REGULARITY CRITERIA WITH ANGULAR INTEGRABILITY
FOR THE NAVIER–STOKES EQUATION

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Abstract. We give new a priori assumptions on weak solutions of the Navier–Stokes equation so as to be able to conclude that they are smooth. The regularity criteria are given in terms of mixed radial-angular weighted Lebesgue space norms.

1. Introduction and main results

We consider the Cauchy problem on $(0, T) \times \mathbb{R}^n$

$$\begin{aligned}
\partial_t u + (u \cdot \nabla) u - \Delta u &= -\nabla P \\
\nabla \cdot u &= 0 \\
 u(x, 0) &= u_0(x).
\end{aligned}$$

(1.1)

It describes the motion of a viscous incompressible fluid in the absence of external forces, where $u$ is the velocity field and $P$ is the pressure.

The first equation is the Newton law while the second follows by the incompressibility of the fluid. In order to require incompressibility at time $t = 0$ it is necessary to restrict to initial data $u_0$ such that $\nabla \cdot u_0 = 0$.

We shall use the same notation for the norm of scalar, vector or tensor quantities, for instance:

$$\|P\|_{L^2}^2 := \int P^2 \, dx, \quad \|u\|_{L^2}^2 := \int \sum_{i=1}^n u_i^2 \, dx, \quad \|\nabla u\|_{L^2}^2 := \int \sum_{i,j=1}^n (\partial_i u_j)^2 \, dx$$

and we often write simply $u \in L^2(\mathbb{R}^n)$ instead of $u \in [L^2(\mathbb{R}^n)]^n$.

The well-posedness of (1.1) is still open even if many partial results have been obtained. In [12, 17] the authors proved global existence of weak solutions for initial data in $L^2$ but a satisfactory well-posedness theory is basically developed only in the case of small initial data or data with a peculiar geometric structure.

In this scenario it is useful to establish a priori conditions under which uniqueness and regularity of the weak solutions are guaranteed. Results of this kind are usually called regularity criteria.

In this paper we focus on some classical regularity criteria [2, 21, 22, 24] and their extension to the setting of weighted Lebesgue spaces [26]. In particular we show how the results in [26] can be improved under the hypothesis of additional angular integrability.

The regularity is basically ensured by boundedness assumptions on quantities like $u, \nabla u, \nabla \times u$ in suitable critical spaces. A simple regularity criterion is for instance

$$\|u\|_{L^s_t L^p_x} := \left( \int_0^T \left( \int_{\mathbb{R}^n} |u(t, x)|^p \, dx \right)^{\frac{s}{p}} \, dt \right)^{\frac{1}{s}} < +\infty, \quad \frac{2}{s} + \frac{n}{p} \leq 1.$$ 

(1.2)

Notice that in the endpoint case (1.2) is invariant with respect to $u(t, x) \to \lambda u(\lambda^2 t, \lambda x)$,

(1.3)
that is the natural scaling of (1.1). In [21] smoothness in space variables has been obtained in the case \( \frac{2}{s} + \frac{n}{p} < 1 \), while the endpoints have been fixed in [9, 11, 22, 24, 28]. We recall the following

**Definition 1.1** ([2]). We say that a point \((\bar{t}, \bar{x}) \in (0, T) \times \mathbb{R}^3\) is regular for a solution \(u(t, x)\) of (1.1) if \(u\) is essentially bounded on a neighborhood of \((\bar{t}, \bar{x})\). (In this case one can prove that \(u(t, x)\) is smooth near \((\bar{t}, \bar{x})\), see for instance [21]). We say that a set is regular if all its points are regular.

Let us also recall that \((0, T) \times \mathbb{R}^n\) is regular provided that (1.2) is satisfied with \(2/s + n/p = 1\) (see for instance [22, 24]).

Then we focus on the weighted norm approach:

**Theorem 1.2** ([26]). Let \(n \geq 3\) and \(u_0 \in L^2(\mathbb{R}^n)\) be a divergence free vector field. Let then \(u\) be a weak solution of (1.1) and \(\bar{x} \in \mathbb{R}^n\) such that

\[
\|x - \bar{x}\|^{1 - \frac{2}{s}}u_0 \|_{L^2} < +\infty,
\]

(1.4)

\[
\|x - \bar{x}\|^{\alpha}u(x, t) \|_{L^\infty_{t}L^p_r} < +\infty,
\]

(1.5)

where

\[
\frac{2}{s} + \frac{n}{p} = 1 - \alpha, \quad -1 \leq \alpha < 1
\]

(1.6)

or

\[
\|x - \bar{x}\|^{\alpha}u(x, t) \|_{L^\infty_{t}L^p_r} < +\infty, \quad -1 < \alpha < 1
\]

(1.7)

or

\[
\sup_{t \in (0, T)} \|x - \bar{x}\|^{\alpha}u(x, t) \|_{L^\infty_{t}L^p_r} < \varepsilon, \quad -1 \leq \alpha \leq 1
\]

(1.8)

with \(\varepsilon\) sufficiently small. Then \((0, T) \times \{\bar{x}\}\) is a regular set.

**Remark 1.1.** The condition \(\frac{2}{s} + \frac{n}{p} = 1 - \alpha\) makes the norm

\[
\|x - \bar{x}\|^{\alpha}u(x, t) \|_{L^\infty_{t}L^p_r}
\]

scaling invariant with respect to

\[
u(t, x - \bar{x}) \rightarrow \lambda u(\lambda^2 t, \lambda(x - \bar{x})).
\]

Our goal is to point out the local aspect of Theorem 1.2: for each \(t \in (0, T)\) there is a neighborhood\(^1\) \(\Omega_{t, \bar{x}}\) of \(\bar{x}\) such that \(u\) is smooth in \(\{t\} \times \Omega_{t, \bar{x}}\).

The restriction to a neighborhood of \(\bar{x}\) can be heuristically explained in the case \(\alpha < 0\): the weight morally localizes the norm around \(\bar{x}\) and a loss of information at infinity occurs.

We shall show how to recover this information by a suitable amount of angular regularity (if \(\alpha < 0\)) and how to do the same in the case \(\alpha > 0\) even if weaker angular regularity is assumed.

By translations it is possible to restrict to the case \(\bar{x} = 0\). All the following results are of course true provided with the norms and weights centered at \(\bar{x} \neq 0\).

In order to quantify precisely our notion of angular regularity we define the norms

\[
\|f\|_{L^p_{t,x}L^\infty_r} := \left( \int_0^{+\infty} \|f(\rho \cdot)\|_{L^\infty_{t}L^p_r(\mathbb{R}^n)} \rho^{n-1} d\rho \right)^{\frac{1}{p}},
\]

\[
\|f\|_{L^\infty_{t,x}L^p_r} := \sup_{\rho > 0} \|f(\rho \cdot)\|_{L^p_{t}L^\infty_r(\mathbb{R}^n)}.
\]

If \(p = \tilde{p}\) the norms reduce to the usual \(L^p\) norms

\[
\|u\|_{L^p_{t,x}L^\infty_r} = \|u\|_{L^p(\mathbb{R}^n)},
\]

\(^1\) We mean a neighborhood in the space variables for each fixed time.
while for radial functions the value of \( \tilde{p} \) is irrelevant

\[
u \: \text{radial} \implies \| u \|_{L^\tilde{p}} \simeq \| u \|_{L^p} \quad \forall p, \tilde{p} \in [1, \infty].
\]

Notice also that the norms (ignoring the constants) are increasing in \( \tilde{p} \).

The idea of distinguishing radial and angular directions is not new and has proved successful in the context of Strichartz estimates and dispersive equations (see for instance \([1],[3],[10],[18],[19],[27]\)).

