Directed momentum current induced by the $\mathcal{PT}$-symmetric driving

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We investigate the directed momentum current in the quantum kicked rotor model with $\mathcal{PT}$ symmetric driving potential. For the quantum non-resonance case, the values of quasi-energy becomes to be complex when the strength of imaginary part of the kicking potential exceeds a threshold value, which opens the opportunity for investigating the directed transport of ultracold atoms in complex optical lattices [25–31], which opens the opportunity for investigating the directed transport of quantum particle driven by the $\mathcal{PT}$ symmetric potential. More recent study on the QKR with $\mathcal{PT}$ symmetry reports the rich behaviour, such as the spontaneous $\mathcal{PT}$ symmetry breaking [32] and the ratchets acceleration in the quantum resonance condition [33].

Motivated by these studies, we investigate the wavepackets dynamics of the QKR with $\mathcal{PT}$ symmetry. We concentrate on the directed motion in momentum space when the system is in the quantum non-resonance condition. We find that the broken of the $\mathcal{PT}$ symmetry can induce rich transport behaviour in momentum space. Specially, in the vicinity of phase-transition points, the momentum current increases linearly with time. Detailed investigation reveals that the platform of the momentum current is determined by mean momentum of some eigenstates of the Floquet operator whose imaginary parts of the quasi-energy are significantly large. Above the phase-transition point, the momentum current increases linearly with time. Interestingly, its acceleration rate exhibits a kind of “quantized” increment with the kicking strength. We propose a modified classical acceleration mode of the kicked rotor model to explain such an intriguing phenomenon. Our theoretical prediction is in good agreement with numerical results.

I. INTRODUCTION

An intrinsic property of $\mathcal{PT}$ symmetric quantum systems is that such system possess spontaneous phase transition that is the real energy eigenvalues become to be complex when the strength of imaginary part of the complex potential exceeds a threshold value [1]. Wavepackets transport behavior of those systems has received intense investigations in recent years [2, 3]. In the broken $\mathcal{PT}$ symmetry phase, the energy band merging leads to peculiar transport behavior of optic wave and matter wave, such as double refraction, nonreciprocal diffraction, bifurcation, and many others [4, 5]. Among those studies, the directed transport of wavepackets is of particular interest [6, 13] as its mechanism is relevant for the construction of nanoscale devices, such as particle separation and electron pumps, and for the understanding of biological molecular motors [14–17].

Cold atoms driven by time-periodical optical lattice is an ideal platform for investigating the directed transport phenomenon [18–22]. A paradigm model of such Floquet-driven systems is the quantum kicked rotor (QKR) which possesses rich and complex physics, such as the Butterfly spectrum [21], the exponentially-fast diffusion [22] and the topological momentum current [23]. Indeed, the atom-optics experiments of this model have reported the directed acceleration of matter wave in momentum space [23]. Moreover, the experimental progress has realized the ultracold atoms in complex optics lattices [25–31], which opens the opportunity for investigating the transport behavior of quantum particle driven by the $\mathcal{PT}$ symmetry.

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ric initial states ensures the appearance of momentum current [11,14,18,19]. The directed momentum current which is leaded by the broken of $PT$ symmetry in the dynamical localization regime will be a new element of the ratchet phenomenon. In addition, our investigation reveals that the transport behaviour of the $PT$ symmetric quantum system exhibits a kind of classical acceleration mode, which may sight a new light on the fundamental problem of quantum-classical correspondence in these systems [32,33].

The paper is organized as follows. In Sec. II we describe the system and show the directed momentum current. In Sec. III we study the “quantized” acceleration mode. Summary is presented in Sec. IV.

II. DIRECTED MOMENTUM CURRENT

We consider the QKR model with the $PT$-symmetry for which the Hamiltonian in dimensionless units takes the following form

$$H = \frac{p^2}{2} + K \left[ \cos(\theta) + i \lambda \sin(\theta) \right] \sum_j \delta(t - j)$$

(1)

where $\theta$ is the angular coordinate, and $p$ is the angle momentum operator written as $p = -i\hbar_{\text{eff}} \partial/\partial \theta$, with $\hbar_{\text{eff}}$ being the effective Planck constant. The parameter $K$ is the kicking strength, while $\lambda$ is induced here to describe the strength of the imaginary part of the complex kicking potential. We denote the angle momentum eigenstate by $|\varphi_n\rangle$, with $p|\varphi_n\rangle = n\hbar_{\text{eff}}|\varphi_n\rangle$ and $(\theta|\varphi_n\rangle = e^{in\theta}/\sqrt{2\pi}$. In the basis of $|\varphi_n\rangle$, an arbitrary state can be expressed as $\psi(\theta,t) = \sum_{n=-\infty}^{+\infty} \psi_n(t)(\theta|\varphi_n\rangle$, with $\psi_n(t)$ being the wavefunction in the momentum representation. In our numerical simulations, the initial state is taken as the ground state of the unperturbed Hamiltonian $H_0 = \frac{p^2}{2}$, i.e., $\psi(\theta,0) = 1/\sqrt{2\pi}$.

