A NOTE ON FINITE ABELIAN GERBES
OVER TORIC DELIGNE-MUMFORD STACKS

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Abstract. Any toric Deligne-Mumford stack is a \( \mu \)-gerbe over the underlying toric orbifold for a finite abelian group \( \mu \). In this paper we give a sufficient condition so that certain kinds of gerbes over a toric Deligne-Mumford stack are again toric Deligne-Mumford stacks.

1. Introduction

Let \( \Sigma := (N, \Sigma, \beta) \) be a stacky fan of \( \text{rank}(N) = d \) as defined in [4]. If there are \( n \) one-dimensional cones in the fan \( \Sigma \), then modelling the construction of toric varieties [5], [6], the toric Deligne-Mumford stack \( \mathcal{X}(\Sigma) = [Z/G] \) is a quotient stack, where \( Z = C^n - V \), the close subvariety \( V \subset C^n \) is determined by the ideal \( J_\Sigma \) generated by \( \prod_{\rho, \sigma} z_i : \sigma \in \Sigma \) and \( G \) acts on \( Z \) through the map \( \alpha : G \longrightarrow (C^x)^n \) in the following exact sequence determined by the stacky fan (see [4]):

\[
1 \longrightarrow \mu \longrightarrow G \xrightarrow{\alpha} (C^x)^n \longrightarrow T \longrightarrow 1.
\]

Let \( \overline{G} = \text{Im}(\alpha) \). Then \( [Z/G] \) is the underlying toric orbifold \( \mathcal{X}(\Sigma_{\text{red}}) \). The toric Deligne-Mumford stack \( \mathcal{X}(\Sigma) \) is a \( \mu \)-gerbe over \( \mathcal{X}(\Sigma_{\text{red}}) \).

Let \( \mathcal{X}(\Sigma) \) be a toric Deligne-Mumford stack associated with the stacky fan \( \Sigma \). Let \( \nu \) be a finite abelian group, and let \( \mathcal{G} \) be a \( \nu \)-gerbe over \( \mathcal{X}(\Sigma) \). We give a sufficient condition so that \( \mathcal{G} \) is also a toric Deligne-Mumford stack. We have the following theorem:

**Theorem 1.1.** Let \( \mathcal{X}(\Sigma) \) be a toric Deligne-Mumford stack with stacky fan \( \Sigma \). Then every \( \nu \)-gerbe \( \mathcal{G} \) over \( \mathcal{X}(\Sigma) \) is induced by a central extension

\[
1 \longrightarrow \nu \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1;
\]

i.e., we have a Cartesian diagram:

\[
\begin{array}{ccc}
\mathcal{G} & \longrightarrow & B\tilde{G} \\
\downarrow & & \downarrow \\
\mathcal{X}(\Sigma) & \longrightarrow & BG.
\end{array}
\]
In general, the $\nu$-gerbe $G$ is not a toric Deligne-Mumford stack. But if the central extension is abelian, then we have:

**Corollary 1.2.** If the $\nu$-gerbe $G$ is induced from an abelian central extension, then it is a toric Deligne-Mumford stack.

This small note is organized as follows. In Section 2 we construct the new toric Deligne-Mumford stack from an abelian central extension and prove the main results. In Section 3 we give an example of a $\nu$-gerbe over a toric Deligne-Mumford stack.

In this paper, by an orbifold we mean a smooth Deligne-Mumford stack with trivial stabilizers at the generic points.

2. The proof of the main results

We refer the reader to [4] for the construction and notation of toric Deligne-Mumford stacks. For the general theory of stacks, see [2].

Let $\Sigma := (N, \Sigma, \beta)$ be a stacky fan. From Proposition 2.2 in [4], we have the following exact sequences:

$$0 \to DG(\beta)^* \to \mathbb{Z}^n \to N \to \text{Coker}(\beta) \to 0,$$

$$0 \to N^* \to \mathbb{Z}^n \to DG(\beta) \to \text{Coker}(\beta^\vee) \to 0,$$

where $\beta^\vee$ is the Gale dual of $\beta$. As a $\mathbb{Z}$-module, $\mathbb{C}^\times$ is divisible, so it is an injective $\mathbb{Z}$-module and hence the functor $\text{Hom}_{\mathbb{Z}}(-, \mathbb{C}^\times)$ is exact. We get the exact sequence:

$$1 \to \text{Hom}_{\mathbb{Z}}(\text{Coker}(\beta^\vee), \mathbb{C}^\times) \to \text{Hom}_{\mathbb{Z}}(DG(\beta), \mathbb{C}^\times) \to \text{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, \mathbb{C}^\times) \to \text{Hom}_{\mathbb{Z}}(N^*, \mathbb{C}^\times) \to 1.$$

