Perturbation Bounds for Orthogonally Decomposable Tensors and Their Applications in High Dimensional Data Analysis*

Arnab Audity and Ming Yuan†
Columbia University

(July 20, 2020)

Abstract

We develop deterministic perturbation bounds for singular values and vectors of orthogonally decomposable tensors, in a spirit similar to classical results for matrices. Our bounds exhibit intriguing differences between matrices and higher-order tensors. Most notably, they indicate that for higher-order tensors perturbation affects each singular value/vector in isolation. In particular, its effect on a singular vector does not depend on the multiplicity of its corresponding singular value or its distance from other singular values. Our results can be readily applied and provide a unified treatment to many different problems involving higher-order orthogonally decomposable tensors. In particular, we illustrate the implications of our bounds through three connected yet seemingly different high dimensional data analysis tasks: tensor SVD, tensor regression and estimation of latent variable models, leading to new insights in each of these settings.

*This research was supported by NSF Grant DMS-1803450. Part of the work was done while the second author was visiting the Institute for Theoretical Studies at ETH Zürich, Switzerland, and he wishes to thank the institute for their hospitality.

†Address for Correspondence: Department of Statistics, Columbia University, 1255 Amsterdam Avenue, New York, NY 10027.
1 Introduction

Singular value decomposition (SVD) is routinely performed to process data organized in the form of matrices, thanks to its optimality for low-rank approximation, and relationship with principal component analysis; and perturbation analysis of SVD plays a central role in studying the performance of these procedures. More and more often, however, multidimensional data in the form of higher-order tensors arise in applications. While higher-order tensors provide us a more versatile tool to encode complex relationships among variables, how to perform decompositions similar to SVD and how these decompositions behave under perturbation are often the most fundamental issues in these applications. In general, decomposition of higher-order tensors is rather delicate and poses both conceptual and computational challenges. See Kolda and Bader (2009); Cichocki et al. (2015) for recent surveys of some of the difficulties as well as existing techniques and algorithms to tackle them. In particular, we shall focus here on a class of tensors that allows for direct generalization of SVD. The so-called orthogonally decomposable (odeco) tensors have been previously studied by Kolda (2001); Chen and Saad (2009); Robeva (2016); Belkin et al. (2018) among others, and commonly used in high dimensional data analysis (see, e.g., Anandkumar et al., 2014a, b, c; Liu et al., 2017). The main goal of this work is to study the effect of the perturbation on the singular values and vectors of an odeco tensor or odeco approximations of a nearly odeco tensor, and demonstrate how it provides a powerful and unifying treatment to many different problems in high dimensional data analysis.

Let $\mathcal{T}$ and $\mathcal{\tilde{T}}$ be two $p$th ($p \geq 3$) order odeco tensor with singular value/vector tuples $\{(\lambda_k, u_k^{(1)}, \ldots, u_k^{(p)}) : 1 \leq k \leq d\}$ and $\{({\tilde{\lambda}}_k, {\tilde{u}}_k^{(1)}, \ldots, {\tilde{u}}_k^{(p)}) : 1 \leq k \leq d\}$, respectively. We are interested in how the difference between the two sets of singular values and vectors is characterized by the spectral norm of the difference $\mathcal{\tilde{T}} - \mathcal{T}$. The spectral norm of a tensor $\mathcal{A} \in \mathbb{R}^{d \times \cdots \times d}$ is defined by

$$\|\mathcal{A}\| = \max_{u^{(i)} \in \mathcal{S}^{d-1}} \langle \mathcal{A}, u^{(1)} \otimes \cdots \otimes u^{(p)} \rangle,$$

and $\mathcal{S}^{d-1}$ denotes the unit sphere in $\mathbb{R}^d$. More specifically, we show that there exist a numerical constant $C \geq 1$ and a permutation $\pi : [d] \to [d]$ such that for all $k = 1, \ldots, d$,

$$|\lambda_k - {\tilde{\lambda}}_{\pi(k)}| \leq C\|\mathcal{T} - \mathcal{\tilde{T}}\|,$$  

(1)
and
\[
\max_{1 \leq q \leq p} \sin \angle(\mathbf{u}_k^{(q)}, \tilde{\mathbf{u}}_{\pi(k)}^{(q)}) \leq C \cdot \frac{\|\tilde{\mathcal{T}} - \mathcal{T}\|}{\lambda_k},
\]
under the convention that 1/0 = +\infty. Here and in what follows \(\angle(\mathbf{u}, \tilde{\mathbf{u}})\) is the angle between two vectors \(\mathbf{u}\) and \(\tilde{\mathbf{u}}\) taking value in \([0, \pi/2]\).

These bounds can be viewed as a generalization of classical results for matrices by Weyl (Weyl, 1912), Davis-Kahan (Davis and Kahan, 1970), Wedin (Wedin, 1972) among others. However, there are also crucial distinctions. In particular, the \(\sin \Theta\) theorems of Davis-Kahan-Wedin bound the perturbation effect on the \(k\)th singular vector by \(C \|\tilde{\mathcal{T}} - \mathcal{T}\|/\min_{j \neq k} |\lambda_j - \lambda_k|\). The dependence on the gap \(\min_{j \neq k} |\lambda_j - \lambda_k|\) between \(\lambda_k\) and other singular values is unavoidable for matrices. This is not the case for higher-order odeco tensors where perturbation affects the singular vectors in separation.

It is also worth noting that for either (1) or (2) to hold, it is necessary that \(C \geq 1\). While in general one must take \(C > 1\), if a singular value \(\lambda_k\) is sufficiently large relative to the size of perturbation \(\|\tilde{\mathcal{T}} - \mathcal{T}\|\), then we can take the constant \(C = 1\) in (1) and arbitrarily close to 1 in (2) so that (1) and (2) are sharp in that the constant factor cannot be further reduced.

In general, a perturbed odeco tensor may no longer be odeco. However, we show that when the perturbation is small, a perturbed odeco tensor is nearly odeco and as such, any of its odeco approximations would have similar singular values and vectors. Also, these singular values and vectors are necessarily close to those of the unperturbed odeco tensor, and the effect of perturbation can be be similarly bounded.

Given the importance of perturbation analysis in fields such as machine learning, numerical analysis, and statistics, it is conceivable that our analysis and algorithms can prove useful in many situations. For illustration, we shall consider three specific examples from high dimensional data analysis, namely tensor SVD, tensor regression, and method of moments for estimation of latent variable models. Our general perturbation analysis offers a unified treatment to these seemingly different problems and leads to novel results that are either simpler or sharper than those in the literature. In particular, we establish minimax optimal rates for estimating the singular vectors of an odeco tensor when contaminated with Gaussian noise. Our result indicates that any of its singular vectors can be estimated as well
as if all other singular values are zero, or in other words, as in the rank one case. Similarly, for tensor regression with Gaussian design, we show that a simple spectral method leads to minimax optimal estimate of a low-rank or nearly low-rank odeco coefficient tensor. Finally, as an example of latent variable model estimation via methods of moment, we develop minimax optimal rates for independent component analysis and show that they can be attained by odeco approximations, again using the tools we developed.

Our development is related to fast-growing literature on using tensor methods in statistics and machine learning, especially a fruitful line of research in developing algorithm dependent bounds for perturbed odeco tensors. See, e.g., Anandkumar et al. (2014a); Mu et al. (2015, 2017); Belkin et al. (2018). The practical value of perturbation bounds (1) and (2) hinges upon our ability to compute \{(\tilde{\lambda}_i, \tilde{u}^{(1)}_i, \ldots, \tilde{u}^{(k)}_i) : 1 \leq i \leq r\}. While this can be done effectively when \(\tilde{T}\) is odeco, computing perturbed singular values and vectors for a general higher-order tensor is NP-hard. See, e.g., Hillar and Lim (2013). Nonetheless, as these recent works reveal, when a tensor is nearly odeco, it is possible to develop efficient algorithms for computing odeco approximations as well as their singular value/vectors. The deterministic and algorithm independent perturbation bounds we provide here complements these earlier developments in at least two different ways. First of all, our bounds could be readily used for perturbation analysis of any algorithm that produces an odeco approximation, allowing us to derive bounds on the singular values and vectors from those on the approximation error of the tensor itself. On the other hand, they can also serve as a benchmark on how well any procedure, computationally feasible or not, could perform. Indeed as we can see from the three high dimensional data analysis examples, our perturbation bounds often yield tight information theoretical limits for statistical inferences.

The rest of the paper is organized as follows. In the next section, we derive general perturbation bounds for odeco tensors. Section 3 extends these bounds to nearly odeco tensors. In Section 4, we investigate the implications of these bounds for three specific tensor data analysis tasks. Proofs of the main results are presented in Section 5.
2 Tensor Perturbation Bounds

2.1 Odeco Tensors

We say a $p$th order tensor $\mathbf{T} \in \mathbb{R}^{d_1 \times \cdots \times d_p}$ is odeco if it can be expressed as

$$\mathbf{T} = \sum_{k=1}^{d_{\text{min}}} \lambda_k u_k^{(1)} \otimes \cdots \otimes u_k^{(p)}$$

for some scalars $\lambda_1 \geq \ldots \geq \lambda_{d_{\text{min}}} \geq 0$ sorted in a nonincreasing order, and unit vectors $u_k^{(q)}$s such that $\langle u_k^{(q)} , u_k^{(q')} \rangle = \delta_{k,k'}$ where $d_{\text{min}} = \min\{d_1, \ldots, d_p\}$ and $\delta$ is the Kronecker’s delta. See, e.g., Kolda (2001); Robeva (2016) for further discussion of orthogonally decomposable tensors. We can view (3) as a high order generalization of the SVD and shall refer to $\lambda_i$s and $u_k^{(q)}$s as the singular values and vectors of $\mathbf{T}$. For brevity, we shall write

$$\mathbf{T} = \left[\{\lambda_k : 1 \leq k \leq d_{\text{min}}\} ; U^{(1)}, \ldots, U^{(p)}\right]$$

if (3) holds. Here $U^{(q)} \in \mathbb{R}^{d_q \times d_{\text{min}}}$ with $u_k^{(q)}$ as its $k$th column.

In the case when all singular values $\lambda_i$s are distinct, the SVD for odeco tensors as defined above coincides with the so-called higher-order SVD (HOSVD) which applies SVD after flattening a higher-order tensor to a matrix, for example, by collapsing all indices except the first one. See, e.g., De Lathauwer et al. (2000a,b). However, when the singular values have multiplicity more than one, there is a subtle difference between the two. It is easy to verify that, by Kruskal’s Theorem (Kruskal, 1977), the decomposition (3) is essentially unique in that

$$\sum_{k=1}^{d_{\text{min}}} \lambda_k u_k^{(1)} \otimes \cdots \otimes u_k^{(p)} = \sum_{k=1}^{d_{\text{min}}} \tilde{\lambda}_k \tilde{u}_k^{(1)} \otimes \cdots \otimes \tilde{u}_k^{(p)}$$

implies that there exists a permutation $\pi : [d_{\text{min}}] \to [d_{\text{min}}]$ such that $\lambda_k = \tilde{\lambda}_{\pi(k)}$ for all $1 \leq i \leq d_{\text{min}}$; and in addition, if $\lambda_k > 0$, then

$$\langle u_k^{(q)} , u_{\pi(k)}^{(q)} \rangle = \pm 1, \quad \text{and} \quad \prod_{q=1}^{p} \langle u_k^{(q)} , u_{\pi(k)}^{(q)} \rangle = 1.$$ 

This is to be contrasted with the HOSVD where only the singular space associated with a singular value can be identified. This subtle difference also has important practical implications. In general, the SVD of odeco tensors cannot be computed via HOSVD unless all singular values are distinct.
Nonetheless computing the singular value/vectors for an odeco tensor is feasible. For example, it can be computed via Jennrich’s algorithm when \( p = 3 \). See, e.g., Harshman (1970); Leurgans et al. (1993). More generally, efficient algorithms also exist to take full advantage of the orthogonal structure. In particular, if an odeco tensor is symmetric so that \( d_1 = \cdots = d_p =: d \), and \( u_k^{(1)} = \cdots = u_k^{(p)} =: u_k \) for all \( k = 1, \ldots, d \), Belkin et al. (2018) showed that \( \pm u_k \) are the only local maxima of

\[
F(a) := |\langle \mathcal{T}, a \otimes \cdots \otimes a \rangle|
\]

over \( S^{d-1} \). In addition, there is a full measure set \( U \subset S^{d-1} \times \cdots \times S^{d-1} \) such that a gradient iteration algorithm with initial value arbitrarily chosen from \( U \) converges to one of the \( u_k \)s. In light of these properties, one can enumerate all the singular values and singular vectors by repeatedly applying the gradient iteration algorithm with an initial value randomly chosen from the orthogonal complement of the linear space spanned by those already identified local maxima. Interested readers are referred to Belkin et al. (2018) for further details.

The argument presented in Belkin et al. (2018) relies heavily on the hidden convexity of \( F \), which no longer holds when \( T \) is not symmetric. However, their main observations remain valid for general odeco tensors. More specifically, write

\[
F(a^{(1)}, \ldots, a^{(p)}) := |\langle \mathcal{T}, a^{(1)} \otimes \cdots \otimes a^{(p)} \rangle| \tag{4}
\]

with slight abuse of notation. Denote by

\[
G(a^{(1)}, \ldots, a^{(p)}) := \left( \frac{\mathcal{T} \times_2 a^{(2)} \cdots \times_p a^{(p)}}{\| \mathcal{T} \times_2 a^{(2)} \cdots \times_p a^{(p)} \|}, \ldots, \frac{\mathcal{T} \times_1 a^{(1)} \cdots \times_{p-1} a^{(p-1)}}{\| \mathcal{T} \times_1 a^{(1)} \cdots \times_{p-1} a^{(p-1)} \|} \right). \tag{5}
\]

the gradient iteration function for \( F \) so that

\[
G_n = G \circ G \circ \cdots \circ G \underbrace{\text{n times}}_{n \text{ times}}
\]

maps from a set of initial values to the output from running the gradient iteration \( n \) times. Similar to the symmetric case, we have the following result for general odeco tensors:

**Theorem 2.1.** Let \( \mathcal{T} \) be an odeco tensor, and \( F \) and \( G \) be defined by (4) and (5) respectively. Then the set \( \{ (\pm u_k^{(1)}, \ldots, \pm u_k^{(p)} ) : \lambda_k > 0 \} \) is a complete enumeration of all local maxima of \( F \). Moreover, there exists a full measure set \( U \subset S^{d_1-1} \times \cdots \times S^{d_p-1} \) such that for any \( (a^{(1)}, \ldots, a^{(p)}) \in U \), \( G_n(a^{(1)}, \ldots, a^{(p)}) \to (\sigma_1 u_k^{(1)}, \ldots, \sigma_p u_k^{(p)}) \) for some \( 1 \leq k \leq d_{\text{min}} \), and \( \sigma_1, \ldots, \sigma_p \in \{ \pm 1 \} \), as \( n \to \infty \).
The main architect of the proof of Theorem 2.1 is similar to that for symmetric cases. See, e.g., Belkin et al. (2018). For completeness, a detailed proof is included in the Appendix. In light of Theorem 2.1, we can then compute all the singular value/vector tuples of an odeco tensor sequentially by applying gradient iterations and random initializations, in the same manner as the symmetric case.

2.2 Perturbation Bounds via Matricization

Let \( \mathcal{T} \) and \( \tilde{\mathcal{T}} \) be two odeco tensors with SVD:

\[
\mathcal{T} = \left[ \{ \lambda_k : 1 \leq i \leq d_{\min} \}; U^{(1)}, \ldots, U^{(p)} \right],
\]

and

\[
\tilde{\mathcal{T}} = \left[ \{ \tilde{\lambda}_k : 1 \leq k \leq d_{\min} \}; \tilde{U}^{(1)}, \ldots, \tilde{U}^{(p)} \right],
\]

respectively. We are interested in characterizing the difference between the two sets of singular values and vectors in terms of the “perturbation” \( \tilde{\mathcal{T}} - \mathcal{T} \).

It is instructive to first briefly review classical results in the matrix case, i.e., \( p = 2 \). Note that every matrix is odeco. Perturbation analysis of the singular vectors and spaces for matrices is well studied. See, e.g., Bhatia (1987); Stewart and Sun (1990), and references therein. In particular, Weyl’s perturbation theorem indicates that

\[
\max_{1 \leq k \leq d_{\min}} |\lambda_k - \tilde{\lambda}_k| \leq \| \mathcal{T} - \tilde{\mathcal{T}} \|. 
\] (6)

When a singular value \( \lambda_k \) has multiplicity more than one, its singular space has dimension more than one and singular vectors \( u_k \) and \( v_k \) are no longer uniquely identifiable. But if it is simple, i.e., \( \lambda_{k-1} > \lambda_k > \lambda_{k+1} \), then the Davis-Kahan-Wedin sin \( \Theta \) theorem states that

\[
\sin \angle (u_k^{(q)}, \tilde{u}_k^{(q)}) \leq \frac{\| \mathcal{T} - \tilde{\mathcal{T}} \|}{\min\{ \lambda_k - \lambda_{k-1}, \lambda_k - \lambda_{k+1} \}},
\] (7)

provided that the denominator on the righthand side is positive. It is oftentimes more convenient to consider a modified version of the above bound for the singular vectors in terms of the gap between singular values of \( \mathcal{T} \):

\[
\sin \angle (u_k^{(q)}, \tilde{u}_k^{(q)}) \leq \frac{2\| \mathcal{T} - \tilde{\mathcal{T}} \|}{\min\{ \lambda_k - \lambda_{k-1}, \lambda_k - \lambda_{k+1} \}},
\] (8)
which follows immediately from (6) and (7).

It is worth noting that the dependence of any general perturbation bounds on the gap between singular values is unavoidable for matrices and can be illustrated by the following simple example from Bhatia (2013):

$$
\mathcal{T} = \begin{pmatrix}
1 + \delta & 0 \\
0 & 1 - \delta
\end{pmatrix}, \quad \text{and} \quad \tilde{\mathcal{T}} = \begin{pmatrix}
1 & \delta \\
\delta & 1
\end{pmatrix}.
$$

(9)

It is not hard to see that $$\| \mathcal{T} - \tilde{\mathcal{T}} \| = \sqrt{2}\delta$$ and can be made arbitrarily small at the choice of $$\delta > 0$$. Yet the singular vectors of $$\mathcal{T}$$ and $$\tilde{\mathcal{T}}$$ are $$\{ (0, 1)^\top, (1, 0)^\top \}$$ and $$\{ (1/\sqrt{2}, 1/\sqrt{2})^\top, (1/\sqrt{2}, -1/\sqrt{2})^\top \}$$ respectively so that

$$\sin \angle (u_k^{(q)}, \tilde{u}_k^{(q)}) = \frac{\| \mathcal{T} - \tilde{\mathcal{T}} \|}{\lambda_1 - \lambda_2},$$

for $$k = 1, 2$$ and $$q = 1, 2$$.

These classical perturbation bounds can be applied to higher-order tensors through matricization or flattening, as for HOSVD. More specifically, write $$\text{Mat}_q : \mathbb{R}^{d_1 \times \cdots \times d_p} \to \mathbb{R}^{d_q \times d_{-q}}$$ by collapsing all indices other than the $$q$$th one and therefore converting a $$p$$th order tensor into a $$d_q \times d_{-q}$$ matrix where $$d_{-q} = d_1 \cdots d_{q-1}d_{q+1} \cdots d_p$$. For an odeco tensor $$\mathcal{T}$$, its SVD determines that of $$\text{Mat}_q(\mathcal{T})$$. More specifically,

$$\text{Mat}_q(\mathcal{T}) = U^{(q)}(\text{diag}(\lambda_1, \ldots, \lambda_{d_{\min}}))(V^{(q)})^\top,$$

where

$$V^{(q)} = U^{(1)} \odot \cdots \odot U^{(q-1)} \odot U^{(q+1)} \odot \cdots \odot U^{(p)}.$$

Here $$\odot$$ stands for the Khatri-Rao product. This immediately implies that

$$\max_{1 \leq k \leq d_{\min}} |\lambda_k - \tilde{\lambda}_k| \leq \min_{1 \leq q \leq p} \| \text{Mat}_q(\mathcal{T}) - \text{Mat}_q(\tilde{\mathcal{T}}) \|,$$

and

$$\sin \angle (u_k^{(q)}, \tilde{u}_k^{(q)}) \leq \frac{2\| \text{Mat}_q(\mathcal{T}) - \text{Mat}_q(\tilde{\mathcal{T}}) \|}{\min\{\lambda_{k-1} - \lambda_k, \lambda_k - \lambda_{k+1}\}},$$

in light of (6) and (8). These bounds, however, are suboptimal and can be significantly improved in a couple of directions that highlight fundamental differences between matrices and higher-order tensors.
First of all, we can derive perturbation bounds in terms of the tensor operator norm \( \|T - \tilde{T}\| \). Note that for any tensor \( T \), \( \|T\| \leq \|\text{Mat}_q(T)\| \) and in general the difference could be large. For example, when \( T \) is a \( p \)th order Gaussian ensemble of dimension \( d \times \cdots \times d \) whose entries are independent standard normal random variables. With high probability, \( \|T\| \asymp \sqrt{d} \) yet for any \( 1 \leq q \leq p \), \( \|\text{Mat}_q(T)\| \asymp d^{(p-1)/2} \). Although it is true that \( \|T\| = \|\text{Mat}_q(T)\| \) for \( q = 1, \ldots, p \) for odeco tensors, the difference between two odeco tensors is not necessarily odeco and as a result \( \|T - \tilde{T}\| \) and \( \|\text{Mat}_q(T) - \text{Mat}_q(\tilde{T})\| \) can be quite different. As a simple example, consider the case when \( T = u \otimes u \otimes u \) and \( \tilde{T} = u \otimes v \otimes v \) where \( u = (0, 1)^\top \) and \( v = (1, 0)^\top \). It is easy to see that \( \|T - \tilde{T}\| = 1 \) and yet \( \|\text{Mat}_1(T) - \text{Mat}_1(\tilde{T})\| = \sqrt{2} \). Indeed, as we shall show, in many applications, perturbation bounds in terms of \( \|T - \tilde{T}\| \) are much sharper and more powerful.

