ON FINITE DIFFERENCE SCHEMES FOR PARTIAL INTEGRO-DIFFERENTIAL EQUATIONS OF LÉVY TYPE

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Abstract. In this article we introduce a finite difference approximation for integro-differential operators of Lévy type. We approximate solutions of integro-differential equations, where the second order operator is allowed to degenerate. In the existing literature, the Lévy operator is treated as a zero/first order operator outside of a centered ball of radius $\delta$, leading to error estimates of order $\xi(\delta) + N(\delta)(h + \sqrt{\tau})$, where $h$ and $\tau$ are the spatial and temporal discretization parameters respectively. In these estimates $\xi(\delta) \downarrow 0$, but $N(\delta) \uparrow \infty$ as $\delta \downarrow 0$. In contrast, we treat the integro-differential operator as a second order operator on the whole unit ball. By this method we obtain error estimates of order $(h + \tau^k)$ for $k \in \{1/2, 1\}$, eliminating the additional errors and the blowing up constants. Moreover, we do not pose any conditions on the Lévy measure.

1. Introduction

In the present article we consider a finite difference approximation scheme for the partial integro-differential equation (PIDE)

$$du_t(x) = [(L_t + J)u_t(x) + f_t(x)] \, dt, \quad (t, x) \in [0, T] \times \mathbb{R},$$

$$u_0(x) = \psi(x), \quad x \in \mathbb{R}, \quad (1.2)$$

where the operators are given by

$$L_t\phi(x) = a_t(x)\partial_x^2\phi(x) + b_t(x)\partial_x\phi(x) + c_t(x)\phi(x),$$

$$J\phi(x) = \int_{\mathbb{R}} \left( \phi(x + z) - \phi(x) - I_{|z| \leq 1} z\partial_x\phi(x) \right) \nu(dz).$$

and the coefficient of the second derivative in $L_t$ is allowed to degenerate. Here $\nu$ denotes a Lévy measure on $\mathbb{R}$, that is a Borel measure on $\mathbb{R}$ such that

$$\nu(\{0\}) = 0, \quad \int_{\mathbb{R}} 1 \wedge z^2 \nu(dz) < \infty.$$ 

Equations of this form are of importance, since are satisfied by certain functionals of jump-diffusion Markov processes, that are known to be of interest in mathematical finance (for further reading on the subject we refer to [1]).

Finite difference schemes for equations of this form have previously been studied in [2], [6] and [14]. In these articles the integro-differential operator is either truncated, or approximated by a second order difference operator in a neighborhood around the origin of radius $\delta > 0$, and the remaining operator
(the integral over \( \{|z| \geq \delta\} \)) is treated as a zero/first order operator. In [2], the solution \( u \) is first approximated by \( u^\delta \), the solution of the corresponding equation where the integral operator over \( \{|z| \leq \delta\} \) is replaced by a second order operator (resulting to a non-degenerate equation), and \( u^\delta \) is in turn approximated using a finite difference scheme by \( u^{\delta,h} \), the solution of the corresponding equation where the integral operator over \( \{|z| \leq \delta\} \) is replaced by a second order operator (resulting to a non-degenerate equation), and \( u^{\delta,h} \) is in turn approximated using a finite difference scheme by \( u^{\delta,h,\tau} \), where \( h \) and \( \tau \) are the spatial and temporal discretization parameters respectively. This leads to estimates of the form \( \|u - u^{\delta,h,\tau}\| \leq N f(\delta) + N(\delta)(\sqrt{\tau} + h) \) where \( f(\delta) = \int_{|z| \leq \delta} z^3 \nu(dz)/\int_{|z| \leq \delta} z^2 \nu(dz) \). In this estimate, the constant \( N(\delta) \) depends on \( \delta \) and blows up as \( \delta \to 0 \) at a rate of \( \nu(\{|z| \geq \delta\}) \), which is a consequence of the fact that the integro-differential operator is treated as a first/zero order operator away from the ball \((-\delta,\delta)\). In a similar manner in [14], \( \delta \) is a function of \( h \), and the corresponding convergence rate for the spatial approximation is of order \( h^{\kappa(h/2)} \) where \( \kappa(\delta) := \int_{[-1,1] \setminus (-\delta,\delta)} |z| \nu(dz) \). If then for example the Lévy measure has a density of the form \( |z|^{-(2+\alpha)} \) for some \( \alpha \in (0,1) \), then the convergence is of order \( h^{(1-\alpha)} \), which can be very slow, depending on \( \alpha \). The approach in [6] is also similar (truncation of the integro-differential operator near zero). Under some technical conditions posed on the Lévy measure (it is assumed to have a density of a particular form, that is twice continuously differentiable and has a prescribed behavior near zero), similar estimates are obtained (with constants blowing up as the truncation parameter \( \delta \to 0 \)).

In contrast to these works, in the present article we do not truncate the operator near the origin. We introduce an approximation that treats the integro-differential operator as a second order operator on the whole unit ball. Our approximation is similar to the one that we introduced in [3], [5]. However, in these works the results and their proofs rely on the non-degeneracy of the second order differential operator. We show that the approximate operator \( J^h \) that we suggest here is negative semi-definite, and this combined with estimates obtained in [9] for the difference operators lead to apriori estimates of the solution of the scheme independent of the discretization parameters, without posing a non-degeneracy condition. This, combined with consistency estimates for the operators lead to estimates of the form \( \|u - u^{h,\tau}\| \leq N(h + \sqrt{\tau}) \) where \( N \) depends only on the data of the equation. We also show, under some more spatial regularity of the data, that \( \|u - u^{h,\tau}\| \leq N(h + \tau) \). Also, let us note here that we do not pose any additional assumption on the Lévy measure \( \nu \).

The analysis of the spatial approximation is done in the spirit of [15]. The equations are first discretized in space and solved as equations in Sobolev spaces over \( \mathbb{R} \) \((u^h)\) and as equations on the grid \((v^h)\). Error estimates are obtained in Sobolev norms for the difference \( u - u^h \). By embedding theorems, the restriction of \( u^h \) on the grid is shown to agree with \( v^h \). Hence, the error estimates in Sobolev norm imply pointwise error estimates for the difference \( u - v^h \), by virtue of Sobolev embedding theorems. The discretized equations are further discretized in time (see also [8]), they are solved in Sobolev spaces.
(u^{h,\tau}) and on the grid (v^{h,\tau}), and estimates are obtained for u^h - u^{h,\tau}, which in turn imply estimates for v^h - v^{h,\tau}.

