TROPICAL AND ORDINARY CONVEXITY COMBINED

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Abstract. A polytrope is a tropical polytope which at the same time is convex in the ordinary sense. A \(d\)-dimensional polytrope turns out to be a tropical simplex, that is, it is the tropical convex hull of \(d + 1\) points. This statement is equivalent to the known fact that the Segre product of two full polynomial rings (over some field \(K\)) has the Gorenstein property if and only if the factors are generated by the same number of indeterminates. The combinatorial types of polytropes up to dimension three are classified.

1. Introduction

In [8] Develin and Sturmfels defined tropical polytopes, and they showed that tropical polytopes, or rather configurations of \(n\) tropical points in the tropical affine space \(\mathbb{T}A^d\), are equivalent to regular subdivisions of the product of simplices \(\Delta_{n-1} \times \Delta_d\). It is important that there is a natural way to identify \(\mathbb{T}A^d\) with \(\mathbb{R}^d\); this way it is possible to carry geometric concepts from \(\mathbb{R}^d\) to \(\mathbb{T}A^d\). A key result [8], Theorem 15, says that each tropical polytope comes naturally decomposed into ordinary polytopes which are also convex in the tropical sense. These objects are the topic of this paper, and we call them polytropes.

Each polytrope \(P\) is a tropical simplex, that is, it is the tropical convex hull of \(d + 1\) points, where \(d\) is the dimension of \(P\). It turns out that this statement is equivalent to the known fact from Commutative Algebra that the Segre product of two full polynomial rings (over some field \(K\)) has the Gorenstein property if and only if the factors are generated by the same number of indeterminates.

Polytropes are not new. Postnikov and Stanley studied deformations of the Coxeter hyperplane arrangement of type \(A_d\), that is, arrangements of affine hyperplanes in \(\mathbb{R}^d\) with normal vectors \(e_i - e_j\) for \(i \neq j\) [20]; here \(e_1, e_2, \ldots, e_d\) are the standard basis vectors of \(\mathbb{R}^d\). Their bounded cells are precisely the polytropes. In a paper by Lam and Postnikov [17] the same objects are called the alcoved polytopes of type \(A\). More recently, polytropes occurred as the bounded intersections of apartments in Bruhat–Tits buildings of type \(\tilde{A}_d\), see Keel and Tevelev [16] or Joswig, Sturmfels, and Yu [15], as the inversion domains of Alessandrini [1], and as the max-plus definite closures of Sergeev [24]. An additional motivation to study polytropes comes from the fact that each tropical polytope \(P\) has a canonical decomposition into polytropes.

The paper is structured as follows. We begin with a short section gathering the relevant facts about tropical polytopes. Then we prove our main result, and this section also contains more information about the interplay between the tropical and the ordinary convexity of a polytrope. The subsequent section lists specific examples, among which are the associahedra and order polytopes. One application of our main result is that it allows for a fairly efficient (compared with other more obvious approaches) enumeration of all combinatorial types of polytropes.

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2. Tropical convexity

This section is meant to collect basic facts about tropical convexity and to fix the notation.

Defining tropical addition $x \oplus y := \min(x, y)$ and tropical multiplication $x \odot y := x + y$ yields the tropical semi-ring $(\mathbb{R}, \oplus, \odot)$. Component-wise tropical addition and tropical scalar multiplication

$$\lambda \odot (\xi_0, \ldots, \xi_d) := (\lambda \odot \xi_1, \ldots, \lambda \odot \xi_d) = (\lambda + \xi_0, \ldots, \lambda + \xi_d)$$

equips $\mathbb{R}^{d+1}$ with a semi-module structure. For $x, y \in \mathbb{R}^{d+1}$ we let

$$[x, y]_\text{trop} := \{(\lambda \odot x) \oplus (\mu \odot y) \mid \lambda, \mu \in \mathbb{R}\}$$

be the tropical line segment between $x$ and $y$. A subset of $\mathbb{R}^{d+1}$ is tropically convex if it contains the tropical line segment between any two of its points. A direct computation shows that if $S \subset \mathbb{R}^{d+1}$ is tropically convex then $S$ is closed under tropical scalar multiplication. This leads to the definition of the tropical affine space as the quotient semi-module

$$\mathbb{T}A^d := \mathbb{R}^{d+1}/(\mathbb{R} \odot (0, \ldots, 0)) .$$

Note that $\mathbb{T}A^d$ was called “tropical projective space” in [8], [14], [9], and [15]. Tropical convexity gives rise to the hull operator $\text{tconv}$. A tropical polytope is the tropical convex hull of finitely many points in $\mathbb{T}A^d$.

Like an ordinary polytope each tropical polytope $P$ has a unique set of generators which is minimal with respect to inclusion; these are the tropical vertices of $P$.

There are several natural ways to choose a representative coordinate vector for a point in $\mathbb{T}A^d$. For instance, in the coset $x + (\mathbb{R} \odot (0, \ldots, 0))$ there is a unique vector $c(x) \in \mathbb{R}^{d+1}$ with non-negative coordinates such that at least one of them is zero; we refer to $c(x)$ as the canonical coordinates of $x \in \mathbb{T}A^d$. Moreover, in the same coset there is also a unique vector $(\xi_0, \ldots, \xi_d)$ such that $\xi_0 = 0$. Hence the map

$$c_0 : \mathbb{T}A^d \to \mathbb{R}^d, (\xi_0, \ldots, \xi_d) \mapsto (\xi_1 - \xi_0, \ldots, \xi_d - \xi_0)$$

is a bijection. Often we will identify $\mathbb{T}A^d$ with $\mathbb{R}^d$ via this map. This is also sound from the topological point of view: The maximum norm on $\mathbb{R}^{d+1}$ induces a metric on $\mathbb{T}A^d$ and, in this way, a natural topology; the map $c_0$ is a homeomorphism.

The tropical determinant $\text{tdet} M$ of a matrix $M = (\mu_{ij}) \in \mathbb{R}^{(d+1) \times (d+1)}$ is given as

$$\text{tdet} M := \bigoplus_{\sigma \in \text{Sym}_{d+1}} \mu_{0, \sigma(0)} + \cdots + \mu_{d, \sigma(d)} ,$$

where $\text{Sym}_{d+1}$ denotes the symmetric group of degree $d + 1$ acting on the set $\{0, 1, \ldots, d\}$. In the literature this is also called the “min-plus permanent” of $M$. The matrix $M \in \mathbb{R}^{(d+1) \times (d+1)}$ is tropically singular if the minimum in (2) is attained at least twice.

