Optimizing interferometer experiments for CMB B mode measurement

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ABSTRACT

The sensitivity of interferometers with linear polarizers to the CMB E and B mode are variant under the rotation of the polarizer frame, while interferometer with circular polarizers are equally sensitive to E and B mode. We present analytically and numerically that the diagonal elements of window functions for CMB E/B power spectra are maximized in interferometric measurement of linear polarization, when the polarizer frame is in certain rotation from the associated baseline. We also present the simulated observation to show that the 1σ errors on E/B mode power spectrum estimation are variant under the polarizer frame rotation in the case of linear polarizers, while they are invariant in the case of circular polarizers. Simulation of the configuration similar to the DASI shows that minimum 1σ error on B mode in interferometer measurement with linear polarizers is 26 per cent of that in interferometric measurement with circular polarizers. The simulation also shows that the E/B mixing in interferometer measurement with linear polarizers can be as low as 23 per cent of that in interferometric measurement with circular polarizers. It is not always possible to physically align the polarizer frame with all the associated baselines in the case of an interferometer array \( N > 2 \). There exist certain linear combinations of visibilities, which are equivalent to visibilities of the optimal polarizer frame rotation. We present the linear combinations, which enables B mode optimization for an interferometer array \( N > 2 \).

Key words: – cosmology: cosmic microwave background – techniques: interferometric

1 INTRODUCTION

The Cosmic Microwave Background (CMB) is expected to be linearly polarized by Thomson scattering at the last scattering surface and after re-ionization. The detection of the CMB polarization has been reported by the Degree Angular Scale Interferometer (DASI) [Kovac et al 2002] and recently by the Wilkinson Microwave Anisotropy Probe (WMAP) satellite [Page et al 2006]. Measurements of the CMB temperature anisotropy with interferometers are made in the Very Small Array (VSA), the Cosmic Background Imager (CBI) and many other experiments. The CMB polarization measurements with interferometers are on-going and planned in the experiments such as DASI, CBI and the Millimeter-wave Bolometric Interferometer (MBI) [Tucker et al 2003; Korotkov et al 2006]. With many desirable features of an interferometer, interferometers are more and more employed in CMB polarization experiments.

The CMB polarization can be decomposed into gradient-like E mode and curl-like B mode [Zaldarriaga and Seljak 1997]. B mode polarization is not induced by scalar density perturbation but only tensor perturbation, while E mode polarization is induced by both [Seljak and Zaldarriaga 1997]. Since tensor-to-scalar ratio is much smaller than one in most inflationary models, B mode polarization is expected to be much smaller than E mode polarization. Though there is complication by gravitational lensing [Okamoto and Hu 2003], measurement of B mode polarization makes it possible probing the Universe on the energy scale at inflationary period [Dodelson 2003].

The 1σ error on the parameter estimation can be forecast from the Fisher matrix [Dodelson 2003]. It will be shown that in interferometric measurement of Stokes parameter Q or U, the 1σ error on E and B power spectra estimation varies with rotation of its polarizer frame. We will show that certain rotation of the polarizer frame from the associated baseline minimize the 1σ error on either E or B mode power spectra estimation. For a feedhorn array \( N > 2 \), it is not always possible to realize the specific rotation of the polarizer frame from all the associated baselines. We will show that forming certain linear combinations of polarimetric visibilities is identical with physically rotating the polarizer frame to the optimal orientation, thereby en-
abling the B mode measurement optimization for a feedhorn array (N>2).

This paper is organized as follows. We discuss Stokes parameters in §2. The formalism for interferometric CMB polarization measurement on a spherical sky is presented in §3. In §4, with flat sky approximation, we show that interferometric measurement of Stokes parameter Q or U can be configured to suppressing E mode, leading to smaller leakage between E and B mode. In §5, we discuss the variation of 1σ error on power spectra estimation and show that B mode sensitivity is maximized, when the polarizer frame is in certain rotation from the baseline. In §6, we show that there exists certain linear combinations of visibilities, which enables the B mode optimization for a feedhorn array (N>2).

In §7, the summary and conclusion are given. In appendix, we show analytically that the diagonal element of window functions are maximized, when the polarizer frame is rotated from the baseline by certain angles.

2 STOKES PARAMETERS

There are Stokes parameters, which describe the state of polarization (Krains[1986]), which are measured in reference to (\(\hat{e}_\theta, \hat{e}_\phi\)) (Zaldarriaga and Seljak[1997]). \(\hat{e}_\theta\) and \(\hat{e}_\phi\) are the unit vectors of the spherical coordinate system and given by (Arfken and Weber[2000])

\[
\hat{e}_\theta = \hat{i} \cos \theta \cos \phi + \hat{j} \cos \theta \sin \phi - \hat{k} \sin \theta,
\]

\[
\hat{e}_\phi = -\hat{i} \sin \phi + \hat{j} \cos \phi.
\]

Since Thomson scattering does not generate circular polarization but linear polarization in early universe, the phase difference between \(E_\theta\) and \(E_\phi\) is zero and circular polarization state \(V\) is redundant in the study on the CMB polarization. Stokes parameters \(Q\) and \(U\) are as follows:

\[
Q = \langle E_\theta^2 - E_\phi^2 \rangle,
\]

\[
U = \langle 2 E_\theta E_\phi \rangle,
\]

where \(\langle \ldots \rangle\) indicates time average. \(Q\) and \(U\) transform under rotation of an angle \(\psi\) on the plane perpendicular to direction \(\hat{n}\) as

\[
Q'(\hat{n}) = Q(\hat{n}) \cos 2\psi + U(\hat{n}) \sin 2\psi,
\]

\[
U'(\hat{n}) = -Q(\hat{n}) \sin 2\psi + U(\hat{n}) \cos 2\psi,
\]

with which the following quantities can be constructed (Zaldarriaga and Seljak[1997]):

\[
Q'(\hat{n}) \pm i U'(\hat{n}) = e^{\mp 2i\psi}(Q(\hat{n}) \pm i U(\hat{n})).
\]

For all-sky analysis, \(Q\) and \(U\) are expanded in terms of spin \(\pm 2\) spherical harmonics (Zaldarriaga and Seljak[1997]) as follows:

\[
Q(\hat{n}) + i U(\hat{n}) = \sum_{l,m} (a_{E,l,m} + i a_{B,l,m}) 2 Y_{l,m}(\hat{n}),
\]

\[
Q(\hat{n}) - i U(\hat{n}) = \sum_{l,m} (a_{E,l,m} - i a_{B,l,m}) -2 Y_{l,m}(\hat{n}),
\]

