A CALCULUS FOR CONFORMAL HYPERSURFACES AND NEW HIGHER WILLMORE ENERGY FUNCTIONALS

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Abstract. The invariant theory for conformal hypersurfaces is studied by treating these as the conformal infinity of a conformally compact manifold. For a given conformal hypersurface embedding, a distinguished ambient metric is found (within its conformal class) by solving a singular version of the Yamabe problem. Using existence results for asymptotic solutions to this problem, we develop the details of how to proliferate conformal hypersurface invariants. In addition we show how to compute the solution’s asymptotics. We also develop a calculus of conformal hypersurface invariant differential operators and in particular, describe how to compute extrinsically coupled analogues of conformal Laplacian powers. Our methods also enable the study of integrated conformal hypersurface invariants and their functional variations. As a main application we develop new higher dimensional analogues of the Willmore energy for embedded surfaces. This complements recent progress on the existence and construction of such functionals.

Keywords: Conformally compact, conformal geometry, holography, hypersurfaces, Willmore energy, Yamabe problem.

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1. Introduction

The data for our study is a $d$-dimensional Riemannian manifold $(M, g)$ equipped with a smoothly embedded, for simplicity oriented, codimension 1 submanifold $\Sigma$, commonly termed a hypersurface:

Our aim is to develop a calculus for the study of conformal hypersurfaces including the natural invariant differential operators associated with these and conformal hypersurface invariants. The latter are natural density-valued tensor fields defined along $\Sigma$ and determined by the data $(M, g, \Sigma)$, such that, as densities, they are unchanged when $g$ is replaced by a conformally related metric $\Omega^2 g$ where $\Omega$ is a positive function. Among such invariants there are some distinguished invariants [18] that, in a precise sense, provide higher dimensional analogues of the celebrated Willmore equation studied in e.g. [28, 34]. Recently energy functionals for these objects have been constructed from conformal anomalies in a renormalised volume expansion [20] (see also [19]). A second main aim here is to apply tools developed in [17, 18] to provide a construction of manifestly conformally invariant energies with the same leading order functional gradient (with respect to variation of embedding) as the anomaly functionals. Not only do these new energies yield alternative conformally invariant higher Willmore equation, the nature of these suggests they will also be useful for analysing and even altering the functionals in [19, 20]. Alterations may be useful because the positivity of these higher “energies” is not established. It is also shown in [19] that these global invariants are related to a notion of $Q$-curvature for conformal hypersurfaces.

It is by now well-established that aspects of the intrinsic conformal geometry of a hypersurface $\Sigma$ can be effectively treated by taking, at least in some collar neighborhood of $\Sigma$, the bulk metric $g$ to be the Poincaré–Einstein metric of Fefferman–Graham (FG) [9]. This amounts to solving Einstein’s equations for metrics that are singular along $\Sigma$. Unfortunately this approach is not suitable for a study of hypersurface invariants since it essentially forces the embedding of $\Sigma$ to be totally umbilic [27, 13], i.e., everywhere vanishing trace-free second fundamental form. However, in a companion paper [18], we showed that the singular Yamabe problem provides exactly the right weakening of the Poincaré–Einstein condition to yield a powerful “holographic” framework for the study of conformal hypersurface invariants.

Problem 1.1 (Singular Yamabe). Given an oriented hypersurface $\Sigma$, find a smooth function $\sigma$ such that

(i) $\sigma$ is a defining function for $\Sigma$ (so $\Sigma$ is the zero locus $Z(\sigma)$ and $d\sigma \neq 0$ along $\Sigma$); and
(ii) the singular metric $g^\sigma = g/\sigma^2$ has scalar curvature $Sc^{g^\sigma} = -d(d-1)$. 

The second part of this problem is governed by the non-linear pde
\begin{equation}
S(g, \sigma) := |d\sigma|^2 - \frac{2}{d} \sigma [\Delta^g + \frac{\text{Se}_g}{2(d-1)}] \sigma = 1.
\end{equation}
Here $d$ is the exterior derivative and $\Delta^g$ is the (negative energy) Laplacian. Clearly, since the metric-defining function pair $(\Omega^2 g, \Omega \sigma)$ define the same singular metric $g^0$, the above equation is conformally invariant; $S(\Omega^2 g, \Omega \sigma) = S(g, \sigma)$. Therefore the above problem can be treated using conformal geometry.

1.1. Elements of tractor calculus and the singular Yamabe problem. A key tool for studying problems in conformal geometry is the tractor calculus of $\text{[3]}$ (see also $\text{[15]}$). The standard tractor bundle and its connection are equivalent to the normal conformal Cartan connection $\text{[6, 7]}$, and are related to objects first developed by Thomas $\text{[31]}$.

Recall that a conformal structure $c$ is an equivalence class of Riemannian metrics where any two metrics $g, g' \in c$ are related by a conformal rescaling; that is $g' = \Omega^2 g$ with $C^\infty M \ni \Omega > 0$. Locally each $g \in c$ determines a volume form and, squaring this, a section of $(\Lambda^d TM)^2$. So, on a conformal manifold $(M, c)$ there is a canonical section $g$ of $\mathcal{C}^2 TM \otimes \mathcal{E} M^2$ called the conformal metric. Here $\mathcal{E} M[w]$, for any $w \in \mathbb{R}$, denotes the conformal density bundle. This is the natural (oriented) line bundle equivalent, via the conformal structure $c$, to $\left((\Lambda^d TM)^2\right)^2$.

On a conformal manifold $(M, c)$, there is no distinguished connection on the tangent bundle $TM$. However there is a canonical tractor metric $h$ and linear connection $\nabla^T$ (preserving $h$; the superscript $T$ will often be suppressed) on a related higher rank vector bundle known as the tractor bundle $T M$, which yields a simplified treatment of Problem $\text{[11]}$. The tractor bundle $T M$ is not irreducible but has a composition series summarised via a semi-direct sum notation
\begin{equation}
T M = \mathcal{E} M[1] \oplus T^* M[1] \oplus \mathcal{E} M[-1].
\end{equation}
Here $T^* M[w] := T^* M \otimes \mathcal{E} M[w]$. A choice of metric $g \in c$, or equivalently a nowhere vanishing section $\tau$ of $\mathcal{E} M[1]$ by setting $g = \tau^{-2} g$, determines an isomorphism
\begin{equation}
T M \cong^g \mathcal{E} M[1] \oplus T^* M[1] \oplus \mathcal{E} M[-1].
\end{equation}
Computations relying on this isomorphism will be referred to as “working in a scale” and the section $\tau$ is called a true scale (later the term scale will be used for more general sections of $\mathcal{E} M[1]$). We will employ an abstract index notation both for sections of tensor bundles in general and for sections $V^A$ of $T M$, and thus write $V^A \cong (v^+, v_a, v^-) =: [V^A]^g_0$ to denote the image of $V^A$ under the above isomorphism. We denote $h(V, V)$ by $V^2$, and in this scale the squared length of $V$ with respect to the tractor metric is given by
\begin{equation}
V^2 \cong 2v^+v^- + g_{ab}v^av^b.
\end{equation}

It is propitious to reformulate the notion of a defining function in terms of densities: A section $\sigma$ of $\mathcal{E} M[1]$ is said to be a defining density for a hypersurface $\Sigma$ if $\Sigma = Z(\sigma)$ and $\nabla \sigma$ is nowhere vanishing along $\Sigma$ where $\nabla$ is the Levi-Civita connection for some, equivalently any, $g \in c$. For a defining density $\sigma$, we may define a corresponding scale tractor
\begin{equation}
T M \ni I^A_{\sigma} := (\sigma, \nabla_\sigma \sigma, -\frac{1}{d}(\Delta^g + J)\sigma) =: \hat{D}^A \sigma.
\end{equation}
Here $\nabla^g$ is the Levi-Civita connection of $g$ and $\Delta^g$ its Laplacian, while $J := -\text{Se}_g / (d(d-1))$. In Riemannian signature, it follows immediately that for any defining density $\sigma$ we
have that
\[ I_\sigma^2 > 0 \]
holds in a neighbourhood of \( \Sigma \). (We will implicitly use this fact in formulae involving the reciprocal function \( 1/I^2 \).) Moreover,
\[ I_\sigma^2 \equiv S(g, \sigma). \]

In words, the singular Yamabe Problem 1.1 amounts to finding a defining density whose scale tractor has squared length equalling unity.

It is worthwhile observing that any FG Poincaré–Einstein metric \( g_o \) solves the singular Yamabe problem. However, for general boundary conformal geometries, the problem of finding a smooth FG Poincaré–Einstein metric is obstructed, and a similar statement holds for the singular Yamabe problem [2]. Therefore, we formulate an asymptotic version of Problem 1.1:

**Problem 1.2.** Find a smooth defining density \( \sigma \) such that
\[ (1.3) \quad I_\sigma^2 = 1 + \sigma^d B, \]
for some smooth \( A_\ell \in \Gamma(\mathcal{E}M[-\ell]) \), where \( \ell \in \mathbb{N} \cup \{\infty\} \) is as high as possible.

Building on the foundational work [2], a solution to this problem was given in [18]:

**Theorem 1.3.** Given a defining density \( \sigma_0 \), there exists an improved defining density
\[ (1.4) \quad \sigma = \sigma_0 \left( 1 + \alpha_1 \hat{\sigma} + \cdots + \alpha_{d-1} \hat{\sigma}^{d-1} \right), \]
where \( \hat{\sigma} = \sigma_0 / \sqrt{I_{\sigma_0}^2} \) in a neighborhood of \( \Sigma \), and \( \alpha_k \) are smooth densities, such that
\[ (1.5) \quad I_{\sigma}^2 = 1 + \sigma^d B. \]
Moreover, the restriction of the weight \( w = -d \) density \( B \) to the hypersurface \( \Sigma = Z(\sigma) \), denoted \( B := B|_{\Sigma} \) and termed the “obstruction density”, is a natural conformal hypersurface invariant which depends only on the data of the conformal embedding \( \Sigma \hookrightarrow (M, c) \).

The improved defining density \( \sigma \) of the theorem is unique modulo the addition of terms of order \( \sigma^{d+1} \) and any such defining density is termed a **conformal unit defining density**. Sections of conformal (possibly tensor-valued) density bundles expressible in a choice of scale in terms of the metric and polynomials built from jets of \( \sigma \) are termed termed **coupled conformal invariants** (see [18, Section 6.1] for a precise definition). The existence of conformal unit defining densities allows us to proliferate conformal hypersurface invariants as encapsulated by the following theorem:

**Theorem 1.4** (See [18]). Suppose that \( \sigma \) is a conformal unit defining density and \( P(c, \sigma) \) is a weight \( w \) coupled conformal invariant depending pointwise on at most the \( d \)-jet of \( \sigma \). Then the restriction of \( P \) to \( \Sigma \) is a weight \( w \) conformal hypersurface invariant.

Application of this theorem requires the construction of the needed coupled conformal invariants. A main direction of this paper is to explain how to systematically produce these by the application of tractor calculus.

Another main outcome of our approach is the construction of invariant differential operators determined by the conformal embedding. Notable among these are the extrinsically coupled conformal Laplacian powers \( P_k \) of [18]; for \( k \) even these take the form \( \Delta^{k/2} \) plus lower order curvature terms and generalise the Laplacian powers of [21]. An application of these is the construction of scalar invariants that cannot be directly reached
from the above theorem. In particular, we can produce invariants of the weight \( w = 1 - d \)
that allows them to be integrated over a hypersurface. This exploited in the following result:

**Theorem 1.5.** Given a closed embedded hypersurface \( \Sigma \) in \( M \), the functional

\[
\int_\Sigma N^A P_{d-1} N_A
\]

is a conformal invariant of \( \Sigma \). With respect to variation of the embedding, the gradient of
this functional is a conformal hypersurface invariant. For even dimensional hypersurfaces
this is a conformal hypersurface invariant with linear leading term in agreement with the
obstruction density.

Here \( N_A \) is the hypersurface normal tractor of [3], see Equation (3.16).

The last statement of Theorem 1.5 shows that for \( \Sigma \) of even dimension, these energy
functionals are genuinely quadratic at leading order, meaning that their variational
gradients are linear at linear leading order. Thus these gradients also provide higher
dimensional analogues of the Willmore invariant and this proves that the functionals are higher
dimensional analogues of the Willmore energy. Exact agreement (not just leading
order) between the gradient and the obstruction density is verified for surfaces in Example
6.2 and for 3-dimensional hypersurfaces in [10]. Physically, these functionals (in both
dimension parities) are candidate actions for rigid membrane dynamics.

1.2. **Structure of the article.** Apart from the new results established here, this paper
is strongly linked to [18]. In one direction, an objective here is to show how the formalism introduced in [18] gives an effective calculus for the computation and treatment
of conformal hypersurface invariants. In the other direction, many of the results in [18]
can only be fully appreciated and exploited when reinterpreted in terms of basic Riemannian geometry formulae; producing these involves considerable subtlety, and so a second
goal is illustrate how such formulae may be extracted.

In Section 2, we review the theory of Riemannian hypersurface invariants, and show
how these may be treated via a Riemannian analog of the singular Yamabe problem. In Section 3 we show how existence of conformal unit defining densities alone allows
us to proliferate conformal hypersurface invariants. As an application, we compute the
obstruction density in low dimensions. Then in Section 4 we develop the tractor calculus
of conformal hypersurface invariants. This allows powerful tractor techniques to be
applied to these problems. Section 5 takes up the problem of constructing invariant
differential operators acting on conformal hypersurface invariants. As an application, we
calculate extrinsically coupled conformal Laplacian powers in low dimensions. The final
Section 6 treats Theorem 1.5 and gives low dimensional examples.

1.3. **Notation.** Our notations for standard objects in Riemannian geometry, hypersurface
theory and the conformal tractor calculus coincides with that of [18] Sections 2.1, 2.3
and 3.1], but we will also remind readers of key definitions at the appropriate junctures.

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2. HYPERSURFACE INVARIANTS

To prepare for our study of conformal hypersurface invariants we first demonstrate how Riemannian hypersurface invariants can be efficiently treated via an analog of the singular Yamabe Problem [12]. Since locally any hypersurface is the zero set of some defining function, there is no loss of generality in restricting to hypersurfaces $\Sigma$ which are the zero locus $Z(s)$ of some defining function $s$. To further simplify our discussion we also assume that $M$ is oriented with volume form $\omega$. Given a hypersurface in $M$, it has an orientation determined by $s$ and $\omega$, as $ds$ is a conormal field. Different defining functions are compatibly oriented if they determine the same orientation on $\Sigma$.

**Definition 2.1.** For hypersurfaces, a scalar Riemannian pre-invariant is a function $P$ which assigns to each pair consisting of a Riemannian $n$-manifold $(M,g)$ and hypersurface defining function $s$, a function $P(s;g)$ such that:

(i) $P(s;g)$ is natural, in the sense that for any diffeomorphism $\phi : M \to M$ we have $P(\phi^*s;\phi^*g) = \phi^*P(s;g)$.

(ii) The restriction of $P(s;g)$ is independent of the choice of oriented defining functions, meaning that if $s$ and $s'$ are two compatibly oriented defining functions such that $Z(s) = Z(s') =: \Sigma$ then $P(s;g)|_{\Sigma} = P(s';g)|_{\Sigma}$.

(iii) $P$ is given by a universal polynomial expression such that, given a local coordinate system $(x^a)$ on $(M,g)$, $P(s;g)$ is given by a polynomial in the variables

$$g_{ab}, \quad \partial_1 g_{bc}, \quad \ldots, \quad \partial_1 \partial_2 \ldots \partial_1 g_{bc}, \quad (\det g)^{-1},$$

$$s, \quad \partial_1 s, \quad \ldots, \quad \partial_1 \partial_2 \ldots \partial_1 s, \quad ||ds||_{g}^{-1}, \omega_{a_1 \ldots a_d},$$

for some positive integers $k, \ell$.

A scalar Riemannian invariant of a hypersurface $\Sigma$ is the restriction $P(\Sigma;g) := P(s;g)|_{\Sigma}$ of a pre-invariant $P(s;g)$ to $\Sigma := Z(s)$.

