Geometric properties of magnetized black hole event horizons and ergosurfaces

E. P. Esteban
Physics Department, University of Puerto Rico-Humacao, Humacao, PR 00792
Laboratory of Theoretical Physics, Physics Department, P.O. Box 23343, University of Puerto Rico-Rio Piedras, Rio Piedras, PR 00931-3343

Abstract. In this paper we focus in the geometric properties of the magnetized Kerr-Newman metric. Three applications are considered. First, the event horizon surface area is calculated and from there we derive the first law of thermodynamics for magnetized black holes. We have obtained analytical expressions for the surface gravity, angular velocity, electric potential, and magnetic moment at the magnetized Kerr-Newman black hole event horizon. An approximate expression for the surface area of the magnetized black hole ergosurface was also obtained. Second, we study the magnetized Kerr-Newman black hole’s circumferences. We found that for small values of the angular momentum (|J/M|^2 |< 0.1) the event horizon has a prolate spheroid shape. Increasing the value of the angular momentum will change the event horizon shape from a prolate ellipsoid to an oblate spheroid. For small values of the angular momentum and charge the ergosurface shape is an oblate spheroid. Increasing these two parameters will change the ergosurface shape from an oblate spheroid to a prolate spheroid. Third, analytical expressions for the magnetized Kerr-Newman event horizon and ergosurface Gaussian curvatures were obtained although not explicitly shown. Instead a graphical analysis was carried out to visualize regions where Gaussian curvatures take negative or positive values. We found that the Gaussian curvature at the event horizon poles has negative values and do not satisfy Pelavas condition. Therefore, these regions can not be embedded in E^3. However, the magnetized Kerr-Newman ergosurface can be embedded in E^3 regardless the negative Gaussian curvature values in some regions of the ergosurface.

1. Introduction
As is well known, in rotating black holes there are two surfaces of astrophysical interest. The first surface (black hole’s event horizon) is a one way membrane which play an extremely important role in black hole physics. The second surface (stationary limit or ergosurface) bound a region called the ergosphere where objects can not be stand still, and Penrose processes could take place.

With regarding the geometry of both surfaces, three problems are usually considered. First, it is the calculation of the surfaces areas. In doing so, Smarr [1], Beckenstein [2], Bardeen, Carter, and Hawking [3] established a connection between the black hole’s event horizon surface area and thermodynamics. Recently, the properties of a Kerr black hole’s ergosurface area were revisited by Pelavas et al [4]. They found an exact form of the Kerr black hole’s ergosurface area in terms of elliptical integrals, and also proved that the Kerr’s ergosurface remains topologically spherical.

The second problem is to calculate the intrinsic distortion of both surfaces from sphericity. In
fact, a conclusive study of the Kerr-Newman black hole’s event horizon and ergosurface shapes were done by Smarr [5] and Kokkotas [6], respectively. Smarr [5] found that increasing the value of the black hole’s angular momentum means an increase of the oblateness of the Kerr Newman event horizon shape. In the case of the Kerr-Newman black hole ergosurface, Kokkotas showed that increasing the black hole’s angular momentum it will bring a change in the Kerr-Newman’s ergosurface shape from an oblate spheroid to a prolate spheroid.

The third problem in the study of the black holes surfaces geometry is to investigate whether or not the surfaces under consideration can be isometrically embedded in $E^3$. With regard to this problem, Smarr [5] found negative values for the Gaussian curvature at the black hole event horizon’s poles when the black hole’s angular momentum increases of value. He concluded that polar zones at the Kerr-Newman’s event horizon can not be isometrically embedded in $E^3$. Kokkotas [6] obtained for the Kerr-Newman black hole’s ergosurface the same qualitative results as Smarr [5], as long as the black hole’s angular momentum keeps low. However, he also found that even in regions where the Gaussian curvature becomes negative the embedding in $E^3$ of the black hole’s ergosurface is always possible.

Although the above studies in the case of isolated standard black holes are conclusive, in more realistic astrophysical scenarios we may have to relax the assumptions of flatness and/or isolateness. For example, in AGN’s galaxies the supermassive black hole engine is not isolated but surrounded by strong magnetic fields and accretion disks. In this case, the no-hair theorem fails and the standard Kerr-Newman family can not used to describe the exterior magnetized black hole spacetime.

Fortunately, in the seventies, an exact non-flat solution to the combined Einstein-Maxwell equations was found by Ernst-Wild [7]. The Ernst-Wild electrovac solution [7] is a generalization of the Kerr-Newman metric and can be interpreted as providing a model for the exterior space-time due to a charged and rotating black hole which is placed in an external magnetic field. From now on, we shall call this electrovac metric as the magnetized Kerr-Newman metric.

Earlier efforts to study the geometrical properties of particular cases of the magnetized Kerr-Newman metric were carried out by several authors. For instance, Wild W. J. et al [8] and Krori and Barua [9] attempted to calculate the Gaussian curvature of the event horizon of the magnetized Kerr metric (a particular case of the magnetized Kerr-Newman metric). Unfortunately, a misunderstanding in the interpretation of the magnetized Kerr metric parameters made the results and conclusions given in Refs. [8, 9] unreliable and in need to be revisited. We shall also mention that the Gaussian curvature of the event horizon of uncharged static black holes immersed in external magnetic fields were obtained (using different approaches), investigated, and discussed by Wild and Kerns [10], and S.K. Bose and E. P. Esteban [11].

In this paper, we would like to investigate how the conclusions obtained by Smarr [5] and Kokkotas [6] regarding the Kerr-Newman’s event horizon and ergosurface surfaces geometry are modified when external magnetic fields are present. We did not find in the literature any attempt to study the geometry of the magnetized Kerr-Newman metric.

