Truthful Data Acquisition via Peer Prediction

Yiling Chen
Harvard University
yiling@seas.harvard.edu

Yiheng Shen
Tsinghua University
shen-yh17@mails.tsinghua.edu.cn

Shuran Zheng
Harvard University
shuran_zheng@seas.harvard.edu

Abstract

We consider the problem of purchasing data for machine learning or statistical estimation. The data analyst has a fixed budget to purchase datasets from multiple data providers. She does not have any test data that can be used to evaluate the collected data and can assign payments to data providers solely based on the collected datasets. We consider the problem in the standard Bayesian paradigm and in two settings: (1) data are only collected once; (2) data are collected repeatedly and each day’s data are drawn independently from the same distribution. For both settings, our mechanisms guarantee that truthfully reporting one’s dataset is always an equilibrium, by adopting techniques from peer prediction: pay each provider the mutual information between his reported data and other providers’ reported data. Depending on the data distribution, the mechanisms can also discourage misreports that would lead to inaccurate predictions. Our mechanisms also guarantee individual rationality and budget feasibility for certain underlying distributions in the first setting and for all distributions in the second setting.

1 Introduction

Data has been the fuel of the success of machine learning and data science, which is becoming a major driving force for technological and economic growth. An important question is how to acquire high-quality data to enable learning and analysis when data are private possessions of data providers. Naively, we could issue a constant payment to data providers in exchange for their data. But data providers can report more or less data than they actually have or even misreport values of their data without affecting their received payments. Alternatively, if we have a test dataset, we could reward data providers according to how well the model trained on their reported data performs on the test data. However, if the test dataset is biased, this could potentially incentivize data providers to bias their reported data toward the test set, which will limit the value of the acquired data for other learning or analysis tasks. Moreover, a test dataset may not even be available in many settings.

In this work, we explore the design of reward mechanisms for acquiring high-quality data from multiple data providers when a data buyer doesn’t have access to a test dataset. The ultimate goal is that, with the designed mechanisms, strategic data providers will find that truthfully reporting their possessed dataset is their best action and manipulation will lead to lower expected rewards. To make the mechanisms practical, we also require our mechanisms to always have non-negative and bounded payments so that data providers will find it beneficial to participate in (a.k.a. individual rationality) and the data buyer can afford the payments.

In a Bayesian paradigm where data are generated independently conditioned on some unknown parameters, we design mechanisms for two settings: (1) data are acquired only once, and (2) data are...
acquired repeatedly and each day’s data are independent from the previous days’ data. For both settings, our mechanisms guarantee that truthfully reporting the datasets is always an equilibrium. For some models of data distributions, data providers in our mechanisms receive strictly lower rewards in expectation if their reported dataset leads to an inaccurate prediction of the underlying parameters. While this doesn’t strictly discourage manipulations of datasets that do not change the prediction of the parameters, it is a significant step toward achieving strict incentives for truthful reporting one’s datasets, an ideal goal, especially because finding a manipulation without affecting the prediction of the parameters can be difficult. Our mechanisms guarantee IR and budget feasibility for certain underlying distributions in the first setting and for any underlying distributions in the second setting.

Our mechanisms are built upon recent developments \cite{15} \cite{14} in the peer prediction literature. The insight is that if we reward a data provider the mutual information \cite{15} between his data and other providers’ data, then by the data processing inequality, if other providers report their data truthfully, this data provider will only decrease the mutual information, hence his reward, by manipulating his dataset. We extend the peer prediction idea developed by \cite{15} to the data acquisition setting, and to further guarantee IR and budget feasibility.

2 Related Work

The problem of purchasing data from people has been investigated with different focuses, e.g. privacy concerns \cite{12} \cite{9} \cite{11} \cite{21} \cite{7} \cite{26}, effort and cost of data providers \cite{23} \cite{4} \cite{1} \cite{5}, reward allocation \cite{10} \cite{2}. Our work is the first to consider rewarding data without (good) test data that can be used to evaluate the quality of reported data. Similar to our setting, \cite{10} \cite{2} consider paying to multiple data providers in a machine learning task. They use a test set to assess the contribution of subsets of data and then propose a fair measurement of the value of each data point in the dataset, which is based on the Shapley value in game theory. Both of the works do not formally consider the incentive compatibility of payment allocation. \cite{26} proposes a market framework that purchases hypotheses for a machine learning problem when the data is distributed among multiple agents. Again they assume that the market has access to some true samples and the participants are paid with their incremental contributions evaluated by these true samples.

The main techniques of this work come from the literature of peer prediction \cite{17} \cite{22} \cite{8} \cite{24} \cite{15} \cite{14} \cite{16} \cite{13}. Peer prediction is the problem of information elicitation without verification. The participants receive correlated signals of an unknown ground truth and the goal is to elicit the true signals from the participants. In our problem, the dataset can be viewed as a signal of the ground truth. What makes our problem more challenging than the standard peer prediction problem is that (1) the signal space is much larger and (2) the correlation between signals is more complicated. Standard peer prediction mechanisms either require the full knowledge of the underlying signal distribution, or make assumptions on the signal distribution that are not applicable to our problem. \cite{14} applies the peer prediction method to the co-training problem, in which two participants are asked to submit forecasts of latent labels in a machine learning problem. Our work is built upon the main insights of \cite{14}. We discuss the differences between our model and theirs in the model section, and show how their techniques are applied in the result sections.

Our work is also related to Multi-view Learning (see \cite{27} for a survey). But our work focuses on the data acquisition, but not the machine learning methods used on the (multi-view) data.

3 Model

A data analyst wants to gather data for some future statistical estimation or machine learning tasks. There are \(n\) data providers. The \(i\)-th data provider holds a dataset \(D_i\) consisting of \(N_i\) data points \(d_i^{(1)}, \ldots, d_i^{(N_i)}\) with support \(D_i\). The data generation follows a standard Bayesian process. For each data set \(D_i\), data points \(d_i^{(j)} \in D_i\) are i.i.d. samples conditioned on some unknown parameters \(\theta \in \Theta\). Let \(p(\theta, D_1, \ldots, D_n)\) be the joint distribution of \(\theta\) and \(n\) data providers’ datasets. We consider two types of spaces for \(\Theta\) in this paper: (1) \(\theta\) has finite support, i.e., \(|\Theta| = m\) is finite, and (2) \(\theta\) has continuous support, i.e. \(\Theta \subseteq \mathbb{R}^m\). For the case of continuous support, \footnote{This means that a data provider can report a different dataset without changing his reward as long as the dataset leads to the same prediction for the underlying parameters as his true dataset.}
to alleviate computational issues, we consider a widely used class of distributions, the *exponential family*.

The data analyst’s goal is to incentivize the data providers to give their true datasets with a budget $B$. She needs to design a payment rule $r_i(D_1, \ldots, D_n)$ for $i \in [n]$ that decides how much to pay data provider $i$ according to all the reported datasets $D_1, \ldots, D_n$. The payment rule should ideally incentivize truthful reporting, that is, $D_i = D_i$ for all $i$.

Before we formally define the desirable properties of a payment rule, we note that the analyst will have to leverage the correlation between people’s data to distinguish a misreported dataset from a true dataset because all she has access to is the reported datasets. To make the problem tractable, we thus make the following assumption about the data correlation: parameters $\theta$ contains all the mutual information between the datasets. More formally, the datasets are independent conditioned on $\theta$.

**Assumption 3.1.** $D_1, \ldots, D_n$ are independent conditioned on $\theta$, $p(D_1, \ldots, D_n|\theta) = p(D_1|\theta) \cdots p(D_n|\theta)$.

This is definitely not an assumption that would hold for arbitrarily picked parameters $\theta$ and any datasets. One can easily find cases where the datasets are correlated to some parameters other than $\theta$. So the data analyst needs to carefully decide what to include in $\theta$ and $D_i$, by either expanding $\theta$ to include all relevant parameters or reducing the content of $D_i$ to exclude all redundant data entries that can cause extra correlations.

**Example 3.1.** Consider the linear regression model where provider $i$’s data points $d_{ij} = (x_{ij}, y_{ij})$ consist of a feature vector $x_{ij}$ and a label $y_{ij}$. We have a linear model

$$y_{ij} = \theta^T x_{ij} + \epsilon_{ij}.$$  

Then datasets $D_1, \ldots, D_n$ will be independent conditioning on $\theta$ as long as (1) different data providers draw their feature vectors independently, i.e., $x_{ij}^{(1)}, \ldots, x_{ij}^{(n)}$ are independent for all $j_1 \in [N_1], \ldots, j_n \in [N_n]$, and (2) the noises are independent.

We further assume that the data analyst has some insight about the data generation process.

**Assumption 3.2.** The data analyst possesses a commonly accepted prior $p(\theta)$ and a commonly accepted model for data generating process so that she can compute the posterior $p(\theta|D_i)$, $\forall i, D_i$.

When $|\theta|$ is finite, $p(\theta|D_i)$ can be computed as a function of $p(\theta|d_i)$ using the method in Appendix B. For a model in the exponential family, $p(\theta|D_i)$ can be computed as in Definition 4.2.

Note that we do not always require the data analyst to know the whole distribution $p(D_i|\theta)$, it suffices for the data analyst to have the necessary information to compute $p(\theta|D_i)$.

**Example 3.2.** Consider the linear regression model in Example 3.1. We use $x_i$ to represent all the features in $D_i$ and use $y_i$ to represent all the labels in $D_i$. If the features $x_i$ are independent from $\theta$, the data analyst does not need to know the distribution of $x_i$. It suffices to know $p(y_i|x_i, \theta)$ to compute $p(\theta|D_i)$ because

$$p(\theta|x_i, y_i) \propto p((x_i, y_i)|\theta)p(\theta) = p(y_i|x_i, \theta)p(\theta|x_i)p(\theta) = p(y_i|x_i, \theta)p(\theta|x_i)p(\theta).$$

Finally we assume that the identities of the providers can be verified.

**Assumption 3.3.** The data analyst can verify the data providers’ identities, so one data provider can only submit one dataset and get one payment.

We now formally introduce some desirable properties of a payment rule. We say that a payment rule is *truthful* if reporting true datasets is a weak equilibrium, that is, when the others report true datasets, it is also (weakly) optimal for me to report the true dataset (based on my own belief).

**Definition 3.1 (Truthfulness).** Let $D_{-i}$ be the datasets of all providers except $i$. A payment rule $r(D_1, \ldots, D_n)$ is truthful if: for any (commonly accepted model of) underlying distribution $p(\theta, D_1, \ldots, D_n)$, for every data provider $i$ and any realization of his dataset $D_i$, when all other data providers truthfully report $D_{-i}$, truthfully reporting $D_i$ leads to the highest expected payment, where the expectation is taken over the distribution of $D_{-i}$ conditioned on $D_i$, i.e.,

$$\mathbb{E}_{D_{-i} \sim p(D_{-i}|D_i)}[r_i(D_i, D_{-i})] \geq \mathbb{E}_{D_{-i} \sim p(D_{-i}|D_i')}[r_i(D_i', D_{-i})], \quad \forall i, D_i, D_i'.$$
Furthermore, let \( \Delta \) be the analyst’s budget \( B \).

A more ideal property would be that the expected payment is strictly lower for any dataset \( D \), when the reported data does not give the accurate prediction of \( \theta \).

Because truthfulness is defined as a weak equilibrium, it does not necessarily discourage misreporting. So, we want a stronger guarantee than truthfulness. We thus define sensitivity: the expected payment should be strictly lower when the reported data does not give the accurate prediction of \( \theta \).

**Definition 3.2 (Sensitivity).** A payment rule \( r(D_1, \ldots, D_n) \) is sensitive if for any (commonly accepted model of) underlying distribution \( p(\theta, D_1, \ldots, D_n) \), for any provider \( i \) and any realization of his dataset \( D_i \), when all other providers \( j \neq i \) report \( \tilde{D}_j \) with accurate posterior \( p(\theta|\tilde{D}_j(D_j)) = p(\theta|D_j) \), we have (1) truthfully reporting \( D_i \) leads to the highest expected payment

\[
E_{D_i \sim p(D_i|D_j)}[r_i(D_i, \tilde{D}_{-i}(D_{-i}))] \geq E_{D_i \sim p(D_i|D_j)}[r_i(D_i', \tilde{D}_{-i}(D_{-i}))], \forall D_i'
\]

and (2) reporting a dataset \( D_i' \) with inaccurate posterior \( p(\theta|D_i') \neq p(\theta|D_i) \) is strictly worse than reporting a dataset \( \tilde{D}_i \) with accurate posterior

\[
E_{D_i \sim p(D_i|D_j)}[r_i(D_i, \tilde{D}_{-i}(D_{-i}))] > E_{D_i \sim p(D_i|D_j)}[r_i(D_i', \tilde{D}_{-i}(D_{-i}))],
\]

Furthermore, let \( \Delta_i = p(\theta|D_i') - p(\theta|D_i) \), a payment rule is \( \alpha \)-sensitive for agent \( i \) if

\[
E_{D_i \sim p(D_i|D_j)}[r_i(D_i, \tilde{D}_{-i}(D_{-i}))] - E_{D_i \sim p(D_i|D_j)}[r_i(D_i', \tilde{D}_{-i}(D_{-i}))] \geq \alpha \|\Delta_i\|,
\]

for all \( D_i, D_i' \) and reports \( \tilde{D}_{-i}(D_{-i}) \) that give the accurate posteriors.

