Quenched Survival of Bernoulli Percolation on Galton–Watson Trees

Marcus Michelen \(^1\) · Robin Pemantle \(^2\) · Josh Rosenberg \(^3\)

Received: 12 August 2019 / Accepted: 20 August 2020 / Published online: 28 August 2020
© Springer Science+Business Media, LLC, part of Springer Nature 2020

Abstract
We explore the survival function for percolation on Galton–Watson trees. Letting \( g(T, p) \) represent the probability a tree \( T \) survives Bernoulli percolation with parameter \( p \), we establish several results about the behavior of the random function \( g(T, \cdot) \), where \( T \) is drawn from the Galton–Watson distribution. These include almost sure smoothness in the supercritical region; an expression for the \( k \)th-order Taylor expansion of \( g(T, \cdot) \) at criticality in terms of limits of martingales defined from \( T \) (this requires a moment condition depending on \( k \)); and a proof that the \( k \)th order derivative extends continuously to the critical value. Each of these results is shown to hold for almost every Galton–Watson tree.

Keywords Supercritical · Quenched survival · Random tree · Branching process

1 Introduction

Let \( GW \) denote the measure on locally finite rooted trees induced by the Galton–Walton process for some fixed progeny distribution \( \{p_n\} \) whose mean will be denoted \( \mu \). A random tree generated according to the measure \( GW \) will be denoted as \( T \). Throughout, we let \( Z \) denote a random variable with distribution \( \{p_n\} \) and assume that \( P[Z = 0] = 0 \); passing to the reduced tree as described in [2, Chap. 1.D.12], no generality is lost for any of the questions in the paper.
The growth rate and regularity properties of both random and deterministic trees can be analyzed by looking at the behavior of a number of different statistics. The Hausdorff dimension of the boundary and the escape speed of random walk are almost surely constant for a fixed Galton–Watson measure. Quantities that are random but almost surely well defined include the martingale limit $W := \lim Z_n/\mu^n$, the resistance to infinity when edges at level $n$ carry resistance $x^n$ for a fixed $x < \mu$, and the probability $g(T, p)$ that $T$ survives Bernoulli-$p$ percolation, i.e., the probability there is a path of open edges from the root to infinity, where each edge is declared open with independent probability $p$. In this paper we seek to understand $GW$-almost sure regularity properties of the survival function $g(T, \cdot)$ and to compute its derivatives at criticality.

The properties of the Bernoulli-$p$ percolation survival function have been studied extensively in certain other cases, such as on the deterministic $d$-dimensional integer lattice, $\mathbb{Z}^d$. When $d = 2$, the Harris–Kesten Theorem [9,11] states that the critical percolation parameter $p_c$ is equal to 1/2 and that critical percolation does not survive: $g(\mathbb{Z}^2, 1/2) = 0$; more interesting is the nondifferentiability from the right of the survival function at criticality [12]. When $d \geq 3$, less is known, despite the high volume of work on the subject. The precise value of the critical probability $p_c(d)$ is unknown for each $d \geq 3$; for $d \geq 19$, mean-field behavior has been shown to hold, implying that percolation does not occur at criticality [8]. This has recently been upgraded with computer assistance and shown to hold for $d \geq 11$ [6], while the cases of $3 \leq d \leq 10$ are still open. Lower bounds on the survival probability of $\mathbb{Z}^d$ in the supercritical region are an area of recent work [5], but exact behavior near criticality is not known in general. On the question of regularity, the function $g(\mathbb{Z}^d, p)$ is smooth on $(p_c(d), 1]$ for each $d \geq 2$ [7, Theorem 8.92].

There is less known about the behavior of $g(T, \cdot)$ for random trees than is known on the integer lattice. We call the random function $g(T, \cdot)$ the quenched survival function to distinguish it from the annealed survival function $g$, where $g(x)$ is the probability of survival at percolation parameter $x$ averaged over the $GW$ distribution. For the regular $d$-ary tree, $T_d$, the classical theory of branching processes implies that the critical percolation parameter $p_c$ is equal to $1/d$, that $g(T_d, 1/d) = 0$ (that is, there is no percolation at criticality), and that for $p > p_c$, the quantity $g(T_d, p)$ is equal to the largest fixed point of $1 - (1 - px)^d$ in $[0, 1]$ (see, for instance, [2] for a treatment of this theory).

For Galton–Watson trees, a comparison of the quenched and annealed survival functions begins with the following classical result of Lyons, showing that $p_c$ is the same in both cases.

**Theorem 1.1** [13] Let $T$ be the family tree of a Galton–Watson process with mean $E[Z] =: \mu > 1$, and let $p_c(T) = \sup \{ p \in [0, 1] : g(T, p) = 0 \}$. Then $p_c(T) = \frac{1}{\mu}$ almost surely. Together with the fact that $g(T, 1/\mu) = 0$, this implies $g(T, p_c) = 0$ almost surely. □

To dig deeper into this comparison, observe first that the annealed survival probability $g(x)$ is the unique fixed point on $[0, 1)$ of the function $1 - \phi(1 - px)$ where $\phi(z) = E_Z Z$ is the probability generating function of the offspring distribution. In the next section we show that the annealed survival function $g(p)$ is smooth on $(p_c, 1)$ and, under moment conditions on the offspring distribution, the derivatives extend continuously to $p_c$. This motivates us to ask whether the same holds for the quenched survival function. Our main results show this to be the case, giving regularity properties of $g(T, p)$ on the supercritical region.

Let $r_j$ be the coefficients in the asymptotic expansion of the annealed function $g$ at $p_c$. These are shown to exist in Proposition 2.6 below. In Theorem 3.1, under appropriate moment conditions, we will construct for each $j \geq 1$ a martingale $\{ M_n^{(j)} : n \geq 1 \}$ with an almost sure limit $M^{(j)}$, that is later proven to equal the $j$th coefficient in the asymptotic expansion of the
quenched survival function \( g \) at \( p_c \). Throughout the analysis, the expression \( W \) denotes the martingale limit \( \lim Z_n/\mu^n \).

**Theorem 1.2** (main results)

(i) For \( GW \) a.e. tree \( T \), the quantity \( g(T, x) \) is smooth as a function of \( x \) on \((p_c, 1)\).

(ii) If \( \mathbb{E}Z^{2k+1+\beta} < \infty \) for some positive integer \( k \) and some \( \beta > 0 \), then we have the \( k \)-th order approximation

\[
g(T, p_c + \varepsilon) = \sum_{j=1}^{k} M^{(j)} \varepsilon^j + o(\varepsilon^k)
\]

for \( GW \) a.e. tree \( T \), where \( M^{(i)} \) is the quantity given explicitly in Theorem 3.1. Additionally, \( M^{(1)} = \mathbb{W}_1 \) and \( \mathbb{E}[M^{(j)}] = r_j \), where \( W \) is the martingale limit for \( T \) and \( j!r_j \) are the derivatives of the annealed survival function, for which explicit expressions are given in Proposition 2.6.

(iii) If \( \mathbb{E}Z^{2k^3+3+\beta} < \infty \) for some \( \beta > 0 \), then \( GW \)-almost surely \( g(T, \cdot) \) is of class \( C^k \) from the right at \( p_c \) and \( g^{(j)}(T, p_c^+) = j!M^{(j)} \) for all \( j \leq k \); see the beginning of Sect. 2.1 for calculus definitions.

**Remarks** Smoothness of \( g(T, \cdot) \) on \((p_c, 1)\) does not require any moment assumptions, in fact even when \( \mathbb{E}Z = \infty \) one has \( p_c = 0 \) and smoothness of \( g(T, \cdot) \) on \((0, 1)\). The moment conditions relating to expansion at criticality given in (ii) are probably not best possible, but are necessary in the sense that if \( \mathbb{E}Z^k = \infty \) for some \( k \) then not even the annealed survival function is smooth (see Proposition 2.4 below).

The proofs of the first two parts of Theorem 1.2 are independent of each other. Part (ii) is proved first, in Sect. 3. Part (i) is proved in Sect. 4.2 after some preliminary work in Sect. 4.1. Finally, part (iii) is proved in Sect. 4.3. The key to these results lies in a number of different expressions for the probability of a tree \( T \) surviving \( p \)-percolation and for the derivatives of this with respect to \( p \). The first of these expressions is obtained via inclusion-exclusion. The second, Theorem 4.1 below, is a Russo-type formula [14] expressing the derivative in terms of the expected branching depth

\[
\frac{d}{dp} g(T, p) = \frac{1}{p} \mathbb{E}_T |B_p|
\]

for \( GW \)-almost every \( T \) and every \( p \in (p_c, 1) \), where \( |B_p| \) is the depth of the deepest vertex \( B_p \) whose removal disconnects the root from infinity in \( p \)-percolation. The third generalizes this to a combinatorial construction suitable for computing higher moments.

A brief outline of the paper is as follows. Section 2 contains definitions, preliminary results on the annealed survival function, and a calculus lemma. Section 3 writes the event of survival to depth \( n \) as a union over the events of survival of individual vertices, then obtains bounds via inclusion-exclusion. Let \( X_n^{(j)} \) denote the expected number of cardinality \( j \) sets of surviving vertices at level \( n \), and let \( X_n^{(j,k)} \) denote the expected \( k \)th falling factorial of this quantity. These quantities diverge as \( n \rightarrow \infty \) but inclusion-exclusion requires only that certain signed sums converge as \( n \rightarrow \infty \). The Bonferroni inequalities give upper and lower bounds on \( g(T, \cdot) \) for each \( n \). Strategically choosing \( n \) as a function of \( \varepsilon \) and using a modified Strong Law argument allows us to ignore all information at height beyond \( n \) (Proposition 3.9).

Each term in the Bonferroni inequalities is then individually Taylor expanded, yielding an expansion of \( g(T, p_c + \varepsilon) \) with coefficients depending on \( n \). Letting \( T \sim GW \) and \( n \rightarrow \infty \), the
variables $X_n^{(j,k)}$ separate into a martingale part and a combinatorial part. The martingale part converges exponentially rapidly (Theorem 3.5). The martingale property for the coefficients themselves (Lemma 3.12) follows from some further analysis (Lemma 3.11) eliminating the combinatorial part when the correct signed sum is taken.

Section 4.1 proves the above formula for the derivative of $g$ (Theorem 4.1) via a Markov property for the coupled percolations as a function of the percolation parameter $p$. Section 4.2 begins with a well-known branching process description of the subtree of vertices with infinite lines of descent. It then goes on to describe higher order derivatives in terms of combinatorial gadgets denoted $D$ which are moments of the numbers of edges in certain rooted subtrees of the percolation cluster and generalize the branching depth. We then prove an identity for differentiating these (Lemma 4.12), and apply it repeatedly to $g(T, p) = p^{−1}EB_p$, to write $(\partial/\partial p)^kg(T, p)$ as a finite sum $\sum_\alpha D_\alpha$ of factorial moments of sets of surviving vertices. This suffices to prove smoothness of the quenched survival function on the supercritical region $p_c < p < 1$.

For continuity of the derivatives at $p_c$, an analytic trick is required. If a function possessing an order $N$ asymptotic expansion at the left endpoint of an interval $[a, b]$ ceases to be of class $C^k$ at the left endpoint for some $k$, then the $k + 1$st derivative must blow up faster than $(x − a)^{−N/k}$ (Lemma 2.1). This is combined with bounds on how badly things can blow up at $p_c$ (Proposition 4.14) to prove continuity from the right at $p_c$ of higher order derivatives.

The paper ends by listing some questions left open, concerning sharp moment conditions and whether an asymptotic expansion ever exists without higher order derivatives converging at $p_c$.

2 Constructions, Preliminary Results, and Annealed Survival

2.1 Smoothness of Real Functions at the Left Endpoint

Considerable work is required to strengthen conclusion $(ii)$ of Theorem 1.2 to conclusion $(iii)$. For this reason, we devote a brief subsection to making the calculus statements in Theorem 1.2 precise. We begin by recalling some basic facts about one-sided derivatives at endpoints, then state a useful lemma.

Let $f$ be a smooth function on the nonempty real interval $(a, b)$. We say that $f$ is of class $C^k$ from the right at $a$ if $f$ and its first $k$ derivatives extend continuously to some finite values $c_0, \ldots, c_k$ when approaching $a$ from the right. This implies, for each $j \leq k$, that $\varepsilon^{-1} \left( f^{(j-1)}(a + \varepsilon) – f^{(j-1)}(a) \right) \to c_j$ as $\varepsilon \downarrow 0$, and therefore that $c_j$ is the $j$th derivative of $f$ from the right at $a$. It follows from Taylor’s theorem that there is an expansion with the same coefficients:

$$f(x) = f(a) + \sum_{j=1}^{k} \frac{c_j}{j!}(x - a)^j + o(x - a)^k$$

(2.1)
as $x \to a^+$.

The converse is not true: the classical example $f(x) = x^k \sin(1/x^m)$ shows that it is possible to have the expansion (2.1) with $f$ ceasing to be of class $C^j$ once $j(m + 1) \geq k$. On the other hand, the ways in which the converse can fail are limited. We show that if $f$ possesses an order-$k$ expansion as in (2.1) and if $j \leq n$ is the least integer for which $f \notin C^j$, then $f^{(j+1)}$ must oscillate with an amplitude that blows up at least like a prescribed power.
of $(x - a)^{-1}$. Here, we use the usual “little oh” and “little omega” notation: $a_n = o(b_n)$ indicates that $|a_n|/|b_n| \to 0$ and $a_n = \omega(b_n)$ indicates $|a_n|/|b_n| \to \infty$. We also make use of the similar “big oh” notation: $a_n = O(b_n)$ means $|a_n| \leq C|b_n|$ for some $C > 0$; $|a_n| = \Omega(|b_n|)$ means $|a_n| \geq c|b_n|$ for $c > 0$; and $a_n = \Theta(b_n)$ means $c|b_n| \leq |a_n| \leq C|b_n|$ for some $c, C > 0$.

**Lemma 2.1** Let $f : [a, b] \to \mathbb{R}$ be $C^\infty$ on $(a, b)$ with

$$f(a + \varepsilon) = c_1 \varepsilon + \cdots + c_k \varepsilon^k + \cdots + c_N \varepsilon^N + o(\varepsilon^N)$$

for some $k, N$ with $1 \leq k < N$, and assume

$$\lim_{\varepsilon \to 0} f^{(j)}(a + \varepsilon) = j! c_j$$

for all $j$ such that $1 \leq j < k$. If $f^{(k)}(a + \varepsilon) \not\to k! c_k$ as $\varepsilon \to 0^+$, then there must exist positive numbers $u_n \downarrow 0$ such that

$$|f^{(k+1)}(u_n)| = o\left(u_n^{-\frac{N}{k}}\right).$$

**Proof** Step 1: Assume without loss of generality that $a = 0$. Also, replacing $f$ by $f - q$ where $q$ is the polynomial $q(x) := \sum_{j=1}^{N} c_j x^j$, we may assume without loss of generality that $c_j = 0$ for $j \leq N$.

Step 2: Fixing $f$ satisfying (2.2)–(2.3), we claim that

$$\liminf_{x \downarrow 0} f^{(k)}(x) \leq 0.$$  

(2.4)

To see this, assume to the contrary and choose $c, \delta > 0$ with $f^{(k)}(x) \geq c$ on $(0, \delta)$. Using $f^{(k-1)}(0) = 0$ and integrating gives

$$f^{(k-1)}(x) = \int_0^x f^{(k)}(t) \, dt \geq cx$$

for $0 < x < \delta$. Repeating this argument and using induction, we see that $f^{(k-j)}(x) \geq cx^j/j!$, whence $f(x) \geq cx^k/k!$ on $(0, \delta)$ contradicting $f(x) = o(x^N)$.

Step 3: An identical argument shows that $\limsup_{x \downarrow 0} f^{(k)}(x) \geq 0$. Therefore,

$$\liminf_{x \downarrow 0} f^{(k)}(x) \leq 0 \leq \limsup_{x \downarrow 0} f^{(k)}(x).$$

Assuming as well that $f^{(k)}(x) \not\to 0$ as $x \downarrow 0$, at least one of these inequalities must be strict. Without loss of generality, we assume for the remainder of the proof that it is the second one. Because $f^{(k)}$ is continuous, we may fix $C > 0$ and intervals with both endpoints tending to zero such that $f^{(k)}(x)$ is equal to $C$ at one endpoint, $C/2$ at the other endpoint, and is at least $C/2$ everywhere on the interval.

Step 4: Let $J = [c, d]$ denote one of these intervals. We next claim that we can find nested subintervals $J = J_0 \supseteq J_1 \supseteq \cdots \supseteq J_k$ with lengths $|J_j| = 4^{-j} |J_0|$ such that

$$|f^{(k-j)}| > \frac{C}{2 \cdot 4^{j(j+1)/2}} (d - c)^j$$

on $J_j$.

This is easily seen by induction on $j$. The base case $j = 0$ is already done. Assume for induction we have chosen $J_j$. The function $f^{(k-j)}$ does not change sign on $J_j$, hence $f^{(k-j-1)}$...
is monotone on $J_j$. If $f^{(k-j-1)}$ has a zero in the first half of $J_j$, we let $J_{j+1}$ be the last 1/4 of $J_j$ and let $I$ denote the second 1/2 of $J_j$. If $f^{(k-j-1)}$ has a zero in the second half of $J_j$, we let $J_{j+1}$ be the first 1/4 of $J_j$ and $I$ denote the first half of $J_j$. If $f^{(k-j-1)}$ does not vanish on $J_j$ we let $J_{j+1}$ be the first or last 1/4 of $J_j$, whichever contains the endpoint at which the absolute value of the monotone function $f^{(k-j-1)}$ is maximized over $J_j$, letting $I$ denote the half of $J_j$ containing $J_{j+1}$. Figure 1 shows an example of the case where $f^{(k-j-1)}$ has a zero in the second half of $J_j$.

In all of these cases, $f^{(k-j-1)}$ is monotone and does not change sign on $I$, and its minimum modulus on $J_{j+1}$ is at least $|f^{(k-j-1)}(I)|$ times the minimum modulus of $f^{(k)}$ on $J_j$, which is equal to $C/(2 \cdot 4^{(j+1)/2})$, establishing the claim by induction.

Step 5: Let $[a_n, b_n]$ be intervals tending to zero as in Step 3 and let $\xi_n$ denote the right endpoint of the $(k + 1)$th nested sub-interval $J_k$ of $[a_n, b_n]$ as in Step 4. Denoting $C' := C/(2 \cdot 4^{(k+1)/2})$, we have $|f(\xi_n)| \geq C'(b_n - a_n)^k$; together with the hypothesis that $f(x) = o(x^N)$, this implies $(b_n - a_n)^k = o(\xi_n^N)$ as $n \to \infty$. Seeing that $b_n - a_n = o(\xi_n)$ for some $\xi_n \in [a_n, b_n]$ is enough to conclude that $a_n \sim b_n \sim \xi_n$. By the Mean Value Theorem, because $f^{(k)}$ transits from $C/2$ to $C$ on $[a_n, b_n]$, there is some point $u_n \in [a_n, b_n]$ with

$$|f^{(k-1)}(u_n)| \geq \frac{C/2}{b_n - a_n}.$$

We have seen that $(b_n - a_n)^{-1} = o((\xi_n^N/k)$. Because $\xi_n \sim a_n \sim b_n \sim u_n$, this implies $|f^{(k-1)}(u_n)| = o(u_n^{-N/k})$, proving the lemma. \hfill \Box

### 2.2 Galton–Watson Trees

Since we will be working with probabilities on random trees, it will be useful to explicitly describe our probability space and notation. We begin with some notation we use for all trees, random or not. Let $U$ be the canonical Ulam-Harris tree [1]. The vertex set of $U$ is
the set \( V := \bigcup_{n=1}^{\infty} \mathbb{N}^n \), with the empty sequence \( \mathbf{0} = \emptyset \) as the root. There is an edge from any sequence \( a = (a_1, \ldots, a_n) \) to any extension \( a \uplus j = (a_1, \ldots, a_n, j) \). The depth of a vertex \( v \) is the graph distance between \( v \) and \( \mathbf{0} \) and is denoted \(|v|\). We work with trees \( T \) that are locally finite rooted subtrees of \( \mathcal{U} \). The usual notations are in force: \( T_n \) denotes the set of vertices at depth \( n \); \( T(v) \) is the subtree of \( T \) at \( v \), canonically identified with a rooted subtree of \( \mathcal{U} \), in other words the vertex set of \( T(v) \) is \( \{ w : v \sqcup w \in V(T) \} \) and the least common ancestor of \( v \) and \( w \) is denoted \( v \wedge w \).

