RANK 3 QUADRATIC GENERATORS OF VERONese EMBEDDINGS

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Abstract. Let $L$ be a very ample line bundle on a projective scheme $X$ defined over an algebraically closed field $k$ with char $k \neq 2$. We say that $(X, L)$ satisfies property QR($k$) if the homogeneous ideal of the linearly normal embedding $X \subset \mathbb{P}H^0(X, L)$ can be generated by quadrics of rank $\leq k$. Many classical varieties such as Segre-Veronese embeddings, rational normal scrolls and curves of high degree satisfy property QR($4$).

In this paper, we first prove that if char $k \neq 3$ then $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ satisfies property QR($3$) for all $n \geq 1$ and $d \geq 2$. We also investigate an asymptotic behavior of property QR($3$) for any projective scheme. Namely, we prove that (i) if $X \subset \mathbb{P}H^0(X, L)$ is $m$-regular then $(X, L^d)$ satisfies property QR($3$) for all $d \geq m$ and (ii) if $A$ is an ample line bundle on $X$ then $(X, A^d)$ satisfies property QR($3$) for all sufficiently large even number $d$. These results provide an affirmative evidence for the expectation that property QR($3$) holds for all sufficiently ample line bundles on $X$, as in the cases of Green-Lazarsfeld’s condition $N_p$ and Eisenbud-Koh-Stillman’s determinantal presentation in [EKS88]. Finally, when char $k = 3$ we prove that $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2))$ fails to satisfy property QR($3$) for all $n \geq 3$.

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1. Introduction

The study about the interaction between geometric properties of a projective variety $X \subset \mathbb{P}^n$ and structural properties of its defining ideal $I(X)$ is one of the central issues in algebraic geometry. For the last few decades, the problem of giving conditions to guarantee that $I(X)$ is generated by quadrics and the first few syzygy modules are generated by linear syzygies have attracted considerable attention (cf. [Gr84a], [Gr84b], [GL88], [EL93], [GP99], [Ina97], etc). Another important direction to study structures of defining equations of $X$ is to examine whether $I(X)$ is defined by 2-minors of one or several linear
determinantal presentations (cf. [EKS88], [Puc98], [Ha02], [B08], [SS11], etc). In such a case, \( I(X) \) admits a generating set consisting of quadrics of rank \( \leq 4 \). The main purpose of this paper is to exhibit many cases where \( I(X) \) is generated by quadrics of rank 3. In particular, we prove that every Veronese embedding of \( \mathbb{P}^n \) by \( \mathcal{O}_{\mathbb{P}^n}(d) \) has such a property unless \( \text{char} \ k = 3 \).

To state our results precisely, we begin with some notation and definitions. Let \( X \subset \mathbb{P}^n \) be any projective scheme over an algebraically closed field \( k \). Through this paper, we assume that \( \text{char} \ k \neq 2 \) since the rank of a quadric is not well-defined over characteristic 2 (e.g. \( xy \)). We say that \( X \subset \mathbb{P}^n \) satisfies property \( QR(k) \) if \( I(X) \) is generated by quadrics of rank at most \( k \). For a very ample line bundle \( L \) on a projective scheme \( X \), we will say that \( L \) satisfies property \( QR(k) \) when so does the linearly normal embedding \( X \subset \mathbb{P}H^0(X, L) \).

We can reinterpret property \( QR(k) \) as follows. Let \( \mathbb{P}^N, N = \binom{n+2}{2} - 1 \), be the space of quadrics in \( \mathbb{P}^n \) and let \( \Phi_k, 1 \leq k \leq n \), denote the variety of all quadrics of rank at most \( k \). Now, consider the subspace \( \mathbb{P}(I(X)_2) \) of \( \mathbb{P}^N \) and \( \Phi_k(X) := \Phi_k \cap \mathbb{P}(I(X)_2) \) as a projective algebraic set in \( \mathbb{P}(I(X)_2) \). In this framework, \( X \) satisfies property \( QR(k) \) if and only if \( \Phi_k(X) \) is nondegenerate in \( \mathbb{P}(I(X)_2) \).

There are lots of examples of property \( QR(4) \) in the literature. First, let \( L \) be a line bundle of degree \( d \) on a smooth curve \( C \) of genus \( g \). In [St.D72], B. Saint-Donat proved that if \( d \geq 2g + 2 \) then \( C \subset \mathbb{P}H^0(C, L) \) satisfies property \( QR(4) \). When \( C \) is non-hyperelliptic, non-trigonal and not isomorphic to a plane quintic, M. Green [Gr84] reproved the classical Torelli’s Theorem by showing that the canonical embedding \( \phi_{K_C}(C) \subset \mathbb{P}^{g-1} \) satisfies property \( QR(4) \). See also [Pe23] [St.D73] [AH81]. Many classical constructions in projective geometry such as rational normal scrolls, Veronese varieties and Segre varieties of two projective spaces are determinantly presented in the sense that their homogeneous ideals are generated by 2-minors of a 1-generic matrix of linear forms (see e.g. [EKS88], [Harr92], [Puc98], [Ha02]). Furthermore, any Segre-Veronese variety is defined ideal-theoretically by 2-minors of several linear determinantal presentations, which are also called ‘flattenings’ (see [B08]). Recently, Sidman and Smith in [SS11] proved that every sufficiently ample line bundle on a projective connected scheme is determinantly presented. So, they all satisfy property \( QR(4) \).

Our first main theorem is about the property \( QR(3) \) of the Veronese variety \( V_{n,d} := \nu_d(\mathbb{P}^n) \subset \mathbb{P}^N, N = \binom{n+d}{n} - 1 \), which is unexpected(!).

**Theorem 1.1.** Let \( d, n \) be any positive integers and suppose that \( \text{char} \ k \neq 2, 3 \). Then \( (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \) satisfies property \( QR(3) \).

For the proof of this result, see Theorem 4.2.

To prove Theorem 1.1, the first step will be to find many quadratic equations of rank 3 in \( I(V_{n,d}) \). To this aim, we use two methods. To explain the first one, let \( f : H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \rightarrow H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)) \) be the natural isomorphism. Then we obtain the following map

\[
Q : H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \times H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \times H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d-2)) \rightarrow I(V_{n,d})_2
\]
defined by
\[ Q(s, t, h) = f(s \otimes s \otimes h) f(t \otimes t \otimes h) - f(s \otimes t \otimes h)^2. \]
This map is well-defined since \( Q(s, t, h) \) is either 0 or else a rank 3 quadratic equation of \( V_{n,d} \). The second method is the use of the natural group action of \( \text{PGL}_n(k) \) on \( I(V_{n,d})_2 \), by which the rank is preserved. That is, any \( Q \in I(V_{n,d})_2 \) of rank 3 and \( \sigma \in \text{PGL}_n(k) \) give us a new element \( \sigma(Q) \in I(V_{n,d})_2 \) of rank 3.

We prove Theorem 1.1 by using the double induction on \( n \geq 1 \) and \( d \geq 2 \). So, we first prove our theorem for the cases where \( n = 1 \) (Corollary 2.4) and \( d = 2 \) (Theorem 3.1), respectively. Then we complete the proof by combining the induction hypothesis and the above two methods. In fact, this step is not simple and we must deal with more than 10 partial cases.

Let \( W \) be the subspace of \( I(V_{n,d}) \) spanned by the image of the above \( Q \)-map. In Theorem 2.2, we provide an explicitly defined finite subset \( \Gamma \) of the image of the \( Q \)-map which spans \( W \). See Notation and Remarks 2.1 for the definition of \( \Gamma \). In the proof of Theorem 1.1, it is shown that \( I(V_{n,d}) \) is generated by \( \Gamma \). In particular, the property \( QR(3) \) of \((\mathbb{P}^n, O_{\mathbb{P}^n}(d))\) is induced by the decomposition \( O_{\mathbb{P}^n}(d) = O_{\mathbb{P}^n}(1)^2 \otimes O_{\mathbb{P}^n}(d-2) \).

Our second main result is

**Theorem 1.2.** Suppose that \( \text{char } k = 3 \). Then
1. \((\mathbb{P}^1, O_{\mathbb{P}^1}(d))\) satisfies property \( QR(3) \) for all \( d \geq 2 \).
2. \((\mathbb{P}^n, O_{\mathbb{P}^n}(2))\) satisfies property \( QR(3) \) if and only if \( n \leq 2 \).

For the proof of this result, see Theorem 3.1.

For the cases \( n = 1 \) and \( n = d = 2 \), our proof of Theorem 1.1 is indeed characteristic free. To prove the failure of property \( QR(3) \) of \((\mathbb{P}^n, O_{\mathbb{P}^n}(2))\) for \( n \geq 3 \), a crucial point is that any quadratic equation of \( \nu_2(\mathbb{P}^n) \) of rank 3 is obtained from the above \( Q \)-map. Then, for \( n = 3 \) we find a subset \( \Gamma \) of \( I(\nu_2(\mathbb{P}^n)) \) with \( |\Gamma| = 19 \) which generates the subspace spanned by the image of the \( Q \)-map. Since \( I(\nu_2(\mathbb{P}^n))_2 \) is of 20-dimension, this shows that \((\mathbb{P}^3, O_{\mathbb{P}^3}(2))\) fails to satisfy property \( QR(3) \). For \( n \geq 4 \), the proof comes from the fact that \( \nu_2(\mathbb{P}^n) \) contains \( \nu_2(\mathbb{P}^3) \) as an ideal-theoretic linear section (see Remark 5.1).

When \( \text{char } k = 3 \), \( n \geq 2 \) and \( d \geq 3 \), we do not know yet whether \((\mathbb{P}^n, O_{\mathbb{P}^n}(d))\) satisfies property \( QR(3) \) or not.

Our third main result is about the asymptotic nature of the rank of quadratic equations of the Veronese re-embedding of \((X, L)\) when \( X \) is an arbitrary projective scheme and \( L \) is a very ample line bundle on \( X \). In this direction, the first result is due to D. Mumford, who proved in [M70, Theorem 1] that if \( X \subset \mathbb{P}^n \) is a nondegenerate irreducible projective variety, then

(i) the \( d \)th Veronese re-embedding of \( X \) is set-theoretically defined by quadrics, and
(ii) those quadrics can be chosen as quadrics of rank at most 4 for all \( d \geq \text{deg}(X) \). Since [M70] had appeared, there have been several interesting generalizations. In [Gr84a] and [Gr84b], Green considered the quadratic generation of the homogeneous ideal as the first step towards understanding higher syzygies. So far, numerous results have been reported in this direction. Due to Green-Lazarsfeld in [GL88],
we say that $L$ satisfied condition $N_p$ for some $p \geq 1$ if $X \subset \mathbb{P} H^0(X, L)$ is projectively normal and ideal-theoretically cut out by quadrics such that the first $(p-1)$ steps of the minimal free resolution of $I(X)$ are linear. In [EL93], Ein and Lazarsfeld proved that if $X$ is a complex smooth variety and $L$ is a very ample line bundle on $X$ of degree $d_0$ then $L^d$ satisfies condition $N_{d+d_0}$ (see also [GP99]). Thus this result generalizes the statement (i) above. Also the statement (ii) is widely extended by many results on the determinantal presentation of projective varieties, as mentioned above.

Along this line, our third main result is

**Theorem 1.3.** Suppose that $\text{char } k \neq 2, 3$ and let $L$ be a very ample line bundle on a projective scheme $X$ defining the linearly normal embedding $X \subset \mathbb{P} H^0(X, L)$.

If $m$ is an integer such that $X$ is $j$-normal for all $j \geq m$ and $I(X)$ is generated by forms of degree $\leq m$, then $(X, L^d)$ satisfies property $QR(3)$ for all $d \geq m$.

For the proof of this result, see the beginning of §5.

In Theorem 1.3, we can take $m$ to be the regularity of $X \subset \mathbb{P} H^0(X, L)$ in the sense of Castelnuovo-Mumford.

As an immediate application of Theorem 1.3 suppose that $(X, L)$ satisfies Green-Lazarsfeld’s condition $N_1$. Then one could take $m = 2$ and hence $(X, L^d)$ satisfies property $QR(3)$ for all $d \geq 2$ (see Corollary 5.2). This can be applied to the Grassmannian manifolds (see Example 5.5). In §5 we provide a few examples which illustrate how to apply Theorem 1.3 to specific varieties.

