ALLARD TYPE BOUNDARY REGULARITY THEOREM
FOR VARIFOLDS WITH $C^{1,\alpha}$ BOUNDARY

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Abstract. In this paper we show that Allard’s boundary regularity theorem for general varifolds [All75] can be generalized in the case of $C^{1,\alpha}$ boundaries, (Theorem 3.14); in [All75] it is required that the boundary is $C^{1,1}$. The proof presented here is along the lines of the proof of Allard’s interior regularity theorem [All72] found in [Sim83]. In particular we give the proof in the case of a rectifiable varifold. In this way, as in [Sim83], we simplify the notation and the computations needed for the proof, without however weakening the original hypotheses in Allard’s paper [All75], because of the rectifiability theorem (cf. [Sim83, Theorem 32.1]).

1. Notation and Preliminaries

Let $B$ be a $C^{1,\alpha}$ closed $(k-1)$-dimensional submanifold of $\mathbb{R}^n$ passing through the origin, with $k < n$ and $0 < \alpha \leq 1$. Then there exists a non-negative constant $\kappa$ and radius $R > 0$ with $\kappa R^\alpha < 1$ such that

$$|\text{proj}_{N_x B}(y - b)| \leq \kappa |y - b|^{1+\alpha}$$

1.1

$$\|\text{proj}_{N_x B} - \text{proj}_{N_b B}\| \leq \kappa |y - b|^{\alpha}$$

for all $y, b \in B \cap B_R(0)$.

We use the notation $T_x B$ for the tangent space of $B$ at $x$, $N_x B$ for the normal space of $B$ at $x$ and $\text{proj}_{T_x B}, \text{proj}_{N_x B}$ for the projections onto the two spaces respectively. Finally, $B_r(x) \subset \mathbb{R}^n$ will denote the $n$-dimensional ball of radius $r$ centered at $x \in \mathbb{R}^n$ and for any $m \in \mathbb{N}$, $\omega_m$ will denote the $m$-dimensional area of the open unit ball centered at the origin in $\mathbb{R}^m$.

Remark 1.2. Since $\kappa R^\alpha < 1$, for any $x \in B$ and $r > 0$ such that $B_r(x) \subset B_R(0)$, $B \cap B_r(x)$ is the graph of a $C^{1,\alpha}$ function above $T_x B$, i.e

$$B \cap B_r(x) = \text{graph } \psi_x \cap B_r(x)$$

where

$$\psi_x : B_r(x) \cap T_x B \to N_x B$$

is a $C^{1,\alpha}$ function such that

$$\frac{||\psi_x||_0}{r} + ||D\psi_x||_0 + r^\alpha |D\psi_x|_\alpha \leq \kappa.$$

Definition 1.3. For any $x \in B_R(0)$, we define $\rho(x)$ to be the distance of $x$ from $B$ and $\bar{x}$ will denote a point on $B$ such that $|x - \bar{x}| = \rho(x)$. (Note that there is not necessary a unique such point $\bar{x}$).

Under the above assumptions it is easy to check that the following inequalities hold
Remark 1.4. Let \( x \in B_{R/4}(0) \setminus B \) and \( y \in B_{\rho(x)/2}(x) \). Then

\[
| \text{proj}_{N_x B} (y - \bar{x}) - (y - \bar{y}) | \leq c \kappa \rho(y)^{1+\alpha}
\]

and

\[
| \rho(x) D \rho(x) - (x - \bar{x}) | \leq c \kappa \rho(x)^{1+\alpha}
\]

for some absolute constant \( c \).

We consider a rectifiable \( k \)-varifold, \( V = (M, \theta) \), where \( M \) is a countably \( k \)-rectifiable, \( \mathcal{H}^k \)-measurable subset of \( \mathbb{R}^n \) and \( \theta \) a locally \( \mathcal{H}^k \)-integrable function on \( M \) and we let \( \mu_V = \mathcal{H}^k \llcorner \theta \) be the weight measure of \( V \) (cf. [Sim83] Chapter 4). For \( V \) we assume that it satisfies the following 3 properties, which we will denote by \( P_1 \)-\( P_3 \):

\( P_1 \) \( 0 \in \text{spt} V, \mu_V(B) = 0. \)

\( P_2 \) \( \theta(x) = \Theta_V(x) \geq 1 \) for \( \mu_V \)-almost every \( x \in \mathbb{R}^n. \)

\( P_3 \) There exist constants \( a \geq 0, q < \frac{k}{k-1} \), such that

\[
\delta V(X) \leq a \left( \int_{B_R(0) \setminus B} |X|^q d\mu_V \right)^{\frac{1}{q}}
\]

whenever \( X \) is a smooth vector field with compact support in \( B_R(0) \setminus B \) and where \( \delta V(X) \) denotes the first variation of \( V \) with respect to \( X \) i.e.

\[
\delta V(X) = \int_{B_R(0)} \text{div}_M X d\mu_V.
\]

Notice that property \( P_3 \) implies the following, which we will denote by \( P_3' \):

\( P_3' \) There exists a \( \mu_V \)-measurable function \( H : B_R(0) \setminus B \to \mathbb{R}^n \) with \( |H(x)| = D_{\mu_V} \| \delta V \|(x) \) (where \( \| \delta V \| \) is the total variation of \( \delta V \), cf. [All72] Chapter 4) for all \( x \in B_R(0) \setminus B \) such that

\[
\delta V(X) = \int_{B_R(0)} X \cdot H d\mu_V
\]

for any smooth vector field \( X \) with compact support in \( B_R(0) \) and such that \( X(y) = 0 \) for all \( y \in B \). For \( H \) we also have that

\[
\left( \int_{B_R(0) \setminus B} |H|^p d\mu_V \right)^{\frac{1}{p}} \leq a
\]

where \( p > k \) is such that \( \frac{1}{q} + \frac{1}{p} = 1. \)

Remark 1.5. By the definition of a \( k \)-rectifiable varifold and Rademacher’s theorem (cf. [Sim83] Theorem 5.2), the function \( \rho(x) = \text{dist}(x, B) \) is a \( \mu_V \)-almost everywhere \( C^1 \) function on \( M \) and therefore \( \nabla \rho = \nabla^M \rho \) is well defined.

Throughout this paper the letter \( c \) will denote a constant which possibly depends on the given variables \( n, k, p, \alpha \). When different constants appear in the course of a proof we will keep the same letter \( c \) unless the constant depends on some different parameters.
2. First Variation and Monotonicity

Throughout this section we assume that $B$, $V$ are as defined above, i.e. they satisfy $\text{[\text{I}] and properties P}_1$-$P_3$ (and hence $\text{P}_3'$) and we let $U = B_{R/4}(0)$.

**Lemma 2.1** (First Variation Formula). For any smooth vector field $X$ with compact support in $U$:

\[ \delta V(X) = \int_{U \setminus B} X \cdot H \, d\mu_V + \int_B X \cdot \eta dV_{\text{sing}} \]

where $\eta = \eta(y) \in N_y B$ for all $y \in B$.

**Proof.** We will first prove that $V$ has locally bounded variation in $U$, i.e. we will show that for any $W \Subset U$ there exists a constant $c$ (depending on $W$) such that

\[ \delta V(X) \leq c \sup_U |X| \]

for any smooth vector field $X$ with support in $W$.

For any smooth function $\phi : \mathbb{R} \to \mathbb{R}$ we can write $\delta V(X)$ as follows:

\[ \delta V(X) = \int_U \text{div}_M[(1 - \phi(\rho))X] d\mu_V + \int_U \phi'(\rho) \nabla \rho \cdot X d\mu_V \]

\[ + \int_U \phi(\rho) \text{div}_M X d\mu_V. \]

Let $\{\phi_h\}_{0 < h < 1}$ be a family of smooth functions such that

\[ \phi_h(\rho) = \begin{cases} 1 & \text{for } \rho \leq h/2 \\ 0 & \text{for } \rho \geq h \end{cases}, \quad \phi'_h(\rho) \leq 0 \]

and such that $\phi_h \xrightarrow{h \to 0} \chi(-\infty, 0]$, the indicator function of $(-\infty, 0]$. Then, by property $\text{P}_3'$

\[ \int_U \text{div}_M[(1 - \phi_h(\rho))X] d\mu_V = \int_U (1 - \phi_h(\rho))X \cdot H d\mu_V \xrightarrow{h \to 0} \int_{U \setminus B} X \cdot H d\mu_V \]

and by property $\text{P}_1$

\[ \int_U \phi_h(\rho) \text{div}_M X d\mu_V \xrightarrow{h \to 0} 0. \]

Hence, by using (3) with $\phi = \phi_h$ and letting $h \to 0$, we have that (2) is equivalent to

\[ \lim_{h \to 0} \frac{1}{h} \int_{T_h} \nabla \rho \cdot X d\mu_V \leq c \sup_U |X| \]

where $T_h = \{x : \rho(x) < h\}$. Therefore for proving that $V$ has locally bounded variation in $U$ it suffices to show that for any $W \Subset U$ there exists a constant $c$ (depending on $W$) such that

\[ \lim_{h \to 0} \frac{1}{h} \int_{T_h} \chi d\mu_V \leq c \]

where $\chi$ is a smooth function such that $\chi = 1$ on $W$, $0 \leq \chi \leq 1$ and with compact support in $U$.

Let $W$ be a Whitney partition of $\mathbb{R}^n \setminus B$ (cf. [KP99]). Then

\[ B_{R/4}(0) \setminus B \subset \bigcup_{C \in W} C \]
where the elements $C$ of the collection $W$ are cubes with the following property: If
$\rho_C = \text{dist}(C, B)$, then $\rho_C > 0$ and
$$\text{diam } C \leq \rho_C \leq 3 \text{ diam } C.$$ For each $C \in W$, we let $x_C \in C$ be the center of the cube $C$ and $y_C \in B$ be such that $|x_C - y_C| = \rho(x_C)$. Finally let $\phi_C$ be a partition of unity subordinate to the covering $W$ and such that
$$|D\phi_C(x)| \leq C\rho(x)^{-1}$$
where $C$ is an absolute constant.

Given $\chi : U \to \mathbb{R}$ a smooth, non negative function with compact support we define the vector field
$$X(x) = \sum_{C \in W} \phi_C(x)\psi(\rho(x))\chi(x)X_C(x).$$
where $\psi : \mathbb{R} \to \mathbb{R}$ is a smooth non negative function and for each $C \in W$, $X_C$ is defined by
$$X_C(x) = \text{proj}_{N_{y_C}B}(x - y_C).$$
$X$ is then a smooth vector field that vanishes on $B$ and thus by property $P'_3$
$$\delta V(X) = \int_U \text{div}_M X\,d\mu_V = \int_U X \cdot H\,d\mu_V.$$ By Remark 1.4 for any $x \in C$
$$X_C(x) = \rho D\rho + Y_C(x),$$
with $Y_C$ satisfying $|Y_C(x)| \leq c\kappa\rho(x)^{1+\alpha}$.
Furthermore
$$\text{div}_M X_C = \text{trace}(e(x)DX_C) \geq 1$$
where $e(x)$ denotes the matrix of the projection onto $T_xM$ and $DX_C$ is the matrix of the projection onto $N_{y_C}B$. Therefore
$$\text{div}_M X = \sum_{C \in W} \text{div}_M (\phi_C \psi(\rho)\chi X_C)$$
$$\geq \sum_{C \in W} \phi_C \chi(\psi(\rho) + \rho \psi'(\rho)) + \sum_{C \in W} \phi_C \psi(\rho) \left(\nabla \rho \cdot Y_C - \rho |\nabla \rho|^2\right)$$
$$+ \sum_{C \in W} \phi_C \psi(\rho) \left(\rho \nabla X \cdot D\rho + \nabla \chi \cdot Y_C\right) + \sum_{C \in W} \psi(\rho)\chi \nabla \phi_C \cdot X_C.$$ We introduce the quantities
$$Y = \sum_{C \in W} \phi_C Y_C, \quad g = \sum_{C \in W} \nabla \phi_C \cdot X_C$$
for which, by (10) and Remark 1.4 we have the following estimates:
$$|Y(x)| \leq c\kappa\rho(x)^{1+\alpha}$$
and
$$|g(x)| = \sum_{C \in W} \nabla \phi_C(x) \cdot (X_C(x) - (x - \bar{x})) \leq c\kappa\rho^\alpha.$$
Substituting for the quantities $Y$ and $g$, inequality \((14)\) now reads
\[
\text{div}_M X \geq \chi(\psi(\rho) + \rho\psi'(\rho)) + \chi\psi'(\rho)\nabla\rho \cdot Y - \chi\psi'(\rho)\rho|\nabla^\perp\rho|^2
+ \psi(\rho)\nabla\chi \cdot (\rho D\rho + Y) + \psi(\rho)\chi.
\]

Applying this inequality in \((9)\) we get:
\[
\int_U \chi(\psi(\rho) + \rho\psi'(\rho))d\mu_V \leq \int_U \chi\psi(\rho)\rho D\rho \cdot Hd\mu_V + \int_U \chi\psi(\rho)Y \cdot Hd\mu_V
- \int_U \psi(\rho)\chi gd\mu_V - \int_U \psi(\rho)\nabla\chi (\rho D\rho + Y)
- \int_U \chi\psi'(\rho)\nabla\rho \cdot Y d\mu_V + \int_U \chi\psi'(\rho)\rho|\nabla^\perp\rho|^2d\mu_V.
\]