3 INTERFEROMETRIC MEASUREMENT

The discussion in this section is for an ideal interferometer. An interferometer measures time-averaged correlation of two electric field from a pair of identical apertures positioned at \(r_1\) and \(r_2\). The separation, \(B = r_1 - r_2\), of two apertures is called the ‘baseline’ and the measured correlation is called ‘visibility’ (Lawson[1999]). Depending on the instrumental configuration, visibilities are associated with \((E_\theta^2 - E_\phi^2), (2 E_\theta E_\phi)\) and \((E_\theta^2 - E_\phi^2) \pm 2 E_\theta E_\phi\) respectively, where \(\hat{x}\) and \(\hat{y}\) are axes of the polarizer frame. As discussed in §2 Stokes parameters at angular coordinate \((\theta, \phi)\) are defined in respect to two basis vectors \(\hat{e}_\theta\) and \(\hat{e}_\phi\). Consider the polarization observation, whose antenna pointing is in the direction of angular coordinate \((\theta_A, \phi_A)\). The polarizers and baselines are assumed to be on the aperture plane. Then, the global frame coincides with the polarizer frame after Euler rotation \((\phi_A, \theta_A, \psi)\) on the global frame, where \(\psi\) is the rotation around the axis in the direction of antenna pointing. Most of interferometer experiments for the CMB observation employ feedhorns for beam collection. After passing through the feedhorn system, an incoming off-axis ray becomes on-axis ray. Then, the basis vectors \(\hat{e}_\theta\) and \(\hat{e}_\phi\) of the ray after the feedhorn system are related to the basis vectors \(\hat{e}_x\) and \(\hat{e}_y\) of the polarizer frame as follows:

\[
\hat{e}_x + i \hat{e}_y = e^{-i\psi}(\hat{e}_{\theta_A} + i \hat{e}_{\phi_A}) = e^{i(\Phi - \psi)}(\hat{e}_\theta + i \hat{e}_\phi),
\]

where \(\Phi\) is given by

\[
\Phi = \tan^{-1} \left[ \frac{\sin \theta \sin(\phi - \phi_A)}{\sin \theta \cos \phi - \cos \theta \sin \phi_A} \right] + \tan^{-1} \left[ \frac{\sin \theta_A \sin(\phi - \phi_A)}{\sin \theta \cos \phi + \cos \theta \sin \phi_A} \right].
\]

Refer to Appendix [A] for the details on the derivation of \(\Phi\).

With Eq. [8] we can easily show that

\[
(E_\theta^2 - E_\phi^2) + i(2 E_\theta E_\phi) = e^{-i(2\psi - 2\Phi)}((E_\theta^2 - E_\phi^2) + i(2 E_\theta E_\phi)).
\]

With the employment of linear polarizers, the visibilities associated with \((E_\theta^2 - E_\phi^2)\) or \((2 E_\theta E_\phi)\) are as follows:

\[
V_Q = f(\nu) \int d\Omega A(\hat{n}) \cdot \hat{A}_A,
\]

\[
V_U = f(\nu) \int d\Omega A(\hat{n}) \cdot \hat{A}_A.
\]

With the employment of circular polarizers, the visibilities associated with \((E_\theta^2 - E_\phi^2) \pm 2 E_\theta E_\phi\) are as follows:

\[
V_{RL} = f(\nu) \int d\Omega A(\hat{n}) \cdot \hat{A}_A,
\]

\[
V_{LR} = f(\nu) \int d\Omega A(\hat{n}) \cdot \hat{A}_A.
\]
With Eqs. 9 and 10, visibilities are expressed in terms of E/B mode as follows:

\[
V_Q' = -f(\nu) \int d\Omega A(\hat{n} - \hat{n}_{A}) e^{i(2\pi u \cdot \hat{n})} \times \text{Re} \left[ e^{-i(2\pi - 2\Phi(\hat{n}))} (a_{E,lm} + i a_{B,lm}) 2Y_{lm} \right],
\]

\[
V_U' = -f(\nu) \int d\Omega A(\hat{n} - \hat{n}_{A}) e^{i(2\pi u \cdot \hat{n})} \times \text{Im} \left[ e^{-i(2\pi - 2\Phi(\hat{n}))} (a_{E,lm} + i a_{B,lm}) 2Y_{lm} \right],
\]

\[
V_{RL} = -f(\nu) \int d\Omega A(\hat{n} - \hat{n}_{A}) \times (a_{E,lm} + i a_{B,lm}) 2Y_{lm} e^{i(2\pi u \cdot \hat{n} + 2\pi - 2\Phi(\hat{n}))},
\]

4 REDUCING LEAKAGE BETWEEN E AND B MODE

For the observation of small patch of sky, flat sky approximation in small angle limit can be made. In flat sky approximation, Eqs. 9, 10, 11 and 12 are as follows:

\[
V_Q' = f(\nu) \int d^2\Omega' \hat{A}(u - u') \times \text{Re} \left[ e^{-i(2\pi - 2\Phi(\hat{n}'))} (a_{E,lm} + i a_{B,lm}) 2Y_{lm} \right],
\]

\[
V_U' = f(\nu) \int d^2\Omega' \hat{A}(u - u') \times \text{Im} \left[ e^{-i(2\pi - 2\Phi(\hat{n}'))} (a_{E,lm} + i a_{B,lm}) 2Y_{lm} \right],
\]

\[
V_{RL} = f(\nu) \int d^2\Omega' \hat{A}(u - u') \times (a_{E,lm} + i a_{B,lm}) 2Y_{lm} e^{i(2\pi u \cdot \hat{n} + 2\pi - 2\Phi(\hat{n}'))},
\]

where \( \phi_u \) is the direction angle of a vector \( u \). With Eqs. 17 and 18, visibilities are expressed in terms of E and B mode as follows:

\[
V_Q' = f(\nu) \int d^2\Omega' \hat{A}(u - u') \times [\cos(2(\psi - \phi_u)) \hat{E}(u') + \sin(2(\psi - \phi_u)) \hat{B}(u')],
\]

\[
V_U' = f(\nu) \int d^2\Omega' \hat{A}(u - u') \times [-\sin(2(\psi - \phi_u)) \hat{E}(u') + \cos(2(\psi - \phi_u)) \hat{B}(u')],
\]

\[
V_{RL} = f(\nu) \int d^2\Omega' \hat{A}(u - u') \times \left[ e^{-i2(\psi - \phi_u)} (\hat{E}(u') + i \hat{B}(u')) \right],
\]

\[
V_{LR} = f(\nu) \int d^2\Omega' \hat{A}(u - u') \times \left[ e^{i2(\psi - \phi_u)} (\hat{E}(u') - i \hat{B}(u')) \right].
\]

