1 Introduction

Harish-Chandra modules play an important role in the theory of representations of semi-simple Lie-groups over \( \mathbb{R} \), in a certain sense they are the algebraic skeleton of a certain class of representations of semi simple real Lie groups.

In this note we show that certain classes of Harish-Chandra modules have in a natural way a structure over \( \mathbb{Z} \). The Lie group is replaced by a split reductive group scheme \( G/\mathbb{Z} \), its Lie algebra is denoted by \( \mathfrak{g}_\mathbb{Z} \). On the
group scheme $G/\mathbb{Z}$ we have a Cartan involution $\Theta$ which acts by $t \mapsto t^{-1}$ on the split maximal torus and the fixed point group scheme $K/\mathbb{Z}$ of $\Theta$ is a flat group scheme over $\mathbb{Z}$. A Harish-Chandra module over $\mathbb{Z}$ is a $\mathbb{Z}$-module $V$ which comes with an action of the Lie algebra $g_{\mathbb{Z}}$, an action of the group scheme $K$, and we require some compatibility conditions between these two actions. Finally we require that $V$ is a union of finitely generated $\mathbb{Z}$ modules $V_I$ which are $K$ invariant.

The definitions are imitating the definition of a Harish-Chandra modules over $\mathbb{R}$ or over $\mathbb{C}$. (See for instance [1] 0.2.5, there these modules are called $(g,K)$ modules.)

For these $(g_{\mathbb{Z}}, K)$ modules $V$ we define cohomology modules $H^\bullet(g_{\mathbb{Z}}, K, V)$ and these will be finitely generated $\mathbb{Z}$ modules provided the module $V$ satisfies suitable finiteness conditions. We construct some simple examples, especially we construct the $\mathbb{Z}$- version of the discrete series representations of $\text{Gl}_2(\mathbb{R})$ and compute their cohomology.

In the next section we discuss the process of induction: For a parabolic subgroup $P/\mathbb{Z}$ and a $(m, K^M)$- module $V$ for its reductive quotient $M/\mathbb{Z}$ we define the induced module $\text{Ind}_G^P V$.

In the final section we study intertwining operators between some specific induced Harish-Chandra modules $\text{Ind}_G^{Q_D} \mu, \text{Ind}_G^{Q_D'}$ where $P, Q$ are maximal parabolic subgroups of $\text{Gl}_N/\mathbb{Z}$. Here we have to introduce some twisting, we achieve such a twisting by extending the scalars from $\mathbb{Z}$ to the function field $\mathbb{Q}(s)$ and define $\text{Ind}_G^{Q_D} \mu \otimes s$ over $\mathbb{Q}(s)$. Then our intertwining operators are defined as integrals. We can not expect that they are defined over $\mathbb{Q}(s)$. But it turns out (and this is certainly not surprising) that they can be written down in terms of the form $\Gamma(s) R(s)$ with $R(s) \in \mathbb{Q}(s)$ and $\Gamma(s)$ is of course the $\Gamma$-function. If the intertwining operator is holomorphic at $s = 0$ we can evaluate at $s = 0$ and it turns out that our intertwining operator, which is defined by the transcendental process of integration, is essentially a power of $\pi$ times a non zero rational number (Theorem 4.1).

This rationality result is used in [7] Thm. 7.48, it can be formulated without reference to rational integral structures on Harish-Chandra modules, we just have to choose the ”right” basis in certain one dimensional vector spaces.

The main reason why we develop these concepts is an intriguing question concerning the cohomology of these modules and its behavior under the intertwining operators. It turns out that the cohomology in certain situations is a free module of rank one over a small ring $R$ (for instance $\mathbb{Z}$, $\mathbb{Z}[i, \frac{1}{2}], \ldots$). Then the intertwining operator divided by the appropriate power of $\pi$ induces an isomorphism between these cohomology modules after we tensor them by the quotient field of $R$. This isomorphism depends on some data, for instance some highest weights. Our question is whether this isomorphism is already an isomorphism over the basic ring $R$ independently of the data.

This question has been investigated in [5] in a special case and reduced
to an combinatorial identity, which then was proved by D. Zagier (see [12]) in an appendix to [9]. This gives a positive answer to the question above in this special case. This is the only evidence I have that the question makes sense, except that it seems to be a very natural one.

In this note we work with certain specific choices of Cartan involutions. Such a choice provides the so called maximal definite group schemes \( K / \mathbb{Z} \), these group schemes are flat over \( \text{Spec}(\mathbb{Z}) \) and they are even reductive if we invert the prime 2. But we can also choose other maximal definite group schemes \( K' \) which are reductive at the prime 2 and perhaps non reductive at some other places. This suggests that we should speak of sheaves of Harish-Chandra modules over \( \text{Spec}(\mathbb{Z}) \).

2 Harish-Chandra modules over \( \mathbb{Z} \)

2.1 The general setup

For any affine group scheme \( H / \text{Spec}(\mathbb{Z}) \) we denote by \( A(H) \) its algebra of regular functions. The affine algebra of the multiplicative group scheme \( \mathbb{G}_m \) is \( A(\mathbb{G}_m) = \mathbb{Z}[x, x^{-1}], \) i.e. we choose the generator \( \gamma_1 \) of the character module \( X^*(\mathbb{G}_m) \), it is given by the identity. Let \( \mathbb{G}_a \) be the one dimensional additive group scheme \( A(\mathbb{G}_a) = \text{Spec}(\mathbb{Z}[X]) \).

Let \( G / \text{Spec}(\mathbb{Z}) \) be a reductive connected group scheme, we assume that the derived group \( G^{(1)} / \text{Spec}(\mathbb{Z}) \) is a simply connected Chevalley scheme, the central torus \( C / \text{Spec}(\mathbb{Z}) \) should be split. Let \( \mathfrak{g}, \mathfrak{g}^{(1)}_Z \) be the Lie algebras of \( G / \text{Spec}(\mathbb{Z}), G^{(1)} / \text{Spec}(\mathbb{Z}) \) respectively, let \( \mathfrak{c}_Z \) be the Lie algebra of \( C \). We have the split maximal torus \( T / \text{Spec}(\mathbb{Z}) \), let \( T^{(1)} / \text{Spec}(\mathbb{Z}) = T \cap G^{(1)} \). We choose a Borel subgroup \( B / \text{Spec}(\mathbb{Z}) \supset T / \text{Spec}(\mathbb{Z}) \). As usual we denote the character module \( \text{Hom}(T, \mathbb{G}_m) \) by \( X^*(T) \), we have the direct sum decomposition

\[
X^*_\mathbb{Q}(T) = X^*(T) \otimes \mathbb{Q} = X^*_\mathbb{Q}(T^{(1)}) \oplus X^*_\mathbb{Q}(C),
\]

we will always write \( \gamma = \gamma^{(1)} + \delta \), this is the decomposition of a character \( \gamma \in X^*_\mathbb{Q}(T) \) into its semi simple and its abelian part.

Let \( \Delta (\text{ resp. } \Delta^+ \subset X^*(T)) \) be the set of roots (resp. positive roots), let \( \pi = \{ \alpha_1, \alpha_2, \ldots, \alpha_r \} \subset \Delta \) be the set of simple positive roots. Let \( \gamma_1, \gamma_2, \ldots, \gamma_r \in X^*(T^{(1)}) \) be the dominant fundamental weights, we extend them to elements in \( X^*_\mathbb{Q}(T) \) by putting the abelian part equal to zero. The element \( \rho \in X^*_\mathbb{Q}(T) \) is the half sum of positive roots.

For any root \( \alpha \) we have the root subgroup scheme \( U_\alpha / \text{Spec}(\mathbb{Z}) \), we assume that for all simple roots we have fixed an isomorphism

\[
\tau_\alpha : \mathbb{G}_a / \text{Spec}(\mathbb{Z}) \overset{\sim}{\longrightarrow} U_\alpha / \text{Spec}(\mathbb{Z}),
\]

i.e. we have selected a generator \( e_\alpha \) of the abelian group \( U_\alpha(Z) \overset{\sim}{\longrightarrow} \mathbb{Z} \).
From our simple root $\alpha$ we also get a subgroup scheme $H_\alpha \subset G^{(1)}/\text{Spec}(\mathbb{Z})$ which is "generated" by $U_\alpha, U_{-\alpha}$ and which is isomorphic to $\text{Sl}_2/\text{Spec}(\mathbb{Z})$. It has a maximal torus $T_\alpha/\text{Spec}(\mathbb{Z}) \subset T^{(1)}/\text{Spec}(\mathbb{Z})$ which is the intersection of the kernels of the fundamental weights $\gamma_\beta$ where $\beta \neq \alpha$. The choice of $\tau_\alpha$ is the same as the choice of an isomorphism

$$\tilde{\tau}_\alpha : \text{Sl}_2/\text{Spec}(\mathbb{Z}) \to H_\alpha$$

which sends the diagonal torus to $T_\alpha$ and on the $\mathbb{Z}$-valued points

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \to e_\alpha$$

The derivative of $\tau_\alpha$ defines a generator $E_\alpha \in \text{Lie}(U_\alpha)$. Finally we define the coroot $\alpha^\vee : G_m \xrightarrow{\sim} T_\alpha$ which is defined by the rule $<\alpha^\vee, \alpha> = 2$.

Let $\Theta$ be the unique automorphism of $G^{(1)}/\text{Spec}(\mathbb{Z})$ which induces $t \mapsto t^{-1}$ on $T^{(1)}$ and restricted to $H_\alpha$ and composed with $\tilde{\tau}_\alpha^{-1}$ is the inner automorphism given by the element

$$s_\alpha = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$ 

We call the pair $(G^{(1)}, \Theta)$ an Arakelov Chevalley scheme. The automorphism restricted to $G^{(1)}(\mathbb{R})$ is of course a Cartan involution and the fixed point set $G^{(1)}(\mathbb{R})^\Theta = K^{(1)}_\infty$ is a maximal compact subgroup. Here we use this automorphism to give the structure of a group scheme over $\text{Spec}(\mathbb{Z})$ to $K^{(1)}_\infty$. To be more precise: The group scheme of fixed points $K^{(1)}/\text{Spec}(\mathbb{Z}) = (G^{(1)})^\Theta/\text{Spec}(\mathbb{Z})$ is a flat group scheme over $\text{Spec}(\mathbb{Z})$, it is smooth and connected over $\text{Spec}(\mathbb{Z})[\frac{1}{2}]$.) We call $K^{(1)}$ a maximal definite connected subgroup scheme of $G^{(1)}/\mathbb{Z}$. We denote by $\mathfrak{k}_{Z[\frac{1}{2}]}$ its Lie algebra over $\mathbb{Z}[\frac{1}{2}]$. We put $\mathfrak{k}_Z = \mathfrak{g}_Z \cap \mathfrak{k}_{Z[\frac{1}{2}]}$. Here $\mathfrak{k}_Z$ is a maximal sub algebra for which the restriction of the Killing form is negative definite. This justifies the terminology.

If we have an extension of the Cartan involution to $G/\mathbb{Z}$ then we can also look on the fixed point scheme $G^\Theta/\mathbb{Z}$ and define $K/\mathbb{Z} = G^\Theta$. We are mostly interested in cases where this extension induces $t \mapsto t^{-1}$ on $C/\mathbb{Z}$, then $K^{(1)}$ is the connected component of the identity of $K/\mathbb{Z}$. In general we denote by $K$ a group scheme lying between $K^{(1)}$ and $G^\Theta$. Then $K/K^{(1)}$ is a finite constant group scheme which is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^s$. We also consider larger subschemes of the form $K = K^{(1)} \cdot C'$, where $C'$ is any subtorus of the split maximal torus $C$. We call them essentially maximal definite subgroup schemes. They are also smooth over $\mathbb{Z}[\frac{1}{2}]$ and the Lie algebra is denoted by $\mathfrak{k}_{Z[\frac{1}{2}]}$. Again we define $\tilde{\mathfrak{k}}_Z = \mathfrak{k}_{Z[\frac{1}{2}]} \cap \mathfrak{g}_Z$.

For any ring $\mathbb{Z} \subset R$ we define the notion of a Harish-Chandra module over $R$, or equivalently a $(\mathfrak{g}_Z, K)$- module over $R$. 

4
1) This will be a projective $R$–module $V$ which is the union of finitely generated projective submodules $V_I$, $I \in \mathcal{I}$ such that $V/V_I$ is torsion free. We have an action of $K$ on $V$ which respects the $V_I$.

2) If $L$ is the quotient field of $R$ then every irreducible finite dimensional representation $\vartheta$ of $K \times L$ occurs with finite multiplicity in this module and we have the isotypical decomposition

$$V \otimes L = \bigoplus \vartheta(\vartheta)$$

where $\vartheta(\vartheta)$ is the $\vartheta$ isotypical component.

3) We have a Lie-algebra action of $\mathfrak{g}_Z \otimes R$ on $V$.

4) The group scheme $K$ acts by the adjoint action on $\mathfrak{g}_Z$ and the $R$-module homomorphism

$$(\mathfrak{g}_Z \otimes R) \otimes V \to V,$$

which is given by 3), is $K$ invariant.

5) The restriction of the Lie-algebra action of $\mathfrak{g}_Z$ to the Lie-algebra $\mathfrak{k}_Z = \text{Lie}(K)$ is the differential of the action of $K$.

Finally we formulate a finiteness condition

6) For any $I \in \mathcal{I}$ we find an $I_1 \in \mathcal{I}$ such that $V_I \subset V_{I_1}$ such that the Lie algebra action of $\mathfrak{g}_Z$ on $V$ induces an $R$-bilinear map

$$\mathfrak{g}_Z \times V_I \to V_{I_1}$$

We say that the $(\mathfrak{g}_Z, K)$–module has a central character if the Lie algebra of the center $\mathfrak{c}_Z = \text{Lie}(C)$ acts by a linear map $z_V : \mathfrak{c}_Z \to R$.

2.2 Some comments

This is almost the same as the usual definition of a Harish-Chandra module except that the field of scalars $C$ has been replaced by $R$ and the action of the maximal compact group $K_\infty$ is replaced by the action of the group scheme $K$.

We want to remind the reader what it means that the group scheme $K/\text{Spec}(Z)$ acts upon $V$ and $V_I$. We recall that by definition $K/\text{Spec}(Z)$ is a functor from the category of affine schemes $Y \to \text{Spec}(R)$ to the category of groups. This means that for any commutative ring $R_1$ containing $R$ we get an abstract group of $R_1$–valued point $G(R_1)$ which depends functorially on $R_1$. Then the action of $K/\text{Spec}(Z)$ on the $R$ module $V$ provides for any $R_1$
an action of $\mathcal{K}(R_1)$ on the $R_1$ module $\mathcal{V} \otimes_R R_1$. We require that for all our finitely generated submodules the module $\mathcal{V}_I \otimes R_1$ is invariant under $\mathcal{K}(R_1)$.

In all examples which will be discussed below we take for $\mathcal{I}$ the set of finite sets of isomorphism classes of irreducible representations of the group scheme $\mathcal{K}$. If $I = \{\vartheta_1, \ldots, \vartheta_r\}$ then

$$\mathcal{V}_I = \mathcal{V} \cap \oplus_{\nu=1}^r \mathcal{V}(\vartheta_\nu)$$

(3)

In this case the requirement 6) is superfluous.

We call $\mathcal{V}$ irreducible if $\mathcal{V} \otimes L$ does not contain a proper $(\mathfrak{g}_Z, \mathcal{K})$ submodule, we call it absolutely irreducible if $\mathcal{V} \otimes L_1$ stays irreducible for any finite extension $L_1/L$.

We saw already that we have some flexibility in the choice of $\mathcal{K}$. If we replace $\mathcal{K}$ by the connected component of the identity $\mathcal{K}^{(1)}$ then we can restrict the $(\mathfrak{g}_Z, \mathcal{K})$ module to $(\mathfrak{g}_Z, \mathcal{K}^{(1)})$. It may happen that the restriction of an irreducible module is not irreducible anymore.

2.3 Motivation for this concept

This may look a little bit artificial. Let us choose a dominant weight $\lambda \in X^*(T)$ and construct a highest weight module $M_{\lambda, Z}$. This highest weight module has a central character $\zeta_\lambda \in X^*(C)$. We are looking for absolutely irreducible Harish-Chandra modules $\mathcal{V}$ (over $Z$ or a slightly larger ring) having the central character $z_V = -d\zeta_\lambda$, and which have non trivial cohomology with coefficients in $M_{\lambda, Z}$. The cohomology is defined as the cohomology of the complex

$$\text{Hom}_{\mathcal{K}}(\Lambda^\bullet(\mathfrak{g}_Z/\mathfrak{k}_Z), \mathcal{V} \otimes M_{\lambda, Z})$$

where the definition of the complex is exactly the same as in the traditional situation (See for instance [4] Chap. 3, section 4). Hence we define

$$H^\bullet(\mathfrak{g}_Z, \mathcal{K}, \mathcal{V} \otimes M_{\lambda, Z}) = H^\bullet(\text{Hom}_{\mathcal{K}}(\Lambda^\bullet(\mathfrak{g}_Z/\mathfrak{k}_Z), \mathcal{V} \otimes M_{\lambda, Z}))$$

(4)

It easy to see that only the semi-simple component is relevant for the computation of the cohomology, we have

$$H^\bullet(\mathfrak{g}_Z, \mathcal{K}, \mathcal{V} \otimes M_{\lambda, Z}) = H^\bullet(\mathfrak{g}_Z^{(1)}, \mathcal{K}^{(1)}, \mathcal{V} \otimes M_{\lambda, Z})^{\mathcal{K}/\mathcal{K}^{(1)} \otimes \Lambda^\bullet(\zeta)}$$

(5)

We will see that that factor $\Lambda^\bullet(\zeta)$ is rather uninteresting. If we replace $\mathcal{K}$ by a larger group $\tilde{\mathcal{K}} = \mathcal{K}^{(1)} \cdot C'$ then we define more generally

$$H^\bullet(\mathfrak{g}_Z, \tilde{\mathcal{K}}, \mathcal{V} \otimes M_{\lambda, Z}) = H^\bullet(\text{Hom}_{\mathcal{K}^{(1)}}(\Lambda^\bullet(\mathfrak{g}_Z/\tilde{\mathfrak{k}}_Z), \mathcal{V} \otimes M_{\lambda, Z}))$$

(6)

(Observe the subscript at the Hom is $\mathcal{K}^{(1)}$ and not $\tilde{\mathcal{K}}$ as one might expect.) If we choose $C' = C$ and replace $\mathcal{K}$ in $[5]$ by $\tilde{\mathcal{K}}$ then the factor $\Lambda^\bullet(\zeta)$ is replaced by $\Lambda^0(\zeta) = Z$. 6
We will be mainly concerned with the group scheme \( G = \text{Gl}_n/\text{Spec}(\mathbb{Z}) \), the involution \( \Theta \) will be the usual involution \( g \mapsto g^{-1} \). Our first aim will be to construct for a given highest weight module \( M_{\lambda} \), an absolutely irreducible \((\text{g}_R, K)\) module \( D_{\lambda} \) which has non trivial cohomology.