Turning now to Galton–Watson trees, let \( \phi(z) := \sum_{n=1}^\infty p_n z^n \) be the offspring generating function for a supercritical branching process with no death, i.e., \( \phi(0) = 0 \). We recall,

\[
\phi'(1) = \mathbb{E} Z =: \mu \\
\phi''(1) = \mathbb{E}[Z(Z-1)]
\]

where \( Z \) is a random variable with probability generating function \( \phi \). We will work on the canonical probability space \( (\mathcal{O}, \mathcal{F}, \mathbb{P}) \) where \( \mathcal{O} = (\mathbb{N} \times [0,1])^V \) and \( \mathcal{F} \) is the product Borel \( \sigma \)-field. We take \( \mathbb{P} \) to be the probability measure making the coordinate functions \( \omega_v = (\deg_v, U_v) \) i.i.d. with the law of \( (Z, U) \), where \( U \) is uniform on \([0,1]\) and independent of \( Z \). The variables \( \{\deg_v\} \), where \( \deg_v \) is interpreted as the number of children of vertex \( v \), will construct the Galton–Watson tree, while the variables \( \{U_v\} \) will be used later for percolation. Let \( T \) be the random rooted subtree of \( \mathcal{U} \) which is the connected component containing the root of the set of vertices that are either the root or are of the form \( v \sqcup j \) such that \( 0 \leq j < \deg_v \). This is a Galton–Watson tree with offspring generating function \( \phi \). Let \( T := \sigma(\{\deg_v\}) \) denote the \( \sigma \)-field generated by the tree \( T \). The \( \mathbb{P} \)-law of \( T \) on \( T \) is \( \mathbb{GW} \).

As is usual for Galton–Watson branching processes, we denote \( Z_n := |T_n| \). Extend this by letting \( Z_n(v) \) denote the number of offspring of \( v \) in generation \(|v|+n \); similarly, extend the notation for the usual martingale \( W_n := \mu^{-n} Z_n \) by letting \( W_n(v) := \mu^{-n} Z_n(v) \). We know that \( W_n(v) \to W(v) \) for all \( v \), almost surely and in \( L^q \) if the offspring distribution has \( q \) moments. This is stated without proof for integer values of \( q \geq 2 \) in [10], p. 16] and [2, p. 33, Remark 3]; for a proof for all \( q > 1 \), see [3, Theorems 0 and 5]. Further extend this notation by letting \( v^{(i)} \) denote the \( i \)th child of \( v \), letting \( Z_n^{(i)}(v) \) denote \( n \)th generation descendants of \( v \) whose ancestral line passes through \( v^{(i)} \), and letting \( W_n^{(i)}(v) := \mu^{-n} Z_n^{(i)}(v) \). Thus, for every \( v \), \( W(v) = \sum_i W^{(i)}(v) \). For convenience, we define \( p_c := 1/\mu \), and recall that \( p_c \) is in fact \( \mathbb{GW} \)-a.s. the critical percolation parameter of \( T \) as per Theorem 1.1.

Bernoulli Percolation

Next, we give the formal construction of Bernoulli percolation on random trees. For \( 0 < p < 1 \), simultaneously define Bernoulli\((p)\) percolations on rooted subtrees \( T \) of \( \mathcal{U} \) by taking the percolation clusters to be the connected component containing \( \mathbf{0} \) of the induced subtrees of \( T \) on all vertices \( v \) such that \( U_v \leq p \). Let \( \mathcal{F}_v \) be the \( \sigma \)-field generated by the variables \( \{U_v, \deg_v : |v| < n\} \). Because percolation is often imagined to take place on the edges rather than vertices, we let \( U_e \) be a synonym for \( U_v \), where a is the farther of the two endpoints of \( v \) from the root. Write \( v \leftrightarrow_{T,p} w \) if \( U_e \leq p \) for all edges \( e \) on the geodesic from \( v \) to \( w \) in \( T \). Informally, \( v \leftrightarrow_{T,p} w \) iff \( v \) and \( w \) are both in \( T \) and are connected in the \( p \)-percolation. The event of successful \( p \)-percolation on a fixed tree \( T \) is denoted \( H_T(p) := \{ 0 \leftrightarrow_{T,p} \infty \} \). The event of successful \( p \)-percolation on the random tree \( T \), is denoted \( H_T(p) \) or simply \( H(p) \). Let \( g(T,p) := \mathbb{P}[H_T(p)] \) denote the probability of \( p \)-percolation on the fixed tree \( T \). Evaluating at \( T = T \) gives the random variable \( g(T,p) \) which is easily seen to equal the conditional expectation \( \mathbb{P}(H(p) \mid T) \). Taking unconditional expectations we see that \( g(p) = \mathbb{E} g(T,p) \).
2.3 Smoothness of the Annealed Survival Function $g$

By Lyons’ theorem, $g(p_c) = \mathbb{E}_g(T, p_c) = 0$. We now record some further properties of the annealed survival function $g$.

**Proposition 2.2** The derivative from the right $K := \partial_+ g(p_c)$ exists and is given by

\[
K = \frac{2}{p_c^2 \phi''(1)}.
\]  

(2.5)

where $1/\phi''(1)$ is interpreted as $\lim_{\xi \to 1} -1/\phi''(\xi)$.

**Proof** Let $\phi_p(z) := \phi(1 - p + pz)$ be the offspring generating function for the Galton–Watson tree thinned by $p$-percolation for $p \in (p_c, 1)$. The fixed point of $\phi_p$ is $1 - g(p)$. In other words, $g(p)$ is the unique $s \in (0, 1)$ for which $1 - \phi_p(1 - s) = s$, i.e. $1 - \phi(1 - ps) = s$.

By Taylor’s theorem with Mean-Value remainder, there exists a $\xi \in (1 - pg(p), 1)$ so that

\[
1 - \phi(1 - pg(p)) = pg(p)\phi'(1) - \frac{p^2 g(p)^2}{2}\phi''(\xi) = \frac{p}{p_c} g(p) - \frac{p^2 g(p)^2}{2}\phi''(\xi).
\]

Setting this equal to $g(p)$ and solving yields

\[
\frac{g(p)}{p - p_c} = \frac{2}{p_c^2 \phi''(\xi)}.
\]

Taking $p \downarrow p_c$ and noting $\xi \to 1$ completes the proof. 

**Corollary 2.3** (i) The function $g$ is analytic on $(p_c, 1)$. (ii) If $\sum_n p_n (1 + \delta)^n < \infty$ for some $\delta > 0$ then $g$ is analytic on $[p_c, 1)$, meaning that for some $\varepsilon > 0$ there is an analytic function $\tilde{g}$ on $(p_c - \varepsilon, 1)$ such that $g(p) = \tilde{g}(p) 1_{p > p_c}$.

**Proof** Recall that for $p \in (p_c, 1)$, $g(p)$ is the unique positive $s$ that satisfies $s = 1 - \phi(1 - ps)$. It follows that for all $p \in (p_c, 1)$, $g(p)$ is the unique $s$ satisfying

\[
F(p, s) := s + \phi(1 - ps) - 1 = 0.
\]

Also note that since $\phi(1 - ps)$ is analytic with respect to both variables for $(p, s) \in (p_c, 1) \times (0, 1)$, this means $F$ is as well.

We aim to use the implicit function theorem to show that we can parametrize $s$ as an analytic function of $p$ on $(p_c, 1)$; we thus must show $\frac{\partial F}{\partial s} \neq 0$ at all points $(p, g(p))$ for $p \in (p_c, 1)$. Direct calculation gives

\[
\frac{\partial F}{\partial s} = 1 - p\phi'(1 - ps).
\]

Because $\phi$ is strictly convex on $(p_c, 1)$, we see that $\frac{\partial F}{\partial s}$ is positive for $p \in (p_c, 1)$ at the fixed point. Therefore, $g(p)$ is analytic on $(p_c, 1)$.

To prove (ii), observe that $\phi$ extends analytically to $[0, 1 + \delta]$ by hypothesis, which implies that $1 - \phi(1 - ps)$ is analytic on a real neighborhood of zero. Also $1 - \phi(1 - ps)$ vanishes at $s = 0$, therefore $\psi(p, s) := (1 - \phi(1 - ps))/s$ is analytic near zero and for $(p, s) \in (p_c, 1) \times (0, 1)$, the least positive value of $s$ satisfying $\psi(p, s) = 1$ yields $g(p)$. Observe that

\[
\frac{\partial \psi}{\partial p}(p_c, 0) = \lim_{s \to 0} \frac{s\phi'(1 - p_c s)}{s} = \phi'(1) = \mu.
\]
By implicit differentiation,

\[ \partial_+ g(p_c) = -\frac{\partial \psi / \partial s}{\partial \psi / \partial p}(p_c, 0) \]

which is equal to $1/K$ by Proposition 2.2. In particular, $(\partial \psi / \partial s)(p_c, 0) = -\mu/K$ is non-vanishing. Therefore, by the analytic implicit function theorem, solving $\psi(p, s) = 1$ for $s$ defines an analytic function $\tilde{g}$ taking a neighborhood of $p_c$ to a neighborhood of zero, with $\tilde{g}(p) > 0$ if and only if $p > p_c$. We have seen that $\tilde{g}$ agrees with $g$ to the right of $p_c$, proving $(ii)$. \hfill \Box

In contrast to the above scenario in which $Z$ has exponential moments and $g$ is analytic at $p_c^+$, the function $g$ fails to be smooth at $p_c^+$ when $Z$ does not have all moments. The next two results quantify this: no $k$th moment implies $g \notin C^k$ from the right at $p_c$, and conversely, $EZ^k < \infty$ implies $g \in C^j$ from the right at $p_c$ for all $j < k/2$.

**Proposition 2.4** Assume $k \geq 2$, $E[Z^k] < \infty$, and $E[Z^{k+1}] = \infty$. Then $g^{(k+1)}(p)$ does not extend continuously to $p_c$ from the right.

**Proof** If $\lim_{p \to p_c^-} g^{(j)}(p)$ does not exist for any $j \leq k$, then we are done. Hence, we will assume all such limits exist. Next we take the $(k + 1)$th derivative of each side of the expression $1 - \phi(1 - pg(p)) = g(p)$ in order to get an equality of the form

\[
g^{(k+1)}(p) = \sum_{j=1}^{N} p^{b_j} g^{(b_j,j)}(p)^{\ell_{j,j}} \cdot \phi^{(d_j)}(1 - pg(p)) + (-1)^k \left[ g(p) + pg'(p) \right]^{k+1} \phi^{(k+1)}(1 - pg(p)) + \phi'(1 - pg(p))g^{(k+1)}(p) \tag{2.6} \]

where the $b_j, i$’s and $d_j$’s are all less than or equal to $k$. Denoting the sum in (2.6) as $S_{k+1}(p)$ and solving for $g^{(k+1)}(p)$, we now get

\[
g^{(k+1)}(p) = \frac{S_{k+1}(p) + (-1)^k \left[ g(p) + pg'(p) \right]^{k+1} \phi^{(k+1)}(1 - pg(p))}{1 - p\phi'(1 - pg(p))}. \]

Since we’re assuming that $g^{(j)}(p) = O(1)$ as $p \downarrow p_c$ for all $j \leq k$, and because $E[Z^j] < \infty \implies \phi^{(j)}(1 - pg(p)) = O(1)$ as $p \downarrow p_c$, it follows that $S_{k+1}(p) = O(1)$. Combining this with the fact that $E[Z^{k+1}] = \infty \implies \phi^{(k+1)}(1 - pg(p)) \to \infty$ as $p \downarrow p_c$, and that $g(p) + pg'(p) \to p_c K > 0$ as $p \downarrow p_c$, we now see that the numerator in the above expression for $g^{(k+1)}(p)$ must go to infinity as $p \downarrow p_c$, which means $g^{(k+1)}(p)$ must as well. \hfill \Box

In Sect. 2.4 we will prove the following partial converse.

**Proposition 2.5** For each $k \geq 1$, if $E[Z^{2k+1}] < \infty$, then $g \in C^k$ from the right at $p_c$.

Note that here we require $2k + 1$ moments rather than $k$ moments. This is because Proposition 2.6 shows that for each $j$, $j + 1$ moments suffice to obtain a $j$-order expansion of $g(p_c + \varepsilon)$ and we will use Lemma 2.1 for $N = 2k$. \hfill \star
2.4 Expansion of the Annealed Survival Function $g$ at $p_c^+$

A good part of the quenched analysis requires only the expansion of the annealed survival function $g$ at $p_c^+$, not continuous derivatives. Proposition 2.6 below shows that $k + 1$ moments are enough to give the order $k$ expansion. Moreover, we give explicit expressions for the coefficients. We require the following combinatorial construction: let $C_j(k)$ denote the set of compositions of $k$ into $j$ parts, i.e., ordered $j$-tuples of positive integers $(a_1, \ldots, a_j)$ with $a_1 + \cdots + a_j = k$; for a composition $a = (a_1, \ldots, a_j)$, define $\ell(a) = j$ to be the length of a, and $|a| = a_1 + \cdots + a_j$ to be the weight of a. Let $C(\leq k)$ denote the set of compositions with weight at most $k$.

**Proposition 2.6** Suppose $E[Z^{k+1}] < \infty$. Then there exist constants $r_1, \ldots, r_k$ such that $g(p_c + \varepsilon) = r_1 \varepsilon + \cdots + r_k \varepsilon^k + o(\varepsilon^k)$. Moreover, the $r_j$’s are defined recursively via

$$
r_1 = g'(p_c) = \frac{2}{p_c^2 \phi''(1)};
$$

$$
r_j = \frac{2}{p_c^2 \phi''(1)} \sum_{a \in C(\leq j), a \neq (j)} r_{a_1} \cdots r_{a_{\ell(a)}} \binom{\ell(a) + 1}{j - |a|} p_c^{|a|+\ell(a)+1-j} (-1)^{\ell(a)} \frac{\phi^{(\ell(a)+1)}(1)}{(\ell(a)+1)!}.
$$

(2.7)

**Proof** To start, we utilize the identity $1 - \phi(1 - pg(p)) = g(p)$ for $p = p_c + \varepsilon$, and take a Taylor expansion:

$$
\sum_{j=1}^{k+1} (p_c + \varepsilon)^j g(p_c + \varepsilon)^j (-1)^{j-1} \frac{\phi^{(j)}(1)}{j!} + o\left( ((p_c + \varepsilon)g(p_c + \varepsilon))^{k+1} \right) = g(p_c + \varepsilon).
$$

Divide both sides by $g(p_c + \varepsilon)$ and bound $g(p_c + \varepsilon) = O(\varepsilon)$ to get

$$
\sum_{j=1}^{k+1} (p_c + \varepsilon)^j g(p_c + \varepsilon)^j (-1)^{j-1} \frac{\phi^{(j)}(1)}{j!} - 1 = o(\varepsilon^k).
$$

(2.8)

Proceeding by induction, if we assume that the proposition holds for all $j < k$ for some $k \geq 2$, and we set

$$
p_k(\varepsilon) := \frac{g(p_c + \varepsilon) - \sum_{j=1}^{k-1} r_j \varepsilon^j}{\varepsilon^k},
$$

then (2.8) gives us

$$
o(\varepsilon^k) = \sum_{j=1}^{k+1} (p_c + \varepsilon)^j g(p_c + \varepsilon)^j (-1)^{j-1} \frac{\phi^{(j)}(1)}{j!} - 1
$$

$$
= \sum_{j=1}^{k+1} (p_c + \varepsilon)^j \left( \sum_{i=1}^{k-1} r_i \varepsilon^i + p_k(\varepsilon) \varepsilon^k \right)^{j-1} (-1)^{j-1} \frac{\phi^{(j)}(1)}{j!} - 1.
$$

(2.9)

Noting that the assumption that the proposition holds for $j = k - 1$ implies that $p_k(\varepsilon) = o(\varepsilon^{-1})$, we find that the expression on the right hand side in (2.9) is the sum of a polynomial in $\varepsilon$, the value $-\frac{p_c^2 \phi''(1)}{2} p_k(\varepsilon) \varepsilon^k$, and an error term which is $o(\varepsilon^k)$. This implies that all terms of this polynomial that are of degree less than $k$ must cancel, and that the sum of the term of

\[ Springer \]
order $k$ and $-p_c^2p''(1)\frac{p_k(\varepsilon)\varepsilon^k}{2}$ must be $o(\varepsilon^k)$. This leaves only terms of degree greater than $k$. It follows that $p_k(\varepsilon)$ must be equal to $C + o(1)$, for some constant $C$.

To complete the induction step, it remains to show that $C = r_k$. To do so we must find the coefficient of $\varepsilon^k$ in each term. We use the notation $[\varepsilon^j]$ to denote the coefficient of $\varepsilon^j$. For any $j$, we calculate

$$[\varepsilon^k] \left( p_c + \varepsilon \right)^j \left( \sum_{i=1}^{k-1} r_i \varepsilon^i \right) = \sum_{r=1}^{k} \left( \sum_{i=1}^{k-1} [\varepsilon^r] \left( \sum_{i=1}^{k-1} r_i \varepsilon^i \right)^{j-1} \right) \left( \varepsilon^{k-r} \right) (p_c + \varepsilon)^j \]

$$

$$= \sum_{r=1}^{k} \left( \sum_{a \in C_{j-1}(r)} r_{a_1} \cdots r_{a_{j-1}} \right) \left( \frac{j}{k-r} \right) p_c^{j+r-k}.$$

Putting this together with (2.9) we now obtain the desired equality $C = r_k$. Finally, noting that the base case $k = 1$ follows from Proposition 2.2, we see that the proposition now follows by induction.

**Proof of Proposition 2.5** Induct on $k$. For the base case $k = 1$, differentiate both sides of the expression $1 - \phi(1 - pg(p)) = g(p)$ and solve for $g'(p)$ to get

$$g'(p) = \frac{g(p)\phi'(1 - pg(p))}{1 - p\phi'(1 - pg(p))},$$

The numerator and denominator converge to zero as $p \downarrow p_c$. We would like to apply L'Hôpital's rule but we have to be careful because all we have *a priori* is the expansion at $p_c$, not continuous differentiability. We verify that the denominator satisfies

$$1 - p\phi'(1 - pg(p)) \sim \mu \cdot (p - p_c) \quad (2.10)$$

as $p = p_c + \varepsilon \downarrow p_c$ by writing the denominator as $1 - (p_c + \varepsilon)(\mu - \phi''(1)p_r c + \varepsilon) = \varepsilon(\mu - p_c^2 p_1 \phi''(1) + o(1))$, then plugging in $r_1 = 2/(p_c^2 \phi''(1))$ to obtain $\varepsilon(\mu - o(1))$. Similarly, the numerator is equal to $(r_1 \varepsilon + o(1))(\mu + o(1)) = (\mu r_1 + o(1))\varepsilon$. Dividing yields $g'(p_c + \varepsilon) = r_1 + o(\varepsilon)$, verifying the proposition when $k = 1$.

