Properties of stepwise irregular graphs

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Abstract Stepwise irregular (SI) graphs were introduced by Ivan Gutman recently in 2018 and in these graphs the difference between the degrees of any two adjacent vertices is exactly one. In this work, we show the existence of connected bicyclic SI graphs of order 5 and 9 which has been claimed to be non-existent by Gutman. We also show the existence of SI graphs of different order and cyclomatic numbers when there exist a SI graph with a vertex of degree 1 or 2. We give a characterization on the order of the connected tricyclic SI graphs. At the end, we investigate some properties of the SI graphs under elementary graph operations along with the lower and upper bound of the irregularity of SI graphs.

Keywords Graph irregularity · Stepwise irregular graph · Line graph · Total graph · Albertson index

1 Introduction

Let $G = G(V, E)$ be a connected simple graph with $|V(G)| = n$ (n number of vertices) and $|E(G)| = m$ (m number of edges). The order of a graph $G$ is the number of vertices in $G$. The neighbourhood of a vertex $v$ in $G$ is denoted by $nbd(v)$ and it contains all the vertices adjacent to $v$. Its cardinality is called the degree of the vertex $v$. The degree of a vertex $v$ in $G$ is often represented by $d_G(v)$. The maximum and minimum degree of the vertices in $G$ is denoted by $\Delta(G)$ and $\delta(G)$ respectively. A path of length $(n - 1)$ with $n$ vertices is denoted by $P_n$. 

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A graph is regular if all its vertices are of same degree, otherwise it is irregular. To observe the irregularity of a given graph has been an important direction of research. There are several articles \cite{1,2,11,13,14} in literature to investigate the irregularity measures of a graph. Also, irregularity of graphs has widely been used for analyzing topological structures of deterministic and random networks occurring in chemistry, bio-informatics and social networks \cite{4,7}.

If \( u \) and \( v \) are two adjacent vertices in \( G \), then the edge between \( u \) and \( v \) is denoted by \( uv \). The imbalance of an edge \( uv \in E(G) \), defined by Albertson \cite{3}, is \( |d_G(u) - d_G(v)| \) and the irregularity of \( G \), which is the most popular and most frequently used graph index, is \( irr(G) = \sum_{uv \in E(G)} |d_G(u) - d_G(v)| \).

Among many degree based topological graph indices, two popular indices are the first Zagreb index \( M_1(G) \) and the second Zagreb index \( M_2(G) \) where

\[
M_1(G) = \sum_{v \in V(G)} d_G(v)^2 \quad \text{and} \quad M_2(G) = \sum_{uv \in E(G)} d_G(u)d_G(v).
\]

There are another multiplicative versions of Zagreb indices such as

\[
\Pi_1(G) = \prod_{v \in V(G)} d_G(v)^2 \quad \text{and} \quad \Pi_2(G) = \prod_{uv \in E(G)} d_G(u)d_G(v).
\]

The cyclomatic number of a connected graph \( G \) with order \( n \) and \( m \) edges is \( \gamma(G) = m - n + 1 \). The graphs with cyclomatic number \( \gamma = 0, 1, 2, 3, 4 \) are called tree, unicyclic, bicyclic, tricyclic and tetracyclic graphs respectively.

In this work, we investigate the properties of the stepwise irregular graphs introduced by Ivan Gutman \cite{9}. We show the existence of stepwise irregular graphs of order 5 and 9. The existence of different stepwise irregular graphs and cyclomatic numbers also have been shown when the precondition of existence of stepwise irregular graph with at least one vertex of degree 1 or 2 is satisfied. Moreover, a characterization on the order of the vertices of a tricyclic graph has been provided using the previous results. We further have shown that every integer in the interval \([\delta(G), \Delta(G)]\) must be degree of some vertex in \( G \). Next we show that the stepwise irregular graphs are not closed under edge deletion, vertex deletion and complementation. The subdivision, line and total graph of a SI graph is also not SI but the conditions under which these graphs becomes SI have been investigated. The paper ends with the lower and upper bound of the irregularity of stepwise irregular graphs.

