JOHNSON PSEUDO-CONTRACTIBILITY AND PSEUDO-AMENABILITY OF $\theta$-LAU PRODUCT

M. ASKARI-SAYAH$^1$, A. POURABBAS$^1$, AND A. SAHAMI$^2$

Abstract. Given Banach algebras $A$ and $B$ and $\theta \in \Delta(B)$. We shall study the Johnson pseudo-contractibility and pseudo-amenability of the $\theta$-Lau product $A \times_\theta B$. We show that if $A \times_\theta B$ is Johnson pseudo-contractible, then both $A$ and $B$ are Johnson pseudo-contractible and $A$ has a bounded approximate identity. In some particular cases, a complete characterization of Johnson pseudo-contractibility of $A \times_\theta B$ is given. Also, we show that pseudo-amenability of $A \times_\theta B$ implies the approximate amenability of $A$ and pseudo-amenability of $B$.

1. Introduction

Let $A$ and $B$ be two Banach algebras and $\theta \in \Delta(B)$, where $\Delta(B)$ is the character space of $B$. Then the Banach space $A \times B$ with the product

$$(a,b)(c,d) = (ac + \theta(d)a + \theta(b)c, bd), \quad a,c \in A, b,d \in B,$$

and $\ell^1$-norm becomes a Banach algebra, which is called the $\theta$-Lau product of $A$ and $B$ which is denoted by $A \times_\theta B$. The $\theta$-Lau product was first introduce by A. T. Lau [14] for $F$-algebras. Recently, this product was extended to general Banach algebras by M. Monfared [15] for every Banach algebras $A$ and $B$ and every character $\theta \in \Delta(B)$. One may regard $A \times_\theta B$ as a closed two sided ideal (Banach subalgebra) of $A \times_\theta B$ by identifying it with $A \times \{0\}$ ($\{0\} \times B$), respectively. Therefore, if there is no ambiguity, we may simply write $a (b)$ instead of $(a,0)$ ($(0,b)$) for every $a \in A$ ($b \in B$), respectively. Monfared studied several properties of $A \times_\theta B$ including semisimplicity, Arens regularity, existence of approximate identity and amenability. We recall that the concept of an amenable Banach algebra was introduced by Johnson in 1972. Indeed,
A Banach algebra $A$ is called amenable if there is an element $M \in (A \otimes_p A)^{**}$ such that $a \cdot M = M \cdot a$ and $\pi_A^*(M)a = a$ for every $a \in A$, where $\pi : A \otimes_p A \to A$ is the product morphism and $A \otimes_p A$ is the projective tensor product of $A$. Motivated by this construction of Johnson, some authors introduce several modifications of this notion by relaxing some conditions in different versions of definitions of amenability. The notion of pseudo-amenability was introduced by F. Ghahramani and Y. Zhang [13]. A Banach algebra $A$ is called pseudo-amenable if there is a net $(m_\alpha) \subseteq A \otimes_p A$ such that $a \cdot m_\alpha - m_\alpha \cdot a \to 0$ and $\pi_A(m_\alpha)a \to a$ for every $a \in A$. The concept of approximately amenable Banach algebras was introduced by F. Ghahramani and R. J. Loy in [11], see also [12]. A Banach algebra $A$ is called approximately amenable if there are nets $(M_\alpha) \subseteq A \otimes_p A$, $(F_\alpha) \subseteq A$ and $(G_\alpha) \subseteq A$ such that for every $a \in A$

(i) $a \cdot M_\alpha - M_\alpha \cdot a + F_\alpha \otimes a - a \otimes G_\alpha \to 0$;
(ii) $aF_\alpha \to a$, $G_\alpha a \to a$ and
(iii) $\pi_A(M_\alpha)a - F_\alpha a - G_\alpha a \to 0$.

Recently the second and third authors [19] have defined a new concept related to amenability called Johnson pseudo-contractibility. Indeed, a Banach algebra $A$ is called Johnson pseudo-contractible if there is a not necessarily bounded net $(M_\alpha) \subseteq (A \otimes_p A)^{**}$ such that $a \cdot M_\alpha = M_\alpha \cdot a$ and $\pi_A^*(M_\alpha)a - a \to 0$ for every $a \in A$.

