Avoiding long Berge cycles II, exact bounds for all $n$

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Abstract

Let $E_G(n,k)$ denote the maximum number of edges in an $n$-vertex $r$-uniform hypergraph with no Berge cycles of length $k$ or longer. In the first part of this work [5], we have found exact values of $E_G(n,k)$ and described the structure of extremal hypergraphs for the case when $k - 2$ divides $n - 1$ and $k \geq r + 3$.

In this paper we determine $E_G(n,k)$ and describe the extremal hypergraphs for all $n$ when $k \geq r + 4$.

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1 Definitions, Berge $F$ subhypergraphs

An $r$-uniform hypergraph, or simply $r$-graph, is a family of $r$-element subsets of a finite set. We associate an $r$-graph $H$ with its edge set and call its vertex set $V(H)$. Usually we take $V(H) = [n]$, where $[n]$ is the set of first $n$ integers, $[n] = \{1, 2, 3, \ldots, n\}$. We also use the notation $H \subseteq \binom{[n]}{r}$.

Definition 1.1. For a graph $F$ with vertex set $\{v_1, \ldots, v_p\}$ and edge set $\{e_1, \ldots, e_q\}$, a hypergraph $H$ contains a Berge $F$ if there exist distinct vertices $\{w_1, \ldots, w_p\} \subseteq V(H)$ and edges $\{f_1, \ldots, f_q\} \subseteq E(H)$, such that if $e_i = v_\alpha v_\beta$, then $\{w_\alpha, w_\beta\} \subseteq f_i$.

Of particular interest to us are Berge cycles and Berge paths.

Definition 1.2. A Berge cycle of length $\ell$ in a hypergraph is a set of $\ell$ distinct vertices $\{v_1, \ldots, v_\ell\}$ and $\ell$ distinct edges $\{e_1, \ldots, e_\ell\}$ such that $\{v_i, v_{i+1}\} \subseteq e_i$ with indices taken modulo $\ell$.

A Berge path of length $\ell$ in a hypergraph is a set of $\ell + 1$ distinct vertices $\{v_1, \ldots, v_{\ell+1}\}$ and $\ell$ distinct edges $\{e_1, \ldots, e_\ell\}$ such that $\{v_i, v_{i+1}\} \subseteq e_i$ for all $1 \leq i \leq \ell$.

Let $H$ be a hypergraph and $p$ be an integer. The $p$-shadow, $\partial_p H$, is the collection of the $p$-sets that lie in some edge of $H$. In particular, we will often consider the 2-shadow $\partial_2 H$ of a $r$-uniform hypergraph $H$. Each edge of $H$ yields in $\partial_2 H$ a clique on $r$ vertices.

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## 2 Graphs without long cycles

**Theorem 2.1** (Erdős and Gallai [1]). Let \( k \geq 3 \) and let \( G \) be an \( n \)-vertex graph with no cycle of length \( k \) or longer. Then \( e(G) \leq (k-1)(n-1)/2 \).

This bound is the best possible if \( n-1 \) is divisible by \( k-2 \). A matching lower bound can be obtained by gluing together complete graphs of sizes \( k-1 \).

Let \( EG(n,k) \) denote the maximum size of a graph on \( n \) vertices such that it does not contain any cycle of length \( k \) or longer. Write \( n \) in the form of \( (k-2)\lfloor \frac{n-1}{k-2} \rfloor + m \) where \( 1 \leq m \leq k-2 \). Considering an \( n \)-vertex graph whose 2-connected blocks are complete graphs of size \( k-1 \) except one which is a \( K_m \) we get

\[
EG(n,k) \geq f(n,k) := \left\lfloor \frac{n-1}{k-2} \right\rfloor \binom{k-1}{2} + \binom{m}{2}. \tag{1}
\]

It took some 15 years to prove that equality holds in (1) for all \( n \) and \( k \geq 3 \) (Kopylov [7] and independently Woodall [9]). One of the difficulties is, as Faudree and Schelp [3, 4] observed, that for odd \( k \) there are infinitely many extremal graphs very different from the ones above.

**Construction 2.2.** Fix \( k \geq 4, n \geq k, \frac{k}{2} > a \geq 1 \). Define the \( n \)-vertex graph \( H_{n,k,a} \) as follows. The vertex set of \( H_{n,k,a} \) is partitioned into three sets \( A, B, C \) such that \( |A| = a, |B| = n - k + a \) and \( |C| = k - 2a \) and the edge set of \( H_{n,k,a} \) consists of all edges between \( A \) and \( B \) together with all edges in \( A \cup C \). \( B \) is taken to be an independent set.

![Figure 1: H_{14,11,3}](image)

When \( a \geq 2 \), \( H_{n,k,a} \) is 2-connected, has no cycle of length \( k \) or longer, and

\[
e(H_{n,k,a}) = \binom{k-a}{2} + a(n-k+a).
\]

Kopylov and Woodall ([7] and [9]) characterized the structure of the extremal graphs. Namely, either

- the blocks of \( G \) are \( p \) complete graphs \( K_{k-1} \) and a \( K_m \), where \( p := \left\lfloor \frac{n-1}{k-2} \right\rfloor \), or
- \( k \) is odd, \( m = (k+1)/2 \) or \( (k-1)/2 \) and \( q \) of the blocks of \( G \) are \( K_{k-1} \)'s and one block is a copy of an \( H_{n-q(k-2),k,(k-1)/2} \).
3 Main result: Hypergraphs with no long Berge cycles

Let \( \text{EG}_r(n, k) \) denote the maximum size of an \( r \)-uniform hypergraph on \( n \) vertices that does not contain any Berge cycle of length \( k \) or longer. In \cite{5}, we proved an analogue of the Erdős–Gallai theorem on cycles for \( r \)-graphs.

**Theorem 3.1** \((\cite{5})\). Let \( r \geq 3 \) and \( k \geq r + 3 \), and suppose \( H \) is an \( n \)-vertex \( r \)-graph with no Berge cycle of length \( k \) or longer. Then \( e(H) \leq \frac{n-1}{r-2} \binom{k-1}{r-1} \). Moreover, equality is achieved if and only if \( \partial_2 H \) is connected and for every block \( D \) of \( \partial_2 H \), \( D = K_{k-1} \) and \( H[D] = K_{k-1}^{(r)} \).

Since a Berge cycle can only be contained in a single block of the 2-shadow \( \partial_2 H \), the construction in Theorem 3.1 cannot contain Berge cycles of length \( k \) or longer. Thus Theorem 3.1 determines \( \text{EG}_r(n, k) \) and describes extremal \( r \)-graphs when \( k-2 \) divides \( n-1 \) and \( k \geq r + 3 \). Ergemlidze, Győri, Methuku, Salia, Tompkins, and Zamora \cite{2} proved similar results for \( k \in \{r + 1, r + 2\} \). The case of short cycles, \( k \leq r \), is different, see \cite{8}.

Our goal in this paper is to determine \( \text{EG}_r(n, k) \) for all \( n \) when \( r \geq 3 \) and \( k \geq r + 4 \). We also describe the extremal hypergraphs. We conjecture that our results below holds for \( k = r + 3 \) too.

The tools used here do not seem to be sufficient to verify the conjecture (see the remark at the end of Section 6). The case \( n \leq k - 1 \) is trivial, \( \text{EG}_r(n, k) = \binom{n}{r} \). Let \( n = (k - 2) \left\lfloor \frac{n-1}{k-2} \right\rfloor + m \) where \( 1 \leq m \leq k - 2 \). Define

\[
 f_r(n, k) := \left\lfloor \frac{n-1}{k-2} \right\rfloor \binom{k-1}{r} + \begin{cases} 
 \frac{m-1}{r} & \text{for } 1 \leq m \leq r, \\
 \binom{m}{r} & \text{for } r + 1 \leq m \leq k-2.
\end{cases}
\]  

**Theorem 3.2.** Let \( r \geq 3 \) and \( k \geq r + 4 \), and suppose \( H \) is an \( n \)-vertex \( r \)-graph with no Berge cycle of length \( k \) or longer. Then \( e(H) \leq f_r(n, k) \). Moreover, equality is achieved if and only if \( H \) has the structure described in Constructions 4.1 and 4.2 in the next section.

The structure of the paper is as follows.

In the next section (Section 4) we prove the lower bound \( \text{EG}_r(n, k) \geq f_r(n, k) \). In Section 5 we recall some tools we developed in \cite{5}: the notion of representative pairs and Kopylov’s Theorem in a useful form. In Section 6 we introduce one more tool, the notion of \( (2, r) \) mixed hypergraphs and propose a more general problem. In Section 7 we prepare the proof by proving a handy upper bound in the case of a 2-connected \( \partial_2 H \), and finally in Section 8 we prove our main result, Theorem 3.2.
4 Constructions

In this section we define two classes of $r$-graphs avoiding Berge cycles of length $k$ or longer (for $k \geq r + 2$). Write $n$ in the form of $(k - 2)\left\lfloor \frac{n - 1}{k - 2} \right\rfloor + m$ where $1 \leq m \leq k - 2$. Let $p := \left\lfloor \frac{n - 1}{k - 2} \right\rfloor$. Let $V = [n]$ be an $n$-element set (the set of vertices).

Construction 4.1. In case of $m \geq r + 1$, let $V_1, \ldots, V_{p+1}$ be a sequence of subsets of $[n]$ satisfying

$$|(V_1 \cup \cdots \cup V_{i-1}) \cap V_i| = 1, \quad (3)$$

for all $1 < i \leq p + 1$ such that one $V_i$ has $m$ elements and each other has $(k - 1)$-elements. Then replace each $V_i$ with a copy of $K_{r\{V_i\}}$, the complete $r$-uniform hypergraph on it.

Each Berge cycle in the $r$-uniform families in Construction 4.1 must be contained in one of the $V_i$’s so its length is at most $k - 1$. Hence

$$\text{EG}_r(n, k) \geq p \binom{k - 1}{r} + \binom{m}{r}$$

for all $n$, $k$, and $r$. We will see in Section 8 that in case of $m \geq r + 1$ (and $k \geq r + 4 \geq 7$) these are the only extremal hypergraphs.

Construction 4.2. In case of $m \leq r$, let $\mathcal{V} := \{V_1, \ldots, V_p\}$ be a sequence of $(k - 1)$-element subsets of $[n]$ such that

$$|(V_1 \cup \cdots \cup V_{i-1}) \cap V_i| \leq 1 \quad (4)$$

for every $i \geq 2$. Let $H$ be the graph whose edge set is the union of the edge sets of complete graphs on $V_i \in \mathcal{V}$, so $|E(H)| = \binom{k - 1}{2}$. Then $H$ has a forest-like structure of cliques (i.e., every block of $H$ is a clique), and in particular every cycle is contained in some $V_i \in \mathcal{V}$.

The graph $H$ necessarily consists of $m$ (nonempty) components, let $C_1, \ldots, C_m$ be the vertex sets of them. Some $C_{\alpha}$’s could be singletons, and $\bigcup_{\alpha=1}^m C_\alpha = V$. Let $H_\alpha := H|C_{\alpha}$. Define $B_i$ as the complete $r$-graph with vertex set $V_i$, and set $\mathcal{H}_\alpha := \mathcal{H}_{\alpha} := \bigcup\{B_i : V_i \subseteq \mathcal{V}, V_i \subseteq C_{\alpha}\}$, $\mathcal{H} := \bigcup_{\alpha=1}^m \mathcal{H}_\alpha$.

If $m > 1$, let $T$ be a tree with vertex set $[m]$, such that a pair $e = \{\alpha(e), \alpha'(e)\}$ is in $E(T)$ only if the components $C_{\alpha}$ and $C_{\alpha'}$ in $H$ satisfy $|V(C_{\alpha})| + |V(C_{\alpha'})| \geq r$. For each such edge $e$, we “blow up” $e$ into an $r$-edge containing vertices of $C_\alpha$ and $C_{\alpha'}$ as follows:

Select the non-empty sets $A(e) \subseteq C_\alpha$ and $A'(e) \subseteq C_{\alpha'}$ so that $|A(e)| + |A'(e)| = r$ and if $|V(C_{\alpha})| > 1$ (resp. $|V(C_{\alpha'})| > 1$), then $A(e) \subseteq V_i \subseteq C_{\alpha}$ for some $V_i \in \mathcal{V}$ (resp. $A'(e) \subseteq V_i' \subseteq C_{\alpha'}$ for some $V_i' \in \mathcal{V}$). Let $\mathcal{D} := \{A(e) \cup A'(e) : e \in E(T)\}$. Our construction is $\mathcal{H} \cup \mathcal{D}$ (see Figure 2).

By definition, $\mathcal{H} \cup \mathcal{D}$ has no long Berge cycle yielding

$$\text{EG}_r(n, k) \geq |\mathcal{H}| + |\mathcal{D}| = p \binom{k - 1}{r} + m - 1$$

for all $n$, $k$, and $r$. Indeed, every edge of $\mathcal{D}$ is a cut-edge of the hypergraph $\mathcal{H} \cup \mathcal{D}$, every Berge cycle of $\mathcal{H} \cup \mathcal{D}$ is contained in a single component $C_{\alpha}$, even more, it is contained a single $V_i$.

We will see in Section 8 that in the case of $m \leq r$ (and $k \geq r + 4 \geq 7$) these are the only extremal hypergraphs.
5 Representative pairs, the structure of Berge $F$-free hypergraphs

In this section we collect some tools and statements developed and used in [5]. We do not repeat their proofs.

**Definition 5.1.** For a hypergraph $H$, a system of distinct representative pairs (SDRP) of $H$ is a set of distinct pairs $A = \{ \{x_1, y_1\}, \ldots, \{x_s, y_s\} \}$ and a set of distinct hyperedges $A = \{f_1, \ldots, f_s\}$ of $H$ such that for all $1 \leq i \leq s$

- $\{x_i, y_i\} \subseteq f_i$, and
- $\{x_i, y_i\}$ is not contained in any $f \in H - \{f_1, \ldots, f_s\}$.

**Lemma 5.2.** Let $H$ be a hypergraph, let $(A, A)$ be an SDRP of $H$ of maximum size. Let $B := H \setminus A$ and let $B = \partial_2 B$ be the 2-shadow of $B$. For a subset $S \subseteq B$, let $B_S$ denote the set of hyperedges that contain at least one edge of $S$. Then for all nonempty $S \subseteq B$, $|S| < |B_S|$.

Note that $|H| = |A| + |B|$.

**Lemma 5.3.** Let $H$ be a hypergraph and let $(A, A)$ be an SDRP of $H$ of maximum size. Let $B := H \setminus A$, $B = \partial_2 B$, and let $G$ be the graph on $V(H)$ with edge set $A \cup B$. If $G$ contains a copy of a graph $F$, then $H$ contains a Berge $F$ on the same base vertex set.

In this paper, we only use the previous lemma in the case that $F$ is a cycle or path. I.e., if the longest Berge cycle (path) in $H$ is of length $\ell$, then the longest cycle (path) in $G$ is also of length at most $\ell$.

**Definition:** For a natural number $\alpha$ and a graph $G$, the $\alpha$-disintegration of a graph $G$ is the process of iteratively removing from $G$ the vertices with degree at most $\alpha$ until the resulting graph has minimum degree at least $\alpha + 1$ or is empty. This resulting subgraph $H(G, \alpha)$ will be called the $(\alpha + 1)$-core of $G$. It is well known (and easy) that $H(G, \alpha)$ is unique and does not depend on the order of vertex deletion.

The following theorem is a consequence of Kopylov’s Theorem [7] on the structure of graphs without long cycles. We state it in the form that we need.

**Theorem 5.4 (7, see also Theorem 5.1 in [5]).** Let $k \geq 5$ and let $t = \lfloor \frac{k-1}{2} \rfloor$. Suppose that $G$ is an $n$-vertex graph with no cycle of length at least $k$. If $G$ is 2-connected and $n \geq k$ then there exists a subset $S \subset V(G)$, $s := |S|$, $k - t \leq s \leq k - 2$ (i.e., $2 \leq k - s \leq t$), such that the vertices of $V \setminus S$ can be removed by a $(k - s)$-disintegration.
Lemma 5.5 ([5], Lemma 5.3). Let \( w, r \geq 2 \) and let \( \mathcal{H} \) be a \( w \)-vertex \( r \)-graph. Let \( \overline{\mathcal{H}} \) denote the family of pairs of \( V(\mathcal{H}) \) not contained in any member of \( \mathcal{H} \) (i.e., the complement of the 2-shadow). Then

\[
|\mathcal{H}| + |\overline{\mathcal{H}}| \leq \begin{cases} 
\binom{w}{2} & \text{for } 2 \leq w \leq r + 2, \\
\binom{w}{r} & \text{for } r + 2 \leq w.
\end{cases}
\]

Moreover, equality holds if and only if

- \( w > r + 2 \) and \( \mathcal{H} \) is complete, or
- \( w = r + 2 \) and either \( \mathcal{H} \) or \( \overline{\mathcal{H}} \) is complete.

We say a graph \( G \) is hamilton-connected if for any \( x, y \in V(G) \), \( G \) contains a path from \( x \) to \( y \) that covers \( V(G) \).

Lemma 5.6 ([6], Theorem 5). Let \( G \) be an \( n \)-vertex graph with minimum degree \( \delta(G) \geq 2 \). If \( e(G) \geq \left(\frac{n-1}{2}\right) + 2 \) then \( G \) is hamilton-connected unless \( G \) is obtained from \( K_{n-1} \) by adding a vertex of degree 2.

6 Maximal mixed hypergraphs

One of our tools is the notion of mixed hypergraphs. For \( r \geq 3 \), a \((2, r)\) mixed hypergraph is a triple \( \mathcal{M} = (A, B, V) \), where \( V \) is a vertex set, \( A \) is the edge set of a graph, \( B \) is an \( r \)-graph (i.e., \( A \subseteq \binom{V}{2} \), \( B \subseteq \binom{V}{r} \)) such that \( A \cup B \) satisfies the Sperner property: there is no \( a \in A \), \( b \in B \) with \( a \subset b \). We often will denote the 2-shadow \( \overline{\mathcal{B}} \) by \( B \).

Let \( m_r(n, k) \) denote the maximum size of a mixed hypergraph \( \mathcal{M} \) on \( n \) vertices such that \( \overline{\mathcal{B}} \) does not contain any cycle of length \( k \) or longer.

Lemma 6.1.

\[
\text{EG}_r(n, k) \leq m_r(n, k).
\]

Proof. Let \( \mathcal{H} \) be an \( r \)-uniform hypergraph on \( n \) vertices with no Berge cycle of length \( k \) or longer \((k \geq r + 3 \geq 6)\) with \( \text{EG}_r(n, k) \) edges. Let \( (A, A) \) be an SDRP of \( \mathcal{H} \) of maximum size. Let \( B := \mathcal{H} \setminus A, B = \overline{\mathcal{B}} \). By definition, \( \mathcal{M} := (A, B, V) \) is a \((2, r)\) mixed hypergraph \((r \geq 3)\) with vertex set \( V \). By Lemma 5.3 the graph \( G \) with edge set \( A \cup B \) does not contain a cycle of length \( k \) or longer. Hence

\[
\text{EG}_r(n, k) = |\mathcal{H}| = |A| + |B| \leq m_r(n, k).
\]

We will show that these two functions are very close to each other and determine \( m_r(n, k) \) for all \( n \) (when \( k \geq r + 4, r \geq 3 \)). We need more definitions and constructions.

A sequence of sets \( \mathcal{S} = (V_1, \ldots, V_p), V_i \subseteq V \), is called a (linear) hypergraph forest with vertex set \( V \) if

\[
|(V_1 \cup \cdots \cup V_{i-1}) \cap V_i| \leq 1
\]

holds for each \( 2 \leq i \leq p \). To avoid trivialities we usually suppose that \( |V_i| \geq 2 \) for each \( i \). If \( \sum_i(|V_i| - 1) = |V| - 1 \) then equality holds in (5) for all \( i \), and we call \( \mathcal{S} \) a hypergraph tree.
Construction 6.2. Write $n$ in the form of $(k-2)\left\lceil \frac{n-1}{k-2} \right\rceil + m$ where $1 \leq m \leq k-2$. Let $p := \left\lfloor \frac{n-1}{k-2} \right\rfloor$. In case of $m = 1$, let $V_1, \ldots, V_p$ be a sequence of $(k-1)$-element subsets of $[n]$ forming a hypergraph tree. In case of $2 \leq m \leq k-2$, let $V_1, \ldots, V_{p+1}$ be a sequence of subsets of $[n]$ satisfying (5) such that one $V_i$ has $m$ elements and each other has $(k-1)$-elements. Finally, put either a copy of $K^{(r)}_{p+1}$ or $K_{|V_i|}$ into each $V_i$.

Each cycle in the 2-shadow of any $(2, r)$ mixed family in Construction 6.2 must be contained in one of the $V_i$’s, so its length is at most $k-1$. Taking the largest possible mixed hypergraph of this type we get

$$m_r(n, k) \geq f^+_r(n, k) := \left\lceil \frac{n-1}{k-2} \right\rceil \binom{k-1}{r} + \begin{cases} \binom{m}{2} & \text{for } 1 \leq m \leq r+1, \\ \binom{m}{r} & \text{for } r+2 \leq m \leq k-2. \end{cases}$$

(6)

Theorem 6.3. Let $r \geq 3$ and $k \geq r+4$, and suppose $\mathcal{M}$ is an $n$-vertex $(2, r)$ mixed hypergraph with no cycle of length $k$ or longer in $\partial_2 \mathcal{M}$. Then $|\mathcal{M}| \leq f^+_r(n, k)$. Moreover, equality is achieved if and only if $\mathcal{M}$ has the structure described in Construction 6.2 above.

Remark. This is one point that does not hold for $k = r + 3$, because in that case every SDRP is simply a graph, $\mathcal{B} = \emptyset$, and according to Kopylov’s Theorem 5.4, there are more extremal graphs than in Construction 6.2.

7 Inequalities

Let $k \geq 5$ and let $t = \left\lfloor \frac{k-1}{2} \right\rfloor$, $r \geq 3$, and $k \geq r + 3$. In this section most of the time we suppose that $k \geq r + 4$, but almost all inequalities hold for the case $k = r + 3$, too.

Let $\mathcal{M} = (A, \mathcal{B}, V)$ be a $(2, r)$ mixed hypergraph such that $G := A \cup \mathcal{B}$ is an $n$-vertex graph with no cycle of length at least $k$.

Suppose that $A \cup \mathcal{B}$ is 2-connected and $n \geq k$. Theorem 5.4 implies that for some $k-t \leq s \leq k-2$ (i.e., $2 \leq k-s \leq t$) there exist an $s$-element set $S \subset V$ such that

$$\text{the vertices of } A \cup \mathcal{B} \setminus S \text{ can be removed by a } (k-s)\text{-disintegration.}$$

(7)

For the edges of $A$ and $\mathcal{B}$ contained in $S$ we use Lemma 5.5 to see that

$$|A[S]| + |\mathcal{B}[S]| \leq \max \left\{ \binom{s}{2}, \binom{s}{r} \right\}.$$
Since \( \ell \leq k - s \) we obtain that

\[
|A| + |B| \leq u_r(n, k, s) := \max \left\{ \binom{s}{2}, \binom{s}{r} \right\} + (n - s) \max \left\{ k - s, \frac{k - s}{r - 1} \right\}. \tag{8}
\]

In the rest of this section we give upper bounds for \( u_r(n, k, s) \). The following inequalities can be obtained by some elementary estimates on binomial coefficients. The main result of this section is the following lemma.

**Lemma 7.1.** If \( r \geq 3, k \geq r + 4, k - t \leq s \leq k - 2 \), and \( n \geq k \), then \( u_r(n, k, s) \leq f_r(n, k) - \left( \frac{r}{2} \right) \).

**Proof.** When \( s \) is a variable taking only nonnegative integer values, and \( r, n \) and \( k \) are fixed, the functions \( \binom{s}{2}, \binom{s}{r} \), \( (n - s)(k - s) \) and \( (n - s)(\frac{k - s}{r - 1}) \) are convex. So their maximums and sums, in particular, \( u_r(n, k, s) \), are convex, too. We obtain that

\[
\max_{k - t \leq s \leq k - 2} u_r(n, k, s) = \max \left\{ u_r(n, k, k - 2), u_r(n, k, k - t) \right\}.
\]

Our first observation is that (for \( r \geq 3, k \geq r + 3 \)) if \( n \geq k - 2 + s \), then

\[
u_r(n, k, s) = u_r(n - k + 2, k, s) + (k - 2) \max \left\{ k - s, \frac{k - s}{r - 1} \right\}. \tag{9}\]

**Claim 7.2.** For \( 2 \leq k - s \leq t \) and \( k \geq r + 4 \),

\[
(k - 2) \max \left\{ k - s, \frac{k - s}{r - 1} \right\} \leq \binom{k - 1}{r} - \binom{r}{2}. \tag{10}\]

**Proof of Claim 7.2.** Because of the convexity of the left-hand side (in variable \( s \)), it is enough to check the cases \( s \in \{k - t, k - 2\} \) (i.e., \( k - s \in \{t, 2\} \), respectively). We have three cases to consider: when \( k - s = 2 \), when \( k - s = t \) and \( r \geq k - s \), and finally when \( k - s = t \) and \( 3 \leq r \leq t - 1 \). Substituting \( k - s = 2 \) and \( k - s = t \) into the left-hand side of (10), we get \((k - 2)2\) and \((k - 2)t\), respectively. Then (for \( k \geq 7 \)) we have

\[
(k - 2)t < \binom{k - 1}{3} - \binom{k - 4}{2} \leq \binom{k - 1}{r} - \binom{r}{2}.
\]

This settles the first two cases.

In the case \( k - s = t \) and \( r < k - s \), we need the following inequality (for \( 3 \leq r < t \)):

\[
(k - 2) \binom{t}{r - 1} < \binom{k - 1}{r} - \binom{r}{2}.
\]

We prove the following stronger inequality (for \( 3 \leq r < t \)), because we will use it again.

\[
\binom{k - t}{r} + (k - 3) \binom{t}{r - 1} < \binom{k - 1}{r} - \binom{r}{2}. \tag{11}\]

Since \( \binom{k - t}{r} \geq \binom{t + 1}{r} \geq \binom{t}{r - 1} \) (for \( t \geq r \)), equation (11) completes the proof of (10).
Claim 7.4. For $A$ and $F_k$

Since we have

Proof. Their sizes are $n$.

We will show

Claim 7.3. For $F_k$

Define four $r$-graphs with vertex set $[k - 1]$. $F_0 := \binom{[k - 1]}{r}$, $F_1 := \binom{[k - t]}{r}$,

$F_2 := \{ e \cup \{ i \} : e \in \binom{[t]}{r-1}, i \in [k - t + 1, k - 1] \}$, and

$F_3 := \{ f \cup \{ j \} : f \in \binom{[k - t, k - 1]}{r-1}, j \in [k - t - 1] \}$.

Their sizes are $\binom{k-1}{r}$, $\binom{k-t}{r-1}$, $(t-1)\binom{r}{r-1}$, and $(k-t-1)\binom{r}{r-1}$ respectively. We claim that $F_1$, $F_2$, and $F_3$ are disjoint. Indeed, $|A \cap [k-t]| \leq r - 1$ holds for every $A \in F_2 \cup F_3$, so $A \notin F_1$. Also, if $A \in F_2$ then $|A \cap [k-t-1]| = r - 1 > 1$ so $A \notin F_3$. Since the families $F_1$, $F_2$, and $F_3$ are disjoint subfamilies of $F_0$, we have $|F_1| + |F_2| + |F_3| \leq |F_0|$. This completes the proof of [12].

Claim 7.3. For $k \leq n \leq 2k - 3$ and $k \geq r + 4$, one has $u_r(n, k, k - 2) \leq f_r(n, k) - \binom{r}{2}$.

Proof. We have

$$u_r(n, k, k - 2) = \max \left\{ \binom{k - 2}{2}, \binom{k - 2}{r} \right\} + (n - k + 2) \max \left\{ 2, \binom{2}{r - 1} \right\}$$

$$= \binom{k - 2}{r} + 2(n - k + 2).$$

We will show

$$\binom{k - 2}{r} + 2(n - k + 2) \leq \binom{k - 1}{r} + (n - k + 1) - \binom{r}{2} \leq f_r(n, k) - \binom{r}{2}.$$
Note that the right-hand side is at most $f_r(n,k) - \binom{r}{2}$.

**Proof.** We have

$$u_r(n, k, k-t) = \max \left\{ \binom{k-t}{2}, \binom{k-t}{r} \right\} + (n-k+t) \max \left\{ t, \binom{t}{r-1} \right\}$$

$$= \binom{k-t}{2} + (n-k+t)t.$$ 

Moreover

$$\left( \binom{k-t}{2} + (n-k+t)t \right) - \left( (n-k+1) - \binom{r}{2} \right) = \binom{k-t}{2} + (t-1)(n-k+t) + (t-2) + \binom{r}{2}$$

$$\leq \binom{k-t}{2} + (t-1)(k-3) + (t-2) + \binom{k-4}{2} < \binom{k-1}{3} \leq \binom{k-1}{r}. \quad \Box$$

**Claim 7.5.** For $k \leq n \leq 2k-t-3$, $r < t$ and $k \geq r + 4$,

$$u_r(n, k, k-t) < \binom{k-1}{r} - \binom{r}{2}.$$

Note that the right-hand side is at most $f_r(n,k) - \binom{r}{2}$.

**Proof.** We have

$$u_r(n, k, k-t) = \max \left\{ \binom{k-t}{2}, \binom{k-t}{r} \right\} + (n-k+t) \max \left\{ t, \binom{t}{r-1} \right\}$$

$$= \binom{k-t}{r} + (n-k+t) \binom{t}{r-1} \leq \binom{k-t}{r} + (k-3) \binom{t}{r-1}.$$ 

Here the right-hand side is less than $\binom{k-1}{r} - \binom{r}{2}$ by [11]. \quad \Box

**Proof of Lemma 7.1** Because of the convexity of $u_r(n, k, s)$ (in the variable $s$), it is enough to check the cases $k-s \in \{2, t\}$. Note that for $n_1, n_2 \geq 2$

$$f_r(n_1, k) + f_r(n_2, k) \leq f_r(n_1 + n_2 - 1, k) \quad (13)$$

$$f_r^+(n_1, k) + f_r^+(n_2, k) \leq f_r^+(n_1 + n_2 - 1, k) \quad (14)$$

and here equalities hold for $n_2 = k - 1$. (If we define $f_r(1, k) = f_r^+(1, k) = 0$, then we can use (13), (14) for these values, too).

If $s = 2$ and $k \leq n \leq 2k-3$, then Claim 7.3 yields $u_r(n, k, k-2) \leq f_r(n, k) - \binom{r}{2}$. For $n \geq 2k-2$ we use (9), then the induction hypothesis $u_r(n-k+2, k, k-2) \leq f_r(n-k+2, k)$, and then Claim 7.2
(equation (10)) implies that

\[ u_r(n, k, k - 2) = u_r(n - k + 2, k, k - 2) + (k - 2)2 < f_r(n - k + 2, k) + \binom{k - 1}{r} - \binom{r}{2} \]

\[ = f_r(n - k + 2, k) + f_r(k - 1, k) - \binom{r}{2} = f_r(n, k) - \binom{r}{2}, \]

and we are done.

When \( k - s = t \) the proof is similar. For \( k \leq n \leq 2k - t - 3 \), Claim 7.4 and Claim 7.5 yield \( u_r(n, k, k - t) < f_r(n, k) - \binom{r}{2} \). For \( n \geq 2k - t - 2 \), we use (9), then the induction hypothesis \( u_r(n - k + t, k, k - t) \leq f_r(n - k + t, k) \), and then Claim 7.2 (equation (10)) implies that

\[ u_r(n, k, k - t) = u_r(n - k + 2, k, k - 2) + (k - 2) \max \left\{ t, \left\lfloor \frac{t}{r - 1} \right\rfloor \right\} \]

\[ < f_r(n - k + 2, k) + \binom{k - 1}{r} - \binom{r}{2} \]

\[ = f_r(n - k + 2, k) + f_r(k - 1, k) - \binom{r}{2} = f_r(n, k) - \binom{r}{2}. \]

\[ \square \]

8 Proofs of the main results

In this section we first prove Theorem 6.3 and then Theorem 3.2 for all \( n \geq k \) (and \( r \geq 3, k \geq r + 4 \)).

8.1 Proof of Theorem 6.3 about mixed hypergraphs

Let \( \mathcal{M} = (A, B, V) \) be a \((2, r)\) mixed hypergraph such that \( G := A \cup B \) is an \( n \)-vertex graph with no cycle of length at least \( k \) (\( B := \partial_2 B \) and \( A \cap B = \emptyset \)). Let \( V_1, V_2, \ldots, V_q \) be the vertex sets of the standard (and unique) decomposition of \( G \) into blocks of sizes \( n_1, n_2, \ldots, n_q \). Then the graph \( A \cup B \) restricted to \( V_i \), denoted by \( G_i \), is either a 2-connected graph or a single edge (in the latter case \( n_i = 2 \)), each edge from \( A \cup B \) is contained in a single \( G_i \), and \( \sum_{i=1}^{q} (n_i - 1) \leq (n - 1) \). This decomposition yields a decomposition of \( A = A_1 \cup A_2 \cup \cdots \cup A_q \) and \( B = B_1 \cup B_2 \cup \cdots \cup B_q \), \( A_i \cup B_i = E(G_i) \). If an edge \( e \in B_i \) is contained in \( f \in B \), then \( f \subseteq V_i \) (because \( f \) induces a 2-connected graph \( K_r \) in \( B \)), so the block-decomposition of \( G \) naturally extends to \( B, B_i := \{ f \in B : f \subseteq V_i \} \) and we have \( B = B_1 \cup \cdots \cup B_q \), and \( B_i = \partial_2 B_i \). By definition, \( G \) has no cycle of length \( k \) or longer, so the same is true for each \( G_i \). Suppose that the size of \( A \cup B \) is as large as possible, \( \mathcal{M} \) is extremal, \( |\mathcal{M}| = m_r(n, k) \).

Lemma 5.5 implies that for \( n_i \leq k - 1 \),

\[ |A_i| + |B_i| \leq \max \left\{ \binom{n_i}{2}, \binom{n_i}{r} \right\} = f_r^+(n_i, k), \]

and equality holds only if \( A_i \) is the complete graph (and \( B_i = \emptyset \)) or \( B_i \) is the \( r \)-uniform complete
Lemma 7.1 implies that in the case \( n_i \geq k \)
\[
|A_i| + |B_i| \leq f_r(n_i, k) - \binom{r}{2} < f_r^+(n_i, k).
\] (15)

Adding up these inequalities for all \( 1 \leq i \leq q \) and applying (14), we get
\[
\sum_i (|A_i| + |B_i|) \leq \sum_i f_r^+(n_i, k) \leq f_r^+(1 + \sum_i (n_i - 1), k) \leq f_r^+(n, k).
\] (16)

Since \( f_r^+(n, k) \leq m_r(n, k) \), here equality holds in each term. Consequently \( n_i < k \) for each \( i \), and all but at most one of them should be \( k - 1 \). Otherwise we can use the inequality
\[
\max\left\{\left(\frac{a}{2}\right), \frac{a}{r}\right\} + \max\left\{\left(\frac{b}{2}\right), \frac{b}{r}\right\} < \max\left\{\left(\frac{a - 1}{2}\right), \frac{a - 1}{r}\right\} + \max\left\{\left(\frac{b + 1}{2}\right), \frac{b + 1}{r}\right\}
\]
which holds for all \( 1 < a \leq b < k - 1 \) (and \( 3 \leq r, r + 4 \leq k \)). (The inequality \( f(a) + f(b) \leq f(a - 1) + f(b + 1) \) holds for every convex function \( f \), and here equality holds only if the four points \((a - 1, f(a - 1)), (a, f(a)), (b, f(b)), \) and \((b + 1, f(b + 1))\) are lying on a line). So \( \mathcal{M} \) is a linear tree formed by cliques, as described in Construction 6.2.

8.2 Proof of Theorem 3.2 for \( m > r + 1 \)

Let \( \mathcal{H} \) be an \( r \)-uniform hypergraph on \( n \) vertices with no Berge cycle of length \( k \) or longer (\( r \geq 3, k \geq r + 4 \)). Suppose that \( |\mathcal{H}| \) is maximal, \( |\mathcal{H}| = \text{EG}_r(n, k) \). We have \( f_r(n, k) \leq \text{EG}_r(n, k) \) by Constructions 4.1 and 4.2.

Let \((A, A)\) be an SDRP of \( \mathcal{H} \) of maximum size. Let \( B := \mathcal{H} \setminus A, B = \partial_2 \mathcal{B} \). By Lemma 5.3 the graph \( G \) with edge set \( A \cup B \) does not contain a cycle of length \( k \) or longer. In other words, \( \mathcal{M} = (A, B, V) \) is a \((2, r)\) mixed hypergraph such that \( G := A \cup B \) is an \( n \)-vertex graph with no cycle of length at least \( k \). Then Theorem 6.3 implies that
\[
|A| + |B| \leq f_r^+(n, k).
\] (17)

Since \( n = (k - 2)p + m \) where \( 1 \leq m \leq k - 2 \) and \( m \geq r + 2 \) we have \( f_r^+(n, k) = f_r(n, k) \) by (2) and (6). We obtained that \( \text{EG}_r(n, k) = f_r(n, k) \), as claimed.

Equality can hold in (17) only if \( \mathcal{M} \) has the clique-tree structure, \( V_1, V_2, \ldots, V_{p + 1} \), described in Construction 6.2. In the case of \( m \geq r + 3 \) each block is a complete \( r \)-uniform hypergraph, so Construction 6.2 and Construction 4.1 coincide, and we are done.

In the case \( m = r + 2 \), Theorem 6.3 implies that all but one block define complete \( r \)-graphs and for one of them, say \( V_\ell \), \( \mathcal{M}|V_\ell \) could be either \( K_{r+2} \) or \( K_{r+2}^{(r)} \). If \( \mathcal{M}|V_\ell = K_{r+2} \), then \( \mathcal{H}|V_\ell = K_{r+2}^{(r)} \), so \( A = \emptyset, B = \mathcal{H} \) and we are done. Consider the other case, \( \mathcal{M}|V_\ell = K_{r+2}^{(2)} \), i.e., \( A = G|V_\ell \) is a complete graph (and \( B = \cup_{i \neq \ell} K_{k-1}^{(r)}[V_i] \)). We claim that \( \mathcal{H}|V_\ell = K_{r+2}^{(r)} \) which completes the proof in this subsection.

Suppose, on the contrary, that there exists \( f_i \in A \) such that \( \{x_i, y_i\} \subset V_\ell, \{x_i, y_i, z_i\} \subset f_i \) such
that $z_i \notin V_\ell$. One of the pairs of $x_i z_i$ and $y_i z_i$ is not an edge of $G$, say it is $x_i z_i$. Then removing $x_i y_i$ from $A$ and replacing it by $x_i z_i$, one obtains an SDRP $A'$ ($A$ and $B$ are unchanged). In this case, $E(G') = E(G) \setminus \{x_i y_i\} \cup \{x_i z_i\}$ has a different structure (not a tree of cliques), so it could not be optimal by Theorem 6.3. Therefore such $f_i$ does not exist, i.e., $f_i \subset V_\ell$. In other words $A \subseteq K_{k-1}^{(r)}[V_\ell]$. Since $|A| = \binom{r+2}{2} - \binom{r}{2}$, $A$ is a complete $r$-graph on $V_\ell$. 

8.3 Proof of Theorem 3.2 for $m \leq r+1$, preparations

This is a continuation of the previous two subsections.

Consider an extremal $H$ (i.e., $|H| = EG_r(n, k) \geq f_r(n, k)$) and the blocks $G_1, \ldots, G_q$ of $G$ with vertex sets $V_1, \ldots, V_q$ where $|V_i| = n_i \geq 2$. As we have seen in (15) and (16),

$$|H| = \sum_i (|A_i| + |B_i|) \leq f_r^+(n, k) - \binom{r}{2}$$

(18)

if for any $i$, $n_i \geq k$. For $m = r + 1$, here the right-hand side is

$$p\binom{k-1}{r} + \binom{m}{2} - \binom{r}{2} < p\binom{k-1}{r} + \binom{r+1}{r} = f_r(n, k).$$

Similarly in the case $m \leq r$, the right-hand side is

$$p\binom{k-1}{r} + \binom{m}{2} - \binom{r}{2} < p\binom{k-1}{r} + m - 1 = f_r(n, k).$$

So from now on, we may suppose that $n_i \leq k-1$ for all $i$.

Claim 8.1. There are exactly $p$ blocks $V_i$ of size $k-1$, $n_i = k-1$.

Proof. For $1 \leq x \leq k-1$, define $f(x) := f_r^+(x, k) = \max\{\binom{x}{2}, \binom{x}{r}\}$. Let $f(x_1, \ldots, x_q) := \sum_i f(x_i)$. We want to estimate $f(n_1, \ldots, n_q)$, so define $x_i := n_i$. Let $n' := 1 + \sum_i (n_i - 1)$; we have $n' \leq n$. In case of $2 \leq x_i \leq x_j < k-1$ we are going to replace $x_i$ by $x_i - 1$ and $x_j$ by $x_j + 1$. During this process $f$ never decreases and it ends when all but one $x_i$‘s become $1$ or $k-1$. Then the value of $f$ is exactly $f_r^+(n', k)$ and since $\sum_{1 \leq i \leq q} x_i = \sum_i n_i = n' + q - 1$ is unchanged, in the last step our sequence contains $(k-1)$ exactly $p$ times.

If the number of $(k-1)$’s is unchanged, then there is nothing to prove. Otherwise, after some step the pair $x$ and $k-2$ $(2 \leq x \leq k-2)$ is replaced by $(x-1)$ and $(k-1)$. Then the value of $f$ increased by $f(k-1) + f(x-1) - f(k-2) - f(x)$. Since $f(k-1) = \binom{k-1}{r}$ and $f(k-2) = \binom{k-2}{r}$ the increment is

$$\binom{k-1}{r} + \max\{\binom{x-1}{2}, \binom{x-1}{r}\} - \binom{k-2}{r} - \max\{\binom{x}{2}, \binom{x}{r}\}.$$ 

This is at least

$$\binom{k-2}{r-1} - \max\{x-1, \binom{x-1}{r-1}\} \geq \binom{k-2}{r-1} - \binom{k-3}{r-1} = \binom{k-3}{r-2} \geq \binom{r+1}{r-2} > \binom{r}{2}.$$ 

In this case $|H| < f_r^+(n', k) - \binom{r}{2} \leq f_r(n', k) \leq f_r(n, k)$, a contradiction. 

\hfill \square
Claim 8.2. If a block $V_i$ is of size $k - 1$, then $e(B_i) \geq \binom{k-2}{2} + r - 1$.

Proof. If $|\mathcal{B}_i| > \binom{k-2}{r}$ then the Kruskal-Katona Theorem (or a simple double counting) implies that $|\partial_2 \mathcal{B}_i| \geq \binom{k-2}{2} + r - 1$, and we are done.

If $|\mathcal{B}_i| \leq \binom{k-2}{r}$ then we use Lovász’ version of the Kruskal-Katona theorem. Write $|\mathcal{B}_i|$ in the form of $\left(\frac{x}{r}\right)$, where $x$ is a real number $0 \leq x \leq k - 2$ and (only in this paragraph) $\left(\frac{x}{r}\right)$ is defined as the 
real polynomial $x(x-1)\ldots(x-r+1)/r!$ for $x \geq r-1$ and 0 otherwise. We obtain $|\partial_2 \mathcal{B}_i| \geq \binom{x}{r}$. Since $A_i$ and $B_i$ are disjoint, we have $|A_i| \leq \binom{k-2}{2} - \binom{x}{r}$. So,

$$|A_i| + |\mathcal{B}_i| \leq \binom{k-1}{2} - \binom{x}{2} + \binom{x}{r} \tag{19}$$

holds for some $0 \leq x \leq k - 2$. In this range the right-hand side (as a polynomial of variable $x$) is maximized at $x = k - 2$. Hence (19) yields

$$|A_i| + |\mathcal{B}_i| \leq \binom{k-1}{2} - \binom{k-2}{2} + \binom{k-2}{r}.$$ 

Here the right-hand side is less than $\binom{k-1}{r} - \binom{x}{r}$ which (as we have seen in (18)) leads to the contradiction $|\mathcal{H}| < f_r(n,k)$. \hfill \Box

Claim 8.3. If a block $V_i$ is of size $k - 1$, then $\mathcal{B}_i = K_{k-1}^{(r)}$, a complete $r$-graph.

Proof. Suppose that there exists an $r$-set $f \subset V_i$, $f \notin \mathcal{H}$. Consider the hypergraph $\mathcal{H} \cup \{f\}$. By the maximality of $\mathcal{H}$, $\mathcal{H} \cup \{f\}$ contains a Berge cycle $C$ of length at least $k$, say with base vertices $\{v_1, \ldots, v_\ell\}$ and edges $\{f_1, \ldots, f_\ell\}$ where $f_\ell = f$ (and so $v_1, v_\ell \in V_i$). Since $|V_i| = k - 1$, there is a base vertex of $C$ not contained in $V_i$. Therefore we may pick a segment $P$ of $C$ (a Berge path in $\mathcal{H}$) say $\{v_a, v_{a+1}, \ldots, v_b\}$, $\{f_a, \ldots, f_{b-1}\}$ such that $v_a, v_b \in V_i$ but $\{v_{a+1}, \ldots, v_{b-1}\} \cap V_i = \emptyset$.

Since each $r$-edge in $\mathcal{B}_i$ yields a clique of order $r$ in $B_i$, we have $\delta(B_i) \geq r - 1 \geq 2$. By Claim 8.2 and Lemma 5.6, $B_i$ is hamilton-connected unless $r = 3$ and $B_i$ is a clique on $k - 2$ vertices with a vertex $x$ of degree 2. If the latter holds, then for a neighbor $y$ of $x$, the edge $xy$ is contained in exactly one triangle in $B_i$. But then $xy$ can only be contained in one $r$-edge of $\mathcal{B}$, contradicting Lemma 5.2. So we may assume $B_i$ has a hamilton path between any two vertices, in particular by Lemma 5.3, there is a Berge path $P'$ of length $k - 2$ from $x_a$ to $x_b$ containing all $k - 1$ vertices of $V_i$ as base vertices and using only the edges from $\mathcal{B}_i$. The cycle $P \cup P'$ is a Berge cycle in $\mathcal{H}$ of length at least $k$, a contradiction. Therefore such an edge $f$ cannot exist, $\mathcal{H}|V_i| = K_{k-1}^{(r)}$.

Finally, there is no $A$-edge in $V_i$. If $\{x, y\} \subset V_i$ is an $A$-edge, then no $B$-edge can contain $\{x, y\}$. So all the $\binom{k-2}{r}$ ($\geq k - 3$) subsets of $V_i$ of size $r$ and containing $xy$ should belong to $A$. Therefore $V_i$ must contain at least as many $A$-edges. But $|A_i| \leq k - 1 - r(= \binom{k-1}{2} - \binom{k-2}{2} - r + 1)$ by Claim 8.2. \hfill \Box

8.4 Proof of Theorem 3.2 for $m \leq r + 1$, the end

This is a continuation of the previous three subsections.

Consider an extremal $r$-graph $\mathcal{H}$ on the $n$-element vertex set $V$ (i.e., $|\mathcal{H}| = EG_r(n,k) \geq f_r(n,k)$).

where \( n = p(k - 2) + m, 1 \leq m \leq r + 1 \). Using the previous subsection, we define a different split of \( \mathcal{H} \).

Let \( V := \{ V_i : |V_i| = k-1 \} \). By Claim 8.1 \(|V| = p\). Let \( H \) be the graph whose edge set is the union of the complete graphs on \( V_i \in V \), so \( |E(H)| = p(k-1) \) and it has a forest like structure of cliques (i.e., every cycle in \( H \) is contained in some \( V_i \in V \).) Let \( C_1, \ldots, C_m \) be the vertex sets of the connected components of \( H \). The graph \( H \) necessarily consists of \( m \) (nonempty) components, \( \cup C_\alpha = V \) (\( 1 \leq \alpha \leq m \)), some of them could be singletons. Let \( H_\alpha := H|C_\alpha \), \( \mathcal{H}_\alpha := \cup\{ B_i : V_i \in V, V_i \subset C_\alpha \}, \) and \( D := \mathcal{H} \setminus (\cup \mathcal{H}_\alpha) \). Note that every edge of \( H \) used to be a \( B \)-edge, \( \mathcal{H}_\alpha \subseteq B \) for all \( 1 \leq \alpha \leq m \), and \( D \) is the set of edges in \( \mathcal{H} \) not contained in some \( K^{(r)}_{k-1} \).

Our main observation is the following which is implied by Claim 8.3.

**Claim 8.4.** If \( x, y \in C_\alpha, x \neq y \) then there exists an \( x-y \) Berge path of length at least \( k-2 \) consisting only of \( \mathcal{H}_\alpha \) edges. Moreover, if \( xy \notin E(H_\alpha) \) then there exists such a path of length at least \( 2k - 4 \).  

Suppose that \( f, f' \in D, (f \neq f') \), \( x_\alpha \in C_\alpha \cap f \), \( x'_\alpha \in C_\alpha \cap f' \), \( x_\beta \in C_\beta \cap f \), and \( x'_\beta \in C_\beta \cap f' \), \( (\alpha \neq \beta) \), then

\[
x_\alpha = x'_\alpha \text{ and } x_\beta = x'_\beta.
\]  

(20)

For example, if \( x_\alpha \neq x'_\alpha \) and \( x_\beta \neq x'_\beta \), then there is a Berge path \( P_\alpha \) of length at least \( k-2 \) connecting \( x_\alpha \) with \( x'_\alpha \), \( P_\beta \subseteq \mathcal{H}_\alpha \) and another Berge path \( P_\beta \) of length at least \( k-2 \) connecting \( x_\beta \) with \( x'_\beta \), \( P_\beta \subset \mathcal{H}_\beta \), and these, together with \( f \) and \( f' \) form a Berge cycle of length at least \( 2k-2 \), a contradiction. The case \( \{|x_\alpha, x_\beta\} \cap \{x'_\alpha, x'_\beta\}| = 1 \) is similar: we find a Berge cycle in \( \mathcal{H} \) of length at least \( k \).

The same proof, and the second half of Claim 8.4 imply that

\[
\partial_2 \mathcal{H}|V_\alpha = H_\alpha.
\]

(21)

In other words, if \( f \in \mathcal{H} \setminus \mathcal{H}_\alpha \) then

\[
|f \cap C_\alpha| \geq 2 \text{ implies that } \exists V_i \in V, V_i \subseteq C_\alpha \text{ such that } C_\alpha \cap f = V_i \cap f.
\]

Indeed, otherwise there are \( x, y \in f \) and a Berge \( x, y \)-path in \( H_i \) of length at least \( 2k - 4 \), which together with \( f \) form a Berge cycle of length at least \( 2k - 3 \).

For a subset \( S \subseteq V \), define \( \varphi(S) \) as the set of indices \( 1 \leq \alpha \leq m \) for which \( S \cap C_\alpha \neq \emptyset \). Equation \( \text{(20)} \) can be restated as follows

\[
\text{if } \{ \alpha, \beta \} \subseteq \varphi(f) \cap \varphi(f') \text{ then } C_\alpha \cap f = C_\alpha \cap f' \text{ is a singleton},
\]

(22)

and similarly for \( \beta \). This implies that \( \varphi(f) \neq \varphi(f') \) for \( f \neq f', f, f' \in D \). Even more, the family \( \{ \varphi(f) : f \in D \} \) has the Sperner property. This means that for \( f, f' \in D \) with \( f \neq f' \), one cannot have \( \varphi(f) \subseteq \varphi(f') \). Indeed, \( |\varphi(f)| < r \) implies that there exists a \( C_\alpha \) with \( |C_\alpha \cap f| \geq 2 \), equation \( \text{(21)} \) implies that \( |\varphi(f)| \geq 2 \) for every \( f \in D \), so there exists a \( \beta \in \varphi(f), \alpha \neq \beta \). But then \( \{ \alpha, \beta \} \subseteq \varphi(f) \cap \varphi(f') \) and \( \text{(22)} \) implies that \( |C_\alpha \cap f| = 1 \), a contradiction.

The following claim on the intersection structure of the edges in \( D \) is a generalization of \( \text{(22)} \) which can be considered as the case \( \ell = 2 \). (Technically, two hyperedges sharing at least two vertices form
Let $\mathcal{F} = \{ \varphi(f) : f \in D \}$. Suppose that $\{\alpha_1, \ldots, \alpha_t\} \subset \{1, \ldots, m\}$ and $\varphi(f_1), \ldots, \varphi(f_{\ell}) \in \mathcal{F}$ form a Berge cycle in $\mathcal{F}$. Then for each $j$, the sets $C_{\alpha_j} \cap f_j = C_{\alpha_j} \cap f_{j-1}$ are singletons.

**Proof.** Otherwise, we can relabel $j := 1$ and find two distinct vertices $x_1$ and $x_1'$ such that $x_1 \in C_{\alpha_1} \cap \varphi(f_1)$ and $x_1' \in C_{\alpha_1} \cap \varphi(f_{\ell})$. Furthermore, let $x_i, x_i' \in C_{\alpha_i}$ such that $\{x_{i-1}, x_i\} \subset f_i$ for all $1 \leq i \leq \ell$ ($x_0 := x_{\ell}$, etc.), $P_i$ a Berge path in $\mathcal{H}_i$ connecting $x_i$ with $x_i'$. These paths could be empty (if $x_i = x_i'$) but by Claim 8.4 we can choose $P_i$ so that its length is at least $k - 2$. Then $f_1, P_1, f_2, P_2, \ldots, f_{\ell}, P_{\ell}$ form a cycle of length at least $k$, a contradiction.

**Case 1:** there exists an $f$ such that $|\varphi(f)| = r$. Then $m \geq r$, so $m \in \{r, r+1\}$. If $m = r$, then (because of the Sperner property) $|D| = 1 < m - 1$, a contradiction. So assume $m = r + 1$. Let $\alpha := [m] \setminus \varphi(f)$. We have $\alpha \in \varphi(f')$ for all other $f' \in D$. Since $|D| \geq r + 1 > 3$, there are at least two more $f_2 \neq f_3 \in D \setminus \{f\}$.

Consider first the case that $|C_{\alpha} \cap \varphi(f_2)| \geq 2$ for some $f_2 \in D$. The Sperner property implies that there are distinct $\alpha_2, \alpha_3 \in [m] \setminus \alpha$ such that $\alpha_2 \in \varphi(f_2) \setminus \varphi(f_3)$ and $\alpha_3 \in \varphi(f_3) \setminus \varphi(f_2)$. Then $\alpha, \alpha_2, \alpha_3$ with the hyperedges $\varphi(f_2), \varphi(f), \varphi(f_3)$ form a Berge cycle. However this cycle does not satisfy Claim 8.5. So from now on, we may suppose that $|C_{\alpha} \cap \varphi(f')| = 1$ for all $f' \in D \setminus \{f\}$.

Suppose that there exists an $f_2 \in D$ and an $\alpha_2 \in [m]$ such that $|C_{\alpha} \cap \varphi(f_2)| \geq 2$ (necessarily $\alpha \neq \alpha_2$). Again Sperner property implies that there is an $\alpha_3 \in [m] \setminus \alpha$ such that $\alpha_3 \in \varphi(f_3) \setminus \varphi(f_2)$ (so we have $\alpha_3 \neq \alpha_2$). Then $\alpha, \alpha_2, \alpha_3$ with the hyperedges $\varphi(f_2), \varphi(f), \varphi(f_3)$ form a Berge cycle. However this cycle does not satisfy Claim 8.5. So from now on, we may suppose that $|C_{\alpha} \cap \varphi(f')| = 1$ for all $f' \in D$ and all $\alpha' \in [m]$.

Since $|D| \geq r + 1$ and $[m]$ has exactly $r + 1$ $r$-subsets, $\varphi(D)$ is a complete $r$-graph. Its hyperedges form many Berge cycles, so Claim 8.5 implies that $D$ itself is isomorphic to $K_r^{(r)}$. Thus $\mathcal{H}$ is as in Construction.4.1

**Case 2:** $|\varphi(f)| < r$ for all $f \in D$. In this case every $f \in D$ has an $\alpha(f) \in [m]$ such that $|C_{\alpha(f)} \cap f| \geq 2$. For every $f \in D$, choose another element $\beta(f) \in \varphi(f)$ ($\beta(f) \neq \alpha(f)$) and consider the graph $T := \{\alpha(f), \beta(f)\} : f \in D$. By Claim 8.5 the graph $T$ has no cycle, and the maximality of $|\mathcal{H}|$ implies that $e(T) = |D| \geq m - 1$. So $T$ is a tree. Since $T$ is a tree, one cannot replace an edge $\{\alpha(f), \beta(f)\}$ by the 3-edge $\{\alpha(f), \beta(f), \gamma(f)\}$ without creating a cycle in the resulting hypergraph and thus violating Claim 8.5. So $\varphi(f) = \{\alpha(f), \beta(f)\}$, and (21) implies that the structure of $D$ is as in Construction 4.2. This completes the proof of Theorem 3.2
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