GEOMETRIC DECOMPOSITIONS OF THE SIMPLICIAL LATTICE AND SMOOTH FINITE ELEMENTS IN ARBITRARY DIMENSION

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Abstract. Recently $C^m$-conforming finite elements on simplexes in arbitrary dimension are constructed by Hu, Lin and Wu. The key in the construction is a non-overlapping decomposition of the simplicial lattice in which each component will be used to determine the normal derivatives at each lower dimensional sub-simplex. A geometric approach is proposed in this paper and a geometric decomposition of the finite element spaces is given. Our geometric decomposition using the graph distance not only simplifies the construction but also provides an easy way of implementation.

1. Introduction

In a recent work [9], Hu, Lin and Wu have solved a long-standing open problem in finite element methods: construction of $C^m$-conforming finite elements on simplexes in arbitrary dimension. It unifies the scattered results [3, 18, 1] in two dimensions, [19, 25] in three dimensions, and [26] in four dimensions. In this paper, we provide a geometric decomposition of the finite element spaces constructed in [9] and consequently give a simplified construction different from [9].

A finite element on a geometric domain $K$ is defined as a triple $(K, V, \text{DoF})$ by Ciarlet in [6], where $V$ is the finite-dimensional space of shape functions and the set of degrees of freedom (DoFs) is a basis of the dual space $V'$ of $V$. In this paper $K = T$ is a simplex in $\mathbb{R}^n$ and $V = \mathbb{P}_k(T)$ is the polynomial space with degree bounded by $k$. The difficulty is to identify an appropriate basis of $V'$ to enforce the continuity of the functions across the boundary of the elements so that the global finite element space is a subspace of $C^m(\Omega)$, where $\Omega \subset \mathbb{R}^n$ admits a conforming triangulation $T_h$ consisting of simplexes. It is well known that for a piecewise smooth function to be in $C^m$, it suffices to ensure the continuity of the normal derivative $\partial_n^r u$ for $r = 0, 1, \ldots, m$ across each $(n-1)$-dimensional face of $T_h$. Those $(n-1)$-dimensional faces will meet at lower dimensional sub-simplexes. For example, in three dimensions, faces will share edges and vertices. The continuity of $\partial_n^r u$ on faces will imply stronger smoothness on edges and vertices, which is known as the supersmoothness [17, 7]. Indeed in [9], the authors constructed $C^m$-conforming finite elements under the following requirement on the $C^{m_{\ell}}$ smoothness at $\ell$-dimensional sub-simplexes

$$r_n = 0, \quad r_{n-1} = m \geq 0, \quad r_\ell \geq 2r_{\ell+1} \quad \text{for} \quad \ell = n-2, \ldots, 0,$$

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and the degree of polynomial is exponential in $n$:

$$k \geq 2r_0 + 1 \geq 2^n m + 1.$$  

Notice that such a requirement is sufficient but by no means necessary.

We introduce the simplicial lattice $\mathbb{T}_k^n = \{ \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n) \in \mathbb{N}^{0:n} \mid |\alpha| = k \}$ as the multi-index set with fixed length $k$. Due to the one-to-one mapping between the Bernstein polynomial $\lambda^{\alpha}$, where $\lambda$ is the barycentric coordinate, and the lattice node $\alpha \in \mathbb{T}_k^n$, the key in the construction is a non-overlapping decomposition (partition) of the simplicial lattice in which each component will be used to determine the normal derivatives at each lower dimensional sub-simplex. In [9], a purely algebraic and combinatory approach is used to prove the existence of such partition. In this paper, a geometric approach will be proposed.

We shall introduce a graph distance $\text{dist}(\alpha, f)$ from a lattice node $\alpha \in \mathbb{T}_k^n$ to a $\ell$-dimensional sub-simplex $f \in \Delta_{\ell}(T)$ and reveal the following fact: for $\alpha \in \mathbb{T}_k^n, \beta \in \mathbb{N}^{1:n}$, and $\text{dist}(\alpha, f) > |\beta|$, we have

$$D^\beta \lambda^{\alpha} |_{f} = 0. \tag{1}$$

For $f \in \Delta_{\ell}(T)$, denote by $f^* \in \Delta_{n-\ell-1}(T)$ the sub-simplex of $T$ opposite to $f$. For a lattice node $\alpha \in \mathbb{T}_k^n$, it can be decomposed into two components $\alpha_f, \alpha_{f^*}$, and $\text{dist}(\alpha, f) = |\alpha_{f^*}|$. To give a geometric decomposition of DoFs which ensures the $C^m$-conformity, we reveal the one-to-one mapping of the space $\text{span}\{\lambda^{\alpha} = \lambda_f^{\alpha_f} \lambda_{f^*}^{\alpha_{f^*}}, \alpha \in \mathbb{T}_k^n, |\alpha_{f^*}| = s\}$ with the DoFs

$$\int \frac{\partial^\beta}{\partial n_f} \lambda_f^{\alpha_f} \, ds \quad \forall \alpha \in \mathbb{T}_k^n, |\alpha_f| = k-s, \beta \in \mathbb{N}^{1:n-\ell}, |\beta| = s \tag{2}$$

by mapping $\alpha_{f^*}$ to $\beta$.

Based on (1) and (2), we establish a direct decomposition of the simplicial lattice on an $n$-dimensional simplex $T$:

$$\mathbb{T}_k^n(T) = \bigoplus_{\ell=0}^n \bigoplus_{f \in \Delta_{\ell}(T)} S_{\ell}(f), \tag{3}$$

where

$$S_0(v) = D(v, r_0),$$

$$S_{\ell}(f) = D(f, r_\ell) \setminus \left[ \bigcup_{i=0}^{\ell-1} \bigcup_{e \in \Delta_i(f)} D(e, r_i) \right], \quad \ell = 1, \ldots, n-1,$$

$$S_{n}(T) = \mathbb{T}_k^n(T) \setminus \left[ \bigcup_{i=0}^{n-1} \bigcup_{f \in \Delta_{\ell}(T)} D(f, r_{\ell}) \right].$$

Here $D(f, r) = \{ \alpha \in \mathbb{T}_k^n, \text{dist}(\alpha, f) \leq r \}$ contains lattice nodes at most $r$ distance away from $f$ and will be called the lattice tube of $f$ with radius $r$. The requirement $r_{\ell-1} \geq 2r_\ell$ ensures $\{D(f, r_\ell) \setminus \bigcup_{e \in \Delta_{\ell-1}(f)} D(e, r_{\ell-1}) \mid f \in \Delta_{\ell}(T)\}$ are disjoint so that (3) is a direct decomposition. Geometrically we push all lattice nodes in $S_\ell(f)$ to the face $f$ to determine $r_\ell$ normal derivatives on $f$. Consequently we have the geometric decomposition

$$\mathbb{P}_k(T) = \bigoplus_{\ell=0}^n \bigoplus_{f \in \Delta_{\ell}(T)} \mathbb{P}_k(S_{\ell}(f)).$$
The direct decomposition (3), together with (1) and (2), implies the DoFs

\[
D^\alpha u(v) \quad \alpha \in \mathbb{N}_+^{1 \times n}, |\alpha| \leq r_0, v \in \Delta_0(T_h),
\]

(4)

\[
\int_f \frac{\partial^\beta u}{\partial n^\beta_f} \lambda_{\alpha}^f \, ds \quad \alpha \in S_\ell(f), |\alpha| = k - s, \beta \in \mathbb{N}_+^{1 \times \ell}, |\beta| = s,
\]

(5)

\[
f \in \Delta_\ell(T_h), \ell = 1, \ldots, n - 1, s = 0, \ldots, r_\ell,
\]

(6)

\[
\int_T u\lambda^\alpha \, dx \quad \alpha \in S_n(T), T \in T_h.
\]

DoF (4) means \( u \) is \( C^{r_0} \)-continuous at vertices in \( \Delta_0(T_h) \). By DoFs (4)-(5) and the unsolvence of the finite element on faces, \( \frac{\partial^\beta u}{\partial n^\beta_f} \) is single-valued across face \( f \in \Delta_\ell(T_h) \) for \( |\beta| \leq r_\ell \), that is \( u \) is \( C^{r_\ell} \)-continuous across \( f \) and in particular \( C^{m} \)-continuous across \( (n - 1) \)-dimensional faces as \( r_{n-1} = m \). The interior DoF (6) is included for the unsolvence. Our geometric decomposition using the distance not only simplifies the construction but also provide an easy way of implementation.

Besides standard finite elements on simplexes, \( C^m \)-conforming finite elements on macro-hypercubes and \( C^1 \)-conforming finite elements on macro-simplices in arbitrary dimension are developed in [10] and [8], respectively. In [24], Xu exploits the artificial neural network to devise \( C^m \)-conforming piecewise polynomials and then develops a finite neuron method. On the other side, the lowest order nonconforming finite elements on simplexes were devised in [21, 20, 23] for \( m \leq n \) and \( k = m + 1 \). We refer to [22, 11, 12] for more \( H^m \)-nonconforming finite elements and [4, 13, 14] for \( H^m \)-conforming and nonconforming virtual elements on any shape of polytope \( K \) in \( \mathbb{R}^n \).

The rest of this paper is organized as follows. Some notation and simplicial lattice are introduced in Section 2. We show the geometric decomposition of Lagrange finite elements and Hermite finite elements in arbitrary dimension in Section 3 and Section 4, respectively. In Section 5 the geometric decomposition of \( C^m \)-conforming finite elements in two dimensions is studied. And the geometric decomposition of \( C^m \)-conforming finite elements in arbitrary dimension is developed in Section 6.

2. SIMPLICIAL LATTICE

Let \( T \in \mathbb{R}^n \) be an \( n \)-dimensional simplex with vertices \( v_0, v_1, \ldots, v_n \) in general position. That is

\[
T = \left\{ \sum_{i=0}^{n} \lambda_i v_i \mid 0 \leq \lambda_i \leq 1, \sum_{i=0}^{n} \lambda_i = 1 \right\},
\]

where \( \lambda = (\lambda_0, \lambda_1, \ldots, \lambda_n) \) is called the barycentric coordinate. We will write \( T = \text{Convex}(v_0, \ldots, v_n) \), where Convex stands for the convex combination.

