FROM TAYLOR SERIES OF ANALYTIC FUNCTIONS TO THEIR GLOBAL ANALYSIS

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Abstract. We analyze the conditions on the Taylor coefficients of an analytic function to admit global analytic continuation, complementing a recent paper of Breuer and Simon on general conditions for natural boundaries to form. A new summation method is introduced to convert a relatively wide family of infinite sums and local expansions into integrals. The integral representations yield global information such as analytic continuability, position of singularities, asymptotics for large values of the variable and asymptotic location of zeros.

1. Introduction

Finding the global behavior of an analytic function in terms of its Taylor coefficients is a notoriously difficult problem, in fact one which is impossible in full generality since undecidable statements can be formulated in these terms. However, a very interesting and quite general criterion for the disk of convergence of a Taylor series to coincide with its maximal domain of analyticity was recently discovered by Breuer and Simon [2] (see also [5]). The present paper complements this result by finding criteria on the Taylor coefficients, say at zero, for the associated analytic function not to have natural boundaries and to belong to the class $\mathcal{M}$ of functions analytic in the complex plane with finitely many cuts and with algebraic behavior at infinity (see Definition 2.1 below). Our condition is that the coefficients $c_k$ admit generalized Borel summable (or Ecalle-Borel summable, EB) transseries in $k$. Many general classes of problems in analysis are known to have EB transseries solutions. For details on generalized Borel summability, transseries and resurgence see [4, 8, 10, 11, 12].

In particular it is known [3, 10, 13] that, if the $c_k$ are solutions of generic linear or nonlinear recurrence relations of arbitrary but finite order with analytic coefficients, then they are EB-summable. Recurrence relations exist for instance when the coefficients are obtained by solving differential equations by power series. In a forthcoming paper we show that the Taylor coefficients of the Borel transform of solutions of generic systems of linear or nonlinear ODEs (in the setting of [19]) also admit EB summable transseries. The global analytic structure of the Borel transform is crucial in understanding the monodromy of solutions of such equations.

We also extend our procedure to analyze the global behavior of entire functions, and to formal series, giving criteria directly on the coefficients for the formal series to be Borel summable.

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Globally reconstructing function from its Taylor coefficients, when these admit EB summable transseries is effective, constructive and explicit - in the sense of producing integral representations far easier to analyze than the sums; this provides a new summation procedure, generalizing in some ways the Poisson summation formula.

We recently used this approach to analyze a class of linear PDEs with variable coefficients, [7]. One can obtain explicit integral representations for solutions of ODEs not known to be solvable such as

\begin{equation}
A\eta^2 f^{(4)} + 2A\eta f''' + \frac{1}{2}\eta f'' - (1 + a)f = 0
\end{equation}

arising as the scaling pinching profile \( h \sim (t_c - t)f(x(t_c - t)^{-1/2}) \) of the thin film equation,

\[ h_t + (h_{xx}h)_x = 0, \quad h \sim (t_c - t)f(x(t_c - t)^{-1/2}) \]

where \( t_c \) is the singularity time, where the interest is in the solution \textit{analytic at zero}, \( f_0 \). Let \( A \neq 0 \), and \( a + \frac{1}{2} \notin \mathbb{N} \). While it is not clear how to obtain representations of \( f_0 \) itself, the Taylor coefficients of \( f_0 \) are explicit. From this, our technique introduced in [2,1] by first considering the Laplace transform \( F(p) = \mathcal{L}f(p) \), which is expressible in terms of integrals of Whittaker functions: with

\begin{equation}
F(p) = Ce^{-\frac{p^2}{4\pi^2}}p^{-3/2} \left[ M_{a-\frac{3}{4}, \frac{1}{4}} \left( \frac{p^2}{4A} \right) + \frac{a\Gamma(a)}{2\sqrt{\pi}} W_{a-\frac{3}{4}, \frac{1}{4}} \left( \frac{p^2}{4A} \right) \right]
\end{equation}

we have \( f = \mathcal{L}^{-1}F \). The proof of (1.2) is sketched in [4,3].