More precisely: The lowest degree were we find non trivial cohomology is \( b_n = \left\lceil \frac{n}{2} \right\rceil \) (See \[7\], 3.1.5) and

\[
H^{b_n}(g_\mathbb{Z}, K^{(1)}), D_{\lambda} \otimes M_{\lambda, \mathbb{Z}}) \cong \begin{cases} 
\mathbb{Z}[\frac{1}{2}]\omega_\lambda^+ \oplus \mathbb{Z}[\frac{1}{2}]\omega_\lambda^- & n \text{ even} \\
\mathbb{Z}[\frac{1}{2}]\omega_\lambda & n \text{ odd}
\end{cases} \otimes \Lambda^*(c_\mathbb{Z}) \tag{7}
\]

We still have the action of \( K/\mathbb{Z}(1) = \mathbb{Z}/2\mathbb{Z}(=\pi_0(\text{Gl}_n(\mathbb{R}))) \) on the cohomology. This action is non trivial if \( n \) is even and the cohomology decomposes in a + and a - eigenspace. (See \[43\]). This will be relevant for the definition of the periods in \[7\].

If we take the tensor product \( D_{\lambda} \otimes \mathbb{C} \) then we get the usual Harish-Chandra modules over \( \mathbb{C} \) which are denoted by \( D_{\lambda} \) in \[7\], 3.1.4. We will call these modules over \( \mathbb{C} \) the transcendental Harish-Chandra modules. These special transcendental modules will be the only tempered modules which have cohomology and they contribute to the cuspidal cohomology (See \[7\], Sec. 5).

3 First examples

3.1 The case of the torus \( \mathbb{G}_m \)

For the multiplicative group scheme \( \mathbb{G}_m/\mathbb{Z} \) we have \( \text{Lie}(\mathbb{G}_m)_{\mathbb{Z}} = \mathbb{Z}H \). We may choose for the group scheme \( K \) simply the subscheme \( K = \mu_2 \) of second roots of unity. Then we can construct a \((\mathbb{Z}H, K)\) module \( \mathbb{Z}[\gamma \otimes m] \) for any pair \((\gamma, m)\) where \( \gamma \in X^*(\mathbb{G}_m) \) and where \( m \) is an integer modulo two. If \( \gamma = x^n \) then the generator \( H \) of \( \text{Lie}(\mathbb{G}_m) \) acts by multiplication by \( n \) and the action of \( K_{\mathbb{Z}} \) is given by the sign character \(-1 \mapsto (-1)^m \). Therefore it is clear that these modules \( \mathbb{Z}[\gamma \otimes m] \) are the absolutely irreducible \((\text{Lie}(\mathbb{G}_m)_{\mathbb{Z}}, K)\) modules. The pairs \((\gamma, m)\) are called the characters of Hecke type \(-\gamma \), if \( m = 0 \) then these are the rational characters. We can do essentially the same for any split torus \( C \), for any pair \( \gamma \in X^*(C) \) and any \( \epsilon : K = C(\mathbb{Z}) \to \{\pm\} \) we can construct the \((\text{Lie}(C)_{\mathbb{Z}}, K)\) module \( \mathbb{Z}[\gamma \otimes \epsilon] \).

3.2 The special case \( \text{Gl}_2/\mathbb{Z} \)

We consider the special case \( G = \text{Gl}_2/\text{Spec}(\mathbb{Z}) \) with \( \tilde{\tau}_{\alpha} = \text{Id} \). The group \( \text{Gl}_2(\mathbb{R}) \) has its discrete series representations and the resulting \((g_\mathbb{R}, K_{\mathbb{R}})\) modules. We want to show that these discrete series representations are base extensions of Harish-Chandra modules over \( \mathbb{Z} \).
Inside $G$ we have the subgroup scheme

$$\tilde{K} = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \right\} \subset \text{GL}_2.$$  

The affine algebra of $\tilde{K}$ is $A(\tilde{K}) = \mathbb{Z}[a, b, 1/(a^2 + b^2)]$. Let $\mathcal{O} = \mathbb{Z}[i]$ where $i^2 = -1$. We define the flat group scheme $R_{\mathcal{O}/\mathbb{Z}}(\mathbb{G}_m)$, its $R$ valued points are $R_{\mathcal{O}/\mathbb{Z}}(\mathbb{G}_m)(R) = (\mathcal{O} \otimes \mathbb{Z} R)^\times$. We choose an isomorphism

$$j : \tilde{K} \xrightarrow{\sim} R_{\mathcal{O}/\mathbb{Z}}(\mathbb{G}_m)$$

which is defined by the rule

$$j : I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mapsto i,$$

The group scheme $\tilde{K}/\mathbb{Z}$ is not smooth, but the embedding $\mathbb{Z}[i] \otimes \mathbb{Z}[i] \to \mathbb{Z}[i] \oplus \mathbb{Z}[i]$ induces an embedding

$$\tilde{K} \times \text{Spec}(\mathbb{Z}[i]) \hookrightarrow \mathbb{G}_m \times \mathbb{G}_m$$

which is an inclusion of affine algebras

$$\mathbb{Z}[i][x, x^{-1}] \otimes \mathbb{Z}[i][y, y^{-1}] \hookrightarrow A(\tilde{K}) \otimes \mathbb{Z}[i].$$

Here is $y = \bar{x}$ the complex conjugate of $x$. Then we get

$$a = \frac{1}{2}(x + y), b = \frac{1}{2i}(x - y)$$

This inclusion becomes an isomorphism if we invert 2. We observe that we have the obvious inclusion $i : \mathbb{G}_m \hookrightarrow \tilde{K}$ and we have the restriction of the determinant $\text{det} : \tilde{K} \to \mathbb{G}_m$. The kernel of det is the group scheme

$$\mathcal{K}^{(1)} = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \text{GL}_2 \mid a^2 + b^2 = 1 \right\}$$

The character module $X^*(\mathcal{K}^{(1)} \times \mathbb{Z}[i]) = \mathbb{Z}e$, where

$$e : \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \right\} \mapsto (a + bi).$$

The matrix

$$c_2 = \begin{pmatrix} 1 & 1 \\ -1 & i \end{pmatrix} \in \text{GL}_2(\mathbb{Z}[1, \frac{1}{2}])$$

conjugates the standard diagonal torus $T \times \mathbb{Z}[1, \frac{1}{2}]$ into $\mathcal{K}^{(1)} \cdot \mathbb{G}_m \times \mathbb{Z}[1, \frac{1}{2}]$.  

8
We choose a weight \( \lambda = l\gamma_1 + d\delta \) where \( l \equiv 2d \mod 2 \). We consider the space of regular functions on \( G \) which satisfy

\[
A_\lambda(B\backslash G) = \{ f \in A(G) | f(bg) = \lambda(b)f(g) \}
\]  

(13)

On this space of sections we have the action of \( G \) by right translations. The following is rather obvious and well known

**Proposition 3.1.** The module \( A_\lambda(B\backslash G) \) of regular functions is trivial if \( l > 0 \). If \( l \leq 0 \) it realizes the module \( \mathcal{M}_{\lambda, Z} \) of highest weight \( \lambda^- = -l\gamma_1 + d\delta \).

We can also say that \( \lambda \) defines a line bundle \( L_\lambda \) and

\[
A_\lambda(B\backslash G) = H^0(B\backslash G, L_\lambda).
\]  

(14)

The algebra of regular function is embedded into the larger algebra \( A(G)_e \), these are the function which are regular at the identity element, it is the localization at \( e \). Again we define

\[
A_\lambda(B\backslash G)_e = \{ f \in A(G)_e | f(bg) = \lambda(b)f(g) \}
\]  

(15)

On this module we do not have an action of \( G \), but it is clear that we still have an action of \( g_\mathcal{Z} \).

We consider the morphism of schemes \( m : B \times \tilde{K} \rightarrow G \) given by the multiplication. The intersection \( B \cap \tilde{K} = C = \mathbb{G}_m \) is embedded into the product \( t \mapsto (t, t^{-1}) \). The fiber of the morphism \( m \) are torsors under the action of \( C \). Then \( m \) induces a homomorphism of affine algebras

\[
A(G) \hookrightarrow (A(B) \otimes A(\tilde{K}))^C \hookrightarrow A(G)_e
\]  

(16)

Our character \( \lambda \) defines the rank one module \( Z[\lambda] \). Let \( \lambda_C \) be the restriction of \( \lambda \) to the center \( C \). Then the above embedding defines an inclusion

\[
Z[\lambda] \otimes A_{\lambda_C}(\tilde{K}) \hookrightarrow A(G)_e[\lambda]
\]  

(17)

The left hand hand side is a \( \tilde{K} \)- module, this \( \tilde{K} \) module is invariant under the action of the Lie-algebra \( g_\mathcal{Z} \). This allows us to define the induced module

\[
\mathfrak{Ind}^G_{\tilde{K}} Z[\lambda] = Z[\lambda] \otimes A_{\lambda_C}(\tilde{K})
\]  

(18)

For \( \nu \equiv l \mod 2 \) we define the elements

\[
\Phi_{d,\nu} = \frac{(a + bi)^\nu}{(a^2 + b^2)^{\nu/2}} \in A_{\lambda_C}(\tilde{K}) \otimes Z[i].
\]  

(19)

We get an inclusion

\[
A_{\lambda_C}(\tilde{K}) \otimes Z[i] \supset \bigoplus Z[i]\Phi_{d,\nu}
\]  

(20)
and this inclusion becomes an isomorphism if we invert 2. Then we get a decomposition into eigenspaces under the action of $\mathcal{K}$. The $\Phi_{d,\nu}$ are characters. The complex conjugation $c$ (the non trivial element in $\text{Gal}(\mathbb{Q}((1)/\mathbb{Q}))$ acts on the modules above and $c(\Phi_{d,\nu}) = \Phi_{d,-\nu}$.

We define a submodule

$$A'_{\lambda_C}(\tilde{\mathcal{K}}) = (\bigoplus_{\nu \equiv l \mod 2} \mathbb{Z}[i] \Phi_{d,\nu}) \cap A_{\lambda_C}(\tilde{\mathcal{K}})$$

if we invert 2 it becomes isomorphic to $A_{\lambda_C}(\tilde{\mathcal{K}})$.

The Lie algebra $\mathfrak{g}_Z^{(1)}$ is a direct sum $\mathfrak{g}_Z^{(1)} = \mathbb{Z}H \oplus \mathbb{Z}E_+ \oplus \mathbb{Z}E_-$, where

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$  \hfill (21)

We introduce some more notation

$$V = E_+ + E_-, \quad Y = E_+ - E_-$$

$$P_+ = H + i \otimes V, \quad P_- = H - i \otimes V$$ \hfill (22)

where the elements in the first row are in $\mathfrak{g}_Z^{(1)}$ the elements in the second row are in $\mathfrak{g}_Z^{(1)}$. Under the adjoint action of $\tilde{\mathcal{K}}$ the elements $P_+, P_-$ are eigenvectors. We have

$$\text{Ad}(k)P_+ = \Phi_{0,2}(k)P_+, \quad \text{Ad}(k)P_- = \Phi_{0,-2}(k)P_-$$ \hfill (23)

Some elementary computations yield (the reader may find a more detailed exposition in [4] in the file [sl2neu.pdf]):

$$Y \Phi_{d,\nu} = i \nu \Phi_{d,\nu}, \quad P_+ \Phi_{d,\nu} = (l + \nu) \Phi_{d,\nu+2}, \quad P_- \Phi_{d,\nu} = (l - \nu) \Phi_{d,\nu-2}$$ \hfill (24)

We look at the generators of $\mathfrak{g}_Z/\mathfrak{k}_Z$

$$H = \frac{1}{2}(P_+ + P_-), \quad V = \frac{1}{2i}(P_+ - P_-)$$

and because of the parity conditions it is clear

$$A'_{\lambda_C}(\tilde{\mathcal{K}})$$

is a $(\mathfrak{g}_Z, \tilde{\mathcal{K}})$-module and hence we get that

$$\text{Ind}_{\mathfrak{g}_Z}^{\mathfrak{g}_Z^\mathbb{Z}[\lambda]} \otimes \mathbb{Z}[\frac{1}{2}]$$

is a $(\mathfrak{g}_Z, \tilde{\mathcal{K}})$ module \hfill (25)

Now it becomes clear that $\text{Ind}_{\mathfrak{g}_Z}^{\mathfrak{g}_Z^\mathbb{Z}[\lambda]} \otimes \mathbb{Z}[i, \frac{1}{2}]$ is never irreducible. We have two cases.
Let us first assume \( l \leq 0 \), then we get from our formulas (24) that \( P_+ \Phi_{d,-l} = 0 \), \( P_- \Phi_{d,l} = 0 \) and we find a non trivial invariant submodule

\[
\bigoplus_{l \leq \nu \leq -l, \nu \equiv l \mod 2} \mathbb{Z}[i, \frac{1}{2}] \Phi_{d,\nu}
\]

and if we look a little bit more closely then we see that this is the module \( \mathcal{M}_{\lambda, z}[i, \frac{1}{2}] \). The quotient by this submodule decomposes into a direct sum, i.e. we get an exact sequence

\[
0 \to \mathcal{M}_{\lambda, z}[i, \frac{1}{2}] \to \mathfrak{sl}_2[Z, \mathbb{Z}] \otimes \mathbb{Z}[i, \frac{1}{2}] \to D_+^l \oplus D_-^l \to 0
\]

where

\[
D_+^l = \bigoplus_{\nu \geq -l+2, \nu \equiv l \mod 2} \mathbb{Z}[i, \frac{1}{2}] \Phi_{d,\nu} ; \quad D_-^l = \bigoplus_{\nu \leq -l-2, \nu \equiv l \mod 2} \mathbb{Z}[i, \frac{1}{2}] \Phi_{d,\nu}
\]

is a decomposition into two invariant submodules.

We look at the second case where \( l \geq 0 \). In this case look at the induced module

\[
\mathfrak{sl}_2[Z, \mathbb{Z}] \otimes \mathbb{Z}[i, \frac{1}{2}] \]

here \( 2\rho \) is the sum of the positive roots, in this case we have of course \( 2\rho = \alpha \). In our formula (24) we have to replace \( l \) by \( l+2 \). We have \( P_- \Phi_{d,l+2} = 0 \) and \( P_+ \Phi_{d,-l-2} = 0 \) and hence we see that the two modules in (28) are invariant submodules and we get an exact sequence

\[
0 \to (D_+^l \oplus D_-^l) \otimes \mathbb{Z}[i, \frac{1}{2}] \to \mathfrak{sl}_2[Z, \mathbb{Z}] \otimes \mathbb{Z}[i, \frac{1}{2}] \to \mathcal{M}_\lambda \otimes \mathbb{Z}[i, \frac{1}{2}] \to 0
\]

(29)

For any \( \lambda \) the modules \( D_\lambda^+ \otimes \mathbb{C} \) are the familiar discrete series modules. If we consider the two weights \( \lambda = l\gamma_1 + d\delta, \lambda^- = -l\gamma_1 + d\delta \) and then two discrete series \( D_\lambda^+, D_{\lambda^-} \) are not isomorphic but if we take the tensor product with the rationals then we find isomorphisms

\[
\Psi_{d,l}^\pm : D_\lambda^\pm \otimes \mathbb{Q} \to D_{\lambda^-}^\pm \otimes \mathbb{Q}
\]

(30)

which is uniquely defined by the condition \( \Psi_{d,l}^+(\Phi_{d,\pm(l+2)}) = \Phi_{d,\pm(l+2)} \).

In our notation the discrete series Harish-Chandra modules for \( \mathrm{G}l_2 \) are parametrized by a pair \((\lambda, \text{sign})\) where \( \lambda \) is a highest weight \( \lambda = l\gamma_1 + d\delta, l \geq 11 \).
0. We have seen that with these notation the exact sequences above tell us that
\[ \text{Ext}^1(M_{\lambda,Z[i]}, D^\pm_{\lambda}) \neq 0. \] (31)

If we restrict the action of \( \mathcal{K} \) on \( D^+_\lambda \) to \( \mathcal{K}^{(1)} \) types, then we get a decomposition into \( \mathcal{K}^{(1)} \) types
\[ D^+_\lambda = \bigoplus_{\nu \geq l+2 \nu \equiv l \mod 2} Z[i][\nu] \] (32)
where \( \mathcal{K}^{(1)} \) acts by \( \nu e \) on \( Z[i][\nu] \). The character \( (l+2)e \) is called the minimal \( \mathcal{K}^{(1)} \) type in \( D^+_\lambda \). The character \( -(l+2)e \) is also called the minimal \( \mathcal{K}^{(1)} \) type in \( D^-_\lambda \).

In the following we will work with \( \lambda = l \gamma_1 + d \delta \) and \( l \geq 0 \). We consider the module \( M_{\lambda,Z[i]} \). Let us assume that we realized \( M_{\lambda,Z[i]} \) as the module of homogenous polynomials of degree \( l \) in two variables \( U,V \). (This is actually the module \( H^0(B\setminus G, L_{\lambda-}) \) and we consider the action of \( \mathcal{K}^{(1)} \times Z[i] \) on it and we have the decomposition into eigenspaces
\[ Z[i, \frac{1}{2}](U - iV)^l \oplus \cdots \oplus Z[i, \frac{1}{2}](U + iV)^l. \] (33)
We are only interested in the highest and lowest weight vectors. We abbreviate \( (U - iV)^l = e_{-l}, (U + iV)^l = e_l \). We also put \( D_{\lambda} = D^+_\lambda \oplus D^-_{\lambda} \). The relative Lie-algebra cohomology with coefficients in \( D_{\lambda} \otimes M_{\lambda,Z[i]} \) is the cohomology of the complex
\[ \text{Hom}_{\mathcal{K}^{(1)}}(A^*(gZ/\tilde{t}Z) \otimes Z[i], D_{\lambda} \otimes M_{\lambda,Z[i]}) \] (34)
Here we observe that \( \Lambda^0((gZ/\tilde{t}Z) \otimes Z[i]) = \Lambda^2((gZ/\tilde{t}Z) \otimes Z[i]) = Z[i] \) where we choose \( P_+ \cap P_- \) as generator. Since \( D_{\lambda} \otimes M_{\lambda,Z[i]} \) does not contain the trivial \( \mathcal{K}^{(1)} \) module this complex is zero in degree 0 and 2. In degree one we have
\[ \Lambda^1(gZ/\tilde{t}Z) \otimes Z[i]) = gZ/\tilde{t}Z \otimes Z[i] = Z[i]P_+ \oplus Z[i]P_- \] (35)
We denote by \( P^+_{\lambda}, P^-_{\lambda} \in \text{Hom}((gZ/\tilde{t}Z) \otimes Z[i], Z[i]) \) the dual basis.

