THE TOM DIECK SPLITTING THEOREM IN EQUIVARIANT MOTIVIC HOMOTOPY THEORY

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(Received 22 July 2020; revised 30 June 2021; accepted 3 July 2021; first published online 10 November 2021)

Abstract We establish, in the setting of equivariant motivic homotopy theory for a finite group, a version of tom Dieck’s splitting theorem for the fixed points of a suspension spectrum. Along the way we establish structural results and constructions for equivariant motivic homotopy theory of independent interest. This includes geometric fixed-point functors and the motivic Adams isomorphism.

Keywords: motivic homotopy theory, equivariant homotopy theory, Adams isomorphism, tom Dieck splitting

2020 Mathematics Subject Classification: Primary 14F42, 55P91
Secondary 55P42, 55P92

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1. Introduction

In his 1970 International Congress of Mathematicians talk [36], Segal sketched a computation of the endomorphism ring of the equivariant sphere spectrum for a finite group $G$, identifying this endomorphism ring with the Burnside ring of finite $G$-sets. Using other methods, this computation was recovered and massively generalized by tom Dieck’s splitting theorem [13]. These results form a crucial layer of the foundations on which the successes of equivariant homotopy in the ensuing decades were built, from early foundations [27] to Carlsson’s resolution of the Segal completion conjecture [8] to the Hill–Hopkins–Ravenel solution of the Kervaire invariant one problem [22].

An equivariant version of motivic homotopy theory was introduced by Voevodsky [11] to study quotients of motives by group actions, which played a role in his work on the Bloch–Kato conjecture. A number of authors have subsequently further developed this theory, and variants, including Hu, Kriz, and Ormsby [25], Heller, Krishna, and Østvær [19], Herrmann [21], and Carlsson and Joshua [9]. The state of the art is Hoyois’s [24], where he develops the formalism of Grothendieck’s six operations in this theory.

In this paper we establish an analogue of tom Dieck’s splitting in the context of stable motivic homotopy theory for finite group actions. Throughout, we assume that $G$ is a finite group whose order is coprime to the characteristics of the residue fields of the base scheme $B$; in other words, the group scheme associated to $G$ is linearly reductive over $B$. Our splitting theorem, proved in Theorem 7.4, computes the $N$-fixed points of suspension spectra (more generally of ‘split spectra’) as a motivic $G/N$-spectrum, where $N ⊴ G$ is a normal subgroup. In the case $N = G$, this takes the following form, where $(H)$ denotes the conjugacy class of a subgroup:

**Theorem 1.1** (motivic tom Dieck splitting). Let $G$ be a finite group whose order is invertible on $B$. Let $X$ be a based motivic $G$-space over $B$. There is an equivalence of motivic spectra

$$\Theta_X : \bigoplus_{(H)} (\Sigma^\infty (X^H))_{\text{h}W_H} \overset{\sim}{\to} (\Sigma^\infty X)^G.$$

**Corollary 1.2.** For integers $a$ and $b$, there is a canonical isomorphism

$$\pi_{a,b}^G(1_B) \cong \bigoplus_{(H)} \pi_{a,b}(BW_H).$$

The reader familiar with tom Dieck’s theorem [13] will recognize this result as taking a very similar form as the classical result. The key difference here is that the functor $(-)_{\text{h}G}$ is an algebro-geometric, or motivic, version of the homotopy orbits functor rather than the familiar categorical construction. Recall that the ordinary homotopy orbits functor is defined as follows. A $G$-spectrum $Y$ determines a diagram on the category $BG \simeq B_{\text{Nis}}G$, and the homotopy orbits are the colimit of this diagram: $Y_{\text{h}G} \simeq \text{colim}_{BG} Y$. The motivic version should then be thought of as a motivic, or parameterized, colimit of $Y$ over the category $B_{\text{et}}G$ of étale $G$-torsors. We do not make the version of the definition, as just stated, precise here (this can be done using [6]); instead, we provide a direct construction of the functor $(-)_{\text{h}G}$. First, recall that Morel and Voevodsky [32] introduced a geometric
model for the classifying space of étale $G$-torsors. This construction is distinct from the usual simplicial construction of the classifying space; rather, the simplicial construction is a model for the classifying space of Nisnevich $G$-torsors. The equivariant manifestation of this fact is that the universal free motivic $G$-space $E_G$ is not equivalent to the usual simplicial construction $E_\bullet G$. The motivic homotopy orbits of a $G$-spectrum $Y$ are defined here by a variant on the standard formula $Y_{hG} \cong (E_\bullet G_+ \otimes Y)/G$, obtained by replacing $E_\bullet G$ by $E_G$. That is, we take $Y_{hG} \cong (Y \otimes E_G)/G$ as the definition of the motivic homotopy orbits of $Y$.

Before explaining the intermediate results leading to the splitting theorem, we pause to point out an obvious, but important, difference between ordinary equivariant and motivic equivariant homotopy theory. In the topological case, equivalences are detected by the fixed-point functors for subgroups $H \leq G$. This corresponds to the fact that a set of generators is given by the orbits $G/H$, or that the homotopy theory of $G$-spaces can be presented as presheaves of spaces on the category of $G$-orbits. On the other hand, generators for equivariant motivic homotopy theory are smooth schemes over $B$ with a $G$-action. Orbits $G/H$ are examples of smooth $G$-schemes over $B$, but of course there are many more. Equivalences between motivic $G$-spaces or $G$-spectra are not detected by fixed points, because smooth $G$-schemes cannot in any meaningful way be decomposed into pieces of the form $G/H \times X$ (where $X$ has trivial action). However, by analyzing filtrations of equivariant motivic homotopy theory arising from localizations and colocalizations determined by families of subgroups, as in §3, one can see that equivalences can be detected using only (desuspensions of) smooth $G$-schemes of very special form, namely those of the form $G \times K X$ such that there is a normal subgroup $N \trianglelefteq K$ which acts trivially on $X$ and the quotient $K/N$ acts freely on $X$. These are the $G$-schemes whose stabilizers are concentrated at a single conjugacy class.

In addition to the six-functor formalism established in [24], the proof of Theorem 1.1 relies on several new results for equivariant motivic homotopy theory which should be of independent interest. As with the splitting theorem, there are versions for all of these results established relative to a normal subgroup $N \not\trianglelefteq G$; for simplicity we discuss here in the introduction only the results for $N = G$.

The first key ingredient which we need is the geometric fixed-points functor, constructed in §4.3. The geometric fixed points $X^{\Phi G}$ of a motivic $G$-spectrum $X$ may be obtained as the $G$-fixed points of a suitable localization of $X$, namely one determined by smooth $G$-schemes with trivial action. This functor satisfies analogues of the main features of the geometric fixed-points functor from ordinary equivariant motivic homotopy theory, as follows:

1. It is a symmetric monoidal left adjoint, and $(\Sigma^\infty Y_+)^{\Phi G} \cong \Sigma^\infty (Y^G)$ for any $Y \in \Sm_B$.

2. $X^{\Phi G} \cong (X \otimes \widetilde{E}_P)^G$, where $\widetilde{E}_P$ is the unreduced suspension of the universal motivic $G$-space associated to the family $P$ of proper subgroups of $G$.

---

1Of course, one can define a homotopy theory which has this property, but as pointed out by Herrmann [21], equivariant algebraic $K$-theory is not representable in the resulting homotopy category.
3. $X^{\Phi} \simeq \left( X[a^{-1}] \right)^G$, where $a$ is the Euler class $a : S^0 \to T^G$ and $\bar{\rho}_G$ is the reduced regular representation.

Here, given a representation $V$, we write $T^V$ for the associate motivic sphere (i.e., its Thom space). The connection between items 2 and 3 is provided by a geometric presentation for universal motivic $G$-spaces for families, established in §3, analogous to Morel and Voevodsky’s geometric description of classifying spaces. In particular, for the family of proper subgroups, what we find is that $\tilde{E}P$ may be described by the formula $\tilde{E}P \simeq T^\infty \bar{\rho}_G = \colim_n T^{n\bar{\rho}_G}$, analogous to the familiar formula from topology.

Also of interest is that the construction of the geometric-fixed points functor here permits a motivic version of the Tate square for $C_p$-equivariant motivic spectra in §4.4. This is a homotopy push-out square of motivic spectra,

$$
\begin{array}{ccc}
X^{C_p} & \longrightarrow & X^{\Phi C_p} \\
\downarrow & & \downarrow \\
X^{hC_p} & \longrightarrow & X^{tC_p},
\end{array}
$$

where $X^{hC_p}$ is a motivic version of the homotopy fixed-points functor, defined using $\mathbf{E}C_p$.

A second key ingredient entering into the splitting theorem is the motivic Adams isomorphism, proved in Theorem 6.36. This fundamental result identifies the quotient of a free $G$-spectrum with its fixed points. There is a natural transformation from the former to the latter, and the bulk of the work in §6 is devoted to verifying that this morphism is an equivalence. Our strategy is to first check that this transformation is an equivalence on the full subcategory of dualizable free $G$-spectra. Of course, unless the base is a field of characteristic 0, this does not suffice to conclude the result in general. But since $\mathbf{E}G_+$ is a colimit of dualizable spectra, it does imply that the fixed points of $\mathbf{E}G_+$ coincide with $BG_+$. Using the fact that $BG_+$ contains $\mathbb{1}_B$ as a summand, this lets us define an inverse to the Adams transformation to obtain the general result. It is worth pointing out that if $f : T \to B$ is an étale torsor, then the Adams isomorphism for $f#\mathbb{1}_T$ is a straightforward consequence of ambidexterity, proved in [24], for the finite étale map $f$. An obvious strategy presents itself. If $q : X \to B$ is a smooth $G$-scheme over $B$ with free action, then $g : X \to X/G$ is an étale torsor and $p : X/G \to B$ is smooth. Since $(g#\mathbb{1}_X)^G = (g#\mathbb{1}_X)/G$ and $p#$ commutes with the quotient functor, to verify the Adams isomorphism for $p#(g#\mathbb{1}_X)$ it would suffice to check that the fixed-points functor commutes with $p#$. Establishing this change-of-base formula directly appears to be as difficult as the Adams isomorphism itself, and we actually obtain this base-change formula as a consequence of the Adams isomorphism. It is interesting to note that from the viewpoint of motivic homotopy theory of stacks, this is an instance of a smooth proper base-change formula, along the nonrepresentable map $BG \to B$.

Once all of the foundational results are in place, the proof of the splitting theorem is actually fairly straightforward. It is not hard to write down the map $\Theta_X$ of the statement of the theorem, and to check that it is an equivalence it suffices to check that it is an equivalence when $X$ is concentrated at a single conjugacy class – a case which follows from the analysis in §3 of localizations and colocalizations of equivariant motivic homotopy.
1.1. Outline

We use the language of ∞-categories throughout this paper. We begin in §2 by recalling the construction of the ∞-categories of motivic G-spaces and motivic G-spectra from [24], as well as a few extensions used in this paper. In §3 we study the colocalizations and localizations of equivariant motivic homotopy theory which are determined by a family. In §4 we define fixed-point functors and geometric fixed points. In §5 we define the quotient functor on N-free spectra, and in §6 we prove the motivic Adams isomorphism. Finally, in §7 we prove the motivic tom Dieck splitting theorem.

1.2. Notation

Throughout, B is a quasi-compact, quasi-separated base scheme and G is a finite group whose order is invertible in O_B. We write Sch^G_B for the category of G-schemes which are finitely presented and G-quasi-projective over B. For S ∈ Sch^G_B, write Sch^G_S for the slice category over S and Sm^G_S ⊆ Sch^G_S for the full subcategory whose objects are smooth over S.

If E is a locally free O_S-module, we write

\[ V_S(E) := \text{Spec}(\text{Sym}(E^\vee)) \]  

and

\[ P_S(E) := \text{Proj}(\text{Sym}(E^\vee)), \]

respectively, for the associated vector bundle scheme and the associated projective bundle on S. A representation of G over B will mean a locally free G-module on B (for a recollection of the definition, see [38]). If M is a G-set, we let

\[ \rho_M = O_B[M] := O_B \otimes \mathbb{Z}[M] \]

denote the associated permutation representation. In particular, \( \rho_G \) is the regular representation.

We use the language of ∞-categories in this paper and mostly follow the terminology in [28, 29], with the exception that we write Cat_∞ for the ∞-category of not necessarily small ∞-categories. We write Map^C(x,y) for the space of maps in an ∞-category C and Map^C_G(x,y) for the spectrum of maps in a stable ∞-category. If C is a closed symmetric monoidal ∞-category, we write FC(x,y) for the internal mapping object.

2. Equivariant motivic homotopy theory

We recall definitions and basic properties of equivariant motivic homotopy theory. We will use the ∞-categorical approach to equivariant motivic spectra introduced by Hoyois [24]. Since we are working with finite groups, the unstable homotopy category agrees with those constructed by Voevodsky [11] and Heller, Krishna, and Østvær [19], and the stable homotopy category agrees with the one from [20, Appendix A.4].

2.1. Equivariant geometry

Set S ∈ Sch^G_B. If \( \phi: G \to K \) is a group homomorphism, we write

\[ \phi^{-1}: \text{Sch}_K^S \to \text{Sch}_G^{\phi^{-1}}_S \]
for the restriction functor, which regards a $K$-scheme over $S$ as a $G$-scheme over $S$ via $\phi$. When no confusion should arise, we write $\text{Sch}_{S}^{G}$ instead of $\text{Sch}_{\phi^{-1}S}^{G}$.

Set $X \in \text{Sch}_{S}^{G}$. Say that $G$ acts freely on $X$ if the action of $G(T)$ on $X(T)$ is free for any $T \in \text{Sch}_{B}$. If $G$ acts freely on $X$, then the fppf-quotient $(X/G)_{\text{fppf}}$ is representable by an object $X/G \in \text{Sch}_{B}$ (since all of our schemes are quasi-projective; see [37, Tag 07S7]. Since $G$ is smooth, the fppf-torsor $X \rightarrow X/G$ is an étale torsor (as it is a smooth map and so étale locally admits sections).

**Definition 2.1.** The stabilizer of a point $x \in X$ is the subgroup $\text{Stab}(x) \leq G$, defined by

$$\text{Stab}(x) = \{g \in G \mid g \cdot x = x \text{ and } g \text{ acts as id on } k(x)\}.$$  

Then $G$ acts freely on $X$, provided $\text{Stab}(x) = \{e\}$ for all $x \in X$. More generally, if $H \leq G$ is a subgroup which acts freely on $X$, then the quotient $X/H$ inherits an action of the Weyl group $W(H) = W_{G}(H) := N_{G}(H)/H$, and so defines a functor $(\cdot)/H : \text{Sch}_{B}^{G,H}\text{-free} \rightarrow \text{Sch}_{B}^{W_{H}}$. This is the composite of the restriction functor $\text{Sch}_{B}^{G,H}\text{-free} \rightarrow \text{Sch}_{B}^{N_{H},H}\text{-free}$ and followed by the quotient $\text{Sch}_{B}^{N_{H},H}\text{-free} \rightarrow \text{Sch}_{B}^{W_{H}}$.

Let $N \unlhd G$ be a normal subgroup. Suppose that either

(i) $N$ acts trivially on $S$

(ii) $N$ acts freely on $S$.

In either case, the quotient functor yields a functor

$$\text{Sm}_{S}^{G,N}\text{-free} \rightarrow \text{Sm}_{S/N}^{G/N}.$$  

Write $q : S \rightarrow S/N$ for the quotient map of schemes and $q^{-1} : \text{Sm}_{S/N}^{G/N} \rightarrow \text{Sm}_{S}^{G/N}$ for the functor defined by $q^{-1}(Y) = Y \times_{S/N} S$.

**Proposition 2.2.** Suppose that $N$ acts freely on $S$. Then $(\cdot)/N$ and $\pi^{-1}q^{-1}$ are inverse equivalences

$$(\cdot)/N : \text{Sm}_{S}^{G} \iff \text{Sm}_{S/N}^{G/N} : \pi^{-1}q^{-1}. $$

**Proof.** Let $f : X \rightarrow S$ be in $\text{Sm}_{S}^{G}$. By descent, we have a Cartesian square in $\text{Sm}_{S}$ (hence in $\text{Sm}_{S}^{G}$)

$$\begin{array}{ccc}
X & \longrightarrow & S \\
\downarrow & & \downarrow \\
X/N & \longrightarrow & S/N.
\end{array}$$

It follows that $(\cdot)/N$ is fully faithful. It is also essentially surjective, since if $Y \in \text{Sm}_{S/N}^{G/N}$, then $Y \cong (Y \times_{S/N} S)/N$.  

Let $\iota : H \hookrightarrow G$ be a monomorphism of groups. The induction-restriction adjunction

$$\iota^{*} : \text{Sch}_{S}^{H} \cong \text{Sch}_{S}^{G} : \iota^{-1}$$
restricts to an adjunction
\[ \iota! : \text{Sm}_S^H \rightleftarrows \text{Sm}_S^G : \iota^{-1}. \]

When \( H \leq G \) is a subgroup and \( \iota \) is the inclusion, we often write \( \iota!(X) = G \times_H X \), and this scheme is described concretely as follows. The scheme \( G \times X \) becomes an \( H \)-scheme under the action \( h(g,x) = (gh^{-1},hx) \), and we define
\[ G \times_H X = (G \times X)/H. \]

The scheme \( G \times_H X \) has a left \( G \)-action through the action of \( G \) on itself. We can describe \( G \times_H X \) in slightly more concrete terms as follows. Choose a complete set of left coset representatives \( g_i \); then \( G \times_H X = \bigsqcup_{i} X_i \), each \( X_i \) is a copy of \( X \), and \( g \in G \) acts as
\[ k : X_i \to X_j, \text{ where } k \in H \text{ satisfies } gg_i = g_jk. \]

Set \( X \in \text{Sch}_B^G \). The presheaf of sets \( X^G \) is the presheaf of sets on \( \text{Sch}_B \) defined by
\[ X^G(Y) = \{ y \in X(Y) \mid y \text{ is fixed by } G \}. \]

If \( X \to B \) is seperated, the presheaf \( X^G \) is represented by a closed subscheme of \( X \) which is finitely presented over \( B \), which is moreover smooth over \( B \) if \( X \) is (see [12, Proposition XII.9.2, Corollaire XII.9.8] or [10, Proposition A.8.10]). Note that if \( H \leq G \) is a subgroup, the fixed-point subscheme \( X^H \) comes equipped with an action of the Weyl group \( W(H) \).

Now, suppose that \( N \trianglelefteq G \) is a normal subgroup which acts trivially on \( S \). Write \( \pi : G \to G/N \) for the quotient map. Restricting action along \( \pi \) defines a functor \( \pi^{-1} : \text{Sch}_S^{G/N} \to \text{Sch}_S^G \), which is left adjoint to fixed points. We will usually simply write again \( X \) instead of \( \pi^{-1}X \), whenever context makes the meaning clear. Now, restricting attention to smooth \( S \)-schemes, we obtain the adjunction
\[ \pi^{-1} : \text{Sm}_S^{G/N} \rightleftarrows \text{Sm}_S^G : (-)^N. \]

2.2. Families of subgroups
Families of subgroups provide a convenient way to filter equivariant motivic homotopy theory.

**Definition 2.3.** A family \( \mathcal{F} \) of subgroups of \( G \) is a set of subgroups which is closed under taking subgroups and conjugation.

**Example 2.4.** The following families play an important role:

1. The trivial family \( \mathcal{F}_{\text{triv}} := \{ e \} \).
2. The family of all subgroups \( \mathcal{F}_{\text{all}} := \{ H \leq G \mid H \text{ is a subgroup} \} \).
3. The family of proper subgroups \( \mathcal{P} := \{ H \not\subseteq G \mid H \text{ is a proper subgroup} \} \).
4. For a normal subgroup \( N \trianglelefteq G \), define \( \mathcal{F}[N] := \{ H \leq G \mid N \nleq H \} \). Note that \( \mathcal{P} = \mathcal{F}[G] \).
5. For a normal subgroup \( N \trianglelefteq G \), define \( \mathcal{F}(N) := \{ H \leq G \mid H \cap N = \{ e \} \} \). Note that \( \mathcal{F}(G) = \mathcal{F}_{\text{triv}} \).
If $\mathcal{F}$ is a family, we write $\text{co}(\mathcal{F}) := \mathcal{F}_{\text{all}} - \mathcal{F}$ for its complement. Note that $\text{co}(\mathcal{F})$ is not a family.\footnote{Rather, it is a cofamily, meaning it is closed under conjugation and $K \in \text{co}(\mathcal{F})$ whenever $K$ contains a subgroup in $\text{co}(\mathcal{F})$.}

**Remark 2.5.** A family of subgroups can be equivalently viewed as a sieve on the orbit category $\text{Orb}^G$. The sieve corresponding to $\mathcal{F}$ is the full subcategory $\text{Orb}^G[\mathcal{F}] \subseteq \text{Orb}^G$ of orbits such that $G/H \in \text{Orb}^G[\mathcal{F}]$ if and only if $H \in \mathcal{F}$.

A family $\mathcal{F}$ determines a sieve on $\text{Sm}^G_S$ by letting $\text{Sm}^G_S[\mathcal{F}] \subseteq \text{Sm}^G_S$ be the full subcategory whose objects are smooth $G$-schemes over $S$ such that all stabilizers are contained in $\mathcal{F}$. It is useful to make the following more general definition:

**Definition 2.6.** Let $\mathcal{E}$ be a set of subgroups of $G$ which is closed under conjugacy. Write $\text{Sm}^G_S[\mathcal{E}] \subseteq \text{Sm}^G_S$ for the full subcategory whose objects are those smooth $G$-schemes $X$ over $S$ such that $\text{Stab}(x) \in \mathcal{E}$ for every point $x \in X$.

**Notation 2.7.** Set $X \in \text{Sch}^G_S$. Write

$$X^\mathcal{F} := \bigcup_{H \in \text{co}(\mathcal{F})} X^H$$

and

$$X(\mathcal{F}) := X - X^\mathcal{F}.$$ 

The subset $X(\mathcal{F}) \subseteq X$ is the set of points whose stabilizers are in $\mathcal{F}$. Observe that $X(\mathcal{F}) \subseteq X$ is an open invariant subscheme and $X^\mathcal{F} \subseteq X$ is a closed invariant subscheme, since $X^\mathcal{F}$ is a finite union of closed subschemes.

### 2.3. Motivic $G$-spaces

Recall that the Nisnevich topology can be defined via a cd-structure.

**Definition 2.8 ([40]).** Let $\mathcal{C}$ be a small category which has an initial object $\emptyset$.

1. A cd-structure on $\mathcal{C}$ is a collection $\mathcal{A}$ of commutative squares in $\mathcal{C}$ such that if $Q \in \mathcal{A}$ and if $Q'$ is isomorphic to $Q$, then $Q' \in \mathcal{A}$.
2. Given a cd-structure $\mathcal{A}$ in $\mathcal{C}$, the Grothendieck topology $t_\mathcal{A}$ generated by $\mathcal{A}$ is the smallest topology on $\mathcal{C}$ such that:
   (a) the empty sieve is a covering sieve of $\emptyset$ and
   (b) given any square

   \[
   \begin{array}{ccc}
   V & \longrightarrow & Y \\
   \downarrow & & \downarrow^p \\
   U & \longrightarrow & X \\
   \end{array}
   \]

   in $\mathcal{A}$, the sieve generated by $\{U \to X, Y \to X\}$ is a covering sieve.
**Definition 2.9.** An equivariant map \( f : Y \to X \) is said to be *fixed-point reflecting* at \( y \in Y \) if \( f \) induces an isomorphism \( \text{Stab}(y) \cong \text{Stab}(f(y)) \). If this condition holds at every \( y \in Y \), then \( f \) is simply said to be *fixed-point reflecting*.

Let \( \mathcal{C}_S \subseteq \text{Sch}^G_S \) be a full subcategory containing \( \emptyset \). We often require \( \mathcal{C}_S \) to satisfy one or both of the following properties:

(P) If \( Y \to X \) is fixed-point reflecting and étale, then \( Y \in \mathcal{C}_S \) whenever \( X \in \mathcal{C}_S \).

(H) If \( X \in \mathcal{C}_S \), then so is \( X \times_S \mathbb{A}^1 \).

Our primary examples of interest are the categories \( \text{Sm}^G_S[\mathcal{E}] \), where \( \mathcal{E} \) is a set of subgroups closed under conjugacy. More generally, we could also consider the following property:

(P') If \( Y \to X \) is an equivariant étale map, then \( Y \in \mathcal{C}_S \) whenever \( X \in \mathcal{C}_S \).

The condition (P) on \( \mathcal{C}_S \) guarantees that the fixed-point Nisnevich cd-structure (defined later) is complete, while the condition (P') guarantees that the Nisnevich cd-structure is complete. Some categories of interest in this paper – for example, \( \text{Sm}^G_S[\text{co}(\mathcal{F})] \) for a family \( \mathcal{F} \) – do not satisfy (P') but do satisfy the weaker property (P). We will see in Proposition 2.13 that when \( \mathcal{C}_S \) satisfies (P'), then the topology associated to the fixed-point Nisnevich cd-structure coincides with the Nisnevich topology.

**Definition 2.10.** Let \( \mathcal{C}_S \subseteq \text{Sch}^G_S \) be a full subcategory containing \( \emptyset \).

1. The *Nisnevich cd-structure* \( \text{Nis} \) on \( \mathcal{C}_S \) consists of Cartesian squares

\[
\begin{array}{ccc}
V & \longrightarrow & Y \\
\downarrow & & \downarrow p \\
U & \leftarrow & X,
\end{array}
\tag{2.11}
\]

where \( j \) is open immersion, \( p \) is étale, and the map \( (Y-V)_{\text{red}} \to (X-U)_{\text{red}} \) is an isomorphism in \( \text{Sch}_S \).

2. The *fixed-point Nisnevich cd-structure* \( \text{fpNis} \) on \( \mathcal{C}_S \) consists of Cartesian squares as in the Nisnevich cd-structure, but with the added condition that \( p \) is a fixed-point-reflecting étale map.

**Remark 2.12.** In general, if \( G \) is a flat group scheme over \( B \), one may choose a scheme structure on \( Z := X \setminus U \) so that \( Z \) is invariant under the \( G \)-action [24, Lemma 2.1]. Since \( G \) is a finite discrete group, \( Z_{\text{red}} \) is invariant and the map \( p^{-1}(Z)_{\text{red}} \to Z_{\text{red}} \) is equivariant.

**Proposition 2.13.** Suppose that \( \mathcal{C}_S \) satisfies (P'). The topology \( t_{\text{fpNis}} \) coincides with the Nisnevich topology on \( \mathcal{C}_S \).
Proof. Every $t_{\text{fpNis}}$-cover is a Nisnevich cover. We show that the reverse implication holds. It suffices to show that any Nisnevich square

$$
\begin{array}{c}
V \\ \downarrow \\
U
\end{array} \quad \begin{array}{c}
\longrightarrow \\ \downarrow \\
\longrightarrow \\
X
\end{array} \quad \begin{array}{c}
Y \\ \downarrow \\
U
\end{array}
$$

admits a $t_{\text{fpNis}}$-refinement.

Write $fpr(Y) \subseteq Y$ for the set of points where $p$ is fixed-point reflecting. Since $Y \to X$ is unramified and $\text{Stab}(X) \to X$ is universally closed, [35, Proposition 3.5] applies to show that the set $fpr(Y) \subseteq Y$ is an invariant open subset. We have that $Y \setminus V \subseteq fpr(Y)$. It follows that the outer square

$$
\begin{array}{c}
fpr(V) \\ \downarrow \\
V \\ \downarrow \\
U
\end{array} \quad \begin{array}{c}
fpr(Y) \\ \downarrow \\
Y \\ \downarrow \\
X
\end{array}
$$

(2.14)

is a fixed-point Nisnevich square (as is the top square). In particular, $\{j,pi\}$ is a $t_{\text{fpNis}}$-cover which refines $\{j,p\}$. \hfill $\Box$

Write $\mathcal{P}(\mathcal{C}_S)$ for the $\infty$-category of presheaves of spaces on $\mathcal{C}_S$. Note that $\mathcal{C}_S$ does not necessarily contain a terminal object; in particular, a terminal object $\mathcal{P}(\mathcal{C}_S)$ is in general not representable. We write $\text{pt} \in \mathcal{P}(\mathcal{C}_S)$ for this terminal object. Of course, if $S \in \mathcal{C}_S$, then $\text{pt}$ is representable; it is the presheaf represented by $S$.

Definition 2.15. Say that $F \in \mathcal{P}(\mathcal{C}_S)$ is Nisnevich excisive if:

1. $F(\emptyset)$ is contractible and
2. for any Nisnevich square (2.11) in $\mathcal{C}_S$, the square

$$
\begin{array}{c}
F(X) \\ \downarrow \\
F(V)
\end{array} \quad \begin{array}{c}
\longrightarrow \\ \downarrow \\
\longrightarrow \\
F(Y) \\ \downarrow \\
F(U)
\end{array}
$$

is Cartesian.

Write $\text{Shv}_{\text{Nis}}(\mathcal{C}_S) \subseteq \mathcal{P}(\mathcal{C}_S)$ for the subcategory of Nisnevich excisive presheaves of spaces on $\mathcal{C}_S$.

