We consider bosonic open string field theory in marginally deformed backgrounds, which is obtained by expanding the string field around identity-based solutions associated with marginal deformations. We find a new set of string fields that satisfies the $K B c$ algebra, but the nilpotent kinetic operator is that of the theory expanded around the identity-based marginal solution. By use of these string fields, we construct the tachyon vacuum solution in marginally deformed backgrounds. The vacuum energy density is equivalent to that of the tachyon vacuum without marginal deformations. The gauge invariant overlap is changed according to the effect of marginal deformations, as expected from known results in conformal field theory (CFT). These results suggest that the vacuum energy is zero for the identity-based marginal solutions in the original theory.

Subject Index B28

1. Introduction

Analytic classical solutions corresponding to marginal deformations [1–4] were constructed on the basis of the identity string field in bosonic cubic open string field theory [5]. The classical solutions can reproduce the same effect as Wilson lines in toroidal backgrounds. In addition, the solutions depend on continuous gauge invariant parameters associated with marginal deformations. Since this one-parameter family of solutions is connected to zero string field, the vacuum energy density of the solution is expected to vanish.

Unfortunately, the vacuum energy of the identity-based marginal solutions is difficult to calculate directly due to the apparent divergence. This feature is in contrast to that of other marginal solutions based on a type of wedge state [6–9]. However, such singular behavior does appear in general due to the infinite degrees of freedom of a string field. Indeed, in light-cone type string field theories, the vacuum energy of analytic solutions cannot be calculated explicitly due to divergence [10–12]. Also, for analytic tachyon lump solutions in cubic string field theory, we need a subtraction scheme to evaluate the vacuum energy [13–17]. More importantly, it is necessary to understand this singular nature in order to clarify stringy gauge symmetry [18–22]. Thus, the singularity for the identity-based solution seems to be related to the underlying structure of string field theories.

In this paper, we construct analytic classical solutions in the theory expanded around identity-based marginal solutions. To this end, we make maximal use of the $K B c$ algebra [23,24], especially the method for the Erler–Schnabl solution [25]. For the resulting solutions, we can calculate the vacuum energy and the gauge invariant overlap [26–28] exactly with the help of the $K B c$ algebra.
We find that the vacuum energy is equal to that of the tachyon vacuum with no deformation and the overlap is affected by marginal deformation parameters. The result for the overlap is identical to the effect of coupling between an on-shell closed string state and a general open string field [4]. Consequently, the analytic solutions can be regarded as the tachyon vacuum solution in marginally deformed backgrounds. This result implies that in the original theory the vacuum energy of the identity-based solutions is zero, although the direct calculation gives indefinite results.

This paper is organized as follows. In Sect. 2, we illustrate a point about the identity-based solutions for marginal deformations. Following the convention of Appendix D in Ref. [1], we explain about the identity-based solutions for deformations generated by current operators, including the non-abelian case. Then, we find the theory expanded around the solutions. This theory describes marginally deformed backgrounds and includes the nilpotent kinetic operator $Q'$ depending on the marginal deformation parameters. In Sect. 3, we construct the tachyon vacuum solution in the expanded theory. First, we find a set of operators (string fields) that satisfies the same algebra as that of $K, B, c$, but in which the nilpotent operator is $Q'$ instead of the Kato–Ogawa BRST operator. Having found these operators, it is straightforward to construct the analytic solution in the same manner as the Erler–Schnabl solution. For the analytic solution, we calculate analytically the vacuum energy and the gauge invariant overlap. As a result, we find that the solution is the tachyon vacuum solution in marginally deformed backgrounds. In Sect. 4, we give concluding remarks. Finally, we include two appendices. In Appendix A, we give a detailed calculation of the vacuum energy, and in Appendix B we explain a delta function formula used in the calculation.

2. Marginal deformations in open bosonic string field theory

The action in bosonic cubic open string field theory is given by

$$S[\Psi] = -\int \left( \frac{1}{2} \Psi \ast Q_B \Psi + \frac{1}{3} \Psi \ast \Psi \ast \Psi \right) ,$$

(2.1)

where $Q_B$ is the Kato–Ogawa BRST operator, which is constructed by a conformal field theory (CFT) with the critical dimension 26. From the action, the equation of motion is found to be $Q_B \Psi + \Psi \ast \Psi = 0$. We consider a classical solution using the holomorphic currents $j^a(z)$ associated with a general Lie algebra $G$, including a non-semi-simple case [29]. We suppose that the currents have the operator product expansion (OPE),

$$j^a(z) j^b(w) \sim -g^{ab} \frac{1}{(z-w)^2} + \frac{1}{z-w} f^{ab}_c j^c(w) ,$$

(2.2)

$$g^{ab} = \frac{1}{2} ( f^{ac}_d f^{bd}_c - \Omega^{ab} ) ,$$

(2.3)

where $f^{ab}_c$ is the structure constant of $G$ and $\Omega^{ab}$ is a symmetric, invertible, and invariant matrix.\(^1\)

The currents are primary fields with dimension one for the energy-momentum tensor:

$$T^S(z) = \Omega_{ab} : j^a j^b : (z) ,$$

(2.4)

where $\Omega_{ab}$ is the inverse matrix of $\Omega^{ab}$. The central charge of the Virasoro algebra is given by $c = \text{dim}G - f^{ac}_d f^{bd}_c \Omega_{ab}$.\(^1\)

