HIGHER ARITHMETIC CHERN CHARACTER

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Abstract. A map from the higher arithmetic $K$-group defined by the author to the higher arithmetic Chow group constructed by Burgos and Feliu is given. It is a higher extension of the arithmetic Chern character established by Gillet and Soulé, and it can also be regarded as an analogue of Beilinson’s regulator in Arakelov geometry. It is shown that this map is compatible with the pull-back map and the product structure.

Contents

1. Introduction 1
2. Multi-relative complexes of exact cubes 5
3. The multi-relative $K$-theory and the higher Bott-Chern forms 24
4. Chern form of a hermitian vector bundle on an iterated double 36
5. Several arithmetic $K$-groups 41
6. Arithmetic Chern character of a hermitian vector bundle on an iterated double 46
7. Definition of higher arithmetic Chern character 59
8. Compatibility with pull-back maps 64
9. A tensor product structure on the multi-relative complexes of exact cubes 70
10. $\hat{K}_0(X)$-module structures on arithmetic $K$-groups 82
11. Compatibility with product 89
References 98

1. Introduction

In a celebrated paper [GS2], Gillet and Soulé established a theory of arithmetic Chern character associated with a hermitian vector bundle in Arakelov geometry. This theory is very useful in generating elements of arithmetic Chow groups, and a lot of interesting equalities of numbers have been deduced by calculating the arithmetic intersection of these elements.

The arithmetic Chern characters of hermitian vector bundles on an arithmetic variety $X$ provide a map from the arithmetic $K_0$-group to the arithmetic Chow group

$$\hat{\text{ch}}_0^p : \hat{K}_0(X) \to \hat{CH}^p(X).$$

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Recently, the both sides of this map were extended to higher degree. For instance, in [Ta] the author gave a definition of higher arithmetic $K$-groups $\hat{K}_r(X)$. On the other hand, in [Go] and [BF] two distinct definitions of higher arithmetic Chow groups $\widehat{CH}^p(X,r)$ were proposed when $X$ is defined over an arithmetic field. The both definitions are based on descriptions of the regulator map

$$CH^p(X,r) \rightarrow H_{BD}^{2p-r}(X,\mathbb{R}(p))$$

by means of integral on algebraic cycles, and in [BFT] it was shown that these two definitions are essentially the same.

The aim of this paper is to construct a map from $\hat{K}_r(X)$ to $\widehat{CH}^p(X,r)$ called the higher arithmetic Chern character. It can be seen as an extension of the map (1.1) to higher degree, and also as an analogue in Arakelov geometry of the higher Chern character $\text{ch}_p^r : K_r(X) \rightarrow CH^p(X, r)$.

In order to define the higher Chern character map we usually make use of homotopy theory. Hence we adopt an alternative approach.

In [Le] Levine introduced a scheme called iterated double in order to prove several properties of the higher Chow group, such as localization sequence and contravariant functoriality, without relying on the moving lemma. In doing this, he showed that the higher $K$-group of a scheme can be embedded into the $K_0$-group of the associated iterated double. Let us explain this in detail.

Let $X$ be a regular scheme, and $\Box = \mathbb{P}^1 - \{1\}$ the affine line with a subscheme $\partial \Box = \{0, \infty\}$. Let $\Box^r$ be the cartesian product of $r$ copies of $\Box$. Then the subscheme $\partial \Box^r$ on each component gives a normal crossing divisor $\partial \Box^r \subset \Box^r$. As for the precise definition of $\partial \Box^r$, see §3.6. It is well-known that the $r$-th $K$-group $K_r(X)$ is isomorphic to the multi-relative $K_0$-group of the scheme $X \times \Box^r$ with the normal crossing divisor $X \times \partial \Box^r$.

Denote by $T = D(X \times \Box^r; X \times \partial \Box^r)$ the associated iterated double. Then we obtain closed subschemes $T_1, \ldots, T_r \subset T$ with a closed embedding

$$i_0^r : (X \times \Box^r; X \times \partial \Box^r) \hookrightarrow (T; T_1, \ldots, T_r).$$

Levine showed in [Le] that the pull-back map of multi-relative $K$-groups

$$i_0^r : K_*(T; T_1, \ldots, T_r) \rightarrow K_*(X \times \Box^r; X \times \partial \Box^r)$$

is bijective. Hence we have the following sequence of maps:

$$K_r(X) \cong K_0(X \times \Box^r; X \times \partial \Box^r) \xrightarrow{i_0^r} K_0(T; T_1, \ldots, T_r) \subset K_0(T). \tag{1.2}$$

This means that any element of $K_r(X)$ can be represented by a virtual vector bundle on the iterated double $T$.

In this paper we pursue an analogue of (1.2) in Arakelov geometry. To accomplish this purpose, we define arithmetic analogues of $K$-groups appearing in (1.2). Then we obtain the following sequence:

$$\hat{K}_r(X)_Q \cong \widehat{K}_0(X \times \Box^r; X \times \partial \Box^r)_Q \xrightarrow{\hat{i}_0^r} \widehat{K}_0(T; T_1, \ldots, T_r)_Q \subset \widehat{K}_0(T)_Q, \tag{1.3}$$
where \( A_\mathbb{Q} \) stands for \( A \otimes \mathbb{Q} \). Note that in the arithmetic case this sequence holds only in rational coefficients, and the middle arrow \( \hat{\gamma}_0^* \) is shown to be surjective. We do not know whether \( \hat{\gamma}_0^* \) is bijective or not, nevertheless we know how any element of \( \text{Ker} \hat{\gamma}_0^* \) is described.

We next establish a theory of arithmetic Chern character of a hermitian vector bundle on the iterated double \( T \). This provides a map

\[
(1.4) \quad \hat{c}_T^p : \hat{K}_0(T) \to \hat{CH}^p(X, r).
\]

Combining (1.3) with (1.4), we can obtain the desired map

\[
(1.5) \quad \hat{c}_r^p : \hat{K}_r(X)_{\mathbb{Q}} \to \hat{CH}^p(X, r).
\]

If we ignore the metric structure in the definition of (1.5), we can define

\[
ch^p_r : K_r(X)_{\mathbb{Q}} \to H^{2p-r}(X, \mathbb{R}(p))
\]

which make the following diagram commutative:

\[
\begin{array}{ccc}
\hat{K}_r(X)_{\mathbb{Q}} & \longrightarrow & K_r(X)_{\mathbb{Q}} \\
\downarrow \hat{c}_r^p & & \downarrow ch^p_r \\
\hat{CH}^p(X, r) & \longrightarrow & CH^p(X, r).
\end{array}
\]

However, in this paper we could not prove that the map \( ch^p_r \) agrees with the higher Chern character. Nevertheless we can show that the composite with the regulator map

\[
K_r(X)_{\mathbb{Q}} \xrightarrow{ch^p_r} CH^p(X, r) \to H^{2p-r}(X, \mathbb{R}(p))
\]

agrees with Beilinson’s regulator. This is an evidence that the map \( ch^p_r \) is nothing but the higher Chern character.

We describe the content of each section.

In §2, we first recall a chain complex of exact cubes which computes the rational \( K \)-theory [Mc]. Then we introduce a subcomplex on which the symmetric group acts alternatingly, and show that these two chain complexes are quasi-isomorphic. We next introduce \( C \)-complex [Ha], and we employ it to define a chain complex called multi-relative complex of exact cubes, which computes the multi-relative rational \( K \)-theory. We examine functorial properties of the multi-relative complex of exact cubes.

In §3, we introduce several complexes of differential forms and of currents which compute the Deligne cohomology. We then introduce Wang’s forms and the higher Bott-Chern forms [BW]. We also give a map from the multi-relative complex of exact cubes introduced in §3 to a complex of differential forms, which can be seen as a generalization of the higher Bott-Chern forms to the relative case.

In §4, we introduce iterated double, and establish a theory of Chern forms of hermitian vector bundles on it. In §5, we define relative arithmetic \( K \)-theory, and arithmetic \( K_0 \)-group of an iterated double. We then deduce the sequence (1.3).

In §6, after recalling the definition of the higher arithmetic Chow groups \( \hat{CH}^p(X, r) \) given by Burgos nad Feliu [BF], we give a definition of arithmetic Chern character of hermitian vector bundles on an iterated double. Then we show that it induces the map (1.4). In §7, we deduce the desired map (1.5), and in §8 we show the compatibility of this map with the pull-back map.
The last three sections are devoted to showing the compatibility of \((1.5)\) with product with the arithmetic \(K_0\)-group and the arithmetic Chow group. In \(\S 9\), we put a tensor product structure on the multi-relative complexes of exact cubes. In \(\S 10\), we show that all the arithmetic \(K\)-groups defined in \(\S 5\) admit products with \(\widehat{K}_0(X)\), and the sequence \((1.3)\) respects the \(\widehat{K}_0(X)\)-module structures. Finally in \(\S 11\), we show that the arithmetic Chern character of a hermitian vector bundle on an iterated double has a multiplicative property with the tensor product. Collecting these results, we can show that \((1.5)\) is compatible with the product structures.

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Notations and conventions. Throughout this paper, we fix an universe and assume that all the sets we will consider are contained in this fixed universe. Hence the category of vector bundles on a scheme is a small exact category.

Any small exact category is assumed to have a distinguished zero object denoted by \(0\), and any exact functor between exact categories is assumed to preserve the distinguished zero object. In particular, we fix a distinguished zero object of the category of abelian groups.

For a scheme \(X\), we denote by \(\Psi(X)\) the category of locally free \(\mathcal{O}_X\)-modules of finite rank on \(X\). We identify locally free \(\mathcal{O}_X\)-modules of finite rank with vector bundles of finite rank on \(X\) in the usual way. Then \(\Psi(X)\) is a small exact category, and a distinguished zero object of \(\Psi(X)\) is given as follows: For any open set \(U \subset X\), \(0(U)\) is the distinguished zero object of the category of abelian groups.

For any morphism \(f : X \to Y\) of schemes, there is a pull-back functor \(f^* : \Psi(Y) \to \Psi(X)\), which is an exact functor preserving the distinguished zero object. When \(f\) is the identity \(\text{Id}_X : X \to X\), we identify \(\text{Id}_X^*\) with the identity functor of \(\Psi(X)\).

We fix some conventions on complexes. With a complex of abelian groups \((A^*, d_A)\), we can associate a chain complex \((A_*, \partial_A)\) in the way that \(A_* = A^{-\ast}\) and \(\partial_A = d_A\). In this paper we always identify a complex and the associated chain complex in this way. For a chain complex of abelian groups \(A_* = (A_n, \partial_A)\), let \(A[r]_*\) be a chain complex such that \(A[r]_n = A_{n-r}\) with \(\partial_{A[r]} = (-1)^r \partial_A : A_{n-r} \to A_{n-r-1}\). Similarly, for a complex \(A^* = (A^*, d_A)\), let \(A[r]^*\) be a complex such that \(A[r]_n = A^{n+r}\) with \(d_{A[r]} = (-1)^r d_A\). We define the simple complex \(s(f)\) of a map \(f : A_* \to B_*\) of chain complexes to be

\[ s(f)_n = A_n \oplus B_{n+1} \]

with the boundary map

\[ \partial : s(f)_n \to s(f)_{n-1}, \quad \partial(a, b) = (\partial a, f(a) - \partial b). \]

Then we have a long exact sequence on homology

\[ \cdots \to H_{n+1}(B_\ast) \to H_n(s(f)_\ast) \to H_n(A_\ast) \to H_n(B_\ast) \to \cdots. \]
Throughout this paper, we denote by $[a] \in H_n(A_*)$ the homology class represented by $a \in \operatorname{Ker} \partial$. Moreover, we denote by $\bar{a} \in A_*/\operatorname{Im} \partial$ the element represented by $a \in A_n$.

Finally, we define the truncated relative cohomology groups of a map of complexes $A^* \to B^*$ to be

$$\tilde{H}^n(A^*, B^*) = \{(a, b) \in A^n \oplus B^{n-1} / \text{Im } d; \, da = 0, f(a) = db\}$$

[Bu2 Def.4.2]. We are going to use these groups to define Green objects associated with an algebraic cycle in §6.

2. Multi-relative complexes of exact cubes

2.1. Complexes of exact cubes. Let $\mathfrak{A}$ be a small exact category. We see the ordered set $\{-1, 0, 1\}$ as a category. For $n \geq 0$, an $n$-cube of $\mathfrak{A}$ is a covariant functor from the product $\{-1, 0, 1\}^n$ to $\mathfrak{A}$. Note that a 0-cube of $\mathfrak{A}$ is an object of $\mathfrak{A}$. For an $n$-cube $\mathcal{F}$ of $\mathfrak{A}$, denote by $\mathcal{F}_{a_1, \ldots, a_n}$, the image of the object $(a_1, \ldots, a_n) \in \{-1, 0, 1\}^n$ by $\mathcal{F}$. For $1 \leq j \leq n$ and $-1 \leq i \leq 1$, a face of $\mathcal{F}$ is an $(n-1)$-cube $\partial^j_i \mathcal{F}$ defined by

$$(\partial^j_i \mathcal{F})_{a_1, \ldots, a_{n-1}} = \mathcal{F}_{a_1, \ldots, a_{j-1}, i, a_{j+1}, \ldots, a_{n-1}}.$$ 

For $\alpha = (a_1, \ldots, a_{n-1}) \in \{-1, 0, 1\}^{n-1}$ and for an integer $1 \leq j \leq n$, an edge of $\mathcal{F}$ is an 1-cube $\partial^j_{\alpha} \mathcal{F}$ described as

$$\mathcal{F}_{a_1, \ldots, a_{j-1}, -1, a_{j+1}, \ldots, a_{n-1}} \to \mathcal{F}_{a_1, \ldots, a_{j-1}, 0, a_{j+1}, \ldots, a_{n-1}} \to \mathcal{F}_{a_1, \ldots, a_{j-1}, 1, a_{j+1}, \ldots, a_{n-1}}.$$

An $n$-cube $\mathcal{F}$ of $\mathfrak{A}$ is said to be exact if all the edges of $\mathcal{F}$ are short exact sequences.

For an exact $(n-1)$-cube $\mathcal{F}$ of $\mathfrak{A}$ and for an integer $1 \leq j \leq n$, let $s_j \mathcal{F}$ be an exact $n$-cube such that the edge $\partial^j_{\alpha}(s_j \mathcal{F})$ is $\mathcal{F}_{\alpha} \xrightarrow{Id} \mathcal{F}_{\alpha}$ for any $\alpha \in \{-1, 0, 1\}^{n-1}$. Similarly, let $s_{j-1} \mathcal{F}$ be an exact $n$-cube such that $\partial^{j-1}_{\alpha}(s_{j-1} \mathcal{F})$ is $0 \to \mathcal{F}_{\alpha} \xrightarrow{Id} \mathcal{F}_{\alpha}$ for any $\alpha \in \{-1, 0, 1\}^{n-1}$. These exact $n$-cubes are said to be degenerate.

Denote by $\mathbb{Q}C_n(\mathfrak{A})$ the $\mathbb{Q}$-vector space generated by all exact $n$-cubes of $\mathfrak{A}$. The alternating sum of faces gives a map of $\mathbb{Q}$-vector spaces

$$\partial = \sum_{j=1}^{n} \sum_{i=-1}^{1} (-1)^{i+j} \partial^j_i : \mathbb{Q}C_r(\mathfrak{A}) \to \mathbb{Q}C_{r-1}(\mathfrak{A}),$$

by which $(\mathbb{Q}C_*(\mathfrak{A}), \partial)$ is a chain complex. Denote by $D_n$ the subgroup of $\mathbb{Q}C_n(\mathfrak{A})$ generated by all degenerate $n$-cubes. Then $D_*$ is a subcomplex of $\mathbb{Q}C_*(\mathfrak{A})$.

Theorem 2.1. [Mc] The homology group of the quotient complex

$$\mathbb{Q}C_*(\mathfrak{A}) = \mathbb{Q}C_*(\mathfrak{A}) / D_*$$

is canonically isomorphic to the rational algebraic $K$-theory of $\mathfrak{A}$:

$$H_n(\mathbb{Q}C_*(\mathfrak{A}), \partial) \simeq K_n(\mathfrak{A})_{\mathbb{Q}}.$$
2.2. **The subcomplex** $\tilde{Q}_{C^\text{Alt}}(\mathcal{A})$. Let $\mathcal{A}$ be a small exact category, and $\mathcal{S}_n$ the $n$-th symmetric group. For an exact $n$-cube $F$ of $\mathcal{A}$ and for $\sigma \in \mathcal{S}_n$, let $\sigma F$ be an exact $n$-cube such that $(\sigma F)_{\alpha_1, \ldots, \alpha_n} = F_{\sigma(1), \ldots, \sigma(n)}$. Then $\mathcal{S}_n$ acts on $Q_{C_n}(\mathcal{A})$, and we can obtain the alternating part

$$Q_{C^\text{Alt}}(\mathcal{A}) = \{ x \in Q_{C_n}(\mathcal{A}) ; \sigma(x) = (\operatorname{sgn} \sigma)x \text{ for any } \sigma \in \mathcal{S}_n \}$$

with the section $\operatorname{Alt}_n : Q_{C_n}(\mathcal{A}) \to Q_{C^\text{Alt}}(\mathcal{A})$ defined by

$$(2.1) \quad \operatorname{Alt}_n(x) = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} (\operatorname{sgn} \sigma)\sigma(x).$$

We can show in the same way as [Bl, Lem.1.1] that $\operatorname{Alt}_{n-1}$ is a map of complexes. Since $\sigma F$ is degenerate if so is $F$, the map $\operatorname{Alt}_n$ gives rise to a map of complexes

$$\tilde{\operatorname{Alt}}_n : \tilde{Q}_{C_n}(\mathcal{A}) \to \tilde{Q}_{C^\text{Alt}}(\mathcal{A}),$$

which is also a section of the inclusion $\tilde{Q}_{C^\text{Alt}}(\mathcal{A}) \subset \tilde{Q}_{C_n}(\mathcal{A})$.

**Theorem 2.2.** The inclusion $\tilde{Q}_{C^\text{Alt}}(\mathcal{A}) \hookrightarrow \tilde{Q}_{C_n}(\mathcal{A})$ is a quasi-isomorphism. Hence we have a canonical isomorphism

$$H_n(\tilde{Q}_{C^\text{Alt}}(\mathcal{A})) \simeq K_\mathbb{Q}(\mathcal{A}).$$

To prove this theorem, we need some preparation. For an exact 1-cube $F : A \xrightarrow{f} B \xrightarrow{g} C$ of $\mathcal{A}$, consider the following exact 2-cube:

$$\begin{array}{ccc}
A & \xrightarrow{\operatorname{Id}} & A \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\rho(F) = & A & \xrightarrow{f} B \xrightarrow{g} C \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
C & \xrightarrow{\operatorname{Id}} & C.
\end{array}$$

Suppose $n \geq 1$. For an exact $n$-cube $F$ of $\mathcal{A}$ and for $1 \leq j \leq n$, let $\rho_{n,j}(F)$ be an exact $(n + 1)$-cube such that

$$\partial_{1}^{\alpha_1} \cdots \partial_{j-1}^{\alpha_{j-1}} \partial_{j+1}^{\alpha_{j+1}} \cdots \partial_{n}^{\alpha_n} \rho_{n,j}(F) = \rho \left( \partial_{1}^{\alpha_1} \cdots \partial_{j-1}^{\alpha_{j-1}} \partial_{j+1}^{\alpha_{j+1}} \cdots \partial_{n}^{\alpha_n} F \right).$$

It is obvious that $\rho_{n,j}(F)$ is degenerate if so is $F$. The lemma below follows immediately from the definition:
Lemma 2.3.
\[ \partial_j^0 \rho_{n,j}(\mathcal{F}) = \mathcal{F} = \partial_j^0 \rho_{n,j}(\mathcal{F}), \]
\[ \partial_j^{-1} \rho_{n,j}(\mathcal{F}) = s_j^1 \partial_j^{-1} \mathcal{F} = \partial_j^{-1} \rho_{n,j}(\mathcal{F}), \]
\[ \partial_j^1 \rho_{n,j}(\mathcal{F}) = s_j^{-1} \partial_j^1 \mathcal{F} = \partial_j^1 \rho_{n,j}(\mathcal{F}), \]
and
\[ \partial_k^i \rho_{n,j}(\mathcal{F}) = \begin{cases} 
\rho_{n-1,j-1}(\partial_k^i \mathcal{F}), & k < j, \\
\rho_{n-1,j}(\partial_k^{i-1} \mathcal{F}), & k > j + 1.
\end{cases} \]
Moreover,
\[ \rho_{n+1,j+1}(\rho_{n,j}(\mathcal{F})) = \rho_{n+1,j}(\rho_{n,j}(\mathcal{F})). \]

Let \( \tilde{Q}C_{n,m}(\mathfrak{A}) = \tilde{Q}C_{n+m}(\mathfrak{A}) \) with boundary maps
\[ \partial' = \sum_{j=1}^{n} \sum_{i=-1}^{1} (-1)^{i+j} \partial_j^i : \tilde{Q}C_{n,m}(\mathfrak{A}) \rightarrow \tilde{Q}C_{n-1,m}(\mathfrak{A}), \]
\[ \partial'' = \sum_{j=1}^{m} \sum_{i=-1}^{1} (-1)^{i+j} \partial_n^{i+j} : \tilde{Q}C_{n,m}(\mathfrak{A}) \rightarrow \tilde{Q}C_{n-1,m}(\mathfrak{A}). \]
Then \( (\tilde{Q}C_{n,m}(\mathfrak{A}), \partial', \partial'') \) is a double chain complex.

Lemma 2.4. If \( m \geq 1 \), then \( (\tilde{Q}C_{*,m}(\mathfrak{A}), \partial') \) is acyclic, and if \( n \geq 1 \), then \( (\tilde{Q}C_{n,*}(\mathfrak{A}), \partial'') \) is acyclic.

Proof: For \( m \geq 1 \), define a map
\[ \Phi_{n,m} : \tilde{Q}C_{n,m}(\mathfrak{A}) \rightarrow \tilde{Q}C_{n+1,m}(\mathfrak{A}) \]
by \( \Phi_{n,m}(\mathcal{F}) = (-1)^{n+1} \rho_{n+m,n+1}(\mathcal{F}) \) for an exact \((n + m)\)-cube \( \mathcal{F} \) of \( \mathfrak{A} \). Then \( (\Phi_{*,m}) \) turns out to be a homotopy from the zero map to the identity of the chain complex \( (\tilde{Q}C_{*,m}(\mathfrak{A}), \partial') \). Similarly, if we define
\[ \Psi_{n,m} : \tilde{Q}C_{n,m}(\mathfrak{A}) \rightarrow \tilde{Q}C_{n,m+1}(\mathfrak{A}) \]
for \( n \geq 1 \) by \( \Psi_{n,m}(\mathcal{F}) = -\rho_{n+m,n}(\mathcal{F}) \), then \( (\Psi_{n,*}) \) turns out to be a homotopy from the zero map to the identity of the chain complex \( (\tilde{Q}C_{n,*}(\mathfrak{A}), \partial'') \). \( \square \)

Proof of Thm. 2.2: We will prove that \( \text{Alt}_{*} \) induces an isomorphism on homology. We see \( \mathcal{G}_n \) as a subgroup of \( \mathcal{G}_{n+m} \) in the way that
\[ \mathcal{G}_n = \{ \sigma \in \mathcal{G}_{n+m}; \sigma(j) = j \text{ if } n + 1 \leq j \leq n + m \}. \]
Set
\[ QC_{n,m}^0(\mathfrak{A}) = \{ x \in QC_{n+m}(\mathfrak{A}); \sigma(x) = (\text{sgn} \sigma)x \text{ for any } \sigma \in \mathcal{G}_n \}, \]
and
\[ \tilde{Q}C_{n,m}^0(\mathfrak{A}) = QC_{n,m}^0(\mathfrak{A})/QC_{n,m}^0(\mathfrak{A}) \cap D_{n+m}. \]
Then \((\widetilde{Q}C^0_{n,m}(\mathfrak{A}), \partial', \partial'')\) is a subcomplex of \((\widetilde{Q}C_{n,m}(\mathfrak{A}), \partial', \partial'')\). Define a map

\[
\text{Alt}_{n,m} : \widetilde{Q}C_{n,m}(\mathfrak{A}) \to \widetilde{Q}C^0_{n,m}(\mathfrak{A})
\]

by

\[
\text{Alt}_{n,m}(F) = \frac{1}{n!} \sum_{\sigma \in S_n} (\text{sgn } \sigma) \sigma(F)
\]

for any exact \((n + m)\)-cube \(F\) of \(\mathfrak{A}\). Since the action of \(S_n\) commutes with \(\rho_{n+m,n+1}\), it holds that

\[
(2.2) \quad \text{Alt}_{n+1,m} \Phi_{n,m} = \Phi_{n,m} \text{Alt}_{n,m}.
\]

If \(m \geq 1\), then the chain complex \((\widetilde{Q}C^0_{n,m}(\mathfrak{A}), \partial')\) is proved to be acyclic. In fact, the map

\[
\Phi_{n,m}^\text{alt} = \text{Alt}_{n+1,m} \Phi_{n,m} : \widetilde{Q}C_{n,m}(\mathfrak{A}) \to \widetilde{Q}C^0_{n+1,m}(\mathfrak{A})
\]

turns out to be a homotopy from the zero map to the identity. On the other hand, since \((\widetilde{Q}C^0_{n,m}(\mathfrak{A}), \partial'')\) is a subcomplex of \((\widetilde{Q}C_{n,m}(\mathfrak{A}), \partial'')\) with the splitting map \(\Phi_{n,m}^\text{alt}\), Lem.2.4 implies that \((\widetilde{Q}C^0_{n,m}(\mathfrak{A}), \partial'')\) is also acyclic for \(n \geq 1\). Consequently, the inclusions

\[
\varepsilon_1 : \widetilde{Q}C_* (\mathfrak{A}) = \widetilde{Q}C^0_{0,*} (\mathfrak{A}) \to \text{Tot}(\widetilde{Q}C^0_{*,*} (\mathfrak{A})),
\]

\[
\varepsilon_2 : \widetilde{Q}C^\text{Alt} (\mathfrak{A}) = \widetilde{Q}C^0_{*,0} (\mathfrak{A}) \to \text{Tot}(\widetilde{Q}C^0_{*,*} (\mathfrak{A}))
\]

are quasi-isomorphisms, therefore the composite

\[
(\varepsilon_2)^{-1}_{*} (\varepsilon_1)* : H_* (\widetilde{Q}C_* (\mathfrak{A})) \tilde{\to} H_* (\widetilde{Q}C^\text{Alt}_* (\mathfrak{A}))
\]

is an isomorphism.

Suppose \(m \geq 1\) and let \(x \in \text{Ker } \partial' \subset \widetilde{Q}C^0_{0,m}(\mathfrak{A})\). Then \(x\) is homologous in \(\text{Tot}(\widetilde{Q}C^0_{*,*}(\mathfrak{A}))\) to

\[
x - (\partial' - \partial'') \Phi^\text{alt}_{0,m}(x) = \partial'' \Phi^\text{alt}_{0,m}(x) \in \widetilde{Q}C^0_{1,m-1}(\mathfrak{A}).
\]

It is also homologous to

\[
\partial'' \Phi^\text{alt}_{0,m}(x) - (\partial' + \partial'') \Phi^\text{alt}_{1,m-1} \partial'' \Phi^\text{alt}_{0,m}(x) = \partial'' \Phi^\text{alt}_{0,m-1} \partial' \partial'' \Phi^\text{alt}_{0,m}(x) - \partial'' \Phi^\text{alt}_{1,m-1} \partial'' \Phi^\text{alt}_{0,m}(x)
\]

\[
= -\partial'' \Phi^\text{alt}_{1,m-1} \partial'' \Phi^\text{alt}_{0,m}(x) = \widetilde{Q}C^0_{2,m-2}(\mathfrak{A}).
\]

Repeating this procedure we can say that \(x\) is homologous in \(\text{Tot}(\widetilde{Q}C^0_{*,*}(\mathfrak{A}))\) to

\[
(-1)^{\frac{1}{2}m(m-1)} \partial'' \Phi^\text{alt}_{m-1,1} \partial'' \Phi^\text{alt}_{m-2,2} \cdots \partial'' \Phi^\text{alt}_{0,m}(x) \in \widetilde{Q}C^0_{m,0}(\mathfrak{A}).
\]

This means that the isomorphism \((\varepsilon_2)^{-1}_{*} (\varepsilon_1)*\) is given by

\[
[x] \mapsto (-1)^{\frac{1}{2}m(m-1)} \partial'' \Phi^\text{alt}_{m-1,1} \partial'' \Phi^\text{alt}_{m-2,2} \cdots \partial'' \Phi^\text{alt}_{0,m}(x).
\]
Suppose \( m > 1 \) and \( k \leq m - 2 \). For an exact \( m \)-cube \( \mathcal{F} \) of \( \mathfrak{A} \),

\[
\partial'' \Phi_{k,m-k}(\mathcal{F}) = \sum_{j=1}^{m-k} \sum_{i=-1}^{1} (-1)^{i+j+k+1} \partial_{k+1+j} \rho_{m,k+1}(\mathcal{F})
= (-1)^k \mathcal{F} + \sum_{i=-1}^{m-k} \sum_{j=1}^{1} (-1)^{i+j+k+1} \rho_{m-1,k+1}(\partial^{i,j}_{k+1,j} \mathcal{F}).
\]

This implies that for \( y \in \widetilde{\mathbb{Q}}C^0_{k,m-k}(\mathfrak{A}) \) we have

\[ (2.3) \quad \partial'' \Phi^\text{alt}_{k,m-k}(y) = (-1)^k \text{Alt}_{k+1,m-k-1}(y) - \Phi^\text{alt}_{k,m-k-1} \partial''_{k+1} y, \]

where

\[ \partial''_{k+1} = \sum_{j=1}^{m-k} \sum_{i=-1}^{1} (-1)^{i+j} \partial_{k+1+j}^i. \]

For an exact \((m-1)\)-cube \( \mathcal{G} \) of \( \mathfrak{A} \),

\[ \text{Alt}_{k+2,m-k-1} \Phi_{k+1,m-k-1}(\mathcal{G}) = - \text{Alt}_{k+2,m-k-1} \rho_{m,k+2}\rho_{m-1,k+1}(\mathcal{G}), \]

and it is zero since \( \rho_{m,k+2}\rho_{m-1,k+1}(\mathcal{G}) = \rho_{m,k+1}\rho_{m-1,k+1}(\mathcal{G}) \) is invariant by the transposition \((k + 1, k + 2) \in \mathfrak{S}_{k+2} \). Hence

\[ \text{Alt}_{k+2,m-k-1} \Phi_{k+1,m-k-1} \Phi_{k,m-k-1}(z) = 0 \]

for \( z \in \widetilde{\mathbb{Q}}C^0_{k,m-k}(\mathfrak{A}) \), and it follows from \((2.2)\) that

\[ \Phi^\text{alt}_{k+1,m-k-1} \Phi^\text{alt}_{k,m-k-1}(z) = \text{Alt}_{k+2,m-k-1} \Phi_{k+1,m-k-1} \Phi_{k,m-k-1}(z) = \text{Alt}_{k+2,m-k-1} \Phi_{k+1,m-k-1} \Phi_{k,m-k-1}(z) = 0. \]

Taking the image by \( \Phi^\text{alt}_{k+1,m-k-1} \) of the both sides of \((2.3)\), we have

\[ \Phi^\text{alt}_{k+1,m-k-1} \partial'' \Phi^\text{alt}_{k,m-k}(y) = (-1)^k \Phi^\text{alt}_{k+1,m-k-1} \text{Alt}_{k+1,m-k-1}(y) \]

\[ = (-1)^k \text{Alt}_{k+2,m-k-1} \Phi_{k+1,m-k-1} \text{Alt}_{k+1,m-k-1}(y) \]

for \( y \in \widetilde{\mathbb{Q}}C^0_{k,m-k}(\mathfrak{A}) \), and by \((2.2)\) it is equal to

\[ (-1)^k \text{Alt}_{k+2,m-k-1} \Phi_{k+1,m-k-1}(y) = (-1)^k \Phi^\text{alt}_{k+1,m-k-1}(y). \]

Hence for \( m \geq 1 \) and for \( x \in \widetilde{\mathbb{Q}}C_{0,m}(\mathfrak{A}) \),

\[ (-1)^{\frac{1}{2} m(m-1)} \partial'' \Phi^\text{alt}_{m-1,1} \partial'' \Phi^\text{alt}_{m-2,2} \cdots \partial'' \Phi^\text{alt}_{0,m}(x) \]

\[ = (-1)^{\frac{1}{2} m(m-1)} \partial'' \Phi^\text{alt}_{m-1,1} \partial'' \Phi^\text{alt}_{m-2,2} \cdots \partial'' \Phi^\text{alt}_{1,m-1}(x) \]

\[ = - (-1)^{\frac{1}{2} m(m-1)} \partial'' \Phi^\text{alt}_{m-1,1} \partial'' \Phi^\text{alt}_{m-2,2} \cdots \partial'' \Phi^\text{alt}_{2,m-2}(x) \]

\[ = \cdots \]

\[ = (-1)^{m-1} \partial'' \Phi^\text{alt}_{m-1,1}(x). \]
Since
\[ \partial'' \Phi_{m-1,1} (G) = (-1)^m \partial'' \text{Alt}_{m,1} \rho_{m,m} (G) \]
\[ = (-1)^m \text{Alt}_m \partial'' \rho_{m,m} (G) = (-1)^{m-1} \text{Alt}_m (G) \]
for any exact \( m \)-cube \( G \), we conclude that
\[ (-1)^{\frac{1}{2} (m-1)} \partial'' \Phi_{m-1,1} \partial'' \Phi_{m-2,2} \cdots \partial'' \Phi_{0,m} (x) = \text{Alt}_m (x). \]
This means that the isomorphism
\[ (\varepsilon_2)^{-1} (\varepsilon_1)_* : H_m (\tilde{Q}_C (\mathfrak{A})) \to H_m (\tilde{Q}_C \text{Alt} (\mathfrak{A})) \]
agrees with the map induced by \( \text{Alt}_* : \tilde{Q}_C (\mathfrak{A}) \to \tilde{Q}_C \text{Alt} (\mathfrak{A}) \), which completes the proof. □

2.3. \( \mathcal{C} \)-complexes. In this subsection we will introduce \( \mathcal{C} \)-complexes \( \mathcal{H} \). A \( \mathcal{C} \)-complex \( A = (A^*_m, F^m_{n,m}) \) is a family of chain complexes \( (A^*_m, \partial A^*_m) \) for \( m \in \mathbb{Z} \) such that \( A^*_m = 0 \) except for finitely many \( m \), with maps of abelian groups
\[ F^m_{n,m} : A^*_m \to A^*_m + A^*_{m+n-m-1} \]
for \( m < n \) satisfying the relation
\[ (-1)^m F^m_{n,m} \partial A^*_m + (-1)^n \partial A^*_n F^m_{n,m} + \sum_{m < l < n} F^l_{n,m} F^m_{l,m} = 0. \]
Define the total chain complex \( \text{Tot}(A) \) of the \( \mathcal{C} \)-complex \( A \) by
\[ \text{Tot}(A)_p = \bigoplus_m A^*_p \]
with the boundary map
\[ \partial(x)^p = (-1)^m \partial A^*_m (x^m) + \sum_{l < m} F^l_{A} (x^l) \]
for \( x = (x^m) \in \text{Tot}(A)_p = \bigoplus_m A^*_m \).

Let \( A = (A^*_m, F^m_{n,m}) \) and \( B = (B^*_m, F^m_{n,m}) \) be \( \mathcal{C} \)-complexes. A map of \( \mathcal{C} \)-complexes from \( A \) to \( B \) is a family of maps of abelian groups
\[ f^m_{n,m} : A^*_m \to B^*_m + B^*_{m+n-m} \]
for \( m \leq n \) satisfying the relation
\[ (-1)^n \partial B^*_n f^m_{n,m} + \sum_l F^l_B f^m_{n,m} = (-1)^m f^m_{n,m} \partial A^*_m + \sum_l f^{l,n} F^m_{A} \]
This condition is equivalent to that
\[ \text{Tot}(f) = \bigoplus f^m_{n,m} : \text{Tot}(A) \to \text{Tot}(B) \]
is a map of chain complexes. For two maps \( f = (f^m_{n,m}) : A \to B \) and \( g = (g^m_{n,m}) : B \to C \) of \( \mathcal{C} \)-complexes, define the composite \( gf \) by
\[ (gf)^m_{n,m} = \sum_l g^{l,n} f^m_{n,m} : A^*_m \to C^*_n + C^*_{n+n-m}. \]
It is obvious that \( gf \) is also a map of \( \mathcal{C} \)-complexes.
Let \( f, g : A \to B \) be maps of \( \mathcal{C} \)-complexes. A homotopy \( \Phi = (\Phi^{m,n}) \) from \( f \) to \( g \) is a family of maps of abelian groups

\[
\Phi^{m,n} : A^m_* \to B^{n-m+1}_*
\]

for any \( m \leq n + 1 \) satisfying the relation

\[
(-1)^m \Phi^{m,n} \partial_{A^m} + \sum_l \Phi^{l,n}_{A^l} \mathcal{F}^{l,n} + (-1)^n \partial_{B^n} \Phi^{m,n} + \sum_l \mathcal{F}^{l,n}_B \Phi^{m,n} = g^{m,n} - f^{m,n}.
\]

This condition is equivalent to that

\[
\oplus \Phi^{m,n} : \text{Tot}(A)_* \to \text{Tot}(B)_{*+1}
\]

is a chain homotopy from \( \text{Tot}(f) \) to \( \text{Tot}(g) \).

For a \( \mathcal{C} \)-complex \( A = (A^m_*, \mathcal{F}^{m,n}_A) \) and for \( r \in \mathbb{Z} \), let \( A[r] = (A[r]^m, \mathcal{F}^{m,n}_{A[r]}) \) be the \( \mathcal{C} \)-complex given as follows:

\[
A[r]^m_* = A^m_* + r, \quad \mathcal{F}^{m,n}_{A[r]} = (-1)^r \mathcal{F}^{m+r,n+r} : A^m_* + r \to A^n_* + r.
\]

Then there is a natural isomorphism of chain complexes \( \text{Tot}(A[r]) \cong \text{Tot}(A)[r] \).

Let \( f : A \to B \) be a map of \( \mathcal{C} \)-complexes. Define the simple complex \( s(f) = (C^m_*, \mathcal{F}^{m,n}_C) \) of \( f \) as follows:

\[
C^m_* = A^m_* \oplus B^{m-1}_*, \quad \partial_C(a, b) = \mathcal{F} \partial_{A^m_*}(a) + \partial_B(b),
\]

\[
\mathcal{F}^{m,n}_C(a, b) = \mathcal{F}^{m,n}_{A^m_*}(a) + \mathcal{F}^{m,n}_{B^{m-1}_*}(a) - \mathcal{F}^{m,n}_{B^{m-1}_*}(b).
\]

It is easy to see that \( s(f) \) is a \( \mathcal{C} \)-complex such that its total chain complex is canonically isomorphic to the simple complex of the map \( \text{Tot}(f) : \text{Tot}(A) \to \text{Tot}(B) \). Let \( p^{m,n} : s(f)^m_* = A^m_* \oplus B^{m-1}_* \to A^n_* \) be the map defined by

\[
p^{m,n}(a, b) = \begin{cases} a, & m = n, \\ 0, & m \neq n, \end{cases}
\]

and \( i^{m,n} : B[-1]^m_* = B^{m-1}_* \to s(f)^n_* = A^n_* \oplus B^{n-1}_* \) the map defined by

\[
i^{m,n}(b) = \begin{cases} (0, b), & m = n, \\ (0, 0), & m \neq n. \end{cases}
\]

Then \( p = (p^{m,n}) : s(f) \to A \) and \( i = (i^{m,n}) : B[-1] \to s(f) \) are maps of \( \mathcal{C} \)-complexes. Taking the total chain complexes of a sequence of maps of \( \mathcal{C} \)-complexes \( B[-1] \xrightarrow{i} s(f) \xrightarrow{p} A \xrightarrow{f} B \) yields a distinguished triangle of chain complexes.

The proposition below can be easily verified.

**Proposition 2.5.** Let

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\varphi_A & & \varphi_B \\
\downarrow & & \downarrow \\
A' & \xrightarrow{f'} & B'
\end{array}
\]
be a diagram of maps of $\mathcal{C}$-complexes which is commutative up to homotopy. Let $\Phi_f : A \to B'$ be a homotopy from $\varphi_B f$ to $f \varphi_A$. Then the family of maps $\varphi_{s}^{m,n} : s(f)^m \to s(f')^n$ defined by

$$\varphi_{s}^{m,n}(a,b) = (\varphi_A^{m,n}(a), \varphi_B^{m-1,n-1}(b) + \Phi_f^{m,n-1}(a))$$

forms a map of $\mathcal{C}$-complexes $\varphi_{s} = (\varphi_{s}^{m,n}) : s(f) \to s(f')$. Moreover, the diagram

$$
\begin{array}{ccc}
B[-1] & \xrightarrow{i} & s(f) \\
\downarrow \varphi_B[-1] & & \downarrow \varphi_s \\
B'[-1] & \xrightarrow{i'} & s(f')
\end{array}
$$

is strictly commutative.

**Proposition 2.6.** Let $f : A \to B$ be a map of $\mathcal{C}$-complexes, and $p : s(f) \to A$ the map of $\mathcal{C}$-complexes defined above. Assume that $f$ has a right inverse map up to homotopy, namely, there is a map $g : B \to A$ such that $fg$ is homotopy equivalent to the identity of $B$. Let $\Psi$ be a homotopy from the identity of $B$ to $fg$. Then $t = (t^{m,n}) : A \to s(f)$ defined by

$$t^{m,n}(a) = \left( \delta^{m,n}(a) - \sum_l g^{l,n}f^{m,l}(a), - \sum_l \psi^{l,n-1}f^{m,l}(a) \right),$$

where

$$\delta^{m,n}(a) = \begin{cases} a, & m = n, \\ 0, & m \neq n, \end{cases}$$

is a map of $\mathcal{C}$-complexes such that $pt = Id_A - gf$, and it is a left inverse map of $p$ up to homotopy, namely, $tp$ is homotopy equivalent to the identity of $s(f)$. Moreover, the composite $tg : B \to s(f)$ is homotopy equivalent to the zero map.