A Gaussian beam is a good approximation to many CMB experiments and the Fourier transform of a Gaussian beam is \( \exp[-(u - \hat{u})^2/\sigma^2] \), where \( \sigma = 0.4245 \) FWHM. With such a beam, biggest contribution comes from \( u' = u \) in the integration over \( u' \). We can approximate visibilities as follows:

\[
V_Q' \approx f(\nu) \times \cos(2(\psi - \phi_u)) \hat{E}(u) + \sin(2(\psi - \phi_u)) \hat{B}(u),
\]

\[
V_U' \approx f(\nu) \times [-\sin(2(\psi - \phi_u)) \hat{E}(u) + \cos(2(\psi - \phi_u)) \hat{B}(u)],
\]

\[
V_{RL} \approx f(\nu) \left[ e^{-i2(\psi - \phi_u)} (\hat{E}(u) + i \hat{B}(u)) \right],
\]

\[
V_{LR} \approx f(\nu) \left[ e^{i2(\psi - \phi_u)} (\hat{E}(u) - i \hat{B}(u)) \right].
\]

As seen in Eqs. 23 and 24, the measurement of \( V_{RL} \) and \( V_{LR} \) measures E and B mode equally, independent of the polarizer rotation. From Eqs. 23 and 24, it is seen that \( V_Q' \) and \( V_U' \) gets unequal contribution from E and B mode, when the baseline and the polarizer frame are aligned as follows:

\[
V_Q' \approx \begin{cases} f(\nu) \hat{E}(u) & \psi = \phi_u \\ f(\nu) \hat{B}(u) & \psi = \phi_u + \pi/4 \end{cases}
\]

\[
V_U' \approx \begin{cases} -f(\nu) \hat{B}(u) & \psi = \phi_u \\ f(\nu) \hat{E}(u) & \psi = \phi_u + \pi/4 \end{cases}
\]

Physically, \( \psi = \phi_u \) corresponds to aligning x axis of the polarizer frame with the baseline, and \( \psi = \phi_u + \pi/4 \) is rotating the x axis of the polarizer frame from the baseline by 45° on the aperture plane. Compared with those of \( V_{RL} \) and \( V_{LR} \) in those configuration, either E or B mode in \( V_Q' \) and \( V_U' \) is suppressed while the other mode is intact. The complete separation of E mode from B mode is not possible unless a full-sky map is made with infinite angular resolution (Bunn et al. 2003). The leak from E mode into much weaker B mode causes serious problem. We can reduce E mode leak into B mode by measuring \( V_Q' \) with \( \psi = \phi_u + \pi/4 \) and \( V_U' \) with \( \psi = \phi_u \), in which E mode contribution is suppressed.

We have computed the 2 x 2 leakage matrix (Tegmark 2001) for the simulated observation in 13. \( L_{EB} \) indicates the B mode leakage into E mode measurement and \( L_{BE} \) indicates the E mode leakage into B mode measurement. If \( L_{EB} = L_{BE} = 0 \), there is no leakage at all (Tegmark 2001). \( L_{EB} \) and \( L_{BE} \) are shown for various \( \psi - \phi_u \) in Fig. 1.
where \( \Delta \) is data, \( S \) is signal covariance matrix and \( N \) is noise covariance matrix. By exploring parameter space to maximize likelihood, we can estimate parameters within certain error limits. The diagonal element of \( \partial S / \partial C_l \), where \( C_l \) is the angular power spectrum, shows the sensitivity of the experiment over multipoles. Through this paper, a window function means \( W_l = \partial S / \partial C_l \). In Appendix [B] we have shown analytically that the diagonal \( W_l^{BB} \) of \( V_{QQ} \) is maximized at \( \psi = \phi_u + \pi / 4 \) while that of \( V_{UU} \) is maximized at \( \psi = \phi_u - \pi / 4 \). We have numerically computed \( W_l^{EE} \) and \( W_l^{BB} \) for an interferometer of a 25 cm baseline with 30 to 50 GHz signal frequency range and 3.4° Full Width at Half Maximum (FWHM) beam. They are shown in Fig. 2 and 3, and agree with the analytical result in Appendix [B] The window functions are normalized so that the peak value of the highest window function is a unit value. As also shown in Fig. 2 and 3 an interferometer is sensitive to the multipole range \( l \approx 2\pi u / \Delta l / 2 \), where \( u \) is a baselinelength divided by wavelength and \( \Delta l \) is FWHM of the window function (for a circular Gaussian beam, \( \Delta l = 4\sqrt{2} \ln 2 / \theta_{FWHM} \)). In maximum likelihood estimation, it is usual to estimate the band powers (Knox 1999), which are assumed to be flat over some multipole range. In an interferometer experiment, E and B mode band power, \( \lambda_{EE} \) and \( \lambda_{BB} \) are assumed to be flat over the multipole range the interferometer is sensitive to.

The minimum possible variance on the parameter estimation can be forecast from the Fisher matrix (Dodelson 2003), which is defined as

\[
\mathcal{F}_{ij} = -\frac{\partial^2 (\ln \mathcal{L})}{\partial \lambda_i \partial \lambda_j} + \frac{1}{2} \text{Tr} \left\{ \frac{\partial (S + N)}{\partial \lambda_i} (S + N)^{-1} \frac{\partial (S + N)}{\partial \lambda_j} (S + N)^{-1} \right\}.
\]

(27)