Let $\gamma : \mathbb{R} \to \mathbb{R}$ be a smooth function such that
\[
\gamma(t) = \begin{cases} 1 & \text{for } t \leq 1/2 \\ 0 & \text{for } t \geq 1 \end{cases}, \quad \gamma'(t) \leq 0 \ \forall t.
\]
Then we can use \((14)\) with $\psi(\rho) = \gamma\left(\frac{\rho}{r}\right)$ (for some $r > 0$). Since
\[
\rho\psi'(\rho) = \frac{\rho}{r}\gamma'\left(\frac{\rho}{r}\right) = -r\frac{\partial}{\partial r}\left(\gamma\left(\frac{\rho}{r}\right)\right)
\]
and because of the estimates for $Y$ and $g$ (given in \((12), (13)\)) this gives
\[
\int_U \chi \left((1 - cr\alpha)\gamma\left(\frac{\rho}{r}\right) - (1 + cr\alpha)\frac{\rho}{r}\frac{\partial}{\partial r}\left(\gamma\left(\frac{\rho}{r}\right)\right)\right) d\mu_V
\leq \int_U \gamma\left(\frac{\rho}{r}\right)\chi\rho D\rho \cdot Hd\mu_V + \int_U \gamma\left(\frac{\rho}{r}\right)\chi Y \cdot Hd\mu_V
- \int_U \gamma\left(\frac{\rho}{r}\right)\nabla\chi (\rho D\rho + Y) d\mu_V - \int_U \chi r\frac{\partial}{\partial r}\left(\gamma\left(\frac{\rho}{r}\right)\right) |\nabla^\perp\rho|^2d\mu_V.
\]

Let $r$ be such that $1 - 2cr\alpha > 0$ and set $\Gamma = 2c\alpha^{-1}$, where $c$ is the constant appearing on the LHS of \((15)\). Note that
\[
e^{r\alpha}\frac{\partial}{\partial r}\left(e^{r\alpha}\gamma\left(\frac{\rho}{r}\right)\right) - e^{r\alpha}\frac{1 - 2cr\alpha}{r^2}\gamma\left(\frac{\rho}{r}\right) = \frac{\partial}{\partial r}\left(e^{r\alpha}\gamma\left(\frac{\rho}{r}\right)\right).
\]

Hence after multiplying \((15)\) by $-(1 + cr\alpha)^{-1}e^{r\alpha}r^{-2}$ we get
\[
\frac{\partial}{\partial r}\left(e^{r\alpha}\gamma\left(\frac{\rho}{r}\right)\right)
\geq -\frac{e^{r\alpha}}{r^2}\int_U \gamma\left(\frac{\rho}{r}\right)\chi\rho|H|d\mu_V - \frac{e^{r\alpha}}{r^2}\int_U \gamma\left(\frac{\rho}{r}\right)\chi|Y||H|d\mu_V
- \frac{e^{r\alpha}}{r^2}\int_U \gamma\left(\frac{\rho}{r}\right)\nabla\chi |(\rho + |Y|)| d\mu_V + \frac{1}{2r}\int_U \chi \frac{\partial}{\partial r}\left(\gamma\left(\frac{\rho}{r}\right)\right) |\nabla^\perp\rho|^2d\mu_V.
\]

Letting $\gamma$ increase to the indicator function of $(-\infty, 1)$ and integrating \((16)\) from $\sigma$ to $r$, where $0 < \sigma < r$, we get the following monotonicity inequality for
tubular neighborhoods of $B$:

$$\frac{e^{r\sigma}}{\sigma} \int_{T_\sigma} \chi d\mu_V \leq \frac{e^{r\sigma}}{r} \int_{T_r} \chi d\mu_V$$

(17)

$$+ e^{r\alpha} \int_{T_r} \chi |H| d\mu_V + e^{r\alpha} \int_{T_r} \frac{1}{\rho} |Y| |H| d\mu_V$$

$$+ e^{r\alpha} \int_{T_r} |D\chi| \left( 1 + \frac{1}{\rho} |Y| \right) d\mu_V - \frac{1}{2} \int_{T_r \setminus T_\sigma} \frac{1}{\rho} |\nabla^\perp \rho|^2 d\mu_V$$

Note that the last term on the RHS of (17) is negative so it can be dropped. The other terms, because of the estimate for $Y$ (cf. (12)) and the fact that $H$ is in $L^p$ (cf. $P_{3\gamma}'$), are bounded. Hence we can let $\sigma \to 0$ and this gives

$$\lim_{h \to 0} \frac{1}{h} \int_{T_h} \chi d\mu_V \leq c$$

where the constant $c$ depends on $\kappa, \alpha, a, W$. This proves (2) i.e. that $V$ is of bounded variation or equivalently that $|\delta V|$ is a Radon measure. This implies that for any vector field $X$ with compact support in $U$ we have that

$$\delta V(X) = \int_{U \setminus B} X \cdot H d\mu_V + \int_B X \cdot \eta d\mu_{\text{sing}}$$

where

$$\int_B X \cdot \eta d\mu_{\text{sing}} = \lim_{h \to 0} \frac{1}{h} \int_{T_h} X \cdot \nabla \rho d\mu_V$$

(cf. (3), (4)).

We finally want to prove that $\eta = \eta(y) \in N_y B$, for all $y \in B$.

Let $X$ be a vector field with compact support in $U$ and such that $X(x) \in T_{\bar{x}} B$, $\forall x \in U$, $|X| \leq 1$, where $\bar{x}$ is as in Definition 1.3. Using Remark 1.4 and the fact that $x - \bar{x} \in N_{\bar{x}} B$ we have

$$|X \cdot \nabla \rho|^2 \leq 2 \left| X \cdot \left( D\rho - \frac{x - \bar{x}}{\rho} \right) \right|^2 + 2|X \cdot \nabla^\perp \rho|^2 \leq c\kappa^2 \rho^{2\alpha} + 2|\nabla^\perp \rho|^2$$

and so

$$\frac{1}{h} \int_{T_h} X \cdot \nabla \rho d\mu_V \leq \left( \frac{1}{h} \int_{T_h \cap \text{supp } X} d\mu_V \right)^{\frac{1}{2}} \left( \int_{T_h \cap \text{supp } X} \frac{2|\nabla^\perp \rho|^2}{\rho} d\mu_V + c\rho^{2\alpha - 1} d\mu_V \right)^{\frac{1}{2}}.$$

Using the monotonicity inequality (17) and the fact that $V$ is of bounded variation (in particular inequality (18)) we have that for any $\chi$ with compact support

$$\int_{T_r} \chi \frac{|\nabla^\perp \rho|^2}{\rho} d\mu_V < \infty$$

hence

$$\lim_{h \to 0} \frac{1}{h} \int_{T_h} X \cdot \nabla \rho d\mu_V = 0.$$

Because of (19) this implies that $X(y) \cdot \eta(y) = 0, \forall y \in B$ and so $\eta(y) \in N_y B$, for all $y \in B$. □
We would like to derive now a monotonicity formula for the ratios
\[ r^{-k} m(r) := \frac{1}{r^k} \mu_V(B_r(b)) \]
where \( b \) is a given point on \( B \cap U \) and \( r > 0 \) is such that \( B_r(b) \subset U \). Such estimates can be found for example in \([\text{All75, HS79, DS93b, DS93a, Bro77}]\). However, for completeness, we provide the formulas that will be needed for the proof of the main boundary regularity theorem, Theorem 3.14.

Having established the first variation formula, Lemma 2.1, we can use it with the notation \( X(x) = \phi(d(x))(x - b) \) we have that
\[ X(x) = \phi(d(x))(x - b) \]
where \( d(x) = |x - b| \) and \( \phi : \mathbb{R} \to \mathbb{R} \) is a smooth function with compact support in \([0, R/4 - |b|]\). Then we can argue as in the interior case (i.e. when \( B_r(b) \cap B = \emptyset \), cf. [Sim83, §17]) and by letting \( \phi \) approach the indicator function of \( B_r(b) \), where \( 0 < r < R/4 - |b| \) we get the following **monotonicity identity**:
\[
\frac{d}{dr} \left( r^{-k} \mu_V(B_r(b)) \right) = \frac{d}{dr} \int_{B_r(b)} \frac{|\nabla d|^2}{\sigma} d\mu_V + r^{-k-1} \int_{B_r(b)} (x - b) \cdot H d\mu_V \\
+ r^{-k-1} \int_{B \cap B_r(b)} (x - b) \cdot \eta(x) dV_{\text{sing}}.
\]

Recall that \( \eta(x) \in \mathcal{N}_2 B \) for all \( x \in B \cap U \) and so, using \([1.1] \) we can bound the last term by
\[
\left| \int_{B \cap B_r(b)} (x - b) \cdot \eta(x) dV_{\text{sing}} \right| \leq \kappa r^1 + \alpha \| V_{\text{sing}} \|((B_r(b)).
\]

We now want to estimate \( \| V_{\text{sing}}(B_r(b)) \| \).

In the proof of the first variation formula we have shown that this singular measure is equal to a limit of integration along tubular neighborhoods of \( B \) (equation \([19] \) in the proof of Lemma 2.1). Hence using the monotonicity inequality for tubular neighborhoods \([1.7] \) in the proof of Lemma 2.1 with \( \chi \) approaching the characteristic function of \( B_r(b) \), \( \sigma \to 0 \) and replacing the initial ball \( B_\eta(0) \) with a smaller ball \( B_{R'}(0) \) if necessary (with \( R' \) depending only on \( \alpha, \kappa \), so that the terms \( e^{r\alpha} \) and \( |Y| \) appearing in the monotonicity inequality are sufficiently small), we get the following estimate of the singular measure in terms of \( m(r) \)
\[
\| V_{\text{sing}}(B_r(b)) \| \leq \frac{2}{r} m(r) + 4am(r)^{\frac{1}{2}} + 4m'(r).
\]

Using the notation
\[ \tilde{m}(r) := r^{-k} m(r) \]
we have that
\[
\| V_{\text{sing}}(B_r(b)) \| \leq (2 + 4k)r^{k-1} \tilde{m}(r) + 4ar^{\frac{1}{2}} \tilde{m}(r)^{\frac{1}{2}} + 4r^k \tilde{m}'(r).
\]

**Lemma 2.5.** There exists a function \( \Phi(r) \) and a constant \( \Lambda = \Lambda(k, p, \alpha) \) such that for \( r < R/4 \)
\[
e^{\Phi(r)} \tilde{m}(r)^{\frac{1}{2}} + \Lambda r^{1 - \frac{k}{2}}
\]
is an increasing function of \( r \).
In particular
\[ \Phi(r) = \frac{4k + 2}{pa} kr^\alpha, \quad \Lambda = \frac{a}{p - k} \exp \left( \frac{4k + 2}{pa} \right). \]

**Proof.** By using the estimate for the singular measure \([2.4]\) in the monotonicity identity \([2.2]\) we get the following
\[
\bar{m}'(r) \geq \frac{d}{dr} \int_{B_r(b)} |\nabla d|^2 d\mu_V - (1 + 4k\rho^\alpha)ar^{-\frac{\beta}{\rho}} \bar{m}(r)^{\frac{1}{\rho}}
- (4k + 2)k\alpha r^{-\frac{\beta}{\rho}} \bar{m}(r)^{\frac{1}{\rho}},
\]
and multiplying by \(\bar{m}(r)^{-\frac{1}{\rho}}\) we get
\[
\bar{m}'(r)\bar{m}(r)^{-\frac{1}{\rho}} + (4k + 2)k\alpha r^{-\frac{1}{\rho}} \bar{m}(r)^{\frac{1}{\rho}} \geq -ar^{-\frac{1}{\rho}}.
\]

Let
\[
(2) \quad \Phi(r) = \frac{4k + 2}{pa} kr^\alpha
\]
then by multiplying the above inequality by \(p^{-1}e^{\Phi(r)}\) we have that
\[
\left( e^{\Phi(r)} \bar{m}(r)^{\frac{1}{\rho}} \right)' \geq -\frac{a}{p} e^{\Phi(r)} r^{-\frac{1}{\rho}} \geq -\frac{a}{p} \exp \left( \frac{4k + 2}{pa} \right) r^{-\frac{1}{\rho}}.
\]

Finally, letting
\[
(3) \quad \Lambda = \frac{a}{p - k} \exp \left( \frac{4k + 2}{pa} \right)
\]
we get that
\[
\left( e^{\Phi(r)} \bar{m}(r)^{\frac{1}{\rho}} + \Lambda r^{-\frac{1}{\rho}} \right)' \geq 0
\]
which proves the lemma. \(\square\)

As in the interior case \([\text{All72}, \text{Sim83}]\), immediate consequences of Lemma \(2.5\) are the following results about the density at points on \(B\) and the existence of tangent cones.

**Corollary 2.6.** The density
\[ \Theta_V(b) = \lim_{\rho \downarrow 0} \omega_k^{-1} \bar{m}(r) \]
exists for all \(b \in B \cap U\) and is upper semicontinuous as a function on \(B \cap U\):
\[ \Theta_V(b) \geq \limsup_{y \to b} \Theta_V(y). \]

**Corollary 2.7.** \(V\) has a tangent cone at all points \(b \in B \cap U\).

We also have the following corollary about the density at boundary points, for the proof of which we refer to \([\text{All75}]\) since it is identical to the case when \(B\) is a \(C^{1,1}\) submanifold.
Corollary 2.8. For all \( b \in B \cap U \)
\[ \Theta_V(b) \geq \frac{1}{2}. \]

Finally we want to prove one more monotonicity lemma.

Lemma 2.9. Let \( r_0 > 0 \) be such that \( 4k r_0^\alpha < 1/2, \Lambda r_0^{1-k/p} \leq 1/2(\omega_k/2)^{1/p} \), where \( \Lambda \) is as in Lemma 2.5.

