A Complexity Dichotomy for Permutation Pattern Matching on Grid Classes

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Abstract

Permutation Pattern Matching (PPM) is the problem of deciding for a given pair of permutations π and τ whether the pattern π is contained in the text τ. Bose, Buss and Lubiw showed that PPM is NP-complete. In view of this result, it is natural to ask how the situation changes when we restrict the pattern π to a fixed permutation class C; this is known as the C-Pattern PPM problem. There have been several results in this direction, namely the work of Jelínek and Kynčl who completely resolved the hardness of C-Pattern PPM when C is taken to be the class of σ-avoiding permutations for some σ.

Grid classes are special kind of permutation classes, consisting of permutations admitting a grid-like decomposition into simpler building blocks. Of particular interest are the so-called monotone grid classes, in which each building block is a monotone sequence. Recently, it has been discovered that grid classes, especially the monotone ones, play a fundamental role in the understanding of the structure of general permutation classes. This motivates us to study the hardness of C-Pattern PPM for a (monotone) grid class C.

We provide a complexity dichotomy for C-Pattern PPM when C is taken to be a monotone grid class. Specifically, we show that the problem is polynomial-time solvable if a certain graph associated with C, called the cell graph, is a forest, and it is NP-complete otherwise. We further generalize our results to grid classes whose blocks belong to classes of bounded grid-width. We show that the C-Pattern PPM for such a grid class C is polynomial-time solvable if the cell graph of C avoids a cycle or a certain special type of path, and it is NP-complete otherwise.

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1 Introduction

A permutation is a sequence π = π₁, π₂, ..., πₙ in which each number from the set [n] = {1, 2, ..., n} appears exactly once. We then say that a permutation π = π₁, ..., πₙ contains a permutation σ = σ₁, ..., σₖ, if π has a subsequence of length k whose elements have the same relative order as the elements of σ (see Section 2 for a more formal definition). If π does not contain σ, we say that π avoids σ.
An essential algorithmic problem involving permutations is Permutation Pattern Matching (PPM): Given a permutation \( \pi \) (‘pattern’) of size \( k \) and \( \tau \) (‘text’) of size \( n \), does \( \tau \) contain \( \pi \)? Bose, Buss and Lubiw [10] have shown that PPM is NP-complete. This result has motivated a study of various variants of PPM, in particular to obtain the best possible runtime dependence on \( k \). Guillemot and Marx [15] provided the break-through result in this direction by establishing the fixed-parameter tractability of PPM in terms of the pattern length with algorithm running in \( 2^{O(k^2 \log k)} \cdot n \) time. The first phase of their algorithm finds a suitable decomposition that relies on the proof of Stanley-Wilf conjecture given by Marcus and Tardos [21]. Subsequently, Fox [14] refined the results by Marcus and Tardos and thereby reduced the complexity of the algorithm to \( 2^{O(k^2)} \cdot n \).

Several algorithms also exist whose runtimes depend on different parameters than the length of \( \pi \). Bruner and Lackner [12] described an algorithm for PPM with run time \( O(1.79^{\text{run}(\tau)} \cdot kn) \) where \( \text{run}(\tau) \) is the number of consecutive monotone sequences needed to obtain \( \tau \) via concatenation. Ahal and Rabinovich [1] designed an algorithm for PPM that runs in time \( n^{O(\text{tw}(G_\pi))} \) where \( G_\pi \) is a certain graph associated to the pattern \( \tau \) and \( \text{tw}(G_\pi) \) denotes the treewidth of \( G_\pi \). Later Jelínek, Opler and Valtr [19] introduced a related parameter \( \text{gw}(\pi) \), called the grid-width of \( \pi \), and showed that \( \text{gw}(\pi) \) is equivalent to \( \text{tw}(G_\pi) \) up to a constant and thereby implying an algorithm for PPM running in time \( O(n^{O(\text{gw}(\pi))}) \).

Another approach to tackling the hardness of PPM is to restrict the choice of pattern to a particular permutation class \( \mathcal{C} \), where a permutation class is a set of permutations \( \mathcal{C} \) such that for every \( \sigma \in \mathcal{C} \), every permutation contained in \( \sigma \) belongs to \( \mathcal{C} \) as well.

### C-PATTERN PERMUTATION PATTERN MATCHING (C-PATTERN PPM)

**Input:** A pattern \( \pi \in \mathcal{C} \) of size \( k \) and a permutation \( \tau \) of size \( n \).

**Question:** Does \( \tau \) contain \( \pi \)?

For a permutation \( \sigma \), we let \( \text{Av}(\sigma) \) denote the class of permutations avoiding \( \sigma \). Notice that \( \text{Av}(21) \)-Pattern PPM simply reduces to finding the longest increasing subsequence of \( \tau \), which is a well-known problem and can be solved in time \( O(n \log \log n) \) [20]. Bose, Buss and Lubiw [10] showed that C-PATTERN PPM can be solved in polynomial time if \( \mathcal{C} \) is the class of the so-called separable permutations. Further improvements were given by Ibarra [17], Albert et al. [3] and by Yugandhar and Saxena [28]. Recently, Jelínek and Kynčl [18] completely resolved the hardness of C-PATTERN PPM for classes avoiding a single pattern. They proved that \( \text{Av}(\alpha) \)-Pattern PPM is polynomial-time solvable for \( \alpha \in \{1, 2, 12, 21, 132, 213, 231, 312\} \) and NP-complete otherwise.

Lately, a new type of permutation classes has been gaining a lot of attention. The grid class of a matrix \( \mathcal{M} \) whose entries are permutation classes, denoted by Grid(\( \mathcal{M} \)), is a class of permutations admitting a grid-like decomposition into blocks that belong to the classes \( \mathcal{M}_{i,j} \). If moreover \( \mathcal{M} \) contains only Av(21), Av(12) and \( \emptyset \) we say that Grid(\( \mathcal{M} \)) is a monotone grid class. To each matrix \( \mathcal{M} \) we also associate a graph \( G_\mathcal{M} \), called the cell graph of \( \mathcal{M} \). We postpone their full definitions to Section 2.

Monotone grid classes were introduced partly by Atkinson, Murphy and Ruškuc [6] and in full by Murphy and Vatter [22] who showed that a monotone grid class Grid(\( \mathcal{M} \)) is partially well-ordered if and only if \( G_\mathcal{M} \) is a forest. Brignall [11] later extended their results to a large portion of general grid classes. General grid classes themselves were introduced by Vatter [25] in his paper investigating the growth rates of permutation classes. Since then the grid classes played a central role in most subsequent works on growth rates of permutation classes [4, 27]. Bevan [8, 9] tied the growth rates of grid classes to algebraic graph theory.

Given the prominent role of grid classes in recent developments of the permutation pattern research, it is only natural to investigate the hardness of searching patterns that belong to
a grid class. Neou, Rizzi and Vialette [24] designed a polynomial-time algorithm solving \( C \)-Pattern PPM when \( C \) is the class of the so-called wedge permutations, which can be also described as a monotone grid class. Consequently, Neou [23] asks at the end of his thesis about the hardness of Grid(\( M \))-Pattern PPM for a monotone grid class Grid(\( M \)).

**Our results.** In Section 3, we answer the question of Neou by proving that for a monotone grid class Grid(\( M \)), the problem Grid(\( M \))-Pattern PPM is polynomial-time solvable if the cell graph \( G_M \) is a forest, NP-complete otherwise.

In Section 4, we further extend our results to matrices whose every entry is a permutation class of bounded grid-width. We prove that for such grid classes, Grid(\( M \))-Pattern PPM is polynomial-time solvable if \( G_M \) is a forest which does not contain a certain type of path, and NP-complete otherwise.

## 2 Preliminaries

A permutation of length \( n \) is a sequence in which each element of the set \([n] = \{1, 2, \ldots, n\}\) appears exactly once. When writing out short permutations explicitly, we shall omit all punctuation and write, e.g., 15342 for the permutation 1, 5, 3, 4, 2. The permutation diagram of \( \pi \) is the set of points \( \{(i, \pi_i); \ i \in [n]\} \) in the plane. Note that we use Cartesian coordinates, that is, the first row of the diagram is at the bottom. We blur the distinction between permutations and their permutation diagrams, e.g., we shall refer to ‘the point set \( \pi \)’ rather than ‘the point set of the diagram of the permutation \( \pi \).’

For a point \( p \) in the plane, we denote its first coordinate as \( p.x \), and its second coordinate as \( p.y \). A subset \( S \) of a permutation diagram is isomorphic to a subset \( R \) of a permutation diagram if there is a bijection \( f: R \to S \) such that for any pair of points \( p \neq q \) of \( R \) we have \( f(p).x < f(q).x \) if and only if \( p.x < q.x \), and \( f(p).y < f(q).y \) if and only if \( p.y < q.y \). A permutation \( \pi \) of length \( n \) contains a permutation \( \sigma \) of length \( k \), if there is a subset of \( \pi \) isomorphic to the permutation diagram of \( \sigma \). Such a subset is then an occurrence (or a copy) of \( \sigma \) in \( \pi \). If \( \pi \) does not contain \( \sigma \), we say that \( \pi \) avoids \( \sigma \).

A permutation class is a set \( C \) of permutations with the property that if \( \pi \) is in \( C \), then all the permutations contained in \( \pi \) are in \( C \) as well. For a permutation class \( C \), we let \( Av(\sigma) \) denote the class of \( \sigma \)-avoiding permutations. We shall sometimes use the symbols \( \mathbb{Z} \) and \( \mathbb{N} \) as short-hands for the class of increasing permutations \( Av(21) \) and the class of decreasing permutations \( Av(12) \).

