ON ERDÉLYI-MAGNUS-NEVAI CONJECTURE FOR JACOBI
POLYNOMIALS

ILIA KRASIKOV

ABSTRACT. T. Erdélyi, A.P. Magnus and P. Nevai conjectured that for \( \alpha, \beta \geq -\frac{1}{2} \), the orthonormal Jacobi polynomials \( P^{(\alpha,\beta)}_k(x) \) satisfy the inequality

\[
\max_{x \in [-1,1]} (1-x)^{\alpha+\frac{1}{2}}(1+x)^{\beta+\frac{1}{2}} \left( P^{(\alpha,\beta)}_k(x) \right)^2 = O \left( \max \left\{ 1, (\alpha^2 + \beta^2)^{1/4} \right\} \right),
\]

[Erdélyi et al., Generalized Jacobi weights, Christoffel functions, and Jacobi polynomials, SIAM J. Math. Anal. 25 (1994), 602-614]. Here we will confirm this conjecture in the ultraspherical case \( \alpha = \beta \geq 1 + \sqrt{\frac{3}{2}} \), even in a stronger form by giving very explicit upper bounds. We also show that

\[
\sqrt{\delta^2 - x^2} (1-x^2)\alpha \left( P^{(\alpha,\alpha)}_{2k}(x) \right)^2 < \frac{2}{\pi} \left( 1 + \frac{1}{8(2k+\alpha)^2} \right)
\]

for a certain choice of \( \delta \), such that the interval \( (-\delta, \delta) \) contains all the zeros of \( P^{(\alpha,\alpha)}_{2k}(x) \). Slightly weaker bounds are given for polynomials of odd degree.

Keywords: Jacobi polynomials

1. Introduction

In this paper we will use bold letters for orthonormal polynomials versus regular characters for orthogonal polynomials in the standard normalization \([14]\).

Given a family \( \{p_i(x)\} \) of orthonormal polynomials orthogonal on a finite or infinite interval \( I \) with respect to a weight function \( w(x) \geq 0 \), it is an important and difficult problem to estimate \( \sup_{x \in I} \sqrt{w(x)} |p_i(x)| \), or, more generally, to find an envelope of the function \( \sqrt{w(x)} p_i(x) \) on \( I \). Those two questions become almost identical if we introduce an auxiliary function \( \phi(x) \) such that \( \sqrt{\phi(x)} w(x) p_i(x) \) exhibits nearly equioscillatory behaviour. Of course, the existence of such a function is far from being obvious but it turns out that in many cases one can choose \( \phi = \sqrt{(x-d_m)(d_M-x)} \), with \( d_m, d_M \) being appropriate approximations to the least and the largest zero of \( p_i \) respectively. The simplest example is given by Chebyshev polynomials \( T_i(x) \) and \( \phi = \sqrt{1-x^2} \). This illustrates a classical result of G. Szegö asserting that for a vast class of weights on \([-1,1]\) and \( i \to \infty \), the function \( \sqrt{1-x^2} w(x) p_i(x) \) equioscillates between \( \pm \sqrt{\frac{2}{\pi}} \), \([14]\).

A very general theory for exponential weights \( w = e^{-Q(x)} \) stating that under some technical conditions on \( Q \),

\[
\max_i \left| \sqrt{1-(x-a_i)(a_i-x)} w(x) p_i(x) \right| < C,
\]

1991 Mathematics Subject Classification. 33C45.
where the constant $C$ is independent on $i$ and $a_{\pm i}$ are Mhaskar-Rahmanov-Saff numbers for $Q$, was developed by A.L. Levin and D.S. Lubinsky \[11\]. Recently it has been extended to the Laguerre-type exponential weights $x^{2\rho}e^{-2Q(x)}$ \[11, 12\].

In the case of classical orthogonal Hermite and Laguerre polynomials explicit bounds confirming such a nearly equioscillatory behaviour independently on the parameters involved were given in \[8\] and \[9\] respectively.

The case of Jacobi polynomials $P^{(\alpha, \beta)}_k(x)$, $w(x) = (1 - x)^\alpha(1 + x)^\beta$, is much more difficult. Let us introduce some necessary notation.

We define

$$M^{\alpha,\beta}_k(x, d_m, d_M) = \sqrt{(x - d_m)(d_M - x)} \left(1 - x\right)^\alpha \left(1 + x\right)^\beta \left(P^{(\alpha, \beta)}_k(x)\right)^2,$$

$$M^{\alpha,\beta}_k(d_m, d_M) = \max_{x \in [-1,1]} M^{\alpha,\beta}_k(x; d_m, d_M),$$

what we will abbreviate to $M^{\alpha,\beta}_k(x)$ and $M^{\alpha,\beta}_k$ if $d_m = -1$, $d_M = 1$, that is for $\phi(x) = \sqrt{1 - x^2}$. We will also omit one of the superscripts in the ultraspherical case $\alpha = \beta$ writing, for example, $M^{(\alpha, \alpha)}_k(x)$ instead of $M^{\alpha,\alpha}_k(x)$, and shorten $M^{(\alpha, \alpha)}_k(x, -d, d)$. $M^{(\alpha, \alpha)}_k(-d, d)$ to $M^{\alpha,\alpha}_k(x, d)$, $M^{(\alpha, \alpha)}_k(d)$ respectively.