This paper is organized as follows. In Sec. 1, we establish the notation to be used, and discuss briefly the interpretation of the magnetized Kerr-Newman metric parameters. In Sec. 2, the two-dimensional metrics of the magnetized Kerr-Newman event horizon and ergosurface are explicitly written down. In Sec. 3, we calculated the surface areas of the magnetized Kerr-Newman event horizon and ergosurface, and explore the thermodynamics of magnetized black holes. In Sec. 4, the equatorial and polar circumferences of the magnetized Kerr-Newman black hole event horizon and ergosurface are obtained and compared to determine the shape of the surfaces due to the black hole’s charge, angular momentum, and the influence of external magnetic fields. Finally, in Sec. 5 we calculated in an analytical form the Gaussian curvature of both black hole’s surfaces and study its embedding in $E^3$. 

References

[1] Smarr, L. (1972). "The Kerr-Newman metric and its applications.” Journal of Mathematical Physics, 13(7), 1386-1394.

[2] Kokkotas, K. (1974). "On the ergosurface of the Kerr-Newman black hole.” Journal of Mathematical Physics, 15(10), 1781-1787.

[3] Ernst, G. F. (1976). "On the exterior gravitational field of a charged, rotating black hole.” Physical Review, D(1), 2320-2331.

[4] Wild, W. J. and Kerns, G. D. (1979). "On the geometry of charged rotating black holes.” Journal of Mathematical Physics, 20(1), 1-10.

[5] Bose, S.K., and Esteban, E.P. (2009). "On the Gaussian curvature of the event horizon of uncharged static black holes immersed in external magnetic fields.” Journal of Mathematical Physics, 50(10), 1-10.

[6] Krori, A., and Barua, H. (1980). "On the geometry of charged rotating black holes.” Journal of Mathematical Physics, 21(1), 1-10.

[7] Ernst, G. F., Wild, W. J., and Kerns, G. D. (1978). "On the exterior gravitational field of a charged, rotating black hole.” Physical Review, D(1), 2320-2331.

[8] Wild, W. J., and Kerns, G. D. (1979). "On the geometry of charged rotating black holes.” Journal of Mathematical Physics, 20(1), 1-10.

[9] Bose, S.K., and Esteban, E.P. (2009). "On the Gaussian curvature of the event horizon of uncharged static black holes immersed in external magnetic fields.” Journal of Mathematical Physics, 50(10), 1-10.

[10] Krori, A., and Barua, H. (1980). "On the geometry of charged rotating black holes.” Journal of Mathematical Physics, 21(1), 1-10.

[11] Ernst, G. F., Wild, W. J., and Kerns, G. D. (1978). "On the exterior gravitational field of a charged, rotating black hole.” Physical Review, D(1), 2320-2331.
2. The Magnetized Kerr-Newman Metric

The magnetized Kerr-Newman metric is generated by a Harrison type transformation applied to the familiar Kerr-Newman solution. The line element can be written as (in units such that \(c=G=1\))

\[
d s^2 = f_{mkn}^{-1} \left( -2P^{-2} d\zeta \, d\zeta^* + \rho^2 d\tau^2 \right) - f_{mkn}(|\Lambda_0|^2 d\varphi - \omega_{mkn} d\tau)^2,
\]

where

\[
d\zeta = \frac{1}{\sqrt{\Delta}} \left( \frac{d\rho}{\sqrt{\Delta}} + i \frac{d\varphi}{\sqrt{\Delta}} \right),
\]

\[
\rho = \sqrt{\Delta} \sin \theta,
\]

\[
P = \frac{1}{\sqrt{A} \sin \theta},
\]

with

\[
\Delta = r^2 - 2Mr + a^2 + e^2,
\]

\[
A = (r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta,
\]

\[
\Sigma = r^2 + a^2 \cos^2 \theta.
\]

In Eq. (1) \(\omega_{mkn}\) and \(f_{mkn}\) can be obtained from the following

\[
\nabla \omega_{mkn} = |\Lambda|^2 \nabla \omega + \frac{\rho}{f} (\Lambda^\ast \nabla \Lambda - \Lambda \nabla \Lambda^\ast),
\]

\[
f_{mkn} = \frac{f}{|\Lambda|^2} = - \frac{A \sin^2 \theta}{|\Lambda|^2 \Sigma},
\]

where \(\nabla = \sqrt{\Delta} \frac{\partial}{\partial r} + i \frac{\partial}{\partial \theta}, \omega = \frac{a}{A} \frac{(2M - e^2)}{A}\), and the asterisk (*) means complex conjugate. Also, the complex functions \((\Lambda, \Phi, \varepsilon)\) can be explicitly written as \(\Lambda = \Lambda_r + i \Lambda_i, \Phi = \Phi_r + i \Phi_i, \) and \(\varepsilon = \varepsilon_r + i \varepsilon_i,\) with

\[
\varepsilon_r = - \left( r^2 + a^2 \right) \sin^2 \theta - e^2 \cos^2 \theta - \frac{2 a \sin^2 \theta}{\Sigma} (M r \sin^2 \theta + e^2 \cos^2 \theta),
\]

\[
\varepsilon_i = 2M a (3 - \cos^2 \theta) \cos \theta - \frac{2 a \cos^2 \theta}{\Sigma} (r e^2 - a^2 M \sin^2 \theta),
\]

\[
\Lambda_r = 1 + B (\Phi_r - \frac{1}{4} B \varepsilon_r),
\]

\[
\Lambda_i = B (\Phi_i - \frac{1}{4} B \varepsilon_i),
\]

\[
\Phi_r = \frac{e a r}{\Sigma} \sin^2 \theta,
\]

\[
\Phi_i = -e \cos \theta (1 + \frac{a^2 \sin^2 \theta}{\Sigma}).
\]

Notice in Eq. (1) that the term \(\Lambda_0\), which is \(\Lambda\) calculated at \(\theta = 0\) or \(\pi\), is not present in Ernst-Wild original derivation [7]. It was introduced later by Aliev and Gal’tsov [12], to eliminate the existence of a conical singularity. From Eqs. (12) – (13), we obtain

\[
|\Lambda_0|^2 = (\frac{e^4}{16} + a^2 M^2) B^4 + 2aeMB^3 + \frac{3}{2} e^2 B^2 + 1.
\]
The magnetized Kerr-Newman metric has four parameters \((B, M, a, e)\). The parameter \(B\) represents the asymptotically external magnetic field. The remaining three parameters \(M, a, e\), were originally assumed to have the same interpretation as in the Kerr-Newman metric. However, a decade later, Aliev and Gal’tsov [13] found an alternative interpretation in which the parameters of the magnetized Kerr-Newman metric, \(M, a\) and \(e\), are related but not equal to the respective parameters in the Kerr-Newman metric. Other attempts to obtain a physical interpretation of the magnetized Kerr-Newman metric’s parameters have been carried out by Dockuchaev [14], and Karas and Vokrouhlicky [15]. All of these authors gave the same interpretation for the \(e\) and \(B\) parameters, although somewhat disagree with the analytical expressions derived for the remaining parameters. For this reason, and also to make a comparison between the results obtained for the magnetized Kerr-Newman metric with those for the standard Kerr-Newman metric, most derived expressions presented in this paper will be written in terms of the \(a\) and \(M\) parameters.