The paper is arranged as follows: At first we recall existing results on SI graphs in Section 2. Section 3 investigates several properties of SI graphs and its existence. At the end, we study the properties of the SI graphs under elementary graph operations, along with the bounds of the irregularity of the SI graphs in section 4.
2 Basic results

Followings are some important results on stepwise irregular graphs:

**Lemma 1** [9] The number of edges of a stepwise irregular graph is even.

**Lemma 2** [9] Stepwise irregular graphs are bipartite.

**Lemma 3** [9] Let $G_0$ be an SI graph of order $n_0$ and cyclomatic number $\gamma$, possessing a vertex of degree 1. Then for all $k = 1, 2, ..., $ there exist SI graphs of order $n_0 + 4k$ and cyclomatic number $\gamma$.

**Theorem 1** [9] The order of a stepwise irregular bicyclic graph (graph with $\gamma = 2$) is an odd integer. There exist stepwise irregular bicyclic graphs whose order is any positive odd integer, except 1, 3, 5, 7, 9, and 11.

**Theorem 2** [9] There exist connected stepwise irregular graphs of any order, except 1, 2, 4, 5, and 6.

3 Main Results

In this section, at first we show that first and second Zagreb index and multiplicative versions of Zagreb indices for stepwise irregular graph are even integers. Next we show that there exist connected bicyclic stepwise irregular graph of order 5 and 9 which is a contradiction to Theorem 1 and 2.

**Theorem 3** If $G$ be a SI graph then the graph indices $M_1(G), M_2(G), \Pi_1(G)$ and $\Pi_2(G)$ are even integer.

**Proof** We see that

$$M_1(G) = \sum_{v \in V(G)} d_G(v)^2 = \left( \sum_{v \in V(G)} d_G(v) \right)^2 - 2 \sum_{u \neq v \in V(G)} d_G(u)d_G(v).$$

Now since the sum of degrees of the vertices in $G$ is even number, $M_1(G)$ is even.

Since $G$ is a SI graph, for any edge $uv \in E(G)$, $d_G(u) = d_G(v) \pm 1$ and therefore if $d_G(u)$ is even, then $d_G(v)$ is odd and vice versa. This implies that for any edge $uv \in E(G)$, $d_G(u)d_G(v)$ is even. Thus $M_2(G) = \sum_{uv \in E(G)} d_G(u)d_G(v)$ is even.

On the other hand, the other two multiplicative Zagreb indices $\Pi_1(G)$ and $\Pi_2(G)$ must be even as at least one of the degree of a vertex must be even in a graph.
3.1 On existence of SI graph

In [9], it has been claimed that there do not exist connected bicyclic stepwise irregular graphs of order 5 and 9. In the next result, we show that the claim is in fact not true by constructing examples of connected stepwise irregular graphs of order 5 and 9.

**Theorem 4** There exist connected stepwise irregular bicyclic (with cyclomatic number $\gamma = 2$) graphs of order 5 and 9.

**Proof** The two graphs shown in Figure 1 proves the above claim.

![Bicyclic SI graph of order 5 and 9](image)

**Fig. 1** Bicyclic SI graph of order 5 and 9

Using the above theorem we refine the statements of Theorem 1 and 2 as follows:

**Theorem 5** The order of a stepwise irregular bicyclic graph (graph with $\gamma = 2$) is an odd integer. There exist stepwise irregular bicyclic graphs whose order is any positive odd integer, except 1, 3, 7, and 11.

**Theorem 6** There exist connected stepwise irregular graphs of any order, except 1, 2, 4, and 6.

In the next result, we prove that existence of a SI graph of order $n$ with cyclomatic number $\gamma$ along with a vertex of degree one implies the existence of SI graphs of order $n + 4k + i$ with cyclomatic number $\gamma + i$, $0 \leq i \leq 2$. This result is a different kind of generalization of the SI graphs same in the line of Lemma 3. To prove the result, we need following lemmas.

**Lemma 4** Let $G_0$ be a stepwise irregular graph whose vertex $u$ is of degree 1, cf. Figure 3. The stepwise irregular graph $G_1$ can be constructed by adding 5 new vertices as shown in cf. Figure 3. Moreover, if $G_0$ has cyclomatic number $\gamma$, then $G_1$ has cyclomatic number $\gamma + 1$.

**Proof** From the Figure 3 itself, the proof follows.
Lemma 5 Let $G_0$ be a stepwise irregular graph whose vertex $u$ is of degree 1, cf. Figure 3. A new stepwise irregular graph $G_1$ can be constructed by adding 6 vertices as shown in cf. Figure 3. The graph $G_1$ has cyclomatic number $\gamma + 2$, if $G_0$ has cyclomatic number $\gamma$.

