

Coherent States and Some Topics in Quantum Information Theory : Review

Kazuyuki FUJII *
Department of Mathematical Sciences
Yokohama City University
Yokohama, 236-0027
Japan

Abstract

In the first half we make a short review of coherent states and generalized coherent ones based on Lie algebras su(2) and su(1,1), and the Schwinger's boson method to construct representations of the Lie algebras. In the second half we make a review of recent developments on both swap of coherent states and cloning of coherent states which are important subjects in Quantum Information Theory.

1 Introduction

The purpose of this paper is to introduce several basic theorems of coherent states and generalized coherent states based on Lie algebras su(2) and su(1,1), and to give some applications of them to Quantum Information Theory.

In the first half we make a general review of coherent states and generalized coherent states based on Lie algebras su(2) and su(1,1).

Coherent states or generalized coherent states play an important role in quantum physics, in particular, quantum optics, see [10] and its references, or the books [12], [15]. They also play an important one in mathematical physics, see the textbook [14]. For example, they are very useful in performing stationary phase approximations to path integral, [8], [9], [7].

In the latter half we apply a method of generalized coherent states to some important topics in Quantum Information Theory, in particular, swap of coherent states and cloning of coherent ones.

Quantum Information Theory is one of most exciting fields in modern physics or mathematical physics or applied mathematics. It is mainly composed of three subjects

Quantum Computation, Quantum Cryptography and Quantum Teleportation.

*E-mail address : fujii@yokohama-cu.ac.jp
See for example [11], [16] or [2], [3], [4]. Coherent states or generalized coherent states also play an important role in it.

We construct the swap operator of coherent states by making use of a generalized coherent operator based on \( su(2) \) and moreover show an “imperfect cloning” of coherent states, and last present some related problems.

# 2 Coherent and Generalized Coherent Operators Revisited

We make a some review of general theory of both a coherent operator and generalized coherent ones based on Lie algebras \( su(1, 1) \) and \( su(2) \).

## 2.1 Coherent Operator

Let \( a(a^\dagger) \) be the annihilation (creation) operator of the harmonic oscillator. If we set \( N \equiv a^\dagger a \) (number operator), then

\[
[N, a^\dagger] = a^\dagger, \quad [N, a] = -a, \quad [a^\dagger, a] = -1.
\]

(1)

Let \( \mathcal{H} \) be a Fock space generated by \( a \) and \( a^\dagger \), and \( \{|n\rangle \mid n \in \mathbb{N} \cup \{0\} \} \) be its basis. The actions of \( a \) and \( a^\dagger \) on \( \mathcal{H} \) are given by

\[
a|n\rangle = \sqrt{n}|n-1\rangle, \quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \quad N|n\rangle = n|n\rangle
\]

(2)

where \( |0\rangle \) is a normalized vacuum \( (a|0\rangle = 0 \) and \( \langle 0|0 \rangle = 1 ) \). From (2) state \( |n\rangle \) for \( n \geq 1 \) are given by

\[
|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle.
\]

(3)

These states satisfy the orthogonality and completeness conditions

\[
\langle m|n \rangle = \delta_{mn} \quad \text{and} \quad \sum_{n=0}^{\infty} |n\rangle\langle n| = 1.
\]

(4)

**Definition** We call a state defined by

\[
|z\rangle = e^{za^\dagger - \overline{z}a}|0\rangle \equiv D(z)|0\rangle \quad \text{for} \quad z \in \mathbb{C}
\]

(5)

the coherent state.

## 2.2 Generalized Coherent Operator Based on \( su(1, 1) \)

We consider a spin \( K \) \( (>0) \) representation of \( su(1, 1) \subset sl(2, \mathbb{C}) \) and set its generators \( \{K_+, K_-, K_3\} \ ( (K_+)^\dagger = K_-) \),

\[
[K_3, K_+] = K_+, \quad [K_3, K_-] = -K_-, \quad [K_+, K_-] = -2K_3.
\]

(6)
We note that this (unitary) representation is necessarily infinite dimensional. The Fock space on which \( \{ K_+, K_-, K_3 \} \) act is \( \mathcal{H}_K \equiv \{|K, n\} | n \in \mathbb{N} \cup \{0\} \} \) and whose actions are

\[
K_+|K, n\rangle = \sqrt{(n+1)(2K+n)}|K, n+1\rangle, \quad K_-|K, n\rangle = \sqrt{n(2K+n-1)}|K, n-1\rangle, \\
K_3|K, n\rangle = (K+n)|K, n\rangle,
\]

(7)

where \( |K, 0\rangle \) is a normalized vacuum \( \langle K_-|K, 0\rangle = 0 \) and \( \langle K, 0|K, 0\rangle = 1 \). We have written \( |K, 0\rangle \) instead of \( |0\rangle \) to emphasize the spin \( K \) representation, see [8]. From (7), states \( |K, n\rangle \) are given by

\[
|K, n\rangle = \frac{(K_+)^n}{\sqrt{n!(2K)^n}}|K, 0\rangle,
\]