There exists a function \( \Psi(r) \) such that for all \( 0 < \sigma < r < r_0 \) and \( b \in B \) such that \( B_r(b) \subseteq U \)
\[
e^{-\Psi(\sigma)} \tilde{m}(\sigma) \leq e^{-\Psi(\rho)} \tilde{m}(\rho) - \frac{1}{2} \int_{B_r(b) \setminus B_\sigma(b)} |\nabla^\perp d|^2 d\mu_V
\]
and
\[
e^{-\Psi(\sigma)} \tilde{m}(\sigma) \geq e^{-\Psi(\rho)} \tilde{m}(\rho) - 2 \int_{B_r(b) \setminus B_\sigma(b)} |\nabla^\perp d|^2 d\mu_V.
\]
In particular
\[
\Psi(r) = 4(2k + 1) \left( a \lambda^{-\frac{1}{q}} \left( 1 - \frac{k}{p} \right)^{-1} r^{1-\frac{k}{p}} + \alpha^{-1} \kappa r^\alpha \right)
\]
where \( \lambda = \frac{1}{2p} \exp \left( -\frac{4k+2}{\alpha} \frac{\omega_k}{2} \right). \)

Proof. We first get a lower bound for \( \tilde{m}(r) \) by letting \( \sigma \downarrow 0 \) in the monotonicity formula of Lemma 2.5 and using the lower bound for the density at a boundary point (Corollary 2.8),
\[
\tilde{m}(r)^\frac{1}{p} \geq \frac{1}{2} \exp \left( -\frac{4k+2}{\alpha p} \right) \left( \frac{\omega_k}{2} \right)^{\frac{1}{p}} \Rightarrow m(r) \geq \lambda r^k
\]
where \( \lambda = \frac{1}{2p} \exp \left( -\frac{4k+2}{\alpha} \frac{\omega_k}{2} \right). \) Hence we have that
\[
\int_{B_r(b)} |H| d\mu_V \leq a m(r)^{-\frac{1}{q}} \leq a \lambda^{-\frac{1}{q}} m(r)^{-\frac{1}{q}} \tilde{m}(r)^{-\frac{1}{q}} = a \lambda^{-\frac{1}{q}} \tilde{m}(r)^{-\frac{1}{q}}.
\]

We estimate the singular measure as we did before (in 2.3), using the monotonicity inequality for tubular neighborhoods (inequality 17 in the proof of Lemma 2.1), but now estimating the terms involving \( H \) by [3]
\[
\|V_{\text{sing}}(B_r(b))\| \leq (4k + 2)r^{k-1} \tilde{m}(r) + 4a \lambda^{-\frac{1}{q}} r^{\frac{k}{q}} \tilde{m}(r) + 4r^k \tilde{m}'(r).
\]

Using this estimate in the monotonicity identity 2.2 we get
\[
\tilde{m}'(r) - \left( a \lambda^{-\frac{1}{q}} r^{-\frac{1}{q}} + \kappa r^{\alpha - 1} \right) (4k + 1) \tilde{m}(r) \leq 2 \frac{d}{dr} \int_{B_r(b)} |\nabla^\perp d|^2 d\mu_V
\]
where we have used the assumption \( 4\kappa r^\alpha < 1/2 \), and similarly we get
\[
\tilde{m}'(r) + \left( a \lambda^{-\frac{1}{q}} r^{-\frac{1}{q}} + \kappa r^{\alpha - 1} \right) (4k + 1) \tilde{m}(r) \geq \frac{1}{2} \frac{d}{dr} \int_{B_r(b)} |\nabla^\perp d|^2 d\mu_V.
\]

Let
\[
\Psi(r) = 4(2k + 1) \left( a \lambda^{-\frac{1}{q}} \left( 1 - \frac{k}{p} \right)^{-1} r^{1-\frac{k}{p}} + \alpha^{-1} \kappa r^\alpha \right)
\]
Multiplying the first inequality by $e^{-\Psi(r)}$ and the second one by $e^{\Psi(r)}$ we have that
\[
\left(e^{-\Psi(r)} \bar{m}(r)\right)' \leq 2 \frac{d}{dr} \int_{B_r(b)} \frac{|\nabla d|^2}{d^k} d\mu_V.
\]
and
\[
\left(e^{\Psi(r)} \bar{m}(r)\right)' \geq \frac{1}{2} \frac{d}{dr} \int_{B_r(b)} \frac{|\nabla d|^2}{d^k} d\mu_V
\]
and the lemma then follows by integrating these two inequalities. $\square$

3. Boundary Regularity Theorem

From now on we assume that $V, B$ are as defined in Section 1 satisfying 1.1 and properties $P_1-P_3$ and we take $R$ small enough (depending on $k, \alpha, p$) so that the monotonicity Lemmas 2.5, 2.9 are satisfied in $U = B_R(0)$.

For simplifying computations we make the following assumption:

$$T_0B = \mathbb{R}^{k-1} \times \{0\}^{n-k+1}$$

so that

$$B \cap B_R(0) = \text{graph } \psi \cap B_R(0)$$

where

$$\psi : B_R(0) \cap \mathbb{R}^{k-1} \to \mathbb{R}^{n-k+1}$$

is such that $\psi(0) = 0$, $D\psi(0) = 0$ and

$$\|\psi\|_0 + \|D\psi\|_0 + R^\alpha[D\psi]_\alpha \leq \kappa$$

(cf. Remark 1.2). We will be using the notation $T' = \mathbb{R}^{k-1} \times \{0\}^{n-k+1}$ and $T'^\perp = \{0\}^{k-1} \times \mathbb{R}^{n-k+1}$, so that for the corresponding projections we have that

$$\text{proj}_{T'}(x) = \text{proj}_{T_0B}(x) \text{ and } \text{proj}_{T'^\perp}(x) = \text{proj}_{N_0B}(x) \text{ for all } x \in \mathbb{R}^n.$$

**Definition 3.2.** For each $a \in \mathbb{R}^n \cap B_R(0)$, we define $\omega(a)$, to be the unique point in $B \cap B_R(0)$ such that $a - \omega(a) \in T'^\perp$ and $\zeta(a)$ to be the real valued function

$$\zeta(a) = |a - \omega(a)|, \text{ i.e.}$$

$$\omega(a) = (\text{proj}_{T'}(a), \psi(\text{proj}_{T'}(a)))$$

$$\zeta(a) = |\psi(\text{proj}_{T'}(a)) - \text{proj}_{T'^\perp}(a)|.$$

For all $a \notin B$ we define $\chi(a)$ to be the projection of $\mathbb{R}^n$ onto the subspace $\mathbb{R}^{k-1} \times \{0\}^{n-k+1} + \{t(a - \omega(a)) : t \in \mathbb{R}\}$.

Then the following holds

$$|\text{proj}_{T'}(\omega(x) - \omega(a))|^2 \geq |\omega(x) - \omega(a)|^2 - |\text{proj}_{T'^\perp}(\omega(x) - \omega(a))|^2 \geq (1 - (R^\alpha \kappa)^2) |\omega(x) - \omega(a)|^2$$

and hence

$$|\omega(x) - \omega(a)| \leq (1 - (R^\alpha \kappa)^2)^{-\frac{1}{2}} |\text{proj}_{T'}(x - a)|$$

$$\leq (1 - (R^\alpha \kappa)^2)^{-\frac{1}{2}} |x - a|.$$

We state the following interior regularity lemma, which is a consequence of Al-lard’s interior regularity theorem.
Lemma 3.4. Assume that
\[ R^{-k} \omega_k^{-1} \mu_V (B_R(0)) \leq \frac{1 + \delta}{2} \]
Then for each \(0 < \gamma < 1, \eta > 0\) we can find \(\kappa, a\) and \(\delta\) small enough so that
1. \(N = \text{spt} \mu_V \cap B_{(1-\gamma)} R(0) \setminus B\) is a non-empty continuously differentiable \(k\)-dimensional submanifold of \(\mathbb{R}^n\), which is closed relative to \(B_{(1-\gamma)} R(0) \setminus B\).
2. For each \(a \in N\), \(|\chi(a)^{\perp} (x - a)| \leq \eta |x - a|\), for any \(x \in N \cap B_{\delta}(a)\).
3. \(\Theta_V(a) \leq 1 + \eta\), for any \(a \in N\).

The proof of this lemma is as in the case when \(B\) is a \(C^{1,1}\) submanifold \([75]\) Theorem 4.3] and thus we omit it.

We will finally need the following definition:

Definition 3.5. The \(\text{tilt-excess} E(x, r, T)\) of the varifold \(V\) with respect to a \(k\)-dimensional subspace \(T\) of \(\mathbb{R}^n\) is given by
\[ E(x, r, T) = r^{-k} \int_{B_r(x)} \| \text{proj}_{T_M} \text{ - proj}_T \| d\mu_V (y). \]

We also define the following quantities
\[ E^*(0, R, T) = \max \left\{ E(0, R, T), R^{2(1 - \frac{k}{p})} \left( \int_{B_R(0)} |H|^p \right)^{\frac{2}{p}}, (\kappa R^\alpha)^2 \right\} \]
and
\[ E_*(0, R, T) = \max \left\{ E(0, R, T), R^{-1} R^{2(1 - \frac{k}{p})} \left( \int_{B_R(0)} |H|^p \right)^{\frac{2}{p}}, \kappa R^\alpha \right\}. \]

Often and when there is no confusion we will simply write \(E^*, E_*\) for the above two defined quantities.

Lemma 3.6. For any \(b \in B, r > 0\) such \(B_r(b) \subset B_R(0)\) and \(T\) a \(k\)-dimensional subspace with \(T_b B \subset T\) we have that
\[ E(b, r/2, T) \leq c \left( r^{-k-2} \int_{B_r(b)} \text{dist}(x, T)^2 d\mu_V (x) + r^{2-k} \int_{B_r(b)} |H|^2 d\mu_V + (\kappa r^\alpha)^2 \right) \]
where \(c\) is a constant that depends only on \(n, k, R\).

Remark 3.7. Applying Holder to the RHS of the inequality of Lemma 3.6 we get
\[ E(b, r/2, T) \leq c \left( r^{-k-2} \int_{B_r(b)} \text{dist}(x, T)^2 d\mu_V (x) \right. \]
\[ + \left. r^{2(1 - \frac{k}{p})} \left( \int_{B_r(b)} |H|^p d\mu_V \right)^{\frac{2}{p}} + (\kappa r^\alpha)^2 \right). \]

For the proof of Lemma 3.6 we will need the following notation: Let
\[ P = \{0\}^{k-1} \times \mathbb{R} \times \{0\}^{n-k} = T^\perp \cap (\mathbb{R}^k \times \{0\}^{n-k}) = N_0 B \cap (\mathbb{R}^k \times \{0\}^{n-k}) \]
and let
\[ \bar{B} = B + P. \]
Then $\bar{B}$ is a $C^{1,\alpha}$, $k$-dimensional manifold and $\bar{B} \cap B_R(0)$ can be written as a $C^{1,\alpha}$ graph above $\mathbb{R}^k \times \{0\}^{n-k}$, i.e. there exists a $C^{1,\alpha}$ function

$$\bar{\psi} : \mathbb{R}^k \cap B_R(0) \to \mathbb{R}^{n-k}$$

such that $\bar{B} \cap B_R(0) = \text{graph } \bar{\psi} \cap B_R(0)$ and

$$3.8 \quad \frac{\| \bar{\psi} \|_0}{R} + \| D\bar{\psi} \|_0 + R^\alpha \| D\bar{\psi} \|_\alpha \leq \kappa.$$ 

In particular for $x \in \mathbb{R}^{k-1}$ and $(x, x') \in \mathbb{R}^k$ we have that

$$\bar{\psi}(x, x') = (\psi^2(x), \ldots, \psi^{n-k+1}(x))$$

where $\psi = (\psi^1, \psi^2, \ldots, \psi^{n-k+1}) : \mathbb{R}^{k-1} \to \mathbb{R}^{n-k+1}$ is the function whose graph is equal to $B$ in $B_R(0)$ as defined in 3.1. Note that the smoothness conditions 1.1 and Remark 1.4 still hold with $B$ replaced by $\bar{B}$.

Finally we define $\bar{\rho}$ to be the distance from $\bar{B}$, i.e. $\bar{\rho}(x) = \text{dist}(x, \bar{B})$.

**proof of Lemma 3.6** Without loss of generality we can assume that $B = \mathbb{R}^k \times \{0\}^{n-k}$.

Let $W$ be a Whitney partition of $\mathbb{R}^n \setminus \bar{B}$. Then

$$B_R(0) \setminus \bar{B} \subset \bigcup_{C \in W} C$$

where the elements $C$ of $W$ are cubes such that if $\bar{\rho}_C = \text{dist}(C, \bar{B}) > 0$ then

$$\text{diam } C \leq \bar{\rho}_C \leq 3 \text{ diam } C.$$ 

Let $x_C \in C$ be the center of the cube $C$ and $y_C \in \bar{B}$ be such that $|x_C - y_C| = \bar{\rho}(x_C)$.

Finally let $\phi_C$ be a partition of unity subordinate to $W$ and such that

$$|D\phi_C(x)| \leq C\bar{\rho}(x)^{-1}$$

where $C$ is an absolute constant.