Temporarily, say that $F$ is ‘fixed-point Nisnevich excisive’ if $F(\emptyset) \simeq \text{pt}$ and $F(\mathcal{Q})$ is cartesian for any fixed-point Nisnevich square $\mathcal{Q}$. There is no real need for this extra terminology, by the following proposition:

Proposition 2.16. Set $F \in \mathcal{P}(\mathcal{C}_S)$ and suppose $\mathcal{C}_S$ satisfies $(\mathcal{P})$. Then the following are equivalent:
1. $F$ is fixed-point Nisnevich excisive.
2. $F$ is Nisnevich excisive.
3. $F$ is a sheaf for the Nisnevich topology.

**Proof.** If $C_S$ satisfies $(P')$, then the $cd$-structures $fpNis$ and $Nis$ both satisfy the conditions of [4, Theorem 3.2.5] — that is, they are complete and regular, in the terminology of [40]. Together with Proposition 2.13, it follows that 1, 2, and 3 are equivalent. □

**Definition 2.17.** Let $C_S \subseteq \text{Sch}_S$ be a full subcategory which contains $\emptyset$ and satisfies properties (P) and (H). Say that $F \in \mathcal{P}(C_S)$ is $\mathbb{A}^1$-homotopy invariant if for any $Y \in C_S$, the projection map $\pi: Y \times \mathbb{A}^1 \to Y$ induces an equivalence $F(Y) \simeq F(Y \times \mathbb{A}^1)$. We write $\mathcal{P}_{\mathbb{A}^1}(C_S) \subseteq \mathcal{P}(C_S)$ for the full subcategory consisting of the $\mathbb{A}^1$-homotopy invariant presheaves.

The property that a presheaf is Nisnevich excisive is defined by a small set of conditions. It follows from [28, Section 5.5.4] that the inclusion $\text{Shv}_{Nis}(C_S) \subseteq \mathcal{P}(C_S)$ is an accessible localization. Write $L_{Nis}$ for the resulting localization endofunctor on $\mathcal{P}(C_S)$. Moreover, this localization is left-exact in the sense of [28, Section 6.2.2].

Similarly, the property that a presheaf is $\mathbb{A}^1$-homotopy invariant is defined by a small set of conditions, so that the inclusion $\mathcal{P}_{\mathbb{A}^1}(C_S) \subseteq \mathcal{P}(C_S)$ is also an accessible localization. Write $L_{\mathbb{A}^1}$ for the resulting localization endofunctor. It can be described explicitly by the formula $L_{\mathbb{A}^1} = \text{Sing}_{\mathbb{A}^1}$, where

$$\text{Sing}_{\mathbb{A}^1}(F)(U) := \text{colim}_\Delta (n \mapsto F(U \times \Delta^n)) \quad (2.18)$$

(for details, see, for instance, [2, Theorem 4.25]). A map $f: F_1 \to F_2$ in $\text{Shv}_{Nis}(C_S)$ is a Nisnevich equivalence provided that $L_{Nis}(f)$ is an equivalence, and an $\mathbb{A}^1$-equivalence provided that $L_{\mathbb{A}^1}(f)$ is an equivalence.

**Remark 2.19.** Say that $F \in \mathcal{P}(C_S)$ is strongly $\mathbb{A}^1$-homotopy invariant if for any projection $E \to X$ of a $G$-affine bundle in $C_S$, the induced map is an equivalence $F(X) \simeq F(E)$. Any $X \in \text{Sm}_G^G$ is Nisnevich locally affine. This implies that if $F$ is Nisnevich excisive, then $F$ is strongly $\mathbb{A}^1$-invariant if and only if it is $\mathbb{A}^1$-homotopy invariant (as $E \to X$ always has local sections in this case) [24, Remark 3.13].

In particular, the motivic localization considered here agrees with the one in [24].

**Definition 2.20.**

1. A motivic $G$-space over $S$ is a Nisnevich excisive and $\mathbb{A}^1$-homotopy-invariant presheaf $F \in \mathcal{P}(\text{Sm}_G^G)$.
2. Write $\mathcal{Spc}^G(S)$ for the $\infty$-category of motivic $G$-spaces. The category of based motivic $G$-spaces is $\mathcal{Spc}_\bullet^G(S) = \mathcal{Spc}^G(S)_{pt/}$.
3. More generally, write $\mathcal{Spc}(C_S)$ for the $\infty$-categories of Nisnevich excisive $\mathbb{A}^1$-homotopy-invariant presheaves on $C_S$ and $\mathcal{Spc}_\bullet(C_S) = \mathcal{Spc}(C_S)_{pt/}$. 
When $\mathcal{E}$ is a set of subgroups, closed under conjugacy, we use the notation

$$\text{Sp}_c^{G,\mathcal{E}}(S) := \text{Sp}_c(\text{Sm}_S^{G\uparrow}[\mathcal{E}])$$

$$\text{Sp}_c^{G,\mathcal{E}}(S) := \text{Sp}_c(\text{Sm}_S^{G\uparrow}[\mathcal{E}]).$$

The inclusion $\text{Sp}_c(C_S) \subseteq \mathcal{P}(C_S)$ is an accessible localization, and we write $L_{\text{mot}} : \mathcal{P}(C_S) \to \mathcal{P}(C_S)$ for the corresponding localization endofunctor. The motivic localization functor may be computed by the formula [32, Lemma 3.2.6]

$$L_{\text{mot}}(F) = L_{\text{Nis}} \text{colim}_{n \to \infty} (L_{A^1} \circ L_{\text{Nis}})^n(F). \quad (2.21)$$

**Proposition 2.22.** The motivic localization functor $L_{\text{mot}} : \mathcal{P}(C_S) \to \mathcal{P}(C_S)$ is locally Cartesian\(^3\) and preserves finite products. In particular, colimits in $\text{Sp}_c^{G}(C_S)$ are universal.

**Proof.** The localization functors $L_{\text{Nis}}, L_{A^1}$ satisfy these properties, and therefore so does $L_{\text{mot}}$ using equation (2.21). \qed

We will make use of the notion of the tensor product of presentable $\infty$-categories as defined and studied in [29, Section 4.8.1] (see especially [29, Proposition 4.8.1.17]). The category $\text{Sp}_c(C_S)$ is Cartesian monoidal (i.e., it is symmetric monoidal with respect to the Cartesian product). The symmetric monoidal product on $\text{Sp}_c(C_S)$ extends to one on $\text{Sp}_c(\mathcal{C}_S) \simeq \text{Sp}_c(C_S) \otimes S_*$,

where $S_* := S_{pt}$ denotes the $\infty$-category of pointed spaces (see [15, Lemma 3.6] and [29, Proposition 4.8.2.11]). We write $\wedge$ for the symmetric monoidal product on $\text{Sp}_c(\mathcal{C}_S)$. Sometimes we will need to use the symmetric monoidal product on $\mathcal{P}_*(C_S) := \mathcal{P}(C_S)_{pt/}$, which we denote $\wedge^\mathcal{P}$ to avoid confusion. The symmetric monoidal localization functor $L_{\text{mot}} : \mathcal{P}(C_S) \to \text{Sp}_c(C_S)$ induces a unique symmetric monoidal localization functor $L_{\text{mot}} : \mathcal{P}_*(C_S) \to \text{Sp}_*(C_S)$. In particular, given $X, Y \in \mathcal{P}_*(C_S)$, then

$$L_{\text{mot}}(X) \wedge^\mathcal{P} L_{\text{mot}}(Y) \simeq L_{\text{mot}}(X \wedge Y).$$

Given a functor $u : \mathcal{C} \to \mathcal{D}$, the functor $u^* : \mathcal{P}(D) \to \mathcal{P}(C)$, defined by precomposition with $u$, preserves small limits and colimits, as these are computed objectwise in presheaf $\infty$-categories [28, Corollary 5.1.2.3], which are presentable $\infty$-categories [28, Theorem 5.5.1.1]. It follows from [28, Corollary 5.5.2.9] that $u^*$ admits both a left adjoint $u_!$ and a right adjoint $u_*$,

$$\mathcal{P}(C) \leftarrow \mathcal{P}(D).$$

---

\(^3\)A localization endofunctor $L : \mathcal{D} \to \mathcal{D}$ is *locally Cartesian* if $L(A \times B X) \to A \times B L(X)$ is an equivalence for any maps $A \to B$, $X \to B$ in $\mathcal{D}$, where $A, B \in L(D)$ [16, §1].
A functor $u: C \to D$ between small $\infty$-categories equipped with Grothendieck topologies is called *topologically co-continuous*\(^1\) if for every $Y \in C$ and every covering sieve $R$ on $u(Y)$, the sieve $u^*R \times_{u^*uY} Y \to Y$, consisting of arrows $Z \to Y$ such that $uZ \to uY$ factors through $R$, is a covering sieve on $Y$.

**Lemma 2.23.** Let $u: C \to D$ be a topologically co-continuous functor between small $\infty$-categories equipped with Grothendieck topologies $\tau_C$ and $\tau_D$, respectively. Then $u^*: \mathcal{P}(D) \to \mathcal{P}(C)$ preserves all $\tau_D$-local equivalences.

**Proof.** The set of $\tau_D$-local equivalences in $\mathcal{P}(D)$ is the closure under push-outs, small colimits, and 2-out-of-3 of the set of maps $R \to X$, where $R$ is a covering sieve on $X \in D$. Since $u^*$ preserves colimits, it suffices to show that $u^*R \to u^*X$ is a $\tau_C$-local equivalence. Since colimits are universal in $\mathcal{P}(C)$, it suffices to show that $u^*R \times_{u^*X} Y \to Y$ is an equivalence for any map $Y \to u^*X$, where $Y \in C$. To see this, note that it is in fact a covering sieve. Indeed, $R \times_X uY \to uY$ is a covering sieve, and therefore $u^*(R \times_X uY) \times_{u^*uY} Y \simeq u^*R \times_{u^*X} Y \to Y$ is covering, since $u$ is co-continuous. \(\square\)

For the remainder of this section, \(u: \mathcal{C}_S \subseteq \mathcal{D}_S\) is the inclusion between full subcategories of $\text{Sch}^G_S$, both containing $\emptyset$ and both satisfying properties (P) and (H). Main examples of interest to keep in mind are the inclusions $\text{Sm}^G_S [E] \subseteq \text{Sm}^G_S [E']$.

**Lemma 2.25.** Let $u: \mathcal{C}_S \subseteq \mathcal{D}_S$ be as in formula (2.24). Then $u$ is topologically co-continuous.

**Proof.** Set $X \in \mathcal{C}_S$. Since the $cd$-structure fpNis on each of these categories is complete, the covering sieves on $X$ and on $u(X)$, in both cases, are exactly those which contain the sieve generated by a simple covering [40, Section 2]. Property (P) implies that any simple covering of $X$ in $\mathcal{D}_S$ is a simple covering in $\mathcal{C}_S$. It follows that the pullback $u^*R$ of any covering sieve $R$ is again a covering sieve. \(\square\)

**Proposition 2.26.** Let $u: \mathcal{C}_S \subseteq \mathcal{D}_S$ be as in formula (2.24).

1. The functor $u_!: \mathcal{P}(\mathcal{C}_S) \to \mathcal{P}(\mathcal{D}_S)$ preserves all Nisnevich and all motivic equivalences.
2. The functor $u^*: \mathcal{P}(\mathcal{D}_S) \to \mathcal{P}(\mathcal{C}_S)$ preserves all Nisnevich and all motivic equivalences.

**Proof.** Recall that if $\mathcal{A}$ is a presentable $\infty$-category and $S$ is a set of morphisms in $\mathcal{A}$, then the class of $S$-local equivalences is the closure of $S$ under push-outs, small colimits, and 2-out-of-3 (see, e.g., [28, Proposition 5.5.4.15]).

The first statement then follows from the facts that $u_!$ preserves colimits and fixed-point Nisnevich squares and that $u_!(X \times A^1) \simeq u_!(X) \times A^1$. The second statement follows from Lemma 2.23, Lemma 2.25, and the fact that $u^*(X \times A^1) \simeq u^*(X) \times A^1$. \(\square\)

\(^1\)This is called ‘cocontinuous’ in [3, Definition III.2.1]. We follow the terminology in [26] to avoid confusion with the category theorist’s term ‘cocontinuous functor’.
Corollary 2.27. Let \( u: C_S \subseteq D_S \) be as before. There are natural equivalences \( u^* \circ L_{\text{Nis}} \simeq L_{\text{Nis}} \circ u^* \) and \( u^* \circ L_{\text{mot}} \simeq L_{\text{mot}} \circ u^* \) of functors \( \mathcal{P}(D_S) \to \mathcal{P}(C_S) \).

The adjoint pairs \((u_!, u^*)\) and \((u^*, u_*\)) extend to adjoint pairs on \( \mathbb{A}^1 \)-invariant Nisnevich sheaves. Overloading notation, we continue to write \( u_!, u^*, u_* \) for the induced functors on categories of \( \mathbb{A}^1 \)-invariant Nisnevich sheaves.

Proposition 2.28. Let \( u: C_S \subseteq D_S \) be as in formula (2.24).

1. The restriction functor \( u^*: \mathcal{P}(D_S) \to \mathcal{P}(C_S) \) preserves \( \mathbb{A}^1 \)-invariant Nisnevich sheaves.
2. The induced functor \( u^*: \mathcal{Spc}(D_S) \to \mathcal{Spc}(C_S) \) is symmetric monoidal and has a left adjoint \( u_! \) and a right adjoint \( u_* \).
3. Similarly, \( u^*: \mathcal{Spc}_\bullet(D_S) \to \mathcal{Spc}_\bullet(C_S) \) is symmetric monoidal and has a left and a right adjoint, which we again denote respectively by \( u_! \) and \( u_* \).

Moreover, these functors fit into commutative diagrams

\[
\begin{array}{ccc}
\mathcal{P}(C_S) & \xleftarrow{u_!} & \mathcal{P}(D_S) \\
\xrightarrow{u^*} & & \xrightarrow{u^*} \\
\mathcal{Spc}(C_S) & \xleftarrow{u_!} & \mathcal{Spc}(D_S) \\
\xrightarrow{(\cdot)_*} & & \xrightarrow{(\cdot)_*} \\
\mathcal{Spc}_\bullet(C_S) & \xleftarrow{u_!} & \mathcal{Spc}_\bullet(D_S)
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\mathcal{P}(D_S) & \xleftarrow{u_*} & \mathcal{P}(C_S) \\
\xrightarrow{u^*} & & \xrightarrow{u^*} \\
\mathcal{Spc}(D_S) & \xleftarrow{u_*} & \mathcal{Spc}(C_S) \\
\xrightarrow{(\cdot)_*} & & \xrightarrow{(\cdot)_*} \\
\mathcal{Spc}_\bullet(D_S) & \xleftarrow{u_*} & \mathcal{Spc}_\bullet(C_S)
\end{array}
\]

Proof. Since \( u_! \) preserves fixed-point Nisnevich squares and \( u_! (X \times \mathbb{A}^1) \simeq u_!(X) \times \mathbb{A}^1 \), it follows that \( u^* \) preserves \( \mathbb{A}^1 \)-invariant Nisnevich sheaves on \( C_S \) and thus restricts to a limit-preserving functor \( u^*: \mathcal{Spc}(D_S) \to \mathcal{Spc}(C_S) \). As these are categories are presentable, \( u^* \) has a left adjoint \( u_! \). It follows from Proposition 2.26 that \( u^*: \mathcal{Spc}(D_S) \to \mathcal{Spc}(C_S) \) preserves colimits. In particular, it has a right adjoint \( u_* \). Note that \( u^* \) is symmetric monoidal, since it preserves limits and \( \mathcal{Spc}(D_S) \) is Cartesian monoidal.

The functors \( u^* \) and \( u_* \) preserve final objects and so induce adjoint pairs on based spaces. Since \( u^* \) preserves limits, it has a left adjoint \( u_! \). It is straightforward to verify that these fit into the commutative diagrams displayed.

Remark 2.29. Suppose that \( C_S \) and \( D_S \) are also closed under binary products. Then \( u_!: \mathcal{Spc}(C_S) \to \mathcal{Spc}(D_S) \) preserves binary products (the monoidal product on these categories). However, \( u_! \) is not in general a symmetric monoidal functor, since it does not always preserve the unit object \( \text{pt} \) (because it is not in general representable), but there is always a canonical map \( u_!(\text{pt}) \to \text{pt} \) adjoint to the equivalence \( \text{pt} \simeq u^*(\text{pt}) \). Similarly, \( u_!: \mathcal{Spc}_\bullet(C_S) \to \mathcal{Spc}_\bullet(D_S) \) preserves the smash product, but need not preserve the unit object \( S^0 \simeq \text{pt} \amalg \text{pt} \), though again there is a canonical map \( u_!(S^0) \to S^0 \).
Proposition 2.30. Let $u : C_S \subseteq D_S$ be as before. The functors
\[
  u_!, u_* : \text{Spc}(C_S) \to \text{Spc}(D_S),
  u_!, u_* : \text{Spc}_*(C_S) \to \text{Spc}_*(D_S)
\]
are all full and faithful. In particular, if $E$ is a family of subgroups closed under conjugacy and $u : \text{Sm}^G_S[E] \subseteq \text{Sm}^G_S$ is the inclusion, then $u_!, u_* : \text{Spc}^G, E(S) \to \text{Spc}^G(S)$ and $u_!, u_* : \text{Spc}^G, E(S) \to \text{Spc}^G(S)$ are full and faithful.

Proof. First we note that the unit of the adjunction $Kan$ extensions of $F$ are all full and faithful. In particular, if $E$ is a family of subgroups closed under conjugacy and $u : \text{Sm}^G_S[E] \subseteq \text{Sm}^G_S$ is the inclusion, then $u_!, u_* : \text{Spc}^G, E(S) \to \text{Spc}^G(S)$ and $u_!, u_* : \text{Spc}^G, E(S) \to \text{Spc}^G(S)$ are full and faithful.

Moreover, these functors fit into commutative diagrams
Proof. The functors $i_* : \mathcal{C}_S^H \to \mathcal{C}_S^G$, $i^* : \mathcal{C}_S^G \to \mathcal{C}_S^H$ both send distinguished squares to distinguished squares, and $\iota_1(X) \times \mathbb{A}^1 \simeq \iota_1(X) \times \mathbb{A}^1$. It follows that $\iota_1 : \mathcal{P}(\mathcal{C}_S^H) \to \mathcal{P}(\mathcal{C}_S^G)$ and $\iota^* : \mathcal{P}(\mathcal{C}_S^G) \to \mathcal{P}(\mathcal{C}_S^G)$ preserve all motivic equivalences. It then follows that these induce functors on the category of motivic spaces as displayed. Since $\iota^* : \mathcal{S}_{\text{pc}}(\mathcal{C}_S^G) \to \mathcal{S}_{\text{pc}}(\mathcal{C}_S^G)$ preserves limits and these categories are Cartesian monoidal, it follows that $\iota^*$ is symmetric monoidal.

Since $\iota^*, \iota_*$ preserve final objects, they induce an adjoint pair on based spaces. Since $\iota^*$ preserves limits, it has a left adjoint $\iota_!$. It is straightforward to verify that these fit into commutative diagrams as displayed. \qed

Let $f : T \to S$ be a map in $\text{Sch}_S^G$. Let $\mathcal{C}_T^G \subseteq \text{Sch}_T^G$ and $\mathcal{C}_S^G \subseteq \text{Sch}_S^G$ be full subcategories satisfying (P) and (H), as before. Suppose that the base change $f^{-1} : \text{Sch}_S^G \to \text{Sch}_T^G$ restricts to a functor $f^{-1} : \mathcal{C}_S^G \to \mathcal{C}_T^G$. We write $f_* : \mathcal{P}(\mathcal{C}_T^G) \to \mathcal{P}(\mathcal{C}_S^G)$ for precomposition with $f^{-1}$ – that is, $f_* = (f^{-1})^*$ in the notation from before. The functor $f_*$ preserves Nisnevich excisive presheaves and homotopy-invariant presheaves, and thus restricts to a functor $f_* : \mathcal{S}_{\text{pc}}(\mathcal{C}_T^G) \to \mathcal{S}_{\text{pc}}(\mathcal{C}_S^G)$. This functor preserves limits and thus has a left adjoint, which we write as $f^* : \mathcal{S}_{\text{pc}}(\mathcal{C}_S^G) \to \mathcal{S}_{\text{pc}}(\mathcal{C}_T^G)$. Suppose further that the adjunction $u : \text{Sch}_T^G \rightleftarrows \text{Sch}_S^G : f^{-1}$ restricts to an adjunction

$$u : \mathcal{C}_T^G \rightleftarrows \mathcal{C}_S^G : f^{-1}.$$ \hspace{2cm} (2.33)

Then $f^* : \mathcal{P}(\mathcal{C}_S^G) \to \mathcal{P}(\mathcal{C}_T^G)$ is precomposition with the forgetful functor $u$ and it preserves Nisnevich excisive presheaves and homotopy-invariant presheaves. Thus $f^*$ restricts to a functor $f^* : \mathcal{S}_{\text{pc}}(\mathcal{C}_S^G) \to \mathcal{S}_{\text{pc}}(\mathcal{C}_T^G)$, which is left adjoint to $f_*$. Also in this case, $f^*$ preserves limits and so has a left adjoint $f_* : \mathcal{S}_{\text{pc}}(\mathcal{C}_T^G) \to \mathcal{S}_{\text{pc}}(\mathcal{C}_S^G)$.

The functors $f^*, f_*$ preserve final objects and so induce an adjunction $f^* : \mathcal{S}_{\text{pc}}(\mathcal{C}_S^G) \rightleftarrows \mathcal{S}_{\text{pc}}(\mathcal{C}_T^G) : f_*$. In the case when the forgetful functor restricts to $u : \mathcal{C}_T^G \to \mathcal{C}_S^G$, then there is an adjunction $f^* : \mathcal{S}_{\text{pc}}(\mathcal{C}_S^G) \rightleftarrows \mathcal{S}_{\text{pc}}(\mathcal{C}_T^G) : f^*$.

Remark 2.32. The adjunctions of Lemma 2.31 admit the following alternate description in terms of change-of-base functors. There is an equivalence of categories

$$\text{Sch}_{G \times H S}^G \simeq \text{Sch}_H^S$$ \hspace{2cm} (2.33)

induced by taking the fiber over $\{e\} \times S \subseteq G \times H S$. Write $\mathcal{C}_{G \times H S}^G \subseteq \text{Sch}_{G \times H S}^G$ for the full subcategory corresponding to $\mathcal{C}_S^H$ under formula (2.33). The restriction functor $\iota^*$ corresponds to pullback along $f : G \times H S \to S$. Moreover, the equivalence (2.33) induces an equivalence of motivic spaces

$$\mathcal{S}_{\text{pc}}(\mathcal{C}_{G \times H S}^G) \simeq \mathcal{S}_{\text{pc}}(\mathcal{C}_S^H),$$

and under this equivalence the functors $i_!, i^*, i_*$ are respectively identified with the functors $f^!, f^*, f_*$.

2.4. Motivic $G$-spectra

We recall the construction of categories of motivic $G$-spectra from [24] and some variants.
Let $C^\otimes$ be a presentably symmetric monoidal $\infty$-category and $X$ a set of objects in $C$. If $I = \{x_1, \ldots, x_n\}$ is a finite subset of $X$, write $\otimes I = x_1 \otimes \cdots \otimes x_n$. Write

$$C[X^{-1}] := \colim_{I \subseteq X} C\left[(\otimes I)^{-1}\right],$$

where $C[x^{-1}]$ denotes the symmetric monoidal inversion of an object $x \in C$ in a presentable symmetric monoidal $\infty$-category $C$ [34, §2.1].

Alternatively, one may consider the stabilization in $\Mod_C$, the $\infty$-category of $C$-modules in $\P r_S$. Recall that if $M \in \Mod_C$ and $x \in C$, then $\Stab_x(M)$ is the colimit, in $\Mod_C$, of the sequence

$$M \otimes x \subseteq \cdots \subseteq M \to \cdots.$$

More generally, for a set of objects $X$ in $C$, define

$$\Stab_X(M) := \colim_{I \subseteq X} \Stab_{\otimes I}(M).$$

An object $x \in C$ is $n$-symmetric if the cyclic permutation on $x \otimes n$ is homotopic to the identity. If each $x \in X$ is $n$-symmetric for some $n \geq 2$, then the canonical map of $C[X^{-1}]$-modules is an equivalence

$$M \otimes_C C[X^{-1}] \sim \Stab_X(M)$$

(see [24, Section 6.1] and [34, Corollary 2.22]).

Let $E$ be a finite-rank locally free $G$-module on $S$. Write $T^E \in \Spc_\bullet(S)$ for the associated motivic sphere, defined as the Thom space

$$T^E = \mathcal{V}(E)/\mathcal{V}(E) - z(S),$$

where $z : S \to \mathcal{V}(E)$ is the zero section. We will also write $\Sigma^E$ for the associated endofunctor $\Sigma^E \cong T^E \wedge -$.

We will also be interested in stabilizing the categories $\Spc_\bullet^{G,F}(S)$, for a family $F$. Observe that $\Spc_\bullet^{G,F}(S)$ is an $\Spc_\bullet^G(S)$-module and $u : \Spc_\bullet^{G,F}(S) \to \Spc_\bullet^G(S)$ is a map of $\Spc_\bullet^G(S)$-modules, since if $X \in \Sm_S[F]$ and $Y \in \Sm_S$, then $X \times Y \in \Sm_S[F]$. In particular, even though spheres $T^E \in \Spc_\bullet^G(S)$ are generally not objects of $\Spc_\bullet^{G,F}(S)$, they still determine endofunctors $\Sigma^E : \Spc_\bullet^{G,F}(S) \to \Spc_\bullet^{G,F}(S)$.

Write $\Sph_B^G := \{T^E \mid E \in \Rep_B^G\}$, where $\Rep_B^G$ is the set of finite-rank $G$-vector bundles over $B$.

**Definition 2.34.** A subset $T \subseteq \Sph_B^G$ is **stabilizing** if there is some $T^E \in T$ such that $T^E \cong T \wedge T^E'$, for some locally free $G$-module $E'$.

**Definition 2.35.**

1. Let $p : S \to B$ be a $G$-scheme over $B$ and $T \subseteq \Sph_B^G$ a stabilizing subset. Write

$$\Spt_T^G(S) := \Spc_\bullet^G(S)[(p^*T)^{-1}].$$
If $\mathcal{T} = \{T^{\mathcal{E}}\}$ consists of a single sphere, we write $\text{Spt}_{\mathcal{T}}^G(S)$ in place of $\text{Spt}_{\mathcal{T}}^G(S)$. When $\mathcal{T} = \text{Sph}_B^G$, we simply write $\text{Spt}^G(S) := \text{Spt}_{\text{Sph}_B^G}^G(S)$.

2. Let $\mathcal{F}$ be a family of subgroups. Define

$$\text{Spt}_{\mathcal{T}}^G,\mathcal{F}(S) := \text{Spc}^G_{\mathcal{T}}(S) \otimes \text{Spt}^G(S).$$

Write

$$\Sigma^{\infty}_{\mathcal{T}} : \text{Spc}^G_{\mathcal{T}}(S) \to \text{Spt}_{\mathcal{T}}^G(S)$$

for the stabilization functor. In the case when $\mathcal{T} = \text{Sph}_B^G$, we simply write $\Sigma^{\infty}$. When no confusion should arise, given $X \in \text{Spc}^G_{\mathcal{T}}(S)$ we will write again $X$ for its image in $\text{Spt}_{\mathcal{T}}^G(S)$ instead of $\Sigma^{\infty}_{\mathcal{T}} X$. Given $T^V \in \mathcal{T}$, $p : S \to B$, $E \in \text{Spt}_{\mathcal{T}}^G(S)$, and $k \in \mathbb{Z}$, we typically write $\Sigma^{kV} E$ rather than $\Sigma^{kp^*V} E$ when no confusion should arise.

**Proposition 2.36.** Set $S \in \text{Sch}_B^G$. Then $\text{Spt}_{\mathcal{T}}^G,\mathcal{F}(S)$ is a symmetric monoidal stable $\infty$-category satisfying the following properties:

1. There is a canonical equivalence of $\text{Spc}^G_{\mathcal{T}}(S)$-modules

$$\text{Spt}_{\mathcal{T}}^G,\mathcal{F}(S) \simeq \text{Stab}_{\mathcal{T}}(\text{Spc}^G_{\mathcal{T}}(S)).$$

2. Given a morphism $f : T \to S$ in $\text{Sch}_B^G$, there is an induced adjunction

$$f^* : \text{Spt}_{\mathcal{T}}^G,\mathcal{F}(S) \rightleftarrows \text{Spt}_{\mathcal{T}}^G(T) : f_*$$

such that $f^* \Sigma^{\infty}_{\mathcal{T}} X \simeq \Sigma^{\infty}_{\mathcal{T}} f_* X$. Similarly, if $f$ is smooth, then there is an induced adjunction

$$f^# : \text{Spt}_{\mathcal{T}}^G(T) \rightleftarrows \text{Spt}_{\mathcal{T}}^G(S) : f_*$$

such that $f^# \Sigma^{\infty}_{\mathcal{T}} X \simeq \Sigma^{\infty}_{\mathcal{T}} f^# X$.

3. The $\infty$-category $\text{Spt}_{\mathcal{T}}^G,\mathcal{F}(S)$ is generated under sifted colimits by the compact objects

$$\Sigma^{-kV} \Sigma^{\infty}_{\mathcal{T}} X_+,$$

where $k \geq 0$, $T^V \in \mathcal{T}$, $p : X \to S$ is in $\text{Sm}_S^G[\mathcal{F}]$, and $X$ is affine.