(8)

where \( (a)_n \) is the Pochammer’s notation \( (a)_n \equiv a(a+1) \cdots (a+n-1) \). These states satisfy the orthogonality and completeness conditions

\[
\langle K, m|K, n\rangle = \delta_{mn} \quad \text{and} \quad \sum_{n=0}^{\infty} |K, n\rangle\langle K, n| = 1_K.
\]

(9)

Now let us consider a generalized version of coherent states:

**Definition** We call a state defined by

\[
|z\rangle = e^{zK_+-\bar{z}K_-}|K, 0\rangle \quad \text{for} \quad z \in \mathbb{C}.
\]

(10)

the generalized coherent state (or the coherent state of Perelomov’s type based on \( su(1, 1) \) in our terminology).

Here let us construct an example of this representation. We set

\[
K_+ \equiv \frac{1}{2} \left( a^\dagger \right)^2, \quad K_- \equiv \frac{1}{2} a^2, \quad K_3 \equiv \frac{1}{2} (a^\dagger a + \frac{1}{2}),
\]

(11)

then it is easy to check the relations (13). That is, the set \( \{ K_+, K_-, K_3 \} \) gives a unitary representation of \( su(1, 1) \) with spin \( K = 1/4 \) and \( 3/4 \). We also call an operator

\[
S(z) = e^{\frac{1}{2} [z(a^\dagger)^2 - \bar{z}a^2]} \quad \text{for} \quad z \in \mathbb{C}
\]

(12)

the squeezed operator, see the textbook [14].

### 2.3 Generalized Coherent Operator Based on \( su(2) \)

We consider a spin \( J > 0 \) representation of \( su(2) \subset sl(2, \mathbb{C}) \) and set its generators \( \{ J_+, J_-, J_3 \} \) \( ((J_+)^\dagger = J_-) \),

\[
[J_3, J_+] = J_+, \quad [J_3, J_-] = -J_-, \quad [J_+, J_-] = 2J_3.
\]

(13)

We note that this (unitary) representation is necessarily finite dimensional. The Fock space on which \( \{ J_+, J_-, J_3 \} \) act is \( \mathcal{H}_J \equiv \{|J, n\} | 0 \leq n \leq 2J \} \) and whose actions are

\[
J_+|J, n\rangle = \sqrt{(n+1)(2J-n)}|J, n+1\rangle, \quad J_-|J, n\rangle = \sqrt{n(2J-n+1)}|J, n-1\rangle, \\
J_3|J, n\rangle = (-J+n)|J, n\rangle,
\]

(14)
where $|J,0\rangle$ is a normalized vacuum ($J_-|J,0\rangle = 0$ and $\langle J,0|J,0\rangle = 1$). We have written $|J,0\rangle$ instead of $|0\rangle$ to emphasize the spin $J$ representation, see [8]. From (14), states $|J,n\rangle$ are given by

$$|J,n\rangle = \frac{(J_+)^n}{\sqrt{n!}2^J n^n} |J,0\rangle. \quad (15)$$

These states satisfy the orthogonality and completeness conditions

$$\langle J,m|J,n\rangle = \delta_{mn} \quad \text{and} \quad \sum_{n=0}^{2J} |J,n\rangle\langle J,n| = 1_J. \quad (16)$$

Now let us consider a generalized version of coherent states:

**Definition** We call a state defined by

$$|z\rangle = e^{zJ_- - \bar{z}J_+} |J,0\rangle \quad \text{for} \quad z \in \mathbb{C}. \quad (17)$$

the generalized coherent state (or the coherent state of Perelomov’s type based on $su(2)$ in our terminology).

### 2.4 Schwinger’s Boson Method

Here let us construct the spin $K$ and $J$ representations by making use of Schwinger’s boson method.

Next we consider the system of two-harmonic oscillators. If we set

$$a_1 = a \otimes 1, \quad a_1^\dagger = a^\dagger \otimes 1; \quad a_2 = 1 \otimes a, \quad a_2^\dagger = 1 \otimes a^\dagger, \quad (18)$$

then it is easy to see

$$[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0, \quad [a_i, a_j^\dagger] = \delta_{ij}, \quad i, j = 1, 2. \quad (19)$$

We also denote by $N_i = a_i^\dagger a_i$ number operators.

Now we can construct representation of Lie algebras $su(2)$ and $su(1, 1)$ making use of Schwinger’s boson method, see [8], [9]. Namely if we set

$$su(2) : \quad J_+ = a_1^\dagger a_2, \quad J_- = a_2^\dagger a_1, \quad J_3 = \frac{1}{2} \left( a_1^\dagger a_1 - a_2^\dagger a_2 \right), \quad (20)$$

$$su(1, 1) : \quad K_+ = a_1^\dagger a_2^\dagger, \quad K_- = a_2 a_1, \quad K_3 = \frac{1}{2} \left( a_1^\dagger a_1 + a_2^\dagger a_2 + 1 \right), \quad (21)$$

then it is easy to check that (20) satisfies the relations (8), while (21) satisfies (7).

