ON PLANAR HOLOMORPHIC SYSTEMS

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Abstract. Planar holomorphic systems \( \dot{x} = u(x, y) \), \( \dot{y} = v(x, y) \) are those that \( u = \text{Re}(f) \) and \( v = \text{Im}(f) \) for some holomorphic function \( f(z) \). They have important dynamical properties, highlighting, for example, the fact that they do not have limit cycles and that center-focus problem is trivial. In particular, the hypothesis that a polynomial system is holomorphic reduces the number of parameters of the system. Although a polynomial system of degree \( n \) depends on \( n^2 + 3n + 2 \) parameters, a polynomial holomorphic depends only on \( 2n + 2 \) parameters. In this work, in addition to making a general overview of the theory of holomorphic systems, we classify all the possible global phase portraits, on the Poincaré disk, of systems \( \dot{z} = f(z) \) and \( \dot{z} = 1/f(z) \), where \( f(z) \) is a polynomial of degree 2, 3 and 4 in the variable \( z \in \mathbb{C} \). We also classify all the possible global phase portraits of Möbius systems \( \dot{z} = \frac{Az + B}{Cz + D} \), where \( A, B, C, D \in \mathbb{C} \), \( AD - BC \neq 0 \). Finally, we obtain explicit expressions of first integrals of holomorphic systems and of conjugated holomorphic systems, which have important applications in the study of fluid dynamics.

1. Introduction

The understanding of the phase portrait of planar differential systems involves some questions:

- How is the local behavior around the equilibrium points? A serious problem consists in distinguishing between a focus and a center.
- After equilibrium points the main subjects are limit cycles, i.e., periodic orbits that are isolated in the set of all periodic orbits of a differential system. How many limit cycles are there in the phase portrait?
- A first integral completely determines its phase portrait. Given a vector field on \( \mathbb{R}^2 \), how can one determine if this vector field has a first integral?

These problems are unsolved in general. However for a specific class of systems we get very satisfactory answers. The class we are referring to is the one formed by holomorphic systems.

Planar holomorphic systems \( \dot{x} = u(x, y) \), \( \dot{y} = v(x, y) \) are those that \( u = \text{Re}(f) \) and \( v = \text{Im}(f) \) for some holomorphic function \( f(z) \).

Holomorphic systems have surprising properties:

- The equilibriums are isolated and the topological classification of the local phase portraits is fully known.
- They do not have limit cycles.
- The center–focus problem is totally solved.
- There is a first integral \( H(x, y) \) defined in a subset of total measure in \( \mathbb{R}^2 \).

A holomorphic function \( f \) is a complex-valued function defined in a domain \( \mathcal{V} \subseteq \mathbb{C} \) and satisfying that

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\end{center}
\[ u_x = v_y, \quad u_y = -v_x, \quad \forall z = x + iy \in \mathcal{V}. \]

We remark that Looman–Menchoff’s Theorem states that the above conditions are sufficient to guarantee the analyticity of \( f \). It means that for any \( z_0 \in \mathcal{V} \)

\[
(1) \quad f(z) = A_0 + A_1(z - z_0) + A_2(z - z_0)^2 + \ldots, \quad A_k = a_k + ib_k = \frac{f^{(k)}(z_0)}{k!}
\]

for \( z \in D(z_0, R_{z_0}) \subseteq \mathcal{V} \) where \( D(z_0, R_{z_0}) \) is the largest possible \( z_0 \)-centered disk contained in \( \mathcal{V} \). Unless a translation we can always assume that \( z_0 = 0 \).

If \( f \) is holomorphic in a punctured disc \( D(z_0, R) \setminus \{ z_0 \} \) and it is not derivable at \( z_0 \) we say that \( z_0 \) is a singularity of \( f \). In this case \( f(z) \) is equal to Laurent’s series in \( D(z_0, R) \setminus \{ z_0 \} \)

\[
(2) \quad f(z) = \sum_{k=1}^{\infty} \frac{B_k}{(z - z_0)^k} + \sum_{k=0}^{\infty} A_k(z - z_0)^k,
\]

where \( B_k = \frac{1}{2\pi i} \int_{C_\varepsilon} f(z)(z - z_0)^{k-1}dz \), \( A_k = \frac{1}{2\pi i} \int_{C_\varepsilon} \frac{f(z)}{(z - z_0)^{k+1}}dz \) with \( C_\varepsilon \) parameterized by \( z(t) = \varepsilon e^{it}, \varepsilon \sim 0 \).

If \( B_k \neq 0 \) for an infinite set of indices \( k \) we say that \( z_0 \) is an essential singularity and if there exists \( n \geq 1 \) such that \( B_n \neq 0 \) and \( B_k = 0 \) for every \( k > n \) then we say that \( z_0 \) is a pole of order \( n \). Moreover \( B_1 \) is called residue of \( f \) at \( z_0 \) and it is denoted by \( B_1 = \text{res}(f, z_0) \).

Let \( f : D(0, R) \setminus \{ 0 \} \to \mathbb{C} \) be a holomorphic function as (2) with \( z_0 = 0, B_k = c_k + id_k \) and \( A_k = a_k + ib_k \). Consider the ordinary differential equation

\[
(3) \quad \dot{z}(t) = f(z(t)), \quad t \in \mathbb{R}.
\]

The solution of (3) passing through \( z \in D(0, R) \setminus \{ 0 \} \) at \( t = 0 \) is denoted by \( \varphi_f(t, z) = x(t) + iy(t) \).

**Theorem 1.** The real and imaginary parts of \( \varphi_f(t, z) \) must satisfy the following system

\[
(4) \quad \begin{cases}
\dot{x} &= \sum_{k=1}^{\infty} \left( c_k \frac{p_k}{(x^2 + y^2)^{k}} + d_k \frac{q_k}{(x^2 + y^2)^{k}} \right) + a_0 + \sum_{k=1}^{\infty} (a_k p_k - b_k q_k) \\
\dot{y} &= \sum_{k=1}^{\infty} \left( d_k \frac{p_k}{(x^2 + y^2)^{k}} - c_k \frac{q_k}{(x^2 + y^2)^{k}} \right) + b_0 + \sum_{k=1}^{\infty} (b_k p_k + a_k q_k)
\end{cases}
\]

with \( p_k, q_k \) given in Table (5).
Proof. We have \( f(z) = \sum_{k=1}^{\infty} \frac{c_k + id_k}{z^k} + \sum_{k=0}^{\infty} (a_k + ib_k)z^k \). A direct calculation using Newton’s binomial formula gives us

\[
z^k = (x + iy)^k = p_k + iq_k
\]

with \( p_k \) and \( q_k \) as in the table (5). Thus

\[
(a_k + ib_k)z^k = (a_k p_k - b_k q_k) + i(b_k p_k + a_k q_k)
\]

and

\[
\frac{c_k + id_k}{z^k} = \frac{(c_k + id_k)z^k}{|z|^{2k}} = \frac{(c_k + id_k)}{(x^2 + y^2)^k} (p_k - iq_k) = \frac{1}{(x^2 + y^2)^k} \left( (c_k p_k + d_k q_k) + i(d_k p_k - c_k q_k) \right).
\]

Since \( \dot{x} = \text{Re}(f(z)) \) and \( \dot{y} = \text{Im}(f(z)) \) we conclude the proof.

We refer to system (4) as a holomorphic system if \( f \) is holomorphic in \( D(0,R) \) and as meromorphic system if \( f \) is holomorphic in \( D(0,R) \setminus 0 \) and 0 is a singularity of the kind pole. If \( f \) is holomorphic then the coefficients \( c_k, d_k \) are zero due to Cauchy’s Theorem.

Remark. An easy way to find the polynomials \( p_k \) and \( q_k \) that appear in table (5) is to consider the triangle below. The numbers in bold refer to the coefficients of \( p_k \) and the others refer to \( q_k \). The monomials of the \( k \) line are \( x^k, x^{k-1}y, ..., y^k \).

\[
\begin{array}{cccccccc}
1 & 1 & & & & & & \\
1 & 2 & -1 & & & & & \\
1 & 3 & -3 & -1 & & & & \\
1 & 4 & -6 & -4 & 1 & & & \\
1 & 5 & -10 & -10 & 5 & 1 & & \\
1 & 6 & -15 & -20 & 15 & 6 & -1 & \\
1 & 7 & -21 & -35 & 35 & 21 & -7 & -1 \\
1 & 8 & -28 & -56 & 70 & 56 & -28 & -8 & 1 \\
1 & 9 & -36 & -84 & 126 & 126 & -84 & -36 & 9 & 1
\end{array}
\]
Proposition 2. $E = (0, 0)$ is an equilibrium point of (4) if and only if $f(E) = f(0) = 0$. Moreover, if $f'(0) = a + ib$ then the linear part of (4) at $E$ has jacobian matrix given by

$$J(E) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$ 

Proof. This follows directly from the fact that $f'(0) = u_x(0, 0) + iv_x(0, 0) = a + ib$ and from the Cauchy Riemann equations $u_x = v_y, u_y = -v_x$. $\square$

Proposition 3. If $f = u + iv$ is holomorphic in $D(0, R) \setminus \{0\}$ and it is not identically null then system (4) has a finite number of equilibrium points and all of them are isolated.

Proof. If there exists a sequence of distinct equilibria $(x_n, y_n)$ of (4) then the sequence $z_n = x_n + iy_n$ will be formed by zeros of $f$. Taking $D(0, R)$ if necessary, we can assume that $z_n$ admits a convergent subsequence $z_{n_k}$. In this case $f$ is identically null in a set that has an accumulation point. From the principle of identity of analytic functions it follows that $f \equiv 0$. $\square$

Holomorphic systems have only three kinds of simple equilibrium points, all of them have index +1 (see [13]), they are foci, centers or nodes (see Theorem 2.1 of [1]). Moreover, this class of system do not have limit cycles, see [5, 9, 21, 22, 23, 26, 28].

There are many motivations for study holomorphic vector fields. In addition to those already mentioned, we can also cite [2, 7, 11, 12]. We can highlight the study of parabolic bifurcations, see [10, 27], the study of simultaneous bifurcation of limit cycles [17], the study of dynamical fluids through conjugate systems, see [3, 4, 14, 25], the study of integrability of holomorphic vector fields, more specifically, study the problem of upper bounds of bifurcation of limit cycles relating to Hilbert’s 16th Problem, see [6, 28]. For more details about this problem, we recommend also [18, 19].

In this work, we study the phase portraits of the equations

$$(6) \quad \dot{z} = f(z), \quad \dot{z} = \frac{1}{f(z)} \quad \text{and} \quad \dot{z} = T(z)$$

where $f$ is a complex polynomial of degree 2, 3, 4 and $T(z) = \frac{Az + B}{Cz + D}$ is a Moebius transformation, that is, $A, B, C, D \in \mathbb{C}$ with $AD - BC \neq 0$.

We provide the following possible phase portraits on the Poincaré disk, unless topological equivalence:

- If $f$ is a complex polynomial of degree 2 (resp 3, 4) then there are 3 (resp 9, 22) possible phase portraits of $\dot{z} = f(z)$. See Theorem 16 (resp 21, 28).
- If $f$ is a complex polynomial of degree 2 (resp 3, 4) then there are 3 (resp 11, 8) possible phase portraits of $\dot{z} = \frac{1}{f(z)}$. See Theorem 33 (resp 34, 35).
- If $T$ is a Moebius transformation then there are 9 possible phase portraits of $\dot{z} = T(z)$. See Theorem 39.

The paper is organized as follows. In Section 2 we present some preliminary results. In Section 3 we discuss some aspects of fluid dynamics and integrability.
We show that the phase portrait of \( \dot{z} = \frac{1}{f(z)} \) is equal to the phase portrait of \( \dot{z} = \frac{1}{f(z)} \), where \( f(z) \neq 0 \). In addition, we show that the complex potential of the conjugate holomorphic system is a primitive of \( f(z) \). In Section 4 we present the local phase portraits for the system \( \dot{z} = f(z) \), where \( f(z) \) is a polynomial of degree \( n = 2, 3, 4 \). This section will help us to show all the phase portrait on Poincaré disk. In Section 5, 6, 7, 8 we state and prove Theorems 16, 21, 28, 33, 34, and 35. In Section 9 we study the local dynamics of polynomial conjugate systems \( f_j(z), \) \( j = 2, 3, \) and 4. Finally in Section 10 we state and prove Theorem 39.

