Eilenberg swindles and higher large scale homology of products

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Abstract

We present a geometric method of proving that large scale homology groups of Cartesian products of non-compact spaces vanish and use it to obtain a number of vanishing result for several large scale homology theories in higher degrees. Our method applies for instance to uniformly finite homology, controlled coarse homology, almost equivariant homology and locally finite homology, where the standard Künneth-type theorems are not available. As applications we determine the uniformly finite homology of lattices in products of 2 and 3 trees, show a characterization of amenability in terms of 1-homology and construct aperiodic tilings using higher homology.

1 Introduction

Uniformly finite homology is a coarse homology theory for non-compact metric spaces introduced by Block and Weinberger [2]. It has several interesting applications, in particular in [2] it was shown that the vanishing of the uniformly finite homology in degree 0 characterizes amenability. This fact

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was further applied to construct aperiodic tiles and metrics of positive scalar curvature. Later, in [20], uniformly finite homology was used to prove a geometric version of the von Neumann conjecture. It is also used to characterize those quasi-isometries that are close to bijections, see [7, 20].

While the vanishing in degree 0 is relatively well-understood, uniformly finite homology \( H_{uf}^n \) in higher degrees \( n \geq 1 \), essentially remains uncharted territory. The only results known in this direction are discussed in [3], and include symmetric spaces, non-vanishing results for amenable groups based on the infinite transfer, and recently in [1], where it was shown that higher uniformly finite homology of amenable groups is usually infinite-dimensional.

The main result of this paper, motivated initially by the above problem, is a geometric method for killing homology classes of Cartesian products. Although originally designed for uniformly finite homology, it turned out flexible enough to apply to several large scale homology theories:

(i). uniformly finite homology of Block and Weinberger and its fine simplicial version [2],

(ii). controlled coarse homology of Špakula and the second author [16],

(iii). locally finite homology and Roe’s coarse homology [18],

(iv). and Dranishnikov’s almost equivariant homology [5].

The above homology theories have many applications in group theory, geometric topology and index theory. They are often used to express largeness of manifolds, see [5, 6, 10, 11, 12].

It is worth noting that for any of the above homology theories the homological algebra behind the classical Künneth theorem does not generalize naturally. Indeed, the chains, cycles and boundaries all form infinite-dimensional spaces. These spaces sometimes, as in (i) and (ii), can be equipped with an analytic structure, but in the cases (iii) and (iv) no reasonable analytic structures seem to exist. In such settings tensor products, naturally appearing in Künneth-type theorems, exhibit fundamental difficulties. At present there are no such Künneth-type theorems for any of the above homology theories (i)-(iv).

Our approach is geometric. The main ingredients are 2- and 3-dimensional Eilenberg swindles that we attach to a given cycle. This strategy allows us to gradually reduce any cycle on the product to a cycle of a specific form, representing the same homology class. The final step shows that the cycles of such specific form bound. The general idea is the following. If \( X \) is a locally finite simplicial complex and \( h_\ast \) denotes one of the homology theories mentioned
then by \([X] \in h_0(X)\) we denote the fundamental class of \(X\), represented by the 0-cycle
\[
\sum x,
\]
where \(x\) runs through the vertices of \(X\). Then, heuristically, in dimension 1 our method states that
\[
[X] = 0 \text{ in } h_0(X) \quad \text{and} \quad [Y] = 0 \text{ in } h_0(Y) \implies h_1(X \times Y) = 0.
\]
In the case that motivated our investigations, of uniformly finite homology, the vanishing of the fundamental class \([X]\) corresponds to non-amenability of \(X\) [2] and we obtain the following

**Theorem A.** Let \(X\) and \(Y\) be uniformly contractible locally finite simplicial complexes. If \(X\) and \(Y\) are non-amenable then \(H^{uf}_1(X \times Y; R) = 0\) for \(R = \mathbb{Z}, \mathbb{R}\).

Under the same assumption the conclusion of vanishing of homology in degree 1 also holds for the fine uniformly finite homology \(H^{(\infty)}_1\), and almost equivariant homology. For the controlled coarse homology one can prove a quantitative statement.

Locally finite homology is an important invariant of infinite complexes (see [9, 14]) and its coarsening is the coarse homology introduced by Roe [18]. In this case for the vanishing of \([X]\) it is merely required that \(X\) is infinite and we obtain

**Theorem B.** Let \(X\) and \(Y\) be infinite simplicial complexes and let \(R\) be any abelian group. Then
\[
H^{lf}_1(X \times Y; R) = 0.
\]

The case of a 3-fold product is somewhat more complicated, as we rely on the dimensionality to promote our strategy to this case. Again we reduce arbitrary cycles to cycles of specific forms by attaching 3-dimensional geometric Eilenberg swindles in such a way, that we preserve the homology class. However, this has to be done carefully and in an appropriate order, since an incorrect order will ruin the previous reductions. As a result we obtain vanishing of \(H_2\) for 3-fold products of trees (see Theorem 13) and combining this with the fact that the top-dimensional homology of a product of trees is infinite dimensional (Proposition 21) we obtain

**Theorem C.** Let \(T_i, i = 1, 2, 3\) be infinite trees, whose each vertex has degree at least 3 and let \(R = \mathbb{R}, \mathbb{Z}\). Then
\[
H^{uf}_k(T_1 \times T_2 \times T_3; R) = \begin{cases} 
0 & k \neq 3, \\
\text{infinite dimensional} & k = 3.
\end{cases}
\]
A similar vanishing theorem holds for locally finite homology.

**Theorem D.** Let $T_i$, $i = 1, 2, 3$ be infinite trees, whose each vertex has degree at least 3 and let $R$ be any abelian group. Then

$$H^f_2(T_1 \times T_2 \times T_3; R) = \begin{cases} 0 & k \neq 3, \\ \text{infinite dimensional} & k = 3. \end{cases}$$

Our results have several applications. By the quasi-isometry invariance of uniformly finite homology we obtain the computation of the uniformly finite homology of an important class of groups. Let $\Gamma$ be a lattice in a product of trees. The class of such groups is extremely rich, see for example [4]. Since uniformly finite homology is a quasi-isometry invariant, as a corollary we obtain the computation of the uniformly finite homology of such groups.

**Theorem E.**

1. Let $\Gamma$ be a group acting cocompactly by isometries on a product of 2 trees and let $R \in \mathbb{R}, \mathbb{Z}$. Then

$$H_{i}^{uf}(\Gamma; R) = \begin{cases} 0 & \text{if } i \neq 2 \\ \text{infinite dimensional} & \text{if } i = 2. \end{cases}$$

2. Let $\Gamma$ be a group acting cocompactly by isometries on a product of 3 trees. Then

$$H_{i}^{uf}(\Gamma; R) = \begin{cases} 0 & \text{if } i \neq 3 \\ \text{infinite dimensional} & \text{if } i = 3. \end{cases}$$

Note, that such lattices can be reducible if they split as a product of lattices in the factors, or irreducible. There are examples of lattices in products of trees that are cocompact and irreducible.

Another application is a characterization of amenable groups in terms of 1-homology.

**Corollary F.** Let $G$ be a finitely generated group and let $\Gamma$ denote its Cayley graph. $G$ is amenable if and only if $H_1^{(\infty)}(\Gamma \times T; \mathbb{R}) \neq 0$ for any infinite tree $T$.

We also show a construction of aperiodic tiles using Dranishnikov’s almost equivariant homology, as well as discuss some questions and conjectures that stem from this work. In particular we conjecture that similar vanishing theorems should hold for large scale homology of higher-dimensional products of trees and for thick affine buildings. This is discussed in the last section.
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2 Large scale homology

2.1 Uniformly finite homology

Uniformly finite homology was introduced by Block and Weinberger [2]. Let $X$ be a simplicial complex which is uniformly locally finite. We will assume that $X$ is equipped with a metric $d$, whose restriction to every simplex is the
metric obtained from restricting the Euclidean metric on $\mathbb{R}^n$ to the standard simplex.

Let $R$ be a normed abelian group and define the fine uniformly finite homology with coefficient in $R$ as follows. The chains $C_n^{(\infty)}(X; R)$ are linear combinations

$$c = \sum_{\sigma \in \Delta_n} c(\sigma) \cdot \sigma,$$

where $\Delta_n = \Delta_n(X)$ is the collection of all $n$-simplices in $X$ and $c(\sigma) \in R$ for every $\sigma \in \Delta_n$, satisfying

$$\|c\|_{\infty} = \sup_{\sigma \in \Delta_n} |c(\sigma)|_R < \infty.$$ Together with the standard combinatorial boundary operator the $C_n^{(\infty)}(X; R)$ form a chain complex, whose homology is the (simplicial) fine uniformly finite homology theory $H_n^{(\infty)}(X; R)$.

Now let $X$ be a locally finite discrete metric space. For $d \geq 0$ the Rips complex $P_d(X)$ is the simplicial complex defined as follows. The vertices of $P_d(X)$ are the elements of $X$; $n + 1$ vertices $x_0, \ldots, x_n$ span an $n$-simplex if $d(x_i, x_j) \leq d$ for all $i, j \in \{0, \ldots, n\}$.

For a metric space $X$ a net is a subset $\Gamma \subset X$ such that there is $C > 0$ such that for every $x \in X$ there exists $\gamma \in \Gamma$ with $d(\gamma, x) \leq C$. A (discrete) metric space $X$ has bounded geometry if for every $r > 0$ there exists $N(r) > 0$ such that the cardinality of any ball of radius $r$ in $X$ is at most $N(r)$. See [17].

Given a metric space $X$ containing a net $\Gamma \subseteq X$ of bounded geometry (i.e. a metric space of bounded geometry) the uniformly finite homology of $X$ is the group

$$H_*^{uf}(X; R) = \lim_{d \to \infty} H_*^{(\infty)}(P_d(\Gamma); R).$$

In the case of a uniformly locally finite simplicial complex $X$, this defines a natural coarsening homomorphism

$$c_* : H_*^{(\infty)}(X; R) \to H_*^{uf}(X; R), \quad (1)$$

induced by a natural map $c : X \to P_r(\Gamma)$ for some appropriately chosen sufficiently large $r > 0$ (in this case the net $\Gamma$ can be taken to be the vertex set of $X$). Recall that $X$ is uniformly contractible if for every $r > 0$ there exists $S_r > 0$ such that for every $x \in X$ the ball $B(x, r)$ is contractible inside $B(x, S_r)$. If $X$ is uniformly contractible then $c_*$ is an isomorphism [13, 18].

An important property of $H_*^{uf}$ is that it is invariant under quasi-isometries [2]: if metric spaces $X$ and $Y$ are quasi-isometric then $H_*^{uf}(X; R) \cong H_*^{uf}(Y; R)$. 

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2.2 Other coarse homology theories

We briefly explain how to modify the above definition to obtain other homology theories that are important in large scale geometry.

Coarse homology

If, instead of bounded chains, above we consider chains which are merely locally finite (i.e., the coefficient of every simplex is finite), we obtain the locally finite homology, see e.g. [9, 14]. This homology appears naturally in the study of the topology at infinity of topological spaces, in particular, in the study of ends. The coarsening of the locally finite homology is Roe’s coarse homology [18].

