RADIATION FROM A CHARGE CIRCULATING INSIDE A WAVEGUIDE WITH DIELECTRIC FILLING

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Abstract

The emitted power of the radiation from a charged particle moving uniformly on a circle inside a cylindrical waveguide is considered. The expressions for the energy flux of the radiation passing through the waveguide cross-section are derived for both TE and TM waves. The results of the numerical evaluation are presented for the number of emitted quanta depending on the waveguide radius, the radius of the charge rotation orbit and dielectric permittivity of the filling medium. These results are compared with the corresponding quantities for the synchrotron radiation in a homogeneous medium.

1 Introduction

The wide applications of the synchrotron radiation (see, for example [1, 2, 3] and the references therein) motivate the importance of investigations for various mechanisms of control of the radiation parameters. From this point of view, it is of interest to investigate the influence of a medium on the spectral and angular distributions of the synchrotron emission. This study is also important with respect to some astrophysical problems [4, 5]. The radiation from a charged particle circulating in a homogeneous medium was considered by Tsytovich in [6] (see also [7, 8]). It had been shown that the interference between the synchrotron and Cerenkov radiations leads to interesting effects. The radiation from a charge rotating around a dielectric ball enclosed by a homogeneous medium is investigated in papers [9, 10]. The presence of the ball leads to interesting effects: if for substance of the ball and particle velocity the Cerenkov condition is satisfied, there are strong narrow peaks in the radiation intensity. At these peaks the radiated energy exceeds the corresponding quantity in the case of a homogeneous medium by some orders of magnitude. A similar problem for the case of the cylindrical symmetry we have considered in Refs. [11, 12]. In the paper [11] a recurrent scheme is developed for constructing the Green function of the electromagnetic field for a medium consisting of an arbitrary number of coaxial cylindrical layers. The investigation of the radiation from a charged particle circulating around a dielectric cylinder immersed in a homogeneous medium [12], has shown that under the Cerenkov condition for the material of the cylinder and the velocity of a charge there are narrow peaks in the angular distribution of the number of quanta emitted into the exterior space. For some values of parameters the density of the number of quanta in these peaks exceeds the corresponding quantity for the radiation in the vacuum by several orders.

The present paper is devoted to the radiation from a charge rotating inside a cylindrical waveguide filled by a homogenous dielectric. The contribution of waves of the electric and
magnetic type to the total radiation intensity is determined. The results of the corresponding numerical calculations are presented and a comparison with the intensity of the synchrotron radiation in a homogenous medium is carried out.

2 Electromagnetic fields inside a waveguide

To determine the electromagnetic field of a charged particle rotating inside a cylindrical waveguide with conducting walls and filled by a material with dielectric permittivity \( \varepsilon_0 \), firstly we shall consider a more general problem. Let a charge \( q \) moves with the velocity \( v \) along a circle of radius \( \rho_0 \) in the plane \( z = 0 \) inside a cylinder of radius \( \rho_1 \) and axis \( z \). The permittivity of the medium inside the cylinder is \( \varepsilon_0 \) and it is immersed in a homogeneous medium with permittivity \( \varepsilon_1 \) (for simplicity we shall assume that the permeability is equal to 1). The 4-potential \( A_i = (\varphi, -A_x, -A_y, 0) \) of the corresponding electromagnetic field can be determined via the Green function (being a second rank tensor) by the formula

\[
A_i(x) = -\frac{1}{2\pi c} \int G_{il}(x, x') j_l(x') d^4 x', \quad x = (t, \mathbf{r}), \quad i, l = 0, 1, 2, 3, \tag{1}
\]

where a summation over index \( l \) is assumed, \( c \) is the speed of light in the vacuum, and \( j_l(x) \) is the 4-vector of the current density of the charge. In the cylindrical coordinates \((\rho, \varphi, z)\) the spatial components of the latter are in form

\[
j_l = \frac{v q}{\rho_0} \delta(\rho - \rho_0) \delta(\varphi - \omega_0 t) \delta(z) \delta_{l2}, \quad \omega = \omega_0 \rho_0, \quad l = 1, 2, 3 \tag{2}
\]