**Proof:** It is easy to see that $t$ is a map of $\mathcal{C}$-complexes such that $pt = Id_A - gf$. Let $\Psi_1^{m,n} : s(f)^m \to s(f)^{n+m+1}$ be the map defined by

$$\Psi_1^{m,n}(a,b) = (-g^{m-1,n}(b), -\psi^{m-1,n-1}(b)).$$

Then $\Psi_1 = (\Psi_1^{m,n})$ turns out to be a homotopy from the identity of $s(f)$ to $tp$. Let $\Psi_2^{m,n} : B^m \to s(f)^{n+m+1}$ be the map defined by

$$\Psi_2^{m,n}(b) = \left( -\sum_l g^{l,n}\psi^{m,l}(b), -\sum_l \psi^{l,n-1}\psi^{m,l}(b) \right),$$

then $\Psi_2 = (\Psi_2^{m,n})$ turns out to be a homotopy from the zero map to $tg$. $\square$

Let

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\varphi_A & & \varphi_B \\
A' & \xrightarrow{f'} & B'
\end{array}
$$
be a diagram of maps of \( C \)-complexes which is commutative up to homotopy. Suppose that \( f \) and \( f' \) have right inverse maps \( g : B \to A \) and \( g' : B' \to A' \) up to homotopy such that \( \varphi_A g \) is homotopy equivalent to \( g' \varphi_B \). Let

\[
\begin{align*}
\Phi_f &: A \to B', \\
\Phi_g &: B \to A', \\
\Psi &: B \to B, \\
\Psi' &: B' \to B'.
\end{align*}
\]

be homotopies from \( \varphi_B f \) to \( f' \varphi_A \), from \( \varphi_B g \) to \( g' \varphi_B \), from \( \text{Id}_B \) to \( fg \), and from \( \text{Id}_{B'} \) to \( f'g' \) respectively. Then we have two homotopies

\[
\begin{align*}
\Phi f g + f' \Phi g + \varphi_B \Psi, \Psi' \varphi_B &: B \to B'.
\end{align*}
\]

Definition 2.7. With the above notations, a second homotopy from \( \Phi f g + f' \Phi g + \varphi_B \Psi \) to \( \Psi' \varphi_B \) is a family of maps

\[
\Theta_{m,n} : B_*^m \to B_*^{n-m+2}
\]

for \( m \leq n + 2 \) satisfying the relation

\[
(-1)^n \partial \Theta_{m,n} + \sum_l F_{B'}^{l,n}\Theta_{m,l} - (-1)^m \Theta_{m,n} \partial - \sum_l \Theta_{l,n} F_{B}^{m,l} = \sum_l \varphi_{B'}^{l,n} m_{B'}^{m,l} - \sum_l \phi_{g'}^{l,n} g_{B}^{m,l} - \sum_l f_{l,n} \phi_{B}^{m,l} - \sum_l \varphi_{B}^{l,n} \Psi_{m,l}.
\]

The proposition below is verified by a direct calculation:

Proposition 2.8. With the above notations, let \( t : A \to s(f) \) and \( t' : A' \to s(f') \) be the maps given in Prop.2.6. Then \( \varphi_A t \) is homotopy equivalent to \( t' \varphi_A \), and the homotopy \( \Pi : A \to s(f') \) from \( \varphi_A t \) to \( t' \varphi_A \) is given by

\[
\Pi_{m,n}(a) = \left( -\sum_l \phi_{g}^{l,n} f_{m,l}(a) - \sum_l g_{l,n} \phi_{B}^{m,l}(a), -\sum_l \psi_{l,n}^{-1} \phi_{j}^{m,l}(a) + \sum_l \Theta_{l,n-1} f_{m,l}(a) \right).
\]

2.4. Multi-relative complexes of exact cubes. Let \( X \) be a scheme. As we mentioned in the introduction, the category \( \mathcal{P}(X) \) of vector bundles of finite rank on \( X \) is a small exact category. Hence we can obtain the chain complex of exact cubes of vector bundles \( \widetilde{Q}C_{*}(X) \) and its alternating part \( \widetilde{Q}C_{*}^{\text{Alt}}(X) \) as in \( \S 2.2 \). Then Thm.2.1 and Thm.2.2 imply the following:

Theorem 2.9. The homology groups of the chain complexes \( \widetilde{Q}C_{*}(X) \) and \( \widetilde{Q}C_{*}^{\text{Alt}}(X) \) are canonically isomorphic to the rational K-theory of \( X \):

\[
H_n(\widetilde{Q}C_{*}^{\text{Alt}}(X)) \simeq H_n(\widetilde{Q}C_{*}(X)) \simeq K_n(X)_{\mathbb{Q}}.
\]
Consider a sequence of morphisms of schemes

\[ X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_r} X_r. \]

For a vector bundle \( \mathcal{F} \) on \( X_r \), let us define an exact \((r-1)\)-cube \((f_1, \ldots, f_r)^* \mathcal{F} \) on \( X_0 \) as follows: In the case of \( r \geq 2 \),

\[
((f_1, \ldots, f_r)^* \mathcal{F})_{\alpha_1, \ldots, \alpha_{r-1}} = \begin{cases} 0, & \exists j \text{ such that } \alpha_j = 1, \\ (f_1 \cdots f_1)^* (f_{j_2} \cdots f_{j_{l+1}})^* \cdots (f_{j_l} \cdots f_{j_{l+1}})^* (f_r \cdots f_{j_{l+1}})^* \mathcal{F}, & \text{otherwise}, \end{cases}
\]

where \( \alpha^{-1}(-1) = \{j_1, \ldots, j_l\} \) with \( j_1 < \cdots < j_l \). The maps in \((f_1, \ldots, f_r)^* \mathcal{F}\) are the zero maps or the natural isomorphisms. When \( r = 1 \), define \((f_1)^* \mathcal{F} = f_1^* \mathcal{F}\). We can generalize this procedure to exact cubes in the following way: For an exact \( n \)-cube \( \mathcal{F} \) on \( X_r \), \((f_1, \ldots, f_r)^* \mathcal{F}\) is an exact \((n+r-1)\)-cube on \( X_0 \) so that

\[
\partial_{r}^{\alpha} \cdots \partial_{n+r-1}^{\alpha(n+r-1)} ((f_1, \ldots, f_r)^* \mathcal{F}) = (f_1, \ldots, f_r)^* (\partial_{1}^{\alpha} \cdots \partial_{n}^{\alpha(n)} \mathcal{F}).
\]

**Proposition 2.10.** For an exact \( n \)-cube \( \mathcal{F} \) on \( X_r \), the faces of the exact cube \((f_1, \ldots, f_r)^* \mathcal{F}\) are as follows: For \( 1 \leq j \leq r-1 \),

\[
\partial_j^i ((f_1, \ldots, f_r)^* \mathcal{F}) = \begin{cases} (f_1, \ldots, f_j)^* ((f_{j+1}, \ldots, f_r)^* \mathcal{F}), & i = -1, \\ (f_1, \ldots, f_{j+1} f_j, \ldots, f_r)^* \mathcal{F}, & i = 0, \\ 0, & i = 1 \end{cases}
\]

and for \( r \leq j \),

\[
\partial_j^i ((f_1, \ldots, f_r)^* \mathcal{F}) = (f_1, \ldots, f_r)^* (\partial_{j-r+1}^i \mathcal{F}).
\]

Let \( Y_1, \ldots, Y_r \) be closed subschemes of a scheme \( X \). Let \( Y_J = \bigcap_{j \in J} Y_j \) for \( J \subset \{1, \ldots, r\} \), and \( t_k : Y_{J \cup \{k\}} \hookrightarrow Y_J \) the embedding for \( k \notin J \). For a division \( J = K \bigsqcup I \) such that \( K = \{k_1, \ldots, k_{n-m}\} \) with \( k_1 < \cdots < k_{n-m} \) and for \( \sigma \in S_{n-m} \), we have a sequence of embeddings

\[
Y_J \xrightarrow{t_{k_1}(1)} Y_{J-k_{k_1}(1)} \xrightarrow{t_{k_2}(2)} Y_{J-k_{k_1}(1), k_{k_2}(2)} \xrightarrow{t_{k_3}(3)} \cdots \xrightarrow{t_{k_{n-m}}(n-m)} Y_I.
\]

Note that \((t_{k_1}(1), \ldots, t_{k_{n-m}}(n-m))^* \mathcal{F}\) is degenerate if so is \( \mathcal{F} \). Hence with the above sequence we can associate a map

\[
\Xi_K = \sum_{\sigma \in S_{n-m}} (\text{sgn } \sigma)(t_{k_1}(1), \ldots, t_{k_{n-m}}(n-m))^* : \widetilde{Q}C_*(Y_I) \to \widetilde{Q}C_{*+n-m-1}(Y_J).
\]

Let us consider the composite of \( \Xi_K \) with the boundary map. To do this, we need to introduce signature of a division of subsets of \( \{1, \ldots, r\} \). For a division \( K \bigsqcup I = J \) such that

\[
K = \{k_1, \ldots, k_{n-m}\}, \quad k_1 < \cdots < k_{n-m}, \\
I = \{i_1, \ldots, i_m\}, \quad i_1 < \cdots < i_m, \\
J = \{j_1, \ldots, j_n\}, \quad j_1 < \cdots < j_n,
\]
define the signature as follows:

\[
\operatorname{sgn} \left( \begin{array}{c} K \\ J \end{array} \right) = \operatorname{sgn} \left( \begin{array}{c} k_1 & \cdots & k_{n-m} \\ j_1 & \cdots & j_m \end{array} \right).
\]

The following lemma is easily verified.

**Lemma 2.11.** Let \( J = \bigcup L L' \bigcup I \) be a division and write \( K = \bigcup L \bigcup L' \) and \( P = \bigcup L' \bigcup I \). Then

\[
\operatorname{sgn} \left( \begin{array}{c} K \\ J \end{array} \right) \operatorname{sgn} \left( \begin{array}{c} L \\ L' \end{array} \right) = \operatorname{sgn} \left( \begin{array}{c} L \\ P \end{array} \right) \operatorname{sgn} \left( \begin{array}{c} L' \\ K \end{array} \right).
\]

**Proposition 2.12.** For \( x \in \widetilde{\mathbb{Q}} C_\ast(Y_I) \),

\[
\partial \Xi_K(x) + (-1)^{n-m} \Xi_K(\partial x) = \sum_{a=1}^{n-m-1} (-1)^{a+1} \sum_{\substack{L \bigcup L' = K \\ |L| = a}} \operatorname{sgn} \left( \begin{array}{c} L \\ L' \end{array} \right) \Xi_L \Xi_{L'}(x).
\]

**Proof:** Prop.2.10 implies that

\[
\begin{align*}
\partial \Xi_K(x) &= \sum_{\sigma \in S_{n-m}} (-1)^{n-m-1} \left( \sum_{a=1}^{n-m-1} (-1)^a \left( t_{k_{\sigma(1)}}, \ldots, t_{k_{\sigma(a)}}, t_{k_{\sigma(n-m)}} \right)^\ast \right) \\
&\quad \times \left( \left( t_{k_{\sigma(a+1)}}, \ldots, t_{k_{\sigma(n-m)}} \right)^\ast (x) \right) \\
&\quad + \sum_{\sigma \in S_{n-m}} (-1)^{n-m-1} \left( \sum_{a=1}^{n-m-1} (-1)^a \left( t_{k_{\sigma(1)}}, \ldots, t_{k_{\sigma(a)}, t_{k_{\sigma(a+1)}}, \ldots, t_{k_{\sigma(n-m)}}} \right)^\ast \right) \\
&\quad \times \left( \left( t_{k_{\sigma(a+1)}}, \ldots, t_{k_{\sigma(n-m)}} \right)^\ast (\partial x) \right) \\
&\quad + (-1)^{n-m+1} \sum_{\sigma \in S_{n-m}} \left( \sum_{a=1}^{n-m-1} \left( t_{k_{\sigma(1)}}, \ldots, t_{k_{\sigma(n-m)}} \right)^\ast (\partial x) \right).
\end{align*}
\]

In the above, \((2.4)\) is obviously zero, and \((2.5)\) is equal to \((-1)^{n-m+1} \Xi_K(\partial x)\). Hence

\[
\begin{align*}
\partial \Xi_K(x) + (-1)^{n-m} \Xi_K(\partial x) &= \sum_{\sigma \in S_{n-m}} (-1)^{n-m-1} \left( \sum_{a=1}^{n-m-1} (-1)^a \left( t_{k_{\sigma(1)}}, \ldots, t_{k_{\sigma(a)}}, t_{k_{\sigma(n-m)}} \right)^\ast \right) \\
&\quad \times \left( \left( t_{k_{\sigma(a+1)}}, \ldots, t_{k_{\sigma(n-m)}} \right)^\ast (x) \right) .
\end{align*}
\]

If we denote

\[
L = \{k_{\sigma(1)}, \ldots, k_{\sigma(a)}\} = \{l_1, \ldots, l_a\},
\]
\[
L' = \{k_{\sigma(a+1)}, \ldots, k_{\sigma(n-m)}\} = \{l'_1, \ldots, l'_{n-m-a}\}
\]
with \( l_1 < \cdots < l_a \) and \( l'_1 < \cdots < l'_{n-m-a} \), then

\[
\partial \Xi_K(x) + (-1)^{n-m} \Xi_K(\partial x)
= \sum_{a=1}^{n-m-1} (-1)^{a+1} \sum_{L \sqcup L' = K, |L| = a} \text{sgn} \left( \frac{L}{K} \right) \times \\
\sum_{\tau \in \mathfrak{S}_a} (\text{sgn } \tau)(t_{\tau(l_1)}, \ldots, t_{\tau(l_a)})^* \left( \sum_{\eta \in \mathfrak{S}_{n-m-a}} (\text{sgn } \eta)(t_{\eta(l'_1)}, \ldots, t_{\eta(l'_{n-m-a})})^*(x) \right)
= \sum_{a=1}^{n-m-1} (-1)^{a+1} \sum_{L \sqcup L' = K, |L| = a} \text{sgn} \left( \frac{L}{K} \right) \Xi_L \Xi_{L'}(x),
\]

which completes the proof. \( \square \)

Define a map

\[
F^{m,n} : \bigoplus_{|I|=m} \tilde{Q}C_* (Y_I) \rightarrow \bigoplus_{|J|=n} \tilde{Q}C_{*+n-m-1} (Y_J)
\]

by

\[
F^{m,n}(x)_J = (-1)^n \sum_{K \sqcup I = J, |I|=m} \text{sgn} \left( \frac{K}{J} \right) \Xi_K(x_I)
\]

for \( x = (x_I) \in \bigoplus_{|I|=m} \tilde{Q}C_* (Y_I) \).

**Corollary 2.13.**

\[
\tilde{Q}C_* (X; Y_1, \ldots, Y_r) = \left( \bigoplus_{|I|=m} \tilde{Q}C_* (Y_I), F^{m,n} \right)
\]

is a \( \mathcal{C} \)-complex.
Proof: Let \( x = (x_I) \in \bigoplus \tilde{Q}C_*(Y_I) \). Then it follows from Prop.2.12 and Lem.2.11 that
\[
(-1)^n \partial F^{m,n}(x)J + (-1)^m F^{m,n}(\partial x)_J
= \sum_{K \mid I = J} \frac{\text{sgn}(K)}{|I| = m} \sum_{a=1}^{n-m-1} (-1)^{a+1} \sum_{L \mid I = K} \frac{\text{sgn}(L \wedge K)}{|L| = a} \Xi_L \Xi_L'(x)
= \sum_{a=1}^{n-m-1} (-1)^a \sum_{L \mid I = K} \frac{\text{sgn}(L \wedge K)}{|L| = a} \Xi_L \Xi_L'(x)
= \sum_{a=1}^{n-m-1} (-1)^a \Xi_L \Xi_L'(x_I)
= - \sum_{a=1}^{n-m-1} F^{m,n-a}(x)_J,
\]
which completes the proof.

We next consider pull-back map associated with a morphism of schemes. Let \( T \) be another scheme with closed subschemes \( D_1, \ldots, D_r \), and \( f : X \to T \) a morphism such that \( f(Y_j) \subset D_j \) for \( 1 \leq j \leq r \). In what follows, we denote such a morphism by
\[
f : (X; Y_1, \ldots, Y_r) \to (T; D_1, \ldots, D_r).
\]
Denote the restriction of \( f \) to \( Y_I \to D_I \) by the same symbol \( f \) to simplify the notations. For a division \( K \bigcup I = J \) of subsets of \( \{1, \ldots, r\} \) such that \( K = \{k_1, \ldots, k_{n-m}\} \) with \( k_1 < \cdots < k_{n-m} \), define a map \( \Xi_{K,f} : \tilde{Q}C_*(D_I) \to \tilde{Q}C_{*+n-m}(Y_I) \) by
\[
\Xi_{K,f} = \sum_{p=0}^{n-m} (-1)^p \sum_{\sigma \in S_{n-m}} (\text{sgn } \sigma)(t_{k_\sigma(1)}, \ldots, t_{k_\sigma(n-m)})^* f, t_{k_\sigma(p+1)}, \ldots, t_{k_\sigma(n-m)}.
\]
In particular, in the case that \( K \) is the emptyset, \( \Xi_{\emptyset,f} = f^* : \tilde{Q}C_*(D_I) \to \tilde{Q}C_*(Y_I) \).

Proposition 2.14. For \( x \in \tilde{Q}C_*(D_I), \)
\[
\partial \Xi_{K,f}(x) + (-1)^{n-m+1} \Xi_{K,f}(\partial x)
= - \sum_{a=1}^{n-m} \sum_{L \mid I = K} \frac{\text{sgn}(L \wedge K)}{|L| = a} \Xi_L \Xi_L', f(x) + \sum_{a=0}^{n-m-1} (-1)^a \sum_{L \mid I = K} \frac{\text{sgn}(L \wedge K)}{|L| = a} \Xi_L \Xi_L'(x).
\]

Proof: Prop.2.10 implies that
\[
\partial \Xi_{K,f}(x) + (-1)^{n-m+1} \Xi_{K,f}(\partial x) = \sum_{a=1}^{n-m} (-1)^{a+1} \partial_a^{-1} \Xi_{K,f}(x) + \sum_{a=1}^{n-m} (-1)^a \partial_a^0 \Xi_{K,f}(x),
\]
and
\[
\sum_{a=1}^{n-m} (-1)^a \partial_a^0 \Xi_{K,f}(x) = \sum_{\sigma \in S_{n-m}} (\text{sgn } \sigma) \left\{ \sum_{p=0}^{n-m} \sum_{a=1}^{p} (-1)^{a+p} \left( \cdots, t_{k_{\sigma(a+1)}}, t_{k_{\sigma(a)}}, \cdots, t_{k_{\sigma(p+1)}}, \cdots \right)^* (x) \right. \\
+ \sum_{p=0}^{n-m} \sum_{a=p+1}^{n-m} (-1)^{a+p-1} \left( \cdots, t_{k_{\sigma(p+1)}}, \cdots, t_{k_{\sigma(a+1)}}, t_{k_{\sigma(a)}}, \cdots \right)^* (x) \\
+ \sum_{p=1}^{n-m} \left( \cdots, f_{k_{\sigma(p)}}, \cdots \right)^* (x) \left. \right\} - \sum_{p=0}^{n-m-1} \left( \cdots, t_{k_{\sigma(p+1)}}, f, \cdots \right)^* (x),
\]
which is obviously zero. On the other hand,
\[
\sum_{a=1}^{n-m} (-1)^{a+1} \partial_a^{-1} \Xi_{K,f}(x) = \sum_{p=1}^{n-m} \sum_{a=1}^{p} (-1)^{a+p+1} \sum_{\sigma \in S_{n-m}} (\text{sgn } \sigma) \times \\
\left( t_{k_{\sigma(1)}}, \cdots, t_{k_{\sigma(a)}} \right)^* \left( \left( t_{k_{\sigma(a+1)}}, \cdots, t_{k_{\sigma(p)}}, f, t_{k_{\sigma(p+1)}}, \cdots, t_{k_{\sigma(n-m)}} \right)^* (x) \right) \\
+ \sum_{p=0}^{n-m-1} \sum_{a=p}^{n-m} (-1)^{a+p} \sum_{\sigma \in S_{n-m}} (\text{sgn } \sigma) \times \\
\left( t_{k_{\sigma(1)}}, \cdots, t_{k_{\sigma(p)}}, f, t_{k_{\sigma(p+1)}}, \cdots, t_{k_{\sigma(a)}} \right)^* \left( \left( t_{k_{\sigma(a+1)}}, \cdots, t_{k_{\sigma(n-m)}} \right)^* (x) \right)
\]
\[
= - \sum_{a=1}^{n-m} \sum_{|L|=a} \text{sgn} \left( L' K \right) \Xi_L \Xi_{L',f}(x) + \sum_{a=0}^{n-m-1} \sum_{|L'|=a} \text{sgn} \left( L' K \right) \Xi_{L',f}(x),
\]
which completes the proof. \(\square\)

Define a map
\[
(f^*)^{m,n} : \bigoplus_{|J|=n} \bar{\mathbb{Q}} C_*(D_I) \to \bigoplus_{|J|=n} \bar{\mathbb{Q}} C_{*+n-m}(Y_J)
\]
by
\[
(f^*)^{m,n}(x)_J = \sum_{K \sqcup I = J \atop |I|=n} \text{sgn} \left( K' J \right) \Xi_{K',f}(x_I)
\]
for \(x = (x_I) \in \bigoplus_{|I|=m} \bar{\mathbb{Q}} C_*(D_I)\).
Corollary 2.15. 

\[ f^* = (f^*)^{m,n} : \tilde{QC}_*(T; D_1, \ldots, D_r) \to \tilde{QC}_*(X; Y_1, \ldots, Y_r) \]

is a map of \( \mathbb{C} \)-complexes.

Proof: It follows from Prop.2.14 and Lem.2.11 that

\[ (-1)^n \partial (f^*)^{m,n}(x)_J - (-1)^m (f^*)^{m,n}(\partial x)_J \]

\[ = (-1)^n \sum_{K \prod I = J} \text{sgn} (K, I) \left\{ - \sum_{a=1}^{n-m} \sum_{L \prod L' = K \mid |L'| = a} \text{sgn}(L, L') \Xi_L \Xi_{L', f}(x_I) \right\} \]

\[ + \sum_{a=0}^{n-m-1} (-1)^a \sum_{L \prod L' = K \mid |L'| = a} \text{sgn}(L, L') \Xi_L \Xi_{L', f}(x_I) \]

\[ = (-1)^{n+1} \sum_{a=1}^{n-m} \sum_{L \prod L' = J \mid |L'| = a} \text{sgn}(L, P) \sum_{L' \prod I = P} \text{sgn}(L', I) \Xi_L \Xi_{L', f}(x_I) \]

\[ + \sum_{a=0}^{n-m-1} (-1)^{n+a} \sum_{L \prod L' = J \mid |L'| = a} \text{sgn}(L, P) \sum_{L' \prod I = P} \text{sgn}(L', I) \Xi_L \Xi_{L', f}(x_I) \]

\[ = - \sum_{a=1}^{n-m} F^{n-a,n}(f^*)^{m,n-a}(x)_J + \sum_{a=0}^{n-m-1} (f^*)^{n-a,n} F^{m,n-a}(x)_J, \]

which completes the proof. \( \square \)

Comparing the definition of the simple complex of the pull-back map \( \iota_r^* \) with the definition of \( \tilde{QC}_*(X; Y_1, \ldots, Y_r) \), we obtain the following corollary:

Corollary 2.16. The \( \mathbb{C} \)-complex \( \tilde{QC}_*(X; Y_1, \ldots, Y_r) \) is canonically isomorphic to the simple complex of the map

\[ \iota_r^* : \tilde{QC}_*(X; Y_1, \ldots, Y_{r-1}) \to \tilde{QC}_*(Y_r; Y_1 \cap Y_r, \ldots, Y_{r-1} \cap Y_r) \]

induced by the embedding \( \iota_r : Y_r \hookrightarrow X \). Hence the homology groups of \( \tilde{QC}_*(X; Y_1, \ldots, Y_r) \) are isomorphic to the rational multi-relative K-theory of \( (X; Y_1, \ldots, Y_r) \):

\[ H_n(\tilde{QC}_*(X; Y_1, \ldots, Y_r)) \simeq K_n(X; Y_1, \ldots, Y_r)_\mathbb{Q}. \]

Let

\[ (X; Y_1, \ldots, Y_r) \xrightarrow{f} (T; D_1, \ldots, D_r) \xrightarrow{g} (S; E_1, \ldots, E_r) \]
be morphisms of schemes with closed subschemes. For a division $K \coprod I = J$ of subsets of \{1, \ldots, r\} such that $K = \{k_1, \ldots, k_{n-m}\}$ with $k_1 < \cdots < k_{n-m}$, define a map $\Xi_{K,f,g} : \tilde{QC}_*(E_I) \to \tilde{QC}_{*+n-m+1}(Y_J)$ by

$$
\Xi_{K,f,g} = \sum_{0 \leq p \leq q \leq n-m} (-1)^{p+q} \sum_{\sigma \in \mathfrak{S}_{n-m}} (\text{sgn } \sigma)(\ldots, t_{k_\sigma(p)}, f, t_{k_\sigma(p+1)}, \ldots, t_{k_\sigma(q)}, g, t_{k_\sigma(q+1)}, \ldots)^*.
$$

**Proposition 2.17.** If we write $h = gf$, then for $x \in \tilde{QC}_*(E_I)$,

$$
\partial \Xi_{K,f,g}(x) + (-1)^{n-m} \Xi_{K,f,g}(\partial x)
= \sum_{a=1}^{n-m} \sum_{L \coprod L' = K, |L| = a} (-1)^{a+1} \text{sgn } (L^L_K) \Xi_L \Xi_{L',f,g}(x) + \sum_{a=0}^{n-m} \sum_{L \coprod L' = K, |L| = a} \text{sgn } (L^L_K) \Xi_L \Xi_{L',f,g}(x)
$$

$$
+ \sum_{a=0}^{n-m-1} \sum_{L \coprod L' = K, |L| = a} (-1)^{a+1} \text{sgn } (L^L_K) \Xi_L \Xi_{L',f,g}(x) - \Xi_{K,h}(x).
$$

*Proof:* Prop.2.10 implies that

$$
\partial \Xi_{K,f,g}(x) + (-1)^{n-m} \Xi_{K,f,g}(\partial x)
= \sum_{a=1}^{n-m+1} (-1)^{a+1} \partial_a^0 \Xi_{K,f,g}(x) + \sum_{a=1}^{n-m+1} (-1)^a \partial_a^0 \Xi_{K,f,g}(x).
$$

In the same way as in the proof of Prop.2.12 and Prop.2.14 we can show that

$$
\sum_{a=1}^{n-m+1} (-1)^a \partial_a^0 \Xi_{K,f,g}(x)
= \sum_{p=0}^{n-m} \sum_{\sigma \in \mathfrak{S}_{n-m}} (-1)^{p+1} (\text{sgn } \sigma)(t_{k_\sigma(1)}, \ldots, t_{k_\sigma(p)}, gf, t_{k_\sigma(p+1)}, \ldots, t_{k_\sigma(n-m)})^*(x)
$$

$$
= - \Xi_{K,h}(x).
$$
On the other hand,

\[
\sum_{a=1}^{n-m+1} (-1)^{a+1} a^{-1} \Xi_{K,f,g}(x)
= \sum_{\sigma \in S_{n-m}} (\text{sgn } \sigma) \sum_{0 \leq p \leq q \leq n-m} (-1)^{p+q} \times \\
\left\{ \sum_{a=1}^{p} (-1)^{a+1} (t_{k_{\sigma}(1)}, \ldots, t_{k_{\sigma}(a)})^* \left( (t_{k_{\sigma}(a+1)}, \ldots, t_{k_{\sigma}(p)}, f, \ldots, t_{k_{\sigma}(q)}, g, \ldots)^*(x) \right) \\
+ \sum_{a=p}^{q} (-1)^a (\ldots, t_{k_{\sigma}(p)}, f, \ldots, t_{k_{\sigma}(q)}, g, \ldots)^*(x) \\
+ \sum_{a=q}^{n-m-1} (-1)^{a+1} (\ldots, t_{k_{\sigma}(p)}, f, \ldots, t_{k_{\sigma}(q)}, g, \ldots, t_{k_{\sigma}(n-m)})^*(x) \right\}
\]

\[
= \sum_{a=1}^{n-m} \left\{ \sum_{L \sqcup L' = K \mid |L| = a} (-1)^{a+1} \text{sgn } (L \sqcup L') \Xi_L \Xi_{L',f,g}(x) \right. \\
+ \sum_{a=1}^{n-m} \left. \sum_{L \sqcup L' = K \mid |L| = a} \text{sgn } (L \sqcup L') \Xi_L \Xi_{L',f,g}(x) \right\},
\]

which completes the proof. □

Define a map

\[
\Phi^{m,n} : \bigoplus_{|I|=m} \tilde{Q}\text{C}_*(E_I) \to \bigoplus_{|J|=n} \tilde{Q}\text{C}_{*+n-m+1}(Y_J)
\]

by

\[
\Phi^{m,n}(x)_J = (-1)^n \sum_{K \sqcup I = J \mid |I|=m} \text{sgn } (K \sqcup I) \Xi_{K,f,g}(x_I)
\]

for \(x = (x_I) \in \bigoplus_{|I|=m} \tilde{Q}\text{C}_*(E_I)\).

**Corollary 2.18.**

\(\Phi = \Phi^{m,n} : \tilde{Q}\text{C}_*(S; E_1, \ldots, E_r) \to \tilde{Q}\text{C}_{*+1}(X; Y_1, \ldots, Y_r)\)

is a homotopy from \(h^*\) to \(f^* g^*\).
Proof: It follows from Prop. 2.17 and Lem. 2.11 that

\[
(-1)^n \partial \Phi^m,n(x)_J + (-1)^m \Phi^m,n(\partial x)_J
\]

\[
= \sum_{a=1}^{n-m} (-1)^{a+1} \sum_{\substack{L \parallel P = J \parallel \partial \Phi^m,n(x)_J \\ |L|=a}} \text{sgn}(L \parallel J) \sum_{\substack{L' \parallel I = P \parallel \partial \Phi^m,n(x)_J \\ |I|=m}} \text{sgn}(L' \parallel I) \Xi_L \Xi_{L',f,g}(x_I)
\]

\[
+ \sum_{a=0}^{n-m} \sum_{\substack{L \parallel P = J \parallel \partial \Phi^m,n(x)_J \\ |L|=a}} \text{sgn}(L \parallel J) \sum_{\substack{L' \parallel I = P \parallel \partial \Phi^m,n(x)_J \\ |I|=m}} \text{sgn}(L' \parallel I) \Xi_{L,f,g}(x_I)
\]

\[
+ \sum_{a=0}^{n-m-1} (-1)^{a+1} \sum_{\substack{L \parallel P = J \parallel \partial \Phi^m,n(x)_J \\ |L|=a}} \text{sgn}(L \parallel J) \sum_{\substack{L' \parallel I = P \parallel \partial \Phi^m,n(x)_J \\ |I|=m}} \text{sgn}(L' \parallel I) \Xi_{L,f,g}(x_I)
\]

\[
- \sum_{\substack{K \parallel I = J \parallel \partial \Phi^m,n(x)_J \\ |I|=m}} \text{sgn}(K \parallel J) \Xi_{K,h}(x)
\]

\[
= - \sum_{a=1}^{n-m} F^{n-a,n} \Phi^m,n-a(x_I) + \sum_{a=0}^{n-m} (f^*)^{n-a,n} (g^*)^{m,n-a}(x_I)
\]

\[
- \sum_{a=0}^{n-m-1} \Phi^{n-a,n} F^{m,n-a}(x_I) - (h^*)^{m,n}(x_I),
\]

which completes the proof. \qed

2.5. The \( \mathcal{Q} \)-complex \( \mathcal{Q} \mathcal{C}^{\text{Alt}}_*(X;Y_1,\ldots,Y_r) \). Consider the sequence of embeddings

\[
Y_1 \cap Y_2 \xrightarrow{i_1} Y_2 \xrightarrow{\text{Id}} Y_2 \xrightarrow{i_2} X
\]

and take a vector bundle \( \mathcal{F} \) on \( X \). Then the 2-cube \( (i_1, \text{Id}, i_2)^* \mathcal{F} \) on \( Y_1 \cap Y_2 \) is described as

\[
\begin{array}{ccc}
i_1^* i_2^* \mathcal{F} & \longrightarrow & i_1^* i_2^* \mathcal{F} \\
\downarrow & & \downarrow \\
i_1^* i_2^* \mathcal{F} & \longrightarrow & (i_2 i_1)^* \mathcal{F},
\end{array}
\]

which is not degenerate. This means that \((\text{Id}^*)^{0,2}\) is not the zero map. More generally, the pull-back map by the identity morphism

\[
(\text{Id}_X)^* : \mathcal{Q} \mathcal{C}^{\text{Alt}}_*(X;Y_1,\ldots,Y_r) \to \mathcal{Q} \mathcal{C}^{\text{Alt}}_*(X;Y_1,\ldots,Y_r).
\]

is not the identity, hence we cannot apply Prop. 2.6 and Prop. 2.8 to the multi-relative complexes of exact cubes. However, since the above cube is symmetric, we can overcome this drawback by using the alternating part \( \mathcal{Q} \mathcal{C}^{\text{Alt}}_*(X) \).

Let

\[
X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_r} X_r
\]
be a sequence of morphisms of schemes, and $\mathcal{F}$ an exact $n$-cube on $X_r$. For any $\sigma \in \mathcal{S}_n$, define $\sigma' \in \mathcal{S}_{n+r-1}$ by

$$\sigma' = \begin{pmatrix} 1 & \cdots & r - 1 & r & \cdots & n + r - 1 \\ 1 & \cdots & r - 1 & \sigma(1) + r - 1 & \cdots & \sigma(n) + r - 1 \end{pmatrix}.$$  

Then it holds that

$$(f_1, \ldots, f_r)^* (\sigma(\mathcal{F})) = \sigma'((f_1, \ldots, f_r)^* (\mathcal{F})).$$  

This implies that for a division $K \coprod I = J$ of $\{1, \ldots, r\}$ we have

$$\text{Alt}_* \Xi_K \text{Alt}_* = \text{Alt}_* \Xi_K : \widetilde{QC}_*(Y_J) \to \widetilde{QC}_{*+n-m-1}(Y_J),$$

which leads to the following proposition:

**Proposition 2.19.** If we put

$$\Xi_{K} = \text{Alt}_* \Xi_K : \widetilde{QC}_*(Y_I) \to \widetilde{QC}_{*+n-m-1}(Y_J);$$

and

$$F^{m,n} = (-1)^n \sum_{K \coprod I = J \atop |I| = m} \text{sgn}(K) \Xi_K : \bigoplus_{|I| = m} \widetilde{QC}_*(Y_I) \to \bigoplus_{|J| = n} \widetilde{QC}_{*+n-m-1}(Y_J),$$

then the family

$$\widetilde{QC}_*^\text{Alt}(X; Y_1, \ldots, Y_r) = \left( \bigoplus_{|I| = m} \widetilde{QC}_*(Y_I), F^{m,n} \right)$$

is a $\mathcal{C}$-complex. Similarly, for morphisms

$$(X; Y_1, \ldots, Y_r) \xrightarrow{f} (T; D_1, \ldots, D_r) \xrightarrow{g} (S; E_1, \ldots, E_r),$$

write $\Xi_{K,f} = \text{Alt}_* \Xi_K, f$, $\Xi_{K,f,g} = \text{Alt}_* \Xi_{K,f,g}$ and

$$(f^*)^{m,n} = \sum_{K \coprod I = J \atop |I| = m} \text{sgn}(K) \Xi_{K,f} : \bigoplus_{|I| = m} \widetilde{QC}_*(D_I) \to \bigoplus_{|J| = n} \widetilde{QC}_{*+n-m}(Y_J),$$

$$\Phi^{m,n} = (-1)^n \sum_{K \coprod I = J \atop |I| = m} \text{sgn}(K) \Xi_{K,f,g} : \bigoplus_{|I| = m} \widetilde{QC}_*(E_I) \to \bigoplus_{|J| = n} \widetilde{QC}_{*+n-m+1}(Y_J).$$

Then $f^* = ((f^*)^{m,n})$ is a map of $\mathcal{C}$-complexes and $\Phi = (\Phi^{m,n})$ is a homotopy from $h^*$ to $f^*g^*$.  

**Proposition 2.20.** For the identity map $\text{Id}_X : X \to X$,

$$\text{Id}_X : \widetilde{QC}_*^\text{Alt}(X; Y_1, \ldots, Y_r) \to \widetilde{QC}_*^\text{Alt}(X; Y_1, \ldots, Y_r)$$

is the identity map of the $\mathcal{C}$-complex. Hence if we assume that $(S; E_1, \ldots, E_r) = (X; Y_1, \ldots, Y_r)$ and $gf = \text{Id}_X$ in the above notations, then $f^*g^*$ is
homotopy equivalent to the identity, and a homotopy from the identity to $g^*f^*$ is given by

$$\Phi^{m,n} = (-1)^n \sum_{|K|=m, |I|=n} \text{sgn}(K_I) \Xi_{alt}^{K,f,g} : \bigoplus_{|I|=m} \mathbb{Q}C^*_{alt}(Y_I) \to \bigoplus_{|J|=n} \mathbb{Q}C^*_{alt}(Y_J).$$

**Proof:** Suppose $n - m \geq 1$. It is easy to see that for any exact cube $F$,

$$(t_{k_x(1)}, \ldots, t_{k_x(p)}, \text{Id}_X, t_{k_x(p+1)}, \ldots, t_{k_x(n-m)})^* F$$

is invariant under the action of the transposition $(p, p+1)$ if $1 \leq p \leq n - m - 1$, and it is degenerate if $p = 0$ or $n - m$. Hence

$$\text{Alt}^*(t_{k_x(1)}, \ldots, t_{k_x(p)}, \text{Id}_X, t_{k_x(p+1)}, \ldots, t_{k_x(n-m)})^*(F) = 0,$$

which completes the proof. \qed

3. **The multi-relative $K$-theory and the higher Bott-Chern forms**

3.1. **The cocubical schemes $(\mathbb{P}^1)^*$ and $\square^*$.** A cubical complex is a family of abelian groups \{C_n\}_{n=0,1,\ldots} with maps

$$\partial_i^0, \partial_i^\infty : C_n \to C_{n-1}, \quad 1 \leq i \leq n,$$

$$s_i : C_{n-1} \to C_n, \quad 1 \leq i \leq n,$$

satisfying that

$$\partial_i^k \partial_j^l = \partial_j^{l+1} \partial_i^l, \quad i < j,$$

$$\partial_i^k s_j = \begin{cases} s_{j-1} \partial_i^k, & i < j, \\ \text{Id}, & i = j, \\ s_i \partial_{i-1}^k, & i > j, \\ s_is_j = s_{j+1}s_i, & i \leq j \end{cases}$$

for $k, l = 0$ or $\infty$. Then the same family of groups \{C_n\} with

$$\partial = \sum_{i=1}^{n} (-1)^i(\partial_i^0 - \partial_i^\infty) : C_n \to C_{n-1}$$

forms a chain complex. Set

$$D_n = \sum_{i=1}^{n} \text{Im}(s_i : C_{n-1} \to C_n) \subset C_n.$$

Then $D_*$ is a subcomplex of $C_*$. An element of $D_*$ is said to be degenerate. The subcomplex defined by

$$(NC)_n = \bigcap_{j=1}^{n} \text{Ker} \partial_j^\infty$$

is called the normalized subcomplex of $C_*$. Then there is a direct sum decomposition

$$C_* = D_* \oplus NC_*.$$
Hence, if we write \( \tilde{C}_* = C_*/D_* \), then there is a canonical isomorphism of chain complexes

\[ NC_* \simeq \tilde{C}_*. \]

When an element \( x \in C_* \) is decomposed as \( x = x_0 + x_d \) such that \( x_0 \in NC_* \) and \( x_d \in D_* \), \( x_0 \) is called the normalized component of \( x \), and \( x \) is said to be normalized if \( x_d = 0 \).

Let \( \mathbb{P}^1 \) be the projective line. Let \( z \) be the canonical coordinate of \( \mathbb{P}^1 \), and \( z_j = \pi_j^* z \), where \( \pi_j : (\mathbb{P}^1)^r \to \mathbb{P} \) is the \( i \)-th projection. Define coface and codegeneracy maps

\[
\delta^0_j, \delta^\infty_j : (\mathbb{P}^1)^r \to (\mathbb{P}^1)^{r+1}, \quad 1 \leq j \leq r + 1, \\
\sigma_j : (\mathbb{P}^1)^r \to (\mathbb{P}^1)^{r-1}, \quad 1 \leq j \leq r
\]
as follows:

\[
\delta^0_j(z_1, \ldots, z_r) = (z_1, \ldots, z_{j-1}, 0, z_j, \ldots, z_r), \\
\delta^\infty_j(z_1, \ldots, z_r) = (z_1, \ldots, z_{j-1}, \infty, z_j, \ldots, z_r), \\
\sigma_j(z_1, \ldots, z_r) = (z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_r).
\]

Then \( ((\mathbb{P}^1)^*, \delta^0_j, \sigma_j) \) is a cocubical scheme. For any subset \( J = \{j_1, \ldots, j_n\} \subset \{1, \ldots, r\} \) with \( j_1 < \cdots < j_n \) and for any map \( \iota : J \to \{0, \infty\} \), the closed subscheme

\[ D_{J,\iota} = \text{Im} \left( \delta^{(j_n)}_{j_n} \cdots \delta^{(j_1)}_{j_1} : (\mathbb{P}^1)^{r-n} \to (\mathbb{P}^1)^r \right) \]
is called a face of \( (\mathbb{P}^1)^r \). In other words,

\[ D_{J,\iota} = \{(z_1, \ldots, z_r) \in (\mathbb{P}^1)^r; z_{j_k} = \iota(j_k), k = 1, \ldots, n\}. \]

Let \( D_j = \{z_j = 0 \text{ or } \infty \} \subset (\mathbb{P}^1)^r \) and \( D_J = \bigcap_{j \in J} D_j \). Then \( D_J \) is a disjoint union of faces of \( (\mathbb{P}^1)^r \):

\[ D_J = \bigsqcup_{\iota : J \to \{0, \infty\}} D_{J,\iota} \subset (\mathbb{P}^1)^r. \]

Let \( \square = \mathbb{P}^1 - \{1\} \). Then \( \square^* \subset (\mathbb{P}^1)^r \) with the same coface and codegeneracy maps is also a cocubical scheme. Denote the faces of \( \square^r \) by the same symbol:

\[ D_{J,\iota} = \text{Im} \left( \delta^{(j_1)}_{j_1} \cdots \delta^{(j_n)}_{j_n} : \square^{r-n} \to \square^r \right), \\
D_J = \bigsqcup_{\iota : J \to \{0, \infty\}} D_{J,\iota} \subset \square^r. \]

3.2. Complexes of differential forms. In this subsection we will introduce several complexes of differential forms which we will use throughout the paper. In what follows, by complex algebraic manifold we mean the complex manifold associated with a smooth algebraic variety defined over the complex number field \( \mathbb{C} \). Let \( X \) be a complex algebraic manifold. Denote by \( (E^*_\log, \mathbb{R})(X), d \) the complex of real smooth differential forms on \( X \) with logarithmic singularities along infinity, which is defined in \([Bu1]\). Then \( (E^*_\log, \mathbb{R})(X), d \) with the natural bigrading

\[ E^n_{\log, \mathbb{R}}(X) \otimes \mathbb{C} = \bigoplus_{p+q=n} E^{p,q}_{\log}(X) \]
forms a Dolbeault complex. It is proved in [Bu1] that the complex \( E^{\bullet}_{\log R}(X) \) with the above bigrading computes the cohomology groups of \( X \) with the usual Hodge structure. Denote by \((\mathcal{D}_{\log}^*(X,p), d_D)\) the associated Deligne complex [Bu2, §2], and by \( \tau D^*_\log(X,p) = \tau_{2p} \mathcal{D}^*_\log(X,p) \) the subcomplex of \( \mathcal{D}^*_\log(X,p) \) truncated at the degree \( 2p \). That is to say,

\[
\tau D^n_{\log}(X,p) = \begin{cases} 
E^{n-1}_{\log R}(X)(p-1) \cap \bigoplus_{p'+q'=n-1} E^{p',q'}_{\log R}(X), & n < 2p, \\
E^{2p}_{\log R}(X)(p) \cap E^{p,p}_\log(X) \cap \text{Ker } d, & n = 2p, \\
0, & n > 2p,
\end{cases}
\]

where \( R(p) = (2\pi i)^p R \subset \mathbb{C} \) and \( E^*_{\log R}(X)(p) \) is the space of differential forms on \( X \) with values in \( R(p) \). The differential \( d_D : \tau D^n_{\log}(X,p) \to \tau D^{n+1}_{\log}(X,p) \) is given by

\[
d_D(\omega) = \begin{cases} 
-\pi(d\omega), & n < 2p - 1, \\
-2\overline{\partial}\omega, & n = 2p - 1, \\
0, & n > 2p - 1,
\end{cases}
\]

where \( \pi : E^*_\log(X) \otimes \mathbb{C} \to \tau D^{n+1}_{\log}(X,p) \) is the canonical projection. Then it is shown in [Bu2, Cor.2.7] that if \( n \leq 2p \), then the cohomology groups of \( (\tau D^*_\log(X,p), d_D) \) are canonically isomorphic to the Deligne cohomology of \( X \):

\[
H^n(\tau D^*_\log(X,p), d_D) \simeq H^n_D(X, R(p)).
\]

Let us recall the multiplicative structure of \( \tau D^*_\log(X,p) \) given in [Bu2, §3]. Define a product

\[
\bullet : \tau D_{\log}^n(X,p) \otimes \tau D_{\log}^m(X,q) \to \tau D_{\log}^{n+m}(X,p+q)
\]

as follows: If \( n < 2p \) and \( m < 2q \), then

\[
\omega \bullet \eta = (-1)^n (\partial \omega_{p-1,n-p} - \overline{\partial} \omega_{n-p,p-1}) \wedge \eta + \omega \wedge (\partial \eta_{p-1,m-q} - \overline{\partial} \eta_{m-q,p-1}),
\]

where \( \omega_{p,q} \) is the \((p,q)\)-part of the differential form \( \omega \). If \( n = 2p \) or \( m = 2q \), then define \( \omega \bullet \eta = \omega \wedge \eta \). It is shown in [Bu2, §3] that

\[
d_D(\omega \bullet \eta) = d_D\omega \bullet \eta + (-1)^n \omega \bullet d_D\eta
\]

and the induce map on cohomology agrees with the product in the Deligne cohomology defined in [EV, §2].

Set

\[
\mathcal{D}^{n,-r}_{A}(X,p) = \tau D^n_{\log}(X \times \square^r, p)
\]

and define differentials by

\[
d_D : \mathcal{D}_{A}^{n,-r}(X,p) \to \mathcal{D}^{n+1,-r}_{A}(X,p),
\]

\[
\delta_A = \sum_{j=1}^{r} (-1)^j ((\delta_j^D)^* - (\delta_j^\infty)^*) : \mathcal{D}_{A}^{n,-r}(X,p) \to \mathcal{D}_{A}^{n-r+1}(X,p).
\]
Then $\mathcal{D}_{\Delta}^n_{X,p}(X,p), d_\Delta, \delta_\Delta$ is a double complex. When we fix the first index $n$, $r \mapsto \mathcal{D}_{\Delta}^n_{X,p}(X,p)$ has a cubical structure. Denote by $\mathcal{D}_{\Delta}^n_{X,p}(X,p)$ the subcomplex of degenerate elements with respect to this cubical structure, and set
\[
\mathcal{D}_{\Delta}^n_{X,p}(X,p) = \mathcal{D}_{\Delta}^n_{X,p}(X,p)/\mathcal{D}_{\Delta}^n_{X,p}(X,p).
\]
Denote by $(\mathcal{D}_{\Delta}^n_{X,p}(X,p), d_\Delta)$ the single complex associated with $\mathcal{D}_{\Delta}^n_{X,p}(X,p)$. Then it is shown in \cite{BF} Prop.2.8] that the inclusion
\[
\tau \mathcal{D}_{\Delta}^n_{X,p}(X,p) = \mathcal{D}_{\Delta}^n_{X,p}(X,p) \hookrightarrow \mathcal{D}_{\Delta}^n_{X,p}(X,p)
\]
is a quasi-isomorphism.

