Cones over metric measure spaces and the maximal diameter theorem

Christian Ketterer\textsuperscript{a}

\textsuperscript{a}Institute of Applied Mathematics, Endenicher Allee 60, D-53115 Bonn

Abstract

The main result of this article states that the \((K,N)\)-cone over some metric measure space satisfies the reduced Riemannian curvature-dimension condition \(RCD^*(K,N+1)\) if and only if the underlying space satisfies \(RCD^*(N-1,N)\). The proof uses a characterization of reduced Riemannian curvature-dimension bounds by Bochner’s inequality that was established for general metric measure spaces by Erbar, Kuwada and Sturm in \cite{21} (independently, the same result has been announced by Ambrosio, Mondino and Savaré). As a corollary of our result and the Gigli-Cheeger-Gromoll splitting theorem \cite{25} we obtain a maximal diameter theorem in the context of metric measure spaces that satisfy the condition \(RCD^*\).

Keywords: curvature-dimension condition, metric measure space, cone, maximal diameter

Contents

1 Introduction 2

2 Preliminaries on Dirichlet forms 6

2.1 Dirichlet forms and their \(\Gamma\)-operator 6
2.2 The Bakry-Emery curvature-dimension condition 8
2.3 Some examples of Dirichlet forms 9

3 Skew products and the Bakry-Emery curvature-dimension condition 11

3.1 Skew and \(N\)-skew products between Dirichlet forms 11
3.2 Proof of classical \(\Gamma_2\)-estimates for \(N\)-skew products 13
3.3 A result on essentially self-adjoint operators 17
3.4 Intermezzo 21
3.5 Proof of the Bakry-Emery condition for \((K,N)\)-cones 23

4 Preliminaries on the calculus for metric measure spaces 27

4.1 The curvature-dimension condition 27
4.2 First order calculus for metric measure spaces 28
4.3 The Riemannian curvature-dimension condition 30

5 \((K,N)\)-cones and the Riemannian curvature dimension condition 32

5.1 Warped products and \((K,N)\)-cones for metric measure spaces 32
5.2 On the relation between metric cones and cones in the sense of Dirichlet forms 36
5.3 Proof of the main theorem 39
5.4 Proof of the maximal diameter theorem 41

Email address: ketterer@iam.uni-bonn.de (Christian Ketterer)

Preprint submitted to Elsevier October 10, 2014
1. Introduction

The euclidean cone over a metric space is an important construction in the theory of Riemannian manifolds and metric spaces. Cones appear naturally in numerous situations, e.g., as model spaces or as tangent cones. A special feature is the simplicity of their definition. The distance function of the euclidean cone over a metric space \((F,d_F)\) is induced by the following pseudo-metric:

\[
d_{\text{Con}}((r,x),(s,y)) = \sqrt{r^2 + s^2 - 2rs \cos(d_F(x,y) \wedge \pi)}
\]

for \((r,x)\) and \((s,y)\) in \([0, \infty) \times F\). One of the fundamental results in the context of Alexandrov spaces with curvature bounded from below is that the euclidean cone has a curvature bound equal to 0 if and only if the underlying space has a curvature bound equal to 1. This is a generalization of the observation that the cone over a circle of radius \(\leq 1\) is flat away from the origin. If one weakens the notion of curvature and considers the Ricci curvature of a Riemannian manifold, one could observe the following. A simple computation for the curvature tensors shows that away from the origin, where the cone is a smooth Riemannian manifold in the classical sense, the Ricci tensor is pointwise bounded from below by 0 if and only if the Ricci tensor of the underlying manifold \(F\) is pointwise bounded from below by \(\dim F - 1\). We want to investigate if there are generalizations of this result in a non-smooth context.

Lott/Villani \cite{Lott-Villani} and Sturm \cite{Sturm1, Sturm2} introduced a synthetic notion of lower Ricci curvature bounds for metric measure spaces and initiated a program to study the analytic and geometric properties of these spaces. The idea is to define curvature in terms of convexity of entropy functionals on the \(L^2\)-Wasserstein space of absolutely continuous probability measures. Their approach yields the so-called curvature-dimension condition \(CD(\kappa,N)\) for metric measure spaces. Meanwhile a huge number of results concerning functional inequalities, eigenvalue estimates, regularity and stability properties have been established for \(CD\)-spaces. However, this class is still rather big since it includes also Finsler geometries. The curvature-dimension condition does not imply a linear heat semi group of the corresponding Cheeger energy. For this reason, a refined version of the curvature-dimension condition was introduced by Ambrosio, Gigli and Savaré in \cite{Ambrosio-Gigli-Savare}. They propose so-called Riemannian Ricci curvature bounds that make the linearity of the heat flow part of the definition. Surprisingly, this condition is again equivalent to another unified notion, the so-called evolution variational inequality that holds for gradient flow curves in the \(L^2\)-Wasserstein space. Later, this idea was extended to finite dimensions by Erbar, Kuwada and Sturm.

The main purpose of this article is to generalize the results concerning cones over Riemannian manifolds to arbitrary metric measure spaces that satisfy a reduced Riemannian curvature-dimension condition. We do not restrict ourselves to euclidean cones but consider also spherical and elliptic cones that can be defined by the unified notion of \(K\)-cones for \(K \in \mathbb{R}\) (see Definition \[5.1\]). We will prove our result in the setting of metric measures spaces. Therefore, we also introduce suitable measures. For the euclidean cone a measure is defined by \(d\mu^\kappa(r,x) = r^\kappa \, dr \otimes d\mu_F(x)\), where \(\mu_F\) is the reference measure of the underlying metric measure space. It mimics the Riemannian volume of a cone. The additional parameter \(N \geq 0\) corresponds to a dimension bound for \(F\). In this way, we obtain the notion of \((K,N)\)-cone. Our main results are

**Theorem 1.1.** Let \((F,d_F,m_F)\) be a metric measure space that satisfies \(RCD^*\) for \(N-1,N\) for \(N \geq 1\) and \(\text{diam } F \leq \pi\). Let \(K \geq 0\). Then the \((K,N)\)-cone \(\text{Con}_K^N(F)\) satisfies \(RCD^*\) for \(K,N \geq 1\).

**Theorem 1.2.** Let \((F,d_F,m_F)\) be a metric measure space. Suppose the \((K,N)\)-cone \(\text{Con}_K^N(F)\) satisfies \(RCD^*\) for \(K,N \geq 1\). Then

1. if \(N \geq 1\), \(F\) satisfies \(RCD^*(N-1,N)\) and \(\text{diam } F \leq \pi\),
2. if \(N \in [0,1)\), \(F\) is a point, or, \(N = 0\) and \(F\) consists of exactly two points with distance \(\pi\).

**Corollary 1.3.** Let \((F,d_F,m_F)\) be a metric measure space. \(K \geq 0\) and \(N \geq 1\). Then \(\text{Con}_K^N(F)\) satisfies \(RCD^*\) for \(N-1,N\) if and only if \(F\) satisfies \(RCD^*\) for \(N-1,N\) and \(\text{diam } F \leq \pi\).
The maximal diameter theorem. A corollary of our main theorems and the recently established Gigli-Cheeger-Gromoll splitting theorem in the context of $RCD(0,N)$-spaces is the following maximal diameter theorem for $RCD^*$-spaces.

**Theorem 1.4.** Let $(F, d_F, m_F)$ be a metric measure space that satisfies $RCD^*(N, N+1)$ for $N \geq 0$. If $N = 0$, we assume that $\text{diam} F \leq \pi$. Let $x, y$ be points in $F$ such that $d_F(x, y) = \pi$. Then, there exists a metric measure space $(F', d_{F'}, m_{F'})$ such that $(F, d_F, m_F)$ is isomorphic to $[0, \pi] \times_{\sin} F'$ and

1. if $N \geq 1$, $(F', d_{F'}, m_{F'})$ satisfies $RCD^*(N-1, N)$ and $\text{diam} F' \leq \pi$,

2. if $N \in [0, 1)$, $F'$ is a point, or $N = 0$ and $F$ consists of exactly two points with distance $\pi$.

We remark that a metric measure space that satisfies $RCD^*(N-1, N)$ always has bounded diameter by $\pi$ (see Theorem 1.3 and [47, 56, 21]). In the context of Riemannian manifolds Theorem 1.4 was proven by Cheng. It states that an $n$-dimensional Riemannian manifold that has Ricci curvature bounded from below by $n-1$ and attains its maximal diameter, is the standard sphere $S^n$. We remark that a spherical suspension that is a smooth Riemannian manifold without boundary is a sphere. A result of Anderson [5] shows that for any even dimension $n \geq 4$ and any $\epsilon > 0$ one can find a Riemannian manifold $M'_\epsilon$ that satisfies a Ricci bound of $n-1$ and contains points $x, y \in M'_\epsilon$ such that $d_{M'_\epsilon}(x, y) = \pi - \epsilon = \text{diam} M'_\epsilon$ but which is not homeomorphic to a sphere. In [19] Cheeger and Colding prove that any $n$-dimensional Riemannian manifold with Ricci curvature bounded from below by $n-1$ and almost maximal diameter is close in the Gromov-Hausdorff distance to a spherical suspension over some geodesic metric space. Especially, Cheeger-Colding obtain the following result for Ricci limit spaces.

**Theorem 1.5 (Cheeger-Colding).** If a metric space $(X, d_X)$ is the Gromov-Hausdorff limit of a sequence of $n$-dimensional Riemannian manifolds $M_i$ with $\text{ric}_{M_i} \geq n - 1$ and there are points $x, y \in X$ such that $d_X(x, y) = \pi$, then there exists a length space $(Y, d_Y)$ with $\text{diam} Y \leq \pi$ such that $[0, \pi] \times_{\sin} Y = X$.

In particular, our result is sharp, improves the theorem of Cheeger and Colding by giving additional information on the underlying space $Y$, and provides an alternative proof for Cheng’s theorem. As corollary of Theorem 1.4 we also obtain:

**Corollary 1.6.** Let $(F, d_F, m_F)$ be a metric measure space that satisfies $RCD^*(N-1, N)$ for $N \geq 0$. Assume there are points $x_i, y_i \in F$ for $i = 1, \ldots, n$ with $n > N$ such that $d_F(x_i, y_i) = \pi$ for any $i$ and $d_F(x_i, x_j) = 2$ for $i \neq j$. Then $N = n - 1 \in \mathbb{N}$ and $(F, d_F, m_F) = S^N$.

**Outline of the proof.** Let us briefly sketch the main ideas for the proof of Theorem 1.4. A first step in direction of the result was done by Bacher and Sturm in [11]. They prove Theorem 1.4 when the underlying metric measure space is a smooth Riemannian manifold. Later, the author adapted their ideas in [32] to extend the result to warped products that is a generalization of the concept of metric cone. In some sense the proof in both cases follows the Lagrangian interpretation of curvature-dimension bounds that comes from the theory of optimal transport. One establishes the convexity of the entropy functional along Wasserstein geodesics from bounds for the Ricci tensor. The main problem is to deal with the set of singularity points (in the case of a cone this is just the origin) where the underlying space differs from an ordinary euclidean product. It turns out that the curvature-dimension bound for $F$ guarantees that the optimal transport of absolutely continuous measures does not not see these singularities and consequently, they do not affect the convexity of the entropy. Now, one is tempted to prove the theorem for general metric measure spaces with the same strategy by deducing the convexity of the entropy directly from the convexity of the entropy of the underlying space $F$. The statement that singularities can be neglected holds as well in the general framework. However, as simple the definition of the cone metric might be, the relation of
optimal transport in the cone and optimal transport in the underlying space is rather complicated as can be seen from easy examples. Hence, we need to follow another strategy.

By the work of Ambrosio, Gigli and Savaré \cite{1} we know that spaces that satisfy a Riemannian curvature-dimension condition are directly linked to the theory of Dirichlet forms. Especially, it is the right setting to prove a version of Bochner’s inequality. In the context of a Riemannian manifolds $M$ this is

$$\Gamma_2(u) = \frac{1}{2} \Delta u ||\nabla u||^2 - \langle \nabla u, \nabla \Delta u \rangle \geq \kappa ||\nabla u||^2 + \frac{1}{\kappa} (\Delta u)^2 \quad (1)$$

for $u \in C^\infty(M)$ if the Ricci tensor of $M$ is bounded from below by $\kappa \in \mathbb{R}$ and the dimension is bounded from above by $N \in [1, \infty)$. \cite{1} also characterizes a Ricci curvature and dimension bound of $M$. Recently, \cite{1} was also established for general metric measure spaces in a series of publications by several authors. In a first step Gigli, Kuwada and Ohta proved the Bochner inequality with $N = \infty$ on finite dimensional Alexandrov spaces with curvature bounded from below \cite{2}. Then, Ambrosio, Gigli and Savaré \cite{3} independently announced by Ambrosio, Mondino and Savaré \cite{3} solved the problem for $RCD^+(\kappa, N)$-spaces with finite $N$. Even more, they prove the full equivalence between the reduced Riemannian curvature-dimension condition ($RCD^+(\kappa, N)$) and \cite{1} for any parameters $\kappa \in \mathbb{R}$ and $N \in [1, \infty)$.

Bochner’s inequality captures the Eulerian picture of curvature-dimension bounds. This viewpoint was already used by Bakry, Emery and Ledoux in the setting of diffusion semigroups. They established a so-called Bakry-Emery curvature-dimension condition that exactly mimics inequality \cite{1}. They were able to deduce many results from Riemannian geometry like Sobolev inequalities, Li-Yau gradient estimates or Bonnet-Myers Theorem in a purely abstract setting only relying on \cite{1}. By the work of Ambrosio, Gigli and Savaré \cite{6} we know that spaces that satisfy a Riemannian curvature-dimension condition are directly linked to the theory of Dirichlet forms. Especially, it can be seen from easy examples. Hence, we need to follow another strategy.

Why is this a reasonable strategy? We remind the reader of the situation of smooth Riemannian manifolds. Let us consider general warped products. $(B^d, g_n)$ and $(F^n, g_F)$ are Riemannian manifolds with Ricci curvature bounded from below by $(d - 1)K$ and $(n - 1)K_F$ respectively and $f : B \to (0, \infty)$ is a smooth function such that

$$\nabla^2 f(v, v) \leq -K|v|^2_n \quad \text{for } v \in TB \quad \text{and} \quad |\nabla f|^2_n \leq K_F - K f^2 \quad \text{on } B. \quad (2)$$

We can define the Riemannian warped product $B \times_f F = (B \times F, g_{B \times_f F})$ where $g_{B \times_f F} = \pi_n^* g_n + (\pi_F^* f)^2 \pi_F^* g_F$ is a Riemannian metric on $B \times F$. For example, in the case of the euclidean cone we would choose $B = (0, \infty)$ and $f(r) = r$ and $K_F = 1$ and $K = 0$. The Ricci tensor can be calculated explicitly at any point $(p, x) \in B \times F$ and is given by

$$\text{ric}_{B \times_f F}(\tilde{X}(p, x) + \tilde{V}(p, x)) = \text{ric}_{B}(X_p) - n \frac{\nabla f(X_p)}{f(p)} \nabla f(X_p) + \text{ric}_{F}(V_p) - \left( \frac{\Delta f(p)}{f(p)} + (N - 1) \frac{\nabla f|^2}{f(p)} \right) |\tilde{V}(p, x)|^2 \quad (3)$$

where $\tilde{X}$ and $\tilde{V}$ are horizontal and vertical lifts on $B \times_f F$ of vector fields $X$ and $V$ on $B$ and $F$ respectively. For the precise definitions we refer to \cite{39}. From this formula one can easily deduce our main theorem by applying the curvature properties of $B$ and $F$ and the assumption \cite{2} for $f$. Since $f > 0$, we can forget the issue of singularities for a moment. We can see that calculations can be done pointwise and almost only take place on $B$. Now, we interpret Bochner’s inequality \cite{1} as the functional analog of the corresponding bound for the Ricci tensor. Then, we establish a similar formula as \cite{1} by just using the smooth structure of $B$. In section 3.2 we obtain the following pointwise inequality

$$\Gamma_2^{\times_f f}(u)(p, x) \geq \Gamma_2^{\times_f f}(u^x)(p) + \frac{1}{f(p)} \Delta u^x \frac{(\nabla f u^x)^2}{f(p)} + \frac{1}{f(p)} \Gamma_2^{\times_f f}(u^f)(x) - \left( \frac{\Delta f(p)}{f(p)} + (N - 1) \frac{\nabla f|^2}{f(p)} \right) \frac{1}{f(p)} |\nabla u^f|^2 \text{ m.c. -a.e.} \quad (4)$$
for any \( u \in C^\infty_0(\hat{B}) \otimes \mathcal{A}^r \) where \( \mathcal{A}^r \) is a algebra of functions on \( F \) in the sense of Bakry-Emery calculus where Bochner’s inequality holds with parameter \((N-1)K_r\) and \( N \). Now, if we use again the curvature and concavity conditions for \( B, F \) and \( f \) we obtain a sharp Bochner inequality for \( u \in C^\infty_0(\hat{B}) \otimes \mathcal{A}^r \). This result holds in general for any warped product (Theorem 3.9). At this point, we have proven a sharp \( \Gamma_2 \)-estimate for any \( N \)-warped product provided \( B, F \) and \( f \) satisfy the assumptions of Theorem 3.9. But we do not know if \( C^\infty_0(\hat{B}) \otimes \mathcal{A}^r \) is a dense subset in \( D_\alpha(L^C) \). Indeed, in general, it is false that \( \mathcal{E}^C \) satisfies a curvature-dimension condition even when a \( \Gamma_2 \)-estimate holds on a big class of functions. For example, consider \( F = S^1_2 \) that is a 1-dimensional sphere of diameter \( 2\pi \) and consider the corresponding 1-cone \([0, \infty) \times_s S^1 \). In [11] Bacher and Sturm prove that this cone does not satisfy any curvature-dimension condition. This can be seen from the behavior of optimal transport since the cone in the case of a big circle is a kind of covering and if mass is transported from on sheet to another, the "cheapest" way to do it is to transport all mass through the origin which destroys any convexity of the entropy. Bacher and Sturm observed that this situation can be avoided if and only if the diameter of the underlying space is smaller than \( \pi \). But on the other side, from our result one can see that the \( \Gamma_2 \) estimate holds without any restriction.

Another observation is related to this problem. It is known (see [41], Appendix to Section X.I, Example 4) that the Laplace operator that acts on smooth functions with compact support in \( \mathbb{R}^{N+1} \setminus \{0\} \) is essentially self-adjoint if and only if \( N \geq 3 \) where \( N \in \mathbb{N} \). But this situation exactly corresponds to the case of an euclidean cone over \( S^N \) with admissible algebra \( \mathcal{A}^r = C^\infty(S^N) \). So in this case in general the operator \( L^C \) restricted to \( C^\infty_0((0, \infty)) \otimes C^\infty(S^N) \) will provide more than one self-adjoint extension and the Friedrich’s extension does not need to coincide with the closure of \( C^\infty_0((0, \infty)) \otimes C^\infty(S^N) \) with respect to the graph norm. So, we cannot hope that \( C^\infty_0(\hat{B}) \otimes \mathcal{A}^r \) will be dense in the domain of \( L^C \) in general. But we will see that in the Eulerian picture that is described by the \( \Gamma_2 \)-estimate, the crucial quantity is not the diameter but the first positive eigenvalue of \( L^r \). For metric measure spaces that satisfy \( RCD^*(N-1, N) \) there is a spectral gap \( \lambda_1 \geq \lambda \). This fact together with results from the theory of 1-dimensional essentially self-adjoint operators allows to prove the density of an admissible class of function in the domain of \( L^C \) in the case of \((K, N)\)-cones. Additionally, we obtain a complete picture about how the spectral gap of \( L^r \) enters the proof, and this should be seen in comparison to the Lagrangian viewpoint of Bacher and Sturm.

Hence, we can establish a Bakry-Emery condition for cones. Then, we can use again the equivalence with the \( RCD^*(N-1, N) \) condition to prove Theorem 1.1. The final technical problem at this point is to prove that the intrinsic distance of cones in the sense of Dirichlet forms (see section 3.1) is the corresponding cone metric over the space we started with.

\textit{Plan of the paper.} The paper is roughly divided into two parts. In section 2 and 3 we just consider strongly local and regular Dirichlet forms. Section 2 is an introduction into all necessary notions and definitions concerning this subject. We briefly define symmetric Dirichlet forms in section 2.1 and introduce relevant properties. In section 2.2 we introduce the Bakry-Emery curvature dimension condition in the classical sense and in the sense that was proposed by Ambrosio, Gigli and Savaré in [3]. In section 2.3 we give important smooth examples of Dirichlet forms that satisfy the Bakry-Emery curvature dimension condition and that will be use later. Section 3 is concerned with the proof of the sharp Bakry-Emery condition for cones over Dirichlet forms. In section 3.1, we first introduce skew products that is a well-known construction recipe for Dirichlet forms which was introduced by Fukushima and Oshima. We modify this notion slightly and obtain so-called warped products and cones. In section 3.2 we prove sharp \( \Gamma_2 \)-estimates for warped products on a suitable class of function. In section 3.3 we prove that in the special case of cones this class is dense in the domain of the corresponding self-adjoint operator. Finally, we obtain the full Bakry-Emery curvature-dimension condition for cones in section 3.5.

Section 4 and 5 constitute the second part of the article that is concerned with metric measure spaces that satisfy a Riemannian curvature-dimension condition in the sense of Lott-Villani-Sturm-
Ambrosio-Gigli-Savaré. In section 4 we give a rough overview on classical results and recent developments in the field. We introduce the curvature-dimension condition as proposed by Sturm in section 4.1. In section 4.2 we present important results on Poincaré and Sobolev inequalities on metric measure spaces that are mainly due to Hajlasz and Koskela \[31\] and the first order calculus for metric measure spaces that was developed by Ambrosio, Gigli and Savaré. This allows us to give the definition of Riemannian curvature bounds in section 4.3. Finally, in section 5 we prove our main theorems. In section 5.1 we repeat the notion of metric warped product and cones and we show in section 5.2 that it is consistent with the notion that was developed for Dirichlet forms, at least, if we assume curvature-dimension bounds. In section 5.3 we prove the cone theorem and in section 5.4 we prove the maximal diameter theorem.

Remark 1.7. This article differs slightly from a previous version with the same name including more detailed proofs.

2. Preliminaries on Dirichlet forms

2.1. Dirichlet forms and their \(\Gamma\)-operator

We consider a locally compact and separable Hausdorff space \((X, \mathcal{O}_X)\) and a positive, \(\sigma\)-finite Radon measure \(m_X\) on \(X\) such that \(\text{supp}[m_X] = X\). We denote by \(L^p(X, m_X) \coloneqq L^p(m_X)\) for \(p \in [0, \infty]\) the Lebesgue spaces with respect \(m_X\). Let \((\mathcal{E}^X, D(\mathcal{E}^X))\) be a symmetric Dirichlet form on \(L^2(m_X)\) where \(D(\mathcal{E}^X)\) is a dense subset of \(L^2(m_X)\). A symmetric Dirichlet form is a \(L^2(X, m_X)\)-lower semi-continuous, quadratic form that satisfies the Markov property. Dirichlet forms are closed, i.e. the domain \(D(\mathcal{E}^X)\) is a Hilbert space with respect to the energy norm that comes from the inner product

\[
(u, u)_{D(\mathcal{E}^X)} = (u, u)_{L^2(m_X)} + \mathcal{E}^X(u, u).
\]

There is a self-adjoint, negative-definite operator \((L^X, D_2(L^X))\) on \(L^2(m_X)\). Its domain is

\[
D_2(L^X) = \{v \in D(\mathcal{E}^X) : \exists v \in L^2(X, m_X), -(v, w)_{L^2(m_X)} = \mathcal{E}^X(u, w) \forall u \in D(\mathcal{E}^X)\}.
\]

We set \(v \coloneqq L^x u\). \(D_2(L^X)\) is dense in \(L^2(X, m_X)\) and equipped with the topology given by the graph norm. \(L^x\) induces a strongly continuous Markov semi-group \(\{P^x_t\}_{t \geq 0}\) on \(L^2(X, m_X)\). The relation between form, operator and semi-group is standard (see [22]).

A Dirichlet form is called regular if \(\mathcal{E}^X\) possesses a core. A core of \(\mathcal{E}^X\) is by definition a subset \(\mathcal{C}^x\) of \(D(\mathcal{E}^X) \cap C_0(X)\) such that \(\mathcal{C}^x\) is dense in \(D(\mathcal{E}^X)\) with respect to the energy norm and dense in \(C_0(X)\) with respect to uniform convergence where \(C_0(X)\) is the set of continuous functions with compact support in \(X\). We say that a symmetric form is strongly local if \(\mathcal{E}^X(u, v) = 0\) whenever \(u, v \in D(\mathcal{E}^X)\) and \((u + v) = 0 m_X\)-almost surely in \(X\) for some \(a \in \mathbb{R}\).

