Abstract

In this paper we study coupled dynamical systems and establish several invariance principles relating to the dimensions of the subspace spanned by solutions of each individual system. We consider two types of coupled systems, one with scalar couplings and the other with matrix couplings. Via the rank-preserving flow theory, we prove that scalar-coupled dynamical systems possess the dimensional-invariance principles, in that the dimension of the subspace spanned by the individual systems’ solutions remains invariant. For coupled dynamical systems with matrix coefficients/couplings, necessary and sufficient conditions are given to characterize dimensional-invariance principles. The established invariance principles provide additional characterizations and insights to analyze the transient behaviors and solution evolution for a large family of coupled systems, such as multi-agent consensus dynamics, distributed coordination systems, formation control systems, among others.

I. INTRODUCTION

In this paper we consider the following coupled dynamical systems consisting of $n$ individual systems

$$\dot{x}_i(t) = \sum_{j=1}^{n} \kappa_{ij}(t)x_j(t),$$

where $x_i \in \mathbb{R}^d$ is the state for system $i$, $\kappa_{ij}$ is a coupling weight scalar or matrix between systems $j$ and $i$ (when $i = j$, then it is a coefficient scalar/matrix $\kappa_{ii}$ for system $i$). The coupling/coefficient weights could be constant, time-varying, or state-dependent. The system (1) serves as a very general model to describe many different types of coupled/networked systems, such as formation control systems [1]–[3], network computation systems [4], [5], multi-agent consensus dynamics [6], [7].

Coupled dynamical systems are often operated in a networked manner, where each individual systems interact with other systems to perform a global or common task. Networked control systems in the general model (1) have been attracting increasing attention in the recent decade and can be found in a variety of applications. Depending on the actual control task, the coupling terms can be designed to reflect the information flow between spatially distributed systems, communication requirements or constraints, or cooperative interactions incorporating local tasks to achieve a global task [7].

In this paper, we aim to identify several invariance principles for the solutions of each system arising from different couplings and interactions between individual systems. We discuss two types of coupled dynamical systems that can
be represented by (1), one with scalar couplings, and the other with matrix couplings, respectively. These principles relate to the invariance of the dimensions of the subspaces spanned by the solutions of each individual system, which are thus termed as dimensional-invariance principles. The core concept is based on the rank-preserving flow theory, which is used to characterize the rank of the solution to a matrix differential system. Based on the rank-preserving flow theory and its extensions, we prove that coupled dynamical systems with scalar couplings (i.e., $\kappa_{ij}(t)$ in (1) being scalars), the dimensions of the spanned subspace of individual systems’ solutions remain invariant. For coupled dynamical systems with matrix couplings, necessary and sufficient conditions are given to guarantee such dimensional-invariance principles.

The invariance principles developed in this paper are fundamental yet universal properties for coupled dynamical systems. We note that in most papers on coupled/networked control systems, the focus has been on the stability and convergence analysis, while transient behaviors are largely ignored. The results revealed from the dimensional-invariance principles provide us with additional insights on the transient behaviors and evolutions of all individual solutions, and could assist the convergence and stability analysis of the overall coupled dynamical systems. An example is the distance-based formation control system described by gradient flows from potential functions of interest, which show that an initially collinear formation remains collinear for all time under such flows (see e.g. [2]).

The invariance principles also provide feasible coupling conditions to guarantee that the solutions of individual systems are constrained in some smaller dimensional spaces, which could find particular applications in several practical scenarios. For example, collinear solutions of a coupled dynamical system are of particular interests. In [1], a line formation, in which individual systems’ states are confined in a 1-D subspace, is studied with insights to more general formations on other dimensions. As another example, for a coupled dynamical system that describes the coordination of multiple mobile antennas, collinear solutions have practical significance to align the direction of all antennas in a single line [8]. Motivated by these practical applications, the theory of collinear dynamical systems was studied in [9]. The dimensional-invariance principles established in this paper will provide a unified and general theory and framework to facilitate these applications.

The remaining parts of this paper are structured as follows. In Section II, we prove that the solutions of scalar-coupled dynamical systems have the dimensional-invariance (and furthermore, subspace-preserving) principles. In Section III, matrix-coupled dynamical systems are discussed, for which necessary and sufficient conditions are given to guarantee the dimensional-invariance principle. Applications of the invariance principles in formation control systems are shown in Section IV. Section V presents the conclusions of this paper. In the appendix sections, we present preliminary background on rank-preserving flows, some extensions and proofs, and a brief review of several popular networked dynamical systems that fit in the general model (1).

