Abstract. We show that entropy is globally concave with respect to energy for a rich class of mean field interactions, including regularizations of the point-vortex model in the plane, plasmas and self-gravitating matter in 2D, as well as the higher dimensional logarithmic interactions appearing in conformal geometry and power laws. The proofs are based on a corresponding “microscopic” concavity result at finite $N$, shown by leveraging an unexpected link to Kähler geometry and plurisubharmonic functions. Under more restrictive homogeneity assumptions strict concavity is obtained using a uniqueness result for free energy minimizers, established in a companion paper. The results imply that thermodynamic equivalence of ensembles holds for this class of mean field models. As an application it is shown that the critical inverse negative temperatures - in the macroscopic as well as the microscopic setting - coincide with the asymptotic slope of the corresponding microcanonical entropies. Along the way we also extend previous results on the thermodynamic equivalence of ensembles for continuous weakly positive definite interactions, concerning positive temperature states, to the general non-continuous case. In particular, singular situations are exhibited where, somewhat surprisingly, thermodynamic equivalence of ensembles fails at energy levels sufficiently close to the minimum energy level.

1. Introduction

1.1. General Setup. Let $X$ be a topological space, $W$ a symmetric lower semi-continuous (lsc) function on $X^2$ (the pair interaction potential) and $V$ a lsc function on $X$ (the exterior potential), both taking values in $]-\infty, \infty]$. The corresponding $N$-particle mean field $N$-particle Hamiltonian is defined by

$$H^{(N)}(x_1, ..., x_N) := \frac{1}{2} \frac{1}{N} \sum_{i \neq j \leq N} W(x_i, x_j) + \sum_{i=1}^{N} V(x_i).$$

The self-interactions have, as usual, been excluded in order to render $H^{(N)}$ generically finite in the case when $W$ is singular on the diagonal. The corresponding (macroscopic) energy $E(\mu)$ of a probability measure $\mu$ on $X$, i.e. $\mu \in \mathcal{P}(X)$, is defined by

$$E(\mu) := \frac{1}{2} \int_X W \mu \otimes \mu + \int_X V \mu \in ]-\infty, \infty]$$

when $X$ has compact support. The definition can be extended to non-compactly supported measures (see Section 1.2), but for most purposes it will be enough to consider the restriction of $E(\mu)$ to the space of all probability measures on $X$ with compact support, denoted by $\mathcal{P}(X)_0$.

Now, fix also a measure $\mu_0$ on $X$ (the “prior”). Following [32, 18, 33, 19] the entropy $S(e)$ (at energy $e$) is the function on $\mathbb{R}$ defined by $-\infty$ in the case that $\{E(\mu) = e\}$ is
empty and otherwise

\[
\tag{1.3} S(e) := \sup_{\mu \in \mathcal{P}(X)} \{ S(\mu) : E(\mu) = e \}, \quad S(\mu) := -\int_X \log(\mu/\mu_0) \mu
\]

where \( S(\mu) \) is the entropy of \( \mu \) relative to \( \mu_0 \), which, by definition, is equal to \(-\infty\) if \( \mu \) is not absolutely continuous with respect to \( \mu_0 \). A measure \( \mu^* \) maximizing \( S(\mu) \) above is called a maximum entropy measure. In the case when \( \mu_0 \) is a probability measure, that we shall focus on,

\[
\max_{e \in \mathbb{R}} S(e) = S(e_0) = 0, \quad e_0 := E(\mu_0)
\]

(since \( S(\mu) \leq 0 \) with equality iff \( \mu = \mu_0 \)). This setup is modeled on repulsive Hamiltonians (as in the case of identical point vortices described below), but an equivalent setup of “attractive” Hamiltonians is obtained by replacing \( H(N) \) with \(-H(N) \) and \( e \) with \(-e \).

1.2. Background: concavity of \( S(e) \) and thermodynamic equivalence of ensembles. In the case when \( E(\mu) \) is linear on \( \mathcal{P}(X) \) (i.e. \( W = 0 \) it follows directly from the concavity of \( S(\mu) \) on \( \mathcal{P}(X) \) that the entropy \( S(e) \) is concave with respect to \( e \) (see Section 5.3). This is the standard setup in information theory and statistical inference, going back to Shannon and Jaynes [11], but here we will be concerned with the case when \( E(\mu) \) is quadratic, motivated by mean-field models in statistical mechanics (see 7.2 for a comparison between the classical linear setup and the quadratic setup appearing in the context of plasmas). General, non-linear \( E(\mu) \) also appear naturally in engineering optimization [4]. The concavity properties of the entropy \( S(e) \) for mean-field models and other systems with long-range interactions have been studied extensively from various points of views in the last decades; theoretical as well as experimental and numerical [32, 18, 19, 34, 24]. As stressed in [19] the unusual properties of these systems stem from the lack of additivity (i.e. lack of linearity of \( E(\mu) \)). In particular, the question whether \( S(e) \) is globally concave is crucial in connection to negative temperature states in Onsager’s point-vortex model for the large-time limit of turbulent incompressible non-viscous 2D fluids; classical fluids [32, 18], as well as quantum fluids [38, 12]. In the case of vortices of equal circulation moving in the whole plane \( \mathbb{R}^2 \) the vortex-vortex pair interaction potential \( W(x, y) \) is proportional to the Green function for the Laplacian in \( \mathbb{R}^2 \).

\[
\tag{1.4} W(x, y) = -\log(|x - y|).
\]

As emphasized in [33, 58, 19] the relevance of the global concavity of \( S(e) \) stems from the fact that it equivalently means that \( S(e) \) may (under appropriate regularity assumptions discussed in Section 5) be expressed as the Legendre-Fenchel transform of the Helmholtz (scaled) free energy \( F(\beta) \) at inverse temperature \( \beta \):

\[
\tag{1.5} S(e) = \inf_{\beta \in \mathbb{R}} (-F(\beta) + \beta e),
\]

where \( F(\beta) \) is defined as the infimum of the (scaled) free energy functional \( F_\beta(\mu) \):

\[
\tag{1.6} F(\beta) = \inf_{\mu \in \mathcal{P}(X)} F_\beta(\mu), \quad F_\beta(\mu) := \beta E(\mu) - S(\mu),
\]

(where \( F(\mu) \) is defined to be equal to \(+\infty\) if \( S(\mu) = -\infty \)). Accordingly, when \( S(e) \) is globally concave thermodynamic equivalence of ensembles is said to hold [33, 58, 19] (since it amounts to the equivalence between the microcanonical ensemble at a fixed energy \( e \) and the canonical ensemble at a corresponding fixed inverse temperature \( \beta \), in the large \( N \) limit). More generally, this duality fits into primal-dual formulations of non-linear optimization problems, where the free energy functional appears as the augmented Lagrangian [11]. If thermodynamic equivalence of ensembles holds then \( S(e) \) is differentiable at almost
any energy level \( e \) and, by the concavity of \( F(\beta) \), the infimum over the \( \beta \) in formula (1.5) is attained precisely at the inverse temperature

\[
\beta(e) = \frac{dS(e)}{de}.
\]

Remarkably, as stressed already by Onsager in the late 40s [51], this means, since \( S(e) \) is decreasing for \( e > E(\mu_0) \), that in the “high energy regime”

\[
(1.7) \quad E(\mu_0) < e
\]

an energy level \( e \) should correspond to a negative inverse temperature \( \beta \). As a consequence, the repulsive vortex interaction should then become effectively attractive, resulting in the aggregation of microscopic vortices of equal circulation into large-scale coherent clusters (as observed in oceanic and atmospheric fluids, notably Jupiter’s famous great red spot). A few years after Onsager’s prediction the existence of negative temperature states was experimentally verified in nuclear spin systems [53], while the original prediction was quantitatively experimentally demonstrated only very recently in a 2D quantum superfluid (a Bose Einstein condensate [38, 42]). Note that the high energy region only exists if

\[
E(\mu_0) < \infty.
\]

This is automatically the case if \( X \) is compact and \( W \) and \( V \) are locally integrable, but it also holds in many non-compact situations, for example the vortex model in \( \mathbb{R}^2 \), when \( \mu_0 \) is taken as a Gaussian probability measure (incorporating conservation of angular momentum).

As shown in [32], if \( W(x, y) \) defines a weakly positive definite kernel, as in the point-vortex model, then the concavity of \( S(e) \) holds in the “low energy regime”:

\[
e \leq E(\mu_0),
\]

which corresponds to positive inverse temperature \( \beta \) (more precisely, in [32] it is assumed that \( W \) is continuous; the general case is discussed in Section 5.1). However, the concavity may fail in the “high energy regime” and thus the correspondence with negative temperature then breaks down (leading to the peculiar phenomenon of negative heat-capacity [47, 19]). This is illustrated by the mean-field Blume-Emery-Griffiths spin model in [19, Section 4.2.4]. In the case of the point-vortex model the global concavity of \( S(e) \) has been established when \( X \) is the unit-disc in \( \mathbb{R}^2 \) [18] (or a sufficiently small deformation of the unit-disc) or all of \( \mathbb{R}^2 \) [18, 22, 45], while shown to fail for some domains \( X \) (e.g. a sufficiently thin rectangle). The proofs in [18, 22] exploit that in the case of the point-vortex model any minimizer \( \mu_\beta \) of the free energy functional \( F_\beta(\mu) \) satisfies a second order PDE (the Joyce-Montgomery mean field equation/the Liouville equation). This opens the door for the application of various PDE-techniques (uniqueness results, concentration/compactness alternatives, symmetrization arguments, ...).

1.3. **Summary of the main results.** To the best of the authors knowledge there are, apart from a few special cases - such as the BEG-model and the vortex model recalled above - no general global concavity results for mean field Hamiltonians. Even in the case of the regularized vortex model [32] the question of global concavity of \( S(e) \) raised in [32] appears to have been left open. Similar questions have also been put forward in the context of self-gravitating matter, where regularizations appear naturally [13]. Allowing regularizations is also crucial when comparing theoretical results with numerical simulations (such as [36, 29]) to ensure that the concavity of \( S(e) \) is a robust feature of the models in question. The main aim of the present work is to establish the concavity of \( S(e) \) for a rich class of potentials \( W, V \) and priors \( \mu_0 \), including the point-vortex model in \( \mathbb{R}^2 \), as well as its regularizations.
and regularized plasmas and self-gravitating systems in 2D and power-laws. However, since neither explicit calculations, nor PDE-techniques are available for such interactions $W$ we take a different route. First it is shown that the (upper) microcanonical entropy (at energy $e$).

\begin{equation}
S_+^{(N)}(e) := \frac{1}{N} \log \mu_0^{\otimes N} \{ H^{(N)}/N > e \},
\end{equation}

is concave for any finite $N$ (in the context of the point vortex model this microcanonical entropy appears in [18, Theorem 4.2]). Then, letting $N \to \infty$, and using the asymptotics for $S_+^{(N)}(e)$ from [32], the concavity of the upper entropy $S_+(e)$ is obtained (defined by replacing the condition $E(\mu) = e$ in the definition [13] of $S(e)$ with the condition $E(\mu) \geq e$). Hence, the concavity of $S_+(e)$ in the high energy region, $e > E(\mu_0)$, results from the observation that $S_+(e) = S(e)$ there. This derivation of the concavity of $S_+(e)$ is thus in the spirit of statistical mechanics; the macroscopic property in question emerges from a microscopic one. The proof of the concavity of $S_+^{(N)}(e)$ leverages some developments in Kähler geometry [14, 12], centered around complex analogs of the Brunn-Minkowski inequality. Under more restricted assumptions $S(e)$ is shown to be strictly concave, using a different (macroscopic) approach - which is more in the spirit of [18] - based on a uniqueness result for free energy minimizers of independent interest.

Before turning to a more precise description of the main present results it may be worth emphasizing that the concavity of $S_+^{(N)}(e)$ is considerably stronger than the concavity of $S(e)$ and does not require the mean field scaling (nor the permutation symmetry). Thus it also applies to the microcanonical study of small systems, considered in the physics literature (see, for example, [55, 36, 31]). The relation to the setup in [55, 31] becomes clearer in the equivalent setup of “attractive” Hamiltonians obtained by replacing the Hamiltonian $H^{(N)}$ with $-H^{(N)}$ and $e$ with $-e$. The concavity of $S_+^{(N)}(e)$ then translates into the concavity of

\begin{equation}
S_-^{(N)}(e) := \frac{1}{N} \log \mu_0^{\otimes N} \{ H^{(N)}/N < e \},
\end{equation}

called the bulk-entropy in [55] and the microcanonical Gibbs entropy in [31]. In recent years it has been debated whether this microcanonical entropy is physically more relevant than the microcanonical Boltzmann entropy, obtained by replacing the volume $\mu_0^{\otimes N} \{ H^{(N)}/N < e \}$ with its derivative with respect to $e$ (the surface area of the level-set $H^{(N)}/N = e$); see the discussion in [39] and references therein. The present results may, perhaps, be interpreted as a case for bulk/Gibbs entropy as this entropy is shown to be concave in our class of “attractive” Hamiltonians, while the Boltzmann entropy is not always concave in this class (as discussed in connection to Theorem [57]). On the other hand, in the case when $\mu_0$ is Lesbesgue measure on $\mathbb{R}^{2n}$ the bulk/Gibbs and the Boltzmann entropy coincide in the limit when $N \to \infty$ (in the classical thermodynamical limit this is discussed in [40, Section 6.2] and in the present mean field setup it can be shown that both limits coincide with $S(e)$. [8]

Let now $X$ be a (possible non-compact) subset of $\mathbb{R}^{2n}$ and let $\phi$ be a defining function for $X$, i.e. a continuous function such that

$X = \{ \phi \leq 0 \}$.

Endow $X$ with a probability measure $\mu_0$ which is absolutely continuous wrt Lebesgue measure $d\lambda$

\begin{equation}
\mu_0 = e^{-\Psi_0} d\lambda.
\end{equation}
on $\mathbb{R}^{2n}$. We will identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ in the usual way and denote by $(z_1, \ldots, z_n)$ the standard holomorphic coordinates on $\mathbb{C}^n$.

**Concavity of $S^{(N)}_\pm(e)$ and $S(e)$ in the high energy region $e > e_0$.** The main results, saying that upper microcanonical entropy $S^{(N)}_\pm(e)$ and the entropy $S(e)$ are concave in the high-energy regime \[ \text{(Theorem 6.7 and Theorem 6.9)}, \] are shown to hold under appropriate plurisubharmonicity and symmetry properties of the data. Denoting by $PSH_\alpha$ the class of all plurisubharmonic functions which are invariant under the action
\begin{equation}
(z_1, \ldots, z_n) \mapsto (e^{i\alpha_1 \theta} z_1, \ldots, e^{i\alpha_n \theta} z_n)
\end{equation}
for any $\theta \in \mathbb{R}$, for a given “weight vector” $\alpha \in ]0, \infty[^n$, the main results hold under the following

**Main Assumptions:** $\phi, \Psi_0, -V$ are in the class $PSH_\alpha(\mathbb{C}^n)$ and $-W$ is in $PSH_{\alpha, \alpha}(\mathbb{C}^n \times \mathbb{C}^n)$ for some $\alpha \in ]0, \infty[^n$.

The definition of plurisubharmonicity is recalled in Section 2.3. For the moment we just point out that the class $PSH_\alpha$ is very rich. For example, when the weights $\alpha_i$ are positive integers the class $PSH_\alpha$ contains the functions
\begin{equation}
\psi(z) = \log\left(\sum_{j=1}^{r} |P_j(z)|^2\right)
\end{equation}
where $P_j$ is a polynomial in $z_1, \ldots, z_n$, which is homogeneous wrt the scaling action by $\mathbb{C}^*$ on $\mathbb{C}^n$ with weights $\alpha$. In particular, for any $\alpha$ the class $PSH_\alpha$ contains $\psi(z) = \log |z|$ as well as $\Psi_0(z) := \sum_{i=1}^{\alpha_i} |z_i|^2$, for any positive $\lambda_i$. Hence the Main Assumptions apply to the corresponding Gaussian measures
\begin{equation}
\mu_0 = e^{-\sum_{i=1}^{\alpha_i} |z_i|^2} d\lambda.
\end{equation}

In the case when the data is invariant under rotations of the $z_i$-variables this is - from a physical point of view - the most natural choices of priors, as they incorporate preservation of angular momentum in the $z_i$-variables (see the discussion in Section 3.3).

An important general feature of the class $PSH_\alpha(\mathbb{C}^n)$ is that is closed under scaling by positive numbers, taking sums and maxima, as well as under composition with a complex linear map on $\mathbb{C}^n$ or an increasing convex function, defined on the range of a given $\psi \in PSH_\alpha(\mathbb{C}^n)$. This means, in particular, that the Main Assumptions are stable under a range of different regularizations of the data. For example, the Main Assumptions apply to the point-vortex model in $X := \mathbb{R}^2$ (formula 1.3) endowed with a centered Gaussian measure. But the Main Assumptions also apply to the standard continuous regularization and smooth regularization of the point-vortex model where, for a given positive number $\delta$, the pair interaction $W(x, y)$ is, in the continuous case, modified so that it is constant on $|x - y| \leq \delta$, while the smooth regularization is defined by
\begin{equation}
W_\delta(x, y) = -\frac{1}{2\pi} \log(|x - y| + \delta).
\end{equation}

More generally, they apply to the regularizations obtained by convolution of $-\log |x|$ with a positive sufficiently rapidly decreasing density on $\mathbb{R}^2$, as used in the vortex blob model [18 Section 6.2.1] (or more generally to the convolution of $-\log |x - y|$ with a smooth density on $\mathbb{R}^2 \times \mathbb{R}^2$). An abundance of other examples in $PSH_\alpha$ may be obtained by replacing $\psi$ in formula with $\chi \circ \psi$ for any convex increasing function $\chi$.

Imposing translational and rotational symmetry the Main Assumptions apply, in particular, under the
Homogeneous Assumptions:

- $X$ is either a ball of radius $R$ centered at the origin in $\mathbb{R}^{2n}$ or equal to all of $\mathbb{R}^{2n}$
- $W(x, y) = w(|x - y|), V(x) = v(|x|)$ and $\Psi_0(x) = \psi_0(|x|)$ with $w(r), v(r)$ and $-\psi_0(r)$ concave functions of $\log r$ (when $0 < r \leq 2R$) and bounded from below as $r \to 0$.

In fact, the special assumptions imply that $w(r)$ is decreasing in $r$. In other words, the Homogeneous Assumptions equivalently mean that the pair interaction $W(x, y)$ is repulsive and a concave function of $\log |x - y|$. The special assumptions, apply, for example, to the continuous repulsive power-laws

$$W_\alpha(x, y) := -|x - y|^{\alpha}, \quad \alpha > 0.$$  

Note that the Homogeneous Assumptions apply, in particular, to the standard centered Gaussian probability measure $\mu_0$ on $\mathbb{R}^{2n}$. However, one virtue of the Main Assumptions is that they, as pointed out above, apply to the more general Gaussian measures $1.12$ incorporating conservation of angular momentum in the $z_i$-variables (as discussed in Section 3.3).

Global concavity of $S(e)$ and thermodynamic equivalence of ensembles. In Section 6 it is shown that if the assumption that $W(x, y)$ be weakly positive definite is added to the Main Assumptions, then $S(e)$ is globally concave, i.e. concave on all of $\mathbb{R}$ (Theorem 7.1) and finite on $]e_{\min}, e_{\max}[$. For example, as explained in Section 7.1, this applies to the logarithmic interaction in $\mathbb{R}^{2n}$, as well as the continuous power-laws $1.13$ when $a \in ]0, 2]$ and to the exponential pair-potential

$$W(x, y) = e^{-a|x-y|}, \quad a > 0$$  

when $X$ is taken to be a disc centered at the origin with radius at most $1/2a$ (known as the Born-Mayer potential in chemistry). It should be stressed that neither the power-laws with $a \in ]0, 1]$, nor the exponential pair-potential, are concave wrt $(x, y)$ (otherwise the concavity of $S_+(e)$ could also be deduced from the ordinary Brunn-Minkowski inequality; compare Remark 6.6).

We then deduce that thermodynamic equivalence of ensembles holds for any energy-level $e$ in $]e_{\min}, e_{\max}[$ using a general result (Theorem 7.4), saying that for a general lower semi-continuous convex energy functionals $E(\mu)$ and prior $\mu_0$ thermodynamic equivalence of ensembles holds in the low-energy region $]e_{\min}, e_0[$ iff $E(\mu)$ and $\mu_0$ satisfy a certain compatibility property (the “energy approximation property”). This property has previously appeared in connection to the study of large deviation principles for the corresponding canonical ensembles at positive inverse temperatures $\beta$ [20, 7].

We also show that the global concavity of $S(e)$ holds for singular repulsive power-laws (Prop 6.5). However, in contrast to the continuous power-laws $1.13$ (and the repulsive logarithmic interaction) the singular power-laws do not satisfy the Main Assumptions. In fact, in this case the global concavity of $S(e)$ in high-energy region $e \geq e_0$ holds for a bad reason: $S(e) \equiv S(e_0)$ and, as a consequence, there are no maximum entropy measures $\mu^e$ when $e > e_0$. This means that the equivalence of ensembles at the level of macrostates then breaks down (see Section 4). Similarly, regularized singular power-laws are expected to yield non-equivalent ensembles and thus the corresponding entropies are expected to be non-concave (as discussed in [13 Page 252]).
Critical negative inverse temperatures and existence of maximum entropy measures. The singularity structure of a pair interaction \( W(x, y) \) satisfying the Main Assumptions can be very complicated, even if \( W(x, y) \) is taken to be translationally invariant, i.e.

\[
W(x, y) = -\Psi(x - y)
\]

for a function \( \Psi \) in the class \( \text{PSH}_a(\mathbb{C}^n) \). Still, as shown in Section Section 8.3, the singularities are mild enough to ensure that both the microsopic critical inverse temperature

\[
\beta_{c,N} := \left\{ \beta \in \mathbb{R} : Z_{N,\beta} := \int_{X^N} e^{-\beta H^{(N)}_{\mu}} \mu_0^{\otimes N} < \infty \right\}
\]

and the macroscopic critical inverse temperature

\[
\beta_c := \inf \left\{ \beta \in \mathbb{R} : \inf_{\mu} F_\beta(\mu) > -\infty \right\}
\]

are strictly negative. As a consequence we deduce that, when \( X \) is compact, there exists a maximum entropy measure \( \mu^e \) for any \( e \in ]e_{\min}, e_{\max}[ \). The concavity of \( S_+^{(N)}(e) \) and \( S(e) \) is exploited to establish “dual” formulas for \( \beta_{c,N} \) and \( \beta_c \), which hold under the Main Assumptions (Corollary 8.1 and Corollary 8.4):

\[
\beta_{c,N} = \lim_{e \to \sup_{X^N} E_N} \frac{dS^{(N)}(e)}{de}, \quad \beta_c = \lim_{e \to \sup_{X^N} E_N} \frac{dS(e)}{de}
\]

(1.15)

(which are decreasing limits when using either left or right derivatives). The derivative \( \frac{dS^{(N)}(e)}{de} \) corresponds to the inverse Gibbs temperature at energy \( e \) in the context of small systems [33,39] (when \( H^{(N)} \) is replaced by \( -H^{(N)} \) and \( e \) with \( -e \) so that \( \frac{dS^{(N)}(e)}{de} \) is positive).