Proof We construct the graph $G_1$ from $G_0$ in the following manner as given in Figure 3 which proves the above claim.

Using Lemma 4 and 5 we prove the next theorem.

Theorem 7 Let $G$ be a SI graph of order $n$ and cyclomatic number $\gamma$, possessing a vertex of degree 1. Then for all $k = 1, 2, \ldots$, there exist SI graphs of order $n + 4k + i$ and cyclomatic number $\gamma + i$, for $i = 0, 1, 2$.

Proof
1. $i = 0$ is same as Lemma 3
2. $i = 1$: for $k = 1$ by application of Lemma 4 once we get the required graph.
   For $k \geq 2$, by application of Lemma 4 once and then applying Lemma 3 $(k - 1)$ times, we get the required graph.
3. $i = 2$: for $k = 1$, by one time application of Lemma 5 and for $k = 2$, by two times application of Lemma 4 we get the required graph respectively.
   For $k = 3$, we can get the required graph by two times application of Lemma 4 and one time application of Lemma 3 respectively.
   For $k \geq 4$, by two times application of Lemma 4 and $(k - 2)$ times Lemma 3 we get the required graph.

Corollary 1 Let $G$ be a SI graph of order $n$ and cyclomatic number $\gamma$, possessing a vertex of degree 1. Then for all $k = 2, 3, \ldots$, there exist SI graphs of order $n + 4k + 3$ and cyclomatic number $\gamma + 3$. 
Proof For $k = 2$, first applying Lemma 4 and then Lemma 5 we get the required graph.

For $k = 3$, the required graph can be constructed by application of the Lemma 3 3 times.

For $k \geq 4$, first applying lemma 3 times and then Lemma 3 $(k - 3)$ times, we get the required graph.

In the above results, we proved that the existence of a SI graph with a pendant vertex imply the existence of SI graphs of different order and cyclomatic number. In the following results we show that the existence of a SI graphs with a vertex of degree 2 having neighbours of degree 3 proves the existence of SI graphs of order $n + 7k$ and cyclomatic number $\gamma + k$.

To prove the above result, we will use the following lemma.

**Lemma 6** Let $G_0$ be a stepwise irregular graph whose vertex $u$ is of degree 2 having neighbors each of which are of degree 3, cf. Figure 4. The SI graph $G_1$ can be constructed by adding 7 new vertices as shown in cf. Figure 4. Moreover, if $G_0$ has cyclomatic number $\gamma$, then $G_1$ has cyclomatic number $\gamma + 1$.

![Fig. 4](image)

The next theorem can be shown as a consequence of Lemma 6 and Lemma 3. In fact it can be proved by one time application of Lemma 6 and $(k - 1)$ times application of Lemma 3 to the graph $G_0$.

**Theorem 8** Let $G_0$ be a SI graph of order $n$ and cyclomatic number $\gamma$, possessing a vertex of degree 2 having neighbors each of which are of degree 3. Then for all $k = 1, 2, \ldots$, there exist SI graphs of order $n + 7k$ and cyclomatic number $\gamma + k$.

In [9], a characterization on the order of the connected bicyclic SI graphs was given by Gutman. In this work, we give a characterization on the order of the connected tricyclic SI graphs.
Theorem 10 The order of a SI tricyclic graph is an even integer. Moreover, there exist tricyclic SI graphs whose order is any positive integer except 2, 4, 6 and 8.

Proof By Theorem 5 it can be shown that there exist bicyclic SI graph order 13 + 2k, k ≥ 0. Now by applying Lemma 4 it can further be shown that there exist tricyclic SI graph of order 13 + 2k + 5, i.e., 18 + 2k, k ≥ 0.

Also, there exist bicyclic SI graph of order 5 and 9. Hence, the Lemma 6 proves the existence of tricyclic SI graph of order 12 and 16.

Gutman has already proved the existence of tricyclic SI graphs of order 10 and 14 in [9].

On the other hand, such tricyclic SI graph of order 2, 4, 6 and 8 do not exist.

In the next result, we show that each integer between the minimum and maximum degree of a SI graph, must be a degree of some vertex of the SI graph.

Theorem 11 Let G be a connected SI graph with maximum degree Δ(G) and minimum degree δ(G). Then every number k, δ(G) ≤ k ≤ Δ(G) belongs to the degree set d(G) of G.