As $P^{(\alpha, \beta)}_k(x) = (-1)^k P^{(\beta, \alpha)}_k(-x)$ we may safely assume that $\alpha \geq \beta$.

For $-\frac{1}{2} < \beta \leq \alpha < \frac{1}{2}$, the following is known \[3\]:

$$M^{\alpha,\beta}_k \leq \frac{2^{2\alpha+1} \Gamma(k + \alpha + \beta + 1) \Gamma(k + \alpha + 1)}{\pi k! (2k + \alpha + \beta + 1)^{2\alpha} \Gamma(k + \beta + 1)} = \frac{2}{\pi} + O \left(\frac{1}{k}\right),$$

where $k = 0, 1, \ldots$.

A slightly stronger inequality in the ultraspherical case was obtained earlier by L. Lorch \[13\].

A remarkable result covering almost all possible range of the parameters has been established by T. Erdélyi, A.P. Magnus and P. Nevai, \[5\],

$$M^{\alpha,\beta}_k \leq \frac{2e \left(2 + \sqrt{\alpha^2 + \beta^2}\right)}{\pi},$$

provided $k \geq 0$, $\alpha, \beta \geq -\frac{1}{2}$.

Moreover, they suggested the following conjecture:

**Conjecture 1.**

$$M^{\alpha,\beta}_k = O \left(\max \left\{1, |\alpha|^{1/2}\right\}\right),$$

provided $\alpha \geq \beta \geq -\frac{1}{2}$.

The best currently known bound was given by the author \[7\],

$$M^{\alpha,\beta}_k \leq 11 \left(\frac{(\alpha + \beta + 1)^2 (2k + \alpha + \beta + 1)^2}{4k(k + \alpha + \beta + 1)}\right)^{1/3} = O \left(\alpha^{2/3} \left(1 + \frac{\alpha}{k}\right)^{1/3}\right),$$

provided $k \geq 6$, $\alpha \geq \beta \geq \frac{1+\sqrt{2}}{4}$.

We also brought some evidences in support of the following stronger conjecture

**Conjecture 2.**

$$M^{\alpha,\beta}_k = O \left(\max \left\{1, |\alpha|^{1/3} \left(1 + \frac{|\alpha|}{k}\right)^{1/6}\right\}\right),$$
provided \( \alpha \geq \beta \geq -\frac{1}{2} \).

Here we will confirm this conjecture in the ultraspherical case. Namely we prove the following

**Theorem 1.** Suppose that \( k \geq 6, \alpha = \beta \geq \frac{1+\sqrt{2}}{4} \). Then

\[
M_k^\alpha < \mu \alpha^{1/3} \left(1 + \frac{\alpha}{k}\right)^{1/6},
\]

where

\[
\mu = \begin{cases} 
\frac{10}{7}, & k \text{ even,} \\
22, & k \text{ odd.} 
\end{cases}
\]

We deduce this result from the following two theorems. The first, which has been established in [7], gives a sharp inequality for the interval containing all the local maxima of the function \( M_k^{\alpha, \beta}(x) \). The second one will be proven here and in fact demonstrates equioscillatory behaviour of \( M_k^\alpha(x, d) \) under an appropriate choice of \( d \).

**Theorem 2.** Suppose that \( k \geq 6, \alpha \geq \beta \geq \frac{1+\sqrt{2}}{4} \). Let \( x \) be a point of a local extremum of \( M_k^{\alpha, \beta}(x) \). Then \( x \in (\eta_j - 1, \eta_j) \), where

\[
\eta_j = j \left( \cos(\tau + j\omega) - \theta_j \left( \frac{\sin^4(\tau + j\omega)}{2 \cos \tau \cos \omega} \right)^{1/3} \right) \left(2k + \alpha + \beta + 1\right)^{-2/3}
\]

\[
\sin \tau = \frac{\alpha + \beta + 1}{2k + \alpha + \beta + 1}, \quad \sin \omega = \frac{\alpha - \beta}{2k + \alpha + \beta + 1}, \quad 0 \leq \tau, \omega < \frac{\pi}{2},
\]

and

\[
\theta_j = \begin{cases} 
1/3, & j = -1, \\
3/10, & j = 1.
\end{cases}
\]

In particular, in the ultraspherical case

\[
|x| < \eta = \cos \tau \left(1 - \frac{2^{-1/3}}{3} \left(2k + 2\alpha + 1\right)^{-2/3} \tan^{4/3} \tau\right),
\]

with \( \sin \tau = \frac{2\alpha + 1}{2k + 2\alpha + 1} \).