In the following we define (unitary) generalized coherent operators based on Lie algebras $su(2)$ and $su(1, 1)$.

**Definition** We set

$$su(2) : \quad U_J(z) = e^{za_1^\dagger a_2 - \bar{z}a_2^\dagger a_1} \quad \text{for} \quad z \in \mathbb{C}, \quad (22)$$

$$su(1, 1) : \quad U_K(z) = e^{za_1^\dagger a_2^\dagger - \bar{z}a_2 a_1} \quad \text{for} \quad z \in \mathbb{C}. \quad (23)$$

For the details of $U_J(z)$ and $U_K(z)$ see [14] and [8].
Before closing this section let us make some mathematical preliminaries for the latter sections. We have easily

\[ U_J(t)a_1U_J(t)^{-1} = \cos(|t|)a_1 - \frac{tsin(|t|)}{|t|}a_2, \quad (24) \]
\[ U_J(t)a_2U_J(t)^{-1} = \cos(|t|)a_2 + \frac{\bar{tsin}(|t|)}{|t|}a_1, \quad (25) \]

so the map \((a_1, a_2) \rightarrow (U_J(t)a_1U_J(t)^{-1}, U_J(t)a_2U_J(t)^{-1})\) is

\[ (U_J(t)a_1U_J(t)^{-1}, U_J(t)a_2U_J(t)^{-1}) = (a_1, a_2) \left( \begin{array}{cc} \cos(|t|) & \bar{tsin}(|t|) \\ -\frac{tsin(|t|)}{|t|} & \cos(|t|) \end{array} \right). \]

We note that
\[ \left( \begin{array}{cc} \cos(|t|) & \bar{tsin}(|t|) \\ -\frac{tsin(|t|)}{|t|} & \cos(|t|) \end{array} \right) \in SU(2). \]

On the other hand we have easily

\[ U_K(t)a_1U_K(t)^{-1} = \cosh(|t|)a_1 - \frac{tsinh(|t|)}{|t|}a_2^\dagger, \quad (26) \]
\[ U_K(t)a_2^\daggerU_K(t)^{-1} = \cosh(|t|)a_2^\dagger - \frac{\bar{tsinh}(|t|)}{|t|}a_1, \quad (27) \]

so the map \((a_1, a_2^\dagger) \rightarrow (U_K(t)a_1U_K(t)^{-1}, U_K(t)a_2^\daggerU_K(t)^{-1})\) is

\[ (U_K(t)a_1U_K(t)^{-1}, U_K(t)a_2^\daggerU_K(t)^{-1}) = (a_1, a_2^\dagger) \left( \begin{array}{cc} \cosh(|t|) & -\frac{\bar{tsinh}(|t|)}{|t|} \\ -\frac{tsinh(|t|)}{|t|} & \cosh(|t|) \end{array} \right). \]

We note that
\[ \left( \begin{array}{cc} \cosh(|t|) & -\frac{\bar{tsinh}(|t|)}{|t|} \\ -\frac{tsinh(|t|)}{|t|} & \cosh(|t|) \end{array} \right) \in SU(1, 1). \]

## 3 Some Topics in Quantum Information Theory

In this section we don’t introduce a general theory of quantum information theory (see for example [11]), but focus our mind on special topics in it, that is,

- swap of coherent states
- cloning of coherent states

Because this is just a good one as examples of applications of coherent and generalized coherent states, and our method developed in the following may open a new possibility in quantum information theory.

First let us define a swap operator :

\[ S : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}, \quad S(a \otimes b) = b \otimes a \quad \text{for any } a, b \in \mathcal{H} \quad (28) \]
where $\mathcal{H}$ is the Fock space in Section 2.

It is not difficult to construct this operator in a universal manner, see [3]; Appendix C. But for coherent states we can construct a better one by making use of generalized coherent operators in the preceding section.

Next let us introduce no cloning theorem, [17]. For that we define a cloning (copying) operator $C$ which is unitary

$$C: \mathcal{H} \otimes \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H}, \quad C(h \otimes |0\rangle) = h \otimes h \quad \text{for any } h \in \mathcal{H}. \quad (29)$$

It is very known that there is no cloning theorem

**“No Cloning Theorem”** We have no $C$ above.

The proof is very easy (almost trivial). Because $2h = h + h \in \mathcal{H}$ and $C$ is a linear operator, so

$$C(2h \otimes |0\rangle) = 2C(h \otimes |0\rangle). \quad (30)$$

The LHS of (30) is $C(2h \otimes |0\rangle) = 2h \otimes 2h = 4(h \otimes h)$, while the RHS of (30) is $2C(h \otimes |0\rangle) = 2(h \otimes h)$. This is a contradiction. This is called no cloning theorem.