In (iii) $\omega_a$ means $\partial/\partial x^a$, $g_{ab} = g(\partial_a, \partial_b)$, det $g = \det (g_{ab})$ and $\omega_{a_1 \ldots a_d} = \omega(\partial_{a_1}, \ldots, \partial_{a_d})$. For (i) note that if $\Sigma = Z(s)$, then $\phi^{-1}(\Sigma)$ is a hypersurface with defining function $\phi^*s$. The conditions (i),(ii) and (iii) mean that any Riemannian invariant $P(s;g)|_{\Sigma}$ of $\Sigma$ is entirely determined by the data $(M,g,\Sigma)$. Then in this notation the naturality condition of (i) implies $\phi^*(P(\Sigma, g)) = P(\phi^{-1}(\Sigma), \phi^*g)$. The above definition extends mutatis mutandis to tensor valued hypersurface pre-invariants and invariants.

**Example 2.2.** The quantities

$$P(s;g) = \frac{1}{d-1} \nabla^a \left( \frac{\nabla_a s}{|\nabla s|} \right)$$

and

$$P_{ab}(s;g) = \left( \nabla_a - \frac{(\nabla_a s)(\nabla_c s)}{|\nabla s|^2} \nabla_c \right) \left( \frac{\nabla_b s}{|\nabla s|} \right)$$

are preinvariants, respectively, for the mean curvature $H = P(s;g)|_{\Sigma = Z(s)}$ and second fundamental form $\Pi_{ab} = P_{ab}(s;g)|_{\Sigma = Z(s)}$.

Property (ii) of preinvariants in Definition 2.1 can be exploited to expedite hypersurface invariant computations. For example, a for many purposes simpler mean curvature preinvariant is

$$P(s_1;g) = \frac{\Delta s_1}{d-1},$$

where $s_1 = \frac{s}{|\nabla s|} \left( 1 + \frac{1}{2} \frac{s}{|\nabla s|} (\nabla s).\nabla \log |\nabla s| \right)$. 

\[\ast\]
To see this, one computes $|\nabla s_1|$ and finds that

$$|\nabla s_1|^2 = 1 + s^2 A,$$

where the function $A$ is smooth. This implies that for any defining function $s$, we have that $s_1$ is also a defining function, but with the improved behavior of the length of its gradient quoted above which allows the mean curvature to be computed directly from its Laplacian. Similarly, the second fundamental form preinvariant becomes simply

$$P_{ab}(s_1;g) = \nabla_a \nabla_b s_1.$$

Following this line of reasoning, we pose the following problem for Riemannian hypersurface defining functions:

**Problem 2.3.** Given $\Sigma$, a smooth hypersurface in a Riemannian manifold $(M,g)$ find a defining function $s$ such that $n := \nabla s$ obeys

$$|n|^2 = 1 + s^{\ell + 1} A,$$

for some $A \in C^\infty(M)$ and $\ell \in \mathbb{N} \cup \infty$ as high as possible.

Problem 2.3 can be solved by an explicit recursion to $O(s^\infty)$ [18]. Moreover, the recursion uniquely determines $s$ to any given order. Defining functions obeying $|\nabla s| = 1$ are called *unit defining functions* and we shall also use this terminology in the setting where Problem 2.3 has been solved to sufficiently high order to uniquely determine the jets of $s$ required to evaluate any quantities involved. Note that in fact $|n|^2 = 1$ can be solved in a neighborhood of $\Sigma$, whereby $s$ measures the geodesic distance to the hypersurface. This is a standard manoeuvre in the construction of Gaussian normal coordinates (see for example [33]). For explicit computations the recursion is useful.

**Example 2.4.** Consider the hypersurface in Euclidean space given by the graph of a smooth function $f(x,y)$. To compute the mean curvature we need data of the defining function up to its 2-jet. Thus, beginning with the defining function $z - f(x,y)$, we employ the improvement formula in Equation (2.1) to find

$$s = \frac{z - f}{\sqrt{f_x^2 + f_y^2} + 1} \left( 1 - \frac{1}{2} (z - f) \frac{f_x^2 f_{xx} + 2 f_x f_y f_{xy} + f_y^2 f_{yy}}{(f_x^2 + f_y^2 + 1)^2} \right).$$

It is not difficult to verify that this defining function obeys $|\nabla s|^2 = 1 + s^2 A$ where $A$ is smooth. Moreover, the mean curvature is

$$H = \frac{1}{2} \Delta s|_{z=f} = -\frac{1}{2} \frac{f_{xx} + f_{yy} + f_x^2 f_{xx} - 2 f_x f_y f_{xy} + f_y^2 f_{yy}}{(f_x^2 + f_y^2 + 1)^{3/2}}.$$ 

Readers will recognize the standard mean curvature formula for graphs.

Before developing further the calculus of unit defining functions and applying this to the singular Yamabe problem, we quickly review key ingredients of Riemannian hypersurface theory.

### 2.1. Riemannian hypersurfaces

Given a vector field $\hat{n}^a \in \Gamma(TM)$ such that $\hat{n}^a|_\Sigma$ is a unit normal, we may identify the tangent bundle $T\Sigma$ and the subbundle $TM^\top$ of $TM|_\Sigma$ orthogonal to $\hat{n}^a$. Thus we may employ this isomorphism to identify sections of $T\Sigma$ and $TM^\top$ and use the abstract indices of $TM$ to label these. In particular the projection of tensors on $M$ to hypersurface tensors will be denoted by the symbol $\top$; for a vector $v \in \Gamma(TM)$ we thus have $v^\top := v - \hat{n} \hat{n}.v$. 

Readers will recognize the standard mean curvature formula for graphs.
In general, objects intrinsic to $\Sigma$ will be labeled by a bar. For example, for a vector $\bar{v}^a \in \Gamma(T\Sigma)$ and any extension of this to $v^a \in \Gamma(TM)$ subject to $v|_\Sigma = \bar{v}|_\Sigma = v$, the intrinsic and ambient Levi-Civita connections, $\bar{\nabla}$ and $\nabla$ are related by the Gauß formula
\begin{equation}
\bar{\nabla}_a v^b = (\nabla_a v^b + \hat{n}^b \Pi_{ac} v^c)|_\Sigma,
\end{equation}
where the second fundamental form $\Pi_{ab} \in \Gamma(\hat{\otimes}^2 T^*\Sigma)$ is given by
\begin{equation}
\Pi_{ab} = \nabla_a \hat{n}_b|_\Sigma.
\end{equation}
Identifying $\hat{n}$ and $\nabla s/|\nabla s|$, we see that this formula is the origin of the preinvariant given in Example (2.2).

2.2. **Unit defining functions and Riemannian hypersurface invariants.** Given a unit defining function $s$ we can proliferate Riemannian hypersurface invariants simply by computing all possible tensors built from gradients $\bar{\nabla}_a \bar{\nabla}_b \cdots \nabla_c s$, Riemannian invariants built from Riemann tensors, contractions of these objects and then studying their restriction to $\Sigma$. This methodology also yields efficient derivations of the relations of Gauß, Codazzi, Mainardi and Ricci.

For example, call $n_a := \nabla_a s$. Then from the second fundamental form preinvariant given in Example 2.2, we see immediately that
\begin{equation}
\Pi_{ab} = \nabla_a n_b|_\Sigma.
\end{equation}
However,
\begin{equation}
\nabla_a \nabla_b n_c - \nabla_b \nabla_a n_c = R_{abcd} n^d.
\end{equation}
Restricting the above relation to $\Sigma$, applying the projector $\top$ to the indices $a$, $b$ and $c$, and then using the Gauß formula (2.3), the above relation becomes the well known Codazzi–Mainardi equation
\begin{equation}
\nabla_a \Pi_{bc} - \nabla_b \Pi_{ac} = (R_{abcd} \hat{n}^d)^\top.
\end{equation}
Similar manoeuvres yield the Gauß equation
\begin{equation}
\bar{R}_{abcd} = R_{abcd}^\top + \Pi_{ac} \Pi_{bd} - \Pi_{ad} \Pi_{bc},
\end{equation}
and Ricci relation
\begin{equation}
\Pi_{ab} \Pi^{ab} - (d - 1)^2 H^2 = S c - 2 \text{ Ric}(\hat{n}, \hat{n}) - \hat{S} c.
\end{equation}
For surfaces embedded in three dimensional Euclidean spaces, the above gives Gauß’ *Theorema Egregium*.

We can also compute expressions involving higher jets of $s$: Using the fact $n^a n_a = 1$ to all orders, it follows that $\nabla_a n_b = n^a \nabla_a n_b = n^a \nabla_b n_a = \frac{1}{2} \nabla_b (n^2) = 0$ to all orders along $\Sigma$. Thus, remembering that $n^a|_\Sigma = \hat{n}^a$,
\begin{equation}
\nabla_a \nabla_b \nabla_c s|_\Sigma = \nabla_a \Pi_{bc} - \hat{n}_b \Pi^2_{ac} - \hat{n}_c \Pi^2_{ab} - \hat{n}_a \Pi^2_{bc} - \hat{n}_a R_{d e} \hat{n}^d \hat{n}^e\hat{n}_c.
\end{equation}
Here we have denoted $\Pi^2_{ab} := \Pi^{2}_{ac}$. More generally, we can compute the $(k + 1)^{\text{th}}$ covariant derivative $\nabla_n \nabla_b \cdots \nabla_c s$ in terms of a $\nabla^\top$ derivative of the $k^{\text{th}}$ covariant derivative $\nabla_b \cdots \nabla_c s$ and lower transverse-order derivatives of $s$ (transverse-order counts the number of transverse derivatives $\nabla_n$ in the obvious way, see [4] where it is called normal-order) since
\begin{equation}
\nabla_n \nabla_b \cdots \nabla_c s = \nabla_n \nabla_b \cdots \nabla_c s + n_a \hat{n}^d \nabla_d \nabla_b \cdots \nabla_c s,
\end{equation}
and the fact that $n^a \nabla_b n_a = 0$ to all orders enables us to re-express the second term in terms of $k^{\text{th}}$ derivatives of $s$. Thus by induction the result (2.9) generalises to compute
hypothesis invariants in terms of any number of gradients of a unit defining function \(s\).

We collect some useful identities derived from this observation in the following example:

**Example 2.5.** Expression (2.5) for the second fundamental form implies that the mean curvature obeys

\[
\nabla . n \big|_{\Sigma} = (d - 1)H.
\]

Contracting the immediately subsequent display with \(n^a\) gives

\[
\nabla_a \nabla_b n_c \big|_{\Sigma} = -\Pi^a_{bc} + R_{abcd} \hat{n}^a \hat{n}^d.
\]

The trace of this equation gives

\[
\nabla . n \nabla . n \big|_{\Sigma} = -\Pi^a_{ab} \Pi^b_{ac} + R_{abcd} \hat{n}^a \hat{n}^d.
\]

Finally, for \(f\) any smooth extension of \(\bar{f} \in C^\infty \Sigma\), the ambient and hypersurface Laplacians are related by

\[
(\Delta^g - \nabla . n \nabla_n - \nabla_n^2) f \big|_{\Sigma} = \bar{\Delta} \bar{f}.
\]

\(\star\)

In summary, given a unit defining function, we can proliferate hypersurface invariants by constructing ambient, coupled Weyl invariants (in the sense of Weyl’s classical invariant theory). In fact, the recursion discussed above, establishes the following result:

**Theorem 2.6.** If \(s\) is a unit defining function for a Riemannian hypersurface \(\Sigma\) then, for any integer \(k \geq 1\), the quantity \(\nabla^k s \big|_{\Sigma}\) may be expressed as \(\nabla^{k-2} \Pi\) plus a linear combination of partial contractions involving the conormal \(\hat{n}\), \(\nabla^\ell \Pi\) for \(0 \leq \ell \leq k - 3\), and the Riemannian curvature \(R\) and its covariant derivatives (to order at most \(k - 3\)).

Thus any tensor of the form

\[
\text{Partial-contraction}((\nabla \cdots \nabla s) \cdots (\nabla \cdots \nabla s)(\nabla \cdots \nabla R) \cdots (\nabla \cdots \nabla R)) \big|_{\Sigma},
\]

yields a Riemannian hypersurface invariant. This may be re-expressed as linear combination of tensors built as partial contractions of undifferentiated conormals, as well as the second fundamental form and the Riemann curvature as well as derivatives thereof.

The main thrust of our article is to treat conformal hypersurface invariants by constructing ambient, coupled Weyl invariants (in the sense of Weyl’s classical invariant theory). A dictionary for this analogy is tabulated below:

| Riemannian | Conformal |
|------------|-----------|
| unit defining function | conformal unit defining density |
| \(|\nabla s|_g^2 = 1\) | \(|\nabla \sigma|_g^2 - \frac{2}{d} \sigma \left[ \Delta^g + \frac{S^g}{2(d-1)} \right] \sigma = 1\) |
| Weyl’s invariant theory | Weyl invariants via tractors |

As implied by this table, a complete treatment requires that we introduce a tractor calculus for the computation of ambient coupled conformal invariants. However, simpler aspects of that program can actually be handled with the elementary unit defining function calculus described above.
2.3. Unit defining functions and the singular Yamabe problem. Theorem \[1.3\] ensures that any defining function \(s\) can be improved to a defining density function \(\sigma(s)\) obeying the asymptotic singular Yamabe condition

\[
|\nabla \sigma|^2_g \geq \frac{2}{d} \sigma \left[ \Delta^g + \frac{Sc^g}{2(d-1)} \right] \sigma = 1 + \sigma^d B_{\sigma(s)},
\]

where \(B_{\sigma(s)}\) is smooth. It is possible to directly implement the recursion of [18] to explicitly solve the singular Yamabe problem to the order required for studying the Willmore invariant. This is very useful for applications involving explicit metrics. While this is technically intensive, simplifications arise if one takes \(s\) to be a unit defining function, and in particular if one restricts to the case of a Euclidean ambient space. We record our solution to this problem below:

**Lemma 2.7.** Let \(s\) be a unit defining function for a hypersurface embedded in \(d\)-dimensional Euclidean space, and call \(n = \nabla s\). Then solutions to Equation (2.14) are given by

\[
\begin{align*}
\sigma(s) &= s + \frac{s^2}{4} \nabla n + \frac{s^3}{72} (\nabla n)^2, & d = 3, \\
\sigma(s) &= s + \frac{s^2}{6} \nabla n + \frac{s^3}{18} (\nabla n)^2 + \frac{s^4}{1728} \left( 6 \Delta \nabla n + 4 \nabla n \nabla n \nabla n + \frac{14}{9} (\nabla n)^3 \right), & d = 4,
\end{align*}
\]

with

\[
\begin{align*}
B_{\sigma(s)} &= -\frac{1}{12} \left( 2 \Delta \nabla n + 2 \nabla_n^2 \nabla n + 8 \nabla n \nabla n \nabla n + 3 (\nabla n)^3 \right), & d = 3, \\
B_{\sigma(s)} &= -\frac{1}{108} \left( 9 \nabla_n \Delta \nabla n + 12 \nabla n \Delta^g \nabla n + 6 \nabla n \nabla_n^2 \nabla n + 2 \nabla n \nabla n \nabla n \nabla n + 3 (\nabla n)^3 \right) + 6 (\nabla n)^2 \nabla n \nabla n + 4 (\nabla n)^4, & d = 4.
\end{align*}
\]

**Proof.** The first half of this Lemma can be proved by following the algorithm given in Proposition 4.9 of [18] and thereafter computing \(S(g, \sigma)\) as given in (1.1) (i.e. (2.14)). Alternatively, since the lemma gives explicit formulae for the improved defining function, one can simply directly evaluate \(S(g, \sigma(s))\) for the quoted \(\sigma(s)\). Either method only requires an elementary calculation. \(\square\)

According to Theorem \[1.3\], the quantity \(B_{\sigma(s)}\) yields a natural conformal hypersurface invariant upon restriction to \(\Sigma\). Indeed, \(B_{\sigma(s)}\) equals the obstruction density computed in the scale \(g\). It is interesting therefore to compute this invariant. For that we specialize Equations (2.10), (2.12) and (2.11) to a flat ambient space, and apply the recursion underlying Theorem 2.6 to find

\[
\begin{align*}
\nabla_n \nabla n|_\Sigma &= \Pi^a_a = (d-1)H, \\
\nabla_n \nabla n|_\Sigma &= -\Pi^{ab} \Pi_{ab} = -\Pi^a_a \Pi^b_b - (d-1)H^2, \\
\nabla_n^2 \nabla n|_\Sigma &= 2 \Pi^{ab} \Pi^{bc} \Pi^{ca} = 2 \Pi^{ab} \Pi^{bc} \Pi^{ca} + 6H \Pi^{ab} \Pi^{cb} + 2(d-1)H^3.
\end{align*}
\]

In the above \(\Pi^{ab}\) denotes the trace-free second fundamental form

\[
\Pi_{ab} := \Pi_{ab} - H g_{ab},
\]

which is well known to be a conformal hypersurface invariant. It is not difficult to use these identities and Equation (2.13) to establish that

\[
\Delta \nabla n|_\Sigma = (d-1) \Delta H + 2\Pi^{ab} \Pi^{bc} \Pi^{ca} - (d-7)H \Pi^{ab} \Pi^{bc} + (d-1)(d-3)H^3.
\]

The above results combined with the \(d = 3\) case of Lemma 2.7 give the following:
Proposition 2.8. For surfaces in conformally flat three-manifolds,

\[(2.16)\quad B_{\sigma(0)} = \frac{1}{3} (\bar{\Delta} H + H \bar{\Pi}_{ab} \bar{\Pi}^{ab}) .\]

Remark 2.9. The above result was first obtained in [2]. Using the standard relation between Gauß and mean curvatures in Euclidean 3-space, namely \(K = H^2 - \frac{1}{2} \bar{\Pi}_{ab} \bar{\Pi}^{ab}\), the above display becomes the Willmore invariant, or in other words the functional gradient of the Willmore energy functional (cf. [18, 17]).

