A multiplicative stochastic process deriving the probability distribution in exact form

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Abstract. A simple multiplicative stochastic process is proposed in order to clarify directly the origin of the power–law. In this model, the probability distribution of the stochastic variables is calculated in an exact form. The necessary conditions that the multiplicative stochastic process generates the positive–negative asymmetric power–law distribution are investigated. Finally, comparisons between the distribution of stochastic variables simulated numerically and the stable distribution are discussed.

1. Introduction

Power-law distributions are frequently observed in various kinds of phenomena not only in natural science but also in social science. For instance, distributions of assets, sales, profits, income of firms, the number of employees and personal income (denoted by \(x\)) follow the power–law in the large scale range. The cumulative probability greater than \(x\) is represented as

\[
P_\geq(x) \propto x^{-\alpha}.
\]  

(1)

This is known as Pareto’s Law \([1]\), which has been well investigated by using various models. One of the useful models is a multiplicative stochastic process. This model is given by Langevin equation with multiplicative stochastic noises \(b_n\) and additive stochastic noises \(f_n\):

\[
x_{n+1} = b_n x_n + f_n.
\]  

(2)

The stochastic variables \(x_n\) are updated iteratively, and \(b_n\), \(f_n\) and \(x_n\) are independent each other.

In the stochastic process, the conditions that the distribution of \(x\) follows the power–law are studied\([2, 3]\). And it is also shown that the power–law index \(\alpha\) is estimated by the condition

\[
\langle b_n^\alpha \rangle = 1.
\]  

(3)

Specifically, these conditions are shown by Kesten\([2]\) using renewal theory, and are also shown by Takayasu–Sato–Takayasu\([3]\) by comparing, in terms of the characteristic function, between the multiplicative stochastic process and the general power–law distribution. The mechanism that the multiplicative stochastic process generates the power–law distribution is investigated.
in Refs.[4, 5]. The main purpose of this study is to understand the mechanism by calculating the probability function of $x_n$ directly.

For this aim, the authors introduce the multiplicative stochastic model, the noises of which are set to be simple. In this simple model, the probability function can be represented as an exact form analytically. We directly find that the analytic representation follows the power–law.

Under this simple multiplicative stochastic model, it is possible to investigate the power–law distribution which has the asymmetry of positive and negative. Because the data such as assets, sales and the number of employees take only positive values, the distribution follow the power–law only on the positive side. Therefore it is important to investigate the asymmetrical distribution without negative values. In this study, the necessary conditions that the multiplicative stochastic process generates the asymmetric distribution are first investigated.

The general form of the stable distribution is considered especially with respect to the asymmetric distribution which has the power–law tail.

Moreover, numerical simulations are employed in order to extend the analytical solvable model. It is numerically confirmed that the distribution of stochastic variables agrees with the stable distribution under a certain condition.

2. Analytically solvable multiplicative stochastic process

In this section, the authors propose a model which has an analytic solution in order to clarify the origin of the power–law. Let us consider the simple case \(^1\) that additive noises $f_n$ are not stochastic but equal to some constant $C$ in Eq.(2) as follows

$$x_{n+1} = b_n x_n + C,$$

(4)

where the probability function of multiplicative noises $b_n$ are given by

$$q(b_n) = \begin{cases} 
1 - p & (b_n = 0) \\
p & (b_n = \lambda) \\
0 & \text{(otherwise)}
\end{cases}.$$ 

(5)

This is Bernoulli distribution multiplied by a scale factor $\lambda$. In this case, the power–law index $\alpha$ is estimated to be

$$\alpha = \log_\lambda p^{-1}.$$ 

(6)

by using the necessary condition (3).

From Eq.(4), $x_{n+1}$ is represented as

$$x_{n+1} = C(1 + b_n + b_n b_{n-1} + \cdots + b_n b_{n-1} \cdots b_2 b_1) + b_n b_{n-1} \cdots b_1 b_0 x_0.$$ 

(7)

In this model, by using Eq.(7), probabilities of $x_{n+1}$ can be exactly calculated. The results are shown in Table 1. Here, $p(x_{n+1})$, $P(x_{n+1})$ and $P_\succ(x_{n+1})$ are a probability of $x_{n+1}$, a cumulative probability less than $x_{n+1}$ and a cumulative probability greater than $x_{n+1}$, respectively. Here, for simplicity, we have set $x_0 = 0$. From Table 1, the cumulative probability is represented as

$$P_\succ \left( x_{n+1} = C \frac{\lambda^{i+1} - 1}{\lambda - 1} \right) = p^i \quad (i = 0, \cdots, n).$$ 

