real \((\mathbb{C}\ell_{0,k}, G)\)-bimodules.

By a \(G\)-equivariant extension of the Atiyah–Bott–Shapiro isomorphism, \(R(G)_k \cong KO^{\text{car}}_k(\mathbb{R})\) [30, section 2]. The ground states \(\omega_0\) and \(\omega_1\) are such that \(\text{Ker}(J_0 + J_1)\) is a \((\mathbb{C}\ell_{0,1}, G)\)-module with the Clifford generator \(J_0\) and the unitary/anti-unitary representation \(W\). We can therefore associate the element \([\text{Ker}(J_0 + J_1)] \in R(G)_2\).

**Theorem 3.10 ([11, theorem A], [11, theorem 5.1]).** Let \(G\) be a compact group and \(W\) a unitary/anti-unitary representation on \((\mathcal{H}, \Gamma)\) commuting with \(\Gamma\). Suppose that \(P_0\) and \(P_1\) are basis projections on \((\mathcal{H}, \Gamma)\) such that for all \(g \in G\), \([W_g, P_0] = [W_g, P_1] = 0\). Then the states \(\omega_0^G\) and \(\omega_1^G\) on \(A^{\text{car}}_G(\mathcal{H}, \Gamma)^G\) are equivalent if and only if:

(a) \(P_0 - P_1\) is a Hilbert–Schmidt operator,

(b) \([\text{Ker}(J_0 + J_1)]\) represents the identity in \(R(G)_2\) with \(J_k = i(2P_k - I)\) for \(k \in \{0, 1\}\).
The result in [41] is stated for unitary representations, but the proof also holds for unitary/anti-unitary actions commuting with $\Gamma$ as $W$ restricts to a (linear) orthogonal representation on $H_\mathbb{R} = \{ v \in H : \Gamma v = v \}$.

4. Locality and $KO(A)$-valued indices

Let us now move to a more abstract setting, where we have a $\mathcal{C}^*$-algebra $(A, \tau)$ such that $(\text{Mult}(A), \tau) \subset (\mathcal{B}(H), \text{Ad}_\tau)$. In particular, $a^* = \Gamma a \Gamma^*$ for all $a \in \text{Mult}(A)$. When $A = \mathbb{R}^R(H)$, we recover the results of the previous section. We will assume that $A$ is trivially graded, but will often consider tensor products of the form $A \otimes \mathbb{C}f_{r,s}$, which is $\mathbb{Z}_2$-graded. We assume that $A$ is $\sigma$-unital but make no further assumptions on the specific form of $A$, so $A$ may be a crossed product, groupoid or (uniform) Roe algebra. In section 5 we will consider the case $A = C^*(X)$.

We also note that the content of this section is similar to previous work by Alldridge et al [1].

4.1. Basic construction

Because we also consider unbounded BdG Hamiltonians, we recall the notion of a normalising function.

**Definition 4.1.** We call a continuous odd function $\chi : \mathbb{R} \to [-1, 1]$ a normalising function if $\chi(t) \to \pm 1$ as $x \to \pm \infty$.

We fix a Hilbert space $H$ and real involution $\Gamma$. We wish to consider topological properties of BdG Hamiltonians and quasifree ground states relative to the $\mathcal{C}^*$-algebra $A$. We therefore make the following assumption for this section.

**Assumption 4.2.** Let $H_0$ and $H_1$ act on $(H, \Gamma)$ such that for $k \in \{0, 1\}$, $0 \notin \sigma(H_k)$, and $H_k = H_k^* = -\Gamma H_k \Gamma$.

(a) (Unbounded case) if $H_0$ and $H_1$ are unbounded, we assume that $\chi(H_0), \chi(H_1) \in \text{Mult}(A)$ and $\chi(H_0) - \chi(H_1) \in A$ for any normalising function $\chi$.

(b) (Bounded case) if $H_0$ and $H_1$ are bounded, we assume that $H_0$ and $H_1$ are invertible elements of $\text{Mult}(A)$ and $H_0 - H_1 \in A$.

Physically, assumption 4.2 implies that we are given some Real $\mathcal{C}^*$-algebra $A$ that specifies the allowed deformations of the BdG Hamiltonians within the Nambu space $(H, \Gamma)$.

**Lemma 4.3.**

(a) Given assumption 4.2(a), $J_0 = iH_0[H_0]^{-1}$ and $J_1 = iH_1[H_1]^{-1}$ are Real invertible elements of $\text{Mult}(A)$. Furthermore, $J_0 - J_1$ and $P_0 - P_1 \in A$ for $P_0 = \chi(0, \infty)(H_0)$ and $P_1 = \chi(0, \infty)(H_1)$.

(b) Given assumption 4.2(b), then $\chi(H_0), \chi(H_1) \in \text{Mult}(A)$ for any normalising function $\chi$ and $J_0 - J_1 \in A$.

**Proof.**

(a) For a fixed $k \in \{0, 1\}$, by assumption there is some $\epsilon > 0$ such that $[-\epsilon, \epsilon] \cap \sigma(H_k) = \emptyset$.

Therefore there is a suitable normalising function $\chi$ such that $\chi(H_k) = J_k$. It is then immediate that $J_k \in \text{Mult}(A)$ for any $k \in \{0, 1\}$ and $J_0 - J_1 \in A$. The results for $P_0$ and $P_1$ immediately follow as $J_k = i(2P_k - 1), k \in \{0, 1\}$.

(b) Because $H_0$ and $H_1$ are bounded and invertible, the first statement immediately follows from the continuous functional calculus. Next we write $H_1 = H_0 + a$ for some $a \in A$ and
Using the resolvent identity, we therefore find that

\[ |H_0|^{-1} - |H_1|^{-1} = H_0 \left( |H_0|^{-1} - |H_1|^{-1} \right) + a|H_1|^{-1}. \]

Because \( H_1 \) is invertible in \( \text{Mult}(A) \), \( |H_1|^{-1} \in \text{Mult}(A) \) and so \( a|H_1|^{-1} \in A \). The result will therefore follow if we can show that \( |H_0|^{-1} - |H_1|^{-1} \in A \). We will use the integral formula for fractional powers, where for any self-adjoint and invertible operator \( T \)

\[ |T|^{-1} = (T^2)^{-1/2} = \frac{2}{\pi} \int_0^\infty (T^2 + x^2)^{-1} \, dx. \]

Using the resolvent identity, we therefore find that

\[
|H_0|^{-1} - |H_1|^{-1} = \frac{2}{\pi} \int_0^\infty \left( (H_0^2 + x^2)^{-1} - ((H_0 + a)^2 + x^2)^{-1} \right) \, dx \\
= \frac{2}{\pi} \int_0^\infty (H_0^2 + x^2)^{-1}((H_0 + a)^2 - H_0^2) \\
\times ((H_0 + a)^2 + x^2)^{-1} \, dx.
\]

Because \( (H_0 + a)^2 - H_0^2 = a^2 + aH_0 + H_0a \in A \), the integrand is an element of \( A \). Finally for any \( b \in A \), the continuous functional calculus gives the estimate

\[ \| (H_0^2 + x^2)^{-1}b((H_0 + a)^2 + x^2)^{-1} \|_A \lesssim C(1 + x^2)^{-1}, \]

which then implies that the integral will be norm-convergent in \( A \). \qed

The equivalence condition on the gapped ground states \( \omega_0 \) and \( \omega_1 \) is now turned into a locality-like condition, where we assume that while \( J_0 \) and \( J_1 \) are in \( \text{Mult}(A) \), their difference \( J_0 - J_1 \in A \).

We note that \( J_0 \otimes \rho, J_1 \otimes \rho \in \text{Mult}(A) \otimes \mathbb{C}_\ell_{0,1} \) are Real odd self-adjoint unitaries. Recalling section 2.5, if \( J_0 - J_1 \in A \), then we obtain a class in the relative van Daele \( K \)-theory group

\[ [J_1 \otimes \rho] - [J_0 \otimes \rho] \in DK(\text{Mult}(A) \otimes \mathbb{C}_\ell_{0,1}, \text{Mult}(A)/A \otimes \mathbb{C}_\ell_{0,1}) \cong DK(A \otimes \mathbb{C}_\ell_{0,1}). \]  

Because \( A \) is ungraded, \( DK(A \otimes \mathbb{C}_\ell_{0,1}) \cong KO_2(A^\gamma). \)

Considering \( J_0, J_1 \in \text{Mult}(A) \) as operators on the Hilbert \( C^* \)-module \( A_A \), if \( J_0 - J_1 \in A \), then \( \| J_0 - J_1 \|_{\mathcal{Q}_A} = 0 \) and we can apply proposition 2.8 to obtain the Real unbounded Kasparov module

\[ \left( \mathbb{C}_\ell_{2,0}, \mathcal{O}_{\mathbb{C}(J_0 - J_1)} \otimes \bigoplus \mathbb{C}, \mathcal{O}_{\mathbb{C}(J_1)} \otimes \rho \right) \]

with \( \mathbb{C}_\ell_{2,0} \)-generators \( \{ 1 \otimes \gamma, J_0 \otimes \rho \} \). We denote the corresponding class as \( [\mathcal{O}_A(J_1)] \in KKR(\mathbb{C}_\ell_{2,0}, A) \cong KO_2(A^\gamma). \)
By proposition 2.8, this KKR-theory class can also be represented by the bounded Kasparov module

\[
[C_b(J_1)] = \left[ \left( \mathcal{C} \ell_{n+2,0}, A \otimes \bigotimes C, F_{\mathcal{C}b(J_1)} \otimes \rho \right) \right] \in KKR(\mathcal{C} \ell_{2,0}, A)
\]  

(9)

with \( F_{\mathcal{C}b(J_1)} = C_b(J_1)(1 - C_b(J_1)^2)^{-1/2} \) the bounded transform.

The following result is clear to readers familiar with Kasparov theory, but we state it with a proof for completeness.

**Proposition 4.4.** Suppose that \( \{J_t\}_{t \in [0,1]} \) is a continuous path of skew-adjoint Real unitaries in \( \text{Mult}(A) \) such that \( J_t - J_0 \in A \) for all \( t \in [0,1] \). Then the corresponding class \([C_b(J_1)]\) is trivial in \( KKR(\mathcal{C} \ell_{2,0}, A) \).

