Improved FPT algorithms for weighted independent set in bull-free graphs *

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Abstract. Very recently, Thomassé, Trotignon and Vuskovic [WG 2014] have given an FPT algorithm for Weighted Independent Set in bull-free graphs parameterized by the weight of the solution, running in time $2^{O(k^3)} \cdot n^9$. In this article we improve this running time to $2^{O(k^2)} \cdot n^7$. As a byproduct, we also improve the previous Turing-kernel for this problem from $O(k^5)$ to $O(k^2)$. Furthermore, for the subclass of bull-free graphs without holes of length at most $2p - 1$ for $p \geq 3$, we speed up the running time to $2^{O(k \cdot k^{p-1})} \cdot n^7$. As $p$ grows, this running time is asymptotically tight in terms of $k$, since we prove that for each integer $p \geq 3$, Weighted Independent Set cannot be solved in time $2^{o(k)} \cdot n^{O(1)}$ in the class of $\{ \text{bull, } C_4, \ldots, C_{2p-1} \}$-free graphs unless the ETH fails.

Keywords: parameterized complexity, FPT algorithm, bull-free graphs, independent set, Turing-kernel.

1 Introduction

Motivation. Parameterized complexity deals with problems whose instances $I$ come equipped with an additional integer parameter $k$, and the objective is to obtain algorithms whose running time is of the form $f(k) \cdot \text{poly}(|I|)$, where $f$ is some computable function (see [7, 9, 17] for an introduction to the field). Such algorithms are called Fixed-Parameter Tractable (FPT). A fundamental notion in parameterized complexity is that of kernelization, which asks for the existence of polynomial-time preprocessing algorithms that produce equivalent instances whose size depends exclusively (preferably polynomially) on $k$. We will be only concerned with problems defined on graphs.

In order to obtain efficient FPT algorithms, a usual strategy is to focus on a graph class whose members have a well-defined structure, which can then be exploited to design algorithms. This paradigm has been exhaustively used in the last decades to obtain efficient FPT algorithms for graphs that exclude a fixed graph as a minor, relying on the structural characterization of this graph class given by Robertson and Seymour in their seminal work [19]. Nevertheless, the situation is quite different in graphs that exclude a fixed graph as an induced subgraph, for which the design of FPT algorithms is still in an incipient stage. Quite recently, the structural description of claw-free graphs given by Chudnovsky and Seymour [8] has triggered the design of FPT algorithms in this graph class [4][11][12]. Even more recently, a structural characterization of bull-free graphs has been given by Chudnovsky [1][3]. In this article we focus on this latter graph class.

The bull is the graph defined by the set of vertices $\{x_1, x_2, x_3, y, z\}$ and the set of edges $\{x_1x_2, x_2x_3, x_3x_1, x_1y, x_2z\}$ (see Fig. 1 for an illustration). For a graph $F$, a graph $G$ is said to be $F$-free if $G$ does not contain an induced subgraph isomorphic to $F$. Note that the class of bull-free graphs contains the classes of $P_4$-free and triangle-free graphs, so in particular it contains all bipartite graphs.

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An independent set in a graph is a set of pairwise non-adjacent vertices. In a vertex-weighted graph, the weight of an independent set is the sum of the weights of its vertices. We are interested in the following parameterized problem.

**Weighted Independent Set**

**Input:** An graph $G = (V,E)$ with $|V| = n$, a weight function $w : V \to \mathbb{N}$, and a positive integer $k$.

**Parameter:** The integer $k$.

**Question:** Does $G$ contain an independent set of weight at least $k$?

The above problem is well-known to be $W[1]$-hard in general graphs \cite{7}, and therefore an FPT algorithm is unlikely to exist (see \cite{7,9,17} for the missing definitions). Thus, it is relevant to find graph classes for which the problem admits an FPT algorithm, and for which the non-parameterized version still remains NP-hard. In this direction, Dabrowski, Lozin, Müller and Rautenbach \cite{5} gave an FPT algorithm for Weighted Independent Set in \{$\text{bull}, \overline{P}_5$\}-free graphs, where $\overline{P}_5$ is the complement of a path on 5 vertices. Note that the problem is NP-hard in \{$\text{bull}, \overline{P}_5$\}-free graphs, as it is NP-hard in the subclass of triangle-free graphs \cite{18}. Recently, Thomassé, Trotignon and Vuskovic \cite{20} generalized this result by giving an FPT algorithm for Weighted Independent Set in the class of bull-free graphs, by exploiting the structural results of Chudnovsky \cite{1,2}. This article is the starting point of our work, and its main result is the following.

**Theorem 1 (Thomassé, Trotignon and Vuskovic \cite{20}).** Weighted Independent Set in the class of bull-free graphs can be solved in time $2^{O(k^2)} \cdot n^9$.

**Our results.** Our main contribution is to improve the running time of the FPT algorithm of Thomassé, Trotignon and Vuskovic \cite{20} stated in Theorem 1 specially in terms of the parameter $k$.

**Theorem 2.** Weighted Independent Set in the class of bull-free graphs can be solved in time $2^{O(k^2)} \cdot n^7$.

We would like to point out that we strongly follow the algorithm of \cite{20}, and that our faster algorithm is obtained by improving locally some of the procedures and analyses given in \cite{20}. In particular, one of our main improvements relies on a closer look at the structure of the so-called basic bull-free graphs as described by Chudnovsky in her series of papers \cite{1,2}.

It is shown in \cite{20}, Theorem 7.2] that the FPT algorithm of Theorem 1 actually provides a Turing-kerne of size $O(k^2)$ for Weighted Independent Set in bull-free graphs, and

\footnote{For a function $g : \mathbb{N} \to \mathbb{N}$, a parameterized problem $\Pi$ is said to have a Turing-kernel of size $g(k)$ if there is an algorithm which, given an input $(I,k)$ together with an oracle for $\Pi$ that decides whether $(I,k) \in \Pi$ in constant time whenever $|I| \leq g(k)$, decides whether $(I,k) \in \Pi$ in time polynomial in $|I|$ and $k$.}
that a polynomial kernel is not possible under reasonable complexity hypothesis. Therefore, as our algorithm follows closely that of Theorem 1, from Theorem 2 we immediately obtain the following corollary.