\(^1\) $\Omega^{ab}$ satisfies $\Omega^{ab} = \Omega^{ba}$ and $f^{ab}_c \Omega^{cd} + f^{ad}_c \Omega^{cb} = 0$.\(^1\)
Now, we suppose that the critical CFT, which is used to define the string field theory, separates into two decoupled CFTs and one has the energy momentum tensor (2.4). Then, a classical solution can be constructed as [1–4]

$$\Psi_0 = -V^a_L(F_a)I - \frac{1}{4} g^{ab} C_L(F_a F_b)I,$$

(2.5)

where $I$ is the identity string field. The half-string operators are defined by

$$V^a_L(f) = \int_{C_{	ext{left}}} \frac{dz}{2\pi i} \frac{1}{\sqrt{2}} f(z) c^a(z), \quad C_L(f) = \int_{C_{	ext{left}}} \frac{dz}{2\pi i} f(z) c(z),$$

(2.6)

where $c(z)$ is the ghost operator and $f(z)$ is a function on the unit circle $|z| = 1$. The function $F_a(z)$ in (2.5) has the Lie algebra index and we contract the indices in $V^a_L(F_a)$ and $g^{ab} C_L(F_a F_b)$. Additionally, we must impose the condition $F_a(-1/z) = z^2 F_a(z)$ to satisfy the equation of motion.

The classical solution (2.5) can be expected to correspond to marginal deformations of the associated CFT for the following reasons. First, the solution has arbitrary gauge invariant parameters with a Lie algebra index:

$$f_a = \int_{C_{	ext{left}}} \frac{dz}{2\pi i} F_a(z).$$

(2.7)

Other degrees of freedom of $F_a(z)$ are gauged away by global transformations [1], which are generated by $K_n = L_n - (-1)^n L_{-n}$ [5]. Thus, the physical parameter $f_a$ is related to each marginal deformation generated by the current $j^a(z)$.

The second reason is that the vacuum energy of the solution is expected to be zero, because the solution has continuous parameters $f_a$ and so the vacuum energy is unchanged at zero due to the equation of motion. [2,3,12] Thirdly, if we consider an abelian marginal deformation and introduce Chan–Paton indices in a string field, we can reproduce the effect of background Wilson lines in the theory expanded around the classical solution. [1–3] Hence, we can find the classical solution corresponding to marginal deformations in the string field theory.

If we expand the string field around the classical solution (2.5), we obtain a string field theory in marginally deformed backgrounds. Substituting $\Psi = \Psi_0 + \Phi$ into (2.1), we find that

$$S[\Psi] = S[\Psi_0] + S'[\Phi],$$

(2.8)

$$S'[\Phi] = - \int \left( \frac{1}{2} \Phi \ast Q' \Phi + \frac{1}{3} \Phi \ast \Phi \ast \Phi \right),$$

(2.9)

where the kinetic operator is given by

$$Q' = Q_B - V^a(F_a) - \frac{1}{4} g^{ab} C(F_a F_b),$$

(2.10)

$$V^a(F_a) = \oint \frac{dz}{2\pi i} \frac{1}{\sqrt{2}} F_a(z) c^a(z),$$

(2.11)

$$C(F_a F_b) = \oint \frac{dz}{2\pi i} F_a(z) F_b(z) c(z).$$

(2.12)

Here, $S[\Psi_0]$ corresponds to the vacuum energy of the identity-based marginal solution and $S'[\Phi]$ is the action in a marginally deformed background. Taking the variation of the action (2.9), the equation of motion is given by

$$Q' \Phi + \Phi \ast \Phi = 0,$$

(2.13)

where marginal deformation parameters are included in $Q'$. 

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Here, we note that the kinetic operator $Q'$ seems to be different from the BRST operator found in the first quantization of strings in the marginally deformed background. The operator $Q'$ includes the current as integration over the whole string, although the BRST operator should be affected by a current source inserted at string boundaries in the first quantization. However, we should notice that in string field theories the kinetic operator has various representations, which are connected by gauge transformations. Actually, as mentioned above, we can change $F_a(z)$ in $Q'$ by global gauge transformations. If we take the limit such that $F_a(z)$ approaches a delta function, whose support is located at string boundaries, the operator $Q'$ becomes the BRST operator with boundary source terms. Therefore, we can consider that the BRST operator in the first quantization can be expressed as $Q'$ in a singular limit.

3. Tachyon vacuum solutions in marginally deformed backgrounds

3.1. Tachyon vacuum solutions

We introduce a half-string operator associated with the current $j^a(z)$:

$$J^a_L(f) = \int_{C_{\text{left}}} dz \frac{1}{2\pi i} \frac{1}{\sqrt{2}} f(z) j^a(z), \quad (3.1)$$

where $f(z)$ is a function on the unit circle $|z| = 1$. This operator is transformed into the sliver frame by the conformal mapping $u = \arctan z$. Noting that $dzj^a(z)$ yields no conformal weights, the operator in the sliver frame is written as

$$J^a_L(f) = \int_{-\infty}^{\infty} dy \frac{1}{2\pi i} \frac{1}{\sqrt{2}} f(\tan(\frac{\pi}{4} + iy)) j^a(u), \quad (3.2)$$

where the current $j^a(u)$ is defined on a cylinder of circumference $\pi$. $^3$

Using the calculation method in Ref. [3], we find the anticommutation relations of half-string operators,

$$\{V^a_L(F_a), (B_1)_L\} = J^a_L((1 + z^2)F_a), \quad (3.3)$$

$$\{C^a_L(F_a F_b), (B_1)_L\} = \int_{C_{\text{left}}} dz \frac{1}{2\pi i} (1 + z^2)F_a(z)F_b(z), \quad (3.4)$$

where $(B_1)_L$ is an operator$^4$ appearing in the $KBC$ algebra. [23,24]

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$^2$ This was suggested by T. Erler and C. Maccaferri at the SFT2012 conference in Jerusalem.