**Proof.** The conditions on the path \( \{J_t\}_{t \in [0,1]} \) imply that the pointwise Hilbert \( C^* \)-module \( (J_t - J_0)A \) can be completed to a \( A \otimes \mathcal{C}([0,1]) \)-module, \( (J_t^* - J_0^*)(A \otimes \mathcal{C}([0,1]))_{K\mathcal{C}R} \), \( t \in [0,1] \). The continuity of \( J_t \) also ensures that the pointwise bounded transform \( F_{Cb(J_t)} = \frac{1}{2}J_t(J_0 - 1)(J_tJ_t + 1)^{-1/2}(2 + J_0J_t + J_tJ_0)^{1/2} \) gives a well-defined and skew-adjoint and Fredholm operator \( F_t \) on \( (J_t^* - J_0^*)(A \otimes \mathcal{C}([0,1]))_{K\mathcal{C}R} \). Hence the triple

\[
\left( \mathcal{C} \ell_{2,0}, (J_t^* - J_0^*)(A \otimes \mathcal{C}([0,1])_{K\mathcal{C}R}) \otimes \bigotimes C, F_{\mathcal{C}b(J_t)} \otimes \rho \right)
\]

is a well-defined Kasparov module that gives a homotopy in \( KKR(\mathcal{C} \ell_{2,0}, A) \). However, evaluating at \( t = 0 \), the corresponding Kasparov module \( (\mathcal{C} \ell_{2,0}, 0, 0) \) is degenerate and therefore trivial in \( KKR(\mathcal{C} \ell_{2,0}, A) \). Hence \([C_b(J_1)] = [C_b(J_0)]\) is also trivial. \( \square \)

The contrapositive of proposition 4.4 says that if \([C_b(J_1)]\) is non-trivial in \( KO_2(A^\Lambda) \), then \( J_0 \) and \( J_1 \) cannot be connected by a path \( \{J_t\}_{t \in [0,1]} \) that is local with respect to the algebra \( A \), \( J_t - J_0 \in A \) for all \( t \in [0,1] \).

We now relate our Cayley Kasparov module to the relative van Daele \( K \)-theory class \([J_0 \otimes \rho] - [J_1 \otimes \rho]) \in DR(A \otimes \mathcal{C} \ell_{0,1}) \).

**Proposition 4.5.** The Cayley isomorphism \( C : DR(A \otimes \mathcal{C} \ell_{0,1}) \rightarrow KKR(\mathcal{C} \ell_{2,0}, A) \) from theorem 2.13 is such that

\[
C([J_1 \otimes \rho] - [J_0 \otimes \rho]) = [C_b(J_1)]
\]

**Proof.** The Cayley isomorphism maps \([J_1 \otimes \rho] - [J_0 \otimes \rho])\) to the class of

\[
\left( \mathcal{C} \ell_{1,0}, \left( J_0 - J_1 \right)A \otimes \mathcal{C} \ell_{0,1}, C_b(J_1) \otimes \rho \right)
\]

~ \[
\left( \mathcal{C} \ell_{2,0}, \left( J_0 - J_1 \right)A \otimes \bigotimes C, C_b(J_1) \otimes \rho \right)
\]

Hence we recover \([C_b(J_1)]\). \( \square \)

Proposition 4.5 also gives a simpler proof of proposition 4.4 as a continuous path of Real skew-adjoint unitaries \( \{J_t\}_{t \in [0,1]} \subset \text{Mult}(A) \) with \( J_t - J_0 \in A \) for all \( t \in [0,1] \) immediately implies that the relative van Daele class \([J_0 \otimes \rho] - [J_1 \otimes \rho])\) is trivial.

**Remark 4.6** (Altland–Zirnbauer symmetries relative to \( A \)). Recalling proposition 3.7, let us furthermore assume that there exists a family of mutually anti-commuting, skew-adjoint and Real unitaries \( \{\kappa_j\}_{j=1}^n \subset \text{Mult}(A) \) such that \( J_1 \kappa_j = -\kappa_j J_1 \) for \( j \in \{1, \ldots, n\} \) and \( k \in \{0,1\} \).
We can again apply proposition 2.8 and obtain the Real unbounded Kasparov module
\[
\left( C_{\ell_{n+2,0}}(J_0 - J_1)A_\Lambda \otimes \bigwedge^* C, C_{\ell_0}(J_1) \otimes \rho \right)
\]
with Clifford generators \( \{ 1 \otimes \gamma, J_0 \otimes \rho, \kappa_1 \otimes \rho, \ldots, \kappa_n \otimes \rho \} \), which gives a class \([C_{\ell_0}(J_1)]\) in \( KKR(C_{\ell_{n+2,0}}(A)) \cong KO_{n+2}(A^\prime)\).

As in the case without symmetry, if the class \([C_{\ell_0}(J_1)]\) is non-zero in \( KO_{n+2}(A^\prime)\), then \( J_0 \) and \( J_1 \) cannot be connected by a path \( \{ J_t \}_{t \in [0,1]} \) that anti-commutes with \( \{ \kappa_j \}_{j \in \mathbb{N}} \) and is such that \( J_t - J_0 \in A \) for all \( t \in [0,1] \). Hence the class \([C_{\ell_0}(J_1)]\) represents a topological obstruction to connect the two symmetric BdG Hamiltonians via a path that respects the Altland–Zirnbauer symmetries and is local with respect to the auxiliary algebra \( A \).

We can similarly consider the relative van Daele class, where the odd self-adjoint unitaries \( J_0 \otimes \rho \) and \( J_1 \otimes \rho \) in \( \text{Mult}(A) \otimes C_{\ell_{0,1}} \) anti-commute with \( \{ \kappa_j \otimes \rho \}_{j \in \mathbb{N}} \). Recalling remark 2.12, we have that
\[
[J_0 \otimes \rho] - [J_1 \otimes \rho] \in DK \left( \text{Mult}(A) \otimes C_{\ell_{0,n+1}}, \text{Mult}(A) / A \otimes C_{\ell_{0,n+1}} \right).
\]
and the same basic argument as proposition 4.5 gives that this relative class is represented by \([C_{\ell_0}(J_1)]\) under the isomorphism \( DK(A \otimes C_{\ell_{0,n+1}}) \cong KKR(C_{\ell_{n+2,0}}(A))\).

4.2. Compact G-symmetries

Let us fix a compact group \( G \) and consider symmetries via a linear/anti-linear action \( \beta \) of \( G \) on \( A \) relative to \( \nu : G \to \mathbb{Z}_2 \). That is, for a given \( g \in G \), \( \beta_g \) is linear when \( \nu(g) = 0 \) and anti-linear when \( \nu(g) = 1 \). Such an action has a unique extension to \( \text{Mult}(A) \), which we also denote by \( \beta \). We assume that this action is compatible with the real structure, \( \beta_g(a^\dagger) = [\beta_g(a)]^\dagger \) for all \( a \in \text{Mult}(A) \) and \( g \in G \). Such group actions may be built from a unitary/anti-unitary representation \( W \) of \( G \) on \( (\mathcal{H}, \Gamma) \) relative to \( \nu : G \to \mathbb{Z}_2 \) and such that \( \text{Ad}_{W_g}(\text{Mult}(A)) \subseteq \text{Mult}(A) \) and \( [W_g, \Gamma] = 0 \) for all \( g \in G \).

Let us now consider BdG Hamiltonians \( H_k, k \in \{0, 1\} \), satisfying assumption 4.2. If the BdG Hamiltonians satisfy assumption 4.2(b), then we furthermore assume that \( \beta_g(H_k) = H_k \) for all \( g \in G \) and \( k \in \{0, 1\} \). This then implies that \( \beta_g(J_k) = (-1)^{\nu(g)}J_k \) for all \( g \in G \), \( k \in \{0, 1\} \) and \( J_k = iH_k |H_k\rangle \langle H_k| \). If the BdG Hamiltonians satisfy assumption 4.2(a), then we assume that \( \beta_g(J_k) = (-1)^{\nu(g)}J_k \) for all \( g \in G \) and \( k \in \{0, 1\} \).

Given a \( C^* \)-algebra \( B \) with \( G \)-action \( \beta : G \to \text{Aut}(B) \), recall that a Hilbert \( C^* \)-module \( E_B \) is \( G \)-equivariant if there is an action \( \tilde{\eta} \) of \( G \) on \( E \) such that
\[
\tilde{\eta}_g(e \cdot b) = \tilde{\eta}_g(e) \cdot \beta_g(b), \quad (\tilde{\eta}_g(e_1) \tilde{\eta}_g(e_2))_b = \beta_g((e_1 \otimes e_2)_b)
\]
for all \( e, e_1, e_2 \in E_B, b \in B \) and \( g \in G \). A linear/anti-linear action on \( E_B \) extends to a linear/anti-linear action \( \eta \) on \( \text{End}_d(E) \), where for \( T \in \text{End}_d(E), \eta_g(T) e = \tilde{\eta}_g(T \tilde{\eta}_g^{-1}(e)) \) for all \( e \in E_B \) and \( g \in G \). The following is a simple check.

**Lemma 4.7.** Given \( J_0, J_1 \in \text{Mult}(A) \) and a linear/anti-linear group action \( \beta \) on \( \text{Mult}(A) \) relative to \( \nu : G \to \mathbb{Z}_2 \) as above, the Hilbert \( C^* \)-module \( (J_0 - J_1)A_\Lambda \) is \( G \)-equivariant via the action \( \tilde{\eta}_g((J_0 - J_1)a) = (-1)^{\nu(g)}(J_0 - J_1)\beta_g(a) \) for \( a \in A \) and \( g \in G \).
We note that Dom(\( \mathcal{C}_h(J_1) \)) is invariant under \( \tilde{\beta}_g \) and \( \beta_g(\mathcal{C}_h(J_1)) = (-1)^{\nu(g)}\mathcal{C}_h(J_1) \). In order to construct a \( G \)-equivariant Kasparov module, we would like \( \mathcal{C}_h(J_1) \otimes \rho \) to be \( G \)-invariant. We therefore define the following linear/anti-linear action \( \tilde{\alpha} \) on \( (J_0 - J_1)A_A \otimes \wedge^* \mathbb{C} \),

\[
\tilde{\alpha}_g ((J_0 - J_1)a \otimes w) = (-1)^{\nu(g)}(J_0 - J_1)\beta_a(a) \otimes \gamma^{\nu(g)}w
\]

(10)

for any \( a \in A, \ w \in \wedge^* \mathbb{C} \) and \( g \in G \). We therefore see that under the induced action of \( \tilde{\alpha} \),

\[
\tilde{\alpha}_g \circ (\mathcal{C}_h(J_1) \otimes \rho) \circ \tilde{\alpha}_{g^{-1}} ((J_0 - J_1)a \otimes w)
\]

\[
= (-1)^{\nu(g)}(J_0 + J_0)(\beta_a \circ \beta_{g^{-1}})(a) \otimes \gamma^{\nu(g)}\rho
\]

\[
= J_0(J_1 + J_0)a \otimes w
\]

\[
= (\mathcal{C}_h(J_1) \otimes \rho)((J_1 - J_1)a \otimes w)
\]

for any \( g \in G, \ a \in A \) and \( w \in \wedge^* \mathbb{C} \). Hence \( \mathcal{C}_h(J_1) \otimes \rho \) is \( G \)-invariant under this action.

One can similarly check that the left \( \mathcal{C}_{\ell_2,0} \)-generators on \( (J_0 - J_1)A_A \otimes \wedge^* \mathbb{C} \) are invariant under the induced action on \( \alpha \) on \( \text{End}_{A_A} ((J_0 - J_1)A_A \otimes \wedge^* \mathbb{C}) \) from \( \tilde{\alpha} \),

\[
\alpha_g(1 \otimes \gamma) = 1 \otimes \gamma, \quad \alpha_g(J_0 \otimes \rho) = (-1)^{\nu(g)}J_0 \otimes (-1)^{\nu(g)}\rho = J_0 \otimes \rho.
\]

for any \( g \in G \). We summarise our results.