**Corollary 1.** There exists a Turing-kernel of size $O(k^2)$ for Weighted Independent Set in the class of bull-free graphs.

It is natural to ask whether the algorithm of Theorem 2 can be improved for subclasses of bull-free graphs. We prove that it is the case when, in addition to the bull, we exclude the holes of length at most $2p - 1$ for some integer $p \geq 3$ as induced subgraphs. Note that for each $p \geq 3$, the Weighted Independent Set problem is NP-hard in the class of $\{\text{bull}, C_4, \ldots, C_{2p-1}\}$-free graphs, as for each integer $g \geq 3$, its unweighted version is NP-hard in the class of graphs of girth greater than $g$ [16], that is in $\{C_3, C_4, \ldots, C_g\}$-free graphs, which is a subclass of $\{\text{bull}, C_4, \ldots, C_g\}$-free graphs for $g \geq 4$. More precisely, we prove the following theorem.

**Theorem 3.** For each integer $p \geq 3$, Weighted Independent Set in the class of $\{\text{bull}, C_4, \ldots, C_{2p-1}\}$-free graphs can be solved in time $2^{O(k \cdot k^{\frac{1}{p-1}})} \cdot n^7$.

In the same way as Corollary 1 follows from Theorem 2 from Theorem 3 we obtain the following corollary. It is worth noting that the multipartite construction given in [20, Theorem 7.1] for ruling out the existence of polynomial kernels actually preserves the property of being $\{\text{bull}, C_4, \ldots, C_{2p-1}\}$-free for $p \geq 3$.

**Corollary 2.** For each integer $p \geq 3$, there exists a Turing-kernel of size $O(k \cdot k^{\frac{1}{p-1}})$ for Weighted Independent Set in the class of $\{\text{bull}, C_4, \ldots, C_{2p-1}\}$-free graphs.

Finally, we provide lower bounds on the running time on any FPT algorithm that solves Weighted Independent Set in the class of $\{\text{bull}, C_4, \ldots, C_{2p-1}\}$-free graphs, for $p \geq 3$. These lower bounds rely on the Exponential Time Hypothesis (ETH), which states that there exists a positive real number $s$ such that 3-CNF-Sat with $n$ variables and $m$ clauses cannot be solved in time $2^n \cdot (n + m)^{O(1)}$ (see [15] for more details).

**Theorem 4.** For each integer $p \geq 3$, Weighted Independent Set cannot be solved in time $2^{o(k)} \cdot n^{O(1)}$ in the class of $\{\text{bull}, C_4, \ldots, C_{2p-1}\}$-free graphs unless the ETH fails.

Note that as $p$ grows, the running time of the algorithm of Theorem 3 tends to $2^{O(k)} \cdot n^7$. As the lower bound given by Theorem 4 holds for any fixed integer $p \geq 3$, it follows that, as $p$ grows, the running time of the algorithm of Theorem 3 is asymptotically tight with respect to the parameter $k$.

**Organization of the paper.** In Section 2 we state some definitions and results from [20] that we need in the remaining sections. Section 3 is devoted to the proof of Theorem 2. In Section 4 we focus on bull-free graphs without small holes and prove Theorems 2 and 3. Finally, we conclude with some directions for further research in Section 5. Due to space limitations, the proofs of the results marked with ‘[⋆]’ have been moved to the appendix.

## 2 Preliminaries

All the definitions in this section are taken from [20]. We use standard graph-theoretic notation (see [6] for any undefined terminology). A hole in a graph is an induced cycle of length at least 4.
Trigraphs. We need to work with trigraphs (see [2]), which are a generalization of graphs in which some edges are left “undecided”. Formally, a trigraph consists of a finite set $V(T)$ of vertices and an adjacency function \( \theta : \binom{V(T)}{2} \to \{-1, 0, 1\} \). Two vertices \( u, v \in V(T) \) are strongly adjacent (resp. strongly antiadjacent, resp. semiadjacent) if \( \theta(uv) = 1 \) (resp. \( \theta(uv) = -1 \), \( \theta(uv) = 0 \)), and in that case \( u \) and \( v \) constitute a strong edge (resp. strong antiedge, switchable pair). Two vertices \( u, v \in V(T) \) are adjacent (resp. antiadjacent) if \( \theta(uv) \in \{0, 1\} \) (resp. \( \theta(uv) \in \{-1, 0\} \)), and in that case we say that there is an edge (resp. antiedge) between \( u \) and \( v \). Let \( \eta(T) \) (resp. \( \nu(T), \sigma(T) \)) be the set of strongly adjacent (resp. strongly antiadjacent, semiadjacent) pairs of \( T \). That is, a trigraph \( T \) is a graph if and only if \( \sigma(T) = \emptyset \). For a vertex \( v \in V(T) \), \( N(v) \) (resp. \( \eta(T), \nu(T), \sigma(T) \)) denotes the set of vertices in \( V(T) \setminus \{v\} \) that are adjacent (resp. strongly adjacent, strongly antiadjacent, semiadjacent) to \( v \). The complement \( \overline{T} \) of a trigraph \( T \) is the trigraph with \( V(\overline{T}) = V(T) \) and \( \theta(\overline{uv}) = -\theta(uv) \). A trigraph is monogamous if every vertex belongs to at most one switchable pair. Most trigraphs considered in this paper will be monogamous.

For two disjoint non-empty subsets of vertices \( A, B \) of \( V(T) \), we say that \( A \) is strongly complete (resp. strongly anticOMPLETE) to \( B \) if every vertex in \( A \) is strongly adjacent (resp. strongly antiadjacent) to every vertex in \( B \). A clique (resp. strong clique, independent set, strong independent set) in \( T \) is a set of vertices that are pairwise adjacent (resp. strongly adjacent, antiadjacent, strongly antiadjacent). When we speak about the weighted independent set problem in a trigraph \( T \), we are interested in finding an independent set in \( T \). We denote by \( \alpha(T) \) the maximum weight of an independent set in \( T \) (see [20] for the precise restrictions of the weight functions defined in trigraphs).