In particular, if \( c_k = \varphi(k) \) where the function \( \varphi \), defined in the right half plane, is inverse Laplace transformable and its inverse Laplace transform \( \mathcal{L}^{-1}\varphi \) can be calculated in closed form, the function \( f \) has integral representations in terms of \( \varphi \). We will use some particularly simple examples for illustration. For the first one, the generalized Hurwitz zeta function, our procedure quickly yields one of the known integral representations. For the other three, our procedure yields integral representations while the global behavior of \( \sum_k c_k z^k \) does not follow in any other obvious way:

\begin{align}
c_k^{[1]} &= \frac{1}{(k + a)^b}; \quad c_k^{[2]} = \frac{1}{k^b + \ln k}, \quad c_k^{[3]} = \frac{1}{k^{b+1}}, \quad c_k^{[4]} = e^\sqrt{\pi}, \quad (a, b > 0)
\end{align}

We find that

\begin{equation}
f_1(z) := \sum_{k=1}^{\infty} c_k^{[1]} z^k = \frac{z}{\Gamma(b)} \int_0^\infty \frac{\ln(1 + t)^{b-1} dt}{(1 + t)^a t (t - (z - 1))}
\end{equation}

On the first Riemann sheet \( f_1 \) has only one singularity, at \( z = 1 \), of logarithmic type, and \( f_1 = o(z) \) for large \( z \). General Riemann surface information and monodromy follow straightforwardly from (1.4). A similar complex analytic structure is shared by \( f_2 = \sum_{k=1}^{\infty} c_k^{[2]} z^k \), which has one singularity at \( z = 1 \) where it is analytic in \( \ln(1 - z) \) and \( (1 - z) \); more precisely,

\begin{equation}
f_2(z) = -\frac{1}{2\pi i} \frac{\ln z}{z} \int_0^\infty \frac{e^{-u \ln(z)}}{(-u)^b + \ln(-u)} du + E(z)
\end{equation}

see Definition 2.2, where \( E \) is entire.
The function \( f_3(z) = \sum_{k=1}^{\infty} c_k^3 z^k \) is entire; questions answered regard say the behavior for large negative \( z \) (certainly not obvious from the series) or the asymptotic location of zeros. It will follow that \( f_3 \) can be written as

\[
(1.6) \quad f_3(z) = \int_{0}^{\infty} (1 + u)^{-1} G(\ln(1 + u)) \left[ \exp \left( \frac{ze^{-1}}{1 + u} \right) - 1 \right] \, du
\]

where \( G(p) = s'_2(1 + p) - s'_1(1 + p) \) and \( s_{1,2} \) are two branches of the functional inverse of \( s - \ln s \), cf. [1.2]. Using the integral representation of \( f_3 \), its behavior for large \( z \) can be obtained from (1.6) by standard asymptotics methods; in particular, for large negative \( z \), \( f_3 \) behaves like a constant plus \( z^{-1/2}e^{-z/e} \) times a factorially divergent series (whose terms can be calculated).

For \( c_k^4 \) we find

\[
(1.7) \quad f_4(z) = \sum_{k=1}^{\infty} c_k^4 z^k = -\frac{z}{2\sqrt{\pi}} \int_{C_1} \frac{p^{-3/2}e^{-\frac{z}{p}} \, dp}{e^p - z}
\]

where \( C_1 \) is a spiral \( S_1 \) followed by \([1, \infty)\), where \( S_1 \) starts at 0 and ends at 1, and is given in polar coordinates by \( r = \theta e^{2\pi i \theta}, \, \theta \in [0, 1] \).

We also show that Borel summation of divergent series or transseries of resurgent functions with finitely many Borel-plane singularities, as well as the Abel-Plana version of the Euler-Maclaurin summation formula (see also [6]) can be derived by a natural extension of our analysis. Another illustration is obtaining the closed form Borel summed formula for \( \ln \Gamma \), cf. (2.32) below.

A separate category is represented by lacunary series. Their coefficients do not satisfy our assumption; however a slightly different approach allows for a detailed study of the associated functions as the natural boundary is approached, [9].

2. Main results

A first class of problems is finding the location and type of singularities in \( \mathbb{C} \) and the behavior for large values of the variable of functions given by series with finite radius of convergence (Theorem 2.1), such as the first, second and fourth in (1.3).

The second class of problems amenable to the techniques presented concerns the behavior at infinity (growth, decay, asymptotic location of zeros etc.) of entire functions presented as Taylor series (Theorem 2.2).

The third class of problems is essentially the converse of the two above: given a function that has analytic continuation on some Riemann surface, how is this reflected on \( c_k^2 \)? (Theorem 2.3)

The fourth class of problems is to determine Borel summability of series with zero radius of convergence such as

\[
(2.8) \quad \tilde{f}_5 = \sum_{n=0}^{\infty} n^{n+1} z^n
\]

in which the coefficients of the series are analyzable (Theorem 2.3).