4. The family of functors

$$\{p^* : \text{Spt}_{\mathcal{T}}^G,\mathcal{F}(S) \to \text{Spt}_{\mathcal{T}}^G,\mathcal{F}(X) \mid p : X \to S \text{ is in } \text{Sm}_S^G[\mathcal{F}], \text{ X affine}\}$$

is jointly conservative.

**Proof.** That $\text{Spt}_{\mathcal{T}}^G,\mathcal{F}(S)$ is stable is a consequence of the fact that $\mathcal{T}$ is stabilizing and that $T \simeq S^1 \wedge \mathbb{G}_m$. It is symmetric monoidal by construction. The arguments for the remaining points of (1)–(3) are similar to [24, Section 6]. Since $\text{Spt}_{\mathcal{T}}^G,\mathcal{F}(S)$ is stable, in order to establish (4) it suffices to show that $E \simeq 0$ whenever $p^* E \simeq 0$ for all $p : X \to S$ in $\text{Sm}_S^G[\mathcal{F}]$. 

with $X$ affine. If $T^V \in \mathcal{T}$, then since $V \in \text{Rep}^G_B$, we have $\Sigma^{-kV} \Sigma^\infty_T X_+ \simeq p_# \left( \Sigma^{-kV} 1_X \right)$. Now, if $p^* E \simeq 0$, we have

$$\text{Map}_{\text{Stab}^G_\infty}(\Sigma^{-kV} \Sigma^\infty_T X_+, E) \simeq \text{Map}_{\text{Stab}^G_\infty}(X) \left( \Sigma^{-kV} 1_X, p^* E \right) \simeq 0.$$ 

Given $E$ such that $p^* E \simeq 0$ for all $p : X \to S$ in $\text{Sm}_G[\mathcal{F}]$ with $X$ affine, let $C_E \in \text{Stab}^G_\infty(S)$ be the full subcategory of whose objects are $W$ such that $\text{Map}_{\text{Stab}^G_\infty}(W, E) \simeq 0$ — that is, the $E$-acyclic objects. Then $C_E$ is closed under colimits and it contains $\Sigma^{-kV} \Sigma^\infty_T X_+$ for any $k \geq 0$ and $T^V \in \mathcal{T}$. Thus by (3), $C_E \simeq \text{Stab}^G_\infty(S)$, which implies that $E \simeq 0$ as required.

**Remark 2.37.** Over an affine base, every representation is the quotient of a finite sum of copies of the regular representation $\rho_G$. This implies that for any $S$,

$$\text{Stab}^G_\infty(S) \simeq \text{Stab}^G_{T^V G}(S).$$

Let $N \leq G$ be a normal subgroup and $\pi : G \to G/N$ the quotient homomorphism. This induces a function $\pi^* : \text{Rep}^{G/N}_B \to \text{Rep}^G_B$, and we write

$$N\text{-triv} = \{ T^E \mid E \in \pi^* \left( \text{Rep}^{G/N}_B \right) \} \subseteq \text{Sph}^G_B$$

for the associated set of ‘$N$-trivial $G$-spheres’. This stabilizing set of spheres plays an important role in later sections.

**Lemma 2.38.** Let $\mathcal{F}$ be a family. The adjunction $u : \text{Stab}^G_{\mathcal{T}}(S) \simeq \text{Stab}^G_{\mathcal{T}}(S)$ of $\text{Stab}^G_{\mathcal{T}}(S)$-modules induces an adjoint pair

$$u_! : \text{Stab}^G_{\mathcal{T}}(S) \simeq \text{Stab}^G_{\mathcal{T}}(S) : u_*.$$ 

Moreover, $u_*$ is symmetric monoidal and

$$u_! : \text{Stab}^G_{\mathcal{T}}(S) \to \text{Stab}^G_{\mathcal{T}}(S)$$

is full and faithful with essential image the localizing tensor ideal generated by $\Sigma^{-nV} X_+$, where $T^V \simeq p^* T$ and $X \in \text{Sm}_G[\mathcal{F}]$.

**Proof.** That the adjunction $(u_!, u_*)$ of $\text{Stab}^G_{\mathcal{T}}(S)$-modules induces an adjoint pair on categories of motivic spectra follows from the description of $\text{Stab}^G_{\mathcal{T}}(S)$ and $\text{Stab}^{G, \mathcal{F}}_\infty(S)$, respectively, as $\text{Stab}^G_{\mathcal{T}}(\text{Stab}^G_{\mathcal{T}}(S))$ and $\text{Stab}^G_{\mathcal{T}}(\text{Stab}^{G, \mathcal{F}}_\infty(S))$ (see the discussion preceding [24]). This also implies that $u_!$ is full and faithful, since $u_! : \text{Stab}^{G, \mathcal{F}}_\infty(S) \to \text{Stab}^G_{\mathcal{T}}(S)$ is, by Proposition 2.30.

Let $\mathcal{C}$ be a presentably symmetric monoidal $\infty$-category and $E \in \mathcal{C}$ an idempotent object. That this means there is a map $e : 1 \to E$ such that $id \otimes e : E \simeq E \otimes 1 \to E \otimes E$ and $e \otimes id : E \simeq 1 \otimes E \to E \otimes E$ are equivalences [29, Definition 4.8.2.1]. Tensoring with $E$ is a localization functor $L = E \otimes - : \mathcal{M} \to \mathcal{M}$ on any $\mathcal{M} \in \text{Mod}_\mathcal{C}$ [29, Proposition 4.8.2.4].

**Lemma 2.39.** With notation as before,

$$L \mathcal{M} \simeq L \mathcal{C} \otimes \mathcal{C} \mathcal{M}.$$
Proof. An argument similar to [29, Proposition 4.8.2.10] shows that the forgetful functor $\text{Mod}_E(M) \to M$ determines an equivalence of $C$-modules $\text{Mod}_E(M) \simeq LM$. Since $\text{Mod}_E(M) \simeq \text{Mod}_E(C) \otimes_C M$ by [29, Theorem 4.8.4.6], the lemma follows.

We record the following result which we will use a few times. A similar statement can be found in [5, Lemma 26].

Lemma 2.40. Let $C^\otimes$ be a presentably symmetric monoidal $\infty$-category and $X$ a set of objects. Suppose that $E \in C$ is an idempotent object. Write $L = E \otimes -$ for the associated symmetric monoidal localization endofunctor. Then there is an equivalence

$$L(C[X^{-1}]) \simeq (L C)[X^{-1}]$$

in $\text{CAlg}(\mathcal{P}_{R_{L, \otimes}})$.

Proof. Both of these categories can be identified with $L C \otimes_C C[X^{-1}]$.

Next we record the motivic version of the Wirthmüller isomorphism, which is a special case of the ambidexterity equivalence proved in [24]. If $H \leq G$ is a subgroup and $\iota$ is the inclusion, we sometimes write $G \times_H X := \iota_! X$.

Proposition 2.41 (Wirthmüller isomorphism [24]). Let $\iota : H \hookrightarrow G$ be a group monomorphism. Let $\mathcal{F}, \mathcal{F}'$ be families of subgroups of, respectively, $H$ and $G$ such that the induction-restriction adjunction restricts to $\iota_! : \text{Sm}^H_S[\mathcal{F}] \rightleftarrows \text{Sm}^G_S[\mathcal{F}'] : \iota^{-1}$. Then there is an induced adjunction

$$\iota_! : \text{Spt}^H_{\mathcal{F}}(S) \rightleftarrows \text{Spt}^G_{\mathcal{F}'}(S) : \iota^*,$$

such that $\iota_!(\Sigma^\infty X) \simeq \Sigma^\infty(\iota_! X)$ and $\iota^*(\Sigma^\infty X) \simeq \Sigma^\infty(\iota^{-1} X)$. Moreover, $\iota^*$ admits a right adjoint $\iota_*$ and there is an equivalence

$$\iota_! \simeq \iota_*.$$

Proof. The first statements are straightforward. We explain the last statement. Consider the $G$-equivariant map $f : G \times_H S \to S$. Then $\iota_!, \iota^*, \iota_*$ are identified with $f_\#, f^*, f_\#$, via the equivalence $\text{Spt}^G_{\mathcal{F}}(G \times_H S) \simeq \text{Spt}^H_{\mathcal{F}}(S)$ (see Remark 2.32). Here $\mathcal{F}$ is the family of subgroups of $G$ generated by $\mathcal{F}$.

We have $u^* \iota'_! v_! \simeq \iota_!$ and $u^* \iota'_* v_! \simeq \iota_*$, where we write $u : \text{Sm}^G_S \subseteq \text{Sm}^G_S$ and $v : \text{Sm}^H_S \subseteq \text{Sm}^H_S$ for the inclusions and $\iota'_! : \text{Spt}^H(S) \to \text{Spt}^G(S)$ are the corresponding functors for the family $\mathcal{F}_{all} = \mathcal{F}'$. In particular, the general case follows from the case when $\mathcal{F} = \mathcal{F}_{all}$. But in this case, the Wirthmüller isomorphism is the ambidexterity equivalence [24, Theorem 1.5] for the finite étale morphism $f$.

2.5. Functoriality

We will make use of the functoriality of the $\infty$-categories of equivariant motivic spaces and spectra with respect to the group $G$, the family of subgroups $\mathcal{F}$, and the base scheme $S$. To establish this, it will be convenient to introduce the following notation:
Definition 2.42.

1. The category $\text{Sch}_B$ of equivariant $B$-schemes has objects pairs $(G,S)$ consisting of a finite group $G$ (whose order is invertible in $O_B$) and $S \in \text{Sch}^G_B$. A morphism $(G',S') \to (G,S)$ is a pair $(\phi,f)$, where $\phi : G' \to G$ is a homomorphism of groups and $f : S' \to S$ is a $\phi$-equivariant map of $B$-schemes.

2. The category $\text{Sch}^B_\cdot$ of equivariant $B$-schemes and families has objects consisting of triples $(G,F,S)$, where $(G,S) \in \text{Sch}^B_B$ and $F$ is a family of subgroups of $G$. A morphism $(G',F',S') \to (G,F,S)$ is a triple $(\phi,i,f)$, where $(\phi,f)$ is a morphism in $\text{Sch}_B$ and $i : \phi^{-1}F \subset F'$ is an inclusion of posets.

The inclusion $i : \phi^{-1}F \subset F'$ is unique if it exists. When no confusion should arise, we write $(\phi,f)$ instead of $(\phi,i,f)$ for a morphism in $\text{Sch}^B_\cdot$.

We identify $\text{Sch}_B$ with the full subcategory of $\text{Sch}^B_\cdot$ whose objects are triples of the form $(G,F_{\text{all}},S)$. The inclusion $\text{Sch}_B \subseteq \text{Sch}^B_\cdot$ is left adjoint to the forgetful functor.

In this subsection we extend constructions in the previous sections to functors

$$Spc^\times,Sp^n_G^\times,Sp^n_\cdot : (\text{Sch}^B_\cdot)^{\text{op}} \to \text{CAlg}(\mathcal{P}_{\text{rL}}),$$

whose respective values on $(G,F,S)$ are the symmetric monoidal $\infty$-categories $\text{Spc}^{G,F}(S)$, $\text{Spc}^G_{\cdot,F}(S)$, and $\text{Spt}^{G,F}(S)$, and on morphisms, $(\phi,i,f)^* \simeq i_{\phi^*} f^*$. 

Lemma 2.43. The categories $\text{Sch}_B$ and $\text{Sch}^B_\cdot$ admit finite products. In particular, they are Cartesian symmetric monoidal.

Proof. It is straightforward to check that $(G,S) \times (G',S') \simeq (G \times G',S \times_B S')$ and $(G,F,S) \times (G',F',S') \simeq (G \times G',F \times F',S \times_B S')$, where $F \times F'$ denotes the family of subgroups of $G \times G'$ consisting of those subgroups of the form $H \times H'$ for $H \in F$ and $H' \in F'$.

Corollary 2.44. The assignment $(G,F,S) \mapsto \text{Sm}^G_S[F]$, $(\phi,i,f) \mapsto i_{\phi^{-1}f^{-1}}$, extends to a functor

$$\text{Sch}^B_\cdot^{\text{op}} \to \text{CAlg(\text{Cat})},$$

the category of commutative algebra objects of $\text{Cat}$ with respect to the Cartesian symmetric monoidal structure (equivalently, the category of symmetric monoidal categories).

Composing with the symmetric monoidal presheaves functor $\text{Cat} \to \text{Cat}_{\infty} \to \mathcal{P}_{\text{rL}}$, we obtain a functor

$$(\text{Sch}^B_\cdot)^{\text{op}} \to \text{CAlg}(\mathcal{P}_{\text{rL}}),$$

which sends the object $(G,F,S)$ to $\mathcal{P}(\text{Sm}^G_S[F])$ and the morphism $(\phi,i,f)$ to $i_{\phi^*} f^*$. 

To obtain the necessary functoriality of equivariant motivic spaces and spectra, we follow the techniques of [7, Section 6.1]. Recall from there that we have a commutative diagram of ∞-categories

\[
\begin{array}{cccc}
\mathcal{M}\text{Cat}_\infty & \rightarrow & \mathcal{O}\text{Cat}_\infty & \rightarrow & \mathcal{E} \\
\downarrow & & \downarrow & & \downarrow \\
\text{Cat}_\infty & \xrightarrow{\text{Fun}(\Delta^1,-)} & \text{Cat}_\infty & \rightarrow & \text{Pos},
\end{array}
\]

in which all squares are Cartesian. Here Pos denotes the ∞-category of (not necessarily small) posets, the lower right-hand horizontal arrow sends an ∞-category to the poset of subsets of the set of equivalence classes of objects, and \(\mathcal{E} \rightarrow \text{Pos}\) is the universal co-Cartesian fibration, restricted to posets. The ∞-categories \(\mathcal{O}\text{Cat}_\infty\) and \(\mathcal{M}\text{Cat}_\infty\) are, respectively, the ∞-categories of ∞-categories equipped with a collection of equivalence classes of objects respectively a collection of equivalence classes of arrows.

**Lemma 2.45.** Set \((\mathcal{C},W) \in \mathcal{M}\text{Cat}_\infty\) such that \(\mathcal{C}\) is presentable and \(W\) is of small generation.

1. The partial adjoint to

\[
\text{Cat}_\infty \rightarrow \mathcal{M}\text{Cat}_\infty, \quad \mathcal{C} \mapsto (\mathcal{C},\text{equivalences}),
\]

is defined at \((\mathcal{C},W)\), and the localization \(\mathcal{C}[W^{-1}]\) is again presentable.

2. Suppose that \(\mathcal{C}\) admits a symmetric monoidal structure \(\mathcal{C}^\otimes \in \text{CAlg}(\text{Cat}_\infty)\) and \(W\) is stable under the monoidal product. Then \((\mathcal{C},W)\) lifts to \((\mathcal{C},W)^\otimes \in \text{CAlg}(\mathcal{M}\text{Cat}_\infty)\), and the partial left adjoint to

\[
\text{CAlg}(\text{Cat}_\infty) \rightarrow \text{CAlg}(\mathcal{M}\text{Cat}_\infty), \quad \mathcal{C} \mapsto (\mathcal{C},\text{equivalences}),
\]

is defined at \((\mathcal{C},W)^\otimes\).

**Proof.** The first item follows from [28, Proposition 5.5.4.15, Proposition 5.5.4.20]. It then follows from [29, Proposition 2.2.1.9] that \(\mathcal{C}[W^{-1}]\) inherits a monoidal structure such that \(\mathcal{C} \rightarrow \mathcal{C}[W^{-1}]\) is monoidal. This implies the second item. \(\square\)

Consider the subfunctor of the composition

\[
(\text{Sch}_B[-])^\text{op} \rightarrow \text{Cat} \rightarrow \text{Cat}_\infty \rightarrow \mathcal{P}\text{r}^L \rightarrow \text{Fun}(\Delta^1,\text{Cat}_\infty)
\]

whose value on the object \((G,\mathcal{F},S)\) is the full subcategory

\[
W_{(G,\mathcal{F},S)} \subset \text{Fun}(\Delta^1,\mathcal{P}(\text{Sm}_S^G[\mathcal{F}])),
\]

consisting of the motivic equivalences. Since motivic equivalences are stable under the smash product and are preserved by the functors \(f^*, \phi^*,\) and \(i_!\), the assignments

\[
(G,\mathcal{F},S) \mapsto (\mathcal{P}(\text{Sm}_S^G[\mathcal{F}]),W_{(G,\mathcal{F},S)}), (\mathcal{P}_*(\text{Sm}_S^G[\mathcal{F}]),W_{(G,\mathcal{F},S)})
\]

induce functors

\[
(\text{Sch}_B[-])^\text{op} \rightarrow \mathcal{M}\text{Cat}_\infty.
\]
The images of \( \left( \mathcal{P}(\text{Sm}_S^G[\mathcal{F}]), \mathcal{W}_{(G,\mathcal{F},S)} \right) \) and \( \left( \mathcal{P}_\bullet(\text{Sm}_S^G[\mathcal{F}]), \mathcal{W}_{(G,\mathcal{F},S)} \right) \) under the partial left adjoint to

\[
\text{CAlg} \left( \text{Cat}_\infty \right) \rightarrow \text{CAlg} \left( \mathcal{M} \text{Cat}_\infty \right)
\]

are, respectively, \( \text{Spec}^{G,\mathcal{F}}(S) \) and \( \text{Spec}_\bullet^{G,\mathcal{F}}(S) \).

Write \( \text{Sph}_S^G \) for the set of spheres \( \left\{ T \right\} \), where \( E \) is an equivariant vector bundle over \( S \). The assignment \( (G, \mathcal{F}, S) \mapsto \text{Sph}_S^G \) is a presheaf of sets on \( \text{Sch}_B[\cdot]^\text{op} \), which we write as \( \text{Sph} \). Let

\[
\mathcal{T} : \text{Sch}_B[\cdot]^\text{op} \rightarrow \text{Set}
\]

be a subpresheaf of \( \text{Sph} \), which is closed under the smash product and takes values in stabilizing sets of spheres – that is, there is some \( T^\mathcal{E} \in \mathcal{T}_{(G,\mathcal{F},S)} \) such that \( T^\mathcal{E} \cong T \wedge T^\mathcal{E}' \).

We obtain a functor

\[
\text{Sch}_B[\cdot]^\text{op} \rightarrow \text{CAlg}(\mathcal{O} \text{Cat}_\infty),
\]

which on objects is the assignment \( (G, \mathcal{F}, S) \mapsto (\text{Sph}_S^{G,\mathcal{F}}(S), \mathcal{T}_{(G,\mathcal{F},S)}) \). By the following lemma, we obtain \( \text{Spt}_{\mathcal{T}_{(G,\mathcal{F},S)}}(S) \) as the image of \( (\text{Sph}_S^{G,\mathcal{F}}(S), \mathcal{T}_{(G,\mathcal{F},S)}) \) under the partial left adjoint of \( \text{CAlg} \left( \text{Cat}_\infty \right) \rightarrow \text{CAlg} \left( \mathcal{O} \text{Cat}_\infty \right) \), \( C \mapsto (C, \pi_0 \text{Pic}(C)) \).

**Lemma 2.46.** Let \((C^\text{op}, U)\) be an object of \( \text{CAlg}(\mathcal{O} \text{Cat}_\infty) \) such that \( C \) is presentable symmetric monoidal and \( U \) is small. Then the partial adjoint of

\[
\text{CAlg} \left( \text{Cat}_\infty \right) \rightarrow \text{CAlg} \left( \mathcal{O} \text{Cat}_\infty \right), \quad C \mapsto (C, \pi_0 \text{Pic}(C)),
\]

is defined at \((C^\text{op}, U)\).

**Proof.** This follows from [24, Section 6.1].

**Corollary 2.47.** The assignments \( (G, \mathcal{F}, S) \mapsto \text{Sph}_S^{G,\mathcal{F}}(S) \) and \( (G, \mathcal{F}, S) \mapsto \text{Spt}_{\mathcal{T}_{(G,\mathcal{F},S)}}(S) \) extend to functors \( \text{Spec} : \text{Sch}_B[\cdot]^\text{op} \rightarrow \text{CAlg}(\mathcal{P}_{1^L}) \) and \( \text{Spt} : \text{Sch}_B[\cdot]^\text{op} \rightarrow \text{CAlg}(\mathcal{P}_{1^L}) \), respectively.

3. Filtering by isotropy

In this section, we develop techniques to define and analyze filtrations of motivic \( G \)-spaces and spectra by families of isotropy.

3.1. Universal motivic \( \mathcal{F} \)-spaces

Let \( \mathcal{F} \) be a family of subgroups. In classical equivariant homotopy theory, there is a \( G \)-space \( E_\bullet \mathcal{F} \) characterized by the property that a \( G \)-space \( X \) admits a unique map \( X \rightarrow E_\bullet \mathcal{F} \) if all of the stabilizers of \( X \) are in \( \mathcal{F} \), and no maps from \( X \) to \( E_\bullet \mathcal{F} \) otherwise. The \( G \)-space \( E_\bullet \mathcal{F} \) formally exists as a presheaf on \( \text{Sm}_S^G \) and hence as a motivic \( G \)-space over \( B \), but it does not have the correct universality property.

**Example 3.1.** Let \( G \neq \{e\}, B = \text{Spec}(k) \) a field, and let \( L/k \) be a finite Galois extension such that \( G \subseteq \text{Gal}(L/k) \), and consider \( \text{Spec}(L) \) as a smooth \( G \)-scheme over \( k \) via the
Galois action. Then \( \text{Spec}(L)^H = \emptyset \) for all \( e \neq H \subseteq G \) — that is, it has a free \( G \)-action. However, we claim that

\[
\text{Map}_{\text{Spec}^G(k)}(\text{Spec}(L), E \star G) = \emptyset.
\]

Indeed,

\[
(\pi_0\text{Spec}(L))_0 = \text{Hom}_{\text{Sm}^G_S}(\text{Spec}(L), \coprod_{G} \text{Spec}(k)) = \emptyset,
\]

and so the claim follows, since \( (\pi_0\text{Spec}(L))_0 \) surjects onto \( \pi_0(\text{Map}_{\text{Spec}^G(k)}(\text{Spec}(L), E \star G)) \) (see, e.g., [32, Corollary 2.3.22, Remark 3.2.5]).

**Definition 3.2.** Let \( \mathcal{F} \) be a family of subgroups in \( G \). The **universal motivic** \( \mathcal{F} \)-space over \( S \) is the object \( E_{\mathcal{F}}S \in \mathcal{P}(\text{Sm}^G_S) \) whose value on \( X \in \text{Sm}^G_S \) is

\[
E_{\mathcal{F}}S(X) = \begin{cases} 
\text{pt}, & X \in \text{Sm}^G_S[\mathcal{F}], \\
\emptyset, & \text{else}.
\end{cases}
\]

When the base \( S \) is understood, we simply write \( E_{\mathcal{F}} \).

**Proposition 3.3.** Let \( \mathcal{F} \) be a family of subgroups in \( G \). The presheaf \( E_{\mathcal{F}} \) is a motivic \( G \)-space.

**Proof.** We need to check that \( E_{\mathcal{F}} \) is Nisnevich excisive and \( \mathbb{A}^1 \)-homotopy invariant. From the definition we have that \( E_{\mathcal{F}}(\emptyset) = \text{pt} \). Given a Nisnevich square (2.11), the possible values of \( E_{\mathcal{F}}(X) \) are the squares

\[
\begin{array}{c}
0 & \to & 0 \\
\downarrow & & \downarrow \\
\text{pt} & \to & \text{pt}
\end{array}
\]

which are all Cartesian squares. It follows that \( E_{\mathcal{F}} \) is Nisnevich excisive. Also, \( E_{\mathcal{F}} \) is \( \mathbb{A}^1 \)-homotopy invariant, since \( X \times_S \mathbb{A}^1_S \) is a filtered poset and subject to the following conditions:

**Totaro** [39] and Morel and Voevodsky [32, Section 4.2] constructed a geometric model for the classifying space of an algebraic group. This is generalized by Hoyois in [23, Section 2] to construct a geometric model for certain equivariant classifying spaces. Similar considerations lead to geometric models for the spaces \( E_{\mathcal{F}} \).

**Definition 3.4.** A **system of approximations** to \( E_{\mathcal{F}}S \) is a diagram \((U_i)_{i \in I}\), which is a subdiagram of a diagram \((V_i)_{i \in I}\) of inclusions of \( G \)-equivariant vector bundles over \( S \), where \( I \) is a filtered poset and subject to the following conditions:
1. Each $U_i$ is in $\text{Sm}_S[\mathcal{F}]$, and $U_i \subseteq V_i$ is an open subscheme.
2. For $i \in I$, there exists an element $2i \in I$ with the property that $2i \geq i$ and such that there is an isomorphism $V_{2i} \cong V_i \oplus V_i$ of $G$-vector bundles which identifies the inclusion $V_i \hookrightarrow V_{2i}$ with the inclusion $(id,0) : V_i \hookrightarrow V_i \oplus V_i$.
3. Under the isomorphism $V_{2i} \cong V_i \oplus V_i$, $(U_i \times V_i) \cup (V_i \times U_i) \subseteq U_{2i}$.
4. There is a Nisnevich cover $\{T_j \to S\}$ such that for any affine $X$ in $\text{Sm}_{T_j}[\mathcal{F}]$, there is an $i \in I$ such that $(U_i)_X \to X$ admits a section.

**Example 3.5.** Write $\rho = \rho_G$. Let $U_n \subseteq V_B(n\rho)$ be the open invariant subscheme

$$U_n := V(n\rho) \setminus \bigcup_{H \in \text{co}(\mathcal{F})} V(n\rho)^H.$$

The inclusions $V(n\rho) \subseteq V((n+1)\rho)$ induce maps $U_n \to U_{n+1}$.

Set $f : S \to B$ in $\text{Sch}_B^G$. Then $(f^*U_n)_{n \in \mathbb{N}}$ is a system of approximations to $E\mathcal{F}_S$. Conditions (1)–(3) are clear. To check the last condition, we may assume that $B$ and $S$ are affine (since $S$ is equivariant locally affine). Let $X \in \text{Sm}_S^G[\mathcal{F}]$ be affine. Then for $n$ sufficiently large, there is an equivariant closed immersion of $B$-schemes $X \hookrightarrow V(n\rho)$. For any $H \not\in \mathcal{F}$, we have $X \cap (V(n\rho)^H = \emptyset$, which means that $X \to V(n\rho)$ factors to give a map $X \to U_n$ over $B$. This defines the desired section $X \to f^*U_n$.

**Example 3.6.** Let $N \leq G$ be a normal subgroup. The family $\mathcal{F}[N]$ consists of all subgroups not containing $N$. Write $W := \rho_G/\rho_{G/N}$ for the quotient representation (where $\rho_G/N$ is viewed as a $G$-representation via the quotient homomorphism $G \to G/N$). Let $U_n = V_S(nW) \setminus \{0\}$. This defines a system of approximations to $E\mathcal{F}[N]_S$. Conditions (1)–(3) of the definition are clear. As in the previous example, to check the last condition it suffices to assume that $B$ and $S$ are affine. Let $X \in \text{Sm}_S^G[\mathcal{F}[N]]$ be affine. Then for $n$ large enough, there is an equivariant closed immersion of $B$-schemes $X \hookrightarrow V_B(n\rho_G)$. The preimage of 0 under the projection $p : V(n\rho_G) \to V((n)W)$ is $V(n\rho_G/N)$. Since $N$ is not contained in any stabilizer of $X$, we have $X \cap V_B(n\rho_G/N) = \emptyset$, which implies that restriction of $p$ to $X$ factors through $U_n$. This defines the desired section $X \to U_n$.

If $(U_i)_{i \in I}$ is a system of approximations to $E\mathcal{F}_S$, define

$$U_\infty := \text{colim}_i U_i \in \text{Spc}^G(S).$$

**Proposition 3.7.** Let $\mathcal{F}$ be a family of subgroups in $G$ and $(U_i)_{i \in I}$ be a system of approximations to $E\mathcal{F}_S$. Then there is an equivalence

$$U_\infty \xrightarrow{\sim} E\mathcal{F}_S$$

in $\text{Spc}^G(S)$.

**Proof.** To prove the result, it suffices to work Nisnevich locally on $S$, and so we may assume that $S = T_j$ in the last condition of Definition 3.4. For each $i$, there is a unique map $U_i \to E\mathcal{F}$ which induces the unique map $U_\infty \to E\mathcal{F}$. It suffices to show that $\text{Sing}_{A^1}(U_\infty)(X) \to \text{Sing}_{A^1}(E\mathcal{F})(X)$ is an equivalence for any affine $X$ in $\text{Sm}_S^G$. Both sides are empty if $X \not\in \text{Sm}_S^G[\mathcal{F}]$, so we just need to show that $\text{Sing}_{A^1}(U_\infty)(X)$ is contractible.
for any affine $X$ in $\text{Sm}^G_S[\mathcal{F}]$. By assumption, there is a section of $U_\infty \times S X \to X$, and so the result follows from [23, Lemma 2.6].

**Proposition 3.8.** Let $\mathcal{F}$ be a family of subgroups and let $f : S' \to S$ be a morphism in $\text{Sch}_B^G$. Then

$$f^*(E\mathcal{F}_S) \simeq E\mathcal{F}_{S'}.$$  

**Proof.** This follows from Proposition 3.7, together with Example 3.5. \hfill \Box

Write $i : \text{Sm}_S[\mathcal{F}] \subseteq \text{Sm}_S^G$ for the inclusion of categories.