Exactly the same apparatus can be applied to the second half of Lemma 2.7 to give the analogous four dimensional result:

Proposition 2.10. For hypersurfaces in conformally flat four-manifolds,

\[B_{\sigma(0)} = \frac{1}{6} \left( (\bar{\nabla}_a \bar{\Pi}_{ab})^2 + 2 \bar{\Pi}^{ab} \bar{\Pi}_{ab} + \frac{3}{2} \bar{\nabla}_a \bar{\Pi}_b \bar{\Pi}_b - 2 \bar{\Pi}_{ab} \bar{\Pi}^{ab} + (\bar{\Pi}^{ab} \bar{\Pi}^{ab})^2 \right) .\]

An alternate proof of this proposition based on the holographic formula for the obstruction density \(B\) given in Theorem 7.7 of [18] can be found in [10].

Remark 2.11. The trace-free second fundamental form \(\bar{\Pi}_{ab}\), being conformally invariant, can be extended to an invariant hypersurface tractor \(L_{AB} \in \Gamma(\mathcal{T}(AB)[\Sigma[-1]])\) known as the tractor second fundamental form [23, 30] see Section 4 for details. In these terms, the above display becomes

\[B = \frac{1}{6} ( (\bar{D}_A L_{BC})(\bar{D}^A L^{BC}) + (L_{AB} L^{AB})^2 ) ,\]

where \(\bar{D}^A\) is the Thomas D-operator intrinsic to \(\Sigma\). The above result provides an independent check of conformal invariance, because this quantity is by construction a boundary conformal invariant.

3. Conformal hypersurface invariants

Conformal hypersurface invariants are defined to be the Riemannian invariants (see Definition 2.1) that are distinguished by the property of possessing suitable covariance property under local metric rescalings:

Definition 3.1. A weight \(w\) conformal covariant of a hypersurface \(\Sigma\) is a Riemannian hypersurface invariant \(P(\Sigma, g)\) with the property that \(P(\Sigma, \Omega^2 g) = \Omega^w P(\Sigma, g)\), for any smooth positive function \(\Omega\). Any such covariant determines an invariant section of \(\mathcal{E}[\Sigma][w]\) that we shall denote \(P(\Sigma; g)\), where \(g\) is the conformal metric of the conformal manifold \((M, [g])\). We shall say that \(P(\Sigma; g)\) is a conformal invariant of \(\Sigma\). When \(\Sigma\) is understood by context, the term conformal hypersurface invariant will refer to densities or weighted tensor fields which arise this way.

Example 3.2. Given a defining function \(s\) and \(g \in c\), the quantity

\[P_a(s; g) = \frac{\nabla s}{|\nabla s|_g}\]

is a preinvariant for the Riemannian hypersurface invariant \(P_a(\Sigma; g) = \hat{n}_a\), termed the unit conormal. Since

\[P_a(s; \Omega^2 g) = \Omega P_a(s; g) ,\]
the unit conormal \( \hat{n}_a \) is a weight \( w = 1 \) conformal hypersurface invariant. In contrast, the mean curvature preinvariant \( P(s; g) \) of Example 2.2 obeys
\[
P(\Sigma; \Omega^2 g) = \Omega^{-1} \left( P(\Sigma; g) - \frac{\hat{n}_a \Upsilon}{\delta - 1} \right),
\]
where \( \Upsilon_a := \nabla_a \log \Omega \), so the mean curvature is not a conformal hypersurface invariant. Note however, under metric rescalings \( \Omega \) subject to \( \hat{n}_a \Upsilon = 0 \), i.e., precisely those corresponding to the intrinsic conformal class of metrics \( \bar{c} \) along \( \Sigma \), the mean curvature transforms as a section of \( \mathcal{E} \Sigma[-1] \).

3.1. Computing the obstruction density. Theorem 2.6 describes how to relate the jets of the Riemannian canonical unit defining function \( s \) to the regular invariants of the Riemannian hypersurface that it defines. Here we explain the corresponding algorithm for computing the jets of the canonical conformal unit defining density \( \sigma \) described in Theorem 1.3 and then apply this to computations of the obstruction density. This uses ideas similar to the Riemannian case, but the recursion is more subtle.

In a conformal manifold \((M^d, c)\), \( d \geq 3 \), we consider a hypersurface \( \Sigma \) given as the zero locus of a smooth defining density. We need some key identities. For these we calculate with respect to some metric in the conformal class, \( g \in c \) (but use the conformal metric \( g \) to raise and lower indices). Recall that a conformal unit defining density \( \sigma \) is a defining density satisfying
\[
n^2 = 1 - 2\rho \sigma + \sigma^d B \quad \Leftrightarrow \quad I^d_\sigma = 1 + \sigma^d B,
\]
for some smooth \( B \), where \( n \) is used to denote \( \nabla \sigma \), and (cf. (1.1))
\[
[I^A_\sigma] := [\hat{D}^A \sigma] = \begin{pmatrix} \sigma \\ n_a \\ \rho \end{pmatrix}, \quad \rho := \rho(\sigma) = -\frac{1}{d}(\Delta \sigma + J \sigma).
\]
Such a defining density exists by Theorem 1.3 and is canonical to \( \mathcal{O}(\sigma^d+1) \). Note that in the above, \( n^2 \) is defined via the conformal metric \( n^2 = g^{-1}(\nabla \sigma, \nabla \sigma) \). Display (3.1) gives the failure of \( n \) to be a unit vector field away from \( \Sigma \). Also, as above, we identify \( T\Sigma \cong TM^\perp \) and shall write \( \gamma_{ab} := g_{ab} - n_a n_b \); along \( \Sigma \) this restricts to \( \bar{g} \), the induced metric. We will often denote the scale tractor \( I_\sigma \) of the conformal unit defining density \( \sigma \) simply by \( I \). We also heavily employ the (slightly ambiguous) notation \( =_\Sigma \) to indicate equality along the hypersurface \( \Sigma \). In many instances, one side of such an equation will involve the restriction of an ambient quantity to \( \Sigma \) while the other is a quantity only defined along \( \Sigma \). As a first step, we identify the second fundamental form in terms of the above data:

Lemma 3.3.
\[
(\rho \Sigma) = -H,
\]
and
\[
\nabla_a n_b + \rho n_a n_b \Sigma = \Pi_{ab}, \quad \text{equivalently} \quad \nabla_a n_b \Sigma = \Pi_{ab} + H n_a n_b.
\]
Moreover
\[
\nabla n_a \Sigma = H n_a.
\]
Proof. The proof of the first statement is not essentially different than that of [13, Proposition 3.5], which treats the case of a conformal unit defining density subject to $I_2^2 = 1$ exactly. For the second statement we compute directly, along $\Sigma$, beginning with the definition of the second fundamental form. Since $n$ has unit length along $\Sigma$, we have

$$II_{ab} := \nabla^\top_a n_b = (\nabla_a - n_a \nabla) n_b = \nabla_a n_b + \rho n_a n_b.$$ 

To reach the third line we used (3.1) as follows

$$(3.6) \quad \nabla n n_b = n^c \nabla_c n_b = n^c \nabla_b n_c = \frac{1}{2} \nabla_b n^2 \Sigma = -\rho n_b.$$ 

This last result also gives Equation (3.5). □

Remark 3.4. In fact the Lemma holds when $I_2^2 = 1 + \mathcal{O}(\sigma^2)$. Moreover, this Lemma is the main ingredient needed to recover the result of [13] that the scale tractor for singular Yamabe structures agrees, along $\Sigma$, with the normal tractor.

The algorithm for computing the jets of $\sigma$, and then the obstruction density $B$, now proceeds recursively using two key results. The first of these is a conformal analogue of Proposition 2.6:

**Lemma 3.5.** Suppose that $\sigma$ is a conformal unit defining density for a hypersurface $\Sigma$ in a conformal manifold $(M^d, c)$, with $d \geq 3$. If $g \in c$ and $k \leq d$ is a positive integer, then the quantity $\nabla^k \sigma|_{\Sigma}$, where $\nabla =$ Levi-Civita of $g$, may be expressed as $\nabla^{k-2} II$ plus a linear combination of terms where each term is a homogeneous polynomial in various derivatives $\nabla^m \rho$, with $0 \leq m \leq k - 2$, times a partial contraction involving the conormal $n$, $\nabla^\ell II$ for $0 \leq \ell \leq k - 3$, and the Riemannian curvature $R$ and its covariant derivatives (to order at most $k - 3$).

Proof. We have $n_a = \nabla_a \sigma$, and (according to (3.4)) $\nabla_a \nabla_b \sigma \sum = II_{ab} + H n_a n_b$. The argument is now completed by an induction following exactly the same logic as the proof of Proposition 2.6. Formally the only new features are that rather than $n^2 = 1$, we now have (3.1), i.e., $n^2 = 1 - 2\rho \sigma + \sigma^d B$, and instead of $\nabla_a n_b|_{\Sigma} = II_{ab}$, two derivatives of $\sigma$ are now governed by Equation (3.4). The second of these is a trivial adjustment to substitutions, since it just affects the final evaluation along $\Sigma$. The first means that arguments that previously used $\nabla n^2 = 0$ now incur nonzero terms. These new terms vanish along $\Sigma$, but are picked up by transverse derivatives. In particular, we have $\nabla n n^2 = -2\sigma \nabla n \rho - 2 \rho n^2 + \mathcal{O}(\sigma^{d-1})$. By counting, we see that we encounter $\nabla^\ell n^2$ for $\ell$ at most $k - 1$ ($k \leq d$), so the $\mathcal{O}(\sigma^{d-1})$ contribution never plays a rôle. Similarly, because of the coefficient $\bar{\sigma}$ adjacent to $\rho$ in the Formula (3.1) for $n^2$, it follows that $\nabla^{k-2} \rho$ is the highest $\nabla n$ derivative of $\rho$ that is needed for the expression along $\Sigma$. □

The task of computing $\nabla^k \sigma|_{\Sigma}$ in terms of familiar curvature quantities is not yet complete because derivatives $\nabla^k \rho$ remain. These are dealt with as follows.
In particular, for 

where \( LTOTS \) indicates additional terms involving lower transverse-order derivatives of \( \sigma \).

Proof. Recall that from (3.2) and (1.2) we have

\[
\frac{1}{2} \nabla_n^2 I^2_\sigma = n^a a b n b + \rho n^2 + \sigma \nabla_n \rho
\]

Now by the definition of \( \rho \) in (3.2) we have \( \nabla^a n_a = -d \rho - J \sigma \). Using this, and once again that \( n^2 = 1 - 2 \rho \sigma + \sigma^d B \), we have

\[
\frac{1}{2} \nabla_n^2 I^2_\sigma + (d - 1) \rho - \sigma \nabla_n \rho = -\gamma^{a b} \nabla_a n_b - \sigma (J + 2 \rho^2 - \sigma^{-1} \rho B).
\]

For \( f \) a conformal density on \( M \), and \( k \geq 1 \) an integer, we have

\[
\nabla_n^{k-1}(\sigma f) = \sigma \nabla_n^{k-1} f + (k - 1) n^2 \nabla_n^{k-2} f + \frac{(k-1)(k-2)}{2} \nabla_n (n^2) \nabla_n^{k-3} f + \cdots + (\nabla_n^{k-2} n^2) f.
\]

Applying \( \nabla_n^{k-1} \) to both sides of Expression (3.10), using the last display, and evaluating along \( \Sigma \) gives (3.7). \( \square \)

Note that the last statement of the above proposition is just the \( k = d \) specialisation of (3.7), using also (3.1). Also, the Formula (3.7) extends nicely to the case \( k = 1 \) by the first part of Lemma 3.3. The right-hand-sides of the above three formulæ involve at most a \( k^\text{th} \) transverse derivative of \( \sigma \), all of which can be computed using Lemma 3.5 (for \( k \leq d \)) except for the \( \nabla_{\gamma}^d \rho \) terms appearing explicitly and those produced via Lemma 3.5. In any case these involve \( \ell \) satisfying \( \ell \leq k - 2 \). So, recursively, we have a computational algorithm which yields the following result.

Theorem 3.7. Suppose that \( \sigma \) is a conformal unit defining density for a hypersurface \( \Sigma \) in a conformal manifold \( (M^d, c) \), with \( d \geq 3 \). If \( g \in c \) and \( k \leq d \) is a positive integer, then the quantity

\[
\nabla^k \sigma|_\Sigma, \text{ where } \nabla = \text{Levi-Civita of } g.
\]

may be expressed as \( \nabla^{k-2} II \) plus a linear combination of terms where each term is a homogeneous polynomial in various derivatives \( 0 \leq m \leq k - 2 \), times a partial contraction involving the conormal \( n \), \( \nabla^\ell II \) for \( 0 \leq \ell \leq k - 3 \), and the Riemannian curvature \( R \) and its covariant derivatives (to order at most \( k - 3 \)).

We thus obtain a formula for the obstruction density \( B \) in terms of the undifferentiated conormal and the other quantities listed above.
The last statement may be viewed as following from (3.9) of Proposition 3.6 by using the first part of the Theorem to treat the right-hand-side thereof.

3.1.1. Examples. By applying the algorithm above Theorem 3.7, an explicit computation of the obstruction density in any given dimension is achieved by (i) computing in detail the lower transverse order terms (LTOTs) in the expression (3.7), (ii) evaluating normal derivatives of $\gamma_{ab}\nabla_a n_b$ and (iii) collecting terms involving normal derivatives of ambient curvatures. The terms in (3.7) involving $(k-2)$ normal derivatives of $\rho$ are determined by previous recursions. The $k=1$ step was encapsulated in Lemma 3.3. The case $k=2$, corresponding to dimension $d=2$ is special. We have not treated a tractor calculus when $d=2$ as this requires additional structure. Nevertheless the ASC problem does make sense because we may define $I^2_d$ to be the conformally invariant quantity given by $(\nabla \sigma)^2 - \sigma \Delta \sigma - \sigma^2 J$ in a choice of scale. The existence of a conformal unit defining density $\sigma$ satisfying $I^2_d = 1 + \sigma^2 B$ can be readily verified by explicit computations along the lines of Lemma 2.7. In the following Lemma we interpret $(d-2)P(n,n)$ as zero in dimension $d=2$:

**Lemma 3.8.** Let $\sigma$ be a conformal unit defining density. Then, if $d \geq 2$,

$$\frac{1}{2} \nabla^2_n I^2_d + (d-2)\nabla_n \rho = \nabla_{\Sigma} \Pi_{ab} \Pi_{ab} + (d-2)P(n,n).$$

In particular, for $d=2$ we have

$$B = 0,$$

and for $d > 2$

$$(3.11) \nabla_n \rho = \frac{\nabla_{\Sigma} \Pi_{ab} \Pi_{ab}}{d-2} + P(n,n).$$

**Proof.** Computing one normal derivative of Equation (3.10) and evaluating the result along $\Sigma$ using $\nabla_n \sigma = n^2 \nabla_{\Sigma}$, 1, shows that lower order transverse derivative terms in (3.7) are absent when $k = 2$, so that

$$\frac{1}{2} \nabla^2_n I^2_d + (d-2)\nabla_n \rho \equiv -\nabla_n (\gamma_{ab} \nabla_a n_b) - J - 2\rho^2.$$ 

From the previous $k=1$ step (namely Lemma 3.3) the last term in the above display can be replaced by $-2H^2$, so it only remains to compute the first normal derivative term on the right hand side:

$$\nabla_n (\gamma_{ab} \nabla_a n_b) = -2(\nabla_n n^b)(\nabla_a n_b) + \gamma_{ab}(\nabla_a \nabla_n n_b + R_{cabd} n^c n^d - (\nabla_a n^c)(\nabla_c n_b))$$

$$\equiv -2H^2 + \frac{1}{2} \gamma_{ab} \nabla_a \nabla_b n^2 - \text{Ric}(n,n) - \Pi_{ab} \Pi_{ab}$$

$$= (d-3)H^2 - \text{Ric}(n,n) - \Pi_{ab} \Pi_{ab}.$$ 

Here the second line relied on Lemma 3.3 and Equation (3.6) of its proof, while the third employed the fact that the operator $\gamma^{\alpha\beta}\nabla_{\alpha}$ is tangential along $\Sigma$. Thus

$$\frac{1}{2} \nabla^2_n I^2_d + (d-2)\nabla_n \rho \equiv \Pi_{ab} \Pi_{ab} - (d-1)H^2 + \text{Ric}(n,n) - J = \Pi_{ab} \Pi_{ab} + \text{Ric}(n,n) - J.$$ 

When $d = 2$, $\text{Ric} = g.J$ and the second fundamental form has no trace-free part, so the obstruction density vanishes as claimed. For $d > 2$, the definition of the Schouten tensor implies that $\text{Ric}(n,n) \equiv (d-2)P(n,n) + J$, which completes the proof. \qed
Recall that a conformal manifold equipped with a parallel standard tractor $I \neq 0$ is said to be \emph{almost Einstein} (AE). In this case $I$ is a scale tractor, $\hat{D}\sigma$, for some scale $\sigma$. If $I^2 > 0$, any zero locus $\Sigma$ of $\sigma$ is a totally umbilic, smoothly embedded hypersurface \cite{[13]}. This also follows from a corollary of the above lemma and Lemma 3.3.