2.1. The simplicial lattice. For two non-negative integers \( l \leq m \), we will use the multi-index notation \( \alpha \in \mathbb{N}_+^{l \times m} \), meaning \( \alpha = (\alpha_1, \ldots, \alpha_m) \) with integer \( \alpha_i \geq 0 \). The length of a multi-index is \( |\alpha| := \sum_{i=1}^{m} \alpha_i \) for \( \alpha \in \mathbb{N}_+^{l \times m} \). We can also treat \( \alpha \) as a vector with integer valued coordinates. We define \( \lambda^\alpha := \lambda_0^{\alpha_0} \cdots \lambda_n^{\alpha_n} \) for \( \alpha \in \mathbb{N}_+^{0 \times n} \).

A simplicial lattice of degree \( k \) and dimension \( n \) is a multi-index set of \( n+1 \) components and with fixed length \( k \), i.e.,

\[
T_k^n = \left\{ \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n) \in \mathbb{N}_+^{0 \times n} \mid \alpha_0 + \alpha_1 + \ldots + \alpha_n = k \right\}.
\]
An element $\alpha \in T^n_k$ is called a node of the lattice. We use the convention that: for a vector $\alpha \geq c$ means $\alpha_i \geq c$ for all components $i = 0, 1, \ldots, n$. It holds that

$$|T^n_k| = \binom{n+k}{k} = \dim \mathbb{P}_k(T),$$

where $\mathbb{P}_k(T)$ denotes the set of real valued polynomials defined on $T$ of degree less than or equal to $k$. Indeed the Bernstein basis of $\mathbb{P}_k(T)$ is

$$\{\lambda^\alpha := \lambda_0^{\alpha_0} \lambda_1^{\alpha_1} \ldots \lambda_n^{\alpha_n} \mid \alpha \in T^n_k\}.$$ 

For a subset $S \subseteq T^n_k$, we define

$$\mathbb{P}_k(S) = \text{span}\{\lambda^\alpha, \alpha \in S \subseteq T^n_k\}.$$ 

With such one-to-one mapping between the lattice node $\alpha$ and the Bernstein polynomial $\lambda^\alpha$, we can study properties of polynomials through the simplicial lattice.

Two nodes $\alpha, \beta \in T^n_k$ are adjacent if there exist $0 \leq i_1 < i_2 \leq n$ such that $|\alpha_{i_1} - \beta_{i_1}| = |\alpha_{i_2} - \beta_{i_2}| = 1$ and $|\alpha_{i} - \beta_{i}| = 0$ for $i \neq i_1, i_2$. By assigning edges to all adjacent nodes, the simplicial lattice becomes an undirected graph. The distance of two nodes in the graph is the length of a minimal path connecting them, where the length of a path is defined as the number of edges in the path. One can easily verify that the graph distance $\text{dist}_G(\alpha, \beta) = \frac{1}{2} \sum_{i=0}^{n} |\alpha_{i} - \beta_{i}|$ which is one half of the $L^1$-norm of $\alpha - \beta$ treating $\alpha, \beta \in \mathbb{R}^{n+1}$.

### 2.2. Geometric embedding of a simplicial lattice.

We can embed the simplicial lattice into a geometric simplex by using $\alpha/k$ as the barycentric coordinate of node $\alpha$. Given $\alpha \in T^n_k$, the barycentric coordinate of $\alpha$ is given by

$$\lambda(\alpha) = (\alpha_0, \alpha_1, \ldots, \alpha_n)/k.$$ 

Let $T$ be a simplex with vertices $\{v_0, v_1, \ldots, v_n\}$. The geometric embedding is

$$x : T^n_k \rightarrow T, \quad x(\alpha) = \sum_{i=0}^{n} \lambda_i(\alpha)v_i.$$ 

We will always assume such a geometric embedding of the simplicial lattice exists and write as $T^n_k(T)$. See Fig. 1 for an illustration for a two-dimensional simplicial lattice embedded into a triangle.

The so-called reference simplex $\hat{T}$ is spanned by vertices $v_0 = 0$ and $v_i = e_i = (0, \ldots, 1, \ldots, 0)$, whose barycentric coordinate $\lambda_i = x_i$ for $i = 1, 2, \ldots n$ and $\lambda_0 = 1 - \sum_{i=1}^{n} x_i$. If we embed $T^n_k$ to the scaled reference simplex $k\hat{T}$, then the coordinate of $(\alpha_0, \alpha_1, \ldots, \alpha_n) \in T^n_k(k\hat{T})$ is simply $(\alpha_1, \ldots, \alpha_n)$, where $\alpha_0$ is dropped as the vertex $v_0$ is mapped to the origin. Of course, we can set other vertex as the origin and obtain other embeddings.

A simplicial lattice $T^n_k$ is, by definition, an algebraic set. Through the geometric embedding $T^n_k(T)$, we can use operators for the geometric simplex $T$ to study this algebraic set. For example, for a subset $S \subseteq T$, we use $T^n_k(S) = \{\alpha \in T^n_k, x(\alpha) \in S\}$ to denote the portion of lattice nodes whose geometric embedding is inside $S$. The superscript $n$ will be replaced by the dimension of $S$ when $S$ is a lower dimensional simplex introduced in a moment.
2.3. Sub-simplicial lattices. Following [2], we let \( \Delta(T) \) denote all the sub simplices of \( T \), while \( \Delta_\ell(T) \) denotes the set of sub simplices of dimension \( \ell \), for \( 0 \leq \ell \leq n \). The cardinality of \( \Delta_\ell(T) \) is \( \binom{n+1}{\ell+1} \). Elements of \( \Delta_0(T) = \{ v_0, \ldots, v_n \} \) are \( n+1 \) vertices of \( T \) and \( \Delta_n(T) = T \).

It is the combinatoric structure of the simplex that plays an important role in the construction. For a sub-simplex \( f \in \Delta_\ell(T) \), we will overload the notation \( f \) for both the geometric simplex and the algebraic set of indices. Namely \( f = \{ f(0), \ldots, f(\ell) \} \subseteq \{0,1,\ldots,n\} \) and

\[
\alpha = E(\alpha_f) = E(\alpha_{f*}) = \alpha_f + \alpha_{f*}, \quad \text{and } |\alpha| = |\alpha_f| + |\alpha_{f*}|.
\]

Based on (7), we can write a Bernstein polynomial as

\[
\lambda^\alpha = \lambda_f^{\alpha_f} \lambda_{f*}^{\alpha_{f*}},
\]

where \( \lambda_f = \lambda_{f(0)} \ldots \lambda_{f(\ell)} \in \mathbb{P}_{\ell+1}(f) \) is the bubble function on \( f \).

Define \( T^\ell_{k,\alpha}(f) := \{ \alpha_f \in T^\ell_k(f), \alpha_f \geq \alpha \} \). Then it is easy to see \( T^\ell_{k,1}(f) = T^\ell_k(f) \), where the latter notation indicates the lattices nodes are contained in the interior of \( f \); see the red triangle in Fig. 1. The interior lattice \( T^\ell_k(f) \) is isomorphic to a simplicial lattice with a smaller degree \( k - (\ell + 1) \), denoted by \( T^\ell_{k-(\ell+1)}(f) \).

The one-to-one mapping is

\[
T^\ell_{k-(\ell+1)}(f) \rightarrow T^\ell_{k,1}(f) : \alpha_f \rightarrow \alpha_f + 1.
\]

Denote by \( b_f := \lambda_f \). The interior lattice is related to the so-called bubble polynomial of \( f \):

\[
b_f \mathbb{P}_{k-(\ell+1)}(f) := \text{span}\{ b_f \lambda_f^{\alpha_f} : \alpha_f \in T^\ell_{k-(\ell+1)}(f) \} = \text{span}\{ \lambda_f^{\alpha_f} : \alpha_f \in T^\ell_{k,1}(f) \}.
\]

Geometrically as the bubble polynomial space vanished on the boundary, it is generated by the interior lattice nodes only. In Fig. 1, \( T^\ell_k(f) \) consists of the nodes inside the red triangle, and \( T^\ell_k(f) \) for \( f = \{ 0,1 \} \) is in the blue trapezoid.
In summary, by treating $f$ as a set of indices, we can apply the operators $\cup, \cap, ^*, \setminus$ on sets. While treating $f$ as a geometric simplex, $\partial f, \circ f$ etc can be applied.

2.4. **Distance to a sub-simplex.** Given $f \in \Delta_\ell(T)$, we define the distance of a node $\alpha \in T_n^k$ to $f$ as

$$\text{dist}(\alpha, f) := |\alpha_f^*| = \sum_{i \in f^*} \alpha_i.$$  

Next we present a geometric interpretation of $\text{dist}(\alpha, f)$. We set the vertex $v_{f(0)}$ as the origin and embed the lattice to the scaled reference simplex $k\hat{T}$. Then $|\alpha_f^*| = s$ corresponds to the linear equation

$$x_{f^*(1)} + x_{f^*(2)} + \ldots + x_{f^*(n-\ell)} = s,$$

which defines a hyper-plane in $\mathbb{R}^n$, denoted by $L(f, s)$, with a normal vector $1_{f^*}$. The simplex $f$ can be thought of as a convex combination of vectors $\{e_{f(i)}\}_{i=1}^\ell$. Obviously $1_{f^*} \cdot e_{f(0)f(i)} = 0$ as the zero pattern is complementary to each other. So $f$ is parallel to the hyper-plane $L(f, s)$. The distance $\text{dist}(\alpha, f)$ for $\alpha \in L(f, s)$ is the intercept of the hyper-plane $L(f, s)$; see Fig. 2 for an illustration. In particular $f \in L(f, 0)$ and $\lambda_i|_f = 0$ for $i \in f^*$. Indeed $f = \{x \in T \mid \lambda_i(x) = 0, i \in f^*\}$.

We can extend the definition to the distance between sub-simplexes. For $e \in \Delta_\ell(T), f \in \Delta(T)$, define

$$\text{dist}(e, f) = \min_{\alpha \in \tilde{T}_n^k(e)} \text{dist}(\alpha, f).$$

Then it is easy to verify that: for $e \in \Delta(f^*)$, $\text{dist}(e, f) = k$ and for $e \in \Delta(f)$, i.e., $e \cap f \neq \emptyset$, then $\text{dist}(e, f) = 0$.