Our definition of sensitivity guarantees that at an equilibrium, the reported datasets must give the correct posteriors \( p(\theta|\tilde{D}_i) = p(\theta|D_i) \). We can further show that at an equilibrium, the analyst will get the accurate posterior \( p(\theta|D_1, \ldots, D_n) \).

**Lemma 3.1.** When \( D_1, \ldots, D_n \) are independent conditioned on \( \theta \), for any \( (D_1, \ldots, D_n) \) and \( (\tilde{D}_1, \ldots, \tilde{D}_n) \), if \( p(\theta|D_i) = p(\theta|\tilde{D}_i) \) \( \forall i \), then \( p(\theta|D_1, \ldots, D_n) = p(\theta|\tilde{D}_1, \ldots, \tilde{D}_n) \).

A more ideal property would be that the expected payment is strictly lower for any dataset \( D_i' \neq D_i \). Mechanisms that satisfy sensitivity can be viewed as an important step toward this ideal goal, as the only possible payment-maximizing manipulations are to report a dataset \( \tilde{D}_i \) that has the correct posterior \( p(\theta|\tilde{D}_i) = p(\theta|D_i) \). Arguably, finding such a manipulation can be challenging. Sensitivity guarantees the accurate prediction of \( \theta \) at an equilibrium.

Second, we want a fair payment rule that is indifferent to data providers’ identities.

**Definition 3.3 (Symmetry).** A payment rule \( r \) is symmetric if for all permutation of \( n \) elements \( \pi(\cdot) \),

\[
r_i(D_1, \ldots, D_n) = r_{\pi(i)}(D_{\pi(1)}, \ldots, D_{\pi(n)}) \text{ for all } i.
\]

Third, we want non-negative payments and we want to use all the budget.

**Definition 3.4 (IR and fixed budget).** A payment rule \( r \) is individually rational if \( r_i(D_1, \ldots, D_n) \geq 0 \), \( \forall i, D_1, \ldots, D_n \). A payment rule \( r \) is budget-fixed if \( \sum_{i=1}^n r_i(D_1, \ldots, D_n) = B, \forall D_1, \ldots, D_n \).

We will consider two acquisition settings in this paper:

**One-time data acquisition.** The data analyst collects data in one batch. In this case, our problem is very similar to the single-task forecast elicitation in [14]. But our model considers the budget feasibility and the IR, whereas they only consider the truthfulness of the mechanism.

**Multiple-time data acquisition.** The data analyst repeatedly collects data for \( T \geq 2 \) days. On day \( t \), \( (\theta^{(t)}, D_1^{(t)}, \ldots, D_n^{(t)}) \) is drawn independently from the same distribution \( p(\theta, D_1, \ldots, D_n) \). The analyst has a budget \( B^{(t)} \) and wants to know the posterior of \( \theta^{(t)} \), \( p(\theta^{(t)}|D_1^{(t)}, \ldots, D_n^{(t)}) \). In this case, our setting differs from the multi-task forecast elicitation in [14] because providers can

---

2 A constant payment rule is just a trivial truthful payment rule.

3 Using a fixed test set may encourage misreporting.
decide their strategies on a day based on all the observed historical data before that day. The multi-task forecast elicitation in [14] asks the agents to submit forecasts of latent labels in multiple similar independent tasks. It is assumed that the agent’s forecast strategy for one task only depends on his information about that task but not the information about other tasks.

4 Preliminary

In this section, we introduce some necessary background for developing our mechanisms. We first give the definitions of exponential family distributions. Our designed mechanism will leverage the idea of mutual information between reported datasets to incentivize truthful reporting.

4.1 Exponential Family

Definition 4.1 (Exponential family [19]). A likelihood function \( p(x|\theta) \), for \( x = (x_1, \ldots, x_n) \in \mathcal{X}^n \) and \( \theta \in \Theta \subseteq \mathbb{R}^m \) is said to be in the exponential family in canonical form if it is of the form

\[
p(x|\theta) = \frac{1}{Z(\theta)} h(x) \exp \left[ \theta^T \phi(x) \right] \quad \text{or} \quad p(x|\theta) = h(x) \exp \left[ \theta^T \phi(x) - A(\theta) \right]
\]

(1)

Here \( \phi(x) \in \mathbb{R}^m \) is called a vector of sufficient statistics, \( Z(\theta) = \int_{\mathcal{X}^n} h(x) \exp \left[ \theta^T \phi(x) \right] \) is called the partition function, \( A(\theta) = \ln Z(\theta) \) is called the log partition function.

In Bayesian probability theory, if the posterior distributions \( p(\theta|x) \) are in the same probability distribution family as the prior probability distribution \( p(\theta) \), the prior and posterior are then called conjugate distributions, and the prior is called a conjugate prior for the likelihood function.

Definition 4.2 (Conjugate prior for the exponential family [19]). For a likelihood function in the exponential family \( p(x|\theta) = h(x) \exp \left[ \theta^T \phi(x) - A(\theta) \right] \). The conjugate prior for \( \theta \) with parameters \( \nu_0, \tau_0 \) is of the form

\[
p(\theta) = \mathcal{P}(\theta|\nu_0, \tau_0) = g(\nu_0, \tau_0) \exp \left[ \nu_0 \theta^T \tau_0 - \nu_0 A(\theta) \right].
\]

(2)

Let \( \phi = \frac{1}{n} \sum_{i=1}^n \phi(x_i) \). Then the posterior of \( \theta \) can be represented in the same form as the prior

\[
p(\theta|x) \propto \exp \left[ \theta^T (\nu_0 \tau_0 + n \phi) - (\nu_0 + n) A(\theta) \right] = \mathcal{P}(\theta|\nu_0 + n, \nu_0 \tau_0 + n \phi).
\]

where \( \mathcal{P}(\theta|\nu_0 + n, \nu_0 \tau_0 + n \phi) \) is the conjugate prior with parameters \( \nu_0 + n \) and \( \nu_0 \tau_0 + n \phi \).

A lot of commonly used distributions belong to the exponential family, Gaussian, Multinoulli, Multinomial, Geometric, etc. Due to the space limit, we introduce only the definitions and refer the readers who are not familiar with the exponential family to [19] for more details.

4.2 Mutual Information

We will use the point-wise mutual information defined in [14]. We introduce this notion of mutual information in the context of our problem.

Definition 4.3 (Point-wise mutual information). We define the point-wise mutual information between two datasets \( D_1 \) and \( D_2 \) to be

\[
PMI(D_1, D_2) = \int_{\Theta \subseteq \mathbb{R}^m} \frac{p(\theta|D_1)p(\theta|D_2)}{p(\theta)} d\theta.
\]

(3)

For finite case, we define \( PMI(D_1, D_2) = \sum_{\theta \in \Theta} \frac{p(\theta|D_1)p(\theta|D_2)}{p(\theta)} d\theta. \)

When \( |\Theta| \) is finite or a model in the exponential family is used, the PMI will be computable.

\[\text{This is not to say that the providers will update their prior for } \theta^{(t)} \text{ using the data on first } t - 1 \text{ days. Because we assume that } \theta^{(t)} \text{ is independent from } \theta^{(t-1)}, \ldots, \theta^{(t-1)}, \text{ so the data on first } t - 1 \text{ days contains no information about } \theta^{(t)}. \text{ We use the same prior } p(\theta) \text{ throughout all } T \text{ days. What it means is that when the analyst decides the payment for day } t \text{ not only based on the report on day } t \text{ but also the historical reports, the providers may also use different strategies for different historical reports.}\]
Theorem 5.1. Mechanism 1 is IR, truthful, budget-fixed, symmetric. Works as follows.

In this work we do not assume that payments and guarantee truthfulness. In Appendix C.1, we give an example of such mechanisms.

In this section we apply [14]'s 5 One-time Data Acquisition

Mechanism 1:

One-time data collecting mechanism.

1. Ask all data providers to report their datasets \( D_1, \ldots, D_n \).
2. If \( D_i \in \mathcal{D}_i(D_{-i}) \), we compute a score for his dataset \( s_i = \log PMI(D_i, \bar{D}_{-i}) \).
3. The final payment for agent \( i \) is:

   \[
   r_i(D_1, \ldots, D_n) = \frac{\log \left( 1 + \frac{s_i}{R - L} \sum_{j \neq i} s_j \right)}{n}; \text{ otherwise } r_i(D_1, \ldots, D_n) = 0.
   \]

4. Note that by Proposition 4.1, the expected payment for a data provider is decided by the mutual information between his data and other people’s data. The payments are efficiently computable for finite-size \( \Theta \) and for models in exponential family (Lemma 4.1).

For single-task forecast elicitation, [14] proposes a truthful payment rule.

Definition 4.4 (log-PMI payment [14]). Suppose there are two data providers reporting \( \bar{D}_A \) and \( \bar{D}_B \) respectively. Then the log-PMI rule pays them \( r_A = r_B = \log (PMI(\bar{D}_A, \bar{D}_B)) \).

Proposition 4.1. When the log-PMI rule is used, the expected payment equals the mutual information between \( \bar{D}_A \) and \( \bar{D}_B \), where the expectation is taken over the distribution of \( \bar{D}_A \) and \( \bar{D}_B \).

5 One-time Data Acquisition

In this section we apply [14]'s log-PMI payment rule to our one-time data acquisition problem. The log-PMI payment rule ensures truthfulness, but its payment can be negative or unbounded or even ill-defined. So we mainly focus on the mechanism’s sensitivity, budget-feasibility and IR. To guarantee budget feasibility and IR, our mechanism requires a lower bound and an upper bound of PMI, which may be difficult to find for some models in the exponential family.

If the analyst knows the distribution \( p(D_1 | \theta) \), then she will be able to compute \( p(D_{-i} | D_i) = \sum_\theta p(D_{-i} | \theta) p(\theta | D_i) \). In this case, we can employ peer prediction mechanisms [17] to design payments and guarantee truthfulness. In Appendix C.1 we give an example of such mechanisms.

In this work we do not assume that \( p(D_1 | \theta) \) is known (see Example 3.2). When \( p(D_1 | \theta) \) is unknown but the analyst can compute \( p(\theta | D_i) \), our idea is to use the log-PMI payment rule in [14] and then add a normalization step to ensure budget-fixability and IR. However the log-PMI will be ill-defined if \( PMI = 0 \). To avoid this, for each possible \( D_{-i} \), we define set \( \mathcal{D}_i(D_{-i}) = \{ D_i | PMI(D_i, D_{-i}) > 0 \} \) and the log-PMI will only be computed for \( D_i \in \mathcal{D}_i(D_{-i}) \).

The normalization step will require an upper bound \( R \) and lower bound \( L \) of the log-PMI payment. If \( |\Theta| \) is finite, we can find a lower bound and an upper bound in polynomial time, which we prove in Appendix C.2. When a model in the exponential family is used, it is more difficult to find \( L \) and \( R \). By Lemma 4.1 if the \( g \) function is bounded, we will be able to bound the payment. For example, if we are estimating the mean of a univariate Gaussian with known variance, \( L \) and \( R \) will be bounded if the number of data points is bounded. Details can be found in Appendix C.3. Our mechanism works as follows.

Mechanism 1: One-time data collecting mechanism.

1. Ask all data providers to report their datasets \( D_1, \ldots, D_n \).
2. If \( D_i \in \mathcal{D}_i(D_{-i}) \), we compute a score for his dataset \( s_i = \log PMI(D_i, \bar{D}_{-i}) \).
3. The final payment for agent \( i \) is:

   \[
   r_i(D_1, \ldots, D_n) = \frac{\log \left( 1 + \frac{s_i}{R - L} \sum_{j \neq i} s_j \right)}{n}; \text{ otherwise } r_i(D_1, \ldots, D_n) = 0.
   \]

5.2 Note that by Proposition 4.1, the expected payment for a data provider is decided by the mutual information between his data and other people’s data. The payments are efficiently computable for finite-size \( \Theta \) and for models in exponential family (Lemma 4.1).

Next, we discuss the sensitivity. We first define some notations. When \( |\Theta| \) is finite, let \( Q_{-i} \) be a \((\prod_{j \in [n], j \neq i} |D_j|) \times |\Theta|\) matrix that represents the conditional distribution of \( \theta \) conditioning on \( D_i \).
every realization of $D_{-i}$. So the element in row $D_{-i}$ and column $\theta$ is equal to $p(\theta|D_{-i})$. We also define the data generating matrix $G_i$ with $|D_i|$ rows and $|\theta|$ columns. Each row corresponds to a possible data point $d_i \in D_i$ in the dataset and each column corresponds to a $\theta \in \Theta$. The element in the row corresponding to data point $d_i$ and the column $\theta$ is $p(\theta|d_i)$.