**Proposition 3.2.**
\[ H^1(gZ, \tilde{t}, D_{\lambda} \otimes M_{\lambda,Z[i]}) = Z[i]P^+_{\lambda} \otimes \Phi_{d,l+2} \oplus e_{-l} \bigoplus Z[i]P^-_{\lambda} \otimes \Phi_{d,-l-2} \oplus e_l \]

**Proof.** obvious \( \square \)
If we replace $\tilde{K}$ by $K = \mathcal{K}^{(1)}$, then we get the same, but we have to multiply the right hand side by $\Lambda^*$.

It is clear from the construction, that the element $c$ in the Galois group acts on $\mathcal{D}_\lambda$ and more precisely we have $c(P_+) = P_-, \ c(\Phi_{d,\nu}) = \Phi_{d,-\nu}, \ c(\epsilon_l) = e_{-l}$. Then we put

$$\Omega_{d,l} = P_+^l \otimes \Phi_{d,l+2} \otimes e_{-l}, \ \check{\Omega}_{d,l} = P_-^l \otimes \Phi_{d,-l-2} \otimes \epsilon_l,$$

we may think of these elements as holomorphic and antiholomorphic 1-forms.

We define $\mathcal{D}_{\lambda,Z}$ as the $(g_{Z,K})$ module of elements in $\mathcal{D}_\lambda$ fixed by $c$. Then

$$\text{Hom}_K(\Lambda^1(g_{Z/KZ}), \mathcal{D}_{\lambda,Z} \otimes \mathcal{M}_{\lambda,Z}) = \mathcal{Z}(\Omega_{d,l} + \check{\Omega}_{d,l}) \oplus \mathcal{Z}(i\Omega_{d,l} - i\check{\Omega}_{d,l})$$

We introduce some abbreviations

$$\omega_{d,l}^{(1)} = \Omega_{d,l} + \check{\Omega}_{d,l}, \ \omega_{d,l}^{(2)} = i\Omega_{d,l} - i\check{\Omega}_{d,l}, \ \eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

We still have the action of $O(2)/SO(2) = \mathbb{Z}/2\mathbb{Z} = \pi_0(G(R))$. The nontrivial element is represented by the matrix $\eta$ defined above. Under this action the module $\text{Hom}_K(\Lambda^1(g_{Z/KZ}), \mathcal{D}_{\lambda,Z} \otimes \mathcal{M}_{\lambda,Z})$ decomposes into a + and a − eigenspace. A straightforward computation shows that

$$\eta(P_+^l) = P_-^l(\pm), \ \eta(\Phi_{d,\nu}) = \Phi_{d,-\nu}, \ \eta(\epsilon_{\pm l}) = (-1)^{\frac{d+l}{2}} e_{-(\pm l)}$$

We have generated the + and − eigenspaces (maybe up to a power of 2). We have

$$\eta(\omega_{d,l}^{(1)}) = (-1)^{\frac{d+l}{2}} \omega_{d,l}^{(1)}, \ \eta(\omega_{d,l}^{(2)}) = (-1)^{\frac{d+l}{2}} \omega_{d,l}^{(2)}$$

**Proposition 3.3.** The elements $\omega_{d,l}^{(1)}, \omega_{d,l}^{(2)} \in \text{Hom}_K(\Lambda^1(g_{Z/KZ}), \mathcal{D}_{\lambda,Z} \otimes \mathcal{M}_{\lambda,Z})$ are generators of the ± eigenspaces (maybe up to a power of 2). We have

**Proof.** Again obvious. 

We remember that $d \in \frac{1}{2}\mathbb{Z}$ and satisfies $2d \equiv l \mod 2$, hence it is well defined modulo $\mathbb{Z}$. Which of the two generators $\omega_{d,l}^{(1)}, \omega_{d,l}^{(2)}$ is the generator of the + eigenspace depends on $d$ and they change role if we replace $d$ by $d+1$. This flip plays a decisive role in the definition of the periods in [7].

Our module $\mathcal{D}_{\lambda,Z}$ is irreducible but its base extension $\mathcal{D}_\lambda \otimes \mathbb{Z}[i]$ is reducible, it decomposes into $\mathcal{D}_\lambda^+ \oplus \mathcal{D}_\lambda^-$. If we enlarge $\mathcal{K}^{(1)}$ to $\mathcal{K} = \mathcal{K}^{(1)} \wr < \eta >$ then $\mathcal{D}_{\lambda,Z} \otimes \mathbb{Z}[i]$ becomes an absolutely irreducible $(g_{Z,K})$-module.

Then it is easy to see (see for instance [sl2neu.pdf] that in the case $\lambda$ regular (i.e. $l \neq 0, 1$) $\mathcal{D}_{\lambda,Z}$ is the only irreducible $(g_{Z,K})$ module which has non trivial cohomology with coefficients in $\mathcal{M}_{\lambda,Q}$. If $l = 0$ then the trivial one dimensional $(f_{g_{Z,K}}, \mathcal{K}^{(1)})$-module $\mathbb{Z}$ has non trivial cohomology in degree 0 and 2 and this module completes the list of modules which have non trivial cohomology with coefficients in some $\mathcal{M} - \lambda, \mathbb{Z}$.  

13
3.3 The intertwining operator

We come back to our highest weight \( \lambda = l\gamma + d\delta \), we assume \( l \geq 0 \). In equation (30) we wrote down an intertwining isomorphism \( \Psi_{d,l} \) between the two discrete series representation. If we look at the inverse of this operator and observe that \( \mathcal{D}_\lambda^+ \oplus \mathcal{D}_\lambda^- \) is a quotient of \( \text{Ind}_{G_B Z}[\lambda^-] \) and \( \mathcal{D}_\lambda^+ \oplus \mathcal{D}_\lambda^- \) is a submodule of \( \text{Ind}_{G_B Z}[\lambda + 2\rho] \) then our isomorphism provides an intertwining operator

\[
T_{\lambda}^{alg} : \text{Ind}_{G_B Z}[\lambda^-] \otimes \mathbb{Q} \to \text{Ind}_{G_B Z}[\lambda + 2\rho] \otimes \mathbb{Q}
\]

which is unique up to a scalar and we normalized by fixing its value on a lowest \( K \) type.

By the same token we get an operator in the opposite direction \( T_{\lambda + 2\rho}^{alg} : \text{Ind}_{G_B Z}[\lambda + 2\rho] \otimes \mathbb{Q} \to \text{Ind}_{G_B Z}[\lambda^-] \otimes \mathbb{Q} \).

In this direction the space of homomorphisms is of rank one. The homomorphisms factor over a finite dimensional quotient.

In our situation here the maximal torus \( T = G_m \times G_m \) and so far we only discussed the modules which are induced from rational characters. In this case we also may induce characters \( \lambda \otimes \epsilon \) where \( \epsilon : \tilde{K}^T = \mu_2 \times \mu_2 \to \mu_2 \) is a sign character, it is the form \((\pm 1, \pm 1) \mapsto (\pm 1)^{m_1} (\pm 1)^{m_2} \). Then the induced module \( \text{Ind}_{G_B Z}[\lambda \otimes \epsilon] \) is still reducible if the sign character \( \epsilon = m \) factors over the determinant, i.e. we have \( m_1 = m_2 \). But if the sign character does not factor over the determinant then the induced module \( \text{Ind}_{G_B Z}[\lambda \otimes \epsilon] \) is in fact irreducible.

3.4 Transcendental Harish-Chandra modules

We return briefly to the transcendental theory of Harish-Chandra modules, we tensor everything by \( \mathbb{C} \) and then our modules become Harish-Chandra modules in the traditional sense. The group scheme \( \mathcal{K}^{(1)} \) is replaced by the group \( SO(2) = K_{\infty} = \mathcal{K}^{(1)}(\mathbb{R}) \). The following is of course well known.

The evaluation of the highest weight \( \lambda \) on \( T(\mathbb{R}) \) provides an (algebraic) character \( \lambda_{\mathbb{R}} : T(\mathbb{R}) \to \mathbb{R}^\times \). We define a larger class of (analytic) characters \( \chi : T(\mathbb{R}) \to \mathbb{C}^\times \) which are of the form

\[
\chi = (z, d, m) : \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \mapsto (|t_1 t_2|^{z/2} |t_1 t_2|^d (\frac{t_1}{|t_1|})^{m_1} (\frac{t_2}{|t_2|}^{m_2} (41)
\]

where \( z \) a complex variable and \( m = (m_1, m_2) \) is a pair of integers \( \mod 2 \). The central contribution given by the half integer \( d \) should be fixed. For us it seems to be adequate to distinguish between the character \( \lambda \in X^\times(T) \) and its evaluation \( \lambda_{\mathbb{R}} \). For \( \lambda = l\gamma + d\delta \) we have

\[
\lambda_{\mathbb{R}} = \chi = (z, d, m) \iff z = l \text{ and } m_1 \equiv \frac{l}{2} + d, m_2 \equiv -\frac{l}{2} + d \mod 2
\]
We call such a $\chi$ algebraic, we say that $\chi$ is cohomological if $l \neq 1$. We say that $\chi$ is of algebraic type if $z$ is an integer but the parity conditions may fail.

We define the induced representation

$$I^G_B \chi = \{ f : \text{Gl}_2(\mathbb{R}) \to \mathbb{C} | f \in C_\infty(\text{Gl}_2(\mathbb{R})), f \left( \begin{array}{cc} t_1 & u \\ 0 & t_2 \end{array} \right) g = \chi(t) f(g) \},$$

this is a $\text{Gl}_2(\mathbb{R})$ module. The submodule of $K_\infty$ finite functions is our induced Harish-Chandra module $I^G_B \lambda_R$, for $\chi = \lambda_R$ we have

$$I^G_B \lambda = \mathfrak{m} \Phi^G_B \lambda \otimes \mathbb{C}.$$

Let $m = m_1 + m_2 \mod 2$ then the module is a direct sum

$$I^G_B \chi = \bigoplus_{\nu \equiv m \mod 2} \Phi^G_\chi$$

where

$$\Phi^G_\chi \left( \begin{array}{cc} t_1 & u \\ 0 & t_2 \end{array} \right) \cdot \left( \begin{array}{cc} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{array} \right) = \left( |\frac{t_1}{t_2}|\right)^{z/2} |t_1 t_2|^d \left( \frac{t_1}{|t_1|} \right)^{m_1} \left( \frac{t_2}{|t_2|} \right)^{m_2} e^{2\pi i \nu \phi}$$

We have essentially the same formulae for the action of the Lie algebra

$$Y \Phi^G_\chi = i\nu \Phi^G_\chi, \quad P_+ \Phi^G_\chi = (z + \nu) \Phi^G_\chi, \quad P_- \Phi^G_\chi = (z - \nu) \Phi^G_\chi$$

note that the parity of $\nu$ is equal to the parity of $m_1 + m_2$. (see Slzweineu.pdf)

For $\chi = (z, d, m)$ we put $\chi' = (-z, d, m')$. Then we can write down the classical (standard) intertwining operator

$$T^\text{st}_\chi: I^G_B \chi \to I^G_B (\chi' \otimes \rho^2)$$

which is defined by

$$T^\text{st}_\chi(f)(g) = \int_{-\infty}^{\infty} f\left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \cdot \left( \begin{array}{cc} 1 & u \\ 0 & 1 \end{array} \right) g \, du$$

where $du$ is of course the Lebesgue measure on $\mathbb{R}$. This integral converges for $\Re(z) >> 0$ and has an meromorphic continuation into the entire $z$-plane. We need to locate the poles and we want to show that this operator is never identically zero.

We introduce the notation $\chi^\dagger = \chi' \otimes \rho^2$. We evaluate it at the smallest $K_\infty$ type, which is $\Phi^G_0$ if $m_1 + m_2$ is even and $\Phi^G_1$ if $m_1 + m_2$ is odd. Let us put $\epsilon(m) = 0$ if $m_1 + m_2$ is even and $\epsilon(m) = 1$ else. Then an easy computation shows

$$T^\text{st}_\chi(\Phi^G_\chi) = \frac{\Gamma\left( \frac{z + \epsilon(m)}{2} \right) \Gamma\left( \frac{1}{2} \right)}{\Gamma\left( \frac{z + \epsilon(m)}{2} \right)} \Phi^G_{\epsilon(m)}$$

15
Then we can evaluate $T_{\chi}^{st}$ on any element $\Phi_{\nu}^{\chi}$, we use the recursion provided by the formulae (44). We get for $n \geq 1$

$$P_{+}^{n}(\Phi_{e(m)}^{\chi}) = (z + \epsilon(m)) \cdots (z + \epsilon(m) + 2(n - 1))\Phi_{e(m)+2n}^{\chi}$$

and on the other side (we have to replace $z$ by $2 - z$)

$$P_{+}^{n}(\Phi_{e(m)}^{\chi}) = (2 - z + \epsilon(m)) \cdots (2 - z + \epsilon(m) + 2(n - 1))\Phi_{e(m)+2n}^{\chi}$$

Therefore, if $\nu = \epsilon(m) + 2n$

$$T_{\chi}^{st}(\Phi_{\nu}^{\chi}) = \frac{(2 - z + \epsilon(m)) \cdots (2 - z + \epsilon(m) + \nu - 2)) \Gamma(\frac{z + \epsilon(m) - 1}{2}) \Gamma(\frac{1}{2}) \Phi_{\nu}^{\chi}}{(z + \epsilon(m)) \cdots (z + \nu - 2) \Gamma(\frac{z + \epsilon(m)}{2}) \Phi_{\nu}^{\chi}}$$

(50)

(Note that here $\nu > 1$, the product is empty if $\nu = 0, 1$ and hence it has value one if this is the case. Of course we get a corresponding formula for $\nu \leq 0$).

We say that the intertwining operator $T_{\chi}^{st}$ is holomorphic at $\chi = (z_{0}, d, m)$ if for all $\nu \equiv \epsilon(m) \mod 2$ the function $T_{\chi}^{st}(\Phi_{d,\nu}^{\chi})/\Phi_{d,\nu}^{\chi}$ is holomorphic at $z_{0}$. Otherwise we say that $T_{\chi}^{st}(\Phi_{d,\nu}^{\chi})$ has a pole at $z_{0}$.

**Proposition 3.4.** The intertwining operator $T_{\chi}^{st}$ has its poles at the arguments $z_{0} = 1 - \epsilon(m), -1 - \epsilon(m), \ldots$ and these are first order poles. At these arguments $T_{\chi}^{st}(\Phi_{\nu}^{\chi})$ has a pole for all values $\nu$.

**Proof.** This is essentially an exercise in using the properties of the $\Gamma-$ function. We look at the denominator of the expression in (51). We have

$$\frac{(z + \epsilon(m)) \cdots (z + \nu - 2) \Gamma(\frac{z + \epsilon(m)}{2})}{\Gamma(\frac{z + \epsilon(m)}{2})} = 2^{n} \Gamma(\frac{z + \epsilon(m)}{2} + n - 1)$$

the $\Gamma-$ function has no zeroes, hence the denominator does not contribute to poles. The $\Gamma-$ factor in the numerator has its poles exactly at the above list, these are first order poles and they do not cancel against the product of linear factors in front of the $\Gamma-$ factor.

We can form the composite $T_{\chi}^{st} \circ T_{\chi}^{st}$ and this is an endomorphism of $I_{G}^{c,\chi}$. Since for a general value of $z$ the module is irreducible the operator must be a scalar $\Lambda(\chi)$ and it is not too difficult to write down this scalar.

We define $a(m) = +1$ if $\epsilon(m) = 0$ otherwise $a(m) = -1$.

$$\Lambda(\chi) = \frac{\Gamma(\frac{z-1+\epsilon(m)}{2})\Gamma(\frac{1-z+\epsilon(m)}{2})}{\Gamma(\frac{z+\epsilon(m)}{2})\Gamma(\frac{2-z+\epsilon(m)}{2})} = \frac{2}{z - 1} \left(\frac{\sin(\frac{\pi z}{2})}{\cos(\frac{\pi z}{2})}\right)^{a(m)}$$

For us the important arguments for $\chi$ are the values $\chi = (l + 2, m)$ and $\chi = (-l, m)$ where $l \geq 0$ is an integer and $l \equiv m \mod 2$, we called these
values of $\chi$ cohomological. Our proposition tells us that $T^\text{st}_\chi$ is holomorphic at cohomological arguments. But we also see that $\Lambda(\chi)$ vanishes at these arguments, i.e. $T^\text{st}_\chi \circ T^\text{st}_\chi = 0$. Since it is clear that the linear map $T^\text{st}_\chi$ is never zero it follows that $T^\text{st}_\chi$ maps $I_{GB}^\chi$ to the kernel of $T^\text{st}_\chi^\dagger$.

This is of course consistent with our results in section 3.2, if we tensorize the two exact sequences (27), (29) by the complex numbers and apply our intertwining operator to the terms in the middle of the exact sequences then we get for $\chi = \lambda_{\mathbb{R}}$.

$$T^\text{st}_{\chi} \otimes \rho^2_\mathbb{R} = I_B^G \otimes \rho^2_\mathbb{R} \rightarrow \mathcal{M}_{\lambda,\mathbb{C}} \subset \mathcal{M}_{\lambda,\mathbb{C}}^G = \mathcal{M}_{\lambda,\mathbb{C}}^G$$ (51)

and

$$T^\text{st}_{\chi}^\dagger : I_B^G \otimes \mathcal{M}_{\lambda,\mathbb{C}} \rightarrow \mathcal{M}_{\lambda+2\rho,\mathbb{C}} \subset \mathcal{M}_{\lambda+2\rho,\mathbb{C}}^G$$ (52)

These two intertwining operators are of course multiples of our earlier operators $T^\text{alg}_{\lambda+2\rho} \otimes \mathcal{C}, T^\text{alg}_{\lambda} \otimes \mathcal{C}$. These earlier operators have been normalized such that they gave the “identity” on certain $K_\infty$ types. For the operator $T^\text{alg}_{\lambda+2\rho} \otimes \mathcal{C}$ this is the $K_\infty$-type $\Phi_l$ and for $T^\text{alg}_{\lambda} \otimes \mathcal{C}$ this is $\Phi_{l,0}$. Then a straightforward computation yields for $\chi = \lambda_{\mathbb{R}}$.

$$T^\text{st}_{\chi} \otimes \rho^2_\mathbb{R} = \pi 2 \frac{\Gamma^2(z+1-\epsilon(m))}{\Gamma^2(z+1-\epsilon(m))} T^\text{st}_{\lambda+2\rho}$$

$$T^\text{st}_{\chi} = \pi 2 \frac{\Gamma^2(z+1-\epsilon(m))}{\Gamma^2(z+1-\epsilon(m))} T^\text{st}_{\lambda}$$ (53)

This tells us that the operators $\frac{1}{\pi} T^\text{st}_{\chi}^{-1}, \frac{1}{\pi} T^\text{st}_\chi \otimes \rho^2_\mathbb{R}$ evaluated at cohomological arguments are defined over $\mathbb{Q}(i)$. They even induces $n$ isomorphisms between the $\mathbb{Z}[i, \frac{1}{2}]$ modules of the cohomologically relevant $K_\infty$ types.