We define the lattice tube of $f$ with radius $r$ as

$$D(f, r) = \{\alpha \in T_n^k, \text{dist}(\alpha, f) \leq r\},$$

which contains lattice nodes at most $r$ distance away from $f$. We overload the notation

$$L(f, s) = \{\alpha \in T_n^k, \text{dist}(\alpha, f) = s\},$$

which is defined as a plane early but here is a subset of lattice nodes on this plane. Then by definition,

$$D(f, r) = \bigcup_{s=0}^r L(f, s), \quad L(f, s) = L(f^*, k-s).$$
We have the following characterization of $D(f, r)$.

**Lemma 2.1.** For lattice node $\alpha \in T^n_k$, 
\[ \alpha \in D(f, r) \iff |\alpha_f| \leq r \iff |\alpha_f| \geq k - r, \]
\[ \alpha \notin D(f, r) \iff |\alpha_f| > r \iff |\alpha_f| \leq k - r - 1. \]

**Proof.** By definition of $\text{dist}(\alpha, f)$ and the fact $|\alpha_f| + |\alpha_f^*| = k$. $\square$

For each vertex $v_i \in \Delta_0(T)$, 
\[ D(v_i, r) = \{ \alpha \in T^n_k, |\alpha_i| \leq r \}, \]
which is isomorphic to a simplicial lattice $T^n_r$ of degree $r$; see the green triangle in Fig. 1. For a face $F \in \Delta_{n-1}(T)$, $D(F, r)$ is a trapezoid of height $r$ with base $F$. In general for $f \in \Delta_\ell(T)$, the hyper plane $L(f, r)$ will cut the simplex $T$ into two parts, and $D(f, r)$ is the part containing $f$.

2.5. **Derivative and distance.** The distance of a node $\alpha$ to a sub-simplex $f$ can be used to control the derivative of the corresponding Bernstein polynomial.

**Lemma 2.2.** Let $f \in \Delta_\ell(T)$ be a sub-simplex of $T$. For $\alpha \in T^n_k$, $\beta \in \mathbb{N}^{1:n}$, and $|\alpha_f^*| > |\beta|$, i.e., $\text{dist}(\alpha, f) > |\beta|$, then
\[ D^\beta \lambda^\alpha|_f = 0. \]

**Proof.** For $\alpha \in T^n_k$, we write $\lambda^\alpha = \lambda^{\alpha_f}_f \lambda^{\alpha_f^*}_f$. When $|\alpha_f^*| > |\beta|$, the derivative $D^\beta \lambda^\alpha$ will contain a factor $\lambda^{\alpha_f^*}_f$ with $\gamma \in \mathbb{N}^{1:n-\ell}$, and $|\gamma| = |\alpha_f^*| - |\beta| > 0$. Therefore $D^\beta \lambda^\alpha|_f = 0$ as $\lambda_i|_f = 0$ for $i \in f^*$. $\square$

3. **Lagrangian Finite Elements**

For the polynomial space $\mathbb{P}_k(T)$ with $k \geq 1$ on an $n$-dimensional simplex $T$, we have the following decomposition of Lagrange element [2, (2.6)]

\[ \mathbb{P}_k(T) = \bigoplus_{\ell=0}^n \bigoplus_{f \in \Delta_\ell(T)} b_f \mathbb{P}_{k-(\ell+1)}(f). \]

The function $u \in \mathbb{P}_k(T)$ is uniquely determined by DoFs

\[ \int_f u p \, ds \quad \forall p \in \mathbb{P}_{k-(\ell+1)}(f), f \in \Delta_\ell(T), \ell = 0, 1, \ldots, n. \]
The integral at a vertex is understood as the function value at that vertex. We shall derive (8) and (9) from a geometric decomposition of the simplicial lattice. We use $A \oplus B$ to denote the union of two disjoint sets $A$ and $B$, i.e. $A \oplus B = A \cup B$ with property $A \cap B = \emptyset$. This notation is meant to be suggestive of the fact that $|A \oplus B| = |A| + |B|$, where $|\cdot|$ of a set is its cardinality.

**Lemma 3.1.** It holds that

$$T^k_n(T) = \bigoplus_{v \in \Delta_0(T)} T^0_k(v) \oplus \bigoplus_{\ell=1}^n \bigoplus_{f \in \Delta_\ell(T)} T^\sigma_k(f).$$

**Proof.** As $f$ are disjoint, so is $T^\sigma_k(f)$. Then count the cardinality using the isomorphism $T^\sigma_k(f) \cong T^\sigma_k((\ell+1))$ to finish the proof. \qed

![Figure 3](image-url) \hspace{1cm} Figure 3. Decomposition of Lagrange and Hermite elements in two dimensions.

We can rewrite the decomposition (8) of polynomial and decomposition (9) of DoFs once we have the lattice decomposition (10). The unisolvence depends on a property of the bubble function.

**Lemma 3.2.** Let $f \in \Delta_\ell(T)$ and $b_f = \lambda_f$. For all $e \in \Delta_m(T)$ with $m \leq \ell$ and $e \neq f$ when $m = \ell$, then $b_f|_e = 0$.

**Proof.** We claim $f \cap e^* \neq \emptyset$. Assume $f \cap e^* = \emptyset$. Then $f^* \cup e = \{0, 1, \ldots, n\}$ and thus $f \subseteq e$ which contradicts with either $m < \ell$ or $e \neq f$.

As $f \cap e^* \neq \emptyset$, then $\lambda_f$ contains $\lambda_i$ for some $i \in e^*$ and consequently $\lambda_f|_e = 0$. \qed

**Lemma 3.3.** We have the following decomposition

$$P_k(T) = \bigoplus_{v \in \Delta_0(T)} P_k(T^0_k(v)) \oplus \bigoplus_{\ell=1}^n \bigoplus_{f \in \Delta_\ell(T)} P_k(T^\sigma_k(f)).$$

The function $u \in P_k(T)$ is uniquely determined by DoFs

$$u(v_i) \quad v_i \in \Delta_0(T),$$

$$\int f u^\alpha_f \alpha_f \in T^\sigma_k((\ell+1)) \quad f \in \Delta_\ell(T), \quad \ell = 1, \ldots, n.$$
Proof. For completeness, we adapt the unisolvence proof in [5] to here. We choose a basis \( \{ \phi_i \} \) of \( \mathbb{P}_k(T) \) by the decomposition (11) and denote DoFs as \( \{ N_i \} \). By construction, the dimension of \( \{ \phi_i \} \) matches the number of DoFs \( \{ N_i \} \). The square matrix \( (N_i(\phi_j)) \) is block lower triangular in the sense that for \( \phi_f \in \mathbb{P}_k(\mathcal{T}_h^0(f)) = b_f \mathbb{P}_{k-(\ell+1)}(f) \),
\[
\int_e \phi_f p \, ds = 0, \quad \forall e \in \Delta(T) \text{ with } \dim e \leq \ell, e \neq f, p \in \mathbb{P}_{k-\dim e+1} \quad (e)
\]
by Lemma 3.2. The diagonal block is invertible as \( b_f : \mathbb{P}_{k-(\ell+1)}(f) \to b_f \mathbb{P}_{k-(\ell+1)}(f) \) is an isomorphism and \( b_f > 0 \) in \( f \).
So the unisolvence follows from the invertibility of this lower triangular matrix. \( \square \)

Remark 3.4. The hierarchical polynomial space at vertices can be changed to \( \mathbb{P}_1(T) = \text{span} \{ \lambda_i, i = 0, 1, \ldots n \} \). A hierarchical decomposition sorted by degree can be obtained which may have smaller condition number for the assembled mass and stiffness matrices. But this is not the focus of the current paper.

Let \( \{ T_h \} \) be a family of partitions of \( \Omega \) into nonoverlapping simplexes with \( h_K := \text{diam}(K) \) and \( h := \max_{K \in T_h} h_K \). Let \( \Delta_\ell(T_h) \) be the set of all \( \ell \)-dimensional simplexes of \( T_h \) for \( \ell = 0, 1, \ldots, n \). The mesh \( T_h \) is conforming in the sense that the intersection of any two simplexes is a common lower sub-simplex. The global Lagrange finite element space \( V_h^L \) can be defined as
\[
V_h^L = \bigoplus_{v \in \Delta_0(T_h)} \mathbb{P}_k(\mathcal{T}_h^0(v)) \oplus \bigoplus_{\ell=1}^n \bigoplus_{f \in \Delta_\ell(T_h)} \mathbb{P}_k(\mathcal{T}_h^\ell(f)).
\]
Here we extend the polynomial on \( f \) to each element \( T \) containing \( f \) by the Bernstein form in the barycentric coordinate. Consequently the dimension of \( V_h^L \) is
\[
|V_h^L| = \sum_{\ell=0}^n |\Delta_\ell(T_h)| \binom{k-1}{\ell},
\]
where \( |\Delta_\ell(T_h)| \) is the cardinality of number of \( \Delta_\ell(T_h) \), i.e., the number of \( \ell \)-dimensional simplexes in \( T_h \).

4. HERMITE FINITE ELEMENTS
In this section we will show the geometric decomposition of Hermite finite elements in arbitrary dimension. When \( n > 1 \), the Hermite finite element space is \( C_0 \)-conforming only but \( C^m \) continuous at vertices. The degree of polynomial satisfies \( k \geq 2m + 1 \).