(8)

Furthermore, to examine continuous behavior of the cumulative probability (8), let us define

$$x = C(\lambda^{i+1} - 1)/(\lambda - 1),$$

(9)

$$y = p^i.$$ 

(10)

\(^1\) In Ref.[4], the simple case is considered that multiplicative noises $b_n$ are constant and additive noises $f_n$ are stochastic variables, the distribution of which is power–law.
Table 1. A probability of $x_{n+1}$, a cumulative probability less than $x_{n+1}$ and a cumulative probability greater than $x_{n+1}$

| $x_{n+1}$ | $p(x_{n+1})$ | $P(x_{n+1})$ | $P_>(x_{n+1})$ |
|-----------|--------------|--------------|---------------|
| $C$       | $1-p$        | $1-p$        | 1             |
| $C(1+\lambda)$ | $p(1-p)$    | $1-p^2$      | $p^2$         |
| $C(1+\lambda+\lambda^2)$ | $p^2(1-p)$ | $1-p^3$      | $p^3$         |
| $\cdots$ | $\cdots$     | $\cdots$     | $\cdots$     |
| $C(1+\lambda+\lambda^2+\cdots+\lambda^{n-1})$ | $p^{n-1}(1-p)$ | $1-p^n$      | $p^n$         |
| $C(1+\lambda+\lambda^2+\cdots+\lambda^{n-1}+\lambda^n)$ | $p^n$      | 1            | $p^n$         |

The property of the distribution of $x$ is quite different between in the case of $\lambda > 1$ and in the case of $\lambda < 1$. In the case of $\lambda > 1$, $x$ can be approximated by $x \sim C\lambda^{i+1}/(\lambda - 1)$ in the limit $i \to \infty$. Then the cumulative probability (8) is expressed as the power-law:

$$y \sim Dx^{-\alpha}.$$  \hspace{1cm} (11)

Here $\alpha$ is given by Eq.(6) and $D$ is a constant value as follows

$$D = \left(C\frac{\lambda}{\lambda - 1}\right)^\alpha.$$  \hspace{1cm} (12)

In the case of $\lambda < 1$, there is an upper bound of the distribution of $x$, $C/(1-\lambda)$. Figure 1 shows a numerical simulation result, the cumulative probability (8) and the continuous representation (11) in the case of $\lambda > 1$. The simulation result agrees with the cumulative probability (8) accurately. In this figure, as $x$ is larger, the cumulative probability (8) is quickly closer to the asymptotic continuous power-law (11). Figure 2 shows those in the case of $\lambda < 1$. It is observed that the distribution of $x$ is not power-law.

In addition, in this simple model, the average of $x_{n+1}$ is obtained as follows:

$$\langle x_{n+1} \rangle = C(1-p) \sum_{i=0}^{n-1} p^i S_i + C p^n S_n$$

$$= C \frac{1-(p\lambda)^{n+1}}{1-p\lambda},$$  \hspace{1cm} (13)

where $S_i = (1+\lambda+\lambda^2+\cdots+\lambda^i)$. In the case of $p\lambda \geq 1$, this average diverges in the limit $n \to \infty$. \hspace{1cm} (2)

In the case of $\lambda > 1$, the condition $p\lambda \geq 1$ is rewritten as $1 \geq \log p^{-1}/\log \lambda$. This is identical with $\alpha \leq 1$ (see Eq.(6)). The divergence of the moment of order $m$ $(\geq 2)$ is estimated by using a following inequality:

$$\langle x_{n+1}^m \rangle = C^m(1-p) \sum_{i=0}^{n-1} p^i S_i^m + C^m p^n S_n^m$$

$$> C^m \frac{1-(p\lambda^m)^{n+1}}{1-p\lambda^m}.$$  \hspace{1cm} (14)

At least in the case of $p\lambda^m \geq 1$, the moment of order $m$ diverges in the limit $n \to \infty$. This condition is identical with $\alpha \leq m$ in the case of $\lambda > 1$.

