Nonabelian duality and analytic continuation of instantons

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Abstract

Within the framework of the gauge $O(1,3) \times O(1,3)$-theory, an extension of the Belavin-Polyakov-Schwarz-Tyupkin ansatz is proposed by incorporation there the Levi-Civita tensor. The duality properties of the theory, admitting introduction the complex structure, are such that self-duality condition of the field tensor is an equation for complex-analytic function. New type of duality is found out.

1 Introduction

The instanton theory, having started with the paper [1], looks quite completed. Instantons, which seem to describe quantum vacuum transitions, in the Euclidean theory possess nice topological characteristics.

Such a visible geometric treatise is absent in the physical pseudoeuclidean space-time. Still close relation between the Euclidean instantons and self-dual solutions in the pseudoeuclidean space-time may well exist.

To make it clear we treat nonabelian gauge theory with the gauge group $O(1,3)$ in the physical 4-dimensional pseudoeuclidean space-time. It is the Lorenz group $O(1,3)$ that is the part of the motion group. It is significant for us that real group $O(1,3)$ is related to complex groups $L(C)$ and $SL(2, C)$.

Of course on one hand, self-dual solutions can be obtained from the $O(4)$-instantons just by the Euclid rotation. However, we try to show that the pseudoeuclidean spacetime treatment is far more general for $O(1,3)$-self-dual solutions are complex-analytic manifolds hence they include as a special case Euclidean solutions.

Essential point in our approach is that gauge group is a part of the motion group. It permits all indices (regardless internal or the spacetime one) to treat uniformly. This makes the theory simpler in the end. For example, well-known the ‘tHooft symbols $\eta^a_{\mu\nu}$ in the simple way express by Minkowski space tensors $\eta_{\mu\nu}$. Further we are going to apply these ideas to gravity since it can be treated as $O(1,3)$-gauge theory with the group action in tangent space.

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The approach appears to relate to new kind of duality. It means as follows. Initial theory operates on fields $A_\mu = (1/2)F A_{\alpha \beta \mu} T^{\alpha \beta}$ and $F_{\mu \nu} = (1/2)F_{\alpha \beta \mu \nu} T^{\alpha \beta}$, where $T^{\alpha \beta}$ — the internal symmetry group generators, and the field tensor obeys the Bianchi identity: $D^\nu F^\mu_{\nu \mu} \equiv 0$. One can consider a field $\tilde{A}_\alpha = 1/2 \tilde{A}_{\alpha \mu \nu} J^{\mu \nu}$ to be "dual" to starting field ($J^{\mu \nu}$—the space-time Lorenz group generators). The field $\tilde{F}_{\alpha \beta} = (1/2)\tilde{F}_{\alpha \beta \mu \nu} T^{\alpha \beta}$ corresponds to $\tilde{A}_\alpha$. There should be fulfilled an identity $D^\beta \tilde{F}_{\alpha \beta} \equiv 0$ for this field. It is the case in gravity.

2 Extended Belavin-Polyakov-Schwarz-Tyupkin’s ansatz

Let us consider a nonabelian gauge field in pseudoeuclidean spacetime $R_1^4$ with the group $O(1,3)$. The field is described in routine way with giving a potential $A_\mu(x)$ and calculating field strengths $F_{\mu \nu}(x)$ to take values in the Lie algebra of the Lorenz group:

$$A_\mu = \sum_{\alpha<\beta} A_{\alpha \beta \mu} T^{\alpha \beta}, \quad F_{\mu \nu} = \sum_{\alpha<\beta} F_{\alpha \beta \mu \nu} T^{\alpha \beta}$$

Here $T^{\alpha \beta}$, $\alpha < \beta$ — 6 generators of $so(1,3) \simeq su(2) \oplus su(2)$-algebra, which can be considered antisymmetric in $\alpha, \beta$ for calculations convenience. The generators algebra is:

$$[T^{\alpha \beta}, T^{\gamma \delta}] = \frac{1}{2} (\varepsilon^{\alpha \beta \rho \lambda} \varepsilon^{\gamma \delta \sigma \lambda} - \varepsilon^{\alpha \beta \sigma \lambda} \varepsilon^{\gamma \delta \rho \lambda}) T_{\rho \sigma} = -\delta^{\alpha \gamma} T^{\beta \delta} - \delta^{\beta \gamma} T^{\alpha \delta} + \delta^{\alpha \delta} T^{\beta \gamma} + \delta^{\beta \gamma} T^{\alpha \delta}$$

Here the rotation generators $T^{ab}$ are anti-Hermitian. For convenience, the Minkowski tensor denotes $\delta_{\mu \nu}$ or $\delta^{\mu \nu}$.