For degenerate equations not involving non-local operators we refer to [12], [10], [9] and [7] where acceleration is also obtained in the convergence with respect to the spatial discretization parameter by means of Richardson’s extrapolation. In the last three articles the results are obtained in a more general, stochastic setting, but the results remain optimal for deterministic equations as well.

In conclusion let us introduce some notation. By $u_t(x)$ we denote the value of a function $u : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ at $(t, x) \in [0, T] \times \mathbb{R}$ and when $u$ is understood as a function of $t$ with values in some function space (function of $x \in \mathbb{R}$) we will write $u_t := u_t(\cdot)$ for $t \in [0, T]$. By $\partial_x$ we denote the derivative operator with respect to the spatial variable. The notation $C^\infty_c$ stands for the set of all smooth, compactly supported, real functions on $\mathbb{R}$. We denote by $(\cdot, \cdot)$ and $\| \cdot \|_{L^2}$ the inner product and the norm respectively in $L^2(\mathbb{R})$. For an integer $l \geq 0$, $H^l$ will be the Sobolev space of all function in $L^2(\mathbb{R})$ having distributional derivatives up to order $l$ in $L^2(\mathbb{R})$, with the inner product

$$(f,g)l = \sum_{j=0}^{l} (\partial_x^j f, \partial_x^j g),$$

and we denote the corresponding norm by $\| \cdot \|_{H^l}$. For real number $\alpha, \beta$, we use the notation $\alpha \wedge \beta := \min\{\alpha, \beta\}$. We use the notation $N$ for constants that may change from line to line. In the proofs of lemmas/theorems, the dependence of $N$ to certain parameters is given at the statement of the corresponding lemma/theorem.

2. Formulation of the main results

In this section we introduce our scheme and we state our main results. From now on we will use the following notations

$$\mu_0 := \nu(\mathbb{R} \setminus [-1, 1]), \quad \mu_2 := \int_{|z| \leq 1} z^2 \nu(dz).$$

Assumption 2.1. Let $m \geq 1$ be an integer.

i) The functions $a, b, c : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ are measurable in $(t, x)$. The functions $b, c$ and the function $a$, together with their spatial derivatives up to order $m$ and up to order $\max(m, 2)$ respectively, are continuous in $x \in \mathbb{R}$ and bounded in magnitude by a constant $K$, uniformly in $t \in [0, T]$.

ii) The initial condition $\psi$ belongs to $H^m$ and $f : [0, T] \rightarrow H^m$ is a measurable function such that

$$K_m^2 = \|\psi\|^2_{H^m} + \int_0^T \|f_t\|^2_{H^m} dt < \infty.$$

Assumption 2.2. For all $(t, x) \in [0, T] \times \mathbb{R}$, we have $a_t(x) \geq 0.$
Notice that for $\phi, \varphi \in C_c^\infty$, by virtue of Taylor's formula and integration by parts we have

$$
(J\phi, \varphi) = -\int_{|z| \leq 1} \int_0^1 (1 - \theta)z^2 (\partial_x \phi(\cdot + \theta z), \partial_x \varphi) d\theta \nu(dz)
+ \int_{|z| > 1} (\phi(\cdot + z) + \phi, \varphi) \nu(dz).
$$

The solution of (1.1)-(1.2) is understood in the following sense.

**Definition 2.1.** An $H^1$-valued weakly continuous function $(u_t)_{t \in [0,T]}$ is a solution to (1.1)-(1.2) if for all $\phi \in C_c^\infty$

$$
(u_t, \phi) = (\psi, \phi) + \int_0^t (\partial_x u_s, -\phi \partial_x a_t - a_s \partial_x \phi + b_s \phi) + (c_s u_s, \phi) \, ds
- \int_0^t \int_{|z| \leq 1} \int_0^1 (1 - \theta)z^2 (\partial_x u_s(\cdot + \theta z), \partial_x \phi) d\theta \nu(dz) \, ds
+ \int_0^t \int_{|z| > 1} (u_s(\cdot + z) + u_s, \phi) \nu(dz) \, ds.
$$

The following well-posedness result can be found in [4] and [13].

**Theorem 2.1.** Let Assumptions 2.1 and 2.2 hold. Then (1.1)-(1.2) has a unique solution $u : [0,T] \rightarrow H^1$. Moreover, $u_t$ belongs to $H^m$ for all $t \in [0,T]$, it is weakly continuous as $H^m$-valued function, strongly continuous as function with values $H^{m-1}$, and the following estimate holds

$$
\sup_{t \leq T} \|u_t\|_{H^m}^2 \leq NK^2,
$$

where $N$ is a constant depending only on $T$, $m$, $K$, $\mu_0$ and $\mu_2$.

**Remark 2.1.** If Assumption 2.1 holds with $m \geq 2$ in the above theorem, then the solution is strongly continuous $H^1$ valued function, which by the continuous embedding $H^1 \hookrightarrow C^{0,1/2}$ (space of bounded 1/2-Hölder continuous functions with the usual norm) implies that the solution $u_t(x)$ is a continuous function of $(t,x) \in [0,T] \times \mathbb{R}$.