Applied to the regularized vortex model \( W_\delta \) in \( \mathbb{R}^2 \) the second formula in 1.15 confirms the expectations expressed in [32, Page 855], concerning the slope \( dS_\delta(e)/de \) of the corresponding entropy: on the one hand, as \( e \) converges to the maximum (finite) value of the corresponding regularized energy \( E_\delta(\mu) \) the entropy \( S_\delta(e) \) and its slope \( dS_\delta(e)/de \) both converge towards \(-\infty\). On the other hand, for a fixed \( e \) the slope \( dS_\delta(e)/de \) converges, as \( \delta \to 0 \), to the slope \( dS_0(e)/de \) for the point-vortex model, which, in turn, is close to \(-4\) for large \( e \) (with our normalizations).

1.4. Outlook. In the companion papers [9,8,10] elaborations of the main results stated above are given, which may be summarized as follows.

Attractive classical Hamiltonians and self-gravitating matter in 2D. The concavity results discussed above also apply if \( H^{(N)} \) is replaced by \( -H^{(N)} \) if the energy level \( e \) is replaced by \(-e\). In particular, when \( V = 0 \) this amounts to replacing a “repulsive” pair-interaction \( W(x, y) \) with an “attractive” pair-interaction \( U(x, y) \in \text{PSH}a.a(X \times X) \). It is then natural to include momentum variables \( p \in \mathbb{R}^{2n} \) and consider the corresponding classical mean field Hamiltonian on phase space \((\mathbb{R}_x^{2n} \times \mathbb{R}_p^{2n})^N \) obtained by adding kinetic energy to the potential pair-interaction energy \( U(x, y) := \frac{1}{2N} \sum_{i \neq j \leq N} U(x_i, x_j) + \frac{1}{2} \sum_{i=1}^{N} |p_i|^2 \)

The phase space \( X \times \mathbb{R}_p^{2n} \) is endowed with the standard prior \( \mu_0 \) induced from Lebesgue measure \( dx dp \) on \( \mathbb{R}_x^{2n} \times \mathbb{R}_p^{2n} \). Accordingly, the upper microcanonical entropy \( S_+^{(N)} \) is now replaced by \( S_-^{(N)}(e) \) (formula 1.29). One new feature in this setup is that the prior \( \mu_0 \) has infinite mass. This leads to some technical difficulties appearing in the large \( N \)-limit.
which are deferred to the companion paper [3]. Here we just briefly mention the main results shown in [3]. First the global concavity of \( S^{(N)} \) follows directly from Theorem 6.7. As a consequence, \( S(e) \) is shown to also be globally concave under an additional stability assumption, which ensures the existence of a maximum entropy measure \( \mu^e \). The stability assumption in question is satisfied if, for example, the assumption \( U(x,y) \in PSH_{a,a}(X \times X) \) is complemented with the assumption that \( U(x,y) \) is, up to a bounded term, translationally invariant.

We will in particular specialize to the case when \( U(x,y) \) is the 2D gravitational (Newtonian) pair interaction and its various regularizations, generalizing a result in [1], concerning the unregularized case. This case has been studied extensively in the astrophysics literature, for example, as a model of galactic filaments [29]. The global concavity of \( S(e) \) in for 2D-gravity should be contrasted with the 3D case where \( S(e) \) is identically equal to \( \infty \) [3].

**Strict concavity of \( S(e) \) and the \( Q \)-curvature equation.** In the final Section 9 it is shown that if, under the Homogeneous Assumptions, \( v \) is moreover assumed strictly concave (or, in the case when \( X \) is the ball, \( w \) is strictly decreasing), then \( S(e) \) is strictly concave. This follows from a uniqueness result of free energy minimizers established in the companion paper [9]. Applications to conformal geometry are also given in [9], concerning the case when

\[
W(x,y) = - \log |x - y|
\]

in \( \mathbb{R}^{2n} \). This pair potential is proportional to the Green kernel of the \( n \)th power \( \Delta^n \) of the Laplacian, which, due to its conformal invariance, plays a key role in conformal geometry and mathematical physics [16, 21]. As stressed in [46] the potential of the corresponding maximum entropy measures \( \mu^e \) solve the \( Q \)-curvature equation [21] with prescribed \( Q \)-curvature proportional to \( e^{-\Psi_0} \):

\[
-\Delta^n \phi = e^{-\beta \phi} e^{-\Psi_0}
\]

The relevance of the present results to the \( Q \)-curvature equation are discussed in the companion paper [9].

**A maximum entropy principle for Sasaki-Einstein metrics and AdS/CFT.** The present results arose as a “spin-off effect” of a microcanonical approach to emergent Sasaki-Einstein geometry and the AdS/CFT (gauge/gravity) correspondence between superconformal supersymmetric gauge theories and supergravity. This is the subject of a separate publication [10], but briefly the main results may be summarized as follows. Let \( X \) be an \( n \)-dimensional complex algebraic subvariety of \( \mathbb{C}^m \) which is invariant under the action [1, 10] in \( \mathbb{C}^m \) for some weight-vector \( a \). Then \( X \) is also invariant under the scaling action on \( \mathbb{C}^m \) by \( \mathbb{R}_{>0} \) obtained by replacing \( e^{i\theta} \) with a scalar in \( \mathbb{R}_{>0} \). Denote by \( x_0 \) the point in \( X \) defined by the origin in \( \mathbb{C}^m \) and assume that \( X - \{x_0\} \) is non-singular. This means that the quotient \( M := (Y - \{y_0\})/\mathbb{R}_{>0} \) is a compact manifold, which is diffeomorphic to the intersection of \( Y \) with the unit-sphere in \( \mathbb{C}^m \). In the AdS/CFT correspondence the classical moduli space of vacua of the rank \( N \) gauge theory is parameterized by \( X^N \) and the corresponding supergravity vacuum is encoded by a Sasaki-Einstein metric on the corresponding “horizon” \( M \); the \( \mathbb{R}_{>0} \)-action on \( X \) represents the conformal symmetry of the gauge theory. As observed in [13] the rank \( N \) gauge theory admits a canonical fermionic \( N \)-particle BPS-state \( \Psi_{\text{det}} \), represented by a holomorphic polynomial on \( X^N \). Its self-information

\[
H^{(N)}(x_1, ..., x_N) := - \log |\Psi_{\text{det}}(x_1, ..., x_N)|^2
\]
thus defines an effective symmetric Hamiltonian on $X^N$. Dividing $H^{(N)}$ by $N\lambda_N$ where $\lambda_N$ is the BPS R-charge of $\Psi_{det}$ (i.e. its scaling degree) one obtains, in the limit when $N \to \infty$, a macroscopic energy functional $E(\mu)$ on $\mathcal{P}(X)$, which is non-quadratic when $n > 2$.\footnote{More precisely, the convergence holds in the sense of $\Gamma-$convergence when $X$ is quasi-regular, i.e. $a$ has rational components, but a variant of the argument applies in general, using an approximation argument.} Assuming that the algebraic variety $X$ is Gorenstein there is also a canonical $\mathbb{R}_{>0}-$equivariant volume form $dv_X$ on $X$. This volume form has infinite total volume, but “truncating” it appropriately one obtains a measure $\mu_0$ on $Y$ of finite total volume. For example, one can multiply $dv_X$ with a Gaussian factor $e^{-r^2}$, where $r$ denotes the radial coordinate on $\mathbb{C}^m$, or replace $X$ with its intersection with a ball of a fixed radius $R$ in $\mathbb{C}^m$. Then the corresponding entropy function $S(\epsilon)$ is defined as in formula 1.3. It is shown in [10] that $S(\epsilon)$ is strictly concave and for any fixed energy-level $\epsilon$ there exists a unique maximum entropy measure $\mu^\epsilon$ on $X$. Moreover, as $\epsilon \to \infty$ the push-forward $\nu^\epsilon$ of $\mu^\epsilon$ to $M$, under the natural quotient projection from $X$ to $M$, converges to the volume form of Sasaki-Einstein metric on $M$ - if such a metric exists. Otherwise, the reduced energy of $\nu^\epsilon$ blows up as $\epsilon \to \infty$. The convergence as $\epsilon \to \infty$ is obtained by establishing a macroscopic equivalence of ensembles in this setup, which shows that $\mu^\epsilon$ minimizes the free energy functional introduced in [13], for an appropriate $\beta$.

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1.6. Organization. We start in Section 2 by introducing a very general setup and provide some background on concavity and on plurisubharmonic functions (appearing in the Main Assumptions). In Section 3 general properties of the entropy $S(\epsilon)$ are studied. In particular, finiteness and monotonicity properties of $S(\epsilon)$ are established and relations to the notion of thermodynamic equivalence of ensembles are explored. In the following Section 4 the notion of macrostate equivalence of ensembles is discussed and existence results for maximum entropy measures are provided. Then, in Section 5 we consider the case when $E(\mu)$ is convex and show that thermodynamic equivalence of ensembles holds in the low-energy region $\{\epsilon > e_0\}$ iff the energy approximation property holds. In the remaining sections we specialize to the Main Assumptions. First in Section 6 we deduce the concavity of the upper microcanonical entropy $S^{(N)}_+(\epsilon)$ (Theorem 6.7) from a complex analog of the Brunn-Minkowski inequality. Then, letting $N \to \infty$ the concavity of the entropy $S(\epsilon)$ in the high energy region $\{\epsilon > e_0\}$ (Theorem 6.9) is deduced. In the following Section 7 this is shown to yield global concavity of $S(\epsilon)$ when the Main Assumptions are complemented with weak positive definiteness and some examples are exhibited. In Section 8 applications to slope formulas of critical inverse temperatures are given and some connections to algebraic geometry are explained. In the final Section 9 a strict concavity result for $S(\epsilon)$ is deduced under the Homogeneous Assumptions from a uniqueness result for free energy minimizers, established in the companion paper [9].

2. Setup and preliminaries

2.1. Very General Setup and notation. A very general formulation of the setup that we shall consider, henceforth called the Very General Setup may be formulated as follows. Let $X$ be a topological space endowed with a probability measure $\mu_0$ and $E(\mu)$ a lsc functional $E(\mu)$ on the space $\mathcal{P}(X)$ of all probability measures on $X$. We then define the corresponding entropy $S(\epsilon)$ and free energy $F(\beta)$ as in formula 1.3 and formula [1.6] respectively. Occasionally, when specializing to the General Setup introduced in Section
1.1. the notation $E_{W,V}(\mu)$ will designate an energy functional $E(\mu)$ of the particular form $E(\mu) = \int_W f(x) d\mu(x)$. We will mainly consider the case when $X \subseteq \mathbb{R}^{2n}$ and the Main Assumptions (or the Homogeneous Assumptions) introduced in Section 1.3 hold. These assumptions will be recalled in Section 6.1 but we first provide some preliminaries on concavity and plurisub-

2.2. Concave preliminaries. We will be discussing concavity properties of the entropy $S(\varepsilon)$ we provide some general preliminaries on concave functions. First recall that a function $\phi$ on a convex subset $C$ of $\mathbb{R}^d$ taking values in $]-\infty,\infty]$ is said to be convex on $C$ if for any given two points $x_0$ and $x_1$ and $t \in [0,1]$,

$$\phi(tx_0 + (1-t)x_1) \leq t\phi(x_0) + (1-t)\phi(x_1)$$

and strictly convex on $C$ if the inequality above is strict for any $t \in ]0,1[$. A function $f$ on $C$ is (strictly) concave if $-f$ is (strictly) convex. Here we will be mainly concerned with the case when $d = 1$. In this case, if $f$ is concave and finite on a closed interval $C \subseteq \mathbb{R}$, but not strictly convex, then there exist two points $x_0$ and $x_1$ in $C$ such that $f$ is affine on $[x_0,x_1]$. In Sections 3, 7 we will use some standard properties of convex functions recalled below, translated into the setup of concave functions (for further background see [54] and [59] Section 2.1.3). If $\phi$ is a convex function on $\mathbb{R}^d$ then its subdifferential $(\partial \phi)$ at a point $x_0 \in \mathbb{R}^d$ is defined as the convex set

$$\partial \phi(x_0) := \left\{ y_0 : \phi(x_0) + y_0 \cdot (x - x_0) \leq \phi(x) \ \forall x \in \mathbb{R}^d \right\}$$

In particular, if $\phi(x_0) = \infty$, then $(\partial \phi)(x_0)$ is empty. Similarly, if $f$ is concave on $\mathbb{R}^d$ then its superdifferential $(\partial f)(x_0)$ is defined as above, but reversing the inequality. In other words, $(\partial f)(x_0) := -(\partial (-f))(x_0)$. In the case when $f$ is concave on $\mathbb{R}$ and finite in a neighborhood of $x_0$,

$$(\partial f)(x) = [f'(x+), f'(x-)],$$

where $f'(x+)$ and $f'(x-)$ denote the right and left derivatives of $f$ at $x$, respectively. In particular, $f$ is differentiable at $x$ iff $(\partial f)(x)$ consists of a single point. If $f$ is a function on $\mathbb{R}^d$ taking values in $]-\infty,\infty]$ its (concave) Legendre-Fenchel transform is the usc and concave function on $\mathbb{R}^d$ (taking values in $]-\infty,\infty]$ ) defined by

$$f^*(y) := \inf_{x \in \mathbb{R}} \left( x \cdot y - f(x) \right).$$

It follows readily from the definitions that

$$y \in \partial f(x) \iff x \in \partial f^*(y).$$

Moreover, it is well-known that

$$\partial f(\{ f > -\infty \}) = \partial f^*(\{ f^* > -\infty \}).$$

Note that, in general, $f^{**}$ is the concave envelope of $f$ :

$$(f^{**})(x) = \inf_{a \text{ affine}} \{ a(x) : a \geq f \} = \inf_{g \text{ concave, finite}} \{ g(x) : g \geq f \}.$$
Indeed, the first equality follows directly from the definition and the second one is shown by, for a fixed $x$, taking $a(x)$ to be any affine function coinciding with $g$ at $x$ and with gradient in $\partial g(x)$.

We will also make use of the following lemmas (which are without doubt essentially well-known, but for completeness proofs are provided in the appendix).

**Lemma 2.1.** Let $f$ be a concave function on $\mathbb{R}$ and assume that $f$ is differentiable in a neighborhood of $[x_0, x_1]$. Then $f^*$ is strictly concave in the interior of $[y_0, y_1] := [f'(x_1), f'(x_0)]$.

Note that, in general, $f^{**} \geq f$. Concerning the strict inequality we have the following

**Lemma 2.2.** Let $f$ be a function on $\mathbb{R}$ such that $\sup f < \infty$ and $U \subset \mathbb{R}$ an open set where $f$ is finite and usc. Then $\{f^{**} > f\} \cap U$ is open in $U$ and $f^{**}$ is affine on $\{f^{**} > f\} \cap U$.

2.3. **Background on plurisubharmonicity and the class $PSH_a$.** The Main Assumptions introduced in Section 1.3 involve the notion of plurisubharmonicity. While this notion is central in the fields of several complex variables and complex geometry, it may not be familiar to readers lacking background in these fields. We thus recall the main definitions and properties that we shall use and refer to [26, Section 5.A.] for further background. We will identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ in the standard way. A function $\psi$ on $\mathbb{C}^n$ is said to be plurisubharmonic (psh, for short) if $\psi$ is upper semi-continuous (usc) taking values in $[-\infty, \infty]$ and subharmonic along complex lines, i.e. if $\zeta \mapsto \psi(z_0 + \zeta a_0)$ is a local subharmonic function on $\mathbb{C}$ for any given $z_0, a_0 \in \mathbb{C}^n$, or equivalently that

$$
\psi(z_0) \leq \frac{1}{2\pi} \int \psi(z_0 + e^{i\theta} a_0) d\theta.
$$

In particular, $\psi$ is then subharmonic on $\mathbb{R}^{2n}$. If $\psi$ is smooth then it is psh iff the complex Hessian $\partial\bar{\partial}\psi$ of $\psi$ is a semi-positive Hermitian matrix at any $z$:

$$
\partial\bar{\partial}\psi(z) := \left( \frac{\partial^2 \psi(z)}{\partial z_i \partial \bar{z}_j} \right) \geq 0,
\quad \frac{\partial}{\partial z_i} := \frac{1}{2} \frac{\partial}{\partial x_i} - \frac{i}{2} \frac{\partial}{\partial y_i}.
$$

Equivalently, a function $\psi$ is psh if, locally, it can be expressed as a decreasing limit of smooth psh functions $\psi_j$. In fact, $\psi_j$ may be taken as a convolution of $\psi$ with any suitably scaled smooth probability density with compact support. If $-u$ is plurisubharmonic, then $u$ is called plurisuperharmonic. An open set $\Omega$ in $\mathbb{C}^n$ is said to be pseudoconvex if $\Omega$ admits a continuous psh exhaustion function $\rho$ i.e. $\rho$ is psh on $\Omega$ and such that $\{\rho \leq C\}$ is a compact subset of $\Omega$. We recall the following essentially standard lemma (see the appendix for a proof):

**Lemma 2.3.** Let $\phi$ be a psh function on a pseudoconvex open set $\Omega$. Then $\{\phi < 0\} \cap \Omega$ is also pseudoconvex.

We also recall that the following standard facts [26, Theorem 5.5.], which allows one to construct a range of different types of psh functions:

**Lemma 2.4.** If $\psi_1, ..., \psi_r$ are psh functions and $\chi(t_1, ..., t_r)$ is a convex function on $\mathbb{R}^r$ which is increasing in each $t_i$, then $\chi(\psi_1, ..., \psi_r)$ is psh. In particular, if $\alpha_1, ..., \alpha_r$ are non-negative functions, then

$$
\sum_{i=1}^r \alpha_i \psi_i, \quad \log \sum_{i=1}^r e^{\alpha_i \psi_i} \quad \text{and} \quad \max\{\psi_1, ..., \psi_r\}
$$

are psh functions.
In particular, if $\psi$ is psh and $\chi$ is a convex increasing function on $\mathbb{R}$, then the composed function $\chi(\psi)$ is psh. Since $|f(z)|^2$ is psh for any holomorphic function $f(z)$ on $\mathbb{C}^n$ (as follows, for example directly from the characterization) it follows form the previous lemma that

$$
\psi(z) := \log \left( \sum_{i=1}^r |f_i(z)|^2 \right)
$$

is psh for any given holomorphic functions $f_1, \ldots, f_r$. In particular, $\log |z|^2$ is psh. Moreover, if a function $\psi$ only depends on the absolute values of $z_i$, then $\psi(z)$ is psh iff it is convex with respect to $(\log |z_1|, \ldots, \log |z_n|) \in \mathbb{R}^n$.

2.3.1. The class $PSH_a$. Given $a = (a_1, \ldots, a_m) \in \mathbb{R}^m$ we denote by $V_a$ the vector field on $\mathbb{C}^n$ defined by

$$
V_a := \sum_{i=1}^m a_i \frac{\partial}{\partial \theta_i},
$$

where $\frac{\partial}{\partial \theta_i}$ denotes the generator of the $S^1$-action on $\mathbb{C}^n$ which rotates the $z_i$-coordinate and leaves the other coordinates invariant (i.e. $e^{i\theta} \cdot z := (z_1, \ldots, e^{i\theta}z_i, \ldots, z_n)$). In other words, $V_a$ is the Hamiltonian vector field corresponding to the Hamiltonian

$$
h_a(z) := \sum_{i=1}^m \frac{1}{2a_i} |z_i|^2
$$

on $\mathbb{R}^{2n}$, endowed with its standard symplectic form. Note that the Hamiltonian $h_a$ is plurisubharmonic on $\mathbb{C}^n$ (since $|z|^2$ is). Now if $U$ is an open connected subset of $\mathbb{C}^n$ then the class $PSH_a(U)$ is defined as the class of all psh functions $\psi$ on $U$, not identically $-\infty$, such that $V_a(\psi) = 0$. More generally, if $X$ is closed connected subset of $\mathbb{C}^n$ we denote by $PSH_a(X)$ the class of all functions $\psi$ such that $\psi$ is in $PSH_a(U)$ for some open subset $U$ containing $X$ (depending on $\psi$).

Example 2.5. (The “algebraic and quasi-homogeneous” case). If $P(z_1, \ldots, z_n)$ is a quasi-homogeneous polynomial, i.e. there exists exist positive integer weights $a_1, \ldots, a_n$ such that $P$ is homogeneous of degree $d$ wrt the corresponding $\mathbb{R}^+$-action:

$$
P(c^{a_1}z_1, \ldots, c^{a_n}z_n) = c^dF(z_1, \ldots, z_n)
$$

for any $c \in \mathbb{R}_+$, then $\log |P(z)|$ is in $PSH_a(\mathbb{C}^n)$. More generally, if $P_j$ are polynomials on $\mathbb{C}^n$ which are quasi-homogeneous of degree $d_j$ for the same weights $a_1, \ldots, a_n$ and $\alpha_i > 0$, then

$$
\psi(z) := \log \left( \sum_{j=1}^r |P_j(z)|^{\alpha_j} \right) \in PSH_a(\mathbb{C}^n)
$$

In the particular case when all $\alpha_i = 1$ and $d_i = d$ we call $d$ the degree of $\psi$.

By composing the previous examples with convex increasing functions $\chi$ on $\mathbb{R}$ one may fabricate an abundance of examples of functions in the class $PSH_a(\mathbb{C}^n)$. For example, $\sum_{j=1}^M |P_j(z)|^{\alpha_j}$ is in $PSH_a$ if $P_j(z)$ is a homogeneous polynomial (wrt $a$) and $\alpha_j > 0$. 