(the values of indices \( l = 1, 2, 3 \) correspond to the coordinates \( \rho, \varphi, z \) respectively). A recurrent scheme for the construction of the Green function in a medium consisting of an arbitrary number of coaxial cylindrical layers is developed in [11]. The Green function for the problem under consideration can be derived from the general formulae of this paper. By using the problem symmetry we can write the following Fourier expansion for the Green function:

\[
G_{il}(x, x') = \sum_{m=\pm \infty}^{+\infty} \int dk_z d\omega G_{il}(\omega, m, k_z, \rho, \rho') \exp \left\{ i \left[ m (\varphi - \varphi') + k_z (z - z') - \omega (t - t') \right] \right\}. \tag{3}
\]

In the Lorentz gauge for the components we need here and for \( \rho = \rho_0 < \rho_1, \rho < \rho_1 \) from the general formulae of Ref. [11] one has

\[
G_{il}(\omega, m, k_z, \rho, \rho_0) = \frac{\pi}{4 i^{l-1}} \sum_{\alpha=1,1} \alpha^l \left[ J_{m+\alpha}(\lambda_0 \rho_0) H_{m+\alpha}(\lambda_0 \rho) + B_{m}^{(\alpha)}(\lambda_0 \rho) \right], \quad l = 1, 2, \tag{4}
\]

where \( \rho_0 = \min(\rho, \rho_0), \rho_0 = \max(\rho, \rho_0) \), \( J_m(x) \) is the Bessel function, \( H_m(x) \equiv H_m^{(1)}(x) \) is the Hankel function of the first kind. In [4] the coefficients \( B_{m}^{(\alpha)} \) are defined by the expressions

\[
B_{m}^{(\alpha)} = -J_{m+\alpha}(\lambda_0 \rho_0) \left( \frac{W(J_{m+\alpha}, H_{m+\alpha})}{W(J_{m+\alpha}, H_{m+\alpha})} \right) + \frac{\lambda_1 \lambda_2}{\pi \rho_0^2 \beta_1 W(J_{m+\alpha}, H_{m+\alpha})} \sum_{p=\pm 1} \frac{J_{m+p}(\lambda_0 \rho_0)}{W(J_{m+p}, H_{m+p})}, \tag{5}
\]

where, as in the paper [12], we use notations

\[
\lambda_{1,0} = \frac{m \omega_0}{c} \sqrt{\varepsilon_{1,0} - \frac{e^2 k_z^2}{m^2 \omega_0^2}} \tag{6}
\]

\[
\beta_1 = \frac{\varepsilon_0}{\varepsilon_1 - \varepsilon_0} - \frac{\lambda_0}{2} J_m(\lambda_0 \rho_1) \sum_{l=\pm 1} l \frac{H_{m+l}(\lambda_1 \rho_1)}{W(J_{m+l}, H_{m+l})} \tag{7}
\]
with
\[ W(a, b) = a(\lambda_0 \rho_1) \frac{\partial b(\lambda_1 \rho_1)}{\partial \rho_1} - b(\lambda_1 \rho_1) \frac{\partial a(\lambda_0 \rho_1)}{\partial \rho_1}. \]  
(8)

In formula (8) the summand with \( m = 0 \) does not depend on time and hence does not give a contribution to the radiation field. Therefore in the consideration of the radiation field we can assume \( m \neq 0 \).

Substituting Eq. (2) and expressions (3), (4) for the Green function into formula (1) we receive
\[ A_t(x) = \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{\infty} e^{im(\varphi - \omega t)} dk_z e^{ik_z z} A_{ml}(k_z, \rho), \]  
(9)

where in the Lorentz gauge
\[ A_{ml}(k_z, \rho) = -\frac{\nu q}{4\pi l - 1} \left\{ A_{ml}^{(0)} + \sum_{\alpha = \pm 1} \alpha' B_{m}^{(\alpha)} J_{m+\alpha}(\lambda_0 \rho) \right\}, \quad l = 1, 2, \]  
(10)

\[ A_{m3}(k_z, \rho) = 0. \]

Here the term
\[ A_{ml}^{(0)} = \sum_{\alpha = -1, 1} \alpha' J_{m+\alpha}(\lambda_0 \rho_0) H_{m+\alpha}(\lambda_0 \rho_0), \]  
(11)
corresponds to the field of a charge in a homogeneous medium with permittivity \( \varepsilon_0 \).