The tropical hyperplane $\mathcal{H}_a$ defined by the tropical linear form $a = (\alpha_0, \ldots, \alpha_d) \in \mathbb{R}^{d+1}$ is the set of points $(\xi_0, \ldots, \xi_d) \in \mathbb{T}A^d$ such that the minimum

$$(\alpha_0 \odot \xi_0) \oplus \cdots \oplus (\alpha_d \odot \xi_d)$$

is attained at least twice. The complement of a tropical hyperplane in $\mathbb{T}A^d$ has exactly $d + 1$ connected components, each of which is an open sector. A closed sector is the topological closure of an open sector. The set

$$S_k := \{(\xi_0, \ldots, \xi_d) \in \mathbb{T}A^d \mid \xi_k = 0 \text{ and } \xi_i > 0 \text{ for } i \neq k\},$$

for $0 \leq k \leq d$, is the $k$-th open sector of the tropical hyperplane $\mathcal{Z}$ in $\mathbb{T}A^d$ defined by the zero tropical linear form. Its closure is

$$\bar{S}_k := \{(\xi_0, \ldots, \xi_d) \in \mathbb{T}A^d \mid \xi_k = 0 \text{ and } \xi_i \geq 0 \text{ for } i \neq k\} .$$

We also use the notation $\bar{S}_I := \bigcup \{\bar{S}_i \mid i \in I\}$ for any set $I \subset \{0, \ldots, d\}$.

If $\alpha = (\alpha_0, \ldots, \alpha_d)$ is an arbitrary tropical linear form then the translates $-a + S_k$ for $0 \leq k \leq d$ are the open sectors of the tropical hyperplane $\mathcal{H}_a$. The point $-a$ is the unique point contained...
in all closed sectors of $H_a$, and it is called the \textit{apex} of $H_a$. For each $I \subset \{0,1,\ldots,d\}$ with $1 \leq \#I \leq d$ the set $-a + S_I$ is the \textit{closed tropical halfspace} of $H_a$ of type $I$. The tropical polytopes in $\mathbb{T}A^d$ are exactly the bounded intersections of finitely many closed tropical halfspaces; see \cite{14} and \cite{10}.

The points $v_1, \ldots, v_n \in \mathbb{T}A^d$ are in \textit{tropically general position} if the $n \times (d + 1)$-matrix whose $i$-th row is $v_i$ has no $k \times k$-submatrix which is tropically singular, for $2 \leq k \leq \min(n,d + 1)$.

Note that the integral translates of the hyperplanes $x_i = x_j$ induce a triangulation of $\mathbb{R}^d = c_0(\mathbb{T}A^d)$; this is called the \textit{alcove triangulation} $\mathbb{T}A^d$ of $\mathbb{T}A^d$ by Lam and Postnikov \cite{17}.

A \textit{tropical $d$-simplex} in $\mathbb{T}A^d$ is the tropical convex hull of $d + 1$ points in $\mathbb{T}A^d$ which are not contained in the boundary of a tropical halfspace; see Figure 1. It must be stressed that the vertices of a tropical simplex are not necessarily in tropically general position. For example, see the first tropical triangle in Figure 1.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Four tropical simplices in $\mathbb{T}A^2$ (with their tropical vertices drawn black). The vertices of the first one are not in tropically general position.}
\end{figure}

Let $V := (v_1, \ldots, v_n)$ be a sequence of points in $\mathbb{T}A^d$. The \textit{type} of $x \in \mathbb{T}A^d$ with respect to $V$ is the ordered $(d + 1)$-tuple $\text{type}_V(x) := (T_0, \ldots, T_d)$ where

$T_k := \{i \in \{1, \ldots, n\} \mid v_i \in x + \bar{S}_k\}$.

For a given type $T$ with respect to $V$ the set

$X_V(T) := \{x \in \mathbb{T}A^d \mid \text{type}_V(x) = T\}$

is the \textit{cell} of type $T$ with respect to $V$. With respect to inclusion the types with respect to $V$ form a partially ordered set.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Types and maximal cells with respect to two different triplets of points in $\mathbb{T}A^2$.}
\end{figure}

The symmetric group $\text{Sym}(\{0, \ldots, d\})$ acts on $\mathbb{T}A^d$ by permuting the coordinates. This operation fixes $\mathbb{T}A^d$, and it preserves inclusion of sets as well as ordinary and tropical convexity.
Further, tropical hyperplanes are mapped to tropical hyperplanes, and tropical halfspaces are mapped to tropical halfspaces. This action gives rise to a natural equivalence of point configurations: Two sequences $V = (v_1, \ldots, v_n)$ and $W = (w_1, \ldots, w_n)$ of points in $\mathbb{T}A^d$ are *tropically equivalent* if there is a pair of permutations

$$(\sigma, \tau) \in \text{Sym}(\{1, \ldots, n\}) \times \text{Sym}(\{0, \ldots, d\})$$

such that the map

$$(T_0, \ldots, T_d) \mapsto (U_{\tau(0)}, \ldots, U_{\tau(d)}),$$

where $U_i = \sigma(T_i)$, is a poset isomorphism from the types with respect to $V$ to the types with respect to $W$. Occasionally, it will also be convenient to start the numbering of the vertices with zero rather than one.

**Remark 1.** A decisive difference to ordinary point configurations in $\mathbb{R}^d$ is that each tropical point configuration in $\mathbb{T}A^d$ has a tropically equivalent realization with integral vertices.

Two tropical polytopes are said to be *tropically equivalent* if their tropical vertices are tropically equivalent as point configurations. Figure 2 shows two tropical triangles which are not tropically equivalent as point configurations. Develin and Sturmfels [8], Theorem 1, showed that the tropical equivalence classes of quadruples of points in $\mathbb{T}A^2$ show all 35 tropical equivalence classes of quadruples of points in $\mathbb{T}A^2$.

We will now discuss a link between tropical and ordinary convexity via Puiseux series; for the general picture see Speyer and Sturmfels [25], Theorem 2.1, and Markwig [18]. Let $K = \mathbb{R}((t^{1/\infty}))$ be the field of Puiseux series with real coefficients. It is known that $K$ is real closed which is why its first order theory coincides with the first order theory of the reals; see Salzmann et al. [23], §64.24. In particular, there are ordinary convex polytopes in $K^d$, and they behave much like ordinary polytopes in $\mathbb{R}^d$. An element of $K$ can be written as $f = \sum_{i \geq N} a_i t^{i/n}$ for some $N \in \mathbb{Z}$ and $n \in \mathbb{N}$. In particular, if $f \neq 0$ there is a minimal $d \in \mathbb{Z}$ such that $a_d \neq 0$. We call $d/n$ the (lower) degree of $f$ and denote it by $\text{val} f$. By setting $\text{val}(0) = \infty$ the map $\text{val} : K \to \mathbb{Q} \cup \{\infty\}$ is a valuation. This gives rise to

$$\text{val} : K^d \to (\mathbb{Q} \cup \{\infty\})^d, (f_1, \ldots, f_d) \mapsto (\text{val} f_1, \ldots, \text{val} f_d).$$

In [8], Proposition 2.1, it is shown that each tropical polytope $P$ in $\mathbb{T}A^d$ (identified with $\mathbb{R}^d$ via the map $c_0$ from (11)) with rational coordinates arises as the image of an ordinary convex polytope in $K^d$ under the map $\text{val}$. Each element of the fiber will be called a *Puiseux lifting* of $P$. In the same way tropical hyperplanes are images of ordinary hyperplanes, tropical halfspaces are images of ordinary halfspaces, and tropical point configurations can be lifted to $K^d$.