The time evolution of quantum states during one period is governed by the Floquet operator

$$U(t_j, t_{j-1}) = e^{-i\frac{\lambda}{4\pi} n_{\text{eff}} e^{-i\frac{\lambda}{4\pi} \sum_j}},$$

(2)

where $V(\theta) = K \left[ \cos(\theta) + i \lambda \sin(\theta) \right]$. The eigenequation of the Floquet operator reads

$$U|\psi_\varepsilon\rangle = e^{-i\varepsilon} |\psi_\varepsilon\rangle,$$

(3)

where $\varepsilon$ indicates the quasi-energy. Quantum nonresonance corresponds to irrational values of $\hbar_{\text{eff}}/4\pi$. To numerically investigate the quasi-energy and quasi-eigenstate, we should use the finite truncation to approximate the $U_{m,n}$ matrix of infinite dimension [32]. Such an approximated method is effective, since the matrix $U_{m,n}$ has the band structure [32].

The broken of $PT$ symmetry is quantified by the appearance of the complex quasi-energy, i.e., $\varepsilon = \varepsilon_r + i\varepsilon_i$. To identify such spontaneous symmetry breaking, we numerically calculate the average value of the imaginary part of the quasi-energy

$$|\bar{\varepsilon}_j| = \frac{1}{N} \sum_{j=1}^{N} |\varepsilon_j^*|,$$

where $N$ is the dimension of the Floquet matrix and $\varepsilon_j^*$ denotes the imaginary part of the $j$-th the quasi-energy [33]. Our numerical results show that the average value $\bar{\varepsilon}_j$ is virtually zero for small $\lambda$, and it abruptly increases once the $\lambda$ exceeds a certain critical threshold, i.e., $\lambda > \lambda_c$ (see Fig. 1). This is a clear evidence of the spontaneous $PT$-symmetry breaking controlled by the parameter $\lambda$. In numerical simulations, the truncation of the $U_{m,n}$ matrix is $N = 2048$. In fact, such $PT$-symmetry breaking for the dynamical localization case has been reported in [32].

![FIG. 1. (color online). The average value of the imaginary part of the quasi-energy, i.e., $|\bar{\varepsilon}_j|$ versus the strength of the complex potential $\lambda$ for $\hbar_{\text{eff}} = 1.0$ (squares) and 1.5 (circles). The kick strength is $K = 5$.](image-url)

After studying the properties of eigenvalues of the Floquet operator, we are then to discuss the dynamical behaviour of the system, and we focus on the momentum current here. The momentum current is quantified by the mean momentum

$$\langle p(t) \rangle = \frac{\sum_n p_n |\psi_n(t)|^2}{N},$$

where $N = \sum_n |\psi_n(t)|^2$ is the norm of the quantum state [32]. For $\lambda > \lambda_c$, the appearance of the complex quasi-energies will lead to the exponentially-fast increase of norm with time. The above definition of momentum current drops the contribution from the growth of the norm to the current behaviour. We numerically investigate the momentum current for the irrational values of $\hbar_{\text{eff}}/4\pi$ with different $\lambda$. Our numerical results show that, below the phase-transition point, i.e., $\lambda < \lambda_c$ (see Fig. 2 for $\lambda = 0.06$), the momentum current saturates to a small asymptotic value after the growth during the initially...
short time interval. In fact, in the limit of $\lambda \to 0$, the wavepackets spreads symmetrically in momentum space, thus the average momentum is virtually zero. Interestingly, around the threshold value of spontaneous $\mathcal{PT}$-symmetry breaking, i.e., $\lambda \sim \lambda_c$ (e.g., $\lambda = 0.09$ in Fig. 2), there is the staircase-growth of the momentum current. Moreover, the jump of the momentum current from the lower stair to the upper one is very sharp, which implies that the system changes to different quantum states with time evolution. Above the phase-transition point, i.e., $\lambda > \lambda_c$ (e.g., $\lambda = 0.2$ in Fig. 2), the momentum current linearly increases with time, i.e., $\langle p(t) \rangle = Dt$, for which the growth rate $D$ is independent on $\lambda$ if $\lambda \gg \lambda_c$ (e.g., $\lambda = 0.6$ and 0.9 in Fig. 2). Our investigation on momentum current in dynamical localization regime may sight a new light on the understanding of the unidirectional transport phenomenon.

It is apparent that the components with $\varepsilon_i > 0$ will exponentially grow. Then, the mean momentum $\langle p \rangle$ corresponding to these eigenstates contributes mainly to the momentum current. Interestingly, we find that most of the eigenstates with positive $\varepsilon_i$ are localized in momentum space [see Fig. 3 (a)]. Moreover, the mean momentum of those eigenstates concentrates on several separate values $p_m$ [see Fig. 3 (b)]. Detailed observations show that each $p_m$ is in one-to-one correspondence to the platform of the momentum current in Fig. 2. It is evident that, during the appearance of the platform of the momentum current, the quantum state is the eigentate of the largest $\varepsilon_i$ with the same $\langle p \rangle$. The transition between the eigenstates of different mean momentum leads to the stair-case growth of the momentum current.