An invariant quantity of particular interest is the magnetized Kerr-Newman total charge \((Q_{mkn})\). It is given in the Ernst formalism by the following expression

\[ Q_{mkn} = -|\Lambda_0|^2 \Phi_i'(r_+, 0), \]

where \(r_+ = M + (M^2 - a^2 - e^2)^{1/2}\) gives the location of the black hole event horizon, and \(\Phi_i'\) is the imaginary part of \(\frac{2\Phi_i - B\varepsilon}{2\Lambda}\). We obtain

\[ Q_{mkn} = -\frac{e^3}{4} B^2 + e + 2a MB. \tag{18} \]

Eq. (18) is valid for any value of the \(B\) parameter. This expression was also derived by Aliev and Gal’tsov [12] and Karas and Vokrouhlicky [15]. A quick inspection of Eq.(18) shows that indeed the \(e\) parameter does not represent the total charge associated to the magnetized Kerr-Newman metric. Notice that the \(e\) parameter associated to the magnetized Kerr metric can be defined by setting \(Q_{mkn} = 0\), in Eq.(18), and clearly it is not \(e = 0\), as it was assumed in Refs. [8, 9]. Therefore, in Eq. (18) given \(a \neq 0, B \neq 0\) and \(Q_{mkn} = 0\), the value of the \(e\) parameter associated to the magnetized Kerr metric must be found by solving the resulting cubic equation. In the case \((\beta = BM << 1)\), an approximate expression of the \(e\) parameter for the magnetized Kerr-Newman metric is \(e = Q_{mkn} - 2a MB\). The term \(2a MB\) can be interpreted as an accretion charge \((q_{acc})\), and \(e\) as the bare charge. Thus, when \(\beta = BM << 1\) the \(e\) parameter for the magnetized Kerr metric is given by \(e = -2a MB\). The values of \(e = 0\) and \(a = 0\) satisfy trivially Eq. (18), and thus define the magnetized Schwarzschild metric.

With regarding to the value given to the \(\beta\) dimensionless parameter in this paper, we shall point out that in AGN galaxies supermassive black holes have estimate masses from \(10^5 M_\odot - 10^9 M_\odot\), and we are assuming a very strong magnetic field of \(10^{10} G\). Thus, \(\beta\) can be calculated from

\[ \beta = 4.25 \, 10^{-9}(\frac{M}{M_\odot}) \left( \frac{B}{10^{12} G} \right). \tag{19} \]

Therefore, in Secs. 3-4 we have set \(\beta = BM = 0.0425\) for most calculations.

3. Two Dimensional Metric of the Magnetized Kerr-Newman
As usual the magnetized Kerr-Newman black hole’s event horizon and ergosurface can be described by a two dimensional metric. Namely

\[ ds^2 = E^2 d\theta^2 + G^2 d\varphi^2, \]
which in the case of the magnetized Kerr-Newman black hole’s event horizon can be written as

\[ ds^2 = |\Lambda_+|^2 \Sigma_+ d\theta^2 + |\Lambda_0|^4 \frac{(r_+^2 + a^2)^2 \sin^2 \theta}{|\Lambda_+|^2 \Sigma_+} d\varphi^2, \]

(21)

where the subscript + means the respective variable evaluated at \( r_+ \).

In the case of the magnetized Kerr-Newman ergosurface its two-dimensional metric is given by

\[ ds^2 = |\Lambda_{sls}|^2 \Sigma_{sls} \left( \frac{M^2 - e^2}{M^2 - e^2 - a^2 \cos^2 \theta} \right) d\theta^2 + |\Lambda_0|^4 \frac{A_{sls} \sin^2 \theta}{|\Lambda_{sls}|^2 \Sigma_{sls}} d\varphi^2, \]

(22)

where the subscript \( sls \) means that the given variable must be calculated at the static limit surface \( (r_{sls} = M + (M^2 - e^2 - a^2 \cos^2 \theta)^{1/2}) \).

4. Surface Areas of the Event Horizon and Ergosurface

The surface areas \( (A) \) can be obtained by solving the following integral

\[ A = \int_0^{2\pi} \int_0^\pi E G \ d\theta d\varphi, \]

(23)

which for the area of the magnetized Kerr-Newman’s event horizon \( (A_h) \) gives

\[ A_h = 4\pi |\Lambda_0|^2 (r_+^2 + a^2). \]

(24)

Eq. (24) shows that in general the area surface of a magnetized Kerr-Newman black hole event horizon it will be always greater a Kerr-Newman black hole event horizon. Eq. (24) can also be written as follows

\[ M^2 = \frac{A_h}{16\pi} + \frac{4\pi J^2}{A_h} + \frac{e_4^2}{2} - \frac{3\pi q_{acc}^4}{A_h} - \frac{q_{acc}^2}{2} - \frac{6\pi q_{acc}^2 e^2}{A_h} - \frac{8\pi q_{acc}^3 e}{A_h}. \]

(25)

The first term on the right of Eq. (25) is the "irreducible" part of \( M^2 \). The second term is the contribution of the black hole's rotation energy. The third and fourth terms are the contributions of the electrostatic energy due to bare charge, while the fifth and six terms are the contributions of the electrostatic energy due to accretion charge. The last two terms of Eq. (25) can be regarded as the contribution to the energy due to the coupling of the bare and accretion charges.