Next, inserting our assumption of $2k + 1$ moments into Proposition 2.6 gives the order-2k expansion

$$g(p_c + \varepsilon) = r_1 \varepsilon + \cdots + r_{2k} \varepsilon^{2k} + o(\varepsilon^{2k})$$

as $\varepsilon \downarrow 0$. We assume for induction that $g^{(j)}(p) \rightarrow j! r_j$ as $p \downarrow p_c$ for $1 \leq j < k$ and must prove it for $j = k$. We claim it is enough to show that $g^{(k+1)}(p_c + \varepsilon) = O(\varepsilon^{-2})$. To see why, assume that this holds but that $g^{(k)}(p)$ fails to converge to $k! r_k$ as $p \downarrow p_c$. Setting $N = 2k$, the conclusion of Lemma 2.1 would then yield a sequence of values $\varepsilon_n \downarrow 0$ with $g^{(k+1)}(p_c + \varepsilon_n)\varepsilon_n^{-2}$ tending to infinity, which would be a contradiction.

To show $g^{(k+1)}(p_c + \varepsilon) = O(\varepsilon^{-2})$ we begin by showing that $g^{(k)}(p_c + \varepsilon) = O(\varepsilon^{-1})$; both arguments are similar and the result for $g^{(k)}$ is needed for $g^{(k+1)}$. Again we begin with the identity $g(p) = 1 - \phi(1 - pg(p))$, this time differentiating $k$ times. The result is a sum of terms of the form $C p^a \phi^{(b)}(1 - pg(p)) \prod_{i=1}^{j} [g^{(j)}(p)]^c_i$. To keep track of the proliferation of these under successive differentiation, let the triple $(a, b, c)$ to denote such a term, where $c = (c_0, \ldots, c_k)$ and we ignore the value of the multiplicative constant $C$. For example, before differentiating at all, the right hand side is represented as $\langle 0, 0, \delta_0 \rangle$ where $\delta_i$ denotes the vector with $i$-component equal to 1 and the remaining components equal to 0. Differentiating replaces a term $\langle a, b, c \rangle$ with a sum of terms of four types, where $a, b, i$ and
the entries of the vector $c := (c_0, \ldots, c_k)$ are nonnegative integers and the last type can only occur when $c_i > 0$:

$$\langle a - 1, b, c \rangle, \langle a, b + 1, c + \delta_0 \rangle, \langle a + 1, b + 1, c + \delta_1 \rangle, \text{ and } \langle a, b, c + \delta_{i+1} - \delta_i \rangle.$$  

After differentiating $k$ times, the only term on the right-hand side for which $c_k \neq 0$ will be obtained from the term of the third type in the first differentiation, yielding $\langle 1, 1, \delta_1 \rangle$, followed by the term of the fourth type in each successive differentiation, yielding $\langle 1, 1, \delta_k \rangle$. In other words, the only summand on the right with a factor of $g_\phi$ follows by the term of the fourth type in each successive differentiation, yielding $\langle 1, 1, \delta_k \rangle$. The denominator is still $\Theta(1)$; we conclude this time that the numerator is $O(1)$, and by (2.10), the denominator is $\Theta(\varepsilon)$, proving that $g^{(k)}(p_c + \varepsilon) = O(\varepsilon^{-1})$.

Identical reasoning with $k + 1$ in place of $k$ shows that

$$g^{(k+1)}(p) = \frac{\sum\langle a, b, c \rangle}{1 - p\phi'(1 - pg(p))}$$

where the summands have $a \leq k + 1, b \leq k + 1, c_{k+1} = 0$ and $c_k = 0$ or 1. By (2.10) we know that $g^{(k)}(p_c + \varepsilon) = O(\varepsilon^{-1})$. We conclude this time that the numerator is $O(\varepsilon^{-2})$ and completing the induction.  

\[\Box\]

### 3 Proof of part (ii) of Theorem 1.2: Behavior at Criticality

This section is concerned with the expansion of $g(T, \cdot)$ at criticality. Section 3.1 defines the quantities that yield the expansion. Section 3.2 constructs some martingales and asymptotically identifies the expected number of $k$-subsets of $T_n$ that survive critical percolation as a polynomial of degree $k - 1$ whose leading term is a constant multiple of $W$ (a consequence of Theorem 3.5, below). Section 3.3 finishes computing the $\ell$-term Taylor expansion for $g(T, \cdot)$ at criticality.

#### 3.1 Explicit Expansion

Throughout the paper we use $\{r_j\}$ to denote the coefficients of the expansion of $g$ when they exist, given by the explicit formula (2.7). For $m \geq 1$, the $m$th power of $g$ has a $k$-order expansion at $p_c^+$ whenever $g$ does. Generalizing the notation for $r_j$, we denote the coefficients of the expansion of $g^m$ at $p_c^+$ by $\{r_{m, j}\}$ where

$$g(p_c + \varepsilon)^m = \sum_{j=1}^{\ell} r_{m, j} \varepsilon^j + o(\varepsilon^\ell) \quad (3.1)$$

for any $\ell$ for which such an expansion exists.
We prove part \((ii)\) of Theorem 1.2 by identifying the expansion. To do so, we need a notation for certain expectations. Fix a tree \(T\). For \(n \geq 0\), \(j \geq 1\) and \(v \in T\), define

\[
X_n^{(j)}(v) := \sum_{\{v_1, \ldots, v_j\} \in \binom{T_n}{j}} P_T[v \leftrightarrow p_c \ v_1, v_2, \ldots, v_j]
\]

where \(v \leftrightarrow p_c \ v_1, v_2, \ldots, v_j\) is the event that \(v\) is connected to each of \(v_1, \ldots, v_j\) under critical percolation and we use the notation \(\binom{T_n}{j}\) to denote the set of subsets of \(S\) with \(k\) elements. We omit the argument \(v\) when it is the root and so \(X_n^{(j)} := X_n^{(j)}(0)\). Note that

\[
X_n^{(1)} = W_n, \quad \text{and} \quad X_n^{(2)} = \sum_{\{u,v\} \in \binom{T_n}{2}} p_c^{2n-|u \wedge v|}.
\]

The former is the familiar martingale associated to a branching process, while the latter is related to the energy of the uniform measure on \(T_n\).

Extend this definition further: for integers \(j\) and \(k\), define

\[
X_n^{(j,k)} := \sum_{\{v\} \in \binom{T_n}{k}} \left(|T(v_1, \ldots, v_j)|\right) p_c^{|T(v_1, \ldots, v_j)|}
\]

where \(T(v_1, \ldots, v_j)\) is the smallest rooted subtree of \(T\) containing each \(v_i\) and \(|T(v_1, \ldots, v_j)|\) refers to the number of edges this subtree contains. Note that \(X_n^{(j,0)} = X_n^{(j)}\).

Part \((ii)\) of Theorem 1.2 follows immediately from the following expansion, which is the main work of this section.

**Theorem 3.1** Define

\[
M_n^{(i)} := M_n^{(i)}(T) := \mu^i \sum_{j=1}^i (-1)^{j+1} \sum_{d=j}^{i} p_c^d r_{j,d} X_n^{(j,i-d)}. \tag{3.2}
\]

Suppose that \(E\left[Z^{(2\ell+1)(1+\beta)}\right] < \infty\) for some integer \(\ell \geq 1\) and real \(\beta > 0\). Then for all \(i \leq \ell\):

(i) The quantities \(\{M_n^{(i)} : n \geq 1\}\) are a \(\{T_n\}\)-martingale with mean \(r_i\).
(ii) For \(\mathbb{Q}\)-almost every tree \(T\) the limits \(M^{(i)} := \lim_{n \to \infty} M_n^{(i)}\) exist.
(iii) These limits are the coefficients in the expansion

\[
g(T, p_c + \varepsilon) = \sum_{i=1}^{\ell} M^{(i)} \varepsilon^i + o(\varepsilon^\ell). \tag{3.3}
\]

**Remark** The quantities \(X_n^{(j,i)}\) do not themselves have limits as \(n \to \infty\). In fact for fixed \(i\) and \(j\) the sum over \(d\) of \(X_n^{(j,i-d)}\) is of order \(n^{i-1}\). Therefore it is important to take the alternating outer sum before taking the limit.

### 3.2 Critical Survival of \(k\)-Sets

To prove Theorem 3.1 we need to work with centered variables. Centering at the unconditional expectation is not good enough because these mean zero differences are close to the nondegenerate random variable \(n^{i-1} W\) and therefore not summable. Instead we subtract off a quantity that can be handled combinatorially, leaving a convergent martingale.
Throughout the rest of the paper, the notation $\Delta$ in front of a random variable with a subscript (and possibly superscripts as well) denotes the backward difference in the subscripted variable. Thus, for example,

$$\Delta X_n^{(j,i)} := X_n^{(j,i)} - X_{n-1}^{(j,i)}.$$ 

Let $X_n^{(j,i)} = Y_n^{(j,i)} + A_n^{(j,i)}$ denote the Doob decomposition of the process $\{X_n^{(j,i)} : n = 1, 2, 3, \ldots\}$ on the filtration $\{T_n\}$. To recall what this means, ignoring superscripts for a moment, the $Y$ and $A$ processes are uniquely determined by requiring the $Y$ process to be a martingale and the $A$ process to be predictable, meaning that $A_n \in T_{n-1}$ and $A_0 = 0$. The decomposition can be constructed inductively in $n$ by letting $A_0 = 0$, $Y_0 = \mathbb{E}X_0$, and defining

$$\Delta A_n := \mathbb{E} (\Delta X_n | T_{n-1}) ;$$

$$\Delta Y_n := \Delta X_n - \Delta A_n.$$ 

We begin by identifying the predictable part.

**Lemma 3.2** Let $C_i(j)$ denote the set of compositions of $j$ of length $i$ into strictly positive parts. Let $m_r := \mathbb{E}(Z_r)$ and define constants $c_{j,i}$ by

$$c_{j,i} := \rho^j \sum_{\alpha \in C_i(j)} m_{\alpha_1} m_{\alpha_2} \cdots m_{\alpha_i}.$$ 

Then for each $k \geq 0$,

$$\Delta A_{n+1}^{(j,k)} = -X_n^{(j,k)} + \sum_{i=1}^{j} \sum_{d=0}^{k} \binom{j}{k-d} X_n^{(i,d)}$$

$$= \sum_{d=0}^{k-1} \binom{j}{k-d} X_n^{(j,d)} + \sum_{i=1}^{j} \sum_{d=0}^{k} c_{j,i} \binom{j}{k-d} X_n^{(i,d)}.$$ 

(3.4)

**Proof** For distinct vertices $v_1, \ldots, v_j$ in $T_{n+1}$, their set of parents $u_1, \ldots, u_\ell$ form a subset of $T_n$ with at most $j$ elements. In order to sum over all $j$-sets of $T_{n+1}$, one first sums over all sets of parents. For a fixed parent set $u_1, \ldots, u_\ell$ in $T_{n-1}$, the total number of $j$-sets with parent set $\{u_1, \ldots, u_\ell\}$ is

$$\sum_{\alpha \in C_i(j)} \binom{Z_{\alpha_1}(u_1)}{\alpha_1} \cdots \binom{Z_{\alpha_\ell}(u_\ell)}{\alpha_\ell}.$$ 

Furthermore, we have

$$\binom{|T(v_1, \ldots, v_j)|}{k} = \binom{|T(u_1, \ldots, u_\ell)| + j}{k} = \sum_{d=0}^{k} \binom{j}{k-d} \binom{|T(u_1, \ldots, u_\ell)|}{d}.$$
This gives the expansion
\[
X^{(j,k)}_{n+1} = \sum_{\{v_i\} \in (T_{n+1})} \left( |T(v_1, \ldots, v_j)| \right)^k p_c^{T(v_1, \ldots, v_j)}
\]
\[
= \sum_{\ell=1}^j \sum_{\{u_i\} \in (T_{\ell})} \sum_{d=0}^{k-1} \left( \binom{j}{k-d} \right) p_c^{T(u_1, \ldots, u_\ell)}
\]
\[
\sum_{\alpha \in C_{t(j)}} p_c^j \left( Z_1(u_1) \right) \cdots \left( Z_1(u_\ell) \right).
\]

Taking conditional expectations with respect to \(T_n\) completes the proof of the first identity, with the second following from rearrangement of terms.

The following corollary is immediate from Lemma 3.2 and the fact that \(X^{(j,k)}_0 = Y^{(j,k)}_0\).

**Corollary 3.3** For each \(j\) so that \(E[Z^j] < \infty\) and each \(k\), the terms of the \(Y\) martingale are given by
\[
Y^{(j,k)}_n = Y^{(j,k)}_0 + \sum_{m=1}^n \Delta Y^{(j,k)}_m
\]
\[
= X^{(j,k)}_n - \sum_{m=0}^{n-1} \left[ \sum_{d=0}^{k-1} \binom{j}{k-d} X^{(j,d)}_m + \sum_{i=1}^{j-1} c_{j,i} \sum_{d=0}^k \binom{j}{k-d} X^{(i,d)}_m \right].
\]

We want to show that these martingales converge both almost surely and in some appropriate \(L^p\) space; this will require us to take \(L^{1+\beta}\) norms for some \(\beta \in (0, 1]\). The following randomized version of the Marcinkiewicz–Zygmund inequality will be useful.

**Lemma 3.4** Let \(\{\xi_k\}_{k=1}^\infty\) be i.i.d. with \(E[\xi_1] = 0\) and \(E[|\xi_1|^{1+\beta}] < \infty\) for some \(\beta \in (0, 1]\), and let \(N\) be a random variable in \(\mathbb{N}\) independent from all \(\{\xi_k\}\) and with \(E[N] < \infty\). If we set \(S_n = \sum_{k=1}^n \xi_k\), then there exists a constant \(c > 0\) depending only on \(\beta\) so that
\[
E[|S_N|^{1+\beta}] \leq c E[|\xi_1|^{1+\beta}] E[N].
\]

In particular, if \(\xi(v)\) are associated to vertices \(v \in T_s\), and are mutually independent from \(T_s\), then
\[
\left\| p_c^s \sum_{v \in T_s} \xi(v) \right\|_{L^{1+\beta}} \leq c' p_c^s \left\| \xi(v) \right\|_{L^{1+\beta}}.
\]

**Proof** Suppose first that \(N\) is identically equal to a constant \(n\). The Marcinkiewicz–Zygmund inequality (e.g. [4, Theorem 10.3.2]) implies that there exists a constant \(c > 0\) depending only on \(\beta\) such that
\[
E[|S_n|^{1+\beta}] \leq c \left( \sum_{k=1}^n |\xi_k|^2 \right)^{(1+\beta)/2}.
\]

\(\square\)
Because $1 + \beta \leq 2$ and the $\ell^p$ norms descend, we have $\| (\xi_k)_{k=1}^n \|_{\ell^2} \leq \| (\xi_k)_{k=1}^n \|_{\ell^{1+\beta}}$ deterministically; this completes the proof when $N$ is constant. Writing $\mathbb{E}[|S_N|] = E[\mathbb{E}[|S_N|^{1+\beta}|N]]$ and applying the bound from the constant case completes the proof.

We now show that the martingales $\{Y_n^{(j,k)} : n \geq 0\}$ converge.

**Theorem 3.5** Suppose $\mathbb{E}[Z^{j(1+\beta)}] < \infty$ for some $\beta > 0$. Then

(a) $\| \Delta Y_{n+1}^{(j,k)} \|_{L^{1+\beta}} \leq C e^{-cn}$ where $C$ and $c$ are positive constants depending on $j, k, \beta$ and the offspring distribution.

(b) $Y_n^{(j,k)}$ converges almost surely and in $L^{1+\beta}$ to a limit, which we denote $Y^{(j,k)}$.

(c) There exists a positive constant $c_{j,k}$ depending only on $j, k$ and the offspring distribution so that $X_n^{(j,k)} n^{-(j+k-1)} \to c_{j,k} W$ almost surely and in $L^{1+\beta}$.

**Proof** Step 1: (a) $\implies$ (b). For any fixed $j$ and $k$: the triangle inequality and (a) show that $\sup_n \| Y_n^{(j,k)} \|_{L^{1+\beta}} < \infty$, from which (b) follows from the $L^p$ martingale convergence theorem. Next, we prove an identity representing $X_n^{(j,k)}$ as a multiple sum over values of $X^{(j',k')}$ with $(j', k') < (j, k)$ lexicographically.

Step 2: Some computation. For a set of vertices $\{v_1, \ldots, v_j\}$, let $v = v_1 \wedge v_2 \wedge \cdots \wedge v_j$ denote their most recent common ancestor. In order for $0 \leftrightarrow_{p_c} v_1, \ldots, v_j$ to hold, we must first have $0 \leftrightarrow_{p_c} v$. For the case of $j \geq 2$, looking at the smallest tree containing $v$ and $\{v_i\}$, we must have that this tree branches into some number of children $a \in [2, j]$ immediately after $v$.

We may thus sum over all possible $v$, first by height, setting $s = |v|$, then choosing how many children of $v$ will be the ancestors of the $v_1, \ldots, v_j$. We then choose those children $\{u_i\}$, and choose how to distribute the $\{u_i\}$ among them. In order for critical percolation to reach each $v_1, \ldots, v_j$, it must first reach $v$, then survive to each child of $v$ that is an ancestor of some $\{v_i\}$ and then survive to the $\{u_i\}$ from there. Finally, in order to choose the $k$-element subset corresponding to $[\{T_1, \ldots, T_k\}]$, we may choose $\alpha_0$ elements from the tree $T_1, \ldots, n_a$ and $\alpha_1$ elements from each subtree of $u_1$. Putting this all together, we have the decomposition

$$X_n^{(j,k)} = \sum_{s=0}^{n-1} p_c^s \sum_{v \in T_s} \sum_{a=2}^j \sum_{u \in (T_{a-1}^{(v)})} p_c^d \sum_{\beta \in C_a(j)} \sum_{\alpha_0=0}^k \sum_{\alpha_1=0}^{k-a_0} \left( \sum_{\alpha_1}^{s+a} \right) X_{n-s-1}^{(\beta,\alpha_1)}(u_1) \cdots X_{n-s-1}^{(\beta,\alpha_0)}(u_n)$$

$$= \sum_{s=0}^{n-1} p_c^s \sum_{v \in T_s} \Theta_{n-s-1}^{(j,k)}(v)$$

where $\Theta_{n-s-1}^{(j,k)}(v)$ is defined as the inner quintuple sum in the previous line and $\tilde{C}_a(k)$ denotes the set of weak $\alpha$-compositions of $k$; observe that the notation $\Theta_{n-s-1}^{(j,k)}(v)$ hides the dependence on $s = |v|$.

The difference $\Delta Y_n^{(j,k)}$ can now be computed as follows:

$$\Delta Y_n^{(j,k)} = X_n^{(j,k)} - \sum_{i=1}^j \sum_{d=0}^{k} \binom{j}{k-d} c_{j,i} X_{n-1}^{(j,d)}$$

$$= \sum_{s=0}^{n-1} p_c^s \sum_{v \in T_s} \Theta_{n-s-1}^{(j,k)}(v) - \sum_{s=0}^{n-2} p_c^s \sum_{v \in T_s} \sum_{i=1}^j \sum_{d=0}^{k} \binom{j}{k-d} c_{j,i} \Theta_{n-s-2}^{(j,d)}(v)$$
Additionally, \( Y \) hold for all \( X \) that \( U \) the average of \( \Theta \), where \( U \) the sum of \( 1 \) for \( j \) by induction. In the second stage, we prove \( T \) in two stages (Steps 5 and 6). In the first stage, we fix \( j \) arbitrary. In the second stage, we prove \( U \), thereby also showing \( \Theta \) for \( j = 1 \) and all \( k \).