A realization of a trigraph \( T \) is any trigraph \( T' \) such that \( \eta(T) \subseteq \eta(T'), \nu(T) \subseteq \nu(T'), \) and \( \sigma(T') = \emptyset \) (hence \( T' \) is a graph). Seen as a trigraph, the bull is defined as in Fig. [1] where the corresponding vertices are adjacent or antiadjacent (that is, switchable pairs are allowed). A trigraph is bull-free if no induced subtrigraph of it is a bull.

Decomposition of bull-free trigraphs. The algorithm of [20], hence ours as well, is based on a decomposition theorem of bull-free trigraphs that is a simplified version of the one given by Chudnovsky [1][2], and that we proceed to state. We first need two more definitions that will play a fundamental role.

A set \( X \subseteq V(T) \) is a homogeneous set if \( 1 < |X| < |V(T)| \) and every vertex in \( V(T) \setminus X \) is either strongly complete or strongly anticomplete to \( X \). Thus, \( V(T) \setminus X \) can be partitioned into two (possibly empty) sets \( Y \) and \( Z \) such that \( X \) is strongly complete to \( Y \) and strongly anticomplete to \( Z \); see Fig. [2] for an illustration, where a solid line means that there are all edges, no line means that there are no edges, and a dashed line means that there is no restriction.

A homogeneous pair in \( T \) is a pair \( (A, B) \) of disjoint non-empty subsets of \( V(T) \) such that there exist disjoint (possibly empty) subsets \( C, D, E, F \) of \( V(T) \) such that the following hold:

- \( \{A, B, C, D, E, F\} \) is a partition of \( V(T) \);
- \( |A \cup B| \geq 3 \);
• $|C \cup D \cup E \cup F| \geq 3$;
• $A$ is strongly complete to $C \cup E$ and strongly anticomplete to $D \cup F$;
• $B$ is strongly complete to $D \cup E$ and strongly anticomplete to $C \cup F$; and
• $A$ is not strongly complete nor strongly anticomplete to $B$.

See again Fig. 2 for an illustration. A homogeneous pair is small if $|A \cup B| \leq 6$, and it is proper if $C \neq \emptyset$ and $D \neq \emptyset$.

We now define some classes of so-called basic trigraphs which will also play an important role in the algorithms. Let $T_0$ be the class of monogamous trigraphs $T$ whose vertex set can be partitioned into (possibly empty) sets $X, K_1, \ldots, K_t$ such that $G[X]$ is triangle free, and $K_1, \ldots, K_t$ are strong cliques that are pairwise anticomplete. According to Chudnovsky’s work [1, 2], the trigraphs in $T_0$ satisfy some additional conditions that we will detail in Section 3. This closer look at the class $T_0$ allows us to significantly improve the dependency on $k$ of the algorithm. Finally, let $T_1 = \{T : T \in T_0\}$. A trigraph is basic if it belongs to $T_0 \cup T_1 \cup T_2$. We are ready to state the decomposition theorem.

**Theorem 5 (Chudnovsky [1, 2]).** If $T$ is a bull-free monogamous trigraph, then one of the following holds:

• $T$ is basic;
• $T$ has a homogeneous set;
• $T$ has a small homogeneous pair; or
• $T$ has a proper homogeneous pair.

We say that $(X, Y)$ is a decomposition of a trigraph $T$ if $(X, Y)$ is a partition of $V(T)$ and either $X$ is a homogeneous cut of $T$ or $X = A \cup B$ where $(A, B)$ is a small or proper homogeneous pair of $T$. A decomposition $(X, Y)$ defines two blocks $T_X$ and $T_Y$, whose definition is omitted here, and can be found in [20]. A decomposition $(X, Y)$ is a homogeneous cut if $X$ is a homogeneous set or $X = A \cup B$ where $(A, B)$ is a proper homogeneous pair. A homogeneous cut $(X, Y)$ is minimally-sided if there is no homogeneous cut $(X', Y')$ with $X' \subsetneq X$.

### 3 An improved FPT algorithm in bull-free graphs

In this section we give a proof of Theorem 2. We start by providing a high-level description of the FPT algorithm of [20] in Algorithm 1 below (without giving all the details), which will help us to point out the steps for which we provide an improvement.

**Input:** A bull-free trigraph $T$ with $|V(T)| = n$ and the parameter $k$.

**Output:** ‘Yes’ if $\alpha(T) \geq k$, and an independent set of weight $\alpha(T)$ otherwise.

1. If $T$ is basic, then the problem can be solved in time $O(n^4 m) + 2^{O(k^5)}$, where $m$ is the number of strong edges in $T$.
2. Otherwise, by Theorem 5 $T$ admits a decomposition. Furthermore, it is shown that $T$ admits a so-called extreme decomposition, which is a decomposition $(X, Y)$ such that the block $T_X$ is basic and both $T_X$ and $T_Y$ are bull-free trigraphs. This extreme decomposition can be found in time $O(n^8)$.
   1. First, Step 1 is run on the basic bull-free trigraph $T_X$. If $\alpha(T_X) \geq k$, we answer ‘Yes’ and we stop the algorithm. Otherwise, we use the performed computations to build the weighted trigraph $T_Y$.
   2. The whole algorithm is run recursively on the bull-free trigraph $T_Y$.

**Algorithm 1:** Sketch of the FPT algorithm of [20].
As the size of the trigraph $T_\gamma$ strictly decreases in each recursive step, the overall complexity of Algorithm 1 is easily seen to be upper-bounded by $2^O(k^2) \cdot n^6$. (In fact, the algorithm of \cite{20} starts by trying to find a decomposition of $T$, and if it fails we know by Theorem \cite{5} that $T$ is basic. We reversed the steps in this sketch for the sake of presentation.) Our improvements are the following:

(i) **Improvement in terms of the graph size.** We show that in Step 2, an extreme decomposition $(X,Y)$ of $T$ can be found in time $O(n^6)$.