2. Preliminaries

This section is devoted to state some classic results that will help us to classify the local phase portraits of holomorphic systems. In order to do this, we will start by introducing the concept of conformal conjugation.

Let \( f \) and \( g \) be holomorphic functions defined in some punctured neighborhood of \( 0 \in \mathbb{C} \). We say that \( f \) and \( g \) are \( 0–\text{conformally conjugated} \) if there exist \( R > 0 \) and a conformal map \( \Phi : D(0, R) \to D(0, R) \) such that \( \Phi(0) = 0 \) and \( \Phi(\varphi_f(t, z)) = \varphi_g(t, \Phi(z)) \), for any \( z \in D(0, R) \setminus \{0\} \) and all \( t \) for which the above expressions are well defined and the corresponding points are in \( D(0, R) \).

Let \( f \) and \( g \) be holomorphic functions defined in some punctured neighborhoods of \( z_1 \in \mathbb{C} \) and \( z_2 \in \mathbb{C} \), respectively. We say that \( f \) and \( g \) are \( z_1z_2–\text{conformally conjugated} \) if \( f \circ (z - z_1) \) and \( g \circ (z - z_2) \) are conformally conjugated at 0.

If \( f \) and \( g \) are holomorphic in \( D(0, R) \) then we have:

- If \( f(0) \neq 0 \) and \( g(0) \neq 0 \) then \( f \) and \( g \) are \( 0–\text{conformally conjugated} \);
- If \( f(0) \neq 0 \) and \( g(0) = 0 \) then \( f \) and \( g \) are not \( 0–\text{conformally conjugated} \);
- If \( f(0) = 0 \) and \( g(0) = 0 \) and \( f, g \) are non constant then
  \[ \Phi(\varphi_f(t, z)) = \varphi_g(t, \Phi(z)) \iff \Phi(z)f(z) = g(\Phi(z)), \]
  for \( |z| \) sufficiently small.

Conformal conjugation classes are known in the literature. See for instance [8] and [10]. If \( f \) is a holomorphic function defined in some punctured neighborhood of \( z_0 \in \mathbb{C} \) we have:

(a) If \( f(z_0) \neq 0 \) then \( f \) and \( g(z) \equiv 1 \) are \( z_00–\text{conformally conjugated} \).
(b) If \( f(z_0) = 0 \) and \( f'(z_0) \neq 0 \) then \( f \) and \( g(z) \equiv f'(z_0)z \) are \( z_00–\text{conformally conjugated} \).
(c) If \( f(z_0) = 0 \), \( z_0 \) is a zero of \( f \) of order \( n > 1 \) and \( \text{Res}(1/f, z_0) = 1/\gamma \) then \( f \) and \( g(z) \equiv \gamma z^n/(1 + z^{n-1}) \) are \( z_00–\text{conformally conjugated} \).
(d) If \( f(z_0) = 0 \), \( z_0 \) is a zero of \( f \) of order \( n > 1 \) and \( \text{Res}(1/f, z_0) = 0 \) then \( f \) and \( g(z) \equiv z^n \) are \( z_00–\text{conformally conjugated} \).
(e) If \( z_0 \) is a pole of \( f \) of order \( n \) then \( f \) and \( g(z) \equiv \frac{1}{z^n} \) are \( z_00–\text{conformally conjugated} \).

If \( z_0 \) is an essential singularity of \( f \) then for any direction \( w_0 \), there exists \( z \), arbitrarily close to \( z_0 \), whose flow \( \varphi_f(0, z) \) follows the direction \( w \), with \( w \) being arbitrarily close to \( w_0 \). More precisely we have the following theorem.

**Theorem 4.** Let \( \dot{z} = f(z) \) be a holomorphic system defined in some punctured neighborhood of an essential singularity \( z_0 \in \mathbb{C} \). For \( \varepsilon, \delta > 0 \) sufficiently small and
an arbitrary direction \( w_0 \in \mathbb{C} \) there exist \( z, w \in \mathbb{C} \) such that \( |z - z_0| < \delta, |w - w_0| < \varepsilon \) and \( \frac{d}{dt} \varphi_f(0, z) = w \).

**Proof.** Let \( \varepsilon, \delta, z_0 \) and \( w_0 \) be as in the statement. The Casorati-Weierstrass theorem states that the image of \( D(z_0, \delta) \setminus \{z_0\} \) is dense in \( \mathbb{C} \). Thus there exists \( w \in f(D(z_0, \delta) \setminus \{z_0\}) \) such that \( |w - w_0| < \varepsilon \). So \( w = f(z) \) for some \( z \in D(z_0, \delta) \setminus \{z_0\} \).

Since \( \frac{d}{dt} \varphi_f(0, z) = f(z) = w \), we conclude the proof. \( \square \)

![Figure 1](image-url)  
**Figure 1.** Local dynamics for \( \dot{z} = z^n \exp(1/z^n) \) for \((n,m) = (1,2), (n,m) = (2,3), \) and \((n,m) = (3,4)\), resp.

In the Figure 1 using the first integrals \( H_{1,2} = \exp(-x/(x^2 + y^2))\sin(y/(x^2 + y^2)) \), \( H_{2,3} = \exp(-(x - y)(x + y)/(x^2 + y^2)^2)\sin(2xy/(x^2 + y^2)^2)/2 \), and \( H_{3,4} = \exp(-x(x^2 - 3y^2)/(x^2 + y^2)^3)\sin(y(3x^2 - y^2)/(x^2 + y^2)^3)/3 \), we obtain the local dynamics around the essential singularity of \( \dot{z} = z^n \exp(1/z^n) \) for \((n,m) = (1,2), (n,m) = (2,3), \) and \((n,m) = (3,4)\).

Next proposition, whose proof can be found in [1], gives us important information about how the configuration of the equilibrium points of a polynomial holomorphic system can be if they are all simple.

**Proposition 5.** Consider equation \( \dot{z} = f(z) \) where \( f(z) \) is a complex polynomial of degree \( n \). Assume that all equilibrium points \( z_k, k = 1, \cdots, n \) are simple. Then
a) If \( z_1, \cdots, z_{n-1} \) are centers, then \( z_n \) is also a center.

b) If \( z_1, \cdots, z_{n-1} \) are nodes, then \( z_n \) is also a node.

c) If not all the equilibrium points are centers, then there exist at least two of them that have different stability.

As it can be seen in [1], there are many other results that help us to study the phase portraits. We can cite here some results. Considering the equation (3), with \( f(z) \) a complex polynomial of degree \( n \) and assuming that all their equilibrium points are simple, so we have that if all the equilibrium points are foci, then any geometrical distribution in \( \mathbb{C} \) can be achieved. Moreover, not all the equilibrium points have same stability. Besides we can check that if all the equilibrium points are collinear, then all of them are of the same type and if these points are not center, then they have alternated stability. In the same way, as a direct consequence, we have that \( n \) aligned equilibrium points have alternating stability. Moreover, \( n - 2 \) aligned equilibrium points have alternating stability and the two symmetric with respect to this line and sharing stability.

Another important information to obtain the phase portrait of any system is to study the dynamics at infinity. Consider the equation (3) with \( \deg(f) = n \).
Then, it has exactly \( n - 1 \) equilibrium points at infinity, all of them of saddle type. Moreover, the points at infinity in the Poincaré compactification, have exactly \( n - 1 \) pairs of saddle points, see [1], Theorem 5.1.

**Proposition 6.** Consider equation \( \dot{z} = f(z) \) where \( f(z) \) is a complex polynomial. Then every equilibrium point has a positive index \( n \), where \( n \) is the order of the zero of \( f \). If \( n = 1 \), the point is a source, sink or center, depending on the sign of the real part of \( f'(z) \). If \( n > 1 \), the point is of purely elliptic type with \( 2n - 2 \) elliptic sectors.

For a proof see [28].

Regarding periodic orbits in holomorphic systems, it is possible to calculate the time needed of an orbit to leave a point \( z_0 \) and reach a point \( z_1 \).

**Proposition 7.** Let \( f \) be a complex polynomial function. If \( z = \varphi(t, w) \) and \( c = \text{Res}(1/f, 0) \) then

(a) \( \exp \left( \int_w^z \frac{1}{cf(s)} ds \right) = \exp \left( \frac{t}{c} \right) \) if \( c \neq 0 \).

(b) \( \int_w^z \frac{1}{f(s)} ds = t \) if \( c = 0 \).

See [23] for a proof.

**Example 8.** Consider \( \dot{z} = (-1 + i)z \). The holomorphic system is given by

\[
\dot{x} = -x - y, \quad \dot{y} = x - y.
\]

The equilibrium point \((0, 0)\) is an attracting focus. The solution that passes through \((1, 0)\) at \( t = 0 \) will intersect the \( x \) axis again at the point \((-e^{-\pi}, 0)\) and this will occur after the time \( t = \pi \). Indeed, \( \frac{1}{f(z)} = \frac{1}{(-1 + i)z} \) implies \( \text{res}(1/f, 0) = \frac{1}{-1 + i} \), and thus if \( w = \varphi(t, z) \) we have \( \exp \left( \int_w^z \frac{ds}{-1 + i} \right) = \exp \left( \frac{t}{-1 + i} \right) \), and taking \( t = \pi, z = 1 \) we get \( w = -e^{-\pi} \).

For the beauty of the argument used in [23] to prove the next theorem, let is reproduce its proof here.

**Theorem 9.** Let \( f \) be a holomorphic function defined in a domain \( V \subseteq \mathbb{C} \). The phase portrait of \( \dot{z} = f(z) \) has no limit cycle.

**Proof.** Suppose \( \gamma \) is a periodic orbit of \( \dot{z} = f(z) \) with period \( T \), that is, \( \varphi_f(z, T) = z \) whatever \( z \in \gamma \). Let us fix any point in \( \gamma \) and consider the transition function given \( \xi(z) = \varphi_f(z, T) \). The transition function is analytic and it is equal to identity at all points that are in \( \gamma \). So this function coincides with the identity in a neighborhood of \( z \). This means that the periodic orbit belongs to a continuum of periodic orbits, all with the same period \( T \).

3. **Fluid Dynamics and Integrability**

Consider a perfect, homogeneous and incompressible fluid. In this context, being perfect means that the force is due to pressure only, perpendicular to the separation surface between the parts of the fluid. Being homogeneous and incompressible means that the mass density \( \rho \) is constant at all points of motion. Furthermore, let
is assume that the velocity remains parallel to the plane $xy$ regardless of the third spatial coordinate $z$ such that the motion is the same in all planes parallel to the plane $xy$.

Let $X(x, y) = (X_1(x, y), X_2(x, y))$ be the velocity vector of this movement at the point $(x, y)$. Components $X_1$ and $X_2$ are smooth functions in $\mathbb{R}^2$ and must satisfy the following equations

\begin{equation}
\text{div} \, X = \frac{\partial X_1}{\partial x} + \frac{\partial X_2}{\partial y} = 0. \tag{7}
\end{equation}

Let is also add the hypothesis that the motion is irrotational. We start with the line integral

$$\Gamma = \int_C X \cdot t \, ds$$

where $C$ is a simple closed path and $t$ is the unit tangent vector to $C$. Note that $X \cdot t$ represents the scalar value of the tangential velocity and $\Gamma$ represents a measure of how much particles tend to circulate along the $C$ circuit. A fundamental theorem due to Lord Kelvin states that circulation remains constant over time. Since motion originates from rest, we conclude that circulation is zero for all time. Thus

$$0 = \int_C X \cdot t \, ds = \int_C X_1 \, dx + X_2 \, dy = \iint_R \left( \frac{\partial X_2}{\partial x} - \frac{\partial X_1}{\partial y} \right) \, dxdy,$$

where is the region inside the circuit $C$ and the last equation is due to the Green theorem. Since the integrand is a continuous function and $R$ is an arbitrary region components $X_1$ and $X_2$ must also satisfy the following equations

\begin{equation}
\frac{\partial X_2}{\partial x} - \frac{\partial X_1}{\partial y} = 0. \tag{8}
\end{equation}

Note that the equations (7) and (8) are exactly the Cauchy-Riemann equations for the functions $X_1$ and $-X_2$.