We introduce another double complex. Set
\[
\mathcal{D}_{\mathbb{P}}^n_{r}(X,p) = \tau \mathcal{D}_{\log}^n_{X}(X \times (\mathbb{P}^1)^r, p)
\]
and define differentials by
\[
d_D : \mathcal{D}_{\mathbb{P}}^n_{r}(X,p) \rightarrow \mathcal{D}_{\mathbb{P}}^{n+1}_{r}(X,p),
\]
\[
\delta_\mathbb{P} = \sum_{j=1}^{r} (-1)^j ((\delta_j^0)^* - (\delta_j^\infty)^*) : \mathcal{D}_{\mathbb{P}}^n_{r}(X,p) \rightarrow \mathcal{D}_{\mathbb{P}}^{n-r+1}(X,p).
\]
Then $(\mathcal{D}_{\mathbb{P}}^n_{r}(X,p), d_\mathbb{P}, \delta_\mathbb{P})$ is also a double complex. Let $z$ be the canonical coordinate of $\mathbb{P}^1$, and fix a Kähler form $\Omega = \partial\overline{\partial} \log (1 + |z|^2) \in \mathcal{D}_{\mathbb{P}}^2(X,1)$ on $\mathbb{P}^1$. Set
\[
\mathcal{D}_{\mathbb{P}}^n_{r}(X,p) = \sum_{j=1}^{r} \left( \sigma_j^*(\mathcal{D}_{\mathbb{P}}^n_{r+1}(X,p)) + \pi_j^* \Omega \wedge \sigma_j^*(\mathcal{D}_{\mathbb{P}}^{n-2,r+1}(X,p,1)) \right),
\]
where $\pi_j : X \times (\mathbb{P}^1)^r \rightarrow \mathbb{P}^1$ is the projection to the $j$-th component of $(\mathbb{P}^1)^r$, and $\sigma_j$ is the codegeneracy map of the cocubical scheme $(\mathbb{P}^1)^s$. Set
\[
\mathcal{D}_{\mathbb{P}}^n_{r}(X,p) = \mathcal{D}_{\mathbb{P}}^n_{X,p}(X,p)/\mathcal{D}_{\mathbb{P}}^n_{r}(X,p),
\]
and let $(\mathcal{D}_{\mathbb{P}}^n_{X,p}(X,p), d_\mathbb{P})$ be the associated single complex. Then it is shown in \cite{BW} Prop.1.2] that the inclusion
\[
\tau \mathcal{D}_{\mathbb{P}}^n_{X,p}(X,p) = \mathcal{D}_{\mathbb{P}}^n_{X,p}(X,p) \hookrightarrow \mathcal{D}_{\mathbb{P}}^n_{X,p}(X,p)
\]
is a quasi-isomorphism.

Finally we introduce a triple complex mixing the above construction. Set
\[
\mathcal{D}_{\Delta_{\mathbb{P}}}^n_{r,s}(X,p) = \tau \mathcal{D}_{\log}^n_\Delta(X \times \square^r \times (\mathbb{P}^1)^s, p)
\]
with differentials $d_\Delta, \delta_\Delta, \delta_\mathbb{P}$ defined in the same way as above. Define a subcomplex of degenerate elements as follows:
\[
\mathcal{D}_{\Delta_{\mathbb{P}}}^n_{r,s}(X,p) = \mathcal{D}_{\Delta}^n_{X,p}(X \times (\mathbb{P}^1)^s, p) + \mathcal{D}_{\mathbb{P}}^n_{r,s}(X \times \square^r, p) \subset \mathcal{D}_{\Delta_{\mathbb{P}}}^n_{r,s}(X,p).
\]
Set
\[
\mathcal{D}_{\Delta_{\mathbb{P}}}^n_{r,s}(X,p) = \mathcal{D}_{\Delta_{\mathbb{P}}}^n_{r,s}(X,p)/\mathcal{D}_{\Delta}^n_{X,p}(X,p),
\]
and let $(\mathcal{D}_{\Delta_{\mathbb{P}}}^n_{X,p}(X,p), d_\mathbb{P})$ be the associated single complex. Then the inclusion
\[
\tau \mathcal{D}_{\log}^n_{X,p}(X,p) = \mathcal{D}_{\Delta_{\mathbb{P}}}^n_{X,p}(X,p) \hookrightarrow \mathcal{D}_{\Delta_{\mathbb{P}}}^n_{X,p}(X,p)
\]
forms a Dolbeault complex as well. Denote by $\tau$ truncated at 2
be the map given by the integral
Similarly, we can define products
which satisfies the relation
which satisfy the same relation.

3.3. Complexes of currents. Let $X$ be a complex algebraic manifold of dimension $d_X$. Denote by $E^n_\mathbb{R}(X)_{\text{cpt}}$ the space of real smooth differential forms on $X$ of degree $n$ with compact support, and by $D^n_\mathbb{R}(X)$ the topological dual of $E^{2d_X-n}(X)_{\text{cpt}}(d_X)$. Define $d : D^0_\mathbb{R}(X) \to D^1_\mathbb{R}(X)$ by $dT(\omega) = (-1)^n T(d\omega)$ for $\omega \in E^{2d_X-n-1}(X)_{\text{cpt}}(d_X)$. Denote by $E^{p,q}_\mathbb{R}(X)_{\text{cpt}}$ the space of $(p,q)$-forms on $X$ with compact support, and by $D^{p,q}_\mathbb{R}(X)$ the topological dual of $E^{d_X-p,d_X-q}(X)_{\text{cpt}}$. Then $(D^n_\mathbb{R}(X), d)$ with the bigrading
forms a Dolbeault complex as well. Denote by $\tau D^*_D(X,p)$ the associated Deligne complex truncated at 2p.

Let
be the map given by the integral
for $\omega \in E^{2d_X-n}(X)_{\text{cpt}}$. It satisfies $d[\eta] = [d\eta]$, and gives a quasi-isomorphism of the Dolbeault complexes. Hence it induces a quasi-isomorphism of the associated Deligne complexes:

$$[\quad] : \tau D^*_D(X,p) \to \tau D^*_D(X,p).$$
For an integral subvariety $V \subset X$ of codimension $p$, let $\delta_V \in \tau \mathcal{D}_D^{2p}(X, p)$ be the current given by the integral

$$\delta_V(\omega) = \frac{1}{(2\pi i)^{d_X-p}} \int_V \tau^* \omega$$

for $\omega \in E_R^{2d_X-2p}(X)_{\text{cpt}}(d_X - p)$, where $\tau : \tilde{V} \to V \subset X$ is a disingularization of $V$. We can extend this construction linearly to any cycle $z$ of codimension $p$ and we can define a current $\delta_z \in \tau \mathcal{D}_D^{2p}(X, p)$.

Let $Y$ be a complex algebraic manifold of dimension $d_Y$, and $f : X \to Y$ a proper morphism. We can define the direct image map

$$f_* : \mathcal{D}_D^p(X) \to \mathcal{D}_D^p(Y)(dy - dX)[2dy - 2dX]$$

by $f_*(T)(\omega) = T(f^* \omega)$ for $T \in \mathcal{D}_D^p(X)$ and $\omega \in E_R^{2d_X}(Y)_{\text{cpt}}$. It induces a map of the Deligne complexes

$$f_* : \tau \mathcal{D}_D^p(X, p) \to \tau \mathcal{D}_D^p(Y, p + dy - dX)[2dy - 2dX].$$

### 3.4. Wang’s forms. In this subsection we will introduce Wang’s forms [BW]. Let $z$ be the canonical coordinate of the projective line $\mathbb{P}^1$ and $z_i = \pi_i^* z$, where $\pi_i : (\mathbb{P}^1)^r \to \mathbb{P}^1$ is the $i$-th projection. For $r \leq 1$, define a differential form $W_r$ on $((\mathbb{P}^1)^r)$ by

$$W_r = \frac{1}{2r!} \sum_{i=1}^r (-1)^i S_r^i,$$

where

$$S_r^i = \sum_{\sigma \in \Theta_r} (\text{sgn } \sigma) \log |z_{\sigma(1)}|^2 \frac{dz_{\sigma(2)}}{z_{\sigma(2)}} \wedge \cdots \wedge \frac{dz_{\sigma(i)}}{z_{\sigma(i)}} \wedge \frac{dz_{\sigma(i+1)}}{z_{\sigma(i+1)}} \wedge \cdots \wedge \frac{dz_{\sigma(r)}}{z_{\sigma(r)}}.$$

The form $W_r$ has logarithmic singularities along the codimension one faces of $((\mathbb{P}^1)^r)$, and it is locally integrable on $((\mathbb{P}^1)^r)$ such that $W_r = (-1)^{r-1} W_r$. Hence as a current, $[W_r] \in \tau \mathcal{D}_D^p((\mathbb{P}^1)^r, r)$. When $r = 0$, suppose that $W_0 = 1$.

**Proposition 3.1.** [BFT] Thm.6.7] *When $r \geq 1$, the current $[W_r]$ satisfies the relation*

$$d_\tau [W_r] = \sum_{j=1}^r (-1)^j (\delta_j^0)_*[W_{r-1}] - (\delta_j^\infty)_*[W_{r-1}]$$

We now construct several maps between the Deligne complexes given in §3.2 and §3.3 in the case that $X$ is compact. Let

$$\pi_\mathbb{P} : X \times (\mathbb{P}^1)^r \to (\mathbb{P}^1)^r,$$

$$\pi_X : X \times (\mathbb{P}^1)^r \to X$$

be the projections. For any $\omega \in \mathcal{D}_D^{n-r}(X, p) = \tau \mathcal{D}^n(X \times (\mathbb{P}^1)^r, p)$, consider the integral along fibers

$$\frac{1}{(2\pi i)^r} \int_{(\mathbb{P}^1)^r} \omega \cdot \pi_\mathbb{P}^* W_r \in \tau \mathcal{D}_D^{n-r}(X, p).$$
We can show in the same way as [BW] Lem.6.8 that the above integral is zero if \( \omega \in D^{n-r}_P(X,p) \).

**Proposition 3.2.** The map

\[
\kappa_P : \tilde{D}^*_P(X,p) \to \tau D^{n-r}(X,p)
\]

induced by the above integral is a map of complexes.

**Proof:** For \( \omega \in D^{n-r}_P(X,p) \),

\[
d_D \kappa_P(\omega) = \frac{1}{(2\pi i)^r} d_D \left( \int_{(\mathbb{P}^1)^r} \omega \cdot \pi_P^* W_r \right)
\]

\[
= \frac{1}{(2\pi i)^r} \int_{(\mathbb{P}^1)^r} (d_D - d_D, p)(\omega \cdot \pi_P^* W_r),
\]

where \( d_D, p \) is the differential of \( \tau D^*(X \times (\mathbb{P}^1)^r, p) \) on the component \( (\mathbb{P}^1)^r \). Since \( d_D W_r = 0 \) as a differential form,

\[
\frac{1}{(2\pi i)^r} \int_{(\mathbb{P}^1)^r} d_D(\omega \cdot \pi_P^* W_r) = \frac{1}{(2\pi i)^r} \int_{(\mathbb{P}^1)^r} d_D \omega \cdot \pi_P^* W_r.
\]

On the other hand, it follows from Prop.3.1 that

\[
\frac{1}{(2\pi i)^r} \int_{(\mathbb{P}^1)^r} d_D, p(\omega \cdot \pi_P^* W_r) = \frac{(-1)^{n-1}}{(2\pi i)^{r-1}} \int_{(\mathbb{P}^1)^{r-1}} (\delta_P \omega) \cdot \pi_P^* W_{r-1}.
\]

Hence

\[
d_D \kappa_P(\omega) = \frac{1}{(2\pi i)^r} \int_{(\mathbb{P}^1)^r} d_D \omega \cdot \pi_P^* W_r + \frac{(-1)^n}{(2\pi i)^{r-1}} \int_{(\mathbb{P}^1)^{r-1}} (\delta_P \omega) \cdot \pi_P^* W_{r-1}
\]

\[
= \kappa_P(d_D \omega) + (-1)^n \kappa_P(\delta_P \omega),
\]

which completes the proof. \( \square \)

It is easy to see that the map \( \kappa_P \) is a left inverse of the inclusion \( \tau D^*(X,p) \to \tilde{D}^*_P(X,p) \). In particular, \( \kappa_P \) is a quasi-isomorphism.

Let us next consider a similar map on the complex \( \tilde{D}^*_A(X,p) \). In this case, we can not take the integration along fibers. However, it is shown in [BFT] Prop.6.5 that for any \( \omega \in D^{n-r}_A(X,p) = \tau D^{n}_A(X \times \square^r, p) \) the product \( \omega \cdot \pi_P^* W_r \) is locally integrable on the compactification \( X \times (\mathbb{P}^1)^r \) of \( X \times \square^r \), therefore we can define the current

\[
\pi_{X*}[\omega \cdot \pi_P^* W_r] \in \tau D^{n-r}_D(X,p).
\]

Moreover, [BFT] Prop.6.5 says that

\[
d_D \pi_{X*}[\omega \cdot \pi_P^* W_r] = \pi_{X*}[d_D \omega \cdot \pi_P^* W_r] + (-1)^n \pi_{X*}[\delta_A \omega \cdot \pi_P^* W_{r-1}].
\]

Since \( \pi_{X*}[\omega \cdot \pi_P^* W_r] = 0 \) if \( \omega \in D^{n-r}_A(X,p) \), we conclude that the above current gives a map of complexes

\[
\kappa_A : \tilde{D}^*_A(X,p) \to \tau D^*_D(X,p),
\]
which make the diagram

\[
\begin{array}{c}
\tau \mathcal{D}^* (X,p) \\
\tau \mathcal{D}^*_\mathcal{D} (X,p) \\
\end{array} \quad \xrightarrow{\kappa_A} \quad
\begin{array}{c}
\mathcal{D}^*_A (X,p) \\
\mathcal{D}^*_A ,P (X,p) \\
\end{array}
\]

commutative. In particular, \( \kappa_A \) is also a quasi-isomorphism.

Finally let us consider a map from the complex \( \mathcal{D}^*_A ,P (X,p) \). In the same way as the previous case we can show that for any \( \omega \in \mathcal{D}^{n,-r,-s} (X,p) = \tau \mathcal{D}^{n}_{\text{log}} (X \times \square^r \times (\mathbb{P}^1)^s, p) \), the product \( \omega \cdot \pi^*_P W_{r+s} \) is locally integrable on the compactification \( X \times (\mathbb{P}^1)^{r+s} \) of \( X \times \square^r \times (\mathbb{P}^1)^s \), and that the current \( \pi_X \ast [\omega \cdot \pi^*_P W_{r+s}] \in \tau \mathcal{D}^{n-r-s} (X,p) \) gives a quasi-isomorphism \( \kappa_{A,P} : \mathcal{D}^*_A (X,p) \to \tau \mathcal{D}^*_\mathcal{D} (X,p) \).

Summing up the results in this subsection, we obtain the commutative diagram

\[
\begin{array}{c}
\mathcal{D}^*_A (X,p) \\
\mathcal{D}^*_A ,P (X,p) \\
\end{array} \quad \xrightarrow{\kappa_A} \quad
\begin{array}{c}
\mathcal{D}^*_A ,P (X,p) \\
\tau \mathcal{D}^* (X,p) \\
\end{array}
\]

all the maps in which are quasi-isomorphisms.

3.5. The higher Bott-Chern forms. In this subsection we will recall the higher Bott-Chern forms \([BW]\). Let \( X \) be a complex algebraic manifold and \( \mathcal{F} \) a vector bundle on \( X \). Given a smooth hermitian metric \( h \) on \( \mathcal{F} \), there exists a unique connection on \( \mathcal{F} \) which is compatible both with the complex structure and with the metric. Using the curvature form of this connection, we obtain a differential form

\[
\text{ch}_0 (\mathcal{F}, h) \in \bigoplus_p E^{2p}_R (X)(p) \cap E^{p-p} (X) \cap \text{Ker} \ d
\]

which represents the Chern character of \( \mathcal{F} \). This form is called the Chern form of \( (\mathcal{F}, h) \).

A smooth hermitian metric \( h \) on \( \mathcal{F} \) is said to be smooth at infinity if there is a vector bundle \( \mathcal{F}' \) on a smooth compactification \( \overline{X} \) of \( X \) with a smooth hermitian metric \( h' \) such that the restriction \( (\mathcal{F}'|_{\overline{X}}, h'|_{\overline{X}}) \) is isometric to \( (\mathcal{F}, h) \). In this case the Chern form \( \text{ch}_0 (\mathcal{F}) \) can be extended to a smooth differential form on \( \overline{X} \). In particular,

\[
\text{ch}_0 (\mathcal{F}) \in \bigoplus_p \tau \mathcal{D}^{2p}_{\text{log}} (X,p).
\]

In what follows, any hermitian metric on a vector bundle is supposed to be smooth at infinity.

An exact hermitian \( n \)-cube on \( X \) is an exact \( n \)-cube consisting of vector bundles on \( X \) with smooth hermitian metrics. Denote by \( \mathcal{Q} \mathcal{C}_s (X) \) the chain complex of exact hermitian cubes on \( X \) with smooth at infinity metrics, and by \( \widetilde{\mathcal{Q}} \mathcal{C}_s (X) \) the quotient complex of \( \mathcal{Q} \mathcal{C}_s (X) \) by the subcomplex of degenerate cubes. Denote by \( \widetilde{\mathcal{Q}} \mathcal{C}_s^{\text{Alt}} (X) \) the alternating part of \( \widetilde{\mathcal{Q}} \mathcal{C}_s (X) \). Since the space of smooth at infinity metrics on a vector bundle is
convex, the map $\tilde{Q}\hat{C}_*(X) \to \tilde{Q}C_*(X)$ forgetting metrics is a quasi-isomorphism. Hence Thm.2.9 implies the following:

**Theorem 3.3.** The homology groups of $\tilde{Q}\hat{C}_*(X)$ and $\tilde{Q}C^\text{Alt}_*(X)$ are canonically isomorphic to the rational algebraic $K$-theory of $X$:

$$H_n(\tilde{Q}C^\text{Alt}_*(X)) \simeq H_n(\tilde{Q}\hat{C}_*(X)) \simeq K_n(X)_\mathbb{Q}.$$  

An exact hermitian $n$-cube $\mathcal{F}$ on $X$ is *emi* if for any $1 \leq j \leq n$, the metric on $\partial^{-1}_j \mathcal{F}$ is induced from the metric on $\partial^0_j \mathcal{F}$ by means of the inclusion $\partial^{-1}_j \mathcal{F} \hookrightarrow \partial^0_j \mathcal{F}$. Let $\tilde{Q}C^\text{emi}_*(X)$ denote the subcomplex of $\tilde{Q}\hat{C}_*(X)$ consisting of emi-cubes. As seen in [BW, §3], we can obtain a map of complexes

$$\lambda : \tilde{Q}\hat{C}_*(X) \to \tilde{Q}C^\text{emi}_*(X)$$

such that the composite of $\lambda$ with the inclusion $\tilde{Q}C^\text{emi}_*(X) \hookrightarrow \tilde{Q}\hat{C}_*(X)$ is homotopy equivalent to the identity.

With any emi-$n$-cube $\mathcal{F}$ on $X$ we can associate a hermitian vector bundle $\text{tr}_n(\mathcal{F})$ on $X \times (\mathbb{P}^1)^n$ such that there are isometries

$$\begin{cases} 
\text{tr}_n(\mathcal{F})\big|_{\{z_j=0\}} \simeq \text{tr}_{n-1}(\partial^0_j \mathcal{F}), \\
\text{tr}_n(\mathcal{F})\big|_{\{z_j=\infty\}} \simeq \text{tr}_{n-1}(\partial^{-1}_j \mathcal{F}) \oplus \text{tr}_{n-1}(\partial^1_j \mathcal{F})
\end{cases}$$

for $1 \leq j \leq n$. The hermitian vector bundle $\text{tr}_n(\mathcal{F})$ is called *transgression bundle* of $\mathcal{F}$ [BW, Def.3.8]. Note that the metric on $\text{tr}_n(\mathcal{F})$ is smooth at infinity if each metric on $\mathcal{F}$ is smooth at infinity, as mentioned in [BW, Def.3.8]. Define the *Bott-Chern form* of an exact hermitian $n$-cube $\mathcal{F}$ on $X$ to be the Chern form of $\text{tr}_n(\lambda \mathcal{F})$:

$$\text{ch}_{n}(\mathcal{F})_p = \text{ch}_0(\text{tr}_n(\lambda \mathcal{F}))) \in \bigoplus_p \mathbb{D}_{\mathbb{P}}^{2p-n}(X,p).$$

It follows from the isometries \(3.1\) that $\delta_p \text{ch}_n(\mathcal{F})_p = \text{ch}_{n-1}(\partial \mathcal{F})_p$. Since $\text{ch}_n(\mathcal{F})_p = 0$ if $\mathcal{F}$ is degenerate [BW, Prop.3.11], it induces a map of complexes

$$\text{ch}_{n,p} : \tilde{Q}\hat{C}_*(X) \to \bigoplus_p \mathbb{D}_{\mathbb{P}}^{2p-n}(X,p).$$

In the case that $X$ is compact, we can obtain

$$\text{ch}_n(\mathcal{F}) = \kappa_p(\text{ch}_n(\mathcal{F})_p) \in \bigoplus_p \mathbb{D}^{2p-n}(X,p).$$

Note that in the case that $n=1$, $\text{ch}_1(\mathcal{F}) \in \bigoplus_p \mathbb{D}^{2p-n}(X,p)/ \text{Im} \, d_{\mathbb{D}}$ agrees with the Bott-Chern secondary class of a short exact sequence $\mathcal{F}$ defined by Gillet and Soulé in [GS2].

**Theorem 3.4.** [BW, Thm.5.2] When $X$ is compact, the map on homology induced by the Bott-Chern forms

$$\text{ch}_n = \bigoplus_p \text{ch}_n^p K_n(X)_{\mathbb{Q}} \to \bigoplus_p H^{2p-n}_{\mathbb{D}}(X, \mathbb{R}(p))$$

agrees with Beilinson’s regulator.
Remark: The target of Beilinson’s regulator is the absolute Hodge cohomology, not the Deligne cohomology. Hence in [BW] the higher Bott-Chern forms sit in a complex which computes the absolute Hodge cohomology. However, since the both cohomology theories are canonically isomorphic for compact manifolds, the above theorem follows.

3.6. A multi-relative complex of exact hermitian cubes and the higher Bott-Chern forms. We begin by introducing a metrized version of the complex of exact cubes defined in §2.4. Let $X$ be a complex algebraic manifold and $Y_1, \ldots, Y_r$ closed submanifolds of $X$. Then the same construction as in §2.4 and §2.5 gives a complex

$$
\tilde{\mathcal{QC}}_n(X; Y_1, \ldots, Y_r) = \bigoplus_{|I|=m} \tilde{\mathcal{QC}}_n(Y_I), \quad F^{m,n}
$$

and

$$
\tilde{\mathcal{QC}}^\text{Alt}_n(X; Y_1, \ldots, Y_r) = \bigoplus_{|I|=m} \tilde{\mathcal{QC}}^\text{Alt}_n(Y_I), \quad F^{m,n}
$$

It follows from Cor.2.16 and Thm.3.3 that there are canonical isomorphisms

$$
H_n(\tilde{\mathcal{QC}}^\text{Alt}_n(X; Y_1, \ldots, Y_r)) \simeq H_n(\tilde{\mathcal{QC}}_n(X; Y_1, \ldots, Y_r)) \simeq K_n(X; Y_1, \ldots, Y_r)_\mathbb{Q}.
$$

Let us recall the notations introduced in §3.1. Consider the product $X \times \Box^r$ with the normal crossing divisor

$$
X \times \partial \Box^r = X \times D_1 + \cdots + X \times D_r,
$$

where $D_j = \{ z_j = 0, \infty \} \subset \Box^r$. We identify $X$ with $X \times \{ (\infty, \ldots, \infty) \}$, which is a connected component of $X \times D\{1, \ldots, r\}$. This gives an embedding of chain complexes

$$
i_X: \tilde{\mathcal{QC}}_n(X) \hookrightarrow \tilde{\mathcal{QC}}_n(X \times \Box^r; X \times \partial \Box^r)[r].
$$

The homotopy invariant property of $K$-theory implies that the map (3.2) induces an isomorphism of $K$-groups:

$$
K_{n+r}(X)_\mathbb{Q} \simeq K_n(X \times \Box^r; X \times \partial \Box^r)_\mathbb{Q}.
$$

Let $\mathcal{F}_I$ be an exact hermitian $n$-cube on $X \times D_I$. In other words, $\mathcal{F}_I$ is a family $\{\mathcal{F}_{I,i}\}_{i:I \rightarrow \{0, \infty\}}$ such that $\mathcal{F}_{I,i}$ is an exact hermitian $n$-cube on $X \times D_{I,i}$ with a smooth at infinity metric. We identify $D_{I,i}$ with $\Box^{|I|}$, and let

$$
\text{ch}_n(\mathcal{F}_I)_\mathbb{P} = \sum_{i:I \rightarrow \{0, \infty\}} (-1)^{|i|} \text{ch}_n(\mathcal{F}_{I,i})_\mathbb{P} \in \bigoplus_{p} \mathbb{P}^{2p-n}(X \times \Box^{|I|-|I|}, p),
$$

where $|i|$ is the cardinality of the set $\{i \in I; i(i) = \infty\}$. Then it gives a map

$$
\text{ch}_n,\mathbb{P}: \tilde{\mathcal{QC}}_n(X \times D_I) \rightarrow \bigoplus_{p} \mathbb{P}^{2p-n}(X, p).
$$

Since $\delta_p \text{ch}_n(\mathcal{F}_I)_\mathbb{P} = \text{ch}_{n-1}(\partial \mathcal{F}_I)_\mathbb{P}$, we have the following:

**Proposition 3.5.** For any $x_I \in \tilde{\mathcal{QC}}_n(X \times D_I)$, we have

$$
d_s \text{ch}_n(x_I)_\mathbb{P} = \sum_{|I|=1} (-1)^{r-|I|} \text{ch}_n(x_I|X \times D_I)_\mathbb{P} + (-1)^{r-|I|} \text{ch}_{n-1}(\partial x_I)_\mathbb{P}
$$
in $\widetilde{D}_{X,P}(X,p)$, where $\{i_1, \ldots, i_{r-|I|}\}$ is the complement of $I$ with $i_1 < \cdots < i_{r-|I|}$ and $I_i = I \cup \{i_i\}$.

**Definition 3.6.** An element $x \in \widehat{\mathcal{Q}}\mathcal{C}_n(X)$ is said to be isometrically equivalent to a degenerate element if there is a lift

$$\sum_i r_i [\mathcal{F}_i] \in \mathcal{Q}\mathcal{C}_n(X)$$

of $x$ such that each $\mathcal{F}_i$ is isometric to a degenerate cube.

It is obvious from the definition of the map $\lambda$ in [BW, §3] that $\lambda \mathcal{F}$ is isometrically equivalent to a degenerate element if so is $\mathcal{F}$. Furthermore, it is obvious from the definition of the transgression bundle that an isometry $\mathcal{F} \simeq \mathcal{G}$ of emi- $n$-cubes on $X$ induces an isometry $\text{tr}_n \mathcal{F} \simeq \text{tr}_n \mathcal{G}$ of hermitian vector bundles. Hence if $x \in \widehat{\mathcal{Q}}\mathcal{C}_n(X)$ is isometrically equivalent to a degenerate element, then $\text{ch}_n(x)_\mathbb{F} = 0$.

Consider a sequence of morphisms of complex algebraic manifolds:

$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_r} X_r.$

Then for a hermitian vector bundle $\mathcal{F}$ on $X_r$, $(f_1, \ldots, f_r)^* \mathcal{F}$ is isometrically equivalent to a degenerate element if $r \geq 2$, since all the maps in $(f_1, \ldots, f_r)^* \mathcal{F}$ are isometries or the zero maps. More generally, for an exact hermitian cube $\mathcal{F}$ on $X_r$, $(f_1, \ldots, f_r)^* \mathcal{F}$ is isometrically equivalent to a degenerate element if $r \geq 2$. Hence we have the following:

**Proposition 3.7.** Consider the maps of complex algebraic manifolds with closed submanifolds

$$(X; Y_1, \ldots, Y_r) \xrightarrow{f} (T, D_1, \ldots, D_r) \xrightarrow{g} (S; E_1, \ldots, E_r).$$

Take $x = (x_I) \in \bigoplus_{|I|=m} \widehat{\mathcal{Q}}\mathcal{C}_*(E_I)$ and $J \subset \{1, \ldots, r\}$ with $|J| = n$. Then $F^{m,n}(x)_J$ for $n - m \geq 2$, $g^{m,n}(x)_J$ for $n - m \geq 1$, and $\Phi^{m,n}(x)_J$ for any $m$ and $n$ are isometrically equivalent to degenerate elements. Hence

$$\text{ch}_*(F^{m,n}(x)_J)_\mathbb{F} = 0 \text{ if } n - m \geq 2,$$

$$\text{ch}_*(g^{m,n}(x)_J)_\mathbb{F} = 0 \text{ if } n - m \geq 1,$$

$$\text{ch}_*(\Phi^{m,n}(x)_J)_\mathbb{F} = 0 \text{ for any } m \text{ and } n.$$

**Proposition 3.8.** For

$$x = (x_I) \in \widehat{\mathcal{Q}}\mathcal{C}_n(X \times \partial \vartheta^r; X \times \partial \vartheta^r) = \bigoplus_I \widehat{\mathcal{Q}}\mathcal{C}_n+|I|/(X \times D_I),$$

let

$$\text{ch}_n(x) = \sum_I (-1)^{I/2(|I|+1)+I/2+|I|} \text{ch}_n+|I|(x_I)_\mathbb{F} \in \bigoplus_p \widetilde{D}_{X,p}^{2p-n}(X,p),$$

where $|I|$ is the cardinality of $I$ and $\Sigma I$ is the sum of elements of $I$. Then

$$\text{ch}_*: \widehat{\mathcal{Q}}\mathcal{C}_n(X \times \partial \vartheta^r; X \times \partial \vartheta^r)[n] \to \bigoplus_p \widetilde{D}_{X,p}^{2p-n}(X,p)$$

is an isomorphism.
is a map of chain complexes. Hence it induces a map

\[ ch_n = \bigoplus_p \chi^p_n : H^2_{\mathcal{D}}(X, \mathbb{R}(p)) = \bigoplus_p \mathbb{H}^{2p-n-r}_{\mathcal{D}}(X, \mathbb{R}(p)). \]

Proof: Let \( \{i_1, \ldots, i_{r-|I|}\} \) be the complement of \( I \) with \( i_1 < \cdots < i_{r-|I|} \) and \( I_t = I \cup \{i_t\} \). Then Prop.3.5 implies that

\[ d_s \chi_n(x) = \sum_{l=1}^{r-|I|} (-1)^{\frac{1}{2}|I|(|I|+1)+r|I|+\Sigma I+l} \chi_{n+|I|}(x_I | X \times D_{I_t}) \mathbb{P} \]

\[ + \sum_{l=1}^{r-|I|} (-1)^{\frac{1}{2}|I|(|I|+1)+r|I|+\Sigma I+r-|I|} \chi_{n+|I|-1}(\partial x_I) \mathbb{P}. \]

If we write \( J = I_t = \{j_1, \ldots, j_r\} \) with \( j_1 < \cdots < j_r \) and \( i_t = j_k \), then \( i_t = j_k = l + k - 1 \) and \( \Sigma I + l = \Sigma J - k + 1 \). Hence

\[ \begin{align*}
\chi_{n+|I|-1}(\partial x_J) \mathbb{P} + (-1)^{|J|} \bigoplus \chi_{n+|I|-1}(x_{J-j_k} | X \times D_J) \mathbb{P} \\
= (-1)^{|J|} \sum_{k=1}^{r-|I|} (-1)^{\frac{1}{2}|I|(|I|+1)+r|I|+\Sigma I} \chi_{n+|I|-1}(\partial x_J) \mathbb{P}.
\end{align*} \]

On the other hand, the definition of the boundary map of the total chain complex of the Čech complex \( \check{C}_\ast(X \times \square^r; X \times \partial \square^r) \) together with Prop.3.7 implies that

\[ \chi_{n+|I|-1}(\partial x_J) \mathbb{P} = (-1)^{|J|} \chi_{n+|I|-1}(\partial x_J) \mathbb{P} + (-1)^{|J|} \sum_{k=1}^{r-|I|} (-1)^{k-1} \chi_{n+|I|-1}(x_{J-j_k} | X \times D_J) \mathbb{P}. \]

Hence \((-1)^r d_s \chi_n(x) = \chi_{n-1}(\partial x)\), which completes the proof. \( \square \)

Let \( f : X \to Y \) be a morphism of proper complex algebraic manifolds. It follows from Prop.3.7 that the diagram

\[ \begin{array}{ccc}
\check{C}_\ast(Y \times \square^r; Y \times \partial \square^r)[r] & \xrightarrow{\chi} & \bigoplus_p \mathbb{H}^{2p-r}_{\mathcal{D}, Y}(Y, p) \\
\downarrow f^* & & \downarrow f^* \\
\check{C}_\ast(X \times \square^r; X \times \partial \square^r)[r] & \xrightarrow{\chi} & \bigoplus_p \mathbb{H}^{2p-r}_{\mathcal{D}, X}(X, p)
\end{array} \]
is commutitive. Hence the diagram
\[ K_n(Y × □^r; Y × ∂□^r)_Q \xrightarrow{\text{ch}_n} \bigoplus_p H^{2p-n-r}_D(Y, R(p)) \]
\[ f^* \downarrow \]
\[ K_n(X × □^r; X × ∂□^r)_Q \xrightarrow{\text{ch}_n} \bigoplus_p H^{2p-n-r}_D(X, R(p)) \]
is also commutative.

The proposition below can be easily verified.

**Proposition 3.9.** The diagram
\[ \widetilde{Q}Ch^*(X) \xrightarrow{i_X} \widetilde{Q}Ch^*(X × □^r; X × ∂□^r)[r] \]
\[ \oplus \mathcal{D}^{2p-*}_F(X,p) \xrightarrow{\text{ch}_*} \oplus \mathcal{D}^{2p-*}_R(X,p) \]
is commutative. In particular, the diagram
\[ K_{n+r}(X)_Q \xrightarrow{\text{ch}_{n+r}} K_n(X × □^r; X × ∂□^r)_Q \]
\[ \text{ch}_n \downarrow \]
\[ H^{2p-n-r}_D(X, R(p)) \]
is commutative.

4. **Chern form of a hermitian vector bundle on an iterated double**

4.1. **Hermitian vector bundles on an iterated double.** In this subsection we will introduce a scheme called *iterated double* and construct a theory of Chern forms of hermitian vector bundles on it. Let \( X \) be a scheme and \( Y_1, \ldots, Y_r \) closed subschemes of \( X \). Denote by \( D(X; Y_1, \ldots, Y_r) \) the iterated double defined by Levine in [Le]. As a topological space, it is a union of \( 2^r \) copies of \( X \) indexed by the set of all subsets of \( \{1, \ldots, r\} \). To be more precise, if we denote by \( X_I \) the closed subscheme corresponding to \( I \subset \{1, \ldots, r\} \), then
\[ D(X; Y_1, \ldots, Y_r) = \bigcup_{I \subset \{1, \ldots, r\}} X_I, \]
and for any \( j \notin I \), \( X_I \) is glued with \( X_{I∪\{j\}} \) along \( Y_j \) transversally.

**Definition 4.1.** Suppose that \( X \) is a complex algebraic manifold and that \( Y_1, \ldots, Y_r \) are closed submanifolds of \( X \). Let \( F \) be a vector bundle on the iterated double \( D(X; Y_1, \ldots, Y_r) \). A smooth hermitian metric \( h = (h_I) \) on \( F \) is a family of smooth hermitian metrics \( h_I \) on the restrictions \( F|_{X_I} \) such that \( h_I|_{Y_j} = h_{I∪\{j\}}|_{Y_j} \) for any \( j \notin I \). A smooth hermitian
metric $h$ on $\mathcal{F}$ is said to be smooth at infinity if each $h_I$ is a smooth at infinity metric on $\mathcal{F}|_{X_I}$.

**Proposition 4.2.** If $D(X; Y_1, \ldots, Y_r)$ is quasi-projective, then any vector bundle $\mathcal{F}$ on $D(X; Y_1, \ldots, Y_r)$ admits a smooth hermitian metric which is smooth at infinity.

**Proof:** As shown in [FM] §3.2, for any vector bundle $\mathcal{F}$ there is a morphism $\varphi : D(X; Y_1, \ldots, Y_r) \to G$ to a projective manifold $G$ and a vector bundle $\mathcal{G}$ on $G$ such that $\varphi^* \mathcal{G} \simeq \mathcal{F}$. If we choose a smooth hermitian metric $h$ on $\mathcal{G}$, then the pull-back metric $\varphi^* h$ on $\mathcal{F}$ is smooth at infinity. □

Let $X$ be a complex algebraic manifold and assume $X$ to be projective. Consider the normal crossing divisor

$$X \times \partial \mathbb{D}^r = X \times D_1 + \cdots + X \times D_r \subset X \times \mathbb{D}^r$$

introduced in §3.6. Let $T = D(X \times \mathbb{D}^r; X \times \partial \mathbb{D}^r)$ be the associated iterated double. Then $T$ is quasi-projective, because it is isomorphic to $X \times D(\mathbb{D}^r; \partial \mathbb{D}^r)$ and $D(\mathbb{D}^r; \partial \mathbb{D}^r)$ is affine and of finite type over $\mathbb{C}$. Hence it follows from Prop.4.2 that any vector bundle on $T$ admits a smooth at infinity metric.

Let $(X \times \mathbb{D}^r)_I \subset T$ be the irreducible component corresponding to $I \subset \{1, \ldots, r\}$, and $i_I : X \times \mathbb{D}^r \to T$ the embedding onto $(X \times \mathbb{D}^r)_I$. Let $\mathcal{F}$ be a hermitian vector bundle on $T$ with a smooth at infinity metric. Consider the alternating sum of the Chern form of $i_I^* \mathcal{F}$:

$$\text{ch}_{T,0}(\mathcal{F}) = \sum_I (-1)^{|I|} \text{ch}_0 (i_I^* \mathcal{F}) \in \bigoplus_p \mathbb{T}^{2p-r}(X, p).$$

Then the isometry $(\delta^I_j)^* i_I^* \mathcal{F} \simeq (\delta^I_j)^* i_{I \cup \{j\}}^* \mathcal{F}$ for any $j \notin I$ and for $i = 0$ or $\infty$ implies that $\delta_A \text{ch}_{T,0}(\mathcal{F}) = 0$. Hence $d_s \text{ch}_{T,0}(\mathcal{F}) = 0$. The form $\text{ch}_{T,0}(\mathcal{F})$ is called the Chern form of $\mathcal{F}$.

Let

$$\mathcal{E} : 0 \to \mathcal{F}_{-1} \to \mathcal{F}_0 \to \mathcal{F}_1 \to 0$$

be a short exact sequence of hermitian vector bundles on $T$ with smooth at infinity metrics. Then as shown in [BW] Def.3.8, the metric on the transgression bundle $\text{tr}_1 (\lambda i_I^* \mathcal{E})$ is smooth at infinity, therefore we have

$$\text{ch}_1 (i_I^* \mathcal{E})_p = \text{ch}_0 (\text{tr}_1 (\lambda i_I^* \mathcal{E}))_p \in \bigoplus_p \mathbb{T}^{2p-r-1}(X, p).$$

Let

$$\text{ch}_{T,1}(\mathcal{E}) = \sum_I (-1)^{|I|} \text{ch}_1 (i_I^* \mathcal{E})_p \in \bigoplus_p \mathbb{T}^{2p-r-1}(X, p).$$

Then the isometry $(\delta^I_j)^* i_I^* \mathcal{E} \simeq (\delta^I_j)^* i_{I \cup \{j\}}^* \mathcal{E}$ for any $j \notin I$ and for $i = 0$ or $\infty$ implies that $\delta_A \text{ch}_{T,1}(\mathcal{E}) = 0$. Hence

$$d_s \text{ch}_{T,1}(\mathcal{E}) = (-1)^r \delta_\mathcal{F} \text{ch}_{T,1}(\mathcal{E}) = (-1)^r \left( \text{ch}_{T,0}(\mathcal{F}_{-1}) + \text{ch}_{T,0}(\mathcal{F}_1) - \text{ch}_{T,0}(\mathcal{F}_0) \right).$$

The form $\text{ch}_{T,1}(\mathcal{E})$ is called the Bott-Chern form of $\mathcal{E}$.

Summing up the results obtained in this subsection, we have the following:
Theorem 4.3. The Chern form $\mathrm{ch}_{T,0}(\mathcal{F})$ of a hermitian vector bundle $\mathcal{F}$ on $T$ with a smooth at infinity metric is $d_s$-closed. Moreover, for a short exact sequence
\[
\mathcal{E} : 0 \to \mathcal{F}_{-1} \to \mathcal{F}_0 \to \mathcal{F}_1 \to 0
\]
of hermitian vector bundles with smooth at infinity metrics on $T$,
\[
\mathrm{ch}_{T,0}(\mathcal{F}_{-1}) + \mathrm{ch}_{T,0}(\mathcal{F}_1) - \mathrm{ch}_{T,0}(\mathcal{F}_0) = (-1)^r d_s \mathrm{ch}_{T,1}(\mathcal{E}).
\]
In particular, the element of the Deligne cohomology represented by $\mathrm{ch}_{T,0}(\mathcal{F})$ is independent of the choice of metric, and it induces a map of abelian groups
\[
\mathrm{ch}_{T,0} = \bigoplus_p \mathrm{ch}_{p,T,0} : K_0(T) \to \bigoplus_p H_D^{2p-r}(X, \mathbb{R}(p)).
\]

4.2. Relations with the Beilinson’s regulator. Let
\[
T_j = \bigcup_{j \in I} (X \times \square^r)_I \subset T
\]
for $1 \leq j \leq r$, and $\iota_j : T_j \hookrightarrow T$ the closed embedding. Define a morphism $p_j : T \to T_j$ as follows: For any $I \subset \{1, \ldots, r\}$ with $j \notin I$, $p_j|_{(X \times \square^r)_I}$ and $p_j|_{(X \times \square^r)_{I \cup \{j\}}}$ are given by
\[
p_j|_{(X \times \square^r)_I} : (X \times \square^r)_I \xrightarrow{\mathrm{Id}} (X \times \square^r)_{I \cup \{j\}} \subset T_j,
p_j|_{(X \times \square^r)_{I \cup \{j\}}} : (X \times \square^r)_{I \cup \{j\}} \xrightarrow{\mathrm{Id}} (X \times \square^r)_{I \cup \{j\}} \subset T_j.
\]
Then $p_j$ satisfies $p_j \iota_j = \mathrm{Id}_{T_j}$ and $p_j(T_i) = T_i \cap T_j$, therefore we have maps of $K$-groups
\[
K_*(T; T_1, \ldots, T_{j-1}) \xrightarrow{p_j^*} K_*(T_j; T_1 \cap T_j, \ldots, T_{j-1} \cap T_j)
\]
which satisfy $\iota_j^* p_j^* = \mathrm{Id}$. Hence $\iota_j^*$ is split surjective, and the canonical map
\[
K_*(T; T_1, \ldots, T_j) \to K_*(T; T_1, \ldots, T_{j-1})
\]
is split injective, and the canonical map
\[
(4.1) \quad K_*(T; T_1, \ldots, T_r) \hookrightarrow K_*(T)
\]
is splitting injective.