Controlled coarse homology

If we consider chains, whose absolute values are bounded by a multiple of a non-decreasing function $f : X \to \mathbb{R}$, in the sense that $|c(\sigma)| \leq C f(d(\sigma, x_0))$, where $C > 0$, $x_0$ is a fixed vertex and $d$ is the metric on $X$, then we obtain the controlled coarse homology, $H^f_*(X)$, introduced in [16]. This homology can be used to quantify amenability and thus has several applications through the relation with isoperimetric inequalities on groups. The uniformly finite homology is then the controlled coarse homology with control function $f \equiv 1$.

Dranishnikov’s almost equivariant homology

If in the above chain complex, instead of bounded chains we consider only those chains that take finitely many values, we will obtain the almost equivariant homology $H^{ae}_*(X)$, introduced by Dranishnikov [5] (in Dranishnikov’s work this homology is considered only for a group). In our context it will be useful for constructing aperiodic tiles, see Section 5.2. We remark that the almost equivariant chains do not enjoy the same analytic features as the fine uniformly finite homology, since they form a proper dense subset of the fine uniformly finite chains.

2.3 Eilenberg swindles in degree 0

Let $X$ be a uniformly locally finite simplicial complex and let $[X] \in H^{(\infty)}_0(X; \mathbb{R})$ be the fundamental class of $X$ in the fine uniformly finite homology $H^{(\infty)}_0(X; \mathbb{R})$. 

This is just the class represented by the 0-cycle

$$\sum_{x \in V_X} x,$$

which assigns the coefficient 1 to any vertex $x \in V_X$. We will also view $[X]$ as a class in $H_0^{uf}(X, \mathbb{R})$.

Recall the definition of amenability for metric spaces of bounded geometry:

**Definition 1.** A metric space of bounded geometry is **amenable** if it admits a net $\Gamma \subset X$ with the following property

$$\forall r, \varepsilon > 0 \ \exists U \subset \Gamma \text{ finite s.t. } \frac{|\partial_r U|}{|U|} < \varepsilon$$

where $\partial_r U := \{ x \in \Gamma \mid d(x, U) < r \}$.

The following was proved by Block and Weinberger.

**Theorem 2** ([2]). Let $X$ be a metric space of bounded geometry and let $\Gamma \subset X$ be a net in $X$. The following are equivalent:

1. $X$ is non-amenable,
2. $H_0^{uf}(X; R) = 0$ for $R = \mathbb{R}, \mathbb{Z},$
3. $[\Gamma] = 0$ in $H_0^{uf}(X; R)$ for $R = \mathbb{R}, \mathbb{Z}$.

For the proof we refer to [2, 17]. Suppose that $[X] = 0$ in $H_0^{uf}(X; \mathbb{Z})$, i.e., that there exists a 1-cycle $\psi \in C_1^{(\infty)}(X; \mathbb{Z})$ whose boundary is $\sum_{x \in X} x$. It is possible to decompose $\psi$ as an (infinite) sum of 1-chains of a special form. We now describe this decomposition as it will be the main tool in our further considerations.

For any vertex $x \in V_X$ consider a sequence $(x_k)_{k \in \mathbb{Z}_{<0}}$ of pairwise distinct points such that for any $k \in \mathbb{Z}_{\leq 0}$ we have $[x_{k-1}, x_k] \in \Delta_1(X)$ and such that $x_0 = x$. Now define

$$t_x = \sum_{k \in \mathbb{Z}_{<0}} [x_{k-1}, x_k]. \quad (2)$$

Clearly, $t_x \in C_1^{(\infty)}(X; \mathbb{Z})$ for any $x \in X$. Moreover,

$$\partial t_x = x.$$
We call $t_x$ a tail attached to $x$. Now for any vertex $x \in V_X$ consider a tail $t_x$ constructed as above and consider

$$\sum_{x \in V_X} t_x.$$ 

This is an infinite sum of simplices in $\Delta_1(X)$. For any 1-simplex $\sigma \in \Delta_1(X)$, define

$$T(\sigma) := \{ x \in V_X \mid t_x \text{ passes through } \sigma \}. \quad (3)$$

Clearly every 1-simplex $\sigma \in \Delta_1(X)$ appears in $\sum_{x \in V_X} t_x$ with coefficient equal to the cardinality of $T(\sigma)$. This number might be unbounded. However one can construct tails $t_x$ using only simplices appearing in $\psi \in C^{(\infty)}_1(X;\mathbb{Z})$ (see the proof of Lemma 2.4 [2] for more details). In this way, for any simplex $\sigma \in \Delta_1(X)$ there is a uniformly bounded number of tails passing through it. In particular, in this situation

$$\sum_{x \in V_X} t_x \in C^{(\infty)}_1(X;\mathbb{Z})$$

and $\partial \sum_{x \in V_X} t_x = \sum_{x \in V_X} x$. This construction of tails of 1-simplices attached to points is an instance of an Eilenberg swindle, allowing to push some of the homological information off to infinity. By Theorem 2, it follows that if we consider the fine uniformly finite homology of a simplicial complex $X$, then the Eilenberg swindles construction is possible if and only if $X$ is non-amenable.

### 2.4 Relative homology

Let $X$ be a uniformly locally finite simplicial complex, $A$ be a subcomplex of $X$ and let $R = \mathbb{R}, \mathbb{Z}$. The natural inclusion induces a short exact sequence of chain complexes,

$$0 \longrightarrow C^{(\infty)}_k(A;R) \longrightarrow C^{(\infty)}_k(X;R) \longrightarrow C^{(\infty)}_k(X;R)/C^{(\infty)}_k(A;R) \longrightarrow 0,$$

where as usual,

$$C^{(\infty)}_k(X,A;R) = C^{(\infty)}_k(X;R)/C^{(\infty)}_k(A;R)$$

denotes the relative chains. We get a standard long exact sequence of a pair:

$$\cdots \longrightarrow H^{(\infty)}_k(A;R) \longrightarrow H^{(\infty)}_k(X;R) \longrightarrow H^{(\infty)}_k(X,A;R) \longrightarrow H^{(\infty)}_{k-1}(A;R) \longrightarrow \cdots, \quad (4)$$
As usual, classes in $H^\infty_k(X,A;R)$ are represented $n$-chains $c \in C_n^\infty(X;R)$, satisfying $\partial c \in C_{n-1}^\infty(A;R)$. Such a relative cycle $c$ bounds in $H^\infty_k(X,A;R)$ if and only if

$$c = \partial b + a,$$

for some $b \in C_{n+1}^\infty(X;R)$ and $a \in C_n^\infty(A;R)$.

Consider now a product of $n$ simplicial complexes $X = X_1 \times \cdots \times X_n$. We assume for now that $X$ is equipped with a simplicial structure and by a $k$-cube we will mean a subcomplex which is a product of $k$-edges $e_i \in \Delta_1(X_i)$ and $n-k$ vertices in $X_i$. We additionally assume that the simplicial structure on the product is such that each $k$-cube with the induced simplicial structure is one of finitely many simplicial structures on a cube $[0,1]^k$. An explicit simplicial structure for products, satisfying the above assumptions will be discussed in detail in the next section.

By a boundary of an $k$-cube we denote the subcomplex given by the union of the $2k - 1$-cubes forming its topological boundary.

**Proposition 3.** Let $Y$ be the union of a collection of the $k$-cubes in $X$ and let $A$ be the union of the boundaries of the $k$-cubes in $Y$. Then $H^\infty_i(Y,A;R) = 0$ for $i \leq k-1$ and $R = \mathbb{Z}, \mathbb{R}$.

**Proof:** Let $c$ be a relative cycle; that is $\partial c \in C_i^\infty(A;R), i \leq k-1$. Consider $c_I$, the restriction of $c$ to a $k$-cube $I = I^k \subseteq X$. Denote by $\partial I^k$ the simplicial boundary of $I^k$. For such a cube the standard simplicial homology satisfies

$$H_k(I^k, \partial I^k) \cong H_k(I^k/\partial I^k) \cong H_k(S^k),$$

since the boundary $\partial I^k$ is a deformation retract of its neighborhood in $I^k$. Therefore,

$$H_i(I^k, \partial I^k) = 0,$$

provided $i \leq k-1$.

Now, $c_I$ is a relative cycle in $H_i(I^k, \partial I^k)$, and as such, vanishes. That is,

$$c_I = \partial b_I + a_I,$$

where $b_I \in C_{i+1}(I^k)$ and $a_I \in C_i(\partial I^k)$. Define

$$b = \sum_I \sum_{\sigma \in \Delta_{i+1}(I^k)} b_I(\sigma)\sigma,$$

where $I^k$ runs through all the cubes of dimension $k$ in $Y$. Then $c - \partial b$ is supported on $A = \sum_I \partial I^k$, where again $I^k$ runs through all the cubes of dimension $k$ in $Y$. 



It remains to show that each $b$ and $a$ are bounded. Observe that in the case of $R = \mathbb{Z}$, the boundedness of $c$ and the assumptions on the simplicial structure imply that $c_I$ is one of finitely many possible chains in $C_i(I^k)$. Then there is finitely many possibilities for $b_I$. Consequently, both $b$ and $c - \partial b$ have finitely many possible values, and, in particular, they are uniformly bounded in the $\infty$-norm.

In the case of $R = \mathbb{R}$ it suffices to appeal to the finite-dimensionality of the chain spaces $C_i(\partial I^k; \mathbb{R})$, $C_i(I^k; \mathbb{R})$ and $C_i(I^k, \partial I^k; \mathbb{R})$. We equip these spaces with norms $\| \cdot \|_{\partial I}$, $\| \cdot \|_I$ and $\| \cdot \|_{(I, \partial I)}$, respectively (we omit the degree in the notation). Recall that on finite-dimensional spaces all norms are equivalent and observe that since the image of the differential is closed, there exists a constant $C > 0$ such that for each $c_I$ as above we can choose $b_I$ as above satisfying

$$[c_I] = \partial [b_I]$$

and

$$\|b_I\|_I = \|[b_I]\|_{(I, \partial I)} \leq C^{-1} \|\partial [b_I]\|_{(I, \partial I)} \leq C^{-1} \|[c_I]\|_{(I, \partial I)} \leq C^{-1} \|c_I\|_I,$$

where the constant $C$ is independent of the choice of $I \subseteq Y$. Also,

$$\|c_I - \partial b_I\|_I \leq \|c_I\|_I + \|\partial b_I\|_I.$$

Since every linear operator on finite-dimensional space is continuous, we have

$$\|\partial b_I\|_I \leq \|\partial\| \|b_I\|_I,$$

giving

$$\|c_I - \partial b_I\| \leq \|c_I\|_I \left(1 + C^{-1}\|\partial\|\right).$$

Since in our setting,

$$\|c_I\|_I \leq \|c\|_{\infty},$$

we obtain the claim. \(\square\)

From the exact sequence (4) we obtain

**Corollary 4.** Let $R = \mathbb{Z}, \mathbb{R}$ and let $A, Y$ be as above. The map

$$i_* : H_i^{(\infty)}(A; R) \to H_i^{(\infty)}(Y; R),$$

induced by the inclusion $i : A \to X$, is surjective for $i \leq k - 1$.