For a permutation \( \pi \) of length \( n \) the reverse of \( \pi \) is the permutation \( \pi_n, \pi_{n-1}, \ldots, \pi_1 \), the complement of \( \pi \) is the permutation \( n + 1 - \pi_1, n + 1 - \pi_2, \ldots, n + 1 - \pi_n \), and the inverse of \( \pi \) is the permutation \( \sigma = \sigma_1, \ldots, \sigma_n \) satisfying \( \pi_i = j \iff \sigma_j = i \). We let \( \pi^{-1} \), \( \pi^c \) and \( \pi^{-1} \) denote the reverse, complement and inverse of \( \pi \), respectively. For a permutation class \( C \), we let \( C^r \) denote the set \( \{ \pi^r; \ \pi \in C\} \), and similarly for \( C^c \) and \( C^{-1} \). Note that \( C^r \), \( C^c \) and \( C^{-1} \) are again permutation classes.

A permutation \( \pi \) of length \( n \) is a horizontal alternation if all the even entries of \( \pi \) precede all the odd entries of \( \pi \), i.e., there are no indices \( i < j \) such that \( \pi_i \) is odd and \( \pi_j \) is even. A permutation \( \pi \) is a vertical alternation if \( \pi^{-1} \) is a horizontal alternation.

**Griddings and grid classes.** For two sets \( A \) and \( B \) of numbers, we write \( A < B \) if every element of \( A \) is smaller than any element of \( B \). In particular, both \( \emptyset < A \) and \( A < \emptyset \) holds for any \( A \).
A $k \times \ell$ matrix $\mathcal{M}$ whose entries are permutation classes is called a gridding matrix. Moreover, if the entries of $\mathcal{M}$ belong to the set $\{\mathbb{Z}, \subseteq, \emptyset\}$ then we say that $\mathcal{M}$ is a monotone gridding matrix. A $k \times \ell$-gridding of a permutation $\pi$ of length $n$ are two sequences of (possibly empty) disjoint integer intervals $I_1 < I_2 < \cdots < I_k$ and $J_1 < J_2 < \cdots < J_\ell$ such that both $\bigcup_{i=1}^k I_i$ and $\bigcup_{j=1}^\ell J_j$ are equal to $[n]$. We call the set $I_i \times J_j$ an $(i, j)$-cell of $\pi$. An $\mathcal{M}$-gridding of a permutation $\pi$ is a $k \times \ell$-gridding such that the restriction of $\pi$ to the $(i, j)$-cell is isomorphic to a permutation from the class $M_{i,j}$. If $\pi$ possesses an $\mathcal{M}$-gridding, then $\pi$ is said to be $\mathcal{M}$-griddable and $\pi$ equipped with a fixed $\mathcal{M}$-gridding is called an $\mathcal{M}$-gridded permutation. We let Grid($\mathcal{M}$) be the class of $\mathcal{M}$-griddable permutations. Note that for consistency with our Cartesian numbering convention, we number the rows of a matrix from bottom to top.

The cell graph of the gridding matrix $\mathcal{M}$, denoted $G_{\mathcal{M}}$, is the graph whose vertices are the cells of $\mathcal{M}$ that contain an infinite class, with two vertices being adjacent if they share a row or a column of $\mathcal{M}$ and all cells between them are finite or empty. A proper turning path in $G_{\mathcal{M}}$ is a path $P$ such that no three consecutive cells of $P$ share the same row or column. See Figure 1.

**Grid-width.** An interval family $\mathcal{I}$ is a set of pairwise disjoint integer intervals. The intervallicity of a set $A \subseteq [n]$, denoted by $\text{int}(A)$, is the size of the smallest interval family whose union is equal to $A$. For a point set $S$ in the plane, let $\Pi_x(S)$ denote its projection on the $x$-axis and similarly $\Pi_y(S)$ its projection on the $y$-axis. We write $\text{int}_x(S)$ and $\text{int}_y(S)$ as short for $\text{int}(\Pi_x(S))$ and $\text{int}(\Pi_y(S))$, respectively. For a subset $S$ of the permutation diagram, the grid-complexity of $S$ is the maximum of $\text{int}_x(S)$ and $\text{int}_y(S)$.

A grid tree of a permutation $\pi$ of length $n$ is a rooted binary tree $T$ with $n$ leaves, each leaf being labeled by a distinct point of the permutation diagram. Let $\pi^v_T$ denote the point set of the labels on the leaves in the subtree of $T$ rooted in $v$. The grid-width of a vertex $v$ in $T$ is the grid-complexity of $\pi^v_T$, and the grid-width of $T$, denoted by $\text{gw}(\pi)$, is the maximum grid-width of a vertex of $T$. Finally, the grid-width of a permutation $\pi$, denoted by $\text{gw}(\pi)$, is the minimum of $\text{gw}(\pi)$ over all grid trees $T$ of $\pi$.

We also consider a linear version of this parameter. We say that a rooted binary tree $T$ is a caterpillar if each vertex is either a leaf or has at least one leaf as a child. The path-width of a permutation $\pi$, denoted by $\text{pw}(\pi)$, is the minimum of $\text{gw}(\pi)$ over all caterpillar grid trees $T$ of $\pi$.

We now provide a useful alternative definition of path-width. For permutations $\pi$ and $\sigma$ of length $n$, the path-width of $\pi$ in $\sigma$-ordering, denoted by $\text{pw}^\sigma(\pi)$ is the maximum grid-complexity attained by a set $\{(\sigma_1, \pi_{x_1}), \ldots, (\sigma_i, \pi_{x_i})\}$ for some $i \in [n]$.

**Lemma 2.1.** A permutation $\pi$ of length $n$ has path-width $p$ if and only if the minimum value of $\text{pw}^\sigma(\pi)$ over all permutations $\sigma$ of length $n$ is exactly $p$.

**Proof.** Suppose that $\text{pw}(\pi) = p$ as witnessed by a caterpillar grid tree $T$. Observe that all...
leaves of $T$ except for the deepest pair lie in different depths. Define $\sigma$ to simply order the labels by the depths of their leaves, defining arbitrarily the order of the two deepest leaves. Then every set $\{ (\sigma_1, \pi_{\sigma_1}), \ldots, (\sigma_i, \pi_{\sigma_i}) \}$ corresponds exactly to the set $\pi^T_v$ for some vertex $v$ of $T$.

In order to prove the other direction, we define a sequence of caterpillar trees $T_1, \ldots, T_n$ in the following way. Let $T_1$ be a single vertex labeled by $(\sigma_1, \pi_{\sigma_1})$. For $i > 1$, let $T_i$ be the binary rooted tree with left child a leaf labeled by $(\sigma_i, \pi_{\sigma_i})$ and right child the tree $T_{i-1}$. The tree $T_n$ is a caterpillar grid tree of $\pi$ and for every inner vertex $v$ the set $\pi^T_v$ is equal to $\{ (\sigma_1, \pi_{\sigma_1}), \ldots, (\sigma_i, \pi_{\sigma_i}) \}$ for some $i$. The claim follows.

Ahal and Rabinovich [1] designed an algorithm for PPM that runs in time $n^{O(tw(G_\pi))}$ where $G_\pi$ is a certain graph associated to the pattern $\pi$ and $tw(G_\pi)$ denotes the treewidth of $G_\pi$. The following theorem follows by combining this algorithm with the result of Jelínek et al. [19] who showed that up to a constant, $gw(\pi)$ is equivalent to $tw(G_\pi)$.

\textbf{Theorem 2.2} (Ahal and Rabinovich [1], Jelínek et al. [19]). Let $\pi$ be a permutation of length $k$ and $\tau$ a permutation of length $n$. The problem whether $\tau$ contains $\pi$ can be solved in time $n^{O(gw(\pi))}$.

Importantly, Theorem 2.2 implies that $\mathcal{C}$-Pattern PPM is decidable in polynomial time whenever the class $\mathcal{C}$ has bounded grid-width. In fact, we obtain all the polynomial-time solvable cases of $\mathcal{C}$-Pattern PPM in this paper via showing that $\mathcal{C}$ has bounded grid-width.

### 3 Monotone grid classes

This section is dedicated to proving that the complexity of Grid($\mathcal{M}$)-Pattern PPM is for a monotone gridding matrix $\mathcal{M}$ determined by whether $G_\mathcal{M}$ contains a cycle.

\textbf{Theorem 3.1.} For a monotone gridding matrix $\mathcal{M}$ one of the following holds:

- Either $G_\mathcal{M}$ is a forest, Grid($\mathcal{M}$) has bounded path-width and Grid($\mathcal{M}$)-Pattern PPM can be decided in polynomial time, or
- $G_\mathcal{M}$ contains a cycle, Grid($\mathcal{M}$) has unbounded grid-width and Grid($\mathcal{M}$)-Pattern PPM is NP-complete.

A consistent orientation of a $k \times \ell$ monotone gridding matrix $\mathcal{M}$ is a pair of functions $(c, r)$ such that $c: [k] \to \{-1, 1\}$, $r: [\ell] \to \{-1, 1\}$ and for every $i \in [k]$, $j \in [\ell]$ the value $c(i)r(j)$ is positive if $\mathcal{M}_{i,j} = \square$ and negative if $\mathcal{M}_{i,j} = \blacksquare$.

Intuitively speaking, the purpose of the consistent orientation is to ‘orient’ each nonempty cell of a monotone gridding matrix, so that $\blacksquare$-cells are oriented towards the top-right or towards the bottom-left, while the $\square$-cells are oriented towards the top-left or the bottom-right. The orientation is consistent in the sense that in a given column $i$, either all the nonempty cells are oriented left-to-right (if $c(i) = 1$) or right-to-left (if $c(i) = -1$), while in a row $j$, they are all oriented bottom-to-top (if $r(j) = 1$) or top-to-bottom (if $r(j) = -1$). See Figure 2 (left) for an example of a gridding matrix with a consistent orientation.

