SELF-SIMILAR GROUPS, OPERATOR ALGEBRAS AND SCHUR COMPLEMENTS

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Dedicated to the 70th birthday of D.V. Anosov

ABSTRACT. In the first part of the article we introduce $C^*$-algebras associated to self-similar groups and study their properties and relations to known algebras. The algebras are constructed as sub-algebras of the Cuntz-Pimsner algebra (and its homomorphic images) associated with the self-similarity of the group. We study such properties as nuclearity, simplicity and Morita equivalence with algebras related to solenoids.

The second part deals with the Schur complement transformations of elements of self-similar algebras. We study properties of such transformations and apply them to the spectral problem for Markov type elements in self-similar $C^*$-algebras. This is related to the spectral problem of the discrete Laplace operator on groups and graphs. Application of the Schur complement method in many situations reduces the spectral problem to study of invariant sets (very often of the type of a “strange attractor”) of a multidimensional rational transformation. A number of illustrating examples is provided. Finally we observe a relation between the Schur complement transformations and Bartholdi-Kaimanovich-Virag transformations of random walks on self-similar groups.

1. INTRODUCTION

Self-similar groups is a class of groups which attracts more and more attention of researchers from different areas of mathematics, and first of all from group theory.

Self-similar groups (whose study was initiated by the first named author, N. Gupta, S. Sidki, A. Brunner, P.M. Neumann, and others) posses many nice and unusual properties which allow to solve difficult problems of group theory and related areas, even including problems in Riemannian geometry and holomorphic dynamics. They are closely related to groups generated by finite automata (studied by J. Hořejš, V.M. Glushkov, S. V. Aleshin, V.I. Sushchanski and others) and many of them belong to another interesting class of groups—the class of branch groups. We recommend the following sources for the basic definitions and properties [38, 4, 24, 52].

One of the main features that makes the class of self-similar groups important is that it makes possible treatment of the renorm group in a noncommutative setting. This passage from cyclic to a non-commutative renormalization can be compared with the passage from classical to non-commutative geometry, i.e., passage from commutative $C^*$-algebras of continuous functions to non-commutative $C^*$-algebras, see [10].
Recent research shows that operator algebras also play an important role in the theory of self-similar groups and show interesting connections with other areas (for instance with hyperbolic dynamics). The first appearance of $C^*$-algebras related to self-similar groups was in [3] and was related to the problem of computation of the spectra of Markov type operators on the Schreier graphs related to self-similar groups. In [11] the methods of [3] were interpreted in terms of the Cuntz-Pimsner algebras of the Hilbert bimodules associated with self-similar groups. It was proved in [11] that there exists a smallest self-similar norm (i.e., norm which agrees with the bimodule) and that the Cuntz-Pimsner algebra constructed using the completion of the group algebra by the smallest norm is simple and purely infinite.

This article is a survey of results and ideas on the interplay of self-similar groups, renormalization, self-similar operator algebras, spectra of Markov operators and random walks.

We continue the study of self-similar completions of the group algebra started in [3] and [11]. We prove existence of the largest self-similar norm on the group algebra and define the maximal Cuntz-Pimsner algebra and maximal self-similar completion of the group algebra (Propositions 3.2 and 3.3).

Our starting point is the observation that self-similarities of a Hilbert space $H$ (i.e., isomorphisms between $H$ and its $d$-th, $d \geq 2$ power $H^d$) are in a natural bijection with the representations of the Cuntz algebra $O_d$. Then we consider self-similar unitary representations of a self-similar group and following [11] define the associated universal Cuntz-Pimsner algebra $O_G$. The algebra $O_G$ is universal in the sense that any unitary self-similar representation of the group can be extended to a representation of $O_G$. Homomorphic images of $O_G$ can be constructed using different self-similar representations of $G$. Among important self-similar representations we study the natural unitary representation of $G$ on $L^2(X^\omega, \nu)$, where $X^\omega$ is the boundary of the rooted tree on which $G$ acts and $\nu$ is the uniform Bernoulli measure on it. Another important class of self-similar representations are permutational representations of $G$ on countable $G$-invariant subsets of $X^\omega$.

The sub-$C^*$-algebra of $O_G$ generated by $G$ is denoted $A_{\text{max}}$. For every homomorphic image of the Cuntz-Pimsner algebra $O_G$ we get the respective image of $A_{\text{max}}$. The algebra $A_{\text{min}}$, for instance, is defined as the image of $A_{\text{max}}$ under a permutational representation of $G$ on a generic self-similar $G$-invariant countable set of $X^\omega$ (which can be extended to a permutational representation of $O_G$ in a natural way). The algebra $A_{\text{min}}$ was studied in [3] and [11]. Similarly, the algebra $A_{\text{mes}}$ generated by the natural representation of $G$ (and of $O_G$) on $L^2(X^\omega, \nu)$, is considered. This algebra is particularly convenient for spectral computations (see Subsection 5.2).

It is convenient in many cases to pass to a bigger sub-algebra of $O_G$ than $A_{\text{max}}$. It is the algebra generated by $G$ and the union of the matrix algebras $M_{d \times d}(\mathbb{C})$ naturally constructed inside $O_d \subset O_G$. It is proved in [39] that this algebra is denoted $M_{d^\infty}(G)$ and it is the universal algebra of the groupoid of germs of the action of $G$ on the boundary $X^\omega$ of the rooted tree. This makes it possible to apply the well-developed theory of $C^*$-algebras associated to groupoids to the study of $O_G$ and $A_{\text{max}}$.

For every homomorphic image of $O_G$ (i.e., for every self-similar representation of $G$) we get the corresponding image of $M_{d^\infty}(G)$. 
In Section 2 we show a relation of self-similar groups and algebras to hyperbolic dynamics. If a self-similar group $G$ is contracting, then there is a natural Smale space associated to it (the limit solenoid). It is a dynamical system $(S_G, \hat{s})$ with hyperbolic behavior: the space $S_G$ has a local structure of a direct product such that the homeomorphism $\hat{s}$ is contracting on one factor and expanding on the other. We prove in Theorem 4.8 that the algebra $M_{\hat{s}}(G)$ is Morita equivalent in this case to the convolution algebra of the unstable equivalence relation on the limit solenoid. Such convolution algebras were studied by J. Kaminker, I. Putnam and J. Spielberg in [15, 32, 44].

The algebras $A_{\text{max}}, M_{\hat{s}}(G)$ and their images under self-similar representations of $G$ have a nice self-similarity structure described by matrix recursions, which encode the structure of the Moor diagrams of the Mealy type automata defining the underlying group. This self-similarity is used in the second part of the article in the study of spectra of elements of the involved algebra.

We recall at first the classical Schur complement transformation, which is used in linear algebra for solving systems of linear equations, in statistics for finding conditional variance of multivariate Gaussian random variables and in Bruhat normal form (see [11, 9]). We show in our article that Schur complement is also useful in the study of spectral problems in self-similar algebras and that it can be nicely expressed in terms of the Cuntz algebra. We establish some simple properties of the Schur complement transformations and introduce a semigroup of such transformations.

The method that we use to treat the spectral problem could be a first step in generalization of the method of Malozemov and Teplyaev that they developed for the study of spectra of self-similar graphs related to classical fractals (like the Sierpinski gasket) [54, 36]. In fact they also use (in an implicit form) the Schur complement. Their technique is developed for the case when only one complex parameter is involved. Our technique (which is a development of the technique used in [3, 26, 19]) involves several parameters and therefore necessarily lead to multidimensional rational mappings and their dynamics.

The Schur complements in our situations are renormalization transformations for the spectral problem and related problems. Considered together they generate a noncommutative semigroup which can be called the "Schur renorm group" (observe that in classical situation the renorm group very often is a cyclic semigroup).

We illustrate our method by several examples the most sophisticated among which is the example related to the 3-generated torsion 2-group of intermediate growth constructed in [21].

The transformations that arise in this case are

$$\tilde{S}_1 : \begin{pmatrix} x \\ y \\ z \\ u \\ v \end{pmatrix} \mapsto \begin{pmatrix} x^2(2uy-v(y^2+z^2+u^2+v^2)) \\ (y+z+u+v)((y+z-u-v)(y-z+u+v)-(y+z-u+v)(y-z-u+v)) \\ x^2(2zu-u(y^2+z^2+u^2+v^2)) \\ (y+z+u+v)((y+z-u-v)(y-z+u+v)-(y+z-u+v)(y-z-u+v)) \\ x^2(2uv-z(y^2-z^2+u^2+v^2)) \\ (y+z+u+v)((y+z-u-v)(y-z+u+v)-(y+z-u+v)(y-z-u+v)) \end{pmatrix}.$$
and

\[ 3 \mathcal{S}_2 : \begin{pmatrix} x \\ y \\ z \\ u \\ v \end{pmatrix} \mapsto \begin{pmatrix} \frac{x^2(y+z)}{(u+v+y+z)(u+v-y-z)} \\ u \\ y \\ z \\ v - \frac{x^2(u+v)}{(u+v+y+z)(u+v-y-z)} \end{pmatrix} \]

The dynamical properties of these transformations are not well understood and this is one of intriguing problems.

The examples that appear demonstrate few cases (taken from [8, 19]) leading to easily treatable transformations (when they “integrable” in the sense that there is a nontrivial semi-conjugacy to a one-dimensional map), a couple of examples (taken from [27, 20]) when there is no information about topological nature of the invariant subsets (which according to computer experiments look like “strange attractors”), and a couple of examples when our method does not work (but the corresponding spectral problem is important because of its relation to the problem of finding new constructions of expanders).

In principle, the area of application of our method is much broader but then it requires use of Schur complements in infinite-dimensional spaces of matrices. At the moment we do not have examples of successful applications of the method in the infinite-dimensional case.

One of important properties we are after in the study of self-similar groups and related objects is amenability. Self-similar groups provide a number of examples of amenable but not elementary amenable groups [22, 23]. The fundamental idea how to treat amenability of self-similar groups belongs to Bartholdi and Virag [5]. Roughly speaking it converts self-similarity of a group into self-similarity of a random walk on it. This idea was further developed by V. Kaimanovich (in what he calls the “Münchhausen trick”) using the notion of entropy of a random walk [31]. Kaimanovich introduced transformations of the random walk under the renorm transformations of the group and successfully used them to show that some self-similar groups are amenable. Our observation is that these transformations again can be interpreted as the Schur complement transformations of the measures (or corresponding elements of the group algebra) determining the random walk. More precisely, we show that the transformation considered in [31] is the Schur complement conjugated by the map \( A \mapsto A + I \), where \( I \) is the identity matrix.

We believe that the introduced ideas of self-similar algebras and Schur complement transformations on them will be useful for the study of different aspects of the theory of self-similar groups and its applications.

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2. Self-similar groups

2.1. Definition. Let \( X \) be a finite alphabet and let \( X^* \) denote the set of finite words over the alphabet \( X \). In other terms, \( X^* \) is the free monoid generated by \( X \). We consider \( X^* \) to be the set of vertices of a rooted regular tree with the root coinciding with the empty word \( \emptyset \) and in which a word \( v \) is connected to every word of the form \( vx \) for \( x \in X \).
A rooted tree is a standard model of a self-similar structure. If we remove the empty word from it, then the tree $X^*$ will split into $|X|$ subtrees $xX^*$, where $x \in X$. Each of the subtrees $xX^*$ is isomorphic to the whole tree $X^*$ under the isomorphism $xv \mapsto v$, see Figure 1.

If we consider the boundary of the tree $X^*$, then the self-similarity is even more evident. The boundary of a rooted tree is the set of all infinite paths starting in the root. In our case the boundary of $X^*$ is naturally identified with the set $X^\omega$ of infinite words $x_1x_2\ldots$. The set $X^\omega$ is a disjoint union of the cylindrical sets $xX^\omega = \{xx_2x_3\ldots : x_i \in X\}$, and again, the shift $xw \mapsto w$ is a bijection (and a homeomorphism, if we endow $X^\omega$ with the natural topology of a direct product of discrete sets $X$).

A group acting on the rooted tree $X^*$ is called self-similar, if the action agrees with the described self-similarity structure on the tree $X^*$. Namely, we adopt the following definition.

**Definition 2.1.** A self-similar group $(G, X)$ is a group $G$ acting faithfully on the rooted tree $X^*$ such that for every $g \in G$ and every $x \in X$ there exist $h \in G$ and $y \in X$ such that

$$g(xw) = yh(w)$$

for all $w \in X^*$.

It follows from the definition that if $(G, X)$ is a self-similar group, then for every $v \in X^*$ there exists $h \in G$ uniquely defined by the condition

$$g(vw) = g(v)h(w)$$

for all $w \in X^*$. The element $h$ is called restriction of $g$ in $v$ and is denoted $h = g|_v$.

We have the following obvious properties of restriction

$$g|_{v_1v_2} = (g|_{v_1})|_{v_2}, \quad (gh)|_v = g|h(v)|_v.$$  

It is convenient to identify a letter $x$ with the creation operator on $X^*$ (or on $X^\omega$) given by appending the letter $x$ to the beginning of the word:

$$x \cdot v = xv.$$  

Similarly, we can identify every word $u \in X^*$ with the creation operator

$$u \cdot v = uv.$$
In this case the condition that \( g(vw) = uh(w) \) for all \( w \in X^* \) can be written as equality of compositions of transformations of \( X^* \):

\[
g \cdot v = u \cdot h.
\]

We have the following straightforward corollary of the definitions.

**Proposition 2.1.** The set \( X \cdot G \) of transformations of \( X^* \) of the form \( x \cdot g : w \mapsto xg(w) \) is closed under pre- and post-compositions with action of the elements of \( G \). The obtained left and right actions of \( G \) on \( X \cdot G \) commute and are defined by

\[
h \cdot (x \cdot g) = h(x) \cdot (h | x) g, \quad (x \cdot g) \cdot h = x \cdot (gh),
\]

where \( \cdot \) denotes composition of transformations.

We call the set \( X \cdot G \) the (permutational) \( G \)-bimodule associated with the self-similar group \((G, X)\).

**Definition 2.2.** A self-similar group \((G, X)\) is called self-replicating (recurrent in \[38\]) if it is transitive on the first level \( X^1 \) of the tree \( X^* \) and for any (and thus for every) \( x \in X \) the map \( g \mapsto g|_x \) from the stabilizer of \( x \) in \( G \) to \( G \) is onto. Equivalently, the self-similar action is self-replicating if the left action of \( G \) on the bimodule \( X \cdot G \) is transitive.

It is not hard to prove by induction that if an action is self-replicating, then it is transitive on every level \( X^n \) of the tree \( X^* \) (is level-transitive).

Every self-similar action naturally induces an action on the boundary \( X^\omega \) of the tree \( X^* \). This is an action by measure-preserving homeomorphisms.

**Example 1.** Grigorchuk group. Consider the binary alphabet \( X = \{0, 1\} \), the tree \( X^* \) and let \( G \) be the group generated by the automorphisms \( a, b, c, d \) of \( X^* \) defined recursively by the relations

\[
\begin{align*}
  a(0w) &= 1w, & a(1w) &= 0w \\
  b(0w) &= 0a(w), & b(1w) &= 1c(w) \\
  c(0w) &= 0a(w), & c(1w) &= 1d(w) \\
  d(0w) &= 0w, & d(1w) &= 1b(w).
\end{align*}
\]

This group was defined for the first time in \[21\]. This group is a particularly easy example of a Burnside group (an infinite finitely generated torsion group) and it is the first example of a group of intermediate growth, which answers a question of J. Milnor. Figure 2 shows the Moore diagram of the automaton generating the Grigorchuk group (see a definition of Moore diagrams below).