The vector potential of the electromagnetic field inside a waveguide with conducting walls can be obtained from the formulae given above taking the limit \( \varepsilon_1 \to \infty \). From formulae (3)-(8) it follows that \( \lambda_1(\varepsilon_1) \) enters only in the argument of the Hankel function. By a simple calculation in view of the asymptotic formulae of this function for large values of the argument (13) it can be seen that in the limit \( \varepsilon_1 \to \infty \) the coefficients (3) take the form
\[ B_{m}^{(\alpha)} = -J_{m+\alpha}(\lambda_0 \rho_0) \frac{H_{m+\alpha}(\lambda_0 \rho_1)}{J_{m+\alpha}(\lambda_0 \rho_1)} - \frac{i \alpha J_{m-\alpha}(\lambda_0 \rho_1)}{\pi \rho_1 \lambda_0 J_m(\lambda_0 \rho_0) J_m'(\lambda_0 \rho_1)} \sum_{p=\pm 1} \rho^p J_{m+p}(\lambda_0 \rho_0). \]  
(12)

Thus, the field inside a cylindrical waveguide is determined by formulae (3), (11), where the coefficients \( B_{m}^{(\alpha)} \) are defined according to Eq. (12). Note, that in the limit \( \rho_0 \to \rho_1 \) one has \( B_{m}^{(\alpha)} = -H_{m+\alpha}(\lambda_0 \rho_0) \), and in virtue of Eq. (11) fields (11) tend to zero. We could expect this result as when the charge approaches the wall of a waveguide the particle and its mirror image cancel each other. Before to proceed further calculations we shall consider the analytical properties of \( A_{ml}(k_z, \rho) \) in Eq. (4), as a function on the complex variable \( k_z \). From formulae (3), (11), (12) it can seem, that the points \( k_z = \pm \omega_0 \sqrt{\varepsilon_0}/c \) are branchpoints for this function. However, by using the expansions of the cylindrical functions (see, for example, (14)), it can be seen that actually \( A_{ml}(k_z, \rho) \) is a function on \( \lambda_0^2 \) only: \( A_{ml} = A_{ml}(\lambda_0^2, \rho) \). Note that the logarithmic singularity \( \ln \lambda_0 \) contained in the expansion of the Hankel function in \( A_{ml}^{(0)} \) is cancelled with the corresponding singularity of the first term in Eq. (12). As a result the function \( A_{ml}(k_z, \rho) \) is meromorphic on the complex plane \( k_z \). From the formulae (11), (12) it follows that this function has poles corresponding to the zeros of the Bessel function \( J_m(\lambda_0 \rho_1) \) and its derivative:
\[ k_z = \pm k_n^{(\sigma)} \equiv \pm \sqrt{\frac{m^2 \omega_0^2}{c^2} \varepsilon_0 - \frac{J_m^{(\sigma)}(\lambda_0 \rho_1)^2}{\rho_1^2}}, \quad \sigma = 0, 1, \quad J_m^{(\sigma)}(\lambda_0 \rho_1) = J_m^{(\sigma)}(J_m^{(\sigma)}) = 0, \quad n = 1, 2, \ldots, \]  
(13)
with the notation \( J_m^{(\sigma)}(x) = d^\sigma J_m / dx^\sigma \). Note that the singularities in the first and second terms on the right of formula (12) corresponding to the zeros of the functions \( J_{m\pm 1}(\lambda_0\rho_1) \) are cancelled, and consequently the functions \( B_m^{(\pm 1)} \) are regular at these points. In Eq. (13) \( j_{m,n}^{(\sigma)} \) are positive zeros of the Bessel function \( \sigma = 0 \) and its derivative \( \sigma = 1 \), arranged in ascending order, \( j_{m,n} < j_{m,n+1}^{(\sigma)} \). All these zeros are simple, and hence the values of \( k_z \) given by Eq. (13) correspond to simple poles of the function \( A_{ml}(k_z, \rho) \). They describe the eigenmodes of a cylindrical waveguide and are known as TM modes in the case \( \sigma = 0 \) and TE modes in the case \( \sigma = 1 \).