4.1. Hermite spline in one dimension. For edge \( e \) with vertices \( v_0 \) and \( v_1 \), as \( k \geq 2m + 1, D(v_0, m) \cap D(v_1, m) = \emptyset \). Recall that
\[
\mathcal{T}_k^{m+1}(e) = \{ \alpha_e \in \mathcal{T}_k^1(e), \alpha_e \geq m + 1 \} = \mathcal{T}_k^1(e) \backslash \bigoplus_{i=0}^1 D(v_i, m)\]
We then have the following decomposition of lattice \( \mathcal{T}_k^1(e) \)
\[
\mathcal{T}_k^1(e) = \bigoplus_{i=0}^1 D(v_i, m) \bigoplus \mathcal{T}_k^{m+1}(e).
\]
This leads to the decomposition of polynomial space
\[
\mathbb{P}_k(e) = \bigoplus_{i=0}^1 \mathbb{P}_k(D(v_i, m)) \bigoplus \mathbb{P}_k(\mathcal{T}_k^{m+1}(e)),
\]
and DoFs
\[
D^\alpha u(v) \quad \alpha \in \mathbb{N}, 0 \leq \alpha \leq m, v \in \Delta_0(e),
\]
\[
\int_e u \lambda_e^{\alpha} \, ds \quad \alpha \in T^1_{k,m+1}(e),
\]
which is known as the Hermite spline in one dimension. Notice that as \( t_e^{m+1} = \lambda_e^{m+1} \) is always positive in \( e \), we can simplify the lattice set \( T^1_{k,m+1}(e) \) to \( T^1_{k-2(m+1)} \) and replace (12) by
\[
\int_e u \lambda_e^{\alpha} \, ds \quad \alpha \in T^1_{k-2(m+1)}.
\]
Given a mesh \( T_h \), the decomposition can be naturally extend to the whole mesh and DoFs are single valued at each vertex \( v \in \Delta_0(T) \)
\[
V^H(T_h) = \bigoplus_{v \in \Delta_0(T_h)} \mathbb{P}_k(D(v, m)) \bigoplus_{e \in \Delta_1(T_h)} \mathbb{P}_k(T^1_{k,m+1}(e)).
\]
As the derivative \( D^\alpha u(v) \) is continuous up to order \( m \), \( V^H(T_h) \) defined by (13) is a subspace of \( C^m(\Omega) \). We shall generalize the Hermite spline to arbitrary dimension by imposing the \( C^m \) continuity at vertices. Again we emphasize that, for \( n > 1 \), the corresponding finite element space is no longer \( C^m \)-conforming.

4.2. Derivatives at vertices. Consider a function \( u \in C^m(\Omega) \). The set of derivatives of order up to \( m \) can be written as
\[
\{ D^\alpha u, \alpha \in \mathbb{N}^{1,n}, |\alpha| \leq m \}.
\]
Notice that the multi-index \( \alpha \in \mathbb{N}^{1,n} \). We can add component \( \alpha_0 = m - |\alpha| \). Then the index set forms a simplicial lattice \( T_m \) of degree \( m \). For each vertex, we can use the small simplicial lattice \( T_m \) \( \cong D(v_i, m) \) to determine the derivatives at that vertex; see the green triangle in Fig. 3 (b) and Fig. 4 (a).

**Lemma 4.1.** Let \( i \in \{0, 1, \ldots, n\} \). The polynomial space
\[
\mathbb{P}_k(D(v_i, m)) := \text{span} \{ \lambda_\alpha, \alpha \in T^n_m, \text{dist}(\alpha, v_i) = |\alpha_i^*| \leq m \},
\]
is uniquely determined by the DoFs
\[
\{ D^\beta u(v_i), \beta \in \mathbb{N}^{1,n}, |\beta| \leq m \}.
\]

**Proof.** Obviously the dimensions match. Indeed, a one-to-one mapping is from \( \alpha_i^* \) to \( \beta \).
So it suffices to show that for \( u \in \mathbb{P}_k(D(v_i, m)) \) if DoF (14) vanishes, then \( u = 0 \).
Without loss of generality, we assume \( i = 0 \). Clearly \( \{ \nabla \lambda_1, \ldots, \nabla \lambda_n \} \) forms a basis of \( \mathbb{R}^n \). We choose another basis \( \{ l_i, \ldots, l_n \} \) of \( \mathbb{R}^n \), being dual to \( \{ \nabla \lambda_1, \ldots, \nabla \lambda_n \} \), i.e., \( \nabla \lambda_i \cdot l^j = \delta_{i,j} \). Indeed \( l^j \) is the edge vector \( e_0 \). We can express the derivatives in this non-orthogonal basis and conclude that, for each \( r = 0, 1, \ldots, m \), vanishing \( \{ D^\alpha u, \beta \in T^n_r(F_0) \} \) is equivalent to the vanishing \( \{ D^\beta u, \beta \in T^n_r(F_0) \} \), where
\[
D^\beta u := \frac{\partial \alpha^*}{\partial(l^1)^{\beta_1} \cdots (l^n)^{\beta_n}}. 
\]
By the duality \( \nabla \lambda_i \cdot l^j = \delta_{i,j}, i, j = 1, \ldots, n, \)
\[
D^\beta_0(\lambda^{\alpha_0}_0) = \beta_0 \delta(\alpha, \beta) \quad \text{for } \alpha, \beta \in T^n_0(F_0), 
\]
where \( \beta! = \beta_0! \beta_1! \cdots \beta_n! \) and \( \delta(\alpha, \beta) \) is the Kronecker delta function.
A basis of \( \mathbb{P}_k(D(v_0, m)) \) is given by \( \{ \lambda^{k-|\alpha|}_0, \alpha \in \mathbb{N}^{1,n}, |\alpha| \leq m \} \). The DoF-Function matrix \( D^\beta_0(\lambda^{k-|\alpha|}_0, \lambda^{\alpha}_0) \) is block lower triangular where the lattice nodes are sorted by their length. Then if each block matrix on the diagonal \( (|\alpha| = |\beta| = r) \) is invertible, the whole matrix is invertible which is equivalent to the unisolvency.
Assume \( u = \sum_{\alpha \in \mathbb{N}^{n}} c_{\alpha} \lambda_{0}^{k-|\alpha|} \lambda_{0}^{\alpha} \), with \( c_{\alpha} \in \mathbb{R} \) and \( D^{\beta}u(v_{0}) = 0 \) for all \( \beta \in \mathbb{N}^{1:n} \) satisfying \( |\beta| \leq m \). We prove \( c_{\alpha} = 0 \) by induction with respect to \( |\alpha| \). When \( |\alpha| = 0 \), as \( u(v_{0}) = c_{(0,...,0)} \) and \( u(v_{0}) = 0 \), we conclude \( c_{(0,...,0)} = 0 \). Assume \( c_{\alpha} = 0 \) for all \( \alpha \in \mathbb{N}^{1:n} \) satisfying \( |\alpha| \leq r - 1 \), i.e., \( u = \sum_{\alpha \in \mathbb{N}^{1:n}, |\alpha| = r} c_{\alpha} \lambda_{0}^{k-|\alpha|} \lambda_{0}^{\alpha} \). By Lemma 2.2, the derivative \( D^{\beta}(\lambda_{0}^{k-|\alpha|} \lambda_{0}^{\alpha}) \) vanishes for all \( \beta \in \mathbb{N}^{1:n} \) satisfying \( |\beta| < |\alpha| \). Hence, for \( |\beta| = r \), using (15),

\[
D^{\beta}u(v_{0}) = D^{\beta}_{n} \left( \sum_{\alpha \in \mathbb{N}^{1:n}, |\alpha| = r} c_{\alpha} \lambda_{0}^{k-|\alpha|} \lambda_{0}^{\alpha} \right)(v_{0}) = \beta! c_{\beta} = 0,
\]

which implies \( c_{\beta} = 0 \) for all \( \beta \in \mathbb{N}^{1:n}, |\beta| = r \). Induction for \( r = 1, 2, \ldots, m \) to conclude \( u = 0 \).

\section*{4.3. A decomposition of the simplicial lattice.}

When \( k \geq 2m + 1 \), then \( D(v, m) \) for \( v \in \Delta_{0}(T) \) are disjoint. Denoted by

\[ D(\Delta_{0}(T), m) = \bigoplus_{v \in \Delta_{0}(T)} D(v, m). \]

\begin{lemma}
It holds that
\( T^{n}_{k}(T) = D(\Delta_{0}(T), m) \bigoplus_{f \in \Delta_{0}(T)} T^{n}_{k,1}(f) \big/ D(\Delta_{0}(T), m) \).
\end{lemma}

Consequently

\[ P_{k}(T) = P_{k}(D(\Delta_{0}(T), m)) \bigoplus_{f \in \Delta_{0}(T)} P_{k}(T^{n}_{k,1}(f) \big/ D(\Delta_{0}(T), m)). \]

\begin{proof}
Obviously \( T^{n}_{k}(T) = D(\Delta_{0}(T), m) \bigoplus \big[ T^{n}_{k}(T) \big/ D(\Delta_{0}(T), m) \big] \). Then use the decomposition (10) for \( T^{n}_{k}(T) \) and the fact \( \text{dist}(v, f) = k \geq 2m + 1 \) for \( v \notin \Delta_{0}(f) \) to conclude that \( T^{n}_{k,1}(f) \big/ D(\Delta_{0}(T), m) = T^{n}_{k,1}(f) \big/ D(\Delta_{0}(f), m) \). Then the desired decomposition follows.

The decomposition of polynomial space is a consequence of Bernstein basis and the lattice decomposition (16).
\end{proof}

\section*{4.4. Hermite finite elements.}

\begin{lemma}[Hermite element in \( \mathbb{R}^{n} \)] Let \( k \geq 2m + 1 \) and \( T \) be an \( n \)-dimensional simplex. The shape function space \( P_{k}(T) \) is determined by DoFs
\end{lemma}

\begin{align}
D^{\alpha}u(v_{i}) & \quad \alpha \in \mathbb{N}^{1:n}, |\alpha| \leq m, v_{i} \in \Delta_{0}(T), i = 0, 1, \ldots, n, \\
\int_{f} u \lambda_{f}^{\alpha} \, ds & \quad \alpha \in \mathbb{N}^{1:n}, |\alpha| \leq m, v_{i} \in \Delta_{0}(T), i = 0, 1, \ldots, n.
\end{align}

\begin{proof}
The proof is straightforward in view of decomposition (17). For a polynomial \( u \in P_{k}(T^{n}_{k,1}(f) \big/ D(\Delta_{0}(f), m)) \), as the distance of corresponding lattice nodes to all vertices are greater than \( m \), (18) vanishes which means the DoF-Fun matrix is still block lower triangular. The proof is essentially the same as that for the Lagrange element except using Lemma 4.1 for lattice nodes in \( D(\Delta_{0}(T), m) \).