**Example 2.** Free group. A convenient way to define self-similar groups are Moore diagrams of automata, generating them. Consider, for instance, the Moore diagram shown on Figure 3. The vertices of the diagram correspond to states of the automaton (to generators of the group), the arrows describe transitions between the states and the labels correspond to the output of the automaton. If there is an arrow from a state \( g \) to a state \( h \) labeled by \((x, y)\), then this means that \( g(xw) = yh(w) \) for all \( w \in X^* \). Thus, the automaton shown on Figure 3 describes
the group generated by the elements $a, b, c$ such that
\[
\begin{align*}
a(0w) &= 0b(w), & a(1w) &= 1b(w), \\
b(0w) &= 1a(w), & b(1w) &= 0c(w), \\
c(0w) &= 1c(w), & c(1w) &= 0a(w),
\end{align*}
\]
for all $w \in X^*$.

S. Sidki conjectured in [53] that the three states of the automaton generate a free group of rank 3. This claim was later proved by Y. Vorobets and M. Vorobets in [55].

**Example 3. Basilica.** The group generated by the automaton shown on Figure 4 is called the *Basilica group*, since it is the iterated monodromy group of the polynomial $z^2 - 1$ (see below for the definition).

It is the first example of an amenable group, which can not be constructed from the groups of sub-exponential growth using the operations preserving amenability (taking quotients, extensions, direct limits and passing to a subgroup).

2.2. **Iterated monodromy groups.** Let $M$ be a path connected and locally path connected topological space and let $M_1$ be its open subset. A *d-fold partial self-covering* is a covering map $f : M_1 \longrightarrow M$, i.e., a continuous map such that for
every $x \in \mathcal{M}$ there exists a neighborhood $U \ni x$ whose total preimage $f^{-1}(U)$ is a disjoint union of $d$ open subsets which are mapped homeomorphically onto $U$ by $f$.

For every $n \geq 1$ the iteration $f^n : \mathcal{M}_n \to \mathcal{M}$ is a $d^n$-fold partial self-covering (in general with a smaller domain $\mathcal{M}_n$).

Choose a basepoint $t \in \mathcal{M}$. Then the disjoint union $T = \bigsqcup_{n \geq 0} f^{-n}(t)$ has a natural structure of a $d$-regular tree with the root $t = f^{-0}(t)$ where a vertex $z \in f^{-n}(t)$ is connected to the vertex $f(z) \in f^{-(n-1)}(t)$.

The fundamental group $\pi_1(\mathcal{M}, t)$ acts naturally on every level $f^{-n}(t)$ of the tree $T$. The image of a vertex $z$ under the action of a loop $\gamma \in \pi_1(\mathcal{M}, t)$ is the end of the unique preimage of $\gamma$ under the covering $f^n$, which starts at $z$. It is easy to check that these actions define an action of $\pi_1(\mathcal{M}, t)$ by automorphisms of the rooted tree $T$. This action is called the **iterated monodromy action**.

**Definition 2.3.** The **iterated monodromy group** $\text{IMG}(f)$ of a partial self-covering $f : \mathcal{M}_1 \to \mathcal{M}$ is the quotient of the fundamental group $\pi_1(\mathcal{M}, t)$ by the kernel of the iterated monodromy action.

The iterated monodromy group is self-similar, if we identify the tree of preimages $T$ with $X^*$ in a correct way (see [38, 2]). The obtained self-similar action (called the **standard action**) is described in the following way. Choose a bijection $\Lambda : X \to f^{-1}(t)$ between the alphabet $X$ of size $d$ and the set of preimages of the basepoint. Choose also paths $\ell_x$ from the basepoint $t$ to its preimage $\Lambda(x)$ for every $x$. Then the standard action of $\text{IMG}(f)$ on $X^*$ (which is conjugate to its action on $T$) is given by the formula

\begin{equation}
\gamma(xw) = y(\ell_x \gamma_x \ell_x^{-1})(w),
\end{equation}

where $\gamma_x$ is the $f$-preimage of $\gamma$ starting in $\Lambda(x)$, $y \in X$ is such that $\Lambda(y)$ is the end of $\gamma_x$ and $w \in X^*$ is any word. We multiply here the paths in the natural order (see Figure 5).
3. SELF-SIMILAR ALGEBRAS

3.1. Self-similarities of Hilbert spaces. A \((d\text{-fold})\) similarity of an infinite-dimensional Hilbert space \(H\) is an isomorphism

\[
\psi : H \rightarrow H^d = H \oplus \cdots \oplus H.
\]

**Example 4.** Let \(X\) be an alphabet with \(d\) letters and let \(X^*\) and \(X^\omega\) be the rooted tree and its boundary, respectively. Let \(\nu\) be the uniform Bernoulli measure on \(X^\omega\), i.e., the direct product of uniform probability measures on \(X^\omega\). Then the Hilbert space \(H = L^2(X^\omega, \nu)\) is decomposed into the direct sum

\[
\bigoplus_{x \in X} L^2(xX^\omega),
\]

where the spaces \(L^2(xX^\omega)\) are naturally isomorphic to \(L^2(X^\omega, \nu)\), with the isomorphism the map \(U_x : L^2(xX^\omega) \rightarrow L^2(X^\omega, \nu)\) given by

\[
U_x(f)(w) = \frac{1}{\sqrt{d}} f(xw).
\]

We view \(U_x\) as partial isometries of \(L^2(X^\omega, \nu)\). Hence we get the natural similarity \(\sum_{x \in X} U_x\) of the space \(L^2(X^\omega, \nu)\).

**Example 5.** Let \(W \subset X^\omega\) be a self-similar subset of the boundary of the tree \(X^*\), i.e., such a set that \(W = \bigcup_{x \in X} xW\). Then the space \(\ell^2(W)\) is naturally self-similar, since it can be decomposed into a direct sum

\[
\ell^2(W) = \bigoplus_{x \in X} \ell^2(xW),
\]

where the spaces \(\ell^2(xW)\) are naturally isomorphic to \(\ell^2(W)\), with the isomorphism \(U_x : \ell^2(xW) \rightarrow \ell^2(W)\) given by

\[
U_x(f)(w) = f(xw).
\]

The above two examples will be the main types of self-similarities on Hilbert spaces used in this paper.

3.2. Representations of the Cuntz algebra \(O_d\). Recall that the Cuntz algebra \(O_d\), \(d \geq 2\), is the \(C^*\)-algebra given by the presentation

\[
\langle a_1, a_2, \ldots, a_d : a_1a_1^* + a_2a_2^* + \cdots + a_da_d^* = 1, a_k^*a_k = 1, k = 1, \ldots, d \rangle.
\]
Note that as a corollary of the defining relations we get \( a_k^*a_l = 0 \) for \( k \neq l \). The defining relations can be also written as the following two matrix equalities

\[
(a_1, a_2, \ldots, a_d)(a_1, a_2, \ldots, a_d)^* = 1, \quad (a_1, a_2, \ldots, a_d)^*(a_1, a_2, \ldots, a_d) = I,
\]

where \( I \) is the \( d \times d \) unit matrix.

The Cuntz algebra \( O_d \) is simple (see [12]) and hence any \( d \) isometries \( a_1, \ldots, a_d \) such that \( a_1a_1^* + \cdots + a_da_d^* = 1 \) generate a \( C^* \)-algebra isomorphic to the Cuntz algebra \( O_d \) and determine its representation.

Representations of the Cuntz algebra can be identified with self-similarities of a Hilbert space as the following proposition shows.

**Proposition 3.1.** The relation putting into correspondence to a \( * \)-representation \( \rho : O_d \rightarrow B(H) \) the map

\[
\tau_\rho = (\rho(a_1^*), \rho(a_2^*), \ldots, \rho(a_d^*)) : H \rightarrow H^d
\]

is a bijection between the set of representations of \( O_d \) on \( H \) and the set of \( d \)-fold self-similarities on \( H \).

The inverse of this bijection puts into correspondence to a \( d \)-similarity \( \psi : H \rightarrow H^d \) the representation of \( O_d \) given by \( \rho(a_k) = T_k \), for

\[
T_k(\xi) = \psi^{-1}(0, \ldots, 0, \xi, 0, \ldots, 0),
\]

where \( \xi \) in the right-hand side is at the \( k \)th coordinate of \( H^d \).

**Proof.** If \( \rho \) is a representation of \( O_d \) on \( H \), then \( \rho(a_k) \) are isometries of \( H \) with the subspaces \( H_k = \rho(a_k)(H) \) and \( H = H_1 \oplus \cdots \oplus H_d \). The \( k \)th components of a vector \( \xi \in H \) with respect to this decomposition is its image under the projection \( \rho(a_k a_k^*) \).

Hence we get an isomorphism

\[
H \rightarrow H^d : \xi \mapsto (\rho(a_1)^*(\xi), \rho(a_2)^*(\xi), \ldots, \rho(a_d)^*(\xi))
\]

equal to the composition of the identical map \( H \rightarrow H_1 \oplus \cdots \oplus H_d \) with the isomorphism \( \rho(a_1)^* \oplus \cdots \oplus \rho(a_d)^* : H_1 \oplus \cdots \oplus H_d \rightarrow H^d \).

Conversely, suppose that

\[
\psi : H \rightarrow H^d
\]

is a \( d \)-similarity. The map

\[
\xi \mapsto (0, \ldots, 0, \xi, 0, \ldots, 0)
\]

is an isometry of \( H \) with the \( k \)th direct summand of \( H^d \). Composing it with the isomorphism \( \psi^{-1} \) we get an isometry \( T_k \) of \( H \) with the direct summand \( H_k \) of a decomposition \( H = H_1 \oplus \cdots \oplus H_d \). But tuples of such isometries are precisely the representations of the Cuntz algebra \( O_d \). Note that if \( \psi(\xi) = (\xi_1, \ldots, \xi_d) \), then \( \xi = \sum_k T_k(\xi_k) \), hence \( \xi_k = T_k^*(\xi) \). Consequently, the two bijections are mutually inverse. \( \square \)

**Example 6.** The representation of \( O_d \) associated with the natural \( d \)-similarity of \( L^2(\mathbb{X}^*, \nu) \) is generated by the isometries \( \pi(a_x) = T_x \) on \( L^2(\mathbb{X}^*, \nu) \), given by

\[
T_x(f)(w) = \begin{cases} \sqrt{d}f(w') & \text{if } w = xw' \\ 0 & \text{otherwise.} \end{cases}
\]
Example 7. Let $W$ be a self-similar subset of $X^\omega$. The representation associated to the natural $d$-similarity on $\ell^2(W)$ is generated by the isometries

$$T_x(f)(w) = \begin{cases} f(w) & \text{if } w = xw' \\ 0 & \text{otherwise.} \end{cases}$$

Such representations of $O_d$ are called permutational. Permutational representations of the Cuntz algebra related to self-affine (digit) tilings of the Euclidean space are studied in [6].

3.3. Self-similar groups and their representations. If $(G, X)$ is a self-similar group, then the associated wreath recursion is the embedding $\phi : G \to \text{Symm}(X)_X$ $G = \text{Symm}(X) \rtimes G^X$ given by

$$\phi(g) = \sigma(g|_x)_{x \in X},$$

where $\sigma \in \text{Symm}(X)$ is the action $x \mapsto g(x)$ of $g$ on the first level $X$ of the tree $X^*$ and the components $g|_x$ of $G^X$ are the restrictions of $g$ onto the subtrees $xX^*$, i.e., are given by the condition

$$g(xw) = g(x)g|_x(w)$$

for all $w \in X^*$.

Suppose that we have a self-similar group $(G, X)$ and let $d = |X|$. Let $H$ be a Hilbert space together with a $d$-similarity

$$\psi : H \to H^X.$$ We will denote the summand of $H^X$ corresponding to a letter $x \in X$ by $H_x$. Let $\rho : O_d \to B(H)$ be the associated representation of the Cuntz algebra. It is generated by isometries $T_x = \rho(a_x)$, $x \in X$, such that $H_x = T_x(H)$.

Definition 3.1. A unitary representation $\rho$ of $G$ on $H$ is said to be self-similar (with respect to the $d$-similarity $\psi$) if

$$\rho(g)T_x = T_y\rho(h)$$

whenever $g(xw) = yh(w)$ for all $w \in X^*$, i.e., whenever $g \cdot x = y \cdot h$ in the associated bimodule $X \cdot G$.

Example 8. A self-similar group $G$ acts on the tree $X^*$ by automorphisms and the induced action on the boundary $X^\omega$ is preserving the Bernoulli measure $\nu$. We get hence a unitary representation $\pi$ of $G$ on $L^2(X^\omega, \nu)$. It is easy to see that this representation is self-similar with respect to the natural $d$-similarity on $L^2(X^\omega, \nu)$.

Example 9. Let $W$ be a self-similar $G$-invariant subset of $X^\omega$. Then the permutational representation of $G$ on $W$ is self-similar.

For the general notion of a Cuntz-Pimsner algebra see [14].

Definition 3.2. Let $G$ be a self-similar group acting on $X^*$. The associated (universal) Cuntz-Pimsner algebra $O_G$ is the universal $C^*$-algebra generated by $G$ and $a_x$, $x \in X$, satisfying the following relations

- all relations of $G$;
- Cuntz relations for $a_x$ : $a_x^*a_x = 1$ for all $x \in X$, $\sum_{x \in X} a_x a_x^* = 1$;
- $ga_x = a_y h$ for $g, h \in G$ and $x, y \in X$, if $g(xw) = yh(w)$ for all $w \in X^*$, i.e., if $g(x) = y$ and $h = g|_x$.

The algebra generated by $G$ in $O_G$ is denoted $A_{max}$.
Note that as a corollary of the defining relations we get the relations

\[ g = g \sum_{x \in X} a_x a_x^* = \sum_{x \in X} a_{g(x)} g|_x a_x^*, \]

for every \( g \in G \), where \( g|_x \) is, as usual, the section of \( g \) at \( x \), i.e., such an element of \( G \) that \( g(xw) = g(x)g|_x(w) \) for all \( w \in X^* \).

The next proposition follows directly from the definitions.

**Proposition 3.2.** Let \( \pi : O_d = \langle a_x \rangle_{x \in X} \to B(H) \) be a representation of the Cuntz algebra associated with a \( d \)-similarity of a Hilbert space \( H \).

A unitary representation \( \rho \) of \( G \) on \( H \) is self-similar if and only if \( \rho \) and \( \pi \) generate a representation of the Cuntz-Pimsner algebra \( O_G \).

Consequently, self-similar representations of \( G \) are precisely restrictions onto \( G \) of representations of the Cuntz-Pimsner algebra \( O_G \).

**3.4. Matrix recursions.** A **matrix recursion** on an algebra \( A \) is a homomorphism

\[ \phi : A \to M_{d \times d}(A) \]

of \( A \) into the algebra of matrices over \( A \).

**Example 10.** Let \( \psi : H \to H^d = H^X \) be a \( d \)-similarity on a Hilbert space \( H \) and let \( \rho : G \to B(H) \) be a self-similar unitary representation of a group \( G \). Then every operator \( \rho(g) \) for \( g \in G \) can be written, with respect to the decomposition \( \psi(H) = H^X \), as a \( d \times d \) matrix

\[ \rho(g) = (A_{yx})_{x,y \in X}, \]

where

\[ A_{yx} = \begin{cases} \rho(g|_x) & \text{if } g(x) = y, \\ 0 & \text{otherwise}. \end{cases} \]

For every self-similar group \( G \) we have the associated matrix recursion on the group algebra \( \mathbb{C}[G] \), which is the linear extension of the recursion:

\[ \phi(g) = (A_{yx})_{x,y \in X}, \quad A_{yx} = \begin{cases} g|_x & \text{if } g(x) = y, \\ 0 & \text{otherwise}, \end{cases} \]

which can be interpreted as the **wreath recursion** \( \phi : G \to \text{Symm}(X) \wr G \), as it was defined in Section 3.3.

In terms of the associated representation \( \rho \) of the Cuntz algebra, we have

\[ g|_x = T_{g(x)}^* g T_x, \]

where \( T_x = \rho(a_x) \), which follows from (3.1) (or directly from the defining relations of \( O_G \)).