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3. General properties of $S(e)$ and thermodynamic equivalence of ensembles

In this section general properties of the entropy $S(e)$ are studied and the notion of thermodynamic equivalence of ensembles introduced in [33] is recalled. The main new feature in this section, as compared to the setup in [33], is that $E(\mu)$ is not assumed to be continuous. This leads to some subtle aspects that do not seem to have been addressed before. Throughout the section we will consider the Very General Setup introduced in Section 2.1

3.1. Monotonicity of $S(e)$. The following lemma generalizes [13 Prop 2.2] (with a similar proof) and involves the following ad hoc property:

**Definition 3.1.** Assume that $X$ is compact. Then a functional $E(\mu)$ on $\mathcal{P}(X)$ has the **affine continuity property** if for any $\mu_1 \in \mathcal{P}(X)$ such that $E(\mu_1) < \infty$ and $S(\mu_1) > -\infty$ the function $t \mapsto E(\mu_0(1-t) + t\mu_1)$ is continuous on $[0,1]$. For a general $X$ the affine continuity property is said to hold if it holds for all compact subsets of $X$.

**Lemma 3.2.** (Monotonicity of $S(e)$). Assume that $X$ is compact and $e_0 := E(\mu_0) < \infty$.

- If $E(\mu)$ is convex on $\mathcal{P}(X)$, then $S(e)$ is increasing for $e \leq e_0$ and strictly increasing in the subinterval where $S(e) > -\infty$. In particular,
  \[ S_-(e) := \sup_{E(\mu) \leq e} S(\mu) \]

- If $E(\mu)$ has the affine continuity property, then $S(e)$ is decreasing for $e \geq e_0$ and strictly decreasing in the subinterval where $S(e) > -\infty$. In particular,
  \[ S_+(e) := \sup_{E(\mu) \geq e} S(\mu) \]

More precisely, in the second point there is no need to assume that $E$ is lsc on $\mathcal{P}(X)$ and thus it also follows that $S(e)$ is increasing for $e \leq E(\mu_0)$.

**Proof.** To prove the first point first observe that, since $E$ is lsc and $X$ is compact $\{E(\mu) \leq e\}$ is compact (or empty). We may assume that $S(\mu)$ is not identically equal to $-\infty$ on $\{E(\mu) \leq e\}$ (otherwise we are done). Since $S(\mu)$ is lsc the sup of $S(\mu)$ on the set $\{E(\mu) \leq e\}$ is thus attained at some $\mu_1$ in the set. Assume in order to get a contradiction that $E(\mu_1) < e$. Consider the affine segment $\mu_t$ in $\mathcal{P}(X)$ connecting $\mu_0$ and $\mu_1$: $\mu_t := \mu_0(1-t) + t\mu_1$ for $t \in [0,1]$. By the assumed convexity of $E(\mu)$

\[ E(\mu_t) \leq (1-t)E(\mu_0) + tE(\mu_1) < e \]

for $t$ sufficiently small, using that $E(\mu_0) < \infty$. But, as is well-known, $S(\mu)$ is strictly concave on $\{S(\mu) > -\infty\} \subset \mathcal{P}(X)$ and attains its maximum at $\mu_0$ and hence $S(\mu_t) < S(\mu_1)$ for any $t \in [0,1]$ (as follows from Jensen’s inequality). This contradicts the assumption that $\mu_1$ is a maximizer and hence it must be that $E(\mu_1) = e$, as desired.

To prove the second point it will be enough to show that for any $\mu_1 \in \mathcal{P}(X)$ such that $E(\mu_1) \geq e$ and $S(\mu_1) > -\infty$ there exists $\mu \in \mathcal{P}(X)$ such that $E(\mu) = e$ and $S(\mu) \geq S(\mu_1)$. To this it will, in the light of the previous argument, be enough to show that there exists some $t \in [0,1]$ such that $E(\mu_t) = e$. But, by assumption $E(\mu_0) \leq e$ and $E(\mu_1) \geq e$. We can thus conclude by invoking the assumption that $E(\mu_t)$ is continuous. Since we have not used that $E$ is lsc on $\mathcal{P}(X)$ the same argument applies to $-E$, which proves the last statement of the lemma. □
3.2. **Thermodynamic equivalence of ensembles.** In this section we consider the Very General Setup. It follows readily from the definitions that the Legendre-Fenchel transform $S^*$ of $S$ coincides with the free energy $F(\beta)$:

$$S^* = F.$$ 

Following [33, 58] we make the following

**Definition 3.3.** thermodynamic equivalence of ensembles is said to hold globally if

$$S = F^*$$

and thermodynamic equivalence of ensembles is said to hold at an energy level $e$ if $S(e) > -\infty$ and

$$S(e) = F^*(e).$$

Recall that, in general, a function $S(e)$ is usc and concave iff $S^{**} = S$. It was shown in [33, Prop 3.1a] that $S$ is always usc under the assumption that $X$ is compact and $E(\mu)$ is continuous wrt the weak topology on $P(X)$ (this is the case if $W$ and $V$ are continuous). In this case global thermodynamic equivalence thus holds iff $S$ is concave. But here we need consider the case when the continuity assumptions are not satisfied (and moreover $X$ may be non-compact). We will impose the following compatibility property between $\mu_0$ and $E(\mu)$.

**Definition 3.4.** A measure $\mu_0$ in $X$ is said to has the Energy Approximation Property if for any compactly supported probability measure $\mu$ there exists a sequence $\mu_j \in P(X)$, supported in the same compact set, converging weakly towards $\mu$ with the following properties:

- $\mu_j$ is absolutely continuous with respect to $\mu_0$
- $\lim_{j \to \infty} E(\mu_j) = E(\mu)$

**Remark 3.5.** This property was introduced in the context of large deviation theory in [20] and studied from a potential-theoretic point of view in [7] (see the discussion in the end of Section 3.2).

The energy approximation property ensures that $S(e)$ is finite on $[e_{\min}, e_{max}]$.

**Lemma 3.6.** Assume that $\mu_0$ has the energy approximation property and the affine continuity property on compact subspaces of $X$. Then $S(e)$ is finite on $[e_{\min}, e_{max}]$.

**Proof.** By Lemma 3.2 we just have to verify the claim that there exists some $\mu \in P(X)_0$ such that $E(\mu) \leq e$ and $S(\mu) > -\infty$. To this end take $\delta > 0$ such that $e - \delta > e_{\min}$. By the verify definition of $e_{\min}$ there exists $\mu$ such that $E(\mu) \leq e - \delta$. Moreover, by the monotone convergence theorem $\mu$ may be chosen to have compact support. Now take a sequence $\mu_j(= \rho_j, \mu_0)$ converging weakly towards $\mu$ with the energy approximation property. Replacing $\rho_j$ with $\max\{\rho_j, R\} / \int \{\rho_j, R\} \mu_0$ for a given $R > 0$ and using a diagonal argument we may as well assume that $\rho_j \in L^\infty$. In particular,

$$E(\mu_j) \leq e, \quad S(\mu_j) > -\infty$$

for $j$ sufficiently large, proving the claim when $e \in [e_{\min}, e_0]$. A similar approximation argument applies if instead $e \in [e_0, e_{\max}]$ (again using Lemma 3.2). Finally, if $e = E(\mu_0)$ then $S(\mu) \geq S(\mu_0) = 0$, which concludes the proof of the claim above. $\square$

**Proposition 3.7.** In the Very General Setup the following holds:

- If the entropy $S(e)$ is concave on $[e_{\min}, e_{max}]$ and $\mu_0$ has the energy approximation property and the affine continuity property, then $S(e)$ is continuous on $[e_{\min}, e_{max}]$ and thermodynamic equivalence of ensembles holds for any $e \in [e_{\min}, e_{max}]$. 

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• If the entropy $S(e)$ is concave and continuous on $[e_0, e_{\max}]$ then thermodynamic equivalence of ensembles holds for any $e \in [e_0, e_{\max}]$ and moreover for any $e \in [e_0, e_{\max}]$

\begin{equation}
S(e) = \inf_{\beta \leq 0} (\beta e - F(\beta))
\end{equation}

• If the entropy $S(e)$ is concave and continuous on $[e_{\min}, e_0]$ then thermodynamic equivalence of ensembles holds for any $e \in [e_{\min}, e_0]$

Proof. In order to show that $S(e_1) = S^{**}(e_1)$ at a given point $e_1$ in $[e_{\max}, e_{\min}]$ it is enough to find an affine function $s$ on $\mathbb{R}$ such that $s \geq S$ and $s(e_1) = S(e_1)$ (by formula 2.4). But since $s$ is concave and finite on $[e_{\min}, e_{\max}]$ its superdifferential $\partial S$ is non-empty, i.e. contains some $\beta \in \mathbb{R}$. This means that the affine function

\begin{equation}
s(e) := \beta (e - e_1) + S(e_1)
\end{equation}

coincides with $S$ at $e$ and has the property that $s \geq S$ on $[e_{\min}, e_{\max}]$. Hence, by Lemma 3.2 $s \geq S$ on all of $\mathbb{R}$, which proves the first point.

To prove the second point in the proposition fix $e_1 \in \{E(\mu_0), e_{\max}\}$. By formula 2.4 it will be enough to find an affine function $s$ on $\mathbb{R}$ such that $s \geq S$ and $s(e_1) = S(e_1)$. To this end first define the function $f(e)$ to be equal to $S(e)$ on $[e_0, e_{\max}]$ and $e_0$ when $e < e_0$. Thus $f(e) = \max\{e_0, S(e)\}$ is continuous and convex on $]-\infty, e_{\max}[$. We then obtain the desired affine function $s$ by picking an element $\beta$ in the superdifferential $\partial f$ at $e_1$ and again defining $s(e)$ by formula 3.2. Finally, to prove the last formula we have to show that the infimum in formula 3.1 is attained for some $\beta \leq 0$. But this follows from the fact that, in the previous step, $\beta$ in formula 3.2 is non-positive, since $f$ is decreasing (by Lemma 3.2). The third point is shown in essentially the same way as the second one.

Remark 3.8. If $e_{\max} < \infty$, then it could happen that $S(e_{\max}) \neq S^{**}(e_{\max})$ in the first point of the previous proposition. Also note that in the case when $E(\mu)$ is of the form $E = E_{W,V}$ (as in formula 1.2) then $e_{\max} = \infty$ holds if either there exists $x_0$ such that $V(x_0) = \infty$ or $(x_0, y_0)$ such that $W(x_0, y_0) = \infty$. Indeed, then $E(\mu) = \infty$ for $\mu = \delta_{x_0}/2 + \delta_{x_1}/2$.

As shown in Theorem 5.4 below the energy approximation property is not merely a technical assumption, but essential.

3.3. Priors versus linear constraints. Now consider the Very General Setup in the case when $X$ is a domain in $\mathbb{R}^d$ and $\mu_0 = dx$. Given a continuous function $\psi_0$ and $\lambda \in \mathbb{R}$ we may then replace $\mu_0$ with the prior defined by the probability measure

$$\mu_\lambda := e^{-\lambda \psi_0} dx / Z_\lambda, \quad Z_\lambda := \int_X e^{-\lambda \psi_0} dx,$$

assuming that $Z_\lambda < \infty$. The corresponding corresponding entropy function $S_{\mu_\lambda}(e)$ is closely related to the multi-variable entropy function $S(e, l)$ on $\mathbb{R}^2$ defined by

$$S(e, l) := \sup_{\mu \in P(X)} \{S(\mu) : E(\mu) = e, \ L(\mu) = l\}, \quad L(\mu) := \int_X \psi_0 \mu,$$

obtained by imposing the linear constraint $L(\mu) = l$ (where $S(\mu)$ denotes the entropy of $\mu$ relative to $dx$). Indeed, it follows readily from the definition that, for a fixed $e$, the Legendre-Fenchel transform of the function $\lambda \mapsto S_{\mu_\lambda}(e)$ is given by $-S(e, l) - \log Z_\lambda$. Hence, under the hypothesis that $S(e, l)$ is concave and lower-semicontinuous wrt $l$, inverting the Legendre-Fenchel transform gives

$$S(e, l) = \inf_\lambda \{S_{\mu_\lambda}(e) + \lambda l + \log Z_\lambda\}.$$
As a consequence, if $S_{\mu_\lambda}(e)$ is globally concave with respect to $e$, for any fixed $\lambda$ such that $Z_\lambda$ is finite, then $S(e,l)$ is globally concave on $\mathbb{R}^2$. Multi-variable entropy functions are studied in [33], from the point of view of equivalence of ensembles, but here we will focus on one-variable entropy functions defined with respect to appropriate priors. Note that in the non-compact case when $X = \mathbb{R}^d$ the inclusion of a function $\psi_0$ with sufficient growth at infinity is crucial in order to get a prior measure with finite total mass. In the presence of rotational symmetry the standard choice of a prior is a centered Gaussian measure.

Remark 3.9. More generally, given $r$ functions $\psi_1, ..., \psi_r$ on $\mathbb{R}^d$ and $\lambda_1, ..., \lambda_r \in \mathbb{R}^d$ one can consider the prior $\mu_\lambda = e^{-\sum \lambda_i \psi_i}/Z_\lambda$ and the corresponding entropy function $S(e,l)$ on $\mathbb{R}^{1+d}$. Then the previous considerations still apply if $\lambda$ is replaced by the scalar product between $\lambda$ and $l$.

4. Macrostate equivalence of ensembles and existence of maximum entropy measures

An important motivation for the notion of thermodynamic equivalence of ensembles is that it implies that any maximum entropy measure $\mu^e$ (representing an equilibrium macrostate in the microcanonical ensemble) minimizes the free energy $F_\beta(\mu)$ at an inverse temperature $\beta$ corresponding to the energy level $e$. This is made precise by the following result (essentially contained in [33]).

Lemma 4.1. (Macrostate equivalence of ensembles). Consider the Very General Setup. Assume that $S^{**}(e) = S(e) > -\infty$ and assume that $\partial S(e)$ is non-empty (this is the case if, for example, $S^{**} = S > -\infty$ in a neighborhood of $e$). If $\mu^e$ is a maximal entropy measure with energy $e$, i.e. $S(\mu^e) = S(e)$, then $\mu^e$ minimizes the free energy functional $F_\beta(\mu)$ for any $\beta \in \partial S(e)$.

Proof. By assumption $S(e) > -\infty$. Hence, the assumption that $\beta \in \partial S(e)$ means that $\beta \in (\partial F^*)(e)$. Since $F = (F^*)^*$ it follows from the definition of $\partial F^*$ that

$$F(\beta) = -F^*(e) + \beta e$$

(since $0 \in \partial (-F^*(e) + \beta e)$). In other words,

$$\inf_{\mu \in P(X)} F_\beta(\mu) = -S(\mu^e) + \beta E(\mu^e),$$

which means that $\mu^e$ minimizes $F_\beta(\mu)$, as desired. $\square$

Remark 4.2. Without the property that $S(e) = S^{**}(e)$ a maximal entropy measure $\mu^e$ will, in general, not minimize $F_\beta(\mu)$. This is discussed in the context of BEG-model in the final section of [34] (where it is pointed out that $\mu^e$ may be merely a local minimizer of $F_\beta(\mu)$ or even a saddle point). Moreover, even if $S(e) = S^{**}(e)$ there may, in general, exists minimizers of $F_\beta(\mu)$ for $\beta \in \partial S(e)$, which are not maximum entropy measures (at energy $e$), unless $S(e)$ is strictly concave at $e$ (see [33]).

As shown in [33], the existence of $\mu^e$ is automatic for any $e \in [e_0, e_{\text{max}}]$, when $X$ is compact and $E(\mu)$ is a continuous functional on $P(X)$. However, since we do not impose these assumptions in the Main Assumptions we next provide some general existence result for $\mu^e$, that will be applied to the Main Assumptions in Section 8.3.

4.1. Existence of $\mu^e$ when $X$ is compact. We start with the low-energy region:

Proposition 4.3. Consider the Very General Setup. Assume that $X$ is compact and that the energy approximation property and the affine continuity property holds. Then, for any $e \in [e_{\text{min}}, e_0]$ there exists a maximum entropy measure $\mu^e$. 

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Before turning to the non-compact case we point out that the following concrete bound property does not hold, i.e. that 

$$e^\mu$$

Proof. Fix \(e \in [e_{\text{min}}, e_0]\). First recall that by Lemma 3.6 \(S(e)\) is finite. Next, by Lemma 3.2 (and its proof) it is enough to prove that the functional \(S(\mu)\) admits a maximizer on \(\{E(\mu) \leq e\}\). But since \(E\) is lsc, \(\{E(\mu) \leq e\}\) is closed in the compact space \(\mathcal{P}(X)\), hence compact. The existence of \(\mu^e\) thus follows from the upper-semicontinuity of \(S(\mu)\) on \(\mathcal{P}(X)\).

\[\square\]

In order to ensure the existence of maximum entropy measures in the high-energy region we introduce the following stability property:

Definition 4.4. In the Very General Setup the thermal stability property is said to hold if there exists \(\beta_0 < 0\) such that

$$\inf_{\mathcal{P}(X)} (\beta_0 E - S) > -\infty.$$ 

In other words, this property says that the critical inverse temperature \(\beta_c\) (discussed in Section 3) is strictly negative. Turning to the General Setup we will use the following result, shown in the course of the proof of [3, Lemma 2.13, formula 2.12]):

Lemma 4.5. Consider the General Setup and assume that \(X\) is compact. If the thermal stability property holds, then the functional \(E_{V,W}\) is continuous on \(\{\mu : S(\mu) \geq -C\} \subset \mathcal{P}(X)\) for any given constant \(C > 0\).

The following result generalizes the existence result in [13], concerning the case when \(W(x,y)\) has a logarithmic singularity along the diagonal:

Proposition 4.6. Consider the General Setup. Assume that \(X\) is compact and that the energy approximation property and the thermal stability property hold. Then \(S(e)\) is usc on \([e_{\text{min}}, e_{\text{max}}]\) and for any \(e \in [e_{\text{min}}, e_{\text{max}}]\) there exists a maximum entropy measure \(\mu^e\).

Proof. Take \(e_j \to e \in [e_{\text{min}}, e_{\text{max}}]\) and let \(\mu_j\) be a sequence in \(\mathcal{P}(X)\) such that \(E(\mu_j) = e_j\) and \(S(\mu_j) \geq S(e_j) - 1/j\). In particular, there exists a constant \(C\) such that \(S(\mu_j) \geq -C\). By the previous lemma, we may, after perhaps passing to a subsequence, assume that \(\mu_j \to \mu_{\infty}\) in \(\mathcal{P}(X)\) and \(E(\mu_j) \to E(\mu_{\infty})\). Hence, \(E(\mu_{\infty}) = e\) and since \(S\) is usc on \(\mathcal{P}(X)\) \(S(\mu_{\infty}) \geq \limsup_{j \to \infty} S(\mu_j)\). This shows that \(S(e) \geq S(\mu_{\infty}) \geq \limsup_{j \to \infty} S(e_j)\), i.e. that \(S\) is usc. Similarly, the existence of \(\mu^e\) also follows from the previous lemma, since it shows that \(\{E(\mu) = e\} \cap S(\mu) \geq -C\) is closed (and thus \(S\) attains its maximum value there for \(C\) sufficiently large).

\[\square\]

If the thermal stability property does not hold, then there may not be no maximum entropy measures, where \(S(e)\) is globally concave. In fact, we have the following converse to the previous proposition when \(S(e)\) is concave and continuous on \([e_0, e_{\text{max}}]\).

Proposition 4.7. Consider the Very General Setup and assume that \(X\) is compact and that there exists a maximum entropy measure \(\mu^e\) for some \(e \in [e_0, e_{\text{max}}]\). Then the thermal stability property holds.

Proof. The assumed concavity of \(S(e)\) implies that the right derivative of \(S(e)\) tends to \(\beta_c\) as \(e \to e_{\text{max}}\) (see Cor 3.3 below and its proof). Hence, if we assume that the thermal stability property does not hold, i.e. that \(\beta_c = 0\) it follows, since \(S(e)\) attains its maximum at \(e\) and is assumed continuous and concave on \([e_0, e_{\text{max}}]\) that \(S(e) \equiv S(e_0)\). But \(S(\mu) = S(\mu_0)\) iff \(\mu = \mu_0\) (which implies \(E(\mu) = e_0\)) and hence there exists no maximum entropy measure \(\mu^e\) when \(e > e_0\).

\[\square\]

The previous proposition is illustrated by the case of singular power-laws in Section 5.3. Before turning to the non-compact case we point out that the following concrete bound
implies the thermal stability property (see Lemma 8.5 below):

\[
\text{if } \mu \text{ assumption: (4.2)} \quad - W(x,y) - \frac{1}{2} V(x) - \frac{1}{2} V(y) \leq \frac{1}{2} \phi_0(x) + \frac{1}{2} \phi_0(y) + C_0.
\]

(4.1)

Then we can decompose

\[
E(\mu) = E_{\phi_0}(\mu) - \int \mu \phi_0, \quad E_{\phi_0}(\mu) := \int \left( W(x,y) + \frac{1}{2} V(x) + \frac{1}{2} V(y) + \frac{1}{2} \phi_0(x) + \frac{1}{2} \phi_0(y) \right) \mu \otimes \mu
\]

where the first term has a well-defined value in \([-\infty, \infty]\), since the corresponding integrand is bounded from below. This means that if we replace \(\mathcal{P}(X)\) with the subspace

\[
\mathcal{P}_{\phi_0}(X) := \left\{ \mu \in \mathcal{P}(X) : \int_X \phi_0 \mu < \infty \right\}
\]

then \(S(\epsilon)\) may be expressed as

\[
S(\epsilon) := \sup_{\mu \in \mathcal{P}_{\phi_0}(X)} \left\{ S(\mu) : E(\mu) = \epsilon \right\},
\]

where \(E(\mu)\) is defined by formula (4.3). According to the following result the existence of a maximizer \(\mu^\epsilon\) is guaranteed if \(\phi_0\) has slower growth then an appropriate exhaustion function \(\psi_0\) of \(X\) (i.e. the sub-level sets \(\{\psi_0 \leq R\}\) are compact and exhaust \(X\) when \(R \to \infty\)):

**Proposition 4.8.** Consider the General Setup and assume that there exists a continuous exhaustion function \(\psi_0\) of \(X\) such that the following growth-properties hold:

- \(\int e^{\delta \psi_0} \mu_0 < \infty\) for some \(\delta > 0\)
- The growth-assumption (4.2) holds for a \(\phi_0\) such that \(\phi_0/\psi_0 \to 0\) uniformly as \(\psi_0 \to \infty\) (e.g. for \(\phi_0 = \psi_0^{(1-\epsilon)}\) for some \(\epsilon \in [0, 1]\)).