Definition 2.1 (\(\Gamma\)-operator for Dirichlet forms). Set \(D^\infty(\mathcal{E}^X) = D(\mathcal{E}^X) \cap L^\infty(X, m_X)\). Then \(D^\infty(\mathcal{E}^X)\) is an algebra (see [15]) and for \(u, \phi \in D^\infty(\mathcal{E}^X)\) the following operator is well-defined

\[
\Gamma^x(u; \phi) \coloneqq \mathcal{E}^X(u, \phi) - \frac{1}{2} \mathcal{E}^X(u^2, \phi).
\]

It can be extended by continuity to any \(u \in D(\mathcal{E}^X)\). We call \(\mathcal{G}\) the set of functions \(u \in D(\mathcal{E}^X)\) such that the linear form \(\phi \mapsto \Gamma^x(u; \phi)\) can be represented by an absolutely continuous measure w.r.t \(m_X\) with density \(\Gamma^x(u) \in L^1_b(X, m_X)\). If \(\mathcal{E}^X\) is symmetric, we get the following representation

\[
\mathcal{E}^X(u, u) = \int_X \Gamma^x(v) m\quad \text{for any } u \in \mathcal{G}.
\] (5)

By polarization we can extend the \(\Gamma\)-operator as trilinear form as follows

\[
\Gamma^x(u, v; \phi) = \frac{1}{2} (\Gamma^x(u; \phi) + \Gamma^x(v; \phi) - \Gamma^x(u - v; \phi)) \quad \text{for } u, v \in D(\mathcal{E}^X), \phi \in D^\infty(\mathcal{E}^X)
\]

If \(\mathcal{G} = D(\mathcal{E}^X)\), we say \(\mathcal{E}^X\) admits a “carré du champ” or \(\Gamma\)-operator. Fundamental properties of \(\Gamma^x : D(\mathcal{E}^X) \times D(\mathcal{E}^X) \to L^1(X, m_X)\) are positivity, symmetry, bilinearity and continuity (see [15, Proposition 4.1.3]).
**Leibniz rule.** The strong locality of $\mathcal{E}^X$ implies the strong locality of $\Gamma^X$: \(1_U \cdot \Gamma^X(u,v) = 0\) for all \(u,v \in \mathcal{G}\) and for all open sets $U$ on which $u$ is constant (see [43, Appendix]) and the Leibniz rule: For all $u,v,w \in \mathcal{G}$ such that $v,w \in L^\infty(m_X)$ it holds $v \cdot w \in \mathcal{G}$ and
\[
\Gamma^X(u,v \cdot w) = \Gamma^X(u,v) \cdot w + v \cdot \Gamma^X(u,w)
\] (see [43, Appendix]). One can prove the following

**Lemma 2.2.** Assume $\mathcal{G} = D(\mathcal{E}^X)$. [6] also holds for $u,v,w \in D(\mathcal{E}^X)$ with $v,\Gamma^X(v) \in L^\infty(m_X)$.

**Chain rule.** We say $\mathcal{E}^X$ is of diffusion type if $L^X$ satisfies the following chain rule. Let $\eta$ be in $C^2(\mathbb{R})$ with $\eta(0) = 0$. If $u \in D_2(L^X)$ with $\Gamma(u) \in L^2(X,m_X)$ and $\eta(u) \in D(L^X)$, then
\[
L^X \eta(u) = \eta'(u) L^X u + \eta''(u) \Gamma^X(u).
\] (7)

This is the case when $\mathcal{G} = D(\mathcal{E}^X)$ (see [13, Corollary 6.1.4]).

If $\mathcal{E}^X$ is strongly local and admits a “carré du champ” operator, we can define $D_{loc}(\mathcal{E}^X)$ as follows. $u \in D_{loc}(\mathcal{E}^X)$ if $u \in L^2_{loc}(m_X)$ and for any compact set $K$ there exists $v \in D(\mathcal{E}^X)$ such that $v = u$ on $K$. Hence, for any $u \in D_{loc}(\mathcal{E}^X)$ there exists $\Gamma^X(u) \in L^1_{loc}(m_X)$. The intrinsic distance of $\mathcal{E}^X$ is defined by
\[
d_{\mathcal{E}^X}(x,y) = \sup \{u(x) - (y) : u \in D_{loc}(\mathcal{E}^X) \cap C(X), \ \Gamma^X(u) \leq 1 \text{ m.a.e.}\}.
\]

The intrinsic distance is not metric in general but a pseudo-metric since there can be points $x \neq y$ with $d_{\mathcal{E}^X}(x,y) = 0$. For the rest of this article we always assume that $\mathcal{E}^X$ is a strongly local and regular Dirichlet form with $\mathcal{G} = D(\mathcal{E}^X)$. Then we will call $\mathcal{E}^X$ also admissible.

**Definition 2.3.** Let $\mathcal{E}^X$ be an admissible Dirichlet form.

(i) We say the Dirichlet form $\mathcal{E}^X$ is strongly regular if the topology of $d_{\mathcal{E}^X}$ coincides with the original one.

(ii) We say that $(X,d_{\mathcal{E}^X},m_X)$ satisfies the doubling property if there is constant $2^n = C_0$ for some $n \geq 0$ such that for all $x \in X$ and $0 < r < \text{diam}(X,d_X)$
\[
m_X(B_{2r}(x)) \leq C_0 m_X(B_r(x)).
\]

(iii) We say that $\mathcal{E}^X$ supports a weak local $(q,p)$-Poincaré inequality with $1 \leq p \leq q < \infty$ if there exist constants $C > 0$ and $\lambda \geq 1$ such that for all $u \in D(\mathcal{E}^X)$, any point $x \in X$ and $r > 0$
\[
\left( \int_{B_r(x)} |u - u_{B_r(x)}|^q m_X \right)^{\frac{1}{q}} \leq Cr \left( \int_{B_{\lambda r}(x)} \Gamma^X(u)^{\frac{q}{p}} m_X \right)^{\frac{1}{p}}.
\] (8)

If $\lambda = 1$, we say $\mathcal{E}^X$ supports a strong $(1,p)$-Poincaré inequality. Some authors also use the term Poincaré-Sobolev inequality for the case $q > 1$ and $(1,p)$-Poincaré inequalities are just called $p$-Poincaré inequality (see for example [31]).

**Remark 2.4.**

(i) It is known that under the doubling property for $(X,d_{\mathcal{E}^X},m_X)$ weak local Poincaré inequalities imply strong ones.

(ii) By Hölder’s inequality, a weak local $(1,p)$-Poincaré inequality implies a weak local $(1,p')$-Poincaré inequality for $p' \geq p$.

**Remark 2.5.** A doubling property and Poincaré inequalities are regularity assumptions for Dirichlet forms that were used by Sturm in [43, 44, 45] to derive the following results:

Let $\mathcal{E}^X$ a strongly local and strongly regular Dirichlet form and let $d_{\mathcal{E}^X}$ be its intrinsic distance. Assume that closed balls $B_r(x)$ are compact for any $r > 0$ and $x \in X$. Assume a doubling property holds and $\mathcal{E}^X$ supports a weak local $(2,2)$-Poincaré inequality. Then
(1) $P_t^X$ admits an $\alpha$-continuous kernel and is a Feller semi-group.

(2) $P_t^X$ is $L^2 \to L^\infty$-ultracontractive: $\|P_t^X\|_{L^2 \to L^\infty} \leq 1$.

(3) If $m(X) < \infty$, harmonic functions are constant.

$L^2 \to L^\infty$-ultracontractivity actually comes from an upper bound for the heat kernel (see [28, Chapter 14.1] and [45, Theorem 4.1]). A Feller semi-group is a semi-group that maps bounded continuous functions to bounded continuous functions.

2.2. The Bakry-Emery curvature-dimension condition

In this section we introduce the curvature-dimension condition for Dirichlet forms in the sense of Bakry, Emery and Ledoux. The specific feature of this approach is the existence of an algebra $\mathcal{A}^x$ of bounded measurable functions on $X$ that is dense in $D_2(L^x)$ and in all $L^p$-spaces, stable by $L^x$ and stable by composition with $C^\infty$-functions of several variables that vanish at 0. We call such an algebra admissible. In the context of unbounded operators it is also a core for $(L^x, D_2(L^x))$.

A core for an unbounded operator is a subset of its domain that is dense with respect to the graph norm. The algebra allows to introduce notions of curvature and dimension on a purely algebraic level and provides a calculus that simplifies proofs significantly.

A consequence of the existence of an admissible algebra is that the “carré du champ”-operator

\[ \Gamma^x(u) = \frac{1}{2} L^x(u^2) - u L^x u \quad \text{for all } u \in \mathcal{A}^x. \]

Provided $D(\mathcal{E}^x) = \mathcal{G}$, this rule is consistent with Definition 2.1 (see [15], section I.4). Replacing $L^x$ by $\Gamma^x$ in the definition of the carré du champ we can define the so-called iterated carré du champ or $\Gamma_2$-operator

\[ \Gamma_2^x(u, v) = \frac{1}{2} L^x \Gamma^x(u, v) - \Gamma^x(u, L^x v) \quad \text{for all } u, v \in \mathcal{A}^x. \]

We write $\Gamma^x(u)$ for $\Gamma^x(u, u)$ and similarly for $\Gamma_2^x$.

**Definition 2.6** (Classical Bakry-Emery curvature-dimension condition). Assume there is an admissible algebra $\mathcal{A}^x$ for $\mathcal{E}^x$. Then $\mathcal{E}^x$ satisfies the “classical” Bakry-Emery curvature dimension condition $BE(\kappa, N)$ of curvature $\kappa \in \mathbb{R}$ and dimension $1 \leq N < \infty$ if

\[ \Gamma_2^x(u) \geq \kappa \Gamma^x(u) + \frac{1}{N}(L^x u)^2 \quad \text{for all } u \in \mathcal{A}^x. \]  

(9)

The inequality is understood to hold $m_x$-almost everywhere in $X$. Similar, the condition $BE(\kappa, \infty)$ holds if $\Gamma_2^x(u) \geq \kappa \Gamma^x(u)$ $m_x$-a.e. for all $u \in \mathcal{A}^x$ and $BE(\kappa, N)$ implies $BE(\kappa, \infty)$.

In many situations an algebra $\mathcal{A}^x$ is not available. To overcome this problem, in [3] the Definition 2.3 was reformulated in an “intrinsic” way that also makes sense without the admissible algebra. For the rest of this section we will briefly present this approach and investigate the relation to the previous one. A more detailed description can be found in [3]. We still consider a regular and strongly local Dirichlet form $\mathcal{E}^x$ on some admissible space $X$ like in Section 2.1. The $\Gamma_2$-operator can be defined in a weak sense by

\[ 2 \Gamma_2^x(u, v; \phi) = \Gamma^x(u, v; L^x \phi) - 2 \Gamma^x(u, L^x v; \phi) \quad \text{for } u, v \in D(\Gamma_2) \text{ and } \phi \in D^{h,2}_+(L^x) \]

where $D(\Gamma_2^x) := \{ u \in D_2(L^x) : L^x u \in D(\mathcal{E}^x) \}$ and the set of test functions is denoted by

\[ D^{h,2}_+(L^x) := \{ \phi \in D_2(L^x) : \phi, L^x \phi \in L^\infty(X, m), \phi > 0 \}. \]

$\Gamma_2^x$ is not symmetric in $u$ and $v$. We set $\Gamma_2^x(u, u; \phi) = \Gamma_2^x(u; \phi)$. 

8
Definition 2.7 (Bakry-Emery curvature-dimension condition). Let $\kappa \in \mathbb{R}$ and $N \geq 1$. We say that $\mathcal{E}^X$ satisfies the intrinsic Bakry-Emery curvature-dimension condition (or just Bakry-Emery condition) $BE(\kappa, N)$ if for every $u \in D(\Gamma^X_2)$ and $\phi \in D^{b,2}_+(L^X)$, we have

$$\Gamma^X_2(u; \phi) \geq \kappa \Gamma^X(u; \phi) + \frac{1}{N} \int_X (L^X u)^2 \phi \, dm.$$

(10)

In this case we have that $\mathbb{G} = D(\mathcal{E}^X)$ (see [3, Corollary 2.3]). Hence, $\mathcal{E}^X$ is of diffusion-type. As before we can also define $BE(\kappa, \infty)$ and the implications $BE(\kappa, N) \Rightarrow BE(\kappa, N') \Rightarrow BE(\kappa, \infty)$ for $N' \geq N$ hold as well.

Theorem 2.8 (Bakry-Ledoux gradient estimate). Let $\mathcal{E}^X$ be an admissible Dirichlet form. The estimate $\left( \mathcal{T} \right)$ for $\kappa \in \mathbb{R}$, $N \geq 1$ and any $(u, \phi) \in D(\Gamma^X_2)$ with $\phi \geq 0$ is equivalent to the following gradient estimate. For any $u \in \mathbb{G}$ and $t > 0$, $P_t^X u$ belongs to $\mathbb{G}$ and we have

$$\Gamma^X(P_t^X u) + \frac{1 - e^{-2\kappa t}}{N \kappa} (L^X P_t^X u)^2 \leq e^{-2\kappa t} P_t^X \Gamma^X(u) \quad m\text{-a.e. in } X.$$

(11)

Proof. → The proof of the theorem in this form can be found in [3] (see also [12, 21]).

Remark 2.9. If there is an admissible algebra that is stable with respect to $P_t^X$, the definitions 2.6 and 2.7 are consistent. On the one hand, Definition 2.7 and the existence of an admissible algebra $\mathcal{A}^X$ imply that for any test function $\phi \in D^{b,2}_+(L^X)$ and any $u \in \mathcal{A}^X$

$$\int_X \Gamma^X(u) \phi \, dm \geq \kappa \Gamma(u; \phi) + \frac{1}{N} \int_X (L^X u)^2 \phi \, dm.$$

Then we can replace $\phi \in D^{b,2}_+(L^X)$ by any bounded and measurable function $\phi \geq 0$ by using the mollifying property of $P_t^X$, exactly like in [3] and [21]. This implies the classical Bakry-Emery condition in the sense of Definition 2.6 for $u \in \mathcal{A}^X$. On the other hand, if we assume the Bakry-Emery condition in the sense of Definition 2.6 for some admissible algebra $\mathcal{A}^X$ that is also stable under $P_t^X$, we can apply the following lemma.

Lemma 2.10. Assume there is a subset $\Xi \subset D_2(L^X)$ that is dense with respect to the graph norm and stable under the Markovian semi-group $P_t^X$, and assume we have

$$\Gamma^X_2(u; \phi) \geq \kappa \Gamma^X(u; \phi) + \frac{1}{N} \int_X (L^X u)^2 \phi \, dm \quad \text{if } u \in D(\Gamma^X_2) \cap \Xi \text{ and } \phi \in D^{b,2}_+(L^X).$$

Then $\mathcal{E}^X$ satisfies $BE(\kappa, N)$.

Proof. The proof is straightforward and uses the mollification property of the semigroup and mainly Lemma 1.3.3 from [23] but we omit details here.

2.3. Some examples of Dirichlet forms

In this section we consider some examples in more detail. They will also play an important role later in the article. Let $B$ be a smooth, $d$-dimensional manifold with or without boundary and let $g$ be a Riemannian metric on $B$. $\text{vol}_B = m_B$ is a smooth Radon measure and $(B, m_B)$ is an admissible space. We set $\hat{B} = B \setminus \partial B$, we assume that $(\hat{B}, d_B)$ is geodesically convex, and we consider the standard Dirichlet form with Dirichlet boundary conditions. Its domain is $D(\mathcal{E}^B) = W^{1,2}_0(B, d \text{vol}_B)$. The associated self-adjoint operator is the Dirichlet Laplace operator $\Delta^B$ with domain $D^2(L^B) = W^{2,2}_0(B, d \text{vol}_B)$. In this context, we have $\Gamma^B(u) = |\nabla u|^2_B$. We assume that $(B, g)$ has Ricci curvature bounded from below by $(d - 1)K$. 

9
Proposition 2.12. \(E\) is a polar set in the sense of Grigor’yan and Masamune (see [29]), it is also true. In the weighted case, with Dirichlet boundary conditions. In general, this is not the case. But if the boundary \(B\) is empty, then the form coincides with the form with Dirichlet boundary conditions. In general, this is not the case. But if the boundary \(\partial B\) is a polar set in the sense of Grigor’yan and Masamune (see [29]), it is also true. In the weighted situation that we are considering the boundary is a polar set in this sense. In particular, it follows that \(C_0^\infty(B) \subset D(\mathcal{E}^{n,N})\).

Proposition 2.12. For \(u \in C^\infty(\hat{B}) \cap D^2(L^{n,N})\) there is an explicit formula for the generator of \(\mathcal{E}^{n,N}\) given by

\[
(L^{n,N} u)(p) = (\Delta^u u)(p) + \frac{N}{f(p)} \langle \nabla f, \nabla u \rangle_p \quad \text{for any } p \in \hat{B}.
\]

(13)

Proposition 2.13. Let \((B, g, f^N \text{vol}_B)\) be as above. Then for any \(u \in C^\infty(\hat{B})\) the following \(\Gamma_2\)-estimate holds pointwise everywhere in \(\hat{B}\):

\[
\Gamma_2^{n,N}(u) \geq (d + N - 1)K\|\nabla u\|_g^2 + \frac{1}{d + N}(L^{n,N} u)^2.
\]

(14)

Proof. Since we have (1) for \(\Gamma_2^g = \frac{1}{2}\Delta^g \|\nabla u\|_g^2 - \langle \nabla u, \nabla \Delta^g u \rangle\) and since \(f\) is \(FK\)-concave, we get pointwise for any \(u \in C^\infty(\hat{B})\)

\[
\Gamma_2^{n,N}(u) = \text{ric}_B(\nabla u) + \|\nabla^2 u\|^2_{HS} - \frac{N}{f} \langle \nabla f, \nabla u \rangle \langle \nabla u, \nabla f \rangle \geq (d - 1)K\|\nabla u\|_g^2 + \frac{1}{d}(\Delta^g u)^2 + NK\|\nabla u\|_g^2 + \frac{1}{d} \langle \nabla f, \nabla u \rangle^2 \geq (d + N - 1)K\Gamma^g(u) + \frac{1}{d + N}(L^{n,N} u)^2.
\]

(15)

Details can be found in chapter 14 of [18].

Example 2.14 (1-dimensional model spaces). Let \(B\) be of the form \(I_K = [0, \pi/K]\) for \(K > 0\) and \([0, \infty)\) for \(K \leq 0\). The corresponding operator \(L^{I_K}\) is \(d^2/dx^2\) and its domain is \(H^1_0(I_K, dx)\). It satisfies \(BE(0, 1)\). Consider \(f : I_K \to \mathbb{R}_{\geq 0}\) in \(D_2(L^{I_K})\) that is given by

\[
f(t) = \sin(Kt) = \begin{cases} \frac{1}{\sqrt{K}} \sin(\sqrt{K}t) & \text{for } K > 0 \\ t & \text{for } K = 0 \\ \frac{1}{\sqrt{|K|}} \sinh(\sqrt{|K|}t) & \text{for } K < 0. \end{cases}
\]

We can define \(\mathcal{E}^{I_K,f^N}\) as before.

Proposition 2.15. Let \(K > 0\) and \(N \geq 1\). \(\mathcal{E}^{I_K,\sin^K_N}\) satisfies \(BE(NK, N + 1)\).

Proof. \(\mathcal{E}^{I_K,\sin^K_N}\) is the Cheeger energy of \((I_K, \sin^K_N dr)\) (see Section 12.2) and \((I_K, \sin^K_N dr)\) satisfies the condition \(CD(KN, N + 1)\) by the equivalence of Theorem 12.29 that will be presented later. Hence, the result follows (see [18]).
3. Skew products and the Bakry-Emery curvature-dimension condition

3.1. Skew and N-skew products between Dirichlet forms

In this section we define skew and N-skew products for Dirichlet forms. The notion of skew product is well-known and has been introduced by Fukushima and Oshima in [22]. An N-skew product is a slight modification of that definition where we also change the topology of the underlying space.

Let \( B \) be a \( d \)-dimensional Riemannian manifold with or without boundary such that \( B \) is geodesically convex and \( \text{ric}_p \geq (d - 1)K \) and let \( \mathcal{E}_n \) be its standard Dirichlet form with Dirichlet boundary conditions. Let \( f \in D_2(L^n) \) be smooth and \( FK \)-concave. Let \( \mathcal{E}_p \) be a regular and strongly local Dirichlet form on \( L^2(F, m_p) \) where \( F \) is an admissible space. Consider \((B \times F, \mathcal{O}_n \otimes \mathcal{O}_p)\) with \( m_p = f^n d\text{vol}_n \otimes dm_p \) and the tensor product \( C_0^\infty(B) \otimes D(\mathcal{E}_p) \). \( \mathcal{O}_n \otimes \mathcal{O}_p \) is the product topology.

Remark 3.1. The elements of \( C_0^\infty(B) \otimes D(\mathcal{E}_p) \) are functions of the form \( \sum_{i=1}^{k} u_i^1u_i^2 \) for some finite \( k \in \mathbb{N} \) and \( u_i^1 \in C_0^\infty(B) \) and \( u_i^2 \in D(\mathcal{E}_p) \). We will follow this convention in the rest of the article. In the literature the tensor product between infinite dimensional Hilbert spaces \( \mathcal{H}_i \) for \( i = 1, 2 \) means that one also takes the closure with respect to the induced inner product. Later, this construction will also appear and we use the following notation \( \mathcal{H}_1 \otimes \mathcal{H}_2 \).

**Definition 3.2** (Skew product). Consider the closure of the following densely defined symmetric form on \( L^2(B \times F, f^n d\text{vol}_n \otimes dm_p) \):

\[
\mathcal{E}^C(u) = \int_B \mathcal{E}^n(u^z)d\text{m}_p(x) + \int_B \mathcal{E}^p(u^p)f^{N-2}(p)d\text{vol}_n(p) < \infty
\]  

(16)

for \( u \in C_0^\infty(\hat{B}) \otimes D(\mathcal{E}_p) \) where \( u^z = u(\cdot, x) \) and \( u^p = u(p, \cdot) \) are the horizontal respectively vertical sections of \( u \). \((B \times F, \mathcal{O}_n \otimes \mathcal{O}_p, \mathcal{E}^C)\) is called skew product between \( B, f \) and \( \mathcal{E}^C \).

**Remark 3.3.** \( D(\mathcal{E}^n) \otimes D(\mathcal{E}_p) \subset D(\mathcal{E}^C) \) and

\[
\mathcal{E}^C(u) = \int_B \mathcal{E}^n(u^z)d\text{m}_p(x) + \int_B \mathcal{E}^p(u^p)f^{N-2}(p)d\text{vol}_n(p),
\]  

(17)

holds for any \( u \in D(\mathcal{E}^n) \otimes D(\mathcal{E}_p) \).

The next proposition is a Fubini-type result and was proven by Okura in [38].