For $\lambda \in \mathbb{R} \setminus \{0\}$ we define the following operators

$$
\delta_\lambda \phi(x) := \frac{\phi(x + \lambda) - \phi(x)}{\lambda}, \quad \delta^\lambda \phi(x) := \frac{(\delta_\lambda + \delta_{-\lambda}) \phi(x)}{2}.
$$

We continue with the approximation of the integro-differential operator. For $h \in (0,1)$ we will denote our grid by $\mathbb{G}_h := h\mathbb{Z}$, and for integers $k \geq 1$ we define

$$
B_k^h := ((k-1)h, kh],
$$

while for integers $k \leq -1$ we define

$$
B_k^h := [kh, (k-1)h).
$$
Notice that $B_0^h$ is not defined. From now on we assume that $h \in \{1/n : n \in \mathbb{N}_+\} =: \mathfrak{N}$. We set $A_h := \{m \in \mathbb{Z} : |m| \leq 1/h, m \neq 0\}$ and $B_h := \mathbb{Z} \setminus (A_h \cup \{0\})$. Let us define the operators

$$
J_1^h \phi(x) := \sum_{k \in A_h} \zeta_k^h \sum_{l=0}^{\lfloor |k|/h \rfloor} \theta_k^l \delta_{-h} \delta_h \phi(x + s_k lh),
$$

$$
J_2^h \phi(x) := \sum_{k \in B_h} \left( \phi(x + hk) - \phi(x) \right) \nu(B_k^h),
$$

where

$$
s_k = \frac{k}{|k|}, \quad \zeta_k^h := \int_{B_k^h} z^2 \nu(dz), \quad \theta_k^l := \int_{l/|k|}^{(l+1)/|k|} (1 - \theta) \, d\theta.
$$

We denote $J^h := J_1^h + J_2^h$. The differential operator $L_t$ is approximated by $L_t^h$, given by

$$
L_t^h \phi(x) := a_t(x) \delta^h \phi(x) + b_t(x) \delta^h \phi(x) + c_t \phi(x).
$$

We will write $l_2(G_h)$ for the set of all real valued function $\phi$ on $G_h$ such that

$$
\|\phi\|^2_{l_2(G_h)} := h \sum_{x \in G_h} |\phi(x)|^2 < \infty.
$$

We will denote the corresponding inner product by $(\cdot, \cdot)_{l_2(G_h)}$. Let us now consider in $l_2(G_h)$ the scheme

$$
dv_t^h = \left( (L_t^h + I^h)v_t^h + f_t \right) dt
$$

$$
v_0^h = \psi.
$$

**Remark 2.2.** For $l \geq 1$ we have the continuous embedding $H^l \hookrightarrow l_2(G_h)$ (see [9]). Therefore under Assumption 2.1 we have

$$
\|\phi\|^2_{l_2(G_h)} + \int_0^T \|f_t\|^2_{l_2(G_h)} dt < \infty.
$$

Under the same assumption it is easy to see that $L_t^h + J$ is a bounded linear operator on $l_2(G_h)$ into itself (with norm bounded by a constant uniformly in $t \in [0, T]$). Hence under Assumption 2.1, (2.3)-(2.4) has a unique solution, that is, a continuous function $v : [0, T] \rightarrow l_2(G_h)$ such that for all $t \in [0, T]

$$
v_t^h = \phi + \int_0^t (L_t^h + J)v_s^h + f_s \, ds,
$$

where the equality is understood in $l_2(G_h)$ (hence, also for all $x \in G_h$).

Next is our main result concerning the spatial approximation.
Theorem 2.2. Let Assumptions [2.1] and [2.2] hold with \( m \geq 4 \). Let \( u \) and \( v^h \) be the unique solutions of \([1.1] - [1.2]\) and \([2.3] - [2.4]\) respectively. The following estimate holds,

\[
\sup_{t \in [0,T]} \sup_{x \in G_h} |u_t(x) - v^h_t(x)|^2 + \sup_{t \in [0,T]} \|u_t - v^h_t\|_{L^2(G_h)}^2 \leq N h^2 K_m^2, \]

where \( N \) is a constant depending only on \( m, K, \mu_0, \mu_2 \) and \( T \).

We now move to the temporal discretization. Let \( n \geq 1 \) be an integer and let \( \tau = T/n \). In \( L^2(G_h) \) we consider the implicit scheme

\[
v_{i+1} = v_i + \tau [(L_i^h + J_i^h)v_i + f_{i\tau}], \quad i = 1, ..., n \tag{2.5}
v_0 = \psi. \tag{2.6}
\]

Theorem 2.3. Let Assumptions [2.1] and [2.2] hold. There exists a constant \( N_0 \) depending only on \( K \) and \( T \), such that for any \( h \in \mathfrak{H} \), if \( n > N_0 \), then \([2.5] - [2.6]\) has a unique solution \( (v^h)_{i=0}^n \).

Assumption 2.3. Let \( l \geq 0 \) be an integer. There exist constants \( C \) and \( \gamma > 0 \) such that

\[
|\partial_x^2 a_t(x) - \partial_x^2 a_s(x)|^2 + |\partial_x^2 b_t(x) - \partial_x^2 b_s(x)|^2 + |\partial_x^2 c_t(x) - \partial_x^2 c_s(x)|^2 \leq C|t - s|^{\gamma}
\]

and

\[
\|f_t - f_s\|_{H^l}^2 \leq C|t - s|^{\gamma}
\]

for all \( x \in \mathbb{R}, t, s \in [0, T] \), and \( 0 \leq j \leq l \).

Assumption 2.4. There exists a constant \( K' \), such that for all \( t \in [0,T] \) we have \( \|f_t\|_{H^{m-2}} \leq K' \).

Next is our result concerning the temporal approximation.

Theorem 2.4. Let Assumptions [2.1] - [2.2] hold with \( m \geq 4 \) and let Assumption [2.3] hold with \( l \geq 1 \). Let \((v^h_i)_{i=0}^n\) and \((\psi)_{i=0}^n\) be the unique solutions of equations \([2.3] - [2.4]\) and \([2.5] - [2.6]\) respectively (for \( n > N_0 \)). There exists a constant \( N_0' \) such that if \( n > N_0' \), then:

(i) the following estimate holds,

\[
\max_{1 \leq n} \sup_{x \in G_h} |v^h_i(x) - \psi_i(x)|^2 + \max_{1 \leq n} \|v^h_i - \psi_i\|_{L^2(G_h)}^2 \leq \tau^{1+\gamma} N(K' + K_m^2),
\]

(ii) if moreover \( m \geq 5 \), then

\[
\max_{1 \leq n} \sup_{x \in G_h} |v^h_i(x) - \psi_i(x)|^2 + \max_{1 \leq n} \|v^h_i - \psi_i\|_{L^2(G_h)}^2 \leq \tau^{2+\gamma} N(K' + K_m^2),
\]

where \( N \) is a constant depending only on \( K, C, T, m, \mu_0 \) and \( \mu_2 \).