**Corollary 3.9.** Let $\sigma$ be a conformal unit defining density for a smoothly embedded hypersurface $\Sigma$. If $d \geq 3$,

\[
[\nabla_a I^B]_\Sigma \equiv \begin{pmatrix} 0 \\
 - \frac{1}{\sigma^2} \left[ \nabla_a \hat{\Pi}^b_a - n_a \hat{\Pi}^b_{bc} \right] \end{pmatrix}.
\]

**Proof.** This result follows by directly computing the tractor-coupled gradient of the scale tractor

\[
[\nabla_a I^B] = \begin{pmatrix} \nabla_a n^b + P^b_a \sigma + \rho \delta^b_a \\
 \nabla_a \rho - P^a_b n_b \end{pmatrix},
\]

and evaluating this along $\Sigma$. The result for the middle slot requires only Lemma 3.3. For the bottom slot, one rewrites $\nabla_a \rho = \nabla^\top_a \rho + n_a \nabla n \rho$ the first term of which gives the gradient of mean curvature. Then one uses the following identity obtained from the trace of the Codazzi-Mainardi Equation (2.6) (valid in $d \geq 3$)

\[
(3.12) \quad \nabla \hat{\Pi}_a - (d-2)\nabla_a H = (d-2)(P_{ab} \hat{n}^b) \top,
\]

to obtain the divergence of the trace-free second fundamental form (up to ambient curvatures). Treating the normal derivative of $\rho$ term then requires Lemma 3.8 and yields the result stated. \hfill \Box

The total umbilicity statement mentioned above the corollary follows by observing that the parallel condition implies $I^2$ is constant. Our final example is a computation of the obstruction density for surfaces in three dimensions. First we state the main lemma.

**Lemma 3.10.** Let $\sigma$ be a conformal unit defining density. Then, if $d \geq 3$,

\[
(3.13) \quad \frac{1}{2} \nabla^2 n^2 I_\Sigma^2 + (d-3) \nabla n^2 \rho \Sigma \equiv - \frac{1}{d-2} \left( \nabla \nabla \hat{\Pi} + (d-2) \hat{\Pi}^a_b \hat{\Pi}_b^a [H \hat{\Pi}_ab + P_{ab}] \right)
\]

\[
- 2 \hat{\Pi}^a_b \hat{\Pi}_ac \hat{\Pi}_b^c + 2 \hat{\Pi}^a_b W_{caqb}n^c n^d + \frac{d-3}{d-2} \nabla n G(n,n) + (d-3)(\nabla n + 2H)J.
\]

**Remark 3.11.** In dimensions $d \geq 4$, the Fialkow tensor is defined by \cite{[30]}

\[
(3.14) \quad \mathcal{F}_{ab} := P_{ab} - P_{ab} + H \Pi_{ab} + \frac{1}{2} \hat{g}_{ab} H^2,
\]

and is in fact a weight $w = -2$ tensor density. Using this and that, via Equation (1.5), three normal derivatives of $I_\Sigma^2$ vanishes along $\Sigma$, we may write the above result as

\[
\nabla^2 n \rho \Sigma \equiv - \frac{1}{(d-2)(d-3)} \left( \nabla \nabla \hat{\Pi} + (d-2)(d-4) \hat{\Pi}^a_b \hat{P}_{ab} \right)
\]

\[
- \frac{d-2}{d-3} \hat{\Pi}^a_b \mathcal{F}_{ab} - \nabla^a \left( (\hat{n}^b P_{ba}) \top \right) - H \left[ (d-2)P(n,n) + \hat{\Pi}^a_b \hat{\Pi}_{ab} \right] + (\nabla n + H)J.
\]

\[ \ast \]
In dimensions \( d \geq 4 \) Lemma \[3.10\] determines the second normal derivative of \( \rho \) and when \( d = 3 \) it gives the obstruction density. The result is the generalization of the Willmore invariant \[2.16\] to curved ambient spaces:

**Corollary 3.12.** In dimension \( d = 3 \), the obstruction density is given by

\[
B = -\frac{1}{3}\left( \nabla_a \nabla_b + H \Pi_{ab} + P_{ab}^T \right) \Pi^{ab}.
\]

The proof of Proposition \[3.10\] is involved but conceptually not different to that of Lemma \[3.8\] it is given in Appendix A. This confirms the result of \[2\]. The invariant density \( B \) was also found using tractor methods in \[32\].

### 3.2. Constructing hypersurface conformal invariants and holography.

The conformal defining density on an ambient manifold enables a “holographic” study of extrinsic as well as intrinsic hypersurface conformal geometry: The key ingredient is Theorem \[1.3\], which can be used to proliferate natural invariants of the conformal hypersurface structure \((M, c, \Sigma)\) (see Definition \[3.1\]). Indeed, since the conformal unit defining density \( \sigma \) is determined by the data \((M, c, \Sigma)\), uniquely modulo \( O(\sigma^{d+1}) \), up to the order that \( \sigma \) is uniquely determined, the coupled conformal invariants of the conformal structure and the scale \( \sigma \) are automatically natural invariants of \((M, c, \Sigma)\). Such invariants are easily constructed using the ambient conformal tractor calculus applied to \((M, c)\) and \( \sigma \). Formulas for conformal hypersurface invariants obtained by restricting coupled invariants of the ambient structure \((M, c, \sigma)\) to \( \Sigma \) are termed holographic formulæ.

The simplest example of a tractor-valued holographic formula is the restriction of the scale tractor \( I^A_\sigma \) for a conformal unit defining density \( \sigma \), which is easily computed using see Lemma \[3.3\]

\[
I^A_\sigma |_{\Sigma} = N^A \equiv \begin{pmatrix} 0 \\ \hat{n}_a \\ -H \end{pmatrix}.
\]

The tractor on the right hand side above is the normal tractor of \[3\]. Therefore, the above is a holographic formula for the normal tractor. Another example is the weight \( w = -1 \), trace-free, symmetric tractor

\[
P^{AB} := \tilde{D}^A \tilde{D}^B \sigma = \tilde{D}^A I^B_\sigma,
\]

by which construction, for \( d \geq 4 \), yields a (tractor-valued) hypersurface conformal invariant upon restriction to \( \Sigma \). In the above, we have used the operator \( \tilde{D}^A \), which is defined as a map on section spaces \( \Gamma(T^k M[w]) \to \Gamma(TM \otimes T^k M[w-1]) \) for \( w \neq 1 - d/2 \), where \( T^k M[w] \) denotes a tractor tensor bundle of arbitrary rank. In a scale \( g \),

\[
\tilde{D}^A g = (w, \nabla_a^T, -(d + 2w - 2)^{-1}(g^{ab}\nabla_a^T \nabla_b^T + \omega J)),
\]

and is related to the Thomas D-operator \( D^A \) of \[3\] by \( \tilde{D}^A = (d + 2w - 2)^{-1}D^A \). Equation \[3.17\] can be viewed as the conformal analog of Equation \[2.5\] relating the second fundamental form to a unit defining function. Indeed, in Section \[4.1\] we will use the ambient tractor \( P^{AB} \) to build a holographic formula for the tractor second fundamental form.

We may construct yet further invariants this way, for example in dimensions \( d \geq 4 \), consider the scalar invariant

\[
[W_{ABCD} P^{AC} P^{BD}] |_{\Sigma},
\]
where $W_{ABCD}$ is the $W$-tractor of $\Pi$ (see also [12] [15]). It is an elementary tractor calculus exercise to see that this is simply a multiple of

$$W_{abcd} \Pi^{ac} \Pi^{bd} \mid \Sigma,$$

where $W_{abcd}$ is the ambient Weyl curvature.

It is very easy to make higher order examples. The key point is that the jets of objects such as $P_{AB}$ and $W_{ABCD}$ are now canonically defined (up to the uniqueness bound in the case of $P^{AB}$). In particular, the operator obtained through contraction of the scale tractor $I^A$ and the Thomas-D operator $I^A D_A$ gives at the same time (i) a conformal analog of the ambient Laplace operator and (ii) along $\Sigma$ a conformally invariant Robin type-operator that can be used to differentiate in the normal direction to the hypersurface. Thus $I \cdot D := I^A D_A$ is termed the Laplace–Robin operator, its importance for conformally compact boundary problems is discussed in detail [16]. This enables us to perform conformal analogues of computations such as those leading to Equations (2.11) and (2.12).

3.3. Linking tensor invariants and tractors. There exists a general “splitting technology” (see for example [8]) relating invariant tensor densities and tractors. A particular instance of this is the following construction. First recall that there is a canonical bundle inclusion $E_M[-1] \to TM$ given by the canonical section $X^A \in TM[1]$. In a scale $g$, $[X^A]_g = (0, 0, 1)$, and $X^A$ is termed the canonical tractor. It also induces a surjective bundle map $TM[w] \to E_M[w + 1]$ acting on sections by contraction. We may extend this to a linear map $X^\downarrow : \Gamma(\otimes^2 TM[w]) \to \Gamma(TM[w + 1])$ acting by contraction with the canonical tractor $X$. We now define the canonical map

$$q^* : \ker(X^\downarrow) \to \Gamma(\otimes^2 TM[w + 2]),$$

which, for some $g \in c$ acts as

$$q^* : \begin{pmatrix} 0 & 0 & 0 \\ 0 & t^{ab} & t^{a-} \\ 0 & t^{b-} & t^{--} \end{pmatrix} \mapsto t_{ab}.$$

The map $q^*$ can be used to extract conformal invariants from ambient tractors. When interested in hypersurface conformal invariants, we replace $\ker(X^\downarrow)$ by $\ker_{\Sigma}(X^\downarrow)$ whose elements are tractor sections $T$ such that $X^T = \sigma S$ for some smooth $S$. This gives a map $q_{\Sigma}^* : \ker_{\Sigma}(X^\downarrow) \to \Gamma(\otimes^2 TM[w + 2])$, where $\Gamma_{\Sigma}$ denotes equivalence classes of sections $T \sim \tilde{T} + \sigma S$ with $S$ smooth. We may identify these with their values along $\Sigma$.

An application of this construction is the following result which shows that the tensor $P^{AB}$ of Equation (3.17) is the tractor analog of the trace-free second fundamental form while its normal derivative encodes the invariant Fialkow tensor of Equation (3.14). In the following proposition, we introduce the rigidity density $K := \Pi_{ab} \Pi^{ab} \in \Gamma(E\Sigma[-2]).$

**Proposition 3.13.** Let $\sigma$ be a conformal unit defining density, then if $d \geq 3$,

$$q_{\Sigma}^*(P^{AB}) = \Pi_{ab}$$

and

$$K_{\text{ext}} = P^{AB} P_{AB} \sum K.$$

For $d \geq 4$

$$q_{\Sigma}^* \left( I \cdot \hat{D} P^{AB} + h^{AB} K_{\text{ext}} \right) \sum = - (d - 3) \mathcal{F}_{ab} + \frac{3 \bar{g}_{ab} K}{2(d - 2)}.$$

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Proof. First note that
\[ X_A P^{AB} = 0, \]
because \( I = I_\sigma \) has weight zero and \( X \cdot \tilde{D} T = w T \) for any weight \( w \neq 1 - \frac{d}{2} \) tractor. So, in particular \( P_{AB} \in \ker_\Sigma(X) \). Using that in a choice of scale \( g \), the Thomas D-operator acting on weight \( w \) tractors is given by
\[ [D^A]_g = ((d + 2w - 2)w, (d + 2w - 2)\nabla_a^T, -(\Delta + wJ)), \]
as well as Formula (3.2) for the scale tractor, we see that Equation (3.19) follows from Corollary 3.9. The result for the rigidity density is an immediate consequence.

Next we must verify that \( I \cdot \tilde{D} P^{AB} + h^{AB} \frac{K_{\text{ext}}}{d - 2} \in \ker_\Sigma(X) \). Acting with \( I \cdot \tilde{D} \) on Equation (3.21) we have
\[ 0 \equiv (I \cdot \tilde{D} X_A) P^{AB} + X_A I \cdot \tilde{D} P^{AB} \equiv X_A \left( I \cdot \tilde{D} P^{AB} + h^{AB} \frac{K_{\text{ext}}}{d - 2} \right). \]
The last equality used \( I \cdot \tilde{D} X_A = I_A \) and that
\[ I^A P_{AB} = I^A \tilde{D}_B I_A = \frac{1}{2} \tilde{D}_B I^2 + \frac{1}{d - 2} X_B P^{AC} P_{AC} \equiv X_B \frac{K_{\text{ext}}}{d - 2}, \]
The second step can be easily explicitly verified or follows from Equation (4.14) below, while the final step requires \( d \geq 3 \).

The remainder of the proof is based on the technology introduced in Section 3.1. In particular, computing along \( \Sigma \):
\[ q^*_\Sigma \left( I \cdot \tilde{D} P^{AB} + h^{AB} \frac{K_{\text{ext}}}{d - 2} \right) \\uparrow \equiv q^*_\Sigma \left( [\nabla_n + H] P^{AB} + h^{AB} \frac{K_{\text{ext}}}{d - 2} \right) \\uparrow \]
\[ \equiv \left[ (\nabla_n + H)(\nabla_n n_b + \sigma P_{ab} + \rho g_{ab}) + g_{ab} \frac{K}{d - 2} \right] \\uparrow \]
\[ \equiv \frac{1}{2} \left[ \frac{1}{2} \nabla_a \nabla_b (1 - 2\rho \sigma) + \nabla_n, \nabla_a ] n_b + P_{ab} + g_{ab} \left( \nabla_n \rho + \frac{K}{d - 2} \right) + H \bar{\Pi}_{ab} \right] \\uparrow \]
\[ \equiv 2H \bar{\Pi}_{ab} + R_{c^a b^d} n^a n^d - \Pi_{c^a b^d} + P_{ab} + g_{ab} \left( \nabla_n \rho + H^2 + \frac{K}{d - 2} \right) \]
\[ \equiv -W_{c^a b^d} \hat{n}^a \hat{n}^b - \bar{\Pi}_{ab}^2 + \frac{2g_{ab} K}{d - 2}. \]
The first two equalities above use again the explicit formula for Thomas D-operator (3.22) and for the scale tractor (3.2). The next line relies on the fact that \( \sigma \) is a conformal unit defining density and the line thereafter follows directly the method of Section 3.1 for computing jets of \( \sigma \). The last equality required Lemma 3.8 for the normal derivative of \( \rho \). Finally, tracing the Gauß Equation (2.7) leads to the following identity
\[ \bar{\Pi}_{ab}^2 - \frac{1}{2} \bar{g}_{ab} \bar{\Pi}_{cd} \bar{\Pi}^{cd} - W_{c^a b^d} \hat{n}^a \hat{n}^d = (d - 3) \left( P_{ab}^\top - P_{ab} + H \bar{\Pi}_{ab} + \frac{1}{2} \bar{g}_{ab} H^2 \right). \]
The result follows upon combining the above two displays and the definition of the Fialkow tensor in Equation (3.14). \( \square \)

Corollary 3.14. If \( d \geq 3 \),
\[ I \cdot \tilde{D} K_{\text{ext}} \equiv -2(d - 3) L, \]
where \( L := \hat{\Pi}_{ab} F_{ab} \in \Gamma(\mathcal{E} \Sigma [-3]). \)

Proof. For \( d \geq 4 \), the result follows directly from the proposition using the properties of \( P^{AB} \). For \( d = 3 \), it is easily verified by direct calculation. \( \square \)
Remark 3.15. Since Equation (3.20) exactly matches (3.3), Proposition 3.13 allows us to interpret the Fialkow tensor as the normal derivative of the trace-free second fundamental form canonically defined by the conformal unit defining density. Later, we will see that quantity $L$ plays the rôle of a rigidity density for embedded volumes.