**Proposition 3.4.** Let \( \mathcal{E}^C \) be a skew product like in Definition 3.2. Consider \( u \in D(\mathcal{E}^C) \). Then \( u^z \in D(\mathcal{E}^n) \) for \( \text{m}_p \)-almost every \( x \in F \) and \( u^p \in D(\mathcal{E}_p) \) for \( \text{vol}_n \)-almost every \( p \in B \) and we have

\[
\mathcal{E}^C(u) = \int_B \mathcal{E}^n(u^z)d\text{m}_p(x) + \int_B \mathcal{E}^p(u^p)f^{N-2}(p)d\text{vol}_n(p),
\]  

(18)

Especially \( \mathcal{E}^C \) admits a \( \Gamma \)-operator if and only if \( \mathcal{E}_p \) does so, and in this case we have for \( u \in D(\mathcal{E}^C) \)

\[
\Gamma^C(u)(p, x) = \Gamma^n(u^z)(p) + \frac{1}{f^p(p)} \Gamma^p(u^p)(x) \quad \text{m}_p \text{-a.e. }.
\]

**Corollary 3.5.** Let \( \mathcal{E}^C \) be skew product like in Definition 3.2. Then \( C_0^\infty(\hat{B}) \otimes D^2(L^n) \subset D^2(\mathcal{L}^C) \) and

\[
(\mathcal{L}^C u)(p, x) = (L^n u^z)(p) + \frac{1}{f^p(p)} (L^p u^p)(x) \quad \text{for } \text{m}_p \text{-a.e. } (p, x) \in C.
\]  

(19)
Proof. We consider \( u \in C_0^\infty(\hat{B}) \otimes D^2(L^p) \) and \( v \in D(\mathcal{E}^c) \). Then \( u^x \in C_0^\infty(\hat{B}) \) for \( m_r \)-almost every \( x \) and \( v^x \in D(L^p) \) for every \( p \), and \( v^x \in D(\mathcal{E}^c(x)) \) for \( m_r \)-almost every \( x \) and \( v^x \in D(\mathcal{E}^c) \) for \( \text{vol}_n \) almost every \( p \). Hence

\[
\mathcal{E}^{b,f}(u^x, v^x) = (L^{b,f}u^x, v^x)_{L^2(f^*d\text{vol})} \quad \text{and} \quad \mathcal{E}^c(u^x, v^x) = (L^pu^x, v^x)_{L^2(m_r)}
\]

for \( m_r \)-almost every \( x \) and for \( \text{vol}_n \) almost every \( p \). This and Proposition 3.4 implies

\[
\mathcal{E}^c(u, v) = \int_B \mathcal{E}^{b,f}(u^x, v^x)dm_r(x) + \int_B \frac{1}{f(x)} \mathcal{E}^c(u^x, v^x) f^*(p) d\text{vol}_n(p)
\]

\[
= -\int_B \left( L^{b,f}u^x, v^x \right)_{L^2(f^*d\text{vol})} dm_r(x) + \int_B \frac{1}{f(x)} \left( L^pu^x, v^x \right)_{L^2(m_r)} f^*(p) d\text{vol}_n(p)
\]

\[
= -\int_B \left( L^{b,f}u^x(p) + \frac{1}{f(x)} L^pu^x(p) \right) (v(p,x)) dm_r(p, x).
\]

Then we also see that \( L^{b,f}u^x(p) + f^{-2}(p)L^pu^x(x) \) is \( L^2 \)-integrable with respect to \( m_c \). First, we consider \( u = u_1 \otimes u_2 \in C_0^\infty(\hat{B}) \otimes D^2(L^p) \). Then \( u \) is \( L^2 \)-integrable with respect to \( m_c \) since

\[
\| L^{b,f}u_1 \otimes u_2 \|^2_{L^2(m_c)} < \frac{1}{2} \| L^{b,f}u_1 \|^2 m_c + \frac{1}{2} \| L^pu_2 \|^2_{L^2(f^*d\text{vol})} \| L^pu_2 \|^2 < \infty.
\]

In particular, we used that \( u_1 \) is smooth with compact support in \( \hat{B} \). Hence, \( u_1 \otimes u_2 \in D^2(L^p) \) and (19) holds. In general, any \( u \in C_0^\infty(\hat{B}) \otimes D^2(L^p) \) has the form

\[
u = \sum_{i=1}^k u_i \otimes u_i \]

where \( L^pu_i \) is \( L^2 \)-integrable. Then, by linearity of \( L^{b,f}u_1 \otimes L^pu_2 \) and by triangle inequality also \( L^{b,f}u^x + \frac{1}{f(x)} L^pu^x \) is \( L^2 \)-integrable and (19) holds.

\[N\text{-skew products.}\] We will introduce a slight modification of Definition 3.2. The underlying space of \( \mathcal{E}^c \) is \( B \times F \) equipped with the product topology \( O_n \otimes O_r \) but in general the intrinsic distance \( d_{\mathcal{E}^c} \) induces a different topology that we will describe in more detail. Let us define an equivalence relation on \( B \times F \) as follows:

\[(p, x) \sim (q, y) \iff (p = q \in \partial B) \text{ or } (p = q \in B \text{ and } x = y \in F).\]

Then we can consider the quotient space \( B \times F/\sim =: C \) and the corresponding projection map \( \pi : B \times F \rightarrow C \). Obviously, we have the following decomposition

\[C = \partial B \cup \hat{B} \times F.\]

A subset \( V \subset C \) is open if and only if \( \pi^{-1}(V) \subset B \times F \) is open. We denote the corresponding topology with \( O_c \). If \( u \) is continuous with respect to \( O_c \), then \( u \circ \pi = \tilde{u} \) is continuous with respect to \( O_n \otimes O_r \). By abuse of notation we will also write \( u = \tilde{u} \) when the meaning is clear. If \( \mathcal{E}^c \) is strongly regular, one can define a family of “open balls” that generates the quotient topology, i.e. any open set is a union of elements from this family. First, we pick \((p, x) \in B \times F\) and we consider \( \epsilon_{[p,x]} = \inf_{q \in \partial B} d_{\mathcal{E}^c}(p, q) \). Then, admissible \( \epsilon \)-balls around \([p, x]\) are

\[B^c_\epsilon([p, x]) = \{ (q, y) \in C : d_{\mathcal{E}^c}(q, p) + d_{\mathcal{E}^c}(x, y) < \epsilon \} \subset \hat{B} \times F \subset C \text{ for } 0 < \epsilon < \epsilon_{[p,x]}.\]

For \( p = [p, x] \in \partial B \subset C \) the corresponding \( \epsilon \)-balls are

\[B^c_\epsilon([p, x]) = \{ (q, y) \in C : d_{\mathcal{E}^c}(q, p) < \epsilon \} \subset C \text{ for } 0 < \epsilon < \epsilon_{[p,x]} = \infty.\]
The family of all admissible balls is denoted by

\[ B = \{ B^c_\epsilon([p,x]) : [p,x] \in C, \ 0 < \epsilon < \epsilon([p,x]) \}. \]

It is not hard to check that elements from \( B \) are open with respect to \( O_C \) and that \( B \) is a generator for \( O_C \). We can pushforward the measure \( m_C \) to \( C \) and denote it also with \( m_C \). \( \partial B \subset C \) is a set of measure zero. Hence, \( C \) keeps its product structure \( m_C \)-almost everywhere. Then we can interpret \( \mathcal{E}^C \) also as a Dirichlet form on \( L^2(C,m_C) =: L^2(m_C) \) and the previous Fubini-type results are valid as well.

**Definition 3.6 (N-skew product).** Assume \( \mathcal{E}^F \) is a strongly local, regular and strongly regular Dirichlet form and \( B \) and \( f \) as in Definition 3.2 Consider the admissible space \((C,O_C,m_C)\) and define a Dirichlet form \( \mathcal{E}^C \) on \( L^2(m_C) \) as in Definition 3.2 We call \((C,O_C,m_C,\mathcal{E}^C)\) the \( N \)-skew product between \( B \), \( f \) and \( \mathcal{E}^F \) and we will write \( \mathcal{E}^C = \mathcal{E}^F \times_f \mathcal{E}^F = B \times_f \mathcal{E}^F \). \( B \times_f \mathcal{E}^F \) is strongly local and regular.

Consider the intrinsic distance \( d_{\mathcal{E}^C} \) of \( \mathcal{E}^C = B \times_f \mathcal{E}^F \) on \( C \). The topology that is induced by \( d_{\mathcal{E}^C} \) is denoted by \( O_d \). One can check easily the following Lemma. We omit the proof.

**Lemma 3.7.** If \( \mathcal{E}^F \) is strongly regular, then \( \mathcal{E}^C = B \times_f \mathcal{E}^F \) is strongly regular, i.e. \( O_d = O_C \). 
Closed \( \epsilon \)-balls with respect to \( d_{\mathcal{E}^C} \) are compact if this property holds for \( d_{\mathcal{E}^F} \).

**Definition 3.8 ((K,N)-cones).** Assume the situation of example 2.11 and let \( \mathcal{E}^F \) be a Dirichlet form as before on some space \( F \). Then the \( N \)-skew product with respect to \( I_K \) and \( \sin_K \) is well-defined and called \((K,N)\)-cone over \( \mathcal{E}^F \).

### 3.2. Proof of classical \( \Gamma_2 \)-estimates for \( N \)-skew products

We fix a regular and strongly local Dirichlet form \( \mathcal{E}^F \) on \( L^2(F,m_F) \) for some admissible space \((F,m_F)\). In this subsection we assume there is an admissible algebra \( \mathcal{A}^F \) for \( \mathcal{E}^F \). This enables us to do calculations classically. In subsection 2.4, we will prove analogous results for Dirichlet forms that satisfy the intrinsic Bakry-Emery curvature-dimension condition. The advantages of the classical approach are that calculations can be done pointwise and that the structure of formulas and inequalities becomes more clear.

**Theorem 3.9.** Let \( B \) be a Riemannian manifold (with or without boundary) such that \( B \) is geodesically convex and let \( \Gamma^\alpha \) be the associated standard Dirichlet form with Dirichlet boundary conditions that satisfies \( BE((d-1)K,d) \). Let \( f \in D_2(L^\alpha) \) be smooth and \( FK \)-concave. Let \( \mathcal{E}^F \) satisfy \( BE((N-1)K_F,N) \) for \( N \geq 1 \) and \( K_F \in \mathbb{R} \) such that

\[ \Gamma^\alpha(f) + Kf^2 \leq K_F \] 

Assume there is an admissible algebra \( \mathcal{A}^F \) for \( \mathcal{E}^F \) and set \( \mathcal{A}^C = C_0^\infty(\hat{B}) \otimes \mathcal{A}^F \). Then the \( N \)-skew product \( \mathcal{E}^C = B \times_f \mathcal{E}^F \) satisfies

\[ \Gamma^\alpha_2(u) \geq (N + d - 1)K\Gamma^\alpha(u) + \frac{1}{N + d}(L^\alpha u)^2 \]  \hspace{2cm} (20)

pointwise \( m_C \)-everywhere and for any \( u \in \mathcal{A}^C \).

**Proof.** Every element of \( \mathcal{A}^C \) is of the form \( \sum_{i=1}^k u_i^1 u_i^2 \) for \( k \in \mathbb{N} \), but we will check (20) only for elements of the form \( u + v = u_1 \otimes v_1 + v_2 \otimes v_2 \) where \( u_1,v_1 \in C_0^\infty(\hat{B}) \) and \( u_2,v_2 \in \mathcal{A}^F \). The case of arbitrary finite sum follows in the same way. We compute \( \Gamma^\alpha_2(u), \Gamma^\alpha_2(v), \Gamma^\alpha_2(u,v) \) and \( \Gamma^\alpha_2(v,u) \) explicitly \( m_C \)-a.e. A straightforward calculation yields:

\[ \Gamma^\alpha_2(u,v) + \Gamma^\alpha_2(v,u) = \Gamma^\alpha_2(u_1,v_1) + \Gamma^\alpha_2(v_1,u_1)u_2v_2 + \frac{u_1 v_1}{L^\alpha} (\Gamma^\alpha_2(u_2,v_2) + \Gamma^\alpha_2(v_2,u_2)) \]
\[ + \Gamma^\alpha(u_1,v_2)\frac{1}{L^\alpha}L^\alpha(u_2v_2) - \Gamma^\alpha(v_2,u_1)L^\alpha(u_2v_2) - \Gamma^\alpha(u_1,\frac{v_1}{L^\alpha})u_2L^\alpha(v_2) \]
\[ + (L^\alpha,\frac{v_1}{L^\alpha}) - L^\alpha(v_1)\frac{1}{L^\alpha} - u_1 L^\alpha(v_1)\frac{v_1}{L^\alpha}) \Gamma^\alpha(u_2,v_2) \]  \hspace{2cm} (21)
We set 
\[ \Gamma^{\alpha}_2(u, v) + \Gamma^{\alpha}_2(v, u) - (\Gamma_2^{\alpha}(u_1, v_1) + \Gamma_2^{\alpha}(v_1, u_1))v_2u_2 - \frac{u_2}{\Gamma^{\alpha}_2} (\Gamma_2^{\alpha}(u_2, v_2) + \Gamma_2^{\alpha}(v_2, u_2)) =: (J) \]

When we use the chain rule for \( \Gamma^\alpha \) and \( L^\alpha \) then it yields the following:
\[
\begin{align*}
\Gamma^\alpha \left( \frac{\partial}{\partial f}, v_1 \right) &= -\frac{1}{\Gamma^{\alpha}} \Gamma^\alpha (u_1, v_1) - \frac{2u_1}{\Gamma^{\alpha}} \Gamma^\alpha (f, v_1) \\
L^\alpha \left( \frac{\partial}{\partial f}, v_1 \right) &= -\frac{1}{\Gamma^{\alpha}} L^\alpha (u_1) + \frac{2u_1}{\Gamma^{\alpha}} L^\alpha (v_1) - \frac{2u_1}{\Gamma^{\alpha}} L^\alpha (f) \\
\frac{\partial}{\partial f} \Gamma^\alpha (f, u_1) &= \frac{2u_1}{\Gamma^{\alpha}} \Gamma^\alpha (f, u_1) + \frac{u_2}{\Gamma^{\alpha}} \Gamma^\alpha (u_1, f) - \frac{2u_1}{\Gamma^{\alpha}} \Gamma^\alpha (f, u_1)
\end{align*}
\]

We glue this back into \((J)\) and another straightforward calculation yields.
\[
\begin{align*}
(J) &= \frac{2u_1}{\Gamma^{\alpha}} \Gamma^\alpha (f, u_1) L^\alpha (u_2) v_2 + \frac{2u_1}{\Gamma^{\alpha}} \Gamma^\alpha (f, u_1) L^\alpha (v_2) u_2 - \frac{2u_1}{\Gamma^{\alpha}} \Gamma^\alpha (f) \Gamma^\alpha (u_2, v_2) \\
&\quad + \frac{2u_1}{\Gamma^{\alpha}} \Gamma^\alpha (u_1, f) - \frac{2u_1}{\Gamma^{\alpha}} \Gamma^\alpha (v_1, f) + \frac{u_2}{\Gamma^{\alpha}} \Gamma^\alpha (f) + \frac{u_2}{\Gamma^{\alpha}} \Gamma^\alpha (u_1, v_1) \Gamma^\alpha (u_2, v_2)
\end{align*}
\]

In the case \( u = v \) we obtain the following simplification
\[
(J) = \frac{2u_1}{\Gamma^{\alpha}} \Gamma^\alpha (f, u_1) L^\alpha (u_2) u_2 - \frac{2u_1}{\Gamma^{\alpha}} \Gamma^\alpha (f) \Gamma^\alpha (u_2) + (I)(u_1).
\] (22)

Now, we can compute the \( \Gamma_2 \)-operator for elements of the form \( u_1 \otimes u_2 + v_1 \otimes v_2 = u + v \in C_0^\infty (\bar{B}) \otimes \mathcal{A}^\rho \) for \( m_\rho \)-a.e. point \((p, x) \in \mathcal{C}\):
\[
\begin{align*}
2\Gamma_2^\alpha (u + v)(p, x) &= 2\Gamma_2^\alpha (u^\alpha)(p) + \frac{1}{\Gamma^{\alpha}} 2\Gamma_2^\alpha (u^\rho)(x) \\
&\quad + \frac{1}{\Gamma^{\alpha}} \Gamma^\alpha (u^\rho)(p) L^\alpha (u^\rho)(x) - \frac{2}{\Gamma^{\alpha}} L^\alpha (f)(p) + \frac{u_2}{\Gamma^{\alpha}} \Gamma^\alpha (f)(p) \Gamma^\alpha (u^\rho)(x) \\
&\quad + 2(I)(u_1, v_1)(p) \Gamma^\alpha (u^\rho)(u_2, v_2)(x) + (I)(u_1)(p) \Gamma^\alpha (u^\rho)(u_2)(x) + (I)(v_1)(p) \Gamma^\alpha (u^\rho)(v_2)(x)
\end{align*}
\]

We denote the last line in the previous equation with \((II)\). First, assume \( u_1(p) v_1(p) \neq 0 \). Then, it can be rewritten in the following form
\[
\begin{align*}
(II) &= \frac{2u_1}{\Gamma^{\alpha}} \Gamma^\alpha (u^\rho)(u_2, v_2) \\
&\quad + \frac{u_2}{\Gamma^{\alpha}} (-4 \Gamma^\alpha (\ln |u_1|, \ln f) + 4 \Gamma^\alpha (\ln |v_1|, \ln f) \\
&\quad + 4 \Gamma^\alpha (\ln f) + 4 \Gamma^\alpha (\ln |v_1|, \ln |u_1|)) \Gamma^\alpha (u_2, v_2) \\
&\quad + \frac{u_2}{\Gamma^{\alpha}} (-8 \Gamma^\alpha (\ln |u_1|, \ln f) + 4 \Gamma^\alpha (\ln f) \\
&\quad + 4 \Gamma^\alpha (\ln |u_1|) \Gamma^\alpha (u_2) + \frac{u_2}{\Gamma^{\alpha}} (-8 \Gamma^\alpha (\ln |v_1|, \ln f) + ...) \Gamma^\alpha (v_2)
\end{align*}
\]

We choose an orthonormal basis \((e_i)_{1,\ldots,d}\) with respect to the Riemannian metric at \( TB_p \) and write 
\[
\nabla \ln \left( \frac{f}{|u_1|} \right)_p = \sum_{i=1}^d a^i e_i \quad \text{and} \quad \nabla \ln \left( \frac{f}{|v_1|} \right)_p = \sum_{i=1}^d c^i e_i.
\]

Then we obtain
\[
\begin{align*}
\frac{\partial^2}{\partial u^\alpha} (II) &= \sum_{i=1}^d 2a^i c^i \Gamma^\alpha (u^\rho, v^\rho)(x) + \sum_{i=1}^d (a^i)^2 \Gamma^\alpha (u^\rho)(x) + \sum_{i=1}^d (c^i)^2 \Gamma^\alpha (v^\rho)(x) \\
&= \sum_{i=1}^d \Gamma^\alpha (a^i u^\rho + c^i v^\rho)(x) \geq 0 \quad \text{for } m_\rho \text{-almost every } (p, x)
\end{align*}
\] (23)
We apply the following elementary equality

$$(II) = \frac{4}{p} \langle \nabla \ln(f/|u|), \nabla v_1 \rangle_p \Gamma^p(u^p, v_2)(x)
+ \frac{2}{p} |\nabla \ln(f/|u|)|^2 \Gamma^p(u^p)(x) + \frac{1}{p} |\nabla v_1|^2 \Gamma^p(v_2)(x)$$

and when we set $\nabla v_1_p = \sum_i^d \alpha^i e_i$, similar as before we obtain that

$$\frac{\ell^2(p)}{4}(II) = \sum_i^d \Gamma^p(a^i u^p + \alpha^i v_2)(x) \geq 0. \quad (24)$$

In the same way we can deal with the other cases. If we would consider an arbitrary $u \in C_0^\infty(\hat{B}) \otimes \mathcal{A}^r$ of the form $\sum_j^k \sum_i^d u_{i,j} \otimes u_{2,j} = \sum_j^k u_j$, $\ell^2(p)(II)$ would take the form $\sum_i^d \Gamma^p(\sum_j^k u_j^2) \geq 0$ and all the other calculations are the same. It follows in any case that

$$2 \Gamma_2^p(u, p, x) \geq 2 \Gamma_2^p(u^p)(p) + \frac{2}{p} \Gamma_2^p(u^p)(x)$$

$$+ \frac{2}{p} \Gamma^p(\nabla^p(f, u^x)L^p(u^p) - \frac{2}{p} \left(L^p f(p) + \frac{N-1}{(f(p))} \Gamma^p(f, u^p)\right) \Gamma^p(u^p)(x) \quad m_c \text{-a.e.}$$

for any $u \in C_0^\infty(\hat{B}) \otimes \mathcal{A}^r$. Because of (22) we can see that this estimate is sharp and becomes an equality if $u = f \otimes u_2$. From (15) we have

$$\Gamma_2^p(v^N)(u) \geq \frac{d}{N} + (N - 1)K\Gamma^p(u) + \frac{1}{d} (L^p u^2) + \frac{1}{N} (\frac{N}{d} \Gamma^p(f, u))^2 \quad (25)$$

for any function $u \in C_0^\infty(\hat{B})$. Now we apply the curvature-dimension conditions for $\mathcal{E}^r$ and $\mathcal{E}^n$, inequality (25), and the assumptions on $f$. First, we see that

$$L^p f + \frac{N-1}{f} \Gamma^p(f) \leq -dKf + \frac{N-1}{f} (K - Kf^2) \quad \text{everywhere in } B.$$ 

Then it follows that

$$2 \Gamma_2^p(u) \geq 2(d + N - 1)K\Gamma^p(u^x) + \frac{1}{d} (L^p u^x)^2 + \frac{1}{N} (\frac{N}{d} \Gamma^p(f, u^x))^2$$

$$+ \frac{2}{p} \left((N - 1)K\Gamma^p(u^p) + \frac{1}{N} (\frac{N}{d} \Gamma^p(f, u^p))^2\right)$$

$$+ \frac{1}{d} \Gamma^p(f, u^x)2L^p(u^p) + \frac{2}{p} \left(dKf - \frac{N-1}{d} (K - Kf^2)\right) \Gamma^p(u^p)$$

$$= 2 \left((d + N - 1)K\Gamma^p(u) + \frac{1}{d} (L^p u^x)^2 + \frac{1}{N} (\frac{N}{d} \Gamma^p(f, u^x))^2\right) + \frac{4}{d} (N + d - 1)K\Gamma^p(u^p)$$

$$+ \frac{1}{d} \Gamma^p(f, u^x)2L^p(u^p)$$

$$= 2\left((d + N - 1)K\Gamma^p(u^x) + \frac{1}{d} (L^p u^x)^2 + \frac{4}{d} (N + d - 1)K\Gamma^p(u^p)$$

$$+ \frac{1}{N} (\frac{N}{d} \Gamma^p(f, u^x))^2 + \frac{2}{d} \Gamma^p(f, u^x)2L^p(u^p)\right)$$

$$= 2 \left((N + d - 1)\Gamma^p(u^x) + \frac{1}{d} (L^p u^x)^2\right)$$

$$+ \frac{2}{d} (N + d - 1)K\Gamma^p(u^p) + \frac{2}{d} \left(\frac{N}{d} \Gamma^p(f, u^x) + \frac{1}{d} L^p u^x\right)^2 \quad m_c \text{-a.e.}.$$ 

We apply the following elementary equality

$$\frac{1}{d} a^2 + \frac{1}{N} a b = \frac{1}{N+1} (a + b)^2 + \frac{d}{(N+1)d} (b - \frac{N}{d} a)^2$$ 

(26)
for all $d, N \geq 1$ and for all $a, b \in \mathbb{R}$. Hence

$$2\Gamma_2^C(u) \geq 2(N + d - 1)K\Gamma^p(u^p) + \frac{1}{f^2}2(N + d - 1)K\Gamma^p(u^p)$$

$$+ \frac{2}{d}(L^p u^p)^2 + \frac{2}{N}(\frac{K}{1}\Gamma^p(f, u^p) + \frac{1}{f^2}L^p u^p)^2$$

$$\geq 2(N + d - 1)K\Gamma^C(u) + \frac{2}{N + d}(L^p u^p + \frac{K}{1}\Gamma^p(f, u^p) + \frac{1}{f^2}L^p u^p)^2 \text{ m.c.-a.e.}$$

So we have desired inequality for any $u \in C_0^\infty(\hat{B}) \otimes \mathcal{A}^p$.

\[\square\]

**Theorem 3.10.** Let $\mathcal{E}^p$ be a regular and strongly local Dirichlet form and let $\mathcal{A}^p$ be an admissible algebra for $\mathcal{E}^p$. Assume the $(K, N)$-cone $\mathcal{E}^C = 1_K \times_{\sin_k} \mathcal{E}^p$ satisfies a $\Gamma_2$-estimate of curvature $NK$ and dimension $N + 1$ for $K \in \mathbb{R}$ and $N \geq 1$ on $C_0^\infty(\hat{I}_K) \otimes \mathcal{A}^p$. Then $\mathcal{E}^p$ satisfies $BE(N - 1, N)$.