**Theorem 3.** Suppose that \( \alpha > \frac{1}{2} \), and let

\[
\delta = \sqrt{1 - \frac{4\alpha^2 - 1}{(2k + 2\alpha + 1)^2 - 4}}.
\]

Then

\[
M_k^\alpha(\delta) < \begin{cases} 
\frac{2}{\pi} \left(1 + \frac{1}{\pi(k + \alpha)}\right), & k \geq 2, \text{ even,} \\
\frac{230}{\pi}, & k \geq 3, \text{ odd.}
\end{cases}
\]

Moreover, all local maxima of the function \( M_k^\alpha(x) \) lie inside the interval \((-\delta, \delta)\).
To prove this theorem we construct an envelope of $M_k^{\alpha, \beta}(x; d_m, d_M)$ using so-called Sonin’s function. Then we show that in the ultraspherical case for $\alpha > \frac{1}{2}$ it has the only minimum at $x = 0$ if $\delta_m = -1$, $\delta_M = 1$, whereas for $-d_m = d_M = \delta$ the point $x = 0$ is the only maximum. Sharper bounds for the even case are due to the fact that $x = 0$ is the global maximum of $M_k^{2\alpha}(x, \delta)$ and the value of $P_{2k}^{(\alpha, \alpha)}(0)$ is known.

The paper is organized as follows. In the next section we present a simple lemma being our main technical tool. We will illustrate it by proving that the function $M_k^{\alpha, \beta}(x)$ is unimodal with the only minimum in a point depending only on $\alpha$ and $\beta$. The even and the odd cases of Theorem 3 will be proven in sections 3 and 4 respectively. The last section deals with the proof of Theorem 4.

2. Preliminaries

In his seminal book [14] Szegö presented a few result concerning the behaviour of local extrema of classical orthogonal polynomials based on an elementary approach via so-called Sonin’s function. In particular, he gave a comprehensive treatment of the Laguerre polynomials [14, Sec 7.31, 7.6 ], but did not try to deal with the Jacobi case for arbitrarily values of $\alpha$ and $\beta$. Here we combine his approach with the following very simple idea.

Given a real function $f(x)$, Sonin’s function $S(f; x)$ is

$$S(f; x) = f^2 + f'^2,$$

where $\psi(x) > 0$ on an interval $I$ containing all local maxima of $f$. Thus, they lie on $S$, and if $S$ is unimodal we can locate the global one.

**Lemma 4.** Suppose that a function $f$ satisfies on an open interval $I$ the Laguerre inequality

$$f'^2 - ff'' > 0,$$

and a differential equation

$$f'' - 2Af' + B(x)f = 0,$$

where $A \in C(I)$, $B(x) \in C^1(I)$, and $B$ has at most two zeros on $I$. Let

$$S(f; x) = f^2 + \frac{f'^2}{B},$$

then all the local maxima of $f$ in $I$ are in the intervals defined by $B(x) > 0$, and

$$\text{Sign} \left( \frac{d}{dx} S(f; x) \right) = \text{Sign}(4AB - B').$$

**Proof.** We have $0 < f'^2 - ff'' = f'^2 - 2Af' + Bf^2$, hence $B(x) > 0$ whenever $f' = 0$. Finally,

$$\frac{d}{dx} \left( f^2 + \frac{f'^2}{B} \right) = \frac{4AB - B'}{B^2} f^2(x),$$

and $B(x) \neq 0$ in one or two intervals containing all the extrema of $f$ on $I$. □

Let us make a few remarks concerning the Laguerre inequality (9). Usually it is stated for hyperbolic polynomials, that is real polynomials with only real zeros, and their limiting case, so-called Polya-Laguerre class. In fact, it holds for a much vaster class of functions. Let $L(f) = f'^2 - ff''$, defining $L = \{ f(x) : L(f) > 0 \}$, we observe that $L$ is closed under linear transformations $x \to ax + b$. Moreover, since

$$L(fg) = f^2 L(g) + g^2 L(f),$$
\( \mathcal{L} \) is closed under multiplication as well. Thus, \( L(x^\alpha) = \alpha x^{2\alpha - 2} \), yields the polynomial case and much more. Many examples may be obtain by \( L(e^f) = -e^{2f} f'' \) and obvious limiting procedures.

For our purposes it is enough that (11) holds for the functions
\[
((x - d_m)(d_M - x))^{1/4} (1 - x)^{\alpha/2} (1 + x)^{\beta/2} P_k^{(\alpha, \beta)}(x),
\]
provided \(-1 \leq d_m < x < d_M \leq 1\), and \( \alpha, \beta \geq 0 \).

To demonstrate how powerful this lemma is, we apply it to \( M_k^{\alpha, \beta}(x) \) to show that its local maxima lie on a unimodal curve.