Not every monotone gridding matrix has a consistent orientation: consider, e.g., a $2 \times 2$ gridding matrix with three cells equal to $\square$ and one cell equal to $\blacksquare$. However, Albert et al. [4] observed that we can transform a given gridding matrix $\mathcal{M}$ into another, similar gridding matrix that has a consistent orientation. Let $\mathcal{M}$ be a $k \times \ell$ monotone gridding matrix and $q$ a positive integer. The refinement $\mathcal{M}^{\times q}$ of $\mathcal{M}$ is the $qk \times q\ell$ matrix obtained from $\mathcal{M}$ by replacing each $\blacksquare$-entry by a $q \times q$ diagonal matrix with all the non-empty entries equal to $\blacksquare$, each $\square$-entry by a $q \times q$ anti-diagonal matrix with all the non-empty entries equal to $\square$ and
each empty entry by a \( q \times q \) empty matrix. It is easy to see that \( \text{Grid}(M^{\times q}) \) is a subclass of \( \text{Grid}(M) \). Moreover, if \( G_M \) is a forest then \( \text{Grid}(M^{\times q}) = \text{Grid}(M) \) and \( G_{M^{\times q}} \) is a forest as well.

**Lemma 3.2** (Albert et al. [4, Proposition 4.1]). For every monotone gridding matrix \( M \), the refinement \( M^{\times 2} \) admits a consistent orientation.

We remark that Albert et al. [4] use a slightly different way of defining a permutation class from a given gridding matrix: specifically, their classes only contain permutations in which the entries represented by each cell of the gridding can be placed on a segment with slope \( +1 \) or \( -1 \). However, as Lemma 3.2 is a claim about gridding matrices and not about permutation classes, we can use it here for our purposes.

Now we provide bounds on the width parameters of monotone grid classes depending on the structure of their cell graphs.

**Proposition 3.3.** Let \( M \) be a \( k \times \ell \) monotone gridding matrix that has a consistent orientation. If \( G_M \) is a forest then every permutation of \( \text{Grid}(M) \) has path-width at most \( \max(k, \ell) \).

**Proof.** Let \( \pi \) be a permutation from \( \text{Grid}(M) \) of length \( n \) together with an \( M \)-gridding \( I_1^0 < \cdots < I_k^0 \) and \( J_1^0 < \cdots < J_\ell^0 \). We fix a consistent orientation \((c, r)\) for the matrix \( M \).

For \( I \subseteq I_k^0 \), the extremal point of \( I \) is the rightmost point of \( \pi \) restricted to \( [n] \times [\ell] \) if \( c(i) \) is positive, the leftmost point otherwise. Similarly for \( J \subseteq J_\ell^0 \), the extremal point of \( J \) is the topmost point of \( \pi \) restricted to \( [n] \times [\ell] \) if \( r(j) \) is positive, the bottommost point otherwise. Importantly, the definition of consistent orientation guarantees that if \( I \times J \) contains the extremal points of both \( I \) and \( J \), then these two points must actually be the same point.

We now construct an ordering \( \sigma \) of length \( n \) and a sequence of interval families \((\mathcal{I}_m, \mathcal{J}_m)\) for \( m \in [n] \) such that for every \( m \)

- \( \mathcal{I}_m \) contains \( k \) (possibly empty) intervals \( I_1^m < \cdots < I_k^m \) and \( \mathcal{J}_m \) contains \( \ell \) (possibly empty) intervals \( J_1^m < \cdots < J_\ell^m \),
- \( I_s^m \subseteq I_s^0 \) and \( J_t^m \subseteq J_t^0 \) for every \( s \) and \( t \), and
- for \( P = \{ (\sigma_{m+1}, \pi_{\sigma_{m+1}}), (\sigma_{m+2}, \pi_{\sigma_{m+2}}), \ldots, (\sigma_n, \pi_{\sigma_n}) \} \) we have \( \Pi_s(P) = \bigcup \mathcal{I}_m \) and \( \Pi_t(P) = \bigcup \mathcal{J}_m \).

The third condition then implies that \( \text{pw}^n(\pi) \leq \max(k, \ell) \) which proves the proposition.

Suppose that we have already defined the sequences up to \( m \). Our goal is to find indices \( i \in [k] \) and \( j \in [\ell] \) such that \( I_i^m \times J_j^m \) contains the extremal point of both \( I_i^m \) and \( J_j^m \). To this end, we transform \( G_M \) into a directed graph \( \vec{G}_M \) as follows: suppose that a column \( i \) of \( M \) contains nonempty cells \( c_1, c_2, \ldots, c_p \) ordered bottom to top. These cells form a path in \( G_M \). Assume that the extremal point of \( I_i^m \) is in the cell \( c_j \). We then orient the edges of the path \( c_1 c_2 \cdots c_p \) so that they all point towards \( c_j \), that is, for every \( a < j \) the edge \( \{c_a, c_{a+1}\} \) is oriented upwards, while for \( a \geq j \) it is oriented downwards. If \( I_i^m \) is empty, we remove the edges of the path \( c_1 \cdots c_p \) from \( \vec{G}_M \) entirely. We perform an analogous operation for every row \( j \) of \( M \), orienting the edges of the corresponding path in \( G_M \) so that they point towards the cell containing the extremal point of \( J_j^m \), or removing the edges entirely if \( J_j^m \) is empty.

As \( \vec{G}_M \) is a forest, each of its components has a sink. Ignoring the components of \( \vec{G}_M \) that correspond to intersections of empty rows with empty columns, and choosing a sink of one of the remaining components, we find a column \( i \) and a row \( j \) of \( M \) such that both \( I_i^m \) and \( J_j^m \) have their extremal points in \( I_i^m \times J_j^m \). As we observed, this must be the same point \( p \).
We set \(\sigma_{m+1}\) to be \(p,x\) and we define the interval families \(I^{m+1}\), \(J^{m+1}\) by removing \(p,x\) from \(I^m\) and \(p,y\) from \(J^m\). Observe that \(I^{m+1}\), \(J^{m+1}\) are well-defined because \(p\) was the extremal point of both intervals \(I^m\) and \(J^m\).

Lemma 3.4. Let \(\mathcal{M}\) be a monotone gridding matrix such that \(G_{\mathcal{M}}\) contains a path of length \(k\). Then there exists a permutation \(\pi \in \text{Grid}(\mathcal{M})\) with grid-width at least \(\frac{k}{4}\).

Proof. Without loss of generality, we assume that \(G_{\mathcal{M}}\) is a path of length \(k\), i.e. all the other entries are empty. We construct an \(\mathcal{M}\)-gridding permutation \(\pi\) of length \(n = k^3\) such that \(\text{gw}(\pi) \geq \frac{k}{4}\) in the following way. The permutation \(\pi\) is a union of point sets \(B_1, \ldots, B_k\), called blocks, such that each \(B_i\) has size \(k^2\) and is contained in the cell of the \(\mathcal{M}\)-gridding corresponding to the \(i\)-th vertex of the path. Moreover, for every \(i\), the point set \(B_i \cup B_{i+1}\) forms a horizontal or vertical alternation, depending on whether their respective cells share the same row or column. If the relation between point sets \(B_i\) and \(B_j\) for \(|i-j| \geq 2\) is not fully determined by the position of their respective cells (they share the same row or column) they can be interleaved in arbitrary way as long as \(B_i \cup B_{i+1}\) forms an alternation for every \(i\).

Let \(T\) be the grid-tree of \(\pi\) with minimum grid-width. By standard arguments, there is a vertex \(v \in T\) such that the subtree of \(v\) contains at least \(\frac{n}{4}\) and at most \(\frac{kn}{3}\) leaves. Let \(S\) be the subset of the permutation diagram of \(\pi\) defined by these leaves.

Let the density of a block \(B_i\) be the ratio \(\frac{|S \cap B_i|}{|B_i|}\), denoted by \(d_i\). We claim that the densities of consecutive blocks cannot differ too much, in particular that the difference \(|d_i - d_{i+1}|\) is at most \(\frac{1}{2k}\). Without loss of generality, assume that \(B_i\) and \(B_{i+1}\) share the same row and that \(d_{i+1} > d_i\). If the density of \(B_i\) and \(B_{i+1}\) differed by at least \(\frac{1}{2k}\), there would be at least \(k/4\) more points of \(S\) in \(B_{i+1}\). Since \(B_i \cup B_{i+1}\) forms a vertical alternation, we can pair each point of \(B_i\) with the nearest point to the right of it in \(B_{i+1}\). Then at least \(k/4\) of these pairs would consist of a point of \(B_{i+1}\) in \(S\) and a point of \(B_i\) outside of \(S\). The intervalicity of \(\Pi_x(S \cap (B_i \cup B_{i+1}))\) would thus be at least \(k/4\) and that is a contradiction.

We now show that each block of \(\pi\) contains both a point in \(S\) and a point outside of \(S\). Suppose that the block \(B_i\) is fully contained in \(S\), i.e. the density \(d_i\) is equal to 1. Since the differences in densities of consecutive blocks cannot be larger than \(\frac{1}{2k}\), every block has density at least \(1 - k \cdot \frac{1}{2k} = \frac{3}{4}\). But then \(S\) must contain at least \(\frac{3}{4}n\) points, which is a contradiction, since \(\frac{1}{4}n \leq |S| \leq \frac{2}{3}n\). Similarly, there cannot be any block whose density is equal to 0, since then every block would have density at most \(\frac{1}{4}\), again a contradiction.

