New results on disturbance rejection for energy-shaping controlled port-Hamiltonian systems

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Abstract—In this paper we present a method to robustify energy-shaping controllers for port-Hamiltonian (pH) systems by adding an integral action that rejects unknown additive disturbances. The proposed controller preserves the pH structure and, by adding to the new energy function a suitable cross term between the plant and the controller coordinates, it avoids the unnatural coordinate transformation used in the past. This paper extends our previous work by relaxing the requirement that the systems Hamiltonian is strictly convex and separable, which allows the controller to be applied to a large class of mechanical systems, including underactuated systems with non-constant mass matrix. Furthermore, it is shown that the proposed integral action control is robust against unknown damping in the case of fully-actuated systems.

I. INTRODUCTION

Port-Hamiltonian (pH) systems are a class of nonlinear dynamics that can be written in a form whereby the physical structure related to the system energy, interconnection and dissipation is readily evident [2]. Interconnection and damping assignment passivity-based control (IDA-PBC) is a control design that imposes a pH structure to the closed-loop dynamics [3]. This control method has been successfully applied to a range of nonlinear physical systems such as electrical machines [4], [5], power converters [6], chemical processes [7] and underactuated mechanical systems [8], [9]. Although IDA-PBC is robust to parameter uncertainty and passive perturbations, the presence of (practically unavoidable) external disturbances can significantly degrade its performance by shifting equilibria or even causing instability. In this paper, we focus on the problem of robustifying IDA-PBC vis-à-vis external disturbances via the addition of integral action control (IAC).

A number of IACs have been proposed for pH systems, each conforming to specific control objectives. In [3], an integrator is applied to the passive output of a pH system which has the effect of regulating the passive output to zero. This scheme can be interpreted as a control by interconnection (CbI), studied in [17], and is robust against parameters of the open-loop plant. A fundamentally different approach was taken in [11], [12] where the control objective was to regulate a signal that is not necessarily a passive output of the plant. As such, it was shown that the IAC for passive outputs proposed in [3] was not applicable as it cannot be implemented in a way that preserves the pH form. Rather, a partial state transformation was utilised to ensure that the closed-loop dynamics preserve the pH form. A critical step in this approach is the evaluation of the systems energy function at the transformed coordinates, which is a rather unnatural construction—see Remark 2 in [14]. This IAC was tailored for fully-actuated mechanical systems in [13] and underactuated mechanical systems in [14]. While in both cases the required change of coordinates to preserve the pH form were given explicitly, a number of technical assumptions were imposed to do so in the underactuated case. In both cases, the proposed IACs were shown to preserve the desired equilibrium of the system, rejecting the effect of an unknown matched disturbance.

More recently, an alternate approach to IAC design for pH systems was proposed in [15], [16]. The IAC is designed, as in [11], [12], to regulate signals that are not necessarily passive outputs of the plant. The key advancement over previous solutions is that the IAC does not require any coordinate transformations in the design procedure. Rather, the energy function of the controller depends on both the states of the controller and the plant which allows preservation of the pH structure by construction. It has been shown in [16] that the closed-loop can be interpreted as the power preserving interconnection of the plant and controller, which resembles the CbI technique [17]. It was also shown that, under a number of technical assumptions, the IAC could reject the effects of both a matched and unmatched disturbance.

In this paper, the realm of application of the IAC presented in [16] is significantly extended by relaxing some previously made assumptions. Specifically, the assumption in [16] that the open-loop Hamiltonian is strongly convex and separable is not required here. By relaxing this assumption, our result can be applied to a broader class of pH system. In particular, our result applies to a strictly larger class of underactuated mechanical systems than the ones considered in [14] and [16].

The remainder of the paper is structured as follows: The problem formulation is presented in Section II. A brief summary of previous work is given in Section III. The new IAC scheme is presented in Section IV and the stability properties of the closed-loop are considered in Section V. The IAC is tailored for mechanical systems in Section VI. The control scheme is applied to three examples in Section VII. Finally, the results of the paper are briefly discussed in Section VIII.

Notation. Function arguments are declared upon definition.

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and are omitted for subsequent use. For \( x \in \mathbb{R}^n \) we define \( |x|^2 = x^\top x \) and \( |x|_P^2 = x^\top P x \), for \( P \in \mathbb{R}^{n \times n}, \ P > 0 \). All functions are assumed to be sufficiently smooth. For mappings \( \mathcal{H} : \mathbb{R}^n \to \mathbb{R}, \ C : \mathbb{R}^n \to \mathbb{R}^m \) and \( \mathcal{G} : \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R} \) we denote the transposed gradient as \( \nabla \mathcal{H} := \left( \frac{\partial \mathcal{H}}{\partial x} \right)^\top \), the transposed Jacobian matrix as \( \nabla C := \left( \frac{\partial C}{\partial x} \right)^\top \) and \( \nabla \mathcal{G}(x,y) := \left( \frac{\partial \mathcal{G}}{\partial x} \right)^\top \). For the distinguished \( x^*, \bar{x} \in \mathbb{R}^n \), we define the constant vectors \( \bar{C} := C(\bar{x}), \ C^* := C(x^*), \nabla \mathcal{H}^* := \nabla \mathcal{H}(x^*) \) and \( \nabla \bar{C} := \nabla \mathcal{H}(\bar{x}) \).

II. PROBLEM FORMULATION

A. Perturbed system model

In this paper we consider the scenario where an unperturbed system has been stabilised at a desired constant equilibrium using IDA-PBC and the objective is to add an IAC to reject additive disturbances. More precisely, the dynamics of the system are of the form:

\[
\begin{bmatrix}
\dot{x}_a \\
\dot{x}_u
\end{bmatrix} =
\begin{bmatrix}
J(x) - R(x) & \nabla x_a \mathcal{H} \\
\nabla x_u \mathcal{H} & -d_u(x_a)
\end{bmatrix}
\begin{bmatrix}
x_a \\
x_u
\end{bmatrix}
\begin{bmatrix}
y_a \\
y_u
\end{bmatrix} = \nabla x_a \mathcal{H},
\]

where \( x = \text{col}(x_a,x_u) \in \mathbb{R}^n \) is the state vector, with \( x_a \in \mathbb{R}^m \) and \( x_u \in \mathbb{R}^s, \ s := n - m \), the actuated and unactuated states, respectively, \( y_a \in \mathbb{R}^m, \ y_u \in \mathbb{R}^s \) are the signals to be regulated to zero and \( u \in \mathbb{R}^m \) is the control input. The function \( \mathcal{H} : \mathbb{R}^n \to \mathbb{R}_+ \) is the Hamiltonian of the system. The interconnection and damping matrices are partitioned as

\[
J(x) :=
\begin{bmatrix}
J_{aa}(x) & J_{au}(x) \\
-J_{ua}(x) & J_{uu}(x)
\end{bmatrix},
\]

\[
R(x) :=
\begin{bmatrix}
R_{aa}(x) & R_{au}(x) \\
R_{ua}(x) & R_{uu}(x)
\end{bmatrix},
\]

\[
= R^\top (x) \geq 0,
\]

respectively. The signals \( d_u : \mathbb{R}^m \to \mathbb{R}^m \) and \( d_a : \mathbb{R}^n \to \mathbb{R}^s \) are the, state-dependent, matched and unmatched disturbances of the system, respectively.

We assume that the energy-shaping and damping injection steps of the IDA-PBC design have been accomplished for system (1). This means, on one hand, that a desired equilibrium \( x^* \in \mathbb{R}^n \) satisfies

\[
x^* = \arg \min \mathcal{H}(x)
\]

and is isolated, which implies that the system (1) without disturbances and \( u = 0_{m \times 1} \) has a stable equilibrium at \( x^* \). On the other hand, the damping injection step—consisting of a proportional feedback around the passive output—ensures that

\[
-\nabla^\top \mathcal{H} R \nabla \mathcal{H} \leq -\alpha |y_a|^2,
\]

and \( \alpha > 0 \) is a constant. Consequently, the equilibrium is asymptotically stable if \( y_a \) is a detectable output for the system. See [21] for further details on stability of pH systems.

Notice that we have taken the input matrix of the form

\[
\begin{bmatrix}
I_m \\
0_{s \times m}
\end{bmatrix}.
\]

As is well-known [11] a necessary and sufficient condition to transform—via input and state changes of coordinates—an arbitrary input matrix into this form is that its columns span an involutive distribution.

B. Assumptions

As indicated in the introduction the objectives of the IAC are to preserve the existence of a stable equilibrium and to ensure that the output signal \( \{y_a \text{ and } y_u \text{ when } d_a = 0_{s \times 1} \text{ or } y_u \text{ if } d_a = 0_{m \times 1} \} \) is driven to zero in spite of the presence of disturbances. Also, some stability properties should be preserved when both disturbances act simultaneously. In this subsection we present the assumptions that are imposed on the system (1) to attain these objectives.