Perhaps more importantly, another unsatisfactory aspect of the aforementioned perturbation bounds for higher-order odeco tensors is the dependence on the gap between singular values. While for matrices it is only meaningful to talk about singular spaces when the corresponding singular value is not simple, all singular vectors are identifiable for odeco tensors. The aforementioned bounds for \( \sin(\angle(u^{(q)}_k, \tilde{u}^{(q)}_k)) \) does not tell us anything about the perturbation of the singular vectors at all when a singular value is not simple. Indeed, as we shall show, that the gap \( \min\{\lambda_k - 1, \lambda_k - \lambda_{k+1}\} \) is irrelevant for perturbation analysis of an odeco tensors, and perturbation of each singular vectors is essentially independent of other singular values.

### 2.3 Perturbation Bounds for Odeco Tensors

To appreciate the difference in perturbation effect between matrices and higher-order tensors, we first take a look at the Weyl’s bound for singular values which states that, in the matrix case, i.e., \( p = 2 \),

\[
\max_{1 \leq k \leq d} |\lambda_k - \tilde{\lambda}_k| \leq \|T - \tilde{T}\|. \tag{10}
\]

More generally, when \( p \) is even, asymptotic bounds for simple singular values under infinitesimal perturbation have been studied recently by Che et al. (2016). Their result implies that, in our notation, if \( p \) is even and a simple singular value \( \lambda_j \) is sufficiently far away from \( \lambda_{j-1} \)
and \( \lambda_{j+1} \), then
\[
|\tilde{\lambda}_j - \lambda_j| \leq \|\tilde{T} - T\| + O(\|\tilde{T} - T\|^2),
\]
as \( \|\tilde{T} - T\| \to 0 \). This appears to suggest that it is plausible that (10) could continue to hold for higher-order odco tensors. Unfortunately, this is not the case and (10) does not hold in general for higher-order odco tensors. To see this, let
\[
\mathcal{T} = 2e_1 \otimes e_1 \otimes e_1
\]
and
\[
\tilde{T} = (e_1 + e_2) \otimes (e_1 + e_2) \otimes (e_1 + e_2) + (e_1 - e_2) \otimes (e_1 - e_2).
\]
Obviously \((\lambda_1, \lambda_2) = (2, 0)\) and \((\tilde{\lambda}_1, \tilde{\lambda}_2) = (2\sqrt{2}, 2\sqrt{2})\) so that
\[
\max\{|\lambda_1 - \tilde{\lambda}_1|, |\lambda_2 - \tilde{\lambda}_2|\} = 2\sqrt{2}.
\]
On the other hand, as shown by Yuan and Zhang (2016)
\[
\|\tilde{T} - T\| = 2\|e_2 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_2\| = 4/\sqrt{3} < 2\sqrt{2},
\]
invalidating (10).

At a more fundamental level, for matrices, Weyl’s bound can be viewed as a consequence of Courant-Fischer-Weyl min-max principle which states that
\[
\lambda_k = \min_{S: \text{dim}(S) = d_1 - k + 1} \max_{x^{(1)}(1) \in S^{d_1 - 1} \cap S} \langle \mathcal{T}, x^{(1)} \otimes x^{(2)} \rangle,
\]
and
\[
\lambda_k = \max_{S: \text{dim}(S) = k} \min_{x^{(1)}(1) \in S^{d_1 - 1} \cap S} \max_{x^{(2)}(2) \in S^{d_2 - 1}} \langle \mathcal{T}, x^{(1)} \otimes x^{(2)} \rangle.
\]
Similar characterizations, however, do not hold for higher-order tensors. As an example, consider a \( p \)th order odco tensor of dimension \( d \times \cdots \times d \) and with equal singular values.

The following proposition shows that neither (11) nor (12) holds, in particular for the smallest singular value \( \lambda_d \) where the righthand side of both equations can be expressed as
\[
\min_{x^{(1)}(1) \in S^{d-1}} \max_{x^{(2)}, \ldots, x^{(p)} \in S^{d-1}} \langle \mathcal{T}, x^{(1)} \otimes \cdots \otimes x^{(p)} \rangle.
\]
Proposition 2.2. Let $\mathcal{T}$ be a $p$th ($p \geq 3$) order odeco tensor of dimension $d \times \cdots \times d$. If all its singular values are $\lambda$, then

$$\min_{x^{(1)} \in S^{d-1}} \max_{x^{(2)}, \ldots, x^{(p)} \in S^{d-1}} \langle \mathcal{T}, x^{(1)} \otimes \cdots \otimes x^{(p)} \rangle = \frac{\lambda}{\sqrt{d}}.$$ 

Although straightforward generalizations of Weyl’s bound to higher-order tensor do not hold, perturbation bounds in a similar spirit can still be established. More specifically, we have

Theorem 2.3. Let $\mathcal{T}$ and $\tilde{\mathcal{T}}$ be two $d_1 \times \cdots \times d_p$ ($p > 2$) odeco tensors with SVD:

$$\mathcal{T} = [\{\lambda_k : 1 \leq k \leq d_{\min}\}; U^{(1)}, \ldots, U^{(p)}],$$

and

$$\tilde{\mathcal{T}} = [\{\tilde{\lambda}_k : 1 \leq k \leq d_{\min}\}; \tilde{U}^{(1)}, \ldots, \tilde{U}^{(p)}],$$

respectively where $d_{\min} = \min\{d_1, \ldots, d_p\}$. There exist a numerical constant $C \geq 1$ and a permutation $\pi : [d_{\min}] \to [d_{\min}]$ such that for all $k = 1, \ldots, d_{\min},$

$$|\lambda_k - \tilde{\lambda}_{\pi(k)}| \leq C \|\tilde{\mathcal{T}} - \mathcal{T}\|,$$  \hspace{1cm} (13)

and

$$\max_{1 \leq q \leq p} \sin \angle(\mathbf{u}_k^{(q)} , \tilde{\mathbf{u}}_{\pi(k)}^{(q)}) \leq C \|\tilde{\mathcal{T}} - \mathcal{T}\| \frac{1}{\lambda_k},$$  \hspace{1cm} (14)

with the convention that $1/0 = +\infty$.

As discussed before, despite the similarity in appearance to the bounds for matrices, Theorem 2.3 requires different proof techniques. Moreover, there are several intriguing differences between the bounds given in Theorem 2.3 and classical ones for matrices.

First of all, we do not necessarily match the $k$th singular value/vector tuple $(\lambda_k, \mathbf{u}_k^{(1)}, \ldots, \mathbf{u}_k^{(p)})$ of $\mathcal{T}$ with that of $\tilde{\mathcal{T}}$. This is because we do not restrict that the singular values $\lambda_k$s are distinct and sufficiently apart from each other, and hence the singular vectors corresponding to $\tilde{\lambda}_k$ are not necessarily close to those corresponding to $\lambda_k$. As a simple example, consider the following $2 \times 2 \times 2$ tensors:

$$\mathcal{T} = (1 + \delta)\mathbf{e}_1 \otimes \mathbf{e}_1 \otimes \mathbf{e}_1 + (1 - \delta)\mathbf{e}_2 \otimes \mathbf{e}_2 \otimes \mathbf{e}_2,$$
and
\[ \hat{T} = (1 - \delta)e_1 \otimes e_1 \otimes e_1 + (1 + \delta)e_2 \otimes e_2 \otimes e_2, \]
where \( \delta > 0 \) represents a small perturbation. Obviously, \( \lambda_1 = \tilde{\lambda}_1 = 1 + \delta \) and \( \lambda_2 = \tilde{\lambda}_2 = 1 - \delta \). But the correct way to study the effect of perturbation is to compare \((1 + \delta)e_1 \otimes e_1 \otimes e_1\) with \((1 - \delta)e_1 \otimes e_1 \otimes e_1\), and \((1 - \delta)e_2 \otimes e_2 \otimes e_2\) with \((1 + \delta)e_2 \otimes e_2 \otimes e_2\), and not the other way around. In other words, we want to pair \( \lambda_1 \) with \( \tilde{\lambda}_2 \), and \( \lambda_2 \) with \( \tilde{\lambda}_1 \).

Another notable difference is between the perturbation bound (14) for singular vectors and those from Wedin-Davis-Kahan \( \sin \Theta \) theorems. The gap between singular values is absent in the bound (14). This means that for higher-order odeco tensors, the perturbation affects the singular vectors separately. The perturbation bound (14) depends only on the amount of perturbation relative to their corresponding singular value.

For either (13) or (14) to hold, it is necessary that the constant \( C \geq 1 \). This is clear by considering two rank-one tensors differing only in the nonzero singular value or in one of its corresponding singular vectors. In fact, as we discussed before, the constant \( C \) in the perturbation bounds in Theorem 2.3 needs to be greater than 1 in general. But when the perturbation is sufficiently small, or for large enough singular values, it is, however, possible to take \( C = 1 \) or arbitrarily close to 1 as the following result shows.

**Theorem 2.4.** Let \( T \) and \( \tilde{T} \) be two \( d_1 \times \cdots \times d_p \) (\( p > 2 \)) odeco tensors with SVD:

\[ T = [\{\lambda_k : 1 \leq k \leq d_{\min}\}; U^{(1)}, \ldots, U^{(p)}], \]

and

\[ \hat{T} = [\{\tilde{\lambda}_k : 1 \leq k \leq d_{\min}\}; \tilde{U}^{(1)}, \ldots, \tilde{U}^{(p)}], \]

respectively where \( d_{\min} = \min\{d_1, \ldots, d_p\} \). There exists a permutation \( \pi : [d_{\min}] \to [d_{\min}] \) such that for any \( \varepsilon > 0 \),

\[ |\lambda_k - \tilde{\lambda}_{\pi(k)}| \leq \|\hat{T} - T\|, \tag{15} \]

and

\[ \max_{1 \leq q \leq p} \sin \angle(u_k^{(q)}, \tilde{u}_{\pi(k)}^{(q)}) \leq \frac{(1 + \varepsilon)\|\hat{T} - T\|}{\lambda_k}. \tag{16} \]

provided that \( \lambda_k \geq c_\varepsilon\|\hat{T} - T\| \) for some constant \( c_\varepsilon > 0 \) depending on \( \varepsilon \) only.
In particular, when considering infinitesimal perturbation in that \( \| \tilde{\mathcal{T}} - \mathcal{T} \| = o(\lambda_k) \), we can express the bound \((16)\) for singular vectors as

\[
\max_{1 \leq q \leq p} \sin \angle (u^{(q)}_k, \tilde{u}^{(q)}_{\pi(k)}) \leq \frac{\| \tilde{\mathcal{T}} - \mathcal{T} \|}{\lambda_k} + o\left( \frac{\| \tilde{\mathcal{T}} - \mathcal{T} \|}{\lambda_k} \right),
\]

which is more convenient for asymptotic analysis.

## 3 Perturbation Bounds for Nearly Odeco Tensors

An important consequence of Theorems 2.3 and 2.4 is that if a tensor \( \mathcal{X} \) is “close” to being odeco, then its odeco approximations share similar singular values and vectors, and the perturbation effect can be bounded in a similar fashion as before. More specifically, let \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) be two odeco approximations to an odeco tensor \( \mathcal{A} \). By triangular inequality,

\[
\| \mathcal{A}_1 - \mathcal{A}_2 \| \leq \| \mathcal{A}_1 - \mathcal{A} \| + \| \mathcal{A}_2 - \mathcal{A} \|,
\]

and the righthand side can be made small if \( \mathcal{A} \) is “close” to being odeco, and \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are “good” approximations. As such, in light of Theorem 2.3, the singular values and vectors of \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are necessarily close to each other as well. We now consider two concrete examples that are commonly encountered, and explore this strategy in further details.

### 3.1 General Additive Perturbation

One situation where a nearly odeco tensor may arise is when an odeco tensor \( \mathcal{I} \) is “contaminated” by an additive perturbation \( \mathcal{E} \). In general, \( \mathcal{X} = \mathcal{I} + \mathcal{E} \) is no longer odeco, and we may not be able to define its SVD in the same fashion as \((3)\). However, when \( \| \mathcal{E} \| \) is small, \( \mathcal{X} \) is close to an odeco tenor, namely \( \mathcal{I} \). As a result, a good odeco approximation to \( \mathcal{X} \) is necessarily close to \( \mathcal{I} \) as well and its singular values and vectors to those of \( \mathcal{I} \). More precisely, we have the following result as an immediate consequence of Theorem 2.3.

**Corollary 3.1.** Let \( \mathcal{X} \in \mathbb{R}^{d_1 \times \cdots \times d_p} \) \((p \geq 3)\) be an odeco tensor with SVD

\[
\mathcal{X} = \{[\lambda_k : 1 \leq k \leq d_{\min}]; \mathbf{U}^{(1)}, \ldots, \mathbf{U}^{(p)}\},
\]

and

\[
\mathcal{X}^{\text{odeco}} = \{[\sigma_k : 1 \leq k \leq d]; \mathbf{V}^{(1)}, \ldots, \mathbf{V}^{(p)}\},
\]

13
be an odeco approximation to $\mathcal{X} : = \mathcal{T} + \mathcal{E}$. Then there are a numerical constant $C \geq 1$ and a permutation $\pi : [d] \to [d]$ such that

$$|\lambda_k - \sigma_{\pi(k)}| \leq C(\|\mathcal{X}^{\text{odeco}} - \mathcal{X}\| + \|\mathcal{E}\|), \quad (17)$$

and

$$\max_{1 \leq q \leq p} \sin \angle(u^{(q)}_k, v^{(q)}_{\pi(k)}) \leq \frac{C(\|\mathcal{X}^{\text{odeco}} - \mathcal{X}\| + \|\mathcal{E}\|)}{\lambda_k}. \quad (18)$$

for all $k = 1, \ldots, d_{\min}$.

While Corollary 3.1 holds for any odeco approximation to $\mathcal{X}$, it is oftentimes of interest to “estimate” the singular values and vectors of $\mathcal{T}$ using a “good” odeco approximation. In particular, one may consider the best odeco approximation:

$$\mathcal{X}^{\text{best}} : = \inf_{\mathcal{A} \text{ is odeco}} \|\mathcal{A} - \mathcal{X}\|.$$

It is clear that

$$\|\mathcal{X}^{\text{best}} - \mathcal{X}\| \leq \|\mathcal{T} - \mathcal{X}\| = \|\mathcal{E}\|,$$

so that the bounds (17) and (18) now becomes:

$$|\lambda_k - \sigma_{\pi(k)}| \leq C\|\mathcal{E}\|, \quad (19)$$

and

$$\max_{1 \leq q \leq p} \sin \angle(u^{(q)}_k, v^{(q)}_{\pi(k)}) \leq \frac{C\|\mathcal{E}\|}{\lambda_k}. \quad (20)$$

Indeed (19) and (20) continue to hold for any odeco approximation $\mathcal{X}^{\text{odeco}}$ obeying

$$\|\mathcal{X}^{\text{odeco}} - \mathcal{X}\| \lesssim \|\mathcal{E}\|. \quad (21)$$

It is worth noting that computing an odeco approximation that satisfies (21) is not always straightforward. In fact, development of efficient algorithms that can produce “good” odeco approximation to a nearly odeco tensor is an active research area with fervent interest. A flurry of recent works suggest that finding a $\mathcal{X}^{\text{odeco}}$ satisfying (21) is feasible at least when $\|\mathcal{E}\|$ is sufficiently small. Interested readers are referred to Anandkumar et al. (2014a); Mu et al. (2015, 2017); Belkin et al. (2018) and references therein for more detailed discussions regarding this aspect.
3.2 Incoherent Tensors

Another example of “nearly” odeco tensors are those with incoherent components. More specifically, let

$$\mathcal{X} = \sum_{k=1}^{r} \eta_k a_k^{(1)} \otimes \cdots \otimes a_k^{(p)}$$

where $$\eta_1 \geq \cdots \geq \eta_r > 0$$. Different from odeco tensors, the unit vectors $$a_k^{(q)}$$s in (22) are not required to be orthonormal but assumed to be close to being orthonormal. More specifically, we shall assume that $$A^{(q)}$$s satisfy the isometry condition

$$1 - \delta \leq \lambda_{\min}(A^{(q)}) \leq \lambda_{\max}(A^{(q)}) \leq 1 + \delta, \quad \forall q = 1, \ldots, p,$$

for some $$0 \leq \delta < 1$$, where $$A^{(q)} = [a_1^{(q)}, \ldots, a_r^{(q)}]$$, and $$\lambda_{\min}(\cdot)$$ and $$\lambda_{\max}(\cdot)$$ evaluate the smallest and largest singular values, respectively, of a matrix. Clearly $$\delta = 0$$ if $$A^{(q)}$$ is orthonormal so that $$\delta$$ measures the incoherence of its column vectors. A canonical example of incoherent tensors arises in a probabilistic setting: let $$a_1^{(q)}, \ldots, a_r^{(q)}$$ be independently and uniformly sampled from the unit sphere; then it is not hard to see that $$\delta = O_p(\sqrt{r/d_q})$$.

In light of Kruskal’s Theorem (Kruskal, 1977), the decomposition (22) is essentially unique and therefore $$\mathcal{X}$$ cannot be odeco unless $$\delta = 0$$. However, $$\mathcal{X}$$ is close to being odeco when $$\delta$$ is small. More specifically, let $$A^{(q)} = U^{(q)} P^{(q)}$$ its polar decomposition, and

$$\tilde{\mathcal{X}} = \sum_{k=1}^{r} \eta_k u_k^{(1)} \otimes \cdots \otimes u_k^{(p)}.$$

It is clear that $$\tilde{\mathcal{X}}$$ is odeco and moreover, we can show that

**Theorem 3.2.** Let $$\mathcal{X}$$ be defined by (22) with the unit vectors $$a_k^{(q)}$$s obeying (23), and $$\tilde{\mathcal{X}}$$ by (24). Then

$$\| \tilde{\mathcal{X}} - \mathcal{X} \| \leq (p + 1)\delta \eta_1,$$

and

$$\max_{1 \leq q \leq p} \sin \angle(a_k^{(q)}, u_k^{(q)}) \leq \delta / \sqrt{2}.$$

In light of Theorems 2.3 and 3.2, all “good” odeco approximations to $$\mathcal{X}$$ have singular values and vectors close to $$\eta_k$$s and $$a_k^{(q)}$$s, at least when $$\delta$$ is sufficiently small.
Corollary 3.3. Let $X$ be defined by (22) with the unit vectors $a_k^{(k)}$'s obeying (23), and
\[ X_{odeco} = \{\lambda_k : 1 \leq i \leq d_{\text{min}}\}; U^{(1)}, \ldots, U^{(p)}], \]
an odeco approximation to $X$. Then there exist a numerical constant $C > 0$ and a permutation $\pi : [d_{\text{min}}] \to [d_{\text{min}}]$ such that for any $1 \leq k \leq r$,
\[ |\eta_k - \lambda_{\pi(k)}| \leq C[(p + 1)\delta_1 + \|X_{odeco} - X^\star\|] \]
and
\[ \max_{1 \leq q \leq p} \sin \angle(a_{(q)}^{(k)}, u_{\pi(k)}^{(q)}) \leq C\{(p + 1)\delta_1 + \|X_{odeco} - X^\star\| + \delta\}/\eta_k. \]

We want to point out that Corollary 3.3 can also be viewed as a “robust” version of Theorem 2.3 as the latter can be viewed as a special case of the former when $\delta = 0$.

4 Applications in High Dimensional Data Analysis

The perturbation bounds we derived in the last section are fairly general and can be applied in various settings. We now consider several specific applications in high dimensional data analysis to illustrate how they may lead to new insights for each of them.