A. Assumptions and solution issues of (1)

To address the solution issue of the coupled dynamical system (1), we impose the following mild assumption.

**Assumption 1.** The coefficient/coupling terms $\kappa_{ij}$ are continuous scalar/matrix functions.
The above mild assumption guarantees the existence and uniqueness of the solutions for system (1) [10, Chapter 1.2]. Note that we do not impose additional assumptions on \( \kappa_{ij} \)'s. They can be static, time-varying, state-dependent or other general continuous functions. Note also that the system (1) is not necessarily a time-invariant or time-varying linear system, since the coupling term \( \kappa_{ij} \) may also depend on the state \( x \) which may result in a coupled nonlinear system.

II. Coupled Dynamical Systems with Scalar-Weighted Couplings

Consider the following coupled dynamical systems with scalar couplings

\[
\dot{x}_i(t) = \sum_{j=1}^{n} w_{ij}(t)x_j(t), \tag{2}
\]

where \( w_{ij} \) is a scalar (static or time-varying) coupling weight between agents \( j \) and \( i \). Note that we do not require \( w_{ij} = w_{ji} \), i.e., the coupling weight could be asymmetric.

A. Main results

In this section we show that the coupled dynamical system (2) has the following dimensional-invariance principle.

**Theorem 1.** The coupled dynamical system (2) has the dimensional-invariance principle in the sense that

\[
\text{rank}(X(t)) = r, \quad (r \leq d), \quad \forall t \geq 0,
\]

\[
\text{if } \text{rank}(X(0)) = r. \tag{3}
\]

where \( X = [x_1, x_2, \ldots, x_n] \in \mathbb{R}^{d \times n} \).

**Proof.** Define the composite vector \( x = [x_1^T, x_2^T, \ldots, x_n^T]^T \in \mathbb{R}^{dn} \). In order to obtain a compact form of the system \( \dot{x} \), we define the matrix \( W = \{w_{ij}\} \in \mathbb{R}^{n \times n} \). Therefore, a compact form of (2) can be written as

\[
\dot{x}(t) = (W(t) \otimes I_d)x(t), \tag{4}
\]

where \( \otimes \) denotes the Kronecker product. The vector differential equation (4) on the real vector space \( \mathbb{R}^{dn} \) can be stated equivalently as the following differential flow on the matrix space \( \mathbb{R}^{d \times n} \) (without involving the Kronecker product term)

\[
\dot{X}(t) = X(t)W^\top(t). \tag{5}
\]

Since the solution of (2) is well defined, the existence and uniqueness of the solution to (5) is also well guaranteed. Then according to Lemma 2 (in Appendix I), the rank-preserving property of the matrix flow (5) follows by observing \( B(t) = W^\top \) and \( A(t) = 0 \), which implies the dimensional-invariance property of the solutions to (2) in the sense of (3).

We now show a stronger result, that if initial conditions are chosen from some (affine) subspace, then the solutions of the coupled system (2) will always be in that (affine) subspace.
Corollary 1. In addition to the rank-invariance principle proved above, the solutions of the coupled dynamical systems (2) are subspace-preserving in the sense that

\[
\text{span}([x_1(t), x_2(t), \ldots, x_n(t)]) = \text{span}([x_1(0), x_2(0), \ldots, x_n(0)]) \quad \forall t \geq 0.
\]

Proof. The expression of the system equation \( \dot{X} \) in (5) satisfies the matrix differential equation in (20) with \( W(t) := B^T(t) \). Therefore, the statement is obvious as a direct consequence of Lemma 4 (in Appendix II).

B. Interpretations and implications

The system (2) is a very general form of coupled dynamical systems which encompass many control systems that have been actively studied in the literature. Examples include the distributed system for networked function computation [4], [5], multi-agent consensus systems initialled in [6] (undirected graphs), and developed in e.g. [7] (directed graphs) and [11] (time-varying couplings), and distributed formation control systems [1]–[3]. The results established in this section indicate that, the solutions for individual systems coupled in the form (2) will span a subspace of the same dimension as that spanned by initial conditions, and solutions will be constrained in that subspace over time.

The dimensional-invariance principle for a particular distance-based formation control system [2] has been proved in our previous paper [12]. We note that such a principle also holds for a large family of formation control systems, including those covered in [3]. In a later section we will show, by this example, how this invariance principle could assist our understanding on the evolutions of agents’ positions in a multi-agent formation system.

For some typical coupled dynamical systems with scalar couplings reported in the literature that can be described by the general form (2), see Table I in the Appendix. As a consequence of Theorem 1 and Corollary 1, all systems reviewed in Table I satisfy the dimensional-invariance and subspace-preserving property.