Let us now apply the results obtained in §2 to the maps of $\mathcal{C}$-complexes
\[
\tilde{\mathcal{Q}}\mathcal{C}_*^{\mathrm{Alt}}(T; T_1, \ldots, T_{j-1}) \xrightarrow{p_j^*} \tilde{\mathcal{Q}}\mathcal{C}_*^{\mathrm{Alt}}(T_j; T_1 \cap T_j, \ldots, T_{j-1} \cap T_j).
\]
It follows from Prop.2.20 that there is a homotopy from the identity to $\iota_j^* p_j^*$, and Cor.2.16 says that $\tilde{\mathcal{Q}}\mathcal{C}_*^{\mathrm{Alt}}(T; T_1, \ldots, T_j)$ is isomorphic to the simple complex of $\iota_j^*$. Hence Prop.2.6 says that there is a left inverse map up to homotopy of the canonical map
\[
\tilde{\mathcal{Q}}\mathcal{C}_*^{\mathrm{Alt}}(T; T_1, \ldots, T_j) \to \tilde{\mathcal{Q}}\mathcal{C}_*^{\mathrm{Alt}}(T; T_1, \ldots, T_{j-1}),
\]
which we denote by
\[
t_j : \tilde{\mathcal{Q}}\mathcal{C}_*^{\mathrm{Alt}}(T; T_1, \ldots, T_{j-1}) \to \tilde{\mathcal{Q}}\mathcal{C}_*^{\mathrm{Alt}}(T; T_1, \ldots, T_j).
\]
The description of the map $t_j$ given in Prop.2.6 together with Prop.3.7 implies the following:
Proposition 4.4. Take an element
\[ x = (x_I) \in \bigoplus_{|I|=m} \tilde{\mathbb{Q}} \hat{C}_*^{\text{Alt}}(T_I) \]
and consider its image by the map
\[ t_{m,n}^{j,n} : \bigoplus_{|I|=m} \tilde{\mathbb{Q}} \hat{C}_*^{\text{Alt}}(T_I) \to \bigoplus_{|J|=n} \tilde{\mathbb{Q}} \hat{C}_*^{\text{Alt}}(T_J). \]
If \( m < n \), then \( t_{m,n}^{j,n}(x)_J \) is isometrically equivalent to a degenerate element. On the other hand,
\[ t_{m,m}^{j,m}(x)_J = x_J - p_{j,j}^* x_J. \]

Corollary 4.5. Let
\[ q : \tilde{\mathbb{Q}} \hat{C}_*^{\text{Alt}}(T; T_1, \ldots, T_r) \to \tilde{\mathbb{Q}} \hat{C}_*^{\text{Alt}}(T) \]
be the canonical map, and
\[ t = t_r t_{r-1} \cdots t_1 : \tilde{\mathbb{Q}} \hat{C}_*^{\text{Alt}}(T) \to \tilde{\mathbb{Q}} \hat{C}_*^{\text{Alt}}(T; T_1, \ldots, T_r). \]
Then \( t \) is a left inverse map of the map \( q \) up to homotopy. For \( x \in \tilde{\mathbb{Q}} \hat{C}_*^{\text{Alt}}(T) \), \( t_{0,n}^0(x) \) is isometrically equivalent to a degenerate element if \( 0 < n \), and
\[ t_{0,0}^0(x) = (1 - p_{r,r}^* t_r^*) \cdots (1 - p_{1,1}^* t_1^*) x. \]

Let
\[ t : K_0(T)_Q \to K_0(T; T_1, \ldots, T_r)_Q, \quad q : K_0(T; T_1, \ldots, T_r)_Q \to K_0(T)_Q, \]
be the maps given by \( t \) and \( q \) respectively. Note that \( q \) agrees with the map \((4.1)\) and \( t \) is a left inverse map of \( q \). Cor.4.5 says that
\[ qt = (1 - p_{r,r}^* t_r^*) \cdots (1 - p_{1,1}^* t_1^*): K_0(T)_Q \to K_0(T)_Q. \]

Levine have shown in [Le, Thm.1.10] that the embedding
\[ i_0^* : (X \times \mathbb{R}^r; X \times \partial \mathbb{R}^r) \hookrightarrow (T; T_1, \ldots, T_r) \]
induces an isomorphism of \( K_0 \)-groups:
\[ i_0^* : K_0(T; T_1, \ldots, T_r) \xrightarrow{\sim} K_0(X \times \mathbb{R}^r; X \times \partial \mathbb{R}^r). \]

Proposition 4.6. The diagram
\[
\begin{array}{ccc}
K_0(T)_Q & \xrightarrow{t} & K_0(T; T_1, \ldots, T_r)_Q \\
\text{ch}_{t_0}^p & & \downarrow i_0^* \\
H^p_{\mathbb{D}}(X, \mathbb{R}(p)) & \xleftarrow{\text{ch}_0^p} & K_0(X \times \mathbb{R}^r; X \times \partial \mathbb{R}^r)_Q
\end{array}
\]
is commutative.
Proof: Let $\mathcal{F}$ be a hermitian vector bundle on $T$ with a smooth at infinity metric, and consider

$$i_0^*t(\mathcal{F}) \in \hat{\mathcal{Q}}^\text{Alt}_0(X \times \square^r; X \times \partial \square^r) = \bigoplus_I \hat{\mathcal{Q}}^\text{Alt}_I(X \times D_I).$$

Cor. 4.5 implies that $i_0^*t(\mathcal{F})_I$ is isometrically equivalent to a degenerate element if $I \neq \emptyset$, and that

$$i_0^*t(\mathcal{F})_\emptyset = i_0^*\left(1 - p_r^*\imath_r^*\right) \cdots \left(1 - p_1^*\imath_1^*\right) \mathcal{F} \in \hat{\mathcal{Q}}^\text{Alt}_0(X \times \square^r).$$

The commutative diagram of schemes

(4.2)

implies that $i_0^*(1 - p_r^*\imath_r^*)\mathcal{F}$ is isometric to $(i_0^* - i_I^*\imath_{I\cup\{j\}}^*)\mathcal{F}$ as virtual hermitian vector bundles. Hence $i_0^*(\mathcal{F})_\emptyset$ is isometric to $\sum_I (-1)^{|I|} i_I^*\mathcal{F}$ and

$$\text{ch}_p^0(i_0^*t(\mathcal{F})) = \text{ch}_p^0(i_0^*(\mathcal{F})_\emptyset) = \sum_I (-1)^{|I|} \text{ch}_p^0(i_I^*\mathcal{F}) = \text{ch}_p^0(T; T_1, \ldots, T_r),$$

which completes the proof. \hfill \square

Prop. 3.9 and Prop. 4.6 imply the following:

**Corollary 4.7.** The diagram

$$K_r(X)_Q \xrightarrow{i_0^*} K_0(X \times \square^r; X \times \partial \square^r)_Q \xrightarrow{i_0^*} K_0(T; T_1, \ldots, T_r)_Q \xrightarrow{i_0^*} K_0(T)_Q$$

is commutative.

**Proposition 4.8.** The diagram

$$K_0(T)_Q \xrightarrow{t} K_0(T; T_1, \ldots, T_r)_Q \xrightarrow{q} K_0(T)_Q$$

is commutative.

Proof: Let $\mathcal{F}$ be a hermitian vector bundle on $T$ with a smooth at infinity metric. Then

$$i_I^*t(\mathcal{F}) = i_I^*(1 - p_r^*\imath_r^*) \cdots (1 - p_1^*\imath_1^*)\mathcal{F}.$$
The commutative diagram (1.2) implies that \( i_I^* qt(F) \) is isometric to zero as virtual hermitian vector bundle if \( I \neq \emptyset \), and that \( i_0^* qt(F) \) is isomorphic to \( \sum_I (-1)^{|I|} i_I^* F \). Hence

\[
\text{ch}^p_{\mathcal{C},0}(qt(F)) = \sum_I (-1)^{|I|} \text{ch}^p_0(i_I^* qt(F)) = \sum_I (-1)^{|I|} \text{ch}^p_0(i_I^* F) = \text{ch}^p_{\mathcal{C},0}(F),
\]

which completes the proof. \( \square \)

5. Several arithmetic \( K \)-groups

5.1. Multi-relative arithmetic \( K \)-theory. We begin by introducing some terminology and notations which are used in Arakelov geometry. An arithmetic ring is a triple \( (A, \Sigma, F_\infty) \), where \( A \) is a Noetherian integral domain, \( \Sigma \) is a finite set of embeddings \( A \xrightarrow{i} \mathbb{C} \), and

\[
F_\infty : \bigoplus_{i \in \Sigma} \mathbb{C} = \mathbb{C}^\Sigma \to \mathbb{C}^\Sigma
\]

is a conjugate-linear involution whose restriction to \( A \) is the identity. Here we see \( A \) as a subalgebra of \( \mathbb{C}^\Sigma \) by \( a \mapsto (i(a))_{i \in \Sigma} \). By an arithmetic variety, we mean a separated regular scheme which is flat and of finite type over an arithmetic ring. For an arithmetic variety \( X \) defined over an arithmetic ring \( A \), denote by \( X(\mathbb{C}) \) the complex algebraic manifold associated with the smooth algebraic variety \( X \otimes_A \mathbb{C}^\Sigma \) over \( \mathbb{C} \), and by \( F_\infty : X(\mathbb{C}) \to X(\mathbb{C}) \) the anti-holomorphic involution induced by \( F_\infty : \mathbb{C}^\Sigma \to \mathbb{C}^\Sigma \). A hermitian vector bundle \( F = (F,h) \) on \( X \) is a vector bundle \( F \) equipped with an \( F_\infty \)-invariant smooth hermitian metric \( h \) on \( F(\mathbb{C}) \).

For a differential form \( \eta \) on \( X(\mathbb{C}) \), the involution on the space \( E^n(X(\mathbb{C})) \) given by \( \eta \mapsto \overline{F_\infty(\eta)} = F_\infty^* \eta \) respects the bigrading

\[
E^n(X(\mathbb{C})) = \bigoplus_{p+q=n} E^{p,q}(X(\mathbb{C})).
\]

Hence it induces involutions on the complexes of differential forms and currents introduced in §3. Set

\[
\tau \mathcal{D}^s(X,p) = \tau \mathcal{D}^s(X(\mathbb{C}),p) F_\infty = \text{Id},
\]

\[
\tau \mathcal{D}^D(X,p) = \tau \mathcal{D}^D(X(\mathbb{C}),p) F_\infty = \text{Id},
\]

\[
\mathcal{D}_{\mathcal{A}}^s(X,p) = \mathcal{D}_{\mathcal{A}}^s(X(\mathbb{C}),p) F_\infty = \text{Id},
\]

\[
\mathcal{D}_{\mathcal{B}}^s(X,p) = \mathcal{D}_{\mathcal{B}}^s(X(\mathbb{C}),p) F_\infty = \text{Id},
\]

\[
\mathcal{D}_{A,p}^s(X,p) = \mathcal{D}_{A,p}^s(X(\mathbb{C}),p) F_\infty = \text{Id},
\]
and
\[
\tau D_\ast(X) = \bigoplus_p \tau D^{2p-*}(X,p),
\]
\[
\tau D_{D,\ast}(X) = \bigoplus_p \tau D^{2p-*}_{D}(X,p),
\]
\[
\tilde{D}_{A,\ast}(X) = \bigoplus_p \tilde{D}^{2p-*}_{A}(X,p),
\]
\[
\tilde{D}_{\ast}(X) = \bigoplus_p \tilde{D}^{2p-*}_{\ast}(X,p),
\]
\[
\tilde{D}_{A,\ast}(X) = \bigoplus_p \tilde{D}^{2p-*}_{A,\ast}(X,p).
\]

Note that for an exact hermitian \( n \)-cube \( \mathcal{F} \) on \( X \), we have
\[
\text{ch}_n(\mathcal{F}) \in \tilde{D}_{F,n}(X),
\]
\[
\text{ch}_n(\mathcal{F}) = \kappa_{\mathbb{F}}(\text{ch}_n(\mathcal{F})_{\mathbb{F}}) \in \tau D_n(X).
\]

We now recall the arithmetic \( K_0 \)-group of \( X \) [GS2]. Suppose \( X \) is proper over an arithmetic ring. Define the \textit{arithmetic} \( K_0 \)-group \( \tilde{K}_0(X) \) of \( X \) to be the abelian group generated by pairs \((\mathcal{F}, \tilde{\eta})\) of a hermitian vector bundle \( \mathcal{F} \) on \( X \) with \( \tilde{\eta} \in \tau D_1(X)/\text{Im} \ d_2 \), subject to the relation
\[
(\mathcal{F}_{-1}, \tilde{\eta}_{-1}) + (\mathcal{F}_1, \tilde{\eta}_1) = (\mathcal{F}_0, \tilde{\eta}_{-1} + \tilde{\eta}_1 + \text{ch}_1(\mathcal{F}))
\]
for any short exact sequence \( \mathcal{E} : 0 \to \mathcal{F}_{-1} \to \mathcal{F}_0 \to \mathcal{F}_1 \to 0 \) and for any \( \tilde{\eta}_{-1}, \tilde{\eta}_1 \in \tau D_1(X)/\text{Im} \ d_2 \). There is a map
\[
\text{ch}_0 : \tilde{K}_0(X) \to \tau D_0(X)
\]
which sends a pair \((\mathcal{F}, \tilde{\eta})\) to \( \text{ch}_0(\mathcal{F}) + d_2 \eta \).

We next recall the definition of the higher arithmetic \( K \)-groups. The original definition given in [Ta] requires homotopy theory. But in this paper we will employ a simpler definition, because we only need \( K \)-groups with rational coefficient. For \( r \geq 1 \), define the \( r \)-th higher \textit{rational} arithmetic \( K \)-group of \( X \), which we denote by \( \tilde{K}_r(X)_{\mathbb{Q}} \), to be the homology group of the simple complex of the higher Bott-Chern form, namely,
\[
\tilde{K}_r(X)_{\mathbb{Q}} = H_r \left( s \left( \text{ch}_* : \hat{Q}C_*^\ast(X) \to \tau D_*^\ast(X) \right) \right).
\]
Then there is a long exact sequence
\[
\cdots \to \bigoplus_p H^{2p-*}_D(X, \mathbb{R}(p)) \to \tilde{K}_r(X)_{\mathbb{Q}} \to K_r(X)_{\mathbb{Q}} \xrightarrow{\text{ch}_r} \bigoplus_p H^{2p-*}_D(X, \mathbb{R}(p)) \to \cdots
\]
\[
\cdots \to K_1(X)_{\mathbb{Q}} \to \tau D_1(X)/\text{Im} \ d_2 \to \tilde{K}_0(X)_{\mathbb{Q}} \to K_0(X)_{\mathbb{Q}} \to 0.
\]
For \( r \geq 1 \), set
\[
\tilde{K}_{F,r}(X)_{\mathbb{Q}} = H_r \left( s \left( \text{ch}_* : \hat{Q}C_*^\ast(X) \to \tilde{D}_*^\ast(X) \right) \right).
\]
Since \( \kappa_{\mathbb{F}} : \tilde{D}_{F,\ast}(X) \to \tau D_\ast(X) \) is a quasi-isomorphism and \( \text{ch}_* = \kappa_{\mathbb{F}} \text{ch}_{*,\mathbb{F}} \), there is a natural isomorphism
\[
(5.1) \quad \tilde{K}_{F,r}(X)_{\mathbb{Q}} \to \tilde{K}_r(X)_{\mathbb{Q}}.
\]
Let us now define multi-relative arithmetic $K$-theory. Let $X$ be a proper arithmetic variety and suppose $r \geq 1$. Taking the $F_\infty$-invariant part of the map $\text{ch}_s$ defined in Prop. 3.8 yields the map of complexes

$$\text{ch}_s : \mathbb{Q}[\tilde{C}_s(X \times \Box^r; X \times \partial\Box^r)][r] \to \mathbb{D}_{A,P,*}(X).$$

For $n \geq 0$, define multi-relative rational arithmetic $K$-theory of $(X \times \Box^r; X \times \partial\Box^r)$ to be the homology group of the simple complex of this map:

$$\hat{K}_n(X \times \Box^r; X \times \partial\Box^r)_\mathbb{Q} = H_{n+r}(s(\text{ch}_s)).$$

**Proposition 5.1.** There is a canonical isomorphism

$$\hat{K}_{P,n+r}(X)_\mathbb{Q} \simeq \hat{K}_n(X \times \Box^r; X \times \partial\Box^r)_\mathbb{Q}.$$

**Proof:** The commutative diagram of chain complexes in Prop. 3.9 yields the map $s(\text{ch}_s) \to \text{ch}_s$, which leads to the map of arithmetic $K$-groups

$$\hat{K}_{P,n+r}(X)_\mathbb{Q} \to \hat{K}_n(X \times \Box^r; X \times \partial\Box^r)_\mathbb{Q}$$

which fits into the commutative diagram

$$\cdots \longrightarrow K_{n+r+1}(X)_\mathbb{Q} \longrightarrow \oplus P H_{2p-n-r-1}^p(X, \mathbb{R}(p)) \underline{\downarrow \text{id}} \longrightarrow \cdots$$

$$\cdots \longrightarrow K_{n+1}(X \times \Box^r; X \times \partial\Box^r)_\mathbb{Q} \longrightarrow \oplus P H_{2p-n-r-1}^p(X, \mathbb{R}(p))$$

$$\longrightarrow \hat{K}_{P,n+r}(X)_\mathbb{Q} \longrightarrow \hat{K}_n(X \times \Box^r; X \times \partial\Box^r)_\mathbb{Q} \longrightarrow \longrightarrow \cdots$$

The proposition follows from this diagram and the homotopy invariant property of $K$-theory. \qed

5.2. **Arithmetic $K$-group of an iterated double.** In this subsection we will define arithmetic $K_0$-group of an iterated double $T = D(X \times \Box^r; X \times \partial\Box^r)$ when $X$ is a projective arithmetic variety over an arithmetic ring.

**Definition 5.2.** Define arithmetic $K_0$-group $\hat{K}_0^M(T)$ of the iterated double $T$ to be the abelian group generated by pairs $(\mathcal{F}, \tilde{\eta})$, where $\mathcal{F}$ is a vector bundle on $T$ with a smooth at infinity metric and $\tilde{\eta} \in \mathbb{D}_{A,P,r+1}(X)/\text{Im} d_s$, subject to the relation

$$(\mathcal{F}_{-1}, \tilde{\eta}_{-1}) + (\mathcal{F}_1, \tilde{\eta}_1) - (\mathcal{F}_0, \tilde{\eta}_0 + \tilde{\eta}_1 + (-1)^r \text{ch}_{P,1}(\tilde{\mathcal{E}}))$$

for any short exact sequence $\mathcal{E} : 0 \to \mathcal{F}_{-1} \to \mathcal{F}_0 \to \mathcal{F}_1 \to 0$ and for any $\tilde{\eta}_{-1}, \tilde{\eta}_1 \in \mathbb{D}_{A,P,r+1}(X)/\text{Im} d_s$. 

There is a surjective map
\[ \zeta : \hat{K}_0^M(T) \to K_0(T) \]
which sends \([\mathcal{F}, \tilde{\eta}]\) to \([\mathcal{F}]\). It is easy to see that it satisfies the exact sequence
\[ \tilde{D}_{\mathcal{A}, \mathcal{F}, r+1}(X)/\text{Im} \, d_s \to \hat{K}_0^M(T) \xrightarrow{\zeta} K_0(T) \to 0. \]

**Definition 5.3.** By Thm.4.3 we can define a map
\[ \text{ch}_{T, 0} : \hat{K}_0^M(T) \to \tilde{D}_{\mathcal{A}, \mathcal{F}, r}(X) \]
which sends \((\mathcal{F}, \tilde{\eta})\) to \(\text{ch}_{T, 0}(\mathcal{F}) + d_s \eta\). Let
\[ \hat{K}_0(T) = \text{Ker} \left( \text{ch}_{T, 0} : \hat{K}_0^M(T) \to \tilde{D}_{\mathcal{A}, \mathcal{F}, r}(X) \right). \]
Moreover, define \(\hat{K}_0^M(T; T_1, \ldots, T_r)\) to be the cartesian product of the diagram
\[ \hat{K}_0^M(T) \xrightarrow{\zeta} K_0(T; T_1, \ldots, T_r) \]
\[ \quad \downarrow \quad \downarrow \]
\[ \quad \hat{K}_0(T) \]

In other words, \(\hat{K}_0^M(T; T_1, \ldots, T_r)\) is a subgroup of \(\hat{K}_0^M(T)\) given as follows:
\[ \hat{K}_0^M(T; T_1, \ldots, T_r) = \left\{ x \in \hat{K}_0^M(T); \zeta(x) \in \text{Im}(K_0(T; T_1, \ldots, T_r) \to K_0(T)) \right\}. \]

Finally, let
\[ \hat{K}_0(T; T_1, \ldots, T_r) = \hat{K}_0^M(T; T_1, \ldots, T_r) \cap \hat{K}_0(T) \subset \hat{K}_0^M(T). \]

Let us recall the maps
\[ t : \tilde{Q}C_*^{\text{Alt}}(T) \to \tilde{Q}C_*^{\text{Alt}}(T; T_1, \ldots, T_r), \]
\[ q : \tilde{Q}C_*^{\text{Alt}}(T; T_1, \ldots, T_r) \to \tilde{Q}C_*^{\text{Alt}}(T) \]
given in Cor.4.5. In the same way as in the proof of Prop.4.8, we can show that \(\text{ch}_{T, 1}(qt(\mathcal{E})) = \text{ch}_{T, 1}(\mathcal{E})\) for any short exact sequence \(\mathcal{E}\) of hermitian vector bundles on \(T\). Hence we obtain a map
\[ \tilde{q}t : \hat{K}_0^M(T) \to \hat{R}_0^M(T) \]
which sends \([\mathcal{F}, \tilde{\eta}]\) to \([(qt(\mathcal{F}), \tilde{\eta})]\). It is obvious that the image of this map is contained in \(\hat{K}_0^M(T; T_1, \ldots, T_r)\), hence it induces
\[ (5.2) \quad \hat{t} : \hat{K}_0^M(T) \to \hat{K}_0^M(T; T_1, \ldots, T_r), \]
which turns out to be a splitting map of the inclusion \( \hat{K}_0^M(T; T_1, \ldots, T_r) \subset \hat{K}_0^M(T) \). Prop. 4.8 implies that 

\[
\begin{array}{ccc}
\hat{K}_0^M(T) & \xrightarrow{\hat{q}} & \hat{K}_0^M(T) \\
\downarrow\text{ch}_{T,0} & & \downarrow\text{ch}_{T,0} \\
\hat{D}_{A,\mathbb{P}, r}(X) & & \\
\end{array}
\]

is commutative. Hence the map (5.2) induces

\[ \hat{i}: \hat{K}_0(T) \rightarrow \hat{K}_0(T; T_1, \ldots, T_r), \]

which is also a splitting map of the inclusion \( \hat{K}_0(T; T_1, \ldots, T_r) \subset \hat{K}_0(T) \).

**Proposition 5.4.** The embedding

\[ i_0 : (X \times \square^r; X \times \partial \square^r) \hookrightarrow (T; T_1, \ldots, T_r) \]

induces a surjection

\[ \hat{i}_0 : \hat{K}_0(T; T_1, \ldots, T_r)_\mathbb{Q} \rightarrow \hat{K}_0(X \times \square^r; X \times \partial \square^r)_\mathbb{Q} \]

which fits into the commutative diagram up to sign:

\[
\begin{array}{ccc}
\oplus H_{2p-r-1}^D(X, \mathbb{R}(p)) & \rightarrow & \hat{K}_0(T; T_1, \ldots, T_r)_\mathbb{Q} \\
\downarrow\text{Id} & & \downarrow\hat{i}_0 \\
\oplus H_{2p-r-1}^D(X, \mathbb{R}(p)) & \rightarrow & \hat{K}_0(X \times \square^r; X \times \partial \square^r)_\mathbb{Q} \\
\end{array}
\]

\[
\begin{array}{ccc}
\zeta & \rightarrow & K_0(T; T_1, \ldots, T_r)_\mathbb{Q} \\
\downarrow\zeta & & \downarrow\hat{i}_0 \\
K_0(X \times \square^r; X \times \partial \square^r)_\mathbb{Q} & \rightarrow & \oplus H_{2p-r}^D(X, \mathbb{R}(p)) \\
\downarrow\text{Id} & & \downarrow\text{Id} \\
\end{array}
\]

**Proof:** For a short exact sequence of hermitian vector bundles \( \overline{\mathcal{E}} : 0 \rightarrow \overline{\mathcal{F}}_{-1} \rightarrow \overline{\mathcal{F}}_0 \rightarrow \overline{\mathcal{F}}_1 \rightarrow 0 \) on \( T \), consider the element

\[ (i_0^* t(\overline{\mathcal{E}}), 0) \in \tilde{C}_{\mathbb{Q}}^{1, \text{Alh}}(X \times \square^r; X \times \partial \square^r) \oplus \tilde{D}_{A,\mathbb{P}, r+2}(X) = s(ch_*)_{r+1}. \]

Then we can show in the same way as in the proof of Prop. 4.6 that \( \text{ch}_1(i_0^* t(\overline{\mathcal{E}})) = \text{ch}_{T,1}(\overline{\mathcal{E}}) \).

Hence

\[ (-1)^r \partial(i_0^* t(\overline{\mathcal{E}}), 0) = (-1)^r \text{ch}_1(i_0^* t(\overline{\mathcal{E}})) \]

\[ = (i_0^* t(\overline{\mathcal{F}}_{-1}) + i_0^* t(\overline{\mathcal{F}}_1) - i_0^* t(\overline{\mathcal{F}}_0), (-1)^r \text{ch}_{T,1}(\overline{\mathcal{E}})). \]

Therefore

\[ (\overline{\mathcal{F}}, \eta) \mapsto (i_0^* t(\overline{\mathcal{F}}), -\eta) \in \tilde{C}_0(X \times \square^r; X \times \partial \square^r) \oplus \tilde{D}_{A,\mathbb{P}, r+1}(X) = s(ch_*)_0 \]
gives rise to the map
\[(5.3) \quad \hat{i}_0^* t : \hat{K}_0^M(T) \to s(ch_*)_r / \text{Im} \partial.\]
The equalities
\[\partial(i_0^* t(F), -\eta) = (0, ch_0(i_0^* t(F)) + d_\eta) = (0, ch_T, 0)(F) + d_\eta)\]
imply that the map \((5.3)\) induces
\[(5.4) \quad \hat{i}_0^* t : \hat{K}_0^T \to \hat{K}_0^r(X \times \square^r; X \times \partial \square^r)_Q.\]
Define the map \(\hat{i}_0^*\) to be the composite
\[\hat{i}_0^* : \hat{K}_0^T \to \hat{K}_0^r(X \times \square^r; X \times \partial \square^r)_Q \subset \hat{K}_0^0(T) \to \hat{K}_0^0(X \times \square^r; X \times \partial \square^r)_Q.\]

The commutativity of the diagram is easily verified, and the surjectivity of \(\hat{i}_0^*\) follows from the commutative diagram and the bijectivity of \(i_0^*\) on \(K\)-groups.

Summing up the results in this section, we obtain the following sequence of maps of arithmetic \(K\)-groups:
\[\hat{K}_r(X)_Q \simeq \hat{K}_{r, r}(X)_Q \simeq \hat{K}_0(X \times \square^r; X \times \partial \square^r)_Q \overset{\hat{i}_0^*}{\longrightarrow} \hat{K}_0(T; T_1, \ldots, T_r)_Q \overset{\hat{t}}{\longrightarrow} \hat{K}_0(T)_Q.\]

6. Arithmetic Chern character of a hermitian vector bundle on an iterated double

6.1. Simple complex of a diagram of complexes. In this subsection we will introduce simple complex associated with a diagram of complexes [BF, §1]. Let
\[\mathcal{D} = \begin{pmatrix} B^1_+ & B^2_+ \\ A^1_+ & A^2_+ \end{pmatrix} \]
be a diagram of chain complexes. Consider the map
\[\varphi : A^1_+ \oplus A^2_+ \to B^1_+ \oplus B^2_+\]
defined by \(\varphi(a_1, a_2) = (f_1(a_1) - g_1(a_2), f_2(a_2)).\) Define the simple complex associated with \(\mathcal{D}\), which we denote by \(s(\mathcal{D})_n\), to be the simple complex of \(\varphi\). To be more precise,
\[s(\mathcal{D})_n = A^1_n \oplus A^2_n \oplus B^1_{n+1} \oplus B^2_{n+1}\]
and if we write an element of \(s(\mathcal{D})_n\) in the way that
\[
\begin{pmatrix} b_1 & b_2 \\ a_1 & a_2 \end{pmatrix},
\]
then the boundary map is given by
\[
\partial \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} f_1(a_1) - g_1(a_2) - \partial b_1 & f_2(a_2) - \partial b_2 \\ \partial a_1 & \partial a_2 \end{pmatrix}.
\]

**Proposition 6.1.** [BF, Cor.1.16] If \( g_1 \) is a quasi-isomorphism in the above diagram \( \mathcal{D}_* \), then there is a long exact sequence
\[
\cdots \rightarrow H_n(s(\mathcal{D})_*) \rightarrow H_n(A_1^*) \rightarrow H_{n-1}(B_2^*) \rightarrow H_{n-1}(s(\mathcal{D})_*) \rightarrow \cdots.
\]

### 6.2. Cubical higher Chow groups and higher arithmetic Chow groups.

In [Go] Goncharov constructed a map from the simplicial cycle complex to the complex of currents \( \tau\mathcal{D}^{p}(X,p) \), and defined higher arithmetic Chow groups as the homology groups of the simple complex of this map. Afterward, Burgos, Feliu and the author constructed in [BFT] a similar map by using the cubical cycle complex. In this subsection we will recall their construction.

First we recall the cubical cycle complex. Let \( X \) be an equidimensional variety defined over a field. A closed subscheme of \( X \times \square^r \) is said to be admissible if it intersects properly with \( X \times D_I \) for any \( I \subset \{1, \ldots, r\} \). Denote by \( \mathbb{Z}^p(X, r) \) the \( \mathbb{Q} \)-vector space generated by all admissible and integral subschemes of \( X \times \square^r \) of codimension \( p \). Then the cocubical scheme structure on \( (\square^*)^r \) defined in §3.1 induces the maps
\[
(\delta^0_j)^*, (\delta^\infty_j)^*: \mathbb{Z}^p(X, r) \rightarrow \mathbb{Z}^p(X, r-1),
\]
\[
\sigma^*_j: \mathbb{Z}^p(X, r-1) \rightarrow \mathbb{Z}^p(X, r)
\]
for \( 1 \leq j \leq r \), by which \( (\mathbb{Z}^p(X, *), (\delta^0_j)^*, (\delta^\infty_j)^*, \sigma^*_j) \) is a cubical abelian group. Define the higher Chow groups of \( X \) to be the homology groups of its normalized subcomplex:
\[
CH^p(X, r) = H_r(\mathbb{Z}^p(X, *))_0.
\]

Suppose \( X \) is a compact complex algebraic manifold. For an admissible and integral subscheme \( V \subset X \times \square^r \) of codimension \( p \), let
\[
\mathcal{P}^p(V) = \pi_X^*(\delta^*_V \cap [\pi^*_V W_r]) \in \tau\mathcal{D}_{D}^{2p-r}(X,p),
\]
where \( \overline{V} \) is the closure of \( V \) in \( X \times (\mathbb{P}^1)^r \). In other words, \( \mathcal{P}^p(V) \) is the current on \( X \) defined by the integral
\[
\mathcal{P}^p(V)(\omega) = \frac{1}{(2\pi i)^{2d_X+2r-p}} \int_{\overline{V}} \iota^* \pi_X^* \omega \wedge \iota^* \pi^*_V W_r,
\]
where \( \iota: \overline{V} \rightarrow \overline{V} \) is a desingularization.

**Theorem 6.2.** [BFT, Thm.7.4, Thm.7.8] The above integral is convergent. The map
\[
\mathcal{P}^p: \mathbb{Z}^p(X, *)_0 \rightarrow \tau\mathcal{D}^{2p-\ast}_D(X, p)
\]
given by this integral is a map of complexes, and the induced map on homology
\[
H_r(\mathcal{P}^p): CH^p(X, r) \rightarrow H^{2p-r}(X, \mathbb{R}(p))
\]
agrees with the regulator map of \( X \).
Let $X$ be a smooth proper variety defined over an arithmetic field. Let $\tau \mathcal{D}^*_D(X,p)$ be the complex given by

$$
\tau \mathcal{D}^n_D(X,p) = \begin{cases} 
\tau \mathcal{D}^0_D(X,p), & n < 2p, \\
\tau \mathcal{D}^{2p}_D(X,p) / \tau \mathcal{D}^{2p}_D(X,p), & n = 2p.
\end{cases}
$$

Then taking the $\mathcal{F}_\infty$-invariant part of $\mathcal{P}^p$ yields a map of complexes

$$
(6.1) \quad \mathcal{P}^p : Z^p(X,*_0) \to \tau \mathcal{D}^{2p-*}(X,p).
$$

**Definition 6.3.** Let $\hat{Z}^p_D(X,*_0)$ be the simple complex of the map (6.1), and define the higher arithmetic Chow groups of $X$ to be the homology groups of $\hat{Z}^p_D(X,*_0)$:

$$
\widehat{CH}^p_D(X,r) = H_r(\hat{Z}^p_D(X,*_0)).
$$

It is obvious that $\widehat{CH}^p_D(X,0)$ agrees with the arithmetic Chow groups defined by Gillet and Soulé in [GS1]. The natural map

$$
\tau \mathcal{D}^{2p-r}-1(X,p) \subset \tau \mathcal{D}^{2p-r-1}(X,p) \to \hat{Z}^p_D(X,r)_0
$$

gives

$$
a : H^{2p-r-1}_D(X,\mathbb{R}(p)) \to \widehat{CH}^p_D(X,r)
$$

if $r \geq 1$, and

$$
a : \tau \mathcal{D}^{2p-1}(X,p) / \text{Im } d_D \to \widehat{CH}^p_D(X,0)
$$

when $r = 0$. Concerning these maps the following long exact sequence holds:

$$
\cdots \to H^{2p-r-1}_{D}(X,\mathbb{R}(p)) \to \widehat{CH}^p_D(X,r) \to CH^p(X,r) \to H^{2p-r}_{D}(X,\mathbb{R}(p)) \to \cdots
$$

$$
\cdots \to CH^p(X,1) \to \tau \mathcal{D}^{2p-1}(X,p) / \text{Im } d_D \to \widehat{CH}^p_D(X,0) \to CH^p(X,0) \to 0.
$$

**6.3. Another definition of higher arithmetic Chow groups.** In [GS1], Gillet and Soulé defined intersection product in the arithmetic Chow groups. It is quite natural to seek for a similar product structure in their higher analogues. However, since the definition of $\widehat{CH}^p_D(X,r)$ involves the space of currents, it seems impossible to put a product structure on $\widehat{CH}^p_D(X,r)$. Burgos and Feliu gave in [BF] another definition of higher arithmetic Chow groups in which one can define intersection product. In this subsection we will recall their definition. To do this we first introduce several complexes of differential forms.

For an equidimensional variety $X$ defined over a field, denote by $Z^p_{X,r}$ the set of all admissible subschemes of $X \times \square^r$ of codimension $p$. We abbreviate $Z^p_{X,r}$ to $Z^p_r$ if no confusion occurs. For a complex algebraic manifold $X$, set

$$
\mathcal{D}^*_\log(X \times \square^r - Z^p_r, p) = \lim_{Z \in Z^p_r} \mathcal{D}^*_\log(X \times \square^r - Z, p),
$$

and take the simple complex

$$
\mathcal{D}^*_\log, Z^p_r(X \times \square^r, p) = s(\mathcal{D}^*_\log(X \times \square^r, p) \to \mathcal{D}^*_\log(X \times \square^r - Z^p_r, p))
and its truncated subcomplex
\[ \tau \mathcal{D}^*_\log,\mathbb{Z}_l^p(X \times \Box^r, p) = \tau_{\leq 2p} \mathcal{D}^*_\log,\mathbb{Z}_l^p(X \times \Box^r, p). \]

Set
\[ \mathcal{H}^p(X, r) = H_{\mathcal{D}^*,\mathbb{Z}_l^p}(X \times \Box^r, \mathbb{R}(p)), \]
then it holds that
\[ \mathcal{H}^p(X, r) \simeq H^{2p}(\tau \mathcal{D}^*_\log,\mathbb{Z}_l^p(X \times \Box^r, p)). \]

Since \( \mathcal{H}^p(X, r) \) and \( \tau \mathcal{D}^*_\log,\mathbb{Z}_l^p(X \times \Box^r, p) \) have cubical structures with respect to the index \( r \), we can obtain the normalized subcomplexes, which we denote by \( \mathcal{H}^p(X, *)_0 \) and \( \tau \mathcal{D}^*_\log,\mathbb{Z}_l^p(X \times \Box^r, p)_0 \) respectively. Moreover, the cycle class map in Deligne cohomology gives a map
\[ \chi_1 : \mathbb{Z}^p(X, *)_0 \to \mathcal{H}^p(X, *)_0 \]
such that \( \chi_1 \otimes \mathbb{R} : \mathbb{Z}^p(X, *)_0 \otimes \mathbb{R} \to \mathcal{H}^p(X, *)_0 \) is an isomorphism. Set
\[ \mathcal{D}^{*, -r}_{A, \mathbb{Z}_l^p}(X, p)_0 = \tau \mathcal{D}^*_\log,\mathbb{Z}_l^p(X \times \Box^r, p)_0 \]
and denote by \( \mathcal{D}^{*, -r}_{A, \mathbb{Z}_l^p}(X, p)_0 \) the associated single complex. Then we can write any element of \( \mathcal{D}^{2p - r - i}_{A, \mathbb{Z}_l^p}(X, p)_0 \) as \( (\omega_r, g_r, \ldots, (\omega_0, g_0)) \), where
\[ (\omega_i, g_i) \in \tau \mathcal{D}^{2p - r + i}_{\log, \mathbb{Z}_l^p}(X \times \Box^i, p)_0, \]
in other words,
\[ \omega_i \in \tau \mathcal{D}^{2p - r + i}_{\log, \mathbb{Z}_l^p}(X \times \Box^i, p)_0, \quad g_i \in \tau \mathcal{D}^{2p - r + i - 1}_{\log, \mathbb{Z}_l^p}(X \times \Box^{i - 1}, \mathbb{Z}_l^p, p)_0 \]
such that \( d_{\mathcal{D}} g_r = \omega_r \).

**Proposition 6.4.** [BF, Prop.2.13] The map
\[ \chi_2 : \mathcal{D}^{2p - r - i}_{A, \mathbb{Z}_l^p}(X, p)_0 \to \mathcal{H}^p(X, *)_0 \]
defined by
\[ (\omega_r, g_r, \ldots, (\omega_0, g_0)) \mapsto [(\omega_r, g_r)] \]
is a quasi-isomorphism.

Define a map of complexes
\[ \mathcal{D}^*_A(X, p)_0 \to \mathcal{D}^*_A(X, p)_0 \]
by
\[ (\omega_r, g_r, \ldots, (\omega_0, g_0)) \mapsto (\omega_r, \ldots, \omega_0), \]
and denote by \( \rho \) the composite below:
\[ \rho : \mathcal{D}^*_A(X, p)_0 \to \mathcal{D}^*_A(X, p)_0 \xrightarrow{\sim} \mathcal{D}^*_A(X, p) \to \mathcal{D}^*_A(X, p). \]

Let us now give another definition of higher arithmetic Chow groups [BF]. Let \( X \) be a smooth variety defined over an arithmetic field. Denote by \( \mathcal{H}^p(X, *)_0, \mathcal{D}^*_A(X, p)_0 \) and
The $\mathcal{T}_\infty$-invariant part of the complexes $\mathcal{H}_p(X, \ast)_0, \mathcal{D}_{A, \mathbb{Z}}^*(X, \mathbb{C}, p)_0$ and $\mathcal{D}_{A, \mathbb{P}}^*(X, \mathbb{C}, p)$ respectively. Let $\mathcal{D}_{A, \mathbb{P}}^*(X, \mathbb{P}, \mathbb{C}, p)_0$ be the complex defined by
$$
\mathcal{D}_{A, \mathbb{P}}^n(X, \mathbb{P}, \mathbb{C}, p)_0 = \begin{cases} 
\mathcal{D}_{A, \mathbb{P}}^n(X, \mathbb{P}, \mathbb{C}, p)_0, & n < 2p, \\
0, & n \geq 2p.
\end{cases}
$$

**Definition 6.5.** [BF, Def. 4.2] Let $\mathcal{Z}_p(X, \ast)_0$ be the simple complex of the diagram
$$
\begin{array}{ccc}
\mathcal{H}_p(X, \ast)_0 & \xrightarrow{\chi_1} & \mathcal{D}_{A, \mathbb{P}}^{2p-1}(X, \mathbb{P}, \mathbb{C}, p)_0 \\
\xrightarrow{\chi_2} & & \xrightarrow{\rho} \mathcal{D}_{A, \mathbb{P}}^{2p-1}(X, \mathbb{P}, \mathbb{C}, p)_0.
\end{array}
$$
Define the higher arithmetic Chow groups of $X$ to be the homology groups of this complex:
$$
\widehat{CH}_r^p(X, \ast) = H_r(\mathcal{Z}_p(X, \ast)_0).
$$

**Definition 6.6.** For $r \geq 1$, define a map
$$
a : \mathcal{D}_{A, \mathbb{P}}^{2p-r-1}(X, \mathbb{P}, \mathbb{C}, p)_0 / \text{Im } d_s \to \mathcal{Z}_p(X, \ast)_0 / \text{Im } \partial,
$$
by
$$
a(\eta^p) = \left[ \begin{array}{cc} 0 & -\eta^p \\
0 & 0 \end{array} \right].
$$
Then it induces a map on (co)homology:
$$
a : H_{dD}^{2p-r-1}(X, \mathbb{R}(p)) \to \widehat{CH}_r^p(X, \ast).
$$

If $\eta^p \in \mathcal{D}_{A, \mathbb{P}}^{2p-r-1}(X, \mathbb{P}, \mathbb{C}, p)_0$, then
$$
a(\eta^p) = \left[ \begin{array}{cc} 0 & 0 \\
0 & d_s(\eta^p, 0) \end{array} \right].
$$
In particular, if $\delta_A \eta^p = 0$, then
$$
a(\eta^p) = \left[ \begin{array}{cc} 0 & 0 \\
0 & (d_{dD} \eta^p, \eta^p) \end{array} \right].
$$
In the case that $r = 0$, the canonical inclusion
$$
\tau \mathcal{D}_{A, \mathbb{P}}^{2p-1}(X, \mathbb{P}, \mathbb{C}, p)_0 / \text{Im } d_{dD} \to \mathcal{D}_{A, \mathbb{P}}^{2p-1}(X, \mathbb{P}, \mathbb{C}, p)_0 / \text{Im } d_s
$$
turns out to be an isomorphism. Hence it follows from Prop. 6.1 and Prop. 6.4 that there is a long exact sequence
$$
\to H_{dD}^{2p-r-1}(X, \mathbb{R}(p)) \xrightarrow{a} \widehat{CH}_r^p(X, \ast) \to CH_{dD}^p(X, \ast) \to H_{dD}^{2p-r}(X, \mathbb{R}(p)) \xrightarrow{a} \cdots
$$
$$
\to CH_{dD}^p(X, 1) \to \tau \mathcal{D}_{A, \mathbb{P}}^{2p-1}(X, \mathbb{P}, \mathbb{C}, p)_0 / \text{Im } d_{dD} \xrightarrow{a} \widehat{CH}_0^p(X, 0) \to CH_{dD}^p(X, 0) \to 0.\]
Let us compare two chain complexes \( \hat{Z}_D^p(X,*)_0 \) and \( \hat{Z}_p^p(X,*)_0 \). The quasi-isomorphism \( \kappa = \kappa_{A,p} : \hat{D}_{A,p}^*(X,p) \to \tau D_D^*(X,p) \) defined in §3.4 induces a map of complexes

\[
\kappa : \hat{D}_{A,p}^*(X,p) \to \tau D_D^*(X,p).
\]

Let

\[
\theta : \Omega^p(X,*)_0 \otimes \mathbb{R} \to \tau D_D^{2p-*}(X,p),
\]

which is also a map of complexes. For \( g_i \in \tau D_{log}^{2p-r+i-1}(X \times \mathbb{P}^i - Z^p, p) \), \( g_i \cdot \pi^*_p W_i \) is locally integrable on \( X \times (\mathbb{P}^i)^i \) such that

\[
d \pi_X^*[g_i \cdot \pi^*_p W_i] = \pi_X^*[d_D g_i \cdot \pi^*_p W_i] + (-1)^{r+i-1} \pi_X^* [\delta A g_i \cdot \pi^*_p W_{i-1}]
\]

if \( i < r \), and

\[
d \pi_X^*[g_r \cdot \pi^*_p W_r] = \pi_X^*[d_D g_r \cdot \pi^*_p W_r] - \pi_X^* [\delta_A g_r \cdot \pi^*_p W_{r-1}] - P^p(z)
\]

by [BFT, Prop.7.5, Prop.7.6]. In the above, \( z \in \Omega^p(X,r) \otimes \mathbb{R} \) is the unique element satisfying \( (\chi_1 \otimes \mathbb{R})(z) = [(d_D g_r, g_r)] \) in \( \Omega^p(X,r) \).

**Proposition 6.7.** Assume \( r \geq 1 \). Define a map

\[
\psi : D_{A,Z}^{2p-r}(X,p)_0 \to \tau D_{D}^{2p-r-1}(X,p)
\]

by

\[
\psi((\omega_r, g_r), \ldots, (\omega_0, g_0)) = \sum_{i=0}^r \pi_X^*[g_i \cdot \pi^*_p W_i].
\]

Then

\[
\psi d_s + d_D \psi = \kappa \rho - \theta \chi_2.
\]

**Proof:** Let \( \alpha = ((\omega_r, g_r), \ldots, (\omega_0, g_0)) \in D_{A,Z}^{2p-r}(X,p)_0 \). Then by (6.2) and (6.3),

\[
d_D \psi(\alpha) = \sum_{i=0}^r \pi_X^*[d_D g_i \cdot \pi^*_p W_i] + \sum_{i=1}^r (-1)^{r+i-1} \pi_X^* [\delta_A g_i \cdot \pi^*_p W_{i-1}] - P^p(z),
\]

where \( z \in \Omega^p(X,r)_0 \otimes \mathbb{R} \) such that \( (\chi_1 \otimes \mathbb{R})(z) = [(\omega_r, g_r)] = \chi_2(\alpha) \). Moreover,

\[
\psi d_s(\alpha) = \sum_{i=0}^{r-1} \pi_X^*[(\omega_i - d_D g_i + (-1)^{r-i+1} \delta_A g_{i+1}) \cdot \pi^*_p W_i],
\]

\[
\kappa \rho(\alpha) = \sum_{i=0}^r \pi_X^* [\omega_i \cdot \pi^*_p W_i].
\]

Then

\[
d_D \psi(\alpha) + \psi d_s(\alpha) - \kappa \rho(\alpha) = -P^p(z) = -\theta (\chi_1 \otimes \mathbb{R})(z) = -\theta \chi_2(\alpha),
\]

which completes the proof. \( \square \)

It follows from Prop.6.7 that the map

\[
\Gamma : \hat{Z}_D^p(X,*)_0 \to \hat{Z}_D^p(X,*)_0
\]
given by
\[
\Gamma \left( \begin{array}{cc}
\beta_1 & \beta_2 \\
y & \alpha \\
\end{array} \right) = (y, -\psi(\alpha) + \theta(\beta_1) + \varphi(\beta_2))
\]
is a map of complexes which fits into the commutative diagram
\[
\begin{array}{c}
\cdots \longrightarrow H^{2p-r-1}_D(X, \mathbb{R}(p)) \xrightarrow{\alpha} \widehat{CH}^p(X, r) \\
\downarrow \Id \quad \downarrow \Gamma_* \\
\cdots \longrightarrow H^{2p-r-1}_D(X, \mathbb{R}(p)) \xrightarrow{\alpha} \widehat{CH}^p_D(X, r) \\
\longrightarrow CH^p(X, r) \longrightarrow H^{2p-r}_D(X, \mathbb{R}(p)) \longrightarrow \cdots
\end{array}
\]
This leads to the following theorem:

**Theorem 6.8.** When \( r \geq 1 \),
\[\Gamma_* : \widehat{CH}^p(X, r) \rightarrow \widehat{CH}^p_D(X, r)\]
is an isomorphism.