We also notice that the mixed angular-radial norms have the same scaling of their classical counterparts, in fact

\[
\| |x|^\alpha u(t, x) \|_{L^s_t L^p_x |x|^\theta} \text{ is invariant with respect to } u(t, x) \to \lambda u(\lambda^2 t, \lambda x),
\]

provided that

\[
\frac{2}{s} + \frac{n}{p} = 1 - \alpha.
\]

We obtain new values \( \tilde{p}_G, \tilde{p}_L \) for the angular integrability such that global and local \(^2\) regularity are, respectively, achieved:

\[
\tilde{p}_L := \begin{cases}
\frac{2(n-1)p}{(2\alpha+1)p+2(n-1)} & \text{if } -\frac{1}{2} \leq \alpha < 0 \\
\frac{2(n-1)p}{p+2(n-1)} & \text{if } 0 \leq \alpha < 1,
\end{cases}
\]

\[
\tilde{p}_G := \begin{cases}
\max\left(4, \frac{(n-1)p}{\alpha p + n-1}\right) & \text{if } \frac{1-n}{2} \leq \alpha < 0 \\
\frac{(n-1)p}{\alpha p + n-1} & \text{if } 0 \leq \alpha < 1,
\end{cases}
\]

Notice that neither the quantities are increasing in \( \alpha \),

\[
\tilde{p}_L < \tilde{p}_G, \quad \text{if } \alpha < 1/2, \quad \tilde{p}_L = \tilde{p}_G \quad \text{if } \alpha = 1/2;
\]

and \(^3\)

\[
\tilde{p}_L < \tilde{p}_G < p, \quad \text{if } \alpha > 0;
\]

this is in fact consistent with the previous heuristic. For simplicity we state our results in the case of Schwartz initial data. In Section 4 we show how to refine this assumption.

**Theorem 1.3.** Let \( n \geq 3 \) and \( u_0 \) be a divergence free vector field with each component in the Schwartz class. Let also \( u \) be a weak solution of (1.1). Then \((0, T) \times \mathbb{R}^n\) is a regular set provided that

\[
\alpha \in ((1-n)/2, 0), \quad \max\left(2, \frac{n}{1-\alpha}\right) < p \leq \frac{1-n}{\alpha}, \quad \text{or} \quad p = 2,
\]

and

\[
\| |x|^\alpha u \|_{L^s_t L^p_x |x|^\theta} < +\infty,
\]

with

\[
\frac{2}{s} + \frac{n}{p} = 1 - \alpha,
\]

\[
\max\left(2, \frac{2}{1-\alpha}\right) < s < +\infty, \quad \text{or} \quad s = \frac{2}{1-\alpha},
\]

\(^2\)Here and in the following we mean global and local in space.

\(^3\)Notice that \( \tilde{p}_L = p \) in the endpoint case \( \alpha = -1/2.\)
\[ \tilde{p} \geq \tilde{p}_G := \max \left( 4, \frac{(n-1)p}{\alpha p + n - 1} \right); \quad (1.18) \]

or

\[ \alpha \in [0, 1/2], \quad \max \left( 4, \frac{n}{1 - \alpha} \right) < p \leq +\infty, \quad \text{or} \quad p = 4, \quad (1.19) \]

and

\[ \| |x|^\alpha u \|_{L^2 \cap \mathbb{R}^n} < +\infty, \quad (1.20) \]

with

\[ \frac{2}{s} + \frac{n}{p} = 1 - \alpha, \quad (1.21) \]

\[ \frac{2}{1 - \alpha} \leq s < +\infty, \quad (1.22) \]

\[ \tilde{p} \geq \tilde{p}_G := \frac{(n-1)p}{\alpha p + n - 1}. \quad (1.23) \]

Remark 1.2. Let us point out again that the main information of the Theorem is contained in the assumptions (1.18, 1.23), i.e. the angular integrability necessary in order to get a global regularity result.

It turns out by relations (1.12, 1.13) that in the case of negative weights additional angular integrability (\(\tilde{p}_G > p\)) is necessary in order to get global regularity. On the other hand if we consider \(|x|^\alpha, \alpha > 0\) then the additional information at infinity allows to get global regularity even for weaker angular integrability (\(\tilde{p}_G < p\)).

Remark 1.3. Notice that:

- Our method misses the endpoint \(s = +\infty\);
- If \(n > 3\) we get a gain in the negative values of \(\alpha\) with respect to Theorem 1.2. We have in fact \(\frac{1-n}{p} \leq \alpha\) instead of \(-1 \leq \alpha\). This is also more satisfactory because exhibits a dependence on the dimension. We have, on the other hand, a loss in the positive values of \(\alpha\), i.e. \(\alpha \leq \frac{1}{2}\) instead of \(\alpha < 1\).

Theorem 1.4. Let \(n \geq 3\) and \(u_0\) be a divergence free vector field with each component in the Schwartz class. Let also \(u\) be a weak solution of (1.1). Then \((0, T) \times \{0\}\) is a regular set provided that

\[ \alpha \in [-1/2, 0), \quad \max \left( 2, \frac{n}{1 - \alpha} \right) < p \leq +\infty \quad \text{or} \quad p = 2, \quad (1.24) \]

and

\[ \| |x|^\alpha u \|_{L^2 \cap \mathbb{R}^n} < +\infty, \quad (1.25) \]

with

\[ \frac{2}{s} + \frac{n}{p} = 1 - \alpha, \quad (1.26) \]

\[ \max \left( 2, \frac{2}{1 - \alpha} \right) < s < +\infty, \quad \text{or} \quad s = \frac{2}{1 - \alpha}, \quad (1.27) \]

\[ \tilde{p} \geq \tilde{p}_L := \frac{2(n-1)p}{(2\alpha + 1)p + 2(n-1)}; \quad (1.28) \]

or

\[ \alpha \in [0, 1), \quad \frac{n}{1 - \alpha} \leq p \leq +\infty, \quad (1.29) \]

and

\[ \| |x|^\alpha u \|_{L^2 \cap \mathbb{R}^n} < +\infty, \quad (1.30) \]
with
\[ \frac{2}{s} + \frac{n}{p} = 1 - \alpha, \tag{1.31} \]
\[ \frac{2}{1 - \alpha} \leq s < +\infty, \tag{1.32} \]
\[ \tilde{p} \geq \tilde{p}_L := \frac{2(n-1)p}{p+2(n-1)}. \tag{1.33} \]

Remark 1.4. Notice that:
- Our main assumption is actually weaker than (1.5) because \( \tilde{p}_L < p \) (\( \tilde{p}_L = p \) if \( \alpha = -1/2 \));
- We have a loss in the negative values of \( \alpha \) with respect to 1.2. We assume in fact \( -\frac{1}{2} \leq \alpha \) instead of \( -1 \leq \alpha \).

It is interesting to compare this results with the regularity criteria obtained by working in parabolic Morrey spaces [14, 15, 25]. Consider the norms
\[ \|u\|_{L^p_v((0,T) \times \mathbb{R}^n)} := \sup_{t \in (0,T)} \sup_{x \in \mathbb{R}^n} \frac{1}{r^{\lambda/p}} \|u\|_{L^p(Q_r(t,x))}, \]
where \( Q_r(t,x) \) is the parabolic cylinder of radius \( r \) and centered in \((t,x)\)
\[ Q_r(t,x) := B_r(x) \times (t - r^2, t + r^2) \]
and focus on the formal correspondence
\[ \|u\|_{L^p_v((0,T) \times \mathbb{R}^n)} \leftrightarrow \sup_{x \in \mathbb{R}^n} \|\frac{|x - \bar{x}|}{r} - \lambda/p u\|_{L^p_x L^p_t} \]
Since \( \|\frac{|x - \bar{x}|}{r} - \lambda/p u\|_{L^p_x L^p_t} \geq \sup_{t \in (0,T)} \sup_{r > 0} \frac{1}{r^{\lambda/p}} \|u\|_{L^p(Q_r(t,x))} \) it is clear that boundedness assumptions in weighted spaces are stronger then their counterpart in Morrey spaces. This is heuristically because in the first case the weights provide a residual information even for large \(|x|\). As we have observed this information and angular integrability hypotesis provide a a more satisfactory regularity theory.