III. COUPLED DYNAMICAL SYSTEMS WITH MATRIX-WEIGHTED COUPLINGS

In this section we consider the following dynamical systems with matrix couplings

\[
\dot{x}_i(t) = \sum_{j=1}^{n} W_{ij}(t)x_j(t) = W_{i1}(t)x_1(t) + W_{i2}(t)x_2(t) + \cdots + W_{in}(t)x_n(t),
\]

where \( W_{ij} \in \mathbb{R}^{d \times d} \) is the state coefficient or coupling matrix: i.e., \( W_{ii} \) is the coefficient matrix for system \( i \), and \( W_{ij} \) is the coupling matrix from systems \( j \) to system \( i \).

A. Main results

The main result in this section is the following theorem:
Theorem 2. The coupled dynamical systems (7) have the dimensional-invariance principle in the sense of (3) if and only if the coefficient and coupling matrices $W_{ij}$ satisfy the following condition

$$W_{ii} = A + b_{ii}I_n, i = 1, 2, \ldots, n;$$
$$W_{ij} = b_{ji}I_n, i, j = 1, 2, \ldots, n, i \neq j,$$

for some matrix $A \in \mathbb{R}^{d \times d}$ and scalars $\{b_{ij}\}, i, j = 1, 2, \ldots, n$.

The proof can be found in the Appendix III. We note that in [9], an alternative but equivalent condition for coupled dynamical systems’ solutions to be collinear (or to be $r$-coplanar with dimension $r$) was obtained, via a somewhat more complicated proof.

The following corollary further characterizes the solution property for the case of $n$ coupled dynamical systems.

Corollary 2. Suppose the coupled system (7) consists of $n$ individual systems, and initial conditions $x(0)$ for all the coupled systems are chosen to satisfy $X(0) \in S(n)$ (i.e. the real symmetric matrix space). Then the coupled dynamical systems (7) have both the dimensional-invariance principle and signature-preserving property \(^1\) if and only if the coefficient and coupling matrices $W_{ij}$ satisfy the following condition

$$W_{ii} = A + a_{ii}I_n, i = 1, 2, \ldots, n;$$
$$W_{ij} = a_{ij}I_n, i, j = 1, 2, \ldots, n, i \neq j,$$

for some matrix $A = \{a_{ij}\} \in \mathbb{R}^{d \times d}$.

Proof. By invoking Lemma 3 (in Appendix II), the above condition can be proved by modifying $B$ in the proof of Theorem 2 as $A^\top$.

B. Interpretations and implications

The coupled dynamical systems (7) with matrix weights are also very general that can describe many different types of distributed/networked control systems. Examples include the matrix-weighted consensus dynamics [13], bearing-based formation control systems [14], or networked linear systems for synchronization [15].

To guarantee the invariance of the dimensions of the subspaces spanned by individual systems’ solutions, each individual system should have the same matrix structure in their state coefficients, with the difference being a scalar multiplier of an identity matrix. Furthermore, the couplings should also be a scalar multiplier of an identity matrix. Since the condition is necessary and sufficient, for other types of couplings between individual systems that are not in the forms of (8) and (9), the dimensional-invariance principles cannot be guaranteed.

We also note a difference of the invariance principles between the scalar-coupling case and the matrix-coupling case. As proved in Corollary 1, the solutions of coupled system (7) not only span a subspace of the same rank to that of their initial conditions, but also evolve in that particular subspace spanned by initial conditions. However,

\(^1\)The signature of a real symmetric matrix refers to the number (counted with multiplicity) of its positive, negative and zero eigenvalues.
this subspace-preserving property is not guaranteed for the solutions of the coupled system (7). Theorem 2 only shows the invariance of the dimension of the spanned subspace, while the solutions may also evolve in a different subspace with the same dimension. To be precise, we introduce the concept of Grassmannian subspace to illustrate the difference. The Grassmannian, denoted as \( \text{Gr}(r, d) \), is a space which parameterizes all linear subspaces of a given dimension \( r \) in a vector space \( V \) (in this paper, we restrict our attention of \( V \) to the \( d \)-dimensional Euclidean space \( \mathbb{R}^d \) [16, Page 21]. For example, for \( r = 1 \), the Grassmannian \( \text{Gr}(1, d) \) is the space of lines through the origin in the \( d \)-dimensional space, and it is the same as the projective space of \( d - 1 \) dimensions. For the solutions of coupled dynamical system (7), they will remain collinear if they start collinearly, but the line that passes through the solutions of all individual systems may not be identical over time. In other words, the solutions will evolve in \( \text{Gr}(r, d) \) if they start at a subspace of dimension \( r \).