For each $x \in C$, we define $x'_C$ to be the unique point in $\bar{B} \cap B(y_C, 2\bar{\rho}_C)$ such that $x'_C - x \in N_{y_C}B$. Then by Remark 1.4 we know that if $\bar{x} \in \bar{B} \cap B_R(0)$ is such that $|x - \bar{x}| = \bar{\rho}(x)$ then

$$|x'_C - \bar{x}| \leq c\kappa\bar{\rho}(x)^{1+\alpha} \leq c\kappa\bar{\rho}_C^{1+\alpha}.$$ 

We define the following vector field

$$X = \zeta^2 \sum_{C \in W} \phi_C X_C$$

where

$$X_C(x) = \text{proj}_{\perp}(x - x'_C)$$

and $\zeta$ is a smooth real valued function with compact support in $B_r(0)$ (where $r$ is as in the statement of the lemma) and such that

$$\zeta(x) = 1, \forall x \in B_{r/2}(0) \text{ and } |D\zeta| \leq 3/r.$$ 

Since $X(x) = 0$, for all $x \in B \cap B_R(0)$, the first variation formula implies that

$$\int \text{div}_M X d\mu_V = \int X \cdot H d\mu_V.$$ 

We will estimate

$$3. \quad \text{div}_M X = 2\zeta \sum_{C \in W} \phi_C \nabla^M \zeta \cdot X_C + \zeta^2 \sum_{C \in W} \nabla^M \phi_C \cdot X_C + \zeta^2 \sum_{C \in W} \phi_C \text{div}_M X_C.$$
For each $C \in \mathcal{W}$ we have that
\[ \text{div}_M X_C = \frac{1}{2} |\text{proj}_{T,M} - \text{proj}_T|^2. \]

To see this let $t = (t^{ij})$, $e = (e^{ij})$ denote the matrices of the projections onto $T$ and $T_\perp M$ respectively, then
\[ |\text{proj}_{T,M} - \text{proj}_T|^2 = \sum_{j=1}^n e_j(e + t - 2et)e_j = 2k - 2 \sum_{j=1}^n e_j(eet)e_j \]
(4)
\[ = 2 \sum_{j=1}^n e_j(e(I - t))e_j = 2 \sum_{i=k+1}^n e_i. \]

To estimate the two first terms on the RHS of (3) note that
\[ \sum_{C \in \mathcal{W}} \nabla M \phi_C \cdot X_C = \nabla M \phi_C \cdot (\text{proj}_{T\perp} (x - x_C') - \text{proj}_{T\perp} (x - \bar{x})) \]
\[ = \sum_{C \in \mathcal{W}} D\phi_C \cdot (\text{proj}_{T,M} \circ \text{proj}_{T\perp})(\bar{x} - x_C') \]
and
\[ \nabla M \zeta \cdot X_C(x) = D\zeta \cdot (\text{proj}_{T,M} \circ \text{proj}_{T\perp})(x - x_C'). \]
Hence using (1), (3.8) and the Cauchy-Schwartz inequality, we have that for any $\varepsilon > 0$
\[ |\zeta|^2 \sum_{C \in \mathcal{W}} \nabla M \phi_C \cdot X_C \leq \varepsilon \zeta^2 |\text{proj}_{T,M} - \text{proj}_T|^2 + \frac{C \zeta^2 (kr^\alpha)^2}{\varepsilon} \]
and
\[ |\zeta \nabla M \zeta \cdot X_C| \leq |\zeta||D\zeta||\text{proj}_{T,M} - \text{proj}_T||(|\text{proj}_{T\perp} (x) + |\text{proj}_{T\perp} (x_C')|) \]
\[ \leq \varepsilon \zeta^2 |\text{proj}_{T,M} - \text{proj}_T|^2 + \frac{C}{\varepsilon} r^{-2} (\text{dist}(x, T))^2 + \frac{C}{\varepsilon} (kr^\alpha)^2. \]
For estimating the RHS of the first variation formula (2) we note that
\[ |X_C \cdot H| \leq \frac{\text{dist}(x, T)^2}{r^2} + (kr^\alpha)^2 + 2r^2 |H|^2 \]
Hence using the above estimates in the first variation formula (2) for sufficiently small $\varepsilon > 0$, we get the required estimate of the lemma
\[ E(0, r/2, T) \leq c \left( r^{-k-2} \int_{B_r(0)} \text{dist}(x, T)^2 d\mu_V(x) \right. \]
\[ + \left. r^{2-k} \int_{B_r(0)} |H|^2 d\mu_V + (kr^\alpha)^2 \right). \]

Lemma 3.9 (Lipschitz approximation). Assume that
(1) \[ R^{-k} \omega_k^{-1} \mu_V (B_R(0)) \leq \frac{1 + \delta}{2}. \]

For any $\gamma \in (0, 1/4)$, $l \in (0, 1)$ there exists $\delta > 0$ such that if (1) holds, then there exists a Lipschitz function
\[ f = (f^1, f^2, \ldots, f^{n-k}) : \bar{H} \cap B_{\gamma R}(0) \to \mathbb{R}^{n-k} \]
where $\bar{H}$ is one of the two regions of $T \cap B_R(0)$ defined by $\text{proj}_T(B)$, with $\text{Lip } f \leq l$
and
\begin{equation}
\mathcal{H}^k \left((\text{spt } V \setminus \text{graph } f) \cup (\text{graph } f \setminus \text{spt } V)\right) \cap B_R(0) \leq c \ell^{-2} E^* R^k
\end{equation}
where $c = c(n, k)$ is a constant, $E^* = E^*(0, R, T)$ and $T$ is a $k$-dimensional subspace such that $T' = T_B \subset T$.

\begin{proof}
We can assume that $E^* \leq \delta_0 l^2$, a small multiple of $l^2$, since else we can trivially pick $f = 0$ and then appropriately choose $c$ so that (2) holds.

For any $\gamma \in (0, 1)$ we take $\delta, \delta_0$ small enough, so that the conclusions of the interior regularity lemma, Lemma 3.4, hold in $B_{1, R}(0)$ and with $\eta \leq \eta(l)$ that we will choose later.

We define the following set
\begin{equation}
G = (B \cap B_{1, R}(0)) \cup (\text{spt } V \cap B_{1, R}(0)) \cap \{a : \|\chi(a) - \text{proj}_T a\| \leq 2\eta(l)\}
\end{equation}
where $\chi(a)$ is as in Definition 3.2.

Claim: For all $x, y \in G$:
\begin{equation}
|\text{proj}_{T^\perp} (x - y)| \leq (10\eta(l) + 4E^{*1/2})|x - y|.
\end{equation}

Note first that if both $x$ and $y$ are in $B \cap B_{1, R}(0)$ then since $T^\perp \subset T'$ and by 1.1, we have that
\begin{equation}
|\text{proj}_{T^\perp} (x - y)| \leq 2E^{*1/2}|x - y|.
\end{equation}

So we can assume that one of them, say $y$, is not in $B$.

Assume first that
\begin{equation}
|x - y| < \zeta(y).
\end{equation}

Then by Lemma 3.4(ii) we have that $|\chi(y)^\perp (x - y)| \leq \eta|x - y|$ and so
\begin{equation}
|\text{proj}_{T^\perp} (x - y)| \leq |\text{proj}_{T^\perp} \circ \chi(y)(x - y)| + |\text{proj}_{T^\perp} \circ \chi(y)^\perp (x - y)| \leq 3\eta(l)|x - y|.
\end{equation}

Assume now that
\begin{equation}
|x - y| \geq \zeta(y).
\end{equation}

Note that for all $y \in G$
\begin{equation}
|\text{proj}_{T^\perp} (y - \omega(y))| = |(\text{proj}_{T^\perp} \circ \chi(y))(y - \omega(y))| \leq 2\eta \zeta(y).
\end{equation}

Furthermore for $\delta_0$ small enough (so that $(\kappa R^*)^2 \leq 3/4$), we have by 3.3 that $|\omega(x) - \omega(y)| \leq 2|x - y|$. Hence
\begin{equation}
|\text{proj}_{T^\perp} (x - y)| \leq |\text{proj}_{T^\perp} (x - \omega(x))| + |\text{proj}_{T^\perp} (\omega(x) - \omega(y))| + |\text{proj}_{T^\perp} (\omega(y) - y)|
\leq 2\eta \zeta(x) + |\text{proj}_{T^\perp} (\omega(x) - \omega(a))| + 2\eta \zeta(y)
\leq 2\eta(|x - y| + \zeta(y) + |\omega(y) - \omega(x)|) + 2\kappa R^* |\omega(x) - \omega(y)| + 2\eta|x - y|
\leq 10\eta(l)|x - y| + 4E^{*1/2}|x - y|
\end{equation}

and so the claim is true.

Since
\begin{equation}
|x - y| \leq |\text{proj}_{T^\perp} (x - y)| + |\text{proj}_T (x - y)|
\end{equation}
we have that for all $x, y \in G$
\begin{equation}
|\text{proj}_{T^\perp} (x - y)| \leq (10\eta(l) + 4E^{*1/2})(|\text{proj}_{T^\perp} (x - y)| + |\text{proj}_T (x - y)|).
\end{equation}

So picking
\begin{equation}
\eta(l) = \frac{l}{40}
\end{equation}
and

\[ \delta_0 \leq \frac{1}{16^2} \Rightarrow E^{1/2} \leq \frac{l}{16} \]

we have that

\[ (5) \quad 10\eta(l) + 4E^{1/2} \leq \frac{l}{2} \]

and therefore

\[ |\text{proj}_{T^\perp}(x - y)| \leq l|\text{proj}_T(x - y)|. \]

Hence \( G = \text{graph} f \) where \( f : \text{proj}_T(G) \to T^\perp \) is a Lipschitz function with Lipschitz constant at most \( l \).

Note that by definition of \( G \) and by the interior regularity lemma (Lemma 3.4 (ii)), \( \text{proj}_T(G) \) lies only on one region of \( T \cap B_{\gamma R}(0) \) defined by \( \text{proj}_T(B) \). Calling this region \( \tilde{H} \) and using the extension theorem for Lipschitz functions we can construct a Lipschitz function

\[ f : \tilde{H} \cap B_{\gamma R}(0) \to T^\perp \]

with \( \text{Lip } f \leq l \) and such that \( G \subset \text{graph } f \).

By the above height claim (4) and by the choice of \( \eta(l), \delta_0 \) (cf. (5)) we have that

\[ \frac{d}{d\rho}(x) = \frac{d}{d\rho}(x - 0) \leq l/2|x| \quad \forall x \in G \]

and thus, by truncating the extension if necessary, we also have that

\[ \sup_{\tilde{H} \cap B_{\gamma R}(0)} |f| = \sup_{\text{proj}_T(G)} |f| = \sup_{\text{dist}(x, T)} \{x \in G\} \leq \frac{l\gamma R}{2}. \]

Let

\[ G' = (\text{spt } V \cap B_{\gamma R}(0)) \setminus G \supset (\text{spt } V \cap B_{\gamma R}(0)) \setminus \text{graph } f. \]

We want to estimate \( H^k(G') \).

For \( x \in G' \) we have that

\[ \|\text{proj}_{T_x} - \text{proj}_T\| \geq \|\chi(x) - \text{proj}_T\| - \|\text{proj}_{T_x} - \chi(x)\| > 2\eta(l) - \eta(l) \geq \eta(l) \Rightarrow \]

\[ |\text{proj}_{T_x} - \text{proj}_T|^2 > c\eta(l)^2 \]

where \( c \) depends on \( n, k \). By Lemma 3.4 (i), (iii), \( \text{spt } V \) is smooth away from \( B \) and \( \Theta_V(x) \leq 1 + \eta(l), \forall x \in \text{spt } V \setminus B \). So for every \( x \in G' \), there exists \( \rho_x < \min\{\rho(x), (1 - \gamma)R\} \) such that

\[ \|\text{proj}_{T_y} - \text{proj}_T\|^2 \geq \frac{c}{2}\eta(l)^2, \forall y \in B_{\rho_x}(x) \]

and

\[ \frac{1}{2} \leq \frac{\mu_V(B_{5\rho})(x)}{\omega_k(5\rho)^k} \leq 2(1 + \eta(l)), \forall \rho \leq \rho_x. \]

Hence for all \( \rho \leq \rho_x \) we have

\[ \rho^{-k} \int_{B_{\rho}(x)} |\text{proj}_T - \text{proj}_{T_x}|^2 d\mu_V(y) \geq \frac{c}{2}\eta(l)^2 \rho^{-k} \mu_V(B_{\rho}(x)) \geq c\frac{\eta(l)^2}{4}\omega_k. \]
The collection of balls \( \{ B_{\rho_x}(x) \}_{x \in G'} \) is a cover for \( G' \), so using the 5-times covering lemma we can pick \( \{ x_i \}_{i \in \mathbb{N}} \subset G' \) such that \( \{ B_{\rho_{x_i}}(x_i) \}_{i \in \mathbb{N}} \) are disjoint and \( G' \subset \bigcup_{i \in \mathbb{N}} B_{5\rho_{x_i}}(x_i) \subset B_R(0) \). Then

\[
\mu_V(G') \leq \sum_{i \in \mathbb{N}} \mu_V(B_{5\rho_{x_i}}(x_i)) \leq 2(1 + \eta(l))\omega_k 5^k \sum_{i \in \mathbb{N}} \rho_{x_i}^k \\
\leq c \delta^k \frac{1 + \eta(l)}{\eta(l)^2} \sum_{i \in \mathbb{N}} \int_{B_{\rho_{x_i}}(x_i)} | \text{proj}_T - \text{proj}_{\mathbb{R}^M} |^2 d\mu_V(y) \leq c \frac{1 + \eta(l)}{\eta(l)^2} R^k E^*.
\]

Since \( \Theta_V(x) \geq 1 \) for \( x \in G' \) we get that

\[
H^k(G') \leq \mu_V(G') \leq c l^{-2} R^k E^*.
\]

Hence

\[
H^k((\text{spt } V \cap B_{\gamma R}(0) \setminus \text{graph } f) \leq c l^{-2} E^* R^k.
\]

Now we want to estimate the measure of \((\text{graph } f \setminus \text{spt } V) \cap B_{\gamma R}(0)\). We do this using the following:

Claim: For \( l, \delta_0 \) sufficiently small, the following holds: For each \( x \in (\text{graph } f \setminus \text{spt } V) \cap B_{\gamma R/4}(0) \) there exists \( \tilde{x} \in (\text{graph } f \setminus \text{spt } V) \cap B_{\gamma R/4}(0), \rho_x > 0 \) such that the following hold:

i) \( B_{\rho_x}(\tilde{x}) \cap \text{spt } V = \emptyset, \overline{B}_{\rho_x}(\tilde{x}) \cap \text{spt } V \neq \emptyset \)

ii) \( B_{(1+\frac{1}{2})\rho_x}(\tilde{x}) \cap B = \emptyset \)

iii) \( x \in \overline{B}_{4\rho_x}(\tilde{x}) \)

Notice that it suffices to prove the lemma for small values of \( l \). That is because if it is true for any \( l \leq l_0 \) and \( f_{l_0} \) is a lipschitz function as in the statement of the lemma with \( l = l_0 \), then for any \( l > l_0, f_l \) satisfies the lemma with the constant \( c \) of the lemma (in \( \mu_2 \)) replaced by \( cl_0^{-2} \).