Therefore, every block \(B_i\) contains both a point from \(S\) and a point outside of \(S\). The number of rows and columns of \(S\) spanned by the path is at least \(k\) since every step on the path introduces either a new row or a new column. Let us therefore, without loss of generality, assume that the path spans at least \(\frac{k}{2}\) columns. Since each of the columns contains point both from \(S\) and outside of \(S\), any consecutive interval of \(\Pi_x(S)\) cannot intersect more than 2 of these columns and the grid-complexity of \(S\) is at least \(\frac{k}{4}\). This means that the vertex \(v \in T\) has grid-width at least \(\frac{k}{4}\), therefore the grid-width of \(T\) is at most \(\frac{k}{4}\), and since \(T\) was the optimal grid-tree for \(\pi\), the grid-width of \(\pi\) is at least \(\frac{k}{4}\), as claimed.

We proceed to show that whenever \(G_{\mathcal{M}}\) contains a cycle, \(\text{Grid}(\mathcal{M})\) contains grid subclasses with arbitrarily long paths in their cell graph. In fact, all the properties we show about cyclic grid classes rely structurally only on the existence of these long paths.

Lemma 3.5. Let \(\mathcal{M}\) be a monotone gridding matrix such that \(G_{\mathcal{M}}\) contains a cycle. For every \(p \geq 1\), there is a gridding matrix \(\mathcal{M}_p\) such that \(\text{Grid}(\mathcal{M}_p)\) contains \(\text{Grid}(\mathcal{M}_p)\) as a subclass and moreover, \(G_{\mathcal{M}_p}\) is a proper turning path of length at least \(p\). Furthermore, given \(\mathcal{M}\) and the integer \(p\) we can compute \(\mathcal{M}_p\) in polynomial time.
The staircase classes have been studied by Albert et al. [2] in the context of determining the growth rates of certain permutation classes.

### Proof.

Let $\mathcal{C}$ be a cycle in $G_{\mathcal{M}}$. We can without loss of generality assume that $\mathcal{C}$ is proper turning and that every cell outside of $\mathcal{C}$ is empty, otherwise we could replace all the cells of $\mathcal{M}$ that do not correspond to the turns of $\mathcal{C}$ by empty cells.

The proof is illustrated in Figure 2. Fix a consistent orientation $(c, r)$ of $\mathcal{M}$ and recall the definition of the refinement $\mathcal{M}^{\times p}$. We proceed by labeling the rows and columns of $\mathcal{M}^{\times p}$ using the set $[p]$. The $p$-tuple of columns created from the $i$-th column of $\mathcal{M}$ is labeled $1, 2, \ldots, p$ from left to right if $c(i)$ is positive, and right to left otherwise. Similarly, the $p$-tuple of rows created from the $j$-th row of $\mathcal{M}'$ is labeled $1, 2, \ldots, p$ from bottom to top if $r(j)$ is positive, and top to bottom otherwise. The characteristic of a cell in $\mathcal{M}^{\times p}$ is the pair of labels given to its column and row. The consistent orientation guarantees that each non-empty cell in $\mathcal{M}^{\times p}$ has a characteristic of form $(s, s)$ for some $s \in [p]$. Therefore, $\mathcal{M}^{\times p}$ consists exactly of $p$ components, each being a copy of $\mathcal{M}$.

The $(i, j)$-block of $\mathcal{M}^{\times p}$ is the $p \times p$ submatrix corresponding to the $(i, j)$-cell of $\mathcal{M}$. We pick an arbitrary non-empty cell $(i, j)$ of $\mathcal{M}$ and obtain a matrix $\mathcal{M}_p$ by replacing $(i, j)$-block in $\mathcal{M}^{\times p}$ with the matrix whose only non-empty entries have the characteristic $(s, s + 1)$ for all $s \in [p - 1]$ and are of the same type as $\mathcal{M}_{1,j}$. Grid($\mathcal{M}_p$) is a subclass of Grid($\mathcal{M}$) since the modified $(i, j)$-block corresponds to shifting the original (anti-)diagonal matrix by one row either up or down, depending on the orientation of the $j$-th row of $\mathcal{M}$.

Observe that we connected all the $p$ copies of $\mathcal{M}$ into a single long path. Moreover, we in fact described an algorithm how to compute $\mathcal{M}_p$ in polynomial time.

Combining Lemmas 3.4 and 3.5, we directly obtain the following corollary.

**Corollary 3.6.** Let $\mathcal{M}$ be a monotone gridding matrix such that $G_{\mathcal{M}}$ contains a cycle. Then Grid($\mathcal{M}$) has unbounded grid-width.

In order to state and prove our hardness result that contrasts Proposition 3.3, we need to introduce several definitions. Let $\mathcal{C}$ and $\mathcal{D}$ be permutation classes and $k$ a positive integer. The $(\mathcal{C}, \mathcal{D})$-staircase of $k$ steps is the $(k + 1) \times k$ gridding matrix $\text{St}^k(\mathcal{C}, \mathcal{D})$ such that the $(i, i)$-cell contains $\mathcal{C}$, the $(i + 1, i)$-cell contains $\mathcal{D}$ for every $i \in [k]$, and every other cell is empty. The staircase classes have been studied by Albert et al. [2] in the context of determining the growth rates of certain permutation classes.

Let $\mathcal{M}, \mathcal{N}$ be $k \times \ell$ gridding matrices, let $\pi$ be an $\mathcal{M}$-gridded permutation and let $\tau$ an $\mathcal{N}$-gridded permutation. We say that $\tau$ contains a grid-preserving copy of $\pi$, if there is an...
occurrence of $\pi$ in $\tau$ such that the elements from the $(i,j)$-cell of the $\mathcal{M}$-gridding of $\pi$ are mapped to elements in the $(i,j)$-cell of the $\mathcal{N}$-gridding of $\tau$ for every $i$ and $j$.

To complete the proof of Theorem 3.1, we need to show that Grid($\mathcal{M}$)-PATTERN PPM is NP-complete whenever $G_{\mathcal{M}}$ contains a cycle. Our argument is based upon a construction of Jelínek and Kynčl [18], who proved that that $Av(321)$-PATTERN PPM is NP-complete. The following theorem describes one of the key steps of their proof.

**Theorem 3.7** (Jelínek and Kynčl [18]). Let $\Phi$ be a 3-CNF formula with $v$ variables and $c$ clauses. There is a polynomial time algorithm that outputs a $St^{2c+1}(\mathbb{Z},\mathbb{Z})$-gridded permutation $\pi$ and a $St^{2c+1}(Av(321),\mathbb{Z})$-gridded permutation $\tau$ such that $\Phi$ is satisfiable if and only if there is a grid-preserving copy of $\pi$ in $\tau$. Additionally, the longest increasing subsequences of the $(1,1)$-cell of $\pi$ and the $(1,1)$-cell of $\tau$ are both of length $2v$.

Let us modify $\pi$ and $\tau$ such that any embedding that maps the $(1,1)$-cell of $\pi$ to the $(1,1)$-cell of $\tau$ must already be grid-preserving.

The *lane of $k$ steps* is the $St^k(\mathbb{Z},\mathbb{Z})$-gridded permutation such that each non-empty cell of the staircase contains exactly 2 points and two neighboring cells in the same row form a copy of 1423 while two neighboring cells in the same column form a copy of 1342. The intuition here is that as the lane intersects two adjacent cells in the same row, the two elements in the left cell (corresponding to 1 and 4 of the pattern 1423) are ‘sandwiching’ the two elements of the right cell from above and from below. Similarly, for two cells in the same column, the bottom part of the lane sandwiches the top part from the left and from the right.

For a $St^k(C,D)$-gridded permutation $\pi$, the result of **confining** $\pi$ is the following gridded permutation $\pi'$. Let $\mathcal{N}$ be the gridding matrix obtained by replacing every non-empty entry of $St^k(C,D)$ with a $3 \times 3$ diagonal matrix whose $(1,1)$-cell and $(3,3)$-cell contain $\mathbb{Z}$ and the $(2,2)$-cell is equal to the original entry. Observe that $\mathcal{N}$ consists of 3 components, namely a copy of $St^k(C,D)$ ‘sandwiched’ between two copies of $St^k(\mathbb{Z},\mathbb{Z})$. We obtain $\pi'$ by placing $\pi$ in the middle copy and two lanes of $k$ steps in the outer copies. Finally, we unify each $3 \times 3$ block of the $\mathcal{N}$-gridded permutation $\pi'$ into a single cell. See the left part of Figure 3.

Let $\pi$ and $\tau$ be the gridded permutations from Theorem 3.7 and $\pi'$, $\tau'$ their confined versions. Suppose there is an occurrence of $\pi'$ in $\tau'$ that maps the $(1,1)$-cell of $\pi'$ to the $(1,1)$-cell of $\tau'$. It must map the first two points of the two lanes in $\pi'$ to the first two points of the lanes in $\tau'$ since the longest increasing subsequences in the $(1,1)$-cells of $\pi$ and $\tau$ are of the same length. But then the whole lanes from $\pi'$ map to the respective lanes in $\tau'$; this is because the pair of elements in a given cell of the lane in $\pi'$ (or $\tau'$) sandwiches exactly two other elements of $\pi'$ (or $\tau'$), namely the two elements of the same line belonging to the following cell. This then forces the elements of each non-empty $(i,j)$-cell of $\pi'$ to map to the $(i,j)$-cell of $\tau'$.

**Proposition 3.8.** Let $\mathcal{M}$ be a monotone gridding matrix such that $G_{\mathcal{M}}$ contains a cycle. Then Grid($\mathcal{M}$)-PATTERN PPM is NP-complete.