### 6.4. Chern character of a vector bundle on an iterated double

Let \( X \) be a smooth projective variety defined over a field. Consider the associated iterated double \( T = D(X \times \square^n; X \times \partial^n) \) and a morphism \( \varphi : T \rightarrow G \) to a smooth projective variety \( G \). For any \( I \subset \{1, \ldots, r\} \), denote by \( \varphi_I : X \times \square^n \rightarrow G \) the restriction of \( \varphi \) to the irreducible component corresponding to \( I \). Let \( Z^p(G, n) \) be the subgroup of \( Z^p(G, n) \) such that \( x \in Z^p(G, n) \) if and only if we can take the pull-back cycle of \( x \) by the morphism
\[
X \times D_J \times D_K \hookrightarrow X \times \square^n \times \square^n \xrightarrow{\varphi_I \times \Id} G \times \square^n
\]
for any \( I, J \subset \{1, \ldots, r\} \) and \( K \subset \{1, \ldots, n\} \). The moving lemma [Ha, Thm.1.3] says that the embedding of the normalized subcomplexes
\[Z^p_G(G, *)_0 \hookrightarrow Z^p(G, *)_0\]
is a quasi-isomorphism.

Let \( \mathcal{F} \) be a vector bundle on \( T \). Then we can obtain a morphism \( \varphi : T \rightarrow G \) to a smooth projective variety \( G \) and a vector bundle \( \mathcal{G} \) on \( G \) such that \( \varphi^* \mathcal{G} \cong \mathcal{F} \) [Fu, §3.2]. Let \( y \in Z^p_G(G) \) be a cycle representing the \( p \)-th Chern character \( ch^p_G(G) \in CH^p(G) \), and let us denote
\[
\varphi^* (y) = \sum_I (-1)^{|I|} \varphi^*_I (y) \in Z^p(X \times \square^n).
\]
Since \( \varphi^*_I (y)_{\{z_j = i\}} = \varphi^*_I \cup \{y\}_{\{z_j = i\}} \) for any \( I \subset \{1, \ldots, r\} \) with \( j \notin I \) and for \( i = 0 \) or \( \infty \), it holds that \( (\partial^*_I)^* \varphi^* (y) = 0 \). In particular, \( \varphi^* (y) \in Z^p(X, r)_0 \) such that \( \partial^* \varphi^* (y) = 0 \).
Theorem 6.9. The element $[\varphi^*(y)] \in CH^p(X, r)$ depends only on $F$, and is independent of the choice of morphisms $\varphi$, vector bundles $G$ and $y \in Z^p_{\varphi}(G)$. Hence in what follows we denote

$$[\varphi^*(y)] = \text{ch}^p_{T,0}(F) \in CH^p(X, r).$$

Proof: First we show the independence of the choice of $y \in Z^p_{\varphi}(G)$. Let $y' \in Z^p_{\varphi}(G)$ be another element representing $\text{ch}^p_{0}(G) \in CH^p(G)$. Then there is an element $w \in Z^p_{\varphi}(G, 1)_0$ such that $\partial w = y - y'$. Let

$$\varphi^*(w) = \sum (-1)^{|I|} \varphi^I_*(w) \in Z^p(X \times \square^r, 1).$$

By the identification

$$(X \times \square^r) \times \square^1 \approx X \times \square^{r+1}, \quad ((x, z_1, \ldots, z_r), z') \mapsto (x, z_1, \ldots, z_r, z'),$$

we can see $\varphi^*(w)$ as an element of $Z^p(X \times \square^{r+1})$. Since $\varphi^I_*(w)|_{z_j = i} = \varphi^I_{\cup (I)}(w)|_{z_j = i}$ for any $I \subset \{1, \ldots, r\}$ with $j \notin I$ and for $i = 0$ or $\infty$, $\varphi^*(w) \in Z^p(X, r + 1)_0$ such that

$$\partial \varphi^*(w) = (-1)^r \varphi^*(y) - (-1)^r \varphi^*(y').$$

Hence $[\varphi^*(y)] = [\varphi^*(y')]$ in $CH^p(X, r)$.

We next show the independence of the choice of $\varphi : T \to G$ and vector bundles $G$. Let $\varphi' : T \to G'$ be another morphism to a smooth projective variety $G'$ and $G'$ a vector bundle on $G$ such that there is an isomorphism $\varphi'^* G' \approx \mathcal{F}$. Then as shown in [Fu, §3.2], there is a commutative diagram of morphisms

$$\begin{array}{ccc}
G' & \xrightarrow{\psi} & G'' \\
\varphi' \downarrow & & \downarrow \varphi'' \\
G & \xrightarrow{\psi} & G,
\end{array}$$

where $G''$ is a smooth projective variety on which there is an isomorphism $\psi^* G \approx \psi'^* G'$ which admits the commutative diagram of isomorphisms

$$\begin{array}{ccc}
\varphi^* G & \approx & \varphi'^* \psi^* G \\
\approx & & \varphi'^* \psi'^* G' \\
& & \approx \varphi'^* G' \\
& & \approx \mathcal{F}.
\end{array}$$

Hence we may assume that $G = G''$ and $\psi = \text{Id}$, that is, there is a morphism $\psi : G \to G'$ such that $\psi^* G' \approx G$ and $\varphi = \psi \varphi$.

Set $Z^p_{\psi \varphi}(G') = Z^p_{\psi}(G') \cap Z^p_{\varphi}(G')$. Then there is a cycle $y' \in Z^p_{\psi \varphi}(G')$ representing $\text{ch}^p_{0}(G') \in CH^p(G')$. Moreover, we can take the pull-back cycle $y = \psi^*(y') \in Z^p_{\varphi}(G)$ which represents $\text{ch}^p_{0}(G) \in CH^p(G)$. Then $\varphi^*(y) = \varphi'^*(\psi^*(y')) = \varphi'^*(y')$ in $Z^p(X, r)_0$. This completes the proof. 

Theorem 6.10. The above correspondence $\mathcal{F} \mapsto [\varphi^*(y)]$ gives a map of abelian groups

$$\text{ch}^p_{T,0} : K_0(T) \to CH^p(X, r).$$
Proof: We have only to show that
\[ \text{ch}_{T,0}^p(F_{-1}) + \text{ch}_{T,0}^p(F_1) = \text{ch}_{T,0}^p(F_0) \]
for any short exact sequence \( 0 \to F_{-1} \to F_0 \to F_1 \to 0 \) of vector bundles on \( T \). We can show this equality in the same way as the proof of Thm.6.9, mimicking the argument in [Bu §3.2]. \( \square \)

Definition 6.11. Composing the map \( \text{ch}_{T,0}^p \) with the sequence of maps
\[ K_r(X) \cong K_0(X \times \square^r; X \times \partial \square^r) \cong K_0(T; T_1, \ldots, T_r) \subset K_0(T) \]
given by Levine in [Le], we can define a map
\[ \text{ch}_r^p : K_r(X) \to CH^p(X, r). \]

Remark: The author do not know if the map defined above agrees with the higher Chern character map. However, we will show later in Thm.7.4 that the composite of \( \text{ch}_r^p \) with the regulator map
\[ CH^p(X, r) \to H^{2p-r}_D(X, \mathbb{R}(p)) \]
agrees with Beilinson’s regulator. This is a strong evidence that these two maps agree.

6.5. Arithmetic Chern character of a hermitian vector bundle on an iterated double. In [Bu2] Burgos extended the definition of arithmetic Chow groups to open varieties, and to this end he gave another definition of Green forms. We begin by recalling his construction.

Let \( X \) be a complex algebraic manifold \( X \). The space of Green forms of codimension \( p \) is defined as the truncated cohomology groups of Deligne complexes as follows:
\[ GE^p(X) = \tilde{H}^p(D^*_{\log}(X, p), D^*_{\log}(X - \mathbb{Z}^p, p)). \]

For any subset \( \mathbb{Z}_s^p \subset \mathbb{Z}^p \), set
\[ GE^p_{\mathbb{Z}_s^p}(X) = \tilde{H}^p(D^*_{\log}(X, p), D^*_{\log}(X - \mathbb{Z}_s^p, p)). \]

Any element of \( GE^p(X) \) (resp. \( GE^p_{\mathbb{Z}_s^p}(X) \)) is given by a pair \( (\omega, \tilde{g}) \) of \( \omega \in \tau D^p_{\log}(X, p) \) with \( \tilde{g} \in \tau D^{p-1}_{\log}(X - \mathbb{Z}_s^p) / \text{Im } d_p \) (resp. \( \tilde{g} \in \tau D^{p-1}_{\log}(X - \mathbb{Z}_s^p) / \text{Im } d_p \)) such that \( \omega = d_p \tilde{g} \).

For an arithmetic variety \( X \) defined over an arithmetic ring, let us denote by \( \widehat{CH}^p(X, \mathcal{D}(E_{\log})) \) the arithmetic Chow group of \( X \) by means of \( GE^p(X) \) [Bu2 §7]. Note that when \( X \) is a proper arithmetic variety, \( \widehat{CH}^p(X, \mathcal{D}(E_{\log})) \) is canonically isomorphic to the arithmetic Chow group originally defined by Gillet and Soulé in [GSH], since in this case the definition of arithmetic Chow groups does not depend on the complexes where Green objects lie.

Let \( X \) be a smooth projective variety defined over an arithmetic field, and \( T = D(X \times \square^r; X \times \partial \square^r) \) the iterated double. For a morphism \( \varphi : T \to G \) to a smooth projective variety and for a differential form \( \omega \) on \( G \), let us write
\[ \varphi^*(\omega) = \sum_I (-1)^{|I|} \varphi_I^* (\omega). \]
Since \( \varphi^*_I(\omega)|_{z_j=i} = \varphi^*_{I_0(j)}(\omega)|_{z_j=i} \) for any \( I \subseteq \{1, \ldots, r\} \) with \( j \notin I \) and for \( i = 0 \) or \( \infty \), \( \varphi^*(\omega) \) is normalized such that \( \delta_k \varphi^*(\omega) = 0 \).

Let \( \mathcal{F} \) be a vector bundle with a smooth at infinity metric on \( T \). Take a morphism \( \varphi : T \to G \) to a smooth projective variety \( G \), a vector bundle \( \mathcal{G} \) such that \( \mathcal{F} \simeq \varphi^* \mathcal{G} \), and a cycle \( y \in Z^p_\partial(G) \) representing \( \delta^p_\partial(G) \) in \( CH^p(G) \). Given a smooth hermitian metric \( h \) on \( \mathcal{G} \), we obtain the \( p \)-th arithmetic Chern character \( \hat{\text{ch}}^p_0(\mathcal{G}) \in \overline{CH}^p(G, D(E^*_q)) \) of \( \mathcal{G} = (\mathcal{G}, h) \). Take a Green form \( (\omega, \widetilde{g}) \in G \mathcal{E} \mathcal{Z}^p(G) \) associated with \( y \) in the sense of \([Bu2, \S 5]\) such that the pair \((y, (\omega, \widetilde{g}))\) represents \( \hat{\text{ch}}^p_0(\mathcal{G}) \).

Let \( g \in D_{\log}^{2p-1}(G - 2p, \varphi, p) \) be a lift of \( \widetilde{g} \), and let us denote \((\varphi^*(\omega), \varphi^*(g)) = ((\varphi^*(\omega), \varphi^*(g)), (0, 0), \ldots, (0, 0)) \in D^{2p-r}_{\partial, \varphi}(X, p) \).

Then \( \delta_k(\varphi^*(\omega), \varphi^*(g)) = 0 \) and \( \chi_1(\varphi^*(y)) = \chi_2(\varphi^*(\omega), \varphi^*(g)) \). Moreover, if we take the \((p, p)\)-part of the Bott-Chern form

\[
\begin{align*}
\text{ch}^p_{T, 1}(\mathcal{F}, \varphi^* \mathcal{G}) &= \text{ch}^p_{T, 1}(\mathcal{F} \simeq \varphi^* \mathcal{G} \to 0) \in \overline{D}^{2p-r}_{\partial, \varphi}(X, p), \\
\text{then} \quad d_s \text{ch}^p_{T, 1}(\mathcal{F}, \varphi^* \mathcal{G}) &= (-1)^r \left( \text{ch}^p_{T, 0}(\mathcal{F}) - \varphi^*(\text{ch}^p_{T, 0}(\mathcal{G})) \right). 
\end{align*}
\]

\textbf{Definition 6.12.} Let \( \hat{Z}^p(X, r)_0 \) be the subgroup of \( \hat{Z}^p(X, r)_0 \) defined as follows:

\[
\hat{Z}^p(X, r)_0 = \left\{ x \in \hat{Z}^p(X, r)_0; \partial x = \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix} \right\}.
\]

Then \( \text{Im} \partial \subset \text{Ker} \partial \subset \hat{Z}^p(X, r)_0 \), and the quotient group \( \hat{Z}^p(X, r)_0 / \text{Im} \partial \) can be expressed as a homology group of a chain complex. In fact, if we denote

\[
(s_{<r} A^*)^n = \begin{cases} A^n, & n < r, \\
0, & n \geq r,
\end{cases}
\]

then

\[
\hat{Z}^p(X, r)_0 / \text{Im} \partial = H_p \begin{pmatrix} \mathcal{H}^p(X, r)_0 & \sigma_{<2p-r} \overline{D}^{2p-\ast}_{\partial, X}(X, p) \\
Z^p(X, r)_0 & \overline{D}^{2p-\ast}_{\partial, X}(X, p)_0
\end{pmatrix}.
\]

Consider the element

\[
\text{cl}(\mathcal{F}, \varphi, \mathcal{G}, (y, (\omega, g))) = \begin{pmatrix} 0 & (-1)^{r+1} \text{ch}^p_{T, 1}(\mathcal{F}, \varphi^* \mathcal{G}) \\ \varphi^*(y) & (\varphi^*(\omega), \varphi^*(g)) \end{pmatrix} \in \hat{Z}^p(X, r)_0.
\]

Since

\[
(6.4) \quad \partial \left( \text{cl}(\mathcal{F}, \varphi, \mathcal{G}, (y, (\omega, g))) \right) = \begin{pmatrix} 0 & \text{ch}^p_{T, 0}(\mathcal{F}) \\ 0 & 0 \end{pmatrix},
\]
it holds that $cl(\bar{F}, \varphi, \bar{G}, (y, (\omega, g))) \in \hat{Z}^p(X, r)_0$.

**Theorem 6.13.** In the quotient group, 

$$[cl] = [cl(\bar{F}, \varphi, \bar{G}, (y, (\omega, g)))] \in \hat{Z}^p(X, r)_0 / \text{Im} \partial$$

depends only on $\bar{F}$, in other words, $[cl]$ is independent of the choice of $\varphi, \bar{G}$ and $(y, (\omega, g))$.

**Proof:** First we show that $[cl]$ is independent of the choice of representatives $(y, (\omega, \tilde{g}))$ of $\tilde{c}_h^p(G)$ and lifts $g$ of $\tilde{g}$. Fix a morphism $\varphi : T \to G$ and a hermitian vector bundle $\bar{G}$ such that $\varphi^*G \simeq F$. Denote by $\mathcal{Z}_{\varphi, n}^p$ the set of admissible subschemes of $G \times \square^n$ such that $Y \in \mathcal{Z}_{\varphi, n}^p$ if and only if we can take the pull-back cycle of $[Y]$ by the morphism 

$$X \times D_I \times D_K \leftarrow X \times \square^n \times \square^n \xrightarrow{\varphi \times \text{Id}} G \times \square^n$$

for any $I, J \subset \{1, \ldots, r\}$ and for any $K \subset \{1, \ldots, n\}$. Let $\mathcal{D}_{\log, \mathcal{Z}_{\varphi, n}^p}^*(G \times \square^n, p)$ be the simple complex of the restriction map 

$$\mathcal{D}_{\log, \mathcal{Z}_{\varphi, n}^p}^*(G \times \square^n, p) \to \mathcal{D}_{\log, \mathcal{Z}_{\varphi, n}^p}^*(G \times \square^n - \mathcal{Z}_{\varphi, n}^p, p)$$

and 

$$\tau \mathcal{D}_{\log, \mathcal{Z}_{\varphi, n}^p}^*(G \times \square^n, p) = \tau_{\leq 2p} \mathcal{D}_{\log, \mathcal{Z}_{\varphi, n}^p}^*(G \times \square^n, p)$$

the truncated subcomplex. Moreover, set 

$$\mathcal{H}_{\varphi}^p(G, n) = H^{2p}_{\mathcal{D}_{\log, \mathcal{Z}_{\varphi, n}^p}}(G \times \square^n, \mathbb{R}(p)) = H^{2p}(\tau \mathcal{D}_{\log, \mathcal{Z}_{\varphi, n}^p}^*(G \times \square^n, p)),$$

$$\mathcal{D}_{\mathcal{H}, \mathcal{Z}_{\varphi, n}^p}^s(G, p) = \tau \mathcal{D}_{\mathcal{D}_{\log, \mathcal{Z}_{\varphi, n}^p}}^s(G \times \square^n, p).$$

These complexes have cubical structures with respect to the index $n$, hence we can obtain the normalized subcomplexes, which we denote by $\mathcal{H}_{\varphi}^p(G, \ast)_0$ and $\mathcal{D}_{\mathcal{H}, \mathcal{Z}_{\varphi, n}^p}^{s, -\tau}(G, p)_0$ respectively. Let $\mathcal{D}_{\mathcal{H}, \mathcal{Z}_{\varphi, n}^p}^*(G, p)_0$ be the single complex associated with $\mathcal{D}_{\mathcal{H}, \mathcal{Z}_{\varphi, n}^p}^{s, -\tau}(G, p)_0$. Finally, let $\hat{Z}_{\varphi}^p(G, \ast)_0$ be the simple complex of the diagram

$$\begin{array}{ccc}
\mathcal{H}_{\varphi}^p(G, \ast)_0 & \xrightarrow{\chi_1} & \mathcal{H}_{\varphi}^p(G, \ast)_0 \\
\chi_2 & \xleftarrow{\rho} & \hat{D}_{\mathcal{H}}^{2p - \tau}(G, p)_0
\end{array}$$

where the maps $\chi_1, \chi_2$ and $\rho$ are defined in a similar way to the definition of $\hat{Z}_{\varphi}^p(X, \ast)_0$.

We should note that in the definition of this complex we use $\hat{D}_{\mathcal{H}}^{2p - \tau}(G, p)_0$, not $\hat{D}_{\mathcal{H}, \mathcal{F}}^{2p - \tau}(G, p)_0$. Consider the commutative diagram

$$\begin{array}{ccc}
\mathcal{H}_{\varphi}^p(G, \ast)_0 & \xrightarrow{\chi_1} & \mathcal{H}_{\varphi}^p(G, \ast)_0 \\
\chi_2 & \xleftarrow{\rho} & \hat{D}_{\mathcal{H}}^{2p - \tau}(G, p)_0 \\
\mathcal{D}_{\mathcal{H}, \mathcal{Z}_{\varphi, n}^p}^{s, -\tau}(G, p)_0 & \xrightarrow{\chi_1} & \hat{D}_{\mathcal{H}}^{2p - \tau}(G, p)_0 \\
\chi_2 & \xleftarrow{\rho} & \hat{D}_{\mathcal{H}, \mathcal{F}}^{2p - \tau}(G, p)_0
\end{array}$$
The left vertical arrow in the diagram is a quasi-isomorphism by the moving lemma. Moreover, as shown in [BF Prop.1.31, Prop.2.13], the maps $\chi_2$ and $\chi_1 \otimes \mathbb{R}$ on the both lines are quasi-isomorphisms. Hence all the vertical maps are quasi-isomorphisms, which implies that the natural map
$$\hat{\mathbb{Z}}^p_{\varphi}(G, *)_0 \to \hat{\mathbb{Z}}^p(G, *)_0$$
is also a quasi-isomorphism.

Take another representative $(y', (\omega', \bar{g}'))$ of $\hat{\text{ch}}^p_{\alpha}(G)$ and a lift $g'$ of $\bar{g}'$. Since the map
$$\hat{\mathbb{H}}^p(G, \mathcal{D}(E^*_\log)) \to \hat{\mathbb{H}}^p(G, 0)$$
given by
$$[(y, (\omega, \bar{g}))] \mapsto \begin{pmatrix} 0 & 0 \\ y & (\omega, g) \end{pmatrix}$$
is an isomorphism [BF Thm.4.8], we can obtain an element
$$\begin{pmatrix} \beta_0 & \beta_1 \\ z & \alpha \end{pmatrix} \in \hat{\mathbb{Z}}^p_{\varphi}(G, 1)_0$$
such that
$$\begin{pmatrix} 0 \\ y - y' \\ (\omega - \omega', g - g') \end{pmatrix} = \partial \begin{pmatrix} \beta_0 & \beta_1 \\ z & \alpha \end{pmatrix}.$$ Consider the morphism
$$\varphi^T_I : X \times \square^n \times \square^r \to X \times \square^n \times \square^r, \quad (a, b) \mapsto (b, a).$$
Then the maps $(\varphi^T)^*$ which send $x$ to $\sum (-1)^{|I|}(\varphi^T_I)^*(x)$ form the commutative diagram
$$\begin{array}{cccc}
\hat{\mathbb{Z}}^p_{\varphi}(G, *)_0 & \xrightarrow{\chi_1} & \hat{\mathbb{Z}}^p_{\varphi}(G, *)_0 & \xrightarrow{\chi_2} & \hat{\mathbb{D}}^{2p-*}_{\mathbb{A}, Z_p} (G, p)_0 & \xrightarrow{\rho} & \hat{\mathbb{D}}^{2p-*}_{\mathbb{A}, Z_p} (G, p) \\
(\varphi^T)^* & & (\varphi^T)^* & & (\varphi^T)^* & & (\varphi^T)^* \\
\hat{\mathbb{Z}}^p(X, * + r)_0 & \xrightarrow{\chi_1} & \hat{\mathbb{H}}^p(X, * + r)_0 & \xrightarrow{\chi_2} & \hat{\mathbb{D}}^{2p-*}_{\mathbb{A}, Z_p} (X, p)_0 & \xrightarrow{\rho} & \hat{\mathbb{D}}^{2p-*}_{\mathbb{A}, Z_p} (X, p).
\end{array}$$
The vertical arrows in this diagram are maps of complexes except the right one at $* = 1$. Hence
$$\partial \begin{pmatrix} (\varphi^T)^*(\beta_0) \\ (\varphi^T)^*(\beta_1) \\ (\varphi^T)^*(\alpha) \end{pmatrix} = \begin{pmatrix} 0 \\ (\varphi^*(y) - \varphi^*(y')) (\varphi^*(\omega) - \varphi^*(\omega'), \varphi^*(g) - \varphi^*(g')) \end{pmatrix},$$
which shows that $[cl]$ is independent of the choice of $(y, (\omega, \bar{g}))$ and lifts of $\bar{g}$. 
Then in $\overline{D}_{\mathcal{A},p}(X, p)$ the difference

$$\chi_{T, 1}^p(\varphi^*\mathcal{E}) - (-1)^r \varphi^* \chi_1^p(\mathcal{E})$$

is $d_s$-exact.

Proof: Note that $\chi_1^p(\mathcal{E})_p \in \overline{D}_{\mathcal{P}}^{2p-1}(G, p)$ and $\chi_1^p(\mathcal{E}) \in \overline{D}_{\mathcal{P}}^{2p-1, 0}(G, p)$. Since $\kappa_\mathcal{P}$ is a left inverse of the quasi-isomorphism $\tau \mathcal{D}^*(G, p) \to \overline{D}_{\mathcal{P}}^*(G, p)$, the difference $\chi_1^p(\mathcal{E})_p - \chi_1^p(\mathcal{E})$ is $d_s$-exact in $\overline{D}_{\mathcal{P}}^{2p-1}(G, p)$. Hence there are $\alpha_i \in \overline{D}_{\mathcal{P}}^{2p+i-2, -r, -i}(G, p)$ for $i = 0, 1, 2$ such that

$$\chi_1^p(\mathcal{E}) = -d_2 \alpha_0 + \delta_2 \alpha_1,$$

$$\chi_1^p(\mathcal{E})_p = d_2 \alpha_1 + \delta_2 \alpha_2.$$

Consider $\varphi^* \alpha_i \in \overline{D}_{\mathcal{A}, p}^{2p+i-2, -r, -i}(X, p)$. Since $\delta_\mathcal{A} \varphi^* \alpha_i = 0$,

$$d_s(\varphi^* \alpha_0 + (-1)^r \varphi^* \alpha_1 + \varphi^* \alpha_2) = d_2 \varphi^* \alpha_0 - \delta_2 \varphi^* \alpha_1 + (-1)^r d_2 \varphi^* \alpha_1 + (-1)^r \delta_2 \varphi^* \alpha_2$$

$$= -\varphi^* \chi_1^p(\mathcal{E}) + (-1)^r \varphi^* \chi_1^p(\mathcal{E})_p$$

$$= -\varphi^* \chi_1^p(\mathcal{E}) + (-1)^r \chi_{T, 1}^p(\varphi^*\mathcal{E}),$$

which completes the proof. \qed

Let us go back to the proof of Thm.6.12. Let $\mathcal{G}'$ be the same vector bundle as $\mathcal{G}$ with a different metric. If we denote by

$$\chi_1^p(\mathcal{G}')_\mathcal{G} = \chi_1^p(\mathcal{G} \sim \mathcal{G} \to 0),$$

which is the $(p - 1, p - 1)$-part of the Bott-Chern form, then by [CS2] Thm.4.8 we have

$$\widehat{\chi}_0^p(\mathcal{G}') - \widehat{\chi}_0^p(\mathcal{G}) = a(\widehat{\chi}_1^p(\mathcal{G}', \mathcal{G}))$$

in $\widehat{\mathcal{H}}^p(G, \mathcal{D}(E_{log}^*))$, where

$$a : \mathcal{D}^{2p-1}(G, p)/ \text{Im } d_2 \to \widehat{\mathcal{H}}^p(G, \mathcal{D}(E_{log}^*))$$

is the map which sends $\widetilde{h}$ to $[(0, (d_2 h, \widetilde{h}))]$. This means that if $\widehat{\chi}_0^p(\mathcal{G})$ is represented by $(y, (\omega, \widetilde{g}))$, then $\widehat{\chi}_0^p(\mathcal{G}')$ is represented by $(y, (\omega + d_2 h, \widetilde{g} + \widetilde{h}))$ where $h = \chi_1^p(\mathcal{G}', \mathcal{G})$. Hence

$$[\chi(\mathcal{F}, \varphi, \mathcal{G}, (y, (\omega, g))) - \chi(\mathcal{F}, \varphi, \mathcal{G}', (y, (\omega + d_2 h, g + h)))]$$

$$= \begin{bmatrix} 0 & (-1)^{r+1} \left( \chi_{T, 1}^p(\mathcal{F}, \varphi^*\mathcal{G}) - \chi_{T, 1}^p(\mathcal{F}, \varphi^*\mathcal{G}') \right) \\ 0 & -(d_2 \varphi^* h, \varphi^* h) \end{bmatrix}$$

$$= (-1)^r a \left( \chi_{T, 1}^p(\mathcal{F}, \varphi^*\mathcal{G}) - \chi_{T, 1}^p(\mathcal{F}, \varphi^*\mathcal{G}') \right) - a(\varphi^* \chi_1^p(\mathcal{G}', \mathcal{G})).$$
Similarly denote by \( \text{ch} \) the arithmetic Chow group. First we introduce some notations. For a hermitian vector bundles \( p \) and denote by \( \text{ch}(6.5) \) \( \text{(6.5)} \) and by Lem.6.13 it is equal to \( \psi \) may assume that there is a morphism \( \end{pmatrix} \). This means that \( (6.5) \) is zero. Hence we conclude that \( [c\ell] \) is independent of the choice of metrics on \( G \).

Finally we show that \( [c\ell] \) is independent of the choice of morphisms \( \varphi : T \to G \) and hermitian vector bundles \( \mathcal{G} \) on \( G \). Take another morphism \( \varphi' : T \to G' \) and a hermitian vector bundle \( \mathcal{G}' \) on \( G' \) such that \( \varphi'^*\mathcal{G}' \cong \mathcal{F} \). As shown in the proof of Thm.6.9, we may assume that there is a morphism \( \psi : G \to G' \) such that \( \psi \varphi = \varphi' \) and \( \psi^*\mathcal{G}' \cong \mathcal{G} \). Moreover, since we have shown that \( [c\ell] \) is independent of the choice of metrics on \( \mathcal{G} \), we may assume that the isomorphism \( \psi^*\mathcal{G}' \cong \mathcal{G} \) preserves the metrics. If \( (y', (\omega', \mathcal{G}')) \) is an representative of \( \widehat{\text{ch}}^p(G) \) such that \( y' \in \mathcal{Z}^p_{\psi, \varphi'}(G') \) and \( g' \in \mathcal{D}^{2p-1}_{\log}(G' - \mathcal{Z}^p_{\psi, \varphi'}) \), then \( \widehat{\text{ch}}^p_{(G)} \) is represented by \( (\psi^*(y'), (\psi^*(\omega'), \psi^*(g'))) \). Moreover, the isometry \( \psi^*\mathcal{G}' \cong \mathcal{G} \) implies that \( \widehat{\text{ch}}^p_{(G)}(\mathcal{F}, \varphi^*\mathcal{G}) = \text{ch}_{(G)}(\mathcal{F}, \varphi^*\mathcal{G}) \). Hence it follows that \( [c\ell] \) is independent the choice of morphisms \( \varphi : T \to G \) and hermitian vector bundles \( \mathcal{G} \).

\[ \text{Definition 6.15. We call the element} \]
\[ \text{the p-th arithmetic Chern character of a hermitian vector bundle } \mathcal{F} \text{ on } T. \]

\[ \text{Definition of higher arithmetic Chern character} \]

In this section we construct a map from the higher arithmetic \( K \)-group to the higher arithmetic Chow group. First we introduce some notations. For \( \eta \in \mathcal{D}_{A,F,r}(X) \), write
\[ \eta = \sum_p \eta^p, \quad \eta^p \in \mathcal{D}_{A,F,r}(X,p). \]

Similarly, denote by \( \text{ch}_{(G)}^p \) the component in \( \mathcal{D}_{A,F,r}(X,p) \) of the map \( \text{ch}_{(G)} : K_0(T) \to \mathcal{D}_{A,F,r}(X). \)
Proposition 7.1. We can define a map
\[ \widehat{\text{ch}}_{T,0}^p : \widehat{K}_0^M(T) \to \widehat{Z}^p(X, r)_0 / \text{Im} \partial \]
by
\[ \widehat{\text{ch}}_{T,0}^p(\mathcal{F}, \tilde{\eta}) = \widehat{\text{ch}}_{T,0}^p(\mathcal{F}) + a(\tilde{\eta}^p). \]

Proof: We have only to show that
\[ \widehat{\text{ch}}_{T,0}^p(\mathcal{F}_{-1}) + \widehat{\text{ch}}_{T,0}^p(\mathcal{F}_1) = \widehat{\text{ch}}_{T,0}^p(\mathcal{F}_0) + (-1)^r a(\tilde{\eta}^p) \]
for any short exact sequence of hermitian vector bundles \( \mathcal{E} : 0 \to \mathcal{F}_{-1} \to \mathcal{F}_0 \to \mathcal{F}_1 \to 0 \) on \( T \). It is shown in [Fu, §3.2] that for such \( \mathcal{E} \) there exist a morphism \( \varphi : T \to G \) to a smooth projective variety \( G \) and a short exact sequence of hermitian vector bundles \( \mathcal{E}' : 0 \to \mathcal{G}_{-1} \to \mathcal{G}_0 \to \mathcal{G}_1 \to 0 \) on \( G \) such that \( \varphi^* \mathcal{E}' \simeq \mathcal{E} \). Take representatives \( (y, (\omega, \tilde{\eta})) \) of \( \widehat{\text{ch}}_{0}^p(\mathcal{G}_{-1}) \) and \( (y', (\omega', \tilde{\eta}')) \) of \( \widehat{\text{ch}}_{0}^p(\mathcal{G}_1) \). Since
\[ \widehat{\text{ch}}_{0}^p(\mathcal{G}_{-1}) + \widehat{\text{ch}}_{0}^p(\mathcal{G}_1) = \widehat{\text{ch}}_{0}^p(\mathcal{G}_0) + a(\tilde{\eta}^p) \]
by [GS2] Thm.4.8, \( \widehat{\text{ch}}_{0}^p(\mathcal{G}_0) \) is represented by
\[ (y + y', (\omega + \omega' - d_2 \text{ch}_1^p(\mathcal{E}'), \tilde{\eta} + \tilde{\eta}' - \text{ch}_1^p(\mathcal{E}'))). \]

Then
\[
\begin{align*}
\text{ch}_{T,0}^p(\mathcal{F}_{-1}) &= \begin{pmatrix} 0 & (-1)^{r+1} \text{ch}_{T,1}^p(\mathcal{F}_{-1}, \varphi^* \mathcal{G}_{-1}) \\ \varphi^*(y) & (\varphi^*(\omega), \varphi^*(g)) \end{pmatrix}, \\
\text{ch}_{T,0}^p(\mathcal{F}_1) &= \begin{pmatrix} 0 & (-1)^{r+1} \text{ch}_{T,1}^p(\mathcal{F}_1, \varphi^* \mathcal{G}_1) \\ \varphi^*(y') & (\varphi^*(\omega'), \varphi^*(g')) \end{pmatrix}, \\
\text{ch}_{T,0}^p(\mathcal{F}_0) &= \begin{pmatrix} 0 & (-1)^{r+1} \text{ch}_{T,1}^p(\mathcal{F}_0, \varphi^* \mathcal{G}_0) \\ \varphi^*(y + y') & (\varphi^*(\omega + \omega'), \varphi^*(g + g')) \end{pmatrix} - a(\varphi^* \text{ch}_1^p(\mathcal{E}')).
\end{align*}
\]

Lem.6.13 says that \( \varphi^* \text{ch}_1^p(\mathcal{E}') = (-1)^r \text{ch}_{T,1}^p(\varphi^* \mathcal{E}) \), therefore
\[ \text{ch}_{T,0}^p(\mathcal{F}_{-1}) + \text{ch}_{T,0}^p(\mathcal{F}_1) - \text{ch}_{T,0}^p(\mathcal{F}_0) \\
= (-1)^r a \left( \text{ch}_{T,1}^p(\mathcal{F}_{-1}, \varphi^* \mathcal{G}_{-1}) + \text{ch}_{T,1}^p(\mathcal{F}_1, \varphi^* \mathcal{G}_1) - \text{ch}_{T,1}^p(\mathcal{F}_0, \varphi^* \mathcal{G}_0) + \text{ch}_{T,1}^p(\varphi^* \mathcal{E}') \right). \]

Consider the following exact hermitian 2-cube on \( T \):
\[
\begin{array}{ccc}
\mathcal{E}' & \to & \mathcal{F}_0 \\
\downarrow & & \downarrow \\
\varphi^* \mathcal{G}_{-1} & \to & \varphi^* \mathcal{G}_0 \\
\varphi^* \mathcal{G}_1 & \to & \varphi^* \mathcal{G}_1
\end{array}
\]
Then
\[ d_s \operatorname{ch}_T^p(\mathcal{C}) = (-1)^r \delta_p \operatorname{ch}_T^p(\mathcal{C}) \]
\[ = (-1)^r \left( \operatorname{ch}_{T,1}^p(\mathcal{E}) - \operatorname{ch}_{T,1}^p(\varphi^*\mathcal{E}) - \operatorname{ch}_{T,1}^p(\mathcal{F}_{-1}, \varphi^*\mathcal{G}_{-1}) \right. \]
\[ \left. + \operatorname{ch}_{T,1}^p(\mathcal{F}_0, \varphi^*\mathcal{G}_0) - \operatorname{ch}_{T,1}^p(\mathcal{F}_1, \varphi^*\mathcal{G}_1) \right). \]

This implies that
\[ \widehat{\operatorname{ch}}_{T,0}^p(\mathcal{F}_{-1}) + \widehat{\operatorname{ch}}_{T,0}^p(\mathcal{F}_1) - \widehat{\operatorname{ch}}_{T,0}^p(\mathcal{F}_0) = (-1)^r a(\widehat{\operatorname{ch}}_{T,1}^p(\mathcal{E})), \]
which completes the proof. \(\square\)

**Proposition 7.2.** Suppose \( r \geq 1 \). Define a map
\[ \partial' : \widehat{Z}^p(X, r)^*_0 / \text{Im } \partial \to \widetilde{D}^{2p-r}_{\hat{A}, P}(X, p) \]
so that
\[ \partial x = \left( \begin{array}{cc} 0 & \partial' x \\ 0 & 0 \end{array} \right) \]
for any \( x \in \widehat{Z}^p(X, r)^*_0 \). Then the diagram
\[ \begin{array}{ccc}
\hat{K}_0^M(T) & \xrightarrow{\widehat{\operatorname{ch}}_{T,0}^p} & \widehat{Z}^p(X, r)^*_0 / \text{Im } \partial \\
\downarrow{\operatorname{ch}}_{T,0}^p & & \downarrow{\partial'} \\
\widetilde{D}^{2p-r}_{\hat{A}, P}(X, p) & & 
\end{array} \]
is commutative. Hence taking the kernels of \( \operatorname{ch}_{T,0}^p \) and \( \partial' \) yields the map
\[ \widehat{\operatorname{ch}}_{T,0}^p : \hat{K}_0(T) \to \widetilde{CH}^p(X, r). \]

**Proof:** Let \( (\mathcal{F}, \tilde{\eta}) \) be a pair of a hermitian vector bundle \( \mathcal{F} \) on \( T \) with \( \tilde{\eta} \in \widetilde{D}_{\hat{A}, P, r+1}(X) / \text{Im } d_s \). Then we have seen in (6.34) that
\[ \partial \widehat{\operatorname{ch}}_{T,0}^p(\mathcal{F}) = \left( \begin{array}{cc} 0 & \operatorname{ch}_{T,0}^p(\mathcal{F}) \\ 0 & 0 \end{array} \right). \]
Hence
\[ \partial \widehat{\operatorname{ch}}_{T,0}^p(\mathcal{F}, \tilde{\eta}) = \left( \begin{array}{cc} 0 & \operatorname{ch}_{T,0}^p(\mathcal{F}) + d_s \eta \\ 0 & 0 \end{array} \right), \]
which completes the proof. \(\square\)

**Theorem 7.3.** There is a map
\[ \widehat{\operatorname{ch}}_{r}^p : \hat{K}_r(X) \to \widetilde{CH}^p(X, r) \]
which makes the following diagram commutative:

\[
\begin{array}{ccc}
\hat{K}_r(X)_Q & \xrightarrow{\sim} & \hat{K}_0(X \times \Box^r; X \times \partial\Box^r)_Q \\
\hat{\chi}_r^p & \downarrow & \hat{\chi}_r^p \\
\hat{C}H^p(X, r). & \xrightarrow{\hat{\chi}_{T,0}} & \hat{C}H^p(X, r).
\end{array}
\]

We call this map the higher arithmetic Chern character of \(X\).

**Proof:** We have only to show that the kernel of the surjection \(\hat{i}_0^*\) goes to zero by the map

\[
\hat{K}_0(T; T_1, \ldots, T_r)_Q \subset \hat{K}_0(T)_Q \xrightarrow{\hat{\chi}_r^p} \hat{C}H^p(X, r).
\]

Recall the commutative diagram in Prop.5.4. The exactness of the sequence

\[
K_1(X \times \Box^r; X \times \partial\Box^r)_Q \to \bigoplus_p H^{2p-r-1}_D(X, \mathbb{R}(p)) \to \hat{K}_0(X \times \Box^r; X \times \partial\Box^r)_Q
\]

with the bijectivity of \(i_0^*: K_0(T; T_1, \ldots, T_r) \to K_0(X \times \Box^r; X \times \partial\Box^r)_Q\) implies that \(\text{Ker} \hat{i}_0^*\) agrees with the image of the composite

\[
K_1(X \times \Box^r; X \times \partial\Box^r)_Q \to \bigoplus_p H^{2p-r-1}_D(X, \mathbb{R}(p)) \to \hat{K}_0(T; T_1, \ldots, T_r)_Q.
\]

The commutative diagram in Cor.3.9 and Thm.3.4 imply that any element of \(\text{Ker} \hat{i}_0^*\) is written as \([(0, \tilde{\eta})]\) such that the element \(\tilde{\eta} \in \widetilde{D}_{h, r+1}(X)/\text{Im } d_s\) in contained in the image of Beilinson’s regulator

\[
K_{r+1}(X)_Q \to \bigoplus_p H^{2p-r-1}_D(X, \mathbb{R}(p)) \subset \widetilde{D}_{h, r+1}(X)/\text{Im } d_s.
\]

On the other hand, recall the exact sequence

\[
\hat{C}H^p(X, r + 1) \xrightarrow{H^{r+1}(\mathbb{P}^p)} H^{2p-r-1}_D(X, \mathbb{R}(p)) \to \hat{C}H^p(X, r),
\]

which is shown in [BF Prop.4.4]. Since \(H^{r+1}(\mathbb{P}^p)\) agrees with the regulator map by [BFT Thm.7.8], we conclude that \(\hat{\chi}_{T,0}^p([(0, \tilde{\eta})]) = a(\tilde{\eta})^p\) is zero in \(\hat{C}H^p(X, r)\) if \([(0, \tilde{\eta})] \in \text{Ker} \hat{i}_0^*\), which completes the proof. \(\Box\)
\textbf{Theorem 7.4.} There is a commutative diagram up to sign:

\[
\begin{array}{cccc}
\cdots & \oplus H^{2p-r}_{D}(X, \mathbb{R}(p)) & \rightarrow & K_r(X)_{Q} \\
\downarrow \text{pr} & \downarrow \text{pr} & \downarrow \text{ch}^p_r & \\
\cdots & H^{2p-r}_{D}(X, \mathbb{R}(p)) & \rightarrow & CH^p(X, r) \\
\text{ch}^p_r & \oplus H^{2p-r}_{D}(X, \mathbb{R}(p)) & \rightarrow & \cdots K_1(X)_{Q} \\
\downarrow \text{pr} & \downarrow \text{pr} & \downarrow \text{ch}^p_r & \\
CH^p(X, r) & H^r_{(p^p)} & \rightarrow & CH^p(X, r) & \rightarrow 0 \\
\end{array}
\]

where \(\text{ch}^p_r : K_r(X)_{Q} \rightarrow CH^p(X, r)\) is the map defined in Def.6.11, and \(\text{pr}\) is the canonical projection.

\textbf{Proof:} The commutativity of the diagram is straightforward except (*) and (**), and by Cor.4.7 the commutativity of them follows from the commutativity of the diagram

\[
\begin{array}{cccc}
K_0(T) & \text{ch}^p_{T,0} & \rightarrow & CH^p(X, r) \\
\downarrow \text{ch}^p_{T,0} & \\
H^{2p-r}_{D}(X, \mathbb{R}(p)) & & & \\
\end{array}
\]

Let \(\mathcal{F}\) be a hermitian vector bundle on \(T\). Take a morphism \(\varphi : T \rightarrow G\) and a vector bundle \(\mathcal{G}\) on \(G\) with an isomorphism \(\mathcal{F} \simeq \varphi^*\mathcal{G}\). Moreover, let \(y \in Z^p_{\varphi}(G)\) be a cycle representing \(\text{ch}^p_0(\mathcal{G})\) in \(CH^p(G)\). Then \(\text{ch}^p_{T,0}(\mathcal{F}) = [\varphi^*(y)]\) in \(CH^p(X, r)\).