Evaluated at the maximum of the likelihood, the square root of diagonal element of the inverse Fisher matrix yields the marginalized 1σ error on the parameter estimation. As shown analytically in appendix [A] and numerically in Fig. 2 and 3 diagonal windows functions of \( V_{QQ} \) and \( V_{UU} \) are maximized at certain \( \psi = \phi_u \). Therefore, \( \Delta \lambda_{EE} \) and \( \Delta \lambda_{BB} \) are expected to be smallest with certain \( \psi = \phi_u \). We have numerically computed \( \Delta \lambda_{EE} \) and \( \Delta \lambda_{BB} \) from simulated experiments for various \( \psi = \phi_u \), which are shown in Fig. 4 and 5.
In the simulation, we used the Code for Anisotropies in the Microwave Background (CAMB) \cite{Lewis2000} to compute the power spectra of $\Lambda CDM$ with the tensor-to-scalar ratio ($r = 0.3$). The baseline length 25 [cm] is assumed with the signal frequency range 30–31GHz with 1GHz bandwidth so that the interferometer probes the multipole range of roughly E1/B1 band of DASI \cite{Kovac2002}. The noise covariance matrix is assumed to be diagonal \cite{White1999} and have a uniform value, for which we assumed the sensitivity of DASE: $60 \times 1.2 \pi$ Jy s$^{1/2}$ m$^2$ \cite{Pryke2002}. We assumed the probe of three fields, three hundred sixty five days integration time for each field and simultaneous thirty six orientations for each baseline length. The equatorial coordinates of the assumed three fields are $(80^\circ, 0^\circ)$, $(80^\circ, 120^\circ)$ and $(80^\circ, 240^\circ)$. Fig. 4 and 5 shows $\Delta \lambda_{EE}$ and $\Delta \lambda_{BB}$ for various $\psi - \phi_u$. To be compared with the CAMB power spectra, $\Delta \lambda_{EE}$ and $\Delta \lambda_{BB}$ should be multiplied with $l(l+1)/(2\pi)$, where $l = 162$ is the multipole our assumed interferometer is most sensitive over. As shown in Fig. 4 that $\Delta \lambda_{BB}$ is minimum at $\psi = \phi_u + \pi/4$ in $V_U$ measurement and at $\psi = \phi_u$ in $V_U$ measurement. $\Delta \lambda_{EE}$ and $\Delta \lambda_{BB}$ from $V_{RL}$ or $V_{LR}$ are invariant under the rotation of the polarizer frame. For the same assumed noise variance, the minimum $\Delta \lambda_{BB}$ from $V_Q$ or $V_U$ is 26 per cent of $\Delta \lambda_{BB}$ from $V_{RL}$ or $V_{LR}$. It is shown in Fig. 4 that $\Delta \lambda_{EE}$ is minimum at $\psi = \phi_u$ in $V_Q$ measurement and at $\psi = \phi_u + \pi/4$ in $V_U$ measurement. The minimum $\Delta \lambda_{EE}$ from $V_Q$ or $V_U$ is 1.2 times bigger than the $\Delta \lambda_{EE}$ from $V_{RL}$ or $V_{LR}$. $V_{RL}$ and $V_{LR}$ have information on both of $Q'$ and $U'$, while $V_Q$ and $V_U$ have information on either of $Q'$ and $U'$. It may seem odd that $\Delta \lambda_{BB}$ of $V_Q$ and $V_U$ can be smaller than $\Delta \lambda_{BB}$ of $V_{RL}$ and $V_{LR}$, though more informations are contained in $V_{RL}$ and $V_{LR}$. $\Delta \lambda_{BB}$ is the estimation error marginalized over $\phi_E$, since the likelihood in the simulated observation is the function of two parameters, $\lambda_{EE}$ and $\lambda_{BB}$. $\Delta \lambda_{BB}$ is given by $\lambda_{BB}^0 - \lambda_{BB}^l$, where

$$\frac{\partial L_B(\lambda_{BB})}{\partial \lambda_{BB}} \bigg|_{\lambda_{BB}=\lambda_{BB}^0} = 0, \quad \frac{L_B(\lambda_{BB}^0)}{L_B(\lambda_{BB}^l)} = e^{-1/2}.$$

$L_B(\lambda_{BB})$, which is the likelihood function marginalized over $E$ mode band power, is as follows:

$$L_B(\lambda_{BB}) = \int d\lambda_{EE} L(\lambda_{EE}, \lambda_{BB}, \lambda_{EE} = 0).$$

$L(\lambda_{EE}, \lambda_{BB}, \lambda_{EE} = 0)$ becomes less sensitive to the variation of $\lambda_{EE}$ with reduced contribution of $E$ mode to visibilities. It makes the $L_B(\lambda_{BB})$ more sharply peaked around $\lambda_{BB}^0$, which leads to the reduction of $\Delta \lambda_{BB}$.

The 1σ error on power spectra estimation is the sum of sample variance and noise variance \cite{Park2003};

$$\Delta C_l \sim C_l/(2l+1) + N,$$

where $N$ is noise variance. In the case of circular polarizers, $\Delta \lambda_{BB}$ is smaller than $\Delta \lambda_{EE}$ due to the smaller sample variance of $B$ mode than that of $E$ mode, even though visibilities with circular polarizers have equal sensitivity to both $E$ and $B$ mode.

6 FEEDHORN ARRAY

For optimization, the polarizer frame should be oriented with certain rotation from all the baselines formed by the feedhorn the polarizer is associated with. When there are just two feedhorns, there is a single baseline and orienting the polarizer frame to specific rotation physically is trivial. But in most of real CMB experiments, an array consisting of more than two feedhorns are employed. When all the feedhorns (apertures) are aligned on a straight line, orienting polarizers with specific rotation from all the associated baselines is also trivial even for a $(N > 2)$ feedhorn array. When the feedhorn array is configured on some two dimensional pattern for the optimal uv coverage \cite{Guyon2001, Pryke2002}, it is not possible to realize specific rotation of the polarizer frame from all the associated baselines. We can achieve the optimal polarizer rotation for a array of multiple feedhorns $(N > 2)$ by forming certain linear combination of visibilities as follows. With the employment of orthomode transducer (OMT), each baseline measures the real and imaginary part of $V_Q$ and $V_U$ respectively. Four measured values are obtained such that $V_1 = Re[V_Q^0]$, $V_2 = Im[V_Q^0]$, $V_3 = Re[V_U^0]$, $V_4 = Im[V_U^0]$. With Eq. 4 and
With Eq. 28, 29, 30 and 31, we can easily show that the axis of polarizers is rotated from the baseline by 45°. B mode power spectrum estimation is minimized, when the error on B mode power spectra estimation in \( V_{Q} \) or \( V_{U} \) measurement is equal to the sample variance, the estimation error on the power spectra is minimized (Bowden et al. 2004 [Park et al. 2003]). In parallel with the choice for the polarizer rotation from the baseline, we can optimize the interferometer for B mode with the integration time which makes the sample variance of B mode equal to the noise variance.