In other words, every class in the homology of $Y$ can be represented by a cycle supported only on $A$. 11
3 Products of simplicial complexes

In this section we consider the Cartesian product of two simplicial complexes. We first, endow it with a simplicial structure by defining a suitable triangulation. Then we give a construction of 2-dimensional swindles on the product of two non-amenable simplicial complexes and we use it to prove Theorem A.

3.1 Simplicial structures on products

Following a construction due to Eilenberg and Steenrod ([8, Chapter II.8]), we define a simplicial structure on the product of simplicial complexes. We start by introducing an order.

Definition 5. Let $X$ be a simplicial complex. An order on $X$ is a binary relation $\leq_X$ on the set of vertices $V_X$ satisfying the following conditions:

1. If $x \leq_X x'$ and $x' \leq_X x$ then $x = x'$;
2. Two elements $x, x'$ of $V_X$ are vertices of a given simplex in $X$ if and only if $x \leq_X x'$ or $x' \leq_X x$;
3. If $x, x', x'' \in V_X$ are vertices of a given simplex in $X$ and if $x \leq_X x'$ and $x' \leq_X x''$, then $x \leq_X x''$.

A pair $(X, \leq_X)$ given by a simplicial complex $X$ and a order $\leq_X$ on $X$ is called an ordered simplicial complex.

Definition 6. Let $X$ be a simplicial complex. For any $x \in V_X$, let

$$A_x := \{ x_A \in V_X \mid x_A > x \}$$
$$B_x := \{ x_B \in V_X \mid x_B < x \}.$$

In other words, $A_x$ (resp. $B_x$) denotes the set of vertices $x_A$ which are connected with $x$ by an 1-simplex $[x, x_A] \in \Delta_1(X)$ (resp. $[x_B, x] \in \Delta_1(X)$).

Definition 7. Let $(X, \leq_X)$ and $(Y, \leq_Y)$ be two ordered simplicial complexes. Consider the order defined for any $(x, y), (x', y') \in V_X \times V_Y$ as follows:

$$(x, y) \leq (x', y') \text{ if } x \leq_X x' \text{ and } y \leq_Y y'.$$

The triangulated Cartesian product $X \times_Y Y$ of $X$ and $Y$ is a simplicial complex whose vertices are all the elements $(x, y) \in V_X \times V_Y$; moreover, for any $n \in \mathbb{N}$, a tuple $[(x_0, y_0), \ldots, (x_n, y_n)]$ forms a simplex in $\Delta_n(X \times Y)$ if and only if $(x_0, y_0) \leq \cdots \leq (x_n, y_n)$.
Remark 8. More precisely, for any \( n \in \mathbb{N} \) a \( n \)-simplex in \( X \times Y \) is of the form \([ (x_0, y_0), \ldots, (x_n, y_n) ] \) where \( x_0 \leq x \cdots \leq x_n \) and \( y_0 \leq y \cdots \leq y_n \) (with possible repetitions). In particular, \([x_0, \ldots, x_n]\) is a (possibly degenerate) simplex in \( X \) and \([y_0, \ldots, y_n]\) is a (possibly degenerate) simplex in \( Y \).

For simplicity of notation, we denote the triangulated Cartesian product as \( X \times Y \) and we denote by \( \leq \) any order on \( X, Y \) or \( X \times Y \). In the following sections we will consider the triangulated Cartesian product of uniformly locally finite simplicial complexes. To pass from fine uniformly finite homology to uniformly finite homology, we will need to assume the simplicial complexes to be uniformly contractible. The following lemma is straightforward:

Lemma 9. The triangulated Cartesian product of two uniformly locally finite, uniformly contractible simplicial complexes is a uniformly locally finite and uniformly contractible simplicial complex.

3.2 2-dimensional swindles

Let \( X, Y \) be uniformly locally finite simplicial complexes and let \( X \times Y \) be their triangulated Cartesian product. Any 1-simplex of \( X \times Y \) belongs to one of the following sets:

- The set of \textbf{horizontal edges} \( \Delta_1(X \times Y)_h := \{ [(x, y), (x', y)] \mid x < x' \} \).
  These 1-simplices arise as a product of a 1-simplex of \( X \) and a vertex of \( Y \).

- The set of \textbf{vertical edges} \( \Delta_1(X \times Y)_v := \{ [(x, y), (x, y')] \mid y < y' \} \).
  These 1-simplices arise as a product of a vertex of \( X \) and a 1-simplex of \( Y \).

- The set of \textbf{diagonal edges} \( \Delta_1(X \times Y)_d := \{ [(x, y), (x', y')] \mid x < x', y < y' \} \).

We consider the fine uniformly finite chain complex \( C^{(\infty)}_i(X \times Y) \) with coefficients in \( R = \mathbb{R}, \mathbb{Z} \). Then every chain \( c \in C^{(\infty)}_1(X \times Y) \) can be written as

\[
\sum_{\sigma_h \in \Delta_1(X \times Y)_h} c(\sigma_h) \cdot \sigma_h + \sum_{\sigma_v \in \Delta_1(X \times Y)_v} c(\sigma_v) \cdot \sigma_v + \sum_{\sigma_d \in \Delta_1(X \times Y)_d} c(\sigma_d) \cdot \sigma_d, \tag{5}
\]
where for any $\sigma_b \in \Delta_1(X \times Y)_b, \sigma_v \in \Delta_1(X \times Y)_v$ and any $\sigma_d \in \Delta_1(X \times Y)_d$ we have $c(\sigma_h), c(\sigma_v), c(\sigma_d) \in R$.

We want to reduce to chains in $C_1^{(\infty)}(X \times Y)$ that are supported only on horizontal and vertical edges. In general, the fine uniformly finite homology is defined in terms of triangulations, however we want to operate on cubes, by which we mean products of 1-simplices. Thus, we want to reduce ourself to cycle which are supported only on the faces of those cubes.

**Lemma 10.** Let $\alpha \in H_1^{(\infty)}(X \times Y)$. There exists a cycle $b \in C_1^{(\infty)}(X \times Y)$ such that for any diagonal edge $\sigma_d \in \Delta_1(X \times Y)_d$ we have $b(\sigma_d) = 0$ in $R$ and such that $\alpha = [b]$ in $H_1^{(\infty)}(X \times Y)$.

This lemma can be justified in two ways. One is the use of relative homology and Proposition 3. The second is more elementary and straightforward and we provide it below.

**Proof.** Let $c \in C_1^{(\infty)}(X \times Y)$ be any cycle representing $\alpha$. Any diagonal edge $\sigma_d$ is of the form $[(x, y), (x', y')]$ for some $[x, x'] \in \Delta_1(X)$ and $[y, y'] \in \Delta_1(Y)$ with $x < x'$ in $X$ and $y < y'$ in $Y$. For any diagonal edge $\sigma_d = [(x, y), (x', y')]$, consider the following 2-simplex:

$$\tau_{\sigma_d} = [(x, y), (x', y'), (x, y')].$$

By Definition 7, it is easy to see that $\tau_{\sigma_d}$ is a 2-simplex in $X \times Y$. Now define

$$\phi = \sum_{\sigma_d \in \Delta_1(X \times Y)_d} c(\sigma_d) \cdot \tau_{\sigma_d}.$$

This is a (infinite) sum of 2-simplices in $X \times Y$ with uniformly bounded coefficients: indeed, any simplex $\tau_{\sigma_d}$ appears in the sum with coefficient $c(\sigma_d) \in R$ and since $c$ is in $C_1^{(\infty)}(X \times Y)$, these coefficients must be uniformly bounded. Thus $\phi \in C_2^{(\infty)}(X \times Y)$. Clearly, for any $\sigma_d$ we have

$$\partial \tau_d = [(x, y'), (x', y')] - \sigma_d + [(x, y), (x, y')]$$

where $[(x, y'), (x', y')] \in \Delta_1(X \times Y)_h$ and $[(x, y), (x, y')] \in \Delta_1(X \times Y)_v$. Thus

$$c + \partial \phi = \sum_{\sigma_h \in \Delta_1(X \times Y)_h} b(\sigma_h) \cdot \sigma_h + \sum_{\sigma_v \in \Delta_1(X \times Y)_v} b(\sigma_v) \cdot \sigma_v,$$

where for any $\sigma_h \in \Delta_1(X \times Y)_h, \sigma_v \in \Delta_1(X \times Y)_v$, we have $b(\sigma_h), b(\sigma_v) \in R$. Now let $b := \sum_{\sigma_h \in \Delta_1(X \times Y)_h} b(\sigma_h) \cdot \sigma_h + \sum_{\sigma_v \in \Delta_1(X \times Y)_v} b(\sigma_v) \cdot \sigma_v$. Since $c$ and $b$ differ by the boundary of $\phi$, we have $\alpha = [c] = [b]$ and the claim follows.  

[]
Thus, from Lemma 10 we can assume that any class in $H_1^{(\infty)}(X \times Y)$ is represented by a cycle $c \in C_1^{(\infty)}(X \times Y)$ of the form
\[
\sum_{\sigma_h \in \Delta_1(X \times Y)_h} c(\sigma_h) \cdot \sigma_h + \sum_{\sigma_v \in \Delta_1(X \times Y)_v} c(\sigma_v) \cdot \sigma_v.
\] (6)

Now we take $Y$ to be non-amenable. We want to represent any class in $H_1^{(\infty)}(X \times Y)$ by cycles that are supported only on horizontal edges.

**Lemma 11.** Let $X, Y$ be uniformly locally finite simplicial complexes. Suppose $Y$ is non-amenable and consider the triangulated Cartesian product $X \times Y$. Let $a \in H_1^{(\infty)}(X \times Y)$. There exists a cycle $b \in C_1^{(\infty)}(X \times Y)$ such that for any $\sigma_d \in \Delta_1(X \times Y)_d$ and any $\sigma_h \in \Delta_1(X \times Y)_h$ we have $b(\sigma_d) = 0, b(\sigma_h) = 0$ in $\mathbb{R}$ and such that $a = [b]$ in $H_1^{(\infty)}(X \times Y)$.

**Proof.** Let $c \in C_1^{(\infty)}(X \times Y)$ be any cycle representing $a$. By Lemma 10, we can suppose $c$ to be of the form (6). As in the proof of Lemma 10, we want to construct a chain in $C_2^{(\infty)}(X \times Y)$ whose boundary “kills” all the horizontal edges of $c$. We build up this chain by constructing infinite sums of 2-simplices in $X \times Y$ “attached” to horizontal edges of $c$.