Step 4: \( V \) is always small. Using the identity \((n+1)(n-1) = \sum_{d=0}^k (n-1)k \) and recalling that \( x_{n-1}^{(1,d)} = \left(\frac{n-1}{d}\right) \) \( W_{n-1} \) shows that

\[
V_n^{(j,k)} = \sum_{d=0}^k \left( \frac{j}{k-d} \right) \left( \frac{n-1}{d} \right) p_{c}^{n-1} \sum_{v \in T_n} p_c \left[ \left( \frac{Z_1(v)}{j} \right) - E \left( \frac{Z_1(v)}{j} \right) \right].
\]

Applying Lemma 3.4 shows that the innermost sum, when multiplied by \( p_{c}^{n-1} \), has \( L^{1+\beta} \) norm that is exponentially small in \( n \). With \( k \) fixed and \( d \leq k \), the product with \( \left(\frac{n-1}{d}\right) \) still yields an exponentially small variable, thus

\[
||V_n^{(j,k)}||_{1+\beta} \leq c_{j,k,\beta} e^{-\delta n}
\]

for some \( \delta = \delta(j, k, \beta) > 0 \).

The remainder of the proof is an induction in two stages (Steps 5 and 6). In the first stage we fix \( j > 1 \), assume \( (a)\)–(c) for all \( (j', k') \) with \( j' < j \), and prove \( (a) \) for \( (j, k) \) with \( k \) arbitrary. In the second stage, we prove \( (c) \) for \( (j, k) \) by induction on \( k \), establishing \( (c) \) for \( (j, 1) \) and then for arbitrary \( k \) by induction, assuming \( (a) \) for \( (j, k') \) where \( k' \) is arbitrary and \( (c) \) for \( (j, k') \) where \( k' < k \).

Step 5: Prove \( (a) \) by induction on \( j \). Fix \( j \geq 2 \) and assume for induction that \( (a) \) and \( (c) \) hold for all \( (j', k) \) with \( j' < j \). The plan is this: The quantity \( p_{c}^{n-1} \sum_{v \in T_n} U_n^{(j,k)}(v) \) is \( W_n \) times the average of \( U_n^{(j,k)}(v) \) over vertices \( v \in T_n \). Averaging many mean zero terms will produce
something exponentially small in \( s \). We will also show this quantity to be also exponentially small in \( n - s \), whereby it follows that the outer sum over \( s \) is exponentially small, completing the proof.

Let us first see that \( U^{(j,k)}_n(v) \) has mean zero. Expanding back the \( \Theta \) terms gives

\[
U^{(j,k)}_n(v) = \sum_{a=2}^j \sum_{u \in \mathcal{C}_a(v)} p_c^a \sum_{\alpha_0=0}^k \left( s + a \right) \left( \sum_{\beta \in \mathcal{C}_a(j)} \sum_{\alpha \in \mathcal{C}_a(k-\alpha_0)} X_{n-s-1}^\beta(u_1) \cdots X_{n-s-1}^{(\alpha_a)}(u_a) \right)
- \sum_{i=2}^j \sum_{d=0}^k c_{j,i} \binom{j}{k-d} \sum_{\beta' \in \mathcal{C}_a(i)} \sum_{\alpha' \in \mathcal{C}_a(d-\alpha_0)} X_{n-s-2}^{(\beta_1',\alpha_1')}(u_1) \cdots X_{n-s-2}^{(\beta_a',\alpha_a')}(u_a)
\]

(3.10)

Expanding the first product of \( X \) terms gives

\[
X_{n-s-1}^\beta(u_1) \cdots X_{n-s-1}^{(\alpha_a)}(u_a)
= \prod_{\ell=1}^a \left( \Delta Y_{n-s-1}^{(\alpha_\ell-\alpha'_\ell)}(u_\ell) + \sum_{\beta_\ell'=1}^{\beta_\ell} c_{\beta_\ell,\beta_\ell'} \sum_{\alpha_\ell'=0}^{\alpha_\ell} \left( \beta_\ell \right)_{\alpha_\ell-\alpha'_\ell} X_{n-s-2}^{(\alpha_\ell',\alpha_\ell')}(u_\ell) \right)
\]

(3.11)

The vertices \( u_\ell \) are all distinct children of \( v \). Therefore, their subtrees are jointly independent, hence the pairs \( (\Delta Y_{n-s-1}(u_\ell), X_{n-s-1}(u_\ell)) \) are jointly independent. The product (3.11) expands to the sum of \( a \)-fold products of terms, each term in each product being either a \( \Delta Y \) or a weighted sum of \( X \)'s, the \( \alpha \) terms being jointly independent by the previous observation. Therefore, to see that the whole thing is mean zero, we need to check that the product of the \( a \) different sums of \( X \) terms in (3.11), summed over \( \alpha \) and \( \beta \) to form the first half of the summand in (3.10), minus the subsequent sum over \( i, d, \beta' \) and \( \alpha' \), has mean zero. In fact we will show that it vanishes entirely. For given compositions \( \beta := (\beta_1, \ldots, \beta_a) \) and \( \alpha := (\alpha_1, \ldots, \alpha_a) \), the product of the double sum of \( X \) terms inside the round brackets in (3.11) may be simplified:

\[
\prod_{\ell=1}^a \left( \sum_{\beta_\ell'=1}^{\beta_\ell} c_{\beta_\ell,\beta_\ell'} \sum_{\alpha_\ell'=0}^{\alpha_\ell} \left( \beta_\ell \right)_{\alpha_\ell-\alpha'_\ell} X_{n-s-2}^{(\alpha_\ell',\alpha_\ell')}(u_\ell) \right)
= \sum_{1 \leq \beta' \leq \beta} \sum_{0 \leq \alpha' \leq \alpha} \prod_{\ell=1}^a c_{\beta_\ell,\beta'_\ell} \left( \beta_\ell \right)_{\alpha_\ell-\alpha'_\ell} X_{n-s-2}^{(\alpha_\ell',\alpha_\ell'}(u_\ell).
\]

Applying the identity

\[
\sum_{\beta \in \mathcal{C}_a(j)} \prod_{\ell} c_{\beta_\ell,\beta'_\ell} = c_{j,i}, \quad (3.12)
\]

which follows by regrouping pieces of each composition in \( \mathcal{C}_a(j) \) into smaller compositions each with \( \beta'_\ell \) parts, then summing over \( \alpha \) and \( \beta \) as in (3.10) and simplifying, using (3.12) in the last line, gives
\[ \sum_{\beta \in C(u)} \sum_{1 \leq \beta' \leq \beta} \sum_{a \in C(u') \setminus (k-1)} \sum_{0 \leq \alpha' \leq \alpha} \prod_{t=1}^a c_{\beta_t, \beta'_t} \left( \beta_t \prod_{t=1}^a \alpha_t - \alpha'_t \right) X_{n-s-2}^{(\beta_t, \alpha_t)}(u_t) \]

\[ = \sum_{\beta \in C(u)} \sum_{1 \leq \beta' \leq \beta} \left( \prod_{t=1}^a c_{\beta_t, \beta'_t} \right) \sum_{d=0}^k \sum_{a' \in C(u) \setminus (d-\lambda_0)} \left( \prod_{t=1}^a X_{n-s-2}^{(\beta'_t, \alpha'_t)}(u_t) \right) \left( j \prod_{k-d} \right) \]

This exactly cancels with the quadruple sum on the second line of (3.10), transforming (3.10) into

\[ U_n^{(j,k)}(v) = \sum_{a=2}^j \sum_{a' \in \{T_1(v) - a\}} \sum_{\alpha_0=0}^a \sum_{\beta \in C(u)} \sum_{a' \in C(u) \setminus (d-\lambda_0)} \prod_{t=1}^a(*) \ell, \]

where \((*) = \Delta Y_{n-s-1}^{(\beta_t, \alpha_t)}(u_t)\) for at least one value of \( \ell \) in \([1, a]\), and, when not equal to that, is equal to the last double sum inside the brackets in (3.11).

By the induction hypothesis, the \(\Delta Y\) terms have \((1 + \beta)\) norm bounded above by something exponentially small:

\[ \| \Delta Y_{n-s-1}^{(\beta_t, \alpha_t)}(u_t) \| = O \left( \exp \left[ -\kappa_{\beta_t, \alpha_t}(n - s - 1) \right] \right). \]  

(3.13)

We note that \( a \) and each \( \beta_t \) and \( \alpha_t \) are all bounded above by \( j \) and that in each product

\[ X_{n-s-1}^{(\beta_1, \alpha_1)}(u_1) \cdots X_{n-s-1}^{(\beta_a, \alpha_a)}(u_a), \]

the terms are independent. The inductive hypothesis implies each factor \( X_n^{(j,k)} \) has \( L^{1+\beta} \) norm that is \( O(n^{\lambda(j,k)}) \).

Returning to (3.6), we may apply Lemma 3.4 to see that for each \( s \), the quantity \( p \sum_{v \in T} U_n^{(j,k)}(v) \) is an average of \( |T| \) terms all having mean zero and \( L^{1+\beta} \) bound exponentially small in \( n - s \), and that averaging introduces another exponentially small factor, \( \exp(-vs) \). Because the constants \( \kappa, \lambda \) and \( \mu \) vary over a set of bounded cardinality, the product of these three upper bounds, \( O \left( \exp(-\kappa(n - s)) \cdot \exp(-vs) \cdot n^{\lambda(j,k)} \right) \) decreases exponentially in \( n \).

Step 6: Prove (c) by induction on \((j,k)\). The final stage of the induction is to assume \((a)(c)(j', k')\) lexicographically smaller than \((j, k)\) and prove (c) for \((j, k)\). We use the following easy fact.
Lemma 3.6 If \( a_n \to \infty \) and \( a_n \sim b_n \) then the partials sums are also asymptotically equivalent: \( \sum_{k=1}^n a_k \sim \sum_{k=1}^n b_k \).

We begin the inductive proof of with the case \( k = 0 \). Rearranging the conclusion of Corollary 3.3, we see that

\[
X_n^{(j,0)} = Y_n^{(j,0)} + \sum_{m=0}^{n-1} \sum_{i=1}^{j-1} X_m^{(i,0)}.
\]

Using Lemma 3.6 the induction hypothesis, and the fact that \( Y_n^{(j,0)} = O(1) \) simplifies this to

\[
X_n^{(j,0)} \sim \sum_{m=0}^{n-1} \left[ \sum_{i=1}^{j-1} m^{i-1} c_i' W \right] \\
\sim \sum_{m=0}^{n-1} c_{j-1} m^{j-2} W \\
\sim c_j' n^{j-1} W
\]

where \( c_j' = \lim_{n \to \infty} c_{j-1}' \sum_{m=0}^{n-1} (m/n)^{j-2} = c_{j-1}' / (j - 1) \).

The base case \( k = 0 \) being complete, we induct on \( k \). The same reasoning, observing that the first inner sum is dominated by the \( d = k - 1 \) term and the second by the \( i = j - 1 \) and \( d = k \) term, gives

\[
X_n^{(j,k)} = Y_n^{(j,k)} + \sum_{m=0}^{n-1} \sum_{d=0}^{k-1} \left( \frac{j}{k-d} \right) X_m^{(j,d)} + \sum_{i=1}^{j-1} c_{j,i} \sum_{d=0}^{k-1} \left( \frac{j}{k-d} \right) X_m^{(i,d)} \\
\sim \sum_{m=0}^{n-1} j X_m^{(j,k-1)} + c_{j-1} X_m^{(j-1,k)} \\
\sim \sum_{m=0}^{n-1} \left[ j m^{j+k-2} W c_{j,k-1} + c_{j-1} c_{j-1,k} W m^{j+k-2} \right] \\
\sim W \left( \frac{j c_{j,k-1} + c_{j-1} c_{j-1,k}}{j+k-1} \right) n^{j+k-1}.
\]

Setting \( c_{j,k} := \frac{j c_{j,k-1} + c_{j-1} c_{j-1,k}}{j+k-1} \) completes the almost-sure part of (c) by induction.

The \( L^{1+\beta} \) portion is similar, but we need one more easy fact.

Lemma 3.7 If \( a_n \to \infty \) and \( b_n \to 0 \) then \( \sum_{k=1}^n a_n b_n = o \left( \sum_{k=1}^n a_k \right) \).

This allows us to calculate

\[
\left\| X_n^{(j,k)} - W c_{j,k} \right\|_{L^{1+\beta}} = \left\| Y_n^{(j,k)} + \sum_{m=0}^{n-1} \sum_{d=0}^{k-1} \left( \frac{j}{k-d} \right) X_m^{(j,d)} + \sum_{i=1}^{j-1} c_{j,i} \sum_{d=0}^{k-1} \left( \frac{j}{k-d} \right) X_m^{(i,d)} \right\|_{L^{1+\beta}} - \left| W c_{j,k} \right|_{L^{1+\beta}} \\
\leq o(1) + \sum_{m=0}^{n-1} \left| j X_m^{(j,k-1)} + c_{j-1} X_m^{(j-1,k)} \right| - n^{j+k-1} W c_{j,k} \right\|_{L^{1+\beta}}
\]

\( \square \) Springer
\[ \leq o(1) + n^{-j+k-1} \sum_{m=0}^{n-1} m^{j+k-2} \left( j \left\| \frac{X_{m}}{m^{j+k-2}} - Wc'_{j,k-1} \right\|_{L^{1+\beta}} + c_{j,j-1} \left\| \frac{X_{m}}{m^{j+k-2}} - Wc'_{j-1,k} \right\|_{L^{1+\beta}} \right) = o(1). \]

This completes the induction, and the proof of Theorem 3.5. \qed

### 3.3 Expansion at Criticality

An easy inequality similar to classical Harris inequality \cite{9} is as follows.

**Lemma 3.8** For finite sets of edges \( E_1, E_2, E_3 \), define \( A_j \) to be the event that all edges in \( E_j \) are open. Then

\[ \mathbb{P}[A_1 \cap A_2] \cdot \mathbb{P}[A_1 \cap A_3] \leq \mathbb{P}[A_1] \cdot \mathbb{P}[A_1 \cap A_2 \cap A_3]. \]

**Proof** Writing each term explicitly, this is equivalent to the inequality

\[ p^{\left| E_1 \cup E_2 \right| + \left| E_1 \cup E_3 \right|} \leq p^{\left| E_1 \right| + \left| E_1 \cup E_2 \cup E_3 \right|}. \]

Because \( p \leq 1 \), this is equivalent to

\[ \left| E_1 \cup E_2 \right| + \left| E_1 \cup E_3 \right| \geq \left| E_1 \right| + \left| E_1 \cup E_2 \cup E_3 \right|, \]

which is easily proved for all triples \( E_1, E_2, E_3 \) by inclusion-exclusion. \qed

Before finding the expansion at criticality, we show that focusing only on the first \( n \) levels of the tree and averaging over the remaining levels causes only a subpolynomial error in an appropriate sense.

**Proposition 3.9** Suppose \( E[Z^{(2k-1)(1+\beta)}] < \infty \), and set \( p = p_c + \varepsilon \). Fix \( \delta > 0 \) and let \( n = n(\varepsilon) = \lceil \varepsilon^{-\delta} \rceil \). Then for \( \delta \) sufficiently small and each \( \ell > 0 \),

\[ \sum_{\{u_i\} \in (T^n_p)} \mathbb{P}_{T}[0 \leftrightarrow_p u_1, \ldots, u_k] \left( g(T(u_1), p) \cdots g(T(u_k), q) - g(p)^k \right) = o(\varepsilon^\ell) \quad (3.14) \]

GW-almost surely as \( \varepsilon \to 0^+ \).

**Proof** For sufficiently small \( \delta > 0 \), we note that \((p_c + \varepsilon)^n \leq 2p_c^n\) for each \( m \in [n, kn] \) and for \( \varepsilon \) sufficiently small. This will be of use throughout, and is responsible for the appearance of factors of 2 in the upper bounds.

Next, bound the variance of

\[ \sum_{\{u_i\} \in (T^n_p)} \mathbb{P}_{T}[0 \leftrightarrow_p u_1, \ldots, u_k] \left[ g(T(u_1), q) \cdots g(T(u_k), q) - g(q)^k \right] \]

for a fixed vertex, \( q \). This expression has mean zero conditioned on \( T_n \). Its variance is equal to the expected value of its conditional variance given \( T_n \). We therefore square and take the expectation, where the second sum in the second and third lines are over pairs of disjoint \( k \)-tuples of points.

\[ \begin{align*}
\end{align*} \]
E \left[ \left( \sum_{\{u_i\} \in \binom{T_n}{k}} \mathbf{P}_T[0 \leftrightarrow_p u_1, \ldots, u_k] \left( g(T(u_1), q) \cdots g(T(u_k), q) - g(q)^k \right) \right)^2 \right]_{T_n} \\
= \frac{1}{(k!)^2} \sum_{r=1}^{k} r! \sum_{\{u_i\} \in \binom{T_n}{k}} \left( \frac{k}{r} \right)^2 \mathbf{P}_T[0 \leftrightarrow_p u_1, \ldots, u_k] \mathbf{P}_T[0 \leftrightarrow_p u_1, \ldots, u_r, v_{r+1}, \ldots, v_k] C_r \\
\leq \frac{1}{(k!)^2} \sum_{r=1}^{k} r! \sum_{\{u_i\} \in \binom{T_n}{k}} \left( \frac{k}{r} \right)^2 \mathbf{P}_T[0 \leftrightarrow_p u_1, \ldots, u_r] \mathbf{P}_T[0 \leftrightarrow_p u_1, \ldots, u_k] C_r \\
\leq 4p^n \frac{k}{r} \binom{k}{r} C_r \chi_n^{2k-r}.

Here we have used the bounds \( \mathbf{P}_T[0 \leftrightarrow_p u_1, \ldots, u_r] \leq 2p^n \) and \( \mathbf{P}_T[0 \leftrightarrow_p u_1, \ldots, v_k] \leq 2p^n \mathbf{P}_T[0 \leftrightarrow_p u_1, \ldots, v_k] \) and we have defined

\[
C_r := \mathbb{E} \left[ \left( g(T(u_1), q) \cdots g(T(u_k), q) - g(q)^k \right) \right]_{T_n} \times \left( g(T(u_1), q) \cdots g(T(u_r), q) g(T(v_{r+1}), q) \cdots g(T(v_k), q) - g(q)^k \right).
\]

Taking the expected value and using Theorem 3.5 along with Jensen’s Inequality and induction gives that the variance is bounded above by \( C p^n n^{2k-2} \) for some constant \( C \). This is exponentially small in \( n \), so there exist constants \( c_k, C_k > 0 \) above so that the variance is bounded above by \( C_k e^{-c_k n} \).