Note that the homomorphism \( \phi \) usually is not injective (even if the wreath recursion is and thus the map \( \phi : G \to M_d(\mathbb{C}[G]) \) is injective as well).

**Example 11.** Consider the group \( \mathcal{G} = \langle a, b, c, d \rangle \) from Example 11. Then

\[ \phi(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \phi(b) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}, \phi(c) = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \phi(d) = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}. \]
Denote $\alpha = (b+c+d-1)/2$. Then $\phi(\alpha) = \begin{pmatrix} a & 0 \\ 0 & \alpha \end{pmatrix}$, $\phi(\alpha^2 - 1) = \begin{pmatrix} 0 & 0 \\ 0 & \alpha^2 - 1 \end{pmatrix}$, $\phi(a(\alpha^2 - 1)a) = \begin{pmatrix} \alpha^2 - 1 & 0 \\ 0 & 0 \end{pmatrix}$, and

$$\phi ((\alpha^2 - 1)a(\alpha^2 - 1)a) = 0.$$ 

But

$$(\alpha^2 - 1)a(\alpha^2 - 1)a = \frac{(b+c+d-4)a(b+c+d-4)a}{16} \neq 0$$

in $\mathbb{C}[G]$.

One can find however an ideal $I \subset \mathbb{C}[G]$ such that $\phi$ induces an injective homomorphism $\mathbb{C}[G]/I \rightarrow M_{d \times d}(\mathbb{C}[G]/I)$, see \[51\]. The ideal $I$ is the ascending union of the kernels of the matrix recursions $\mathbb{C}[G] \rightarrow M_{d^n \times d^m}(\mathbb{C}[G])$ describing the action of the group $G$ on the $n$th level of the tree $X^*$.

3.5. **Self-similar completions of the group algebra.** Let $\rho : G \rightarrow B(H)$ be a self-similar representation (with respect to a $d$-similarity $\psi : H \rightarrow H^d$) and let $A_\rho$ be the completion of $\mathbb{C}[G]$ with respect to the norm given by $\rho$. Then the matrix recursion $\phi$ extends to a homomorphism

$$A_\rho \rightarrow M_{d^m \times d^m}(A_\rho),$$

also denoted by $\phi$ (or $\phi_\rho$), which is injective, since it implements the equivalence of the representation $\rho$ with the representation $\psi \circ \rho \circ \psi^{-1}$.

The following description of the completions $A_\rho$ follows directly from Proposition \[52\].

**Definition 3.3.** A completion of $\mathbb{C}[G]$ is called **self-similar** if it is the completion with respect to a self-similar representation.

The following proposition shows that there is a unique maximal completion. We denote by $A_{\text{max}}$ the $C^*$-algebra generated by $G$ in $O_G$.

**Proposition 3.3.** A completion $A$ of $\mathbb{C}[G]$ is self-similar if and only if it is the closure of $\mathbb{C}[G]$ in a homomorphic image of the Cuntz-Pimsner algebra $O_G$. In particular, every such completion is a homomorphic image of the algebra $A_{\text{max}}$.

There also exists the smallest self-similar completion.

**Definition 3.4.** Let $G$ be a countable group of automorphisms of $X^*$. A point $w \in X^\omega$ of the boundary is called $G$-generic if for every $g \in G$ either $g(w) \neq w$ or $w$ is fixed by $g$ together with all points of a neighborhood of $w$.

It is not hard to prove that the set of $G$-generic points is co-meager (i.e., is an intersection of a countable collection of open dense sets), see \[23\] and \[41\]. Let $W \subset X^\omega$ be a non-empty countable $G$-invariant set of $G$-generic points (one can also just take the $G$-orbit of a $G$-generic point). Denote by $\rho_W$ the permutational representation of $G$ on $\ell^2(W)$. The following theorem is proved in \[41\].

**Theorem 3.4.** Let $G$ be a group with a self-similar action on $X^*$. Denote by $\| \cdot \|_{\text{min}}$ the norm on $\mathbb{C}[G]$ defined by the representation $\rho_W$. This norm does not depend on the choice of the set $W$ and $\|a\|_{\text{min}} \leq \|a\|_{\rho}$ for any self-similar representation $\rho$ of $G$ and any $a \in \mathbb{C}[G]$.

The completion of $\mathbb{C}[G]$ with respect to the norm $\| \cdot \|_{\text{min}}$ is the algebra $A_{\text{min}}$ generated by $G$ in a unique simple unital quotient of $O_G$. 

Hence, if $\mathcal{A}$ is a completion of $\mathbb{C}[G]$ with respect to a self-similar representation of $G$, then the identical map on $G$ induces surjective homomorphisms

$$\mathcal{A}_{\text{max}} \rightarrow \mathcal{A} \rightarrow \mathcal{A}_{\text{min}}.$$ 

Let $\mathcal{G}$ be the groupoid generated by the germs of the local homeomorphisms of $X^\omega$ of the form

$$(3.3) \quad T_vg : w \mapsto vg(w), \quad v \in X^*, g \in G.$$ 

Recall that a germ of a local homeomorphism $h : U \rightarrow V$ is a pair $(h, x)$, where $x$ belongs to the domain of $h$, where two pairs $(h_1, x_1)$ and $(h_2, x_2)$ are identified if $x_1 = x_2$ and restrictions of $h_i$ on a neighborhood of $x_1$ coincide. The germs are composed in a natural way and inverse of a germ $(g, x)$ is defined to be equal to $(g^{-1}, g(x))$. The set of germs of a pseudogroup of local homeomorphisms has a natural germ topology defined by the basis consisting of the sets of the form

$${\mathcal{U}}_h, u = \{(h, x) : x \in U\}$$ 

where $h$ is an element of the pseudogroup and $U$ is an open subset of the domain of $h$.

For more details and for the definition of the operator algebras associated with topological groupoids, see [16, 7, 42].

It is not hard to prove that the universal Cuntz-Pimsner algebra $O_G$ coincides with the universal algebra of the groupoid $\mathcal{G}$ (see [39]). One can consider also the reduced $C^*$-algebra of $\mathcal{G}$ (see the definitions in [16, 34]). Let $\mathcal{A}_{\text{red}}$ be the subalgebra of the reduced $C^*$-algebra of $\mathcal{G}$ generated by $G$. Then $\mathcal{A}_{\text{red}}$ is also a completion of $G$ with respect to a self-similar representation (since it comes from a representation of $O_G$, see Proposition 3.2).

A groupoid of germs of an action of a group $G$ on a topological space $X$ is non-Hausdorff if and only if there exists $x \in X$ and an element $g \in G$ such that the germ of $g$ at $x$ can not be separated from the germ of the identity at $x$. The latter is equivalent to the condition that for every neighborhood $U$ of $x$ there exists $y \in U$ such that the germ of $g$ at $y$ is trivial (in particular $g(x) = x$) and there exists $z \in U$ such that $g(z) \neq z$. Consequently, the groupoid of germs is Hausdorff if and only if for every $g \in G$ the interior of the set of fixed points of $g$ is closed. See an example of a self-similar group with non-Hausdorff groupoid of germs of the action on $X^\omega$ in Example 17.

Theorem 3.5. If the groupoid of germs of the action of $G$ on $X^\omega$ is (measurewise) amenable (in particular, if the orbits of the action of $G$ on $X^\omega$ have polynomial growth), then the universal and reduced algebras of the groupoid $\mathcal{G}$ coincide. In particular, their sub-algebras $\mathcal{A}_{\text{max}}$ and $\mathcal{A}_{\text{red}}$ coincide.

If the groupoid of germs of the action of $G$ on $X^\omega$ is Hausdorff, then the algebras $\mathcal{A}_{\text{red}}$ and $\mathcal{A}_{\text{min}}$ coincide.

In particular, if the action of $G$ on $X^\omega$ is free, then the groupoid of germs is Hausdorff and moreover is principal (i.e., is an equivalence relation) and $\mathcal{A}_{\text{red}} = \mathcal{A}_{\text{min}}$.

Another frequently used self-similar completion is the completion $\mathcal{A}_{\text{mes}}$ defined by the natural unitary representation $\pi$ of $G$ on $L^2(X^\omega, \nu)$.

Problem 1. Find conditions when $\mathcal{A}_{\text{mes}} = \mathcal{A}_{\text{min}}$. 
3.6. **Gauge-invariant subalgebra of** $O_G$. Let $M_k$ be the closed linear span in $O_G$ of the elements $a_v a^*_u$ for $g \in G$ and $v, u \in X^k$. Here we use the multi-index notation $a_{x_1, x_2, \ldots, x_n} = a_{x_1} a_{x_2} \cdots a_{x_n}$. In particular, $M_0 = \mathcal{A}_{max}$ is the algebra generated by $G$ in $O_G$.

It is easy to see that for $v_1, v_2, u_1, u_2 \in X^k$ and $g_1, g_2 \in G$

$$a_{v_1} g_1 a^*_u a_{v_2} g_2 a^*_w = \begin{cases} a_{v_1} g_1 g_2 a^*_u & \text{if } u_1 = v_2 \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $a_v a^*_u$ are multiplied in the same way as the matrix units, hence $M_k$ is isomorphic to the algebra $M_{d^k \times d^k} (\mathcal{A}_{max})$ of $d^k \times d^k$-matrices over $\mathcal{A}_{max}$.

Recall that by (3.1) every element $g \in G$ can be written in $O_G$ as a sum

$$g = \sum_{x \in X} a_a g(x) a^*_x.$$

If we apply this formula to the element $g$ in $a_v g a_u^* \in M_k$, we get an element of $M_{k+1}$. Hence $M_{k+1} \supset M_k$.

The algebra $M_{d^\infty} (G)$ is defined as the closure in $O_G$ of the ascending union $\bigcup_{k \geq 0} M_k$.

Every algebra $M_k$ contains the sub-algebra equal to the linear span of the elements $a_v a^*_u$, which is isomorphic to $M_{d^k \times d^k} (\mathbb{C})$. Their union (i.e., the closed linear span of all elements $a_v a^*_u$ for $|v| = |u|$) is isomorphic to the Glimm’s uniformly hyperfinite algebra $M_{d^\infty} (\mathbb{C})$ (see [10, 13]), since the expansion rule (3.1) for $g = 1$ defines the diagonal embeddings $M_{d^k \times d^k} \hookrightarrow M_{d^{k+1} \times d^{k+1}}$. Its diagonal subalgebra, generated by the projections $a_v a^*_u$ is isomorphic to the algebra $C(X^\omega)$ of continuous functions on $X^\omega$. The isomorphism identifies a projection $a_v a^*_u$ with the characteristic function of the cylindrical set $v X^\omega$. It is easy to see that the action of $G \subset M_{d^\infty} (G)$ on the diagonal algebra by conjugation coincides with the action by conjugation of $G$ on $C(X^\omega)$.

It is proved in [11] that if the group $(G, X)$ is self-replicating, then the subalgebra $M_{d^\infty} (G)$ of $O_G$ is generated by $G$ and the subalgebra $C(X^\omega)$, i.e., that $M_{d^\infty} (G)$ is the cross-product of $\mathcal{A}_{max}$ and the algebra of continuous functions on $X^\omega$ induced by the usual action of $G$ on $X^\omega$.

Similarly to the Cuntz algebra (see [12, 13]), we have a natural strongly continuous (gauge) action $\Gamma$ of the circle $T = \{ z \in \mathbb{C} : |z| = 1 \}$ on $O_G$ by

$$\Gamma_z (g) = g$$

$$\Gamma_z (a_x) = z a_x.$$

for $g \in G, x \in X$ and $z \in T$.

Then $\Gamma_z (a_v g a_u^*) = z^{|v| - |u|} a_v g a_u^*$ for $u, v \in X^*$, $g \in G$, thus the integral $\int \Gamma_z (a_v g a_u^*) dz$ is equal to zero for $|v| \neq |u|$ and to $a_v g a_u^*$ for $|v| = |u|$, where $dz$ is the normalized Lebesgue measure on the circle. Consequently the map

$$(3.4) \quad M_0 (a) = \int \Gamma_z (a) dz$$

for $a \in O_G$ is a conditional expectation from $O_G$ onto the subalgebra $M_{d^\infty} (G)$.

**Example 12.** If $G$ is the cyclic group generated by the adding machine $a = \sigma (1, a)$, then the algebra $\mathcal{A}_{max} = \mathcal{A}_{min}$ is isomorphic to the algebra $C(T)$ with the linear recursion $C(T) \rightarrow M_2 (C(T))$ coming from the double self-covering of the circle (if we identify $C(T)$ with $C^* (\mathbb{Z})$ via Fourier series, then this linear recursion is
given by \( z \mapsto \begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix} \), where \( z = e^{2\pi it} \), \( t \in \mathbb{R} \), is the variable of the Fourier series). Consequently, the algebra \( \mathcal{M}_{d \to} (G) \) of the adding machine action is the Bunce-Deddens algebra. It is also isomorphic to the cross-product algebra of the odometer action on the Cantor space \( X^\omega \) (see [13]).

3.7. Overview of algebras associated with self-similar groups. We studied above the universal Cuntz-Pimsner algebra \( O_G \) of a self-similar group \( G \). Representations of this algebra correspond to representations of \( G \) on a Hilbert space \( H \) which are self-similar with respect to some similarity of \( H \) (see Proposition 3.2).

The subalgebra of \( O_G \) generated by \( G \) was denoted \( A_{\text{max}} \). Any self-similar representation of \( G \) extends to a representation of \( A_{\text{max}} \), hence \( A_{\text{max}} \) is the completion of the group algebra with respect to the maximal self-similar norm.

Third algebra is the natural inductive limit of the algebras \( M_{d^n \times d^n} (A_{\text{max}}) \), which we denoted by \( M_{d \to} (G) \). It is a subalgebra of \( O_G \) in a natural way.

Hence we get the following tower of algebras associated to a self-similar group

\[
A_{\text{max}} \subset M_{d \to} (G) \subset O_G.
\]

There exits also the smallest self-similar norm on the group algebra, which corresponds to a unique simple homomorphic image \( O_{G_{\text{min}}} \) of the Cuntz-Pimsner algebra \( O_G \) (see Theorem 3.3). The subalgebra of \( O_{G_{\text{min}}} \) generated by \( G \) is denoted \( A_{\text{min}} \) and it is the completion of the group algebra with respect to the smallest self-similar norm. The image \( M_{d \to} (G)_{\text{min}} \) of \( M_{d \to} (G) \) is also simple.

For any other self-similar representation \( \rho \) of \( G \) we get the corresponding homomorphic image of \( O_G, M_{d \to} (G) \) and \( A_{\text{max}} \). We get hence the following commutative diagram of algebras

\[
\begin{array}{ccc}
A_{\text{max}} & \hookrightarrow & M_{d \to} (G) & \hookrightarrow & O_G \\
\downarrow & & \downarrow & & \downarrow \\
A_{\rho} & \hookrightarrow & M_{d \to} (G)_\rho & \hookrightarrow & O_{G_\rho} \\
\downarrow & & \downarrow & & \downarrow \\
A_{\text{min}} & \hookrightarrow & M_{d \to} (G)_{\text{min}} & \hookrightarrow & O_{G_{\text{min}}}
\end{array}
\]

where all vertical arrows are surjective and all horizontal are embeddings.

An important example of a self-similar representation is the natural unitary representation \( \pi \) of \( G \) on \( L^2(X^\omega, \nu) \), where \( \nu \) is the uniform Bernoulli measure on the boundary \( X^\omega \) of the rooted tree. Together with the natural self-similarity of the space \( L^2(X^\omega, \nu) \) this gives a representation of \( O_G \). We denote the respective homomorphic images by \( A_{\text{mes}} \hookrightarrow M_{d \to} (G)_{\text{mes}} \hookrightarrow O_{G_{\text{mes}}} \).