$^3$ $C_{\text{left}}$ is mapped to the infinite line $u = \pi/4 + iy$ by the mapping $u = \arctan z$.

$^4$ According to the convention of Ref. [25], we defined $(B_1)_L$ and $(K_1)_L$ as

$$(B_1)_L = \int_{C_{\text{left}}} dz \frac{1}{2\pi i} (1 + z^2)b(z), \quad (K_1)_L = \int_{C_{\text{left}}} dz \frac{1}{2\pi i} (1 + z^2)T(z),$$

where $b(z)$ and $T(z)$ are the anti-ghost field and the energy-momentum tensor. The string fields $K$, $B$, and $c$ are defined by

$$K = \frac{\pi}{2} (K_1)_L |I\rangle, \quad B = \frac{\pi}{2} (B_1)_L |I\rangle, \quad c = \frac{1}{\pi} c(1) |I\rangle.$$
Using the relations (3.3) and (3.4) and noting that the left- and right-half operators commute with each other [2,3], we find the following relations with respect to the kinetic operator (2.10):

\[ Q' K' = 0, \quad Q' B = K', \quad Q' c = c K' c, \]  

(3.5)

where \( K' \) is defined by

\[ K' = K + J, \]  

(3.6)

\[ J = -\frac{\pi}{2} J_L^a ((1 + z^2) F_a) |I\rangle - \frac{\pi}{8} \int_{C_{kin}} \frac{dz}{2\pi i} (1 + z^2) g^{ab} F_a(z) F_b(z) |I\rangle, \]  

(3.7)

with the sum on \( a, b \) implicit. Moreover, since \( J \) is independent of the ghost, we find the commutation relation

\[ [B, K'] = 0. \]  

(3.8)

The relations of (3.5) are the same as those of \( K, B, c, \) and \( Q_B \). The commutation relation (3.8) is also the same as that of \( K \) and \( B \). Therefore, we conclude that \( K', B, c, \) and \( Q' \) have the same algebraic structure as that of the \( KBc \) algebra with \( Q_B \).

Having the algebra of \( K', B, \) and \( c \), we now construct a classical solution to (2.13) in marginally deformed backgrounds characterized by \( Q' \). By simply replacing \( K \) with \( K' \) in the Erler–Schnabl solution [25], we can obtain the analytic classical solution,

\[ \Phi_0 = \frac{1}{\sqrt{1 + K'}} [c + c K' B c] \frac{1}{\sqrt{1 + K'}}, \]  

(3.9)

This solution is easily seen to satisfy the equation of motion (2.13).

Similarly, we can find a homotopy operator for \( Q' \phi_0 = Q' + [\Phi_0, \cdot] \):

\[ A = \frac{1}{\sqrt{1 + K'}} B \frac{1}{\sqrt{1 + K'}}. \]  

(3.10)

It follows that \( Q' \phi_0 A = 1 \) from the algebraic structure of \( K', B, \) and \( c \). Then, we expect that the solution (3.9) can be regarded as the tachyon vacuum solution in marginally deformed backgrounds.

3.2. Vacuum energy

In a similar way to the Erler–Schnabl solution, the vacuum energy density of the solution (3.9) can be calculated as

\[ E = \frac{1}{6} \text{Tr} \left( c \frac{1}{1 + K'} c K' c \frac{1}{1 + K'} \right). \]  

(3.11)

This expression is derived from substituting \( \Phi_0 \) into the action (2.9) and using the equation of motion (2.13). Then, we use the fact that a \( Q' \)-exact state vanishes in the trace because \( Q' |I\rangle = 0. \) [2,3]

To evaluate the vacuum energy (3.11), we have to use the following Schwinger representation:

\[ \frac{1}{1 + K'} = \int_0^\infty dte^{-t(1+K')} = \int_0^\infty dte^{-t} U(t), \quad U(t) = e^{-t(K+J)}. \]  

(3.12)

The integrand can easily be rewritten by a path-ordered expression:

\[ U(t) = e^{-tK} T \left[ \exp \left( - \int_0^t dt' J(t') \right) \right], \]  

(3.13)
where the “time-dependent” string field $J(t)$ is defined as

$$J(t) = e^{tK} J e^{-tK}, \quad (3.14)$$

and string fields under the symbol $T$ are arranged from right to left with increasing “time”, i.e. the value of $t$. Here, it should be noted that there is a subtle point in the definition of $J(t)$ itself because it includes $e^{tK} \,(t > 0)$. However, such a negative “time” evolution operator does not emerge in $U(t)$ if we expand the “time”-ordered expression (3.13). Then, $e^{tK}$ in $J(t)$ is not problematic when using $U(t)$.$^5$