**Proof.** We describe the reduction from 3-SAT to Grid($\mathcal{M}$)-PATTERN PPM. Let $\Phi$ be a given 3-CNF formula with $v$ variables and $c$ clauses. Using Theorem 3.7, we compute gridded permutations $\pi$ and $\tau$ such that $\Phi$ is satisfiable if and only if there is a grid-preserving copy of $\pi$ in $\tau$. As we have shown, we can assume that any occurrence of $\pi$ in $\tau$ that maps the $(1,1)$-cell of $\pi$ to the $(1,1)$-cell of $\tau$ must be grid-preserving. Furthermore, we obtain a monotone gridding matrix $\mathcal{M}'$ such that Grid($\mathcal{M}'$) is a subclass of Grid($\mathcal{M}$) and $G_{\mathcal{M}'}$ is a proper turning path $P$ of length $4c + 2$ by application of Lemma 3.5. Without loss of generality, we may assume that the first two cells of this path occupy the same row.
First, we aim to construct gridded permutations \( \pi' \) and \( \tau' \) such that \( \pi' \in \text{Grid}(\mathcal{M}') \) and there is a grid-preserving copy of \( \pi' \) in \( \tau' \) if and only if there is a grid-preserving copy of \( \pi \) in \( \tau \). We essentially aim to generalize the ‘twirl’ operation used by Jelínek and Kynčl [18]. A signed permutation of length \( k \) is a permutation of length \( k \) in which each entry is additionally equipped with a sign. Let \( Q \) be an arbitrary \( k \times \ell \) gridding matrix and let \( f \) be a signed permutation of length \( k \) and \( g \) be a signed permutation of length \( \ell \). The \((f,g)\)-transform of \( Q \) is a gridding matrix \( Q' \) such that \( Q'_{i,j} = (Q_{f(i),g(j)})^o \) where the operation \( o \) is identity if both \( f(i) \) and \( g(j) \) are positive, reversal if only \( f(i) \) is negative, complement if only \( g(j) \) is negative and reverse complement if both \( f(i) \) and \( g(j) \) are negative. In other words, \( Q' \) is created by permuting the rows and columns of \( Q \) according to the permutations \( f \) and \( g \) while also flipping around those with negative signs.

Analogously, we can define a transformation of a \( Q \)-gridded permutation into a \( Q' \)-gridded one. The \((f,g)\)-transform of a \( Q \)-gridded permutation \( \sigma \) is the gridded permutation \( \sigma' \) obtained by permuting the columns of the gridding according to \( f \), rows according to \( g \), replacing the point set of the \( i \)-th column with its reversal if \( f(i) \) is negative and then replacing the point set of the \( j \)-th row with its complement if \( g(j) \) is negative.

For any fixed \( f \) and \( g \), there is indeed a grid-preserving occurrence of \( \pi \) in \( \tau \) if and only if there is a grid-preserving occurrence of the \((f,g)\)-transform of \( \pi \) in the \((f,g)\)-transform of \( \tau \). Let us now define \( f \) and \( g \) such that the \((f,g)\)-transform of \( \text{St}^{2c+1}(\mathbb{Z}, \mathbb{Z}) \) is exactly the gridding matrix \( \mathcal{M}' \). Let \((c,r)\) be a consistent orientation of \( \mathcal{M}' \), let \( s_1, \ldots, s_{2c+2} \) be the sequence of column indices in the order as visited by the path \( P \) and let \( t_1, \ldots, t_{2c+1} \) be the sequence of row indices in the order as visited by \( P \). We set \( f(s_i) = c(i)i \) and \( g(t_j) = r(j)j \).

Observe that the \( i \)-th cell of the path in the staircase is mapped by the \((f,g)\)-transform precisely to the \( i \)-th cell of \( P \). Moreover, the consistent orientation guarantees that the type of the entry obtained via \((f,g)\)-transform agrees with the type of the entry in \( \mathcal{M}' \). Let \( \mathcal{N}' \) be the \((f,g)\)-transform of \( \text{St}^{2c+1}(\text{Av}(321), \mathbb{Z}) \).

We set \( \pi' \) to be the \((f,g)\)-transform of \( \pi \) and \( \tau' \) to be the \((f,g)\)-transform of \( \tau \). As we already argued, \( \pi' \) is an \( \mathcal{M}' \)-gridded permutation, \( \tau' \) is an \( \mathcal{N}' \)-gridded permutation and there is a grid-preserving occurrence of \( \pi' \) in \( \tau' \) if and only if there is a grid-preserving occurrence of \( \pi \) in \( \tau \). Moreover it is still true that if there is an occurrence of \( \pi' \) in \( \tau' \) that maps the \( (s_1,t_1) \)-cell to the \( (s_1,t_1) \)-cell of \( \tau' \) then it must be grid-preserving. By the very same argument as before, in any map of the \( (s_1,t_1) \)-cell of \( \pi' \) the beginning of the \((f,g)\)-transformed lanes must map to the beginnings of the \((f,g)\)-transformed lanes in \( \tau' \). As before, this forces the whole mapping to be grid-preserving.
To summarize the argument so far, we used Theorem 3.7 to reduce the NP-complete 3-SAT problem to the problem of deciding whether a \( \text{St}^{2c+1}(\mathbb{Z}, \mathbb{Z}) \)-gridded permutation \( \pi \) has a grid-preserving occurrence in a \( \text{St}^{2c+1}(\mathbb{Av}(321), \mathbb{Z}) \)-gridded permutation \( \tau \). Then, by means of the \((f, g)\)-transform, we showed that this is equivalent to finding a grid-preserving occurrence of an \( \mathcal{M}' \)-gridded permutation \( \pi' \) in an \( \mathcal{N} \)-gridded permutation \( \tau' \).

As the final step of our argument, we transform \( \pi' \) and \( \tau' \) into permutations \( \pi^* \in \text{Grid}(\mathcal{M}') \) and \( \tau^* \in \text{Grid}(\mathcal{N}) \) (that are no longer gridded) such that \( \tau^* \) contains \( \pi^* \) if and only if there is a grid-preserving copy of \( \pi' \) in \( \tau' \). This will imply that \( \text{Grid}(\mathcal{M}') \)-Pattern PPM is NP-complete, and therefore \( \text{Grid}(\mathcal{M}) \)-Pattern PPM is NP-complete as well.

Let \( p \) be the length of the longest monotone subpermutation of \( \tau' \). Suppose that the \((s_1, t_1)\)-cell of \( \mathcal{M}' \) contains \( \mathbb{Z} \), the other case being symmetric. We obtain \( \pi^* \) by inserting two increasing sequences of length \( p + 1 \) into \( \pi' \), one directly to the left and below the \((s_1, t_1)\)-cell, called a lower anchor, and one directly above and to the right of the \((s_1, t_1)\)-cell, called an upper anchor. We perform the same modification on \( \tau' \) to obtain \( \tau^* \). See Figure 3.

Clearly, if \( \tau' \) contains a grid-preserving copy of \( \pi' \), then \( \pi^* \) contains \( \pi^* \). Let us prove the converse. Fix any embedding \( E \) of \( \pi^* \) into \( \tau^* \). Since there is no increasing subsequence of length \( p + 1 \) in \( \tau' \), at least \( p + 2 \) of the points from the anchors in \( \pi^* \) must map to the anchors of \( \tau^* \). In particular, there is a point of the upper anchor in \( \pi^* \) mapped to the upper anchor in \( \tau^* \) and the same holds for the lower anchors. This implies that \( E \) maps the whole copy of the initial block of \( \pi' \) inside \( \pi^* \) to the initial block of \( \tau^* \). We claim that \( E \) in fact maps the whole initial block of \( \pi' \) inside \( \pi^* \) to the initial block of \( \tau' \) inside \( \tau^* \), except perhaps the rightmost point and the leftmost point of the initial block of \( \pi' \) (which belong to the lanes). Suppose for contradiction that the two leftmost points \( q \) and \( r \) of the initial block of \( \pi' \), i.e. the beginning of one of the lanes, map both to the lower anchor. The second block of the lane must lie in the vertical interval between \( q \) and \( r \) either to the right of both of them or to the left of both of them. But for any two points of the lower anchor in \( \tau^* \) there are no such points to map \( q \) and \( r \) to. The same is true for the beginning of the second lane.

From this argument, it actually follows that we can modify the embedding \( E \) so that it maps the leftmost and rightmost point of the initial block of \( \pi' \) to the leftmost and rightmost point of the initial block of \( \tau' \). By the previous arguments, the restriction of \( E \) to \( \pi' \) yields a grid-preserving embedding of \( \pi' \) into \( \tau' \), which concludes the proof.

We have now completed the proof of Theorem 3.1: the first part of the theorem follows by combining Proposition 3.3 with Theorem 2.2, and the second part follows from Corollary 3.6 and Proposition 3.8.

### 4 General grid classes

In this section, we generalize the results of Theorem 3.1 to any gridding matrix whose every entry has bounded grid-width. Note that a ‘bumper-ended’ path is a certain kind of path in \( G_M \) whose definition we provide later.