\[ V_{1} = f(\nu) \int d\Omega A(\hat{n} - \hat{n}_{A}) \times \text{Re}[e^{-i(2\psi - 2\Phi(\hat{n}))}(Q(\hat{n}) + iU(\hat{n}))] \cos(2\pi u \cdot \hat{n}) \]

\[ V_{2} = f(\nu) \int d\Omega A(\hat{n} - \hat{n}_{A}) \times \text{Re}[e^{-i(2\psi - 2\Phi(\hat{n}))}(Q(\hat{n}) + iU(\hat{n}))] \sin(2\pi u \cdot \hat{n}), \]

\[ V_{3} = f(\nu) \int d\Omega A(\hat{n} - \hat{n}_{A}) \times \text{Im}[e^{-i(2\psi - 2\Phi(\hat{n}))}(Q(\hat{n}) + iU(\hat{n}))] \cos(2\pi u \cdot \hat{n}), \]

\[ V_{4} = f(\nu) \int d\Omega A(\hat{n} - \hat{n}_{A}) \times \text{Im}[e^{-i(2\psi - 2\Phi(\hat{n}))}(Q(\hat{n}) + iU(\hat{n}))] \sin(2\pi u \cdot \hat{n}). \]

With these, we can form two linear combinations \( V_{E} \) and \( V_{B} \) such that

\[ V_{E} = \text{Re}[e^{-i(2\psi - \psi_{0})}(V_{1} + iV_{3})] + \text{i} \text{Re}[e^{-i(2\psi - \psi_{0})}(V_{2} + iV_{4})], \]

\[ V_{B} = \text{Im}[e^{-i(2\psi - \psi_{0})}(V_{1} + iV_{3})] + \text{i} \text{Im}[e^{-i(2\psi - \psi_{0})}(V_{2} + iV_{4})], \]

With Eq. 28, 29, 30 and 31, we can easily show that

\[ V_{E} = f(\nu) \int d\Omega A(\hat{n} - \hat{n}_{A}) \times \text{Re}[e^{-i(2\psi - \Phi(\hat{n}))}(Q(\hat{n}) + iU(\hat{n}))] e^{i2\pi u \cdot \hat{n}} \]

\[ V_{B} = f(\nu) \int d\Omega A(\hat{n} - \hat{n}_{A}) \times \text{Im}[e^{-i(2\psi - \Phi(\hat{n}))}(Q(\hat{n}) + iU(\hat{n}))] e^{i2\pi u \cdot \hat{n}}. \]

We can identify the linear combination \( V_{E} \) with the \( V_{Q} \) and \( V_{B} \) with the \( V_{U} \), whose polarizer frame is aligned with the associated baseline. Hence, the linear combination \( V_{E} \) is optimized for E mode measurement and \( V_{B} \) is for B mode measurement. Linear combination \( V_{E} \) and \( V_{B} \) enable the optimization for a array of multiple feedhorns (N>2).

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APPENDIX A: CMB POLARIZATION BASIS VECTORS AND ANTENNA COORDINATE

In all-sky analysis, the CMB polarization at the angular coordinate \((\theta, \phi)\) are measured in the local reference frame whose axes are \((\hat{e}_\theta, \hat{e}_\phi, \hat{e}_z)\). Let’s call this coordinate frame ‘the local CMBP frame’ from now on. Consider the polarization observation of antenna pointing \((\theta_A, \phi_A)\). A global coordinate frame coincides with the antenna coordinate frame by Euler rotations \(R_y(\alpha)R_z(\phi)\). Since a global coordinate frame coincides with the local CMBP frame by Euler rotations \(R_y(\theta_A)R_z(\phi_A)\), the local CMBP frame is in rotation from the antenna coordinate frame by Euler rotations \(R_y(\theta)R_z(\phi)\). Therefore, the local CMBP frame is in rotation from the antenna coordinate by the Euler angles \((\alpha, \beta, \gamma)\) as follows:

\[
\begin{align*}
\alpha &= \tan^{-1} \left[ \frac{\sin \theta \sin(\phi - \phi_A)}{\sin \theta \cos \theta_A \cos(\phi - \phi_A) - \cos \theta \sin \theta_A} \right], \\
\beta &= \cos^{-1} \left[ \frac{\cos \theta \cos \theta_A + \sin \theta \sin \theta_A \cos(\phi - \phi_A)}{\tan \theta_A \sin(\phi - \phi_A)} \right], \\
\gamma &= \tan^{-1} \left[ \frac{\sin \theta_A \sin(\phi - \phi_A)}{\sin \theta \cos \theta_A + \cos \theta \sin \theta_A \cos(\phi - \phi_A)} \right],
\end{align*}
\]

where the Euler angles \((\alpha, \beta, \gamma)\) can be obtained from \(R_y(\gamma)R_y(\beta)R_z(\alpha) = R_y(\theta_A)R_z(\phi_A)R_z(\theta_A)\). In most CMB polarization experiments, where polarizers are attached to the other side of feedhorns, incoming rays go through polarizers after feedhorns. After passing through a feedhorn, an incoming off-axis ray becomes an on-axis ray. Then the local CMBP frame of the ray after the feedhorn system is simply in azimuthal rotation \(\alpha + \gamma\) from the antenna coordinate. Therefore, \(\Phi\) in Eq.\[9\] is

\[
\Phi = \tan^{-1} \left[ \frac{\sin \theta \sin(\phi - \phi_A)}{\sin \theta \cos \theta_A \cos(\phi - \phi_A) - \cos \theta \sin \theta_A} \right] + \tan^{-1} \left[ \frac{\sin \theta_A \sin(\phi - \phi_A)}{\sin \theta \cos \theta_A + \cos \theta \sin \theta_A \cos(\phi - \phi_A)} \right].
\]

APPENDIX B: WINDOW FUNCTIONS

B1 flat sky approximation in small angle limit

As discussed in \[\text{[4]}\] visibilities with flat sky approximation are as follows:

\[
\begin{align*}
V_{Q'} &= f(v) \int \mathrm{d}^2 u' \tilde{A}(u - u') \times [\cos(2(\psi - \phi_{u'}))\tilde{E}(u') + \sin(2(\psi - \phi_{u'}))\tilde{B}(u')], \\
V_{U'} &= f(v) \int \mathrm{d}^2 u' \tilde{A}(u - u') \times [-\sin(2(\psi - \phi_{u'}))\tilde{E}(u') + \cos(2(\psi - \phi_{u'}))\tilde{B}(u')], \\
V_{RL} &= f(v) \int \mathrm{d}^2 u' \tilde{A}(u - u') \times \left[ e^{i2(\psi - \phi_{u'})}(\tilde{E}(u') + i\tilde{B}(u')) \right], \\
V_{LR} &= f(v) \int \mathrm{d}^2 u' \tilde{A}(u - u') \times \left[ e^{i2(\psi - \phi_{u'})}(\tilde{E}(u') - i\tilde{B}(u')) \right].
\end{align*}
\]