A direct consequence of the theorem above is the following:

Theorem 2.5. Under the assumptions of Theorem 2.4 for all \( n > N_0' \) and all \( h \in \mathfrak{H} \),
(i) the following estimate holds,
\[
\max_{i \leq n} \sup_{x \in G_h} |u_i \tau(x) - v_i^{h, \tau}(x)|^2 + \max_{i \leq n} \|u_i \tau - v_i^{h, \tau}\|_{L_2(G_h)}^2 \leq N (h^2 + \tau^{1+\gamma}) N_m^2
\]

(ii) if moreover \(m \geq 5\), then
\[
\max_{i \leq n} \sup_{x \in G_h} |u_i \tau(x) - v_i^{h, \tau}(x)|^2 + \max_{i \leq n} \|u_i \tau - v_i^{h, \tau}\|_{L_2(G_h)}^2 \leq N (h^2 + \tau^{2+\gamma}) N_m^2,
\]
where \(N_m^2 = K' + K_m^2\), and \(N\) is a constant depending only on \(K, C, T, m, \mu_0\) and \(\mu_2\).

3. Auxiliary Facts

In this section we prove some results that will be used in order to prove the main theorems.

**Lemma 3.1.** For any integer \(l \geq 0\), and for any \(\phi \in H^l\), we have
\[
(\partial_x^j J_h^l \phi, \partial_x^j \phi) \leq 0,
\]
for all integers \(j \in \{0, \ldots, l\}\).

**Proof.** Since \(\partial_x J_h^l \phi = J_h^l \partial_x \phi\), it clearly suffices to show the conclusion with \(l = j = 0\). We have
\[
(J_h^0 \phi(x), \phi(x)) = \sum_{k \in B_h} \left( (\phi(\cdot + hk), \phi) - \|\phi\|_{L_2}^2 \right) \nu(B_k^h)
\]
\[
\leq \sum_{k \in B_h} \left( \|\phi\|_{L_2}^2 - \|\phi\|_{L_2}^2 \right) \nu(B_k^h) = 0,
\]
where the inequality is due to Hölder’s inequality and the translation invariance of the Lebesgue measure. In order to show that \((J_h^l \phi, \phi) \leq 0\), clearly it suffices to show that for each \(k \in A_h\)
\[
\left( \sum_{l=0}^{[k]-1} \theta_k^l \delta_{-h} \delta_h \phi(x + s_k lh), \phi(x) \right) \leq 0. \tag{3.7}
\]
If \(s_k = 1\), then a simple calculation shows that
\[
\sum_{l=0}^{[k]-1} \theta_k^l \delta_{-h} \delta_h \phi(x + s_k lh)
\]
\[
= \sum_{l=0}^{[k]-1} \frac{2[k] - (2l + 1)}{2k^2 h^2} \left[ \phi(x + (l - 1) h) - 2\phi(x + lh) + \phi(x + (l + 1) h) \right]
\]
\[
= \frac{1}{2k^2 h^2} \left( \phi(x + kh) + \phi(x + (k - 1) h) + (2k - 1) \phi(x - h) - (2k + 1) \phi(x) \right),
\]
which combined with Hölder’s inequality imply (3.7). If \( s_k = -1 \), then
\[
\sum_{l=0}^{\lfloor k \rfloor - 1} \theta_k^l \delta_{-k} \phi(x + s_k l h)
\]
\[
= \sum_{l=0}^{\lfloor k \rfloor - 1} \frac{2|k| - (2l + 1)}{2k^2 h^2} \left[ \phi(x - (l + 1)h) - 2\phi(x - lh) + \phi(x - (l - 1)h) \right]
\]
\[
= \frac{1}{2k^2 h^2} \left( \phi(x + kh) + \phi(x + (k + 1)h) + (2|k| - 1)\phi(x + h) - (2|k| + 1)\phi(x) \right),
\]
which again by virtue of Hölder’s inequality implies (3.7). □

Lemma 3.1 combined with Lemma 3.4 from [9] imply the following:

**Lemma 3.2.** Suppose Assumption 2.1 (i) holds. Then for any integer \( l \in \{0, \ldots, m\} \) and any \( \phi \in H^m \), we have
\[
(\partial_x^l (L^h_t + J^h) \phi, \partial_x^l \phi) \leq N \|\phi\|_{H^m}^2,
\]
where \( N \) is a constant depending only on \( K \) and \( m \).

The following is very well known (see e.g. [9], [11]).

**Lemma 3.3.** For each integer \( l \geq 0 \), there is a constant \( N \) depending only on \( l \), such that for all \( u \in H^{l+2} \), \( v \in H^{l+3} \) and \( \lambda \in \mathbb{R} \setminus \{0\} \),
\[
\|\delta^\lambda u - \partial_x u\|_{H^l} + \|\delta^\lambda u - \partial_x u\|_{H^l} \leq N |\lambda| \|u\|_{H^{l+2}},
\]
\[
\|\delta^\lambda v - \partial^2_x v\|_{H^l} + \|\delta^\lambda v - \partial^2_x v\|_{H^l} \leq N |\lambda| \|v\|_{H^{l+3}}.
\]

For our approximation we have the following consistency estimates.

**Lemma 3.4.** For any integer \( l \geq 0 \), and any \( \phi \in H^{l+3} \) we have
\[
\|J^h \phi - J^h \phi\|_{H^l} \leq N h \|\phi\|_{l+3}, \tag{3.8}
\]
where \( N \) is a constant depending only on \( l \), \( \mu_0 \) and \( \mu_2 \).