The underlying physics of the linear growth of the momentum current for $\lambda > \lambda_c$ is due to the gain or loss mechanism induced by the imaginary part of the kicking potential. Such mechanism happens when the Floquet operator of the non-Hermitian term $U_K^\dagger(\theta) = \exp[iK\lambda \sin(\theta)/\hbar_{\text{eff}}]$ operates on the quantum state, i.e., $U_K^\dagger \psi(\theta)$. The action can dramatically cause the annihilation of the quantum state in the region of $\theta \in (-\pi, 0)$, since in this region the value of $U_K^\dagger(\theta)$ is negative. In contrast, the probability of the particle in the region of $(0, \pi)$ will be enhanced as $U_K^\dagger(\theta)$ takes positive value in this region. For $K\lambda/\hbar_{\text{eff}} \gg 1$, the value of the $U_K^\dagger(\theta)$ is extremely large in the position of $\theta = \pi/2$. Therefore, the action of the Floquet operator $U_K^\dagger$ on a quantum state can effectively generate a quantum particle in $\theta = \pi/2$ if the center of the quantum state is not very far from this position. Indeed, our theoretical analysis proves that the wave function after each kicks can be well described by a Gaussian wave packet with the center $\theta_0 = \pi/2$.

In this position, the quantum particle experiences the kicking force of strength $K$. Therefore, the time growth of the mean momentum is roughly in the form of $\langle p(t) \rangle \propto Kt$.

![FIG. 2. (color online). Time dependence of the momentum current $\langle p \rangle$ with $\hbar_{\text{eff}} = 1$ and $K = 5.0$. From bottom to top $\lambda = 0.06$ (black), 0.09 (red), 0.2 (green), 0.6 (blue) and 0.9 (cyan).](image)

The mechanism of the staircase growth of the momentum current can be understood as follows. An arbitrary state can be expanded in the basis of the Floquet eigenstates. At the initial time, the expansion of the quantum state takes the form

$$\langle \psi(t_0) \rangle = \sum_\varepsilon C_\varepsilon |\psi_\varepsilon\rangle,$$

where $|\psi_\varepsilon\rangle$ indicates the eigenstate of the Floquet operator and $C_\varepsilon$ is components of the quantum state. After the $n$-th kicks, the quantum state has the expression

$$\langle \psi(t_n) \rangle = \sum_\varepsilon C_\varepsilon e^{-in\varepsilon t} |\psi_\varepsilon\rangle.$$

In the broken $\mathcal{PT}$ symmetry phase, the quasienergy is complex i.e., $\varepsilon = \varepsilon_r + i\varepsilon_i$. Therefore, the expansion of $|\psi(t_n)\rangle$ can be rewritten as

$$|\psi(t_n)\rangle = \sum_\varepsilon C_\varepsilon e^{in\varepsilon t} e^{-in\varepsilon t} |\psi_\varepsilon\rangle.$$

![FIG. 3. (color online). (a) The Floquet eigenstates in momentum space with $\varepsilon_i = 0.00393$ (red) and 0.008 (black) which corresponds to the two peaks of $\varepsilon_i$ in (b). (b) The imaginary part of the quasi-energy $\varepsilon_i$ versus the mean momentum $\langle p \rangle$ of the corresponding eigenstate. The parameters are $K = 5$, $\hbar_{\text{eff}} = 1$, and $\lambda = 0.09$.](image)
III. “QUANTIZED” ACCELERATION MODE

Further, we numerically investigate the acceleration rate of the momentum current, i.e., $D = \lim_{t_j \to t_f} (p(t_j))/t_f$ for $\lambda \gg \lambda_c$, where $t_f$ is the total time one can track the time evolution. Due to the linear growth of $(p(t))$, the $t_f$ of a scale of hundreds of kicking periods can ensure the precise acceleration rate. Interestingly, we find that the acceleration rate exhibits the “quantized” increment with increasing $K$ (see Fig. [1]), namely, $D = 2n\pi$ for $K \in [2n\pi - \Delta_0, 2n\pi + \Delta_0]$ with $\Delta_0 \approx \pi$ and $n \geq 1$. The mechanism of such an intriguing phenomenon is due to the coexistence of the classical acceleration mode and the “gain-or-loss” effects of the non-Hermitian potential. Remember that for $K\lambda/\hbar_{\text{eff}} \gg 1$ the action of the Floquet operator $U_{\lambda}$ on a quantum state can effectively generate a particle in the position of $\theta = \pi/2$. Indeed, our analytic analysis proves that, after the action of $U^{j}_K$, the particle moves to the position of $\theta(t_{j+1}) = \pi/2 + 2n\pi$. Accordingly, its momentum becomes to be $p(t_{n+1}) = 2n\pi$ for which the actual increment of the momentum during one-period evolution is $D = 2n\pi$. A rough estimation of $\Delta_0$ is a half of the width of the region $[2(n - 1)\pi + \pi/2, 2n\pi + \pi/2]$, i.e., $\Delta_0 \approx \pi$, which is confirmed by our numerical results in Fig [1].