Following the Eilenberg-swindle construction given in Section 2.3, for any $y \in V_Y$, we consider a tail of 1-simplices attached to $y$. More precisely, for any $y \in Y$ consider
\[
t_y = \sum_{k \in \mathbb{Z}_{\geq 0}} [y_{k-1}, y_k] \in C_1^{(\infty)}(Y)
\] (7)
such that for any $k \in \mathbb{Z}_{\geq 0}$ we have $y_{k-1} < y_k$ and $y_0 = y$ in $Y$. We can consider any tail as an infinite sum of vertical edges in $X \times Y$; more precisely, for any $x \in X$ and any $y \in Y$ we can consider the tail
\[
t^Y_{(x,y)} = \sum_{k \in \mathbb{Z}_{\geq 0}} [(x, y_{k-1}), (x, y_k)] \in C_1^{(\infty)}(X \times Y).
\]

Now let $\sigma_h = [(x, y), (x', y)]$ be any horizontal edge in $\Delta_1(X \times Y)_h$. Given two tails $t^Y_{(x,y)}$ and $t^Y_{(x',y)}$ constructed above, attached to the vertices of $\sigma_h$ we define
\[
p_{\sigma_h} = \sum_{k \in \mathbb{Z}_{\geq 0}} [(x, y_{k-1}), (x, y_k), (x', y_k)] - [(x, y_{k-1}), (x', y_{k-1}), (x', y_k)].
\]

We have
\[
\partial p_{\sigma_h} = [(x, y_0), (x', y_0)] + \sum_{k \in \mathbb{Z}_{\geq 0}} [(x, y_{k-1}), (x, y_k)] - \sum_{k \in \mathbb{Z}_{\geq 0}} [(x', y_{k-1}), (x', y_k)]
\]
\[
= \sigma_h + t^Y_{(x,y)} - t^Y_{(x',y)}
\] (8)
Clearly $p_{\alpha_h}$ is an infinite sum of 2-simplices in $X \times Y$. We construct $p_{\sigma_h}$ for any $\sigma_h \in \Delta_1(X \times Y)_h$ and we call it a panel attached to $\sigma_h$ (Figure 1). Then, we consider

$$\phi = \sum_{\sigma_h \in \Delta_1(X \times Y)_h} c(\sigma_h) \cdot p_{\sigma_h},$$

where $c(\sigma_h) \in R$ is the coefficient associated to the simplex $\sigma_h$ in $c$. We observe that $\phi$ is a uniformly finite chain in $C_2^{(\infty)}(X \times Y)$. Indeed, by the construction of the panels $p_{\sigma_h}$ for any horizontal edge $\sigma_h$ it is easy to see the 2-simplices appearing in $\phi$ are either of the form

$$[(x, y), (x, y_A), (x_A, y_A)] \in \Delta_2(X \times Y)$$

for some $x_A \in A_x$ in $X$ and some $y_A \in A_y$ in $Y$, or of the form

$$[(x, y), (x_A, y), (x_A, y_A)] \in \Delta_2(X \times Y)$$

for some $x_A \in A_x$ in $X$ and some $y_A \in A_y$ in $Y$. An easy calculation shows that the coefficient associated to any simplex $[(x, y), (x_A, y), (x_A, y_A)]$ in $\phi$ is

$$\sum_{\tilde{y} \in T(y, y_A)} c(((x, \tilde{y}), (x_A, \tilde{y}))),$$

where $T(y, y_A) := \{ \tilde{y} \in V_Y \mid t_{\tilde{y}} \text{ passes through } [y, y_A] \text{ in } Y \}$ as in (3). In the same way, the coefficient associated to any simplex $[(x, y), (x_A, y), (x_A, y_A)]$ in $\phi$ is

$$-\sum_{\tilde{y} \in T(y, y_A)} c(((x, \tilde{y}), (x_A, \tilde{y}))).$$

Since $Y$ is non-amenable, by Theorem 2 the fundamental class $[Y] = 0$ in $H_0^{(\infty)}(Y; R)$ and by the construction given in Section 2.3, we know that $\partial \sum_{y \in V_Y} t_y = \sum_{y \in V_Y} y$. Thus for any $[y, y_A] \in \Delta_1(Y)$, the number $|T([y, y_A])|$ is uniformly bounded. In particular, the coefficients of $\phi$ are uniformly bounded. Thus, $\phi \in C_2^{(\infty)}(X \times Y)$.

Notice that for any $(x, y) \in V_X \times V_Y$, the tail $t_{(x, y)}^Y$ is a sum of vertical edges in $X \times Y$. By (8), one can easily see that

$$c - \partial \phi = \sum_{\sigma_v \in \Delta(X \times Y)_v} b(\sigma_v) \cdot \partial \phi,$$

where $b(\sigma_v) \in R$ for any $\sigma_v \in \Delta_1(X \times Y)_v$. Let $b = \sum_{\sigma_v \in \Delta_1(X \times Y)_v} b(\sigma_v) \cdot \partial \phi$. Since $c$ and $b$ differ by the boundary of $\phi$, it follows that $b$ is a cycle in $C_1^{(\infty)}(X \times Y)$ and that $a = [c] = [b]$. \qed
Thus, from Lemma 11 we can assume that any class in $H_1^{(\infty)}(X \times Y)$ is represented by a cycle $c \in C_1^{(\infty)}(X \times Y)$ of the form

$$\sum_{\sigma_v \in \Delta_1(X \times Y)} c(\sigma_v) \cdot \sigma_v. \quad (9)$$

We are now ready to prove Theorem A: we just need to show that any class represented by a cycle supported on vertical edges is trivial in $H_1^{(\infty)}(X \times Y)$.

### 3.3 Proof of Theorem A

We first prove the analogous result for the fine uniformly finite homology.

**Theorem 12.** Let $X$ and $Y$ be uniformly locally finite simplicial complexes. If $X$ and $Y$ are non-amenable then $H_1^{(\infty)}(X \times Y) = 0$.

**Proof:** The Cartesian product of two uniformly locally finite simplicial complexes is, as metric space, coarsely equivalent to the triangulated Cartesian product. By the coarse invariance of uniformly finite homology, we can prove the theorem for the triangulated Cartesian product, and the claim will follow for the standard Cartesian product.

Let $X \times Y$ be the triangulated Cartesian product of $X$ and $Y$. Let $\alpha$ be any class in $H_1^{(\infty)}(X \times Y)$. By Lemma 11, we can assume $\alpha$ to be represented
by a cycle $c$ of the form (9) in $C^{(\infty)}_1(X \times Y)$. We want to construct a chain in $C^{(\infty)}_2(X \times Y)$ whose boundary is the cycle $c$; as in Lemma 10, we construct panels of 2-simplices in $X \times Y$ attached to vertical edges in $c$. More precisely, for any $x \in V_X$, consider a tail $t_x$ attached to $x$ in $X$\[ t_x = \sum_{k \in \mathbb{Z}_{\leq 0}} [x_{k-1}, x_k] \in C^{(\infty)}_1(X), \]such that for any $k \in \mathbb{Z}_{\leq 0}$ we have $x_{k-1} < x_k$ and $x_0 = x$ in $X$. We can consider any tail as an infinite sum of horizontal edges in $X \times Y$; more precisely, for any $x \in X$ and any $y \in Y$ we can consider the tail\[ i^X_{(x,y)} = \sum_{k \in \mathbb{Z}_{\leq 0}} [(x_{k-1}, y), (x_k, y)] \in C^{(\infty)}_1(X \times Y). \]Now let $\sigma_v = [(x, y), (x, y')]$ be any vertical edge in $\Delta_1(X \times Y)_v$. and consider\[ p_{\sigma_v} = \sum_{k \in \mathbb{Z}_{\leq 0}} [(x_{k-1}, y), (x_k, y), (x_k, y')] - [(x_{k-1}, y), (x_k, y'), (x_k, y')]. \]We have\[ \partial p_{\sigma_v} = [(x_0, y), (x_0, y')] + \sum_{k \in \mathbb{Z}_{\leq 0}} [(x_{k-1}, y), (x_k, y)] - \sum_{k \in \mathbb{Z}_{\leq 0}} [(x_{k-1}, y'), (x_k, y')] = \sigma_v + i^X_{(x,y)} - i^X_{(x,y')} . \]We construct $p_{\sigma_v}$ for any vertical edge $\sigma_v$ and we call it a panel attached to $\sigma_v$. Then, we consider\[ \phi = \sum_{\sigma_v \in \Delta_1(X \times Y)_v} c(\sigma_v) \cdot p_{\sigma_v}, \]where $c(\sigma_v) \in R$ is the coefficient associated to the simplex $\sigma_v$ in $c$. Now we are left to prove that $\phi$ is a well-defined chain in $C^{(\infty)}_2(X \times Y)$ and that its boundary is $c$. The construction of $\phi$ uses panels of 2-simplices as the construction of the chain $\phi$ in the proof of Lemma 11. Thus, since $X$ is non-amenable it follows that $\phi$ is a well-defined chain in $C^{(\infty)}_2(X \times Y)$. So we are left to prove that $\partial \phi = c$. By (11) it is easy to see that $\partial \phi = c$ if and only if the following holds\[ \sum_{\sigma_v = [(x, y), (x, y')] \in \Delta_1(X \times Y)_v} c(\sigma_v) \cdot i^X_{(x,y)} - c(\sigma_v) \cdot i^X_{(x,y')} = \sum_{x \in V_X} \sum_{y \in V_Y} \left( \sum_{y_A \in A_y} c([(x, y), (x, y_A)]) - \sum_{y_B \in B_y} c([(x, y_B), (x, y)]) \right) \cdot i^X_{(x,y)} = 0. \]
Figure 2: A panel attached to the edge $e$ in a product of trees as in Lemma 11

Notice that since $c$ is a sum of vertical edges, we can write it as

$$c = \sum_{x \in V_X} \sum_{y_{yA} \in \Delta_1(Y)} c([(x, y)(x, y_{yA})] \cdot [(x, y), (x, y_{yA})]].$$

Moreover, since $c$ is a cycle in $C_1^{(\infty)}(X \times Y)$, we have that

$$\partial c = \sum_{x \in V_X} \sum_{y \in V_Y} \left( \sum_{y_{yB} \in B} c([(x, y_{yB}), (x, y)]) - \sum_{y_{yA} \in A} c([(x, y), (x, y_{yA})]) \right) \cdot [(x, y)] = 0.$$

In particular, for any $(x, y) \in V_X \times V_Y$, we have that

$$\sum_{y_{yB} \in B_x} c([(x, y_{yB}), (x, y)]) - \sum_{y_{yA} \in A_y} c([(x, y), (x, y_{yA})]) = 0. \quad (13)$$

Thus (12) follows and we have $\partial \phi = c$. In particular, $\alpha = 0$ in $H_1^{(\infty)}(X \times Y)$. \hfill \qed

We are now in position to prove the main result of this section.

Proof of Theorem A. If $X$ and $Y$ are uniformly contractible then so is $X \times Y$ and

$$H_1^{uf}(X \times Y; R) = H_1^{(\infty)}(X \times Y; R) = 0,$$

by Theorem 12. \hfill \qed
4 Products of trees

In this section we consider a product of three trees. We will always assume that each vertex in the tree has degree at least 3. We also assume that the degrees of all vertices have a uniform upper bound (i.e., they are uniformly locally finite). This last assumption is not crucial but it would need a slightly more general definition of homology. Following the idea presented in the previous section, we use 3-dimensional swindles to prove that the uniformly finite homology of the product of three trees vanishes in degree 2 (Theorem 13). At the end of the section we will give a complete proof of our second main result (Theorem C).

4.1 3-dimensional swindles

The main result of this section is the following

**Theorem 13.** Let $X = T_x \times T_y \times T_z$ be the Cartesian product of trees $T_x, T_y, T_z$. Then $H^2_\infty(X; R) = 0$ for $R = \mathbb{R}, \mathbb{Z}$.

Our method is now to construct 3-dimensional swindles in the product space and to prove that any class in $H^2_\infty(T_x \times T_y \times T_z; R)$ is trivial for $R = \mathbb{R}, \mathbb{Z}$.