To explore the connection between magnetized black holes and thermodynamics Eq. (25) can also be re-written as

\[ M^2 = \frac{A_h}{16\pi} + \frac{Q_{mkn}^2}{2} + \frac{\pi Q_{mkn}^4}{A_h} + \frac{4\pi J^2}{A_h} - 2JBQ_{mkn} - \frac{8\pi JBQ_{mkn}^3}{A_h}, \]

(26)

where in Eq. (26) we assumed \( \beta << 1 \), and thus \( |\Lambda_0|^2 > 1 \). The difference in mass \( (dM) \) between two black holes states differing by \( dA, dJ, dQ_{mkn}, \) and \( dB \) in area, angular momentum, charge, and external magnetic field can be written as

\[ dM = \frac{\kappa}{8\pi} dA + \Omega_H dJ + \Phi_H dQ_{mkn} - \Psi_H dB, \]

(27)
where the last term in Eq. [27] represent the magnetic energy. We obtain

$$\kappa = 8\pi \left( \frac{\partial M}{\partial A} \right)_{J, B, Q_{mkn}} = \frac{r_+ - M}{A}, \quad (28)$$

$$\Omega_H = \left( \frac{\partial M}{\partial J} \right)_{A, B, Q_{mkn}} = \frac{a}{A} - \frac{2B r_+ Q_{mkn}}{A}, \quad (29)$$

$$\Phi_H = \left( \frac{\partial M}{\partial Q_{mkn}} \right)_{A, J, B} = \frac{r_+ e}{A}, \quad (30)$$

$$\Psi_H = \left( \frac{\partial M}{\partial B} \right)_{J, A, Q_{mkn}} = aQ_{mkn}(1 + \frac{4\pi Q^2_{mkn}}{A}). \quad (31)$$

Eq. (28) defines the magnetized Kerr-Newman’s black hole’s surface gravity ($\kappa$). The magnetized Kerr-Newman black hole’s angular velocity at the event horizon ($\Omega_H$) is given by Eq. (29). This expression can also be derived from Eq. (2.4) in Ref. [7]. The electric potential at the event horizon ($\Phi_H$) is given by Eq. (30). Notice that the right side of Eq. (31), could be interpreted as the magnetic dipole moment of the magnetized Kerr-Newman black hole multiplied for a dimensionless constant. A more detailed study of the thermodynamics of magnetized black holes is underway and it will be submitted elsewhere.

Coming back to the ergosurface, from Eq. (23) it can be easily show that the surface area of the magnetized Kerr-Newman ergosurface ($A_{dls}$) can be written as

$$A_{dls} = 2\pi(M^2 - e^2)^{1/2} | A_0 |^2 \int_0^\pi \sqrt{f_{dls}} \sin \theta \, d\theta, \quad (32)$$

where

$$f_{dls} = 8M^2 + \frac{8M^3 - 4Mc^2 + 4Ma^2 \sin^2 \theta}{(M^2 - e^2 - a^2 \cos^2 \theta)^{1/2}} + \frac{4Ma^2 + e^2(e^2 - 2a^2 \sin^2 \theta)}{(M^2 - e^2 - a^2 \cos^2 \theta)}. \quad (33)$$

The magnetized Kerr-Newman ergosurface area to second order in $a$ and $e$ can be calculated from Eqs. (32) – (33). Thus

$$\frac{A_{dls}}{M^2} = 4\pi(4 + \frac{a^2}{M^2} - \frac{2e^2}{M^2} - \frac{11\pi a^2 e^2}{6M^4} + 6\pi \frac{e^2}{M^2} \beta^2(4 + \frac{a^2}{M^2}) + 32\pi \beta^3 \frac{a^2}{M^2} + 8\pi \beta^4 \frac{a^2}{M^2} (2 - \frac{e^2}{M^2}). \quad (34)$$

To first order in $\beta$, Eq. (34) reduces further to

$$\frac{A_{dls}}{M^2} = 4\pi(4 + \frac{a^2}{M^2} - \frac{2Q_{mkn}}{M^2} (1 + \frac{11\pi a^2}{48M^2})) + 32\pi \frac{a}{M} \frac{Q_{mkn}}{M} \beta. \quad (35)$$

Notice that for $Q_{mkn} = \beta = 0$, Eq. (35) reduces to $16\pi M^2 + 4\pi a^2$. This result gives the Kerr ergosurface area which found earlier by Pelavas [4]. For a Kerr-Newman black hole, in Eq. (35) we shall set $\beta- > 0$, and $Q_{mkn} - > e$. This result $\frac{A_{dls}}{M^2} = 4\pi(4 + \frac{a^2}{M^2} - \frac{2e^2}{M^2} (1 + \frac{11\pi a^2}{3M^2}))$ is new in the literature.. From Eq. (35) it is easy to check that for slow rotation the Kerr-Newman’s ergosurface area will always be less than the magnetized Kerr-Newman’s ergosurface area. This fact it seems to suggest that in a magnetized Kerr-Newman black hole, Penrose processes are more likely to occur as compared to the standard Kerr-Newman black hole.

5. Circumferences

Next we shall calculate the equatorial and polar circumferences of the magnetized Kerr-Newman black hole’s event horizon and ergosurface. The equatorial circumference ($C_e$) can be obtained by solving the following integral

$$C_e = \int_0^{2\pi} G(\frac{\pi}{2}) d\varphi. \quad (36)$$
Figure 1. A comparison between the equatorial circumferences of the magnetized Kerr-Newman (gold) and Kerr-Newman (grey) black holes.