4.1 Tensor SVD

Our first example is the so-called tensor SVD problem considered earlier by Richard and Montanari (2014); Liu et al. (2017); Zhang and Xia (2018) among others. SVD is among the most commonly used methods to reduce the dimensionality of the data, and oftentimes serves as a useful first step to capture the essential features in the data for downstream analysis. More specifically, consider observing a $p$th order tensor $X^\star \in \mathbb{R}^{d_1 \times \cdots \times d_p}$ obeying $X^\star = \mathcal{T} + \mathcal{E}$ where the signal tensor
\[ \mathcal{T} = \sum_{k=1}^{d_{\text{min}}} \lambda_k u_k^{(1)} \otimes \cdots \otimes u_k^{(p)}, \]
is odeco and $\mathcal{E}$ is a noisy tensor whose entries are independent standard normal random variables. The goal is to infer about $\mathcal{T}$, especially its singular values and vectors, from $X^\star$. A standard argument yields $\|\mathcal{E}\| = O_p(\sqrt{d_1 + \cdots + d_p})$. See, e.g., Raskutti et al. (2019). We could then apply Corollary 3.1 to derive bounds for estimates of $\lambda_k$s and $u_k^{(q)}$'s.
More specifically, we can estimate $\mathcal{T}$ by the best odeco approximation to $\mathcal{X}$:

$$\hat{\mathcal{T}} = \text{argmin}_{\mathcal{A} \text{ is odeco}} \| \hat{\mathcal{X}} - \mathcal{A} \|.$$ (25)

By triangular inequality,

$$\| \hat{\mathcal{T}} - \mathcal{T} \| \leq \| \hat{\mathcal{T}} - \hat{\mathcal{X}} \| + \| \hat{\mathcal{X}} - \mathcal{X} \| \leq 2\| \mathcal{E} \|.$$ 

Let

$$\hat{\mathcal{T}} = [\{\hat{\lambda}_k : 1 \leq k \leq d_{\text{min}}\}; \hat{U}^{(1)}, \ldots, \hat{U}^{(p)}]$$ (26)

be its singular value decomposition. Theorem 2.3 indicates that there exists a permutation $\pi : [d_{\text{min}}] \to [d_{\text{min}}]$ so that

$$|\hat{\lambda}_{\pi(k)} - \lambda_k| \leq C\| \mathcal{E} \|$$

and

$$\max_{1 \leq q \leq p} \sin \angle (\hat{u}_k^{(q)}, \hat{u}_{\pi(k)}^{(q)}) \leq C\| \mathcal{E} \|/\lambda_k.$$ 

This immediately implies that

**Theorem 4.1.** Consider the tensor SVD model $\mathcal{X} = \mathcal{T} + \mathcal{E}$ where

$$\mathcal{T} = [\{\lambda_k : 1 \leq k \leq d_{\text{min}}\}; U^{(1)}, \ldots, U^{(p)}]$$

is odeco and $\mathcal{E}$ has independent standard normal entries. Let $\hat{\mathcal{T}}$ be the best odeco approximation to $\mathcal{X}$ as defined by (25) with SVD given by (26). Then there exist a numerical constant $C > 0$ and a permutation $\pi : [d_{\text{min}}] \to [d_{\text{min}}]$ such that

$$\mathbb{E} \max_{1 \leq k \leq d_{\text{min}}} |\hat{\lambda}_{\pi(k)} - \lambda_k| \leq C \cdot \sqrt{d_1 + \cdots + d_p},$$ (27)

and

$$\mathbb{E} \max_{1 \leq q \leq p} \sin \angle (u_k^{(q)}, \hat{u}_{\pi(k)}^{(q)}) \leq C \cdot \min \left\{ \frac{\sqrt{d_1 + \cdots + d_p}}{\lambda_k}, 1 \right\},$$ (28)

for any $k = 1, \ldots, d_{\text{min}}$.

While we have focused on estimating the singular values and vectors based on the best odeco approximation to $\mathcal{X}$, it is worth noting that Theorem 4.1 will continue to hold as long as $\mathcal{X}^{\text{odeco}}$ is a “good” approximation to $\mathcal{X}$ in that

$$\| \mathcal{X}^{\text{odeco}} - \mathcal{X} \| \leq C \min_{\mathcal{A} \text{ is odeco}} \| \mathcal{X} - \mathcal{A} \|,$$
for some constant $C > 0$. As noted before, such a relaxation could prove helpful in when computing the best odeco approximation is difficult. To fix ideas, however, we shall focus hereafter on the best odeco approximation for brevity.

Bounds similar to those given by Theorem 4.1 are known when $\mathcal{T}$ is of rank one, that is, $\lambda_2 = \cdots = \lambda_{d_{\min}} = 0$. See, e.g., Richard and Montanari (2014). Theorem 4.1 suggests that the same bounds hold uniformly over all singular values and vectors of an odeco tensor. In other words, we can estimate any singular value and vectors of $\mathcal{T}$ at the same rate as if all other singular values are zero or equivalently as in the rank one case. This also draws contrast with the setting considered by Zhang and Xia (2018). Generalizing the rank-one model of Richard and Montanari (2014), Zhang and Xia (2018) studies efficient estimation strategies of $\mathcal{T}$ and its decomposition when it is of low multilinear ranks. Their analysis requires that $\mathcal{T}$ is nearly cubic, e.g., $d_1 \asymp d_2 \asymp d_3$, and the ranks are of an order up to $d_1^{1/2}$ among other conditions. Odeco tensors have more innate structure and consequently, as Theorem 4.1 indicates, if $\mathcal{T}$ is odeco, its shape and rank are irrelevant for estimating its singular values and vectors.

Moreover, both bounds (27) and (28) can be shown to be minimax optimal in that no other estimates of the singular vectors or values based upon $\mathcal{T}$ could attain a faster rate of convergence.

**Theorem 4.2.** Consider the tensor SVD model $\mathcal{X} = \mathcal{T} + \mathcal{E}$ where

$$\mathcal{T} = \{\{\lambda_k : 1 \leq k \leq d_{\min}\}; \mathbf{U}^{(1)}, \ldots, \mathbf{U}^{(p)}\}$$

is odeco and $\mathcal{E}$ has independent standard normal entries. Then there exists a constant $c > 0$ such that

$$\inf \lambda_k \sup_{u_k^{(q)} \in \mathcal{S}_{d_1 \cdots 1 \leq q \leq p}^{d_2 \cdots d_p}} \mathbb{E} |\tilde{\lambda}_k - \lambda_k| \geq c \cdot \sqrt{d_1 + \cdots + d_p},$$

(29)

and

$$\inf \tilde{u}_k^{(1)}, \ldots, \tilde{u}_k^{(p)} \sup_{u_k^{(q)} \in \mathcal{S}_{d_1 \cdots 1 \leq q \leq p}^{d_2 \cdots d_p}} \mathbb{E} \max \sin \angle (u_k^{(q)}, \tilde{u}_k^{(q)}) \geq c \cdot \min \left\{ \sqrt{d_1 + \cdots + d_p}/\lambda_k, 1 \right\},$$

(30)

where the infimum in (29) and (30) is taken over all estimates based on observing $\mathcal{X}$.
4.2 Tensor Regression

As a second example, we now consider tensor regression where we observe $n$ independent copies of a random pair $(\mathcal{X}, Y) \in \mathbb{R}^{d_1 \times \cdots \times d_p} \times \mathbb{R}$ following a regression model:

$$Y = \langle \mathcal{X}, \mathcal{T} \rangle + \varepsilon,$$

where the unknown regression coefficient tensor $\mathcal{T} \in \mathbb{R}^{d_1 \times \cdots \times d_p}$ is odeco with singular value decomposition

$$\mathcal{T} = \sum_{k=1}^{d_{\min}} \lambda_k u_k^{(1)} \otimes \cdots \otimes u_k^{(p)},$$

and the regression error $\varepsilon$ follows a $N(0, \sigma^2)$ distribution. To fix ideas, we shall focus on scalar responses $Y \in \mathbb{R}$ and Gaussian designs where the covariate tensor $\mathcal{X}$ has independent standard normal entries. Of special interest here is the case when $\mathcal{T}$ is low rank or nearly low rank in that its singular values reside in an $\ell_\alpha$ ($0 \leq \alpha < 1$) ball:

$$B_\alpha(M) = \{(x_1, \ldots, x_{d_{\min}})^\top : |x_1|^\alpha + \cdots + |x_{d_{\min}}|^\alpha \leq M\}.$$  

In particular, when $\alpha = 0$, $(\lambda_1, \ldots, \lambda_{d_{\min}})^\top \in B_0(M)$ implies that $\mathcal{T}$ is of rank up to $M$. In what follows we shall denote the parameter space corresponding to $B_\alpha(M)$ by

$$\Theta(\alpha, M) = \{\mathcal{A} \in \mathbb{R}^{d_1 \times \cdots \times d_p} : \mathcal{A} \text{ is odeco with singular values } (\lambda_1, \ldots, \lambda_{d_{\min}})^\top \in B_\alpha(M)\}.$$  

A general approach to estimate $\mathcal{T}$ in this case is through penalized least squares. Specifically, in our context, one could estimate $\mathcal{T}$ by the solution to

$$\min_{\mathcal{A} \text{ is odeco}} \left\{ \frac{1}{2n} \sum_{i=1}^n (Y_i - \langle \mathcal{X}_i, \mathcal{A} \rangle)^2 + \tau_n \text{Pen}(\mathcal{A}) \right\},$$

where $\tau_n \geq 0$ is a tuning parameter and $\text{Pen}(\cdot)$ is a penalty function that encourages the solution to be of low-rank. See, e.g., Raskutti et al. (2019). Taking advantage of the perturbation bounds we developed before, here we can consider a simpler and more direct approach.

In particular, note that

$$\mathbb{E}(Y \mathcal{X}) = \mathbb{E}(\langle \mathcal{X}, \mathcal{T} \rangle \mathcal{X}) = \mathcal{T}.$$  

This observation leads to the following estimator of $\mathcal{T}$:

$$\hat{\mathcal{T}} = \frac{1}{n} \sum_{i=1}^n Y_i \mathcal{X}_i.$$  

19
This estimate, however, does not exploit the fact that $\mathcal{F}$ is odeco and (nearly) low-rank. To this end, we shall consider a slightly modified estimating strategy.

Our approach is motivated by the observation that $\tilde{\mathcal{F}}$ is nearly odeco in that

$$\|\tilde{\mathcal{F}} - \mathcal{F}\| = O_p\left(\frac{\sqrt{d_1 + \cdots + d_p}}{n}\right).$$

See, e.g., Raskutti et al. (2019). Let $\tilde{\mathcal{F}}^{\text{odeco}}$ be the best odeco approximation of $\tilde{\mathcal{F}}$:

$$\tilde{\mathcal{F}}^{\text{odeco}} = \arg\min_{\mathcal{A} \text{ is odeco}} \|\mathcal{A} - \tilde{\mathcal{F}}\|,$$

Then it is immediate from triangular inequality that

$$\|\tilde{\mathcal{F}}^{\text{odeco}} - \mathcal{F}\| \leq 2\|\tilde{\mathcal{F}} - \mathcal{F}\| = O_p\left(\frac{\sqrt{d_1 + \cdots + d_p}}{n}\right).$$

Similar to before, the following discussion applies as long as $\tilde{\mathcal{F}}^{\text{odeco}}$ is a “good” and not necessarily the best odeco approximation to $\tilde{\mathcal{F}}$. For brevity, we shall not explore this generalization further.

Denote by

$$\tilde{\mathcal{F}}^{\text{odeco}} = \sum_{k=1}^{d_{\min}} \tilde{\lambda}_k \tilde{u}_k^{(1)} \otimes \cdots \otimes \tilde{u}_k^{(p)}$$

its SVD. By Theorems 2.3, there is a numerical constant $C > 0$ and a permutation $\pi : [d_{\min}] \rightarrow [d_{\min}]$ such that

$$|\tilde{\lambda}_{\pi(k)} - \lambda_k| \leq C\|\tilde{\mathcal{F}} - \mathcal{F}\| \leq O_p\left(\frac{\sqrt{d_1 + \cdots + d_p}}{n}\right),$$

and

$$\max_{1 \leq q \leq p} \sin \angle(\tilde{u}_k^{(q)}, \tilde{u}_k^{(q)}) \leq C\|\tilde{\mathcal{F}} - \mathcal{F}\|/\lambda_k = O_p\left(\frac{\sqrt{d_1 + \cdots + d_p}}{n\lambda_k^2}, 1\right),$$

for all $k = 1, \ldots, d_{\min}$. This, in turn, suggests to estimate $\mathcal{F}$ by

$$\tilde{\mathcal{F}} = \sum_{k=1}^{d_{\min}} \hat{\lambda}_k \hat{u}_k^{(1)} \otimes \cdots \otimes \hat{u}_k^{(p)},$$

where $\hat{\lambda}_k = (\tilde{\lambda}_k - \tau_n)_+$ where

$$\tau_n = C_0 \sqrt{\frac{d_1 + \cdots + d_p}{n}}$$
for a sufficiently large constant $C_0 > 0$. Note that in the matrix case, this estimate is also closely related to the nuclear norm regularization. See, e.g., Koltchinskii et al. (2011). In light of (33) and (34), we can derive

$$
\|\widehat{T} - T\|_F^2 = O_p \left( \min \left\{ M \left( \frac{d_1 + \cdots + d_p}{n} \right)^{1-\alpha/2}, d_{\min} \left( \frac{d_1 + \cdots + d_p}{n} \right) \right\} \right),
$$

for any $T \in \Theta(\alpha, M)$. This can be shown to be minimax optimal. To summarize, we have

**Theorem 4.3.** Let $(\mathcal{X}_1, Y_1), \ldots, (\mathcal{X}_n, Y_n)$ be $n$ independent copies of $(\mathcal{X}, Y)$ following the tensor regression model (31) with Gaussian design and coefficient $T \in \Theta(\alpha, M)$ for some $\alpha \in [0, 1)$ and $M > 0$. Let $\widehat{T}$ be defined by (35). Then there exists a constant $C > 0$ such that

$$
E\|\widehat{T} - T\|_F^2 \leq C \cdot \min \left\{ M \left( \frac{d_1 + \cdots + d_p}{n} \right)^{1-\alpha/2}, d_{\min} \left( \frac{d_1 + \cdots + d_p}{n} \right) \right\}.
$$

Conversely, there is a constant $c > 0$ such that

$$
\inf_{\hat{T} \in \Theta(\alpha, M)} \sup_{T} E\|\widehat{T} - T\|_F^2 \geq c \cdot \min \left\{ M \left( \frac{d_1 + \cdots + d_p}{n} \right)^{1-\alpha/2}, d_{\min} \left( \frac{d_1 + \cdots + d_p}{n} \right) \right\},
$$

where the infimum is taken over any estimate $\hat{T}$ based on observations $\{(\mathcal{X}_i, Y_i) : 1 \leq i \leq n\}$.

### 4.3 Independent Component Analysis

Our last example concerns the use of method of moments for latent variable models. It has been shown in a wide range of high dimensional data analysis problems involving latent variables, methods of moments provide an effective strategy for parameter estimation. This includes, among many others, multi-view problem, topic modeling, phylogenetic inferences, and independent component analysis (ICA). See, e.g., Anandkumar et al. (2014a,b); Belkin et al. (2018); Bhaskara et al. (2014). To fix ideas, we shall focus on applying our perturbation bounds to ICA here although similar results can also be derived in the same fashion in other contexts.

The premise of an ICA model is that the coordinates of a random vector $X \in \mathbb{R}^d$ are linear combinations of $d$ independent random variables so that $X = AS$ where $A \in \mathcal{O}(d)$ is a rotation matrix and $S = (S_1, \ldots, S_d)^\top \in \mathbb{R}^d$ is a random vector with independent entries.
For simplicity, we shall follow the convention and assume that the samples are appropriately “whitened” so that we can assume that $\mathbb{E}(\mathbf{S}) = \mathbf{0}$ and $\mathbb{E}(\mathbf{SS}^\top) = I_{d \times d}$. Recall that the fourth order cumulant for a zero-mean and unit-variance $S_k$ is simply its fourth order moment minus 3: $\kappa_4(S_k) = \mathbb{E}S_k^4 - 3$. Therefore the fourth order moment tensor of $\mathbf{X}$ is given by

$$
\mathcal{M}_4(\mathbf{X}) := \mathbb{E}(\mathbf{X} \otimes \mathbf{X} \otimes \mathbf{X} \otimes \mathbf{X}) = \sum_{k=1}^{d} \kappa_4(S_k) \mathbf{a}_k \otimes \mathbf{a}_k \otimes \mathbf{a}_k \otimes \mathbf{a}_k + \mathcal{M}_0,
$$

where $\mathbf{a}_k$ is the $k$th column vector of $\mathbf{A}$ and

$$
\mathcal{M}_0 = \sum_{\{i,j,k,l\} = \{i_1,i_2\}} \sum_{i_1,i_2=1}^{d} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l.
$$

In other words, $\{|\kappa_4(S_k)| : 1 \leq k \leq d\}$ and $(\pm)\mathbf{a}_k$s are the singular values and vectors of fourth order tensor $\mathcal{M}_4(\mathbf{X}) - \mathcal{M}_0$. We can therefore consider estimating $\mathbf{A}$ by first constructing an estimate $\hat{\mathcal{M}}_4$ of $\mathcal{M}_4(\mathbf{X})$, and then estimating its column vectors by the singular vectors of $\hat{\mathcal{M}}_4 - \mathcal{M}_0$.

### 4.3.1 Sample ICA

In particular, after observing $n$ independent copies of $\mathbf{X}$, a natural estimate of $\mathcal{M}_4(\mathbf{X})$ is the corresponding sample moment:

$$
\hat{\mathcal{M}}_4^{\text{sample}}(\mathbf{X}) := \frac{1}{n} \sum_{i=1}^{n} \mathbf{X}_i \otimes \mathbf{X}_i \otimes \mathbf{X}_i \otimes \mathbf{X}_i.
$$

It is known that, under certain regularity conditions, $\hat{\mathcal{M}}_4^{\text{sample}}(\mathbf{X})$ is a consistent estimate of $\mathcal{M}_4(\mathbf{X})$ under the spectral norm, that is,

$$
\|\hat{\mathcal{M}}_4^{\text{sample}}(\mathbf{X}) - \mathcal{M}_4(\mathbf{X})\| \to_p 0,
$$

if and only if $n \gg d^2$. See, e.g., Adamczak et al. (2010); Vershynin et al. (2011) and references therein. In light of Theorem 2.3, this immediately implies that the singular values and vectors of the best odeco approximation to $\hat{\mathcal{M}}_4^{\text{sample}}(\mathbf{X}) - \mathcal{M}_0$ are consistent estimators of $|\kappa_4(S_k)|$s and $\mathbf{a}_k$s. More specifically, let

$$
\hat{\mathcal{M}}_4^{\text{sample,odeco}} = \arg\min_{\mathcal{A} \text{ is odeco}} \|\mathcal{A} - (\hat{\mathcal{M}}_4^{\text{sample}}(\mathbf{X}) - \mathcal{M}_0)\|.
$$

(38)
be the best odeco approximation to \( \hat{\mathcal{M}}_{4,\text{odeco}}^{\text{sample}} - \mathcal{M}_0 \). Indeed, as before, a “good” odeco approximation would also suffice. Write

\[
\hat{\mathcal{M}}_{4,\text{odeco}}^{\text{sample}} = [\{\hat{\lambda}_k^{\text{sample}} : 1 \leq k \leq d\}; \hat{\mathbf{U}}_{(1),\text{sample}}, \ldots, \hat{\mathbf{U}}_{(p),\text{sample}}]
\]

its SVD. Then we have

**Theorem 4.4.** Let \( \varepsilon > 0 \), \( 0 < \delta < 1 \), and \( X_1, \ldots, X_n \) be \( n \) independent copies of \( \mathbf{X} = \mathbf{A} \mathbf{S} \) for a rotation matrix \( \mathbf{A} \in \mathcal{O}(d) \) and a random vector \( \mathbf{S} \) of independent, centered and unit variance components obeying

\[
\|\mathbf{S}\| \leq L_1 \sqrt{d} \quad \text{a.s.,} \quad (\mathbb{E}(\mathbf{S}, \mathbf{x})^8)^{1/8} \leq L_2, \quad \forall \mathbf{x} \in \mathcal{S}^{d-1},
\]

for some constants \( L_1, L_2 > 0 \). Let \( \hat{\lambda}_k^{\text{sample}} \)s and \( \mathbf{u}_{(q),\text{sample}}^{(k)} \)s be defined by (38) and (39). If \( n \geq K d^2 \) for some constant \( K > 0 \) depending only on \( \delta \) and \( \varepsilon \), then there exists a permutation \( \pi : [d] \to [d] \) such that with probability \( 1 - \delta \),

\[
\left| \hat{\lambda}_{\pi(k)}^{\text{sample}} - |\kappa_4(S_k)| \right| \leq \varepsilon,
\]

and

\[
\sin \angle(a_k, \hat{\mathbf{u}}_{\pi(k)}^{(q),\text{sample}}) \leq \frac{\varepsilon}{|\kappa_4(S_k)|},
\]

for any \( 1 \leq k \leq d \) and \( 1 \leq q \leq 4 \).