From the differential equation for Jacobi polynomials
\[
(1 - x^2)y'' = ((\alpha + \beta + 2)x + \alpha - \beta)y' - k(k + \alpha + \beta + 1)y; \quad y = P_k^{(\alpha, \beta)}(x),
\]
we obtain
\[
4(1 - x^2)^2 z'' = 4x(1 - x^2)z' -
\]
\[
[(2k + \alpha + \beta + 1)^2(1 - x^2) - 2(1 + x)\alpha^2 - 2(1 - x)\beta^2 + 1] z;
\]
\[
z = (1 - x)^{\frac{\alpha}{2} + \frac{1}{2}}(1 + x)^{\frac{\beta}{2} + \frac{1}{2}} y, \quad z^2 = M_k^2(x).
\]
Thus, in the notation of Lemma 4
\[
A(x) = \frac{x}{2(1 - x^2)}, \quad B(x) = \frac{(2k + \alpha + \beta + 1)^2(1 - x^2) - 2(1 + x)\alpha^2 - 2(1 - x)\beta^2 + 1}{4(1 - x^2)^2}.
\]

Now we calculate
\[
D = 2(1 - x^2)^3(4AB - B') = (\alpha^2 - \beta^2)(x^2 + 1) + (2\alpha^2 + 2\beta^2 - 1)x.
\]

**Theorem 5.** For \( \alpha \geq \beta > \frac{1}{2} \), the consecutive maxima of the function \( M_k^{\alpha, \beta}(x) \) decrease for \( x < x_0 \) and increase for \( x > x_0 \), where
\[
x_0 = \frac{\sqrt{4\beta^2 - 1} - \sqrt{4\alpha^2 - 1}}{\sqrt{4\beta^2 - 1} + \sqrt{4\alpha^2 - 1}}.
\]

**Proof.** It is enough to show that the function \( S(z; x) \) is unimodal with the only minimum at \( x_0 \).

Since \( B_1 = 4(1 - x^2)B(x) \), the numerator of \( B \), is a quadratic with the negative leading coefficient, by lemma 4 it suffices to verify that \( x_0 \) is the only zero of \( D(x) \) in the region defined by \( B_1(x) > 0 \).

For, we calculate \( B_1(-1) = 1 - 4\beta^2 \leq 0 \), \( B_1(1) = 1 - 4\alpha^2 \leq 0 \), and
\[
B_1 \left( \frac{\beta - \alpha}{\alpha + \beta} \right) = \frac{(2\alpha + 1)(2\beta + 1)}{(\alpha + \beta + 1)^2} \left( (2k_k + 1) + (2\alpha + 2\beta + 1) + 1 \right) > 0.
\]
Since
\[
\frac{\beta - \alpha}{\beta + \alpha + 1} \in [-1, 1],
\]
\( B(x) \) has precisely two zeros on \([-1, 1]\).

It is easy to check that \( D \) has two real zeros for \( \alpha, \beta > \frac{1}{2}, \alpha \neq \beta \). Moreover, for \( \alpha \neq \beta \),
\[
D(-1) = 1 - 4\beta^2 < 0, \quad D(1) = 4\alpha^2 - 1 > 0,
\]
hence only the largest zero of $D$ lies between the zeros of $B_1$. If $\alpha = \beta$, then $D = 0$ implies $x = 0$, and

$$B_1(0) = (2k + 1)(2k + 2\alpha + 2\beta + 1) + 1 > 0,$$

leading to the same conclusion. This completes the proof. \qed

**Remark 1.** Let $-1 < x_1 < \ldots < x_k < 1$, be the zeros of $P^{(\alpha\beta)}_k(x)$. According to Theorem 3 the global extremum of $M^{(\alpha\beta)}_k(x)$ lies in one of the intervals $[\eta_{-1}, x_1]$, $[x_k, \eta_1]$, where $\eta_{\pm 1}$ are given by (4). Rather accurate bounds $\chi_{-1}$ and $\chi_1$ on $x_1$ and $x_k$, such that $x_1 < \chi_{-1} < \chi_1 < x_k$, and $|\eta_j - \chi_j| = O((k + \alpha + \beta)^{-2/3})$, $j = \pm 1$, were given in [10].

### 3. Proof of Theorem 3 even case

In this section we prove Theorem 3 for ultraspherical polynomials of even degree. Without loss of generality we will assume $x \geq 0$.

To simplify some expressions it will be convenient to introduce the parameter $r = 2k + 2\alpha + 1$.

The required differential equation for

$$g = (d^2 - x^2)^{1/4}(1 - x^2)^{\alpha/2}, \quad g^2 = M^\alpha_k(x, -d, d),$$

is

$$g'' - 2A(x)g' + B(x)g = 0,$$

where

$$A(x) = \frac{x(2d^2 - 1 - x^2)}{2(d^2 - x^2)(1 - x^2)},$$

$$B(x) = \frac{(1 - x^2)r^2 - 4\alpha^2}{4(1 - x^2)^2} + \frac{2d^2 - d^4 + (3 - 4d^2)x^2}{4(1 - x^2)(d^2 - x^2)^2}.$$ 

We also find

$$D(x) = \frac{2(d^2 - x^2)^3(1 - x^2)^2}{x} (4AB - B') =$$

$$(4\alpha^2 - (1 - d^2)r^2)(d^2 - x^2)^2 + (3 - 4d^2)x^4 - 2(5d^4 - 9d^2 + 3)x^2 - d^6 + 9d^4 - 9d^2.$$ 

In what follows we choose $d = \delta$, where $\delta$ is defined by (7). Notice that it can be also written as

$$\delta = \sqrt{\frac{r^2 - 4\alpha^2 - 3}{r^2 - 4}}.$$ 

The following lemma shows that $\delta$ is large enough to include all oscillations of $M^\alpha_k(x)$. This fact is crucial for our proof of Theorem 4.