**Lemma 2.** Let $P \subset \mathbb{T}A^d$ be the intersection of $d + 1$ tropical halfspaces. Then one of the following holds:

(i) $P$ is unbounded or
(ii) $P$ is contained in a tropical hyperplane or
(iii) $P$ is a tropical simplex.

Clearly, the properties (i) and (iii) are mutually exclusive, while the two other combinations can occur together.

**Proof.** Let us first assume that the apices of the tropical hyperplanes have rational coordinates. Then the properties above are inherited from ordinary convexity via a Puiseux lifting from $\mathbb{T}A^d$ to $K^d$. If the coordinates of the apices are irrational then we can perturb the situation to rational (or even integer) coordinates in view of Remark 11. □
3. Polytopes

A subset of \( \mathbb{T}A^d \) is \textit{convex in the ordinary sense} if its image in \( \mathbb{R}^d \) under the map \( c_0 \) as in (1) is convex. A \textit{polytrope} is a tropical polytope which is also convex in the ordinary sense. In order to avoid confusion, we call the vertices of a polytrope, seen as an ordinary polytope, its \textit{pseudo-vertices}. A \( d \)-dimensional polytrope \( P \), or \( d \)-polytrope for short, has exactly one bounded cell of dimension \( d \) with respect to its vertices: its interior. This is called the \textit{basic cell}, and its type (with respect to the tropical vertices of \( P \)) is the \textit{basic type} of \( P \).

\textbf{Remark 3.} An ordinary polytope which additionally is tropically convex is not necessarily a polytrope: For example, the ordinary triangle \( \text{conv}\{(0,0), (2,1), (0,1)\} \) is tropically convex. However, this is not a tropical polytope since it is not the tropical convex hull of any finite subset of \( \mathbb{T}A^2 \).

In order to investigate polytopes any further it is useful to look at the root systems of type \( A_d \); see Bourbaki [3] for the complete picture. The relationship to polytopes is the following. The root system of type \( A_d \) consists of the \( d(d+1) \) vectors \( e_i - e_j \) in \( \mathbb{R}^{d+1} \) with \( 0 \leq i, j \leq d \) and \( i \neq j \). Call an ordinary convex polyhedron whose (outer) facet normals (scaled to Euclidean length \( \sqrt{2} \)) form a subset of those roots an \textit{ordinary} \( A_d \)-polyhedron. Since it contains the ray \( \mathbb{R}(1,1,\ldots,1) \) an ordinary \( A_d \)-polyhedron is always unbounded. Its intersection with the coordinate hyperplane \( x_0 = 0 \) has facet normals

\[ \pm e_i \text{ and } e_i - e_j \quad \text{for } 1 \leq i, j \leq d \text{ and } i \neq j. \]

Moreover, the tropical hyperplanes in \( \mathbb{T}A^d \) are formed from pieces of ordinary affine hyperplanes with such normal vectors. It then follows from [8, Lemma 10] that the polytopes are precisely the intersections of ordinary \( A_d \)-polyhedra with the coordinate hyperplane \( x_0 = 0 \). The latter were called \textit{alcoved polytopes of type} \( A \) by Lam and Postnikov [17].

\textbf{Example 4.} The classification of polytopes is the topic of Section \textit{5} below. Here we list the result in the planar case \( d = 2 \). Up to tropical equivalence there are exactly five types of \( 2 \)-polytopes. Considered as ordinary polygons, they have three, four, five, and six pseudo-vertices, respectively; see Figure \textit{5}.

![Figure 3. Four types of polytopes in \( \mathbb{T}A^2 \). The tropical vertices are black, and the pseudo-vertices are grey. The sketches of tropical hyperplanes indicate the facet defining tropical halfspaces.](image)

\textbf{Proposition 5.} Each \( d \)-polytrope has at most \( d(d+1) \) ordinary facets, and this bound is sharp.

\textbf{Proof.} The upper bound is clear since \( d(d+1) \) is the number of roots of type \( A_d \). That this bound is sharp follows from the construction below. \( \square \)

The maximum number of ordinary facets is attained, for instance, by the \( d \)-\textit{pyrope}

\[ \Pi_d := \text{tconv}(-e_0, -e_1, \ldots, -e_d). \]

The name is inspired by the fact that \textit{pyrope} is a mineral whose structure as a pure crystal can take the form of a rhombic dodecahedron, and the latter is combinatorially equivalent to \( \Pi_3 \) as an ordinary polytope; see Figure \textit{4} for a picture. The chemical sum formula of pyrope is \( \text{Mg}_3\text{Al}_2(\text{SiO}_4)_3 \); see Anthony et al. [2] for the mineralogy facts. In general, \( \Pi_d \) is a cubical
zonotope with $d + 1$ zones, which can be written as $\text{conv}([0, 1]^d \cup [-1, 0]^d)$. The number of its pseudo-vertices equals $2^d - 2$.

![Figure 4. The pyrope $\Pi_3$ is a polytrope which, as an ordinary polytope, is a rhombic dodecahedron.](image)

To obtain the exact upper bound for the number of pseudo-vertices of a polytrope is less trivial. A class of polytopes attaining the upper bound on the number of pseudo-vertices will be constructed in the next section.

**Proposition 6.** (Gelfand, Graev, and Postnikov [12], Theorem 2.3(2); Develin and Sturmfels [8], Proposition 19). Each $d$-polytrope has at most $\binom{2^d}{d}$ pseudo-vertices, and this bound is sharp.

The corresponding questions concerning the lower bounds are trivial: The small tropical $d$-simplex

$$t\text{conv}(0, e_1, e_1 + e_2, \ldots, e_1 + e_2 + \cdots + e_d)$$

is also an ordinary simplex, hence the obvious lower bound of $d + 1$ for the number of ordinary facets as well as for the number of pseudo-vertices is actually attained.

Following [8] Proposition 18 we are now going to describe how to obtain the tropical vertices of a polytrope $P$ from an ordinary inequality description. As in [3] we assume that $P$ is the set of points in $\mathbb{T}A^d$, identified with $\mathbb{R}^d$, satisfying the inequalities

$$x_i - x_j \leq c_{ij} \quad \text{for all } (i, j) \in J,$$

where $J$ is a subset of $\{(i, j) \mid i, j \in \{0, \ldots, d\}, i \neq j\}$ and $x_0 = 0$. Since $P$ is bounded, the set of vectors

$$\{e_i - e_j \mid (i, j) \in J\} \cup \{\pm(1, 1, \ldots, 1)\}$$

positively spans $\mathbb{R}^{d+1}$. The last two vectors do not correspond to facet normals, but they make up for the fact that an ordinary $A_d$-polyhedron is always unbounded. We will construct a sequence $V = (v_0, \ldots, v_d)$ of $d + 1$ points which will turn out to be the tropical vertices of $P$. The computation will be organized in a way such that the basic type of $P$ with respect to $V$ is $(0, 1, \ldots, d)$. Each tropical vertex satisfies at least $d$ of the inequalities (6) with equality. This is immediate from the fact that each tropical vertex of $P$ is also a pseudo-vertex, that is, an ordinary vertex of $P$.