Proof Suppose G be a connected SI graph with maximum degree M and minimum degree m. Clearly m and M belong to d(G).

Now if possible let there exists p, m < p < M such that p /∈ d(G).

If p + 1 ∈ d(G), set S be the set of all vertices whose degrees are greater than p. Then none of these vertices are connected to any vertices of degree less than p. This implies that G is disconnected - a contradiction.

If p + 1 /∈ d(G), by a finite number of steps we can always find a k such that p + k ≤ M and p + k ∈ d(G). Using the same argument above, it can be shown that G is disconnected - a contradiction.

Hence our assumption is wrong and the theorem is proved.

Note: There exist SI graphs which have only vertices of degree Δ(G) and δ(G). The complete bipartite graph G = k_{m,m+1}, m ≥ 1 is an example of such kind of graphs.

4 SI graph under elementary graph operations

In this section we investigate the SI property of graphs under several graph operations such as vertex (edge resp.) deleted subgraph, complement graph, line and total graphs etc.

For a given graph G and a vertex v ∈ V(G), the vertex deleted subgraph G − v is obtained by deleting the vertex v and its adjacent edges from G.

Similarly, for a given graph G and an edge e ∈ E(G), the edge deleted subgraph G − e is obtained by deleting the edge e from G.
After deleting either a vertex or an edge from a given connected graph we may get disconnected graph. A disconnected graph $G$ is said to be SI if each component of $G$ is SI.

We show in the next result that edge deletion does not preserve the property of stepwise irregularity.

**Theorem 12** Let $G$ be a SI graph. Then for any edge $e$ of $G$, $G - e$ is no longer SI.

**Proof** Let $e = xy$ be an edge of a SI graph $G$ and $H = G - e$. Also we have the order of $G$ at least 3.

Consider two adjacent vertices $u$ and $v$ in $H$. Then these $u$ and $v$ are also in $G$. Therefore we have $d_G(u) = d_G(v) ± 1$.

There may arise two cases.

1. Neither $u$ nor $v$ is $x$ or $y$. Then $|d_H(u) - d_H(v)| = |d_G(u) - d_G(v)| = 1$.
2. Either $u$ or $v$ (but not both, since we remove the edge $e = xy$) is $x$ or $y$. Without loss of generality let $u = x$ and $v \in V(G) - \{x, y\}$. Then $|d_H(u) - d_H(v)| = |d_G(u) - 1 - d_G(v)| = |d_G(v) ± 1| = 0$ or $2$.

Since the SI graph $G$ has at least 2 edges, there must exist an edge which falls under the second case.

By the second case it follows that $H$ is not SI.

Now in a similar manner we show that, vertex deletion does not preserve the property of stepwise irregularity.

**Theorem 13** Let $G$ be a SI graph. Then for any vertex $v$ of $G$, $G - v$ is not SI.

**Proof** Suppose $G$ be a SI graph and $v$ be a vertex of $G$. Let $H = G - v$.

Consider two adjacent vertices $x$ and $y$ in $H$. Then these $x$ and $y$ are also in $G$. Therefore we have $d_G(x) = d_G(y) ± 1$. There may arise two cases.

1. If $x, y \in V(G) - nbd(v)$, $|d_H(x) - d_H(y)| = |d_G(x) - d_G(y)| = 1$.
2. Either of $x$ or $y$ (but not both) is $v$ (it is because SI graphs are bipartite and if they are both in $nbd(v)$, $\{v, x, y\}$ will form a triangle).

Without loss of generality let $x \in nbd(v)$ and $y \in V(G) - nbd(v)$. Then $|d_H(x) - d_H(y)| = |d_G(x) - 1 - d_G(y)| = |d_G(y) ± 1| = 0$ or $2$.

By the second case it follows that $H$ is not SI.

Moreover, in the next result we show that like in the case of vertex and edge deletion, the complementation operation also does not preserve stepwise irregularity.

**Lemma 7** Let $G$ be a connected SI graph, then $\overline{G}$, the complement graph of $G$ is not SI.

**Proof** Let $G$ be a connected SI graph of order $n$ with maximum degree $M$ and minimum degree $m$. Then there must exist at least two vertices $u, v$ of same degree $k$, for some $k$, $m \leq k \leq M$, in $G$, which are not adjacent to each other. So the vertices $u$ and $v$ are adjacent in $\overline{G}$ having same degree $(n - 1 - k)$. Hence $\overline{G}$ is not SI.
4.1 Subdivision graph of SI graphs

The subdivision graph \([12]\) of a graph \(G\) is denoted by \(S(G)\) and is obtained from \(G\) by inserting a new vertex in each of the edges of \(G\).

