Projective Synchronization and Control of Unified Chaotic System

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The problem of projective synchronization (ps) and control are studied in modified unified chaotic system which possess partially linearity property. The desired ratio factor of corresponding sub-system variable could be obtained by state feedback control. Theoretical analysis and numerical simulations are provided to illustrate the projective synchronization and the feasibility of the proposed control method. The effect on projective synchronization caused by channel noise and parameter mismatch are investigated in detail, the results showed that parameter mismatch has more effect on projective synchronization than channel noise does, which may be applied to chaotic secure communications.

Keywords: unified chaotic system, projective synchronization, parameter mismatch

I. INTRODUCTION

Since the seminal work by Pecora and Carroll, the synchronization of chaotic dynamical systems has been intensively studied. Different types of synchronization phenomena have been numerically observed and experimentally verified in a variety of chaotic systems. In 1999, Mainieri et al. observed a new phenomena in partially linear chaotic system, which they called projective synchronization (ps) — the drive and response vectors synchronize up to a scaling factor. This special synchronization concept attracted increasing attention in past few years.

Xu et al. investigated the stability criterion for projective synchronization in three dimensional chaotic systems, put forward a criteria for the occurrence of projective synchronization in chaotic systems of arbitrary dimension, and a necessary condition of projective synchronization in discrete-time systems of arbitrary dimensions, and applied the projective synchronization technique to chaotic secure communication field. Wang et al. studied the control of projective synchronization and put it into chaotic encryption application.

In 2004, Lu et al. presented a new unified chaotic system with continuous periodic switch (MLCL) which contains the Lorenz and Chen systems as two extremes and the Lü system as a special case. Lu et al. found a similar but non-equivalent attractor in 1999, which is known to be the dual of the Lorenz system. Recently, a chaotic system is presented by Lü et al., which bridged the gap between the Lorenz and Chen systems. And a new unified chaotic system with continuous periodic switch (MLCL) between the Lorenz and Chen system is presented under inspiration of the MLCL chaotic system is described by:

\[
\begin{align*}
\dot{x}_1 &= (25a + 10)(x_2 - x_1) \\
\dot{x}_2 &= (28 - 35a)x_1 - x_1x_3 + (29a - 1)x_2 \\
\dot{x}_3 &= x_3x_2 - \frac{8a}{3}x_3
\end{align*}
\]

where \(a = \sin^2(\omega t)\) and \(\omega\) is an adjustable parameter. The MLCL chaotic system is a non-autonomous system, with the increase of \(t\), system switches continuously between the Lorenz and Chen system, and the switching frequency is determined by the parameter \(\omega\). The largest Lyapunov exponent (LLE) of system increases with parameter \(\omega\) increasing. In the following all the differential equations are solved using fourth-order Runge-Kutta in Matlab. The abundant dynamics for different value of parameter \(\omega\) are displayed in Fig. under initial condition \(x(0) = [0.01, 0.01, 0.01]\).

II. PROJECTIVE SYNCHRONIZATION IN CHAOTIC SYSTEM

A. MLCL chaotic system

Lorenz had found the first classical chaotic attractor in 1963, Chen and Ueta have found a similar but non-equivalent attractor in 1999, which is known to be the dual of the Lorenz system. Recently, a chaotic system is presented by Lü et al., which bridged the gap between the Lorenz and Chen systems. And a new unified chaotic system with continuous periodic switch (MLCL) between the Lorenz and Chen system is presented under inspiration of the MLCL chaotic system is described by:

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B. Projective synchronization in MLCL system

System Eqs. can be rewrite as follow:

\[
\begin{align*}
\dot{x} &= M(x_3)x \\
\dot{x}_3 &= g(x, x_3) = x_1x_2 - \frac{8a}{3}x_3
\end{align*}
\]

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where $M(x) = \begin{bmatrix} -25a - 10 & 25a + 10 \\ 28 - 35a - x_3 & 29a - 1 \end{bmatrix}$, $x = (x_1, x_2)^T$. The two systems are coupled through $z$, the $z$ in the response system will be the $z$ of the drive system. The resulting system is a set of five differential equations:

$$
\begin{align*}
\dot{x}_{1m} &= (25a + 10)(x_{2m} - x_{1m}) \\
\dot{x}_{2m} &= (28 - 35a)x_{1m} - x_{1m}x_{3m} + (29a - 1)x_{2m} \\
\dot{x}_{3m} &= x_{1m}x_{2m} - \frac{8s}{\alpha}x_{3m} \\
\dot{x}_{1s} &= (25a + 10)(x_{2s} - x_{1s}) \\
\dot{x}_{2s} &= (28 - 35a)x_{1s} - x_{1s}x_{3s} + (29a - 1)x_{2s}
\end{align*}