We also have a brief look at the induced modules which are not cohomological, these are the modules induced from $\chi = (l, d, m)_{\mathbb{R}}$ where $l$ is an integer $2d \equiv l \mod 2, l - 1 \equiv \epsilon(m) \equiv 0 \mod 2$. If now $l - 1 + \epsilon(m) \geq 2$ then the operator $T^\text{st}_\chi$ is defined over $\mathbb{Q}(i)$, because $\Gamma(1/2)$ appears in the numerator and in the denominator. If $l + \epsilon(m) - 1 = 0, -2, -4, \ldots$ then the intertwining operator has a pole. But we can modify the operator and define the normalized operator

$$T^\text{norm}_\chi = \frac{1}{\Gamma^2(z+1-\epsilon(m))} T^\text{st}_\chi$$ (54)

This operator is holomorphic everywhere and at the arguments $\chi = (l, d, m)$ which are not cohomological it is an isomorphism and defined over $\mathbb{Q}(i)$. 17
4 Induction of Harish-Chandra modules

4.1 The general context

We pick a standard parabolic subgroup $P/\text{Spec}(\mathbb{Z})$, let $U_P/\text{Spec}(\mathbb{Z})$ be its unipotent radical and $M = P/U_P$ its Levi quotient. We can also view $M/\text{Spec}(\mathbb{Z})$ to be the Levi subgroup which is stable under the Cartan involution, this means that $M = P \cap P^\Theta$. Then $\Theta$ induces a Cartan involution on the semi simple component $M^{(1)}$, it is simply connected. Let $\mathcal{K}^{M,1} \subset M^{(1)}$ be the fixed point scheme. Let $C_M$ be the connected center of $M$ can be projected down to $M$ and yields a (possibly slightly larger) definite subscheme $\mathcal{K}_M \supset \mathcal{K}^{M,1}$.

Let us assume we have a highest weight module $M_\mu, Z$ and a $(\mathfrak{g}_Z, K_M)$ Harish-Chandra module $V$ over some ring $R$ for instance $R = \mathbb{Z}, R = \mathbb{Z}[\frac{1}{2}]$. We then define $V \otimes A(K_M) = \mathfrak{g}_Z, K_M, V \otimes M_\mu, Z$.

We give a construction of the induced $(g_Z, K^{(1)})$ module $\text{Ind}_{G_P} V$. We are interested in the case that $H^\bullet (M_\mu, K_M, V \otimes M_\mu, Z) \neq 0$. In this case we compute the cohomology $H^\bullet (\mathfrak{g}_Z, K^{(1)}, M_\lambda, Z)$ by adapting the method of Delorme.

Let $K_M = P \cap K^{(1)}$. The projection $K_M \to M$ is an injective homomorphism, we identify $K_M$ with its image. This allows us to define the submodule

$$(\mathfrak{g}_Z, K_M, V \otimes M_\mu, Z) \times (\mathfrak{g}_Z, K_M, V \otimes M_\mu, Z) \neq 0.$$
should be. We work with dual numbers $R_1[c]$ then we should have
\[ \epsilon X \left( \sum v_i \otimes f_i \right)(k) = \left( \sum v_i \otimes f_i \right)(k \cdot \exp(\epsilon X)) - \left( \sum v_i \otimes f_i \right)(k) \]
the pity is that the first summand on the right hand side is not yet defined.

To define it we consider the parabolic subgroup $k^{-1}Pk \subset G \times R_1$ and observe that the linear map

\[
\text{Lie}(k^{-1}P) \oplus \mathfrak{t}_Z \otimes R_1 \rightarrow \mathfrak{g}_Z \otimes R_1 \quad (58)
\]
is surjective. Hence we can write $X = V + U$ where $V \in \text{Lie}(k^{-1}P), U \in \text{Lie}(K) \otimes R_1$. Now we can define

\[
\left( \sum v_i \otimes f_i \right)(k \cdot \exp(\epsilon X)) = \left( \sum v_i \otimes f_i \right)(\exp(\epsilon \text{Ad}(k)(V)k \cdot \exp(\epsilon U)) \quad (59)
\]
the expression on the right hand side is defined. If we recall the definition of $(V \otimes A(K))^K$ then we see that it is equal to

\[
e(\sum \text{Ad}(k)(V)v_i \otimes f_i(k) + \sum v_i \otimes Uf_i(k)) + \sum v_i \otimes f_i(k) \quad (60)
\]
It also clear that it does not depend on the decomposition of $X = V + U$.

Hence we can define the induced Harish-Chandra module

\[
\text{Ind}_{G}^{G} \mathcal{V} = (V \otimes A(K))^K \quad (61)
\]
It is not difficult to show that this satisfies all the conditions 1) to 6). Condition 2) may require a longer argument. We will discuss an example in the following section and in this example it becomes clear why condition 2) is fulfilled.

### 4.2 The integral version of $\mathbb{D}_\lambda$

We apply this induction process to a special case of the group $G_{n}/\mathbb{Z}$. We want to construct the $\mathbb{Z}$ structure on the modules which are called $D_\lambda$ in \[7\].

Let $T/\mathbb{Z}$ be the standard split torus and $B \supset T$ the standard Borel subgroup of upper triangular matrices. The parabolic subgroups $P \supset B$ are the standard parabolic subgroups.

Let $\gamma_1, \ldots, \gamma_{n-1} \in X^*_Q(T)$ be the dominant fundamental weights, let $\delta$ be the determinant. For this we choose a self dual highest weight $\lambda = \sum_{i=1}^{n-1} a_i \gamma_i + d \delta$, remember that self dual means $a_i = a_{n-i}$. We use the usual construction to construct the $G/\mathbb{Z}$-module $\mathcal{M}_{\lambda,\mathbb{Z}}$, it is the space of sections $H^0(B\backslash G, \mathcal{L}_\lambda)$ as in equation (14). We use the technique of induction to construct the very specific $(\mathfrak{g}_Z, \mathcal{K}^{(1)})$ modules $D_{\lambda}^{\epsilon}$ (where $\epsilon = \pm 1$) over $\mathbb{Z}[i, \frac{1}{2}]$ which have non trivial cohomology in lowest degree $b_n$

\[
H^{b_n}(\mathfrak{g}_Z, \mathcal{K}^{(1)}, D_{\lambda}^{\epsilon} \otimes \mathcal{M}_{\lambda,\mathbb{Z}}) \rightarrow \mathbb{Z}[i, \frac{1}{2}] \quad (62)
\]
We know that there is only a finite set of isomorphism classes of irreducible Harish-Chandra modules over $\mathbb{C}$ which have non trivial cohomology with coefficients in $\mathcal{M}_\lambda \otimes \mathbb{C}$. If $n$ is even (resp. $n$ is odd) then there are only two (resp. is only one) $(\mathfrak{g}_\mathbb{Z}, \mathcal{K}^{(1)})$ - module(s) which are tempered or which can be the infinite component of a cuspidal representation. (See for instance [7] 3.1, [11] and [9]).

### 4.3 The construction of $D^\lambda_\mathbb{Z}$

We consider the parabolic subgroup $^0P$ whose simple root system is described by the diagram

\[ \circ - \times - \circ - \times - \cdots - \circ(-\times) \]  

(63)

i.e. the set of simple roots $\pi_{^0M}$ of the semi simple part of the Levi quotient $^0M$ consists of those simple which have an odd index. This Levi subgroup can be identified to

\[ \prod_{i: \text{odd}} H_{\alpha_i} = \prod \text{Gl}_2(\times \mathbb{G}_m) \]  

(64)

i.e. each factor is identified to $\text{Gl}_2/\text{Spec}(\mathbb{Z})$, we have an extra factor $\mathbb{G}_m$ if $n$ is odd. Let $m$ be the largest odd integer less than $n$. Note that here we have chosen a splitting of the Levi-quotient to a Levi subgroup, this splitting is unique, since we want that our Levi subgroup is stable under the Cartan involution. Let $^0M^{(1)}$ be the semi simple component, we write as usual $^0M = ^0M^{(1)} \cdot C_{^0M}$.

The standard maximal torus is a product $T = \prod_{i: \text{odd}} T_i(\times \mathbb{G}_m)$ and for each $i = 1, 3, \ldots, m$ we have

\[ X^*(T_i) \otimes \mathbb{Q} = \mathbb{Q}^{\gamma_i^{^0M^{(1)}}} \oplus \mathbb{Q}\delta_i \]  

(65)

where $\gamma_i^{^0M^{(1)}} = \frac{\alpha_i}{2}$ and $\delta_i$ is the determinant on that factor. For $n$ odd let $\delta_n$ be the character which sends the last entry $t_n$ to $t_n$.

Let $B_i \supset T_i$ be the standard Borel subgroup of upper triangular matrices and let $B = \prod_{i: \text{odd}} B_i$ be our Borel subgroup of $^0M$. The $\gamma_i^{^0M^{(1)}}$ are the dominant fundamental weights with respect to the choice of $B$.

We return to the conventions in the first section and apply our considerations in section 3.2 to the factors $H_{\alpha_i}$. The Cartan involution induces the Cartan involution on each of the factors $H_{\alpha_i}$, the group scheme $\prod T_i^{(1)} = T_c$ is a maximal torus in the reductive group $\mathcal{K}^{(1)}$. The character module $X^*(T_c \times \mathbb{Z}[i]) = \oplus i\mathbb{Z}[i]e_i$. The Weyl $W_c$ of this torus acts on
the character module by sending \( e_i \mapsto e_i e_{\sigma(i)} \) where \( \sigma \) is any permutations, where the \( e_i = \pm 1 \) and satisfy \( \prod e_i = 1 \) if \( n \) is even.

Let \( B_c \supset K^{(1)} \times \mathbb{Z}[i] \) be the Borel subgroup of \( K^{(1)} \times \mathbb{Z}[i] \) which contains \( T_c \) and for which the roots \( e_1 - e_3, \ldots, e_{m-2} - e_m, e_{m-2} + e_m \) are the simple positive roots.

We have a very specific Kostant representative \( w_{un} \in W^o P \). The inverse of this permutation it is given by

\[
    w_{un}^{-1} = \{1 \mapsto 1, 2 \mapsto n, 3 \mapsto 2, 4 \mapsto n - 1, \ldots \}.
\]

The length of this element is equal to \( 1/2 \) the number of roots in the unipotent radical of \( ^o P \), i.e.

\[
    l(w_{un}) = \begin{cases} 
        \frac{1}{4}n(n - 2) & \text{if } n \text{ is even} \\
        \frac{1}{4}(n - 1)^2 & \text{if } n \text{ is odd} 
    \end{cases}
\]

Then

\[
    w_{un}(\lambda + \rho) - \rho = \sum_{i: i \text{ odd}} b_i \gamma_i^o M^{(1)} - (2\gamma_2 + 2\gamma_4 + \cdots + 2\gamma_{m-1} + \frac{3}{2}\gamma_{m+1}) + d\delta
\]

(67)

here \( \gamma_2, \gamma_4, \ldots \) are the dominant fundamental weights which have an even index and the \( b_i \) are the cuspidal parameters

\[
    b_{2j-1} = \begin{cases} 
        2a_j + 2a_{j+1} + \cdots + 2a_{\frac{n}{2} - 1} + a_{\frac{n}{2}} + n - 2j & \text{if } n \text{ is even} \\
        2a_j + 2a_{j+1} + \cdots + 2a_{\frac{n}{2} - 1} + n - 2j & \text{if } n \text{ is odd}
    \end{cases}
\]

A simple computation shows that we can rewrite the expression for \( w_{un}(\lambda + \rho) - \rho \)

\[
    w_{un} \cdot \lambda = \sum_{i: i \text{ odd}} (b_i \gamma_i^o M^{(1)} + (c(i, n) + d)\delta_i) + \begin{cases} 
        0 & \text{if } n \text{ even} \\
        \left(-\frac{n-1}{2} + d\right)\delta_n & \text{if } n \text{ odd}
    \end{cases}
\]

(68)

where the coefficients \( c(i, n) \) are given by the formula

\[
    c(i, n) = \begin{cases} 
        \frac{n-i}{2} & \text{if } n \text{ even} \\
        \frac{n-1-i}{2} & \text{if } n \text{ odd}
    \end{cases}
\]

(69)

In this formula the summands \( \mu_i = b_i \gamma_i^o M^{(1)} + (c(i, n) + d)\delta_i \in X^*(T_i) \) and \( -\frac{n-1}{2} + d \in \mathbb{Z} \). The sum of positive roots in the \( i \)-th factor is \( 2\rho_i = \sum_{i: i \text{ odd}} (b_i \gamma_i^o M^{(1)} + (c(i, n) + d)\delta_i) + \left(-\frac{n-1}{2} + d\right)\delta_n \)
\[ \alpha_i = 2^\gamma_i M^{(1)} \]. We take the character \( \mu_i + 2\rho_i = (b_i + 2)\gamma_i M^{(1)} + (c(i,n) + d)\delta_i \), and apply the constructions from (3.2) to it and construct the module \( \mathfrak{Ind}_{B_i}^M(\mu_i + 2\rho_i) \). We know that this module sits in an exact sequence

\[ 0 \to D_{\mu_i} \to \mathfrak{Ind}_{B_i}^M(\mu_i + 2\rho_i) \to M_{\mu_i,z} \to 0 \]  

(70)

We put \( \mu = \omega \cdot \lambda \), this is a character on the maximal torus \( T \) and we can define the induced module \( \mathfrak{Ind}_{B_i}^M(\mu_i + 2\rho_i) \). It is clear that this module is a tensor product

\[ \mathfrak{Ind}_{M}^\mu B_i(\mu_i + 2\rho_i) = \bigotimes_{i: \text{odd}} \mathfrak{Ind}_{M_i}^{\mu_i} B_i(\mu_i + 2\rho_i) \]  

(71)

where the last factor is only there if \( n \) is odd. Then this module contains the submodule

\[ D_{\mu} = \bigotimes_{i: \text{odd}} D_{\mu_i}(\otimes \mathbb{Z}(\frac{n-1}{2} + d)) \to \bigotimes_{i: \text{odd}} \mathfrak{Ind}_{M_i}^{\mu_i}(\mu_i + 2\rho_i)(\otimes \mathbb{Z}(\frac{n-1}{2} + d)) \]  

(72)

We know that \( D_{\mu} \otimes \mathbb{Z}[i, \frac{1}{2}] \) decomposes into the two submodules

\[ D_{\mu} \otimes \mathbb{Z}[i, \frac{1}{2}] = D_{\mu}^{+} \otimes \mathbb{Z}[i, \frac{1}{2}] \oplus D_{\mu}^{-} \otimes \mathbb{Z}[i, \frac{1}{2}], \]  

(73)

hence for any choice of signs we define the module

\[ D_{\mu}^{\pm} = \bigotimes_{i: \text{odd}} D_{\mu_i}^{\pm}(\otimes \mathbb{Z}(\frac{n-1}{2} + d)) \]

and the induced module

\[ D_{\lambda}^{\pm} = \mathfrak{Ind}_{\mathfrak{g}^{\pm}}^{\mathfrak{g}} D_{\mu}^{\pm}. \]  

(74)

The module \( D_{\mu}^{\pm} \) has as minimal \( \mathfrak{k}^{\pm} M \) type the character

\[ (\mathfrak{g}, \mu + 2\rho) = \sum_{i: \text{odd}} \epsilon_i(b_i + 2)e_i - (2\gamma_2 + 2\gamma_4 + \cdots + 2\gamma_{m-1} + \frac{3}{2}\gamma_{m+1}) + d\delta \]

The \( \mathbb{Z}[i, \frac{1}{2}] \) eigenmodule for this character is generated by

\[ D_{\mu}^{\pm}(\mathfrak{g}, \mu + 2\rho) = \mathbb{Z}[i, \frac{1}{2}] \bigotimes_{i: \text{odd}} \Phi_{d,\epsilon_i(b_i+2)}^{(i)} \]  

(75)

so it comes with a canonical generator, let us denote this generator by

\[ \bigotimes_{i: \text{odd}} \Phi_{d,\epsilon_i(b_i+2)}^{(i)} = \Phi_{\mu,\mathfrak{g}}. \]  

(76)
Proposition 4.1. The Weyl group $W_c$ contains a subgroup $S_m$, which acts by sign changes on the generators, i.e. $e_i \mapsto \pm e_i$. Hence $S_m$ acts on the set of characters $(\xi, \mu)$. It acts transitively on this set if $n$ is odd (See [Bou]) and if $n$ is even then we see easily that $(\xi, \mu + 2 \rho)$ and $(\xi', \mu + 2 \rho)$ are equivalent under the Weyl group $W_c$ if and only if $\prod_{i: \text{odd}} e_i = \prod_{i: \text{odd}} e_i'$.

Any of our characters $(\xi, \mu + 2 \rho)$ can be conjugated by an element in $W_c$ into a dominant weight with respect to $B_c$, and an easy computation shows that these dominant weights are

\[
\begin{cases}
(\xi, \mu + 2 \rho) = \sum_{i: \text{odd}, i < m} (b_i + 2)e_i & \text{if } n \text{ odd} \\
(\xi, \mu + 2 \rho) = \sum_{i: \text{odd}, i < m} (b_i + 2)e_i + \varepsilon (b_m + 2)e_m & \text{if } n \text{ even}
\end{cases}
\]

where in the second case $\varepsilon$ assumes the values $+1, -1$. These weights are indeed dominant because $b_{m-2} > b_m$. For $\varepsilon = \pm 1$ we define

\[D^\varepsilon_\lambda = D_{\lambda}^{(1,1,\ldots,\varepsilon)}\]

We have the following

**Proposition 4.1.** The $(\mathfrak{g}_Z, K^{(1)})$ modules $D^\varepsilon_\lambda$ are irreducible. Two such modules are isomorphic if and only if $(\xi, \mu)$ and $(\xi', \mu)$ are conjugate under the Weyl group $W_c$. The module $D^\varepsilon_\lambda$ contains a minimal $K^{(1)}$ type which has highest weight

\[\mu_c(\varepsilon, \lambda) = \sum_{i: \text{odd}, i < m} (b_i + 2)e_i + \varepsilon (b_m + 2)e_m\]

where $\varepsilon = 1$ if $n$ is odd and $\varepsilon = \pm 1$ if $n$ is even. This minimal $K^{(1)}$ type occurs with multiplicity one.