We consider the classical acceleration mode of kicked rotor, which is governed by the classical mapping equation [38]

\[
\begin{align*}
   p(t_{j+1}) &= p(t_j) + K \sin[\theta(t_j)], \\
   \theta(t_{j+1}) &= \theta(t_j) + p(t_{j+1}),
\end{align*}
\]

where $\theta(t_j)$ and $p(t_j)$ denote the angle coordinate and the angular momentum after the $j$-th kick. It is easy to see that a classical trajectory with $[\theta(t_0) = \pi/2, p(t_0) = 2m\pi]$ will be accelerated linearly as time evolves, i.e., $[\theta(t_j) = \pi/2, p(t_j) = p_0 + jK]$ if $K = 2n\pi$, where $m$ and $n$ are all integers. In our model, we should also consider the effects of the imaginary part of the kicking potential on the time evolution of a classical trajectory.

Without loss of generality, we assume that at the time $t = t_j$ the position of a classical trajectory is $[\theta = \pi/2, p = 0]$. Consider the case that the kick strength has some deviation from the ideal value, i.e., $K = 2n\pi + \Delta$ with $|\Delta| < \pi$. According to Eq. (7), after one kick period, the trajectory changes to be

\[
p(t_{j+1}^-) = 2n\pi + \Delta,
\]

and

\[
\theta(t_{j+1}^-) = \frac{\pi}{2} + 2n\pi + \Delta,
\]

where the superscript ‘−’ indicates the time immediately before the action of imaginary part of the kicking potential, i.e., the $U^{j}_K$ operator. Equation (8) reveals that the value of $\Delta$ can be regarded as the distance between the center of the wavepackets and the position of $\theta = \pi/2 + 2n\pi$ which is essentially equal to $\pi/2$ due to the periodical boundary condition. Since the action of the $U^{j}_K$ operator on the wavepackets greatly enhances the probability of a particle in $\theta = \pi/2 + 2n\pi$ if the value of $\Delta$ is smaller than a threshold value, i.e., $\Delta_0$, it is reasonable to believe that, after the action of $U^{j}_K$, the particle moves to the position of $\theta(t_{j+1}) = \pi/2 + 2n\pi$. Accordingly, its momentum becomes to be $p(t_{n+1}) = 2n\pi$ for which the actual increment of the momentum during one-period evolution is $D = 2n\pi$. A rough estimation of $\Delta_0$ is a half of the width of the region $[2(n - 1)\pi + \pi/2, 2n\pi + \pi/2]$, i.e., $\Delta_0 \approx \pi$, which is confirmed by our numerical results in Fig [1].

IV. SUMMARY

In this work, we investigate the directed current of the quantum kicked rotor model whose kicking potential satisfies the $PT$ symmetric condition. We find that in the vicinity of phase-transition point, i.e., $\lambda \approx \lambda_c$, the eigenstates is well localized in momentum space. Moreover, the mean momentum eigenstates with positive real imaginary parts of the quasi-energy concentrates on several separate values. Such property leads to the staircase growth of the momentum current $\langle p(t) \rangle$ with time. When the parameter $\lambda$ is larger than the phase-transition point, i.e., $\lambda > \lambda_c$, the momentum current linearly increases with time, i.e., $\langle p(t) \rangle = Dt$. We make extensive investigations on the acceleration rate $D$ for $\lambda \gg \lambda_c$. Interestingly, we find that, for $K\lambda/\hbar_{\text{eff}} \gg 1$, the acceleration rate exhibits the “quantized” increment with the increase of $K$, i.e., $D = 2n\pi$ for $K \in [2n\pi - \Delta_0, 2n\pi + \Delta_0]$ with $\Delta_0 \approx \pi$ and $n \geq 1$. For $K\lambda/\hbar_{\text{eff}} \gg 1$, our analytic analysis proves that, at any time $t = t_n$, the wavepackets can be well described by the Gaussian function with a center $(\theta = \pi/2, \bar{p} = 2n\pi)$. The motion of the wavepacket in

![Fig. 4. (color online). The acceleration rate $D$ versus $K$ for $b_{\text{eff}} = 0.1$. Dashed-dotted line (in red) indicates the function of the form $D(K) = K$. Dashed lines (in blue) mark the transition points.](image-url)
phase space follows the classical acceleration mode of the trajectory of the kicked rotor model. The theory of the modified acceleration mode of the classical particle by the gain-or-loss mechanism of the complex kicking potential can successfully explain such “quantized” phenomenon of momentum current. Our results are significantly important in the quantum control of the ratchets acceleration of wavepackets.