**Remark 1.4.** Let $X$ be a projective scheme. Our main results in this paper show that there is a significant correlation between some positive nature of the very ample line bundle $L$ on $X$ and the property $QR(3)$ of $(X, L)$, just like in many works on Green-Lazarsfeld’s condition $N_p$ and Eisenbud-Koh-Stillman’s determinantal presentation of very ample line bundles. We will discuss more on this direction in §6.

The paper is structured as follows. In §2 we develop a method to generate rank 3 quadratic equations of $X \subset \mathbb{P} H^0(X, L)$. Also we prove that every rational normal curve satisfies property $QR(3)$. In §3 we study about property $QR(3)$ of the second Veronese embedding $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2))$. In §4 we prove that every Veronese embedding $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ satisfies property $QR(3)$ unless $\text{char } k = 2, 3$. In §5 we study the asymptotic behavior of the property $QR(3)$ of $(X, L^d)$ when $X$ is an arbitrary projective scheme and $L$ is a very ample line bundle on $X$. We also provide some applications of the results to the case of complete intersections, Grassmannian manifolds, Abelian varieties, Enriques surfaces and K3 surfaces. We finish the paper by giving relevant examples, some problems for further direction in §6.

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2. Rank 3 quadratic generators

This section is devoted to introduce our method to generate rank 3 quadrics in the ideal of a projective scheme.

Notation and Remarks 2.1. Let $X$ be a projective scheme and $L$ a very ample line bundle on $X$. Suppose that $L$ is decomposed as $L = L_1^{\otimes 2} \otimes L_2$ where $L_1$ and $L_2$ are line bundles on $X$ such that

$$p := h^0(X, L_1) \geq 2 \quad \text{and} \quad q := h^0(X, L_2) \geq 1.$$  

1. The linearly normal embedding $X \subset \mathbb{P} H^0(X, L) = \mathbb{P}^r$ induces an isomorphism 

$$f : H^0(X, L) \rightarrow H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1)).$$

2. Define the map 

$$Q = Q_{L_1, L_2} : H^0(X, L_1) \times H^0(X, L_1) \times H^0(X, L_2) \rightarrow I(X)_2$$

by 

$$Q(s, t, h) = f(s \otimes s \otimes h)f(t \otimes t \otimes h) - f(s \otimes t \otimes h)^2.$$ 

This map is well-defined since the restriction of $Q(s, t, h)$ to $X$ becomes 

$$(s \otimes s \otimes h)|_X \times (t \otimes t \otimes h)|_X - (s \otimes t \otimes h)|_X^2 = 0$$

(cf. [Eis05] Proposition 6.10). Thus, $Q(s, t, h)$ is either 0 or else a rank 3 quadratic equation of $X$ (see also Lemma 2.3).

3. (A sufficient way to guarantee $QR(3)$) Let $\mathcal{Q} = \mathcal{Q}_{L_1, L_2}$ be the ideal generated by the image of the map $Q_{L_1, L_2}$. Thus $\mathcal{Q} \subseteq I(X)$ and $(X, L)$ satisfies property $QR(3)$ if the equality $\mathcal{Q} = I(X)$ is attained.

4. (Featured subsets of $W_{L_1, L_2}$) Let $\{s_1, s_2, \ldots, s_p\}$ and $\{h_1, h_2, \ldots, h_q\}$ be any chosen bases for $H^0(X, L_1)$ and $H^0(X, L_2)$ respectively. Also, let 

$$\Delta_1 := \{(i, j) \mid i, j \in \{1, \ldots, p\}, i < j\},$$

$$\Delta_2 := \{(i, j, k) \mid (i, j) \in \Delta_1, k \in \{1, \ldots, p\}, k \neq i, j\},$$

$$\Delta_3 := \{(i, j, k, l) \mid (i, j), (k, l) \in \Delta_1, i < k, j \neq k, l\}, \quad \text{and}$$

$$H := \{h_1, \ldots, h_q\} \cup \{h_i + h_j \mid 1 \leq i < j \leq q\}.$$ 

Concerned with the problem of finding a spanning set of $W = W_{L_1, L_2}$, the subspace of $I(X)_2$ spanned by the image of the map $Q_{L_1, L_2}$, we define three types of finite
subsets of $W$ as follows:

\[
\begin{align*}
\Gamma_{11} &= \{ Q(s_i, s_j, h) \mid (i,j) \in \Delta_1, \ h \in H \} \\
\Gamma_{12} &= \{ Q(s_i + s_j, s_k, h) \mid (i,j,k) \in \Delta_2, \ h \in H \} \\
\Gamma_{22} &= \{ Q(s_i + s_j + s_k + s_l, h) \mid (i,j,k,l) \in \Delta_3, \ h \in H \} 
\end{align*}
\]

Let $\Gamma(L_1, L_2)$ be the union $\Gamma_{11} \cup \Gamma_{12} \cup \Gamma_{22}$. Then it can be checked that

\[
|\Gamma(L_1, L_2)| \leq \left( \frac{p+1}{2} \right) \times \left( \frac{q+1}{2} \right). \tag{2.1}
\]

Our main result in this section is

**Theorem 2.2.** Keep the notations in Notation and Remarks 2.1 Then, the subspace $W_{L_1, L_2}$ is spanned by $\Gamma(L_1, L_2)$ as a $\mathbb{k}$-vector space.

To prove Theorem 2.2, we begin with the following

**Lemma 2.3.** Keep the notations in Notation and Remarks 2.1 Then

1. For any $s, t \in H^0(X, L_1), h \in H^0(X, L_2)$ and a constant $\lambda \in \mathbb{k}$, it holds that
   \[Q(s, s, h) = 0, Q(s, t, h) = Q(t, s, h), Q(s, s + t, h) = Q(s, t, h)\]
   and $Q(\lambda s, t, h) = Q(s, t, \lambda h) = \lambda^2 Q(s, t, h)$.

2. For $s, t, u \in H^0(X, L_1), g, h \in H^0(X, L_2)$ and $a, b \in \mathbb{k}$, it holds that
   \[Q(s, at + bu, h) = (a^2 - ab)Q(s, t, h) + (b^2 - ab)Q(s, u, h) + abQ(s, t + u, h)\]
   and
   \[Q(s, t, ag + bh) = (a^2 - ab)Q(s, t, g) + (b^2 - ab)Q(s, t, h) + abQ(s, t, g + h).\]

3. For $m \geq 3$, let $s, t, t_1, \ldots, t_m \in H^0(X, L_1)$ and $h, g_1, \ldots, g_m \in H^0(X, L_2)$. Then
   \[Q(s, t_1 + t_2 + \cdots + t_m, h) = \sum_{1 \leq i < j \leq m} Q(s, t_i + t_j, h) - (m - 2) \sum_{i=1}^m Q(s, t_i, h)\]
   and
   \[Q(s, t, g_1 + g_2 + \cdots + g_m) = \sum_{1 \leq i < j \leq m} Q(s, t, g_i + g_j) - (m - 2) \sum_{i=1}^m Q(s, t, g_i).\]

4. For $s, t, u \in H^0(X, L_1)$ and $h \in H^0(X, L_2)$, it holds that
   \[Q(s + u, t + u, h) = Q(s, u, h) + Q(t, u, h) + Q(s + u, t, h) + Q(t + u, s, h) - Q(s, t, h) - Q(s + t, u, h).\]

**Proof.** (1) Note that $f(s \otimes t \otimes h) = f(t \otimes s \otimes h)$. Then the statement can be easily shown by using the definition of the map $Q$.

(2) One can check that
   \[Q(s, at + bu, h) = a^2Q(s, t, h) + b^2Q(s, u, h) + 2abP(s, t, u, h)\]
   where $P(s, t, u, h) := f(s \otimes s \otimes h)f(t \otimes u \otimes h) - f(s \otimes t \otimes h)f(s \otimes u \otimes h)$. In particular, it holds that
   \[Q(s, t + u, h) = Q(s, t, h) + Q(s, u, h) + 2P(s, t, u, h).\]
Thus we have
\[ Q(s, at + bu, h) = a^2 Q(s, t, h) + b^2 Q(s, u, h) + ab\{ Q(s, t + u, h) - Q(s, t, h) - Q(s, u, h) \}, \]
which shows the first formula. Similarly, one can prove that
\[ Q(s, t, ag + bh) = a^2 Q(s, t, g) + b^2 Q(s, t, h) + ab R(s, t, g, h) \]
where \( R(s, t, g, h) := f(s \otimes s \otimes g) f(t \otimes t \otimes h) + f(s \otimes s \otimes h) f(t \otimes t \otimes g) - 2 f(s \otimes t \otimes g) f(s \otimes t \otimes h). \)
Then it holds that
\[ Q(s, t, g + h) = Q(s, t, g) + Q(s, t, h) + R(s, t, g, h) \]
and hence we get
\[ Q(s, t, ag + bh) = a^2 Q(s, t, g) + b^2 Q(s, t, h) + ab\{ Q(s, t, g + h) - Q(s, t, g) - Q(s, t, h) \} \]
which shows the second formula.

(3) We will prove the first equality by using induction on \( m \geq 3. \) The second one can be shown by a similar way. For simplicity, let \( g(s, t) := f(s \otimes t \otimes h) \) in this proof. For \( m = 3, \)
\[
Q(s, t_1 + t_2 + t_3, h) = g(s, s)g(t_1 + t_2 + t_3, t_1 + t_2 + t_3) - g(s, t_1 + t_2 + t_3)^2 \\
= g(s, s)\{ g(t_1, t_1) + g(t_2, t_2) + g(t_3, t_3) + 2g(t_1, t_2) + 2g(t_1, t_3) + 2g(t_2, t_3) \} - \{ g(s, t_1)^2 + g(s, t_2)^2 + g(s, t_3)^2 + 2g(s, t_1)g(s, t_2) + 2g(s, t_1)g(s, t_3) + 2g(s, t_2)g(s, t_3) \}. 
\]
Since for \( 1 \leq i < j \leq 3 \)
\[
Q(s, t_i + t_j, h) = g(s, s)g(t_i + t_j, t_i + t_j) - g(s, t_i + t_j)^2 \\
= g(s, s)\{ g(t_i, t_i) + g(t_j, t_j) + 2g(t_i, t_j) \} - \{ g(s, t_i)^2 + g(s, t_j)^2 + 2g(s, t_i)g(s, t_j) \},
\]
we can calculate the difference
\[
Q(s, t_1 + t_2, h) + Q(s, t_1 + t_3, h) + Q(s, t_2 + t_3, h) - Q(s, t_1 + t_2 + t_3, h) \\
= g(s, s)\{ g(t_1, t_1) + g(t_2, t_2) + g(t_3, t_3) \} - \{ g(s, t_1)^2 + g(s, t_2)^2 + g(s, t_3)^2 \} \\
= Q(s, t_1, h) + Q(s, t_2, h) + Q(s, t_3, h) \\
\]
Hence, for \( m = 3, \) the statement does hold.

For \( m \geq 4, \) now let us denote \( Q(s, t, h) \) by \( F(t) \) for the simplicity. By induction hypothesis, we have
\[
F(t_1 + \cdots + t_m) = F(t_1 + \cdots + (t_{m-1} + t_m)) \\
= \sum_{1 \leq i < j \leq m-2} F(t_i + t_j) + \sum_{i=1}^{m-2} F(t_i + t_{m-1} + t_m) \\
- (m - 3)\{ \sum_{i=1}^{m-2} F(t_i) + F(t_{m-1} + t_m) \}
\]
and
\[
F(t_i + t_{m-1} + t_m) = F(t_i + t_{m-1}) + F(t_i + t_m) + F(t_{m-1} + t_m) - F(t_i) - F(t_{m-1}) - F(t_m)
\]
for each $1 \leq i \leq m - 2$. Hence, by combining these identities, we get the desired formula for $Q(s, t_1 + t_2 + \cdots + t_m, h)$.

(4) From Lemma 2.3.(1) and (3), we have

$$Q(s + t, u, h) = Q(s + t, s + t + u, h) = Q(s + t, s + u, h) + Q(s + t + u, h) - 2Q(s, t, h) = Q(s + t, u, h)$$

and hence

$$Q(s + t, s + u, h) + Q(s + t + u, h) = 2Q(s, t, h) + 2Q(s + t, u, h). \quad (2.2)$$

By permuting $s, t$ and $u$ in (2.2), it follows that

$$Q(s + t, s + u, h) + Q(s + u, t + u, h) = 2Q(s, u, h) \quad \quad (2.3)$$

and

$$Q(s + t, s + u, h) + Q(s + u, t + u, h) = 2Q(t, u, h) \quad \quad (2.4)$$

Now, the desired identity comes from (2.2), (2.3) and (2.4).

