Fractional fourier series and its applications

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Abstract. In this article, we use a new multiplication to propose a definition of fractional Fourier series, regarding the Jumarie type of modified Riemann-Liouville fractional derivatives. Furthermore, several examples are provided to illustrate the applications of fractional Fourier series.

Keywords: new multiplication, fractional Fourier series, modified Riemann-Liouville fractional derivatives, applications.

1. Introduction
In 1807, Fourier astounded some of his contemporaries by asserting that an arbitrary function could be expressed as a linear combination of sines and cosines. These linear combinations, now called Fourier series, have become an indispensable tool in the analysis of certain periodic phenomena such as vibrations and planetary and wave motion which are studied in physics and engineering. Fourier series is one of the most important tools in applied sciences. For example, one can solve partial differential equations using Fourier series. Further one can find the sum of certain numerical series using Fourier series. For a brief but excellent account of the history of this subject and its impact on the development of mathematics see [1-10].

Fractional derivatives of non-integer orders [11-12] have wide applications in viscoelasticity and damping, diffusion and wave propagation, electromagnetism, chaos and fractals, heat transfer, biology, electronics, signal processing, robotics, system identification, traffic systems, genetic algorithms, percolation, modeling and identification, telecommunications, chemistry, irreversibility, physics, control systems, fractional differential equations, and fractional calculus [13-35]. On the other hand, there are many definitions of fractional derivative. The commonly used definitions are the Riemann-Liouville (R-L) fractional derivative [21], the Caputo definition of fractional derivative [22], the Grunwald-Letnikov (G-L) fractional derivative [21], and the Jumarie’s modified R-L fractional derivative [23].

This paper uses a new multiplication of fractional functions to introduce the definition of fractional Fourier series, regarding Jumarie’s modified R-L fractional derivative. In fact, the fractional Fourier series we defined is the generalization of classical Fourier series. At the same time, some examples are given to demonstrate the applications of fractional Fourier series.

2. Definitions and Methods
Next, we present the fractional calculus used in this paper.

Definition 2.1: Suppose that $\alpha$ is a real number and $p$ is a positive integer. The modified Riemann-Liouville fractional derivatives of Jumarie type ([23]) is defined by
\[ aD_x^\alpha [f(x)] = \begin{cases} \frac{1}{\Gamma(-\alpha)} \int_0^x (x - \tau)^{-\alpha-1} f(\tau) d\tau, & \text{if } \alpha < 0 \\ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_\alpha^x (x - \tau)^{-\alpha} [f(\tau) - f(\alpha)] d\tau, & \text{if } 0 \leq \alpha < 1 \\ \frac{d^p}{dx^p} \left( \frac{aD_x^{\alpha-p}}{\Gamma(\alpha-p)} \right) [f(x)], & \text{if } p \leq \alpha < p + 1 \end{cases} \]

where \( \Gamma(u) = \int_0^\infty t^{u-1} e^{-t} dt \) is the gamma function defined on \( u > 0 \). If \((aD_x^\alpha)^n[f(x)] = \(aD_x^\alpha\)(\(aD_x^\alpha\)) \cdots (aD_x^\alpha)[f(x)]\) exists, then \( f(x) \) is called \( n \)-th order \( \alpha \)-fractional differentiable function, and \((aD_x^\alpha)^n[f(x)]\) is the \( n \)-th order \( \alpha \)-fractional derivative of \( f(x) \). We note that \((aD_x^\alpha)^n \neq (aD_x^\alpha)^n \) in general. On the other hand, we define the fractional integral of \( f(x) \), \((aD_x^{-\alpha})[f(x)] = (aD_x^{-\alpha})[f(x)] \), where \( \alpha > 0 \) and \( f(x) \) is called \( \alpha \)-integral function.

**Proposition 2.2** ([37]): Let \( \alpha, \beta, c \) be real numbers, then

\[ aD_x^\alpha [x^\beta] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, \quad (\beta \geq \alpha > 0) \]

\[ aD_x^\alpha [c] = 0, \quad (\beta \geq \alpha > 0) \]

and

\[ aD_x^\alpha [x^\beta] = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} x^{\beta+\alpha}. \quad (\beta > -1, \alpha > 0) \]

**Definition 2.3** ([36]): The Mittag-Leffler function is defined by

\[ E_\alpha(z) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{k\alpha}}{\Gamma(k\alpha+1)}, \]

where \( \alpha \) is a real number, \( \alpha > 0 \), and \( z \) is a complex variable.