**Proof.** Again we can and we will assume that \( l = 0 \). We have
\[
J^h_2 \phi(x) - J^h_2 \phi(x) = \sum_{k \in B^h_k} \int_{B^h_k} \left( \phi(x + hk) - \phi(x + z) \right) \nu(dz)
\]
\[
= \sum_{k \in B^h_k} \int_0^1 (hk - z) \partial_x \phi(x + z + \theta(hk - z)) d\theta \nu(dz),
\]
which combined with the fact that \( |hk - z| \leq h \) for \( z \in B^h_k \), gives by virtue of Minkowski’s integral inequality
\[
\|J^h_2 \phi - J^h_2 \phi\| \leq h \mu_0 \|\phi\|_{H^1} \tag{3.9}
\]
For $J_1^h - J_1$ we have

$$J_1^h \phi - J_1 \phi = \sum_{k \in A_h} \left[ \frac{|k| - 1}{2} \theta_k \delta_{-h} \delta_h \phi(x + s_k lh) - \int_{|z| \leq 1} \int_0^1 (1 - \theta) z^2 \partial_x^2 \phi \left( x + \theta z \right) d\theta d\nu(dz) \right]$$

$$= \sum_{k \in A_h} \int_{B_h} \int_{|z| = 1} \int_{l/|k|} z^2 (1 - \theta) \left( \delta_{-h} \delta_h \phi(x + s_k lh) - \partial_x^2 \phi(x + \theta z) \right) d\theta d\nu(dz) \tag{3.10}$$

Then we have for the integrand in the above quantity

$$\delta_{-h} \delta_h \phi(x + s_k lh) - \partial_x^2 \phi(x + \theta z)$$

$$= \delta_{-h} \delta_h \phi(x + s_k lh) - \delta_{-h} \delta_h \phi(x + \theta z) + \delta_{-h} \delta_h \phi(x + \theta z) - \partial_x^2 \phi(x + \theta z)$$

$$= \int_0^1 (s_k lh - \theta z) \delta_{-h} \delta_h \partial_x \phi(x + \theta z + \eta(s_k lh - \theta z)) d\eta$$

$$+ \delta_{-h} \delta_h \phi(x + \theta z) - \partial_x^2 \phi(x + \theta z) \tag{3.11}$$

Notice that for $\theta \in [l/|k|, (l + 1)/|k|]$ and for $z \in B_h$ we have

$$|s_k lh - \theta z| \leq |s_k lh - \theta kh| + |\theta kh - \theta z| \leq 2h.$$ 

Hence, for the first term at the right hand side of (3.11) we have for $\|L_h \phi\|_{H^3}$

$$\| \int_0^1 (s_k lh - \theta z) \delta_{-h} \delta_h \partial_x \phi(x + \theta z + \eta(s_k lh - \theta z)) d\eta \|_{L^2} \leq 2h \|\phi\|_{H^3},$$

while for the second one we have by Lemma 3.5

$$\|\delta_{-h} \delta_h \phi(x + \theta z) - \partial_x^2 \phi(x + \theta z)\|_{L^2} \leq h \|\phi\|_{H^3}.$$ 

Therefore,

$$\|\delta_{-h} \delta_h \phi(x + s_k lh) - \partial_x^2 \phi(x + \theta z)\|_{L^2} \leq N h \|\phi\|_{H^2},$$

which combined with 3.10 and Minkowski’s inequality gives

$$\|J_1^h \phi - J_1 \phi\|_{L^2} \leq N h \|\phi\|_{H^3}.$$ 

By this inequality and (3.3) we obtain (3.3). □

**Lemma 3.5.** Let (i) from Assumption 2.1 hold. Then for any $l \leq m$ and for any $\phi \in H^{l+2}$, $t \in [0, T]$ we have

$$\|L_h^l \phi\|_{H^l} + \|J_h^l \phi\|_{H^l} \leq N \|\phi\|_{H^{l+2}},$$

where $N$ is a constant depending only on $K, m, \mu_0$ and $\mu_2$.

**Proof.** Clearly it suffices to show the inequality for $\phi \in C_c^\infty$. We have for $\lambda \neq 0$

$$\delta_\lambda \phi(x) = \int_0^1 \partial_x \phi(x + \theta \lambda) d\theta.$$
Hence by Minkowski’s inequality we get $\| \delta_{\lambda} \phi \|_{L^2} \leq \| \partial_x \phi \|_{L^2}$, which implies $\| \delta_{\lambda} \phi \|_{L^2} \leq N \| \phi \|_{H^1}$, $\| \delta_{\lambda} \delta_{\lambda} \phi \|_{L^2} \leq N \| \phi \|_{H^2}$, $\| \delta_{\lambda} \phi \|_{L^2} \leq N \| \phi \|_{H^2}$. Hence, by Assumption 2.1 (i) we have $\| L^h_t \phi \|_{H^t}^2 \leq N \| \phi \|_{H^t}^2$.

By Minkowski’s inequality we have

$$\| J^h_1 \phi \|_{L^2} \leq \sum_{k \in \delta_h} \zeta^h_k \sum_{l=0}^{|k|-1} \theta^h_l \| \delta_{-h} \delta_h \phi(\cdot + s_k l h) \|_{L^2} \leq \frac{1}{2} \mu_2 \| \phi \|_{H^2}$$

and

$$\| J^h_2 \phi \|_{L^2} \leq \sum_{k \in \mathbb{B}^h} \| \phi(x + h k) - \phi \|_{L^2} \leq 2 \mu_0 \| \phi \|_{L^2}.$$}

These estimates, combined with the fact that $\partial_x J^h = J^h \partial_x$, give

$$\| J^h \phi \|_{H^l} \leq N \| \phi \|_{H^l} + 2 \mu_0.$$}

This finishes the proof.

Next we consider in $L_2(\mathbb{R})$ the following scheme

$$d u^h_t = \left( (L^h_t + I^h) u^h_t + f_t \right) dt \quad (3.12)$$

$$u^h_0 = \psi. \quad (3.13)$$

**Lemma 3.6.** Let Assumption 2.1 hold with some integer $l \geq 1$ instead of $m$. Then (3.12)-(3.13) has a unique $L^2$-solution $(u^h_t)_{t \in [0, T]}$ which is a continuous $H^l$-valued function. If moreover Assumption 2.2 holds, then there exists a constant $N = N(l, T, K)$ such that for all $h \in \mathcal{R}$

$$\sup_{t \leq T} \| u^h_t \|_{H^l}^2 \leq NK_l^2. \quad (3.14)$$

**Proof.** Equation (3.12)-(3.13) is a differential equation on $L^2$ with Lipschitz continuous coefficients and therefore has a unique $L^2$-valued continuous solution $(u^h_t)_{t \in [0, T]}$. Similarly, it is a differential equation on $H^l$ with Lipschitz continuous coefficients and therefore has a unique $H^l$-valued continuous solution $(w^h_t)_{t \in [0, T]}$. Since $H^l \subset L^2$ we have that $w^h = u^h$.