Based on the above, it becomes of our interest to study systems

\begin{equation}
\dot{x} = u(x, y), \quad \dot{y} = -v(x, y). \tag{9}
\end{equation}

where $f(z) = u(x, y) + iv(x, y)$ is a holomorphic function. Let is refer to the above system as the **conjugate system** $\overline{f(z)}$.

The conjugate system is a **gradient system**, that is,

$$\dot{x} = u = \frac{\partial \phi}{\partial x}, \quad \dot{y} = -v = \frac{\partial \phi}{\partial y}.$$

The function $\phi$ satisfies the Laplace equation

$$\Delta \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

and therefore it is a harmonic function. This implies that $\phi$ is the real part of a holomorphic function $F(z) = \phi + \psi i$.

The orthogonality of the level curves of the real and imaginary parts of $F$ implies that the function $\psi$ is in fact a first integral of the conjugate system.
Definition $F(z) = \phi + \psi i$ is called the complex potential function of motion.

**Proposition 10.** Let $f(z)$ be a holomorphic function. A first integral of the conjugate holomorphic system $\dot{z} = f(z)$ is given by $\psi(x, y) = \Re F(z)$ where $F'(z) = f(z)$. In other words, the complex potential of the conjugate holomorphic system is a primitive of $f(z)$.

**Proof.** Indeed, for $f = u + iv$ the holomorphic conjugated system is
\[ \dot{x} = u, \quad \dot{y} = -v \]
which is gradient and thus it exists $\phi$ such that we obtain $u = \phi_x$ and $v = -\phi_y$. The complex potential $F = \phi + \psi i$ is holomorphic and $F'(z) = \phi_x + i\phi_y = u + iv = f(z)$.

**Proposition 11.** Let $f(z)$ be a holomorphic function. The equilibrium points of the conjugate holomorphic system $\dot{z} = \overline{f(z)}$ are the critical points of $\psi$ and those that are not degenerate are of the saddle type.

**Proof.** Since $\dot{x} = u = \phi_x$, $\dot{y} = -v = \phi_y$, the claim about equilibria being the critics of $\psi$ holds true. Indeed, $u = -v = 0$ implies $\psi_x = -\phi_y = 0$, $\psi_y = \phi_x = 0$.

The function $\psi$ is harmonic and $\psi \in C^\infty$, thus $\psi_{xx} + \psi_{yy} = 0$ and, for the Schwartz theorem $\psi_{xy} - \psi_{yx} = 0$. Consequently the determinant of the Hessian matrix $\psi_{xx}\psi_{yy} - \psi_{xy}\psi_{yx} = -\psi_{xx}^2 - \psi_{xy}^2 \leq 0$.

**Example.** Let us consider $f(z) = z^2$. So the conjugate system is
\[ \dot{x} = x^2 - y^2, \quad \dot{y} = -2xy, \]
and the first integral is given by $\psi(x, y) = x^2y - \frac{y^3}{3}$.

![Figure 2. Level curves $x^2y - \frac{y^3}{3} = k$.](image)

**Example.** Let us consider $f(z) = \frac{z^2}{1+z}$. The complex potential is given by $F(z) = \frac{z^2}{2} - z + \log(z + 1)$. So the conjugate system is
\[ \dot{x} = \frac{x^3 + x^2 + xy^2 - y^2}{(1 + x)^2 + y^2}, \quad \dot{y} = \frac{-x^2y + 2xy + y^3}{(1 + x)^2 + y^2}, \]
and the first integral is given by $\psi(x, y) = (x - 1)y + \arctan \frac{y}{x + 1}$. 
3.1. On the integrability of holomorphic systems. Note that the phase portrait of \( \dot{z} = \frac{1}{f(z)} \) is equal to the phase portrait of \( \dot{z} = \frac{1}{\overline{f(z)}} \) where \( f(z) \neq 0 \). In fact, it follows from the fact that \( \frac{1}{f(z)} = \frac{\overline{f(z)}}{|f(z)|^2} \) and \( \frac{1}{|f(z)|^2} > 0 \). In particular, the classification of the possible phase portraits of \( \dot{z} = \frac{1}{f(z)} \) with \( \partial p(z) \leq 4 \), is the same classification of the possible phase portraits of the conjugate systems \( \dot{z} = \overline{p(z)} \) with \( \partial p(z) \leq 4 \). The only difference is that the equilibrium points of \( \dot{z} = \frac{1}{f(z)} \) are polo-type singularities of \( \dot{z} = \frac{1}{p(z)} \).

Now consider \( \dot{z} = f(z) \) with \( z \in A \subseteq \mathbb{C} \). Consider \( g(z) = \frac{1}{f(z)} \). As before, the phase portrait of \( \dot{z} = f(z) \) is equal to the phase portrait of \( \dot{z} = g(z) \) and the equilibrium points of \( \dot{z} = f(z) \) are singularities of \( g(z) \). According Proposition 10 a first integral of \( \dot{z} = f(z) \) is \( H(x, y) = \Im G(z) \) where \( G'(z) = \frac{1}{f(z)} \).

**Theorem 12.** Let \( \dot{z} = f(z) \) be a holomorphic system defined on the open set \( A \subseteq \mathbb{C} \). Thus its trajectories are contained in the level curves of \( H(x, y) = \Im G(z) \) where \( G'(z) = \frac{1}{f(z)} \).

**Proof.** The result follows from the above considerations. However, a very simple way to obtain the same result is to consider the technique of separating variables. In fact, being \( \dot{z} = f(z) \) it follows that

\[
\frac{dz}{dt} = f(z) \implies \frac{dz}{f(z)} = dt
\]

and integrating both sides of the equation we get

\[
G(z) = t + a + bi
\]

where \( G(z) \) is a primitive of \( \frac{1}{f(z)} \) and \( a + bi \) is a complex constant. So the imaginary part on the right side must be equal to the imaginary part on the left side and therefore

\[
\Im G(z) = b.
\]
Example. Consider $\dot{z} = z$. Thus $g(z) = \frac{1}{z}$, $G(z) = \log(z)$ and

$$H(x, y) = \Im(\log(z)) = \arctan \frac{y}{x}.$$ 

Example. Consider $\dot{z} = z^2$. Thus $g(z) = \frac{1}{z^2}$, $G(z) = -\frac{1}{z}$ and

$$H(x, y) = \Im\left(-\frac{1}{z}\right) = \frac{y}{x^2 + y^2}.$$ 

Example. Consider $\dot{z} = (1 + i)z$. Thus $g(z) = \frac{1}{(1 + i)z}$, $G(z) = \frac{1}{1 + i} \log(z)$ and

$$H(x, y) = \Im\left(\frac{1}{1 + i} \log(z)\right) = \arctan \frac{y}{x} - \frac{1}{2} \log(x^2 + y^2).$$ 

See Figure 4.

![Figure 4](image_url)

**Figure 4.** Level curves $\arctan \frac{y}{x} - \frac{1}{2} \log(x^2 + y^2) = k$, $k = 0$ green, $k > 0$ blue, $k < 0$ red.

Example. Consider $\dot{z} = z^2 \exp(1/z)$. Thus $g(z) = \frac{1}{z^2} \exp(-1/z)$, $G(z) = \exp(-1/z)$ and

$$H(x, y) = \Im(\exp(-1/z)) = \exp\left(-x\right) \sin\left(\frac{y}{x^2 + y^2}\right).$$ 

See Figure 5.
4. LOCAL PHASE PORTRAITS AND CENTER-FOCUS PROBLEM

This section is devoted to study the local phase portrait of the systems $\dot{z} = f(z)$ with $f(z)$ holomorphic polynomial of degree $n = 2, 3, 4$. Moreover, we will present the triviality of center-focus problem.

4.1. Center-Focus Problem. Consider $\dot{z} = f(z)$, $f$ polynomial and $f(z_0) = 0$. Let us suppose $f'(z_0) \neq 0$. We know that $f$ and $f'(z_0)z$ are $z_00$–conformally conjugated. So, $z_0$ is a center if $\text{Re}(f'(z_0)) = 0$ and $z_0$ is a focus/node if $\text{Re}(f'(z_0)) \neq 0$. Moreover, if $f'(z_0) = 0$, then there exists $n$ such that $f$ is $z_00$–conformally conjugated to $z^n$ or $(\gamma z^n)/(1 + z^n)$. In both cases, the equilibrium point is neither a center nor a focus. See more details in [8]. Therefore, the center-problem for holomorphic systems is trivial. It is enough analyzing $f'(z_0)$.

4.2. Quartic Polynomial Holomorphic Systems. Let $p(z)$ be a non constant polynomial holomorphic systems with degree $\partial p \leq 4$. Without loss of generality
we assume that \( z_0 = 0 \) is an equilibrium point.

\[
p(z) = A_1 z + A_2 z^2 + A_3 z^3 + A_4 z^4.
\]

• If \( A_1 \neq 0 \) then \( z_0 \) is a simple zero. In this case \( p(z) \) and \( g(z) = A_1 z \) are conformally conjugated.

• If \( A_1 = 0, A_2 \neq 0 \) and \( A_3 \neq 0 \) then \( z_0 \) is a zero of order 2 and \( \text{res} \left( \frac{1}{p}, 0 \right) = \frac{1}{\lambda} = -\frac{A_3}{A_2} \). Thus follow that \( p(z) \) and \( g(z) = \frac{\lambda z^2}{1 + z} \) are conformally conjugated.

• If \( A_1 = 0, A_2 \neq 0 \) and \( A_3 = 0 \) then \( z_0 \) is a zero of order 3. Since \( \text{res} \left( \frac{1}{p}, 0 \right) = 0 \) follow that \( p(z) \) and \( g(z) = \frac{\alpha z^3}{1 + z^2} \) are conformally conjugated, where \( \frac{1}{\alpha} = \frac{A_2^3}{A_3^2} \).

For a general equilibrium point \( z_0, p(z_0) = 0 \), we have:

• If \( p'(z_0) \neq 0 \) then \( p(z) \) and \( g(z) = p'(z_0) z \) are \( z_0 \)-conformally conjugated.

• If \( p'(z_0) = 0, p''(z_0) \neq 0 \) and \( p'''(z_0) \neq 0 \) then \( p(z) \) and \( g(z) = \frac{\lambda z^2}{1 + z} \) are \( z_0 \)-conformally conjugated where \( \frac{1}{\lambda} = \frac{2p'''(z_0)}{3p''(z_0)^2} \).

• If \( p'(z_0) = 0, p''(z_0) \neq 0 \) and \( p'''(z_0) = 0 \) then \( p(z) \) and \( g(z) = \frac{p''(z_0)}{2} z^2 \) are \( z_0 \)-conformally conjugated.

• If \( p'(z_0) = p''(z_0) = 0 \) and \( p'''(z_0) \neq 0 \) then \( p(z) \) and \( g(z) = \frac{\alpha z^3}{1 + z^2} \) are \( z_0 \)-conformally conjugated, where \( \frac{1}{\alpha} = \frac{3p^{(4)}(z_0)^2}{8p''''(z_0)^3} \).

• If \( p'(z_0) = p''(z_0) = p'''(z_0) = 0 \) then \( p(z) \) and \( g(z) = \frac{p^{(4)}(z_0)}{24} z^4 \) are \( z_0 \)-conformally conjugated.
**Figure 6.** Local dynamics: \( \dot{z} = z^2, \) \( \dot{z} = z^3, \) \( \dot{z} = z^4, \) \( \dot{z} = \frac{z^2}{1 + z^2}, \) \( \dot{z} = \frac{z^3}{1 + z^2}, \) and \( \dot{z} = \frac{z^4}{1 + z^3}, \) respectively.