For some typical coupled dynamical systems with matrix coefficients reported in the literature that can be described by the general form (7), see Table II in the Appendix.

IV. APPLICATIONS IN CONVERGENCE ANALYSIS FOR FORMATION CONTROL SYSTEMS WITH GENERALIZED CONTROLLERS

Consider a particular formation control system in the following form

\[
\dot{x}_i = -\sum_{j \in N_i}(\|x_i - x_j\|^2 - d_{k_{ij}}^2)(x_i - x_j), \quad i = 1, \ldots, n
\]  

(12)

where \( x_i \in \mathbb{R}^d \) is the position of agent \( i \) that lives in \( \mathbb{R}^d \), \( N_i \) denotes agent \( i \)’s neighboring set, and \( d_{k_{ij}} \) is the desired distance that agents \( i \) and agent \( j \) aim to achieve. In the literature, the above control system (12) is usually called distance-based formation control system [17], since the target formation shape is described by a set of interagent distances.

The collinearity-preserving property for the solutions of the formation control systems (12) was observed in several previous papers (e.g., see [1], [2], [18]). In [12], we have generalized this collinearity-preserving property, and proved a general dimensional-invariance principle for the formation control system (12). Inspired by the results in Theorem 1, one can also consider the following formation control systems with generalized controllers

\[
\dot{x}_i = -\sum_{j \in N_i} g_{ij}(e_{ij}(x_i, x_j))(x_i - x_j), \quad i = 1, \ldots, n
\]  

(13)

where \( g_{ij} \) is a continuous function of the distance error \( e_{ij}(x_i, x_j) \), which is defined as \( e_{ij} = \|x_i - x_j\|^2 - d_{k_{ij}}^2 \).

The local exponential stability of the general formation control system (13) has been discussed in [3].

The following results are direct consequences of Theorem 1 for system (13).

**Corollary 3.** For 2-D formations, if all the agents start with collinear positions, they will always be in that collinear subspace under the general control law described by (13). Similarly, for 3-D formations, if all the agents start with coplanar (resp. collinear) positions, then they will always be in that coplanar (resp. collinear) subspaces under the control law (13).
Conversely, one can also obtain the following dimensional-invariance principle for formation systems (13) with non-collinear/non-coplanar initial positions.

**Corollary 4.** For 2-D/3-D formations, if all the agents start with non-collinear/non-coplanar positions, they will always be non-collinear/non-coplanar under the general control law described by (13).

Figures 1 and 2 show intuitive explanations of the above two corollaries.

The global analysis of stability and convergence for the formation control system (12) has been discussed in several papers (e.g. [18], [3], [19]), which turns out to be a very challenging problem. The dimensional-invariance (and subspace-preserving) principles as shown in the above two corollaries will hopefully present additional insights for the convergence and stability analysis of general formation control systems (13). In addition, we can conclude that for any formation control system, if it can be written in the form of (2), then agents cannot escape collinear/coplanar positions if they start with collinear/coplanar positions. If one needs to design formation controllers to avoid such invariance property and to enable agents to escape collinear/coplanar positions even if they start collinearly/coplanarly, then one needs to modify the formation controllers such that they cannot be described by (2). For typical examples of formation control systems without the collinear/coplanar invariance property, see [20], [21].

**V. Conclusions**

In this paper we establish several invariance principles for coupled dynamical systems (with scalar couplings and with matrix couplings), in relation to the dimensions of the subspaces spanned by their individual solutions. For coupled dynamical systems with scalar couplings, we prove that their individual solutions satisfy the dimensional-invariance principle (and furthermore, the subspace-preserving principle). For coupled dynamical systems with
matrix coefficients/couplings, necessary and sufficient conditions are given to guarantee the dimensional-invariance principle. The interpretations and implications for the obtained invariance principles are also discussed, with an application to the convergence analysis of formation control systems.

APPENDIX I: BACKGROUND ON RANK-PRESERVING MATRIX FLOW

In this section we will briefly review some background on the rank-preserving flow theory [22, Chapter 5]. For integers \(1 \leq r \leq \min(M, N)\), let

\[
\mathcal{M}(r, M \times N) = \{X \in \mathbb{R}^{M \times N}\mid \text{rank}(X) = r\}
\]  

(14)

denote the set of real \(M \times N\) matrices of fixed rank \(r\). The following results will be useful in later analysis.