Define \( a = a(m, r) = \frac{1}{m} + \frac{r}{m^{l+1}} \) and \( b = b(m, r) = \frac{1}{m} + \frac{r+1}{m^{l+2}} \). For each \( \varepsilon \in (0, 1) \) there exists a unique pair \( (m, r) \) such that \( \varepsilon \in [1/m, 1/(m-1)] \) and \( \varepsilon \in [a, b] \). Assume for now that \( [a^{-\beta}] = [b^{-\beta}] \); the case in which the two differ is handled at the end of the proof. For all \( \varepsilon \in [a, b] \) and \( p = p_c + \varepsilon \), we have

\[
\sum_{\{u_i\} \in \binom{T_n}{k}} \mathbf{P}_T[0 \leftrightarrow_p u_1, \ldots, u_k] g(T(u_1), p) \cdots g(T(u_k), p) \\
\leq \sum_{\{u_i\} \in \binom{T_n}{k}} \mathbf{P}_T[0 \leftrightarrow_{p_c+b} u_1, \ldots, u_k] g(T(u_1), p_c + b) \cdots g(T(u_k), p_c + b).
\]

By Chebyshev’s inequality, the conditional probability that the right-hand side is \( b^{\ell+1} \) greater than its mean, given \( T_n \), is at most \( C_k b^{-2(\ell+2)} e^{-c_k n} \). Because \( n = [b^{-\beta}] \), this is finite when summed over all possible \( m \) and \( r \), implying that all but finitely often

\[
\sum_{\{u_i\} \in \binom{T_n}{k}} \mathbf{P}_T[0 \leftrightarrow_{p_c+b} u_1, \ldots, u_k] g(T(u_1), p_c + b) \cdots g(T(u_k), p_c + b) \\
\leq g(p_c + b)^k \sum_{\{u_i\} \in \binom{T_n}{k}} \mathbf{P}_T[0 \leftrightarrow_{p_c+b} u_1, \ldots, u_k] + b^{\ell+1}.
\]

By a similar argument, we obtain the lower bound

\[
\sum_{\{u_i\} \in \binom{T_n}{k}} \mathbf{P}_T[0 \leftrightarrow_{p_c+a} u_1, \ldots, u_k] g(T(u_1), p_c + b) \cdots g(T(u_k), p_c + b) \\
\geq g(p_c + a)^k \sum_{\{u_i\} \in \binom{T_n}{k}} \mathbf{P}_T[0 \leftrightarrow_{p_c+a} u_1, \ldots, u_k] - b^{\ell+1}.
\]
Letting $(\ast)$ denote the absolute value of the left-hand-side of (3.14), we see that

$$(\ast) \leq g(p_c + b) \sum_{\{u_i\} \in \binom{\mathcal{V}}{k}} \mathbb{P}_T[0 \leftrightarrow_p c+b u_1, \ldots, u_k]$$

$$- g(p_c + a) \sum_{\{u_i\} \in \binom{\mathcal{V}}{k}} \mathbb{P}_T[0 \leftrightarrow_p c+a u_1, \ldots, u_k] + 2b \ell^{\ell+1}$$

$$\leq 2(g(p_c + b) - g(p_c + a)) X_n^{(k)}$$

$$+ g(p_c + b) \left( \mathbb{P}_T[0 \leftrightarrow_p c+b u_1, \ldots, u_k] - \mathbb{P}_T[0 \leftrightarrow_p c+a u_1, \ldots, u_k] \right) + 2b \ell^{\ell+1}$$

$$\leq 2\left( g(p_c + b) - g(p_c + a) \right) X_n^{(k)} + g(p_c + b) \frac{2 \cdot n \cdot k (b-a)}{p_c} X_n^{(k)} + 2b \ell^{\ell+1},$$

where the last inequality is via the Mean Value Theorem.

Dividing by $\varepsilon^\ell$ and setting $C_k = 2k/p_c$, we have

$$2 \frac{g(p_c + b) - g(p_c + a)}{\varepsilon^\ell} X_n^{(k)} + C_k \cdot n \cdot g(p_c + b) \frac{b-a}{\varepsilon^\ell} X_n^{(k)} + 2b(b/a)^\ell$$

$$\leq 2 \frac{b-a}{\varepsilon^\ell} \cdot \frac{g(p_c + b) - g(p_c + a)}{b-a} X_n^{(k)}$$

$$+ C_k \cdot n \cdot g(p_c + b) \frac{b-a}{\varepsilon^\ell} X_n^{(k)} + 2b(b/a)^\ell$$

$$\leq 2k \frac{b-a}{\varepsilon^\ell} \max_{x \in [p_c, 1]} g'(x) X_n^{(k)}$$

$$+ C_k \cdot n \cdot g(p_c + b) \frac{b-a}{\varepsilon^\ell} X_n^{(k)} + 2b \left( \frac{b-a}{a} \right)^\ell$$

again by the Mean Value Theorem.

By Theorem 3.5(c), $n^{-(k-1)} X_n^{(k)}$ converges as $n \to \infty$. By definition of $b, a$ and $n$, $\frac{(b-a)^n}{\varepsilon^\ell} \to 0$ as $\varepsilon \to 0$ for $\delta$ sufficiently small, thereby completing the proof except in the case when $[a-\delta] \neq [b-\delta]$.

When $[a-\delta]$ and $[b-\delta]$ differ, we can split the interval $[a, b)$ into subintervals $[a, c - \delta')$, $[c - \delta', c]$ and $[c, b)$, where $c \in (a, b)$ is the point where $[x-\delta]$ drops. Repeating the above argument for the first and third intervals, taking $\delta'$ sufficiently small, and exploiting continuity of the expression in (3.14) on $[a, c)$ provides us with desired asymptotic bounds for the middle interval, hence the proof is complete. □

As a midway point in proving Theorem 3.1, we obtain an expansion for $g(T, p_c + \varepsilon)$ that for a given $\varepsilon$ is measurable with respect to $T_{n(\varepsilon)}$, where $n(\varepsilon)$ grows like a small power of $\varepsilon^{-1}$.

**Lemma 3.10** Suppose $\mathbb{E} [Z^{(2\ell+1)(1+\beta)}] < \infty$ for some $\ell \geq 1$ and $\beta > 0$. Define $n(\varepsilon) := [\varepsilon^{-\delta}]$. Then for $\delta > 0$ sufficiently small, we have $\mathbb{G}_\mathbb{W}$-a.s. the following expansion as $\varepsilon \to 0^+$:

$$g(T, p_c + \varepsilon) = \sum_{i=1}^\ell \left( \sum_{j=1}^i (-1)^{j+1} \sum_{d=j}^i p_c^{d-j+1} X_{n(\varepsilon)}^{(j+1-d)} \right) \mu_i \varepsilon^i + o(\varepsilon^\ell).$$

**Proof** For each $j$ and $n$, define

$$\text{Bon}^{(j)}_n(\varepsilon) := \sum_{\{v_i\} \in \binom{\mathcal{V}}{j}} \mathbb{P}_T[0 \leftrightarrow_p v_1, \ldots, v_j] g(T(v_1), p) \cdots g(T(v_j), p)$$

Again, by the Mean Value Theorem, we have

$$g(T, p_c + \varepsilon) = \sum_{j=1}^\ell \sum_{d=j}^i p_c^{d-j+1} X_{n(\varepsilon)}^{(j+1-d)} \mu_i \varepsilon^i + o(\varepsilon^\ell).$$

□
and \( \text{Bon}_n^{(j)}(\varepsilon) := \sum_{\{v_i\} \in (T_n^j)} \mathbb{P}_T[0 \leftrightarrow v_1, \ldots, v_j] g(p)^j \)

where we write \( p = p_c + \varepsilon \). Applying the Bonferroni inequalities to the event \( \{0 \leftrightarrow \infty\} = \bigcup_{v \in T_n} \{0 \leftrightarrow v \leftrightarrow \infty\} \) yields

\[
\sum_{i=1}^{2j} (-1)^{i+1} \cdot \text{Bon}_{n(\varepsilon)}^{(i)}(\varepsilon) \leq g(T, p_c + \varepsilon) \leq \sum_{i=1}^{2j+1} (-1)^{i+1} \cdot \text{Bon}_{n(\varepsilon)}^{(i)}(\varepsilon)
\]

for each \( j \), where the \( \pm \) may be either a plus or minus.

For sufficiently small \( \delta > 0 \), Proposition 3.9 allows us to replace each \( \text{Bon}_{n(\varepsilon)}^{(i)}(\varepsilon) \) with \( \text{Bon}_{n(\varepsilon)}^{(i)}(\varepsilon) \), introduce an \( o(\varepsilon^\ell) \) error term, provided \( \mathbb{E}[Z^{(2\ell-1)(1+\beta)}] < \infty \). Moreover, we note

\[
\text{Bon}_{n(\varepsilon)}^{(i)}(\varepsilon) = g(p_c + \varepsilon)^i \sum_{\{v_i\} \in (T_n^i)} \mathbb{P}_T[0 \leftrightarrow p_c + \varepsilon \ v_1, \ldots, v_i] \leq C g(p_c + \varepsilon)^i X_{n(\varepsilon)}^{(i,0)} = o(\varepsilon^{i-1}).
\]

The constant \( C \) is introduced when we bound \( (1 + \varepsilon/p_c)^{|T(v_1, \ldots, v_i)|} \) from above by a constant \( C \) for \( \delta \) sufficiently small; the limit follows from Theorem 3.5(\( \varepsilon \)). For each \( j \), apply (3.15) to show

\[
g(T, p_c + \varepsilon) = \sum_{j=1}^\ell (-1)^{j+1} \text{Bon}_n^{(j)}(\varepsilon) + o(\varepsilon^\ell). \tag{3.16}
\]

Now expand

\[
\text{Bon}_n^{(j)}(\varepsilon) = g(p_c + \varepsilon)^j \sum_{\{v_j\} \in (T_n^j)} (p_c + \varepsilon)^{|T(v_1, \ldots, v_j)|}
\]

\[
= \left( \sum_{i=j}^\ell r_j, i \varepsilon^i + o(\varepsilon^i) \right) \sum_{\{v_i\} \in (T_n^i)} p_c^{|T(v_1, \ldots, v_j)|} (1 + \varepsilon/p_c)^{|T(v_1, \ldots, v_j)|}
\]

\[
= \left( \sum_{i=j}^\ell r_j, i \varepsilon^i + o(\varepsilon^i) \right) \sum_{\{v_i\} \in (T_n^i)} p_c^{|T(v_1, \ldots, v_j)|} \left( \sum_{i=0}^{\ell} \left( \frac{|T(v_1, \ldots, v_j)|}{i} \varepsilon^i \right) \frac{\varepsilon^i}{p_c^i} + O(n^{\ell+1} \varepsilon^{\ell+1}) \right)
\]

\[
= \left( \sum_{i=j}^\ell r_j, i \varepsilon^i + o(\varepsilon^i) \right) \left( \sum_{i=0}^{\ell} \frac{X_{n}^{(j,i)} \varepsilon^i}{p_c^i} + o(\varepsilon^i) \right)
\]

\[
= \sum_{i=j}^\ell r_j, i \varepsilon^i \left( \sum_{d=j} p_c^d r_j, d c_{j,d} + o(\varepsilon^i) \right) + o(\varepsilon^i). \tag{3.17}
\]

Plugging this into (3.16) completes the Lemma. \( \square \)

We are almost ready to prove Theorem 3.1. We have dealt with the martingale part. What remains is to get rid of the predictable part. The following combinatorial identity is the key to making the predictable part disappear.

**Lemma 3.11** Fix \( i \geq 1 \) and suppose \( \mathbb{E}[Z^{i+1}] < \infty \); then for each \( a, b \leq i \) we have

\[
\sum_{d=1}^i \sum_{j=1}^i (-1)^{j-1} p_c^d r_j, d c_{j,a} \left( \frac{j}{b - d} \right) = (-1)^{a+1} p_c^a r_{a,b}.
\]
Proof Begin as in the proof of Proposition 2.6 with the identity
\[
1 - \phi(1 - (p_c + \varepsilon)g(p_c + \varepsilon))^a = g(p_c + \varepsilon)^a.
\]
The idea is to take Taylor expansions of both sides and equate coefficients of \(\varepsilon^b\); more technically, taking Taylor expansions of both sides up to terms of order \(o(\varepsilon^i)\) yield two polynomials in \(\varepsilon\) of degree \(i\) whose difference is \(o(\varepsilon^i)\) thereby showing the two polynomials are equal. The coefficient \([\varepsilon^b]g(p_c + \varepsilon)^a\) of \(\varepsilon^b\) on the right-hand side is \(r_{a,b}\), by definition. On the left-hand-side, we write
\[
\left[1 - \phi(1 - (p_c + \varepsilon)g(p_c + \varepsilon))^a\right]^a = g(p_c + \varepsilon)^a.
\]
\[
= (-1)^a \sum_{j=1}^{i} (-1)^j (1 + \varepsilon/p_c)^j g(p_c + \varepsilon)^j c_{j,a} + o(\varepsilon^i).
\]
The coefficient of \(\varepsilon^b\) of \((1 + \varepsilon/p_c)^j g(p_c + \varepsilon)^j\) is
\[
[e^b](1 + \varepsilon/p_c)^j g(p_c + \varepsilon)^j = \sum_{d=j}^{b} \left( [e^d]g(p_c + \varepsilon)^j \right) \left( [e^{b-d}]g(1 + \varepsilon/p_c)^j \right)
\]
\[
= \sum_{d=j}^{b} r_{j,d} \left( \frac{j}{b-d} \right) p_c^{-(b-d)}.
\]
Equating the coefficients of \(\varepsilon^b\) on both sides then gives
\[
(-1)^a \sum_{j=1}^{i} (-1)^j c_{j,a} \sum_{d=j}^{b} r_{j,d} \left( \frac{j}{b-d} \right) p_c^{-(b-d)} = r_{a,b}.
\]
Multiplying by \(p_c^b(-1)^{a+1}\) on both sides completes the proof. \(\square\)

With Theorem 3.5 and Lemma 3.11 in place, the limits of \(M_n^{(i)}\) fall out easily.

Lemma 3.12 Suppose \(E[Z^{i+1}] < \infty\) for some \(i\) and let \(\beta > 0\) with \(E[Z^{(1+\beta)}] < \infty\). Then
(a) The sequence \((M_n^{(i)})_{n=1}^\infty\) is a martingale with respect to the filtration \((T_n)_{n=1}^\infty\).
(b) There exist positive constants \(C, c\) depending only on \(i, \beta\) and the progeny distribution so that \(||M_n^{(i)} - M_0^{(i)}||_{L^{1+\beta}} \leq C e^{-cn}\).
(c) There exists a random variable \(M^{(i)}\) so that \(M_n^{(i)} \rightarrow M^{(i)}\ both\ almost\ surely\ and\ in\ L^{1+\beta}\).

Proof Note first that (c) follows from (a) and (b) by the triangle inequality together with the \(L^\beta\) martingale convergence theorem. Parts (a) and (b) are proved simultaneously. Write
\[
\mu^{-i} \left( M_n^{(i)} - M_0^{(i)} \right)
\]
\[
= \sum_{j=1}^{i} (1)^j+1 \sum_{d=j}^{i} p_c^d r_{j,d} \left( X_{n+1}^{(j,i-d)} - X_n^{(j,i-d)} \right)
\]
\[
= \sum_{j=1}^{i} (1)^j+1 \sum_{d=j}^{i} p_c^d r_{j,d} \left( \Delta Y_{n+1}^{(j,i-d)} + \sum_{a=1}^{j} \sum_{b=0}^{i-d} \left( \frac{i}{i-d-b} \right) X_n^{(a,b)} - X_n^{(j,i-d)} \right)
\]

\(\diamondsuit\) Springer
This is equivalent to the claim that simply need to handle the second sum in (3.18). We claim that it is identically equal to zero. With the limits of each sum, we recall that Markov’s inequality and Borel–Cantelli shows that By Theorem 3.5, we have that $\Delta Y_{n+1}^{(j,i-d)}$ is exponentially small in $L^{1+\beta}$. This means that we simply need to handle the second sum in (3.18). We claim that it is identically equal to zero. This is equivalent to the claim that

$$
\sum_{j=1}^{i} \sum_{d=j}^{i} \sum_{i-d}^{j} (-1)^{j+1} p_c^d r_{j,d} c_{j,a} \left( \sum_{a=1}^{i} \sum_{b=0}^{i-d} c_{j,a} \left( \sum_{i}^{j-i-d} X_n^{(a,b)} - X_n^{(j,i-d)} \right) \right) X_n^{(a,b)} = \sum_{a=1}^{i} \sum_{b=0}^{i} c_{j,a} \left( \sum_{d=1}^{i} \sum_{j=1}^{i-d} (-1)^{j+1} p_c^d r_{j,d} c_{j,a} \left( \sum_{n=0}^{i-d} X_n^{(a,i-b)} \right) \right) X_n^{(a,i-b)}.
$$

(3.19)

To prove this, we rearrange the sums in the left-hand-side of (3.19). In order to handle the limits of each sum, we recall that $c_{j,a} = 0$ for $j < a$ and $r_{j,d} = 0$ for $d < j$. Relabeling and swapping gives

$$
\sum_{j=1}^{i} \sum_{d=j}^{i} \sum_{i-d}^{j} (-1)^{j+1} p_c^d r_{j,d} c_{j,a} \left( \sum_{a=1}^{i} \sum_{b=0}^{i-d} c_{j,a} \left( \sum_{i}^{j-i-d} X_n^{(a,b)} - X_n^{(j,i-d)} \right) \right) X_n^{(a,b)} = \sum_{a=1}^{i} \sum_{b=0}^{i} c_{j,a} \left( \sum_{d=1}^{i} \sum_{j=1}^{i-d} (-1)^{j+1} p_c^d r_{j,d} c_{j,a} \left( \sum_{n=0}^{i-d} X_n^{(a,i-b)} \right) \right) X_n^{(a,i-b)}.
$$

Lemma 3.11 shows that the term in parentheses is equal to $(-1)^{a+1} p_c^b r_{a,b}$, thereby showing (3.19).

**Proof of Theorem 3.1** Apply Lemma 3.10 to obtain some $\delta > 0$ sufficiently small so that

$$
g(T, p_c + \varepsilon) = \sum_{i=1}^{\ell} M_n^{(i)} \varepsilon^i + o(\varepsilon^{\ell})
$$

(3.20)

with $n = [\varepsilon^{-\delta}]$. The exponential convergence of $M_n^{(i)}$ from Lemma 3.12 together with Markov’s inequality and Borel–Cantelli shows that

$$
|M_n^{(i)} - M^{(i)}| n^N \rightarrow 0
$$

almost surely for any fixed $N > 0$. Because $n = [\varepsilon^{-\delta}]$ implies $n^{-N} = o(\varepsilon^{\ell})$ for $N$ sufficiently large, (3.20) can be simplified to

$$
g(T, p_c + \varepsilon) = \sum_{i=1}^{\ell} M^{(i)} \varepsilon^i + o(\varepsilon^{\ell}).$$
It remains only to show that $E[M^{(i)}] = r_i$. Because $M^{(i)}_n$ converges in $L^{1+\beta}$, it also converges in $L^1$, implying $E[M^{(i)}] = E[M^{(i)}]$. Noting that $E[X^{(j,k)}_1] = (\frac{j}{k})c_{j,1}$, we use Lemma 3.11 with $a = 1$ and $b = i$ in the penultimate line to obtain
\[
p^j_i E[M^{(i)}] + \sum_{j=1}^{i} \sum_{d=j}^{i} (-1)^{j+1} p^d_{i,j,d} E[X^{(i,j-d)}_1] = \sum_{j=1}^{i} \sum_{d=j}^{i} (-1)^{j+1} p^d_{i,j,d} (\frac{j}{i-d})c_{j,i} = (-1)^{j+1} p^j_i r_{1,i} = p^j_i r_i.
\]

\[\square\]

4 Regularity on the Supercritical Region

In this section we prove Russo-type formulas expressing the derivatives of $g(T,p)$ as expectations of quantities measuring the number of pivotal bonds. The first and simplest of these is Theorem 4.1, expressing $g'(T,p)$ as the expected number of pivotal bonds multiplied by $p^{-1}$. In Sect. 4.2 we define some combinatorial gadgets to express more general expectations (Definitions 4.7) and show that these compute successive derivatives (Proposition 4.9). In Sect. 4.3, explicit estimates on these expectations are given in Proposition 4.14, which under suitable moment conditions lead to continuity of the first $k$ derivatives at $p^\pm_c$, which is Theorem 4.13.

4.1 Continuous Differentiability on $(p_c, 1)$

Given $T$ and $p$, let $T_p = T_p(\omega)$ denote the tree obtained from the $p$-percolation cluster at the root by removing all vertices not connected to infinity in $T(v)$. Formally, $v \in T_p$ if and only if $0 \leftrightarrow_{T_p} v$ and $v \leftrightarrow_{T(v),p} \infty$. On the survival event $H_T(p)$ let $B_p$ denote the first node at which $T_p$ branches. Formally, the event $\{B_p = v\}$ is the intersection of three events $\text{Open}(v)$, $\text{NoBranch}(v)$ and $\text{Branch}(v)$ where $\text{Open}(v)$ is the event $0 \leftrightarrow_{T_p} v$ of the path from the root to $v$ being open, $\text{NoBranch}(v)$ is the event that for each ancestor $w < v$, no child of $w$ other than the one that is an ancestor of $v$ is in $T_p$, and $\text{Branch}(v)$ is the event that $v$ has at least two children in $T_p$. We call $|B_p|$ the branching depth. The main result of this subsection is the following.

