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The linear Ising model and its analytic continuation, random walk

B. H. Lavenda

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(Dated: February 8, 2020)

A generalization of Gauss’s principle is used to derive the error laws corresponding to Types II and VII distributions in Pearson’s classification scheme. Student’s $r$-pdf (Type II) governs the distribution of the internal energy of a uniform, linear chain, Ising model, while analytic continuation of the uniform exchange energy converts it into a Student $t$-density (Type VII) for the position of a random walk in a single spatial dimension. Higher dimensional spaces, corresponding to larger degrees of freedom and generalizations to multidimensional Student $r$- and $t$-densities, are obtained by considering independent and identically distributed random variables, having rotationally invariant densities, whose entropies are additive and generating functions are multiplicative.

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A relation between the Ising problem and random walks has been conjectured a number of times over the years [1]. Rather than one of technique, we shall show in this letter that the two are inter convertible through analytic continuation of the uniform exchange energy which transforms a probability density function (pdf) of finite range into one of infinite range. Specifically, a Student $t$-pdf [2], of limited range, which governs the distribution of the internal energy of a linear, uniform Ising chain, is transformed into Student’s $t$-pdf, of unlimited range, for the position of a random walker along an infinite linear chain, through analytic continuation of the uniform exchange energy between spins. These two families of distributions will be derived as error laws from a generalization of Gauss’s principle, which will serve in unifying these two seemingly unrelated phenomena.

Gauss’s principle asserts that if the probability of an error, given the true value, is a function of the error alone, and the log-likelihood function is maximum when the true value coincides with the mean of the observed values, then the law of error must be normal [3]. However, many other error laws exist, such as the beta pdf, in which the probability of an error is not a function of the location parameter [4]. The normal law follows by integration of the log-likelihood equation. The log-likelihood equation is formed by setting deviations from (2) proportional to deviations from the normal law, where $\sigma_1$ is chosen. Then, except for the constant weight, we would get the weighted average [3, pp. 214]

$$\sum_{i=1}^{n} \sigma_i (x_i - \lambda) = 0,$$

where the prime stands for differentiation with respect to $\lambda$.

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$$\sum_{i=1}^{n} \sigma_i (x_i - \lambda) = 0,$$

where the prime stands for differentiation with respect to $\lambda$. In the case of a negative unit weight, we would get the weighted average [3, pp. 214]

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We have identified the weights, $\sigma_i$, as the second derivatives of the entropy. The weights we will consider here (pp. 210–211),

$$\sigma_\pm = - \left(1 \mp \frac{(x - \lambda)^2}{s^2}\right)^{-1},$$

(5)

will lead to Pearson distributions of Types II (negative sign) and VII (positive sign). We assume that the information regarding the values of the location, $\lambda$, and scaling, $s$, parameters depend on (p. 379)

$$u = \frac{x - \lambda}{s}$$

alone.

In the case of Type II distributions a single integration of $\sigma_-$ gives the ‘equation of state’

$$\beta = - \tanh^{-1} u,$$

(6)

where we will soon appreciate $\beta$ as the intensive variable conjugate to the extensive variable $u$. In essence, contains the statement of the second law of thermodynamics

$$\frac{\partial S}{\partial u} = \beta,$$

where $S(u)$ is the entropy. A further integration yields the entropy reduction

$$\Delta S(u) := S(u) - S(0)$$

= $-u \tanh^{-1} u - \frac{1}{2} \log(1 - u^2) + \text{const.},$

(7)

where $S(0)$ is the maximum entropy in the ‘high temperature limit’ $\beta \to 0$, where $u = 0$.

Introducing the equation of state (6) into (7) gives

$$\Delta S = -\beta \tanh \beta + \log \cosh \beta + \text{const.}$$

(8)

Setting $\beta = J/T$, where $J$ is the uniform exchange energy between spins, and $T$ the absolute temperature in energy units where Boltzmann’s constant is unity, $u$ in (6) is identified as the ratio of the internal energy per unit spin to the exchange energy, and $\Delta S$ is the zero-field entropy reduction per spin of the uniform, linear chain, Ising model. Since there are $2^m$ configurations, the maximum entropy per spin is $S(0) = 2$.

For $m$ spins (6) will be $m$ times as large. Viewed as the Legendre transform

$$\Delta S(u) - \frac{\partial \Delta S}{\partial u} u = \log Z(\beta),$$

(9)

the generating function per spin is identified as

$$Z(\beta) = \cosh \beta.$$

(10)

The generating function for $m$ spins is

$$Z_m(\beta) = \cosh^{m-1} \beta,$$

(11)

which is the hallmark of an infinitely divisible distribution.