5. **Quadratic Polynomial Holomorphic Systems**

Consider a quadratic polynomial holomorphic function

\[
f(z) = A_0 + A_1 z + A_2 z^2, \quad A_k = a_k + ib_k, \quad A_2 \neq 0.
\]

Its trajectory are the solutions of the differential system

\[
\begin{align*}
\dot{x} &= a_0 + a_1 x - b_1 y + a_2 (x^2 - y^2) - b_2 (2xy), \\
\dot{y} &= b_0 + b_1 x + a_1 y + b_2 (x^2 - y^2) + a_2 (2xy).
\end{align*}
\]

5.1. **Finite equilibrium points.**

**Proposition 13.** If \( z(t) \) is a periodic orbit of the system \( (11) \) then \( z(t) \) intersects the straight-line \( a_1 + 2a_2 x - 2b_2 y = 0. \)

**Proof.** The divergence of \( (11) \) is

\[
u_x + v_y = 2(a_1 + 2a_2 x - 2b_2 y).
\]

Thus the divergence has constant sign for \( (x, y) \) out of the straight-line \( a_1 + 2a_2 x - 2b_2 y = 0. \) Applying Bendixson’s criteria we conclude the proof. \( \square \)

**Proposition 14.** Let \( f \) be a quadratic holomorphic polynomial function as in \( (10) \) with \( a_0 = b_0 = 0. \) Then system \( (11) \) has at most two equilibrium points. Moreover

(a) If \( a_1 \neq 0 \) then there exist two equilibrium points, one of them being a repelling focus or node and the other being an attracting focus or node.

(b) If \( a_1 = 0 \) and \( b_1 \neq 0 \) then both equilibria are centers.

(c) If \( a_1 = b_1 = 0 \) and \( a_2 \neq 0 \) then there exists only one equilibrium, non-hyperbolic, and the straight-line \( y = -\frac{b_2}{a_2} x \) is invariant.
(d) If \( a_1 = b_1 = a_2 = 0 \) and \( b_2 \neq 0 \) then there exists only one equilibrium, non-hyperbolic, and the straight-line \( x = 0 \) is invariant.

Proof. Considering a translation \( x \mapsto x - a_0, y \mapsto y - b_0 \) if necessary, we can assume that \( a_0 = b_0 = 0 \). We have

\[
f(z) = A_1 z + A_2 z^2 = 0 \iff z = 0 \quad \text{or} \quad z = -\frac{A_1}{A_2}.
\]

So the equilibria are \( E_1 = (0, 0) \) and \( E_2 = \left( \frac{-a_1 a_2 - b_2 b_2}{a_2^2 + b_2^2}, \frac{a_1 b_2 - a_2 b_1}{a_2^2 + b_2^2} \right) \). Since \( f'(0) = A_1 \) and \( f'(\frac{-A_1}{A_2}) = -A \) the jacobian matrices at the equilibrium points of system (11) are

\[
J(E_1) = \begin{bmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{bmatrix} \quad \text{and} \quad J(E_2) = \begin{bmatrix} -a_1 & b_1 \\ -b_1 & -a_1 \end{bmatrix}.
\]

The determinant \( D \) and the trace \( T \) are

\[
D(E_1) = D(E_2) = a_1^2 + b_1^2 \quad \text{and} \quad T(E_1) = -T(E_2) = 2a_1.
\]

We conclude that if \( a_1 > 0 \) then \( E_1 \) is a repelling focus and \( E_2 \) is an attracting focus and if \( a_1 < 0 \) then \( E_1 \) is an attracting focus and \( E_2 \) is a repelling focus. This concludes the proof of (a).

The Lyapunov coefficients (see the appendix) of the equilibrium \( E_1 \) are \( V_1 = a_1, V_2 = 0 \) and \( V_3 = \frac{\pi a_1 (a_1^2 + b_1^2)}{b_1} \). Thus, if \( a_1 = 0 \) and \( b_1 \neq 0 \) then \( E_1 \) is a center. \( \square \)

Similarly we prove that if \( a_1 = 0 \) and \( b_1 \neq 0 \) then \( E_2 \) will also be a center. To do this we move the singularity to the origin considering

\[
x \mapsto x + \frac{b_1 b_2}{a_2^2 + b_2^2} \quad \text{and} \quad y \mapsto y + \frac{a_2 b_1}{a_2^2 + b_2^2}
\]

and we follow the same steps. Note that in this case the equilibria \( E_1 \) and \( E_2 \) are contained in the straight line \( a_1 + 2a_2 x - 2b_2 y = 0 \). This concludes the proof of (b).

If \( a_1 = b_1 = 0 \) we have \( f(z) = \left[ a_2 (x^2 - y^2) - 2b_2 xy \right] + i \left[ b_2 (x^2 - y^2) + 2a_2 xy \right] \) and if \( a_2 \neq 0 \), \( f \left( x - i \frac{b_2}{a_2} x \right) = \left( a_2 + b_2 \right) x^2 - i \left( b_2 + \frac{b_2^2}{a_2} \right) x^2 \).

Since \( \frac{\ln f(z)}{\ln f(0)} = -\frac{b_2}{a_2} \) the straight line \( y = -\frac{b_2}{a_2} x \) is invariant. This concludes the proof of (c).

If \( a_1 = b_1 = a_2 = 0 \) we have \( f(z) = -2b_2 xy + i b_2 (x^2 - y^2) \) and \( f(-iy) = -ib_2 y^2 \), then the straight line \( x = 0 \) is invariant. This concludes the proof of (d). \( \square \)

5.2. Infinite equilibrium points. The formulas we will use in compactification are presented in the appendix.

Proposition 15. Let \( f \) be a quadratic holomorphic polynomial function as in (10) with \( a_0 = b_0 = 0 \).

(a) If \( a_1 \neq 0 \) then there exist two saddle points on the infinite \( S^1 \). One of them is the \( \omega \)-limit of a repelling focus (or node) and the other one is the \( \alpha \)-limit of an attracting focus (or node).

---

\( \overset{1}{\text{We remember that if an equilibrium of a quadratic system has } V_1 = V_2 = V_3 = 0 \text{ then it is a center. Moreover, for holomorphic systems, the calculation of Lyapunov Constants are not necessary. This is not true for general systems. This fact shows one more special property of holomorphic systems. Above, we present the calculation for the reader to compare the results and has more materiality.}} \)
(b) If \( a_1 = 0 \) and \( b_1 \neq 0 \) then there exist two saddle points on the infinite \( S^1 \) which are connected by a finite orbit.

(c) If \( a_1 = b_1 = 0 \) and \( a_2 \neq 0 \) then there exist two saddle points on the infinite \( S^1 \) with finite separatrix contained in the straight line \( y = -\frac{b_2}{a_2}x \).

(d) If \( a_1 = b_1 = a_2 = 0 \) and \( b_2 \neq 0 \) then there exist two saddle points on the infinite \( S^1 \) with finite separatrix contained in the straight line \( x = 0 \).

\textbf{Proof.} The phase portrait on \( U_1 \) is the central projection of the phase portrait of the system

\[
\begin{align*}
\dot{s} &= b_2 + b_1w + a_2s + b_2s^2 + b_1s^2w + a_2s^3, \\
\dot{w} &= -a_2w - a_1w^2 + 2b_2sw + b_1sw^2 + a_2ws^2.
\end{align*}
\]

The equilibrium points in \( S^1 \) are determined by

\[w = 0, \quad b_2 + a_2s + b_2s^2 + a_2s^3 = (b_2 + a_2s)(1 + s^2) = 0.
\]

Thus if \( a_2 \neq 0 \) the equilibrium is \((-\frac{b_2}{a_2}, 0)\) and if \( a_2 = 0 \) either all points are equilibrium points or none point is equilibrium. Since the jacobian matrix at \((-\frac{b_2}{a_2}, 0)\) is

\[
J \left(-\frac{b_2}{a_2}, 0\right) = \begin{bmatrix}
(a_2 + \frac{b_2^2}{a_2^2}) & (b_1 + a_2^2) \\
0 & -(a_2 + \frac{b_2^2}{a_2^2})
\end{bmatrix}
\]

we conclude that \((-\frac{b_2}{a_2}, 0)\) is a saddle because \(\det\left(J \left(-\frac{b_2}{a_2}, 0\right)\right) < 0\).

The phase portrait on \( U_2 \) is the central projection of the phase portrait of the system

\[
\begin{align*}
\dot{s} &= -a_2 - b_2s - b_1w - a_2s^2 - b_2s^3 - b_1s^2w, \\
\dot{w} &= -b_1sw^2 - a_1w^3 - b_2s^2w + b_2w - 2a_2ws.
\end{align*}
\]

The equilibrium points in \( S^1 \) are determined by \(w = 0, \quad -a_2 - b_2s - a_2s^2 - b_2s^3 = (a_2 + b_2s)(-1 - s^2) = 0\). Thus if \( b_2 \neq 0 \) the equilibrium is \((-\frac{a_2}{b_2}, 0)\) and if \( b_2 = 0 \) either all points are equilibrium points or none point is equilibrium. Since the jacobian matrix at \((-\frac{a_2}{b_2}, 0)\) is

\[
J \left(-\frac{a_2}{b_2}, 0\right) = \begin{bmatrix}
-(b_2 + \frac{a_2^2}{b_2^2}) & -(b_1 + a_2^2) \\
0 & (b_2 + \frac{a_2^2}{b_2^2})
\end{bmatrix}
\]

we conclude that \((-\frac{a_2}{b_2}, 0)\) is a saddle because \(\det\left(J \left(-\frac{a_2}{b_2}, 0\right)\right) < 0\). \(\square\)

\textbf{Theorem 16.} Any quadratic holomorphic polynomial system \( \dot{z} = z(z - (a + ib)) \) is topologically equivalent to one of the 3 phase portraits presented in next figure.
6. Cubic Holomorphic Polynomial Systems

In this section, we are interested in the study all the possibilities of phase portrait of a cubic holomorphic polynomial system. We know that the equilibrium points can be foci, centers, nodes or of purely elliptic type with $2n - 2$ elliptic sectors. So, consider the function.

\[ f(z) = A_0 + A_1z + A_2z^2 + A_3z^3, \quad A_k = a_k + ib_k, \quad A_3 \neq 0. \]

**Remark 17.** As we know, by Fundamental Algebra Theorem we can rewrite $f(z)$ as $f(z) = \alpha(z - A_1)(z - A_2)(z - A_3)$, $\alpha \in \mathbb{C}$. Making the change of variables in the form $z = z + A_1$, and making a rescheduling of time, we obtain
\[ \tilde{f} = e^{i\theta}z(z - B_1)(z - B_2). \]

We denote $B_k = a_k + ib_k$.

**Lemma 18.** Let $v_j$ be an eigenvalue of the system $\dot{z} = f(z)$. Then $e^{i\theta}v_j$ is an eigenvalues of $\dot{z} = e^{i\theta}f(z)$.

**Proof.** Let $p_j$ be a complex number $j = 1, \cdots, n$ such that $f(p_j) = 0$. As we know, the eigenvalues of $\dot{z} = f(z)$ are given by $f'(p_j) = v_j$. It is clear that $p_j$ is also an equilibrium point of $\dot{z} = e^{i\theta}f(z)$. Then, their respective eigenvalues are given by $e^{i\theta}f'(p_j) = e^{i\theta}v_j$. □

**Remark 19.** Note that the phase portrait of the system $\dot{z} = f(z)$ and $\dot{z} = e^{i\theta}f(z)$ will not necessarily be the same. For example, take $\dot{z} = iz$ and $\dot{z} = e^{i\pi/2}iz$. For the first system we have a center in the origin. However, taking $\theta = \pi/2$ we obtain an attracting node in the origin. On the other hand, as already mentioned, holomorphic systems have only three kinds of simple equilibrium points that are foci, centers and nodes (see Theorem 2.1 of [1]). Therefore, in order to simplify the study we will consider $\theta = 0$.

To get eigenvalues of (14), it is enough to calculate $f'(z)$ at the equilibrium points $0, B_1$ and $B_2$, that is, the eigenvalues are given by $f'(0), f'(B_1)$ and $f'(B_2)$ and their respective conjugates, see [9]. Therefore, the eigenvalues are given by $v_0 = B_1B_2$ and $\overline{v_0} = B_1B_2^2$, $v_{B_1} = B_1^2 - B_1B_2$ and $\overline{v_{B_1}} = B_1^2 - B_1B_2$, and $v_{B_2} = B_2^2 - B_1B_2$ and $\overline{v_{B_2}} = B_2^2 - B_1B_2$.