The methods used to prove Proposition 3.13 can be employed to generate a set of rank two, symmetric, conformally invariant, extrinsic hypersurface invariants from $(I, \hat{D})^k P^{AB}$, whose first two elements are the trace-free second fundamental form and the Fialkow tensor.

4. Conformal hypersurface tractor calculus

In the previous section we established that conformal hypersurfaces can be naturally treated via tractors. Here we review and extend the known tractor hypersurface calculus using the conformal unit defining density. Key results are tractor analogues of the Gauß formula and second fundamental form. We also show how to relate ambient and hypersurface Thomas D-operators.

4.1. Tractor second fundamental form. We first need a certain differential splitting operator $q$ mapping weighted, trace-free symmetric two-forms into rank two, weight $w \neq 1 - d, -d$, symmetric tractors; this can be viewed as a natural dual of the map $q^*$ defined in Equation (3.18). For dimensions $d \geq 3$ this is given by (see for example [8]):

$$
\Gamma(\bigodot^2 T^* M[w+2]) \ni t_{ab} \mapsto q(t_{ab}) \in \Gamma(T^{(AB)} \circ M[w]).
$$

When in addition $w \neq -\frac{d}{2}$, the conditions $D_A T^{AB} = 0 = X_A T^{AB} = T_{AA}$ characterise the image of this map.

Remark 4.1. When $t_{ab} \in \Gamma(\bigodot^2 T^* M[3-d])$, the weight $-d - 1$ density

$$(\nabla_a \nabla_b + P_{ab}) t^{ab},
$$

appearing as the residue of the pole at $w = 1 - d$ in the above display, is conformally invariant.

On the conformal manifold $(\Sigma, c_{\Sigma})$, applying the map $q$ to the trace-free second fundamental form $\Pi_{ab} \in \Gamma(\bigodot^2 T^* \Sigma[1])$ gives the tractor second fundamental form [23, 30]:

**Definition 4.2.** Let $d \geq 4$. The tractor second fundamental form $L^{AB} \in \Gamma(T^{(AB)} \circ \Sigma[-1])$ is defined by

$$
[L^{AB}] := q(\Pi_{ab}) \begin{pmatrix}
0 & 0 & 0 \\
0 & \Pi_{ab} & -\frac{\nabla \hat{\Pi}_{ab}}{d-2} \\
0 & -\frac{\nabla \hat{\Pi}_{ab}}{d-2} & \nabla \nabla \hat{\Pi}_{+(d-2)} P_{ab} \hat{\Pi}_{ab} - \hat{\Pi}_{ab} \hat{\Pi}_{ac} \hat{\Pi}_{bd} - \hat{\Pi}_{ab} \hat{\Pi}_{ad} \hat{\Pi}_{bd} - 3)
\end{pmatrix}.
$$

Remark 4.3. A dimensional continuation argument can be used to obtain the $d = 3$ obstruction density from the tractor second fundamental form: In dimensions $d \geq 4$, the Fialkow–Gauß equation (3.3) implies

$$
\hat{P}_{ab} \Pi_{ab} = P_{ab} \Pi_{ab} + H \Pi_{ab} \Pi_{ab} + \frac{\Pi_{ab} W_{cab} \hat{\Pi}_{cd} \hat{\Pi}_{ad} \hat{\Pi}_{bd}}{d-3}.
$$
so that the part of $L^{AB}$ singular when $d = 3$ can be rewritten as

$$\frac{\nabla \nabla \hat{H} + (d - 2)(P_{ab}^\top + H\hat{H}_{ab})\hat{H}^{ab}}{(d - 2)(d - 3)} + \frac{\hat{H}_{ab}W_{cabd}\hat{n}^c\hat{n}^d - \hat{H}_{ab}\hat{H}^{bc}\hat{H}^{ab}}{(d - 3)^2}.$$  

The numerator of the second term in this expression vanishes identically in $d = 3$ while the first numerator evaluated at $d = 3$ is

$$(\nabla_a \nabla_b + P_{ab}^\top + H\hat{H}_{ab})\hat{H}^{ab};$$

this is precisely the obstruction density (3.15).

Corollary 3.9 and Proposition 3.13 suggest that a holographic formula for the tractor second fundamental form can be built from $P^{AB} = \hat{D}^A I^B$. For that, we need the following result:

**Lemma 4.4.** Let $\sigma$ be a conformal unit defining density and $d \geq 4$, then for $g \in c$,

$$[\Delta I^A] \Sigma \equiv \begin{pmatrix} \nabla_a \hat{H}_{a} - \hat{n}_a K \\ \hat{n}_a \nabla_a \hat{H}^{ab} \hat{H}^{ab} + 2\hat{H} K - \frac{(3d-8)L}{d-3} \end{pmatrix}. $$

**Proof.** Firstly, recall that in a choice of scale $g$, the tractor connection acts on a standard tractor $V^A$ according to (see for example [3])

$$\begin{pmatrix} v^+ \\ v_b \\ v^- \end{pmatrix} \Rightarrow \begin{pmatrix} \nabla_a v^+ - v_a \\ \nabla_a v_b + g_{ab} v^+ + P_{ab} v^+ \\ \nabla_a v^- - P_{ac} v^c \end{pmatrix}.$$  

Applying the above equation to the scale tractor twice and then contracting with the inverse metric yields

$$[g^{ab}\nabla_a \nabla_b I^A] \Sigma \equiv \begin{pmatrix} \Delta n_a + 2\nabla_a \rho \\ (\Delta - J)\rho - 2P^{ab}\nabla_a n_b - \nabla_n J \end{pmatrix}.$$  

Along $\Sigma$ we have

$$\Delta n_a + 2\nabla_a \rho = \nabla^b \nabla_a n_b + 2\nabla_a \rho = \nabla_a \nabla_n + \text{Ric}_{ab} n^b + 2\nabla_a \rho = -(d - 2)(\nabla_a \rho - P_{ab} \hat{n}^b) = -(d - 2)(\nabla_a \rho + n_a \nabla_n \rho - P_{ab} \hat{n}^b) = -n_a \hat{H}_{bc} \hat{H}^{bc} + (d - 2) (\nabla_a H + (P_{ab} \hat{n}^b)^\top).$$

The last line was obtained using Equations (3.3) and (3.11). The traced Codazzi-Mainardi equation (3.12) establishes the middle slot of the right hand side of the displayed result. Note that this result could also be obtained from Corollary 3.9 and symmetry of $P^{AB}$.

Also, computing along $\Sigma$ (using Lemma A.2 to handle the ambient Laplace operator and Equation (3.4) for the gradient of the normal vector)

$$(\Delta - J)\rho - 2P^{ab}\nabla_a n_b - \nabla_n J = \Delta H + \nabla_n^2 \rho + (d - 2) H \nabla_n \rho - 2P^{ab} \hat{H}_{ab} - (\nabla_n + H) J.$$
Normal derivatives of \( \rho \) are given by Lemma 3.8 and Proposition 3.10. Furthermore, a simple consequence of the Codazzi–Mainardi Equation (2.6) is the following identity
\[
(4.3) \quad \bar{\Delta} H = \frac{1}{d-1} \nabla^a (\nabla_a H_a - (\text{Ric}_{ab} \hat{n}^b) \nabla^a) = \frac{1}{d-2} \nabla^a (\nabla_a H_a - (d-2)(P_{ab} \hat{n}^b)) ,
\]
which allows the Laplacian of the mean curvature to be traded for divergences of the trace-free second fundamental form. In addition, normal derivatives of the normal components of the Einstein tensor are given by Lemma A.7 and the ambient Schouten tensor can be eliminated using the Fialkow–Gauß equation (3.3). Orchestrating those maneuvers gives the bottom slot of the displayed result and completes the proof.

The above lemma combined with Corollary 3.9 determine \( \hat{D}^A I^B \) along \( \Sigma \). This, together with Corollary 3.14, gives the following holographic formula for the tractor second fundamental form (up to a slight modification):

**Proposition 4.5.** Let \( \sigma \) be a conformal unit defining density and \( d \geq 4 \). Then
\[
(4.4) \quad \left[ \hat{D}^A I^B - \frac{2}{d-2} (A X^B) K_{\text{ext}} + \frac{X^A X^B I \cdot \hat{D} K_{\text{ext}}}{(d-2)(d-3)} \right] = \Sigma L^A B + \frac{X^A X^B L}{d-3} .
\]

**Remark 4.6.** The first term on the left hand side of (4.4) is \( P^{AB} \) as promised in Section 3.2. It follows from Equation (3.23) that, along \( \Sigma \), the first two terms are the orthogonal projection of \( P^{AB} \) to hypersurface tractors (meaning sections of the tractor subbundle consisting of tractors orthogonal to the normal tractor). The failure of this to be a holographic formula for the tractor second fundamental form is measured by \( I \cdot \hat{D} K_{\text{ext}} |_{\Sigma} = -2(d-3) L \), which equals the contraction of the Fialkow tensor and the trace-free second fundamental form, see Corollary 3.14.

### 4.2. Thomas D-operator.

Here, given a defining density for a hypersurface \( \Sigma \), we construct a general family of tangential operators (this notion was introduced in [16]) to describe ambient operators that descend to hypersurface operators upon restriction; see Definition 5.1 below) that relate the ambient and intrinsic Thomas D-operators along \( \Sigma \). The following Proposition was proved in [14] for the special case of the AE setting:

**Proposition 4.7.** Let \( \sigma \) be a conforming density for a hypersurface \( \Sigma \) and denote \( \hat{I}^A := I^A / \sqrt{\Sigma^2} \). Then, if \( w + \frac{d}{2} \neq 1, \frac{3}{2}, 2 \), the operator
\[
(4.5) \quad \hat{D}^T_A := \hat{D}_A - \hat{I}^A \hat{D} + \frac{1}{h(h-1)(h-2)} X_a (\frac{1}{T^2} I \cdot D)^2 , \quad h + 2 := d + 2w ,
\]

mapping \( \Gamma(T^\Phi M[w]) \to \Gamma(T_A M \otimes T^\Phi M[w-1]) \), is tangential.

**Proof.** The proof of this fact only requires that we establish the operator relation
\[
\hat{D}^T_A \circ \sigma = 0 .
\]

This follows from two facts: (i) The \( \mathfrak{sl}(2) \) algebra (see [16])
\[
(4.6) \quad [d + 2w, \sigma] = 2\sigma , \quad \left[ \frac{1}{T^2} I \cdot D, \sigma \right] = d + 2w , \quad \left[ d + 2w, \frac{1}{T^2} I \cdot D \right] = -\frac{2}{T^2} I \cdot D ,
\]
spanned by \( \sigma \) (viewed as a multiplicative operator on sections), \( d + 2w \) where \( w \) is the linear operator that returns the weight of a tractor, and \( \frac{1}{T^2} I \cdot D \). (ii) The commutator
of \( \hat{D}^A \) and \( \sigma \) (again viewed as a multiplicative operator)
\[
[\hat{D}^A, \sigma] = I^A - \frac{2}{h(h-2)} X^A I \cdot D,
\]
valid acting on tractors of weight \( w \neq -\frac{d}{2}, 1 - \frac{d}{2} \) which is easily verified by direct computation in a choice of scale. \( \square \)

**Remark 4.8.** In fact we will also need a replacement of the tangential Thomas D-operator at the missing weight \( w = 1 - \frac{d}{2} \). Given a weight \( w' \) tractor \( V^A \in \Gamma(T^A M \otimes T^\Phi M[w']) \) subject to \( X_A V^A = 0 \), and a boundary Yamabe weight \( w \neq -\frac{d}{2}, 1 - \frac{d}{2} \), we can construct a tangential analog of the operator \( V^A \hat{D}^T_A \) at the Yamabe weight \( w = 1 - \frac{d}{2} \) as follows: First, calling \( [V^A]_g = (0, v_a, v) \), it is easy to check that the operator, given by
\[
V^A \hat{D}^T_A := v^a \nabla_a + [1 - \frac{d}{2}] v,
\]
for some \( g \in c \) defines a mapping \( \Gamma(T^\Phi M[1 - \frac{d}{2}]) \to \Gamma(T^{\Phi'} M[w'] \otimes T^\Phi M[-\frac{d}{2}]) \). However, for any defining density \( \sigma \), we have \( I_v V = \sigma u \) for some smooth, weight \( w' - 1 \) density \( u \). Thus, for some \( g \in c \) we have \( n^a v_a + \sigma v = \sigma u \) and hence \( V \cdot \hat{D}^T \triangleq v^a \nabla_a^T + v [1 - \frac{d}{2}] + O(\sigma) \nabla_n \), which is clearly tangential.

Proposition 4.7 suggests that when expressed in terms of a scale, the tangential Thomas D-operator
\[
D^T_A := \begin{cases}
(d + 2w - 2) \hat{D}^T_A, & w \neq 1 - \frac{d}{2}, \frac{3}{2} - \frac{d}{2}, 2 - \frac{d}{2}, \frac{d}{2} \choose 2

D_A - \hat{I}_A \hat{I} \cdot D + X_A I \cdot D \circ \frac{1}{27^T} \circ I \cdot D, & w = 1 - \frac{d}{2}, \frac{3}{2} - \frac{d}{2}, 2 - \frac{d}{2}
\end{cases}
\]
depends on the tractor-coupled connection only through the tangential combination \( \nabla_a^T := \nabla_a - \hat{n}_a \nabla_n \). For the case where the defining density is conformal, it follows immediately that the operator \( \hat{D}^T_A \) is independent of any choices. For that case we call the operator \( D^T_A \) the *tangential Thomas D-operator*. We will verify that this operator indeed factors through \( \nabla_a^T \) in the sense mentioned, see Equation (4.9) of the following Lemma. That Lemma also collects a number of critical results and details important for later developments. Let us point out some interesting features: In Equation (4.7), the general formula for \((I \cdot D)^2\) along \( \Sigma \) is given; for boundary Yamabe weight \( w = \frac{3}{2} - \frac{d}{2} \), all normal derivatives drop out, implying that this operator then becomes tangential. This is the first example of the extrinsic conformal Laplacians discussed in Section 5 and Remark 4.11. The Lemma’s next equation specialises Equation (4.5) to conformal unit defining densities. The formula for this in a choice of scale, given in Equation (4.9), should be compared with the general result for the Thomas D-operator in Equation (3.22), keeping in mind that the orthogonal subbundle \( N^\perp \) of \( TM |\Sigma \) and the intrinsic hypersurface tractor bundle \( T \Sigma \) are isomorphic (see [23] as well as [18] Section 3.2 for details). This shows that, along \( \Sigma \), the tangential Thomas D-operator yields an extrinsic hypersurface Thomas D-operator with ambient tractor-coupled connection save for a modification by the operator
\[
\frac{w^A}{2(d-2)(d+2w-3)}.
\]
Lemma 4.9. Let $\sigma$ be a conformal unit defining density and $d \geq 3$. Then, acting on weight $w$ tractors, the following operator identity holds along $\Sigma$, in a choice of scale $g$,

\[(I-D)^2 \equiv -\{(d + 2w - 4)\left\{\nabla a^2 + w\left[\frac{1}{2} \frac{\bar{\Pi}_{ab} \Pi_{ab}}{d - 2}\right]\right\} - (d + 2w - 3)\left[\nabla a^2 - w\left(2H\nabla_n - P(n, n) - \frac{\bar{\Pi}_{ab} \Pi_{ab}}{d - 2} - \frac{(2w - 1)H^2}{2}\right)\right]\},\]

where $\Delta^\top := \bar{g}^{ab} \nabla_a \nabla_b^\top$. Moreover, specializing to a conformal unit defining density, the operator $\hat{D}_A^I$, as defined in Proposition 4.7, is given by

\[(4.8) \quad \hat{D}_A^I = \hat{D}_A - I_A I \hat{D} + \frac{1}{h(h - 1)(h - 2)} X_A (I-D)^2, \quad h + 2 := d + 2w.\]

It is determined up to terms of order $O(\sigma^{d-1})$ times a smooth differential operator, and is subject to the same weight restrictions as in Proposition 4.7. In a choice of scale $g$,

\[(4.9) \quad \hat{D}_A^I \equiv \hat{D}_A - I_A I \hat{D} + \frac{1}{h(h - 1)(h - 2)} X_A (I-D)^2, \quad h + 2 := d + 2w.\]

Proof. For the first statement, we first use that

\[I-D \equiv (d + 2w - 2)(\nabla_n + w\rho) - \sigma(\Delta + wJ)\]

and

\[(4.10) \quad I-D \equiv \nabla_n - wH\]

to compute the operator statement (acting on weight $w$ objects) along $\Sigma$ directly

\[I-D^2 \equiv (d + 2w - 4)\left[\nabla_n - (w - 1)H\right]\left[(d + 2w - 2)\left(\nabla_n + w\rho\right) - \sigma(\Delta + wJ)\right] = -(d + 2w - 4)\left[\Delta + wJ - (d + 2w - 2)\left(\nabla_n^2 - (2w - 1)H\nabla_n + w(\nabla_n \rho) + w(w - 1)H^2\right)\right].\]

On the second line we used the operator product identity $\nabla_n \circ \sigma = 1$ valid along $\Sigma$ and $\rho|_\Sigma = -H$ as per Lemma 3.3. To obtain the quoted result we used Equation 2.8, Lemma 3.8 as well as the operator identity for the tractor-coupled Laplacian

\[(4.11) \quad \Delta \equiv \Delta^\top + \nabla_n^2 + (d - 2)H\nabla_n,\]

which can easily be established along the same lines used to prove Lemma A.2.