For real values of \( \varepsilon_0 \) poles (13) are situated on the real axis of the complex plane \( k_z \), for \( j_{m,n}^{(\sigma)} \leq m\omega_0\rho_1\sqrt{\varepsilon_0/c} \), and are purely imaginary otherwise. In formula (11) to derive unambiguous result for the integral over \( k_z \) it is necessary to specify the rules to escape the real poles (see Section 3). For this purpose note that in physically realistic situations the permittivity is complex: \( \varepsilon_0 = \varepsilon_0' + i\varepsilon_0'' \), where the imaginary part \( \varepsilon_0'' > 0 \) describes the absorption in the medium. From here it follows that the radicals \( \pm k_n^{(\sigma)} \) from Eq. (13) with a positive/negative real part are situated in the upper/lower half of the complex plane \( k_z \). This leads to the following rule to circle the poles in (j): in the integral over \( k_z \) the positive poles should be circled from below, and negative ones – from above. Note also that if the walls of the waveguide have finite conductivity, then to the wave number \( k_z \) an additional imaginary part is added (see, for example, (14)), which leads to the same rule to escape the poles.

3 Radiation intensity

In this section we shall consider the radiation field travelling inside the cylinder at large distances from the charge. First of all let us show that in Eq. (11) the fields \( A_{ml}^{(0)} \) and the term corresponding to the first summand in definition (12) do not contribute to the radiation field. This immediately follows from the estimate of the integral over \( k_z \) in Eq. (1) by using the stationary phase method (see, for example, (15)). As in this integral the phase \( k_z z \) has no stationary points, for large \( |z| \) the integral tends to zero faster than any power of \( 1/|z| \), provided that the coefficient of the exponential function belongs to the class \( C^\infty(R) \). It follows from here that the singularities of this coefficient can only give the contribution to the radiation field. As it has been mentioned in the previous section, for the integral over \( k_z \) in Eq. (1) the singularities correspond to the poles of the second summand in Eq. (12) at \( k_z = \pm k_n^{(\sigma)} \), defined by relations (13). To determine the corresponding contributions to the integral we note that, as it has been mentioned above, by taking into account the absorption, the points \( k_z = k_n^{(\sigma)} ( -k_n^{(\sigma)} ) \) are situated in the upper (lower) half of the complex plane \( k_z \). Therefore the integration contour over \( k_z \) in Eq. (1) can be closed by a semicircle of a large radius in the upper (lower) half-plane. This choice is stipulated by the fact that for large \( |z| \) the integrand exponentially tends to zero in the upper (lower) half-plane for \( z > 0 ( < 0 ) \). As a result the corresponding integral vanishes when the radii of the semicircles go to infinity. Thus, for large \( |z|, z > ( < ) 0 \) the integral over \( k_z \) in (1), according to the residue theorem, is equal to the sum of the residues at poles \( k_z = k_n^{(\sigma)} ( -k_n^{(\sigma)} ) \), multiplied by \( 2\pi i \). For large \( n \), when \( j_{m,n}^{(\sigma)} > m\omega_0\rho_1\sqrt{\varepsilon_0/c} \), the poles \( \pm k_n^{(\sigma)} \) are purely imaginary in the limit \( \varepsilon_0'' \to 0 \) and the corresponding contribution tends to zero exponentially for \( z \to \infty \). As a result these poles do not contribute to the radiation field. Therefore, the radiation field far from the charge takes the form

\[
A_l(r, t) = 2\pi i \sum_{m=-\infty}^{+\infty} e^{im(\varphi - \omega_0 t)} \sum_{\sigma=0,1}^{n_{\max}} \sum_{n=1}^{n_{max}} \text{Res}_{k_z=k_n^{(\sigma)}} A_{ml} e^{ik_z z}, \tag{14}
\]
where the maximal value \( n_{\text{max}}^{(\sigma)} \) is defined by the conditions
\[
j_{m,n_{\text{max}}^{(\sigma)}}^{(\sigma)} \leq \sqrt{\varepsilon_0 m v \rho_1 / c \rho_0}, \quad j_{m,n_{\text{max}}^{(\sigma)}}^{(\sigma)} > \sqrt{\varepsilon_0 m v \rho_1 / c \rho_0}.
\]

(15)