Now we are ready to give a

**Proof of Theorem 2.2.** Consider the general member

$$G := Q(a_1s_1 + \cdots + a_ps_p, b_1s_1 + \cdots + b_ps_p, c_1h_1 + \cdots + c_qh_q)$$

of the image of the map $Q_{L_1, L_2}$ where $a_1, \ldots, a_p, b_1, \ldots, b_p, c_1, \ldots, c_q$ are from $k$. By applying Lemma 2.3.(1) and (3) repeatedly, one can see that $G$ is a $k$-linear combination of the quadratic equations of the form

$$Q(\alpha_is_i + \beta_1s_j, \alpha_2s_k + \beta_2s_l, \alpha_3h_m + \beta_3h_n)$$

where $\alpha_u, \beta_v \in k$. Then, by Lemma 2.3.(2), we may assume that $\alpha_u, \beta_v \in \{0, 1\}$. Finally, Lemma 2.3.(4) enables us to exclude the cases where $\{i, j\} \cap \{k, l\}$ is not empty.

We finish this section by applying Theorem 2.2 to the rational normal curves.

**Corollary 2.4.** For every $d \geq 2$, let $C_d \subset \mathbb{P}^d$ be the standard rational normal curve of degree $d$. Then $I(C_d)$ is generated by $\Gamma(\mathcal{O}_{\mathbb{P}^1}(1), \mathcal{O}_{\mathbb{P}^1}(d - 2))$. In particular, every rational normal curve satisfies property $\text{QR}(3)$.

**Proof.** Consider the $Q$-map about the decomposition

$$\mathcal{O}_{\mathbb{P}^1}(d) = \mathcal{O}_{\mathbb{P}^1}(1)^{\otimes 2} \otimes \mathcal{O}_{\mathbb{P}^1}(d - 2).$$

Let $\{s, t\}$ be a basis for $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$. So, $\{s^{d-2-i}t^i \mid 0 \leq i \leq d - 2\}$ is a basis for $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d - 2))$. Then $\Gamma := \Gamma(\mathcal{O}_{\mathbb{P}^1}(1), \mathcal{O}_{\mathbb{P}^1}(d - 2))$ is equal to

$$\{Q(s, t, s^{d-2-i}t^i) \mid 0 \leq i \leq d - 2\} \cup \{Q(s, t, s^{d-2-i}t^i + s^{d-2-j}t^j) \mid 0 \leq i < j \leq d - 2\}.$$

Note that $\Gamma(\mathcal{O}_{\mathbb{P}^1}(1), \mathcal{O}_{\mathbb{P}^1}) = \{Q(s, t, 1) = f(s^2)f(t^2) - f(st)^2 = z_0z_2 - z_1^2\}$. Now, we will show that $\Gamma$ has exactly $\binom{d}{3}$ quadrics of rank 3 and that they are $k$-linearly independent. Let $z_0, z_1, \ldots, z_d$ be the homogeneous coordinates of $\mathbb{P}^d$. Then

$$F_i := Q(s, t, s^{d-2-i}t^i) = z_iz_{i+2} - z_{i+1}^2.$$
and
\[ G_{i,j} := Q(s, t, s^{d-2-i}t^i + s^{d-2-j}t^j) = (z_i + z_j)(z_{i+2} + z_{j+2}) - (z_{i+1} + z_{j+1})^2. \]

Let \( G'_{i,j} := G_{i,j} - F_i - F_j = z_iz_{i+j} + z_iz_{i+2} - 2z_{i+1}z_{j+1} \). The leading terms of \( F_i \)'s are \( z_iz_{i+j} \) and those of \( G'_{i,j} \)'s are \( z_iz_{i+j} \) in the standard lexicographic order. They are all distinct, because \( i < j \). It means that all \( F_i \)'s and \( G'_{i,j} \)'s are \( k \)-linearly independent and
\[ |\Gamma(\mathcal{O}_{\mathbb{P}^1(1)}, \mathcal{O}_{\mathbb{P}^1(d-2)})| = (d-1) + \binom{d-1}{2} = \frac{d}{2} = \dim_k I(C_d). \]

In consequence, it is shown that \( \Gamma \) generates \( I(C_d) \).

3. Second Veronese embeddings

This section is devoted to solving the problem whether the second Veronese variety satisfies property \( QR(3) \) or not. It is interesting that the answer for this question depends on the characteristic of the base field \( k \).

**Theorem 3.1.** Let \( n \) be any positive integer. Then

1. Suppose that \( \text{char} \ k \neq 2, 3 \). Then \( I(V_{n,2}) \) is generated by \( \Gamma(\mathcal{O}_{\mathbb{P}^n(1)}, \mathcal{O}_{\mathbb{P}^n}) \). In particular, \( (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n(2)}) \) satisfies property \( QR(3) \).
2. If \( \text{char} \ k = 3 \), then \( (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n(2)}) \) satisfies property \( QR(3) \) if and only if \( n \leq 2 \).

We begin with fixing a few notation related to the Veronese embedding of projective spaces, which we use throughout the remaining part of this paper.

**Notation and Remarks 3.2.** By \( \mathbb{N}_0 \) we denote the set of non-negative integers. For the integers \( n \geq 1 \) and \( d \geq 2 \), consider the Veronese variety
\[ V_{n,d} := \nu_d(\mathbb{P}^n) \subset \mathbb{P}^{N(n,d)} \quad \text{where} \quad N(n, d) = \binom{n + d}{n} - 1. \]

For \( A(n, d) := \{(a_0, \ldots, a_n) \mid a_i \in \mathbb{N}_0 \quad \text{and} \quad a_0 + \cdots + a_n = d\} \), let
\[ B(n, d) := \{z_I \mid I \in A(n, d)\} \]
be the set of standard homogeneous coordinates of the projective space \( \mathbb{P}^{N(n,d)} \).

1. The homogeneous ideal of \( V_{n,d} \) in \( \mathbb{P}^{N(n,d)} \) is generated by the set of quadrics
\[ Q(n, d) := \{z_IZ_J - z_KZ_L \mid I, J, K, L \in A(n, d) \quad \text{and} \quad I + J = K + L\}. \]
In particular, \( (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n(d)}) \) does always satisfy \( QR(4) \).
2. For \( I = (a_0, a_1, \ldots, a_n) \in A(n, d) \) and \( k \in \{0, 1, \ldots, n\} \), we will denote \( a_k \) by \( I_k \) and \( \sum_{i=0}^n a_i \) by \(|I|\).
3. For each \( 0 \leq k \leq n \), consider the inclusion map from \( A(n-1, d) \) into \( A(n, d) \) defined by inserting 0 to the \( k \)th location. This map identifies \( B(n-1, d) \) with a subset of \( B(n, d) \). Also this map induces an inclusion map
\[ \iota_k : I(V_{n-1,d}) \rightarrow I(V_{n,d}). \]
Moreover, \( \iota_k \) maps the subset \( \Gamma(\mathcal{O}_{\mathbb{P}^{n-1}(1)}, \mathcal{O}_{\mathbb{P}^{n-1}(d-2)}) \) into \( \Gamma(\mathcal{O}_{\mathbb{P}^n(1)}, \mathcal{O}_{\mathbb{P}^n(d-2)}) \).
(4) Suppose that $d \geq 3$. For each $0 \leq k \leq n$, consider the inclusion map from $A(n, d - 1)$ into $A(n, d)$ defined by adding 1 to the $k$th location. This corresponds to the multiplication map by $x_i$ of the homogeneous coordinate ring of $\mathbb{P}^n$. This map identifies $B(n, d - 1)$ with a subset of $B(n, d)$. Also this map induces the map

$$\delta_k : I(V_{n,d-1}) \to I(V_{n,d}).$$

Moreover, $\delta_k$ maps the subset $\Gamma(\mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n}(d-3))$ into $\Gamma(\mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n}(d-2)).$

The following proposition plays a crucial role in the proof of Theorem 3.1.

**Proposition 3.3.** If $I(V_{2d-1,d})$ is generated by $\Gamma(\mathcal{O}_{\mathbb{P}^{2d-1}}(1), \mathcal{O}_{\mathbb{P}^{2d-1}}(d-2))$, then $I(V_{n,d})$ is generated by $\Gamma(\mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n}(d-2))$ for all $n \geq 2d - 1$.

**Proof.** We use induction on $n \geq 2d - 1$. The case $n = 2d - 1$ is done by our assumption.

Suppose that $n \geq 2d$ and let $Q = z_1 z_2 - z_3 z_4$ be an element of $\mathcal{O}(n, d)$. We need to show that $Q$ is a linear combination of quadratic equations in $\Gamma(\mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n}(d-2))$. Since $n \geq 2d$, by pigeonhole principle, there exists $0 \leq k \leq n$ such that $I_k = J_k = K_k = L_k = 0$. Now, by using the inclusion map

$$\iota_k : I(V_{n-1,d}) \to I(V_{n,d}),$$

we can regard $Q$ as an element of $I(V_{n-1,d})$. By induction hypothesis, $Q$ is a linear combination of elements in $\Gamma(\mathcal{O}_{\mathbb{P}^{n-1}}(1), \mathcal{O}_{\mathbb{P}^{n-1}}(d-2))$. Then, by Notation and Remarks 3.2(3), $Q$ is also a linear combination of elements in $\Gamma(\mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n}(d-2))$. This completes the proof. \qed

Now, we give the

**Proof of Theorem 3.1.** (1) By Proposition 3.3, it is enough to check the statement for the cases where $n = 2$ and $n = 3$. So, consider the $Q$-map about the decomposition

$$\mathcal{O}_{\mathbb{P}^n}(2) = \mathcal{O}_{\mathbb{P}^n}(1)^{\otimes 2} \otimes \mathcal{O}_{\mathbb{P}^n}.$$ Let $\{x_0, \ldots, x_n\}$ and $\{1\}$ be respectively bases of $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ and $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n})$. Thus the $k$-vector space $W_{\mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n}}$ defined in Notation and Remarks 2.1(3) is generated by the union $\Gamma := \Gamma(\mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n}) = \Gamma_{11} \cup \Gamma_{12} \cup \Gamma_{22}$ where

$$\begin{cases}
\Gamma_{11} = \{ F_{i,j} := Q(x_i, x_j, 1) \mid (i, j) \in \Delta_1 \}, \\
\Gamma_{12} = \{ G_{i,j,k} := Q(x_i, x_j + x_k, 1) \mid (i, j, k) \in \Delta_2 \} \quad \text{and} \\
\Gamma_{22} = \{ H_{i,j,k,l} := Q(x_i + x_j, x_k + x_l, 1) \mid (i, j, k, l) \in \Delta_3 \}.
\end{cases}$$

For $n = 2$, $\Gamma(\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2})$ consists of the following six elements:

$$\begin{cases}
F_{0,1} := Q(x_0, x_1, 1) = z_{200} z_{020} - z_{110}^2 \\
F_{0,2} := Q(x_0, x_2, 1) = z_{200} z_{002} - z_{101}^2 \\
F_{1,2} := Q(x_1, x_2, 1) = z_{020} z_{002} - z_{011}^2 \\
G_{0,1,2} := Q(x_0, x_1 + x_2, 1) = z_{200} (z_{020} + 2 z_{011} + z_{002}) - (z_{110} + z_{101})^2 \\
G_{1,0,2} := Q(x_1, x_0 + x_2, 1) = z_{200} (z_{020} + 2 z_{101} + z_{002}) - (z_{110} + z_{011})^2 \\
G_{2,0,1} := Q(x_2, x_0 + x_1, 1) = z_{002} (z_{200} + 2 z_{110} + z_{020}) - (z_{101} + z_{011})^2
\end{cases}$$
One can easily check that the following three identities hold:

\[
\begin{align*}
G_{0,1,2} &= F_{0,1} + F_{0,2} + 2(z_{200}z_{011} - z_{110}z_{101}) \\
G_{1,0,2} &= F_{0,1} + F_{1,2} + 2(z_{020}z_{101} - z_{110}z_{011}) \\
G_{2,0,1} &= F_{0,2} + F_{1,2} + 2(z_{002}z_{110} - z_{101}z_{011})
\end{align*}
\]

Since \( \text{char } k \neq 2 \), this verifies that the homogeneous ideal

\[ I(V_{2,2}) = \langle F_{0,1}, F_{0,2}, F_{1,2}, z_{200}z_{011} - z_{110}z_{101}, z_{020}z_{101} - z_{110}z_{011}, z_{002}z_{110} - z_{101}z_{011} \rangle \]

is generated by \( \Gamma(O_{\mathbb{P}^2}(1), O_{\mathbb{P}^2}) \).

