Some aspects of $N = (2, 2)$, $D = 2$ supersymmetry

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Abstract: The off-shell description of $(2,2)$ supersymmetric non-linear $\sigma$-models is reviewed. The conditions for ultra-violet finiteness are derived and T-duality is discussed in detail.

1 Introduction

The close relation between supersymmetry and complex geometry becomes particularly rich in two dimensions. We focus on non-linear $\sigma$-models with $N = (2, 2)$ supersymmetry. These are not only building blocks for type II superstrings, they describe the matter sector of $N = (2, 2)$ string theories as well.

Any bosonic non-linear $\sigma$-model can be extended to an $N = (1, 1)$ supersymmetric model. The model is completely determined once a target manifold, $M$, its metric, $g$, and a closed 3-form, $T$, are specified [1]. Classically, $(2, 2)$ supersymmetry requires the existence of two covariantly constant complex structures, which are such that the metric is hermitean w.r.t. both of them. These models do have an off-shell description which becomes manifest in $(2, 2)$ superspace. Such a formulation yields the surprising result that the local geometry of all manifolds which allow for $(2, 2)$ supersymmetry is encoded in a potential, the Lagrange density. This generalizes the most important property of Kähler manifolds to a large class of complex manifolds. The superspace description also facilitates the study of the ultra-violet properties of these models. Finally $T$-duality transformations can be studied while keeping the $(2, 2)$ supersymmetry manifest. The present paper reviews the off-shell description of $N = (2, 2)$ non-linear $\sigma$-models. Subsequently, we study their ultra-violet properties and finally the classical, abelian $T$-duality transformations are presented.

2 $N = (2, 2)$ non-linear $\sigma$-models

As already mentioned, $N = (2, 2)$ supersymmetry for a $\sigma$-model in $(1, 1)$ superspace is equivalent to the existence of two complex structures $J_+$ and $J_-$, which are covariantly constant, $\nabla^\pm J^a_{+b} = \nabla^\pm J^a_{-b} = 0$, where $\nabla^\pm$ denotes covariant differentiation using the $\Gamma^a_{\pm bc} \equiv \{^a_{bc}\} \pm T^a_{bc}$ connection, and which are such that the metric is hermitean w.r.t. both of them, $J_{\pm ab} = -J_{\pm ba}$. On-shell, one gets the standard $N = (2, 2)$ supersymmetry algebra, while off-shell closure is only achieved along $\ker[J_+, J_-] = \ker(J_+ - J_-) \oplus \ker(J_+ + J_-)$.  

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Building on the results of [2], it was argued in [3] that, in order to achieve a manifest $(2,2)$ supersymmetric description of these models, i.e. a formulation in $(2,2)$ superspace, chiral, twisted chiral and semi-chiral fields are sufficient. More explicitly: the tangent space at any point of the target manifold can be decomposed into three subspaces: $\ker(J_+ - J_-) \oplus \ker(J_+ + J_-) \oplus (\ker[J_+, J_-])^\perp$. These subspaces are conjectured to be integrable to chiral, twisted chiral and semi-chiral coordinates respectively.

In order to obtain the off-shell formulation of these models, we describe it in $N=(2,2)$ superspace. We call its coordinates $x^a$, $\bar{x}^{\dot{a}}$; $a, \dot{a} \in \{1, \ldots, d_e\}$. The fermionic derivatives satisfy

\begin{align*}
D_+ z^a &= D_- \bar{z}^{\dot{a}} = D_\perp \bar{z}^{\dot{a}} = 0. \\
D_+ w^m &= D_\perp \bar{w}^{\tilde{m}} = D_- \bar{w}^{\tilde{m}} = 0. \\
D_+ \bar{r}^a &= D_- \bar{s}^{\dot{a}} = D_\perp \bar{s}^{\dot{a}} = 0. \\
D_+ \bar{s}^{\dot{a}} &= D_- \bar{r}^a = D_\perp \bar{r}^a = 0.
\end{align*}