For the case of matched disturbances it is possible to preserve in closed-loop the original equilibrium \( x^* \). This, clearly, implies that if this equilibrium is asymptotically stable then \( |y_a(t)| \to 0, |y_u(t)| \to 0 \) as desired. In order to ensure the former property for state-dependent disturbances the following assumption is imposed.

**Assumption 1**: The disturbance \( d_a \) can be written in the form

\[
d_a = G_d(x)d_a,
\]

where \( d_a \in \mathbb{R}^m \) is constant and \( G_d : \mathbb{R}^n \to \mathbb{R}^{m \times m} \). Moreover, \( G_d \) is full rank and sign definite. Without loss of generality, it is assumed that \( G_d < 0 \).

Similarly to the case above, to handle the case of state-dependent unmatched disturbances it is necessary to assume they satisfy a structural condition, which is articulated as follows.

**Assumption 2**: The disturbance \( d_u \) can be written in the form

\[
d_u = (J_{au} + R_{au})^\top \bar{d}_u,
\]

where \( \bar{d}_u \in \mathbb{R}^m \) is constant.

An additional difficulty for the unmatched disturbance case is that it is not possible to preserve the original equilibrium, even in the case when \( d_u \) is constant—see [12] for a detailed discussion. Therefore, it is necessary to consider another value for \( x \) in the closed-loop to be stabilized, that we denote \( \bar{x} = (\bar{x}_a, \bar{x}_u) \in \mathbb{R}^n \). The new equilibrium \( \bar{x} \) should belong to the set

\[
\mathcal{E} := \{ x \in \mathbb{R}^n | - (J_{au} + R_{au})^\top (\nabla x_a \mathcal{H} + \bar{d}_u) + (J_{au} - R_{au}) \nabla x_a \mathcal{H} = 0_{s \times 1} \},
\]

which is the set of assignable equilibria. The following assumption guarantees the existence of such an \( \bar{x} \) and is utilised later for stability analysis.

**Assumption 3**: There exists an isolated \( \bar{x} \in \mathbb{R}^n \) satisfying

\[
\bar{x} = \arg \min \mathcal{H}(x)
\]

where

\[
\mathcal{H}(x) := \mathcal{H}(x) + x_a^\top \bar{d}_u.
\]

The following remarks regarding the assumptions are in order.

**R1.** Consistent with the internal model principle, the assertion that \( \bar{d}_u \in \mathbb{R}^m \) in Assumption [2] implies that there are enough control actions to reject the unmatched disturbance. See [12] for further discussions on the need for this condition.

**R2.** Unmatched disturbances of the form considered in Assumption 2 can be equivalently described by a matched
disturbance and a “shifted” Hamiltonian. This property is utilised to analyse the effects of unmatched disturbances.

R3. To verify that \( \hat{x} \) is indeed an assignable equilibrium, notice that as it minimises the function \( H \), it satisfies \( \nabla_x H = -d_u, \nabla_x \hat{H} = 0_{m \times 1} \) which implies that \( \hat{x} \in \mathcal{E} \). Moreover, as \( \nabla_x \hat{H} = 0_{s \times 1} \), if the point \( \hat{x} \) is asymptotically stable, then \( |y_a(t)| \to 0 \), which is part of the control objective.

R4. The particular form of \( H \) in Assumption 3 is necessary to construct a Lyapunov function to study the “shifted” equilibrium \( \hat{x} \). Clearly, the assumption is satisfied if \( H \) is convex (at least locally in the domain of interest).

C. Problem statement

Consider the pH system (1) verifying Assumption 1 when \( d_u = 0_{s \times 1} \) and Assumptions 2-3 when \( d_u = 0_{m \times 1} \). Define mappings \( \hat{u} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m \) and \( F : \mathbb{R}^n \to \mathbb{R}^m \) such that the IAC

\[
\dot{u} = \hat{u}(x, x_c) \quad \dot{x}_c = F(x)
\]

ensures the closed-loop is an, unperturbed, pH system with an (asymptotically) stable equilibrium at \( (x^*, \bar{x}_c) \) when \( d_u = 0_{s \times 1} \) and at \( (\bar{x}, \bar{x}_c) \) when \( d_u = 0_{m \times 1} \), for some \( \bar{x}_c \in \mathbb{R}^m \). Moreover, give conditions under which a stable equilibrium exists in the presence of, both, matched and unmatched disturbances.

III. Previous Work and Contributions of the Paper

A. Integral action on passive outputs \( y_a \)

For the case of constant, matched disturbances \( \bar{d}_a \) it is well-known [3], [12] that adding an IAC around the passive outputs \( y_a \) of the form

\[
\begin{align*}
\dot{x}_c &= K_i y_a \\
\dot{u} &= -x_c,
\end{align*}
\]

(7)

where \( K_i > 0 \), ensures the closed-loop is a pH system with a stable equilibrium at \( (x^*, \bar{d}_a) \) and guarantees that \( |y_a(t)| \to 0 \). The equilibrium is, moreover, asymptotically stable if \( y_a \) is a detectable output, which ensures that the non-passive output \( y_a \), also converges to zero.

Unfortunately, the detectability condition is rather restrictive, hence, the need to propose alternative IACs even in the case \( d_u = 0_{s \times 1} \). In fact, the IAC (7) cannot ensure detectability when used for mechanical systems [13]. Moreover, this simple output-feedback construction is applicable only to the passive output. Indeed, it is shown in [13] that it is not possible to use an IAC of the form (7) around \( y_a \), preserving the pH form—see also the discussion in [11].

B. Integral action of \( y_a \) via coordinate transformations

To reject unmatched disturbances it seems reasonable to add an IAC around the output \( y_a \). An approach to carry out this task was proposed in [11] and further investigated in [12], [13], [14]. The key step in those papers is the solution of a nonlinear algebraic equation that ensures the existence of a change of coordinates

\[
z = \text{col}(z_1, z_2, z_3) = \text{col}(\psi(x_a, x_u, x_c), x_u, x_c)
\]

(8)

and an IAC

\[
u = \hat{u}(x, x_c)
\]

\[
\dot{x}_c = K_i \nabla_x \mathcal{H}(\psi(x_a, x_u, x_c), x_u)
\]

(9)

such that, the closed-loop system written in the new coordinates has a pH structure.

\[
\dot{z} = \left[ \begin{array}{c}
J(x) - R(x) |_{x = (\mu(z), z_2)} \\
0_{s \times m}
\end{array} \right]
\]

\[
0_{s \times s}
\]

\[
\begin{bmatrix}
\nK_i \\
0_{s \times s}
\end{bmatrix}
\]

(10)

\[
\mathcal{H}_{cl}(z) = \mathcal{H}(z_1, z_2) + \frac{1}{2} |z_3 - \bar{d}_a|^2_{K^{-1}}
\]

where \( J \) and \( R \) are the open-loop interconnection and damping matrices defined in [2], \( \mu : \mathbb{R}^{n+m} \to \mathbb{R}^m \) is a left inverse of \( \psi \) in the sense that

\[
\psi(\mu(z), z_2, z_3) = z_1,
\]

(11)

and \( \mathcal{H}_{cl} \) is the new Hamiltonian function given by

As discussed in Subsection II-B in the case of unmatched disturbances it is not possible to preserve the open-loop equilibrium of the unperturbed system \( x^* \). To ensure that an equilibrium exists for the IAC (9) it is necessary to impose an assumption, similar to Assumption 3 (see remark R3. of Subsection II-B), namely

\[
\exists \hat{x} \in \mathcal{E} \cap \{ x \in \mathbb{R}^n \mid \nabla_x \mathcal{H}(\psi(x_a, x_u, \bar{d}_a), x_u) = 0_{s \times 1} \},
\]

(11)

with \( \mathcal{E} \) defined in (4). Under this assumption it can be shown that the closed-loop system (10) has a stable equilibrium at

\[
\hat{z} := (\psi(\hat{x}_a, \hat{x}_u, \bar{d}_a), \hat{x}_u, \bar{d}_a).
\]

(11)

Furthermore, if the output \( y_a \) is detectable the equilibrium is asymptotically stable and \( |y_a(t)| \to 0 \). Interestingly, it can be shown that if the system (1) satisfies Assumption 3 and there exists a suitable transformation (8), then the condition (11) is satisfied—the proof of this fact is provided in Lemma 2 in the Appendix.

In addition to the above IAC, the coordinate transformation (8) has been utilised for damping injection into unactuated coordinates (see [13] and [14]). This extension has allowed the construction of strict Lyapunov functions and verification of additional stability properties such as exponential stability and input-to-state stability.

In spite of the interesting stability properties of the IAC (9) quoted above, there are several limitations of this approach. First, the solution of the aforementioned algebraic equation—which should satisfy some restrictive rank and matching conditions—is often problematic. Interestingly, it has been solved for permanent magnet synchronous motors (PMSM) in [11] and for fully actuated and a class of underactuated mechanical systems in [13] and [14], respectively. Second, as indicated in the introduction, the construction of the new
Hamiltonian function is rather unnatural and, in contrast with the technique proposed in this paper, is not amenable for a physical interpretation of the IAC. Finally, as remarked in [13] and [14], the resulting expressions for the controllers are, in general, quite involved.