Let $T_x, T_y$ and $T_z$ be trees. We consider the Cartesian product $T_x \times T_y \times T_z$ and following Definition 7, we endow it with the structure of a simplicial complex. More precisely, the triangulated Cartesian product $T_x \times T_y \times T_z$ is a 3-dimensional simplicial complex having vertex set $V_{T_x} \times V_{T_y} \times V_{T_z}$ and with simplices given by totally ordered tuples by the product order. In particular, any 2-simplex on $X$ is of the form

$$[(x, y, z), (x', y', z'), (x'', y'', z'')]$$

with $x \leq x' \leq x''$ in $V_{T_x}$, $y \leq y' \leq y''$ in $V_{T_y}$, $z \leq z' \leq z''$ in $V_{T_z}$. Notice that, since $T_x$ is a 1-dimensional simplicial complex, by Definition 5 we have $x \leq x' \leq x''$ in $V_{T_x}$ if and only if one of the following situation occurs:

1. $x = x' = x''$ in $V_{T_x}$.
2. $x < x' = x''$ in $V_{T_x}$ (in particular, $[x, x']$ in $E_{T_x}$).
3. $x = x' < x''$ in $V_{T_x}$ (in particular, $[x', x'']$ in $E_{T_x}$).

So, either the vertices $x \leq x' \leq x''$ in $V_{T_x}$ are all equal or they change only once. The same holds for $T_y$ and for $T_z$. Thus, any $\sigma \in \Delta_2(X)$ belongs to one of the classes described in the following definition:
Definition 14. Let \( X = T_x \times T_y \times T_z \). For any (non-degenerate) \( \sigma \in \Delta_2(X) \), one of the following situations occurs:

- The simplex \( \sigma \) is an \textbf{x-simplex}, i.e., \( \sigma = [(x,y,z),(x,y',z'),(x,y'',z'')] \) with
  \[
  y < y' = y'' \quad \text{or} \quad y = y' < y'' \in V_Ty, \\
  z < z' = z'' \quad \text{or} \quad z = z' < z'' \in V_Tz.
  \]

- The simplex \( \sigma \) is a \textbf{y-simplex}, i.e., \( \sigma = [(x,y,z),(x',y,z'),(x'',y,z'')] \) with
  \[
  x < x' = x'' \quad \text{or} \quad x = x' < x'' \in V_Tx, \\
  z < z' = z'' \quad \text{or} \quad z = z' < z'' \in V_Tz.
  \]

- The simplex \( \sigma \) is a \textbf{z-simplex}, i.e., \( \sigma = [(x,y,z),(x',y',z),(x'',y'',z')] \) with
  \[
  x < x' = x'' \quad \text{or} \quad x = x' < x'' \in V_Tx, \\
  y < y' = y'' \quad \text{or} \quad y = y' < y'' \in V_Ty.
  \]

- Otherwise the simplex \( \sigma \) is a \textbf{diagonal simplex}.

![Figure 3: The different types of simplices in a product: the x-simplices (left), y-simplices (center) and z-simplices (right).](image)

In particular, \( \sigma \) is a diagonal simplex if and only if it does not lay on the boundary of any cube in \( X \). We denote the set of \textbf{x-simplices} (resp. \textbf{y-simplices} and \textbf{z-simplices}) by \( \Delta_2(X)_x \) (resp. \( \Delta_2(X)_y \) and \( \Delta_2(X)_z \)) and the set of diagonal simplices by \( \Delta_2(X)_d \).

We consider the fine uniformly finite chain complex \( C^{(\infty)}_*(X) \) of the triangulated Cartesian product \( X = T_x \times T_y \times T_z \) with coefficients in \( R = \mathbb{R}, \mathbb{Z} \). We can classify elements of \( C^{(\infty)}_2(X) \) as follow:
Definition 15. 1. Any chain of the form \( c_x = \sum_{\sigma_x \in \Delta_2(X)_x} c(\sigma_x) \cdot \sigma_x \in C_2^{(\infty)}(X) \) is an \textbf{x-chain}.

2. Any chain of the form \( c_y = \sum_{\sigma_y \in \Delta_2(X)_y} c(\sigma_y) \cdot \sigma_y \in C_2^{(\infty)}(X) \) is a \textbf{y-chain}.

3. Any chain of the form \( c_z = \sum_{\sigma_z \in \Delta_2(X)_z} c(\sigma_z) \cdot \sigma_z \in C_2^{(\infty)}(X) \) is a \textbf{z-chain}.

4. Any chain of the form \( c_d = \sum_{\sigma_d \in \Delta_2(X)_d} c(\sigma_d) \cdot \sigma_d \in C_2^{(\infty)}(X) \) is a \textbf{diagonal chain}.

A class in \( H_2^{(\infty)}(X) \) is called an \textbf{x-class} (resp. \textbf{y-class} and \textbf{z-class}) if it is represented by an \textbf{x-chain} (reps. \textbf{y-chain}, \textbf{z-chain}) that is a cycle in \( C_2^{(\infty)}(X) \).

Remark 16. Clearly every chain \( c \in C_2^{(\infty)}(X) \) can be written as \( c = c_x + c_y + c_z + c_d \), for some \textbf{x-chain} \( c_x \), some \textbf{y-chain} \( c_y \), some \textbf{z-chain} \( c_z \) and some diagonal chain \( c_d \).

The following lemma shows that we can restrict to classes in \( H_2^{(\infty)}(X) \) represented by cycles supported on the boundary of any cube in \( X \). In other words, we can restrict to cycles \( c \) with \( c_d = 0 \).

Lemma 17. Let \( a \in H_2^{(\infty)}(X) \). There is a cycle \( b \in C_2^{(\infty)}(X) \) of the form \( b = b_x + b_y + b_z \) such that \( a = [b] \) in \( H_2^{(\infty)}(X) \).

As for Lemma 10, this can be justified in two ways. One is using relative homology and Proposition 3. The second one is more explicit and we provide it below.

Proof: Let \( c = c_x + c_y + c_z + c_d \in C_2^{(\infty)}(X) \) be any cycle representing \( a \). We want to find a chain \( \phi \in C_2^{(\infty)}(X) \) such that \( c - \partial \phi \) is of the form \( b_x + b_y + b_z \) for some \textbf{x-chain} \( b_x \in C_2^{(\infty)}(X) \), some \textbf{y-chain} \( c_y \in C_2^{(\infty)}(X) \), and some \textbf{z-chain} \( b_z \in C_2^{(\infty)}(X) \). For any \( (x, y, z) \in V_X \), consider some \((x', y', z') \in V_X \) with \( x' > x \) in \( T_X \), \( y' > y \) in \( T_Y \) and \( z' > z \) in \( T_Z \). It is easy to see that the edge \( e = [(x, y, z), (x', y', z')] \in \Delta_1(X) \) is the face of 6 different diagonal 2-simplices contained in a cube in \( X \) (Figure 4). More precisely, the only simplices in \( \Delta_2(X) \) having \( e \) as a face are:

\[\sigma_0 = [(x, y, z), (x', y, z), (x', y', z')] \in \Delta_2(X)_d;\]
\[\sigma_1 = [(x, y, z), (x', y', z), (x', y', z')] \in \Delta_2(X)_d;\]
\[\sigma_2 = [(x, y, z), (x, y', z), (x', y', z')] \in \Delta_2(X)_d;\]
\[\sigma_3 = [(x, y, z), (x, y', z'), (x', y', z')] \in \Delta_2(X)_d;\]
\[ \sigma_4 = [(x, y, z), (x, y, z'), (x', y', z')] \in \Delta_2(X) \]
\[ \sigma_5 = [(x, y, z), (x', y, z'), (x', y', z')] \in \Delta_2(X). \]

Suppose that for any \( i \in \{0, \ldots, 5\} \) the simplex \( \sigma_i \) appears in \( c \) with coefficient \( c_i \). Then, since \( c \) is a cycle, it is easy to see that
\[ c_0 + c_1 + c_2 + c_3 + c_4 + c_5 = 0. \quad (14) \]

We “connect” the diagonal simplices listed above with 3-simplices in \( X \). More precisely, consider the following simplices in \( \Delta_3(X) \):
\[ \tau_0 = [(x, y, z), (x', y, z), (x', y', z')] \in \Delta_3(X): \text{ this simplex has } \sigma_0 \text{ and } \sigma_1 \text{ as faces}; \]
\[ \tau_1 = [(x, y, z), (x', y, z), (x', y', z')] \in \Delta_3(X): \text{ this simplex has } \sigma_1 \text{ and } \sigma_2 \text{ as faces}; \]
\[ \tau_2 = [(x, y, z), (x, y', z), (x, y', z')] \in \Delta_3(X): \text{ this simplex has } \sigma_2 \text{ and } \sigma_3 \text{ as faces}; \]
\[ \tau_3 = [(x, y, z), (x, y', z), (x', y', z')] \in \Delta_3(X): \text{ this simplex has } \sigma_3 \text{ and } \sigma_4 \text{ as faces}; \]
\[ \tau_4 = [(x, y, z), (x, y', z), (x', y', z')] \in \Delta_3(X): \text{ this simplex has } \sigma_4 \text{ and } \sigma_5 \text{ as faces}. \]

Now define
\[ d_{[(x, y, z),(x', y', z')]} = d_0 \cdot \tau_0 - d_1 \cdot \tau_1 + d_2 \cdot \tau_2 - d_3 \cdot \tau_3 + d_4 \cdot \tau_4 \]
for \( d_0 = c_0 \in R, d_1 = c_0 + c_1 \in R, d_2 = c_0 + c_1 + c_2 \in R, d_3 = c_0 + c_1 + c_2 + c_3 \in R, d_4 = c_0 + c_1 + c_2 + c_3 + c_4 \in R \). Notice that, by the cycle condition \((14)\), we have \( d_4 = -c_5 \).

Consider
\[ \psi = \sum_{(x, y, z) \in \mathcal{V}_X} \sum_{x' > x, y' > y, z' > z} d_{[(x, y, z),(x', y', z')]} \]

An easy computation shows that
\[ c'_{[(x, y, z),(x', y', z')]} := \sum_{i=0}^{5} c_i \cdot \sigma_i - d_{[(x, y, z),(x', y', z')]} \quad (15) \]
does not contain any diagonal simplex. It follows that \( \partial \psi \) is supported only on \( x \)-simplices, \( y \)-simplices and \( z \)-simplices in \( \Delta_2(X) \). From \((15)\) it follows that \( c - \partial \psi = b_x + b_y + b_z \) for some \( x \)-chain \( c_x \), \( y \)-chain \( c_y \) and \( z \)-chain \( c_z \) in \( C_2^{(\infty)}(X) \). Thus, we have \( \alpha = [c] = [b] \) for \( b = b_x + b_y + b_z \) and the claim follows. \( \square \)
In the following lemma we prove that any class in $H_2^{(\infty)}(X)$ can be represented by a cycle supported only on $y$-simplices and $z$-simplices. More precisely, we have:

**Lemma 18.** Let $\alpha \in H_2^{(\infty)}(X)$. There is a cycle $b \in C_2^{(\infty)}(X)$ of the form $b = b_y + b_z$ such that $\alpha = [b]$ in $H_2^{(\infty)}(X)$.