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Appendix A: The acceleration of momentum current

The Floquet operator of the non-Hermitian QKR reads

$$U = U_f U_K^i U_K^i,$$  \hspace{1cm} (A1)

with the free evolution operator \( U_f = \exp(-ip^2/2\hbar_{\text{eff}}) \), the evolution operator of the real part of the kicking potential

$$U_K^r = \exp(-iK\hbar_{\text{eff}} \cos(\theta)),$$  \hspace{1cm} (A2)

and that of the imaginary part

$$U_K^i = \exp(iK\hbar_{\text{eff}} \sin(\theta)).$$  \hspace{1cm} (A3)

The maximum value of \( U_K^i \) corresponds to \( \theta_0 = \pi/2 \). In condition that \( \hbar_{\text{eff}} \to 0 \) (with \( \lambda K/\hbar_{\text{eff}} \gg 1 \) and \( K/\hbar_{\text{eff}} \gg 1 \)), the expansions of first order for \( U_K^r \) and \( U_K^i \) around \( \theta_0 \) take the form

$$U_K^r \approx \exp\left[-\frac{\lambda K(\theta - \theta_0)^2}{2\hbar_{\text{eff}}}\right] \exp\left(\frac{\lambda K}{\hbar_{\text{eff}}}\right),$$  \hspace{1cm} (A4)

and

$$U_K^i \approx \exp\left[iK(\theta - \theta_0)/\hbar_{\text{eff}}\right].$$  \hspace{1cm} (A5)

As a further step, we consider the time evolution of a quantum state under the action of above operators.

Without loss of generality, we assume that the initial state is a Gaussian wavepacket

$$\psi(\theta, t_0) = \frac{1}{(\sigma^2\pi)^{1/4}} \exp\left(-\frac{\theta^2}{2\sigma^2} + \frac{ip_0\theta}{\hbar_{\text{eff}}}\right),$$  \hspace{1cm} (A6)

for which the uncertainty relation is such that \( \delta\theta\delta p = \hbar_{\text{eff}}/2 \). The evolution of quantum states from \( t_0 \) to \( t_1 = t_0 + 1 \) is governed by \(|\psi(t_1)\rangle = U_K|\psi(t_0)\rangle\). The action of \( U_K^r \) on \( \psi(\theta, t_0) \) yields that

$$\tilde{\psi}^r(\theta, t_0) \propto \exp\left[-\frac{\lambda K(\theta - \theta_0)^2}{2\hbar_{\text{eff}}} - \frac{\theta^2}{2\sigma^2} + \frac{ip_0\theta}{\hbar_{\text{eff}}}\right].$$  \hspace{1cm} (A7)

In condition that \( \lambda K/\hbar_{\text{eff}} \gg 1/\sigma^2 \), we can neglect the contribution of the second term in right-hand side of the above equation. Then, the quantum state can be approximated as

$$\tilde{\psi}^r(\theta, t_0) \propto \exp\left[-\frac{\lambda K(\theta - \theta_0)^2}{2\hbar_{\text{eff}}} + \frac{ip_0\theta}{\hbar_{\text{eff}}}\right].$$  \hspace{1cm} (A8)

For the convenience of analysis, hereafter, we use this state as the initial state for the time evolution. That is to say the Floquet operator is redefined as

$$U = U_K^i U_f U_K^i.$$  \hspace{1cm} (A9)

By comparison Eq. (A9) with Eq. (A1), one can see that the sequence of the action of \( U_K^i \) and \( U_f \) in time evolution is rearranged, which actually has no effect on physical results.

Consider the action of \( U_K^i \) in Eq. (A3) on \( \tilde{\psi}(\theta, t_0) \),

$$\psi(x, t_0^+) = U_K^i \tilde{\psi}(\theta, t_0) \propto \exp\left[-\frac{\lambda K(\theta - \theta_0)^2}{2\hbar_{\text{eff}}} + \frac{ip_0\theta}{\hbar_{\text{eff}}}\right] \times \exp\left(\frac{i}{\hbar_{\text{eff}}} p_0\theta\right),$$  \hspace{1cm} (A10)

where the superscript ‘+’ indicates the time immediately after the action of the real part of the kicking potential. Next step is the action of the free evolution operator \( U_f \) on the quantum state. Before that we should transform the state to momentum space,

$$\psi(p, t_0^+) = \int_{-\infty}^{\infty} \psi(x, t_0^+) \exp(-ipx/\hbar_{\text{eff}}) dx \propto \exp\left[-\frac{(p - p_K)^2}{2\hbar_{\text{eff}}\lambda K} - \frac{ip_0\theta}{\hbar_{\text{eff}}}\right],$$  \hspace{1cm} (A11)

where \( p_K = p_0 + K \). Then, the action of \( U_f \) on \( \psi(p, t_0^+) \) yields

$$\psi(p, t_1^-) = U_f \psi(p, t_1^+) \propto \exp\left[-\frac{(p - p_K)^2}{2\hbar_{\text{eff}}\lambda K} - \frac{ip_0(\theta + \pi/2)}{\hbar_{\text{eff}}}\right],$$  \hspace{1cm} (A12)

where the superscript ‘-’ indicates the time immediately before the action of the kicking operator \( U_K^i \).