The condition \( k \geq 2m + 1 \) is required so that the disks \( D(v_{i}, m) \) are disjoint. The set

\[ T^{n}_{k,1}(f) \big/ D(\Delta_{0}(T), m) = \{ \alpha \in T^{n}_{k}(f), 1 \leq \alpha \leq k - m - 1 \}, \]

which can be verified as follows: for any \( v \in \Delta_{0}(f) \), \( \alpha_{v} \geq 1 \) as \( \alpha_{f} \in T^{n}_{k,1}(f) \). The condition \( \alpha \notin D(\Delta_{0}(f), m) \) is equivalent to \( \text{dist}(\alpha_{f}, v) = |\alpha_{v,|} > m \) which implies the upper bound \( \alpha_{v} \leq k - m - 1 \).

Then we set $\alpha_f = \alpha_f - 1$ to get the lattice set in DoF (19). The degree of polynomial is reduced from $k$ to $k - (\ell + 1)$ as $b_f = \lambda_f \in P_{\ell+1}(f)$ is always positive in $\tilde{f}$. □

**Remark 4.4.** The index set in (19) can be written in another form. For $\alpha_f \in T_{k-(\ell+1)}^f$, $\alpha_f \leq |\alpha_f| = k - (\ell + 1) \leq k - m - 2$ if $\ell \geq m + 1$, i.e., there is no need to impose the constraint $\alpha_f \leq k - m - 2$ when $\ell \geq m + 1$. Consider the case $\ell \leq m$. Take $\alpha \in T_{k-(\ell+1)}^f$. The bound $\alpha_v \leq k - m - 2$ is equivalent to $|\alpha_v| \geq m - \ell + 1$, i.e., $\alpha \notin D(v, m - \ell)$. Therefore

$$\mathbb{T}_{k,1}^f(f) \setminus D(\Delta_0(f), m) \cong \mathbb{T}_{k-(\ell+1)}^f(f) \setminus D(\Delta_0(f), m - \ell).$$

Geometrically we consider the inner simplicial lattice and subtract vertex disks with a smaller radius.

Given a mesh $\mathcal{T}_h$, the decomposition can be naturally extend to the whole mesh

$$V^H(\mathcal{T}_h) = \bigoplus_{v \in \Delta_0(\mathcal{T}_h)} P_k(D(v, m)) \oplus \bigoplus_{\ell=1}^n \bigoplus_{f \in \Delta_\ell(\mathcal{T}_h)} P_k(\mathbb{T}_{k,1}^f) \setminus D(\Delta_0(f), m)).$$

And DoFs are single valued at each sub-simplex (symbolically change $\Delta_\ell(T)$ to $\Delta_\ell(\mathcal{T}_h)$). The obtained space $V^H(\mathcal{T}_h)$ is $C^0$-conforming only for $n > 1$ but $C^m$ continuous at vertices. The dimension of $V^H(\mathcal{T}_h)$ is

$$\dim V^H(\mathcal{T}_h) = |\Delta_0(\mathcal{T}_h)| \left(\binom{n+m}{m} + \sum_{\ell=1}^n |\Delta_\ell(\mathcal{T}_h)| \left(\binom{k-1}{\ell} - (\ell + 1) \binom{m}{\ell}\right)\right).$$

When computing the dimension of $P_k(\mathbb{T}_{k,1}^f) \setminus D(\Delta_0(f), m))$, it is easier to use the equivalent index set in (20).

Compared with the Lagrange elements, more DoFs are accumulated to vertices and may reduce the dimension of the finite element space. For example, in two dimensions, moving edge-wise and element-wise DoFs to vertices will reduce the dimension of the finite element space around one half less, which is considered as an advantage of using Hermite elements vs Lagrange elements.

**5. Smooth Finite Elements in Two Dimensions**

We shall re-construct the $C^m$-conforming finite element on two-dimensional triangular grids, firstly constructed by Bramble and Zlámal [3], by a decomposition of the simplicial lattice. We start from a Hermite finite element space which ensures the tangential derivatives across edges are continuous. By adding degrees of freedom on the normal derivative, we can impose the continuity of derivatives across triangles. We use two-dimensional case as an introductory example for the so-called super-smoothness at lower sub-simplexes: the smoothness at vertices is $C^{2m}$ which is sufficient but may not be necessary.

We use a pair of integers $r = (r_0, r_1)$ for the smoothness at 0-dimensional sub-simplex (vertex) and at 1-dimensional sub-simplex (edge), respectively. To be $C^m$-conforming, $r_1 = m$ is the minimum requirement for edges and $r_0 \geq 2r_1$ for vertices.

**5.1. Normal derivatives.** Given an edge $e$, we first identify lattice nodes to determine the normal derivative

$$\left\{\frac{\partial^\beta u}{\partial n_e^\beta} | e, 0 \leq \beta \leq m\right\}.$$

By Lemma 2.2, if the lattice node is $r_1 + 1$ away from the edge, then the corresponding Bernstein polynomial will have vanishing normal derivatives up to order $r_1$. 
On the two vertices, we have used $D(\Delta_0(e), r_0)$ for the derivative at vertices. So we will use the rest, i.e., $D(e, r_1)\setminus D(\Delta_0(e), r_0)$ for the normal derivative.

**Lemma 5.1.** Let $r_0 \geq r_1 \geq 0$ and $k \geq 2r_0 + 1$. Let $e \in \Delta_1(T)$ be an edge of a triangle $T$. The polynomial function space $\mathbb{P}_k(D(e, r_1)\setminus D(\Delta_0(e), r_0))$ is determined by DoFs

$$\int_e \frac{\partial^\beta u}{\partial n_e^\beta} \lambda_e^{\alpha_e} \, ds \quad \alpha_e \in \mathbb{T}^1_{k-2(r_0+1)+\beta}, \beta = 0, 1, \ldots, r_1.$$ 

**Proof.** Without loss of generality, we take $e = e_{0,1}$. By definition $D(e, r_1) = \bigoplus_{i=0}^{r_1} L(e, i)$, where recall that

$L(e, i) = \{\alpha \in \mathbb{T}^2_k, \text{dist}(\alpha, e) = i\} = \{\alpha \in \mathbb{T}^2_k, \alpha_2 = i\} = \{\alpha \in \mathbb{T}^2_k, \alpha_0 + \alpha_1 = k - i\}$

consists of lattice nodes parallel to $e$ and with distance $i$. Then $L(e, i) \cong \mathbb{T}^1_{k-i}(e)$ by keeping $(\alpha_0, \alpha_1)$ only.

Now we use the requirement $\alpha \notin D(\Delta_0(e), r_0)$ to figure out the range of the nodes. Using Lemma 2.1, we derive from $\text{dist}(\alpha, v_0) > r_0$ that $\alpha_0 < k - r_0$. Together with $\alpha_0 + \alpha_1 = k - i$, we get the lower bound $\alpha_1 \geq r_0 - i + 1$. Similarly $\alpha_0 \geq r_0 - i + 1$. Therefore the line segment

$L(e, i)\setminus D(\Delta_0(e), r_0) = \{(\alpha_0, \alpha_1, i), \alpha_0 + \alpha_1 = k - i, \min\{\alpha_0, \alpha_1\} \geq r_0 - i + 1\},$

which can be identified with the lattice $\mathbb{T}^1_{k-2(r_0+1)+i}$ without inequality constraint.

Applying the same argument in the proof of Lemma 4.1, it follows from Lemma 2.2 that matrix $(\frac{\partial^{\beta}u}{\partial n_e^{\beta}}(\lambda^{\alpha_1}_e \lambda^{\alpha_2}_e) e^{\beta}_e)$ is lower triangular. Hence it suffices to prove the polynomial function space $\mathbb{P}_k(L(e, i)\setminus D(\Delta_0(e), r_0))$ is determined by DoFs

$$\int_e \frac{\partial^i u}{\partial n_e^i} \lambda_e^{\alpha_e} \, ds \quad \alpha_e \in \mathbb{T}^1_{k-2(r_0+1)+i}.$$ 

Take $u = \sum_{\alpha_e \in \mathbb{T}^1_{k-2(r_0+1)+i}} c_{\alpha_e} \lambda_e^{\alpha_e} \lambda_0^{r_0-i+1} \lambda^i_2 \in \mathbb{P}_k(L(e, i)\setminus D(\Delta_0(e), r_0))$ with coefficients $c_{\alpha_e} \in \mathbb{R}$. Then

$$\frac{\partial^i u}{\partial n_e^i} \big|_e = i!(n_e \cdot \nabla \lambda_2)^i \lambda_0^{r_0-i+1} \sum_{\alpha_e \in \mathbb{T}^1_{k-2(r_0+1)+i}} c_{\alpha_e} \lambda_e^{\alpha_e}.$$ 

Noting that $n_e \cdot \nabla \lambda_2$ is a constant and $\lambda_e$ is always positive in the interior of $e$, the vanishing DoF (21) means $c_{\alpha_e} = 0$ for all $\alpha_e \in \mathbb{T}^1_{k-2(r_0+1)+i}$. $\square$

Geometrically we push all lattice nodes in $D(e, r_1)\setminus D(\Delta_0(e), r_0)$ to the edge to determine normal derivatives on $e$ up to order $r_1$.

### 5.2. The geometric decomposition.

The requirement $r_1 \leq r_0$ in Lemma 5.1 is due to the fact that the smoothness of the normal derivative is less than or equal to that of the vertices. In a triangle, a vertex will be shared by two edges and to have enough lattice nodes for each edge, $r_0 \geq 2r_1$ is required.

**Lemma 5.2.** Let $r_1 = m$, $r_0 \geq 2r_1$, and $k \geq 2r_0 + 1 \geq 4m + 1$. Let $T$ be a triangle. Then it holds that

$$\mathbb{T}^n_k(T) = \mathbb{S}_0(T) \oplus \mathbb{S}_1(T) \oplus \mathbb{S}_2(T),$$

where finish the proof.
For a node \( \alpha \) the geometric decomposition of a Hermite element.

Proof.\( (23) \) Then set \( S_0(T) = D(\Delta_0(T), r_0), S_1(T) = \bigoplus_{e \in \Delta_1(T)} (D(e, r_1) \setminus S_0(T)), S_2(T) = T^e_k(T) \setminus (S_0(T) \oplus S_1(T)). \)

This leads to the decomposition of the polynomial space

\[
P_k(T) = P_k(S_0(T)) \oplus P_k(S_1(T)) \oplus P_k(S_2(T)).
\]

Proof. As \( k \geq 2r_0 + 1 \), the sets \( \{D(v, r_0), v \in \Delta_0(T)\} \) are disjoint. As \( \text{dist}(v, e) = k \geq 2r_0 + 1 > r_1 \) for \( v \notin \Delta_0(e) \), \( D(e, r_1) \setminus D(\Delta_0(T), r_0) = D(e, r_1) \setminus D(\Delta_0(e), r_0) \).