Definition 2.1. Let \( \{a_j : 1 \leq j \leq N\} \) be a set of nonzero complex numbers with distinct arguments. Let \( \mathcal{M} \) consist of the functions algebraically bounded at \( \infty \) and analytic in \( \mathbb{C} \setminus \bigcup_{j=1}^{N} \{a_j t : t \geq 1\} \). By dividing by a power of \( z \) and subtracting out the principal part (i.e., the negative powers of \( z \)) we can assume that
A representation of the form (2.9) exists for Hilbert-transform-like integrals, such as
\begin{equation}
\int_{\Gamma} g(s) ds
\end{equation}
the integral of \( g \) over \( \Gamma \). Since \( g(s) \) vanishes at \( \infty \), the integral is independent of the choice of \( \epsilon \) as long as it is small enough.

The following observations will simplify our proofs.

**Note 2.3.** Let \( g(s), U_\delta \) and \( \Gamma_\epsilon \) be as in Definition 2.2. \( \Gamma_\epsilon \) separates \( \mathbb{C} \setminus \Gamma_\epsilon \) into two regions. We denote the region containing \( \mathbb{R}^+ \) by \( S_1 \) and the other by \( S_2 \). Let
\begin{align}
G_1(z) &= \int_{\Gamma_\epsilon} \frac{g(s) ds}{s - z} \quad (z \in S_1) \\
G_2(z) &= \int_{\Gamma_\epsilon} \frac{g(s) ds}{s - z} \quad (z \in S_2)
\end{align}
Then \( G_1 \) is analytic in \( S_1 \) and \( G_2 \) is analytic in \( S_2 \). By slightly deforming \( \Gamma_\epsilon \) we are able to see that each \( G_i \) can be analytically continued to \( S_i \cup \Gamma_\epsilon, i = 1, 2 \). On \( \Gamma_\epsilon \) their analytic continuations satisfy
\begin{equation}
G_2(z) - G_1(z) = 2\pi i g(z) \quad (z \in \Gamma_\epsilon)
\end{equation}
Hence \( G_2(z) \) can be analytically continued to \( \mathbb{C} \setminus [0, \infty) \) and \( G_1(z) \) can be analytically continued to at least \( U_\delta \) and in regions where \( g \) is analytic. For each \( z \in \mathbb{C} \setminus [0, \infty) \)
\begin{equation}
G_2(z) = \int_{0}^{\infty} \frac{g(s) ds}{s - z}
\end{equation}
**Note 2.4.** A representation of the form (2.9) exists for Hilbert-transform-like integrals such as \( h(t) = \int_{0}^{\infty} (s - t)^{-1} H(s) ds \) with \( H \) analytic at zero, for instance \( h(t) = -(2\pi i)^{-1} \int_{0}^{\infty} (s - t)^{-1} H(s) \ln s ds \).

**Note 2.5.** Consider the composition of \( g \) with \( s \mapsto \ln(1 + s) \), the branch cut of which is chosen to be \( (-\infty, -1] \). If \( g \) is analytic in \( U_\delta \setminus [0, \infty) \), then \( g(\ln(1 + s)) \) is analytic in the set \( -1 + \exp(U_\delta \setminus [0, \infty)) \). If in addition we have the decay condition \( g(\ln(1 + s)) = o(|s|^{-\alpha}) \) for some \( \alpha > 0 \) as \( |s| \to \infty \), then there exists a \( \delta \) small enough such that \( U_\delta \subseteq -1 + \exp(U_\delta \setminus [0, \infty)) \). It is easy to see from the decay condition that
\[ \int_{\Gamma_\delta} g(p) dp = \int_{\exp(\Gamma_\epsilon) - 1} \frac{g(\ln(1 + s)) ds}{1 + s} = \int_{\Gamma_\delta} \frac{g(\ln(1 + s))}{1 + s} ds \]
and thus we can make the change of variable

\[ \oint_0^\infty g(p)dp = \oint_0^\infty g(\ln(1 + s)) \frac{ds}{1 + s} \]

While providing integral formulae in terms of functions with known singularities which are often rather explicit, the following result can also be interpreted as a duality of resurgence.

