New formulation of \((g - 2)_\mu\) hadronic contribution

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In frames of agreement to consider the annihilation of electron-positron pair to hadrons cross section to be including the virtual photon polarization effects a new formulation of hadron contribution to muon anomalous magnetic moment is suggested. It consists in using the experimentally observed cross section converted with the known kernels. The lowest order kernel remains to be the same but some modification of radiative corrected kernel is needed. The explicit form of this new kernel is given. We estimate the accuracy of new formulation on the level \(\delta a_{\mu, \text{hadr}} / a_{\mu, \text{hadr}} \sim 10^{-5}\).

I. MOTIVATION

Anomalous magnetic moment of muon \(a_{\mu}\) is very sensitive laboratory to search new physics beyond the Standard Model (SM) (see \[1\] and the references therein). However before driving any premature conclusions about new physics some caution at the level of precision required of hadronic uncertainties is SM should be paid \[2\]. The estimation of theoretical and experimental ones becomes very important.

It is the motivation of our paper to suggest a new, more natural form of inclusion of hadronic vacuum polarization effects. The theoretical as well the systematic experimental uncertainties we expect to be considerably reduced.

The SM contributions are usually split into three parts: \(a_{\mu} = a_{\mu}^{QED} + a_{\mu}^{EW} + a_{\mu}^{\text{hadr}}\). The part of \(a_{\mu}^{\text{hadr}}\) taking only vacuum polarization effects (we don’t consider hadronic contributions of light-by-light type) usually is presented in form (see for instance \[3\] and references therein):

\[
a_{\mu}^{\text{hadr}} = \frac{1}{3} \left( \frac{\alpha}{\pi} \right)^2 \int_{4m_e^2}^{\infty} \frac{ds}{s} R(s) \left[ K^{(1)}(s) + \frac{\alpha}{\pi} K^{(2)}(s) \right],
\]

\[1\]

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with

\[ K^{(1)}(s) = \int_0^1 dx \frac{x^2(1-x)}{x^2 + \rho(1-x)}, \quad \rho = \frac{s}{M^2}, \tag{2} \]

where \( M \) is the muon mass and

\[ R(s) = \frac{\sigma_{e^+e^-\rightarrow hadr}(s)}{\sigma_{e^+e^-\rightarrow \mu^+\mu^-}(s)} = 12\pi Im_h \Pi(s), \quad \sigma_{e^+e^-\rightarrow \mu^+\mu^-}(s) = \frac{4\pi\alpha^2}{3s}. \tag{3} \]

The quantity \( \sigma_{e^+e^-\rightarrow hadr}(s) \) which enters the quantity \( R(s) \) is rather unphysical one as well as it does not takes into account the effects of vacuum polarization of virtual photon \( (e^+ + e^- \rightarrow \gamma^* \rightarrow hadr) \). The physical one can be obtained by the replacement

\[ Im_h \Pi(s) \rightarrow Im_h \left( \frac{\Pi(s)}{1 - \Pi(s)} \right) = \frac{Im_h \Pi(s)}{|1 - \Pi(s)|^2}, \quad \Pi(s) = \Pi_l(s) + \Pi_h(s), \tag{4} \]

where \( \Pi_l(s), \Pi_h(s) \) are leptonic and hadronic contributions to the vacuum polarization operator. Namely the quantity \( \sigma_{exp. e^+e^-\rightarrow hadr}(s) \), defined as:

\[ \sigma_{exp. e^+e^-\rightarrow hadr}(s) = \frac{\sigma_{e^+e^-\rightarrow hadr}(s)}{|1 - \Pi(s)|^2}, \tag{5} \]

is more relevant to experiment, contrary to Born one \( \sigma_{e^+e^-\rightarrow hadr}(s) \). In the region of narrow resonances the application of this formula must be performed with some care [4].