Let us return to the case of coherent states. For coherent states $|\alpha\rangle$ and $|\beta\rangle$ the superposition $|\alpha\rangle + |\beta\rangle$ is no longer a coherent state, so that coherent states may not suffer from the theorem above.

**Problem** Is it possible to clone coherent states?

At this stage it is not easy, so we will make do with approximating it (imperfect cloning in our terminology) instead of making a perfect cloning.

We write notations once more.

| Coherent States | $|\alpha\rangle = D(\alpha)|0\rangle$ for $\alpha \in \mathbb{C}$ |
|-----------------|--------------------------------------------------|
| Squeezed–like States | $|\beta\rangle = S(\beta)|0\rangle$ for $\beta \in \mathbb{C}$ |

### 3.1 Some Useful Formulas

We list and prove some useful formulas in the following. Now we prepare some parameters $\alpha$, $\epsilon$, $\kappa$ in which $\epsilon, \kappa$ are free ones, while $\alpha$ is unknown one in the cloning case. Let us unify the notations as follows.

$$\alpha: \text{unknown} \quad \alpha = |\alpha| e^{i\chi}; \quad \epsilon: \text{known} \quad \epsilon = |\epsilon| e^{i\phi}; \quad \kappa: \text{known} \quad \kappa = |\kappa| e^{i\delta}. \quad (31)$$

(i) First let us calculate

$$S(\epsilon)D(\alpha)S(\epsilon)^{-1}. \quad (32)$$

For that we can show

$$S(\epsilon)\alpha S(\epsilon)^{-1} = cosh(|\epsilon|)\alpha - e^{i\phi} sinh(|\epsilon|)\alpha^\dagger. \quad (33)$$

From this it is easy to check

$$S(\epsilon)D(\alpha)S(\epsilon)^{-1} = D\left(\alpha S(\epsilon)\alpha^\dagger S(\epsilon)^{-1} - \alpha S(\epsilon)\alpha S(\epsilon)^{-1}\right)$$

$$= D\left(cosh(|\epsilon|)\alpha + e^{i\phi} sinh(|\epsilon|)\alpha\right). \quad (34)$$
Therefore
\[ S(\epsilon)D(\alpha)S(\epsilon)^{-1} = \begin{cases} D(e^{i|\alpha|}) & \text{if } \phi = 2\chi \\ D(e^{-i|\alpha|}) & \text{if } \phi = 2\chi + \pi \end{cases} \] (35)

This formula is a bit delicate in the cloning case. That is, if we could know \( \chi \) the phase of \( \alpha \) in advance, then we can change a scale of \( \alpha \) by making use of this one.

(ii) Next let us calculate
\[ S(\epsilon)S(\alpha)S(\epsilon)^{-1}. \] (36)

From the definition
\[ S(\epsilon)S(\alpha)S(\epsilon)^{-1} = S(\epsilon)\exp \left\{ \frac{1}{2} (\alpha(a^\dagger)^2 - \bar{\alpha}a^2) \right\} S(\epsilon)^{-1} = e^{Y/2} \]
where
\[ Y = \alpha \left( S(\epsilon)a^\dagger S(\epsilon)^{-1} \right)^2 - \bar{\alpha} \left( S(\epsilon)aS(\epsilon)^{-1} \right)^2. \]

From (36) and after some calculations we have
\[ Y = \left\{ \cosh^2(|\epsilon|)\alpha - e^{2i\phi}\sinh^2(|\epsilon|)\bar{\alpha} \right\} (a^\dagger)^2 - \left\{ \cosh^2(|\epsilon|)\bar{\alpha} - e^{-2i\phi}\sinh^2(|\epsilon|)\alpha \right\} a^2 \\
+ (-e^{-i\phi}\alpha + e^{i\phi}\bar{\alpha})\sinh(2|\epsilon|)(a^\dagger a + 1/2), \] (37)
see (36). This is our second formula. Now
\[ -e^{-i\phi}\alpha + e^{i\phi}\bar{\alpha} = |\alpha|(-e^{-i(\phi-\chi)} + e^{i(\phi-\chi)}) = 2i|\alpha|\sin(\phi - \chi), \]
so if we choose \( \phi = \chi \), then \( e^{2i\phi}\bar{\alpha} = e^{2i\chi}e^{-i\chi}|\alpha| = \alpha \) and
\[ \cosh^2(|\epsilon|)\alpha - e^{2i\phi}\sinh^2(|\epsilon|)\bar{\alpha} = \left( \cosh^2(|\epsilon|) - \sinh^2(|\epsilon|) \right)\alpha = \alpha, \]
and finally \( Y = \alpha(a^\dagger)^2 - \bar{\alpha}a^2. \) That is,
\[ S(\epsilon)S(\alpha)S(\epsilon)^{-1} = S(\alpha) \iff S(\epsilon)S(\alpha) = S(\alpha)S(\epsilon). \]
The operators \( S(\epsilon) \) and \( S(\alpha) \) commute if the phases of \( \epsilon \) and \( \alpha \) coincide.