2 This model means St. Petersburg paradox in the case of $p = 1/2$, $\lambda = 2$ and $C = 1$. 
3. Effect of additive noises

In the previous section, in order to carry out the analytical calculation, additive noises and multiplicative noises are set to be constant and 0 or \( \lambda \), respectively. In such a case, \( x \) follows the discrete distribution (8), the intervals of which are \( C\lambda^i \) \( (i = 1, \cdots, n) \). In this section, by taking various distributions of positive additive noises \( f_n \) \( (\geq 0) \), the effects of additive noises are examined. The purpose is to investigate how continuous power–law distributions are obtained. In the investigation, multiplicative noises \( b_n \) are given by the same distributions in the previous section (5) with \( \lambda = e (> 1) \) and \( p = e^{-1.2} \).

Firstly, let us consider the case that \( f_n \) follow two-values distributions. The probability function is given by

\[
r(f_n) = \begin{cases} 
  p_1 & (f_n = C_1) \\
  p_2 & (f_n = C_2) \\
  0 & \text{(otherwise)}
\end{cases} \,
\]

(15)

where \( p_1 + p_2 = 1 \). Although it is difficult to obtain an analytic representation of probabilities of \( x \), the distribution of \( x \) can be investigated by a numerical simulation. Figure 3 shows the simulation result. The resulting distribution have been denser compared with the discrete distribution in the previous section (Fig. 1).

Secondarily, let us consider the case that \( f_n \) follow continuous uniform distributions. The probability density function is given by

\[
r(f_n) = \begin{cases} 
  \frac{1}{\lambda_f} & (0 \leq f_n \leq \lambda_f) \\
  0 & \text{(otherwise)}
\end{cases} \,
\]

(16)

Figure 4 shows the simulation result, in which the distribution of \( x \) seems to be continuous. In the power–law range in Fig. 4, the distribution of \( x \) oscillates. This oscillation is attributed to the discontinuity of additive noises at \( f_n = \lambda_f \).
Figure 3. The simulation result in the case that additive noises follow two-values distributions (15). Black dots, which seem to be meandering curve, are simulation results for $C_1 = 1$, $C_2 = 10$ and $p_1 = p_2 = 0.5$. The gray solid straight line follows Eqs.(11)-(12), in which the parameter $C$ is replaced by $\langle f_n \rangle$. The lower gray dashed line follows Eqs.(11)-(12), in which the parameter $C$ is replaced by $C_1$. The upper gray dashed line follows Eqs.(11)-(12), in which the parameter $C$ is replaced by $C_2$.

Figure 4. The simulation result in the case that additive noises follow continuous uniform distributions (16). Black dots, which seem to be meandering curve, are simulation results for $\lambda_f = e$. The gray straight line follows Eqs.(11)-(12), in which the parameter $C$ is replaced by $\langle f_n \rangle$.

Thirdly, let us consider the case that $f_n$ follow exponential distributions. The probability density function is given by

$$r(f_n) = \begin{cases} \frac{1}{\lambda_f} e^{-f_n/\lambda_f} & (0 \leq f_n) \\ 0 & \text{otherwise} \end{cases}.$$  \hspace{1cm} (17)

Figure 5 shows the simulation result. The distribution of $x$ have been smooth and continuous power–law.

Finally, let us consider the case that $f_n$ follow Weibull distributions. The probability density function is given by

$$r(f_n) = \begin{cases} \frac{k_f}{\lambda_f} \left(\frac{f_n}{\lambda_f}\right)^{k_f-1} e^{-(f_n/\lambda_f)^{k_f}} & (0 \leq f_n) \\ 0 & \text{otherwise} \end{cases}.$$  \hspace{1cm} (18)

Figure 6 shows the simulation result. The distribution of $x$ oscillates. Furthermore, it dose not seem to be continuous in the large $x$ range. These reasons are attributed to a sharp peak of additive noises distributions. Weibull distribution with large degrees of freedom $k_f$ has a sharp peak in a certain positive value. Because the sharp peak is similar to the $\delta$–function, it is thought that the result is similar to the results observed in the case we take constant additive noises.
Figure 5. The simulation result in the case that additive noises follow exponential distributions (17). Black dots, which seem to be solid curve, are simulation results for $\lambda_f = e$. The gray straight line follows Eqs.(11)-(12), in which the parameter $C$ is replaced by $\langle f_n \rangle$.