II. SECOND ORDER KERNEL MODIFICATION

Keeping this definition in mind one must revise the formulae for \( a_{\mu}^{hadr} \), cited above. Really, one must replace in integrands of \( a_{\mu}^{hadr} : \sigma_0^{e^+e^-\rightarrow hadr}(s) \rightarrow \sigma_{exp.}^{e^+e^-\rightarrow hadr}(s) \). Kernel \( K^{(1)}(s) \) remains the same, but the kernel \( K^{(2)}(s) \) must be modified to avoid the double counting. The modification consists in eliminating of contributions of all Feynman diagrams containing two polarization of vacuum insertions (both hadronic, leptonic sort and the mixed ones). It results in omitting the contributions of \( K^{(2b,2c)}(s) \) in terminology of ref. [3]. As for \( K^{(2a)}(s) \) it must be modified, in such a way to extract the contribution of such Feynman diagram (see Fig. 1).
which contains polarization operator for muon case with hadronic one. So our result consists in replacement $K^{(2a)}(s)$ 

$$
K^{(2a)}(s) = 2 \left\{ \frac{-139}{144} + \frac{115}{72} \rho + \left( \frac{19}{12} - \frac{7}{36} \rho + \frac{23}{144} \rho^2 + \frac{1}{\rho - 4} \right) L + \frac{1}{\Delta} \left( -\frac{4}{3} + \frac{127}{36} \rho - \frac{115}{72} \rho^2 + \frac{23}{144} \rho^3 \right) \ln y + \left( \frac{9}{4} + \frac{5}{24} \rho - \frac{1}{2} \rho^2 - \frac{2}{\rho} \right) \xi_2 + \frac{5}{96} \rho^2 L^2 + \frac{1}{\Delta} \left( -\frac{1}{2} \rho + \frac{17}{24} \rho^2 - \frac{7}{48} \rho^3 \right) L \ln y + \left( \frac{19}{24} + \frac{53}{48} \rho - \frac{29}{96} \rho^2 - \frac{1}{3} \rho + \frac{2}{\rho - 4} \right) \ln^2 y + \frac{1}{\Delta} \left( -2 \rho + \frac{17}{6} \rho^2 - \frac{7}{12} \rho^3 \right) D_p(\rho) + \frac{1}{\Delta} \left( \frac{13}{3} - \frac{7}{6} \rho + \frac{1}{4} \rho^2 - \frac{1}{6} \rho^3 - \frac{4}{\rho - 4} \right) D_m(\rho) + \left( \frac{1}{2} - \frac{7}{6} \rho + \frac{1}{2} \rho^2 \right) T(\rho) \right\}, \quad (6)
$$

with $L = \ln(s/M^2)$, $\Delta = \sqrt{\rho(\rho - 4)}$, $\xi_2 = \pi^2/6$ and

$$
y = \frac{\sqrt{\rho} - \sqrt{\rho - 4}}{\sqrt{\rho} + \sqrt{\rho - 4}},
$$

$$
D_p(\rho) = Li_2(y) + \ln y \ln(1 - y) - \frac{1}{4} \ln^2 y - \xi_2,
$$

$$
D_m(\rho) = Li_2(-y) + \frac{1}{4} \ln^2 y + \frac{1}{2} \xi_2,
$$

$$
T(\rho) = -6Li_3(y) - 3Li_3(-y) + \ln^2 y \ln(1 - y) + \frac{1}{2} (\ln^2 y + 6\xi_2) \ln(1 + y) + 2 \ln y (Li_2(-y) + 2Li_2(y)),
$$

$$
Li_2(y) = -\int_0^y \frac{dx}{x} \ln(1 - x), \quad Li_3(y) = \int_0^y \frac{dx}{x} Li_2(x), \quad (7)
$$

by the new one:

$$
K^{(2)}(s) = K^{(2a)}(s) - K^{(2b)}(s) \bigg|_{m_f = M}, \quad (8)
$$

with

$$
K^{(2b)}(s)_{m_f = M} = 2 \int_0^1 dx \frac{x^2(1 - x)}{x^2 + \rho(1 - x)} \Pi(1, x),
$$

$$
\Pi(1, x) = \frac{8}{9} + \frac{b^2}{3} - b \left( \frac{1}{2} - \frac{b^2}{6} \right) \ln b - \frac{1}{b + 1},
$$

$$
b = \frac{2 - x}{x}. \quad (9)
$$

The quantity $K^{(2b)}(s)_{m_f = M}$ can be calculated analytically:

$$
K^{(2b)}(s)_{m_f = M} = 2 \rho \left[ \frac{8}{9} \rho^2 + \frac{35}{36} \rho - \frac{4}{3} \xi_2 - \frac{1}{\Delta} [L_- P_1(x_-) - L_+ P_1(x_+) - \frac{1}{\Delta} [L_- P_2(x_-) - L_+ P_2(x_+)] \right], \quad (10)
$$
with \( x_\pm = (\rho \pm \Delta)/2, \Delta = \sqrt{\rho(\rho - 4)} \) and

\[
L_\pm = \ln \frac{x_\pm}{x_\mp - 1}, \quad Li_\pm = Li_2(1 - x_\mp),
\]

\[
P_1(z) = -\frac{5}{9}z^4 - \frac{2}{3}z^3 + \frac{2}{3}z^2, \quad P_2(z) = \frac{1}{3}z^4 - 2z^2 + \frac{4}{3}z. \tag{11}
\]

For expansion into series by powers of \( \rho^{-1} \) we have:

\[
\bar{K}^{(2)}(s) = 2 \frac{1}{\rho} \left[ \bar{a}_1 + \bar{b}_1 L + \frac{1}{\rho} (\bar{a}_2 + \bar{b}_2 L + \bar{c}_2 L^2) + \frac{1}{\rho^2} (\bar{a}_3 + \bar{b}_3 L + \bar{c}_3 L^2) + \right. \]

\[
\left. + \frac{1}{\rho^3} (\bar{a}_4 + \bar{b}_4 L + \bar{c}_4 L^2) + \frac{1}{\rho^4} (\bar{a}_5 + \bar{b}_5 L + \bar{c}_5 L^2) \right] + O(\rho^{-6}), \tag{12}
\]

with

\[
\begin{align*}
\bar{a}_1 & = \frac{50}{27} - \frac{2}{3} \xi_2, \\
\bar{a}_2 & = \frac{9241}{1152} - \frac{103}{24} \xi_2, \\
\bar{a}_3 & = \frac{15256601}{432000} - \frac{803}{40} \xi_2, \\
\bar{a}_4 & = \frac{64452261}{432000} - \frac{10829}{120} \xi_2, \\
\bar{a}_5 & = \frac{18433084459}{2783000} - \frac{19877}{35} \xi_2.
\end{align*}
\]

\[
\begin{align*}
\bar{b}_1 & = -\frac{23}{36}, \\
\bar{b}_2 & = -\frac{487}{216}, \\
\bar{b}_3 & = -\frac{29279}{3600}, \\
\bar{b}_4 & = -\frac{57917}{1800}, \\
\bar{b}_5 & = -\frac{34443349}{264000},
\end{align*}
\]

\[
\begin{align*}
\bar{c}_2 & = \frac{43}{144}, \\
\bar{c}_3 & = \frac{221}{80}, \\
\bar{c}_4 & = \frac{3763}{240}, \\
\bar{c}_5 & = \frac{47651}{630}.
\end{align*}
\]

So our final result have a form for hadronic contribution to anomalous magnetic moment of muon is:

\[
a_{\mu\text{had}} = \frac{1}{3} \left( \frac{\alpha}{\pi} \right)^2 \int \frac{ds}{s} R_{\text{exp.}}^h(s) \left[ K^{(1)}(s) + \frac{\alpha}{\pi} \bar{K}^{(2)}(s) \right], \tag{14}
\]

where \( R_{\text{exp.}}^h(s) = \sigma_{\text{exp.}}^{e^+e^\to\text{had}}(s)/\sigma_{\text{exp.}}^{e^+e^\to\mu^+\mu^-}(s) \) and \( \bar{K}^{(2)}(s) \) is given above (see (8), (12)).

### III. DISCUSSION

The set of FDs contribution with lepton and hadron vacuum polarization associated with different virtual photon lines cannot be considered with the method discussed above. Their contribution (see Fig. 2) enhanced by logarithmical factor can be estimated as \( \delta a_{\mu\text{had}} \sim (\alpha/\pi)^2 (1/3) \ln M^2/m_e^2 \approx 2 \cdot 10^{-5} a_{\mu\text{had}} \). Fortunately this is beyond the modern experimental possibilities.

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Fig. 2: Typical contribution with lepton and hadron vacuum polarization associated with different virtual photon lines enhanced by logarithmical factor.

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