**Lemma 6.** The interval $(-\delta, \delta)$ contains all local maxima of $M^\alpha_k(x)$, provided $\alpha > \frac{1}{2}$.

**Proof.** The assumption $\alpha > \frac{1}{2}$ implies that $\delta$ is real for $k \geq 0$. It is an immediate corollary of a general result given in [7] (eq. (17) for $\lambda = 0$), that in the ultraspherical case and $k, \alpha \geq 0$, all local maxima of $M^\alpha_k(x)$ lie between the zeros of the equation

$$A_0(x) = 4k(k + 2\alpha + 1) - ((2k + 2\alpha + 1)^2 + 4\alpha + 2)x^2 = 0.$$ 

Since, as easy to check, $A_0(\delta) > 0$, the local maxima are confined to the interval $(-\delta, \delta)$.
To apply Lemma 7 we shall check the relevant properties of \( B \) and \( D \), what will be accomplished in the following to lemmas.

**Lemma 7.** Let \( \alpha > \frac{1}{2}, \ k \geq 1 \), then for \( d = \delta \) the equation \( B(x) = 0 \) has the only real positive zero \( x_0 \), \( \delta < x_0 < 1 \). In particular, \( B(x) > 0 \) for \( 0 < x < \delta \).

**Proof.** It is easy to check that \( r^2 - 4\alpha^2 > 3, \ r^2 > 4, \) for \( \alpha > \frac{1}{2}, \ k \geq 1 \). The numerator \( B_1 \) of \( B(x) \) is

\[
B_1(x) = -r^2x^6 + ((1 + 2\delta^2)r^2 + 4\delta^2 - 4\alpha^2 - 3)x^4 - \\
(\delta^4 + 2\delta^2)r^2 - \delta^4 - 8\alpha^2\delta^2 + 6\delta^2 - 3)x^2 + (\delta^2r^2 - 4\alpha^2\delta^2 - \delta^2 + 2)\delta^2.
\]

Using Mathematica we find the discriminant of this polynomial in \( x \),

\[
\text{Dis}_x(B_1) = \frac{(r^2 - 4\alpha^2 - 3)((r^2 - 4\alpha^2 - 2)^2 + 2r^2 - 9)(24\alpha^2 - 6)^6r^8}{(r^2 - 4)^{14}} R^2(\alpha, r),
\]

where

\[
R(\alpha, r) = 100(r^2 - 4\alpha^2)^2\alpha^2r^2 + 7r^6 - (976\alpha^2 + 90)r^4 + (5456\alpha^4 + 3180\alpha^2 + 375)r^2 - 4(12\alpha^2 + 5)^3.
\]

Under our assumptions the expressions \( r^2 - 4\alpha - 3 \) and \( (r^2 - 4\alpha^2 - 2)^2 + 2r^2 - 9 \) are positive. Furthermore, rewriting \( R(\alpha, r) \) in terms of \( k \) and \( \alpha \) one can checks that the substitution \( \alpha \to \alpha + \frac{1}{2} \) gives a polynomial consisting of monomials of the same sign. Thus, for any \( k > 0 \) and \( \alpha > \frac{1}{2} \) the discriminant does not vanish and the equation \( B_1(x) = 0 \) has the same number of real zeros. For \( \alpha = k = 1 \) we obtain the following test equation with just two real zeros,

\[
804 - 2733x^2 + 3150x^4 - 1225x^6 = 0.
\]

It is left to demonstrate that the only positive zero \( x_0 \) of the equation \( B_1(x) = 0 \), is in the interval \((\delta, 1)\). For, we verify

\[
B_1(\delta) = 5(1 - \delta^2)^2\delta^4 > 0, \quad B_1(1) = -4\alpha^2(1 - \delta^2)^2 < 0.
\]

This completes the proof. \( \square \)

**Lemma 8.** Let \( \alpha > \frac{1}{2}, \ k \geq 1 \) and \( 0 < x < \delta \), then \( D(x) < 0 \).

**Proof.** We find

\[
\frac{(r^2 - 4)^3}{3(4\alpha^2 - 1)} D(x) = 2(r^2 - 4)(2r^2 - 12\alpha^2 - 5)x^2 - (r^2 - 4\alpha^2 - 3)(4r^4 - 4\alpha^2 - 15).
\]

Then

\[
D(0) < 0, \quad D(\delta) = -5(4\alpha^2 - 1)(r^2 - 4\alpha^2 - 3) < 0,
\]

and the result follows. \( \square \)

Applying two previous lemmas and Lemma 4 we obtain the following result.