Substituting Eqs. (3.12) and (3.13) into (3.11) and rewriting the trace into a correlation function on the cylinder, the vacuum energy can be expressed as

$$E = \lim_{t_2 \to 0} -\frac{1}{6} \frac{\partial}{\partial t_2} \int_0^\infty dt_1 \int_0^\infty dt_3 \left( \frac{2}{\pi} \right)^3 e^{-t_1-t_3} \begin{align*} &\times \left( \frac{\pi}{2} (t_1 + t_2 + t_3) \right) c \left( \frac{\pi}{2} (t_2 + t_3) \right) c \left( \frac{\pi}{2} t_3 \right) e^{-\int_0^{t_1+t_2+t_3} 3(t') dt'} \right) \mathcal{C}_l^{2(t_1+t_2+t_3)}, \quad (3.15) \end{align*}$$

where $\langle \cdot \rangle_{C_l}$ is the correlation function of a cylinder of circumference $l$. Noting that $J_L^a$ is expressed as (3.2) in the sliver frame, we find that the operator $\mathcal{J}(t)$ is given as follows:

$$\mathcal{J}(t) = \mathcal{J}(t) - \frac{\pi}{8} \int_{-\infty}^{\infty} \frac{dy}{2\pi} \cos \left( \frac{\pi}{4} + iy \right) g^{ab} F_a \left( \tan \left( \frac{\pi}{4} + iy \right) \right) F_b \left( \tan \left( \frac{\pi}{4} + iy \right) \right),$$

$$\mathcal{J}(t) = -\frac{\pi}{2} \int_{-\infty}^{\infty} \frac{dy}{2\sqrt{2}\cos^2 \left( \frac{\pi}{4} + iy \right)} F_a \left( \tan \left( \frac{\pi}{4} + iy \right) \right) j^a \left( \frac{\pi}{2} t + iy \right), \quad (3.16)$$

where $\mathcal{J}(t)$ is defined as the term including the current operator $j^a$. We note that the path-ordered exponential in (3.13) becomes the conventional exponential in the CFT correlator in the above sense.

To calculate the vacuum energy, let us consider the matter part of the correlation function in (3.15). In the abelian case (i.e. $f^{ab}_c = 0$), writing $t = t_1 + t_2 + t_3$ and using (3.16), we can easily find that the current correlator is reduced to a two-point function:

$$\langle e^{-\int_0^t 3(t') dt'} \rangle_{C_{2t}} = \exp \left\{ \frac{\pi t}{8} \int_{-\infty}^{\infty} \frac{dy}{2\pi} \cos \left( \frac{\pi}{4} + iy \right) g^{ab} F_a \left( \tan \left( \frac{\pi}{4} + iy \right) \right) F_b \left( \tan \left( \frac{\pi}{4} + iy \right) \right) \right\} \times \exp \left\{ \frac{1}{2} \int_0^t dt_1 \int_0^t dt_2 \langle \mathcal{J}(t_1) \mathcal{J}(t_2) \rangle_{C_{2t}} \right\}. \quad (3.17)$$

This correlator can be calculated by using a current–current correlation function in the sliver frame. The answer is

$$\langle e^{-\int_0^t 3(t') dt'} \rangle_{C_{2t}} = 1. \quad (3.18)$$

Moreover, we can obtain the same result even for general currents associated with non-abelian algebra. We give a detailed derivation in Appendix A.

$^5$ A similar situation occurs if we consider gauge transformations including a family of wedge states. [30]
According to the result (3.18), we find that the vacuum energy (3.15) is unaffected by the matter correlator. Consequently, we conclude that the tachyon vacuum energy in the marginally deformed backgrounds is unchanged from that of the original background, namely,\[ E = -1/2\pi^2. \]

3.3. Gauge invariant overlaps

The gauge invariant overlap is defined by

\[ O(V, \Phi) = \text{Tr} (V \Phi), \tag{3.19} \]

where \( \Phi \) is a string field and \( V \) is a closed string vertex operator. We consider the case that \( V \) is given as

\[ V = N c(i)c(-i)\varphi(i, -i), \tag{3.20} \]

where \( N \) is a normalization constant and \( \varphi(z, \bar{z}) \) is the matter part of \( V \). The gauge invariant overlap is also an observable in marginally deformed backgrounds characterized by \( Q' \).\[ 4\]

Again, by replacing \( K \) with \( K' \) in the Erler–Schnabl solution, the overlap of the classical solution (3.9) becomes

\[ O(V, \Phi_0) = \text{Tr} \left( V c \frac{1}{1 + K'} \right). \tag{3.21} \]

From (3.12) and (3.13), it can be rewritten by using the correlator on the cylinder:

\[ O(V, \Phi_0) = \int_0^\infty dt e^{-t^2/2} \left\langle V(i\infty, -i\infty)c(0)e^{-\int_0^t dt' \beta(t')} \right\rangle_{\mathbb{C}_{\pi^2/2}} \]

\[ = \int_0^\infty dt e^{-t^2/2N/\pi} \lim_{M \to \infty} \frac{2N}{\pi} \times \left\langle c(iM)c(-iM)c(0) \right\rangle_{\mathbb{C}_{\pi^2/2}} \left\langle \varphi(iM, -iM)e^{-\int_0^t dt' \beta(t')} \right\rangle_{\mathbb{C}_{\pi^2/2}}. \tag{3.22} \]

In contrast to the vacuum energy, there is a possibility that the overlap is changed by marginal deformations.

Let us explicitly evaluate the effect of marginal deformations. We expand the matter correlation function in (3.22) up to the first order with respect to the function \( F_a \):

\[ \left\langle \varphi(iM, -iM)e^{-\int_0^t dt' \beta(t')} \right\rangle_{\mathbb{C}_{\pi^2/2}} \]

\[ = \left\langle \varphi(iM, -iM) \right\rangle_{\mathbb{C}_{\pi^2/2}} + \frac{\pi}{2} \int_0^t dt' \int_{-\infty}^\infty dy \frac{1}{2\pi \sqrt{2} \cos^2 \left( \frac{\pi}{4} + iy \right)} \times F_a \left( \tan \left( \frac{\pi}{4} + iy \right) \right) \left\langle \varphi(iM, -iM) j^\alpha \left( \frac{\pi}{2} t' + iy \right) \right\rangle_{\mathbb{C}_{\pi^2/2}} + \cdots. \tag{3.23} \]