For \( n = 3 \), note that \( \dim_k I(V_{3,2}) = 20 \) and \( I(V_{3,2}) \) is generated by the 2-minors of the \( 4 \times 4 \) symmetric matrix

\[
A = \begin{pmatrix}
z_{2000} & z_{1100} & z_{1010} & z_{1001} \\
z_{1100} & z_{0200} & z_{0110} & z_{0101} \\
z_{1010} & z_{0110} & z_{0020} & z_{0011} \\
z_{1001} & z_{0101} & z_{0011} & z_{0002}
\end{pmatrix}.
\]

For each \( i \in \{1, 2, 3, 4\} \), let \( A_i \) be the principal submatrix of \( A \) obtained by eliminating the \( i \)th row and column. Then 2-minors of \( A_i \) are contained in \( \nu_i(I(V_{2,2})) \). So they can be handled by the previous case. The remaining generators not coming from these principal submatrices are

\[ R_1 = z_{1100}z_{0011} - z_{1010}z_{0110}, \quad R_2 = z_{1100}z_{0011} - z_{0101}z_{0110} \quad \text{and} \quad R_3 = z_{1010}z_{0110} - z_{1001}z_{0110}. \]

Note that \( R_3 = -R_1 + R_2 \). Thus it suffices to show that \( R_1 \) and \( R_2 \) are \( k \)-linear combinations of elements in \( \Gamma = \Gamma(O_{\mathbb{P}^3}(1), O_{\mathbb{P}^3}) \). Now, consider the set

\[ \Gamma_{22} = \{ H_{0,1,2,3}, H_{0,2,1,3}, H_{0,3,1,2} \}. \]

Letting \( H'_{0,1,2,3} = 4z_{1100}z_{0011} - 2z_{1010}z_{0110} - 2z_{1001}z_{0110} \), we have

\[
H'_{0,1,2,3} = (z_{2000} + 2z_{1100} + z_{0200})(z_{0020} + 2z_{0111} + z_{0002}) - (z_{1010} + z_{1001} + z_{0110} + z_{0101})^2
\]

\[ = G_{0,2,3} + G_{1,2,3} + G_{2,0,1} + G_{3,0,1} - F_{0,2} - F_{0,3} - F_{1,2} - F_{1,3} + H'_{0,1,2,3}. \]

Then it holds that the elements \( H'_{0,1,2,3} \) is contained in \( W_{O_{\mathbb{P}^3}(1), O_{\mathbb{P}^3}} \). Similarly, one can show that two elements

\[ H'_{0,2,1,3} = 4z_{1010}z_{0110} - 2z_{1100}z_{0011} - 2z_{1001}z_{0110} \]

and

\[ H'_{0,3,1,2} = 4z_{1001}z_{0110} - 2z_{1100}z_{0011} - 2z_{1010}z_{0110} \]

are also contained in \( W_{O_{\mathbb{P}^3}(1), O_{\mathbb{P}^3}} \). Since

\[ H'_{0,1,2,3} = 2R_1 + 2R_2, \quad H'_{0,2,1,3} = -4R_1 + 2R_2 \quad \text{and} \quad H'_{0,3,1,2} = 2R_1 - 4R_2 \quad (3.1) \]

and since \( \text{char } k \neq 2, 3 \), it follows that

\[ R_1 = \frac{1}{6}(H'_{0,1,2,3} - H'_{0,2,1,3}) \quad \text{and} \quad R_2 = \frac{1}{6}(H'_{0,1,2,3} - H'_{0,3,1,2}) \]

are contained \( W_{O_{\mathbb{P}^3}(1), O_{\mathbb{P}^3}} \). This completes the proof that the ideal \( I(V_{3,2}) \) is generated by \( \Gamma \) and hence \( (\mathbb{P}^3, O_{\mathbb{P}^3}(2)) \) satisfies property QR(3).
(2) Suppose that $\text{char}\ k = 3$. First note that Corollary 2.4 for $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))$ and the proof in (1) for $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ are still valid in $\text{char}\ k = 3$. Thus, it is enough to show that $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2))$ fails to satisfy property $QR(3)$ if $n \geq 3$.

When $n = 3$, we will first show that $W := W_{\mathcal{O}_{\mathbb{P}^3}(1), \mathcal{O}_{\mathbb{P}^3}}$ is a proper subset of $I(V_{3,2})$. Indeed, $W = \langle \Gamma \rangle$ by Theorem 2.2 and $\Gamma$ has at most 21 elements by (2.1). Furthermore, $\Gamma_{22} = \{H_{0,1,2,3}, H_{0,2,1,3}, H_{0,3,1,2}\}$ and $\Gamma_{11} \cup \Gamma_{12}$ has at most 18 elements. From (3.1), we obtain $H_{0,1,2,3} = H_{0,2,1,3} = H_{0,3,1,2}$.

since $\text{char}\ k = 3$. In particular, the image of $\Gamma_{22}$ in the quotient space $W/\langle \Gamma_{11} \cup \Gamma_{12} \rangle$ has at most one nonzero element. In consequence, we have

$$\dim_k W \leq |\Gamma_{11} \cup \Gamma_{12}| + 1 = 19.$$ 

This shows that $W$ is strictly smaller than $I(V_{3,2})$ since $\dim_k I(V_{3,2}) = 20$. Next, let $W'$ be the $k$-vector space spanned by all rank 3 elements in $I(V_{3,2})$. We claim that $W' = W$ and hence any quadric of rank 3 in $I(V_{3,2})$ is contained in $W$. Obviously, this completes the proof. That is, $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$ fails to satisfy property $QR(3)$ since $W'$ is a proper subspace of $I(V_{3,2})$.

To see this claim, consider the ring homomorphism

$$\nu_2^\#: k[z_0, z_1, \ldots, z_9] \to k[x_0, \ldots, x_3]$$

corresponding to the Veronese embedding $\mathbb{P}^3 \leftarrow \nu_2^\# V_{3,2} \subset \mathbb{P}^9$. Note that there is no rank 2 element in $I(V_{3,2})$. If there exists a nonzero rank 2 element $L_1^2 + L_2^2 \in I(V_{3,2})$, then $\nu_2^\#(L_1 + L_2^2) = q_1^2 + q_2^2 = 0$ and since $k[x_0, \ldots, x_3]$ is a UFD, it means that $q_1 = \pm i q_2$ for $i \in k$ such that $i^2 = -1$. This implies that $L_1 = \pm i L_2$ and it contradicts to the assumption that $L_1 + L_2^2$ is nonzero. Let $\Theta = L_1^2 - L_2 L_3 \in I(V_{3,2})$ be a nonzero quadric of rank 3 where $L_1, L_2, L_3 \in k[z_0, z_1, \ldots, z_9]$ are linear forms. We may assume that every rank 3 quadric is of this form via the transform $L_1^2 + L_2^2 + L_3^2 \mapsto L_1^2 - (L_2 + i L_3)(-L_2 + i L_3)$. Then $\nu_2^\#(L_1) = q_1, \nu_2^\#(L_2) = q_2, \nu_2^\#(L_3) = q_3$ for some quadratic forms $q_1, q_2, q_3$ in $k[x_0, \ldots, x_3]$ such that

$$\nu_2^\#(\Theta) = q_1^2 - q_2 q_3 = 0.$$ 

We can check that $q_1$ is not proportional to $q_2$ (and also to $q_3$) since if it is, then the original quadric $L_1^2 - L_2 L_3$ has rank 2 which is a contradiction. From the UFD property of $k[x_0, \ldots, x_3]$, it follows that $q_1 = l_1 l_2, q_2 = l_1^2$ and $q_3 = l_2^2$ for some linear forms $l_1, l_2, l_3$. In consequence, $\Theta = -Q(l_1, l_2, 1)$ is an element of $W$.

For $n \geq 4$, choose a hyperplane $H = \mathbb{P}^{n-1}$ of $\mathbb{P}^n$. Then one can check that $V_{n-1,2} = \nu_2^\#(H)$ in $\mathbb{P}_{\mathbb{P}^n}^{N(n-1,2)}$ is an ideal-theoretic linear section of $V_{n,2} \subset \mathbb{P}_{\mathbb{P}^n}^{N(n,2)}$ (cf. [CP15] Proposition 2.1). In particular, if $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(2))$ fails to satisfy property $QR(3)$ then so does $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2))$. This completes the proof that $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2))$ fails to satisfy property $QR(3)$ for all $n \geq 3$. □

Remark 3.4. From Theorem 2.2 one may ask if all the members of $\Gamma$ are $k$-linearly independent or not.

(1) For $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$, it is shown in the proof of Theorem 3.1 that $\Gamma$ is a basis for $I(V_{2,2})$. 


(2) For \((\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))\), suppose that \(\text{char } k \neq 2, 3\). In the proof of Theorem 3.1, it is shown that \(\Gamma\) spans the 20-dimensional vector space \(I(V_{3,2})_2\). Since \(|\Gamma| \leq 21\), there exists at most one non-trivial \(k\)-linear relation among the elements of \(\Gamma\). Note that the identity
\[
H'_{0,1,2,3} + H'_{0,2,1,3} + H'_{0,3,1,2} = 0
\] (3.2)
provides one such relation. In consequence, (3.2) is essentially the unique linear relation among the elements of \(\Gamma\).

4. Higher Veronese embeddings

This section is devoted to the proof of Theorem 1.1. So, we assume that \(\text{char } k \neq 2, 3\). We require a few more notation and definition.

Notation and Remarks 4.1. (Continued from Notation and Remarks 3.2) Let \(V_{n,d} := \nu_d(\mathbb{P}^n) \subset \mathbb{P}^N\) where \(N = N(n, d) = \left(\frac{n+d}{n}\right) - 1\) be the \(d\)th Veronese embedding of \(\mathbb{P}^n\) for some \(n \geq 1\) and \(d \geq 2\).

(1) For \(0 \leq i \leq n\), let \(e_i\) denote the \(i\)th coordinate vector. Thus \(A(n, 1) = \{e_i \mid 0 \leq i \leq n\}\).

(2) For \(I \in A(n, d)\), we define \(\text{Supp}(I)\) as the set \(\{k \mid 0 \leq k \leq n, \ I_k > 0\}\).

(3) For simplicity, we denote the monomial \(z_I z_J\) by \([I, J]\). For example, an element \(z_I z_J - z_K z_L\) of \(Q(n, d)\) is denoted by \([I, J] - [K, L]\).

(4) Throughout this section, we use the map \(Q\), the ideal \(Q\) and the finite set \(\Gamma := \Gamma(L_1, L_2)\) in Notation and Remarks 2.1 for the pair \((L_1, L_2) = (\mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n}(d-2))\). By Theorem 2.2 we know that \(\Omega\) is generated by \(\Gamma\).

(5) Let \(q\) and \(q'\) be two homogeneous quadratic equations. We say that they are equivalently related and write \(q \sim q'\) if \(q - q'\) can be represented by the sum of elements in \(\Gamma\), or equivalently, \(q - q'\) is contained in \(Q\). So a quadratic equation \(q\) is equivalent to 0 if and only if \(q \in Q\).

(6) From Notation and Remarks 3.2(1), one can see that the following three statements are equivalent.
\[
(i) \ [I, J] \sim [K, L] \text{ for any } I, J, K, L \in A(n, d) \text{ satisfying } I + J = K + L;
(ii) \ I(V_{n,d}) = \Omega;
(iii) \ I(V_{n,d}) \text{ is generated by } \Gamma.
\]
In particular, \((\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))\) satisfies property \(QR(3)\) if one of \(i) \sim (iii)\) holds.

We will prove the following

Theorem 4.2. \(I(V_{n,d})\) is generated by \(\Gamma = \Gamma(\mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n}(d-2))\).

As is mentioned in Notation and Remarks 4.1(6), Theorem 4.2 implies Theorem 1.1.