Proof of Claim: Given \( x \in (\text{graph } f \setminus \text{spt } V) \cap B_{\gamma R/4}(0) \), let

\[
\sigma = \max\{ \rho : \exists y \in \text{graph } f \cap B_{\gamma R/4} \text{ such that } B_{\rho}(y) \cap \text{spt } V = \emptyset, \overline{B}_{\rho}(y) \cap \text{spt } V \neq \emptyset \text{ and } x \in \overline{B}_{\rho/4}(y) \}.
\]

Let \( y = (y_1, f(y_1)) \in \text{graph } f \setminus \text{spt } V \) be such that \( B_{\sigma}(y) \) satisfies the above properties and let \( \tilde{y} = (\tilde{y}_1, f(\tilde{y}_1)) \in B \) be such that \( |y - \tilde{y}| = \text{dist}(y, B) \). Because of the choice of \( \sigma \) there exists \( z = (z_1, f(z_1)) \in \text{spt } V \cap \partial B_{\sigma}(y) \) such that \( \text{dist}(z_1, \partial H) \geq |y_1 - \tilde{y}_1| - \beta \) with \( \beta \to 0 \) as \( l, \delta_0 \to 0 \), so that for \( l, \delta_0 \) sufficiently small \( \text{dist}(z_1, \partial H) \geq \sigma/4 \).

Then we can pick \( \tilde{x}_1 \in B_{\rho_x}(y_1) \), so that for \( \tilde{x} = (\tilde{x}_1, f(\tilde{x}_1)) \) we have the following:

\[
y \in B_{\sigma/2}(\tilde{x}) \subset B_{\sigma}(y), z \in B_{\sigma/4}(\tilde{x}) \text{ and } B_{11\sigma/10}(\tilde{x}) \cap B = \emptyset \text{ (e.g. pick } \tilde{x}_1 \text{ in the direction of } y_1 - \tilde{y}_1 \text{ and such that } |\tilde{x}_1 - y_1| < \sigma/10 \).
\]

So that the claim is true with \( \rho_x \in (\sigma/3, 5\sigma/4) \).

Now let \( x \in (\text{graph } f \setminus \text{spt } V) \cap B_{\gamma R/4}, B_{\rho_x}(\tilde{x}) \) be as in the above claim and let \( y \) be such that \( y \in \text{spt } V \cap \partial B_{\rho_x}(\tilde{x}) \). Then by the interior monotonicity formula \( \text{Sim83 Theorem 17.6} \) and for \( \delta_0 \) sufficiently small we have that

\[
(\frac{11}{10})^{-k} \mu_V(B_{\rho_x}(\tilde{x})) \geq \frac{1}{11^k} \left( \frac{1}{10} \rho_x \right)^{-k} \mu_V(B_{\rho_x}(y)) \geq \omega_k 11^{-k-1}
\]
Using the 5-times covering lemma we can take a disjoint collection 

\[ \{ \} \]

such that

\[ \text{we get that} \]

\[ (9) \]

Since \( x \in \text{graph } f \) and \( \text{Lip } f \leq l \) we have that for \( z \in \text{graph } f \)

\[ \left| \text{proj}_T\left( \frac{z-x}{\sigma} \right) \right|^2 \leq l^2 \]

hence

\[ (9) \]

We can take \( l \) small enough so that \( cl^2 \leq 1/2 \), where \( c \) as in (9). Using (8) we have that

\[ \rho_x^k \leq c \left( \mu_V(B_{4\rho_x}(x)) \setminus \text{graph } f \right) + \int_{B_{4\rho_x}(x)} \left| \text{proj}_{T^*M} - \text{proj}_T \right|^2 \mu_V(z) \) .

Hence for any \( x = (x_1, f(x_1)) \in (\text{graph } f \setminus \text{spt } V) \cap B_{\gamma R/4} \) we can find \( \bar{x} = (\bar{x}_1, \bar{f}(x_1)) \in (\text{graph } f \setminus \text{spt } V) \cap B_{\gamma R/4} \) and \( \rho = \rho(x_1) \) so that \( x \in B_{4\rho_x}(\bar{x}) \) and

\[ (9) \]

Using the 5-times covering lemma we can take a disjoint collection \( \{ B_{4\rho(x_1)}(\bar{x}_1) \} \) such that

\[ \text{proj}_T\left( (\text{graph } f \setminus \text{spt } V) \cap B_{\gamma R/4}(0) \right) \subset \bigcup x_1 B_{20\rho(x_1)}(\bar{x}_1) \]
that the following holds: If \( f \) satisfies (3), then \( \rho(\alpha) \leq \omega_k^{-1/2} E^{1/2k} R \). Hence the lemma is true with \( \gamma = \gamma/4 \).

\[ \| f \|_{L^p} \leq C (\omega_k^{-1/2} E^{1/2k} R)^{1/2} \]

Remark 3.10. Let \( f : \tilde{H} \cap B_{cR}(0) \to \mathbb{R}^{n-k} \) be the Lipschitz function constructed in Lemma 3.3 and suppose that \( E^* \leq \varepsilon \). Then there is small enough the following holds: If \( a \in (\text{spt} V \cap B_{cR}(0)) \) then \( \rho(a) \leq \omega_k^{-1/k} E^{1/2k} R \).

To see this recall that for \( \delta E^* \) small enough it holds that \( \rho(a) \geq 1/2 \omega_k \) and therefore if \( \rho(a) \geq \omega_k^{-1/k} E^{1/2k} R \) then

\[ |T - \chi(a)|^2 \leq 2 \omega_k^{-1} \left( \frac{R}{\rho(a)} \right)^k E + \eta^2 \leq 4 \eta^2 \]

provided that \( \varepsilon^{1/2} \leq \eta^2 \). But this implies that \( a \in G \), where \( G \) is the set defined in (3) of the proof of Lemma 3.9 and satisfies \( G \subset \text{graph } f \).

Lemma 3.11 (Harmonic approximation lemma). Let \( \Omega \) be a domain in \( \mathbb{R}^k \) such that \( \partial \Omega \) is \( C^1 \) and \( 0 \in \partial \Omega \). For any \( r > 0 \), \( \varepsilon > 0 \) there exists \( \delta = \delta(k, \varepsilon) > 0 \) such that the following holds: If \( f \in W^{1,2}(\Omega \cap B_r(0)) \) is such that

\[ r^{-k} \int_{\Omega \cap B_r(0)} |Df|^2 \, dH^k \leq 1, \]

\[ r^{-1} \sup_{\partial \Omega \cap B_r(0)} |f| \leq \delta \]

and

\[ r^{-k} \left| \int_{\Omega \cap B_r(0)} Df \cdot D\zeta \, dH^k \right| \leq \delta \sup |D\zeta| \]

for any \( \zeta \in C_c^\infty(B_r(0) \cap \Omega) \), and if

\[ |\nu_{\partial \Omega}(x) \cdot (x - y)| \leq \delta |x - y|, \ \forall x, y \in \partial \Omega \cap B_r(0) \]
where $\nu_{\partial \Omega}$ denotes the outward pointing unit normal to $\partial \Omega$. Then there exists a function $w \in W^{1,2}(B_r(0))$ with the following properties: for some halfspace $H$ of $\mathbb{R}^k$, $w$ is harmonic in $H \cap B_r(0)$,

$$w(x) = 0, \forall x \in B_r(0) \cap \partial H,$$

$$r^{-k} \int_{H \cap B_r(0)} |Dw|^2 d\mathcal{H}^k \leq 1$$

and

$$r^{-k-2} \int_{\Omega \cap B_r(0)} (w - f)^2 d\mathcal{H}^k \leq \varepsilon.$$

**Proof.** Suppose that the lemma is not true. Then for some $\varepsilon > 0$ we can find a sequence of domains $\{\Omega_i\}_{i \in \mathbb{N}}$ with $0 \in \Omega_i, \forall i$ and corresponding functions $f_i \in W^{1,2}(\Omega_i \cap B_r(0))$, such that for all $i$:

1. $|\nu_{\partial \Omega_i}(x) \cdot (x - y)| \leq \frac{1}{i} |x - y|, \forall x,y \in \partial \Omega_i \cap B_r(0)$

2. $r^{-k} \int_{\Omega_i \cap B_r(0)} |Df_i|^2 d\mathcal{H}^k \leq 1$

3. $r^{-1} \sup_{\partial \Omega_i \cap B_r(0)} |f_i| \leq \frac{1}{i} \Rightarrow \omega_{k-1}^{-1} r^{-k} \int_{\partial \Omega_i \cap B_r(0)} |f_i| d\mathcal{H}^{k-1} \leq \frac{2}{i}$

and

4. $r^{-k} \left| \int_{\Omega_i \cap B_r(0)} Df_i \cdot D\zeta d\mathcal{H}^k \right| \leq \frac{1}{i} \sup |D\zeta|$

for any $\zeta \in C^\infty_c(\overline{B_r(0)} \cap \Omega_i)$, but such that

$$r^{-k-2} \int_{\Omega_i \cap B_r(0)} (w - f_i)^2 d\mathcal{H}^k > \varepsilon$$

for any $w \in W^{1,2}(B_r(0))$, such that for some halfspace $H$ of $\mathbb{R}^k$, $w$ is harmonic in $H \cap B_r(0)$, $w = 0$ on $\partial H \cap B_r(0)$ and

$$r^{-k} \int_{H \cap B_r(0)} |Dw|^2 d\mathcal{H}^k \leq 1.$$

Let

$$\bar{f}_i = \frac{1}{|\Omega_i \cap B_r(0)|} \int_{\Omega_i \cap B_r(0)} f_i d\mathcal{H}_k.$$

Then [2] and [3] imply that $\|f_i\|_{L^1(\Omega_i \cap B_r(0))} \leq cr^{k+1}$ and so $\|\bar{f}_i\|_{L^2(\Omega_i \cap B_r(0))} \leq cr^{k+2}$, with the constants $c$ being uniform in $i$.

By the Poincare inequality we get

$$r^{-k-2} \int_{\Omega_i \cap B_r(0)} |f_i - \bar{f}_i|^2 d\mathcal{H}^k \leq cr^{-k} \int_{\Omega_i \cap B_r(0)} |Df_i|^2 d\mathcal{H}^k \leq c$$

and so

$$r^{-k-2} \int_{\Omega_i \cap B_r(0)} f_i^2 d\mathcal{H}^k \leq c$$

for all $i \in \mathbb{N}$. 

Let \(Ef_i \in W^{1,2}(\mathbb{R}^n)\) denote an extension of \(f_i\) to \(\mathbb{R}^n\). Since \(\|f\|_{W^{1,2}(\Omega \cap B_r(0))}\) are uniformly bounded we can also take \(Ef_i\) to have uniformly bounded \(W^{1,2}\) norm and hence we can apply Rellich’s theorem to conclude that after passing to a subsequence
\[
\int_{B_r(0)} |Ef_i - w|^2 d\mathcal{H}^k \to 0
\]
for some \(w \in W^{1,2}(B_r(0))\).

Because of (1), after passing to a further subsequence, if necessary, we have that
\[
\partial \Omega_i \cap B_r(0) \to \partial H \cap B_r(0)
\]
for some halfspace \(H\) of \(\mathbb{R}^k\) and where the convergence is in the \(C^\beta\) sense for all \(\beta < 1\). Hence we have that
\[
\int_{B_r(0) \cap \Omega_i} |f_i - w|^2 d\mathcal{H}^k \to 0
\]
and also, by (2) (4) we get respectively
\[
r^{-k} \int_{B_r(0) \cap \partial H} |Dw|^2 d\mathcal{H}^k \leq 1
\]
and
\[
r^{-k} \int_{B_r(0) \cap \partial H} Dw \cdot D\zeta d\mathcal{H}^k = 0
\]
for all \(\zeta \in C^\infty(\partial B_r(0) \cap H)\), i.e. \(w\) is harmonic in \(H \cap B_r(0)\).

Without loss of generality we can assume that \(H = \{x \in \mathbb{R}^k : x^k \geq 0\}\). For any \(\zeta \in C^\infty(\partial B_r(0))\), we have that
\[
\int_{B_r(0) \cap \partial H} D_k f_i \zeta d\mathcal{H}^k = -\int_{B_r(0) \cap \partial H} f_i D_k \zeta d\mathcal{H}^k + \int_{\partial \Omega_i \cap B_r(0)} f_i \zeta D_k v_{\partial \Omega_i} d\mathcal{H}^{k-1}.
\]
Taking the limit for \(i \to \infty\) and using (3) (5) we get
\[
\int_{B_r(0) \cap \partial H} D_k w \zeta d\mathcal{H}^k = -\int_{B_r(0) \cap \partial H} w D_k \zeta d\mathcal{H}^k
\]
and so \(w = 0\) on \(B_r(0) \cap \partial H\).

Hence we have a contradiction. \(\square\)

**Theorem 3.12** (Tilt excess decay theorem). There exist \(\varepsilon, \delta > 0\) and \(\theta \in (0,1)\) such that if
\[
R^{-k} \omega_k^{-1} \mu_V(B_R(0)) \leq \frac{1+\delta}{2}
\]
and
\[
E_* (0, R, T) \leq \varepsilon, \text{ where } T = \mathbb{R}^k \times \{0\}^{n-k}
\]
then
\[
E_* (0, \theta R, S) \leq \max \{\theta^{2\alpha}, \theta^{2(1-k/p)}\} E_* (0, R, T)
\]
for some \(k\)-dimensional subspace \(S \subset \mathbb{R}^n\), such that \(T' = \mathbb{R}^{k-1} \times \{0\}^{n-k+1} \subset S\).

**Proof.** By Lemmas 5.4 5.9 for any \(\gamma \in (0,1/4)\) and \(\eta \in (0,1)\) we can take \(\delta, \varepsilon\) small enough so that
\[
(1) \quad \Theta_V (x) \leq 1 + \eta \text{ for all } x \in B_{\gamma R}(0) \cap \text{spt } V
\]
and there exists a Lipschitz function \(f = (f^1, f^2, \ldots, f^{n-k}) : \tilde{H} \cap B_{\gamma R}^k(0) \to \mathbb{R}^{n-k}\)
such that \(\text{Lip } f \leq 1\) and
\[
(2) \quad \mathcal{H}^k \left( ((\text{graph } f \setminus \text{spt } V) \cup (\text{spt } V \setminus \text{graph } f)) \cap B_{\gamma R}(0) \right) \leq c E_* R^k
\]
where \( c = c(n,k) \), \( E^* = E^*(0,R,T) \) and recall that \( \bar{H} \) is one of the two regions of \( T \) defined by \( \text{proj}_T B \).