Letting $\mu := \text{Hom}_{\mathbb{Z}}(\text{Coker}(\beta^\vee), \mathbb{C}^\times)$, we have the exact sequence $1.1$. Let $\Sigma(1) = n$ be the set of one-dimensional cones in $\Sigma$ and $V \subset \mathbb{C}^n$ the closed subvariety defined by the ideal generated by

$$J_\Sigma = \left< \prod_{\rho \notin \sigma} z_i : \sigma \in \Sigma \right>.$$

Let $Z := \mathbb{C}^n \setminus V$. From [3], the complex codimension of $V$ in $\mathbb{C}^n$ is at least 2. The toric Deligne-Mumford stack $X(\Sigma) = [Z/G]$ is the quotient stack where the action of $G$ is through the map $\alpha$ in $1.1$.

**Lemma 2.1.** If $\text{Codim} \subset \mathbb{C}^n \geq 2$, then $H^1(Z, \nu) = H^2(Z, \nu) = 0$, where $\nu$ is a finite abelian group.

**Proof.** Consider the following exact sequence:

$$0 \to H^0(\mathbb{C}^n, \nu) \to H^0(\mathbb{C}^n, \nu) \to H^0(Z, \nu) \to$$

$$\to H^1(\mathbb{C}^n, \nu) \to H^1(\mathbb{C}^n, \nu) \to H^1(Z, \nu) \to$$

$$\to H^2(\mathbb{C}^n, \nu) \to \cdots.$$

Since $\text{Codim} \subset \mathbb{C}^n \geq 2$, so the real codimension is at least 4 and $H^i(\mathbb{C}^n, \nu) = 0$ for $i = 1, 2, 3$, so from the exact sequence and $H^i(\mathbb{C}^n, \nu) = 0$ for all $i > 0$ we prove the lemma. □
2.1. **The Proof of Theorem 1.1.** Consider the following diagram:

\[
\begin{array}{ccc}
Z & \longrightarrow & pt \\
\downarrow & & \downarrow \\
[Z/G] & \longrightarrow & BG
\end{array}
\]

which is Cartesian. Consider the Leray spectral sequence for the fibration \(\pi\):

\[
H^p(BG, R^q\pi_*\nu) \Rightarrow H^{p+q}([Z/G], \nu).
\]

We compute

\[
H^2([Z/G], \nu) = \bigoplus_{p+q=2} H^p(BG, R^q\pi_*\nu).
\]

First we have that \(R^q\pi_*\nu = [H^q(Z, \nu)/G]\). There are three cases.

1. When \(p = 2, q = 0\), \(R^0\pi_*\nu = \nu\) because \(Z\) is connected, so

\[
H^p(BG, R^q\pi_*\nu) = H^2(BG, \nu).
\]

2. When \(p = 1, q = 1\), \(R^1\pi_*\nu = [H^1(Z, \nu)/G]\), so

\[
H^p(BG, R^q\pi_*\nu) = H^1(BG, H^1(Z, \nu)),
\]

and by Lemma 2.1, \(H^1(Z, \nu) = 0\), so we have \(H^p(BG, R^q\pi_*\nu) = 0\).

3. When \(p = 0, q = 2\), \(R^2\pi_*\nu = [H^2(Z, \nu)/G]\), so

\[
H^p(BG, R^q\pi_*\nu) = H^0(BG, H^2(Z, \nu));
\]

also from Lemma 2.1, \(H^2(Z, \nu) = 0\), and so we have \(H^p(BG, R^q\pi_*\nu) = 0\).

So we get that

\[
H^2([Z/G], \nu) \cong H^2(BG, \nu).
\]

Since for the finite abelian group \(\nu\), the \(\nu\)-gerbes are classified by the second cohomology group with coefficient in the group \(\nu\), and Theorem 1.1 is proved.

2.2. **The Proof of Corollary 1.2.** Let \(X(\Sigma) = [Z/G]\). The \(\nu\)-gerbe \(G\) over \([Z/G]\) is induced from a \(\nu\)-gerbe \(\widetilde{G}\) over \(BG\) in the following central extension:

\[
1 \longrightarrow \nu \longrightarrow \widetilde{G} \xrightarrow{\varphi} G \longrightarrow 1,
\]

where \(\widetilde{G}\) is an abelian group. So the pullback gerbe over \(Z\) under the map \(Z \longrightarrow [Z/G]\) is trivial. So we have

\[
G = BG \times_{BG} [Z/G] = [Z/\widetilde{G}].
\]