First we may assume that each inequality in the description (6) is tight, that is, that the corresponding ordinary affine hyperplane supports $P$. Second we may assume that each root
vector of type $A_d$ actually gives one inequality in the description \([6]\). If this assumption is not initially given it can explicitly be established as follows. Let \((i, k) \notin J\), that is, the corresponding inequality is initially not given. If \((i, j_1), (j_1, j_2), \ldots, (j_{m-1}, j_m), (j_m, k)\) are in $J$ then the equation
\[
x_i - x_k = x_i - x_{j_1} + x_{j_1} - x_{j_2} + x_{j_2} - \cdots + x_{j_{m-1}} - x_{j_m} + x_{j_m} - x_k
\]
leads to the definition
\[
c_{ik} := c_{i j_1} + c_{j_1 j_2} + \cdots + c_{j_{m-1} j_m} + c_{j_m k},
\]
and $x_i - x_k \leq c_{ik}$ is a new tight inequality. Iterating this procedure gives all the inequalities desired since \(\{e_i - e_j \mid (i, j) \in J\} \cup \{\pm (1, 1, \ldots, 1)\}\) positively spans $\mathbb{R}^d$. Now the coordinates \((v_{i0}, \ldots, v_{id})\) of the point $v_i$ are uniquely determined by setting $v_{i0} = 0$ and $d(d+1)$ more equations. An equivalent but more symmetric requirement is
\[
(7)
v_{ii} = 0 \quad \text{and} \quad v_{ik} = c_{ki} \quad \text{for} \ i \neq k.
\]
This computation is equivalent to the Floyd–Warshall algorithm for computing all shortest paths in a directed graph \([7]\). Lemma 10 of \([8]\) proves the following.

**Theorem 7.** The $d+1$ points in the sequence $V = (v_0, v_1, \ldots, v_d)$ defined in \([7]\) are the tropical vertices of the $d$-polytrope $P$. In particular, each polytrope is a tropical simplex.

**Example 8.** We wish to give an example of how to compute the tropical vertices of a polytrope from an ordinary inequality description. Let $P$ be the 2-polytrope described by the inequalities $x_1 \leq 2, -x_1 \leq 0, x_2 \leq 2, -x_2 \leq 0, x_1 - x_2 \leq 1$; this looks like the third tropical triangle in Figure 3 which is an ordinary pentagon. All inequalities are tight. The unique initially missing inequality corresponds to $e_2 - e_1$. We compute $x_2 - x_1 = x_2 - x_0 + x_0 - x_1$ and
\[
c_{21} = c_{20} + c_{01} = 2 + 0 = 2.
\]
Hence the missing inequality is $x_2 - x_1 \leq 2$. From this we infer that $v_0 = (0, c_{10}, c_{20}) = (0, 2, 2)$, $v_1 = (c_{01}, 0, c_{21}) = (0, 0, 2)$, and $v_2 = (c_{02}, c_{12}, 0) = (0, 1, 0)$. With respect to these generators the type of the basic cell reads $(0, 1, 2)$.

The tropical halfspaces containing a tropical polytope $P$ are partially ordered by inclusion. A tropical halfspace which is minimal with respect to this partial ordering and which has the additional property that its apex is a pseudo-vertex, is called facet defining for $P$. It is known that $P$ is the intersection of its (finitely many) facet defining tropical halfspaces. Notice that the proof in \([14]\) Theorem 3.6 uses \([14]\) Proposition 3.3 which is wrong. A corrected statement is due to Gaubert and Katz \([10]\) Proposition 1, and this suffices to prove \([14]\) Theorem 3.6; see also \([10]\) Theorem 2. As in Lemma 2 one can use Puiseux liftings to show that if $P$ is a full-dimensional polytrope $P$ in $\mathbb{T}^d$ it has exactly $d+1$ facet defining tropical halfspaces. Here we give a direct and constructive proof.

For an arbitrary sequence $V = (v_1, \ldots, v_n)$ of points in $\mathbb{T}^d$ and $k \in \{0, \ldots, d\}$ let
\[
c_k(V) := (-v_{1,k} \odot v_1) \oplus (-v_{2,k} \odot v_2) \oplus \cdots \oplus (-v_{n,k} \odot v_n)
\]
be the $k$-th corner of $P = \text{tconv}(V)$. By construction each corner belongs to the tropical convex hull $P$. It is also obvious that the cornered tropical halfspace $c_k + \mathbb{S}_k$ contains $P$. Notice that the corners of $P$ do not depend on the choice of the set of generators $V$. We say that $P = \text{tconv}(V) \subset \mathbb{T}^d$ is full-dimensional if its dimension as an ordinary polytopal complex in $\mathbb{R}^d$ equals $d$. Here we do not assume that $P$ is a polytrope.

**Proposition 9.** Suppose that $P$ is a full-dimensional tropical polytope. Then the $d+1$ cornered tropical halfspaces are facet defining tropical halfspaces of $P$.  

**Proof.** The $k$-th corner $c_k = (c_{k0}, \ldots, c_{kd})$ is contained in the $d$ ordinary affine hyperplanes $x_k - x_l = c_{lk} - c_{kl}$ for all $l \in \{0, \ldots, d\} \setminus \{k\}$. The corresponding $d$ normal vectors $e_k - e_l$ are skew to the vector $(1, 1, \ldots, 1)$, and hence they linearly span the quotient $\mathbb{R}^d = \mathbb{T}^d$. Therefore, the intersection of these hyperplanes is a point. This implies that $c_k$ is a vertex of the maximum tropical hyperplane arrangement induced by $V$, which means that $c_k$ is a pseudo-vertex.
Suppose that $c_k + \tilde{S}_k$ is not minimal. Then there must be some other tropical halfspace $w + \tilde{S}_K$ contained in $c_k + \tilde{S}_k$ which still contains $P$. Without loss of generality we can assume that $w + \tilde{S}_K$ is minimal and thus $w \in P$. Since $c_k + \tilde{S}_k$ consists of a single closed sector it follows that $K = \{0\}$. Moreover, since $c_k$ is contained in $P$, we have $c_k - w \in \tilde{S}_k$. However, we also have $w \in c_k + \tilde{S}_k$ since $c_k + \tilde{S}_k$ contains all points of $P$. We conclude that $w = c_k$, and this proves that each cornered halfspace is facet defining.

**Proposition 10.** If $P$ is a polytrope then the cornered tropical halfspaces are the only facet defining tropical halfspaces of $P$.