Theorem 4.1 The derivative of the quenched survival function is given by $g'(T,p) = p^{-1}E_T|B_p|$, which is finite and continuous on $(p_c, 1)$.

We begin with two annealed results. Although it may be obvious, we point out that the notation $T_p$ means to take the random tree $T$ defined by the deg$_v$ variables and apply the random map $T \mapsto T_p$ defined by the $U_v$ variables (see Sect. 2.2 for relevant definitions). Recall that $g(p)$ denotes the annealed survival function. The following fact is elementary and follows from taking the thinned o.g.f. $\psi(z,s) := \phi(1-s(1-z))$, setting $s = pg(p)$ to obtain the survivor tree, then conditioning on being non-empty, i.e., $(\psi - \psi(0))/(1 - \psi(0))$:

Proposition 4.2 For any $p > p_c$ define an offspring generating function
\[
\phi_p(z) := \frac{\phi(1 - pg(p)(1 - z)) - \phi(1 - pg(p))}{g(p)}.
\]
Then the conditional distribution of \( T_p \) given \( H(p) \) is Galton–Watson with offspring generating function \( \phi_p \), which we will denote \( GW_p \). □

**Lemma 4.3** (annealed branching depth has exponential moments) Let

\[
A_p = A_p(\phi) := \phi'_p(0)
\]

(4.2)

denote the probability under \( GW_p \) that the root has precisely one child. Suppose \( r > 0 \) and \( p > p_c \) satisfy \( (1 + r)A_p < 1 \). Then \( E(1 + r)^{|B_p|} < \infty \).

**Proof** The result is equivalent to finiteness of \( \sum_{n=1}^{\infty} (1 + r)^n P(|B_p| \geq n) \). Proposition 4.2 implies that \( P(|B_p| \geq n) = A''_p \), showing \( E(1 + r)^{|B_p|} \) to be the sum of a convergent geometric series.

Next we recast the \( p \)-indexed stochastic process \( \{T_p : p \in [0, 1]\} \) as a Markov chain. Define a filtration \( \{G_p : 0 \leq p \leq 1\} \) by \( G_p = \sigma(T, \{U_e \vee p\}) \). Clearly if \( p > p' \) then \( G_p \subseteq G_{p'} \), thus \( \{G_p\} \) is a filtration when \( p \) decreases from 1 to 0. Informally, \( G_p \) knows the tree, knows whether each edge \( e \) is open at “time” \( p \), and if not, “remembers” the time \( U(e) \) when \( e \) closed.

**Lemma 4.4** Fix any tree \( T \). The edge processes \( \{1_{U(e) \leq p}\} \) are independent left-continuous two-state continuous time Markov chains. They have initial state 1 when \( p = 1 \) and terminal state 0 when \( p = 0 \), and they jump from 1 to 0 at rate \( p^{-1} \). The process \( \{T_p\} \) is a function of these and is also Markovian on \( \{G_p\} \).

**Proof** Independence and the Markov property for \( \{1_{U(E) \leq p}\} \) are immediate. The jump rate is the limit of \( \varepsilon^{-1}P(U(e) \in (p - \varepsilon, p)|U(e) < p) \) which is \( p^{-1} \). The Markov property for \( T_p \) on \( \{G_p\} \) is immediate because it is a function of Markov chains on \( \{G_p\} \).

Next we define the quantity \( \beta \) as \( \beta := \inf\{p : T_p \text{ is infinite}\} \). Thus \( g(T, p) = P_T(\beta \leq p) \) and \( g'(T, p) \) is the density, if it exists, of the \( P_T \)-law of \( \beta \). Before proceeding to the proof of Theorem 4.1, we will need to establish one additional lemma.

**Lemma 4.5** With probability 1, at \( p = \beta \) the root of \( T_p \) is connected to infinity, \( |B_p| < \infty \) (i.e. \( T_p \) does branch somewhere for \( p = \beta \)), and there is a vertex \( v \leq B_p \) with \( U_v = \beta \). Consequently, the event \( H(p) \) is, up to measure 0, a disjoint union of the events \( \{B_\beta = v\} \cap \{\beta \leq p\} \).

**Proof** As \( p \) decreases from 1 to 0, the vertex \( B_p \) can change only by jumping to a descendant or to zero. Either zero is reached after finitely many jumps, in which case \( B_\beta \) is the value just before jumping to zero, or there is a countable sequence of jumps. The decreasing limit of the jump times must be at least \( p_c \) because below \( p_c \) the value of \( B_p \) is always zero. The set of infinite paths, also known as \( \partial T \), is compact, which means a decreasing intersection of closed subsets of \( \partial T \) is non-empty. It follows that if \( \beta \) is the decreasing limit of jump times for \( B_p \) then \( H(\beta) \) occurs, that is \( 0 \leftrightarrow \infty \) for \( p = \beta \). Because \( g(p_c) = 0 \), we conclude the probability of a countable sequence of jump times decreasing to \( p_c \) is zero. To prove the lemma therefore, it suffices to rule out a sequence of jump times decreasing to some value \( y > p_c \). Now for the annealed process, \( \{B_p : p > p_c\} \) is a time-inhomogeneous Markov chain, the jump rate depends only on \( p \) and not \( |B_p| \), and the jump rate is bounded for \( p \in [y, 1] \). It follows that for almost surely every \( T \), the probability of infinitely many jumps in \( [y, 1] \) is zero for any \( y > p_c \). □
Proof of Theorem 4.1} By Lemma 4.5 \( H_T(p) \) is equal to the union of the disjoint events \( \{ B_\beta = v \} \cap H_T(p) \). On \( \{ B_\beta = v \} \) the indicator \( 1_{H(p)} \) jumps to zero precisely when \( \text{Open}(v) \) does so, which occurs at rate \( p^{-1} |v| \). Because all jumps have the same sign, it now follows that

\[
\frac{d}{dp} g(T, p) = \frac{1}{p} \sum_{v \in T} |v| \mathbb{P}(B_p = v) = \frac{1}{p} \mathbb{E}[|B_p|],
\]

which may be \(+\infty\). Summing by parts, we also have

\[
\frac{d}{dp} g(T, p) = \frac{1}{p} \sum_{v \neq \emptyset} \mathbb{P}(B_p \geq v)
\]  

(4.3)

where \( B_p \geq v \) denotes \( B_p = w \) for some descendant \( w \) of \( v \).

To see that this is finite and continuous on \((p_c, 1)\), consider any \( p' > p_c \) and \( r > 0 \) with \((1 + r)p' Ap' < 1\). For any \( p \in (p', (1 + r)p') \) we have

\[
\mathbb{P}_T(B_p \geq v) = \mathbb{P}_T(\text{Open}(v)) \mathbb{P}((\text{NoBranch}(v, p)) g(T(v), p) \leq (1 + r)^{|v|} (p')^{|v|} \mathbb{P}_T(\text{NoBranch}(v, p')).
\]

Taking the expectation of the expression on the right and multiplying by \( g(p') \) we observe that

\[
g(p') \mathbb{E} \left[ \sum_{v \in T} (1 + r)^{|v|} (p')^{|v|} \mathbb{P}_T(\text{NoBranch}(v, p')) \right]
\]

\[
= \mathbb{E} \left[ \sum_{v \in T} (1 + r)^{|v|} (p')^{|v|} \mathbb{P}_T(\text{NoBranch}(v, p'))) g(T(v), p') \right]
\]

\[
= \mathbb{E} \left[ \sum_{n=0}^{\infty} (1 + r)^n \mathbb{P}(B_{p'} \geq n) \right] < \infty
\]

where the last inequality follows from Lemma 4.3. This now implies that for \( G_W \)-almost every \( T \), the right-hand-side of (4.3) converges uniformly for \( p \in (p', (1 + r)p') \), thus implying continuity on this interval. Covering \((p_c, 1)\) by countably many intervals of the form \((p', (1 + r)p')\), the theorem follows by countable additivity. \(\square\)

### 4.2 Smoothness of \( g \) on the Supercritical Region

Building on the results from the previous subsection, we establish the main result concerning the behavior of the quenched survival function in the supercritical region.

**Theorem 4.6** For \( G_W \)-a.e. \( T \), \( g(T, p) \in C^\infty((p_c, 1)) \).

In order to prove this result, we define quantities generalizing the quantity \( \mathbb{E}_T |B_p| \) and show that the derivative of a function in this class remains in the class. We begin by presenting several definitions. Throughout the remainder of the paper, our trees are rooted and ordered, meaning that the children of each vertex have a specified order (usually referred to as left-to-right rather than smallest-to-greatest) and isomorphisms between trees are understood to preserve the root and the orderings.

**Definition 4.7** (i) Collapsed trees. Say the tree \( \mathcal{V} \) is a collapsed tree if no vertex of \( \mathcal{V} \) except possibly the root has precisely one child.
Fig. 2 Illustrations showing how $\Phi$ acts on a tree $T$

(ii) Initial subtree. The tree $\widetilde{T}$ is said to be an initial subtree of $T$ if it has the same root, and if for every vertex $v \in \widetilde{T} \subseteq T$, either all children of $v$ are in $\widetilde{T}$ or no children of $v$ are in $\widetilde{T}$, with the added proviso that if $v$ has only one child in $T$ then it must also be in $\widetilde{T}$.

(iii) The collapsing map $\Phi$. For any ordered tree $T$, let $\Phi(T)$ denote the isomorphism class of ordered trees obtained by collapsing to a single edge any path along which each vertex except possibly the last has only one child in $T$ (see figure below).

(iv) Notations $T(\mathcal{V})$ and $\mathcal{V} \preceq T$. It follows from the above definitions that any collapsed tree $\mathcal{V}$ is isomorphic to $\Phi(\widetilde{T})$ for at most one initial tree $\widetilde{T} \subseteq T$. If there is one, we say that $\mathcal{V} \preceq T$ and denote this subtree by $T(\mathcal{V})$. We will normally use this for $T = T_p$. For example, when $\mathcal{V}$ is the tree with one edge then $\mathcal{V} \preceq T_p$ if and only if $T_p$ has precisely one child of the root, in which case $T_p(\mathcal{V})$ is the path from the root to $B_p$.

(v) The embedding map $\iota$. If $e$ is an edge of $\mathcal{V}$ and $\mathcal{V} \preceq T$, let $\iota(e)$ denote the path in $T(\mathcal{V})$ that collapses to the edge carried to $e$ in the above isomorphism. For a vertex $v \in \mathcal{V}$ let $\iota(v)$ denote the last vertex in the path $\iota(e)$ where $e$ is the edge between $e$ and its parent; if $v$ is the root of $\mathcal{V}$ then by convention $\iota(v)$ is the root of $T$.

(vi) Edge weights. If $\mathcal{V} \preceq T$ and $e \in E(\mathcal{V})$, define $d(e) = d_{T(\mathcal{V})}(e)$ to be the length of the path $\iota(e)$.

(v) Monomials. A monomial in (the edge weights of) a collapsed tree $\mathcal{V}$ is a set of non-negative integers $\{F(e) : e \in E(\mathcal{V})\}$ indexed by the edges of $\mathcal{V}$, identified with the product

$$\langle T, \mathcal{V}, F \rangle := \begin{cases} \prod_{e \in E(\mathcal{V})} d(e)^{F(e)} & \text{if } \mathcal{V} \preceq T, \\ 0 & \text{otherwise}. \end{cases} \quad (4.4)$$

A monomial $F$ is only defined in reference to a weighted collapsed tree $\mathcal{V}$. As an example, if $T$ is as in Fig. 2 with $\mathcal{V} = \Phi(T)$, and if we take $F(e) = 1$ for the three edges down the left, $F(e) = 3$ for the rightmost edge and $F(e) = 0$ for the other four edges, then $\langle T, \mathcal{V}, F \rangle = 2 \cdot 3 \cdot 2 \cdot 2^3$. 

\[\text{Springer}\]
Then there exists a collection of collapsed trees \( \mathcal{V} \), and a monomial \( F \), define functions \( \mathcal{R} = \mathcal{R}(T, r, \mathcal{V}, p) \) and \( \mathcal{D} = \mathcal{D}(T, F, \mathcal{V}, p) \) by

\[
\mathcal{R} := \mathbb{E}_T \left[ (1 + r)^{|E(T_p(\mathcal{V}))|} \mathbf{1}_{\mathcal{V} \leq T_p} \right] \quad (4.5)
\]

\[
\mathcal{D} := \mathbb{E}_T \left[ (T_p, \mathcal{V}, F) \right] . \quad (4.6)
\]

For example, if \( \mathcal{V}_1 \) is the tree with a single edge \( e \) and \( F_1(e) = 1 \), then \( \langle T_p, \mathcal{V}_1, F_1 \rangle = |B_p| \)

and the conclusion of Theorem 4.1 is that for \( p > p_c \),

\[
\frac{d}{dp} g(T, p) = \frac{1}{p} \mathbb{E}_T |B_p| = \frac{1}{p} \mathcal{D}(T, F, \mathcal{V}_1, p). \quad (4.7)
\]

The main result of this section, from which Theorem 4.6 follows without too much further work, is the following representation.

**Proposition 4.9** Let \( \mathcal{V} \) be a collapsed tree and let \( F \) be a monomial in the variables \( d_T, \forall e \). Then there exists a collection of collapsed trees \( \mathcal{V}_1, \ldots, \mathcal{V}_m \) for some \( m \geq 1 \) and monomials \( F_1, \ldots, F_m \), given explicitly in (4.12) below, such that

\[
\frac{d}{dp} \mathbb{E}_T \langle T_p, \mathcal{V}, F \rangle = \frac{1}{p} \sum_{i=1}^{m} \mathbb{E}_T \langle T_p, \mathcal{V}_i, F_i \rangle \quad (4.8)
\]

on \( (p_c, 1) \) and is finite and continuous on \( (p_c, 1) \) for \( \mathbb{GW}-a.e. \) tree \( T \). Furthermore, each monomial \( F_i \) on the right-hand side of (4.8) satisfies \( \deg(F_i) = 1 + \deg(F) \) and each of the edge sets \( E(\mathcal{V}_i) \) satisfies \( |E(\mathcal{V}_i)| \leq 2 + |E(\mathcal{V})| \).

The content of this result is twofold: that the derivative of a random variable expressible in the form (4.8) is also expressible in this form, and that all derivatives are continuous on \( (p_c, 1) \). The proof of (4.8) takes up some space due to the bulky sums involved. We begin with two finiteness results that are the analogues for \( \mathcal{R} \) and \( \mathcal{D} \) of the exponential moments of \( B_p \) proved in Proposition 4.3. Recall the notation \( A_p \) for the probability of exactly one child of the root having an infinite descendant tree given that at least one does.

**Lemma 4.10** If \( \mathcal{V} \) is a collapsed tree, \( p \in (p_c, 1) \), and \( r > 0 \) with \( (1 + r)A_p < 1 \), then \( \mathbb{E}_R(T, r, \mathcal{V}, p) < \infty \).

**Proof** Let \( Z_p \) denote a variable distributed as the first generation of \( T_p \) conditioned on \( H(p) \), in other words, having PGF \( \phi_p \). The annealed collapsed tree \( \Phi(T_p) \), with weights, has a simple description. The weights \( d(par(v), v) \) of the edges will be IID geometric variables with mean \( 1/(1 - A_p) \) because the length of the descendant path needed to reach a node with at least two children is geometric with success probability \( 1 - A_p \). Let \( G \) denote a geometric variable with this common distribution. The number of children of each node other than the root will have offspring distribution \( (Z_p | Z_p \geq 2) \), whereas the offspring distribution at the root will be \( Z_p \) because the root is allowed to have only one child in \( T_p \).

The cases where \( \mathcal{V} \) is empty or a single vertex being trivial, we assume that \( \mathcal{V} \) has at least one edge. The event \( \mathcal{V} \leq T_p \) is the intersection of the event \( H(p) \) with the events \( \deg_{T_p}(v) = \deg_{\mathcal{V}}(v) \) as \( v \) varies over the interior vertices of \( \mathcal{V} \) including \( 0 \). The branching structure makes these events (when conditioned on \( H(p) \)) independent with probabilities \( \mathbb{P}(Z_p = \deg(v) | Z_p \geq 2) \), except at the root where one simply has \( \mathbb{P}(Z_p = \deg(0)) \) (with
\[ \deg(v) \text{ representing the number of children of } v. \] It follows from this and the IID geometric edge weights that
\[ \mathbf{E}(T, r, V, p) = g(p) \left[ \mathbf{E}(1 + r)^G \right]^{(E(V))} \mathbf{P}(Z_p = \deg(0)) \prod_v \mathbf{P}(Z_p = \deg(v) - 1 | Z_p \geq 2), \]
(4.9)

where the product is over vertices \( v \in V \setminus (\partial V \cup \{0\}) \). Finiteness of \( \mathbf{E}(T, r, V, p) \) then follows from finiteness of \( \mathbf{E}(1 + r)^G \), which was Lemma 4.3.

Before proceeding to give the proof of Proposition 4.9, we present one final result establishing finiteness and continuity of quenched monomial expectations.

**Lemma 4.11** If \( V \) is a collapsed tree and \( F \) is a monomial, then for \( \mathbb{G}_\infty \)-almost every \( T \), the quantity \( \mathcal{D}(T, F, V, p) \) is finite and varies continuously as a function of \( p \) on \((p_c, 1)\).

**Proof** For finiteness and continuity, as well as for computing the explicit representation in (4.12) below, we need to decompose the monomial expectation according to the identity
\[ \mathbf{E} \mathbf{R}(T, r, V, p) = \left[ \mathbf{E}(1 + r)^G \right]^{(E(V))} \mathbf{P}(Z_p = \deg(0)) \prod_v \mathbf{P}(Z_p = \deg(v) - 1 | Z_p \geq 2), \]

where the product is over vertices \( v \in V \setminus (\partial V \cup \{0\}) \). Finiteness of \( \mathbf{E} \mathbf{R}(T, r, V, p) \) then follows from finiteness of \( \mathbf{E}(1 + r)^G \), which was Lemma 4.3.

Letting \( \text{Open} = \bigcap_{v \in \tilde{T}} \text{Open}(v) \) be the event that all of \( \tilde{T} \) is open under \( p \)-percolation;
\( \text{NoBranch} = \bigcap_{v \in V \setminus \partial V} \text{NoBranch}(v) \) be the event that no interior vertex \( v \in \tilde{T} \), has a child not in \( \tilde{T} \);
\( \text{LeafBranch} = \bigcap_{v \in \partial B} \text{Branch}(v) \) be the event that the \( p \)-percolation cluster branches at every leaf of \( T \).