The algebra \( A_{\text{mes}} \) is residually finite-dimensional, since the representation \( \pi \) of \( G \) is a direct sum of finite-dimensional representations (coming from the action of \( G \) on the levels of the tree). These finite-dimensional representations give an additional tool to the study of the algebra \( A_{\text{mes}} \).

Therefore the next questions about the epimorphisms \( A_{\text{max}} \to A_{\text{mes}} \to A_{\text{min}} \) are natural.

**Problem 2.** When are the epimorphisms \( A_{\text{mes}} \to A_{\text{min}} \) and \( A_{\text{max}} \to A_{\text{mes}} \) isomorphisms?
Problem 3. Under which conditions the algebra $A_{mes}$ isomorphic to the reduced $C^*$-algebra of the group $G$? Is it so when $G$ is the free group generated by the automaton shown on Figure 3 in Example 2?

4. Contracting groups and limit solenoids

4.1. Matrix recursion for contracting groups.

Definition 4.1. A self-similar group $(G, X)$ is said to be contracting if there exists a finite set $N \subset G$ such that for every $g \in G$ there exists $n_0$ such that $g|_v \in N$ for all words $v \in X^*$ of length greater than $n_0$. The smallest set $N$ with this property is called the nucleus of the self-similar group.

Example 13. The adding machine action of $\mathbb{Z}$ is contracting with the nucleus $N = \{1, a, a^{-1}\}$, since $a^n|_0 = a^{|n/2|}$ and $a^n|_1 = a^{|n/2|}$.

Example 14. It is also not hard to prove that the torsion group from Example 1 is contracting with the nucleus $\{1, a, b, c, d\}$.

Example 15. The free group considered in Example 2 is not contracting. In this example any section $g|_v$ has the same length as $g$.

The nucleus $N$ of a contracting group can be interpreted as an automaton, i.e., for every $g \in N$ and every letter $x \in X$ the restriction $g|_x$ belongs to $N$. Consequently, if we denote by $N$ the linear span of $N$ in $\mathbb{C}[G]$ and $\phi : \mathbb{C}[G] \rightarrow M_{d \times d}(\mathbb{C}[G])$ is the associated matrix recursion, then $\phi(N)$ is a subspace of the space $M_{d \times d}(N)$ of matrices with entries in $N$. Moreover, the following is true.

Theorem 4.1 ([39]). If the group $(G, X)$ is contracting and $N$ is its nucleus, then the algebra $O_G$ is generated by $\{a_x\}_{x \in X} \cup N$ and is defined by the following finite set of relations

1. Cuntz relations
   $$a_x^*a_x = 1$$

2. decompositions
   $$g = \sum_{x \in X} a_{g(x)} g|_x a_x^*$$
   for every $g \in N$ (this includes the remaining Cuntz algebra relation
   $$\sum_{x \in X} a_x a_x^* = 1$$
   in the case $g = 1$).

3. all relations $g_1 g_2 g_3 = 1$ of length at most three which are true for the elements of the nucleus $N$ in the group $G$ and relations $g g^* = g^* g = 1$ for $g \in N$.

Moreover, the groupoid of germs of the action of $G$ on $X^\omega$ is amenable in the case of a contracting group $G$. This follows from the fact that the orbits of the of the action on $X^\omega$ have polynomial growth (see [1] and [38]). This implies the following corollary of Theorem 3.5.

Proposition 4.2. If the group $G$ is contracting and level-transitive, then the algebras $A_{max}$ and $A_{red}$ coincide. If, additionally, for every element $g$ of the nucleus of $G$ the interior of the set of fixed points of $g$ is closed, then all self-similar completions of $G$ are isomorphic to $A_{max} = A_{min} = A_{red}$.
Examples 16. The adding machine action, the Basilica group \[3\] satisfy all the conditions of the corollary, hence they have unique self-similar completions.

Example 17. Consider the group generated by the transformations \(a, b, c\) given by
\[
\begin{align*}
  a(0w) &= 1w, & a(1w) &= 0w \\
  b(0w) &= 0a(w), & b(1w) &= 1c(w) \\
  c(0w) &= 0w, & c(1w) &= 1b(c).
\end{align*}
\]
This is one of the Grigorchuk groups \(G_w\) studied in [22] (for \(w = \ldots 11\)). Its growth was studied by A. Erschler in [14].

This group is contracting with the nucleus \(\{1, a, b, c, bc = cb\}\), but its groupoid of germs is not Hausdorff. Namely, the set of fixed points of \(c\) is equal to \(\{111\ldots\} \cup \bigcup_{k=0,1,\ldots} \{11\ldots 10X^\omega\}\), its interior is \(\bigcup_{k=0,1,\ldots} \{11\ldots 10X^\omega\}\), which is not closed.

Let us show that in this case the element \(b + c - bc - 1\) belongs to the kernel of the epimorphism \(A_{\text{max}} = A_{\text{red}} \to A_{\text{min}}\).

We have
\[
\phi(b + c - bc - 1) = \left( \begin{array}{cc} a & 0 \\ 0 & c \end{array} \right) + \left( \begin{array}{cc} 1 & 0 \\ 0 & b \end{array} \right) - \left( \begin{array}{cc} a & 0 \\ 0 & bc \end{array} \right) - \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) = \left( \begin{array}{cc} 0 & 0 \\ 0 & b + c - bc - 1 \end{array} \right),
\]
which implies that \(\pi(b + c - bc - 1) = 0\) for the permutation representation \(\pi\) on any orbit of \(G\) on \(X^\omega\). Hence \(b + c - bc - 1\) is equal to zero in \(A_{\text{min}}\).

On the other hand, it follows from contraction that the isotropy group of the point \(111\ldots \in X^\omega\) in the groupoid of germs of the action of \(G\) on \(X^\omega\) contains \(4\) elements (the germs of \(1, b, c, bc\)). Consequently, for any germ \(\gamma\) with range \(111\ldots\) the germs \(\gamma, b \cdot \gamma, c \cdot \gamma\) and \(bc \cdot \gamma\) are pairwise different (here the range of a germ \((h, x)\) is the point \(h(x)\)). This implies that the element \(b + c - bc - 1\) is non-zero in the regular representation of the groupoid, hence it is non-zero in \(A_{\text{red}}\).

Consequently, the homomorphism \(A_{\text{red}} \to A_{\text{min}}\) is not an isomorphism in general. In particular, \(A_{\text{red}}\) is not simple, even though the action of the group \(G\) is minimal. (If the action is minimal and the groupoid of germs is Hausdorff, then the reduced algebra of the groupoid of germs is simple, see [46] Proposition 4.6.)

Problem 4. Describe the kernel of the epimorphism \(A_{\text{red}} \to A_{\text{min}}\). Is it true that for a contracting group \(G\) it is generated (in \(O_G\)) by linear combinations of the elements of the nucleus?

4.2. Limit solenoid. Let us fix some contracting self-similar group \((G, X)\) with the nucleus \(N\). Consider the space \(X^2\) of bi-infinite sequences of the form
\[
\ldots x_{-2}x_{-1} \cdot x_0x_1 \ldots
\]
of letters \(x_i \in X\). Here the dot marks the place between the coordinates number 0 and number \(-1\). We consider \(X^2\) to be a topological space with the direct product topology of discrete sets \(X\).
Definition 4.2. Two sequences \( \ldots x_{-2}x_{-1} \cdot x_0x_1 \ldots \) and \( \ldots y_{-2}y_{-1} \cdot y_0y_1 \ldots \) are said to be asymptotically equivalent (with respect to the action of \( G \)) if there exists a finite set \( N \subset G \) and a sequence \( g_i \in N \) such that

\[
g_i(x_{i}x_{i+1}x_{i+2} \ldots) = y_{i}y_{i+1}y_{i+2} \ldots
\]

for all \( i \in \mathbb{Z} \).

It is proved in [38] that we can take \( N \) equal to the nucleus of \( G \) and that the asymptotic equivalence relation can be described in the following way.

Proposition 4.3. The sequences \( \ldots x_{-2}x_{-1} \cdot x_0x_1 \ldots \) and \( \ldots y_{-2}y_{-1} \cdot y_0y_1 \ldots \) are asymptotically equivalent if and only if there exists a sequence \( g_i \in N \) of elements of the nucleus such that \( g_i \cdot x_i = y_i \cdot g_{i-1} \), i.e., such that \( y_i = g_i(x_i) \) and \( g_{i-1} = g_i^2 | x_i \).

Definition 4.3. The limit solenoid \( S_G \) of the self-similar group \((G, X)\) is the quotient of the topological space \( X^\omega \) by the asymptotic equivalence relation.

The next proposition follows from the definition and Proposition 4.3 (see details in [38]).

Proposition 4.4. The limit solenoid \( S_G \) is a compact metrizable finite-dimensional space. If the action of \( G \) on \( X^\omega \) is level-transitive, then \( S_G \) is connected. The shift

\[
\ldots x_{-2}x_{-1} \cdot x_0x_1 \ldots \mapsto \ldots x_{-3}x_{-2} \cdot x_{-1}x_0 \ldots
\]

induces a homeomorphism \( \hat{s} : S_G \rightarrow S_G \).

Definition 4.4. Let \( X^-^\omega \) be the space of sequences \( x_2x_1 \), over the alphabet \( X \) with the direct product topology. Two sequences \( x_2x_1, y_2y_1 \in X^-^\omega \) are asymptotically equivalent if there exists a finite set \( N \subset G \) and a sequence \( g_k \in N \) such that \( g_k(x_k \ldots x_1) = y_k \ldots y_1 \) for all \( k \geq 1 \). The quotient of \( X^-^\omega \) by the asymptotic equivalence relation is called the limit space of the group \((G, X)\) and is denoted \( J_G \).

Here also we can take \( N \) to be equal to the nucleus (see [38]).

We have a natural continuous projection \( S_G \rightarrow J_G \) induced by the map

\[
\ldots x_{-2}x_{-1} \cdot x_0x_1 \ldots \mapsto \ldots x_{-2}x_{-1} \cdot x_0x_1 \ldots
\]

This projection semiconjugates the map \( \hat{s} : S_G \rightarrow S_G \) with the map \( s : J_G \rightarrow J_G \) induced by the one-sided shift \( \ldots x_2x_1 \mapsto \ldots x_3x_2 \). We call \((J_G, s)\) the limit dynamical system.

The following theorem is proved in [38], where a more general formulation can be found.

Theorem 4.5. If \( f : M_1 \rightarrow M \) is an expanding partial covering (where \( M_1 \) is an open subset of \( M \) with the induced Riemann metric), then the iterated monodromy group \( IMG(f) \) is contracting and the limit dynamical system \( s : J_{IMG(f)} \rightarrow J_{IMG(f)} \) is topologically conjugate to the action of \( f \) on its Julia set.

The limit solenoid \( S_G \) can be reconstructed from the limit dynamical system as the inverse limit of the sequence

\[
J_G \leftarrow^s J_G \leftarrow^s \ldots
\]

The map \( s \) induces a homeomorphism of the inverse limit, which is conjugate with \( \hat{s} : S_G \rightarrow S_G \).
Example 18 (Lyubich-Minsky laminations). Let \( f \in \mathbb{C}(z) \) be a post-critically finite rational function (i.e., the orbits of its critical points are finite). Consider the inverse limit \( \hat{\mathcal{S}}_f \) of the backward iteration

\[
\hat{\mathbb{C}} \leftarrow \hat{\mathbb{C}} \leftarrow \ldots,
\]

where \( \hat{\mathbb{C}} \) is the Riemann sphere. The shift along the inverse sequence (i.e., along the action of \( f \)) defines a homeomorphism \( \hat{f} : \hat{\mathcal{S}}_f \rightarrow \hat{\mathcal{S}}_f \), called the natural extension of \( f \). We have the natural projection \( P \) of \( \hat{\mathcal{S}}_f \) onto \( \hat{\mathbb{C}} \) (onto the first term of the inverse sequence).

Let \( \mathcal{S}_f \) be the preimage of the Julia set in \( \hat{\mathcal{S}}_f \) under the projection \( P \). Then the space \( \mathcal{S}_f \) is homeomorphic to the limit solenoid of \( \text{IMG}(f) \) and the action of \( \hat{f} \) on \( \mathcal{S}_f \) is topologically conjugate to the action of the shift on the limit solenoid.

The space \( \hat{\mathcal{S}}_f \) and the homeomorphism \( \hat{f} \) were studied in [35].

4.3. The solenoid as a hyperbolic dynamical system. Let us have a look at the dynamical system \( \hat{s} : \mathcal{S}_G \rightarrow \mathcal{S}_G \) more carefully. To avoid technicalities, we will assume that the self-similar group \( (G, \mathcal{X}) \) is finitely generated, contracting, self-replicating and that it is regular, which means that for every \( g \in G \) and \( w \in \mathcal{X}^\omega \) either \( g(w) \neq w \), or \( w \) is fixed by \( g \) together will all points of a neighborhood of \( w \) (i.e., for some beginning \( v \) of \( w \) the restriction \( g|_v \) is trivial). In other words, a group \( G \) is said to be regular if every point of \( \mathcal{X}^\omega \) is \( G \)-regular in the sense of Definition 3.4. It is sufficient to check the regularity condition for the elements \( g \) belonging to the nucleus of \( G \).

If \( \hat{f} : \mathcal{M} \rightarrow \mathcal{M} \) is an expanding self-covering of a complete compact geodesic space, then the iterated monodromy group \( G = \text{IMG}(f) \) is contracting and the dynamical system \( \hat{s} : \hat{\mathcal{J}}_{\text{IMG}(f)} \rightarrow \hat{\mathcal{J}}_{\text{IMG}(f)} \) is topologically conjugate to \( (\mathcal{M}, f) \). Moreover, \( \text{IMG}(f) \) satisfies the regularity condition in this case (this easily follows from the description of the orbispace structure on \( \hat{\mathcal{J}}_{\text{IMG}(f)} \) given in [38]).

We say that two points \( \xi, \zeta \in \mathcal{S}_G \) are stably (unstably) equivalent if for every neighborhood of the diagonal \( U \subset \mathcal{S}_G \times \mathcal{S}_G \) there exists \( n_U \geq 0 \) such that

\[
(\hat{s}^n(\xi), \hat{s}^n(\zeta)) \in U
\]

for all \( n \geq n_U \) \((n \leq -n_U)\), respectively.

Proposition 4.6. Let the points \( \xi, \zeta \in \mathcal{S}_G \) be represented by sequences \( (x_n)_{n \in \mathbb{Z}} \) and \( (y_n)_{n \in \mathbb{Z}} \), respectively.

The points \( \xi, \zeta \) are stably equivalent if and only if there exists \( n \in \mathbb{Z} \) such that the sequences \( \ldots x_{n-1} x_n \) and \( \ldots y_{n-1} y_n \in \mathcal{X}^\omega \) are asymptotically equivalent, i.e., represent the same point of \( \mathcal{J}_G \).

The points \( \xi, \zeta \) are unstably equivalent if and only if there exists \( n \in \mathbb{Z} \) such that

\[
g(x_n x_{n+1} \ldots) = y_n y_{n+1} \ldots
\]

for some element \( g \) of the nucleus.

Proof: It is easy to see that stable and unstable equivalence follows from the conditions in the proposition.

Let us show that the converse implications hold. Let \( k \) be such that for every two elements \( g, h \in \mathcal{N} \) of the nucleus and every word \( v \in \mathcal{X}^n \) either \( g(v) \neq h(v) \) or \( g(v) = h(v) \) and \( g|_v = h|_v \). Such \( k \) exists by the regularity condition.
Denote by $U_k$ the set of pairs $((x_n)_{n \in \mathbb{Z}}, (y_n)_{n \in \mathbb{Z}})$ such that $g(x_{-k} \ldots x_k) = y_{-k} \ldots y_k$ for some element $g \in \mathcal{N}$ of the nucleus. Let $U_k$ be the image of $U_k$ in $S_G \times S_G$.