**Proof.** We assume a $\Gamma_2$-estimate for $(L^p, \mathcal{A}^C)$ and deduce the curvature dimension condition for $(L^p, \mathcal{A}^p)$. We have to show that the $\Gamma_2$-estimate holds pointwise a.e. in $F$ for any $u^p \in \mathcal{A}^p$. From calculations in the previous proof we have the identity (22) for $\Gamma_2^C$ in the following form

$$\Gamma_2^C(u_1 \otimes u_2) = \Gamma_2^C(i_K, \sin_k)(u_1) u_2^2 + \frac{u_1^2}{\sin_k^4} \Gamma_2^p(u_2)$$

$$+ \frac{u_1}{\sin_k^4} \cos_k \cdot u_1^2 L^p(u_2)u_2 - \frac{u_1^2}{\sin_k^4} \left(-K \sin_k + \frac{N - 1}{\sin_k} \cos_k^2\right) \Gamma^p(u_2)$$

$$+ \left(\frac{2}{\sin_k^4}(u_1')^2 - \frac{4u_1}{\sin_k^4} \cos_k \cdot u_1^2 + \frac{2u_1^2}{\sin_k^4} \cos_k^2\right) \Gamma^p(u_2).$$

(27)

for any $u_1 \in C_0^\infty(\hat{B})$ and any $u_2 \in \mathcal{A}^p$ m.c-a.e. in $I_K \times F$. We consider some open set $U \subset \hat{I}_K$ and we choose $u_1 \in C_0^\infty(\hat{I}_K)$ such that $u_1 = \sin_k$ on $U$. By the special choice of $u_1$ (27) reduces at $(r, x) \in U \times F$ to

$$\Gamma_2^C(u_1 \otimes u_2) = \Gamma_2^C(i_K, \sin_k)(u_1) u_2^2 + \frac{u_1^2}{\sin_k^4} \Gamma_2^p(u_2)$$

$$+ \frac{u_1}{\sin_k^4} \cos_k \cdot u_1^2 L^p(u_2)u_2 - \frac{u_1^2}{\sin_k^4} \left(-K \sin_k + \frac{N - 1}{\sin_k} \cos_k^2\right) \Gamma^p(u_2)$$

for m.c-a.e. $(r, x) \in U \times F$. In the case of $(I_K, \sin_k^N)$ the identity (15) in the proof of Proposition 213 implies

$$\Gamma_2^C(i_K, \sin_k^N)(u_1) = (u_1')^2 + \frac{N}{\sin_k^2} K \sin_k \cdot (u_1')^2 + \frac{N}{\sin_k^2} (\cos_k \cdot u_1')^2$$

everywhere in $\hat{I}_K$ for $u_1 \in C_0^\infty(\hat{I}_K)$. Hence, we obtain

$$\frac{u_1^2}{\sin_k^4} \Gamma_2^p(u_2) = \Gamma_2^C(u_1 \otimes u_2) - \frac{1}{\sin_k^4} \Gamma_2^C(i_K, \sin_k)(u_1) u_2^2$$

$$- \frac{u_1}{\sin_k^4} \cos_k \cdot u_1' \cdot L^p(u_2)u_2 + \frac{u_1^2}{\sin_k^4} \left(-K \sin_k + \frac{N - 1}{\sin_k} \cos_k^2\right) \Gamma^p(u_2)$$

$$= \Gamma_2^C(u_1 \otimes u_2) - (u_1')^2 u_2^2 - \frac{N}{\sin_k^2} K \sin_k \cdot (u_1')^2 u_2^2 - \frac{N}{\sin_k^2} (\cos_k \cdot u_1')^2 u_2^2$$

$$- \frac{u_1}{\sin_k^4} \cos_k \cdot u_1' \cdot L^p(u_2)u_2 + \frac{u_1^2}{\sin_k^4} \left(-K \sin_k + \frac{N - 1}{\sin_k} \cos_k^2\right) \Gamma^p(u_2)$$

16
m_c-a.e. in $U \times F$. On the other side the $\Gamma_2$-estimate for $E^r$ gives for any $u_1 \in C_0^\infty(\hat{B})$ and any $u_2 \in \mathcal{A}^r$

$$
\Gamma_2^r(u_1 u_2) \geq N K ((u'_1)^2 w_2^2 + \frac{2}{\sin^2 \kappa} \Gamma^r(u_2)) + \frac{1}{N+1} \left( L^r u_1 u_2 + \frac{N}{\sin^2 \kappa} \Gamma^r (\sin \kappa, u_1) u_2 + \frac{1}{\sin^2 \kappa} L^r u_2 \right)^2
$$

m_c-a.e. . Since $u_1(r) = \sin_s r$ and $u'_1(r) = \cos_s r$, we get after some cancellations

$$
\frac{1}{\sin^2 \kappa} \Gamma_2^r(u_2) \geq (N - 1)(\cos^2 \kappa + K \sin^2 \kappa) \frac{1}{\sin^2 \kappa} \Gamma^r(u_2) - \frac{\cos^2 \kappa}{\sin^2 \kappa} L^r(u_2) u_2
$$

$$
- (K \sin^2 \kappa)^2 u_2 - \frac{N}{\sin^2 \kappa} (\cos \kappa \cdot \cos \kappa)^2 u_2^2
$$

$$
+ \frac{1}{N+1} \left( -K \sin^2 \kappa u_2 + \frac{N}{\sin^2 \kappa} \cos^2 \kappa u_2 + \frac{1}{\sin^2 \kappa} L^r u_2 \right)^2
$$

m_c-a.e. in $U \times F$. (28)

We consider the last term on right side in (28) in more detail. From the identity (26) we deduce

$$
\frac{1}{N+1} \left( -K \sin^2 \kappa u_2 + \frac{N}{\sin^2 \kappa} \cos^2 \kappa u_2 + \frac{1}{\sin^2 \kappa} L^r u_2 \right)^2
$$

$$
= (-K \sin^2 \kappa u_2)^2 + \frac{1}{N} \left( \frac{N}{\sin^2 \kappa} \cos^2 \kappa u_2 \right)^2 + \frac{1}{N} \left( \frac{1}{\sin^2 \kappa} L^r u_2 \right)^2 + \frac{2}{\sin^2 \kappa} \cos^2 \kappa u_2 L^r u_2
$$

$$
- \frac{1}{(N+1)N} \left( \frac{N}{\sin^2 \kappa} \cos^2 \kappa u_2 + \frac{1}{\sin^2 \kappa} L^r u_2 + NK \sin^2 \kappa u_2 \right)^2
$$

It follows

$$
\Gamma_2^r(u_2) \geq (N - 1) \Gamma^r(u_2) + \frac{1}{(N+1)N} (L^r u_2 + Nu_2)^2
$$

(29)

at $x$ for $m_c$-almost every $(r, x) \in U \times F$. But since (29) does not depend on $r \in U$ anymore, we can conclude it holds for $m_c$-a.e. $x \in F$.

We fix such a $x$. $E^r$ is strongly local. So we can add constants without affecting $L^r$, $\Gamma^r$ and $\Gamma_2^r$. Thus we can replace $u_2$ by $\tilde{u}_2 := u_2 + C$, where $C = -u_2(x) - \frac{1}{r} L^r u_2(x)$. Then $(L^r u_2)(x) = (L^r \tilde{u}_2)(x)$, $\Gamma(u_2) = \Gamma(\tilde{u}_2)$ and $\Gamma_2^r(u_2) = \Gamma(\tilde{u}_2)$ and $(L^r \tilde{u}_2 + Nu_2)^2$ vanishes at $x$. Hence, we obtain the desired estimate for $u_2$ at $m_c$-almost every $x$ and we obtain the condition $BE(N-1, N)$ for $F$.

\[ \square \]

3.3. A result on essentially self-adjoint operators

**Definition 3.11** (direct sum). Suppose $(\mathcal{H}_i, \| \cdot \|_{\mathcal{H}_i})_{i \in \mathbb{N}}$ is a sequence of Hilbert spaces. Its direct sum is a Hilbert space that is given by

$$
\mathcal{H} = \bigoplus_{i=1}^{\infty} \mathcal{H}_i = \left\{ v := (v_i)_{i \in \mathbb{N}} : v_i \in \mathcal{H}_i \text{ such that } \| v \|_{\mathcal{H}}^2 := \sum_{i=1}^{\infty} \| v_i \|_{\mathcal{H}_i}^2 < \infty \right\}
$$

where the inner product is $\sum_{i=1}^{\infty} (v_i, u_i)_{\mathcal{H}_i} = (v, u)_{\mathcal{H}}$. Additionally, we introduce

$$
\sum_{i=1}^{\infty} \mathcal{H}_i = \left\{ v := (v_i)_{i \in \mathbb{N}} : v_i \in \mathcal{H}_i \text{ and } v_i = 0 \text{ except for finitely many } i \in \mathbb{N} \right\}.
$$
Theorem 3.12. Let $\mathcal{E}^F$ be a regular and strongly local Dirichlet form. Assume the spectrum of $-L^F$ is discrete and its first positive eigenvalue satisfies $\lambda_1 \geq N$. Let $E_i \subset D^2(L^F)$ be the eigenspace that corresponds to the $i$th eigenvalue $\lambda_i$. Let $E^C = I_K \times_{\sin^N} \mathcal{E}^F$ be the $(K,N)$-cone for $K \in \mathbb{R}$ and $N \geq 1$ over $\mathcal{E}^F$ and let $L^C$ be the corresponding self-adjoint operator. Let $A$ be dense in the domain of $L^{I_K, \sin^N}$. Then

$$\Xi = [A \otimes E_0] \oplus \sum_{i=1}^{\infty} C_0^\infty(\hat{I}_K) \otimes E_i$$

is dense in the domain of $L^C$ with respect to the graph norm.

Proof. Since $L^F$ has a discrete spectrum, there is a spectral decomposition of $L^2(m_F)$ with respect to its eigenvalues.

$$L^2(F, m_F) = E_0 \oplus \bigoplus_{i=1}^{\infty} E_i = E_0 \oplus E_\perp.$$ 

We denote the restriction of $\mathcal{E}^F$ and $(\cdot, \cdot)_{L^2(m_F)}$ to $E_\perp$ by $\mathcal{E}^F = \mathcal{E}^F|_{E_\perp \times E_\perp}$ and $(\cdot, \cdot)_{E_\perp}$ respectively. Then there is a densely defined, self-adjoint operator $L^F$ on $(E_\perp \times E_\perp)$ that corresponds to $\mathcal{E}^F$. It is easy to see that $D^2(L^F) = D^2(L^F) \cap E_\perp$. Also $L^F(C, m_F)$ can be decomposed orthogonally into $L^2(C, m_F) = U_0 \oplus U_\perp$ where $U_0 = L^2(I_K, \sin^N \, dr) \otimes E_0$ and $U_\perp = L^2(I_K, \sin^N \, dr) \otimes E_\perp$. For $u = u_1 \perp u_2 \in U_0$ and $v = v_1 \perp v_2 \in U_\perp$ with $u_1, v_1 \in L^2(I_K, \sin^N \, dr)$, $u_2 \in E_0$ and $v_2 \in E_\perp$, we have

$$\mathcal{E}^F(u, v) = \mathcal{E}^{I_K, \sin^N}(u_1, u_2)[(u_2, v_2)_{L^2(m_F)} + (u_1, u_2)_{L^2(I_K, \sin^N \, dr)}] \mathcal{E}^F(u_2, v_2) = 0$$

since $\mathcal{E}^F(u_2, v_2) = -(L^F u_2, v_2)_{L^2(m_F)} = 0$ and $E_0 \perp E_\perp$ in $L^2(m_F)$. Thus we can decompose $\mathcal{E}^C$ orthogonally as follows

$$\mathcal{E}^C = \mathcal{E}^C|_{U_0 \times U_0} + \mathcal{E}^C|_{U_\perp \times U_\perp} =: \mathcal{E}_0^C + \mathcal{E}_{\perp}^C.$$ 

One checks that $u = u_0 + u_\perp \in D(\mathcal{E}_0^C)$ if and only if $u_0 \in D(\mathcal{E}^F)$ and $u_\perp \in D(\mathcal{E}^F)$ and that $u = u_0 + u_\perp \in D(\mathcal{E}_\perp^C)$ if and only if $u_0 \in D(L_0^C)$ and $u_\perp \in D(L_\perp^C)$ and we have $L^C = L_0^C + L_\perp^C$. $L_\perp^C$ is a densely defined operator on $U_\perp$ with

$$D^2(L_\perp^C) = D^2(L^F) \cap U_\perp = D^2(L^C) \cap L^2(I_K, \sin^N \, dr) \otimes E_\perp.$$ 

$C_0^\infty(\hat{I}_K) \otimes D^2(L^F)$ is a subset of $D^2(L^C)$, hence, $C_0^\infty(\hat{I}_K) \otimes D^2(L_\perp^C)$ is a subset of $D^2(L_\perp^C)$. For $u_1 \in C_0^\infty(\hat{I}_K)$ and $u_2 \in D^2(L_\perp^C)$ we have

$$L_\perp^C(u_1 \otimes u_2) = L^{I_K, \sin^N} u_1 \otimes u_2 + \frac{u_1}{\sin^2} \otimes L^F_{u_2} u_2.$$ 

For all $i \in \mathbb{N} \setminus \{0\}$ we set $\hat{U}_i = U_i \cap C_0^\infty(\hat{I}_K) \otimes D^2(L_\perp^C) = C_0^\infty(\hat{I}_K) \otimes E_i$ and consider the restriction of $L_\perp^C$ to $\hat{U}_i$

$$L_{\perp, i}^C|_{\hat{U}_i} = L^C|_{C_0^\infty(\hat{I}_K) \otimes E_i} = (L^{I_K, \sin^N} + \frac{N}{\sin^2})|_{C_0^\infty(\hat{I}_K) \otimes E_i} I|_{E_i} = (L^I_K + \frac{N}{\sin^2} \Gamma^{I_K}(\sin, \cdot) - \frac{N}{\sin^2})|_{C_0^\infty(\hat{I}_K) \otimes E_i} I|_{E_i} =: L_\perp^C.$$ 

$L_{\perp, i}^C$ is a densely defined operator on $L^2(\sin^N \, dr) \otimes E_i$ and we define

$$\sum_{i=1}^{\infty} L_{\perp, i}^C := \hat{L}_\perp^C \text{ on } \sum_{i=1}^{\infty} C_0^\infty(\hat{I}_K) \otimes E_i.$$
We will show that \( \tilde{L}_+^C \) is essentially self-adjoint. Then, the unique self adjoint extension of \( \tilde{L}_+^C \) has to coincide with \( L_+^C \). In particular, \( \sum_{i=1}^{\infty} C_0^\infty(I_K) \otimes E_i \) is dense in \( D^2(L_+^C) \) with respect to the graph norm. It is sufficient to show that the operator \( (L_{IK} + \frac{N}{\sin r} \Gamma(\sin r) - \frac{\lambda}{\sin^2 r})|C_0^\infty(I_K) \) is essentially self-adjoint for every \( \lambda \in \text{spec } L^r \) (see [41, ch. X, problem 1.a]).

We follow the proof of Theorem X.11 in [41]. Consider the unitary transformation

\[
U : L^2(I_K, \sin^2 r) dr \rightarrow L^2(I_K, dr), \quad \phi(r) \mapsto \sqrt{\sin r} \phi(r).
\]

\( C_0^\infty((0, \infty)) \) is invariant under \( U \) and \( L_{IK, \sin^2 r} - \frac{\lambda}{\sin^2 r} \) takes the form

\[
U \left( \frac{d^2}{dr^2} + \frac{N}{\sin r} \frac{d}{dr} \frac{d}{dr} - \frac{\lambda}{\sin^2 r} \right) U^{-1} = \frac{d^2}{dr^2} + \left( \frac{N^2}{4} \cos^2 r - \frac{N}{2} - \lambda_i \right) \frac{1}{\sin^2 r}.
\]

We get a Schrödinger-type operator defined on \( C_0^\infty(\hat{I}_K) \). The question, if such an operator is essentially self-adjoint, is a classical problem from quantum mechanics. It was answered by Hermann Weyl who analyzed the solutions of the following ordinary differential equation \(-\phi'' + V \phi = \lambda \phi\). One says that \( V(r) \) is in the limit circle case at \( r \in \partial I_K \) (we assume that \( \partial I_K = \{0, \infty\} \) for \( K \leq 0 \)) if for some \( \lambda \), all solutions are locally square integrable around \( r \). Otherwise we say \( V \) is in the limit point case at \( r \).

**Theorem 3.13** (Weyl’s limit point-limit circle criterion). Let \( V \) be a continuous real-valued function on \( \hat{I}_K \). Then, the operator \( H = -d^2/dr^2 + V \) is essentially self-adjoint on \( C_0^\infty(\hat{I}_K) \) if and only if \( V \) is in the limit point case at any \( r \in \partial I_K \).

For the particular case that we consider, the limit point case at \( \infty \) for \( K \leq 0 \) is easy to check (see Theorem X.8 in [41] and the next corollary). The case \( r = 0 \) for \( K \leq 0 \) and \( r = 0 \) and \( r = \pi/\sqrt{K} \) for \( K > 0 \) follows from the next theorem.

**Theorem 3.14.** Let \( V \) be continuous on \( \hat{I}_K \) and positive near \( r_0 \). If \( V(r) \geq \frac{N}{4} \frac{1}{(r-r_0)^2} \) for \( r \rightarrow r_0 \in \partial I_K \setminus \{ \infty \} \) then \( H = -d^2/dr^2 + V(r) \) is in the limit point case at \( r_0 \). If for some \( \epsilon > 0 \), \( V(r) \leq \left( \frac{N}{4} - \epsilon \right) \frac{1}{(r-r_0)^2} \) near \( r_0 \), then \( H \) is in the limit circle case at \( r_0 \).

**Proof.** \( \rightarrow \) Theorem X.10 in [41].

So far we did not use the spectral gap of \(-L^r\). We have \( \lambda_i \geq N \) for any \( \lambda_i \in \text{spec } L^r \). But then, the operator \(-\frac{d^2}{dr^2} - \frac{\lambda_i}{\sin^2 r} \) is essentially self-adjoint on \( C_0^\infty((0, \infty)) \) since for \( r \rightarrow 0 \)

\[
\left( \frac{N^2}{4} \cos^2 r - \frac{N}{2} - \lambda_i \right) \frac{1}{\sin^2 r} \sim \left( \frac{N(N-2)}{4} + \lambda \right) \frac{1}{r^2} \geq \left( \frac{N(N-2)}{4} + N \right) \frac{1}{r^2} \geq \frac{3}{4r^2}
\]

where the last inequality holds if \( N \geq 1 \). Analogously for \( r \rightarrow \pi \).

For \( C_0^\infty \) we can not follow this strategy (see the next Remark). But since \( A \) is assumed to be dense in the domain of \( L_{IK, \sin^2 r} \), we obtain that

\[
\Xi = \left[ A \otimes E_0 \right] \oplus \sum_{i=1}^{\infty} C_0^\infty(\hat{I}_K) \otimes E_i \subset D^2(L^C) \oplus D^2(L^C) = D^2(L^C)
\]

is dense in \( D^2(L^C) \).

**Remark 3.15.** For \( u_1 \otimes c \in C_0^\infty(\hat{I}_K) \otimes \mathbb{R} \) we have that \( L_+^C(u_1 \otimes c) = L_{IK, \sin^2 r} \mathbb{R} \). But in general, \( L_{IK, \sin^2 r}|C_0^\infty(I_K) \) is not essentially self-adjoint. More precisely, under the transformation \( U \) it becomes

\[
\hat{L} = L_{IK, \sin^2 r}|C_0^\infty(I_K) = \frac{d^2}{dr^2} - \left( \frac{N^2}{4} \cos^2 r - \frac{N}{2} \right) \frac{1}{\sin^2 r}
\]

is not essentially self-adjoint. More precisely, under the transformation \( U \) it becomes

\[
\hat{L} = L_{IK, \sin^2 r}|C_0^\infty(I_K) = \frac{d^2}{dr^2} - \left( \frac{N^2}{4} \cos^2 r - \frac{N}{2} \right) \frac{1}{\sin^2 r}
\]
We see that
\[
\left( \frac{N^2}{4} \cos^2 \frac{N}{2} \right) \frac{1}{\sin^2} - \frac{N(N-2)}{4} \frac{1}{r^2} \geq \frac{3}{4r^2} \quad \text{if } N \geq 3
\]
for \( r \to 0 \) and
\[
\left( \frac{N^2}{4} \cos^2 \frac{N}{2} \right) \frac{1}{\sin^2} \leq \frac{3}{4r^2} - \epsilon \quad \text{for some } \epsilon > 0 \text{ if } N < 3
\]
for \( r \to 0 \). Analogously for \( r \to \pi \). Hence, in the case \( N \geq 3 \) we can choose \( C_0^\infty(\tilde{I}_k) \) as dense subset \( A \) in \( D^2(\hat{L}^K,\sin^N_K) \) since the closure of \( C_0^\infty(\tilde{I}_k) \) is the only self-adjoint extension. On the other hand, in the case \( 1 \leq N < 3 \) the operator \( \hat{L} \) is not essentially self-adjoint. There are more than one self-adjoint extensions \( A_\eta \) of \( \hat{L} \) and
\[
D^2(\text{Cl}(\tilde{L})) \subseteq D(\text{A}_\eta) \subset D((\hat{L})^*)
\]
where \( \text{Cl}(\tilde{L}) \) denotes the closure with respect the graph norm and \((\hat{L})^* \) is the adjoint operator. Hence \( C_0^\infty(\tilde{I}_k) \) can not be dense in the domain \( D^2(\hat{L}^K,\sin^N_K) \) if \( 1 \leq N < 3 \). But we can choose
\[
A_0 := \bigcup_{t>0} P_t^{I^K,\sin^N_K} [C^\infty(\tilde{I}_k) \cap L^2(\sin^N_K r dr)] \subset D(\Gamma_2^{I^K,\sin^N_K}) \cap C^\infty(\tilde{I}_k).
\]
where \( P_t^{I^K,\sin^N_K} \) is the induced semi-group of \( L^{I^K,\sin^N_K} \). This is a dense subset in the domain of \( L^{I^K,\sin^N_K} \) since it is stable with respect the semigroup, and it consists of functions that are smooth in \( \tilde{I}_k \) since they solve a parabolic PDE with smooth coefficients in \( \tilde{I}_k \).

Remark 3.16. In the following the set \( \Xi \) will play an important role. Therefore, we will consider the cases \( \lambda_1 \geq 3 \) and \( \lambda_1 \in [1,3) \), that appear in the previous theorem, separately. In the first case we choose \( A = C_0^\infty(\tilde{I}_k) \), in the second case we choose \( A = A_0 \).

Lemma 3.17. Consider \( E^F \) and \( I^K \times \sin^N_K E^F \) as before. Assume \( L^C \) has a discrete spectrum and let \( E_i \) be the eigenspace for the \( i \)-th eigenvalue. Consider \( u = \sum_{i=1}^k u_1^i \otimes u_2^i \in \Xi \). Then
\[
P_t^C u = \sum_{i=0}^k P_t^{I^K,\sin^N_K,\lambda_i} u_1^i \otimes u_2^i \quad \text{for any } t > 0.
\]
where \( P_t^{I^K,\sin^N_K,\lambda_i} \) is the semi-group that is generated by \( L^{I^K,\sin^N_K,\lambda_i} \).

Proof. \( \rightarrow [38, \text{proof of Lemma 3.3}] \)

Remark 3.18. In any case \( \lambda_1 \geq 1 \) we also define
\[
\Xi' := \bigcup_{t>0} P_t^C \Xi = [A_0 \otimes E_0] \oplus \left[ \sum_{i=1}^\infty P_t^{I^K,\sin^N_K,\lambda_i} C_0^\infty(\tilde{I}_k) \otimes E_\lambda \right].
\]
\( \Xi' \) is dense in \( D^2(L^C) \) and stable with respect to \( P_t^C \). \( \Xi' \) will provide a suitable class of test function.

Lemma 3.19. Consider \( I^K \times \sin^N_K E^F \), the corresponding operator \( L^C \) and \( u \in \Xi' \) of the form
\[
u = u_1 \otimes u_2 = P_t^C (\tilde{u}_1 \otimes u_2) = P_t^{I^K,\sin^N_K,\lambda_1} \tilde{u}_1 \otimes u_2 \in D^2(L^C)
\]
where \( u_2 \) is an eigenfunction for the eigenvalue \( \lambda \in \{0\} \cup [N,\infty) \) of \(-L^F \) with \( N \geq 1 \), and \( \tilde{u}_1 \in C_0^\infty(\tilde{I}_k) \) if \( \lambda > 0 \) and \( \tilde{u}_1 \in A_0 \) if \( \lambda = 0 \). Then
\[
L^C u = L^{I^K,\sin^N_K,\lambda_1} u_1 \otimes u_2.
\]
Proof. We choose $v = v_1 \otimes v_2$ where $v_1 \in C^0_0(\tilde{I}_K) \subset D(\mathcal{E}^{I_K, \sin^N}_L)$ and $v_2 \in D(\mathcal{E}^r)$. Then

$$-(L^r u, v)_{L^2(m_c)} = \int_F \mathcal{E}^{I_K, \sin^N}_L (u^r, v^r) d m_r(x) + \frac{1}{\tilde{p}(p)} \mathcal{E}^r (u^r, v^r) \sin^N_r r dr$$

$$= \int_F \left[ \mathcal{E}^{I_K, \sin^N}_L (u_1, v_1) + \int_{I_K} \frac{\tilde{m}_{1}}{\sin^N_r} \sin^N_r r dr \right] u_2 v_2 d m_r$$

$$= \mathcal{E}^{I_K, \sin^N}_L (u_1, v_1) \int_F u_2 v_2 d m_r = -(L^{I_K, \sin^N}_L u_1 \otimes u_2, v)_{L^2(m_c)}$$

Since $C^0_0(\tilde{I}_K) \otimes D(\mathcal{E}^r)$ is dense in $L^2(m_c)$, the last identity holds for any $v \in L^2(m_c)$ and the statement follows.

3.4. Intermezzo

Now, we want to understand the regularity of $P^{I_K, \sin^N}_L u$ for $\lambda \geq N \geq 1$, $K > 0$ and $u \in C^0_0(\tilde{I}_K)$. This might be done by studying the corresponding Sturm-Liouville operator

$$L^{I_K, \sin^N}_K u |_{C^0_0(\tilde{I}_K)} = \frac{d^2}{dr^2} + N \frac{\cos K}{\sin^N_r} \frac{d}{dr} - \frac{\lambda}{\sin^N_r}$$

and its eigenfunctions. We will go another way and use the result by Bacher and Sturm from [11] that states that Theorem 1.1 holds if the underlying space is a weighted Riemannian manifold. Then, we also use Theorem 4.26 which connects the Bakry-Emery condition for Dirichlet forms with the Riemannian-curvature-dimension condition to deduce $L^\infty$-bounds for the gradient of $P^{I_K, \sin^N}_L u$.

**Proposition 3.20.** Let $\lambda \geq N \geq 1$ and $K > 0$. Consider the essentially self-adjoint operator $L^{I_K, \sin^N}_K u |_{C^0_0(\tilde{I}_K)}$ and the corresponding semi-group $P^{I_K, \sin^N}_L u$. Then

$$\Gamma^{I_K} (P^{I_K, \sin^N}_L u) = \left( (P^{I_K, \sin^N}_L u)^t \right)^2 \in L^\infty(\tilde{I}_K, \sin^N_r r dr).$$

for $u \in C^0_0(\tilde{I}_K)$.

Proof. Let us assume that $K = 1$ and we consider the metric measure space $F = (I_K, \sin^N_r dr)$ for $\tilde{K} \geq 1$ such that $\tilde{K}N = \lambda$. $F$ satisfies the condition $RCD^*(\tilde{K}(N-1), N)$. We have the Dirichlet form

$$\mathcal{E}^{I_K, \sin^{N-1}} = \text{Ch}^r$$

on $L^2(\sin^{N-1}_r dr)$. By Theorem 2.15 it satisfies the Bakry-Emery condition $BE(\tilde{K}(N-1), N)$.