Letting \( \text{SubTree}(T, V) \) denote the set of subtrees \( \tilde{T} \subseteq T \) for which \( \tilde{T} \) has the same root as \( T \), \( \Phi(\tilde{T}) \) is isomorphic to \( V \), and which satisfy the property that if \( v \in \tilde{T} \) has only one child in \( T \) then that child is also in \( \tilde{T} \), we may write
\[ \langle T_p, V, F \rangle = \sum_{\tilde{T} \in \text{SubTree}(T, V)} F(V, \tilde{T}) F(T_p(V) = \tilde{T}) \]
(note that \( \tilde{T} \) does not need to be an initial subtree of \( T \), since we are not making the assumption that all vertices in \( T \) are either in \( \tilde{T} \) or are descendants of vertices in \( \tilde{T} \)). Taking expectations now yields
\[ \mathcal{D}(T, F, V, p) = \sum_{\tilde{T} \in \text{SubTree}(T, V)} \mathcal{D}|_{\tilde{T}} \]

where
\[ \mathcal{D}|_{\tilde{T}} := \mathbf{P}(T_p(V) = \tilde{T}) F(V, \tilde{T}) \]
\[ = \mathbf{P}_T(\text{Open}) \mathbf{P}_T(\text{NoBranch}) \mathbf{P}_T(\text{LeafBranch}) F(V, \tilde{T}) \]
\[ = p^{|E(\tilde{T})|} \prod_{w \in \partial \tilde{T}} (1 - pg(T(w), p)) \prod_{v \in \partial \tilde{T}} g_2(T(v), p) F(V, \tilde{T}) \]
(4.10)

with \( g_2(T, p) \) denoting the probability \( T_p \) branches at the root. Each summand \( \mathcal{D}|_{\tilde{T}} \) in (4.10) is continuous in \( p \), so it suffices to show that for \( \mathbb{G}_\infty \)-almost every \( T \), the sum
converges uniformly on any compact subinterval \([a, b] \subseteq (p_c, 1)\). For any \(r > 0\) there are only finitely many \(\tilde{T} \in \text{SubTree}(T, \mathcal{V})\) for which \(F(\mathcal{V}, \tilde{T}) > (1 + r)^{\|E(\tilde{T})\|}\). We may therefore choose \(C_r\) such that \(F(\mathcal{V}, \tilde{T}) \leq C_r(1 + r)^{\|E(\tilde{T})\|}\) for all \(\tilde{T}\). The result now follows similarly to as in the proof of Theorem 4.1. Cover \((p_c, 1)\) by countably many intervals \((p', (1 + r)p')\) on which

\[
\mathcal{D}|_{\tilde{T}} \leq \left( C_r(1 + r)^{\|E(\tilde{T})\|} \right) \left( p' \right)^{\|E(\tilde{T})\|} (1 + r)^{\|E(\tilde{T})\|} \prod_{w \notin \tilde{T}} \prod_{v \in \partial \tilde{T}} (1 - p' g(T(w), p') )
\]

Next note that

\[
\mathcal{E}R(T, 2r + r^2, \mathcal{V}, p') = \mathbb{E} \left[ \sum (p')^{\|E(\tilde{T})\|} (1 + 2r + r^2)^{\|E(\tilde{T})\|} \prod_{w \notin \tilde{T}} \prod_{v \in \partial \tilde{T}} (1 - p' g(T(w), p') ) \prod_{v \in \tilde{T}} g_2(T(v), p') \right]
\]

\[
= \left( g_2(p') \right)^{\|\mathcal{V}\|} \mathbb{E} \left[ \sum (p')^{\|E(\tilde{T})\|} (1 + 2r + r^2)^{\|E(\tilde{T})\|} \prod_{w \notin \tilde{T}} \prod_{v \in \partial \tilde{T}} (1 - p' g(T(w), p') ) \right]
\]

where the two sums are taken over all \(\tilde{T} \in \text{SubTree}(T, \mathcal{V})\). By Lemma 4.10, we know that if \((1 + 2r + r^2)A_p < 1\), then the above expectations are all finite. It then follows that the sum over \(\text{SubTree}(T, \mathcal{V})\) of the expression on the bottom in (4.11) is finite for \(\mathbb{G}\)-almost every \(T\), thus implying uniform convergence of the sum of the \(\mathcal{D}|_{\tilde{T}}\) expressions on \((p', (1 + r)p')\).

Now covering \((p_c, 1)\) by a countable collection of such intervals, we see that \(\mathcal{D}(T, F, \mathcal{V}, p)\) is both finite and continuous \(\mathbb{G}\)-almost surely, thus completing the proof. \(\square\)

**Proof of Proposition 4.9** Step 1: Notation to state precise conclusion. Given a finite collapsed tree \(\mathcal{V}\) and a monomial \(F\), we define several perturbations of \(\mathcal{V}\) and corresponding perturbations of \(F\) and \(\mathcal{D}\).

(i) Let \(v\) be an interior vertex of \(\mathcal{V}\), for which \(L \geq 0\) of its children are also interior vertices. Define \(V_i(v + 1)\) (for \(0 \leq i \leq L\)) to be \(\mathcal{V}\) with a child \(v_i\) added to \(v\) via an edge \(e_s\) placed anywhere between the \(i\)th and \((i + 1)\)th of the \(L\) children of \(v\) that are interior vertices (where we are counting from left to right). Define the monomial \(F_{v+1}^{(i)}(e_s) = 1\) and \(F_{v+1}^{(i)}(e) = F(e)\) for \(e \neq e_s\). Define

\[
\mathcal{D}_{v+1}(T, F, \mathcal{V}, p) := \sum_{i=1}^{L} \mathcal{D}(T, F_{v+1}^{(i)}, V_i(v + 1), p).
\]

(ii) If \(v\) is a vertex in \(\partial \mathcal{V}\), i.e. a leaf, let \(\mathcal{V}(v + 2)\) denote the result of adding two children \(v_0\) and \(v_1\) to \(v\) connected by edges \(e_0\) and \(e_1\) respectively. For \(j = 0, 1\), define a monomial \(F_j\) on the edges of \(\mathcal{V}(v + 2)\) by \(F_j = F\) on edges of \(\mathcal{V}\), \(F_j = 1\) on \(e_j\) and \(F_j = 0\) on \(e_{1-j}\). Define

\[
\mathcal{D}_{v+2}(T, F, \mathcal{V}, p) := \mathcal{D}(T, F_0, \mathcal{V}(v + 2), p) + \mathcal{D}(T, F_1, \mathcal{V}(v + 2), p).
\]
(iii) If $e$ is any edge in $E(V)$, define two collapsed trees $V(e; L)$ and $V(e; R)$ by subdividing $e$ at a new midpoint $m$ into edges $e_0$ and $e_1$, then adding a new child of $m$ with an edge $e_*$ going to the left of $e_1$ in $V(e; L)$ and to the right of $e_1$ in $V(e; R)$. For $j = 0, 1$, define monomials $F_{j,L}$ on the edges of $V(e; L)$ by $F_{j,L}(e') = F(e')$ for $e' \in E(V)$ not equal to $e$, $F_{j,L}(e_1) = 1$, $F_{j,L}(e_{1-j}) = 0$, and $F_{j,L}(e_*) = 1$. Define $F_{j,R}$ analogously on the edges of $V(e; R)$ and define

$$D_{e+2}(T, F, V, p) := \sum_{j=0}^{1} \sum_{A \in \{L, R\}} D(T, F_{j,A}, V(e; A), p).$$

(iv) Finally, for any edge $e \in E(V)$, we define $F_e$ by $F_e(e') = F(e')$ when $e' \neq e$ and $F_e(e) = F(e) + 1$. In other words, the power of $dT(e)$ in the monomial is bumped up by one and nothing else changes. Define

$$D_e(T, F, V, p) := D(T, F_e, V, p).$$

The explicit version of (4.8) that we will prove is

$$\frac{d}{dp} D(T, F, V, p) = \frac{1}{p} \left[ \sum_{e \in E(V)} D_e(T, F, V, p) + \sum_{v \in V} D_{v+2}(T, F, V, p) \right.
\left. - \left( \sum_{e \in E(V)} D_{e+2}(T, F, V, p) + \sum_{v \in V \setminus \partial V} D_{v+1}(T, F, V, p) \right) \right]$$

(4.12)

Step 2: Convergence argument. The proof of (4.12) relies on the following fact allowing us to interchange a derivative with an infinite sum. If $B = \sum_{n=1}^{\infty} b_n$ converges everywhere on $(a, b)$ and $B_n$ are differentiable with derivatives $b_n$ such that $\sum_{n=1}^{\infty} b_n$ is continuous and $\sum_{n=1}^{\infty} |b_n(x)|$ converges to a function which is integrable on any interval $(r, s)$ for $a < r < s < b$, then $B$ is differentiable on $(a, b)$ and has derivative $\sum_{n=1}^{\infty} b_n$. This follows, for example, by applying the dominated convergence theorem to show that $\int_{c}^{x} \sum_{n=1}^{M} b_n(t) \, dt$ converges to $B(x) - B(c)$ for $a < c < x < b$. The point of arguing this way is that it is good enough to have convergence of $\sum b_n$ to a continuous function and $\sum |b_n|$ to an integrable function; we do not need to keep demonstrating uniform convergence.

Step 3: Differentiating $D|_{\tilde{T}}$. We will apply this fact to (4.10) with $B = D(T, F, V, p)$, and $\{B_n\} = \{D|_{\tilde{T}}\}$. Each component function $D|_{\tilde{T}}$ is continuously differentiable on $(p_c, 1)$ because $g$ and $g_2$ themselves are. Differentiating $D|_{\tilde{T}}$ via the product rule yields three terms, call them $b_{\tilde{T}, j}$ for $1 \leq j \leq 3$:

$$\frac{d}{dp} D|_{\tilde{T}} = b_{\tilde{T}, 1} + b_{\tilde{T}, 2} + b_{\tilde{T}, 3}$$

$$= \left[ \frac{d}{dp} P_T(\text{Open}) \right] P_T(\text{NoBranch}) P_T(\text{LeafBranch}) F(V, \tilde{T})$$

$$+ P_T(\text{Open}) \left[ \frac{d}{dp} P_T(\text{NoBranch}) \right] P_T(\text{LeafBranch}) F(V, \tilde{T})$$

$$+ P_T(\text{Open}) P_T(\text{NoBranch}) \left[ \frac{d}{dp} P_T(\text{LeafBranch}) \right] F(V, \tilde{T}).$$
Quenched Survival of Bernoulli Percolation on Galton–Watson Trees

The three sums $b_j := \sum_{T \in \text{SubTree}(T, V)} b_{T, j}$ will be identified as the three terms on the right-hand side of (4.12), each of which is known, by Lemma 4.11, to be finite and continuous on $(p_c, 1)$. All the terms in $\sum_T b_{T, j}$ have the same sign, namely positive for $j = 1, 2$ and negative for $j = 3$. Therefore their absolute values sum to an integrable function. It follows that $D$ is continuously differentiable on $(p_c, 1)$ with the explicit derivative given by (4.12). It remains only to identify the sums $b_1, b_2$ and $b_3$.

Step 4: Identifying jump rates. Recall from Lemma 4.4 the Markov chain $\{T_p : 1 \geq p \geq 0\}$. This chain enters the set $\{T(V) = \tilde{T}\}$, which is equal to $\text{Open} \cap \text{NoBranch} \cap \text{LeafBranch}$, exactly when the function NoBranch, which is decreasing in $p$, jumps from 0 to 1 and the chain is already in the set $\text{Open} \cap \text{LeafBranch}$. The chain leaves the set $\{T(V) = \tilde{T}\}$ exactly when it is in the set and either $\text{Open}$ or $\text{LeafBranch}$ jumps from 1 to 0. The three terms in the product rule correspond to these three possibilities. Figure 3 shows an example of this: $V$ is the illustrated 5-edge tree; $T_p(V)$ and $T_{p+h}(V)$ are respectively $\tilde{T}$ and $\tilde{T}'$; the jump into $\tilde{T}$ as $p$ decreases occurs because NoBranch at the parent of $u$ flips from 0 to 1, while the jump out of $\tilde{T}'$ is attributed to LeafBranch flipping from 1 to 0 at the right child of $u$; the edge weight on the edge between $u$ and its parent also changes.

Fig. 3 $T$ is in black while $T_p$ and $T_{p+h}$ are in red. The trees $T_p(V)$ and $T_{p+h}(V)$ are initial subtrees of the red tree. Between times $p$ and $p + h$, some pivotal bond in the right subtree of $\Phi^{-1}_{p+h}(u)$ flips (this having previously been the right subtree of the parent of $\Phi^{-1}_{p}(u)$), causing this portion of $T_{p+h}$ to branch earlier than $T_p$. Further portions of $T$ are omitted in the figure of $T_{p+h}$ to stress that they do not contribute to $T_{p+h}(V)$.

The jump rate out of Open, call it $\rho_1$, is the easiest to compute. Jumps out of Open occur precisely when an edge in $\tilde{T}$ closes, which happens at rate $p^{-1}E(\tilde{T})$. Writing $|E(\tilde{T})|$ as $\sum_{e \in E(V)} d(e)$ shows that $\rho_1 F(V, \tilde{T}) = \sum_{e \in E(V)} p^{-1} F_e(V, \tilde{T})$. Multiplying $\rho_1 F(V, \tilde{T})$ by $P(T_p(V) = \tilde{T})$ and summing over $\tilde{T} \in \text{SubTree}(V, \tilde{T})$ gives $\sum_{e \in E(V)} p^{-1} D(T, F_e, V, p)$, which is the first term in (4.12).

Next we compute the jump rate out of LeafBranch. If such a jump takes place, let $q$ represent the time at which it occurs. Because LeafBranch is an intersection over $v \in \partial V$ of events that $\iota(v)$ has at least two children in $T_p$ (recall that $\iota$ refers to the embedding map defined near the beginning of this section), with probability 1 there is a unique $v_s \in \partial V$ such that $\iota(v_s)$ has at least two children in $T_q$ and at most one child in $T_{q-1}$, which means $\iota(v_s)$ has precisely two children in $T_q$. It follows, recalling the construction of $D_{v+2}$ at the beginning of this proof, that $V(v_s + 2) \leq T_q$ and that closure of an edge $e \in T_q(\iota(v_s))$ causes an exit from LeafBranch if and only if $e \in \iota(e_0) \cup \iota(e_1)$; the edge must occur before the subtree $T_q(\iota(v_s))$ branches again. The rate $\rho_2$ is therefore equal to $\sum_{v_s \in \partial V} p^{-1} (d(e_0) + d(e_1))$, where $e_j$ are the edges added to $V$ in $V(v_s + 2)$. Summing this last expression for $\rho_2$ over all the possible
\(\tau(e_0), \tau(e_1)\) combinations (multiplied by their individual probabilities) and then multiplying by \(F(V, \tilde{T})P(T_q(V) = \tilde{T})\) and summing over \(\tilde{T}\) gives the second term in (4.12).

Finally, we compute the jump rate into NoBranch. Such a jump can only occur in the following manner. After the jump, \(T_{q-1}(V) = \tilde{T}\); at the time of the jump, \(T_q(w)\) is infinite for precisely one child \(w\) outside of \(\tilde{T}\) whose parent \(v\) is an interior vertex of \(\tilde{T}\); after the jump, one of the edges in \(T_q(V)\), either leading back to the parent or forward but somewhere before the first branch, closes.

There are two possibilities. First, the parent \(v\) of \(w\) might have at least two children in \(T_q\) other than \(w\). In that case, \(\nu_i(\Phi(v) + 1) \leq T_q\) where the index \(i\) refers to the relative placement of \(w\) among the other children of \(v\) (see definition (i) at the beginning of the proof), and the edge that closes at time \(q\) is in \(\iota(e_s)\), where \(e_s\) is the edge added to \(V\) to obtain \(\nu_i(\Phi(v) + 1)\). The rate at which this happens is \(\rho_3 = p^{-1}d(e_s)\). Summing over all possible \(\iota(e_s)\) (multiplied by their probabilities), and then multiplying by \(F(V, \tilde{T})P(T_q(V) = \tilde{T})\) and summing over \(i\) and then \(\tilde{T}\) gives \(p^{-1}D_{\Phi(v)+1}(T, F, V, p)\). The other possibility is that the parent of \(w\) has only one other child in \(T_q\). Let \(e\) denote the descending edge from \(\Phi(v)\). In this case, depending on whether \(w\) is to the left or right of that child, \(\nu(e; L) \leq T_q\) or \(\nu(e; R) \leq T_q\). In either case, the jump rate is \(\rho_4 := p^{-1}d(e_s)\), where as in construction (iii), the edge \(e_s \in E(\nu(e; L/R))\) corresponds to the path in \(T_q\) containing the edge connecting \(v\) to \(w\). Now taking the sum of \(\rho_4\) over all possible paths \(\iota(e_s)\) (multiplied by their probabilities) and then multiplying by \(F(V, \tilde{T})P(T_q(V) = \tilde{T})\) and summing over the left and right cases and then over all \(\tilde{T} \in \text{SubTree}\), we obtain \(p^{-1}D_{e+2}(T, F, V, p)\). Next taking the sum over all \(v \in V \setminus \partial V\) for the former case (where \(v\) had at least two children other than \(w\)) and summing over all \(e \in E(V)\) for the latter, we find that the jump rate into NoBranch is indeed equal to

\[
p^{-1}\left( \sum_{e \in E(V)} D_{e+2}(T, F, V, p) + \sum_{v \in V \setminus \partial V} D_{v+1}(T, F, V, p) \right)
\]

which is the subtracted term in (4.12). \(\square\)

The next to last step in proving Theorem 4.6 is to apply Proposition 4.9 to repeated derivatives to obtain the following representation of the higher order derivatives of \(g(T, p)\) on \((p_c, 1)\).

**Lemma 4.12** Under the same hypotheses, for any \(k \geq 1\), there exists a finite set of monomials of degree at most \(k + \deg(F)\), call them \(\{F_\alpha : \alpha \in \mathcal{A}\}\), constants \(\{C_\alpha : \alpha \in \mathcal{A}\}\), and corresponding collapsed trees \(\{V_\alpha : \alpha \in \mathcal{A}\}\) of size at most \(2k + |E(V)|\), such that for \(\mathcal{G}\)-almost every tree \(T\),

\[
\left( \frac{d}{dp} \right)^k D(T, F, V, p) = \sum_{\alpha \in \mathcal{A}} C_\alpha p^{-k-1 + \deg(F_\alpha) - \deg(F)} D(T, F_\alpha, V_\alpha, p) \quad (4.13)
\]

on \((p_c, 1)\).

**Proof** Differentiate (4.8) a total of \(k - 1\) more times, using Proposition 4.9 to simplify each time. Each time a derivative is taken, either it turns \(p^{-j}\) into \(-jp^{-j-1}\) for some \(j\), or else a term \(D(T, F', V', p)\) is replaced by a sum of terms \(D(T, F'', V'', p)\) where \(\deg(F'') = 1 + \deg(F')\) and \(|E(V'')| \leq 2 + |E(V')|\). The lemma follows by induction. \(\square\)
Proof of Theorem 4.6 Applying Lemma 4.12 to (4.7), we see that there are a pair of finite index sets $A$ and $B$, constants $\{C_\alpha : \alpha \in A\}$ and $\{C_\beta : \beta \in B\}$, collapsed trees $\{V_\alpha : \alpha \in A\}$ and $\{V_\beta : \beta \in B\}$, and corresponding monomials $\{F_\alpha : \alpha \in A\}$ and $\{F_\beta : \beta \in B\}$, such that

\[
\left(\frac{d}{dp}\right)^{k+1} g(T, p) = \left(\frac{d}{dp}\right)^k p^{-1} D(T, F_1, V_1, p)
\]

\[
= \sum_\alpha C_\alpha p^{-k-2+\deg F_\alpha} D(T, F_\alpha, V_\alpha, p)
\]

\[
+ \sum_\beta C_\beta p^{-k-3+\deg F_\beta} D(T, F_\beta, V_\beta, p)
\]

(note the need for the pair of distinct sums is on account of the $p^{-1}$ term in front of $D(T, F_1, V_1, p)$). It follows from this that the $k$th derivative from the right of $g(T, p)$ exists for $GW$-almost every tree $T$ and is given by an expression which is continuous in $p$. □

4.3 Continuity of the Derivatives at $p_c$

We now address the part of Theorem 1.2 concerning the behavior of the derivatives of $g$ near criticality. We restate this result here as the following Theorem.

Theorem 4.13 If $E[Z^{(2k^2+3)(1+\beta)}] < \infty$ for some $\beta > 0$, then

\[
\lim_{p \to p_c^+} g^{(j)}(T, p) = j!M^{(j)}
\]

for every $j \leq k$ $GW$-a.s. where $M^{(j)}$ are as in Theorem 3.1.

To prove Theorem 4.13 we need to bound how badly the monomial expectations $D(T, F, V, p_c + \epsilon)$ can blow up as $\epsilon \downarrow 0$, then use Lemma 2.1 to see that they can’t blow up at all.