**Proposition 3.9.** There is a canonical equivalence of endofunctors

$$i_! i^* \simeq E\mathcal{F} \times - : \text{Sp}c^G(S) \to \text{Sp}c^G(S)$$

and

$$i_! i^* \simeq E\mathcal{F}^+ \land - : \text{Sp}c^G_*(S) \to \text{Sp}c^G_*(S).$$

**Proof.** We treat the unbased case; the based case then follows. Set $X \in \text{Sp}c^G(S)$. We have $(i^* X)(W) \simeq X(W)$ for $W \in \text{Sm}^G_S[\mathcal{F}]$. In particular, the projection $E\mathcal{F} \to \text{pt}$ induces the equivalences $i^*(E\mathcal{F} \times X) \simeq i^*(X)$ and thus

$$i_! i^*(E\mathcal{F} \times -) \sim i_! i^*(-).$$

Now $E\mathcal{F}$ is equivalent to $\text{colim}_i U_i$, where $U_i \in \text{Sm}^G_S[\mathcal{F}]$. If $W \in \text{Sm}^G_S$, then each $U_i \times W$ is in $\text{Sm}^G_S[\mathcal{F}]$. It follows that if $X$ is any object of $\text{Sp}c^G(S)$, then, writing $X$ as a colimit of objects of $\text{Sm}^G_S$, we see that $E\mathcal{F} \times X \simeq i_!(E)$ for some $E \in \text{Sp}c^G_\mathcal{F}(S)$. In particular, since $\eta : i_! \simeq i^* i_!$ is an equivalence by Proposition 2.30, we have $i_! i^* i_!(E) \simeq i_! i^* i_!(E)$. From the triangle identity for the unit and counit, we have that $\epsilon_1 : i_! i^* (i_!(E)) \simeq i_!(E)$ is also an equivalence. It follows that we have equivalences

$$E\mathcal{F} \times - \simeq i_! i^*(E\mathcal{F} \times -) \simeq i_! i^*.$$ \hfill \Box

**Corollary 3.10.** Let $\mathcal{F}$ be a family of subgroups.

1. The essential images of $E\mathcal{F} \times -$ and $E\mathcal{F}^+ \land -$ are, respectively, the subcategories $\text{Sp}c^G_\mathcal{F}(S) \subseteq \text{Sp}c^G(S)$ and $\text{Sp}c^G_\mathcal{F}(S) \subseteq \text{Sp}c^G_*(S)$.
2. The projection $E\mathcal{F} \times X \to X$ is an equivalence for $X \in \text{Sp}c^G(S)$ if and only if $X \simeq i_!(X')$ for some $X' \in \text{Sp}c^G_\mathcal{F}(S)$.
3. The projection $E\mathcal{F}^+ \land Y \to Y$ is an equivalence for $Y \in \text{Sp}c^G_*(S)$ if and only if $Y \simeq i_!(\tilde{Y})$ for some $\tilde{Y} \in \text{Sp}c^G_\mathcal{F}(S)$.
4. The canonical maps are equivalences

$$\text{Map}_{\text{Sp}c^G(S)}(E\mathcal{F} \times X, X') \simeq \text{Map}_{\text{Sp}c^G(S)}(E\mathcal{F} \times X, E\mathcal{F} \times X')$$

$$\text{Map}_{\text{Sp}c^G_*(S)}(E\mathcal{F}^+ \land Y, Y') \simeq \text{Map}_{\text{Sp}c^G_*(S)}(E\mathcal{F}^+ \land Y, E\mathcal{F}^+ \land Y').$$
Recall that $i_! : \mathcal{S}pt^G_{T^F}(S) \to \mathcal{S}pt^G_T(S)$ is full and faithful with essential image the localizing tensor ideal generated by $T^F \otimes X$, where $X \in Sm^G_S[F]$ and $T^F \in T$ (see Lemma 2.38).

**Proposition 3.11.** There is an equivalence of colocalization endofunctors

$$E_F \otimes - \simeq i_! i_* : \mathcal{S}pt^G_T(S) \to \mathcal{S}pt^G_T(S).$$

In particular, there are natural equivalences

$$i_! i_* \simeq i_! i_*(1_S) \otimes \text{id}$$

and

$$i_* i^* \simeq F_{\mathcal{S}pt^G_T(S)}(E_F, \text{id}).$$

**Proof.** The first statements follow from Proposition 3.9. The last statement follows from the natural equivalences

$$\text{Map}_{\mathcal{S}pt^G_T(S)}(X, i_* i^* Y) \simeq \text{Map}_{\mathcal{S}pt^G_T(S)}(i_! i_*(X), Y) \simeq \text{Map}_{\mathcal{S}pt^G_T(S)}(E_F \otimes X, Y).$$

\[\square\]

### 3.2. Filtrations by adjacent families

We recall the definition of adjacent families.

**Definition 3.12.** Let $\mathcal{F} \subseteq \mathcal{F}'$ be an inclusion of families of subgroups of $G$.

1. We say that $\mathcal{F}$ and $\mathcal{F}'$ are **adjacent** if there is a subgroup $H \leq G$ such that $\mathcal{F}' \setminus \mathcal{F} = \{ (H) \}$.

2. If $N \trianglelefteq G$ is a normal subgroup, say that $\mathcal{F}$ and $\mathcal{F}'$ are **$N$-adjacent** if there is a subgroup $H \leq N$ such that $\mathcal{F}' \setminus \mathcal{F} = \{ K \leq G \mid (K \cap N) = (H) \}$ (where as before, $(A)$ denotes the $G$-conjugacy class of a subgroup $A$).

3. Say that $\mathcal{F}$ and $\mathcal{F}'$ are **$N$-adjacent at $H \leq N$** if the families are $N$-adjacent and $\mathcal{F}' \setminus \mathcal{F}$ is the set of subgroups $K \leq G$ such that $(K \cap N) = (H)$.

Of course, if $N = G$ then $N$-adjacent families are exactly adjacent families. Since $G$ is finite, one can always find a filtration

$$\varnothing = \mathcal{F}_{-1} \subseteq \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_n = \mathcal{F}_{\text{all}},$$

such that each pair $\mathcal{F}_i \subseteq \mathcal{F}_{i+1}$ is $N$-adjacent. For example, one can be produced as follows. Define the sequence of families

$$\{ e \} = \text{Fil}^G_0 \subset \text{Fil}^G_1 \subset \cdots \subset \text{Fil}^G_i \subset \cdots \subset \text{Fil}^G_N = \mathcal{F}_{\text{all}}$$

by setting $\text{Fil}_0 = \{ e \}$ and inductively defining $\text{Fil}_i^G$ by

$$\text{Fil}_i^G := \{ H \leq G \mid \text{each proper subgroup } K < H \text{ is in } \text{Fil}_{i-1}^G \}.$$
(Since $G$ is finite, this sequence terminates at a finite stage.) Each $\text{Fil}_i^G$ is a family. More generally, define
\[ \text{Fil}_i^{N \triangleright G} := \{ K \leq G \mid K \cap N \in \text{Fil}_i^N \}. \]
The families just defined are not adjacent, but $\text{Fil}_i^{N \triangleright G} \setminus \text{Fil}_{i-1}^{N \triangleright G}$ is a finite union of conjugacy classes. Let $\{(H_i)\}$ be the set of these conjugacy classes. Then the families
\[ \text{Fil}_{i-1}^{N \triangleright G} \subseteq \text{Fil}_i^{N \triangleright G} \cup \{(H_1)\} \subseteq \text{Fil}_{i-1}^{N \triangleright G} \cup \{(H_1),(H_2)\} \subseteq \cdots \subseteq \text{Fil}_i^{N \triangleright G} \]
are all $N$-adjacent.

In any case, a filtration $\emptyset \subseteq F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n$ gives rise to a filtration
\[ * \rightarrow E_{F_0} \wedge X \rightarrow E_{F_1} \wedge X \rightarrow \cdots \rightarrow E_{F_n} \wedge X \simeq X \tag{3.13} \]
of an object $X \in \mathcal{C}$ whenever $\mathcal{C}$ is an $\text{Spc}_G(S)$-module, for example, $\text{Spt}_G(S)$.

To use this filtration, we need to analyze the filtration quotients, which we do in Proposition 3.27 and Proposition 4.12.

### 3.3. Universal spaces for pairs

**Definition 3.14.** Let $F \subseteq F'$ be a subfamily. Define the based motivic $G$-space $E(F',F)$ so that it sits in the cofiber sequence
\[ E_{F_0} \rightarrow E_{F_1} \rightarrow E(F',F). \]
If $F' = \text{Fil}_{all}$, define $\tilde{E}_{F} := E(F_{all},F)$.

Note that $E((F',\emptyset)) \simeq E(F')_+$. At the other extreme, since $E_{F_{all}} \simeq \text{pt}$, the space $\tilde{E}_{F}$ sits in the cofiber sequence
\[ E_{F_+} \rightarrow S^0 \rightarrow \tilde{E}_{F} \tag{3.15} \]
in $\text{Spc}_G(S)$.

**Proposition 3.16.** Let $F \subseteq F'$ be a subfamily. There is a canonical equivalence $E(F',F) \simeq \tilde{E}_{F} \wedge E_{F'}$. In particular, $\tilde{E}_{F} \wedge E_{F_+} \simeq \text{pt}$.

**Proof.** This follows from the commutative diagram
\[
\begin{array}{ccc}
E_{F_+} & \longrightarrow & E_{F'_+} \\
\downarrow & & \downarrow \\
(\tilde{E}_{F} \times E_{F'})_+ & \longrightarrow & (E_{F_{all}} \times E_{F'})_+ \longrightarrow \tilde{E}_{F} \wedge E_{F'_+}
\end{array}
\]
induced by inclusions of families, in which the rows are each cofiber sequence and the left and middle vertical arrows are equivalences.

**Corollary 3.17.** Set $X \in \text{Spc}_G(S)$. The map $X \rightarrow \tilde{E}_{F} \wedge X$ induced by $S^0 \rightarrow \tilde{E}_{F}$ induces an equivalence
\[ \text{Map}_{\text{Spc}_G(S)}(\tilde{E}_{F} \wedge X, \tilde{E}_{F} \wedge Y) \simeq \text{Map}_{\text{Spc}_G(S)}(X, \tilde{E}_{F} \wedge Y). \]
Proof. This follows from the previous proposition, Corollary 3.10, and the fiber sequence $\text{Map}(\tilde{E}F \wedge X, \tilde{E}F \wedge Y) \to \text{Map}(X, \tilde{E}F \wedge Y) \to \text{Map}(E F_+ \wedge X, \tilde{E}F \wedge Y)$. Alternatively, simply note that $E F_+ \otimes -$ is a colocalization and $\tilde{E}F \otimes -$ is a localization endofunctor.

Lemma 3.18. Let $\mathcal{F} \subseteq \mathcal{F}'$ be an inclusion of families and $\mathcal{E}$ a family such that $\mathcal{E} \cap \mathcal{F} = \mathcal{E} \cap \mathcal{F}'$. Then the map $S^0 \to \tilde{E} \mathcal{E}$ induces an equivalence of based motivic $G$-spaces

$$E(\mathcal{F}', \mathcal{F}) \to E(\mathcal{F}', \mathcal{F}) \wedge \tilde{E} \mathcal{E}.$$ 

Proof. Smashing $E \mathcal{E}_+$ with the defining cofiber sequence for $E(\mathcal{F}', \mathcal{F})$ yields the cofiber sequence

$$E F_+ \wedge E \mathcal{E}_+ \to E F_+ \wedge E \mathcal{E}_+ \to E(\mathcal{F}', \mathcal{F}) \wedge E \mathcal{E}_+ .$$

Smashing $E(\mathcal{F}', \mathcal{F})$ with the defining cofiber sequence for $\tilde{E} \mathcal{E}$ yields the cofiber sequence

$$E(\mathcal{F}', \mathcal{F}) \wedge E \mathcal{E}_+ \to E(\mathcal{F}', \mathcal{F}) \overset{i}{\to} E(\mathcal{F}', \mathcal{F}) \wedge \tilde{E} \mathcal{E}.$$ 

By the hypothesis and Proposition 3.21, $f$ is an equivalence, and so $E(\mathcal{F}', \mathcal{F}) \wedge E \mathcal{E}_+$ is contractible. This implies that $i$ is an equivalence.

To continue the analysis, we pass to the $\infty$-categorical stabilization – that is, $S^1$-spectra – of the various categories of motivic $G$-spaces. We write

$$S\text{pt}_{S^1}^{G, \mathcal{E}}(S) := \text{Stab}(S\text{pc}_{S^1}^{G, \mathcal{E}}(S)).$$

Recall that $S\text{pt}_{S^1}^{G, \mathcal{E}}(S)$ can be identified with the category of $\mathbb{A}^1$-invariant Nisnevich sheaves of spectra. We write $\text{Map}_C(X, Y)$ for the spectrum of maps in a stable $\infty$-category $C$.

Let $u : \mathcal{C}_S \subseteq \mathcal{D}_S$ be as in formula (2.24). Then we have induced functors

$$S\text{pt}_{S^1}(\mathcal{C}_S) \xleftarrow{u^*} S\text{pt}_{S^1}(\mathcal{D}_S).$$

Since $u^* : \mathcal{P}(\mathcal{D}_S) \to \mathcal{P}(\mathcal{C}_S)$ is a symmetric monoidal left adjoint, it follows that $S\text{pt}_{S^1}(\mathcal{D}_S) \to S\text{pt}_{S^1}(\mathcal{C}_S)$ is as well. Since $u_* : \mathcal{C}_S \subseteq \mathcal{D}_S$ are full and faithful by Proposition 2.30, it follows that $u_* : S\text{pt}_{S^1}(\mathcal{C}_S) \to S\text{pt}_{S^1}(\mathcal{D}_S)$ are as well.

We introduce a minor technical condition on pairs $\mathcal{F} \subseteq \mathcal{F}'$ over $S$, which we sometimes require:

Condition 3.19. Set $\mathcal{F} \subseteq \mathcal{F}'$. Suppose there is a normal subgroup $N \subseteq G$ such that:

1. $N$ and the elements of $(\mathcal{F}' \cap \mathcal{F}[N]) \setminus \mathcal{F}$ act trivially on $S$ and
2. $\mathcal{F} \subseteq \mathcal{F}' \cap \mathcal{F}[N]$.

We will say that $\mathcal{F}$ satisfies this condition if the pair $\mathcal{F} \subseteq \mathcal{F}$ all satisfies the condition.
Remark 3.20. The pair $\mathcal{F} \subseteq \mathcal{F}'$ satisfies Condition 3.19 in the following two important cases, which cover all of the cases relevant to this paper:

1. The base $S$ has trivial action (in which case we take $G = N$).
2. The normal subgroup $N$ acts trivially on $S$ and $\mathcal{F} = \mathcal{F}' \cap \mathcal{F}[N]$.

Proposition 3.21. Let $\mathcal{F} \subseteq \mathcal{F}'$ be a subfamily satisfying Condition 3.19, $\mathcal{E} = \mathcal{F}' \setminus \mathcal{F}$, and $u : \text{Sm} \normalsize_{G}^{S}[\mathcal{E}] \subseteq \text{Sm} \normalsize_{G}^{S}$ be the inclusion. Let $f : X_1 \to X_2$ be a map in $\text{Sp} \normalsize_{S_{1}}^{G_{E}}(S)$. Suppose that $S$ has trivial action. Then the following are equivalent:

1. The map $f_* : X_1(W) \to X_2(W)$ is an equivalence, for any $W \in \text{Sm} \normalsize_{G}^{S}[\mathcal{E}]$.
2. The map $u^*(f) : u^*(X_1) \to u^*(X_2)$ is an equivalence in $\text{Sp} \normalsize_{S_{1}}^{G_{E}}(S)$.
3. The map

$$E(\mathcal{F}', \mathcal{F}) \otimes X_1 \to E(\mathcal{F}', \mathcal{F}) \otimes X_2$$

is an equivalence in $\text{Sp} \normalsize_{S_{1}}^{G_{E}}(S)$.

Proof. The first two items are immediately equivalent.

To see that (3) implies (2), we have $u^*(E(\mathcal{F}', \mathcal{F}) \otimes X) \simeq u^*(E(\mathcal{F}', \mathcal{F})) \otimes u^*(X)$ and it is straightforward that $u^*(E(\mathcal{F}', \mathcal{F})) \simeq S^0$ in $\text{Sp} \normalsize_{S_{1}}^{G_{E}}(S)$.

Now we show that (2) implies (3). First, we assume that all elements of $\mathcal{E}$ act trivially on $S$. Filter the inclusion $\mathcal{F} \subseteq \mathcal{F}'$ by adjacent families and consider the resulting sequence

$$E\mathcal{F}_+ = E\mathcal{F}_{0+} \to E\mathcal{F}_{1+} \to E\mathcal{F}_{2+} \to \cdots \to E\mathcal{F}_{n-1+} \to E\mathcal{F}_{n+} = E\mathcal{F}_+'$$

An inductive argument shows that it suffices to establish that

$$E(\mathcal{F}_i, \mathcal{F}_{i-1}) \otimes X_1 \to E(\mathcal{F}_i, \mathcal{F}_{i-1}) \otimes X_2$$

is an equivalence for $1 \leq i \leq n$. Thus, we may assume that $\mathcal{F}', \mathcal{F}$ are adjacent and say that $\mathcal{F}' \setminus \mathcal{F} = \{(H)\}$ and $H$ acts trivially on $S$.

By Proposition 3.9 and Proposition 3.16, in order to show that the map displayed here is an equivalence, it suffices to show that

$$\text{Map}_{\text{Sp} \normalsize_{S_{1}}^{G_{E}}(S)}(W_+, E\mathcal{F} \otimes f)$$

is an equivalence for any $p : W \to S$ in $\text{Sm} \normalsize_{G}^{S}[\mathcal{F}']$, with $W$ affine. From the gluing sequence [24, Proposition 5.2], we have the cofiber sequence

$$W(\mathcal{F})_+ \to W_+ \to p_{\#}i_*(W_+^{\mathcal{F}})$$

in $\text{Sp} \normalsize_{S_{1}}^{G_{E}}(S)$, where $i : W^{\mathcal{F}} \subseteq W$ is the inclusion (see Notation 2.7). By Proposition 3.16 we have

$$\text{Map}_{\text{Sp} \normalsize_{S_{1}}^{G_{E}}(S)}(W_+, E\mathcal{F} \otimes f) \simeq \text{Map}_{\text{Sp} \normalsize_{S_{1}}^{G_{E}}(S)}(E\mathcal{F} \otimes W(\mathcal{F})_+, E\mathcal{F} \otimes f) \simeq \text{Map}_{\text{Sp} \normalsize_{S_{1}}^{G_{E}}(S)}(W_+, E\mathcal{F} \otimes f)$$

We have $E\mathcal{F} \otimes W(\mathcal{F})_+ \simeq \text{pt}$, using Corollary 3.10 and Proposition 3.16, and so we conclude that

$$\text{Map}_{\text{Sp} \normalsize_{S_{1}}^{G_{E}}(S)}(p_{\#}i_*(W_+^{\mathcal{F}}), E\mathcal{F} \otimes f) \simeq \text{Map}_{\text{Sp} \normalsize_{S_{1}}^{G_{E}}(S)}(W_+, E\mathcal{F} \otimes f).$$
The canonical map $G \times S(H) W^H \to W^F$ is an isomorphism. Indeed, it suffices to check that $W^H \cap W^gHg^{-1} = \emptyset$ if $H \neq gHg^{-1}$. Now, if $w \in W^H \cap W^gHg^{-1}$, then $\text{Stab}(w)$ contains both of these subgroups. But since $\text{Stab}(w) \in F'$ and $(H)$ is a maximal element of this poset, we have that $H = \text{Stab}(w) = gHg^{-1}$. In particular, $W^F$ is smooth over $S$, since $W^H \to S$ is smooth. It follows that $W^F = W^H$. Indeed, we have $\text{Th}(V) \simeq G_+ \times_{N_H} \text{Th}(V|_{W^H})$, so via the induction-restriction adjunction we can replace $G$ by $NH$ and $F,F'$ by $F|_{NH}, F'|_{NH}$, if necessary. Since $G$ is linearly reductive, we can write $V \simeq V' \oplus V^H$. Since $(V')^H = 0$, all stabilizers of $V' \setminus \{0\}$ are in $F$ and so $\overline{E}F \otimes V' \setminus \{0\} \simeq \ast$, which implies that $\overline{E}F \otimes \text{Th}(V') \simeq \overline{E}F \otimes W^F$.

Now it follows that

$$\text{Map}_{\text{Spt}^G_{S^1}(S)}(\text{Th}(V), \overline{E}F \otimes f) \simeq \text{Map}_{\text{Spt}^G_{S^1}(S)}(\text{Th}(V') \otimes \text{Th}(V^H), \overline{E}F \otimes f)$$

$$\simeq \text{Map}_{\text{Spt}^G_{S^1}(S)}(W^F_+ \otimes \text{Th}(V^H), \overline{E}F \otimes f)$$

$$\simeq \text{Map}_{\text{Spt}^G_{S^1}(S)}(W^F_+ \otimes \text{Th}(V^H), u^*(\overline{E}F \otimes f))$$

$$\simeq \text{Map}_{\text{Spt}^G_{S^1}(S)}(W^F_+ \otimes \text{Th}(V^H), u^*(f)),$$

which is an equivalence, as needed.

Next we consider the case when $F = F' \cap F[N]$, where $N$ is a normal subgroup of $G$ which acts trivially on $S$. Again we consider $p : W \to S$ in $\text{Sm}_{S}[F']$ and the cofiber sequence

$$W(F[N])_+ \to W_+ \to p_\# i_* (W_+^N)$$

in $\text{Spt}^G_{S^1}(S)$, where now $i : W^N \subseteq W$ is the inclusion. Since $W(F[N])_+ \subseteq \text{Sm}_{S}[F]$ we have that $\overline{E}F \otimes W(F[N])_+ \simeq \text{pt}$, and we conclude that

$$\text{Map}_{\text{Spt}^G_{S^1}(S)}(p_\# i_* (W_+^N), \overline{E}F \otimes f) \simeq \text{Map}_{\text{Spt}^G_{S^1}(S)}(W_+, \overline{E}F \otimes f).$$

Since $N$ acts trivially on $S$, $W_+^N \to S$ is again smooth, and so we have $p_\# i_* W_+^N \simeq \text{Th}(N_i)$. A similar argument as in the previous paragraph shows that

$$\text{Map}_{\text{Spt}^G_{S^1}(S)}(\text{Th}(N_i), \overline{E}F \otimes f)$$

is an equivalence.

Now we consider the general case. Let $N$ be the normal subgroup as in Condition 3.19 and consider the inclusions $F \subseteq F'' \subseteq F'$, where we write $F'' = F' \cap F[N]$. From the cofiber sequence

$$\mathbf{E}(F'', F) \to \mathbf{E}(F', F) \to \mathbf{E}(F', F''),$$
we see that it suffices to show that \( E(F'', F) \otimes f \) and \( E(F', F'') \otimes f \) are both equivalences. The case \( F'' \subseteq F' \) is covered by the previous paragraph, and since all elements of \( F'' \setminus F \) act trivially on \( S \), this case is covered by the first.

\[ \square \]

**Corollary 3.22.** Let \( F \subseteq F' \) be a subfamily which satisfies **Condition 3.19**. Write \( E = F' \setminus F \), and \( u : \text{Sm}_S^G[E] \subseteq \text{Sm}_S^G \) the inclusion. Then the map

\[
E(F', F) \otimes u^* X \rightarrow E(F', F) \otimes X
\]

is an equivalence for any \( X \in \text{Spt}_S^G(S) \).

**Proof.** By **Proposition 3.21**, it suffices to show that \( u^* \epsilon : u^*(w_1 u^* X) \rightarrow u^*(X) \) is an equivalence. This follows from the triangle identity for the unit and counit, since the unit \( \eta : \text{id} \simeq u^* u_! \) is an equivalence, as \( u_! \) is full and faithful.

\[ \square \]

### 3.4. Localization at a cofamily

**Proposition 3.23.** Let \( F \) be a family of subgroups which satisfies **Condition 3.19** and write \( j : \text{Sm}_S^G[\text{co}(F)] \subseteq \text{Sm}_S^G \) for the inclusion. Then the natural equivalence \( j^* \left( \tilde{E} \mathcal{F} \otimes - \right) \simeq j^* \) induces an equivalence of localization endofunctors

\[
\tilde{E} \mathcal{F} \otimes - \simeq j_* j^* : \text{Spt}_S^G(S) \rightarrow \text{Spt}_S^G(S).
\]

**Proof.** We have natural equivalences

\[
\text{Map}(W, j_* j^* X) = \text{Map}(j^* W, j^* X) = \text{Map}(j^* W, \tilde{E} \mathcal{F} \otimes X) = \text{Map}(\tilde{E} \mathcal{F} \otimes j^* W, \tilde{E} \mathcal{F} \otimes X) = \text{Map}(\tilde{E} \mathcal{F} \otimes W, \tilde{E} \mathcal{F} \otimes X) = \text{Map}(W, \tilde{E} \mathcal{F} \otimes X),
\]

where the second follows by adjunction and the equivalence \( j^* X \simeq j^* (\tilde{E} \mathcal{F} \otimes X) \), the fourth from **Corollary 3.22**, and the third and fifth from **Corollary 3.17**.

\[ \square \]

Write \( L_{\text{co}(F)} \text{Spt}_T^G(S) \) for the essential image of \( \tilde{E} \mathcal{F} \wedge - : \text{Spt}_T^G(S) \rightarrow \text{Spt}_T^G(S) \). Since \( E \mathcal{F}_+ \wedge - \) is a colocalization endofunctor, \( \tilde{E} \mathcal{F} \wedge - \) is a localization endofunctor. In particular, \( \tilde{E} \mathcal{F} \) is an idempotent object of \( \text{Spt}_T^G(S) \) [29, Proposition 4.8.2.4], and so \( \text{Spt}_T^G(S) \rightarrow L_{\text{co}(F)} \text{Spt}_T^G(S) \) is a symmetric monoidal localization.

We will abuse notation and terminology slightly by saying that a stabilizing set of spheres \( \{ T^E \in \text{Sp}c_{\bullet}^G(\mathcal{F}) \} \) is in \( \text{Sp}c_{\bullet, \text{co}(\mathcal{F})}^G(S) \) if it is in the essential image of \( j_i \). Suppose that \( T \subseteq \text{Sph}_B^G \) is a set of spheres such that \( p^* T \) is in \( \text{Sp}c_{\bullet, \text{co}(\mathcal{F})}^G(S) \). In this case, we can stabilize \( \text{Sp}c_{\bullet, \text{co}(\mathcal{F})}^G(S) \) with respect to \( T \) and we define

\[
\text{Spt}_{_T^*}^{G, \text{co}(\mathcal{F})}(S) := \text{Sp}c_{\bullet, \text{co}(\mathcal{F})}^G(S)[(p^* T)^{-1}].
\]

Equivalently, \( \text{Spt}_{_T^*}^{G, \text{co}(\mathcal{F})}(S) \simeq \text{Spt}_{S_1}^{G, \text{co}(\mathcal{F})}(S)[(p^* T)^{-1}] \).
Proposition 3.24. Let \( \mathcal{F} \) be a family which satisfies Condition 3.19 and write \( j : \text{Sm}_S^G[\text{co}(\mathcal{F})] \subseteq \text{Sm}_S^G \) for the inclusion.

1. The functor \( j_1 : \text{Spt}_{S_1}^{G,\text{co}(\mathcal{F})}(S) \to \text{Spt}_{S_1}^G(S) \) is symmetric monoidal.

2. There is an equivalence \( j_* \simeq \tilde{\mathbb{E}}\mathcal{F} \land j_1 \). Moreover, \( j_* \) induces a symmetric monoidal equivalence \( \tilde{j}_* : \text{Spt}_{S_1}^{G,\text{co}(\mathcal{F})}(S) \to L_{\text{co}(\mathcal{F})}\text{Spt}_{S_1}^G(S) \).

Proof. Note that \( S \in \text{Sm}_S^G[\text{co}(\mathcal{F})] \), so that \( j_1 \) preserves terminal objects. That it is symmetric monoidal then follows from the fact that \( j \) preserves products. Since \( j_1 \) is a fully faithful left adjoint, the unit map \( \text{id} \to j^*j_1 \) is an equivalence. Precomposing \( j_1 \) with the equivalence \( \tilde{\mathbb{E}}\mathcal{F} \land (-) \simeq j_1j^* \) of Proposition 3.23, we obtain equivalences \( \tilde{\mathbb{E}}\mathcal{F} \land j_1(-) \simeq j_*j^*j_1 \) and consequently a commutative diagram

\[
\begin{array}{ccc}
\text{Spt}_{S_1}^{G,\text{co}(\mathcal{F})}(S) & \xrightarrow{j_1} & \text{Spt}_{S_1}^G(S) \\
\downarrow{\tilde{j}_*} & & \downarrow{L_{\text{co}(\mathcal{F})}\text{Spt}_{S_1}^G(S)} \\
& & \\
& & \end{array}
\]

In particular, the functor \( \tilde{j}_* : \text{Spt}_{S_1}^{G,\text{co}(\mathcal{F})}(S) \to L_{\text{co}(\mathcal{F})}\text{Spt}_{S_1}^G(S) \) is a composite of symmetric monoidal functors, hence symmetric monoidal itself. It is fully faithful, since the composite functor \( \text{Spt}_{S_1}^{G,\text{co}(\mathcal{F})}(S) \to L_{\text{co}(\mathcal{F})}\text{Spt}_{S_1}^G(S) \subseteq \text{Spt}_{S_1}^G(S) \) (also denoted \( j_* \)) is fully faithful, by Proposition 2.30. To see that \( j_* \) is also essential surjective, suppose \( X \in \text{Spt}_{S_1}^G(S) \) and consider \( j^*X \in \text{Spt}_{S_1}^{G,\text{co}(\mathcal{F})}(S) \). By the previous equivalence \( \tilde{\mathbb{E}}\mathcal{F} \land j_1(-) \simeq j_* \) and Proposition 3.23, we see that

\[
\tilde{\mathbb{E}}\mathcal{F} \land j_1j^*X \simeq \tilde{\mathbb{E}}\mathcal{F} \land X,
\]

since both are equivalent to \( j_*j^*X \). Thus \( \tilde{j}_* : \text{Spt}_{S_1}^G(S) \to \text{Spt}_{S_1}^{G,\text{co}(\mathcal{F})}(S) \) is an equivalence of symmetric monoidal \( \infty \)-categories. \( \square \)

Proposition 3.25. Let \( \mathcal{F} \) be a family which satisfies Condition 3.19 and write \( j : \text{Sm}_S^G[\text{co}(\mathcal{F})] \subseteq \text{Sm}_S^G \) for the inclusion. Suppose that \( p^*\mathcal{T} \) is in \( \text{Sp}_{\text{co}(\mathcal{F})}(S) \).