In the Section 2 we deal with Johnson pseudo-contractible Banach algebras. We show that if $A \times_\theta B$ is Johnson pseudo-contractible, then $A$ is Johnson pseudo-contractible and has a bounded approximate identity and $B$ is Johnson pseudo-contractible. Moreover, we show that in particular cases, for example when $A$ is Arens regular and weakly sequentially complete or when $A$ is a dual Banach algebra, Johnson pseudo-contractibility of $A \times_\theta B$ is equivalent with amenability of $A$ and Johnson pseudo-contractibility of $B$. Some example are given at the end of the section.

In the Section 3 we focus on pseudo-amenability of $A \times_\theta B$. Pseudo-amenability of $A \times_\theta B$ was studied by E. Ghaderi et al. [10]. They showed that pseudo-amenability of $A \times_\theta B$ implies pseudo-amenability of $B$, and implies pseudo-amenability of $A$ whenever $A$ has a bounded approximate identity. We show that the existence of bounded approximate identity in this result is not a necessary condition. Indeed, we show that if $A \times_\theta B$ is pseudo-amenable, then $A$ is approximately amenable and $B$ is pseudo-amenable.

2. Johnson Pseudo-Contractibility of $A \times_\theta B$

We state a result from [2] that will be used frequently in this section.

**Theorem 2.1.** Let $A$ be a Johnson pseudo-contractible Banach algebra with an identity. Then $A$ is amenable.

**Lemma 2.1.** Let $A$ be a Johnson pseudo-contractible Banach algebra and let $I$ be a two sided closed ideal of $A$. If $I$ has a bounded approximate identity, then $I$ is Johnson pseudo-contractible.
Proof. By hypothesis there is a net \((M_\alpha) \subseteq (A \otimes_p A)^{**}\) such that \(a \cdot M_\alpha = M_\alpha \cdot a\) and \(\pi_\alpha^{**}(M_\alpha) a - a \to 0\) for every \(a \in A\). Let \((e_\beta)\) be a bounded approximate identity for \(I\) and let \(E\) be a weak* cluster point of \((e_\beta)\) in \(I^{**}\). Then by setting \((N_\alpha) = (E \cdot M_\alpha \cdot E) \subseteq (I \otimes_p I)^{**}\), we have

\[ x \cdot N_\alpha = N_\alpha \cdot x, \]

and

\[ \pi_I^{**}(N_\alpha)x = \pi_\alpha^{**}(E \cdot M_\alpha \cdot E)x = \pi_\alpha^{**}(M_\alpha)x \to x, \]

for every \(x \in I\). It follows that \(I\) is Johnson pseudo-contractible.

\(\Box\)

**Theorem 2.2.** Let \(A\) and \(B\) be two Banach algebras and \(\theta \in \Delta(B)\). If \(A \times_\theta B\) is Johnson pseudo-contractible, then the following statements hold.

(a) \(A\) is Johnson pseudo-contractible and has a bounded approximate identity.

(b) \(B\) is Johnson pseudo-contractible.

Proof. Suppose that \(\Phi : (A \times_\theta B) \otimes_p (A \times_\theta B) \to A \times_\theta B\) is the linear map determined by

\[ \Phi((a, b) \otimes (c, d)) = \theta(d)(a, b), \quad a, c \in A, \ b, d \in B. \]

Let \((U_\alpha) \subseteq ((A \times_\theta B) \otimes_p (A \times_\theta B))^{**}\) be such that

\[ (a, b) \cdot U_\alpha = U_\alpha \cdot (a, b), \quad \pi_{A \times_\theta B}^{**}(U_\alpha)(a, b) \to (a, b), \]

for every \(a \in A\) and \(b \in B\). Then by Goldstine’s theorem for every \(\alpha\) there exists a net \((u_{\alpha, \beta})\) in \((A \times_\theta B) \otimes_p (A \times_\theta B)\) such that \(w^* - \lim_{\beta} u_{\alpha, \beta} = U_\alpha\). Suppose that