Counting components, one finds that chiral and twisted chiral fields contain as many components as a $(1,1)$ superfield, while semi-chiral fields have twice as many, half of which turn out to be auxiliary. This is not so surprising as chiral and twisted chiral superfields parametrize $\ker(J_+ - J_-)$ and $\ker(J_+ + J_-)$ resp., where the $N=(2,2)$ algebra closes off-shell. On the other hand, semi-chiral fields parametrize $(\ker[J_+, J_-])^\perp$ where there is no off-shell closure and where additional auxiliary fields should be introduced. Note that there is also a complex linear superfield $f$ satisfying the quadratic constraint $D_+ D_- f = D_\perp D_\perp f = 0$ and a twisted complex linear superfield $g$ defined by $D_+ D_- g = D_\perp D_\perp g = 0$. They can be represented in terms of semi-chiral fields, $f = r + \bar{s}$ and $g = r' + \bar{s}'$. Potentials which depend on semi-chiral fields through these particular combinations were excluded in the analysis in [3], as the resulting models exhibit gauge invariances. Aspects of semi-chiral models with gauge invariances were studied in [4]. Complex linear and twisted linear fields are alternatives to chiral and twisted chiral fields resp. A detailed study of their properties is given in [5].

The action in $(2,2)$ superspace is

$$S = \int d^2 x d^4 \theta K(z, \bar{z}, w, \bar{w}, r, \bar{r}, s, \bar{s}),$$

with $K$ a real potential. The potential is determined modulo a generalized Kähler transformation, $K \simeq K + f(z, w, r) + \bar{f}(\bar{z}, \bar{w}, \bar{r}) + g(z, \bar{w}, \bar{s}) + \bar{g}(\bar{z}, w, s)$. Starting from eq. (1), one passes to $N=(1,1)$ superspace. Upon elimination of the auxiliary fields, one finds expressions for the metric, torsion and complex structures in terms of the potential $K$. Various explicit examples are known. Kähler manifolds are described using chiral fields only. The $SU(2) \times U(1)$ Wess-Zumino-Witten (WZW) model can be described either in terms of a chiral and a twisted chiral multiplet or in terms of one semi-chiral multiplet depending on the choice one makes for the complex structures. The WZW model on $SU(2) \times SU(2)$ is described in terms of a semi-chiral and a chiral field. Hyper-Kähler manifolds too can be described in terms of semi-chiral coordinates. Indeed, choosing $J_+$ and $J_-$ such that $\{J_+, J_-\} = 0$, one gets $\ker[J_+, J_-] = \emptyset$.

3 Ultra-violet properties

Requiring conformal invariance severely restricts the allowed potentials $K$ in eq. (1). The one loop $\beta$-function of a general non-linear $\sigma$-model vanishes if $\nabla^a \nabla_b \Phi = 0$, with $\Phi$ the dilaton and
$R^+_{ab}$ is the Ricci tensor computed using the $\Gamma_+$ connection. When only chiral and twisted chiral fields are present, the explicit expressions for metric and torsion, obtained from eq. $[3]$, are used in the analysis of the $\beta$-function $[12]$. Once semi-chiral fields enter the game, the expressions for metric and torsion (see e.g. $[3]$ or $[4]$) are so complicated that this program becomes technically unfeasable. The way out is to recompute the $\beta$-functions, but now directly in $(2,2)$ superspace. At present, there is no good understanding of the dilaton in the presence of semi-chiral superfields, so we limit ourselves to the study of a necessary condition for conformal invariance: ultra-violet finiteness.