For (3.14), we have for any $t \in [0, T]$

$$\| u^h_t \|_{H^l}^2 = \| \psi \|_{H^l}^2 + \int_0^t \left[ \left( (L^h_s + I^h) u^h_s + u^h_s \right)_{H^l} + (f_s, u^h_s)_{H^l} \right] ds \leq \| \psi \|_{H^l}^2 + N \int_0^t \| u^h_s \|_{H^l}^2 ds + \int_0^T \| f_s \|_{H^l}^2 ds < \infty,$$

where the last inequality is by virtue of Lemma 3.2 and Young’s inequality. Gronwall’s lemma finishes the proof.
**Theorem 3.7.** Let Assumptions 2.1 and 2.2 with $m \geq 4$, and let $u^h$ and $u$ be the unique solutions of (3.12)-(3.13) and (1.1)-(1.2) respectively. Then for any $h \in \mathcal{H}$ we have
\[
\sup_{t \leq T} \|u_t - u^h_t\|_{H^m-3}^2 \leq NK_m^2 h, \tag{3.15}
\]
where $N$ is a constant depending only on $m, T, \mu_0, \mu_2$ and $K$.

*Proof.* We have that $u^h - u$ satisfies the conditions of Lemma 3.6 with $l = m - 3$, $\psi = 0$ and $f_l = (L^h_t - L_t)u_t + (I^h - I)u_t$. Therefore we have
\[
\sup_{t \leq T} \|u^h_t - u_t\|_{H^m-3}^2 \leq N \int_0^T \|(L^h_t - L_t)u_t + (J^h - J)u_t\|_{H^m-3}^2 dt \tag{3.16}
\]
\[
\leq Nh \int_0^T \|u_t\|_{H^m}^2 dt \leq NhK_m^2. \tag{3.17}
\]
where the second inequality follows from Lemmata 3.3 and 3.4. This finishes the proof. \hfill \Box

Next we continue with the time discretization. Let us consider on $L_2(\mathbb{R})$ the following implicit scheme,
\[
u_i = u_{i-1} + \tau [(L^h_{i\tau} + J^h)u_i + f_{i\tau}], \quad i = 1, \ldots, n \tag{3.18}
\]
\[
u_0 = \psi. \tag{3.19}
\]

The following is very well known.

**Lemma 3.8.** Let $\mathcal{D}$ be a bounded linear operator on a Hilbert space $X$ into itself. If there exists $\delta > 0$ such that $(\mathcal{D}\phi, \phi)_X \geq \delta \|\phi\|^2_X$, for all $\phi \in X$, then for any $f \in X$, there exists a unique $g \in X$ such that $\mathcal{D}g = f$.

**Theorem 3.9.** Let Assumptions 2.1 and 2.2 hold. Then there exists a constant $N'$ depending only on $K, T$ and $m$, such that if $n > N'$, for any $h \in \mathcal{H}$ there exists a unique $L_2$-solution $(u^h_{i\tau})_{i=0}^n$ of (3.18)-(3.19). Moreover $u^h_{i\tau} \in H^m$ for each $i = 0, \ldots, n$.

*Proof.* Let us write (3.18) in the form
\[
\mathcal{D}_i u_i = F_i, \quad i = 1, \ldots, n,
\]
where
\[
\mathcal{D}_i = I - \tau (L^h_{i\tau} + J^h), \quad F_i = u_{i-1} + f_{(i-1)\tau}.
\]

For each $i = 1, \ldots, n$, $\mathcal{D}_i$ is a bounded linear operator from $H^k$ to $H^k$ for all $k = 0, \ldots, m$. By Lemma 3.2 we have
\[
(\mathcal{D}_i \phi, \phi)_k = \|\phi\|^2_{H^k} - \tau \left( (L^h_{i\tau} + J^h)\phi, \phi \right)_k \geq \|\phi\|^2_{H^k} - \tau N \|\phi\|^2_{H^k},
\]
for all $k = 0, \ldots, m$, with $N$ depending only on $K$ and $m$. Hence, if $n > TN$, then we have with $\lambda := 1 - (\tau/N) > 0$
\[
(\mathcal{D}_i \phi, \phi)_k \geq \lambda \|\phi\|^2_{H^k}.
\]

The conclusion follows from the lemma above.
Theorem 3.10. Let Assumptions 2.1, 2.2 and 2.4 hold with $m \geq 4$ and let $(u^{h}_{i})_{i \in [0, T]}$ and $(u^{h, \tau}_{i})_{i = 0}$ be the unique solutions of equations (3.12)–(3.13) and (3.18)–(3.19) respectively (for $n > N'$). There exists a constant $N_{1}$ such that if $n > N_{1}$, then:

(i) if Assumption 2.3 holds with $l = m - 3$, then

$$\max_{i \leq n} \| u^{h}_{i\tau} - u^{h, \tau}_{i} \|^{2}_{H^{m-3}} \leq \tau^{1-\gamma} N(K' + K_{m}^{2})$$

(3.20)

(ii) if Assumption 2.3 holds with $l = m - 4$, then

$$\max_{i \leq n} \| u^{h}_{i\tau} - u^{h, \tau}_{i} \|^{2}_{H^{m-4}} \leq \tau^{2-\gamma} N(K' + K_{m}^{2}),$$

(3.21)

where $N$ is a constant depending only on $K, C, T, m, \mu_{0}$ and $\mu_{2}$.