\section{Integral action of \( y_a \) without coordinate transformations}

An alternate approach to apply IAC to \( y_a \) without using coordinate transformations was proposed in [16]. The paper considers a plant of the form (1) with \( R_{au} = 0_{m \times s} \), constant disturbances and \( \mathcal{H} \) strongly convex and separable. The control objective is to design an IAC that preserves in closed-loop the component of the original equilibrium associated to the unactuated coordinate, that is, the desired closed-loop equilibrium is of the form \((\bar{x}_a, \bar{x}_u, \bar{x}_c)\). Under the assumption of separability of \( \mathcal{H} \) the asymptotic stability of this equilibrium implies \(|y_a(t)| \to 0\).

The IAC proposed in [16] is given by

\[
\begin{align*}
\dot{u} &= (J_{aa} - R_{aa})\nabla_{x_a} \mathcal{H}_c \\
\dot{x}_c &= E^\top J_{aa} \nabla_{x_a} \mathcal{H}_c,
\end{align*}
\]

where \( x_a \in \mathbb{R}^s \), \( E \in \mathbb{R}^{m \times s} \) is a constant matrix and \( \mathcal{H}_c : \mathbb{R}^s \to \mathbb{R} \) is a strictly convex function with argument \( E^\top x_a - x_c \). Thanks to this particular choice of argument of \( \mathcal{H}_c \) it is possible to prove that the closed-loop dynamics is a pH system of the form

\[
\begin{bmatrix}
\dot{x}_u \\
\dot{x}_c
\end{bmatrix} =
\begin{bmatrix}
J_{aa} - R_{aa} & J_{aa} \\
-J_{aa} & -R_{aa} - J_{aa}^\top E
\end{bmatrix}
\begin{bmatrix}
0_{m \times s} \\
E^\top J_{aa} & 0_{s \times s}
\end{bmatrix}
\begin{bmatrix}
\nabla_{x_a} \mathcal{H}_c \\
\nabla_{x_u} \mathcal{H}_c
\end{bmatrix} -
\begin{bmatrix}
d_a \ u \\
0_{s \times 1}
\end{bmatrix},
\]

with

\[
\mathcal{H}_c(x, x_c) := \mathcal{H}(x_a, x_u) + \mathcal{H}_c(E^\top x_a - x_c),
\]

which has the property that

\[
\nabla \mathcal{H}_c = \begin{bmatrix}
\nabla_{x_a} \mathcal{H} + E \nabla \mathcal{H}_c \\
\nabla_{x_u} \mathcal{H} \\
- \nabla \mathcal{H}_c
\end{bmatrix}.
\]

It was shown that if \( J_{aa} - R_{aa} \) is constant, the desired closed-loop equilibrium is asymptotically stable for any constant matched disturbance \( d_a \). In the case of unmatched disturbances, it was shown that the desired closed-loop equilibrium is asymptotically stable provided that there exists a suitable equilibrium for the system, \( J_{aa} \) is constant and \( d_a \) is in the range of \( J_{aa}^\top \). When matched and unmatched disturbances coexist, stability of the closed-loop was guaranteed and \(|y_a(t)| \to 0\), provided that a suitable equilibrium exists and the matrices \( J_{aa} - R_{aa} \) and \( J_{aa} \) are constant.

\section{Integral action control for mechanical systems}

Applying an IDA-PBC to a perturbed mechanical systems yields a closed-loop of the form [8]

\[
\begin{bmatrix}
\dot{q} \\
\dot{p}
\end{bmatrix} =
\begin{bmatrix}
0_{\ell \times \ell} & M^{-1}(q)M_d(q) \\
-M_d(q)M^{-1}(q) & J_2(q, p) - R_d(q)
\end{bmatrix}
\nabla \mathcal{H}_d +
\begin{bmatrix}
d_a \\
G(q)(u - d_a)
\end{bmatrix},
\]

where \( q, p \in \mathbb{R}^\ell, \ell := \frac{\ell}{2} \) are the generalised coordinate and momentum vectors, \( G \) is the full rank input matrix, \( M \) and \( M_d \) are the positive definite inertia matrices of the open and closed-loop, respectively, \( R_d \geq 0 \) is the dissipation matrix, \( J_2 = -J_2^\top \) is a designer chosen matrix capturing the effect of gyroscopic forces, \( \mathcal{H}_d \) is the desired Hamiltonian function, which is of the form

\[
\mathcal{H}_d(q, p) = \frac{1}{2} \| p \|^2_{M_d^{-1}(q)} + V_d(q),
\]

where \( V_d \) is the closed-loop potential energy.

IDA-PBC ensures that \( V_d \) has an isolated minimum at a desired value \( q^\ast \). The latter fact, together with positivity of the matrix \( M_d \), implies that \((q^\ast, 0)\) is a stable equilibrium of (14) when \( u, d_a \) and \( d_u \) are zero. The robust regulation control objective is to design an IAC that ensures stability of the desired equilibrium \((q^\ast, 0, \bar{x}_c)\).

In [13] it is proven that the addition of an IAC of the form (7) to mechanical systems has catastrophic effects—making the achievement of the control objectives a zero measure event. IACs using the coordinate transformation technique of Subsection III-B have been reported for fully actuated systems in [13] and extended in [14] to underactuated systems with \( M_d \) constant and

\[
G^\perp(q) \nabla q(p^\top M^{-1}(q)p) = 0,
\]

where \( G^\perp : \mathbb{R}^\ell \to \mathbb{R}^{(\ell - m) \times m} \) is a full-rank left annihilator of \( G \). In both papers, stability of the desired equilibrium is ensured in the absence of unmatched disturbances. Notice that Assumption [16] implies that the inertia matrix is independent of the non-actuated coordinates that, together with the condition of constant \( M_d \), rules out many systems of practical interest. More recently, the approach of Subsection III-C has been used in [16] assuming full actuation and constant inertia matrices.

\section{Contributions of the paper}

The main contributions of the present paper may be summarized as follows:

\textbf{C1.} Compared with the IAC of Subsection III-A

(i) the method is applicable for both matched and unmatched disturbances; and (ii) the restrictive assumption of detectability of \( y_a \) is significantly relaxed by the addition of damping between the actuated coordinates and the dynamic extension.

\textbf{C2.} Compared with the IAC of Subsection III-B

(i) the need to solve a nonlinear algebraic equation is obviated; (ii) the construction of the closed-loop Hamiltonian function is more natural, and admits a physical interpretation; and (iii) the resulting IACs are considerably simpler.

\textbf{C3.} Compared with the IAC of Subsection III-C

(i) the assumption of convexity and separability of the systems Hamiltonian \( \mathcal{H} \) are removed; (ii) the result is extended to a class of state-dependent matched and unmatched disturbances; and (iii) the detectability assumption required for asymptotic stability is relaxed—see point (ii) of C1 above.
C4. Compared with the IACs for mechanical systems of Subsection [III-D], the assumptions of $M_g$ constant and (16) are removed, significantly enlarging the class of underactuated mechanical systems for which the method is applicable. Moreover, the resulting IACs are much simpler.

IV. NEW CLOSED-LOOP PORT-HAMILTONIAN STRUCTURE

In this section we present a novel IAC that does not rely on changes of coordinates nor assumes separability of the systems Hamiltonian. This IAC generates a new pH closed-loop system that, with a suitable selection of the tuning gains, provides the solutions to the problems formulated in Subsection [II-C].

A. New closed-loop pH dynamics

**Proposition 1:** Consider the system (1) in closed-loop with the controller

\[
\dot{u} = [-J_{aa} + R_{aa} + J_{c1}(x) - R_{c1}(x) - R_{c2}(x)]\nabla x_a \mathcal{H} + [J_{c1}(x) - R_{c1}(x)]K_i(x_a - x_c) + 2R_{au} \nabla x_a \mathcal{H}
\]

where $x_c \in \mathbb{R}^m$ and the (possibly state-dependent) matrices $J_{c1} = -J_{c1}^T$, $R_{c1} > 0$ and $R_{c2} \geq 0$ are chosen by the designer. Then, the closed-loop dynamics expressed in the coordinates

\[
w = \begin{bmatrix} w_a \\ w_u \\ w_c \end{bmatrix} := \begin{bmatrix} x_a \\ x_u \\ x_a - x_c \end{bmatrix},
\]

can be written in the pH form

\[
w = [J_{cl}(w) - R_{cl}(w)] \nabla \mathcal{H}_{cl}(w) - \begin{bmatrix} d_a \\ d_a \end{bmatrix}, \tag{18}
\]

with new interconnection and damping matrices given by

\[
J_{cl} := \begin{bmatrix} J_{c1} \\ -(J_{aa} + R_{aa})^T \\ J_{aa} + R_{aa} \\ J_{c1} \end{bmatrix}, \quad R_{cl} := \begin{bmatrix} R_{c1} + R_{c2} \\ 0_{m \times m} \\ 0_{m \times m} \\ R_{c1} \end{bmatrix},
\]

and $\mathcal{H}_{cl} : \mathbb{R}^{2m+n} \to \mathbb{R}$ is the closed-loop Hamiltonian defined as

\[
\mathcal{H}_{cl}(w) := \mathcal{H}(w_a, w_u) + \frac{1}{2}||w_c||^2_{K_i}.
\]