The first non-negative eigenvalue of the corresponding self-adjoint operator equals $\tilde{K}N = \lambda$. An eigenfunction is given by $\cos^r$ what easily can be checked. Since $1 \leq \tilde{K}$, $F$ also satisfies $RCD^*(N-1, N)$ and we can consider the metric $(1, N)$-cone $[0, \pi] \times_{\sin^N} F$. By the result of Bacher and Sturm from [11] it satisfies $CD^*(N, N + 1)$ but also $RCD^*(N, N + 1)$ because of Corollary 5.15. By Theorem 4.26 the Cheeger energy $\text{Ch}^{\text{Con}, N, K(r)}$ of $[0, \pi] \times_{\sin^N} F$ satisfies $BE(N, N + 1)$. It implies a Bakry-Emery gradient estimate

$$|\nabla P^{\text{Con}, N, K(r)}_t u|_{\infty}^2 \leq e^{-2Nt} P^{\text{Con}, N, K(r)}_t |\nabla u|_{\infty}^2$$

for $u \in D(\text{Ch}^{\text{Con}, N, K(r)})$. By the main results from Section 5.2 the Cheeger energy of the metric cone coincides with the $N$-skew product $I_1 \times_{\sin^N} \text{Ch}^r$ in the sense of Dirichlet forms and we have

$$|\nabla u|_{\infty}^2 = ((u^x)^t)^2 + \frac{1}{\sin^N r} ((u^x)^t)^2 = \Gamma^{0, \pi}(u^x) + \frac{1}{\sin^N r} I^r_K (u^r).$$

21
In particular, the curvature-dimension condition implies that the metric \((1,N)\)-cone satisfies volume doubling and supports a Poincaré inequality. Hence, \(P^c_t\) is \(L^2 \to L^\infty\)-ultracontractive by Remark 2.3.

We choose \(u = u_1 \otimes u_2\) where \(u_1 \in C^\infty_0((0,\pi))\) and \(u_2 \in E_1\). \(E_1\) denotes the eigenspace of \(\lambda\). Lemma 3.17 implies that \(P^c u = P^{[0,\pi],\sin^N,\lambda} u_1 \otimes u_2\) and (31) becomes

\[
\Gamma^{[0,\pi]}(P_t^{[0,\pi],\sin^N,\lambda} u_1) u_2^2 + \frac{1}{\sin^2 u}(P_t^{[0,\pi],\sin^N,\lambda} u_1)^2 \Gamma^I_K (u_2) \leq e^{-2Nt} P_t^c \Gamma^C(u) \in L^\infty(m_c).
\]

This implies that

\[
\Gamma^{[0,\pi]}(P_t^{[0,\pi],\sin^N,\lambda} u_1) = ((P_t^{[0,\pi],\sin^N,\lambda} u_1))^2 \in L^\infty(\sin^N r dr)
\]

that is the statement in the case \(K = 1\).

**Remark 3.21.** At this point we can make an important remark on the regularity of test functions \(u \in \mathcal{E}'\). Consider a strongly local, regular and strongly regular Dirichlet form \(\mathcal{E}'\) that satisfies \(BE(N-1,N)\) and a volume doubling property and supports a local Poincaré inequality. Assume that closed balls are compact. Then remark 2.3 implies \(L^2 \to L^\infty\)-ultracontractivity for \(P^c_t\) and it follows that \(P^c_t \Gamma^C(u) \in L^\infty(m_c)\) for any \(u \in D(\mathcal{E}')\). Hence, if we consider eigenfunctions of \(L^c\), the Bakry-Ledoux gradient estimate implies

\[
\Gamma^c(P^c_t u) = e^{-\lambda t} \Gamma^c(u) \leq P^c_t \Gamma^c(u) \in L^\infty(m_c)
\]

and especially \(u, \Gamma^c(u) \in L^\infty(m_c)\). Then, the previous proposition implies for

\[ u = u_1 \otimes u_2 \in P^c_t [C^\infty_0(\hat{I}_K) \otimes E_\lambda] = P^c_t^{I_K,\sin^N,\lambda} C^\infty_0(\hat{I}_K) \otimes E_\lambda \quad \text{and} \quad \lambda \geq 1 \]

that \(u, \Gamma^C(u), L^c u \in L^\infty(m_c)\). The same conclusion holds for \(u = u_1 \otimes u_2 \in \mathcal{A}_0 \otimes \mathcal{E}_0\) because \((I_K, \sin^N r dr)\) satisfies \(RCD(N,N+1)\). Hence, for any \(u \in \mathcal{E}'\) we have \(u, \Gamma^c(u), L^c u \in L^\infty(m_c)\).

**Remark 3.22.** Consider \(u \in P^c_t^{I_K,\sin^N,\lambda} C^\infty_0(\hat{I}_K)\) for \(\lambda \geq 1\). We know that \(\Gamma^I_K(u) \in L^\infty\). Especially

\[
\int_{I_K} \Gamma^I_K(u, \phi) \sin^N r dr \leq \|u\|_{L^\infty(\sin^N r dr)} < \infty
\]

for any \(\phi \in C^\infty_0(\hat{I}_K)\) with \(\|\sqrt{\Gamma^I_K(\phi)}\|_{L^\infty} \leq 1\). Hence, there exists a Radon measure \(-\Delta u\) on \(I_K\) such that

\[
\int_{I_K} d(-\Delta u) = \int_{I_K} \Gamma^I_K(u, \phi) \sin^N r dr = \int_{I_K} L^{I_K,\sin^N,\phi} u \sin^N r dr.
\]

\(L^{I_K,\sin^N,\lambda} u \in L^2(\sin^N r dr)\) and for \(\phi \in C^\infty_0(\hat{I}_K)\) we obtain

\[
\int_{I_K} L^{I_K,\sin^N,\lambda} u \phi \sin^N r dr = \mathcal{E}^{I_K,\sin^N,\lambda}(u, \phi)
\]

\[
= \int_{I_K} \Gamma^I_K(u, \phi) \sin^N r dr + \int_{I_K} \frac{\lambda}{\sin^N} u \phi \sin^N r dr
\]

\[
= \int_{I_K} \phi d(-\Delta u) + \int_{I_K} \frac{\lambda}{\sin^2} u \phi \sin^N r dr.
\]

In the case \(u \in \mathcal{A}_0\) and \(\lambda = \lambda_0 = 0\) these identities are already true where \(d\Delta u = L^{I_K,\sin^N,\lambda} u \sin^N r dr\).
3.5. Proof of the Bakry-Emery condition for \((K, N)\)-cones

We assume \(K > 0\). Then, the underlying cone measure \(m_C\) is finite and constants are integrable.

**Theorem 3.23.** Let \(\mathcal{E}^r\) be a strongly local, regular and strongly regular Dirichlet form that satisfies the Bakry-Emery curvature-dimension condition \(\text{BE}(N - 1, N)\) in the sense of Definition 2.7.

Assume \(\mathcal{E}^r\) satisfies a volume doubling property, it admits a local \((2, 2)\)-Poincaré inequality and closed balls are compact. Assume the spectrum of \(L^r\) is discrete and the first positive eigenvalue of \(-L^r\) satisfies \(\lambda_1 > N\). Let \(\mathcal{E}^c = I_K \times \mathcal{E}^r\) be the \((K, N)\)-cone for \(K > 0\) and \(N \geq 1\) over \(\mathcal{E}^r\) and let \(L^c\) be the corresponding self-adjoint operator. Assume also that \(\mathcal{E}^c\) satisfies a volume doubling property and admits a local \((2, 2)\)-Poincaré inequality. Then \(I_K \times \mathcal{E}^r\) satisfies \(\text{BE}(KN, N + 1)\).

**Proof.** By Lemma 4.4 \(\mathcal{E}^c\) is a strongly regular Dirichlet form and closed balls with respect to the intrinsic distance \(d_C\) are compact. Hence, we can make use of Sturm’s result from [43, 44, 45]. Especially, there are the properties of Remark 4.5. Consider

\[
\Xi = [\mathcal{A} \otimes E_0] \oplus \left[ \sum_{i=1}^{\infty} C^{\infty}_i(\hat{I}_K) \otimes E_i \right]
\]

from Theorem 3.12. In the case \(N \geq 3\) we set \(\mathcal{A} = C^{\infty}_0(\hat{I}_K)\), and in the case \(N < 3\) we set \(\mathcal{A} := \mathcal{A}_0 \subset D(I^r_{K, \sin^N K}) \cap C^{\infty}(\hat{I}_K)\) like in Remark 3.13. Consider \(u = \sum_{i=0}^{k} u^1_i \otimes u^2_i \in \Xi\). We have

\[
L^c u = L_{n, \sin^N K} u^r + \frac{1}{\sin K} L^r u^p = \sum_{i=0}^{k} \left( L_{n, \sin^N K} u^1_i u^2_i + \frac{1}{\sin K} L^r u^1_i u^2_i \right) = \sum_{i=0}^{k} \left( L_{n, \sin^N K} u^1_i + \frac{\lambda}{\sin K} u^1_i \right) u^2_i \in [L^c_{K, \sin^N K} \mathcal{A} \otimes E_0] \oplus \sum_{i=1}^{\infty} C^{\infty}_i(\hat{I}_K) \otimes E_i \subset D(\mathcal{E}^c).
\]

In particular, \(\Xi \subset D(\Gamma^c_\lambda)\). We remind on the regularity properties of eigenfunctions of \(L^r\) and of test function \(\phi \in \Xi'\) (see Remark 3.21). Hence, \(\Gamma_\lambda^c(u, v; \phi)\) is well-defined for any \(u, v \in \Xi\) and any test function \(\phi \in \Xi'\).

1. First, we assume \(N \geq 3\). Let \(u = u_1 \otimes u_2 \in C^{\infty}_0(\hat{I}_K) \otimes E_i\) and \(v = v_1 \otimes v_2 \in C^{\infty}_0(\hat{I}_K) \otimes E_j\) for \(i, j > 0\). We take a test function \(\phi \in \Xi'\) of the form

\[
\phi = \phi_1 \otimes \phi_2 \in P_t^{\Gamma^c(\hat{I}_K, \sin^N K)} \otimes C^{\infty}_0(\hat{I}_K) \otimes E_\lambda \subset \Xi' \text{ if } \lambda > 0
\]

or \(\phi \in \mathcal{A}_0 \otimes E_0\) if \(\lambda = 0\) (see the definition of \(\Xi'\)). We know that \(\phi, \Gamma^c(\phi), L^c \phi \in L^\infty(m_C)\).

\[
\Gamma_\lambda^c(u, v; \phi) = \int_{\hat{I}_K} \frac{1}{2} \Gamma^c(u, v)L^c \phi d m_C - \int_{\hat{I}_K} \Gamma^c(u, L^c v) \phi d m_C
\]

\[
= \int_{\hat{I}_K} \frac{1}{2} \left( \Gamma^c(u_1, v_1)u_2v_2 + \frac{m_C}{\sin K} \Gamma^c(u_2, v_1) \right) L^c \phi d m_C - \int_{\hat{I}_K} \Gamma^c(u, L_{K, \sin^N K}^r v_1v_2 + \frac{m_C}{\sin K} L^r v_2) \phi d m_C
\]

We consider (I):

\[
(I) := \int_{\hat{I}_K} \int_{\hat{I}_K} \frac{1}{2} \Gamma^c(u_1, v_1)u_2v_2L^c \phi \sin^N K r dr d m_C + \int_{\hat{I}_K} \int_{\hat{I}_K} \frac{1}{2} \frac{m_C}{\sin K} \Gamma^c(u_2, v_2) L^c \phi \sin^N K r dr d m_C
\]

\[
= (I)_1 + (I)_2
\]

23
One can easily check that the integrals are well-defined. For example, we see that $(I)_1 < \infty$ since $\Gamma^I(u_1, v_1) \in C_0^\infty(I_k)$ and $u_2v_2 \in L^1(m_r)$. We can calculate $(I)_1$ explicitly because of Proposition 3.22 and Remark 4.2.2.

We consider $(I)_1$:

$$2(I)_1 = \int \int L^{I_k}(u_1, v_1) L^{I_k} \sin^N \phi_1 \sin^N \phi d \tau d m_r$$

$$= \left( \int \int L^{I_k}(u_1, v_1) \phi_1 \sin^N \phi d \tau \right) \int u_2v_2 d m_r$$

Then, we consider $(\grave{\text{II}})_2$:

$$\Gamma^C_2(u, v, \phi) = \int \int \int L^{I_k}(u_1, v_1) u_2v_2 \phi d m_r$$

Finally, we obtain

$$\Gamma^C_2(u, v, \phi) = \int \int \int L^{I_k}(u_1, v_1) u_2v_2 \phi d m_r$$
\( \Gamma_2^c(u; v; \phi) \) and also the right hand side in the last equation is linear in \( \phi \). Hence, the last equation also holds for general \( \phi = \sum_{i=1}^k \phi^*_i \otimes \phi^*_i \in \Xi' \). \( \Gamma_2^c(u; v; \phi) + \Gamma_2^c(v; u; \phi) \) looks exactly like the weak version of equation (21) in the proof of Theorem 3.9 where we prove the classical \( \Gamma_2 \)-estimate. Hence, we can proceed exactly like in the proof of Theorem 3.9 and we obtain the sharp \( \Gamma_2 \)-estimate in a weak form for \( u \in \Xi \) and test functions \( \phi \in \Xi' \) with \( \phi \geq 0 \).

2. Now, we deal with the case \( 1 \leq N < 3 \). We compute \( \Gamma_2^c(u; v; \phi) \) exactly like in the case \( N \geq 3 \) but we have to consider the case when \( u_1 \otimes u_2 \in A_0 \otimes E_0 \) and \( v_1 \otimes v_2 \in C_0^\infty(I_K) \otimes E_i \) for \( i > 0 \) separately. Any other case is already covered by the previous paragraph. We recall that \( u_2 = \text{const} = m \in \mathbb{R} \) because of Remark 2.3. We can compute \( \Gamma_2^c(u; v; \phi) \) for \( \phi \in \Xi' \) exactly like in the previous paragraph since terms of the form \( u_1 v_1, \Gamma^I_K(u_1, v_1) \) and \( L^{I_K, \sin K}_c(u_1 v_1) \) are in \( C_0^\infty(I_K) \). We obtain again formula (33).

The only case that we still have to check is \( u = v \in A_0 \otimes E_0 \). It is not covered, yet, since \( u_1^2 \notin C_0^\infty(I_K) \). First, let \( \phi = \phi_1 \otimes \phi_2 \in P^{I_K, \sin K}_c \otimes E_i \). We know that \( u = u_1 \otimes m \in D(\mathcal{E}^c) \), \( u_1 \in A_0 \subset D(\Gamma^{I_K, \sin K}_c) \) and \( \Gamma^c(u) = \Gamma^{I_K}(u_1)m^2 \). Hence,

\[
\Gamma_2^c(u; \phi) = \int_C \frac{1}{2} L^c \phi \Gamma^c(u)d m_c - \int_C \Gamma^c(u, L^c u) \phi d m_c
\]

\[
= \int_C \frac{1}{2} L^{I_K, \sin K}_c \phi_1 \phi_2 \Gamma^{I_K}(u_1)m^2 \sin^N r dr dm_r
\]

\[
- \int_C \Gamma^{I_K}(u_1, L^{I_K, \sin K}_c u_1)m^2 \phi_1 \phi_2 \sin^N r dr dm_r
\]

\[
= \int_C m^2 \phi_2 d m_r \int_C \Gamma^{I_K}(u_1) L^{I_K, \sin K}_c \phi_1 - \Gamma^{I_K}(u_1, L^{I_K, \sin K}_c u_1) \phi_1 \sin^N r dr
\]

Since \( \phi_2 \) is an eigenfunction of \( L^c \), the right hand side is 0 unless \( \lambda_1 = 0 \) and \( \phi_2 \neq 0 \). We conclude that \( \Gamma_2^c(u; \phi) \neq 0 \) for \( \phi \in \Xi' \) only if \( \phi_2 = \text{const} \neq 0 \). In any case:

\[
\Gamma_2^c(u; \phi) = \int_C m^2 \phi_2 d m_r \Gamma^{I_K, \sin K}_c(u_1; \phi_1).
\] (34)

This is just (33) where we replace \( \Gamma^{I_K, \sin K}_c(u_1) \phi_1 \) by \( \Gamma^{I_K, \sin K}_c(u_1; \phi_1) \). But we can proceed like at the end of the previous paragraph. Because \( \mathcal{E}^{I_K, \sin K}_c \) satisfies \( BE(K, N + 1) \) we can bound (34) by

\[
\Gamma_2^c(u; \phi) \geq m^2 \int_C \int_K \left( KN \phi \Gamma^{I_K, \sin K}_c(u_1) + \frac{1}{N + 1} (L^{I_K, \sin K}_c u_1)^2 \phi \right) \sin^N r dr dm_r
\]

if \( \phi \geq 0 \). Hence, for \( u \in \Xi \) and \( \phi \in \Xi' \) with \( \phi \geq 0 \) we have the desired \( \Gamma_2 \)-estimate.

3. We extend this estimate to any function \( u \in D(\Gamma_2) \). We choose a sequence \( u_n \in \Xi \) that converges to \( u \in D(\Gamma_2) \) in \( D^2(L^c) \). Then we obtain that

\[
\int_C \Gamma^c(u_n, u_n) L^c \phi d m_c \rightarrow \int_C \Gamma^c(u, u) L^c \phi d m_c, \quad \int_C \Gamma^c(u_n, u_n) \phi d m_c \rightarrow \int_C \Gamma^c(u, u) \phi d m_c, \quad \int_C L^c u_n \phi d m_c \rightarrow \int_C L^c u \phi d m_c.
\] (35)

for \( \phi \in D^{I_K}_c(L^c) \cap \Xi \). We still need to show convergence of \( \int_C \Gamma^c(u_n, L^c u_n) \phi d m_c \). Since \( u_n, L^c u_n, \phi \in D(\mathcal{E}^c) \) and \( \phi, \Gamma^c(\phi) \in L^\infty(m_c) \), we can apply the Leibniz rule (6) for \( \Gamma^c \). We
obtain
\[ \int_C \Gamma^e(u_n, L^e u_n) \phi d m_c = \int_C \Gamma^e(u_n, L^e u_n \phi) d m_c - \int_C \Gamma^e(u_n, \phi) L^e u_n d m_c \]
\[ = - \int_C (L^e u_n)^2 \phi d m_c - \int_C \Gamma^e(u_n, \phi) L^e u_n d m_c. \]

Consider the second term on the right hand side.
\[ \left| \int_C \Gamma^e(u_n, \phi) L^e u_n d m_c - \int_C \Gamma^e(u, \phi) L^e u d m_c \right| \]
\[ \leq \int_C |\Gamma^e(u_n, \phi) L^e (u_n - u)| + |\Gamma^e(u_n - u, \phi) L^e u| d m_c \]
\[ \leq \|\Gamma^e(\phi)\|_{L^\infty} \left( \int_C \Gamma^e(u_n) d m_c \int_C (L^e(u_n - u))^2 d m_c + \int_C \Gamma^e(u_n - u) d m_c \int_C (L^e u)^2 d m_c \right) \]

Since \( \phi \in \Xi' \), we have that \( \|\Gamma^e(\phi)\|_{L^\infty} < \infty \). It follows that
\[ \int_C \Gamma^e(u_n, \phi) L^e u_n d m_c \rightarrow \int_C \Gamma^e(u, \phi) L^e u d m_c \quad \text{for} \quad u_n \rightarrow u \quad \text{in} \quad D^2(L^e) \]
and consequently
\[ \int_C \Gamma^e(u_n, L^e u_n) \phi d m_c \rightarrow \int_C \Gamma^e(u, L^e u) \phi d m_c \quad \text{for} \quad u_n \rightarrow u \quad \text{in} \quad D^2(L^e) \]
for any \( u \in D(\Gamma^e_2) \) and for any test function \( \phi \in \Xi' \) with \( \phi \geq 0 \).

4. Finally, we show that the \( \Gamma_2 \)-estimate holds for any admissible test function \( \phi \in D^{b,2}_+(L^e) \).

Since we assume \( K > 0 \), the measure \( m_e \) is finite and we can assume that \( \phi \geq M > 0 \) for some positive constant \( M \in D^2(L^e) \). Consider a sequence \( \phi_n \in \Xi \) that converges to \( \phi \) in \( D^2(L^e) \). Then, we also have \( P^e_t \phi \geq M \) and \( P^e_t \phi_n \rightarrow P^e_t \phi \) in \( D^2(L^e) \) for all \( t > 0 \). Since we assume that \( \mathcal{E}^c \) satisfies volume doubling and supports a Poincaré inequality, is strongly regular and admits that closed balls are compact (see Lemma \[ \text{[37]} \] there is an upper bound for the heat kernel (see \[ \text{[42]} \], Corollary 4.2, Remark \[ \text{[25]} \] that is equivalent to \( L^2 \rightarrow L^\infty \)-ultracontractivity of the semigroup \( P^e_t \) (see \[ \text{[28]} \], Chapter 14.1) and Remark \[ \text{[2.3]} \]). Hence, \( P^e_t \phi_n \rightarrow P^e_t \phi \) and \( L^e P^e_t \phi_n \rightarrow L^e P^e_t \phi \) in \( L^\infty(m_c) \). Since \( P^e_t \phi \geq M > 0 \), we deduce that \( P^e_t \phi_n \in D^{b,2}_+(L^e) \cap \Xi' \) for \( n \) sufficiently big. Then, the results from the previous paragraphs state that
\[ \int_C \left( \frac{1}{2} \Gamma^e(u)L^e P^e_t \phi_n - \Gamma^e(u, L^e u)P^e_t \phi_n \right) d m_c \geq \int_C \left( K N \Gamma^e(u) + \frac{1}{N+1} (L^e u)^2 \right) P^e_t \phi_n m_c. \]

Hence, if \( n \rightarrow \infty \)
\[ \int_C \left( \frac{1}{2} \Gamma^e(u)L^e P^e_t \phi - \Gamma^e(u, L^e u)P^e_t \phi \right) d m_c \geq \int_C \left( K N \Gamma^e(u)P^e_t \phi + \frac{1}{N+1} (L^e u)^2 P^e_t \phi \right) m_c \]
for \( u \in D(\Gamma_2) \) and \( \phi \in D^{b,2}_+ \) with \( \phi \geq M > 0 \) because of the \( L^\infty \)-convergence of \( P^e_t \phi_n \) and \( P^e_t L^e \phi_n \). Then we also let \( M \rightarrow 0 \) and the inequality holds for any test function of the form \( P^e_t \phi \) where \( \phi \in D^{b,2}_+ \). Finally, by application of Lebesgue’s dominated convergence theorem one can check that \( P^e_t \phi \) and \( P^e_t L^e \phi \) converges to \( \phi \) and \( L^e \phi \) respectively, w.r.t. weak-\( * \) convergence if \( \phi \in D^{b,2}_+(L^e) \), and we obtain the \( \Gamma_2 \)-estimate for any \( u \in D(\Gamma_2) \) and for any \( \phi \in D^{b,2}_+(L^e) \).
4. Preliminaries on the calculus for metric measure spaces

4.1. The curvature-dimension condition

**Assumption 4.1.** Let \((X, d_X)\) be a complete and separable metric space, and \(m_X\) a locally finite Borel measure on \((X, d_X)\) with full support. That is, for all \(x \in X\) and all sufficiently small \(r > 0\) the volume \(m_X(B_r(x))\) of balls centered at \(x\) is positive and finite. We assume that \(X\) has more than one point. A triple \((X, d_X, m_X)\) will be called metric measure space.

\((X, d_X)\) is called intrinsic or length space if \(d_X(x, y) = \inf L(\gamma)\) for all \(x, y \in X\), where the infimum runs over all curves \(\gamma\) in \(X\) connecting \(x\) and \(y\). \((X, d_X)\) is called strictly intrinsic or geodesic space if every two points \(x, y \in X\) are connected by a curve \(\gamma\) with \(d_X(x, y) = L(\gamma)\). Distance minimizing curves of constant speed are called geodesics. \((X, d_X)\) is called non-branching if for every tuple \((z, x_0, x_1, x_2)\) of points in \(X\) for which \(z\) is a midpoint of \(x_0\) and \(x_1\) as well as of \(x_0\) and \(x_2\), it follows that \(x_1 = x_2\). For basic facts about optimal transport and Wasserstein geometry we refer to [48]. An intrinsic metric space which is complete and locally compact, is strictly intrinsic, i.e. a geodesic space [10, Theorem 2.5.23].