Proof. In order to ease the notation, let us introduce $e_{i} = u^{h}_{i\tau} - u^{h, \tau}_{i}$. We have that $(e_{i})_{i = 0}^{n}$ satisfies

$$e_{i} = e_{i-1} + \tau \mathbb{R} e_{i} + F_{i}, \quad i = 1, \ldots, n,$$

$$e_{0} = 0,$$

where

$$\mathbb{R} = L^{h}_{\infty} + J^{h}, \quad F_{i} := \int_{(i-1)\tau}^{i\tau} F_{t} \, dt.$$

$$F_{i} := (L^{h}_{i} + J^{h})u^{h}_{i\tau} - (L^{h}_{k(t)} + J^{h})u^{h}_{k(t)} + f_{i} - f_{k(t)},$$

and $k(t) = \lfloor nt \rfloor / n$. By the identity $\| b \|^{2} - \| a \|^{2} = 2(b, b - a) - \| b - a \|^{2}$, we have for $j \leq m - 3$ and $i \geq 1$,

$$\| \partial^{j}_{x} e_{i} \|^{2} - \| \partial^{j}_{x} e_{i-1} \|^{2} \leq 2\tau(\partial^{j}_{x} e_{i}, \mathbb{R} e_{i} \partial^{j}_{x} e_{i}) + 2(\partial^{j}_{x} e_{i}, \partial^{j}_{x} F_{i})$$

(3.22)

By Lemma 3.2 we have

$$2\tau(\partial^{j}_{x} e_{i}, \mathbb{R} \partial^{j}_{x} e_{i}) \leq \tau N \| \partial^{j}_{x} e_{i} \|^{2},$$

while by Young's inequality we have

$$2(\partial^{j}_{x} e_{i}, \partial^{j}_{x} F_{i}) \leq \tau \| \partial^{j}_{x} e_{i} \|^{2} + \tau^{-1} \| \int_{(i-1)\tau}^{i\tau} \partial^{j}_{x} F_{i} \, dt \|^{2}$$

$$\leq \tau \| \partial^{j}_{x} e_{i} \|^{2} + \int_{(i-1)\tau}^{i\tau} \| \partial^{j}_{x} F_{i} \|^{2} \, dt.$$

By using these inequalities and summing up (3.22) over $0 \leq j \leq q$, where $q \in \{m - 4, m - 3\}$, and over $i \leq \rho \leq n$, we get

$$\| e_{\rho} \|^{2}_{H^{q}} \leq \tau N \sum_{i=1}^{\rho} \| e_{i} \|^{2}_{H^{q}} + N \int_{0}^{T} \| F_{i} \|^{2}_{H^{q}} \, dt < \infty,$$
where \( N \) is a constant depending only on \( m \) and \( K \). Let us set \( N_1 := TN \). By the discrete Gronwall inequality we have for \( n > N_1 \) (i.e. for \( \tau < 1/N \))

\[
\max_{\rho \leq n} \|e_{\rho}\|^2_{H^q} \leq N \int_0^T \|F_t\|^2_{H^q} \, dt,
\]

where \( N \) depends only on \( m, K \) and \( T \). We only have to estimate the term at the right hand side of the above inequality.

\[
\begin{align*}
\int_0^T \|F_t\|^2_{H^q} \, dt &\leq N \int_0^T \|(L_t^h - L_{k(t)}^h)u_t^h\|^2_{H^r} \, dt \\
&+ N \int_0^T \|(J^h + L_{k(t)}^h)(u_t^h - u_{k(t)}^h)\|^2_{H^r} \, dt \\
&+ N \int_0^T \|f_t - f_{k(t)}\|^2_{H^r} \, dt.
\end{align*}
\]

(3.23)

Let us show first (3.20) under Assumption 2.3 with \( l = m - 3 \). By Assumption 2.3 and (3.14) we have with \( q = m - 3 \)

\[
\begin{align*}
\int_0^T \|(L_t^h - L_{k(t)}^h)u_t^h\|^2_{H^r} \, dt &\leq \tau^\gamma N \int_0^T \|u_t^h\|^2_{H^{q+2}} \, dt \leq \tau^\gamma N K_2^2 (3.24) \\
\int_0^T \|f_t - f_{k(t)}\|^2_{H^r} \, dt &\leq \tau^\gamma T. \quad (3.25)
\end{align*}
\]

By Lemma 3.5 we have

\[
\int_0^T \|(J^h + L_{k(t)}^h)(u_t^h - u_{k(t)}^h)\|^2_{H^r} \, dt \leq N \int_0^T \|u_t^h - u_{k(t)}^h\|^2_{H^{q+2}} \, dt.
\]

Therefore, in order to show (i) we only need to show that

\[
\int_0^T \|u_t^h - u_{k(t)}^h\|^2_{H^{m-1}} \, dt \leq N\tau(K_2^2 + K'). \quad (3.26)
\]

For \( \phi \in H^{m-1} \), and \( \phi' \in H^m \), one has \( |\langle \phi', \phi \rangle_m| \leq \|\phi'\|_{H^m} \|\phi\|_{H^{m-2}} \). Using this and Young’s inequality we obtain for \( s, t \in [0, T] \) with \( s \leq t \)

\[
\|u_t^h - u_s^h\|_{m-1}^2 = 2 \int_s^t \left( u_r^h - u_s^h, (L_r^h + J^h)u_r + f_r \right)_{m-1} \, dr \\
\leq N \int_s^t \|u_r^h - u_s^h\|^2_{H^m} + \|(L_r^h + J^h)u_r\|^2_{H^{m-2}} + \|f_r\|^2_{H^{m-2}} \, dr \\
\leq N \int_s^t \sup_{t' \leq T} \|u_r^h\|^2_{H^m} + \|f_r\|^2_{H^{m-2}} \, dr \\
\leq N(K_2^2 + K')(t - s),
\]

where the last inequality follows by Lemma 3.6 and Assumption 2.4. This shows (3.20), which combined with (3.24) and (3.25) (with \( q = m - 3 \)), imply (3.20) by virtue of (3.23). In order to show (3.24) under Assumption 2.3
with \( l = m - 4 \), by virtue of \( 3.23 \), \( 3.24 \) and \( 3.25 \), with \( q = m - 4 \), it suffices to show
\[
\int_0^T \| u^h_t - u^k_h(t) \|_{H^{m-2}}^2 dt \leq N\tau^2(K_m^2 + K').
\]
For \( t, s \in [0, T] \) we have
\[
\| u^h_t - u^h_s \|_{H^{m-2}}^2 \leq \left\| \int_s^t (L^h_r + J^h_r) u^h_r + f_r \ dr \right\|_{H^{m-2}}^2 \\
\leq \left( \int_s^t N \sup_{t' \leq T} \| u^h_{t'} \|_{H^m} + \| f_r \|_{H^{m-2}} \ dr \right)^2 \\
\leq N(t - s)^2(K_m^2 + K').
\]
This brings the proof to an end.

\[\square\]

4. PROOFS OF THE MAIN RESULTS

We are now ready to prove the main theorems.