**Theorem 2.1.** (i) Assume that \( f(z) = \sum_{k=0}^\infty c_k z^k \) is a series with positive, finite radius of convergence, with \( c_k \) having Borel sum-like representations of the form

\[ c_k = \sum_{j=1}^N a_j^{-k} \oint_0^\infty e^{-kp} F_j(p) dp \quad (k \geq 1) \]

with \( a_j \) as in Definition 2.1, \( F_j \) analytic in \( U_\delta \setminus [0, \infty) \) for some \( \delta > 0 \) and algebraically bounded at \( \infty \). Then, \( f \) is given by

\[ f(z) = f(0) + z \oint_0^\infty \frac{N}{\sum_{j=1}^N (1 + s)((1 + s)a_j - z)} F_j(\ln(1 + s)) ds \]

(ii) Furthermore, \( f \in \mathcal{M}' \). The behavior of \( f \) at \( a_j \) and is of the same type as the behavior of \( F_j(\ln(1 + s)) \) at 0. More precisely, for small \( z \notin [0, \infty) \),

\[ f(a_j(z + 1)) = 2\pi i F_j(\ln(1 + z)) + G(z) \]

where \( G(z) \) is analytic at 0.

(iii) Conversely, assume \( f \in \mathcal{M}' \), and has finitely many singularities located at \( \{a_j t_{j,l}\} \), \( (1 \leq j \leq N, 1 \leq l) \), with \( 1 = t_{j,1} \) and \( t_{j,l} < t_{j,l+1} \) for all \( j,l \). Let \( c_k = f^{(k)}(0)/k! \); then \( c_k \) have Borel sum-like representations of the form

\[ c_k = \frac{1}{2\pi i} \sum_{j=1}^N (a_j)^{-k} \oint_0^\infty e^{-ks} f(a_j e^s) ds, \quad k \geq 1 \]

The behavior at \( a_j \) and at \( \infty \) will follow from the proof.

As it will be clear from the proofs, the method and results would apply, with minor adaptations to functions of several complex variables.

### 2.1. Entire functions.

We restrict the analysis to entire functions of exponential order one, with complete information on the Taylor coefficients. Such functions include of course the exponential itself, or expressions such as \( f^3 \). It is useful to start with \( f^3 \) as an example. The analysis is brought to the case in Theorem 2.1 by first taking a Laplace transform. Note that

\[ \int_0^\infty e^{-xz} f(z) dz = \frac{1}{x} \sum_{n=1}^\infty \frac{n!}{n^{n+1} x^n} \]

The study of entire functions of exponential order one likely involves the factorial, and then a Borel summed representation of the Stirling formula is needed; this is provided in the Appendix.

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(1) After developing these methods, it has been brought to our attention that a duality between resurgent functions and resurgent Taylor coefficients has been noted in an unpublished manuscript by Ecalle. This will be further explored in a forthcoming paper.
Theorem 2.2. Assume that the entire function $f$ is given by

(2.20) \[ f(z) = \sum_{k=1}^{\infty} \frac{c_k z^k}{k!} \]

with $c_k$ as in Theorem 2.1 (i). Then,

(2.21) \[ f(z) = \oint_{0}^{N} \sum_{j=1}^{N} \left[ \left( e^{\zeta_j (s+1)} - 1 \right) \frac{F_j(\ln(1+s))}{(1+s)} \right] ds \]

As in the simple example, the behavior at infinity follows from the integral representation by classical means.

2.2. Borel summation. We obtain from Theorem 2.1, in the same way as above, the following.

Theorem 2.3. Consider the formal power series

(2.22) \[ \tilde{f}(z) = \sum_{k=1}^{\infty} c_k k! x^{-k-1} \]

with coefficients $c_k$ as in Theorem 2.1 (i). Then the series (2.22) is (generalized) Borel summable to

(2.23) \[ \int_{0}^{N} dpe^{-pz} \sum_{j=1}^{N} \frac{F_j(\ln(1+s))}{(1+s)(a_js+a_j-p)} ds \]
\[ = \sum_{j=1}^{N} \int_{0}^{\infty} F_j(\ln(1+s)) \left( -\frac{1}{x} + a_j(s+1)e^{-a_j(s+1)x} \text{Ei}(a_j(s+1)x) \right) ds \]

The proof proceeds as in the previous sections, taking now a Borel transform followed by Laplace transform.