**Proof.** Let $c$ be any cycle in $C_2^{(\infty)}(X)$ representing $\alpha$. By Lemma 17 we can assume $c$ to be of the form $c = c_x + c_y + c_z$, for some $x$-chain $c_x$, some $y$-chain $c_y$ and some $z$-chain $c_z$ in $C_2^{(\infty)}(X)$. We want to construct a chain $\psi \in C_3^{(\infty)}(X)$ such that $c - \partial \psi = b_y + b_z$ for some $y$-chain $b_y$ and some $z$-chain $b_z$ in $C_2^{(\infty)}(X)$.

From Definition 14, we have that any (non-degenerate) $\sigma_x \in \Delta_2(X)_x$ is of one of the following forms:

1. $[(x, y, z), (x, y_A, z), (x, y_A, z_A)]$ for some $(x, y, z) \in V_X$, some $y_A \in A_y$ and some $z_A \in A_z$;

2. $[(x, y, z), (x, y, z_A), (x, y_A, z_A)]$ for some $(x, y, z) \in V_X$, some $y_A \in A_y$ and some $z_A \in A_z$.

For some $(x, y, z) \in V_X$, fix $y' \in A_y$ and $z' \in A_z$ and take $\sigma_x \in \Delta_2(X)_x$ of the form $\sigma_x = [(x, y, z), (x, y', z), (x, y', z')]$. Following the 0-dimensional Eilenberg swindles construction on $T_x$ (Section 2.3), we consider a tail of simplices attached...
to any vertex $x$ in $T_x$. More precisely, for any $x \in V_{T_x}$ let

$$t_x = \sum_{k \in \mathbb{Z}_{\leq 0}} [x_{k-1}, x_k] \in C_1^{(\infty)}(T_x)$$

such that, for all $k \in \mathbb{Z}_{\leq 0}$, we have $x_{k-1} < x_k$ and for $k = 0$, we have $x_0 = x$ in $T_x$. For any $k \in \mathbb{Z}_{\leq 0}$, consider the following simplices in $\Delta_3(X)$:

$\tau_0^k = [(x_{k-1}, y, z), (x_k, y, z), (x_k, y', z'), (x_k, y', z')]$,

$\tau_1^k = [(x_{k-1}, y, z), (x_{k-1}, y', z), (x_k, y', z), (x_k, y', z')]$,

$\tau_2^k = [(x_{k-1}, y, z), (x_{k-1}, y', z), (x_{k-1}, y', z'), (x_k, y', z')]$,

and define:

$$p_\sigma = \sum_{k \in \mathbb{Z}_{\leq 0}} \tau_0^k - \tau_1^k + \tau_2^k.$$  \hspace{1cm} (16)

In other words, $p_\sigma$ is an infinite sums of 3-simplices in $X$ which “follow” the direction of the tail $t_x$ attached to each vertex of $\sigma$ (Figure 5).

![Figure 5: A (rectangular) beam attached to the face of a cube (dark gray). It consists of two beams attached to two 2-simplices which share an edge. Thus, the beams intersect along a panel, whose one segment is shaded light gray.](image)

Notice that the boundary of $p_\sigma$, consists of $\sigma_x$ and 2-dimensional panels attached to each face of $\sigma_x$ (Figure 6). More precisely, consider the faces of $\sigma_x$:

$\sigma_0^0 = [(x, y', z), (x, y', z')]$,

$\sigma_1^1 = [(x, y, z), (x, y', z')]$,

$\sigma_2^2 = [(x, y, z), (x, y', z)]$.
and the following panels of 2-simplices in \( X \):
\[
p_{\sigma_0^0} = \sum_{k \leq z \leq 0} \left[ (x_{k-1}, y', z), (x_k, y', z), (x_k, y', z') \right] - \left[ (x_{k-1}, y', z), (x_k, y', z'), (x_k, y', z') \right],
\]
\[
p_{\sigma_1^1} = \sum_{k \leq z \leq 0} \left[ (x_{k-1}, y, z), (x_k, y, z), (x_k, y', z') \right] - \left[ (x_{k-1}, y, z), (x_k, y', z'), (x_k, y', z') \right],
\]
\[
p_{\sigma_2^2} = \sum_{k \leq z \leq 0} \left[ (x_{k-1}, y, z), (x_k, y, z), (x_k, y', z) \right] - \left[ (x_{k-1}, y, z), (x_k, y', z), (x_k, y', z) \right].
\]
(17)

An easy calculation shows that
\[
\partial p_{\sigma_x} = \sigma_x - p_{\sigma_0^0} + p_{\sigma_1^1} - p_{\sigma_2^2}.
\]
(18)

Following the construction above, after choosing a tail \( t_x \) for any \( x \in V_T \), we construct \( p_{\sigma_x} \) as in (16) for any \( \sigma_x \in \Delta_2(X)_x \) and we call it a beam attached to \( \sigma_x \). Then, we define:
\[
\psi = \sum_{\sigma_x \in \Delta_2(X)_x} c(\sigma_x) \cdot p_{\sigma_x}.
\]

By the Eilenberg-swindle construction given in Section 2.3, for any \( x \in V_T \), we can choose a tail \( t_x \) such that \( \sum_{x \in V_T} t_x \) is a well-defined element in \( C_{-1}^{(\infty)}(X) \).

It follows that any simplex in \( \Delta_2(X) \) is contained in a uniformly bounded number of beams of type \( p_{\sigma_x} \). Thus, \( \psi \) is a well-defined chain in \( C_{-1}^{(\infty)}(X) \).

From (18), it follows that
\[
\partial \psi = c_x - \sum_{\sigma_x \in \Delta_2(X)_x} c(\sigma_x) \cdot p_{\sigma_x} + \sum_{\sigma_x \in \Delta_2(X)_x} c(\sigma_x) \cdot p_{\sigma_x^1} - \sum_{\sigma_x \in \Delta_2(X)_x} c(\sigma_x) \cdot p_{\sigma_x^2}.
\]
(19)

Consider, again, \( \sigma_x = [(x, y, z), (x, y', z), (x, y', z')] \in \Delta_2(X)_x \). Notice that \( \sigma_x \) shares the face \( \sigma_1^1 \) with a unique simplex of the form \( \tilde{\sigma}_x = [(x, y, z), (x, y, z'), (x, y, z')] \).

By the cycle condition on \( c \), it follows that \( c(\sigma_x) = c(\tilde{\sigma}_x) = 0 \). Moreover, since \( \sigma_x = \tilde{\sigma}_x^1 \), by the construction of the panels, we have that \( p_{\sigma_x^1} = p_{\tilde{\sigma}_x^1} \). For any \( \sigma_x \in \Delta_2(X)_x \) we can find a unique \( \tilde{\sigma}_x \) such that \( \sigma_x^1 = \tilde{\sigma}_x^1 \) and such that \( c(\sigma_x) + c(\tilde{\sigma}_x) = 0 \). It follows that
\[
\sum_{\sigma_x \in \Delta_2(X)_x} c(\sigma_x) \cdot p_{\sigma_x^1} = 0.
\]

So, from (19) it follows that
\[
\partial \psi = c_x - \sum_{\sigma_x \in \Delta_2(X)_x} c(\sigma_x) \cdot p_{\sigma_x^1} - \sum_{\sigma_x \in \Delta_2(X)_x} c(\sigma_x) \cdot p_{\sigma_x^2}.
\]
In particular, we have
\[ c - \partial \varphi = c_y + \sum_{\sigma_x \in \Delta_2(X)_x} c(\sigma_x) \cdot p_{\sigma_x^0} + c_z + \sum_{\sigma_z \in \Delta_2(X)_z} c(\sigma_z) \cdot p_{\sigma_z^2} \]

From (17), it is easy to see that for any \( \sigma_x \in \Delta_2(X)_x \), the panel \( p_{\sigma_x^0} \) is a \( y \)-chain in \( C^2(X) \), while \( p_{\sigma_z^2} \) is a \( z \)-chain in \( C^2(X) \). In particular, \( b_y := c_y + \sum_{\sigma_x \in \Delta_2(X)_x} c(\sigma_x) \cdot p_{\sigma_x^0} \) is a \( y \)-chain in \( C^2(X) \) and \( b_z := c_z + \sum_{\sigma_z \in \Delta_2(X)_z} c(\sigma_z) \cdot p_{\sigma_z^2} \) is a \( z \)-chain in \( C^2(X) \). Since \( c - \partial \varphi = b_y + b_z \), we have that \( \alpha = [c] = [b] \) for \( b = b_y + b_z \) and the claim follows.

Now we want to prove that every class in \( H^2(X) \) can be represented by a \( z \)-chain (or \( x \)-chain or \( y \)-chain) that is a cycle in \( C^2(X) \). The technique used in the next lemma is similar to the one just seen in Lemma 18: we use beams in the \( y \)-direction to “kill” all the \( y \)-simplices in a cycle.

**Lemma 19.** Let \( \alpha \in H^2(X) \). There is a cycle \( b_z \in C^2(X) \) such that \( \alpha = [b_z] \) in \( H^2(X) \).
Proof. Let \( c \in C_2^{(\infty)}(X) \) be any cycle representing \( a \). From Lemma 18, we can assume \( c \) to be of the form \( c = c_y + c_z \), for some \( y \)-chain \( c_y \) and some \( z \)-chain \( c_z \) in \( C_2^{(\infty)}(X) \). From Definition 14 it follows that any \( y \)-simplex if of one of the following forms:

- \([ (x, y, z), (x, y, z_A), (x_A, y, z_A) ] \) for some \( (x, y, z) \in V_X \), some \( x_A \in A_x \) and some \( z_A \in A_z \).

- \([ (x, y, z), (x_A, y, z), (x_A, y, z_A) ] \) for some \( (x, y, z) \in V_X \), some \( x_A \in A_x \) and some \( z_A \in A_z \).