To study the infinite equilibrium points, we use the Poincaré Compactification. The expression of Poincaré Compactification in the chart $U_1$ is given by

\[ \dot{s} = w^3 \left( -su \left( \frac{1}{w}, \frac{s}{w} \right) + u \left( \frac{1}{w}, \frac{s}{w} \right) \right), \quad \dot{w} = -w^4u \left( \frac{1}{w}, \frac{s}{w} \right). \]

And the expression in $U_2$ is given by

\[ \dot{s} = w^3 \left( -u \left( \frac{s}{w}, \frac{1}{w} \right) - su \left( \frac{s}{w}, \frac{1}{w} \right) \right), \quad \dot{w} = -w^4v \left( \frac{s}{w}, \frac{1}{w} \right). \]

Note that, in this charts, the point $(s, w)$ at infinity has its coordinate in $(s, 0)$. So, for the chart $U_1$ we must study the system $\dot{s} = 2s(s^2 + 1), \dot{w} = 0$.

Note that $s = 0$ is a saddle of the system above. For the other charts, following the same steps, we also found a saddle point. We call the four points at infinity as $I_1, I_2, I_3$ and $I_4$. 

Remark 20. We would like to emphasize, as already mentioned in [23], that conformal conjugacy is stronger than topological conjugacy or topological equivalence. Moreover, the angles in the tangent space are preserved.

Theorem 21. If we distinguish nodes and foci, there are only nine different topologically phase portraits of the cubic holomorphic system. Without this distinction, there are only six phase portraits.
Remark 22. Without distinction between nodes and foci, the phase portraits represented by the Figures c2), c4) and c9), are topologically equivalent. The phase portraits c6) and c7) are also topologically equivalent.

Proof. To prove this result is enough to show that among all configurations possible, some of them are no-realizable and the remaining is obtained. All possible configurations are given by

(a) 2 centers and 1 node/focus, (g) 1 center and 1 double,
(b) 2 nodes and 1 center/focus, (h) 1 node and 1 double,
(c) 3 centers, (i) 1 focus and 1 double,
(d) 3 nodes, (j) 1 center and 2 foci,
(e) 1 triple, (k) 1 node and 2 foci,
(f) 3 foci, (l) 1 center, 1 node and 1 focus.

The items (a) and (b) are no-realizable. From Proposition 5 it is impossible to obtain two centers and one focus/node. In the same way, it is impossible to obtain two nodes and one focus/center. The remaining cases, all of them are realizable and we are going to show examples.

According [2], there exists only possibility for three centers, see the phase portrait c1). Using Proposition 5 we have Figure c2). Using Proposition 6 we obtain Figure c3). For the remaining cases, it is sufficient to analyze the possibilities of the separatrices. Before we continue the proof, let us analyze Figure c2).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig8.png}
\caption{Impossible orbit paths.}
\end{figure}
Since $p_2$ is an attracting node, it is the $\omega - limit$ of $I_1$ and $I_3$. Then, there is no orbit coming out from $p_1$ and arriving at $I_2$. Analogously, the same conclusion is true for any orbit coming out from $p_3$ and arriving at $I_4$. Therefore, the only phase portrait possible is given by $c2$.

In the case $(f)$ we have that $p_1$ is a repelling focus and $p_2$ and $p_3$ are attracting foci. The orbits coming from $p_1$ have $I_2$, $I_4$, $p_2$, and $p_3$ as $\omega - limit$. The orbits coming from $I_1$ and $I_3$ have $p_2$ and $p_3$ as $\omega - limit$. See, Figure $c4$.

In the case $(g)$, we have one double point formed by two elliptic sectors separated by parabolic sectors and one center. See Figure $c5$.

In the case $(h)$, we have one node and one double point. The double point is formed by two elliptic sectors separated by parabolic sectors. The attracting node is the $\omega - limit$ of the orbits coming from the infinity and from the double point. See $c6$. The same happens for the case $(i)$. See Figure $c7$.

In the case $(j)$, we have one center $p_1$, and two foci. The point $p_2$ is repelling, and the point $p_3$ is attracting. Since $p_3$ is attracting, the orbits coming from $I_1$ and $p_2$ have $p_3$ as $\omega - limit$. We also have that $I_2$ is the $\omega - limit$ of the orbits coming from $p_2$. See Figure $c8$.

In the case $(k)$, we have that $p_1$ is an attracting node, $p_2$ is an attracting focus, and $p_3$ is a repelling focus. The orbits coming from $p_3$ have, $I_2$, $I_4$, $p_1$, and $p_2$ as $\omega - limit$. The orbits coming from $I_3$ and $I_1$ have $p_1$ and $p_2$ as $\omega - limit$. See Figure $c9$.

Finally, in the case $(l)$, we have a center $p_1$, a repelling node $p_2$ and an attracting focus $p_3$. The point $p_3$ is the $\omega - limit$ of the orbits coming from $I_1$ and $p_2$. We also have that $I_2$ is $\omega - limit$ of the orbit coming from $p_2$. See Figure $c8$. \qed

Remark 23. This result is similar to the result showed in [24], where the authors study the phase portrait of Abel polynomial systems, that is, $F(z) = (z - A)(z - B)(z - C)$, where $A$, $B$ and $C$ are complex numbers. However, using remark (17), to studying $F(z) = (z - A)(z - B)(z - C)$ is the same to as studying $F(z) = z(z - A)(z - B)$. In [24] the authors say that are not possible the case $l$. Since we work with two less parameters, we can find conditions to exist 1 center, 1 node and 1 focus. Moreover, here we use $F'(0), F'(A)$ and $F'(B)$ to find the eigenvalues.

Remark 24. In order to obtain the phase portrait in Figures 1) – 9), we take the following parameter values:

(a) 2 centers and 1 node/focus: no-realizable.
b) 2 nodes and 1 center/focus: no-realizable.
(c) 3 centers: $\dot{z} = z(z - (1 + i))(z - (3 + 3i))$.
(d) 3 nodes: $\dot{z} = z(z - 1)(z - 2)$.
e) 1 triple: $\dot{z} = z^3$.
f) 3 foci: $\dot{z} = z(z - (1 - 3i))(z - (2 + 4i))$.
g) 1 center and 1 double: $\dot{z} = z(z - (1 - i))(z - (1 - i))$.
h) 1 node and 1 double: $\dot{z} = z(z + i)(z + i)$.
i) 1 focus and 1 double: $\dot{z} = z(z - (1 + 3i))(z - (1 + 3i))$.
j) 1 center and 2 foci: $\dot{z} = z(z - (3/2 - i))(z - (2 - 3i))$.
k) 1 node and 2 foci: $\dot{z} = z(z - (2/3 - i))(z - (2 - 3i))$.
l) 1 center, 1 node and 1 focus: $\dot{z} = z(z - (2 - 3/2i))(z - (36/25 - 48/25i))$. 
7. Quartic Holomorphic Polynomial System

Consider a quartic holomorphic polynomial function

$$F(z) = A_0 + A_1 z + A_2 z^2 + A_3 z^3 + A_4 z^4, \quad A_k = a_k + ib_k, \quad A_4 \neq 0.$$  

By the Remark 17, it is enough study the system $$F(z) = z(z - B_1)(z - B_2)(z - B_3).$$

It is clear that the roots of $$F(z) = z(z - B_1)(z - B_2)(z - B_3)$$ are 0, B_1, B_2 and B_3 and the associated eigenvalues are $$v_0 = -B_1 B_2 B_3,$$ $$v_{B_1} = B_3^3 - B_2^2 B_3 - B_1^2 B_3 + B_1 B_2 B_3,$$ $$v_{B_2} = -B_1 B_3^2 + B_1 B_2 B_3 + B_2^3 - B_2^2 B_3$$ and $$v_{B_3} = B_1 B_2 B_3 - B_1 B_2^3 - B_2 B_3^2 + B_3^3$$ and their respective conjugates.

Remark 25. As in Section 6, it is easy to find six saddle points at infinity.

![Figure 9](image_url)

**Figure 9. Saddle points at infinity for a quartic system.**

Remark 26. As we can see in [1], when we have four equilibrium points of type focus, the points can take four different geometrical distributions.

- Collinear: all of them are aligned.
- Triangle: three on the vertices of a triangle and the other one inside.
- Border: three aligned and the other one not.
- Quadrilateral: on the vertices of a quadrilateral.

Proposition 27. If a quartic system has four equilibrium points, all of them of focus type, then its phase portrait is given one of the figures Q22)-Q27).

See [1] for a proof.

Theorem 28. If we distinguish nodes and foci, there are twenty-nine topologically different phase portraits of the quartic holomorphic system. Without this distinction, there are twenty two phase portraits. See figures below.

![Figure 10](image_url)

**Figure 10. Geometrical distributions.**
Proof. Following the same scheme of Theorem 21, to prove this result is enough to show that, among all configurations, some of them are no-realizable and the remaining is obtained. All possible configurations are given by:

(a) 3 centers and 1 node/focus, 
(b) 3 nodes and 1 center/focus, 
(c) 1 quadruple, 
(d) 4 centers, 
(e) 4 node, 
(f) 4 foci, 
(g) 1 node, 2 foci and 1 center, 
(h) 2 foci and 1 double, 
(i) 3 foci and 1 center, 
(j) 2 centers and 1 double, 
(k) 1 focus and 2 double, 
(l) 1 double and 2 node, 
(m) 1 center and 2 double, 
(n) 1 node and 2 double, 
(o) 2 node and 1 double, 
(p) 1 center, 1 double and 1 focus, 
(q) 1 center, 1 double and 1 node, 
(r) 1 double, 1 focus and 1 node, 
(s) 2 foci and 2 centers, 
(t) 1 node, 1 center and 2 foci, 
(u) 2 nodes, 1 center and 1 focus, 
(v) 4 foci.

From the Proposition it is impossible to obtain items (a) and (b). All the remaining cases are realizable and we are going to show examples.

For the case (c), using Proposition we obtain Figure Q1. The items (d) and (e) were analysed in and correspond to Figures Q2, Q4 and Q28.

For the case (f), using Remark we have 4 possibilities that correspond to Figures Q22–Q27. According , the phase Q25 to Q27 are obtained considering the system ̇z = ((−1+2i)z(z−3)(z−2i))(z−(η+2i)). If η = η^± = (−5±√89)/2, we have a center. If η ∈ (η^−, η^+) we have a repelling focus and otherwise an attracting focus. The phase portraits Q25 to Q27 are obtained taking η = 2, η ≈ 2.2 and η = 3 respectively.
For the case \((g)\) we have that \(p_2\) is an attracting node, \(p_3\) is a center and \(p_1\) and \(p_4\) are repelling foci. \(p_2\) is the \(\omega\)-limit of the orbits coming from \(I_4, I_6, p_1,\) and \(p_4\). In the same way, since \(p_1\) and \(p_4\) are repelling, their respective \(\omega\)-limit are \(I_3\) and \(I_5\). See Figure Q6.

For the case \((h)\) we have that \(p_2\) is an attracting focus and \(p_1\) is a repelling focus. Moreover, we have one double point. The orbits coming from \(p_1\) have as \(\omega\)-limit \(I_3, I_1\), the double point, and the point \(p_2\). The orbit coming from \(I_2\) has \(p_2\) as \(\omega\)-limit. See Figure Q7.

For the case \((i)\) we have three foci, the points \(p_2\) and \(p_3\) are attracting, and the point \(p_1\) is repelling. Moreover, we also have a center \(p_4\). This center is contained in a region limited by the orbit coming from \(I_3\), and arriving at \(I_4\). Since the points \(p_2\) and \(p_3\) are attracting, they are \(\omega\)-limit for the orbits coming from \(I_2, I_6\) and the orbits coming from \(p_1\). Furthermore, the orbits coming from \(p_1\) also have \(I_1\) and \(I_5\) as \(\omega\)-limit. See Figure Q8.