Since the marginal deformation parameter \( f_a \) is given by the integration of \( F_a \) as in (2.7), the second term is the first-order correction with respect to \( f_a \). Now suppose that the OPE of \( \varphi \) with the current is given by

\[ j^\alpha(w)\varphi(z, \bar{z}) \sim \left( \frac{1}{w - z} - \frac{1}{w - \bar{z}} \right) A^\alpha\varphi(z, \bar{z}). \tag{3.24} \]
Here $A^a$ is a constant. From this OPE, we can calculate the first-order correction in (3.23):

$$\frac{\pi}{2} \int_0^t dt' \int_{-\infty}^{\infty} dy \frac{1}{2\pi \sqrt{2} \cos^2 \left(\frac{\pi}{4} + iy\right)} F_a \left(\tan \left(\frac{\pi}{4} + iy\right)\right) \frac{1}{t} \cos \left(\frac{\pi t'}{t} + i \frac{2y}{t}\right) \times \left\{ \frac{\cos \left(\frac{2iM}{t}\right)}{\sin \left(\frac{\pi t'}{t} + i \frac{2y}{t} - \frac{2iM}{t}\right)} - \frac{\cos \left(\frac{2iM}{t}\right)}{\sin \left(\frac{\pi t'}{t} + i \frac{2y+2M}{t}\right)} \right\} A^a \langle \varphi(iM, -iM)\rangle_{C_{\pi t}^2}$$

$$\rightarrow \sqrt{2} \pi i \int_{-\infty}^{\infty} dy \frac{1}{2\pi \cos^2 \left(\frac{\pi}{4} + iy\right)} F_a \left(\tan \left(\frac{\pi}{4} + iy\right)\right) A^a \langle \varphi(iM, -iM)\rangle_{C_{\pi t}^2} \quad (M \rightarrow \infty)$$

$$= \sqrt{2} \pi i f_a A^a \langle \varphi(iM, -iM)\rangle_{C_{\pi t}^2}, \quad (3.25)$$

where the parameter $f_a$ is given by (2.7).\(^6\)

Now we consider an abelian current algebra for the marginal solution (2.5). In this case, since the structure constant is zero, the higher-order terms can be easily computed and then the correlation function turns out to be

$$\left[\langle \varphi(iM, -iM)\rangle \right]_{C_{\pi t}^2} \rightarrow e^{\sqrt{2} \pi i f_a A^a} \langle \varphi(iM, -iM)\rangle_{C_{\pi t}^2} \quad (M \rightarrow \infty). \quad (3.27)$$

Consequently, the marginal deformation causes the phase shift of the gauge invariant overlap for the tachyon vacuum:

$$O(V, \Phi_0) = e^{\sqrt{2} \pi i f_a A^a} \times O(V, \Phi_0)\big|_{f_a=0}. \quad (3.28)$$

As a concrete example, let us consider the marginally deformed background for the $U(1)$ current,

$$j(z) = \frac{i}{\sqrt{2} \alpha'} \partial X^{25}(z), \quad (3.29)$$

where $X^{25}$ is one of the string coordinates.\(^7\) We then consider the case that the direction $X^{25}$ is compactified on a circle of radius $R$ and the matter part of the closed string vertex operator is given as

$$\varphi(z, \bar{z}) = \bar{\varphi}(z, \bar{z}) e^{ik_L X^{25}(z) + ik_R X^{25}(\bar{z})}, \quad (3.31)$$

where $\bar{\varphi}$ is the operator containing no $X^{25}$. This vertex operator corresponds to a closed string state with the momentum $k_L + k_R = m/R$ ($m = 0, \pm 1, \pm 2, \ldots$) and the winding number $(k_L - k_R)\alpha'/R = w$ ($w = 0, \pm 1, \pm 2, \ldots$) in the $X^{25}$ direction. Let us consider the zero momentum sector, namely, $k_L = k/2, k_R = -k/2$, because the solution (3.9) has zero momentum. In this case, the

\(^6\) With the mapping $u = \arctan z$, the parameter $f_a$ can be rewritten as

$$f_a = \int_{zz} \frac{dz}{2\pi i} F_a(z) = \int_{-\infty}^{\infty} dy \frac{1}{2\pi \cos^2 \left(\frac{\pi}{4} + iy\right)} F_a \left(\tan \left(\frac{\pi}{4} + iy\right)\right), \quad (3.26)$$

where $u = \pi/4 + iy$ on the left-half of a string.

\(^7\) The OPE of $X^{25}$ is given by

$$X^{25}(z)X^{25}(z') \sim -2\alpha' \log(\bar{z} - \bar{z}'). \quad (3.30)$$
Comparing this OPE with the result of (3.28), we find that the overlap has the following phase factor due to the marginal deformation:

$$\exp \left( i \sqrt{\kappa} k \int_{C_{\text{left}}} \frac{dz}{2\pi} F(z) \right).$$  (3.33)

As the simplest form of $F(z)$, we choose

$$F(z) = \lambda \left( z + \frac{1}{z} \right) \frac{1}{z}. \quad (3.34)$$

Note that the function satisfies $F(-1/z) = z^2 F(z)$ and $\lambda$ should be a real parameter due to the reality condition imposed on the marginal solution (2.5). For the function, the marginal deformation parameter is given by

$$\int_{C_{\text{left}}} \frac{dz}{2\pi i} F(z) = \lambda \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} 2 \cos \theta = \frac{2}{\pi} \lambda. \quad (3.35)$$

From (3.33), (3.35), and $k = w R/\alpha'$, we find that, due to the marginal deformation generated by the $U(1)$ current, the overlap is changed as

$$O(V, \Phi_0) = \exp \left( \frac{2wR}{\sqrt{\alpha'}} \lambda \right) \times O(V, \Phi_0)_{\lambda=0}. \quad (3.36)$$

This phase factor completely agrees with the effect of a Wilson line for open-closed string couplings in Ref. [4], which is also derived by conformal field theories in Ref. [31].