**Lemma 1.** \(\mathcal{M}(r, M \times N)\) is a smooth and connected manifold of dimension \(r(M + N - r)\), if \(\max(M, N) > 1\). The tangent space of \(\mathcal{M}(r, M \times N)\) at an element \(X\) is

\[
T_X\mathcal{M}(r, M \times N) = \{\Delta_1 X + X \Delta_2 \mid \Delta_1 \in \mathbb{R}^{M \times M}, \Delta_2 \in \mathbb{R}^{N \times N}\}. 
\]

(15)

The proof can be found in [22, Page 133]. A matrix differential equation \(\dot{X} = F(t, X)\) evolving on the matrix space \(\mathbb{R}^{M \times N}\) is said to be rank-preserving if the rank of every solution \(X(t)\) is constant as a function of \(t\), that is, \(\text{rank}(X(t)) = \text{rank}(X(0))\) for all \(t \geq 0\). The following lemma characterizes such rank-preserving flows (cf. Lemma 1.22 in Chapter 5 of [22]).

**Lemma 2.** Let \(I \subset \mathbb{R}\) be an interval and let \(A(t) \in \mathbb{R}^{M \times M}, B(t) \in \mathbb{R}^{N \times N}\) with \(t \in I\) be a continuous time-varying family of matrices. Then

\[
\dot{X}(t) = A(t)X(t) + X(t)B(t), \quad X(0) \in \mathbb{R}^{M \times N}
\]

(16)

is rank-preserving. Conversely, every rank-preserving differential equation on \(\mathbb{R}^{M \times N}\) is of the form (16) for matrices \(A(t)\) and \(B(t)\).

The proof of Lemma 2 is based on the fact that (16) defines a time varying vector field on the subset of the tangent space of \(\mathcal{M}(r, M \times N)\) described by (15). The full proof can be found in [22, Page 139].

**Remark 1.** Note that the above lemma on rank-preserving flows also implies that the limit value \(X(\infty)\) (if it exists) has rank less than or equal to \(\text{rank}(X(0))\). To avoid ambiguity, in this paper we only consider the case that \(I\) is a finite time interval. When we say \(t \geq 0\), we implicitly exclude the case of \(t = \infty\).

---

2 One typical example of \(\text{rank}(X(\infty)) < \text{rank}(X(0))\) comes from the formation control problem with unrealizable shapes [23]: If the triangle inequality does not hold for the desired distances in a triangular shape control problem, then all the agents will converge to a stable collinear equilibrium for which \(\text{rank}(X(\infty)) = 1\), even if they start with noncollinear positions with \(\text{rank}(X(0)) = 2\). Note that for such flows the rank-preserving property still holds for any finite time but at the limit \(t = \infty\) the rank reduces.
APPENDIX II: EXTENSIONS ON RANK-PRESERVING MATRIX FLOW

This section presents some extensions on the rank-preserving flow theory. The following lemma further characterizes rank-reserving flows on a symmetric matrix space. Let $S(N)$ denote the $N \times N$ real symmetric matrix space. For integers $r \in [1, N]$, let

$$S(r, N) = \{ X \in \mathbb{R}^{N \times N} | \text{rank}(X) = r \}$$

(17)

denote the set of real symmetric $N \times N$ matrices of fixed rank $r$.

**Lemma 3.** Let $I \subset \mathbb{R}$ be an interval and let $A(t) \in \mathbb{R}^{N \times N}$ with $t \in I$ be a continuous time-varying family of matrices. Then

$$\dot{X}(t) = A(t)X(t) + X(t)A^\top(t), \quad X(0) \in S(N)$$

(18)

is a rank-preserving (and hence signature-preserving) flow on $S(N)$. Conversely, every rank-preserving (and hence signature-preserving) differential equation on $S(N)$ is of the form (18).

**Proof.** The rank-preserving property of $X(t)$ is obvious from Lemma 2. The tangent space of $S(r, N)$ at an element $X$ is

$$T_X S(r, N) = \{ \Delta X + X \Delta^\top | \Delta \in \mathbb{R}^{N \times N} \}.$$  

(19)

Therefore (18) defines a time varying vector field on each subset of the tangent space of $S(r, N)$. Thus for any initial condition $X(0) \in S(N)$, the solution $X(t)$ of (18) satisfies $X(t) \in S(N)$, for $t \in I$. The signature-preserving property is therefore a direct consequence of the rank-preserving property and the fact that $X(t) \in S(N)$. Conversely, suppose $X(t)$ is rank-preserving and $X(t) \in S(N)$ (and therefore is signature-preserving). Then it defines a vector field $F(t, X)$ on $S(r, N)$, with $F(t, X) \in T_X S(r, N)$ as in (19). Letting $\Delta := A(t) \in \mathbb{R}^{N \times N}$ completes the proof. \hfill \Box

In the following we present a more refined principle, termed subspace-preserving principle, for matrix differential systems.