For multipole \(\ell > 60\), the following relations between the flat sky power spectra and the exact power spectra from spherical sky works within one percent error\[\text{[White et al. 1999]}]:

\[
\begin{align*}
\langle E(u) E^*(u') \rangle &\approx C_{l}^{EE}\delta(u - u') |_{l = 2\pi u}, \\
\langle B(u) B^*(u') \rangle &\approx C_{l}^{BB}\delta(u - u') |_{l = 2\pi u}, \\
\langle E(u) B^*(u') \rangle &= 0.
\end{align*}
\]

With the correspondence of the power spectra between flat sky and spherical sky, it can be easily derived that diagonal elements of E/B window functions and their derivatives with respect to the rotation of the polarizer frame, \(\psi\), are as follows:

\[
\begin{align*}
(i) \quad &\langle V_{Q'}(u) V_{Q'}(u') \rangle, \\
& W_{EE}(u, u') = f^2(v) u' \int \cos^2(2\psi - 2\phi_{u'}) \tilde{A}(u - u') \tilde{A}(u - u') \mathrm{d}\phi_{u'}, \\
& W_{BB}(u, u') = f^2(v) u' \int \sin^2(2\psi - 2\phi_{u'}) \tilde{A}(u - u') \tilde{A}(u - u') \mathrm{d}\phi_{u'}, \\
& \frac{\partial W_{EE}(u, u')}{\partial \psi} = f^2(v) u' \int \mathrm{d}\phi_{u'} \tilde{A}^2(u - u') \\
& \times [-2\sin 4\psi \cos 4\phi_{u'} + 2\cos 4\psi \sin 4\phi_{u'}], \\
& \frac{\partial W_{BB}(u, u')}{\partial \psi} = f^2(v) u' \int \mathrm{d}\phi_{u'} \tilde{A}^2(u - u') \\
& \times [2\sin 4\psi \cos 4\phi_{u'} - 2\cos 4\psi \sin 4\phi_{u'}], \\
(ii) \quad &\langle V_{U'}(u) V_{U'}(u') \rangle, \\
& W_{EE}(u, u') = f^2(v) u' \int \sin^2(2\psi - 2\phi_{u'}) \tilde{A}(u - u') \tilde{A}(u - u') \mathrm{d}\phi_{u'}, \\
& W_{BB}(u, u') =
\end{align*}
\]
\[ f^2(\nu) \int \cos^2(2\psi - 2\phi_L) A(u - u') A(u - u') d\phi_L. \]

\[ \frac{\partial W^{EE}(u, u')}{\partial \psi} = f^2(\nu) \int d\phi_L A^2(u - u') \times [2 \sin 4\psi \cos 4\phi_L - 2 \cos 4\psi \sin 4\phi_L], \]

\[ \frac{\partial W^{BB}(u, u')}{\partial \psi} = f^2(\nu) \int d\phi_L A^2(u - u') \times [-2 \sin 4\psi \cos 4\phi_L + 2 \cos 4\psi \sin 4\phi_L]. \]

(iii) \( \langle V_{RL}(u)V_{RL}(u') \rangle \) and \( \langle V_{LR}(u)V_{LR}(u') \rangle \).

\[ W^{EE}(u, u') = W^{BB}(u, u') = f^2(\nu) u' \int A(u - u') A(u - u') d\phi_L, \]

\[ \frac{\partial W^{EE}(u, u')}{\partial \psi} = \frac{\partial W^{BB}(u, u')}{\partial \psi} = 0. \]

From (iii), we can see that the diagonal E and B mode window functions in \( V_{RL} \) and \( V_{LR} \) measurement are invariant under the rotation of the polarizer frame. From (i) and (ii), we can see that the derivatives of the diagonal E/B window functions in \( V_{Q'} \) and \( V_{U'} \) measurement are zero, when

\[ \sin 4\psi \int d\phi_L A^2(u - u') \cos 4\phi_L \]

\[ = \cos 4\psi \int d\phi_L A^2(u - u') \sin 4\phi_L. \]  

\[ \psi \text{, which satisfies Eq. } [B4] \text{ is } \psi = \frac{1}{4} \tan^{-1} \left( \frac{\int d\phi_L \cos 4\phi_L A^2(u - u')}{\int d\phi_L \sin 4\phi_L A^2(u - u')} \right). \]  

A Gaussian beam is a good approximation to many CMB experiments and the Fourier transform of a Gaussian beam is

\[ \tilde{A}(u - u') = \exp\left[ -\frac{|u - u'|^2 \sigma^2}{2} \right] = \exp\left[ -\frac{u^2 + u'^2 - 2uu' \cos(\phi_L - \phi_u)}{2} \right], \]

where \( \sigma = 0.4245 \) FWHM. With a Gaussian beam, the argument of \( \tan^{-1} \) in Eq. [B5] is

\[ \int_0^{2\pi} d\phi_L \sin 4\phi_L \exp[2uu' \cos(\phi_L - \phi_u) \sigma^2] \]

\[ \int_0^{2\pi} d\phi_L \cos 4\phi_L \exp[2uu' \cos(\phi_L - \phi_u) \sigma^2] \]

\[ = \int_0^{2\pi} d\phi_L \sin 4(\phi + \phi_u) \exp[2uu' \cos(\phi + \phi_u) \sigma^2] \]

\[ \int_0^{2\pi} d\phi_L \cos 4(\phi + \phi_u) \exp[2uu' \cos(\phi + \phi_u) \sigma^2] \]

\[ = \int_0^{2\pi} d\phi_L \sin 4(\phi + \phi_u) \exp[2uu' \cos(\phi + \phi_u) \sigma^2] \]

\[ \int_0^{2\pi} d\phi_L \cos 4(\phi + \phi_u) \exp[2uu' \cos(\phi + \phi_u) \sigma^2] \]

\[ = \sin 4\phi_u \int_0^{2\pi} d\phi_L \cos 4\phi_L \exp[2uu' \cos(\phi_L + \phi_u) \sigma^2] \]

\[ \cos 4\phi_u \int_0^{2\pi} d\phi_L \sin 4\phi_L \exp[2uu' \cos(\phi_L + \phi_u) \sigma^2] \]

\[ = \tan 4\phi_u \]

From the third line to the fourth line in Eq. [B6] \( \int_0^{2\pi} d\phi \sin 4\phi \exp[2uu' \cos(\phi) \sigma^2] = 0 \) was used. By plugging Eq. [B6] into Eq. [B5] we get

\[ \psi = \frac{1}{4} \tan^{-1} (\tan 4\phi_u) \]

\[ = \phi_u + \frac{n\pi}{4} \quad (n = \ldots, -2, -1, 0, 1, 2, \ldots), \]

where \( \phi_u \) is the orientation of the baseline. The diagonal element of E and B mode window functions of \( V_{Q'} \) and \( V_{U'} \) is maximized or minimized, when \( x \)-axis of the polarizer is in rotation from the baseline by \(-90^\circ, -45^\circ, 0^\circ, 45^\circ, 90^\circ\). Since the second derivative of diagonal element of E mode window function has the opposite sign of that of B mode window function, the diagonal element of B mode window function is minimized at the polarizer rotation which maximizes the diagonal element of E mode window function, and vice versa.