▶ **Theorem 4.1.** Let \( M \) be a gridding matrix such that every entry of \( M \) has bounded grid-width. Then one of the following holds:

- Either \( G_M \) is a forest that avoids a bumper-ended path, \( \text{Grid}(M) \) has bounded grid-width and \( \text{Grid}(M) \)-Pattern PPM can be decided in polynomial time, or
- \( G_M \) contains a bumper-ended path or a cycle, \( \text{Grid}(M) \) has unbounded grid-width and \( \text{Grid}(M) \)-Pattern PPM is NP-complete.
Unlike monotone grid classes, the gridding matrices of general grid classes may contain entries corresponding to finite nonempty classes. However, we can ignore these entries without affecting the properties we are interested in. To see this, let $\mathcal{M}'$ be the gridding matrix obtained by removing all finite entries from a gridding matrix $\mathcal{M}$. Note that the cell graph of $\mathcal{M}$ is equal to the cell graph of $\mathcal{M}'$. Moreover, the NP-completeness of $\text{Grid}(\mathcal{M}')$-Pattern PPM trivially implies the NP-completeness of $\text{Grid}(\mathcal{M})$-Pattern PPM. Finally, $\text{Grid}(\mathcal{M}')$ has bounded grid-width if and only if $\text{Grid}(\mathcal{M})$ has bounded grid-width since inserting a constant number of points into a permutation increases its grid-width at most by a constant. Thus, if $\mathcal{M}'$ satisfies one of the two options in Theorem 4.1, then $\mathcal{M}$ satisfies this option as well. From now on, we will assume that $\mathcal{M}$ contains only infinite (or empty) entries.

One natural way to generalize the notion of monotone classes is to consider classes that have bounded path-width in left-to-right ordering or in the bottom-to-top ordering. For permutation $\pi$, the horizontal path-width is $\text{pw}^\pi(i)$ where $\sigma_i = i$, and the vertical path-width is $\text{pw}^\pi(\pi)$ where $\sigma_i = \pi_i^{-1}$. The horizontal path-width was introduced independently by Ahal and Rabinovich [1] and Albert et al. [3] in the context of designing permutation pattern matching algorithms. Moreover, there is a connection to the so-called insertion-encodable classes which appear often in the area of permutation classes enumeration, see e.g. [5, 7, 26].

A horizontal monotone juxtaposition is a monotone grid class $\text{Grid}(C \ D)$ where both $C$ and $D$ are non-empty. Similarly, a vertical monotone juxtaposition is a monotone grid class $\text{Grid}(C \ D)$. The following two lemmas are stated for the horizontal path-width, but their symmetric versions hold for the vertical path-width. The next lemma was also proved, in a different form, by Albert et al. [5] in the context of regular insertion encodings.

**Lemma 4.2.** For a permutation class $C$ the following are equivalent:

(a) $C$ has unbounded horizontal path-width,

(b) $C$ contains arbitrarily large horizontal alternations, and

(c) $C$ contains a horizontal monotone juxtaposition as a subclass.

**Proof.** Suppose (a) holds and for every integer $k$ there is a permutation $\pi^k \in C$ such that the horizontal path-width of $\pi^k$ is at least $k$. Then there is $i$ such that the set $\{\pi_1^k, \ldots, \pi_i^k\}$ has intervalicity at least $k$. Each pair of neighboring intervals is separated by $\pi_j^k$ for some $j > i$. Therefore, $\pi^k$ contains a horizontal alternation of size at least $2k - 1$ which proves (b).

On the other hand, a horizontal alternation of size $2k$ must have a horizontal path-width at least $k$ and thus (b) implies (a).

Now suppose that (b) holds. A monotone horizontal alternation is a horizontal alternation whose set of odd entries and set of even entries both form monotone sequences. We claim that every horizontal alternation $\pi$ of size $2k^4$ contains a monotone horizontal alternation of size $2k$. By applying the Erdős–Szekeres theorem [13] on the odd entries of $\pi$ we obtain a horizontal alternation $\pi'$ of size at least $2k^4$ whose odd entries form a monotone sequence. Applying the Erdős–Szekeres theorem again on the even entries of $\pi'$ yields a monotone horizontal alternation $\pi''$ of size at least $2k$. Therefore, $C$ contains arbitrarily large monotone horizontal alternations. There are only four possible types of such alternations depending on the type of the monotone sequences. Therefore, $C$ also contains arbitrarily large alternations belonging to a horizontal monotone juxtaposition $\text{Grid}(D_1 \ D_2)$ for some choice of $D_1$, $D_2 \in \{\mathbb{Z}, \mathbb{S}\}$. Since every $\sigma \in \text{Grid}(D_1 \ D_2)$ is contained in a sufficiently large monotone alternation, in fact $C$ must contain the whole class $\text{Grid}(D_1 \ D_2)$ as a subclass.

On the other hand, if (c) holds then $C$ contains arbitrarily large monotone horizontal alternations which trivially implies (b).
Lemma 4.3. Let $\pi$ be a permutation from a class $\mathcal{C}$ with bounded horizontal path-width and let $S$ be a subset of $\pi$ such that $\text{int}_p(S) = k$. Then $\text{int}_p(S) \leq \alpha k$ where the constant $\alpha$ depends only on $\mathcal{C}$.

Proof. By Lemma 4.2, there exists an $l$ such that $\mathcal{C}$ does not contain any vertical alternation of size $l$. Let $I$ be the interval family of size $k$ such that $\bigcup I = \Pi_p(S)$ and let $I$ be an interval of $I$. Let $S_I$ be the subset of $S$ such that $\Pi_p(S_I) = I$ and let $J$ be the smallest interval family such that $\Pi_p(S_I) = \bigcup J$. We claim that $J$ contains at most $2l - 1$ intervals. For contradiction, suppose that the size of $J$ is at least $2l$. Then between each pair of consecutive intervals of $J$ there is a value $j$ such that $\pi_j^{-1}$ lies outside the interval $I$. There is at least $2l - 1$ gaps between intervals of $J$ and therefore by the pigeon-hole principle either $l$ of them contain a point to the right of $I$ or at least $l$ of the gaps contain a point to the left of $I$. Either way, we obtain a horizontal alternation of size $l$, which is a contradiction.

For each interval $I \in I$, we showed that the intervalicity of $\Pi_p(S_I)$ is at most $2l - 1$ and thus the intervalicity of $\Pi_p(S)$ is at most $k(2l - 1)$.

An ordered pair $(p, q)$ of vertices in $G_M$ is a bumper if either $M_q$ has unbounded horizontal path-width and shares the same row with $M_p$, or if $M_q$ has unbounded vertical path-width and shares the same column with $M_p$. A bumper-ended path is a path $P = p_1, \ldots, p_k$ in $G_M$ such that both $(p_2, p_1)$ and $(p_k, p_{k-1})$ are bumpers.

Lemma 4.4. If $G_M$ contains a bumper-ended path then Grid($\mathcal{M}$) has unbounded grid-width and the problem Grid($\mathcal{M}$)-PATTERN PPM is NP-complete.

Proof. We aim to show that Grid($\mathcal{M}$) must contain a cyclic monotone grid class as its subclass. The proof is illustrated in Figure 4. Consider the bumper-ended path $p_1, p_2, \ldots, p_k$. Let us assume that both $M_{p_1}$ and $M_{p_k}$ have unbounded horizontal path-width as the other cases can be proved in an analogous way. Each of the infinite classes $M_{p_i}$ contains a monotone subclass $\mathcal{C}_i$ due to the Erdős–Szekeres theorem [13]. Moreover, the classes $M_{p_1}$ and $M_{p_k}$ contain a monotone juxtaposition by Lemma 4.2. Let Grid($\mathcal{C}_1 \mathcal{D}_1$) be the juxtaposition contained in $M_{p_1}$ and Grid($\mathcal{C}_k \mathcal{D}_k$) the juxtaposition contained in $M_{p_k}$. We define the monotone gridding matrix $\mathcal{M}'$ by replacing every entry of $\mathcal{M}$ with the following $2 \times 2$ matrix:

- entry $M_{p_i}$ for $i$ between 2 and $k - 1$ is replaced with $\mathcal{C}_i^{\times 2}$
- entry $M_{p_i}$ for $i \in \{1, k\}$ is replaced with $\left( \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right)$, and
- every other entry is replaced with an empty $2 \times 2$ matrix.

Clearly, Grid($\mathcal{M}'$) is still a subclass of Grid($\mathcal{M}$). The $(i, j)$-block of $\mathcal{M}'$ is the $2 \times 2$ submatrix obtained from the $(i, j)$-cell in $\mathcal{M}$. If we forget about the blocks of $p_1$ and $p_k$ we are left with two disjoint copies of the original path. Adding back the blocks connects the endpoints of both paths together and creates a cycle since $p_2$ shares the same column with $p_1$. 

![Figure 4](image-url)
and $p_{k-1}$ shares the same column with $p_k$. Thus, $\text{Grid}(\mathcal{M}')$ is a monotone grid subclass of $\text{Grid}(\mathcal{M})$ whose cell graph contains a cycle and the claim follows from Proposition 3.8. ▶

Proposition 4.5. Let $\mathcal{M}$ be a gridding matrix such that every entry of $\mathcal{M}$ has bounded grid-width, and there are no nonempty finite entries. If $G_\mathcal{M}$ is a forest that avoids a bumper-ended path then $\text{Grid}(\mathcal{M})$ has bounded grid-width and $\text{Grid}(\mathcal{M})$-Pattern PPM can be decided in polynomial time.