**Proposition 4.8.** Let \( H_0 \) and \( H_1 \) be BdG Hamiltonians satisfying assumption 4.2 and \( J_k = \text{sgn}(H_k), \ k \in \{0,1\} \). Let \( \nu : G \rightarrow \mathbb{Z}_2 \) be a homomorphism and suppose that there is a linear/anti-linear action \( \beta \) of \( G \) on \( \text{Mult}(A) \) relative to \( \nu \) and such that \( \beta_g(J_k) = (-1)^{\nu(g)}J_k \) for all \( g \in G \) and \( k \in \{0,1\} \). Then for \( \tilde{\alpha} \) the linear/anti-linear group action on the Hilbert \( C^* \)-module given by equation (10), the triple

\[
\left( \mathcal{C}_{\ell_2,0}, (J_0 - J_1)A_A \otimes \wedge^* \mathbb{C}, \mathcal{C}_h(J_1) \otimes \rho \right)
\]

is a Real unbounded Kasparov module with Clifford generators \( \{1 \otimes J_0 \otimes \rho\} \) that restricts to a real \( G \)-equivariant unbounded Kasparov module on the elements fixed by the real structure.

Once again, the class \( \left[ \mathcal{C}_h(J_1) \right] \in K_{O^2}(A^*) \) of the unbounded Real Kasparov module from proposition 4.8 gives an obstruction to the existence of a \( G \)-equivariant path of Real skew-adjoint unitaries \( \{J_j\}_{j \in [0,1]} \subset \text{Mult}(A) \) such that \( J_i - J_0 \in A \). A bounded representative of \( \left[ \mathcal{C}_h(J_1) \right] \) can be written as

\[
\left( \mathcal{C}_{\ell_2,0}, A_A \otimes \wedge^* \mathbb{C}, F\mathcal{C}_h(J_1) \otimes \rho \right)
\]

with with Clifford generators \( \{1 \otimes J_0 \otimes \rho\} \) and group action \( \tilde{\alpha}_g(a \otimes w) = \beta_g(a) \otimes \gamma^{\nu(g)}w \) for all \( a \otimes w \in A \otimes \wedge^* \mathbb{C} \) and \( g \in G \).

**4.3. Defects and short exact sequences**

Many defects of interest in condensed matter physics such as codimension one boundaries, screw-dislocations etc have a description via a short exact sequence of observable \( C^* \)-algebras. With these examples in mind, we assume that there is a semi-split short exact sequence of \( C^* \)-algebras

\[
0 \rightarrow B \rightarrow E \xrightarrow{\phi} A \rightarrow 0.
\]

(11)
We will assume that $E$ is unital. Using that $\text{Mult}(A) = \text{End}_A(A)$, the map $\phi : E \to A$ induces a unital $^*$-homomorphism (also denoted by $\phi$), $\phi : E \to \text{Mult}(A)$. Given skew-adjoint unitaries $J_0, J_1 \in \text{Mult}(A)$, we will assume a triviality condition on $J_0$, namely that a lift $T \in \phi^{-1}(J_0)$ is invertible. Therefore, the phase $\tilde{J}_0 = T|T|^{-1}$ is a Real skew-adjoint unitary in $E$ that will act as a basepoint. We also fix a Real skew-adjoint lift $\tilde{J}_1 \in E$ of $J_1$.

Recalling proposition 3.7, let us also assume that we have Altland–Zirnbauer symmetries that are compatible with respect the short exact sequence of equation (11). That is, we assume that there are mutually anti-commuting skew-adjoint unitaries $\{\kappa_j\}_{j=1}^n$ in $E$ such that

$$\kappa_j = \kappa_j^* = -\kappa_j^0, \quad \kappa_j \tilde{J}_0 = -\tilde{J}_0 \kappa_j, \quad \kappa_j \tilde{J}_1 = -\tilde{J}_1 \kappa_j$$

for all $j \in \{1, \ldots, n\}$. Such a condition ensures that $\{\phi(\kappa_j)\}_{j=1}^n \subset \text{Mult}(A)$ act as Altland–Zirnbauer symmetries in the sense of remark 4.6.

A simple way to obtain such Altland–Zirnbauer symmetries is if there is a vector space $\mathcal{W}$ such that $\{\kappa_j\}_{j=1}^n \subset \text{End}(\mathcal{W})$ and there is a factorisation of the short exact sequence

$$0 \to B' \otimes \text{End}(\mathcal{W}) \to E' \otimes \text{End}(\mathcal{W}) \to A' \otimes \text{End}(\mathcal{W}) \to 0.$$

Assuming the condition that $J_0 - J_1 \subset A$, we can construct a bounded or unbounded Kasparov module that gives a class in $\text{KKR}(\mathcal{C}_\ell^{n+2}, A)$. Our task is to examine the image of this class under the boundary map induced from equation (11).

**Proposition 4.9.** The image of $[C_0(J_1)]$ under the boundary map $\partial : \text{KKR}(\mathcal{C}_\ell^{n+2}, A) \to \text{KKR}(\mathcal{C}_\ell^{n+1+0}, B)$ is represented by the bounded Kasparov module

$$(\mathcal{C}_\ell^{n+1+0}, B_B \otimes \bigwedge \mathcal{C}, \tilde{J}_1 \otimes \rho)$$

with $\mathcal{C}_\ell^{n+1+0}$-generators $\{1 \otimes \gamma, \kappa_1 \otimes \rho, \ldots, \kappa_n \otimes \rho\}$.

**Proof.** The class $[C_0(J_1)]$ can equivalently be represented by the van Daele $K$-theory element $[J_1 \otimes \rho] - [J_0 \otimes \rho] \in \text{DK}(A \otimes \mathcal{C}_\ell^{n+1+0})$. Taking $J_0$ and the lift $J_0$ as a basepoint, we can therefore apply proposition 2.14 to obtain the result. \(\Box\)

Let us also note the special situation with no additional symmetries.

**Proposition 4.10.** The image of $[C_0(J_1)] \in \text{KO}_2(A')$ under the boundary map $\partial : \text{KO}_2(A') \to \text{KO}_1(B')$ is given by $[e^{\pi \tilde{J}_1}]$.

**Proof.** Applying [9, proposition 5.7], the boundary map can be represented by the unbounded Real Kasparov module

$$(\mathcal{C}, \coth \left(\frac{\pi \tilde{J}_1}{2}\right) B \otimes \mathcal{C}_{\ell_0,1}\mathcal{C}_{\ell_0,3}, \tanh \left(\frac{\pi \tilde{J}_1}{2}\right) \otimes \rho).$$

Then the skew-adjoint Cayley transform $(T + 1)(T - 1)^{-1} \in \text{Inv}(\mathcal{C}, \coth(\pi \tilde{J}_1)B)$ for $T = \tanh \left(\frac{\pi \tilde{J}_1}{2}\right)$ gives an isomorphism to $\text{KO}_1(B')$ (cf [9, example 5.1]), where we see that

$$\left[\left(\tanh \left(\frac{\pi \tilde{J}_1}{2}\right) + 1\right) \left(\tanh \left(\frac{\pi \tilde{J}_1}{2}\right) - 1\right)^{-1}\right] = [e^{\pi \tilde{J}_1}] \in \text{KO}_1(B').$$

\(\Box\)
Lastly we consider the $G$-equivariant case, where the same proof as proposition 4.9 applies with the extra considerations done in section 4.2.

**Proposition 4.11.** Suppose that the short exact sequence of equation (11) is $G$-equivariant under linear/anti-linear actions on $B$, $E$ and $A$ relative to $\eta : G \to \mathbb{Z}_2$ and such that $\beta_b^g(J_k) = (-1)^{|\eta(g)|}J_k$, $k \in \{0, 1\}$. Then the image of $[C_b(J_1)] \in KK^{G}(\mathbb{C}\ell_{2,0}, A)$ under the boundary map $\partial : KK^{G}(\mathbb{C}\ell_{2,0}, A) \to KK^{G}(\mathbb{C}\ell_{1,0}, B)$ is represented by the $G$-equivariant Real Kasparov module

$$\left( \mathbb{C}\ell_{1,0}, B_b \otimes \bigwedge^* \mathbb{C}, J_1 \otimes \rho \right)$$

with Clifford generator $\mathbf{1} \otimes \gamma$ and group action $\alpha_{\gamma}(b \otimes v) = \beta_{\gamma}^b(b) \otimes \gamma^{(\eta)}w$ for all $b \in B$, $w \in \bigwedge^* \mathbb{C}$ and $g \in G$.

While the $G$-equivariant case follows quite easily from our general boundary map computations, a more careful analysis is required when the extension and defect algebras $E$ and $B$ carry different linear/anti-linear group symmetries from $A$, as is often the case when considering systems with defects.

**5. Local equivalence, $K$-homology and coarse assembly**

Our aim for this section is to connect the $K$-theoretic indices constructed in section 4 to $K$-homology in a physically meaningful way. Due to its close connection with duality theory [25, chapters 5 and 6], coarse geometry is a natural setting to consider this connection.

**5.1. Coarse geometry and indices**

We give a brief overview of Roe $C^*$-algebras and the coarse index. A more comprehensive introduction can be found in [25, 43, 44]. We consider a complex Hilbert space $\mathcal{H}$ with a real involution $\Gamma = \Gamma^* = \Gamma^{-1}$ and mutually anti-commuting skew-adjoint unitaries $\{ \kappa_i \}_{i=1}^n \subset B(\mathcal{H})$ that are invariant under the real structure $A\Gamma$ on $B(\mathcal{H})$. Now suppose there is a skew-adjoint operator $F \in B(\mathcal{H})$ such that

$$\Gamma FT = F, \quad 1 + F^2 \in \mathbb{K}(\mathcal{H}), \quad F\kappa_j = -\kappa_j F.$$  

Then $F$ is Fredholm and we construct a Real Fredholm module and $K$-homology class

$$[F] = \left[ \left( \mathbb{C}\ell_{n+1,0}, \mathcal{H} \otimes \bigwedge^* \mathbb{C}, F \otimes \rho \right) \right] \in KO^{-n-1}(\mathbb{R})$$

with Clifford generators $\{ 1 \otimes \gamma, \kappa_1 \otimes \rho, \ldots, \kappa_n \otimes \rho \}$. We can also define an ungraded Clifford module index $[\text{Ker}(F)] \in M_n/M_{n+1} \cong KO_{n+1}(\mathbb{R})$. If $[F] = [F'] \in KO^{-n-1}(\mathbb{R})$, then $[\text{Ker}(F)] = [\text{Ker}(F')] \in KO_{n+1}(\mathbb{R})$. Therefore the skew-adjoint Fredholm index can be seen as a map $KO^{-n-1}(\mathbb{R}) \to KO_{n+1}(\mathbb{R})$. The coarse assembly map generalises this basic idea.

Let $X$ be a second countable, metrizable and locally compact space with proper coarse structure. For the reader unfamiliar with coarse structures, it will suffice to consider $X$ as a second countable, locally compact and proper metric space (closed and bounded subsets of $X$ are compact). We also assume that $C_0(X)$ has a real structure $\tau$, e.g. $f(x) = f(\tau(x))$ with $\tau$ an order-2 automorphism on $X$. Fix a Hilbert space $\mathcal{H}$ and real involution $\Gamma$ and suppose there is
a non-degenerate Real representation \( \varphi : C_0(X) \to B(\mathcal{H}) \), \( \varphi(f^*) = \text{Ad}_T \circ \varphi(f) \). We say that \( \varphi \) is ample if no non-zero element of \( C_0(X) \) acts compactly on \( \mathcal{H} \).