In the classical paper [1] (BPST), in order to find an explicit solution in the Euclidean $O(4)$-gauge theory, $O(4)$-symmetric ansatz was introduced:

$$A_{\alpha \beta \mu} = f(r) x_\lambda \delta_{\alpha \beta \lambda \mu}, \quad \delta_{\alpha \beta \mu \lambda} = \delta_{\alpha \mu} \delta_{\beta \lambda} - \delta_{\alpha \lambda} \delta_{\beta \mu}.$$  \hspace{1cm} (1)

This was the most general central-symmetric representation for $O(4)$-gauge potential $A_{\mu}$, to have mixed the internal and space-time indices.

Here more general ansatz is proposed:

$$A_{\alpha \beta \mu} = x^\lambda (f(r) \delta_{\alpha \beta \mu \lambda} + g(r) \varepsilon_{\alpha \beta \mu \lambda}),$$ \hspace{1cm} (2)

and gauge group is a Lorenz that $O(1,3)$ to be considered as an internal symmetry group. Such an approach had already been treated in [3].

Motivations of the ansatz extension are quite clear. Technically the use of fully antisymmetric tensor $\varepsilon_{\alpha \beta \mu \lambda}$ together with or instead of $\delta_{\alpha \beta \mu \lambda}$ for representation of the potential $A_{\alpha \beta \mu}$ is quite acceptable and not worse. Additionally [2] gives an opportunity to find a new solution, distinctive feature of that will be possible asymmetry under reflections.
What extent are functions \( u(r) \) and \( v(r) \) independent in?

We’ll see that these functions are parts of a single whole complex-analytic function, if strength field tensor is selfdual.

Now we pass from the real potential \( A_{\alpha\beta\mu} \) of the gauge Lorenz group on to complex potential \( a_{\alpha\beta\mu} \) of \( SL(2, C) \times SL(2, C) \)-group (or \( L(C) \)):

\[
a_{\alpha\beta\mu} = A_{\alpha\beta\mu} + i\ast A_{\alpha\beta\mu}, \quad \ast A_{\alpha\beta\mu} = \frac{1}{2} \varepsilon_{\alpha\beta\rho\sigma} A^{\rho\sigma\mu}
\]

Then with taking into account (2):

\[
a_{\alpha\beta\mu} = (f - ig) x^\lambda (\delta_{\alpha\beta\mu\lambda} + i\varepsilon_{\alpha\beta\mu\lambda}) \quad (4)
\]

By now one can be seen that the potential of \( SL(2, C) \times SL(2, C) \)-group (hence an entire theory) contains complex-analytic function \( f - ig \). We can treat the functions to depend on four variables \( x^i \). Then further one can say of holomorphic continuation of solutions.

It is a tensors of the \( SL(2, C) \)-group appeared here that are t’Hooft’s symbols in the space \( R^4 \):

\[
\delta_{\alpha\mu\nu} + i\varepsilon_{\alpha\mu\nu} = \eta_{\alpha\mu\nu}
\]

Later we are going to consider tensors \( \eta_{\alpha\beta\mu\nu} = \delta_{\alpha\beta\mu\nu} + i\varepsilon_{\alpha\beta\mu\nu} \), main property of which is double sided (anti)selfduality (see below):

\[
\ast \eta_{\alpha\beta\mu\nu} = \eta_{\ast \alpha\beta\mu\nu} = -i\eta_{\alpha\beta\mu\nu}
\]

We’ll be back to this later, and for now go on to work out the ansatz (2). Our actual aim is to calculate (real) field strength tensor. But first let us usher in some notations and compose a list of formulas to require next.