(ii) **Improvement in terms of the parameter.** We show that in Step 1, the problem can be solved in basic trigraphs in time $O(n^4m) + 2^O(k^2)$.

The two improvements above yield the running time given in Theorem \cite{2} We now proceed to explain these improvements in detail.

**Theorem 6.** There is an algorithm running in time $O(n^6)$ whose input is a trigraph $T$. The output is a small homogeneous pair of $T$ if some exists. Otherwise, if $G$ has a homogeneous cut, then the output is a minimally-sided homogeneous cut. Otherwise, the output is: "$T$ has no small homogeneous pair, no proper homogenous pair, and no non-homogenous set.”

The proof of \cite{20} Theorem 4.3] starts by enumerating all sets of vertices of size at most 6 and then it checks whether they define a small homogeneous pair. This procedure takes time $O(n^8)$. Our first improvement is a simple algorithm that finds small homogeneous pairs $(A,B)$ in time $O(n^6)$, if there exists one. Without loss of generality, we can assume that $|A| \geq |B|$. The main idea is to fix the vertices of $A$ and then try to find a suitable $B$ verifying $|A \cup B| \leq 6$. While we have not found a small homogeneous pair, we execute Algorithm 2 below for all possible pairs of positive integers $(i,j)$ such that $3 \leq i + j \leq 6$ and $j \leq i$ (note that there are at most 8 such pairs), in lexicographic order for $i \in \{2, \ldots, 5\}$ and $j \in \{1, \ldots, \min\{1, 6 - i\}\}$.

**Lemma 1.** Algorithm 2 is correct and runs in time $O(n^6)$. That is, a small homogeneous pair in a trigraph $T$ can be found in time $O(n^6)$, if it exists.

**Proof:** Suppose that $T$ contains a small homogeneous pair $(A,B)$ such that $|A| = i$ and $|B| = j$, and that $T$ does not contain a small homogeneous pair $(A',B')$ with $|A'| = i$ and $|B'| < j$ (such a pair would have been found in previous iterations). We claim that there exists a vertex $v \in R$ that is neither strongly complete nor strongly anticomplete to $A$, or neither strongly complete nor strongly anticomplete to $B$. Indeed, otherwise $(A,B \setminus \{v\})$ would be a small homogeneous pair, contradicting the conditions of the algorithm. Let $B' = B \setminus \{v\}$. At some point, the algorithm will consider the pair $(A,B')$, and then it will find the corresponding $v$ and check that the found pair is indeed homogeneous. Since $|A| + |B| \leq 6$, these two operations can be done in linear time. Since $i + j - 1$ vertices are guessed, the complexity of the algorithm is $O(n^{i+j}) = O(n^6)$, as $i + j \leq 6$. \hfill $\square$

The second bottleneck in the proof of \cite{20} Theorem 4.3] is a subroutine that finds a minimally-sided proper homogeneous pair, if it exists, in time $O(n^7)$. We prove the following lemma.

**Lemma 2.** \cite{3} There exists an algorithm running in time $O(n^6)$ that finds a minimally-sided homogeneous cut in a trigraph $T$, provided that $T$ has some homogeneous cut.

Lemmas 1 and 2 together clearly imply Theorem 6.
Input: A trigraph $T$ on $n$ vertices, two positive integers $i$ and $j$ such that $3 \leq i + j \leq 6$ and $i \geq j$, and such that $T$ does not contain a small homogeneous pair $(A', B')$ with $|A'| = i$ and $|B'| < j$.
Output: A small homogeneous pair $(A, B)$ with $|A| = i$ and $|B| = j$, if it exists.

begin
forall the subsets $A \subsetneq V$ of size $i$ do
forall the subsets $B' \subsetneq V \setminus A$ of size $j - 1$ do
$B = B'$, $R = V \setminus (A \cup B')$.
while $|B| \neq j$ and $R \neq \emptyset$ do
pick a new vertex $v \in R$ and remove it from $R$.
if $v$ is neither strongly complete nor strongly anticomplete to $A$, or neither strongly complete nor strongly anticomplete to $B$ then
$\Leftarrow$ add $v$ to $B$.
if $|B| = j$ and all vertices of $V \setminus (A \cup B)$ are either strongly complete or strongly anticomplete to $A$ and either strongly complete or strongly anticomplete to $B$ then
$\Leftarrow$ return $(A, B)$.

Algorithm 2: Algorithm for finding a small homogeneous pair of size $i + j$.

Improvement in terms of the parameter. We now focus on the improvement in Step 1 of Algorithm 1. It is shown in the proof [20, Lemma 6.1] that Weighted Independent Set restricted to the class $T_1$ admits a kernel of size $O(k^5)$, and this is what gives the function $2O(k^5)$ in the algorithm of Theorem 1 as well as the Turing-Kernel of Corollary 1. In the following we will show that the kernel in the class $T_1$ can be improved to $f(k) = O(k^2)$, concluding the proof of Theorem 2 and of Corollary 1. This improvement is detailed in the following lemma, which should be compared to [20, Lemma 6.1]. More precisely, in [20, Lemma 6.1] the function $f$ is defined as $f(x) = g(x) + (x - 1)((^{x+1}_{\frac{x-2}}) - 1)$. We redefine $f$ as $f(x) = 5g(x)$, yielding the desired upper bound.

Lemma 3. There is an $O(n^4 \cdot m)$-time algorithm with the following specifications.

Input: A weighted monogamous basic trigraph $T$ on $n$ vertices and $m$ strong edges, in which all vertices have weight at least 1 and all switchable pairs have weight at least 2, with no homogeneous set, and a positive integer $k$.
Output: One of the following true statements:
1. $n \leq f(k)$;
2. the number of maximal independent sets in $T$ is at most $n^3$; or
3. $\alpha(T) \geq k$.