The first variation formula, applied for the vector field \( X = \zeta e_{k+j} \), where \( 1 \leq j \leq n-k \) and \( \zeta \in C_c^\infty(B\gamma R(0) \setminus B) \) gives:

\[
\int_{B_n(0)} \zeta e_{k+j} \cdot H d\mu_V = \int_{B_R(0)} \nabla x^{k+j} \cdot \nabla \zeta d\mu_V.
\]

Let \( \bar{f}^j(x) = f^j(x^1, \ldots, x^n) = f^j(x^1, \ldots, x^k) \) for \( j = 1, \ldots, n-k \) and let \( M_1 = M \cap \text{graph } f, \) where as previously we use the notation \( M = \text{spt } V \). Then on \( M_1 \) we have that \( x^{k+j} = \bar{f}^j \), so

\[
\int_{M_1} \nabla \bar{f}^j \cdot \nabla \zeta d\mu_V = - \int_{M \setminus M_1} \nabla x^{k+j} \cdot \nabla \zeta d\mu_V + \int_{M} \zeta e_{k+j} \cdot H d\mu_V
\]

Since \( E^* \leq E_\ast \), by \([2]\) we have that

\[
\mathcal{H}^k(M \setminus M_1 \cap B\gamma R(0)) \leq c R^k E_\ast.
\]

Using the assumption on \( \mu_V |B_R(0)\) we get

\[
\int_M \zeta e_{k+j} \cdot H d\mu_V \leq \|H\|_{L^p(B_R(0))} (\mu_V(B\gamma R(0)))^{1-\frac{j}{p}} \sup |\zeta| \leq c E^\frac{j}{k} \varepsilon \sup |D\zeta|
\]

where \( c = c(k,p) \). Hence

\[
(3) \quad \int_{M_1} \nabla \bar{f}^j \cdot \nabla \zeta d\mu_V \leq c R^k (E_\ast + E^\frac{j}{k}) \varepsilon \sup |D\zeta|.
\]

Furthermore, since

\[
\frac{1}{2} |\text{proj}_{T^c} M - \text{proj}_T|^2 = \sum_{j=1}^{n-k} |\nabla x^{k+j}|^2
\]

(cf. \([4]\) in proof of Lemma \(3.6\)) we have that

\[
(4) \quad \int_{M_1} |\nabla \bar{f}^j|^2 d\mu_V = \int_{M_1} |\nabla x^{k+j}|^2 d\mu_V \leq R^k E \leq R^k E_\ast.
\]

Now let \( \zeta_1 \in C^1_c(B_{\gamma R/2}(0) \cap \bar{H}) \). Then there exists a function \( \bar{\zeta}_1 \in C^1_c(B\gamma R(0) \setminus B) \) such that \( \bar{\zeta}_1(x^1, x^2, \ldots, x^n) = \zeta_1(x^1, x^2, \ldots, x^k) \) in a neighborhood of \((B_{\gamma R/2}(0) \times \mathbb{R}^{n-k}) \cap \text{spt } V \cap B\gamma R \). This is true because for \( \varepsilon \) small enough we have that \( \partial B\gamma R \cap \text{spt } V \cap (B_{\gamma R/2} \times \mathbb{R}^{n-k}) = \emptyset \)

(cf. Remark \(5.10\)).

Note that on \( M \cap B\gamma R(0) \)

\[
D\bar{f}^j = \text{proj}_T(D\bar{f}^j) \text{ and } D\bar{\zeta}_1 = \text{proj}_T(D\bar{\zeta}_1)
\]

so for all \( x \in M_1 \cap B\gamma R(0) \) we have

\[
|\nabla \bar{f}^j \cdot \nabla \bar{\zeta}_1 - D\bar{f}^j \cdot D\bar{\zeta}_1| \leq |\text{proj}_{T^c} M - \text{proj}_T|^2 |D\bar{f}^j||D\bar{\zeta}_1|
\]

and

\[
\left| \left( \nabla \bar{f}^j \right)^2 - \left( D\bar{f}^j \right)^2 \right| \leq |\text{proj}_{T^c} M - \text{proj}_T|^2 |D\bar{f}^j|^2.
\]
Using these two estimates as well as (3) with \( \zeta \) replaced by \( \tilde{\zeta}_1 \) and (4) we get that
\[
R^{-k} \int_{M_1} D\tilde{f}^j : D\tilde{\zeta}_1 d\mu_V \leq c(E^*_\varepsilon^{1/2} + E_*) \sup |D\tilde{\zeta}_1|
\]
and
\[
R^{-k} \int_{M_1} |D\tilde{f}^j|^2 d\mu_V \leq c E_*
\]
where we have used the fact that \( |D\tilde{f}^j| < 1 \) on \( M_1 \).

Define \( F : \mathbb{R}^k \to \mathbb{R}^{n+k} \) to be \( F(x) = (x, f(x)) \), \( \forall x \in B_{\gamma R}(0) \). Using the area formula for \( F \), along with the above estimates (5), (6) and also the density and the measure estimate (1) and (2) we get that
\[
R^{-k} \int_{B_{\gamma R}(0) \cap \tilde{H}} Df^j \cdot D\tilde{\zeta}_1(\theta \circ F) J(F) d\mathcal{H}^k
\leq R^{-k} \int_{M_1} D\tilde{f}^j : D\tilde{\zeta}_1 d\mu_V \\
+ R^{-k} \int_{\text{graph } \tilde{f} \setminus M_1} Df^j \cdot D\tilde{\zeta}_1 \theta d\mathcal{H}^k
\leq c(E^*_\varepsilon^{1/2} + E_*) \sup |D\tilde{\zeta}_1|
\]
and
\[
R^{-k} \int_{B_{\gamma R/2}(0) \cap \tilde{H}} |Df^j|^2 (\theta \circ F) J(F) d\mathcal{H}^k \leq c E_*
\]
where \( J(F) \) denotes the Jacobian of \( F \).

Since for \( J(F) \) we have that \( 1 \leq J(F) \leq \sqrt{1 + c|Df^j|^2} \) and for the density \( 1 \leq \theta \leq 1 + \eta \) for \( \mu_V \)-almost every \( x \in M \), we get that
\[
R^{-k} \int_{B \cap B_{\gamma R/2}} |Df^j|^2 d\mathcal{H}^k \leq R^{-k} \int_{B_{\gamma R/2}} |Df^j|^2 (\theta \circ F) J(F) d\mathcal{H}^k \leq c E_*
\]
and
\[
R^{-k} \int_{B \cap B_{\gamma R/2}(0)} Df^j \cdot D\tilde{\zeta}_1 d\mathcal{H}^k \leq c(E^*_\varepsilon^{1/2} + E_*) \sup |D\tilde{\zeta}_1| \\
+ c \sup |D\tilde{\zeta}_1| R^{-k} \int_{B_{\gamma R/2}(0)} \eta |Df^j| + |Df^j|^2 d\mathcal{H}^k
\leq c((\eta + \varepsilon^{1/2}) E^*_\varepsilon^{1/2} + E_*) \sup |D\tilde{\zeta}_1|
\]

Note also that by (1) we have that
\[
|\text{proj}_{N_x \partial \tilde{H}}(x - y)| \leq \varepsilon^{1/2} E^*_\varepsilon^{1/2} |x - y|, \forall x, y \in \partial \tilde{H} \cap B_{\gamma R/2}(0)
\]
and
\[
R^{-1} \sup_{\partial H \cap B_{\gamma R/2}(0)} |f^j| \leq \varepsilon^{1/2} E^*_\varepsilon^{1/2}.
\]

Because of the above estimates (7), (10), (9) and (8) we can apply the harmonic approximation lemma (Lemma 3.11) to the functions \( g^j = E_*^{-1/2} f^j \). Hence for any \( \sigma > 0 \) we can choose \( \varepsilon, \eta > 0 \) small enough so that there exist functions \( w^j \in W^{1,2}(B_{\gamma R/2}(0)) \) that are harmonic on \( H \cap B_{\gamma R/2}(0) \), where \( H \) is one of the halfspaces
on $T$ defined by $\{ x \in \mathbb{R}^k : x_k = 0 \}$ (recall that $T' = T_0B = \mathbb{R}^{k-1} \times \{0\}^{n-k+1}$) and such that

$$w^j = 0 \text{ on } \partial H \cap B^k_{\gamma R/2}(0)$$

(11)

$$R^{-k} \int_{H \cap B^k_{\gamma R/2}(0)} |Dw^j|^2 d\mathcal{H}^k \leq cE_*$$

and

(12)

$$R^{-k-2} \int_{\tilde{H} \cap B^k_{\gamma R/2}(0)} |w^j - f^j|^2 d\mathcal{H}^k \leq \sigma E_*.$$

Without loss of generality we can assume that $H = \{ x \in T : x_k \geq 0 \}$. For each $j = 1, 2, \ldots, n-k$ we define

$$u^j(x) = u^j(x^1, \ldots, x^k) = \begin{cases} w^j(x^1, \ldots, x^k), & \text{for } x^k \geq 0, x \in B^k_{\gamma R/2}(0) \\ -w^j(x^1, \ldots, -x^k), & \text{for } x^k < 0, x \in B^k_{\gamma R/2}(0). \end{cases}$$

Then $u^j$ are odd harmonic functions on $B^k_{\gamma R/2}(0)$ that agree with $w^j$ on $H \cap B^k_{\gamma R/2}(0)$. So for $\lambda < \gamma/2$ and any multiindex $\alpha$:

$$\sup_{B^k_{\gamma R/2}(0)} |D^\alpha u^j| \leq \left( \frac{k|\alpha|}{(\gamma/2 - \lambda)R} \right)^{|\alpha|-1} \wedge^{-1} R^{-k/2} \| Du^j \|_{L^2(B^k_{\gamma R/2}(0))}$$

(13)

$$\leq \left( \frac{k|\alpha|}{(\gamma/2 - \lambda)R} \right)^{|\alpha|-1} \sup_{B^k_{\gamma R/2}(0)} |Du^j|$$

We define the linear function $l^j(x) = u^j(0) + x \cdot Du^j(0) = x \cdot Dw^j(0)$ and using the Taylor expansion of $u^j$ we get that

$$\sup_{B^k_{\gamma R/2}(0)} |u^j - l^j| \leq c\lambda^2 R^{-k/2+2} \| Du^j \|_{L^2(B^k_{\gamma R/2}(0))}.$$  

(14)

Using the estimates (14), (12) and (9) we get:

$$\int_{\tilde{H} \cap B^k_{\gamma R/2}(0)} |f^j - l^j|^2 d\mathcal{H}^k \leq \int_{H \cap B^k_{\gamma R/2}(0)} |f^j - w^j|^2 d\mathcal{H}^k$$

(15)

$$+ \int_{\tilde{H} \cap B^k_{\gamma R/2}(0)} |u^j - l^j|^2 d\mathcal{H}^k$$

$$+ \int_{(\tilde{H} \setminus H) \cap B^k_{\gamma R/2}(0)} |f^j - w^j|^2 d\mathcal{H}^k$$

$$\leq c\lambda^{-k-2} \sigma E_+ + \lambda^2 E_* + \varepsilon E_*.$$  

Let $l = (l^1, l^2, \ldots, l^{n-k}) : \mathbb{R}^k \to \mathbb{R}^{n-k}$. Note that $l(0) = 0$ and since all $l^j$ are linear we have that $S = \text{graph } l$ is a $k$ dimensional subspace of $\mathbb{R}^n$. Also since $u^j$ is an odd function we have that $u^j(x^1, \ldots, x^{k-1}, 0) = 0, \forall j \in \{1, 2, \ldots, n-k\}$ and so

$$l^j(x^1, \ldots, x^{k-1}, 0) = (x^1, \ldots, x^{k-1}, 0) \cdot Du^j(x^1, \ldots, x^{k-1}, 0) = 0.$$
hence \( T' = \mathbb{R}^{k-1} \times \{0\}^{n-k+1} \subset S \). Furthermore:

\[
(\lambda R)^{-k-2} \int_{B_{\lambda R}(0)} \text{dist}(x, S)^2 \, d\mu_V \leq (\lambda R)^{-k-2} \int_{B_{\lambda R}(0) \cap M_1} \text{dist}(x, S)^2 \, d\mu_V \\
+ (\lambda R)^{-k-2} \int_{B_{\lambda R}(0) \cap (M \setminus \text{graph } f)} \text{dist}(x, S)^2 \, d\mu_V \\
\leq c(\lambda R)^{-k-2} \int_{B_{\lambda R}(0) \cap H} \sum_{j=1}^{n-k} |f_j - \bar{v}_j|^2 \, d\mathcal{H}^k \\
+ (\lambda R)^{-k-2} \int_{B_{\lambda R}(0) \cap (M \setminus \text{graph } f)} \text{dist}(x, S)^2 \, d\mu_V.
\]

By Remark \( \textbf{3.10} \) and since \( R^{-k} \mathcal{H}^k((M \setminus \text{graph } f) \cap B_{\lambda R}(0)) \leq cE_* \), we have that

\[
(\lambda R)^{-k-2} \int_{B_{\lambda R}(0) \cap (M \setminus \text{graph } f)} \text{dist}(x, S)^2 \, d\mu_V \leq c\lambda^{k-2} \varepsilon^{1/k} E_*.
\]

Hence

\[
(\lambda R)^{-k-2} \int_{B_{\lambda R}(0)} \text{dist}(x, S)^2 \, d\mu_V \leq c\varepsilon E_* \lambda^{-k-2} + c\lambda^2 E_*
\]

provided that \( \varepsilon^{1/k} \leq \lambda^{4+k} \).