1. \( j_1 \) induces a symmetric monoidal functor \( j_1 : \text{Spt}_{\mathcal{T}}^{G,\text{co}(\mathcal{F})}(S) \to \text{Spt}_{\mathcal{T}}^G(S) \).

2. \( \tilde{\mathbb{E}}\mathcal{F} \land j_1 \) induces a symmetric monoidal equivalence

\[
\tilde{\mathbb{E}}\mathcal{F} \land j_1 : \text{Spt}_{\mathcal{T}}^{G,\text{co}(\mathcal{F})}(S) \to L_{\text{co}(\mathcal{F})}\text{Spt}_{\mathcal{T}}^G(S).
\]

Proof. Since \( j_1 \) is a symmetric monoidal left adjoint, it induces a symmetric monoidal functor on \( p^*\mathcal{T} \)-stabilizations.

By Proposition 3.24, the induced map

\[
\text{Spt}_{\mathcal{T}}^{G,\text{co}(\mathcal{F})}(S) \simeq \text{Spt}_{S_1}^{G,\text{co}(\mathcal{F})}(S)[p^*\mathcal{T}^{-1}] \to (L_{\text{co}(\mathcal{F})}\text{Spt}_{S_1}^G(S))[p^*\mathcal{T}^{-1}]
\]

is an equivalence in \( \text{CAlg}(\mathcal{P}_{1^L,8}) \). The result now follows from the equivalence from Lemma 2.40:

\[
(L_{\text{co}(\mathcal{F})}\text{Spt}_{S_1}^G(S))[p^*\mathcal{T}^{-1}] \simeq L_{\text{co}(\mathcal{F})}(\text{Spt}_{S_1}^G(S))[p^*\mathcal{T}^{-1}].
\] \( \square \)
3.5. Adjacent pairs

Lemma 3.26. Suppose that $\mathcal{F} \subseteq \mathcal{F}'$ is $N$-adjacent at $H$ and that $H$ acts trivially on $S$. Set $X \in \text{Sm}_{S}^{G}[\mathcal{F}' \setminus \mathcal{F}]$. Then the canonical map

$$f : G \times_{N_{G}H} X^{H} \to X$$

is an isomorphism.

Proof. The map $f$ is identified with the map $\coprod_{[g] \in G/N_{G}(H)} X^{gHg^{-1}} \to X$ induced by the inclusions $X^{gHg^{-1}} \subseteq X$. We show that the induced map $f_{s}$ on the fiber over any $s \in S$, is an isomorphism. By [18, Corollary 17.9.5], this implies that $f$ is an isomorphism, as it is a map between smooth (in particular flat) finitely presented $S$-schemes. The stabilizer $\text{Stab}(x)$ of any $x \in X_{s}$ contains a subgroup conjugate to $H$, which implies that $\coprod X_{s}^{gHg^{-1}} \to X_{s}$ is surjective. On the other hand, the closed subschemes $X_{s}^{gHg^{-1}} \subseteq X_{s}$ are pairwise disjoint, since all stabilizers of $X_{s}$ are in $\{K \leq G \mid (K \cap N) = (H)\}$. Indeed, if $\text{Stab}(x)$ contains both $H$ and $gHg^{-1}$, $\text{Stab}(x) \cap N$ contains both of these subgroups, which means that $H = gHg^{-1}$. Thus $\coprod X_{s}^{gHg^{-1}} \to X_{s}$ is a bijective, closed immersion. Since these are smooth over $\text{Spec}(k(s))$, in particular reduced, the map $f_{s}$ is an isomorphism.

Let $H \leq N$ be a subgroup. Then $W_{N}H$ is a normal subgroup of $W_{G}H$, and we will often write $WH = W_{G}H/W_{N}H$ for the quotient of Weyl groups. We will often write $E_{W_{N}H}(W_{G}H)$ instead of $E_{\mathcal{F}}(W_{N}H)$ for the universal $W_{N}H$-free motivic $W_{G}H$-motivic space, in order to emphasize the ambient group, where $\mathcal{F}(W_{N}H)$ is the family of subgroups $\{K \leq W_{G}H \mid K \cap W_{N}H = \{e\}\}$.

Proposition 3.27. Suppose that $\mathcal{F} \subseteq \mathcal{F}'$ is $N$-adjacent at $H$. Then the following are true:

1. $(G \times_{N_{G}H} E_{W_{N}H}(W_{G}H))_{+} \wedge E(\mathcal{F}', \mathcal{F}) \to E(\mathcal{F}', \mathcal{F})$ is an equivalence in $\text{Sp}_{S_{1}}^{G}(S)$.
2. $E(\mathcal{F}', \mathcal{F})|_{N_{G}H} \to \overline{\mathcal{F}}[H] \wedge E(\mathcal{F}', \mathcal{F})|_{N_{G}H}$ in $\text{Sp}_{S_{1}}^{N_{G}H}(S)$.

Proof. It suffices to prove these equivalences in the case $S = B$; the general case follows by applying $f^{*}$, where $f : S \to B$ is the structure map. For the first item, by Proposition 3.21 it suffices to show that the projection

$$p : (G \times_{N_{G}H} E_{W_{N}H}(W_{G}H))_{+} \to S^{0}$$

induces equivalences $\text{Map}_{\text{Sp}_{S_{1}}^{G}(B)}(X, p)$, for any $X \in \text{Sm}_{S}^{G}[\mathcal{F}' \setminus \mathcal{F}]$. But for such an $X$ we have, by Lemma 3.26, that $X \cong G \times_{N_{G}H} X^{H}$, so we have

$$\text{Map}_{\text{Sp}_{S_{1}}^{G}(B)}(X_{+}, (G \times_{N_{G}H} E_{W_{N}H}(W_{G}H))_{+})$$

$$\cong \text{Map}_{\text{Sp}_{S_{1}}^{N_{G}H}(B)}(X^{H}_{+}, (G \times_{N_{G}H} E_{W_{N}H}(W_{G}H))_{+})$$

$$\cong \text{Map}_{\text{Sp}_{S_{1}}^{W_{N}H}(B)}(X^{H}_{+}, (G \times_{N_{G}H} E_{W_{N}H}(W_{G}H))_{+}^{H})$$

$$\cong \text{Map}_{\text{Sp}_{S_{1}}^{W_{G}H}(B)}(X^{H}_{+}, (S^{0})^{H})$$
\begin{equation}
\simeq \text{Map}_{\text{Sp}\mathbb{S}^1_G}(B, X^H) \simeq \text{Map}_{\text{Sp}\mathbb{S}^1_G}(B, (G \times N_G H X^H) +, S^0).
\end{equation}

For the second item, we have that $\mathcal{F}_{B} \cap \mathcal{F}_{N_G} = \mathcal{F}_{B} \cap \mathcal{F}_{N_G}$, and so this follows from Lemma 3.18.

4. Fixed-point functors

We define fixed-point functors on motivic spaces and spectra. Throughout this section, $N \trianglelefteq G$ is a normal subgroup, and we write $\pi: G \to G/N$ for the quotient homomorphism. Unless noted otherwise, we assume that $N$ acts trivially on $S$.

4.1. Fixed-point motivic spaces

We begin by extending the adjunction

\begin{equation}
\pi^{-1}: \text{Sm}_S^{G/N} \simeq \text{Sm}_S^G: (-)^N
\end{equation}

to motivic $G$-spaces. Note that $\pi^{-1}$ is full and faithful and induces an equivalence

\begin{equation}
\text{Sm}_S^{G/N} \simeq \text{Sm}_S^G[\text{co}((\mathcal{F}[N])] \in \text{Sm}_S^G
\end{equation}

with the subcategory of smooth $G$-schemes over $S$ on which $N$ acts trivially.

The functor $\pi_*$ in the following propositions is the $N$-fixed-point functor on motivic $G$-spaces. In keeping with standard notation, we sometimes write $(\cdot)^N:= \pi_*$. 

**Proposition 4.2.** The functor $\pi^{-1}: \text{Sm}_S^{G/N} \to \text{Sm}_S^G$ induces adjoint pairs of functors

$\pi^*: \text{Sp}^{G/N}(S) \simeq \text{Sp}_*^G(S): \pi_*$

$\pi^*: \text{Sp}^{G/N}_*(S) \simeq \text{Sp}_*^G(S): \pi_*$

such that the following diagrams commute:

\[
\begin{array}{ccc}
\text{Sm}_S^{G/N} & \xrightarrow{\pi^{-1}} & \text{Sm}_S^G \\
\downarrow & & \downarrow \\
\text{Sp}^{G/N}_*(S) & \xrightarrow{\pi^*} & \text{Sp}_*^G(S) \\
(-)_* & \downarrow & (-)_* \\
\text{Sp}^{G/N}_*(S) & \xrightarrow{\pi^*} & \text{Sp}_*^G(S)
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\text{Sm}_S^G & \xrightarrow{(-)^N} & \text{Sm}_S^{G/N} \\
\downarrow & & \downarrow \\
\text{Sp}_*^G(S) & \xrightarrow{\pi_*} & \text{Sp}_*^{G/N}(S) \\
(-)_* & \downarrow & (-)_* \\
\text{Sp}_*^G(S) & \xrightarrow{\pi_*} & \text{Sp}_*^{G/N}(S).
\end{array}
\]

Moreover, $\pi^*$ and $\pi_*$ are symmetric monoidal, and $\pi_*$ preserves colimits.

**Proof.** Nisnevich topologies correspond under the equivalence (4.1), so the adjunction can be obtained as a special case of Proposition 2.28, where we set $\pi^* := (\pi^{-1})_*$ and $\pi_* := (\pi^{-1})_*$. The remaining claims about $\pi^*$ and $\pi_*$ are straightforward. $\square$
4.2. Fixed-point spectra

We view $G/N$-equivariant vector bundles on $S$ as $G$-equivariant vector bundles via the quotient homomorphism $\pi : G \to G/N$. Given a stabilizing subset $T \subseteq \text{Sph}^G_B$, we have the stabilizing subset

$$\pi^* T = \{ T\pi^* \mathcal{E} \mid \mathcal{E} \in T \} \subseteq \text{Sph}^G_B,$$

which for simplicity we usually write again as $T$. We call a $G$-sphere of the form $\pi^* (T\mathcal{E})$ an $N$-trivial $G$-sphere. Recall also that we write $N$-triv $= \pi^* (\text{Sph}^G_B)$.

**Proposition 4.3.** Let $T \subseteq \text{Sph}^G_B$ be a stabilizing set of spheres. There is an adjoint pair of symmetric monoidal functors

$$\pi^* : \text{Spt}^{G/N}_T (S) \rightleftarrows \text{Spt}^G_T (S) : \pi_*$$

such that $\pi_* (\Sigma^\infty_T Y) \simeq \Sigma^\infty_T (Y^N)$ for $Y \in \text{Sp}_G(S)$.

**Proof.** The adjoint pair is the stabilization of the adjoint pair in Proposition 4.2, using $\pi^* (X \otimes T\mathcal{E}) \simeq \pi^* (X) \otimes \Sigma^\infty \mathcal{E}$ and $(Y \otimes \Sigma^\infty \mathcal{E})^N \simeq Y^N \otimes T\mathcal{E}$.

That the fixed-point functor in this proposition commutes with stabilization is a consequence of the fact that we have only stabilized with respect to a set of $N$-trivial spheres. In general, fixed-point functors do not commute with stabilization; rather, this is a key feature of the geometric fixed-points functor, defined later.

**Lemma 4.4.** Set $T \subseteq \text{Sph}^G_B$. The equivalence (4.1) induces inverse equivalences

$$\tilde{\pi}^* : \text{Spt}^{G/N}_T (S) \simeq \text{Spt}^{G, co(F[N])}_T (S) : \tilde{\pi}_*$$

which fit into commutative diagrams

$$\xymatrix{ \text{Spt}^{G/N}_T (S) \ar[rr]^{\tilde{\pi}^*} \ar[dr]^\pi & & \text{Spt}^{G, co(F[N])}_T (S) \ar[dl]_{\pi_*} \ar[dd]^{j_!} \ar@/_2pc/[ddll]_{\tilde{\pi}_*} \ar@/^2pc/[ddll]_{\pi_*} \\
\text{Spt}^G_T (S) & \text{Spt}^{G, co(F[N])}_T (S) \ar[ll]_{j_!} & \text{Spt}^G (S) \\
& \text{Spt}^G (S) }$$

**Proof.** Nisnevich topologies correspond under the equivalence (4.1), so that $\pi^* : \text{Sp}^{G/N}_T (S) \simeq \text{Sp}^{G, co(F[N])}_T (S)$ is an equivalence of symmetric monoidal $\infty$-categories with inverse $\pi_*$. The first statement follows. The commutativity of the displayed diagrams is straightforward to check.

Let $T \subseteq \text{Sph}^G_B$ be a stabilizing set of $G$-spheres and $T^N = \{ T\mathcal{E}^N \mid \mathcal{E} \in T \}$ be the associated stabilizing set of $G/N$-spheres. Continuing to overload notation, we simply write $\pi^* : \text{Spt}^{G/N}_T (S) \to \text{Spt}^G_T (S)$ again for composite

$$\xymatrix{ \text{Spt}^{G/N}_T (S) \ar[r]^{\pi^*} & \text{Spt}^{G/N}_T (S) \ar[r]^{\pi^*} & \text{Spt}^G_T (S). }$$
Since $\pi^*$ preserves colimits, we obtain the fixed-points adjunction

$$\pi^*: \Spt_{G/N}^G(S) \rightleftarrows \Spt_G^G(S): \pi_*.$$  

The stabilizing set of spheres $\mathcal{T}$ does not appear in the notation for the fixed-points functor $\pi_*$. We usually consider stabilization with respect to all spheres, and will always make the domain of $\pi_*$ explicit in other cases. When $\mathcal{T} = \text{Sph}_B^G$, in keeping with standard notation, we will sometimes write $(-)^N := \pi_*$.  

We also have not made reference to the base scheme in the notation. We show in the next section, after proving the Adams isomorphism, that the fixed-points functor is compatible with the various change-of-base functors in motivic homotopy.

### 4.3. Geometric fixed points

Recall that $\mathcal{F}[N] := \{H \leq G \mid N \not\subseteq H\}$ and that we write

$$L_{\co(\mathcal{F}[N])}\Spt_{\mathcal{T}}(S) \subseteq \Spt_{\mathcal{T}}(S)$$

for the essential image of the endofunctor $\tilde{\mathcal{E}}\mathcal{F}[N] \otimes - : \Spt_{\mathcal{T}}(S) \to \Spt_{\mathcal{T}}(S)$. Write $W = \rho_G/\rho_G/N$. By Example 3.6, we have

$$\tilde{\mathcal{E}}\mathcal{F}[N] \simeq \colim_n T^nW.$$  

(4.5)

**Lemma 4.6.** The map $S^0 \to T^W$ induces an equivalence

$$(\tilde{\mathcal{E}}\mathcal{F}[N]) \simeq T^W \otimes (\tilde{\mathcal{E}}\mathcal{F}[N])$$

in $\Spc^G(S)$.

**Proof.** This follows from formula (4.5), together with the fact that the cyclic permutation acts as the identity on $T^W \wedge T^W \wedge T^W$ [24, Lemma 6.3]. □

**Proposition 4.7.** The stabilization functor $\Spt_{N-\text{triv}}^G(S) \to \Spt^G(S)$ induces an equivalence

$$L_{\co(\mathcal{F}[N])}\Spt_{N-\text{triv}}^G(S) \rightleftarrows L_{\co(\mathcal{F}[N])}\Spt^G(S)$$

of symmetric monoidal stable $\infty$-categories.

**Proof.** Write $L = L_{\co(\mathcal{F}[N])}$. We make use of the equivalences $\Spt_{T^\rho G/N}^G(S) \simeq \Spt_{N-\text{triv}}^G(S)$ and $\Spt_{T^\rho G}^G(S) \simeq \Spt^G(S)$ and the equivalence (4.5). Since $T^{\rho_G} = T^{\rho_G/N} \otimes T^W$, we have an equivalence

$$\Spt^G(S) \simeq \Spt_{N-\text{triv}}^G(S) \left[(T^W)^{-1}\right],$$

and under this equivalence, $L\Spt^G(S) \simeq \left[L\Spt_{N-\text{triv}}^G(S)\right] \left[(T^W)^{-1}\right]$ by Lemma 2.40. But it follows from Lemma 4.6 that $\Sigma^W$ is an autoequivalence of $L\Spt_{N-\text{triv}}^G(S)$, and therefore
the stabilization induces an equivalence
\[ \mathrm{LSp}^G_{N-\text{triv}}(S) \simeq (\mathrm{LSp}^G_{N-\text{triv}}(S))[T^W]^{-1}. \]

The result follows, since \((\mathrm{LSp}^G_{N-\text{triv}}(S))[T^W]^{-1} \simeq \mathrm{L}(\mathrm{Sp}^G_{N-\text{triv}}(S))[T^W]^{-1}\) \(\square\).

Write \(\psi\) for the composite
\[ \mathrm{Sp}^{G/N}(S) \xrightarrow{\pi^*} \mathrm{Sp}^G_{N-\text{triv}}(S) \to \mathrm{Sp}^G(S) \to \mathrm{L}_{\text{co}(F[N])} \mathrm{Sp}^G(S). \]

**Proposition 4.8.** The functor \(\psi: \mathrm{Sp}^{G/N}(S) \to \mathrm{L}_{\text{co}(F[N])} \mathrm{Sp}^G(S)\) is an equivalence of symmetric monoidal stable \(\infty\)-categories.

**Proof.** By Proposition 4.7, this composite is equivalent to
\[ \mathrm{Sp}^{G/N}(S) \xrightarrow{\pi^*} \mathrm{Sp}^G_{N-\text{triv}}(S) \xrightarrow{\tilde{\mathcal{E}}F[N]\otimes^\pi -} \mathrm{L}_{\text{co}(F[N])} \mathrm{Sp}^G_{N-\text{triv}}(S), \]
which is an equivalence by Lemma 4.4 and Proposition 3.25. \(\square\)

**Definition 4.9.** Let \(N \triangleleft G\) be a normal subgroup. The **motivic geometric fixed-points functor**
\[ (-)^{\Phi_N}: \mathrm{Sp}^G(S) \to \mathrm{Sp}^{G/N}(S) \]
is the composite \(\mathrm{Sp}^G(S) \to \mathrm{L}_{\text{co}(F[N])} \mathrm{Sp}^G(S) \xrightarrow{\psi^{-1}} \mathrm{Sp}^{G/N}(S)\).

**Proposition 4.10.** The functor \((-)^{\Phi_N}: \mathrm{Sp}^G(S) \to \mathrm{Sp}^{G/N}(S)\) satisfies the following properties:

1. It is a symmetric monoidal left adjoint; moreover, its right adjoint \(\tilde{\mathcal{E}}\mathcal{F}[N] \otimes -\) is full and faithful.
2. There is a natural equivalence \((\Sigma^\infty X)^{\Phi_N} \simeq \Sigma^\infty (X^N)\) for \(X \in \mathrm{Sp}^G(S)\).
3. \((-)^{\Phi_N} \simeq (\tilde{\mathcal{E}}\mathcal{F}[N] \otimes -)^N\).
4. There is a natural transformation \((-)^N \to (-)^{\Phi_N}\).
5. There is a natural equivalence \(\tilde{\mathcal{E}}\mathcal{F}[N] \otimes Y \simeq \tilde{\mathcal{E}}\mathcal{F}[N] \otimes Y^{\Phi_N}\) for \(Y \in \mathrm{Sp}^G(S)\).

**Proof.** This is an easy consequence of the foregoing results. \(\square\)

In §6.2 and §6.4, we show that fixed points commute with arbitrary base change. We note here the easier fact that geometric fixed points commute with arbitrary base change.

**Proposition 4.11.** Let \(p: T \to S\) be a map in \(\text{Sch}^{G/N}_B\). There is a natural equivalence \(p^*(Y^{\Phi_N}) \simeq (p^*Y)^{\Phi_N}\).
Proof. Since $p^* \tilde{E}[N]_S \cong \tilde{E}[N]_T$, we have a commutative diagram

$$
\begin{array}{ccc}
S\text{pt}^{G/N}(S) & \xrightarrow{\pi^*} & S\text{pt}^G(S) \\
\downarrow p^* & & \downarrow p^* \\
S\text{pt}^{G/N}(T) & \xrightarrow{\pi^*} & S\text{pt}^G(T)
\end{array}
$$

so that $p^* \psi \simeq \psi p^*$, which implies the result.

Recall that for a subgroup $H \leq N$, we write $W_H = W_G/H/W_N H$ for the quotient of Weyl groups. We also write $E_{W_N H}(W_G H)$ for the universal $W_N H$-free $W_G H$-motivic space (also denoted $E_{\mathcal{F}}(W_N H)$). See §3.2 for a recollection of $N$-adjacency.

Proposition 4.12. Suppose that $F \subseteq F'$ is $N$-adjacent at the subgroup $H \leq N$. Then for $X \in S\text{pt}^G(S)$, there is a natural equivalence

$$(E(F',F) \otimes X)^N \simeq G/N \cdot \Phi^H \left( E_{W_N H}(W_G H) \otimes X^H \right)^{W_N H}.$$ 

Proof. Consider the commutative diagram of group homomorphisms

$$
\begin{array}{ccc}
N_G H & \xleftarrow{\lambda} & G \\
\xrightarrow{\pi'} & & \xrightarrow{\pi} \\
W_G H & \xrightarrow{\pi''} & W \xleftarrow{\kappa} G/N.
\end{array}
$$

There is a natural equivalence $\pi'_* \lambda! \simeq \bar{\lambda}! \pi''_*$, since both are seen to be right adjoint to the restriction along $N_G H \to G/N$. By Proposition 3.27, the canonical maps

$$
\begin{array}{ccc}
\lambda_! \left( E_{W_N H}(W_G H)_+ \otimes \lambda^* \left( E(F',F) \otimes X \right) \right) & \xrightarrow{\sim} & E(F',F) \otimes X \\
\downarrow \sim & & \\
\lambda_! \left( E_{W_N H}(W_G H)_+ \otimes \tilde{E}[H] \otimes \lambda^* \left( E(F',F) \otimes X \right) \right)
\end{array}
$$

are equivalences. Applying $\pi'_*$, the result follows from the equivalence given and the fact that $(-)^\Phi$ is symmetric monoidal.

4.4. Homotopy fixed points and the Tate construction

Homotopy fixed points together with the Tate spectrum, introduced in [17], play critical roles in computations and applications of equivariant homotopy theory. Using the constructions we have already reviewed, it is straightforward to define the motivic version of these functors and establish the analogue of the Tate diagram for $G = C_p$.

Definition 4.13. Set $X \in S\text{pt}^G(S)$ and $N \leq G$ a normal subgroup.

1. The motivic homotopy fixed-point spectrum of $X$ is

$$
X^{hN} := \pi_* i_* i^*(X) \cong F(E\mathcal{F}(N)_+,X)^N,
$$

where $i: Sm_{G,N\text{-free}}^S \subseteq Sm_G^S$ is the inclusion.
2. The motivic Tate spectrum of \( X \) is
\[
X^{tN} := (\tilde{E}F(N) \wedge F(EF(N), X))^N.
\]

**Proposition 4.14** (motivic Tate diagram). Suppose that \( p \) is invertible on \( B \). Set \( X \in \text{Spt}^{C_p}(B) \). There is a natural push-out square
\[
\begin{array}{ccc}
X^{C_p} & \longrightarrow & X^{\Phi C_p} \\
\downarrow & & \downarrow \\
X^{hC_p} & \longrightarrow & X^{tC_p}
\end{array}
\]
in \( \text{Spt}(B) \).

**Proof.** Consider the following commutative diagram with exact rows:
\[
\begin{array}{cccc}
(E_{C_p+} \wedge X)^{C_p} & \longrightarrow & X^{C_p} & \longrightarrow & (\tilde{E}C_p \wedge X)^{C_p} \\
\downarrow & & \downarrow & & \downarrow \\
(E_{C_p+} \wedge F(E_{C_p+}, X))^{C_p} & \longrightarrow & F(E_{C_p+}, X)^{C_p} & \longrightarrow & (\tilde{E}C_p \wedge F(E_{C_p+}, X))^{C_p}.
\end{array}
\]

The proposition follows by noting that the left vertical map is an equivalence. To see this, note that by **Proposition 3.11**, the map \( X \to F(E_{C_p+}, X) \) is identified with the natural transformation \( i_i^* \eta : i_i^* i_i^* \to i_i^* i_i^* \), where \( \eta \) is the unit \( \text{id} \to i_i^* \) of the adjunction. Since \( i_i^* \) is full and faithful, so is \( i_i^* \). In particular, the counit \( i_i^* i_i^* \to \text{id} \) is an equivalence. That \( i_i^* \eta \) is an equivalence now follows from the triangle identity.

5. Quotient spectra

As in the previous section, we let \( N \leq G \) be a normal subgroup and \( \pi : G \to G/N \) the quotient homomorphism. It turns out (as in classical equivariant homotopy) that the functor \( \pi^* : \text{Spt}^{G/N}(S) \to \text{Spt}^G(S) \) does not have a left adjoint, except in the trivial case when \( G = \{ e \} \). It does, however, have a partial left adjoint, constructed in this section, defined on the full subcategory of \( N \)-free \( G \)-spectra.

5.1. Stabilization of free objects

We write \( \text{Spt}^{G,N\text{-free}}_T(S) := \text{Spt}^{G,F(N)}_T(S) \). In this section, we show that the stabilization
\[
\lambda^* : \text{Spt}^{G,N\text{-free}}_{N\text{-triv}}(S) \to \text{Spt}^{G,N\text{-free}}(S)
\]
is an equivalence of stable \( \infty \)-categories. We are grateful to Tom Bachmann for suggestions which streamlined the argument we originally gave here.

Recall that the quotient functor induces an equivalence of categories \((-)/N : \text{Sm}^G_S \simeq \text{Sm}^{G/N}_{S/N} \) if \( N \) acts freely on \( S \). Under this equivalence, Nisnevich topologies correspond, so we obtain an equivalence of symmetric monoidal \( \infty \)-categories
\[
\text{Spc}^G(S) \simeq \text{Spc}^{G/N}(S/N).
\]
Lastly, stabilizing with respect to $\text{Sph}^{G/N}_{S/N}$, we obtain an equivalence of symmetric monoidal $\infty$-categories

$$\pi_! : \text{Spt}^{G}_{N,\text{triv}}(S) \sim \text{Spt}^{G/N}(S/N).$$

**Lemma 5.1.** If $N$ acts freely on $S$, then $\lambda^* : \text{Spt}^{G,N,\text{free}}_{N,\text{triv}}(S) \rightarrow \text{Spt}^{G,N,\text{free}}(S)$ is an equivalence.

**Proof.** It suffices to see that if $E \rightarrow S$ is an equivariant vector bundle on $S$, then $T_E$ is invertible in $\text{Spt}^{G,N,\text{free}}_{N,\text{triv}}(S) \simeq \text{Spt}^{G,N,\text{free}}_{N,\text{triv}}(S)$. Under the equivalence $\pi_! : \text{Spt}^{G}_{N,\text{triv}}(S) \simeq \text{Spt}^{G/N}(S/N)$ of symmetric monoidal $\infty$-categories, we have that $\pi_!(T_E) \simeq T_{E/N}$, where $E/N \rightarrow S/N$ is the quotient, which is a $G/N$-equivariant vector bundle. In particular, $T_{E/N}$ is invertible, so the result follows.

**Theorem 5.2.** Let $N \triangleleft G$ be a normal subgroup. The stabilization functor

$$\lambda^* : \text{Spt}^{G,N,\text{free}}_{N,\text{triv}}(S) \rightarrow \text{Spt}^{G,N,\text{free}}(S)$$

is an equivalence of stable $\infty$-categories.