(iii) Third formula is: For \( V(t) = e^{itN} \) where \( N = a^\dagger a \) (a number operator)
\[ V(t)D(\alpha)V(t)^{-1} = D(e^{it}\alpha). \] (38)
The proof is as follows.
\[ V(t)D(\alpha)V(t)^{-1} = \exp \left( \alpha V(t)a^\dagger V(t)^{-1} - \bar{\alpha}V(t)aV(t)^{-1} \right) \\
= \exp \left( \alpha e^{it}a^\dagger - \bar{\alpha}e^{-it}a \right) = D(e^{it}\alpha), \] (39)
where we have used
\[ V(t)aV(t)^{-1} = e^{itN}ae^{-itN} = e^{-it}a. \]
This formula is often used as follows.
\[ |\alpha\rangle \rightarrow V(t)|\alpha\rangle = V(t)D(\alpha)V(t)^{-1}V(t)|0\rangle = D(e^{it}\alpha)|0\rangle = |e^{it}\alpha\rangle, \] (40)
where we have used \( V(t)|0 \rangle = |0 \rangle \). That is, we can add a phase to \( \alpha \) by making use of this formula.

(iv) Fourth formula is: Let us calculate the following

\[
U_J(t)S_1(\alpha)S_2(\beta)U_J(t)^{-1} = U_J(t)e^{\frac{1}{2} \left\{ \overline{\alpha}^2(a_1^\dagger)^2 - \overline{\alpha}^2(a_1)^2 + \overline{\beta}^2(a_2^\dagger)^2 - \overline{\beta}^2(a_2)^2 \right\}}U_J(t)^{-1} = e^X \tag{41}
\]

where

\[
X = \frac{\alpha}{2} (U_J(t)a_1^\dagger U_J(t)^{-1})^2 - \frac{\overline{\alpha}}{2} (U_J(t)a_1 U_J(t)^{-1})^2 + \frac{\overline{\beta}}{2} (U_J(t)a_2^\dagger U_J(t)^{-1})^2 - \frac{\beta}{2} (U_J(t)a_2 U_J(t)^{-1})^2.
\]

From (24) and (25) we have

\[
X = \frac{1}{2} \left\{ \cos^2(|t|) \alpha + \frac{t^2 \sin^2(|t|)}{|t|^2} \beta \right\} (a_1^\dagger)^2 - \frac{1}{2} \left\{ \cos^2(|t|) \overline{\alpha} + \frac{\overline{t}^2 \sin^2(|t|)}{|t|^2} \overline{\beta} \right\} a_1^2 \\
+ \frac{1}{2} \left\{ \cos^2(|t|) \beta + \frac{t^2 \sin^2(|t|)}{|t|^2} \alpha \right\} (a_2^\dagger)^2 - \frac{1}{2} \left\{ \cos^2(|t|) \overline{\beta} + \frac{\overline{t}^2 \sin^2(|t|)}{|t|^2} \overline{\alpha} \right\} a_2^2 \\
+ (\beta t - \alpha \overline{t}) \sin(2|t|) \frac{a_1^\dagger a_2^\dagger}{2|t|} - (\overline{\beta} \overline{t} - \overline{\alpha} t) \sin(2|t|) \frac{a_1 a_2}{2|t|}.
\tag{42}
\]

If we set

\[
\beta t - \alpha \overline{t} = 0 \iff \beta t = \alpha \overline{t}, \tag{43}
\]

then it is easy to check

\[
\cos^2(|t|) \alpha + \frac{t^2 \sin^2(|t|)}{|t|^2} \beta = \alpha, \quad \cos^2(|t|) \beta + \frac{\overline{t}^2 \sin^2(|t|)}{|t|^2} \alpha = \beta,
\]

so, in this case,

\[
X = \frac{1}{2} \alpha (a_1^\dagger)^2 - \frac{1}{2} \overline{\alpha} a_1^2 + \frac{1}{2} \beta (a_2^\dagger)^2 - \frac{1}{2} \overline{\beta} a_2^2.
\]

Therefore

\[
U_J(t)S_1(\alpha)S_2(\beta)U_J(t)^{-1} = S_1(\alpha)S_2(\beta). \tag{44}
\]

That is, \( S_1(\alpha)S_2(\beta) \) commutes with \( U_J(t) \) under the condition (13).