In order to prove Theorem 4.2 we will use the double induction on \(n\) and \(d\). Indeed, Theorem 4.2 is already proven for \(n = 1\) in Corollary 2.4 and for \(d = 2\) in Theorem 3.1 respectively. From now on, let \(n \geq 2\) and \(d \geq 3\). Also we assume that
Lemma 4.3. Suppose that \( d \geq 3 \) and let \( I \) and \( J \) be two elements in \( A(n, d - 2) \). Then

\[
(1) \quad [2e_i + I, 2e_j + I] \sim [e_i + e_j + I, e_i + e_j + I]
\]

\[
(2) \quad [2e_i + I, e_j + e_k + I] \sim [e_i + e_j + I, e_i + e_k + I]
\]

\[
(3) \quad [e_i + e_j + I, e_k + e_l + I] \sim [e_i + e_k + I, e_j + e_l + I]
\]

\[
(4) \quad [2e_i + I, 2e_j + J] + [2e_j + I, 2e_i + J] \sim 2[e_i + e_j + I, e_i + e_j + J]
\]

\[
(5) \quad [2e_i + I, e_j + e_k + J] + [e_j + e_k + I, 2e_i + J]
\]

\[
\sim [e_i + e_j + I, e_i + e_k + J] + [e_i + e_k + I, e_i + e_j + J]
\]

\[
(6) \quad [e_i + e_j + I, e_k + e_l + J] + [e_k + e_l + I, e_i + e_j + J]
\]

\[
\sim [e_i + e_k + I, e_j + e_l + J] + [e_j + e_l + I, e_i + e_k + J]
\]

Proof. To use the Q-map about the decomposition \( \mathcal{O}_{\mathbb{P}^n}(d) = \mathcal{O}_{\mathbb{P}^n}(1)^{\otimes 2} \otimes \mathcal{O}_{\mathbb{P}^n}(d - 2) \), let \( \{x_0, x_1, \ldots, x_n\} \) be a basis of \( H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \).

(1) For an element in \( \Gamma_{11} \),

\[
Q(x_i, x_j, x^l) = f(x_i^2 x^l) f(x_j^2 x^l) - f(x_i x_j x^l)^2 = [2e_i + I, 2e_j + I] - [e_i + e_j + I, e_i + e_j + I] \in \Omega
\]

Hence, the relation (1) holds.

(2) For an element in \( \Gamma_{12} \),

\[
Q(x_i, x_j + x_k, x^l) = f(x_i^2 x^l) f((x_j + x_k)^2 x^l) - f(x_i(x_j + x_k) x^l)^2
\]

\[
= [2e_i + I, 2e_j + I] + 2[2e_i + I, e_j + e_k + I] + 2[e_i + I, 2e_k + I] - [e_i + e_j + I, e_i + e_j + I] - [e_i + e_k + I, e_i + e_k + I].
\]

Using the equivalence relation (1) for each pair \((i, j)\) and \((i, k)\), this is equivalent to

\[
2[2e_i + I, e_j + e_k + I] - 2[e_i + e_j + I, e_i + e_k + I].
\]

Since \( \text{char}(k) \neq 2 \), it follows the above relation (2).

(3) For an element in \( \Gamma_{22} \),

\[
Q(x_i + x_j, x_k + x_l, x^l) = f((x_i + x_j)^2 x^l) f((x_k + x_l)^2 x^l) - f((x_i + x_j)(x_k + x_l) x^l)^2
\]

\[
= [2e_i + I, 2e_k + I] - [e_i + e_k + I, e_i + e_k + I] + [2e_i + I, 2e_l + I] - [e_i + e_l + I, e_i + e_l + I] + [2e_j + I, 2e_k + I] - [e_j + e_k + I, e_j + e_k + I] + [2e_j + I, 2e_l + I] - [e_j + e_l + I, e_j + e_l + I] + [2e_i + e_j + I, 2e_k + I] - 2[e_i + e_j + I, e_i + e_k + I] + 2[e_i + e_j + I, 2e_l + I] - 2[e_i + e_j + I, e_j + e_l + I] + 2[e_i + e_k + I, 2e_l + I] - 2[e_i + e_k + I, e_j + e_l + I] + 2[e_i + e_l + I, 2e_k + I] - 2[e_i + e_l + I, e_j + e_k + I] + 4[e_i + e_j + I, e_k + e_l + I] - 2[e_i + e_k + I, e_j + e_l + I] - 2[e_i + e_l + I, e_j + e_k + I].
\]

We can erase the above 8 equations using the relation (1) and (2). Then the only remaining equivalent part is

\[
4[e_i + e_j + I, e_k + e_l + I] - 2[e_i + e_k + I, e_j + e_l + I] - 2[e_i + e_l + I, e_j + e_k + I]. \quad (4.1)
\]
By exchanging $j$ and $k$, we also obtain the relation

$$Q(x_i + x_k, x_j + x_l, x^I) \sim 4[e_i + e_k + I, e_j + e_l + I] - 2[e_i + e_j + I, e_k + e_l + I] - 2[e_i + e_l + I, e_j + e_k + I].$$

(4.2)

By subtracting (4.2) from (4.1) and dividing it by 6, it can be shown that

$$[e_i + e_j + I, e_k + e_l + I] \sim [e_i + e_l + I, e_j + e_k + I].$$

To this aim, we require the assumption that $\text{char}(k) \neq 2, 3$.

(4) For an element in $\Gamma_{11}$,

$$Q(x_i, x_j, x^I_k + x^J) = f(x_i^2(x^I_k + x^J))f(x_j^2(x^I_k + x^J)) - f(x_i x_j(x^I_k + x^J))^2$$

$$= [2e_i + I, 2e_j + I] - [e_i + e_j + I, e_i + e_j + I] - 2[e_i + e_j + I, e_i + e_j + I].$$

Using the relation (1) for $I$ and $J$, one can show that the above equation is equivalent to

$$[2e_i + I, 2e_j + I] - 2[e_i + e_j + I, e_i + e_j + I].$$

Hence the relation (4) holds.

(5) For an element in $\Gamma_{12}$,

$$Q(x_i, x_j, x^I_k + x^J)$$

$$= f(x_i^2(x^I_k + x^J))f((x_j + x_k)^2(x^I_k + x^J)) - f(x_i(x_j + x_k)(x^I_k + x^J))^2$$

$$= [2e_i + I, 2e_j + I, 2e_k + I] - [e_i + e_j + I, e_i + e_j + I] - 2[e_i + e_j + I, e_i + e_j + I].$$

Thus, we have

$$[2e_i + I, e_j + e_k + J] + [e_j + e_k + I, 2e_i + J] - [e_i + e_j + I, e_i + e_k + J] - [e_i + e_k + I, e_i + e_j + J] \in \mathfrak{Q}.$$ 

Hence the relation (5) holds.
(6) For an element in $\Gamma_{22}$, 
\[
Q(x_i + x_j, x_k + x_l, x^I + x^J) = f((x_i + x_j)(x_k + x_l)(x^I + x^J)) - f((x_i + x_j)(x_k + x_l)(x^I + x^J))^2
\]
\[
= [2e_i + I, 2e_k + I] - [e_i + e_k + I, e_i + e_k + I] + [2e_i + I, 2e_i + I] - [e_i + e_i + I, e_i + e_i + I]
\]
\[
+ [2e_j + I, 2e_j + I] - [e_j + e_j + I, e_j + e_j + I] + [2e_j + I, 2e_i + I] - [e_i + e_i + I, e_i + e_i + I]
\]
\[
+ [2e_i + I, 2e_i + I] - [e_i + e_k + I, e_i + e_k + I] + [2e_i + I, 2e_i + I] - [e_i + e_i + I, e_i + e_i + I]
\]
\[
+ [2e_j + I, 2e_j + J] - [e_i + e_k + J, e_i + e_k + J] + [2e_i + J, 2e_i + J] - [e_i + e_i + J, e_i + e_i + J]
\]
\[
+ [2e_j + J, 2e_j + J] - [e_j + e_k + J, e_j + e_k + J] + [2e_j + J, 2e_i + J] - [e_i + e_i + J, e_i + e_i + J]
\]
\[
+ [2e_i + J, 2e_i + J] - [e_i + e_k + J, e_j + e_k + J] + [2e_i + J, 2e_i + J] - [e_i + e_i + J, e_j + e_i + J]
\]
\[
+ [2e_j + J, 2e_j + J] - [e_j + e_k + J, e_j + e_k + J] + [2e_j + J, 2e_j + J] - [e_j + e_j + J, e_j + e_i + J]
\]
\[
\text{(first 4 lines equivalent to 0 by (1))}
\]
\[
+ 2[2e_i + I, e_i + e_i + I] - 2[e_i + e_k + I, e_i + e_k + J] + 2[2e_i + J, e_k + e_k + J] - [e_k + e_k + J, e_i + e_k + J]
\]
\[
+ 2[2e_j + I, e_k + e_i + I] - 2[e_j + e_k + I, e_j + e_i + J] + 2[2e_i + J, e_k + e_k + J] - [e_j + e_k + J, e_j + e_k + J]
\]
\[
+ 2[e_i + e_j + I, 2e_i + I] - 2[e_i + e_k + I, e_j + e_k + J] + 2[2e_i + J, e_k + e_k + J] - [e_j + e_k + J, e_j + e_k + J]
\]
\[
+ [2e_i + J, 2e_i + I] - 2[e_i + e_k + I, e_j + e_i + J] + 2[e_i + J, 2e_i + J] - [e_k + e_i + J, e_i + e_i + J]
\]
\[
+ [2e_j + J, 2e_j + J] - [e_j + e_k + J, e_j + e_k + J] + [2e_j + J, 2e_i + J] - [e_i + e_i + J, e_j + e_i + J]
\]
\[
\text{(next 4 lines equivalent to 0 by (2))}
\]
\[
+ 2[2e_i + I, e_k + e_i + I] - 2[e_i + e_k + I, e_i + e_k + J] + 2[2e_i + J, e_k + e_k + J] - [e_i + e_i + I, e_i + e_i + J]
\]
\[
+ 2[2e_j + I, e_k + e_i + I] - 2[e_j + e_k + I, e_j + e_i + J] + 2[2e_i + J, e_k + e_k + J] - [e_j + e_i + I, e_j + e_i + J]
\]
\[
+ 2[e_i + e_j + I, 2e_i + I] - 2[e_i + e_k + I, e_j + e_k + J] + 2[2e_i + J, e_k + e_k + J] - [e_j + e_k + J, e_j + e_k + J]
\]
\[
+ [2e_i + J, 2e_i + I] - 2[e_i + e_i + I, e_j + e_i + J] + 2[e_i + J, 2e_i + J] - [e_i + e_i + J, e_j + e_i + J]
\]
\[
\text{(and 4 lines equivalent to 0 by (4))}
\]
\[
+ 2[2e_i + I, e_k + e_i + I] - 2[e_i + e_k + I, e_i + e_i + J] + 2[2e_i + J, e_k + e_k + J] - [e_i + e_i + I, e_i + e_i + J]
\]
\[
+ 2[2e_j + I, e_k + e_i + I] - 2[e_j + e_k + I, e_j + e_i + J] + 2[2e_i + J, e_k + e_k + J] - [e_j + e_i + I, e_j + e_i + J]
\]
\[
+ 2[e_i + e_j + I, 2e_i + I] + 2[e_i + e_j + I, 2e_i + I] - 2[e_i + e_i + I, e_j + e_i + J] + 2[e_i + J, 2e_i + J] - [e_j + e_i + J, e_j + e_i + J]
\]
\[
\text{(and 4 lines equivalent to 0 by (5))}
\]
\[
+ 4[e_i + e_j + I, e_k + e_i + I] - 2[e_i + e_k + I, e_j + e_i + J] - 2[e_i + e_i + I, e_j + e_k + I]
\]
\[
+ 4[e_i + e_j + J, e_k + e_i + J] + 2[e_i + e_j + I, e_j + e_i + J] - 2[e_i + e_j + I, e_j + e_i + J]
\]
\[
+ 4[e_i + e_j + I, e_k + e_i + J] + 4[e_i + e_i + I, e_j + e_k + J] - 2[e_i + e_k + I, e_j + e_k + J]
\]
\[
\text{(and 2 lines equivalent to 0 by (3))}
\]
\[
+ 4[e_i + e_j + I, e_k + e_i + J] + 4[e_i + e_j + I, e_i + e_j + J] - 2[e_i + e_k + I, e_j + e_k + J]
\]
\[
- 2[e_i + e_j + I, e_i + e_k + J] - 2[e_i + e_i + I, e_j + e_k + J] - 2[e_j + e_k + I, e_i + e_i + J]
\]
\[
\text{(the only remaining part)}
\]

Hence we see that $Q(x_i + x_j, x_k + x_l, x^I + x^J)$ is equivalent to 
\[
4[e_i + e_j + I, e_k + e_l + J] + 4[e_i + e_l + I, e_i + e_j + J] - 2[e_i + e_k + I, e_j + e_l + J]
\]
\[
- 2[e_j + e_l + I, e_i + e_k + J] - 2[e_i + e_l + I, e_j + e_k + J] - 2[e_j + e_k + I, e_i + e_l + J].
\]
\[
(4.3)
\]