The second statement follows from the defining property of a conformal unit defining density in Equation (1.5). For the third we first use Equation (3.22) as well as Equation (4.10) to find the operator statement for the first two terms of Equation 4.8,

\[\hat{D}_A^I \equiv \hat{D}_A - I_A I \hat{D} + \frac{1}{h(h - 1)(h - 2)} X_A (I-D)^2, \quad h + 2 := d + 2w.\]

Remembering that $\nabla_n^\top \equiv \nabla - n_a \nabla_n$, it is easy to verify that the top two slots on the right hand side of the above display agree with those quoted in Equation 4.9. Thus it only remains to verify the bottom slot of Equation 4.9. Using the computation of $I.D^2$ along $\Sigma$ shown above, as well as the bottom slot in the above display, one can employ
Equation (2.8) to trade the ambient $J$ for its intrinsic counterpart $\bar{J}$, Equation (4.11) to exchange $\Delta$ for $\Delta^\top$ and Equation (3.11) to handle $\nabla_n \rho$. This yields the result quoted in the bottom slot of Equation (4.8).

The Thomas D-operator identity (4.12)

$$D_A \circ X^A = (d + w)(d + 2w + 2)$$

is useful in many contexts; the tangential Thomas D-operator obeys an analog of this:

**Corollary 4.10.** Let $T \in \Gamma(T^0 M[w])$ where $w + \frac{d}{2} \neq 1, \frac{3}{2}, 2$. Then

$$\hat{D}_A^T(X^A T) \equiv \frac{(d + 2w + 1)(d + w - 1) T}{d + 2w - 1}.$$

**Proof.** Noting that in a choice of scale,

$$[\nabla^\top_b(X^A T)]_g \equiv \begin{pmatrix} 0 \\ g_{ab} T \\ 0 \end{pmatrix} \quad \text{and} \quad [\Delta^\top(X^A T)]_g \equiv \begin{pmatrix} (d - 1)T \\ * \\ * \end{pmatrix},$$

the result follows directly by application of Equation (4.9). □

**Remark 4.11.** As mentioned above (see also [17, 14]) at weight $w = \frac{3}{2} - \frac{d}{2}$, the terms in Equation (4.7) above involving $\nabla_n$ are absent, and the operator

$$I \cdot D^2 \equiv \Delta^\top + \left(\frac{3}{2} - \frac{d}{2}\right) \left[ J - \frac{1}{2} \bar{\Pi}_{ab} \bar{\Pi}^{ab} \right] =: \Box^\top_Y,$$

is tangential. Specializing to densities, $\Delta^\top$ becomes the intrinsic Laplace operator $\bar{\Delta}$ along $\Sigma$ and $\Box^\top_Y$ is the intrinsic Yamabe Laplacian modified by the rigidity density. *

Our first application of the canonical tangential Thomas D-operator is to compute its action on the scale tractor. This gives another holographic formula for the tractor second fundamental form (again up to known terms) that can be regarded as a conformal analog of the Riemannian result for the second fundamental form in terms of the ambient Levi-Civita connection acting on a unit normal vector in Equation (2.4).

**Proposition 4.12.** Let $d \geq 4$. Then

$$\hat{D}_A^T N_B \equiv L_{AB} + \frac{X_A(N_B K + X_B L)}{d - 3},$$

where $N_B$ is any smooth extension of the normal tractor off $\Sigma$.

**Proof.** Let $\sigma$ be a conformal unit defining density and $I_A := \hat{D}_A \sigma$. Then, since $\hat{D}_A^T$ is tangential, we may replace the left hand side of the above display by $\hat{D}_A^T I_B$, which we shall now compute. From Corollary 3.9 we have

$$[\nabla^\top_a I^B] \equiv \begin{pmatrix} 0 \\ \bar{\Pi}_{ab} \\ -\bar{\Pi}_{\bar{\alpha} \bar{\beta}} \frac{d - 3}{d - 2} \end{pmatrix}. $$

Using that $\nabla^\top_a \bar{\Pi}^a_{\bar{b}} = \bar{\nabla} \bar{\Pi}_b - n_b \bar{\Pi}_{ac} \bar{\Pi}^{ac} = \bar{\nabla} \bar{\Pi}_b - n_b K$, we compute

$$[\Delta^\top I^B] = \begin{pmatrix} 0 \\ \frac{d - 3}{d - 2} \bar{\nabla} \bar{\Pi}_b - n_b K \\ \bar{\nabla} \bar{\nabla} \bar{\Pi} + (d - 2) P_{ab} \bar{\Pi}^{ab} \frac{d - 3}{d - 2} \end{pmatrix}.$$
We now have the main ingredients required to employ Equation (4.9) of Lemma 4.9 and find
\[
\begin{pmatrix}
\hat{D}^A I B \\
\end{pmatrix} \equiv \begin{pmatrix}
0 & 0 & 0 \\
0 & \Pi_{ab} & -\frac{\nabla_{\hat{n}}}{d-2} \\
0 & \frac{\nabla_{\hat{n}}}{d-2} + \frac{n_k K}{d-3} & \frac{\nabla_{\hat{n}}}{(d-2)(d-3)} \\
\end{pmatrix}.
\]

The final result is obtained upon using the Fialkow–Gauß Equation (3.3) to give
\[
P_{ab} \Pi^{ab} = \bar{P}_{ab} \Pi^{ab} + L -HK.
\]

Remark 4.13. Since Propositions 4.5 and 4.12 both give holographic formulæ for the tractor second fundamental form, we can use the former to given an alternate proof of the latter, without recourse to explicit expressions in a choice of scale. One begins by using Equations (3.17) and (4.8) to give \( \hat{D} A I B = P_{AB} - I A I P_B + (d-3)^{-1} X_A I \hat{D} I P_B \) (for \( d \geq 4 \)). Then employing Equation (3.23) in concert with Corollary 3.14 and Proposition 4.5, the result of Proposition 4.12 can easily be obtained by applying the fundamental calculus of the Thomas D-operator expressed by the modified Leibniz rule 26.

(4.14) \( \hat{D}^A(T_1T_2) - (\hat{D}^A T_1) T_2 - T_1(\hat{D}^A T_2) = -\frac{2}{d+2w_1 + 2w_2 - 2} X^A (\hat{D}^A T_1)(\hat{D}^B T_2), \)
valid for \( T_{1,2} \in \Gamma(\hat{TM}|w_{1,2} \neq -d/2) \) and \( w_1 + w_2 \neq 1 - d/2 \), and the resulting operator commutator relation (see [13] Section 3.6)
\[
[\hat{D}^A, \sigma^k] = k \sigma^{k-1} I^A - \frac{2k}{(d+2k+2w-2)(d+2w-2)} X^A X^{k-1} I^D - \frac{k(k-1)}{d+2k+2w-2} \quad k \in \mathbb{Z}_{\geq 0},
\]
valid for any scale \( \sigma \) and acting on tractors of weight \( w \neq 1 - d/2, 1 - k - d/2 \).

To complete the relationship between the tangential Thomas-D operator \( \hat{D}^A \) and the intrinsic Thomas-D operator \( D^A \) of the hypersurface \( \Sigma \), we need a generalization of the Gauß formula (2.3) relating the projected tractor connection \( \nabla^T \) to its intrinsic hypersurface counterpart \( \nabla \) (this result was also developed in [23, 30, 32, 33]).

**Proposition 4.14** (Fialkow–Gauß formula). Let \( V^A \in \Gamma(\hat{TM}) \) be such that along \( \Sigma \) it lies in \( \Gamma(N^\perp) \) and denote by \( \Sigma_B^A := \delta_B^A - N^A N_B \) the projector mapping \( \Gamma(\hat{TM}|\Sigma) \to \Gamma(N^\perp) \). Then, for \( d \geq 4 \),
\[
\Sigma_B^A \nabla_c^T V^B = \nabla_c^T V^A + N^A L_c^B V_B = \nabla_c^T V^A + F_c^A B^B V_B = \nabla_c^F V^A.
\]

Here \( F \) is a conformally invariant, one-form valued, boundary tractor endomorphism given in a boundary splitting by
\[
[F_c^A B^B] = \begin{pmatrix}
0 & 0 & 0 \\
F_{ca} & 0 & 0 \\
0 & -F_{cb} & 0 \\
\end{pmatrix}.
\]

**Proof.** Let us fix an ambient scale \( g \in c \). This induces a boundary scale \( \bar{g} \in c_\Sigma \). Now recall (see [13] Section 3.2]) that the isomorphism between the subbundle \( N^\perp \) orthogonal to the normal tractor (with respect to the tractor metric \( h \)) along \( \Sigma \) and the intrinsic hypersurface tractor bundle \( T\Sigma \) gives a map between sections expressed in scales \( g \) and \( \bar{g} \),
respectively:

\[(4.16) \quad [V^A]_g := \left( \begin{array}{c} v^+ \\ v_a \\ v^- \end{array} \right) \xrightarrow{\sim} \left( \begin{array}{c} v^+ \\ v_a - \hat{n}_a H v^+ \\ v^- + \frac{1}{2} H^2 v^+ \end{array} \right) = [U^A B]_g [V^B]_g =: [\bar{V}^A]_g,\]

where \( V^A \in \Gamma(N^+) \) and \( \bar{V}^A \in \Gamma(T\Sigma) \). Here the \( SO(d+1,1) \)-valued matrix

\[[U^A B]_g [\Sigma_D^C \nabla_a^T V^D]_g = [U^B C]_g [\nabla_a^T V^C + N^C L_a^D V^D]_g = [\nabla_a \bar{V}^B + \mathcal{F}_a^B C V^C]_g.\]

Now, along \( \Sigma \), we have (using the expression for the normal tractor in Equation (3.16)) that \( V_A = (v^-, v_a, v^+) \in \Gamma(N^+) \) obeys \( \bar{n} v = H v^+ \) while the isomorphism between \( \Gamma(N^+) \) and \( \Gamma(T\Sigma)_{|\Sigma} \), given in scales \((g, \bar{g})\) in Equation (4.16), maps \( V_A \) to \( \bar{V}_A = (\bar{v}^-, \bar{v}_a, v^+) \) where

\[\bar{v}_a \equiv v_a^+, \quad \bar{v}^- \equiv v^- + \frac{1}{2} H^2 v^+ .\]

Using the expression for the tractor connection acting on a standard tractor in Equation (4.12) applied to our choice of ambient scale \( g \) we have

\[\nabla_a \bar{V}^B \equiv \left( \begin{array}{c} \nabla_a^T v^+ - v_a^+ \\ \nabla_a^T v_b + \left( P_{ab} - \hat{n}_a P(\hat{n}, b) \right) v^+ + \left( g_{ab} - \hat{n}_a \hat{n}_b \right) v^- \\ \nabla_a^T v^- - P(a, v) + \hat{n}_a P(\hat{n}, v) \end{array} \right) .\]

We now simplify, slot by slot, each expression on the right hand side, beginning at the top:

\[\nabla_a^T v^+ - v_a^+ \equiv \nabla_a v^+ - \bar{v}_a .\]

For the middle slot we have

\[\nabla_a v_b + \left( P_{ab} - \hat{n}_a P(\hat{n}, b) \right) v^+ + \left( g_{ab} - \hat{n}_a \hat{n}_b \right) v^- \]

\[= \nabla_a \bar{v}_b - \hat{n}_b \Pi_a^c \bar{v}_c + \Pi_{ab} H v^+ + \hat{n}_b (\nabla_a H) v^+ + \hat{n}_b \nabla_a v^+ + P_{ab}^\top v^+ + \hat{n}_b \hat{n}_b \nabla_a v^- - \hat{n}_a \hat{n}_b P(\hat{n}, \hat{n}) v^+ + \hat{g}_{ab} v^- \]

\[= \nabla_a \bar{v}_b - \hat{n}_b \left( \Pi_{ac} \bar{v}_c - \frac{\nabla_a^{\bar{v}} H}{d} \right) \nabla_a v^+ + \hat{n}_b H \nabla_a v^- + \hat{n}_a \]

\[+ \left( P_{ab}^\top + H \Pi_{ab} + \frac{1}{2} \hat{g}_{ab} H^2 \right) v^+ + \hat{g}_{ab} \bar{v}^- .\]

Here we have used the traced-Gauß–Mainardi Equation (3.12) to handle gradients of mean curvature. Observe that for \( d \geq 4 \), the last term in brackets, by virtue of the
Gauß–Fialkow Equation (3.3), becomes simply \( \hat{P}_{ab} + F_{ab} \). So the middle slot is

\[
\nabla_a \bar{v}_b + (\hat{P}_{ab} + F_{ab}) \bar{v}^a - \hat{g}_{ab} \left( \hat{\Pi}_{ac} \bar{v}^c - \frac{(\nabla \hat{\Pi}_a) v^+}{d - 2} \right) + \hat{n}_b H (\nabla_a v^+ - \bar{v}_a).
\]

For the bottom slot we have, using the same method at \( d \geq 4 \),

\[
\nabla_a^+ v^+ - P(a, v) + \hat{n}_a P(\hat{n}, v)
\]

Putting the three slots back together, we find that \( \nabla_a^+ V^B \) (along \( \Sigma \)) is

\[
\begin{pmatrix}
1 & 0 & 0 \\
\hat{n}_b H & \delta^c_b & 0 \\
-\frac{H^2}{2} & -\hat{\omega}^c & 1
\end{pmatrix}
\begin{pmatrix}
\nabla_a v^+ - \bar{v}_a \\
\nabla_a \bar{v}_c + \hat{P}_{ac} v^+ + \hat{g}_{ac} \bar{v}^c \\
\nabla_a \bar{v}^c - \hat{P}_{bab} \bar{v}^b
\end{pmatrix}
- \begin{pmatrix}
0 \\
\hat{n}_c L^D_D + \left( F_{ac} v^+ \right) \\
0
\end{pmatrix},
\]

where (according to Equation (4.1)) \( L^D_D = \hat{\Pi}_{ac} \bar{v}^d - \frac{(\nabla \hat{\Pi}_a) v^+}{d - 2} \). This establishes the second equality displayed at the beginning of the proof. It remains to establish the first equation shown there. For that note that

\[
\Sigma^A_D \nabla^c V^B \stackrel{\Sigma}{=} \nabla^c V^A + N^A (\nabla^c N^B) V^B.
\]

Corollary 3.9 combined with Equation (4.1) implies that \( \nabla^c N^B = L^B_c \) and this completes the proof.