We exploit again this viewpoint through a really interesting example, i.e. the weighted counterpart of the following

**Theorem 1.5** ([2]). Let \( n = 3 \) and \( u \) be a suitable weak solution of (1.1). There is an absolute constant \( \varepsilon \) such that if
\[ \limsup_{r \to 0} \frac{1}{r^2} \int_{Q_r(t,x)} |u|^3 + |p|^{3/2} \leq \varepsilon, \tag{1.34} \]
where
\[ Q_r(t,0) := \{(\tau, y) : y < r, \quad \bar{t} - 7/8 r^2 < \tau < \bar{t} + 1/8 r^2\} ; \]
then \((\bar{t}, 0)\) is a regular point.

We focus on the condition
\[ \|\frac{|x|}{r} - \lambda/p u\|_{L^p_x L^p_t} < \infty. \tag{1.35} \]
A little work is necessary in order to show that (1.35) is actually stronger than (1.34). We just sketch the argument that is classical in the context of the Navier–Stokes theory. At first notice that the pressure can be recovered by \( u \) through \(^4\)
\[ P = \sum_{i,j=1}^3 R_i R_j u_i u_j, \]
\(^4\)See also the next Section.
where $R_i$ is the Riesz transform in the $i$-th direction. So the second term in (1.34) can be bounded by using the Calderon-Zygmund inequality (see [23])

$$|||x|^3 P||_{L'(R^3)} \leq C|||x|^6 u^2||_{L'(R^3)},$$

$r \in (1, \infty), \ -\frac{3}{r} < \delta < 1 - \frac{3}{r},$

with the choice $(\delta, r) = \left(-\frac{4}{3}, \frac{3}{2}\right)$. Then the smallness assumption in (1.34) easily follows by (1.35) provided that $T > \bar{t}$:

$$\limsup_{r \to 0} \frac{1}{r^2} \int_{Q^*_+(\bar{t},0)} |u|^3 = \limsup_{r \to 0} \frac{1}{r^2} \int_{t-7/8r^2}^{t+1/8r^2} \int_{B(0,r)} |u|^3 \leq \limsup_{r \to 0} \int_{t-7/8r^2}^{t+1/8r^2} \int_{B(0,r)} |x|^{-2}|u|^3 \leq \limsup_{r \to 0} \int_{t-7/8r^2}^{t+1/8r^2} \int_{\mathbb{R}^n} |x|^{-2}|u|^3 = 0.$$

Then also notice that $(\alpha, p, s) = \left(-\frac{2}{3}, 3, 3\right)$ is an admissible choice of indexes in Theorem 1.2.

Theorems 1.3, 1.4 suggest that it is possible to

1. get global regularity (in $(0, T) \times \mathbb{R}^n$) by a suitable amount of angular integrability in (1.35);

2. get regularity in $(0, T) \times \{0\}$ even by weaker angular integrability in (1.35).

The first point is achieved by applying Theorem 1.3 with

$$(\alpha, s, p, \tilde{p}) = \left(-\frac{2}{3}, 3, 3, +\infty\right),$$

i.e. by assuming

$$|||x|^{-2/3} u||_{L^3_{[x]} L^3_{\tilde{p}}} L^\infty < +\infty;$$

notice that in this case the indexes satisfy the endpoint relation $p = \frac{1-n}{\alpha}$ so we have to require $L^\infty$ boundedness in the angular direction.

Otherwise it is interesting to notice that Theorem 1.4 can not give a positive answer to the second point because the value $\alpha = -2/3$ is not permitted. This is of course due to our method and in particular to the fact that we never work directly with the energy estimate as in [26]. Otherwise a more direct proof of Theorem 1.4 requires a delicate analysis of the properies of the Riesz transform in mixed radial-angular spaces. This is also a topic of independent interest and we hope to reexamine it in a future work.

The rest of the paper is organized as follows: in the second Section we recall the well known integral formulation of (1.1); in the third Section we prove time decay estimates for the heat and Oseen kernels in weighted $L^p_{[x]} L^\tilde{p}_{\theta}$ spaces; in the fourth Section we prove the main Theorems.

2. Integral formulation of the problem

We recall the integral formulation of the Navier–Stokes problem. By taking the divergence of the first equation in (1.1) and by using the incompressibility:

$$-\Delta P = \sum_{i=1}^{n} \partial_i \sum_{j=1}^{n} u_j \partial_j u_i$$

(2.1)

$$= \sum_{i,j=1}^{n} \partial_i \partial_j (u_i u_j),$$

(2.2)
so $P$ can be, at least formally, recovered by $u$ through
\[ P = -\Delta^{-1} \sum_{j=1}^{n} \partial_i \partial_j (u_i u_j). \] (2.3)
Thus (1.1) becomes
\[
\begin{cases}
    u &= e^{t \Delta} u_0 - \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (u \otimes u) \, ds \\ 
    \nabla \cdot u &= 0
\end{cases}
\quad \text{in } [0, T) \times \mathbb{R}^n,
\] (2.4)
where $(u \otimes u)_{i,j} := u_i u_j$ and $\mathbb{P}$ is formally defined by
\[ \mathbb{P} f := f - \nabla \Delta^{-1} (\nabla \cdot f). \] (2.5)
This operator is a really useful tool in the study of the Navier–Stokes problem. It is actually a projection on the subspace of the divergence free vector fields $(\mathbb{P} f = f \Leftrightarrow \nabla \cdot f = 0)$. If $f \in [L^2(\mathbb{R}^n)]^n$ then $\mathbb{P}$ is rigorously defined by
\[ \mathbb{P} f := f + R \otimes R f, \]
where $R$ is the vector of the Riesz transformations. On the other hand $\mathbb{P}$ can be defined on larger Banach spaces as a Calderon-Zygmund operator. Furthermore we are basically interested in the operator $\mathbb{P}(\nabla \cdot)$ that, thanks to the differentiation, can be actually defined on $[L^1_{\text{loc}}(\mathbb{R}^n)]^{n \times n}$, i.e., the space of uniformly locally integrable functions (see [16] for further details).

Now we focus on some properties of the Osseen kernel. At first we need the following

**Lemma 2.1** ([16]). Let $1 \leq i, j \leq n$. The operator $\Delta^{-1} \sum_{j=1}^{n} \partial_i \partial_j e^{t \Delta}$ is a convolution operator $\sum_{j=1}^{n} O_{i,j} \ast f_j$ with
\[ O_{i,j}(t,x) := \frac{1}{t^\frac{n}{2}} a_{i,j} \left( \frac{x}{\sqrt{t}} \right) \]
and for each multi-index $\eta$
\[ a_{i,j} \in C^\infty(\mathbb{R}^n), \quad (1 + |x|)^{n+|\eta|} |\partial^\eta a_{i,j}| \in L^\infty(\mathbb{R}^n). \]
This is the main technical tool necessary in order to study the properties of $e^{t \Delta} \mathbb{P}(\nabla \cdot)$, it holds in fact the following

**Proposition 2.2** ([16]). Let $1 \leq i, j, k \leq n$. The operator $e^{t \Delta} \mathbb{P}(\nabla \cdot)$ is a convolution operator $\sum_{k=1}^{n} K_{i,j,k}(t) \ast f_{j,k}$ with
\[ K_{i,j,k}(t,x) := \frac{1}{t^\frac{n}{2}} k_{i,j,k} \left( \frac{x}{\sqrt{t}} \right) \]
and for each multi-index $\eta$
\[ k_{i,j,k} \in C^\infty(\mathbb{R}^n), \quad (1 + |x|)^{n+1+|\eta|} |\partial^\eta k_{i,j,k}| \in L^\infty(\mathbb{R}^n). \]
We conclude the section with an useful equivalence result:

**Theorem 2.3** ([16]). Let $u \in \cap_{s<T} \left(L^2_{\text{loc}}(0, T) \times \mathbb{R}^n) \right)$. Then the following are equivalent:

1. $(u, P)$ is a weak solution of
\[
\begin{cases}
    \partial_t u + (u \cdot \nabla) u - \Delta u &= -\nabla P \\ 
    \nabla \cdot u &= 0 \\ 
    u &= u_0
\end{cases}
\quad \text{in } [0, T) \times \mathbb{R}^n
\] (2.6)
2. $(u, P)$ is a solution of the integral problem
\[
\begin{cases}
    u &= e^{t \Delta} u_0 - \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (u \otimes u) \, ds \\ 
    \nabla \cdot u &= 0 \\ 
    P &= R \otimes R (u \otimes u)
\end{cases}
\quad \text{in } [0, T) \times \mathbb{R}^n
\] (2.7)
3. Time decay estimates for the heat and Oseen kernels

We prove time decay estimates for the operators $e^{t\Delta}$ and $e^{t\Delta}P(\nabla \cdot \cdot)$. These estimates turn out to be fundamental in the study of the Navier–Stokes problem with small data since the pioneering work of Kato [13]. Following the same philosophy we take advantage of them in order to get regularity criteria. This is natural by working with the integral formulation (2.4).