Consider \( \sigma_y = \{ (x, y, z), (x, y, z'), (x', y, z') \} \) for some \( (x, y, z) \in V_X \), and some \( x' \in A_x \), \( z' \in A_z \). For any \( y \in V_{T_y} \), consider a tail of simplices attached to \( y \) as given in Section 2.3. So, for any \( y \in V_{T_y} \), consider

\[
t_y = \sum_{k \in \mathbb{Z}_{\leq 0}} [y_{k-1}, y_k] \in C_2^{(\infty)}(T_y)
\]

such that for any \( k \in \mathbb{Z}_{\leq 0} \), we have \( y_{k-1} < y_k \) and for \( k = 0 \), we have \( y_0 = y \) in \( T_y \). For any \( k \in \mathbb{Z}_{\leq 0} \), consider the following simplices in \( \Delta_3(X) \):

\[
\tau_k^0 = [(x, y_{k-1}, z), (x, y_{k}, z), (x, y_k, z'), (x', y_k, z')],
\]

\[
\tau_k^1 = [(x, y_{k-1}, z), (x, y_{k-1}, z'), (x, y_k, z'), (x', y_k, z')],
\]

\[
\tau_k^2 = [(x, y_{k-1}, z), (x, y_{k-1}, z'), (x', y_{k-1}, z'), (x', y_k, z')],
\]

and define:

\[
p_{\sigma_y} = \sum_{k \in \mathbb{Z}_{\leq 0}} \tau_k^0 - \tau_k^1 + \tau_k^2. \tag{20}
\]

As in (16), this is just a beam attached to simplex \( \sigma_y \) and constructed by choosing 3-simplices in \( X \) “following” the tails in the \( y \)-direction. Now consider the faces of \( \sigma_y \):

\[
\sigma_y^0 = [(x, y, z'), (x', y, z')],
\]

\[
\sigma_y^1 = [(x, y, z), (x', y, z')],
\]

\[
\sigma_y^2 = [(x, y, z), (x, y, z')].
\]

As in Lemma 18, we have that

\[
\partial p_{\sigma_y} = \sigma_y - p_{\sigma_y^0} + p_{\sigma_y^1} - p_{\sigma_y^2}, \tag{21}
\]
where \( p_{\sigma^0_y}, p_{\sigma^1_y} \) and \( p_{\sigma^2_y} \) are the three panels attached to \( \sigma^0_y, \sigma^1_y \) and \( \sigma^2_y \) respectively and defined as follows:

\[
\begin{align*}
 p_{\sigma^0_y} &= \sum_{k \in \mathbb{Z}_{\geq 0}} [(x, y_{k-1}, z'), (x, y_k, z'), (x', y_k, z')] - [(x, y_{k-1}, z'), (x', y_{k-1}, z'), (x', y_k, z')], \\
p_{\sigma^1_y} &= \sum_{k \in \mathbb{Z}_{\geq 0}} [(x, y_{k-1}, z), (x, y_k, z), (x', y_k, z')] - [(x, y_{k-1}, z), (x', y_{k-1}, z), (x', y_k, z')], \\
p_{\sigma^2_y} &= \sum_{k \in \mathbb{Z}_{\geq 0}} [(x, y_{k-1}, z), (x, y_k, z), (x, y_k, z')] - [(x, y_{k-1}, z), (x, y_k, z'), (x, y_k, z')].
\end{align*}
\]

For any \( y \in V_T \), we construct a beam \( p_{\sigma_y} \) as above and we consider:

\[
\psi = \sum_{\sigma_y \in \Delta_2(X)_y} c(\sigma_y) \cdot p_{\sigma_y}.
\]

For any \( y \in V_T \), we can choose a tail \( t_y \) in such a way that \( \sum_{y \in V_T} t_y \) is a well-defined element in \( C_1^{(\infty)}(X) \). In particular, it follows that \( \psi \in C_2^{(\infty)}(X) \). By \( (21) \), it follows that

\[
\partial \psi = c_y - \sum_{\sigma_y \in \Delta_2(X)_y} c(\sigma_y) \cdot (p_{\sigma^0_y} - p_{\sigma^1_y} + p_{\sigma^2_y})
\]

Notice that for any \( \sigma_y \in \Delta_2(X)_y \), the panel \( p_{\sigma^0_y} \) is a z-chain in \( C_2^{(\infty)}(X) \) so to prove the statement it suffices to show that \( \sum_{\sigma_y \in \Delta_2(X)_y} c(\sigma_y) \cdot p_{\sigma^0_y} = 0 \) and \( \sum_{\sigma_y \in \Delta_2(X)_y} c(\sigma_y) \cdot p_{\sigma^2_y} = 0 \). Indeed, in this case we have

\[
c - \partial \psi = c_z + \sum_{\sigma_y \in \Delta_2(X)_y} c(\sigma_y) \cdot p_{\sigma^0_y}.
\]

In particular, \([c] = [b_z] \) where \( b_z = c_z + \sum_{\sigma_y \in \Delta_2(X)_y} c(\sigma_y) \cdot p_{\sigma^0_y} \) is a y-chain and the claim follows. As in Lemma 18, notice that \( \sigma_y = [(x, y, z), (x, y, z'), (x', y, z')] \) shares the face \( \sigma^1_y \) with \( \tilde{\sigma}_y = [(x, y, z), (x', y, z), (x', y, z')] \). By the cycle condition on \( c \), it follows that \( c(\sigma_y) + c(\tilde{\sigma}_y) = 0 \). Since \( \sigma^1_y = \tilde{\sigma}^1_y \), by the construction of the panels, we have that \( p_{\sigma^1_y} = p_{\tilde{\sigma}^1_y} \). Thus for any \( \sigma_y \in \Delta_2(X)_y \) we can find a unique simplex \( \tilde{\sigma}_y \in \Delta_2(X)_y \) such that \( \sigma^1_y = \tilde{\sigma}^1_y \) and such that \( c(\sigma_y) + c(\tilde{\sigma}_y) = 0 \). We, thus, have

\[
\sum_{\sigma_y \in \Delta_2(X)_y} c(\sigma_y) \cdot p_{\sigma^1_y} = 0.
\]

It remains to prove that \( \sum_{\sigma_y \in \Delta_2(X)_y} c(\sigma_y) \cdot p_{\sigma^2_y} = 0 \). Notice that every panel \( p_{\sigma^2_y} \) appears in \( \sum_{\sigma_y \in \Delta_2(X)_y} c(\sigma_y) \cdot p_{\sigma^2_y} \) with coefficient given by the sum of all the y-simplices having \( \sigma^2_y \) as a face.
Consider, again, $\sigma_y = [(x,y,z),(x,y,z'),(x',y,z')]$. Since $c$ is a cycle, the coefficients of the 2-simplices that have $\sigma_y^2$ as a face must sum up to zero. More precisely, we can make a list of all the 2-simplices in $X$ which have $\sigma_y^2$ as a face (Figure 7). They are:

1. $[(x,y,z),(x,y,z'),(x_A,y,z')] \in \Delta_2(X)_y$ for all $x_A \in A_x$ (see Definition 6);
2. $[(x_B,y,z),(x,y,z),(x,y,z')] \in \Delta_2(X)_y$ for all $x_B \in B_x$;
3. $[(x,y,z),(x,y,z'),(x,y_A,z')] \in \Delta_2(X)_x$ for all $y_A \in A_y$;
4. $[(x,y_B,z),(x,y,z),(x,y,z')] \in \Delta_2(X)_x$ for all $y_B \in B_y$;
5. $[(x,y,z),(x,y,z'),(x_A,y_A,z')] \in \Delta_2(X)_d$ for all $x_A \in A_x$ and all $y_A \in A_y$;
6. $[(x_B,y_B,z),(x,y,z),(x,y,z')] \in \Delta_2(X)_d$ for all $x_B \in B_x$ and all $y_B \in B_y$.

Notice that all the simplices in 3,4,5 and 6 are $x$-simplices or diagonal simplices. Since $c$ is of type $c = c_y + c_z$, these simplices do not appear in $c$, in particular, they have zero coefficient. On the other hand, the simplices in 1 and 2 might appear in $c$ with non-zero coefficient. Let

$$A_{\sigma_y^2} := \{(x,y,z),(x,y,z'),(x_A,y,z') \in \Delta_2(X)_y \mid x_A \in A_x\},$$

$$B_{\sigma_y^2} := \{(x_B,y,z),(x,y,z),(x,y,z') \in \Delta_2(X)_y \mid x_B \in B_x\}.$$
Figure 8: Three different beams along a tail in $T_x$ attached to $y$-simplices (brown) and $z$-simplices (not shown). The coefficients of the beams cancel out on the boundary on the wall shared by the beams (blue). The 2-dimensional skeleton of such a beam resembles closely the tail of an infinitely long Chinese dragon kite.

By the cycle condition on $c$, we have that

$$\sum_{\sigma \in A_y \cup B_y} c(\sigma) = 0. \tag{22}$$

We can repeat the operation for all $\sigma_y \in \Delta_2(X)_y$: we can consider the face $\sigma^2_y$ and all the $y$-simplices sharing the face $\sigma^x_y$ and their corresponding coefficient in $c$. It is easy to see that, for any $\sigma_y \in \Delta_2(X)_y$, the panel $p_{\sigma^2_y}$ appears in the sum $\sum_{\sigma_y \in \Delta_2(X)_y} c(\sigma_y) \cdot p_{\sigma^2_y}$ with coefficient

$$c(p_{\sigma^2_y}) := \sum_{\sigma \in A_y \cup B_y} c(\sigma).$$

Thus, by (22), for any $\sigma^2_y \in \Delta_2(X)_y$ we have that $c(p_{\sigma^2_y}) = 0$ (Figure 8). It follows that $\sum_{\sigma_y \in \Delta_2(X)_y} c(\sigma_y) \cdot p_{\sigma^2_y} = 0$. So we have that $c - \partial \psi = b_z$ where $b_z = c_z + \sum_{\sigma_y \in \Delta_2(X)_y} c(\sigma_y) \cdot p_{\sigma^2_y}$ and $\alpha = [c] = [b_z]$.

The last step before the proof of Theorem 13 is to prove that any cycle which are supported only on $z$-simplices gives a trivial class in homology, as the next lemma shows:
Lemma 20. Any \( \mathbf{x} \text{-class, } \mathbf{y} \text{-class and } \mathbf{z} \text{-class is trivial in } H_2^{(\infty)}(X) \).

Proof. We will prove the lemma for \( \mathbf{x} \)-classes, the other cases are similar. Let \( \alpha \in H_2^{(\infty)}(X) \) be a \( \mathbf{z} \)-class and let \( c_z = \sum_{\Delta_2(X)_k} c(\sigma_z) \cdot \sigma_z \) be a cycle representing \( \alpha \). To prove the lemma, we provide a chain in \( C_3^{(\infty)}(X) \) that bounds \( c_z \). We use, again, 0-dimensional Eilenberg swindles to construct such a chain. More precisely, as in the proof of Lemma 18 and Lemma 19, for any \( \mathbf{z} \)-simplex \( \sigma_z \), we construct a beam \( p_{\sigma_z} \) attached to \( \sigma_z \) and we define the chain

\[
\phi = \sum_{\sigma_z \in \Delta_2(X)_k} c(\sigma_z) \cdot p_{\sigma_z}.
\]

As in (18), for any \( \sigma_z \in \Delta_2(X)_k \) we have that

\[
\partial p_{\sigma_z} = \sigma_z - p_{\sigma_z}^0 + p_{\sigma_z}^1 - p_{\sigma_z}^2.
\]

and

\[
\partial \phi = c - \sum_{\sigma_z \in \Delta_2(X)_k} c(\sigma_z) \cdot (p_{\sigma_z}^0 - p_{\sigma_z}^1 + p_{\sigma_z}^2).
\]

(23)

So, to prove the statement it suffices to show that

\[
\sum_{\sigma_z \in \Delta_2(X)_k} c(\sigma_z) \cdot (p_{\sigma_z}^0 - p_{\sigma_z}^1 + p_{\sigma_z}^2) = 0.
\]

By a similar argument than the one used in Lemma 18 and Lemma 19, we have

\[
\sum_{\sigma_z \in \Delta_2(X)_k} c(\sigma_z) p_{\sigma_z}^1 = 0.
\]

By a similar argument as the one given in the proof of Lemma 19, using the cycle condition on \( c \), we can prove that \( \sum_{\sigma_z \in \Delta_2(X)_k} c(\sigma_z) \cdot p_{\sigma_z}^0 = 0 \). Indeed every panel \( p_{\sigma_z}^0 \) appears in the sum \( \sum_{\sigma_z \in \Delta_2(X)_k} c(\sigma_z) \cdot p_{\sigma_z}^0 \) with coefficient given by the sum of the coefficients of all the \( \mathbf{z} \)-simplices which have \( \sigma_z^0 \) as a face. Since \( c \) is a cycle and it does not contain \( \mathbf{x} \)-simplices and \( \mathbf{y} \)-simplices or diagonal simplices, it follows that the coefficients of the \( \mathbf{z} \)-simplices that have \( \sigma_z^0 \) as a face must sum up to zero. Thus, \( \sum_{\sigma_z \in \Delta_2(X)_k} c(\sigma_z) \cdot p_{\sigma_z}^0 = 0 \). In a similar way we can prove that \( \sum_{\sigma_z \in \Delta_2(X)_k} c(\sigma_z) \cdot p_{\sigma_z}^2 = 0 \). From (23), it follows that \( c = \partial \phi \) and, thus, \( \alpha = 0 \).