Put a hermitian metric on \(\mathcal{G}\), and take the pull-back metric on \(\mathcal{F}\) by means of the isomorphism \(\mathcal{F} \simeq \varphi^*\mathcal{G}\). Denote by \(\mathcal{G}\) and \(\mathcal{F}\) the hermitian vector bundles obtained in this way. Let \((\omega, \widehat{G}) \in GE^p_{\varphi}(G)\) be a Green form associated with \(y\) such that \(\text{ch}^p_0(\mathcal{G})\) is represented by \((y, (\omega, \widehat{G}))\). Then since \(d_Dg = \omega\) and \(\delta_A\varphi^*(g) = 0\), it follows from (6.3) that

\[
d_D\pi_X[\varphi^*(g) \cdot \pi^p_{\varphi}W_r] = \pi_X[\varphi^*(\omega) \cdot \pi^p_{\varphi}W_r] - \mathcal{P}(\varphi^*(y)).
\]

Since \(\omega = \text{ch}^p_0(\mathcal{G})\), it follows that \(\varphi^*(\omega) = \text{ch}^p_{T,0}(\mathcal{F})\), therefore

\[
\pi_X[\varphi^*(\omega) \cdot \pi^p_{\varphi}W_r] = \kappa_A(\text{ch}^p_{T,0}(\mathcal{F})).
\]

This means that \(\mathcal{P}(\varphi^*(y))\) and \(\kappa_A(\text{ch}^p_{T,0}(\mathcal{F}))\) give the same cohomology class in \(H^{2p-r}_{D}(X, \mathbb{R}(p))\). Since \(\kappa_A\) induces the identity on cohomology, we conclude that \(\mathcal{P}(\varphi^*(y))\) and \(\text{ch}^p_{T,0}(\mathcal{F})\) give the same cohomology class in \(H^{2p-r}_{D}(X, \mathbb{R}(p))\). This completes the proof. \(\square\)
8. Compatibility with pull-back maps

8.1. Pull-back maps on arithmetic $K$-groups. Let $f : X \to Y$ be a morphism of smooth projective varieties defined over an arithmetic field. Consider the pull-back map

$$f^* : \widetilde{QC}_*^{\text{Alt}}(Y \times \square^r; Y \times \partial \square^r) \to \widetilde{QC}_*^{\text{Alt}}(X \times \square^r; X \times \partial \square^r)$$

defined in Prop.2.19. Prop.3.7 says that $(f^*)^{m,n}(x)$ is isometrically equivalent to a degenerate element for $m < n$. Hence the diagram

$$\begin{array}{ccc}
\widetilde{QC}_*^{\text{Alt}}(Y \times \square^r; Y \times \partial \square^r)[r] & \xrightarrow{\text{ch}_*} & \widetilde{D}_{\text{K},*}(Y) \\
\downarrow f^* & & \downarrow f^* \\
\widetilde{QC}_*^{\text{Alt}}(X \times \square^r; X \times \partial \square^r)[r] & \xrightarrow{\text{ch}_*} & \widetilde{D}_{\text{K},*}(X)
\end{array}$$

is commutative. This diagram gives the pull-back map

$$\hat{f}^* : \hat{K}_0(Y \times \square^r; Y \times \partial \square^r)_Q \to \hat{K}_0(X \times \square^r; X \times \partial \square^r)_Q.$$

For two morphisms $f : X \to Y$ and $g : Y \to Z$, we have given in Prop.2.19 a homotopy $\Phi$ from $(gf)^*$ to $g^*f^*$. Prop.3.7 says that $\Phi^{m,n}(x)$ is isometrically equivalent to a degenerate element for any $m$ and $n$. This implies that $\hat{g}^* \hat{f}^* = \hat{f}^* \hat{g}^*$. Moreover, Prop.2.20 says that the identity morphism of $X$ induces the identity of $\widetilde{QC}_*^{\text{Alt}}(X \times \square^r; X \times \partial \square^r)$, which implies that $\hat{1}_X^* = \text{Id}$. Since the alternating part of the commutative diagram in Cor.3.9

$$\begin{array}{ccc}
\widetilde{QC}_*^{\text{Alt}}(X) & \xrightarrow{i_X} & \widetilde{QC}_*^{\text{Alt}}(X \times \square^r; X \times \partial \square^r)[r] \\
\downarrow \text{ch}_* & & \downarrow \text{ch}_* \\
\widetilde{D}_{\text{K},*}(X) & \rightarrow & \widetilde{D}_{\text{K},*}(X)
\end{array}$$

is compatible with the pull back maps $f^*$, the diagram

$$\begin{array}{ccc}
\hat{K}_r(Y)_Q & \xrightarrow{\hat{f}^*} & \hat{K}_{r,r}(Y)_Q & \xrightarrow{\hat{f}^*} & \hat{K}_0(Y \times \square^r; Y \times \partial \square^r)_Q \\
\downarrow \hat{f}^* & & \downarrow \hat{f}^* & & \downarrow \hat{f}^* \\
\hat{K}_r(X)_Q & \xrightarrow{\hat{f}^*} & \hat{K}_{r,r}(X)_Q & \xrightarrow{\hat{f}^*} & \hat{K}_0(X \times \square^r; X \times \partial \square^r)_Q
\end{array}$$

is commutative.

Consider the iterated doubles $T = (X \times \square^r; X \times \partial \square^r)$ and $U = D(Y \times \square^r; Y \times \partial \square^r)$. Then $f$ induces a morphism $f_D : T \to U$. It is obvious from the definition that the Chern form $\text{ch}_{U,0}(\mathcal{F}) \in \widetilde{D}_{\text{K},*,r}(Y)$ of a hermitian vector bundle $\mathcal{F}$ on $U$ satisfies $f^* \text{ch}_{U,0}(\mathcal{F}) = \text{ch}_{T,0}(f_D^* \mathcal{F})$, and that the Bott-Chern form $\text{ch}_{U,1}(\mathcal{E}) \in \widetilde{D}_{\text{K},*,r+1}(Y)$ of a short exact sequence $\mathcal{E}$ of hermitian vector bundles on $U$ satisfies $f^* \text{ch}_{U,1}(\mathcal{E}) = \text{ch}_{T,1}(f_D^* \mathcal{E})$. Hence we can define pull-back map of arithmetic $K$-groups

$$\hat{f}_D^* : \hat{K}_0^M(U) \to \hat{K}_0^M(T)$$
by $[(\mathcal{F}, \eta)] \mapsto [(f_D^* \mathcal{F}, f_D^* (\eta))]$. Let $T_1, \ldots, T_r \subset T$ and $U_1, \ldots, U_r \subset U$ be the closed subschemes introduced in §4.2, and 

$$
\iota_j : U_j \hookrightarrow U, \quad \iota_j : T_j \hookrightarrow T, \quad p_j : U \to U_j, \quad p_j : T \to T_j
$$

the morphisms defined in §4.2. It is obvious that the pull-back map defined above induces

$$
\tilde{f}_D^* : \tilde{K}_0^M(U; U_1, \ldots, U_r) \to \tilde{K}_0^M(T; T_1, \ldots, T_r),
$$

$$
\tilde{f}_D : \tilde{K}_0(U) \to \tilde{K}_0(T),
$$

$$
\tilde{f}_D^* : \tilde{K}_0(U; U_1, \ldots, U_r) \to \tilde{K}_0(T; T_1, \ldots, T_r).
$$

For $1 \leq j \leq r$, consider the diagram

$$
\begin{array}{ccc}
\tilde{Q}\tilde{C}_s^{\text{Alt}}(U; U_1, \ldots, U_{j-1}) & \xrightarrow{i_j^*} & \tilde{Q}\tilde{C}_s^{\text{Alt}}(U; U_1 \cap U_j, \ldots, U_{j-1} \cap U_j) \\
\downarrow f_D & & \downarrow f_D \\
\tilde{Q}\tilde{C}_s^{\text{Alt}}(T; T_1, \ldots, T_{j-1}) & \xrightarrow{i_j^*} & \tilde{Q}\tilde{C}_s^{\text{Alt}}(T; T_1 \cap T_j, \ldots, T_{j-1} \cap T_j).
\end{array}
$$

Then Prop.2.19 implies that the family of maps

$$
\Phi_i^{m,n}(x)_J = (-1)^n \sum_{K \mid I = J} \text{sgn}(K) \left( \tilde{\Xi}_{K,i,j}^{\text{Alt}}(x_I) - \tilde{\Xi}_{K,f_D,i,j}^{\text{Alt}}(x_I) \right)
$$

for any $m$ and $n$ gives a homotopy

$$
\Phi_i : \tilde{Q}\tilde{C}_s^{\text{Alt}}(U; U_1, \ldots, U_{j-1}) \to \tilde{Q}\tilde{C}_s^{\text{Alt}}(T; T_1 \cap T_j, \ldots, T_{j-1} \cap T_j)
$$

from $f_D^{i_j}f_D^*$ to $i_j^*f_D^*$. The diagram

$$
\begin{array}{ccc}
\tilde{Q}\tilde{C}_s^{\text{Alt}}(U; U_1 \cap U_j, \ldots, U_{j-1} \cap U_j) & \xrightarrow{p_j^*} & \tilde{Q}\tilde{C}_s^{\text{Alt}}(U; U_1, \ldots, U_{j-1}) \\
\downarrow f_D & & \downarrow f_D \\
\tilde{Q}\tilde{C}_s^{\text{Alt}}(T; T_1 \cap T_j, \ldots, T_{j-1} \cap T_j) & \xrightarrow{p_j^*} & \tilde{Q}\tilde{C}_s^{\text{Alt}}(T; T_1, \ldots, T_{j-1}),
\end{array}
$$

is also commutative up to homotopy, and a homotopy

$$
\Phi_p : \tilde{Q}\tilde{C}_s^{\text{Alt}}(U; U_1 \cap U_j, \ldots, U_{j-1} \cap U_j) \to \tilde{Q}\tilde{C}_s^{\text{Alt}}(T; T_1, \ldots, T_{j-1})
$$

from $f_D^{i_j}p_j^*$ to $p_j^*f_D^*$ is given by

$$
\Phi_p^{m,n}(x)_J = (-1)^n \sum_{K \mid I = J} \text{sgn}(K) \left( \tilde{\Xi}_{K,p,j}^{\text{Alt}}(x_I) - \tilde{\Xi}_{K,f_D,p,j}^{\text{Alt}}(x_I) \right).
$$

Moreover, let

$$
\Psi_U : \tilde{Q}\tilde{C}_s^{\text{Alt}}(U; U_1 \cap U_j, \ldots, U_{j-1} \cap U_j) \to \tilde{Q}\tilde{C}_s^{\text{Alt}}(U; U_1 \cap U_j, \ldots, U_{j-1} \cap U_j),
$$

$$
\Psi_T : \tilde{Q}\tilde{C}_s^{\text{Alt}}(T; T_1 \cap T_j, \ldots, T_{j-1} \cap T_j) \to \tilde{Q}\tilde{C}_s^{\text{Alt}}(T; T_1 \cap T_j, \ldots, T_{j-1} \cap T_j)
$$

be the homotopies from the identity to $i_j^*p_j^*$ given in Prop.2.20. Then we have the following:
Proposition 8.1. There is a second homotopy

$$\Theta : \tilde{Q}\tilde{C}_*^{\text{Alt}}(U_j; U_1 \cap U_j, \ldots, U_{j-1} \cap U_j) \to \tilde{Q}\tilde{C}_*^{\text{Alt}}(T_j; T_1 \cap T_j, \ldots, T_{j-1} \cap T_j)$$

from $\Phi_4 p^* + \iota_j^* \Phi_1 + f_2^* \Psi_U$ to $\Psi_T f^*_{I}$ in the sense of Def.2.7.

Proof: We begin by introducing a map of chain complexes of exact cubes associated with a sequence of morphisms

$$(X_1; Y_1, 1, \ldots, Y_{r-1}, Y_r) \xrightarrow{f_1} (X_2; Y_2, 1, \ldots, Y_{r-1}, Y_r) \xrightarrow{f_2} (X_3; Y_3, 1, \ldots, Y_{r-1}, Y_r) \xrightarrow{f_3} (X_4; Y_4, 1, \ldots, Y_{r-1}, Y_r).$$

For $1 \leq i \leq 4$ and $J \subset \{1, \ldots, r\}$, with $k \notin J$, denote by $\iota_k$ the embedding $Y_i \hookrightarrow Y_i$. Define $\Xi_{K,f_1,f_2,f_3} : \tilde{Q}\tilde{C}_*(Y_{i,1}) \to \tilde{Q}\tilde{C}_{*+n-m+2}(Y_{i,1})$ by

$$\Xi_{K,f_1,f_2,f_3} = \sum_{0 \leq p \leq q \leq r \leq n-m} (-1)^{p+q+r} \sum_{\sigma \in S_{n-m}} (\text{sgn} \sigma)(\ldots, \iota_{k_{\sigma(p)}}, f_1, \ldots, \iota_{k_{\sigma(q)}}, f_2, \ldots, \iota_{k_{\sigma(r)}}, f_3, \ldots)^*.$$

Then we can show in the same way as the proof of Prop.2.12 that

$$(8.1) \quad \partial \Xi_{K,f_1,f_2,f_3}(x) + (-1)^{n-m+1} \Xi_{K,f_1,f_2,f_3}(\partial x)$$

$$= - \sum_{a=1}^{n-m} \sum_{[L]=a} \text{sgn}(L,L') \Xi_L \Xi_{L',f_1,f_2,f_3}(x)$$

$$+ \sum_{a=0}^{n-m} (-1)^a \sum_{[L]=a} \text{sgn}(L,L') \Xi_{L,f_1} \Xi_{L',f_2,f_3}(x)$$

$$- \sum_{a=0}^{n-m} \sum_{[L]=a} \text{sgn}(L,L') \Xi_{L,f_1,f_2} \Xi_{L',f_3}(x)$$

$$+ \sum_{a=0}^{n-m-1} (-1)^a \sum_{[L]=a} \text{sgn}(L,L') \Xi_{L,f_1,f_2,f_3} \Xi_{L'}(x)$$

$$- \Xi_{K,f_1,f_2,f_3}(x) + \Xi_{K,f_1,f_3}(x).$$

Denote $\Xi_{K,f_1,f_2,f_3}^{\text{alt}} = \text{Alt}_* \Xi_{K,f_1,f_2,f_3} : \tilde{C}_*^{\text{Alt}}(Y_{i,1}) \to \tilde{C}_*^{\text{Alt}}_{*+n-m+2}(Y_{i,1}).$

Let us go back to the proof of the proposition. Define a map of $\mathcal{C}$-complexes

$$\Theta : \tilde{Q}\tilde{C}_*^{\text{Alt}}(U_j; U_1 \cap U_j, \ldots, U_{j-1} \cap U_j) \to \tilde{Q}\tilde{C}_*^{\text{Alt}}(T_j; T_1 \cap T_j, \ldots, T_{j-1} \cap T_j)$$

by

$$\Theta^{m,n}(x) = - \sum_{K \mid I=J} \text{sgn}(K,I) \left( \Xi_{K,I,D,j,p_j}(x_I) - \Xi_{K,I,J,p}(x_I) + \Xi_{K,I,J,p}(x_I) \right).$$
Note that $\Xi^{alt}_{D,Id} = 0$, which we can show in the same way as Prop.2.20. Using the equality (8.1), we have

$$(-1)^n \partial \Theta^{m,n}(x)_j - (-1)^m \Theta^{m,n}(\partial x)_j$$

$$= - \sum_{a=1}^{n-m} F^{n-a,n}_\theta \Theta^{m,n-a}(x)_j + \sum_{a=0}^{n-m-1} \Theta^{n-a,n} F^{m,n-a}(x)_j$$

$$- \sum_{a=0}^{n-m} (f^*_D)^{n-a,n} \Psi^{m,n-a}_U(x)_j - \sum_{a=0}^{n-m} (t^*_j)^{n-a,n} \Psi^{m,n-a}_T(x)_j$$

$$- \sum_{a=0}^{n-m} \phi^{n-a,n}_i (p^*_j)^{m,n-a}(x)_j + \sum_{a=0}^{n-m} \psi^{n-a,n}_T (f^*_D)^{m,n-a}(x)_j,$$

which says that $\Theta$ is a second homotopy from $\Phi_i p^*_j + t^*_j \phi^*_p + f^*_D \psi_U$ to $\psi_T f^*_D$. □

It follows from Prop.8.1 and Prop.2.8 that the diagram

$$\begin{array}{ccc}
\tilde{Q}C^*_{\text{Alt}}(U;U_1,\ldots,U_{j-1}) & \xrightarrow{t_j} & \tilde{Q}C^*_{\text{Alt}}(U;U_1,\ldots,U_j) \\
\downarrow f^*_D & & \downarrow f^*_D \\
\tilde{Q}C^*_{\text{Alt}}(T;T_1,\ldots,T_{j-1}) & \xrightarrow{t_j} & \tilde{Q}C^*_{\text{Alt}}(T;T_1,\ldots,T_j)
\end{array}$$

(8.2)

is commutative up to homotopy. Denote by $\Pi_j$ the homotopy from $f^*_D t_j$ to $t_j f^*_D$ given in Prop.2.8. It is obvious that $\Phi^*_i p^*_j(x), \Phi^*_p m,n(x), \Psi^{m,n}_U(x), \Psi^{m,n}_T(x)$ and $\Theta^{m,n}(x)$ are isometrically equivalent to degenerate elements for any $x \in \oplus \tilde{Q}C^*_{\text{Alt}}(U_I)$. Hence $\Pi_j^{m,n}(x)$ is also isometrically equivalent to a degenerate element. Connecting the diagram (8.2) for all $j$, we obtain the following diagram

$$\begin{array}{ccc}
\tilde{Q}C^*_{\text{Alt}}(U) & \xrightarrow{t} & \tilde{Q}C^*_{\text{Alt}}(U;U_1,\ldots,U_r) \\
\downarrow f^*_D & & \downarrow f^*_D \\
\tilde{Q}C^*_{\text{Alt}}(T) & \xrightarrow{t} & \tilde{Q}C^*_{\text{Alt}}(T;T_1,\ldots,T_r)
\end{array}$$

which is commutative up to homotopy, and a homotopy from $f^*_D t$ to $t f^*_D$ is given by

$$\Pi = \sum_{j=1}^n t_n \cdots t_{j+1} \Pi_j t_{j-1} \cdots t_1.$$ 

It is obvious that $\Pi^{0,n}(x)$ is isometrically equivalent to a degenerate element for $x \in \tilde{Q}C^*_{\text{Alt}}(U)$.

By Prop.2.19 the diagram

$$\begin{array}{ccc}
\tilde{Q}C^*_{\text{Alt}}(U;U_1,\ldots,U_r) & \xrightarrow{i^*_\hat{\phi}} & \tilde{Q}C^*_{\text{Alt}}(Y \times \square^r; Y \times \partial \square^r) \\
\downarrow f^*_D & & \downarrow f^*_\hat{\phi} \\
\tilde{Q}C^*_{\text{Alt}}(T;T_1,\ldots,T_r) & \xrightarrow{i^*_\hat{\phi}} & \tilde{Q}C^*_{\text{Alt}}(X \times \square^r; X \times \partial \square^r)
\end{array}$$
is commutative up to homotopy, and a homotopy from $f^*i_0$ to $i_0^*f_D^*$ is given by $\Psi = \Phi_{i_0^*f_D} - \Phi_{f^*i_0}$. Then $\Psi^{m,n}(x)$ is isometrically equivalent to a degenerate element for any $x \in \mathbb{Q}C^\text{Alt}_{s}(U_I)$. Hence the diagram

$$
\begin{array}{ccc}
\mathbb{Q}C^\text{Alt}_{s}(U) & \xrightarrow{i_0^*f_D} & \mathbb{Q}C^\text{Alt}_{s}(Y \times \Box^r; Y \times \partial \Box^r) \\
\downarrow f_D & & \downarrow f^* \\
\mathbb{Q}C^\text{Alt}_{s}(T) & \xrightarrow{i_0^*f_D} & \mathbb{Q}C^\text{Alt}_{s}(X \times \Box^r; X \times \partial \Box^r)
\end{array}
$$

is also commutative up to homotopy, and a homotopy from $f^*i_0^*t$ to $i_0^*f_D^*$ is given by $\Pi' = \Psi t + i_0^*\Pi$. It is obvious that $\Pi'^{0,n}(x)$ is isometrically equivalent to a degenerate element for any $x \in \mathbb{Q}C^\text{Alt}_{s}(U)$. In particular, $\text{ch}_s(\Pi'^{0,n}(x)) = 0$.

Consider the diagram:

$$
\begin{array}{ccc}
\tilde{K}_0(U)_Q & \xrightarrow{i_0^*t} & \tilde{K}_0(Y \times \Box^r; Y \times \partial \Box^r)_Q \\
\downarrow \tilde{f}_D & & \downarrow \tilde{f}^* \\
\tilde{K}_0(T)_Q & \xrightarrow{i_0^*t} & \tilde{K}_0(X \times \Box^r; X \times \partial \Box^r)_Q,
\end{array}
$$

(8.3)

where the maps $i_0^*t$ are defined in (5.4). Let $\mathcal{F}$ be a virtual hermitian vector bundle on $U$ and $\eta \in \mathcal{D}_{\mathbb{A},r+1}(Y)$ such that $\text{ch}_{T,0}(\mathcal{F}) + d_s\eta = 0$. Then

$$
\begin{align*}
\tilde{f}^*i_0^*t(\mathcal{F}, \eta) & = [(f^*i_0^*t(\mathcal{F}), -f^*(\eta))], \\
i_0^*t \tilde{f}_D^*(\mathcal{F}, \eta) & = [(i_0^*f_D^*t(\mathcal{F}), -f_D^*(\eta))].
\end{align*}
$$

On the other hand, since $\text{ch}_s(\Pi'^{0,n}(x)) = 0$ for any $x \in \mathbb{Q}C^\text{Alt}_{s}(U)$,

$$
\partial(\Pi'(\mathcal{F}), 0) = (f_D^*i_0^*t(\mathcal{F}) - i_0^*f_D^*t(\mathcal{F}), 0)
$$

in $s(\text{ch}_s)_r$. This means that the diagram (8.3) is commutative. Restricting (8.3) to the relative $K$-theories, we have the commutative diagram

$$
\begin{array}{ccc}
\tilde{K}_0(U; U_1, \ldots, U_r)_Q & \xrightarrow{i_0^*t} & \tilde{K}_0(Y \times \Box^r; Y \times \partial \Box^r)_Q \\
\downarrow \tilde{f}_D & & \downarrow \tilde{f}^* \\
\tilde{K}_0(T; T_1, \ldots, T_r)_Q & \xrightarrow{i_0^*t} & \tilde{K}_0(X \times \Box^r; X \times \partial \Box^r)_Q.
\end{array}
$$

To Sum up, we obtain the following proposition:
Proposition 8.2. The diagram

\[
\begin{array}{ccl}
\hat{K}_r(Y)_Q & \sim & \hat{K}_0(Y \times \square^r; Y \times \partial \square^r)_Q \\
\| & \| & \| \\
\hat{K}_r(X)_Q & \sim & \hat{K}_0(X \times \square^r; X \times \partial \square^r)_Q \\
\end{array}
\]

is commutative.

8.2. The main theorem. We begin by recalling the pull-back map of higher arithmetic Chow groups [BF]. Let \( f : X \to Y \) be a morphism of smooth projective varieties defined over an arithmetic field. Let \( Z^p_f(Y,*) \) be the subcomplex of \( Z^p(Y,*) \) such that \( y \in Z^p_f(Y,*) \) if and only if one can take the pull-back cycle \( f^* (y) \in Z^p(X,*) \). Then the moving lemma says that the inclusion \( Z^p_f(Y,*)_0 \hookrightarrow Z^p(Y,*)_0 \) is a quasi-isomorphism. We can define complexes \( H^p_f(Y,*)_0 \) and \( D^*_p, \bar{\partial}_f(Y,*,p)_0 \) in the same way as in §6.3. Let \( \hat{Z}^p_f(Y,*)_0 \) be the simple complex of the diagram

\[
\begin{array}{ccc}
& & \\
& & \\
& & \\
\end{array}
\]

Then the natural inclusion

\[
\hat{Z}^p_f(Y,*)_0 \hookrightarrow \hat{Z}^p(Y,*)_0
\]

is a quasi-isomorphism, and collecting the pull-back maps on the complexes in (8.4) yields the map

\[
f^*: \hat{Z}^p_f(Y,*)_0 \to \hat{Z}^p(X,*)_0.
\]

Taking the maps on homology we obtain the pull-back map on higher arithmetic Chow groups:

\[
f^*: \hat{CH}^p(Y,r) \sim H_r(\hat{Z}^p_f(Y,*)_0) \xrightarrow{f^*} \hat{CH}^p(X,r).
\]

Theorem 8.3. The diagram

\[
\begin{array}{ccc}
\hat{K}_r(Y)_Q & \sim & \hat{K}_0(Y \times \square^r; Y \times \partial \square^r)_Q \\
\| & \| & \| \\
\hat{K}_r(X)_Q & \sim & \hat{K}_0(X \times \square^r; X \times \partial \square^r)_Q \\
\end{array}
\]

is commutative.
Proof: Substituting the complex $\sigma_{<2p-r} \bar{D}_{k,p}^2(Y,p)_{0}$ for $\bar{D}_{k,p}^2(Y,p)_{0}$ in the diagram (8.4), we obtain a map

\[ \hat{f}^* : \bar{Z}^p(Y,r)_{0}/\text{Im} \partial \sim \bar{Z}^p_{f}(Y,r)_{0}/\text{Im} \partial \]

which is an extension of (8.3). Then the definition of $\hat{\text{ch}}_p$ and Prop.8.2 imply that the theorem follows from the commutativity of the diagram

\[ \begin{array}{ccc}
\hat{K}_0^M(U) & \xrightarrow{\hat{\text{ch}}^p_{U,0}} & \bar{Z}^p(Y,r)_{0}/\text{Im} \partial \\
\hat{f}_D & & \hat{f}^* \\
\hat{K}_0^M(T) & \xrightarrow{\hat{\text{ch}}^p_{T,0}} & \bar{Z}^p(X,r)_{0}/\text{Im} \partial,
\end{array} \]

and it is equivalent to that $\hat{f}^* \hat{\text{ch}}^p_{U,0}(\mathcal{F}) = \hat{\text{ch}}^p_{T,0}(f_D^* \mathcal{F})$ for any hermitian vector bundle $\mathcal{F}$ on $U$.

Take a morphism $\varphi : U \to G$ to a smooth projective variety and a hermitian vector bundle $\mathcal{G}$ on $G$ with an isomorphism $\varphi^* \mathcal{G} \simeq \mathcal{F}$. Let $Z^p_{f,\varphi}(G)$ be the subgroup of $Z^p(G)$ such that $z \in Z^p_{f,\varphi}(G)$ if and only if one can define the pull-back cycle of $z$ by the morphism

\[ Y \times D_J \hookrightarrow Y \times \square^r \varphi^* \mathcal{G}, \]

and also by the morphism

\[ X \times D_J \hookrightarrow X \times \square^r \xrightarrow{f^* \varphi^*} Y \times \square^r \varphi^* \mathcal{G} \]

for any $I, J \subset \{1, \ldots, r\}$. Let $(y, (\omega, \tilde{g}))$ be a pair of $y \in Z^p_{f,\varphi}(G)$ and a Green form $(\omega, \tilde{g}) \in G\mathcal{L}_{f,\varphi}^p(G)$ associated with $y$ representing $\hat{\text{ch}}^p_{U,0}(\mathcal{G}) \in \hat{\text{CH}}^p(G, \mathcal{D}(E_{\log}^*)).$ Then

\[ \hat{\text{ch}}^p_{U,0}(\mathcal{F}) = \begin{bmatrix}
0 & \varphi^* (y) \\
(-1)^{r+1} \chi^p_{U,1}(\mathcal{F}, \varphi^* \mathcal{G}) & \varphi^* (\omega), \varphi^* (g)
\end{bmatrix}, \]

which is an element of $\bar{Z}^p(Y,r)_{0}/\text{Im} \partial$. Since $f_D^* \varphi^* \mathcal{G} \simeq f_D^* \mathcal{F}$,

\[ \hat{f}^* \hat{\text{ch}}^p_{U,0}(\mathcal{F}) = \begin{bmatrix}
0 & f^* \varphi^* (y) \\
(-1)^{r+1} \chi^p_{T,1}(f_D^* \mathcal{F}, f_D^* \varphi^* \mathcal{G}) & f^* \varphi^* (\omega), f^* \varphi^* (g)
\end{bmatrix}, \]

which completes the proof. \qed

9. A TENSOR PRODUCT STRUCTURE ON THE MULTI-RELATIVE COMPLEXES OF EXACT CUBES

9.1. An exact cube $(\mathcal{F}_0, \ldots, \mathcal{F}_l)$. Let $\mathfrak{A}$ be a small exact category. For a sequence of isomorphisms $\mathcal{F}_0 \simeq \mathcal{F}_1 \simeq \cdots \simeq \mathcal{F}_l$ of objects of $\mathfrak{A}$, define an exact $l$-cube $(\mathcal{F}_0, \ldots, \mathcal{F}_l)$ of
\(A\) as follows: If \(\alpha_j = 1\) for some \(j\), then
\[
\langle F_0, \ldots, F_l \rangle_{\alpha_1, \ldots, \alpha_l} = 0.
\]
On the other hand, if \(\alpha_j = -1\) and \(\alpha_{j+1} = \cdots = \alpha_l = 0\), then
\[
\langle F_0, \ldots, F_l \rangle_{\alpha_1, \ldots, \alpha_l} = F_{l-j}.
\]
The maps in \(\langle F_0, \ldots, F_l \rangle\) are the zero maps, the identities, or composites of the isomorphisms in the sequence. When \(l = 0\), \(\langle F_0 \rangle\) is supposed to be the 0-cube \(F_0\). For instance, \(\langle F_0, F_1, F_2 \rangle\) is described as
\[
\begin{array}{cccc}
& F_0 & \rightarrow & F_1 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
F_0 & \rightarrow & F_2 & \rightarrow 0 \\
\downarrow & \downarrow & \downarrow & \\
0 & \rightarrow & 0 & \rightarrow 0 \\
\end{array}
\]
Let us consider the faces of \(\langle F_0, \ldots, F_l \rangle\). Firstly,
\[
\partial^1_j \langle F_0, \ldots, F_l \rangle = 0
\]
for any \(1 \leq j \leq l\). Moreover,
\[
\begin{align*}
\partial^{-1}_1 \langle F_0, \ldots, F_l \rangle &= \langle F_0, \ldots, F_{l-1} \rangle, \\
\partial^0_j \langle F_0, \ldots, F_l \rangle &= \langle F_0, \ldots, F_{l-2}, F_l \rangle,
\end{align*}
\]
and if \(j \geq 2\), then \(\partial^{-1}_j \langle F_0, \ldots, F_l \rangle\) is a degenerate cube and
\[
\partial^0_j \langle F_0, \ldots, F_l \rangle = \langle F_0, \ldots, F_{l-j-1}, F_{l-j+1}, \ldots, F_l \rangle.
\]
Hence it holds in \(\tilde{Q}C_*(A)\) that
\[
\partial \langle F_0, \ldots, F_l \rangle = \sum_{j=0}^{l} (-1)^j \langle F_0, \ldots, F_{l-j-1}, F_{l-j+1}, \ldots, F_l \rangle.
\]
We can generalize this construction to a sequence of exact cubes. If \(F_0 \simeq \cdots \simeq F_l\) is a sequence of isomorphisms of exact \(n\)-cubes of \(A\), then we can obtain an exact \((n + l)\)-cube \(\langle F_0, \ldots, F_l \rangle\) so that
\[
\partial^\alpha_{l+1} \cdots \partial^\alpha_{l+n} \langle F_0, \ldots, F_l \rangle = \langle (F_0)_{\alpha_1, \ldots, \alpha_n}, \ldots, (F_l)_{\alpha_1, \ldots, \alpha_n} \rangle.
\]
Then it holds in \(\tilde{Q}C_*(A)\) that
\[
\partial \langle F_0, \ldots, F_l \rangle = \sum_{j=0}^{l} (-1)^j \langle F_0, \ldots, F_{l-j-1}, F_{l-j+1}, \ldots, F_l \rangle
\]
\[
+ \sum_{j=1}^{n} \sum_{i=-1}^{1} (-1)^{l+i+j} \langle \partial^j_0 F_0, \ldots, \partial^j_i F_l \rangle.
\]
9.2. An exact cube \((\mathcal{F}; \varphi_1, \varphi_2, \ldots, \varphi_l) (\mathcal{G})\). We begin by recalling the definition of tensor product of exact cubes. Let \(\mathcal{A}\) be a small exact category, and assume that \(\mathcal{A}\) is equipped with tensor product. Given an exact \(n\)-cube \(\mathcal{F}\) and an exact \(m\)-cube \(\mathcal{G}\) of \(\mathcal{A}\), define an exact \((n+m)\)-cube \(\mathcal{F} \otimes \mathcal{G}\) by

\[
(\mathcal{F} \otimes \mathcal{G})_{\alpha_1, \ldots, \alpha_{n+m}} = \mathcal{F}_{\alpha_1, \ldots, \alpha_n} \otimes \mathcal{G}_{\alpha_{n+1}, \ldots, \alpha_{n+m}}.
\]

Then it gives a product of chain complexes

\[
\otimes : \tilde{\mathcal{Q}}C_s(\mathcal{A}) \otimes \tilde{\mathcal{Q}}C_s(\mathcal{A}) \to \tilde{\mathcal{Q}}C_s(\mathcal{A}).
\]

Let \(\mathcal{A}_0, \ldots, \mathcal{A}_l\) and \(\mathcal{B}\) be small exact categories, and consider the diagram of functors

\[
\begin{array}{ccc}
\mathcal{A}_0 & \xrightarrow{\varphi_1} & \mathcal{A}_1 \\
\downarrow{\pi_0} & & \downarrow{\pi_1} \\
\mathcal{B} & & \mathcal{A}_l \xleftarrow{\varphi_l} \mathcal{B}
\end{array}
\]

where each \(\pi_p\) is an exact functor and each \(\varphi_p\) is an exact functor from \(\mathcal{A}_p\) to the category of exact \(s_p\)-cubes of \(\mathcal{A}_{p-1}\). Then we can extend \(\varphi_p\) to an exact functor from the category of exact \(n\)-cubes of \(\mathcal{A}_{p+1}\) to that of exact \((s_p+n)\)-cubes of \(\mathcal{A}_{p-1}\) in the way that

\[
\partial^n_{s_p+1} \cdots \partial_{s_p+n}^{\mathcal{G}} \varphi_p(\mathcal{G}) = \varphi_p(\mathcal{G}_{\alpha_1, \ldots, \alpha_n})
\]

for any exact \(n\)-cube \(\mathcal{G}\) of \(\mathcal{A}_p\).

Suppose that each \(\mathcal{A}_p\) is equipped with tensor product. Fix an object \(\mathcal{F}\) of \(\mathcal{B}\), and we abbreviate \(\pi_p(\mathcal{F}) \otimes \mathcal{G}\) to \(\mathcal{F} \otimes \mathcal{G}\) for any exact cube \(\mathcal{G}\) of \(\mathcal{A}_p\). Moreover, assume that there is a natural transformation

\[
\mathcal{F} \otimes \varphi_p(\mathcal{G}) \simeq \varphi_p(\mathcal{F} \otimes \mathcal{G}).
\]

Set \(s = \sum_{p=0}^l s_p\). Then for any exact \(n\)-cube \(\mathcal{G}\) of \(\mathcal{A}_l\), we have a sequence of isomorphisms of exact \((n+s)\)-cubes

\[
\mathcal{F} \otimes \varphi_1 \varphi_2 \cdots \varphi_l(\mathcal{G}) \simeq \varphi_1(\mathcal{F} \otimes \varphi_2 \cdots \varphi_l(\mathcal{G})) \simeq \cdots \simeq \varphi_1 \varphi_2 \cdots \varphi_l(\mathcal{F} \otimes \mathcal{G})
\]

of \(\mathcal{A}_0\), and the associated exact \((n+s+l)\)-cube of \(\mathcal{A}_0\):

\[
\langle \mathcal{F}; \varphi_1, \varphi_2, \ldots, \varphi_l \rangle (\mathcal{G})
\]

\[
= \langle \mathcal{F} \otimes \varphi_1 \varphi_2 \cdots \varphi_l(\mathcal{G}), \varphi_1(\mathcal{F} \otimes \varphi_2 \cdots \varphi_l(\mathcal{G})), \ldots, \varphi_1 \varphi_2 \cdots \varphi_l(\mathcal{F} \otimes \mathcal{G}) \rangle.
\]

When \(l = 0\), \(\langle \mathcal{F}; \varphi_1, \varphi_2, \ldots, \varphi_l \rangle (\mathcal{G})\) is supposed to be the tensor product \(\mathcal{F} \otimes \mathcal{G}\). Since \(\langle \mathcal{F}; \varphi_1, \varphi_2, \ldots, \varphi_l \rangle (\mathcal{G})\) is degenerate if so is \(\mathcal{G}\), it induces a map

\[
\langle \mathcal{F}; \varphi_1, \varphi_2, \ldots, \varphi_l \rangle : \tilde{\mathcal{Q}}C_s(\mathcal{A}_l) \to \tilde{\mathcal{Q}}C_{s+s+l}(\mathcal{A}_0).
\]

We can generalize this construction to the case that \(\varphi_p\) is a linear sum of exact functors from \(\mathcal{A}_p\) to the category of exact cubes of \(\mathcal{A}_{p-1}\). In particular we have

\[
\langle \mathcal{F}; \varphi_1, \ldots, \varphi_p, \ldots, \varphi_l \rangle
\]

\[
= \sum_{j=1}^{s_p} \sum_{i=-1} \langle \mathcal{F}; \varphi_1, \ldots, \partial^j_{p} \varphi, \ldots, \varphi_l \rangle : \tilde{\mathcal{Q}}C_s(\mathcal{A}_l) \to \tilde{\mathcal{Q}}C_{s+s+l-1}(\mathcal{A}_0).
Let us consider the faces of the exact cube $\langle \mathcal{F}; \varphi_1, \varphi_2, \ldots, \varphi_l \rangle (\mathcal{G})$. As we have seen in the previous subsection, $\partial^l_j \langle \mathcal{F}; \varphi_1, \varphi_2, \ldots, \varphi_l \rangle (\mathcal{G})$ is degenerate if $1 \leq j \leq l$ and $i = 1$, or if $2 \leq j \leq l$ and $i = 0$. On the other hand,

$$
\partial^{-1}_l \langle \mathcal{F}; \varphi_1, \ldots, \varphi_l \rangle (\mathcal{G}) = \langle \mathcal{F}; \varphi_1, \ldots, \varphi_{l-1} \rangle (\varphi_l(\mathcal{G})),
$$
and for $1 \leq j \leq l - 1$,

$$
\partial^j \langle \mathcal{F}; \varphi_1, \ldots, \varphi_l \rangle (\mathcal{G}) = \langle \mathcal{F}; \varphi_1, \ldots, \varphi_{l-j}\varphi_{l-j+1}, \ldots, \varphi_l \rangle (\mathcal{G}).
$$

However, $\partial^j_1 \langle \mathcal{F}; \varphi_1, \ldots, \varphi_l \rangle (\mathcal{G})$ is not equal to $\varphi_1(\langle \mathcal{F}; \varphi_2, \ldots, \varphi_l \rangle (\mathcal{G}))$. In fact,

$$
\partial^j_1 \langle \mathcal{F}; \varphi_1, \ldots, \varphi_l \rangle (\mathcal{G}) = \sigma(\varphi_1(\langle \mathcal{F}; \varphi_2, \ldots, \varphi_l \rangle (\mathcal{G})))
$$

where $\sigma \in \mathcal{G}_{n+s+l-1}$ is the transposition of the sequence $\{1, 2, \ldots, l - 1\}$ with the adjacent one $\{l, \ldots, l + s_1 - 1\}$. Hence we have

$$
\partial(\langle \mathcal{F}; \varphi_1, \ldots, \varphi_l \rangle (\mathcal{G})) = \langle \mathcal{F}; \varphi_1, \ldots, \varphi_{l-1} \rangle (\varphi_l(\mathcal{G}))
$$

$$
+ \sum_{j=1}^{l-1}(-1)^j \langle \mathcal{F}; \varphi_1, \ldots, \varphi_{l-j}\varphi_{l-j+1}, \ldots, \varphi_l \rangle (\mathcal{G})
$$

$$
+ (-1)^j \sigma(\varphi_1(\langle \mathcal{F}; \varphi_2, \ldots, \varphi_l \rangle (\mathcal{G})))
$$

$$
+ (-1)^l \sum_{j=1}^{l} (-1)^{s_1+\cdots+s_{j-1}} \langle \mathcal{F}; \varphi_1, \ldots, \varphi_j, \ldots, \varphi_l \rangle (\mathcal{G})
$$

$$
+ (-1)^{s+l} \langle \mathcal{F}; \varphi_1, \ldots, \varphi_l \rangle (\partial \mathcal{G}).
$$

If we use the chain complex $\overline{C}^\text{Alt}_{*}(\mathfrak{A})$, then we can get rid of the action of $\sigma \in \mathcal{G}_{*}$ from the above expression. Let

$$
\varphi_p^\text{alt} = \text{Alt}_{*} \varphi_p : \overline{C}^\text{Alt}_{*}(\mathfrak{A}_p) \to \overline{C}^\text{Alt}_{*+s_p}(\mathfrak{A}_{p-1}),
$$

and

$$
\langle \mathcal{F}; \varphi_1, \varphi_2, \ldots, \varphi_l \rangle^\text{alt} = \text{Alt}_{*} \langle \mathcal{F}; \varphi_1, \varphi_2, \ldots, \varphi_l \rangle : \overline{C}^\text{Alt}_{*}(\mathfrak{A}_l) \to \overline{C}^\text{Alt}_{*+s_l}(\mathfrak{A}_0).
$$

Since the signature of $\sigma$ is $(-1)^{s_1(l-1)}$, we have the following:

**Proposition 9.1.** For $x \in \overline{C}^\text{Alt}_{*}(\mathfrak{A}_l)$,

$$
\partial \left( \langle \mathcal{F}; \varphi_1, \ldots, \varphi_l \rangle^\text{alt}(x) \right) = \langle \mathcal{F}; \varphi_1, \ldots, \varphi_{l-1} \rangle^\text{alt} (\varphi_l^\text{alt}(x))
$$

$$
+ \sum_{j=1}^{l-1}(-1)^j \langle \mathcal{F}; \varphi_1, \ldots, \varphi_{l-j}\varphi_{l-j+1}, \ldots, \varphi_l \rangle^\text{alt}(x)
$$

$$
+ (-1)^{s_1(l-1)+l} \varphi_1^\text{alt}(\langle \mathcal{F}; \varphi_2, \ldots, \varphi_l \rangle^\text{alt}(x))
$$

$$
+ (-1)^l \sum_{j=1}^{l} (-1)^{s_1+\cdots+s_{j-1}} \langle \mathcal{F}; \varphi_1, \ldots, \varphi_j, \ldots, \varphi_l \rangle^\text{alt}(x)
$$

$$
+ (-1)^{s+l} \langle \mathcal{F}; \varphi_1, \ldots, \varphi_l \rangle^\text{alt}(\partial x).\]
9.3. A tensor product structure on $\tilde{\mathbb{Q}}^{\text{Alt}}_{C_*(X; Y_1, \ldots, Y_r)}$. Let $X$ be a scheme and $Y_1, \ldots, Y_r$ closed subschemes of $X$. Assume that $X$ is defined over a base scheme $S$. Fix a vector bundle $F$ on $S$. For any exact cube $G$ on $Y_I$, we write $F \otimes G$ for $\pi_I^* F \otimes G$, where $\pi_I : Y_I \to S$ is the structure map. In this subsection, we will construct a map of $C$-complexes

$$(\mathcal{F} \otimes ) : \tilde{\mathbb{Q}}^{\text{Alt}}_{C_*(X; Y_1, \ldots, Y_r)} \to \tilde{\mathbb{Q}}^{\text{Alt}}_{C_*(X; Y_1, \ldots, Y_r)}$$

such that

$$(\mathcal{F} \otimes)^{m,m} : \bigoplus_{|I|=m} \tilde{\mathbb{Q}}^{\text{Alt}}_{C_*(Y_I)} \to \bigoplus_{|I|=m} \tilde{\mathbb{Q}}^{\text{Alt}}_{C_*(Y_I)}$$

is given by the tensor product with $F$.

Let $K \bigsqcup I = J$ be a division of subsets $\{1, \ldots, r\}$ with $|I| = m$ and $|J| = n$, and write $K = \{k_1, \ldots, k_{n-m}\}$ with $k_1 < \cdots < k_{n-m}$. Assume that $K$ is not empty. Let us recall the map defined in §2.4:

$$\Xi_K = \sum_{\sigma \in S_{n-m}} (\text{sgn } \sigma)(t_{k_{\sigma(1)}}, \ldots, t_{k_{\sigma(n-m)}})^* : \tilde{\mathbb{Q}}_{C_*(Y_I)} \to \tilde{\mathbb{Q}}_{C_+ n-m-1(Y_J)}.$$ 

If we see this map as a linear sum of exact functors from $\mathfrak{P}(Y_I)$ to the category of exact $(n-m-1)$-cubes of $\mathfrak{P}(Y_J)$, then Prop.2.12 says that

$$(9.1) \quad \partial \Xi_K = \sum_{a=1}^{n-m-1} (-1)^{a+1} \sum_{L \bigsqcup L' = K \atop |L| = a} \text{sgn } \binom{L}{K} \Xi_L \Xi_{L'}.$$

Consider a division $K = K_1 \bigsqcup K_2 \bigsqcup \cdots \bigsqcup K_l$ such that each $K_j$ is not empty. Let $|K_p| = s_p$. Then we have the following diagram of functors:

Here we see $\Xi_{K_p}$ as a linear sum of exact functors from $\mathfrak{P}(Y_{K_p, J})$ to the category of exact $(s_p-1)$-cubes of $Y_{K_p, J}$. With this diagram we can associate a map

$$(\mathcal{F}; \Xi_{K_1}, \ldots, \Xi_{K_l})^{\text{alt}} : \tilde{\mathbb{Q}}^{\text{Alt}}_{C_*(Y_I)} \to \tilde{\mathbb{Q}}^{\text{Alt}}_{C_+ n-m}(Y_J).$$
Then it follows from Prop.9.1 that for \( x_I \in \tilde{\mathbb{Q}}C^\text{Alt}_s(Y_I) \),

\[
\partial \left( \langle F; \Xi_{K_1}, \ldots, \Xi_{K_l} \rangle_{\text{alt}} (x_I) \right) = \langle F; \Xi_{K_1}, \ldots, \Xi_{K_{l-1}} \rangle_{\text{alt}} (\Xi_{K_l}^\text{alt}(x_I)) \\
+ \sum_{p=1}^{l-1} (-1)^p \langle F; \Xi_{K_1}, \ldots, \Xi_{K_{l-p}} \Xi_{K_{l-p+1}}, \ldots, \Xi_{K_l} \rangle_{\text{alt}} (x_I) \\
+ (-1)^{s_1(l-1)+1} \Xi_{K_1} \langle F; \Xi_{K_2}, \ldots, \Xi_{K_l} \rangle_{\text{alt}} (x_I) \\
+ (-1)^l \sum_{p=1}^l (-1)^{s_1 + \cdots + s_{p-1} - p + 1} \langle F; \Xi_{K_1}, \ldots, \partial \Xi_{K_p}, \ldots, \Xi_{K_l} \rangle_{\text{alt}} (x_I) \\
+ (-1)^{n-m} \langle F; \Xi_{K_1}, \ldots, \Xi_{K_l} \rangle_{\text{alt}} (\partial x_I).
\]

For any division \( K = K_1 \coprod K_2 \coprod \cdots \coprod K_l \), define the signature

\[
\text{sgn} \left( \begin{array}{c}
K_1 \\
\vdots \\
K_l
\end{array} \right)
\]

as follows: If we write

\[
K_p = \{ k_{p,1}, \ldots, k_{p,s_p} \}, \quad k_{p,1} < \ldots < k_{p,s_p}, \\
I = \{ i_1, \ldots, i_m \}, \quad i_1 < \ldots < i_m, \\
J = \{ j_1, \ldots, j_n \}, \quad j_1 < \ldots < j_n,
\]

then

\[
\text{sgn} \left( \begin{array}{c}
K_1 \\
\vdots \\
K_l
\end{array} \right) = \text{sgn} \left( \begin{array}{cccc}
k_{1,1} & \cdots & k_{1,s_1} & k_{2,1} \\
& & & \cdots \\
& & & k_{l,s_l}
\end{array} \right)
\]

Define a map

\[
\varphi^{m,n} = (F \otimes)^{m,n} : \bigoplus_{|I|=m} \tilde{\mathbb{Q}}C_s^\text{Alt}(Y_I) \rightarrow \bigoplus_{|J|=n} \tilde{\mathbb{Q}}C^\text{Alt}_{s+n-m}(Y_J)
\]

as follows: For \( x = (x_I) \in \bigoplus_{|I|=m} \tilde{\mathbb{Q}}C_s^\text{Alt}(Y_I) \),

\[
\varphi^{m,n}(x)_J = \sum_{K_1 \coprod \cdots \coprod K_l \coprod J} \text{sgn} \left( \begin{array}{c}
K_1 \\
\vdots \\
K_l \\
J
\end{array} \right) (-1)^{b(s_1, \ldots, s_l)} \langle F; \Xi_{K_1}, \ldots, \Xi_{K_l} \rangle_{\text{alt}} (x_I),
\]

where \( s_j = |K_j| \) and

\[
b(s_1, \ldots, s_l) = \begin{cases} 
    s_{l-1} + s_{l-3} + \cdots + s_2, & \text{if } l \text{ is odd}, \\
    s_{l-1} + s_{l-3} + \cdots + s_1, & \text{if } l \text{ is even}.
\end{cases}
\]

The lemma below follows easily from the definition of \( b(s_1, \ldots, s_l) \).