**B2 spherical sky**

Visibilities from spherical sky are as follows:

\[ V_{Q'} = \frac{1}{2} f(\nu) \int d\Omega \hat{n} \cdot \hat{A} e^{i(2\nu u \cdot \hat{n})} \]

\[ \times [e^{-i(2\nu - 2\phi(\hat{n}))} (a_{E,lm} + ia_{B,lm}) 2Y_{lm} + e^{i(2\nu - 2\phi(\hat{n}))} (a_{E,lm} - ia_{B,lm}) - 2Y_{lm}] \]

\[ V_{U'} = \frac{i}{2} f(\nu) \int d\Omega \hat{n} \times \hat{A} e^{i(2\nu u \cdot \hat{n})} \]

\[ \times [e^{-i(2\nu - 2\phi(\hat{n}))} (a_{E,lm} + ia_{B,lm}) 2Y_{lm} - e^{i(2\nu - 2\phi(\hat{n}))} (a_{E,lm} - ia_{B,lm}) - 2Y_{lm}] \]

\[ V_{RL} = -f(\nu) \int d\Omega \hat{n} \cdot \hat{A} \]

\[ \times (a_{E,lm} + i a_{B,lm}) 2Y_{lm} e^{i(2\nu u \cdot \hat{n} + 2\phi(\hat{n}))}, \]

\[ V_{LR} = -f(\nu) \int d\Omega \hat{n} \times \hat{A} \]

\[ \times (a_{E,lm} - i a_{B,lm}) -2Y_{lm} e^{i(2\nu u \cdot \hat{n} - 2\phi(\hat{n}))}. \]

With Eq. [B7], [B8], [B9] and [B10] visibilities from spherical sky can be expressed as follows:

\[ V_{Q'}(\hat{n}, u) = \sum_{l,m} (e^{-i2\nu l} R_{lm} + e^{i2\nu l} L_{lm}) a_{E,lm} \]

\[ + i (e^{-i2\nu l} R_{lm} - e^{i2\nu l} L_{lm}) a_{B,lm}, \]  

\[ V_{U'}(\hat{n}, u) = \sum_{l,m} -i(e^{-i2\nu l} R_{lm} - e^{i2\nu l} L_{lm}) a_{E,lm} \]

\[ + (e^{-i2\nu l} R_{lm} + e^{i2\nu l} L_{lm}) a_{B,lm}, \]  

\[ V_{RL}(\hat{n}, u) = \sum_{l,m} 2(a_{E,lm} + i a_{B,lm}) e^{-i2\nu l} R_{lm}, \]  

\[ V_{LR}(\hat{n}, u) = \sum_{l,m} 2(a_{E,lm} + i a_{B,lm}) e^{i2\nu l} L_{lm}, \]  

where

\[ R_{lm} = \frac{-f(\nu)}{2} \int d\Omega \hat{n} \cdot \hat{A} \]

\[ 2Y_{lm} e^{i(2\nu u \cdot \hat{n} + 2\phi(\hat{n}))}, \]

\[ L_{lm} = \frac{-f(\nu)}{2} \int d\Omega \hat{n} \times \hat{A} \]

\[ -2Y_{lm} e^{i(2\nu u \cdot \hat{n} - 2\phi(\hat{n}))}, \]

and \( \hat{n}_A \) indicates the direction of antenna pointing.
Spin \pm 2 spherical harmonics have the following form (Zaldarriaga 1998).

\[ 2 Y_{lm}(\hat{n}) = \sqrt{\frac{2l+1}{4\pi}} (F_{1,lm}(\theta) + F_{2,lm}(\theta)) e^{im\phi}, \]
\[ -2 Y_{lm}(\hat{n}) = \sqrt{\frac{2l+1}{4\pi}} (F_{1,lm}(\theta) - F_{2,lm}(\theta)) e^{im\phi}, \]
where \( F_{1,lm} \) and \( F_{2,lm} \) can be computed in terms of Legendre functions as follows (Kamionkowski et al. 1997):

\[ F_{1,lm}(\theta) = 2 \sqrt{\frac{(l-2)(l-m)}{(l+2)(l+m)}} [(l+m) \cos \theta P_{l-1}^m(\cos \theta) - \frac{l-m}{\sin^2 \theta} + \frac{1}{2}(l-1)P_{l-1}^m(\cos \theta)], \]
\[ F_{2,lm}(\theta) = 2 \sqrt{\frac{(l-2)(l-m)}{(l+2)(l+m)}} \frac{m}{\sin^2 \theta} [(l+m) P_{l-1}^m(\cos \theta) - (l-1) \cos \theta P_{l-1}^m(\cos \theta)]. \]

The covariance properties of the E and B mode are given by

\[ a_{E,lm}a_{E,l'm'}^* = C_{l_l} \delta_{ll'} \delta_{mm'}, \]
\[ a_{B,lm}a_{B,l'm'}^* = C_{l_l} \delta_{ll'} \delta_{mm'}, \]
\[ a_{E,lm}a_{B,l'm'}^* = 0. \]

With these covariance properties, Eq. \[ B11 B12 B13 \text{ and } B14 \] it can be easily shown that diagonal elements of E/B window functions and their derivatives with respect to the rotation of the polarizer frame, \( \psi \), are as follows:

(i) \( \langle V_{O'}(\mathbf{u})V_{O'}(\mathbf{u})^{*} \rangle \),

\[ W_i^{EE} = \sum_l R_{i,lm} R_{i,lm}^{*} + L_{i,lm} L_{i,lm}^{*}, \]
\[ W_i^{BB} = \sum_l R_{l,lm} R_{l,lm}^{*} + L_{l,lm} L_{l,lm}^{*}, \]
\[ \frac{\partial W_i^{EE}}{\partial \psi} = \sum_l i4e^{i4\psi} L_{l,lm} R_{l,lm}^{*} - i4e^{-i4\psi} R_{l,lm} L_{l,lm}^{*}, \]
\[ \frac{\partial W_i^{BB}}{\partial \psi} = \sum_l i4e^{i4\psi} L_{l,lm} R_{l,lm}^{*} - i4e^{-i4\psi} R_{l,lm} L_{l,lm}^{*}. \]