We investigate the connection between homogeneous weights and angular regularity by working in $L^p_{|x|}L^{\tilde{p}}_\theta$ spaces. In particular we show that higher angular integrability allows to consider a larger set of weights.

As mentioned the idea of distinguish radial and angular integrability often occurs in harmonic analysis and PDE’s. In particular we refer to [4] where this technology has been applied to recover in a more general setting the improvements to Sobolev embeddings and Caffarelli-Kohn-Nirenberg inequalities known in the radial case by [5, 6, 7, 8, 20].

We need the following

Lemma 3.1 ([4]). Let $n \geq 2$ and $1 \leq p \leq q \leq \infty$, $1 \leq \tilde{p} \leq \tilde{q} \leq \infty$. Assume $\alpha, \beta, \gamma$ satisfy the set of conditions

$$\beta > -\frac{n}{q}, \quad \alpha < \frac{n}{p'}, \quad \alpha - \beta \geq (n-1) \left( \frac{1}{q} - \frac{1}{p} + \frac{1}{p} - \frac{1}{\tilde{q}} \right),$$

(3.1)

$$\alpha - \beta + \gamma > n \left( 1 + \frac{1}{q} - \frac{1}{p} \right).$$

(3.2)

Then

$$|||x|^\beta S_r \phi||_{L^p_{|x|}L^{\tilde{p}}_\theta} \leq C|||x|^\alpha \phi||_{L^p_{|x|}L^{\tilde{q}}_\theta},$$

(3.3)

where

$$S_r \phi := \int_{\mathbb{R}^n} K(x-y)\phi(y) \, dy,$$

and the kernel $K$ satisfies

$$|K(x)| \leq \frac{\text{Const}}{(1 + |x|^2)^{\gamma/2}}.$$

Remark 3.1. Let point out that:

- The assumptions $\beta > -\frac{n}{q}, \alpha < \frac{n}{p'}$ are necessary to ensure local integrability;
- The assumption (3.2) is due to the smoothness of the kernel in the origin. It is less restrictive than its counterpart in the homogeneous case (see [4])

$$\alpha - \beta + \gamma = n \left( 1 + \frac{1}{q} - \frac{1}{p} \right),$$

that follows by scaling;

- The assumption

$$\alpha - \beta \geq (n-1) \left( \frac{1}{q} - \frac{1}{p} + \frac{1}{p} - \frac{1}{\tilde{q}} \right)$$

follows by testing the inequality under translations.

It is useful to define the quantity

$$\Lambda(\alpha, p, \tilde{p}) := \alpha + \frac{n-1}{p} = \frac{n-1}{\tilde{p}}.$$  

(3.4)

Notice that

$$\alpha - \beta \geq (n-1) \left( \frac{1}{q} - \frac{1}{p} + \frac{1}{p} - \frac{1}{\tilde{q}} \right) \Leftrightarrow \Lambda(\alpha, p, \tilde{p}) \geq \Lambda(\beta, q, \tilde{q}).$$
This notation is more convenient for our purposes; we use also simply \( \Lambda_\alpha \) when the values of \( p, \tilde{p} \) will be clear by the context.

**Proposition 3.2.** Let \( n \geq 2, 1 \leq p \leq q \leq +\infty \) and \( 1 \leq \tilde{p} \leq \tilde{q} \leq +\infty \). Assume further that \( \alpha, \beta \) satisfy the set of conditions

\[
\beta > -\frac{n}{q}, \quad \alpha < \frac{n}{p'}, \quad \Lambda(\alpha, p, \tilde{p}) \geq \Lambda(\beta, q, \tilde{q}).
\]

Then for each multi-index \( \eta \)

\[
(1) \quad |||x|^{\beta} \partial^n e^{t\Delta} u_0|||_{L^p_{\eta} L^q_{\eta}} \leq \frac{c_0}{t^{(k+q-\frac{n}{2})/2}} |||x|^{\alpha} u_0|||_{L^p_{\eta} L^q_{\eta}}, \quad t > 0,
\]

provided that \( |\eta| + \frac{n}{p} - \frac{n}{q} + \alpha - \beta \geq 0, \)

\[
(2) \quad |||x|^{\beta} \partial^n e^{t\Delta} \mathbb{P} \nabla : F|||_{L^p_{\eta} L^q_{\eta}} \leq \frac{d_0}{t^{(k+q-\frac{n}{2})/2}} |||x|^{\alpha} F|||_{L^p_{\eta} L^q_{\eta}}, \quad t > 0,
\]

provided that \( 1 + |\eta| + \frac{n}{p} - \frac{n}{q} + \alpha - \beta > 0. \)

**Proof.** The proof follows by Lemma (3.1) and scaling considerations. At first notice

\[
e^{t\Delta} \phi = S_{\sqrt{t}\Delta} S_{1/\sqrt{t}\phi},
\]

where \( S_\lambda \) is defined by

\[
(S_\lambda \phi)(x) = \phi \left( \frac{x}{\lambda} \right).
\]

Then

\[
|||x|^{\beta} \partial^n S_\lambda \phi|||_{L^p_{\eta} L^q_{\eta}} = \lambda^{\alpha + \beta - |\eta|} |||x|^{\beta} \phi|||_{L^p_{\eta} L^q_{\eta}}.
\]

We get

\[
|||x|^{\beta} \partial^n e^{t\Delta} u_0|||_{L^p_{\eta} L^q_{\eta}} = |||x|^{\beta} \partial^n \sqrt{t} \Delta S_{1/\sqrt{t}\nabla u_0}|||_{L^p_{\eta} L^q_{\eta}}
\]

\[
= \frac{c_0}{t^{(k+q-\frac{n}{2})/2}} |||x|^{\alpha} \partial^n S_{1/\sqrt{t}\nabla u_0}|||_{L^p_{\eta} L^q_{\eta}}
\]

\[
\leq \frac{c_0}{t^{(k+q-\frac{n}{2})/2}} |||x|^{\alpha} S_{1/\sqrt{t}\nabla u_0}|||_{L^p_{\eta} L^q_{\eta}}
\]

\[
= \frac{c_0}{t^{(k+q-\frac{n}{2})/2}} |||x|^{\alpha} u_0|||_{L^p_{\eta} L^q_{\eta}};
\]

provided that

\[
\Lambda(\alpha, p, \tilde{p}) \geq \Lambda(\beta, q, \tilde{q}).
\]

Notice that the third condition in (3.1) is trivially satisfied by the heat kernel. To prove (3.7) we have to work with the operator \( e^{t\Delta}\mathbb{P}(\nabla) \) that is (see Lemma 2.2) a convolution operator with a kernel \( K \) such that

\[
K_{j,k,m}(t, x) := k_{j,k,m} \left( \frac{x}{\sqrt{t}} \right),
\]

and

\[
(1 + |x|)^{1+n+|\mu|} \partial^n k_{j,k,m} \in L^\infty(\mathbb{R}^n),
\]

for each multi-index \( \mu \). By (3.11) follows

\[
K(t) * \phi = \frac{1}{\sqrt{t}} S_{\sqrt{t}k} * S_{1/\sqrt{t}\phi},
\]