**Proof.** Write $\lambda_*$ for the right adjoint of $\lambda^*$. We first show that $\text{id} \rightarrow \lambda_* \lambda^*$ is an equivalence – that is, $\lambda^*$ is fully faithful. By Proposition 2.36 it suffices to check that $p^* \rightarrow p_* \lambda_* \lambda^*$ is an equivalence for any $p : X \rightarrow S$ in $\text{Sm}_{S}^{G,N,\text{free}}$ (indeed, the family of such $p^*$ is conservative). But $\lambda^*$ commutes with $p^*$, and since $p$ is smooth, so does $\lambda_*$. Therefore this transformation is identified with $p^* \rightarrow \lambda_* \lambda^* p^*$, and since $X$ is $N$-free, $\lambda_* \lambda^* p^* \simeq p^*$ by Lemma 5.1, as required.

To see that $\lambda^*$ is essentially surjective, by Proposition 2.36 it suffices to show that $T^{-V} \otimes X_+$ is in the image of $\lambda^*$, where $T^{-V} \in \text{Sph}_{G}^{G}$ and $p : X \rightarrow S$ is in $\text{Sm}_{S}^{G,N,\text{free}}$. But $T^{-V} \otimes X_+ \simeq p_{\#} p^*(T^{-V})$, and by Lemma 5.1 we have that $p^*(T^{-V})$ is in the essential image of $\lambda^*$; since $\lambda^*$ and $p_{\#}$ commute, we are done.

**Remark 5.3.** It follows that when $S$ has $N$-free action, there is an induced equivalence of symmetric monoidal $\infty$-categories

$$\pi_! : \text{Spt}^{G}(S) \simeq \text{Spt}^{G/N}(S/N).$$

Let $p : X \rightarrow S$ be a map. The exceptional push-forward on $N$-free $G$-spectra $p_! : \text{Spt}^{G,N,\text{free}}(X) \rightarrow \text{Spt}^{G,N,\text{free}}(S)$ can be defined as the composite

$$i^* p^! : \text{Spt}^{G,N,\text{free}}(X) \rightarrow \text{Spt}^{G}(X) \rightarrow \text{Spt}^{G}(S) \rightarrow \text{Spt}^{G,N,\text{free}}(S).$$

If $p$ is smooth, then $p_! \simeq p_{\#} \sum_{\Omega_f}$, where $\Omega_f$ is the sheaf of differentials of $X$ over $S$.

**Corollary 5.4.** The stable $\infty$-category $\text{Spt}^{G,N,\text{free}}(S)$ is generated under colimits by any of the following sets:

1. $\Sigma^{-k_{\rho_{G/N}}} p_{\#} \mathbb{1}_{X}$, where $k \geq 0$ and $p : X \rightarrow S$ is in $\text{Sm}_{S}^{G,N,\text{free}}$ with $X$ affine.
2. $\Sigma^{-k_{\rho_{G/N}}} p_{\#} \mathbb{1}_{X}$, where $k \geq 0$ and $p : X \rightarrow S$ is in $\text{Sm}_{S}^{G,N,\text{free}}$ with $X$ affine.
3. $\Sigma^{-k\rho_{G/N}}q_*\mathbb{1}_X$, where $k \geq 0$ and $q: X \to S$ is in $\text{Sch}_S^G$ with $q$ projective.

4. $\Sigma^{-k\rho_{G/N}}q_*\mathbb{1}_X$, where $k \geq 0$ and $q: X \to S$ is in $\text{Sch}_S^{G,N}$.

**Proof.** The objects in (1) are generators of $\text{Spt}^{G,N}_{\text{free}}(S)$, by Proposition 2.36.

If $X$ is affine, then there is a surjection $p^*(n\rho_G) \to \Omega_p$ for some $n > 0$. Let $\mathcal{E}$ be the kernel of this map. Then in $\text{Spt}^{G,N}_{\text{free}}(X)$ we have an equivalence $T^{\Omega_p} \wedge T^\mathcal{E} \simeq T^{n\rho_{G/N}}$. Let $r: V(\mathcal{E}) \to X$ be the projection. We then have

$$\Sigma^{n\rho_{G/N}}(p \circ r): (\mathbb{1}_{V(\mathcal{E})}) \simeq p_\# \mathbb{1}_X.$$ 

Therefore the generators in (1) are contained in the category generated under colimits by the objects in (2).

Next, we check that the set in (2) is contained in the category generated under colimits by the objects $\Sigma^{-k\rho_{G/N}}q_*\mathbb{1}_X$ of (3).

Let $p: Y \to S$ be in $\text{Sm}_S^{G,N}$-free. Since $p$ is $G$-quasi-projective, there is an equivariant compactification

$$Y \xleftarrow{u} \bar{Y} \xleftarrow{t} Z \xrightarrow{p} S,$$

where $f$ is an equivariant projective morphism, $u$ is an invariant open, and $t$ is an invariant closed complement. From the gluing sequence, using the fact that $f,g$ are proper, we obtain an exact sequence in $\text{Spt}^G(S)$ of the form

$$\Sigma^{-k\rho_{G/N}}p_\# \mathbb{1}_Y \to \Sigma^{-k\rho_{G/N}}f_*\mathbb{1}_{\bar{Y}} \to \Sigma^{-k\rho_{G/N}}g_*\mathbb{1}_Z.$$ 

Applying $i^*$ finishes the third case.

Write $\mathcal{E}F(N) \simeq \text{colim}_n U_n$, where $U_n \in \text{Sm}_S^{G,N}$-free. Now (4) follows by noting that if $q: X \to S$ is projective, then $i^*q_*(\mathbb{1}_X)$ is the colimit of $q_{n*}(\mathbb{1}_{X \times U_n})$, where $q_n: X \times U_n \to S$.

5.2. Quotient functors

**Proposition 5.5.** Let $N \trianglelefteq G$ be a normal subgroup which acts trivially on $S$.

1. There are colimit-preserving functors

$$(-)/N: \text{Spc}^{G,N}_{\text{free}}(S) \to \text{Spc}^{G/N}(S/N)$$

$$(-)/N: \text{Spc}^{G,N}_{\bullet}(S) \to \text{Spc}^{G/N}_{\bullet}(S/N)$$
such that the following diagram commutes:

\[
\begin{array}{ccc}
\text{Sm}_{S}^{G,N\text{-free}} & \xrightarrow{(-)/N} & \text{Sm}_{S/N}^{G/N} \\
\downarrow & & \downarrow \\
\text{Sp}_{S}^{G,N\text{-free}}(S) & \xrightarrow{(-)/N} & \text{Sp}_{S/N}^{G/N}(S/N) \\
\downarrow & & \downarrow \\
\text{Sp}_{S}^{G,N\text{-free}}(S) & \xrightarrow{(-)/N} & \text{Sp}_{S/N}^{G/N}(S/N). \\
\end{array}
\]

2. The functor \((-)/N\) satisfies a projection formula \((X \times Y)/N \simeq (X/N) \times Y\) for \(X \in \text{Sp}_{S}^{G,N\text{-free}}(S)\) and \(Y \in \text{Sp}_{S}^{G/N}(S/N)\). Similarly, if \(A,B\) are based, then \((A \wedge \pi^{*}B)/N \simeq (A/N) \wedge B\).

**Proof.** Write \(q: \text{Sm}_{S}^{G} \to \text{Sm}_{S/N}^{G/N}\) for the quotient functor, \(q(W) = W/N\). Since \(q\) sends Nisnevich squares in \(\text{Sm}_{S}^{G,N\text{-free}}\) to Nisnevich squares in \(\text{Sm}_{S}^{G/N}\) and

\[
q(W \times \mathbb{A}^{1}) \simeq q(W) \times \mathbb{A}^{1},
\]

the functor \(q^{*}: \mathcal{P}(\text{Sm}_{S/N}^{G/N}) \to \mathcal{P}(\text{Sm}_{S}^{G})\) defined by precomposition restricts to a functor \(q^{*}: \text{Sp}_{S}^{G,N\text{-free}}(S/N) \to \text{Sp}_{S}^{G,N\text{-free}}(S)\). Since \(q_{*}\) preserves limits, it admits a left adjoint

\[
(-)/N: \text{Sp}_{S}^{G,N\text{-free}}(S) \to \text{Sp}_{S/N}^{G/N}(S/N)
\]

with the stated properties. Similarly, since \(q_{*}\) preserves the terminal object, it induces a limit-preserving functor \(q_{*}: \text{Sp}_{S/N}^{G,N}(S/N) \to \text{Sp}_{S}^{G,N\text{-free}}(S)\) on based spaces and therefore admits a left adjoint \((-)/N: \text{Sp}_{S}^{G,N\text{-free}}(S) \to \text{Sp}_{S/N}^{G/N}(S/N)\). The last statement follows from Proposition 2.2. \(\square\)

If \(X \in \text{Sm}_{S}^{G}\) does not have free action, the scheme \(X/G\) need not be smooth. It is still possible to define a quotient functor on motivic \(G\)-spaces. However, this functor does not stabilize to give a quotient functor on all of \(\text{Spt}^{G}(S)\).

**Proposition 5.6.** Let \(N \trianglelefteq G\) be a normal subgroup which acts trivially on \(S\).

1. There is a colimit-preserving functor

\[
\pi_{!}: \text{Spt}_{S}^{G,N\text{-free}}(S) \to \text{Spt}_{S/N}^{G/N}(S/N)
\]

such that the following diagram commutes:

\[
\begin{array}{ccc}
\text{Sm}_{S}^{G,N\text{-free}} & \xrightarrow{\pi_{!}} & \text{Sm}_{S/N}^{G/N} \\
\downarrow & & \downarrow \\
\text{Spt}_{S}^{G,N\text{-free}}(S) & \xrightarrow{\pi_{!}} & \text{Spt}_{S/N}^{G/N}(S/N). \\
\end{array}
\]

The right adjoint of \(\pi_{!}\) is the composite \(i^{*}\pi^{*}\), where \(i: \text{Sm}_{S}^{G,N\text{-free}} \subseteq \text{Sm}_{S}^{G}\) is the inclusion.
2. The functor \( \pi_! \) satisfies a projection formula. That is, if \( X \in \text{Spt}^{G,N\text{-free}}(S) \) and \( Y \in \text{Spt}^{G/N}(S/N) \), then \( \pi_!(X \otimes \pi^*Y) \simeq \pi_!(X) \otimes Y \).

**Proof.** If \( V \to S \) is a \( G/N \)-equivariant vector bundle, we have \( (\Sigma^V X)/N \simeq \Sigma^V X/N \) by Proposition 5.5. It follows that \((-)/N\) extends to a colimit-preserving functor

\[
\pi_! : \text{Spt}_{N\text{-triv}}^{G,N\text{-free}}(S) \to \text{Spt}^{G/N}(S).
\]

The first statement then follows from Theorem 5.2, and the second statement from Proposition 5.5. \( \square \)

6. The motivic Adams isomorphism

In classical homotopy, the Adams isomorphism identifies the \( N \)-fixed points of an \( N \)-free \( G \)-spectrum \( X \) with the quotient of \( X \) by the \( N \)-action. This equivalence was established by Adams for \( N = G \) in [1, Theorem 5.3] and generalized in [27, Theorem II.7.1]. A recent modern take on this, in terms of orthogonal spectra, appears in [33]. In this section, we establish a version for \( N \)-free motivic \( G \)-spectra.

6.1. The Adams transformation

For the remainder of this section we suppose that \( S \) has trivial \( N \)-action and we let \( \pi : G \to G/N \) denote the quotient homomorphism. As before, we write \( i : \text{Sm}_S^{G,N\text{-free}} \subseteq \text{Sm}_S^G \) for the inclusion. The Adams isomorphism is a comparison of the two functors

\[
\pi_!, \pi_* i_! : \text{Spt}^{G,N\text{-free}}(S) \to \text{Spt}^{G/N}(S).
\]

We first construct a comparison transformation \( \tau : \pi_! \to \pi_* i_! \). Consider the following Cartesian square of surjective homomorphisms:

\[
\begin{array}{ccc}
G \times_{G/N} G & \xrightarrow{pr_1} & G \\
\downarrow{pr_2} & & \downarrow{\pi} \\
G & \xrightarrow{\pi} & G/N.
\end{array}
\]  

(6.1)

Write \( G' = G \times_{G/N} G \) and \( N' = \ker(pr_2) \). Observe that if \( X \in \text{Sm}_S^{G,N\text{-free}} \), then \( pr_1^*X \) is in \( \text{Sm}_S^{G',N'\text{-free}} \). We have a commutative diagram

\[
\begin{array}{ccc}
\text{Spt}^{G,N\text{-free}}(S) & \xrightarrow{i_!} & \text{Spt}^G(S) \\
\downarrow{j^*pr_1^*i_*} & & \downarrow{pr_*} \\
\text{Spt}^{G',N'\text{-free}}(S) & \xrightarrow{j_!} & \text{Spt}^{G'}(S)
\end{array}
\]

of colimit-preserving functors, where \( j : \text{Sm}_S^{G',N'\text{-free}} \subseteq \text{Sm}_S^{G'} \) is the inclusion.
Proposition 6.2. With notation as before, the following diagram commutes:

\[
\begin{array}{ccc}
\text{Sp}^G, N\text{-free}(S) & \xrightarrow{j^*pr_1^*i_!} & \text{Sp}^{G', N'}\text{-free}(S) \\
\downarrow{\pi_*} & & \downarrow{pr_2!} \\
\text{Sp}^{G/N}(S) & \xrightarrow{\pi_*} & \text{Sp}^G(S).
\end{array}
\]

Proof. We have a transformation \(\nu : pr_{2!}j^*pr_1^*i_! \to \pi_*\pi_!\), defined as the composite

\[pr_{2!}j^*pr_1^*i_! \to pr_{2!}j^*pr_1^*i_! \pi^*\pi_! \to pr_{2!}j^*pr_2^*\pi^*\pi_! \to \pi^*\pi_!,\]

where the first arrow is induced by the unit of the adjunction \((\pi_!, i^*\pi_*)\) and the last by the counit of \((pr_{2!}, j^*pr_2^*)\). To check that this is an equivalence, by Corollary 5.4 it suffices to check that \(\nu\) is an equivalence on \(\Sigma^{-\rho}_{G/N}X_+\), where \(X \in \text{Sm}_S^{G,N\text{-free}}\). Via the projection formula in Proposition 5.6, we see that \(\nu_{\Sigma^{-\rho}_{G/N}X_+}\) is identified with \(\Sigma^{-\rho}_{G/N}X_+\), and so it suffices to check that \(\nu\) is an equivalence on \(X_+\). But this case follows from the isomorphism of \(G\)-schemes \((pr_1^{-1}X)/N' \simeq \pi^{-1}(X/N)\).

Remark 6.3. It is sometimes convenient to write again \(pr_1^*\) for \(j^*pr_1^*i_!\), so that the equivalence \(\pi^*\pi_! \simeq pr_{2!}j^*pr_1^*i_!\) can be expressed compactly as

\[\pi^*\pi_! \simeq pr_{2!}pr_1^*.\]

Since \(pr_1^*\) takes \(N\)-free spectra to \(\ker(pr_2)\)-free spectra, this is only a minor overloading of notation, and no confusion should arise.

We obtain a transformation

\[\hat{\tau} : \pi^*\pi_! \to i_!\]

via

\[\pi^*\pi_! \simeq pr_{2!}j^*pr_1^*i_! \to pr_{2!}j^*\Delta^*pr_1^*i_! \simeq i_!,\]

where \(\Delta : G \to G \times_{G/N} G\) is the diagonal and we use the fact that \(\Delta_1 \simeq \Delta_+\) by the Wirthmüller isomorphism (Proposition 2.41).

Definition 6.5. The Adams transformation

\[\tau : \pi_! \to \pi^*i_!\]

is the transformation induced by adjunction from \(\hat{\tau} : \pi^*\pi_! \to i_!,\) just constructed.

In a certain sense, the Adams transformation is ‘smashing’, as made precise in the next proposition. First note that there is a canonical transformation

\[\pi^*i_!i^* (I_S) \otimes \id \to \pi^*i_!i^* \pi^*\]

between endofunctors of \(\text{Sp}^{G/N}(S)\) obtained as the adjoint of

\[\pi^*\pi^*i_!i^* (I_S) \otimes \pi^* \to i_!i^* (I_S) \otimes \pi^*\pi_! \simeq i_!i^* \pi^*\].
Since \( i^*i^* \cong i^*i^*(1_S) \otimes \text{id} \cong 1_{E,F(N)} \otimes \text{id} \) by Proposition 3.11, transformation (6.6) can equivalently be written as \( \pi_* (1_{E,F(N)}) \otimes \text{id} \to \pi_* (1_{E,F(N)} \otimes \pi^*) \).

**Proposition 6.7.** Set \( X \in \text{Spt}^{G,N,\text{free}}(S) \). The diagram

\[
\begin{array}{ccc}
\pi_1 X & \xrightarrow{\tau} & \pi_* i_1^* \pi_1 X \\
\downarrow & & \uparrow \\
\pi_1 i_1^*(1_S) \otimes \pi_1 X & \xrightarrow{\tau \otimes \text{id}} & \pi_* i_1^* (1_S) \otimes \pi_1 X
\end{array}
\]

commutes, where the left vertical map is obtained from the oplax monoidality of \( \pi_1 \), the right one is formula (6.6), and the top right horizontal arrow comes from the unit of the adjunction \((\pi_1, i_* \pi_1^*)\).

**Proof.** Consider the following diagram:

\[
\begin{array}{ccc}
\pi^* \pi_1 X & \xrightarrow{\sim} & \pi^* \pi_1 i_1^* (1_S) \otimes \pi^* \pi_1 X \\
\downarrow & & \uparrow \\
\text{pr}_2 j^* \text{pr}_1^* i_1 X & \xrightarrow{\sim} & \text{pr}_2 j^* \text{pr}_1^* i_1^* (1_S) \otimes \text{pr}_2 j^* \text{pr}_1^* i_1 X
\end{array}
\]

The functors \( \pi^*, \text{pr}_1^*, j^* \) are symmetric monoidal, \( i_1 \) is nonunital symmetric monoidal, and \( i^*(1_S) = 1_{E,F(N)} \) is the unit of \( \text{Spt}^{G,N,\text{free}}(S) \). It is straightforward to check that the top square commutes. Using the natural equivalence \( \Delta^* \otimes \text{id} \cong \Delta^* \Delta^* (1_S) \otimes \text{id} \), it is straightforward to check that the remaining squares commute. This implies the result by adjointness. \( \square \)

### 6.2. Changing the base

Our next goal is to verify that the Adams transformation \( \tau \) is compatible with the various change-of-base functors in motivic homotopy. First, however, we recall some basic facts about manipulating natural transformations and adjunctions that we will use.

Let

\[
\begin{array}{ccc}
A & \xrightarrow{f^*} & B \\
g^* \downarrow & \xRightarrow{\phi} & \downarrow k^* \\
C & \xrightarrow{h^*} & D
\end{array}
\]
be a diagram of $\infty$-categories, where $\phi: k^*f^* \to h^*g^*$ is a natural transformation. Suppose that $f^*$, $h^*$ admit respective left adjoints $f_!$ and $h_!$. The left mate of $\phi$ is a natural transformation

\[
\begin{array}{c}
A & \xleftarrow{f_!} & B \\
\downarrow{g^*} & \nearrow{\phi_L} & \downarrow{k^*} \\
C & \xleftarrow{h_!} & D
\end{array}
\]

Explicitly, $\phi_L: h_!k^* \to g^*f_!$ is defined to be the composite

\[
h_!k^* \to h_!k^*f^*f_! \xrightarrow{\phi} h_!h^*g^*f_! \to g^*f_!.
\]

Similarly, if $g^*$, $k^*$ admit respective right adjoints $g_*$ and $k_*$, then the right mate of $\phi$ is a transformation

\[
\begin{array}{c}
A & \xrightarrow{f^*} & B \\
\downarrow{g_*} & \nearrow{\phi_R} & \downarrow{k_*} \\
C & \xrightarrow{h^*} & D
\end{array}
\]

Explicitly, $\phi_R: f^*g_* \to k_*h^*$ is defined to be the composite

\[
f^*g_* \to k_*k^*f^*f_* \xrightarrow{\phi} k_*h^*g^*g_* \to k_*h^*.
\]

If $\psi$ and $\phi$ are natural transformations, then $\psi \simeq \phi_L$ if and only if $\phi \simeq \psi_R$.

**Lemma 6.8.** Suppose we are given a diagram of $\infty$-categories

\[
\begin{array}{c}
A & \xrightarrow{f^*} & B \\
\downarrow{g^*} & \nearrow{\phi} & \downarrow{k^*} \\
C & \xrightarrow{h^*} & D
\end{array}
\]

where $f^*, h^*$ have left adjoints and $g^*, k^*$ have right adjoints. Then $\phi_L$ is an equivalence if and only if $\phi_R$ is an equivalence.

**Proof.** This is a straightforward check. \qed

**Remark 6.9.** It often happens that $\phi$ is invertible. Care should be taken to not confuse the mates of $\phi$ with those of $\phi^{-1}$ (assuming all requisite adjoints exist). For example, it is not the case the mates of $\phi$ are equivalences exactly when the mates of $\phi^{-1}$ are equivalences.

We will need to know that units and counits of adjunctions are compatible across equivalences induced by mates.

**Lemma 6.10.** Let $L: C \xrightarrow{\simeq} D: R$ and $L': C' \xrightarrow{\simeq} D': R'$ be two adjoint pairs and $F: C \to C'$ and $G: D \to D'$ be functors. Let $\phi: FR \to RG$ and $\psi: LF \to GL$ be mates. Write $\eta, \epsilon$ for
the unit and counit of \((L,R)\) and \(\eta', \epsilon'\) for the unit and counit of \((L', R')\). Then for \(X \in \mathcal{D}, Y \in \mathcal{C}\), the diagrams

\[
\begin{array}{ccc}
L'R'GX & \xrightarrow{L'\phi} & L'FRX \\
& \Downarrow{\epsilon'G} & \Downarrow{G\epsilon} \\
& GX & \xleftarrow{\psi R} \\
& G\epsilon & \\
\end{array}
\]

and

\[
\begin{array}{ccc}
FRLY & \xrightarrow{\phi L} & R'GLY \\
& \Downarrow{\eta'F} & \Downarrow{R'\psi} \\
FY & \xleftarrow{\eta F} & R'L'FY.
\end{array}
\]

commute.

**Proof.** The first claim follows from the commutativity of the diagram

\[
\begin{array}{ccc}
L'FRX & \xrightarrow{L'F_\eta R} & L'FRLRX \\
& \Downarrow{id} & \Downarrow{L'F \epsilon e} \\
& L'FRX & \xrightarrow{L'\phi LR} L'R'GLRX \\
& \Downarrow{L'G \epsilon e} & \Downarrow{G \epsilon} \\
& GX & \xrightarrow{\psi R} GLRX \\
\end{array}
\]

since the top composite is \(\psi R\). The second claim follows from the commutativity of the diagram

\[
\begin{array}{ccc}
R'L'FY & \xrightarrow{R'L'F_\eta} & R'L'FRLY \\
& \Downarrow{\eta'F} & \Downarrow{R'L'\phi L} \\
FY & \xleftarrow{\eta F} & FRLY \\
& \Downarrow{\eta R'GL} & \Downarrow{id} \\
& R'GLY & R'GLY,
\end{array}
\]

where the top composite is \(R'\psi\).

Let \(p : T \to S\) be a map in \(\text{Sch}_B^G\), on which \(N\) acts trivially. Write \(\pi : G \to G/N\) for the quotient. We will use \(i\) to denote both the inclusion \(\text{Sm}_{S}^{G,N\text{-free}} \subseteq \text{Sm}_{S}^{G}\) and the inclusion \(\text{Sm}_{T}^{G,N\text{-free}} \subseteq \text{Sm}_{T}^{G}\). Fix equivalences

\[
\alpha : \pi^* p^* \sim p^* \pi^*
\]

and

\[
\gamma : i^* p^* \sim p^* i_!.
\]

The right mate of \(\alpha\) is a transformation \(\alpha_R : p^* \pi_\ast \to \pi_* p^*\). We write \(\nu\) for the right mate of \(\alpha_R\),

\[
\nu = (\alpha_R)_R : \pi_* p_* \sim p_* \pi_!,
\]

which is an equivalence \(\alpha \simeq (\alpha_R)_L\) by Lemma 6.8, since \(\alpha \simeq (\alpha_R)_L\) is an equivalence. Write \(\nu' = \nu(\gamma^{-1})_R\), which is an equivalence

\[
\nu' : \pi_* i_* p_* \sim p_* \pi_* i_!.
\]
We have an equivalence
\[ \alpha^{-1} \gamma_R : p^* \pi^* \cong i^* \pi^* p^*. \]

Write \( \beta = (\alpha^{-1} \gamma_R)_L \) for the left mate of \( \alpha^{-1} \gamma_R \). Then \( \beta \) is an equivalence
\[ \beta : \pi! p^* \cong p^* \pi!, \]
and we write \( \phi = (\beta^{-1})_R \), which is a transformation
\[ \phi : \pi! p^* \to p^* \pi!. \]

If \( p \) is smooth, then \( \alpha \) has a left mate \( \alpha_L : p^! \pi^* \cong \pi^* p^! \), which is an equivalence. It follows that \( \alpha_R \) is an equivalence by Lemma 6.8. In Corollary 6.37 we see that \( \alpha_R \) is more generally an equivalence even when \( p \) is not smooth. Write
\[ \alpha : p^! \pi^* \to \pi^* p^! \]
for the left mate of \( \alpha^{-1}_R : \pi^* p^! \cong p^* \pi^! \) and
\[ \alpha \gamma : p^# \pi^* i^! \to \pi^* i^! p^# \]
for the left mate of \( \alpha^{-1}_R \gamma \).

For the next lemmas it is convenient to fix some further exchange transformations. Set \( \kappa_i : \text{pr}^*_i p^* \cong p^* \text{pr}^*_i \) and \( \lambda : j^* p^* \cong p^* j^* \). Set \( \nu = ((\lambda \kappa_2)^{-1})_L : \text{pr}_{2!} p^* \cong p^* \text{pr}_{2!} \).

We use the following basic consequence of the fact that \( \text{Sp}^{G,F}(S) \) is the value of a functor \( \text{Sch}_B[\cdot]^{op} \). Let \( f : T \to S \) be a scheme map and \( \phi : G \to K \) a homomorphism. The exchange \( \phi^* f^* \cong f^* \phi^* \) expresses the fact that \( (\text{id}, f)(\phi, \text{id}) \) and \( (\phi, \text{id})(\text{id}, f) \) are both equal to \( (\phi, f) \). In particular, these exchanges can be chosen compatibly. That is, if \( \phi : G \to K \) and \( \psi : K \to H \) are homomorphisms, then the diagram
\[
\begin{aligned}
\phi^* \psi^* f^* & \sim \phi^* f^* \psi^* \sim f^* \phi^* \psi^* \\
(\psi \phi)^* f^* & \sim f^* (\psi \phi)^*
\end{aligned}
\]
commutes, and similarly for a composite of scheme maps.

**Lemma 6.11.** Set \( X \in \text{Sp}^{G,N-\text{free}}(S) \). The following diagram commutes:
\[
\begin{array}{ccc}
\pi^* \pi^* p^* X & \xrightarrow{\alpha \beta} & p^* \pi^* \pi^* X \\
\downarrow \sim & & \downarrow \sim \\
\hat{\gamma} p^* X & \xrightarrow{\sim} & p^* \hat{\gamma} X.
\end{array}
\]
**Proof.** We have to see that each rectangle of the following diagram commutes:

\[
\begin{array}{ccc}
\pi^* \pi_1 p^* X & \xrightarrow{\beta} & \pi^* p^* \pi_1 X \\
\sim & & \sim \\
pr_2 j^* \pi_1^* i^* p^* X & \xrightarrow{\kappa_1 \gamma} & pr_2 j^* p^* \pi_1^* i^* X \\
\downarrow & & \downarrow \\
pr_2 j^* \Delta^* \pi_1^* i^* p^* X & \xrightarrow{\nu \lambda} & p^* pr_2 j^* \pi_1^* i^* X \\
\downarrow \gamma & & \downarrow \pi \\
i^* p^* & \xrightarrow{\gamma} & p^* i^*.
\end{array}
\]

To see that the top rectangle commutes, consider the following diagram:

\[
\begin{array}{ccc}
pr_2 j^* \pi_1^* i^* p^* X & \xrightarrow{} & pr_2 j^* \pi_1^* i^* p^* \pi_1^* i^* X \\
\downarrow & & \downarrow \\
pr_2 j^* \pi_1^* i^* i^* \pi_1^* p^* X & \xrightarrow{} & pr_2 j^* \pi_1^* i^* i^* \pi_1^* \pi_1^* X \\
\downarrow & & \downarrow \\
pr_2 j^* \pi_1^* i^* \pi_1^* i^* \pi_1^* p^* X & \xrightarrow{} & pr_2 j^* \pi_1^* i^* \pi_1^* i^* \pi_1^* \pi_1^* X \\
\downarrow & & \downarrow \\
\pi^* \pi_1 p^* X & \xrightarrow{} & \pi^* \pi_1 p^* \pi_1^* X \\
\downarrow & & \downarrow \\
\pi^* \pi_1 p^* X & \xrightarrow{} & \pi^* \pi_1 p^* \pi_1^* X \\
\downarrow & & \downarrow \\
p^* \pi^* \pi_1^* X & \xrightarrow{} & p^* \pi^* \pi_1^* X \\
\end{array}
\]

The outer composites of this diagram yield the diagram of the lemma, and a straightforward inspection suffices to see that most of the pieces of this diagram commute. The remaining pieces involve moving \( p^* \) either across the unit for the adjunction \((\pi_1, \pi_1^* \pi_1^*)\) or across the counit for the adjunction \((pr_2, j^* pr_2^*)\). In either case, that this results in a commutative diagram follows from Lemma 6.10, since the pairs of exchange equivalences \( \pi_1 p^* \simeq \pi_1 p^* \), \( p^* \pi_1 \simeq i^* \pi_1^* p^* \) and \( pr_2 p^* \simeq p^* pr_2, \) \( p^* j^* pr_2^* \simeq j^* pr_2^* p^* \) are mates.