Proof. First, suppose that $G_\mathcal{M}$ contains more than one component. In that case, choose a component of $G_\mathcal{M}$, and let $\mathcal{M}_1$ be the submatrix of $\mathcal{M}$ spanned by the rows and columns containing the vertices of the chosen component, while $\mathcal{M}_2$ is the submatrix spanned by the remaining rows and columns of $\mathcal{M}$. An $\mathcal{M}$-gridded permutation $\pi$ can be partitioned into two subpermutations $\pi_1$ and $\pi_2$ where $\pi_1$ is the $\mathcal{M}_1$-gridded subpermutation of $\pi$ consisting of the rows and columns of $\mathcal{M}_1$. Let $T_1$ be the optimal grid tree of $\pi_1$. We define a grid tree $T$ of $\pi$ by taking a root vertex with children $T_1$ and $T_2$. The grid-complexity of any vertex in $T_1$ or $T_2$ has increased at most by $\max(k, \ell)$ where $k$ and $\ell$ are the dimensions of $\mathcal{M}$. Therefore $\text{gw}(\pi) \leq \max(\text{gw}(\pi_1), \text{gw}(\pi_2)) + \max(k, \ell)$. Applying this argument inductively shows that $\text{Grid}(\mathcal{M})$ has bounded grid-width if and only if the grid-width of $\text{Grid}(\mathcal{M}')$ is bounded for every submatrix $\mathcal{M}'$ of $\mathcal{M}$ spanned by a connected component $G_\mathcal{M}$. In the rest of the proof, we assume that $G_\mathcal{M}$ is a tree. The proof is based on a sequence of claims that will be stated and proven independently.

Claim 4.6. The tree $G_\mathcal{M}$ contains a vertex $r$ such that for any other vertex $q \neq r$, the path from $r$ to $q$ does not end in a bumper.

Assume for contradiction that the claim fails. Let $r$ be any vertex of $G_\mathcal{M}$. By assumption, there is a vertex $q \neq r$ such that the path from $r$ to $q$ ends in a bumper $(p, q)$. Choose such a vertex $q$ as far as possible from $r$. Applying our assumption for $q$ in the role of $r$, there is a vertex $q' \neq q$ such that the path from $q$ to $q'$ ends in a bumper $(p', q')$. If the path from $q$ to $q'$ contains the vertex $p$, then it is a bumper-ended path, which is impossible. If the path from $q$ to $q'$ avoids $p$, it means that the path from $r$ to $q'$ ends in the bumper $(p', q')$ and is strictly longer than the path from $r$ to $q$, which contradicts the choice of $q$. This proves Claim 4.6.

Let $r$ be the vertex of $G_\mathcal{M}$ whose existence is guaranteed by Claim 4.6. We now define a rooted tree $T_\mathcal{M}$ on the same vertex set as $G_\mathcal{M}$ as follows (see Figure 5). The vertex $r$ is the root $T_\mathcal{M}$. For a vertex $v \neq r$ in $G_\mathcal{M}$, we set the parent of $v$ in $T_\mathcal{M}$ to be the furthest vertex on the $vr$-path in $G_\mathcal{M}$ that shares the same row or column with $v$. Observe that whenever a vertex $v$ shares the same column with its parent $w$ in $T_\mathcal{M}$ then the entry $\mathcal{M}_v$.
We remark that if \( \pi \) with all its children in \( T_M \) has bounded vertical path-width. The dominant cell of a row, or a column, is the cell \( v \) such that all the other cells in the row, or column, are its children in \( T_M \).

Let \( \pi \) be an \( M \)-gridded permutation, and let \( \pi_v \) denote the subset of points contained in a cell \( v \). Assume that \( \pi_v \) is nonempty whenever \( M_v \) is a nonempty cell of \( M \). We define an auxiliary directed graph \( G_\pi \) on the points of \( \pi \) whose every connected component is a tree rooted in some point of \( \pi_v \). Suppose that the vertex \( v \) of \( T_M \) shares the same column with its parent \( w \). The parent of a point \( p \) in \( \pi_v \) is the nearest point in \( \pi_w \) to the right of \( p \), and if there is no such point \((p \text{ lies to the right of all the points of } \pi_w)\) then the rightmost point in \( \pi_w \). If \( v \) and \( w \) share the same row, then the parent of \( p \) is the nearest point in \( \pi_w \) above \( p \), or if there is no such point \((p \text{ lies above all the points of } \pi_w)\) then the topmost point in \( \pi_w \). Let \( P \) be a subset of the permutation diagram of \( \pi \). The point set \( P \) contains \( P \) and every point that lies in \( \pi \) in a subtree of some point \( p \in P \).

Let \( v \) be a non-empty cell such that \( \pi_v \) contains \( m \) points. The permutation \( \tilde{\pi}_v \) is the standardized version of \( \pi_v \), i.e. the point set inside \([m] \times [m]\) that is isomorphic to \( \pi_v \). The construction of the graph \( G_\pi \) guarantees the following property and its symmetric version.

\begin{itemize}
  \item \textbf{Observation 4.7.} Let \( v \) be the dominant cell of its row. Let \( S \) be a subset of \( \pi_v \), let \( \tilde{S} \) be the corresponding subset of the standardized \( \tilde{\pi}_v \) and let \( S' \) be the set containing \( S \) and all its children in \( \pi \) that lie in the same row. Then \( \text{int}_y(\tilde{S}) = \text{int}_y(S') \). Symmetrically, if \( v \) is dominant in its column, \( S \) and \( \tilde{S} \) are as above, and \( S' \) contains \( S \) and all its children in \( \pi \) in the same column, then \( \text{int}_x(\tilde{S}) = \text{int}_x(S') \).
\end{itemize}

We inductively define a function \( h \) on the vertex set of \( T_M \), which will later serve us as an upper bound for the grid-width of any \( \pi \in \text{Grid}(M) \). For any leaf \( u \) of \( T_M \) we set \( h(u) = 1 \). For any other vertex \( v \), we let \( W \) be the set of children of \( v \) and define \( h(v) = 1 + \sum_{w \in W} \alpha_w h(w) \) where \( \alpha_w \) is the constant obtained as follows. If \( v \) shares a column with \( w \), then \( \alpha_w \) is the constant from Lemma 4.3 applied on the class \( M_w \), otherwise it is the constant from the ‘vertical’ version of Lemma 4.3 applied on the class \( M_w \). We state only one of the symmetric versions of the following two claims. However, we are proving both of them simultaneously by induction.

\begin{itemize}
  \item \textbf{Claim 4.8.} Let \( S \) be a subset of the \( i \)-th column of the \( M \)-gridding of \( \pi \) such that \( \text{int}_y(S) = 1 \). Let \( v \) be the dominant cell of the \( i \)-th column and let \( S_v = S \cap \pi_v \). Then \( \text{int}_z((S \setminus S_v) \cup S_v) \leq h(v) \) and \( \text{int}_y(S \setminus S_v) \leq h(v) - 1 \).
\end{itemize}

We remark that if \( v \) is not equal to the root \( r \), then the set \( (S \setminus S_v) \cup S_v \) from the claim above is actually equal to \( S_v \).

To prove Claim 4.8, suppose first that \( v \) is the only nonempty cell in its column, and therefore \( v \) is a leaf of \( T_M \). Then \( S = S_v \), and hence \( \text{int}_y((S \setminus S_v) \cup S_v) = \text{int}_y(S) = 1 = h(v) \) and \( \text{int}_y(S \setminus S_v) = \text{int}_y(0) = 0 = h(v) - 1 \), as claimed.

Now suppose that \( v \) is not the only nonempty cell in its column, and let \( C \) be the set of nonempty cells different from \( v \) in the same column as \( v \). Note that each cell in \( C \) is a child of \( v \) in \( T_M \), and if \( v \neq r \), then \( C \) is precisely the set of children of \( v \). Observe that \( S \setminus S_v \) is a disjoint union of the sets \( S_w \) over all \( w \in C \). For a cell \( w \in C \), let \( S_w \) be the set \( S \cap \pi_v \), let \( \tilde{\pi}_w \) be the standardization of \( \pi_w \), and let \( \tilde{S}_w \) be the subset of \( \tilde{\pi}_w \) that corresponds to \( S_w \). From Lemma 4.3 we get \( \text{int}_y(\tilde{S}_w) \leq \alpha_w \), and in particular, \( \tilde{S}_w \) can be partitioned into sets \( \tilde{S}^1_w, \tilde{S}_w^2, \ldots, \tilde{S}_w^\ell_w \) for some \( \ell_w \leq \alpha_w \), where \( \text{int}_y(\tilde{S}_w^i) = 1 \) for each \( i \). Let \( S'_w \) be the subset of \( \pi_w \) that corresponds to \( \tilde{S}_w^i \) in the standardization \( \tilde{\pi}_w \). Let \( R_w^j \) be the set \( S'_w \) together with all its children in \( G_\pi \) that lie in the same row. By Observation 4.7, for each \( i \) we have
int_y(R_w) = 1. Using the symmetric version of Claim 4.8 with each $R_w$ in the role of $S$ shows that

$$\int_y(S \setminus S_v) \leq \sum_{w \in C} \int_y(S_w)$$

$$\leq \sum_{w \in C} \sum_{i=1}^{\ell_w} \int_y(S_w)$$

$$= \sum_{w \in C} \sum_{i=1}^{\ell_w} \int_y((R_w \setminus S_w) \cup S_w)$$

$$\leq \sum_{w \in C} \alpha_w h(w)$$

$$\leq h(v) - 1,$$

and similarly,

$$\int_x((S \setminus S_v) \cup S_v) = \int_x\left(S \cup \bigcup_{w \in C} S_w\right)$$

$$= \int_x\left(S \cup \bigcup_{w \in C} \bigcup_{i=1}^{\ell_w} R_w \setminus S_w\right)$$

$$\leq 1 + \sum_{w \in C} \sum_{i=1}^{\ell_w} \int_x(R_w \setminus S_w)$$

$$\leq 1 + \sum_{w \in C} \alpha_w (h(w) - 1)$$

$$\leq h(v),$$

proving Claim 4.8.