Figure 6. The simulation result in the case that additive noises follow Weibull distributions (18). Black dots, which seem to be meandering curve, are simulation results for $\lambda_f = e$ and $k_f = 10$. The gray straight line follows Eqs.(11)-(12), in which the parameter $C$ is replaced by $\langle f_n \rangle$.

4. The asymmetric stable distribution

In the previous section, the multiplicative stochastic process which generates continuous power–law distributions is obtained by introducing continuous additive noises distributions. In this section, let us consider the necessary condition that the multiplicative stochastic process derives the stable distribution. The purpose of this section is to investigate whether the distribution in previous section satisfies the condition for the stable distribution.

A stable distribution does not change the function form by the convolution, and the characteristic function is given by

$$\phi(t) = \exp \{ it\delta - \gamma|t|^\alpha [1 - i\beta \text{sgn}(t)\Phi(\alpha)] \}, \quad (19)$$

where $\text{sgn}(t)$ is the sign of $t$ and $\Phi(\alpha)$ is given by

$$\Phi(\alpha) = \left\{ \begin{array}{ll} \tan(\alpha \pi/2) & \text{for } \alpha \neq 1 \\ -(2/\pi) \log |t| & \text{for } \alpha = 1 \end{array} \right. \quad (20)$$

Here, $\alpha \in (0, 2]$ gives the power–law index of the tail of the distribution, $\beta \in [-1, 1]$, called the skewness parameter, is a measure of asymmetry, $\gamma \in [0, \infty)$ is a scale parameter and $\delta \in (-\infty, \infty)$ is a location parameter.

The necessary condition that the multiplicative stochastic process generates the stable distribution has been given by Takayasu–Sato–Takayasu (TST) [3]. Conditions $\langle f_n^2 \rangle > 0$ and $\langle b_n^2 \rangle > 1$ are obtained from the requirement that the second order moment of $x$ diverges. The condition $\langle b_n^2 \rangle = 1$ is obtained from the comparison between the characteristic functions. Conditions $\langle \ln b_n \rangle > 0$ and $\langle b_n^2 \rangle > 1$ are obtained by restricting $\alpha$ to the range of $(0, 2)$ under the condition of $\langle b_n^2 \rangle = 1$. The condition $\langle b_n^2 \rangle = 1$ is extended to $\langle |b_n|^\alpha \rangle = 1$ for Gaussian multiplicative noises in Ref.[6]. These arguments were restricted to the case of $\beta = 0$ and
δ = 0, which means that the distribution of x has positive–negative symmetry. The distribution obtained in the previous section takes only positive values and, therefore, the distribution has left-right asymmetry. In order to compare such the distribution with the stable distribution, it is necessary to examine whether these conditions given by TST can be applied to the case of β ≠ 0 and δ ≠ 0.

The probability density functions of the distribution of x_n, b_n and f_n are expressed as p(x_n), q(b_n) and r(f_n), respectively. The characteristic functions of x_n and f_n are expressed as ϕ_n(t) and ψ(t), respectively. The characteristic function on the left hand side of Eq.(2) is

$$\phi_{n+1}(t) = \langle e^{itx_{n+1}} \rangle = \int_{-\infty}^{\infty} e^{itx_{n+1}} p(x_{n+1}) dx_{n+1}. \quad (21)$$

The characteristic function on the right hand side of Eq.(2) is

$$\langle e^{it(b_n x_n + f_n)} \rangle = \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} e^{itb_n x_n} p(x_n) dx_n \right\} q(b_n) db_n \times \int_{-\infty}^{\infty} e^{itf_n r(f_n)} df_n$$

$$= \int_{-\infty}^{\infty} \phi_n(b_n t) q(b_n) db_n \times \psi(t). \quad (22)$$

Then the expression

$$\phi_{n+1}(t) = \int_{-\infty}^{\infty} \phi_n(b_n t) q(b_n) db_n \times \psi(t) \quad (23)$$

is obtained. In the limit n → ∞, supposing x_{n+1} and x_n become the stable distributions with same parameters, these characteristic functions are written by ϕ(t) (19). Moreover, f_n is assumed to be the distribution, the characteristic function of which is expanded only by the integer order in terms of t. The expansion of each characteristic function in Eq.(23) around t = 0 is given by

$$\phi_{n+1}(t) = ϕ(t) = 1 + itδ - γ |t|^α [1 - iβ \text{sgn}(t) \Phi(α)] + \cdots, \quad (24)$$

$$\phi_n(b_n t) = ϕ(b_n t) = 1 + ib_n tδ - γ |b_n t|^α [1 - iβ \text{sgn}(b_n t) \Phi(α)] + \cdots, \quad (25)$$

$$ψ(t) = 1 + it(f_n) + \cdots. \quad (26)$$

By substituting Eqs.(24)-(26) into Eq.(23) and comparing coefficients in terms of t, necessary conditions that the distribution of x is stable are obtained.