(3.13)
So
\[
\| |x|^\beta \partial^\alpha e^{t^\Delta P \nabla \cdot F} | F \|_{L_{t,x}^q L_{t}^p} \leq \frac{C_q R^{-\Lambda_{\alpha,\beta}}}{t^{(1+|\eta|)+\frac{n}{p}-\frac{\alpha-\beta}{2}+\frac{n}{2}}+\frac{n}{2}} \| |x|^\alpha u_0 \|_{L_{t,x}^p L_{t}^q}, \quad t > 0,
\]
provided that \(|\eta| + \frac{\alpha}{p} + \frac{n}{q} + \alpha - \beta \geq 0, \Lambda_{\alpha,\beta} < 0,
\]
(1)
\[
\| |x|^\beta \partial^\alpha e^{t^\Delta P \nabla \cdot F} | F \|_{L_{t,x}^q L_{t}^p} \leq \frac{d_q R^{-\Lambda_{\alpha,\beta}}}{t^{(1+|\eta|)+\frac{n}{p}-\frac{\alpha-\beta}{2}+\frac{n}{2}}+\frac{n}{2}} \| |x|^\alpha F \|_{L_{t,x}^p L_{t}^q}, \quad t > 0,
\]
provided that \(1 + |\eta| + \frac{n}{p} - \frac{\alpha}{q} + \alpha - \beta > 0, \Lambda_{\alpha,\beta} < 0.
\]
(2)

**Proposition 3.3.** Let \(n \geq 2, 1 \leq p \leq q < +\infty \) and \(1 \leq \tilde{p} \leq \tilde{q} \leq +\infty \). Assume further that \(\alpha, \beta \) satisfy the set of conditions

\[
\beta > -\frac{n}{q}, \quad \alpha < \frac{n}{p'}, \quad \Lambda(\alpha, p, \tilde{p}) < \Lambda(\beta, q, \tilde{q}),
\]
and define

\[
\Lambda_{\alpha,\beta} := \Lambda(\alpha, p, \tilde{p}) - \Lambda(\beta, q, \tilde{q}).
\]

Let then

\[
\Pi(R) := \left\{ (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n : \frac{|x|}{\sqrt{t}} \leq R \right\},
\]

for each multi index \(\eta\)

\[
(1)
\]

\[
\Pi(R) | |x|^\beta \partial^\alpha e^{t^\Delta P \nabla \cdot F} | F \|_{L_{t,x}^q L_{t}^p} \leq \frac{d_q R^{-\Lambda_{\alpha,\beta}}}{t^{(1+|\eta|)+\frac{n}{p}-\frac{\alpha-\beta}{2}+\frac{n}{2}}+\frac{n}{2}} \| |x|^\alpha F \|_{L_{t,x}^p L_{t}^q}, \quad t > 0,
\]

(2)

\[
\Pi(R) | |x|^\beta \partial^\alpha e^{t^\Delta P \nabla \cdot F} | F \|_{L_{t,x}^q L_{t}^p} \leq \frac{d_q R^{-\Lambda_{\alpha,\beta}}}{t^{(1+|\eta|)+\frac{n}{p}-\frac{\alpha-\beta}{2}+\frac{n}{2}}+\frac{n}{2}} \| |x|^\alpha F \|_{L_{t,x}^p L_{t}^q}, \quad t > 0,
\]

**Proof.** Let us write simply \(\Lambda\) instead of \(\Lambda_{\alpha,\beta}\). Of course

\[
\Lambda < 0 \quad \Rightarrow \quad R^{-\Lambda} \left| \frac{x}{\sqrt{t}} \right|^\alpha \geq 1, \quad \text{if} \quad (t, x) \in \Pi(R).
\]
then
\[\|\Pi(R)[|x|^\beta \partial_y e^{t\Delta} u_0]\|_{L^q_t L^\infty_y} \leq \|\Pi(R)[|x|^\beta \partial_y e^{t\Delta} S_1/\sqrt{t}u_0]\|_{L^q_t L^\infty_y} \leq \frac{R^{-\Lambda}}{\|x|^{\beta+\Lambda}} \|\Pi(R)[|x|^\beta \partial_y e^{t\Delta} S_1/\sqrt{t}u_0]\|_{L^q_t L^\infty_y} \leq \frac{c_0}{\bar{\ell}(|\eta|+\frac{n}{q})^{2}} \|\Pi(R)[|x|^\beta \partial_y e^{t\Delta} S_1/\sqrt{t}u_0]\|_{L^q_t L^\infty_y},\]
where the indexes relationships are consistent because
\[\Lambda_\alpha \geq \Lambda(\Lambda_{\alpha,\beta} + \beta, p, \bar{q}) = \Lambda(\Lambda_\alpha - \Lambda_{\beta} + \beta, p, \bar{q}) = \Lambda_\alpha.\]

The proof of (3.16) is analogous. \hfill \Box

Remark 3.2. We have observed observed that the inequalities hold with an additional factor \(R^{-\Lambda}\) after localization in the interior of a space-time parabola. Notice that this factor goes to 1 as \(\Lambda \to 0^+\). To get a constant independent on \(\Lambda\) it is instead necessary to restrict the size of the parabola. If we chose the constant equal to \(K\), we need to restrict to
\[\Pi(K) \equiv \left\{ |x| \leq \frac{K^{-1/2}}{\sqrt{t}} \right\}.
\]
Notice that \(\Pi(K)\) fills the whole space-time as \(\Lambda \to 0^+\).

Then integral estimates can be obtained by the time decay properties. Let us introduce another useful notation
\[\Omega(\alpha, p, s) := \alpha + \frac{n}{p} + \frac{2}{s},\] (3.17)

Proposition 3.4. Let \(n \geq 2, 1 \leq p \leq q < \frac{np}{(n+\alpha) p + n - 2}, r \in (1, +\infty)\) and \(1 \leq \bar{q} \leq \bar{q}' \leq +\infty\). Assume further that \(\alpha, \beta\) satisfy
\[\beta > -\frac{n}{q}, \quad \alpha < \frac{n}{p'},\] (3.18)

Then for each multi-index \(\eta\)
\[\|\Pi(R)[|x|^\beta \partial_y e^{t\Delta} u_0]\|_{L^q_t L^\infty_y} \leq c_0 \|\Pi(R)[|x|^\alpha u_0]\|_{L^p_t L^{\bar{q}}_y}, \quad t > 0,\] (3.19)

provided that
\[|\eta| + \Omega(\alpha, p, \infty) = \Omega(\beta, q, r), \quad \Lambda(\alpha, p, \bar{q}) \geq \Lambda(\beta, q, \bar{q});\] (3.20)
and
\[\|\Pi(R)[|x|^\beta \partial_y e^{t\Delta} u_0]\|_{L^q_t L^\infty_y} \leq c_0 \Lambda^{-\Lambda_{\alpha,\beta}} \|\Pi(R)[|x|^\alpha u_0]\|_{L^p_t L^{\bar{q}}_y}, \quad t > 0,\] (3.21)

provided that
\[|\eta| + \Omega(\alpha, p, \infty) = \Omega(\beta, q, r), \quad \Lambda_{\alpha,\beta} := \Lambda(\alpha, p, \bar{q}) - \Lambda(\beta, q, \bar{q}) < 0,\] (3.22)