2.3. Other applications; the examples in the introduction.

2.3.1. Other growth rates. Series with coefficients with growth rates precluding a straightforward inverse Laplace transform can be accommodated, for instance by analytic continuation. We have for positive $\gamma$,

(2.24) \[ e^{-\gamma \sqrt{n}} = \frac{\gamma}{2\sqrt{\pi}} \int_{0}^{\infty} p^{-3/2} e^{-\frac{\gamma^2}{4p}} e^{-np} dp \]

which can be analytically continued in $\gamma$. We note first that the contour cannot be, for this function, detached from zero. Instead, we keep $[1, \infty)$ as part of the original contour fixed and, deform the part $[0, 1]$ by simultaneously rotating $\gamma$ and $p$ to maintain $\gamma^2/p$ real and positive near the origin. We get

(2.25) \[ e^{\sqrt{n}} = -\frac{1}{2\sqrt{\pi}} \int_{C_1} p^{-3/2} e^{-\frac{1}{4p}} e^{-np} dp \]

and (1.7) follows, for the same reason Theorem 2.1 (i) holds. In particular,

(2.26) \[ \lim_{z \to -1^+} \sum_{n=1}^{\infty} e^{\sqrt{n}z^n} = \frac{1}{2\sqrt{\pi}} \int_{C_1} \frac{p^{-3/2} e^{-\frac{1}{4p}}}{ep + 1} dp \]
The sum (2.26) is unwieldy numerically, while the integral (2.24) can be evaluated accurately by standard means. In a similar way we get

\[(2.27) \sum_{k=0}^{\infty} \frac{e^{i\sqrt{k}}}{k^a} = -\frac{\gamma^{2a-1/2}}{\sqrt{\pi}} \int_{C} e^{-\frac{1}{\sqrt{4p}}} U(2a + 1/2; \frac{1}{\sqrt{4p}}) \frac{1}{p^{a-1}(e^p + 1)} dp\]

for \(a > 1/2\) for which the series converges. Here \(C\) is a contour consisting of \(S_2\) followed by \([1, \infty)\), where \(S_2\) starts at 0 and ends at 1, and is given by \(r = 1 - \theta/\pi\), \((\theta \in [0, \pi])\) in polar coordinates. \(U\) is the parabolic cylinder function \([1]\).

The coefficients \(c_k^{(1)}\) in (1.3). We have

\[(2.28) \mathcal{L}^{-1}\left[\frac{1}{(n + a)^b}\right] = \Gamma(b)^{-1} p^{b-1} e^{-ap}\]

The rest follows in the same way (1.7) was obtained, after changing variables to \(1 + t = e^p\).

The coefficients \(c_k^{(2)}\). We let \(x = n\) and take the inverse Laplace transform in \(x\):

\[(2.29) G(p) = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \frac{e^{xp}}{x^b + \ln x} dx\]

where the contour can be bent backwards for \(p \in \mathbb{R}^+\), to hang around \(\mathbb{R}^-\). Then, with the change of variable \(x = -u\) (2.29) becomes

\[(2.30) G(p) = \frac{1}{2\pi i} \int_{0}^{\infty} \frac{e^{-up}}{(-u)^b + \ln(-u)} du\]

and thus

\[c_k = (\mathcal{L}G)(k) = \int_{0}^{\infty} G(p) e^{-kp} dp = \int_{0}^{\infty} \left[-\frac{G(p) \ln p}{2\pi i}\right] e^{-kp} dp\]

We see that \(F_1(p) = (-G(p) \ln p)/2\pi i\) and by Theorem 2.1

\[f_2(z) = \int_{0}^{\infty} \frac{\tilde{G}(s)}{s - (z - 1)} ds\]

where \(\tilde{G}(s) = F_1(\ln(1 + s))/(1 + s)\). Hence the singularity of \(f_2(z)\), at \(z = 1\) on the first Riemann sheet, according to Note 2.3 is that of \(\phi(z) = 2\pi i \tilde{G}(z - 1)\), as in (1.5).

The example of the coefficients \(c_k^{(3)}\) is studied in a similar way as Theorem 2.2 related calculations can be found in (4.2).

The coefficients \(c_k^{(4)}\) were treated at the beginning of this section.