**Lemma 4.** Let $I \subset \mathbb{R}$ be an interval and let $B(t) \in \mathbb{R}^{N \times N}$ with $t \in I$ be a continuous time-varying family of matrices. Then

$$\dot{X}(t) = X(t)B(t), \quad X(0) \in \mathbb{R}^{M \times N}$$

(20)

is subspace-preserving in the sense that $\text{span}(X(t)) = \text{span}(X(0))$. Conversely, every subspace-preserving differential equation on $\mathbb{R}^{M \times N}$ is of the form (16) for some matrices $B(t)$.

**Proof.** We rewrite (20) as $\dot{X}^\top(t) = B^\top(t)X^\top(t)$, which has a unique solution given by $X^\top(t) = \Phi_{B^\top(t)}(t, 0)X^\top(0)$, where $\Phi_{B^\top(t)}(t, 0)$ is the state transition matrix associated with the coefficient matrix $B^\top(t)$ (see [10, Chapter 1.3]). Therefore, the solution to the system (20) can be written as $X(t) = X(0)\Phi_{B^\top(t)}^\top(t, 0)$. Since the state transition
matrix $\Phi^\top B^\top(t)$ is non-singular [10, Chapter 1.3], this implies that $\text{span}(X(t)) = \text{span}(X(0))$, $\forall t \in I$. For the converse statement, note that $\text{span}(X(t)) = \text{span}(X(0))$ implies that there exists a non-singular matrix $\Phi$ such that $X(t) = X(0)\Phi$. In the context of matrix differential equation, the transpose of the matrix $\Phi$ is the state transition matrix associated with a matrix $B^\top(t)$ in a matrix differential equation in the form of (20). 

\begin{remark}
Correspondingly, one can also show that a matrix differential equation in the form $\dot{X}(t) = A(t)X(t)$, $X(0) \in \mathbb{R}^{M \times N}$, where $A(t) \in \mathbb{R}^{M \times M}$ is a continuous matrix, is row-subspace-preserving, in the sense that $\text{span}(X^\top(t)) = \text{span}(X^\top(0))$. The proof is similar to that of Lemma 4 and is omitted here.
\end{remark}

\section*{Appendix III: Proofs of Theorem 2}

In this section we present the proof for Theorem 2.

\textbf{Proof.} Define $X = [x_1, x_2, \ldots, x_n] \in \mathbb{R}^{d \times n}$. We now determine conditions for the coefficient/coupling matrix $W_{ij}$ such that the coupled linear dynamical system (7) possesses the required dimensional-invariance property. From Lemma 2, this is equivalent to saying that the matrix differential system $\dot{X}$ should take the following form

$$\dot{X} = [\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n]$$

$$= A[x_1, x_2, \ldots, x_n] + [x_1, x_2, \ldots, x_n]B$$

(21)

for some $A \in \mathbb{R}^{d \times d}$ and $B = \{b_{ij}\} \in \mathbb{R}^{n \times n}$.

Expanding the expression of $\dot{X}$ in (21), one can obtain the equivalent formula in (23) (in the next page).

Note also that from (7) the matrix differential system (21) can be written as (24) (in the next page). In order to guarantee the dimensional-invariance principle, each coefficient term in the system (24) should take the identical form as in (23), which implies

$$W_{11} = A + b_{11}I_n,$$

$$W_{12} = b_{21}I_n,$$

$$W_{13} = b_{31}I_n,$$

$$\vdots$$

$$W_{ii} = A + b_{ii}I_n,$$

$$W_{ij} = b_{ji}I_n, i, j = 1, 2, \ldots, n, i \neq j,$$

$$\vdots$$

$$W_{nn} = A + b_{nn}I_n,$$

(22)

which is the necessary and sufficient condition to guarantee the dimensional-invariance property for the coupled dynamical system (7). \qed
\[ \dot{X} = A[x_1, x_2, \ldots, x_n] + [x_1, x_2, \ldots, x_n] \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} \]
\[ = [(A + b_{11})x_1 + b_{21}x_2 + \cdots + b_{n1}x_n, b_{12}x_1 + (A + b_{22})x_2 + \cdots + b_{n2}x_n, \ldots, b_{1n}x_1 + b_{2n}x_2 + \cdots + (A + b_{nn})x_n] \]  
\[ (23) \]

\[ \dot{X} = [\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n] = [W_{11}x_1 + W_{12}x_2 + \cdots + W_{1n}x_n, W_{22}x_2 + W_{21}x_1 + \cdots + W_{2n}x_n, \ldots, W_{n1}x_1 + W_{n2}x_2 + \cdots + W_{nn}x_n] \]  
\[ (24) \]