First of all, we introduce dual conjugation operations. For any pair of indices \( \alpha, \beta \), in which a tensor is antisymmetric, we define a dual tensor. For instance, gauge o(1,3)-potential \( A_{\alpha\beta\mu} \) is antisymmetric in \( \alpha, \beta \). Dual tensor \( \ast A_{\alpha\beta\mu} \) is

\[
\ast A_{\alpha\beta\mu} = \frac{1}{2} \varepsilon_{\alpha\beta\rho\sigma} A^{\rho\sigma\mu}
\]

For strength tensor \( F_{\alpha\beta\mu\nu} \) (and for others) it defines the left dual conjugate \( \ast (F_{\alpha\beta\mu\nu}) \), the right dual conjugate \( (F_{\alpha\beta\mu\nu}) \), twice dual conjugate \( \ast (F_{\alpha\beta\mu\nu}) \):

\[
\ast F_{\alpha\beta\mu\nu} = \frac{1}{2} \varepsilon_{\alpha\beta\rho\sigma} F^{\rho\sigma\mu\nu}, \quad F_{\ast \alpha\beta\mu\nu} = \frac{1}{2} F_{\alpha\beta\rho\sigma} \varepsilon^{\rho\sigma\mu\nu}, \quad \ast F_{\ast \alpha\beta\mu\nu} = \frac{1}{2} \varepsilon_{\alpha\beta\rho\sigma} F^{\rho\sigma\gamma\delta} \varepsilon_{\gamma\delta\mu\nu}
\]

E. g.

\[
\ast \delta_{\alpha\beta\mu\nu} = \delta_{\ast \alpha\beta\mu\nu} = \varepsilon_{\ast \alpha\beta\mu\nu}, \quad \varepsilon_{\ast \alpha\beta\mu\nu} = \varepsilon_{\alpha\beta\mu\nu} = -\delta_{\alpha\beta\mu\nu}, \quad \ast \delta_{\ast \alpha\beta\mu\nu} = -\delta_{\alpha\beta\mu\nu}, \quad \ast \varepsilon_{\ast \alpha\beta\mu\nu} = -\varepsilon_{\alpha\beta\mu\nu}
\]

Double left (right) dual conjugation operation gives:

\[
\ast \ast F_{\alpha\beta\mu\nu} = -F_{\alpha\beta\mu\nu} = F_{\ast \ast \alpha\beta\mu\nu}
\]
Twice dual tensor \( *F_{\alpha\beta}^{\mu\nu} \) can be expressed by \( F_{\alpha\beta}^{\mu\nu} \) and its contractions \([4]\):

\[
*F_{\alpha\beta}^{\mu\nu} = -F_{\mu\nu\alpha\beta} + F_{\alpha\mu\delta\beta\nu} + F_{\beta\nu\delta\alpha\mu} - F_{\alpha\nu\delta\beta\mu} - \frac{1}{2} F\delta_{\alpha\beta}^{\mu\nu} \quad (7)
\]

Here

\[
F_{\alpha\mu} = F_{\alpha\nu\mu\nu}, \quad F = F_{\mu}^{\mu}
\]

Professor D. Fairlie kindly informed me of the remarkable Lanczos paper \([5]\), where betweenness relation \( R_{\alpha\beta}^{\mu\nu} \) and \( *R_{\alpha\beta}^{\mu\nu} \) probably first had been obtained for the Euclidean signature space-time.

Further tensor

\[
X_{\alpha\beta}^{\mu\nu} = x_{\alpha}x_{\mu}\delta_{\beta\nu} + x_{\beta}x_{\nu}\delta_{\alpha\mu} - x_{\alpha}x_{\nu}\delta_{\beta\mu} - x_{\beta}x_{\mu}\delta_{\alpha\nu} \quad (8)
\]

often occurs. Here is some properties of the tensor:

\[
*X_{\alpha\beta}^{\mu\nu} = (x_{\nu}\epsilon_{\alpha\beta\mu\lambda} - x_{\mu}\epsilon_{\alpha\beta\nu\lambda})x^{\lambda}, \quad X_{\alpha\beta}^{\mu\nu} = (x_{\beta}\epsilon_{\alpha\mu\nu\lambda} - x_{\alpha}\epsilon_{\beta\mu\nu\lambda})x^{\lambda}, \quad (9)
\]

\[
*X_{\alpha\beta}^{\mu\nu} = X_{\alpha\beta}^{\mu\nu} - r^{2}\delta_{\alpha\beta}^{\mu\nu}, \quad *X_{\alpha\beta}^{\mu\nu} + X_{\alpha\beta}^{\mu\nu} = r^{2}\epsilon_{\alpha\beta}^{\mu\nu},
\]

\[
X_{\alpha\beta} = X_{\beta\mu}^{\mu} = r^{2}\delta_{\alpha\beta}, \quad X = X_{\mu}^{\mu} = 6r^{2}
\]