Another example is provided by the log of the Gamma function, \(\ln \Gamma(n) = \sum_{k=1}^{n} \ln k\). It is convenient to first subtract out the leading behavior of the sum to arrange that the summand is inverse Laplace transformable. With

\[g_n = \ln \Gamma(n + 1) - \left((n + 1) \ln(n + 1) - n - \frac{1}{2} \ln(n + 1)\right)\]

we get

\[(2.31) g_N = \sum_{i=1}^{N} \left[1 - \left(\frac{1}{2} + n\right) \ln \left(1 + \frac{1}{n}\right)\right] = \sum_{i=1}^{N} \int_{0}^{\infty} e^{-np} \frac{1}{p} - \left(\frac{p}{2} + 1\right) e^{-p} dp\]
Figure 1. Singularities of $f$, cuts, direction of integration and Cauchy contour deformation.

where $L^{-1}$ of the summand in the middle term is most easily obtained by noting that its second derivative is a rational function. Summing as usual $e^{-np}$ we get

$$\ln \Gamma(n) = n(\ln n - 1) - \frac{1}{2} \ln n + \frac{1}{2} \ln(2\pi) + \int_0^\infty \left(1 - \frac{p}{2} - \frac{(p + 1)}{p^2(e^p - 1)}\right)e^{-np}dp$$

Obviously, if the behavior of the coefficients is of the form $A_k c_k$ where $c_k$ satisfies the conditions in the paper, one simply changes the independent variable to $z' = Az$.

3. **Proof of Theorem 2.1**

If $f \in M'$ we write the Taylor coefficients in the form

$$(3.33) \quad c_k = \frac{1}{2\pi i} \oint f(p)dp \quad (k \geq 1)$$

where the contour of integration is a small circle of radius $r$ around the origin. We attempt to increase $r$ without bound. In the process, the contour will hang around the singularities of $f$ as shown in Figure 1. Each integral over a curve that wraps around a ray $\{a_j t : t \geq 1\}$ converges by the decay assumptions and the contribution of the arcs at large $r$ vanishes, since $f(z) = o(z)$ as $z \to \infty$.

To be more precise, let $C_{j,\epsilon}$ be the part of the image of $\Gamma_r$ under the mapping $s \to a_j e^s$, let $C_{j,\epsilon,R}$ be the part of $C_{j,\epsilon}$ inside the disk $|s| \leq R$, and $C_R$ be the part of the contour on $|s| = R$. Then for $\epsilon$ small enough and $R$ large enough we have

$$c_k = \frac{1}{2\pi i} \oint f(p)dp = \frac{1}{2\pi i} \left(\sum_{j=1}^N \int_{C_{j,\epsilon,R}} f(p)dp + \int_{C_R} f(p)dp\right)$$
By the change of variable \( p = a_j e^s \) and letting \( R \to \infty \) we get

\[
\int_{a_j C_{-R}} \frac{f(p) dp}{p^{k+1}} = \int_{\Gamma_+} a_j e^s f(a_j e^s) ds = \int_0^\infty a_j^{-k} e^{-ks} f(a_j e^s) ds
\]

and the integral over \( C_R \) vanishes as \( R \to \infty \) by decay condition for \( k \geq 1 \).

In the opposite direction, first let \( \epsilon \) be small enough so that for all \( j \) and \( k = 1 \)

\[
(3.34) \quad \int_0^\infty e^{-kp} F_j(p) dp = \int_{\Gamma_+} e^{-kp} F_j(p) dp
\]

Then for all \( 1 \leq j \leq N, k \geq 1 \) \((3.34)\) is true. Also let \( z \) be small so that

\[
(3.35) \quad |a_j^{-1} e^{-pz}| \leq \delta^2 < 1
\]

for all \( j \) and \( p \in \Gamma_+ \).

Then, by the dominated convergence theorem (which applies in this case, see \((3.37)\)) we have

\[
(3.36) \quad f(z) - f(0) = \sum_{k=1}^\infty c_k z^k = \sum_{k=1}^\infty \left( \sum_{j=1}^N a_j^{-k} \int_{\Gamma_+} e^{-kp} F_j(p) dp \right) z^k
\]

\[
= \int_{\Gamma_+} \sum_{j=1}^N \left( \sum_{k=1}^N \left( a_j^{-1} e^{-pz} \right)^k \right) F_j(p) dp = \int_{\Gamma_+} \sum_{j=1}^N \left( \frac{a_j^{-1} e^{-pz}}{1 - a_j^{-1} e^{-pz}} \right) F_j(p) dp
\]

\[
= \sum_{j=1}^N \int_0^\infty \frac{a_j^{-1} e^{-pz}}{1 - a_j^{-1} e^{-pz}} F_j(p) dp = z \sum_{j=1}^N \int_0^\infty \frac{F_j(\ln(1+s)) ds}{(1+s)(sa_j + a_j - z)}
\]

as stated. The third equality holds because we have, in view of \((3.35)\),

\[
\left| \sum_{j=1}^N \left( a_j^{-1} e^{-pz} \right)^k F_j(p) \right| \leq \sum_{j=1}^N |a_j^{-1} e^{-pz}|^{k/2} \left( \left| a_j^{-1} e^{-pz} \right|^{k/2} |F_j(p)| \right)
\]

\[
(3.37) \quad \leq \sum_{j=1}^N \delta^k \left( |a_j^{-1} e^{-pz}|^{k/2} |F_j(p)| \right)
\]