We give the sufficient condition for the mechanism to be sensitive.

**Theorem 5.2.** When $|\Theta|$ is finite, Mechanism $\mathcal{M}$ is sensitive if for all $i$, $Q_{-i}$ has rank $|\Theta|$.

Since the size of $Q_{-i}$ can be exponentially large, it may be computationally infeasible to check the rank of $Q_{-i}$. We thus give a simpler condition that only uses $G_i$, which has a polynomial size. This simpler condition also shows that it is easy for Mechanism $\mathcal{M}$ to be sensitive.

**Definition 5.1.** The Kruskal rank (or $k$-rank) of a matrix $M$, denoted by $\text{rank}_k(M)$, is the maximal number $r$ such that any set of $r$ columns of $M$ is linearly independent.

**Corollary 5.1.** When $|\Theta|$ is finite, Mechanism $\mathcal{M}$ is sensitive if for all $i$, $\sum_{j \neq i} \text{rank}_k(G_j) (N_j - 1) + 1 \geq |\Theta|$, where $N_j$ is the number of data points in $D_j$.

In Appendix [C.5.1] we also give a lower bound for $\alpha$ so that Mechanism $\mathcal{M}$ is $\alpha$-sensitive.

When $\Theta \subseteq \mathbb{R}^m$, it becomes more difficult to guarantee sensitivity. Suppose the data analyst uses a model from the exponential family so that the prior and all the posterior of $\theta$ can be written in the form in Lemma [4.1]. The sensitivity of the mechanism will depend on the normalization term $g(\nu, \tau)$ (or equivalently, the partition function) of the pdf. More specifically, define

$$h_{D_{-i}}(\nu_i, \tau_i) = \frac{g(\nu_i, \tau_i)}{g(\nu_i + \nu_{-i} - \nu_0, \nu_i + \nu_{-i} - \nu_0)}, \quad (4)$$

then we have the following sufficient and necessary conditions for the sensitivity of the mechanism.

**Theorem 5.3.** When $\Theta \subseteq \mathbb{R}^m$, if the data analyst uses a model in the exponential family, then Mechanism $\mathcal{M}$ is sensitive if and only if for any $(\nu_i', \tau_i') \neq (\nu_i, \tau_i)$, we have $\Pr_{D_{-i}}[h_{D_{-i}}(\nu_i', \tau_i') \neq h_{D_{-i}}(\nu_i, \tau_i)] > 0$.

The theorem basically means that the mechanism will be sensitive if any pairs of different reports that will lead to different posteriors of $\theta$ can be distinguished by $h_{D_{-i}}(\cdot)$ with non-zero probability. However, for different models in the exponential family, this is not always true. For example, if we estimate the mean $\mu$ of a univariate Gaussian with a known variance and the Gaussian conjugate prior is used, then the normalization term only depends on the variance but not the mean, so in this case $h(\cdot)$ can only detect the change in variance, which means that the mechanism will be sensitive to replication and withholding, but not necessarily other types of manipulations. But if we estimate the mean of a Bernoulli distribution whose conjugate prior is the Beta distribution, then the partition function will be the Beta function, which can detect different posteriors and thus the mechanism will be sensitive. See Appendix [C.4] for more details. The missing proofs can be found in Appendix [C.5].

## 6 Multiple-time Data Acquisition

Now we consider the case when the data analyst needs to repeatedly collect data for the same task. At day $t$, the analyst has a budget $B(t)$ and a new ensemble $(\theta^{(t)}, D_1^{(t)}, \ldots, D_n^{(t)})$ is drawn from the same distribution $p(\theta, D_1, \ldots, D_n)$, independent of the previous data. Again we assume that the data generating distribution $p(D_i | \theta)$ can be unknown, but the analyst is able to compute $p(\theta | D_i)$ after seeing the data. (See Example [3.2].) The data analyst can use the one-time purchasing mechanism (Section [5]) at each round. But we show that if the data analyst can give the payment one day after the data is reported, a broader class of mechanisms can be used to guarantee the desirable properties, which guarantees bounded payments without any assumptions on the underlying distribution.

Our method uses the $f$-mutual information gain in [14] for multi-task forecast elicitation. The payment is specified by a differentiable convex function $f : \mathbb{R} \rightarrow \mathbb{R}$ and its convex conjugate $f^*$.

**Definition 6.1 (Convex conjugate).** For any function $f : \mathbb{R} \rightarrow \mathbb{R}$, define the convex conjugate function of $f$ as

$$f^*(y) = \sup_x xy - f(x).$$
On day $t$, the data providers are first asked to report their data for day $t$. Then for each provider $i$, we use the other providers’ reported data on day $t - 1$ and day $t$ to evaluate provider $i$’s reported data on day $t - 1$, that is, use $\tilde{D}_{-i}^{(t-1)}$ and $\bar{D}_{-i}^{(t)}$ to evaluate $\tilde{D}_{i}^{(t-1)}$. A score $s_i$ will be computed for each provider’s reported dataset $\tilde{D}_{i}^{(t-1)}$ on day $t - 1$.

The score $s_i$ for dataset $\tilde{D}_{i}^{(t-1)}$ is defined as the difference between its point-wise mutual information (defined in Definition 4.3) with $\tilde{D}_{-i}^{(t-1)}$ and its point-wise mutual information with $\bar{D}_{i}^{(t)}$, where the difference is specified by a pair of functions $(f'(\cdot), f^*(f'(\cdot)))$,

$$s_i = f'(PMI(\tilde{D}_{i}^{(t-1)}, \tilde{D}_{-i}^{(t-1)})) - f^*(f'(PMI(\bar{D}_{i}^{(t)}, \tilde{D}_{-i}^{(t)}))).$$

(5)

The pair $(f'(\cdot), f^*(f'(\cdot)))$ serves as a pair of distinguishers for the two distributions $p(D_{-i}|D_i)$ and $p(D_{-i})$. The score $s_i$ basically represents the difference (between $p(D_{-i}|D_i)$ and $p(D_{-i})$) that is identified by this pair of distinguishers based on the reported data.

According to the definition (5), if we carefully choose the convex function $f$ to be a differentiable convex function with a bounded derivative $f' \in [0, U]$ and with the convex conjugate $f^*$ bounded on $[0, U]$ then the scores $s_1, \ldots, s_n$ will always be bounded. We can then normalize $s_1, \ldots, s_n$ so that the payments are non-negative and sum up to $B^{(t-1)}$. Here we give one possible choice of $f'$ that can guarantee bounded scores: the Logistic function $\frac{1}{1+e^{-x}}$.

| $f(x)$   | $f'(x)$ | range of $f'(x)$ | $f^*(x)$ | range of $f^*(x)$ |
|----------|---------|------------------|----------|------------------|
| $\ln(1 + e^x)$ | $\frac{1}{1+e^{-x}}$ | $[\frac{1}{2}, 1]$ on $\mathbb{R}^+$ | $x \ln x + (1-x) \ln(1-x)$ | $[-\ln 2, 0]$ on $[\frac{1}{2}, 1]$ |

Finally, if day $t$ is the last day, we adopt the one-time mechanism to pay for day $t$’s data as well.

**Mechanism 2:** Multi-time data collecting mechanism.

Given a differentiable convex function $f$ with $f' \in [0, U]$ and $f^*$ bounded on $[0, U]$ for $t = 1, \ldots, T$ do

1. On day $t$, ask all data providers to report their datasets $\tilde{D}_{1}^{(t)}, \ldots, \tilde{D}_{n}^{(t)}$.
2. If $t$ is the last day $t = T$, use the payment rule of Mechanism 1 to pay for day $T$’s data or just give each data provider $B^{(T)}/n$.
3. If $t > 1$, give the payments for day $t - 1$ as follows. First compute all the scores $s_1, \ldots, s_n$ as in (5). Then normalize the scores so that the total payment is equal to the budget $B^{(t-1)}$. Let the range of the scores be $[L, R]$. Assign payments

$$r_i(\tilde{D}_{1}^{(t-1)}, \ldots, \tilde{D}_{n}^{(t-1)}) = \frac{B^{(t-1)}}{n} \left(1 + \frac{s_i - L}{R - L} \Sigma_{j \neq i} s_j \right).$$

end for

Our first result is that Mechanism 2 guarantees all the basic properties of a desirable mechanism.

**Theorem 6.1.** Given any differentiable convex function $f$ that has (1) a bounded derivative $f' \in [0, U]$ and (2) the convex conjugate $f^*$ bounded on $[0, U]$, Mechanism 2 is IR, budget-fixed, truthful and symmetric in all $T$ rounds.

If we choose computable $f'$ and $f^*$ (e.g. $f'$ equal to the Logistic function), the payments will also be computable for finite-size $\Theta$ and for models in exponential family (Lemma 4.1). Mechanism 2 has basically the same sensitivity guarantee as Mechanism 1 in the first $T - 1$ rounds. We defer the sensitivity analysis to Appendix 12.1. The missing proofs in this section can be found in Appendix 12.2.

**7 Discussion**

The limitations of our method point towards future directions. First, our one-time data acquisition mechanism requires a lower bound and an upper bound of PMI, which may be difficult to find for some models in the exponential family. A natural question is: can we find a mechanism that would work for any data distribution, just as our multi-time data acquisition mechanism? Another interesting direction is to design stronger mechanisms to strengthen the sensitivity guarantees in this
work. Finally, our method incentivizes truthful reporting, but it is not guaranteed that datasets that give more accurate posteriors will receive higher payments (in expectation). It would be fairer if the mechanism could have this property as well.
References

[1] Jacob Abernethy, Yiling Chen, Chien-Ju Ho, and Bo Waggoner. Low-cost learning via active data procurement. In Proceedings of the Sixteenth ACM Conference on Economics and Computation, EC ’15, pages 619–636, New York, NY, USA, 2015. ACM.

[2] Anish Agarwal, Munther Dahleh, and Tuhin Sarkar. A marketplace for data: An algorithmic solution. In Proceedings of the 2019 ACM Conference on Economics and Computation, pages 701–726, 2019.

[3] Glenn W Brier. Verification of forecasts expressed in terms of probability. Monthly weather review, 78(1):1–3, 1950.

[4] Y. Cai, C. Daskalakis, and C. H. Papadimitriou. Optimum statistical estimation with strategic data sources. In Proceedings of the 28th Conference on Learning Theory, pages 280–296, 2015.

[5] Yiling Chen, Nicole Immorlica, Brendan Lucier, Vasilis Syrgkanis, and Juba Ziani. Optimal data acquisition for statistical estimation. In Proceedings of the 2018 ACM Conference on Economics and Computation, EC ’18, pages 27–44, New York, NY, USA, 2018. ACM.

[6] Yiling Chen and Shuran Zheng. Prior-free data acquisition for accurate statistical estimation. In Proceedings of the 2019 ACM Conference on Economics and Computation, pages 659–677, 2019.

[7] Rachel Cummings, Stratis Ioannidis, and Katrina Ligett. Truthful linear regression. In Proceedings of the 28th Conference on Learning Theory, pages 448–483, 2015.

[8] Anirban Dasgupta and Arpita Ghosh. Crowdsourced judgement elicitation with endogenous proficiency. In Proceedings of the 22nd international conference on World Wide Web, pages 319–330, 2013.

[9] Lisa Fleischer and Yu-Han Lyu. Approximately optimal auctions for selling privacy when costs are correlated with data. CoRR, abs/1204.4031, 2012.

[10] Amirata Ghorbani and James Zou. Data shapley: Equitable valuation of data for machine learning. arXiv preprint arXiv:1904.02868, 2019.

[11] Arpita Ghosh, Katrina Ligett, Aaron Roth, and Grant Schoenebeck. Buying private data without verification. CoRR, abs/1404.6003, 2014.

[12] Arpita Ghosh and Aaron Roth. Selling privacy at auction. In Proceedings of the 12th ACM Conference on Electronic Commerce, EC ’11, pages 199–208, New York, NY, USA, 2011. ACM.

[13] Yuqing Kong. Dominantly truthful multi-task peer prediction with a constant number of tasks. In Proceedings of the Fourteenth Annual ACM-SIAM Symposium on Discrete Algorithms, pages 2398–2411. SIAM, 2020.

[14] Yuqing Kong and Grant Schoenebeck. Water from two rocks: Maximizing the mutual information. In Proceedings of the 2018 ACM Conference on Economics and Computation, pages 177–194, 2018.

[15] Yuqing Kong and Grant Schoenebeck. An information theoretic framework for designing information elicitation mechanisms that reward truth-telling. ACM Transactions on Economics and Computation (TEAC), 7(1):1–33, 2019.

[16] Yang Liu, Juntao Wang, and Yiling Chen. Surrogate scoring rules. arXiv preprint arXiv:1802.09158, 2018.

[17] Nolan Miller, Paul Resnick, and Richard Zeckhauser. Eliciting informative feedback: The peer-prediction method. Management Science, 51(9):1359–1373, 2005.

[18] Kevin P Murphy. Conjugate bayesian analysis of the gaussian distribution. 1(2σ2):16.

[19] Kevin P Murphy. Machine learning: a probabilistic perspective. MIT press, 2012.