By Lemma \( \textbf{3.6} \) we can estimate the tilt-excess using the above height integral, which gives us

\[
E(0, \lambda R/2, S) \leq c\sigma E_* \lambda^{-k-2} + c\lambda^2 E_* + \lambda^{2(1-k/p)} E_0 E_* + \lambda^{2\alpha} E_*.
\]

So pick \( \lambda \) so that \( c\lambda^2 \leq 1/4(\lambda/2)^{2(1-k/p)} \), take \( \sigma \) so that \( c\sigma \lambda^{-k-2} \leq 1/4(\lambda/2)^{2\alpha} \) and then pick \( \varepsilon \) small enough so that \( \lambda^{2(1-k/p)} \varepsilon \leq 1/4(\lambda/2)^{2(1-k/p)} \), \( \lambda^{2\alpha} \varepsilon \leq 1/4(\lambda/2)^{2\alpha} \), \( \varepsilon^{1/k} \leq \lambda^{4+k} \). Finally we need to take \( \varepsilon, \eta \) small enough so that we can apply the harmonic approximation lemma with the selected \( \sigma \) (as we did above when constructing the harmonic functions \( w_j \)). Then for \( \theta = \lambda/2 \)

\[
E(0, \theta R, S) \leq \left( \frac{\theta^{2(1-k/p)}}{2} + \frac{\theta^{2\alpha}}{2} \right) E_*(0, R, T).
\]

Finally since

\[
(\theta R)^{2(1-k/p)} \| H \|_{L^p(B_{R}(0))}^2 \leq \theta^{2(1-k/p)} R^{2(1-k/p)} \| H \|_{L^p(B_{R}(0))}^2
\]

and

\[
(\kappa \theta R)^{\alpha} = \theta^{2\alpha} (\kappa R^\alpha)^2
\]

we have

\[
E_*(0, \theta R, S) \leq \max \{ \theta^{2\alpha}, \theta^{2(1-k/p)} \} E_*(0, R, T).
\]

\[ \square \]

**Theorem 3.13** (Boundary regularity theorem). We assume that

\[
R^{-k} \omega_k^{-1} \mu_V(B_{R}(0)) \leq \frac{1 + \delta}{2}
\]

and

\[
E_*(0, R, T) \leq \varepsilon
\]

where \( T = \mathbb{R}^k \times \{0\}^{n-k} \).
Then for any $\gamma \in (0, 1/32)$ there exist $\varepsilon, \delta$ such that under the above hypotheses there exists a $C^{1,m}$ function

$$f : \tilde{H} \cap B^k_{\gamma R}(0) \to \mathbb{R}^{n-k}$$

where $m = \min\{\alpha, 1 - k/p\}$, $\tilde{H}$ is one of the two regions of $T$ defined by $\text{proj}_T(B)$ (as in the lipschitz approximation lemma) such that $f(0) = 0$,

$$\text{spt } V \cap B_{\gamma R}(0) = \text{graph } f \cap B_{\gamma R}(0)$$

and

$$(\gamma R)^n \frac{|Df(x) - Df(y)|}{|x - y|^m} \leq cE^*(0, R, T)$$

for all $x, y \in B^k_{\gamma R}(0) \cap \tilde{H}$.

**Proof:** Fix $\gamma \in (0, 1/32)$. We first show that the tilt excess decay theorem (Theorem 3.12) is applicable in the ball $B_{\gamma R}(b)$ for any $b \in B_{\gamma R}(0) \cap B$.

We note that

$$E(b, \gamma R, T) \leq \gamma^{-k}E(0, R, T) \Rightarrow E_*(b, \gamma R, T) \leq \gamma^{-k}E_*$$

for any $b \in B_{\gamma R}(0) \cap B$.

Let $T_b = T_b B + \{0\}^{k-1} \times \mathbb{R} \times \{0\}^{n-k}$. Then, by [1.1]

$$(\gamma R)^{-k} \omega_k^{-1} |\text{proj}_{T_b} - \text{proj}_T|^2 \leq c\varepsilon E_*$$

where $c$ depends only on $n, k$ and hence

$$E_*(b, \gamma R, T_b) \leq E_* (b, \gamma R, T) + \frac{1}{(\gamma R)^k} \int_{B_{\gamma R}(b)} |\text{proj}_{T_b} - \text{proj}_T|^2 d\mu V$$

$$\leq c\gamma^{-k}E_*(0, R, T).$$

**Claim:** For any $\delta_1 > 0$, there exist $\varepsilon, \delta$ small enough so that for any $b \in B_{\gamma R}(0) \cap B$:

$$|\Theta_V(x) - 1| \leq \varepsilon, \forall x \in B_{2\gamma R}(0) \cap \text{spt } V$$

and there exists a Lipschitz function

$$f : \tilde{H} \cap B^k_{2\gamma R}(0) \to \mathbb{R}^{n-k}$$

with $\text{Lip } f \leq l$ and such that

$$H^k \left( (\text{graph } f \setminus \text{spt } V) \cup (\text{spt } V \setminus \text{graph } f) \right) \cap B_{2\gamma R}(0) \leq cl^{-2}E^* R^k.$$  

By Remark 3.10 we have that

$$(B_{2\gamma R}(0) \cap \text{spt } V) \setminus T_{cE^*} \subset \text{graph } f$$

where recall that

$$T_\rho = \{x \in \mathbb{R}^n : \rho(x) \leq \rho\}.$$
Therefore, by the area formula for the function \( F(x) = (x, f(x)) \) and using the fact that \( J(F) \leq \sqrt{1 + d^2} \), \( \theta(x) \leq 1 + \eta \) as well as \( \mu(V) \) we have that for \( b \in B_{\gamma R}(0) \)

\[
\mu(V(B_{\gamma R}(b) \setminus T_{cE^{1/2k} R}) \leq (1 + \eta)\sqrt{1 + d^2} (1 + \varepsilon) \frac{\gamma^k R_k}{2} \omega_k.
\]

By the tubular monotonicity inequality \( \text{(17)} \) in the proof of Lemma 2.4 we have that

\[
\mu(V(B_{\gamma R}(b) \cap T_{cE^{1/2k} R}) \leq cE^{1/2k} (\gamma R)^k \omega_k
\]

Hence

\[
(\omega_k^{-1}(\gamma R)^{-k}) \mu(V(B_{\gamma R}(b)) \leq \frac{1}{2}(1 + \eta)\sqrt{1 + d^2}(1 + \varepsilon) + cE^{1/2k} \leq \frac{1 + \delta_1}{2}
\]

provided that \( l, \eta, \varepsilon \) are small enough. In particular we first pick \( \varepsilon \) small enough so that \( c\varepsilon^{1/2k} \leq \delta_1/4 \), then pick \( l \) so that for small enough \( \varepsilon, \eta \\
\frac{1}{2}(1 + \eta)\sqrt{1 + d^2}(1 + \varepsilon) \leq 1/2 + \delta_1/4
\]

where \( c \) in both inequalities is as in \( \text{(6)} \). Hence the claim is true.

Because of \( \text{(2)} \) \( \text{(3)} \) we can apply the the tilt excess decay theorem (Theorem 3.12) in \( B_{\gamma R}(b) \) for any \( b \in B_{\gamma R}(0) \) and with \( T \) replaced by \( T_b \). Hence using the notation

\[
m = \min \left\{ \alpha, \left(1 - \frac{k}{p}\right) \right\}
\]

we get that for \( \varepsilon, \delta \) small enough there exists \( \theta \in (0, 1) \) such that

\[
E_*(b, \theta^{-1} R, S_1) \leq \theta^{2m} E_*(b, \gamma R, T_b) \leq c\theta^{2m} E_*(0, R, T)
\]

for some \( k \)-dimensional subspace \( S_1 \), depending on \( b \), with \( T_b B \subset S_1 \) (note that the last inequality follows by \( \text{(2)} \) ).

Because of \( \text{(7)} \) \( \text{(9)} \) we can apply again the the tilt excess decay theorem in \( B_{\theta^{-1} R}(b) \) and by induction we get \( k \)-dimensional subspaces \( S_1, S_2, \ldots \) all of which contain \( T_b B \) and such that

\[
E_*(b, \theta^{j^{-1}} R, S_j) \leq \theta^{2m} E_*(b, \theta^{j^{-1}} R, S_{j-1}) \\
\]

\[
\leq \theta^{2jm} E_*(b, \gamma R, T^b) \leq c\theta^{2jm} E_*(0, R, T).
\]

Using the monotonicity at the boundary (cf. \( \text{(1)} \) in Lemma 2.4) we have that

\[
|\text{proj}_{S_j} - \text{proj}_{S_{j-1}}|^2 \leq c \left(E(b, \theta^{j} R, S_j) + E(b, \theta^{j^{-1}} R, S_{j-1})\right) \\
\leq c\theta^{2(j-1)m} E_*(0, R, T).
\]

Hence for \( i > j \)

\[
|\text{proj}_{S_i} - \text{proj}_{S_j}|^2 \leq c\theta^{2jm} E_*(0, R, T)
\]

and thus the subspaces \( S_j \) converge to a \( k \)-dimensional subspace \( S(b) \), for which \( T_b B \subset S(b) \) and

\[
|\text{proj}_{S(b)} - \text{proj}_{S_j}|^2 \leq c\theta^{2jm} E_*(0, R, T) \quad \forall j \geq 0.
\]

For \( j = 0 \), the above inequality yields

\[
|\text{proj}_{S(b)} - \text{proj}_{T_b}|^2 \leq cE_*(0, R, T).
\]
For any $\sigma \leq \gamma R$ we can pick $j \geq 0$ such that $\theta^{j+1}\gamma R < \sigma \leq \theta^j \gamma R$. Then by (10) and the monotonicity formula at the boundary, (2) in Lemma 2.9 we have
\begin{equation}
E(b, \sigma, S(b)) \leq C_0(\gamma R)^m E_*(0, R, T) \leq C_0(\gamma R)^{2m} E_*(0, R, T)
\end{equation}
and by (11) we furthermore have
\begin{equation}
E_*(b, \sigma, T) \leq C_r \varepsilon.
\end{equation}
We claim that $\text{spt } V \cap B_{\gamma R/2}(0) \subset \text{graph } f$, where $f$ is the Lipschitz function defined in (4).

Because of (5) we only need to show that
$\text{spt } V \cap T_{x, \varepsilon} \subset \text{graph } f$.

Let $\rho = C_{E1/2k} R$ and let $\varepsilon$ be small enough so that $4\rho < \gamma R$. Let $x \in \text{spt } V \cap B_{\gamma R/2} \cap \text{T}_\rho$ and let $\bar{x} \in B \cap B_{\gamma R}(0)$ be such that $|x - \bar{x}| = \rho(x)$. Then by (12)
\begin{equation}
E(x, \rho(x)/2, S(\bar{x})) \leq C_{E}(\bar{x}, 2\rho(x), S(\bar{x})) \leq C_r \varepsilon.
\end{equation}
Using this, the interior monotonicity [Sim83, 17.6] and Lemma 3.4 (ii) we have that
\begin{equation}
|\text{proj}_{\chi(z)} - \text{proj}_{S(\bar{x})}|^2 \leq (1 - c\varepsilon)^{-1} \left(E(x, \rho(x)/2, S(\bar{x})) + E(x, \rho(x)/2, \chi(x))\right) \leq C(\varepsilon + \eta^2).
\end{equation}
Hence by the above estimate, (11) and (1) we get
\begin{equation}
\|\text{proj}_{\chi(z)} - \text{proj}_{S(\bar{x})}\|^2 \leq C(\eta^2 + \varepsilon).
\end{equation}
Picking $\varepsilon$ small enough so that Lemma 3.4 holds with $\eta$ such that $(C(\eta^2 + \varepsilon))^{1/2} < 2\eta(l) = l/20$ (with $c$ as in (14)) we have that
$\text{spt } V \cap B_{\gamma R/2}(0) \cap \text{T}_\rho \subset G \Rightarrow \text{spt } V \cap B_{\gamma R/2}(0) \cap \text{T}_\rho \subset \text{graph } f$.
where $G$ is the set constructed in (3) of the proof of Lemma 3.9.

We also claim that the inverse is true, i.e.
$\text{graph } f \cap B_{\gamma R/2}(0) \subset \text{spt } V$.

Assume it is not true. Then we could find a point $(z, f(z)) \in (\text{graph } f \cap B_{\gamma R/2}(0)) \setminus \text{spt } V$ and $\sigma > 0$ such that
\begin{equation}
(B_\sigma^k(z) \times \mathbb{R}^{n-k}) \cap B_{\gamma R/2}(0) \cap \text{spt } V = \emptyset
\end{equation}
and
\begin{equation}
(\theta B_\sigma^k(z) \times \mathbb{R}^{n-k}) \cap B_{\gamma R/2}(0) \cap \text{spt } V \neq \emptyset.
\end{equation}

Let $Y = (y, f(y)) \in (\overline{B_\sigma^k(z) \times \mathbb{R}^{n-k}}) \cap B_{\gamma R/2}(0) \cap \text{spt } V$, then
\begin{equation}
\omega_k \Theta_V(Y) = \lim_{r \to 0} r^{-k} \int_{B_r(Y)} d\mu_V = \lim_{r \to 0} r^{-k} \int_{B_r(Y)} \theta(x) d\mathcal{H}^k(x)
\end{equation}
and because $\text{spt } V \cap B_{\gamma R/2}(0) \subset \text{graph } f$ we can estimate $\Theta_V(Y)$ by using the area formula for the function $F(x) = (x, f(x))$:
\begin{equation}
\int_{B_r(Y)} \theta(x) d\mathcal{H}^k(x) = \int_{\overline{B_r^k(y)} \cap \overline{H}} \theta \circ F(x) JF d\mathcal{H}^k(x).
\end{equation}
Since $\theta \circ F(x) = 0$ for $x \in B_r^k(y) \cap B_r^k(z)$ we have that
\begin{equation}
\Theta_V(Y) < 1
\end{equation}
and in the special case when \((y, f(y)) \in \text{spt} \ V \cap B\), we similarly get
\[
\Theta_V(Y) < 1/2
\]
which in both cases is a contradiction (cf. Corollary \ref{cor:contradiction}).