**Theorem 14** If \(G\) is a SI graph, then the subdivision graph \(S(G)\) is not SI.

**Proof** Let \(G\) be a SI graph and \(e = uv\) be an edge of \(G\). Then \(d_G(u) = d_G(v) = 1\). Now in the subdivision graph \(S(G)\), a new vertex \(w\) (say) is inserted in \(e\). We see that \(d(w) = 2\) in \(S(G)\). Hence, either \(|d_{S(G)}(u) - d_{S(G)}(w)| \neq 1\) or \(|d_{S(G)}(v) - d_{S(G)}(w)| \neq 1\). This implies that \(S(G)\) is not SI.

**Example 1** There exist non-SI graphs whose subdivision graph is SI. A 3-regular graph is an example of such kind of graph.

Moreover, we can show that there exist only three types of graph for which the subdivision graph can be a SI graph.

**Theorem 15** The subdivision graph of any connected graph other than \(K_2\), \(K_{1,3}\) and 3-regular graphs, is not SI.

**Proof** If \(G = K_2\), \(S(G) = K_{1,2}\) which is SI. If \(G = K_{1,3}\), it can be directly verified that \(S(G)\) is SI. By Example \([1]\) the subdivision graph of 3-regular graph is SI.

Assume that \(G\) is a graph other than \(K_2\), \(K_{1,3}\) and 3-regular graphs. Then there exists at least one edge \(uw \in E(G)\) whose one of adjacent vertices has degree other than 1 and 3. In fact let the vertex \(u \in V(G)\) be such that \(d(u) \neq 1, 3\). Now in the subdivision graph \(S(G)\), a new vertex \(w\) (say) is inserted onto this edge and the degree of this new vertex is 2 which is also adjacent to the vertex \(u\). Clearly, \(|d(u) - d(w)| \neq 1\). Therefore \(S(G)\) can not be SI.

4.2 Line and total graph of SI graph

The line graph \(L(G)\) of a graph \(G\) is the graph whose vertices correspond to the edges of \(G\) and any two vertices in \(L(G)\) are adjacent if and only if the edges corresponding to the vertices are adjacent in \(G\) \([12]\). Moreover, if \(e = uv\) is an edge of \(G\) then \(deg_{L(G)}(e) = deg_G(u) + deg_G(v) - 2\).

**Theorem 16** Let \(G\) be a SI graph, then \(L(G)\), the line graph of \(G\), is not SI.

**Proof** Let \(G\) be a SI graph. Then \(G\) has at least two edges. Let \(e_1 = uv\) and \(e_2 = uw\) be two adjacent edges in \(G\). Then we have, \(d_G(v) = d_G(u) + 1\) and \(d_G(w) = d_G(u) + 1\).

Since \(e_1\) and \(e_2\) are adjacent in \(G\), they are adjacent in \(L(G)\). We also have, \(d_{L(G)}(e_1) = d_G(u) + d_G(v) - 2\) and \(d_{L(G)}(e_2) = d_G(u) + d_G(w) - 2\).

Therefore \(|d_{L(G)}(e_1) - d_{L(G)}(e_2)| = |d_G(v) - d_G(w)| = |d_G(u) + 1| - |d_G(u) + 1| = 0\) or 2. This implies that \(L(G)\) is not SI.
Note: There exist non-SI graphs whose line graph can be SI. The graph $P_4$ is not an SI graph, but the line graph of $P_4$ is $P_3$ which is a SI graph.

In fact, in the following result we show that $P_4$ is the only graph whose line graph is a SI graph.

**Theorem 17** $G = P_4$ if and only if $L(G)$ is a SI graph.

**Proof** If $G = P_4$, then $L(G) = P_3$ which is SI.

To prove the converse, suppose $L(G) = H$ (say) is a SI graph. If $L(G) = P_3$, we are done. If possible let, $H \neq P_3$. Then we claim that $H$ has $K_{1,3}$ as induced subgraph.