**Lemma 9.2.** For \( l \geq 2 \) and \( 1 \leq p \leq l-1 \),

\[
b(s_1, \ldots, s_l) + b(s_1, \ldots, s_p + s_{p+1}, \ldots, s_l) + \sum_{a=1}^{p} s_a
\]
is an even number. Moreover,
\[ b(s_1, \ldots, s_l) + b(s_1, \ldots, s_{l-1}) = m - n - s_l, \]
\[ b(s_1, \ldots, s_l) + b(s_2, \ldots, s_l) = \begin{cases} 2b(s_2, \ldots, s_l), & \text{if } l \text{ is odd,} \\ 2b(s_2, \ldots, s_l) + s_1, & \text{if } l \text{ is even.} \end{cases} \]

Using Lem.9.2 and (9.1) we can show that
\[
\sum_{K_1 \cdots K_l \mid l = J} \text{sgn} \left( \sum_{K_1 \cdots K_l \mid l = J} (-1)^b(s_1, \ldots, s_l) \times \left( \sum_{p=1}^{l-1} (-1)^p \left( F; \Xi K_1, \ldots, \Xi K_{l-p}, \Xi K_{l-p+1}, \ldots, \Xi K_l \right) \right) \right) \]
\[
+ (-1)^{l} \sum_{p=1}^{l} (-1)^{s_1 + \cdots + s_{l-1} - p + 1} \left( F; \Xi K_1, \ldots, \partial \Xi K_{l-p}, \ldots, \Xi K_l \right) \right) \]
is equal to zero. Hence
\[
\partial \varphi^{m,n}(x)_J + (-1)^{n-m-1} \varphi^{m,n} \partial (x)_J
\]
\[
= \sum_{K_1 \cdots K_l \mid l = J} \text{sgn} \left( \sum_{K_1 \cdots K_l \mid l = J} (-1)^b(s_1, \ldots, s_l) \times \left( \sum_{p=1}^{l-1} (-1)^p \left( F; \Xi K_1, \ldots, \Xi K_{l-p}, \Xi K_{l-p+1}, \ldots, \Xi K_l \right) \right) \right) \]
\[
+ \sum_{K_1 \cdots K_l \mid l = J} \text{sgn} \left( \sum_{K_1 \cdots K_l \mid l = J} (-1)^b(s_1, \ldots, s_l) + s_1(l-1)+1 \sum_{p=1}^{l} (-1)^{s_1 + \cdots + s_{l-1} - p + 1} \Xi K_1 \left( F; \Xi K_2, \ldots, \Xi K_l \right) \right) \]
Applying Lem.9.2 to this equality we have
\[
\partial \varphi^{m,n}(x)_J + (-1)^{n-m-1} \varphi^{m,n} \partial (x)_J
\]
\[
= \sum_{K_1 \cdots K_l \mid l = J} \text{sgn} \left( \sum_{K_1 \cdots K_l \mid l = J} (-1)^b(s_1, \ldots, s_l) + m+n+s_1 \left( F; \Xi K_1, \ldots, \Xi K_{l-1} \right) \right) \times \Xi K_1 \left( F; \Xi K_2, \ldots, \Xi K_l \right) \right) \]
\[
+ \sum_{K_1 \cdots K_l \mid l = J} \text{sgn} \left( \sum_{K_1 \cdots K_l \mid l = J} (-1)^b(s_2, \ldots, s_l) + 1 \Xi K_1 \left( F; \Xi K_2, \ldots, \Xi K_l \right) \right) \]
\[
= (-1)^n \sum_{s_1=1}^{n-m-1} \varphi^{m+s_1,n}(F_{m+s_1}(x))_J + (-1)^{n+1} \sum_{s_1=1}^{n-m-1} F_{n-s_1,n}(\varphi^{m,n-s_1}(x))_J, \]
which leads to the following theorem:

**Proposition 9.3.**

\[ (\mathcal{F} \otimes ) = \varphi = (\varphi^{m,n}) : \mathcal{Q}C^\text{Alt}_s(X; Y_1, \ldots, Y_r) \to \mathcal{Q}C^\text{Alt}_s(X; Y_1, \ldots, Y_r) \]

is a map of $\mathcal{C}$-complexes.
9.4. A homotopy from $(\mathcal{F} \otimes f^* \mathcal{F})$ to $f^*(\mathcal{F} \otimes \mathcal{F})$. Let $T$ be another scheme with closed subschemes $D_1, \ldots, D_r$, and $f : (X; Y_1, \ldots, Y_r) \to (T; D_1, \ldots, D_r)$ a morphism. Assume that $X$ and $T$ are defined over a base scheme $S$, and $f$ is defined over $S$. Let $\mathcal{F}$ be a vector bundle on $S$. We abbreviate $\pi_X^* \mathcal{F} \otimes \mathcal{F}$ and $\pi_T^* \mathcal{F} \otimes \mathcal{F}$ to $\mathcal{F} \otimes \mathcal{F}$, where $\pi_X : X \to S$ and $\pi_T : T \to S$ are the structure morphisms. Then we have the diagram

\[
\begin{array}{c}
\tilde{Q}C^\text{Alt}_*(T; D_1, \ldots, D_r) \\ (\mathcal{F} \otimes) \downarrow \\
\tilde{Q}C^\text{Alt}_*(T; D_1, \ldots, D_r) \xrightarrow{f^*} \tilde{Q}C^\text{Alt}_*(X; Y_1, \ldots, Y_r) \\
\end{array}
\]

The aim of this subsection is to construct a homotopy $\Phi f$ from $(\mathcal{F} \otimes \mathcal{F}) f^*$ to $f^*(\mathcal{F} \otimes \mathcal{F})$.

Let $K \coprod I = J$ be a division of subsets of $\{1, \ldots, r\}$ and $K = K_1 \coprod \cdots \coprod K_l$ a division of $K$. Consider the following diagram:

\[
\begin{array}{c}
\mathfrak{P}(Y_J) \xrightarrow{\Xi_{K_1}} \cdots \xrightarrow{\Xi_{K_p-1}} \mathfrak{P}(Y_{K_p \cup \cdots \cup K_l \cup I}) \xrightarrow{\Xi_{K_p,f}} \mathfrak{P}(D_{K_p+1 \cup \cdots \cup K_l \cup I}) \xrightarrow{\Xi_{K_{p+1}}} \cdots \xrightarrow{\Xi_{K_1}} \\
\pi_X \downarrow \quad \pi_X^* \quad \pi_T \downarrow \quad \pi_T^* \\
\mathfrak{P}(S),
\end{array}
\]

where $\Xi_{K_p,f}$ is the linear sum of exact functors defined in §2.4. Unlike the previous case, $K_p$ may be empty in this case. Define a map

\[
\Phi^m_n : \bigoplus_{|I|=m} \tilde{Q}C^\text{Alt}_*(D_I) \to \bigoplus_{|J|=n} \tilde{Q}C^\text{Alt}_{*+n-m+1}(Y_J)
\]

by

\[
\Phi^m_n(x)_J = \sum_{K_1 \coprod \cdots \coprod K_l \coprod I = J} \text{sgn}(K_1 \cdots K_l I) \times \\
\sum_{p=1}^{l} (-1)^{b(s_1,\ldots,s_{p-1}+s_p,\ldots,s_l)+n+p+1+1} \langle \mathcal{F}; \Xi_{K_1}, \ldots, \Xi_{K_{p-1}+s_p, \cdots, s_l} \rangle^\text{alt}(x_I)
\]

for $x = (x_I) \in \bigoplus_{|I|=m} \tilde{Q}C^\text{Alt}_*(D_I)$. In the above, $|K_j| = s_j$ and $s_0$ is supposed to be zero.
Let us calculate $\partial \Phi_{f}^{m,n}(x)$ using Prop.9.1 and Lem.9.2. Since a similar cancellation of terms to (9.2) occurs in this case,

$$( -1 )^{n} \partial \Phi_{f}^{m,n}(x)_{J} + (-1)^{m} \Phi_{r}^{m,n}(\partial x)_{J} = \sum_{K_{1} \cdots \cdots K_{i}\cdots K_{l} I = J} \text{sgn} ( K_{1} \cdots K_{i} I ) \times \\
\left\{ \sum_{p=1}^{l-1} (-1)^{b(\ldots, s_{p-1} + s_{p}, \ldots) + p + l + 1} \langle \mathcal{F}_{m}, \ldots, \Xi_{K_{p}, f}, \ldots, \Xi_{K_{l-1}} \rangle_{alt} \mathcal{Z}_{K_{l}}(x_{I}) \\
+ (-1)^{b(s_{1}, \ldots, s_{l-1} + s_{l}) + 1} \langle \mathcal{F}_{m}, \ldots, \Xi_{K_{1}, f}, \ldots, \Xi_{K_{l-1}} \rangle_{alt} \mathcal{Z}_{K_{l}}(x_{I}) \\
+ (-1)^{b(s_{1}, \ldots, s_{l}) + s_{1}(l-1)} \mathcal{Z}_{K_{l}, f}(x_{I}) \right\} \\
= - \sum_{s_{1}=1}^{n-m} \Phi_{f}^{m+s_{1}, n} F^{m, m+s_{1}}(x)_{J} - \sum_{s_{1}=0}^{n-m} \langle \mathcal{F}_{m} \rangle_{alt}^{m+s_{1}, n}(f^{*})^{m, m+s_{1}}(x)_{J} \\
+ \sum_{s_{1}=0}^{n-m} \langle f^{*} \rangle_{alt}^{n-s_{1}, n}(\mathcal{F}_{m})^{m, n-s_{1}}(x)_{J} - \sum_{s_{1}=1}^{n-m} F^{m-s_{1}, n} \Phi_{f}^{m, n-s_{1}}(x)_{J}.$$

Hence we have the following:

**Proposition 9.4.**

$$\Phi_{f} : \mathcal{Q}C_{s}^{alt}(T; D_{1}, \ldots, D_{r}) \rightarrow \mathcal{Q}C_{s+1}^{alt}(X; Y_{1}, \ldots, Y_{r})$$

is a homotopy from $(\mathcal{F} \otimes_{s}) f^{*}$ to $f^{*}(\mathcal{F} \otimes_{s})$.

The closed immersion $\iota_{r} : Y_{r} \hookrightarrow X$ induces a map of $\mathcal{C}$-complexes

$$\iota_{r}^{*} : \mathcal{Q}C_{s}^{alt}(X; Y_{1}, \ldots, Y_{r-1}) \rightarrow \mathcal{Q}C_{s}^{alt}(Y_{1} \cap Y_{r}, \ldots, Y_{r-1} \cap Y_{r})$$

and a homotopy $\Phi_{\iota_{r}}$ from $(\mathcal{F} \otimes_{s}) \iota_{r}^{*}$ to $\iota_{r}^{*}(\mathcal{F} \otimes_{s})$. If we identify the simple complex of $\iota_{r}^{*}$ with the complex $\mathcal{Q}C_{s}^{alt}(X; Y_{1}, \ldots, Y_{r})$ by Cor.2.16, then Prop.2.5 says that the homotopy $\Phi_{\iota_{r}}$ gives a map of $\mathcal{C}$-complexes

$$\varphi_{s} : \mathcal{Q}C_{s}^{alt}(X; Y_{1}, \ldots, Y_{r}) \rightarrow \mathcal{Q}C_{s}^{alt}(X; Y_{1}, \ldots, Y_{r}).$$

**Proposition 9.5.** The map $\varphi_{s}$ agrees with $(\mathcal{F} \otimes_{s})$.

**Proof.** Let $x = (x_{I}) \in \bigoplus_{|I| = m} \mathcal{Q}C_{s}^{alt}(Y_{I})$. It follows from the definition of $\varphi_{s}$ that $\varphi_{s}^{m,n}(x)$ is written as

$$\varphi_{s}^{m,n}(x)_{J} = \sum_{I \subseteq J} \varphi_{s}^{I,J}(x_{I}).$$
In the case that \( r \notin J \) or \( r \in I \), \( \varphi_{s}^{I,J}(x_I) \) comes from \((\mathcal{F} \otimes )\), that is,

\[
\varphi_{s}^{I,J}(x_I) = \sum_{K_{1} \cdots K_{I} \cap I = J} \text{sgn} \langle K_{1} \cdots K_{I} \rangle (-1)^{b(s_1, \ldots, s_l)} \langle \mathcal{F}; \Xi_{K_{1}}, \ldots, \Xi_{K_{I}} \rangle^{\text{alt}}(x_I),
\]

where \( |K_j| = s_j \). Assume \( r \in J \) and \( r \notin I \). In this case, \( \varphi_{s}^{I,J}(x_I) \) comes from the homotopy \( \Phi_{t_r} \). To be more precise, if we write \( J = J - \{ r \} \), then

\[
\varphi_{s}^{I,J}(x_I) = \sum_{K_{1} \cdots K_{I} \cap I = J^r} \text{sgn} \langle K_{1} \cdots K_{I} \rangle \times \sum_{p=1}^{l} (-1)^{b(\ldots, s_{p-1} + s_p \ldots) + (n-1)p + l} \langle \mathcal{F}; \Xi_{K_{1}}, \ldots, \Xi_{K_{p}, t_r}, \ldots, \Xi_{K_{I}} \rangle^{\text{alt}}(x_I).
\]

Write \( K'_p = K_p \cup \{ r \} \) and \( s'_p = |K'_p| = s_p + 1 \). Then the following equalities hold:

\[
\Xi_{K_{p}, t_r} = (-1)^{s_p} \Xi_{K'_p},
\]

\[
\text{sgn} \langle K_{1} \cdots K'_{p} \cdots K_{I} \rangle = (-1)^{m + s_{p+1} + \ldots + s_l} \text{sgn} \langle K_{1} \cdots K_{p} \cdots K_{I} \rangle.
\]

Hence using Lem.9.2 we can show that

\[
\varphi_{s}^{I,J}(x_I) = \sum_{K_{1} \cdots K_{I} \cap I = J} \sum_{p=1}^{l} \text{sgn} \langle K_{1} \cdots K'_{p} \cdots K_{I} \rangle \times (-1)^{b(s_1, \ldots, s_l) + p + l} \langle \mathcal{F}; \Xi_{K_{1}}, \ldots, \Xi_{K'_p}, \ldots, \Xi_{K_{I}} \rangle^{\text{alt}}(x_I).
\]

It follows from the definition of \( b(s_1, \ldots, s_l) \) that

\[
(-1)^{b(s_1, \ldots, s_l) + p + l} = (-1)^{b(s_1, \ldots, s'_p, \ldots, s_l)}.
\]

Hence if we change the symbol \( K'_p \) to \( K_p \) and \( s'_p \) to \( s_p \), then

\[
\varphi_{s}^{I,J}(x_I) = \sum_{K_{1} \cdots K_{I} \cap I = J} \text{sgn} \langle K_{1} \cdots K_{I} \rangle (-1)^{b(s_1, \ldots, s_l)} \langle \mathcal{F}; \Xi_{K_{1}}, \ldots, \Xi_{K_{I}} \rangle^{\text{alt}}(x_I),
\]

which completes the proof. \( \square \)

9.5. **A second homotopy arising from a section of a closed immersion.** Let \( f : (X; Y_1, \ldots, Y_r) \to (T; D_1, \ldots, D_r) \) be a closed immersion defined over a base scheme \( S \), and suppose that there is a morphism \( g : (T; D_1, \ldots, D_r) \to (X; Y_1, \ldots, Y_r) \) also defined over \( S \) such that \( gf = \text{Id}_X \). Then we have the diagram of \( \mathbb{C} \)-complexes

\[
\begin{array}{ccc}
\tilde{Q}C_{s}^{\text{Alt}}(X; Y_1, \ldots, Y_r) & \xrightarrow{g^*} & \tilde{Q}C_{s}^{\text{Alt}}(T; D_1, \ldots, D_r) \\
\downarrow (\mathcal{F} \otimes ) & & \downarrow (\mathcal{F} \otimes ) \\
\tilde{Q}C_{s}^{\text{Alt}}(X; Y_1, \ldots, Y_r) & \xrightarrow{f^*} & \tilde{Q}C_{s}^{\text{Alt}}(X; Y_1, \ldots, Y_r)
\end{array}
\]
and the homotopies

\[ \Phi_g : \widetilde{QC}^\text{Alt}_s(X; Y_1, \ldots, Y_r) \to \widetilde{QC}^\text{Alt}_s(T; D_1, \ldots, D_r), \]

\[ \Phi_f : \widetilde{QC}^\text{Alt}_s(T; D_1, \ldots, D_r) \to \widetilde{QC}^\text{Alt}_s(X; Y_1, \ldots, Y_r), \]

\[ \Psi : \widetilde{QC}^\text{Alt}_s(X; Y_1, \ldots, Y_r) \to \widetilde{QC}^\text{Alt}_s(X; Y_1, \ldots, Y_r) \]

from \((F \otimes g^*)\) to \(g^*(F \otimes \cdot)\), from \((F \otimes f^*)\) to \(f^*(F \otimes \cdot)\), and from the identity to \(f^*g^*\) respectively. Hence we have two homotopies

\[ \Phi_f g^* + f^* \Phi_g + (F \otimes \cdot) \Psi, \Psi(F \otimes \cdot) : \widetilde{QC}^\text{Alt}_s(X; Y_1, \ldots, Y_r) \to \widetilde{QC}^\text{Alt}_{s+1}(X; Y_1, \ldots, Y_r) \]

from \((F \otimes \cdot)\) to \(f^*g^*(F \otimes \cdot)\). In this subsection we will construct a second homotopy between them which admits the condition of Def.2.7.

Define a map

\[ \Theta_1^{m,n} : \bigoplus_{|I| = m} \widetilde{QC}^\text{Alt}_s(Y_I) \to \bigoplus_{|J| = n} \widetilde{QC}^\text{Alt}_{s+n-m+2}(Y_J) \]

by

\[ \Theta_1^{m,n}(x)_I = \sum_{K_1 \cdots K_l} \operatorname{sgn}(K_1 \cdots K_l) \times \sum_{l=1}^{|I|} (-1)^{b(s_1, \ldots, s_l) + 1} (F; \Xi_{K_1}, \ldots, \Xi_{K_p} f_g, \ldots, \Xi_{K_l})^\text{alt} (x_I) \]

for \(x = (x_I) \in \bigoplus_{|I| = m} \widetilde{QC}^\text{Alt}_s(Y_I)\), where \(|K_I| = s_j\). In the above, \(K_p\) may be empty. Then a similar cancellation of terms to (9.2) occurs, therefore

\[ (-1)^n \partial \Theta_1^{m,n}(x)_I - (-1)^m \Theta_1^{m,n}(\partial x)_I \]

\[ = \sum_{s_1 = 1}^{n-m} \Theta_1^{m+s_1,n} F^{m+s_1,n}(x)_I - \sum_{s_1 = 0}^{n-m} (F \otimes )^{m+s_1,n} \Psi^{m+s_1,n}(x)_I \]

\[ + \sum_{s_1 = 0}^{n-m} \Psi^{n-s_1,n} (F \otimes )^{m,n-s_1}(x)_I - \sum_{s_1 = 1}^{n-m} F^{n-s_1,n} \Theta_1^{m,n-s_1}(x)_I \]

\[ + \sum_{K_1 \cdots K_l} \operatorname{sgn}(K_1 \cdots K_l) \times \sum_{l=1}^{l-1} (-1)^{b(s_1, \ldots, s_l) + n + p + l + s_p} (F; \ldots, \Xi_{K_p} f \Xi_{K_p+1} g, \ldots)^\text{alt} (x_I). \]

Meanwhile, define a map

\[ \Theta_2^{m,n} : \bigoplus_{|I| = m} \widetilde{QC}^\text{Alt}_s(Y_I) \to \bigoplus_{|J| = n} \widetilde{QC}^\text{Alt}_{s+n-m+2}(Y_J) \]
by

\[
\Theta_{2}^{m,n}(x)_{J} = \sum_{K_{1} \cdots \cdots K_{l} I = J} \sum_{1 \leq p < q \leq l} (-1)^{c_{p,q}} \langle F; \Xi_{K_{1}}, \ldots, \Xi_{K_{p}},f, \ldots, \Xi_{K_{q}},g, \ldots, \Xi_{K_{l}} \rangle^{alt} (x_{I})
\]

for \(x = (x_{I}) \in \bigoplus_{|I| = m} \tilde{Q}C_{*}^{Alt}(Y_{I})\), where

\[
c_{p,q} = b(s_{1}, \ldots, s_{l}) + \sum_{j=p}^{q-1} s_{j} + p + q + 1
\]

and \(|K_{j}| = s_{j}\). Here \(K_{p}\) and \(K_{q}\) may be empty. Then

\[
(-1)^{n} \partial \Theta_{2}^{m,n}(x)_{J} - (-1)^{m} \Theta_{2}^{m,n}(\partial x)_{J}
\]

\[
= \sum_{s_{1}=1}^{n-m} \Theta_{2}^{m+s_{1},n} F_{m,m+s_{1}}(x)_{J} - \sum_{s_{1}=0}^{n-m} \Phi_{f}^{m+s_{1},n}(g^{*})_{m,m+s_{1}}(x)_{J}
\]

\[
- \sum_{s_{1}=0}^{n-m} (f^{*})^{n-s_{1},n} \Phi_{g}^{m,n-s_{1}}(x)_{J} - \sum_{s_{1}=1}^{n-m} F_{n-s_{1},n} \Theta_{2}^{m,n-s_{1}}(x)_{J}
\]

\[
+ \sum_{K_{1} \cdots \cdots K_{l} I = J} \sum_{p=1}^{l-1} (-1)^{b(s_{1}, \ldots, s_{l}) + n + p + l + s_{p}} \langle F; \ldots, \Xi_{K_{p}},f, \Xi_{K_{p+1}},g, \ldots \rangle^{alt} (x_{I}).
\]

Hence we have the following:

**Proposition 9.6.** If we set

\[
\Theta^{m,n} = \Theta_{1}^{m,n} + \Theta_{2}^{m,n} : \bigoplus_{|I|=m} \tilde{Q}C_{*}^{Alt}(Y_{I}) \rightarrow \bigoplus_{|J|=n} \tilde{Q}C_{*+n-m+2}^{Alt}(Y_{J}),
\]
then for $x = (x_I) \in \bigoplus_{|I|=m} \widetilde{Q}C_{\ast}^{\text{Alt}}(Y_I)$,

$$(-1)^n \partial \Theta^{m,n}(x) + \sum_{k_1=1}^{n-m} F^{n-k_1,n} \Theta^{m,n-k_1}(x),$$

$$- (-1)^m \Theta^{m,n}(x) - \sum_{k_1=1}^{n-m} \Theta^{m+k_1,n} F^{m,m+k_1}(x)$$

$$= \sum_{k_1=0}^{n-m} \varphi^{n-k_1,m} \Theta(x),$$

$$- \sum_{k_1=0}^{n-m} \Phi^{n+k_1,m} \Theta(x) - \sum_{k_1=0}^{n-m} \Phi^{n-k_1,m} \Theta(x).$$

In other words, $\Theta = (\Theta^{m,n})$ is a second homotopy from $\Phi g^* + f^* \Phi + (\mathcal{F} \otimes)\Psi$ to $\Psi(\mathcal{F} \otimes)$.

10. $\hat{K}_0(X)$-module structures on arithmetic $K$-groups

10.1. $\hat{K}_0(X)$-module structures on $\hat{K}_0(X)$ and on $\hat{K}_0(T)$. Let $X$ be a smooth proper variety defined over an arithmetic ring. Let us first recall the $\hat{K}_0(X)$-module structure on $\hat{K}_0(X)$ given in [Ta §5]. Define an operation on the Deligne complexes

$$\Delta: \tau \mathcal{D}^{2p-n}(X,p) \times \tau \mathcal{D}^{2q-m}(X,q) \to \tau \mathcal{D}^{2p+2q-n-m-1}(X,p+q)$$

as follows: Let

$$a_{i,j}^{n,m} = 1 - 2(n+m)^{-1} \sum_{\alpha=0}^{i-1} \binom{n+m-i-j+1}{n-\alpha} \binom{i+j-1}{\alpha},$$

and for $\omega \in \tau \mathcal{D}^{2p-n}(X,p)$ and $\tau \in \tau \mathcal{D}^{2q-m}(X,q)$,

$$\omega \triangle \tau = \sum_{1 \leq i \leq n} \sum_{1 \leq j \leq m} a_{i,j}^{n,m} \omega^{(p-n+i-1,p-i)} \wedge \tau^{(q-m+j-1,q-j)}$$

if $n,m \geq 1$, and $\omega \triangle \tau = 0$ if $n = 0$ or $m = 0$.

Proposition 10.1. [Ta Thm 5.2] Let $\mathcal{F}$ and $\mathcal{G}$ be an exact hermitian $n$-cube and $m$-cube on $X$. Then

$$\text{ch}_{n+m}(\mathcal{F} \otimes \mathcal{G}) = \text{ch}_n(\mathcal{F}) \bullet \text{ch}_m(\mathcal{G}) + (-1)^{n+1} d_\mathcal{D}(\text{ch}_n(\mathcal{F}) \triangle \text{ch}_m(\mathcal{G}))$$

$$+ (-1)^n d_\mathcal{D} \text{ch}_n(\mathcal{F}) \triangle \text{ch}_m(\mathcal{G}) - \text{ch}_n(\mathcal{F}) \triangle d_\mathcal{D} \text{ch}_m(\mathcal{G}).$$

Suppose $r \geq 1$. Let $(\mathcal{F}, \eta)$ be a pair of a hermitian vector bundle $\mathcal{F}$ on $X$ with $\eta \in \tau \mathcal{D}_1(X)/\text{Im} \ d_\mathcal{D}$ and

$$(y, \tau) \in \widetilde{Q}C_r(X) \otimes \tau \mathcal{D}_{r+1}(X) = s(\text{ch}_r)$$
such that $\partial(y, \tau) = 0$. Note that $\text{ch}_0(F) \cdot \tau = \text{ch}_0(F) \wedge \tau$ by definition, and in what follows we write $\text{ch}_0(F) \cdot \tau$ instead of $\text{ch}_0(F) \wedge \tau$.

**Proposition 10.2.** Define a product of above pairs by

$$(F, \eta) \cdot (y, \tau) = (F \otimes y, \text{ch}_0(F) \cdot \tau) \in s(\text{ch})_r.$$  

Then this product gives rise to a map of arithmetic $K$-groups

$$\tilde{K}_0(X) \times \tilde{K}_r(X)_0 \to \tilde{K}_r(X)_0,$$

by which $\tilde{K}_r(X)_0$ is a $\tilde{K}_0(X)$-module.

**Proof:** It is obvious that $\partial(F \otimes y, \text{ch}_0(F) \cdot \tau) = 0$ and that $\partial(F \otimes y', \text{ch}_0(F) \cdot \tau') = (F \otimes y, \text{ch}_0(F) \cdot \tau)$ if $(y, \tau) = (y', \tau')$. Let $E : 0 \to F_{-1} \to F_0 \to F_1 \to 0$ be a short exact sequence of hermitian vector bundles on $X$. We see $E$ as an element of $\mathbb{Q} \tilde{C}_1(X)$. Then Prop.10.1 implies that

$$\text{ch}_{r+1}(E \otimes y) = \text{ch}_1(E) \cdot \text{ch}_r(y) + d_2(\text{ch}_1(E) \wedge \text{ch}_r(y)) - \text{ch}_1(E) \wedge d_2 \text{ch}_r(y).$$

Since $\text{ch}_r(y) = d_2 \tau$, we have

$$\partial(E \otimes y, - \text{ch}_1(E) \cdot \tau + \text{ch}_1(E) \wedge \text{ch}_n(y))$$

$$= (\partial(F \otimes y, \text{ch}_{r+1}(E \otimes y) + d_2(\text{ch}_1(E) \cdot \tau - \text{ch}_1(E) \wedge \text{ch}_n(y)))$$

$$= (F_{-1} \otimes y + F_1 \otimes y - F_0 \otimes y, \text{ch}_0(F_{-1}) \cdot \tau + \text{ch}_0(F_1) \cdot \tau - \text{ch}_0(F_0) \cdot \tau),$$

which means that the map (10.1) is well-defined.

We next show that the above product gives a $\tilde{K}_0(X)$-module structure on $\tilde{K}_r(X)_0$. Take two elements $[(F_1, \eta_1), (F_2, \eta_2)] \in \tilde{K}_0(X)$. Then their product in $\tilde{K}_0(X)$ is given by

$$[(F_1 \otimes F_2, \text{ch}_0(F_1) \wedge \eta_2 + \eta_1 \wedge \text{ch}_0(F_2) + d_2 \eta_1 \wedge \eta_2)] \in \tilde{K}_0(X).$$

For any exact hermitian $r$-cube $\mathcal{G}$ on $X$, consider the following exact hermitian $(r+1)$-cube

$$\langle F_1, F_2 \rangle (\mathcal{G}) = (F_1 \otimes (F_2 \otimes \mathcal{G}) \to (F_1 \otimes F_2) \otimes \mathcal{G} \to 0).$$

Then it gives a map

$$\langle F_1, F_2 \rangle : \mathbb{Q} \tilde{C}_r(X) \to \mathbb{Q} \tilde{C}_{r+1}(X)$$

which satisfies

$$\partial (\langle F_1, F_2 \rangle (x)) = F_1 \otimes (F_2 \otimes x) - (F_1 \otimes F_2) \otimes x - \langle F_1, F_2 \rangle (\partial x)$$

for $x \in \mathbb{Q} \tilde{C}_r(X)$. Note that $\langle F_1, F_2 \rangle (x)$ is isometrically equivalent to a degenerate element since the isomorphism in (10.2) is an isometry. Hence we have $\text{ch}_{r+1}(\langle F_1, F_2 \rangle (x)) = 0$.

Let $(y, \tau) \in s(\text{ch})_r$, such that $\partial(y, \tau) = 0$. Then

$$(F_1, \eta_1) \cdot ((F_2, \eta_2) \cdot (y, \tau)) - (F_1 \otimes F_2, \text{ch}_0(F_1) \wedge \eta_2 + \eta_1 \wedge \text{ch}_0(F_2) + d_2 \eta_1 \wedge \eta_2) \cdot (y, \tau)$$

$$= (F_1 \otimes (F_2 \otimes y) - (F_1 \otimes F_2) \otimes y, 0).$$

Since $\partial y = 0$ and $\text{ch}_{r+1}(\langle F_1, F_2 \rangle (y)) = 0$, it is equal to $\partial (\langle F_1, F_2 \rangle (y), 0)$. This means that the product (10.1) is associative. The identity condition $[(\mathcal{O}, 0)] \cdot [(y, \tau)] = [(y, \tau)]$,
where $\mathcal{O}$ is the structure sheaf with the canonical metric, can be shown in a similar way by using the canonical isometry $\mathcal{O} \otimes \mathcal{G} \simeq \mathcal{G}$.

We next consider a $\tilde{K}_0(X)$-module structure on $\tilde{K}_{P,r}(X)_Q$. Let $(\mathcal{F}, \tilde{\eta})$ be a pair of a hermitian vector bundle $\mathcal{F}$ on $X$ with $\tilde{\eta} \in \tau \mathcal{D}_1(X)/\text{Im} \, d_\delta$. Suppose $r \geq 1$ and let

$$(y, \tau) \in \tilde{Q} \tilde{C}(X) \oplus \tilde{D}_{P,r+1}(X) = s(\text{ch}_{xP})_r,$$

then define their product by a similar expression:

$$(10.3) \quad (\mathcal{F}, \tilde{\eta}) \cdot (y, \tau) = (\mathcal{F} \otimes y, \text{ch}_0(\mathcal{F}) \cdot_P \tau),$$

where $\cdot_P$ is the product introduced in §3.2. It is easy to show that the canonical isomorphism $\tilde{K}_{P,r}(X)_Q \to \tilde{K}(X)_Q$, which is given in §5.1, is compatible with the product with the pair $(\mathcal{F}, \tilde{\eta})$, hence the product (10.3) gives a map

$$\tilde{K}_0(X) \times \tilde{K}_{P,r}(X)_Q \to \tilde{K}_{P,r}(X)_Q,$$

by which $\tilde{K}_{P,r}(X)_Q$ is a $\tilde{K}_0(X)$-module.

Here we assume that $X$ is projective over an arithmetic field. Let $T = D(X \times \Box^r; X \times \partial \Box^r)$ be the associated iterated double. Let $(\mathcal{F}, \tilde{\eta})$ be a pair of a hermitian vector bundle $\mathcal{F}$ on $X$ with $\tilde{\eta} \in \tau \mathcal{D}_1(X)/\text{Im} \, d_\delta$, and $(\mathcal{G}, \tilde{\tau})$ a pair of a hermitian vector bundle $\mathcal{G}$ on $T$ with $\tilde{\tau} \in \tilde{D}_{A,P,r+1}(X)/\text{Im} \, d_s$. Define a product of such pairs by

$$(10.4) \quad (\mathcal{F}, \tilde{\eta}) \cdot (\mathcal{G}, \tilde{\tau}) = (\mathcal{F} \otimes \mathcal{G}, \text{ch}_0(\mathcal{F}) \cdot_{A,P} \tilde{\tau} + \tilde{\eta} \cdot_{A,P} \text{ch}_{T,0}(\mathcal{G}) + d_s \tilde{\eta} \cdot_{A,P} \tilde{\tau}),$$

where $\cdot_{A,P}$ is the product introduced in the end of §3.2.

**Proposition 10.3.** The above product induces a pairing of arithmetic $K$-groups

$$\tilde{K}_0(X) \times \tilde{K}_0^M(T) \to \tilde{K}_0^M(T),$$

by which $\tilde{K}^M_0(T)$ is a $\tilde{K}_0(X)$-module.

**Proof:** It is easy to see that (10.3) is compatible with the relation coming from a short exact sequence of hermitian vector bundles on $T$. Let $\mathcal{E}: 0 \to \mathcal{F}_{-1} \to \mathcal{F}_0 \to \mathcal{F}_1 \to 0$ be a short exact sequence of hermitian vector bundles on $X$. Then

$$(\mathcal{F}_1 + \mathcal{F}_{-1} - \mathcal{F}_0, -\text{ch}_1(\mathcal{E})) \cdot (\mathcal{G}, \tilde{\tau}) = (\mathcal{F}_1 \otimes \mathcal{G} + \mathcal{F}_{-1} \otimes \mathcal{G} - \mathcal{F}_0 \otimes \mathcal{G}, -\text{ch}_1(\mathcal{E}) \cdot_{A,P} \text{ch}_{T,0}(\mathcal{G})).$$

Note that $\text{ch}_1(\mathcal{E}) \cdot_{A,P} \text{ch}_{T,0}(\mathcal{G})$ is in $\tilde{D}_{A,r+1}(X)$. On the other hand,

$$\text{ch}_{T,1}(\mathcal{E} \otimes \mathcal{G}) = \sum_I (-1)^{|I|} \text{ch}_1(i_I^*(\mathcal{E} \otimes \mathcal{G}))_F$$

$$= \sum_I (-1)^{|I|} \pi_{X,F}^* \text{ch}_1(\mathcal{E})_F \wedge \pi_{X,A}^* \text{ch}_0(i_I^*(\mathcal{G}))$$

$$= \pi_{X,F}^* \text{ch}_1(\mathcal{E})_F \wedge \pi_{X,A}^* \text{ch}_{T,0}(\mathcal{G}),$$

where $\pi_{X,F}: X \times \Box^r \times \mathbb{P}^1 \to X \times \mathbb{P}^1$ and $\pi_{X,A}: X \times \Box^r \times \mathbb{P}^1 \to X \times \Box^r$ are the projections. Moreover, the difference $\text{ch}_1(\mathcal{E})_F - \text{ch}_1(\mathcal{G})$ is $d_x$-exact in $\tilde{D}_{P,2}(X)$ by Lem.6.13, and it follows that $\text{ch}_1(\mathcal{E}) \cdot_{A,P} \text{ch}_{T,0}(\mathcal{G}) = \pi_X^* \text{ch}_1(\mathcal{E}) \wedge \text{ch}_{T,0}(\mathcal{G})$ in $\tilde{D}_{A,r+1}(X)$. Then taking the
The splitting maps Proposition 10.4. Since \( d_{K} \)-closedness of \( \text{ch}_{T,0}(\mathcal{G}) = 0 \) into consideration, we can show in the same way as the proof of Lem.6.13 that

\[
\chi_{T,1}(\varepsilon \otimes \mathcal{G}) = (-1)^{r} \chi_{1}(\varepsilon) \cdot_{A} \chi_{T,0}(\mathcal{G})
\]

\[
= \pi_{X}^{*} \chi_{1}(\varepsilon) \wedge \pi_{X,\mathcal{A}}^{*} \chi_{T,0}(\mathcal{G}) - (-1)^{r} \pi_{X}^{\mathcal{A}} \chi_{1}(\varepsilon) \wedge \chi_{T,0}(\mathcal{G})
\]

is \( d_{K} \)-exact in \( \tilde{D}_{A,\mathcal{P},r+1}(X) \). This means that

\[
(\mathcal{F}_{1} + \mathcal{F}_{-1} - \mathcal{F}_{0}, -\tilde{c}_{1}(\mathcal{E})) \cdot (\mathcal{G}, \mathcal{T})
\]

\[
= (\mathcal{F}_{1} \otimes \mathcal{G} + \mathcal{F}_{-1} \otimes \mathcal{G} - \mathcal{F}_{0} \otimes \mathcal{G}, -(1)^{r} \tilde{c}_{T,1}(\varepsilon \otimes \mathcal{G}))
\]

therefore the product (10.4) is compatible with the relation coming from a short exact sequence of hermitian vector bundles on \( X \).

Finally we show that the product admits the associative law. We should note that the product \( \cdot \) on \( \tau D_{log}^{*}(X, \ast) \) does not satisfies the associative law in general, whereas if at least one of three elements \( \omega, \eta, \tau \) is in \( \tau D_{log}^{2p}(X \times \mathbb{P}^{r} \times (\mathbb{P}^{1})^{\mathcal{A}}; p) \), then \( \omega \cdot (\eta \cdot \tau) = (\omega \cdot \eta) \cdot \tau \) holds. Using this fact we can show the associativity of the product in the same way as the proof of the associativity of the product in \( \hat{K}_{0}(X) \) in [GS2 Thm.7.3.2].

It is easy to see that the diagram

\[
\begin{array}{ccc}
\tilde{K}_{0}(X) \times \hat{K}_{0}^{M}(T) & \longrightarrow & \hat{K}_{0}^{M}(T) \\
\chi_{0} \times \chi_{T,0} & & \chi_{T,0} \\
\tau D_{0}(X) \times \tilde{D}_{A,\mathcal{P},r}(X) \longrightarrow \tilde{D}_{A,\mathcal{P},r}(X)
\end{array}
\]

is commutative. This implies that \( \tilde{K}_{0}(T) \subset \hat{K}_{0}^{M}(T) \) is a \( \hat{K}_{0}(X) \)-submodule. Moreover, since

\[
\begin{array}{ccc}
\tilde{K}_{0}(X) \times \hat{K}_{0}^{M}(T) & \longrightarrow & \hat{K}_{0}^{M}(T) \\
\zeta \times \zeta & & \zeta \\
K_{0}(X) \times K_{0}(T) & \longrightarrow & K_{0}(T)
\end{array}
\]

is commutative, \( \hat{K}_{0}^{M}(T; T_{1}, \ldots, T_{r}) \) and \( \hat{K}_{0}(T; T_{1}, \ldots, T_{r}) \) are also \( \hat{K}_{0}(X) \)-submodules of \( \hat{K}_{0}^{M}(T) \).

**Proposition 10.4.** The splitting maps

\[
\hat{t}: \hat{K}_{0}^{M}(T) \rightarrow \hat{K}_{0}^{M}(T; T_{1}, \ldots, T_{r}),
\]

\[
\hat{t}: \hat{K}_{0}(T) \rightarrow \hat{K}_{0}(T; T_{1}, \ldots, T_{r})
\]

respect \( \hat{K}_{0}(X) \)-module structures.

**Proof:** Since the second map is the restriction of the first map, it suffices to proof the claim for the first one. We will prove that the composite

\[
\hat{K}_{0}^{M}(T) \xrightarrow{\hat{t}} \hat{K}_{0}^{M}(T; T_{1}, \ldots, T_{r}) \xrightarrow{\hat{q}} \hat{K}_{0}^{M}(T)
\]
is a map of $\hat{K}_0(X)$-modules. Let $\mathcal{F}$ be a hermitian vector bundle on $X$, and $\mathcal{G}$ a hermitian vector bundle on $T$. Since $qt(\mathcal{G})$ is equal to $(1 - p^*_r t^*_r) \cdots (1 - p^*_1 t^*_1)\mathcal{G}$, there is a canonical isometry $qt(\mathcal{F} \otimes \mathcal{G}) \simeq \mathcal{F} \otimes qt(\mathcal{G})$ as virtual hermitian vector bundles on $T$. Hence the proposition follows. \hfill $\Box$

10.2. A $\hat{K}_0(X)$-module structure on $\hat{K}_0(X \times \Box^r; X \times \partial \Box^r)_\mathbb{Q}$. Let $X$ be a smooth proper variety defined over an arithmetic field, and $\mathcal{F}$ a hermitian vector bundle on $X$. Applying the construction in §9.3 we can obtain a tensor product functor

$$(\mathcal{F} \otimes ) : \tilde{\mathcal{C}}^\text{Alt}_\ast(X \times \Box^r; X \times \partial \Box^r) \rightarrow \tilde{\mathcal{C}}^\text{Alt}_\ast(X \times \Box^r; X \times \partial \Box^r).$$

**Proposition 10.5.** The diagram

$$
\begin{array}{ccc}
\tilde{\mathcal{C}}^\text{Alt}_\ast(X \times \Box^r; X \times \partial \Box^r)[r] & \xrightarrow{\mathcal{F} \otimes} & \tilde{\mathcal{C}}^\text{Alt}_\ast(X \times \Box^r; X \times \partial \Box^r)[r] \\
\downarrow \text{ch}_\ast & & \downarrow \text{ch}_\ast \\
\tilde{D}_{A,P,\ast}(X) & \xrightarrow{\text{ch}_0(\mathcal{F}) \ast_{A,P}} & \tilde{D}_{A,P,\ast}(X)
\end{array}
$$

is commutative.

*Proof:* For any $x_I \in \tilde{\mathcal{C}}^\text{Alt}_\ast(X \times D_I)$ with $I \subset \{1, \ldots, r\}$, it holds that $\text{ch}_n(\mathcal{F} \otimes x_I)_P = \text{ch}_0(\mathcal{F}) \ast_{A,P} \text{ch}_n(x_I)_P$ in $\tilde{D}_{A,P,r+n}(X)$. Moreover, for $I \subset J$ with $I \neq J$ and for a division $K_1 \cdots K_l \cdots K_l \cdots K_l \cdots K_l = J$, $\text{ch}_\ast(\langle \mathcal{F}; \Xi_{K_1}, \ldots, \Xi_{K_l} \rangle (x_I)) = 0$ because $\langle \mathcal{F}; \Xi_{K_1}, \ldots, \Xi_{K_l} \rangle (x_I)$ is isometrically equivalent to a degenerate element. Hence the proposition follows. \hfill $\Box$

**Corollary 10.6.** Let $(\mathcal{F}, \eta)$ be a pair of a hermitian vector bundle $\mathcal{F}$ on $X$ with $\eta \in \tau D_1(X) / \text{Im } d_2$. Then for

$$(y, \tau) \in \tilde{\mathcal{C}}^\text{Alt}_0(X \times \Box^r; X \times \partial \Box^r) \oplus \tilde{D}_{A,P,\ast+1}(X) = s(\text{ch}_\ast)_r,$$

the product

$$(10.5) \quad (\mathcal{F}, \eta) \cdot (y, \tau) = (\mathcal{F} \otimes y, \text{ch}_0(\mathcal{F}) \ast_{A,P} \tau),$$

gives a map of complexes

$$(\mathcal{F}, \eta) \cdot : \hat{K}_0(X \times \Box^r; X \times \partial \Box^r)_\mathbb{Q} \rightarrow \hat{K}_0(X \times \Box^r; X \times \partial \Box^r)_\mathbb{Q}.$$

**Proposition 10.7.** The isomorphism

$$\hat{K}_P,\ast(X)_\mathbb{Q} \simeq \hat{K}_0(X \times \Box^r; X \times \partial \Box^r)_\mathbb{Q}$$

given in Prop.5.1 is compatible with the product with the pair $(\mathcal{F}, \eta)$. Hence the product

$$(10.5)$$

gives a $\hat{K}_0(X)$-module structure on $\hat{K}_0(X \times \Box^r; X \times \partial \Box^r)_\mathbb{Q}$, and the above isomorphism respects the $\hat{K}_0(X)$-module structures.