**Proof:** First, we underscore the presence of two key terms in the control signal (17); first, the term $-(J_{aa} - R_{aa})\nabla x_a \mathcal{H}$ than cancels the (1,1)-block of the open-loop system matrix; second, the term $2R_{au} \nabla x_a \mathcal{H}$ that changes the sign of the of the (1,2)-block of the open-loop dissipation matrix. Hence, the dynamics of $x_a$ reduces to

\[
\dot{x}_a = (J_{c1} - R_{c1} - R_{c2})\nabla x_a \mathcal{H} + (J_{au} + R_{au})\nabla x_a \mathcal{H} + (J_{c1} - R_{c1})K_i(x_a - x_c) - d_a,
\]

Now, notice that

\[
\nabla \mathcal{H}_{cl} = \begin{bmatrix} \nabla x_a \mathcal{H} \\ \nabla x_u \mathcal{H} \\ K_i(x_a - x_c) \end{bmatrix}, \tag{22}
\]

which replaced in (21) yields

\[
\dot{x}_a = (J_{c1} - R_{c1} - R_{c2})\nabla w_a \mathcal{H}_{cl} + (J_{au} + R_{au})\nabla w_a \mathcal{H}_{cl} + (J_{c1} - R_{c1})\nabla w_c \mathcal{H}_{cl} - d_a,
\]

which is the dynamics of $w_a$ in (18).

The proof follows noticing that the second rows of (1) and (18) match. Finally, writing the controller dynamics (17) as

\[
\dot{x}_c = -R_{c2}\nabla w_c \mathcal{H}_{cl} + (J_{au} + R_{au})\nabla w_a \mathcal{H}_{cl}
\]

and subtracting it from the expression of $\dot{x}_a$ above yields the last row of (18). Thus, the closed-loop dynamics can be written in the pH form (18) as claimed.

B. Discussion

The following remarks are in order.

R5. The significance of the new IAC is easily appreciated comparing the closed-loop damping matrices of the new IAC [19] with the ones of the IACs of Subsections [III-B] and [III-C] namely, (10) and (13), respectively. While the two latter IACs leave this matrix unaltered (with respect to the open-loop system) the new IAC adds, via $R_{c1}$ and $R_{c2}$, damping to the new dissipation matrix. Notice that this is achieved “swapping” the (1,2) block $R_{au}$ to the interconnection matrix [19]. This additional damping injection is, of course, fundamental to enhance the performance of the IAC.

R6. For simplicity we have proposed the quadratic function $\frac{1}{2}||w_c||^2_{K_i}$ as the energy of the controller dynamics. Proposition 1 and subsequent results of the paper, can be easily extended to any convex function $H_c(w_c)$, replacing in (17) the term $K_i(x_a - x_c)$ by $\nabla H_c$.

R7. As indicated in the proof, the control signal in (17) contains a partial linearizing term $-(J_{aa} + R_{aa})\nabla x_a \mathcal{H}$ and replaces the (1,1) block of the $J - R$ matrix by $J_{c1} - R_{c1} - R_{c2}$. The first two terms are introduced to be able to satisfy Assumption [1] while the third one adds damping when $d_a = 0$.

R8. A drawback of the proposed controller is that it requires the knowledge of open-loop dissipation matrix $R$ that, in general, is uncertain. In the case that $R_{au}$ is constant and positive definite and $R_{au} = 0_{m \times m}$, the IAC (17) can be made independent of $R$. Indeed, by choosing the controller gains as

\[
J_{c1} = 0_{m \times m}, \quad R_{c1} = R_{aa}, \quad K_i = \kappa R_{aa}^{-1},
\]

where $\kappa > 0$ is a tuning parameter, the IAC simplifies to

\[
u = [-J_{aa} - R_{c2}(x)]\nabla x_a \mathcal{H} - \kappa(x_a - x_c)
\]

\[
\dot{x}_c = -R_{c2}\nabla x_a \mathcal{H} - J_{au}\nabla w_a \mathcal{H},
\]
which is clearly independent of the open-loop damping $R$.

**R9.** The IAC (17) can be equivalently expressed with integrator state $\dot{w}_c$ as

$$u = [-J_{aa} + R_{aa} + J_{c1} - R_{c1} - R_{c2}] \nabla_x \mathcal{H}$$

$$+ [J_{c1} - R_{c1}]\dot{K}_i \dot{w}_c + 2R_{au} \nabla_x \mathcal{H}$$

$$\dot{w}_c = [J_{c1} - R_{c1}] \nabla_x \mathcal{H} + K_i \dot{w}_c - d_a,$$  \hspace{1cm} (25)

In the case of unmatched disturbances only ($d_a = 0_{m \times 1}$) the IAC can be implemented in this form. This may be advantageous in applications where the IAC can be implemented in this form. This may be advantageous in applications where $R$ is small.

Advantageous in applications where $R$ is small.

Finally, looking at the time derivative of the Hamiltonian function (26) and define

$$\nabla \mathcal{H}_{cl} = \begin{bmatrix} 0_{m \times 1} \\ 0_{s \times 1} \\ d_a \end{bmatrix},$$

since $x^*$ is a minimum of $\mathcal{H}$. Replacing this equation in (18) with $d_a = 0_{s \times 1}$ we get

$$\dot{\tilde{w}}|_{w = \tilde{w}} = \begin{bmatrix} (\tilde{J}_{c1} - \tilde{R}_{c1})K_i \tilde{w}_c - d_a(\tilde{x}) \\ 0_{s \times 1} \\ G_d(K_i \tilde{w}_c - d_a) \end{bmatrix} = \begin{bmatrix} 0_{(n+m) \times 1} \\ 0_{s \times 1} \\ 0_{s \times 1} \end{bmatrix}$$

where we have invoked Assumption 1 and (28) to get the second identity and $K_i \tilde{w}_c = \tilde{d}_a$ for the third one.

To prove that (29) is stable we evaluate the shifted Hamiltonian function (26) and define

$$\mathcal{W}(w) := \mathcal{H}_{cl}(w_a, w_u) - \tilde{d}_a^T (w_c - \tilde{w}_c) - \mathcal{H}_{cl},$$

which is clearly positive-definite, therefore qualifies as a Lyapunov candidate for the closed-loop system. The time derivative of $\mathcal{H}_{cl}$ verifies

$$\mathcal{H}_{cl} = \nabla^T \mathcal{H}_{cl}(J_{cl} - R_{cl}) \nabla \mathcal{H}_{cl} - (\nabla w_a \mathcal{H}_{cl} + \nabla w_c \mathcal{H}_{cl})^T d_a$$

$$= - \left\| \nabla w_a \mathcal{H}_{cl} \right\|^2_{R_{c2}} + \left\| \nabla w_a \mathcal{H}_{cl} + K_i \dot{w}_c \right\|^2_{G_d}$$

$$- \left\| \nabla w_a \mathcal{H}_{cl} \right\|^2_{R_{w_a}} - \left( \nabla w_a \mathcal{H}_{cl} + K_i \dot{w}_c \right)^T G_d \tilde{d}_a$$

$$\leq - \left\| \nabla w_a \mathcal{H}_{cl} \right\|^2_{R_{c2}} + \left\| \nabla w_a \mathcal{H}_{cl} + K_i \dot{w}_c \right\|^2_{G_d}$$

$$- \left( \nabla w_a \mathcal{H}_{cl} + K_i \dot{w}_c \right)^T G_d \tilde{d}_a,$$

where we used the fact that $R_{w_a} \geq 0$ to get the inequality. Likewise, the time derivative of the term $\tilde{d}_a^T \dot{w}_c$ satisfies

$$\tilde{d}_a^T \dot{w}_c = \tilde{d}_a G_d(\nabla w_a \mathcal{H}_{cl} + K_i \dot{w}_c - \tilde{d}_a).$$

Combining these two equations we complete the squares and get the bound

$$\mathcal{W} \leq - \left\| \nabla w_a \mathcal{H}_{cl} \right\|^2_{R_{c2}} - \left\| \nabla w_a \mathcal{H}_{cl} + K_i \dot{w}_c \right\|^2_{R_{c1}} \leq 0,$$ \hspace{1cm} (31)

which proves claim (i).

The proof of claim (ii) is established with LaSalle’s invariance principle and the following implication

$$\mathcal{W} = 0 \iff |Y_a| = 0.$$

Finally, to verify (iii), notice that if $\mathcal{H}$ is radially unbounded in $x$, then $\mathcal{H}_{cl}$ and $\mathcal{W}$ are radially unbounded in $w$, which implies that the stability properties of the equilibrium are global.

\begin{thebibliography}{10}

\end{thebibliography}
B. The case of unmatched disturbances

Our attention is now turned to considering the dynamics (18) with unmatched disturbances. That is, $d_a = 0_{m \times 1}$ and $d_u$ is non-zero. The approach taken here is to utilize Assumption 2 to transform the dynamics into a similar system subject to a matched disturbance. Stability analysis of the transformed system then follows in much the same way as the matched disturbance case in Proposition 2.