Now we are ready to prove Theorem 13.

Proof of Theorem 13. By Lemma 19, every class in \( H_2^{(\infty)}(T_x \times T_y \times T_z; R) \) is represented by a \( \mathbf{z} \)-chain \( b_z \) that is a cycle in \( C_2^{(\infty)}(T_x \times T_y \times T_z; R) \); in other words, every class in \( H_2^{(\infty)}(T_x \times T_y \times T_z; R) \) is a \( \mathbf{z} \)-class. On the other hand, by Lemma 20, every \( \mathbf{z} \)-class is trivial in \( H_2^{(\infty)}(T_x \times T_y \times T_z; R) \). Thus the claim follows. \[ \square \]
4.2 Proof of Theorem C

We now complete the proof of Theorem C. By the coarse invariance of uniformly finite homology we can consider the triangulated Cartesian product. Notice that the triangulated Cartesian product $X = T_1 \times T_2 \times T_3$ is a uniformly locally finite, uniformly contractible 3-dimensional simplicial complex, thus the coarsening homomorphism gives $H^{uf}_*(X;R) \cong H^{(\infty)}_*(X;R)$. Observe that since $X$ is non-amenable we have $H^{uf}_0(X;R) = 0$ for $R = \mathbb{R}, \mathbb{Z}$. Writing $X = T_1 \times (T_2 \times T_3)$ we conclude from Theorem A that $H^{uf}_1(X;R) = 0$. Since $X$ is 3-dimensional we obtain $H^{uf}_k(X;R) = 0$ for $k \geq 3$. Theorem 13 now gives $H^{uf}_3(X;R)$ is infinite dimensional.

Let $T_1, \ldots, T_k$ be trees. For any normed abelian group $R$ we have:

$$H^{uf}_*(\prod_{i=1}^k T_i; R) \cong H^{(\infty)}_*(\prod_{i=1}^k T_i; R).$$

Since $\prod_{i=1}^k T_i$ is a $k$-dimensional complex, clearly $H^{uf}_i(\prod_{i=1}^k T_i; R) = 0$ for $i \geq 3$. Theorem C follows from Proposition 21.

**Proposition 21.** $H^{(\infty)}_k(\prod_{i=1}^k T_i; R)$ is infinite-dimensional.

**Proof.** It suffices to show that the space of uniformly finite $k$-cycles is infinite-dimensional. Consider an bi-infinite geodesic $\sigma_i$ in $T_i$. The product $\prod_{i=1}^k \sigma_i$ is a uniformly finite $k$-cycle. If we choose $\sigma_i$ in $T_i$ such that $\sigma_i$ and $\sigma'_i$ lying in different branches of the tree and are disjoint, then the resulting $k$-cycles $\prod \sigma_i$ and $\prod \sigma'_i$ have disjoint supports.

The same proof gives a similar statements for the other three homology theories listed in the introduction.

4.3 Other large scale homology theories

We will now indicate how the above constructions apply to other large scale homology theories.

4.3.1 Coarse homology

In the case of locally finite $H^{lf}_0(X) = 0$ as soon as $X$ is infinite. Indeed, observe that in that case we can always arrange tails $t_x$, described in section 2.3, such that there is only finitely many passing through each edge. When inspecting the proof of Theorems A and 13 the reader can observe that the process of
attaching panels and beams, respectively, preserves local finiteness of coefficients of the cycles involved in this process. Therefore the same method of proof also proves Theorem B and Theorem D.

4.3.2 Almost equivariant homology

Almost equivariant homology is obtained by considering only those locally finite chains, that attain finitely many values. Such chains are automatically chains in the fine uniformly finite homology. We again observe that the process of attaching panels and beams preserves the property that a chain has finitely many values. Therefore, we can conclude that Theorems A and C hold when the uniformly finite homology $H_{uf}$ is replaced with the almost equivariant homology $H_{ae}$.

4.3.3 Controlled coarse homology

Chains in the controlled coarse homology $H^f_*$ are locally finite chains, whose growth is controlled by a fixed, non-decreasing function $f$, see [16] for details. In this case, the process of attaching panels and beams can influence the control functions, however again in a controlled way. For instance, in the case of a product for which $[X] = 0$ in $H^f_0(X)$ and $[Y] = 0$ in $H^g_0(Y)$, our method gives

$$H^f_1(X \times Y) = 0.$$  

We leave the details to the reader.

5 Applications

5.1 A characterization of amenable groups

Here we prove a characterization of amenability in terms of 1-homology.

Proof of Corollary F. If $G$ is non-amenable, then by Theorem A we have that $H^1_{(\infty)}(\Gamma \times T; \mathbb{R}) = 0$.

Assume now that $G$ is amenable. Let $c$ be a cycle in $C_1^{(\infty)}(\Gamma \times T; \mathbb{R})$. Then, as in the proof of Theorem 12, we can choose $c'$ representing the same class in uniformly finite homology, such that $c'$ vanishes on all horizontal edges; that is, on edges of the form $e \times p$, for an edge $e$ in $\Gamma$ and a vertex $p$ in $T$. Then, averaging $c'$ over $\Gamma$ using the invariant mean on $G$, we obtain a new 1-cycle, $d$. There is also a natural map

$$i : H^1_{(\infty)}(T; \mathbb{R}) \to H^1_{(\infty)}(\Gamma \times T; \mathbb{R}),$$

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defined by copying a cycle in $T$ onto every vertical edge. The composition of $i$ with the averaging map is the identity on the cycles in $C^{(\infty)}_1(G;\mathbb{R})$. It follows that the infinite-dimensional $H^{(\infty)}_1(T;\mathbb{R})$ injects into $H^{(\infty)}_1(G \times T;\mathbb{R})$.  

5.2 Aperiodic tiles

This section owes much to discussions of the second author with Shmuel Weinberger.

Let $X$ be an infinite simplicial complex equipped with a metric. A set of tiles for $X$ is a triple $\{T, W, m\}$, where $T$ is a finite collection of finite polygons with boundary, called prototiles or simply tiles, each of which has distinguished faces, $W$ is the set of all faces of the prototiles in $T$ and $m: W \to W$ is a matching function, determining which tiles can be neighboring tiles in a tiling. A tiling of $X$ by the set of tiles $T$ is a cover $X = \bigcup T_i$, where each $T_i$ is simplicially isomorphic to one of the prototiles, every non-empty intersection of two distinct $T_i$ and $T_j$ is identified with faces $w_i$ and $w_j$ of the corresponding tiles and satisfies $m(w_i) = w_j$. Such a tiling is aperiodic if no group acting on $X$ cocompactly by simplicial automorphisms preserves the tiling. An aperiodic set of tiles of $X$ is a set of tiles admitting only aperiodic tilings. Block and Weinberger used uniformly finite homology to construct a periodic tiles for every non-amenable space [2], see also [17]. More recently coarse homology was also used to construct aperiodic tiles for certain amenable manifolds [15].

The result of this paper allows us to construct aperiodic tiles for products as in [2], but using 1-homology instead of 0-homology. Let $M$ and $N$ be finite simplicial complexes, such that $\pi_1(M)$ and $\pi_1(N)$ are both non-amenable and $H_1(M \times N;\mathbb{R}) \neq 0$. The universal cover $\tilde{M} \times \tilde{N}$ of $M \times N$ satisfies

$$H^{ae}_1(\tilde{M} \times \tilde{N}) = 0,$$

by Theorem A.

Consider the infinite transfer

$$\tau: H_1(M \times N;\mathbb{R}) \to H^{ae}_1(\tilde{M} \times \tilde{N};\mathbb{R}) = 0$$

into the almost equivariant homology of the universal cover $\tilde{M} \times \tilde{N}$ of $M \times N$. Given a chain $a$ on $M \times N$ the map $\tau$ assigns coefficient $a(\sigma)$, where $\sigma$ is a simplex in $M \times N$, to every simplex $\tilde{\sigma}$ laying over $\sigma$ in $\tilde{M} \times \tilde{N}$.

We choose a fundamental polytope for the action of $\Gamma = \pi_1(M) \times \pi_1(N)$ and consider $\tau(\sigma) = [a]$ for some class $0 \neq a \in H_1(\Gamma,\mathbb{R})$. Then $a$ is $\Gamma$-equivariant and

$$a = \partial \psi,$$
for some almost equivariant 2-chain $\psi$ on $\tilde{M} \times \tilde{N}$. Since $\psi$ has finitely many values, there are finitely many types of such decoration and the rule we impose is that tiles match if the restrictions of $\psi$ to the tiles give $a$ as a boundary on neighboring tiles. In this way we obtain a finite set of tiles $\mathcal{T}$ of $\tilde{M} \times \tilde{N}$.

**Proposition 22.** The set $\mathcal{T}$ is an aperiodic set of tiles of $\tilde{M} \times \tilde{N}$.

**Proof.** Consider a tiling of $\tilde{M} \times \tilde{N}$ by tiles from $\mathcal{T}$ and assume that it is periodic; that is, it would be preserved by a finite index normal subgroup $H \subseteq \Gamma$. The restrictions of $\psi$ to the tiles now form a new almost equivariant chain, call it $\phi$, but the matching rule guarantees that $\partial \phi = a$. Additionally, both $\phi$ and $a$ are $H$-equivariant, and thus pass down to the homology group $H_1((\tilde{M} \times \tilde{N})/H; \mathbb{R})$, giving $\tau_H(a) = 0$, where $\tau_H : H_1(M \times N; \mathbb{R}) \to H_1((\tilde{M} \times \tilde{N})/H; \mathbb{R})$ is the standard finite transfer map. However, this is impossible, since the standard finite transfer with coefficients in $\mathbb{R}$ is always an injection on homology. \qed

6 Some questions and final remarks

In the case of a product of trees we believe that a higher dimensional vanishing up to the “rank” should also be true.

**Conjecture 23.** Let $T$ be a tree. Then for every $k \in \mathbb{N}$ we have $H^u_i(T^{k+1}) = 0$ for $i = 0, 1, \ldots, k$.

Our current method for $k = 2$ relies essentially on the features of low-dimensional products, in particular on the difference 1 between the degree of homology and the number of factors.

A second case, in which we believe similar vanishing should take place is the case of affine buildings. Recall that thick affine buildings exhibit branching: every edge is common to three 2-simplices. For example, a product of two trees is a building. This branching allows to make some reductions of general cycles to cycles of specific form, similarly as in the case of products.

**Conjecture 24.** Let $X$ be a thick affine building. Then $H^u_k(X) = 0$ for $k = 0, \ldots, \text{rank} X - 1$.

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