The corresponding quantum state in real space is

$$\psi(\theta, t_1^-) \propto \int_{-\infty}^{\infty} dp \psi(p, t_1^-) \exp(ip\theta/\hbar_{\text{eff}}) \propto \exp\left[-\frac{\theta - \theta_0 - p_K}{2\hbar_{\text{eff}}\lambda K}\right] \times \exp\left(i\frac{\theta - \theta_0 + \frac{p_K}{\lambda K\sigma^2}}{\hbar_{\text{eff}}}\right).$$  \hspace{1cm} (A13)

We assume that

$$p_K = 2n_0\pi + \Delta,$$  \hspace{1cm} (A14)
with $-\pi < \Delta < \pi$. Then, the quantum state in Eq. (A13) is rewritten as

$$
\psi(\theta, t_1^-) \propto \exp \left\{ \frac{[\theta - (\theta_0 + 2n_0\pi) - \Delta]^2}{2\hbar_{\text{eff}}\lambda\hbar} \right\} \times \exp \left[ i \left( \theta - \theta_0 + \frac{\pi K}{\lambda\hbar} \right)^2 \right].
$$

(Apparently, Eq. (A16) indicates a Gaussian wavepackets with the center $\theta = \theta_0 + 2n_0\pi + \Delta$. If the distance between $\theta$ and $\theta_0 + 2n_0\pi$ is smaller than a threshold value $\Delta_0$, the action of $U_{K}^{\pm}$ on the quantum state $\psi(\theta, t_1^-)$ can effectively enhance the probability of a particle in the position of $\theta = \theta_0 + 2n_\pi$. Then, we get

$$
\psi(\theta, t_1^-) = U_{K}^{\pm} \psi(\theta, t_1^-)
$$

(A16)

where the superscript ‘+’ indicates the time immediately after the action of $U_{K}^{+}$. A rough estimation of $\Delta_0$ is a half of the width of the region $[2(n-1)\pi + \theta_0, 2n\pi + \theta_0]$, i.e., $\Delta_0 \approx \pi$. Now, the time evolution of a quantum state during an entire kicking period ends.

It is apparent that the center of this wavepacket in real space is

$$
\bar{\theta}_t = 2n_0\pi + \theta_0.
$$

(A17)

with the second moment

$$
\delta\theta = \sqrt{\bar{\theta}^2 - (\bar{\theta})^2} = \sqrt{\frac{\hbar_{\text{eff}}\lambda\hbar}{2K}}.
$$

(A18)

To obtain the momentum center of the wavepacket $\psi(\theta, t_1^-)$, we should transform it to momentum space,

$$
\psi(p, t_1^+) \propto \int d\theta e^{-ip\theta/\hbar} \psi(\theta, t_1^-)
$$

(A19)

$$
\propto \exp \left[ \frac{(p - 2n_0\pi)^2}{2\hbar_{\text{eff}}\lambda\hbar} + if(p) \right],
$$

where $f(p)$ is an unimportant function of momentum. The function $f(p)$ does not determine the mean momentum, hence it is no need to know the specific form of $f(p)$. From the above quantum state, we can get the mean momentum

$$
\bar{p}_t = 2n_0\pi.
$$

(A20)

with the second moment

$$
\delta p = \sqrt{\bar{p}^2 - (\bar{p})^2} = \sqrt{\frac{\hbar_{\text{eff}}\lambda K}{2}}.
$$

(A21)

One can find that the quantum state $|\psi(t_1)\rangle$ satisfies the uncertainty relation

$$
\delta\theta\delta p = \frac{\hbar_{\text{eff}}}{2},
$$

(A22)

which is same as that of the initial wavepacket.

As a brief summary, the quantum state after the evolution of first kicking period can be well described by a Gaussian wavepacket,

$$
\psi(\theta, t_1) \approx \exp \left[ \frac{-\lambda K (\theta - \bar{\theta}_t)^2}{2\hbar_{\text{eff}}} + i\bar{p}_t \theta \right],
$$

(A23)

where

$$
\begin{align*}
\bar{\theta}_t &= 2n_0\pi + \theta_0, \\
\bar{p}_t &= 2n_0\pi,
\end{align*}
$$

(A24)

with $\theta_0 = \pi/2$. The time evolution of quantum state for $t > t_1$ just repeats the above procedure. Accordingly, the quantum state at any time $t = t_n$ can be well approximated by a Gaussian wavepacket. Moreover, we can get the its center $(\bar{\theta}_{t_n}, \bar{p}_{t_n})$ by using the iterative method.