Every set $U_k$ is a neighborhood of the diagonal and for every neighborhood of the diagonal $U \subset S_G \times S_G$ there exists $k$ such that $U_k \subset U$.

Let $\xi$ and $\zeta$ be the points of $S_G$ represented by the sequences $(x_i)_{i \in \mathbb{Z}}$ and $(y_i)_{i \in \mathbb{Z}}$.

Suppose that $\xi$ and $\zeta$ are stably equivalent. Then for any $k$ there exists $n_k$ such that the pair $(\tilde{S}^n((x_i)_{i \in \mathbb{Z}}), \tilde{S}^n((y_i)_{i \in \mathbb{Z}}))$ belongs to $U_k$ for all $n \geq n_k$. Then for every $n > n_k + k$ there exists $g_n \in \mathcal{N}$ such that $g_n(x_{-n}x_{-n+1} \ldots x_{-n+2k}) = y_{-n}y_{-n+1} \ldots y_{-n+2k}$. Consequently, for $h_n = g_n|_{x_{-n}x_{-n+k+1} \ldots x_{-n+2k}}$ we have

$$h_n(x_{-n+k} \ldots x_{-n+2k}) = y_{-n+k} \ldots y_{-n+2k}.$$ 

Note that by the choice of $k$ the element $h_n$ is determined uniquely by the pair of words $(x_{-n}x_{-n+1} \ldots x_{-n+2k}, y_{-n}y_{-n+1} \ldots y_{-n+2k})$ (i.e., does not depend on the choice of $g_n$). The uniqueness of $g_n$ implies that $h_n \cdot x_{-n+k} = y_{-n+k} \cdot h_n$, which finishes the proof.

The proof of the statement about the unstable equivalence is analogous. □

Hence we get the following.

**Corollary 4.7.** In conditions of Proposition 4.4 the points $\xi$ and $\zeta$ are unstably equivalent if and only if $x_0x_1 \ldots$ and $y_0y_1 \ldots$ belong to one $G$-orbit.

In other words, the unstable equivalence relation $U \subset S_G \times S_G$ is a union of the sets $U_g$ equal to the images in $S_G \times S_G$ of the sets

$$\{(\ldots -x_{-2}x_{-1} \cdot x_0x_1 \ldots \cdot y_{-2}y_{-1} \cdot g(x_0x_1 \ldots)) : x_i, y_i \in X\}.$$ 

Let us introduce a topology on $U$ equal to the (direct limit) topology of the union $U = \bigcup_{g \in G} U_g$, where $U_g$ are taken with the induced topology of the subset $U_g \subset S_G \times S_G$ and the direct limit is taken with respect to the identical maps between the sets $\bigcup_{g \in A} U_t$, where $A$ runs through the finite subsets of $G$.

This topology converts the unstable equivalence relation into a Hausdorff topological groupoid, which we call the unstable groupoid. A natural Haar system on it comes from the push forward of the uniform Bernoulli measure $\nu$ on $X^{-\omega}$.

The shift $\hat{s} : S_G \rightarrow S_G$ defines automorphisms of the stable and unstable groupoids.

The next theorem can be seen as a formulation of the fact that $G$ is the holonomy group of the unstable foliation of $S_G$ in terms of $C^*$-algebras.

**Theorem 4.8.** The universal convolution algebra of the unstable groupoid is strongly Morita equivalent to $\mathcal{M}_{d^\infty}(G)$.

**Proof.** Let us construct an equivalence bimodule (in the sense of [37]) between the groupoid $U$ of the unstable equivalence relation and the groupoid $\mathcal{G}$ of germs of the semigroup of transformations $T_vgT^*_v$ for $u, v \in X^*, |u| = |v|$ and $g \in G$. The universal convolution algebra of $\mathcal{G}$ is $\mathcal{M}_{d^\infty}(G)$ (see [39]). Note also that regularity of $(G, X)$ implies that the groupoid $\mathcal{G}$ is principal (i.e., is an equivalence relation), since the isotropy groups of $\mathcal{G}$ are trivial.

Let us consider the bimodule $Z$ consisting of the set of pairs $(\zeta, w)$, where $\zeta \in S_G$ and $w \in X^*$ are such that $\ldots x_{-2}x_{-1} \cdot w$ represents a point unstably equivalent to $\zeta$ (for some and hence for any $\ldots x_{-2}x_{-1}$). Let $Z_g \subset Z$ for $g \in G$ be the set of pairs $(\zeta, w)$ such that $\zeta$ is represented by a sequence $\ldots x_{-2}x_{-1} \cdot g(w)$. Endow $Z_g$
with the induced topology from $S_G \times X^\omega$. The set $Z$ is a union of $Z_g$ for all $g \in G$.

Consider the direct limit topology on $Z$ coming from this union.

We have a natural right action of $U$ on $Z$:

$$(\zeta_2, w) \cdot (\zeta_1, \zeta_2) = (\zeta_1, w)$$

and a left action of $G$ on $Z$:

$$(w_2, w_1) \cdot (\zeta, w_1) = (\zeta, w_2),$$

which satisfy the conditions of the equivalence bimodule (see Definition 2.1 of [37]).

One can find another proof of this theorem in [39].

The dynamical system $\hat{s} : S_G \to S_G$ is an example of a Smale space (for a definition see [32]). The $C^*$-algebras associated with hyperbolic dynamical systems were studied in [45, 32, 44, 47, 48].

5. Schur Complements and Rational Multidimensional Dynamics

5.1. Schur complements of operators. Schur complements are widely used in linear algebra (usually without knowledge of the name). The term “Schur complement” was apparently introduced for the first time by E. V. Haynsworth [29]. See a survey [11] of the use of Schur complement in algebra and statistics. It is also related to Bruhat normal form of matrices over a skew field (see [9]), which is used to define the Dieudonné determinant.

Schur complements are quite often used in different situations where renormalization principle can be used. Here we describe one such situation, which is related to computation of spectra of Hecke type operators and discrete Laplace operators attached to self-similar groups and their Schreier graphs. The material written here summarizes the ideas and results of the articles [3, 25, 26, 17, 20].

Let $H$ be a Hilbert space decomposed into a direct sum $H = H_1 \oplus H_2$ of two non-zero subspaces. Let $M \in B(H)$ be a bounded operator and let

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

be an operator matrix representing $M$ according to this decomposition.

Definition 5.1. (i) Assume that $D \in B(H_2)$ is invertible. Then the first Schur complement is the operator

$$S_1(M) = A - BD^{-1}C.$$

(ii) Assume that $A \in B(H_1)$ is invertible. Then the second Schur complement is the operator

$$S_2(M) = D - CA^{-1}B.$$

The reason why Schur complements are useful for the spectral problem is the following well known statement.

Theorem 5.1. Suppose that $D$ is invertible. Then $M$ is invertible if and only if $S_1(M)$ is invertible. Similarly, if $A$ is invertible, then $M$ is invertible if and only if $S_2(M)$ is invertible.

The inverse is computed then by the formula

$$M^{-1} = \begin{pmatrix} S_1^{-1} & -S_1^{-1}BD^{-1} \\ -D^{-1}CS_1^{-1} & D^{-1}CS_1^{-1}BD^{-1} + D^{-1} \end{pmatrix},$$

(5.1)
where $S_1 = S_1(M)$.

Proof. Consider

$$L = \begin{pmatrix} 1 & 0 \\ -D^{-1}C & D^{-1} \end{pmatrix},$$

where 0 and 1 represent the zero and identity linear maps between the corresponding subspaces of $H$.

Then

$$L^{-1} = \begin{pmatrix} 1 & 0 \\ C & D \end{pmatrix}, \quad ML = \begin{pmatrix} A - BD^{-1}C & BD^{-1} \\ 0 & 1 \end{pmatrix}.$$ A triangular operator matrix

$$R = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$$

represents a (right) invertible operator if and only if $x$ is (right) invertible, hence invertibility of $ML$ is equivalent to invertibility of $S_1(M)$ and the result follows.

The second part of the theorem is proved similarly.

Direct computations show that (5.1) takes place.

Note that if $A$, $D$ and $M$ are invertible, then

$$M^{-1} = \begin{pmatrix} S_1^{-1} & B' \\ C' & S_2^{-1} \end{pmatrix},$$

where $S_1 = S_1(M)$, $S_2 = S_2(M)$ and

$$B' = -S_1^{-1}BD^{-1} = -A^{-1}BS_2^{-1},$$
$$C' = -D^{-1}CS_1^{-1} = -S_2^{-1}CA^{-1}.$$ The last two equalities follow from

$$S_1A^{-1}B = B - BD^{-1}CA^{-1}B = BD^{-1}S_2$$
and

$$CA^{-1}S_1 = C - CA^{-1}BD^{-1}C = S_2D^{-1}C.$$ Formula (5.1) is called sometimes Frobenius formula, see, for instance [15]. We see that taking Schur complement of $M$ is equivalent to inverting the left top corner of the matrix $M^{-1}$.

The following corollary is known as Schur formula [49] and easily follows from the proof of Theorem 5.1.

**Corollary 5.2.** Let $H$ be finite dimensional and suppose that the determinant $|D|$ is not equal to zero. Then

$$|M| = |S_1(M)| \cdot |D|.$$ There is nothing special in decomposition of $H$ into a direct sum of two subspaces.

If $H = H_1 \oplus H_2 \oplus \cdots \oplus H_d$ and

$$M = \begin{pmatrix} M_{11} & \cdots & M_{1d} \\ \vdots & \ddots & \vdots \\ M_{d1} & \cdots & M_{dd} \end{pmatrix},$$

(5.2)
where \( M_{ij} : H_j \rightarrow H_i \), then we can write \( H \) as \( H = H_1 \oplus H_1^\perp \), where \( H_1^\perp = H_2 \oplus \cdots \oplus H_d \) and get

\[
M = \begin{pmatrix} M_{11} & B \\ C & D \end{pmatrix},
\]

where \( B, C \) and \( D \) are operators represented by \( 1 \times (d - 1), (d - 1) \times 1 \) and \( (d - 1) \times (d - 1) \)-sized operator matrices coming from \((5.2)\). Then \( S_1(M) \) is defined as before under the condition that \( D \) is invertible.

Changing the order of the summands (putting \( H_i \) on the first place, say using a cyclic permutation) we define the \( i \)th Schur complement \( S_i(M) \).

We will use these definitions in situations when \( H_i \) for \( i = 1, \ldots, m \) are isomorphic. There are two cases.

(I) Finite dimensional, when \( H = H' \oplus \cdots \oplus H' \) for \( \text{dim} \ H' < \infty \).

We will apply the Schur complements to the sequence of operators \( M_n \in B(H_n) \), where \( \text{dim} \ H_n = d^n \), and

\[
H_{n+1} = H_n \oplus \cdots \oplus H_n, \quad \text{d times}
\]

for all \( n = 0, 1, \ldots \) and \( \text{dim} \ H_0 = 1 \).

(II) Infinite dimensional case. Suppose that we have an infinite dimensional (separable) Hilbert space \( H \) and a fixed \( d \)-similarity \( \psi : H \rightarrow H^d \).

Let

\[
\bar{S}_i = \psi^{-1}S_i\psi : B(H) \rightarrow B(H), \quad i = 1, \ldots, d
\]

be partially defined transformations, where \( S_i \), as before, is the \( i \)th Schur complement.

In the sequel we will write just \( S_i \) instead of \( \bar{S}_i \), as this will not lead to a confusion.

**Proposition 5.3.** Let \( \rho \) be a representation of the Cuntz algebra \( O_d \) associated with the \( d \)-similarity \( \psi : H \rightarrow H^d \) and let \( T_i = \rho(a_i) \) be the images of the generators of the Cuntz algebra. Then the Schur complements are given by

\[
S_i(M) = (T_i^*M^{-1}T_i)^{-1}.
\]

**Proof.** Follows directly from the definition of \( T_i \) and inversion formula \((5.1)\) from Theorem \((5.1)\). Recall (see Proposition \((3.3)\)) that \( T_i \) are given by

\[
T_i(\xi) = \psi^{-1}(0, \ldots, 0, \xi, 0, \ldots, 0),
\]

where \( \xi \) is on the \( i \)th coordinate.

**Corollary 5.4.** We have the following formula for the composition of Schur complements

\[
S_{i_k} \circ \cdots \circ S_{i_1}(M) = ((T_{i_k} \cdots T_{i_1})^* \cdot M^{-1} \cdot (T_{i_k} \cdots T_{i_1}))^{-1}.
\]

We see that a composition of \( k \) Schur complements associated with the \( d \)-similarity \( \psi : H \rightarrow H^d \) is a Schur complement associated with the corresponding \( d^k \)-similarity of \( H \) obtained by iteration of \( \psi \).

Consider the semigroup \( S = \langle S_1, \ldots, S_m \rangle \) of partial transformations of \( B(H) \) generated by the transformations \( S_i \). We will call it the Schur renorm-semigroup.

In the examples that will follow, we will restrict \( S \) (or its particular elements \( s \in S \))
ont to some $S$-invariant (or $s$-invariant) subspaces $B' \subset B(H)$. In the examples that we consider $B'$ will be finite dimensional.

We will see in Example 25 how nicely Schur complement behaves on the space of resolvents.

The Schur transformations $S_i$ are homogeneous:

$$S_i(t M) = t S_i(M)$$

for all $t \in \mathbb{C}$. This allows to define the corresponding transformations $\hat{S}_i$ on the “projective space” $PB(H) = B(H)/\mathbb{C}^\times$ of $B(H)$ (here $\mathbb{C}^\times$ is the multiplicative group of $\mathbb{C}$). Restricting to invariant finite-dimensional subspaces $B' \subset B(H)$ (real or complex) we get transformations on the corresponding projective spaces $\mathbb{R}P^n$ or $\mathbb{C}P^n$, where $n + 1 = \dim B'$.

5.2. Schur complements in self-similar algebras. If $B$ is a unital Banach algebra together with a unital embedding $B \rightarrow M_{d \times d}(B)$ (we call such algebras self-similar), then we can compute the Schur complements $S_i(a)$ of the elements of $B$.

Let $(G, X)$ be a self-similar group, let $\phi : G \rightarrow \text{Symm}(X) \times G$ be the corresponding wreath recursion and let $O_d$ be the universal Cuntz-Pimsner algebra of $G$. Then the Schur complements $S_i$ are defined on $A_{\text{max}} \subset O_d$ and can be computed using Proposition 5.3 or directly using the matrix recursion $\phi : \mathbb{C}[G] \rightarrow M_{d \times d}(\mathbb{C}[G])$.

If $\psi : H \rightarrow H^d$ is a fixed $d$-similarity on $H$ (i.e., a representation of the Cuntz algebra $O_d$, then, as it was observed in Proposition 3.4 every self-similar representation $\rho$ of $G$ extends to a representation $\rho : O_d \rightarrow B(H)$. The corresponding Schur complements on the closure $A_\rho$ of $\mathbb{C}[G]$ are also computed by the formula in Proposition 5.3.

Let us consider, for instance, the space $L^2(X^\omega, \nu)$, the natural unitary representation $\pi$ of a self-similar group $(G, X)$ on it and the natural $d$-similarity (and the associated representation of the Cuntz algebra) on $L^2(X^\omega, \nu)$, see Example 4. Consider the respective self-similar completion $A_{\text{mes}}$ of $\mathbb{C}[G]$.

The partition $\xi_n$ of the boundary $\partial X^\omega = X^\omega$ into a disjoint union of $d^n$ cylinder subsets $v X^\omega$ ($v \in X^n$), corresponding to the vertices of the $n$th level of the tree is $G$-invariant (since it is invariant under the action of the whole automorphism group of the tree).