If the thermal stability property holds (i.e. \(\beta_c < 0\)), then there exists a measure \(\mu^\epsilon\) realizing the sup in formula (4.4) for any given \(\epsilon \in [\epsilon_0, \epsilon_{\text{max}}]\).

**Proof.** Setting \(\tilde{W}(x,y) := W(x,y) + \frac{1}{2} V(x) + \frac{1}{2} V(y) - \frac{1}{2} \phi_0(x) + \frac{1}{2} \phi_0(y)\) we can express \(E_{\phi_0}(\mu) = \int \tilde{W}(x,y) \mu \otimes \mu\). Now fix \(\epsilon \in [\epsilon_0, \epsilon]\) and recall that \(S(\epsilon)\) is finite. Since, by assumption, \(\tilde{W}(x,y)\) is lsc on \(X \times X\) and bounded from below it extends to a lsc function on \(\tilde{X} \times \tilde{X}\), where \(\tilde{X}\) denotes the one-point compactification of \(X\). Moreover, we identify \(\psi_0\) with a lsc function on \(\tilde{X}\), taking the value \(\infty\) at the point at infinity and \(\mu_0\) with a probability measure on \(\tilde{X}\), not charging the point at infinity. Accordingly, we can identify \(E_{\phi_0}(\mu)\) and \(S(\mu)\) with functionals on \(\mathcal{P}(\tilde{X})\). Denote by \(\tilde{S}(\epsilon)\) the corresponding entropy
function. Since \( \int_X \mu \psi_0 < \infty \) implies that \( \mu \) does not charge the point at infinity it will, in order to prove the proposition, be enough to show that the sup defining \( \bar{S}(e) \) is attained. To this end take a sequence \( \mu_j \in \mathcal{P}(X) \) such that \( E(\mu_j) = e \) and \( S(\mu_j) \) increases to \( \bar{S}(e) \).

Decompose \( \mu = e^{-\delta \Psi_0} \mu_\delta \) for \( \delta > 0 \) such that \( \mu_\delta := e^{\delta \Psi_0} \mu_0 \) has finite total mass. Then there exists a constant \( C \) such that

\[
S(\mu_j) = S_{\mu_\delta}(\mu) - \delta \int \Psi_0 \mu_j \geq -C.
\]

Since \( S_{\mu_\delta}(\mu) \) is uniformly bounded from above on \( \mathcal{P}(X) \) (using that \( \mu_\delta \) has total finite mass) this means that there exists a finite constant \( C_\delta \) such that

\[
\int \psi_0 \mu_j \leq C_\delta < \infty.
\]

Now, since \( \bar{X} \) compact we may, after perhaps passing to a subsequence, assume that \( \mu_j \to \mu_\infty \) weakly in \( \mathcal{P}(X) \) for some \( \mu_\infty \) (which, by the bound \( 4.6 \) does not charge the point at infinity). Moreover, combining the bound \( 4.6 \) with the growth-assumption on the continuous function \( \phi_0 \) gives (using Markov’s inequality) that

\[
\lim_{j \to \infty} \int \phi_0 \mu_j = \int \phi_0 \mu_\infty.
\]

Since \( S(\mu) \) is usc on \( \mathcal{P}(\bar{X}) \) all that remains is to verify that

\[
\lim_{j \to \infty} E_{\phi_0}(\mu_j) = E_{\phi_0}(\mu_\infty)
\]

To this end we rewrite the assumed thermal stability property as

\[
\beta_0 E_{\phi_0}(\mu) - \beta_0 \int \phi_0 \mu - S(\mu) \geq -C_0, \quad \beta_0 < 0
\]

Note that

\[
-\beta_0 \int \phi_0 \mu - S(\mu) = -S_{\mu_{\beta_0}}(\mu), \quad \mu_{\beta_0} := e^{\beta_0 \phi_0} \mu_0,
\]

where the measure \( \mu_{\beta_0} \) has finite mass (since \( \beta_0 \leq 0 \) and \( \phi \geq 0 \)) and thus identifies with a measure on \( \bar{X} \). Accordingly, can view \( 4.8 \) as an inequality on \( \mathcal{P}(\bar{X}) \), saying that lsc functional \( E_{\phi_0}(\mu) \) has the thermal stability property wrt the measure \( \mu_{\beta_0} \) on the compact space \( \bar{X} \). Thus, it follows from Lemma \( 4.5 \) that \( E_{\phi_0} \) is continuous on \( \{ S_{\mu_{\beta_0}}(\mu) \geq -C \} \).

Finally, combining \( 4.9 \) and \( 4.8 \) reveals that \( S_{\mu_{\beta_0}}(\mu_j) \geq -C \) for some constant \( C \) and hence the desired convergence \( 4.7 \) follows. \( \square \)

**Remark 4.9.** To see that the growth-properties in the previous proposition are essential consider the case when \( X = \mathbb{R}^d, \mu_0 = e^{-|x|^2} dx \) and \( V(x) = -|x|^p \) for \( p > 0 \). Then the thermal stability property does hold (in fact, \( \beta_c = -\infty \), since \( Z_\beta := \int e^{-\beta V} \mu_0 < \infty \) for any \( \beta < 0 \)). Moreover, \( \int e^{\beta \phi_0} \mu_0 < \infty \) for \( \psi_0 := |x|^2 \). However, for \( e \leq e_0 \) a maximum entropy measure \( \mu^e \) only exists under the assumption that \( p < 2 \), i.e. precisely when \( -V/\psi_0 \to \infty \) (indeed, if \( \mu^e \) exists, then \( \mu^e = e^{-\beta V} / \int e^{-\beta V} \mu_0 \) for some \( \beta > 0 \) (see Section 5.1.1)).

5. **Concavity of \( S(e) \) in the low-energy region for convex \( E(\mu) \)**

5.1. **Concavity and monotonicity of \( S(e) \) in the low-energy region \( e \leq e_0 \) when \( E(\mu) \) is convex.** We now consider the entropy \( S(e) \) in the low-energy region \( e \leq e_0 \) under the assumption that \( E(\mu) \) is convex. By way of motivation we start with the case when \( E(\mu) \) is affine.
5.1.1. The case of $E(\mu)$ affine. In the case when $E(\mu)$ is affine on $\mathcal{P}(X)$ it follows directly from the definition of $S(e)$ that $S(e)$ is globally concave, using the concavity of $S(\mu)$ on $\mathcal{P}(X)$. Moreover, if $X$ is compact and $E(\mu) = \langle V, \mu \rangle$ for $V \in C^0(X)$, then a duality argument reveals that $S(e)$ is finite and strictly concave on $]e_{\min}, e_{\max}[$. In fact,

$$S(e) = F^*_V(e), \quad F_V(\beta) = -\log \int_X e^{-\beta V} \mu_0,$$

where $F_V(\beta) < \infty$ for all $\beta$, since $X$ is compact and $V$ is bounded. Indeed, in this case it follows from Jensen’s inequality that the free energy $F(\beta)$ is of the form $F_V(\beta)$ above \footnote{This is an instance of the classical Gibb’s variational principle}. Since $F_V(\beta)$ is differentiable on all of $\mathbb{R}$ and its derivative tends to $\inf_X V (= e_{\min})$ and $\sup_X V (= e_{\max})$ as $\beta \to \infty$ and $\beta \to -\infty$, respectively, it thus follows from Lemma 5.2 below that $S_V(e)$ is strictly concave on $]e_{\min}, e_{\max}[$. However, if $X$ is non-compact, then the strict concavity of $S_V(e)$ may fail as illustrated by the following simple example:

$$X = \mathbb{R}, \quad \mu_0 = e^{-|x|} dx, \quad V(x) = |x|^2.$$

In this case $E(\mu_0) < \infty$, but $\int e^{-\beta V} \mu_0 < \infty$ ifff $\beta \geq 0$. It follows that $S(e) = S(e_0) = 0$ for $e > e_0$ and thus $S(e)$ is not strictly concave. Indeed, applying the second point in Prop 3.7 we get, for $e \geq e_0$

$$S(e) = \inf_{\beta \leq 0} (\beta e - F_V(\beta)).$$

But, since $F_V(\beta) = \infty$ for $\beta < 0$ the rhs above is attained at $\beta = 0$, showing that $S(e) = 0$. Also note that replacing $V$ with $-V$ yields an example where $S(e)$ fails to be strictly concave in the low-energy region. Note also that in this example, the sup defining $S(e)$ is not attained in the region where $S(e) = S(e_0)$, if $e \neq e_0$. Indeed, if the sup is attained at $\mu^e$ satisfying $E(\mu) = e$, then $S(\mu^e) = S(\mu_0)$ and hence $\mu^e = \mu_0$, which forces $e = E(\mu_0) := e_0$.

5.1.2. The case of $E(\mu)$ convex. Using Lemma 5.2 we observe that similar arguments apply in the low-energy region when $E(\mu)$ is convex, under some further regularity assumptions:

**Proposition 5.1.** Let $X$ be a topological space and $E(\mu)$ a lsc convex functional on $\mathcal{P}(X)$ and $\mu_0 \in \mathcal{P}(X)$.

- If $X$ is compact and $e_0 := E(\mu_0) < \infty$, then $S(e)$ is concave on $]-\infty, e_0]$. 
- If $X$ is $\sigma$–compact (i.e. a countable union of compact space) and $E(1_{K\mu_0}) < \infty$ for any compact subspace $K$ of $X$, then, if the energy approximation property holds, $S(e)$ is concave, increasing and finite (hence continuous) on $]e_{\min}, e_0[$.

**Proof.** Given $e_1$ and $e_2$ in $]-\infty, e_0]$ and $t \in [0,1]$ set $e_t := (1-t)e_0 + te_1$. Let $\mu_1$ and $\mu_2$ be contenders for the sup defining $S(e_1)$ and $S(e_2)$, respectively. Set $\mu_t := (1-t)\mu_1 + t\mu_2$. Since $E(\mu)$ is assumed convex, $E(\mu_t) \leq e_t$. Hence, if $X$ is compact and $E(\mu_0) < \infty$, then Lemma 3.2 gives, $S(e_1) \geq S(\mu_t) \geq (1-t)S(\mu_1) + tS(\mu_2)$, using that $S$ is concave on $\mathcal{P}(X)$. This proves the first point. To prove the second one we write $X$ is an increasing union of compact subspaces $X_R$. Denoting by $S_R$ the entropy corresponding to $X_R$ it follows directly from the definition that $S_R(e) \leq S(e)$. Now, by the energy approximation property in Lemma 3.6 $-\infty < S_R(e) \leq S(e)$. A slight variant of the argument in the end of the proof of Theorem 6.9 then shows that $S_R(e)$ increases towards $S(e)$ as $R \to \infty$. Hence, we can conclude by invoking the first point. \hfill $\square$

Next, a different duality argument yields strict concavity and continuity up to $e = e_0$ when $X$ is compact. The proof uses the following duality criterion:
Lemma 5.2. Consider the Very general setup and assume that $X$ is compact and that the energy approximation property holds. If $F(\beta)$ is differentiable in a neighborhood of $[\beta_0, \beta_1]$ and $[F'(\beta_1), F'(\beta_0)] \subset [e_{\min}, e_{\max}]$, then $S(e)$ is strictly concave and equal to $F^*$ on $[F'(\beta_1), F'(\beta_0)]$. Moreover, in general, if $F$ is differentiable at $\beta$, then $F'(\beta) = E(\mu_\beta)$ for any minimizer of $F_\beta$.

Proof. Since $F(\beta)$ is concave and $F = S^*$ Lemma [24] implies that $S^{**}$ is strictly concave on $[F'(\beta_1), F'(\beta_0)]$. Next, by Prop [11] $S$ is usc on $U := [e_{\min}, e_{\max}]$ and hence Lemma [22] forces $S^{**} = S$ on $[F'(\beta_1), F'(\beta_0)]$, which concludes the proof of the first statement. The last statement follows directly from letting $\delta$ tend to zero (from left and from right) in the inequality

$$F(\beta_0) - F(\beta) \leq F_{\beta_0}(\mu_\beta) - F(\mu_\beta) = \delta E(\mu_\beta).$$

□

Proposition 5.3. Assume that $X$ is compact, $E(\mu)$ is lsc and convex on $P(X)$. Then $S(e)$ is strictly concave and $S(e) = F^*(e)$ on $[e_{\min}, e_0]$. Moreover, $S(e)$ is continuous on $[e_{\min}, e_0]$.

Proof. The concavity was shown in [22] under the extra assumption that $E(\mu)$ is continuous on $P(X)$. Here we note that an alternative argument yields strict concavity under the more general assumptions in the proposition. The starting point is the observation that $F'_{\beta}(\mu)$ is convex on $P(X)$ for $\beta \geq 0$ and strictly convex on $\{F_\beta < \infty\}$. Indeed, since $E(\mu)$ is assumed convex this follows directly from the corresponding property of $-S(\mu)$ (i.e. from the case $\beta = 0$), which is well-known [24]. It then follows from general principles that $F(\beta)$ is differentiable with derivative at $\beta$ given by $e(\beta) := E(\mu_\beta)$, where $\mu_\beta$ is the unique minimizer of $F_\beta$. Indeed, this follows from the general statement in the appendix of [11], using that $E(\mu_\beta)$ is continuous in $\beta$ by the argument below. Hence, by Lemma [5.2] $S(e)$ is strictly concave and equal to $F^*$ on the interval $\lim_{\beta \to 0} e(\beta), \lim_{\beta \to 0} e(\beta)]$. By the concavity of $F(\beta)$ the function $e(\beta)$ is decreasing. Moreover, the energy approximation property implies, in a rather straightforward manner, that

$$\lim_{\beta \to 0} e(\beta) = e_{\min}$$

(see [7]). All that remains is thus to verify that

$$\lim_{\beta \to 0} e(\beta) = e_0.$$  

But since $e(\beta)$ is decreasing this follows readily from the lower-semi continuity of $E(\mu)$ (see [4]). To prove that $S(e)$ is continuous on $[e_{\min}, e_0]$ it will be enough, by the previous step, to show that $F^*(e)$ is continuous on $[e_{\min}, e_0]$ and $F^*(e_0) = 0$. Since $F^*$ is concave it is enough to show that $F^*(e)$ is finite on $[e_{\min}, e_{\max}]$. But

$$S \leq S^{**} = F^* \leq 0,$$

where the last inequality follows from restricting the inf defining $F^*$ to $\beta = 0$. Since $S$ is finite (by the previous proposition) it follows that $F^*$ is also finite and thus continuous on $[e_{\min}, e_{\max}]$. Hence, by the continuity of $F^*$ at $e_0$ we get $S(e) \to F^*(e_0)$ as $e \to e_0$. But

$$F^*(e_0) = \inf_{\beta \in \mathbb{R}} (\beta e_0 - F(\beta)), \quad F(\beta) = \inf_{\mu \in P(X)} \beta E(\mu) - S(\mu) \leq \beta E(\mu_0) - S(\mu_0) = \beta e_0$$

Hence,

$$F^*(e_0) = \inf_{\beta \in \mathbb{R}} (\beta e_0 - F(\beta)) \geq \inf_{\beta \in \mathbb{R}} (\beta e_0 - \beta e_0) = 0.$$

which gives $\lim_{e \to e_0} S(e) = 0$. Since, trivially, $S(e) \leq S(e_0) = 0$ it follows that $S(e) \to 0 = S(e_0)$, as desired. □
5.2. The necessity of the energy approximation property for thermodynamic equivalence of ensembles. We next show that the assumption that $\mu_0$ has the energy approximation property, used in the previous section is necessary for having thermodynamic equivalence of ensembles:

**Theorem 5.4.** Let $X$ be a compact topological space endowed with a measure $\mu_0$ such that $E(\mu_0) < \infty$ and assume that $E(\mu)$ is a lsc convex functional on $P(X)$ and $V \in C^0(X)$. Denote by $S_V(e)$ entropy $S_V(e)$ associated to $E_V(\mu) := E(\mu) + (V, \mu)$ and the measure $\mu_0$. Then $S_V(e)$ is concave and finite on $[e_{\text{min}}, e_0]$ for any $V \in C^0(X)$ iff $\mu_0$ has the energy approximation property. In other words, thermodynamic equivalence of ensembles holds in the low-energy regions $[e_{\text{min}}, e_0]$ for all $V \in C^0(X)$ iff $\mu_0$ has the energy approximation property.

*Proof.* First assume that $\mu_0$ has the energy approximation property. Since $E_V(X)$ is lsc and convex it then follows from the previous proposition that $S_V(e)$ is concave on $[e_{\text{min}}, e_0]$ for any $V \in C^0(X)$. To prove the converse first note that, by the third point in Prop. 3.7, the restriction of $S_V$ to $[e_{\text{min}}, e_0]$ is equal to the Legendre-Fenchel transform of $F_V(\beta)$. Hence, since $S_V(e)$ is assumed finite on $[e_{\text{min}}, e_0]$ it follows from the property of gradient images in formula (2.3) that $dF_V(\beta)/d\beta \to e_{\text{min}}$ as $\beta \to \infty$ (using either left or right derivatives). Since $F_V(\beta)$ is concave this means that

$$\lim_{\beta \to \infty} F_V(\beta)/\beta = \inf_{P(X)} E(\mu).$$

Now, by definition, $F_V(\mu)/\beta = E(\mu) - S(\mu)/\beta$ and hence

$$\lim_{\beta \to \infty} \inf_{P(X)} (E(\mu) - S(\mu)/\beta) = \inf_{P(X)} E(\mu).$$

But as shown in [2] the latter convergence holds for all $V \in C^0(X)$ iff $\mu_0$ has the approximation property (briefly, the point is that the convergence in question is, since $E(\mu)$ is convex equivalent to the $\Gamma-$convergence of $E(\mu) - S(\mu)/\beta$ towards $E(\mu)$, which, in turn, is equivalent to the energy approximation property of $\mu_0$).

In the case when $W(x, y)$ is the repulsive logarithmic interaction in $\mathbb{R}^2$ or $W(x, y) = |x - y|^{-s}$ in $\mathbb{R}^n$ for $s \in [d - 2, d]$ (specializing to the Coulomb interaction when $s = d - 2$) a potential-theoretic characterization of measures $\mu_0$ satisfying the energy approximation property was given in [2]. In particular, it was shown that any compact domain $X$ with smooth boundary admits probability measures $\mu_0$ with support $X$ and a density in $L^1(X, dx)$, for which the energy-approximation property fails. Hence, by the previous theorem thermodynamic equivalence of ensembles also fails. On the other hand, Lebesgue measure on a compact domain $X$ has the energy approximation property, if $X$ is non-thin at all boundary points, in the sense of classical potential theory. For example, this is the case if any point $x \in \partial X$ is the vertex of a cone contained in $X$ (e.g. if $X$ is a Lipschitz domain).

5.3. The catastrophic case of singular power-laws. Consider now the case when $X$ is compact and $W(x, y)$ is a repulsive power-law singularity:

$$W(x, y) := |x - y|^{-\alpha} + H(x, y), \quad \alpha > 0$$

for $H$ continuous on $X \times X$. In particular, $e_\infty = \infty$. We will say that a compact set $X$ is strictly star-shaped if for any point $x \in X$ and $c \in [0, 1]$ the scaled point $cx$ is contained in the interior of $X$. 

![Image](image-url)
Proposition 5.5. Consider a repulsive power-law singularity \( W \) on a compact strictly star-shaped subset \( X \) of \( \mathbb{R}^d \) and let \( \mu_0 \) be proportional (or comparable) to Lebesgue measure on \( X \). Then \( S(e) \) is concave on \( \mathbb{R} \) and finite (hence continuous) on \( [e_{\text{min}}, \infty[ \). Moreover, \( S(e) = S(e_0) \) for any \( e \geq e_0 \) and, as a consequence, there exists no maximum entropy measure \( \mu^e \) when \( e > e_0 \).

Proof. For simplicity we will assume that \( H = 0 \), but the general case is shown in essentially the same way. First observe that \( E \) has the energy approximation property. Indeed, using that \( X \) is assumed strictly star-shaped and \( W(e^t) \) is monotone in \( t \) it is, by the argument in the proof of Lemma 6.3 enough to show this when the support of \( \mu \) is contained in the interior of \( X \). Let \( \mu_\epsilon \) be defined as in formula 6.2. First using that \( E \) is convex and then that \( E \) is translationally invariant gives

\[
E(\mu_\epsilon) \geq \int_{a \in B_\epsilon} \sigma_\epsilon E((T_\epsilon)_*\mu) = \int_{a \in B_\epsilon} \sigma_\epsilon E(\mu) = E(\mu).
\]

The reversed asymptotic inequality follows directly from the lower semi-continuity of \( E \), resulting from the assumed lower semi-continuity of \( \psi \). Next note that the affine continuity property appearing in Lemma 3.2 holds, as is seen by modifying the proof of Lemma 6.2. Indeed, using the Cauchy-Schwartz inequality the finiteness in formula 6.1 follows from the property appearing in Lemma 3.2 holds, as is seen by modifying the proof of Lemma 6.2.

Next, we will show that \( S(e) = S(e_0) \) for any \( e > e_0 \). To this end it will, thanks to Lemma 3.2, be enough to show that there exists a family \( \nu_\epsilon \in \mathcal{P}(X) \) parametrized by \( \epsilon > 0 \) such that, as \( \epsilon \to 0 \),

\[
(i) \ E(\nu_\epsilon) \to \infty, \quad (ii) \ S(\nu_\epsilon) \to S(\mu_0) = 0.
\]

We will take \( \nu_\epsilon = e^{\alpha/4}(T_\epsilon)_*\mu_0 + (1 - e^{\alpha/4})\mu_0 \), where, as before, \( T_\epsilon \) denotes the scaling map \( x \mapsto \epsilon x \). First observe that since \( S((T_\epsilon)_*\mu_0) = d \log \epsilon \) for some (as seen by making the change of variables \( x \mapsto T_\epsilon(x) \) in the integrals) we get, using the concavity of \( S(\mu) \) on \( \mathcal{P}(X) \),

\[
S(\nu_\epsilon) \geq e^{\alpha/4}S((T_\epsilon)_*\mu_0) + (1 - e^{\alpha/4})S(\mu_0) \geq e^{\alpha/4}d \log \epsilon + 0,
\]

which verifies the second item in formula 5.3. To prove the first one observe that, making the change of variables \( x \mapsto T_\epsilon(x) \) in the integrals, reveals that \( E((T_\epsilon)_*\mu_0) = e^{-\alpha}E(\mu_0) \) and hence \( E(\nu_\epsilon) = e^{\alpha/2}E((T_\epsilon)_*\mu_0) = e^{\alpha/2}e^{-\alpha}E(\mu_0) \), which proves the first item in formula 5.3.