By exchanging $j$ and $k$, we also obtain the relation 
\[
Q(x_i + x_k, x_j + x_l, x^I + x^J) \sim
\]
\[
4[e_i + e_k + I, e_j + e_l + J] + 4[e_j + e_l + I, e_i + e_k + J] - 2[e_i + e_j + I, e_k + e_l + J]
\]
\[
- 2[e_k + e_l + I, e_i + e_j + J] - 2[e_i + e_l + I, e_j + e_k + J] - 2[e_j + e_k + I, e_i + e_l + J].
\]
\[
(4.4)
\]
By subtracting (4.4) from (4.3) and dividing it by 6, it can be shown that
\[ [e_i + e_j + I, e_k + e_l + J] + [e_k + e_i + I, e_j + e_l + J] \sim [e_i + e_k + I, e_j + e_l + J] + [e_j + e_l + I, e_i + e_k + J]. \]

Here, we require again the assumption that \( \text{char}(\mathbb{k}) \neq 2, 3. \)

**Lemma 4.4.** Let \( I, J, K, L \in A(n, d) \) be such that \( I + J = K + L. \) Under the assumption (\dagger) (just after Theorem 4.2), if
\[ \text{Supp}(I) \cap \text{Supp}(J) \cap \text{Supp}(K) \cap \text{Supp}(L) \text{ or } \text{Supp}(I)^c \cap \text{Supp}(J)^c \cap \text{Supp}(K)^c \cap \text{Supp}(L)^c \]

is nonempty, then \([I, J] \sim [K, L].\)

**Proof.** The case where \( \text{Supp}(I)^c \cap \text{Supp}(J)^c \cap \text{Supp}(K)^c \cap \text{Supp}(L)^c \) is nonempty is already dealt with in the proof of Proposition 3.3, by the induction on \( n. \)

Next, suppose that \( \text{Supp}(I) \cap \text{Supp}(J) \cap \text{Supp}(K) \cap \text{Supp}(L) \) has an element, say \( i. \) Note that the four elements
\[ I' = I - e_i, \quad J' = J - e_i, \quad K' = K - e_i \quad \text{and} \quad L' = L - e_i \]

of \( A(n, d-1) \) satisfies the condition \( I' + J' = K' + L'. \) By the induction hypothesis on \( d, \)

it holds that \( (I', J') \sim (K', L'). \) Now, by using the map \( \delta_i : I(V_{n,d-1})_2 \rightarrow I(V_{n,d})_2 \) induced from multiplication by \( x_i, \) we can conclude that \([I, J] \sim [K, L].\) \( \square \)

Using the above two lemmas, we can prove the following lemmas.

**Lemma 4.5.** Suppose that \( I_0 \geq 3 \) and \( J_1 \geq 1, J_2 \geq 1. \) Then
\[ [I, J] \sim [-2e_0 + e_1 + e_2 + I, 2e_0 - e_1 - e_2 + J]. \]

**Proof.** By the Lemma 4.3 (5), we have
\[ [2e_0 + (I - 2e_0), e_1 + e_2 + (J - e_1 - e_2)] + [e_1 + e_2 + (I - 2e_0), 2e_0 + (J - e_1 - e_2)] \sim [e_0 + e_1 + (I - 2e_0), e_0 + e_2 + (J - e_1 - e_2)]. \]

Since \( I_0 - 2e_0 > 0, \) it holds by Lemma 4.4 that
\[ [e_1 + e_2 + (I - 2e_0), 2e_0 + (J - e_1 - e_2)] \sim [e_0 + e_1 + (I - 2e_0), e_0 + e_2 + (J - e_1 - e_2)] \]

and
\[ [e_1 + e_2 + (I - 2e_0), 2e_0 + (J - e_1 - e_2)] \sim [e_0 + e_2 + (I - 2e_0), e_0 + e_1 + (J - e_1 - e_2)]. \]

In consequence,
\[ [I, J] = [2e_0 + (I - 2e_0), e_1 + e_2 + (J - e_1 - e_2)] \sim [e_1 + e_2 + (I - 2e_0), 2e_0 + (J - e_1 - e_2)]. \]

Here we use the property that if \( a + b \sim c + d, b \sim c, \) and \( b \sim d, \) then \( a \sim b. \) \( \square \)

**Lemma 4.6.** If \( I_0, I_1, J_2, J_3 \geq 1 \) and \( I_1 \) or \( J_2 \) is \( \geq 2, \) then
\[ [I, J] \sim [-e_0 + e_3 + I, e_0 - e_3 + J]. \]
Proof. Let $I' = e_2 + e_3 + (I - e_0 - e_1)$ and $J' = e_0 + e_1 + (J - e_2 - e_3)$. Then it follows by Lemma 4.3(6) that

$$[I, J] + [I', J'] = [e_0 + e_1 + (I - e_0 - e_1), e_2 + e_3 + (J - e_2 - e_3)] + [e_2 + e_3 + (I - e_0 - e_1), e_0 + e_1 + (J - e_2 - e_3)]$$

$$\sim [e_0 + e_2 + (I - e_0 - e_1), e_1 + e_3 + (J - e_2 - e_3)] + [e_1 + e_3 + (I - e_0 - e_1), e_0 + e_2 + (J - e_2 - e_3)]$$

$$= [-e_1 + e_2 + I, e_1 - e_2 + J] + [-e_0 + e_3 + I, e_0 - e_3 + J].$$

Since $I_1 \geq 2$ or $J_2 \geq 2$, it holds that $I'_1 = I_1 - 1 \geq 1$ and $J'_1 = J_1 + 1 \geq 1$, or $I'_2 = I_2 + 1 \geq 1$ and $J'_2 = J_2 - 1 \geq 1$. Therefore we have

$$[I', J'] \sim [-e_1 + e_2 + I, e_1 - e_2 + J]$$

by Lemma 4.4. In conclusion, it is shown that

$$[I, J] \sim [-e_0 + e_3 + I, e_0 - e_3 + J]$$

by the above equivalence relation of $[I, J] + [I', J']$. \hfill \Box

Recall that the automorphism group $\text{Aut}(V_{n,d}, \mathbb{P}^N)$ of the Veronese variety $V_{n,d}$ in $\mathbb{P}^N$ is defined as

$$\text{Aut}(V_{n,d}, \mathbb{P}^N) := \{ \sigma \in \text{PGL}_N(k) \mid \sigma(V_{n,d}) = V_{n,d} \}. \tag{4.4}$$

There is a natural group action of $\text{Aut}(V_{n,d}, \mathbb{P}^N)$ on the homogeneous ideal $I(V_{n,d})$. In particular, it acts on the $k$-vector space $I(V_{n,d})_2$ and the rank is preserved under this action. Since there is a natural isomorphism between $\text{PGL}_n(k)$ and $\text{Aut}(V_{n,d}, \mathbb{P}^N)$, the group $\text{PGL}_n(k)$ acts on the homogeneous coordinate rings of $\mathbb{P}^n$ and $\mathbb{P}^N$. Also, it acts on $I(V_{n,d})_2$ by which the rank is preserved. Moreover, this action commutes with the $Q$-map in the sense that for any $\sigma \in \text{PGL}_n(k)$,

$$\sigma(Q(s, t, h)) = Q(\sigma(s), \sigma(t), \sigma(h)).$$

By using this observation, we can prove the following

**Lemma 4.7.** If $I_0 = I_1 = K_0 = K_1 = 1$ and $J_0 = J_1 = L_0 = L_1 = 0$, then $[I, J] \sim [K, L]$.

Proof. Let $I' = (2, 0, I_2, \ldots, I_n, I'' = (0, 2, I_2, \ldots, I_n)$, $K' = (2, 0, K_2, \ldots, K_n)$ and $K'' = (0, 2, K_2, \ldots, K_n)$. By Lemma 4.4 we have

$$[I', J] \sim [K', L] \quad \text{and} \quad [I'', J] \sim [K'', L].$$

Now, consider the automorphism $\sigma$ of $\mathbb{P}^n$ induced from the homogeneous coordinate change

$$(x_0, x_1, \ldots, x_n) \mapsto (x_0 + x_1, x_1, \ldots, x_n).$$

From this change of coordinates $x_i^2$ is sent to $(x_0 + x_1)^2 = x_0^2 + 2x_0x_1 + x_1^2$, which shows $\sigma(z_{I'}) = z_{I'} + 2z_I + z_{I''}$. Then it holds that

$$0 \sim \sigma([I', J] - [K', L]) = [I', J] + 2[I, J] + [I'', J] - [K', L] - 2[K, L] - [K'', L].$$

In consequence, it holds that $[I, J] \sim [K, L]$. \hfill \Box
Now, we are ready to give a

**Proof of Theorem 4.2.** As mentioned in Notation and Remarks 4.1.(5), we need to show that \([I, J] \sim [K, L]\) for any choice of \(I, J, K, L \in A(n, d)\) with \(I + J = K + L\). There are the following three cases.

**Case 1.** \(\text{Supp}(I) \cap \text{Supp}(J) \neq \emptyset\) and \(\text{Supp}(K) \cap \text{Supp}(L) \neq \emptyset\)

**Case 2.** \(\text{Supp}(I) \cap \text{Supp}(J) = \emptyset\) and \(\text{Supp}(K) \cap \text{Supp}(L) \neq \emptyset\)

**Case 3.** \(\text{Supp}(I) \cap \text{Supp}(J) = \emptyset\) and \(\text{Supp}(K) \cap \text{Supp}(L) = \emptyset\)

We will deal with these 3 cases in turn.

**Case 1.** In Lemma 4.4 we've already dealt with the case \(\text{Supp}(I) \cap \text{Supp}(J) \cap \text{Supp}(J) \cap \text{Supp}(K)\) is nonempty. So, without loss of generality, we can assume that \(0 \in \text{Supp}(I) \cap \text{Supp}(J)\) and \(1 \in \text{Supp}(K) \cap \text{Supp}(L)\).

If \(K_0\) and \(L_0\) are both nonzero, by Lemma 4.4 we are done. So, one of them is zero. Similarly, one of \(I_1\) and \(J_1\) is zero. So, we may assume that

\[
J_1 = L_0 = 0 \quad \text{and hence} \quad K_0 = I_0 + J_0 \quad \text{and} \quad I_1 = K_1 + L_1.
\]

Then it is true that

\[
J_2 + \cdots + J_n = d - J_0 = d - K_0 + I_0 = I_0 + K_1 + \cdots + K_n > K_1
\]

since \(I_0 > 0\). Thus there exist non-negative integers \(J_2', \ldots, J_n'\) such that

(i) \(J_i \geq J_i'\) for every \(2 \leq i \leq n\), and

(ii) \(J_2' + \cdots + J_n' = K_1\).

Now, we define two elements \(M\) and \(N\) in \(A(n, d)\) as

\[
M := (I_0, L_1, I_2 + J_2', \ldots, I_n + J_n') \quad \text{and} \quad N := (J_0, K_1, J_2 - J_2', \ldots, J_n - J_n').
\]

Then it holds that

\[
|M| = I_0 + I_2 + \cdots + I_n + L_1 + J_2' + \cdots + J_n' = d - I_1 + L_1 + K_1 = d,
\]

\[
|N| = J_0 + J_2 + \cdots + J_n + K_1 - (J_2' + \cdots + J_n') = J_0 + J_2 + \cdots + J_n = d
\]

and

\[
M + N = (I_0 + J_0, L_1 + K_1, I_2 + J_2, \ldots, I_n + J_n) = I + J = K + L.
\]

Therefore \([I, J] - [M, N]\) and \([K, L] - [M, N]\) are contained in \(I(V_n,d)\). Also, \([I, J] \sim [M, N]\) (resp. \([K, L] \sim [M, N]\)) since \(I_0, J_0, M_0\) and \(N_0\) (resp. \(K_1, L_1, M_1\) and \(N_1\)) are nonzero (cf. Lemma 4.4). In consequence, it holds that \([I, J] \sim [K, L]\).

**Case 2.** In this case, without loss of generality, we may assume that \(K_0 \geq 1, L_0 \geq 1\) and \(I_0 = K_0 + L_0, J_0 = 0\). Then \(I_0 \geq 2\).

**Case 2-1.** Suppose that \(|\text{Supp}(I)| = 1\). Since \(I_0\) is already nonzero, \(I_0 = d \geq 3\) and \(I_i = 0\) for all \(1 \leq i \leq n\).