**Proof of Theorem 2.2.** Let \( \mathcal{F} \) denote the continuous embeddings \( H^{m-3} \hookrightarrow l_2(\mathbb{G}_h) \) and \( H^{m-3} \hookrightarrow C^{0,1/2} \). Let \( u^h \) and \( v^h \) denote the solutions of \( 3.12-3.13 \) and \( 2.3-2.4 \) (the same equation, considered on \( l_2(\mathbb{G}_h) \) and \( L^2(\mathbb{G}_h) \)). By applying \( \mathcal{F} \) to both sides of \( 3.12 \) we see that \( \mathcal{F} u^h \) satisfies \( 2.3-2.4 \). Therefore \( \mathcal{F} u^h = v^h \) by uniqueness. Notice also that \( \mathcal{F} u^h_t = \mathcal{F} u^h_t \), and \( u^h_t(x) = \mathcal{F} u^h_t(x) = \mathcal{F} u^h_t(x) = \mathcal{F} u^h_t(x) \), for all \( t \in [0, T] \) and \( x \in \mathbb{G}_h \). Hence
\[
\sup_{x \in \mathbb{G}_h} |v^h_t(x) - u^h_t(x)| = \sup_{x \in \mathbb{G}_h} |\mathcal{F} u^h_t(x) - u^h_t(x)| \\
= \sup_{x \in \mathbb{G}_h} |\mathcal{F} u^h_t(x) - \mathcal{F} u^h_t(x)| \\
\leq N \| u^h_t - u^h_t \|_{H^{m-3}},
\]
and
\[
\| v^h_t - u^h_t \|_{l_2(\mathbb{G}_h)} = \| \mathcal{F} u^h_t - \mathcal{F} u^h_t \|_{l_2(\mathbb{G}_h)} \\
\leq N \| u^h_t - u^h_t \|_{H^{m-3}},
\]
where \( N \) depends only on \( m \). The conclusion now follows from Theorem 3.7.

**Proof of Theorem 2.3.** Notice that the existence part follows easily from Theorem 3.9. Namely, if \( u^{h,\tau} \) solves \( 3.18-3.19 \), then \( \mathcal{F} u^{h,\tau} \) solves \( 2.3-2.4 \). Also, the uniqueness part is immediate if for example one poses a CLF condition on \( \tau \) and \( h \). However such a condition is obviously not necessary, and therefore, in order to prove Theorem 2.3 we will proceed as in the proof of Theorem 3.9. Hence, we need the following, whose proof is essentially given in [10], but we give a sketch for the convenience of the reader.
Lemma 4.1. Let Assumptions 2.1 and 2.2 hold with \( m = 1 \). Then for any \( \phi \in l_2(G_h) \) we have

\[
(L^h_t + J^h)\phi, \phi \|_{l_2(G_h)} \leq N\|\phi\|_{l_2(G_h)}^2,
\]

where \( N \) depends only on \( K \).

Proof. One can replace \( (\cdot, \cdot) \) with \((\cdot, \cdot)_{l_2(G_h)}\) in the proof of Lemma 3.1 to obtain

\[
(J^h \phi, \phi)_{l_2(G_h)} \leq 0.
\]

Consequently we only need that \((L^h_t \phi, \phi)_{l_2(G_h)} \leq N\|\phi\|_{l_2(G_h)}^2\). This is proved in [10]. In the proof of Lemma 3.3 in that article, one can replace \((\cdot, \cdot)\) with \((\cdot, \cdot)_{l_2(G_h)}\) to obtain

\[
|((\delta^h \phi, (\delta^h a_t)T^h \phi)_{l_2(G_h)})| + |(b_t \delta^h \phi, \phi)_{l_2(G_h)}| + |(c_t \phi, \phi)_{l_2(G_h)}| \leq N\|\phi\|_{l_2(G_h)}^2,
\]

(4.27)

where \( T^h \phi(x) = (\phi(x + h) + \phi(x - h))/2 \), and \( N \) depends only on \( K \). It is shown also in [10] (see (3.3)) that for functions \( u, v \)

\[
\delta^h(\phi) = (\delta^h u)T^h v + (\delta^h v)T^h u.
\]

Therefore,

\[
(a_t \delta^h \phi_t, u_t)_{l_2(G_h)} = - (\delta^h u_t, (\delta^h a_t)T^h \phi_t)_{l_2(G_h)} = - (\delta^h u_t, (\delta^h a_t)T^h u_t)_{l_2(G_h)} - (\delta^h u_t, (T^h a_t)\delta^h u_t)_{l_2(G_h)},
\]

(4.28)

Notice that by virtue of Assumption 2.2, we have

\[-(\delta^h u_t, (T^h a_t)\delta^h u_t)_{l_2(G_h)} \leq 0.\]

Hence, (4.28) and (4.27) imply

\[
(L^h_t \phi, \phi)_{l_2(G_h)} \leq N\|\phi\|_{l_2(G_h)}^2.
\]

\[
\Box
\]

Proof of Theorem 2.3. The proof is the same as the one of Theorem 2.2, this time using Lemma 4.1 instead of Lemma 3.2.

Proof of Theorem 2.4. The conclusion follows by Sobolev embeddings and Theorem 3.9 similarly to the proof of Theorem 2.2.

\[
\Box
\]

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