Hence, \( S(e) = S(e_0) \) for any \( e > e_0 \). Since we have shown that \( S(e) \) is concave, increasing and continuous on \( [e_{\text{min}}, e_0] \) and \( S(e) = S(e_0) \) it follows that \( S(e) \) is concave and continuous on \( [e_{\text{min}}, e_0] \). Moreover, since \( S(\mu) = S(\mu_0) \) iff \( \mu = \mu_0 \) (which implies \( E(\mu) = e_0 \)) it follows that there exists no maximum entropy measure for \( e > e_0 \). \( \square \)

More precisely, the proof of the previous proposition reveals that, for any given \( e > e_0 \) there exists \( \mu \in \mathcal{P}(X) \) with energy \( e \), i.e. \( E(\mu) = e \), whose entropy \( S(\mu) \) can be taken to be arbitrarily close to the maximal entropy and such that \( \mu \) has a “core-halo” structure, i.e. \( \mu \) is a convex combination

\[
(1 - \lambda)\mu_0 + \lambda \mu_1
\]

for some \( \lambda \in [0, 1[ \), where \( \mu_1 \) (the “core”) can be taken to be a uniform measure of arbitrarily large density on a ball with arbitrarily small radius, centered at a given point in the interior of \( X \).

Remark 5.6. The assumption the \( X \) be star-shaped was imposed to ensure the energy approx property and can certainly be relaxed. For example, if \( W(x, y) \) is the Coulomb...
interaction in $\mathbb{R}^n$, then, as pointed out in Section 5.2, the energy approximation property in question holds if the interior of $X$ is non-thin at all boundary points.

The previous proposition also applies to the corresponding singular power-laws obtained by switching the sign of $W$, if at the same time $e$ is replaced by $-e$ (using that $S(-e)$ is concave iff $S(e)$ is). In the case of the Newtonian pair-interaction in $\mathbb{R}^3$ the non-existence of the corresponding maximum entropy measure is closely related to the gravitational catastrophe (Antonov instability) which plays a central role in astrophysics [16] Section 4.10.1] (as explained in [8]).

6. Concavity in the High Energy Region under the Main Assumptions

We start by recalling the Main/Homogeneous Assumptions stated in the introduction of the paper.

6.1. The Main and Homogeneous Assumptions. Let $X$ be a (possible non-compact) subset of $\mathbb{R}^{2n}$ and let $\phi$ be a defining function for $X$, i.e. a continuous function such that

$$X = \{\phi \leq 0\}.$$ 

Endow $X$ with a measure $\mu_0$ which is absolutely continuous wrt Lebesgue measure $d\lambda$:

$$\mu_0 = e^{-\Psi_0}d\lambda.$$ 

on $\mathbb{R}^{2n}$. As pointed out above we will identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ and denote by $(z_1, ..., z_n)$ the standard holomorphic coordinates on $\mathbb{C}^n$.

Main Assumptions: $\phi \in PSH_a(\mathbb{C}^n)$, $\Psi_0, -V \in PSH_a(X)$ and $-W \in PSH_{a,a}(X \times X)$ for some $a \in ]0, \infty[^n$.

The class $PSH_a(X)$ was defined in Section 2.3 and the class $PSH_{a,a}(X \times X)$ is defined similarly, by identifying $\mathbb{C}^n \times \mathbb{C}^n$ with $\mathbb{C}^{2n}$ and using the weight vector $(a, a)$. Recall that we also introduced the Homogeneous Assumptions in Section 1.3 which according to the following lemma is a special case of the Main Assumptions:

Lemma 6.1. If the Homogeneous Assumptions are satisfied, then so are the Main Assumptions.

Proof. First note that $-v(r)$ is increasing in $r$. Indeed, since $\phi(t) := v(e^t)$ is convex in $t$ the limit, denoted by $\phi(-\infty)$, of the one sided derivative $\phi'(t^+)$ exists as $t \rightarrow \infty$. Since $\phi(t)$ is assumed bounded from above as $t \rightarrow -\infty$ it follows that $\phi(-\infty) \geq 0$. Hence, by convexity, $\phi'(t^+) \geq 0$ for all $t$, showing that $\phi(t)$ is increasing in $t$, as desired. Since $\log |z|$ is psh this means that $V(z)$ is an increasing convex function of the psh function $\log |z|$ when $z \neq 0$ and bounded from above in a punctured neighborhood of the origin in $\mathbb{C}^n$. But any psh function which is locally bounded from above on the complement of a pluripolar set $A$ (i.e. a set which is locally the $-\infty$-set of a psh function) extends over $A$ to a unique psh function [26] Thm 5.24]. Thus $-V$ indeed defines a psh function on $X$ and the same argument applies to $\Psi_0(z)$. Similarly since $(z, \zeta) \mapsto (z - \zeta)$ is holomorphic the function $\log |z - \zeta|$ is psh on $\mathbb{C}^{2n}$ and thus $-W(z, \zeta)$ is an increasing convex function of a psh function when $\log |z - \zeta| \neq -\infty$ and thus psh. All in all this means that the Main Assumptions are satisfied with, for example, $a_0 = ... = a_n = 1$. □

We next show that the "affine continuity property" and energy approximation property introduced in Section 3 both hold under the Main Assumptions.

Lemma 6.2. Under the Main Assumptions the affine continuity property holds.
Proof. Since $X$ may be assumed compact and $W$ is lsc we may after perhaps replacing $W$ with $W + C$ i.e. $E$ with $E + C$, as well assume that $W \geq 0$ on $X \times X$. Hence, by the dominated convergence theorem it is enough to verify that if $E(\mu) < \infty$, then

$$
(6.1) \quad \int_{X \times X} W \mu \otimes \mu_0 < \infty.
$$

Set $u_\mu(x) := \int_X W(x, y) \mu(y)$. Since $-W$ is psh on a neighborhood of $X \times X$ the function $-u_\mu(x)$ psh on $X$. Now since $X$ is connected, as shown in the course of the proof of Theorem 6.7 any psh function (or more generally, subharmonic function) is either identically equal to $-\infty$ or in $L^1_{\text{loc}}$ (as follows from the submean property of subharmonic functions). But, by assumption, $\int_X u_\mu \mu = E(\mu) < \infty$ and hence $-u_\mu$ cannot be identically $-\infty$. Since $\mu_0 = e^{-\Psi_0} d\lambda$ 6.1 thus follows directly in the case when $\Psi_0$ is bounded. In the general case we can use that by Cor 8.1 below, there exists $q > 1$ such that $\int_X e^{-q\Psi} d\lambda < \infty$ for any psh function $\psi$ (not identically $-\infty$) and apply Hölder’s inequality to conclude. \hfill $\Box$

**Lemma 6.3.** Assume that the Main Assumptions hold. Then the corresponding energy approximation property is satisfied.

**Proof.** First consider the case when the support of $\mu$ is contained in the interior of $X$. Set

$$
(6.2) \quad \mu_\epsilon := \int_{a \in B_\epsilon} \sigma_\epsilon(T_a)^* \mu,
$$

where, for a given $a \in \mathbb{R}^d$, $T_a$ is the map $x \mapsto x + a$ and $B_\epsilon$ denotes the ball of radius $\epsilon$ centered at the origin. For $\epsilon$ sufficiently small $\mu_\epsilon$ is also supported in $X$. It is a standard fact that $\mu_\epsilon$ is absolutely continuous wrt Lebesgue measure and $\mu_\epsilon \to \mu$ weakly as $\epsilon \to 0$. Moreover,

$$
(6.3) \quad \lim_{\epsilon \to 0} E(\mu_\epsilon) = E(\mu).
$$

Indeed, setting $\Psi := -W(x, y) + V(x) + V(y)$ and changing the order of integration gives

$$-E(\mu_\epsilon) = \int_{B_\epsilon \times B_\epsilon} \sigma_\epsilon \otimes \sigma_\epsilon \int_{X \times X} \Psi(T_a)^* \mu \otimes (T_b)^* \mu = \int_{X \times X} \mu(x) \otimes \mu(y) \int_{B_\epsilon \times B_\epsilon} \sigma_\epsilon \otimes \sigma_\epsilon \Psi(x + a, x + b).$$

Recall that, in general, if $\psi(x)$ is a subharmonic function, then $\int_{B_\epsilon} \sigma_\epsilon \psi(x + a)$ decreases to $\psi(x)$, as $\epsilon$ decreases to 0. Hence, the convergence 6.3 follows from the monotone convergence theorem. Finally, for a general $\mu \in \mathcal{P}(X)$ we consider for $\tau \in \mathcal{C}$ the holomorphic action $\tau \otimes z$ defined by formula 6.9. If $z \in X$ and $\Re \tau < 0$, then it follows readily from the definitions that $\tau \otimes z$ is contained in the interior of $X$ (compare the proof of Theorem 6.7). Hence, fixing $t < 0$ and setting $F_t(z) := t \otimes z$ the probability measure $\mu^t := (F_t)_* \mu$ is supported in the interior of $X$. Moreover, $\mu^t$ converges weakly towards $\mu$ when $t \to 0$ and

$$
(6.4) \quad \lim_{t \to 0} E(\mu^t) = E(\mu).
$$

Indeed, proceeding as above

$$-E(\mu^t) = \int \Psi(e^t x, e^t y) \mu(x) \otimes \mu(y)
$$

where $t \mapsto \Psi(e^t x, e^t y)$ is increasing (as shown in the course of the proof of Theorem 6.7). Hence, the convergence 6.4 follows from the monotone convergence theorem. We can thus conclude the proof by combining the convergence in 6.3 and 6.4 using a standard diagonal argument.

Combining the previous two lemmas with Lemma 3.6 we arrive at the following \hfill $\Box$

**Proposition 6.4.** Under the Main Assumptions $S(e)$ is finite on $[e_{\min}, e_{\max}]$. 

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6.2. Concavity of the microcanonical entropy $S^{(N)}_\mu (e)$. The following result is a slight generalization of a result shown in the course of the proof of \cite[Theorem 2.3]{12}, which, in turn, is based on the main result in \cite{14}.

**Proposition 6.5.** Let $Y$ be a pseudoconvex domain, $\Psi$ a psh function on $Y$ and $\mu_0$ a measure on $Y$ such that $\mu_0 = e^{-\psi_0} d\lambda$ for a psh function $\psi_0$ on $Y$. Assume that $Y$ is endowed with a holomorphic action of by compact group $G$ such that if $\Psi$ and $\mu_0$ are $G$–invariant and assume also that for any $t \in \mathbb{R}$ any $G$–invariant holomorphic function on $\{ \Psi < t \}$ is constant. Then either the function

$$t \mapsto \log \mu_0 \{ \Psi < t \}$$

is identically equal to $+\infty$ or concave. In particular, if $\mu_0$ is moreover assumed to have finite total mass, then the concavity in question holds.

**Proof.** First recall the main result in \cite{14}. Consider $\mathbb{C}^{n+1}$ with holomorphic coordinates $(z,t) \in \mathbb{C}^n \times \mathbb{C}$. Let $D$ be a pseudoconvex domain in $\mathbb{C}^{n+1}$ endowed with a psh function $\psi(z,t)$. Denote by $D_t$ the subset of $D$ obtained by fixing the $t$–coordinate. According to the main result of \cite{14} the function $B_t(z)$ on $D$ defined by

$$B_t(z) := \sup \left\{ |f(z)|^2 : f \text{ holomorphic on } D_t \text{ and } \int_{D_t} |f|^2 e^{-\psi(z,t)} d\lambda \leq 1 \right\}$$

has the property that either $\log B_t(z)$ is subharmonic in $t$ or identically equal to $-\infty$. In the present case we take

$$D := \{ \Psi(z) - \Re(t) < 0 \} \subset Y \times \mathbb{C}$$

and $\psi(z,t) = \psi_0(z)$. Note that $D$ is a pseudoconvex domain in $\mathbb{C}^{n+1}$. Indeed, this follows from Lemma \ref{2.2} using that $\Psi(z) - \Re(t)$ is psh in $Y \times \mathbb{C}$ and $Y \times \mathbb{C}$ is pseudoconvex (since $Y$ is). Now, by assumption, the group $G$ acts holomorphically on $D_t$. In particular, if $dG$ denotes a $G$–invariant measure on $G$ (i.e. Haar measures) and $f$ is holomorphic on $D_t$ then the function

$$f_G(z) := \int_{g \in G} f(g \cdot z) dG$$

is holomorphic and $G$–invariant. Hence, replacing $f$ with $f_G$ and using the “triangle inequality” the sup in formula (6.5) may as well be restricted to all $G$–invariant holomorphic functions $f$. But, by assumption, any such functions is constant and hence we may as well take $f$ to be identically equal to 1. But this means that $B_t(z) = 1/ \int_{Y \cap \{ \psi(z,t) < t \}} e^{-\psi_0} d\lambda$. The theorem thus follows from the main result of \cite{14}, recalled above (also using that if $\phi(t)$ is subharmonic in $t$ and only depends on $\Re(t)$ then $\phi(t)$ is convex wrt $t \in \mathbb{R}$). \hfill $\Box$

**Remark 6.6.** (Brunn-Minkowski inequality) This proposition can be viewed as a generalization of the classical fact that the logarithm of the volume $\mu_0 \{ \{ \phi \leq t \} \}$ is concave if $\phi$ is a convex function on $\mathbb{R}^n$ and $\mu_0$ is a log concave measures i.e. $\mu_0 = 1_C e^{-\phi_0} d\lambda$ for $C \subset \mathbb{R}^n$ a convex body and $\phi_0$ a convex function. This is a consequence of the Brunn-Minkowski inequality, but it also follows from the previous proposition by considering the map $L(z) = (\log |z_1|, \ldots, \log |z_n|)$ from $(\mathbb{C} \setminus \{0\})^n$ onto $\mathbb{R}^n$ which has the property that $\phi(x)$ is convex iff $\psi := L^* \phi$ is psh and hence $C$ is convex iff $Y := L^{-1}(C)$ is pseudoconvex in $\mathbb{C}^n$ (using that $L^* \phi$ is bounded from above and this extends to a psh function on $\mathbb{C}^n$). The classical fact in question then follows by taking $G$ as the $n$–dimensional compact torus, acting on $\mathbb{C}^n$ in the standard way (and thus preserving the fibers of the map $L$).

It should be stressed that plurisubharmonicity alone is not enough to ensure the concavity in the previous proposition, as illustrated by the case when $Y$ is the unit-disc in
$\mathbb{C}$ and $\Psi(z)$ is the Green function for the Laplacian with a pole at $w \in Y$, for a given non-zero $w$ in the interior of $Y$, i.e. $\Psi(z) = \log \left( \frac{\vert z - w \vert}{1 - \bar{w}z} \right)$ Indeed, as shown in the proof of [12, Thm 2.3] the concavity in question is then equivalent to the subharmonicity of the Schwartz symmetrization of $\Psi(z)$, which only holds when $a$ is zero, i.e. when $\Psi$ is $S^1$-invariant, as pointed out in the introduction of [12].

We next apply the previous proposition to the case when the $N$-particle Hamiltonian $H^{(N)}$ on $X^N$ comes from a pair-interaction $W$ and exterior potential $V$ such that $-W$ and $-V$ and also $\phi$ and $\Psi_0$ satisfy the Main Assumptions (but there is no need to assume that $W(x,y)$ is symmetric or that the indices $i,j$ range over all of $\{1,2,...,N\}$):

**Theorem 6.7.** If the Main Assumptions hold and $H^{(N)}$ is the function on $X^N$ defined by

$$H^{(N)}(x_1,...,x_N) := a_{ij} \sum_{(i,j) \in \mathcal{I}} W(x_i, x_j) + b_i \sum_{j \in \mathcal{J}} V(x_j)$$

for some subsets $\mathcal{I}$ of $\{1,...,N\}^2$ and $\mathcal{J}$ of $\{1,...,N\}$ and non-negative constants $a_{ij}$ and $b_i$. Then

$$S^+(N)(e) := \log \mu^\oplus_{0} \left\{ H^{(N)} > e \right\}$$

is concave in $e$ and finite when $e > \sup_{X^N} H^{(N)}$. In particular, $S^+(N)(e)$ is concave when $H^{(N)}$ is the mean field Hamiltonian $\mathcal{H}$.

**Proof.** It is a standard fact that the closure of the orbits of the vector field $\mathcal{V}_a$ coincides with the orbits of a compact torus $G$ acting holomorphically on $\mathbb{C}^n$. The assumptions imply that $\phi, V, W$ and $\mu_0$ are invariant under the action of $G$ (using the diagonal action on $X \times X$). By assumption $X$ admits a continuous psh exhaustion function $\rho$. By the construction in Lemma 2.3, $\rho$ may as well be assumed to be $G$-invariant. Since the maximum of a finite number of psh functions is still psh the function $\rho_N$ on $X^N$ defined by

$$\rho(z_1,...,z_N) := \max_{i=1,...,N} \rho(z_i)$$

is psh and $G$-invariant wrt the diagonal action of $G$ on $X^N$ and thus defines a $G$-invariant continuous psh exhaustion function of $X^N$. The corollary will thus follow from the previous theorem applied to $X^N$, $\Psi_N := -H^{(N)}$ and the measure $\mu^\oplus_{0}$ if $X^N$, once the assumptions on $G$ have been verified. To this end consider the holomorphic action of the additive group $\mathbb{C}$ on $\mathbb{C}^n$ defined as follows: given $\tau \in \mathbb{C}$ and $z \in \mathbb{C}^n$

$$\tau \otimes z := (e^{a_1 \tau} z_1, ..., e^{a_n \tau} z_n).$$

Note that, for $z$ fixed, $\tau \otimes z \to 0$ as the real part $\Re \tau \to -\infty$. Moreover, if $\phi$ is a psh function on $\mathbb{C}^n$ then

$$\phi(\tau \otimes z) \leq \phi(z) \quad \text{if} \quad \Re \tau \leq 0$$

To see this first observe that since $\phi$ is psh and the orbits of the $\mathbb{C}$-action define holomorphic curves the function $\phi(\tau) := \phi(\tau \otimes z)$ is subharmonic for a fixed $z$. Moreover, since $\phi \in \text{PSH}_a$ the function $\phi(\tau)$ is independent of the imaginary part $\Im \tau$ and hence $\phi(\tau)$ is convex wrt the real part $t$ of $\Re \tau$. Since $\phi$ is bounded from above close to the origin it follows that there exists a constant $C$ such that $\phi(\tau) \leq C$ as $\Re \tau \to -\infty$. But then the convexity of $\phi(t)$ implies that $d\phi(t)/dt \to 0$ as $\tau \to -\infty$ in $\mathbb{R}$. Hence, by convexity, $\phi(\tau)$ is increasing in the real part of $\tau$, proving the inequality. As a consequence, $X$ is $G$-invariant, connected and the origin $0$ is contained in the interior of $X$ (using that the action by $\mathbb{C}$ is locally free). Moreover, the same thing goes for the sublevel sets $\{\Psi_N(z_1,...,z_N) < t\}$. Indeed, by the previous argument $\Psi_N(\tau \otimes z_N, ..., \tau \otimes z_N)$ is increasing wrt the real part.
of $\tau$. In particular, the minimum of $\Psi$ is attained at the origin in $X^N$, which implies that
0 is an interior point of $\{\Psi_N(z_1, ..., z_N) < t\}$, as long as $t > \inf_{X^N} \Psi_N$. Thus if $f$ is a
holomorphic function on $\{\Psi_N(z_1, ..., z_N) < t\}$, then in order to verify that $f$ is constant on
$\{\Psi_N(z_1, ..., z_N) < t\}$ it is enough to verify that its Taylor expansion at the origin 0 in $\mathbb{C}^n$ is
a constant. To simplify the notation we will prove this when $N = 1$ (but the general case
if the same up to a change of notation). Using multinomial notation the action of $\nu_\alpha$
on $f(z)$ close to the origin in $\mathbb{C}^n$ gives, by Taylor expansion of $f$,
$$\nu_\alpha(f) = \nu_\alpha\left(\sum_{\alpha_i \geq 0} c_\alpha z_1^{\alpha_1} \cdots z_n^{\alpha_n}\right) = \sum_{\alpha_i \geq 0} i\alpha \cdot \nu_\alpha z_1^{\alpha_1} \cdots z_n^{\alpha_n}.$$Since $a_i > 0$ the scalar product $a \cdot \alpha$ is non-vanishing for $\alpha \neq 0$. Hence, $\nu_\alpha f = 0$ can only
hold if the Taylor coefficients $c_\alpha$ vanish for $\alpha \neq 0$. Since $X$ is connected it follows that $f$
is identically constant (by the identity principle for holomorphic functions).

As for the finiteness of $S^1_N(e)$, for $e > \sup_{X^N} H^{(N)}$, it follows directly from the fact that
any psf function is usc, hence the subset where $H^{(N)} > e$ is open. $\square$

When $H^{(N)}$ is replaced by the “attractive” Hamiltonian $-H^{(N)}$ the previous theorem
also shows that $S^1_N(e)$ (formula 6.9) is concave. The following simple example illustrates
the relevance of the plurisubharmonicity assumption in the previous theorem. Consider
the “attractive” Hamiltonian obtained by taking $W = 0$ in formula 1.1 and assume that $V$
is $S^1$-invariant. Then $V$ is psf iff $V = \phi(|z|)$ for a convex increasing function $\phi$ on $\mathbb{R}$.
Assuming that $\phi(x)$ is strictly increasing and $N = 1$ we get

$$\mu_0^{\otimes N} \{H^{(N)} \leq e\} = C_{\text{vol}} e^{2nf(e)},$$

where $f(e)$ is the function defined, on the image of $\phi$, as the inverse of $\phi(x)$ and $C_{\text{vol}}$
is the volume of the unit-ball in $\mathbb{R}^2$. Hence, the logarithm of $\mu_0^{\otimes N} \{H^{(N)} \leq e\}$ is concave iff
$f$ is concave iff its inverse $\phi$ is convex iff $V$ is psf. In the case when $N \geq 1$ an illustrative
class of “attractive” Hamiltonians is given by the case when $V$ is a power-law, $V = |z|^\alpha$
for $\alpha > 0$ (i.e. $\phi(x)$ is the convex function $x^\alpha$). Then a simple scaling argument reveals
that the volume $\mu_0^{\otimes N} \{H^{(N)} \leq e\}$ is of the form 6.8 for $f(e) = \log e$ when $n$ is replaced
by $nN/\alpha$, if $e \geq 0$. Hence, the logarithm of $\mu_0^{\otimes N} \{H^{(N)} \leq e\}$ is indeed concave. Note that
in this example the logarithm of the surface area of $\{H^{(N)} = e\}$, i.e. of the derivative of
$\mu_0^{\otimes N} \{H^{(N)} \leq e\}$, is not concave unless $N$ is taken sufficiently large; $N \geq \alpha/2n$ (otherwise
it is convex).