In order to transform the unmatched disturbance problem into a similar matched disturbance problem, first consider the closed-loop dynamics with the unmatched disturbance satisfying Assumption 2 and $R_2 = 0_{m \times m}$. Define

$$\ddot{w} = (J_{cl} - R_{cl}) \nabla H_{cl} - \left[ \begin{array}{c} 0_{m \times m} \\ (J_{uu} + R_{uu})^T \\ 0_{m \times m} \end{array} \right] \tilde{d}_u.$$  \hspace{1cm} (32)

The term $\tilde{d}_u$ can be "reflected" through the (2,1) block of $J_{cl} - R_{cl}$ where it then appears alongside the first element of $\nabla H_{cl}$. However, this process results in an additional term which enters the system as a matched disturbance of the form described by Assumption 1. The transformed dynamics are given by

$$\ddot{w} = (J_{cl} - R_{cl}) \nabla H_{cl} - \left[ \begin{array}{c} 0_{m \times m} \\ (J_{uu} + R_{uu})^T \\ 0_{m \times m} \end{array} \right] \tilde{d}_u.$$  \hspace{1cm} (33)

By defining

$$H_{cl}(w) = H(w_a, w_u) + \frac{1}{2} \|w_c\|^2_{K_1},$$  \hspace{1cm} (34)

with $H$ given in (6), the dynamics are of the form

$$\ddot{w} = (J_{cl} - R_{cl}) \nabla H_{cl} - \left[ \begin{array}{c} 0_{m \times m} \\ (J_{uu} + R_{uu})^T \\ 0_{m \times m} \end{array} \right] \tilde{d}_u.$$  \hspace{1cm} (35)

Proposition 3: Consider the system (1) with $d_a = 0_{m \times 1}$, $d_u$ satisfying Assumption 2 and the open-loop system (1) satisfying Assumption 3. Let the IAC be given by (17) with the controller parameter $R_{2} = 0_{m \times m}$.

(i) The $(n + m)$-dimensional vector

$$\bar{w} = (\bar{x}_{a}, \bar{x}_{c}, K_i^{-1} \tilde{d}_u)$$  \hspace{1cm} (36)

is a stable equilibrium of the closed-loop system.

(ii) If the signal

$$Y_u := \nabla w_a H + K_i \tilde{w}_c$$  \hspace{1cm} (37)

is a detectable output, the equilibrium is asymptotically stable.

(iii) The stability properties are global if the Hamiltonian function $H$ is radially unbounded.

Proof: As the disturbance satisfies Assumption 2 and $R_{2} = 0_{m \times m}$, the closed-loop dynamics can be written as in (35). To verify that $\bar{w}$ is indeed an equilibrium point, consider the gradient of $H_{cl}$:

$$\nabla H_{cl} = \begin{bmatrix} \nabla_x H \\ K_i w_c \end{bmatrix}$$  \hspace{1cm} (38)

By Assumption 3 $\nabla H(\bar{x}) = 0_{n \times 1}$. This leads to

$$\nabla H_{cl} = \begin{bmatrix} 0_{m \times 1} \\ 0_{n \times 1} \\ K_i w_c \end{bmatrix}.$$  \hspace{1cm} (39)

Substitution of the gradient (39) into (35) yields

$$\ddot{w} |_{w = \bar{w}} = \begin{bmatrix} J_{cl} - R_{cl} \\ 0_{n \times 1} \end{bmatrix} (K_i \bar{w}_c - \tilde{d}_u) = 0_{(n + m) \times 1},$$

which verifies that $\bar{w}$ is an equilibrium.

The proof of stability follows from similar argument as Proposition 2. The shifted Hamiltonian function (26) becomes

$$W(w) := H_{cl}(w_a, w_u) - \tilde{d}_u (w_c - \bar{w}_c) - H_{cl},$$

which is clearly positive-definite, therefore qualifies as a Lyapunov candidate for the closed-loop system. The time derivative of $H_{cl}$ verifies

$$\dot{H}_{cl} \leq \|\nabla w_a H_{cl} + K_i \tilde{w}_c\|^2_{(J_{cl} - R_{cl})}$$

where we used the fact that $R_{uu} > 0$ to get the inequality. Likewise, the time derivative of the term $\tilde{d}_u w_c$ satisfies

$$\dot{\tilde{d}}_u \tilde{w}_c = \tilde{d}_u (J_{cl} - R_{cl}) (\nabla w_a H_{cl} + K_i \tilde{w}_c - \tilde{d}_u).$$

Combining these two equations we complete the squares and get the bound

$$W \leq -\|\nabla w_a H_{cl} + K_i \tilde{w}_c\|_{R_{cl}} \leq 0,$$  \hspace{1cm} (40)

which proves claim (i).

The proof of claim (ii) is established with LaSalle’s invariance principle and the following implication

$$W = 0 \iff |Y_u| = 0.$$

Finally, to verify (iii), notice that if $H$ is radially unbounded in $x$, then $H_{cl}$ and $W$ are radially unbounded in $w$, which implies that the stability properties of the equilibrium are global.

C. The case of matched and unmatched disturbances

To close this section, we note that although the cases of matched and unmatched disturbances have been treated separately, the controller is able to reject the effects of both simultaneously. Indeed, if Assumptions 13 are satisfied and the controller parameters are chosen such that $R_{2} = 0_{m \times m}$ and $J_{cl}, R_{cl}$ satisfy (27), then the closed-loop is stable.

The proof of this claim follows from the same line of reasoning as in the unmatched disturbance case in subsection 3.8.3. The unmatched disturbance, which satisfies Assumption 2, is again "reflected" through the (2,1) block of $J_{cl} - R_{cl}$ so that the term $d_u$ appears alongside the first element of $\nabla H_{cl}$. However, this process of moving the unmatched disturbance again creates a similar matched disturbance. Recalling that $J_{cl} - R_{cl} = G_d$, the resulting dynamics have the form

$$\ddot{w} = (J_{cl} - R_{cl}) \nabla H_{cl} - \begin{bmatrix} \tilde{d}_u + \tilde{d}_a \\ 0_{m \times n} \end{bmatrix}.$$  \hspace{1cm} (41)
where $H_d$ is defined in (34).

**Proposition 4:** Consider the system (1) with $d_a$ satisfying Assumption 1, $d_a$ satisfying Assumption 2, and the open-loop system (1) satisfying Assumption 3. Let the IAC be given by (17) with the controller parameter $R_c = 0_{m \times m}$.

(i) The $(n+m)$-dimensional vector

$$\tilde{w} = (\tilde{x}_a, \tilde{x}_u, \tilde{K}_i^{-1} [\tilde{d}_a + \tilde{d}_u])$$

is a stable equilibrium of the closed-loop system.

(ii) If the signal $Y_u$ defined in (17) is a detectable output, the equilibrium is asymptotically stable.

(iii) The stability properties are global if the Hamiltonian function $H$ is radially unbounded.

**Proof:** The proof follows from the same procedure as the proof of Proposition 3 and is omitted for brevity.

\section{VI. APPLICATION TO MECHANICAL SYSTEMS}

In this section, the IAC (17) is applied to robustify energy-shaping controlled underactuated mechanical systems of the form (14) with respect to constant matched disturbances. The problem considered here has been previously considered in [13] and [14] (see [14] for the detailed explanation and motivation of the problem). For convenience, we repeat the dynamics (14) here:

$$\begin{bmatrix}
\dot{q} \\
\dot{p}
\end{bmatrix} =
\begin{bmatrix}
0_{l \times l} & M^{-1}(q)M_d(q) \\
-M_d(q)M^{-1}(q) & J_2(q, p) - R_d(q)
\end{bmatrix}
\nabla H_d
+ \begin{bmatrix}
d_a \\
G_q(q)(u - d_m)
\end{bmatrix},$$

$$H_d(q, p) = \frac{1}{2} \| p \|_{M_d^{-1}(q)}^2 + V_d(q).$$

**A. Problem formulation**

Consider the system of an energy-shaping controlled mechanical system (43) subject to a constant matched disturbance. That is, $d_a = d_m$ for some constant $d_m \in \mathbb{R}^m$. Define mappings $u : \mathbb{R}^l \times \mathbb{R}^l \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ and $F : \mathbb{R}^l \times \mathbb{R}^l \rightarrow \mathbb{R}^m$ such that the IAC

$$u = u(q, p, x_c)$$
$$\dot{x}_c = F(q, p),$$

where $x_c \in \mathbb{R}^m$ is the state of the controller, that ensures the closed-loop is a pH system with an (asymptotically) stable equilibrium at $(q, p, x_c) = (q^*, 0, x_c^*)$ for some $x_c^* \in \mathbb{R}^m$.