**Lemma 9.** For \( x \geq 0 \) the local maxima of \( M_k^\alpha(x, \delta) \) form a decreasing sequence. In particular, \( M_k^\alpha(\delta) = M_k^\alpha(0, \delta) \).
Remark 2. The value of $\delta$ has been found as a solution of the equation $D\delta_x D = 0$. Surprisingly, it is split into linear and biquadratic factors. Besides trivial zeros $d = 0, 1$, this equation has four positive roots $d_1 < d_2 < d_3 < d_4$, where $d_1$ is of order $O\left(\frac{1}{\sqrt{k(k+\alpha)}}\right)$. The other three are very close, in fact

$$d_3 - d_2 = O\left(\frac{1}{k^{3/2}\sqrt{k+\alpha}}\right), \quad d_4 - d_3 = O\left(\frac{\alpha^2}{k^{3/2}(k+\alpha)^{5/2}}\right).$$

We have chosen the simplest one $\delta = d_3$.

To prove the inequality

$$M_k^\alpha(\delta) < \frac{2}{\pi} \left(1 + \frac{1}{8(k+\alpha)^2}\right),$$

we have to find $M_k^\alpha(0, \delta)$. The value of $P_k^{(\alpha,\alpha)}(0)$ for even $k$ is (see e.g. [1]),

$$P_k^{(\alpha,\alpha)}(0) = (-1)^{k/2} \frac{\Gamma(k+\alpha+1)}{2^k k^{3/2}! \Gamma\left(\frac{k+\alpha+1}{2}\right)}.$$

This yields

$$P_k^{(\alpha,\alpha)}(0) = (-1)^{k/2} \frac{\sqrt{r} k! \Gamma(r-k)}{2^{r/2} \left(\frac{k}{2}\right)! \Gamma\left(\frac{r-k+1}{2}\right)}.$$

To simplify this expression we use the following inequality (see e.g. [2]),

$$\frac{\Gamma(x+1)}{\Gamma^2\left(\frac{x}{2}+1\right)} < \frac{2x+\frac{1}{2}}{\pi(x+\frac{1}{2})}, \quad x \geq 0,$$

what yields for $k+2\alpha \geq 0$,

$$\left(\frac{P_k^{(\alpha,\alpha)}(0)}{\pi \sqrt{(2k+1)(r+2\alpha)}}\right)^2 < \frac{2r}{\pi \sqrt{(2k+1)(r+2\alpha)}}.$$

Hence, for $|x| \leq \delta$, we have

$$M_k^\alpha(\delta) = M_k^\alpha(0, \delta) = \delta \left(P_k^{(\alpha,\alpha)}(0)\right)^2 < \frac{1}{\pi \sqrt{(2k+1)(r+2\alpha)}}.$$

It is an easy exercise to check that for $k \geq 2$, $\alpha \geq \frac{1}{2}$, the last expression does not exceed

$$\frac{2}{\pi} \left(1 + \frac{1}{8(k+\alpha)^2}\right).$$

This proves the even case of Theorem [3].

Remark 3. In [5] the following pointwise bound on $M_k^{\alpha,\beta}(x)$ is given.

$$M_k^{\alpha,\beta}(x) < \frac{2e}{\pi} \frac{(2k+2\alpha+2\beta+1)(2k+2\alpha+2\beta+2)}{(2k+2\alpha+2\beta+2)^2 - \frac{2\alpha^2}{1-x} - \frac{2\beta^2}{1+x}}.$$

For the ultraspherical case this yields

$$M_k^\alpha(0) < \frac{2e}{\pi} \left(1 + O\left(\frac{\alpha^2}{k(k+\alpha)}\right)\right).$$

Thus, (17) is quite precise, provided $\alpha = O(k)$.
4. Proof of Theorem 3, odd case

In this section we will establish the odd case of Theorem 3 by reducing it to the previous one. We also give slightly more accurate bounds under the assumptions $k \geq 7, \quad \alpha \geq \frac{1 + \sqrt{7}}{4}$. They will be used in the proof of Theorem 4 in the next section.

As $\delta$ is a function of $k$ and $\alpha$, to avoid ambiguities or a messy notation arising when they vary, throughout this section we will use $\delta(k, \alpha)$ instead of $\delta$ and set $F^\alpha = M^\alpha(k, \alpha)$, and $F^\alpha(k, x) = M^\alpha_k(x, \delta)$.

Since the value of the first, nearest to zero, maximum of $F^\alpha_k(x)$, which we assume is attained at $x = \xi$, is unknown for odd $k$, we need some technical preparations. First of all we have to find an upper bound on $\xi$.

Let $0 = \xi < x_1 < ... < x_i$, be the nonnegative zeros of $P^{(\alpha, \alpha)}_k(x)$. Obviously, $0 < \xi < x_1$, so we can use an upper bound on $x_1$ instead. An appropriate estimate for zeros of ultraspherical polynomials has been given in [4], in particular

$$x_1 < \left(\frac{2k^2 + 1}{4k + 2 + \alpha}\right)^{-1/2} h_k,$$

where $h_k$ is the least positive zero of the Hermite polynomial $H_k(x)$.