Firstly, we can get the center $(\bar{\theta}_{t_2}, \bar{p}_{t_2})$ at the time $t = t_2$. Taking accounting to $\bar{p}_{t_1} = 2n_0\pi$ [see Eq. (A20)], for $K = 2n_\pi + \Delta (-\pi < \Delta < \pi)$, then we arrive at $\bar{p}_{t_1} + K = 2(n_0 + n)\pi + \Delta$. Note that the value of $\Delta$ is smaller than a threshold value $\Delta_0$. By repeating the procedure of the derivation for $(\bar{\theta}_{t_1}, \bar{p}_{t_1})$, one obtains that

$$
\begin{align*}
\bar{\theta}_{t_2} &= 2(n_0 + n)\pi + \theta_0, \\
\bar{p}_{t_2} &= 2(n_0 + n)\pi.
\end{align*}
$$

(A25)

By using the same method, one can find

$$
\begin{align*}
\bar{\theta}_{t_j} &= 2[n_0 + (j-1)n]\pi + \theta_0, \\
\bar{p}_{t_j} &= 2[n_0 + (j-1)n]\pi.
\end{align*}
$$

(A26)

It is evident that the acceleration rate is

$$
D = \bar{p}_{t_j} - \bar{p}_{t_{j-1}} = 2n_\pi,
$$

for $K \in [2n_\pi - \Delta_0, 2n_\pi + \Delta_0]$ with $\Delta_0 \approx \pi$. Our analytic analysis is confirmed by numerical results.

[1] C. M. Bender and S. Boettcher, Real Spectra in Non-Hermitian Hamiltonians Having $\mathcal{PT}$ Symmetry, Phys. Rev. Lett. 80, 5243 (1998).
[2] V. V. Konotop, J. Yang, D. A. Zezyulin, Nonlinear waves in $\mathcal{PT}$-symmetric systems, Rev. Mod. Phys. 88, 035002 (2016).

[3] M. Kozlov, and G. Tsironis, Control of power in parity-time symmetric lattices, New J. Phys. 17 105004 (2015).

[4] K. G. Makris, R. El-Ganainy, D. N. Christodoulides, and Z. H. Musslimani, Beam Dynamics in $\mathcal{PT}$ Symmetric Optical Lattices, Phys. Rev. Lett. 100, 103904 (2008).

[5] S. Longhi, Dynamic localization and transport in complex crystals, Phys. Rev. B80, 235102 (2009).

[6] P. Hänggi and F. Marchesoni, Artificial Brownian motors: Controlling transport on the nanoscale, Rev. Mod. Phys. 81, 387 (2009).

[7] J. Gong, and P. Brumer, Generic Quantum Ratchet Accelerator with Full Classical Chaos, Phys. Rev. Lett. 97, 240602 (2006).

[8] J. Pelc, J. Gong, Paul Brumer, Chaos and Correspondence in Classical and Quantum Hamiltonian Ratchets: A Heisenberg Approach, Phys. Rev. E79, 066207 (2009).

[9] J. Gong and P. Brumer, Directed anomalous diffusion without a biased field: A ratchet accelerator, Phys. Rev. E 70, 016202 (2004).

[10] J. Wang, and J. Gong, Quantum Ratchet Accelerator without a Bichromatic Lattice Potential, Phys. Rev. E78, 036219 (2008).

[11] M. Sadgrove, M. Horikoshi, T. Sekimura, and K. Nakagawa, Rectified Momentum Transport for a Kicked Bose-Einstein Condensate, Phys. Rev. Lett. 99, 043002 (2007).

[12] S. Dadras, A. Gresch, C. Groisenu, S. Winberger, and G. Summy, Quantum Walk in Momentum Space with a Bose-Einstein Condensate, Phys. Rev. Lett. 121, 070402 (2018).

[13] G. Summy, and S. Winberger, Quantum random walk of a Bose-Einstein condensate in momentum space, Phys. Rev. A93, 023638 (2016).

[14] J. Ni, S. Dadras, W. Lam, R. K. Shrestha, M. Sadgrove, S. Winberger, and G. Summy, Hamiltonian Ratchets with Ultra-Cold Atoms, Ann. Phys. 520(8), 1600335 (2017).

[15] S. Kohler, J. Lehmann, and P. Hänggi, Driven quantum transport on the nanoscale, Phys. Rep. 406, 379 (2005).

[16] F. Jülicher, A. Ajdari, and J. Prost, Modeling molecular motors, Rev. Mod. Phys. 69, 1269 (1997).

[17] J. Lehmann, S. Kohler, P. Hänggi, and A. Nitzan, Molecular Wires Acting as Coherent Quantum Ratchets, Phys. Rev. Lett. 88, 228305 (2002).

[18] A. Kenfack, J. Gong, and A. K. Pattanayak, Controlling the Ratchet Effect for Cold Atoms, Phys. Rev. Lett., 100, 044104 (2008).