**B. Momentum transformation**

The system (43) is similar to the system (1) while the problem formulation of subsection VI-A mirrors that of subsection II-C. Thus, it makes sense to apply the IAC (17) as a solution to the disturbed mechanical system problem.

The key difference between the systems (43) and (1) is that in (43), the control input $u$ is pre-multiplied by a matrix $G(q)$. To compensate for this difference, we present a momentum transformation that allows the dynamics (43) to be expressed in a similar form where the input is pre-multiplied by the identity matrix. Slightly different forms of the following lemma have been used in the literature (e.g. [22] Lemma 2, [23] Proposition 1 and [24] Theorem 1).

**Lemma 1:** Consider the system (43) under the change of momentum

$$(q, p) = (q, T(q)p),$$

where $T(q) \in \mathbb{R}^{l \times l}$ is invertible. Then, the dynamics (43) can be equivalently expressed as

$$\begin{bmatrix}
\dot{q} \\
\dot{p}
\end{bmatrix} =
\begin{bmatrix}
0_{l \times l} & Q \\
-Q^T & C - D
\end{bmatrix}
\nabla H + \begin{bmatrix}
0_{l \times m} \\
0_{l \times m}
\end{bmatrix}
\left u - d_m \right ,$$

with

$$H(q, p) := \frac{1}{2} \| p \|_{M_a^{-1}(q)}^2 + V_d,$$

and

$$M_a^{-1}(q) = T^{-T} M_d^{-1} T^{-1},$$

$$Q(q) = M^{-1} M_d T,$$

$$C(q, p) = \left( \nabla H \right)_T(Tp) M^{-1} M_d - M_d M^{-1} \nabla q(Tp) + T J_u T^T \left| p = T^{-1}(q) p \right)$$

$$D(q, p) = T R_d T^T \left| p = T^{-1}(q) p \right.$$}

**Proof:** The proof of this lemma follows the same as the proof in [22] Lemma 2, [23] Proposition 1 and [24] Theorem 1, therefore the full proof is omitted.

Consider now the transformation matrix $T(q)$ defined as

$$T(q) = \left( G(q) \right)^T G^T(q)$$

where $G^\perp(q)$ is a full-rank left annihilator of $G(q)$. It follows that

$$T(q) G(q) = \begin{bmatrix} I_m \\ 0_{r \times m} \end{bmatrix},$$

where $r := l - m$. Considering the momentum vector in (46) as $p = \text{col}(p_a, p_u)$, where $p_a \in \mathbb{R}^m$ and $p_u \in \mathbb{R}^r$, and the matrices $C, D$ and $Q$ as

$$C = \begin{bmatrix} C_{aa} & C_{au} \\ -C_{au}^T & C_{uu} \end{bmatrix},$$

$$D = \begin{bmatrix} D_{aa} & D_{au} \\ D_{au}^T & D_{uu} \end{bmatrix},$$

$$Q = [Q_a \quad Q_u],$$

where $C_{aa}, D_{aa} \in \mathbb{R}^{m \times m}, C_{au}, D_{au} \in \mathbb{R}^{m \times r}, C_{uu}, D_{uu} \in \mathbb{R}^{r \times r}$, $Q_a \in \mathbb{R}^{l \times m}, Q_u \in \mathbb{R}^{l \times r}$, the system (43), under the change of coordinates (45) where $T(q)$ is defined by (49), can be expressed as

$$\begin{bmatrix}
\dot{p}_a \\
\dot{p}_u
\end{bmatrix} =
\begin{bmatrix}
C_{aa} - D_{aa} & C_{au} - D_{au} \\
-C_{au}^T & C_{uu} - D_{uu}
\end{bmatrix}
\begin{bmatrix}
q_a \\
q_u
\end{bmatrix}
\begin{bmatrix}
\nabla p_a \nabla H \\
\nabla q_u \nabla H
\end{bmatrix}
+ \begin{bmatrix}
I_m \\
0_{r \times m}
\end{bmatrix}
\left u - d_m \right ,$$

(52)

Notice that the system (52) is in the form (1) with $x_a = p_a, x_u = \text{col}(p_u, q)$,

$$J_a(x_a, x_u) = C_{aa},$$

$$J_u(x_a, x_u) = \begin{bmatrix} C_{au} - Q_a \\ Q_u \end{bmatrix},$$

$$J_{uu}(x_a, x_u) = \begin{bmatrix} C_{uu} - Q_u \\ Q_u \end{bmatrix}$$
\[
R_{aa}(x_a, x_u) = D_{aa} \\
R_{au}(x_a, x_u) = \begin{bmatrix} D_{au} & 0_{m \times 1} \end{bmatrix} \\
R_{uu}(x_a, x_u) = \begin{bmatrix} 0_{1 \times r} & 0_{1 \times 1} \end{bmatrix}
\]  
(53)

C. Integral action controller

In the previous section it was shown that the system (43) can be equivalently written as (52), which falls into the class of systems (1). Thus, we can now utilise the control law (17) to solve the disturbance rejection problem. The following proposition formalises the stability properties of the closed-loop.

**Proposition 5:** Consider the dynamics (52), or equivalently (43), in closed-loop with the controller

\[
u = (-C_{aa} + D_{aa} + J_{c1} - R_{c1} - R_{c2})\nabla_p u + (J_{c1} - R_{c1})K_i(p_a - x_c) + 2D_{au}\nabla_p u
\]

\[
\dot{x}_c = -R_{c2}\nabla_p u + (C_{au} + D_{au})\nabla_p u - Q_{d1}\nabla_q u
\]

(54)

where \(J_{c1}, R_{c1}, R_{c2}, K_i \in \mathbb{R}^{m \times m}\) are constant tuning parameters chosen to satisfy \(J_{c1} = -J_{c1}^*\) and \(R_{c1}, R_{c2}, K_i > 0\).

(i) The \((2l + m)\)-dimensional vector

\[
(q, p, 0_{c}) = (q^*, 0_{l \times 1}, K_i^{-1}(J_{c1} - R_{c1})^{-1}\tilde{d}_m).
\]

(55)

is a stable equilibrium of the closed-loop system.

(ii) If the output

\[
Y_m := \begin{bmatrix} \nabla_p u & \nabla_p \end{bmatrix} = \begin{bmatrix} p_a - x_c & K_i^{-1}(J_{c1} - R_{c1})^{-1}\tilde{d}_m \end{bmatrix}
\]

(56)

is detectable, then the equilibrium is asymptotically stable.

(iii) The stability results are global if \(V_d\) is radially bounded.

**Proof:** The proof follows from direct application of Propositions 1 and 2. As (52) is a matched disturbance problem, it must be verified that the disturbances satisfy Assumption 1. This can be seen to be true by taking \(G_d = J_{c1} - R_{c1}\) for any constant \(J_{c1}, R_{c1}\). Then, the disturbance can be written as

\[
\tilde{d}_m = (J_{c1} - R_{c1}) (J_{c1} - R_{c1})^{-1}\tilde{d}_m := (J_{c1} - R_{c1}) (J_{c1} - R_{c1})^{-1}\tilde{d}_m.
\]

(57)

verifying Assumption 1.

As the system (52) is in the form (1) and the controller (54) has the form (17), then, by Proposition 1, the closed-loop dynamics can be written in the form (18). Claims (i) and (ii) then follow from direct application of Proposition 2. Also by Proposition 2, (iii) is true if \(H_d\) is radially unbounded. Considering the definition of \(H_d\) in (47) and noting that \(M_d > 0\), \(H_d\) is radially unbounded if \(V_d\) is radially unbounded as desired.

The result in Proposition 5 can be tailored to fully-actuated mechanical systems, and since detectability can be easily shown in that case, then asymptotic stability of the desired equilibrium \((q^*, p^*, x_c^*)\) is ensured.

VII. EXAMPLES

In this section, the proposed IAC is implemented on three examples. First, the IAC is applied to the PMSM with unknown load torque and unknown mechanical friction. Interestingly, in this example, the IAC can be implemented without knowledge of the motors angular velocity. The second example is a 2-degree of freedom (DOF) manipulator with unknown mechanical friction. Interestingly, in this example, the IAC can be implemented without knowledge of the motors angular velocity. The third example is the vertical take-off and landing (VTOL) system, an example of an underactuated mechanical system subject to a matched disturbance.

A. PMSM with mechanical friction and unknown load torque

The PMSM is described by the dynamics (4):

\[
\begin{align*}
L_d^i_d &= -R_{si_d} + \omega L_{qi_d} + v_d \\
L_d^i_q &= -R_s i_d - \omega L_{di_d} - \omega \Phi + v_q \\
J\dot{\omega} &= n_p [(L_d - L_q)i_d i_q + \Phi i_q] - R_m \omega - \tau_L
\end{align*}
\]

(58)

where \(i_d, i_q\) are currents, \(v_d, v_q\) are voltage inputs, \(n_p\) is the number of pole pairs, \(L_d, L_q\) are the stator inductances, \(\Phi\) is the back emf constant, \(J\) is the moment of inertia, \(R_m\) is the electrical resistance, \(R_m\) is the mechanical friction coefficient and \(\tau_L\) is a constant load torque.