### 3.2 Swap of Coherent States

The purpose of this section is to construct a swap operator satisfying

\[
|\alpha_1 \rangle \otimes |\alpha_2 \rangle \longrightarrow |\alpha_2 \rangle \otimes |\alpha_1 \rangle. \tag{45}
\]

Let us remember \( U_J(\kappa) \) once more

\[
U_J(\kappa) = e^{\kappa a_1^\dagger a_2 - \overline{\kappa} a_1 a_2^\dagger} \quad \text{for} \quad \kappa \in \mathbb{C}.
\]
We note an important property of this operator:

\[ U_J(\kappa)|0\rangle \otimes |0\rangle = |0\rangle \otimes |0\rangle. \tag{46} \]

The construction is as follows.

\[ U_J(\kappa)|\alpha_1\rangle \otimes |\alpha_2\rangle = U_J(\kappa)D(\alpha_1) \otimes D(\alpha_2)|0\rangle \otimes |0\rangle = U_J(\kappa)D_1(\alpha_1)D_2(\alpha_2)|0\rangle \otimes |0\rangle = U_J(\kappa)D_1(\alpha_1)D_2(\alpha_2)U_J(\kappa)^{-1}|0\rangle \otimes |0\rangle \quad \text{by (46)}, \tag{47} \]

and

\[ U_J(\kappa)D_1(\alpha_1)D_2(\alpha_2)U_J(\kappa)^{-1} = \exp \left\{ \alpha_1 a_1^\dagger - \bar{\alpha}_1 a_1 + \alpha_2 a_2^\dagger - \bar{\alpha}_2 a_2 \right\} U_J(\kappa)^{-1} \]

\[ = \exp \left\{ \alpha_1 (U_J(\kappa)a_1 U_J(\kappa)^{-1})^\dagger - \bar{\alpha}_1 U_J(\kappa)a_1 U_J(\kappa)^{-1} \right\} \]

\[ + \alpha_2 (U_J(\kappa)a_2 U_J(\kappa)^{-1})^\dagger - \bar{\alpha}_2 U_J(\kappa)a_2 U_J(\kappa)^{-1} \} \]

\[ \equiv \exp(X). \tag{48} \]

From (24) and (25) we have

\[ X = \left\{ \cos(|\kappa|)\alpha_1 + \frac{\kappa \sin(|\kappa|)}{|\kappa|} \alpha_2 \right\} a_1^\dagger - \left\{ \cos(|\kappa|)\bar{\alpha}_1 + \frac{\bar{\kappa} \sin(|\kappa|)}{|\kappa|} \bar{\alpha}_2 \right\} a_1 \]

\[ + \left\{ \cos(|\kappa|)\alpha_2 - \frac{\bar{\kappa} \sin(|\kappa|)}{|\kappa|} \bar{\alpha}_1 \right\} a_2^\dagger - \left\{ \cos(|\kappa|)\bar{\alpha}_2 - \frac{\kappa \sin(|\kappa|)}{|\kappa|} \alpha_1 \right\} a_2, \]

so

\[ \exp(X) = D_1 \left( \cos(|\kappa|)\alpha_1 + \frac{\kappa \sin(|\kappa|)}{|\kappa|} \alpha_2 \right) \otimes D_2 \left( \cos(|\kappa|)\alpha_2 - \frac{\bar{\kappa} \sin(|\kappa|)}{|\kappa|} \alpha_1 \right) \]

\[ = D \left( \cos(|\kappa|)\alpha_1 + \frac{\kappa \sin(|\kappa|)}{|\kappa|} \alpha_2 \right) \otimes \frac{\bar{\kappa} \sin(|\kappa|)}{|\kappa|} \alpha_1 \right) \]

Therefore we have from (18)

\[ |\alpha_1\rangle \otimes |\alpha_2\rangle \longrightarrow |\cos(|\kappa|)\alpha_1 + \frac{\kappa \sin(|\kappa|)}{|\kappa|} \alpha_2 \rangle \otimes |\cos(|\kappa|)\alpha_2 - \frac{\bar{\kappa} \sin(|\kappa|)}{|\kappa|} \alpha_1 \rangle. \]

If we write \( \kappa = |\kappa| e^{i\delta} \) from (31), then the above formula reduces to

\[ |\alpha_1\rangle \otimes |\alpha_2\rangle \longrightarrow |\cos(|\kappa|)\alpha_1 + e^{i\delta} \sin(|\kappa|) \alpha_2 \rangle \otimes |\cos(|\kappa|)\alpha_2 - e^{-i\delta} \sin(|\kappa|) \alpha_1 \rangle. \tag{49} \]

This is a central formula. Here if we choose \( \sin(|\kappa|) = 1 \), then

\[ |\alpha_1\rangle \otimes |\alpha_2\rangle \longrightarrow |e^{i\delta} \alpha_2 \rangle \otimes |e^{-i\delta} \alpha_1 \rangle = |e^{i\delta} \alpha_2 \rangle \otimes |e^{-i(\delta+\pi)} \alpha_1 \rangle. \]