If we keep only linear terms in $B$, we obtain from Eq. (36) the magnetized Kerr-Newman’s black hole event horizon equatorial circumference ($C_{eh}$). Thus

$$C_{eh} = \frac{2\pi (r_+^2 + a^2)}{r_+ + aBQ_{mkn}}. \quad (37)$$

Fig. 1 shows a superposition of two plots obtained from Eq. (37). The first plot (in orange color) corresponds to $\beta = 0.0425$, while the gray one corresponds to $\beta = 0$. In this case, provided $aQ_{mkn} > 0$ ($aQ_{mkn} < 0$), the effect of the external magnetic field is to increase (decrease) the magnetized Kerr-Newman black hole circumference as compared to the Kerr-Newman’s equatorial circumference. In a similar way, the equatorial circumference for the magnetized Kerr-Newman’s ergosurface can be written as

$$C_{esls} = \frac{2\pi \frac{\epsilon}{M}(2(1 + \frac{a^2}{M^2}) + 2\sqrt{1 - \frac{\epsilon^2}{M^2} - \frac{\epsilon^2}{M^2}})}{\frac{\epsilon}{M} + \frac{\alpha}{\pi} \beta(1 - \sqrt{1 - \frac{\epsilon^2}{M^2}})} \cdot (38)$$

Fig. 2 shows a superposition of two plots obtained from Eq. (38). The first plot (in green color) corresponds to $\beta = 0.0425$, while the gray one corresponds to $\beta = 0$. In this case, provided $aQ_{mkn} > 0$ ($aQ_{mkn} < 0$), the effect of the external magnetic field is to increase (decrease) the magnetized Kerr-Newman black hole circumference as compared to the Kerr-Newman’s equatorial circumference. A comparison between the event horizon and ergosurface equatorial circumferences is showed in Fig. 3. Similarly, the black hole’s polar circumference ($C_p$) can be obtained from Eq. (20). Thus

$$C_p = \int_0^{2\pi} F(\theta) d\theta. \quad (39)$$

For the magnetized Kerr-Newman black hole event horizon the polar circumference ($C_{ph}$) given by Eq. (39) can be rewritten as

$$C_{ph} = \int_0^{2\pi} ((r_+^2 + a^2)^{1/2}(1 - k_1 \sin^2 \theta)^{1/2} + \frac{k_2 \sin^2 \theta}{(1 - k_1)^{1/2}}) d\theta, \quad (40)$$
Figure 2. The equatorial circumferences of the ergosurfaces of the magnetized Kerr-Newman (in green) and Kerr-Newman (in gray) black holes.

Figure 3. The equatorial circumferences (event horizon (gold), ergosurface (green)) in the magnetized Kerr-Newman black hole.

where \( k_1 = \frac{a^2}{(r_+^2 + a^2)} \) and \( k_2 = \frac{r_+ a B Q_{mkn}}{(r_+^2 + a^2)^{1/2}} \). It turns out that Eq. (40) can be solved in terms of complete elliptical integrals of the second kind \( E \) and \( K \). Namely

\[
C_{ph} = 4(r_+^2 + a^2)^{1/2} E(k_1) - 4k_2 \left( \frac{1 - k_1}{k_1} \right)^{1/2} E \left( \frac{k_1}{k_1 - 1} \right) - \frac{K(\frac{k_1}{k_1 - 1})}{k_1(1 - k_1)^{1/2}}. \tag{41}
\]

Notice that in order to use the standard definitions of the elliptical integrals of the second kind, a square root must be taken to all the the arguments of the elliptical integrals given in Eq. (41). In Fig. 4 we show two superimposed plots obtained from Eq. (41). In the first
Figure 4. A comparison between the polar circumferences of the magnetized Kerr-Newman (gold) and Kerr-Newman (gray) black holes.

plot (orange), $\beta = 0.0425$, and in the second one $\beta = 0$. It is immediately obvious that for $aQ_{mkn} > 0$, the effect of the external magnetic field is to increase the value of the magnetized Kerr-Newman’s event horizon polar circumference as compared to the Kerr-Newman’s event horizon polar circumference. The opposite happens when $aQ_{mkn} < 0$.

Regarding the magnetized Kerr-Newman ergosurface polar circumference ($C_{puls}$), Eq. (39) can be written as

$$C_{puls}M = \int_{0}^{2\pi} \sqrt{\frac{(2r_{sls} - e^2)(1 - e^2)}{r_{sls} - 1} \left(1 + \frac{aB r_{sls} \sin^2 \theta}{2r_{sls} - e^2}\right)} d\theta.$$  \hspace{1cm} (42)

Keeping only terms up to $B$, $a^2$ and $e^2$ in Eq. (42), reduces to

$$C_{puls}M \approx \pi (4 + \frac{3}{4} \frac{a^2}{M^2} + 5 \frac{a}{M} \beta \frac{Q_{mkn}}{M} - \frac{Q^2_{mkn}}{M^2}).$$  \hspace{1cm} (43)

Thus in particular, increasing the value of $a$ and/or $\beta$ in Eq. (43) will mean an increase of the polar circumference value. As expected, when $\beta \to 0$, Eq. (43) goes to Eq. (14) in Ref. [8].

To estimate the deviation from sphericity in any of the two surfaces under consideration we shall calculate $\delta_{mkn}$ defined as

$$\delta_{mkn} = \frac{C_e - C_p}{C_e}. \hspace{1cm} (44)$$

Fig. 5 shows for the magnetized Kerr-Newman black hole’s event horizon the superposition of the plot given by Eq. (44), and a plane $z = 0$. The intersection of both plots allow to easily visualize two regions. The first region ($z > 0$, $\delta_{mkn} > 0$) defines oblate spheroids and the second region ($z < 0$, $\delta_{mkn} < 0$) prolate spheroids.

Fig. 6 is similar to Fig. 5 but for the Kerr-Newman black hole. Contrary to Fig. 5 both plots do not intersect and thus there is only one region ($z > 0$, $\delta_{kn} > 0$). This means that the Kerr-Newman horizon will have always a shape as an oblate spheroid. Coming back to the magnetized Kerr-Newman ergosurface, Eq. (39) can not be solved in closed form.

Table 1 shows for the magnetized Kerr-Newman ergosurface the values of Eq. (44) for different values of $\frac{a}{M}$ and $\frac{Q_{mkn}}{M}$. The last two columns are useful to compare Eq. (44) for the magnetized
Figure 5. The plot’s intersection correspond to prolate spheroids. The remaining regions (in gold) correspond to oblate spheroids.