Remark 6.8. In the case of the mean field Hamiltonian $H^{(N)}$ the concavity of $S^1_N(e)$ holds
more generally if the assumptions on $W$ and $V$ are replaced by the weaker assumption that
the negative of $W(x, y) + \frac{N}{N-1} (V(x) + V(y))$ is in $PSH_{\text{psf}}(X \times X)$. Indeed, rewriting

$$H^{(N)}(x_1, ..., x_N) = \frac{1}{2N} \sum_{i \neq j \leq N} \left( W(x_i, x_j) + \frac{N}{N-1} (V(x_i) + V(x_j)) \right)$$

we can then apply the previous theorem to the mean field Hamiltonian corresponding to the
pair-interaction $W(x, y) + \frac{N}{N-1} (V(x) + V(y))$.

6.2.1. Incorporating constraints. Prop 6.5 may be generalized by replacing $\Psi$ with a finite
number of functions $\psi_1, ..., \psi_r$ on $Y$ satisfying the same assumptions as $\Psi$ and replacing
the sublevel $\{\Psi < t\}$ with the intersection of the sub-level sets $\{\psi_i < t_i\}$, $i = 1, ..., r$
for a given $t = (t_1, ..., t_r) \in \mathbb{R}^r$. Then the logarithm of the corresponding volume defines
a concave function of $t \in \mathbb{R}^r$. Indeed, one simply replaced the domain $\mathcal{D}$ in the proof

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with the intersection of the pseudo-convex domains \( \{ \psi_1(z) - \Re(t_1) < 0 \} \) in \( \mathbb{C}^n \times \mathbb{C}^r \). Since the intersection of pseudo-convex domains is pseudo-convex the main result in [11] then implies that the corresponding function \( \log B(t) \) is a psh function of \( t \in \mathbb{C}^r \) and thus, by translational invariance in the imaginary arguments, it defines a convex function on \( \mathbb{R}^r \). As a consequence if one assumes given \( \psi_1, \ldots, \psi_r \) as above then Theorem [5,4] may be generalized to the statement that the “constrained microscopic entropy”

\[
\log \mu_0 \cap N \left\{ H(N) (z_1, \ldots, z_N) > e, \sum_{i=1}^N \psi_i(z_i) \leq l_1, \ldots, \sum_{i=1}^N \psi_i(z_i) \leq l_r \right\}
\]

is a concave function of \((e, l_1, \ldots, l_r) \in \mathbb{R}^{1+r}\). In particular, when \( X \subset \mathbb{C}^n \) this applies to \( \psi_i(z_1, \ldots, z_n) = \lambda_i |z_i|^2 \) for given positive numbers \( \lambda_1, \ldots, \lambda_n \), as in the Gaussian case discussed in Section 6.3. Anyhow, in this paper we will, for simplicity, stick to the non-constrained setup. On the other hand, as pointed out in Section 3.3 the constraints may be incorporated in the prior measure.

### 6.3. Concavity of the entropy \( S(e) \) when \( e_0 \leq e \)

Now assume that \( X \in \mathbb{R}^{2n} \) that we identify with \( \mathbb{C}^n \), as usual.

**Theorem 6.9.** If the Main Assumptions holds then the entropy \( S(e) \) is a decreasing concave continuous function on \([e_0, e_{\text{max}}]\).

In order to prove this we first assume that \( X \) is compact and invoke the following

**Proposition 6.10.** [32] Assume that \( W \) and \( V \) are continuous and \( X \) is compact. Then

\[
\lim_{N \to \infty} S^{(N)}_+(e) = S_+(e) := \sup_{E(\mu) \geq e} S(\mu)
\]

**Proof.** Consider the open interval \( \Delta := \{ t > e \} \) in \( \mathbb{R} \). By [32, Thm 2.1] the limsup and liminf of \( S^{(N)}_+(e) \) is equal to the sup of \( S(\mu) \) over all \( \mu \in \mathcal{P}(X) \) such that \( E(\mu) \in \Delta \) and \( E(\mu) \in \Delta \), respectively. But by Lemma 3.2 below both these quantities are equal to \( S_+(e) \). \( \square \)

When \( W \) and \( V \) are continuous and \( X \) is compact Theorem 6.7 thus shows that \( S_+(e) \) is a limit of concave functions on \( \mathbb{R} \) and thus concave on \( \mathbb{R} \). Next, we invoke the monotonicity properties shown in Lemma 3.2 below which show that

\[
S_+(e) = \max\{e_0, S(e)\}
\]

and hence \( \max\{e_0, S(e)\} \) is is concave. But by Lemmas 3.1, 6.3 \( S(e) \) is finite for any \( e \in [e_{\text{min}}, e_{\text{max}}] \). Hence, \( \max\{e_0, S(e)\} \) is concave and finite on \([e_{\text{min}}, e_{\text{max}}]\) and thus concave and continuous on \([e_{\text{min}}, e_{\text{max}}]\). This proves the theorem in the case when \( W \) and \( V \) are continuous and \( X \) is compact.

#### 6.3.1. Conclusion of the proof of Theorem 6.9

Still assuming that \( X \) is compact we will next show that \( S(e) \) is decreasing, concave and continuous when \( e \in [e_0, e_{\text{max}}] \), i.e. when

\[
(6.9) \quad E(\mu_0) \leq e < \sup_{\mathcal{P}(X)} E(\mu)
\]

We will proceed by an approximation argument and exploit that \( S(e) > -\infty \) (Prop 6.4). Take a sequence \( W_{\delta}(x, y) \) of continuous pair-interactions increasing to \( W \) satisfying the Main Assumptions. For example, \( W_{\delta} \) may be defined as a convolution of \( W \) with a compactly supported smooth density \( \rho_{\delta} \). First observe that since, by assumption, \( E(\mu_0) < e \), we get \( E_{\delta}(\mu_0) < e \) for \( \delta \) sufficiently small (by the monotone convergence theorem). Thus,
by the concavity and continuity of $S_{+,\delta}(e)$ on $\mathbb{R}$ established in the previous section, we just have to verify that

\begin{equation}
\lim_{\delta \to 0} S_\delta(e) = S(e)
\end{equation}

for any fixed $e$ satisfying the inequalities in formula \ref{6.9}. To this end first note that

\begin{equation}
S_\delta(e) \leq S(e).
\end{equation}

Indeed, by Lemma \ref{3.2} it is enough to prove the corresponding inequality for the upper entropies, where it follows directly from the assumption that $E_\delta(\mu) \leq E_0(\mu)$. Now fix a candidate $\mu$ for the sup defining $S_0(e)$ and set $e_\delta := E_\delta(\mu)$. Then

\[ S_0(\mu) \leq S_\delta(e_\delta) = S(\mu_\delta), \]

where $\mu_\delta$ realizes the sup defining $S_\delta(e_\delta)$. Moreover, fixing a positive number $\epsilon$ we have

\[ e_\delta \geq e - \epsilon \]

for $\delta$ sufficiently small ($\delta < \delta_0$). Hence, since $S_\delta$ is decreasing when $e > E_\delta(\mu_0)$ (by Lemma \ref{3.2}), we get

\[ S_0(\mu) \leq S_\delta(e - \epsilon) \]

for $\delta < \delta_0$. Using that $S_\delta(e)$ is concave we thus deduce that

\[ S_\delta(e_\delta) \leq S_\delta(e) + \epsilon \frac{d}{de} S_\delta(e). \]

Combining the latter inequality with the inequality \ref{6.11} reveals that all that remains, in order to prove the convergence \ref{6.10} is to verify that

\begin{equation}
\left| \frac{d}{de} S_\delta(e) \right| \leq C
\end{equation}

as $\delta \to 0$. To this end first observe that, using again that $S_\delta(e)$ is decreasing and concave yields for any fixed $e' > e$

\[ \left| \frac{d}{de} S_\delta(e) \right| = - \frac{d}{de} S_\delta(e) \leq \frac{S_\delta(e) - S_\delta(e')}{e' - e} \leq \frac{-S_\delta(e')}{e' - e}. \]

In particular, if $e'$ is a fixed number satisfying $e < e' < \sup_{\mathcal{P}(X)} E(\mu)$ we get (by Lemma \ref{3.2}) that

\[ S_\delta(e') \geq S(\mu') \]

for any $\mu' \in \mathcal{P}(X)$ such that $E_\delta(\mu') \geq e'$. Now, by Lemmas \ref{3.3}, \ref{3.4} below $\mu'$ can be chosen, independently of $\delta$, so that $E(\mu') \geq e' + \epsilon$ and $S(\mu') > -\infty$. We then get, for any $\delta$ sufficiently small, that $E_\delta(\mu') \geq e'$ (by the monotone convergence theorem) and thus the uniform bound \ref{6.12} follows.

This concludes the proof of Theorem \ref{3.5} in the case when $X$ is compact. In the general case we fix $R > 0$ and denote by $X_R$ the intersection of $X$ with a ball $B_R$ of radius $R$ centered at the origin. Then $X_R$ is also pseudoconvex (as follows from Lemma \ref{2.3} applied to $\phi(z) = |z|^2 - R$). Thus, as shown in the previous section, the entropy $S_R(e)$ associated to the restrictions to $B_R$ of $W, V$ and $\mu_0$ is concave in $e$ for $e > E(\mu_0)$. Hence all that remains is to verify that

\begin{equation}
\lim_{R \to \infty} S_R(e) = S(e)
\end{equation}

for any fixed $e$ satisfying the inequalities \ref{6.9}. To this end first note that, since $X_R \subset X$, it follows immediately that $S_R(e) \leq S(e)$. Now assume that $e > E(\mu_0)$ and fix a candidate $\mu$ for the sup defining $S(e)$. Set

\[ \mu_R := 1_{B_R} \frac{\mu}{\mu(B_R)}. \]
Then

\[ S(\mu) \leq S_R(\varepsilon_R), \quad \varepsilon_R := E(\mu_R). \]

By the monotone convergence theorem \( E(\mu_R) \to E(\mu) \). Hence, using that \( S_R(\varepsilon) \) is decreasing and concave for \( R \) sufficiently large (by the previous step) we can proceed essentially as when approximating \( W \) with \( W_\delta \) above, to get

\[ S(\mu) \leq \limsup_{R \to \infty} S_R(\varepsilon), \]

which concludes the proof of the convergence \ref{prop:6.13} and thus the concavity in Theorem \ref{thm:6.9}.

7. Global concavity of \( S(\varepsilon) \) and examples

Recall that, in classical terminology, a symmetric function \( W(x, y) \) is a weakly positive definite kernel, i.e. that for any positive integer \( N \)

\[ \sum_{i,j \leq N} W(x_i, x_j)a_i a_j \geq 0 \quad \forall (a_i) \in \mathbb{R}^N : \sum_{i=1}^N a_i = 0 \]

If the first inequality holds for any sequence \( (a_i)_{i=1}^N \), then \( W(x, y) \) is called a positive definite kernel. \footnote{\( W(x, y) \) is a weakly positive kernel iff \( -W(x, y) \) is a negative definite kernel in the terminology of \cite{5}.}

Now assume that \( W(x, y) \) is weakly positive definite and satisfies the Main Assumptions in a neighborhood of \( X \times X \). Then \( W \) can be expressed as increasing limit of continuous (and even smooth) such functions \( W_\delta(x, y) \). Indeed, if \( \rho \) is a smooth compactly supported probability density on \( \mathbb{R}^{2n} \times \mathbb{R}^{2n} \) we can take

\[ (7.1) \quad W_\delta := (W * \rho_\delta) := \int W(\cdot + a, \cdot + b) \rho_\delta(a, b) d\lambda(a) d\lambda(b), \quad \rho_\delta(x, y) = \rho(\delta^{-1}x, \delta^{-1}y) \delta^{4n}. \]

Since \(-W\) is psh \( W_\delta \) indeed increases to \( W \). Moreover, since \( W(\cdot + a, \cdot + b) \) is weakly positive definite and satisfies the Main assumptions for any \((a, b)\) so does \( W_\delta \).

**Theorem 7.1.** If the Main Assumptions hold and moreover \( W(x, y) \) is assumed weakly positive definite, then the entropy \( S(\varepsilon) \) is globally concave and hence thermodynamic equivalence of ensembles holds for any \( \varepsilon \in [\varepsilon_{\min}, \varepsilon_{\max}] \).

**Proof.** By Theorem \ref{thm:6.9} \( S(\varepsilon) \) is concave and continuous on \( [\varepsilon_{\min}, \varepsilon_{\max}] \). Next, recall the classical fact that a weakly positive definite kernel defines a convex functional \( E_W(\mu) \) on \( \mathcal{P}(X) \) (and vice versa). Hence, by Prop \ref{prop:5.3} \( S(\varepsilon) \) is concave and continuous on \( [\varepsilon_{\min}, 0] \) when \( X \) is compact and \( W \) is continuous. The theorem thus follows, in the compact and continuous case, from the second and third point in Prop \ref{prop:5.7}. Next assume that \( X \) is still compact and define \( W_\delta \) to be a regularization as in formula \ref{eq:7.1}. Then the corresponding entropy \( S_\delta(\varepsilon) \) is concave on \( [\varepsilon_{\min,\delta}, \varepsilon_{\max,\delta}] \). Moreover, by the approximation argument used in the proof of Theorem \ref{thm:6.9} and the finiteness of \( S(\varepsilon) \) the function \( S_\delta(\varepsilon) \) converge point-wise to \( S(\varepsilon) \) on \( [\varepsilon_{\min}, 0] \). Thus \( S(\varepsilon) \) is also concave on \( [\varepsilon_{\min}, 0] \). Finally, the general non-compact case is deduced from the compact case using again the approximation arguments in the proof of Theorem \ref{thm:6.9} and the finiteness of \( S(\varepsilon) \). 

\[ \square \]

Recall that, by Bochner’s classical theorem, a translationally invariant kernel \( W(x, y) = W(x-y) \) is positive definite iff the function \( W \) on \( \mathbb{R}^d \) is the Fourier transform of a (positive) measure on \( \mathbb{R}^d \). In the case of translationally and rotationally invariant kernels the following classical result holds \cite{5}:

\[ 3W(x, y) \text{ is a weakly positive kernel iff } -W(x, y) \text{ is a negative definite kernel in the terminology of } \cite{5}. \]
Lemma 7.2. (Bernstein+Schoenberg). Let \( w(r) \) be a continuous function on \([0, \infty[\) which is smooth on \([0, \infty[\). Then \( W(x, y) := w(|x - y|) \) is a positive definite kernel iff \( f(r) := w(r^{1/2}) \) is completely monotone, i.e. \((-1)^m \partial^m f(r)/dr^m \geq 0\) for all non-negative integers \( m \).

The previous lemma implies that if \( w \) is non-negative on \([0, \infty[\) (but possibly equal to \( \infty \) at \( r = 0 \)) and \( w(r^{1/2}) \) is completely monotone for \( r > 0 \), then \( w(|x - y|) \) is still positive definite. Indeed, one can apply the previous lemma to

\[
w_\epsilon(r) := w \left( (r^2 + \epsilon)^{1/2} \right)
\]

and then let \( \epsilon \to 0 \).

Corollary 7.3. Under the Homogeneous Assumptions together with the assumption that \( w(r^{1/2}) \) is completely monotone for \( r > 0 \) the entropy \( S(\epsilon) \) is globally concave and thermodynamic equivalence of ensembles holds at all energies.

It should be pointed out that assumptions in the previous corollary are preserved if \( w \) is replaced by \( w_{\epsilon}(r) \) above (using that \( \log(|z|^2 + \epsilon) \) is psh) and similarly for \( v \) and \( \psi_0 \). This gives a convenient explicit regularization procedure preserving the property that \( S(\epsilon) \) is globally concave.

7.1. Examples where \( S(\epsilon) \) is globally concave. We next provide some examples where Theorem 7.1 applies and thus \( S(\epsilon) \) is globally concave. More examples may, for example, be obtained by taking convolutions (as in formula 7.1). Note also that if the entropy \( S_{W,V}(\epsilon) \) corresponding to the interactions \( W \) and \( V \) is globally concave, then so is \( S_{-W,-V}(\epsilon) \), since \( S_{-W,-V}(\epsilon) = S_{W,V}(-\epsilon) \). In this way one may thus go from a situation of repulsive interactions to attractive ones.

Theorem 7.1 applies to the case when \( W(x, y) = -\log |z - w| \) when, for example, \( \mu_0 \) is Lebesgue measure on for example a ball in \( \mathbb{R}^{2n} \) or a centered (possibly non-standard) Gaussian measure in \( \mathbb{R}^{2n} \) (as in formula 1.12). Indeed, then \( W \) satisfies the Homogeneous Assumptions and the positive definiteness follows, for example, from the fact that \( W \) is the Green kernel on \( \mathbb{R}^{2n} \) of the \( n \)th power of the Laplacian, which is positive definite as a formally self-adjoint operator. More generally, the ball may be replaced with any domain \( X \) satisfying the Main Assumptions, for example

\[
X = \{ z \in \mathbb{R}^{2n} : \sum_{i=1}^r |P_i(z)|^{\alpha_i} \leq 1 \},
\]

for a quasi-homogeneous polynomials \( P_1, \ldots, P_r \) and \( \alpha_i > 0 \) (see Example 2.5).

Theorem 7.1 also applies to the continuous repulsive power-laws with exponent in \([0, 2]\)

\[
W(x, y) = -|x - y|^a, \ a \in [0, 2],
\]

as well as to

\[
W(x, y) = e^{-a|x - y|^a}, \ a \in [0, 2],
\]

when \( X \) is taken to be a disc centered at the origin with radius at most \((1/2a)^{1/a}\). Indeed, a direct computation reveals that \( w(r) \) satisfies the Homogeneous Assumptions for any \( a, \alpha > 0 \) (by a scaling it is enough to verify the case when \( a = \alpha = 1 \)). Moreover, by [5, Cor 3.3] (and its proof) the kernels in question are weakly positive definite when \( a \in [0, 2] \). Note that in the case of the repulsive logarithmic interaction, as well as for repulsive power-laws with \( a \in [0, 2] \), Prop 4.8 ensures the existence of maximum entropy measures \( \mu^\epsilon \), when \( \mu_0 \) is a centered Gaussian measure (by taking \( \psi_0 = |x|^2 \)).
7.1.1. **The point vortex model.** Consider the point vortex model (for vortices with identical circulations) on a domain \( X \) in \( \mathbb{R}^2 \). In the case when \( X = \mathbb{R}^2 \)

\[
W(x, y) = -\log |x - y|, \quad V(x) = 0
\]

(with our normalizations). As discussed in the previous section, \( S^{(N)}_+(e) \) and \( S(e) \) are both globally concave (and thermodynamic equivalence of ensemble holds) if \( \mu_0 \) is taken to be a centered Gaussian measure. As indicated in [18 Section 5], the concavity of \( S(e) \) also follows from the results in [18], using completely different techniques (see also [35] where the concavity of the corresponding multi-variable entropy \( S(e, l) \), discussed in Section 3.3 is shown). But, as discussed in the introduction of the paper, the main point of the present technique is that it also applies to regularizations of \( W \).

In the case of when \( X \) is a compact domain with smooth boundary \( W(x, y) \) is defined as the negative of Green function \( G_X(x, y) \) for the Laplacian on \( X \) with Dirichlet boundary conditions and \( V(x) = \gamma(x)/N \) where \( \gamma \) is the restriction to the diagonal of \( G_X(x, y) + \log |x - y| \) \([17, 18, 49]\). In particular, when \( X \) is the unit-disc

\[
W(z, w) = -\log \frac{|z - w|}{|1 - zw|}, \quad V(x) = \frac{1}{N} \log |1 - |z|^2|
\]

In this case Theorem 6.7 implies that \( S^{(N)}_+(e) \) is globally concave when \( N \leq 3 \), as follows from combining Remark 6.8 with the following lemma, proved in the appendix.

**Lemma 7.4.** Denote by \( D \) the interior of the unit-disc in \( \mathbb{C} \) and set

\[
\psi(z, w) := \log \frac{|z - w|^2}{|1 - zw|^2}, \quad \phi(z) = -\log \left( |1 - |z|^2|^2 \right)
\]

The function \( \psi(z, w) + \lambda (\phi(z) + \phi(w)) \) is psh in \( D \times D \) iff \( \lambda \geq 1/2 \).

We leave open the question whether \( S^{(N)}_+(e) \) is concave also when \( N > 3 \). As for \( S(e) \) it was shown to be concave in [18], using a completely different method. In the case when a rotationally invariant exterior potential \( V_e \) is added to \( V(x) \) in formula (4.2) the previous lemma shows that \( S^{(N)}_+(e) \) is concave for any \( N \) (and hence also \( S(e) \)) if \( -\partial \bar{\partial} V_e \geq \partial \bar{\partial} \phi/2 \) in \( D \) i.e. if the Laplacian of \( V_e \) is sufficiently negative:

\[
\frac{1}{4} \Delta V_e (z) \leq -\frac{1}{(1 - |z|^2)^2}.
\]

This should be contrasted with the fact that the global concavity of \( S(e) \) may fail if the Laplacian is positive, e.g. in the case when \( V_e(z) = \omega |z|^2 \), for \( \omega > 0 \), studied in [57] and [18 Lemma 8.2].