**Proof.** Let $(v_0, v_1, \ldots, v_d)$ be the tropical vertices of the $d$-polytrope $P$. Up to a transformation we can assume that the basic type of $P$ is $(0,1,\ldots,d)$. Moreover, we assume that the coordinates are chosen such that $v_{ki} = 0$ holds for all $i \in \{0,\ldots,d\}$. We have to show that there are no other facet defining tropical halfspaces for $P$. By construction the cornered hull

$$ (c_0 + \tilde{S}_0) \cap (c_1 + \tilde{S}_1) \cap \cdots \cap (c_d + \tilde{S}_d) $$

of $P$ is the convex polyhedron subject to the $d(d+1)$ ordinary inequalities $x_i - x_j \geq c_{ij}$ for all $i \neq k$. Equivalently, we have

$$ x_i - x_k \leq -c_{ki} \quad \text{for } i \neq k $$

Since the cornered hull is bounded it is a polytrope. We can apply the procedure (7) to get at the tropical vertices of the cornered hull. These are exactly the points $v_0, \ldots, v_d$, and the claim follows.

**Remark 11.** Since $c_k + \tilde{S}_k$ contains all points in $V$ we have that $T_k = \{1,\ldots,n\}$ where $T = (T_0, \ldots, T_d)$ is the type of $c_k$ with respect to $V$. The $k$-th corner is the unique pseudo-vertex of $P$ with this property.

**Remark 12.** If $V \in \mathbb{R}^{(d+1)\times(d+1)}$ is a matrix (with zero diagonal) whose rows correspond to the tropical vertices of a polytrope $P$ then the rows of its negative transpose $-V^t$ yield the corners. It follows that the corners are the tropical vertices of $P$, seen as a max-tropical polytope. The map $V \mapsto -V^t$ is an instance of the duality of tropical polytopes discussed in [8, Theorem 23].

4. Constructions and examples

4.1. **Associahedra.** Studying expansive motions Rote, Santos, and Streinu arrived at interesting new realizations of the associahedron [22], §5.3. They consider the polyhedron in $\mathbb{R}^n$ which is defined by

$$ x_j - x_i \geq (i - j)^2 \quad \text{for } 1 \leq i < j \leq n $$

$$ x_1 = 0 $$

$$ x_n = (n - 1)^2. $$

This turns out to be an ordinary polytope which is combinatorially equivalent to the $(n - 2)$-dimensional associahedron, which is a secondary polytope of a convex $(n + 1)$-gon. The $(\binom{n}{2}) - 1$ inequalities $x_j - x_i \geq (i - j)^2$ for $(i,j) \neq (1,n)$ are all facet defining.

If we project the polytope defined in (9) orthogonally onto the subspace of $\mathbb{R}^n$ spanned by the standard basis vectors $e_2, e_3, \ldots, e_{n-1}$ we obtain a full-dimensional realization $\text{Assoc}_{n-2} \subset \mathbb{R}^{n-2}$ which is tropically convex (via the identification from [11]). That is, $\text{Assoc}_{n-2}$ is a polytrope.

We can apply the procedure from (7) to determine the tropical vertices of $\text{Assoc}_{n-2}$. Each tropical vertex will be described by listing the $n - 1$ ordinary facets containing it. If the ordinary facet $x_j - x_i = (i - j)^2$ from (9) is denoted as $(i,j)$ then the $j$-th tropical vertex of $\text{Assoc}_{n-2}$, where $1 \leq j \leq n - 1$, is the intersection of the facets $(1,j), (2,j), \ldots, (j-1,j), (j+1,n), (j+2,n), \ldots, (n-1,n)$. For example, the tropical vertices of $\text{Assoc}_3$ are

$$ (2,5), (3,5), (4,5) = (7,12,15), $$

$$ (1,2), (3,5), (4,5) = (1,12,15), $$

$$ (1,3), (2,3), (4,5) = (3,4,15), \quad \text{and} $$

$$ (1,4), (2,4), (3,4) = (5,8,9). $$
On the right hand side are the coordinates in $\mathbb{R}^3$. The polytrope $\text{Assoc}_2$ is an ordinary pentagon like in Figure 3 (third).

4.2. Polytopes with many pseudo-vertices. We want to construct a class of polytopes which attain the upper bound on the number of pseudo-vertices from Proposition 6. This construction is an explicit instance of what arises from the proof of [8], Proposition 19. The following lemma says that we can perturb the vertices of the pyrope $\Pi_d$ construction is an explicit instance of what arises from the proof of [8], Proposition 19. The following lemma says that we can perturb the vertices of the pyrope $\Pi_d$ from [1] quite a bit, and we still have a polytrope.

Lemma 13. For an arbitrary matrix $E = (\varepsilon_{ik})_{i,k} \in [0, \frac{1}{2})^{(d+1) \times (d+1)}$ the tropical polytope

$$\Pi_d^E := \text{tconv}(\varepsilon_0 + \varepsilon_{01}, -\varepsilon_1 + \varepsilon_{11}, \ldots, -\varepsilon_d + \varepsilon_{d1})$$

is a polytrope.

Proof. A direct computation shows that the generators $-\varepsilon_0 + \varepsilon_{01}, \ldots, -\varepsilon_d + \varepsilon_{d1}$, in fact, are the tropical vertices of $\Pi_d$. For the rest of the proof we fix this particular vertex ordering.

Observe that the type of the origin is $(0, 1, \ldots, d)$. Now we compute the type $(T_0, T_1, \ldots, T_d)$ of the vertex $-\varepsilon_i + \varepsilon_{ij}$. We claim that $T_k = \{i, k\}$ if $i \neq k$ and $T_i = \{i\}$. Indeed, for $i \neq j$ we have $-\varepsilon_j + \varepsilon_{ji}, \varepsilon_{i} - \varepsilon_j + \varepsilon_{ji}, -\varepsilon_i, \varepsilon_{i}, \varepsilon_{i} - \varepsilon_i, \varepsilon_{i}, \varepsilon_{i} - \varepsilon_i + \varepsilon_{ji}, -\varepsilon_i, \varepsilon_{i}$ if and only if $e_i - e_j + \varepsilon_{ji} \leq e_i$, $\varepsilon_{i} - \varepsilon_i, \varepsilon_{i}$ if and only if $j = k$ since $0 \leq \varepsilon_{ik}, \varepsilon_{jk} < \frac{1}{2}$.

From this we learn that each vertex is contained in the closure of the cell of type $(0, 1, \ldots, d)$, and hence there is only one bounded cell.

For a random matrix $E$ Lemma 13 would yield a polytrope with the maximal number of vertices (almost surely). The following is a deterministic solution.