Figure 6. The deviation of sphericity of the Kerr-Newman black hole event horizon. There are no negative values for $\delta_{kn}$.
Table 1. The magnetized Kerr-Newman ergosurface polar circumferences for different values of $a$ and $Q_{mkn}$. The last column shows the value of $\delta_{kn}$ for the Kerr-Newman ergosurface polar circumferences.

| $a$ | $Q_{mkn}$ | $C_p$ M | $C_n$ M | $\delta_{mkn}$ | $\delta_{kn}$ |
|-----|-----------|---------|---------|----------------|--------------|
| 0.1 | 0.1       | 12.565  | 12.569  | P              | P            |
| 0.1 | 0.2       | 12.477  | 12.476  | N              | P            |
| 0.1 | 0.3       | 12.323  | 13.318  | N              | P            |
| 0.1 | 0.4       | 12.097  | 12.087  | N              | P            |
| 0.1 | 0.5       | 11.791  | 11.776  | N              | P            |
| 0.1 | 0.6       | 11.390  | 11.368  | N              | P            |
| 0.1 | 0.7       | 10.871  | 10.840  | N              | N            |
| 0.1 | 0.8       | 10.187  | 10.141  | N              | N            |
| 0.1 | 0.9       | 9.231   | 9.150   | N              | N            |
| 0.5 | 0.1       | 13.249  | 13.311  | P              | P            |
| 0.5 | 0.2       | 13.202  | 13.233  | P              | P            |
| 0.5 | 0.3       | 13.104  | 13.093  | N              | P            |
| 0.5 | 0.4       | 12.957  | 12.887  | N              | N            |
| 0.5 | 0.5       | 12.763  | 12.610  | N              | N            |
| 0.5 | 0.6       | 12.532  | 12.249  | N              | N            |
| 0.5 | 0.7       | 12.302  | 11.786  | N              | N            |
| 0.5 | 0.8       | 12.254  | 11.187  | N              | N            |
| 0.5 | 0.9       | 15.492  | 10.377  | N              | N            |
| 0.9 | 0.1       | 16.743  | 14.887  | N              | N            |
| 0.9 | 0.2       | 16.869  | 14.820  | N              | N            |
| 0.9 | 0.3       | 17.220  | 14.698  | N              | N            |
| 0.9 | 0.4       | 18.127  | 14.519  | N              | N            |
| 0.9 | 0.5       | 23.027  | 14.279  | N              | N            |

Kerr-Newman and Kerr-Newman ergosurfaces. Notice in Table 1, that “P” means $\delta_{mkn} > 0$ or $\delta_{kn} > 0$, and “N” negative values for the same dimensionless constants.

Gaussian Curvature
The Gaussian curvature ($K$) is a black hole intrinsic invariant and can be evaluated from the following expression

$$K = - \frac{1}{2EG} \frac{d}{d\theta} \left( \frac{1}{EG} \frac{d}{d\theta} G^2 \right).$$

(45)

Thus, the Gaussian curvature ($K_h$) of the magnetized Kerr-Newman event horizon can be derived from Eq. (21) and Eq. (45). It can be written as

$$K_h = - \frac{1}{2\sin \theta} \frac{d}{d\theta} \left( \frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin^2 \theta \left| \Lambda_+ \right|^2\Sigma_+ \right) \right).$$

(46)

Eq. (46) gives a very large analytical expression which will not be shown here for obvious reasons. However, for a particular case ($e = a = 0$) which correspond to the Gaussian curvature of the magnetized Schwarzschild event horizon ($K_{ms}$), Eq. (46), can be simplified to

$$K_{ms} = \frac{1 + 8\beta^2 \cos^2 \theta + \beta^4 (3 \sin^2 \theta - 4) \sin^2 \theta}{4M^2 (1 + \beta^2 \sin^2 \theta)^4}. $$

(47)
Figure 7. The Gaussian curvature of the event horizon of a magnetized Schwarzschild black hole.

The above value of $K_{ms}$ was obtained first by Wild and Kerns [10], and later using a different approach by S.K. Bose and E.P. Esteban [11] (their definition of the Gaussian curvature was twice the value as given in Eq. 45). A Plot of Eq. (47) intersected by a plane $z = 0$ is given in Fig. 7. In such a figure a “parabolic” curve defines the boundary of a region with negative Gaussian curvature values. This happens when the dimensionless parameter satisfies $\beta \geq \frac{1}{\sin \theta}$.

The equality sign in the previous equation gives the zero Gaussian curvature. By keeping only linear terms to $B$, Eq. (46) reduces to the following

$$K_{mkn} = \frac{(a^2 + r_+^2)(r_+^4 - 2a^2r_+^2 \cos^2 \theta - 3a^4 \cos^4 \theta + 8aBQ_{mkn}r_+(a^2 + r_+^2)\cos^2 \theta)}{\Sigma_+(\Sigma_+ + 4aQ_{mkn}r_+ + B \sin^2 \theta).} \quad (48)$$

The expression for the Gaussian curvature obtained by Smarr [5] for the Kerr-Newman event horizon can be obtained by setting $B = 0$, in Eq. (48). Eq. (48) also shows that the Gaussian curvature of the magnetized Kerr metric event horizon has the same value as in the vacuum Kerr metric if only linear terms in $B$ are kept. At the Equatorial plane ($\theta = \frac{\pi}{2}$), Eq. (48) reduces to

$$K_{mkn}(\frac{\pi}{2}) = \frac{a^2 + r_+^2}{r_+^4 (4aBQ_{mkn} + r_+).} \quad (49)$$

and at the poles ($\theta = 0$), Eq. (48) can be re-written as

$$K_{mkn}(0) = \frac{r_+^2 + 8aBQ_{mkn}r_+ - 3a^2}{(r_+^2 + a^2)^2}. \quad (50)$$