For each \( j \), \( |a_j^{-1} e^{-pz}|^{k/2} F_j(p) \) is integrable over \( \Gamma_+ \) since \( F_j \) is algebraically bounded at \( \infty \), so we may interchange the order of integration and summation over \( k \). The last equality holds because \( F_j(\ln(1+s))/(sa_j + a_j - z) = o(|s|^{-\alpha}) \) for some \( \alpha > 1 \) so we can make the change of variable \( p = \ln(1+s) \), see Note \(2.5\). Hence \((2.15)\) holds for \( z \) small.

Given \( j \in \{1, ..., N\} \) we may write

\[
(3.38) \quad I_j(z) := \int_0^\infty \frac{F_j(\ln(1+s)) ds}{(1+s)(sa_j + a_j - z)} = \int_0^\infty \frac{F_j(\ln(1+s)) / (a_j(1+s)) ds}{(s - (z/a_j - 1))}
\]

Since \( F_j \) is analytic in \( \mathbb{C} \setminus [0, \infty) \) and is algebraically bounded at \( \infty \), \( F_j(\ln(1+s))/(a_j(1+s)) \) is analytic in \( \mathbb{C} \setminus [0, \infty) \) and vanishes uniformly as \( \text{Re}(s) \to \infty \). Then it becomes obvious from Note \(2.3\) that the integral \( I_j \) in \((3.38)\) is analytic in \( \mathbb{C} \setminus \{a_j t : t \geq 1\} \). Thus \( f(z) \) can be analytically continued to \( \mathbb{C} \setminus \bigcup_{j=1}^N \{a_j t : t \geq 1\} \).
To see that \( f(z) = o(z) \) as \( |z| \to \infty \), it suffices to show that for each \( j \), \( I_j(z) = o(1) \). Assume \( 3\epsilon < \delta \). We use the contours \( \exp(\Gamma_\epsilon) - 1 \) and \( \exp(\Gamma_{3\epsilon}) - 1 \). If \( |\arg(z/a_j)| \geq 2\epsilon \) and \( |z/a_j| \) is large enough we write:

\[
I_j(z) = \int_{\exp(\Gamma_\epsilon) - 1} \frac{F_j(\ln(1+s))/(a_j(1+s))}{(s - (z/a_j - 1))} \, ds
\]

(3.39)

Then there exist some positive number \( \rho_1 \) such that \( |s - (z/a_j - 1)| \geq \rho_1 |s| \) for each \( s \in \exp(\Gamma_\epsilon) - 1 \), so we can use dominated convergence to obtain \( I_j(z) \to 0 \) as \( |z| \to \infty \). If \( |\arg(z/a_j)| \leq 2\epsilon \) we use Note 2.3 to write:

\[
I_j(z) = \int_{\Gamma_{3\epsilon}} \frac{F_j(\ln(1+s))/(a_j(1+s))}{(s - (z/a_j - 1))} + 2\pi i F_j(\ln(z/a_j))/z \, ds
\]

(3.40)

Then there exist some positive number \( \rho_2 \) such that \( |s - (z/a_j - 1)| \geq \rho_2 |s| \) for each \( s \in \exp(\Gamma_{3\epsilon}) - 1 \), so we can use dominated convergence to prove the integral in the right hand side of (3.40) is \( o(1) \). By assumption it is obvious that \( F_j(\ln(z/a_j))/z = o(1) \), so in this case we also have \( I_j(z) \to 0 \) as \( |z| \to \infty \).