Using the energy shaping controller proposed in (3), the closed-loop has an asymptotically stable equilibrium at \((0, 0, \omega^*)\) when \(\tau_L = 0\) and \(R_m = 0\). The closed-loop dynamics have the \(\mathbb{p}\) representation

\[
\begin{bmatrix}
i_q \\
i_d \\
\dot{\omega}
\end{bmatrix} = \begin{bmatrix}
-C_{12}(i_d, i_q) & -C_{23} & C_{13}(i_q) \\
-C_{12}(i_d, i_q) & -C_{23} & C_{13}(i_q) \\
0 & 0 & R_m \gamma_2
\end{bmatrix} \begin{bmatrix}
\nabla_i q \H \\
\nabla_i d \H \\
\nabla_\omega \H
\end{bmatrix} + \begin{bmatrix}
u \\
0 \\
0
\end{bmatrix} - \begin{bmatrix}
0 \\
0 \\
\frac{1}{\tau_L + R_m \omega^*}
\end{bmatrix},
\]

(59)

where

\[
H(i_d, i_q, \omega) = \frac{1}{2} \gamma_1 i_d^2 - \frac{1}{2C_{23}} n_p \Phi i_q^2 + \frac{1}{2} \gamma_2 (\omega - \omega^*)^2.
\]

(60)

\(r_1, r_2, \gamma_1, \gamma_2, C_{23} \in \mathbb{R}\) are tuning parameters satisfying \(r_1, r_2, \gamma_1, \gamma_2 > 0, C_{23} < 0, C_{12} : \mathbb{R}^2 \to \mathbb{R}\) is a free function, \(C_{13}(i_q) := -\frac{n_p}{\gamma_1}(L_d - L_q)i_q\), and \(u \in \mathbb{R}\) is an additional voltage input for IAC design.

Taking \(x_d = i_q, x_u = \text{col}(i_d, \omega)\), the system is of the form (1) with

\[
\begin{bmatrix}
J_{aa} \\
J_{au} \\
J_{uu}
\end{bmatrix} = \begin{bmatrix}
0 & C_{12}(i_d, i_q) & C_{23} \\
-C_{12}(i_d, i_q) & C_{13}(i_q) & 0 \\
0 & C_{13}(i_q) & 0
\end{bmatrix} \quad \text{and} \quad d_u = \text{col}\left(0, \frac{1}{\tau_L + R_m \omega^*}\right).
\]

(61)

Notice that the disturbance is unmatched. Our objective is now to apply the IAC (17) to guarantee that the disturbed system
has a stable equilibrium satisfying $\nabla_{x_c} H = 0_{2 \times 1}$. That is, the shifted Hamiltonian (6) takes the form expressed in different coordinates. Interestingly, the IAC (67) has a stable equilibrium satisfying $d_u = 0$ for some constant $\bar{d}_u$. This is only possible if $C_{12} = 0$. As this is a free function, we make this selection for the energy shaping controller. With this choice,

$$d_u = \frac{1}{JC_{23}} (\tau_L + R_m \omega^r). \quad (63)$$

A3 The shifted Hamiltonian (6) takes the form

$$H = \frac{1}{2} \gamma_1 \bar{t}_d^2 - \frac{1}{2} \frac{n_p}{J} \Phi q^2 + \frac{1}{2} \gamma_2 (\omega - \omega^r)^2$$

$$+ i \frac{1}{JC_{23}} (\tau_L + \tau_{L_m} \omega^r) = \frac{1}{2} \gamma_1 \bar{t}_d^2 - \frac{1}{2} \frac{n_p}{J} \Phi \left( \frac{i - \tau_L + \tau_{L_m} \omega^r}{n_p \Phi} \right)^2$$

$$+ \frac{1}{2} \gamma_2 (\omega - \omega^r)^2 + (\tau_L + \tau_{L_m} \omega^r)^2 \frac{1}{2n_pJC_{23} \Phi} \quad (64)$$

which is clearly minimised at the desired point

$$(i, \bar{t}_d, \omega) = \left( \frac{\tau_L + \tau_{L_m} \omega^r}{n_p \Phi}, 0, \omega^r \right). \quad (65)$$

As the system satisfies Assumptions 2 and 3 integral action can be applied using the control law (17). Taking $R_{c_1} = R_{aa}$, $J_c = 0$, $R_{c_2} = 0$, the control law simplifies to

$$u = -r_2 K_i (i - x_c)$$

$$\dot{x}_c = C_{23} \nabla_{\omega} H. \quad (66)$$

Since the system (59) is subject to an unmatched disturbance only, the IAC can be implemented in the $u_c$ coordinates as per (25). Using this realisation, the controller simplifies to be

$$u = -r_2 K_i w_c$$

$$\dot{w}_c = -r_2 (\nabla_{i} H_d + K_i w_c). \quad (67)$$

where $w_c \in \mathbb{R}$ is the state of the controller. To reiterate, the expressions (66) and (67) describe the same IAC but are expressed in different coordinates. Interestingly, the IAC (67) is independent of the rotor speed $\omega$ whereas the realisation (66) is not.

By Proposition 3 the point (65) is a stable equilibrium of the closed-loop system. Further, the system is asymptotically stable if the output

$$Y_u = \nabla_{x_c} H + K_i \bar{w}_c$$

$$= \nabla_{i} H + \bar{d}_u + K_i (w_c - K_i^{-1} \bar{d}_u)$$

$$= \nabla_{i} H + K_i \bar{w}_c \quad (68)$$

is detectable. To verify this fact, we set $Y_u = 0$ identically and investigate whether this implies that $|w(t)| \to 0$, with $w = (i, q, \omega, w_c)$. Defining the set $Y_u = \{w \in \mathbb{R}^4 | Y_u = 0\}$, the dynamics of $w_c$ restricted to $Y_u$ satisfy $\dot{w}_c|_{Y_u = 0} = 0$ which implies that $w_c|_{Y_u = 0}$ is constant. By the dynamics of $w_c$ in (67), $i_q|_{Y_u = 0}$ must be constant and $i_d|_{Y_u = 0} = 0$. Considering the dynamics of $i_q$ in (59) and recognising that $u|_{Y_u = 0} = -r_2 K_i w_c$, $i_q|_{Y_u = 0} = 0$ implies that $\omega|_{Y_u = 0} = \omega^r$ and $\bar{d}_i|_{Y_u = 0} = 0$. The dynamics of $i_d$ in (59) now reduce to $i_d|_{Y_u = 0} = -r_1 \nabla_{i} H_d$ which implied that $i_d, i_d$ tend to zero. To complete the argument, consider the dynamics of $\omega$ in (59) which implies that $-C_{23} - \nabla_{i} H_d|_{Y_u = 0} \omega \to \frac{1}{2} (\tau_L + \tau_{L_m} \omega^r)$, which can be used to recover the equilibrium (65).

B. 2-degree of freedom manipulator with unknown damping

The 2-degree of freedom (DOF) planar manipulator in closed-loop with energy shaping control has the dynamic equations

$$\dot{\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix}} = \begin{bmatrix} 0_{2 \times 2} & I_{2 \times 2} \\ -I_{2 \times 2} & -R_d \end{bmatrix} \begin{bmatrix} \nabla_q H \\ \nabla_p H \end{bmatrix} + \begin{bmatrix} 0_{2 \times 2} \\ I_{2 \times 2} \end{bmatrix} (u + \bar{d}_a)$$

$$\begin{bmatrix} \nabla_q H \\ \nabla_p H \end{bmatrix}$$

$$H = \frac{1}{2} p^T M^{-1}(q)p + \frac{1}{2} (q - \bar{q})^T K_p (q - \bar{q})$$

where

$$M(q) = \begin{bmatrix} a_a + a_u + 2b \cos \theta_u & a_u + b \cos \theta_u \\ a_u + b \cos \theta_u & a_u \end{bmatrix} \quad (69)$$

$q = (\theta_a, \theta_u), \ p = M(q) \hat{q}, \ a_u, a_u, b$ are constant system parameters, $K_p \in \mathbb{R}^{2 \times 2}$ is positive definite and $\bar{d}_a \in \mathbb{R}^2$ is a constant disturbance (25). $R_d \in \mathbb{R}^{2 \times 2}$ is an unknown, positive definite matrix arising from physical damping. Taking $p_1, q_2 = q$, this system is of the form (1) with

$$J_{aa} = 0_{2 \times 2} \quad R_{aa} = R_d$$

$$J_{aa} = -I_{2 \times 2} \quad R_{aa} = 0_{2 \times 2} \quad (70)$$

$$J_{aa} = 0_{2 \times 2} \quad R_{aa} = 0_{2 \times 2} \quad (71)$$

As discussed in R8. of subsection IV-B as $R_{aa}$ is constant and $R_{aa} = 0_{2 \times 2}$, the IAC can be applied to the system without knowledge of the damping parameter $R_d$. The simplified IAC is given by

$$u = -r_2 c (x) M^{-1}(q)p - \kappa (p - x_c)$$

$$\dot{x}_c = -r_2 c (x) M^{-1}(q) p - \nabla_q H \quad (72)$$

where $r_2 c > 0$ and $\kappa > 0$ are free to be chosen.