It is worth noting that \( \mathcal{M}_4(\mathbf{X}) - \mathcal{M}_0 \) is strongly symmetric. We can therefore require that \( \hat{\mathcal{M}}_{4,\text{odeco}}^{\text{sample}} \) be chosen strongly symmetric as well, in which case its singular vectors can also be made the same across different modes, e.g.,

\[
\hat{\mathcal{M}}_{4,\text{odeco}}^{\text{sample}} = [\{\hat{\sigma}_k^{\text{sample}} : 1 \leq k \leq d\}; \hat{\mathbf{U}}^{\text{sample}}, \ldots, \hat{\mathbf{U}}^{\text{sample}}].
\]

In doing so, however, we may need to flip the sign of some “singular values” in the above decomposition. See, e.g., Friedland and Ottaviani (2014); Friedland (2016). Similarly, it is possible to show that

\[
\left| \hat{\sigma}_{\pi(k)}^{\text{sample}} - \kappa_4(S_k) \right| \leq \varepsilon,
\]

and

\[
\sin \angle(a_k, \hat{\mathbf{u}}_{\pi(k)}^{\text{sample}}) \leq \frac{\varepsilon}{|\kappa_4(S_k)|}.
\]
Note that while $\hat{\lambda}_k^{\text{sample}}$ only estimates the absolute value $|\kappa_4(S_k)|$, $\hat{\sigma}_k^{\text{sample}}$ estimates $\kappa_4(S_k)$ itself. We can alternatively derive symmetric estimates from the (asymmetric) odeco approximation 

$$\hat{\mathcal{M}}_4^{\text{sample,odeco}} = \{[\hat{\lambda}_k^{\text{sample}} : 1 \leq k \leq d]; \hat{U}^{(1),\text{sample}}, \ldots, \hat{U}^{(p),\text{sample}}\},$$

using

$$\hat{U}^{\text{sample}} = \frac{1}{p \sum_{q=1}^p \hat{U}^{(q),\text{sample}}}.$$

By Theorem 4.4, there exists a permutation $\pi$ such that $\sin \angle (a_k, \hat{u}_{\pi(k)}^{\text{sample}}) \leq \varepsilon / |\kappa_4(S_k)|$.

### 4.3.2 Minimax Optimal ICA

Theorem 2.3 suggests that consistent estimates of $\kappa_4(S_k)$s and $A$ can be obtained as soon as we can estimate $\mathcal{M}_4(X)$ consistently under the tensor spectral norm. This turns out to be possible with a sample size smaller than that required by Theorem 4.4, and in general under less restrictive assumptions.

The main idea is to first construct a reliable estimate of

$$\theta_u := E[\langle u, X \rangle^4] = \langle \mathcal{M}_4(X), u \otimes u \otimes u \otimes u \rangle$$

for a set of $u$s and then look for an estimate $\hat{\mathcal{M}}_4$ of $\mathcal{M}_4(X)$ so that

$$\langle \hat{\mathcal{M}}_4, u \otimes u \otimes u \otimes u \rangle$$

is close to these estimates.

**Estimating $\theta_u$.** We first discuss how to estimate $\theta_u$ for a fixed $u \in S^{d-1}$. Note that this amounts to estimating the mean of $\langle u, X \rangle^4$ and we shall appeal to a strategy developed by Catoni (2012). More specifically, let $\psi : \mathbb{R} \mapsto \mathbb{R}$ be a non-decreasing influence function such that

$$-\log(1 - x + x^2/2) \leq \psi(x) \leq \log(1 + x + x^2/2).$$

Denote by

$$\psi_\alpha(x) = \psi(\alpha x).$$

See Catoni (2012) for details. For any $u \in S^{d-1}$, denote by $\hat{\theta}_u$ the solution to

$$\sum_{i=1}^n \psi_\alpha \{ (u^\top X_i)^4 - \theta_u \} = 0.$$
Assuming that
\[ \sup_{u \in S^{d-1}} E(u, X)^8 \leq L \]
for some constant \( L > 0 \), as shown by Catoni (2012), for any \( t > \exp(-n/2) \),
\[ |\hat{\theta}_u - E(u^\top X_i)^4| \leq \sqrt{\frac{2L \log(t^{-1})}{n(1 - 2 \log(t^{-1})/n)}}, \]
with probability at least \( 1 - 2t \).

**Estimating \( \mathcal{M}_4(X) \).** Now let \( \mathcal{N} \) be a 1/4 covering set of \( S^{d-1} \) with \( |\mathcal{N}| \leq 9^d \). We then estimate \( \mathcal{M}_4(X) \) by the solution, denoted by \( \hat{\mathcal{M}}_4 \) hereafter, to
\[
\min_{\mathcal{A} : \mathcal{A} = \mathcal{M}_0 \text{ is odeco}} \max_{u \in \mathcal{N}} |\hat{\theta}_u - \langle \mathcal{A}, u \otimes u \otimes u \otimes u \rangle|.
\]
By union bound, with probability at least \( 1 - 2(9/e^3)^d \),
\[ \max_{u \in \mathcal{N}} |\hat{\theta}_u - \langle \mathcal{M}_4(X), u \otimes u \otimes u \otimes u \rangle| \leq \sqrt{\frac{6Ld}{n(1 - 6d/n)}}, \]
Under this event, we have
\[ \|\hat{\mathcal{M}}_4(X) - \mathcal{M}_4(X)\| = O_p \left( \sqrt{\frac{d}{n}} \right). \]
More precisely, we have

**Theorem 4.5.** Suppose that \( X_1, \ldots, X_n \) are \( n \) independent copies of a random vector \( X \) such that \( n > L_1 d \) and \( \sup_{u \in S^{d-1}} E(u, X)^8 \leq L_2 \) for some constants \( L_1 > 4 \) and \( L_2 > 0 \). Let \( \hat{\mathcal{M}}_4 \) be a solution to (40). Then
\[ \|\hat{\mathcal{M}}_4 - \mathcal{M}_4(X)\| = O_p \left( \sqrt{\frac{d}{n}} \right). \]

In light of Theorem 4.5, \( \hat{\mathcal{M}}_4 - \mathcal{M}_0 \) and \( \mathcal{M}_4 - \mathcal{M}_0 \) are two odeco tensors \( O_p(\sqrt{d/n}) \)-far from each other. We can then estimate \( \mathbf{A} \) by the singular vectors of \( \hat{\mathcal{M}}_4 - \mathcal{M}_0 \). As before, we can also replace \( \hat{\mathcal{M}}_4 - \mathcal{M}_0 \) by any “good” odeco approximation to \( \mathcal{M}_4 - \mathcal{M}_0 \). Denote by \( \hat{\mathcal{M}}_{\text{odeco}} \) its SVD. Immediately following Theorems 2.3 and 4.5, we get
Theorem 4.6. Suppose that $X_1, \ldots, X_n$ are $n$ independent copies of $X = AS$ for a rotation matrix $A \in O(d)$ and a random vector $S$ of independent, centered and unit variance components obeying
\[ \mathbb{E}(S, x)^8 \leq L, \quad \forall x \in S^{d-1}, \]
for some constant $L > 0$. Let $\hat{\lambda}_k$ s and $\hat{\mu}_k^{(q)}$ s be defined by (40) and (41). Then there exists a permutation $\pi : [d] \rightarrow [d]$ such that
\[ |\kappa_4(S_k)| - \hat{\lambda}_{\pi(k)}| = O_p \left( \sqrt{\frac{d}{n}} \right) \]
and
\[ \sin \angle(\hat{\mu}_k^{(q)} \pi(k), a_k) = O_p \left( \min \left\{ \sqrt{\frac{d}{n \kappa_4^2(S_k)}}, 1 \right\} \right), \]
for all $1 \leq k \leq d$ and $1 \leq q \leq 4$.

Note that as for sample ICA, we could impose symmetry for $\hat{M}_4$ and $\hat{M}_\text{deco}$ so that we can estimate $\kappa_4(S_k)$ s, not just their absolute values, from the spectral decomposition of $\hat{M}_\text{deco}$. We shall omit the details for brevity.

In addition, the next result shows that the convergence rates given by Theorem 4.6 is minimax optimal and no other estimates of the mixing matrix $A$ could achieve a faster rate of convergence.

Theorem 4.7. For a given rotation matrix $A \in O(d)$ and constant $K > 1$, denote by $\mathcal{P}_\text{ICA}(A, K)$ the collection of all probability laws in $\mathbb{R}^d$ so that a random vector $X \sim P \in \mathcal{P}_\text{ICA}(A, K)$ can be expressed as $X = AS$ where $S$ is a random vector with independent, centered and unit variance components obeying
\[ K^{-1} \leq |\kappa_4(S_k)| \leq K, \quad \forall k = 1, \ldots, d. \]

Then there exits a constant $c > 0$ such that
\[ \inf_{\hat{A}} \sup_{P \in \mathcal{P}_\text{ICA}(A, K)} \mathbb{E} \max_{1 \leq k \leq d} \sin \angle(\hat{a}_k, a_k) \geq c \sqrt{\frac{d}{n}}, \]
where the infimum is taken over all estimators of $A$ based on observing $n$ independent copies $X_1, \ldots, X_n$ of $X$. 

26
5 Proofs

Proof of Proposition 2.2. For brevity, we shall assume that $\lambda = 1$. For any $x^{(1)} \in S^{d-1}$, there exists $c = (c_1, \ldots, c_d)^T \in S^{d-1}$ such that

$$x^{(1)} = c_1 u^{(1)}_1 + \ldots + c_d u^{(1)}_d.$$ 

Therefore

$$\min_{x^{(1)} \in S^{d-1}} \max_{x^{(2)}, \ldots, x^{(p)} \in S^{d-1}} \langle \mathcal{T}, x^{(1)} \otimes \cdots \otimes x^{(p)} \rangle = \min_{c \in S^{d-1}} \max_{x^{(2)}, \ldots, x^{(p)} \in S^{d-1}} \langle \mathcal{T}, (c_1 u^{(1)}_1 + \ldots + c_d u^{(1)}_d) \otimes x^{(2)} \otimes \cdots \otimes x^{(p)} \rangle$$

$$= \min_{c \in S^{d-1}} \max_{x^{(2)}, \ldots, x^{(p)} \in S^{d-1}} \left( \sum_{k=1}^d c_k u^{(2)}_k \otimes \cdots \otimes u^{(p)}_k, x^{(2)} \otimes \cdots \otimes x^{(p)} \right)$$

$$= \min_{c \in S^{d-1}} \left\| \sum_{k=1}^d c_k u^{(2)}_k \otimes \cdots \otimes u^{(p)}_k \right\|.$$ 

Note that since

$$\sum_{k=1}^d c_k u^{(2)}_k \otimes \cdots \otimes u^{(p)}_k$$

is a $(p-1)$th order deco tensor, we get

$$\left\| \sum_{k=1}^d c_k u^{(2)}_k \otimes \cdots \otimes u^{(p)}_k \right\| = \max_{1 \leq k \leq d} c_k,$$

so that

$$\min_{x^{(1)} \in S^{d-1}} \max_{x^{(2)}, \ldots, x^{(p)} \in S^{d-1}} \langle \mathcal{T}, x^{(1)} \otimes \cdots \otimes x^{(p)} \rangle = \min_{c \in S^{d-1}} \max_{1 \leq k \leq d} c_k = \frac{1}{\sqrt{d}}.$$ 

The proof is now completed. \qed

Proof of Theorem 2.3. In fact, Theorem 2.3 follows from Theorem 2.4. We shall now describe how we may proceed to prove Theorem 2.3 in light of Theorem 2.4. Indeed, note that Theorem 2.4 already shows that (13) and (14) hold for any $\lambda_k \geq c_\epsilon \| \mathcal{T} - \mathcal{F} \|$ with an appropriate choice of constant $C = 1 + \epsilon$. When $\lambda_k < c_\epsilon \| \mathcal{T} - \mathcal{F} \|$, then for any $k' \in [d]$,

$$\max_{1 \leq q \leq p} \sin \angle(u^{(q)}_k, \tilde{u}^{(q)}_{k'}) \leq 1 \leq \frac{c_\epsilon (1 + \epsilon) \| \mathcal{T} - \mathcal{F} \|}{\lambda_k},$$

so that (14) holds with $C = c_\epsilon (1 + \epsilon)$.
A careful inspection of the proof to Theorem 2.4 also shows that (13) holds with $C = 1$ not only for any $\lambda_k \geq c_\varepsilon \| \mathcal{T} - \tilde{T} \|$ but also for any $\tilde{\lambda}_k \geq c_\varepsilon \| \mathcal{T} - \tilde{T} \|$. On the other hand, for any $\lambda_k < c_\varepsilon \| \mathcal{T} - \tilde{T} \|$ and $\lambda_{k'} < c_\varepsilon \| \mathcal{T} - \tilde{T} \|$, we must have

$$|\lambda_k - \tilde{\lambda}_k| < c_\varepsilon \| \mathcal{T} - \tilde{T} \|,$$

which again suggests that (13) holds with $C = c_\varepsilon$.

\[ \tag{13} \]

**Proof of Theorem 2.4.** For brevity, we shall assume that $d_1 = \cdots = d_p =: d$. The proof proceeds by induction. To this end, we first consider the basic case when $k = 1$.

**Basic case.** Recall that

$$\lambda_1 = \max_{\mathbf{a}(q) \in S^{d-1}, q=1,\ldots,p} \langle \mathcal{T}, \mathbf{a}(1) \otimes \cdots \otimes \mathbf{a}(p) \rangle$$

and

$$\tilde{\lambda}_1 = \max_{\mathbf{a}(q) \in S^{d-1}, q=1,\ldots,p} \langle \tilde{T}, \mathbf{a}(1) \otimes \cdots \otimes \mathbf{a}(p) \rangle$$

We consider separately the cases when $\tilde{\lambda}_1 \leq \lambda_1$ and $\tilde{\lambda}_1 > \lambda_1$.

**Basic case (a): $\tilde{\lambda}_1 \leq \lambda_1$.** Observe that

$$\lambda_1 = \langle \mathcal{T}, \mathbf{u}_1^{(1)} \otimes \cdots \otimes \mathbf{u}_1^{(p)} \rangle$$

$$\leq \langle \tilde{T}, \mathbf{u}_1^{(1)} \otimes \cdots \otimes \mathbf{u}_1^{(p)} \rangle + \| \tilde{T} - \mathcal{T} \|$$

$$= \sum_{k=1}^d \tilde{\lambda}_k \prod_{q=1}^p \langle \mathbf{u}_1^{(q)}, \tilde{u}_k^{(q)} \rangle + \| \tilde{T} - \mathcal{T} \|.$$ 

The first term can be further bounded by

$$\sum_{k=1}^d \tilde{\lambda}_k \prod_{q=1}^p \langle \mathbf{u}_1^{(q)}, \tilde{u}_k^{(q)} \rangle$$

$$\leq \max_{1 \leq k \leq d} \left\{ \tilde{\lambda}_k \prod_{q=1}^p \| \mathbf{u}_1^{(q)} \otimes \tilde{u}_k^{(q)} \|^{(p-2)/p} \right\} \times \left( \sum_{k=1}^d \prod_{q=1}^p \| \mathbf{u}_1^{(q)} \otimes \tilde{u}_k^{(q)} \|^2/p \right)^{1/p}$$

$$\leq \max_{1 \leq k \leq d} \left\{ \tilde{\lambda}_k \prod_{q=1}^p \| \mathbf{u}_1^{(q)} \otimes \tilde{u}_k^{(q)} \|^{(p-2)/p} \right\} \times \left( \prod_{q=1}^p \left( \sum_{k=1}^d \| \mathbf{u}_1^{(q)} \otimes \tilde{u}_k^{(q)} \|^2 \right) \right)^{1/p}$$

$$= \max_{1 \leq k \leq d} \left\{ \tilde{\lambda}_k \prod_{q=1}^p \| \mathbf{u}_1^{(q)} \otimes \tilde{u}_k^{(q)} \|^{(p-2)/p} \right\},$$

28
where the second inequality follows from Holder’s inequality. Denote by $\pi(1)$ the index that maximizes the rightmost hand side. When there are more than one maximizers, we take $\pi(1)$ to be an arbitrary maximizing index. Then

$$\lambda_1 \leq \tilde{\lambda}_{\pi(1)} + \|\tilde{T} - T\|,$$

which, together with the fact that $\tilde{\lambda}_{\pi(1)} \leq \tilde{\lambda}_1 \leq \lambda_1$, implies that

$$|\lambda_1 - \tilde{\lambda}_{\pi(1)}| \leq \|\tilde{T} - T\|.$$

In addition,

$$\lambda_1 \leq \tilde{\lambda}_{\pi(1)} \prod_{q=1}^{p} \|\langle u_{1}^{(q)}, \tilde{u}_{\pi(1)}^{(q)} \rangle \|^{(p-2)/p} + \|\tilde{T} - T\| \leq \lambda_1 \prod_{q=1}^{p} \|\langle u_{1}^{(q)}, \tilde{u}_{\pi(1)}^{(q)} \rangle \|^{(p-2)/p} + \|\tilde{T} - T\|.$$

Thus,

$$\|\langle u_{1}^{(q)}, \tilde{u}_{\pi(1)}^{(q)} \rangle \| \geq (1 - \lambda_1^{-1} \|\tilde{T} - T\|)^{p/(p-2)}$$

for all $q = 1, \ldots, p$. Now recall that

$$\sin^2 \angle(u_{1}^{(q)}, \tilde{u}_{\pi(1)}^{(q)}) = \langle u_{1}^{(q)} - \langle u_{1}^{(q)}, \tilde{u}_{\pi(1)}^{(q)} \rangle \tilde{u}_{\pi(1)}^{(q)}, u_{1}^{(q)} \rangle = \sum_{k \neq \pi(1)} \langle u_{1}^{(q)}, \tilde{u}_{k}^{(q)} \rangle^2.$$  

We get

$$\max_{1 \leq q \leq p} \sin^2 \angle(u_{1}^{(q)}, \tilde{u}_{\pi(1)}^{(q)}) \leq 1 - (1 - \lambda_1^{-1} \|\tilde{T} - T\|)^{2p/(p-2)}.$$  

We shall now use this to derive a sharper bound for the lefthand side.