The location parameter δ is obtained by comparing the coefficient of t in both sides of Eq.(23) as follows

$$δ = \int_{-\infty}^{\infty} b_n δ q(b_n) db_n + \langle f_n \rangle \quad (27)$$

$$= \frac{\langle f_n \rangle}{1 - \langle b_n \rangle}. \quad (28)$$

From t^α order term in Eq.(23), the condition

$$γ |t|^α [1 - iβ \text{sgn}(t) \Phi(α)] = \int_{-\infty}^{\infty} γ |b_n t|^α [1 - iβ \text{sgn}(b_n t) \Phi(α)] q(b_n) db_n \quad (29)$$

is obtained. In the case of t > 0, two conditions

$$γ = γ \left[ \int_{-\infty}^{0} (-b_n)^α q(b_n) db_n + \int_{0}^{\infty} b_n^α q(b_n) db_n \right], \quad (30)$$

$$iγβ\Phi(α) = iγβ\Phi(α) \left[ - \int_{-\infty}^{0} (-b_n)^α q(b_n) db_n + \int_{0}^{\infty} b_n^α q(b_n) db_n \right] \quad (31)$$
are derived from the real and imaginary parts of Eq.(29). In order to satisfy both Eqs.(30)-(31), at least one of the following four conditions is necessary:

- $\gamma = 0$ ($x$ is constant $\delta$)
- $\beta = 0$ (the distribution of $x$ is symmetric)
- \(\Phi(\alpha) = 0\) (\(\alpha = 2\))
- $q(b_n) = 0$ for $b_n < 0$ (the distribution of $b_n$ is positive)

In the case of $\gamma \neq 0(x$ is not constant), $\beta = 0$(the distribution of $x$ has positive-negative symmetry) and $\alpha \neq 2$(the distribution of $x$ is not the normal distribution), it is allowed that $b_n$ take negative values, and the power–law index $\alpha$ can be estimated by the condition $\langle |b_n|^\alpha \rangle = 1$ [6] by Eq.(30). In the case of $\gamma \neq 0, \beta \neq 0$(the distribution of $x$ does not have positive–negative symmetry) and $\alpha \neq 2$, it is not allowed that $b_n$ take negative values, and the power–law index $\alpha$ can be estimated by the condition $\langle b_n^\alpha \rangle = 1$ [3] by Eq.(30). These arguments are valid even if the distribution of $b_n$ is discrete.

From these results, it is considered that the negative multiplicative noises $b_n$ have an effect which makes the distribution of $x$ symmetrical. This is consistent with the analytic form of the distribution of $x$ in section 2. Let us take $\lambda < -1$ in the Table 1. In such a case, the sign of $x_{n+1}$ takes the value of positive and negative alternately. Therefore the distribution of $x_{n+1}$ has the positive–negative symmetry.

In the previous section, the distribution of $x$ does not have positive–negative symmetry and $b_n$ take positive values. In such the case, the condition $\langle b_n^\alpha \rangle = 1$ should be taken. Since the conditions obtained in this section are the necessary conditions, the resulting distributions are not always stable even if these conditions are satisfied. This point will be investigated in the following section.

5. Comparison between the simulation result and the stable distribution
The multiplicative stochastic process, which can express the probability of stochastic variables exactly, is introduced in section 2. The model is extended to generate the continuous distribution in section 3. The necessary conditions for the asymmetric stable distribution are derived in section 4. In this section, it will be confirmed that the distribution generated by the multiplicative stochastic process is the stable distribution. In this discussion, Eq.(5) and Eq.(17) are adopted for the distribution of $b_n$ and $f_n$, respectively.

The stable distribution is characterized by four parameters $\alpha$, $\beta$, $\gamma$ and $\delta$ in the characteristic function (19). It is necessary to identify the parameters in order to compare the simulation result with the stable distribution. The parameter $\alpha$ is identified by Eq.(6). In the case of $\alpha \geq 1$, the parameter $\delta$ is identified by Eq.(28). In order to decide other parameters $\beta$, $\gamma$ and $\delta$(for $\alpha < 1$), we use universal properties known about the stable distribution [7, 8].