Since $h_k \leq \sqrt{\frac{2k}{4k+2}}$, [14, sec. 6.3], we obtain

$$(18) \quad \xi \leq \sqrt{\frac{21}{2k^2 + 4\alpha k + 2\alpha + 1}} := \xi_0.$$  

Using the formula

$$\frac{d}{dx} P^{(\alpha, \beta)}_k(x) = \frac{k + \alpha + \beta + 1}{2} P^{(\alpha+1, \beta+1)}_{k-1}(x),$$

which for the ultraspherical orthonormal case yields

$$\frac{d}{dx} P^{(\alpha, \alpha)}_k(x) = \sqrt{(r - k)k} P^{(\alpha+1, \alpha+1)}_{k-1}(x)$$

and the simplest Taylor expansion around zero,

$$P^{(\alpha, \alpha)}_k(\xi) = \sqrt{(r - k)k} P^{(\alpha+1, \alpha+1)}_{k-1}(\xi) \xi, \quad 0 < \xi < 1,$$

what reduces the problem to the even case, we obtain

$$F^\alpha_k(\xi) < \sqrt{\delta^2(k, \alpha) - \xi^2} (1 - \xi^2) \alpha \left(P^{(\alpha+1, \alpha+1)}_{k-1}(\epsilon \xi)\right)^2 (r - k)k \xi^2 <$$

$$\frac{\sqrt{\delta^2(k, \alpha) - \xi^2} (1 - \xi^2) \alpha}{\sqrt{\delta^2(k - 1, \alpha + 1) - \epsilon^2 \xi^2} (1 - \epsilon^2 \xi^2) \alpha} F^{\alpha+1}_{k-1}(\epsilon \xi)(r - k)k \xi_0^2 <$$

$$\frac{\sqrt{\delta^2(k, \alpha) - \xi^2}}{(1 - \xi^2) \sqrt{\delta^2(k - 1, \alpha + 1) - \xi^2} \xi^2} F^{\alpha+1}_{k-1} (r - k)k \xi_0^2.$$  

The last function increases in $\xi$ and substituting $\xi_0$ we have

$$(19) \quad F^\alpha_k(\xi) < v(k, \alpha) F^{\alpha+1}_{k-1},$$

where

$$v(k, \alpha) = \frac{(r - k)k \xi_0^2 \sqrt{\delta^2(k, \alpha) - \xi_0^2}}{(1 - \xi_0^2) \sqrt{\delta^2(k - 1, \alpha + 1) - \xi_0^2}}.$$
We have checked using Mathematica that

\[ v_1(k, \alpha) = \left(1 + \frac{1}{8(k + \alpha)^2}\right) v(k, \alpha) \]

is a decreasing function in \( k \) and \( \alpha \), provided \( k \geq 3 \) and \( \alpha \geq \frac{1}{2} \) (an explicit expression for \( v \) is somewhat messy and is omitted). In fact, this is much easier than one may expect as the numerator and the denominator of \( \frac{d}{d\alpha} v_1^2(k + 3, \alpha + \frac{1}{2}) \) and \( \frac{d}{dk} v_1^2(k + 3, \alpha + \frac{1}{2}) \) consist of the monomials of the same sign.

Calculations yield

\[
v_1(3, \frac{1}{2}) < 115, \quad v_1(7, \frac{1 + \sqrt{2}}{4}) < \frac{29}{2}.
\]

Finally, applying (14) and (19) and coming back to the usual notation, we conclude

Lemma 10. Let \( k \) be odd, then

\[
\mathcal{M}_k^\alpha(\delta) \leq \begin{cases} 230 \pi^2 & k \geq 3, \quad \alpha > \frac{1}{2}, \\ 29 \pi^2 & k \geq 7, \quad \alpha > 1 + \sqrt{2} \end{cases}
\]

This completes the proof of Theorem 3.

5. Proof of Theorem 1

First, we will establish the following bounds which are slightly better than these of Theorem 1 but stated in terms of \( r = 2k + 2\alpha + 1 \), and \( \tau = \frac{2\alpha + 1}{r} \). It is worth noticing that in some respects \( r \) and \( \tau \) are more natural parameters than \( k \) and \( \alpha \) (see [7]).

Lemma 11.

\[
\mathcal{M}_k^\alpha < \begin{cases} 12 \frac{1}{15} r^{1/3} \tan^{1/3} \tau & k \geq 6, \quad \text{even}, \\ 14 \frac{1}{15} r^{1/3} \tan^{1/3} \tau & k \geq 7, \quad \text{odd}. \end{cases}
\]

provided \( k \geq 6, \quad \alpha \geq \frac{1 + \sqrt{2}}{2} \).

Proof. Let \( \epsilon = \frac{2^{1/3}}{3} r^{-2/3} \tan^{4/3} \tau \). It is easy to check that \( \epsilon < \frac{1}{15} \), (the extremal case corresponds to \( k = 6, \alpha = \infty \)).