Proof. By the time decay
\[\|\Pi(R)[|x|^\beta \partial_y e^{t\Delta} u_0]\|_{L^q_t L^\infty_y} \leq c_0 \frac{1}{\bar{\ell}(|\eta|+\frac{n}{q})^{2}} \|\Pi(R)[|x|^\alpha u_0]\|_{L^p_t L^{\bar{q}}_y},\] (3.23)
follows that $\partial^\alpha e^{t\Delta} u_0$ is bounded in the Lorentz space $L^{r,\infty}(\mathbb{R}^+; L^q_{|x|^\alpha} L^p_{\theta})$ provided that $|\eta| + \Omega(\alpha, p, \infty) = \Omega(\beta, q, r)$. In fact
\[
\| \| |x|^\beta \partial^\alpha e^{t\Delta} u_0 \|_{L^q_{|x|} L^p_{\theta}} \|_{L^{r,\infty}} \leq c_\eta \| \| |x|^\alpha u_0 \|_{L^p_{|x|} L^p_{\theta}} \|_{L^{r,\infty}} \leq c_\eta \| u_0 \|_{L^p_{|x|} L^p_{\theta}},
\]
when
\[
(|\eta| + \frac{n}{p} + \frac{n + \alpha - \beta}{2})/2 = \frac{1}{p} + |\eta| + \Omega(\alpha, p, \infty) = \Omega(\beta, q, r).
\]

Let now consider $(\alpha_0, \beta_0, p_0, p_1, q_0, q_1, r_0)$, $(\alpha_1, \beta_1, p_1, p_1, q_1, q_1, r_1)$ such that the assumptions of the Theorem are satisfied. We have the bounded operators
\[
\partial^\alpha e^{t\Delta} : \begin{cases} 
L^p_{|x|^{\alpha_0} d|x|} \rightarrow L^{\alpha_0,\infty} L^q_{|x|^{\beta_0} d|x|} L^{p_0}_{\theta}, 

L^p_{|x|^{\alpha_1} d|x|} \rightarrow L^{\alpha_1,\infty} L^q_{|x|^{\beta_1} d|x|} L^{p_1}_{\theta},
\end{cases} \tag{3.23}
\]
and we can use real interpolation with parameters $(\xi, r_\xi), 0 \leq \xi \leq 1$ provided that
\[
p_\xi < r_\xi, \tag{3.24}
\]
\[
\frac{1}{p_\xi} = (1 - \xi) \frac{1}{p_0} + \frac{\xi}{p_1},
\]
\[
\frac{1}{q_\xi} = (1 - \xi) \frac{1}{q_0} + \frac{\xi}{q_1},
\]
\[
\frac{1}{r_\xi} = (1 - \xi) \frac{1}{r_0} + \frac{\xi}{r_1},
\]
\[
\frac{1}{p_\xi} = (1 - \xi) \frac{1}{p_0} + \frac{\xi}{p_1},
\]
\[
\alpha_\xi = (1 - \xi) \alpha_0 + \xi \alpha_1,
\]
\[
\beta_\xi = (1 - \xi) \beta_0 + \xi \beta_1.
\]

Then since
\[
(L^{\alpha_0,\infty} L^{q_0}_{|x|^{\beta_0} d|x|} L^{p_0}_{\theta}, L^{\alpha_1,\infty} L^{q_1}_{|x|^{\beta_1} d|x|} L^{p_1}_{\theta})_{\xi, r_\xi} = L^{\alpha_\xi}_{|x|^{\alpha_\xi} d|x|} L^{p_\xi}_{\theta},
\]
we get the bounded operators
\[
\partial^\alpha e^{t\Delta} u_0 : L^{p_\xi}_{|x|^{\alpha_\xi} d|x|} L^{p_\xi}_{\theta} \rightarrow L^{p_\xi}_{|x|^{\alpha_\xi} d|x|} L^{p_\xi}_{\theta}.
\]

It is now straightforward to check that the indexes satisfy (3.18, 3.20) and the other assumptions. In particular (3.24) is ensured by $q_\xi < \frac{n}{|\eta| + \alpha_0 - \beta_0 p_0 + \frac{n}{p_0} - 2}$. Of course this method misses the endpoint $r = 1$. The estimates (3.21) can be proved in the same way by using the localized time decay.

Then we bound the Duhamel term.
Proposition 3.5. Let \( n \geq 2, 2 \leq p \leq 2q \leq +\infty, 2 < s \leq 2r < +\infty \) and \( 2 \leq \tilde{p} \leq 2\tilde{q} \leq +\infty \). Assume further that \( \alpha, \beta \) satisfy
\[
\beta > \frac{n}{q}, \quad \alpha < \frac{n}{2} - \frac{n}{p},
\]
then for each multi-index \( \eta \)
\[
\left\| x^\beta \partial^\eta \int_0^t e^{(t-s)\Delta} \nabla \cdot (u \otimes u) \, ds \right\|_{L_t^q L_x^\tilde{q}} \lesssim d_\eta \| x^\alpha u \|_{L_t^p L_x^\tilde{p}}^2, \quad t > 0,
\]
provided that
\[
2\Omega(\alpha, p, s) = \Omega(\beta, q, r) + 1 - |\eta|, \quad 2\Lambda(\alpha, p, \tilde{p}) \geq \Lambda(\beta, q, \tilde{q});
\]
in particular
\[
\left\| x^\beta \int_0^t e^{(t-s)\Delta} \nabla \cdot (u \otimes u) \, ds \right\|_{L_t^q L_x^\tilde{q}} \lesssim d_\eta \| x^\beta u \|_{L_t^p L_x^\tilde{p}}^2, \quad t > 0,
\]
provided that
\[
\frac{2}{r} + \frac{n}{q} = 1 - \beta, \quad \Lambda(\beta, q, \tilde{q}) \geq 0.
\]

Proof. By Minkowski inequality and (3.6)
\[
\left\| x^\beta \partial^\eta \int_0^t e^{(t-s)\Delta} \nabla \cdot F(x, s) \, ds \right\|_{L_t^q L_x^\tilde{q}} \lesssim d_\eta \left\| \int_{\mathbb{R}^+} \| x^\beta \partial^\eta e^{(t-s)\Delta} \nabla \cdot F \|_{L_t^q L_x^\tilde{q}} \, ds \right\|_{L_t^{\tilde{q}}},
\]
provided that
\[
\tilde{p}_0 \leq \tilde{q}, \quad p_0 \leq q \quad 1 + |\eta| + \frac{n}{p_0} + \frac{n}{q} + \alpha_0 - \beta > 0, \quad \Lambda_{\alpha_0} \geq \Lambda_{\beta}.
\]
Let then
\[
1 + \frac{1}{r} = \frac{1}{s_0} + \frac{1}{k},
\]
and use the Young inequality in Lorentz spaces
\[
\| \cdot \|_{L^r} \leq \| \cdot \|_{L^{s_0}} \| \cdot \|_{L^{k, \infty}},
\]
that is allowed if \( 1 < r, s_0, k < +\infty \). We get
\[
\left\| x^\beta \partial^\eta \int_0^t e^{(t-s)\Delta} \nabla \cdot F(x, s) \, ds \right\|_{L_t^q L_x^\tilde{q}} \lesssim d_\eta \| x^\alpha F \|_{L_t^{s_0} L_x^{p_0}} \| \int_{\mathbb{R}^+} \frac{dt}{(1 + |\eta| + \frac{n}{p_0} - \frac{n}{q} + \alpha_0 - \beta)/2} \|_{L_t^{k, \infty}} \lesssim d_\eta \| x^\alpha F \|_{L_t^{s_0} L_x^{p_0}},
\]
provided that
\[
p_0 \leq q \quad (1 + |\eta| + \frac{n}{p_0} - \frac{n}{q} + \alpha_0 - \beta)/2 = \frac{1}{k}, \quad \Lambda_{\alpha_0} \geq \Lambda_{\beta},
\]
since
\[
\| \int_{\mathbb{R}^+} \frac{dt}{t^{1/k}} \|_{L_t^{k, \infty}} = 1.
\]
By \((3.31)\) and the second in \((3.32)\)
\[
\Omega(\alpha_0, p_0, s_0) = 1 - |\eta| + \Omega(\beta, q, r).
\]  
(3.33)

We now specify \(F = u \otimes u\)
\[
\left\| \frac{1}{x^\beta} \partial_x^{\alpha} \int_0^t e^{(t-s)\Delta} \nabla \cdot (u \otimes u) \, ds \right\|_{L_t^\alpha L_x^s L^{\tilde{p}_0}_\theta} \\
\leq c_\eta \|x^{\alpha_0} \|_2^2 \|L_t^{\alpha_0} L_x^{p_0} L^{\tilde{p}_0}_\theta \|

\leq c_\eta \|x^{\alpha_0/2} \|_2^2 \|L_t^{\alpha_0} L_x^{2p_0} L^{\tilde{p}_0}_\theta \|

\leq c_\eta \|x^{\alpha} \|_2^2 \|L_t^{\alpha} L_x^{p} L^{\tilde{p}}_\theta ,
\]  
(3.34)

where we have set \((\alpha_0/2, 2s_0, 2p_0, 2\tilde{p}_0) = (\alpha, s, p, \tilde{p})\).