### 4. Concluding remarks

We have constructed the tachyon vacuum solution in marginally deformed backgrounds. The background is characterized by the nilpotent kinetic operator $Q'$, which is given by expanding the action around the identity-based marginal solution $\Phi_0$. To construct the tachyon vacuum solution, we have used the string fields $K'$, $B$, and $c$, which satisfy the $K B c$ algebra. In particular, we have investigated the Erler–Schnabl type solution $\Phi_0$. The vacuum energy and the gauge invariant overlap for $\Phi_0$ are exactly calculable by the current correlation function and the $K B c$ algebra.

The vacuum energy is the same as that for the tachyon vacuum solution in the undeformed background $\Psi_{\text{ES}}$, [25] namely, $S'[\Phi_0] = S[\Psi_{\text{ES}}]$, where $S'[\Phi]$ is defined by (2.9). Now let us introduce a parameter $s$ in the weighting functions as $F_a(z) = s F_a(z)$; we denote the corresponding identity-based marginal solution and tachyon vacuum solution in the marginal background as $\Psi_0$ and $\hat{\Phi}_0$, respectively. We note that the sum of them, $\Psi_0 + \hat{\Phi}_0$, satisfies the conventional equation of motion: $Q_B(\Psi_0 + \hat{\Phi}_0) + (\Psi_0 + \hat{\Phi}_0)^2 = 0$, and then we have

$$\frac{d}{ds} S[\Psi_0 + \hat{\Phi}_0] = - \int \frac{d}{ds}(\Psi_0 + \hat{\Phi}_0) * (Q_B(\Psi_0 + \hat{\Phi}_0) + (\Psi_0 + \hat{\Phi}_0)^2) = 0. \quad (4.1)$$

Noting $\lim_{s \to 0}(\Psi_0 + \hat{\Phi}_0) = 0 + \Psi_{\text{ES}} = \Psi_{\text{ES}}$, we obtain $S[\Psi_0 + \Phi_0] = S[\Psi_{\text{ES}}]$ by integrating the above from $s = 0$ to $s = 1$. Therefore, combining this with our result, $S'[\Phi_0] = S[\Psi_{\text{ES}}]$, we get
$S[\Phi_0] = S[\Psi_0 + \Phi_0]$, which implies that the vacuum energy of the identity-based marginal solution vanishes: $S[\Psi_0] = 0$. Actually, we can also show it in the same way:

$$S[\Psi_0] = \int_0^1 ds \frac{d}{ds} S[\hat{\Psi}_0] = -\int_0^1 ds \int \frac{d}{ds} \hat{\Psi}_0 \ast \left( Q_B \hat{\Psi}_0 + (\hat{\Psi}_0)^2 \right) = 0, \quad (4.2)$$

but it is difficult to calculate $S[\Psi_0]$ directly because of the singular property of an identity-based solution. In this sense, our result gives further evidence of the vanishing vacuum energy for the identity-based marginal solution.

As for the gauge invariant overlap, we have obtained a current dependent expression. For an on-shell closed tachyon vertex $e^{i \frac{1}{2} X^{25(2)}} - i \frac{1}{2} X^{25(2)}$ and a marginal current $\partial X^{25}$, the value of the gauge invariant overlap is changed by a phase factor, which coincides with the previous result obtained by other methods.

If we take a graviton vertex $\partial X^0 \partial X^0$ and a marginal current $\partial X^{25}$, noting the relation (3.18), we find that the value of the gauge invariant overlap is the same as that of the undeformed background: $O(V, \Phi_0) = O(V, \Psi_E)$. By normalizing $V$ appropriately, we have $S[\Phi_E] = O(V, \Psi_E)$, and then $O(V, \Phi_0) = S[\Phi_E] = S[\Psi_0 + \Phi_0]$ holds using (4.1). Recently, it was proved in Ref. [32] that the gauge invariant overlap with a graviton vertex is proportional to the vacuum energy of classical solutions. If we apply this relation to a solution $\Psi_0 + \Phi_0$ in the undeformed theory, $S[\Psi_0 + \Phi_0] = O(V, \Psi_0 + \Phi_0)$ is suggested. Combining the above, we have $O(V, \Phi_0) = O(V, \Psi_0 + \Phi_0)$, which implies that the gauge invariant overlap for the identity-based marginal solution vanishes: $O(V, \Psi_0) = 0$ although it is difficult to compute $O(V, \Psi_0)$ straightforwardly due to the inner product of identity states. We emphasize that this result also agrees with previous indirect calculations [4].

We have another type of identity-based solution, which is constructed in terms of the BRST current and the ghost field [3]. This identity-based solution is regarded to correspond to the tachyon vacuum due to various facts about cohomology and vacuum energy [33–36]. As an application of our method, it seems to be an interesting problem to construct analytic solutions in the theory expanded around this identity-based solution. There is a possibility of finding a solution that is regarded as the perturbative vacuum with calculable vacuum energy. If we construct such a solution, we will understand more about identity-based solutions in string field theories.