**Definition 4.2** ([10], Reduced curvature-dimension condition). A metric measure space \((X, d_X, m_X)\) satisfies the condition \(CD^*(K, N)\) for \(K \in \mathbb{R}\) and \(N \in [1, \infty)\) if for each pair \(\mu_0, \mu_1 \in P^2(X, d_X, m_X)\) there exists an optimal coupling \(q\) of \(\mu_0 = \rho_0 m_X\) and \(\mu_1 = \rho_1 m_X\) and a geodesic \(\mu_t = \rho_t m_X\) in \(P^2(X, d_X, m_X)\) connecting them such that

\[
\int_X \rho_t^{-1/N'} \rho_0 d m_X \geq \int_{X \times X} [\sigma_{K,N'}^{(1)}(d_X)p_0^{-1/N'}(x_0) + \sigma_{K,N'}^{(t)}(d_X)p_1^{-1/N'}(x_1)] dq(x_0, x_1)
\]

for all \(t \in (0, 1)\) and all \(N' \geq N\) where \(d_X := d_X(x_0, x_1)\). In the case \(K > 0\), the volume distortion coefficients \(\sigma_{K,N}^{(t)}(\cdot)\) for \(t \in (0, 1)\) are defined by

\[
\sigma_{K,N}^{(t)}(\theta) = \frac{\sin \left(\frac{\sqrt{K/N} \theta}{\sin \left(\sqrt{K/N} \theta\right)}\right)}{\sin \left(\frac{\sqrt{K/N} \theta}{\sin \left(\sqrt{K/N} \theta\right)}\right)}
\]

if \(0 \leq \theta < \frac{\sqrt{N}}{K/N - 1}\) and by \(\sigma_{K,N}^{(t)}(\theta) = \infty\) if \(\theta \geq \frac{\sqrt{N}}{K/N - 1}\). In the case \(K \leq 0\) an analogous definition applies with an appropriate replacement of \(\sin \left(\sqrt{N/K} \theta\right)\).

**Remark 4.3** ([17]). A metric measure space \((X, d_X, m_X)\) satisfies the curvature-dimension condition \(CD(K, N)\) for \(K \in \mathbb{R}\) and \(N \in [1, \infty)\) if we replace in Definition 4.2 the coefficients \(\sigma_{K,N}^{(t)}(\theta)\) by

\[
\tau_{K,N}^{(t)}(\theta) = \begin{cases} 
\infty & \text{if } K \theta^2 > (N-1)\pi^2, \\
\frac{t^{1/N} \cdot \sigma_{K,N}^{(t)}(\theta)^{1-1/N}}{t} & \text{if } K \theta^2 \leq (N-1)\pi^2 \text{ & } N > 1, \\
\frac{1}{t} & \text{if } K \theta^2 \leq 0 \text{ & } N = 1.
\end{cases}
\]

This is the original condition that was introduced by Sturm in [47]. By definition a single point satisfies \(CD^*(K, 1)\) for any \(K > 0\), and \(K > 0\) and \(N = 1\) can only appear in this case.

**Theorem 4.4** (Doubling property, [19]). For a metric measure space \((X, d_X, m_X)\) that satisfies \(CD^*(K, N)\) for some \(K \in \mathbb{R}\) and \(N \geq 1\), the doubling property holds on each bounded subset \(X' \subset \text{supp} \, m\). In particular each bounded closed subset is compact and \((X, d_X, m_X)\) is locally compact. If \(K \geq 0\) or \(N = 1\) the doubling constant is \(\leq 2^N\).

**Theorem 4.5** (Hausdorff dimension, [10]). For a metric measure space \((X, d_X, m_X)\) that satisfies \(MCP(K, N)\) for some \(K \in \mathbb{R}\) and \(N \geq 1\), the support of \(m_X\) has Hausdorff dimension \(\leq N\).

**Remark 4.6.** The condition \(CD^*\) implies that the metric space \((X, d_X)\) is geodesic.
contraction property by Ohta in [37] and by Sturm in [47]. The latter is more restrictive and implies the former. In a non-branching situation the definitions coincide. We give the definition in the sense of Ohta.

**Definition 4.7** (Measure contraction property, [36]). Let \((X, d_x, m_x)\) be a non-branching metric measure space. Then it satisfies the measure contraction property \(MCP(K, N)\) if for any \(x \in X\), for any measurable subset \(A \subset X\) with \(m_x(A) < \infty\) (and \(A \subset B_{\pi}^{\sqrt{m_x}}(x)\) if \(K > 0\)), there exists a \(L^2\)-Wasserstein geodesic II such that \(\delta_x = (e_0)_I\) and \(m_x(A)^{-1} m_x((e_1)_I)\) and

\[
d m_x \geq (e_t)_* \left( \frac{\tau_t^{(t)}}{K,N} (L(\gamma))^{N} m_x(A) d II(\gamma) \right).
\]

By definition a single point satisfies \(MCP(K, 1)\) for any \(K > 0\), and \(K > 0\) and \(N = 1\) can only appear in this case.

A corollary of the measure contraction property is the Bonnet-Myers Theorem.

**Theorem 4.8** (Generalized Bonnet-Myers Theorem, [36]). Assume that a metric measure space \((X, d_x, m_x)\) satisfies satisfies \(MCP(K, N)\) for some \(K \in \mathbb{R}\) and \(N \geq 1\). Then the diameter of \((X, d_x)\) is bounded by \(\pi \sqrt{N - 1/K}\). In particular, a metric measure space that is nonbranching and satisfies the reduced curvature-dimension condition \(CD^*(K, N)\) for \(K > 0\) has bounded diameter by \(\pi \sqrt{N - 1/K}\).

**Remark 4.9.** One can check that the generalized Bonnet-Myers Theorem is an almost immediate consequence of the condition \(CD(K, N)\). But despite this fact the reduced curvature-dimension is more suitable for many applications. If the metric measure space is a Riemannian manifold, the reduced and non-reduced condition are equivalent and one conjectures that this should hold also in a more general setting. In any case, there are the following implications

\[
CD(K, N) \Rightarrow CD^*(K, N), \ CD^*(K, N) \Leftrightarrow CD_{loc}(K, N), \ CD^*(0, N) \Leftrightarrow CD(0, N)
\]

(see [10]) where the definition of \(CD_{loc}(K, N)\) can be found for example in [47].

### 4.2. First order calculus for metric measure spaces

Let \((X, d_x, m_x)\) be a metric measure space. We recall the concept of upper gradient. Let \(\gamma : J \to (X, d_x)\) be absolutely continuous curve in the sense of [2]. Then, \(\gamma\) has a well-defined metric speed \(|\dot{\gamma}(t)| = \lim_{h \to 0} \frac{1}{h} d_x(\gamma_t, \gamma_{t+h})\) such that \(|\dot{\gamma}(\cdot)| \in L^1(J, dt)\). We denote with \(AC^p(J, X)\) the set of all absolutely continuous curves that are defined on \(J\) and such that the metric speed is in \(L^p(J, dt)\). Then \(AC^1(J, X)\) is the set of absolutely continuous curves. A Borel function \(g : X \to [0, \infty]\) is an upper gradient of a continuous function \(u : X \to \mathbb{R}\) if for any absolutely continuous curve \(\gamma : [0, 1] \to X\) we have

\[
|u(\gamma_0) - u(\gamma_1)| \leq \int_0^1 g(\dot{\gamma}(t))|\dot{\gamma}(t)| dt.
\]

We say that a metric measure space \((X, d_x, m_x)\) supports a weak local \((q, p)\)-Poincaré inequality with \(1 \leq p \leq q < \infty\) if (3) in Definition 2.3 holds for any continuous \(u\) on \(X\), for any \(x \in X\) and \(r > 0\) such that \(m_x(B_r(x)) > 0\), and any upper gradient \(g\) of \(u\). We just have to replace \(\sqrt{\Gamma(u)}\) by upper gradients \(g\) of \(u\). The statements of remark 2.4 hold as well.
**Definition 4.10.** Let $u : X \to \mathbb{R}$ be a continuous function. The local slope (or local Lipschitz constant or pointwise Lipschitz constant) is the Borel function Lip given by
\[
\text{Lip } u(x) = \limsup_{y \to x} \frac{|u(y) - u(x)|}{d_X(x, y)}.
\]
Lip $u$ is an upper gradient for $u$ \cite[proposition 1.11]{18}.

T. Rajala and M.-K. von Renesse proved the following results that we state only for $K \geq 0$.

**Theorem 4.11 (\cite{10}).** Suppose that $(X, d_X, m_X)$ satisfies $CD(K, N)$ with $K \geq 0$ and $N \geq 1$. Then $(X, d_X, m_X)$ supports a weak local $(1, 1)$-Poincaré inequality.

**Theorem 4.12 (\cite{49}).** Suppose that $(X, d_X, m_X)$ is nonbranching and satisfies $MCP(K, N)$ with $K \geq 0$ and $N \geq 1$. Then $(X, d_X, m_X)$ supports a weak local $(1, 1)$-Poincaré inequality.

**Remark 4.13.** If a metric measure spaces satisfies a doubling property, Hajlasz and Koskela proved in \cite{31} that a weak local $(1, p)$-Poincaré inequality also implies a $(q, p)$-Poincaré inequality for $q \leq \frac{2p}{p-1}$ if the doubling constant satisfies $C \leq 2^N$. This is the case if the space satisfies the condition $CD(0, N)$. In particular, $(X, d_X, m_X)$ supports a weak local $(2, 2)$-Poincaré inequality. In the following we say that a metric measure space $X$ supports a (weak) local Poincaré inequality if it supports a weak local $(1, 1)$-Poincaré inequality.

We want to define Sobolev spaces and a notion of modulus of a gradient on a suitable class of functions. There are several authors that gave different definitions \cite[see also \cite{1}, \cite{12}, \cite{30}]. Here, we follow the approach of Ambrosio, Gigli and Savaré. Their main result from \cite{5} (see also \cite{4}) states that under the Assumption 4.1 most of the different approaches coincide and give us the same notion of Sobolev space and modulus of a gradient. The key is a non-trivial approximation by Lipschitz functions that we will use as starting point for our presentation. For any Borel function $u : X \to \mathbb{R}$ in $L^2(X, m_X)$ the Cheeger energy $\text{Ch}^X(u)$ is defined by
\[
\text{Ch}^X(u) = \frac{1}{2} \inf_{h \to \infty} \left\{ \liminf_{h \to \infty} \int_X (\text{Lip } u_h)^2 d m_X : u_h \text{ Lipschitz, } \|u_h - u\|_{L^2(X, m_X)} \to 0 \right\}. \tag{38}
\]
Then the $L^2$-Sobolev space is given by $D(\text{Ch}^X) = \{ u \in L^2(m_X) : \text{Ch}^X(u) < \infty \}$. The associated norm is $\|u\|_{L^2(D(\text{Ch}^X))} = \|u\|_{L^2}^2 + 2\text{Ch}^X(u)$. An important fact is that Ch is not a quadratic form in general.

**Definition 4.14.** Let $(X, d_X, m_X)$ be a metric measure space. If the Cheeger energy $\text{Ch}^X$ is a quadratic form, we call $(X, d_X, m_X)$ infinitesimal Hilbert.

Another result from \cite{3} is that $\text{Ch}^X$ can be represented by
\[
\text{Ch}^X(u) = \frac{1}{2} \int_X |\nabla u|^2_w d m_X \quad \text{if } u \in D(\text{Ch}) \tag{39}
\]
and $+\infty$ otherwise where $|\nabla u|_w : X \to [0, \infty]$ is Borel measurable and called the minimal weak upper gradient of $u$. The notion of minimal weak upper gradient is motivated by the following definitions that we take from \cite{3}.

We say that $u : X \to \mathbb{R} \cup \{\infty\}$ is “Sobolev along almost every curve” if $u \circ \gamma$ coincides a.e. in $[0, 1]$ and in $\{0, 1\}$ with an absolutely continuous map $u_\gamma : [0, 1] \to \mathbb{R}$ for almost every curve $\gamma$. The definition of the property ”for almost every curve $\gamma$” can be found in \cite{3}. We will not state it because it will not be used in the sequel.

**Definition 4.15.** For $u$ that is Sobolev along almost every curve, a $m_X$-measurable function $G : X \to [0, \infty]$ is a weak upper gradient of $u$ if
\[
|u(\gamma_0) - u(\gamma_1)| \leq \int_0^1 G(\gamma(t))|\gamma'(t)| dt \quad \text{for almost every curve } \gamma.
\]
Then any function \( u \in D(\text{Ch}) \) is Sobolev along a.e. curve and \(|\nabla u|_w\) is a minimal weak upper gradient in the following sense: If \( G \) is a weak upper gradient of \( u \), then \(|\nabla u|_w \leq G\) m\(_X\)-a.e. in \( X \).

**Remark 4.16.** An upper gradient \( g \) for some continuous \( u \) is also a weak upper gradient in the sense of the previous definition. The converse is in general not true. Hence, we have \(|\nabla u|_w \leq |\text{Lip } u|\) a.e., but no equality in general. If we assume a doubling property and a local Poincaré inequality, there is the following result of Cheeger.

**Theorem 4.17.** If \((X, d_X, m_X)\) is a complete and intrinsic measure space that provides a doubling property and a local \((1, 2)\)-Poincaré inequality, then for any function \( u : X \to \mathbb{R} \) that is locally Lipschitz, we have \( \text{Lip } u = |\nabla u|_w \) m\(_X\)-a.e.

**Proof.** \(\rightarrow [18]\).

The minimal weak upper gradient provides a stability property.

**Theorem 4.18** (Stability theorem, [5]). Let \((X, d_X, m_X)\) be a complete and separable metric measure space. Let \( u_n \in D(\text{Ch}) \) such that \( u_n \to u \in L^2(m_X) \) pointwise a.e. and assume \(|\nabla u_n|_w \in L^2(m_X)\) converges weakly to \( g \in L^2(m_X) \). Then \( u \in D(\text{Ch}) \) and \( g \geq |\nabla u|_w \) m\(_X\)-a.e.

**Proof.** \(\rightarrow \) Theorem 5.3 in [5].

**Remark 4.19.** In the introduction of [5] the authors remark that a complete and separable metric measure space \((X, d_X, m_X)\) whose balls have finite measure (hence, it fits in our Assumption 4.1), supports a weak local \((1, 1)\)-Poincaré inequality with constants \( C > 0 \) and \( \lambda \geq 1 \) if and only if it holds for any Lipschitz function \( u \) and upper gradient \( \text{Lip } u \).

**Remark 4.20.** If we assume the Cheeger energy \( \text{Ch}^X \) of \((X, d_X, m_X)\) to be a quadratic form, it yields a strongly local Dirichlet form \((\text{Ch}^X, D(\text{Ch}^X))\) on \( L^2(X, m_X) \) where the set of Lipschitz functions is dense in \( D(\text{Ch}^X) \) with respect to the Energy norm (see Proposition 4.10 in [6]). Additionally, if we assume that the space \( X \) is compact, Lipschitz function are dense in \( C_0(X) \) with respect to uniform convergence by application of the Stone-Weierstraß Theorem. Hence, \( \text{Ch}^X \) is a regular Dirichlet form and Lipschitz functions are a core.

### 4.3. The Riemannian curvature-dimension condition

**Definition 4.21** ([21, 24], Riemannian curvature-dimension condition). A metric measure space \((X, d_X, m_X)\) satisfies the Riemannian curvature-dimension condition \(\text{RCD}^*(K, N)\) if it is infinitesimal Hilbert and satisfies the condition \(\text{CD}^*(K, N)\).

The definition of Riemannian curvature bounds was first introduced by Ambrosio, Gigli and Savaré in [6] for infinite dimension bounds in terms of the evolution variational inequality. The finite dimensional counterpart was first considered by Gigli in [24] where Laplace comparison estimates have been proved. The coherence of the finite and infinite dimensional setting was proved in [2]. Finally, Erbar, Kuwada and Sturm established a unified definition of Riemannian curvature bounds in [21] in terms of a modified EV inequality that depends also on a dimensional parameter. We will not give the definition of EVI since it will not be used in this article.

**Proposition 4.22** ([6]). Assume \((X, d_X, m_X)\) satisfies \(\text{RCD}^*(K, N)\) for \(K \in \mathbb{R}\) and \(N \geq 1\). Then the intrinsic distance \(d_{\text{Ch}^X}\) coincides with \(d_X\).

**Theorem 4.23.** Let \((X, d_X, m_X)\) be a metric measure space that satisfies \(\text{RCD}^*(K, N)\) for \(K \in \mathbb{R}\) and \(N \geq 1\). Then the space satisfies the measure contraction property \(\text{MCP}(K, N)\).

**Proof.** The theorem is a corollary of several results by Cavalletti, Gigli, Sturm and Rajala and can be found in this form in [27].
Remark 4.24. Under the condition \( RCD^*(K, N) \) several regularity properties for the Markov semi-group \( P_t \) have been obtained in [6]. If \( u \in D(\mathcal{E}^x) \) and \( \Gamma(u) \in L^\infty \), \( P_t u \) has a Lipschitz representative, denoted by \( \tilde{P}_t u \) ([6, Theorem 6.1, Theorem 6.2]). Especially, any \( u \in D(\mathcal{E}^x) \) with \( \Gamma(u) \in L^\infty \) has a Lipschitz representative \( \tilde{u} \) such that \( |\nabla \tilde{u}| \leq \|\nabla u\|_{L^\infty} \). Under stronger conditions, namely \( L^2 \to L^\infty \)-ultracontractivity, we even have that \( P_t u \) is Lipschitz for any \( f \in L^2 \) ([6, Remark 6.4]). Especially, this is the case when the space satisfies \( RCD^*(K, N) \).

We introduce the following regularity assumption for metric measure spaces \((X, d_X, m_X)\). Because of Remark 4.23 and Remark 4.26 these properties are necessarily satisfied by \( RCD^* \)-spaces.

**Assumption 4.25.** \((X, d_X, m_X)\) is a geodesic metric measure space satisfying \( \text{supp} \, m_X = X \). In addition, every \( u \in D(\text{Ch}^X) \) with \( \nabla u \) a.e. admits a 1-Lipschitz representative.

The main result of Erbar, Kuwada and Sturm in [21] is

**Theorem 4.26.** Let \((X, d_X, m_X)\) be a metric measure space that satisfies the condition \( RCD^*(K, N) \). Then

1. \( \text{BE}(K, N) \) holds for \((\text{Ch}^X, D(\text{Ch}^X))\).

Moreover, if \((X, d_X, m_X)\) is a metric measure space that is infinitesimal Hilbert, satisfies the Assumption 4.25 and \((\text{Ch}^X, D(\text{Ch}^X))\) satisfies the condition \( \text{BE}(K, N) \) then

2. \((X, d_X, m_X)\) satisfies \( CD^*(K, N) \), i.e. the condition \( RCD^*(K, N) \).

**Proof.** \( \rightarrow [21, \text{Theorem 7}, \text{Theorem 4.1}, \text{Theorem 4.3}, \text{Theorem 4.8}, \text{Proposition 4.9}] \). The following theorem of Koskela and Zhou in [31] will be important later.

**Theorem 4.27.** Let \( \mathcal{E}^x \) be a regular, strongly local and strongly regular symmetric Dirichlet form on \( L^2(X, m_X) \). Suppose \((X, d_X, m_X)\) satisfies a doubling property. Then \( \text{Lip}(X) \subset D_{\text{loc}}(\mathcal{E}) \), \( \Gamma^x(u) \) exists for any \( u \in \text{Lip}(X) \) and \( \Gamma^x(u) \leq \text{Lip}(u)^2 \, m_X \)-a.e.

**Proposition 4.28.** Let \((X, d_X, m_X)\) be a metric measure space that satisfies \( RCD^*(K, N) \) for \( K > 0 \) and \( N > 1 \). Then the spectrum of the associated Laplace operator \( L^x \) is discrete and the first non-zero eigenvalue is \( \geq K \frac{N}{N-1} \).

**Proof.** The condition \( CD(K, N) \) implies a \((2^*, 2)\)-Sobolev inequality of the form

\[
\left( \int_{B_x(R)} u^{2^*} \, d m_X \right)^{\frac{1}{2^*}} \leq A \int_{B_x(R)} u^2 \, d m_X + B \int_{B_x(R)} (\text{Lip} u)^2 \, d m_X,
\]

where \( 2^* = \frac{2N}{N-2} \) for any Lipschitz function \( u \) that is supported in a ball \( B_x(R) \) (Theorem 30.23 in [45]). Then, a Rellich-Kondrachov compactness Theorem is implied by results of Haiman and Koskela [31, Theorem 8.1]. Finally, the proof of the first statement works by induction exactly as for Riemannian manifolds (see [12]).

The second statement directly comes from the Bakry-Emery characterization of the Riemannian curvature-dimension condition. Choose any eigenfunction \( u \) with eigenvalue \( \lambda \). Since \( X \) is compact, an admissible test function is \( \phi = 1 \). Then the condition \( BE(K, N) \) implies

\[
0 \geq \int_X \Gamma^x(u, L^x u) \, d m_X + K \int_X \Gamma^x(u) \, d m_X + \frac{1}{N} \int_X (L^x u)^2 \, d m_X
\]

\[
= -\lambda \int_X \Gamma^x(u, u) \, d m_X + \lambda K \int_X u^2 \, d m_X + \frac{\lambda^2}{N} \int_X u^2 \, d m_X = \int_X u^2 \, d m_X \left( -\lambda^2 + \lambda K + \frac{\lambda^2}{N} \right)
\]

from which follows \( \lambda \geq K \frac{N}{N-1} \). \( \square \)
Remark 4.29. The conclusion of the previous theorem is also true if $K = 0$ and $N = 1$. Then $\lambda_1 \geq 1$. It follows since in this case $F \simeq \lambda S^1$ or $F \simeq \lambda [0, \pi]$ for some $0 < \lambda \leq 1$. The diameter bound implies that $F$ is compact and there are points $x, y \in F$ such that $\text{diam}_F = d_F(x, y)$ and there is at least one geodesic between $x$ and $y$. Hence, the Hausdorff dimension has to be 1 and $F$ consists of finitely many geodesic segments that connect $x$ and $y$ since the measure is assumed to be locally finite. But the curvature-dimension condition implies that there can be at most two geodesics.

5. ($K, N$)-cones and the Riemannian curvature dimension condition

5.1. Warped products and ($K, N$)-cones for metric measure spaces

Let $(B, d_B)$ and $(F, d_F)$ be metric spaces that are complete, locally compact and (strictly) intrinsic. Let $f : B \to \mathbb{R}_{\geq 0}$ be a locally Lipschitz function. We call a curve $\gamma = (\alpha, \beta)$ in $B \times F$ admissible if $\alpha$ and $\beta$ are absolutely continuous in $B$ and $F$ respectively. In that case we define

$$L(\gamma) = \int_0^1 \sqrt{\dot{\alpha}(t)^2 + (f \circ \alpha)^2(t)\dot{\beta}(t)^2} dt.$$ 

$L$ is a length-structure on the class of admissible curves (for details see [16] and [1]). Then we can define a pseudo-distance between $(p, x)$ and $(q, y)$ by

$$\inf L(\gamma) =: d_\gamma((p, x), (q, y)) \in [0, \infty)$$

where the infimum is taken over all admissible curves $\gamma$ that connect $(p, x)$ and $(q, y)$. The induced metric space $B \times_f F$ is called warped product of $(B, d_B)$, $(F, d_F)$ and $f : B \to \mathbb{R}_{\geq 0}$. One can see that its topology coincides with the topology that was introduced for warped products in the setting of Dirichlet forms (see section 3.1). By definition $B \times_f F$ is an intrinsic metric space. Completeness and local compactness follow from the corresponding properties of $B$ and $F$. A suitable measure is given by $dm= f^s dm_B \otimes dm_F$ and the corresponding metric measure space $(B \times_f F, m_\gamma) = B \times_f^\gamma F$ is called $N$-warped product.

Definition 5.1. (($K, N$)-cones). For a metric measure space $(F, d_F, m_F)$ the $(K, N)$-cone is a metric measure space defined as follows:

- $\text{Con}_K(F) := \begin{cases} F \times [0, \pi/\sqrt{K}] / (F \times \{0, \pi/\sqrt{K}\}) & \text{if } K > 0 \\ F \times [0, \infty) / (F \times \{0\}) & \text{if } K \leq 0 \end{cases}$

- For $(x, s), (x', t) \in \text{Con}_K(F)$

$$d_{\text{Con}_K}((x, s), (x', t)) := \begin{cases} \cos^{-1}(\cos_K(s) \cos_K(t) + K \sin_K(s) \sin_K(t) \cos(d(x, x') \wedge \pi)) & \text{if } K \neq 0 \\ \sqrt{s^2 + t^2 - 2st \cos(d(x, x') \wedge \pi)} & \text{if } K = 0. \end{cases}$$

- $m_{\text{Con}_K}^N = \sin_K^N t dt \otimes m_F = m_C$

where $\sin_K(t)$ is defined as in Example [2,13] and $\cos_K(t) = \cos(\sqrt{K} t)$ for $K > 0$ and $\cos_K(t) = \cosh(\sqrt{-K} t)$ for $K < 0$. The triple $(\text{Con}_K(F), d_{\text{Con}_K}, m_{\text{Con}_K}^N)$ is denoted by $\text{Con}_{K,N}(F)$. If $\text{diam} F \leq \pi$ and $F$ is a length space, the $(K, N)$-cone coincides with the $N$-warped product $I_K \times_{\sin_K} F$.