\[ u_{\alpha, \beta} = \sum_{i=1}^\infty (a_i^{\alpha, \beta}, b_i^{\alpha, \beta}) \otimes (c_i^{\alpha, \beta}, d_i^{\alpha, \beta}) \]

for sequences \((a_i^{\alpha, \beta})\), \((c_i^{\alpha, \beta})\) \(\subseteq A\) and \((b_i^{\alpha, \beta}), (d_i^{\alpha, \beta})\) \(\subseteq B\), where

\[ \sum_{i=1}^\infty ||(a_i^{\alpha, \beta}, b_i^{\alpha, \beta})|| \cdot ||(c_i^{\alpha, \beta}, d_i^{\alpha, \beta})|| < \infty. \]

Note that \(\theta\) has an extension \(\tilde{\theta} \in \Delta(B^{**})\) given by \(\tilde{\theta}(F) = F(\theta)\) for every \(F \in B^{**}\). Since \(\Phi\) and \(\theta\) are bounded, \(\Phi^{**}\) and \(\tilde{\theta}\) are weak* continuous maps. Now we have

\[ \langle (0, \tilde{\theta}), \Phi^{**}(U_\alpha) \rangle = w^* - \lim_{\beta} \langle (0, \theta), \Phi(u_{\alpha, \beta}) \rangle \]

\[ = w^* - \lim_{\beta} \langle (0, \theta), \pi_{A \times_\theta B}^{**}(u_{\alpha, \beta}) \rangle \]

\[ = \langle (0, \tilde{\theta}), \pi_{A \times_\theta B}^{**}(U_\alpha) \rangle \to 1. \]

Set \(\Phi^{**}(U_\alpha) = (\phi_\alpha, \psi_\alpha)\), where \(\phi_\alpha \in A^{**}\) and \(\psi_\alpha \in B^{**}\). We can see that \(\tilde{\theta}(\psi_\alpha) \to 1\). Take \(\alpha_0\) such that \(\tilde{\theta}(\psi_{\alpha_0}) \neq 0\), for every \(a \in A\) we have

\[ a\Phi^{**}(U_{\alpha_0}) = \Phi^{**}(a \cdot U_{\alpha_0}) = \Phi^{**}(U_{\alpha_0} \cdot a) = 0. \]

Also, we have

\[ a\Phi^{**}(U_{\alpha_0}) = (a, 0)(\phi_{\alpha_0}, \psi_{\alpha_0}) = (a\phi_{\alpha_0} + \tilde{\theta}(\psi_{\alpha_0})a, 0). \]
Therefore \( a\phi_\alpha + \tilde{\theta}(\psi_\alpha) a = 0 \), so \( a(-\tilde{\theta}(\psi_\alpha)^{-1}\phi_\alpha) = a \), where \( -\tilde{\theta}(\psi_\alpha)^{-1}\phi_\alpha \in A^{**} \). This shows that \( A \) has a bounded right approximate identity. A similar argument shows that \( A \) has a bounded left approximate identity. Since \( A \) is a two sided closed ideal of \( (A \times B) \) and has a bounded approximate identity, by Lemma 2.1 it is Johnson pseudo-contractible.

It is well known that \( (A \times B)/A \cong B \) and there is a surjective homomorphism from \( A \times B \) onto \( (A \times B)/A \). So, [19, Proposition 2.9] implies Johnson pseudo-contractibility of \( B \). □

We remark that the converse of the previous theorem does not hold in general. For example, \( A(H) \), the Fourier algebra on the integer Heisenberg group \( H \), is Johnson pseudo-contractible and has a bounded approximate identity and \( M(H) \), the measure algebra over \( H \), is Johnson pseudo-contractible (\( H \) is discrete and amenable). But \( A(H) \times_B M(H) \) is not Johnson pseudo-contractible for every \( \theta \in \Delta(M(H)) \). Indeed, \( A(H) \times_B M(H) \) has an identity [15, Proposition 2.3]. If \( A(H) \times_B M(H) \) is Johnson pseudo-contractible, then, by Theorem 2.1, \( A(H) \times_B M(H) \) is amenable and [15, page 285] implies the amenability of \( A(H) \). It gives a contradiction that \( H \) has an abelian subgroup of finite index, see [9, Theorem 2.3].