Computing the one-loop counterterms using the background field formalism and the techniques described in $[3]$ and $[4]$, is quite standard. Full details are given in $[12]$. We first consider a potential which only depends on $d_s$ semi-chiral fields, $K(r, \bar{r}, s, \bar{s})$. The one-loop counterterm reads

$$S^{(1)} = \frac{1}{2\pi \bar{\varepsilon}} \int d^2x d^4 \theta \ln \frac{\det N_2}{\det (\sqrt{-3 N_1})},$$  

where

$$N_1 \equiv \begin{pmatrix} K_{\alpha \beta} & K_{\alpha \beta} \\ K_{\bar{\alpha} \bar{\beta}} & K_{\bar{\alpha} \bar{\beta}} \end{pmatrix}, \quad N_2 \equiv \begin{pmatrix} K_{\alpha \beta} & K_{\alpha \beta} \\ K_{\bar{\alpha} \bar{\beta}} & K_{\bar{\alpha} \bar{\beta}} \end{pmatrix}. \quad (3)$$

In $[3]$, it was shown that non-degeneracy of the target manifold metric is equivalent to $\det N_1 \neq 0$ and $\det N_2 \neq 0$. From eq. $[3]$ we get the condition for UV finiteness at one loop:

$$\det N_2 = (-1)^{d_s} |F(r)|^2 |G(s)|^2 \det N_1, \quad (4)$$

where $F$ and $G$ are arbitrary functions of $r$ and $s$ resp. Note that there is no coordinate transformation compatible with the constraints, which can remove $|F(r)|^2 |G(s)|^2$.

One checks this result by working through some examples which are known to be UV finite. The WZW model on $SU(2) \times U(1)$ is described by a semi-chiral multiplet $[3, 2]$ with potential

$$K = -(r + \bar{s})(\bar{r} + s) + \frac{1}{2}(s + \bar{s})^2 - 2 \int d^4 x \ln(1 + \exp \frac{x}{2}), \quad (5)$$

and we find $F = 1$ and $G = \exp(-s/2)$. Another class of interesting examples are the 4-dimensional hyper-Kähler manifolds where $J_+$ and $J_-$ are chosen to be anti-commuting. The potential then satisfies $[3] |K_{rs}|^2 + |K_{sr}|^2 = 2K_{s\bar{s}}K_r \bar{r}$ so that eq. $[3]$ is satisfied with $F = G = 1$.

Finally, one can also repeat the calculation for a general potential, eq. $[1]$, which depends on all three types of superfields. One finds the counterterm

$$S^{(1)} = \frac{1}{2\pi \bar{\varepsilon}} \int d^2x d^4 \theta \ln \frac{\det(-K_{\bar{m} \bar{n}})}{\det K_{\bar{a} \bar{b}} \det(\sqrt{-3 N'_1})}, \quad (6)$$

where $N'_1$ and $N'_2$ are similar to $N_1$ and $N_2$ in eq. $[3]$ except that in $N'_1$ one has to write the $d_s \times d_s$ matrices $(K_{AB}) - (K_{Ab})(K_{ab})^{-1}(K_{bA})$ instead of the matrices $K_{AB}$, while in $N'_2$ one has $(K_{AB}) - (K_{Am})(K_{mn})^{-1}(K_{nB})$ instead of $K_{AB}$. In these expressions, the capital letters denote the indices appearing in eq. $[3]$, while the small indices denote derivatives w.r.t. chiral or twisted chiral fields, following the notation introduced in the definition of these fields. Again, this result can be verified using a non-trivial example, e.g. using the potential which describes the $SU(2) \times SU(2)$ WZW model in terms of a semi-chiral and a chiral multiplet $[4]$.

### 4 Duality transformations

The study of $T$-duality in $N = (2,2)$ superspace has the obvious advantage that the extended supersymmetry remains manifest in the dual model. Several aspects of dualities in $(2,2)$ superspace have been
previously studied. We refer the reader to [3] and the references therein. We present here a different approach starting from semi-chiral fields and imposing further conditions on the prepotentials or assuming the existence of certain isometries.

We begin with the first order lagrangian

$$K(V, \bar{V}, W, \bar{W}, \cdots) = fV - \bar{f}\bar{V} - sW - \bar{s}\bar{W},$$

(7)

where $V$ and $W$ are unconstrained complex prepotentials, $r$, $s$ are semi-chiral fields and the dots stand for any kind of other superfields which do not participate in the duality transformation. When no isometries are present, one gets four dual formulations of the model, depending on whether one integrates over $r$ and $s$, over $V$ and $W$, over $r$ and $W$ or over $s$ and $V$.