**Example 5.1.** Let \( M \) be a complete Riemannian manifold and \( V \) a complex Hermitian vector bundle. Suppose that \( C_0(M) \) has a real structure \( f(x)^T = \overline{f(\tau(x))} \) with \( \tau \) an order-2 automorphism on \( M \). Then taking \( \mathcal{C} \) pointwise complex conjugation on \( V \), we can define a real structure on the space of \( L^2 \)-sections \( L^2(M, V) \) by \( \psi(x)^T = \mathcal{C} \circ \psi(\tau(x)) \) for all \( \psi \in L^2(M, V) \). The multiplication representation of \( C_0(M) \) on \( L^2(M, V) \) is then a Real and ample representation. To see this, we remark that the spectrum of \( \phi(f) \) is the closure of \( \text{Ran}(f) \), which is a non-discrete set for \( f \) non-zero and \( M \) complete.

**Definition 5.2.** Let \( T \in B(\mathcal{H}) \) and \( \varphi : C_0(X) \to B(\mathcal{H}) \) a representation.

(a) We say that \( T \) is pseudolocal with respect to \( \varphi : C_0(X) \to B(\mathcal{H}) \) if \( \varphi(f_1)T\varphi(f_2) \) is compact for all \( f_1, f_2 \in C_0(X) \) such that \( f_1 \) or \( f_2 \) have compact support and \( \text{supp}(f_1) \cap \text{supp}(f_2) = \emptyset \).

(b) We say that \( T \) has finite propagation (or \( T \) is controlled) with respect to \( \varphi : C_0(X) \to B(\mathcal{H}) \) if there exists an \( R > 0 \) such that \( \varphi(f_1)T\varphi(f_2) = 0 \) for any \( f_1, f_2 \in C_0(X) \) such that \( d(\text{supp}(f_1), \text{supp}(f_2)) > R \).

The \( \mathcal{C}^* \)-subalgebra of \( B(\mathcal{H}) \) generated operators that are pseudolocal and have finite propagation with respect to \( \varphi : C_0(X) \to B(\mathcal{H}) \) is denoted by \( D^*(X) \).

The real involution \( \Gamma \) on \( \mathcal{H} \) gives a real structure \( \text{Ad}_\Gamma \) on \( D^*(X) \) that makes it a Real \( \mathcal{C}^* \)-algebra.

**Lemma 5.3 (Kasparov’s lemma, [25, lemma 5.4.7]).** A bounded operator \( T \in B(\mathcal{H}) \) is pseudolocal with respect to \( \varphi : C_0(X) \to B(\mathcal{H}) \) if and only if \( [T, \varphi(f)] \in \mathbb{K}(\mathcal{H}) \) for all \( f \in C_0(X) \).

**Definition 5.4.** We say that \( T \in B(\mathcal{H}) \) is locally compact with respect to \( \varphi : C_0(X) \to B(\mathcal{H}) \) if for all \( f \in C_0(X) \), \( \varphi(f)T \) and \( T\varphi(f) \) are compact. We define the Roe algebra \( \mathcal{C}^*(X) = \{ T \in D^*(X) : T \text{ locally compact} \} \).

It follows from the definition that \( \mathcal{C}^*(X) \) is a closed two-sided ideal in \( D^*(X) \). Note also that the definition of \( D^*(X) \) and \( \mathcal{C}^*(X) \) rely on a choice of Hilbert space \( \mathcal{H} \) and representation \( \varphi \). If \( \varphi_1 \) and \( \varphi_2 \) are ample representations, then \( \mathcal{C}^*(X, \varphi_1) \cong \mathcal{C}^*(X, \varphi_2) \) [20, theorem 1].

If \( X \) is discrete and we take \( \varphi : C_0(X) \to L^2(X) \otimes \mathbb{C}^n \) as \( \varphi(f)(\psi(x) \otimes v) = f(x)(\psi(x) \otimes v) \) for \( \psi \otimes v \in L^2(X) \otimes \mathbb{C}^n \), then this representation is not ample. In this setting, a natural object of study is the uniform Roe algebra \( \mathcal{C}^u(X) \), which embeds in \( \mathcal{C}^*(X) \).

Because we consider both the discrete and continuous settings, we will work primarily with \( \mathcal{C}^*(X) \) with the knowledge that \( \mathcal{C}^u(X) \) can be embedded in this algebra if \( X \) is discrete. At the level of \( K \)-theory, \( KO_*(\mathcal{C}^u(X)^c) \) is much richer than \( KO_*(\mathcal{C}^*(X)^c) \), though the \( K \)-theory of \( \mathcal{C}^*(X) \) will still capture the large scale properties of \( X \). See [20, 35] for further discussion on this point.

Let us now consider the coarse index, which is a map \( KO^{-}(\mathcal{C}_0(X)^c) \to KO_*(\mathcal{C}^*(X)^c) \). We first note the following.

**Proposition 5.5.** Suppose that \( F = F^* = -F^\ast \in B(\mathcal{H}) \) and assume one of the following:

(a) There is an ample Real representation \( \varphi : C_0(X) \to B(\mathcal{H}) \) and \( \{ \kappa_j \}_{j=1}^n \subset B(\mathcal{H}) \) are mutually anti-commuting Real skew-adjoint unitaries with infinite-dimensional eigenspaces
such that for all \( f \in C_0(X) \) and \( j = 1, \ldots, n \),
\[
[F, \varphi(f)], \quad \varphi(f)(1 + F^2), \quad [\kappa_j, \varphi(f)] \in \mathcal{H}, \quad F\kappa_j = -\kappa_jF.
\]

(b) There is a Real representation \( \varphi : C_0(X) \to \mathcal{B}(\mathcal{H}) \) and \( \{\kappa_j\}_{j=1}^n \subset C_0(X) \) are mutually anti-commuting Real skew-adjoint unitaries such for all \( f \in C_0(X) \) and \( j = 1, \ldots, n \),
\[
[F, \varphi(f)], \quad \varphi(f)(1 + F^2) \in \mathcal{H}, \quad [\kappa_j, \varphi(f)] = 0, \quad F\kappa_j = -\kappa_jF.
\]

Then the triple
\[
\left( C_0(X) \otimes \mathbb{C} \ell_{n+1,0}, \varphi \mathcal{H} \otimes \bigwedge^\bullet \mathbb{C}, F \otimes \rho \right)
\]
is a Real Fredholm module (Real \( C_0(X) \)-\( \mathbb{C} \) Kasparov module) with Clifford generators \( \{1 \otimes \gamma, \kappa_1 \otimes \rho, \ldots, \kappa_n \otimes \rho\} \).

**Proof.** Given condition (b), the result holds immediately from the definition of a Real Fredholm module. For condition (a), we need to check that the specified representation of \( C_0(X) \otimes \mathbb{C} \ell_{n+1,0} \) is well-defined. Because \( \varphi \) is ample, \( \varphi(f) = q \circ \varphi(f) \) for \( q : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})/\mathbb{K}(\mathcal{H}) \) the quotient onto the Calkin algebra. For \( f \in \{1, \ldots, n\} \), the skew-adjoint unitary \( \kappa_j \) has infinite-dimensional eigenspaces, which implies that \( q(\kappa_j) = \kappa_j \). We therefore have that
\[
[\varphi(f), \kappa_j] = [q \circ \varphi(f), q(\kappa_j)] = q ([\varphi(f), \kappa_j]) = 0, \quad \text{for all } j \in \{1, \ldots, n\}.
\]
Therefore the representation of \( C_0(X) \otimes \mathbb{C} \ell_{n+1,0} \) is indeed well-defined. The remaining conditions required to obtain a Real Fredholm module are immediate from the assumptions. \( \square \)

Given the setting of proposition 5.5, we let \( [F] \) denote the corresponding class in \( \text{KKR}(C_0(X) \otimes \mathbb{C} \ell_{n+1,0}, \mathbb{C}) \cong \text{KO}^{-n-1}(C_0(X))^\gamma \). When \( f(x)^{\mathbb{C}} = f(x) \), we have that \( \text{KO}^{-n}(C_0(X)^{\mathbb{C}}) \cong \text{KO}_n(X) \), the topological \( K \)-homology of the space \( X \).

Let us now define the coarse index or coarse assembly map. Assume the setting of proposition 5.5. If \( F \in \mathcal{B}(\mathcal{H}) \) does not have finite propagation with respect to \( \varphi : C_0(X) \to \mathcal{B}(\mathcal{H}) \), take a Real partition of unity \( \{\eta_i\} \) of \( X \) subordinate to a locally finite open cover and define
\[
F' = \sum_i \eta_i^{1/2} F \eta_i^{1/2},
\]
which converges in the strong topology. The operator \( F' \) also determines a Real Fredholm module and \( [F] = [F'] \in \text{KO}^{-n-1}(C_0(X)) \) [25, page 354]. Therefore \( F' \) is an element of \( D^*(X) \) such that \( (F')^\gamma + 1 \in C^*(X) \). Hence \( q(F') \) is a skew-adjoint unitary in the quotient \( D^*(X)/C^*(X) \). In particular, \( q(F') \otimes \rho \in D^*(X)/C^*(X) \otimes \mathbb{C} \ell_{0,1} \) is an odd self-adjoint unitary that anti-commutes with the odd self-adjoint unitaries \( \{q(\kappa_j) \otimes \rho\}_{j=1}^n \). Recalling remark 2.12 and fixing a basepoint odd Real self-adjoint unitary \( e \in D^*(X)/C^*(X) \otimes \mathbb{C} \ell_{0,1} \) that anti-commutes with \( \{q(\kappa_j) \otimes \rho\}_{j=1}^n \), we therefore obtain a class in the van Daele \( K \)-theory group \( [q(F') \otimes \rho] \in \text{DKR}(D^*(X)/C^*(X) \otimes \mathbb{C} \ell_{0,n+1}) \). Following Roe [45, 46], the coarse index \( \mu_X : \text{KO}^{-n-1}(C_0(X)^{\mathbb{C}}) \to \text{KO}_{n+1}(C^*(X)^{\mathbb{C}}) \) can be defined by the composition
\[
\text{KO}^{-n-1}(C_0(X)^{\mathbb{C}}) \to \text{DKR}(D^*(X)/C^*(X) \otimes \mathbb{C} \ell_{0,n+1}) \overset{\text{adj}}{\longrightarrow} \text{KKR}(\mathbb{C} \ell_{n+1,0}, C^*(X))
\]
where we have identified $KKR(\mathbb{C}[\ell^1, C^\ast(X)])$ with $KO_{n+1}(C^\ast(X))$ and $\mathcal{C} \circ \partial$ is the boundary map in van Daele K-theory composed of the equivalence between $DK$ and $KKR$. This map can also be defined in the equivariant setting (see sections 5.5 and 5.6 below).