Proposition 4.14 Let $V$ be a collapsed tree with $\ell$ leaves and $E$ edges and let $F$ be a monomial in the edges of $V$. Suppose that the offspring distribution has at least $m$ moments, where $m \geq \max_e F(e)$ and also $m \geq 3$. Then

\[
D(T, F, V, p_c + \epsilon) = O(\epsilon^\lambda)
\]

for any $\lambda < 2\ell - E - \deg(F)$ and $GW$-almost every $T$.

To prove this, we first record the asymptotic behavior of the following annealed quantities at $p_c + \epsilon$, where $K = 2/(p_c^3\phi''(1))$ as in Proposition 2.2.

Lemma 4.15 Assume $\phi$ has at least three moments. Then as $\epsilon \to 0^+$

\[
g(p_c + \epsilon) \sim K\epsilon
\]

\[
1 - A_p \sim \mu \epsilon
\]

\[
g_2(p_c + \epsilon) \sim K \mu \epsilon^2.
\]

Proof The first of these is Proposition 2.2. For the second, we recall from the definition that

\[
A_p = \left. \frac{d}{dz} \phi'(z) \right|_{z=0}
\]

and differentiate (4.1) at $z = 0$ to obtain

\[
A_p = p\phi'(1 - pg(p)).
\]
Differentiating with respect to \( p \) at \( p = p_c \) and using \( g(p_c) = 0 \), \( \phi'(1) = \mu \) and \( g'(p_c) = K \) then gives

\[
\frac{d}{dp} A_p \bigg|_{p_c} = \phi'(1 - p_c g(p_c)) - [g(p_c) + p_c g'(p_c)]p_c \phi''(1 - p_c g(p_c)) \\
= \phi'(1) - p_c^2 K \phi''(1) \\
= -\mu
\]

which proves the second estimate. The third we can obtain by first calculating \( \mathbf{P}(Z_p \geq 2) \) (recall \( Z_p \) represents the size of the first generation of \( T_p \) conditioned on \( H(p) \)), which is simply equal to \( 1 - A_p \). Now multiplying by \( g(p) \) and using the first two estimates for \( 1 - A_p \) and \( g(p) \) near \( p_c \), we obtain the third estimate.

\[ \text{Proof of Proposition 4.14} \] We begin by computing annealed expectations. Observe that

\[ \ell = 1 + \sum_{v \in \mathcal{V} \setminus \partial \mathcal{V}} (\deg_v - 1) \]

where \( \deg_v \) is the number of children of \( v \). Observe next that

\[ \mathbf{P}(Z_p \geq k) \leq \frac{1}{g(p_c + \varepsilon)} \mathbf{E}\left(\frac{Z_p}{k}\right)^{(p_c + \varepsilon) g(p_c + \varepsilon)} = O(\varepsilon^{k-1}) \]

under the assumption of at least \( m \) moments with \( k \leq m \) (where \( Z \) represents the unconditioned offspring r.v.). If \( k > m \) then the above equality still gives us \( \mathbf{P}(Z_p \geq k) \leq \mathbf{P}(Z_p \geq m) = O(\varepsilon^{m-1}) \). Putting these together we now obtain the expression

\[ \mathbf{P}(Z_p \geq k) = O(\varepsilon^{(k \wedge m)-1}) \] (4.14)

whenever we have at least \( m \) moments. For future use, we also note here that if we condition on \( Z_p \geq 2 \) in the above inequalities, then the denominator in the expression above \((4.14)\) becomes \( g_2(p_c + \varepsilon) \), and the expression on the right in \((4.14)\) becomes \( O(\varepsilon^{(k \wedge m)-2}) \). Finally, recall as well that the collapsed annealed tree \( \Phi(T_{p_c + \varepsilon}) \) has edge weights that are IID geometric random variables with means \( 1/(1 - A_{p_c + \varepsilon}) \sim \mu^{-1}\varepsilon^{-1} \).

Using these, we estimate the annealed expectation of \( \mathcal{D}(T, F, \mathcal{V}, p_c + \varepsilon) \) by using the branching process description of \( \Phi(T_{p_c + \varepsilon}) \) to write the quantity \( ED(T, F, \mathcal{V}, p_c + \varepsilon) \) as the probability of the event \( H(p) \) that \( T_p \) is nonempty, multiplied by the product of the following expectations which, once we condition on \( H(p) \), are jointly independent. Assume that \( Z \) has at least \( m \) moments for \( m = \max \varepsilon F(e) \). The event \( H(p_c + \varepsilon) \) has probability \( g(p_c + \varepsilon) = O(\varepsilon) \) for \( T_p \). Conditional on \( H(p_c + \varepsilon) \), there is a factor of \( \mathbf{E}G^F(e) = O(\varepsilon^{-F(e)}) \) for each edge. Lastly, because the non-root vertices of \( \Phi(T_{p_c + \varepsilon}) \) consist precisely of those non-root vertices of \( T_{p_c + \varepsilon} \) that posses more than one child, this amounts to conditioning on having at least two offspring. Therefore, there is a factor of \( O(\varepsilon^{-\deg_v - 1}) \) for \( v \) equal to the root, and a factor of \( O(\varepsilon^{-\deg_v - 2}) \) for every other interior vertex of \( \mathcal{V} \). Multiplying all of these gives

\[ ED(T, F, \mathcal{V}, p_c + \varepsilon) = O\left(\varepsilon^{2\ell - \deg F}\right) \]

(note the exponent is simply a different way of writing \( \ell - |\mathcal{V} \setminus \partial \mathcal{V}| + 1 - \deg F \)).

The intuition behind the rest of the proof is as follows. By Markov’s inequality, the quenched expectation cannot be more than \( \varepsilon^{-\delta} \) times this except with probability \( \varepsilon^\delta \). As \( \varepsilon \) runs over powers of some \( r < 1 \), these are summable, hence by Borel–Cantelli, this threshold
is exceeded finitely often. This only proves the estimate along the sequence $p_c + r^n$. To complete the argument, one needs to make sure the quenched expectation does not blow up between powers of $r$. This is done by replacing the monomial expectation with an expression $\Psi(T, F, V, \epsilon_1, \epsilon_2)$ that is an upper bound for the quenched expectation $D(T, F, V, x)$ as $x$ varies over an interval $[p_c + \epsilon_1, p_c + \epsilon_2]$.

Accordingly, fix $\epsilon_1 < \epsilon_2$ and let $x$ vary over the interval $[p_c + \epsilon_1, p_c + \epsilon_2]$. Extend the notation in an obvious manner, letting $P_T(\text{Open}(\tilde{T}, x))$ denote the probability of all edges of $\tilde{T}$ being open at parameter $x$, and similarly for NoBranch and LeafBranch. Fixing $T$, by monotonicity,

$$P_T(T_x(V) = \tilde{T}) = P_T(\text{Open}(\tilde{T}, x))P_T(\text{NoBranch}(\tilde{T}, x))P_T(\text{LeafBranch}(\tilde{T}, x)) \leq P_T(\text{Open}(\tilde{T}, \epsilon_2))P_T(\text{NoBranch}(\tilde{T}, \epsilon_1))P_T(\text{LeafBranch}(\tilde{T}, \epsilon_2)).$$

Thus we may define the upper bound $\Psi$ by

$$\Psi(T, F, V, \epsilon_1, \epsilon_2) = \sum_{\tilde{T} \in \text{SubTree}(T, V)} P_T(\text{Open}(\tilde{T}, \epsilon_2))P_T(\text{NoBranch}(\tilde{T}, \epsilon_1))P_T(\text{LeafBranch}(\tilde{T}, \epsilon_2))F(\tilde{T}, V).$$

Taking the expectation,

$$E\Psi(T, F, V, \epsilon_1, \epsilon_2) = \int \left[ \sum_{\tilde{T} \in \text{SubTree}(T, V)} P_T(\text{Open}(\tilde{T}, \epsilon_2))P_T(\text{NoBranch}(\tilde{T}, \epsilon_1))P_T(\text{LeafBranch}(\tilde{T}, \epsilon_2)) \right] F(\tilde{T}, V) d\mathcal{GW}(T).$$

In each summand, first integrate over the variables $\text{deg}_w$ for $w \geq v$ with $v \in \partial \tilde{T}$. This replaces $P_T(\text{LeafBranch}(\tilde{T}, \epsilon_2))$ by $g_2(p_c + \epsilon_2)^{|\partial \tilde{T}|}$. Similarly, integrating $P_T(\text{LeafBranch}(\tilde{T}, \epsilon_1))$ over just these variables would replace this with a factor of $g_2(p_c + \epsilon_1)^{|\partial \tilde{T}|}$. Therefore, using that $|\partial \tilde{T}| = |\partial V|$ and noticing also that $P_T(\text{Open}(\tilde{T}, \epsilon_2)) = ([p_c + \epsilon_2]/(p_c + \epsilon_1))^{|E(\tilde{T})|}P_T(\text{Open}(\tilde{T}, \epsilon_1))$, we see that

$$E\Psi(T, F, V, \epsilon_1, \epsilon_2) = \int \left[ \sum_{\tilde{T} \in \text{SubTree}(T, V)} \left( \frac{g_2(p_c + \epsilon_2)}{g_2(p_c + \epsilon_1)} \right)^{|\partial V|} \left( \frac{p_c + \epsilon_2}{p_c + \epsilon_1} \right)^{|E(\tilde{T})|} P_T(\text{Open}(\tilde{T}, \epsilon_1)) \cdot P_T(\text{NoBranch}(\tilde{T}, \epsilon_1)) \cdot P_T(\text{LeafBranch}(\tilde{T}, \epsilon_1)) \right] F(\tilde{T}, V) d\mathcal{GW}(T)$$

$$= \int \left[ \sum_{\tilde{T} \in \text{SubTree}(T, V)} \left( \frac{g_2(p_c + \epsilon_2)}{g_2(p_c + \epsilon_1)} \right)^{|\partial V|} \left( \frac{p_c + \epsilon_2}{p_c + \epsilon_1} \right)^{|E(\tilde{T})|} P_T(T_{p_c+\epsilon_1}(V) = \tilde{T}) \right] F(\tilde{T}, V) d\mathcal{GW}(T).$$

To integrate over $d\mathcal{GW}(T)$, recall that the edge weights $d(e)$ in $\Phi(T_{p_c+\epsilon_1})$ are IID geometrics with mean $1/(1 - A_{p_c+\epsilon_1})$ and independent from the degrees $\text{deg}_v$. Let $C$ denote any upper bound for $P(\mathcal{H}(p))$ times the product over interior vertices $v \in V$ of the quantity $\epsilon_1^{\text{deg}_v - j}p_j^{Z_p = \text{deg}_v} |Z_p \geq j|$, which is finite by (4.14) (note we’re assuming here that $j = 1$ for the root, which means no conditioning, and 2 for all other interior vertices). Let $G$ denote such a geometric random variable as referenced above, let $\alpha$ denote an upper bound for $g_2(p_c + \epsilon_2)/g_2(p_c + \epsilon_1)$ and $1 + \delta$ denote an upper bound for $(p_c + \epsilon_2)/(p_c + \epsilon_1)$. Integrating against $d\mathcal{GW}(T)$ now yields

$$E\Psi(T, F, V, \epsilon_1, \epsilon_2) \leq \alpha^{|\partial V|} \cdot C \epsilon_1^{2\ell-E} \cdot \prod_{e \in E(V)} [E(1 + \delta)^GG^{F(e)}].$$
Because $P(G = k) = (1 - A_p)A_p^{k-1}$, we see that for any $f$, $(1 + \delta)^G f(G)$ may be computed as

$$E(1 + \delta)^G f(G) = \sum_{k=1}^{\infty} (1 + \delta)^k (1 - A_p)A_p^{k-1} f(k) = \frac{(1 - A_p)(1 + \delta)}{1 - (1 + \delta)A_p} \sum_{k=1}^{\infty} (1 - (1 + \delta)A_p)((1 + \delta)A_p)^{k-1} f(k)$$

$$= \frac{(1 - A_p)(1 + \delta)}{1 - (1 + \delta)A_p} E f(G')$$

where $G'$ is a geometric r.v. with parameter $(1 + \delta)A_p$. Thus,

$$E\Psi(T, F, V, \varepsilon_1, \varepsilon_2) \leq C\alpha^{[\beta]_1} \varepsilon_e^{2\ell - E} \frac{(1 - A_p)(1 + \delta)}{1 - (1 + \delta)A_p} \prod_{e \in E(V)} E(G')^{F(e)}. \tag{4.15}$$

If we choose $\varepsilon_2 < \frac{3}{2}\varepsilon_1$ then $\delta = (p_c + \varepsilon_2)/(p_c + \varepsilon_1) - 1 < \frac{p_c}{p_c + \varepsilon_1}$ which implies $1 - (1 + \delta)A_p > (1/2)(1 - A_p)$ as $\varepsilon_1 \to 0$. This in turn implies that as $\varepsilon_1 \to 0$ we have $E(G')^k \leq 2^k E G^k = O(\varepsilon^{-k})$, and hence

$$E\Psi(T, F, V, \varepsilon_1, \varepsilon_2) = O\left(\varepsilon_e^{-2\ell - E - \deg(F)}\right). \tag{4.16}$$

Now the Borel–Cantelli argument is all set up. Let $\varepsilon_n := \left(\frac{4}{3}\right)^{-n}$ and apply the previous argument with $\varepsilon_2 = \varepsilon_n$ and $\varepsilon_1 = \varepsilon_{n+1}$. We see that

$$E\Psi(T, F, V, \varepsilon_{n+1}, \varepsilon_n) = O\left(\varepsilon_e^{-2\ell - E - \deg(F)}\right)$$

and hence by Markov’s inequality,

$$P\left(\Psi(T, F, V, \varepsilon_{n+1}, \varepsilon_n) > \varepsilon_e^{2\ell - E - \deg(F) - t}\right) = O\left(\left(\frac{4}{3}\right)^{-tn}\right).$$

This is summable, implying these events occur finitely often, implying that the quenched survival function satisfies

$$D(T, F, V, p_c + \varepsilon) = O\left(\varepsilon_e^{2\ell - E - \deg(F) - t}\right)$$

for $\mathbb{G}_\varepsilon$-almost every $T$. \hfill \Box

Before presenting the proof of Theorem 4.13, we just need to establish one final lemma.

**Lemma 4.16** Every non-empty collapsed tree $V$ satisfies the inequality $2\ell - E \geq 1$.

**Proof** Certainly the lemma holds for any $V$ of height one. Now if $V$ represents any non-empty collapsed tree for which the lemma applies and we add $k$ children to one of the boundary vertices of $V$ (note we can assume $k \geq 2$ since all non-root interior vertices of a collapsed tree must have at least 2 children), then the value of $2\ell - E$ is increased by $2(k - 1) - k = k - 2 \geq 0$, which means the lemma still applies to the new collapsed tree. Since any collapsed tree can be obtained from a height one tree via finitely many of these steps, the lemma then follows. \hfill \Box

**Proof of Theorem 4.13** Presume the result holds for all $j < k$. Lemma 4.12 expresses $(d/dp)^{k+1} g(T, p)$ on $(p_c, 1)$ as a sum of terms of the form $D(T, F, V_\alpha, p)$ with $\deg(F_\alpha) \leq k + 1$ and $|E(V_\alpha)| \leq 2k + 1$. By Proposition 4.14, because our moment assumption implies the weaker moment assumption of $k + 1$ moments, each summand on the right-hand side of (4.13) is $O(\varepsilon_e^{2\ell - E - \deg(F)})$. The worst case is $\deg(F) = k + 1$ and $2\ell - E = 1$ (see lemma above). Therefore, the whole sum satisfies

$$\left|g^{(k+1)}(T, p)\right| = O\left(\varepsilon^{k-1}\right) \tag{4.17}$$
for any $t > 0$ and $\text{GW}$-almost every $T$.

Suppose now for contradiction that $g^{(k)}(T, p_c + \varepsilon)$ does not converge to $k! c_k$ as $\varepsilon \downarrow 0$. We apply Lemma 2.1 with $N = k^2 + 1$. To check the hypotheses, note that the existence of the order-$N$ expansion follows from Theorem 3.1 with $\ell = k^2 + 1$ and our assumption that $\mathbb{E}[Z^{(1+\beta)(2k^2+3)}] < \infty$. The induction hypothesis implies hypothesis $(2.3)$ of Lemma 2.1. Our assumption for proof by contradiction completes the verification of the hypotheses of Lemma 2.1. Seeing that $N/k = k + 1/k$, the conclusion of the lemma directly contradicts (4.17) when $t < 1/k$. Since the proof of the induction step also establishes the base case $k = 1$, the proof of the theorem is complete.

5 Open Questions

We conclude with a couple of open questions. Propositions 2.4 and 2.5 are converses of a sort but they leave a gap as to whether $g \in C^j$ from the right at $p_c^+$ for $k/2 \leq j \leq k$.

**Question 1** Do $k$ moments of the offspring distribution suffice to imply that the annealed survival function is $k$ times differentiable at $p_c^+$? More generally, is there a sharp result that $k$ moments but not $k + 1$ imply $j$ times differentiability but not $j + 1$ for some $j \in \lfloor k/2 \rfloor, k$?

Another question is whether the expansion of either the quenched or annealed survival function at $p_c$ has any more terms than are guaranteed by the continuity class.

**Question 2** Does it ever occur that $g$ has an order-$k$ expansion at $p_c^+$ but is not of class $C^k$ at $p_c^+$? Does this happen with positive probability for $g(T, \cdot)$?

Recall that the annealed survival function is analytic on $(p_c, 1)$ whenever the offspring generating function extends beyond 1. We do not know whether the same is true of the quenched survival function.

**Question 3** When the offspring distribution has exponential moments, is the quenched survival function $g(T, p)$ almost surely analytic on $(p_c, 1)$?

Acknowledgements The authors would like to thank Michael Damron for drawing our attention to [6]. We would also like to thank Yuval Peres for helpful conversations.

References

1. Addario-Berry, L., Ford, K.: Poisson–Dirichlet branching random walks. Ann. Appl. Prob. 23, 283–307 (2013)
2. Athreya, K., Ney, P.: Branching Processes. Springer, New York (1972)
3. Bingham, N., Doney, R.: Asymptotic properties of supercritical branching processes I: the Galton–Watson process. Adv. Appl. Prob. 6, 711–731 (1974)
4. Chow, Y.S., Teicher, H.: Probability Theory. Springer Texts in Statistics, 3rd edn. Springer, New York (1997). Independence, interchangeability, martingales
5. Duminil-Copin, H., Tassion, V.: A new proof of the sharpness of the phase transition for Bernoulli percolation and the Ising model. Commun. Math. Phys. 343(2), 725–745 (2016)
6. Fitzner, R., van der Hofstad, R.: Mean-field behavior for nearest-neighbor percolation in $d > 10$. Electron. J. Probab. 22(43), 65 (2017)
7. Grimmet, G.: Percolation. Grundlehren der mathematischen Wissenschaften, vol. 321, 2nd edn. Springer, New York (1999)
8. Hara, T., Slade, G.: Mean-field behaviour and the lace expansion. Probability and Phase Transition (Cambridge, 1993). NATO Advanced Science Institute Series C Mathematical and Physical Science, vol. 420, pp. 87–122. Kluwer Academic Publisher, Dordrecht (1994)
9. Harris, T.: A lower bound for the percolation probability in a certain percolation process. Proc. Camb. Philos. Soc. 56, 13–20 (1960)
10. Harris, T.: The Theory of Branching Processes. Springer, Berlin (1963)
11. Kesten, H.: The critical probability of bond percolation on the square lattice equals $\frac{1}{2}$. Commun. Math. Phys. 74(1), 41–59 (1980)
12. Kesten, H., Zhang, Y.: Strict inequalities for some critical exponents in two-dimensional percolation. J. Stat. Phys. 46(5–6), 1031–1055 (1987)
13. Lyons, R.: Random walks and percolation on trees. Ann. Probab. 18, 931–958 (1990)
14. Russo, L.: On the critical percolation probabilities. Z. Wahrsch. Verw. Gebiete 56(2), 229–237 (1981)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.