[20] XuanLong Nguyen, Martin J Wainwright, and Michael I Jordan. Estimating divergence functionals and the likelihood ratio by convex risk minimization. IEEE Transactions on Information Theory, 56(11):5847–5861, 2010.
[21] Kobbi Nissim, Salil Vadhan, and David Xiao. Redrawing the boundaries on purchasing data from privacy-sensitive individuals. In Proceedings of the 5th Conference on Innovations in Theoretical Computer Science, ITCS ’14, pages 411–422, New York, NY, USA, 2014. ACM.

[22] Dražen Prelec. A bayesian truth serum for subjective data. science, 306(5695):462–466, 2004.

[23] Aaron Roth and Grant Schoenebeck. Conducting truthful surveys, cheaply. In Proceedings of the 13th ACM Conference on Electronic Commerce, EC’12, 2012.

[24] Victor Shnayder, Arpit Agarwal, Rafael Frongillo, and David C Parkes. Informed truthfulness in multi-task peer prediction. In Proceedings of the 2016 ACM Conference on Economics and Computation, pages 179–196, 2016.

[25] Nicholas D Sidiropoulos and Rasmus Bro. On the uniqueness of multilinear decomposition of n-way arrays. Journal of Chemometrics: A Journal of the Chemometrics Society, 14(3):229–239, 2000.

[26] Bo Waggoner, Rafael Frongillo, and Jacob D Abernethy. A market framework for eliciting private data. In Advances in Neural Information Processing Systems, pages 3510–3518, 2015.

[27] Chang Xu, Dacheng Tao, and Chao Xu. A survey on multi-view learning. arXiv preprint arXiv:1304.5634, 2013.
A Mathematical Background

Our mechanisms are built with some important mathematical tools. First, in probability theory, an f-divergence is a function that measures the difference between two probability distributions.

**Definition A.1** (f-divergence). Given a convex function f with f(1) = 0, for two distributions over \( \Omega \), \( p, q \in \Delta \Omega \), define the f-divergence of p and q to be

\[
D_f(p, q) = \int_{\omega \in \Omega} p(\omega) f \left( \frac{q(\omega)}{p(\omega)} \right).
\]

In duality theory, the convex conjugate of a function is defined as follows.

**Definition A.2** (Convex conjugate). For any function \( f : \mathbb{R} \to \mathbb{R} \), define the convex conjugate function of f as

\[
f^*(y) = \sup_x xy - f(x).
\]

Then the following inequality (20, 14) holds.

**Lemma A.1** (Lemma 1 in [20]). For any convex function f with f(1) = 0, any two distributions over \( \Omega \), p, q \( \in \Delta \Omega \), let \( \mathcal{G} \) be the set of all functions from \( \Omega \) to \( \mathbb{R} \), then we have

\[
D_f(p, q) \geq \sup_{g \in \mathcal{G}} \int_{\omega \in \Omega} g(\omega)p(\omega) - f^*(g(\omega))q(\omega) = \sup_{g \in \mathcal{G}} E_p g - E_q f^*(g).
\]

A function \( g \) achieves equality if and only if

\[
g(\omega) \in \partial f \left( \frac{p(\omega)}{q(\omega)} \right), \quad \forall \omega
\]

where \( \partial f \left( \frac{p(\omega)}{q(\omega)} \right) \) represents the subdifferential of f at point \( p(\omega)/q(\omega) \).

The f-mutual information of two random variables is a measure of the mutual dependence of two random variables, which is defined as the f-divergence between their joint distribution and the product of their marginal distributions.

**Definition A.3** (Kronecker product). Consider two matrices \( A \in \mathbb{R}^{m \times n} \) and \( B \in \mathbb{R}^{p \times q} \). The Kronecker product of \( A \) and \( B \), denoted as \( A \otimes B \), is defined as the following \( pm \times qn \) matrix:

\[
A \otimes B = \begin{bmatrix}
a_{11}B & \cdots & a_{1n}B \\
\vdots & \ddots & \vdots \\
a_{m1}B & \cdots & a_{mn}B
\end{bmatrix}.
\]

**Definition A.4** (f-mutual information and pointwise MI). Let \( (X, Y) \) be a pair of random variables with values over the space \( X \times Y \). If their joint distribution is \( p_{X,Y} \) and marginal distributions are \( p_X \) and \( p_Y \), then given a convex function f with f(1) = 0, the f-mutual information between X and Y is

\[
I_f(X;Y) = D_f(p_{X,Y}, p_X \otimes p_Y) = \int_{x \in X, y \in Y} p_{X,Y}(x, y) f \left( \frac{p_X(x) \cdot p_Y(y)}{p_{X,Y}(x, y)} \right).
\]

We define pointwise mutual information \( K(x, y) \) as the reciprocal of the ratio inside f,

\[
K(x, y) = \frac{p_{X,Y}(x, y)}{p_X(x) \cdot p_Y(y)}.
\]

If two random variables are independent conditioning on another random variable, we have the following formula for the pointwise mutual information.

**Lemma A.2.** When random variables X, Y are independent conditioning on \( \Theta \), for any pair of \( (x, y) \in X \times Y \), we have pointwise mutual information

\[
K(x, y) = \sum_{\theta \in \Theta} \frac{p(\theta|x)p(\theta|y)}{p(\theta)},
\]

if \( |\Theta| \) is finite, and

\[
K(x, y) = \int_{\theta \in \Theta} \frac{p(\theta|x)p(\theta|y)}{p(\theta)} d\theta
\]

if \( \Theta \subseteq \mathbb{R}^m \).
Proof. We only prove the second equation for $\Theta \subseteq \mathbb{R}^m$ as the proof for finite $\Theta$ is totally similar.

$$K(x, y) = \frac{p(x, y)}{p(x) \cdot p(y)} = \frac{\int_{\Theta \in \Theta} p(x|\theta)p(y|\theta) \, d\theta}{p(x) \cdot p(y)} = \int_{\Theta \in \Theta} \frac{p(\theta|x)p(\theta|y)}{p(\theta)} \, d\theta,$$

where the last equation uses Bayes’ Law.

**Definition A.5** (Exponential family [19]). A probability density function or probability mass function $p(x|\theta)$, for $x = (x_1, \ldots, x_n) \in X^n$ and $\theta \in \Theta \subseteq \mathbb{R}^m$ is said to be in the exponential family if it is of the form

$$p(x|\theta) = h(x) \exp \left[ \theta^T \phi(x) - A(\theta) \right]$$

where $A(\theta) = \log \int_{X^n} h(x) \exp \left[ \theta^T \phi(x) \right]$. The conjugate prior with parameters $\nu_0, \tau_0$ for $\theta$ has the form

$$p(\theta) = P(\theta|\nu_0, \tau_0) = g(\nu_0, \tau_0) \exp \left[ \nu_0 \theta^T \tau_0 - \nu_0 A(\theta) \right].$$

Let $\bar{s} = \frac{1}{n} \sum_{i=1}^{n} \phi(x_i)$. Then the posterior of $\theta$ is of the form

$$p(\theta|x) \propto \exp \left[ \theta^T (\nu_0 \tau_0 + n \bar{s}) - (\nu_0 + n) A(\theta) \right]$$

$$= P(\theta|\nu_0 + n, \frac{\nu_0 \tau_0 + n \bar{s}}{\nu_0 + n}),$$

where $P(\theta|\nu_0 + n, \frac{\nu_0 \tau_0 + n \bar{s}}{\nu_0 + n})$ is the conjugate prior with parameters $\nu_0 + n$ and $\frac{\nu_0 \tau_0 + n \bar{s}}{\nu_0 + n}$.

**Lemma A.3.** Let $\theta$ be the parameters of a pdf in the exponential family. Let $P(\theta|\nu, \tau) = g(\nu, \tau) \exp \left[ \nu \theta^T \tau - \nu A(\theta) \right]$ denote the conjugate prior for $\theta$ with parameters $\nu, \tau$. For any three distributions of $\theta$,

$$p_1(\theta) = P(\theta|\nu_1, \tau_1),$$

$$p_2(\theta) = P(\theta|\nu_2, \tau_2),$$

$$p_0(\theta) = P(\theta|\nu_0, \tau_0),$$

we have

$$\int_{\Theta \in \Theta} \frac{p_1(\theta)p_2(\theta)}{p_0(\theta)} \, d\theta = \frac{g(\nu_1, \tau_1)g(\nu_2, \tau_2)}{g(\nu_0, \tau_0)g(\nu_1 + \nu_2 - \nu_0, \frac{\nu_1 \tau_1 + \nu_2 \tau_2 - \nu_0 \tau_0}{\nu_1 + \nu_2 - \nu_0}).$$

**Proof.** To compute the integral, we first write $p_1(\theta), p_2(\theta)$ and $p_3(\theta)$ in full,

$$p_1(\theta) = P(\theta|\nu_1, \tau_1) = g(\nu_1, \tau_1) \exp \left[ \nu_1 \theta^T \tau_1 - \nu_1 A(\theta) \right],$$

$$p_2(\theta) = P(\theta|\nu_2, \tau_2) = g(\nu_2, \tau_2) \exp \left[ \nu_2 \theta^T \tau_2 - \nu_2 A(\theta) \right],$$

$$p_0(\theta) = P(\theta|\nu_0, \tau_0) = g(\nu_0, \tau_0) \exp \left[ \nu_0 \theta^T \tau_0 - \nu_0 A(\theta) \right].$$

Then we have the integral equal to

$$\int_{\Theta \in \Theta} \frac{p_1(\theta)p_2(\theta)}{p_0(\theta)} \, d\theta$$

$$= \int_{\Theta \in \Theta} \frac{g(\nu_1, \tau_1)g(\nu_2, \tau_2)}{g(\nu_0, \tau_0)} \exp \left[ \nu_0 \theta^T \tau_0 - \nu_0 A(\theta) \right] \, d\theta$$

$$= \frac{g(\nu_1, \tau_1)g(\nu_2, \tau_2)}{g(\nu_0, \tau_0)} \int_{\Theta \in \Theta} \left[ \frac{\theta^T (\nu_1 \tau_1 + \nu_2 \tau_2 - \nu_0 \tau_0)}{\nu_1 + \nu_2 - \nu_0} \right] \, d\theta$$

$$= \frac{g(\nu_1, \tau_1)g(\nu_2, \tau_2)}{g(\nu_0, \tau_0)} \cdot \frac{1}{\nu_1 + \nu_2 - \nu_0, \frac{\nu_1 \tau_1 + \nu_2 \tau_2 - \nu_0 \tau_0}{\nu_1 + \nu_2 - \nu_0}}.$$
The mechanism is as follows.

\[ g \left( \nu_1 + \nu_2 - \nu_0, \frac{\nu_1 \bar{\tau}_1 + \nu_2 \bar{\tau}_2 - \nu_0 \bar{\tau}_0}{\nu_1 + \nu_2 - \nu_0} \right) \exp \left[ \theta^T (\nu_1 \bar{\tau}_1 + \nu_2 \bar{\tau}_2 - \nu_0 \bar{\tau}_0) - A(\theta)(\nu_1 + \nu_2 - \nu_0) \right] \]

is the pdf

\[ p \left( \theta | \nu_1 + \nu_2 - \nu_0, \frac{\nu_1 \bar{\tau}_1 + \nu_2 \bar{\tau}_2 - \nu_0 \bar{\tau}_0}{\nu_1 + \nu_2 - \nu_0} \right) \]

and thus has the integral over \( \theta \) equal to 1.

\[ \square \]

**B Missing proof for Lemma 3.1**

**Lemma B.1 (Lemma 3.1).** When \( D_1, \ldots, D_n \) are independent conditioned on \( \theta \), for any \((D_1, \ldots, D_n)\) and \((\tilde{D}_1, \ldots, \tilde{D}_n)\), if \( p(\theta|D_i) = p(\theta|\tilde{D}_i) \) \( \forall i \), then \( p(\theta|D_1, \ldots, D_n) = p(\theta|\tilde{D}_1, \ldots, \tilde{D}_n) \).

**Proof.** Suppose \( \forall i, p(\theta|D_i) = p(\theta|D_i') \), then we have

\[
p(\theta|D_1, D_2, \ldots, D_n) = \frac{p(D_1, D_2, \ldots, D_n, \theta)}{p(D_1, D_2, \ldots, D_n)}
= \frac{p(D_1, \ldots, D_n|\theta) \cdot p(\theta)}{p(D_1, \ldots, D_n)}
= \frac{p(D_1|\theta) \cdot p(D_2|\theta) \cdots p(D_n|\theta) \cdot p(\theta)}{p(D_1, D_2, \ldots, D_n)}
= \frac{p(\theta|D_1) \cdot p(\theta|D_2) \cdots p(\theta|D_n) \cdot p(\theta)}{p(D_1, D_2, \ldots, D_n) \cdot p^{n-1}(\theta)}
\]

Similarly, we have

\[
p(\theta|D'_1, D'_2, \ldots, D'_n) \propto \frac{p(\theta|D'_1) \cdot p(\theta|D'_2) \cdots p(\theta|D'_n)}{p^{n-1}(\theta)},
\]

since the analyst calculate the posterior by normalize the terms, we have

\[
p(\theta|D_1, D_2, \ldots, D_n) = p(\theta|D'_1, D'_2, \ldots, D'_n).
\]

\[ \square \]

**C One-time data acquisition**

**C.1 An example of applying peer prediction**

The mechanism is as follows.