Proof: The proof follows closely that of [20, Lemma 6.1]. Let $G$ be the realization of $T$ where all switchable pairs are set to “strong antiedge”. We first check whether $n \leq f(k)$ in constant time. If this is not the case, we apply [20, Theorem 5.4] to $G$, and check whether Output 2 is true. If not, it just remains to prove that Output 3 is a true statement. The running time of the algorithm is $O(n^4 \cdot m)$.

Since $T$ is basic, there are three cases to consider. Assume first that $T \in T_0$. If $k \geq 2$, then $f(k) > 8 \geq n$, so the algorithm should have given Output 1, a contradiction. Thus, $k \leq 1$, and Output 3 is true. If $T \in T_1$, then by [20, Lemma 5.9] $T$ has at most $n^3$ maximal independent sets, so the algorithm should have given Output 2, a contradiction.

Thus, necessarily $T \in T_2$. Suppose for contradiction that $\alpha(T) < k$. We consider the decomposition of $T$ into a triangle-free trigraph $X$ and a disjoint union of $t$ strong cliques $K_1, \ldots, K_t$. In contrast to the proof of [20, Lemma 6.1], we will use the following two properties of the class $T_1$, as described by Chudnovsky [1, 2]:
(i) Each vertex of $X$ has neighbors in at most two distinct cliques.

(ii) For each clique $K \in \{K_1, \ldots, K_t\}$, with $K = \{v_1, \ldots, v_r\}$, the neighborhood of $K$ in $T$ is a bipartite trigraph, with bipartition $(A, B)$, such that for all $i \in \{1, \ldots, r\}$, $A_{i+1} \subseteq A_i$ and $B_i \subseteq B_{i+1}$, where $A_i = A \cap N(v_i)$ and $B_i = B \cap N(v_i)$ (see Fig. 3).

![Fig. 3. Adjacency between a clique $K$ and the set $X$ in the proof of Lemma 3.](image)

We can suppose that $|X| \leq g(k)$, otherwise as $T[X]$ is triangle-free, by Ramsey Theorem it follows that $\alpha(G) \geq k$, so we would have that $\alpha(T) \geq \alpha(G) \geq k$.

For $1 \leq i \leq t$, let us denote by $N(K_i)$ the subset of vertices of $X$ that are adjacent to at least one vertex of $K_i$. By Property (i) above, it holds that

$$\sum_{i=1}^{t} |N(K_i)| \leq 2|X|. \quad (1)$$

**Claim 1** For each clique $K \in \{K_1, \ldots, K_t\}$, it holds that $|K| \leq 2|N(K)|$.

**Proof:** Consider an arbitrary $K \in \{K_1, \ldots, K_t\}$, and let $K = \{v_1, \ldots, v_r\}$. Consider the set $N(K)$ as described by Property (ii) above. Let us consider $K' = \{v_{i_1}, \ldots, v_{i_{r'}}\}$, for $1 \leq i_1 < i_2 < \cdots < i_{r'} \leq r$, the set of vertices in $K$ that do not belong to any switchable pair. Since $T$ is monogamous, we have that $r - r' \leq |N(K)|$.

Let us note $V_j = A_j \cup B_j$, where $A_j = A \cap N(v_j)$. Note that any two vertices in $K'$ must have a distinct neighborhood, otherwise they form a homogeneous set, a contradiction. Together with Property (ii), this implies that for all $j \in \{1, \ldots, r' - 1\}$, $V_j \subseteq V_{j+1}$.

Since $B_i \neq \emptyset$ for all $i \in \{1, \ldots, r\}$, we have that $|V_{r'}| \geq r'$. And since $V_{r'} \subseteq N(K')$, we have that $|N(K')| \geq |V_{r'}| \geq r' = |K'|$.

Therefore, $|K| = r = (r - r') + r' \leq |N(K)| + |N(K')| \leq 2|N(K)|$, and the claim follows.

Equation (1) and Claim 1 imply that $\sum_{i=1}^{t} |K_i| \leq 4|X|$, and therefore
\[ |V(T)| = |X| + \sum_{i=1}^{t} |K_i| \leq |X| + 4|X| = 5|X|, \] (2)

that is, \( n \leq 5|X| \), and since \( |X| \leq g(k) \), the algorithm should have given Output 1, a contradiction.

\[ \square \]

4 Independent set in bull-free graphs without small holes

In this section we deal with bull-free graphs without small holes. Namely, we provide a faster FPT algorithm in Subsection 4.1 and we prove the lower bound in Subsection 4.2.

4.1 Faster FPT algorithm in \{bull, \( C_4, \ldots, C_{2p-1} \}\}-free graphs

In this subsection we prove Theorem 3. We use the same algorithm described in Section 3 for general bull-free graphs, and the improvement in the time bound for \{bull, \( C_4, \ldots, C_{2p-1} \}\}-free graphs consists in a more careful analysis of the kernel size for the basic class \( \mathcal{T}_1 \). More precisely, we will prove that the function \( g(x) = x(x^{\frac{1}{3}} + 2) \) yields a kernel of size \( O(k \cdot k^{\frac{1}{3}}) \) for the class \( \mathcal{T}_1 \). Indeed, in the proof of Lemma 3 if \( T \) is a \{bull, \( C_4, \ldots, C_{2p-1} \}\}-free trigraph that belongs to the basic class \( \mathcal{T}_1 \), the following lemma implies that in this case it holds that \( |X| \leq g(k) \), hence proving Theorem 3. The proof is inspired from classical arguments in Ramsey theory [6] (see also [14] for recent results on the independence number of triangle-free graphs in terms of several parameters).

Lemma 4. Let \( p, k \geq 2 \) be two integers and let \( G \) be a graph of girth \( g(G) \geq 2p \). If \( |V(G)| \geq k(k^{\frac{1}{3p}} + 2) \), then \( \alpha(G) \geq k \).

Proof: Let \( G' = G \) and \( S = \emptyset \). While there exists a vertex \( v \in V(G') \) such that \( \deg_G(v) < (k^{\frac{1}{3p}} + 1) \), we do the following:

- Add \( v \) to \( S \); and
- Remove \( N[v] \) from \( G' \).