**Case 2-1-1.** If \(J_i = 0\) for some \(i \neq 0\), then \(I_i = J_i = 0\) and hence we are done by Lemma 4.4.
Case 2-1-2. Suppose that $J_i \neq 0$ for all $1 \leq i \leq n$. Since $n \geq 2$, we have $J_1 \geq 1$ and $J_2 \geq 1$. By Lemma 4.5, it holds that

$[I, J] = [I', J'] = [-2e_0 + e_1 + e_2 + I, 2e_0 - e_1 - e_2 + J]$.

Since $I'_0 = I_0 - 2e_0 \geq 1$, $J'_0 = J_0 + 2e_0 \geq 1$, $K_0 \geq 1$ and $L_0 \geq 1$, it follows by Lemma 4.3 that $[I', J'] \sim [K, L]$. Therefore it is true that $[I, J] \sim [K, L]$.

Case 2-2. Suppose that $|\text{Supp}(I)| \geq 2$. Without loss of generality, we may assume that $I_1 \geq 1$ and $J_1 = 0$.

Case 2-2-1. Suppose that $|\text{Supp}(J)| = 1$. Then we may assume that $J_2 = d \geq 3$. By Lemma 4.5 with indices $I, J$ permuted, it holds that

$[I, J] = [-e_0 - e_1 + 2e_2 + I, e_0 + e_1 - 2e_2 + J]$.

Since $I_0 - 1 \geq 1$, $J_0 + 1 \geq 1$, $K_0 \geq 1$ and $L_0 \geq 1$, it follows by Lemma 4.4 that $[I, J] \sim [K, L]$.

Case 2-2-2. Suppose that $|\text{Supp}(J)| \geq 2$. Then we may assume that $J_2 \geq 1$ and $J_3 \geq 1$.

Case 2-2-2-1. Suppose that $I_1 \geq 2$ or $J_2 \geq 2$. By Lemma 4.6, it holds that $[I, J] = [-e_0 + e_3 + I, e_0 - e_3 + J]$. Since $I_0 - 1 \geq 1$, $J_0 + 1 \geq 1$, $K_0 \geq 1$ and $L_0 \geq 1$, it follows by Lemma 4.4 that $[I, J] \sim [K, L]$.

Case 2-2-2-2. Now, the only remaining case is where all nonzero entries of $I$ and $J$ other than $I_0$ are equal to 1. After reordering, we can obtain the following form, where $\vec{I}_a$ and $\vec{O}_b$ mean respectively the list of $a$ 1’s and $b$ 0’s.

$I = (I_0, \vec{I}_s, \vec{I}_t, \vec{O}_u, \vec{O}_v), \hspace{1em} J = (0, \vec{O}_s, \vec{O}_t, \vec{I}_u, \vec{I}_v)$

$K = (K_0, \vec{I}_s, \vec{I}_t, \vec{O}_u, \vec{O}_v), \hspace{1em} L = (L_0, \vec{O}_s, \vec{I}_t, \vec{O}_u, \vec{I}_v)$

Case 2-2-2-2-1. If one of $s, t, u$ and $v$ is greater than or equal to 2, then we can apply Lemma 4.7 to show that $[I, J] \sim [K, L]$.

Case 2-2-2-2-2. If $s, t, u, v \leq 1$, then $d = |J| = u + v \leq 2$. So, it reduces to the case when $d \leq 2$ and it is contradict to the assumption.

Case 3. For the third case, by reordering the indices, we can represent $I, J, K, L$ by the tuples $S, T, U, V$ with nonzero entries with length $s, t, u, v$, respectively, as below:

$I = (S, T, \vec{O}_u, \vec{O}_v), \hspace{1em} J = (\vec{O}_s, \vec{O}_t, U, V), \hspace{1em} K = (S, \vec{O}_t, U, \vec{O}_v), \hspace{1em} L = (\vec{O}_s, T, \vec{O}_u, V)$

(4.5)

where the maximum entry of each $S, T, U$ and $V$ is moved to the first place in each $S, T, U, V$ (namely $S_0, T_0, U_0$ and $V_0$). Note that we can freely exchange roles of $S, T, U, V$ in this format (4.5) as permuting indices.

Case 3-1. Suppose that one of $s, t, u, v$ is zero. Without loss of generality, we may assume that $s = 0$. Then $d = |I| = |T|$ and $d = |L| = |T| + |V| = d + |V|$. Hence $|V| = 0$. It means that $I = (T, \vec{O}_u) = L$ and $J = (\vec{O}_t, U) = K$. This is a trivial case.
This completes the proof. □

Case 3-2. Suppose that $s, t, u, v \geq 1$. First, note that (after further re-indexing, if needed) we can write the relation $[I, J] \sim [K, L]$ as

$$[(S_0, T_0, 0, 0, \tilde{O}_{n-3}) + I', (0, 0, U_0, V_0, \tilde{O}_{n-3}) + J'] \sim [(S_0, 0, U_0, 0, \tilde{O}_{n-3}) + K', (0, T_0, 0, V_0, \tilde{O}_{n-3}) + L']$$

(4.6)

where $I', J', K'$ and $L'$ are the remaining parts of $I, J, K$ and $L$ whose first 4 entries are all zeros.

Case 3-2-1. Now, let us treat the case of at least two of $S_0, T_0, U_0$ and $V_0$ being greater than or equal to 2. As exchanging roles of $S, T, U, V$, we may assume that $S_0 \geq 2, T_0 \geq 2$ or $S_0 \geq 2, U_0 \geq 2$ in (4.5). For $S_0 \geq 2, T_0 \geq 2$, by Lemma 4.6 first we have

$$[(S_0, T_0, 0, 0, \tilde{O}_{n-3}) + I', (0, 0, U_0, V_0, \tilde{O}_{n-3}) + J'] \sim [(S_0 - 1, T_0, 0, 1, \tilde{O}_{n-3}) + I', (1, 0, U_0, V_0 - 1, \tilde{O}_{n-3}) + J'].$$

And, again by Lemma 4.6 with indices permuted as $(0, 1, 2, 3) \mapsto (0, 2, 1, 3)$ we obtain

$$[(S_0, 0, U_0, 0, \tilde{O}_{n-3}) + K', (0, T_0, 0, V_0, \tilde{O}_{n-3}) + L'] \sim [(S_0 - 1, 0, U_0, 1, \tilde{O}_{n-3}) + K', (1, T_0, 0, V_0 - 1, \tilde{O}_{n-3}) + L'].$$

Since $S_0 - 1 \geq 1$, we have

$$[(S_0 - 1, T_0, 0, 1, \tilde{O}_{n-3}) + I', (1, 0, U_0, V_0 - 1, \tilde{O}_{n-3}) + J'] \sim [(S_0 - 1, 0, U_0, 1, \tilde{O}_{n-3}) + K', (1, T_0, 0, V_0 - 1, \tilde{O}_{n-3}) + L'].$$

by Lemma 4.4 since the first entry belongs to the common support. From the expression (4.6), using equivalence relation, we can deduce that $[I, J] \sim [K, L]$. For the other case $S_0 \geq 2, U_0 \geq 2$, similarly we can apply Lemma 4.6 with permuted indices and Lemma 4.4 to obtain the same result.

Case 3-2-2. Suppose that at most one of $S_0, T_0, U_0$ and $V_0$ is greater than or equal to 2. Without loss of generality, we may assume that $T_0, U_0, V_0 \leq 1$. Then, since the maximum entry is less than or equal to 1, all the entries of $T, U, V$ are 1’s in (4.5).

Case 3-2-2-1. If $t, u$ or $v$ is greater than or equal to 2 (i.e. at least two consecutive indices corresponding to 0’s and 1’s), then by re-indexing such indices to the first places, we can get that $[I, J] \sim [K, L]$ by Lemma 4.7

Case 3-2-2-2. When $t, u$ and $v$ are all less than or equal to 1, we have $d = |J| = u + v \leq 2$. This contradicts to our assumption $n \geq 2, d \geq 3$.

This completes the proof. □

5. Property $\text{QR}(3)$ of arbitrary projective schemes

Our purpose in this section is to prove Theorem 4.3 about the asymptotic behavior of the rank of quadratic generators of the Veronese re-embedding and to provide its various
Theorem 1.3. Suppose that char $\mathbb{k} \neq 2, 3$ and let $L$ be a very ample line bundle on a projective scheme $X$ defining the linearly normal embedding

$$X \subset \mathbb{P}H^0(X, L).$$

If $m$ is an integer such that $X$ is $j$-normal for all $j \geq m$ and $I(X)$ is generated by forms of degree $\leq m$, then $(X, L^d)$ satisfies property $QR(3)$ for all $d \geq m$.

Recall that $X$ is $j$-normal if the natural map $\text{Sym}^d H^0(X, L) \to H^0(X, L^d)$ is surjective.

Proof of Theorem 1.3. Let $X_d \subset \mathbb{P}^{r(d)} := \mathbb{P}H^0(X, L^d)$ be the linearly normal embedding of $X$ by the complete linear series $|L^d|$. Since $X \subset \mathbb{P}^n := \mathbb{P}H^0(X, L)$ is $d$-normal and $\mathbb{P}^n = \mathbb{P}\text{Sym}^d H^0(X, L)$, we can regard $\mathbb{P}^{r(d)}$ as a subspace of $\mathbb{P}^{n(d)}$ by the surjective map $\text{Sym}^d H^0(X, L) \to H^0(X, L^d)$. Then $X_d$, the embedding of $X$ by $|L^d|$, is precisely equal to the image $\nu_d(X)$ where $\nu_d : \mathbb{P}^n \to \mathbb{P}^{n(d)}$ denotes the $d$th Veronese embedding of $\mathbb{P}^n$ (cf. [Hart77, Exercise II.5.13]). We first show that $X_d$ is ideal-theoretically the intersection of $V_{n,d} = \nu_d(\mathbb{P}^n)$ and $\mathbb{P}^{r(d)}$ in $\mathbb{P}^{n(d)}$. Let $\mathcal{J}$ be the ideal sheaf of $V_{n,d}$ in $\mathbb{P}^{n(d)}$. Also let $\mathcal{I}_d$ and $\mathcal{J}_d$ be respectively the ideal sheaves of $X_d$ in $V_{n,d}$ and in $\mathbb{P}^{n(d)}$. Thus there is an exact sequence

$$0 \to \mathcal{J} \to \mathcal{J}_d \to \mathcal{I}_d \to 0 \quad (5.1)$$

of coherent sheaves on $\mathbb{P}^{n(d)}$. Since $V_{n,d}$ is projectively normal, we obtain the short exact sequence

$$0 \to I(V_{n,d}/\mathbb{P}^{n(d)}) \to I(X_d/\mathbb{P}^{n(d)}) \to E := \bigoplus_{j \geq 0} H^0(\mathbb{P}^n, \mathcal{I}_X(dj)) \to 0$$

of graded modules on $\mathbb{P}^{n(d)}$. Note that $E$ is generated by $E_1$ as a graded module on $\mathbb{P}^{n(d)}$ since $I(X)$ is generated by forms of degree $\leq d$. Also $I(\mathbb{P}^{r(d)}/\mathbb{P}^{n(d)})$ is contained in $I(X_d/\mathbb{P}^{n(d)})$ and

$$I(\mathbb{P}^{r(d)}/\mathbb{P}^{n(d)})_1 = I(X_d/\mathbb{P}^{n(d)})_1 \cong E_1.$$

Obviously, $I(\mathbb{P}^{r(d)}/\mathbb{P}^{n(d)})$ is generated by its degree one piece. Therefore it holds that

$$I(X_d/\mathbb{P}^{n(d)}) = I(V_{n,d}/\mathbb{P}^{n(d)}) + I(\mathbb{P}^{r(d)}/\mathbb{P}^{n(d)}),$$

which shows exactly that $X_d$ is the ideal-theoretic intersection of $V_{n,d} = \nu_d(\mathbb{P}^n)$ and $\mathbb{P}^{r(d)}$.

Now, $X_d \subset \mathbb{P}^{r(d)}$ satisfies property $QR(3)$ since the Veronese variety $V_{n,d} \subset \mathbb{P}^{n(d)}$ satisfies property $QR(3)$ and $X_d$ is its linear section. 