**Mechanism 3: One-time data collecting mechanism by using Brier Score.**

1. Ask all data providers to report their datasets \( \tilde{D}_1, \ldots, \tilde{D}_n \).
2. For all \( D_{-i} \), calculate probability \( p(D_{-i}|D_i) \) by the reported \( D_i \) and \( p(D_i|\theta) \).
3. The Brier score for agent \( i \) is \( s_i = 1 - \frac{1}{|D_{-i}|} \sum_{D_{-i}} (p(D_{-i}|\tilde{D}_i) - \|D_{-i} = \tilde{D}_{-i}\|)^2 \),
   where \( \|D_{-i} = \tilde{D}_{-i}\| = 1 \) if \( D_{-i} \) is the same as the reported \( \tilde{D}_{-i} \) and 0 otherwise.
4. The final payment for agent \( i \) is \( r_i = \frac{\theta}{\theta - \frac{1}{n-1} \sum_{j \neq i} s_i} \).

This payment function is actually the mean square error of the reported distribution on \( D_{-i} \). It is based on the Brier score which is first proposed in [3] and is a well-known bounded proper scoring rule. The payments of the mechanism are always bounded between 0 and 1.
Then we have will still maximize the agent’s revenue and the mechanism is truthful.

\textbf{Theorem C.1.} Mechanism 3 is IR, truthful, budget-bounded, symmetric.

\textbf{Proof.} The symmetric property is easy to verify. Moreover, since the payment for each agent is in the interval \([0, 1]\), the mechanism is then budget-bounded and IR. We only need to prove the truthfulness. Suppose that all the other agents except \(i\) reports truthfully. Agent \(i\) has true dataset \(D_i\) and reports \(\tilde{D}_i\). Since in the setting, the analyst is able to calculate \(p(D_{-i} | D_i)\), then if the agent receives \(s_i\) as their payment, from agent \(i\)’s perspective, his expected revenue is then:

\[
\text{Rev}'_i = \sum_{D_{-i}} p(D_{-i} | D_i) \cdot \left(1 - \sum_{D'_{-i}} (p(D'_{-i} | \tilde{D}_i) - \|D'_{-i} = D_{-i}\|)^2 \right)
\]

\[
= - \sum_{D_{-i}} p(D_{-i} | D_i) \left(\sum_{D'_{-i}} (p(D'_{-i} | \tilde{D}_i))^2 - 2p(D_{-i} | \tilde{D}_i) \right)
\]

\[
= \sum_{D_{-i}} \left(-p(D_{-i} | \tilde{D}_i)^2 + 2p(D_{-i} | \tilde{D}_i)p(D_{-i} | D_i) \right)
\]

Since the function \(-x^2 + 2ax\) is maximized when \(x = a\), the revenue \(\text{Rev}'_i\) is maximized when \(\forall D_{-i}, p(D_{-i} | D_{-i}) = p(D_{-i} | D_i)\). Since the real payment \(r_i\) is a linear transformation of \(s_i\) and the coefficients are independent of the reported datasets, reporting the dataset with the true posterior will still maximize the agent’s revenue and the mechanism is truthful. \(\square\)

\section*{C.2 Bounding log-PMI: discrete case}

In this section, we give a method to compute the bounds of the log-PMI score when \(|\Theta|\) is finite. First we give the upper bound of the PMI. We have for any \(i, D_i \in D_i(D_{-i})\)

\[
\text{PMI}(D_i, D_{-i}) \leq \max_{i, D'_i, D'_{-i} \in D_i(D'_{-i})} \{ \text{PMI}(D'_i, D'_{-i}) \}
\]

\[
= \max_{i, D'_i, D'_{-i} \in D_i(D'_{-i})} \left\{ \sum_{\theta \in \Theta} \frac{p(\theta | D'_i)p(\theta | D'_{-i})}{p(\theta)} \right\}
\]

\[
\leq \max_{i, D'_i} \left\{ \sum_{\theta \in \Theta} \frac{p(\theta | D'_i)}{\min_{\theta} \{p(\theta)\}} \right\}
\]

\[
\leq \frac{1}{\min_{\theta} \{p(\theta)\}}.
\]

The last inequality is because we have \(\sum_{\theta} p(\theta | D'_i) = 1\).

Since we have assumed that \(p(\theta)\) is positive, the term \(\frac{1}{\min_{\theta} \{p(\theta)\}}\) could then be computed and is finite. Thus we just let \(R\) be \(\log \left(\frac{1}{\min_{\theta} \{p(\theta)\}} \right)\). Then we need to calculate a lower bound of the score. We have for any \(i, D_{-i}\) and \(D_i \in D_i(D_{-i})\)

\[
\text{PMI}(D_i, D_{-i}) = \sum_{\theta \in \Theta} \frac{p(\theta | D_i)p(\theta | D_{-i})}{p(\theta)} \geq \sum_{\theta \in \Theta} p(\theta | D_i)p(\theta | D_{-i}). \quad \text{(8)}
\]

\textbf{Claim C.1.} Let \(D = \{d^{(1)}, \ldots, d^{(N)}\}\) be a dataset with \(N\) data points that are i.i.d. conditioning on \(\Theta\). Let \(D\) be the support of the data points \(d\). Define

\[
T = \frac{\max_{\theta \in \Theta} p(\theta)}{\min_{\theta \in \Theta} p(\theta)}, \quad U(D) = \max_{\theta \in \Theta, d \in D} \left\{ p(\theta | d) \middle/ \min_{\theta, d \in D: p(\theta | d) > 0} p(\theta | d) \right\},
\]

Then we have

\[
\frac{\max_{\theta \in \Theta} p(\theta | D)}{\min_{\theta: p(\theta | D) > 0} p(\theta | D)} \leq U(D)^N \cdot T^{N-1}.
\]
Proof. By Lemma 3.1 we have
\[ p(\theta | D) \propto \frac{\prod_j p(\theta | D^{(j)})}{p(\theta)^{N-1}}, \]
for a fixed \( D \), it must hold that
\[ \frac{\max_{\theta \in \Theta} p(\theta | D)}{\min_{\theta : p(\theta | D) > 0} p(\theta | D)} \leq U(D)^N \cdot T^{N-1}. \]

Claim C.2. For any two datasets \( D_i \) and \( D_j \) with \( N_i \) and \( N_j \) data points respectively, let \( D_i \) be the support of the data points in \( D_i \) and let \( D_j \) be the support of the data points in \( D_j \). Then
\[ \frac{\max_{\theta \in \Theta} p(\theta | D_i, D_j)}{\min_{\theta : p(\theta | D_i, D_j) > 0} p(\theta | D_i, D_j)} \leq U(D_i)^{N_i} \cdot U(D_j)^{N_j} \cdot T^{N_i+N_j-1}. \]

Proof. Again by Lemma 3.1 we have
\[ p(\theta | D_i, D_j) \propto \frac{p(\theta | D_i)p(\theta | D_j)}{p(\theta)}. \]
Combine it with Claim C.1, we prove the statement.

Then for any \( D_i \), since \( \sum_{\theta \in \Theta} p(\theta | D_i) = 1 \), by Claim C.1
\[ \min_{\theta : p(\theta | D_i) > 0} p(\theta | D_i) \geq \frac{1}{1 + |\Theta| \cdot U(D_i)^{N_i} \cdot T^{N_i-1}} \triangleq \eta(D_i, N_i). \]
And for any \( D_{-i} \), since \( \sum_{\theta \in \Theta} p(\theta | D_{-i}) = 1 \), by Claim C.2
\[ \min_{\theta : p(\theta | D_{-i}) > 0} p(\theta | D_{-i}) \geq \frac{1}{1 + |\Theta| \cdot \Pi_{j \neq i} U(D_j)^{N_j} \cdot T^{\sum_{j \neq i} N_j-1}} \triangleq \eta(D_{-i}, N_{-i}). \]
Finally, for any \( i, D_{-i} \), and \( D_i \in \mathcal{D}_i(D_{-i}) \), according to 16,
\[ PMI(D_i, D_{-i}) \geq \sum_{\theta \in \Theta} p(\theta | D_i)p(\theta | D_{-i}) \geq \eta(D_i, N_i) \cdot \eta(D_{-i}, N_{-i}). \]
The last inequality is because \( D_i \in \mathcal{D}_i(D_{-i}) \) and there must exists \( \theta \in \Theta \) so that both \( p(\theta | D_i) \) and \( p(\theta | D_{-i}) \) are non-zero. Both \( \eta(D_i, N_i) \) and \( \eta(D_{-i}, N_{-i}) \) can be computed in polynomial time. Take minimum over \( i \), we find the lower bound for PMI.

C.3 Bounding log-PMI: continuous case

Consider estimating the mean \( \mu \) of a univariate Gaussian \( N(x | \mu, \sigma^2) \) with known variance \( \sigma^2 \). Let \( D = \{ x_1, \ldots, x_N \} \) be the dataset and denote the mean by \( \bar{x} = \frac{1}{N} \sum_j x_j \). We use the Gaussian conjugate prior,
\[ \mu \sim N(\mu | \mu_0, \sigma_0^2). \]
Then according to [18], the posterior of \( \mu \) is equal to
\[ p(\mu | D) = N(\mu | \mu_N, \sigma_N^2), \]
where
\[ \frac{1}{\sigma_N^2} = \frac{1}{\sigma^2} + \frac{N}{\sigma_0^2} \]
only depends on the number of data points.

By Lemma 4.1 we know that the payment function for exponential family is in the form of
\[ PMI(D_i, D_{-i}) = \frac{g(\nu, \bar{x}_i)g(\nu_{-i}, \bar{x}_{-i})}{g(\nu_0, \bar{x}_0)g(\nu + \nu_{-i} - \nu_0, \nu_{i} \bar{x}_i + \nu_{-i} \bar{x}_{-i} - \nu_0 \bar{x}_0 \bar{x}_0 \bar{x}_0).} \]
The normalization term for Gaussian is 
\[
\frac{1}{\sqrt{2\pi} \sigma^2},
\]
so we have
\[
PMI(D_i, D_{-i}) = \frac{1}{\sigma_0^2} \frac{\sqrt{\frac{N_i}{\sigma^2} + \frac{N_{-i}}{\sigma^2}}}{\sqrt{\frac{1}{\sigma_0^2} + \frac{N_{i} + N_{-i}}{\sigma^2}}}
\]

When the total number of data points has an upper bound \(N_{\text{max}}\), each of the square root term should be bounded in the interval
\[
\left[\frac{1}{\sigma_0}, \sqrt{\frac{1}{\sigma_0^2} + \frac{N_{\text{max}}}{\sigma^2}}\right]
\]
Therefore \(PMI(D_i, D_{-i})\) is bounded in the interval
\[
\left[(1 + N_{\text{max}}\sigma_0^2/\sigma^2)^{-1/2}, 1 + N_{\text{max}}\sigma_0^2/\sigma^2\right].
\]

### C.4 Sensitivity analysis for the exponential family

If we are estimating the mean \(\mu\) of a univariate Gaussian \(\mathcal{N}(x|\mu, \sigma^2)\) with known variance \(\sigma^2\). Let \(D = \{x_1, \ldots, x_N\}\) be the dataset and denote the mean by \(\bar{x} = \frac{1}{N} \sum_j x_j\). We use the Gaussian conjugate prior,
\[
\mu \sim \mathcal{N}(\mu_0, \sigma_0^2).
\]
Then according to [18], the posterior of \(\mu\) is equal to
\[
p(\mu|D) = \mathcal{N}(\mu|\mu_N, \sigma_N^2),
\]
where
\[
\frac{1}{\sigma_N^2} = \frac{1}{\sigma_0^2} + \frac{N}{\sigma^2}
\]
only depends on the number of data points. Since the normalization term \(\frac{1}{\sqrt{2\pi} \sigma^2}\) of Gaussian distributions only depends on the variance, function \(h(\cdot)\) defined in [12]
\[
h_{D_{-i}}(N_i, \bar{x}_i) = \frac{g(\nu_i, \bar{x}_i)}{g(\nu_i + \nu_{-i} - \nu_0, \nu_{i} + \nu_{-i} - \nu_0)}
\]
\[
= \frac{1}{\sigma_i^2} \sqrt{\frac{1}{\sigma_0^2} + \frac{N_i}{\sigma^2}} \sqrt{\frac{1}{\sigma_0^2} + \frac{N_i + N_{-i}}{\sigma^2}}
\]
will only be changed if the number of data points \(N_i\) changes, which means that the mechanism will be sensitive to replication and withholding, but not necessarily other types of manipulations.