The Legendre transform (11) is asymptotically equivalent to the Laplace transform

$$Z_m(\beta) = \int_{-1}^{0} e^{-\beta u - \Delta S(u)} \, du$$

(12)

in the limit as $m \to \infty$. The function, $\beta u - \Delta S(u)$, has a minimum on $[-1, 0]$ at the interior point (13). Employing Laplace’s method, the integral (12) can be evaluated by developing the exponent in a Taylor series to second order about the minimum (13), and using the fact that $m$ is so large that the value of the resulting Gaussian integral is barely changed when the lower limit of integration is set equal to $-\infty$. We then obtain

$$Z_m(\beta) = \sqrt{\frac{\pi}{2m}} \cosh^{m-1} \beta$$

for the expression of the generating function of $m$ spins. Consequently, (10) and (11) are logarithmically equivalent.

The entropy reduction determines the univariate pdf through Boltzmann’s principle (3):

$$p(u) = K \, e^{\Delta S(u)},$$

(13)

where $K$ is a constant. Introducing (14) into (15) gives

$$p(u) = K e^{-u \tanh^{-1} u} (1 - u^2)^{\frac{3}{2}} = K (1 - u^2)^{\frac{3}{2}} + O (u^4).$$

Consider a uniform distribution of mass inside a hypersphere of $m$-dimensions with unit radius. The energies $U_1, \ldots, U_m$ of $m$ spins are supposed to be independent and identically distributed (iid), and, in addition, they have rotationally invariant densities. Such a hypersphere will be spanned by the coordinates, $u_1, \ldots, u_m$. The Euclidean distance from the origin to any point

$$|u| = (u_1^2 + \cdots + u_m^2)^{\frac{1}{2}}.$$  

(14)

The section at $|u|$ has radius $\sqrt{1 - u^2}$, and an ‘area’ proportional to $(1 - u^2)^{\frac{3}{2}(m-1)}$. Hence, it has a density

$$p_m(u) = K e^{-(m-1) \Delta S(u)}$$

(15)

$$\simeq K (1 - u^2)^{\frac{3}{2}(m-1)} + O (u^4).$$

The multidimensional pdf, (15), shows that we are essentially dealing with a beta pdf

$$p_m(u) = K (1 + u)^{\frac{1}{2}(m-1)} (1 - u)^{\frac{1}{2}(m-1)} + O (u^4),$$

defined over $[-1, +1]$, rather than over $[-1, 0]$. In fact, the entropy reduction, (8), can be written as an entropy of mixing

$$\Delta S(u) = -\frac{1}{2} \left\{ (1 + u) \log(1 + u) + (1 - u) \log(1 - u) \right\}.$$
The distribution of mass inside a spherical layer whose radius varies from \((1 - \epsilon)\) to 1, with \(\epsilon\) an infinitesimal quantity is \([8, p. 191]\)

\[
p_m(u) = K \left\{ (1 - u^2)^{\frac{1}{2}(m-1)} - [(1 - \epsilon)^2 - u^2]^\frac{1}{2}(m-1) \right\}
\]

\[
\simeq K (m - 1)\epsilon (1 - u^2)^{\frac{1}{2}(m-3)} = K (1 - u^2)^{\frac{1}{2}(m-3)},
\]

where the constants have been absorbed into the new definition of \(K\). This is the same pdf as \([8]\), except that \(m\) has been reduced by a factor of 2. A change of variable, \(u = \cos \theta\), converts it into

\[
p_m(\theta) = K \sin^{m-2} \theta.
\]  

(16)

For a sphere, \(m = 3\), and it is well-known that two points on its surface subtend an angle between \(\theta\) and \(\theta + d\theta\), at the center of the sphere, if and only if the second point falls in a ring of area \(2\pi \sin \theta \, d\theta\). This occurs with probability \(2\pi \sin \theta \, d\theta / 4\pi = \frac{1}{2} \sin \theta \, d\theta\) \([10]\).

The approach to the normal pdf can be viewed in the following terms. As \(m\) increases, jumps to the left and right will tend to equalize, like heads and tails, so that the distribution will tend to concentrate around the origin. To avoid this, and derive the shape of the asymptotic distribution, we introduce the scaling \(u = u / \sqrt{m}\) \([8, p. 59]\) so that \((1 - u^2/m)^{\frac{1}{2}(m-1)} \rightarrow e^{-\frac{1}{2}u^2}\). The normal distribution is obtained in the limit without having to invoke any assumptions regarding the central limit theorem.