The stack \([Z/\widetilde{G}]\) is this \(\nu\)-gerbe \(G\) over \([Z/G]\). Consider the commutative diagram:

\[
\begin{array}{ccc}
\widetilde{G} & \xrightarrow{\varphi} & G \\
\alpha \downarrow & & \downarrow \\
(C^\times)^n & \xrightarrow{\cong} & (C^\times)^n
\end{array}
\]

where \(\alpha\) is the map in (1.1). So we have the following exact sequences:

\[
1 \longrightarrow \nu \longrightarrow ker(\alpha) \longrightarrow \mu \longrightarrow 1
\]

and

\[
1 \longrightarrow ker(\alpha) \longrightarrow \widetilde{G} \xrightarrow{\hat{\alpha}} (C^\times)^n \longrightarrow T \longrightarrow 1,
\]

\[
H^2([Z/G], \nu) \cong H^2(BG, \nu).
\]
where $T$ is the torus of the simplicial toric variety $X(\Sigma)$. Since the abelian groups $\tilde{G}$, $G$ and $(\mathbb{C}^\times)^n$ are all locally compact topological groups, taking Pontryagin duality and the Gale dual, we have the following diagrams:

$$
\begin{array}{cccccccc}
0 & \longrightarrow & N^* & \longrightarrow & Z^n & \xrightarrow{\beta^\vee} & DG(\beta) & \longrightarrow & Coker(\beta^\vee) & \longrightarrow & 0 \\
\downarrow & & id & & \downarrow & & p_\varphi & & \downarrow & & \\
0 & \longrightarrow & \tilde{N}^* & \longrightarrow & Z^n & \xrightarrow{\tilde{\beta}^\vee} & DG(\tilde{\beta}) & \longrightarrow & Coker((\tilde{\beta})^\vee) & \longrightarrow & 0, \\
0 & \longrightarrow & DG(\tilde{\beta})^* & \longrightarrow & Z^n & \xrightarrow{\tilde{\beta}} & \tilde{N} & \longrightarrow & Coker(\tilde{\beta}) & \longrightarrow & 0 \\
\downarrow & & id & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & DG(\beta)^* & \longrightarrow & Z^n & \xrightarrow{\beta} & N & \longrightarrow & Coker(\beta) & \longrightarrow & 0,
\end{array}
$$

where $p_\varphi$ is induced by $\varphi$ in $\Sigma$ under the Pontryagin duality. Suppose $\tilde{\beta} : Z^n \longrightarrow \tilde{N}$ is given by $\{\tilde{b}_1, \cdots, \tilde{b}_n\}$, then $\Sigma := (\tilde{N}, \Sigma, \tilde{\beta})$ is a new stacky fan. The toric Deligne-Mumford stack $\mathcal{X}(\Sigma) = [Z/\tilde{G}]$ is the $\nu$-gerbe $G$ over $\mathcal{X}(\Sigma)$.

**Remark 2.2.** From Proposition 4.6 in [3], any Deligne-Mumford stack is a $\nu$-gerbe over an orbifold for a finite group $\nu$. Our results are the toric case of that general result.

In particular, given a stacky fan $\Sigma = (N, \Sigma, \beta)$, let $\Sigma_{\text{red}} = (\overline{N}, \Sigma, \overline{\beta})$ be the reduced stacky fan, where $\overline{N}$ is the abelian group $N$ modulo torsion, and $\overline{\beta} : Z^n \longrightarrow \overline{N}$ is given by $\{\overline{b}_1, \cdots, \overline{b}_n\}$, which are the images of $\{b_1, \cdots, b_n\}$ under the natural projection $N \longrightarrow \overline{N}$. Then the toric orbifold $\mathcal{X}(\Sigma_{\text{red}}) = [Z/\overline{G}]$. From [1,1], let $\overline{G} = \text{Im}(\alpha)$. Then we have the following exact sequences:

$$
\begin{align*}
1 & \longrightarrow \overline{G} \longrightarrow (\mathbb{C}^\times)^n \longrightarrow T \longrightarrow 1, \\
1 & \longrightarrow \mu \longrightarrow G \longrightarrow \overline{G} \longrightarrow 1.
\end{align*}
$$

So $G$ is an abelian central extension of $\overline{G}$ by $\mu$. $\mathcal{X}(\Sigma)$ is a $\mu$-gerbe over the toric orbifold $\mathcal{X}(\Sigma_{\text{red}})$. Any $\mu$-gerbe over the toric orbifold coming from an abelian central extension is a toric Deligne-Mumford stack. This is a special case of the main results and is the toric case of rigidification construction in [1].

**Remark 2.3.** From the proof of Corollary 1.2 we see that if a $\nu$-gerbe over $\mathcal{X}(\Sigma)$ comes from a gerbe over $BG$ and the central extension is abelian, then we can construct a new toric Deligne-Mumford stack.