Thus, at large distances from the charge the radiation field inside the waveguide is presented in the form of waves with discrete set of values of the wave vector projection on the waveguide axis, \( k_z = k_{n}^{(\sigma)}, n = 1, 2, ..., n_{\text{max}}^{(\sigma)} \), defined by formula (13). Having \( A_{ml} \) we can find the scalar potential by using the Lorentz gauge condition. As a result the electric and magnetic fields can be presented in the form of sums of the TE and TM waves:
\[
F_l(r, t) = \sum_{m=-\infty}^{+\infty} e^{im(\varphi - \omega t)} \sum_{\sigma=0,1} \sum_{n=1}^{n_{\text{max}}^{(\sigma)}} F_{ml}^{(\sigma)}(j_{m,n}^{(\sigma)}, \rho), \quad F = E, H.
\]

(16)

For the \( z \)-components of the fields corresponding to the elementary waves one has
\[
E_{m3}^{(0)} = -2q \frac{J_{m}(j_{m,n}^{(0)} \rho_0 / \rho_1)}{\varepsilon_0 \rho_1^2} J_{m}^{(0)}(j_{m,n}^{(0)}), \quad H_{m3}^{(0)} = 0, \quad \text{for TM waves,}
\]
\[
H_{m3}^{(1)} = \frac{2qvi j_{m,n}^{(1)} J_{m}(j_{m,n}^{(1)} \rho_0 / \rho_1)}{c \rho_1^2} J_{m}^{(1)}(j_{m,n}^{(1)}), \quad E_{m3}^{(1)} = 0, \quad \text{for TE waves.}
\]

(17)

(18)

The transverse components can be found from the formulae (see, for example, [14])
\[
E_{ml}^{(0)} = \frac{ik_{m}^{(0)}}{j_{m,n}^{(0)} \rho_1^2} \nabla_{l} \Psi, \quad H_{ml}^{(0)} = \frac{\varepsilon_0 m \omega_0}{c k_{m}^{(0)}} [ e_3 E_{ml}^{(0)},
\]
\[
H_{ml}^{(1)} = \frac{ik_{m}^{(1)}}{j_{m,n}^{(1)} \rho_1^2} \nabla_{l} \Psi, \quad F_{ml}^{(1)} = \frac{m \omega_0}{c k_{m}^{(1)}} [ e_3 F_{ml}^{(1)}],
\]

(19)

where \( \Psi = E_{m3}^{(0)} \) for TM waves and \( \Psi = H_{m3}^{(1)} \) for TE waves, \( \nabla_{l} = (\partial / \partial \rho, im / \rho, 0) \), and \( e_3 \) is the unit vector along the axis \( z \). The energy flux through the cross section of the waveguide per unit time interval is given by the Poynting’s vector \( \mathbf{S} \):
\[
I = \int_{0}^{\rho_1} \rho d\rho \int_{0}^{2\pi} d\varphi (e_3 \cdot \mathbf{S}), \quad \mathbf{S} = \frac{c}{4\pi} [E H],
\]

(20)

where the fields are defined by expansions (16). Substituting these expansions into Eq. (20) and using the formulae for the integrals involving products of the Bessel functions [13] it can be seen that the contribution of the terms \( [E_{m}^{(\sigma)}(j_{m,n}, \rho) H_{ml}^{(\sigma)}(j_{m,n}^{*}, \rho)] \) is proportional to \( \delta_{mn} \delta_{mn'} \delta_{\sigma\sigma'} \).

As a result the intensity \( I \) can be presented as a sum of the radiation intensities on separate modes:
\[
I = \sum_{m=1}^{\infty} \left( \sum_{n=1}^{n_{\text{max}}^{(1)}} I_{mn}^{(\text{TE})} + \sum_{n=1}^{n_{\text{max}}^{(0)}} I_{mn}^{(\text{TM})} \right),
\]

(21)

where the energy radiated on the frequency \( \omega = m \omega_0 \) in the form of the TE and TM modes per per unit time interval is determined by the formulae
\[
I_{m}^{(\text{TM})} = \sum_{n=1}^{n_{\text{max}}^{(0)}} I_{mn}^{(\text{TM})}, \quad I_{mn}^{(\text{TM})} = \frac{2g^2vm}{\varepsilon_0 \rho_0} \frac{k_{n}^{(0)} j_{m}^{(0)}(j_{m,n}^{(0)} \rho_1 / \rho_0)}{J_{m}^{(0)}(j_{m,n}^{(0)})},
\]