We then show that the sets \( \{D(e, r_1) \setminus D(\Delta_0(e), r_0), e \in \Delta_1(T)\} \) are disjoint. A node \( \alpha \in D(e_{01}, r_1) \) implies \( \alpha_2 \leq r_1 \) and \( \alpha \in D(e_{02}, r_1) \) implies \( \alpha_1 \leq r_1 \). Therefore \( |\alpha_0| = \alpha_1 + \alpha_2 \leq 2r_1 \leq r_0 \), i.e., \( (D(e_{01}, r_1) \cap D(e_{02}, r_1)) \subseteq D(v_0, r_0) \). Repeat the argument for each pair of edges to conclude \( \{D(e, r_1) \setminus D(\Delta_0(e), r_0), e \in \Delta_1(T)\} \) are disjoint. Then \((22)\) follows.

We give another description of \( S_1(T) \) to rewrite the index set. For a lattice node \( \alpha \in D(e_{01}, r_1) \setminus D(\Delta_0(e_{01}), r_0) \), it satisfies the constraint

\[
\alpha_2 \leq r_1, \alpha_0 + \alpha_2 > r_0, \alpha_1 + \alpha_2 > r_0, \alpha_0 + \alpha_1 + \alpha_2 = k.
\]

We let \( \alpha_2 = i \) for \( i = 0, 1, \ldots, r_1 = m \). Then

\[
\lambda_0^{\alpha_0} \lambda_1^{\alpha_1} \lambda_2^{\alpha_2} = \lambda^i(\lambda_0 \lambda_1)\lambda_0^{\alpha_0 - (r_0 + 1) + i} \lambda_1^{\alpha_1 - (r_0 + 1) + i} = b^i b^{r_0 - 2i + 1} c^{\alpha_{e}}
\]

with \( \alpha_{e} \in \mathbb{N}^{0,1}_{k - 2(r_0 + 1) + i} \). So we have

\[
P_k(S_1(T)) = \bigoplus_{e \in \Delta_1(T)} \bigoplus_{i = 0}^{m} b^{i} b^{r_0 + 1 - 2i} \bigoplus_{k - 2(r_0 + 1) + i(e)}
\]

For a node \( \alpha \in S_2(T) \), it satisfies the constraint

\[
\alpha_0 + \alpha_1 + \alpha_2 = k, \alpha > r_1 = m, \alpha < k - r_0.
\]

Then set \( \tilde{\alpha} = \alpha - (m + 1) \). We can write

\[
\lambda^\alpha = \lambda^{m+1} \lambda^{\tilde{\alpha}}, |\tilde{\alpha}| = k - 3(m + 1), \tilde{\alpha} \leq k - r_0 - m - 2.
\]
Therefore
\begin{equation}
\mathbb{P}_k(S_2(T)) = b_T^{m+1}P_{k-3(m+1)}^0(T),
\end{equation}
where
\[ P_{k-3(m+1)}^0(T) := \text{span} \{ \lambda^\alpha | \alpha \in T_{k-3(m+1)}^2, \alpha \leq k - r_0 - m - 2 \}. \]

When \( r_0 = 2r_1 = 2m \) or \( r_0 = 2r_1 + 1 = 2m + 1 \), \( P_{k-3(m+1)}^0(T) = P_{k-3(m+1)}(T) \) as the constraint automatically holds.

Such simplification is not needed in implementation. Distance to a vertex or an edge is computable and a logic array can be used to represent \( S_k(T) \).

5.3. Smooth finite elements in two dimensions.

**Theorem 5.3.** Let \( r_1 = m, r_0 \geq 2r_1, \) and \( k \geq 2r_0 + 1 \geq 4m + 1 \). Let \( T \) be a triangle. The shape function space \( \mathbb{P}_k(T) \) is determined by the DoFs
\begin{align}
D^\alpha u(v) & \quad \alpha \in \mathbb{N}_0, |\alpha| \leq r_0, v \in \Delta_0(T), \quad (26) \\
\int_e \frac{\partial^\beta u}{\partial n_e} \lambda_e^\alpha \, ds & \quad \alpha \in T_{k-2(r_0+1)+\beta}, \beta = 0, 1, \ldots, r_1, \quad (27) \\
\int_T u \lambda^\alpha \, dx & \quad \alpha \in T_{k-3(m+1)}, \alpha \leq k - r_0 - m - 2. \quad (28)
\end{align}

**Proof.** By the decomposition (23) of \( \mathbb{P}_k(T) \) and characterization (24)-(25), the dimension of \( \mathbb{P}_k(T) \) matches the number of DoFs. Let \( u \in \mathbb{P}_k(T) \) satisfying all the vanishing DoFs (26)-(28) vanish. Thanks to Lemma 4.1, Lemma 5.1 and Lemma 5.2, it follows from the vanishing DoFs (26) and (27) that \( u \in \mathbb{P}_k(S_2(T)) \). As \( b_T \) is always positive in the interior of \( T \), \( u = 0 \) holds from the vanishing DoF (28).

When \( r_1 = m = 1 \) and \( r_0 = 2 \), this is known as Argyris element [1, 15]. When \( r_1 = m, r_0 = 2m \) and \( k = 4m + 1 \), \( C^m \)-continuous finite elements are constructed in [3, 18], whose interior DoFs are different from (28). In our notation, the edge and interior DoFs in [9] are adopted as
\begin{align}
\int_e \frac{\partial^\beta u}{\partial n_e} \lambda_e^\alpha \, ds & \quad \alpha \in T_{k-\beta+1}, \alpha \geq r_0 - \beta + 1, \beta = 0, 1, \ldots, r_1, \\
\int_T u \lambda^\alpha \, dx & \quad \alpha \in S_2(T),
\end{align}
which are slightly different from (27)-(28). We further remove the edge and element bubbles in the test function space in DoFs.

With mesh \( \mathcal{T}_h \), define the global \( C^m \)-continuous finite element space
\begin{equation}
V(\mathcal{T}_h) = \{ u \in C^m(\Omega) : u|_T \in \mathbb{P}_k(T) \text{ for all } T \in \mathcal{T}_h, \text{ and all the DoFs (26) and (27) are single-valued} \}.
\end{equation}

Then \( V(\mathcal{T}_h) \) admits the following geometric decomposition
\begin{align}
V(\mathcal{T}_h) = & \bigoplus_{v \in \Delta_0(\mathcal{T}_h)} \mathbb{P}_k(D(v, r_0)) \oplus \bigoplus_{e \in \Delta_1(\mathcal{T}_h)} \mathbb{P}_k(D(e, r_1) \setminus S_0(e)) \\
& \oplus \bigoplus_{T \in \mathcal{T}_h} \mathbb{P}_k(S_2(T)),
\end{align}
where \( S_0(e) = D(\Delta_0(e), r_0) \). The dimension of \( V(T_h) \) is
\[
\dim V(T_h) = |\Delta_0(T_h)| \left( \frac{r_0 + 2}{2} \right) + |\Delta_1(T_h)| (m + 1) \left( k - 2r_0 - 1 + m/2 \right)
+ |\Delta_2(T_h)| \left( \frac{k - 3m - 1}{2} - 3 \frac{r_0 - 2m}{2} \right).
\]
In particular, for the minimum degree case: \( r_1 = m, r_0 = 2m, k = 4m + 1 \), we denoted by \( V^{BZ}(T_h) \) and the dimension
\[
\dim V^{BZ}(T_h) = |\Delta_0(T_h)| \left( \frac{2m + 2}{2} \right) + |\Delta_1(T_h)| \left( \frac{m + 1}{2} \right) + |\Delta_2(T_h)| \left( \frac{m - 2}{2} \right).
\]
When \( m \leq 1 \), there is no interior moments as \( k = 4m + 1 \) is small.

6. Smooth Finite Elements in Arbitrary Dimension

In this section we shall generalize the construction to arbitrary dimension. The smoothness at sub-simplexes is exponentially increasing as the dimension decreases
\[
r_n = 0, \quad r_{n-1} = m, \quad r_\ell \geq 2r_{\ell+1} \quad \text{for}\; \ell = n - 2, \ldots, 0.
\]
And the degree of polynomial \( k \geq 2r_0 + 1 \geq 2^n m + 1 \). The key in the construction is a non-overlapping decomposition of the simplicial lattice in which each component will be used to determine the normal derivatives.

When \( n = 3, m = 1, r_1 = 2, r_0 = 4 \) and \( k \geq 9 \), it is the \( C^1 \) element on tetrahedron constructed by Zhang in [25]. When \( n = 4, m = 1, r_2 = 2, r_1 = 4, r_0 = 8 \) and \( k \geq 17 \), it is the \( C^1 \) element on simplex in four dimensions constructed by Zhang in [26]. Neilan’s Stokes element [16] is a \( C^0 \) element with parameters \( n = 3, m = r_2 = 0, r_1 = 1, r_0 = 2 \), and \( k \geq 5 \).

6.1. A decomposition of the simplicial lattice. We explain the requirement \( r_{\ell-1} \geq 2r_\ell \).

**Lemma 6.1.** Let \( T \) be an \( n \)-dimensional simplex. For \( \ell = 1, \ldots, n - 1 \), if \( r_{\ell-1} \geq 2r_\ell \), the sub-sets \( \{D(f, r_\ell) \setminus \bigcup_{e \in \Delta_{\ell-1}(f)} D(e, r_{\ell-1})\}, f \in \Delta_\ell(T) \) are disjoint.

**Proof.** Consider two different sub-simplexes \( f, \tilde{f} \in \Delta_\ell(T) \). The dimension of their intersection is at most \( \ell - 1 \). Therefore \( f \cap \tilde{f} \subset e \) for some \( e \in \Delta_{\ell-1}(f) \). Then \( e^* \subset (f \cap \tilde{f})^* = f^* \cup \tilde{f}^* \). For \( \alpha \in D(f, r_\ell) \cap D(\tilde{f}, r_\ell) \), we have \( |\alpha_{e^*}| \leq |\alpha_{f^*}| + |\alpha_{\tilde{f}^*}| \leq 2r_\ell \leq r_{\ell-1} \). Therefore we have shown the intersection region \( D(f, r_\ell) \cap D(\tilde{f}, r_\ell) \subset \bigcup_{e \in \Delta_{\ell-1}(f)} D(e, r_{\ell-1}) \) and the result follows. \( \square \)

Next we remove \( D(e, r_i) \) from \( D(f, r_\ell) \) for all \( e \in \Delta_\ell(T) \) and \( i = 0, 1, \ldots, \ell - 1 \).