From [15, page 285] and Theorem 2.1, we have the following corollary.

**Corollary 2.1.** If \( B \) has an identity, then the following statements are equivalent:

(a) \( A \times_B B \) is Johnson pseudo-contractible;
(b) \( A \times_B B \) is amenable;
(c) \( A \) and \( B \) are amenable.

**Corollary 2.2.** If \( A \) has an identity, then \( A \times_B B \) is Johnson pseudo-contractible if and only if \( A \) is amenable and \( B \) is Johnson pseudo-contractible.

**Proof.** In view of [3] \( A \times_B B \) is nothing but the \( \ell^1 \)-direct sum \( A \oplus B \) with coordinatewise product whenever \( A \) has an identity. If \( A \) is amenable and \( B \) is Johnson pseudo-contractible, then \( A \oplus B \) is Johnson pseudo-contractible by [19, Theorem 2.11]. The converse comes immediately from Theorem 2.2 and Theorem 2.1. □

A Banach algebra \( A \) is called dual if it is a dual space such that multiplication in \( A \) is separately \( w^* \)-continuous. It is well known that a dual Banach algebra with a bounded approximate identity has an identity [18, Proposition 1.2], so we have the following corollary from Theorem 2.2 and Corollary 2.2.

**Corollary 2.3.** Let \( B \) be a Banach algebra and let \( A \) be a dual Banach algebra and \( \theta \in \Delta(B) \). Then \( A \times_B B \) is Johnson pseudo-contractible if and only if \( A \) is amenable and \( B \) is Johnson pseudo-contractible.

A Banach algebra \( A \) is called Arens regular if the first and the second Arens products on \( A^{**} \) coincide. Also, a Banach algebra \( A \) is called weakly sequentially complete if every weakly Cauchy sequence in \( A \) is weakly convergent.
Proposition 2.1. Suppose that $A$ and $B$ are two Banach algebras and $\theta \in \Delta(B)$. If $A$ is Arens regular and weakly sequentially complete, then $A \times_\theta B$ is Johnson pseudo-contractible if and only if

(a) $A$ is amenable and has an identity;
(b) $B$ is Johnson pseudo-contractible.

Proof. If $A \times_\theta B$ is Johnson pseudo-contractible, then, by Theorem 2.1, $A$ has a bounded approximate identity. Using Ülger theorem [4, Theorem 2.9.39], $A$ has an identity. Now apply Corollary 2.2. □

It seems that Johnson pseudo-contractibility of $A \times_\theta B$ is related with amenability of $A$. We believe that Corollary 2.2 holds without the assumption that $A$ has an identity. However, it remains as a conjecture. We left it as an open problem in the following questions.

Question 1. Does Johnson pseudo-contractibility of $A \times_\theta B$ implies the amenability of $A$?

Question 2. Suppose that $A$ is an amenable Banach algebra and $B$ is a Johnson pseudo-contractible Banach algebra and $\theta \in \Delta(B)$. Is $A \times_\theta B$ a Johnson pseudo-contractible Banach algebra?

We finish this section with some examples. First we recall some concepts and notations from semigroup theory. A semigroup $S$ is called regular if for every $s \in S$ there exists an element $t \in S$ such that $sts = s$ and $tst = t$. A semigroup $S$ is an inverse semigroup if for every $s \in S$ there exists a unique element $t \in S$ such that $sts = s$ and $tst = t$. The set of idempotents of a semigroup $S$ is denoted by $E(S)$, which is a partially ordered set with the following order

$$p \leq q \iff p = pq = qp, \quad p, q \in E(S).$$

For $p \in E(S)$, we set $\{p\} = \{x : x \leq p\}$. An inverse semigroup $S$ is called uniformly locally finite if $\sup\{|(p) : p \in E(S)\} < \infty$. It is well known that the discrete semigroup algebra $\ell^1(S)$ is weakly sequentially complete [4, Theorem A.4.4]. Our main reference for semigroup theory is [5].