Now by operating the operator \( V = e^{-i\delta N} \otimes e^{i(\delta+\pi)N} \) where \( N = a^\dagger a \) from the left (see (40)) we obtain the swap

\[ |\alpha_1\rangle \otimes |\alpha_2\rangle \longrightarrow |\alpha_2\rangle \otimes |\alpha_1\rangle. \]

A comment is in order. In the formula (19) we set \( \alpha_1 = \alpha \) and \( \alpha_2 = 0 \), then (19) reduces to

\[ |\alpha\rangle \otimes |0\rangle \longrightarrow |\cos(|\kappa|)\alpha \rangle \otimes |e^{-i\delta} \sin(|\kappa|) \alpha \rangle = |\cos(|\kappa|)\alpha \rangle \otimes |e^{-i(\delta+\pi)} \sin(|\kappa|) \alpha \rangle. \tag{50} \]
3.3 Imperfect Cloning of Coherent States

We cannot clone coherent states in a perfect manner like

$$|\alpha\rangle \otimes |0\rangle \rightarrow |\alpha\rangle \otimes |\alpha\rangle \quad \text{for} \quad \alpha \in \mathbb{C}. \quad (51)$$

Then our question is: is it possible to approximate? Here let us note once more that $\alpha$ is in this case unknown. We show that we can at least make an “imperfect cloning” in our terminology against the statement of [1]. The method is almost same with one in the preceding subsection. By (50)

$$|\alpha\rangle \otimes |0\rangle \rightarrow |\cos(|\kappa|)|\alpha\rangle \otimes |e^{-i(\delta+\pi)}\sin(|\kappa|)|\alpha\rangle,$$

we have by operating the operator $1 \otimes e^{i(\delta+\pi)N}$ (see (40))

$$|\alpha\rangle \otimes |0\rangle \rightarrow |\cos(|\kappa|)|\alpha\rangle \otimes |\sin(|\kappa|)|\alpha\rangle. \quad (52)$$

Here if we set $|\kappa| = \pi/4$ in particular, then we have

$$|\alpha\rangle \otimes |0\rangle \rightarrow |\frac{\alpha}{\sqrt{2}}\rangle \otimes |\frac{\alpha}{\sqrt{2}}\rangle. \quad (53)$$

This is the “imperfect cloning” which we have called.

A comment is in order. The authors in [1] state that the “perfect cloning” (in their terminology) for coherent states is possible. But it is not correct as shown in [3]. Nevertheless their method is simple and very interesting, so it may be possible to modify their “proof” more subtly by making use of (53).

3.4 Swap of Squeezed–like States

We would like to construct an operator like

$$|\beta_1\rangle \otimes |\beta_2\rangle \rightarrow |\beta_2\rangle \otimes |\beta_1\rangle. \quad (54)$$

In this case we cannot use an operator $U_J(\kappa)$. Let us explain the reason. Similar to (17)

$$U_J(\kappa)|\beta_1\rangle \otimes |\beta_2\rangle = U_J(\kappa)S(\beta_1) \otimes S(\beta_2)|0\rangle \otimes |0\rangle = U_J(\kappa)S_1(\beta_1)S_2(\beta_2)|0\rangle \otimes |0\rangle$$

$$= U_J(\kappa)S_1(\beta_1)S_2(\beta_2)U^{-1}_J(\kappa)|0\rangle \otimes |0\rangle. \quad (55)$$

On the other hand by (11)

$$U_J(\kappa)S_1(\beta_1)S_2(\beta_2)U^{-1}_J(\kappa) = e^X,$$

where

$$X = \frac{1}{2} \left\{ \cos^2(|\kappa|)|\beta_1\rangle + \frac{\kappa^2 \sin^2(|\kappa|)|\beta_2\rangle}{|\kappa|^2} \right\} (a_1^\dagger)^2 - \frac{1}{2} \left\{ \cos^2(|\kappa|)|\beta_1\rangle + \frac{\kappa^2 \sin^2(|\kappa|)|\beta_2\rangle}{|\kappa|^2} \right\} a_1^2$$

$$+ \frac{1}{2} \left\{ \cos^2(|\kappa|)|\beta_2\rangle + \frac{\kappa^2 \sin^2(|\kappa|)|\beta_1\rangle}{|\kappa|^2} \right\} (a_2^\dagger)^2 - \frac{1}{2} \left\{ \cos^2(|\kappa|)|\beta_2\rangle + \frac{\kappa^2 \sin^2(|\kappa|)|\beta_1\rangle}{|\kappa|^2} \right\} a_2^2$$

$$+ (\beta_2\kappa - \beta_1\bar{\kappa})\frac{\sin(2|\kappa|)}{2|\kappa|}a_1^\dagger a_2 - (\beta_2\bar{\kappa} - \beta_1\kappa)\frac{\sin(2|\kappa|)}{2|\kappa|}a_1 a_2.$$
Here an extra term containing $a_1^\dagger a_2^\dagger$ appeared. To remove this we must set $\beta_2 \bar{\kappa} - \beta_1 \bar{\kappa} = 0$, but in this case we meet

$$U_J(\kappa)S_1(\beta_1)S_2(\beta_2)U_J(\kappa)^{-1} = S_1(\beta_1)S_2(\beta_2)$$

by (44). That is, there is no change.