(ii) \( \langle V_{U'}(\mathbf{u})V_{U'}(\mathbf{u})^{*} \rangle \),

\[ W_i^{EE} = \sum_l R_{i,lm} R_{i,lm}^{*} + L_{i,lm} L_{i,lm}^{*}, \]
\[ W_i^{BB} = \sum_l R_{l,lm} R_{l,lm}^{*} + L_{l,lm} L_{l,lm}^{*}, \]
\[ \frac{\partial W_i^{EE}}{\partial \psi} = \sum_l -i4e^{-i4\psi} L_{l,lm} R_{l,lm}^{*} + i4e^{i4\psi} R_{l,lm} L_{l,lm}^{*}. \]

From (iii) and (iv) we can see that the diagonal E and B mode window functions in \( V_{RL} \) and \( V_{LR} \) measurement are invariant under the rotation of the polarizer frame. From (i) and (ii) we can see that the derivatives of the diagonal E and B mode window functions in \( V_{O'} \) and \( V_{U'} \) measurement are zero, when

\[ e^{i4\psi} L_{l,lm} R_{l,lm}^{*} - e^{-i4\psi} R_{l,lm} L_{l,lm}^{*} = 0. \]  

Since the left side of Eq. \[ B15 \] is

\[ -i2 \text{Im}[e^{-i4\psi} R_{l,lm} R_{l,lm}^{*}] = -i2(- \sin 4\psi \text{Re}[R_{l,lm} L_{l,lm}^{*}] + \cos 4\psi \text{Im}[R_{l,lm} L_{l,lm}^{*}]}, \]

the following \( \psi \) satisfies Eq. \[ B15 \]

\[ \psi = \frac{1}{4} \tan^{-1} \left( \frac{\text{Im}[R_{l,lm} L_{l,lm}^{*}]}{\text{Re}[R_{l,lm} L_{l,lm}^{*}]} \right). \]

The diagonal elements of window function are independent of the choice of the reference coordinate (White and Srednicki 1993). So we can choose antenna pointing at \( z \) axis without loss of generality. In the reference frame of our choice, \( \Phi(\hat{n}) \) is reduced to azimuthal angle \( \phi \). In most of CMB interferometer experiments, primary beam pattern are azimuthally symmetric and baselines are coplanar. Then \( R_{i,lm} \) and \( L_{i,lm} \) are

\[ R_{i,lm} = -\frac{f(\nu)}{2} \int_0^{2\pi} d(\theta) \sin \theta A(\theta) \int_0^{2\pi} d\phi \]
\[ \times 2Y_{lm}(\theta, \phi) e^{i(2\nu u \sin \theta \cos(\phi - \phi_u) - 2\phi + \phi_u)}, \]
\[ = -\frac{f(\nu)}{2} \sqrt{\frac{2l+1}{4\pi}} \]
\[ \times \int_0^{2\pi} d(\theta) \sin \theta A(\theta)[F_{1,lm}(\theta) + F_{2,lm}(\theta)] \]
\[ \times \int_0^{2\pi} d\phi e^{i(2\nu u \sin \theta \cos(\phi - \phi_u) - 2\phi + (m+2)\phi)}, \]
\[ = e^{i\frac{\pi}{2} m} \epsilon_{l,m+2}(\nu, \phi_u) \Theta_{lm}, \]
\[ L_{i,lm} = -\frac{f(\nu)}{2} \int_0^{2\pi} d(\theta) \sin \theta A(\theta) \int_0^{2\pi} d\phi \]
\[ \times 2Y_{lm}(\theta, \phi) e^{i(2\nu u \sin \theta \cos(\phi - \phi_u) - 2\phi + \phi_u)}, \]
\[ = -\frac{f(\nu)}{2} \sqrt{\frac{2l+1}{4\pi}} \]
\[ \times \int_0^{2\pi} d(\theta) \sin \theta A(\theta)[F_{1,lm}(\theta) - F_{2,lm}(\theta)] \]
\[ \times \int_0^{2\pi} d\phi e^{i(2\nu u \sin \theta \cos(\phi - \phi_u) - 2\phi - (m+2)\phi)}, \]
\[ = e^{i\frac{\pi}{2} m} \epsilon_{l,m+2}(\nu, \phi_u) \Theta_{lm}, \]
\[
\times -2Y_{lm}(\theta, \phi)e^{i(2\pi u \sin \theta \cos(\phi - \phi_u) + 2\psi + 2\phi)},
\]

\[
= -\frac{f(\nu)}{2} \sqrt{\frac{2l + 1}{4\pi}}
\times \int_0^\pi d\theta \sin \theta A(\theta)\left[F_{1,lm}(\theta) - F_{2,lm}(\theta)\right]
\times \int_0^{2\pi} d\phi e^{i(2\pi u \sin \theta \cos(\phi - \phi_u) + 2\psi + (m-2)\phi)},
\]

\[
= e^{i\frac{m}{2}(m-2)\phi_u - 2\Theta_{lm}}, \quad (B18)
\]

where

\[
2\Theta_{lm} = -f(\nu)\sqrt{\frac{(2l + 1)\pi}{4}}\int_0^\pi \sin(\theta)A(\theta)
\times J_{m+2}(2\pi u \sin(\theta))(F_{1,lm}(\theta) + F_{2,lm}(\theta)),
\]

\[
-2\Theta_{lm} = -f(\nu)\sqrt{\frac{(2l + 1)\pi}{4}}\int_0^\pi \sin(\theta)A(\theta)
\times J_{m-2}(2\pi u \sin(\theta))(F_{1,lm}(\theta) - F_{2,lm}(\theta)).
\]

By plugging Eq. [B17] and [B18] into Eq. [B16] we get

\[
\psi = 1\tan^{-1}\frac{\text{Im}[R_{i,lm}(L_{i,lm}^*)]}{\text{Re}[R_{i,lm}(L_{i,lm}^*)]}
= 1\tan^{-1}\frac{\text{Im}[e^{i\phi_u}2\Theta_{lm} - 2\Theta_{lm}]}{\text{Re}[e^{i\phi_u}2\Theta_{lm} - 2\Theta_{lm}]}
= \phi_u + \frac{n\pi}{4}. \quad (n = \ldots, -2, -1, 0, 1, 2, \ldots)
\]

\(\psi = \phi_u + \frac{n\pi}{4}\) maximizes or minimizes the diagonal element of E and B mode window functions of \(V_Q^\prime\) and \(V_U^\prime\), which is consistent with the result obtained with flat sky approximation.