The argument for the commutativity of the remaining squares is similar. \( \square \)

**Proposition 6.12.** Set \( X \in Spt_{G,N\text{-free}}(S) \). The diagram

\[
\begin{array}{ccc}
p_1^* p^* X & \xrightarrow{\beta} & p^* \pi_1^* X \\
\tau p^* & \downarrow & \downarrow \pi^* \\
p_1^* i^* p^* X & \xleftarrow{\gamma \alpha R} & p^* \pi_1^* i^* X
\end{array}
\]

commutes.
Proof. To see this, consider the diagram

\[
\begin{array}{cccccc}
\pi^* \pi_! p^* X & \xrightarrow{\pi^* \beta} & \pi^* p^* \pi_! X & \xrightarrow{\alpha} & p^* \pi^* \pi_! X \\
\pi^* \tau p^* & \downarrow & \pi^* p^* \tau & \downarrow & p^* \pi^* \tau \\
\pi^* \pi_* i^! p^* X & \xleftarrow{\pi^* \gamma^{-1} \alpha R} & \pi^* p^* \pi_* i_! X & \xrightarrow{\alpha} & p^* \pi^* \pi_* i_! X \\
\end{array}
\]

By adjointness, it suffices to see that the top-left square commutes. The right square commutes, and the lower rectangle commutes by Lemma 6.10, so it suffices to see that the outer diagram commutes. This follows from Lemma 6.11.

Proposition 6.13. Set \( Y \in \mathcal{S}pt^{G,N,\text{free}}(T) \). The following diagram commutes:

\[
\begin{array}{ccc}
p_# \pi_! Y & \xrightarrow{\beta_L} & \pi_! p_# Y \\
p_# \tau & \downarrow & \tau p_# \\
p_# \pi_* i_! Y & \xrightarrow{\pi \alpha} & \pi_* i_! p_# Y.
\end{array}
\]

Proof. The diagram of the lemma is the composite of the squares

\[
\begin{array}{ccc}
p_# \pi_! Y & \xrightarrow{\beta} & p_# p^* \pi_! Y & \xrightarrow{\pi \alpha} & \pi_! p_# Y \\
p_# \pi_* i_! Y & \xrightarrow{\pi \alpha} & p_# i_* p_# Y & \xrightarrow{\pi \alpha} & \pi_* i_! p_# Y.
\end{array}
\]

The first and the third square commute by functoriality. The second square commutes by Proposition 6.12.

Proposition 6.14. Set \( Y \in \mathcal{S}pt^{G,N,\text{free}}(T) \). The following diagram commutes:

\[
\begin{array}{ccc}
\pi_! p_* Y & \xrightarrow{\phi} & p_* \pi_! Y \\
\tau p_* & \downarrow & \tau p_* \\
\pi_* i_! p_* Y & \xrightarrow{\nu} & p_* \pi_* i_! Y.
\end{array}
\]

Proof. Consider the following diagram. By adjointness, we need to see that the combined top rectangles commute:
Here $\epsilon$ is the counit of the adjunction $(p^*, p_*)$. The left triangle commutes by Proposition 6.12, the bottom triangle commutes by Lemma 6.10, and the top composite is equivalent to $\pi_! \epsilon$, also by Lemma 6.10. It follows that the outer diagram commutes, and since $\beta$ is an equivalence, the combined rectangles commute as desired.

Lemma 6.15. Let $f : T \to S$ be a map in $\text{Sch}_B^G$. The diagram

$$
\begin{array}{ccc}
\text{Sp}^{G,N\text{-free}}(T) & \xrightarrow{i^!} & \text{Sp}^G(T) \\
\downarrow f_* & & \downarrow f_* \\
\text{Sp}^{G,N\text{-free}}(S) & \xrightarrow{i^!} & \text{Sp}^G(S)
\end{array}
$$

commutes.

Proof. The functor $f_* : \text{Sp}^{G,N\text{-free}}(T) \to \text{Sp}^{G,N\text{-free}}(S)$ may be computed as the composite

$$
\text{Sp}^{G,N\text{-free}}(T) \xrightarrow{i^!} \text{Sp}^G(T) \xrightarrow{f_*} \text{Sp}^G(S) \xrightarrow{i^*} \text{Sp}^{G,N\text{-free}}(S).
$$

By Corollary 6.32, we have $i_! i^* \simeq i_! i^*(1) \otimes -$ and $i_! i^*(1) \simeq \colim_n U_n$, where $U_n$ is dualizable over $S$. Therefore,

$$
i_! i^* f_* i^! \simeq \colim_n F(D(U_n), f_* i^! ) \simeq f_* F(f^* D(U_n), i^! ) \simeq f_* \colim_n f^* D(U_n, i^! ) \simeq f_* \colim_n f^* U_n \otimes i^! \simeq f_*(i_! i^*(1) \otimes i^! ) \simeq f_* i_! ,
$$

where we use the fact that $f^* i_! i^*(1) \simeq i_! i^*(1)$ by Proposition 3.8, since $i_! i^*(1) \simeq E\mathcal{F}(N)$.

6.3. $\tau$ is an equivalence

In this subsection, we show that the Adams transformation $\tau$ is an equivalence. By Corollary 5.4, it suffices to show that $\tau$ is an equivalence on $\Sigma^{-k_{BG\!/N}} q_# 1_X$, where $q : X \to S$ is in $\text{Sm}_S^{G,N\text{-free}}$, which in turn would follow by showing that $\tau$ is an equivalence on all $q_# 1_X$. Unfortunately, we do not know how to show this directly. Instead, our strategy is to first show that it is an equivalence on those $q_# 1_X$ which are dualizable. This immediately implies that $\tau$ is an equivalence on the subcategory of $\text{Sp}^{G,N\text{-free}}(S)$ generated under colimits by $\langle N\text{-trivial desuspensions of} \rangle$ such $q_# 1_X$. Of course, this is only a proper subcategory of $\text{Sp}^{G,N\text{-free}}(S)$, unless $S$ is the spectrum of a field of characteristic 0. However, since $1_{E\mathcal{F}(N)}$ has dualizable skeleta by Corollary 6.32, we can at least conclude that

$$
\tau : \pi_! 1_{E\mathcal{F}(N)} \to \pi_* 1_{E\mathcal{F}(N)}
$$
is an equivalence. This now allows us to define an inverse $\tau^{-1}$ using Proposition 6.7 and conclude that $\tau$ is an equivalence in general.

**Lemma 6.16.** Set $E \in \mathcal{S}pt_{G,N}^G(S)$ and suppose that $i^*_i E$ is dualizable. Then $E$ is also dualizable, and $i^*_i FSpt_{G,N}^G(S)(E,D) \simeq FSpt_{G,N}^G(i^*_i E,i^*_i D)$.

In particular, the dual of $E$, regarded as an object of $\mathcal{S}pt_{G,N}^G(S)$, agrees via $i^*_i$ with the dual of $E$, regarded as an object of $\mathcal{S}pt^G(S)$.

**Proof.** The first claim follows formally from the facts that $E \simeq i^*_i i^*_i E$ and that $i^*_i$ is a symmetric monoidal functor. The second claim follows by applying $i^*_i$ to the equivalence $i^*_i F_{\mathcal{S}pt_{G,N}^G(S)}(i^*_i E,i^*_i D) \simeq F_{\mathcal{S}pt_{G,N}^G(S)}(E,D)$, and by the fact that $i^*_i i^*_i \simeq i^*_i i^*_i (1_S) \otimes \text{id}$ (see Proposition 3.11), together with the commutative square

\[
\begin{array}{c}
\downarrow \\
i^*_i i^*_i 1_S \otimes DS(i^*_i E) \otimes i^*_i D & \longrightarrow & i^*_i i^*_i 1_S \otimes F(i^*_i E,i^*_i D) \\
\downarrow & & \downarrow \\
DS(i^*_i E) \otimes i^*_i D & \longrightarrow & F(i^*_i E,i^*_i D),
\end{array}
\]

where the vertical maps are induced by the projection $i^*_i i^*_i (1_S) \to 1_S$ and the left-hand vertical map is an equivalence, since $i^*_i 1_S \otimes i^*_i D \to i^*_i D$ is an equivalence.

The last statement follows because $F_{\mathcal{S}pt_{G,N}^G(S)}(i^*_i E,1_S) \simeq F_{\mathcal{S}pt_{G,N}^G(S)}(i^*_i E,i^*_i (i^*_i 1_S))$.

**Lemma 6.17.** Let $p : Y \to S$ be a smooth equivariant map. For any $E \in \mathcal{S}pt^G(Y)$, the diagram

\[
D_S(p^#E) \otimes S p^#E \xrightarrow{ev} 1_S
\]

\[
p^#(p^*DS(p^#E) \otimes Y E) \xrightarrow{ev'}
\]

commutes, where the vertical equivalence is the projection formula and the diagonal map is the composite

\[
p^#(p^*DS(p^#E) \otimes Y E) \to p^#(DY(p^*p^#E) \otimes Y E) \to p^#(DY(E) \otimes Y E)
\]

\[
\xrightarrow{p^#ev} p^#1_Y \simeq p^#p^*1_S \to 1_S
\]

of the canonical map followed by maps induced by the unit and counit of the adjunction $(p^#,p^*)$, respectively.

**Proof.** The composite of the vertical and horizontal map is adjoint to the composite map in the diagram

\[
p^*DS(p^#E) \otimes Y E \longrightarrow p^*(DS(p^#E) \otimes Y p^#E) \longrightarrow p^*1_S \simeq 1_Y
\]

\[
p^*DY(p^#E) \otimes Y p^#p^#E \longrightarrow DY(p^#p^#E) \otimes Y p^#p^#E,
\]
where the left diagonal is induced by the unit of the adjunction \((p\# , p^*)\) and the square commutes by Lemma 6.10 (with \(F = p^* = G, R = F(p\# E, -)\), and \(R' = F(p^* p\# E, 0)\)). Applying \(p\#\) everywhere, we obtain the commutative diagram

\[
\begin{array}{ccc}
p\# (p^* D_S(p\# E) \otimes_Y E) & \rightarrow & p\# (p^* D_S(p\# E) \otimes_Y p^* p\# E) \\
p\# (D_Y(p^* p\# E) \otimes_Y E) & \rightarrow & p\# (D_Y(p^* p\# E) \otimes_Y p^* p\# E) \\
& & \rightarrow p\# p^* 1_S \rightarrow 1_S,
\end{array}
\]

where the lower piece commutes for formal reasons: given a map \(\varphi : M \rightarrow N\) in any symmetric monoidal category, the induced diagram

\[
\begin{array}{ccc}
D(N) \otimes M & \rightarrow & D(M) \otimes M \\
\downarrow & & \downarrow \\
D(N) \otimes N & \rightarrow & 1
\end{array}
\]

commutes. This proves the claim.

Let \(q : X \rightarrow S\) be an object in \(\text{Sm}^{G,N\text{-free}}_S\) and write \(\overline{X} = X/N\), \(f : X \rightarrow \overline{X}\), for the quotient. Since \(N\) acts trivially on \(S\), the structure map factors through the quotient and we have the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & \overline{X} \\
\downarrow{q} & & \downarrow{p} \\
S & &
\end{array}
\]

(6.18)

in \(\text{Sm}^G_S\), where \(f\) is finite étale. Our first goal is to establish Theorem 6.25, which says that when \(q\# 1_X\) is dualizable, its dual is computed as \(\pi_1 D_S(q\# 1_X)\).

We define a candidate evaluation map

\[
\epsilon : \pi_1 D_S(p\# f\# 1_X) \otimes_S p\# 1_\overline{X} \rightarrow 1_S
\]

(6.19)
as follows:

\[
\begin{align*}
\pi_1 D_S(p\# f\# 1_X) \otimes_S p\# 1_\overline{X} & \sim p^* \pi_1 D_S(p\# f\# 1_X) \\
& \rightarrow p\# \pi_1 D_{X}(p^* p\# f\# 1_X) \\
& \rightarrow p\# \pi_1 f\# 1_X \simeq p\# 1_\overline{X} \\
& \rightarrow 1_S.
\end{align*}
\]

Here the first equivalence is the projection formula, the second arrow is the equivalence \(p^* \pi_1 \simeq \pi_1 p^*\) together with the exchange \(p^* D_S \rightarrow D_{\overline{X}} p^*\), the third arrow is induced by the unit of the adjunction \((p\#, p^*)\) together with the equivalence \(D_{\overline{X}}(f\# 1_X) \simeq f\# 1_X\), and the last arrow is induced by the counit of the adjunction \((p\#, p^*)\).
Remark 6.20. The adjoint of $\epsilon$ is the map
\[ e : \pi_! D_S(q\#1_X) \to D_S(p\#1_X). \] 
(6.21)
The map $e$ can also be described as the adjoint of the map
\[ e' : D_S(q\#1_X) \to \pi_* D_S(p\#1_X), \]
which is the dual of the composite $p\#1_X \to p\#f_*f^*1_X \simeq p\#f_1_X$.

Note that the evaluation map for $q\#1_X$ factors canonically as
\[ D_S(q\#1_X) \otimes_S q\#1_X \to i_! E_F(N) \to 1_S, \]
where $1_E_F(N) \in \text{Sp}^{G,N,\text{free}}(S)$ is the unit.

Lemma 6.22. The following diagram commutes:
\[ \begin{array}{ccc}
\pi_! (D_S(q\#1_X) \otimes_S q\#1_X) & \longrightarrow & \pi_! 1_{E,F}(N) \\
\downarrow & & \downarrow \\
\pi_! (D_S(q\#1_X)) \otimes_S p\#1_X & \longrightarrow & 1_S.
\end{array} \]

Proof. The left vertical map comes from the oplax monoidality of $\pi_!$. It may also be described as the map induced by $p\#f_1_X \to p\#1_X$ together with the equivalence
\[ \pi_! (D_S(p\#f_1_X) \otimes_S p\#1_X) \simeq \pi_! D_S(p\#f_1_X) \otimes_S p\#1_X. \]
Consider the diagram
\[ p_\# \pi_! (p^*D(q\#1_X) \otimes_X f_1_X) \longrightarrow p_\# \pi_! (D(f_1_X) \otimes_X f_1_X) \longrightarrow p\#1_X \longrightarrow 1_S, \]
where the top and bottom horizontal arrows of the square are the composite of the exchange $p^*D_S \to D_X p^*$ with the map induced by the unit of the adjunction $(p\#, p^*)$.
Via Lemma 6.17 and the equivalences $\pi_! p\# = p\# \pi_!$, we see that the top row is identified with the composite of the top horizontal and right vertical arrows in the diagram of the lemma. The composite around the lower part of this diagram is identified with the composite of the left vertical and bottom horizontal arrows of the diagram in the lemma. That this diagram commutes follows from the next lemma together with the equivalence $D_X(f_1_X) \simeq f_1_X$. \qed

Lemma 6.23. The following diagram commutes:
\[ \begin{array}{ccc}
f_1_X \otimes f_1_X & \longrightarrow & f_1_X, \\
\downarrow & & \downarrow \\
f_1_X \otimes 1_X & \simeq & f_1_X \otimes 1_X.
\end{array} \]
where the horizontal map is the projection formula $f_\# \mathbb{1}_X \otimes f_\# \mathbb{1}_X \simeq f_\# f^* f_\# \mathbb{1}_X$ followed by the counit (using the ambidexterity equivalence $f_\# \simeq f_*$).

**Proof.** Let $p_i : X \times X \rightarrow X$ be the projection to the $i$th factor and $\Delta : X \rightarrow X \times X$ the diagonal. Under the ambidexterity equivalence $f_\# \simeq f_*$, the counit of the adjunction $(f^*, f_*)$ is the arrow $f^* f_\# \rightarrow \text{id}$, defined as the composite

$$f^* f_\# \simeq p_2 p^* \rightarrow p_2 \Delta_\ast \Delta^* p_1^* \simeq p_2 \Delta \ast \simeq \text{id}$$

(see [24, Theorem 6.9]). In particular, together with the equivalence $f_\# p_2 \simeq f_\# p_1$, we see that the morphism $f_\# f^* f_\# \mathbb{1}_X \rightarrow f_\# \mathbb{1}_X$ in $\text{Sp} \mathbb{G}_* (\mathcal{X})$ can be represented by the projection

$$f_\# (X \times X)_{+} \rightarrow f_\# \left( \frac{(X \times X)_{+}}{(X \times X - \Delta(X))_{+}} \right) \simeq f_\# X_+,$$

where $X \times X$ is an $X$-scheme via the first coordinate. The lemma follows from the commutativity of the diagram where the rows are the cofiber sequences in $\text{Sp} \mathbb{G}_* (\mathcal{X})$ associated to the closed immersions $\Delta(X) \subseteq X \times X$ and $\Delta(f) \subseteq X \times X$, the diagonal, and the graph of $f$, respectively:

$$
\begin{array}{ccc}
(X \times X - \Delta(X))_{+} & \longrightarrow & (X \times X)_{+} \quad \xrightarrow{p_1} \quad X_+ \\
\downarrow & & \downarrow \\
(X \times X - \Delta(f))_{+} & \longrightarrow & (X \times X)_{+} \quad \xrightarrow{\sim} \quad X_+.
\end{array}
$$

Now we suppose that $q_\# \mathbb{1}_X$ is dualizable in $\text{Sp} \mathbb{G}, \mathbb{N}$-free and $G$ is isomorphic to a semidirect product of $N$ and $G/N$. We define a ‘coevaluation’

$$\eta : \mathbb{1}_S \rightarrow p_\# \mathbb{1}_X \otimes_\mathcal{S} \pi_! D_S (p_\# f_\# \mathbb{1}_X) \quad (6.24)$$

as follows. Since $G$ is isomorphic to a semidirect product of $G/N$ and $N$, there is a map $\mathbb{1}_S \rightarrow \pi_! \mathbb{1}_E F(N)$ which splits the canonical map $\pi_! \mathbb{1}_E F(N) \rightarrow \mathbb{1}_S$, obtained by taking the $N$-quotient of the (unstable) map $N'_s \rightarrow \mathbb{1}_E F(N)$, where $N'$ is the $G$-set $G/(G/N)$. Now, since $q_\# \mathbb{1}_X$ is dualizable in $\text{Sp} \mathbb{G}(S)$, it is dualizable in $\text{Sp} \mathbb{G}, \mathbb{N}$-free($S$) by Lemma 6.16, and the duals in both categories agree under the standard inclusion. This means we have a coevaluation map

$$\text{coev} : \mathbb{1}_E F(N) \rightarrow p_\# f_\# \mathbb{1}_X \otimes_\mathcal{S} D_S (p_\# f_\# \mathbb{1}_X).$$

Now, $\eta$ is defined as the composite

$$
\begin{array}{l}
\mathbb{1}_S \rightarrow \pi_! \mathbb{1}_E F(N) \xrightarrow{\pi_! \text{coev}} \pi_! (p_\# \mathbb{1}_X \otimes_\mathcal{S} D_S (p_\# f_\# \mathbb{1}_X)) \\
\rightarrow p_\# \mathbb{1}_X \otimes_\mathcal{S} \pi_! D_S (p_\# f_\# \mathbb{1}_X).
\end{array}
$$

**Theorem 6.25.** Let $q : X \rightarrow S$ be an object of $\text{Sm} \mathbb{G}, \mathbb{N}$-free and $f : X \rightarrow \mathcal{X}$, $p : \mathcal{X} \rightarrow S$ as in diagram (6.18). Suppose that $q_\# \mathbb{1}_X \in \text{Sp} \mathbb{G}(S)$ is dualizable. Then $p_\# \mathbb{1}_\mathcal{X}$ is dualizable in
\( \Sigma pt^{G/N}(S) \) and formula (6.21) is an equivalence

\[ \pi_1 D_S (q_\# I_X) \sim D_S (\pi_1 q_\# I_X). \]

**Proof.** First we explain why it suffices to assume that \( G \) is a semidirect product of \( N \) and \( G/N \). Suppose that the result is true in this case. Given an arbitrary \( G \) and \( N \), and letting \( X \) be a dualizable \( N \)-free smooth \( G \)-scheme over \( S \), consider the Cartesian square (6.1). Since \( X \) is dualizable over \( S \), as a smooth \( G \)-scheme, the \( \ker(pr_2) \)-free \( G \times_{G/N} G \)-scheme \( pr_1^*X \) is dualizable over \( S \). Since \( G \times_{G/N} G \) is a semidirect product of \( G \) and \( \ker(pr_2) \), our hypothesis implies that the composite is an equivalence

\[ pr_2 pr_1^* D_S (q_\# I_X) \sim pr_2 D_S (pr_1^* q_\# I_X) \xrightarrow{\sim} D_S (pr_2 pr_1^* q_\# I_X), \]

where we again write \( e \) for an instance of formula (6.21) for the group \( G \times_{G/N} G \) and we use the fact that \( pr_1^* D_S (q_\# I_X) \sim D_S (pr_1^* q_\# I_X) \), since \( q_\# I_X \) is dualizable and \( pr_1^* \) is symmetric monoidal. We claim that this implies that \( p_\# I_X \) is dualizable in \( \Sigma pt^{G/N}(S) \). Indeed, \( pr_2 pr_1^* q_\# X \simeq \pi^*_1 \pi_1 q_\# I_X \) is dualizable in \( \Sigma pt^G(S) \) and therefore \( \Phi^N(\pi^*_1 \pi_1 q_\# I_X) \simeq \pi_1 q_\# I_X \simeq p_\# I_X \) is dualizable in \( \Sigma pt^{G/N}(S) \), as \( \Phi^N \) is symmetric monoidal. In this case, \( \epsilon \) and \( \eta \) are evaluation and coevaluation maps for the duality pairing.

The diagram

\[
\begin{array}{ccc}
\pi^*_1 D_S (q_\# I_X) & \xrightarrow{\pi^*_e} & \pi^* D_S (\pi_1 q_\# I_X) \\
\uparrow & & \downarrow \sim \\
pr_2 pr_1^* D_S (q_\# I_X) & \xleftarrow{\sim} & pr_2 D_S (pr_1^* q_\# I_X) \\
\end{array}
\]

is an equivalence. The functor \( \pi^*_1 \) is conservative, since \( \Phi^N \circ \pi^*_1 \simeq \id \) (see Proposition 4.10). It follows that \( \epsilon \) is an equivalence.

It remains to establish the result under that assumption that \( G \) is a semidirect product. We show that in this case the evaluation and coevaluation maps (6.19) and (6.24) satisfy the triangle identities. To compactify notation, we write \( X = p_\# f_\# I_X \), \( X = p_\# I_X \), \( 1 = I_S \), and \( 1_{E,F} = 1_{E,F(N)} \). We have to check that the following two composites are the identity:

\[
\begin{align*}
X \otimes 1 & \xrightarrow{\id \otimes \eta} X \otimes \pi_1 D(X) \otimes X \xrightarrow{\epsilon \otimes \id} 1 \otimes X \\
1 \otimes \pi_1 D(X) & \xrightarrow{\eta \otimes \id} \pi_1 D(X) \otimes X \otimes \pi_1 D(X) \xrightarrow{\id \otimes \epsilon} \pi_1 D(X) \otimes 1.
\end{align*}
\]
To establish the first identity we observe that we have a commutative diagram

$$
\begin{array}{ccc}
X \otimes 1 & \xrightarrow{id \otimes \eta} & X \otimes \pi_! D(X) \otimes X \\
\downarrow id & & \downarrow \epsilon \otimes id \\
X \otimes \pi_! I_{E,F} & \xrightarrow{\pi_!(id \otimes coev)} & X \otimes \pi_!(D(X) \otimes X) \\
\downarrow \pi_!(X \otimes I_{E,F}) & & \downarrow \pi_!(ev \otimes id) \\
\pi_!(X \otimes D(X) \otimes X) & \xrightarrow{\pi_!(ev \otimes id)} & \pi_!(I_{E,F} \otimes X),
\end{array}
$$

in which $X \simeq X \otimes I_{E,F}$ and the upper right-hand square commutes by Lemma 6.22. The second identity is established by a similar diagram.

The following well-known result is purely categorical, and applies to our particular functor $\pi^*$, as $\pi^*$ is symmetric monoidal by Proposition 4.3 and satisfies the projection formula of Proposition 5.6. See [14] for detailed derivations of these results in the 1-categorical context. In the $\infty$-categorical context, the natural transformations are derived via adjunction in exactly the same way, and whether or not a natural transformation is a natural equivalence is a property of the induced natural transformation between homotopy 1-categories.

**Lemma 6.27.** Let $\pi^* : C \to D$ be a symmetric monoidal functor of closed symmetric monoidal $\infty$-categories which admits both a left adjoint $\pi_!$ and a right adjoint $\pi_*$ and satisfies the following projection formula: For all objects $Y \in C$ and $Z \in D$, the map

$$
\pi_!(Z \otimes \pi^* Y) \to (\pi_! Z) \otimes Y,
$$

induced by the unit map $Z \to \pi^* \pi_! Z$, is an equivalence. Then $\pi^*$ is closed symmetric monoidal; that is, the map

$$
\pi^* F_C(Y, X) \to F_D(\pi^* Y, \pi^* X),
$$

induced by the evaluation $Y \otimes F_C(Y, X) \to X$, is an equivalence.

**Proof.** Let $Z$ be an object of $D$. We have a commutative square

$$
\begin{array}{ccc}
\text{Map}(Z, \pi^* F(Y, X)) & \longrightarrow & \text{Map}(Z, F(\pi^* Y, \pi^* X)) \\
\downarrow & & \downarrow \\
\text{Map}((\pi_! Z) \otimes Y, X) & \longrightarrow & \text{Map}(\pi_!(Z \otimes \pi^* Y), X),
\end{array}
$$

in which the vertical maps are equivalences by adjunction. But the bottom horizontal map is induced by the equivalence $\pi_!(Z \otimes \pi^* Y) \to (\pi_! Z) \otimes Y$, so the top horizontal map is an equivalence as well. Since $Z$ is arbitrary, the result follows.

**Corollary 6.28.** For any given object $X$ of $C$, there is a canonical equivalence of functors

$$
F(\pi_!(-), X) \simeq \pi_* F(-, \pi^* X) : D^{op} \to C.
$$
In particular, specializing to the case in which $X$ is the unit of $\mathcal{C}$, we obtain an equivalence $D\pi_! \sim \pi_* D$.

**Proof.** The adjoint of the composite natural transformation of functors

$$\pi^* F_C(\pi_!(-), X) \to F_D(\pi^* \pi_!(-), \pi^* X) \to F_D(-, \pi^* X)$$

is the desired natural transformation

$$F_C(\pi_!(-), X) \to \pi_* F_D(-, \pi^* X).$$

Let $Z$ be any object of $\mathcal{C}$. This transformation is an equivalence if and only if it is an equivalence upon evaluation at any object $Y$ of $D$. We have equivalences

$$\text{Map}(Z, F(\pi_! Y, X)) \simeq \text{Map}(Z \otimes \pi_! Y, X)$$

and

$$\text{Map}(Z, \pi_* F(Y, \pi^* X)) \simeq \text{Map}(\pi_!(\pi^* Z \otimes Y), \pi_* X).$$

Hence the result follows from the projection formula $\pi_!(\pi^* Z \otimes Y) \simeq Z \otimes \pi_! Y$. $\square$

We now return to the special case of our particular symmetric monoidal functor $\pi^*$, and we write $h : D_{S}\pi_! \sim \pi_* D_{S}$ for the canonical equivalence of Corollary 6.28.

**Proposition 6.29.** Let $q : X \to S$ be an object of $\text{Sm}^{G,N}_{S}$-free such that that $q_\# \mathbb{1}_X \in Spt^G(S)$ is dualizable. The diagram

$$\begin{array}{c}
D_S D_{S}(\pi_! q_\# \mathbb{1}_X) \xrightarrow{D(e)} D_{S}(\pi_! D_{S}(q_\# \mathbb{1}_X)) \xrightarrow{h} \pi_* D_{S} D_{S}(q_\# \mathbb{1}_X) \\
\pi_! q_\# \mathbb{1}_X \xrightarrow{\tau} \pi_* q_\# \mathbb{1}_X
\end{array}$$

commutes, where $e$ is the map from Remark 6.20.