Example 2.1. Suppose that $B$ is a Banach algebra and $\theta \in \Delta(B)$.

(i) Let $S$ be a uniformly locally finite inverse semigroup. Then Johnson pseudo-contractibility of $\ell^1(S) \times_\theta B$ implies that $\ell^1(S)$ is Johnson pseudo-contractible and has a bounded approximate identity. From [16, Proposition 2.1] $E(S)$ must be finite and from [20, Theorem 2.3] every maximal subgroup of $S$ is amenable, in other word $\ell^1(S)$ is amenable, see [7].

(ii) Suppose that $S$ is regular and $\ell^1(S)$ is Arens regular. If $\ell^1(S) \times_\theta B$ is Johnson pseudo-contractible, then, by Proposition 2.1, $\ell^1(S)$ is amenable and has an identity. So, by [7], $E(S)$ is finite. Now [5, Theorem 12.2] implies that $S$ is a unital finite semigroup. Indeed, $\ell^1(S) \times_\theta B$ is Johnson pseudo-contractible if and only if $S$ is a unital finite semigroup and $B$ is Johnson pseudo-contractible.
Example 2.2. Using [8, Theorem 3.1] one can see that $M_I(\mathbb{C})$ (the Banach algebra of $I \times I$-matrices over $\mathbb{C}$, with finite $\ell^1$-norm and matrix multiplication) has no bounded approximate identity unless $I$ is finite, but in this case $M_I(\mathbb{C})$ is amenable and has an identity. So, for Banach algebra $B$ and $\theta \in \Delta(B)$, $M_I(\mathbb{C}) \times_\theta B$ is Johnson pseudo-contractible if and only if $I$ is finite and $B$ is Johnson pseudo-contractible.

A linear subspace $S^1(G)$ of $L^1(G)$ is said to be a Segal algebra on $G$ if it satisfies the following conditions:

(i) $S^1(G)$ is dense in $L^1(G)$;
(ii) $S^1(G)$ with a norm $\|\cdot\|_{S^1(G)}$ is a Banach space and $\|f\|_{L^1(G)} \leq \|f\|_{S^1(G)}$ for every $f \in S^1(G)$;
(iii) $S^1(G)$ is left translation invariant (that is, $L_y f \in S^1(G)$ for every $f \in S^1(G)$ and $y \in G$) and the map $y \mapsto L_y f$ from $G$ into $S^1(G)$ is continuous, where $L_y(f)(x) = f(y^{-1}x)$;
(iv) $\|L_y(f)\|_{S^1(G)} = \|f\|_{S^1(G)}$, for every $f \in S^1(G)$ and $y \in G$.

Example 2.3. Suppose that $B$ is a Banach algebra and $\theta \in \Delta(B)$. Let $S^1(G)$ be a Segal algebra on $G$. If $S^1(G) \times_\theta B$ is Johnson pseudo-contractible, then $S^1(G) = L^1(G)$.

3. PSEUDO-AMENABILITY OF $A \times_\theta B$

Remark 3.1. Note that if $U \in (A \times_\theta B) \otimes_p (A \times_\theta B)$, then there are $M \in A \otimes_p A$, $N \in A \otimes_p B$, $L \in B \otimes_p A$ and $H \in B \otimes_p B$ such that

$U = M + N + L + H$

and

$\|U\|_{(A \times_\theta B) \otimes_p (A \times_\theta B)} = \|M\|_{A \otimes_p A} + \|N\|_{A \otimes_p B} + \|L\|_{B \otimes_p A} + \|H\|_{B \otimes_p B}$.

Theorem 3.1. Suppose that $A$ and $B$ are Banach algebras and $\theta \in \Delta(B)$. If $A \times_\theta B$ is pseudo-amenable, then

(a) $A$ is approximate amenable and
(b) $B$ is pseudo-amenable.

Proof. It is well known that $(A \times_\theta B)/A \cong B$ and there is a surjective homomorphism from $A \times_\theta B$ onto $(A \times_\theta B)/A$. So [13, Proposition 2.2] implies pseudo-amenability of $B$.