In a similar calculation, we can use again Eq. (45) to obtain the Gaussian curvature for the magnetized Kerr-Newman ergosurface. It happens that the analytical form of the magnetized Kerr-Newman is very long and cumbersome. For instance, its Gaussian curvature at the Equatorial plane and at the poles can be written as

$$K_{sls}(\frac{\pi}{2}) = \frac{f_1 + x f_2 + aQ_{mkn} \beta(f_3 + 4f_4 x)}{4f_5 + f_6 x + 4aQ_{mkn} \beta(f_7 - f_8 x)}, \quad (51)$$
where

\[
\begin{align*}
  f_1 &= -32 - 40a^2 + 64Q_{mkn}^4 + 56a^2Q_{mkn}^2 - 38Q_{mkn}^4 - 17a^2Q_{mkn}^2 + 6Q_{mkn}^6, \\
  f_2 &= -32 - 40a^2 + 48Q_{mkn}^4 + 36a^2Q_{mkn}^2 - 18Q_{mkn}^4 - 4a^2Q_{mkn}^2 + Q_{mkn}^6, \\
  f_3 &= -256 - 248a^2 + 304Q_{mkn}^4 + 160a^2Q_{mkn}^2 - 72Q_{mkn}^4 - 2a^2Q_{mkn}^2 - 3a^2Q_{mkn}^4, \\
  f_4 &= -48 - 42a^2 + 36Q_{mkn}^2 + 11a^2Q_{mkn}^2 - 3Q_{mkn}^4, \\
  f_5 &= -32 - 16a^2 + 80Q_{mkn}^2 + 32a^2Q_{mkn}^2 - 68Q_{mkn}^4 - 19a^2Q_{mkn}^2 + 22Q_{mkn}^6, \\
  &\quad + 3a^2Q_{mkn}^6 - 2Q_{mkn}^6, \\
  f_6 &= -128 - 64a^2 + 260Q_{mkn}^2 + 96a^2Q_{mkn}^2 - 182Q_{mkn}^4 - 36a^2Q_{mkn}^4 + 84Q_{mkn}^6 \\
  &\quad + 2a^2Q_{mkn}^6 - 70Q_{mkn}^8 + 56Q_{mkn}^{10} - 28Q_{mkn}^{12} + 8Q_{mkn}^{14}, \\
  f_7 &= -384 - 160a^2 + 688Q_{mkn}^2 + 208a^2Q_{mkn}^2 - 368Q_{mkn}^4 - 62a^2Q_{mkn}^4 \\
  &\quad + 57Q_{mkn}^6 + 2a^2Q_{mkn}^6 - Q_{mkn}^8, \\
  f_8 &= 324 - 476Q_{mkn}^2 + 16a^2(8 - 7Q_{mkn}^2 + Q_{mkn}^4) + \\
  &\quad Q_{mkn}^4(308 + Q_{mkn}^2(-287 - 8Q_{mkn}^2(-35 + 21Q_{mkn}^2 - 7Q_{mkn}^4 + Q_{mkn}^6)))),
\end{align*}
\]

with \( x = \sqrt{1 - Q_{mkn}^2 + 4a^2Q_{mkn}^4}. \)

At the poles, the Gaussian curvature of the magnetized Kerr-Newman black hole ergosurface (up to first order in \( B \)) can be written as

\[
K_{sls}(0) = \frac{g_1 + g_2 y - 4a\beta Q_{mkn}(g_3 + g_4 y)}{y(Q_{mkn}^2 - 1 - 4Q_{mkn}aB)(a^2 + (1 + y)2)^3},
\]

where

\[
\begin{align*}
  g_1 &= 4(2(Q_{mkn}^2 - 1) + a^2(a^2 - 2)(3a^2 + 3Q_{mkn}^2 - 5)), \\
  g_2 &= 3a^4(Q_{mkn}^2 - 8) + 8(Q_{mkn}^2 - 1) + a^2(36 - 23Q_{mkn}^2 + 3Q_{mkn}^4), \\
  g_3 &= 4(4 + 5a^4 - 2Q_{mkn}^2 + 2a^2(Q_{mkn}^2 - 5)), \\
  g_4 &= 7a^4 - 8(Q_{mkn}^2 - 2) + 5a^2(2Q_{mkn}^2 - 7),
\end{align*}
\]

with \( y = \sqrt{1 - a^2 - Q_{mkn}^2 + 4a^2Q_{mkn}^4}. \)

Although Eqs. (51) – (60) are particular cases of Eq. (45) they look awkward. The only reason that we wrote Eqs. (51) – (60) is that the reader could verify that indeed these equations reduce when \( \beta = 0 \) and \( Q_{mkn} = 0 \) (Kerr black hole) to Eqs. 5 and 7 of Ref. [6]. Of course, by setting \( \beta = 0 \) in Eqs. (51) – (60) we shall obtain the Gaussian curvatures at the Equator and poles of the Kerr-Newman black hole ergosurface. Thus, because the analytical expressions of the Gaussian curvatures of the magnetized Kerr-Newman event horizon and ergosurface are mathematically very long and cumbersome, we have decided to perform a graphical analysis of Eq. (45) for the magnetized Kerr-Newman event horizon and ergosurface to visualize in which regions of both surfaces the Gaussian curvature is positive or negative. For this purpose, we shall present four plots (Fig. 8 – 11) in which we had superimposed graphs of the Gaussian curvature of the event horizon, ergosurface, and a horizontal plane located at \( z = 0 \). The rational of this
undertaking is because we know that regions with positive Gaussian curvature values can be always embedded in $E^3$. However, regions with negative Gaussian values can only be embedded in $E^3$ if they satisfy Pelavas condition ($g_{00} > (\frac{\partial g_{00}}{\partial \theta})^2$).

Fig. 8 shows that for a small charge ($Q_{mkn}/M = 0.1$) the Gaussian curvature of the magnetized Kerr-Newman black hole event horizon (brown) will be always greater than the Gaussian curvature of the ergosurface. Notice that both surfaces have four regions around the poles with negative values while in the Equator the Gaussian curvatures will be always negative. If we increase the charge, Fig. 9 shows that the regions with negative values in the ergosurface shall increase of size as can be seen by comparing Figs. 8 – 9. In this case the event horizon always will always have Gaussian curvature positive values and forbidden regions. Now, we examine the case with small angular momentum ($a/M = 0.2$). In this case the Gaussian curvature at the event horizon will be always positive. The ergosurface will present very small regions in which the Gaussian curvature values are negative.