**TABLE I**
**COUPLED DYNAMICAL SYSTEMS THAT CAN BE DESCRIBED BY (2)**

| Ref. | Coupled/networked control systems | System dynamics equation | Coefficient/coupling term |
|------|-----------------------------------|--------------------------|----------------------------|
| [4], [5], etc. | Network distributed computation | $\dot{x}_i = \sum_{j \in N_i} w_{ij} x_j, \quad i = 1, \ldots, n$ | $w_{ij}$ |
| [6], [7], [11], etc. | Multi-agent consensus | $\dot{x}_i = \sum_{j \in N_i} a_{ij} (x_j - x_i), \quad i = 1, \ldots, n$ | $w_{ij} = a_{ij}, i \neq j$, $w_{ii} = -\sum_{j \in N_i} a_{ij}$ |
| $a_{ij}$ : weighted adjacency matrix including undirected/directed graphs, static/time-varying/switching topologies |
| [1], [2], [3], etc. | Distance-based formation shape control | $\dot{x}_i = -\sum_{j \in N_i} g_{ij} (x_i - x_j), \quad i = 1, \ldots, n$ | $w_{ij} = g_{ij}, i \neq j$, $w_{ii} = -\sum_{j \in N_i} g_{ij}$ |
| $e_{ij} = \|x_i - x_j\|^2 - d_{ij}^2$, $g_{ij} = e_{ij}$ or $g_{ij} = g_{ij}(e_{ij})$ |
| [24], [25], [26], etc. | Multi-agent coordination/affine formation/swarming | $\dot{x}_i = -\sum_{j \in N_i} u_{ij}(t,x)(x_i - x_j), \quad i = 1, \ldots, n$ | $w_{ij} = u_{ij}, i \neq j$, $w_{ii} = -\sum_{j \in N_i} u_{ij}$ |

**APPENDIX IV: A BRIEF REVIEW OF COUPLED SYSTEMS THAT CAN BE DESCRIBED BY (2) AND (7)**

We review and summarize in Table I and Table II several popular coupled dynamical systems reported in the vast literature, which can be described by (2) and (7), respectively. As a consequence of Theorem 1 and Corollary 1, for all the coupled or networked control systems with scalar couplings reviewed in Table I, dimensional-invariance (and furthermore, subspace-preserving) principles are guaranteed. For coupled/networked control systems reviewed in Table II, if the matrix condition in Theorem 2 is satisfied, then they also possess the dimensional-invariance property. For example, for the synchronization control of *identical* networked linear systems with matrix coefficients/couplings [15] (i.e., for the Type I system, with $A_i := A, \forall i$ and $W_{ij} = b_{ij}I_d$), the matrix condition of Theorem 2 is satisfied and the solutions of such networked control systems possess the dimensional-invariance principle. In contrast, for the Type II coupled systems for linear system synchronization, the condition (11) in Theorem 2 would be violated and thus the dimensional-invariance property is not guaranteed.
### References

1. B. D. O. Anderson and U. Helmke, “Counting critical formations on a line,” *SIAM Journal on Control and Optimization*, vol. 52, no. 1, pp. 219–242, 2014.

2. L. Krick, M. E. Broucke, and B. A. Francis, “Stabilisation of infinitesimally rigid formations of multi-robot networks,” *International Journal of Control*, vol. 82, no. 3, pp. 423–439, 2009.

3. Z. Sun, S. Mou, B. D. O. Anderson, and M. Cao, “Exponential stability for formation control systems with generalized controllers: A unified approach,” *Systems & Control Letters*, vol. 93, pp. 50–57, 2016.

4. Z. Costello and M. Egerstedt, “The degree of nonholonomy in distributed computations,” in *Proc. of the 53rd Annual Conference on Decision and Control (CDC)*, pp. 6092–6098, IEEE, 2014.

5. Z. Costello and M. Egerstedt, “From global, finite-time, linear computations to local, edge-based interaction rules,” *IEEE Transactions on Automatic Control*, vol. 60, no. 8, pp. 2237–2241, 2015.

6. R. Olfati-Saber and R. M. Murray, “Consensus problems in networks of agents with switching topology and time-delays,” *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1520–1533, 2004.