Let $H_n = \{ X_{A_i^{(n)}}, 1 \leq i \leq d^n \}$ be the finite dimensional subspace spanned by the characteristic functions of the atoms $A_i^{(n)}$ of the partition $\xi_n$. Then $H_n$ is a $\pi(G)$-invariant subspace and $\{ H_n \}_{n=1}^\infty$ is a nested sequence with canonical embeddings $H_n \hookrightarrow H_{n+1}$ and

$$H = \bigcup_{n=1}^\infty H_n.$$  \hfill (5.3)

Note that

$$H_{n+1} = \bigoplus_{1 \leq i \leq d} T_i(H_n),$$

where $T_i = \pi(a_i)$ are the generators of the Cuntz algebra associated to the natural $d$-similarity of $L^2(X^\omega, \nu)$. The space $T_i(H_n)$ is the linear span of the characteristic functions of cylindrical sets of the level $n$ with the first letter $i$. 

SELF-SIMILAR GROUPS, OPERATOR ALGEBRAS AND SCHUR COMPLEMENTS 25


Let $M = \sum_{g \in G} \lambda_g g \in \ell^1(G)$ be an arbitrary element of the Banach group algebra and let $\pi(M)$ be its image in $A_{mes}$. Denote $M_n = \pi(M)|_{H_n}$. Relation (5.3) and $G$-invariance of the spaces $H_n$ implies that

$$
(5.4) \quad Sp(\pi(M)) = \bigcup_{n=1}^{\infty} Sp(M_n).
$$

The formula (5.4) reduces the problem of computation of the spectrum of $M$ to finite dimensional problems of finding $Sp(M_n)$.

The latter problem requires computation of the $d^n \times d^n$-size matrices $g^{(n)}$ given by the recursive relations:

$$
g^{(0)} = 1, \quad g^{(n+1)} = (A_{yx})_{x,y \in X}, \quad A_{yx} = \begin{cases} g^{(n)}(x) & \text{if } g^{(n)}(x) = y, \\ 0 & \text{otherwise}, \end{cases}
$$

repeating the matrix recursion (3.2) and finding spectra of their linear combinations. There is no general tool for solving this problem. Nevertheless, we will show how the problem can be solved using the Schur map in some particular cases.

We will start with some examples which have a complete solution of the spectral problem, then show a few examples when the spectral problem is reduced to a problem in Dynamical Systems of finding invariant sets for a multidimensional rational mapping (it looks that these invariant sets are indeed “strange attractors” for the corresponding maps, at least it is so in many cases). Finally we will show some examples for which the spectral problem is extremely interesting and has links with other topics, but it looks like the method of Schur complements doesn’t work for these examples.

We perform some computations in homogeneous coordinates of the space of parameters, since the spectrum of a pencil (i.e., the set of values of parameters corresponding to degenerate matrices) is invariant under multiplication by a non-zero number and the Schur complement transformations are homogeneous. But the obtained transformations are written in non-homogeneous coordinates as transformations in the Euclidean space.

In all examples that follow we use parametric family of elements of a group algebra (usually the sum with coefficients-parameters of generators and identity) or a matrix with entries of this type and consider the corresponding operator in $L^2(X^\omega, \nu)$ given by the representation $\pi$ (so that the operators belong to the algebra $A_{mes}$).

Example 19. Let $\mathcal{G} = \langle a, b, c, d \rangle$ be the group from Example 1. Since the generators are of order 2, the operator $M = \pi(a + b + c + d)$ is a self-adjoint operator in $B(L^2(\mathcal{G}, \nu))$, being also a self-adjoint element of $A_{mes}$ with the same spectrum. In what follows we omit $\pi$. We have $a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $b = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$, $c = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, $d = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$, and

$$
M = a + b + c + d = \begin{pmatrix} 2a + 1 & 1 \\ 1 & b + c + d \end{pmatrix}.
$$
Also \(a_{n+1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\), \(b_{n+1} = \begin{pmatrix} a_n & 0 \\ 0 & c_n \end{pmatrix}\), \(c_{n+1} = \begin{pmatrix} a_n & 0 \\ 0 & d_n \end{pmatrix}\), \(d_{n+1} = \begin{pmatrix} 1 & 0 \\ 0 & b_n \end{pmatrix}\) are recursive relations determining matrices of generators viewed as operators in the spaces \(H_{n+1}\) and \(H_n\) respectively (0 and 1 represent here the zero operators and the identity operator of the corresponding size). Instead of computing spectrum of \(M_n = a_n + b_n + c_n + d_n\) directly, one can try to find spectrum of the whole pencil \(M_n(x,y,z,u) = xa_n + yb_n + zc_n + ud_n\) of matrices, where \(x, y, z, u \in \mathbb{C}\).

If we find spectrum of each of the pencils \(M_n(x,y,z,u)\), we will solve the problem of finding spectrum of the pencil \(M(x,y,z,u) = xa + yb + zc + ud\) of infinite dimensional operators, since spectrum \(M(x,y,z,u)\) is the closure of the unions of spectra of \(M_n(x,y,z,u)\).

The 5-dimensional space of operators \(M(x,y,z,u,v) = xa + yb + zc + ud + v\cdot1\) is invariant with respect to the Schur complements which in this case are given by the following formulae.

\[
\begin{array}{c}
\hat{S}_1: \begin{pmatrix} x \\ y \\ z \\ u \\ v \end{pmatrix} \mapsto \begin{pmatrix}
\frac{z + y}{x^2(2yuv - u(y^2 + z^2 - u^2 + v^2))} \\
\frac{(y + z - u - v)(y - z - u - v)(-y + z - u - v)}{x^2(2yuv - u(y^2 + z^2 - u^2 + v^2))} \\
\frac{(y + z + u + v)(y + z - u - v)(y - z + u + v)}{x^2(2yuv - u(y^2 + z^2 - u^2 + v^2))} \\
\frac{u + v + (y + z + u + v)(y + z - u - v)(-y + z - u - v)}{x^2(2yuv - u(y^2 + z^2 - u^2 + v^2))}
\end{pmatrix}
\end{array}
\]

and

\[
\begin{array}{c}
\hat{S}_2: \begin{pmatrix} x \\ y \\ z \\ u \\ v \end{pmatrix} \mapsto \begin{pmatrix}
\frac{u + v + (y + z - u - z)}{x^2(u + v)} \\
\frac{(u + v + y + z)(u + v - y - z)}{x^2(u + v)} \\
\frac{y}{u + v + y + z} \\
\frac{z}{x^2(u + v)} \\
\frac{v - (u + v + y + z)(u + v - y - z)}{x^2(u + v)}
\end{pmatrix}
\end{array}
\]

The complement \(\hat{S}_2\) is not defined when \((y + z)a + (u + v)\cdot1\) is not invertible, i.e., if
\[
y + z + u + v = 0 \tag{5.5}
\]
or
\[
y + z - u - v = 0, \tag{5.6}
\]
as \(Sp(a) = \{\pm1\}\).

The complement \(\hat{S}_1\) is not defined for the same conditions \(5.5\) and \(5.6\) and also the conditions obtained from them by application of the cyclic permutation \((u,y,z) \mapsto (z,u,y) \mapsto (y,z,u)\), as

\[
yc + zd + ub + v \cdot 1 = y \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} + z \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} + u \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} + v \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} (y + u)a + (z + v)1 & 0 \\ 0 & yd + zb + uc + v1 \end{pmatrix}.
\]

Note that in both cases the complement \(\hat{S}_i\) is not defined if and only if the denominator in the corresponding formulae is equal to zero.
Note that the third iteration of the map $\hat{S}_2$ fixes the three middle coordinates, hence we actually get a family of maps $C^2 \to C^2$ depending on three parameters.

It is not known at the moment if the spectrum of the pencil $M(x, y, z, u, v)$ is invariant with respect to these maps and what are its topological properties. Further investigation in this direction would be interesting.

But let us simplify the problem and consider the pencil $M(x) = xa + b + c + d$. The spectral problem for $M(x)$ consists in finding those values of the parameters $x$ and $\lambda$ for which the operator $M(x) - \lambda \cdot 1$ is not invertible. We will call this set the spectrum of the pencil. This terminology will be used also for further examples. To simplify transformations let us change the system of coordinates and consider the pencil $R(\lambda, \mu) = -\lambda a + b + c + d - (\mu + 1)1$ and its finite dimensional approximations $R_n(\lambda, \mu) = -\lambda a_n + b_n + c_n + d_n - (\mu + 1)1_n$ represented by a 2-parametric family of $2^n \times 2^n$-matrices.

Let

$$\Sigma = \{(\lambda, \mu) : R(\lambda, \mu) \text{ is not invertible}\}$$

and

$$\Sigma_n = \{(\lambda, \mu) : R_n(\lambda, \mu) \text{ is not invertible}\}.$$

Define $F, G : \mathbb{R}^2 \to \mathbb{R}^2$ be two rational maps

$$F : \begin{pmatrix} \lambda \\ \mu \end{pmatrix} \mapsto \begin{pmatrix} \frac{2(4-\mu^2)}{\lambda^2} \\ -\mu - \frac{\mu(4-\mu^2)}{\lambda^2} \end{pmatrix},$$

$$G : \begin{pmatrix} \lambda \\ \mu \end{pmatrix} \mapsto \begin{pmatrix} \frac{2\lambda^2}{4-\mu^2} \\ \mu + \frac{\mu\lambda^2}{4-\mu^2} \end{pmatrix}$$

and observe, following Y. Vorobets, that $H \circ F = G$, $H \circ G = F$, where

$$H : \begin{pmatrix} \lambda \\ \mu \end{pmatrix} \mapsto \begin{pmatrix} \frac{4}{\lambda} \\ -2\mu/\lambda \end{pmatrix}$$

is an involutive map $\mathbb{R}^2 \to \mathbb{R}^2$.

**Theorem 5.5.** I. The maps $F$ and $G$ seen as maps on the projective space are respectively the first and the second Schur maps restricted on the pencil $R(\lambda, \mu)$.

II. The set $\Sigma$ is invariant with respect to $F$ and $G$, i.e., $F^{-1}\Sigma = \Sigma$, $G^{-1}\Sigma = \Sigma$ and relations

$$\Sigma_{n+1} = F^{-1}\Sigma_n = G^{-1}\Sigma_n$$

hold.

III. The set $\Sigma$ is the set shown on Figure [4]

IV. The set $\Sigma_n$ is the union of the line $\lambda + \mu - 2 = 0$ and hyperbolas $H_\theta = 0$, $\theta \in \bigcup_{i=0} \alpha^{-i}(0)$, where $H_\theta = 4 - \mu^2 + \lambda^2 + 4\lambda\theta$. See Figure [5,2] for the spectrum $\Sigma_5$.

**Proof.** We restrict ourselves here only to computation of the Schur maps. The rest can be found in [3].

We have

$$R(\lambda, \mu) = \begin{pmatrix} 2a - \mu & -\lambda \\ -\lambda & b + c + d - \mu - 1 \end{pmatrix}.$$
For getting the first Schur map observe that $t = (b + c + d - 1)/2$ is an idempotent. Hence

$$(b + c + d - \mu - 1)^{-1} = (2t - \mu)^{-1} = \frac{2t + \mu}{4 - \mu^2}$$

and

$$\hat{S}_1(R(\lambda, \mu)) = 2a - \mu - \frac{\lambda^2(b + c + d + \mu - 1)}{4 - \mu^2},$$
which is proportional to $-\lambda' a + b + c + d - (\mu' + 1)$ for $(\lambda', \mu') = F(\lambda, \mu)$.

Similarly

$$\hat{S}_2(R(\lambda, \mu)) = b + c + d - \mu - 1 - \lambda^2(2a - \mu)^{-1}$$

$$= b + c + d - \mu - 1 - \frac{\lambda^2(2a + \mu)}{4 - \mu^2}$$

$$= -\frac{2\lambda^2}{4 - \mu^2} a + b + c + d - \left(\mu + \frac{\mu\lambda^2}{4 - \mu^2} + 1\right)$$

$$= -\lambda'' a + b + c + d - (\mu'' + 1),$$

where $(\lambda'', \mu'') = G(\lambda, \mu)$. □

**Remark.** The equations of hyperbolas $H_\theta$ are related by the rule

$$H_\theta(F(\lambda, \mu)) = 1 - \mu^2 \frac{1}{\lambda^4} H_{\frac{\sqrt{2\pi}}{2\pi}}(\lambda, \mu) H_{\frac{\sqrt{2\pi}}{2\pi}}(\lambda, \mu).$$

The maps $F$ and $G$ are semi-conjugate to the map $x \mapsto x^2 - 1$ via the maps $\psi_F(x, y) = \frac{4 - y^2 + x^2}{4x}$ and $\psi_G(x, y) = \frac{4 - x^2 + y^2}{4y}$, respectively. This is the crucial point in computation of the spectrum of $R_n(\lambda, \mu)$.

**Example 20.** Let $H = H^3$ be the Hanoi Towers group (on three pegs) studied in [19] and [18]. It is a self-similar group acting on $X^*$ for $X = \{0, 1, 2\}$ with generators $a, b, c$ satisfying the following matrix recursions

$$a = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & a \end{pmatrix},$$

$$b = \begin{pmatrix} 0 & 0 & 1 \\ 0 & b & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$c = \begin{pmatrix} c & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Consider the two-parametric family of elements of the $C^*$-algebra $A_{mes}$.

$$\Delta(x, y) = \begin{pmatrix} c - x & y & y & y \\ y & b - x & y & y \\ y & y & a - x \\ c - x & 0 & 0 & y \\ 0 & -x & 1 & 0 \\ 0 & 1 & -x & 0 \\ y & 0 & 0 & -x \\ 0 & y & 0 & 0 \\ 0 & 0 & y & -x \\ y & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & y & 0 \\ y & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & y & 0 \\ y & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & y & 0 \\ y & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & y & 0 \end{pmatrix}.$$
Permuting rows and columns and dividing them into blocks we get the matrix

\[
\begin{pmatrix}
  c - x & 0 & 0 & y & 0 & 0 & y & 0 & 0 \\
  0 & b - x & 0 & y & 0 & 0 & y & 0 & 0 \\
  0 & 0 & a - x & 0 & y & 0 & 0 & y & 0 \\
  y & 0 & 0 & -x & 0 & 1 & y & 0 & 0 \\
  0 & y & 0 & 0 & -x & 0 & 0 & y & 1 \\
  0 & 0 & y & 1 & 0 & -x & 0 & 0 & y \\
  y & 0 & 0 & y & 0 & 0 & -x & 1 & 0 \\
  0 & y & 0 & 0 & y & 0 & 1 & -x & 0 \\
  0 & 0 & y & 0 & 1 & y & 0 & 0 & -x \\
\end{pmatrix}.
\]

Computation of Schur complement with respect to the given partition of the matrix yields

\[ S_1(\Delta(x, y)) = \Delta(x', y'), \]

where

\[ x' = x - \frac{2(x^2 - x - y^2)y^2}{(x - y - 1)(x^2 - 1 + y - y^2)} \]

and

\[ y' = \frac{(x + y - 1)y^2}{(x - y - 1)(x^2 - 1 + y - y^2)}. \]

These rational functions were calculated in [19] using a different base. As it was observed in [19], the map \( F : (x, y) \mapsto (x', y') \) is semiconjugate to the map \( f : \mathbb{R} \to \mathbb{R} : x \mapsto x^2 - x - 3 \). The spectrum of \( \Delta(x, y) \) is in this case the union

\[ \bigcup_{\theta \in \mathbb{Z}} \mathcal{H}_\theta \cup L_0 \cup L_1 \cup L_2, \]

where \( S = \{-2, 0\} \), and \( \mathcal{H}_\theta \) is the hyperbola \( x^2 - xy - 2y^2 - \theta y = 1 \) and \( L_0, L_1, L_2 \) are the lines given by the equations

\[
\begin{align*}
  x - 1 - 2y &= 0, \\
  x + 1 + y &= 0, \\
  x - 1 + y &= 0.
\end{align*}
\]

A part of the spectrum is drawn on Figure 8.