**Example 5.6.** Suppose that $X$ is a second countable, metrizable and compact space with proper coarse structure. Then $C^\ast(X) \cong \mathbb{K}(\mathcal{H})$ and the assembly map $\mu_X : KO^{-n-1}(C(X)^\ast) \to KO_{n+1}(C^\ast(X))$ reduces to the Clifford module index map $KO_{n+1}(\mathbb{R}) \cong \mathbb{R}$ considered at the beginning of this section.

5.2. **Locally equivalent ground states**

Let us now fix a Real non-degenerate representation $\varphi : C_0(X) \to \mathcal{B}(\mathcal{H})$. We are interested in ground states on $A^n_{\text{eq}}(\mathcal{H}, \Gamma)$ with respect to the quasi-free action $\alpha : \mathbb{R} \to \text{Aut}(A^n_{\text{eq}}(\mathcal{H}, \Gamma))$ such that $\alpha_t(\sigma(v)) = e^{i\theta t} \sigma(v)$ for $t \in \mathbb{R}, v \in \mathcal{H}$ and $H = H^* = -\Gamma H^\ast$ a BdG Hamiltonian. We will now restrict to BdG Hamiltonians and dynamics that are local with respect to the representation $\varphi$.

**Assumption 5.7.** We consider BdG Hamiltonians $H = H^* = -\Gamma H^\ast$ such that $0 \notin \sigma(H)$.

- (Unbounded case) if $H$ is unbounded, we assume that $\chi(H) \in \text{Mult}(C^\ast(X))$ for any normalising function $\chi$ and $\varphi(f), i\text{sgn}(H) \in \mathbb{K}(\mathcal{H})$ for all $f \in C_0(X)$.
- (Bounded case) if $H$ is bounded, we assume that $H$ is invertible in $D^\ast(X)$.

If $\chi(H) \in D^\ast(X)$ for any regularising function $\chi$, then assumption 5.7(a) is satisfied. Our slightly weaker condition allows us to accommodate Hamiltonians with higher-order terms, see example 5.10.

The functional calculus gives us the following.

**Lemma 5.8.** Let $H \in \mathcal{B}(\mathcal{H})$ be gapped and pseudolocal with respect to $\varphi : C_0(X) \to \mathcal{B}(\mathcal{H})$. Then for any regularising function $\chi$, $\chi(H), J = iH[H]^\ast$ and $P = \chi(0, \infty)(H)$ are pseudolocal with respect to $\varphi : C_0(X) \to \mathcal{B}(\mathcal{H})$.

**Example 5.9 (bounded/discrete examples).** Let $\Lambda$ be a proper and discrete metric space, e.g. $\Lambda = \mathbb{Z}^d$. In this case, $C_0(\Lambda)$ acts naturally on $\mathcal{V} = \ell^2(\Lambda) \otimes \mathbb{C}^\ast$, which we can extend to $\mathcal{H} = \mathcal{V} \otimes \mathbb{C}^\ast$ with real involution $\mathcal{C}(1 \otimes \sigma_1)$. We consider BdG Hamiltonians as in equation (6),

$$H = \begin{pmatrix} \hbar & \delta \\ \delta^\ast & -\tau \end{pmatrix}, \quad h, \delta \in \mathcal{B}[\ell^2(\Lambda) \otimes \mathbb{C}^\ast], \quad \tau = \mathcal{C} \Lambda \mathcal{C}, \quad \delta^\ast = -\delta.$$

The representation $\varphi : C_0(\Lambda) \to \mathcal{B}(\ell^2(\Lambda) \otimes \mathbb{C}^\ast)$ can be decomposed into a sum of terms of $\{\delta_x\}_{x \in \Lambda}$, the projection onto the site $\{x\} \otimes \mathbb{C}^\ast$. With this in mind, we impose the following conditions for $T = h$ or $\delta$:

- There exists $R > 0$ such that $p_x T p_x = 0$ if $d(x, y) > R$.
- The operators $p_x T$ and $T p_x$ are compact for any $x \in \Lambda$.

If these conditions are satisfied for $h$ and $\delta$, then $H$ is an element of the uniform Roe algebra, $C_0^\ast(\Lambda)$, which can be naturally embedded in $C^\ast(\Lambda)$. In particular, if $H$ is invertible in $C_0^\ast(\Lambda)$, then it easily follows that $\chi(H)$ and $iH[H]^\ast$ are elements in $C_0^\ast(\Lambda) \subset D^\ast(\Lambda)$ for any normalising function $\chi$. As such, for two BdG Hamiltonians $H_0, H_1$ satisfying these conditions, we obtain that $H_0 - H_1$, $J_0 - J_1 \in C^\ast(\Lambda)$.

**Example 5.10 (unbounded/continuous examples).** Let $M$ be a complete Riemannian manifold and consider the multiplication representation of $C_0(M)$ acting diagonally on
\[ \mathcal{H} = L^2(M, C^n) \otimes C^2 \] with real involution \( \mathcal{C}(1 \otimes \sigma_1) \) and \( \mathcal{C} \) complex conjugation. As before, we restrict to gapped BdG Hamiltonians of the form \( H = \begin{pmatrix} h & \delta \\ \delta^* & -\frac{h}{\hbar} \end{pmatrix} \) with \( \delta^* = -\frac{\pi}{\hbar} \). In typical examples, \( h \) and \( \delta \) are differential operators (cf example 3.3).

To apply our general framework, we would like to find gapped BdG Hamiltonians such that \( J = i \text{sgn}(H) \) is pseudolocal and an element of \( \text{Mult}(C^*(M)) \). We also wish to consider pairs of BdG Hamiltonians such that \( J_0 - J_1 \in C^*(M) \). We will examine these conditions separately.

To show \( J = i \text{sgn}(H) \) is pseudolocal, some care needs to be taken if \( H \) is a second or higher-order differential operator as the commutator \([H, \varphi(f)]\) will generally not be bounded for \( f \in C_c^\infty(M) \). Such BdG Hamiltonians will not give rise to a spectral triple (unbounded \( C_0(M) - \mathbb{C} \) Kasparov module), whose properties can then be used to show that \( \text{sgn}(H) \) is pseudolocal. However, following [22, appendix A] and fixing \( \varepsilon > 0 \), we ask for BdG Hamiltonians to have \( \varepsilon \)-bounded commutators in the sense that for all \( f \in C_c^\infty(M) \),

(a) \( \varphi(f)\text{Dom}(H) \subset \text{Dom}(H) \),

(b) The operators \([H, \varphi(f)](1 + H^2)^{-\varepsilon/2} \) and \((1 + H^2)^{-\varepsilon/2}[H, \varphi(f)]\) extend to bounded operators on \( \mathcal{H} \).

The basic intuition behind the parameter \( \varepsilon \) is as follows. Suppose that \( H \) is an order-\( m \) differential operator and take \( f \in C_c^\infty(M) \). Then the commutator \([H, \varphi(f)]\) will have order \( m - 1 \). Therefore \([H, \varphi(f)](1 + H^2)^{-\varepsilon/2} \) and \((1 + H^2)^{-\varepsilon/2}[H, \varphi(f)]\) are order-0 pseudodifferential operators and so are bounded on \( L^2(M, C_{\mathfrak{m}}) \). Hence \( H \) is \( \varepsilon \)-bounded with \( \varepsilon = \frac{1}{m} \).

More generally, the parameter \( \varepsilon^{-1} \) can be thought of as the order of the operator \( H \).

For BdG Hamiltonians with \( \varepsilon \)-bounded commutators and whose resolvent is locally compact with respect to \( \varphi \), the bounded transform \( F_H = (1 + H^2)^{-1/2} \) is such that the triple

\[ (C_0(M) \otimes \mathcal{C}_{\ell^1,0}, \mathcal{H} \otimes \bigwedge \mathcal{C}, iF_H \otimes \rho) \]

will give a Real Fredholm module [22, theorem A.6]. Furthermore, because \( H \) is invertible, \([0, 1] \ni t \mapsto F_t = H((1 - t) + H^2)^{-1/2} \) is a norm-continuous path between \( F_H \) and \( \text{sgn}(H) \). Therefore for any \( f \in C_0(M) \), \([\varphi(f), \text{sgn}(H)]\) is a norm-limit of compact operators and so is also compact. Hence \( J = i \text{sgn}(H) \) is pseudolocal with respect to \( \varphi \).

Let us now consider the condition that \( J \in \text{Mult}(C^*(M)) \) for BdG Hamiltonians \( H = \begin{pmatrix} h & \delta \\ \delta^* & -\frac{h}{\hbar} \end{pmatrix} \). If \( \delta \) is a first-order differential operator and \( h = \Delta_M - \mu \) with \( \Delta_M \) the Laplace–Beltrami operator, it is in general quite non-trivial to show that \( J \in \text{Mult}(C^*(M)) \).

In the case that \( M = \mathbb{R}^d \), the operator \( \Delta_{\text{def}} + V \) is affiliated to the Roe algebra \( C^*(\mathbb{R}^d) \) for any \( V \in L^\infty(\mathbb{R}^d) \) [20, proposition 1]. The first-order coupling term \( \delta \) will also be affiliated to \( C^*(\mathbb{R}^d) \). Therefore, provided there is sufficient compatibility with \( h \) and \( \delta \) (e.g. \( h \) and \( \delta \) commute) and \( H \) is gapped, \( H^{-1} \) will then be a matrix over \( \text{Mult}(C^*(\mathbb{R}^d)) \). Ignoring matrix degrees of freedom, we then obtain that \( J \in \text{Mult}(C^*(\mathbb{R}^d)) \). For the case of more general \( M \) we will simply assume that \( J \in \text{Mult}(C^*(M)) \).

Given a pair of BdG Hamiltonians \( H_0 \) and \( H_1 \) such that \( J_0 \) and \( J_1 \) are pseudolocal elements in \( \text{Mult}(C^*(M)) \), we further assume that \( \text{Dom}(H_0) = \text{Dom}(H_1) \) and \( \varphi(f)(H_0 - H_1) \) \((i + H_0)^{-1} \in \mathbb{H}(H) \) for all \( f \in C_0(M) \). Then \( J_0 - J_1 \in C^*(M) \) and we are in the setting of section 4. In more general examples, the framework of [12] provides technical conditions on \( H_0 \) and \( H_1 \) so that \( J_0 - J_1 \in C^*(M) \).
Definition 5.11.

(a) We say that the representation $\varphi : C_0(X) \to B(H)$ is locally compatible with $\Gamma$ if $\Gamma$ restricts to a real involution on $\text{Ran}(\varphi(f)) \subset H$ for all $f \in C_0(X)$.

(b) We say that two pure and quasifree states $\omega_0$ and $\omega_1$ of $A_\text{al}(H, \Gamma)$ are locally equivalent with respect to $(X, \varphi)$ if $\varphi$ is locally compatible with $\Gamma$ and there is a dense $^*$-algebra $A \subset C_0(X)$ such that $\omega_0$ is equivalent to $\omega_1$ as a state on $A_\text{al}(\text{Ran}(\varphi(f)), \Gamma)$ for all $f \in A$.