Notice that \(2\Omega(\alpha, s, p) = \Omega(\alpha_0, s_0, p_0), 2\Lambda_\alpha = \Lambda_{\alpha_0}\) so \((3.33), (3.35)\) and the last in \((3.32)\) lead to
\[
2\Omega(\alpha, p, s) = \Omega(\beta, q, r) + 1 - |\eta|, \quad 2\Lambda_\alpha \geq \Lambda_.
\]

Finally notice that \((3.31)\) and \((3.32)\) imply
\[
r \geq s_0 = s/2, \quad q \geq p_0 = p/2, \quad \tilde{q} \geq \tilde{p}_0 = \tilde{p}/2.
\]

These conditions are furthermore consistent with the choice \((\alpha, s, p, \tilde{p}) = (\beta, q, \tilde{q})\), in such a way we recover inequality \((3.28)\)
\[
\left\| \frac{1}{x^\beta} \partial_x^{\alpha} \int_0^t e^{(t-s)\Delta} \nabla \cdot (u \otimes u) \, ds \right\|_{L_t^\alpha L_x^s L^{\tilde{p}_0}_\theta} \leq d_\eta \|x^{\alpha} \|_2^2 \|L_t^\alpha L_x^p L^{\tilde{p}}_\theta ,
\]
provided that
\[
\Omega(\beta, q, r) = 1 - |\eta|, \quad \Lambda(\beta, q, \tilde{q}) \geq 0.
\]

\square

4. Proof of the main results

We refer to the relations
\[
\frac{2}{s} + \frac{n}{p} = 1 - \alpha, \quad \alpha_0 = 1 - \frac{n}{p_0},
\]
as scaling assumptions.

As mentioned Theorems \ref{3.4}, \ref{3.5} actually hold under weaker assumptions on \(u_0\), we prove in fact:

**Theorem 4.1.** Theorem \ref{3.4} holds if \(u_0 \in L^2(\mathbb{R}^n)\) is a divergence free vector field and
\[
\left\| \frac{x^{\alpha_0}}{|x^{\alpha_0}} u_0 \right\|_{L_t^{\alpha_0} L_x^{p_0} L^{\tilde{p}_0}_\theta} < +\infty
\]
with
\[
\alpha_0 \in [(2 - n)/2, 2/(2 + n)], \quad \alpha_0 = 1 - \frac{n}{p_0}, \quad \tilde{p}_0 \leq \tilde{p}_G/2,
\]  
(4.1)

or
\[
\begin{align*}
2 \leq p_0 & \leq \tilde{p}_G/2 \quad \text{if} \quad \tilde{p}_G \leq 2n
\end{align*}
\]
(4.2)

\[
\begin{align*}
2 \leq p_0 & \leq \tilde{p}_G/2 \quad \text{if} \quad \tilde{p}_G > 2n;
\end{align*}
\]  
(4.3)

or
\[
\begin{align*}
2 \leq p_0 & \leq p/2 \quad \text{if} \quad p \leq 2n
\end{align*}
\]  
(4.4)
while \( u \) has to satisfy \((1.14, 1.15, 1.16, 1.17, 1.18)\), or \((1.19, 1.20, 1.21, 1.22, 1.23)\).

Proof. Since we want to use the regularity condition \((1.2)\) we need to show that
\[
\|u\|_{L^q_t L^r_x} < +\infty, \quad \text{with} \quad \frac{2}{r} + \frac{n}{q} = 1. \tag{4.5}
\]
Let's start by the integral representation
\[
u = e^{t\Delta} u_0 - \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (u \otimes u)(s) \, ds
\]
and distinguish the cases \( \alpha \in [(1-n)/2, 0) \) and \( \alpha \in [0, 1/2] \).

**Case** \( \alpha \in [(1-n)/2, 0) \).
\[
\|u\|_{L^q_t L^r_x} \leq \left\|e^{t\Delta} u_0\right\|_{L^q_t L^r_x} + \left\|\int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (u \otimes u)(s) \, ds\right\|_{L^q_t L^r_x}
\]
\[
= I + II.
\]
By the scaling assumption and Proposition 3.4
\[
I \leq c_0 \|x|^{\alpha_0} u_0\|_{L^q_{|x|^{\alpha_0}}} \tag{4.6}
\]
provided that
\[
p_0 \leq q < \frac{n p_0}{p_0 - 2}, \quad \tilde{p}_0 \leq q, \quad \Lambda(\alpha_0, p_0, \tilde{p}_0) \geq 0. \tag{4.7}
\]
Actually the condition \( \Lambda(\alpha_0, p_0, \tilde{p}_0) \geq 0 \) is not necessary in order to prove the Theorem, we assume it for now in order to avoid some technicalities in the proof. We will show how to remove it at the end of the proof. We use Proposition 3.5 and scaling to bound
\[
II \leq d_0 \|x|^{\alpha_0} u\|_{L^q_{|x|^{\alpha_0}}}^2 \tag{4.8}
\]
provided that
\[
\Lambda(\alpha, p, \tilde{p}) \geq 0, \tag{4.8}
\]
\[
2 \leq p \leq +\infty, \quad 2 < s < +\infty, \quad p/2, \tilde{p}/2 \leq q, \quad s/2 \leq r. \tag{4.9}
\]
Condition \((4.8)\) is ensured by
\[
\tilde{p} \geq \frac{(n-1)p}{\alpha p + n - 1}. \tag{4.10}
\]
Notice also that \((4.10)\), the scaling and \( \alpha < 0 \) imply \( \frac{n}{1-\alpha} < p \leq \frac{1-n}{\alpha} \), so the widest range for \( p \) is attained as \( \alpha \to 0^- \). Then we need a couple \((r, q)\) such that \((4.9)\) is consistent with \( \frac{2}{r} + \frac{n}{q} = 1 \). We choose \( q = \tilde{p} G = \max \left(4, \frac{(n-1)p}{\alpha p + n - 1}\right) \). This is allowed by \((1-n)/2 \leq \alpha\), in fact if \( \frac{(n-1)p}{\alpha p + n - 1} \geq 4 \) we get
\[
\frac{2}{r} = 1 - \frac{n}{q} = 1 - \frac{2n \alpha}{n - 1} + \frac{2n}{p} \geq \frac{2}{r} - \frac{4}{s} = \frac{1-n-2\alpha}{n-1},
\]
so
\[
(1-n)/2 \leq \alpha \Rightarrow s/2 \leq r;
\]
while if \( \frac{(n-1)p}{\alpha p + n - 1} < 4 \) we get \( p < 4 \) and
\[
\frac{2}{r} = 1 - \frac{n}{4} = 2 - \frac{2n}{p} \geq \frac{2}{r} - \frac{4}{s} < -1 + \frac{2n}{p} - \frac{n}{4} < 0.
\]
Finally \((4.7)\) becomes
\[
p_0 \leq \frac{\tilde{p}_0 G}{2} < \frac{np_0}{p_0 - 2}.
\]
that by a straightforward calculation leads to (4.2) and \( \alpha_0 \in [(2-n)/2, 2/(2+n)] \).