Remark 5.2. The $(K, N)$-cone with respect to $(F, d_F)$ is an intrinsic (resp. strictly intrinsic) metric spaces if and only if $(F, d_F)$ is intrinsic (resp. strictly intrinsic) at distances less than $\pi$ (see [16, Theorem 3.6.17] for $K = 0$). We also remark that away from the singularity points the $(K, N)$-cone metric is locally bi-Lipschitz equivalent to the euclidean product metric between $| \cdot - \cdot |$ and $d_F$.
Remark 5.3. The $(K, N)$-cone is a metric measure space and we can consider its Cheeger energy that we denote with $\text{Ch}^{\sin N, K}(F)$. In general the intrinsic distance of $\text{Ch}^{\sin N, K}(F)$ does not need to coincide with $d_{\text{Con}, K}$. Similar, we denote the Cheeger energy of $I_K \times_{\sin N}^{\sin N} F$ with $\text{Ch}^{I_K \times_{\sin N}^{\sin N} F}$. It is not clear if $\text{Ch}^{I_K \times_{\sin N}^{\sin N} F}$ will coincide with the $(K, N)$-cone $\mathcal{E}^C = B \times_{\sin N}^{\sin N} \text{Ch}^F$ in the sense of Dirichlet forms. In the first case, we define the length structure that yields an intrinsic distance that determines the Cheeger energy. In the second case, we define a Dirichlet form on $L^2(B \times F, f^s d\text{vol}_N m_F)$ that determines the intrinsic distance. We will address this problem in the next section.

5.2. On the relation between metric cones and cones in the sense of Dirichlet forms

Let $\text{Ch}^F$ be the Cheeger energy of $(F, d_F, m_F)$ that is a locally compact, length metric measure space. If we assume that $\text{diam} F \leq \pi$, then we have $I_K \times_{\sin N}^{\sin N} F = \text{Con}, K(F)$. We can also consider $I_K \times_{\sin N}^{\sin N} \text{Ch}^F = \mathcal{E}^C$ in the sense of Dirichlet forms like in section 2 and 3 where $\text{Ch}^F = \mathcal{E}^F$. We denote with $\Gamma$ the $\Gamma$-operator of $\mathcal{E}^C$ and with $|\nabla u|_w$ the minimal weak upper gradient with respect to $\text{Ch}^{I_K \times_{\sin N}^{\sin N} F}$. The underlying topological space of $\mathcal{E}^C$ is by definition $I_K \times F/\sim$ as it was defined in section 3.1. If $\mathcal{E}^F$ is strongly local, regular and strongly regular and closed balls are compact, the same properties also hold for $\mathcal{E}^C$. We want to analyze the intrinsic distance of $\mathcal{E}^C$ in more detail. The key result is the following proposition.

Proposition 5.4. Let $(F, d_F, m_F)$ be a locally compact length metric measure space that satisfies volume doubling and supports a local Poincaré inequality. Assume $\text{diam} F \leq \pi$. Then $D(I_K \times_{\sin N}^{\sin N} \text{Ch}^F) \subset D(\text{Ch}^{I_K \times_{\sin N}^{\sin N} F})$ and for any $u \in D(I_K \times_{\sin N}^{\sin N} \text{Ch}^F)$ we have

$$|\nabla u|_w^2 \leq \Gamma^{I_K \times_{\sin N}^{\sin N} F}(u^t) + \frac{1}{m_{I_K}^C} |\nabla u^t|_w \text{ m}_C - a.e.}$$

where $u^t(r) = u(r, x)$ and $u^t(x) = u(r, x)$. Especially, the result holds if $(F, d_F, m_F)$ satisfies the condition $RCD(N - 1, N)$.

Proof. We follow the proof of Lemma 6.12 in [6] and use the following elementary lemma in [7]:

Lemma 5.5. Let $d(s, t) : (0, 1)^2 \to \mathbb{R}$ be a map that satisfies

$$|d(s, t) - d(s', t')| \leq |v(s) - v(s')|, \ |d(s, t) - d(s, t')| \leq |v(t) - v(t')|$$

for any $s, t, s', t' \in (0, 1)$, for some locally absolutely continuous map $v : (0, 1) \to \mathbb{R}$ and let $\delta(t) := d(t, t)$. Then $\delta$ is locally absolutely continuous in $(0, 1)$ and

$$\frac{d}{dt} [\delta]_{t} \leq \limsup_{h \to 0} \frac{d(t, t) - d(t - h, t)}{h} + \limsup_{h \to 0} \frac{d(t, t + h) - d(t, t)}{h} \text{ dt-a.e. in } (0, 1).$$

Proof. $\Rightarrow$ [7, Lemma 4.3.4]

1. Consider $u \in C^0(I_K) \cap \text{Lip}(F)$. $u$ is also Lipschitz with respect to $\text{Con}_{N, K}(F)$. Let $\gamma = (\alpha, \beta) : [0, 1] \to (I_K)_N \times F$ be a curve in $AC^2(\text{Con}_{N, K}(F))$ where $(I_K)_N = I_K \setminus \partial(I_K)_N$. Then, one can check that $\alpha \in AC^2(I_K)$ and $\beta \in AC^2(F, d_F)$ and there is $g \in L^2((0, 1), dt)$ such that

$$d_F(\beta_t, \beta_s) \leq \int_s^t g(\tau) d\tau \text{ and } |\alpha_t - \alpha_s| \leq \int_s^t g(\tau) d\tau$$

For $K > 0$ we have the following estimates (and similar for any $K \in \mathbb{R}$).

$$d_{\text{Con}}((r, y), (r, x)) = \cos_K^{-1}(\cos^2 r + \sin^2 r \cos |x, y|)$$

$$= \cos_K^{-1}(1 - \sin^2 r(1 - \cos |x, y|)) \leq \cos_K^{-1}(1 - \frac{1}{2} \sin^2 r |x, y|^2)$$

$$\leq x + o(x^2) \text{ for } x \to 0$$

$$\cos_K^{-1}(1 - \frac{1}{2} x^2) \text{ for } x \to 0$$

$$\text{ (42) \text{ (43) } }$$

33
Then we can see that

\[ |u(\alpha_s, \beta_t) - u(\alpha_s, \beta_t')| \leq L d_{Con_K}((\alpha_s, \beta_t), (\alpha_s, \beta_t')) \]

\[ \leq L \cos^{-1}_K(1 - \frac{1}{2} \sin^2 \alpha_s d_F(\beta_t, \beta_t')) \leq C d_F(\beta_t, \beta_t') \leq C \int_t^\tau g(\tau) d\tau \]

where \( L \) is a Lipschitz constant of \( u \) and \( C > 0 \) is another constant. Hence, we can apply the lemma from above and obtain

\[ \left| \frac{d}{dt}(u \circ \gamma)(t) \right| \leq \limsup_{h \to 0} \frac{|u(\alpha - h, \beta_t) - u(\alpha, \beta_t)|}{h} + \limsup_{h \to 0} \frac{|u(\alpha, \beta_t + h) - u(\alpha, \beta_t)|}{h} \]

for a.e. \( t \in [0, 1] \). By definition of the local Lipschitz constant \( \text{Lip} \) and by the elementary estimate

\[ 2ab \leq a^2 + b^2 \]

for any \( a, b \in \mathbb{R} \), it follows that

\[ \left| \frac{d}{dt}(u \circ \gamma)(t) \right| \leq \text{Lip} u^{\alpha_t}(\alpha_t)|\dot{\alpha}(t)| + \text{Lip} u^{\alpha_t}(\beta_t)|\dot{\beta}(t)| \]

\[ \leq \sqrt{\left(\text{Lip} u^{\alpha_t}(\alpha_t) + \frac{1}{\sin^2 K(t)}\right)^2 + \sin^2 K(t)|\dot{\beta}(t)|^2} =: G(\gamma(t))|\dot{\gamma}(t)| \]

for a.e. \( t \in [0, 1] \). If we want to check that \( G \) is a weak upper gradient of \( u \), we only need to consider curves like above since \( u \) has compact support in \( I_K \times_{\sin K} F \). Hence, integration with respect to \( t \) on both sides shows that \( G \) is a weak upper gradient of \( u \). It follows

\[ |\nabla u|_w (r, x) \leq G(r, x) \text{ m}_C \text{-a.e. } \]

Since \((F, d_F, m_F)\) satisfies a volume doubling property and supports a local Poincaré inequality, Cheeger's theorem (Theorem 4.17) states that \( \text{Lip} u^r = |\nabla u^r|_w \). Then the square of the right hand side of (44) \text{ m}_C \text{-a.e. equals}

\[ ((u^r)^2)^2 + \frac{1}{\sin^2 K}|\nabla u^r|_w^2 = \Gamma^u_{K, \sin K} (u^r) + \frac{1}{\sin^2 K}|\nabla u^r|_w^2 = \Gamma^C (u). \]

2. By the definition of skew products, \( C^\infty_0 (\tilde{I}_K) \otimes D(\mathcal{E}^r) \) is a dense subset of \( D(I_K \times_{\sin K} \mathcal{E}^r) \). Hence, for any \( u \in D(I_K \times_{\sin K} \mathcal{E}^r) \) there is a sequence \( u_n \in C^\infty_0 (\tilde{I}_K) \otimes D(\mathcal{E}^r) \) that converges to \( u \) with respect to the energy norm of \( I_K \times_{\sin K} \mathcal{E}^r \), and since \( \Gamma^C \) is a continuous bilinear form, we will find a subsequence such that

\[ \Gamma^u_{K, \sin K} (u_{n_i}) + \frac{1}{\sin^2 K}|\nabla u^r|_{w_i} = \Gamma^C (u_{n_i}) \rightarrow \Gamma^C (u) = \Gamma^u_{K, \sin K} (u^r) + \frac{1}{\sin^2 K}|\nabla u^r|_w \text{ m}_C \text{-a.e.} \]

The left hand side of (44) converges weakly in \( L^2 (m_c) \) (after taking another subsequence) and the limit is a weak upper gradient of \( u \). This follows from the stability theorem for minimal weak upper gradients in \( F \) (see Theorem 4.18). More precisely, we can argue as follows. Since \( |\nabla u|_w \in L^2 (m_c) \) is a bounded sequence, we find a subsequence that converges weakly to \( g \geq |\nabla u|_w \in L^2 (m_c) \). In particular, we have convergence for any test function \( \phi \in L^2 (m_c) \) such that \( \phi \geq 0 \). Hence, inequality (44) is preserved \text{ m}_C \text{-a.e. an we have}

\[ |\nabla u|_w^2 (r, x) \leq \Gamma^u_{K, \sin K} (u^r) (r) + \frac{1}{\sin^2 K}|\nabla u^r|_w^2 (x) \text{ m}_C \text{-a.e.} \]

and in particular, \( u \in D(I_K \times_{\sin K} \mathcal{E}^r) \) implies \( u \in D(\mathcal{E}^r) \).

\[ \square \]

**Lemma 5.6.** Let \((F, d_F, m_F)\) satisfy RCD*(N-1, N) for \( N \geq 1 \) and \( \text{diam } F \leq \pi \). Let \( \text{Con}_{N,K}(F) \) be the corresponding \((K, N)\)-cone for \( K \geq 0 \). Then \( \text{Con}_{N,K}(F) \) satisfies a volume doubling property and supports a local Poincaré inequality.
Proof. We assume $N > 1$ since the case $N = 1$ is already clear by Remark 1.29. We will use the following theorem of Ohta from [37].

**Theorem 5.7.** If the metric measure space $(F, d_F, m_F)$ satisfies $MCP(N - 1, N)$ in the sense of Ohta and if $diam_F \leq \pi$ then the associated $(0, N)$-cone satisfies $MCP(0, N + 1)$ in the sense of Ohta.

Hence, in the case $K = 0$ we proceed as follows. When $(F, d_F, m_F)$ satisfies $RCD^+(N - 1, N)$, Theorem 4.23 implies $MCP(N - 1, N)$ that implies a measure contraction property $MCP(0, N + 1)$ for $\text{Con}_{N,1}(F)$ in the sense of Ohta. In particular, $\text{Con}_{N,1}(F)$ satisfies a volume doubling property by results of Ohta in [37] and supports a local Poincaré inequality by Theorem 4.12 and Theorem 5.7. The latter follows since the condition $RCD^+(N - 1, N)$ implies that for every $x \in F$ and $m_\cdot$-a.e. $y \in F$ there is a unique geodesic. This property is inherited by the cone since $diam_F \leq \pi$. Hence, $MCP \sim \text{la Ohta}$ is the same as $MCP \sim \text{la Sturm}$ and we can apply Theorem 4.12 by von Reneses.

The case $K > 0$ can be covered in the same way. Assume without loss of generality that $K = 1$. By following straightforwardly Ohta’s proof of Theorem 5.7 in [37] we can prove the analogous result for $(1, N)$-cones where one should use the following formula for the projection of a geodesic $\gamma = (\alpha, \beta) : [0, 1] \to \text{Con}_{N,1}(F) \setminus \{\text{singularities}\}$ to $[0, \pi]$.

$$\cos \alpha(t) = a_{1,1}^{1-t}(L(\gamma)) \cos \alpha(0) + a_{1,1}^t(L(\gamma)) \cos \alpha(1).$$

Alternatively, one can use Theorem 5.7 directly and compare the metric and the measure of the spherical cone around the origin with the metric of the Euclidean cone around the origin. More precisely, one can find constants $m, M > 0$ such that

$$\frac{1}{M} d_{\text{Con}_K} \leq d_{\text{Con}_0} \leq \frac{1}{m} d_{\text{Con}_K} \quad \text{and} \quad \frac{1}{M} \sin_K^N r \leq r^N \leq \frac{1}{m} \sin_K^N r.$$

From this estimates one can easily deduce the doubling property and the Poincaré inequality in a neighborhood of the origin from the corresponding results for the 0-cone. Away from the singularities the same argument works by comparison with the direct product $(I_K \times F, d_{\text{Eucl}} \times d_F, L^1 \otimes m_F)$.

**Theorem 5.8.** Let $(F, d_F, m_F)$ be a metric measure space satisfying $RCD^+(N - 1, N)$ for $N \geq 1$ and $diam_F \leq \pi$. Then the intrinsic distance $d_{E^C}$ of $E^C = I_K \times_{\sin_K}^N \text{Ch}^\pi$ coincides with $d_{\text{Con}_K}$.

Proof. By remark 5.10 we know that in any case $diam_F \leq \pi$, thus $I_K \times_{\sin_K}^N F = \text{Con}_{N,1}(F)$. We only check the case $K > 0$.

1. We know from Proposition 5.4 that $D(I_K \times_{\sin_K}^N \text{Ch}^\pi) \subset D(\text{Ch}_{\text{Con}_{N,1}}^\pi(F))$ and for any $u \in D(I_K \times_{\sin_K}^N \text{Ch}^\pi)$

$$|\nabla u|_w^2 = \Gamma_{I_K, \sin_K}^N (u^*) + \frac{1}{\sin_K} |\nabla u^*|_w \quad m_\cdot \text{-a.e.}$$

(46)

where $u^*(r) = u(r, x)$ and $u^*(x) = u(r, x)$. For the intrinsic distance of $E^C$ we need to consider $u \in L_{G,\text{loc}} = C(I_K \times F^\pi, \mathcal{O}_C) \cap L_{\text{loc}}$ where

$$L_{\text{loc}} := \{ \psi \in L_{0,\text{loc}}(I_K \times_{\sin_K}^N \text{Ch}^\pi) : \sqrt{\Gamma_{I_K}^C(\psi)} \leq 1 m_\cdot \text{-a.e. in } I_K \times F^\pi \}.$$

One has to prove that $u$ is 1-Lipschitz with respect to $d_{\text{Con}_K}$. We will follow an argument that was suggested to the author by Tapio Rajala.

First, $\Gamma_{I_K}^C(u) \leq 1 m_\cdot \text{-a.e.}$ implies $|\nabla u|_w \leq 1 m_\cdot \text{-a.e.}$ by (46). $|\nabla u|_w$ is a weak upper gradient and $\text{Con}_{N,1}(F)$ satisfies the measure contraction property $MCP(0, N + 1)$ by the proof of the previous lemma. Consider two points $p, q \in \text{Con}_{N,1}(F)$, $B_r(q) \subset \text{Con}_{N,1}(F)$, $\mu_0 = \ldots$
By the definition of the Cheeger energy this implies the result.

\[ \int |u(\gamma) - u(\gamma_0)| d\Pi(\gamma) \leq \int \int_0^1 |\nabla u|_{u(\gamma(t))} L(\gamma) dt d\Pi(\gamma) \leq d_W(\delta_0, \mu_1). \]

where \( d_W \) is the \( L^2 \)-Wasserstein metric of \( \text{Con}_{N,K}(F) \). In the last inequality we just use that \( \mu_t \leq C(t) m_c \) for some \( C(t) > 0 \) and any \( t < 0 \) and \( |\nabla u|_{u} \leq 1 \) m_c-a.e. If \( \epsilon \to 0 \), we obtain

\[ |u(p) - u(q)| \leq d_W(\delta_0, \delta_q) = d_{\text{Con}}(p,q). \]

This yields

\[ d_{EC}((s,y),(r,x)) = \sup \{ u(s,y) - u(r,x) : u \in \mathcal{L}_{C,\text{loc}} \} \leq d_{\text{Con}}((r,x),(s,y)) \] (47)

for all \( (r,x),(s,y) \in \text{Con}_{N,K}(F) \).

2. On the other hand, we define \( g((p,x)) = d_{\text{Con}}((p,x),(q,y)) \) for some \((q,y) \in I_K \times_{\text{sin} K} F \) where

\[ d_{\text{Con}}((p,x),(q,y)) = \cos^{-1}\left(\frac{\cos_K(p) \cos_K(q) + K \sin_K(p) \sin_K(q) \cos d_F(x,y)}{h(p,x)}\right). \]

If \( h \in D_{\text{loc}}(\mathcal{E}^{I_K,\text{sin} K}) \otimes D(\text{Ch}^r) \) since \( \cos_K, \sin_K \in D_{\text{loc}}(\mathcal{E}^{I_K,\text{sin} K}) \) and \( \cos d_F(\cdot, q), 1 \in D(\text{Ch}^r) \). We can calculate \( \Gamma^c(g) \) explicitly. We get

\[ \Gamma^c(g) = \left(\cos^{-1}\left(\frac{h(p,x)}{1 - h^2(p,x)}\right)\right)^2 \frac{1}{1 - h^2(p,x)} \Gamma^c(h(p,x)) \]

Then, a straightforward calculation using the chain rule and \( \Gamma^c(d_F(\cdot, y)) \leq 1 \) yields

\[ \Gamma^c(h)(p,x) = \Gamma^c(\cos_K p \cos_K q) + 2 \Gamma^c(\cos_K p \cos_K q, \sin_K p \sin_K q) \cos d_F(x,y) \]

\[ + \Gamma^c(\sin_K p \sin_K q) \cos^2 d_F(x,y) + \frac{\sin^2 q \sin^2 p}{\sin^2 p} \Gamma^c(\cos d_F(x,y)) \leq 1 - h^2(p,x) \]

Hence \( \Gamma^c(g) \leq 1, g \in \mathcal{L}_{C,\text{loc}} \) and

\[ g((p,x)) - g((q,y)) = d_{\text{Con}}((p,x),(q,y)) \leq d_{EC}((p,x),(q,y)) \]

by definition of \( d_{EC} \). Hence, we obtain that \( d_{EC} = d_{\text{Con}} \).

\[ \square \]

**Corollary 5.9.** Let \( (F,d_F,m_F) \) be a metric measure space satisfying \( RCD^*(N-1,N) \) for \( N \geq 1 \) and \( \text{diam} F \leq \pi \). Then \( I_K \times_{\text{sin} K} \text{Ch}^r = \text{Ch}^{N,K}(F) \).

**Proof.** \( d_{\text{Con}} = d_{EC} \) by Theorem 5.8 and \( d_{\text{Con}} \) induces the topology of the underlying space \( I_K \times F/ \sim \). Theorem 3.8 implies the doubling property for \( \text{Con}_{N,K}(F) \). Then, by Theorem 4.22 and Proposition 5.3, we get that any Lipschitz function \( u \) with respect to \( d_{\text{Con}} \) is in \( D_{\text{loc}}(I_K \times_{\text{sin} K} \text{Ch}^r) \) and

\[ \Gamma^c(u) = \text{Lip}(u) = |\nabla u|_{m_c}^2 \text{ m}_c \text{-a.e.} \] (48)

By the definition of the Cheeger energy this implies the result.

\[ \square \]
The case when $\text{Con}_{N,K}(F)$ satisfies $\text{RCD}^*(KN,N+1)$.

Remark 5.10. We remind the reader on a result by Bacher and Sturm from [11]. They show the following. If the $(K,1)$-cone over some 1-dimensional space satisfies $CD(0,2)$ then the diameter of the underlying space is bounded by $\pi$. It is easy to see that their proof can be extended to any $(K,N)$-cone of any dimension bound $N$ and any parameter $K$. Thus, if $\text{Con}_{N,K}(F)$ satisfies $\text{RCD}^*(KN,N+1)$, then $\text{diam} F \leq \pi$, and again $\text{Con}_{N,K}(F) = I_K \times_{\sin K}^N F$.

Lemma 5.11. Let $(F,d_F,m_F)$ be a metric measure space that satisfies a volume doubling property and supports a Poincaré inequality. Assume $\text{Con}_{N,K}(F)$ satisfies $\text{RCD}^*(KN,N+1)$ for $K \geq 0$ and $N \geq 1$. Then the intrinsic distance $d_{\text{C}}$ of $\mathcal{E}^C = I_K \times_{\sin K}^N \text{Ch}^F$ coincides with $d_{\text{Con}_K}$.

Proof. Since $F$ satisfies a volume doubling property, supports a local Poincaré inequality and is infinitesimal Hilbertian, we can apply Proposition 5.3. Then, we have for any $u \in D(I_K \times_{\sin K}^N \text{Ch}^F)$
\[
|\nabla u|^2 \leq C^C(u) \ m_c \text{-a.e.} .
\] (49)

$\text{Con}_{N,K}(F)$ satisfies a Riemannian curvature-dimension condition. Hence, $|\nabla u| \leq \sqrt{C^C(u)} \leq 1$ $m_c$-a.e. implies $u$ is 1-Lipschitz and [47] holds. We can proceed as in the proof of Theorem 5.8 and obtain that $d_{\text{C}} = d_{\text{Con}_K}$.

Corollary 5.12. Let $(F,d_F,m_F)$ be a metric measure space that satisfies a volume doubling property and supports a Poincaré inequality. Assume $\text{Con}_{N,K}(F)$ satisfies $\text{RCD}^*(KN,N+1)$ for $K \geq 0$ and $N \geq 1$. Then $I_K \times_{\sin K}^N \text{Ch}^F = \text{Ch}^{\text{Con}_{N,K}(F)}$.

Proof. We can follow the proof of Corollary 5.9.

Lemma 5.13. Let $(F,d_F,m_F)$ be a metric measure space. Assume the $(K, N)$-cone $\text{Con}_{N,K}(F)$ satisfies $\text{RCD}^*(KN,N+1)$ for $N \geq 1$ and $K \geq 0$. Then $(F,d_F,m_F)$ satisfies a volume doubling property, supports a local Poincaré inequality and $(F,d_F,m_F)$ is infinitesimal Hilbertian.

Proof. We prove the result for $K > 0$. The general case follows in the same way. Consider
\[ x \in F \mapsto \{1, x\} \times F \subset \text{Con}_{N,K}(F) \]

We can find constants $M > m > 0$ such that
\[ \frac{1}{M} d_{\text{Con}_K} \leq \max \{ |\cdot|, d_F \} \leq \frac{1}{m} d_{\text{Con}_K} \& \ \frac{1}{M} \sin_K^N \leq 1 \leq \frac{1}{m} \sin_K^N \]
in an $\epsilon$-neighborhood of $\{1\} \times F \subset \text{Con}_{N,K}(F)$. On the one hand, from this we can easily deduce the volume doubling property for $F$. Pick a point $x \in F$ and let $r > 0$. Then
\[
4r m_F(B_{2r}(x)) \leq \frac{1}{m} \sin_K^N r dr \otimes m_F([-2r+1, 2r+1] \times B_{2r}(x))
\]
\[
\leq \frac{1}{m} m_c(B_{2Mr}(1,x)) \leq \frac{1}{m} C \left( \frac{2M}{m} \right)^N m_c(B_{Mr}(1,x))
\]
\[
\leq \frac{1}{m} C \left( \frac{2M}{m} \right)^N m_c([-r+1, r+1] \times B_{r}(x))
\]
\[
\leq 2^N C \left( \frac{M}{m} \right)^{N+1} 2r m_F(B_{2r}(x)).
\]

We used the volume doubling property of $\text{Con}_{N,K}(F)$ in the third inequality. We also obtain that the space $F$ supports a weak local Poincaré inequality because of the bi-Lipschitz invariance of this property. For example, we can follow the method that is provided in Section 4.3 of [14].
Now, we will check that \( F \) is infinitesimal Hilbertian. For any Lipschitz function \( u \) on \( \text{Con}_{N,K}(F) \) we see

\[
(\text{Lip } u)(r,x) = \limsup_{(s,y) \to (r,x)} \frac{|u(s,y) - u(r,x)|}{d_{\text{Con}_K}((s,y),(r,x))} \\
> \limsup_{(s,y) \to (r,x), r = s} \frac{|u(r,y) - u(r,x)|}{d_{\text{Con}_K}((r,y),(r,x))} \\
> \limsup_{y \to x} \frac{|u'(y) - u'(x)|}{\sin_K(r)|x,y|} = \frac{1}{\sin_K(r)} \text{Lip } u'(x).
\]

The second last inequality comes from (12) and (13). Following the steps in paragraph 1 of the proof of Proposition 5.4 we can see that (44) holds for \( C^\infty_K(\bar{I}_K) \otimes u \) where \( u \in \text{Lip}(F) \). There, we did not use that \( F \) is infinitesimal Hilbertian. By locality of the minimal weak upper gradient (13) also holds for \( 1 \otimes u \). Then (14) and (50) imply

\[
\text{Lip}(1 \otimes u)(r,x) = \frac{1}{\sin_K(r)} \text{Lip } u(x) \quad \text{for a.e. } r \in [0, \pi/\sqrt{K}] \text{ and } m_\mu \text{-a.e. } x \in F.
\]

Then, \( \int_F \text{Lip } ud\mu \) is an infinitesimal Hilbertian. By definition the Cheeger energy \( \text{Ch}^\mu \).

\[\Box\]

**Lemma 5.14.** Let \( (F, d_F, m_\mu) \) be a metric measure space and \( \text{Con}_{N,K}(F) \) satisfies \( RCD^*(KN,N+1) \) for \( K \geq 0 \) and \( N \geq 1 \). Then \( d_{\text{Ch}^\mu} = d_F \). In particular, \( \text{Ch}^\mu \) is strongly regular.