**Lemma 6.2.** Given integer \( m \geq 0 \), let non-negative integer array \( r = (r_0, r_1, \ldots, r_n) \) satisfy
\[
r_n = 0, \quad r_{n-1} = m, \quad r_\ell \geq 2r_{\ell+1} \quad \text{for}\; \ell = n - 2, \ldots, 0.
\]
Let \( k \geq 2r_0 + 1 \geq 2^n m + 1 \). For \( \ell = 1, \ldots, n - 1 \),
\[
\bigcup_{\ell=0}^{\ell-1} \bigcup_{e \in \Delta_\ell(f)} D(e, r_i) = D(f, r_\ell) \setminus \bigcup_{i=0}^{\ell-1} \bigcup_{e \in \Delta_\ell(T)} D(e, r_i).
\]
In (29), the relation \( \subseteq \) is obvious as \( \Delta_i(f) \subseteq \Delta_i(T) \).

To prove \( \subseteq \), it suffices to show for \( \alpha \in D(f, r_\ell) \setminus \left( \bigcup_{i=0}^{\ell-1} \bigcup_{e \in \Delta_i(f)} D(e, r_i) \right) \), it is not in \( D(e, r_i) \) for \( e \in \Delta_i(T) \) and \( e \notin \Delta_i(f) \).

By definition,

\[
|\alpha_{e, \ell}| \leq r_\ell, \quad |\alpha_e| \leq k - r_i - 1 \quad \text{for all} \quad e \in \Delta_i(f), i = 0, \ldots, \ell - 1.
\]

For each \( e \in \Delta_i(T) \) but \( e \notin \Delta_i(f) \), the dimension of the intersection \( e \cap f \) is at most \( i - 1 \). It follows from \( r_j \geq 2r_{j+1} \) and \( k \geq 2r_0 + 1 \) that: when \( i > 0 \),

\[
|\alpha_e| = |\alpha_{e \cap f}| + |\alpha_{e \cap f^*}| \leq k - r_i - 1 + r_\ell \leq k - r_i - 1,
\]

and when \( i = 0 \),

\[
|\alpha_e| = |\alpha_{e \cap f^*}| \leq k - r_i - 1.
\]

So \( |\alpha_{e, \ell}| > r_i \). We conclude that \( \alpha \notin D(e, r_i) \) for all \( e \in \Delta_i(T) \) and (29) follows.

We are in the position to present our main result.

**Theorem 6.3.** Given integer \( m \geq 0 \), let non-negative integer array \( r = (r_0, r_1, \cdots, r_n) \) satisfy

\[
r_n = 0, \quad r_{n-1} = m, \quad r_\ell \geq 2r_{\ell+1} \quad \text{for} \quad \ell = n - 2, \ldots, 0.
\]

Let \( k \geq 2r_0 + 1 \geq 2^n m + 1 \). Then we have the following direct decomposition of the simplicial lattice on an \( n \)-dimensional simplex \( T \):

\[
T_k^n(T) = \bigoplus_{\ell=0}^{n} \bigoplus_{f \in \Delta_i(T)} S_\ell(f),
\]

where

\[
S_0(v) = D(v, r_0),
\]

\[
S_\ell(f) = D(f, r_\ell) \setminus \left( \bigcup_{i=0}^{\ell-1} \bigcup_{e \in \Delta_i(f)} D(e, r_i) \right), \quad \ell = 1, \ldots, n - 1,
\]

\[
S_n(T) = T_k^n(T) \setminus \bigcup_{i=0}^{n-1} \bigcup_{f \in \Delta_i(T)} D(f, r_i).
\]

Consequently we have the following geometric decomposition of \( \mathbb{P}_k(T) \)

\[
\mathbb{P}_k(T) = \bigoplus_{\ell=0}^{n} \bigoplus_{f \in \Delta_i(T)} \mathbb{P}_\ell(S_\ell(f)).
\]

**Proof.** First we show that the sets \( \{ S_\ell(f), f \in \Delta_i(T), \ell = 0, \ldots, n \} \) are disjoint. Take two vertices \( v_1, v_2 \in \Delta_0(T) \). For \( \alpha \in D(v_1, r_0) \), we have \( \alpha_{v_1} \geq k - r_0 \). As \( v_1 \subseteq v_2 \) and \( k \geq 2r_0 + 1 \), \( \alpha_{v_2} \geq k - r_0 \geq r_0 + 1 \), i.e., \( \alpha \notin D(v_2, r_0) \). Hence \( \{ S_0(v), v \in \Delta_0(T) \} \) are disjoint and \( \bigoplus_{v \in \Delta_0(T)} S_0(v) \) is a disjoint union. By Lemma 6.1 and (29), we know \( \{ S_\ell(f), f \in \Delta_i(T), \ell = 0, \ldots, n \} \) are disjoint.

Next we inductively prove

\[
\bigoplus_{i=0}^{\ell} \bigoplus_{f \in \Delta_i(T)} S_i(f) = \bigcup_{i=0}^{\ell} \bigcup_{f \in \Delta_i(T)} D(f, r_i) \quad \text{for} \quad \ell = 0, \ldots, n - 1.
\]
Obviously (32) holds for \( \ell = 0 \). Assume (32) holds for \( \ell < j \). Then

\[
\bigoplus_{f \in \Delta_s(T)} S_\ell(f) = \bigoplus_{f \in \Delta_s(T)} S_j(f) \oplus \bigcup_{i=0}^{-1} \bigcup_{e \in \Delta_s(T)} D(e, r_i)
\]

\[
= \bigoplus_{f \in \Delta_s(T)} \left( D(f, r_j) \setminus \bigcup_{i=0}^{-1} \bigcup_{e \in \Delta_s(T)} D(e, r_i) \right) \bigoplus_{i=0}^{-1} \bigcup_{e \in \Delta_s(T)} D(e, r_i)
\]

By induction, (32) holds for \( \ell = 0, \ldots, n - 1 \). Then (30) is true from the definition of \( S_n(T) \) and (32).

We can write out the inequality constraints in \( S_\ell(f) \). For \( \ell = 1, \ldots, n \),
\[
S_\ell(f) = \{ \alpha \in \mathbb{T}_n^\ell : |\alpha_f| \leq r, |\alpha_e| \leq k - r - 1, \forall e \in \Delta_s(f), i = 0, \ldots, \ell - 1 \}
\]

where \( \mathbb{T}_n^\ell \) consists of lattice nodes \( s \) away from \( f \). The following unisolvence is the generalization of Lemma 4.1 from a vertex to a sub-simplex \( f \).

\textbf{Lemma 6.4.} Let \( \ell = 0, \ldots, n-1 \) and \( s \leq r \). Given \( f \in \Delta_s(T) \), let \( n_f = \{n_1^f, n_2^f, \ldots, n_n^{n-\ell}_f\} \) be \( n - \ell \) vectors spanning the normal plane of \( f \). The polynomial space \( \mathbb{P}_k(S_\ell(f) \cap L(f, s)) \) is uniquely determined by DoFs

\[
\int_f \frac{\partial^j u}{\partial n_f^\beta} \lambda_f^\alpha \, ds \quad \forall \alpha \in S_\ell(f), |\alpha_f| = k - s, \beta \in \mathbb{N}^{1-n-\ell}, |\beta| = s.
\]

\textbf{Proof.} A basis of \( \mathbb{P}_k(S_\ell(f) \cap L(f, s)) \) is \( \{\lambda^\alpha = \lambda_f^\alpha (x_f^\alpha)^\beta, \alpha \in S_\ell(f), |\alpha_f| = s \} \) and thus the dimensions match (by mapping \( x_f^\alpha \) to \( \beta \)).

We choose a basis of the normal plane \( \{n_1^f, n_2^f, \ldots, n_n^{n-\ell}_f\} \) s.t. it is dual to the vectors \( \{\nabla \lambda_f^{(1)}, \nabla \lambda_f^{(2)}, \ldots, \} \), i.e., \( \nabla \lambda_f^{(i)} \cdot n_j^f = \delta_{i,j} \) for \( i, j = 1, \ldots, n - \ell \). Then we have the duality

\[
\frac{\partial}{\partial n_f^\beta} (\lambda_f^{\alpha'}) = \beta \delta(\alpha_f, \beta), \quad \alpha_f, \beta \in \mathbb{N}^{1-n-\ell}, |\alpha_f| = |\beta| = s,
\]

which can be proved easily by induction on \( s \). When \( T \) is the reference simplex \( \tilde{T} \), \( \lambda_i = x_i \) and \( \nabla \lambda_i = -e_i \), (36) is the calculus result \( D_{n_f}^{\beta} (\alpha_f^{\gamma'}) \beta \delta(\alpha_f, \beta) \).

Assume \( u = \sum c_{\alpha_f, \beta} \lambda_f^{\alpha'} \lambda_f^{\beta'} \in \mathbb{P}_k(S_\ell(f) \cap L(f, s)) \). If the derivative is not fully applied to the component \( \lambda_f^{\alpha'} \), then there is a term \( \lambda_f^{\gamma'} \) with \( |\gamma| > 0 \) left and \( \lambda_f^{\beta'} \) remains. So for any \( \beta \in \mathbb{N}^{1-n-\ell} \) and \( |\beta| = s \),

\[
\frac{\partial^j u}{\partial n_f^\beta} = \beta! \sum_{\alpha \in S_\ell(f), |\alpha_f| = k-s} c_{\alpha_f, \beta} \lambda_f^{\alpha'}.
\]
The vanishing DoF (35) implies \[ \sum_{\alpha \in S_\ell(f), |\alpha_f| = k-s} c_{\alpha_f, \beta} \lambda_{\alpha_f}^\alpha |f| = 0. \] Hence \( c_{\alpha_f, \beta} = 0 \) for all \( |\alpha_f| = k-s, \alpha \in S_\ell(f) \). As \( \beta \) is arbitrary, we conclude all coefficients \( c_{\alpha_f, \alpha_f^*} = 0 \) and thus \( u = 0 \).  