**Case \( \alpha \in [0,1/2] \).** The only difference is in the choice of \( (r,q) \). Here we set \( q = p/2 \).

In such a way (4.9) is ensured by \( \alpha \leq 1/2 \), in fact

\[
\frac{2}{r} = 1 - \frac{2n}{p} \Rightarrow \frac{2}{r} - \frac{4}{s} = -1 + 2\alpha,
\]

so

\[
\alpha \leq 1/2 \Rightarrow s/2 \leq r.
\]

Notice that in this case we do not have the restriction \( p \leq \frac{1-n}{\alpha} \). Then (4.7) becomes

\[
p_0 \leq \frac{q}{2} < \frac{np_0}{p_0 - 2},
\]

that by a straightforward calculation leads to (4.4), \( \alpha_0 \in [(2-n)/2, 2/(2+n)] \) and suppose \( \Lambda < 0 \). We can use the localized estimate (3.21) to get the bound

\[
\| \Pi(R) u \|_{L^2 L^q} \leq R^{-\Lambda} c_0 \| |x|^{\alpha_0} u \|_{L^{p_0} L^{p_0}} + d_0 \| |x|^{\alpha} u \|_{L^2 L^q}.
\]

where

\[
\Pi(R) := \left\{ (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n : \frac{|x|}{\sqrt{t}} \leq R \right\}.
\]

So \( (0,T) \times \mathbb{R}^n \) is a regular set by taking the limit \( R \to +\infty \).

\[
\square
\]

**Theorem 4.2.** Theorem 1.4 holds if \( u_0 \in L^2 \cap L^2_{[x^2-n} dx \) is a divergence free vector field such that

\[
\| |x|^{\alpha_0} u \|_{L^{p_0} L^{p_0}} < +\infty, \quad \Lambda(p_0,0,\tilde p_0) \geq 0,
\]

with

\[
\alpha_0 \in \left[ 1 - n, \frac{2-n}{2+n} \right], \quad \alpha_0 = 1 - \frac{n}{p_0}, \quad \tilde p_0 \leq \frac{p}{2}, \tag{4.11}
\]

\[
\left\{\begin{array}{ll}
1 \leq p_0 \leq p/2, & p \leq n \\
1 \leq p_0 \leq p/2, & p < \frac{p}{p-n} \quad \text{if } p > n;
\end{array}\right.
\]

\[
\alpha_0 \in \left[ 1 - (1-\alpha)n, 1 - (1-\alpha)\frac{2n}{2+n} \right], \quad \alpha_0 = 1 - \frac{n}{p_0}, \quad \tilde p_0 \leq \frac{p}{2}, \tag{4.12}
\]

\[
\frac{1}{1 - \alpha} \leq p_0 \leq \frac{p}{2}, \quad p_0 < \frac{p}{(1-\alpha)p-n}; \tag{4.13}
\]

while \( u \) has to satisfy (1.24, 1.25, 1.26, 1.27, 1.28), or (1.29, 1.30, 1.31, 1.32, 1.33).

**Proof.** Since we want to use directly Theorem 1.2 so we need to show that

\[
\| |x|^{\beta} u \|_{L^2 L^q} < +\infty, \quad \text{with} \quad \frac{2}{r} + \frac{n}{q} = 1 - \beta. \tag{4.15}
\]

Let’s start by the integral representation

\[
u = e^{t\Delta} u_0 - \int_0^t e^{(t-s)\Delta} \mathcal{P} \nabla \cdot (u \otimes u)(s) \, ds.
\]

and distinguish the cases \( \alpha \in [-1/2,0) \) and \( \alpha \in [0,1) \).
Case $\alpha \in [-1/2, 0)$.

\[ \| |x|^\beta u\|_{L^p \mathcal{L}^q} \leq \| |x|^\beta e^{t\Delta} u_0\|_{L^p \mathcal{L}^q} \]

\[ + \left\| |x|^\beta \int_0^t e^{(t-s)\Delta} \nabla \cdot (u \otimes u)(s) \, ds \right\|_{L^p \mathcal{L}^q} \]

= $I + II$.

By the scaling assumption and Proposition 3.4

\[ I \leq c_0 \| |x|^\alpha u_0\|_{L^{p_0} \mathcal{L}^{p_0}} \] (4.16)

provided that

\[ p_0 \leq q < \frac{np_0}{(\alpha_0 - \beta)p + n - 2}, \quad \tilde{p}_0 \leq q, \quad \Lambda(\alpha_0, p_0, \tilde{p}_0) \geq 0. \] (4.17)

We use Proposition 3.5 and scaling to bound

\[ II \leq d_0 \| |x|^\alpha u\|^2_{L^p \mathcal{L}^p} \]

provided that

\[ 2\Lambda(\alpha, p, \tilde{p}) \geq \beta, \] (4.18)

\[ 2 \leq p \leq +\infty, \quad 2 < s < +\infty, \quad p/2, \tilde{p}/2 \leq q, \quad s/2 \leq r. \] (4.19)

Condition (4.18) is ensured by

\[ \tilde{p} \geq \frac{2(n - 1)}{2\alpha - \beta + 2(n - 1)}. \] (4.20)

Then we need a triple $(\beta, r, q)$ such that (4.19) is consistent with $\frac{2}{p} + \frac{s}{q} = 1 - \beta$.

We are using Theorem 1.2 so it is necessary to restrict to $-1 \leq \beta$ and, in order to get the lowest value for $\tilde{p}$, we choose $\beta = -1$. In such a way (4.20) becomes (1.28).

By this choice we have

\[ \tilde{p} \leq p \quad \text{if} \quad -1/2 \leq \alpha, \]

that is in fact the range of $\alpha$ we have restricted on. Then we choose $q = p/2$ so by the scaling relation

\[ \frac{2}{r} - \frac{4}{s} = 2\alpha - 2 \leq 0, \]

that is consistent with $s/2 \leq r$. Because of the choice $q = p/2$ and the scaling we have to require

\[ \max \left( \frac{2}{1 - \alpha}, \frac{n}{1 - \alpha} \right) < p, \quad \text{or} \quad p = 2. \]

Then (4.17) becomes

\[ p_0 \leq q < \frac{np_0}{2p_0 - 2}, \]

that by a straightforward calculation leads to (4.12) and $\alpha_0 \in \left[ 1 - n, \frac{2 - n}{2 + n} \right]$.

Case $\alpha \in [0, 1)$. The only difference is again in the choice of $(\beta, r, q)$. Since $\alpha \geq 0$ we can reach smaller values for $\tilde{p}$ by setting $2\alpha - \beta = 1$ in (4.20), in such a way

\[ \tilde{p} \geq \frac{2(n - 1)p}{p + 2(n - 1)}. \]

We actually choose

\[ (\beta, r, q) = (2\alpha - 1, s/2, p/2). \]

It is easy to check that this is consistent with the scaling relation. Now by (4.17) and scaling we have

\[ p_0 \leq q < \frac{np_0}{(2 - 2\alpha)p_0 - 2}. \]
that by a straightforward calculation leads to (4.14) and
\[ \alpha_0 \in \left[ 1 - (1 - \alpha)n, 1 - (1 - \alpha) \frac{2n}{2 + n} \right). \]
\[ \square \]

5. OUTLOOKS AND REMARKS

In this paper we develop a technique that makes able to get new regularity criteria for weak solutions of (1.1) from a given one. In principle this machinery could be applied to many different criteria known in literature even if we basically focus on (1.2) and on Theorem 1.2.

The relations between the indexes in the main theorems are not the most general possible, for instance different choices are allowed than \( q = p/2 \) in the proofs. Anyway we prefer to lose a little in generality in order to get simpler statements.

In the second section we prove time decay estimates for the heat and Oseen kernels that we consider of independent interest. In particular we plan to use them to study the small data problem for (1.1) in future works.

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