Proof of the claim: Since H is SI, there are two cases:

1. If $H = K_{m,m+1}$, $m \geq 2$, then clearly it has $K_{1,3}$ as induced subgraph.
2. If $H \neq K_{m,m+1}$, $m \geq 2$, then let the degree set of $H$ be $\{m, m+1, ..., M\}$ with minimum degree $m \geq 1$ and maximum degree $M$. Now consider the induced subgraph by the set of a vertex of degree $k+1$ ($\geq 3$) where $m < k < M$ and its any 3 adjacent vertices. We see that the induced subgraph forms $K_{1,3}$.

Therefore using the fact ([12], pp 74–75) that a graph having $K_{1,3}$ as induced subgraph can not be a line graph, we get a contradiction that $H$ is a line graph.

Hence the only possibility is $L(G) = P_3$.

This implies that $G = P_3$.

Using the definition of line graph the following can be shown.

**Theorem 18** Let $G$ be a SI graph. Then the degree of each vertex of the line graph $L(G)$ is an odd integer.

The total graph $T(G)$ of a graph $G$ is a graph whose vertex set is $V(G) \cup E(G)$ and two vertices in $T(G)$ are adjacent if and only if the corresponding elements are adjacent or incident in $G$ ([12]). If $u$ is a vertex of $G$, then $deg_{T(G)}(u) = 2deg_G(u)$. If $e = uv$ is an edge of $G$ then $deg_{T(G)}(e) = deg_G(u) + deg_G(v)$.

Further more, it is noted that every edge in a graph $G$ (say) produces at least one triangle in $T(G)$. Therefore, the total graph of any graph is not bipartite. This implies the following lemmas.

**Lemma 8** Let $G$ be a SI graph, then $T(G)$, the total graph of $G$, is not SI.

**Lemma 9** Let $G$ be a graph, then the total graph $T(G)$ can never be SI.

In [3], Albertson claimed that the irregularity of a bipartite graph is maximum if it is a complete bipartite graph. Using this result we give a lower and upper bound for the number of edges of a SI graph of given order and show that both the bounds are tight.

**Theorem 19** Let $G$ be a connected SI graph of order $n$ and $m$ be the number of edges of $G$. Then $m$ is even and $n - 1 \leq m \leq \frac{n^2 - 1}{2}$. Moreover, for odd $n$, the lower and upper bound is attained when $G$ is a SI tree graph and $K_{n-1, n+1}$ respectively.
Proof Let $G$ be a connected SI graph of order $n$ and $m$ edges. Since $G$ is connected, it must have $n-1$ edges. Therefore, $m \geq n-1$. And the equality holds for all SI tree graphs.

To prove the upper bound, since $G$ is SI, it is bipartite and the irregularity of $G$ is same as the number of edges of $G$. It is known that the irregularity of a bipartite graph is maximum if it is a complete bipartite graph. The maximum irregularity of $G$ can be obtained by partitioning $n$ vertices into two vertex sets in such a way that the number of edges is maximum. Let the partition of $n$ be $n = \frac{n-k}{2} + \frac{n+k}{2}$, for some $k \geq 1$. The complete bipartite graph $K_{\frac{n-k}{2}, \frac{n+k}{2}}$ has $\frac{n-k}{2} \cdot \frac{n+k}{2}$ edges. Thus $m = \text{irr}(G) \leq \frac{n-k}{2} \cdot \frac{n+k}{2} \leq \frac{n^2-1}{4}$.

The equality holds if $n$ is odd and $k = 1$. In this case we get $G = K_{\frac{n-1}{2}, \frac{n+1}{2}}$, which is SI. Therefore for complete bipartite SI graph $K_{\frac{n-1}{2}, \frac{n+1}{2}}$, the equality holds.

Note: By the definition of SI graphs, it is clear that the disjoint union of SI graphs is SI. The join of two graphs $G$ and $H$, denoted by $G+H$, is the graph obtained by $V(G+H) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. Using the definition of join of two graphs it can be shown that for any two non trivial graphs $G$ and $H$, the join $G+H$ is not bipartite. Hence the join can never be SI graph unless $G = K_m, H = K_{m+1}$.

Product graphs are very useful to construct large graphs from small graphs and has been used to design interconnection networks. Some of products of two graphs are lexicographic, direct, Cartesian and strong product [15]. Among these products, lexicographic and strong product of two non trivial graphs is not bipartite, therefore not SI. On the other hand, it can be shown that the direct and Cartesian product of two graphs need not be SI.

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