(22)

\[
I_{m}^{(\text{TE})} = \sum_{n=1}^{n_{\text{max}}^{(1)}} I_{mn}^{(\text{TE})}, \quad I_{mn}^{(\text{TE})} = \frac{2g^2v^3m}{c^2 \rho_1^2 \rho_0} \frac{j_{m,n}^{(1)}}{k_{n}^{(1)}(j_{m,n}^{(1)} \rho_1 / \rho_0)} \frac{J_{m}^{(1)}(j_{m,n}^{(1)} \rho_1 / \rho_0)}{J_{m}^{(1)}(j_{m,n}^{(1)} \rho_1 / \rho_0)}.\]

(23)
In these expressions the terms with fixed \( n \) correspond to the waves with a given value \( k_z = k_n^{(\sigma)} \) of the wave vector projection on the waveguide axis. Note that in the limit \( \rho_0 \to \rho_1 \) the radiation goes to zero. Expanding the Bessel functions in the numerators of Eqs. (22) and (23) in this limit we receive

\[
I_{mn}^{(\text{TM})} \approx \frac{2q^2\omega_0m}{\varepsilon_0} \left( 1 - \frac{\rho_0}{\rho_1} \right)^2, \quad (24)
\]

\[
I_{mn}^{(\text{TE})} \approx \frac{2q^2\omega_0^3m}{c^2} \left( \frac{j_{m,n}^{(1)} - m^2}{k_n^{(1)}} \right) \left( 1 - \frac{\rho_0}{\rho_1} \right)^2, \quad (25)
\]

under the assumption \( j_{m,n}^{(\sigma)}(1 - \rho_0/\rho_1) \ll 1 \). As a necessary condition for the presence of a radiation of a given type on the frequency \( \omega = m\omega_0 \) one has

\[
n\omega_0 > \omega_{m,1}^{(\sigma)}, \quad \omega_{m,n}^{(\sigma)} = \frac{j_{m,n}^{(\sigma)}}{\rho_1 \sqrt{\varepsilon_0}}, \quad (26)
\]

where we have introduced the boundary frequency \( \omega_{m,n}^{(\sigma)} \) for given \( m, n \) and for a given type of waves \((\sigma = 0, 1)\). In terms of the charge velocity it looks like \( v\sqrt{\varepsilon_0}/c > (j_{m,n}^{(\sigma)}/m)(\rho_0/\rho_1) \). In particular, by taking into account the inequality \( j_{m,1}^{(\sigma)} \geq m \) (see, for example, [13]), as a necessary condition for the presence of a radiation flux through the cross-section of the waveguide we have \( v\sqrt{\varepsilon_0}/c > \rho_0/\rho_1 \). If this condition does not satisfied we have an interesting situation when the rotating particle does not radiate. Note that the boundary frequency \( \omega_{m,n}^{(\sigma)} \) is a characteristic of the waveguide and does not depend on the parameters of the charge (energy, radius of the orbit). In formulae (22) the quantities \( k_n^{(\sigma)} \) are expressed through the boundary frequency by the formula

\[
k_n^{(\sigma)} = \frac{\sqrt{\varepsilon_0}}{c} \sqrt{m^2\omega_0^2 - \omega_{m,n}^{(\sigma)^2}}. \quad (27)
\]

From Eqs. (22), (27) it follows that if the charge is orbiting with frequency \( \omega_0 = \omega_{m,n}^{(\sigma)}/m \), i.e. if the frequency of radiation coincides with the one of boundary frequencies (20), the radiation intensity for the TE waves goes to infinity. Note that in this limit the projection \( k_z = k_n^{(\sigma)} \) goes to zero. However, it is necessary to take into account that under these conditions the absorption in the medium and in the walls of the waveguide becomes important, and consequently it is necessary to take into account the imaginary part \( \varepsilon_0'' \) of the permittivity of the filling medium and finite conductivity of the waveguide walls (the damping constant in the waveguide walls depending on the conductivity is presented, for example, in [14]). Thus, formulae (22) are valid for frequencies, not too close to the boundary ones, when \( k_n^{(\sigma)} \gg \beta_\lambda, m\omega_0\sqrt{\varepsilon''_0}/c \), where \( \beta_\lambda \) is the damping constant due to the ohmic losses in the walls of the waveguide.