**Definition 2.4** ([37]): If \( 0 < \alpha \leq 1 \) and \( x \) is a real variable. Then \( E_\alpha(x^\alpha) \) is called \( \alpha \)-fractional exponential function, and the \( \alpha \)-fractional cosine and sine function are defined as follows:

\[ \cos_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k\alpha}}{\Gamma(2k\alpha+1)}, \]

and

\[ \sin_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)}. \]

**Proposition 2.5** (fractional Euler’s formula)[27]: Let \( 0 < \alpha \leq 1 \), then

\[ E_\alpha(ix^\alpha) = \cos_\alpha(x^\alpha) + isin_\alpha(x^\alpha). \]

**Remark 2.6**: If \( \alpha = 1 \), we obtain the traditional Euler’s formula \( e^{ix} = \cos x + i\sin x \). On the other hand, the smallest positive real number \( T_\alpha \) such that \( E_\alpha(iT_\alpha) = 1 \), is called the period of \( E_\alpha(ix^\alpha) \).

The following is a new multiplication of fractional functions adopted in this paper.

**Definition 2.7** ([15]): If \( \lambda, \mu, z \) are complex numbers, \( 0 < \alpha \leq 1 \), \( j, l, k \) are non-negative integers, and \( a_k, b_k \) are real numbers, \( p_k(z) = \frac{1}{\Gamma(k\alpha+1)} z^k \) for all \( k \). The \( \vee \) multiplication is defined by

\[ p_j(\lambda x^\alpha) \vee p_l(\mu y^\alpha) = \frac{1}{\Gamma(j/\alpha+1)} (\lambda x^\alpha)^j \vee \frac{1}{\Gamma(l/\alpha+1)} (\mu y^\alpha)^l = \frac{1}{\Gamma(j/\alpha+1)} \left( \frac{j+l}{j} \right) (\lambda x^\alpha)^j (\mu y^\alpha)^l, \]

where \( \left( \frac{j+l}{j+l} \right) \) is defined above.

If \( f(\lambda x^\alpha) \) and \( g(\mu y^\alpha) \) are two fractional functions,

\[ f(\lambda x^\alpha) = \sum_{k=0}^{\infty} a_k p_k(\lambda x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)} (\lambda x^\alpha)^k, \quad (10) \]

\[ g(\mu y^\alpha) = \sum_{k=0}^{\infty} b_k p_k(\mu y^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)} (\mu y^\alpha)^k, \quad (11) \]

then we define

\[ f(\lambda x^\alpha) \vee g(\mu y^\alpha) = \sum_{k=0}^{\infty} a_k p_k(\lambda x^\alpha) \vee \sum_{k=0}^{\infty} b_k p_k(\mu y^\alpha) \]
\[
\sum_{k=0}^{\infty} \left( \sum_{m=0}^{k} a_{k-m} b_m p_{k-m} (\lambda x^\alpha) \otimes p_m (\mu y^\alpha) \right).
\]  
(12)

**Proposition 2.8:** \( f(\lambda x^\alpha) \otimes g(\mu y^\alpha) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)} \sum_{m=0}^{k} a_{k-m} b_m (\lambda x^\alpha)^{k-m} (\mu y^\alpha)^m. \)

(13)

**Definition 2.9:** Let \((f(\lambda x^\alpha))^\otimes n = f(\lambda x^\alpha) \otimes \cdots \otimes f(\lambda x^\alpha)\) be the \(n\) times product of the fractional function \(f(\lambda x^\alpha).\) If \(f(\lambda x^\alpha) \otimes g(\lambda x^\alpha) = 1,\) then we call \(g(\lambda x^\alpha)\) is the \(\otimes^{-1}\) reciprocal of \(f(\lambda x^\alpha),\) and is denoted as \(\left( f(\lambda x^\alpha) \right)^{\otimes^{-1}}.\)