Hence
\[
\text{(15) \hspace{2cm} graph } f \cap B_{\gamma R/2}(0) = \text{spt} \ V \cap B_{\gamma R/2}(0).
\]
For the subspace \(S(b)\) as constructed above, let
\[
\ell_b : (\ell^1_b, \ldots, \ell^k_b) : \mathbb{R}^k \to \mathbb{R}^{n-k}
\]
be the linear function such that \(\text{graph } \ell_b = S(b)\).

For any \(b \in B \cap B_{\gamma R/2}(0) \cap \text{spt} \ V\) we have that \(b = (b', f(b'))\) for some \(b' \in \partial H \cap B_{\gamma R/2}(0)\). Let \(0 < \sigma \leq \gamma R/2\) be such that \(B_\sigma(b) \subset B_{\gamma R/2}(0)\). Using the area formula along with the fact that \(1 \leq JF \leq \sqrt{1 + c|Df|^2}, \theta(x) \geq 1\) for \(\mu_V\)-almost every \(x \in \text{spt} \ V\) as well as \(\ref{lem:area-formula}\) we get that
\[
(\sigma/2)^{-k} \int_{\tilde{H} \cap B_{\gamma R/2}(b')} \sum_{i=1}^{n-k} |Df^i - D\ell^i_b|^2 d\mathcal{H}^k \leq cE(b, \sigma, S(b)) \leq c \left( \frac{\sigma}{\gamma R} \right)^{2m} E_*(0, R, T).
\]

Since \(\Theta_V(b) \geq 1/2\) (Corollary \ref{cor:contradiction}), letting \(\sigma \to 0\), we have that
\[
\text{(17) \hspace{2cm} } Df^i(\text{proj}_T(b)) = D\ell^i_b
\]
for all \(i = 1, \ldots, n-k\).

Let \(x \in (\text{spt} \ V \setminus B) \cap B_{\gamma R/2}(0)\) and let \(\bar{x} \in \text{spt} \ V \cap B \cap B_{\gamma R}\) be such that \(|x - \bar{x}| = \rho(x)\). Then, using \(\ref{lem:area-formula}\)
\[
E(x, \rho(x), S(\bar{x})) \leq 2^k E(\bar{x}, 2\rho(x), S(\bar{x})) \Rightarrow
\]
\[
E_*(x, \rho(x), S(\bar{x})) \leq c \left( \frac{\rho(x)}{\gamma R} \right)^{2m} E_*(0, R, T).
\]

Furthermore, since \(\text{spt} \ V \cap B_\sigma(x) \subset \text{graph } f\) for \(\sigma \leq \rho(x)\), by the area formula and Lemma \ref{lem:area-formula} we can estimate:
\[
\omega_1^{-1} \sigma^{-k} \mu_V(B_\sigma(x)) \leq (1 + \eta)\sqrt{1 + c\ell^2}, \forall \sigma \leq \rho(x).
\]

Hence we can apply the tilt excess decay lemma for the interior \cite[22.5]{Sim83} in the ball \(B_{\rho(x)}(x)\) and arguing as in \(\ref{lem:decay-lemma}\) \cite{Sim83} we conclude that there exists a \(k\)-dimensional subspace \(S(x)\) such that for all \(\sigma < \rho(x)\)
\[
\text{(18) \hspace{2cm} } E_*(x, \rho(x), S(\bar{x})) \Rightarrow
\]
\[
E_*(x, \rho(x), S(x)) \leq c \left( \frac{\sigma}{\rho(x)} \right)^{2m} E_*(0, R, T).
\]

Letting as before
\[
\ell_x : (\ell^1_x, \ldots, \ell^{n-k}_x) : \mathbb{R}^k \to \mathbb{R}^{n-k}
\]
to be the linear function such that graph $\ell_x = S(x)$, we can argue as in the case of a boundary point (cf. [16]) to conclude that

$$
(\sigma/2)^{-k} \int_{\tilde{H} \cap B^k_{n/2}(x)} |Df_i - D\ell_x|^2 d\mathcal{H}^k \leq c E(x, \sigma, S(x))
$$

(19)

and letting $\sigma \to 0$ we have

$$
Df_i(\text{proj}_T(x)) = D\ell_x
$$

(20)

for $x \in (\text{spt} \ V \setminus B) \cap B_{\gamma R/2}(0)$ and all $i = 1, \ldots, n - k$.

Now let $x, y \in \tilde{H} \cap B^k_{\gamma R/4}(0)$ and let $X, Y \in \text{spt} \ V \cap B_{\gamma R/2}(0)$ be such that $X = (x, f(x))$ and $Y = (y, f(y))$. We will show that for all $j = 1, 2, \ldots, n - k$

$$
|Df^i(x) - Df^i(y)| \leq c \left( \frac{|x - y|}{\gamma R} \right)^m E^*(0, R, T)^{1/2}.
$$

(21)

Let $r = |x - y|$ and $r' = |X - Y|$. Note that $r, r' \leq \gamma R$ and

$$
r^2 \leq r'^2 \leq (1 + l^2)r^2 \leq 2r^2.
$$

Case 1: $X, Y \in B \Rightarrow x, y \in \partial \tilde{H}$.

Using (17) we have

$$
\sum_{i=1}^{n-k} |Df_i(x) - Df_i(y)|^2 \leq c r^{-k} \int_{\tilde{H} \cap B^k_{n/2}(x)} |D\ell_x^i - Df^i(z)|^2 d\mathcal{H}^k(z)
$$

$$
+ c(2r)^{-k} \int_{\tilde{H} \cap B^k_{n/2}(y)} |D\ell_y^i - Df^i(z)|^2 d\mathcal{H}^k(z)
$$

and by (16)

$$
|Df^i(x) - Df^i(y)| \leq c \left( \frac{|x - y|}{\gamma R} \right)^m E^*(0, R, T)^{1/2}
$$

(22)

for all $i = 1, 2, \ldots, n - k$.

Case 2: $X \in B, Y \notin B$.

For $Y = (y, f(y))$, let $Y_0 = (y_0, f(y_0)) \in B$ be such that $\rho(Y) = |Y - Y_0|$ and $y_0$ such that $Y_0 = (y_0, f(y_0))$. Using (17) and (20) we have

$$
\sum_{i=1}^{n-k} |Df_i(y_0) - Df_i(y)|^2 \leq c |y - y_0|^{-k} \int_{\tilde{H} \cap B^k_{|y-y_0|}(y_0)} |D\ell_y^i - Df^i(z)|^2 d\mathcal{H}^k(z)
$$

$$
+ c(2|y - y_0|)^{-k} \int_{\tilde{H} \cap B^k_{2|y-y_0|}(y_0)} |Df^i(z) - D\ell_{y_0}^i|^2 d\mathcal{H}^k(z)
$$

hence by (16), (19)

$$
|Df^i(y_0) - Df^i(y)| \leq c \left( \frac{|y - y_0|}{\gamma R} \right)^m E^*(0, R, T)^{1/2}.
$$
Using (20) we have

\[ |Df^i(y_0) - Df^i(x)| \leq c \left( \frac{|x - y_0|}{\gamma R} \right)^m E_*(0, R, T)^\frac{1}{2}. \]

Hence, since \(|x - y_0| \leq |x - y| + |y - y_0|\) and \(|y - y_0| \leq |y - x|\) we have that

\[ |Df^i(x) - Df^i(y)| \leq c \left( \frac{|x - y|}{\gamma R} \right)^m E_*(0, R, T)^\frac{1}{2}. \]

for all \(i = 1, 2, \ldots, n - k\).

Case 3: \(X, Y \notin B\).

Let \(X_0 = (x_0, f(x_0)), Y_0 = (y_0, f(y_0)) \in B\) be such that \(|X - X_0| = \rho(X)\) and \(|Y - Y_0| = \rho(Y)\).

Case 3 (i): \(\min\{|x - x_0|, |y - y_0|\} > 2|x - y|\).

Using (20) we have

\[
\sum_{i=1}^{n-k} |Df^i(x) - Df^i(y)|^2 \leq c r^{-k} \int_{B_{r}(x)} \sum_{i=1}^{n-k} |Df^i_x - Df^i(z)|^2 d\mathcal{H}^k(z) \\
+ c(2r)^{-k} \int_{B_{2r}(y)} \sum_{i=1}^{n-k} |Df^i_y - Df^i(z)|^2 d\mathcal{H}^k(z)
\]

and by (19)

\[ |Df^i(x) - Df^i(y)| \leq c \left( \frac{|x - y|}{\gamma R} \right)^m E_*(0, R, T)^\frac{1}{2}. \]

Case 3 (ii): \(\min\{|x - x_0|, |y - y_0|\} \leq 2|x - y|\).

Then by Cases 1 and 2 we have that

\[
|Df^i(x_0) - Df^i(y_0)| \leq c \left( \frac{|x_0 - y_0|}{\gamma R} \right)^m E_*(0, R, T)^\frac{1}{2} \\
|Df^i(x) - Df^i(x_0)| \leq c \left( \frac{|x - x_0|}{\gamma R} \right)^m E_*(0, R, T)^\frac{1}{2} \\
|Df^i(y) - Df^i(y_0)| \leq c \left( \frac{|y - y_0|}{\gamma R} \right)^m E_*(0, R, T)^\frac{1}{2}.
\]

Note that \(|x_0 - y_0| \leq c|x - y|\) and without loss of generality we can assume that \(|x - x_0| \leq 2|x - y|\). Then we also have \(|y - y_0| \leq |x - y| + |x - x_0| + |x_0 - y_0| \leq c|x - y|\). Hence

\[ |Df^i(x) - Df^i(y)| \leq c \left( \frac{|x - y|}{\gamma R} \right)^m E_*(0, R, T)^\frac{1}{2} \]

for all \(i = 1, 2, \ldots, n - k\).

Hence \(f\) is a \(C^{1, m}\) function with \(f(0) = 0\) and so the theorem is true. \(\square\)

The regularity theorem still holds without the assumption \(E_*(0, R, T) \leq \varepsilon\). In particular we have the following.

**Theorem 3.14.** We assume that

\[
a = R^{1-k/p} \|H\|_{L_p(B_R(0))} \leq \delta , \; \kappa R^m \leq \delta
\]

and

\[ R^{-k} \mu_V(B_R(0)) \leq \frac{1 + \delta}{2}. \]
Then for any $\gamma \in (0,1/32)$ there exists $\delta$ such that under the above hypotheses there exists a linear isometry $q$ and a $C^{1,m}$ function $f : \tilde{H} \cap B_{\gamma R}(0) \to \mathbb{R}^{n-k}$ with $m = \min\{\alpha, 1 - k/p\}$ and $\tilde{H}$ a $C^{1,\alpha}$ domain such that $0 \in \partial \tilde{H}$, such that $f(0) = 0$,

$spt V \cap B_{\gamma R}(0) = q(\text{graph } f) \cap B_{\gamma R}(0)$

and

$$\left(\gamma R\right)^m \frac{|Df(x) - Df(y)|}{|x - y|^m} \leq c\eta^{\frac{1}{2}} \text{ for all } x, y \in B_{\gamma R}(0)$$

where $\eta = \eta(\delta)$ is as in Theorem 3.3.

Proof. We will prove that given $\gamma \in (0,1)$ we can take $\delta$ small enough so that for a $k$-dimensional subspace $T$ such that $T_q B = T' \subset T$, $E(0, \gamma R, T) \leq \eta^{1/2k}$. Then the theorem will be an immediate consequence of Theorem 3.13 with $\varepsilon = \eta^{1/2k}$.

Given $\gamma \in (0,1)$, take $\delta$ small enough so that the interior regularity theorem, Theorem 3.4, holds in $B_{\gamma R}(0)$ for some $\eta \in (0,1)$.

For $\sigma \leq 1 - \gamma$, let $\mathcal{O} = \{B_{\sigma R/5}(x)\}$ be a collection of disjoint balls of radii $\sigma R/5$ and with centers in $spt V \cap B_{\gamma R}(0) \setminus T_{\sigma R}$. For any ball in $\mathcal{O}$ we have that $B_{\sigma R}(x) \subset B_{R}(0) \setminus B$ so by the interior monotonicity inequality with $\varepsilon = \eta^{1/2k}$ we get that

$$\omega_k^{-1}(\sigma R/5)^{-k} \mu_V(B_{\sigma R/5}(x)) \geq (1 + c\delta)^{-1}.$$ 

Hence for $N$ distinct balls in $\mathcal{O}$, we have that

$$c N \sigma^k \leq \frac{1 + \delta}{2} \Rightarrow N \leq c \sigma^{-k}.$$ 

Taking $\mathcal{O}$ to be maximal we also have that

$$(spt V \cap B_{\gamma R}(0)) \setminus T_{\sigma R} \subset \bigcup_{B_{\sigma R/5}(x) \in \mathcal{O}} B_{\sigma R}(x).$$

For any $x \in spt V \setminus T_{\sigma R}$, by Theorem 3.4 we have that

$$\|\text{proj}_{T_{\sigma R}^\perp} - \text{proj}_{T_{\sigma R}}\| \leq c\eta \quad \forall y, z \in B_{\sigma R}(x).$$

Let now $y, z \in B_{\gamma R}(0) \setminus T_{\sigma R}$. Joining $y, z$ with balls of the form $B_{\sigma R}(x)$ such that $B_{\sigma R/5}(x) \in \mathcal{O}$, we get that

$$\|\text{proj}_{T_{\sigma R}^\perp} - \text{proj}_{T_{\sigma R}}\| \leq c N \eta \leq c \sigma^{-k} \eta \quad \forall y, z \in B_{\gamma R}(0) \setminus T_{\sigma R}.$$ 

By the tubular monotonicity inequality, [17] in the proof of Lemma 2.1 we get

$$\mu_V(B_{\gamma R}(0) \cap T_{\sigma R}) \leq c \sigma R^k.$$ 

Hence picking $T = \chi(x)$, for any $x \in B_{\gamma R}(0) \setminus T_{\sigma R}$ we have that

$$E(0, \gamma R, T) \leq c \left(\sigma^{-2k} \eta^2 + \sigma\right)$$

so picking $\sigma = \eta^{1/2k}$ we have that

$$E(0, \gamma R, T) \leq c \eta^{1/2k}.$$ 

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