5. Extrinsic Conformal Laplacian Powers

An important component in our calculus is the construction of extrinsically coupled invariant differential operators. The key notion here are tangential operators as defined in [16].

**Definition 5.1.** Let \( \sigma \) be a defining density and \( \mathcal{O} \) be a smooth map on tractor bundle section spaces \( \Gamma(\mathcal{T}^k M[w]) \rightarrow \Gamma(\mathcal{T}^{k'} M[w']) \). Then if

\[
\mathcal{O} \circ \sigma = \sigma \circ \mathcal{O}'
\]

where here \( \sigma \) denotes the multiplicative operator sending \( \Gamma(\mathcal{T}^k M[w]) \rightarrow \Gamma(\mathcal{T}^k M[w + 1]) \) (for any \( \Phi \)) and \( \mathcal{O}' \) is any smooth section map \( \Gamma(\mathcal{T}^k M[w + 1]) \rightarrow \Gamma(\mathcal{T}^{k'} M[w' + 1]) \), we call the operator \( \mathcal{O} \) tangential.

The above definition extends to vector bundles where multiplication of sections by a defining density \( \sigma \) is well-defined.

**Example 5.2.** The map \( \Gamma(\wedge^k M[w]) \rightarrow \Gamma(\wedge^k M[w + 1]) \), defined in a choice of scale by

\[
\omega \mapsto \sigma d\omega - \omega \varepsilon(n)\omega
\]

with as usual \( n = \nabla \sigma \), is tangential.

**Remark 5.3.** Tangential operators are of particular interest because we may define

\[
\overline{\mathcal{O}} : \Gamma(\mathcal{T}^k M[w] | \Sigma) \rightarrow \Gamma(\mathcal{T}^{k'} M[w'] | \Sigma) \quad \text{by} \quad \overline{\mathcal{O}} T := (\mathcal{O} T) | \Sigma,
\]

where \( T \in \Gamma(\mathcal{T}^k M[w]) \) and \( \overline{T} = T | \Sigma \).
In [16], it is proved that for any defining density the operator
\[
P_k : \Gamma\left( T^k M \left[ \frac{k - d + 1}{2} \right] \right) \to \Gamma\left( T^{k-1} M \left[ \frac{k - d + 1}{2} \right] \right), \quad k \in \mathbb{Z}_{\geq 1},
\]
defined by
\[
(5.2) \quad P_k := \left( -\frac{1}{I^2} I \partial I \right)^k,
\]
is tangential. Moreover, for AE structures it is shown that the above gives a holographic formula for the conformally invariant Laplacian powers of [21]. In [18, Section 7.1] it is shown that, by taking \( \sigma \) to be a conformal unit defining density, the above construction gives extrinsically coupled analogues \( P_k \) of conformally invariant Laplacian powers determined by the conformal embedding \( \Sigma \hookrightarrow M \). An interesting feature is that for \( k \) odd, the construction naturally produces a leading term in which the trace-free second fundamental form partially replaces the role of the inverse metric. Here we will exploit our conformal calculus to compute explicit formulæ for \( P_k := P_k^\sigma \) with \( k = 2, 3 \).

**Proposition 5.4.** Acting on tractors of weight \( \frac{3-d}{2} \) and \( \frac{1-d}{2} \), respectively,
\[
P_2 \equiv \Delta^\top + \frac{3-d}{2} \left[ J - \frac{K}{2(d-2)} \right], \quad d \geq 3,
\]
\[
P_3 \equiv -8 \left[ \nabla^a \nabla^b \nabla_a \nabla^b + (\nabla \nabla^b - n^a \nabla^b) \nabla_b - \frac{1}{2} n^a (\nabla^b \nabla^a_{ab}) \right. \\
\left. - \frac{1}{2} \frac{2-d}{d-3} \left( \nabla \nabla \nabla - (d-4) \nabla^a \nabla^b \nabla_a \right) \right], \quad d \geq 4.
\]

**Proof.** The result for \( P_2 \) was proven in Lemma [4.9]. For \( P_3 \), we initially assume only \( d \geq 3 \) and now compute along \( \Sigma \):
\[
\frac{1}{4} P_3 = I_{\sigma} \partial I_{\sigma} \partial I_{\sigma} \partial
\]
\[
= \left( \nabla_n + \frac{d}{2} H \right) \left( -\sigma [\Delta + (1 - \frac{d}{2}) J] \right) \left( \nabla_n + (2 - \frac{d}{2}) \rho - \frac{\sigma}{r} [\Delta + (2 - \frac{d}{2}) J] \right)
\]
\[
= \left[ \nabla_n, \Delta \right] + 2 H \Delta + \left( J + (d-4)(\nabla_n \rho) \right) \nabla_n - (d-4)(\nabla^a H) \nabla^a_n
\]
\[
+ \frac{d-4}{2} \left( -\left( \Delta H \right) + (\nabla^2 \rho) + (d-2) \left( H(\nabla_n \rho) - (\nabla_n J) - H J \right) \right).
\]

To obtain the last line we used the operator identities \( [\nabla_n, \sigma] \equiv 1 \) and \( [\Delta, \sigma] \equiv 2[\nabla_n + \frac{d}{2} H] \) (these follow from Lemma [3.3]) and then Lemma [A.2] to handle the Laplace operator acting on densities along \( \Sigma \). To expedite the following computations we introduce the notation
\[
\mathcal{R}_{ab} := [\nabla_a, \nabla_b],
\]
for the operator given by the commutator of connections acting on mixed tensor-tractor quantities. So in particular, for any \( \Gamma(TM) \)-valued operator \( v^c \), we have the operator identity
\[
\mathcal{R}_{ab} \circ v^c = \mathcal{R}_{ab} \circ v^c + R_{ab}^\varepsilon d v^d.
\]
We now focus on the first two operators in the last line of the first display of the proof above.

\[
[\nabla_n, \Delta] + 2H \Delta = \{ n^a \mathcal{R}_{ab}^d - (\nabla_b n_a) \nabla^a, \nabla^b \} + 2H \Delta \\
\leq 2n^a \mathcal{R}_{a}^{ab} \nabla_b \nabla^a - n^b \text{Ric}_{ba} \nabla^a + (n^a \nabla^b \mathcal{R}_{ab}) \\
-2(\Pi^{ab} + n^a n^b H) \nabla_a \nabla_b + 2H \Delta - (\Delta n_a) \nabla^a \\
= -2P_{ab} \nabla_a \nabla^b + (2n^a \mathcal{R}_{a}^{bd} - (d - 2)(n^a \mathcal{P}_{a}^{b}) \nabla^b - (d - 2)P(n, n) + J + 2\Pi_{ab} \mathcal{P}^{ab}) \nabla_n + (n^a \nabla^b \mathcal{R}_{ab}) \\
-\nabla_b (\Pi^{ab} + n^a n^b H) \nabla_a - n_b (\nabla_n \nabla_b n^a) \nabla a \\
= -2P_{ab} \nabla_a \nabla^b + (2n^a \mathcal{R}_{a}^{bd} - (d - 2)(n^a \mathcal{P}_{a}^{b}) \nabla^b - (d - 2)P(n, n) + J + 2\Pi_{ab} \mathcal{P}^{ab}) \nabla_n + (n^a \nabla^b \mathcal{R}_{ab}) \\
-\nabla_b (\Pi^{ab} + n^a n^b H) \nabla_a - n_b (\nabla_n \nabla_b n^a) \nabla a \\
+J + (d - 4)\nabla_n \rho + (d - 2)H(\nabla_n \rho) - (\nabla_n J - HJ).
\]

In the first line, note that \{\cdot, \cdot\} denotes the operator anticommutator while the second and third lines employed Lemma [3.3] Using the antisymmetry of the Riemann tensor in its first two slots and the conformal unit defining density property (3.1), since \(d \geq 3\), the very last line of the above display becomes

\[
(\nabla_n \nabla^a (\rho \sigma)) \nabla_a + (\nabla_n \nabla^a c) (\nabla_a \nabla c) + \nabla_a \nabla^a H \nabla_a + 2(\nabla_n \rho) \nabla_n.
\]

Using this and the traced-Codazzi–Mainardi Equation (3.12) we obtain

\[
[\nabla_n, \Delta] + 2H \Delta = -2P_{ab} \nabla^a \nabla_b \nabla^b + (2n^a \mathcal{R}_{a}^{bd} - 2\nabla \mathcal{P}^{b} + (d - 4) \nabla^{b} H) \nabla^b \\
-J + (d - 4)\nabla_n \rho \nabla_n + (n^a \nabla^b \mathcal{R}_{ab}).
\]

Putting the above identity together with the first display of this proof we have

\[
\frac{1}{4} P_3 = -2P_{ab} \nabla^a \nabla^b \nabla^b - 2(\nabla \mathcal{P}^{b} - n^a \mathcal{R}_{a}^{bd} \nabla^b) \nabla^b + (n^a \nabla^b \mathcal{R}_{ab}) \\
+ \frac{d - 4}{2} \left( - (\Delta H) + (\nabla J^2) + (d - 2)H(\nabla_n \rho) - (\nabla_n J) - HJ \right).
\]

The term \(\nabla_n^2 \rho\) involves four normal derivatives of the conformal unit defining density \(\sigma\) so is only determined by the hypersurface embedding when \(d \geq 4\) which we henceforth assume. Using Equation (4.3) to handle \(\Delta H\) and Lemmas [3.8, A.6, A.7] for \(\nabla_n \rho, \nabla_n^2 \rho\), as well as the Fialkovsky–Gauss Equation (3.3), we obtain the quoted result for \(P_3\).

\[\Box\]

**Remark 5.5**. Note that \(P_2\) is a Laplace-type operator in the usual sense. On the other hand, viewing the trace-free second fundamental form as a proxy for the inverse metric, the leading term of \(P_3\) is an “extrinsic Laplacian”. For \(k = 2, 3\), the explicit formulae above for \(P_k\) have a pole at \(d = k\). Hence, a dimensional continuation argument along the lines given in Remark 4.3 implies that the residue of these poles is separately conformally invariant in dimension \(d = k\). For \(P_2\), this quantity vanishes but in dimension \(d = 3\), a computation similar to that given in the remark, shows that the residue is precisely the obstruction density. It is natural to conjecture that this property will persist for higher dimensional extrinsic conformal Laplacian powers and thus provide an alternate method to compute obstruction densities (at least modulo conformally invariant densities with lower order leading derivative structure). 

\[\ast\]

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The operator $P_2$ is seen to be the intrinsic tractor-coupled Yamabe operator plus an invariant extrinsic term (proportional to the rigidity density). Since it is the result of a lengthy computation, it is worthwhile demonstrating conformal invariance of $P_3$. This is done in the following lemma and proposition:

**Lemma 5.6.** Let $F_{ab} \in \Gamma(\Lambda^2 M)$ and view $F_{ab}$ as a weight zero operator on tractors, acting by multiplication. Then $\nabla^a \circ F_{ab} + F_{ab} \circ \nabla^a$ is an invariant operator on weight $2 - \frac{d}{2}$ tractors mapping $\Gamma(T^* M[2 - \frac{d}{2}]) \to \Gamma(T^* M \otimes T^* M[-\frac{d}{2}])$.

**Proof.** Writing $\nabla^a \circ F_{ab} + F_{ab} \circ \nabla^a = 2F_{ab} \circ \nabla^a + (\nabla^a F_{ab})$, one only needs to compute each term for $\Omega^2 g \in \mathfrak{c}$ acting on $T \in \Gamma(T^* M[2 - \frac{d}{2}])$. For the first we have $(2F_{ab} \nabla^b T) |_{\Omega^2 g} = 2F_{ab} \nabla^b + [2 - \frac{d}{2}] T^b T$ where $\Omega = \Gamma^{-1} d\Omega$ and the right hand side is given for $g \in \mathfrak{c}$. For the term $(\nabla^a F_{ab})$, one must compute the transformation of the Levi-Civita connection acting on a two form. It is not difficult to verify that this exactly cancels the inhomogeneous term produced by the first term. $\square$

This lemma implies that the operator $n^a [\mathscr{A}^b_a \circ \nabla^b_T + \nabla^b_T \circ \mathscr{A}^b_a]$ is conformally invariant. The following proposition expresses $P_3$ as a sum of this operator, an invariant extrinsic term $L$ and a manifestly invariant tractor operator:

**Proposition 5.7.** When $d \geq 4$,

$$P_3 \equiv -8 \left[ L^{AB} + \frac{X^A X^B L}{d-3} \right] \tilde{D}_A \tilde{D}_B + 4 n^a [\mathscr{A}^b_a \circ \nabla^b_T + \nabla^b_T \circ \mathscr{A}^b_a] - 4 (d-4) L.$$

**Proof.** We use the following: (i) Remark 4.8 to define $L^{AB} \tilde{D}_A \tilde{D}_B$ at interior Yamabe weight, (ii) the result for the tractor second fundamental form in (4.1), (iii) the canonical tangential Thomas $D$-operator (4.9) and (iv) the tractor connection as given in (4.2), to compute an operator identity on weight $2 - \frac{d}{2}$ tractors. This gives

$$L^{AB} \tilde{D}_A \tilde{D}_B = \Pi^{ab} \left[ - \nabla^a_T \left( \nabla^b_T + \left[ 2 - \frac{d}{2} \right] n_b H \right) + \left[ 2 - \frac{d}{2} \right] P_{ab} - \frac{\nabla^a \Pi^b - n^a \Pi^b}{2(d-2)} \right] = \Pi^{ab} \left[ - \nabla^a_T \left( \nabla^b_T + \left[ 2 - \frac{d}{2} \right] P_{ab} - \frac{\nabla^a \Pi^b - n^a \Pi^b}{2(d-2)} \right) \right].$$

Noting that $n^a [\mathscr{A}^b_a \circ \nabla^b_T + \nabla^b_T \circ \mathfrak{A}^b_a] = \frac{1}{2} n^a [\mathscr{A}^b_a \circ \nabla^b_T + \nabla^b_T \circ \mathfrak{A}^b_a]$ and comparing with the formula for $P_3$ in Proposition 5.4, completes the proof. $\square$

Proposition 5.4 gives a compact formula for the extrinsic Laplacian appearing above when $d = 4$, thus the remaining difficulty in computing the $d = 4$ obstruction density in curved ambient spaces is calculating two normal derivatives of the canonical extension $K_{ext}$ of the rigidity density $\Pi_{ab} \Pi^{ab}$. Since one normal derivative of the canonically extended trace-free second fundamental form is closely related to the Fialkow tensor, this boils down to computing one normal derivative of the corresponding extension of the Fialkow tensor. This is the next natural example in the general program of proliferating natural invariants of the conformal hypersurface structure $(M, c, \Sigma)$ discussed in Section 3.2. That computation has been performed in [10].
6. FUNCTIONALS FOR CRITICAL CONFORMAL HYPERSURFACE INVARIANTS

We now consider the construction of critical weight Lagrangian densities along the conformal hypersurface and thus seek Riemannian hypersurface invariants, of weight $-d+1$, that yield conformally invariant integrals. While it is straightforward to construct examples (see [24]), the most interesting cases give action functionals that, with respect to variation of the hypersurface embedding, yield Euler–Lagrange equations with a linear leading term. For hypersurfaces embedded in 5 dimensional Euclidean space, such a functional has been constructed [25] by writing down a linear combination of all possible integrated Riemannian hypersurface invariants and then fixing coefficients by demanding invariance under rigid conformal motions (see also [22]). These functionals are also of considerable interest since they may appear as contributions to extrinsically coupled renormalized volume anomalies [19, 20]. Constructing integrated conformal hypersurface invariants with leading derivative term quadratic in curvatures is rather difficult, but a resolution is provided via the extrinsic conformal Laplacians powers $P_k$ described in Section 5 and encapsulated by Theorem 1.5.

**Proof of Theorem 1.5.** Recall that conformal densities of weight $-n$ may integrated on conformal $n$-manifolds. Theorem 7.1 of [18] establishes that the operator $P_{d-1}$ is determined naturally by $(M, c, \Sigma)$ and thus that $N^A P_{d-1} N_A$ is a weight $-d + 1 = \text{dim}(\Sigma)$ density along $\Sigma$, which yields the first statement of the Theorem.