**Definition 2.10:** If \(f(x) = \sum_{k=0}^{\infty} a_k x^k,\) \(g(\mu x^\alpha) = \sum_{k=0}^{\infty} b_k p_k (\mu x^\alpha),\) then

\[
f(\mu^{1/\alpha} x) = \sum_{k=0}^{\infty} a_k \left( g(\mu x^\alpha) \right)^{\otimes k}.
\]

Proposition 2.11 (fractional DeMoivre’s formula) \[24\]: If \(0 < \alpha \leq 1,\) and \(n\) is an integer, then

\[
\cos_{\alpha}(x^\alpha) + i\sin_{\alpha}(x^\alpha) \right)^{\otimes n} = \cos_{\alpha}(nx^\alpha) + i\sin_{\alpha}(nx^\alpha). \]

(15)

**Remark 2.12:** The case \(\alpha = 1\) is the classical DeMoivre’s formula \([\cos x + i\sin x]^n = \cos(nx) + i\sin(nx).\]

The following is the main method used in this article.

**Theorem 2.13 (integration by parts for fractional calculus):** Suppose that \(0 < \alpha \leq 1,\) \(a, b\) are real numbers, then

\[
\left( a^{\frac{\alpha}{2}} D_x^\alpha \right) \left( f(x) \otimes (a D_x^\alpha)[g(x)] \right) = f(x) \otimes \left( a^{\frac{\alpha}{2}} D_x^\alpha \right)[g(x)] - \left( a^{\frac{\alpha}{2}} D_x^\alpha \right)\left[ g(x) \otimes \left( a^{\frac{\alpha}{2}} D_x^\alpha \right)[f(x)] \right].\]

(16)

Next, we use the new multiplication to define the fractional Fourier series of fractional functions.

**Definition 2.14 (fractional Fourier series):** Let \(0 < \alpha \leq 1,\) and \(f(x^\alpha)\) be a continuous \(\alpha\)-fractional periodic function defined on an interval with the same period \(T_\alpha\) of \(E_\alpha(\alpha x^\alpha).\) Then the \(\alpha\)-fractional Fourier series expansion of \(f(x^\alpha)\) is

\[
\frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos_\alpha(nx^\alpha) + b_n \sin_\alpha(nx^\alpha),
\]

(17)

where

\[
\begin{align*}
a_0 &= \frac{2}{T_\alpha} \left( \frac{\alpha}{2} \frac{T_\alpha^2}{a} \right) [f(x^\alpha)] \\
a_n &= \frac{2}{T_\alpha} \left( \frac{\alpha}{2} \frac{T_\alpha^2}{a} \right) [f(x^\alpha) \otimes \cos_\alpha(nx^\alpha)] \\
b_n &= \frac{2}{T_\alpha} \left( \frac{\alpha}{2} \frac{T_\alpha^2}{a} \right) [f(x^\alpha) \otimes \sin_\alpha(nx^\alpha)],
\end{align*}
\]

(18)

for all positive integers \(n.\)

**3. Examples and Illustrations**

For the fractional Fourier series, we will use the following applications to illustrate.

**Example 3.1:** Let \(0 < \alpha \leq 1,\) and assume that \(f(x^\alpha) = \frac{1}{\Gamma(2\alpha+1)} x^{2\alpha}\) defined on \([-\frac{T_\alpha}{2}, \frac{T_\alpha}{2}].\) Then we have

\[
a_0 = \frac{4}{T_\alpha} \left[ \frac{1}{\Gamma(2\alpha+1)} x^{2\alpha} \right] = \frac{T_\alpha^2}{2\Gamma(3\alpha+1)}.
\]

(19)

And using integration by parts for fractional calculus yields

\[
a_n = \frac{4}{T_\alpha} \left[ \frac{1}{\Gamma(2\alpha+1)} x^{2\alpha} \otimes \cos_\alpha(nx^\alpha) \right] = \frac{4}{n^2 T_\alpha} \left( \frac{1}{\Gamma(\alpha+1)} \left( \frac{T_\alpha}{2} \right) \otimes \cos_\alpha\left( \frac{n}{2} T_\alpha \right) \right).
\]