**Proof.** For the irreducibility we tensor by $\mathbb{C}$ and refer to [9] and [11]. Any element in the Weyl group $W_c$ can be represented by an element $w \in G(Z)$ which normalizes $T_c = K^{e,M}$. Then the multiplication from the left by $w$ induces an isomorphism

\[(D^\varepsilon_\mu \otimes A(K^{(1)}))^eM \simto (D^w_{\mu} \otimes A(K^{(1)}))^eM\]

and this is an isomorphism of $(\mathfrak{g}_Z, K^{(1)})$ modules. We prove the assertion concerning the $K^{(1)}$-types. We have a decomposition of $D^\varepsilon_\mu$ into $T_c$-types

\[D^\varepsilon_\mu = \bigoplus_{k_1 \geq 0, \ldots, k_m \geq 0} \mathbb{Z}[i, \frac{1}{2}] ((b_1 + 2 + 2k_1)e_1 + \cdots + \varepsilon (b_m + 2 + 2k_m)e_m)\]

The character $(b_1 + 2 + 2k_1)e_1 + (b_3 + 2 + 2k_3)e_3 + \cdots + \varepsilon (b_m + 2 + 2k_m)e_m$ may not be in the positive chamber ($k_m$ may be too large) but we can conjugate it to $\mu_c(k)$ in the positive chamber. For this character it is easy to see that

\[\mu_c(k) = (b_1 + 2)e_1 + (b_3 + 2)e_3 + \cdots + \varepsilon (b_m + 2)e_m + \sum_i m_i \alpha_i \epsilon \]
where the \( m_l \geq 0 \). The highest weight \( \mu_c(\epsilon, \lambda) = \mu_c(0) \).

Now we have the classical formula that

\[
(Z[i, 1/2](\mu_c(k)) \otimes A(K^{(1)}))^M \otimes \mathbb{Q} = \bigoplus \vartheta A(K^{(1)} \otimes Z[i, 1/2])(\mu_c(k), \vartheta) \otimes \mathbb{Q}
\]

(82)

where \( \vartheta \) runs over the isomorphism classes of irreducible \( K^{(1)} \otimes Z[i, 1/2] \) modules and where \( A(K^{(1)} \otimes Z[i, 1/2])(\mu_c(k)) = \{ f \mid R_{t(1-\epsilon)} f = \mu_c(k) \} f \) for all \( t \in T_c(R_1) \). Then it is well known that the multiplicity of \( \vartheta \) in \( A(K^{(1)} \otimes Z[i, 1/2])(\mu_c(k)) \) is equal to the multiplicity of \( \mu_c(k) \) in \( \vartheta \). We get that the representation \( \vartheta \) occurs with multiplicity one. (We notice that our argument also implies that for a given \( \vartheta \), the number of those \( k \) for which \( \mu_c(k) \) occurs in \( \vartheta \) with positive multiplicity, is finite. This settles the condition (2) in the definition of Harish-Chandra modules in section 2.1 for \( \text{Ind}_{\text{P}^\pm}^G \mu \) but this argument works in the general case too.)

\[
\square
\]

4.4 The cohomology \( H^\bullet(g_Z, K^{(1)}, \mathbb{D}_\lambda^c \otimes M_{\lambda, Z[i, 1/2]}) \)

We define as usual the \( (g_Z, K^{(1)}) \)-cohomology as the cohomology of the complex

\[
\text{Hom}_{K^{(1)}}(\Lambda^\bullet(g_Z/k_Z), \mathbb{D}_\lambda^c \otimes M_{\lambda, Z})
\]

(83)

If we tensor by the complex numbers then we know that \( \mathbb{D}_\lambda^c \otimes \mathbb{C} \) is unitary and since \( M_{\lambda, Z} \otimes \mathbb{C} \) is dual to its conjugate, it follows that all the differentials in the above complex are trivial, i.e the complex is equal to its cohomology.

We apply the Delorme method (or Frobenius reciprocity). Let \( ^0m_Z \) be the Lie algebra of \( ^0M \), let \( ^0m_Z^{(1)} \) be the Lie algebra of \( ^0M^{(1)} \). Let \( u_Z \) be the Lie-algebra of the unipotent radical of \( ^0P \) and finally let \( c_Z \) be the Lie algebra of \( C^c \cdot M \). Then

\[
g_Z[1/2] \otimes g_Z[1/2] = ^0m_Z^{(1)} \otimes u_Z[1/2] \oplus c_Z[1/2] \oplus (u_Z[1/2] \otimes u_Z(1))
\]

(84)

where now the right hand side is a \( (\mathfrak{m}, K^{c \cdot M}) \) -module. The group scheme \( K^{c \cdot M} \) acts by the adjoint action. It acts trivially on \( c_Z \), and the adjoint action of \( K^{c \cdot M} \) on \( u_Z \) extends to the adjoint action of \( ^0M \). (Remember that \( ^0M \) is a subgroup of \( ^0P \).) We get an isomorphism of complexes

\[
\text{Hom}_{K^{(1)}}(\Lambda^\bullet (g_Z/k_Z), \mathbb{D}_\lambda^c \otimes M_{\lambda, Z}) = \text{Hom}_{K^{c \cdot M}}(\Lambda^\bullet (^0m_Z^{(1)} \otimes u_Z), \mathbb{D}_\mu^c \otimes \text{Hom}(\Lambda^\bullet (u_Z), M_{\lambda, Z}) \otimes \Lambda^\bullet (c_Z))
\]

(85)

(86)
In the following we concentrate on the $\Lambda^0(c_\mathbb{Z})$ component. We claim that the $\mathbb{Z}[i, \frac{1}{2}]$ module

$$\text{Hom}_{K^M}(\Lambda^\bullet(\mathcal{O}^{(1)}_{\mathbb{Z}[i, \frac{1}{2}]}/\mathcal{O}\mathcal{Z}[i, \frac{1}{2}]), \mathcal{D}_\mu \otimes \text{Hom}(\Lambda^\bullet(\mathcal{U}_{\mathbb{Z}[i, \frac{1}{2}]})/\mathcal{M}_{\lambda, \mathbb{Z}[i, \frac{1}{2}]})$$

is free of rank one. We will be more precise: We decompose the three $K^M$-modules $\Lambda^\bullet(\mathcal{O}^{(1)}_{\mathbb{Z}[i, \frac{1}{2}]}/\mathcal{O}\mathcal{Z}[i, \frac{1}{2}]), \mathcal{D}_\mu$, and $\text{Hom}(\Lambda^\bullet(\mathcal{U}_{\mathbb{Z}[i, \frac{1}{2}]})/\mathcal{M}_{\lambda, \mathbb{Z}[i, \frac{1}{2}]})$ into eigenspaces with respect to characters in $X^*(K^M \otimes \mathbb{Z}[i, \frac{1}{2}])$ and show that there is exactly one triple of characters which contributes non trivially to the $\text{Hom}_{K^M}$, i.e. which satisfies $\eta_m = \eta_D + \eta_h$.

The module $\Lambda^\tau(\mathcal{O}^{(1)}_{\mathbb{Z}[i, \frac{1}{2}]}/\mathcal{O}\mathcal{Z}[i, \frac{1}{2}])$ contains the submodule

$$\bigoplus \mathbb{Z}[i, \frac{1}{2}] P_{1}^{\epsilon_1} \wedge P_{2}^{\epsilon_2} \wedge \cdots \wedge P_{m}^{\epsilon_m}$$

and on the individual summand our torus $T_c \cdot C_{\mathcal{O}M}$ acts by characters $\nu(\xi) = 2(\epsilon_1 \epsilon_2 + \cdots + \epsilon_m \epsilon_n) \cdot C_{\mathcal{O}M}$ We choose for $\xi$ the value $\xi_0 = (+, +, \ldots, \epsilon)$ hence $\Lambda^\tau(\mathcal{O}^{(1)}_{\mathbb{Z}[i, \frac{1}{2}]}/\mathcal{O}\mathcal{Z}[i, \frac{1}{2}])$ contains the direct summand

$$\Lambda^\tau(\mathcal{O}^{(1)}_{\mathbb{Z}[i, \frac{1}{2}]}/\mathcal{O}\mathcal{Z}[i, \frac{1}{2}])(\nu(\xi_0)) = \mathbb{Z}[i, \frac{1}{2}] (\nu(\xi_0))$$

The character $\nu(\xi_0)$ will be our $\eta_m$. The action of $C_{\mathcal{O}M}$ on $\Lambda^\tau(\mathcal{O}^{(1)}_{\mathbb{Z}[i, \frac{1}{2}]}/\mathcal{O}\mathcal{Z}[i, \frac{1}{2}])$ is trivial.

The module $\mathcal{D}_{\mu}$ contains the submodule $\mathcal{D}_{\mu}(\mu + 2\rho, \xi_0)$ with multiplicity one, hence

$$\mathcal{D}_{\mu}(\mu + 2\rho, \xi_0) = \mathbb{Z}[i, \frac{1}{2}] \Phi_{\mu, \epsilon} \subset \mathcal{D}_{\mu}$$

The center $C_{\mathcal{O}M}$ acts on $\mathcal{D}_{\mu}(\mu + 2\rho, \xi_0)$ by the character

$$-\zeta(\mu) = (2\gamma_2 + 2\gamma_4 + \cdots + 2\gamma_{m-1} + \frac{3}{2} \gamma_{m+1}) - d\delta$$

Finally investigate the structure of $\text{Hom}(\Lambda^\bullet(\mathcal{U}_{\mathbb{Z}[i, \frac{1}{2}]})/\mathcal{M}_{\lambda, \mathbb{Z}[i, \frac{1}{2}]})$. The conjugation by the matrices $c_{2, i}$ or better conjugation by the product $\tilde{c} = \prod_{i: \text{odd}} c_{2, i}$ provides an identification

$$\tilde{c} : X^*(K^M \otimes \mathbb{Z}[i, \frac{1}{2}]) \rightarrow X^*(T).$$

Note that $\tilde{c}(\epsilon_i) = \gamma_i^{M(1)}$ and for even indices $i$ we have $\tilde{c}(\gamma_i) = \gamma_i$.

This suggests that we consider $\text{Hom}(\Lambda^\bullet(\mathcal{U}_{\mathbb{Z}[i, \frac{1}{2}]})/\mathcal{M}_{\lambda, \mathbb{Z}[i, \frac{1}{2}]})$ as a module for $\mathcal{O}M$ and we even restrict our attention to the action of the center $C_{\mathcal{O}M}$. We have the following proposition which must be already in $[8]$. 

25
Proposition 4.2. The character $\zeta(\mu)$ occurs only in degree $l(w_{un})$ and the eigenspace

$$H^l(w_{un})(u_Z, M_{\lambda,Z}) = \text{Hom}(\Lambda^l(w_{un})(u_Z), M_{\lambda,Z})(\zeta(\mu))$$

is irreducible with highest weight $w_{un}(\lambda + \rho) - \rho$. The homomorphism

$$\text{Hom}(\Lambda^l(w_{un})(u_Z), M_{\lambda,Z})(\zeta(\mu)) \to H^l(w_{un})(u_Z, M_{\lambda,Z})$$

is an isomorphism.

Proof. The Lie algebra $u_Z$ has the basis $e_\beta$, where $\beta \in \Delta^+ \setminus \{\alpha_1, \alpha_3, \ldots, \alpha_m\}$. Let us denote by $e_\beta^\vee$ the dual basis. For any subset $J = \{\beta_1, \beta_2, \ldots, \beta_s\} \subset \Delta^+_o$ we define $e_J^\vee = e_{\beta_1}^\vee \wedge e_{\beta_2}^\vee \wedge \ldots e_{\beta_s}^\vee$. The element $e_J^\vee$ is an eigenvector for the standard maximal torus $T \subset o^M$, the eigenvalue is the character $\chi_J = -\sum \beta_i$. For any Kostant representative $w \in W^{o^p}$ we define the set $\Delta^+(w) = \{\alpha | w^{-1} \alpha < 0\}$. Then we know that the restriction of $w_{un}(\lambda + \rho) - \rho = J_{\Delta^+(w)}$ to $C^s M$ is $\zeta(\mu)$. The weight $J_{\Delta^+(w)}$ is the highest weight of an irreducible $o^M$ submodule $N$ in $\text{Hom}(\Lambda^l(w_{un})(u_Z), M_{\lambda,Z})(\zeta(\mu))$ and the weight subspaces in $N$ are of multiplicity one and of the form $\chi_J$. Now a simple computation shows that a subset $J_1 \subset \Delta^+_o$ for which the restriction of $\chi_{J_1}$ to $C^s M$ is equal to $\zeta(\mu)$ must be on of the $\chi_J$ occurring in $N$ and hence it follows that $\text{Hom}(\Lambda^l(w_{un})(u_Z), M_{\lambda,Z})(\zeta(\mu))$ is irreducible. \[\square\]

This implies that

$$\text{Hom}_{K^o M}(\Lambda^*(o^M m_{\frac{1}{2}}^Z / o^M Z_{\frac{1}{2}}^Z), D^w_{\mu} \otimes H^l(w_{un})(u_Z, M_{\lambda,Z})) =$$

$$\text{Hom}_{K^o M}(\Lambda^*(o^M m_{\frac{1}{2}}^Z / o^M Z_{\frac{1}{2}}^Z), D^w_{\mu} \otimes \text{Hom}(\Lambda^*(u_Z / Z^M_{\frac{1}{2}}), M_{\lambda,Z})) \quad (93)$$

If we choose a generator $x_\lambda$ of the highest weight module $M_{\lambda,Z}(\lambda)$ then

$$\xi(w_{un} \cdot \lambda) = e_{\Delta^+(w_{un})}^\vee \otimes w_{un} \lambda \in H^l(w_{un})(u_Z, M_{\lambda,Z}) \quad (94)$$

is a generator of the highest weight module $H^l(w_{un})(u_Z, M_{\lambda,Z}))(w_{un} \cdot \lambda)$ it is actually unique up to a sign. We can modify our Borel subgroup $B \subset o^M$ by flipping into the opposite in some of the factors. Then the highest weight with respect to such a Borel subgroup will be

$$\lambda(w_{un}, \epsilon) = \sum_{i \text{ odd}} \epsilon_i b_i \gamma_i o^M \quad (95)$$

where of course again $\epsilon_i = \pm 1$ and the indices $i$ with $\epsilon_i = -1$ tell us where we flipped the Borel subgroup. To such a weight we have a generating weight vector $\xi(w_{un} \cdot \lambda, \epsilon)$. Let us call these weight vectors $\xi(w_{un} \cdot \lambda, \epsilon)$ the extremal weight vectors.
We replace the split torus $T$ by $\tilde{K}^\circ M$, these two tori have $T_{\text{split}} = C^\circ M$ in common. Then we see that that we have extremal weight spaces
\[ \mathbb{H}(w_{un})(u_{\mathbb{Z}[i, \frac{1}{2}]}, \mathcal{M}_{\lambda, \mathbb{Z}[i, \frac{1}{2}]})(w_{un} \cdot \lambda)(\tilde{c}^{-1}(\lambda(w_{un}, \xi))) = \mathbb{Z}[i, \frac{1}{2}]\tilde{c}^{-1}(\xi(w_{un} \cdot \lambda, \xi)) \] (96)
and on this weight space the torus $\tilde{K}^\circ M$ acts by the character $\tilde{c}^{-1}(\lambda(w_{un}, -\xi)) = -b_1 e_1 - b_3 e_3 - \ldots - b_m e_m + \zeta(\mu)$.

$D^\circ_\mu$ contains the rank one submodule $D^\circ_\mu(\mu_\epsilon(\lambda, \epsilon)) = \mathbb{Z}[i, \frac{1}{2}]\Phi_{\mu, \xi_0}$ and $\tilde{K}^\circ M$ acts by the character $\mu_\epsilon(\lambda, \epsilon) = \sum_{i: \text{odd}, j < m} (b_1 + 2)e_i + \epsilon(b_m + 2)e_m - \zeta(\mu)$
and this is a minimal $\tilde{K}^\circ M$ type (see [80]). The sum of these two characters is $\nu(\xi_0) = 2e_1 + 2e_2 + \ldots + 2e_m$ and hence we see
\[ \text{Hom}_{\tilde{K}^\circ M}(\Lambda^\bullet (\mathcal{M}_{\lambda, \mathbb{Z}[i, \frac{1}{2}]}, D^\circ_\mu \otimes \text{Hom}(\Lambda^\bullet (u_{\mathbb{Z}[i, \frac{1}{2}]}, \mathcal{M}_{\lambda})) = \text{Hom}_{\tilde{K}^\circ M}(\Lambda^\bullet (\mathcal{M}_{\lambda, \mathbb{Z}[i, \frac{1}{2}]}, D^\circ_\mu(\mu_\epsilon(\lambda, \epsilon)) \otimes \text{Hom}(\Lambda^\bullet (u_{\mathbb{Z}[i, \frac{1}{2}]}, \mathcal{M}_{\lambda}))(\tilde{c}^{-1}(\lambda(w_{un}, -\xi_0)) \] (97)
Each of the modules in the argument on the right hand side is of rank one and we have chosen a generator for each of them. Hence we see
\[ \text{Hom}_{\tilde{K}^\circ M}(\Lambda^\bullet (\mathcal{M}_{\lambda, \mathbb{Z}[i, \frac{1}{2}]}, D^\circ_\mu \otimes \text{Hom}(\Lambda^\bullet (u_{\mathbb{Z}[i, \frac{1}{2}]}, \mathcal{M}_{\lambda})) = \mathbb{Z}[i, \frac{1}{2}]\Omega(\lambda, \epsilon) \] (98)
where $\Omega(\lambda, \epsilon)$ is the tensor product of the generators and $\epsilon = \pm 1$. The generator sits in degree $b_n = \circ r + l(w_{un})$.

If $n$ is odd then the choice of $\xi$ is irrelevant, if $n$ is even we get two irreducible $(g_\mathbb{Z}, \mathcal{K}^{(1)})$ modules. As we did in the case $G = \text{Gl}_2$ we can enlarge the connected group scheme $\mathcal{K}^{(1)}$ to the larger group scheme $\mathcal{K} = \mathcal{K}^{(1)} \rtimes \{\eta\}$ where $\eta$ is the diagonal matrix which has 1 on the diagonal up the $(n-1)$-th entry and $-1$ the $n$-th entry. Then $\mathbb{D}_\lambda = \mathbb{D}_\lambda^{(n-1)} \oplus \mathbb{D}_\lambda^{(1)}$ is an irreducible $(g_\mathbb{Z}, \mathcal{K})$ module over $\mathbb{Z}[i, \frac{1}{2}]$, the element $\eta$ yields an isomorphism between the two summands.