Note that

$$T(I, u_{1}^{(2)}, \ldots, u_{1}^{(p)}) = \lambda_1 u_{1}^{(1)}.$$

We get

$$\lambda_1 \sin^2 \angle(u_{1}^{(1)}, \tilde{u}_{\pi(1)}^{(1)}) = T(u_{1}^{(1)} - \langle u_{1}^{(1)}, \tilde{u}_{\pi(1)}^{(1)} \rangle \tilde{u}_{\pi(1)}^{(1)}, u_{1}^{(2)}, \ldots, u_{1}^{(p)}) \leq \tilde{T}(u_{1}^{(1)} - \langle u_{1}^{(1)}, \tilde{u}_{\pi(1)}^{(1)} \rangle \tilde{u}_{\pi(1)}^{(1)}, u_{1}^{(2)}, \ldots, u_{1}^{(p)}) + \|\tilde{T} - T\| \cdot \sin \angle(u_{1}^{(1)}, \tilde{u}_{\pi(1)}^{(1)}).$$

29
The first term on the rightmost hand side can be further bounded by

\[
\mathcal{F}(u_1^{(1)} - \langle u_1^{(1)}, \tilde{u}_{\pi(1)}^{(1)} \rangle \tilde{u}_{\pi(1)}^{(1)}, u_1^{(2)}, \ldots, u_1^{(p)})
\]

\[
= \sum_{k \neq \pi(1)} \lambda_k \prod_{q=1}^p \langle u_1^{(q)}, \tilde{u}_k^{(q)} \rangle
\]

\[
\leq \lambda_1 \sum_{k \neq \pi(1)} \prod_{q=1}^p |\langle u_1^{(q)}, \tilde{u}_k^{(q)} \rangle|
\]

\[
\leq \lambda_1 \sin \angle(u_1^{(1)}, \tilde{u}_{\pi(1)}^{(1)}) \cdot \left( \sum_{k \neq \pi(1)} \prod_{q=2}^p |\langle u_1^{(q)}, \tilde{u}_k^{(q)} \rangle|^2 \right)^{1/2}
\]

\[
\leq \lambda_1 \prod_{q=1}^p \sin \angle(u_1^{(q)}, \tilde{u}_{\pi(1)}^{(q)}),
\]

where the second inequality follows from Cauchy-Schwarz inequality. This gives

\[
\lambda_1 \sin^2 \angle(u_1^{(1)}, \tilde{u}_{\pi(1)}^{(1)}) \leq \lambda_1 \prod_{q=1}^p \sin \angle(u_1^{(q)}, \tilde{u}_{\pi(1)}^{(q)}) + \|\mathcal{F} - \mathcal{F}\| \cdot \sin \angle(u_1^{(1)}, \tilde{u}_{\pi(1)}^{(1)}).
\]

Similarly, we can derive

\[
\lambda_1 \sin^2 \angle(u_1^{(q)}, \tilde{u}_{\pi(1)}^{(q)}) \leq \lambda_1 \prod_{j=1}^p \sin \angle(u_1^{(j)}, \tilde{u}_{\pi(1)}^{(j)}) + \|\mathcal{F} - \mathcal{F}\| \cdot \sin \angle(u_1^{(q)}, \tilde{u}_{\pi(1)}^{(q)}),
\]

for \(q = 2, \ldots, p\). Together we have

\[
\lambda_1 \max_{1 \leq q \leq p} \sin^2 \angle(u_1^{(q)}, \tilde{u}_{\pi(1)}^{(q)})
\]

\[
\leq \lambda_1 \prod_{q=1}^p \sin \angle(u_1^{(q)}, \tilde{u}_{\pi(1)}^{(q)}) + \|\mathcal{F} - \mathcal{F}\| \cdot \max_{1 \leq q \leq p} \sin \angle(u_1^{(q)}, \tilde{u}_{\pi(1)}^{(q)})
\]

\[
\leq \lambda_1 \left( \max_{1 \leq q \leq p} \sin \angle(u_1^{(q)}, \tilde{u}_{\pi(1)}^{(q)}) \right)^p + \|\mathcal{F} - \mathcal{F}\| \cdot \max_{1 \leq q \leq p} \sin \angle(u_1^{(q)}, \tilde{u}_{\pi(1)}^{(q)}).
\]

In light of (42), the first term on the rightmost hand side can be bounded by

\[
\lambda_1 \left( \max_{1 \leq q \leq p} \sin \angle(u_1^{(q)}, \tilde{u}_{\pi(1)}^{(q)}) \right)^p \leq \lambda_1 [1 - (1 - \lambda_1^{-1} \|\mathcal{F} - \mathcal{F}\|^{2p/(p-2)} \|\mathcal{F} - \mathcal{F}\|^{2p/(p-2)} \|\mathcal{F} - \mathcal{F}\|^{2p/(p-2)} \|\mathcal{F} - \mathcal{F}\|^{2p/(p-2)})^{-1} \cdot \|\mathcal{F} - \mathcal{F}\| \cdot \lambda_1.
\]

Thus, rearranging terms in the above expression gives

\[
\max_{1 \leq q \leq p} \sin \angle(u_1^{(q)}, \tilde{u}_{\pi(1)}^{(q)}) \leq \left( 1 - [1 - (1 - \lambda_1^{-1} \|\mathcal{F} - \mathcal{F}\|^{2p/(p-2)} \|\mathcal{F} - \mathcal{F}\|^{2p/(p-2)} \|\mathcal{F} - \mathcal{F}\|^{2p/(p-2)} \|\mathcal{F} - \mathcal{F}\|^{2p/(p-2)})^{-1} \cdot \|\mathcal{F} - \mathcal{F}\| \cdot \lambda_1.
\]

Note that the function

\[
h_1(x) = \left( 1 - [1 - (1 - x)^{2p/(p-2)} \|\mathcal{F} - \mathcal{F}\|^{2p/(p-2)} \|\mathcal{F} - \mathcal{F}\|^{2p/(p-2)} \|\mathcal{F} - \mathcal{F}\|^{2p/(p-2)} \|\mathcal{F} - \mathcal{F}\|^{2p/(p-2)})^{-1}
\]

is unbounded.
is monotonically increasing and continuously differentiable at 0 with \( h_1(0) = 1 \). We get
\[
\max_{1 \leq q \leq p} \sin \angle(u_1^{(q)}, \bar{u}_{\pi(1)}^{(q)}) \leq \frac{(1 + \varepsilon)\| \bar{F} - \mathcal{F} \|}{\lambda_1},
\]
provided that \( \| \bar{F} - \mathcal{F} \| \leq c_{\varepsilon, p}\lambda_1 \) for a sufficiently small constant \( c_{\varepsilon, p} > 0 \).

**Basic case (b):** \( \tilde{\lambda}_1 > \lambda_1 \). Next consider the case when \( \lambda_1 < \tilde{\lambda}_1 \). As in the previous case, we can derive that
\[
\lambda_1 \leq \max_{1 \leq k \leq d} \left\{ \tilde{\lambda}_k \prod_{q=1}^{p} \| \langle u_1^{(q)}, \bar{u}_k^{(q)} \rangle \|^{(p-2)/p} \right\} + \| \bar{F} - \mathcal{F} \| = \tilde{\lambda}_{\pi(1)} \prod_{q=1}^{p} \| \langle u_1^{(q)}, \bar{u}_{\pi(1)}^{(q)} \rangle \|^{(p-2)/p} + \| \bar{F} - \mathcal{F} \|.
\]
On the other hand,
\[
\tilde{\lambda}_{\pi(1)} \leq \tilde{\lambda}_1 \leq \lambda_1 + \| \mathcal{F} - \bar{F} \|,
\]
where the second inequality follows from triangular inequality. Therefore
\[
\tilde{\lambda}_{\pi(1)} \left( 1 - \prod_{q=1}^{p} \| \langle u_1^{(q)}, \bar{u}_{\pi(1)}^{(q)} \rangle \|^{(p-2)/p} \right) \leq 2\| \bar{F} - \mathcal{F} \|,
\]
leading to
\[
\max_{1 \leq q \leq p} \sin^2 \angle(u_1^{(q)}, \bar{u}_{\pi(1)}^{(q)}) \leq 1 - \left( 1 - 2\lambda_1^{-1}\| \bar{F} - \mathcal{F} \| \right)^{2p/(p-2)}.
\]
Now following an identical argument as in the previous case, we can get
\[
\lambda_1 \max_{1 \leq q \leq p} \sin^2 \angle(u_1^{(q)}, \bar{u}_{\pi(1)}^{(q)}) \leq (\lambda_1 + \| \bar{F} - \mathcal{F} \|) \left[ 1 - \left( 1 - 2\lambda_1^{-1}\| \bar{F} - \mathcal{F} \| \right)^{2p/(p-2)} \right]^{\frac{p-2}{2}} \times \\
\times \max_{1 \leq q \leq p} \sin^2 \angle(u_1^{(q)}, \bar{u}_{\pi(1)}^{(q)}) + \| \bar{F} - \mathcal{F} \| \cdot \max_{1 \leq q \leq p} \sin \angle(u_1^{(q)}, \bar{u}_{\pi(1)}^{(q)}),
\]
leading to
\[
\max_{1 \leq q \leq p} \sin \angle(u_1^{(q)}, \bar{u}_{\pi(1)}^{(q)}) \leq \left( 1 - (1 + \lambda_1^{-1}\| \bar{F} - \mathcal{F} \|) \left[ 1 - \left( 1 - 2\lambda_1^{-1}\| \bar{F} - \mathcal{F} \| \right)^{2p/(p-2)} \right]^{\frac{p-2}{2}} \right) \times \frac{\| \bar{F} - \mathcal{F} \|}{\lambda_1}.
\]
Note that the function
\[ h_2(x) = \left[ 1 - (1 + x) \left[ 1 - \left( \frac{1 - 2x}{1 + 2x} \right)^{2p/(p-2)} \right]^{2/2} \right]^{-1} \]
is continuously differentiable at 0 with \( h_2(0) = 1, h'_2(0) > 0 \). We get
\[
\max_{1 \leq q \leq p} \sin \angle(u^{(q)}_i, \tilde{u}^{(q)}_{\pi(i)}) \leq \frac{(1 + \varepsilon)\|\tilde{\mathcal{F}} - \mathcal{F}\|}{\lambda_1}
\]
provided that \( \|\tilde{\mathcal{F}} - \mathcal{F}\| \leq c_{\varepsilon,p} \lambda_1 \) for a sufficiently small constant \( c_{\varepsilon,p} > 0 \).

Induction. Next we treat the more general case by induction. To this end, assume that
there exists an injective map \( \pi : [l] \to [d] \) such that for all \( k \leq l(< r) \),
\[
|\lambda_k - \tilde{\lambda}_{\pi(k)}| \leq \|\tilde{\mathcal{F}} - \mathcal{F}\| \tag{43}
\]
and
\[
\max_{1 \leq q \leq p} \sin \angle(u^{(q)}_k, \tilde{u}^{(q)}_{\pi(k)}) \leq \frac{(1 + \varepsilon)\|\tilde{\mathcal{F}} - \mathcal{F}\|}{\lambda_k} \tag{44}
\]
we shall now argue they continue to hold for \( k = l + 1 \).

Induction (a): \( \lambda_{l+1} \geq \max_{k \notin \pi([l])} \tilde{\lambda}_k \). Consider first the case that \( \lambda_{l+1} \geq \max_{k \notin \pi([l])} \tilde{\lambda}_k \). Similar to before,
\[
\lambda_{l+1} = \langle \mathcal{F}, u^{(1)}_{l+1} \otimes \cdots \otimes u^{(p)}_{l+1} \rangle \\
\leq \langle \tilde{\mathcal{F}}, u^{(1)}_{l+1} \otimes \cdots \otimes u^{(p)}_{l+1} \rangle + \|\tilde{\mathcal{F}} - \mathcal{F}\| \\
\leq \max_{1 \leq k \leq d} \left\{ \tilde{\lambda}_k \prod_{q=1}^p \|u^{(q)}_{l+1}, \tilde{u}^{(q)}_{\pi(k)}\|^{{(p-2)/p}} \right\} + \|\tilde{\mathcal{F}} - \mathcal{F}\|.
\]
We first argue that the index maximizing the rightmost hand side is not from \( \pi([l]) \). To this end, note that by the induction hypothesis, for any \( k \in [l] \),
\[
\tilde{\lambda}_{\pi(k)} \prod_{q=1}^p \|u^{(q)}_{l+1}, \tilde{u}^{(q)}_{\pi(k)}\|^{(p-2)/p} \leq (\lambda_k + \|\tilde{\mathcal{F}} - \mathcal{F}\|) \left( \max_{1 \leq q \leq p} \sin \angle(u^{(q)}_k, \tilde{u}^{(q)}_{\pi(k)}) \right)^{p-2} \\
\leq (\lambda_k + \|\tilde{\mathcal{F}} - \mathcal{F}\|) \left( (1 + \varepsilon)\|\tilde{\mathcal{F}} - \mathcal{F}\|/\lambda_k \right) \\
\leq (1 + \varepsilon)(1 + c_{\varepsilon,p})\|\tilde{\mathcal{F}} - \mathcal{F}\|.
\]
Therefore,

\[
\max_{1 \leq k \leq l} \left\{ \tilde{\lambda}_{\pi(k)} \prod_{q=1}^{p} \left| \langle \mathbf{u}_{l+1}, \tilde{\mathbf{u}}_{\pi(k)}^{(q)} \rangle \right|^{(p-2)/p} \right\} + \| \tilde{T} - T \|
\]

\[
\leq [1 + (1 + \varepsilon)(1 + c_{\varepsilon,p})] \| \tilde{T} - T \|
\]

\[
\leq c_{\varepsilon,p}[1 + (1 + \varepsilon)(1 + c_{\varepsilon,p})] \lambda_{l+1} < \lambda_{l+1},
\]

by taking \( c_{\varepsilon,p} > 0 \) small enough. Thus the index, hereafter denoted by \( \pi(l+1) \), that maximizes

\[
\tilde{\lambda}_{k} \prod_{q=1}^{p} \left| \langle \mathbf{u}_{l+1}, \tilde{\mathbf{u}}_{k}^{(q)} \rangle \right|^{(p-2)/p}
\]

must be different from \( \{ \pi(1), \ldots, \pi(l) \} \). In addition, because

\[
\tilde{\lambda}_{\pi(l+1)} \leq \lambda_{l+1} \leq \tilde{\lambda}_{\pi(l+1)} \prod_{q=1}^{p} \left| \langle \mathbf{u}_{l+1}, \tilde{\mathbf{u}}_{\pi(l+1)}^{(q)} \rangle \right|^{(p-2)/p} + \| \tilde{T} - T \| \leq \tilde{\lambda}_{\pi(l+1)} + \| \tilde{T} - T \|,
\]

we immediately deduce that

\[
|\tilde{\lambda}_{\pi(l+1)} - \lambda_{l+1}| \leq \| \tilde{T} - T \|,
\]

and

\[
|\langle \mathbf{u}_{l+1}, \tilde{\mathbf{u}}_{\pi(l+1)}^{(1)} \rangle| \geq \left( 1 - \lambda_{l+1}^{-1} \| \tilde{T} - T \| \right)^{p/(p-2)}.
\]

(45)

Similar to before, we can derive

\[
\lambda_{l+1} \sin^2 \angle(\mathbf{u}_{l+1}, \tilde{\mathbf{u}}_{\pi(l+1)}^{(1)}) = \tilde{T}(\mathbf{u}_{l+1}^{(1)} - \langle \mathbf{u}_{l+1}^{(1)}, \tilde{\mathbf{u}}_{\pi(l+1)}^{(1)} \rangle \tilde{\mathbf{u}}_{\pi(l+1)}^{(1)}, \mathbf{u}_{l+1}^{(2)}, \ldots, \mathbf{u}_{l+1}^{(p)}) \leq \tilde{T}(\mathbf{u}_{l+1}^{(1)} - \langle \mathbf{u}_{l+1}^{(1)}, \tilde{\mathbf{u}}_{\pi(l+1)}^{(1)} \rangle \tilde{\mathbf{u}}_{\pi(l+1)}^{(1)}, \mathbf{u}_{l+1}^{(2)}, \ldots, \mathbf{u}_{l+1}^{(p)}) + \| \tilde{T} - T \| \sin \angle(\mathbf{u}_{l+1}^{(1)}, \tilde{\mathbf{u}}_{\pi(l+1)}^{(1)}).
\]

Moreover, because \( \lambda_{l+1} \geq \max_{k \notin \pi(l+1)} \tilde{\lambda}_{k} \), we get

\[
\tilde{T}(\mathbf{u}_{l+1}^{(1)} - \langle \mathbf{u}_{l+1}^{(1)}, \tilde{\mathbf{u}}_{\pi(l+1)}^{(1)} \rangle \tilde{\mathbf{u}}_{\pi(l+1)}^{(1)}, \mathbf{u}_{l+1}^{(2)}, \ldots, \mathbf{u}_{l+1}^{(p)})
\]

\[
= \sum_{k \neq \pi(l+1)} \tilde{\lambda}_{k} \prod_{q=1}^{p} \left| \langle \mathbf{u}_{l+1}, \tilde{\mathbf{u}}_{k}^{(q)} \rangle \right|
\]

\[
\leq \sum_{k=1}^{l} \tilde{\lambda}_{\pi(k)} \prod_{q=1}^{p} \left| \langle \mathbf{u}_{l+1}, \tilde{\mathbf{u}}_{\pi(k)}^{(q)} \rangle \right| + \lambda_{l+1} \sum_{k \notin \pi([l+1])} \prod_{q=1}^{p} \left| \langle \mathbf{u}_{l+1}, \tilde{\mathbf{u}}_{k}^{(q)} \rangle \right|
\]

\[
\leq \sum_{k=1}^{l} \tilde{\lambda}_{\pi(k)} \prod_{q=1}^{p} \left| \langle \mathbf{u}_{l+1}, \tilde{\mathbf{u}}_{\pi(k)}^{(q)} \rangle \right| + \lambda_{l+1} \left( \max_{1 \leq q \leq p} \angle(\mathbf{u}_{l+1}, \tilde{\mathbf{u}}_{\pi(l+1)}^{(q)}) \right)^{p}.
\]
The first term on the rightmost hand side can be bounded by

\[
\sum_{k=1}^{l} \tilde{\lambda}_{\pi(k)} \prod_{q=1}^{p} \langle \mathbf{u}_{i+1}^{(q)} , \tilde{\mathbf{u}}_{\pi(k)}^{(q)} \rangle
\]

\[
\leq \max_{1 \leq k \leq l} \{ \tilde{\lambda}_{\pi(k)} \sin \angle(u_{k}^{(1)} , \tilde{u}_{\pi(k)}^{(1)}) \} \left( \max_{1 \leq q \leq p} \sin \angle(u_{i+1}^{(q)} , \tilde{u}_{\pi(i+1)}^{(q)}) \right)^{p-1}
\]

\[
\leq \max_{1 \leq k \leq l} \{ (\lambda_{k} + \| \tilde{T} - T \|) \sin \angle(u_{k}^{(1)} , \tilde{u}_{\pi(k)}^{(1)}) \} \left( \max_{1 \leq q \leq p} \sin \angle(u_{i+1}^{(q)} , \tilde{u}_{\pi(i+1)}^{(q)}) \right)^{p-1}
\]

\[
\leq (1 + c_{\varepsilon,p})(1 + \varepsilon)\| \tilde{T} - T \| \left( \max_{1 \leq q \leq p} \sin \angle(u_{i+1}^{(q)} , \tilde{u}_{\pi(i+1)}^{(q)}) \right)^{p-1}
\]

On the other hand, as before, we can derive from (45) that

\[
\max_{1 \leq q \leq p} \sin^{2} \angle(u_{i+1}^{(q)} , \tilde{u}_{\pi(i+1)}^{(q)}) \leq 1 - \left( 1 - \frac{\| \tilde{T} - T \|}{\lambda_{i+1}} \right)^{2p/(p-2)}
\]

so that the second term can be bounded by

\[
\lambda_{i+1} \left( \max_{1 \leq q \leq p} \sin \angle(u_{i+1}^{(q)} , \tilde{u}_{\pi(i+1)}^{(q)}) \right)^{p}
\]

\[
\leq \lambda_{i+1} \left[ 1 - \left( 1 - \frac{\| \tilde{T} - T \|}{\lambda_{i+1}} \right)^{2p/(p-2)} \right]^{\frac{p-2}{2}} \max_{1 \leq q \leq p} \sin^{2} \angle(u_{i+1}^{(q)} , \tilde{u}_{\pi(i+1)}^{(q)})
\]

Denote by

\[
h_{3}(x; \varepsilon, p) = \left[ 1 - (1 - x)^{2p/(p-2)} \right]^{\frac{p-2}{2}} + (1 + c_{\varepsilon,p})(1 + \varepsilon)x.
\]

Then

\[
\lambda_{i+1} \sin^{2} \angle(u_{i+1}^{(1)} , \tilde{u}_{\pi(i+1)}^{(1)})
\]

\[
\leq \lambda_{i+1} h_{3} \left( \frac{\| \tilde{T} - T \|}{\lambda_{i+1}} ; \varepsilon, p \right) \max_{1 \leq q \leq p} \sin^{2} \angle(u_{i}^{(q)} , \tilde{u}_{\pi(i)}^{(q)}) + \| \tilde{T} - T \| \cdot \max_{1 \leq q \leq p} \sin \angle(u_{i}^{(q)} , \tilde{u}_{\pi(i)}^{(q)}),
\]

implying

\[
\sin \angle(u_{i+1}^{(1)} , \tilde{u}_{\pi(i+1)}^{(1)}) \leq \frac{1}{1 - h_{3} \left( \frac{\| \tilde{T} - T \|}{\lambda_{i+1}} ; \varepsilon, p \right) \cdot \frac{\| \tilde{T} - T \|}{\lambda_{i+1}}} \cdot \frac{1}{\lambda_{i+1}}.
\]

Observe that \( h_{3} \) is a continuous and increasing function of \( x \) and \( h_{3}(0) = 0 \). The desired claim then follows by taking \( c_{\varepsilon,p} > 0 \) small enough.
Induction (b): $\lambda_{l+1} < \max_{k \notin \pi([l])} \tilde{\lambda}_k$. Now consider the case when $\lambda_{l+1} < \max_{k \notin \pi([l])} \tilde{\lambda}_k$.