Since

\[ \delta > \cos \tau > \eta = (1 - \epsilon) \cos \tau, \]

where \( \eta \) is defined in (9), it follows by Theorem 2 that all local maxima of \( \mathcal{M}_k^\alpha(x) \) are inside the interval \((-\delta, \delta)\). Now we have

\[
\max_{|x| \leq 1} \left(1 - x^2\right)^{\alpha + \frac{1}{2}} \left(P_k^{(\alpha, \alpha)}(x)\right)^2 = M_k^\alpha(\delta) \max_{0 \leq x \leq \eta} \sqrt{\frac{1 - x^2}{\delta^2 - x^2}} = \]

\[
M_k^\alpha(\delta) \sqrt{\frac{1 - \eta^2}{\delta^2 - \eta^2}}.
\]

By the explicit expression for \( \epsilon \) given by (9), one can check that the function \( \sqrt{2 - \epsilon} \) increases in \( k \) and decreases in \( \alpha \). We obtain by \( \epsilon < \frac{1}{15} \),

\[ \sqrt{\delta^2 - \eta^2} > \sqrt{\cos^2 \tau - \eta^2} = \sqrt{\epsilon(2 - \epsilon) \cos \tau} > \frac{7}{5} \sqrt{\epsilon} \cos \tau. \]
Using the restrictions $k \geq 6$, $\alpha \geq 1 + \frac{\sqrt{2}}{4}$, and a simple trigonometric inequality, we find

$$\sqrt{1 - \eta^2} = \sqrt{1 - (1 - \epsilon)^2 \cos^2 \tau} \leq \sin \tau (1 + \epsilon \cot^2 \tau) = \left(1 + \frac{1}{3} \left(\frac{2k(k+2\alpha+1)}{(2\alpha+1)^2(2k+2\alpha+1)^2}\right)^{1/3}\right)^{1/3} \sin \tau < \frac{37}{32} \sin \tau.$$

Thus, we obtain

$$\sqrt{1 - \eta^2} < \frac{185 \tan \tau}{224 \sqrt{\epsilon}} = \frac{185 \sqrt{\frac{3}{224}}}{224} r^{1/3} \tan^{1/3} \tau < \frac{13}{9} r^{1/3} \tan^{1/3} \tau,$$

and the result follows by (22) and (14) for $k$ even, and (20) for $k$ odd. \qed

Now Theorem 11 is an immediate corollary of (21) and

$$r^{1/3} \tan^{1/3} \tau = \left((2\alpha+1)^2(2k+2\alpha+1)^2\right)^{1/6} \leq \left(4\sqrt{2} - 2\right)^{1/3},$$

for $\alpha \geq 1 + \frac{\sqrt{2}}{4}$. This completes the proof.

**Acknowledgement.** I am grateful to D.K. Dimitrov for a helpful discussion and, especially, for bringing my attention to Sonin’s function.

**REFERENCES**

[1] M. Abramowitz, I.A. Stegun, *Handbook of Mathematical functions*, Dover, New Yorl, 1964.
[2] J. Bustoz, M.E.H. Ismail, *On gamma function inequality*, Math. Comp. Vol. 47 (1986), 659-667.
[3] Y. Chow; L. Gatteschi, R. Wong, *A Bernstein-Type Inequality for the Jacobi Polynomial*, Proc. Amer. Math. Soc. Vol. 121, No. 3. (1994), pp. 703-709.
[4] Á. Elbert, P.D. Siafarikas, *Monotonicity Properties of the Zeros of Ultraspherical Polynomials*, J. Approx. Theory, Vol. 96, (1999), 31-39.
[5] T. Erdélyi, A.P. Magnus, P. Nevai, *Generalized Jacobi weights, Christoffel functions, and Jacobi polynomials*, SIAM J. Math. Anal. 25 (1994), 602-614.
[6] T. Kasuga, R. Sakai, *Orthonormal polynomials with generalized Freud-type weights*, J. Approx. Theory 121 (2003), 13-53.
[7] I. Krasikov, *On the maximum of Jacobi Polynomials*, J. Approx. Theory, Vol. 136 (2005), 1-20.
[8] I. Krasikov, *Sharp inequalities for Hermite polynomials*, proceeding CTF-2005, to appear.
[9] I. Krasikov, *Inequalities for orthonormal Laguerre Polynomials*, J. Approx. Theory, to appear.
[10] I. Krasikov, *On extreme zeros of classical orthogonal polynomials*, J. Comp. Appl. Math., Vol.193, (2006), 168-182.
[11] E. Levin, D.S. Lubinsky, *Orthogonal polynomials for exponential weights*, CMS Books in Math., Vol.4, Springer-Verlag, New-York, 2001.
[12] A.L. Levin and D.S. Lubinsky, *Orthogonal polynomials for exponential weights $x^2 e^{-Q(x)}$ on $[0, a]$, J. Approx. Theory, 134 (2005) 199-256.
[13] L. Lorch, *Inequalities for ultraspherical polynomials and the gamma function*, J. Approx. Theory; 40, (1984), 115-120.
[14] G. Siegö, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ., v.23, Providence, Rl, 1975.

**DEPARTMENT OF MATHEMATICAL SCIENCES, BRUNEL UNIVERSITY, UXBRIDGE UB8 3PH UNITED KINGDOM**

**E-mail address:** mastiik@brunel.ac.uk