(3)

where subscript $m$ denotes the drive part and $s$ the response part. Let $\frac{\dot{x}_{ps}}{x_{ps}} = \frac{x_{1ps}}{x_{1ps}} = \frac{x_{2ps}}{x_{2ps}} = \alpha$, where $\alpha$ is the ratio factor of state variable amplitude. We define error function as, $e_{ps} = x_{1m}x_{2s} - x_{1s}x_{2m}$, and construct the Lyapunov function $V(x) = \frac{1}{2}e_{ps}^2$. The time derivative of $V(x)$ along system $\mathbf{3}$ is

$$
\dot{V} = e_{ps} \dot{e}_{ps} = e[(\dot{x}_{1m}x_{2s} + x_{1m}\dot{x}_{2s} - \dot{x}_{1s}x_{2m} - x_{1s}\dot{x}_{2m})] = (4\sin^2(\omega t) - 11)e_{ps}^2 = (8\sin^2(\omega t) - 22)V

(4)

Since $\sin^2(t) \in [0, 1]$, $8\sin^2(\omega t) - 22 < 0$, then we have,

$$
V(t) = V(0)e^{(8\sin^2(\omega t) - 22)t} \to 0
$$

Therefore, the following equation is obtained:

$$
\lim_{t \to \infty} e = \lim_{t \to \infty} (x_{1m}x_{2s} - x_{1s}x_{2m}) = 0

(5)
$$

Error function asymptotically converges to zero, which means the projective synchronization occurs.

In order to explain this clearly, we study the evolution of the angular phase and ratio factor $\alpha$ in the cylindrical coordinates. Firstly, $x_1 = r \cos \phi, x_2 = r \sin \phi, x_3 = x_3$, from Eqs. $\mathbf{3}$, we have, $\lim_{t \to \infty} r_m \cos \phi_m \sin \phi - \cos \phi, \sin \phi_m = r_m \sin \phi - \phi_m = 0$. Since $r_s, r_m$ could not be zero all the time (if $r_s = 0$, the stabilized phenomena occurs explain in following context). if $r_m$ becomes nearly zero, some jumping points will occur along with $\alpha$ evolution, both cases have no substantial impact upon subsequent analysis and conclusion, so we have, $\lim_{t \to \infty} \sin(\phi_s - \phi_m) = 0$. It shows that frequency locking phenomena occurs in coupled system after some time of evolution.

Secondly, The time derivative of $\alpha$ along system $\mathbf{3}$ is:

$$
\dot{\alpha} = \alpha \left( \frac{\dot{r}_s}{r_s} - \frac{\dot{r}_m}{r_m} \right) = \alpha \left[ 54\sin^2(\omega t) - 9 \right] \sin(\phi_m + \phi_s) + \left( 38 - 10\sin^2(\omega t) - x_{3m} \right) \cos(\phi_m + \phi_s) \sin(\phi_s - \phi_m)

\alpha = \alpha \ast \text{func}(x_{3m}, \phi_m, \phi_s) \sin(\phi_s - \phi_m)

(6)
$$

since $\lim_{t \to \infty} \sin(\phi_s - \phi_m) = 0$, then we have, $\lim_{t \to \infty} \dot{\alpha} = 0$, which means that factor $\alpha$ converges to a constant after some time of evolution.

Fig 2 denotes the ratio factor $\alpha$ for different value of parameter $\omega$ under initial condition $[x_m(0), x_s(0)] = [0, 1, 0 - 0.1 - 1, 1]$. It showed that the parameter $\omega$ has little effect on factor $\alpha$ provided $\omega < 1$. When $\omega > 1$, $\alpha$ increases with parameter $\omega$ increasing. when $\omega_c \approx 8.935$, $\alpha$ nearly becomes zero (which means the response system was equivalently stabilized to origin, and the value $\omega_c$ is greatly dependent on initial condition), and the phase changes from negative to positive after $\alpha$ increasing continuously. In the whole process, the projective synchronization can achieve up to $10^{-6} \sim 10^{-8}$ precision level regardless of the transient time, and higher $\omega$ higher precision. Here the precision is defined as:

$$
p = \frac{\text{std}(\alpha - \text{mean}(\alpha))}{\text{mean}(\alpha)}

(7)
$$

where $\text{std}$ denotes the standard deviation and $\text{mean}$ the average. But the value of $\alpha$ could not be predicted. The varying tendency of factor $\alpha$ is well consistent with the results in literature $\mathbf{10}$.