If we are estimating the mean \(\mu\) of a Bernoulli distribution \(Ber(x|\mu)\). Let \(D = \{x_1, \ldots, x_N\}\) be the data points. Denote by \(\alpha = \sum_i x_i\) the number of ones and denote by \(\beta = \sum_i 1 - x_i\) the number of zeros. The conjugate prior is the Beta distribution,
\[
p(\mu) = \text{Beta}(\mu|\alpha_0, \beta_0) = \frac{1}{B(\alpha_0, \beta_0)} \mu^{\alpha_0 - 1} (1 - \mu)^{\beta_0 - 1}.
\]
where \(B(\alpha_0, \beta_0)\) is the Beta function
\[
B(\alpha_0, \beta_0) = \frac{(\alpha_0 + \beta_0 - 1)!}{(\alpha_0 - 1)! (\beta_0 - 1)!}.
\]
The posterior of \(\mu\) is equal to
\[
p(\mu|D) = \text{Beta}(\mu|\alpha_0 + \alpha, \beta_0 + \beta).
\]
Then we have
\[
h_{D_{-i}}(\alpha, \beta) = \frac{B(\alpha_0 + \alpha_i + \alpha_{-i}, \beta_0 + \beta_i + \beta_{-i})}{B(\alpha_0 + \alpha_i, \beta_0 + \beta_i)}
\]
\[
= \frac{(\alpha_0 + \beta_0 + N_i + N_{-i} - 1)! (\alpha_0 + \alpha_i - 1)! (\beta_0 + \beta_i - 1)!}{(\alpha_0 + \alpha_i + \alpha_{-i} - 1)! (\beta_0 + \beta_i + \beta_{-i} - 1)! (\alpha_0 + \beta_0 + N_i - 1)!}.
\]
Define $A_i = \alpha_0 + \alpha_i - 1$ and $B_i = \beta_0 + \beta_i - 1$, since $N_i = \alpha_i + \beta_i$ and $N_{-i} = \alpha_{-i} + \beta_{-i}$, we have

$$h_{D_{-i}}(\alpha, \beta) = h_{\alpha_{-i}, \beta_{-i}}(A_i, B_i) = \frac{A_i!B_i!(A_i + B_i + \alpha_i + \beta_i - 1)!}{(A_i + \alpha_i - 1)!(B_i + \beta_i - 1)!(A_i + B_i + \alpha_i + \beta_i)!}$$

Now we are going to prove that for any two different pairs $(A_i, B_i)$ and $(A'_i, B'_i)$, there should always exist a pair $(\alpha'_{-i}, \beta'_{-i})$ selected from the four pairs: $(\alpha_{-i}, \beta_i), (\alpha_{-i} + 1, \beta_i), (\alpha_{-i}, \beta_i + 1), (\alpha_{-i} + 1, \beta_i + 1)$, such that $h_{\alpha'_{-i}, \beta'_{-i}}(A_i, B_i) \neq h_{\alpha'_{-i}, \beta'_{-i}}(A'_i, B'_i)$.

Suppose that this does not hold, then there should exist two pairs $(A_i, B_i)$ and $(A'_i, B'_i)$ such that for each $(\alpha'_{-i}, \beta'_{-i})$ in the four pairs, $h_{\alpha'_{-i}, \beta'_{-i}}(A_i, B_i) = h_{\alpha'_{-i}, \beta'_{-i}}(A'_i, B'_i)$.

Then by the two cases when $(\alpha'_{-i}, \beta'_{-i}) = (\alpha_{-i}, \beta_{-i})$ and $(\alpha_{-i} + 1, \beta_{-i})$ we can derive that

$$\frac{h_{\alpha_{-i}+1, \beta_{-i}}(A_i, B_i)}{\alpha_{-i}, \beta_{-i}}(A_i, B_i) = \frac{h_{\alpha_{-i}+1, \beta_{-i}}(A'_i, B'_i)}{A'_i + \alpha_{-i} + 1 + \beta_{-i} + 1} = \frac{(A_i + B_i - A'_i - B'_i)(\alpha_{-i} + 1) + (A'_i - A_i)(\alpha_{-i} + \beta_{-i} + 2) + A'_i B_i - A_i B'_i = 0}{A_i + B_i - A'_i - B'_i}$$

Replacing $\beta_{-i}$ with $\alpha_{-i} + 1$, we could get

$$(A_i + B_i - A'_i - B'_i)(\alpha_{-i} + 1) + (A'_i - A_i)(\alpha_{-i} + \beta_{-i} + 3) + A'_i B_i - A_i B'_i = 0$$

Subtracting the last equation from this, we get $A'_i - A_i = 0$. Symmetrically, when $(\alpha'_{-i}, \beta'_{-i}) = (\alpha_{-i}, \beta_{-i})$ and $(\alpha_{-i}, \beta_{-i} + 1)$ and replacing $\alpha_{-i}$ with $\alpha_{-i} + 1$, we have $B'_i - B_i = 0$ and thus $(A_i, B_i) = (A'_i, B'_i)$. This contradicts to the assumption that $(A_i, B_i) \neq (A'_i, B'_i)$. Therefore for any two different pairs of reported data in the Bernoulli setting, at least one in the four others’ reported data $(\alpha_{-i}, \beta_i), (\alpha_{-i} + 1, \beta_i), (\alpha_{-i}, \beta_i + 1), (\alpha_{-i} + 1, \beta_i + 1)$ would make the agent strictly faithfully report his posterior.

C.5 Missing proofs

C.5.1 Proof for Theorem 5.1 and Theorem 5.2

Theorem C.2 (Theorem 5.1). Mechanism $\mathcal{I}$ is IR, truthful, budget-bounded, symmetric.

We suppose that the dataset space of agent $i$ is $\mathcal{D}_i$. We first give the definitions of several matrices. These matrices are essential for our proofs, but they are unknown to the data analyst. Since the dataset $D_i$ consists of $N_i$ i.i.d data points drawn from the data generating matrix $G_i$, we define prediction matrix $P_i$ of agent $i$ to be a matrix with $|\mathcal{D}_i| = |\mathcal{D}|^N_i$ rows and $|\Theta|$ columns. Each column corresponds to a $\theta \in \Theta$ and each row corresponds to a possible dataset $D_i \in \mathcal{D}_i$. The matrix element on the column corresponding to $\theta$ and the row corresponding to $D_i$ is $p(D_i|\theta)$. Intuitively, this matrix is the posterior of agent $i$’s dataset conditioned on the parameter $\theta$.

Similarly, we define the out-prediction matrix $P_{-i}$ of agent $i$ to be a matrix with $\prod_{j \neq i} |\mathcal{D}_j|$ rows and $|Y|$ columns. Each column corresponds to a $\theta \in \Theta$ and each row corresponds to a possible dataset $D_{-i} \in \mathcal{D}_{-i}$. The element corresponding to $D_{-i}$ and $\theta$ is $p(D_{-i}|\theta)$. In the proof, we also give a lower bound on the sensitiveness coefficient $\alpha$ related to these out-prediction matrices.

Theorem C.3 (Theorem 5.2). Mechanism $\mathcal{I}$ is sensitive if either condition holds:

1. $\forall i, Q_{-i}$ has rank $|\Theta|$;
2. $\forall i, \sum_{i' \neq i} \text{rank}_2(G_{i'}) \cdot (N_{i'} - 1) + 1 \geq |\Theta|$.

Moreover, it is $e_i \cdot \frac{B}{n(B - L)}$ - sensitive for agent $i$, where $e_i$ is the smallest singular value of matrix $P_{-i}$.

Proof. We first prove that the payment $r_{ij}$ should be bounded. By definition, we have for all $j \in [n]$

$$R \geq s_j \geq L.$$

We denote the term $\frac{1}{n-1} \sum_{j \neq i} s_j$ as $\pi_{-i}$. Then $\frac{2n - \pi_{-i}}{R - L}$ should be bounded in $[-1, 1]$ and the final payment for each agent should be bounded in the interval $[0, \frac{2B}{n}]$.
We suppose that agent $i$’s expected revenue of Mechanism 1 is $\text{Rev}_i$. Then we have

$$\text{Rev}_i = \frac{B}{n} \left( 1 + \sum_{D_{-i} \in \mathcal{D}_{-i}(D_{-i})} p(D_{-i}|D_i) \cdot \log PMI(\tilde{D}_i, D_{-i}) - \pi_{-i} \right).$$

We consider another revenue $\text{Rev}'_i \triangleq \sum_{D_{-i}} p(D_{-i}|D_i) \cdot \log \left( \sum_{\theta} \frac{p(\theta|D_i) \cdot p(\theta|D_{-i})}{p(\theta)} \right)$ assuming that $0 \cdot \log 0 = 0$. Then we have

$$\text{Rev}'_i = \sum_{D_{-i}} p(D_{-i}|D_i) \cdot \log \left( \sum_{\theta} \frac{p(\theta|D_i) \cdot p(\theta|D_{-i})}{p(\theta)} \right)
= \sum_{D_{-i}, D_i \in \mathcal{D}_{-i}(D_{-i})} p(D_{-i}|D_i) \cdot \log PMI(\tilde{D}_i, D_{-i})
+ \sum_{D_{-i}, D_i \notin \mathcal{D}_{-i}(D_{-i})} p(D_{-i}|D_i) \cdot \log PMI(\tilde{D}_i, D_{-i})
= \sum_{D_{-i}, D_i \in \mathcal{D}_{-i}(D_{-i})} p(D_{-i}|D_i) \cdot \log PMI(\tilde{D}_i, D_{-i}) + \sum_{D_{-i}, D_i \notin \mathcal{D}_{-i}(D_{-i})} 0 \cdot \log 0
= \sum_{D_{-i}, D_i \in \mathcal{D}_{-i}(D_{-i})} p(D_{-i}|D_i) \cdot \log PMI(\tilde{D}_i, D_{-i})
= \text{Rev}_i \cdot \frac{n}{B} \cdot (R - L) - R + L + \pi_{-i}.
$$

$\text{Rev}'_i$ is a linear transformation of $\text{Rev}_i$. The coefficients $L$, $R$, $\frac{n}{B}$ and $\pi_{-i}$ do not depend on $\tilde{D}_i$. The ratio $\frac{n}{B} \cdot (R - L)$ is larger than 0. Therefore, the optimal reported $\tilde{D}_i$ for $\text{Rev}_i$ should be the same as that for $\text{Rev}'_i$. If the payment rule with revenue $\text{Rev}'_i$ is $e_i$-sensitive for agent $i$, then the Mechanism 1 would then be $e_i \cdot \frac{B}{n(R - L)}$-sensitive. In the following part, we prove that real dataset $D_i$ would maximize the revenue $\text{Rev}'_i$ and the $\text{Rev}'_i$ is $e_i \cdot \frac{B}{n(R - L)}$-sensitive for all the agents. Thus in the following parts we prove the revenue $\text{Rev}'_i$ is $e_i$-sensitive for agent $i$.

$$\text{Rev}'_i = \sum_{D_{-i}} p(D_{-i}|D_i) \cdot \log \left( \sum_{\theta} \frac{p(\theta|\tilde{D}_i) \cdot p(\theta|D_{-i})}{p(\theta)} \right)
= \sum_{D_{-i}} p(D_{-i}|D_i) \cdot \log \left( \sum_{\theta} \frac{p(\theta|\tilde{D}_i) \cdot p(\theta, D_{-i})}{p(\theta)} \right) - \sum_{D_{-i}} p(D_{-i}|D_i) \cdot \log (p(D_{-i}))
= \sum_{D_{-i}} p(D_{-i}|D_i) \cdot \log \left( \sum_{\theta} \frac{p(\theta|\tilde{D}_i) \cdot p(\theta, D_{-i})}{p(\theta)} \right) - C.
$$

Since the term $\sum_{D_{-i}} p(D_{-i}|D_i) \cdot \log (p(D_{-i}))$ does not depend on $\tilde{D}_i$, agent $i$ could only manipulate to modify the term $\sum_{D_{-i}} p(D_{-i}|D_i) \cdot \log \left( \sum_{\theta} \frac{p(\theta|\tilde{D}_i) \cdot p(\theta, D_{-i})}{p(\theta)} \right)$. Since we have

$$\sum_{D_{-i}, \theta} \frac{p(\theta|\tilde{D}_i) \cdot p(\theta, D_{-i})}{p(\theta)} = \sum_{\theta} \frac{1}{p(\theta)} \left( \sum_{D_{-i}} p(\theta|\tilde{D}_i) \cdot p(\theta, D_{-i}) \right) = \sum_{\theta} \frac{1}{p(\theta)} \left( p(\theta|\tilde{D}_i) \cdot p(\theta) \right) = \sum_{\theta} p(\theta|\tilde{D}_i) = 1,$$

Since we have $\sum_{\theta} p(\theta|\tilde{D}_i) p(\theta, D_{-i}) = 1$, we could view the term $\sum_{\theta} p(\theta|\tilde{D}_i) p(\theta, D_{-i})$ as a probability distribution on the variable $D_{-i}$. Since it depends on $\tilde{D}_i$, we denote it as $\tilde{p}(D_{-i}|\tilde{D}_i)$. 
Since if we fix a distributions \( p(\sigma) \), then the distribution \( q(\sigma) \) that maximizes \( \sum_\sigma p(\sigma) \log q(\sigma) \) should be the same as \( p \). (If we assume that \( 0 \cdot \log 0 = 0 \), this still holds.) When agent \( i \) report truthfully,

\[
\sum_\theta \frac{p(\theta|D_i) \cdot p(\theta, D_{-i})}{p(\theta)} = \sum_\theta \frac{p(D_i, \theta) \cdot p(D_{-i}, \theta)}{p(D_i) \cdot p(\theta)} = \sum_\theta \frac{p(D_i, \theta) \cdot p(D_{-i}, \theta)}{p(D_i)} = \sum_\theta \frac{p(D_{-i}, \theta)}{p(D_i)} = p(D_{-i}|D_i).
\]

The data provider can always maximize \( \text{Rev}'_i \) by truthfully reporting \( D_i \). And we have proven the truthfulness of the mechanism.