For \( u \in \mathbb{P}_k(S_\ell(f) \cap L(f, s)) \) and \( \beta \in \mathbb{N}^{1:n-\ell} \) with \(|\beta| < s\), by Lemma 2.2, \( \partial^\beta |u|_f = 0 \). Applying the operator \( \partial^{\beta}(\cdot) |_f \) to the direct decomposition \( \mathbb{P}_k(S_\ell(f)) = \bigoplus_{s=0}^r \mathbb{P}_k(S_\ell(f) \cap L(f, s)) \) will possess a block lower triangular structure and leads to the following unisolvent result.

**Lemma 6.5.** Let \( \ell = 0, \ldots, n-1 \). The polynomial space \( \mathbb{P}_k(S_\ell(f)) \) is uniquely determined by DoFs

\[
\int f \frac{\partial^\beta u}{\partial \alpha_f^\beta} \lambda_{\alpha_f}^\alpha \, ds \quad \forall \alpha \in S_\ell(f), |\alpha_f| = k-s, \beta \in \mathbb{N}^{1:n-\ell}, \beta = s, s = 0, \ldots, r_\ell.
\]

Together with decomposition (31) of the polynomial space, we obtain the following result.

**Theorem 6.6.** Given integer \( m \geq 0 \), let non-negative integer array \( r = (r_0, r_1, \ldots, r_n) \) satisfy

\[
r_n = 0, \quad r_{n-1} = m, \quad r_\ell \geq 2r_{\ell+1} \text{ for } \ell = n-2, \ldots, 0.
\]

Let \( k \geq 2r_0 + 1 \geq 2^n m + 1 \). Then the shape function \( \mathbb{P}_k(T) \) is uniquely determined by the following DoFs

\[
D^\alpha u(v) \quad \alpha \in \mathbb{N}^{1:n}, |\alpha| \leq r_0, v \in \Delta_0(T),
\]

\[
\int f \frac{\partial^\beta u}{\partial \alpha_f^\beta} \lambda_{\alpha_f}^\alpha \, ds \quad \alpha \in S_\ell(f), |\alpha_f| = k-s, \beta \in \mathbb{N}^{1:n-\ell}, \beta = s,
\]

\[
f \in \Delta_\ell(T), \ell = 1, \ldots, n-1, s = 0, \ldots, r_\ell,
\]

\[
\int_T u \lambda^\alpha \, dx \quad \alpha \in S_n(T).
\]

**Proof.** Thanks to the decomposition (31), the dimensions match. Take \( u \in \mathbb{P}_k(T) \) satisfy all the DoFs (38)-(40) vanish. We are going to show \( u = 0 \).

For \( \alpha \in S_\ell(f) \) and \( e \in \Delta_\ell(T) \) with \( i \leq \ell \) and \( e \neq f \), by (33) and (34) we have \( |\alpha_e^*| \geq r_i + 1 \), hence \( \frac{\partial^\beta u}{\partial \alpha_e^\beta} |_e = 0 \) for \( \beta \in \mathbb{N}^{1:n-i} \) with \(|\beta| \leq r_i \). Again this tells us that applying the operator \( \frac{\partial^\beta(\cdot) |_f}{\partial \alpha_f^\beta} \) to the direct decomposition \( \mathbb{P}_k(T) = \bigoplus_{\ell=0}^n \bigoplus_{f \in \Delta_\ell(T)} \mathbb{P}_k(S_\ell(f)) \) will produce a block lower triangular structure. Then apply Lemma 6.5, we conclude \( u \in \mathbb{P}_k(S_n(T)) \), which together with the vanishing DoF (40) gives \( u = 0 \).  

**Remark 6.7.** For \( \alpha \in S_\ell(f) \), by (33) we have \( |\alpha_e| \leq k - r_{\ell-1} - 1 \) for all \( e \in \Delta_{\ell-1}(f) \), then \( \alpha_f \geq r_{\ell-1} + 1 - |\alpha_f^*| \), and

\[
\lambda^\alpha = \lambda_{\alpha_f}^\alpha, \quad \lambda_{f}^\alpha = \lambda_{\alpha_f}^\alpha \lambda_{f}^{r_{\ell-1}+1-|\alpha_f^*|} \lambda_{f}^{-r_{\ell-1}+1+|\alpha_f^*|}.
\]
As the two-dimensional case, using \( \alpha_f - (r_{\ell - 1} + 1) + |\alpha_f| \) as the new index, DoFs (39)-(40) can be replaced by

\[
\int_f \frac{\partial^\beta u}{\partial n^\beta_f} \lambda^{\alpha_f}_f \, ds \quad \beta \in \mathbb{N}^{1:n-\ell}, |\beta| = s, s = 0, \ldots, r_\ell, \quad \alpha \in \mathbb{N}^{1:m}\kappa_{-(\ell+1)(r_{\ell - 1} + 1) + s},
\]

\[
|\alpha_f| \leq k - r_i - 1 - (i + 1)(r_{\ell - 1} + 1 - s), \forall \varepsilon \in \Delta_i(f), i = 0, \ldots, \ell - 2,
\]

\[
f \in \Delta_\ell(T), \ell = 1, \ldots, n - 1,
\]

\[
\int_T u^\lambda \alpha \, dx \quad \alpha \in \mathbb{T}_{k-(n+1)(m+1)}^n,
\]

\[
|\alpha| \leq k - r_i - 1 - (i + 1)(m + 1), \forall \varepsilon \in \Delta_i(T), i = 0, \ldots, n - 2.
\]

Namely, we can remove bubble functions in the test function space.

6.3. Smooth finite elements in arbitrary dimension. Given a triangulation \( T_h \), the finite element space is obtained by asking the DoFs depending on the sub-simplex only.

**Theorem 6.8.** Given integer \( m \geq 0 \), let non-negative integer array \( r = (r_0, r_1, \cdots, r_n) \) satisfy

\[
r_n = 0, \quad r_{n-1} = m, \quad r_\ell \geq 2r_{\ell+1} \text{ for } \ell = n - 2, \ldots, 0.
\]

Let \( k \geq 2r_0 + 1 \geq 2^nm + 1 \). The following DoFs

\[(41) \quad D^\alpha u(v) \quad \alpha \in \mathbb{N}^{1:n}, |\alpha| \leq r_0, v \in \Delta_0(T_h), \]

\[(42) \quad \int_f \frac{\partial^\beta u}{\partial n^\beta_f} \lambda^{\alpha_f}_f \, ds \quad \alpha \in S_\ell(f), |\alpha| = k - s, \beta \in \mathbb{N}^{1:n-\ell}, |\beta| = s, s = 0, \ldots, r_\ell,
\]

\[
f \in \Delta_\ell(T_h), \ell = 1, \ldots, n - 1,
\]

\[(43) \quad \int_T u^\lambda \alpha \, dx \quad \alpha \in S_n(T), T \in T_h,
\]

will define a finite element space

\[
V_h = \{ u \in C^m(\Omega) \mid \text{DoFs (41) - (42) are single valued}, u|_T \in \mathbb{P}_k(T), \forall T \in T_h \}.
\]

**Proof.** Restricted to one simplex \( T \), by Theorem 6.6, DoFs (41)-(43) will define a function \( u \) s.t. \( u|_T \in \mathbb{P}_k(T) \). We only need to verify \( u \in C^m(\Omega) \). It suffices to prove \( \frac{\partial^\alpha u}{\partial n^\alpha} |_F \in \mathbb{P}_{k-i}(F) \), for all \( i = 0, \ldots, m \) and all \( F \in \Delta_{n-1}(T) \), are uniquely determined by (41)-(42) on \( F \).

Let \( w = \frac{\partial^\alpha u}{\partial n^\alpha} |_F \in \mathbb{P}_{k-i}(F) \). Consider the modified index sequence \( r^i_F = (r_0 - i, r_1 - i, \ldots, r_{n-2} - i, 0) \) and degree \( k^i = k - i \). Then \( k^i, r^i_F \) satisfies the condition in Theorem 6.3 and we obtain a direct decomposition of \( T_{k-i}^{n-1}(F) = \bigoplus_{r=0}^{n-1} \bigoplus_{f \in \Delta_i(F)} S^F_{r}(f) \), where

\[
S^F_0(v) = D(v, r_0 - i) \cap T_{k-i}^{n-1}(F),
\]

\[
S^F_{r_f}(f) = (D(f, r_\ell - i) \cap T_{k-i}^{n-1}(F)) \setminus \left( \bigoplus_{r=0}^{n-2} \bigoplus_{f \in \Delta_i(F)} S^F_r(f) \right), \quad \ell = 1, \ldots, n - 2,
\]

\[
S^F_{n-1} = T_{k-i}^{n-1}(F) \setminus \left( \bigoplus_{r=0}^{n-2} \bigoplus_{f \in \Delta_i(F)} S^F_r(f) \right).
\]
The DoFs (41)-(42) related to \( w \) are
\[
D^\alpha_F w(v) = \left. \frac{\partial^{|\alpha|}}{\partial n_{F,f}^{|\alpha|}} \lambda^j_f \right|_{s} ds, \quad \alpha \in \mathbb{N}^{1:n-1}, |\alpha| \leq r_0 - i, v \in \Delta_0(F),
\]
where \( D^\alpha_F w \) is the tangential derivatives of \( w \), and \( n_{F,f} \) is the normal vector of \( f \) but tangential to \( F \). Clearly the modified sequence \( \mathbf{r}_f \) still satisfies constraints required in Theorem 6.6. We can apply Theorem 6.6 with the shape function space \( \mathcal{P}_{k-i}(F) \) to conclude \( w \) is uniquely determined on \( F \). Thus the result \( u \subset C^m(\Omega) \) follows.

Counting the dimension of \( V_h \) is hard and not necessary. The cardinality of \( S_\ell(f) \) is difficult to determine due to the inequality constraints. In the implementation, compute the distance of lattice nodes to sub-simplexes and use a logic array to find out \( S_\ell(f) \). Our geometric decomposition using the distance not only simplifies the construction but also provide an easy way of implementation.

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