Note that by construction the set \( S \) is an independent set in \( G \). When there is no such vertex \( v \in V(G') \) anymore, there are two possibilities:

- If \( |S| \geq k \), we are done.
- Otherwise, since at each step we removed strictly less than \( k(k^{\frac{1}{3p}} + 2) \) vertices from \( G \) and by hypothesis \( |V(G)| \geq k(k^{\frac{1}{3p}} + 2) \), we have that \( V(G') \neq \emptyset \). Note that for all \( v \in V(G') \), it holds that \( \deg_{G'}(v) \geq (k^{\frac{1}{3p}} + 1) \).

In the second case, consider an arbitrary vertex \( v \in V(G') \). Let us note \( N_i \) the set of vertices at distance \( i \) from \( v \) in \( G' \); see Fig. 4 for an illustration. We shall prove the following two properties by induction for \( i \in \{1, \ldots, p-1\} \):

(i) \( N_i \) is an independent set in \( G' \); and
(ii) \( |N_i| \geq (k^{\frac{1}{3p}})^{i-1}(k^{\frac{1}{3p}} + 1) \).

For \( i = 1 \), \( N_1 \) is an independent set because \( G' \) is triangle-free, as it is an induced subgraph of a graph of girth at least \( 2p \geq 4 \). And we have that \( |N_1| = \deg_{G'}(v) \geq k^{\frac{1}{3p}} + 1 \).

Suppose that these two properties are true at level \( i \), for \( 1 \leq i < p-1 \). Let us show that they are also true at level \( i + 1 \). Note first that \( N_{i+1} \) is an independent set, as otherwise
there would be a cycle in $G'$ of length at most $2i + 3 \leq 2p - 1$, a contradiction (see Fig. 4). On the other hand, two vertices in $N_i$ cannot have a common neighbor in $N_{i+1}$, as otherwise there would be a cycle in $G'$ of length at most $2i + 2 \leq 2p - 2$, a contradiction (see Fig. 4). That is, each vertex in $N_i$ has exactly one neighbor in $N_{i-1}$, and since all vertices in $N_i$ have degree at least $k$ in $G'$, it follows that

$$|N_{i+1}| \geq |N_i| \cdot k^{\frac{1}{p-1}} \geq (k^{\frac{1}{p-1}})^i(k^{\frac{1}{p-1}} + 1).$$

Thus, by induction, $N_{p-1}$ is an independent set in $G$ and $|N_{p-1}| \geq (k^{\frac{1}{p-1}})^{p-2}(k^{\frac{1}{p-1}} + 1) \geq k$, as we wanted to prove.

We conclude this subsection with a subtlety that we overlooked so far for the sake of simplicity. In order to have an FPT algorithm for \{bull, $C_4$, ..., $C_{2p-1}$\}-free graphs, as we claim, we need to make sure that in Algorithm 1 we do not create small holes in the recursive steps. One can check that the block $T_Y$, in which the recursive call is made, does not contain small holes. Nevertheless, the block $T_X$ may contain an induced $C_4$ when the switchable pair \{c, d\} is added (see [20] for the precise definition of $T_X$). Fortunately, we can obtain the same asymptotic upper bound of $O(k \cdot k^{\frac{1}{p-1}})$ on the size of $T_X$ in Step 1 of Algorithm 1 when it belongs to the class $T_1$, by using the same arguments, and just distinguishing one more case: if $T_X$ contains a $C_4$, then we apply Lemma 4 to the graph $T_X \setminus \{c\}$ (or $T_X \setminus \{d\}$), which can be easily seen to be \{bull, $C_4$, ..., $C_{2p-1}$\}-free, and we just have to add one more vertex (c or d) to the upper bound given by Lemma 4.

### 4.2 A lower bound in \{bull, $C_4$, ..., $C_{2p-1}$\}-free graphs

In this subsection we prove Theorem 4. In fact, we show the lower bound holds even for unweighted Independent Set. We will reduce from the following problem.

**Sparse-3-Sat**

**Input:** A set of variables $\{x_1, \ldots, x_n\}$ and a set of 3-variable clauses $\{c_1, \ldots, c_m\}$ such that each literal appears at most $c$ times in the clauses, for some constant $c$.

**Question:** Is there an assignment of the variables such that all the clauses are satisfied?
The \textsc{Sparse-3-Sat} problem cannot be solved in time \(2^{o(n)}\) unless the ETH fails (see for instance [13]). Our reduction consists of a modification of the classical reduction to show the \textsc{NP-hardness} of \textsc{Independent Set} [10].

**Proof of Theorem 4** We will show that if we could solve \textsc{Independent Set} restricted to \{bull, \(C_4\), \ldots, \(C_{2p-1}\)}-free graphs in time \(2^{o(k)} \cdot n^{O(1)}\), the we could solve \textsc{Sparse-3-SAT} in time \(2^{o(n)}\), which is impossible unless the ETH fails.

We first define a transformation from an instance \(\phi\) of \textsc{Sparse-3-Sat} to a graph \(G_\phi\).

With each clause \(c_j\), for \(1 \leq j \leq m\), we associate a triangle where each vertex corresponds to a literal of the clause. For each variable \(x \in \{x_1, \ldots, x_n\}\), we add all the edges between the vertices corresponding to \(x\) and all the vertices corresponding to \(\overline{x}\).

Observe that all the clauses \(\phi\) can be satisfied if and only if the graph \(G_\phi\) has an independent set of size \(m\), and that since each literal appears in at most \(c\) clauses in \(\phi\), the degree of each vertex of \(G_\phi\) is bounded by \(c + 2\), hence \(|E(G_\phi)| \leq 3m(c+2)\).

We now transform the graph \(G_\phi\) into a \{bull, \(C_4\), \ldots, \(C_{2p-1}\}\}-free graph \(G'_{\phi}\) by replacing each edge of \(G_\phi\) with a path on \(q\) vertices, where \(q\) is the smallest even integer such that \(3(q+1) \geq 2p\). See Fig. 5 for an illustration. The newly added vertices are called internal, and the other ones are called original.