Then we need to prove the relation between the sensitiveness of the mechanism and the out-prediction matrices. When Alice reports \( \tilde{D}_i \) the revenue difference from truthfully report is then

\[
\Delta_{\text{Rev}'}_i = \sum_{D_{-i}} p(D_{-i}|D_i) \log p(D_{-i}|D_i) - \sum_{D_{-i}} p(D_{-i}|D_i) \log \tilde{p}(D_{-i}|D_i)
\]

\[
= \sum_{D_{-i}} p(D_{-i}|D_i) \log \frac{p(D_{-i}|D_i)}{\tilde{p}(D_{-i}|D_i)}
\]

\[
= D_{KL}(p||\tilde{p})
\]

\[
\geq \sum_{D_{-i}} \|p(D_{-i}|D_i) - \tilde{p}(D_{-i}|D_i)\|^2.
\]

We let the distribution difference vector be \( \Delta_i \) (Note that here \( \Delta_i \) is a \(|\Theta|\)-dimension vector), then we have

\[
\Delta_{\text{Rev}'}_i \geq \sum_{D_{-i}} \|p(D_{-i}|D_i) - \tilde{p}(D_{-i}|D_i)\|^2 \geq \sum_{D_{-i}} \left( \sum_\theta (p(\theta|D_i) - \tilde{p}(\theta|D_i)) \cdot p(D_{-i}|\theta) \right)^2
\]

\[
= \|P_{-i}\Delta_i\|^2.
\]

Since \( e_i \) is the minimum singular value of \( P_{-i} \) and thus \( P_{-i}^T e_i I - e_i I \) is semi-positive, we have

\[
\|P_{-i}\Delta_i\|^2 = \Delta_i^T P_{-i}^T P_{-i} \Delta_i
\]

\[
= \Delta_i^T (P_{-i}^T P_{-i} - e_i I) \Delta_i + \Delta_i^T e_i I \Delta_i
\]

\[
\geq \Delta_i^T e_i I \Delta_i
\]

\[
\geq e_i \Delta_i^T \Delta_i
\]

\[
= \|\Delta_i\| \cdot e_i.
\]

Finally get the payment rule with revenue \( \text{Rev}'_i \) is \( e_i \)-sensitive for agent \( i \). If all \( P_{-i} \) has rank \(|\Theta|\), then all the singular values of the matrix \( P_{-i} \) should have positive singular values and for all \( i, e_i > 0 \). By now we have proven that if all the \( P_{-i} \) has rank \(|\Theta|\), then the mechanism is sensitive.

Since \( p(\theta|D_i) = p(D_i|\theta) \cdot \frac{p(\theta)}{p(D_i)} \), we have the matrix equation:

\[
Q_{-i} = \Lambda_{D_{-i}}^{-1} \cdot P_{-i} \cdot \Lambda^\theta,
\]

where \( \Lambda_{D_{-i}}^{-1} = \begin{bmatrix} \frac{1}{p(D_1^i)} & \frac{1}{p(D_2^i)} & \cdots & \frac{1}{p(D_q^i)} \end{bmatrix} \) and \( \Lambda^\theta = \begin{bmatrix} p(\theta_1) & p(\theta_2) & \cdots & p(\theta_q) \end{bmatrix} \).

\( p(D_i^j) \) is the probability that agent \( i \) gets the dataset \( D_i^j \). \( p(\theta_k) \) is the probability of the prior of
the parameter $\theta$ with index $k$. Both are all diagonal matrices. Both of the diagonal matrices well-defined and full-rank. Thus the rank of $P_{-i}$ should be the same as $Q_{-i}$ and we have proved the first condition.

The proof for the second sufficient condition is directly derived from the paper [25] and the condition 1. We first define a matrix $G'_r$ with the same size as $G_r$ while its elements are $p(d_i|\theta)$ rather than $p(\theta|d_i)$. Since for all $i' \in [n]$ the prediction matrix $P_{i'}$ is the columnwise Kronecker product (defined in Lemma 1 in [25] which is shown below) of $N_i$ data generating matrices. By using the following Lemma in [25], if the k-rank of $G'_r$, is $r$, then each time we multiply(columnwise Kronecker product) a matrix by $G'_r$, the k-rank would increase by at least $\text{rank}_k(G'_r)$, or reach the cap of $|\Theta|$.

**Lemma C.1.** Consider two matrices $A = [a_1, a_2, \ldots, a_F] \in \mathbb{R}^{I \times F}, B = [b_1, b_2, \ldots, b_F] \in \mathbb{R}^{P \times F}$ and $A \otimes_c B$ is the columnwise Kronecker product of $A$ and $B$ defined as:

$$A \otimes_c B \triangleq [a_1 \otimes b_1, a_2 \otimes b_2, \ldots, a_F \otimes b_F],$$

where $\otimes$ stands for the Kronecker product. It holds that

$$\text{rank}_k(A \otimes_c B) \geq \min\{\text{rank}_k(A) + \text{rank}_k(B) - 1, F\}.$$

Therefore the final k-rank of the $N_i$ would be no less than $\min\{N_i \cdot (r - 1) + 1, |\Theta|\}$. We then need to calculate the k-rank of the out-prediction matrix of each agent $i$ and verify whether it is $|\Theta|$. Similarly, the out-prediction matrix of agent $i$ is the columnwise Kronecker product of all the other agent’s prediction matrices. By the same lower bound tool in [25], the k-rank of $P_{-i}$ should be at least $\min\{\sum_{i' \neq i} \text{rank}_k(G'_r) \cdot (N_i - 1) + 1, |\Theta|\}$ and by Theorem 5.2 if the k-rank of all prediction matrices are all $|\Theta|$, Mechanism should be sensitive.

**C.5.2 Missing Proof for Theorem 5.2**

When $\Theta \subseteq \mathbb{R}^m$ and a model in the exponential family is used, we prove that the mechanism will be sensitive if and only if for any $(\nu', \bar{\nu}) \neq (\nu, \bar{\nu})$,

$$\Pr_{D_{-i}|D_i}[h_{D_{-i}}(\nu', \bar{\nu}) \neq h_{D_{-i}}(\nu, \bar{\nu})] > 0. \quad (9)$$

We first show that the above condition is equivalent to that for any $(\nu', \bar{\nu}) \neq (\nu, \bar{\nu})$,

$$\Pr_{D_{-i}|D_i}[h_{D_{-i}}(\nu', \bar{\nu}) \neq h_{D_{-i}}(\nu, \bar{\nu})] > 0, \quad (10)$$

where $D_{-i}$ is drawn from $p(D_{-i}|D_i)$ but not $p(D_{-i})$. This is because, by conditional independence of the datasets, for any event $\mathcal{E}$, we have

$$\Pr_{D_{-i}|D_i}[\mathcal{E}] = \int_{\theta \in \Theta} p(\theta|D_i) \Pr_{D_{-i}|\theta}[\mathcal{E}] d\theta$$

and

$$\Pr_{D_{-i}}[\mathcal{E}] = \int_{\theta \in \Theta} p(\theta) \Pr_{D_{-i}|\theta}[\mathcal{E}] d\theta.$$ 

Since both $p(\theta)$ and $p(\theta|D_i)$ are always positive because they are in exponential family, it should hold that

$$\Pr_{D_{-i}|D_i}[\mathcal{E}] > 0 \iff \Pr_{D_{-i}}[\mathcal{E}] > 0.$$

Therefore (9) is equivalent to (10), and we only need to show that the mechanism is sensitive if and only if (10) holds.

When we’re using a (canonical) model in exponential family, the prior $p(\theta)$ and the posteriors $p(\theta|D_i), p(\theta|D_{-i})$ can be represented in the standard form (7).

$$p(\theta) = \mathcal{P}(\theta|\nu_0, \bar{\nu}_0),$$

$$p(\theta|D_i) = \mathcal{P}(\theta|\nu_i, \bar{\nu}_i),$$

$$p(\theta|D_{-i}) = \mathcal{P}(\theta|\nu_{-i}, \bar{\nu}_{-i}),$$

$$p(\theta|\bar{D}_i) = \mathcal{P}(\theta|\nu'_i, \bar{\nu}'_i),$$

$$p(\theta|\bar{D}_{-i}) = \mathcal{P}(\theta|\nu'_{-i}, \bar{\nu}'_{-i}).$$
where \( \nu_0, \tau_0 \) are the parameters for the prior \( p(\theta) \), \( \nu_i, \tau_i \) are the parameters for the posterior \( p(\theta|D_i) \), \( \nu_{-i}, \tau_{-i} \) are the parameters for the posterior \( p(\theta|D_{-i}) \), and \( \nu'_i, \tau'_i \) are the parameters for \( p(\theta|\bar{D}_i) \).

From the proof for Theorem 5.1, we know that the difference between the expected score of reporting \( D_i \) and the expected score of reporting \( \bar{D}_i \neq D_i \) is equal to

\[
\Delta_{\text{Rev}} = D_{KL}(p(D_{-i}|D_i)||p(D_{-i}|\bar{D}_i)).
\]

Therefore if \( p(D_{-i}|D_i) \) differs from \( p(D_{-i}|\bar{D}_i) \) with non-zero probability, that is,

\[
\Pr_{D_{-i}|D_i} [p(D_{-i}|D_i) \neq p(D_{-i}|\bar{D}_i)] > 0,
\]

then \( \Delta_{\text{Rev}} > 0 \). By Lemma 4.2 and Lemma 4.3

\[
p(D_{-i}|D_i) = \int_{\theta \in \Theta} \frac{p(\theta|D_i)p(\theta|D_{-i})}{p(\theta)} d\theta = \frac{g(\nu_i, \tau_i)g(\nu_{-i}, \tau_{-i})}{g(\nu_0, \tau_0)g(\nu_i + \nu_{-i} - \nu_0, \nu_i \tau_{-i} + \nu_{-i} \tau_i - \nu_i \tau_{-i} - \nu_0 \tau_0)},
\]

\[
p(D_{-i}|\bar{D}_i) = \int_{\theta \in \Theta} \frac{p(\theta|\bar{D}_i)p(\theta|D_{-i})}{p(\theta)} d\theta = \frac{g(\nu'_i, \tau'_i)g(\nu_{-i}, \tau_{-i})}{g(\nu_0, \tau_0)g(\nu'_i + \nu_{-i} - \nu_0, \nu'_i \tau_{-i} + \nu_{-i} \tau'_i - \nu'_i \tau_{-i} - \nu_0 \tau_0)}.
\]

Therefore (11) is equivalent to

\[
\Pr_{D_{-i}|D_i} [h_{D_{-i}}(\nu_i, \tau_i) \neq h_{D_{-i}}(\nu'_i, \tau'_i)] > 0.
\]

Therefore if for all \((\nu'_i, \tau'_i) \neq (\nu_i, \tau_i)\), we have

\[
\Pr_{D_{-i}|D_i} [h_{D_{-i}}(\nu_i, \tau_i) \neq h_{D_{-i}}(\nu'_i, \tau'_i)] > 0,
\]

then reporting any \((\nu'_i, \tau'_i) \neq (\nu_i, \tau_i)\) will lead to a strictly lower expected score, which means the mechanism is sensitive. To prove the other direction, if the above condition does not hold, i.e., there exists \((\nu'_i, \tau'_i) \neq (\nu_i, \tau_i)\) with

\[
\Pr_{D_{-i}|D_i} [h_{D_{-i}}(\nu'_i, \tau'_i) \neq h_{D_{-i}}(\nu_i, \tau_i)] = 0,
\]

then reporting \((\nu'_i, \tau'_i) \neq (\nu_i, \tau_i)\) will give the same expected score as truthfully reporting \((\nu_i, \tau_i)\), which means that the mechanism is not sensitive.

**D Multiple-time data acquisition**

**D.1 Sensitivity analysis**

We first give the sensitivity analysis for finite-size \(|\Theta|\). The results are basically the same as the ones for the one-time data acquisition mechanism except that we do not give a lower bound for \(\alpha\).

**Theorem D.1.** When \(|\Theta|\) is finite, if \(f\) is strictly convex, then Mechanism \(I\) is sensitive in the first \(T - 1\) rounds if either of the following two conditions holds,

1. \(\forall i, Q_{-i} \text{ has rank } |\Theta|\),
2. \(\forall i, \sum_{i' \neq i} \text{ rank}_k(G_{i'}) \cdot (N_{i' - 1}) + 1 \geq |\Theta|\).