(20)

\(b_n = 0\) for all positive integers \(n.
\)

Therefore, using the \(\alpha\)-fractional Fourier series of \(f(x^\alpha)\) yields
\[
\frac{1}{\Gamma(2\alpha+1)} x^{2\alpha} = \frac{T_{\alpha}^2}{4\Gamma(3\alpha+1)} + \frac{4}{T_{\alpha}} \sum_{n=1}^{\infty} \frac{1}{n^2} \left( \frac{1}{\Gamma(\alpha+1)} \left( \frac{T_{\alpha}}{2} \right) \otimes \cos_{\alpha} \left( \frac{n T_{\alpha}}{2} \right) \right) \cos_{\alpha} \left( n x^\alpha \right). \tag{21}
\]

for \(-\frac{T_{\alpha}}{2} < x^\alpha \leq \frac{T_{\alpha}}{2}\).

In particular, if we take \( x^\alpha = 0 \) into Eq. (21), then we obtain
\[
\sum_{n=1}^{\infty} \frac{1}{n^2} \left( \frac{1}{\Gamma(\alpha+1)} \left( \frac{T_{\alpha}}{2} \right) \otimes \cos_{\alpha} \left( \frac{n T_{\alpha}}{2} \right) \right) = - \frac{T_{\alpha}^3}{16\Gamma(3\alpha+1)}. \tag{22}
\]

On the other hand, taking \( x^\alpha = \frac{T_{\alpha}}{2} \) into Eq. (21) yields
\[
\sum_{n=1}^{\infty} \frac{1}{n^2} \left( \frac{1}{\Gamma(\alpha+1)} \left( \frac{T_{\alpha}}{2} \right) \otimes \cos_{\alpha} \left( \frac{n T_{\alpha}}{2} \right) \right) \cos_{\alpha} \left( \frac{n T_{\alpha}}{2} \right) = \frac{T_{\alpha}^3}{16} \left( \frac{1}{\Gamma(2\alpha+1)} - \frac{1}{\Gamma(3\alpha+1)} \right). \tag{23}
\]

Example 3.2: If \( 0 < \alpha \leq 1 \), and let \( g(x^\alpha) = \frac{1}{\Gamma(\alpha+1)} x^\alpha \) defined on \(-\frac{T_{\alpha}}{2}, \frac{T_{\alpha}}{2}\). Then \( a_0 = 0 \) and for all positive integers \( n \), \( a_n = 0 \). Using integration by parts for fractional calculus yields
\[
b_n = \frac{4}{T_{\alpha}} \left( \left[ \frac{1}{\Gamma(\alpha+1)} x^\alpha \otimes \sin_{\alpha} \left( n x^\alpha \right) \right] - \frac{4}{n^2} \left( \frac{T_{\alpha}}{2} \right) \otimes \cos_{\alpha} \left( \frac{n T_{\alpha}}{2} \right) \right). \tag{24}
\]

Thus,
\[
\frac{1}{\Gamma(\alpha+1)} x^\alpha = - \frac{4}{T_{\alpha}} \sum_{n=1}^{\infty} \frac{1}{n^2} \left( \frac{1}{\Gamma(\alpha+1)} \left( \frac{T_{\alpha}}{2} \right) \otimes \cos_{\alpha} \left( \frac{n T_{\alpha}}{2} \right) \right) \sin_{\alpha} \left( n x^\alpha \right), \tag{25}
\]

for \(-\frac{T_{\alpha}}{2} < x^\alpha < \frac{T_{\alpha}}{2}\).

4. Conclusions

As mentioned above, integration by parts for fractional calculus is the major technique to obtain the fractional Fourier series of some fractional functions. In fact, by using the Jumarie type of modified R-L fractional derivatives and a new multiplication defined in this paper, we can generalize the traditional Fourier series. Furthermore, the applications of fractional Fourier series are extensive, and can be used to solve many fractional differential equations. In the future, we will use fractional Fourier series to expand our research topics to the problems of applied mathematics and fractional calculus.

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