The form of the pdf \([8]\) agrees with that found from the log-likelihood equation for the centering parameter \(\lambda\) \([8, p. 210]\)

\[
\frac{d}{d\lambda} \log L(\lambda) = \sum_{i=1}^{n} \frac{x_i - \lambda}{1 - (x_i - \lambda)^2/m}.
\]

Integrating gives the likelihood function

\[
L(\lambda) = \prod_{i=1}^{n} \left\{ 1 - \frac{(x_i - \lambda)^2}{m} \right\}^{\frac{1}{2}m},
\]

of a Type II law. Apart from the norming constant, the likelihood function is seen to be the product of pdfs of the form \([8]\).

Analytic continuation of the exchange energy, \((i.e., J \rightarrow iJ)\) converts the thermal equation of state \([9]\) into

\[
\beta = \tan^{-1} u,
\]  

(17)

where \(\beta \rightarrow i\beta\), and \(u \rightarrow -iu\) in \([9]\). What was an inverse temperature has now become an angle, \(\beta\). Such a process simulates a symmetric walk on an infinite 1-dimensional lattice, where what was the dimensionless internal energy per unit spin, \(u\), is now the position of the walker along the chain. Since the interval for \(\beta\) is \([-\frac{\pi}{2}, \frac{\pi}{2}]\), the walker can wander off to \(\pm \infty\).

On account of the properties of concavity and convexity, the equation of state \([17]\) is the negative derivative of the entropy and the positive derivative of the logarithm of the generating function, respectively. Differentiating a second time shows that we are dealing with a Type VII distribution, whose weight is given by \(\sigma_+\) in \([9]\). Integrating \([17]\) gives the entropy reduction

\[
\Delta S(u) = -u \tan^{-1} u + \frac{1}{2} \log (1 + u^2) + \text{const.}
\]  

(18)

For small \(u\), the entropy reduction \([18]\) reduces to the negative quadratic form of the normal law,

\[
\Delta S(u) \simeq -\frac{1}{2} \log (1 + u^2) \rightarrow -\frac{1}{2} u^2.
\]

Introducing \([17]\) into \([18]\) enables the comparison of

\[
\Delta S = -\beta \tan \beta - \log \cos \beta + \text{const.},
\]  

(19)

and \([8]\). Under \(\beta \rightarrow i\beta\), one becomes the negative of the other. On the strength of the Legendre transform, \([9]\), the logarithm of the generating function can be identified from \([19]\) as

\[
\log Z(\beta) = -\log \cos \beta.
\]  

(20)

The characteristic function of a random walk model with equal \(a \text{ priori}\) probabilities of a jump to the left or to the right is \(\cos \beta\) \([3]\). This symmetric Bernoullian

\[n\text{ity of light along lines of constant } v, \text{ the axis orthogonal to } u \text{ in the } u, v\text{-plane. The convolution property of the Cauchy pdf, according to Feller, is a statement of Hughen's principle which states that intensity of light is the same for a source located at the origin along the line } v = 1, \text{ as it would be if the source were distributed along this line and the intensity were measured along } v = 2. \text{ Moreover, if we make } n \text{ measurements, the average, } (U_1 + U_2 + \ldots + U_n)/n, \text{ has exactly the same pdf as that of a single measurement, so that nothing is gained by making repeated measurements.}

\[1\text{ A similar transformation leads from aperiodic to periodic behavior in Newton's reiteration scheme. An approximate root of the equation } g(x) = 0 \text{ has } x_{n+1} = x_n - g(x_n)/g'(x_n) \text{ as its next approximation. For } g(x) = x^2 - 1, \text{ this reads } x_{n+1} = \frac{1}{2}(x_n + 1/x_n). \text{ Setting } x_n = \cot \theta_n \text{ gives } \theta_{n+1} = 2\theta_n, \text{ without any trace of periodicity. However, by making } x \text{ purely imaginary, } x = iy, \text{ the iterative scheme } y_{n+1} = \frac{1}{2}(y_n - 1/y_n) \text{ has the solution } y = -\cot(x\pi), \text{ which demands that } \theta_{n+1} = 2\theta_n \text{ mod } 1. \text{ Now, initial aperiodic behavior can lead to preperiodic orbits by an appropriate choice of a rational number for } \theta_n, \text{ instead of to a fixed point }\]

\[2\text{ Under } \beta \rightarrow i\beta, \text{ one becomes the negative of the other. On the strength of the Legendre transform, }\]

\[3\text{ The generating function exists for all real } \beta \text{ in all cases where the pdf has a finite range, and in the case of the normal law }\]

\[p. 87\]. In cases where the integral exists, the logarithm of the generating function will be equal in magnitude and opposite in sign to the logarithm of the characteristic function.
distribution has iid, the characteristic function of $m$ jumps is simply $\cos^m \beta$. The inverse Fourier transform

$$p(u) = \frac{2^{-m}}{\pi} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} e^{-i\beta u} \left( e^{i\beta} + e^{-i\beta} \right)^m d\beta$$

$$= \frac{1}{2^m} \left( \frac{m}{\frac{1}{2} (m + u)} \right) \rightarrow e^{-u^2/2m}$$

gives the pdf, which in the large $m$-limit, transforms into the normal pdf. The approach to the normal pdf can be obtained from Boltzmann’s principle, \[13\].