7. W. Ren, R. W. Beard, and E. M. Atkins, “Information consensus in multivehicle cooperative control,” *IEEE Control Systems*, vol. 27, no. 2, pp. 71–82, 2007.

8. W. L. Stutzman and G. A. Thiele, *Antenna theory and design*. John Wiley & Sons, 2012.

9. J. M. Montenbruck and S. Zeng, “Collinear dynamical systems,” in *Proc. of the 2017 American Control Conference*, 2017.

10. R. W. Brockett, *Finite dimensional linear systems*. SIAM, 1970.

11. S. Martin and A. Girard, “Continuous-time consensus under persistent connectivity and slow divergence of reciprocal interaction weights,” *SIAM Journal on Control and Optimization*, vol. 51, no. 3, pp. 2568–2584, 2013.

12. Z. Sun, U. Helmke, and B. D. O. Anderson, “Rigid formation shape control in general dimensions: an invariance principle and open problems,” in *Proc. of the IEEE 54th Annual Conference on Decision and Control (CDC)*, pp. 6095–6100, IEEE, 2015.

13. S. E. Tuna, “Synchronization under matrix-weighted laplacian,” *Automatica*, vol. 73, pp. 76–81, 2016.

14. S. Zhao and D. Zelazo, “Bearing rigidity and almost global bearing-only formation stabilization,” *IEEE Transactions on Automatic Control*, vol. 61, no. 5, pp. 1255–1268, 2016.

15. L. Scardovi and R. Sepulchre, “Synchronization in networks of identical linear systems,” *Automatica*, vol. 45, no. 11, pp. 2557–2562, 2009.

16. J. M. Lee, *Manifolds and differential geometry*, vol. 107. American Mathematical Society, 2009.

17. K.-K. Oh, M.-C. Park, and H.-S. Ahn, “A survey of multi-agent formation control,” *Automatica*, vol. 53, pp. 424–440, 2015.

18. K.-K. Oh and H.-S. Ahn, “Distance-based undirected formations of single-integrator and double-integrator modeled agents in n-dimensional space,” *International Journal of Robust and Nonlinear Control*, vol. 24, no. 12, pp. 1809–1820, 2014.
[19] X. Chen, M.-A. Belabbas, and T. Basar, “Global stabilization of triangulated formations,” *SIAM Journal on Control and Optimization*, vol. 55, no. 1, pp. 172–199, 2017.

[20] M.-C. Park, K.-K. Oh, and H.-S. Ahn, “Modified gradient control for acyclic minimally persistent formations to escape from collinear position,” in *Proc. of the IEEE 51st Annual Conference on Decision and Control (CDC)*, pp. 1423–1427, IEEE, 2012.

[21] H. Garcia de Marina, Z. Sun, M. Cao, and B. D. O. Anderson, “Controlling a triangular flexible formation of autonomous agents,” in *Proc. of the 20th IFAC World Congress*, 2017.

[22] U. Helmke and J. B. Moore, *Optimization and dynamical systems*. Springer, 1994.

[23] Z. Sun, S. Mou, U. Helmke, and B. D. O. Anderson, “Convergence analysis for rigid formation control with unrealizable shapes: The 3 agent case,” in *Proc. of the 2014 Australian Control Conference*, pp. 1–6, 2014.

[24] X. Chen, “Swarm aggregation under fading attractions,” *IEEE Transactions on Automatic Control*, DOI: 10.1109/TAC.2016.2626979, 2017.

[25] X. Chen, M. A. Belabbas, and T. Basar, “Controllability of formations over directed time-varying graphs,” *IEEE Transactions on Control of Network Systems*, DOI: 10.1109/TCNS.2015.2504034, 2016.

[26] Z. Lin, L. Wang, Z. Chen, M. Fu, and Z. Han, “Necessary and sufficient graphical conditions for affine formation control,” *IEEE Transactions on Automatic Control*, vol. 61, no. 10, pp. 2877–2891, 2016.

[27] M. H. Trinh and H.-S. Ahn, “Theory and applications of matrix-weighted consensus,” *arXiv preprint arXiv:1703.00129*, 2017.

[28] Z. Li, Z. Duan, G. Chen, and L. Huang, “Consensus of multiagent systems and synchronization of complex networks: A unified viewpoint,” *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 57, no. 1, pp. 213–224, 2010.

[29] P. Wieland, R. Sepulchre, and F. Allgöwer, “An internal model principle is necessary and sufficient for linear output synchronization,” *Automatica*, vol. 47, no. 5, pp. 1068–1074, 2011.