Example 14. For any fixed positive $\varepsilon$ with $\varepsilon < \frac{1}{2}$ let

$$E = \begin{pmatrix}
0 & \varepsilon & \varepsilon^2 & \ldots & \varepsilon^{d-1} & \varepsilon^d \\
\varepsilon^d & 0 & \varepsilon & \varepsilon^2 & \ldots & \varepsilon^{d-1} \\
\varepsilon^{d-1} & 0 & \varepsilon & \varepsilon^2 & \ldots & \varepsilon^{d-2} \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\varepsilon^2 & \ldots & \varepsilon^{d-1} & \varepsilon^d & 0 & \varepsilon \\
\varepsilon & \varepsilon^2 & \ldots & \varepsilon^{d-1} & \varepsilon^d & 0
\end{pmatrix}.$$ 

Then the perturbed pyrope $\Pi_d^E$ is a polytrope with $\binom{2d}{d}$ pseudo-vertices, which is the upper bound from Proposition 6.

There is only one tropical type of 2-polytope attaining the upper bound six on the number of pseudo-vertices, shown in Figure 3 (fourth). Already in dimension 3, however, there are five distinct types of polytopes with 20 vertices, which are also pairwise not combinatorially equivalent as ordinary polytopes. All of them are simple and share the same $f$-vector $(20, 30, 12)$. For each of the five types we give a $4 \times 4$-matrix such that the tropical convex hull of the rows gives the corresponding polytrope; these are also shown in Figure 5.

$$\begin{pmatrix}
0 & 0 & 2 & 2 \\
4 & 0 & 4 & 3 \\
4 & 3 & 0 & 4 \\
6 & 4 & 4 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 2 & 2 & 2 \\
4 & 0 & 4 & 2 \\
4 & 3 & 0 & 4 \\
6 & 4 & 4 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 10 & 11 & 14 \\
14 & 0 & 10 & 11 \\
11 & 14 & 0 & 10 \\
10 & 11 & 14 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 1 & 2 & 2 \\
8 & 0 & 8 & 7 \\
10 & 6 & 0 & 8 \\
6 & 5 & 4 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 6 & 6 & 2 \\
6 & 0 & 2 & 3 \\
11 & 10 & 0 & 10 \\
8 & 8 & 9 & 0
\end{pmatrix}$$

5. Enumerating all polytopes

We want to explain how to enumerate all polytopes in $T \mathbb{A}^d$ for fixed $d$. Since their number of pseudo-vertices (and ordinary facets) is bounded by Propositions 5 and 6 it is clear that there are only finitely many distinct tropical types. Of course, in principal, it is possible to enumerate all regular subdivisions of $\Delta_d \times \Delta_d$ and to sort out those which are dual to a polytrope; see [21, 19]. But this does not seem to be practically feasible even for $d = 3$ due to the sheer size of the secondary polytope of $\Delta_3 \times \Delta_3$. However, there is a more efficient approach which
will be the subject of the discussion now. The efficiency will be underlined by being able to achieve a complete classification of the tropical types of 3-polytropes; we think that even the 4-dimensional case is within reach.

In view of Remark 1 we can restrict our attention to enumerating *lattice polytropes*, that is, polytropes whose pseudo-vertices have integral coordinates. Since the alcove triangulation $\mathcal{T}\Delta^d$ induces a triangulation on any lattice polytrope, and since the small tropical simplex from (5) is a maximal face of $\mathcal{T}\Delta^d$, it suffices to enumerate integral polytropes which contain the small tropical simplex. This means that we can obtain each (tropical type of) polytrope by successively adding generators outside the small tropical simplex.

Throughout the following we look at a $d$-polytrope $P = \text{tconv}(v_0, \ldots, v_d) \subset \mathcal{T}\Delta^d$, and we assume that the basic type of $P$ is $(0, 1, \ldots, d)$, which is equivalent to requiring that $\text{tdet}(v_0, \ldots, v_d) = v_0 + \cdots + v_d$. Our type computations will be with respect to this ordering of the vertices of $P$.

Let $(T_{i0}, T_{i1}, \ldots, T_{id}) = \text{type}_V(v_i)$. Since $v_i$ is a tropical vertex of $P$ we have $T_{ik} = \{i\}$. Moreover, as the basic type is $(0, 1, \ldots, d)$ we have $k \in T_{ik}$ for all $i, k$. In this situation the tropical halfspace $v_i + \bar{S}_i$ intersects $P$ only in the vertex $v_i$. The set $v_i + \bar{S}_i$ is always contained in the normal cone of $v_i$ seen as a vertex of the ordinary polytope $P$.

In Figure 6 the light regions form the tropical halfspaces $v_i + \bar{S}_i$. For a new point $x$ the tropical polytope $P(x) := \text{tconv}(v_0, \ldots, v_d, x)$ will be convex in the ordinary sense or not, depending on the type of $x$. 

**Figure 5.** The five tropical types of 3-polytropes with 20 vertices.

**Figure 6.** The light/yellow regions are the tropical halfspaces $v_i + \bar{S}_i$, the dark/red ones are the cells $X_{i,j}$ of type $(T_{i,j,0}, T_{i,j,1}, T_{i,j,2})$ as in (10). See also Figure 2 (right).
It can be shown that the point \( P \) cannot be a polytrope.

As above, we have that the tropical polytope \( P(x) = \text{tconv}(v_0, \ldots, v_d, x) \) is convex in the ordinary sense if and only if

\[
x \in \bigcup_{i=0}^{d} (\overline{X_i} \cap (v_i + \tilde{S}_i)),
\]

where \( \overline{X_i} \) is the topological closure of \( X_i \). Moreover, in this case we have \( P(x) \supseteq P \), so \( v_i \) is redundant in \( P(x) \).

In order to give it a concise name we call the set \( \overline{X_i} \cap (v_i + \tilde{S}_i) \) the \( i \)-th valid region with respect to \( V \).

**Proof.** First let us assume that \( x \in X_{i,j} \cap (v_i + \tilde{S}_i) \). By symmetry we can assume that \( i = 0 \). This is to say that

\[
\text{type}_T(x) = (\emptyset, T_1^0, T_2^0, \ldots, T_{j-1}^0, T_j^0 \cup \{0\}, T_{j+1}^0, \ldots, T_d^0).
\]

As \( x \) is contained in \( v_0 + \tilde{S}_0 \) it follows that \( P(x) = \text{tconv}(x, v_1, \ldots, v_d) \).

We have to show that \( P(x) \) is convex in the ordinary sense. To this end we fix a point \( z \) in the basic cell of \( P \). Then \( \text{type}_V(z) = (0, 1, \ldots, d) \). Since \( x \in v_0 + \tilde{S}_0 \), and since the other vertices remain the same we conclude that the type of \( z \) with respect to \( V(x) := (x, v_1, \ldots, v_d) \) is also \( (0, 1, \ldots, d) \). Now we compute the type \( (U_0, U_1, \ldots, U_d) \) of \( x \) with respect to \( V(x) \). Clearly, \( 0 \in U_0 \). If we can show that for all \( k \in \{1, \ldots, d\} \) we have \( k \in U_k \) then it follows that \( x \) is in the boundary of the cell of type \( (0, 1, \ldots, d) \), and hence \( P(x) \) is a polytrope.