**Proof.** Write $D = D_{S}$ and $X = q_\# \mathbb{1}_X$. The commutativity of the diagram of the proposition is equivalent, by adjointness, to that of

$$\begin{array}{c}
\pi^* DD(\pi_! X) \xrightarrow{\pi^* h D(e)} \pi^* \pi_* DD(X) \xrightarrow{\epsilon} DD(X) \\
\pi^* \pi_! X \xrightarrow{\hat{\pi}} X
\end{array}$$

\[\text{can}\]
where \( \epsilon : \pi^*\pi_* \to \text{id} \) is the counit. The outer composite of this diagram agrees with the outer composite of the following diagram, which we will show to be commutative:

Here, \( \text{pr}_i \) and \( \Delta \) are as in §6.1, \( \phi : \text{id} \to \Delta_* \Delta^* \) arises from the Wirthmüller isomorphism \( \Delta_* \simeq \Delta^* \) together with the unit \( \text{id} \to \Delta_* \Delta^* \), \( \beta : \Delta_* \Delta^* \to \text{id} \) is the counit, and we identify \( \pi_* \pi^! X \simeq \text{pr}_2^! \pi^! X \) as in Remark 6.3. The map \( \gamma \) is induced by the composite of exchanges

\[
\Delta_1 D \xrightarrow{\psi} D \Delta_* \quad \text{and} \quad D \Delta^* \xrightarrow{\psi_R} \Delta^* D
\]

as follows. First, the exchange \( \psi \) is the equivalence fitting in the commutative diagram

\[
\begin{array}{ccc}
\Delta_1 D & \xrightarrow{\psi} & D \Delta_* \\
\downarrow & & \downarrow \\
\Delta_* D & \xleftarrow{\sim} & D \Delta_1 
\end{array}
\]

where the vertical equivalences are the Wirthmüller isomorphism and the bottom one is the standard equivalence. In particular, \( \psi \) is an equivalence. The exchange \( \psi_R \) is its right mate and \( \psi_R \) is an equivalence on dualizable objects, so in particular \( \psi_R \) applied to \( X \) is
invertible. By Lemma 6.10, the diagram
\[
\begin{array}{ccc}
D\Delta_{!}\Delta^{*}q_{#}\mathbb{1}_{X} & \xleftarrow{\gamma} & D\Delta_{!}\Delta^{*}q_{#}\mathbb{1}_{X} \\
\downarrow & & \downarrow \\
Dq_{#}\mathbb{1}_{X} & \xrightarrow{\text{unit}_{D}} & Dq_{#}\mathbb{1}_{X}
\end{array}
\]
(6.30)
commutes, where \(\gamma\) is by definition the upper horizontal composite. This implies that the subdiagram marked (2) in the large diagram of this proof commutes.

Next, to see that (1) commutes, we need to see that the diagram
\[
\begin{array}{ccc}
\text{pr}_{2}!D(\Delta_{!}\Delta^{*}\text{pr}_{2}^{*}q_{#}\mathbb{1}_{X}) & \xrightarrow{e} & D(\text{pr}_{2}!\Delta_{!}\Delta^{*}\text{pr}_{2}^{*}q_{#}\mathbb{1}_{X}) \\
\uparrow & & \uparrow \\
\text{pr}_{2}!\Delta_{!}\Delta^{*}\text{pr}_{2}^{*}D(q_{#}\mathbb{1}_{X}) & \xrightarrow{\sim} & D(q_{#}\mathbb{1}_{X})
\end{array}
\]
commutes, where \(\gamma'\) is the composite of \(\gamma\) with the exchange \(\text{pr}_{1}^{*}D \to D\text{pr}_{1}^{*}\). Using Remark 6.20 and the fact that \(\Delta_{!}\Delta^{*}\text{pr}_{1}^{*}q_{#}\mathbb{1}_{X} \simeq m_{#}\mathbb{1}_{G'/G\times X}\), where \(m : G'/G \times X \to S\) is the \(G'\)-equivariant structure map, we see that the arrow \(e : \text{pr}_{2}!D(\Delta_{!}\Delta^{*}\text{pr}_{1}^{*}q_{#}\mathbb{1}_{X}) \to D(\text{pr}_{2}!\Delta_{!}\Delta^{*}\text{pr}_{1}^{*}q_{#}\mathbb{1}_{X})\) is obtained as the adjoint of the map
\[
D(\Delta_{!}\Delta^{*}\text{pr}_{1}^{*}q_{#}\mathbb{1}_{X}) \simeq D(\Delta_{!}\Delta^{*}\text{pr}_{2}^{*}q_{#}\mathbb{1}_{X}) \xrightarrow{\text{unit}_{D}} D(\text{pr}_{2}^{*}q_{#}\mathbb{1}_{X}) \simeq \text{pr}_{2}^{*}D(q_{#}\mathbb{1}_{X}).
\]
It now follows, using diagram (6.30), that this diagram, and hence (1), commutes.

That (3) commutes follows from the commutativity of the diagram
\[
\begin{array}{ccc}
W & \xrightarrow{\pi^{*}\pi_{1}} & \pi_{1}W \\
\downarrow & & \downarrow \\
\text{pr}_{2}!\Delta_{!}\Delta^{*}\text{pr}_{1}^{*}W & \xrightarrow{\beta} & \text{pr}_{2}!\text{pr}_{1}^{*}W
\end{array}
\]
for \(W \in \text{Sp}^{G,N\text{-free}}(S)\). This commutativity can be checked for \(W\) the suspension spectrum of a smooth \(N\)-free \(G\)-scheme over \(S\), which is straightforward to verify. Using Lemma 6.10, we see that subdiagram (4) commutes. That subdiagram (5) commutes follows by applying \(D\) to diagram (6.26). The remaining subdiagrams are easily seen to commute. \(\square\)

Next, we verify that \(E\mathcal{F}[N]_{+}\) is a colimit of dualizable spectra. Let \(\mathcal{F}\) be a family. If \(X \in \text{Sm}^{G}_{B}\), recall that we write
\[
X^{\mathcal{F}} = \bigcup_{H \in \text{co}(\mathcal{F})} X^{H} \quad \text{and} \quad X(\mathcal{F}) = X \setminus X^{\mathcal{F}}.
\]

Write \(f : X \to B\) and \(g : X(\mathcal{F}) \to B\) for the structure maps.

**Lemma 6.31.** Suppose that \(f_{#}(\mathbb{1}_{X})\) is dualizable in \(\text{Sp}^{G}(B)\). Then \(g_{#}(\mathbb{1}_{X(\mathcal{F})})\) is dualizable.
**Proof.** Filter the inclusion \( \mathcal{F} \subseteq \mathcal{F}_{\text{all}} \) by adjacent families \( \mathcal{F} = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_n = \mathcal{F}_{\text{all}} \). Write \( \alpha_i : X(\mathcal{F}_i) \subseteq X \) and \( \beta_i : X(\mathcal{F}_i)^{\mathcal{F}_{i-1}} \subseteq X(\mathcal{F}_i) \) for the inclusions and write \( f_i : X(\mathcal{F}_i) \to B \) for the induced structure map. From the gluing sequence, we obtain exact sequences

\[
  f_i# (\mathbb{1}_{X(\mathcal{F}_i)}) \to f_{i+1#} (\mathbb{1}_{X(\mathcal{F}_{i+1})}) \to f_{i+1#} \beta_{i+1} (\mathbb{1}_{X(\mathcal{F}_{i+1})}^{\mathcal{F}_i}).
\]

Let \( H_i \leq G \) be a subgroup such that \( \mathcal{F}_{i+1} \setminus \mathcal{F}_i = \{ (H_i) \} \). Then since \( X(\mathcal{F}_{i+1})^{\mathcal{F}_i} \) is concentrated at \( (H_i) \), we have that \( X(\mathcal{F}_{i+1})^{\mathcal{F}_i} \cong G \times_{NH_i} X(\mathcal{F}_{i+1})^{H_i} \) by Lemma 3.26. We then have that

\[
  f_{i+1#} (\beta_{i+1})^* (\mathbb{1}_{X(\mathcal{F}_{i+1})}^{\mathcal{F}_i}) \cong G^+ \cap NH_i (f_{H_i})_{#}^{*} (\mathbb{1}_{X(\mathcal{F}_{i+1})}^{H_i}),
\]

where \( f_{H_i} : X(\mathcal{F}_{i+1})^{H_i} \to B \) is the structure map.

Now suppose that \( f_{i+1#} (\mathbb{1}_{X(\mathcal{F}_{i+1})}) \) is dualizable. It follows that \( (f_{H_i})_{#}^{*} (\mathbb{1}_{X(\mathcal{F}_{i+1})}^{H_i}) \) is also dualizable, since this is obtained by applying the geometric fixed-points functor. Therefore \( f_{i+1#} (\beta_{i+1})^* (\mathbb{1}_{X(\mathcal{F}_{i+1})}^{\mathcal{F}_i}) \) is dualizable and we conclude that \( f_{i#} (\mathbb{1}_{X(\mathcal{F}_i)}) \) is dualizable as well. Since we have assumed that \( f_{#} (\mathbb{1}_{X}) = f_{n#} (\mathbb{1}_{X(\mathcal{F}_n)}) \) is dualizable, the result follows by (finite) induction. \( \square \)

**Corollary 6.32.** Set \( S \in \text{Sch}^G_B \). Given a family \( \mathcal{F} \), \( \mathbb{1}_{E_{\mathcal{F}}} \in \text{Spt}^G(S) \) can be expressed as

\[
  \mathbb{1}_{E_{\mathcal{F}}} \cong \text{colim}_{n \in \mathbb{N}} q_n# \mathbb{1}_{U_n},
\]

where \( q_n : U_n \to S \) is in \( \text{Sm}_S^G[\mathcal{F}] \) and \( q_{n#} \mathbb{1}_{U_n} \) is dualizable in \( \text{Spt}^G(S) \).

**Proof.** It suffices to show this when \( S = B \). Use the previous lemma together with the presentation of Example 3.5. \( \square \)

**Corollary 6.33.** Suppose that \( q_# \mathbb{1}_X \in \text{Spt}^G(S) \) is dualizable, where \( q : X \to S \) is an object of \( \text{Sm}_S^G, N\text{-free} \). Then the Adams transformation is an equivalence

\[
  \tau : \pi_! q_! \mathbb{1}_X \xrightarrow{\sim} \pi_* \iota_! q_# \mathbb{1}_X.
\]

In particular,

\[
  \tau : \pi_! \mathbb{1}_{E_{\mathcal{F}(N)}} \xrightarrow{\sim} \pi_* \iota_! \mathbb{1}_{E_{\mathcal{F}(N)}}.
\]

**Proof.** The first statement is immediate from the Proposition 6.29. The second statement then follows, using Corollary 6.32. \( \square \)

Next we will define a transformation

\[
  \nu : \pi_* \iota_1 \to \pi_1,
\]

which we will show is inverse to \( \tau \). We begin with the following observation. Recall that there is a transformation \( \pi_* \iota_1^* (\mathbb{1}_S) \otimes \text{id} \to \pi_* \iota_1^* \pi^* \) (see formula (6.6)).

**Lemma 6.34.** The map (6.6) is an equivalence.
**Proof.** Set $Y \in \mathcal{S}pt^{G/N}(S)$ and write $E\mathcal{F}(N) \simeq \colim_n U_n$, with $q_n \# \mathbb{1}_{U_n}$ dualizable over $S$ (see Corollary 6.32). Then map (6.6) evaluated on $Y$ is the colimit of the transformations

$$\pi_*(q_n \# \mathbb{1}_{U_n}) \otimes Y \to \pi_*(q_n \# \mathbb{1}_{U_n} \otimes \pi^*Y).$$

We see that each of these transformations is an equivalence since, each fits into the commutative diagram

$$\begin{array}{c}
\pi_*(DD(q_n \# \mathbb{1}_{U_n}) \otimes \pi^*Y) \\
\uparrow \\
\pi_*DD(q_n \# \mathbb{1}_{U_n}) \otimes Y \\
\downarrow \\
\pi_!q_n \# \mathbb{1}_{U_n} \otimes Y \\
\uparrow \\
F(D(q_n \# \mathbb{1}_{U_n}) \otimes \pi^*Y) \\
\end{array} \cong \pi_!DD(q_n \# \mathbb{1}_{U_n}) \otimes Y \cong F(D(q_n \# \mathbb{1}_{U_n}) \otimes Y).$$

Now define $\nu : \pi_* i! \to \pi_!$ as the composite

$$\begin{array}{c}
\pi_* i! \xrightarrow{\text{unit}} \pi_* i! i^* \pi^* \pi_! \cong \pi_*(i! i^*(\mathbb{1}_S) \otimes \pi^* \pi_!)
\\
\cong \pi_*(i! i^*(\mathbb{1}_S)) \otimes \pi_! \xrightarrow{\tau^{-1}} \pi_! i^*(\mathbb{1}_S) \otimes \pi_! \to \pi_!.
\end{array}$$

**Lemma 6.35.** The composite $\pi_! \xrightarrow{\tau} \pi_* i! \xrightarrow{\nu} \pi_!$ is an equivalence.

**Proof.** The composite $\nu \tau$ agrees with the composite around the following diagram

$$\begin{array}{c}
\pi! \xrightarrow{\tau} \pi_* i! \\
\downarrow \\
\pi_! i^*(\mathbb{1}_S) \otimes \pi_! \xrightarrow{\tau \otimes \text{id}} \pi_* i! i^* \pi^* \pi_! \xrightarrow{\sim} \pi_! i^*(\mathbb{1}_S) \otimes \pi_! \xrightarrow{\tau^{-1} \otimes \text{id}} \pi_! i^*(\mathbb{1}_S) \otimes \pi_!.
\end{array}$$

We are now in a position to prove that the Adams transformation $\tau : \pi_! \to \pi_* i!$, as defined in Definition 6.5, is an equivalence. Recall that $\tau$ is obtained via adjunction from the natural transformation (6.4), where $i_* : \mathcal{S}pt^{G,N\text{-free}}(S) \to \mathcal{S}pt^G(S)$ is the inclusion functor, $\pi_! : \mathcal{S}pt^{G,N\text{-free}}(S) \to \mathcal{S}pt^{G/N}(S)$ is the quotient functor, and $\pi_* : \mathcal{S}pt^G(S) \to \mathcal{S}pt^{G/N}(S)$ is the fixed-point functor.

**Theorem 6.36 (Adams isomorphism).** The Adams transformation $\tau : \pi_! \to \pi_* i!$ is an equivalence.

**Proof.** By Corollary 5.4, it suffices to show that $\tau$ is an equivalence on $\Sigma^{-V}q_* \mathbb{1}_X$, where $V$ is an $N$-trivial representation and $q : X \to S$ is an $N$-free (not necessarily smooth) $G$-scheme over $S$. Write $f : X \to \overline{X}$ for the quotient and $p : \overline{X} \to S$ for the induced map. Consider the diagram

$$\begin{array}{c}
\pi_! p_* f_* \mathbb{1}_X \xrightarrow{\tau} \pi_* i! p_* f_* \mathbb{1}_X \xrightarrow{\nu} \pi_! p_* f_* \mathbb{1}_X \\
\downarrow \nu' \sim \downarrow \\
p_* \pi_! f_* \mathbb{1}_X \xrightarrow{\tau} p_* \pi_* i! f_* \mathbb{1}_X \xrightarrow{\nu} p_* \pi_! f_* \mathbb{1}_X.
\end{array}$$
The left-hand square commutes by Proposition 6.14 and Lemma 6.15, and the right-hand square commutes by a similar argument. Both horizontal composites are equivalences by Lemma 6.35, and therefore the outer vertical arrows are equivalences, as the middle one is. Finally, \( \tau : \pi_1 f_* \mathbb{1}_X \to \pi_* i_* f_* \mathbb{1}_X \) is an equivalence by Corollary 6.33 (since \( f\# \mathbb{1}_X \simeq f_* \mathbb{1}_X \), by ambidexterity [24], as \( f \) is finite étale).

6.4. Applications

In this section, we present a few applications of the Adams isomorphism.

Let \( B^G \) denote the classifying stack of the group \( G \) and and \( f : B^G \to B^\left( G/N \right) \) the resulting proper map of stacks. We have an equivalence \( \text{Spt}(B^G) \simeq \text{Spt}^G(S) \), and from this perspective the fixed-point functor \( \pi_* \) becomes identified with the push-forward functor \( f_* \). The following base-change results are an instance of the proper and the smooth proper base-change formula, but with the curious feature that \( f \) is not a representable morphism. These do not follow immediately from the six-functor formalism in [24], precisely because \( f \) is not representable.

We use the names for exchange morphisms given in the first part of the previous subsection.

**Corollary 6.37** (proper base change). Let \( p : T \to S \) be a morphism in \( \text{Sch}_{B}^{G/N} \). The exchange

\[
\alpha_R : p^* \pi_* \to \pi_* p^*
\]

is an equivalence.

**Proof.** Choose a filtration \( \mathcal{F} = \mathcal{F}_{-1} = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_n = \mathcal{F}_{all} \) such that each pair \( \mathcal{F}_i \subseteq \mathcal{F}_{i+1} \) is \( N \)-adjacent (see §3.2). This gives rise to the filtration of \( X \in \text{Spt}^G(S) \),

\[
* \simeq \mathbf{E}\mathcal{F}_{i-1} \otimes X \to \mathbf{E}\mathcal{F}_0 \otimes X \to \cdots \to \mathbf{E}\mathcal{F}_{n-1} \otimes X \to \mathbf{E}\mathcal{F}_n \otimes X \simeq X.
\]

It thus suffices to check that \( \alpha_R \) is an equivalence on each filtration quotient \( \mathbf{E}(\mathcal{F}_{i+1}, \mathcal{F}_i) \otimes X \). Suppose that \( \mathcal{F} \subseteq \mathcal{F}' \) is \( N \)-adjacent at \( H \leq N \). By Proposition 4.12, we find that

\[
p^* \pi_* (\mathbf{E}(\mathcal{F}', \mathcal{F}) \otimes X) \simeq p^* \left( G/N_+ \times_{\mathcal{W}} \left( \mathbf{E}\mathcal{F}(W_N H)_+ \otimes X^{\Phi_H} \right)^{W_N H} \right)
\]

\[
\simeq G/N_+ \times_{\mathcal{W}} p^* \left( \left( \mathbf{E}\mathcal{F}(W_N H)_+ \otimes X^{\Phi_H} \right)^{W_N H} \right)
\]

\[
\simeq G/N_+ \times_{\mathcal{W}} \left( p^* \left( \mathbf{E}\mathcal{F}(W_N H)_+ \otimes X^{\Phi_H} \right)^{W_N H} \right)
\]

\[
\simeq G/N_+ \times_{\mathcal{W}} \left( \left( \mathbf{E}\mathcal{F}(W_N H)_+ \otimes (p^* X)^{\Phi_H} \right)^{W_N H} \right)
\]

\[
\simeq \pi_* p^* (\mathbf{E}(\mathcal{F}', \mathcal{F}) \otimes X).
\]

Here, the third equivalence follows from Theorem 6.36 and Proposition 6.12, since \( \mathbf{E}\mathcal{F}(W_N H)_+ \otimes X^{\Phi_H} \) is \( W_N H \)-free. The fourth follows from Proposition 4.11.

**Corollary 6.38** (smooth proper base change). Let \( p : T \to S \) be a smooth morphism in \( \text{Sch}_{B}^{G/N} \). The exchange

\[
\overline{\alpha} : p\# \pi_* \to \pi_* p\#
\]

is an equivalence.
Proof. This is similar to the proof of the proper base change. □

Corollary 6.39 (projection formula). Set $S \in \text{Sch}_B^{G/N}$. There is a canonical equivalence

$$\pi_*(X) \otimes Y \cong \pi_*(X \otimes \pi^*(Y))$$

for $X \in \text{Spt}^G(S)$ and $Y \in \text{Spt}^{G/N}(S)$.

Proof. Consider the map $\pi_*(X) \otimes Y \to \pi_*(X \otimes \pi^*(Y))$ adjoint to the map

$$\pi^*(\pi_*(X) \otimes Y) \cong \pi^* \pi_*(X) \otimes \pi^* Y \to X \otimes \pi^* Y,$$

where the equivalence follows from the symmetric monoidality of $\pi^*$ and the map is given by tensoring the the counit $\pi^* \pi_* X \to X$ with $\pi^* Y$.

Let $\mathcal{F} \subseteq \mathcal{F}'$ be an $N$-adjacent pair of families, say at $H \leq N$. Then by Proposition 4.12,

$$(\mathbf{E}(\mathcal{F}', \mathcal{F}) \otimes X)^N \cong G/N_+ \ltimes_W \left( (\mathbf{E}_W (W_N H)_+ \otimes X^\Phi H)^{W_N H} \right).$$

Under this equivalence, the transformation is identified with $G/N_+ \ltimes_W$ – applied to the transformation

$$(\mathbf{E}_W (W_N H)_+ \otimes X^\Phi)^{W_N H} \otimes Y \to (\mathbf{E}_W (W_N H)_+ \otimes \Phi^H X \otimes \pi^* Y)^{W_N H}.$$

That this is an equivalence follows from the commutativity of the diagram

$$
\begin{array}{ccc}
\pi_! W \otimes V & \longrightarrow & \pi_! (W \otimes \pi^* V) \\
\tau \otimes \text{id} & & \tau \\
\pi_* W \otimes V & \longrightarrow & \pi_* (W \otimes \pi^* V),
\end{array}
$$

where $W \in \text{Spt}^{G,N\text{-free}}(S)$, $V \in \text{Spt}^{G/N}(S)$; the commutativity can be checked by an argument similar to ones from before – for example, in Proposition 6.7.

The general case follows by choosing a filtration $\varnothing = \mathcal{F}_{-1} = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_n = \mathcal{F}_{\text{all}}$ such that each pair $\mathcal{F}_i \subseteq \mathcal{F}_{i+1}$ is $N$-adjacent and considering the induced filtration $\mathbf{E}_W \otimes X$ on $X$. □

7. Splitting motivic $G$-spectra à la tom Dieck

Throughout this section, $N \subseteq G$ is a normal subgroup and we assume that $N$ acts trivially on $S$.

For a subgroup $H \leq N$, we write $W H = W_G H/W_N H$ for the quotient of Weil groups. Let $\mathcal{F}(W_N H)$ be the family of subgroups $\{ K \leq W_G H \mid K \cap W_N H = \{ e \} \}$. As before, we write $\mathbf{E}_{W_N H}(W_G H) = \mathbf{E}_W (W_N H)$ for the universal $W_N H$-free $W_G H$-motivic space, to emphasize the ambient group.
Definition 7.1. Let $X$ be a motivic $G$-spectrum and $H \leq G$ a subgroup. A splitting of $X$ at $H$ is a map

$$X^H \xrightarrow{f_H} X^{\Phi H}$$

in $\mathcal{Spt}^{WGH}(S)$ which splits the canonical map $X^H \to X^{\Phi H}$. An $N$-splitting of $X$ is a choice of splitting of $X$ at each subgroup $H \leq N$.

Example 7.2.

1. Set $Y \in \mathcal{Sp}^G_S(S)$. The suspension spectrum $\Sigma^\infty Y$ has a canonical splitting at any subgroup $H \leq G$, defined as follows. Write $\pi : NGH \to WG_H$ for the quotient. The counit of the adjunction $\pi^* \dashv (-)^H$ on based motivic $NGH$-spaces yields the map $\pi^*(Y^H) \to Y$ of spaces and thus a map of $NGH$-spectra $\pi^* \Sigma^\infty (Y^H) \to \Sigma^\infty Y$. Its adjoint is the map $\Sigma^\infty (Y^H) \to (\Sigma^\infty Y)^H$.

   By Proposition 4.10, this induces the desired splitting.

2. If $X \in \mathcal{Sp}^{G/N}(S)$, then $\phi^*(X)$ is split.

3. If $X$ and $Y$ are split at $H$, then $X \otimes Y$ is canonically split via the composition $(X \otimes Y)^{\Phi H} \simeq X^{\Phi H} \otimes Y^{\Phi H} \to X^H \otimes Y^H \to (X \otimes Y)^H$.

Write $i : \text{Sm}^G_{S,N}$-free $\subseteq \text{Sm}^G_{S}$ for the inclusion.

Definition 7.3. The motivic homotopy orbit point spectrum of $X$ is

$$X_{hN} := \pi_* i^* (X) \simeq (\mathcal{E}_F(N) \wedge X)/N.$$

Let $X$ be an $N$-split motivic $G$-spectrum. Let $H \leq N$ be a subgroup and consider the composition, where for notational brevity, we write simply $\mathcal{E}_H = \mathcal{E}_{W_N H}(WG_H)$. Define the map $\Theta_{X,H}$ as the following composite, where the maps are explained later:

$$G/N_+ \times_{WH} \left( X^{\Phi H} \right)_{hW_N H} \simeq G/N_+ \times_{WH} \left( (\mathcal{E}_{H+} \otimes X)^{\Phi H} \right)^{W_N H}$$

$$\simeq G/N_+ \times_{WH} \left( (\mathcal{E}_{H+} \otimes X)^{\Phi H} \right)^{W_N H}$$

$$\xrightarrow{f_H} G/N_+ \times_{WH} \left( (\mathcal{E}_{H+} \otimes X)^H \right)^{W_N H}$$

$$\simeq ((G_+ \times_{NGH} \mathcal{E}_{H+}) \otimes X)^N$$

$$\rightarrow X^N.$$ 

The map $f_H$ is the splitting of $X$ at $H$, and the last map is induced by the projection $G_+ \times_{NGH} \mathcal{E}_{H+} \simeq (G \times_{NHG} \mathcal{E}_H) \rightarrow S^0$.

The first equivalence comes from the Adams isomorphism. The second comes from the monoidality of geometric fixed points and the fact that $H$ acts trivially on $\mathcal{E}_F(W_N H)_+$. The fourth map, which is an equivalence, comes from a canonical exchange.
of functors (see the proof of Proposition 4.12) together with the projection formula for induction restriction. Note that this map depends only on the $G$-conjugacy class of the subgroup $H$.

Now define the map of motivic $G/N$-spectra

$$\Theta_X : \bigoplus_{(H)} G/N_+ \times_{WH} (X^{\Phi H})_{hW N H} \to X^N,$$

where the index is over the set of $G$-conjugacy classes of subgroups of $N$, to be the sum over the maps $\Theta_{X, H}$ defined before. The map $\Theta_X$ is natural with respect to maps of $N$-split spectra which are compatible with splitting.

**Theorem 7.4** (motivic tom Dieck splitting). Let $X \in \text{Spt}^G(S)$ be an $N$-split motivic $G$-spectrum. The map

$$\Theta_X : \bigoplus_{(H)} G/N_+ \times_{WH} (X^{\Phi H})_{hW N H} \to X^N$$

is an equivalence of motivic $G/N$-spectra. In particular, for integers $a$ and $b$ there is a canonical isomorphism

$$\pi_{a, b}^G(1_B) \cong \bigoplus_{(H)} \pi_{a, b}(BW H_+).$$

**Proof.** Since $G$ is finite, there is a sequence of families

$$\emptyset = \mathcal{F}_{-1} \subseteq \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_n = \mathcal{F}_{\text{all}}$$

such that each pair $\mathcal{F}_i \subseteq \mathcal{F}_{i+1}$ is $N$-adjacent (see §3.2). This gives rise to the filtration of the identity functor

$$* \cong E\mathcal{F}_{-1+} \otimes \to E\mathcal{F}_{0+} \otimes \to \cdots \to E\mathcal{F}_{n-1+} \otimes \to E\mathcal{F}_{n+} \otimes \cong \text{id}.$$ 

It thus suffices to show that $\Theta_{X \otimes E(\mathcal{F}', \mathcal{F})}$ is an equivalence whenever $\mathcal{F} \subseteq \mathcal{F}'$ is an $N$-adjacent pair.

But if $\mathcal{F} \subseteq \mathcal{F}'$ is $N$-adjacent at $H \leq N$, then all summands of the domain of $\Theta$ vanish except the summand corresponding to the conjugacy class $(H)$, and $\Theta_{X \otimes E(\mathcal{F}', \mathcal{F})}$ is an equivalence by Proposition 4.12. \qed

**Remark 7.5.** Let $\text{Fin}_G$ denote the category of finite $G$-sets. It is not difficult to see that the functor $c: \text{Fin}_G \to \text{Sm}^G_B$, defined by $A \mapsto \coprod A B$, induces a functor $c^*: \text{Spt}^G \to \text{Spt}^G(B)$, which is colimit preserving and symmetric monoidal. Segal [36] showed that $\pi^G_0(1) \cong A(G)$ is the Burnside ring, and so we obtain a canonical ring map $c^*: A(G) \to \pi^G_{0, 0}(1_B)$. We also have a canonical ring map $\pi^*: \pi_{0, 0}(1_B) \to \pi_{0, 0}(BW H_+)$ induced by the projection $\pi: G \to \{c\}$. We thus obtain a ring map

$$A(G) \otimes \pi_{0, 0}(1_B) \to \pi^G_{0, 0}(1_B).$$

Using the splitting theorem and the fact that $\pi^G_{0, 0}(BW H_+) \cong \pi_{0, 0}(BW H) \oplus \pi_{0, 0}(1_B)$, we see that this is an injective ring map.

When $B = \text{Spec}(k)$ is the spectrum of a perfect field, Morel [31] showed that $\pi_{0, 0}(1_k) \cong GW(k)$ is the Grothendieck–Witt ring of symmetric bilinear forms. We thus have in this
case a canonical (injective) ring map 

\[ A(G) \otimes GW(k) \to \pi^G_{0,0}(1_k). \]

This map is in general not surjective, since \( \pi_{0,0}(BWH) \) is in general nonzero [30].

Acknowledgements. This project was begun a while ago. During its long gestation period we have had the pleasure and benefit of many interesting and helpful conversations on the material in this paper. We especially thank Tom Bachmann, Elden Elmanto, Christian Haesemeyer, Marc Hoyois, Niko Naumann, Markus Spitzweck, and Paul Arne Østvær, as well as the 2016 WiT team: Agnes Beaudry, Kathryn Hess, Magda Kedziorek, Mona Merling, and Vesna Stojanoska. Finally, we are grateful to the anonymous referee for a careful reading and helpful comments, which improved a previous draft of this paper.

The first author was supported by DFG award GE 2504/1-1 and NSF award DMS-1714273. The second author was supported by DFG award HE6740/1-1 and NSF award DMS-1710966.

Competing Interest. The author(s) declare none.

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