By assumption there is a net $(U_\alpha) \subseteq (A \times_\theta B) \otimes_p (A \times_\theta B)$ such that

$(x, y) \cdot U_\alpha - U_\alpha \cdot (x, y) \to 0, \quad \pi(U_\alpha)(x, y) \to (x, y)$

for every $x \in A$, $y \in B$. Particularly for every $x \in A$ we have

(3.1) $x \cdot U_\alpha - U_\alpha \cdot x \to 0, \quad \pi(U_\alpha)x \to x$.

Suppose that $U_\alpha = \sum_{i=1}^\infty (a_i^\alpha, b_i^\alpha) \otimes (c_i^\alpha, d_i^\alpha)$ for sequences $(a_i^\alpha), (c_i^\alpha) \subseteq A$ and $(b_i^\alpha), (d_i^\alpha) \subseteq B$, where $\sum_{i=1}^\infty \|((a_i^\alpha, b_i^\alpha)) \cdot (((c_i^\alpha, d_i^\alpha))\| < \infty$. Set $M_\alpha = \sum_{i=1}^\infty a_i^\alpha \otimes c_i^\alpha$, $F_\alpha = -\sum_{i=1}^\infty \theta(d_i^\alpha)a_i^\alpha$, $\|$}$ I$ and $\mathbb{C}$,
\( G_\alpha = -\sum_{i=1}^{\infty} \theta(b_i^\alpha)c_i^\alpha \) and \( H_\alpha = \sum_{i=1}^{\infty} b_i^\alpha \otimes d_i^\alpha \). One can easily see that
\[
\pi_{A\times B}(U_\alpha) = (\pi_A(M_\alpha) - F_\alpha - G_\alpha, \pi_B(H_\alpha)).
\]
For an arbitrary element \( b \) in \( B \), we have
\[
\pi_{A\times B}(U_\alpha)(0, b) = (\theta(b)(\pi_A(M_\alpha) - F_\alpha - G_\alpha), \pi_B(H_\alpha)b) \to (0, b),
\]
so
\[
\pi_A(M_\alpha) - F_\alpha - G_\alpha \to 0, \quad \theta(\pi_B(H_\alpha)) \to 1.
\]
Note that
(3.2)
\[
x \cdot U_\alpha = \sum_{i=1}^{\infty} (x, 0)(a_i^\alpha, 0) \otimes (c_i^\alpha, 0) + \sum_{i=1}^{\infty} (x, 0)(b_i^\alpha) \otimes (c_i^\alpha, 0)
\]
\[
+ \sum_{i=1}^{\infty} (x, 0)(a_i^\alpha, 0) \otimes (0, d_i^\alpha) + \sum_{i=1}^{\infty} (x, 0)(b_i^\alpha) \otimes (0, d_i^\alpha)
\]
\[
= x \cdot \left( \sum_{i=1}^{\infty} (a_i^\alpha \otimes c_i^\alpha) \right) + \sum_{i=1}^{\infty} (x \otimes (b_i^\alpha)c_i^\alpha) + \sum_{i=1}^{\infty} (xa_i^\alpha \otimes d_i^\alpha) + \sum_{i=1}^{\infty} (\theta(b_i^\alpha)x \otimes d_i^\alpha)
\]
\[
= x \cdot M_\alpha - x \otimes G_\alpha + \sum_{i=1}^{\infty} (xa_i^\alpha \otimes d_i^\alpha) + \sum_{i=1}^{\infty} (\theta(b_i^\alpha)x \otimes d_i^\alpha).
\]
Similarly we have
(3.3)
\[
U_\alpha \cdot x = M_\alpha \cdot x - F_\alpha \otimes x + \sum_{i=1}^{\infty} (b_i^\alpha \otimes c_i^\alpha x) + \sum_{i=1}^{\infty} (b_i^\alpha \otimes \theta(d_i^\alpha)x).
\]
From (3.2), (3.3) and (3.1), by using Remark 3.1 we obtain
(a) \( x \cdot M_\alpha - M_\alpha \cdot x + F_\alpha \otimes x - x \otimes G_\alpha \to 0; \)
(b) \( \sum_{i=1}^{\infty} (xa_i^\alpha \otimes d_i^\alpha) + \sum_{i=1}^{\infty} (\theta(b_i^\alpha)x \otimes d_i^\alpha) \to 0; \)
(c) \( \sum_{i=1}^{\infty} (b_i^\alpha \otimes c_i^\alpha x) + \sum_{i=1}^{\infty} (b_i^\alpha \otimes \theta(d_i^\alpha)x) \to 0. \)
Define a bounded linear map \( \phi : A \otimes_p B \to A \) by \( \phi(a \otimes b) = \theta(b)a \). From (b) we have
\[
-xF_\alpha + \theta(\pi_B(H_\alpha))x = x \sum_{i=1}^{\infty} \theta(d_i^\alpha)a_i^\alpha + \sum_{i=1}^{\infty} \theta(b_i^\alpha d_i^\alpha)x
\]
\[
= \phi \left( \sum_{i=1}^{\infty} (xa_i^\alpha \otimes d_i^\alpha) + \sum_{i=1}^{\infty} (\theta(b_i^\alpha)x \otimes d_i^\alpha) \right) \to 0,
\]
now \( \theta(\pi_B(H_\alpha)) \to 1 \) implies that \( xF_\alpha \to x \). Similarly, by using (c) we have \( G_\alpha x \to x \).
So we find \( (M_\alpha) \subseteq A \otimes_p A, (F_\alpha) \subseteq A \) and \( (G_\alpha) \subseteq A \) such that
(a) \( x \cdot M_\alpha - M_\alpha \cdot x + F_\alpha \otimes x - x \otimes G_\alpha \to 0; \)
(b) \( xF_\alpha \to x, \quad G_\alpha x \to x; \)
(c) \( \pi_A(M_\alpha)x - F_\alpha x - G_\alpha x \to 0, \)
for every \( x \in A \). It follows that \( A \) is approximately amenable. \( \square \)