For the case \((j)\) we have two centers \(p_1\) and \(p_2\) and one double point. The centers are contained in the regions limited by the orbits coming from \(I_4\) and arriving at \(I_5\) and the orbits coming from \(I_2\) and arriving at \(I_1\), respectively. See Figure Q9.

For the case \((k)\) we have one repelling focus \(p_1\) and two double points. On the other hand, since \(p_1\) is a repelling focus, the \(\omega\)-limit of the orbits coming from \(p_1\) only can be the double point and \(I_3\). See Figure Q10.

For the case \((l)\) we have an attracting node \(p_1\), a repelling node \(p_2\) and one double point. The orbits coming from the double point of the left side have the point \(p_1\) and \(I_5\) as \(\omega\)-limit. The point \(p_1\) is also \(\omega\)-limit of the orbits coming from \(I_6\). For the right side of the double point, we have that \(p_2\) is repelling with \(\omega\)-limit being \(I_3\) and the double point. See Figure Q11.

For the case \((m)\) we have one center \(p_1\) and two double points. The center is contained in a region limited by the orbits coming from \(I_2\) and arriving at \(I_1\). See Figure Q12.

For the case \((n)\) we have one attracting node \(p_1\) and two double points. We have 4 orbits coming from double points with \(\omega\)-limit being \(I_1, p_1, I_5\) and \(I_3\). And we also have the double point being \(\omega\)-limit of the orbits coming from \(I_2\) and \(I_4\). Lastly, we can see that \(p_1\) is \(\omega\)-limit of the orbits coming from \(I_6\). See Figure Q13.

For the case \((o)\) we have two repelling foci \(p_1\) and \(p_2\) which are the \(\omega\)-limit of the orbits coming from \(I_1\) and \(I_3\) respectively. Moreover, the \(\omega\)-limit of the orbits coming from \(I_2, I_4, I_5,\) and \(I_6\) is the double point. See Figure Q29.

For the case \((p)\), we have a center \(p_2\), an attracting focus \(p_1\), and a double point. \(p_1\) is the \(\omega\)-limit of the orbits coming from the double point, and of the orbits coming from \(I_2\). See Figure Q15.

For the case \((q)\) we have a center \(p_1\), an attracting node \(p_2\) and a double point. \(p_2\) is the \(\omega\)-limit of the orbits coming from double point, and of the orbits coming from \(I_2\). See Figure Q16.

For the case \((r)\) we have a repelling focus \(p_1\), an attracting node \(p_2\) and a double point. The orbits coming from \(p_1\) have \(\omega\)-limit in the double point, \(I_3, I_5,\) and \(p_2\). Lastly, \(p_2\) is \(\omega\)-limit of the orbits coming from \(I_4\). See Figure Q17.

For the case \((s)\) we have two foci. The point \(p_3\) is an attracting focus and the point \(p_4\) is a repelling focus. Furthermore, we also have two center \(p_1\) and \(p_2\). Each center is contained in a limited region formed for a separatrix coming \(I_3\) and
arriving at $I_4$ and coming from $I_6$ and arriving $I_1$. The orbits coming from $p_4$ have $\omega$–limit in $I_5$ and $p_2$. The orbits coming from $I_2$ are arriving at $p_2$. See Figure Q18.

For the case (t) we have an attracting node $p_2$, a center $p_1$, an attracting focus $p_3$ and a repelling focus $p_4$. The orbits coming from $p_4$ have, $p_2$, $p_3$, $I_1$, and $I_3$ as $\omega$–limit. On the other way, the only possibility for the orbits coming from $I_2$ and $I_6$ is arriving at $p_3$ and $p_2$ respectively. This case is Figure Q20.

For the case (u) we have an attracting node $p_2$, a repelling node $p_3$, a center $p_1$ and a repelling focus $p_4$. $p_2$ is the $\omega$–limit from the orbits coming from $p_3$, $p_4$, $I_2$, and $I_4$. On the other hand, $I_1$ and $I_3$ are $\omega$–limit of the orbits coming from $p_3$ and $p_4$ respectively. This case is Figure Q21.

\begin{remark}
To obtain the phase portrait in Figures (a) to (t), we take the following systems.
\end{remark}

\begin{enumerate}
\item 1 quadruple: \( z = z^4 \).
\item 4 centers: \( z = (z - i)(z - 2i)(z - 3i) \).
\item 4 foci: \( z = (z - (1 - 3i))(z - (2 - 2i))(z - (3 - 3i)) \).
\item 4 nodes: \( z = z(z - 1)(z - 2)(z - 3) \).
\item 4 foci and 1 center: \( z = (z - (3 + 2i))(z - 5/3)(z - (3 - 2i)) \).
\item 1 node, 2 foci and 1 center: \( z = (z - (1 - 3i))(z - (2 - 2i))(z - i) \).
\item 2 foci and 1 double: \( z = z^2(z - (2 - 2i))(z - (3 - i)) \).
\item 3 foci and 1 center: \( z = (z - (1 - 3i))(z - 2i)(z - 3) \).
\item 2 centers and 1 double: \( z = z^2(z - 2i)(z + i) \).
\item 1 focus and 2 double: \( z = z^3(z - (3 + i)) \).
\item 1 double and 2 node: \( z = z^2(z - 1)(z + 1) \).
\item 1 center and 2 double: \( z = z^2(z - i) \).
\item 1 node and 2 double: \( z = z^3(z + 1) \).
\item 2 foci and 1 double: \( z = z^2(z - 1 - 3i)(z - (2 - 2i)) \).
\item 1 center, 1 double and 1 focus: \( z = z^2(z - (1 + 3i))(z - 1) \).
\item 1 center, 1 double and 1 node: \( z = z^2(z - (1 - i))(z - 1) \).
\item 1 double, 1 focus and 1 node: \( z = z^2(z - 2 + 2i)(z - 2i) \).
\item 2 foci and 2 centers: \( z = (z - (1 + i))(z - (3/5 + 3i))(z - (3 - 2i)) \).
\item 1 node, 1 focus and 1 double: \( z = z^2(z - (1 - 3i))(z - (1 - 1/3 - 2i)) \).
\item 1 node, 1 center and 2 foci: \( z = (z - (2 + 2i))(z - (4 - 2i))(z - (3 - i)) \).
\item 2 nodes, 1 center and 1 focus: \( z = (z - (3 - i))(z - (39/25 - 52/25i))(z - (507/125 - 169/125i)) \).
\item 4 foci colinear: \( z = (-1 + 3i)(z - 1)(z - 4)(z - 8) \).
\item 4 foci triangle: \( z = (-1 + 3i)(z - (1 + 3i))(z - (2 - 2i)) \).
\item 4 foci border: \( z = (-1 + 3i)(z - 1)(z - 2)(z = (3 + 12i)) \).
\item 4 foci Quadrilateral: \( z = (1 + 2i)(z - 3)(z - 2i)(z + (22/10 + 2i)) \).
\item 4 foci Quadrilateral: \( z = (1 + 2i)(z - 3)(z - 2i)(z - (3 + 2i)) \).
\item 4 centers: \( z = z^3 - i/3 \).
\end{enumerate}

\begin{remark}
The examples given in phase portraits Q22) – Q27 was given in [1] and Figure 28) was given in [2].
8. Phase portrait of the family $\dot{z} = \frac{1}{f(z)}$

In this Section, we study the phase portraits of the family $\dot{z} = \frac{1}{f(z)}$, with $f(z)$ polynomial of degree 2, 3, and 4.

**Proposition 31.** Let $f(z)$ be a rational function, i.e., $f(z) = \frac{P(z)}{Q(z)}$, where $P(z) = a_nz^n + \cdots + a_0$ and $Q(z) = b_mz^m + \cdots + b_0$ are polynomials in $z$ of degree $n$ and $m$ respectively. Let $c$ be the residue of $g(z) = \frac{Q(1/z)}{z^2P(1/z)}$ at $z = 0$. Then, there exists $R > 0$ such that the corresponding equation $\dot{z} = f(z)$ is conformally conjugated, in $\C \setminus \mathcal{D}(0, R)$, to

(a) $\dot{z} = \frac{1}{z^{m-n}} + c\frac{1}{z^{2(m-n)+1}}$, if $n < m + 1$,
(b) $\dot{z} = \frac{a_n/b_m}{z}$, if $n = m + 1$,
(c) $\dot{z} = z^2$, if $n = m + 2$,
(d) $\dot{z} = (z)^{n-m}$, if $n > m + 2$.

See [23] for a proof.

**Proposition 32.** Consider the equations given by $\dot{z}_1 = f_1(z)^{-1}$, $\dot{z}_2 = f_2(z)^{-1}$ and $\dot{z}_4 = f_4(z)^{-1}$, where $f_1(z), f_2(z), f_3(z)$ are polynomials of degree 2, 3 and 4, respectively. Then, in a neighborhood of infinity, the phase portrait of each equations above is conformally conjugated to the phase portrait in a neighborhood of $\dot{z} = (1/z)^2$, $\dot{z} = (1/z)^3$ and $\dot{z} = (1/z)^4$, respectively.

Figure 11. Dynamic near infinity of $\dot{z} = \frac{1}{z^2}$, $\dot{z} = \frac{1}{z^3}$ and $\dot{z} = \frac{1}{z^4}$ respectively.

**Proof.** Applying Proposition 31, item (a), we get $n = 0$ and $m = 2$ for $\dot{z}_1$, $n = 0$ and $m = 3$ for $\dot{z}_2$ and $n = 0$ and $m = 4$ for $\dot{z}_3$. Then, we have $\dot{z} = (1/z)^2 + c_1(1/z)^3$, $\dot{z} = (1/z)^3 + c_2(1/z)^7$ and $\dot{z} = (1/z)^4 + c_3(1/z)^9$. As we have $c_1 = c_2 = c_3 = 0$, we obtain $\dot{z} = (1/z)^2$, $\dot{z} = (1/z)^3$ and $\dot{z} = (1/z)^4$ respectively. □

In this section, we study the phase portrait of the systems

\begin{align*}
(16) & \quad \dot{z} = \frac{1}{z(z - A_1)}, \\
(17) & \quad \dot{z} = \frac{1}{z(z - A_1)(z - A_2)}.
\end{align*}
and

\[ \dot{z} = \frac{1}{z(z - A_1)(z - A_2)(z - A_3)}. \]

Applying the change of variables \( w = \frac{1}{z} \) in \( \dot{z} = \frac{1}{z^2} \), \( \dot{z} = \frac{1}{z^3} \) and \( \dot{z} = \frac{1}{z^4} \), we obtain

\[ \dot{w} = -\frac{1}{z^2} \dot{z} = -\frac{1}{z^2} \cdot \frac{1}{z^2} = -\frac{1}{z^4} = -w^4, \]
\[ \dot{w} = -\frac{1}{z^2} \dot{z} = -\frac{1}{z^2} \cdot \frac{1}{z^3} = -\frac{1}{z^5} = -w^5, \]
\[ \dot{w} = -\frac{1}{z^2} \dot{z} = -\frac{1}{z^2} \cdot \frac{1}{z^4} = -\frac{1}{z^6} = -w^6, \]

respectively.

We know that the phase portrait of \( \dot{w} = -w^5 \) is topologically equivalent in a neighborhood of the origin of \( w = 0 \), which is equivalent to \( z = \infty \), to that of Figure 6.(1). Applying a blow-up of \( w = 0 \), and a simple inversion we obtain Figure 8.

![Figure 8](image_url)

**Figure 12.** Topological phase portrait of \( w = -w^5 \) at \( w = 0 \) and blow-up of the origin.

Now, we study the phase portrait in the whole Poincaré disc. The next theorems provides the phase portrait of the systems (17) and (18).

**Theorem 33.** For the system \( \dot{z} = \frac{1}{z(z - A_1)} \), we have, through topological equivalence, the following phase portraits.