Another possible application of our method is construction of classical solutions in superstring field theories in marginally deformed backgrounds. In superstring field theories, we found identity-based solutions corresponding to marginal deformations [1,37]. Additionally, several analytic solutions were found in terms of the supersymmetric extension of the $KBc$ algebra [38–41]. Hence, it is possible to consider analytic solutions with calculable vacuum energy in the background expanded around identity-based supersymmetric marginal solutions [42].

We have found that the action (2.9) is useful for analyzing marginal deformed backgrounds. Also, we know that the level truncation scheme works well for calculating the vacuum energy of identity-based tachyon vacuum solutions. [33,35,43] Thus, it is natural to ask whether we can analyze the vacuum structure of the action (2.9) by using the level truncation approximation. In this regard, it is known that in the level truncation analysis of marginal deformations, there are two branches of the solution for a finite range of the marginal field [44,45]. To gain a deeper understanding of the vacuum structure, it is interesting to calculate numerically the tachyon vacuum energy by using the action (2.9) and to clarify the dependence of the marginal parameters (2.7) (I. Kishimoto and T. Takahashi, manuscript in preparation).
In this paper, we have succeeded in combining the $KBC$ algebraic technique with methods for investigating identity-based solutions. We expect that the combination of the two methods will potentially open up new ways to investigate string field theories.

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Appendix A  A proof of Eq. (3.18)

For the current correlation functions on a cylinder of circumference $\pi$, we define the following two quantities:

\[
\mathcal{F}_n = \int_{-\infty}^{\infty} dy_1 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx_1 f_{a_1}(y_1) \cdots \int_{-\infty}^{\infty} dy_n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx_n f_{a_n}(y_n) \langle j^{a_1}(u_1) \cdots j^{a_n}(u_n) \rangle_{C_\pi}, \quad (A.1)
\]

\[
\mathcal{G}^a_n(u) = \int_{-\infty}^{\infty} dy_1 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx_1 f_{a_1}(y_1) \cdots \int_{-\infty}^{\infty} dy_n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx_n f_{a_n}(y_n) \langle j^{a}(u) j^{a_1}(u_1) \cdots j^{a_n}(u_n) \rangle_{C_\pi}, \quad (A.2)
\]

where we set $u = x + iy$. The function $f_{a}(y)$ has the Lie algebra index $a$ and depends only on the imaginary part of $u$. By definition, we have

\[
\mathcal{F}_{n+1} = \int_{-\infty}^{\infty} dy \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx f_{a}(y) \mathcal{G}^a_n(u). \quad (A.3)
\]

One- and two-point correlation functions on the cylinder are given by

\[
\langle j^a(u) \rangle_{C_\pi} = 0, \quad \langle j^a(u) j^b(u') \rangle_{C_\pi} = \frac{-g^{ab}}{\sin^2(u - u')}. \quad (A.4)
\]

From these results, we find

\[
\mathcal{F}_1 = 0, \quad (A.5)
\]

\[
\mathcal{G}^a_1(u) = -2\pi g^{ab} f_{b}(y), \quad (A.6)
\]

where to derive (A.6) we used the formula (see Appendix B)

\[
\int_0^\pi dx \frac{1}{\sin^2[x + i(y - y')]} = 2\pi \delta(y - y'). \quad (A.7)
\]

The current correlation function satisfies the Ward–Takahashi identity in the sliver frame:

\[
\langle j^a(u) j^{a_1}(u_1) \cdots j^{a_n}(u_n) \rangle_{C_\pi} = \sum_{k=1}^{n} \frac{-g^{a}_a}{\sin^2(u - u_k)} \langle j^{a_1}(u_1) \cdots j^{a_k}(u_k) \cdots j^{a_n}(u_n) \rangle_{C_\pi} 
\]

\[
+ \sum_{k=1}^{n} \frac{\cos u_k}{\cos u} \frac{f^{a_1 b}}{\sin(u - u_k)} \langle j^{b}(u_k) j^{a_1}(u_1) \cdots j^{a_k}(u_k) \cdots j^{a_n}(u_n) \rangle_{C_\pi}, \quad (A.8)
\]
where the caret above $j^a$ (i.e. $\hat{j}^a$) means that it is to be omitted from the correlator. Substituting (A.8) into (A.2) and using (A.7), we can calculate $G_a^{n+1}(u)$ ($n \geq 1$) as

$$G_a^{n+1}(u) = -2\pi(n+1)g^{ab}f_b(y)\mathcal{F}_n + (n+1)\int_{-\infty}^{\infty} dy' \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx' f_b(y') \frac{\cos u'}{\cos u} \frac{f^{ab}c}{\sin(u-u')} G_c^{n}(u').$$

(A.9)

Now let us prove

$$G_a^n(u) = -2\pi n g^{ab}f_b(y)\mathcal{F}_{n-1}, \quad \mathcal{F}_0 = 1,$$

(A.10)

for $n = 1, 2, 3, \ldots$. It is true for $n = 1$ from (A.6). Assume that it holds for $n < N$. From (A.9), it follows that

$$G_a^{N+1}(u) = -2\pi (N+1)g^{ab}f_b(y)\mathcal{F}_N - 2\pi N(N+1)\int_{-\infty}^{\infty} dy' \times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx' \frac{\cos u'}{\cos u} \frac{f^{ab}c}{\sin(u-u')} f_b(y') f_d(y')\mathcal{F}_{N-1}.$$

(A.11)

Here, $f^{ab}c g^{cd}$ is antisymmetric on indices $b$ and $d$ because the associativity of the OPE between the currents shows that [29]

$$f^{ab}c g^{cd} + f^{ad}c g^{cb} = 0.$$  

(A.12)

Then the second term in (A.11) vanishes. Therefore, (A.10) is true also for $n = N + 1$ and the result follows by induction.