Write $\tilde{U}_l^{(1)} = (\tilde{u}_{\pi(1)}^{(1)}, \ldots, \tilde{u}_l^{(1)})$. Then

$$\max_{k \notin \pi([l])} \tilde{\lambda}_k = \max_{\tilde{U}_l^{(1)} \in S^{d-1}, \tilde{a}^{(1)} \in S^{p}} \langle \tilde{\mathcal{T}}, \tilde{a}^{(1)} \otimes \cdots \otimes \tilde{a}^{(p)} \rangle$$

Observe that

$$\max_{\tilde{U}_l^{(1)} \in S^{d-1}, \tilde{a}^{(1)} \in S^{p}} \langle \tilde{\mathcal{T}}, \tilde{a}^{(1)} \otimes \cdots \otimes \tilde{a}^{(p)} \rangle = \max_{\tilde{a}^{(1)} \in S^{p}} \langle \mathcal{T}, \tilde{a}^{(1)} \otimes \cdots \otimes \tilde{a}^{(p)} \rangle$$

$$= \max_{\tilde{a}^{(1)} \in S^{p}} \mathcal{T}(I - \tilde{U}_l^{(1)}(\tilde{U}_l^{(1)})^\top, I, \ldots, I) \tilde{a}^{(1)} \otimes \cdots \otimes \tilde{a}^{(p)}$$

$$= \max_{\tilde{a}^{(1)} \in S^{p}} \left( \sum_{k=1}^d \lambda_k (I - \tilde{U}_l^{(1)}(\tilde{U}_l^{(1)})^\top) u_k^{(1)} \otimes \cdots \otimes u_k^{(p)} \tilde{a}^{(1)} \otimes \cdots \otimes \tilde{a}^{(p)} \right)$$

$$= \max_{1 \leq k \leq d} \{ \lambda_k \|(I - \tilde{U}_l^{(1)}(\tilde{U}_l^{(1)})^\top) u_k^{(1)} \| \}.$$

By the induction hypothesis, for any $k \leq l$,

$$\lambda_k \|(I - \tilde{U}_l^{(1)}(\tilde{U}_l^{(1)})^\top) u_k^{(1)} \| \leq \lambda_k \|(I - \tilde{u}_{\pi(k)}^{(1)}(\tilde{u}_{\pi(k)}^{(1)})^\top) u_k^{(1)} \|$$

$$\leq (1 + \varepsilon) \| \mathcal{T} - \mathcal{T} \|

< \lambda_{l+1} - \| \mathcal{T} - \mathcal{T} \|,$$

by taking $c_{\varepsilon, p} > 0$ small enough. Hence

$$\max_{k \notin \pi([l])} \tilde{\lambda}_k \leq \max_{k \notin \pi([l])} \{ \lambda_k \|(I - \tilde{U}_l^{(1)}(\tilde{U}_l^{(1)})^\top) u_k^{(1)} \| \} + \| \mathcal{T} - \mathcal{T} \| \leq \lambda_{l+1} + \| \mathcal{T} - \mathcal{T} \|.$$

This suggests that the index, denoted by $\pi(l+1)$, that maximizes

$$\tilde{\lambda}_{\pi(l+1)} = \max_{q=1}^p \| \langle u_{\pi(l+1)}^{(q)}, \tilde{u}_k^{(q)} \rangle \| \quad \text{where} \quad \sum_{q=1}^p \langle u_{\pi(l+1)}^{(q)}, \tilde{u}_k^{(q)} \rangle = \lambda_{l+1}$$

is distinct from $\pi([l])$. Moreover, following the same argument as the previous case, we can derive that

$$\tilde{\lambda}_{\pi(l+1)} - \| \mathcal{T} - \mathcal{T} \| \leq \lambda_{l+1} \leq \tilde{\lambda}_{\pi(l+1)} \prod_{q=1}^p \| \langle u_{\pi(l+1)}^{(q)}, \tilde{u}_k^{(q)} \rangle \|^{(p-2)/p} + \| \mathcal{T} - \mathcal{T} \|.$$
so that
$$|\bar{\lambda}_{\pi(t+1)} - \lambda_{t+1}| \leq \|\bar{T} - T\|,$$
and
$$|\langle \tilde{u}_{\pi(t+1)}^{(q)}, \tilde{u}_{\pi(t+1)}^{(q)} \rangle| \geq \left(1 - 2\lambda_{t+1}^{-1}\|\bar{T} - T\|\right)^{p/(p-2)}.$$  

The rest of the proof is identical to the previous case and is therefore omitted for brevity.

Note that although for preciseness, in the proof, we take the constant $c_{\varepsilon,p} > 0$ depending on the order of the tensor, it can be taken to be strictly increasing with $p$ so that the argument holds if we take $c_{\varepsilon,3}$ for all $p \geq 3$. \hfill $\square$

**Proof of Theorem 3.2.** Recall that the polar factor of $A^{(q)}$ is the unitary matrix
$$U^{(q)} = A^{(q)}[A^{(q)^\top}A^{(q)}]^{-1/2}.$$

It is not hard to see that
$$\|A^{(q)} - U^{(q)}\| = \|[(A^{(q)^\top}A^{(q)})^{1/2} - I]\| \leq \max_{1 \leq i \leq d} |\lambda_i((A^{(q)^\top}A^{(q)})^{1/2} - I)| \leq \delta. \quad (46)$$

We can then consider approximating $X$ by
$$X^{\text{odeco}} = \sum_{i=1}^{d} \eta_i u_i^{(1)} \otimes \cdots \otimes u_i^{(p)}.$$

Recall that
$$\|X - X^{\text{odeco}}\| = \sup_{x^{(q)} \in S^{d-1}:1 \leq q \leq p} \langle X - X^{\text{odeco}}, x^{(1)} \otimes \cdots \otimes x^{(p)} \rangle.$$

For any fixed $x^{(q)}$s,
$$\langle X - X^{\text{odeco}}, x^{(1)} \otimes \cdots \otimes x^{(p)} \rangle$$
$$= \sum_{i=1}^{d} \eta_i \left( \prod_{q=1}^{p} \langle x^{(q)}, a_i^{(q)} \rangle - \prod_{q=1}^{p} \langle x^{(q)}, u_i^{(q)} \rangle \right)$$
$$= \sum_{i=1}^{d} \eta_i \left( \langle x^{(q)}, a_i^{(q)} \rangle - \langle x^{(1)}, u_i^{(1)} \rangle \right) \prod_{q=2}^{p} \langle x^{(q)}, a_i^{(q)} \rangle$$
$$+ \sum_{i=1}^{d} \eta_i \langle x^{(1)}, u_i^{(1)} \rangle \left( \langle x^{(2)}, a_i^{(2)} \rangle - \langle x^{(2)}, u_i^{(2)} \rangle \right) \prod_{q=3}^{p} \langle x^{(q)}, a_i^{(q)} \rangle$$
$$+ \ldots \ldots \ldots +$$
$$+ \sum_{i=1}^{d} \eta_i \prod_{q=1}^{p-1} \langle x^{(q)}, u_i^{(q)} \rangle \left( \langle x^{(p)}, a_i^{(p)} \rangle - \langle x^{(p)}, u_i^{(p)} \rangle \right).$$

36
Each term on the rightmost hand side can be bounded via Cauchy-Schwarz inequality:

\[
\sum_{i=1}^{d} \eta_i \prod_{k=1}^{d-k+2} \left( \langle x^{(q)}, u_i^{(q)} \rangle \frac{(\langle x^{(k)}, a_i^{(k)} \rangle - \langle x^{(k)}, u_i^{(k)} \rangle)}{\prod_{q=k+1}^{\beta} \langle x^{(q)}, a_i^{(q)} \rangle} \right) \leq \|A^{(k)} - U^{(k)}\| \left[ \sum_{i=1}^{d} \left( \eta_i \prod_{k=1}^{d-k+2} \left( \frac{1}{\prod_{q=k+1}^{\beta} \langle x^{(q)}, a_i^{(q)} \rangle} \right) \right) \right]^{1/2}
\]

Note that

\[
\left| \langle x^{(q)}, u_i^{(q)} \rangle \right|, \quad \left| \langle x^{(q)}, a_i^{(q)} \rangle \right| \leq 1,
\]

and

\[
\sum_{i=1}^{d} \langle x^{(q)}, u_i^{(q)} \rangle^2 = 1, \quad \sum_{i=1}^{d} \langle x^{(q)}, a_i^{(q)} \rangle^2 \leq 1 + \delta.
\]

We immediately get

\[
\sum_{i=1}^{d} \eta_i \left( \langle x^{(q)}, a_i^{(q)} \rangle - \langle x^{(1)}, u_i^{(1)} \rangle \right) \prod_{q=2}^{\beta} \langle x^{(q)}, a_i^{(q)} \rangle \leq \delta(1 + \delta)\eta_1,
\]

and for \( k \geq 2, \)

\[
\sum_{i=1}^{d} \eta_i \prod_{k=1}^{d-k+2} \left( \langle x^{(q)}, u_i^{(q)} \rangle \left( \langle x^{(k)}, a_i^{(k)} \rangle - \langle x^{(k)}, u_i^{(k)} \rangle \right) \prod_{q=k+1}^{\beta} \langle x^{(q)}, a_i^{(q)} \rangle \right) \leq \delta\eta_1.
\]

Hence

\[
\|\mathcal{X} - \mathcal{X}^{\text{deco}}\| \leq (p + 1)\delta\eta_1.
\]

Note also that for any \( 1 \leq q \leq p, \) using equation (46)

\[
\max_{1 \leq j \leq d} \sin \angle \left( a_j^{(q)}, u_{\pi(j)}^{(q)} \right) \leq \sqrt{1 - \min_j (a_j^{(q)}, u_j^{(q)})^2} \leq \sqrt{1 - (1 - \|A^{(q)} - U^{(q)}\|^2/2)} \leq \delta/\sqrt{2}.
\]

The desired claim then follows from Theorem 2.4. \( \square \)

Proof of Theorems 4.1 and 4.2. As shown by Raskutti et al. (2019),

\[
\mathbb{E}\|\mathcal{E}\| \leq 4 \log(4p) \left( \sqrt{d_1 + \cdots + \sqrt{d_p}} \right).
\]
Theorem 4.1 then follow immediately from Theorem 2.3.

To prove the lower bound, first note that a lower bound for a special case is also a lower bound for the more general case. Therefore,

\[ \inf_{\tilde{\mathbf{u}}_k^{(1)}, \ldots, \tilde{\mathbf{u}}_k^{(p)}} \sup_{\mathbf{u}_k^{(q)} \in S_{dq-1}, 1 \leq q \leq p} \mathbb{E} \max_{1 \leq q \leq p} \sin \angle(\mathbf{u}_k^{(q)}, \tilde{\mathbf{u}}_k^{(q)}) \geq \inf_{\tilde{\mathbf{u}}_k^{(1)}, \ldots, \tilde{\mathbf{u}}_k^{(p)}} \max_{1 \leq q \leq p} \mathbb{E} \max_{1 \leq q \leq p} \sin \angle(\mathbf{u}_k^{(q)}, \tilde{\mathbf{u}}_k^{(q)}). \]

The special case was simply the rank one case where \( \mathcal{T} \) has only one nonzero singular value \( \lambda_k \). It was shown by Zhang and Xia (2018) that for this case,

\[ \inf_{\tilde{\mathbf{u}}_k^{(1)}, \ldots, \tilde{\mathbf{u}}_k^{(p)}} \sup_{\mathbf{u}_k^{(q)} \in S_{dq-1}, 1 \leq q \leq p} \mathbb{E} \max_{1 \leq q \leq p} \sin \angle(\mathbf{u}_k^{(q)}, \tilde{\mathbf{u}}_k^{(q)}) \geq c \cdot \sqrt{d_1 + \cdots + d_p} \lambda_k, \]

and thus (30) follows. The lower bound (29) for estimating the singular value follows by the same argument.

Proof of Theorem 4.3. We shall first prove the upper bound. In light of Theorem 2.3, there exists a permutation \( \pi : [d_{\min}] \to [d_{\min}] \) such that for any \( 1 \leq k \leq d_{\min} \),

\[ |\lambda_k - \tilde{\lambda}_{\pi(k)}| \leq C \| \mathcal{F} - \mathcal{T} \| \]

and

\[ \max_{1 \leq q \leq p} \sin \angle(\mathbf{u}_k^{(q)}, \tilde{\mathbf{u}}_{\pi(k)}^{(q)}) \leq \frac{C \| \mathcal{F} - \mathcal{T} \|}{\lambda_k}. \]

For brevity, we shall first assume in the rest of the proof, without loss of generality, that \( \pi \) is the identity.

Observe that

\[ \| \mathcal{F} - \mathcal{T} \|_F = \| \sum_{k: \lambda_k \geq \tau_n/2} (\tilde{\lambda}_k \tilde{\mathbf{u}}_k^{(1)} \otimes \cdots \otimes \tilde{\mathbf{u}}_k^{(p)} - \lambda_k \mathbf{u}_k^{(1)} \otimes \cdots \otimes \mathbf{u}_k^{(p)}) \|_F \]

\[ - \sum_{k: \lambda_k < \tau_n/2} (\tilde{\lambda}_k \tilde{\mathbf{u}}_k^{(1)} \otimes \cdots \otimes \tilde{\mathbf{u}}_k^{(p)} - \lambda_k \mathbf{u}_k^{(1)} \otimes \cdots \otimes \mathbf{u}_k^{(p)}) \|_F \]

\[ \leq \| \sum_{k: \lambda_k \geq \tau_n/2} (\tilde{\lambda}_k \tilde{\mathbf{u}}_k^{(1)} \otimes \cdots \otimes \tilde{\mathbf{u}}_k^{(p)} - \lambda_k \mathbf{u}_k^{(1)} \otimes \cdots \otimes \mathbf{u}_k^{(p)}) \|_F \]

\[ + \| \sum_{k: \lambda_k < \tau_n/2} (\tilde{\lambda}_k \tilde{\mathbf{u}}_k^{(1)} \otimes \cdots \otimes \tilde{\mathbf{u}}_k^{(p)} - \lambda_k \mathbf{u}_k^{(1)} \otimes \cdots \otimes \mathbf{u}_k^{(p)}) \|_F \]

\[ =: \Delta_1 + \Delta_2. \]
We now bound $\Delta_1$ and $\Delta_2$ separately.

For $\Delta_1$, first observe that

$$\Delta_1 = \| \text{Mat}_1 \left( \sum_{k: \lambda_k \geq \tau_n/2} (\lambda_k \hat{u}_k^{(1)} \otimes \cdots \otimes \hat{u}_k^{(p)} - \lambda_k u_k^{(1)} \otimes \cdots \otimes u_k^{(p)}) \right) \|_F.$$ 

Denote by

$$r_n = |\{k : \lambda_k \geq \tau_n/2\}|.$$

It is not hard to see that for any $T \in \Theta(\alpha, M)$,

$$r_n \leq \min\{d_{\min}, M/(\tau_n/2)^\alpha\} \leq \min\{d_{\min}, CM\tau_n^{-\alpha}\}.$$ 

Because

$$\text{Mat}_1 \left( \sum_{k: \lambda_k \geq \tau_n/2} (\lambda_k \hat{u}_k^{(1)} \otimes \cdots \otimes \hat{u}_k^{(p)} - \lambda_k u_k^{(1)} \otimes \cdots \otimes u_k^{(p)}) \right)$$

has rank up to $2r_n$,

$$\Delta_1 \leq \sqrt{2} \left\| \text{Mat}_1 \left( \sum_{k: \lambda_k \geq \tau_n/2} (\lambda_k \hat{u}_k^{(1)} \otimes \cdots \otimes \hat{u}_k^{(p)} - \lambda_k u_k^{(1)} \otimes \cdots \otimes u_k^{(p)}) \right) \right\|$$

$$\leq \sqrt{2r_n} \max_{k: \lambda_k \geq \tau_n/2} \left\| \text{Mat}_1 \left( (\lambda_k \hat{u}_k^{(1)} \otimes \cdots \otimes \hat{u}_k^{(p)} - \lambda_k u_k^{(1)} \otimes \cdots \otimes u_k^{(p)}) \right) \right\|.$$ 

Now note that

$$|\tilde{\lambda}_k - \lambda_k| = |(\tilde{\lambda} - \tau_n)_+ - \lambda_k| \leq \tau_n + |\tilde{\lambda}_k - \lambda_k| \leq \tau_n + C\|\hat{T} - T\|.$$

and

$$\sin \angle(u_k^{(1)}, \hat{u}_k^{(1)}) \leq \frac{C\|\hat{T} - T\|}{\lambda_k}.$$ 

In addition,

$$\sin^2 \angle(u_k^{(2)} \circ \cdots \circ u_k^{(p)}, \hat{u}_k^{(1)} \circ \cdots \circ \hat{u}_k^{(p)})$$

$$= 1 - \cos^2 \angle(u_k^{(2)} \circ \cdots \circ u_k^{(p)}, \hat{u}_k^{(1)} \circ \cdots \circ \hat{u}_k^{(p)})$$

$$= 1 - \prod_{q=2}^{p} \cos^2 \angle(u_k^{(q)}, \hat{u}_k^{(q)})$$

$$= 1 - \prod_{q=2}^{p} \left[ 1 - \sin^2 \angle(u_k^{(q)}, \hat{u}_k^{(q)}) \right]$$

$$\leq \sum_{q=2}^{p} \sin^2 \angle(u_k^{(q)}, \hat{u}_k^{(q)})$$

$$\leq \frac{C(p - 1}\|\hat{T} - T\|^2}{\lambda_k^2}.$$ 

39
Therefore,
\[
\Delta_1 \leq C \min\{\sqrt{d_{\min}}, M^{1/2} \tau_n^{-\alpha/2}\} \|\widehat{F} - F\|,
\]
for some constant \(C > 0\).

Now consider \(\Delta_2\). Under the event that
\[
\|\widehat{F} - F\| < \tau_n/2,
\]
we have that \(\tilde{\lambda}_k \leq \tau_n\) for any \(k\) such that \(\lambda_k \leq \tau_n/2\), using theorem 2.4. This means \(\tilde{\lambda}_k = 0\) and hence
\[
\Delta_2 = \| \sum_{k: \lambda_k < \tau_n/2} \lambda_k \mathbf{u}_k^{(1)} \otimes \cdots \otimes \mathbf{u}_k^{(p)} \|_F
\]
\[
\leq \left( \sum_{k: \lambda_k < \tau_n/2} \lambda_k^2 \right)^{1/2}
\leq \min \left\{ \sqrt{d_{\min}} (\tau_n/2), \left( (\tau_n/2)^{2-\alpha} \sum_{1 \leq k \leq d_{\min}} \lambda_k^\alpha \right)^{1/2} \right\}
\leq \min \{ \sqrt{d_{\min}} (\tau_n/2), CM^{1/2} \tau_n^{1-\alpha/2} \},
\]
and thus
\[
\mathbb{E} \left[ \|\widehat{F} - F\|^2_\mathbb{P}(\|\widehat{F} - F\| < \tau_n/2) \right] \leq \min \{ d_{\min}(\tau_n/2)^2, CM \tau_n^{2-\alpha} \}.
\]

On the other hand,
\[
\mathbb{E} \left[ \|\widehat{F} - F\|^2_\mathbb{P}(\|\widehat{F} - F\| \geq \tau_n/2) \right]
\leq \mathbb{E} \left[ \|\widehat{F} - F\|^2_\mathbb{P}(\|\widehat{F} - F\| \geq \tau_n/2) \right] + d_{\min}\mathbb{E} \left[ \|\widehat{F} - F\|^2_\mathbb{P}(\|\widehat{F} - F\| \geq \tau_n/2) \right]
\leq \tau_n^2 d_{\min} \mathbb{P}(\|\widehat{F} - F\| \geq \tau_n/2) + d_{\min} \mathbb{E} \left[ \|\widehat{F} - F\|^2_\mathbb{P}(\|\widehat{F} - F\| \geq \tau_n/2) \right].
\]

It remains to estimate the expectation on the right hand side. Note that
\[
\|\widehat{F} - F\| \leq 2 \left\| \frac{1}{n} \sum_{i=1}^n Y_i \mathcal{X}_i - F \right\|
= 2 \left\| \frac{1}{n} \sum_{i=1}^n \varepsilon_i \mathcal{X}_i + \frac{1}{n} \sum_{i=1}^n (\mathcal{X}_i, F) \mathcal{X}_i - F \right\|
\]
For any fixed \(\mathbf{v}_i \in \mathcal{S}^{d-1}\) for \(1 \leq i \leq p\), an application of Bernstein inequality for subexponential random variables yields that
\[
\mathbb{P} \left\{ \left\langle \frac{1}{n} \sum_{i=1}^n \varepsilon_i \mathcal{X}_i + \frac{1}{n} \sum_{i=1}^n (\mathcal{X}_i, F) \mathcal{X}_i - F, \mathbf{v}_1 \otimes \cdots \otimes \mathbf{v}_p \right\rangle > t \right\} \leq 2 \exp(-2n \min\{t^2, t\}).
\]
We now consider \((1/2)\)-nets over \(S_{d_q-1}^q\), each of which has cardinality at most \(5^d_q\), for \(q = 1, \ldots, p\). Thus for
\[
t > \sqrt{\frac{2(d_1 + \cdots + d_p)}{n}}.
\]
By union bound, we get that
\[
P\left( \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i X_i^* + \frac{1}{n} \sum_{i=1}^{n} (X_i^*, \mathcal{T}) X_i - \mathcal{T} \right\| > t \right) \leq 2 \exp(-0.3n \min\{t^2, t\}).
\]
With this tail bound, we now have
\[
\mathbb{E} \left[ \left\| \tilde{\mathcal{T}} - \mathcal{T} \right\|^2 \mathbb{I}(\left\| \tilde{\mathcal{T}} - \mathcal{T} \right\| \geq \tau_n/2) \right] \leq 2 \int_{\tau_n/2}^{\infty} 2 \exp(-2n \min\{t^2, t\}) dt \leq C \exp(-c(d_{\min})),
\]
for
\[
\tau_n = \sqrt{8(d_1 + \cdots + d_p)/n}.
\]
Consequently,
\[
\mathbb{E} \left[ \left\| \tilde{\mathcal{T}} - \mathcal{T} \right\|_F^2 \mathbb{I}(\left\| \tilde{\mathcal{T}} - \mathcal{T} \right\| \geq \tau_n/2) \right] \leq C \tau_n^2 d_{\min} \exp(-c d_{\min}/2).
\]
The upper bound (36) then follows by combining the bounds under the two events.