Since our weight $\lambda$ is essentially self dual, i.e. $a_i = a_{n-i}$ we have the constraint $a_{\frac{n}{2}} \equiv 2d \mod 2$. Then it is clear that
\[ \eta\Omega(\lambda, \epsilon) = (-1)^{\frac{a_{\frac{n}{2}} - 2d}{2}}\Omega(\lambda, -\epsilon) \] (99)
We form again the elements
\[ \omega^{(1)}_\lambda = \Omega(\lambda, +1) + \Omega(\lambda, -1), \omega^{(2)}_\lambda = \Omega(\lambda, +1) - \Omega(\lambda, -1) \] (100)
and then these two elements are the generators for the \( \pm \) eigenspaces under the action of \( \eta \). We have the decomposition
\[ H^*(gZ, K^{(1)}, D_e \otimes M_{Z, i[\frac{1}{2}]}) = H^*_{i=1}(gZ, K^{(1)}, D_e \otimes M_{Z, i[\frac{1}{2}]}) \oplus H^*_{i=2}(gZ, K^{(1)}, D_e \otimes M_{Z, i[\frac{1}{2}]}) \]
and
\[ H^*(gZ, K^{(1)}, D_e \otimes M_{Z, i[\frac{1}{2}]}) \overset{\sim}{\longrightarrow} H^*_{i=1}(gZ, K^{(1)}, D_e \otimes M_{Z, i[\frac{1}{2}]}) \).
In degree \( \bullet = b_n \) the cohomology is the free \( Z[\frac{1}{2}] \)-module generated by a class \( \omega^{(e)}_\lambda \) where \( e = 1, 2 \).

4.5 The arithmetic of the intertwining operator

In this section we apply the above considerations to study the arithmetic properties of an intertwining operator between two induced modules.

We start from the group scheme \( \text{Gl}_N/Z \), let \( B \) be the standard Borel subgroup and consider the standard parabolic subgroups \( P \supset B \) (resp. \( P' \supset B \)) with reductive quotient \( M = \text{Gl}_n \times \text{Gl}_{n'} = M_1 \times M_2 \) (resp. \( M' = \text{Gl}_{n'} \times \text{Gl}_{n} \)). Let \( U_P \) resp. \( U_{P'} \) be the unipotent radicals. Let \( \pi = \{ \alpha_1, \ldots, \alpha_{N-1} \} \subset X^*(T) \) be the set of positive (with respect to \( B \)) simple roots. We identify the set of simple roots with the set of indices \( \{1, 2, \ldots, N-1\} \). Let us denote by \( w_N^- \) the permutation which reverses the order by \( i \leftrightarrow i' \), i.e. \( i' = N - i \). Then the positive simple roots for \( M \) are \( \pi_M = \{ \alpha_1, \ldots, \alpha_{n-1} \} \cup \{ \alpha_{n+1}, \ldots, \alpha_{N-1} \} \), and accordingly we denote the system of simple roots of \( M' \) by \( \pi_M' = w_N^-(\pi_M) = \{ \alpha_1, \ldots, \alpha_{n'-1} \} \cup \{ \alpha_{n'+1}, \ldots, \alpha_{N-1} \} \). Let \( \gamma_n \) resp. \( \gamma_{n'} \) be the fundamental weight attached to the missing root \( \alpha_n \) resp. \( \alpha_{n'} \). If \( \Delta^+_U \) resp. \( \Delta^+_U' \) are the positive roots occurring in these radicals, let \( \rho_U, \rho_{U'} \) be the half sums over these roots. Then
\[ \rho_U = \frac{N}{2} \gamma_n, \rho_{U'} = \frac{N}{2} \gamma_{n'} \] (101)
We choose a highest weight \( \lambda \) for \( \text{Gl}_N \), let \( M_\lambda \) be the resulting \( \text{Gl}_N \)-module.

We pick a Kostant representative \( w \in WP \), and we write
\[ \tilde{\mu} = w(\lambda + \rho) - \rho = \sum_{i=1}^{n-1} a_i' \gamma_i^M + \sum_{i=n+1}^{N-1} a_i' \gamma_i^M + a(w, \lambda) \gamma_n + d \det_N \]
\[ = \sum_{i=1}^{n-1} a_i' \gamma_i^M + d(w, \lambda) \det_n + \sum_{i=n+1}^{N-1} a_i' \gamma_i^M + d'(w, \lambda) \det_{n'} = \mu_1 + \mu_2 \] (102)
Here $\mu_1, (\text{resp.} \mu_2)$ are highest weights on $M_1(\text{resp. } M_2)$. In some situations it is more convenient to look at

$$\tilde{\mu} + \rho = w(\lambda + \rho) = \sum_{i=1}^{n-1} b_i \gamma_i + \sum_{i=n+1}^{N-1} b_i \gamma_i + b(w, \lambda) \gamma_n + d\det N$$  \hspace{1cm} (103)$$

Then we have the relations $b_i = a'_i + 1$ and $b(w, \lambda) = a(w, \lambda) + \frac{N}{2}$. We define the weight of $\tilde{\mu}$:

$$w(\tilde{\mu}) = w(\mu_1) + w(\mu_2) = \sum_{i=1}^{n-1} b_i + \sum_{i=n+1}^{N-1} b_i$$  \hspace{1cm} (104)$$

We make assumptions on $w$:

a) The length $l(w) = 1/2 \dim U_P$ - this means that $w$ is balanced.

b) Both weights $\mu_1, \mu_2$ are essentially self dual.

c) The weight $\tilde{\mu}$ is in the negative chamber, this means that $a(w, \lambda) \leq -\frac{N}{2}$ or $b(w, \lambda) \leq 0$.

We have the two longest Kostant representatives $w_P \in W^P (\text{resp. } w_Q \in W^Q)$ which send all the roots in $\Delta^+_U$ (resp. $\Delta^+_U$) into negative roots. If $s_i \in W$ is the reflection attached to the simple root $\alpha_i$ then we can write any element $w \in W^P$ as a product of reflections $w = s_n s_i s_j ... s_k$. We can always complete this product of reflections to get the longest element (it always starts with $s_n$ and stops with $s_n'$)

$$s_n s_i s_j ... s_k s_{k'} ... s_{\nu} s_{\mu'} = w_P$$  \hspace{1cm} (105)$$

Then $w' = s_{\nu'} ... s_{k'} \in W^Q$ and we get a one to one correspondence between $W^P$ and $W^Q$ which is defined by

$$w = w_P w' \text{ or } w' = w_Q w$$  \hspace{1cm} (106)$$

(See 5.3.7) We have $l(w) + l(w') = \dim U_P$, since $w$ is balanced we see that $w'$ is also balanced. For $w = e$ the identity element we get $w' = w_Q e$.

A presentation of $w_P$ as in $\text{(105)}$ yields a sequence of roots in $\Delta^+_U$: The first element in this sequence is $\beta_1 = \alpha_n$. Then we find a root $\beta_2 \in \Delta^+_U$ such that $s_n \beta_2 = \alpha_i$ is a simple root. Then $s_i s_n$ sends the roots $\beta_1$ and $\beta_2$ into the set of negative roots and we find a root $\beta_3$ such that $s_n s_i \beta_3$ is a simple root $\alpha_{\nu}$ and $s_{\nu}$ is the next factor in $w_P = s_n s_i s_{\nu} ...$. To say this in different words: We get an ordering $\{\beta_1, \beta_2, ..., \beta_{d_U}\} = \Delta^+_U$ such that $x_k^{-1}$ conjugates exactly the first $k$ roots $\{\beta_1, ..., \beta_k\}$ into negative roots where

$$x_k = s_n s_i ... s_{\mu} \text{ is the product of the first } k \text{ factors in } \text{(105)}. \hspace{1cm} (107)$$
We can write
\[ wp = x_1 y_d u_{-1} = s_n(s_i \cdots s_n') = x_2 y_2 = (s_n s_i)(s_j \cdots s_n') = x_k y_k \] (108)
and the \( x_k, y_k \) are corresponding elements. We also define a function \( r_w : \{1, 2, \ldots, d_U\} \to \{1, 2, \ldots, N - 1\} = \pi = \{\alpha_1, \ldots, \alpha_{N-1}\} \) by the rule
\[ x_k = x_{k-1} s_{r_w(k)} \text{ or } x_{k-1}^{-1} a_{r_w(k)} = \beta_k \] (109)

For our element \( w \) above the corresponding element \( w' \) has also length \( \frac{1}{2} \dim_{U_v} \) and we have the corresponding formula
\[
\tilde{\mu}' = w'(\lambda + \rho) = \sum_{i=1}^{n-1} b_i \gamma_i^M + \sum_{i=n+1}^{N-1} b_i \beta_i^M + b'(w', \lambda) \gamma_n' + d(w', \lambda) \det N
\]
\[
= \sum_{i=n+1}^{N-1} b_i \gamma_i^M + d(w', \lambda) \det_n + \sum_{i=n+1}^{N-1} b_i \beta_i^M + d(w', \lambda) \det_n = \mu'_1 + \mu'_2.
\] (110)

The formula tells us that the semi simple components \( \mu'^{(1)}_1 = \mu^{(1)}_2, \mu'^{(1)}_2 = \mu^{(1)}_1 \). Here we use our assumption that \( \mu_i \) are essentially self dual. The coefficients of \( \gamma_n, \gamma_n' \) are related by
\[ d(w, \lambda) + d(w', \lambda) = -N \] (111)

The weights \( \mu_1, \mu_2 \) yield Harish-Chandra modules \( \mathbb{D}_{\mu_1} \) (for \( \text{Gl}_n \)) and \( \mathbb{D}_{\mu_2} \) for \( \text{Gl}_{n'} \). (See section [4.3]) and hence \( \tilde{\mu} = \mu_1 + \mu_2 \) provides a Harish-Chandra module \( \mathbb{D}_{\tilde{\mu}} \) for \( M \). By the same argument we get a Harish-Chandra module \( \mathbb{D}_{\tilde{\mu}'} \) for \( M' \). These two \((\mathfrak{m}, \mathcal{K}^M)\) (resp. \((\mathfrak{m}', \mathcal{K}^M')\)) modules both have a minimal \( \mathcal{K}^M \) type \( \mu_c(\epsilon, \tilde{\mu}) = \mu_c(\epsilon_1, \mu_1) + \mu_c(\epsilon_2, \mu_2) \) (resp. \( \mathcal{K}^M' \) type \( \mu_c(\epsilon, \tilde{\mu}') \)).

We consider the \((g_z, \mathcal{K}^{(1)})\) modules \( \mathfrak{Ind}_{\mathfrak{D}_{\tilde{\mu}}}^G \mathbb{D}_{\tilde{\mu}} \) and \( \mathfrak{Ind}_{Q}^G \mathbb{D}_{\tilde{\mu}'} \). Both these modules contain the irreducible module \( \mathfrak{Ind}_{\mu_c(\epsilon, \tilde{\mu})}^G \mathbb{D}_c \) as minimal \( \mathcal{K}^{(1)} \) type and this \( \mathcal{K}^{(1)} \)-type has multiplicity one.

We take the base extensions to \( \mathbb{C} \) and twist them by a holomorphic variable. We introduce the abbreviating notation
\[ \mathfrak{Ind}_{\mathfrak{D}_{\tilde{\mu}}}^G \mathbb{D}_c \otimes \mathbb{C}(|\gamma_n|^z) = \mathfrak{Ind}_{\mathfrak{D}_{\tilde{\mu}}}^G \mathbb{D}_c \otimes (z) \]
Then we can write down the usual intertwining operator
\[ T^{w, \gamma_n}(z) : \mathfrak{Ind}_{\mathfrak{D}_{\tilde{\mu}}}^G \mathbb{D}_c \otimes \mathbb{C}(|\gamma_n|^z) \to \mathfrak{Ind}_{\mathfrak{D}_{\tilde{\mu}'}}^G \mathbb{D}_c \otimes \mathbb{C}(|\gamma_n'|^{-z}) \] (112)
which is given by the integral
\[ \{g \mapsto f(g)\} \to \{g \mapsto \int_{U_v(\mathbb{R})} f(w p u g) d_{\infty} u\} \]
where \( d_{\infty} u \) is the Haar measure obtained from the epinglage. This integral converges if \( \Re(z) >> 0 \). The action of \( \mathcal{K}^{(1)}(\mathbb{R}) \) on these modules is independent of \( z \) especially it is clear that the above lowest \( \mathcal{K}^{(1)}(\mathbb{R}) \)-type occurs in the deformed modules with multiplicity one.
Theorem 4.1. i) This operator extends to a meromorphic operator in the entire \( s \)-plane and it is holomorphic at \( z = 0 \).

ii) At \( z = 0 \) it is an isomorphism and modified by the factor \( \frac{1}{\pi^{d_U/2}} \) it is defined over \( \mathbb{Q} \), i.e. we get an isomorphism

\[
\frac{1}{\pi^{d_U/2}} T^{w_p,\text{st}}(0) : \mathfrak{f}^G_B D_{\bar{\mu}} \otimes \mathbb{Q} \xrightarrow{\sim} \mathfrak{f}^G_B D_{\mu'} \otimes \mathbb{Q} \tag{113}
\]

Proof. The first assertion is over \( \mathbb{C} \) and follows from results of Speh (See [11]).

For the second we use the standard strategy and write the operator as a product of operators induced from intertwining operators on \( \text{Gl}_2 \). Here we have to deal with the problem that some of the operators will not be defined because we encounter poles in the backwards operators.

We apply the consideration from section (4.2) to \( M \) and \( M' \), i.e. to the two factors \( \text{Gl}_n \) and \( \text{Gl}_{n'} \). Especially we introduce the subgroup scheme \( \circ M \subset M \) as the product of the two corresponding groups in the two factors. This also yields the element \( w^M_{un} \) in the Weyl group of \( M \). The module \( \mathfrak{f}^G_B w^M_{un} \cdot (\tilde{\mu} + 2\rho^M) \) is induced from a module \( \mathfrak{f}^G_B w^M_{un} \cdot (\tilde{\mu} + 2\rho^M) \), Hence we get \( D_{\mu} \subset \mathfrak{f}^G_B w^M_{un} \cdot (\tilde{\mu} + 2\rho^M) \) and finally

\[
\mathfrak{f}^G_B D_{\bar{\mu}} \hookrightarrow \mathfrak{f}^G_B w^M_{un} \cdot (\tilde{\mu} + 2\rho^M). \tag{114}
\]

Now we can try to extend the intertwining operator \( T^{w_p,\text{st}}(0) \) to this larger module. This may not be possible. Therefore we apply the usual technique and deform the induced module by a character \( \gamma^z \) where \( z \in \mathbb{C}^{N-1} \). We consider the extension

\[
T^{w_p,\text{st}}(z) : \mathfrak{f}^G_B w^M_{un} \cdot (\tilde{\mu} + 2\rho^M) \otimes |\gamma|^z \rightarrow \mathfrak{f}^G_B w^M_{un} \cdot (\tilde{\mu} + 2\rho^M') \otimes |\gamma|^z' \tag{115}
\]

which is given by the following integral: We write \( U_P \) as product of one parameter subgroups (note that \( \alpha_{n'} = \beta_{d_U}, \alpha_n = \beta_1 \))

\[
U_P = U_{\alpha_{n'}} \times U_{\beta_{d_U-1}} \times \cdots \times U_{\alpha_n} \tag{116}
\]

\[
T^{w_p,\text{st}}(z)(f)(g) = \int_{U(\mathbb{R})} f(w_p u g) du = \int \cdots \int f(w_p u_{\alpha_{n'}} \cdots u_{\alpha_n} g) du_{\alpha_{n'}} \cdots du_{\alpha_n} \tag{117}
\]

where the measure is the Lebesgue measure which is normalized by the epinglage. This integral converges for \( \Re(z_i) >> 0 \) and has a meromorphic extension into the entire complex plane.
We have by definition
\[
w_P u_n u_1 = s_n u_{\alpha_n} y_d u_{t-1} u_1,
\]
where \( u_1 \in U_1(\mathbb{R}) = U_{\beta d_{t-1}} \times \cdots U_{\alpha_t} \). Hence our integral becomes
\[
T^{w_P, st}(g)(f) = \int_{U_1(\mathbb{R})} \int_{U_{\alpha_n}(\mathbb{R})} f(s_n u_{\alpha_n} y_d u_{t-1} u_1) du_{\alpha_n} du_1
\]
(118)

The inner integral is an intertwining operator. We write our induced modules now as induced from characters \( \chi \) on the \( \mathbb{R} \) valued points of the torus \( T(\mathbb{R}) \), let \( \chi = w_m \cdot (\mu + 2\rho_M)_{\mathbb{R}} \) then
\[
\{ g \mapsto f(g) \} \mapsto \{ g \mapsto \int_{U_{\alpha_n}(\mathbb{R})} f(s_n u_{\alpha_n} g) du_{\alpha} \}
\]
(120)
is an intertwining operator
\[
T^{\alpha_n}(s_n, \chi, \tilde{z}) : I^G_B \chi \otimes |\gamma|^{\tilde{z}} \rightarrow I^G_B (s_n \cdot (\chi \otimes |\gamma|^{\tilde{z}})).
\]
(121)

where of course \( s_n \cdot (\chi \otimes |\gamma|^{\tilde{z}}) = s_n(\chi \otimes |\gamma|^{\tilde{z}}) + s_n(|\rho|) - |\rho| \).

This intertwining operator is now induced from an intertwining operator between two \( SL_2 \) modules. Let \( \bar{H}_{\alpha_n} \) be the reductive subgroup \( H_{\alpha_n} \cdot T \), the group scheme \( \bar{H}_{\alpha_n} \) is then the Levi quotient of a parabolic subgroup \( P_{\alpha_n} \).