So we assume that there is some \( k \in \{1, \ldots, d\} \) with \( k \notin U_k \). Since \( v_0 \) is the only point that is now missing in the sequence of generators we know that \( U_k \supseteq T_k^0 \setminus \{0\} \). Actually, since \( x \) is a tropical vertex of \( P(x) \), we even have \( U_k \supseteq T_k^0 \). By construction \( j \in T_j^0 \subseteq U_j \), and also \( k \in T_k^0 \subseteq U_k \) for \( k \in \{1, 2, \ldots, d\} \setminus \{j\} \) because \( v_0 \) is contained in the boundary of the basic cell of \( P \).

It remains to prove the converse: We have to show that if \( x \notin \bigcup_{i=0}^{d} (X_i \cap v_i + \tilde{S}_i) \) then \( P(x) \) is not convex in the ordinary sense. We distinguish two cases. If \( x \notin P \cup \bigcup_{i=0}^{d} (v_i + \tilde{S}_i) \) then none of the generators of \( P(x) \) is redundant, and, due to Theorem 7, the tropical polytope \( P(x) \) cannot be a polytrope.

Finally, let \( x \in \bigcup_{i=0}^{d} (v_i + \tilde{S}_i) \setminus \bigcup_{i=0}^{d} X_i \). Again, by symmetry we can assume that \( x \in v_0 + \tilde{S}_0 \). As above \( P(x) \supseteq P \). Then if \( (U_0, U_1, \ldots, U_d) := \text{type}_V(x) \) there is some \( j \neq 0 \) such that \( U_j = \emptyset \). It can be shown that the point \( y := \frac{1}{2}(x + v_j) \) lies outside \( P(x) \), whence \( P(x) \) is not convex in the ordinary sense. \( \square \)

With the aid of Proposition 15 we can enumerate all tropical equivalence types of polytopes. Consider a polytrope \( P = \text{tconv}(v_0, \ldots, v_d) \) in \( \mathbb{T} \mathbb{A}^d \) and its valid regions \( \overline{X_i} \cap (v_i + \tilde{S}_i) \). Simultaneously choosing \( d + 1 \) points \( v_i' \in \overline{X_i} \cap (v_i + \tilde{S}_i) \) with \( i \in \{0, \ldots, d\} \) the tropical convex hull \( \text{tconv}(v_0', \ldots, v_d') \) is a polytrope because the valid regions with respect to the old points \( v_0, \ldots, v_d \) are contained in the valid regions of the new points \( v_0', \ldots, v_d' \). Moreover, if the types of \( (v_0', \ldots, v_d') \) are the same as \( (v_0'', \ldots, v_d'') \) then the resulting polytopes \( \text{tconv}(v_0', \ldots, v_d') \) and \( \text{tconv}(v_0'', \ldots, v_d'') \) are tropically equivalent.

For our initial points \( v_0, \ldots, v_d \) we take the (tropical) vertices of the small tropical \( d \)-simplex scaled by \( d \). The advantage of this scaling is that each cell in the valid regions contains (at least) one integral point. The tropical convex hulls of \( d + 1 \) such points, one from each valid region, yield all the tropical types of polytopes in \( \mathbb{T} \mathbb{A}^d \). In order to enumerate all tropical equivalence classes it suffices to consider one (integral) point per cell within each valid region.
For an efficient procedure it is essential to take symmetries into account. We implemented this enumeration scheme in polymake [11], and the result of the computation for \( d = 3 \) is given in Table 1. Here \( t_3(m) \) is the number of tropical equivalence classes of 3-polytopes with exactly \( m \) pseudo-vertices, and \( o_3(m) \) is the corresponding number of combinatorial types of ordinary polytopes. We necessarily have \( o_d(m) \leq t_d(m) \) for all choices of \( m \) and \( d \). From Proposition 6 we know that the maximum number of pseudo-vertices equals \( \binom{6}{3} = 20 \).

| \( m \) | \( t_3(m) \) | \( o_3(m) \) |
|---|---|---|
| 4 | 1 | 1 |
| 5 | 1 | 1 |
| 6 | 4 | 2 |
| 7 | 3 | 3 |
| 8 | 20 | 6 |
| 9 | 14 | 6 |
| 10 | 39 | 13 |
| 11 | 43 | 14 |
| 12 | 68 | 27 |
| 13 | 54 | 22 |
| 14 | 74 | 31 |
| 15 | 53 | 30 |
| 16 | 43 | 31 |
| 17 | 21 | 20 |
| 18 | 17 | 17 |
| 19 | 8 | 8 |
| 20 | 5 | 5 |

The total numbers are \( \sum_{m=4}^{20} t_3(m) = 468 \) and \( \sum_{m=4}^{20} o_3(m) = 237 \). To locate some special examples in Table 1 that occurred above: The (up to tropical equivalence) unique 3-polytope with 4 pseudo-vertices is the small tropical tetrahedron. The 3-polytope \( \Pi_3 \) from Figure 4 has \( 2^4 - 2 = 14 \) pseudo-vertices; the associahedron \( \text{Assoc}_3 \) from Section 4.1 also has 14 pseudo-vertices, but it is not even combinatorially equivalent to \( \Pi_3 \). The five classes of 3-polytopes with 20 pseudo-vertices are shown in Figure 5.

6. GORENSTEIN SIMPLICIAL COMPLEXES AND GORENSTEIN POLYTOPES

From Theorem 7 and [8], Proposition 24, we know that \( d \)-polytopes in \( \mathbb{T}A^d \), identified with the tropical point configuration of their tropical vertices, are dual to triangulations of the product of simplices \( \Delta_d \times \Delta_d \). The purpose of this section is to view these triangulations as abstract simplicial complexes and to interpret them in terms of Commutative Algebra. In particular, this way we will obtain an alternate proof of Theorem 7.

A standard construction of new simplicial complexes from old ones is iterative coning. For the following it is crucial to determine if a given simplicial complex has been obtained in such a way. Let \( \Delta \) be an arbitrary simplicial complex on a finite vertex set \( V = \{x_1, \ldots, x_n\} \). As usual we let

\[
\text{st}_\Delta \sigma := \{ \tau \in \Delta \mid \sigma \cup \tau \in \Delta \},
\]

\[
\text{lk}_\Delta \sigma := \{ \tau \in \text{st}_\Delta \sigma \mid \sigma \cap \tau = \emptyset \},
\]

\[\text{core} V := \{ v \in V \mid \text{st}_\Delta v \neq \Delta \}, \text{ and} \]

\[\text{core} \Delta := \Delta_{\text{core} V},\]

where \( \Delta_W \) is the subcomplex of \( \Delta \) induced on the vertices \( W \subseteq V \). By construction \( \Delta_{V \setminus \text{core} V} \) is a simplex, and \( \Delta \) is the join of \( \text{core} V \) with \( \Delta_{V \setminus \text{core} V} \).
For a field $K$ let $K[\Delta]$ be the Stanley–Reisner ring of $\Delta$, that is,
$$K[\Delta] := K[x_1, \ldots, x_n]/I_{\Delta},$$
where $I_{\Delta}$ is the ideal generated by the monomials whose exponent vectors correspond to characteristic functions of the (minimal) non-faces of $\Delta$. A direct computation shows that
$$K[\Delta] = K[\text{core } \Delta][x \mid x \in V \setminus \text{core } V],$$
that is, $K[\Delta]$ is the full polynomial ring with coefficients $K[\text{core } \Delta]$ and indeterminates indexed by $V \setminus \text{core } V$. A simplicial complex $\Delta$ is called Gorenstein if $K[\Delta]$ is a Gorenstein ring. Further, a positively $\mathbb{Z}^d$-graded ring $R$ is Gorenstein if it is Cohen–Macaulay, and the Matlis dual of the top local cohomology is isomorphic to a $\mathbb{Z}^d$-graded translate of $R$; see [26], Section I.12. More useful for our purposes is the following characterization.