With the entropy reduction given by \[13\], the pdf for a single jump of length $u$ is given by Boltzmann’s principle, \[13\]:

$$p(u) = K (1 + u^2)^{\frac{1}{2}} e^{-u \tan^{-1} u}.$$  

Expanding the exponent, we get

$$p(u) = K (1 + u^2)^{\frac{1}{2}} + O (u^4). \quad \text{(21)}$$

If the steps, $U_1, U_2, \ldots, U_m$, are iid, then the length,

$$u = ||U|| = (u_1^2 + \cdots + u_m^2)^{\frac{1}{2}},$$

will have an $m$-dimensional pdf of the form \[4\]

$$p_m(u) = K (1 + u^2)^{-\frac{m}{2}} + O(u^4). \quad \text{(22)}$$

The change of variable, $u = \cot \theta$ will transform \[22\] into \[10\] \[3\] p. 193\).

The form of the pdf \[22\] is, again, supported by the log-likelihood equation for the centering parameter $\lambda$ \[3\] p. 210\]

$$\frac{d}{d\lambda} \log \mathcal{L}(\lambda) = \sum_{i=1}^{n} \frac{x_i - \lambda}{1 + (x_i - \lambda)^2/m}.$$  

|Footnote|The distinction between \[24\] and the Cauchy distribution in $\mathbb{R}^m$ is the following. For $m = 2$, the probability that a vector in a randomly chosen direction with a length $R$ will be greater than $r$ is $Pr(R > r) = (1 + r^2)^{-\frac{1}{2}}$ \[12\] p. 71\]. The extreme observations that $m - 1$ will be greater than $r$ while the remaining one will lie in the range $dr$ will have a density proportional to $r(1 + r^2)^{-\frac{1}{2}(m+1)}$. The change of variable, $r = \sqrt{x}$ will convert it into a beta density of the second kind, proportional to $(1 + x)^{-\frac{1}{2}(m+1)}$. Likewise, the property of extreme observations for $Pr(R > r) = (1 - r^2)^{\frac{1}{2}}$ will lead to a beta density of the first kind, which is closely allied to order statistics. For extreme observations of largest value, Boltzmann’s principle \[13\] must be modified in such a way that the logarithm of the tail of the distribution is proportional to the entropy reduction \[13\].

By integration we find

$$\mathcal{L}(\lambda) = \prod_{i=1}^{n} \left\{ 1 + \left( \frac{x_i - \lambda}{m} \right)^2 \right\}^{-\frac{1}{2}},$$

for the likelihood function of a Type VII law. To within a constant factor, the likelihood function is a product of pdfs of the form \(22\).

The pdf \[22\] is a generalization of the 1-dimensional Student $t$-pdf with $m$ degrees of freedom to an $m$-dimensional distribution having the same number of degrees of freedom \[3\] p. 194\]. In the limit as $m \rightarrow \infty$, \[22\] transforms into the normal pdf,

$$p(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}, \quad \text{(23)}$$

where the norming constant $K$

$$\lim_{m \rightarrow \infty} \left\{ m^{-\frac{1}{2}} \left[ \begin{array}{c} \frac{1}{2} \left( \frac{1}{2} (m - 1) \right) \end{array} \right]^{-1} \right\} = (2\pi)^{-\frac{1}{2}},$$

and the scaling $u \rightarrow u/\sqrt{m}$ has been introduced. Student’s $t$-distribution \[22\] can, therefore, rightly be considered the last stop before normality for large, but finite, $m$.

We have derived Student’s $t$-pdf from Boltzmann’s principle, \[13\], where the entropy \[13\] is independent of the number of degrees of freedom. As a consequence of its statistical independence, the entropy is additive. For $m$ iid random variables, or degrees of freedom, the total entropy is $m$ times as great as the entropy of a single variate, while the joint pdf is the product of functions, and this can only be true if there is a logarithmic relation between the two. If this were not the case, it would be impossible to talk about the entropy of a single constituent of the system, or the probability of a single event. Consequently, any putative expression for the entropy which is defined in terms of the degrees of freedom of the system \[14\] must be considered not as an entropy, but, rather, as an interpolation formula between two bona fide expressions of the entropy. A prototype is the Rényi entropy, which is an interpolation formula connecting the Boltzmann-Hartley and the Gibbs-Shannon entropies as a parameter ranges over its permitted values (i.e., those that preserve the property of Schur-concavity \[13\]).

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