Imposing the constraint $V = W$ in eq. (7) yields $K(V, \bar{V}) = fV - \bar{f}\bar{V}$, where we identified the complex linear field $f = r + \bar{s}$. This is the first order lagrangian for the well known duality which interchanges a chiral for a complex linear superfield. Alternatively, assuming the existence of an isometry such that the potential in eq. (7) depends only on the prepotentials through the combination $V + W$, we can rewrite eq. (7) as $K(V + W, V + W, \cdots) - \frac{1}{2}(r + \bar{s})(V + W) - \frac{1}{2}(\bar{r} + s)(V + W) - \frac{1}{2}(r - \bar{s})(V - W) - \frac{1}{2}(\bar{r} - s)(V - W)$. Integrating over $V - \bar{W}$, requires that $r = \bar{s}$, which implies that $r$ is a chiral field $z$. Calling $V' \equiv V + W$, we get the first order lagrangian $K(V', V', \cdots) - zV' - \bar{z}\bar{V}'$, which makes it possible to pass from a complex linear field to a chiral field. The twisted version of these dualities are obtained by interchanging the roles of $W$ and $V$. This time a twisted chiral and a twisted complex linear superfield will be dual dual to each other.

Continuing to restrict our class of models, we assume an isometry in the previously considered potential, $K(V + \bar{V}) = fV - \bar{f}\bar{V}$, we perform the integration over $V - \bar{V}$ and obtain that $f$ is real. Consistency of the constraints gives $f = f = w + \bar{w}$. Introducing the real prepotential $V' \equiv V + W$, we get the first order lagrangian $K(V', \cdots) - (w + \bar{w})V'$, which makes it possible to pass from a chiral to a twisted chiral field. Both descriptions have an Abelian isometry. On the other hand, the lagrangian $K(V, \bar{V}, \cdots) - zV - \bar{z}\bar{V}$ allows for a further reality constraint on the prepotential: $V = \bar{V}$ and we get $K(V, \cdots) - (z + \bar{z})V$. The resulting model allows one to pass from a twisted chiral to a chiral field.

Finally, the potential in eq. (7) can allow for other constraints or have more subtle isometries. Requiring that the prepotentials in eq. (7) satisfy $V + \bar{V} = W + \bar{W}$, we can parametrize the lagrangian by $K(V' + W', V' + W', V' + W', V' + W', \cdots) - r(V' + W') - \bar{r}(\bar{V}' + \bar{W}') - s(V' + W') - \bar{s}(\bar{V}' + \bar{W}')$. Integrating over the chiral fields forces $V'$ to be chiral, $V' = z$, and $W'$ to be twisted chiral, $W' = w$. Indeed the equations of motion for $r$ and $s$, $D_+(V' + W') = 0$ and $D_-(V' + W') = 0$, imply that $D_+(V' - \bar{V}') = D_+(V' - \bar{V}') = D_+(W' - \bar{W}') = D_+(W' - \bar{W}') = 0$. The resulting potential is given by $K(z + w, \bar{z} + \bar{w}, \bar{z} + w, \cdots)$ and exhibits an Abelian isometry, $z \to z + \varepsilon, w \to w - \varepsilon$ with $\varepsilon$ a real constant. Integrating first over the prepotentials, yields the dual model with a potential of the form $K_{dual}(r - \bar{r}, r + \bar{s}, \bar{r} + s, \cdots)$, which again has an Abelian isometry.

The inverse transformation can be obtained by requiring that the potential in eq. (7) has an isometry $K(V + \bar{V}, V + W, \bar{V} + W, V + W, \cdots)$. Integrating over the semi-chiral fields yields the model in terms of a semi-chiral multiplet with potential, $K(r + \bar{r}, r + \bar{s}, \bar{r} + s, \cdots)$. In order to obtain the dual model, we rewrite eq. (7) as $K(x, u, u, \cdots) - x(r + \bar{r} - s - \bar{s})/2 - iy(r - \bar{r} + s - \bar{s})/2 - us - \bar{u}s$, where $x \equiv V + \bar{V}, y \equiv -i(V - \bar{V}), u \equiv \bar{V} + W$ and $\bar{u} \equiv V + W$. Integrating over $y$ requires that $r + s$ is real. Compatibility of this with the constraints on $r$ and $s$ gives that we can express $r$ and $s$ in terms of a chiral and a twisted chiral field: $r = z + w$ and $s = \bar{z} - w$. Subsequent integration over $x, u$ and $\bar{u}$ gives the dual potential $K_{dual}(w + \bar{w}, \bar{z} - w, z - \bar{z}, \cdots)$.