Let \( B_{\alpha_n} \) be the Borel subgroup in \( H_{\alpha_n} \). Then the integral in (120) also defines an intertwining operator
\[
T_{\alpha_n}^{\alpha_n}(s_n, \chi, \tilde{z}) : I^{\bar{H}_{\alpha_n}}_{B_{\alpha_n}} \chi \otimes |\gamma|^{\tilde{z}} \rightarrow I^{\bar{H}_{\alpha_n}}_{B_{\alpha_n}} (s_n \cdot (\chi \otimes |\gamma|^{\tilde{z}})).
\]
(122)

Our two induced modules can by written as two step induction
\[
I^G_B \chi \otimes |\gamma|^{\tilde{z}} = I^G_{\bar{P}_{\alpha_n}} I^{\bar{H}_{\alpha_n}}_{B_{\alpha_n}} (\chi \otimes |\gamma|^{\tilde{z}}), I^G_B s_n \cdot (\chi \otimes |\gamma|^{\tilde{z}}) = I^G_{\bar{P}_{\alpha_n}} I^{\bar{H}_{\alpha_n}}_{B_{\alpha_n}} (s_n \cdot (\chi \otimes |\gamma|^{\tilde{z}}))
\]
(123)

and then our intertwining operator is induced
\[
T^{\alpha_n}(s_n, \chi, \tilde{z}) = I^G_{\bar{P}_{\alpha_n}} T_{\alpha_n}^{\alpha_n}(s_n, \chi, \tilde{z}).
\]
(124)

Hence we can write equation (119)
\[
T^{w_P, st}(g)(f) = \int_{U_1(\mathbb{R})} I^G_{\bar{P}_{\alpha_n}} (T_{\alpha_n}^{\alpha_n}(s_n, \chi, \tilde{z})(f))(y_d u_{t-1} u_1) du_1
\]
(125)

We iterate this process. We have \( y_d u_{t-1} = s_i s_j \cdots s_{n_t} \), we apply the above process again and eventually we get
\[
T^{w_P, st}(g)(f) = I^G_{\bar{P}_{\alpha_n}} T_{\beta d_{t-1}}^{\beta d_{t-1}} (s_{n_t} x_{d_{t-1}}^{-1} \cdot \chi, x_{d_{t-1}}^{-1} \tilde{z}) \circ \cdots \circ I^G_{\bar{P}_{\alpha_n}} T_{\alpha_n}^{\alpha_n}(s_n, \chi, \tilde{z})(f)
\]
(126)
We understand the $\text{Gl}_2$ intertwining operators

$$T_{tw,(k)}^{st}(s_{tw,(k)}, x_{k-1}^{-1} \mathbf{1}_{\mathbb{Z}}) : I_{B_{tw,(k)}}(x_{k-1}^{-1} \cdot \chi \otimes \gamma | \tilde{z}) \rightarrow I_{B_{tw,(k)}}(x_{k-1}^{-1} \cdot \chi \otimes \gamma | \tilde{z})$$

(127)

In section 3.3, we defined the algebraic intertwining operator $T_{\lambda}^{alg}$ by fixing their value on the lowest $\mathcal{K}$ type. If we put $\chi = \lambda_{\mathbb{R}}$ then we can extend these operators to the twisted modules

$$T_{\lambda}^{alg}(s_{\alpha}, \chi, z) : I_{B_{\alpha}}^{\text{Gl}_2} \chi \otimes |\gamma| z \rightarrow I_{B_{\alpha}}^{\text{Gl}_2} s_{\alpha} \cdot (\chi \otimes |\gamma| z)$$

With respect to the basis $\Phi_{\nu}$, the operator $T_{\lambda}^{alg}(s_{\alpha}, \chi, z)$ acts as a diagonal matrix with entries in the rational function field $\mathbb{Q}(\tilde{z})$ and we know the factor that compares $T_{\lambda}^{alg}(s_{\alpha}, \chi, z)$ to $T_{\lambda}^{st}(s_{\alpha}, \chi, z)$, it is a ratio of $\Gamma$ values. To compute $T_{tw,(k)}(s_{tw,(k)}, x_{k-1}^{-1} \cdot \chi)$ we need to know the restriction of $x_{k-1}^{-1} \cdot \chi$ to the torus $T_{tw,(k)}$, we recall that the coroot $\alpha_{tw,(k)}^{\vee} : \mathbb{G}_m \rightarrow T_{tw,(k)}$ provides an identification. Hence the restriction of $x_{k-1}^{-1} \cdot \chi$ to $T_{tw,(k)}$ is a character on $\mathbb{G}_m(\mathbb{R}) = \mathbb{R}^\times$ and an easy computation shows that this character is

$$t \mapsto t^{<\alpha_{tw,(k)}^{\vee}, x_{k-1}^{-1} \chi>} |t|^{h(\beta_k)+<\alpha_{tw,(k)}^{\vee}, \tilde{z}>}$$

We still can manipulate the exponent. We have $x_{k-1} \alpha_{tw,(k)}^{\vee} = \beta_k^{\vee}$. Then the first exponent becomes $<\beta_k^{\vee}, \chi>$ and for the second one we get $<\beta^{\vee} - \alpha_{tw,(k)}^{\vee}, \rho> = h(\beta_k)$, where for $\beta = \alpha_{\nu} + \cdots + \alpha_{\nu+h}$ we put $h(\beta) = h$.

Hence our character is

$$t \mapsto t^{<\beta_k^{\vee}, \chi>} |t|^{h(\beta_k)+<\alpha_{tw,(k)}^{\vee}, \tilde{z}>}$$

(128)

Then we put $\epsilon_{w}(k, \chi) = 0$ if $<\beta_k^{\vee}, \chi> \equiv 0 \mod 2$ and $\epsilon_{w}(k, \chi) = 1$ else.

Then we get from our formulae in section 3.3

$$T_{tw,(k)}^{st}(s_{tw,(k)}, x_{k-1}^{-1} \cdot \chi, x_{k-1} \tilde{z}) = \frac{\Gamma(\frac{<\beta_k^{\vee}, \chi> + \epsilon_{w}(k, \chi) + h(\beta_k) - 1 + <\alpha_{tw,(k)}^{\vee}, \tilde{z}>}{2})}{\Gamma(\frac{<\beta_k^{\vee}, \chi> + h(\beta_k) + <\alpha_{tw,(k)}^{\vee}, \tilde{z}>}{2})} \Gamma(\frac{1}{2}) M_{k}(\tilde{z})$$

(129)

where $M_{k}(\tilde{z})$ is a diagonal matrix with entries in the field of rational functions $\mathbb{Q}(\tilde{z})$, it may have a pole at the hyperplane $<\alpha_{tw,(k)}^{\vee}, \tilde{z}> = 0$ but the ratio

$$M_{k}^{*}(\tilde{z}) = \frac{M_{k}(\tilde{z})}{\Gamma(\frac{<\beta_k^{\vee}, \chi> + h(\beta_k) + <\alpha_{tw,(k)}^{\vee}, \tilde{z}>}{2})}$$

is holomorphic on this hyperplane and hence can be evaluated at $\tilde{z} = 0$.

33
By definition the number $\langle \beta_k^\vee, \chi \rangle + \epsilon_w(k, \chi)$ is even and hence this operator is holomorphic at $z = 0$ if $h(\beta_k)$ is even. In this case the character restricted to the torus $T_w(k)$ is cohomological. We find

$$T_{r_w(k)}^{\text{st}}(s_{r_w(k)}, x_{k-1}^{-1} \cdot \chi, x_{k-1}^{-1} z) \big|_{z=0} = \Gamma \left( \frac{\langle \beta_k^\vee, \chi \rangle + \epsilon_w(k, \chi) + h(\beta_k) - 1}{2} \right) \Gamma \left( \frac{1}{2} \right) M_k^*(0)$$

(130)

where $M_k^*(0)$ is a matrix with rational entries and the factor in front is $\pi \times$ a rational number. This tells us that

$$T_{r_w(k)}^{\text{st}}(s_{r_w(k)}, x_{k-1}^{-1} \cdot \chi) \big|_{z=0} = \pi \times T_{r_w(k)}^{\text{alg}}(s_{r_w(k)}, x_{k-1}^{-1} \cdot \chi) q_k,$$  

(131)

where $q_k \in \bar{\mathbb{Q}}^\times$.

If $h(\beta_k)$ is odd then the hyperplane $\langle \alpha_{r_w(k)}^\vee, z \rangle = 0$ may be a first order pole, this happens exactly when

$$\langle \beta_k^\vee, \chi_{\text{alg}} \rangle + \epsilon_w(k, \chi) + h(\beta_k) - 1 = 0, -2, -4, \ldots$$

We put $m_k = 1$ if $h(\beta_k)$ is odd and we encounter a pole and $m_k = 0$ else. Then we manipulate the right hand side in equation (129) and change it to

$$\langle \alpha_{r_w(k)}^\vee, z \rangle^{m_k} \Gamma \left( \frac{\langle \beta_k^\vee, \chi_{\text{alg}} \rangle + \epsilon_w(k, \chi) + h(\beta_k) - 1 + \langle \alpha_{r_w(k)}^\vee, z \rangle}{2} \right) \Gamma \left( \frac{1}{2} \right) \frac{M_k(z)}{\langle \alpha_{r_w(k)}^\vee, z \rangle^{m_k}}$$

(132)

the last factor to the right is still a diagonal matrix with entries in the field $\mathbb{Q}(z)$. The expression in values of the Gamma-function can be evaluated at $z = 0$ and the result is a rational number, the two contributions of $\sqrt{\pi}$ cancel.

We return to our factorization of the intertwining operator $T_{w,r}^{\text{st}}(z)$. It is an intertwining operator between two Harish-Chandra modules with a $\mathbb{Q}$-structure. They have a decomposition into $K^{(1)}$ types (which are of course $\mathbb{Q}$ vector spaces $\otimes \mathbb{C}$). We consider the restriction to a $K^{(1)}$ type $\vartheta$ which is of course finite dimensional. Then our product decomposition yields

$$\left( \prod_k \langle \alpha_{r_w(k)}^\vee, z \rangle^{m_k} \right) \frac{\Gamma \left( \frac{\langle \beta_k^\vee, \chi \rangle + \epsilon_w(k, \chi) + h(\beta_k) - 1 + \langle \alpha_{r_w(k)}^\vee, z \rangle}{2} \right)}{\Gamma \left( \frac{\langle \beta_k^\vee, \chi \rangle + \epsilon_w(k, \chi) + h(\beta_k) + \langle \alpha_{r_w(k)}^\vee, z \rangle}{2} \right)} \Gamma \left( \frac{1}{2} \right) M(\vartheta, z)$$

(133)

where

$$M(\vartheta, z) \in \text{Hom}_{K^{(1)}}(\mathfrak{h} \otimes \mathbb{C}, (\mu + 2\rho_M) \otimes \mathbb{C}, \mathfrak{h} \otimes \mathbb{C}, (\mu' + 2\rho_{M'}) \otimes \mathbb{C}) \otimes \mathbb{Q}(z)$$

34
The factor in front can be evaluated at $z = 0$. Each factor contributes by a non zero rational number or $\pi$ times a non zero rational number. We get a factor $\pi$ in the cases where $h(\beta_k)$ is even and this happens $d_U/2$ number of times. So we would be finished with the proof if we evaluate $M(\vartheta, z)$ at $z = 0$ and observe that this is a matrix with entries rational numbers. But we do not know whether $M(\vartheta, z)$ can be evaluated at zero, we have moved the poles in the Gamma-factors into $M(\vartheta, z)$, the extended intertwining operator in equation (115) may not be regular at $z = 0$.

We are only interested in the restriction of the operator to $\text{Ind}_{\mu}^G \mathbb{D}_{\mu} \otimes \mathbb{C}[\gamma_n]$ [112], i.e. we restrict it to the line $z|_{\gamma_n} \subset \mathbb{C}^{N-1}$. We notice that this line is not contained in any of the hyperplanes $<\alpha_{\gamma_n}>(k) z > k \in \mathbb{Z}$. Hence we see that our operator $M(\vartheta, z)$ is a meromorphic function in the variable $z$.

The modules $\text{Ind}_{\mu}^G \mathbb{D}_{\mu}$ and $\text{Ind}_{\mu'}^G \mathbb{D}_{\mu'}$ contain the special irreducible $\mathcal{K}^{(1)}$-module $\mathcal{X}[\mu]$ with highest weight $\mu_\alpha(\epsilon, \tilde{\mu})$ with multiplicity one. This $\mathcal{K}^{(1)}$ module occurs with higher multiplicity $t$ in $\text{Ind}_{\tilde{\mu}}^G \mathbb{D}_{\tilde{\mu}}$, $(\tilde{\mu} + 2\rho_{\mathcal{M}}) \otimes \mathbb{C}|\gamma_n|^z$. Restriction to this $\mathcal{K}^{(1)}$ type yields a diagram

$$
\begin{array}{c}
\mathcal{X}[\tilde{\mu}] \otimes (z) \\
\downarrow \\
(\mathcal{X}[\tilde{\mu}])^t \otimes (z)
\end{array}
\begin{array}{c}
\xrightarrow{T^{w,p,\text{st}}(z)} \\
\xrightarrow{T^{w,p,\text{st}}(z)}
\begin{array}{c}
\mathcal{X}[\tilde{\mu}] \otimes (-z) \\
\downarrow \\
(\mathcal{X}[\tilde{\mu}])^t \otimes (-z)
\end{array}
\end{array}
\tag{134}
$$

where the downarrows are the inclusion by the first coordinate. Then our matrix $M(\vartheta_{\mu_\alpha(\epsilon, \tilde{\mu})}, z)$ will be an $t \times t$ matrix with entries $C_{l,m}(\vartheta_{\mu_\alpha(\epsilon, \tilde{\mu})}, z)$ where $C_{l,m}(\vartheta_{\mu_\alpha(\epsilon, \tilde{\mu})}, z) \in \mathbb{Q}(z)$. We look at this first row, which tells us what happens to the first coordinate under $T^{w,p,\text{st}}(z)$. This first row is $(C_{1,1}(\vartheta_{\mu_\alpha(\epsilon, \tilde{\mu})}, z), 0, \ldots, 0)$. The rational function $(C_{1,1}(\vartheta_{\mu_\alpha(\epsilon, \tilde{\mu})}, z) \in \mathbb{Q}(z)$ is regular at $z = 0$. (See [13], Prop. 7.44). Therefore we can evaluate the first row at $z = 0$. The result will be $((C_{1,1}(\vartheta_{\mu_\alpha(\epsilon, \tilde{\mu})}, 0), 0, \ldots, 0))$ where $C_{1,1}(\vartheta_{\mu_\alpha(\epsilon, \tilde{\mu})}, 0) \in \mathbb{Q}^\times$. 

\[ \square \]

It is clear that the operator $\frac{1}{\pi} T^{w,p,\text{st}}(0)$ induces an isomorphism in cohomology

$$
T^{w,p,*} : H^\bullet(\mathfrak{g}, \mathcal{K}, \text{Ind}_{\mu}^G \mathbb{D}_{\mu} \otimes \mathcal{M}_\lambda) \otimes \mathbb{Q} \rightarrow H^\bullet(\mathfrak{g}, \mathcal{K}, \text{Ind}_{\mu'}^G \mathbb{D}_{\mu'} \otimes \mathcal{M}_\lambda) \otimes \mathbb{Q}
\tag{135}
$$

We can compute these cohomology groups using Delorme. Let $\mathcal{K}^{(1),M} \subset M$ be the connected maximal definite group scheme then

$$
H^\bullet(\mathfrak{g}, \mathcal{K}, \mathcal{M}_\lambda) \rightarrow H^+_+(\mathfrak{m}, \mathcal{K}^{(1),M}, \mathbb{D}_{\mu} \otimes \mathcal{M}_{\text{wun},\lambda}) \oplus H^\bullet(\mathfrak{m}, \mathcal{K}^{(1),M}, \mathbb{D}_{\tilde{\mu}} \otimes \mathcal{M}_{\text{wun},\lambda})
$$
It is certainly possible to prove the necessary rationality result at the place infinity with somewhat lesser effort. In [7] the relative period is defined after we make some choices of basis vectors in various vector spaces (most of the time one dimensional). Then the computation of the intertwining operator

\[ H^\bullet(g_\mathbb{Z}, \mathcal{K}, 3\mathfrak{m}^{\xi}_Q \mathcal{D}_{\mathfrak{g}} \otimes \mathcal{M}_\lambda') \rightarrow H^\bullet(m, \mathcal{K}^{(1),M'}, \mathcal{D}_{\mathfrak{g}} \otimes \mathcal{M}_{\nu_{\text{un}},\lambda}) \]

and the inclusion is always an isomorphism to the + component. Now we remember that \( M = M_1 \times M_2 \) one of the factors is a \( \text{Gl}_n \) with \( n \) even the other is \( \text{Gl}_{n'} \) with \( n' \) odd. Then (in the lowest degree) the cohomology \( H^\bullet(m, \mathcal{K}^{(1),M'}, \mathcal{D}_{\mathfrak{g}} \otimes \mathcal{M}_{\nu_{\text{un}},\lambda}) \) is generated by an element \( \omega_{\mu_1}^{(e)} \otimes \omega_{\mu_2} \) where \( e = 1 \) or 2. The cohomology \( H^\bullet(m, \mathcal{K}^{(1),M'}, \mathcal{D}_{\mathfrak{g}} \otimes \mathcal{M}_{\nu_{\text{un}},\lambda}) \) is generated by \( \omega_{\mu_2'} \otimes \omega_{\mu_1'}^{(e')} \) where \( 1' = 2, 2' = 1 \). (This introduces the relative period). Then we get

\[ T^{w_p,b_n+b_{n'}}(\omega_{\mu_1}^{(e)} \otimes \omega_{\mu_2}) = C_{1,1}(g_{\mu_2}(e,\bar{\mu}),0)\omega_{\mu_2'} \otimes \omega_{\mu_1'}^{(e')} \]  

(136)

This rationality result is applied in [7] to prove a rationality result for ratios of critical values of Rankin-Selberg \( L \)-functions at consecutive critical arguments. We recall from the work of Shahidi [10] that we can attach a local \( L \)-function \( L^\text{coh}_{\infty}(\mathbb{D}_{\bar{\mu}},s) \) to our (representation) Harish-Chandra module \( \mathbb{D}_{\bar{\mu}} \). (This local \( L \)-function differs by the usual shift from Shahidis \( L \) function.) Then we can rewrite the formula above into

\[ T^{w_p,s}(0)(\omega_{\mu_1}^{(e)} \otimes \omega_{\mu_2}) = c_{\infty}(\bar{\mu}) \frac{L^\text{coh}_{\infty}(\mathbb{D}_{\bar{\mu}},w(\bar{\mu}) + b(\bar{\mu},\lambda))}{L^\text{coh}_{\infty}(\mathbb{D}_{\bar{\mu}},w(\bar{\mu}) + b(\bar{\mu},\lambda) + 1)}(\omega_{\mu_2'} \otimes \omega_{\mu_1'}^{(e')}) \]

(137)

The local \( L \)-function can be expressed in terms of products of functions \( \Gamma_C(z) = 2(2\pi)^{-s} \Gamma(s) \) (See [7],7.2.1 ) and using this expression we find

\[ \frac{L^\text{coh}_{\infty}(\mathbb{D}_{\bar{\mu}},w(\bar{\mu}) + b(\bar{\mu},\lambda))}{L^\text{coh}_{\infty}(\mathbb{D}_{\bar{\mu}},w(\bar{\mu}) + b(\bar{\mu},\lambda) + 1)} = \frac{\pi^{d_{\text{un}}/2}}{\prod N_i(w,\bar{\mu})} \]

(138)

where \( N_i(w,\bar{\mu}) \) are certain integers (See Cor. 7.33 in [7]). The combinatorial lemma in [7] (Appendix by Weselmann) implies that under the given conditions the \( N_i(w,\bar{\mu}) \neq 0 \).