**Example 3.1.** Let \( S \) be a uniformly locally finite inverse semigroup and let \( B \) be a Banach algebra and \( \theta \in \Delta(B) \). If \( \ell^1(S) \times_\theta B \) is pseudo-amenable, then by Theorem 3.1 \( \ell^1(S) \) is approximately amenable. Theorem 4.3 of [17] shows that \( \ell^1(S) \) is amenable.

**Example 3.2.** Let \( G = SU(2) \) be the \( 2 \times 2 \) unitary group, and suppose that \( S^1(G) \neq L^1(G) \) is a Segal algebra on \( G \). In [1] Alaghmandan showed that \( S^1(G) \) is not approximately amenable. Thus, by Theorem 3.1, \( S^1(G) \times_\theta B \) is not pseudo-amenable for every Banach algebra \( B \) and \( \theta \in \Delta(B) \).

**Example 3.3.** Let \( G \) be an infinite abelian compact group and let \( B \) be a Banach algebra and \( \theta \in \Delta(B) \). We claim that \( L^2(G) \times_\theta B \) is not pseudo-amenable. To see this, suppose that \( L^2(G) \times_\theta B \) is pseudo-amenable. Then Theorem 3.1 implies that \( L^2(G) \) is approximately amenable. But by the Plancherel theorem \( L^2(G) \) is isometrically isomorphism to \( \ell^2(\hat{G}) \), where \( \hat{G} \) is the dual group of \( G \) and \( \ell^2(\hat{G}) \) is equipped with the pointwise product. So \( \ell^2(\hat{G}) \) is approximately amenable which is a contradiction with the main result of [6].

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1Faculty of Mathematics and Computer Science, Amirkabir University of Technology, 424 Hafez Avenue, 15914 Tehran, Iran
Email address: mehdi17@aut.ac.ir
Email address: arpakabas@aut.ac.ir

2Department of Mathematics, Faculty of Basic Sciences, Ilam University, P.O. Box 69315-516, Ilam, Iran
Email address: a.sahami@ilam.ac.ir