We will now define, for every $p \in \pi$, a grid tree $T_p$ whose leaves are exactly the points in $[p]$. The definition proceeds inductively on the size of $[p]$. If $p$ has no children in $G_\pi$, that is if $[p] = \{p\}$, then $T_p$ consists of the single vertex $p$. Suppose now that $p$ has at least one child in $G_\pi$. Recall that each child of $p$ belongs to a cell in the gridding of $\pi$ which is in the same row or in the same column as the cell of $p$. Let $C$ and $R$ denote, respectively, the set of children of $p$ in the same column and the set of children of $p$ in the same row. Note that $C$ and $R$ are disjoint, and if $p$ does not belong to the root cell $\pi_r$ then one of $C$ and $R$ is empty.

Recall that a caterpillar is a binary tree whose every internal node has at least one leaf child. Note that the leaves of a caterpillar can be ordered top to bottom by their distance from the root, where the order of the bottommost pair of leaves is irrelevant.

If $C$ is nonempty, we construct a tree $T_p^C$ in the following two steps:

- Construct a caterpillar whose leaves are the points from $C \cup \{p\}$, and the top-to-bottom order of the leaves in the caterpillar coincides with the left-to-right order of the points in $\pi$.

- In the caterpillar constructed above, for each $q \in C$ replace the leaf $q$ with a copy of the tree $T_q$. Call the resulting tree $T_p^C$.

Symmetrically, if $R$ is nonempty, construct a tree $T_p^R$ by first taking the caterpillar whose
leaves in top-to-bottom order are the points of \( R \cup \{ p \} \) in top-to-bottom order, and then for each \( q \in R \) replace the leaf \( q \) with a copy of \( T_q \).

If the set \( R \) is empty, we define \( T_p = T_p^C \), and if \( C \) is empty, we define \( T_p = T_p^R \). If both \( C \) and \( R \) are nonempty (which may only happen when \( p \) is in \( \pi_v \)), we let \( T_p \) be the tree obtained by replacing the leaf \( p \) in \( T_p^C \) by a copy of \( T_p^R \). Note that in all the cases, the leaves of \( T_p \) form precisely the set \( \{ p \} \).

**Claim 4.9.** Let \( v \) be a nonempty cell of \( \mathcal{M} \). If \( v \neq r \), then for every \( p \in \pi_v \), the tree \( T_p \) has grid-width at most \( h(v) \). For every \( p \in r \), the tree \( T_p \) has grid-width at most \( 2h(r) \).

We prove the claim by induction on the size of \( T_p \). The claim clearly holds when \( T_p \) is the single vertex \( p \). Suppose now that \( T_p \) has more vertices, and that \( v \neq r \). In such case \( T_p \) is equal to \( T_p^C \) or to \( T_p^R \). Suppose that \( T_p = T_p^C \), the other case being symmetric. Let \( C \) be again the set of children of \( p \) in \( G_\pi \) (necessarily, they are all in the same column of the gridding as the point \( p \), since \( v \) is not the root vertex). Let \( u \) be a node of \( T_p \), and let \( L_u \) be the set of leaves of the subtree of \( T_p \) rooted at \( u \). Our goal is to show that the grid complexity of \( L_u \) is at most \( h(v) \). If \( u \) is the leaf \( p \) or \( u \) is inside a copy of \( T_q \) for some \( q \in C \), the claim follows by induction. Suppose that \( u \) is a node of the caterpillar from which \( T_p \) was constructed. Let \( S \) be the set of points in \( L_u \) that are in the same gridding column as \( p \).

By the construction of \( T_p \), the set \( S \) satisfies \( \text{int}_x(S) = 1 \), and \( L_u \) is equal to \( \overline{S} \). Note that the set \( S_u = S \cap \pi_u \) is either empty or contains the single point \( p \). By Claim 4.8,

\[
\text{int}_x(L_u) = \text{int}_x(\overline{S}) = \text{int}_x((\overline{S} \setminus S_u) \cup S_u) \leq h(v)
\]

and

\[
\text{int}_y(L_u) = \text{int}_y(S_u) + \text{int}_y(\overline{S} \setminus S_u) \leq \text{int}_y(\{p\}) + \text{int}_y(\overline{S} \setminus S_u) \leq 1 + (h(v) - 1) = h(v).
\]

This shows that \( L_u \) has grid complexity at most \( h(v) \), and therefore \( T_p \) has grid-width at most \( h(v) \).

It remains to deal with the case when \( p \) belongs to the root cell \( r \). Using the same argument as in the first part of the proof, we again see that both \( T_p^C \) and \( T_p^R \) have grid-width at most \( h(r) \). Moreover, for each node \( u \) of \( T_p \) the subtree of \( T_p \) rooted at \( u \) is either equal to a subtree of \( T_p^C \), or it is equal to a subtree of \( T_p^R \), or it contains the entire tree \( T_p^R \) together with a subtree of \( T_p^C \). In the former two cases, the set of leaves of the subtree has grid complexity at most \( h(r) \), in the last case it has grid complexity at most \( 2h(r) \). This proves Claim 4.9.

We are ready to construct a grid tree \( T \) of the permutation \( \pi \) and provide a bound on its grid-width. By assumption, the entries of \( \mathcal{M} \) have bounded grid-width, and we let \( g \) be the grid-width of the root entry \( M_r \). Let \( \pi^*_r \) be the standardization of \( \pi_r \), and let \( T_r \) be the optimum grid tree of \( \pi^*_r \); in particular, \( T_r \) has grid-width at most \( g \). A grid tree \( T \) of the whole permutation \( \pi \) is obtained by taking \( T_r \) and replacing the leaf corresponding to a point \( p \in \pi \), with the tree \( T_p \). We claim that \( T \) has grid-width at most \( 4gh(r) \). The tree \( T \) contains every point of \( \pi \), and we showed in Claim 4.9 that the grid-width of any node contained in a copy of some \( T_p \) is at most \( 2h(r) \).

Let now \( u \) be a node of \( T \) that is not contained in any copy of the tree \( T_p \), in other words, \( u \) is an internal node of \( T_r \). Let \( L^* \subseteq \pi^*_r \) be the set of leaves of \( T_r \) in the subtree rooted at \( u \), and let \( L \) be the subset of \( \pi_r \) that is mapped to \( L^* \) by the standardization that maps \( \pi_r \) to \( \pi^*_r \). Then the subset of \( \pi \) contained in the subtree of \( T \) rooted at \( u \) is precisely \( L \).

Applying Observation 4.7, we see that \( L \) together with its neighbors in \( G_r \) spans at most \( g \) consecutive intervals in the row and column of the \( r \)-cell. By applying Claim 4.8
individually on each of these $2g$ intervals, we get that the grid-complexity of $\bar{L}$ is at most $4gh(r)$. It follows that $\text{Grid}(\mathcal{M})$ has bounded grid-width, and therefore $\text{Grid}(\mathcal{M})$-PPM can be decided in polynomial time.

To complete the proof of Theorem 4.1, it suffices to point out that if $G_{\mathcal{M}}$ contains a cycle then $\text{Grid}(\mathcal{M})$ contains a monotone grid subclass $\text{Grid}(\mathcal{M}')$ where $\mathcal{M}'$ is obtained by replacing every infinite class in $\mathcal{M}$ by its monotone subclass. Applying Proposition 3.8 then wraps up the proof of Theorem 4.1.

5 Concluding remarks and open problems

The $\mathcal{C}$-PPM is the problem of determining whether a permutation $\pi \in \mathcal{C}$ is contained in a permutation $\tau \in \mathcal{C}$. Even though $\text{Av}(321)$-PPM is NP-complete, $\text{Av}(321)$-PPM can be decided in polynomial time [16]. This leads to the natural question of whether the same can happen in the universe of grid classes.

► **Open problem 1.** Is there any (monotone) gridding matrix $\mathcal{M}$ such that $\text{Grid}(\mathcal{M})$-PPM is NP-complete, while $\text{Grid}(\mathcal{M})$-PPM can be decided in polynomial time?

A path class of order $k$ is a monotone grid class whose cell graph is a path on $k$ vertices. Let us say that a permutation class $\mathcal{C}$ contains paths of all orders, if for every $k$, the class $\mathcal{C}$ contains as a subclass a path class of order $k$. Note that by Lemma 3.4, such a class $\mathcal{C}$ has unbounded grid-width. In fact, all the known examples of classes with unbounded grid-width contain paths of all orders. We may therefore ask whether this property precisely characterizes the classes with unbounded grid-width.

► **Open problem 2.** Does every class with unbounded grid-width contain paths of all orders?

The existence of paths of all orders may also help with establishing the NP-completeness of $\mathcal{C}$-PPM. Suppose that $\mathcal{C}$ is a class that contains paths of all orders. It is not hard to argue that such a class $\mathcal{C}$ necessarily contains, for every $k$, a monotone grid subclass whose grid graph is a properly turning path on $k$ vertices. If we additionally assume that for a given integer $k$ we are able to construct, in time polynomial in $k$, such a properly turning path class of order $k$ contained in $\mathcal{C}$, then we can adapt the hardness reduction from the proof of Proposition 3.8 to show that $\mathcal{C}$-PPM is NP-complete.

The results of Ahal and Rabinovich [1] imply that $\mathcal{C}$-PPM is polynomial whenever $\mathcal{C}$ has bounded grid-width. On the other hand, in all the known examples of a class $\mathcal{C}$ with unbounded grid-width where the complexity of $\mathcal{C}$-PPM is known, the $\mathcal{C}$-PPM problem is NP-complete. We wonder whether bounded grid-width might be the property characterizing the complexity of $\mathcal{C}$-PPM.

► **Open problem 3.** Is it true that $\mathcal{C}$-PPM is NP-complete whenever $\mathcal{C}$ has unbounded grid-width, and polynomial otherwise?

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