Asymptotic behavior of the cumulative distribution of the stable distribution is given by

$$P_>(x) \sim \gamma^\alpha c_\alpha (1 + \beta)x^{-\alpha}$$

(32)

as $x \to \infty$ in the case of $-1 < \beta \leq 1$, where $c_\alpha = \sin(\pi \alpha/2)\Gamma(\alpha)/\pi$ [7, 8]. Asymptotic behavior in the opposite side of the distribution of $x$ is given by

$$P_<(x) \sim \gamma^\alpha c_\alpha (1 - \beta)(-x)^{-\alpha}$$

(33)

as $x \to -\infty$ in the case of $-1 \leq \beta < 1$ [7, 8]. For $\beta = \pm 1$, the stable distribution has one power–law tail. For $\beta \neq \pm 1$, the stable distribution has power–law tails in both sides. Since

\footnote{In the case of $t < 0$, the same conditions are derived from the coefficient of $(-t)^\alpha$.}
the multiplicative stochastic process considered in this study has the power–law tail only in a positive side, we take $\beta = 1$. The parameter $\gamma$ is fixed by Eq.(6), $\beta = 1$ and Eq.(32).

According to Ref.[7, 8], the support of the stable distribution is

$$
\begin{align*}
&x \in [\delta - \gamma \tan(\pi \alpha/2), \infty) \quad \alpha < 1 \text{ and } \beta = 1 \\
&x \in (-\infty, \delta + \gamma \tan(\pi \alpha/2)] \quad \alpha < 1 \text{ and } \beta = -1 \\
&x \in (-\infty, \infty) \quad \text{otherwise}
\end{align*}
$$

(34)

In the case of $\alpha < 1$ and $\beta = 1$, the support of the stable distribution is $x \in [\delta - \gamma \tan(\pi \alpha/2), \infty)$. Since the support of the distribution of the simulation result is $x \in [0, \infty)$, the parameter $\delta$ is identified by $\delta = \gamma \tan(\pi \alpha/2)$.

Firstly, let us compare a simulation result with the stable distribution in the case of $0 < \alpha < 1$. In order to set $\alpha = 0.5$, the parameters of Eq.(5) are adopted for $\lambda = e$ and $p = e^{-0.5}$. The parameter of Eq.(17) is adopted for $\lambda_f = 1$. Figure 7 shows the comparison between the cumulative distribution of the simulation result and the stable distribution in the case of $\alpha = 0.5$. Black points are the simulation results. The gray solid curve is the stable distribution. The simulation result seems to agree with the stable distribution in terms of the cumulative distribution.

Figure 8 shows the same kind of the comparison by using of the probability density. Black points are the simulation results. The gray solid curve is the stable distribution. The gray dashed curve is the distribution of additive noises $f_n$. The simulation result seems to agree with the stable distribution in the range of $10^{-1} < x$. However, the simulation data do not agree with the stable distribution in the range of $x < 10^{-1}$. This reason is considered that the influence of additive noises becomes considerable in the small $x$.

Secondly, let us compare a simulation result with the stable distribution in the case of $1 < \alpha < 2$. In order to set $\alpha = 1.5$, the parameters of Eq.(5) are adopted for $\lambda = e$ and $p = e^{-1.5}$. Moreover, in order to set $\delta = 1$, the parameter of Eq.(17) are adopted for $\lambda_f = 1/(1 - \langle b_n \rangle) = 1/(1 - e^{-0.5})$. The average value measured by simulation data is $\langle x \rangle = 0.9712$. This average almost agrees with $\delta$. Figure 9 shows the comparison between the cumulative distribution of the simulation result and the stable distribution in the case of $\alpha = 1.5$. Black points are the simulation results. The gray solid curve is the stable distribution.

The simulation result seems to agree with the stable distribution only for $x > 10^1$ in the range of power–law. On the one hand, the support of the stable distribution $\alpha = 1.5$ and $\beta = 1.0$ is $x \in (-\infty, \infty)$. On the other hand, the support of the distribution of the simulation result is $x \in [0, \infty)$. Therefore it is hard to consider two distributions to be identified with each other. Figure 10 shows the same kind of the comparison by using the probability density function.