Though a detailed discussion of the geometric properties of these duality transformations will be given elsewhere, we do give here some explicit examples. Particularly interesting is the $SU(2) \times U(1)$ WZW model. Using the description in terms of a chiral and a twisted chiral field with potential $K = \frac{1}{2}(z + \bar{z})^2 - f w + \bar{w} - \bar{z} - z dx \ln(1 + e^x)$, we get by dualizing the twisted chiral field $K_{dual} = z'\bar{z}' - \frac{1}{2}(z + \bar{z})^2 - \frac{1}{2}(z + \bar{z})^2 - \frac{1}{2}(z + \bar{z})^2 - y f' dx \ln(1 + e^x)$, while dualizing the chiral field yields $K_{dual}' = -w'\bar{w}' + \frac{1}{2}(w + \bar{w})^2 - (w + \bar{w})' y' f' dx \ln(1 + e^x)$, where $y = \ln(e^{-z - \bar{z}} - 1)$ and $y' = \ln(e^{w + \bar{w}} - 1)$. In both cases, we obtain a free field and the disk $SU(2)/U(1)$. The two descriptions of $SU(2)/U(1)$ are each others dual.
Turning to the semi-chiral description of this model, eq. (5), we notice that it has an Abelian isometry of the type previously discussed. This allows us to dualize it to a chiral and a twisted field with potential
\[
K_{\text{dual}} = z \bar{z} + 2 \ln(1 - e^{w + \bar{w}})^2 - 2 \int (w + \bar{w} - 2 \ln(1 - e^{w + \bar{w}})) dx \ln(1 + e^x).
\]
Again we notice a factorization of the dual model into \((U(1))^2 \times SU(2)/U(1)\).

5 Conclusions

The off-shell description of \(N = (2, 2)\) non-linear \(\sigma\)-models seems now to be under control, though a precise geometric characterization of the semi-chiral sector is still lacking. This would have particularly interesting applications. E.g. as all hyper-Kähler manifolds allow for a semi-chiral description, we obtain a potential, not the Kähler potential, which allows for the computation of not only the metric, but all three, mutually anti-commuting complex structures as well.

Our formulation reveals a rich variety of duality transformations with the additional bonus that \((2, 2)\) supersymmetry remains manifest. We summarize how the various duality transformations act on superfields.

1. Dualities without isometries
   - 4 dual formulations of a semi-chiral multiplet
   - complex linear ↔ chiral
   - complex twisted linear ↔ twisted chiral

2. Dualities with isometries
   - chiral ↔ twisted chiral
   - semi-chiral ↔ 1 chiral + 1 twisted chiral

The previous analysis was purely classical. In order to get the quantum mechanical properties of these duality transformations, a manifest \((2, 2)\) supersymmetric description of the dilaton coupling is needed. This necessitates a good understanding of \(N = (2, 2)\) supergravity and its coupling to chiral, twisted chiral and semi-chiral superfields. In [14], \(N = (2, 2)\) supergravity was investigated for backgrounds consisting of chiral and twisted chiral superfields. It was found that the geometry of the \((2, 2)\) super worldsheet implies the existence of four types (anti-)chiral and twisted (anti-)chiral) of worldsheet curvatures, which couple to fields satisfying the same constraints. A generalization of this in the presence of semi-chiral superfields is presently under study. Though the ultra-violet properties of \((2, 2)\) non-linear \(\sigma\)-models are well understood, a good control over the dilaton would also enable one to investigate the superconformal invariance of these models in detail.

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