We now consider the lower bound. We shall proceed with two separate cases.

**Case 1:** \(n < (d_{\min}/M)^\frac{\alpha}{2} \cdot (d_1 + \cdots + d_p)\)

Define
\[
r = \frac{M}{2} \left( \frac{d_1 + \cdots + d_p}{n} \right)^{-\alpha/2}
\]
and take
\[
\lambda_1 = \cdots = \lambda_r = \sqrt{\frac{d_1 + \cdots + d_p}{n}}
\]
and \(\lambda_i = 0\) for \(i = r + 1, \ldots, d_{\min}\). This can be done since in this case we have \(r \leq d_{\min}/2\). We assume without of loss of generality, that \(\sigma = 1\) and \(d_1 = \max\{d_1, \ldots, d_p\}\).

Using results on packing numbers of orthogonal spaces (see, e.g., Szarek, 1997), we can get a set \(\mathcal{A}\) of \(d_1 \times r\) orthonormal matrices \(A_i\) such that \(|\mathcal{A}| \geq 2^{r(d_1 - r)}\) and
\[
r/2p \geq \|A_1 - A_2\|_F^2 \geq r/4p
\]
for any \( A_1 \neq A_2, A_1, A_2 \in \mathcal{A} \). Finally we construct the coefficient tensors

\[ T_j = \sum_{l=1}^{r} \lambda_l \mathbf{a}_l^{(j)} \otimes \mathbf{e}_1 \cdots \otimes \mathbf{e}_l, \]

for \( A_j \in \mathcal{A} \). For any two \( A_1 \neq A_2 \in \mathcal{A} \), the probability models \( P_1 \) and \( P_2 \) of \((X_i, Y_i)\) under the given tensor regression model have Kullback-Leibler divergence

\[ \text{KL}(P_1 | P_2) = \mathbb{E}_{P_1} \log \left( \frac{\exp(- (y_i - \langle \mathcal{T}_1, X_i \rangle)^2/2))}{\exp(- (y_i - \langle \mathcal{T}_2, X_i \rangle)^2/2))} \right) \]

\[ = \| \mathcal{T}_1 - \mathcal{T}_2 \|_F^2 \]

\[ = \frac{d_1 + \cdots + d_p}{n} \| A_1 - A_2 \|_F^2 \]

\[ \leq \frac{1}{2p} M \left( \frac{d_1 + \cdots + d_p}{n} \right)^{1-\alpha/2} \leq \frac{rd_1}{2n}. \]

Now using generalized Fano’s lemma, we have for some constant \( c > 0 \), that

\[ \inf_{\mathcal{T}} \sup_{\mathcal{F} : \lambda \in B_{\alpha}(M)} \| \mathcal{T} - \mathcal{T} \|_F^2 > cM \left( \frac{d_1 + \cdots + d_p}{n} \right)^{1-\alpha/2} \left( 1 - \frac{nrd_1/2n}{d_1 r} \right) \]

\[ \geq cM \left( \frac{d_1 + \cdots + d_p}{n} \right)^{1-\alpha/2}. \]

**Case 2: \( n > (d_{\min}/M)\hat{\alpha} \cdot (d_1 + \cdots + d_p) \)**

With the same framework as before, we can now simply take

\[ \lambda_i = \sqrt{\frac{d_1 + \cdots + d_p}{n}} \]

for \( i = 1, \ldots, d_{\min} \). By the assumption on \( n \), the vector of singular values \( \tilde{\lambda} \in B_{\alpha}(M) \).

We can again construct a set of \( d_1 \times d_{\min}/2 \) orthonormal matrices \( \mathcal{A} = \{ A_i \} \) with \(|\mathcal{A}| \geq 2^{d_{\min}/(d_1-d_{\min}/2)} \) and \( d_{\min}/2p \geq \| A_1 - A_2 \|_F^2 \geq d_{\min}/4p \) for any \( A \neq A_2 \in \mathcal{A} \). We can then derive using Fano’s lemma, that there is a constant \( c > 0 \) for which

\[ \inf_{\mathcal{T}} \sup_{\mathcal{F} : \lambda \in B_{\alpha}(M)} \| \mathcal{T} - \mathcal{T} \|_F^2 > c d_{\min} \left( \frac{d_1 + \cdots + d_p}{n} \right). \]

Combining the two cases, it is easy to see that we have a constant \( c > 0 \) such that

\[ \inf_{\mathcal{T}} \sup_{\mathcal{F} : \lambda \in B_{\alpha}(M)} \| \mathcal{T} - \mathcal{T} \|_F^2 > c \cdot \min \left\{ d_{\min} \cdot \frac{d_1 + \cdots + d_p}{n}, M \left( \frac{d_1 + \cdots + d_p}{n} \right)^{1-\alpha/2} \right\}, \]

thus showing (37). □
Proof of Theorem 4.4. A careful inspection of the proof of proposition 4.3 from Vershynin et al. (2011) reveals that with probability $1 - \delta$, we have

$$\|\hat{\mathcal{M}}_{\text{sample}}(X) - \mathcal{M}(X)\| \leq \varepsilon.$$ 

The claim then follows using Theorem 2.4. \hfill \square

Proof of Theorem 4.5. It is clear that $M_4(X)$ is in the feasible set of the optimization. By the union bound applied Catoni (2012)’s result to each $u \in \mathcal{N}$, we have with probability at least

$$1 - |\mathcal{N}| \exp(-3d) = 1 - \exp(-(3 - \log 9)d) \geq 1 - 2.2^{-d},$$

that

$$\max_{u \in \mathcal{N}} |\hat{\theta}_u - \langle M_4(X), u \otimes u \otimes u \otimes u \rangle| \leq \sqrt{\frac{6Ld}{n(1 - 6d/n)}}.$$ 

Under this event, for any $u \in \mathcal{N}$,

$$\langle \hat{M}_4(X) - M_4(X), u \otimes u \otimes u \otimes u \rangle$$

$$\leq |\hat{\theta}_u - \langle \hat{M}_4(X), u \otimes u \otimes u \otimes u \rangle| + |\hat{\theta}_u - \langle M_4(X), u \otimes u \otimes u \otimes u \rangle|$$

$$\leq 2 \max_{u \in \mathcal{N}} |\hat{\theta}_u - \langle M_4(X), u \otimes u \otimes u \otimes u \rangle|$$

$$\leq \frac{5Ld}{n(1 - 6d/n)},$$

from which we can conclude that

$$\|\hat{M}_4(X) - M_4(X)\|$$

$$= \sup_{u \in S^{d-1}} |\langle \hat{M}_4(X) - M_4(X), u \otimes u \otimes u \otimes u \rangle|$$

$$\leq \sup_{u \in \mathcal{N}} |\langle \hat{M}_4(X) - M_4(X), u \otimes u \otimes u \otimes u \rangle| + \frac{1}{4} \sup_{u \in S^{d-1}} |\langle \hat{M}_4(X) - M_4(X), u \otimes u \otimes u \otimes u \rangle|$$

which means that

$$\|\hat{M}_4(X) - M_4(X)\| \leq 7 \sqrt{\frac{Ld}{n(1 - 6d/n)}}$$

with probability at least $1 - 1.4^{-d}$. \hfill \square
Proof of Theorem 4.7. For $1 \leq k \leq d$, we take independent random variables $S_k = \alpha S_{k,1} + \sqrt{1 - \alpha^2} S_{k,2}$ where $S_{k,1} \sim N(0,1)$ and $S_{k,2}$ is a Radamacher random variable. It is not hard to see that

$$|\kappa_4(S_k)| = 2(1 - \alpha^2)^2.$$ 

By taking

$$\max \left\{ 1 - \left( \frac{K}{2} \right)^{1/2}, 0 \right\} \leq \alpha^2 \leq 1 - \left( \frac{1}{2K} \right)^{1/2},$$

we can ensure that

$$K^{-1} \leq |\kappa_4(S_k)| \leq K.$$ 

In order to prove the lower bound, we consider an overcomplete ICA model as $X = AS$, where $A$ is a $2d \times d$ orthogonal matrix, while $S$ is a $d$-dimensional vector consisting of iid random variables $S_k$ as defined above. We will show that the best approximation of each column of $A$ satisfies the given lower bound. It is well known that a subset $\mathcal{V} \subset S^{d-1}$ can be constructed such that for any $v_i \neq v_j \in \mathcal{V}$,

$$\|v_i v_i^T - v_j v_j^T\| \geq \frac{1}{2}$$

and $|\mathcal{V}| \geq 2^d$.

We next define

$$\mathcal{W} := \left\{ \begin{pmatrix} \sqrt{1 - \delta} \\ \sqrt{\delta} v \end{pmatrix} : v \in \mathcal{V} \right\}.$$ 

It is clear that $w_i \in S^d$. Moreover,

$$\|w_i w_i^T - w_j w_j^T\| \geq \sqrt{\delta (1 - \delta)} \|v_i v_i^T - v_j v_j^T\| \geq \frac{1}{2} \sqrt{\delta (1 - \delta)}$$

whenever $w_i \neq w_j \in \mathcal{W}$.

Fix any $2d \times (d - 1)$ orthonormal matrix $M$. We define

$$\mathcal{U} = \{ u_i : u_i = M^i w_i \text{ for } w_i \in \mathcal{W} \}.$$ 

Note that any $u_i \neq u_j \in \mathcal{U}$,

$$\|u_i - u_j\| \geq \frac{1}{2} \cdot \sqrt{\delta (1 - \delta)}.$$ 

On the other hand,

$$\|u_i - u_j\| = \sqrt{\delta} \|v_i - v_j\| \leq 2\sqrt{\delta}.$$
and $|\mathcal{U}| = |\mathcal{V}| \geq 2^d$. Finally, we construct a collection of mixing matrices $A^{(i)} \in \mathcal{O}(2d, d)$ as $A^{(i)} = [u_i, M]$. The density function of each $S_k$ is

$$f(s) = \frac{1}{2(2\pi\alpha^2)^{d/2}} \exp(-(s - \sqrt{1 - \alpha^2})^2/2\alpha^2) + \frac{1}{2(2\pi\alpha^2)^{d/2}} \exp(-(s + \sqrt{1 - \alpha^2})^2/2\alpha^2).$$

It follows that for two mixing matrices $A^{(1)} \neq A^{(2)}$, for $i = 1, 2$ we have $f_i(x) = \prod f(a^{(i)}_k x)$, which immediately means $f_1(x) / f_2(x) \leq \exp\left(\frac{(u^T_1 x)^2 - (u^T_2 x)^2}{2\alpha^2} + \left|x^T(u_1 - u_2)\right|\right)$. When $X$ has the distribution $f_1(x)$, $u^T_2 x = u_2 A^{(1)} S = (u^T_2 u_1) S_1$, so that the ratio above is bounded by

$$\exp\left(\frac{(u^T_1 u_2)^2 - 1}{2\alpha^2} S_1^2 + \left|(1 - u^T_1 u_2) S_1\right|\right).$$

By our construction,

$$(1 - u^T_1 u_2) = \|u_1 - u_2\|^2 \leq \delta.$$     

Consequently, the Kullback-Leibler divergence, when we observe $x_1, \ldots, x_n$, can be bounded as

$$KL(f_1; f_2) = n\mathbb{E}_{X \sim f_1} \left(\log \left(\frac{f_1(x)}{f_2(x)}\right)\right) \leq n\delta.$$  

By generalized Fano’s inequality, we then get two constants $c, C > 0$ such that

$$\inf \sup \mathbb{E}_{\hat{u}, u \in \mathcal{U}} \left\|\hat{u} \hat{u}^T - u_i u_i^T\right\|_2 \geq c\sqrt{\delta(1 - \delta)} \left(1 - \frac{Cn\delta + \log 2}{d \log 2}\right).$$

We now take $\delta = cd/n$ to get that

$$\inf \sup \mathbb{E}_{\hat{u}, u \in \mathcal{U}} \left\|\hat{u} \hat{u}^T - u_i u_i^T\right\|_2 \geq c\sqrt{\frac{d}{n}}.$$

The claim in theorem 4.7 now follows by noting that given $S$, any estimator constructed from $OS$ where $O \in \mathcal{O}(d)$ can also be constructed from $AS$ where $A = BO$ for some $B \in \mathcal{O}(2d, d)$. $\square$
References

Radosław Adamczak, Alexander Litvak, Alain Pajor, and Nicole Tomczak-Jaegermann. Quantitative estimates of the convergence of the empirical covariance matrix in log-concave ensembles. *Journal of the American Mathematical Society*, 23(2):535–561, 2010.

Animashree Anandkumar, Rong Ge, Daniel Hsu, Sham M Kakade, and Matus Telgarsky. Tensor decompositions for learning latent variable models. *The Journal of Machine Learning Research*, 15(1):2773–2832, 2014a.

Animashree Anandkumar, Rong Ge, and Majid Janzamin. Sample complexity analysis for learning overcomplete latent variable models through tensor methods. *arXiv preprint arXiv:1408.0553*, 2014b.

Animashree Anandkumar, Rong Ge, and Majid Janzamin. Guaranteed non-orthogonal tensor decomposition via alternating rank-1 updates. *arXiv preprint arXiv:1402.5180*, 2014c.

Mikhail Belkin, Luis Rademacher, and James Voss. Eigenvectors of orthogonally decomposable functions. *SIAM Journal on Computing*, 47(2):547–615, 2018.

Aditya Bhaskara, Moses Charikar, and Aravindan Vijayaraghavan. Uniqueness of tensor decompositions with applications to polynomial identifiability. In *Conference on Learning Theory*, pages 742–778, 2014.

Rajendra Bhatia. *Perturbation bounds for matrix eigenvalues*, volume 53. Siam, 1987.

Rajendra Bhatia. *Matrix analysis*, volume 169. Springer Science & Business Media, 2013.

Olivier Catoni. Challenging the empirical mean and empirical variance: a deviation study. In *Annales de l'IHP Probabilités et statistiques*, volume 48, pages 1148–1185, 2012.

Maolin Che, Liqun Qi, and Yimin Wei. Perturbation bounds of tensor eigenvalue and singular value problems with even order. *Linear and multilinear algebra*, 64(4):622–652, 2016.

Jie Chen and Yousef Saad. On the tensor svd and the optimal low rank orthogonal approximation of tensors. *SIAM Journal on Matrix Analysis and Applications*, 30(4):1709–1734, 2009.
Andrzej Cichocki, Danilo Mandic, Lieven De Lathauwer, Guoxu Zhou, Qibin Zhao, Cesar Caiafa, and Huy Anh Phan. Tensor decompositions for signal processing applications: From two-way to multiway component analysis. IEEE Signal Processing Magazine, 32(2):145–163, 2015.

Chandler Davis and William Morton Kahan. The rotation of eigenvectors by a perturbation. iii. SIAM Journal on Numerical Analysis, 7(1):1–46, 1970.

Lieven De Lathauwer, Bart De Moor, and Joos Vandewalle. On the best rank-1 and rank-\((r_1, r_2, \ldots, r_n)\) approximation of higher-order tensors. SIAM journal on Matrix Analysis and Applications, 21(4):1324–1342, 2000a.

Lieven De Lathauwer, Bart De Moor, and Joos Vandewalle. A multilinear singular value decomposition. SIAM journal on Matrix Analysis and Applications, 21(4):1253–1278, 2000b.

Shmuel Friedland. Remarks on the symmetric rank of symmetric tensors. SIAM Journal on Matrix Analysis and Applications, 37(1):320–337, 2016.

Shmuel Friedland and Giorgio Ottaviani. The number of singular vector tuples and uniqueness of best rank-one approximation of tensors. Foundations of Computational Mathematics, 14(6):1209–1242, 2014.

R Harshman. Foundations of the parafac procedure: Model and conditions for an explanatory factor analysis. Technical Report UCLA Working Papers in Phonetics 16, University of California, Los Angeles, Los Angeles, CA, 1970.

Christopher J Hillar and Lek-Heng Lim. Most tensor problems are np-hard. Journal of the ACM (JACM), 60(6):45, 2013.

Tamara G Kolda. Orthogonal tensor decompositions. SIAM Journal on Matrix Analysis and Applications, 23(1):243–255, 2001.

Tamara G Kolda and Brett W Bader. Tensor decompositions and applications. SIAM review, 51(3):455–500, 2009.
Vladimir Koltchinskii, Karim Lounici, Alexandre B Tsybakov, et al. Nuclear-norm penalization and optimal rates for noisy low-rank matrix completion. *The Annals of Statistics, 39* (5):2302–2329, 2011.

Joseph B Kruskal. Three-way arrays: rank and uniqueness of trilinear decompositions, with application to arithmetic complexity and statistics. *Linear algebra and its applications, 18* (2):95–138, 1977.

Sue E Leurgans, Robert T Ross, and Rebecca B Abel. A decomposition for three-way arrays. *SIAM Journal on Matrix Analysis and Applications, 14*(4):1064–1083, 1993.

Tianqi Liu, Ming Yuan, and Hongyu Zhao. Characterizing spatiotemporal transcriptome of human brain via low rank tensor decomposition. *arXiv preprint arXiv:1702.07449, 2017.*

Cun Mu, Daniel Hsu, and Donald Goldfarb. Successive rank-one approximations for nearly orthogonally decomposable symmetric tensors. *SIAM Journal on Matrix Analysis and Applications, 36*(4):1638–1659, 2015.

Cun Mu, Daniel Hsu, and Donald Goldfarb. Greedy approaches to symmetric orthogonal tensor decomposition. *SIAM Journal on Matrix Analysis and Applications, 38*(4):1210–1226, 2017.

Garvesh Raskutti, Ming Yuan, Han Chen, et al. Convex regularization for high-dimensional multiresponse tensor regression. *The Annals of Statistics, 47*(3):1554–1584, 2019.

Emile Richard and Andrea Montanari. A statistical model for tensor pca. In *Advances in Neural Information Processing Systems*, pages 2897–2905, 2014.

Elina Robeva. Orthogonal decomposition of symmetric tensors. *SIAM Journal on Matrix Analysis and Applications, 37*(1):86–102, 2016.

Gilbert W Stewart and Ji-guang Sun. *Matrix perturbation theory.* Academic Press, 1990.

Stanislaw J Szarek. Metric entropy of homogeneous spaces. *arXiv preprint math/9701213, 1997.*
Appendix – Proof of Theorem 2.1.

We first show that if \( \lambda_k > 0 \), then \( (\pm u_k^{(1)}, \ldots, \pm u_k^{(p)}) \) is a local maximum of \( F \). Consider the Lagrange form of \( F \):

\[
F_{\lambda_k}(a^{(1)}, \ldots, a^{(p)}) := F(a^{(1)}, \ldots, a^{(p)}) + \lambda_k \sum_{q=1}^{p} \left( 1 - \|a^{(q)}\|^2 \right).
\]

It is easy to see that \( (u_k^{(1)}, \ldots, u_k^{(p)}) \) satisfies the first order condition of \( F_{\lambda_k} \):

\[
\mathcal{T} \times_q \neq q a^{(q)} = \lambda_k a^{(q)}, \quad \forall q = 1, \ldots, p.
\]

Moreover, it can also be derived that the Hessian of \( F_{\lambda_k} \) is

\[
\begin{bmatrix}
-\lambda_k I_{d_1} & \mathcal{T} \times_{q' \neq \{1,2\}} a^{(q')} & \cdots & \mathcal{T} \times_{q' \neq \{1,p\}} a^{(q')}

(\mathcal{T} \times_{q' \neq \{1,2\}} a^{(q')})^\top & -\lambda_k I_{d_2} & \cdots & \mathcal{T} \times_{q' \neq \{2,p\}} a^{(q')}

\vdots & \vdots & \ddots & \vdots

(\mathcal{T} \times_{q' \neq \{1,p\}} a^{(q')})^\top & (\mathcal{T} \times_{q' \neq \{2,p\}} a^{(q')})^\top & \cdots & -\lambda_k I_{d_p}
\end{bmatrix}.
\]

When evaluated at \( (u_k^{(1)}, \ldots, u_k^{(p)}) \), the Hessian becomes

\[
H = \lambda_k v_k \otimes v_k - \lambda_k \text{diag}(I_{d_1} + u_k^{(1)} \otimes u_k^{(1)}, I_{d_2} + u_k^{(2)} \otimes u_k^{(2)}, \ldots, I_{d_p} + u_k^{(p)} \otimes u_k^{(p)}),
\]
where \( \mathbf{v}_k = [(u^{(1)}_k)^\top, \ldots, (u^{(p)}_k)^\top]^\top \). For any \((a^{(1)}, \ldots, a^{(p)}) \neq (u^{(1)}_k, \ldots, u^{(p)}_k)\), write
\[
a^{(q)} = \langle a^{(q)}, u^{(q)}_k \rangle u^{(q)}_k + \tilde{a}^{(q)}.
\]

It can be verified that
\[
([a^{(1)}]^\top, \ldots, (a^{(p)})^\top]H([a^{(1)}]^\top, \ldots, (a^{(p)})^\top]^\top < 0
\]
if
\[
\prod_{1 \leq q \leq p} |\langle a^{(q)}, u^{(q)}_k \rangle| < 1.
\]
This implies that \((u^{(1)}_k, \ldots, u^{(p)}_k)\) is a local maximum of \(F\).