Our model does not completely reproduce the stable distribution. In the case of $0 < \alpha < 1$, the simulation result almost agrees with the stable distribution excluding the range of small $x$. In the case of $1 < \alpha < 2$, the simulation result dose not agree with the stable distribution, because the support of the distributions is quite different from the stable distribution.

$^4$ The parameter $\gamma$ is estimated by fitting the cumulative distribution (32). The fitting range is taken from 99% to 99.9% of the distribution of $x$ obtained by the simulation. As a result, the value of $\gamma$ is estimated to be $1.6246 \pm 0.0065$. In the case of $\alpha = 0.5$, $\delta$ is the same value as $\gamma$, because $\delta = \gamma \tan(\pi \alpha/2) = \gamma$. For the parameters of the stable distribution, $\alpha = 0.5$, $\beta = 1.0$, $\gamma = 1.6246$ and $\delta = 1.6246$ are adopted. A program STABLE[9] is used to plot the stable distribution.

$^5$ The parameter $\gamma$ is estimated by fitting the probability density distribution (32). The fitting range is taken from 99% to 99.9% of the distribution of $x$ obtained by the simulation. As a result, the value of $\gamma$ is estimated to be $\gamma = 0.74980 \pm 0.00053$. For the parameter of the stable distribution, $\alpha = 1.5$, $\beta = 1.0$, $\gamma = 0.7480$ and $\delta = 1.0$ are adopted.
Figure 7. The cumulative probability greater than $x$: $P_x(x)$. The simulation result in the case of $\alpha = 0.5$ is shown. Black dots, which seem to be solid curve, are simulation results for $\lambda = e$, $p = e^{-0.5}$ and $\lambda_f = 1$. A gray curve is the stable distribution, the parameters of which are $\alpha = 0.5$, $\beta = 1.0$, $\gamma = 1.6246$ and $\delta = 1.6246$.

Figure 8. The probability density function: $p(x)$. The simulation result in the case of $\alpha = 0.5$ is shown. Black dots, which seem to be solid curve, are simulation results for $\lambda = e$, $p = e^{-0.5}$ and $\lambda_f = 1$. A solid gray curve is the stable distribution, the parameters of which are $\alpha = 0.5$, $\beta = 1.0$, $\gamma = 1.6246$ and $\delta = 1.6246$. A dashed gray curve is additive noises $f_n$ (17).

6. Conclusion

The purpose of this study is to investigate the emergence of the power–law by the multiplicative stochastic process.

In section 2, the authors propose a model which derives the probability distribution of stochastic variables in exact form to clarify the origin of the power–law. In order to carry out an analytical calculation, additive noises and multiplicative noises are set to be constant and 0 or $\lambda$, respectively. The property of the distribution of $x$ is quite different between in the case of $\lambda > 1$ and in the case of $\lambda < 1$. In the case of $\lambda > 1$, we find that the distribution is power–law. In the case of $\lambda < 1$, the distribution is not power–law.

In section 3, effects of additive noises are examined by numerical simulations. By taking various distributions of positive additive noises $f_n (\geq 0)$, the continuous power–law distribution is obtained.

In section 4, the necessary conditions that the multiplicative stochastic process generates the positive–negative asymmetric stable distribution are first shown. The most important condition for the asymmetric distribution is that multiplicative noises take only positive values.

In section 5, the comparison between the distribution obtained by the numerical simulation and the stable distribution is examined. In the case of the power–law index $\alpha < 1$, the simulation result almost agrees with the stable distribution.

In this paper, the roles of $b_n$ are clarified. The multiplicative noises of $b_n > 1$ extend the tail of the power–law distribution. The noises $b_n < 1$ limit the distribution finitely. The negative noises $b_n$ make the distribution positive–negative symmetric. Let us consider that $x_n$ are data of velocity. The noises $b_n > 1$ have the effect corresponding to acceleration. The noises $b_n < 1$ have
the effect corresponding to deceleration. The negative noises $b_n$ have the effect corresponding to reflection.

In the case of the general multiplicative stochastic process, various distributions are taken as the multiplicative noises and the additive noises. Although in such a complicated case, it is considered that the similar mechanism observed in this study produces the power–law distribution. For various applications of the multiplicative stochastic process, the authors believe that the knowledge obtained in this study will be useful significantly.

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