**Example 21.** Let \( B = \text{IMG} \left( z^2 - 1 \right) \) be the Basilica group, studied in [27, 28, 5, 2, 38]. It is realized as a self-similar group acting on the binary tree and generated by \( a, b \), which are given by the matrix recursions

\[
\begin{align*}
  a &= \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}, \\
  b &= \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}.
\end{align*}
\]

Consider the pencil

\[
R(\lambda, \mu) = a + a^{-1} + \lambda(b + b^{-1}) - \mu = \begin{pmatrix} 2 - \mu & \lambda(1 + a^{-1}) \\ \lambda(1 + a) & b + b^{-1} - \mu \end{pmatrix}.
\]

We have

\[
\hat{S}_2(R(\lambda, \mu)) = b + b^{-1} - \mu - \frac{\lambda^2(2 + a + a^{-1})}{2 - \mu} = \frac{\lambda^2}{\mu - 2}(a + a^{-1}) + b + b^{-1} - \mu + \frac{2\lambda^2}{\mu - 2} \sim a + a^{-1} + \frac{\mu - 2}{\lambda^2}(b + b^{-1}) - \left( \frac{\mu(\mu - 2)}{\lambda^2} - 2 \right),
\]
where “∼” means “proportional”. We get hence the rational map

\begin{align*}
\lambda &\mapsto \frac{\mu-2}{\lambda^2}, \\
\mu &\mapsto -2 + \frac{\mu(\mu-2)}{\lambda^2}.
\end{align*}

(5.7)

This rational map is quite complicated and the structure of the spectrum of the pencil $R(\lambda, \mu)$ is unclear. Computer experiments suggest that it has to have a structure of a “strange attractor”. Most probably the map (5.7) is not semiconjugate to any one-dimensional map.

The pencil $R(\lambda, \mu)$ is not invariant with respect to the first Schur complement (because the inverse of $b + b^{-1} - \mu$ is not a finite sum).

**Example 22.** Let $G = \text{IMG}(z^2 + i)$ be the group studied in [2, 38, 20]. It is generated by $a, b, c$ given by the matrix recursions

\[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}, \quad
\begin{bmatrix}
a & 0 \\
0 & c
\end{bmatrix}, \quad
\begin{bmatrix}
b & 0 \\
0 & 1
\end{bmatrix}.
\]

It is a branch group of intermediate growth. For the notion of branch groups see [4] and [24]; the proof of intermediate growth of $\text{IMG}(z^2 + i)$ is given in [8]. Consider the pencil

\[
M(y, z, \lambda) = a + yb + zc - \lambda = M(y, z, \lambda) = \begin{pmatrix}
ya + zb - \lambda & 1 \\
1 & yc + z - \lambda
\end{pmatrix},
\]

then

\[
\mathcal{S}_1(M(y, z, \lambda)) = M(y', z', \lambda'),
\]
where \( \Phi : (y, z, \lambda) \mapsto (y', z', \lambda') \) is the map
\[
\Phi : \begin{cases}
y \mapsto \frac{y'}{y}, \\
z \mapsto \frac{z'}{y(z-\lambda-y)(z-\lambda+y)}, \\
\lambda \mapsto \frac{-\lambda y^2 + \lambda (z-\lambda)^2 + z - \lambda}{y(z-\lambda-y)(z-\lambda+y)}.
\end{cases}
\]

It is not known if \( \Phi \) is semiconjugate to a map of a smaller dimension and therefore there is no information about the structure of the spectrum of the pencil \( M(y, z, \lambda) \).

In this example, as also in the previous one, we have difficulty to apply \( S_2 \), since in these cases we do not get finite combinations of the group elements.

**Example 23.** Consider now the free group from Example 2. The corresponding matrix recursion is
\[
a_{n+1} = \begin{pmatrix} b_n & 0 \\ 0 & b_n \end{pmatrix}, \quad b_{n+1} = \begin{pmatrix} 0 & c_n \\ a_n & 0 \end{pmatrix}, \quad c_{n+1} = \begin{pmatrix} 0 & a_n \\ c_n & 0 \end{pmatrix}.
\]

For the inverses we have
\[
a_{n+1}^{-1} = \begin{pmatrix} b_n^{-1} & 0 \\ 0 & b_n^{-1} \end{pmatrix}, \quad b_{n+1}^{-1} = \begin{pmatrix} 0 & a_n^{-1} \\ c_n^{-1} & 0 \end{pmatrix}, \quad c_{n+1}^{-1} = \begin{pmatrix} 0 & c_n^{-1} \\ a_n^{-1} & 0 \end{pmatrix}.
\]

The problem is to find the spectrum of the matrix
\[
M_n = a_n + a_n^{-1} + b_n + b_n^{-1} + c_n + c_n^{-1} = a_{n-1} + a^{-1}_{n-1} + b_{n-1} + b_n^{-1} + c_{n-1} + c^{-1}_{n-1},
\]
but introduction of new parameters and computation of Schur complements unfortunately does not lead us to a success.

**Problem 5.** What is the spectrum of the matrix \( M_n \)? Does there exist \( \epsilon > 0 \) such that for every \( n \) the spectrum of \( M_n \) belongs to the set \( \{-6 + \epsilon, 6 - \epsilon\} \cup \{6, -6\} \)? If the answer is “yes”, then the sequence of the Schreier graphs of the action of \( \langle a, b, c \rangle \) on the levels of the tree \( X^* \) is a sequence of expanders. (Here a Schreier graph of a group generated by a finite set \( S \) and acting on a set \( M \) is the graph with the set of vertices \( M \) in which two vertices \( x, y \in M \) are connected by an edge if one is the image of the other under the action of an element of \( S \).)

**Example 24.** Consider the group generated by the transformations \( a, b, c \) of \( X^* \) for \( X = \{0, 1\} \) given by the recurrent relations
\[
a(0w) = 1b(w), \quad a(1w) = 0b(w),
\]
\[
b(0w) = 0a(w), \quad b(1w) = 1c(w),
\]
\[
c(0w) = 0c(w), \quad c(1w) = 1a(w),
\]
i.e., the group generated by the automaton shown on Figure 9.

This group is isomorphic (by a result of Y. Muntyan and D. Savchuk) to the free product \( C_2 * C_2 * C_2 \) of three groups of order 2, see [38] Theorem 1.10.2. The elements \( a, b, c \) are of order 2.

The corresponding matrix recursions are
\[
a_{n+1} = \begin{pmatrix} 0 & b_n \\ b_n & 0 \end{pmatrix}, \quad b_{n+1} = \begin{pmatrix} a_n & 0 \\ 0 & c_n \end{pmatrix}, \quad c_{n+1} = \begin{pmatrix} c_n & 0 \\ 0 & a_n \end{pmatrix}.
\]
Problem 6. What is the spectrum of the matrices
\[ M_n = a_n + b_n + c_n = \begin{pmatrix} a_{n-1} + c_{n-1} & b_n \\ b_n & a_{n-1} + c_{n-1} \end{pmatrix} \]?
In particular, is the spectrum of \( M \) a subset of \((-2\sqrt{2}, 2\sqrt{2}) \cup \{3\}\)? If it is, then the Schreier graphs of the action of the group on the levels of the tree \( X^* \) are Ramanujan.

6. Analytic families

Let \( \psi: H \rightarrow H_1 \oplus H_2 \) be an isomorphism and let
\[ M(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix} \]
be an analytic on some domain \( \Omega \) operator valued function. Assume that \( D(z) \) is invertible on \( \Omega \). Then
\[ S_1(M(z)) = A(z) - B(z)D^{-1}(z)C(z) \]
is analytic on \( \Omega \). We get therefore a partially defined Schur map between the spaces of analytic operator valued functions.

If \( H \) is a Hilbert space with a \( d \)-similarity \( \psi: H \rightarrow H^d \), then we can define in a similar way partially defined Schur transformations \( \tilde{S}_i \) on the space of analytic operator valued functions on \( H \).

Example 25. Let \( M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) and consider the function
\[ M - z = \begin{pmatrix} A - z & B \\ C & D - z \end{pmatrix}. \]
It is mapped by the Schur map to \( S_1 = A - z - B(D - z)^{-1}C \) and we have
\[ (M - z)^{-1} = \begin{pmatrix} S_1^{-1} & -S_1^{-1}B(D - z)^{-1} \\ -(D - z)^{-1}CS_1^{-1} & -(D - z)^{-1}CS_1^{-1}B(D - z)^{-1} + (D - z)^{-1} \end{pmatrix}. \]
It follows from the proof in Theorem 5.1 that \( S_1((M - z)^{-1}) = (A - z)^{-1} \) and, similarly, \( S_2((M - z)^{-1}) = (D - z)^{-1} \).

Consider now an operator-valued holomorphic function of \( n \) complex variables
\[ M: \mathbb{C}^n \rightarrow B(H) \]
and let again \( \psi: H \rightarrow H^d \) be a \( d \)-similarity. We will denote the respective Schur transformations \( \tilde{S}_i \) just by \( S_i \).
Definition 6.1. The function $M(z)$ is self-similar (with respect to $\widetilde{S}_1$) if there is a function $F : \mathbb{C}^n \to \mathbb{C}^n$ such that

$$S_i(M(z)) = M(F(z)).$$

We can consider functions $M$ from the projective space $\mathbb{CP}^{n-1}$ to $B(H)$. Self-similarity of such functions is defined analogically.

Example 26. Let $G$ be a self-similar group generated by $\{a_1, \ldots, a_k\}$ and acting on a rooted $d$-regular tree $X^*$. Consider the pencil of Hecke type operators

$$M(z) = M(z_1, \ldots, z_k) = \sum_{i=1}^{k} z_i \pi(a_i + a_i^{-1}),$$

where $\pi$ is the self-similar representation of $G$ on $L^2(X^*, \nu)$. The operators $M(z)$ belong to the algebra $A_{mes}$.

Assume that $M(z)$ is a self-similar function with respect to $S_i$. This means that

$$S_i(M(z_1, \ldots, z_k)) = \sum_{i=1}^{k} z_i' \pi(a_i + a_i^{-1})$$

where $z_i' = Z_j(z_1, \ldots, z_k)$ for $j = 1, \ldots, k$ are some functions. We get hence a map $F = (Z_1, \ldots, Z_k) : \mathbb{C}^k \to \mathbb{C}^k$ and the projectivized map $\widetilde{F} : \mathbb{CP}^{k-1} \to \mathbb{CP}^{k-1}$.

Proposition 6.1. The hyperspace $L = \left\{ \sum_{i=1}^{k} z_i = 0 \right\}$ is forward $S_i$-invariant for all $i$.

Here forward $S_i$-invariance is the condition $S_i(L \cap \text{Dom}(S_i)) \subset L$.

Proof.

Lemma 6.2. If $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ belongs to the kernel of $M$, then $x_1 \in \text{ker} S_1(M)$. Conversely, if $x_1 \in \text{ker} S_1(M)$ then there is $x_2$ such that $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \text{ker} M$.

Proof. We have

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} Ax_1 + Bx_2 \\ Cx_1 + Dx_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$ 

We get $x_2 = -D^{-1}Cx_1$ if $D$ has right inverse. Consequently, $(A - BD^{-1}C)x_1 = 0$.

Conversely, if $x_1 \in \text{ker} S_1(M)$, then $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, where $x_2 = -D^{-1}Cx_1$. \(\square\)

Since $\sum_{i=1}^{k} z_i = 0$ the constant function $c \neq 0$ on $\partial T$ belongs to ker $M$. Its restriction onto the cylindrical set of words starting with $i$ (where $i$ is the same as in $S_i$) is also a constant function and by the above lemma, we have $S_i(M)c = 0$, hence $\sum z_i' = 0$. That means that $S_i(L \cap \text{Dom}(S_i)) \subset L$. \(\square\)

Let

$\Sigma M(z) = \{ z : M(z) \text{ is not invertible} \}$

be the spectrum (critical set) of the pencil $M(z)$. We have then the following corollary of Theorem 5.1.
Corollary 6.3. Let $M(z)$ and $A(z), B(z), C(z), D(z)$ be as before. Then
$$\Sigma M(z) \setminus \Sigma D(z) = \Sigma S_1(M(z)) \setminus \Sigma D(z).$$

If $M(z)$ is self-similar, i.e., if $S_1(M(z)) = M(F(z))$, then
$$\Sigma S_1(M(z)) = \Sigma M(F(z)) = F^{-1}(\Sigma(M(z))),$$
and we get

Corollary 6.4. The spectrum $\Sigma M(z)$ is backward-invariant under $F$, i.e.,
$$F^{-1}(\Sigma M(z)) = \Sigma M(z)$$
if and only if
$$\Sigma D(z) \cap \Sigma M(z) = \Sigma D(z) \cap F^{-1}(\Sigma M(z)).$$

The condition of the corollary is not easy to check if $\Sigma = \Sigma M(z)$ is unknown.

In all examples that were treated, though, we have $F^{-1}\Sigma = \Sigma$.

Problem 7. Under what natural and easy-to-check conditions the equality $F^{-1}\Sigma = \Sigma$ is true? Under what conditions we have $\Sigma = \bigcup_{n=0}^{\infty} F^{-n}L$?

In all treated examples the map $F$ is rational. Therefore the problem of finding the spectrum of a pencil in a self-similar group is related to the problem of description of invariant subsets of multidimensional rational mappings. This subject is of independent interest in dynamical systems, multidimensional complex analysis, etc (see [50]). Spectra of Hecke type elements in $C^*$-algebras related to self-similar groups is a big source of interesting examples of dynamical systems on the complex projective space.

7. Schur maps and random walks

Consider the map $J : B(H) \to B(H) : A \mapsto A + I$ where $I$ is the identity operator and suppose that we have fixed a $d$-similarity $\psi : H \to H^d$. Consider the conjugates $k_i = JS_iJ^{-1}$ of the Schur maps. We call $k_i$ the probabilistic Schur maps (the reason will be clarified later). If $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, then

(7.1) $k_1(M) = A + B(I - D)^{-1}C,$

where $I$ is the identity operator (or a matrix of the same size as $D$).

Following Bartholdi, Virag [5] and Kaimanovich [31], we are going to apply the probabilistic Schur maps to study random walks on self-similar groups.

Let $(G, X)$ be a self-similar group acting on a $d$-regular tree and let $\mu$ be a probability distribution on $G$. Thus, every element $g$ of $G$ has a mass $\mu_g$, $0 \leq \mu_g \leq 1$ and $\sum_{g \in G} \mu_g = 1$. The set $\text{supp} \mu = \{g : \mu_g \neq 0\}$ is called the support of $\mu$. The measure $\mu$ is non-degenerate if $\text{supp} \mu$ generates $G$. We will identify $\mu$ with the element

$$\mu = \sum_{g \in G} \mu_g g$$

of $\ell^1(G)$. Moreover, $\mu$ belongs to the simplex

$$\ell^1_+ (G) = \{ \mu \in \ell^1(G) : 0 \leq \mu_g \leq 1, \sum_{g \in G} \mu_g \}.$$

The left random walk on $G$ generated by $\mu$ starts at some element of $G$ (usually at the identity) and at each step makes the move $g \mapsto hg$ with probability $\mu(h)$.

When started at $e$ (the identity element of $G$) the distribution at the moment $n$ is given by the $n$th convolution $\mu^{(n)} = \mu \ast \cdots \ast \mu$ of $\mu$ and the probability of return

$P_{e,e}^{(n)}$ is equal to $\mu^{(n)}(e)$. Left random walk on $G$ is invariant with respect to the right action of $G$ on itself.

The main topics that interest specialist in random walks are:

1. norm of the Markov operator

$$M = \sum_{g \in G} \mu_g L_g$$

in $\ell^2(G)$, where $L_g$ are the operators of the left regular representation. We identify here the elements of $\ell^1(G)$ with the left convolution operators on $\ell^2(G)$;

2. spectrum of $M$;

3. asymptotic behavior of $P_{e,e}^{(n)}$ when $n \to \infty$;

4. Liouville property (i.e., when all harmonic functions are constant);

and other asymptotic characteristics of the random walks.