3. **An example**

**Example 3.1.** Let $\Sigma$ be the complete fan of the projective line, $N = \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$, and $\beta : Z^2 \longrightarrow Z \oplus \mathbb{Z}/3\mathbb{Z}$ be given by the vectors $\{b_1 = (1, 0), b_2 = (-1, 1)\}$. Then $\Sigma = (N, \Sigma, \beta)$ is a stacky fan. We compute that $(\beta)^\vee : Z^2 \longrightarrow DG(\beta) = \mathbb{Z}$ is given by the matrix $[3,3]$. So we get the following exact sequence:

$$
(3.1) \quad 1 \longrightarrow \mu_3 \longrightarrow \mathbb{C}^\times \xrightarrow{[3,3]^\vee} (\mathbb{C}^\times)^2 \longrightarrow \mathbb{C}^\times \longrightarrow 1.
$$

The toric Deligne-Mumford stack is $\mathcal{X}(\Sigma) = [(\mathbb{C}^2 - \{0\})/\mathbb{C}^\times]$, where the action is given by $\lambda(x, y) = (\lambda^3 x, \lambda^3 y)$. So $\mathcal{X}(\Sigma)$ is the nontrivial $\mu_3$-gerbe over $\mathbb{P}^1$ coming
from the canonical line bundle over \( \mathbb{P}^1 \). Let \( G \to \mathcal{X}(\Sigma) \) be a \( \mu_2 \)-gerbe such that it comes from the \( \mu_2 \)-gerbe over \( BC^\times \) given by the central extension

\[
1 \to \mu_2 \to C^\times \xrightarrow{(j)^2} C^\times \to 1.
\]

From the sequences (3.1) and (3.2), we have

\[
1 \to \mu_3 \otimes \mu_2 \to C^\times \xrightarrow{[6,0]^t} (C^\times)^2 \to C^\times \to 1.
\]

The Pontryagin dual of \( C^\times \xrightarrow{[6,0]^t} (C^\times)^2 \) is \((\beta)^\vee: \mathbb{Z}^2 \to \mathbb{Z}\), which is given by the matrix \([6,0]\). Taking the Gale dual we have

\[
\tilde{\beta}: \mathbb{Z}^2 \to \mathbb{Z} \oplus \mathbb{Z}_6,
\]

which is given by the vectors \( \{\tilde{b}_1 = (1,0), \tilde{b}_2 = (-1,1)\} \). Let \( \Sigma = (\tilde{N},\Sigma,\tilde{\beta}) \) be the new stacky fan. Then we have the toric Deligne-Mumford stack \( \mathcal{X}(\tilde{\Sigma}) = [C^2 - \{0\}/C^\times] \), where the action is given by \( \lambda(x,y) = (\lambda^6x, \lambda^6y) \). So \( \mathcal{X}(\Sigma) \) is the canonical \( \mu_6 \)-gerbe over \( \mathbb{P}^1 \).

If the \( \mu_2 \)-gerbe over \( BC^\times \) is given by the central extension

\[
1 \to \mu_2 \to C^\times \times \mu_2 \xrightarrow{\alpha} C^\times \to 1,
\]

where \( \alpha \) is given by the matrix \([1,0]\), then from (3.1) and (3.3), we have

\[
1 \to \mu_3 \otimes \mu_2 \to C^\times \times \mu_2 \xrightarrow{\varphi} (C^\times)^2 \to C^\times \to 1,
\]

where \( \varphi \) is given by the matrix \( \begin{bmatrix} 3 & 0 \\ 3 & 0 \end{bmatrix} \). The Pontryagin dual of \( \varphi \) is: \((\beta')^\vee: \mathbb{Z}^2 \to \mathbb{Z} \oplus \mathbb{Z}_2\), which is given by the transpose of the above matrix. Taking the Gale dual we get

\[
\tilde{\beta}: \mathbb{Z}^2 \to \mathbb{Z} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_2,
\]

which is given by the vectors \( \{\tilde{b}_1 = (1,0,0), \tilde{b}_2 = (-1,1,0)\} \). So \( \Sigma' = (\tilde{N}',\Sigma,\tilde{\beta}') \) is a stacky fan. The toric Deligne-Mumford stack is \( \mathcal{X}(\Sigma') = [C^2 - \{0\}/C^\times \times \mu_2] \), where the action is \( (\lambda_1, \lambda_2): (x,y) = (\lambda_1^3x, \lambda_2^3y) \). So \( \mathcal{G}' = \mathcal{X}(\Sigma') \) is the trivial \( \mu_2 \)-gerbe over \( \mathcal{X}(\Sigma) \) and \( \mathcal{X}(\Sigma) \cong \mathcal{X}(\Sigma') \).

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