When \(\Theta \subseteq \mathbb{R}^m\) is a continuous space, the results are entirely similar to the ones for Mechanism \(I\) but with slightly different proofs.

Suppose the data analyst uses a model from the exponential family so that the prior and all the posterior of \(\theta\) can be written in the form in Lemma 4.1. The sensitivity of the mechanism will depend on the normalization term \(g(\nu, \tau)\) (or equivalently, the partition function) of the pdf. Define

\[
h_{D_{-i}}(\nu_i, \tau_i) = \frac{g(\nu_i, \tau_i)}{g(\nu_i + \nu_{-i} - \nu_0, \nu_i \tau_{-i} + \nu_{-i} \tau_i - \nu_i \tau_{-i} - \nu_0 \tau_0)}.
\]

then we have the following sufficient and necessary conditions for the sensitivity of the mechanism.

**Theorem D.2.** When \(\Theta \subseteq \mathbb{R}^m\), if the data analyst uses a model in the exponential family and a strictly convex \(f\), then Mechanism \(I\) is sensitive in the first \(T - 1\) rounds if and only if for any \((\nu'_i, \tau'_i) \neq (\nu_i, \tau_i)\), we have \(\Pr_{D_{-i}} [h_{D_{-i}}(\nu'_i, \tau'_i) \neq h_{D_{-i}}(\nu_i, \tau_i)] > 0\).

See Section 5 for interpretations of this theorem.
D.2 Missing proofs

The following part are the proofs for our results.

**Proof of Theorem 6.1.** It is easy to verify that the mechanism is IR, budget-fixed and symmetric. We prove the truthfulness as follows.

Let’s look at the payment for day $t$. At day $t$, data provider $i$ reports a dataset $\bar{D}_i^{(t)}$. Assuming that all other data providers truthfully report $\bar{D}_j^{(t)}$, data provider $i$’s expected payment is decided by his expected score

$$\mathbb{E}_{\bar{D}_i^{(t)}, \bar{D}_j^{(t+1)}|\bar{D}_i^{(t)}}[s_i] = \mathbb{E}_{\bar{D}_i^{(t)}|\bar{D}_i^{(t)}}f'(v(\tilde{q}_i, p(\theta|D_i^{(t)}))) - \mathbb{E}_{\bar{D}_j^{(t+1)}}f^*(f'(v(\tilde{q}_i, p(\theta|D_j^{(t+1)})))) .$$

(13)

The second expectation is taken over the marginal distribution $p(D_j^{(t+1)})$ without conditioning on $D_i^{(t)}$ because $D_i^{(t+1)}$ is independent from $D_i^{(t)}$, so we have $p(D_j^{(t+1)}|D_i^{(t)}) = p(D_j^{(t+1)}|D_i^{(t)})$.

We then use Lemma A.1 to get an upper bound of the expected score (13) and show that truthfully reporting $D_i$ achieves the upper bound. We apply Lemma A.1 on two distributions of $D_i$, the distribution of $D_{-i}$ conditioning on $D_i$, $p(D_{-i}|D_i)$, and the marginal distribution $p(D_{-i})$. Then we have

$$D_f(p(D_{-i}|D_i), p(D_{-i})) \geq \sup_{g \in \mathcal{G}} \mathbb{E}_{D_{-i}|D_i}[g(D_{-i})] - \mathbb{E}_{D_{-i}}[f^*(g(D_{-i}))],$$

(14)

where $f$ is the given convex function, $\mathcal{G}$ is the set of all real-valued functions of $D_{-i}$. The supremum is achieved and only achieved at function $g$ with

$$g(D_{-i}) = f'(\frac{p(D_{-i}|D_i)}{p(D_{-i})}) \quad \text{for all } D_{-i}.$$  

(15)

Consider function $g_{\tilde{q}_i}(D_{-i}) = f'(v(\tilde{q}_i, p(\theta|D_{-i})))$. Then (14) gives an upper bound of the expected score (13) as

$$D_f(p(D_{-i}|D_i), p(D_{-i})) \geq \mathbb{E}_{D_{-i}|D_i}[g_{\tilde{q}_i}(D_{-i})] - \mathbb{E}_{D_{-i}}[f^*(g_{\tilde{q}_i}(D_{-i}))]$$

$$= \mathbb{E}_{D_{-i}}[f'(v(\tilde{q}_i, p(\theta|D_{-i}))))] - \mathbb{E}_{D_{-i}}[f^*(f'(v(\tilde{q}_i, p(\theta|D_{-i}))))].$$

By (15), the upper bound is achieved only when

$$g_{\tilde{q}_i}(D_{-i}) = f'(\frac{p(D_{-i}|D_i)}{p(D_{-i})}) \quad \text{for all } D_{-i},$$

that is

$$f'(v(\tilde{q}_i, p(\theta|D_{-i}))) = f'(\frac{p(D_{-i}|D_i)}{p(D_{-i})}) \quad \text{for all } D_{-i}.$$  

(16)

Then it is easy to prove the truthfulness. Truthfully reporting $D_i$ achieves (16) because by Lemma A.2 for all $D_i$ and $D_{-i}$,

$$v(p(\theta|D_i), p(\theta|D_{-i})) = \frac{p(D_i, D_{-i})}{p(D_i)p(D_{-i})} = \frac{p(D_{-i}|D_i)}{p(D_{-i})}.$$  

Again, let $Q_{-i}$ be a $(\prod_{j \in [n], j \neq i}|D_j|^{N_j}) \times |\Theta|$ matrix with elements equal to $p(\theta|D_{-i})$ and let $G_i$ be the $|D_i| \times |\Theta|$ data generating matrix with elements equal to $p(\theta|d_i)$. Then we have the following sufficient conditions for the mechanism’s sensitivity.

**Proof of Theorem D.1.** We then prove the sensitivity. For discrete and finite-size $\Theta$, we prove that when $f$ is strictly convex and $Q_{-i}$ has rank $|\Theta|$, the mechanism is sensitive. When $f$ is strictly convex, $f'$ is a strictly increasing function. Then condition (16) is equivalent to

$$v(\tilde{q}_i, p(\theta|D_{-i})) = \frac{p(D_{-i}|D_i)}{p(D_{-i})} \quad \text{for all } D_{-i}.$$  

(17)
We show that when matrix \( Q_{-i} \) has rank \(|\Theta|\), \( \bar{q}_i = p(\theta|D_i) \) is the only solution of (17), which means that the payment rule is sensitive. By definition of \( v(\cdot) \),

\[
v(\bar{q}_i, p(\theta|D_{-i})) = \sum_{\theta \in \Theta} \bar{q}_i(\theta)p(\theta|D_{-i}) = (Q_{-i}\Lambda\bar{q}_i)_{D_{-i}}
\]

where \( \Lambda \) is the \(|\Theta| \times |\Theta| \) diagonal matrix with \( 1/p(\theta) \) on the diagonal. Then if \( \bar{q}_i = p(\theta|D_i) \) and \( \bar{q}_i = q \) are both solutions of (17), we must have

\[
Q_{-i}\Lambda p(\theta|D_i) = Q_{-i}\Lambda q \quad \Longrightarrow \quad Q_{-i}\Lambda(p(\theta|D_i) - q) = 0.
\]

Since \( Q_{-i}\Lambda \) must have rank \(|\Theta|\), which means that the columns of \( Q_{-i}\Lambda \) are linearly independent, we must have

\[
p(\theta|D_i) - q = 0,
\]

which completes our proof of sensitivity for finite-size \( \Theta \).

**Proof of Theorem D.2.** When \( \Theta \subseteq \mathbb{R}^m \) and a model in the exponential family is used, we prove that when \( f \) is strictly convex, the mechanism will be sensitive if and only if for any \((\nu'_i, \tau'_i) \neq (\nu_i, \tau_i)\),

\[
\Pr_{D_{-i}|D_i}[h_{D_{-i}}(\nu'_i, \tau'_i) \neq h_{D_{-i}}(\nu_i, \tau_i)] > 0. \tag{18}
\]

We first show that the above condition is equivalent to that for any \((\nu'_i, \tau'_i) \neq (\nu_i, \tau_i)\),

\[
\Pr_{D_{-i}|D_i}[h_{D_{-i}}(\nu'_i, \tau'_i) \neq h_{D_{-i}}(\nu_i, \tau_i)] > 0, \tag{19}
\]

where \( D_{-i} \) is drawn from \( p(D_{-i}|D_i) \) but not \( p(D_{-i}) \). This is because, by conditional independence of the datasets, for any event \( E \), we have

\[
\Pr_{D_{-i}|D_i}[E] = \int_{\theta \in \Theta} p(\theta|D_i) \Pr_{D_{-i}|\theta}[E] \, d\theta
\]

and

\[
\Pr_{D_{-i}}[E] = \int_{\theta \in \Theta} p(\theta) \Pr_{D_{-i}|\theta}[E] \, d\theta.
\]

Since both \( p(\theta) \) and \( p(\theta|D_i) \) are always positive because they are in exponential family, it should hold that

\[
\Pr_{D_{-i}|D_i}[E] > 0 \quad \iff \quad \Pr_{D_{-i}}[E] > 0.
\]

Therefore (18) is equivalent to (19), and we only need to show that the mechanism is sensitive if and only if (19) holds.

We then again apply Lemma [A.1] By Lemma [A.1] and the strict convexity of \( f \), \( \bar{q}_i \) achieves the supremum if and only if

\[
v(\bar{q}_i, p(\theta|D_{-i})) = \frac{p(D_{-i}|D_i)}{p(D_{-i})} \quad \text{for all } D_{-i}.
\]

By the definition of \( v \) and Lemma [A.2] the above condition is equivalent to

\[
\int_{\theta \in \Theta} \bar{q}_i(\theta)p(\theta|D_{-i}) \, d\theta = \int_{\theta \in \Theta} \frac{p(\theta|D_i)p(\theta|D_{-i})}{p(\theta)} \, d\theta \quad \text{for all } D_{-i}. \tag{20}
\]

When we’re using a (canonical) model in exponential family, the prior \( p(\theta) \) and the posteriors \( p(\theta|D_i), p(\theta|D_{-i}) \) can be represented in the standard form (7),

\[
p(\theta) = \mathcal{P}(\theta|\nu_0, \tau_0),
\]

\[
p(\theta|D_i) = \mathcal{P}(\theta|\nu_i, \tau_i),
\]

\[
p(\theta|D_{-i}) = \mathcal{P}(\theta|\nu_{-i}, \tau_{-i}),
\]

\[
\bar{q}_i = \mathcal{P}(\theta|\nu'_i, \tau'_i),
\]

24
where $\nu_0, \tau_0$ are the parameters for the prior $p(\theta)$, $\nu_i, \tau_i$ are the parameters for the posterior $p(\theta|D_i)$, $\nu_i, \tau_i$ are the parameters for the posterior $p(\theta|D_{-i})$, and $\nu_i', \tau_i'$ are the parameters for $\tilde{q}_i$. Then by Lemma A.3 the condition that $\tilde{q}_i$ achieves the supremum (20) is equivalent to

$$\frac{g(\nu_i', \tau_i')}{g(\nu_i + \nu_i' - \nu_0, \tau_i + \tau_i' - \nu_0 \tau_0)} = \frac{g(\nu_i, \tau_i)}{g(\nu_i + \nu_i' - \nu_0, \tau_i + \tau_i' - \nu_0 \tau_0)},$$

for all $D_{-i}$, (21)

which, by our definition of $h(\cdot)$, is just

$$h_{D_{-i}}(\nu_i', \tau_i') = h_{D_{-i}}(\nu_i, \tau_i),$$

for all $D_{-i}$.

Now we are ready to prove Theorem D.2. Since (18) is equivalent to (19), we only need to show that the mechanism is sensitive if and only if for all $(\nu_i', \tau_i') \neq (\nu_i, \tau_i)$,

$$\Pr_{D_{-i}|D_i} [h_{D_{-i}}(\nu_i', \tau_i') \neq h_{D_{-i}}(\nu_i, \tau_i)] > 0.$$

If the above condition holds, then $\tilde{q}_i$ with parameters $(\nu_i', \tau_i') \neq (\nu_i, \tau_i)$ should have a non-zero loss in the expected score (13) compared to the optimal solution $p(\theta|D_i)$ with parameters $(\nu_i, \tau_i)$, which means that the mechanism is sensitive. For the other direction, if the condition does not hold, i.e., there exists $(\nu_i', \tau_i') \neq (\nu_i, \tau_i)$ with

$$\Pr_{D_{-i}|D_i} [h_{D_{-i}}(\nu_i', \tau_i') \neq h_{D_{-i}}(\nu_i, \tau_i)] = 0,$$

then reporting $(\nu_i', \tau_i') \neq (\nu_i, \tau_i)$ will give the same expected score as truthfully reporting $(\nu_i, \tau_i)$, which means that the mechanism is not sensitive.