![Fig. 5. Construction of the graph \(G'_{\phi}\) in the proof of Theorem 4.](image)

\begin{enumerate}
\item \textbf{Claim 2} \([\star]\) \(G_\phi\) has an independent set of size \(m\) if and only if \(G'_{\phi}\) has an independent set of size \(|E(G_\phi)| \cdot \frac{\sqrt{3}}{2} + m\). That is, all the clauses \(\phi\) can be satisfied if and only if the graph \(G'_{\phi}\) has an independent set of size \(|E(G_\phi)| \cdot \frac{\sqrt{3}}{2} + m\).
\end{enumerate}

To conclude, assume that we can solve \textsc{Independent Set} in \{bull, \(C_4\), \ldots, \(C_{2p-1}\}\}-free graphs on \(t\) vertices in time \(2^{o(k)} \cdot t^{O(1)}\), and let \(k = |E(G_\phi)| \cdot \frac{\sqrt{3}}{2} + m\). Then, by Claim \([\star]\) by solving \textsc{Independent Set} in \(G'_{\phi}\) we could solve \textsc{Sparse-3-Sat} in time \(2^{o(|E(G_\phi)| \cdot \frac{\sqrt{3}}{2} + m)} \cdot (3m + |E(G_\phi)| \cdot q)^{O(1)} = 2^{o(n)}\), where we have used that \(|E(G_\phi)| \leq \frac{3m(c+2)}{2}\) and that \(m \leq 2c \cdot n\). This is impossible unless the ETH fails.

5 Conclusions and further research

We showed in Theorem 2 that \textsc{Weighted Independent Set} in bull-free graphs can be solved in time \(2^{O(k^2)} \cdot n^4\), and the lower bound of Theorem 4 states that the problem cannot be solved in time \(2^{o(k)} \cdot n^{O(1)}\) in bull-free graphs unless the ETH fails. Closing this complexity gap (in terms of \(k\)) is an interesting avenue for further research.
It is tempting to try to apply similar techniques for obtaining FPT algorithms for other (NP-hard) problems in bull-free graphs. The Independent Feedback Vertex Set problem may be a natural candidate.

Feghali, Abu-Khzam and Müller [8] have recently shown that the problem of deciding whether the vertices of a graph can be partitioned into a triangle-free subgraph and a disjoint union of cliques is NP-complete in planar and perfect graphs. Note that this problem is closely related to deciding whether a given graph belongs to the class \( T_1 \) of basic bull-free graphs. Is this problem NP-complete when restricted to bull-free graphs? The recognition of the class \( T_1 \) has also been left as an open question in [20].

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A Proof of Lemma 2

We shall present an algorithm to find a minimally-sided homogeneous cut in a trigraph $T$ that runs in time $O(n^6)$. The algorithm first tries to find a minimally-sided homogeneous set. For doing this, we reuse the same algorithm described in [20, Lemma 4.2], which runs in time $O(n^2)$. Then, in order to find a minimally-sided proper homogeneous pair, the approach in [20] makes $O(n^2)$ calls to the the algorithm of [20, Lemma 4.1], which runs in time $O(n^3)$, yielding an overall complexity of $O(n^5)$. We proceed to improve this part.

We describe in Algorithm 3 below how to find minimally-sided proper homogeneous pairs. This algorithm is strongly inspired from [20, Lemma 4.1], but even if its complexity is still quadratic, the difference lies on the fact that we will need to run Algorithm 3 $n^6$ times instead of $O(n^3)$, because we will only need to guess 4 vertices.

More precisely, in order to find a minimally-sided proper homogeneous pair, we run Algorithm 3 for all quadruples of vertices $(a_1, a_2, c, d)$ such that $a_1$ and $a_2$ are strongly adjacent to $c$ and strongly antiadjacent to $d$. Therefore, we have an algorithm running in time $O(n^6)$.

We would like to point out that the algorithm does not always output a proper homogeneous pair which is minimally-sided. Namely, the algorithm outputs the following: either a proper homogeneous pair that may be minimally-sided, or it guarantees that there is no minimally-sided proper homogeneous pair $(A, B)$ such that $a_1, a_2 \in A$ and $c, d \notin A \cup B$.

For the readability of the algorithm, let $\mathcal{P}$ be the following property:

**Property $\mathcal{P}$:** There is no minimally-sided proper homogeneous pair $(A, B)$ such that $a_1, a_2 \in A$ and $c, d \notin A \cup B$.

We are now ready to provide a formal description of Algorithm 3.

We want to prove that Algorithm 3 considers all minimally-sided proper homogeneous pairs, as these pairs are the only ones that may define a minimally-sided homogeneous cut. Let $(A_m, B_m)$ be a minimally-sided proper homogeneous pair, and let $(A_m, B_m, C_m, D_m, E_m, F_m)$ be the corresponding partition. Without loss of generality, we may assume that $|A_m| \geq 2$.

**Claim 3** The pair $(A_m, B_m)$ is returned by Algorithm 3 for a certain quadruple $(a_1, a_2, c, d)$, with $a_1, a_2 \in A_m$, $c \in C_m$, and $d \in D_m$.

**Proof:** We proceed to show inductively that by construction, the vertices in $A \cup B$ at the end of the algorithm necessarily belong to all proper homogeneous pairs $(A', B')$ with $a_1, a_2 \in A'$ and $c, d \notin A' \cup B'$.

Let $A_i$ and $B_i$ be the sets $A$ and $B$, respectively, at the end of step $i$ of the algorithm, with $i \leq n$. Let us show that at each step $i$, the sets $A_i$ and $B_i$ satisfy $A_i \subseteq A'$ and $B_i \subseteq B'$.