*Proof:* Let

$$i_X : \tilde{\mathcal{C}}^\text{Alt}_\ast(X) \rightarrow \tilde{\mathcal{C}}^\text{Alt}_\ast(X \times \Box^r; X \times \partial \Box^r)[r]$$
be the restriction to the alternating part of the map \([3.2]\) defined in §3.6. Then we can easily verify that \(\iota_X\) commutes with the tensor product functor, from which the proposition follows. \(\square\)

10.3. **Comparison of the \(\hat{K}_0(X)\)-module structures on \(\hat{K}_r(X)\) and on \(\hat{K}_0(T)\).** In this subsection we will prove the following theorem:

**Proposition 10.8.** The map

\[
\tilde{\iota}_0^r : \hat{K}_0(T; T_1, \ldots, T_r)_Q \to \hat{K}_0(X \times \Box^r; X \times \partial \Box^r)_Q
\]

respects the \(\hat{K}_0(X)\)-module structure.

**Proof:** It suffices to show that the map \([5.4]\)

\[
\tilde{\iota}_0^r : \hat{K}_0(T) \to \hat{K}_0(X \times \Box^r; X \times \partial \Box^r)_Q
\]

respects the \(\hat{K}_0(X)\)-module structures. Recall the morphisms

\[
\iota_j : (T_j; T_1 \cap T_j, \ldots, T_{j-1} \cap T_j) \to (T; T_1, \ldots, T_{j-1}),
\]

\[
p_j : (T; T_1, \ldots, T_{j-1}) \to (T_j; T_1 \cap T_j, \ldots, T_{j-1} \cap T_j)
\]

introduced in §4.2, and the maps of complexes induced by them

\[
\tilde{\Theta}^r_0(T; T_1, \ldots, T_{j-1}) \xrightarrow{\iota_j^*} \tilde{\Theta}^r_0(T_j; T_1 \cap T_j, \ldots, T_{j-1} \cap T_j).
\]

Let \(\mathcal{F}\) be a hermitian vector bundle on \(X\). Prop.9.4 says that we can obtain a homotopy \(\Phi_i\) from \((\mathcal{F} \otimes)\iota_j^*\) to \(\iota_j^*(\mathcal{F} \otimes)\) and a homotopy \(\Phi_p\) from \((\mathcal{F} \otimes)p_j^*\) to \(p_j^*(\mathcal{F} \otimes)\). Let \(\Psi\) be the homotopy from the identity to \(\iota_j^*p_j^*\) given in Prop.2.20. Then by Prop.9.6 we have a second homotopy

\[
\Theta : \tilde{\Theta}^r_0(T_j; T_1 \cap T_j, \ldots, T_{j-1} \cap T_j) \to \tilde{\Theta}^r_0(T_j; T_1 \cap T_j, \ldots, T_{j-1} \cap T_j)
\]

from \(\Phi_i p_j^* + \iota_j^* \Phi_p + (\mathcal{F} \otimes) \Psi\) to \(\Psi (\mathcal{F} \otimes)\). Hence it follows from Prop.2.8 that the diagram

\[
\begin{array}{ccc}
\tilde{\Theta}^r_0(T; T_1, \ldots, T_{j-1}) & \xrightarrow{\iota_j} & \tilde{\Theta}^r_0(T; T_1, \ldots, T_j) \\
(\mathcal{F} \otimes) & & (\mathcal{F} \otimes) \\
\tilde{\Theta}^r_0(T; T_1, \ldots, T_{j-1}) & \xrightarrow{\iota_j} & \tilde{\Theta}^r_0(T; T_1, \ldots, T_j)
\end{array}
\]

(10.6)

is commutative up to homotopy. Denote by \(\Pi_j\) the homotopy from \((\mathcal{F} \otimes)\iota_j\) to \(\iota_j(\mathcal{F} \otimes)\) given in Prop.2.8. Since the images of the homotopies \(\Phi_i, \Phi_p, \Psi\) and the second homotopy \(\Theta\) are isometrically equivalent to degenerate elements, \(\Pi_j^{m,n}(x)\) is isometrically equivalent to a degenerate element for any \(m\) and \(n\). Connecting (10.6) for all \(j\), we can obtain the following commutative diagram up to homotopy

\[
\begin{array}{ccc}
\tilde{\Theta}^r_0(T) & \xrightarrow{t} & \tilde{\Theta}^r_0(T; T_1, \ldots, T_r) \\
(\mathcal{F} \otimes) & & (\mathcal{F} \otimes) \\
\tilde{\Theta}^r_0(T) & \xrightarrow{t} & \tilde{\Theta}^r_0(T; T_1, \ldots, T_r),
\end{array}
\]

(10.7)
and a homotopy $\Pi$ from $(\mathcal{F} \otimes )t$ to $t(\mathcal{F} \otimes )$ is given by

$$\Pi = \sum_{j=1}^{r} t_r \cdots t_{j+1} \Pi_j t_{j-1} \cdots t_1.$$  

It is obvious that $\Pi^{0,n}(x)$ is isometrically equivalent to a degenerate element for any $n$. On the other hand, Prop.9.4 says that the diagram

$$\begin{array}{ccc}
\widetilde{QC}_A^t(T; T_1, \ldots, T_r) & \xrightarrow{i^*_0} & \widetilde{QC}_A^t(X \times \square^r; X \times \partial \square^r) \\
(\mathcal{F} \otimes ) & \downarrow & (\mathcal{F} \otimes ) \\
\widetilde{QC}_A^t(T; T_1, \ldots, T_r) & \xrightarrow{i^*_0} & \widetilde{QC}_A^t(X \times \square^r; X \times \partial \square^r)
\end{array}$$

(10.8)

is also commutative up to homotopy, and if we denote by $\Phi_{i_0}$ the homotopy given in Prop.9.4, then $\Phi_{i_0}^{m,n}(x)$ is isometrically equivalent to a degenerate element for any $m$ and $n$. Combining (10.7) with (10.8) yields the diagram

$$\begin{array}{ccc}
\widetilde{QC}_A^t(T) & \xrightarrow{i^*_0 t} & \widetilde{QC}_A^t(X \times \square^r; X \times \partial \square^r) \\
(\mathcal{F} \otimes ) & \downarrow & (\mathcal{F} \otimes ) \\
\widetilde{QC}_A^t(T) & \xrightarrow{i^*_0 t} & \widetilde{QC}_A^t(X \times \square^r; X \times \partial \square^r)
\end{array}$$

which is also commutative up to homotopy, and a homotopy from $(\mathcal{F} \otimes )i^*_0 t$ to $i^*_0 t(\mathcal{F} \otimes )$ is given by $\Pi' = \Phi_{i_0} t + i^*_0 \Pi$. It is obvious that $\Pi^{0,n}(x)$ is isometrically equivalent to a degenerate element as well.

Any element of $\widetilde{K}(T)$ is represented by a pair $(\mathcal{G}, \tau)$ of a virtual hermitian vector bundle $\mathcal{G}$ on $T$ with $\tau \in \tilde{D}_{h,p} r + 1(X) / \text{Im } d_{\tau}$ such that $\text{ch}_{1,0}(\mathcal{G}) + d_{\tau} = 0$. Since the map (5.4)

$$i^*_0 t : \widetilde{K}(T) \to \widetilde{K}(X \times \square^r; X \times \partial \square^r)$$

sends $[(\mathcal{G}, \tau)]$ to $[(i^*_0 t(\mathcal{G}), -\tau)]$, we have

$$[((\mathcal{F}, \eta)) \cdot i^*_0 t (([\mathcal{G}, \tau])] = [(\mathcal{F} \otimes i^*_0 t(\mathcal{G}), -\text{ch}_0(\mathcal{F} \cdot A, p, \tau)].$$

On the other hand, since $\text{ch}_{1,0}(\mathcal{G}) + d_\tau = 0$, we have

$$i^*_0 t ([((\mathcal{F}, \eta))] \cdot ([\mathcal{G}, \tau])] = i^*_0 t ([((\mathcal{F} \otimes \mathcal{G}, \text{ch}_0(\mathcal{F} \cdot A, p, \tau)) = [(i^*_0 t(\mathcal{G} \otimes \mathcal{F}), -\text{ch}_0(\mathcal{F} \cdot A, p, \tau)].$$

Since $\text{ch}_0(\Pi^{0,n}(\mathcal{G})) = 0$, we have

$$(i^*_0 t(\mathcal{F} \otimes \mathcal{G}) - \mathcal{F} \otimes i^*_0 t(\mathcal{G}), 0) = \partial(\Pi'(\mathcal{G}), 0),$$

which means that $[(\mathcal{F}, \eta)] \cdot i^*_0 t ([\mathcal{G}, \tau])] = i^*_0 t ([([\mathcal{F}, \eta]) \cdot ([\mathcal{G}, \tau])]$. This completes the proof. □
11. Compatibility with product

11.1. Product of \( \widehat{CH}^q(X, r) \) with \( \widehat{CH}^p(X) \). In [BF], Burgos and Feliú constructed a product in higher arithmetic Chow groups. In this subsection, we will recall their construction in the case of the product with \( \widehat{CH}^p(X) \). We begin by fixing some notations.

Throughout this section, fix \( r \geq 1 \). For a variety \( X \) defined over a field, denote by \( Z^p_X \) the set of all closed subschemes of \( X \) of codimension \( p \), and by \( Z^q_{X,r} \) the set of all admissible subschemes of \( X \times \square^r \) of codimension \( q \). For another variety \( Y \), write \( Z^p_{X,Y,r} = Z^p_X \times Z^q_Y \).

We regard \( Z^p_{X,Y,r} \) as a subset of \( Z^{p+q}_{X \times Y,r} \) in the way that \( (Z, W) \mapsto Z \times W \). In what follows, we identify \( Z^p_X \) with \( Z^p_{X,Y,r} \) by \( Z \mapsto Z \times Y \times \square^r \) and \( Z^q_Y \) with \( Z^q_{X,Y,r} \) by \( W \mapsto X \times W \).

Suppose \( X \) and \( Y \) are compact complex algebraic manifolds. Write \( \square^r \) for short. Then it is shown in [BF, Lem.5.5] that \( X \), the set of all closed subschemes of \( X \times \square^r \) of codimension \( p \), is compact complex algebraic manifold. We regard \( X \times Y \) as a subset of \( X \times Y \times \square^r \).

Throughout this section, fix \( r \). To be more precise, \( X,Y,r \). To be more precise, \( X,Y,r \).

Let \( D \) be a quasi-isomorphism. Take the truncated subcomplex \( D \).

Then it is shown in [BF, Lem.5.5] that

\[
0 \to D^\ast_{\log}(\square^r X,Y - Z^p_{X,Y,r}, p + q) \to D^\ast_{\log}(\square^r X,Y - Z^q_{Y,r}, p + q) \to 0
\]

is a short exact sequence, where the map \( j^p_{X,Y,r} \) is given by \((\omega, \tau) \mapsto -\omega + \tau r \). This implies that the natural map

\[
\begin{aligned}
D^\ast_{\log}(\square^r X,Y - Z^p_{X,Y,r}, p + q) \to s(-j^p_{X,Y,r})^\ast
\end{aligned}
\]

is a quasi-isomorphism. Let

\[
\begin{aligned}
j^p_{X,Y,r} : D^\ast_{\log}(\square^r X,Y - Z^p_{X,Y,r}, p + q) \to s(-j^p_{X,Y,r})^\ast
\end{aligned}
\]

be the composite of the restriction to \( D^\ast_{\log}(\square^r X,Y - Z^p_{X,Y,r}, p + q) \) with (11.1), and take the simple complex \( s(i^p_{X,Y,r})^\ast \). To be more precise,

\[
\begin{aligned}
s(i^p_{X,Y,r})^n = D^0_{\log}(\square^r X,Y - Z^p_{X,Y,r}, p + q) \oplus D^0_{\log}(\square^r X,Y - Z^q_{X,Y,r}, p + q) \\
\oplus D^{n-1}_{\log}(\square^r X,Y - Z^p_{X,Y,r}, p + q) \oplus D^{n-2}_{\log}(\square^r X,Y - Z^p_{X,Y,r}, p + q) \\
\oplus D^{n-1}_{\log}(\square^r X,Y - Z^q_{X,Y,r}, p + q) \oplus D^{n-2}_{\log}(\square^r X,Y - Z^q_{X,Y,r}, p + q)
\end{aligned}
\]

with the differential given by

\[
ds(\omega_0, (\omega_1, \omega_2), \omega_3) = (dD\omega_0, (\omega_0 - dD\omega_1, \omega_0 - dD\omega_2), -\omega_1 + \omega_2 + dD\omega_3).
\]

If we set

\[
D^\ast_{\log, Z^p_{X,Y,r}}(\square^r X,Y,p + q) = s(D^\ast_{\log}(\square^r X,Y - Z^p_{X,Y,r}, p + q) \to D^\ast_{\log}(\square^r X,Y - Z^p_{X,Y,r}, p + q)),
\]

then the natural map

\[
D^\ast_{\log, Z^p_{X,Y,r}}(\square^r X,Y,p + q) \to s(i^p_{X,Y,r})^\ast
\]

is a quasi-isomorphism. Take the truncated subcomplex

\[
D^\ast_{\log, Z^p_{X,Y,r}}(X \times Y,p + q) = \tau_{\leq 2(p + q)} D^\ast_{\log, Z^p_{X,Y,r}}(\square^r X,Y,p + q).
\]

Let \( D^\ast_{\log, Z^p_{X,Y}}(X \times Y,p + q) \) be the single complex of the normalized subcomplex of \( D^\ast_{\log, Z^p_{X,Y,r}}(X \times Y,p + q) \) with respect to the cubical structure on the index \( r \). In the same
way, we can obtain the single complex of the normalized subcomplex of \( \tau_{\leq 2(p+q)} s(p,q) \) with respect to the cubical structure on the index \( r \), which we denote by \( s_{\mathcal{A}}(p,q) \). Then the map \( \text{(11.2)} \) induces a quasi-isomorphism

\[
\mathcal{D}_{\mathcal{A}, \mathcal{Z}_{X,Y}^{p,q}}(X \times Y, p + q) \rightarrow s_{\mathcal{A}}(p,q).
\]

Let

\[
\pi_X : \Box_{X,Y} = X \times Y \times \Box^r \rightarrow X,
\]
\[
\pi_Y : \Box_{X,Y} = X \times Y \times \Box^r \rightarrow Y \times \Box^r
\]

be the projections, and define an exterior product of \( \omega \in \tau \mathcal{D}^*(X, p) \) with \( \omega' \in \tau \mathcal{D}_{\log}^*(Y \times \Box^r, q) \) as follows:

\[
\omega *_{\mathcal{A}} \omega' = \pi_X^* \omega \bullet \pi_Y^* \omega' \in \tau \mathcal{D}_{\log}^*(\Box_{X,Y}^r, p + q).
\]

Then it induces a map of complexes

\[
*_{\mathcal{A}} : \tau \mathcal{D}^*(X, p) \times \tau \mathcal{D}_{\log}^*(Y \times \Box^r, q) \rightarrow \tau \mathcal{D}_{\log}^*(\Box_{X,Y}^r, p + q).
\]

Similarly, for \( g \in \tau \mathcal{D}_{\log}(X - \mathcal{Z}_X^{p,q}, p) \) and \( g' \in \tau \mathcal{D}_{\log}(Y \times \Box^r - \mathcal{Z}_Y^{q}, q) \), define

\[
g *_{\mathcal{A}} g' = \pi_X^* g \bullet \pi_Y^* g' \in \tau \mathcal{D}_{\log}^*(\Box_{X,Y} - \mathcal{Z}_X^{p} \cup \mathcal{Z}_Y^{q}, p + q),
\]

\[
\omega *_{\mathcal{A}} \omega' = \pi_X^* \omega \bullet \pi_Y^* \omega' \in \tau \mathcal{D}_{\log}^*(\Box_{X,Y} - \mathcal{Z}_Y^{q}, p + q).
\]

Using these products, we can define

\[
(11.3) \quad \circ : \tau \mathcal{D}_{\log}^2(X, p) \times \tau \mathcal{D}_{\log}^m(\Box_{X,Y}^r, q) \rightarrow \tau \mathcal{D}_{\log}^n(\Box_{X,Y}^r, p + q)
\]

by

\[
(\omega, g) \circ (\omega', g') \mapsto (\omega *_{\mathcal{A}} \omega', (g *_{\mathcal{A}} \omega', (-1)^n \omega *_{\mathcal{A}} g'), (-1)^{n-1} g *_{\mathcal{A}} g'),
\]

which turns out to be a map of complexes. Taking the cohomology yields

\[
H^{2p+2q}_{\mathcal{D}_{\log}^*}(X, \mathbb{R}(p)) \times H^{2q}_{\mathcal{D}_{\log}^*}(Y \times \Box^r, \mathbb{R}(q)) \rightarrow H^{2p+2q}_{\mathcal{D}_{\log}^*}(\mathcal{Z}_{X,Y}^{p,q} \times \mathcal{Z}_{Y,Y}^{q}).
\]

Combining this map with

\[
H^{2p+2q}_{\mathcal{D}_{\log}^*}(\mathcal{Z}_{X,Y}^{p,q} \times \mathcal{Z}_{X,Y}^{q}) \simeq H^{2p+2q}_{\mathcal{D}_{\log}^*}(\mathcal{Z}_{X,Y}^{p,q} \times \mathbb{R}(p + q)) \rightarrow H^{2p+2q}_{\mathcal{D}_{\log}^*}(\Box_{X,Y}^r, \mathbb{R}(p + q))
\]

induced by \( \text{(11.2)} \) and the inclusion \( \mathcal{Z}_{X,Y}^{p,q} \subset \mathcal{Z}_{X,Y}^{p+q} \), and taking the normalized subcomplexes, we obtain a map

\[
\times : \mathcal{H}^p(X, 0) \times \mathcal{H}^q(Y, r)_0 \rightarrow \mathcal{H}^{p+q}(X \times Y, r)_0.
\]

Prop. 5.13 in \[BF\] says that

\[
Z^p(X, 0) \times Z^q(Y, r)_0 \xrightarrow{\times} Z^{p+q}(X \times Y, r)_0
\]

\[
\mathcal{H}^p(X, 0) \times \mathcal{H}^q(Y, r)_0 \xrightarrow{\times} \mathcal{H}^{p+q}(X \times Y, r)_0
\]
is commutative, where the upper horizontal arrow is the exterior product of cycles. On the other hand, taking the single complexes of the normalized subcomplexes on the both sides of (11.3), we have

\[ \diamond : \tau D^*_\mathbb{Z}(X, p) \times \mathbb{Z}^*_\mathbb{A}(Y, q)_0 \to s_\mathbb{A}(i_{X,Y}^{p,q})^* \]

An element \( a \in s_\mathbb{A}(i_{X,Y}^{p,q})_0^{2p+2q-r} \) is written as

\[ a = \sum_{j=0}^r a_j, \quad a_j \in \tau_{\leq 2p+2q-r} s(i_{X,Y,j}^{p,q})_0^{2p+2q-r+j}. \]

Then by the quasi-isomorphism (11.2) we have the cohomology class

\[ [a_r] \in H^{2(p+q)}_{D,\mathbb{Z}^*_\mathbb{A}(Y,r)} (\square_{X \times Y} ; p + q)_0. \]

Since \( \mathbb{Z}^{p,q}_{\mathbb{A}, \mathbb{P}^r} \subset \mathbb{Z}^{p,q}_{\mathbb{A} \times \mathbb{P}^r} \), the correspondence \( a \mapsto [a_r] \) gives

\[ \chi_2 : s_\mathbb{A}(i_{X,Y}^{p,q})_0^{2p+2q-r} \to H^{p+q}(X 	imes Y, *)_0, \]

which turns out be a map of complexes. Cor.5.14 in [BF] says that

\[ \tau D^*_{\mathbb{Z}^p}(X, p) \times D^*_{\mathbb{A}, \mathbb{Z}^q}(Y, q)_0 \overset{\circ}{\longrightarrow} s_\mathbb{A}(i_{X,Y}^{p,q})_0^{2p+2q-r} \]

\[ \begin{align*}
\tau^p(X, 0) \times \tau^q(Y, *)_0 & \xrightarrow{\times} H^{p+q}(X \times Y, *)_0 \\
\chi_2 \times \chi_2 & \end{align*} \]

is commutative.

Denote the following canonical maps by the same symbol \( \rho \):

\[ \rho : \tau D^*_\mathbb{Z}(X, p) \to \tau D^*_\mathbb{X}(X, p), \]

\[ \rho : D^*_\mathbb{A}(Y, q)_0 \to D^*_\mathbb{A}(Y, q)_0 \simeq \overline{D}^*_\mathbb{A}(Y, q) \to \overline{D}^*_\mathbb{A}(X, Y, p + q), \]

\[ \rho : s_\mathbb{A}(i_{X,Y}^{p,q})_0 \to \overline{D}^*_\mathbb{A}(X \times Y, p + q) \simeq \overline{D}^*_\mathbb{A}(X \times Y, p + q). \]

Moreover, define a map of complexes

\[ (11.4) \quad \ast_{\mathbb{A}, \mathbb{P}} : \tau D^*_\mathbb{X}(X, p) \times \overline{D}^*_\mathbb{A}(Y, q) \to \overline{D}^*_\mathbb{A}(X \times Y, p + q) \]

by \( \ast_{\mathbb{A}, \mathbb{P}} \omega' = \pi_X^* \omega \bullet \pi_Y^* \omega' \), where

\[ \pi_X : X \times Y \times \square^r \times (\mathbb{P}^1)^s \to X, \]

\[ \pi_Y : X \times Y \times \square^r \times (\mathbb{P}^1)^s \to Y \times \square^r \times (\mathbb{P}^1)^s \]

are the projections. Then the diagram

\[ \begin{align*}
\tau D^*_{\mathbb{Z}^p}(X, p) \times D^*_{\mathbb{A}, \mathbb{Z}^q}(Y, q)_0 & \overset{\circ}{\longrightarrow} s_\mathbb{A}(i_{X,Y}^{p,q})_0^* \\
\rho \times \rho & \\
\tau D^*(X, p) \times \overline{D}^*_\mathbb{A}(X \times Y, p + q) & \longrightarrow \overline{D}^*_\mathbb{A}(X \times Y, p + q) \end{align*} \]

is commutative.
Under the above preparation, we can define an exterior product of \( \widetilde{CH}^p(X) \) with \( \widetilde{CH}^q(Y, r) \). Let \((y, (\omega, \bar{g}))\) be a pair of \( y \in Z^p(X) \) with a Green form \((\omega, \bar{g})\) associated with \( y \), and take a lift \( g \in \tau D_{\log}^{2p-1}(X - Z^p, p) \). Let
\[
\left( \begin{array}{c}
\beta_1 \\
\beta_2 \\
z \\
\alpha
\end{array} \right) \in \widehat{Z}^q(Y, r)_0.
\]
Define their product as follows:
\[
(11.5) \quad (y, (\omega, g)) \cdot \left( \begin{array}{c}
\beta_1 \\
\beta_2 \\
z \\
\alpha
\end{array} \right) = \left( \begin{array}{c}
\chi_1(y) \times \beta_1 \\
\chi_1(y) \times \beta_2 \\
\omega \ast \beta_1 \\
\omega \ast \beta_2 \\
(y \times z) \times \alpha
\end{array} \right).
\]
Then as shown in [BF] §5, this product gives a map
\[
\widetilde{CH}^p(X) \times \widetilde{CH}^q(Y, r) \rightarrow H_r
\]
\[
(11.6) \quad \begin{array}{c}
\mathcal{H}^{p+q}(X \times Y, *)_{0} \\
\mathcal{H}^{2p+2q-\ast}(X \times Y, p + q)
\end{array}
\]
\[
\begin{array}{c}
\chi_1 \\
\chi_2 \\
\rho
\end{array}
\]
\[
\begin{array}{c}
Z^{p+q}(X \times Y, *)_{0} \\
\mathcal{H}^{2p+2q-\ast}(X \times Y, p + q)
\end{array}
\]
where \( \mathcal{D}_{\ast}^\ast(X \times Y, p + q) \) is the complex defined in §6.3. On the other hand, it is obvious that the diagram
\[
(11.7) \quad s_{\mathcal{H}(p,q)}^\ast : (Y, r) \rightarrow \mathcal{D}_{\ast}^\ast(X \times Y, p + q) \rightarrow \mathcal{D}_{\ast}^\ast(X \times Y, p + q)
\]
\[
\rho \quad \rho
\]
is commutative, hence it gives
\[
(11.8) \quad \approx H_r \begin{array}{c}
\mathcal{H}^{p+q}(X \times Y, *)_{0} \\
\mathcal{H}^{2p+2q-\ast}(X \times Y, p + q)
\end{array}
\]
\[
\begin{array}{c}
\chi_1 \\
\chi_2 \\
\rho
\end{array}
\]
\[
\begin{array}{c}
Z^{p+q}(X \times Y, *)_{0} \\
\mathcal{D}_{\ast}^\ast(X \times Y, p + q)
\end{array}
\]
\[
\begin{array}{c}
\mathcal{H}^{2p+2q-\ast}(X \times Y, p + q)
\end{array}
\]
\[
\rightarrow H_r \begin{array}{c}
\mathcal{H}^{p+q}(X \times Y, *)_{0} \\
\mathcal{H}^{2p+2q-\ast}(X \times Y, p + q)
\end{array}
\]
\[
\begin{array}{c}
\chi_1 \\
\chi_2 \\
\rho
\end{array}
\]
\[
\begin{array}{c}
Z^{p+q}(X \times Y, *)_{0} \\
\mathcal{D}_{\ast}^\ast(X \times Y, p + q)
\end{array}
\]
Define an exterior product of higher arithmetic Chow groups

\[ \cup : \widehat{CH}^p(X) \times \widehat{CH}^q(Y, r) \to \widehat{CH}^{p+q}(X \times Y, r) \]

to be the composite of \([11.6]\) with \([11.5]\). Substituting \(\sigma_{<2q-r} \widehat{D}^s_{h,p}(Y, q)\) for \(\widehat{D}^s_{h,p}(Y, q)\) and \(\sigma_{<2p+2q-r} \widehat{D}^s_{h,p}(X \times Y, q)\) for \(\widehat{D}^s_{h,p}(X \times Y, q)\), we can also define an extended exterior product

\[ \cup : \widehat{CH}^p(X) \times \widehat{CH}^q(Y, r) \to \widehat{CH}^{p+q}(X, r) \]

to be the composite of the exterior product in the case that \(X = Y\) with the pull-back map \(\widehat{\Delta}^*\).

**Proposition 11.1.** Let \(\eta^p \in \tau D^{2p-1}(X, p)\). If \(r \geq 1\), then for any \(x \in \widehat{CH}^q(Y, r)\) the exterior product \(a(\eta^p) \cup x\) is equal to zero.

**Proof:** Suppose that \(x\) is represented by

\[ \begin{pmatrix} \beta_1 & \beta_2 \\ y & \alpha \end{pmatrix} \in \text{Ker } \partial \subset \widehat{Z}^p(Y, r)_0. \]

Then

\[
\begin{pmatrix}
H_{p}^p \left( \begin{array}{c}
\mathcal{H}^{p+q}(X \times Y, *)_0 \\
\mathcal{D}_{h,p}^{2p+2q-\ast}(X \times Y, p + q)
\end{array} \right)
\end{pmatrix}
\]

the product of \((0, (d_D \eta^p, \eta^p))\) with the element \([11.9]\) is given by

\[ \begin{pmatrix} 0 & d_D \eta^p \ast_{h,p} \beta_2 \\ 0 & (d_D \eta^p, \eta^p) \circ \alpha \end{pmatrix}. \]

If we write \(a = (\alpha_1, \alpha_2) \in D_{h,2q-r}^*(Y, q)_0\) with \(\alpha_1 \in D_{h,2p-r}^*(Y, q)_0\) and \(\alpha_2 \in D_{h,2p-r-1}^*(Y - Z^q, q)_0\), then it holds in \(s(i_{X,Y,r}^{p,q}, 2p+2q-r)\) that

\[
(d_D \eta^p, \eta^p) \circ \alpha = (d_D \eta^p, \eta^p) \circ (\alpha_1, \alpha_2) = (d_D \eta^p \ast_{h} \alpha_1, \eta^p \ast_{h} a_1, d_D \eta^p \ast_{h} \alpha_2), -\eta^p \ast_{h} \alpha_2).
\]

Since \(\eta^p \in \tau D^{2p-1}(X, p)\) and \(d_s \alpha = 0\), it is equal to

\[ d_s(\eta^p \ast_{h} \alpha_1, (0, -\eta^p \ast_{h} \alpha_2), 0). \]
It follows from the definition of \( \chi_2 \) that \( \chi_2(\eta^p \ast_{\mathbb{A}} \alpha_1, (0, -\eta^p \ast_{\mathbb{A}} \alpha_2), 0) = 0 \). Hence (11.10) is homologous to

\[
\begin{pmatrix}
0 & d_\Omega \eta^p \ast_{\mathbb{A}, \mathbb{F}} \beta_2 - \eta^p \ast_{\mathbb{A}, \mathbb{F}} \alpha_1 \\
0 & 0
\end{pmatrix}.
\]

The \( \partial \)-closedness of (11.9) implies that \( d_s \beta_2 = \rho(\alpha) = \alpha_1 \), therefore

\[d_\Omega \eta^p \ast_{\mathbb{A}, \mathbb{F}} \beta_2 - \eta^p \ast_{\mathbb{A}, \mathbb{F}} \alpha_1 = d_s(\eta^p \ast_{\mathbb{A}, \mathbb{F}} \beta_2),\]

which means that (11.11) is \( \partial \)-exact. Hence the proposition follows. \( \square \)

11.2. The main theorem. In this subsection we prove the following theorem:

**Theorem 11.2.** Let \( X \) be a smooth projective variety defined over an arithmetic field. Then

\[
\hat{K}_0(X) \times \hat{K}_r(X)_{\mathbb{Q}} \longrightarrow \hat{K}_r(X)_{\mathbb{Q}}
\]

is commutative.

**Proof:** We have shown in §10 that the sequence of maps

\[
\hat{K}_r(X)_{\mathbb{Q}} \simeq \hat{K}_0(X \times \square^r; X \times \partial \square^r)_{\mathbb{Q}} \leftarrow \hat{K}_0(T; \{T_1, \ldots, T_r\})_{\mathbb{Q}} \subset \hat{K}_0(T)_{\mathbb{Q}}
\]

respects the \( \hat{K}_0(X) \)-module structures. Hence we have only to show that the diagram

\[
\begin{array}{ccc}
\hat{K}_0(X) \times \hat{K}_r(T) & \longrightarrow & \hat{K}_0(T) \\
\oplus \hat{\chi}^p_0 \times \hat{\chi}^q_0 & \downarrow \hat{\chi}^n_0 & \\
\oplus \hat{\chi}^p(T) \times \hat{\chi}^q(T) & \downarrow \hat{\chi}^n_0 & \\
\end{array}
\]

(11.12)

is commutative.

First we consider the product with \( x_1 = [(0, \tilde{\eta})] \in \hat{K}_0(X) \) where \( \eta \in \tau \Omega_1(X) \). In this case it follows from the definition of the product of arithmetic \( K_0 \)-groups described as (10.2) that \( x_1 \cdot x_2 = 0 \) for any \( x_2 \in \hat{K}_0(T) \), and Prop.11.1 says that \( a(\tilde{\eta}^p) \cdot \hat{\chi}^q_{T,0}(x_2) = 0 \). Hence the theorem follows in this case.

Let us now introduce exterior product of arithmetic \( K_0 \)-groups. Let \( Y \) be a smooth projective variety, and \( T = D(Y \times \square^r; Y \times \partial \square^r) \) the associated iterated double. We identify \( X \times T \) with \( T' = T(X \times Y \times \square^r; X \times Y \times \partial \square^r) \). Define an exterior product

\[
\cup : \hat{K}_0(X) \times \hat{K}_0^M(T) \rightarrow \hat{K}_0^M(T')
\]

by

\[
[(\mathcal{F}, \tilde{\eta})] \cup [(\mathcal{F}', \tilde{\eta}')] = [(\mathcal{F} \boxtimes \mathcal{F}', \chi_{0}(\mathcal{F}) \ast_{\mathbb{A}, \mathbb{F}} \tilde{\tau} + \tilde{\eta} \ast_{\mathbb{A}, \mathbb{F}} \chi_{T,0}(\mathcal{F}) + d_\Omega \eta \ast_{\mathbb{A}, \mathbb{F}} \tilde{\tau})].
\]

In the above, \( \mathcal{F} \boxtimes \mathcal{F}' = \pi_X^* \mathcal{F} \otimes \pi_T^* \mathcal{F}' \), where \( \pi_X : T' = X \times T \rightarrow X \) and \( \pi_T : |PRT| = X \times T \rightarrow T \) are the projections, and the product \( \ast_{\mathbb{A}, \mathbb{F}} \) is defined in (11.3). We can show
in the same way as the proof of Prop.10.3 that this product is well-defined, and it gives a map
\[ \cup : \check{K}_0(X) \times \check{K}_0(T) \to \check{K}_0(T'). \]

Here we assume that \( X = Y \). Let \( \Delta : X \to X \times X \) be the diagonal morphism, and \( \Delta_D : T \to T' \) the morphism between the iterated doubles induced by \( \Delta \). Then the diagram (11.12) is decomposed as follows:

\[
\begin{align*}
\check{K}_0(X) \times \check{K}_0(T) \quad &\quad \cup \quad \quad \check{K}_0(T') \quad \quad \Delta_D \quad \quad \check{K}_0(T) \\
\oplus \hat{\text{ch}}_{T,0}^{p-q} \quad &\quad \downarrow \quad \quad \hat{\text{ch}}_{T',0}^{n} \quad \quad \downarrow \quad \quad \hat{\text{ch}}_{T,0}^{n} \\
\oplus \check{CH}^p(X) \times \check{CH}^q(X, r) \quad &\quad \cup \quad \quad \check{CH}^n(X \times X, r) \quad \quad \Delta^* \quad \quad \check{CH}^n(X, r).
\end{align*}
\]

The right diagram is commutative by (8.6). Hence the theorem follows from the lemma below. \( \square \)

**Lemma 11.3.** For a hermitian vector bundle \( \mathcal{F} \) on \( X \), denote by
\[ \mathcal{F} \boxtimes : \check{K}_0(T) \to \check{K}_0(T') \]
the exterior product with \( [(\mathcal{F}, 0)] \in \check{K}_0(X) \). Then the diagram
\[
\begin{align*}
\check{K}_0(T) \quad &\quad \mathcal{F} \boxtimes \quad \quad \check{K}_0(T') \\
\oplus \check{ch}_{T,0}^{n-p} \quad &\quad \downarrow \quad \quad \check{ch}_{T',0}^{n} \quad \quad \downarrow \quad \quad \check{ch}_{T,0}^{n} \\
\oplus \check{CH}^{n-p}(X, r) \quad &\quad \sum_{p} \check{ch}_{(\mathcal{F})}^{p} \quad \quad \check{CH}^{n}(X \times Y, r)
\end{align*}
\]
is commutative.

**Proof:** In the previous subsection we have shown that the exterior product is extended to
\[ \check{CH}^p(X) \times \check{Z}^q(Y, r)_0^*/\text{Im} \partial \to \check{Z}^{p+q}(X \times Y, r)_0^*/\text{Im} \partial. \]

Hence it suffices to prove the commutativity of the extended diagram
\[
\begin{align*}
\check{K}_0^M(T) \quad &\quad \mathcal{F} \boxtimes \quad \quad \check{K}_0^M(T') \\
\oplus \check{ch}_{T,0}^{n-p} \quad &\quad \downarrow \quad \quad \check{ch}_{T',0}^{p} \quad \quad \downarrow \quad \quad \check{ch}_{T,0}^{n} \\
\oplus \check{Z}^{n-p}(Y, r)_0^*/\text{Im} \partial \quad &\quad \sum_{p} \check{ch}_{(\mathcal{F})}^{p} \quad \quad \check{Z}^{n}(X \times Y, r)_0^*/\text{Im} \partial.
\end{align*}
\]

First we consider the case that \( x_2 = [(0, \tilde{\tau})] \in K_0^M(T) \) with \( \tilde{\tau} \in \check{D}_{A, p, r+1}(X) \). In this case,
\[
\check{ch}_{T',0}^{n}(\mathcal{F} \boxtimes x_2) = \check{ch}_{T',0}^{n}([(0, \check{ch}_{0}(\mathcal{F}) *_{A, p} \tilde{\tau})]) = \sum_{p+q=n} a(ch_{0}(\mathcal{F}) *_{A, p} \tilde{\tau}^q). \]
On the other hand, if \( \hat{\text{ch}}_0^p(\mathcal{F}) \) is represented by \((y^p, (\omega^p, g^p))\) such that \( y^p \in Z^p(X) \) and \((\omega^p, g^p)\) is a Green form associated with \( y^p \), then it follows from the definition of the product that

\[
\hat{\text{ch}}_0^p(\mathcal{F}) \cup a(\tau^q) = \begin{pmatrix}
0 & -\omega^p \ast_{A,P} \tau^q \\
0 & 0
\end{pmatrix} = a(\hat{\text{ch}}_0^p(\mathcal{F}) \ast_{A,P} \tau^q)
\]

since \( \hat{\text{ch}}_0^p(\mathcal{F}) = \omega^p \). Hence the lemma follows in this case.

Finally let us assume that \( x_2 = [(\mathcal{F}, 0)] \). In this case what we need to show is the equality

\[
(11.13) \quad \hat{\text{ch}}_{T,0}^n(\mathcal{F} \boxtimes \mathcal{F}') = \sum_{p+q=n} \hat{\text{ch}}_0^p(\mathcal{F}) \cup \hat{\text{ch}}_0^q(\mathcal{F}')
\]

in \( \tilde{Z}^n(X \times Y, r)^0/\text{Im} \partial \). Take a morphism \( \varphi : T \to G \) to a smooth projective variety \( G \) and a hermitian vector bundle \( \mathcal{G}' \) on \( G \) such that \( \varphi^* \mathcal{G}' \simeq \mathcal{F}' \). Let \((y^p, (\omega^p, g^p)) \) and \((z^q, (\tau^q, h^q)) \) be representatives of \( \hat{\text{ch}}_0^p(\mathcal{F}) \) and \( \hat{\text{ch}}_0^q(\mathcal{G}') \) respectively. Assume that \( z^q \in Z^q(G) \). Then the exterior product of \( \hat{\text{ch}}_0^p(\mathcal{F}) \) with \( \hat{\text{ch}}_0^q(\mathcal{G}') \) in

\[
H_r \left( \begin{array}{c}
\mathcal{H}^{p+q}(X \times Y, *)_0 \\
Z^{p+q}(X \times Y, *)_0
\end{array} \right)
\]

\[
\sigma_{<2p+2q-r} \mathcal{G}_h^{2p+2q-r}(X \times Y, p + q)
\]

\[
s_h(i_{X,Y})^{2p+2q-\ast}
\]

is equal to

\[
(11.14) \quad \left[ \begin{array}{cc}
0 & (-1)^{r+1} \hat{\text{ch}}_0^p(\mathcal{F}) \ast_{A,P} \hat{\text{ch}}_0^q(\mathcal{F}', \varphi^* \mathcal{G}')
\end{array} \right].
\]

On the other hand, since there is an isomorphism \((\text{Id}_X \times \varphi)^*(\mathcal{F} \boxtimes \mathcal{G}') \simeq \mathcal{F} \boxtimes \mathcal{F}'\),

\[
\hat{\text{ch}}_n^n(\mathcal{F} \boxtimes \mathcal{F}') = \left[ \sum_{p+q=n} (1 \times \varphi)^*(y^p \times z^q) \right] \left( 1 \times \varphi \right)^* \left( \pi_X^* (\omega^p, g^p) \ast \pi_G^* (\tau^q, h^q) \right)
\]

where \( \pi_X^* (\omega^p, g^p) \ast \pi_G^* (\tau^q, h^q) \in \mathcal{D}_A^{2n-2X \times G, 1 \times \varphi} (X \times G, n)_0 \) is a representative of the star product of the Green forms \( \pi_X^* (\omega^p, g^p) \) and \( \pi_G^* (\tau^q, h^q) \). It is easy to see that

\[
\sum_{p+q=n} \hat{\text{ch}}_0^p(\mathcal{F}) \ast_{A,P} \hat{\text{ch}}_0^q(\mathcal{F}', \varphi^* \mathcal{G}) = \hat{\text{ch}}_{T,0}^n(\mathcal{F} \boxtimes \mathcal{F}', \varphi^* \mathcal{G}).
\]

Let \( Z'^{p,q}_{X,G,\varphi} = Z^p_X \times Z^q_{G,\varphi} \subset Z^n_{X \times G} \), and we identify \( Z^p_X = Z^p_{X,G,\varphi} \) and \( Z^q_G = Z^0_{G,\varphi} \), in the same way as in 811.1. Let

\[
J_{X,G,\varphi} : \mathcal{D}_{\log}^*(X \times G - Z^p_X, n) \oplus \mathcal{D}_{\log}^*(X \times G - Z^q_G, n) \to \mathcal{D}_{\log}^*(X \times G - Z^p_X \cup Z^q_G, n)
\]
be the map given by \((\omega, \tau) \mapsto -\omega + \tau\). Then the natural map
\[
(11.15) \quad \mathcal{D}^*_\log (X \times G - Z^{p,q}_{X,G,\varphi}, n) \to s(-j^{p,q}_{X,G,\varphi})^* 
\]
is a quasi-isomorphism \cite[Lem.5.5]{BF}. Let
\[
il_{\pi} : \mathcal{D}^*_\log (X \times G, n) \to s(-j^{p,q}_{X,G,\varphi})^*
\]
be the composite of the restriction map to \(\mathcal{D}^*_\log (X \times G - Z^{p,q}_{X,G,\varphi}, n)\) with (11.15), and take the truncated subcomplex of the simple complex of this map, which we denote by \(\tau \leq 2n s(i_{X,G,\varphi}^{p,q})^*\). Then in the same way as the definition of the map (11.3) we can define a map of complexes
\[
\circ : \tau \mathcal{D}^*_\log (X, p) \times \tau \mathcal{D}^*_\log (G, q) \to \tau \leq 2n s(i_{X,G,\varphi}^{p,q})^* 
\]
which induces an exterior product of Green forms
\[
\circ : GE^p(X) \times GE^q(G) \to \hat{H}^{2n}(\mathcal{D}^*(X \times G, n), s(-j^{p,q}_{X,G,\varphi})^*). 
\]
Moreover, we have a commutative diagram
\[
\begin{array}{ccc}
\hat{H}^{2n}(\mathcal{D}^*(X \times G, n), s(-j^{p,q}_{X,G,\varphi})^*) & \xleftarrow{\sim} & GE^n_{\pi X,G,\varphi}(X \times G) \xrightarrow{\sim} GE^n_{\pi X,G,\varphi}(X \times G) \\
\uparrow & & \uparrow \\
\tau \leq 2ns(i_{X,G,\varphi}^{p,q})^{2n} & \xleftarrow{\sim} & \tau \mathcal{D}^{2n-\varphi}_{\pi X,G,\varphi}(X \times G, n) \xrightarrow{\sim} \tau \mathcal{D}^{2n-\varphi}_{\pi X,G,\varphi}(X \times G, n) \\
1 \times \varphi^* & \downarrow & 1 \times \varphi^* \\
\tau \leq 2n s(i_{X,Y}^{p,q})^{2n-r} & \xleftarrow{\sim} & \tau \mathcal{D}^{2n-r}_{\kappa Z_{X,Y}}(X \times Y, n_0) \xrightarrow{\sim} \tau \mathcal{D}^{2n-r}_{\kappa Z_{X,Y}}(X \times Y, n_0), \\
\end{array}
\]
where the upper horizontal maps are surjective, and Thm.5.12 in \cite{Bu2} says that the upper horizontal map sends the exterior product \((\omega^p, g^p) \circ (\tau^q, h^q)\) to the star product of \(\pi^*_X(\omega^p, g^p)\) with \(\pi^*_G(\tau^q, h^q)\). Since
\[
(\omega^p, g^p) \circ (\varphi^*(\tau^q), \varphi^*(h^q)) = (1 \times \varphi)^* ((\omega^p, g^p) \circ (\tau^q, h^q)),
\]
the sequence of maps (11.8) sends the element (11.14) to
\[
\begin{pmatrix}
0 & (-1)^{r+1} \text{ch}^p_0(\mathcal{F}) \star_{\mathbb{A}, \mathbb{P}} \text{ch}^q_1(\mathcal{F}) \star_{\mathbb{F}} \varphi^*(\tau^q, h^q) \\
g^p \times \varphi^*(z^q) & \pi^*_X(\omega^p, g^p) \star \pi^*_G(\tau^q, h^q)
\end{pmatrix},
\]
which means that
\[
\sum_{p+q=n} \hat{\text{ch}}^p_0(\mathcal{F}) \cup \hat{\text{ch}}^q_{T,0}(\mathcal{F}) = \hat{\text{ch}}^n_{T,0}(\mathcal{F} \boxtimes \mathcal{F}).
\]
This completes the proof of (11.13). \(\square\)
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