The simplicity of this IAC should be compared with the ones proposed in (13) to reject constant matched disturbances that, moreover, are not applicable in the presence of unknown damping.

C. Damped VTOL aircraft

The dynamics of the VTOL aircraft with physical damping in closed loop with the IDA-PBC law proposed in (8), (26) are of the form

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0_{3 \times 3} & M^{-1}M_d \\ -M_d M^{-1} & J_u - G K_v G^+ + R_0 M_d \end{bmatrix} \begin{bmatrix} \nabla_{H_d} \end{bmatrix} + \begin{bmatrix} 0_{3 \times 2} \\ \frac{G(q)}{G(q)} \end{bmatrix} (u + \bar{d}_m)$$

$$H_d = \frac{1}{2} \| p \|^2_{M_a^{-1}(q)} + V_d(q), \quad (73)$$
where $q = (x, y, \theta)$ with $x, y$ denoting the translational position and $\theta$ the angular position of the aircraft, $p = \dot{q}$ is the momentum,

$$
V_d(q) = \frac{g(1 - \cos \theta)}{k_1 - k_2} + \frac{1}{2}(z(q) - z(q^*))^\top P(z(q) - z(q^*))
$$

$$
z(q) = \begin{bmatrix}
x - x^* - \frac{k_1}{k_1 - k_2} \sin \theta \\
y - y^* - \frac{k_1}{k_1 - k_2} \cos \theta - 1
\end{bmatrix}
$$

$$
M_d(q) = \begin{bmatrix}
k_1 \epsilon \cos^2 \theta + k_3 & k_1 \epsilon \cos \theta \sin \theta & k_1 \cos \theta \\
k_1 \epsilon \cos \theta \sin \theta & -k_1 \epsilon \cos \theta + k_3 & k_1 \sin \theta \\
k_1 \cos \theta & k_1 \sin \theta & k_2
\end{bmatrix}
$$

$$
R_0(q) = \text{diag}(r_1(q), r_2(q), r_3(q))
$$

$$
\dot{p}(q, p) = M_d^{-1}(q)p
$$

$$
\alpha_1(q) = \frac{1}{2}k_1 \gamma_{30} \begin{bmatrix} 2 \epsilon \cos \theta & 2 \epsilon \sin \theta & 1 \end{bmatrix}^\top
$$

$$
\alpha_2(q) = \frac{1}{2}k_1 \gamma_{30} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^\top
$$

$$
\alpha_3(q) = \frac{1}{2}k_1 \gamma_{30} \begin{bmatrix} -1 & 0 & 0 \end{bmatrix}^\top
$$

$$
\gamma_{30} = k_1 - \epsilon k_2
$$

$$
G(q) = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 & \frac{1}{\epsilon} \sin \theta
\end{bmatrix},
$$

(74)

$g$ is the acceleration due to gravity, $\epsilon$ is a constant that describes the coupling effect between the translational and rotational dynamics, $r_i(q) > 0$ are the damping coefficients, $\bar{d}_m \in \mathbb{R}^2$ is a constant disturbance and $u \in \mathbb{R}^2$ is a control input for additional control design. The parameters $K_v, P \in \mathbb{R}^{2 \times 2}$ and $k_1, k_2, k_3 \in \mathbb{R}$ are tuning parameters of the IDA-PBC law that should be chosen to satisfy the conditions given in Section 7.1 of [26] to ensure stability of the closed-loop.

In order to enhance the robustness of the control system with an integral action controller, we perform the momentum transformation (45) with $T(q)$ defined in (49), which yields

$$
T(q) = \begin{bmatrix}
x^2 + \sin^2 q_3 + \frac{\sin(2q_3)}{2(x^2 + 1)} & -\frac{\epsilon \cos(q_3)}{x^2 + 1} & \frac{\epsilon \sin(q_3)}{x^2 + 1} \\
-\frac{\sin(2q_3)}{2(x^2 + 1)} & x^2 - \sin^2 q_3 + \frac{\epsilon \sin(q_3)}{x^2 + 1} & -\frac{\epsilon \cos(q_3)}{x^2 + 1} \\
0 & \frac{\epsilon \sin(q_3)}{x^2 + 1} & -\epsilon
\end{bmatrix}.
$$

(75)

Then, the dynamics (73) written in coordinates $(q, p)$ take the form (52). The integral action control law (54) can be applied and the closed-loop system is stable by Proposition 5. Since $H_0$ is positive definite and proper, as discussed in [8], the dynamics (73) in closed-loop with the integral controller (54) ensures that the desired equilibrium $(q, p, x_c) = (q^*, 0, x_c, K_i^{-1} \bar{d}_m)$ is globally stable. Furthermore, noting that $R_0 > 0$, $p(t) \to 0$. Combining this fact with the detectability arguments contained within Proposition 8 of [8], the closed-loop is almost globally asymptotically stable.

D. Damped VTOL aircraft simulation

The energy-shaping controlled VTOL system subject to constant disturbance in closed-loop with the IAC (54) was numerically simulated under the following scenario: the initial conditions are $q(0) = (-5, 0, 0.1)$, $p(0) = (-0.1, -0.1, 0.1)$ and $x_c = (0, 0)$, the desired configuration is $q^* = (5, 0, 0)$ and $p^* = (0, 0, 0)$. The values of the model parameters and controller gains are $R_0 = I_{3 \times 3}$, $\epsilon = 1$, $k_1 = 2$, $k_2 = 1.1$, $k_3 = 30$, $J_{c_1} = 0_{2 \times 2}$, $K_i = I_{2 \times 2}$,

$$
K_v = \begin{bmatrix}
10 & 5 \\
5 & 10
\end{bmatrix},
$$

$$
P = \begin{bmatrix}
0.03 & 0 \\
0 & 0.02
\end{bmatrix},
$$

$$
R_{c_1} = \begin{bmatrix}
10 & 5 \\
5 & 10
\end{bmatrix},
$$

$$
R_{c_2} = \begin{bmatrix}
10 & 0 \\
0 & 10
\end{bmatrix}.
$$

Figure 1 shows the time histories of the configuration variables, the momentum, the controller states and Lyapunov function. On the time interval $[0, 30]$, the system operates without disturbance and tends towards the desired configuration, as expected. At $t = 30s$, a matched disturbance $\bar{d}_m = (5, -5)$ was applied to the system. The IAC rejects the effect of the disturbance and causes the VTOL system to tend towards the desired configuration.

VIII. CONCLUSION

In this paper, a method for designing IAC for pH system subject to matched and unmatched disturbances is presented. The proposed design extends our previous work by relaxing the restrictive assumptions of a strongly convex and separable Hamiltonian function for the open-loop system. By relaxing these assumptions, the proposed IAC is shown to be applicable to a more general class of mechanical systems that strictly contains the class considered in previous works. The method is illustrated on three examples: a PMSM with unknown load torque and mechanical friction, 2-DOF manipulator with unknown damping and a VTOL aircraft.
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APPENDIX

Lemma 2: Consider the system (1) and assume there exists some \( x \) verifying Assumption 3. If there exists a suitable coordinate transformation \( \xi \), then there exists some \( \tilde{x} \) satisfying (11).

Proof: By Assumption 3 there exists some \( x \) satisfying \( \nabla_{x_a} \mathcal{H} = -\tilde{a}_x \), \( \nabla_{x_a} \mathcal{H} = 0_m \times 1 \). Using this point, we define \( (\tilde{z}_1, \tilde{z}_2) = (\tilde{x}_1, \tilde{x}_2) \). Using this definition, the point \( \tilde{x} \) can be defined using the functions \( \psi \) and its inverse \( \mu \):

\[
(\tilde{x}_a, \tilde{x}_u) = \left( \mu(\tilde{z}_1, \tilde{z}_2, \tilde{d}_u), \tilde{z}_2 \right).
\]

It must now be verified that \( \tilde{x} \) satisfies (11). To see that \( \tilde{x} \in \mathcal{E} \), first notice that \( \nabla_{x_a} \mathcal{H} |_{x_1} = \nabla_{x_a} \mathcal{H} |_{x_2} = -\tilde{a}_x \) and \( \nabla_{x_a} \mathcal{H} |_{x_1} = \nabla_{x_a} \mathcal{H} |_{x_2} = 0_m \times 1 \). Thus, \( \tilde{x}_a \in \mathcal{E} \). Now it must be shown that \( \nabla_{x_a} \mathcal{H}(\psi(\tilde{x}_a, \tilde{x}_u, \tilde{d}_u), \tilde{u}) = 0 \). By definition, this expression is equivalent to \( \nabla_{x_a} \mathcal{H}(\tilde{z}_1, \tilde{z}_2) = \nabla_{x_a} \mathcal{H}(\tilde{x}_a, \tilde{x}_u, \tilde{d}_u) \), which is equal to zero by Assumption 3.

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