A physically reasonable choice for a dense $^*$-subalgebra $A \subset C_0(X)$ is $C_c(X)$, the algebra of compactly supported functions. In our examples of interest, $C_0(X)$ acts diagonally on $H = \mathcal{V} \otimes \mathbb{C}^2$ with $\Gamma = \mathcal{E}(1 \otimes \sigma_1)$. Letting $\Pi_{\varphi(f)}$ denote the projection onto $\text{Ran}(\varphi(f))$, $\Pi_{\varphi(f)} \Gamma = \Gamma \Pi_{\varphi(f)}$ and $\varphi$ is locally compatible with $\Gamma$ when $\mathcal{E}$ is pointwise complex conjugation for example. Recalling theorem 3.1, the basis projections $P_0$ and $P_1$ will give locally equivalent quasifree states $\omega_0$ and $\omega_1$ if and only if $\Pi_{\varphi(f)}(P_0 - P_1) \Pi_{\varphi(f)}$ is Hilbert–Schmidt for all $f \in A$.

Remarks 5.12.

(a) If $\omega_0$ is unitarily equivalent to $\omega_1$, then it is locally equivalent for any $(X, \varphi)$ as in definition 5.11.

(b) If the space $X$ is compact, then our notion of local equivalence reduces to equivalence of the states $\omega_0$ and $\omega_1$.

(c) If the representation $\varphi : C_0(X) \to B(H)$ commutes with the BdG Hamiltonians, $\varphi(f)H_k = H_k \varphi(f)$ for $k \in \{0, 1\}$, then $\omega_0$ and $\omega_1$ are locally equivalent if and only if they are equivalent.

(d) A similar notion of local quasi-equivalence is defined for states of nets of $C^*$-algebras $\mathcal{O} \to \mathfrak{A}(\mathcal{O})$ that appear in algebraic quantum field theory, see [14, 38] for example. Here we work with a different class of spaces.

Example 5.13. Let us revisit the case of a discrete and proper metric space $\Lambda$ from example 5.9. Take two invertible BdG Hamiltonians $H_0, H_1 \in C^*_c(\Lambda) \subset C^*(\Lambda)$ acting on $\ell^2(\Lambda, \mathbb{C}^n) \otimes \mathbb{C}^2$ and $\Gamma = \mathcal{E}(1 \otimes \sigma_1)$ with $\mathcal{E}$ component-wise complex conjugation. We can consider $C_c(\Lambda)$ as a dense $^*$-subalgebra of $C_0(\Lambda)$. In particular, any function $f \in C_c(\Lambda)$ will be supported on a finite set $Y \subset \Lambda$. Hence, the restriction of $H_0$ and $H_1$ to $\text{Ran}(\varphi(f))$ is the restriction to $\ell^2(Y) \otimes \mathbb{C}^{2n} \cong \mathbb{C}^{2n} \otimes \mathbb{C}^{2n}$. Because we are now in a finite-dimensional Hilbert space, all pure states are unitarily equivalent to each other. Therefore we see that in discrete examples, local equivalence is satisfied without issue.

Proposition 5.14. If the gapped ground states $\omega_0$ and $\omega_1$ are locally equivalent with respect to $(X, \varphi)$, then $\varphi(f)(J_0 - J_1) \in \mathbb{K}(H)$ for all $f \in C_0(X)$, i.e. $J_0 - J_1 \in C^*(X)$.

Proof. Choose $f \in C_0(X)$ with an approximating sequence $f_n \in A$. Without loss of generality, we can assume that $f$ is real-valued. As $\omega_0$ and $\omega_1$ are locally equivalent with respect to $(X, \varphi)$, $\varphi(f_n)(P_0 - P_1)\varphi(f_n)$ maps the unit ball of $H$ to a precompact set. Hence it is compact and so is $\varphi(f_n)(J_0 - J_1)\varphi(f_n) = 2i\varphi(f_n)(P_0 - P_1)\varphi(f_n)$.

We will first show that $|\varphi(f)(J_0 - J_1)| \in \mathbb{K}$, where we compute

\[
(\varphi(f)(J_0 - J_1))^* \varphi(f)(J_0 - J_1) = -(J_0 - J_1)\varphi(f)^2(J_0 - J_1)
\]

\[
= -\varphi(f)(J_0 - J_1)\varphi(f)(J_0 - J_1) + \mathbb{K}
\]

\[
= \lim_{n \to \infty} -\varphi(f_n)(J_0 - J_1)\varphi(f_n)(J_0 - J_1) + \mathbb{K},
\]
where in the second line we used that \([\varphi(f), (\mathcal{J}_0 - \mathcal{J}_1)] \in \mathbb{K}\) as \(\mathcal{J}_0 - \mathcal{J}_1\) is pseudolocal with respect to \(\varphi : C_0(\mathcal{X}) \rightarrow \mathcal{B}(\mathcal{H})\). Hence we have a limit of compact operators which is compact. Because the compact operators are closed under square root, this implies that \(|\varphi(f)(\mathcal{J}_0 - \mathcal{J}_1)| \in \mathbb{K}\). By the polar decomposition, \(\varphi(f)(\mathcal{J}_0 - \mathcal{J}_1)\) is compact. □

5.3. Local equivalence to K-homology class

Building from our work in the previous subsection, we will extend assumption 5.7 and consider pairs of BdG Hamiltonians that are local with respect to the Real representation \(\varphi : C_0(\mathcal{X}) \rightarrow \mathcal{B}(\mathcal{H})\).

**Assumption 5.15.** We assume that the BdG Hamiltonians \(\mathcal{H}_0, \mathcal{H}_1\) are such that for all \(k \in \{0, 1\}\),

(a) \(\not\in \sigma(\mathcal{H}_k)\) and \(\mathcal{H}_k = H_k^* = -\Gamma H_k^\Gamma\),

(b) \(\chi(\mathcal{H}_k) \in \text{Mult}(C^*(\mathcal{X}))\) for any normalising function \(\chi\) and \(J_k = \text{sgn}(\mathcal{H}_k)\) is pseudolocal with respect to \(\varphi : C_0(\mathcal{X}) \rightarrow \mathcal{B}(\mathcal{H})\),

(c) \(\varphi(f)(\mathcal{J}_0 - \mathcal{J}_1) \in \mathbb{K}(\mathcal{H})\) for all \(f \in C_0(\mathcal{X}); \text{i.e. } \mathcal{J}_0 - \mathcal{J}_1 \in C^*(\mathcal{X})\).

If \(\mathcal{H}_0\) and \(\mathcal{H}_1\) are bounded, we assume they are invertible as elements of \(D^*(\mathcal{X})\).

**Remarks 5.16.**

(a) By proposition 5.14, pairs of invertible BdG Hamiltonians \(\mathcal{H}_0, \mathcal{H}_1\) satisfying assumption 5.7 and whose ground states are locally equivalent will satisfy the conditions of assumption 5.15.

(b) If \(\mathcal{X}\) is a compact space, then \(C^*(\mathcal{X}) = \mathbb{K}(\mathcal{H})\) and we recover the setting of section 3.

We now use assumption 5.15 and the relative Cayley transform \(C_{\mathcal{H}_0}(\mathcal{J}_1)\) to construct a K-homology class from the pair of gapped ground states \(\omega_0\) and \(\omega_1\).

**Proposition 5.17.** Suppose that \(\mathcal{H}_0\) and \(\mathcal{H}_1\) satisfy assumption 5.15. Further assume one of the following:

(a) The Real representation \(\varphi : C_0(\mathcal{X}) \rightarrow \mathcal{B}(\mathcal{H})\) is ample,

(b) The Hamiltonian \(\mathcal{H}_0\) is trivially local with respect to \((\mathcal{X}, \varphi)\) in the sense that \([\varphi(f), \mathcal{J}_0] = 0\) for all \(f \in C_0(\mathcal{X})\).

Then the triple

\[
\left( C_0(\mathcal{X}) \otimes \mathcal{C}l_{2,0}, \varphi, \mathcal{H} \otimes \bigwedge^* \mathbb{C}, F_{C_{\mathcal{H}_0}(\mathcal{J}_1)} \otimes \rho \right)
\]

is a Real Fredholm module with Clifford generators \(\{1 \otimes \gamma, \mathcal{J}_0 \otimes \rho\}\) and \(F_{C_{\mathcal{H}_0}(\mathcal{J}_1)}\) the bounded transform of the Cayley transform \(C_{\mathcal{H}_0}(\mathcal{J}_1) = \mathcal{J}_0(\mathcal{J}_1 + \mathcal{J}_0)(\mathcal{J}_1 - \mathcal{J}_0)^{-1}\).

**Proof.** We first consider case (a). Because \(\mathcal{J}_0\) has infinitely degenerate \(\pm i\) eigenspaces, it generates an ample representation of the ungraded Clifford algebra \(\mathcal{C}l_{0,1}\) on \(\mathcal{H}\). Therefore, denoting \(q : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})/\mathbb{K}(\mathcal{H})\) the quotient map, \([\varphi(f), \mathcal{J}_0] = q([\varphi(f), \mathcal{J}_0]) = 0\) for all \(f \in C_0(\mathcal{X})\) as \(\varphi\) is ample and \([\varphi(f), \mathcal{J}_0]\) is compact. Hence the representation of \(C_0(\mathcal{X}) \otimes \mathcal{C}l_{2,0}\) is well-defined.

Recall the bounded transform \(F_{C_{\mathcal{H}_0}(\mathcal{J}_1)} = \frac{1}{2} \mathcal{J}_0(\mathcal{J}_1(\mathcal{J}_0 - 1)(\mathcal{J}_0 \mathcal{J}_1 + 1))^{-1}(2 + \mathcal{J}_0 \mathcal{J}_1 + \mathcal{J}_1 \mathcal{J}_0)^1/2\) and its basic properties from equations (4) and (5) on page 10. Because \(\mathcal{J}_1 \mathcal{J}_0\) is normal, we can write \(F_{C_{\mathcal{H}_0}(\mathcal{J}_1)} = J_0 \eta(\mathcal{J}_1 \mathcal{J}_0, (\mathcal{J}_1(\mathcal{J}_1)^0))\) with \(\eta(z, \overline{z}) = \frac{1}{2}(z - 1)(\overline{z} + 1)^{-1}(2 + z + \overline{z})^{1/2}\) a continuous function on the relevant domain. Because \(\mathcal{J}_0\) and \(\mathcal{J}_1\) have compact commutators with \(\varphi(f)\),
Given the Real representation \( \varphi \) and the assumption 5.15, we assume that we are in one of the following settings:

(a) \( \varphi \) is ample,

(b) \([\varphi(f), J_0] = 0 \) for all \( f \in C_0(X) \).

In either setting, we obtain a \( K \)-homology element \( [F_{C(J_1)}] \in KO^{-2}(C_0(X)^*) \) by proposition 5.17. Because \( J_0, J_1 \in \text{Mult}(C^*(X)) \) with \( J_0 - J_1 \in C^*(X) \), we can also consider the \( K \)-theory elements constructed in section 4.1 for \( A = C^*(X) \). Namely, by proposition 2.8, we have the class

\[
[C_{J_0}(J_1)] = \left[ \left( \mathbb{C} \langle \ell, J_0 - J_1 \rangle C_0(X)^* \right) \otimes \Lambda, [C_{J_0}(J_1) \otimes \rho] \right] \in KO_{2}(C^*(X)^2),
\]

which by proposition 4.5 is equivalent to the relative van Daele \( K \)-theory element

\[
[J_1 \otimes \rho] - [J_0 \otimes \rho] \in DK(\text{Mult}(C^*(X)) \otimes \mathbb{C} \ell_{0,1}, \text{Mult}(C^*(X)) / C^*(X) \otimes \mathbb{C} \ell_{0,1}),
\]

where we recall that \( DK(A, A/I) \cong DK(I) \).