We could not construct operators as in the subsection 3.2 in spite of very our efforts, so we present

**Problem** Is it possible to find an operator such as $U_J(\kappa)$ in the preceding subsection for performing the swap?

### 3.5 Squeezed–Coherent States

We introduce interesting states called squeezed–coherent ones :

$$|\beta, \alpha\rangle = S(\beta)D(\alpha)|0\rangle$$

for $\beta, \alpha \in \mathbb{C}$. (56)

$|\beta, 0\rangle$ is a squeezed–like state and $|0, \alpha\rangle$ is a coherent one. These states play a very important role in Holonomic Quantum Computation, see for example [5], [6] or [18], [13].

**Problem** Is it possible to find some operators for performing the swap or imperfect cloning?

### Appendix Universal Swap Operator

Let us construct the swap operator in a universal manner

$$U : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}, \quad U(a \otimes b) = b \otimes a$$

for $a, b \in \mathcal{H}$

where $\mathcal{H}$ is an infinite–dimensional Hilbert space. Before constructing it we show in the finite–dimensional case, [4].

For $a, b \in \mathbb{C}^2$ then

$$a \otimes b = \begin{pmatrix} a_1 \bar{b} \\ a_2 \bar{b} \end{pmatrix} = \begin{pmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{pmatrix}, \quad b \otimes a = \begin{pmatrix} b_1 a_1 \\ b_1 a_2 \\ b_2 a_1 \\ b_2 a_2 \end{pmatrix} = \begin{pmatrix} a_1 b_1 \\ a_2 b_1 \\ a_1 b_2 \\ a_2 b_2 \end{pmatrix},$$

so it is easy to see

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{pmatrix} = \begin{pmatrix} a_1 b_1 \\ a_2 b_1 \\ a_1 b_2 \\ a_2 b_2 \end{pmatrix}.$$
That is, the swap operator is

\[
U = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\] (57)

This matrix can be written as follows by making use of three Controlled–NOT matrices (gates)

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix},
\] (58)

See for example [4].

It is not easy for us to conjecture its general form from this swap operator. Let us try for \( n = 3 \). The result is

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
a_1 b_1 \\
a_1 b_2 \\
a_1 b_3 \\
a_2 b_1 \\
a_2 b_2 \\
a_2 b_3 \\
a_3 b_1 \\
a_3 b_2 \\
a_3 b_3
\end{pmatrix}.
\]

Here we rewrite the swap operator above as follows.

\[
U = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}. 
\] (59)

Now, from the above form we can conjecture the general form of the swap operator.

We note that

\[
(\mathbf{1} \otimes \mathbf{1})_{ij,kl} = \delta_{ik}\delta_{jl},
\] (60)

so after some trials we conclude

\[
U : \mathbb{C}^n \otimes \mathbb{C}^n \rightarrow \mathbb{C}^n \otimes \mathbb{C}^n
\]
as
\[ U = (U_{ij,kl}) \ ; \ U_{ij,kl} = \delta_{il} \delta_{jk}, \] (61)
where \( ij = 11,12,\ldots,1n,21,22,\ldots,2n,\ldots, n1,n2,\ldots, nn. \)

The proof is simple and as follows.

\[ (a \otimes b)_{ij} = a_i b_j \rightarrow \{ U(a \otimes b) \}_{ij} = \sum_{kl=1}^{nn} U_{ij,kl} a_k b_l = \sum_{kl=1}^{nn} \delta_{il} \delta_{jk} a_k b_l \]
\[ = \sum_{i=1}^{n} \delta_{il} b_l \sum_{k=1}^{n} \delta_{jk} a_k = b_i a_j = (b \otimes a)_{ij}. \]

At this stage there is no problem to take a limit \( n \to \infty. \)

Let \( \mathcal{H} \) be a Hilbert space with a basis \( \{ e_n \} \ (n \geq 1). \) Then the universal swap operator is given by
\[ U = (U_{ij,kl}) \ ; \ U_{ij,kl} = \delta_{il} \delta_{jk}, \] (62)
where \( ij = 11,12,\ldots,\ldots. \)

We note that this is not a physical construction but only a mathematical (abstract) one, so we have a natural question.

**Problem** Is it possible to realize this universal swap operator in Quantum Optics?

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