Remark 5.1 (An ideal-theoretic version of Mumford’s fundamental observation in [M70]). Let $X \subset \mathbb{P}^n$, $m$ and $X_d \subset \mathbb{P}^{r(d)}$ be as in Theorem 1.3 and its proof. So, it is shown in the above proof that $X_d$ is an ideal-theoretic linear section of the Veronese variety $V_{n,d}$ for all $d \geq m$. Obviously, this implies that various structural properties of $V_{n,d}$ is inherited by $X_d$. For example, it is shown in [Puc98, Corollary 3.5] that $V_{n,d}$ is determinantal presented by $(1, d-1)$-type symmetric flattening. Therefore $(X, L^d)$ is also determinantal presented.
for all $d \geq m$. More precisely, let $\Omega(L, L^{d-1})$ be the matrix of linear forms on $\mathbb{P}^{r(d)}$ obtained from the natural map

$$H^0(X, L) \otimes H^0(X, L^{d-1}) \to H^0(X, L^d).$$

Then the homogeneous ideal of $X_d \subset \mathbb{P}^{r(d)}$ is generated by 2-minors of $\Omega(L, L^{d-1})$ (cf. Theorem 1.1 in [SS1].)

Theorem 1.3 and Remark 5.1 imply several geometric consequences for which we are aiming.

**Corollary 5.2.** Suppose that $\text{char } \mathbb{k} \neq 2, 3$. Let $L$ be a very ample line bundle on a projective scheme $X$ which satisfies condition $N_1$. Then $(X, L^d)$ is determinantly presented and satisfies property $QR(3)$ for all $d \geq 2$.

**Proof.** By our assumption on $L$, the linearly normal embedding $X \subset \mathbb{P} H^0(X, L)$ is $j$-normal for all $j \geq 2$ and its homogeneous ideal is generated by forms of degree $\leq 2$. Thus the assertions come immediately from Remark 5.1 and Theorem 1.3 respectively. □

In a similar manner, we can obtain a more general statement as follows:

**Corollary 5.3.** Suppose that $\text{char } \mathbb{k} \neq 2, 3$. Let $A$ be an ample line bundle on a projective connected scheme $X$. Then there exists a number $d_0 = d_0(X, A)$ such that if $d$ has a proper divisor $\geq d_0$, then $A^d$ satisfies property $QR(3)$. In particular, $A^d$ satisfies property $QR(3)$ if $d$ is even and $d \geq 2d_0$.

**Proof.** Note that a sufficiently ample line bundle on $X$ satisfies condition $N_1$ (cf. [SS1, Proof of Theorem 1.1]). This implies that there exists a number $d_0$ such that $A^d$ satisfies condition $N_1$ for every $d \geq d_0$. Now, suppose that $d$ has a proper divisor $\ell \geq d_0$ and write $d = \ell \times k$ where $k \geq 2$ is an integer. Then $A^d = (A^\ell)^k$ satisfies property $QR(3)$ since $A^\ell$ satisfies condition $N_1$ (cf. Corollary 5.2). □

We finish this section by providing a couple of applications of Theorem 1.3 to some classical varieties in the literature.

**Example 5.4.** Let $X \subset \mathbb{P}^n$ be a smooth complete intersection of hypersurfaces $F_1, \ldots, F_c$ of degrees $d_1 \geq \cdots \geq d_c \geq 2$. Also let $L := \mathcal{O}_X(1)$. Thus $X$ is projectively normal and its homogeneous ideal is generated by forms of degree $\leq d_1$.

1. By Theorem 1.3, it holds that $L^d$ satisfies property $QR(3)$ for all $d \geq d_1$.
2. Recall that $K_X = \mathcal{O}_X(d_1 + \cdots + d_c - n - 1)$. Thus, if $c \geq 2$ and $d_2 + \cdots + d_c \geq n + 1$ then the canonical embedding of $X$ satisfies property $QR(3)$ by (1).
3. If $n \geq 5$ and $X$ is a curve, then it holds always that $d_2 + \cdots + d_{n-1} \geq n + 1$. Thus, by (2), we can see that there are $\infty$-many canonical curves which satisfy property $QR(3)$ (see also Example 6.2).

**Example 5.5.** Let $X = \text{Gr}(k, V)$ be the Grassmannian manifold of $k$-dimensional subspaces of the $n$-dimensional $k$-vector space $V$ where $n \geq 3$ and $1 \leq k \leq n - 2$. Also let $L$ be the generator of Pic $X$ which defines the Plücker embedding of $X$.

1. Let $\text{Gr}(k, n) \subset \mathbb{P}(\wedge^k V)$ be the Plücker embedding. Then it contains $\text{Gr}(2, 4)$ as a linear section (cf. [Harr92, Chapter 6]). Thus $L$ fails to satisfy property $QR(5)$ since
Gr(2, 4) ⊂ ℙ^5 is a hyperquadric of rank 6. On the other hand, it is shown in [KPRS08] that Gr(k, n) is set-theoretically cut out by quadratic equations of rank 6.

(2) It is well-known that L satisfies condition N_2 (cf. Proposition 3.8 and Remark 3.9.1 in [EGHP03]). Therefore it follows by Theorem 1.3 that (Gr(k, n), L^d) satisfies property QR(3) for all d ≥ 2.

For the remaining part of this section, we assume that the characteristic of k̄ is zero.

Example 5.6. Let X be an abelian variety and let A be an ample line bundle on X. The main theorem in [Pare00] implies that A^d satisfies condition N_d for all d ≥ 4. Then it follows by Corollary 5.3 and its proof that (X, A^d) satisfies property QR(3) whenever d ≥ 8 and neither d ≠ 9 nor d is a prime (that is, d has a proper divisor ≥ 4).

Example 5.7. Let S ⊂ ℙ^{g−1} be a linearly normal Enriques surface and let L := O_S(1). It is well-known that g ≥ 6. Also (S, L) is called a Reye polarization if g = 6 and S fails to be 2-normal, or equivalently it lies on a quadric. In [GLM02], the authors prove the following results:

a. If (S, L) is a Reye polarization, then S ⊂ ℙ^5 is 4-regular and its homogeneous ideal is generated by forms of degree ≤ 3.

b. If (S, L) is not a Reye polarization, then S ⊂ ℙ^{g−1} is 3-regular.

Thus Theorem 1.3 shows that (S, L^d) satisfies property QR(3) for all d ≥ 3.

Example 5.8. Let S ⊂ ℙ^g be a linearly normal K3 surface and let L := O_S(1). Recall that a general hyperplane section C ⊂ ℙ^{g−1} of S is a canonical curve of genus g. Thus S is projectively normal and its homogeneous ideal is generated by quadratic and cubic equations. Now, Theorem 1.3 shows that (S, L^d) satisfies property QR(3) for all d ≥ 3. In particular, any general hyperplane section of S ⊂ ℙH^0(S, L^d) is a canonical curve satisfying property QR(3).

6. Further discussions and Open problems

In the final section, we will discuss some open questions related to our main results in the present paper.

Positivity and Rank 3 quadratic generation Let L be a very ample line bundle on a projective scheme X. There have been many interesting results showing that the positivity of L is reflected in the defining equations of X ⊂ ℙH^0(X, L) and the syzygies among them. To state more precisely, due to [Gr84b, Definition 3.1], we say that a property P holds for every sufficiently ample line bundle on X if there exists a line bundle A on X such that P holds for all line bundles L ∈ Pic(X) for which L ⊗ A is ample. For example, let p be a positive integer. Theorem 1 in [EL93] shows that when X is a smooth complex variety, every sufficiently ample line bundle on X satisfies condition N_p. Recently, Sidman and Smith in [SS11] prove that every sufficiently ample line bundle on a connected scheme is determinantal presented. In this direction, our main results in this paper allude to a significant correlation between some positive nature of the very ample line bundle L on X and the property QR(3) of (X, L), just like in many works on condition N_p and determinantal presentation of projective embeddings. For instance, a sufficiently ample
line bundle on a projective connected scheme $X$ satisfies condition $N_1$ (cf. [SS11, Proof of Theorem 1.1]). For all those sufficiently ample line bundles $L$, it follows by Corollary 5.2 that $(X, L^2)$ satisfies property $QR(3)$. This observation leads us naturally to formulate

**Conjecture 6.1.** Every sufficiently ample line bundle on a projective scheme satisfies property $QR(3)$.

(1) The above conjecture is shown to be true by Theorem 1.1 and Theorem 1.3 when $Pic(X)$ is generated by a very ample line bundle (e.g. $X = \mathbb{P}^n$ or a Grassmannian manifold).

(2) In [Park20], the above conjecture is proved for any projective integral curve $C$ of arithmetic genus $g$ by showing that property $QR(3)$ holds for $(C, \mathcal{L})$ whenever $\deg(\mathcal{L}) \geq 4g + 4$. Note that if $\mathcal{L} = \mathcal{M}^2$ and $\deg(\mathcal{M}) \geq 2g + 2$, then $(C, \mathcal{L})$ satisfies property $QR(3)$ by Corollary 5.2 since $(C, \mathcal{M})$ satisfies condition $N_1$.

(3) Among aforementioned classical varieties, we intend to treat Segre-Veronese embeddings and rational normal scrolls in a forthcoming paper [MP20]. Let $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ where $k \geq 2$ and $n_1, \ldots, n_k \geq 1$. Every very ample line bundle on $X$ satisfies property $QR(4)$. In [MP20], we prove that $O(d_1, \ldots, d_k)$ (with $d_1, \ldots, d_k \geq 1$) fails to satisfy property $QR(3)$ if $d_i = 1$ for some $1 \leq i \leq k$. This means that all line bundles on the boundary of the ample cone of $X$ fail to satisfy property $QR(3)$.

**Property $QR(3)$ of canonical curves** Let $C \subset \mathbb{P}^{g-1}$ be a smooth canonical curve of genus $g$ satisfying condition $N_1$. M. Green in [Gr84] proved that property $QR(4)$ holds. Regarding property $QR(3)$, we have a computational example of some canonical curve of $g = 6$ which fail to satisfy property $QR(3)$. See Example 6.2. On the other hand, our main results produce infinitely many examples of canonical curves satisfying property $QR(3)$ (cf. Example 5.4 and Example 5.8). Thus it seems an interesting open question whether property $QR(3)$ holds for canonical curves of sufficiently large genus.

**Example 6.2** (A canonical curve without property $QR(3)$). Let $S := \mathbb{K}[x_1, x_2, x_3, x_4, x_5, x_6]$ be the homogeneous coordinate ring of $\mathbb{P}^5$. Consider the homogeneous ideal $I =$
\[ \langle Q_1, Q_2, Q_3, Q_4, Q_5, Q_6 \rangle \] of \( S \) where

\[\begin{align*}
Q_1 &= -x_2^2 + 3x_1 + x_3^2 - 2x_4x_2 + 2x_5x_1 + 2x_5x_3 - 2x_5x_4 - 3x_6x_1 - x_6x_2 \\
    &\quad -3x_6x_3 + 4x_6x_4 + 3x_6x_5 - 8x_6^2, \\
Q_2 &= -x_3x_2 + 2x_3^2 + 4x_3 - 3x_4x_2 + 4x_5x_1 + 4x_5x_3 + 2x_5x_4 + x_5x_6 + 5x_6x_2 \\
    &\quad -8x_6x_3 + 7x_6x_4 + 2x_6x_5 + 5x_6^2, \\
Q_3 &= -2x_3^2 + 2x_4x_2 + x_4x_3 - 3x_5x_1 - 4x_5x_3 - 3x_5x_4 - 4x_6x_1 - 6x_6x_2 \\
    &\quad + 6x_6x_3 - 4x_6x_4 - x_6x_5 - 10x_6^2, \\
Q_4 &= -x_3^2 + x_4x_2 + x_4^2 - x_5x_1 - 3x_5x_3 - 3x_5x_4 - 6x_6x_1 - 5x_6x_2 + 3x_6x_3 \\
    &\quad - x_6x_4 - x_6x_5 - 6x_6^2, \\
Q_5 &= -x_3^2 + x_4x_2 - x_5x_1 + x_5x_2 + 2x_5x_4 + 7x_6x_1 + 3x_6x_2 + 4x_6x_3 \\
    &\quad - 6x_6x_4 - 2x_6x_5 + 6x_6^2, \\
Q_6 &= x_5^2 - x_5x_1 + 2x_6x_2 + x_6x_3 - 2x_6x_4. 
\end{align*}\]

Then \( C := Z(I) \subset \mathbb{P}^5 \) is a curve of genus 6 and degree 10. Indeed, it is the canonical embedding of the modular curve \( X_0(58) \). Furthermore, up to scalar multiplication, \( Q_6 \) is the only quadratic equation of rank 3 in \( I \). Therefore \( C \) fails to satisfy property \( QR(3) \). All computations are provided by the computer algebra systems Macaulay2 [GS10].

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