**Proof.** We assume \( K > 0 \). The case \( K = 0 \) follows in the same way. Consider \( u(\cdot) = d_F(x, \cdot) \in D(\text{Ch}^\mu) \). \( u \) satisfies \( |\nabla u|_{w} \leq 1 \). Hence, \( d_{\text{Ch}^\mu} \geq d_F \). The converse inequality is obtained as follows. Consider \( u \in D_{\text{loc}}(\text{Ch}^\mu) \cap C(F) \) with \( |\nabla u|_{w} \leq 1 \). Let \( \epsilon > 0 \). We choose \( \delta > 0 \) such that \( \frac{1}{\sin_K(x,y)} \leq 1 + \epsilon \) if \( r \in B_\delta(\pi/2) \). Let \( u_1 \in C^\infty_K(\bar{I}_K) \) such that \( u_1 \leq 1 \) and \( u_1|_{B_\delta(\pi/2)} = 1 \). \( u_1 \otimes u \in D_{\text{loc}}(E^\mu) \) and

\[
|\nabla (u_1 \otimes u)|^2 = (u_1')^2 u^2 + \frac{u^2}{\sin_K} |\nabla u|_{w}^2 \leq \varepsilon \cdot 1 + \epsilon.
\]

In particular, it follows that \( |\nabla (u_1 \otimes u)| = \frac{1}{\sin_K} |\nabla u|_{w} \leq 1 + \epsilon \) on \( B_\delta(\pi/2) \times F \). Since \( \text{Con}_{N,K}(F) \) satisfies \( RCD^*(KN,N+1) \), this implies that \( u_1 \otimes u \) admits a Lipschitz representative and the Lipschitz constant is locally less than \( 1 + \epsilon \) on some neighborhood of \( \pi/2 \times F \). This can be seen from standard arguments like in paragraph 1 of the proof of Proposition 5.8. Hence, for any \( x, y \in F \) such that \( d_F(x,y) \) is small, we have

\[
|u(x) - u(y)| \leq (1 + \epsilon) d_{\text{Con}_K}((\pi/2,x)), (\pi/2,y)) \leq (1 + \epsilon) d_F(x,y).
\]

It follows that \( d_{\text{Ch}^\mu} \leq (1 + \epsilon) d_F \) locally. Now, \( d_F \) is geodesic by the remark directly after Definition 5.1. We can conclude that \( d_{\text{Ch}^\mu} \leq (1 + \epsilon) d_F \) globally, and since \( \epsilon > 0 \) was arbitrary, we have \( d_{\text{Ch}^\mu} \leq d_F \).

\[\Box\]

We summarize the results of this section in the following corollary.

**Corollary 5.15.** Let \( (F, d_F, m_\mu) \) be a metric measure space and \( K \geq 0 \). Assume

1. \( (F, d_F, m_\mu) \) satisfies \( RCD^*(N-1,N) \) for \( N \geq 1 \) and \( \text{diam}_F \leq \pi \), or
2. \( \text{Con}_{N,K}(F) \) satisfies \( RCD^*(KN,N+1) \) for \( N \geq 1 \).

Then \( \text{Ch}^\text{Con}_{N,K}(F) = I_K \times \text{Ch}^\mu \), \( d_{\text{Ch}^\mu} = d_F \) and \( d_{\text{Ch}^\text{Con}_{N,K}(F)} = d_{\text{Con}_{N,K}(F)} \).
5.3. Proof of the main theorem

Proof of Theorem 1.2. The Cheeger energy \( \text{Ch}^r \) of \((F, d_F, m_F)\) is a strongly local, regular and strongly regular Dirichlet form that satisfies \( BE(N-1, N) \) by Theorem 4.26. By Proposition 4.28 the spectrum of the associated Laplace operator \( L^F \) is discrete and the first positive eigenvalue of \(-L^F\) satisfies \( \lambda_1 \geq N \). By Theorem 5.8 and Corollary 5.9 we know that \( I_K \times \sin_K \text{Ch}^r = \text{Ch}^{C_{0}^{\infty}(\mathcal{E}^c)} \) and \( d_{\text{Con}} = d_{\text{Ch}^{C_{0}^{\infty}(\mathcal{E}^c)}} \). Lemma 5.3 states that \( \text{Con}_{N,K}(F) \) satisfies a volume doubling property and supports a local Poincaré inequality. By Lemma 5.4 \( \text{Ch}^{C_{0}^{\infty}(\mathcal{E}^c)} \) is strongly regular and closed balls in \( \text{Con}_{N,K}(F) \) are compact since \((F, d_F, m_F)\) and its Cheeger energy \( \text{Ch}^r \) satisfy the corresponding properties. Hence, if \( K > 0 \), we can apply Theorem 4.24 and \( I_K \times \sin_K \text{Ch}^r \) satisfies \( BE(KN, N+1) \).

Finally, we want to apply the backward direction of Theorem 4.26. Results of Sturm from [14] (see Remark 2.5 and Remark 4.13) state a Feller property for the corresponding semigroup \( P^F \) of \( \text{Con}_{K,N}(F) \). Thus, we can apply Theorem 3.15 from [3] that states that in this case any \( u \in D(\mathcal{E}^c) \) with \( \sqrt{1-u}(u) \in L^\infty(m_c) \) has a continuous representative. Consequently, any such \( u \in D(\mathcal{E}^c) \) is Lipschitz continuous with respect to the intrinsic distance of \( I_K \times \sin_K \text{Ch}^r \) that coincides with \( d_{\text{Con}} \) by Corollary 5.15. Thus, the regularity Assumption 4.25 is satisfied and Theorem 4.26 yields the condition \( RCD^*(NK, N+1) \) if \( K > 0 \).

The case \( K = 0 \) follows from the case \( K > 0 \). The rescaled space \( \text{Con}_{N,K,\nu}(F) \) converges with respect to pointed measured Gromov-Hausdorff convergence to \( \text{Con}_{K,N}(F) \) if \( n \to \infty \). Hence, \( \text{Con}_{N,0}(F) \) satisfies \( RCD^*(0, N+1) = RCD(0, N+1) \) by the stability property of the condition \( RCD \) under measured Gromov-Hausdorff convergence.

Proof of Theorem 1.1. First, let us consider the case \( N \geq 1 \). Remark 5.10 states that \( \text{diam} F \leq \pi \) and \( \text{Con}_{1,\nu}(F) = I_K \times \sin_K F \) in any case when \( N \geq 1 \). We need to check the condition \( RCD^*(N-1, N) \) for \((F, d_F, m_F)\). Corollary 5.15 implies that \((F, d_F, m_F)\) is infinitesimal Hilbertian. By Proposition 5.11 and Corollary 5.12 the intrinsic distance of \( \mathcal{E}^c = I_K \times \sin_K \text{Ch}^r \) is the K-cone distance \( d_{\text{Con}} \) and the Cheeger energy of the \((K, N)\)-cone coincides with \( \mathcal{E}^c \). Theorem 4.26 implies the condition \( BE(KN, N+1) \) for \( I_K \times \sin_K \mathcal{E}^c \).

One can check that \( C_{0}^{\infty}(\tilde{I}_K) \otimes D(\Gamma_F^2) \subset D(\Gamma_F^2) \) and \( 1 \otimes D_{+}^{h,2}(L^F) \subset D_{+}^{h,2}(L^c) \). Hence, we can again derive formula (33) in precisely the same way as in the proof of Theorem 3.24 for \( u_1 \otimes u_2 \in C_{0}^{\infty}(\tilde{I}_K) \otimes D(\Gamma_F^2) \) and \( 1 \otimes \phi_2 \in 1 \otimes D_{+}^{h,2}(L^c) \). Now, we can follow the proof of Theorem 4.26 and we obtain

\[
\int_F L^F \phi \Gamma^F(u) d m_F - \int_F \Gamma^F(u, L^F u) \phi d m_F \\
\geq (N-1) \int_F \Gamma^F(u) \phi d m_F + \frac{1}{N} \int_F (L^F u^2)^2 \phi d m_F - \frac{1}{(N+1)N} \int_F (L^F u_2 + NK_F u_2)^2 \phi d m_F
\]

for any \( u \in D(\Gamma_F^2) \) and any \( \phi \in D_{+}^{h,2}(L^c) \). We want to deduce \( RCD^*(N-1, N) \) for \( F \). However, we cannot apply the argument of Theorem 5.11 directly since pointwise estimates for the Bochner inequality do not make sense. But like in the proof of Theorem 4.26 (more precisely, see Proposition 4.7 in [21]), we get a gradient estimate of the following type:

\[
|\nabla P^r_t u_2|^2 + \frac{c(t)}{N} \left( |L^F P^r_t u_2|^2 - \frac{1}{(N+1)} P^r_t (L^F u_2 + NK_F u_2)^2 \right) \leq e^{-2K t} P^r_t |\nabla u_2|^2
\]

for \( m_F \)-a.e. in \( F \) for any \( u_2 \in D^2(L^c) \). We sketch the argument briefly. Consider

\[
h(s) := e^{-2(N-1)s} \int_F P^r_s \phi |\nabla P^r_{t-s} u_2|^2 d m_F.
\]
One estimates the derivative of $h$ as:

$$h'(s) = 2e^{-2(N-1)s} \int \phi (L^s P_{t-s} u_2)^2 + \frac{1}{2} L^s P_s \phi (\nabla P_{t-s} u_2)^2 - P_s \phi (L^s P_{t-s} u_2) d m_P$$

$$\geq 2e^{-2(N-1)s} \int P_s \phi \left( \frac{1}{N} (L^s P_{t-s} u_2)^2 \right) - \frac{1}{(N+1)^N} (L^s P_{t-s} u_2 + N K_P P_{t-s} u_2)^2 \right) d m_P$$

$$\geq 2e^{-2(N-1)s} \int P_s \phi \left( \frac{1}{N} (L^s P_{t-s} u_2)^2 \right) - \frac{1}{(N+1)^N} (L^s P_{t-s} u_2 + N K_P P_{t-s} u_2)^2 \right) d m_P$$

where we used [31] in the first and Jensen's inequality in the second inequality. Finally, we integrate $h'$ from 0 to $t$ and the rest of the proof is exactly the same as in Proposition 4.9 in [21].

We remark that $F$ satisfies a doubling property and supports a local Poincaré inequality and by Lemma [5,11] we have that $d_{Ch^r} = d_P$, which implies that $Ch^r$ is strongly local. Thus, by the results of Remark [6,3] the associated semigroup is Feller and has a continuous kernel. Then we proceed as follows. For $u_2 \in D^2(L^r)$ we consider $P^r_{s-} (L^r u_2 + N K_P u_2) = L^r P^r_{s-} u_2 + N K_P P^r_{s-} u_2 = v_2$ and for $x \in F$ we define $v_{2,x} = v_2 - v_2(x)$. $v_{2,x}$ is continuous on $F$ and $v_{2,x}(x) = 0$. We consider $P^r_{s-} (v_{2,x}^2)$ that is jointly continuous in $z \in F$ and $t \geq 0$. For instance, this follows since $v_{2,x}^2 \in C(F) \cap L^\infty (m_P)$ and since we have a nice upper bound for the heat kernel associated to $Ch^r$. Then, to prove that $P^r_{s-} (v_{2,x}^2)$ is jointly continuous, we can copy the proof of the corresponding result in $\mathbb{R}^n$. It holds that $P^r_{s-} (v_{2,x}^2)(x) = v_{2,x}(x) = 0$. Hence, for any $\epsilon > 0$ and any $x \in F$ there is $\delta_x > 0$ and $\tau_x > 0$ such that $|P^r_{s-} (v_{2,x}^2)(y)| < \epsilon$ for any $y \in B_\delta(x)$ and $0 < t < \tau_x$. Since $F$ is compact, there is a finite collection $(x_i)_{i=1}^k$ of points such that $B_{\delta_x}(x_i) = 1,\ldots,k$. Now we choose $x_i \in F$ with $B_\delta(x_i)$ and we set $\delta_i = \delta_{x_i}$. Consider

$$P^r_{s-} u_2 - N K_P P^r_{s-} u_2(x_i) = v_{2,x_i} \in D^2(L^r)$$

and insert it in (22) for $t < \tau$.

$$|\nabla P^r_{s-} v_{2,x_i}|^2 + c(t) \frac{1}{N} (L^r P^r_{s-} v_{2,x_i})^2$$

$$= c(t) \frac{1}{N} (L^r P^r_{s-} v_{2,x_i})^2$$

We can see that

$$\text{(s)} = P^r_{s-} (L^r P^r_{s-} u_2 + N K_P P^r_{s-} u_2 - N K_P P^r_{s-} u_2(x_i) - L^r P^r_{s-} u_2(x_i))^2$$

$$= P^r_{s-} (v_2 - v_2(x_i))^2$$

For any $y \in B_{\delta}(x_i)$ we get $|(\text{(s})| = |P^r_{s-} (v_{2,x_i}^2)(y)| < \epsilon$. From that and since $v_{2,x_i}$ differs form $P^r_{s-} u_2$ only by a constant, we get for any $0 < t < \tau$ and $m_P$-a.e. $y \in B_{\delta}(x_i)$

$$|\nabla P^r_{s-} P^r_{s-} u_2|^2 (y) + c(t) \frac{1}{N} (L^r P^r_{s-} P^r_{s-} u_2)^2 (y) - \frac{1}{N+1}$$

$$\leq e^{-2Kt} (1/\phi) \left| v_{2,x_i}^2 \right|$$

The last inequality does not depend on $x_i$ anymore and since $\epsilon > 0$ is arbitrary, we obtain

$$|\nabla P^r_{s-} P^r_{s-} u_2|^2 + c(t) \frac{1}{N} (L^r P^r_{s-} P^r_{s-} u_2)^2 \leq e^{-2Kt} (1/\phi) \left| v_{2,x_i}^2 \right|$$

for $0 < t < \tau$ and $m_P$-a.e. for $u_2 \in D^2(L^r)$. Then we can also let $s$ go to 0

$$|\nabla P^r_{s-} u_2|^2 + c(t) \frac{1}{N} (L^r P^r_{s-} u_2)^2 \leq e^{-2Kt} (1/\phi) \left| v_{2,x_i}^2 \right|$$

for $0 < t < \tau$.
and finally, we can follow the proof of Theorem 4.8 in [21] to obtain the condition $BE(N-1, N)$. Now, similar like in the previous theorem, this implies $RCD^*(N-1, N)$ for $(F, d_F, m_F)$. We only need to check the Assumption 1.25. The condition $RCD^*(KN, N+1)$ for $\text{Con}_{N,K}(F)$ implies that $u \in D(\text{Ch}^{\text{Con},K}(F))$ with $\Gamma^e(u) \in L^\infty(m_F)$ admits a Lipschitz representative, and Theorem 5.9 says that $I_K \times_\sin^N Ch^F = \text{Ch}^{\text{Con},N,K}(F)$. These two statements imply that also $v \in D(\text{Ch}^F)$ with $\Gamma^e(v) \in L^\infty(m_F)$ admits a Lipschitz representative with respect to $d_F$.

For the case $N \in [0, 1)$ we argue by contradiction. First, $F$ has to be discrete. Otherwise, we would find a geodesic $\gamma$ in $F$, and consequently the cone over $\text{Im}(\gamma)$ would be a 2-dimensional subset of $\text{Con}_{N,K}(F)$. This contradicts the condition $RCD^*(KN, N+1)$ for $\text{Con}_{N,K}(F)$ that implies that the Hausdorff dimension of $\text{Con}_{N,K}(F)$ cannot be bigger than $N+1 < 2$.

The second observation is that any pair of points $(x, y) \in F$ satisfies $d_F(x, y) < \pi$. Otherwise, there are points $x, y \in F$ with $d_F(x, y) = \pi$. It follows that there is no continuous curve between $(1, x)$ and $(1, y)$ in $\text{Con}_{N,K}(F)$ such that its length is $\epsilon$-close to $d_{\text{Con},K}((1, x), (1, y))$ for $\epsilon > 0$ sufficiently small. A continuous curve that connects $(1, x)$ and $(1, y)$ consists of the segments that connect each of this points with the nearest origin and its length is $d_{\text{Con},K}(o, (1, x)) + d_{\text{Con},K}(o, (1, y)) > d_{\text{Con},K}((1, x), (1, y))$. But $\text{Con}_{N,K}(F)$ satisfies a curvature-dimension condition. Therefore, it has to be an intrinsic metric space what contradicts the previous observation.

We observe that $F$ can have at most two points. Otherwise we will find an optimal transport between absolutely continuous measures in $\text{Con}_{N,K}(F)$ that is essentially branching what contradicts the $RCD^*$-condition. For instance, assume there are three points. The geodesics between $(s, x)$, $(t, y)$ and $(r, z)$ for $s, t, r \leq 1$ consist exactly of segments that connect the origin. Hence, one can consider an absolutely continuous measure that is concentrated on one segment and transported to an absolutely continuous measure that is concentrated equally on the two other segments. Finally, in the case where $F$ is just one point we see that $I_K \times_\sin^N Ch^F = (I_K, \sin^N)$. Otherwise, if $F$ has two points with distance $\pi$, $N$ has to be 0 and we see that $\text{Con}_{0,K}(F) = \frac{1}{\sqrt{N}}S^1$.

5.4. Proof of the maximal diameter theorem

As an application of the previous results we can prove a spherical splitting theorem for $RCD^*$-spaces by application of the splitting theorem for $RCD^*(0, N)$ spaces. For Riemannian manifolds with non-negative Ricci curvature bounds this was proven by Cheeger and Gromoll in [20]. Recently, N. Gigli proved the result for general $RCD^*(0, N)$-spaces:

**Theorem 5.16** (N. Gigli, [23]). Let $(X, d_X, m_X)$ be a metric measure space that satisfies $RCD(0, N+1)$ for $N \geq 0$ and contains a geodesic line. Then $(X, d_X, m_X)$ is isomorphic to the euclidean product of the Euclidean line $(\mathbb{R}, d_{\text{Eucl}}, L^1)$ and another metric measure space $(X', d_{X'}, m_{X'})$ such that

1. $(X', d_{X'}, m_{X'})$ is $RCD(0, N)$ if $N \geq 1$,
2. $X'$ is just a point if $N \in [0, 1)$.

Here “isomorphic” means that there is a measure preserving isometry.

**Proof of Theorem 5.16**. $(F, d_F, m_F)$ is compact with $\text{diam } F \leq \pi$. Therefore, $\text{Con}_{0}^{N+1}(F) = [0, \infty) \times_0^{N+1} F$ and by Theorem 1.1 $\text{Con}_{0}^{N+1}(F)$ satisfies $RCD^*(0, N+2) = RCD(0, N+2)$. Since $d_F(x, y) = \pi$, there is a geodesic line in $\text{Con}_{0}^{N+1}(F)$ by definition of $d_{\text{Con}}$. Thus, by the first part of Gigli’s Theorem, $\text{Con}_{N+2,0}(F) =: X$ splits into $X = \mathbb{R} \times X'$ where $X' = (X', d_{X'}, m_{X'})$ denotes a metric measure space that satisfies $RCD^*(0, N+1)$. One can easily see that $X'$ is a metric cone over $F' = F \cap X'$, that $F'$ is a geodesic space and that $F'$ embeds geodesically in $F$.

Consider $(1, f), (1, g)$ in $\{1\} \times F$. We find $r, s > 0, i, j \in [-1, 1]$ and $f', g' \in F'$ such that

$$d_{X'}((1, f), (1, g))^2 = 2 - 2 \cos d_F(f, g) = r^2 + s^2 - 2rs \cos d_{X'}(f', g') + |i - j|^2.$$
Because the metric on $X$ is precisely given by the metric $l^2$-product of $|\cdot - \cdot|$ and $d_{X^*}$, the Pythagorean theorem holds. Hence $i^2 + j^2 = 1$. It follows that

$$\cos d_{X}(f,g) = i^2 + (1 - i^2)\frac{x^2}{2} \cos d_{X^*}(f',g').$$

There are unique numbers $\theta, \phi \in [0, \pi]$ such that $i = \cos \theta$ and $j = \cos \phi$. Thus, there is an isometry between $(F, d_{X})$ and the metric 1-cone with respect to $F'$. In particular, $F$ is also a topological suspension in the sense of Ohta’s topological splitting result in [37] and the measure has the form $d\mu = \sin^\theta d\mu'$ for some Borel measure $d\mu'$ on $F'$. Hence, $F$ is a $(K, N)$-cone over $(F', d_{X'}, \mu')$ in the sense of Definition 5.1. Finally, Theorem 1.2 yields the result.

**Proof of Corollary 1.6.** First, we consider $x_0, y_0 \in F$ with $d_{X}(x_0, y_0) = \pi$. The maximal diameter theorem implies that $F$ is a spherical suspension with respect to some metric measure spaces $F'_n$ that satisfies $RCD(N-2, N-1)$ where the pair $(x_0, y_0)$ corresponds to the two origins of $I_{X^*} \times_{\sin} F'$. If we consider another pair $(x_1, y_1)$, we obtain another suspension structure. Hence, we find a loop $s : [0, 2\pi]/(0 \sim 2\pi) \to S$ in $F$ that is geodesic for small distances and intersects with $F'$ at $x_1$ since $d_{X}(x_0, x_1) = \frac{\pi}{2}$. But this also implies that $d_{X}(y_0, y_1) = \frac{\pi}{2}$ and $y_1 \in F'$. Since $F'$ embeds geodesically into $F$, we have $d_{X}(x_i, y_i) = \pi$ for $i \geq 1$. Hence, we can proceed by induction and the second part of the maximal diameter theorem tells us that that after finitely many steps no further decomposition is possible and $F \cong \mathbb{S}^k$ for some $k \in \mathbb{N}$. But then, $n - 1 = N = k$.

**Acknowledgements.** The author wants to thank Karl-Theodor Sturm for proposing this interesting problem, stimulating discussions during the work on this article and important comments and remarks. I also want to thank the anonymous reviewer for carefully reading the article and for helpful comments.

**References**

[1] Stephanie B. Alexander and Richard L. Bishop. Warped products of Hadamard spaces. *Manuscripta Math.*, 96(4):487–505, 1998.

[2] Luigi Ambrosio, Nicola Gigli, Andrea Mondino, and Tapio Rajala. Riemannian ricci curvature lower bounds in metric measure spaces with $\pi$-finite measure. http://arxiv.org/abs/1007.1924.

[3] Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré. Bakry-Emery curvature-dimension condition and Riemannian Ricci curvature bounds. http://arxiv.org/abs/1009.5786.

[4] Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré. Calculus and heat flow in metric measure spaces and applications to spaces with Ricci bounds from below. http://arxiv.org/abs/1106.2090.

[5] Luigi Ambrosio, Andrea Mondino, and Giuseppe Savaré. Density of Lipschitz functions and equivalence of weak gradients in metric measure spaces. http://arxiv.org/abs/1111.3730.

[6] Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré. Metric spaces with Riemannian Ricci curvature bounded from below. http://arxiv.org/abs/1109.0222.

[7] Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré. Gradient flows in metric spaces and in the space of probability measures. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, second edition, 2008.

[8] Luigi Ambrosio, Andrea Mondino, and Giuseppe Savaré. Nonlinear diffusion equations and curvature conditions in metric measure spaces. In preparation.

[9] Michael T. Anderson. Metrics of positive Ricci curvature with large diameter. *Manuscripta Math.*, 68(4):405–415, 1990.

[10] Kathrin Bauer and Karl-Theodor Sturm. Localization and tensorization properties of the curvature-dimension condition for metric measure spaces. *J. Funct. Anal.*, 259(1):28–56, 2010.

[11] Kathrin Bauer and Karl-Theodor Sturm. Ricci bounds for euclidean and spherical cones. *Singular Phenomena and Scaling in Mathematical Models*, pages 3–23, 2014.

[12] Dominique Bakry and Michel Ledoux. A logarithmic Sobolev form of the Li-Yau parabolic inequality. *Rev. Mat. Iberoam.*, 22(2):683–702, 2006.

[13] Pierre H. Bérard. *Spectral geometry: direct and inverse problems*, volume 1207 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986. With appendixes by Gérard Besson, and by Béard and Marcel Berger.

[14] Anders Björn and Jana Björn. *Nonlinear potential theory on metric spaces*, volume 17 of *EMS Tracts in Mathematics*. European Mathematical Society (EMS), Zürich, 2011.

[15] Nicolas Bouleau and Francis Hirsch. *Dirichlet forms and analysis on Wiener space*, volume 14 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1991.