The nature of the singularities of \( f \) is derived from Note 2.3. Let \( z \notin [0, \infty) \) be small, then for each \( j \in \{1, \ldots, N\} \)

\[
f((z + 1)a_j) = f(0) + ((z + 1)a_j) \sum_{l \neq j} I_l((z + 1)a_j) + ((z + 1)a_j) I_j((z + 1)a_j)
\]

(4.42)

\[
= f(0) + ((z + 1)a_j) \sum_{l \neq j} I_l((z + 1)a_j) + ((z + 1)a_j) \int_{0}^{\infty} \frac{F_j(\ln(1+s))/(a_j(1+s))}{s - z} ds
\]

where \( A(t) \) is analytic at \( z = 0 \). The last equality is obtained by Note 2.3. It is obvious from Note 2.3 that each \( I_l((z + 1)a_j) \) (\( l \neq j \)) is analytic on \([0, \infty)\). Thus

\[
f((z + 1)a_j) = \hat{G}(z) + 2\pi i F_j(\ln(1 + z))
\]

where \( \hat{G}(z) \) is analytic at \( z = 0 \). Hence (2.17) follows.

4. Appendix

4.1. Simple integral representation of \( 1/n! \). We have

\[
\frac{1}{\Gamma(z)} = -\frac{i e^{-\pi iz}}{2\pi} \int_{0}^{\infty} s^{-z} e^{-s} ds = -\frac{i e^{-\pi iz} z^{-z}}{2\pi} \int_{0}^{\infty} s^{-z} e^{-zs} ds
\]

(4.43)

with our convention of contour integration. From here, one can proceed as in (4.2)

4.2. The Gamma function and Borel summed Stirling formula. We have

\[
n! = \int_{0}^{\infty} t^n e^{-t} dt = n^{n+1} \int_{0}^{\infty} e^{-n(s-\ln s)} ds
\]

(4.44)

\[
= n^{n+1} \int_{0}^{1} e^{-n(s-\ln s)} ds + n^{n+1} \int_{1}^{\infty} e^{-n(s-\ln s)} ds
\]
On $(0, 1)$ and $(1, \infty)$ separately, the function $s - \ln(s)$ is monotonic and we may write, after inverting $s - \ln(s) = t$ on the two intervals to get $s_{1,2} = s_{1,2}(t) \tag{4.47}$

$$n! = n^{n+1} \int_1^\infty e^{-nt}(s_{1,2}^n(t) - s_{1,2}'(t))dt = n^{n+1}e^{-n} \int_0^\infty e^{-np}G(p)dp$$

where $G(p) = s_{2}'(1+p) - s_{1}'(1+p)$. From the definition it follows that $G$ is bounded at infinity and $p^{1/2}G$ is analytic in $p$ at $p = 0$. Using now \((4.44)\) and Theorem 2.1 in \((2.19)\) we get

$$\int_0^\infty e^{-xz}f_3(z)dz = \frac{1}{x^2} \int_0^\infty G(\ln(1+t)) \frac{G(1+t)}{(te+(e-x^{-1}))(t+1)}dt \tag{4.45}$$

Upon taking the inverse Laplace transform we obtain \((1.6)\).

4.3. Solution of \((1.1)\). Assume the solution to \((1.1)\) which is analytic at $\eta = 0$ has Taylor expansion $f(\eta) = \sum_{k=0}^\infty c_k\eta^k$. Then

$$c_k = \frac{c_1(-1)^{k/2-1/2}\Gamma(k/2-1-a)k}{A\Gamma(-1/2-a)\Gamma(k+1)^2} (k \text{ is odd}) \tag{4.46}$$

$$c_k = 0 \quad (k \text{ is even}) \tag{4.47}$$

It is obvious that $f(\eta)$ is entire. Consider the Laplace transform $F(p) = \mathcal{L}f(p)$. Then $F(p) = \sum_{k=0}^\infty \Gamma(k+1)c_k p^{-k-1}$. Let $G(p) = F(1/p)$; then $G(z)$ is a solution to the differential equation

$$G''(z) + \left( -\frac{4}{z} + \frac{z}{2A} \right) G'(z) + \left( \frac{6}{z^2} - \frac{3/2 + a}{A} \right) G(z) = 0 \tag{4.48}$$

and $G(z)$ is analytic at $z = 0$.

The normalization transformation $G(z) = z^{3/2}e^{-z^2/A^2}h(z^2/A)$ (cf. \([19]\) for a general description of normalization methods) yields

$$h'' - \frac{7}{4}h' + \frac{1}{16} \left( 12 - \frac{3 + 4a}{s} + \frac{3}{s^2} \right) h = 0 \tag{4.49}$$

solvable in terms of Whittaker functions \([1]\). Substituting back, by straightforward algebra, this yields \((4.1)\).

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\(^{2}\)The functions $s_{1,2}$ are given by branches of $-W(-e^t)$, where $W$ is the Lambert function.
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