Now we begin to calculate the strength tensor of \( o(1,3) \)-field \( F_{\alpha\beta}^{\mu\nu} \):

\[
F_{\alpha\beta}^{\mu\nu} = \partial_{\mu}A_{\alpha\beta}^{\nu} - \partial_{\nu}A_{\alpha\beta}^{\mu} + [A_{\mu}, A_{\nu}]_{\alpha\beta}
\]

Using (9), it directly obtains:

\[
\partial_{\mu}A_{\alpha\beta}^{\nu} - \partial_{\nu}A_{\alpha\beta}^{\mu} = -2(f\delta_{\alpha\beta}^{\mu\nu} + g\epsilon_{\alpha\beta}^{\mu\nu}) - \frac{1}{r}(fX_{\alpha\beta}^{\mu\nu} + g'*X_{\alpha\beta}^{\mu\nu})
\]

The commutator calculation require a little more complicated computations. Using the generators algebra and formulas (2)–(9), we find:

\[
[A_{\mu}, A_{\nu}]_{\alpha\beta} = A_{\lambda\beta}^{\alpha\nu}A_{\lambda}^{\mu\nu} - A_{\lambda\mu}^{\alpha\nu}A_{\lambda}^{\beta\nu} = (f^{2} - g^{2})(X_{\alpha\beta}^{\mu\nu} - r^{2}\delta_{\alpha\beta}^{\mu\nu}) - 2fgX_{\alpha\beta}^{\mu\nu}
\]

At last, the strength tensor is obtained:

\[
F_{\alpha\beta}^{\mu\nu} = (-2f - r^{2}(f^{2} - g^{2}))\delta_{\alpha\beta}^{\mu\nu} - (2g + rg')\epsilon_{\alpha\beta}^{\mu\nu} + (f^{2} - g^{2}) - \frac{f'}{r}X_{\alpha\beta}^{\mu\nu} -
\]

\[
(2fg - \frac{g'}{r})X_{\alpha\beta}^{\mu\nu} \quad (10)
\]

So, the field \( F_{\alpha\beta}^{\mu\nu} \) is represented in the form of decomposition in tensors \( \delta_{\alpha\beta}^{\mu\nu}, X_{\alpha\beta}^{\mu\nu} \) and their dual conjugates. This decomposition is rather convenient in order to study the field properties. For example, by means of the use \([6]–[9]\) properties, one can easily find different dual conjugates.

The presence of the tensor \( \epsilon_{\alpha\beta}^{\mu\nu} \) in decomposition \((10)\) suggests that \( T \)- and \( P \)-symmetries are not conserved for the field \((10)\). However, for \( sl(2,C) \)-field (see below) these symmetries are not broken.
Note, that for $g = 0$ a property
\[ F_{\alpha\beta\mu\nu} = F_{\mu\nu\alpha\beta} \]
takes place like in gravity. But it is not the case for (10) because of the last term $X_{*\alpha\beta\mu\nu}$.

The Bianchi identities are of the form:
\[ D_\nu F_{*\alpha\beta\mu} \equiv 0, \tag{11} \]
or as follows:
\[ D_\nu *F_{*\alpha\beta\mu} \equiv 0, \tag{12} \]

Now we pay attention to a new type of nonabelian duality to appear here. High symmetry of the ansatz (2) allows to operate on dual potential $\tilde{A}_\alpha = (1/2)\tilde{A}_{\alpha\mu\nu}J^{\mu\nu}$ with correspondent dual strength $\tilde{F}_{\alpha\beta} = (1/2)\tilde{F}_{\alpha\beta\mu\nu}T^{\alpha\beta}$. