Geometric inequalities for free boundary hypersurfaces in a ball

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Abstract
In this paper, we prove a family of sharp geometric inequalities for free boundary hypersurfaces in a ball in space forms.

Keywords Free boundary hypersurfaces · Sub-static · Geometric inequalities · Reilly formula

Mathematics Subject Classification 53C21 · 53C24

1 Introduction

The sharp geometric inequalities for closed hypersurfaces in Riemannian manifolds have attracted a lot of attention during the past few decades. By means of curvature flows, the sharp geometric inequalities such as quermassintegral inequalities, Minkowski-type inequalities, Alexandrov–Fenchel-type inequalities and more general weighted inequalities have been established for closed hypersurfaces in space forms or warped product spaces, see [2–4, 10,11,13–15,19,29,32,35], etc. Another powerful approach to prove the sharp geometric inequalities is the (weighted) Reilly formula, which can be used to prove Heintze–Karcher-type inequality and Minkowski-type inequalities for closed hypersurfaces in Riemannian manifolds, see, e.g., [21,23,25,26,36].

Influenced by the pioneering works due to De Lellis-Müller [6] and De Lellis-Topping [7], Perez [22, Thm. 3.1] proved the following geometric inequality for closed hypersurfaces in \(\mathbb{R}^{n+1}\). The equality case was characterized by Cheng and Zhou [5, Thm. 1.3].
Theorem A ([5,22]) Let \( n \geq 2 \) and \( \Sigma \) be a closed immersed oriented hypersurface in \( \mathbb{R}^{n+1} \) with nonnegative Ricci curvature. Then
\[
\int_{\Sigma} |H - \overline{H}|^2 \leq \frac{n}{n-1} \int_{\Sigma} |\hat{h}|^2,  \tag{1.1}
\]
where \( \hat{h} = h - \frac{H}{n} g \) and \( \overline{H} = \int_{\Sigma} H / |\Sigma| \). Equality holds in \( (1.1) \) if and only if it is a sphere.

Inequality \((1.1)\) is equivalent to
\[
\int_{\Sigma} \left| h - \frac{\overline{H}}{n} g \right|^2 \leq \frac{n}{n-1} \int_{\Sigma} |\hat{h}|^2. \tag{1.2}
\]

It was pointed out by De Lellis and Topping [7] that Perez’s inequality \((1.1)\) holds even for the closed hypersurfaces with nonnegative Ricci curvature in Einstein manifolds, see also [5, Thm. 1.2]. Recently, Perez’s inequality \((1.2)\) was generalized by Agostiniani, Fogagnolo and Mazzieri [1, Thm. 6.1] to the bounded outward minimizing open domain with smooth boundary \( \Sigma \) in \( \mathbb{R}^3 \) by nonlinear potential theory.

Kwong [16, Thms. 2.2 & 4.1] generalized Perez’s inequality \((1.1)\) to the higher-order mean curvatures for closed immersed submanifolds in space forms. Let \( M^{n+1}(K) \) be the simply connected space form with constant sectional curvature \( K \in \{-1, 0, 1\} \). Then \( M^{n+1}(0) = \mathbb{R}^{n+1}, M^{n+1}(1) = \mathbb{S}^{n+1} \) and \( M^{n+1}(-1) = \mathbb{H}^{n+1} \). Let \( H_k \) be the \( k \)th mean curvature and \( T_k \) the \( k \)th Newton tensor, respectively, see (2.1) and (2.2) in Sect. 2. We denote by \( \tilde{T}_k = T_k - \frac{1}{n} (\text{tr} T_k) g \) the tracefree part of \( T_k \), in particular, \( \tilde{T}_1 = -\hat{h} \). For closed immersed hypersurfaces with nonnegative Ricci curvature in space forms, Kwong’s results can be stated as follows:

Theorem B ([16]) Let \( n \geq 2 \) and \( \Sigma \) be a closed immersed oriented hypersurface in \( M^{n+1}(K) \) with nonnegative Ricci curvature. Then for \( k \in \{1, \ldots, n-1\} \), we have
\[
\int_{\Sigma} (H_k - \overline{H}_k)^2 \leq \frac{n(n-1)}{(n-k)^2} \int_{\Sigma} |\tilde{T}_k|^2, \tag{1.3}
\]
where \( \overline{H}_k = \int_{\Sigma} H_k / |\Sigma| \). Equality is attained at geodesic spheres. Conversely, if \( \Sigma \) has positive Ricci curvature, then equality in \( (1.3) \) implies that \( \Sigma \) is a geodesic sphere.

Due to Fraser-Schoen’s important work on the first Steklov eigenvalue and minimal free boundary surfaces [8,9], it is natural to establish the sharp geometric inequalities for free boundary hypersurfaces in a ball. In [33], Wang and Xia utilized the (weighted) Reilly formula to prove the Heintze–Karcher-type inequality for such hypersurfaces. More recently, by introducing a specifically designed flow, Scheuer, Wang and Xia [28] proved a family of Alexandrov–Fenchel-type inequalities for free boundary hypersurfaces in a unit Euclidean ball. See also [31,34].

We use \( x : M^n \to B \subset M^{n+1}(K) \) to denote an isometric immersion of an oriented \( n \)-dimensional compact manifold with boundary \( \partial M \) into a (closed) geodesic ball \( B \) in \( M^{n+1}(K) \) such that
\[
x(\text{int} M) \subset \text{int} B \quad \text{and} \quad x(\partial M) \subset \partial B.
\]
The hypersurface \( \Sigma = x(M) \) is called free boundary if it meets the geodesic sphere \( \partial B \) orthogonally. It is convenient to use the following models to represent the space forms \( M^{n+1}(K) \).

(i) If \( K = 0 \), then \( (M^{n+1}(0), \overline{g}) = (\mathbb{R}^{n+1}, \delta) \) and \( B = B_R \) is a Euclidean ball of radius \( R \), where \( \delta \) is the Euclidean metric.
(ii) If $K = -1$, then we use the Poincaré ball model $(B^{n+1}, e^{2u} \delta)$ to represent the hyperbolic space $\mathbb{H}^{n+1}$, where $B^{n+1}$ is the open unit ball centered at the origin in $\mathbb{R}^{n+1}$ and $e^{2u} = \frac{4}{(1-|x|^2)^2}$. Abuse of notation, here we also use $x$ to denote the position vector with respect to the center of $B^{n+1}$ and $|x| = \delta(x, x)^{1/2}$ is its Euclidean norm. Let $B = B^R$ be a geodesic ball in $(B^{n+1}, e^{2u} \delta)$ with radius $R \in (0, \infty)$ centered at the center of $B^{n+1}$.

(iii) If $K = 1$, then we use the model $(\mathbb{R}^{n+1}, e^{2u} \delta)$ to represent the unit sphere without the south pole $S^{n+1}(\delta)$ and $e^{2u} = \frac{4}{(1+|x|^2)^2}$. Let $B = B^R_S$ be a geodesic ball in $S^{n+1}$ with radius $R \in (0, \pi)$ centered at the north pole.

For simplicity, we call a point $p \in \Sigma$ an elliptic point, if its second fundamental form satisfies $h_{ij} > c g_{ij}$ at this point, where $c = 0$ when $K = 0, 1$ and $c = 1$ when $K = -1$.

Our first result of this paper is the following.

**Theorem 1.1** Let $n \geq 2$ and $\Sigma$ be a compact immersed oriented hypersurface with free boundary in a ball in space forms. Assume that $\Sigma$ has nonnegative Ricci curvature. Then for $k \in \{1, \cdots, n-1\}$, we have

$$
\int_{\Sigma} |H_k - H| k|^2 \leq \frac{n(n-1)}{(n-k)^2} \int_{\Sigma} |\tilde{f}_k|^2. \tag{1.4}
$$

Equality is attained at spherical caps or a flat disk. Conversely, if there exists an elliptic point in the interior of $\Sigma$, then equality in (1.4) implies that $\Sigma$ is a spherical cap.

The condition of nonnegative Ricci curvature on hypersurface in space form $M^{n+1}(K)$ can be expressed in terms of its second fundamental form. By Gauss equations, we have

$$\text{Ric}^\Sigma_{ij} = H h_{ij} - h_{ik} h_{kj} + (n-1) K g_{ij}. \tag{1.5}$$

Then $\text{Ric}^\Sigma \geq 0$ provided that either (i) $K = 0, 1$ and $\Sigma$ is convex, i.e., $h_{ij} \geq 0$; or (ii) $K = -1$ and $\Sigma$ is horospherical convex, i.e., $h_{ij} \geq g_{ij}$.

In particular, we obtain the following result for convex embedded free boundary hypersurfaces in a ball in Euclidean space.

**Corollary 1.2** Let $n \geq 2$ and $\Sigma$ be a convex embedded free boundary hypersurface in a ball in $\mathbb{R}^{n+1}$. Then for $k \in \{1, \cdots, n-1\}$, we have

$$
\int_{\Sigma} |H_k - H| k|^2 \leq \frac{n(n-1)}{(n-k)^2} \int_{\Sigma} |\tilde{f}_k|^2. \tag{1.5}
$$

Equality holds in (1.5) if and only if $\Sigma$ is a spherical cap or a flat disk.

In [28], Scheuer, Wang and Xia found a natural counterpart of quermassintegrals $W_k$ for convex embedded free boundary hypersurfaces $\Sigma$ in the unit Euclidean ball. Let $\partial \Sigma$ be the boundary of $\Sigma$ in $S^n$. We use $|\partial \Sigma|$ to denote the $(n-1)$-dimensional Hausdorff measure of $\partial \Sigma$, and $|\Sigma|$ to denote the $n$-dimensional Hausdorff measure of $\Sigma$. As an application of Corollary 1.2, we obtain the following geometric inequalities.

**Corollary 1.3** (i) Let $\Sigma^2$ be a convex embedded free boundary surface in $\mathbb{B}^3$. Then

$$
\frac{1}{4} \left( \frac{\int_{\Sigma} H}{|\Sigma|} \right)^2 + |\partial \Sigma| \geq 2\pi. \tag{1.6}
$$

Equality holds in (1.6) if and only if it is a flat disk or a spherical cap.
(ii) Let \( \Sigma^3 \) be a convex embedded free boundary hypersurface in \( \mathbb{R}^4 \). Then
\[
\frac{1}{12} \left( \frac{1}{3} \left( f_3 \left( \frac{H^2}{|\Sigma|} \right) + |\partial \Sigma| \right) \right) \geq f_3 \circ f_1^{-1} \left( \frac{1}{4} |\Sigma| \right),
\]
where \( f_k(r) = W_k(\hat{C}_r) \) is a strictly increasing function of \( r \) (see Sect. 2 for details), and \( f_1^{-1} \) is the inverse of \( f_1 \). Equality holds in (1.7) if and only if it is a flat disk or a spherical cap.

**Remark 1.4**

(1) By Hölder inequality, inequality (1.6) implies the Willmore-type inequality
\[
\frac{1}{4} \int_\Sigma H^2 + |\partial \Sigma| \geq 2\pi,
\]
which was proved by Volkman [30, Cor. 5.8].

(2) By Hölder inequality, inequality (1.7) implies the Willmore-type inequality
\[
\frac{1}{12} \left( \frac{1}{3} \int_\Sigma H^2 + |\partial \Sigma| \right) \geq f_3 \circ f_1^{-1} \left( \frac{1}{4} |\Sigma| \right),
\]
which was proved by Scheuer, Wang and Xia [28, Cor. 1.4].

To further generalize the inequality (1.4), we consider a Riemannian triple \((\Sigma, g, V)\) which constitutes an \( n \)-dimensional smooth connected manifold \( \Sigma \), a Riemannian metric \( g \) and a smooth nonzero function \( V \) on \( \Sigma \). \( V \) is called a potential function. A Riemannian triple \((\Sigma, g, V)\) is called sub-static if
\[
\Delta V g - \nabla^2 V + V \text{Ric}^\Sigma \geq 0, \quad \text{on} \quad \Sigma,
\]
where \( \Delta, \nabla^2 \) and \( \text{Ric}^\Sigma \) are the Laplacian, Hessian and Ricci curvature of \((\Sigma, g)\), respectively. We also call it strictly sub-static if the strict inequality holds in (1.8) on \( \Sigma \). The potential functions associated to free boundary hypersurfaces in a ball in space forms are the functions \( V_a \) which are defined as follows:
\[
V_a = \begin{cases} 
\langle x, a \rangle, & \text{if } K = 0; \\
\frac{2\langle x, a \rangle}{1-|a|^2}, & \text{if } K = -1; \\
\frac{2\langle x, a \rangle}{1+|a|^2}, & \text{if } K = 1,
\end{cases}
\]
where \( a \in \mathbb{R}^{n+1} \) is a constant vector and \( \langle x, a \rangle = \delta(x, a) \). For simplicity, we use \( B_a^+ = \{V_a > 0\} \cap B \) to denote the half ball of \( B \) in the direction of \( a \). We obtain the following weighted geometric inequalities.

**Theorem 1.5**

Let \( n \geq 2 \) and \( x : M \to \mathbb{M}^{n+1}(K) \) be a compact immersed oriented hypersurface \( \Sigma \) with free boundary in a ball \( B \). Assume \( \Sigma \) lies in \( B_a^+ \) and \((\Sigma, g, V_a)\) is sub-static. Then for \( k \in \{1, \ldots, n-1\} \), we have
\[
\int_\Sigma \left| V_a \right| H_k - \overline{H}_k V_a \right|^2 \leq \frac{n(n-1)}{(n-k)^2} \int_\Sigma V_a |\overline{T}_k|^2,
\]
where \( \overline{H}_k = \int_\Sigma \frac{V_a H_k}{\int_\Sigma V_a} \). Equality is attained at spherical caps. Conversely, if \((\Sigma, g, V_a)\) is strictly sub-static and there exists an elliptic point in the interior of \( \Sigma \), then equality in (1.9) implies that \( \Sigma \) is a spherical cap.

As a direct application, we obtain the following corollary.
Corollary 1.6 Let \( n \geq 2 \) and \( x : M \to \mathbb{R}^{n+1} \) be an embedded oriented hypersurface \( \Sigma \) with free boundary in a ball \( B \). If \( \Sigma \) is strictly convex, then there exists a unit vector \( a \in \mathbb{R}^{n+1} \) such that \( V_a > 0 \) and for \( k \in \{1, \cdots, n-1\} \), we have
\[
\int_{\Sigma} V_a |H_k - \overline{H}_k|^2 \leq \frac{n(n-1)}{(n-k)^2} \int_{\Sigma} V_a |\tilde{T}_k|^2.
\]
Equality holds if and only if \( \Sigma \) is a spherical cap.

The paper is organized as follows: In Sect. 2, we collect basic facts about free boundary hypersurfaces and weighted Reilly formula. In Sect. 3, we give the proof of Theorems 1.1, 1.5 and Corollaries 1.2, 1.3, 1.6.

2 Preliminaries

Let \( n \geq 2 \). Let \((\Sigma, g)\) be a free boundary hypersurface given by an immersion \( x : M^n \to B \subset \mathbb{R}^{n+1}(K) \). We denote by \( \nabla, \Delta \) and \( \nabla^2 \) the gradient, the Laplacian and the Hessian on \((\mathbb{R}^{n+1}(K), \bar{g})\), respectively, while by \( \nabla, \Delta \) and \( \nabla^2 \) the gradient, the Laplacian and the Hessian on \((\Sigma, g)\), respectively. We choose one of the unit normal vector fields along the immersion \( x \) and denote it by \( \nu \). Denote by \( h \) and \( H \) the second fundamental form and the mean curvature of the hypersurface \( \Sigma \), respectively. Precisely, \( h(X, Y) = \bar{g}(\nabla_X \nu, Y) \) and \( H = \text{tr} h \). Since the mean curvature vector is independent of the choice of \( \nu \), we make a convention on \( \nu \) to be the opposite direction of mean curvature vector. We denote \( \mu \) the outward unit normal of \( \partial \Sigma \) in \( \Sigma \), \( \bar{N} \) the outward normal of \( \partial B \) in \( B \) and \( \overline{\nu} = \nu \) along the boundary \( \partial \Sigma \). See Fig. 1.

The principal curvatures \( \kappa = (\kappa_1, \cdots, \kappa_n) \) are the eigenvalues of the Weingarten matrix \( \mathcal{W} = (h^i_j) = (g^{ik} h_{kj}) \) on \( \Sigma \), i.e., the eigenvalues of the second fundamental form \( h \) with respect to the induced metric \( g \). For \( k \in \{1, 2, \cdots, n\} \), the \( k \)-th mean curvature \( H_k \) is defined by
\[
H_k = \frac{1}{k!} \sum_{1 \leq i_1, \cdots, i_k \leq n} \delta_{i_1,i_2,\cdots,i_k}^{j_1,j_2,\cdots,j_k} h_{i_1 j_1} h_{i_2 j_2} \cdots h_{i_k j_k} = \sum_{i_1 < \cdots < i_k} \kappa_{i_1} \cdots \kappa_{i_k},
\]

Fig. 1 \( \Sigma = x(M) \) and \( \partial \Sigma = x(\partial M) \)
where $\delta_{j_1\cdots j_k}^{i_1\cdots i_k}$ is the generalized Kronecker delta function. For $m \in \{0, 1, \cdots, n - 1\}$, the $m$-th Newton tensor $(T_m)^j_i$ is defined as the derivative of $H_{m+1}$ with respect to its argument $h^{i_j}$, which can be expressed by

$$(T_m)^j_i = \frac{1}{m!} \sum_{1 \leq i_1, \cdots, i_m \leq n} \delta_{i_1 i_2 \cdots i_m}^{j_1 j_2 \cdots j_m} h^{i_1}_{j_1} h^{i_2}_{j_2} \cdots h^{i_m}_{j_m}. \quad (2.2)$$

By Codazzi equations for hypersurfaces in space forms, we have

**Lemma 2.1** ([24]) Let $\Sigma$ be an immersed hypersurface in space forms. Let $\text{div}$ be the divergence on $\Sigma$. Then for $m \in \{0, \cdots, n - 1\}$, we have

$\text{div}(T_m) = 0, \quad \text{tr}(T_m) = (n - m)H_m. \quad (2.3)$

Let $\tilde{T}_m = T_m - \frac{1}{n} \text{tr}(T_m)g$. It follows from (2.3) that

$\text{div}(\tilde{T}_m) = -\frac{n - m}{n} \nabla H_m. \quad (2.4)$

We collect the basic facts on $H_k$ (see, e.g., [12]). The $k$-th Garding cone is defined by $\Gamma_k^+ = \{ \kappa \in \mathbb{R}^n \mid H_i(\kappa) > 0, i = 1, \cdots, k \}$.

**Lemma 2.2** Let $k \in \{1, 2, \cdots, n - 1\}$. If $\kappa \in \Gamma_k^+$, the following Newton-MacLaurin inequality holds

$$\frac{n - k}{n} H_1 H_k \geq (k + 1)H_{k+1}. \quad (2.5)$$

Equality holds in (2.5) if and only if $\kappa = cI$ for some constant $c > 0$, where $I = (1, \cdots, 1)$.

If $\tilde{T}_1 = 0$, then $\kappa = cI$ follows from $\tilde{T}_1 = -\tilde{h}$. For $k \in \{2, \cdots, n\}$, the following lemma characterizes the traceless property of $T_k$.

**Lemma 2.3** Let $k \in \{2, \cdots, n - 1\}$. If $\kappa \in \Gamma_k^+$, then $\tilde{T}_k = 0$ if and only if $\kappa = cI$ for some constant $c > 0$.

**Proof** If $\tilde{T}_k = 0$, we have $(T_k)^j_i = \frac{n - k}{n} H_k \delta^i_j$. Multiplying by $h^{i_j}$ on the both sides, we get

$$(k + 1)H_{k+1} = \frac{n - k}{n} H_k H_1,$$

which is an equality in (2.5). If $\kappa \in \Gamma_k^+$, then $\kappa = cI$ for some constant $c > 0$. The converse is obvious. \qed

The following proposition is a well-known fact for free boundary hypersurfaces in a geodesic ball in space forms.

**Lemma 2.4** ([20,27,33]) Let $x : M \rightarrow \mathbb{R}^{n+1}$ be an immersed oriented hypersurface $\Sigma$ with free boundary in a ball $B$. Then $\mu$ is a principal direction of $\partial \Sigma$ in $\Sigma$. Namely,

$$h(e, \mu) = 0, \quad \text{for any } e \in T(\partial \Sigma).$$

In turn, $\nabla_\mu v = h(\mu, v)$.$v$. Moreover, for any tangential vector field $Z \in T(\partial \Sigma)$, we have

$$T_k(\mu, Z) = 0, \quad k \in \{0, 1, \cdots, n - 1\}. \quad (2.6)$$
In [28], Scheuer, Wang and Xia defined the following geometric quantities for free boundary hypersurfaces in a unit Euclidean ball, which are expected to be the counterparts to the quermassintegrals for closed hypersurfaces in space forms:

\[
W_0(\hat{\Sigma}) = |\hat{\Sigma}|, \quad W_1(\hat{\Sigma}) = \frac{1}{n+1} |\Sigma|, \\
W_k(\hat{\Sigma}) = \frac{1}{n+1} \int_{\Sigma} H_{k-1} \left( \frac{n}{k-1} \right) + \frac{k-1}{(n+1)(n-k+2)} W_{k-2}^{\Sigma}(\partial \hat{\Sigma}), \quad 2 \leq k \leq n+1, 
\]

where \( |\hat{\Sigma}| \) denotes the \((n+1)\)-dimensional Hausdorff measure of \( \hat{\Sigma} \), and \( W_{k-2}^{\Sigma}(\partial \hat{\Sigma}) \) denotes the \((k-2)\)-th quermassintegral of the closed hypersurface \( \partial \Sigma \subset \mathbb{S}^n \), see [28, (1.1)]. Furthermore, the spherical cap of radius \( R \) around \( a \in \mathbb{S}^n \) is defined by

\[
C_R(a) = \{ y \in \mathbb{B}^{n+1} : |y - \sqrt{R^2 + 1} a| = R \}, \quad R < \infty, 
\]

and \( \hat{C}_R(a) \) is the convex domain enclosed by the spherical cap \( C_R(a) \) in \( \mathbb{B}^{n+1} \), where the argument \( a \) will be dropped in cases where it is not relevant. They proved that if \( \Sigma \) is a spherical cap or a flat disk, \( W_3(\hat{\Sigma}) \) is a strictly increasing function \( f_k(r) \) is the strictly increasing function \( f_k(r) = W_k(\hat{C}_r) \). Equality holds if and only if \( \Sigma \) is a spherical cap or a flat disk.

In particular, if \( n = 2 \),

\[
W_3(\hat{\Sigma}) = \frac{1}{3} \int_{\Sigma} H_2 + \frac{2}{3} W_1^{\Sigma}(\partial \hat{\Sigma}) = \frac{1}{3} \int_{\Sigma} H_2 + \frac{1}{3} |\partial \Sigma|, 
\]

then the equality (2.7) becomes

\[
\frac{1}{3} \int_{\Sigma} H_2 + \frac{1}{3} |\partial \Sigma| = \frac{\omega_2}{6} = \frac{2\pi}{3}. 
\]

If \( n = 3 \),

\[
W_3(\hat{\Sigma}) = \frac{1}{12} \int_{\Sigma} H_2 + \frac{1}{4} W_1^{\Sigma}(\partial \hat{\Sigma}) = \frac{1}{12} \int_{\Sigma} H_2 + \frac{1}{12} |\partial \Sigma|, \quad W_1(\hat{\Sigma}) = \frac{1}{4} |\Sigma|, 
\]

then the inequality (2.8) with \( k = 1 \) becomes

\[
\frac{1}{12} \left( \int_{\Sigma} H_2 + |\partial \Sigma| \right) \geq (f_3 \circ f_1^{-1}) \left( \frac{1}{4} |\Sigma| \right). 
\]
Laplacian and the Hessian on \( (\partial \Omega, g) \), respectively. In this paper, we will use the following special form of the weighted Reilly formula.

**Proposition 2.5** [21, Cor. 5.1] Let \( (\Omega, g) \) be a compact Riemannian manifold with smooth connected boundary \( \partial \Omega \). Let \( V \in C^\infty(\Omega) \) be a smooth positive function on \( \Omega \) such that \( \frac{\nabla^2 V}{V} \) is continuous up to \( \partial \Omega \). Then for any \( f \in C^\infty(\Omega) \), the following integral identity holds:

\[
\int_\Omega V \left( \Delta f - \frac{\Delta V}{V} f \right)^2 - V \left| \nabla^2 f - \frac{\nabla^2 V}{V} f \right|^2 \, d\Omega \\
= \int_\Omega \left( \Delta V g - \nabla^2 V + V \text{Ric} \right) \left( \nabla f - \frac{\nabla V}{V} f, \nabla f - \frac{\nabla V}{V} f \right) \, d\Omega \\
+ \int_{\partial \Omega} \left( h - \frac{\nabla V}{V} g \right) \left( \nabla f - \frac{\nabla V}{V} f, \nabla f - \frac{\nabla V}{V} f \right) \, dA \\
+ \int_{\partial \Omega} VH \left( f v - \frac{\nabla V}{V} f \right)^2 + 2V \left( f V - \frac{\nabla V}{V} f \right) \left( \Delta f - \frac{\Delta V}{V} f \right) \, dA. \tag{2.11}
\]

Here \( v \) is the unit outward normal of \( \partial \Omega \) and \( v = \nabla_v V \), \( f v = \nabla_v f \), \( h(\cdot, \cdot) \) and \( H \) are the second fundamental form and the mean curvature of \( \partial \Omega \), and \( \text{Ric} \) is Ricci curvature of \( \Omega \), respectively.

### 3 Proof of main results

**Proof of Theorem 1.1** We first give the proof of the inequality (1.4) in Theorem 1.1 and the inequality (1.9) in Theorem 1.5 in a unified way. We choose \( V = 1 \) in the proof of Theorem 1.1, \( V = V_a \) in the proof of Theorem 1.5, respectively. In Theorem 1.5, the hypersurface \( \Sigma \) is assumed to be contained in an open half-ball \( B_{a^+} \), which is equivalent to \( V_a > 0 \).

Consider the Neumann boundary value problem

\[
\begin{cases}
\Delta f - \frac{\Delta V}{V} f = H_k - \overline{H}_k^V, & \text{in } \Sigma, \\
f_\mu - \frac{V_\mu}{V} f = 0, & \text{on } \partial \Sigma,
\end{cases} \tag{3.1}
\]

where \( \mu \) is the unit outward normal of \( \partial \Sigma \) in \( \Sigma \). The existence and uniqueness (up to an additive \( a V \)) of the solution to (3.1) follows from the Fredholm alternative. For simplicity, we denote by

\[
A_{ij} = f_{ij} - \frac{V_{ij}}{V} f, \quad A = \Delta f - \frac{\Delta V}{V} f.
\]

Let \( \tilde{A}_{ij} = A_{ij} - \frac{1}{n} g_{ij} \) be the tracefree part of \( A_{ij} \).

Using Eq. (3.1), we have

\[
\int_\Sigma V \left( H_k - \overline{H}_k^V \right)^2 = \int_\Sigma \left( H_k - \overline{H}_k^V \right) \left( \Delta f - f \Delta V \right) \\
= \int_{\partial \Sigma} \left( H_k - \overline{H}_k^V \right) \left( V f_\mu - f V_\mu \right) - \int_\Sigma \left( \nabla H_k, V \nabla f - f \nabla V \right) \\
= \frac{n}{n-k} \int_\Sigma \left( \text{div}(\tilde{T}_k), V \nabla f - f \nabla V \right)
\]
\[
\frac{n}{n-k} \left( \int_{\partial \Sigma} \hat{T}_k(V \nabla f - f \nabla V, \mu) - \int_{\Sigma} V \sum_{i,j} (\hat{T}_k)_{ij} \hat{A}_{ij} \right) \\
\leq \frac{n}{n-k} \left( \int_{\Sigma} V |\hat{T}_k|^2 \right)^{1/2} \left( \int_{\Sigma} V \sum_{i,j} |\hat{A}_{ij}|^2 \right)^{1/2}.
\]

(3.2)

In the second equality, we used integration by parts, and in the third equality we used \( f_\mu - \frac{V}{\sqrt{V}} f = 0 \) to eliminate the boundary integral on \( \partial \Sigma \) and (2.4). In the fourth equality, we used integration by parts again. In the last line, the boundary integral on \( \partial \Sigma \) vanishes due to (2.6) and the inequality follows from the Hölder inequality.

We use \( \hat{V} \) to denote the gradient on \( \partial \Sigma \). We take \( \Omega = \Sigma \) and \( V = 1 \) or \( V = V_\alpha \) in the weighted Reilly formula (2.11), and in view of the boundary condition \( f_\mu - \frac{V}{\sqrt{V}} f = 0 \), we obtain

\[
\int_{\Sigma} V \left( A^2 - \sum_{i,j} |A_{ij}|^2 \right) = \int_{\partial \Sigma} V \left( h^{\Sigma} - \frac{V_\mu}{\sqrt{V}} g^{\Sigma} \right) \left( \hat{V} f - \frac{\hat{V}}{\sqrt{V}} f, \hat{V} f - \frac{\hat{V}}{\sqrt{V}} f \right) \\
+ \int_{\Sigma} \left( \Delta V g - \nabla^2 V + V \operatorname{Ric}^\Sigma \right) \left( \nabla f - \frac{\nabla V}{V} f, \nabla f - \frac{\nabla V}{V} f \right).
\]

(3.3)

By the free boundary condition, we have \( \mu = \sqrt{V} \). Then the induced metric \( g^{\Sigma} \) from (\( \Sigma, g \)) coincides with the metric \( g^{\partial B}|_{T(\partial \Sigma) \otimes T(\partial \Sigma)} \), where \( g^{\partial B} \) is the metric of \( \partial B \). Furthermore, the second fundamental form \( h^{\Sigma} \) coincides with \( h^{\partial B}|_{T(\partial \Sigma) \otimes T(\partial \Sigma)} \), where \( h^{\partial B} \) is the second fundamental form of \( \partial B \) given by

\[
h^{\partial B} = \begin{cases}
(1/R)g^{\partial B}, & \text{if } K = 0, \\
(\cot R)g^{\partial B}, & \text{if } K = -1, \\
(\csc R)g^{\partial B}, & \text{if } K = 1.
\end{cases}
\]

(3.4)

We have the following identity

\[
h^{\partial B} - \frac{(V_\alpha)_{\sqrt{V}}}{V_\alpha} g^{\partial B} = 0, \quad \text{on } \partial B,
\]

see [33, (5.12)]. For convenience of readers, we give a proof of (3.5) in Remark 3.1.

In case of Theorem 1.1, \( \Delta V g - \nabla^2 V + V \operatorname{Ric}^\Sigma = \operatorname{Ric}^\Sigma \geq 0 \) and \( h^{\Sigma} - \frac{V_\mu}{\sqrt{V}} g^{\Sigma} = h^{\Sigma} \geq 0 \). In case of Theorem 1.5, \( \Delta V g - \nabla^2 V + V \operatorname{Ric}^\Sigma \geq 0 \) and \( h^{\Sigma} - \frac{(V_\alpha)_{\mu}}{V_\alpha} g^{\Sigma} = 0 \) due to (3.5). It follows from (3.3) that \( \int_{\Sigma} V A^2 - V \sum_{i,j} |A_{ij}|^2 \geq 0 \), which yields

\[
\int_{\Sigma} V \sum_{i,j} |\hat{A}_{ij}|^2 \leq \frac{n-1}{n} \int_{\Sigma} V A^2.
\]

(3.6)

Substituting (3.6) into (3.2), we obtain

\[
\int_{\Sigma} V (H_k - \overline{H}_k^V)^2 \leq \frac{n}{n-k} \left( \int_{\Sigma} V |\hat{T}_k|^2 \right)^{1/2} \left( \frac{n-1}{n} \int_{\Sigma} V A^2 \right)^{1/2} \\
\leq \frac{n}{n-k} \sqrt{\frac{n-1}{n}} \left( \int_{\Sigma} V |\hat{T}_k|^2 \right)^{1/2} \left( \int_{\Sigma} V |H_k - \overline{H}_k^V|^2 \right)^{1/2}.
\]

(3.7)
Finally, if \( \int_{\Sigma} V |H_k - \overline{H}_k|^2 = 0 \), then the inequality (1.4) or (1.9) holds trivially. Otherwise, by eliminating \( \int_{\Sigma} V |H_k - \overline{H}_k|^2 \) on both sides of (3.7), we obtain
\[
\int_{\Sigma} V (H_k - \overline{H}_k)^2 \leq \frac{n(n-1)}{(n-k)^2} \int_{\Sigma} V |\tilde{T}_k|^2.
\]
To complete the proof of Theorem 1.1, we define the subset \( \Sigma_+ \) of \( \Sigma \) which consists of elliptic points in \( \Sigma \), i.e.,
\[
\Sigma_+ = \{ x \in \text{int} \Sigma : h_{ij} > cg_{ij} \},
\]
where \( c = 0 \) when \( K = 0, 1 \) and \( c = 1 \) when \( K = -1 \). By assumption, \( \Sigma_+ \) is nonempty and it is obviously open. Now we show that \( \Sigma_+ \) is also closed, by showing that
\[
h_{ij}|_{\Sigma_+} \geq (c + \varepsilon)g_{ij}|_{\Sigma_+}
\]
for some constant \( \varepsilon > 0 \).

In view of (3.3), the equality in (3.6) implies that
\[
\text{Ric}^\Sigma (\nabla f, \nabla f) = 0. \tag{3.8}
\]
On the other hand, the equality in (3.2) implies that there exists a constant \( \beta \in \mathbb{R} \) such that
\[
(\dot{T}_k)_{ij} = \beta \dot{A}_{ij} = \beta (f_{ij} - \Delta f/n g_{ij}). \tag{3.9}
\]
Let \( p \in \Sigma_+ \), then by Gauss equation
\[
\text{Ric}^\Sigma_{ij} = (Hh_{ij} - h_{ik}h^k_j) + (n - 1)Kg_{ij} > 0,
\]
we have \( \text{Ric}^\Sigma > 0 \) in an open neighborhood \( U \subset \Sigma_+ \) of \( p \). Then by (3.8), we have \( f \equiv \alpha \) for some \( \alpha \in \mathbb{R} \) and hence \( H_k \equiv \overline{H}_k > H_k(cI) \) in \( U \) by Eq. (3.1). So \( \dot{T}_k \equiv 0 \) in \( U \) by (3.9), which implies
\[
h_{ij} = \frac{1}{(n-k)} H_k g_{ij} = \frac{1}{(n-k)} \overline{H}_k g_{ij}, \quad \text{in} \ U,
\]
due to Lemma 2.3. Thus, we have
\[
h_{ij} \geq (c + \varepsilon)g_{ij}, \quad \text{in} \ U.
\]
Therefore, \( \Sigma_+ \) is closed and \( \Sigma_+ = \text{int} \Sigma \) by connectedness. It follows that \( \Sigma \) is umbilical and hence it is a spherical cap. The proof of Theorem 1.1 is completed. \( \square \)

**Remark 3.1** Since \( \partial B \) is a geodesic sphere of radius \( R \) in \( M^{n+1}(K) \), we have
\[
N = \begin{cases} 
x/R, & \text{if } K = 0, \\
x/\sinh R, & \text{if } K = -1, \\
x/\sin R, & \text{if } K = 1.
\end{cases} \tag{3.10}
\]
On the other hand, \( \partial B \) can be viewed as the Euclidean sphere \( |x| = R_\mathbb{R} \), where \( R_\mathbb{R} \) is given by
\[
R_\mathbb{R} = \begin{cases} 
R, & \text{if } K = 0; \\
\sqrt{\frac{\cosh R - 1}{\cosh R + 1}}, & \text{if } K = -1; \\
\sqrt{\frac{1 - \cos R}{1 + \cos R}}, & \text{if } K = 1.
\end{cases} \tag{3.11}
\]
Let $\{E_i\}_{i=1}^{n+1}$ be an orthonormal basis in $(\mathbb{R}^{n+1}, \delta)$ and $\overline{E}_i = e^{-u} E_i$. Then $\{\overline{E}_i\}_{i=1}^{n+1}$ is an orthonormal basis in $(M^{n+1}(K), \overline{g} = e^{2u}\delta)$. In view of the models of the space forms, any function $f$ defined on $(M^{n+1}(K), \overline{g})$ can be also considered as a function defined on $\mathbb{H}^{n+1}$ or $\mathbb{R}^{n+1}$, respectively. Then there holds

$$\nabla^\mathbb{R} f = \sum_{i=1}^{n+1} E_i(f) E_i = e^{2u} \overline{E}_i(f) \overline{E}_i = e^{2u} \nabla f.$$ 