**Theorem 16** (Stanley [26], Theorem 5.1). A simplicial complex $\Delta$ is Gorenstein (over a field $K$) if and only if for all $\sigma \in \text{core } \Delta$ we have
$$\tilde{H}_i(\Gamma; K) = \begin{cases} K & \text{if } i = \dim \Gamma \\ 0 & \text{otherwise,} \end{cases}$$
where $\Gamma = \text{lk}_{\text{core } \Delta} \sigma$.

Here $\tilde{H}_i(\Gamma; K)$ is the $i$-th reduced (simplicial) homology of $\Gamma$ with coefficients in $K$. The characterization requires $\Gamma$ to have the same homology (with coefficients in $K$) as the sphere of dimension $\dim \Gamma$. The tight span of a triangulation is its dual cell complex.

**Proposition 17.** A regular triangulation of an ordinary polytope is Gorenstein (over an arbitrary field $K$) if and only if its tight span has a unique maximal cell.

**Proof.** Let $\Delta$ be a regular triangulation of an ordinary polytope $P$. First suppose that the tight span $\Delta^*$ consists of a single maximal cell. Hence there is a simplex $\sigma \in \Delta$ in the interior of $P$ which is contained in each maximal simplex of $\Delta$. The vertices of $\sigma$ are precisely the cone points of $\Delta$, and $\Delta$ is the join of $\sigma$ with $\text{lk}_\Delta \sigma = \text{core } \Delta$. The link of an interior face in a triangulated manifold (with or without boundary) is a simplicial sphere. By Theorem 16 it follows that $\Delta$ is Gorenstein.

Conversely, let $\Delta$ be Gorenstein. Again, by Theorem 16 we know that
$$\Delta = \Delta_{V \setminus \text{core } V} \ast \text{core } \Delta,$$
where $V$ is the vertex set of $P$ (and $\Delta$), $V \setminus \text{core } V \neq \emptyset$, and $\text{core } \Delta$ is an orientable pseudomanifold. Then $\Delta_{V \setminus \text{core } V}$ is an interior simplex contained in all maximal simplices of $\Delta$, and hence $\Delta_{V \setminus \text{core } V}$ corresponds to the unique maximal cell of $\Delta^*$.

Let $P$ be an ordinary lattice $d$-polytope embedded into the affine hyperplane $\mathbb{R}^d \times \{1\}$ of $\mathbb{R}^{d+1}$. Then $M(P) = \text{pos } P \cap \mathbb{Z}^{d+1}$ is the set of lattice points in the positive cone spanned by $P$ in $\mathbb{R}^{d+1}$. Now $P$ is a Gorenstein polytope if there exists $u \in \text{int } M(P)$ such that
$$(11) \quad \text{int } M(P) = u + M(P),$$
see Bruns and Herzog [4], Chapter 6. Here $\text{int } M(P) = (\text{pos } P \setminus \partial(\text{pos } P)) \cap \mathbb{Z}^{d+1}$ denotes the set of interior lattice points of $M(P)$. Gorenstein polytopes and their Gorenstein triangulations are related as follows; see also Conca, Hoşten, and Thomas [6].

**Theorem 18** (Bruns and Römer [5], Corollary 8). Let $P$ be an ordinary lattice $d$-polytope with some regular and unimodular triangulation using all the lattice points in $P$. Then $P$ is a Gorenstein polytope if and only if it has some regular triangulation which is Gorenstein.

Now there is the following well-known result; for far generalizations see Goto and Watanabe [13], Theorem 4.4.7. For the sake of completeness we give a simple proof.

**Theorem 19.** The product of simplices $\Delta_m \times \Delta_n$ is a Gorenstein polytope if and only if $m = n$. 
Proof. The simplex $\Delta_n = \text{conv}(0, e_1, \ldots, e_n)$ is a Gorenstein polytope by the criterion \cite{[11]}, since the scaled simplex $k\Delta_n$ contains precisely one interior lattice point, namely $e_1 + \cdots + e_n$, if $k = n + 1$. This yields that $k(\Delta_m \times \Delta_n)$ contains exactly one interior lattice point if and only if $m = n = k - 1$. The claim now follows from \cite{[11]}. \hfill $\square$

The ring $K[\Delta_n]$ is isomorphic to the full polynomial ring in $n + 1$ indeterminates with coefficients in $K$. The ring $K[\Delta_m \times \Delta_n]$ is isomorphic to the Segre product of polynomial rings (with their natural gradings). Therefore, Theorem \ref{thm:main} translates into the language of Commutative Algebra as follows: The Segre product of $K[x_0, \ldots, x_m]$ and $K[x_0, \ldots, x_n]$ (with their natural gradings) is Gorenstein if and only if $m = n$.

The point of this section is that this can be used to give an alternate proof of our main result.

Alternate proof of Theorem \ref{thm:main} Let $P$ be a $d$-polytope in $\mathbb{T} \mathbb{A}^d$ with tropical vertices $v_1, \ldots, v_n$. We have to show that $n = d + 1$.

Now $P$ coincides with the tight span of the regular triangulation of $\Delta_{n-1} \times \Delta_d$ dual to the point configuration $(v_1, \ldots, v_n)$. In particular, this triangulation is a Gorenstein simplicial complex by Proposition \cite{[17]} So $\Delta_{n-1} \times \Delta_d$ is an ordinary polytope with a Gorenstein triangulation. Since products of simplices do admit a regular and unimodular triangulation, for instance, the staircase triangulation, the result of Bruns and Römer, Theorem \ref{thm:bruns-roemer} can be applied. We derive that $\Delta_{n-1} \times \Delta_d$ is a Gorenstein polytope and hence $n = d + 1$ by Theorem \ref{thm:main} \hfill $\square$

It is worth to mention that Theorem \ref{thm:main} can be read both ways. This means that, by reversing the argument above, Theorem \ref{thm:main} is, in fact, equivalent to Theorem \ref{thm:main}.

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