Let $N^A$ be any smooth extension to $M$ of the normal tractor. For $d$ odd, Theorem 7.1 of [18] also ensures that the operator $P_{d-1}$ has non-zero leading term proportional to $(\Delta^T)^{\frac{d-1}{2}}$. We will show below, that when $N^A P_{d-1} N_A$ is integrated over the hypersurface $\Sigma$, this leading term of $P_{d-1}$ contributes a term of the form $\tilde{H}_{ab} \tilde{\Delta}^{\frac{d-3}{2}} \tilde{H}_{ab}$ to the integrand. In particular this involves $d-3$ derivatives and is quadratic in the second fundamental form. We will show that the lower order terms of $P_{d-1}$ cannot contribute terms of this order to the integral $\int_{\Sigma} N^A P_{d-1} N_A$. To see this, firstly note that from Proposition 7.3 of [18] we have

$$P_{d-1} = G^b \circ \nabla_b^T,$$

for some smooth operator $G^b$. Hence

$$N^A P_{d-1} N_A \Sigma \sum = [N^A, G^b] \circ \nabla_b^T N_A,$$

because $N^A \nabla^T N_A \Sigma \sum = 0$. But, because the operator $\nabla^T$ is tangential, we may use Corollary 3.9 to see that $\nabla_b^T N_A$ is linear in curvatures. The operator $G^b$ can only fail to commute with $N^A$ when a $\nabla^T$, in the expression for $G^b$, hits $N^A$ and produces a second curvature. Apart from its leading derivative term, the operator $G^b$ is necessarily at least linear in curvatures. Hence, only the leading derivative term of $P_{d-1}$ yields a term quadratic in curvatures, as required.

Thus we can now focus on the leading term of $P_{d-1}$ in the density $N^A P_{d-1} N_A$ which can be rewritten as $N^A \nabla^b \tilde{\Delta}^{\frac{d-1}{2}} \nabla_b^T N_A$, because reordering derivatives yields subleading terms involving curvatures. Here, again up to subleading curvature terms, the operator $\nabla^T$ equals $\nabla$ twisted by the ambient tractor connection. Thus, discarding a divergence because $\Sigma$ is closed, the functional $\int_{\Sigma} N^A P_{d-1} N_A$ has leading term proportional to

$$\int_{\Sigma} (\nabla^b N^A) \tilde{\Delta}^{\frac{d-3}{2}} \nabla_b^T N_A.$$
We may use Corollary 3.9 again to see that \([\nabla_ b^\top N_A]\) has the form
\[
\nabla_ b^\top N_A = \Sigma \begin{pmatrix} 0 \\ -\Pi_b^a \\ * \end{pmatrix}.
\]
It follows that the leading term of the functional, as claimed, is a non-zero multiple of
\[
\int \Pi_{ab} \Delta^{\frac{d-3}{2}} \Pi_{ab}.
\]
It is not difficult to check that when varying an embedding of a functional \(\int \Pi_{ab} K_{ab}\), the contribution to the Euler-Lagrange equation from the variation of the (explicit) trace-free second fundamental form is \(\nabla^a \nabla^b K_{(ab)}\), where \(K_{(ab)}\) denotes the trace-free symmetric part of the tensor \(K\). Since varying the measure or the operator \(\Delta^{\frac{d-3}{2}}\) necessarily leads to contributions quadratic in \(\Pi\), it follows that the functional in the above display contributes only \(2 \nabla^a \nabla^b \Delta^{\frac{d-3}{2}} \Pi_{ab}\), at linear order in \(\Pi\), to the Euler–Lagrange equation. Employing the identity (4.3) we thus obtain the Euler–Lagrange equation
\[
\Delta^{\frac{d-3}{2}} H + \text{lower order terms} = 0,
\]
in agreement with the result of Theorem 5.1 of [18] for the leading order contribution to the obstruction density.

**Remark 6.1.** As we discuss in the following example, the last statement of the above theorem also holds for embedded volumes, except that the Euler–Lagrange equation is now quadratic in the second fundamental form as it must be to agree with the leading term of the corresponding obstruction density. It seems plausible that a similar statement holds for all higher, odd dimensional embedded hypersurfaces.

**Example 6.2.** A simple application of our extrinsic Laplacian formulæ is to compute low dimensional examples of the action functional density (1.6). The easiest case is dimension \(d = 3\) for which we find
\[
N_A P_2 N_A = N_A g^a I_A = N_A \Delta I_A = -\Pi_{ab} \tilde{\Pi}^{ab}.
\]
The second step above used Proposition 5.4 while the last step of this computation relied on Corollary 3.9 to evaluate \(\nabla_a I_A\) as well as Equations (4.2) and (2.3), respectively, for the tractor connection and the relation between tangential and boundary Levi-Civita connections. Hence the functional
\[
\int \Sigma N_A P_2 N_A = -\int \Pi_{ab} \Pi^{ab} = -\int \Sigma K,
\]
recovers the well-known Willmore energy [34] or (extended to Lorentzian signature) the rigid string action of [29] which justifies calling \(K\) the rigidity density.

The above functional appears in the formula for the renormalized area of a minimal surface embedded in a hyperbolic 3-manifold [1]. It is interesting to note that the above functional also appears as the log term coefficient in the asymptotic expansion for the volume associated with a 2-brane in the AdS/CFT correspondence (and is linked to the anomaly for boundary observables) [22]; the corresponding anomaly functionals for hypersurfaces of arbitrary dimensions have recently been computed in [19].
In the next dimension $d = 4$, the computation of $N_A P_3 N^A$ is more involved, but remains simple for conformally flat structures: From Proposition 5.4 we have in this case that $\nabla^T_c N^A \equiv -8 \left[ \nabla^b \nabla^T_b + (\nabla_b \nabla^b) \right] \nabla^T_c$. We again use Corollary 3.9, which gives

$$[\nabla^T_c N^A] \equiv \begin{pmatrix} 0 \\ \nna^a_c \\ \gamma_n^a_c \end{pmatrix}.$$ 

Since $N_A \nabla^T_c N^A = 0$, we only need to compute the leading double derivative term which again requires using Equation (4.2) for the tractor connection. This yields

$$N_C \nabla_a^T \nabla^T_b N^C = -\nabla^a \nabla^b N^C = -\nabla^a F^a_{bc} = -L.$$ 

Hence, as promised, $L$ plays the rôle of a rigidity density for embedded volumes. Indeed, for conformally flat structures, it is straightforward to compute the embedding variation of the functional

$$\int_{\Sigma} N_A P_3 N^A = 8 \int_{\Sigma} \nabla^a \nabla^b \nabla^c \nabla^d N^A = 8 \int_{\Sigma} L.$$ 

(Functionals constructed from powers of $\nabla$ have been studied in [24].) The resulting Euler–Lagrange equation is $\mathcal{B} = 0$ with $\mathcal{B}$ given by the conformally-flat, four-manifold, obstruction density quoted in Proposition 2.10. Details of this computation and its extension to generally curved conformal structures is presented in [10]. There it is shown that, for hypersurfaces in general 4-manifolds, the functional gradient of (1.6) agrees precisely with the obstruction density. We note that the functional $\int_{\Sigma} L$ in Lorentzian signature could be of interest for a rigid membrane theory.

**Appendix A. Proof of Lemma 3.10**

In this section, we employ the notations of section 3.1 and break the proof of Proposition 3.10 into several smaller pieces. The first of these explicates the terms “LTOTs” of Equation (3.7).

**Lemma A.1.** Let $\sigma$ be a conformal unit defining density, then

$$\frac{1}{2} \nabla^a_n r^2 + (d - 3) \nabla^a_n \rho = -\nabla^a_n (\rho_n^a \nabla^b r) - \nabla^a_n (5 \rho^2 + 2 J) + 4 \rho^3 + 2 \rho J.$$

The proof of the above Lemma is, by now, elementary. Of the terms on the right hand side of (A.1), only the the first has not been computed from previous steps in the recursion. This is somewhat involved. Firstly, we need a lemma relating the ambient and hypersurface Laplacians.

**Lemma A.2.** Let $f$ be a (smooth) extension of any function $\bar{f}$ defined along $\Sigma$. Then

$$\Delta f \equiv (\Delta - \nabla^2_n - (d - 2) H \nabla_n) f.$$ 

**Proof.** The proof is a simple (double) application of the formula (2.3) relating ambient and hypersurface Levi-Civita connections

$$\Delta f = \nabla^a \nabla^b \nabla^c \nabla^d f + \nabla^a (\nabla^b n^c \nabla^d f) \equiv (\nabla^a - n^a \nabla_n) (\nabla_a - n_a \nabla_n) f$$

$$\equiv (\Delta - \nabla^2_n + (\nabla_n n^a) \nabla_a - (\nabla^a_t n^a) \nabla_n) f.$$ 

Finally, note that $\nabla^a_n n^a \equiv \Pi^a_n = (d - 1) H$ and $\nabla_n n_a = \frac{1}{2} \nabla_a n^2 \equiv \nabla_n (\rho \sigma) \equiv H.$
This result allows us to compute a quantity required for handling the troublesome term $-\nabla^2_n \gamma^{ab} \nabla_a n_b$ in Equation \eqref{A.1}.

**Lemma A.3.** Let $\sigma$ be a conformal unit defining density, and $d > 2$, then
\[
\gamma^{ab} \nabla_a \nabla_b n_c \Sigma \equiv \Delta H - 2(d - 1)H \nabla_n \rho + (d - 1)H^3.
\]

**Proof.** Again, we compute explicitly along $\Sigma$ using the techniques developed in section 3.1 and, at the last step, the preceding lemma:
\[
\gamma^{ab} \nabla_a \nabla^2_n n_b = \frac{1}{2} \gamma^{ab} \nabla_a \nabla_b (1 - 2\rho \sigma + O(\sigma^2))
\]
\[
= -\gamma^{ab} \nabla_a \nabla_b (\sigma \nabla_b \rho + \rho n_b)
\]
\[
= -\gamma^{ab} \nabla_a (\nabla_b (1 - 2\rho \sigma) + \nabla_n \rho n_b + \rho \nabla_n n_b)
\]
\[
= -\gamma^{ab} \nabla_a \nabla_b \rho - (d - 1)H \nabla_n \rho - \gamma^{ab} \rho \nabla_a (-\sigma \nabla_b \rho - \rho n_b)
\]
\[
= \Delta H - 2(d - 1)H \nabla_n \rho + (d - 1)H^3.
\]
\[\square\]

**Remark A.4.** This result ensures that the leading term of the $d = 3$ obstruction density coincides with the leading Laplacian term of the Willmore invariant (\ref{2.16}).

To use Lemma A.3, we still need to commute the operators $\nabla^2_a$ and $\gamma^{ab} \nabla_a$. This calculation is encoded in the following result.

**Lemma A.5.** Let $\sigma$ be a conformal unit defining density and $d > 2$, then
\[
\nabla^2_n \gamma^{ab} \nabla_a n_b - \gamma^{ab} \nabla^2_n \nabla_a n_b \Sigma \equiv 12H \nabla_n \rho - 4H^3,
\]
\[
\gamma^{ab} (\nabla^2_n \nabla_a n_b - \nabla_a \nabla^2_n n_b) \equiv -((\nabla_n - H) \text{Ric}(n, n) + 2 \Pi^{ab} \Pi_{ac} \Pi^c_b + 3H \Pi^{ab} \Pi_{ab})
\]
\[
- (d - 1)H^3 - 2\Pi^{ab} R_{cabd} n^c n^d.
\]

**Proof.** Again, both these results can be obtained computing along $\Sigma$ using the techniques developed in section 3.1
\[
[\nabla^2_n, \gamma^{ab}] \nabla_a n_b = -2n^a(\nabla^2_n n^b) \nabla_a n_b - 2(\nabla_n n^a)(\nabla_n n^b) \nabla_a n_b - 4n^a (\nabla_n n^b) \nabla_n \nabla_a n_b
\]
\[
= -2n^a ((-\nabla^b \rho - n^b \nabla_n \rho + H^2 n^b) (\Pi_{ab} + H n_a n_b) - 2H^2 n^a n^b (\Pi_{ab} + H n_a n_b)
\]
\[
- 4n^a H n^b (\nabla_a \nabla_n n_b + R_{cabd} n^c n^d - \nabla_a n^c \nabla_c n_b)
\]
\[
= 2H (2\nabla_n \rho - H^2) - 2H^3 - 4H n^a \nabla^2_n n_a + 4H^3
\]
\[
= 12H \nabla_n \rho - 4H^3,
\]
and
\[
\gamma^{ab} (\nabla^2_n \nabla_a n_b - \nabla_a \nabla^2_n n_b) = \gamma^{ab} ([\nabla_n, \nabla_a] \nabla_n n_b + \nabla_n (R_{cabd} n^c n^d - (\nabla_a n_c) \nabla^c n_b))
\]
\[
= \gamma^{ab} (R_{dabc} n^d \nabla_n n^c - (\nabla_a n_c) \nabla^c n_b)
\]
\[
- \nabla_n \text{Ric}(n, n) + 2n^a(\nabla^b n^b) R_{cabd} n^c n^d - 2\gamma^{ab} (\nabla_a n_c) \nabla_n \nabla^c n_b
\]
\[
= -H \text{Ric}(n, n) - 3 \Pi^{ab} \nabla_a \nabla_n n_b - \nabla_n \text{Ric}(n, n) - 2 \Pi^{ab} (R_{cabd} n^c n^d - \nabla_a n_c \nabla^c n_b)
\]
\[
= -((\nabla_n + H) \text{Ric}(n, n) - 3H \Pi^{ab} \Pi_{ab} - 2 \Pi^{ab} R_{cabd} n^c n^d + 2 \Pi^{ab} \Pi_{ab} H^2)
\]
\[
= -((\nabla_n - H) \text{Ric}(n, n) + 2 \Pi^{ab} \Pi_{ac} \Pi^c_b + 3H \Pi^{ab} \Pi_{ab} - (d - 1)H^3 - 2\Pi^{ab} R_{cabd} n^c n^d).
\]
Orchestrating Lemmas A.1, A.3 and A.5 plus the results of section 3.1 for the previous steps of the recursion involving $\rho$ and $\nabla_n\rho$ along $\Sigma$, gives immediately our first formula for the $d = 3$ obstruction density and $\nabla^2\rho$.

**Lemma A.6.** Let $\sigma$ be a conformal unit defining density, then if $d > 2$,

$$
\frac{1}{2} \nabla^3\rho^2 + (d - 3)\nabla^2\rho = -\Delta H - H\hat{\Pi}^{ab}\hat{\Pi}_{ab} - 2\hat{\Pi}^{ab}\hat{\Pi}_{ab}\hat{\Pi}_b^c + \nabla_n G(n,n) + (d - 3)(\nabla_n + 2H)G + 2\hat{\Pi}^{ab}R_{cabcd}n^c n^d + H Ric(n,n).
$$

To complete the proof of Proposition 3.10 we need to (i) express the ambient Riemann tensor in terms of its Weyl and Schouten tensor constituents, (ii) trade the Laplacian of mean curvature for the second fundamental form divergence using the hypersurface identity (4.3) and (iii) rewrite the normal derivative of the normal components of the ambient Einstein tensor $\nabla_n G(n,n)$ in terms of hypersurface quantities. Only step (iii) is non-trivial, it relies on one more Lemma.

**Lemma A.7.**

$$
\nabla_n G(n,n) \equiv -\nabla^a (\text{Ric}_{ab} n^b)^\top + \hat{\Pi}^{ab} \text{Ric}_{ab} - (d - 2)H \text{Ric}(n,n).
$$

**Proof.** This computation relies on the algebraic Bianchi identity for the ambient Riemann tensor:

$$
\nabla_a ((\text{Ric}_{ab} n^b)^\top) = \gamma^{ab} \nabla_a (R_{dcb} n^d - n_b R_{dec} n^c n^d n^e)
$$

$$
= \Pi^{ab} \text{Ric}(a,b) - \gamma^{ab} n^d (\nabla_d R_{cab}) + \nabla_c R_{dab} - (d - 1)H \text{Ric}(n,n)
$$

$$
= \Pi^{ab} \text{Ric}(a,b) + \frac{1}{2} \nabla_n R - \nabla_n \text{Ric}(n,n) + \nabla_n (n^a n^b) \text{Ric}_{ab} - (d - 1)H \text{Ric}(n,n)
$$

$$
= -\nabla_n G(n,n) + \Pi^{ab} \text{Ric}_{ab} - (d - 2)H \text{Ric}(n,n).
$$

\[\square\]

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