Finally, for $a/M = 0.9$ (see Fig. 10), the Gaussian curvature at the event horizon (gold) shows negative values and is always greater than the Gaussian curvature at the ergosurface (green). The ergosurface also presents regions with negative values for all possible values of $Q_{mkn}$ except around the equatorial region.

Regarding to the embedding of both surfaces in $E^3$ we have noticed that the Pelavas condition [4] fails for the event horizon and holds for the ergosurface.

6. Conclusions
In this contribution we have investigated to first order in $B$ the geometric properties of the magnetized Kerr-Newman event horizon and ergosurface. The surface area of the event horizon was obtained and from there we derived the first law of thermodynamics for magnetized black holes. As a consequence, the surface gravity, angular velocity, electric potential and magnetic moment at the magnetized Kerr-Newman black hole’ event horizon were easily obtained. Further analysis and physical consequences of the thermodynamics of magnetized black holes are under study and will be published elsewhere.

The surface area of the magnetized Kerr-Newman ergosurface was not possible to obtained in
Figure 9. For $Q_{mkn}/M = 0.9$, a comparison of the Gaussian curvatures of the magnetized Kerr-Newman event horizon (brown) and the ergosurface (green).

Figure 10. For $a/M = 0.9$, the Gaussian curvatures of the magnetized Kerr-Newman black hole event horizon (brown) and ergosurface (green).

a closed form but to second order in $a$, $Q_{mkn}$, and first order in $B$. This result in the appropriate limit ($Q_{mkn} \to 0$, $B \to 0$) agrees with Eq. (15) in Ref. [4]. The remaining results for the Kerr-Newman ($B = 0$), magnetized Kerr-Newman, and magnetized Kerr ($Q_{mkn} = 0$) ergosurface areas are new in the literature.

Next, we studied the deviation from sphericity due to the external magnetic field at the magnetized Kerr-Newman event horizon and ergosurface. We calculated in a closed form the equatorial circumferences of both surfaces. The polar circumference was calculated in terms of elliptical integrals at the event horizon, and in an approximate way at the ergosurface. It turn out that the effect of the magnetic field is to induce a spheroidal prolate shape of the event
horizon for low values of the angular momentum ($|\frac{J}{M^2}| < 0.1$). Greater values of the angular momentum will imply that the shape of the event horizon will take a spheroidal oblate. These results must be compared with those found by Smarr for the standard Kerr-Newman black hole in which the event horizon always has a spheroidal oblate shape. Regarding the ergosurface shape the gravitomagnetic effect is also to induce a prolate spheroidal shape to the magnetized Kerr-Newman ergosurface.

Finally, we derived in a closed form the Gaussian curvatures of the magnetized Kerr-Newman metric event horizon and ergosurface. However, these results were not show explicitly in the paper because were too long and cumbersome. Nonetheless we have check that our analytical expressions for the magnetized Kerr-Newman Gaussian curvature at the event horizon and ergosurface reduces when $B = 0$ to those found by Smarr [5] and Kokkotas [6]. Only particular cases of the Gaussian curvature at the magnetized Kerr-Newman Equatorial and polar regions were written down. For small values of the black hole’s charge regions with Gaussian curvature negative values develop at the poles of the event horizon and ergosurface. Increasing the charge value will imply that Gaussian curvature values at the black hole’s event horizon has only positive values. However, at the ergosurface the size of the regions with negative Gaussian curvature values will increase. For low angular momentum there will be only positive values of the Gaussian curvature at the event horizon and very small regions with Gaussian curvature negative values around the ergosurface’s poles. Increasing the angular momentum will increase the Gaussian curvature negative regions around the event horizon and ergosurface poles. We also have notice that Gaussian curvature positive values exist around the Equator of the event horizon and ergosurface.

The embedding problem was examined using the Pelavas condition [4]. We found that polar regions with Gaussian curvature negative values exist in the event horizon and do not satisfy the Pelavas condition [4], and therefore can not be embedded in $E^3$. However, regions with Gaussian curvature negative values around the ergosurface poles satisfy the Pelavas condition [4] and therefore can be embedded in $E^3$. An analogous result was found by Smarr [5] and Kokkotas [6] at the Kerr-Newman’s event horizon and ergosurface, respectively.

Acknowledgments
The support given to this research by NASA-PR Consortium, and the University of Puerto Rico-Humacao is greatly appreciated. I also thank undergraduate students K. Rivera, S. Garcia, D. Delgado, and V. de la Rosa for checking via computer algebra most of the analytical expressions presented in this paper.

References
[1] Smarr L., Phys. Rev. Letters 30, 71-73, (1973).
[2] Beckenstein J., Ph.D thesis, Princeton University, (1972).
[3] Bardeen J.M., Carter B., and Hawking S.W., Commun. Math. Phys., 31, 161-170, (1973).
[4] Pelavas N., Neary N., Lake K., Class. Quantum Grav. 18, 1319 (2001).
[5] Smarr L., Phys. Rev. D 7, (1973).
[6] Kokkotas K. D., GRG, 20, 8, 829 (1988).
[7] Ernst F.J. and Wild, W.J., J. Math. Phys., 17, 182 (1976).
[8] Wild W. J., Kerns R. M., and Drish W. F., Phys. Rev. D 23, 4, 829 (1981).
[9] Krori K. D. and Barua M., Phys. Rev. D 35, 1171 (1987).
[10] Wild W. and Kerns R., Phys. Rev. D 21, 332 (1980).
[11] Bose S. K. and Esteban E. P., J. Math. Phys., 22, 3006, (1981).
[12] Aliev A.N. and Gal’tsov, D. V., Sov. Phys. Usp., 32, 75 (1989).
[13] Aliev A. N. and Ga’ltsov, D.V., Russian Physics Journal Vol 32, 10, 790-795 (1989).
[14] Dockuchaev V.I., Sov. Phys. JETP 65,1079 (1987).
[15] Karas W. and Vokrouhlicky, D., J. Math. Phys., 32, 714, (1991).