Let $u$ be one of the vertices of the first level of the tree and let $H = St_G(u)$ be its stabilizer in $G$. As $H$ has finite index in $G$, the random walk hits $H$ with probability 1. Let $\mu_H$ be the distribution on $H$ given by the probability of the first hit, i.e.,

$$\mu_H(h) = \sum_{n=0}^{\infty} f_{e,h}^{(n)},$$

where $f_{e,h}^{(n)}$ is the probability to hit $H$ at the element $h$ for the first time at step $n$. As $H < G$ is a recurrent set as a subgroup of finite index in $G$, we have $\sum_{h \in H} \mu_H(h) = 1$. Let $p_i : H \to G$ be the $i$th projection map $h \to h|_i$ for $1 \leq i \leq d$, and let $\mu_i$ be the image of $\mu_H$ under $p_i$.

The next theorem and its proof are analogous to Theorem 2.3 of V. Kaimanovich [31], but they are formulated a bit differently.

**Theorem 7.1.** In the above conditions

$$\mu_i = k_i(\mu).$$

**Proof.** Let $M = (m_{ij})_{1 \leq i,j \leq d}$ be the matrix representation of the Markov operator $M$ coming from the wreath recursion $\phi : G \to \text{Symm}(X) \rtimes G$. Let us extend it to the map $\ell^1_+(G) \to M_\infty(\ell^1_+(G))$.

Then the measures $\mu_i$ can be expressed as

$$(7.2) \quad \mu_i = m_{ii} + M_{i\infty} (I + M_{i\infty} + M_{i\infty}^2 + \cdots) M_{i\infty} = \mu_{ii} + M_{i\infty} (I - M_{i\infty})^{-1} M_{i\infty},$$

where $M_{i\infty}$ (respectively, $M_{\infty i}$) denotes the row $(m_{ij})_{j \neq i}$ (respectively, the column $(m_{ji})_{j \neq i}$) of the matrix $M$ with deleted element $m_{ii}$, and $M_{i\infty}$ is the $(d-1) \times (d-1)$-matrix obtained from $M$ by removing its $ith$ row and $ith$ column.

The first term in (7.2) corresponds to staying at the point $i$ (in the random walk on $X$ induced by the random walk on $G$), while the first factor of the second term corresponds to moving from $i$ to $X \setminus \{i\}$, the second its factor corresponds to staying
in $X \setminus \{i\}$ and the third factor of the second term corresponds to moving back from $X \setminus \{i\}$ to $i$.

Comparing (7.2) with (7.1) we get the statement of the theorem. \hfill $\Box$

In case $\text{St}_G(i) = \text{St}_G(1)$ for all $i \in X$ (for instance, if $X^*$ is a binary tree, or if $G$ acts on the first level by powers of a transitive cycle $\sigma$) we can interpret the above fact as that we have a sequence of stopping times $\tau(n)$ such that if

$$\phi(Y^{(n)}) = (Y_1^{(n)}, Y_2^{(n)}, \ldots, Y_d^{(n)})$$

is image of the $n$th step $Y^{(n)}$ of the random walk under the wreath recursion $\phi$, then

$$\phi(Y^{\tau(n)}) = (Y_1^{(\tau(n))}, Y_2^{(\tau(n))}, \ldots, Y_d^{(\tau(n))}),$$

i.e., the random element at time $\tau(n)$ belongs to the stabilizer of the first level, and $Z_i^{(n)} = Y_i^{(\tau(n))}$ is the random walk on $G$ determined by the measure $\mu_i$. Thus we can treat asymptotic characteristics of $\mu$-random walk on $G$ via $\mu_i$-random walks on $G$. Of course for complete reconstruction of $(G, \mu)$ we also need to know the joint distributions of the processes $(G, \mu_i)$ (as they are usually not independent). Nevertheless, some information about the random walk can be obtained without the independence.

The maps $k_i : M \longrightarrow M : \mu \mapsto \mu_i$ on the simplex of measures on $G$ are continuous and their fixed points are of a special interest for us. The fixed points always exist (for example the unit mass concentrated on the identity), but we are interested only in non-degenerate fixed points (i.e., with support generating the group).

**Remark.** Invariance of the hyperplane $L = \{x \in \ell^1(G) : \sum_{g \in G} x_g = 0\}$ under the Schur map (the proof of which is analogous to the proof of Proposition [5.1]) implies $k_i$-invariance of the hyperplane $L = \{\lambda : \sum_{g \in G} \lambda_g = 1\}$. The probabilistic meaning of the maps $k_i$ shows that the simplex of probability measures $M \subset L$ is also $k_i$-invariant.

Now we are going to describe briefly V. Kaimanovich’s approach to testing amenability of $G$ by entropy method (which develops the ideas previously expressed in [5]).

Having a left random walk $g_{n+1} = h_{n+1}g_n$, given by $(G, \mu)$, where $(h_n)_{n \geq 0}$ is a sequence of independent $\mu$-distributed random variables, we consider the induced Markov chain, denoted $(X \cdot G, \mu)$, on the $G$-bimodule $X \cdot G$ seen as a left $G$-space:

$$x_{n+1} \cdot g_{n+1} = h_{n+1} \cdot (x_n \cdot g_n) = h_{n+1}(x_n) \cdot h_{n+1}|x_n \cdot g_n.$$ 

If we start the Markov chain $(X \cdot G, \mu)$ at the point $x = x \cdot 1 \in X \cdot G$, we get a projection $\Pi_x : g_n \mapsto g_n \cdot x = g_n(x) \cdot g_n|x$ of the random walk on $G$ onto the Markov chain on $X \cdot G$. If the action $(G, X)$ is self-replicating (Definition [2.2]), then the map $\Pi_x : G \rightarrow G \cdot X : g \mapsto g \cdot x$ is onto.

The tuple $(\Pi_1(g), \ldots, \Pi_d(g)) = (g(1) \cdot g_1|1, \ldots, g(d) \cdot g_d|d)$ determines $g$ uniquely, since it means that the image of $g$ in $\text{Symm}(X) \ltimes G$ is $\pi(g_1|1, \ldots, g_d|d)$, where $\pi$ is the permutation $i \mapsto g(i)$, $i \in X$.

The right action of $G$ on the bimodule $X \cdot G$ commutes with the left action, hence the Markov chain $(X \cdot G, \mu)$ is invariant under the right action of $G$. The quotient of $X \cdot G$ by the right $G$-action is naturally identified with $X$ (the orbit of $x \cdot g$ is
labeled by \( x \). Consequently, we get a Markov chain on \( X \) equal to the quotient of the chain \((X \cdot G, \mu)\) by the right \( G \)-action. It is easy to see that this chain is given by the action of \( G \) on \( X \):

\[
  x \mapsto y, \quad \text{with probability } \sum_{g(x) = y} \mu(g).
\]

This chain is irreducible when \( G \) acts transitively on the first level \( X \) and \( \mu \) is non-degenerate.

Consider now the trace of the Markov chain \((X \cdot G, \mu)\) on the right \( G \)-orbit \( x \cdot G = \{ x \cdot g : g \in G \} \) for a fixed letter \( x \in X \), which is a recurrent set (due to irreducibility of the quotient chain), i.e., consider the Markov chain with the first return transitions. We can identify the points of the orbit \( x \cdot G \) with the group \( G \) using the map \( x \cdot g \mapsto g \) and get in this way a Markov chain on \( G \). It is not hard to check that this Markov chain is the left random walk defined by the measure \( \mu_x = k_x(\mu) \), constructed above (see [31]).

**Example 27.** Let \( B = \langle a, b \rangle \) be the Basilica group generated by the states of the automaton shown on Figure 4. We have \( \phi(a) = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} \) and \( \phi(b) = \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix} \).

Then \( \phi(a^{-1}) = \begin{pmatrix} 1 & 0 \\ 0 & b^{-1} \end{pmatrix} \) and \( \phi(b^{-1}) = \begin{pmatrix} 0 & a^{-1} \\ a^{-1} & 1 \end{pmatrix} \). Take \( \mu = p(a + a^{-1}) + q(b + b^{-1}) \in \ell^1(B) \), where \( 2(p + q) = 1 \).

The transition moves for the Markov chain on \( \{0, 1\} \times B \) are

\[
  0 \cdot g \xrightarrow{a} a(0) \cdot a \| g = 0 \cdot g, \\
  0 \cdot g \xrightarrow{a^{-1}} a^{-1}(0) \cdot a^{-1} \| g = 0 \cdot g, \\
  1 \cdot g \xrightarrow{a} a(1) \cdot a(0) \| g = 1 \cdot b \| g, \\
  1 \cdot g \xrightarrow{a^{-1}} a^{-1}(1) \cdot a^{-1} \| g = 1 \cdot b^{-1} \| g,
\]

and

\[
  0 \cdot g \xrightarrow{b} b(0) \cdot b \| g = 1 \cdot g, \\
  0 \cdot g \xrightarrow{b^{-1}} b^{-1}(0) \| 1 \cdot a^{-1} g, \\
  1 \cdot g \xrightarrow{b} b(1) \cdot b \| 1 \cdot a g, \\
  1 \cdot g \xrightarrow{b^{-1}} b^{-1}(1) \cdot b^{-1} \| 1 \cdot g = 0 \cdot g.
\]

See a graphical description of the random walk on \( \{0, 1\} \cdot B \) on Figure 10. Figure 11 shows the graph of the induced random walk on \( \{0, 1\} = X \) with transition probabilities 1/2.

The main characteristics of a random walk \((G, \mu)\) on groups are:

(i) the spectral radius of the random walk

\[
  r = \limsup_{n \to \infty} \sqrt[n]{\mu^{(n)}(e)};
\]

(ii) the drift

\[
  \lambda = \lim_{n \to \infty} \frac{|g_n|}{n},
\]

where \( \{g_n\}_{n \geq 1} \) is a random trajectory and \( |g_n| \) is the length of an element;
(iii) entropy

\[ h = \lim_{n \to \infty} \frac{1}{n} H(\mu(n)), \]

where \( H \) is the entropy of a probability distribution.

Vanishing of \( \lambda \) implies vanishing of the entropy, which implies amenability of \( G \) (in case of a non-degenerate measure \( \mu \) \([30]\)), while \( r = 1 \) for a symmetric measure \( \mu \) is equivalent to amenability (see \([33]\)).

Kaimanovich’s approach to amenability (called in his paper “Münchhausen trick”) is based on the inequalities (see \([31]\) Theorem 3.1)

\[ h(G, \mu) \leq h(G, \mu_i) \leq |X|h(G, \mu) \]

which holds for the projections \( \mu_i, \ i = 1, \ldots, d \), described above. His main observation based on these inequalities and properties of the entropy is that if \( \mu \) is non-degenerate and self-affine, i.e., if there is \( \alpha > 0 \) such that

\[ \mu_i = (1 - \alpha)\delta_e + \alpha \mu \]

for some \( i \), then the inequalities imply \( h(\mu) = 0 \) and hence that \( G \) is amenable (and moreover, has Liouville property).
Kaimanovich calls the measure $\mu$ satisfying the self-affinity condition self-similar. Self-similar measures are fixed points of the maps

$$
\alpha_i : \mu \mapsto \frac{k_i(\mu) - k_i(\delta_e)}{1 - k_i(\delta_e)},
$$

which are continuous maps of the simplex of probability measures on $G$. The simplex is compact with respect to the weak topology. For all $i$ there is an $\alpha_i$-invariant measure (indeed, the measure $\delta_e$ concentrated in identity is such a measure) but the problem is to find a non-degenerate fixed point if such exist.

The next few examples show when this is indeed the case. Here we follow [31].

**Example 28.** Take $G = \langle a, b, c, d \rangle$ — the group of Example 1. Consider a one-dimensional family of measures

$$
\mu = 2\alpha m_1 + 4\beta m_2 = \alpha a + \beta b + \beta c + \beta d + \alpha + \beta,
$$

where $m_1 = (1 + a)/2$, $m_2 = (1 + b + c + d)/4$ and $2\alpha + 4\beta = 1$ and $0 \leq \alpha, \beta \leq 1$.

Then

$$
\mu_1 = \frac{\alpha}{1 - \alpha} + 4\beta m_1 + \frac{4\alpha\beta}{1 - \alpha} m_2
$$

$$
\mu_2 = \frac{\alpha}{1 - \alpha} + \frac{4\alpha\beta}{1 - \alpha} m_1 + 4\beta m_2.
$$

In terms of the parameter $\alpha \in (0, 1/2)$ the corresponding transformations $\phi_i$ take the form

$$
\phi_1 : \alpha \mapsto \frac{2\beta}{1 - \alpha} = \frac{1 - \alpha}{2},
$$

$$
\phi_2 : \alpha \mapsto \frac{2\alpha\beta}{1 - \alpha} \left(1 - \frac{\alpha}{1 - \alpha}\right) = \frac{2\alpha\beta}{1 - 2\alpha} = \frac{\alpha}{2}.
$$

The only fixed point for $\phi_2$ corresponds to $\alpha = 0$ and $\mu$ is degenerate in this case, while for $\phi_1$ the value $\alpha = 1/3$ gives a fixed point corresponding to a non-degenerate self-affine measure

$$
\mu = \frac{2}{3} m_1 + \frac{1}{3} m_2 = \frac{5}{12} + \frac{1}{3} a + \frac{1}{12} (b + c + d).
$$

Removing the atom at the identity we obtain a self-affine measure concentrated on the generating set $\{a, b, c, d\}$

$$
\tilde{\mu} = \frac{4}{7} + \frac{1}{7} (b + c + d)
$$

with self-similarity coefficient $1/2$

$$
\tilde{\mu}_1 = \frac{1}{2} + \frac{1}{2} \tilde{\mu}.
$$

**Example 29.** Basilica group, $B = \langle a, b \rangle$ is defined by

$$
a(0w) = 0w, \quad a(1w) = 1b(w)
$$

$$
b(0w) = 1w, \quad b(1w) = 0a(w),
$$

see Example 3. Consider the one-parameter family of measures

$$
\mu = \frac{a + a^{-1} + rb + rb^{-1}}{2(r + 1)}, \quad r \geq 0.
$$
Then
\[
k_2(\mu) = \frac{b + b^{-1}}{2(r+1)} + \frac{r^2(2 + a + a^{-1})}{4(1 + r)^2} \left(1 - \frac{1}{r+1}\right)^{-1}
\]
\[= \frac{r}{4(r+1)}(a + a^{-1}) + \frac{b + b^{-1}}{2(r+1)} + \frac{r}{2(r+1)}
\]
\[= \frac{r + 2}{2(r+1)} \left(\frac{r}{2(r+2)}(a + a^{-1}) + \frac{1}{r+2}(b + b^{-1})\right) + \frac{r}{2(r+1)}
\]
\[= p(r)\phi_2(\mu) + q(r),
\]
where \(p(r) = \frac{r+2}{2(r+1)}\), \(q(r) = \frac{r}{2(r+1)}\) and
\[
\phi_2 \left( \frac{1}{2(r+1)} : \frac{r}{2(r+1)} \right) = \left( \frac{r}{2(r+2)} : \frac{1}{r+2} \right)
\]
is the projectivization of the corresponding map, which represents the map \(\Lambda: \frac{1}{r} \mapsto \frac{r}{2} \) of \(\mathbb{R}\) or \(z \mapsto \frac{1}{2z}\) if \(z = \frac{1}{r}\). This map has no fixed points, but is periodic of period 2; \(\Lambda^2 = \text{id}\). This fact is used in [5] to prove amenability of \(B\).

At the same time, as it is shown in [31], the map \(\phi_1\) has a fixed point, which represents a non-degenerate measure and this gives another way to prove amenability of \(B\).

**Problem 8.** Using the Münchhausen trick construct new interesting examples of self-similar amenable but not elementary amenable groups.

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