Our rationality result is then equivalent to the assertion that \( c_{\infty}(\bar{\mu}) \in \mathbb{Q}_\times \). It enters in the proof of the main theorem in [7].

### 4.6 An intriguing question

It is certainly possible to prove the necessary rationality result at the place infinity with somewhat lesser effort. In [7] the relative period is defined after we make some choices of basis vectors in various vector spaces (most of the time one dimensional). Then the computation of the intertwining operator
comes down to see its effect on these basis elements and this computation can be carried out quite directly.

We develop the concept of Harish-Chandra modules over \( \mathbb{Z} \) because this gives us some motivation for the choice of these basis elements. But we can get more profit out of it. We have seen that the cohomology modules

\[
H^\bullet(g_{\mathbb{Z}}, \mathcal{K}, \mathfrak{Ind}_{\mathbb{Q}}^G D_{\tilde{\mu}} \otimes M_{\lambda}) \otimes \mathbb{Z}[\frac{1}{2}], \quad H^\bullet(g_{\mathbb{Z}}, \mathcal{K}, \mathfrak{Ind}_{\mathbb{G}}^G D_{\mu'} \otimes M_{\lambda}) \otimes \mathbb{Z}[\frac{1}{2}]
\]

in lowest degree are free of rank one and the intertwining multiplied by \( 1/\pi d_u/2 \) induces an isomorphism if we tensorize by \( \mathbb{Q} \). But we may also consider the slightly modified operator

\[
\tilde{T}^{w_P}(\tilde{\mu}) = L_{c_\infty}^\text{coh}((D_{\tilde{\mu}}, w(\tilde{\mu}) + b(w, \lambda) + 1)) T^{w_P, \text{st}}(0)
\]

which also induces an isomorphism between the two modules after we tensor them by \( \mathbb{Q} \). We ask the question

Is the modified operator

\[
\tilde{T}^{w_P}(\tilde{\mu} : H^\bullet(g_{\mathbb{Z}}, \mathcal{K}, \mathfrak{Ind}_{\mathbb{G}}^G D_{\tilde{\mu}} \otimes M_{\lambda}) \otimes \mathbb{Z}[\frac{1}{2}] \rightarrow H^\bullet(g_{\mathbb{Z}}, \mathcal{K}, \mathfrak{Ind}_{\mathbb{Q}}^G D_{\mu'} \otimes M_{\lambda}) \otimes \mathbb{Z}[\frac{1}{2}]
\]

an isomorphism ?

This is of course equivalent with the assertion \( c_\infty(\tilde{\mu}) \in \mathbb{Z}[\frac{1}{2}]^\times \). The only non trivial case where we know that this true is the case \( N = 3 \). (See [12].)

A similar question is discussed in my preprint "Secondary Operations in the Cohomology of Harish-Chandra Modules" ([7], folder "Eisenstein", SecOps.pdf)

4.7 Fixing the periods

In [7] the authors prove a rationality result for ratios of consecutive special values of Rankin-Selberg \( L \)- functions. In this rationality result a certain relative period \( \Omega(\tilde{\sigma}_f) \) enters, this period is basically a non zero complex number which is defined modulo \( E^{\times} \) where \( E \) is a number field over which \( \tilde{\sigma}_f \) is defined. We will show here that we can make this choice of periods more precise so that they are essentially defined modulo the units \( \mathcal{O}_E^\times \). (For a more precise statement see further down.) This allows us to speak of the prime factorization of the ratios of critical values (divided by the period) and this is of arithmetic interest. These considerations are not included into [7] because the authors where concerned that the paper may become too long.

Assume \( n \) even and \( G = \text{GL}_n/\mathbb{Z} \), for simplicity we assume \( F = \mathbb{Q} \). We consider the inner cohomology \( H^\bullet_{f!}(\mathcal{S}_{K_f}^G, M_{\lambda, \mathbb{Z}}) \). This is a finitely generated
The module $H^\bullet_{\text{int}}(S^G_{K_f}, \mathcal{M}_{\lambda, \mathbb{Z}})$ is its quotient by torsion. We have an action of the "integral" Hecke algebra on these cohomology groups (See [4], Chap.3, 2.3.) If we extend $\mathbb{Q}$ to finite extension $E/\mathbb{Q}$ and tensor by the ring of integers $\mathcal{O}_E$ then we get a decomposition up to isogeny (See [4], 2.3.9.)

$$H^\bullet_{\text{int}}(S^G_{K_f}, \mathcal{M}_{\lambda, \mathcal{O}_E}) \supset \bigoplus_{\pi_f \in \text{Col}(G, \lambda, K_f)} H^\bullet_{\text{int}}(S^G_{K_f}, \mathcal{M}_{\lambda, \mathcal{O}_E})(\pi_f) \quad (142)$$

We have the action of $\pi_0(\text{Gl}_n(\mathbb{R}))$ on these cohomology groups and after inverting 2 we get an isomorphism

$$H^\bullet_{\text{int}}(S^G_{K_f}, \mathcal{M}_{\lambda, \mathcal{O}_E[\frac{1}{2}]}) \cong \bigoplus_{\pi_f \in \text{Col}(G, \lambda, K_f)} H^\bullet_{\text{int}}(S^G_{K_f}, \mathcal{M}_{\lambda, \mathcal{O}_E[\frac{1}{2}]}) \cong (\pi_f)(+) \oplus H^\bullet_{\text{int}}(S^G_{K_f}, \mathcal{M}_{\lambda, \mathcal{O}_E[\frac{1}{2}]}) \cong (\pi_f)(-) \quad (142)$$

If a summand $\pi_f$ is strongly inner (See [7], 5.1) and if we tensor by $\mathbb{Q}$ the two summands become isomorphic $H^G_{K_f}$ modules. If $S$ is a finite set of primes containing the primes where $K_f$ is ramified then $H^G_{K_f} = \prod_{p \in S} H^G_{K_p}$ is a central sub algebra of $H^G_{K_f}$. (See [4], 2.3.2). We say that $\pi_f$ is weakly split by $E$ if the restriction of $\pi_f$ to $H^G_{K_f}$ is a homomorphism $\psi^S(\pi_f) : H^G_{K_f} \to \mathcal{O}_E$.

The eigenvalues of the Hecke operators outside $S$ lie in $\mathcal{O}_E$.) We define

$$H^\bullet_{\text{int}}(S^G_{K_f}, \mathcal{M}_{\lambda, \mathcal{O}_E[\frac{1}{2}]}) = \{ \xi \in H^\bullet_{\text{int}}(S^G_{K_f}, \mathcal{M}_{\lambda, \mathcal{O}_E[\frac{1}{2}]}) | h\xi = \psi(\pi_f)(h)\xi \text{ for all } h \in H^G_{K_f} \}.$$ 

Since our group is $\text{Gl}_n$ we have strong multiplicity one. It follows that the isomorphism type $\pi_f$ is uniquely determined by $\psi(\pi_f)$ and it is absolutely irreducible, more precisely

$$H^\bullet_{\text{int}}(S^G_{K_f}, \mathcal{M}_{\lambda, \mathcal{O}_E[\frac{1}{2}]}) = H^\bullet_{\text{int}}(S^G_{K_f}, \mathcal{M}_{\lambda, \mathcal{O}_E[\frac{1}{2}]}) \cong (\psi(\pi_f), \epsilon) \quad (143)$$

If we define $E(\pi_f) \subset E$ to be the subfield of $E$ which is generated by the values $\psi(\pi_f)(h)$ (it is independent of the choice of $S$) then

$$H^\bullet_{\text{int}}(S^G_{K_f}, \mathcal{M}_{\lambda, \mathcal{O}_E[\frac{1}{2}]}) = H^\bullet_{\text{int}}(S^G_{K_f}, \mathcal{M}_{\lambda, \mathcal{O}_E[\psi(\pi_f)]}) \cdot \mathcal{O}_E \quad (144)$$

If $\pi_f$ is strongly inner then the cohomology modules in lowest degree

$$H^0_{\text{int}}(S^G_{K_f}, \mathcal{M}_{\lambda, \mathcal{O}_E})(\pi_f, \epsilon)$$

are absolutely irreducible $H^G_{K_f}$ modules (See [7], 3.3.3) (This also means that the homomorphism $H^G_{K_f} \to \text{End}_E(H^0_{\text{int}}(S^G_{K_f}, \mathcal{M}_{\lambda, \mathcal{O}_E})(\pi_f, \epsilon))$ is surjective.)

The module of homomorphisms

$$\mathcal{T}^{\text{alg}}(\pi_f, \epsilon) = \text{Hom}_{H^G_{K_f}}(H^0_{\text{int}}(S^G_{K_f}, \mathcal{M}_{\lambda, \mathcal{O}_E})(\pi_f, \epsilon), H^0_{\text{int}}(S^G_{K_f}, \mathcal{M}_{\lambda, \mathcal{O}_E})(\pi_f, -\epsilon)) \quad (145)$$

38
is a finitely generated, torsion free \( \mathcal{O}_E \) module of rank one. We consider it as an invertible sheaf for the Zariski topology on \( \text{Spec}(\mathcal{O}_E) \). For any open subset \( U \subset \text{Spec}(\mathcal{O}_E) \) we use the usual notation \( T^{\text{alg}}(\pi_f, \epsilon)(U) \) for the module of sections over \( U \), this is a module for \( \mathcal{O}(U) \). It is clear that we find a covering of \( \text{Spec}(\mathcal{O}_E) \) by two open sets \( U_1, U_2 \) such that

\[
T^{\text{alg}}(\pi_f, \epsilon)(U_i) = \mathcal{O}_E(U_i)T^{\text{alg}}(\pi_f, \epsilon)
\]

where

\[
T^{\text{alg}}_i(\pi_f, \epsilon) \in \text{Hom}_{\text{finite}}(H^b_{\text{int}}(S_{K_f}^G, \mathcal{M}_{\lambda, \mathcal{O}_E})(\pi_f, \epsilon), H^b_{\text{int}}(S_{K_f}^G, \mathcal{M}_{\lambda, \mathcal{O}_E})(\pi_f, -\epsilon)).
\]

These homomorphisms are unique up to an element in \( \mathcal{O}_E(U_i)^\times \).

For a given level \( K_f \) we can find a finite Galois extension \( E/\mathbb{Q} \) such that all \( \pi_f \) which occur in \( H^*(S_{K_f}^G, \mathcal{M}_{\lambda, E}) \) are weakly split or what amounts to the same absolutely irreducible. We have the action of the Galois group \( \text{Gal}(E/\mathbb{Q}) \) on the set of isomorphism classes \( \text{Coh}(G, \lambda, K_f) \).

For a given \( \pi_f \) we choose a covering \( \text{Spec}(\mathcal{O}_{E(\pi_f)}) \) by open subsets \( U_1, U_2 \) and generators \( T^{\text{alg}}_i(\pi_f, \epsilon)(U_i) \). Let us call such a choice a \textit{local trivialization} of \( T^{\text{alg}}(\pi_f, \epsilon) \). Then it is clear that we choose our trivialization such that it is invariant under the action of the Galois group: To get this we choose a \( \pi_f \) in an orbit and our local trivialization \( \{T^{\text{alg}}_i(\pi_f, \epsilon)\}_{i=1,2} \) over \( \mathcal{O}_E(\pi_f) \) as above. For \( \tau \in \text{Gal}(E/\mathbb{Q}) \) we define

\[
T^{\text{alg}}_i(\tau(\pi_f), \epsilon) = \tau(T^{\text{alg}}_i(\pi_f, \epsilon))
\]

and then this system of local trivializations

\[
\{T^{\text{alg}}_i(\tau(\pi_f), \epsilon)\}_{\pi_f \in \text{Coh}(G, \lambda, K_f), \tau, i}
\]

is defined over \( \mathbb{Q} \), i.e. invariant under the Galois group \( \text{Gal}(E/\mathbb{Q}) \).

We return to the transcendental level. We assume \( \pi_f \in \text{Coh}(G, \lambda, K_f) \) we choose a model space for \( \pi_f \) say \( H_{\pi_f} = H^b_{\text{int}}(S_{K_f}^G, \mathcal{M}_{\lambda, \mathcal{O}_E})(\pi_f, +) \). For any \( \iota : E \to \mathbb{C} \) we get an inclusion

\[
\Phi(\lambda, \pi_f, \iota) : \mathbb{D}_\lambda \otimes H_{\pi_f} \otimes E, \iota \mathbb{C} \to \mathcal{A}(G(\mathbb{Q}) \backslash G(\mathbb{A}))
\]

which is Hecke equivariant and therefore unique up to a scalar. This map provides an isomorphism for the cohomology

\[
\Lambda^*(\Phi(\lambda, \pi_f, \iota)) : H^*(\mathbb{G}_{\mathbb{Z}}, \mathbb{D}_\lambda \otimes \mathcal{M}_{\lambda}) \otimes H_{\pi_f} \otimes E, \iota \mathbb{C} \to H^*(S_{K_f}^G, \mathcal{M}_{\lambda}, \mathbb{C})(\pi_f)
\]

which respects the action of \( \pi_0(\text{Gl}_n(\mathbb{R})) \). We have seen in section 4.3 that we have canonical generators for the \( + \) and \( - \) eigenspaces (See 100)

\[
H^b_{\text{int}}(\mathbb{G}_{\mathbb{Z}}, \mathbb{D}_\lambda \otimes \mathcal{M}_{\lambda}, \mathbb{C}) = \mathbb{C}\omega_{\lambda}^{(+)} \oplus \mathbb{C}\omega_{\lambda}^{(-)}.
\]

39
The choice of the generators was motivated by integrality considerations and in this sense they are canonical. But using the explicit description we could just write them down in an ad hoc manner. The actual choice is not so important, what really matters is that they are "entangled", this means once we choose \( \omega^{(+)}_\lambda \) the choice of \( \omega^{(-)}_\lambda \) is forced upon us. See [7], 5.2.

For \( \epsilon = \pm \) we have the two isomorphisms

\[
H_{\pi_f} \otimes_{E,\epsilon} C \xrightarrow{\Psi(\lambda, \pi_f, \epsilon, \epsilon)} H^{bn}_{S G K_f, M_{\lambda, E}}(\pi_f, \epsilon) \otimes_{E,\epsilon} C
\]

which are given by the composition \( \psi_f \mapsto \omega^{(\epsilon)}_\lambda \times \psi_f \mapsto \Lambda^{bn}(\Phi(\lambda, \pi_f, \epsilon))(\omega^{(\epsilon)}_\lambda \times \psi_f) \). We get a composition \( T^{\text{trans}}(\pi_f, \epsilon, \epsilon) : H^{bn}_{S G K_f, M_{\lambda, E}}(\pi_f, \epsilon) \otimes_{E,\epsilon} C \rightarrow H^{bn}_{S G K_f, M_{\lambda, E}}(\pi_f, -\epsilon) \otimes_{E,\epsilon} C \)

and this isomorphism does not depend on the choice of the embedding \( \Phi(\lambda, \pi_f, \epsilon) \). For \( i = 1, 2 \) we define the periods by comparing the two isomorphism between the \( \pm \) eigenspaces (See also [7], 5.2.3):

\[
\Omega_i(\pi_f, \epsilon) T^{\text{trans}}(\pi_f, \epsilon, \epsilon) = T^{\text{alg}}_i(\pi_f, \epsilon) \otimes_{E,\epsilon} C
\]

The periods \( \Omega_i(\pi_f, \epsilon) \) are complex numbers which are well defined modulo \( \iota(\mathcal{O}(E(\pi_f))(U_i))^\times \) and the ratio \( \Omega_1(\pi_f, \epsilon) / \Omega_2(\pi_f, \epsilon) \) is an element in \( \iota(\mathcal{O}(E(\pi_f)))(U_1 \cap U_2)^\times \).

If we now work with this refined definition of the periods the assertion in Theorem 7.39 in [7] remains unchanged but now it makes sense to ask for the decomposition of these numbers into prime ideals. We have evidence that this decomposition into prime factors has some influence on the structure of cohomology of arithmetic groups. The prime factors should be related to denominators of Eisensteinclasses but this relationship could be spoiled by primes dividing the factor \( c_\infty(\tilde{\mu}) \) above. (See also the above reference to SecOps.pdf in [4]).

**References**

[1] A. Borel and N. Wallach, *Continuous cohomology, discrete subgroups, and representations of reductive groups*. Second edition. Mathematical Surveys and Monographs, 67. American Mathematical Society, Providence, RI, 2000.

[2] H. Bruinier, G. van der Geer, G. Harder, D. Zagier *The 1-2-3 of Modular Forms* Springer, Universitext
[3] M. Demazure and A. Grothendieck, *Structure des schémas en groupes réductifs*. SGA 3, TOME III, Lecture Notes in Math., 153, Springer-Verlag, Berlin 1970.

[4] G. Harder, *Cohomology of arithmetic groups*, Book in preparation. Preliminary version available at http://www.math.uni-bonn.de/people/harder/Manuscripts/buch/

[5] G. Harder, *Arithmetic aspects of Rank one Eisenstein Cohomology*, in Cycles, Motives and Shimura Varieties TIFR 2010 p. 131–190 (2010).

[6] G. Harder and A. Raghuram, *Eisenstein cohomology and ratios of critical values of Rankin–Selberg L-functions*. C. R. Math. Acad. Sci. Paris 349 (2011), no. 13-14, 719–724.

[7] G. Harder- A. Raghuram, *Eisenstein Cohomology for GL_{N} and ratios of critical values of Rankin–Selberg L-functions - I*. arXiv:1405.6513

[8] B. Kostant *Lie algebra cohomology and the generalized Borel-Weil theorem*. Ann. of Math. (2) 74 1961 329?387.

[9] C. Mœglin *Representations of GL(n) over the real field* Proceedings of Symposia in Pure Mathematics, Vol. 61 (1997), pp. 157-166

[10] F. Shahidi *Whittaker models for real groups* Duke Mathematical Journal Vol. 47, No. 1

[11] B. Speh *Some results on principal series for GL(n, R)* Ph. D. Dissertation, M.I.T.

[12] D. Zagier, *Appendix: On Harder’s SL(2, R) – SL(3, R)-identity* in Cycles, Motives and Shimura Varieties TIFR 2010 p. 191–195 (2010).