7.1.2. **Insulated plasmas and self-gravitating matter in 2D.** The point vortex model on a compact domain \( X \) is physically equivalent to a one-component Coulomb plasma if inertial effects are ignored (i.e. the limit of infinite damping is considered) and the boundary of \( X \) is assumed to be conductive [57]. On the other hand, the case when the boundary of \( X \) is non-conducting, i.e. \( X \) is insulated, corresponds to the mean field Hamiltonian on \( X \) with Coulomb pair-interaction \(-\log |x - y| \) (and \( V \equiv 0 \) [35]). In this case the Main Assumptions apply when \( \mu_0 \) is the uniform measure on the \( X \) unit-disc \( X \), as discussed in the beginning of Section 7.1. More generally, the Main assumptions apply when the exterior potential \( V \) is radial and \( \Delta V \leq 0 \), i.e. \( V \) is the potential induced by a distribution of fixed particles with the same charge as the plasma. Switching the sign of the Coulomb interaction yields a system of self-gravitating matter, studied in [4] with inertial effects included. Generalizations to regularized self-gravitating matters are given in [8] (briefly outlined in Section 1.4).
8. CRITICAL INVERSE TEMPERATURES AND EXISTENCE OF MAXIMUM ENTROPY MEASURES

In the Very General Setup the macroscopic inverse temperatures is defined by

\begin{equation}
\beta_c := \inf \left\{ \beta \in \mathbb{R} : \inf_{\mu} F_\beta(\mu) > -\infty \right\}.
\end{equation}

The microscopic inverse temperature \( \beta_{c,N} \) is, in the General Setup, defined by

\[
\beta_{c,N} := \left\{ \beta \in \mathbb{R} : Z_{N,\beta} := \int_{X^N} e^{-\beta H^{(N)}} (e^{-\Psi_0} dx)^\otimes N < \infty \right\}
\]

and respectively, where \( H^{(N)} \) denotes the mean field Hamiltonian \( 1.1 \) corresponding to \( W \) and \( V \).

8.1. Dual expressions for the critical inverse temperatures. We start with the following dual “slope formula” for \( \beta_{N,c} \), under the Main Assumptions, which also shows that \( \beta_{N,c} < 0 \).

**Corollary 8.1.** Under the same assumptions as in Prop 6.5 the following holds if \( \mu_0 \) has finite mass on \( Y \) and \( \Psi \) is not identically constant:

\[
c(\Psi, \mu_0)(\Psi) := - \inf \left\{ \beta \in ]-\infty, 0] : \int_Y e^{\beta \Psi} \mu_0 < \infty \right\} = \lim_{\epsilon \to -\inf_Y \Psi} \frac{d}{d\epsilon} \log (\mu_0 \left\{ \Psi < -\epsilon \right\}),
\]

using either right or left derivatives in the rhs. As a consequence, the set of all negative \( \beta \) such that \( \int_Y e^{\beta \Psi} \mu_0 < \infty \) is open. In particular, under the Main Assumptions

\[
\beta_{N,c} = \lim_{\epsilon \to -\sup_{X^N} F_N} \frac{dS^{(N)}(\epsilon)}{d\epsilon} < 0, \quad Z_{N,\beta_{N,c}} = \infty
\]

**Proof.** By Prop 6.5 (and Theorem 6.7) the function

\[
\phi(t) := - \log \mu(t), \quad \mu(t) := (\mu_0 \left\{ \Psi < -t \right\})
\]

is convex wrt \( t \in \mathbb{R} \). Consider first the case when \( t_0 := \inf_Y \Psi > -\infty \). Then, trivially, \( \beta_c = -\infty \). Moreover, \( \phi(t) \) is convex and finite for \( t > t_0 \) and \( \phi(t) \to \infty \) as \( t \) decreases to \( t_0 \). But this forces \( d\phi(t)/dt \to -\infty \) as \( t \) decreases to \( t_0 \). Indeed, by the convexity of \( \phi \) the limit of \( d\phi(t)/dt \) decreases to \( M_0 \in [-\infty, \infty] \) as \( t \) decreases to \( t_0 \). Assume, to get a contradiction, that \( M_0 > -\infty \). Then, fixing \( t_1 > t_0 \) gives \( \phi(t) \leq \phi(t_1) + |M||t_1 - t_0| < \infty \) as \( t \to t_0 \), which contradicts that \( \phi(t) \to \infty \) as \( t \) decreases to \( t_0 \).

Next, assume that \( \inf_Y \Psi = -\infty \). Since \( \beta \leq 0 \) we have

\[
\int_Y e^{\beta \Psi} \mu_0 \leq \int_{\{\Psi < 0\}} e^{\beta \Psi} \mu_0 + \mu_0(Y),
\]

where, by assumption, the second term is finite. Pushing forward the measure \( \mu_0 \) on \( Y \) to \( \mathbb{R} \) under the map \( z \to \Psi(z) \) gives

\[
\int_{\{\Psi < 0\}} e^{\beta \Psi} \mu_0 = \int_{-\infty}^0 e^{\beta V(t)} dt = -\beta Z(\beta) + V(0), \quad Z(\beta) := \int_{-\infty}^0 e^{\beta V(t)} dt,
\]

where the second equality follows from integrating by parts. We may then conclude the proof of the first formula in the corollary by expressing

\[
Z(\beta) := \int_{-\infty}^0 e^{\beta - \phi(t)} dt
\]
and applying Lemma 8.3 below to the convex function \( \Phi = \beta t - \phi(t) \), which implies that

\[
\int_Y e^{\beta \Phi} \mu_0 < \infty \iff -\beta < \lim_{t \to \infty} \frac{d\phi}{dt},
\]

concluding the proof of formula in question. To prove that \( \beta_{N,c} < 0 \) note that \( \phi(t) \to \infty \) as \( t \to -\infty \) and \( \phi(t) \to 0 \) as \( t \to \infty \). Since \( \phi(t) \) is convex if follows that, using either left or right derivatives, \( \lim_{t \to -\infty} d\phi(t)/dt \leq 0 \) and \( \lim_{t \to \infty} d\phi(t)/dt = 0 \). But if \( \beta_{N,c} = 0 \), then, by the previous step, \( \lim_{t \to -\infty} d\phi(t)/dt = 0 \) and hence, by convexity, \( \phi(t) \) is constant. But this can only happen if \( \Psi \) is constant, which is excluded by the assumptions. Thus \( \beta_{N,c} < 0 \), as desired. Finally, to prove the last openness statement we just have to verify that if \( \int_Y e^{\beta \Phi} \mu_0 < \infty \), then there exists \( \delta > 0 \) such that \( \int_Y e^{(\beta - \delta) \Phi} \mu_0 < \infty \). But this follows directly from the strict inequality in the right hand side of formula 8.2.

\[ \square \]

Remark 8.2. In the case when \( Y \) is compact and \( \Psi_0 = 0 \) (or, equivalently, bounded) the number \( c_Y(\Psi) \) is called the integrability threshold of \( \Psi \) on \( Y \) (or the complex singularity exponent) in the complex geometry literature (whose inverse is the Arnold multiplicity) \[27\]. It follows from Skoda’s local integrability inequality that \( c_Y(\Psi) > 0 \) for any function which is psh on a neighborhood of \( Y \) and not identically \( -\infty \). Moreover, \( \int_Y e^{\beta \Phi} d\lambda = \infty \) in the critical case \( \beta = -c_Y(\Psi) \), by the resolution of the openness conjecture in \[15\] (see also \[37\] for the resolution of the strong openness conjecture). The proof above yields a simplification of the proof in \[15\] under the symmetry assumption that \( \Psi \in PSH(Y)_\alpha \) (anyhow, just like \[15\], it is based on \[14\]).

In the proof above the following elementary fact was used:

**Lemma 8.3.** Let \( \Phi(t) \) be a convex function on \( ]-\infty, 0[ \) such that \( \Phi(t) \) is bounded as \( t \to 0 \). Then

\[
\int_{-\infty}^0 e^{-\Phi(t)} dt < \infty
\]

iff \( \lim_{t \to -\infty} d\Phi(t)/dt < 0 \), using either left or right derivatives.

**Corollary 8.4.** Consider the Main Assumptions and assume also that \( S(\epsilon_{\max}) = -\infty \), if \( \epsilon_{\max} < \infty \). Then, as \( \epsilon \) increases strictly towards \( \epsilon_{\max} \)

\[
\beta_{c} = \lim_{\epsilon \to \epsilon_{\max}} \frac{dS(\epsilon_{\pm})}{d\epsilon},
\]

where \( dS(\epsilon_{\pm})/d\epsilon \) denotes either the left or the right derivative of the concave function \( S(\epsilon_{\pm}) \).

**Proof.** By Theorem 3.9 \( S(\epsilon) \) is concave and continuous on \( [\epsilon_0, \epsilon_{\max}] \). Denote by \( \bar{F} \) the usc concave function defined as \( F \) when \( \beta \leq 0 \) and as \( -\infty \) when \( \beta > 0 \). By Prop 3.7 \( S = (\bar{F})^* \) on \( [\epsilon_0, \epsilon_{\max}] \). Set \( g := (\bar{F})^* \). Thus \( g \) is constant for \( \epsilon \leq \epsilon_0 \) and on \( [\epsilon_0, \epsilon_{\max}] \) it coincides with \( S(\epsilon) \) (by \[37\]). Moreover, \( g^* = \bar{F} \) and hence \( \{g^* < \infty\} = [\beta_c, \infty[ \). Thus, by formula 2.3

\[ [\beta_c, \infty[ = \partial g(\{g > -\infty\}) = \partial S(\{\epsilon_0, \epsilon_{\max}\}), \]

which proves formula 8.3 using that \( dS(\epsilon+)/d\epsilon \leq dS(\epsilon)/d\epsilon \) and \( dS(\epsilon+)/d\epsilon \) and \( dS(\epsilon+)/d\epsilon \) are both decreasing (by concavity). \[ \square \]

8.2. **Concrete expressions in the homogeneous case.** It seems natural to expect that, under rather general assumptions, \( \beta_{N,c} \to \beta_c \) as \( N \to \infty \). Here we will show that this is the case under the Homogeneous Assumptions; in fact, \( \beta_{N,c} = \beta_c \) for any \( N \). The starting point is the following essentially well-known consequence of the Gibbs variational principle (compare \[44, 17, 6\]):
Lemma 8.5. Let $H^{(N)}$ be a mean field Hamiltonian of the form (44, 17). Then
\[ Z_{N,\beta} \leq \int_X e^{-\beta V(x)} \mu_0(x) \left( \int e^{-\beta \left( \frac{1}{2} W(x,y) + V(y) \right)} \mu_0(y) \right)^{N-1} \]
and
\[ -\frac{1}{N \beta} \log Z_{N,\beta(N-1)N} \leq \inf_{\mu \in \mathcal{P}(X)} F(\beta) =: F(\beta) \]

As a consequence, $\beta_0 \leq \limsup_{N \to \infty} \beta_{N,e}$ and if there exists $\beta_0 < 0$ such that
\[ \sup_{x \in X} \int e^{-\beta_0 \left( \frac{1}{2} W(x,y) + V(y) \right)} \mu_0(y) < \infty, \quad \int_X e^{-\beta_0 V} \mu_0 < \infty \]
then $\beta_{N,e} < \beta_0$ and $\beta_c < \beta_0$.

Proof. First observe that it will be enough to consider the case when $V = 0$ (otherwise we just replace $\mu_0$ with $e^{-\beta V} \mu_0$). Decompose $-\beta H^{(N)} = \frac{1}{N} \sum_{i=1}^N f_i$, where $f_i$ is the sum of $\frac{1}{2} W(x_i, x_j)$ over all $j$ such that $j \neq i$. The arithmetic-geometric means inequality gives
\[ \int_X e^{-\beta H^{(N)}} \mu_0^\otimes N \leq \sum_{i=1}^N \frac{1}{N} \int_X e^{f_i} \mu_i^\otimes N = \int_X \mu_0 \left( \int e^{-\beta \frac{1}{2} W(x,y)} \mu_0(y) \right)^{N-1} \]
Hence, estimating the latter integral over $X$ with the sup over $X$ proves the first inequality in the proposition. To prove the second one first note Gibbs variational principle (Jensen’s inequality) gives: for any given $\mu \in \mathcal{P}(X)$
\[-\frac{1}{N \beta} \log Z_{N,\beta} := \int_X e^{-\beta N E^{(N)}} \mu_0^\otimes N \leq \beta \int_X E^{(N)} \mu_0^\otimes N - S(\mu), \quad E^{(N)} := H^{(N)}/N \]
as long as the right hand side is well-defined. In the case when $H^{(N)}$ is of the form in the lemma
\[ \int_X E^{(N)} \mu_0^\otimes N = \frac{1}{N(N-1)} N(N-1) E(\mu) = \frac{N-1}{N} E(\mu), \]
which proves (8.4) by taking the infimum over $\mu$. \hfill \square

The following result generalizes the case of the logarithmic interaction considered in [14, 17].

Proposition 8.6. Under the Homogeneous Assumptions in $\mathbb{R}^d$ (but allowing $d$ to be odd)
\[ \beta_c = \beta_{c,N} = \frac{2d}{w}, \quad w := \lim_{t \to -\infty} \frac{dw(\ell)}{dt} = \lim_{t \to -\infty} \frac{w(\ell)}{t} \]
if $v$ and $\psi_0$ are assumed bounded in a neighborhood of $0$. Moreover, $Z_{N,\beta} = \infty$ when $\beta = \frac{4N}{w}$.

Proof. To simplify the notation we will prove the proposition in the case when $V = 0$ (but the proof in the general case is essentially the same). First observe that
\[ \sup_X \int_X e^{-\frac{\beta}{2} W(x,y)} \mu_0(y) < \infty \iff \int_0^1 e^{-\frac{\beta}{2} w(r)} r \frac{dr}{r} < \infty \iff \beta > \frac{2d}{w} \]
Indeed, since $w$ is decreasing, $w(r) \leq C$ if $r \geq 1$ and hence, using that $\mu_0$ is a probability measure,
\[ \int_X e^{-\frac{\beta}{2} W(x,y)} \mu_0(y) = \int_X e^{-\frac{\beta}{2} w(|x-y|)} \mu_0(y) \leq \int_{X \cap \{|x-y| \leq 1\}} e^{-\frac{\beta}{2} w(|x-y|)} \mu_0(y) + e^{-\frac{\beta}{2} C} \]
Changing variables in the integral above and setting \( \gamma := -\beta \) yields

\[
\int_{X \cap \{|x-y| \leq 1\}} e^{\frac{\gamma}{2}w(|x-y|)} \mu_0(y) = \int_{\{|z| \leq 1\}} e^{\frac{\gamma}{2}W(|z|)} e^{-\psi_0(x+z)} d\lambda(z) \leq C' \int_{\{|z| \leq 1\}} e^{\frac{\gamma}{2}W(|z|)} d\lambda(z)
\]

using that \( \psi_0 \) is bounded from below. This proves \(8.6\) using Lemma \(8.3\) in the last equivalence (by setting \( t := \log r \)). Hence, applying the previous lemma gives

\[
(8.7)
\]

To prove that \( \beta_{N,c} \geq 2d/w \) we restrict the integration over \( X^N \) to a ball \( B_R \) of radius \( R \) centered at the origin and use that \( w \) is decreasing to get

\[
Z_{N,\beta} \geq \int_{B_R^2} e^{-\frac{N(N-1)}{2w}w(R)} \mu_0^N \geq C e^{-\frac{\beta N}{2}w(R)} (R^d)^N \geq C'
\]

Setting \( R = e^t \) thus gives

\[
(Z_{N,\beta})^{1/N} \geq C^{1/N} e^{-t\left(\frac{1}{2}w(e^t) - d\right)}
\]

Hence, if \( \beta < 2d/w \), then as \( R \to 0 \), i.e. \( t \to -\infty \) we get \( (Z_{N,\beta})^{1/N} \geq C^{1/N} e^{-t\delta} \) for some \( \delta > 0 \). This means that \( Z_{N,\beta} = \infty \), which proves \( \beta_{N,c} = 2d/w \). Moreover, if \( \beta = 2d/w \) then the argument shows that the integral of \( e^{-\beta NE(X)} \mu_0^N \) over \( B_R^d \) does not tend to zero as \( R \to 0 \). Since \( \mu_0 \) does not charge single points it follows that \( Z_{N,\beta} = \infty \) (for \( d \) even this is a special case of the last statement in Cor \(8.1\)).

Next, thanks to the second inequality in Lemma \(8.5\) the inequality \(8.7\) implies that

\[
\beta_c \leq \frac{2d}{w}
\]

All that remains is thus to verify the reversed inequality. To this end fix \( \beta \) such that \( F(\beta) > -\infty \), i.e. such that there exists a constant \( C \) such that

\[
(8.8)
\]

For \( \epsilon > 0 \) set \( \nu_\epsilon = (T_\epsilon)_* \nu_0 \) where \( \nu_0 \) is any fixed probability measure such that \( S(\nu_0) > -\infty \). Then, on the one hand, as \( t := (\log \epsilon) \) tends to \( -\infty \)

\[
\frac{1}{t} E(\nu_\epsilon) = \frac{1}{2} \int_{X^2} \frac{1}{2t} w(e^t|x-y|) \nu_0(x) \nu_0(y) \to \frac{1}{2} \bar{w}
\]

by the monotone convergence theorem (using that the integrand is monotone in \( t \), by concavity). On the other hand,

\[
S(\nu_\epsilon) = S(\nu_0) + d \log \epsilon
\]

Hence, applying the inequality \(8.8\) to \( \nu_\epsilon \) and dividing both sides with \( t \) implies, by letting \( t \to -\infty \), that

\[
\frac{\beta}{2} \bar{w} - d \leq 0.
\]

This shows that \( \beta_c \geq \frac{2d}{w} \), as desired. \( \square \)

Remark 8.7. Remarkably, it is always the case that \( F(\beta_c) < \infty \) when \( X \) is a compact domain in \( \mathbb{R}^d \) and \( W(x,y) = -\log(|x-y|) \). Indeed, this follows from Adam’s generalization of the Moser-Trudinger inequality in \( \mathbb{R}^2 \), as discussed in \(9\) (see also \[44\,17\]). This finiteness should be contrasted with the general divergence \( Z_{N,\beta_c} = -\infty \) for any \( N \) (see Cor \(8.1\)).
Note that if $X$ is compact and $W$ is finite, then $\beta_c = \beta_{c,N} = -\infty$, but the converse does not hold, as illustrated by an application of the previous proposition to the case when
\[ W(x, y) = \log (\log 1/|x - y|). \]

8.3. The anisotropic case. Consider now the case when the Main Assumptions hold and $W$ is translationally invariant
\[ W(z, w) = -\Psi(z - w), \quad \Psi \in PSH_a(\mathbb{C}^n), \]
but not necessarily isotropic. More generally, since we will only be concerned with integrability properties we allow that (8.7) only holds up to a bounded term.

**Proposition 8.8.** Consider the Main Assumptions and assume moreover that $W$ is translationally invariant (up to a bounded term). Then there exists a positive number $\gamma$ such that
\[ \max \{\beta_{N,c}, \beta_c\} = -\gamma < 0 \]

**Proof.** First assume that $V = 0$. Then the first integral appearing in the uniform integrability property 8.5 may, after making the change of variables $z = y - z$, be estimated as
\[ \int e^{\frac{\beta}{2} \Psi(y - x)} e^{-\Psi_0(y)} dy \leq \left( \int e^{\frac{\beta}{2} \Psi(z)} dz \right)^{1/p} \left( \int e^{-q\Psi_0(z + x)} dz \right)^{1/q}, \]
using Hölder’s inequality with conjugate exponents $p$ and $q$. By the translational invariance of Lebesgue measure the integral in the second factor is given by the integral of
\[ \int e^{-q\Psi_0(z + x)} dz \]
and thus independent of $x$. Moreover, it follows from the openness statement in Corollary 8.1 that the integral is finite for $q$ sufficiently close to 1. Similarly, we can then make the integral in the first factor finite by taking $\beta$ negative, but sufficiently close to 0. Finally, in the case when $V$ is not identically zero we first apply the Cauchy-Schwartz inequality to estimate
\[ \int e^{-\beta \left( \frac{1}{2} W(x,y) + V(y) \right)} \mu_0(y) \leq \int e^{-2\beta \frac{1}{2} W(x,y)} \mu_0(y) \int e^{-2\beta V(y)} \mu_0(y) \]
and then repeat the previous argument to both integrals appearing in the right hand side.

Next, consider the case when $\Psi$ has an isolated singularity at the origin, i.e. $\Psi$ is locally bounded on the complement of the origin. Then one gets the following concrete bound, expressed in terms of the integrability threshold $c_0(\Psi)$ of $\Psi$ on a ball $B_\epsilon$ centered at the origin in $\mathbb{C}^n$ of sufficiently small radius $\epsilon$ (discussed in Remark 8.2).

**Proposition 8.9.** Let $X$ be a compact subspace of $\mathbb{C}^n$ and assume that $\Psi$ has an isolated singularity at the origin and that $V$ and $\Psi_0$ are bounded. Then, for any sufficiently small $\epsilon$
\[ \max_{N \geq 2} \{\beta_{N,c}, \beta_c\} = -\frac{1}{2} c_0(\Psi) < 0. \]

**Proof.** The assumptions ensure that the bounds 8.5 in Lemma 8.3 hold iff the sup is replaced with an integral i.e. iff $Z_{2, 3\epsilon} < \infty$ iff $\int_{B_\epsilon} e^{3\epsilon \frac{1}{2} \Psi} d\lambda < \infty$ (as seen by changing variables as in the first equality in formula 8.10). Hence, we can conclude using the very definition of $c_0(\Psi)$. \[ \square \]
In fact, as discussed in Remark 8.2 it is enough to assume that $PSH(\mathbb{C}^n)$. The invariant $c_0(\Psi)$ plays a key role in current complex geometry and can be estimated from below in terms of certain multiplicities (expressed as local intersection numbers) [28]. In the “algebraic” case in formula \[1.1\] the integrability threshold $c_0(\Psi)$ coincides with the log canonical threshold at $0 \in \mathbb{C}^n$ of the ideal in the polynomial ring $\mathbb{C}[z_1, ..., z_n]$ generated by the corresponding polynomials $P_j(z)$ [50].

**Example 8.10.** The log canonical threshold can be computed using algebro-geometric techniques. For example, when $\Psi(z) = \log \left( |z_1|^{2\alpha_1} + ... + |z_n|^{2\alpha_n} \right)$ for positive real numbers $\alpha_i$ one gets $c_0(\psi) = 1/\alpha_1 + ... + 1/\alpha_n$ [50, Example 1.9].