We now argue that \(F\) has no local maximum other than \(\{ (\pm u^{(1)}_k, \ldots, \pm u^{(p)}_k) : \lambda_k > 0 \}\).
We shall prove this by contradiction. Assume the contrary that unit length vectors \(a^{(q)}\)'s are a local maximum but \((a^{(1)}, \ldots, a^{(p)}) \notin \{ (\pm u^{(1)}_k, \ldots, \pm u^{(p)}_k) : \lambda_k > 0 \}\). By first order condition, there exists a \(\lambda \in \mathbb{R}\) such that
\[
\mathcal{F} \times_{q' \neq q} a^{(q')} = \lambda a^{(q)}, \quad \forall q = 1, \ldots, p.
\]
We can assume that \(\lambda > 0\) without loss of generality. This implies
\[
\lambda_k \prod_{q' \neq q} \langle u^{(q')}_k, a^{(q')} \rangle = \mathcal{F} \times_{q} u^{(q)}_k \times_{q' \neq q} a^{(q')} = \lambda \langle u^{(q)}_k, a^{(q)} \rangle
\]
and hence if \(\langle u^{(q)}_k, a^{(q)} \rangle \neq 0\),
\[
\frac{\prod_{q' \neq q} \langle u^{(q')}_k, a^{(q')} \rangle}{\langle u^{(q)}_k, a^{(q)} \rangle} = \frac{\lambda}{\lambda_k} \quad \forall q = 1, \ldots, p \text{ and } \forall k : \lambda_k > 0.
\]
On the other hand, \(\langle u^{(q)}_k, a^{(q)} \rangle = 0\) for some \(q\) implies that \(\langle u^{(q)}_k, a^{(q)} \rangle = 0\) for all \(q\). By symmetry, for each \(k : \lambda_k > 0\), we get that,
\[
|\langle u^{(q)}_k, a^{(q)} \rangle| = \gamma_k = \begin{cases} \left( \frac{\lambda}{\lambda_k} \right)^{\frac{1}{p-2}}, & \forall q = 1, \ldots, p \text{ if } \prod_{q=1}^{p} \langle u^{(q)}_k, a_q \rangle \neq 0 \\ 0, & \forall q = 1, \ldots, p \text{ if } \prod_{q=1}^{p} \langle u^{(q)}_k, a_q \rangle = 0. \end{cases}
\]
Moreover, we have
\[
\prod_{q} \langle u^{(q)}_k, a^{(q)} \rangle \geq 0, \quad \forall k \text{ such that } \lambda_k > 0.
\]
We first consider the case where
\[ S := \{ k : \lambda_k \prod_q \langle u_k^{(q)}, a^{(q)} \rangle \neq 0 \} \]
has at most 1 element. We pick \( j \in S \) if it exists, otherwise let \( j \) be an arbitrary element from \( \{ k : \lambda_k > 0 \} \). Since \( a^{(q)} \neq u_j^{(q)} \) for at least one \( q \), we can construct a new vector \( b^{(q)} \in S^{d-1} \), which has
\[ \left| \langle u_j^{(q)}, b^{(q)} \rangle \right| > \left| \langle u_j^{(q)}, a^{(q)} \rangle \right|, \]
while
\[ \prod_q \left| \langle u_k^{(q)}, b^{(q)} \rangle \right| = 0 \quad \forall k \neq j \text{ such that } \lambda_k > 0. \]
It is now easy to see that \( F(b^{(1)}, \ldots, b^{(p)}) > F(a^{(1)}, \ldots, a^{(p)}) \). Since we can take \( b^{(q)} \) arbitrarily close to \( a^{(q)} \), it is clear that \( (a^{(1)}, \ldots, a^{(p)}) \) cannot be a maximum.

Henceforth, we assume that \( S \) has at least two elements, say \( j_1 \) and \( j_2 \). Let us define
\[ \eta := \min \left\{ \left| \langle u_j^{(q)}, a^{(q)} \rangle \right| : 1 \leq q \leq p, \ i = 1, 2 \right\} / 2. \]
For \( 0 < \delta < \eta \), for each \( 1 \leq q \leq p \), we construct \( b^{(q)} \) as follows:
\[ b^{(q)}(\delta) = s_1 \left( \sqrt{\langle u_{j_1}^{(q)}, a^{(q)} \rangle^2 + \delta} \right) u_{j_1}^{(q)} + s_2 \left( \sqrt{\langle u_{j_2}^{(q)}, a^{(q)} \rangle^2 - \delta} \right) u_{j_2}^{(q)} + \sum_{k \neq j_1, j_2} \langle u_k^{(q)}, a^{(q)} \rangle u_k^{(q)}, \]
where \( s_i = \text{sign}(\langle u_{j_i}^{(q)}, a^{(q)} \rangle) \) for \( i = 1, 2 \). Evidently, \( b^{(q)}(\delta) \in S^{d-1} \), and \( \text{sign}(\langle u_k^{(q)}, b^{(q)} \rangle) = \text{sign}(\langle u_k^{(q)}, a^{(q)} \rangle) \) for all \( k \) and \( q \). Since \( (a^{(1)}, \ldots, a^{(p)}) \) is a critical point, we get using (47) and (48) that
\[
F(b^{(1)}, \ldots, b^{(p)}) - F(a^{(1)}, \ldots, a^{(p)}) \\
= \lambda_{j_1} \prod_{q=1}^p \left[ \langle u_{j_1}^{(q)}, a^{(q)} \rangle^2 + \delta \right]^{1/2} - \lambda_{j_1} \prod_{q=1}^p \left| \langle u_{j_1}^{(q)}, a^{(q)} \rangle \right| + \lambda_{j_2} \prod_{q=1}^p \left[ \langle u_{j_2}^{(q)}, a^{(q)} \rangle^2 - \delta \right]^{1/2} - \lambda_{j_2} \prod_{q=1}^p \left| \langle u_{j_2}^{(q)}, a^{(q)} \rangle \right| \\
= \lambda_{j_1} [\gamma_{j_1}^2 + \delta]^{p/2} - \lambda_{j_1} \gamma_{j_1}^p + \lambda_{j_2} [\gamma_{j_2}^2 - \delta]^{p/2} - \lambda_{j_2} \gamma_{j_2}^p \\
= \frac{p}{2} \times \lambda_{j_1} \delta \xi_{j_1}^{p/2-1} - \frac{p}{2} \times \lambda_{j_2} \delta \xi_{j_2}^{p/2-1}. \]
for some $\gamma^2_{j_1} \leq \xi_1 \leq \gamma^2_{j_1} + \delta$ and $\gamma^2_{j_1} - \delta \leq \xi_2 \leq \gamma^2_{j_2}$. Since $f(x) = x^{p/2-1}$ is monotonically increasing for $p > 2$, 

$$F(b^{(1)}, \ldots, b^{(p)}) - F(a^{(1)}, \ldots, a^{(p)}) > \frac{p}{2} \delta (\lambda_{j_1} \gamma_{j_1}^{p-2} - \lambda_{j_2} \gamma_{j_2}^{p-2})$$

$$= \frac{p \delta}{2} \left[ \lambda_{j_1}/\|u^{(q)}_{j_1}, a^{(q)}\| - \lambda_{j_2}/\|u^{(q)}_{j_2}, a^{(q)}\| \right]$$

$$= \frac{p \delta}{2} [\lambda - \lambda] \text{ using (47).}$$

Since we can take $\delta$ to be arbitrarily close to zero, it is clear that $(a^{(1)}, \ldots, a^{(p)})$ is not a local maximum.

**Global attraction of the hidden basis:** We will follow the outline in section 4.2.2 of Belkin et al. (2018). For brevity, we assume $d_1 = \cdots = d_p = d$. For $(a_1, \ldots, a_p) \in S^{d-1} \times \cdots \times S^{d-1}$, the tangent space of the cross-product of $p$ spheres is

$$T_{a_1, \ldots, a_p} S^{d-1} \times \cdots \times S^{d-1} = a_1^\perp \times \cdots \times a_p^\perp.$$

We define the exponential map $\phi : T_{a_1, \ldots, a_p} S^{(d-1)\otimes p} \to S^{(d-1)\otimes p}$ as:

$$\phi(x_1, \ldots, x_p) = \left[ a_1 \cos \|x_1\| + \frac{x_1}{\|x_1\|} \sin \|x_1\|, \ldots, a_p \cos \|x_p\| + \frac{x_p}{\|x_p\|} \sin \|x_p\| \right].$$

It can be checked that $D\phi = D\phi^{-1} = \text{diag}[P_{a_1^\perp}, \ldots, P_{a_p^\perp}]$.

We first determine the local convergence manifold of $(a_1, \ldots, a_p)$, that is, the set

$$L_{local} = \{ \hat{x}(0) : \lim_{t \to \infty} x(t) = a_i, x(t) \in U_i \forall t \in \mathbb{N} \}$$

for some local neighborhoods $U_i$ of $a_i$. To disprove global attraction to a particular critical point, note that it is enough to determine $L_{local} \cap Q_{a_1} \times \cdots \times Q_{a_p}$, where

$$Q_{a_q} = \{ v \in S^{d-1} : \text{sign}(\langle v, u^{(q)}_i \rangle) = \text{sign}(\langle a_q, u^{(q)}_i \rangle) \forall i \text{ such that } \langle a_q, u^{(q)}_i \rangle \neq 0 \}$$

for $1 \leq q \leq p$.

Let $S = \{ i : \Pi_q \langle a_q, u^{(q)}_i \rangle \neq 0 \}$. We will use $P^{(q)}_S = \sum_{i \in S} u^{(q)}_i u^{(q)T}_i$, and similarly $P^{(q)}_{\tilde{S}}$ for $\tilde{S} = [d]/S$. Using (47) it is easy to see that if $\lambda > 0$, $S$ is in fact same as $S_k = \{ i : \langle a_k, u^{(k)}_i \rangle \neq 0 \}$ for all $1 \leq k \leq p$. We then have the following lemma.
Lemma 1. \( D[\phi \circ G \circ \phi^{-1}]_{\phi(a_1 \ldots a_p)} \) is a matrix with the following properties:
1. \( D[\phi \circ G \circ \phi^{-1}] \) is the 0 map on \( K = \{(x_1 \ldots x_p) : x_q \in \text{Range}(\mathcal{P}_S^{(q)}) \text{ for all } q\} \).
2. If \( |S| > 1 \), there is a space

\[
\mathcal{L} = \text{(Range}(\mathcal{P}_S^{(1)} \cap \mathcal{P}_{a_1^+}) \cap \mathcal{Q}_{a_1}) \times \cdots \times \text{(Range}(\mathcal{P}_S^{(p)} \cap \mathcal{P}_{a_p^+}) \cap \mathcal{Q}_{a_p})
\]

of positive dimension on which \((x_1^T \ldots x_p^T)[D[\phi \circ G \circ \phi^{-1}] - I](x_1^T \ldots x_p^T)^T > 0\).

Proof of Lemma 1. Since \((a_1, \ldots, a_p)\) is a fixed point of \( G \), we have using chain rule that

\[
D[\phi \circ G \circ \phi^{-1}]_{\phi(a_1 \ldots a_p)} = D\phi_G(a_1^+)DG_{a_1}D\phi^{-1}_{\phi(a_1^+)} = \text{diag}[\mathcal{P}_{a_1^+} \ldots \mathcal{P}_{a_p^+}]DG_{a_1} \text{diag}[\mathcal{P}_{a_1^+} \ldots \mathcal{P}_{a_p^+}].
\]

Since \( T \times_q \neq a^{(q)} \lambda a_q \) for \( q = 1, \ldots, p \), \( \|T \times_q \neq a^{(q)}\| = \lambda \). Hence after some calculation we obtain

\[
DG = \frac{1}{\lambda}
\begin{bmatrix}
0 & \mathcal{P}_{a_1^+} T \times_q \neq 1,2 a^{(q)} & \cdots & \mathcal{P}_{a_1^+} T \times_q \neq 1, p a^{(q)} \\
\mathcal{P}_{a_2^+} (T \times_q \neq 1, 2) a^{(q)} & 0 & \cdots & \mathcal{P}_{a_2^+} (T \times_q \neq 2, p) a^{(q)} \\
\vdots & \vdots & \ddots & \vdots \\
\mathcal{P}_{a_p^+} (T \times_q \neq 1, p) a^{(q)} & \mathcal{P}_{a_p^+} (T \times_q \neq 2, p) a^{(q)} & \cdots & 0
\end{bmatrix}.
\]

Now for \((x_1 \ldots x_p) \in K\), for any \( k, l \in [p]\),

\[
x_k^T \mathcal{P}_{a_k^+} T \times_q \neq 1, p a^{(q)} \mathcal{P}_{a_l^+} T \times_q \neq k, l x_l = x_k^T T \times_q \neq k, l x_l
\]

\[
= \sum_{i \in S} \lambda_i \langle x_k, u_i^{(k)} \rangle \langle x_l, u_i^{(l)} \rangle \prod_{q \neq k, l} \langle a^{(q)}, u_i^{(q)} \rangle = 0.
\]

The first claim is now proved. For the rest, note similarly that for \((x_1, \ldots, x_p) \in \mathcal{L}\),

\[
(x_1^T, \ldots, x_p^T)[D[\phi \circ G \circ \phi^{-1}] - I](x_1^T, \ldots, x_p^T)
\]

\[
= \frac{1}{\lambda} \sum_{k \neq l} \sum_{i \in S} \lambda_i \langle x_k, u_i^{(k)} \rangle \langle x_l, u_i^{(l)} \rangle \prod_{q \neq k, l} \langle a_q, u_i^{(q)} \rangle
\]

\[
= \frac{1}{\lambda} \sum_{k \neq l} \sum_{i \in S} \langle x_k, u_i^{(k)} \rangle \langle x_l, u_i^{(l)} \rangle \times \frac{\lambda_i \prod_{q} \langle a_q, u_i^{(q)} \rangle}{\langle a_k, u_i^{(k)} \rangle \langle a_l, u_i^{(l)} \rangle}
\]

\[
= \frac{1}{\lambda} \sum_{k \neq l} \sum_{i \in S} \langle x_k, u_i^{(k)} \rangle \langle x_l, u_i^{(l)} \rangle \times \lambda
\]

\[
\geq \sum_{k \neq l} x_k^T x_l = \sum_{k \neq l} x_k^T x_l - \sum_{q=1}^{p} x_q^T x_q = p^2 - p
\]

\[
> \sum_{q=1}^{p} x_q^T x_q = p,
\]

53
where we use definition of $\mathcal{L}$ in the second equality and (47) in the third equality. Claim 2 then follows since $p > 2$. \hfill \Box

Lemma 1 implies that the space spanned by the eigenvectors of $D[\phi \circ G \circ \phi^{-1}]$ with absolute eigenvalues less than 1, has dimension at most $(d-1)^p - (|S|-1)^p$. Using theorem 4.17 from Belkin et al. (2018) we obtain that, if $|S| > 1$, the local convergence manifold

$$\mathcal{L}_{local} = \{ \tilde{x}(0) : \lim_{t \to \infty} x_i(t) = a_i, \ x_i(t) \in U_i \ \forall t \in \mathbb{N} \}$$

has dimension strictly lower than that of $\mathcal{S}^{(d-1)^p}$. On the other hand, it is immediate that the convergence manifold is full dimensional whenever $|S| = 1$.

**Local to global:** Arguing along the lines of theorems 4.21-4.24 in Belkin et al. (2018), using the continuity and injectivity of $G$, we can get a measure zero set $M$ such that for any $\tilde{x} = (x_1, \ldots, x_p) \in \mathcal{S}^{(d-1)^p}/M$, we have $\eta > 0$ and a critical point $(a_1, \ldots, a_p)$ with $|S| > 1$ such that $(G_n(\tilde{x}))_q \in Q_{a_q}$,

$$\max_{1 \leq q \leq p} \left\| \mathcal{P}_S^{(q)}(G_n(\tilde{x}))_q \right\| \to 0, \ \text{and} \ \|G_n(x_1 \otimes \cdots \otimes x_p) - a_1 \otimes \cdots \otimes a_p\|_F \geq \eta,$$

for all sufficiently large $n$. To reduce notation, we use $\mathcal{U}_i$, $\mathcal{A}$ and $\mathcal{D}$ to mean $(u^{(1)}_i \otimes \cdots \otimes u^{(p)}_i)$, $(a_1 \otimes \cdots \otimes a_p)$ and $G_n(x_1 \otimes \cdots \otimes x_p)$ respectively. It can be checked by one application of $G$ that there exist $\varepsilon > 0$ and $i \neq j \in S$ such that

$$\frac{\langle \mathcal{X}, \mathcal{U}_i \rangle / \langle \mathcal{A}, \mathcal{U}_i \rangle}{\langle \mathcal{X}, \mathcal{U}_j \rangle / \langle \mathcal{A}, \mathcal{U}_j \rangle} > 1 + \varepsilon.

We already have that with probability one, any starting point for the gradient iteration satisfies the claim above. We will now see that with each step of the iteration, large inner products (between the estimate tensor and the hidden basis elements) become larger. Because of the norm constraint, this means that the estimate becomes more and more correlated with a particular basis element, eventually converging to it.

**Lemma 2.** Suppose we have $\varepsilon > 0$ and $(x_1 \ldots x_p)$ satisfying $\max_{i,j} \frac{\langle \mathcal{X}, \mathcal{U}_i \rangle / \langle \mathcal{A}, \mathcal{U}_i \rangle}{\langle \mathcal{X}, \mathcal{U}_j \rangle / \langle \mathcal{A}, \mathcal{U}_j \rangle} > 1 + \varepsilon$. Then,

$$\max_{i,j} \frac{\langle G(\mathcal{X}), \mathcal{U}_i \rangle / \langle G(\mathcal{X}), \mathcal{U}_j \rangle}{\langle \mathcal{X}, \mathcal{U}_j \rangle / \langle \mathcal{A}, \mathcal{U}_j \rangle} \geq (1 + \varepsilon)^{p-2} \max_{i,j} \frac{\langle \mathcal{X}, \mathcal{U}_i \rangle}{\langle \mathcal{X}, \mathcal{U}_j \rangle}.$$
Proof of Lemma 2. Let $i,j$ be the indices that maximize $\frac{\langle X_i, U_i \rangle}{\langle X_j, U_j \rangle}/\langle A_i, U_i \rangle$. By the definition of $G$, we have

$$\frac{G(X), U_i}{G(X), U_j} = \lambda_i^p \left( \frac{\langle A_i, U_i \rangle}{\langle A_j, U_j \rangle} \right)^{p-2} \times \frac{\langle X, U_i \rangle}{\langle X, U_j \rangle}$$

$$\geq (1 + \varepsilon)^{p-2} \cdot \lambda_i^p \left( \frac{\langle A_i, U_i \rangle}{\langle A_j, U_j \rangle} \right)^{p-2} \cdot \frac{\langle X, U_i \rangle}{\langle X, U_j \rangle}$$

$$= (1 + \varepsilon)^{p-2} \cdot \frac{\lambda_i^p}{\lambda_j^p} \cdot \frac{\langle X, U_i \rangle}{\langle X, U_j \rangle},$$

where we use equation (47) in the last step.

By lemma 2, we obtain at least one $j \in S$ such that $\langle G_n(X), U_j \rangle \to 0$. By the definition of $G$ it can be checked that this in turn implies that $\max_{1 \leq q \leq p} |\langle G_n(X), U_j^{(q)} \rangle| \to 0$ for some $j \in S$.

Repeated use of lemma 2 gives us that for any starting point $X(0)$ in a probability one set, we have a critical point $(a_1, \ldots, a_p)$ with $S = \{i \in [d] : \langle A_i, U_i \rangle \neq 0\}$ satisfying $|S| > 1$, such that $\|P_S^{(q)}(G_n(X))_q\| \to 0$ and there is at least one $j \in S$ for which $\langle G_n(X), U_j \rangle \to 0$ as $n \to \infty$. We can now repeat the entire argument to get a decreasing sequence $S = S_0 \supset S_1 \supset \ldots S_k$ such that $|S_k| = |S| - k$ and $\langle G_n(X), U_i \rangle \to 0$ for all $i \notin S_k$. Therefore $G_n(X) \to (u_i^{(1)}, \ldots, u_i^{(p)})$ for some $i$ with $\lambda_i > 0$. This finishes the proof of theorem 2.1.