![Phase portraits](image_url)

S1) S2) S3)
\begin{proof}
This Theorem can also be found in \cite{15} and, for completeness sake we will prove it below. Firstly, let us consider two cases. If $A_1 = 0$ then the origin is a pole of order 2 and by Proposition \ref{prop:A10} we have Figure S1). If $A_1 \neq 0$, then $A_1$ and the origin are distinct poles of order 1. By Proposition \ref{prop:A1neq0} we know the phase portrait near infinity, so it must be given by Figure S2) or S3) depending on the separatrices connection. Moreover, each case is realizable. \qed

\textbf{Theorem 34.} For the system \( \dot{z} = \frac{1}{z(z-A_1)(z-A_2)} \), we have, through topological equivalence, the following phase portraits.

\begin{center}
\textbf{Sc1)}
\end{center}

\begin{center}
\textbf{Sc2)}
\end{center}

\begin{center}
\textbf{Sc3)}
\end{center}

\begin{center}
\textbf{Sc4)}
\end{center}

\begin{proof}
If $A_1 = A_2 = 0$, then the origin is a pole of order 3 and by Proposition \ref{prop:A1A20} we have Figure Sc1. For the remaining cases, the phase portraits depend on separatrices connections. If $A_1 = A_2 \neq 0$, we have two poles, one of order 1 and other of order 2. Then, we have one possibility to connect and it is realizable, see figure Sc2. For the last case, we have $A_1 \neq A_2$ and both distinct of zero. In this case we have 3 polos of order one and we have 2 possibilities to connect and they are realizable, see figures Sc3 and Sc4.

To obtain the phase portrait in Figures Sc1 to Sc4, we take the following systems.

\begin{align*}
(1) \quad & \dot{z} = \frac{1}{z^3}, \\
(2) \quad & \dot{z} = \frac{1}{z(z-1)^2}, \\
(3) \quad & \dot{z} = \frac{1}{z(z-1)(z-2)}, \\
(4) \quad & \dot{z} = \frac{1}{z(z-i)(z-2)}. 
\end{align*}

\qed
\end{proof}
Theorem 35. For the system \( \dot{z} = \frac{1}{z(z - A_1)(z - A_2)(z - A_3)} \), we have, through topological equivalence, the following phase portraits.

Proof. For this system, we have to consider four cases.
(1) \(A_1 = A_2 = A_3 = 0\).
(2) \(A_2 = A_3 = 0\).
(3) \(A_3 = 0\).
(4) \(A_i \neq 0, i = 1 \cdots 3\).

In case (1) the origin is a pole of order 4 and by Proposition 6 we have Figure Sq1. For (2), we have two poles of order two. Then the possibilities to the phase portrait is given by Figures Sq2, Sq5, Sq9 and Sq10. Analyzing the case (3), we have three polos, being one of order two and two poles of order one. This case corresponds to the phase portraits Sq4, Sq6, Sq8 and Sq11. For the last case (4), we have four poles of order one and it corresponds to Figures 3) and Sq7.

To obtain the phase portrait in Figures Sq1 to Sq11, we take the following systems:

\[
(1) \dot{z} = \frac{1}{z^4}.
(2) \dot{z} = \frac{1}{z(z-3)^3}.
(3) \dot{z} = \frac{1}{z(z-1)(z-2)(z-3)}.
(4) \dot{z} = \frac{1}{z(z-1)(z-2)^2}.
(5) \dot{z} = \frac{1}{z(z-i)^3}.
(6) \dot{z} = \frac{1}{z(z-i)(z-2)^2}.
(7) \dot{z} = \frac{1}{z(z-1)(z-2)(z-i)}.
(8) \dot{z} = \frac{1}{z(z-i)(z-2i)^2}.
(9) \dot{z} = \frac{1}{z^2(z-2)^2}.
(10) \dot{z} = \frac{1}{z^2(z-i)^2}.
(11) \dot{z} = \frac{1}{z(z-3)(z-2)^2}.
\]
\[ j = 2 \cdots 4, \text{ in cartesian coordinates it follows that} \]
\[
\begin{align*}
\phi^2(x, y) & = \frac{1}{3}a_2x^3 - a_2y^2x - b_2x^2y + \frac{1}{2}a_1x^2 - b_1yx + a_0x + \frac{1}{3}b_2y^3 \\
-\frac{1}{2}a_1y^2 - b_0y, \\
\psi^2(x, y) & = a_2x^2y - \frac{1}{3}a_2y^3 + \frac{1}{3}b_2x^3 - b_2xy^2 + a_1xy + \frac{1}{2}b_1x^2 \\
-\frac{1}{2}b_1y^2 + a_0y + b_0x.
\end{align*}
\]
\[
\begin{align*}
\phi^3(x, y) & = \frac{1}{3}a_2x^3 - \frac{3}{2}a_3x^2y^2 - b_3x^3y + \frac{a_2}{3}x^3 + b_2y^3 \\
-\frac{1}{2}a_1y^2 - \frac{1}{2}a_3y^3 - b_1x + a_0x - b_0y, \\
\psi^3(x, y) & = \frac{1}{4}b_2x^4 + \frac{1}{2}b_4x^2y^2 - \frac{1}{2}a_3x^3y - \frac{3}{2}a_2x^2y - b_1xy \\
+\frac{1}{2}b_1y^2 + a_0x + \frac{1}{2}b_2x^2 + b_1x^2 - b_1y^2 + b_0y + b_0x.
\end{align*}
\]
\[
\begin{align*}
\phi^4(x, y) & = \frac{1}{5}a_4x^4 - 2a_4x^3y^2 + a_4y^4x - \frac{1}{4}a_4x^4y + \frac{1}{2}a_4x^2y^3 + \frac{1}{4}a_3x^4 \\
-3a_2x^3y^2 - b_3x^3y + \frac{1}{2}a_3x^3 - a_2y^3x - b_2x^2y \\
+1/2a_1x^2 - b_3x^3y + a_0x - \frac{1}{2}b_0x^2 + \frac{1}{2}a_2x^2y + \frac{1}{2}a_3x^3 \\
-\frac{1}{2}a_1y^2 - b_0y, \\
\psi^4(x, y) & = a_4x^4y - 2a_4x^3y^2 + \frac{1}{2}a_4y^5 + \frac{1}{2}a_4x^4y + b_0x^2y^2 \\
+\frac{1}{2}a_3x^3y + a_3y^3 + \frac{1}{2}a_2x^2y - 3b_3x^2y^3 + \frac{1}{2}a_2x^2y \\
-\frac{1}{2}a_1x^2 - \frac{1}{2}a_2x^3 - \frac{1}{2}b_2xy^2 + a_1xy + \frac{1}{2}b_1x^2 - \frac{1}{2}b_2y^2 \\
+\frac{1}{2}a_3x^3 - \frac{1}{2}a_2x^2y + b_0y + b_0x.
\end{align*}
\]

Let us consider now the conjugate of \( A_1z, A_2z^2, A_3z^3 \) and \( A_4z^4 \). As we get the explicit equations \( \psi^j, j = 2 \cdots 4 \), it is enough to show the level curves of \( \psi \) to obtain the local dynamic. Moreover, following the reasoning of Poincaré Compactification, section \( \text{[5]} \) it is easy to show that the infinity of \( A_1z, A_2z^2, A_3z^3 \) and \( A_4z^4 \) are node points (4, 6, 8 and 10 nodes points respectively) with alternating stability and considering the Poincaré Disk, diametrically opposite points are connected by a separatrix. As seen above, the finite singular points are the saddle type. Then, these facts allow us to make the local study and the global phase portraits of conjugates of \( A_1z, A_2z^2, A_3z^3 \) and \( A_4z^4 \). Note that there is a relation between the degree of \( z \) and the number of sectors. That is, the number of sectors around the origin is given by \( 2n + 2 \), where \( n \) is degree of \( z^n \).

**Remark 36.** We notice that the local dynamics of \( z^n \) and of \( z^n/(1 + z) \) are topologically equivalent. In fact, this also occur to the conjugate of \( z^3 \) and the conjugate of \( z^3/(1 + z^3) \), and the conjugate of \( z^4 \) and conjugate of \( z^4/(1 + z^4) \). In all cases, the local dynamics is formed by \( 2n + 2 \) sectors, where \( n \) is the degree of \( z \), and the origin is of saddle type. This phenomenon does not occur for \( z^n \) and \( z^n/(1 + z^n)^{-1} \), that is, it is a peculiarity of conjugate systems. Naturally, the local dynamics being topological equivalent does not implies the same phase portrait in all Poincaré Disk. This fact can be checked if we take the systems \( (1 + 4i)z^2 \) and \( z^2/(1 + z) \). In both cases, the local dynamics around the origin is formed by \( 6 \) sectors and the origin is saddle type. However, as we can seen in the Figure 12, the phase portrait in all Poincaré disk is different.
Now, let us consider equations
Figure 16. Phase Portraits $\dot{z} = z^2$ and $\dot{z} = z^2/(1 + z)$.

$$
p_2(z) = A_0 + A_1 z + A_2 z^2,
p_3(z) = B_0 + B_1 z + B_2 z^2 + B_3 z^3,
p_4(z) = C_0 + C_1 z + C_2 z^2 + C_3 z^3 + C_4 z^4.
$$

with $A_0 = B_0 = C_0 = 0$, that is, let us consider equations with the form

$$
f_2(z) = A_1 z + A_2 z^2,
f_3(z) = B_1 z + B_2 z^2 + B_3 z^3,
f_4(z) = C_1 z + C_2 z^2 + C_3 z^3 + C_4 z^4.
$$

Compactifying on $f_2, f_3$ and $f_4$, we obtain that the infinity are nodes (4, 6, 8 and 10 node points respectively) with alternating stability. Moreover, $f_2, f_3$ and $f_4$ have 2, 3 and 4 saddle points. Below, we show one example of phase portrait for each $f_j(z)$.

Figure 17. Phase Portraits of $\dot{z} = (1 + i)z + (3 + 4i)z^2$, $\dot{z} = (1 + 3i + z)(2 + 2i + z)(3 + i + z)$, and $\dot{z} = (1 + i)z + (2 - i)z^2 + i z^3 + 2 z^4$, resp.

10. Moebius Systems

Moebius transformations form a class of conformal maps that have a very strong geometric appeal and that have surprising properties. A Moebius transformation is a map of the form $T(z) = \frac{Az + B}{Cz + D}$ satisfying that $AD - BC \neq 0$.

The derivative of a Moebius transformation is given by $T'(z) = \frac{AD - BC}{(Cz + D)^2}$. They are invertible and their inverse is still a Moebius transformation. Moreover they can be obtained by composing translations $z \mapsto z + K$, rotations $z \mapsto \alpha z, |\alpha| = 1$, inversion $z \mapsto \frac{1}{z}$ and homotheties $z \mapsto \alpha z, |\alpha| \neq 1, 0$. 


The most natural way to consider $T$ is as a function of the plane extending into itself $T: \mathbb{C} \rightarrow \mathbb{C}$ defining $T \left( \frac{z}{z} \right) = \infty$ and $T(\infty) = \frac{z}{z}$.

Two important properties of Moebius transformations are as follows:

- Given three distinct points $z_1, z_2, z_3$ in $\mathbb{C}$ and another three distinct points in $w_1, w_2, w_3$ in $\mathbb{C}$ there is only one Moebius transformation satisfying that $T(z_j) = w_j, j = 1, 2, 3$.
- The $\mathcal{F}$ family, formed by the circles of $\mathbb{C}$ is preserved by Moebius $T$ transformations. From the point of view of the plane, this means that the image of circles and straight lines by Moebius transformations is still circles and straight lines.