Our task is to relate the \( K \)-homology element \( [F_{C(J_1)}] \) to the \( K \)-theory classes in equations (12) and (13). To do this, we will use the coarse assembly map \( \mu_X : KO^{-2}(C_0(X)^*) \rightarrow KO_{4}(C^*(X)^5) \).

**Theorem 5.19.** The coarse assembly map \( \mu_X : KO^{-2}(C_0(X)^*) \rightarrow KO_{4}(C^*(X)^5) \) is such that \( \mu_X \left( [F_{C(J_1)}] \right) = [C_{J_0}(J_1)] \) with \( [F_{C(J_1)}] \) the \( K \)-homology class from proposition 5.17 and \( [C_{J_0}(J_1)] \) the \( K \)-theory class from equation (12).

**Proof.** We use a duality theory approach to the assembly map as developed by Roe [45, 46]. By the naturality of the long-exact sequence in \( K \)-theory (including van Daele \( K \)-theory), it
suffices to consider the assembly map via the boundary map that arises from the short exact sequence

\[ 0 \to C^*(X) \to \text{Mult}(C^*(X)) \to \text{Mult}(C^*(X))/C^*(X) \to 0 \]

rather than the short exact sequence from the ideal \( C^*(X) \subset D^*(X) \) [44, proposition 5.11]. Let \( \mathcal{Q}(C^*(X)) = \text{Mult}(C^*(X))/C^*(X) \) and \( q : \text{Mult}(C^*(X)) \to \mathcal{Q}(C^*(X)) \) the quotient map. Proposition 5.17 implies that \( q(C_{\phi_t}(J_1)) \otimes \rho \in \mathcal{Q}(C^*(X)) \otimes \mathbb{C}_{\ell_{0,1}} \) is an odd self-adjoint unitary that anti-commutes with \( q(J_0) \otimes \rho \). Let us now fix a reference odd self-adjoint unitary \( e \in \mathcal{Q}(C^*(X)) \otimes \mathbb{C}_{\ell_{0,1}} \) that lifts to an odd self-adjoint unitary in \( \text{Mult}(C^*(X)) \otimes \mathbb{C}_{\ell_{0,1}} \) that anti-commutes with \( J_0 \otimes \rho \). Then \( [F_{C_{\phi_t}(J_1)} \otimes \rho] \in DK_*(\mathcal{Q}(C^*(X)) \otimes \mathbb{C}_{\ell_{0,2}}) \), where the degree shift is because \( F_{C_{\phi_t}(J_1)} \otimes \rho \) and \( e \) anti-commute with \( J_0 \otimes \rho \) (cf. remark 2.12). The coarse assembly map is given by the composition

\[ KO^{-2}(C_0(X)^5) \to DK_*(\mathcal{Q}(C^*(X)) \otimes \mathbb{C}_{\ell_{0,2}}) \xrightarrow{\partial} DK_*(C^*(X) \otimes \mathbb{C}_{\ell_{0,1}}) \xrightarrow{q} KO_2(C^*(X)^5), \]

where the first map is given by \( [F_{C_{\phi_t}(J_1)}] \mapsto [q(F_{C_{\phi_t}(J_1)}) \otimes \rho] \) and \( \partial \) is the boundary map in van Daele \( K \)-theory. Applying proposition 2.14, the boundary map composed with the equivalence between \( DK \) and \( KKR \) is represented by

\[
\partial \left( [q(F_{C_{\phi_t}(J_1)}) \otimes \rho] \right) = \left( \mathbb{C}_{\ell_{1,0}}, (C^*(X) \otimes \mathbb{C}_{\ell_{0,1}})_{\mathbb{C}^\infty} \otimes \mathbb{C}_{\ell_{0,1}}, F_{C_{\phi_t}(J_1)} \otimes \rho \right) = \left( \mathbb{C}_{\ell_{2,0}}, C^*(X) \otimes \mathbb{C}_{\ell_{0,1}} \otimes \bigwedge^\infty \mathbb{C}, F_{C_{\phi_t}(J_1)} \otimes \rho \right)
\]

with left Clifford generators \( J_0 \otimes \rho \) and \( \{J_0 \otimes \rho, 1 \otimes \gamma \} \) in the first and second lines respectively. Recalling proposition 2.8 and equation (9) on page 20, this Kasparov module is a representative of \([C_{\phi_t}(J_1)]\) as required. \( \square \)

Theorem 5.19 shows that the coarse index associated to the pair of BdG Hamiltonians \( H_0 \) and \( H_1 \) satisfying assumption 5.15 encodes a topological obstruction to locally connect the skew-adjoint unitaries \( J_0 \) and \( J_1 \) with respect to \( (X, \varphi) \), as is explained in the following corollary.

**Corollary 5.20.** Suppose that there is a continuous path \( \{J_t\}_{t \in [0,1]} \) of Real skew-adjoint unitaries in \( \text{Mult}(C^*(X)) \) such \( \varphi(f)J_0 - J_1 \in \mathbb{E}(H) \) for all \( f \in C_0(X) \) and \( t \in [0,1] \). Then the coarse index \( \mu_X \left[ [F_{C_{\phi_t}(J_1)}] \right] \) is trivial in \( KO_2(C^*(X)^5) \).

**Proof.** If such a path \( \{J_t\}_{t \in [0,1]} \) exists, then the class \( [C_{\phi_t}(J_1)] \in KO_2(C^*(X)^5) \) is trivial by proposition 4.4. Hence the coarse index vanishes. \( \square \)

Let us also briefly consider the case of Altland–Zirnbauer symmetries, i.e. we assume that there exist mutually anti-commuting Real skew-adjoint unitaries \( \{\kappa_j\}_{j=1}^n \subset \text{Mult}(C^*(X)) \) that anti-commute with \( J_0 \) and \( J_1 \). We additionally assume that the eigenspaces of \( \kappa_j \) are infinite dimensional for all \( j \in \{1, \ldots, n\} \). Such a circumstance trivially happens when \( \mathcal{H} = \mathcal{H}' \otimes W \), where \( \mathcal{H}' \) is infinite-dimensional, \( \varphi = \varphi' \otimes 1_W \) and \( \kappa_j = 1_{W'} \otimes \kappa'_j \) with \( \{\kappa'_j\}_{j=1}^n \) ungraded Clifford generators in \( \text{End}(W) \).
Proposition 5.17 along with our additional assumptions on \( \{ \kappa_i \}_{i=1}^n \) imply that the triple
\[
(C_0(X) \otimes \mathcal{C}(n, 2, 0) \otimes \mathcal{H} \otimes \bigwedge^n \mathbb{C}, F_{C_0}(\ell_1) \otimes \rho)
\] (14)
is a Real Fredholm module with left Clifford generators \( \{ 1 \otimes \gamma, J_0 \otimes \rho, \kappa_1 \otimes \rho, \ldots, \kappa_n \otimes \rho \} \). We therefore obtain a \( K \)-homology class \( [F_{C_0}(\ell_1)] \in KO^{-n-2}(C_0(X)^v) \). We similarly have a \( K \)-theory class from the Kasparov module constructed in remark 4.6 with \( A = C^*(X) \),
\[
(C_\ell(n+2, 0), [J_0 - J_1]C^*(X)C^*(X) \otimes \bigwedge^n \mathbb{C}, C_{\ell}(J_1) \otimes \rho)
\] (15)
with Clifford generators \( \{ 1 \otimes \gamma, J_0 \otimes \rho, \kappa_1 \otimes \rho, \ldots, \kappa_n \otimes \rho \} \). Like the case for \( n = 0 \), our result is that the coarse assembly map relates the equivalence classes of the Kasparov modules in equations (14) and (15).

**Theorem 5.21.** Let \( [F_{C_0}(\ell_1)] \in KO^{-n-2}(C_0(X)^v) \) and \( [C_{\ell}(J_1)] \in KO^{n+2}(C^*(X)^v) \) denote the classes from the Real Kasparov modules in equations (14) and (15) respectively. Then \( \mu_X \left( [F_{C_0}(\ell_1)] \right) = [C_{\ell}(J_1)] \).

Taking into account the extra Clifford symmetries (cf remark 2.12), the proof follows the same argument as theorem 5.19, where we apply proposition 2.14 to the map \( DK(\mathcal{Q}(C^*(X))) \otimes \mathcal{C}(n, 2, 0) \stackrel{\partial \ell} \longrightarrow KKR(\mathcal{C}(n+2, 0), C^*(X)) \).

### 5.5. Compact \( G \)-symmetry

Let \( G \) be a compact group and \( W \) a unitary/anti-unitary representation on \( \mathcal{H} \) relative to a homomorphism \( \nu : G \to \mathbb{Z}_2 \), i.e. \( W_g \) is unitary if \( \nu(g) = 0 \) and anti-unitary if \( \nu(g) = 1 \). We also assume that \( W_g^* \Gamma = \Gamma W_g \). Let us similarly assume that there is a left-action \( G \times X \to X \) which gives rise to an linear/anti-linear action \( \eta \) on \( C_0(X) \) with respect to \( \nu \). We therefore consider representations \( \varphi : C_0(X) \to B(\mathcal{H}) \) such that \( \varphi \circ \eta(f) = \text{Ad}_{W_g} \circ \varphi(f) \) for all \( f \in C_0(X) \) and \( g \in G \). Given BdG Hamiltonians \( H_0 \) and \( H_1 \) satisfying assumption 5.15, we assume that \( W_g \left( \text{Dom}(H_1) \right) \subset \text{Dom}(H_0) \) and \( W_g H_1 W_g^* = H_1 \) for all \( g \in G \) and \( k \in \{ 0, 1 \} \). In particular, this implies that, for all \( g \in G \), \( \text{Ad}_{W_g}(\ell_k) = (-1)^k \rho_k \ell_k \) for \( k \in \{ 0, 1 \} \) and \( \text{Ad}_{W_g}(C_{\ell}(J_1)) = (-1)^{\nu(g)} C_{\ell}(J_1) \).