Introducing \( \gamma_{m,n}^{(\sigma)} = j_{m,n}^{(\sigma)}/m\rho_1 \), the arguments of the Bessel functions in the numerators of Eqs. (22), (23) can be written as \( \gamma_{m,n}^{(\sigma)} \). Note that for these quantities one has the following inequalities \( \rho_0/\rho_1 < \gamma_{m,n} < v\sqrt{\varepsilon_0}/c \). Let us consider the radiation intensities (22), (23) for large values of \( m \). From the well known properties of the Bessel functions for large values of the order it follows that the behavior of these intensities essentially depends on the sign of \( 1 - \gamma_{m,n} \).

First of all assume that the Cerenkov condition is not satisfied, \( v\sqrt{\varepsilon_0}/c < 1 \), and synchrotron radiation is present only. For \( 1 - v\sqrt{\varepsilon_0}/c \ll 1 \) the radiation intensity has maximum for \( \omega \sim \omega_c = m_c\omega_0 \), with \( m_c = (1 - (\gamma_{m,n}^{(s)})^2)^{-3/2} \), and as it follows from Eq. (15) \( m_c < (1 - v^2\varepsilon_0/c^2)^{-3/2} \). For the frequencies \( \omega \gg \omega_c \) the emitted power exponentially decreases. If the Cerenkov condition is satisfied, for the modes with \( \gamma_{m,n}^{(\sigma)} > 1 \) in addition to the synchrotron radiation we also have Cerenkov radiation. Note that in the waveguide the sufficient condition for this radiation at a
given harmonic $m$ is $\gamma_m^{(\sigma)} \lambda_{max} > 1$ instead of $v\sqrt{\varepsilon_0/c} > 1$. For this modes the radiation intensity for large $m$ approximately linearly increases in the mean with frequency. This is a characteristic feature of the Cerenkov radiation. In this case to obtain a finite result for the total intensity [21] it is necessary to take into account the dispersion for the permittivity. Consider now the limiting case of large values of the waveguide radius $\rho_1 \to \infty$. In this limit the main contribution to the radiation intensity is due to large values $j_m^{(\sigma)}$, when we can use the asymptotic formula [13]

$$j_m^{(\sigma)} \sim \pi \left( n + \frac{m - 1}{2} + \frac{(-1)^n}{4} \right).$$

From this relation it follows that $\Delta \lambda = (j_{m,n+1}^{(\sigma)} - j_m^{(\sigma)})/\rho_1 = \pi/\rho_1$. Now dividing and multiplying expressions [22] by $\pi/\rho_1$ and taking the limit $\rho_1 \to \infty$ the sum over $n$ can be replaced by the integration over $\lambda_0$. As a result one has

$$I_{0m}^{(TM)} = \frac{q^2 v \lambda_m^{(\sigma)}}{\varepsilon_0 \rho_0} \int_0^{\omega_m \lambda_0/c} \frac{\sqrt{m^2 \omega^2_0 \varepsilon_0/c^2 - \lambda_0^2}}{\lambda_0} J_m^2(\lambda_0 \rho_0) d\lambda_0,$$

$$I_{0m}^{(TE)} = \frac{q^2 v^3 \lambda_m^{(\sigma)}}{\varepsilon_0 \rho_0^2} \int_0^{\omega_m \lambda_0/c} \frac{\lambda_0}{\sqrt{m^2 \omega^2_0 \varepsilon_0/c^2 - \lambda_0^2}} J_m^2(\lambda_0 \rho_0) d\lambda_0.\ (30)$$

By introducing a new integration variable $\theta$ according to $\lambda_0 \equiv (m\omega_0 \sqrt{\varepsilon_0/c}) \sin \theta$ for the total radiation intensity (including the regions $z > 0$ and $z < 0$) with the frequency $\omega = m\omega_0$ in the limit $\rho_1 \to \infty$ we receive

$$I_{0m} = 2(I_{0m}^{(TE)} + I_{0m}^{(TM)}) = \frac{2q^2 m^2 \omega^2_0}{c^2 \varepsilon_0} \times$$

$$\times \int_0^{\pi/2} \left[ \cot \theta J_m^2(mv \sqrt{\varepsilon_0} \sin \theta/c) + \frac{\gamma^2}{c^2} \varepsilon_0 J_m^2(mv \sqrt{\varepsilon_0} \sin \theta/c) \right] \sin \theta d\theta\ (31)$$

This formula coincides with the expression for the radiation intensity of a point charge circulating in a homogeneous medium [3, 4].