Combining the results of (A.3) and (A.10), we can find that, for integer $n$,

$$\mathcal{F}_{2n} = (2n - 1)!! \left(-2\pi^2 \int_{-\infty}^{\infty} dy g^{ab}f_a(y) f_b(y)\right)^n,$$

(A.13)

and $\mathcal{F}_{2n-1} = 0$.

Finally, we can derive (3.18) from these results. For $\mathcal{J}(t)$ defined in (3.16), we find

$$\left\langle e^{-\int_0^t \mathcal{J}(t') dt'} \right\rangle_{C_T}^{\frac{n}{2}} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\pi}{2}\right)^n \int_{-\infty}^{\infty} dy_1 \int_{0}^{t} dt_1 f_{a_1}(y_1) \cdots \int_{-\infty}^{\infty} dy_n \times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dt_n f_{a_n}(y_n) \left\langle j^{a_1}(u_1) \cdots j^{a_n}(u_n) \right\rangle_{C_T},$$

where $u_k = \frac{\pi t_k}{2} + i y_k$ and $f_a(y)$ is given by

$$f_a(y) = \frac{1}{2\pi \sqrt{2} \cos^2 \left(\frac{\pi}{4} + iy\right)} F_a \left(\tan \left(\frac{\pi}{4} + iy\right)\right).$$

(A.15)

In the integrand, CFT correlators on $C_{\frac{n}{2}}$ can be rewritten as those on $C_{\pi}$:

$$\left\langle j^{a_1}(u_1) \cdots j^{a_n}(u_n) \right\rangle_{C_{\frac{n}{2}}} = \left(\frac{2}{t}\right)^n \left\langle j^{a_1}\left(\frac{2u_1}{t}\right) \cdots j^{a_n}\left(\frac{2u_n}{t}\right) \right\rangle_{C_{\pi}}.$$  

(A.16)

Then, by a change of variables as $\pi t_k/2 \rightarrow x_k$ and $2y_k/t \rightarrow y_k$, the correlation function is computed as

$$\left\langle e^{-\int_0^t \mathcal{J}(t') dt'} \right\rangle_{C_{\frac{n}{2}}} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{t}{2}\right)^n \int_{-\infty}^{\infty} dy_1 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx_1 f_{a_1} \left(\frac{ty_1}{2}\right) \cdots \int_{-\infty}^{\infty} dy_n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx_n f_{a_n} \times \left(\frac{ty}{2}\right)^n \left\langle j^{a_1}(u_1) \cdots j^{a_n}(u_n) \right\rangle_{C_{\pi}},$$

(A.17)
where \( u_k = x_k + iy_k \). From the result (A.13), the above correlation function becomes

\[
= \exp \left( -\frac{\pi^2 t}{2} \int_{-\infty}^{\infty} dy \frac{g^{ab}}{2}\left( f_a(y) f_b(y) \right) \right) \\
= \exp \left\{ -\frac{\pi t}{8} \int_{-\infty}^{\infty} dy \frac{1}{2\pi \cos^4 \left( \frac{\pi}{4} + iy \right)} g^{ab} \left( \tan \left( \frac{\pi}{4} + iy \right) \right) F_a \left( \tan \left( \frac{\pi}{4} + iy \right) \right) \right\}. \tag{A.18}
\]

This cancels the first factor on the right-hand side of (3.17). As a result, we can find that (3.18) is derived from (A.18).

### Appendix B  A delta function formula

Let us derive the expression of the delta function (A.7). First, setting \( z = e^{2ix} \), we find that

\[
\int_0^\pi \frac{dx}{\sin^2(x + iy)} = \oint_{|z|=1} \frac{dz}{2\pi i (z - e^{2iy})^2}. \tag{B.1}
\]

It turns out that the integration (B.1) is zero for \( y \neq 1 \), but it diverges at \( y = 0 \). To evaluate the singularity at \( y = 0 \), we rewrite (B.1) as

\[
= \frac{d}{dy} \oint_{|z|=1} \frac{dz}{2\pi i (z - e^{2iy})} = -2\pi \frac{d}{dy} \theta(-y) = 2\pi \delta(y). \tag{B.2}
\]

Thus, we find the formula (A.7).

Alternatively, the formula can be understood as the principal value integral. Using the periodicity for \( x \) and extracting \( x = 0 \), we define the integration as

\[
\int_0^\pi \frac{dx}{\sin^2(x + iy)} = \lim_{\epsilon \to 0^+} \left( \int_\epsilon^{\frac{\pi}{2}} dx + \int_{-\frac{\pi}{2}}^{-\epsilon} dx \right) \frac{1}{\sin^2(x + iy)}. \tag{B.3}
\]

The integration is easily calculated as

\[
= \lim_{\epsilon \to 0^+} \left( \cot(\epsilon + iy) + \cot(\epsilon - iy) \right). \tag{B.4}
\]

If \( y \neq 0 \), it becomes zero for taking the limit. To evaluate the singularity, we expand the cotangent as a Laurent series.

\[
= \lim_{\epsilon \to 0^+} \left\{ \frac{1}{\epsilon + iy} + \frac{1}{\epsilon - iy} + \cdots \right\} \\
= i \lim_{\epsilon \to 0^+} \left\{ \frac{1}{y + i\epsilon} - \frac{1}{y - i\epsilon} \right\} \\
= 2\pi \delta(y). \tag{B.5}
\]

We again find the formula (A.7).

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