The action \( \text{Ad}_{W} \) on \( B(\mathcal{H}) \) gives a well-defined linear/anti-linear action on \( C^*(X) \), \( \text{Multi}(C^*(X)) \) and the quotient \( \mathcal{Q}(C^*(X)) = \text{Multi}(C^*(X))/C^*(X) \) relative to the homomorphism \( \nu : G \to \mathbb{Z}_2 \) that commutes with the real structure \( \text{Ad}_{\Gamma} \). We denote these actions on \( C^*(X) \) and \( \mathcal{Q}(C^*(X)) \) by \( \beta_{\nu}^{C^*(X)} \) and \( \beta_{\nu}^{\mathcal{Q}(X)} \) respectively. We can then define the action \( \tilde{\alpha} \) of \( G \) on the Hilbert \( C^* \)-module \( C^*(X) \otimes \bigwedge^n \mathbb{C} \)
\[
\tilde{\alpha}_g(T \otimes v) = \beta_{\nu}^{C^*(X)}(T) \otimes \gamma_{\nu(g)} v, \quad g \in G, \ T \otimes v \in C^*(X) \otimes \bigwedge^n \mathbb{C}.
\] (16)

One then checks that the induced action \( \alpha \) of \( G \) on \( \text{End}_{C^*(X)}(C^*(X) \otimes \text{End}(\bigwedge^n \mathbb{C})) \) is such that for all \( g \in G \),
\[
\alpha_g(J_0 \otimes \rho) = J_0 \otimes \rho, \quad \alpha_g(F_{C_0}(\ell_1) \otimes \rho) = F_{C_{\ell}(J_1)}(\ell_1) \otimes \rho, \quad \alpha_g(1 \otimes \gamma) = 1 \otimes \gamma.
\]

Analogously, we can define a group action on \( \mathcal{H} \otimes \bigwedge^n \mathbb{C} \) via the unitary/anti-unitary representation \( W \) such that \( W_g = W_g \otimes \gamma_{\nu(g)} \) for all \( g \in G \). The operators \( F_{C_{\ell}(J_1)}(\ell_1) \otimes \rho, J_0 \otimes \rho \) and \( 1 \otimes \gamma \)
will then be $G$-invariant under the induced action $\text{Ad}_{\tilde{g}}$. With the preliminaries established, we can now state the result.

**Theorem 5.22.** The triple

$$\left(\mathcal{C}(\ell_2, \gamma) \otimes \bigwedge^* \mathbb{C}, F_{\tilde{c}_0}(J_1) \otimes \rho\right)$$

is a Real $G$-equivariant Fredholm module with $G$-action by $\tilde{W}$ and Clifford generators $\{1 \otimes \gamma, J_0 \otimes \rho\}$. Under the equivariant assembly map $\mu_0^g : KO^{-2}(C_0(X)) \rightarrow KO^2_2(C^*(X))$, the class of this Real Fredholm module is mapped to the K-theory class represented by the unbounded $G$-equivariant Real Kasparov module from proposition 4.8 with $A = C^*(X)$,

$$\left(\mathcal{C}(\ell_2, (J_0 - J_1)C^*(X)) \otimes \bigwedge^* \mathbb{C}, C_{\tilde{c}_0}(J_1) \otimes \rho\right)$$

with $G$-action by $\tilde{\alpha}$ from equation (16) and Clifford generators $\{1 \otimes \gamma, J_0 \otimes \rho\}$.

**Proof.** Because all relevant operators are $G$-invariant, the same proof as proposition 5.17 and theorem 5.19 gives the result. \hfill \square

### 5.6. Discrete $\Upsilon$-symmetries

A full discussion of discrete symmetries and the $\Upsilon$-equivariant assembly map deserves a separate treatment and so we will only give a basic overview. We use the notation $\Upsilon$ for a discrete group to distinguish the setting from the case of a compact group action. We will furthermore restrict ourselves to *linear* group actions ($\nu(g) = 0$ for all $g \in \Upsilon$).

Fix a discrete group $\Upsilon$ and a proper, isometric and cocompact left-action $\Upsilon \times X \rightarrow X$ giving a Real action $\eta : G \rightarrow \text{Aut}(C_0(X))$. We similarly take a unitary representation $V : \Upsilon \rightarrow \mathcal{U}(\mathcal{H})$ such that $[V_g, J] = 0$ and $\varphi \circ \eta_g(f) = \text{Ad}_{V_g} \circ \varphi(f)$ for all $g \in \Upsilon$ and $f \in C_0(X)$. We will furthermore assume that $\mathcal{H}$ is a $\Upsilon$-adequate $X$-module in the sense of [44, definition 5.13] (this condition can always be guaranteed).

Once again we take BdG Hamiltonians $H_0$ and $H_1$ acting on $\mathcal{H}$ that satisfy assumption 5.15 and furthermore for all $g \in \Upsilon$ and $k \in \{0, 1\}$,

$$V_g \cdot \text{Dom}(H_k) \subset \text{Dom}(H_k), \quad V_g H_k V_g^* = H_k.$$

Let $\text{Mult}(C^*(X))^\Upsilon$ and $C^*(X)^\Upsilon$ denote the subalgebras of $\text{Mult}(C^*(X))$ and $C^*(X)$ respectively consisting of elements that are fixed by $\text{Ad}_{V_g}$ for all $g \in \Upsilon$. Then because $V$ is a unitary representation,

$$J_0, J_1 \in \text{Mult}(C^*(X))^\Upsilon, \quad J_0 - J_1 \in C^*(X)^\Upsilon.$$

In [45, section 2], Roe constructs a full right Hilbert $C^*_r(\Upsilon)$-module $L^2(\Upsilon)(X)_{C^*_r(\Upsilon)}$ such that $C^*(X)^\Upsilon$ is isomorphic to $\mathbb{K}_{C^*_r(\Upsilon)}(L^2(\Upsilon)(X))$. That is, $C^*(X)^\Upsilon$ is Morita equivalent to $C^*_r(\Upsilon)$ and we obtain an invertible element

$$[\{C^*(X)^\Upsilon, L^2(\Upsilon)(X)_{C^*_r(\Upsilon)}, 0\}] \in \text{KKR}(C^*(X)^\Upsilon, C^*_r(\Upsilon)).$$
We can therefore construct a class in the $K$-theory of $C_\gamma(T)$ by composing this Morita equivalence with our generic $K$-theory construction from proposition 2.8,

$$
\left( C\ell_{2,0}, (J_0 - J_1)C^*(X)^\gamma \otimes \bigwedge^* C, C_{\delta_0}(J_1) \otimes \rho \right) \otimes_{C^*(X)^\gamma} \left( C^*(X)^\gamma, L^2_{C}(X)_{C^*(X)^\gamma} \otimes \bigwedge^* C, C_{\delta_0}(J_1) \otimes \rho \right)
$$

with left Clifford generators $\{J_0 \otimes \rho, 1 \otimes \gamma\}$. Noting that $J_0$, $F_{C_{\delta_0}(J_1)} \in \text{Mult}(C^*(X)^\gamma \subset \text{End}_{C^*(X)^\gamma}(L^2_{C}(X)))$ and using the properties of the bounded transform $F_{C_{\delta_0}(J_1)}$ from equations (4) and (5) on page 10, a bounded representative of this Kasparov module is given by

$$
\left( C\ell_{2,0}, L^2_{C}(X)_{C^*(X)^\gamma} \otimes \bigwedge^* C, F_{C_{\delta_0}(J_1)} \otimes \rho \right)
$$

with left Clifford generators $\{J_0 \otimes \rho, 1 \otimes \gamma\}$.

We can define a representation of $\gamma$ on $\mathcal{H} \otimes \bigwedge^* C$ by $\tilde{V}_g(v \otimes w) = V_g v \otimes w$ for all $v \otimes w \in \mathcal{H} \otimes \bigwedge^* C$ and $g \in \gamma$. Because $F_{C_{\delta_0}(J_1)}$ and $J_0$ are invariant under $\text{Ad} V_g$ for all $g \in \gamma$, proposition 5.17 once again gives that

$$
\left( C_0(\gamma) \otimes C\ell_{2,0}, \hat{\chi} \mathcal{H} \otimes \bigwedge^* C, F_{C_{\delta_0}(J_1)} \otimes \rho \right)
$$

is a $\gamma$-equivariant Real Fredholm module with group action by $\tilde{V}$ and left Clifford generators $\{1 \otimes \gamma, J_0 \otimes \rho\}$.

**Theorem 5.23.** The coarse assembly map $\rho_X^T : KO_T^2(C_0(\gamma)^T) \rightarrow KO_T^2(C_\gamma(T)^T)$ maps the class of the Real Fredholm module from equation (18) to the $K$-theory class represented by the Real Kasparov module from equation (17).

**Proof.** Once again, we use Roe’s description of the coarse assembly map via duality theory [45]. The coarse assembly map is the composition

$$
KO_T^2(C_0(\gamma)) \rightarrow DK_0(Q(C^*(X)^\gamma \otimes C\ell_{2,0}) \longrightarrow DK_0(C^*(X)^\gamma \otimes C\ell_{2,0}) \longrightarrow KO^2(C^*(X)^\gamma)
$$

with $Q(C^*(X)^\gamma) = \text{Mult}(C^*(X)^\gamma \otimes C^*(X)^\gamma)$ and in the last step we use the Morita equivalence of $C^*(X)^\gamma \cong C^*(X)^\gamma$ with $C^*(X)^\gamma$. The class of the Fredholm module from equation (18) is initially mapped to $[\rho(F_{C_{\delta_0}(J_1)} \otimes \rho)]$. Then by the same argument as theorem 5.19, $\partial([\rho(F_{C_{\delta_0}(J_1)} \otimes \rho)]$ combined with the isomorphism $DK_0(C^*(X)^\gamma \otimes C\ell_{2,0}) \cong KKR(C\ell_{2,0}, C^*(X)^\gamma)$ is represented by the Kasparov module $\left( C\ell_{2,0}, C^*(X)^\gamma \otimes \bigwedge^* C, F_{C_{\delta_0}(J_1)} \otimes \rho \right)$ with $C\ell_{2,0}$-generators $\{1 \otimes \gamma, J_0 \otimes \rho\}$. Applying the Morita equivalence,

$$
\left( C\ell_{2,0}, C^*(X)^\gamma \otimes \bigwedge^* C, F_{C_{\delta_0}(J_1)} \otimes \rho \right)
$$

is represented by the Kasparov module $\left( C\ell_{2,0}, C^*(X)^\gamma \otimes \bigwedge^* C, F_{C_{\delta_0}(J_1)} \otimes \rho \right)$ with $C\ell_{2,0}$-generators $\{1 \otimes \gamma, J_0 \otimes \rho\}$. Applying the Morita equivalence,
with $\mathbb{C}l_{2,0}$-generators $\{1 \otimes \gamma, J_0 \otimes \rho\}$. We therefore obtain the Real Kasparov module from equation (17).

**Remarks 5.24.**

(a) The $\Upsilon$-equivariant assembly map, interpreted as a higher index, provides a topological obstruction to the existence of a $\Upsilon$-invariant path of Real skew-adjoint unitaries $\{J_t\}_{t \in [0,1]} \subseteq \text{Mult}(C^*(X))^\Upsilon$ such that $\varphi(f)(J_0 - J_t) \in \mathbb{K}(H)$ for all $f \in C_0(X)$ and $t \in [0,1]$. As in the non-equivariant setting (corollary 5.20), the existence of such a path implies that $\mu_X^\Upsilon$ applied to the $K$-homology class from equation (18) will be trivial in $KO_2(C_\Upsilon(T))$.

(b) In the case that $\Upsilon = S$, a space group embedded in Euclidean space, the group $KO_*(C_\Upsilon(S))$ has been extensively studied as way to classify free-fermionic topological insulators and superconductors, see [23, 34] for example. A more comprehensive comparison between the $K$-homology invariants that arise from locally equivalent quasifree ground states with the $K$-theory invariants from free-fermionic topological phases would be interesting to consider. We leave this question to another place.

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No new data were created or analysed in this study.

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