This property is true for $A_0 = \{a_1, a_2\}$ and $B_0 = \emptyset$. Suppose it is true at step $i < n$, that is, $A_i \subseteq A'$ and $B_i \subseteq B'$, and let us prove that it is also true at step $i + 1$. Let $x_{i+1}$ be the vertex that is added to $A_i$ or to $B_i$ at step $i + 1$. As $x_{i+1} \in R$, either $x_{i+1}$ is not strongly adjacent or strongly antiadjacent to $A_i$, or $x_{i+1}$ is not strongly adjacent or strongly antiadjacent to $B_i$. As $A_i \subseteq A'$ and $B_i \subseteq B'$, necessarily $x_{i+1}$ belongs to $A \cup B$. Thus, $x_{i+1}$ is either strongly adjacent to $c$ (if $x_{i+1} \in A$) and then $x_{i+1}$ is marked $\alpha$ and belongs to $A_{i+1}$, or strongly adjacent to $d$ (if $x_{i+1} \in B$) and then $x_{i+1}$ is marked $\beta$ and belongs to $B_{i+1}$. In both cases, we have that $A_{i+1} \subseteq A'$ and $B_{i+1} \subseteq B'$.

Therefore, $A \subseteq A'$ and $B \subseteq B'$, and in particular $A \subseteq A_m$ and $B \subseteq B_m$. But since $(A_m, B_m)$ is a minimally-sided proper homogeneous set, it follows that $A = A_m$ and $B = B_m$, hence the pair $(A_m, B_m)$ is indeed returned by Algorithm 3.

$\square$
**Input:** A trigraph $T$, 4 vertices $a_1, a_2, c,$ and $d$ such that $a_1$ and $a_2$ are strongly adjacent to $c$ and strongly antiadjacent to $d$.

**Output:** A smallest proper homogeneous pair $(A, B)$ such that $a_1, a_2 \in A$ and $c, d \notin A \cup B$, if it exists, or Property $P$ otherwise.

begin

$R = \{a_1, a_2\}$, $S = V \setminus R$, $A = \emptyset$, $B = \emptyset$.

We mark the vertices of $V(T)$ as follows:

- $\alpha$ for the vertices strongly adjacent to $c$ and strongly antiadjacent to $d$;
- $\beta$ for the vertices strongly adjacent to $d$ and strongly antiadjacent to $c$; and
- $\varepsilon$ for the remaining vertices.

while there is a marked vertex $x$ in $R$ do

if $x$ is marked $\varepsilon$ then

Output $P$.

if $x$ is marked $\alpha$ then

Move the following sets from $S$ to $R$: $\sigma(x) \cap S$, $(\eta(x) \cap S) \setminus \eta(a)$ and $(\eta(a) \cap S) \setminus \eta(x)$. Move $x$ from $R$ to $A$.

if $x$ is marked $\beta$ then

if $B$ is empty then

Let $b := x$. Move $\sigma(b) \cap S$ from $S$ to $R$.

Move $b$ from $R$ to $B$.

else

Move the following sets from $S$ to $R$: $\sigma(x) \cap S$, $(\eta(x) \cap S) \setminus \eta(b)$ and $(\eta(b) \cap S) \setminus \eta(x)$. Move $x$ from $R$ to $B$.

if $B$ is empty then

$A$ is a homogeneous set: output $P$.

else

if $B$ is either strongly complete or strongly anticomplete to $A$ then

$A$ is a homogeneous set: Output $P$.

else

if $|S| \geq 3$ then

Output $(A, B)$.

else

Output $P$.

end

**Algorithm 3:** Algorithm for finding minimally-sided proper homogeneous pairs.
B Proof of Claim 2

First, if $G_\phi$ has an independent set $S$ of size $m$, we take all the vertices of $S$ and we add $\frac{q}{2}$ internal vertices for each original edge. We can add so many vertices since at most one vertex of each original edge of $G_\phi$ can be in $S$.

Conversely, suppose that $G_\phi'$ has an independent set $S'$ of size $|E(G_\phi)| \cdot \frac{q}{2} + m$. As $S'$ cannot contain more than $|E(G_\phi)| \cdot \frac{q}{2}$ internal vertices, there are at most $m$ vertices of $V(G_\phi)$ in $S'$. Let $\eta$ be the number of edges $xy \in E(G_\phi)$ such that both $x$ and $y$ are in $S'$.

If $\eta = 0$, then $S' \cap V(G_\phi)$ is an independent set of $G$ of size at least $m$, and we are done.

We now now that if $\eta > 0$, there exists an independent set $S''$ in $G_\phi'$ such that $|S''| = |S'|$ and with strictly less than $\eta$ edges $xy \in E(G_\phi)$ such that both $x$ and $y$ are in $S''$.

Let $x, y \in S'$ be such that $xy \in E(G)$, and let us note $(x = x_0, x_1, \ldots, x_q, y)$ the path between $x$ and $y$ in $G_\phi'$ induced by the subdivision of the edge $xy$. Let $i$ be the smallest integer in $\{1, \ldots, q\}$ such that $x_i$ and $x_{i+1}$ are not in $S'$. Note that such an integer $i$ exists since $q$ is an even number, and observe that $i$ is an odd number; see Fig. 6 for an illustration, where the red vertices belong to $S'$.

We now construct $S''$ as follows: we initialize $S'' = S'$, and for all $j \in \{0, \ldots, \frac{q-1}{2}\}$, we remove $x_{2j}$ from $S''$ and we add $x_{2j+1}$. Observe that since $x_{i+1}$ is not in $S'$, $S''$ is indeed an independent set of size $|E(G_\phi)| \cdot \frac{q}{2} + m$ such that the parameter $\eta$ has strictly decreased; see the lower part of Fig. 6.

Repeating this procedure while $\eta > 0$, we eventually obtain an independent set $S'_0$ of $G_\phi'$ of size $|E(G_\phi)| \cdot \frac{q}{2} + m$ such that there are no two vertices $x, y \in S'_0$ such that $xy \in E(G_\phi)$. Therefore, $S'_0 \cap V(G_\phi)$ is an independent set in $G_\phi$. Furthermore, it has size at least $m$ since there cannot be more than $|E(G_\phi)| \cdot \frac{q}{2}$ internal vertices in $S'_0$. 

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**Fig. 6.** Decreasing the parameter $\eta$ in the proof of Theorem 4.