We carried out numerical calculations for the number of the emitted quanta per one period of the particle orbiting,

$$N_{0m}^{(TE)} = \frac{2\pi}{\hbar m \omega^2_0} I_{0m}^{(TE)}, \quad N_{0m}^{(TM)} = \frac{2\pi}{\hbar m \omega^2_0} I_{0m}^{(TM)},\ \ (32)$$

for various values of the parameters $\varepsilon_0, \rho_1/\rho_0$. In figures 1 and 2 the quantities $N_{0m}^{(TE)}, N_{0m}^{(TM)}$ for the harmonic $m = 24$ are presented as functions on the ratio $\rho_1/\rho_0$ for an electron with the energy $2MeV$ and for the media with permittivities $\varepsilon_0 = 3$ and $\varepsilon_0 = 1$, respectively. The narrow peaks in the graphs of the number of quanta for TE waves correspond to the singularities on the boundary frequencies $\omega_m^{(\sigma)}$. For the values of parameters corresponding to figure 1 one has $v\sqrt{\varepsilon_0/c} \approx 1.67$ and the Cerenkov condition is satisfied. As $J_{m,1}^{(0)}/m \approx 1.24, J_{m,1}^{(1)}/m \approx 1.1$, then the condition (20) is satisfied for all values of the ratio $\rho_1/\rho_0 \geq 1$. As a result the radiation is nonzero for any $\rho_1/\rho_0 > 1$. For values of the parameters corresponding to fig. 2 the Cerenkov’s condition is not satisfied ($v\sqrt{\varepsilon_0/c} \approx 0.97$) and the radiation corresponds to the synchrotron emission. Note that in this case the radiation is much more weaker in comparison to the case of figure 1. Now there is a range of values of the ratio $\rho_1/\rho_0$ for which condition (26) is not satisfied and in this region the radiation is absent. It is well seen in fig. 2. We also carried out numerical calculations for an energy of an election equal to 0.6 MeV and for the same values of the remaining parameters, as in figure 1. In this case the Cerenkov condition is not satisfied and,
Figure 1: The number of quanta emitted on the harmonic $m = 24$ in the form of the TE and TM waves per circulating period of an electron, multiplied by $\frac{\hbar c}{q^2}, \frac{\hbar c N_{m}^{(TE)}}{q^2}$ and $\frac{\hbar c N_{m}^{(TM)}}{q^2}$, versus the ratio $\rho_1/\rho_0$. The energy of an electron is 2 MeV, and the dielectric permittivity for the filling medium is $\varepsilon_0 = 3$.

Figure 2: The same as in figure 1 for $\varepsilon_0 = 1$. 
as calculations have shown, the qualitative behaviour of the $N_m^{(TM)}$ and $N_m^{(TE)}$ in dependence of the ratio $\rho_1/\rho_0$ is similar to the case presented in figure 2 and the radiation intensity is much more weaker to compared with the case of figure 1.

For comparison we have presented in figure 3 the dependences on the permittivity of the number of quanta emitted from an electron in the form of the TM and TE waves in a waveguide and in a homogeneous medium on the harmonic $m = 24$. The energy of an electron is $2\,\text{MeV}$, and $\rho_1/\rho_0 = 1.5$. Note that the intensities of the both types of radiation increase when the permittivity increases.

![Figure 3](image-url)

Figure 3: The number of emitted quanta $\frac{hcN_m}{q^2}$ for the TM and TE waves on the harmonic $m = 24$ as a function on the dielectric permittivity $\varepsilon_0$ in the waveguide and in the homogeneous medium (dashed lines). The energy of an electron is equal to $2\,\text{MeV}$, and $\rho_1/\rho_0 = 1.5$.

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