Then it follows from (3.10) and (3.11) that

$$\frac{(V_a)_N}{V_a} = \overline{g}(\nabla \log V_a, \overline{N}) = \langle \nabla^\mathbb{R} \log V_a, \overline{N} \rangle$$

$$= \begin{cases} \langle \overline{N}, a \rangle = \frac{1}{R}, & \text{if } K = 0, \\ \langle \overline{N}, a \rangle + \frac{2}{1-|x|^2} \overline{x}, & \text{if } K = -1, \\ \langle \overline{N}, a \rangle - \frac{2}{1+|x|^2} \overline{x}, & \text{if } K = 1. \end{cases}$$

Then the identity (3.5) follows from (3.4) and (3.12).

As a direct application, we give the proof of Corollaries 1.2, 1.3.

**Proof of Corollary 1.2** For convex hypersurfaces with free boundary in a ball of $\mathbb{R}^{n+1}$, the nonnegativity of Ricci curvature follows directly from Gauss equation. Then the inequality follows immediately. To prove the rigidity part, we may assume that $\Sigma$ is not the flat disk, otherwise we are done. Then by [Lem.3.1] [18], there exists a strictly convex point in the interior of $\Sigma$. Then it follows from Theorem 1.1 that $\Sigma$ is a spherical cap. This completes the proof of Corollary 1.2. \(\square\)

**Proof of Corollary 1.3** Inequality (1.5) with $k = 1$ is equivalent to

$$\int_{\Sigma} H_2 \leq \frac{n-1}{2n} \left( \int_{\Sigma} H \right)^2.$$ 

(3.13)

Then substituting the above inequality (3.13) into (2.9) and (2.10), we obtain the desired inequalities (1.6) and (1.7), respectively. The equality characterization follows from the equality case of (3.13) in Corollary 1.2. This completes the proof of Corollary 1.3. \(\square\)

Next, we complete the proof of Theorem 1.5.

**Proof of Theorem 1.5** The inequality (1.9) in Theorem 1.5 has been proved. In view of (3.3), the equality in (3.6) implies that

$$(\Delta V_a g - \nabla^2 V_a + V_a \text{Ric}_\Sigma) \left( \nabla f - \frac{\nabla V_a}{V_a} f, \nabla f - \frac{\nabla V_a}{V_a} f \right) = 0.$$ 

(3.14)

On the other hand, the equality in (3.2) implies that there exists a constant $\beta \in \mathbb{R}$ such that

$$(\hat{T}_k)_{ij} = \beta \hat{A}_{ij} = \beta \left( f_{ij} - \frac{(V_a)_{ij}}{V_a} f - \frac{1}{n} \left( \Delta f - \frac{\Delta V_a}{V_a} f \right) g_{ij} \right).$$

(3.15)
By assumption that \( \Delta V_g - \nabla^2 V_a + V_a \text{Ric}_\Sigma > 0 \) on \( \Sigma \), by (3.14) we have \( f = \alpha V_a \) for some \( \alpha \in \mathbb{R} \) on \( \Sigma \) and hence \( H^V_k = H_k \) on \( \Sigma \) in view of Eq. (3.1). The remaining proof is almost the same as that in the proof of Theorem 1.1. The proof of Theorem 1.5 is completed.

\[ \square \]

**Remark 3.2** The sub-static condition of \((\Sigma, g, V_a)\) can be expressed in terms of certain convexity of the hypersurface. Since \( \nabla^2 V_a = -K V_a g \), it follows from the Gauss equations and Gauss formula that

\[
\Delta V_a = \nabla^2 V_a (v, v) - H (V_a)_v = -nK V_a - H (V_a)_v,
\]

\[
(V_a)_{ij} = (V_a)_{ij} - (V_a)_v h_{ij} = -K V_a g_{ij} - (V_a)_v h_{ij},
\]

\[
\text{Ric}_{ij}^\Sigma = H h_{ij} - h_{ik} h_{jk} + (n - 1) K g_{ij},
\]

where we use \((V_a)_{ij} = \nabla_i \nabla_j V_a\) and \((V_a)_v = \nabla_i \nabla_j V_a\). Then we have

\[
\Delta V_a g_{ij} - (V_a)_{ij} + V_a \text{Ric}_{ij}^\Sigma = (V_a)_{ik} h_{jk} - (V_a)_v g_{ik} (H g_{kj} - h_{kj}).
\]

(3.16)

So if \( \Sigma \) is convex and \( h_{ij} \geq (V_a)_v g_{ij} \), then \((\Sigma, g, V_a)\) is sub-static.

**Proof of Corollary 1.6** Since \( \Sigma \) is a strictly convex embedded free boundary hypersurface in a Euclidean ball, there exists a constant unit vector \( a \in \mathbb{R}^{n+1} \) such that \( V_a = \langle x, a \rangle > 0 \) and \( (V_a)_v = \langle a, v \rangle < 0 \), see [17, Lems. 11 & 12]. Then \((\Sigma, g, V_a)\) is strictly sub-static in view of (3.16). Applying Theorem 1.5, we complete the proof of Corollary 1.6.

\[ \square \]

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