[19] E. Lundh, and M. Wallin, Ratchet Effect for Cold Atoms in an Optical Lattice, Phys. Rev. Lett. 94, 110603 (2005).

[20] W. L. Zhao, L. B. Fu, and J. Liu, Nonlinearity effects on the directed momentum current, Phys. Rev. E90, 022907 (2014).

[21] J. Wang, and J. Gong, Proposal of a Cold-atom Realization of Quantum Maps with Hofstadter’s Butterfly Spectrum, Phys. Rev. A77, 034105(R) (2008).

[22] J. Wang, I. Guarnieri, G. Casati, J. Gong, Long-lasting Exponential Spreading in Periodically Driven Quantum Systems, Phys. Rev. Lett. 107, 234104 (2011); H. L. Wang, J. Wang, I. Guarnieri, G. Casati, J. Gong, Exponential Quantum Spreading in a Class of Kicked Rotor Systems near High-Order Resonances, Phys. Rev. E88, 052919 (2013); W. Zhao, J. Gong, W. Wang, G. Casati, J. Liu and L. Fu, Exponential wave-packet spreading via self-interaction time modulation, Phys. Rev. A94 053631 (2016).

[23] D. Y. H. Ho, and J. Gong, Quantized Adiabatic Transport in Momentum Space, Phys. Rev. Lett. 109 010601 (2012); H. Wang, D. Y. H. Ho, W. Lawton, J. Wang, J. Gong, Kicked-Harper model versus On-Resonance Double Kicked Rotor Model: From Spectral Difference to Topological Equivalence, Phys. Rev. E88, 052920 (2013).

[24] M. Sadgrove, T. Schell, K. Nakagawa, and S. Winberger, Engineering quantum correlations to enhance transport in cold atoms, Phys. Rev. A87, 013631 (2013).

[25] C. Keller, M. K. Oberthaler, R. Abfalterer, S. Bernet, J. Schmiedmayer, and A. Zeilinger, Tailored Complex Potentials and Friedel’s Law in Atom Optics, Phys. Rev. Lett. 79, 3327 (1997).

[26] Z. Zhang, Y. Zhang, J. Sheng, L. Yang, M. Miri, D. N. Christodoulides, B. He, Y. Zhang, and M. Xiao, Observation of Parity-Time Symmetry in Optically Induced Atomic Lattices, Phys. Rev. Lett. 117, 123601 (2016).

[27] J. Li, A. K. Harter, J. Liu, L. d. Melo, Y. N. Joglekar, L. Luo, Observation of parity-time symmetry breaking transitions in a dissipative Floquet system of ultracold atoms, quant-ph, arXiv:1608.05601

[28] M. Kreibich, J. Main, H. Cartarius, and G. Wunner, Tilted optical lattices with defects as realizations of $\mathcal{PT}$ symmetry in Bose-Einstein condensates, Phys. Rev. A93, 023624 (2016), and references therein.

[29] M. Kreibich, J. Main, H. Cartarius, and G. Wunner, Hermitian four-well potential as a realization of a $\mathcal{PT}$-symmetric system, Phys. Rev. A87, 051601(R) (2013).

[30] M. Kreibich, J. Main, H. Cartarius, and G. Wunner, Realizing $\mathcal{PT}$-symmetric non-Hermiticity with ultracold atoms and Hermitian multwell potentials, Phys. Rev. A90, 033630 (2014).

[31] C. Hang, and G. Huang, Parity-time symmetry along with nonlocal optical solitons and their active controls in a Rydberg atomic gas, Phys. Rev. A98, 043840 (2018).

[32] C. T. West, T. Kottos, and T. Prosen, $\mathcal{PT}$-Symmetric Wave Chaos, Phys. Rev. Lett. 104, 054102 (2010).

[33] S. Longhi, Localization, quantum resonances, and ratchet acceleration in a periodically kicked $\mathcal{PT}$-symmetric quantum rotor, Phys. Rev. A95 012125 (2017).

[34] C. M. Bender, H. F. Jones, Quantum counterpart of spontaneously broken classical $\mathcal{PT}$ symmetry, J. Phys. A 44 015301 (2011).

[35] A Mostafazadeh, Pseudo-Hermitian Representation of Quantum Mechanics, Int. J. Geom. Meth. Mod. Phys. 7, 1191 (2010).

[36] F. M. Izrailev, Chaotic structure of eigenfunctions in systems with maximal quantum chaos, Phys. Lett. A 125, 250 (1987); F. M. Izrailev, Simple models of quantum chaos: spectrum and eigenfunctions, Phys. Rep. 196 299 (1990).

[37] See Appendix for detailed discussions about the properties of the quasi-eigenstates and for the derivation of the “quantized” acceleration rate of the momentum current.

[38] B. V. Chirikov, A universal instability of many-dimensional oscillator systems, Phys. Rev. 52 263 (1979).