In the simplest case when $\Psi$ is “algebraic quasi-homogeneous” of degree $d$ (Example 2.5) with an isolated singularity at the origin (i.e. the zero-locus of corresponding polynomials $P_j$ only intersect at the origin) we have, by homogeneity, that $\Psi = d \log |z|^2 + \varphi(z)$, for a positive number $d$ and a continuous function $\varphi$, which descends to the compact quotient $(\mathbb{C}^{n+1} + \{0\})/\mathbb{C}_a^*$ and is thus bounded. In this case it thus follows from Prop 8.6 that $\beta_{N,e} = \beta_e = \frac{4n}{d}$.

A wide variety of such $\Psi$ may be obtained by taking $P_i = \partial f(z)/\partial z_i$ for given quasi-homogeneous polynomial $f$ with an isolated degenerate zero at the origin in $\mathbb{C}^n$. Then $\Psi(z)$ can be expressed in terms of a Ginzburg-Landau type potential:

$$\Psi(z) = \log \left( \sum_i |\frac{\partial f}{\partial z_i}(z)|^2 \right),$$

so that $W(z, w)$ is the standard logarithmic interaction precisely when $f$ is proportional $z_1^2 + ... + z_n^2$.

**8.4. Existence of maximum entropy measures.** Combining Prop 8.8 with the results in Section 4.1 yields the following existence result:

**Proposition 8.11.** Consider the Main Assumptions when $X$ is compact. Then, for any $e \in [e_{\min}, e_0]$ there exists a maximum entropy measure $\mu^e$. If moreover $W(x, y)$ is assumed translationally invariant (up to a bounded term) then there exists a maximum entropy measure $\mu^e$ for any $e \in [e_0, e_{\max}]$. In particular, this is the case under the Homogeneous Assumptions.

Turning to the non-compact case we recall that, under the Main Assumptions,

$$\mu_0 = e^{-\Psi_0} d\lambda$$

for $\Psi_0 \in PSH_a(X)$. As a consequence, if $\Psi_0$ is also assumed to be a continuous exhaustion function (which is automatically the case if $\Psi_0$ is rotationally invariant), then Prop 4.8 implies the following

**Proposition 8.12.** Consider the Main Assumptions and assume that $\Psi_0$ is continuous exhaustion function and that the growth-assumption \[4.2\] holds for a $\phi_0$ such that $\phi_0/\Psi_0 \to 0$ uniformly as $|x| \to \infty$. If $W(x, y)$ is assumed translationally invariant (up to a bounded term), then there exists a maximum entropy measure $\mu^e$ for any $e \in [e_{\min}, e_{\max}]$.

**Proof.** According to Prop Prop 4.8 we just have to verify that $\int e^{\delta \Psi_0} \mu_0 < \infty$ for some $\delta > 0$. But this follows from openness property in Cor 8.1. □
2.5. Gaussian measure,

Theorem 9.3. under the Homogeneous Assumptions, using a uniqueness result for minimizers of $F_\beta$ shown in the companion paper [9]. The starting point is the following criterion for the strict concavity of $S(e)$ in the high energy region:

**Proposition 9.1.** Assume that $X$ is compact and that $F_\beta$ has a unique minimizer on $\mathcal{P}(X)$ for any $\beta \in ]\beta_c,0[$. If the energy approximation property holds and $E(\mu_\beta) \to e_{\text{max}}$ as $\beta \to \beta_c$, then $S(e)$ is strictly concave on $]e_0,e_{\text{max}}[$ (in particular, this is the case if $E(\mu)$ is continuous on $\mathcal{P}(X)$).

**Proof.** As pointed out in the proof of Prop 5.3 the uniqueness assumption implies that $F(\beta)$ is differentiable. Thus we can conclude by applying Lemma 1.1.

In the case of the point-vortex model the uniqueness assumption in the previous proposition (and the energy approximation property) holds on any simply connected compact domain $X$ [18]. Moreover, by the concentration/compactness alternative established in [18], the blow-up property holds iff $\mu_\beta$, converges weakly towards a Dirac mass (such domains $X$ are called *domains of the first kind* in [18]). The following result is shown in [9]:

**Theorem 9.2.** (Uniqueness) Let $X$ be a ball centered at the origin in $\mathbb{R}^{2n}$ or all of $\mathbb{R}^{2n}$. Assume that $W$ and $V$ satisfy the Homogeneous Assumptions and that $v + \beta \psi_0$ is strictly concave wrt log $r$ when $r > 0$ for a given $\beta < 0$. Then any minimizer of $F_\beta(\mu)$ is uniquely determined. If the latter assumption is replaced by the assumption that $W(x,y)$ is a weakly positive definite kernel and that $w(r)$ is strictly increasing, then minimizers are uniquely determined modulo translation when $X = \mathbb{R}^{2n}$ and unique when $X$ is a ball.

We finally arrive at the following

**Theorem 9.3.** Under the Homogeneous Assumptions, the entropy $S(e)$ is concave for $e > E(\mu_0)$ and strictly concave if $X$ is a ball and either $v$ is strictly concave wrt log $r$ or $w$ is strictly increasing for $r \in ]0,\infty[$. If moreover $W(x,y)$ is a weakly positive definite kernel, then $S(e)$ is strictly concave on $]e_{\text{min}},e_{\text{max}}[$.

**Proof.** First consider the case when $X$ is compact and $W, V$ and $\Psi_0$ are continuous and $v$ is strictly concave wrt log $r$, when $r > 0$. Then the strict concavity of $S(e)$ follows directly from combining the previous theorem with Lemma 6.9 (and similarly if $w$ is strictly increasing), using that the energy approximation property holds under the Main Assumptions and hence also under the Homogeneous Assumptions. Next, if $v$ is not assumed strictly concave wrt log $r$, we replace $v$ with $v + \epsilon r$. Then the corresponding entropy $S(\epsilon)$ is concave and letting $\epsilon \to 0$ reveals that $S(e)$ is also concave. The general case is then deduced from the previous case using the approximation arguments employed in the proof of Theorem 6.9. Finally, if $W(x,y)$ is weakly positive definite, then by Prop 5.3 $S(e)$ is also strictly concave on $]e_{\text{min}},e_0[$ and continuous on $]e_{\text{min}},e_0]$. This means that $S(e)$ is strictly concave on both $]e_{\text{min}},e_0[$ and $]e_0,e_{\text{max}}[$. Since $S$ is continuous on $]e_{\text{min}},e_{\text{max}}[$ it follows that $S$ is strictly concave on $]e_{\text{min}},e_{\text{max}}[$ (indeed, otherwise it would be affine on some open interval in $]e_{\text{min}},e_{\text{max}}[$ which would contradict the strict concavity on $]e_{\text{min}},e_0[$ or $]e_0,e_{\text{max}}[$. 


10. Appendix

In this appendix we provide, for the convenience of the reader, some proofs of essentially well-known results stated in Section 2 and the proof of of Lemma 7.4.

10.1. Proof of Lemma 2.1. Assume that $f$ is not strictly concave in the interior of $[y_0, y_1]$. Then there exists an open interval $I \subset [y_0, y_1]$ such that $f^*$ is affine on $I$. In particular, there exists a number $a$ such that $f'(y) \equiv a$ on $I$. Note that $a \in [x_0, x_1]$. Indeed, since $f$ is concave $(\partial f([−∞, x_0])) \in \{y \geq f'(x_0)\}$ and $(\partial f([x_1, ∞)) \in \{y \leq f'(x_1)\}$. Hence, by $(\partial f)(y)$ for any $y \in I$ it follows from 2.2 that $I \subset (\partial f)(a)$, showing that $f$ is not differentiable at $a$.

10.2. Proof of Lemma 2.2. First observe that $f^*$ is continuous on $U$. Indeed, $f^*$ is concave (since it is an inf of affine functions) and hence it is enough to check that $f$ is finite on $U$. But $f^* \geq f$ and by assumption $f > −\infty$ on $U$. Moreover, by assumption there exists a constant $C$ such that $f < C$. Since the constant function $C$ is a contender for the inf in formula 2.3 it follows that $f^* \leq C$, showing that $f^*$ is finite and thus continuous on $U$. As a consequence, $Ω := \{f^* > f\} \cap U$ is open in $U$. Now fix a point $x_0 \in Ω$. Since $Ω$ is open there exists $x_{0,±}$ in $Ω$ such that $x_{0,−} < x_0 < x_{0,+}$. Moreover, since $f$ is use we may assume that the affine function $a(x)$ on $[x_{0,−}, x_{0,+}]$ satisfies $f(x) < a(x)$ on $[x_{0,−}, x_{0,+}]$. Hence, the continuous function $f^*$ defined as $f$ on the complement of $[x_{0,−}, x_{0,+}]$ and as $a(x)$ on $[x_{0,−}, x_{0,+}]$ is concave on $\mathbb{R}$ and satisfies $f^* \geq f$. But then it follows that $f^* = f$, by formula 2.3.

10.3. Proof of Lemma 2.3. The function $ρ := −\log(−\phi)$ on $Ω$ is, clearly, an exhaustion function. Moreover, $ρ$ is psh. Indeed, fix $z_0 \in Ω$ and $z \in \mathbb{C}^n$ and consider the restriction of $ρ$ to the complex line $w \mapsto z_0 + wz$, parametrized by $w \in \mathbb{C}$. First assume that $ϕ$ is smooth at $z_0$ and factorize the Laplacian on $\mathbb{C}_w$ in the standard way, $Δ = 4∂w∂\bar{w}$. We get

$$∂w∂\bar{w}ρ = −∂w(\frac{∂w(−ϕ)}{−ϕ}) = ∂w(\frac{∂wϕ}{−ϕ}) − ∂wϕ∂w(−ϕ)^{-1} = \frac{∂w∂wϕ}{−ϕ} + ∂wϕ∂wϕ ≥ 0 + 0.$$ 

Hence, $w \mapsto ρ(z_0 + wz)$ is subharmonic close to $z_0$. The subharmonicity in the general case is shown in a similar way using either distributional derivatives or a regularization algorithm. Finally, denoting by $ρ_y$ a continuous psh exhaustion function of $Y$ the maximum of $ρ$ and $ρ_y$ defines a psh exhaustion function of $\{ϕ < 0\} \cap Y$. Indeed, in general, the maximum of two psh functions is still psh (as follows from the corresponding standard result for subharmonic functions).

10.4. Proof of Lemma 7.4. Since $ψ(z, w)$ is locally bounded from above it will be enough to consider the complement of the diagonal in $D × D$ (using that the diagonal is pluripolar [20, Thm 5.24]). In this region $\log(\|z − w\|^2)$ is pluriharmonic, i.e. its complex Hessian vanishes (since $\log \|ξ\|^2$ is harmonic when $ξ \neq 0$). Hence, the complex Hessian $∂∂\bar{w}ρ$ coincides with $∂∂$ applied to $−\log(1 − z\bar{w})^2$, i.e. to $−\log(1 − z\bar{w}) − \log(1 − z\bar{w})$. Accordingly, a direct computation yields

$$∂∂\bar{w}(\psi(z, w)) = \begin{pmatrix} 0 & (1 − z\bar{w})^{-2} \\ (1 − z\bar{w})^{-2} & 0 \end{pmatrix}, \partial∂(ϕ(z) + ϕ(w)) = \begin{pmatrix} 2(1 − z\bar{z})^{-2} & 0 \\ 0 & 2(1 − w\bar{w})^{-2} \end{pmatrix}.$$ 

In particular, when $λ = 1/2$ we get

$$∂∂(ψ(z, w) + λ(ϕ(z) + ϕ(w))) = \begin{pmatrix} (1 − z\bar{z})^{-2} & (1 − z\bar{w})^{-2} \\ (1 − z\bar{w})^{-2} & (1 − w\bar{w})^{-2} \end{pmatrix}.$$
Since the trace of this Hermitian matrix is manifestly non-negative the matrix is semi-positive definite iff its determinant is non-negative. But the determinant is non-negative iff
\[(1 - z\bar{z})(1 - w\bar{w}) \leq (1 - z\bar{w})(1 - \bar{z}w),\]
which in turn is equivalent to \(-z\bar{z} - w\bar{w} \leq -z\bar{w} - \bar{z}w\) and hence also to the trivial inequality \(0 \leq |(z - w)|^2\). The same computation also reveals that when \(\lambda < 1/2\) the determinant is negative at \((z, w) = (0, 0)\) and hence also at any point in the complement of the diagonal in \(D \times D\) which is sufficiently close to \((0, 0)\).

References
[1] Aly, J.J: Thermodynamics of a two-dimensional self-gravitating system. Phys. Rev. E 49, 3771 (1994)
[2] Ambrosio, L; Gigli, N; Savaree,G: Gradient flows in metric spaces and in the space of probability measures. Lectures in Mathematics ETH Zurich. Birkhauser Verlag, Basel, 2005.
[3] V. A. Antonov, Original inVest. Leningrad Univ.7, 135 (1962); English translation in Dynamics of Star Clusters (IAU Symposium, Vol. 113), J. Goodman and P. Hut, ed.(Reidel,Dordrecht,1985),p.525.
[4] Arora ,J.S.,Chahande ,A.I. and Paeng ,J.K.: Multiplier methods for engineering optimization. International J. for Numerical Methods in Engineering. 32 (1991),1485–1525
[5] Berg,C.,Christensen ,J.P.R; Ressel, P. (1984): Harmonic Analysis on Semigroups. Graduate Texts in Mathematics100. Springer, New York. Theory of positive definite and related functions.
[6] Berman, R.J: On Large Deviations for Gibbs Measures, Mean Energy and Gamma-Convergence. Constructive Approximation volume 48, pages 3–30 (2018).
[7] Berman, R.J: The Coulomb gas, potential theory and phase transitions. preprint arXiv:1811.10249(2018)
[8] Berman, R.J: Concavity of the microcanonical Gibbs entropy for small systems and 2D-gravity (in preparation).
[9] Berman, R.J: The Q-curvature equation on even-dimensional conical spheres and uniqueness of free energy minimizers (in preparation).
[10] Berman, R.J: A maximum entropy principle for Sasaki-Einstein metrics and AdS/CFT. (in preparation)
[11] Berman, R.J; Berndtsson, B: Real Monge-Ampère equations and Kähler-Ricci solitons on toric log Fano varieties. Ann. Fac. Sci. Toulouse Math. (6) 22 (2013), no. 4, 649–711.
[12] Berman, R.J; Berndtsson, B: Symmetrization of Plurisubharmonic and Convex Functions. Indiana Univ. Math. J. Vol. 63, No. 2 (2014), pp. 345-365
[13] RJ Berman, TC Collins, D Persson: Emergent Sasaki-Einstein geometry and AdS/CFT. arXiv:2008.12004, 2020
[14] Berndtsson, B:Subharmonicity properties of the Bergman kernel and some other functions associated to pseudoconvex domains., Ann Inst Fourier 56 (2006) pp 1633-1662.
[15] B. Berndtsson.The openness conjecture and complex Brunn-Minkowski inequalities. Complex geometry and dynamics, 29–44, Abel Symp.,10, Springer, Cham, 2015
[16] J.Binney; S.Tremaine: Galactic Dynamics. Princeton Series in Astrophysics (2008).
[17] Caglioti,E; Lions, P-L; Marchioro.C; Pulvirenti.M: A special class of stationary flows for two-dimensional Euler equations: a statistical mechanics description. Communications in Mathematical Physics (1992) Volume 143, Number 3, 501-525
[18] Caglioti,E; Lions, P-L; Marchioro.C; Pulvirenti.M: A special class of stationary flows for two-dimensional Euler equations: a statistical mechanics description. Part II. Communications in Mathematical Physics (1995) Volume 174, 229-260
[19] A Campa, T Dauxois, S Ruffo: Statistical mechanics and dynamics of solvable models with long-range interactions. Physics Reports, 2009 - Elsevier
[20] Chafaï, D; Gozlan, N; Zitt, P-A: First-order global asymptotics for confined particles with singular pair repulsion. Ann. Appl. Probab. 24 (2014), no. 6, 2371–2413.
[21] S-Y A. Chang: Conformal invariants and partial differential equations. Bull. Amer. Math. Soc. 42 (2005), 365-393
[22] Chanillo, S.;Kiesling, M. K.-H.: Rotational symmetry of solutions to some nonlinear problems in statistical mechanics and geometry. Comm. Math. Phys.160(1994), 217–238
[23] Chavanis, P-H: Statistical mechanics of two-dimensional vortices and stellar systems. Pages 208-292 in [24].
[24] T. Dauxois, S. Ruffo, E. Arimondo, and M. Wilkens, eds. Dynamics and Thermodynamics of Systems with Long Range Interactions, vol. 602 of Lecture Notes in Physics (Springer, New York, 2002).
[25] Dembo, A; Zeitouni O: Large deviation techniques and applications. Jones and Bartlett Publ. 1993
[26] Demayuali, J: Complex Analytic and Differential Geometry. https://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf
[27] Demayuali, J-P ; Kollár, J.: Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds. Annales scientifiques de l'École Normale Supérieure, Série 4, Tome 34 (2001) no. 4, pp. 525-55
[28] Jean-Pierre Demayuali, Hoàng Hiep Pham: A sharp lower bound for the log canonical threshold. Acta Math. 212 (1) 1 - 9, 2014
[29] P Di Cintio, S Gupta, L Casetti: Dynamical origin of non-thermal states in galactic filaments. Monthly Notices of the Royal Astronomical Society, Volume 475, Issue 1, March 2018, Pages 1137–1147
[30] C.F. Driscoll and K.S. Fine: Experiments on vortex dynamics in pure electron plasmas. Phys. Fluids B 2 , 1359 (1990)
[31] J. Dunkel and S. Hilbert: Consistent thermostatistics forbids negative absolute temperatures. Nature Physics.10,67(2014)
[32] Eyink, G.L., Spohn, H.: Negative-temperature states and large-scale, long-lived vortices in two-dimensional turbulence J. Stat. Phys.70, Nov. 3/4 (1993).
[33] Ellis, R. S.; Haven, K; Turkington, B: Large deviation principles and complete equivalence and nonequivalence results for pure and mixed ensembles. J. Statist. Phys. 101 (2000), no. 5-6, 999–1064.
[34] RS Ellis, H Toulouse, B Turkington: Thermodynamic versus statistical nonequivalence of ensembles for the mean-field Blume–Emery–Griffiths model. Physica A: Statistical Mechanics and its Applications Volume 335, Issues 3–4, 15 2004, Pages 518-538
[35] A. J. M. Garrett: Maximum Entropy with Nonlinear Constraints: Physical Examples. In Maximum Entropy and Bayesian Methods pp 243-249, Fundamental Theories of Physics book series (FTPH, volume 39).
[36] D.H. E. Gross: Microcanonical Thermodynamics: Phase Transitions in "small" Systems. World scientific lecture notes in physics. Vol 66 (2001).
[37] Q. Guan, X. Zhou:A proof of Demayuali's strong openness conjecture. Ann. of Math. (2)182(2015), no.2, 605–616
[38] Gauthier et al: Giant vortex clusters in a two-dimensional quantum fluid. Science. 364 (6447): (2019) 1264–1267. arXiv:1801.06951
[39] S Hilbert, P Hänggi, J Dunkel: Thermodynamic laws in isolated systems. Physical Review E, 2014 - APS
[40] Huang, K: Statistical mechanics. J (Wiley, 1987)
[41] Jaynes, E. T.: Information Theory and Statistical Mechanics. Phys. Rev., 106, 620 (1957)
[42] Johnstone et al: Evolution of large-scale flow from turbulence in a two-dimensional superfluid . Science. 365 (6447) (2019) 1267–1271. [arXiv:1801.06952]
[43] Kiessling M.K.H.: On the equilibrium statistical mechanics of isothermal classical self-gravitating matter. J Stat Phys 55, 203-257 (1989).
[44] Kiessling M.K.H.: Statistical mechanics of classical particles with logarithmic interactions. Comm. Pure Appl. Math. 46 (1993), 27-56.
[45] Kiessling, M. K.-H.: The unbounded 2D guiding center plasma.J. Plasma Phys.54(1995),11–29.
[46] Kiessling, Michael K.-H.: Statistical mechanics approach to some problems in conformal geometry. Statistical mechanics: from rigorous results to applications. Phys. A 279 (2000), no. 1-4, 353–368.
[47] D.Lynden-Bell: Negative specific heat in astronomy, physics and chemistry. Physica A: Statistical Mechanics and its Applications. Volume 263, Issues 1–4, 1 (1999) Pages 293-304
[48] A. J. Majda; A. L. Bertozzi: Vorticity and incompressible flow. Cambridge University Press (2010).
[49] C. Marchioro, M. Pulvirenti, Mathematical Theory of Incompressible Nonviscous Fluids, in: Applied Mathematical Sciences, vol. 96, Springer-Verlag, New York, 1994
[50] Mustata, M; IMPANGA lecture notes on log canonical thresholds. In Contributions to Algebraic Geometry: Impanga Lecture Notes. Editor: P.Pragasz, EMS 2012.
[51] Onsager, L.: Statistical hydrodynamics, Nuovo Cim. Suppl.6, 279–287 (1949).
[52] Padmanabhan, T: Statistical mechanics of gravitating systems. Phys. Rev.188, 285 (1990)
[53] Ramsey, N. F., Thermodynamics and statistical mechanics at negative absolute temperatures, Phys. Rev.103, 20–28 (1956).
[54] Rockafellar, R. T: Convex analysis. Reprint of the 1970 original. Princeton Landmarks in Mathematics. Princeton Paperbacks. Princeton University Press, Princeton, NJ, 1997.

43
[55] H.H. Rugh: Microthermodynamic formalism. Phys. Rev. E64, 055101~2001.
[56] C.H. Silvestre T.M. Rocha Filho: Ergodicity in a two-dimensional self-gravitating many-body system. Physics Letters A Volume 380, Issue 3, 28 January 2016, Pages 337-348
[57] R. A. Smith and T. M. O’Neil: Nonaxisymmetric thermalequilibria of a cylindrically bounded guiding center plasma or discrete vortex system. Phys. Fluids B2(1990), 2961–2975.
[58] Touchette, H.; Ellis, R.S.; Turkington, B: An introduction to the thermodynamic and macrostate levels of nonequivalent ensembles. Physica A: Statistical Mechanics and its Applications Volume 340, Issues 1–3, 1 2004, Pages 138-146
[59] Villani, C.: Topics in optimal transportation. Graduate Studies in Mathematics, 58. American Mathematical Society, Providence, RI, 2003. xvi+370 pp