In this section we describe the phase portrait of holomorphic systems of the type

$$\dot{z} = \frac{Az + B}{Cz + D}, \quad AD - BC \neq 0.$$  \hfill (19)

Equivalently, writing $z = x + iy, T = u + iv, A = a_1 + a_2i, B = b_1 + b_2i, C = c_1 + c_2i$ and $D = d_1 + d_2i$ we get the planar system

$$\dot{u} = \frac{u_20x^2 + u_10x + u_02y^2 + u_01y + u_00}{q(x, y)}, \quad \dot{v} = \frac{v_20x^2 + v_10x + v_02y^2 + v_01y + v_00}{q(x, y)},$$

with

$$q(x, y) = (c_1^2 + c_2^2)x^2 + (2c_1d_1 + 2c_2d_2)x + (c_1^2 + c_2^2)y^2 + (2c_1d_2 - 2c_2d_1)y + d_1^2 + d_2^2$$

and

$$
\begin{align*}
    u_{20} &= a_1c_1 + a_2c_2 \\
    u_{10} &= a_1d_1 + a_2d_2 + b_1c_1 + b_2c_2 \\
    u_{02} &= a_1c_1 + a_2c_2 \\
    u_{01} &= a_1d_2 - a_2d_1 - b_1c_2 + b_2c_1 \\
    u_{00} &= b_2d_2 + b_1d_1
\end{align*}
\quad
\begin{align*}
    v_{20} &= -a_1c_2 + a_2c_1 \\
    v_{10} &= -a_1d_2 + a_2d_1 - b_1c_2 + b_2c_1 \\
    v_{02} &= -a_1c_2 + a_2c_1 \\
    v_{01} &= a_1d_1 + a_2d_2 - b_1c_1 + b_2c_2 \\
    v_{00} &= -b_1d_2 + b_2d_1
\end{align*}
\hfill (20)

**Proposition 37.** The equilibrium points and poles of the Moebius systems (19) are described in the following table.

| $A$ | $B$ | $C$ | $D$ | Equilibrium | Pole |
|-----|-----|-----|-----|-------------|------|
| 0   | 0   | 0   | 0   | -           | $z_0 = 0$ |
| 0   | 0   | 0   | 0   | -           | $z_0 = -\frac{B}{A}$ |
| 0   | 0   | 0   | 0   | $z_0 = 0$   | $z_0 = -\frac{B}{A}$ |
| 0   | 0   | 0   | 0   | $z_0 = 0$   | $z_0 = -\frac{B}{A}$ |
| 0   | 0   | 0   | 0   | $z_0 = -\frac{B}{A}$ | $z_0 = 0$ |
| 0   | 0   | 0   | 0   | $z_0 = -\frac{B}{A}$ | $z_0 = -\frac{B}{A}$ |
| 0   | 0   | 0   | 0   | $z_0 = -\frac{B}{A}$ | $z_0 = -\frac{B}{A}$ |

**Proposition 38.** The Moebius system (19) is conformally conjugated to $f(z) = \frac{A^2}{AD - BC}z$ in a neighborhood of the equilibrium $z_0 = -\frac{B}{A}$ and conformally conjugated to $f(z) = \frac{B}{A}$ in a neighborhood of the pole $z_0 = -\frac{C}{D}$.

**Proof.** In fact, it follows of the derivative of the Moebius map $T$ at $-\frac{B}{A}$ which is given by

$$T'' \left( -\frac{B}{A} \right) = \frac{A^2}{AD - BC}$$
and from the limit
\[ \lim_{z \to -D} \left( z + \frac{D}{C} \right) \frac{Az + B}{Cz + D} = \frac{BC - AD}{C} \neq 0. \]

\[ \square \]

**Proposition 39.** Let \( \dot{z} = T(z) \) be the Moebius system (19) with \( A \neq 0 \).

\( a) \) Its trajectories are contained in the level curves of
\[ H(x, y) = 3 \left( \frac{ACz + (-BC + AD) \log(B + Az)}{A^2} \right). \]

\( b) \) For the Moebius system, we have, through topological equivalence, the following phase portraits.

\[ \text{Figure 18. Topological phase portrait of Moebius System} \]

**Proof.**  
\( a) \) It follows from the primitive
\[ \int \frac{1}{T(z)} \, dz = \int \frac{Cz + D}{Az + B} \, dz = \frac{ACz + (-BC + AD) \log(B + Az)}{A^2}. \]

\( b) \) We have the following cases:

- \( A = 0 \) and \( B \neq 0 \). In this case, the system \( \dot{z} = \frac{B}{Cz} \) is conformally conjugated with \( \dot{z} = \frac{1}{z} \). Then, we have a simple pole singularity at \( z = 0 \). Using Poincaré Compactification, we obtain 4 noodle points at the infinity with alternated stability. This case corresponds the Figure 18-M1).
• $C = 0$ and $D \neq 0$. In this case, the system $\dot{z} = \frac{Az + B}{D}$ is conformally conjugated with $\dot{z} = \eta z$, $\eta \in \mathbb{C}$. If $\Re(\eta) = 0$ and $\Im(\eta) \neq 0$, we obtain the Figure 18-M2). If $\Re(\eta) \neq 0$ and $\Im(\eta) \neq 0$, then we have the Figure 18-M3) and the Figure 18-M4). If $\Re(\eta) \neq 0$ and $\Im(\eta) = 0$, we obtain the Figure 18-M5) and the Figure 18-M6).

• $A \neq 0$, $B \neq 0$, and $C \neq 0$. We have a simple pole at $z = -\frac{D}{C}$ and i) one center or ii) one focus/node. According [23], we have two nodes at infinity with alternated stability. Then, the possibilities for the phase portrait are given by Figures 18-M7), 18-M8) and Figure 18-M9). All this cases are realizable and we can obtain with the following systems:

- $\dot{z} = \frac{(3/13 - 2/13)(1+i)z - 3/2 + i}{z} - \text{case M7},$
- $\dot{z} = \frac{(3/13 - 2/13)(1+i)z - 3/2 + i}{z} - \text{case M8},$
- $\dot{z} = \frac{(12/37 + 2i/37)(1-2i)z - 27/14 + i}{z} - \text{case M9}).$

□

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12. Appendix

12.1. Lyapunov Constants. Consider a differential system of the kind

(21) $\dot{x} = \beta y + \varphi(x, y), \quad \dot{y} = -\beta x + \psi(x, y)$

with $\beta > 0$ and $\varphi, \psi$ analytical functions which vanish together with their first derivatives at the point $(0, 0)$.

Consider the polar change $x = r \cos \theta$, $y = r \sin \theta$. Thus system (21) becomes

(22) $\dot{r} = P(r, \theta), \quad \dot{\theta} = Q(r, \theta)$.

The phase portrait of (22) is composed by the graphics of

$r = f_\rho(\theta), \quad f_\rho(0) = \rho$, solutions of

(23) $\frac{dr}{d\theta} = R(r, \theta) = \frac{P(r, \theta)}{Q(r, \theta)} = R_1(\theta)r + R_2(\theta)r^2 + ...$

with

$R_k(\theta) = \frac{1}{k!} \frac{\partial^k R(r, \theta)}{\partial r^k} |_{r=0}.$

The series expansion of $f_\rho(\theta)$ is denoted by

$f_\rho(\theta) = u_1(\theta)\rho + u_2(\theta)\rho^2 + ...$
Put \( r = f_\rho(\theta) \) in (23) to get
\[
u_1'(\theta)\rho + u'_2(\theta)\rho^2 + \ldots = R_1(\theta)(u_1(\theta)\rho + \ldots) + R_2(\theta)(u_1(\theta)\rho + \ldots)^2 + \ldots
\]
The returning map \( \pi : [0, +\infty) \to \mathbb{R} \) is \( \pi(\rho) = f_\rho(2\pi) - \rho \). The Lyapunov values are given by
\[
V_k = \frac{\pi^{(k)}(0)}{k!}
\]
If there exists \( n \in \mathbb{N} \) such that
\[
\pi'(0) = \ldots = \pi^{(n-1)}(0) = 0, \pi^n(0) \neq 0
\]
then \( n \) is odd. In this case we say that \( \frac{n-1}{2} \) is the multiplicity of the focus.

If \((0, 0)\) is a multiple focus of multiplicity \( k > 1 \) of (21) then
(a) There exist \( \varepsilon_0 > 0 \) and \( \delta_0 > 0 \) such that for any system \( \delta_0 \)-close to system (21) has at most \( k \) limit cycles in a \( \varepsilon_0 \)-neighborhood of \((0, 0)\).
(b) For any \( \varepsilon < \varepsilon_0, \delta < \delta_0 \) and \( 1 \leq s \leq k \) there exists a system which is \( \delta \)-close to (21) and has precisely \( s \) limit cycles in a \( \varepsilon \)-neighborhood of \((0, 0)\).

We can compute the Lyapunov values using the following algorithm.

1. \( R \)
2. \( r_1 = \text{diff}(R, r) \)
3. \( R_1 = \text{subs}(r = 0, R1) \)
4. \( r_2 = \text{diff}(r_1, r) \)
5. \( R_2 = \frac{1}{2}\text{subs}(r = 0, r2) \)
6. \( r_3 = \text{diff}(r_2, r) \)
7. \( R_3 = \frac{1}{4}\text{subs}(r = 0, r3) \)
8. \( r_4 = \text{diff}(r_3, r) \)
9. \( R_4 = (1/24)\text{subs}(r = 0, r4) \)
10. \( r_5 = \text{diff}(r_4, r) \)
11. \( R_5 = (1/120)\text{subs}(r = 0, r5) \)
12. \( r_6 = \text{diff}(r_5, r) \)
13. \( R_6 = (1/720)\text{subs}(r = 0, r6) \)
14. \( r_7 = \text{diff}(r_6, r) \)
15. \( R_7 = (1/5040)\text{subs}(r = 0, r7) \)
16. \( U_2 = \text{int}(R2, \theta = 0..k) \)
17. \( V_2 = \text{subs}(k = 2\pi, U2) \)
18. \( u_2 = \text{subs}(k = \theta, U2) \)
19. \( U_3 = \text{int}(2U2R2 + R3, \theta = 0..k) \)
20. \( V_3 = \text{subs}(k = 2\pi, U3) \)
21. \( u_3 = \text{subs}(k = u, U3) \)
22. \( V_4 = \text{simplify}(\text{subs}(k = 2\pi, U4)) \)
23. \( u_4 = \text{subs}(k = u, U4) \)
24. \( U_5 = \text{int}(R2(2u2u3 + 2u4) + R3(3u2^2 + 3u3) + 4R4u2 + R5, u = 0..k) \)
25. \( V_5 = \text{simplify}(\text{subs}(k = 2\pi, U5)) \)
26. \( u_5 = \text{subs}(k = u, U5) \)
27. \( U_6 = \text{int}(R2(2u2u4 + u3^2 + 2u5) + R3(u2^3 + 6u2u3 + 3u4) + R4(6u2^2 + 4u3) + 5R5u2 + R6, u = 0..k) \)
28. \( V_6 = \text{simplify}(\text{subs}(k = 2\pi, U6)) \)
29. \( V_7 := \text{simplify}(\text{subs}(k = 2\pi, U7)) \)
12.2. Poincaré Compactification. Now we present the formulas concerning the Poincaré compactification for a polynomial differential system with degree $n$ in $\mathbb{R}^2$. More precisely we consider the polynomial differential system

$$\dot{x} = u(x, y), \quad \dot{y} = v(x, y).$$

This polynomial system is extended to an analytic system on a closed disk of radius one, whose interior is diffeomorphic to $\mathbb{R}^2$ and its boundary, the 1–dimensional circle $S^1$; plays the role of the infinity. This closed ball is denoted by $D^1$. We consider 4 open charts on $S^1$

(a) $U_1 = \{(x, y) : x > 0\}$ and $V_1 = \{(x, y) : x < 0\}$,
(b) $U_2 = \{(x, y) : y > 0\}$ and $V_2 = \{(x, y) : y < 0\}$.

The phase portrait on $U_1$ is the central projection of the phase portrait of the system

$$\dot{s} = w^n(-su + v), \quad \dot{w} = w^n(-wu)$$

where $s$ is the coordinate of the tangent line $TS_{(1,0)}^1$ at $(1,0) \in S^1$, and $u,v$ are evaluated at $(1/w, s/w)$. Moreover we consider $w = 0$.

The flow on $U_2$ is determined by the system

$$\dot{s} = w^n(-sv + u), \quad \dot{w} = w^n(-wv)$$

where $s$ is the coordinate of the tangent plane $TS_{(0,1)}^1$ at $(0,1) \in S^1$, and $u,v$ are evaluated at $(s/w, 1/w)$. Moreover we consider $w = 0$.

The expression for the extend differential system in the local chart $V_i$, $i = 1, 2$ is the same as in $U_i$ multiplied by $(-1)$.

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