In addition, an interesting phenomena is found, if the initial condition happens to satisfy the condition, $\frac{x_{1m}(0)}{x_{2m}(0)} = \frac{x_{1s}(0)}{x_{2s}(0)} = \alpha_0$, then factor $\alpha$ will keep unchanged during the evolution.

when parameter $\omega$ is fixed at 20, factor $\alpha$ varies totally randomly under different initial conditions, other than the prediction of value of $\alpha$. The above results accord with the fact that chaotic systems are highly sensitive to initial condition and system parameters.

### III. CONTROL ON THE PROJECTIVE SYNCHRONIZATION

In this section, we presented a continuous control method to obtain desired factor $\alpha_d$ based on system states feedback technique under initial condition of $\omega = 20$ and $[x_m(0), x_s(0)] = [0.1, 0 - 0.1 - 1, 1]$. Two schemes,
linear feedback and nonlinear feedback, are taken into consideration.

In case of linear feedback, let control item $U = k(x_s - \alpha_d x_m)$, where $k$ denotes the strength of feedback, adding $U$ to the drive part of coupled system (8) (if adding $U$ to response part, similar results can also be obtained):

$$\begin{align*}
\dot{x}_{1m} &= (25a + 10)(x_{2m} - x_{1m}) + k(x_{1s} - \alpha_d x_{1m}) \\
\dot{x}_{2m} &= (28 - 35a)x_{1m} - x_{1m}x_{3m} + (29a - 1)x_{2m} \\
&\quad + k(x_{2s} - \alpha_d x_{2m}) \\
\dot{x}_{3m} &= x_{1m}x_{2m} - \frac{8 + a}{3}x_{3m} \\
\dot{x}_{1s} &= (25a + 10)(x_{2s} - x_{1s}) \\
\dot{x}_{2s} &= (28 - 35a)x_{1s} - x_{1s}x_{3m} + (29a - 1)x_{2s}
\end{align*}$$

(8)

The projective synchronization curve is displayed in Fig.3(a).

In case of nonlinear feedback, $U = \alpha_d (x_s - \alpha_d x_m)$, and other parameters keep the same, the projective synchronization curve is plotted in Fig.3(b).

From Fig.3 and much simulation, we found:

1. Projective synchronization occurs in short time and desired ratio factor $\alpha_d$ is obtained using both methods.

2. In comparison, under small value of $\alpha$, linear feedback method has faster control response speed than nonlinear feedback does, while under large value of $\alpha$, the control response speed is comparative.

3. The synchronization precision both degrade slightly compared with the uncontrolled coupled system and wave error occurs, but the precision is still able to achieve $10^{-3} \sim 10^{-4}$ level, and larger value of $\alpha$, higher precision.

4. Large value of $\alpha$ could be obtained successfully by both methods, while such large $\alpha$ can not be observed in uncontrolled coupled system. In order to add evidence, we have plotted the drive and response subsystem phase portrait for ratio factor $\alpha = 3$.

IV. EFFECT ON PS CAUSED BY CHANNEL NOISE AND PARAMETER MISMATCH

In practical scenario, channel noise and system parameter mismatch are unavoidable. In this section, we investigate the effect on ps caused by such imperfection. the following figures are based on condition of $k = 20, \omega = 20$.

From Fig.4 and much numerical simulation, we found that under certain SNR ($SNR >= 10dB$), channel noise has less effect on ps, the precision is still able to achieve $10^{-2} \sim 10^{-3}$ level. PS could occur if taken some noise-reduction method. In addition, the effect from parameter $k$ mainly lies in control time $t_{ps}$, and $t_{ps}$ decreases while $k$ increases. The effect from parameter $\omega$ put little influence on ps partially owing to channel noise containing much frequency ingredient.

From Fig.5, we found that when parameter mismatch is limited to range of $[0, 10\%]$, ps can be achieved successfully with $10^{-2} \sim 10^{-3}$ synchronization precision. The effect from parameter $k$ lies in the control time $t_{ps}$ mentioned above. Parameter $\omega$ has more effect on ps compared with aforementioned case.
In general, parameter mismatch has much influence on ps than channel noise does, ps could be achieved with certain precision provided taken some effective measure, this property can be applied to chaos control and chaotic secure communications.

V. CONCLUSIONS

In this paper, we have investigated the projective synchronization properties in MLCM chaotic system, and present control scheme based continuous state feedback to control the MLCL system to obtain desired ratio factor \( \alpha \). The effectiveness and feasibility of our methods have been verified by computer simulation. The effect on projective synchronization caused by channel noise and parameter mismatch are investigated, the results showed that parameter mismatch has more effect on projective synchronization than channel noise does, which may be applied to chaotic secure communications.

To our best knowledge, this is the first report on the projective synchronization of modified unified chaotic system. From the viewpoint of system energy, the small ratio factor are desired to obtained, and discrete impulsive control scheme are considered to replace the continuous feedback control counterpart.

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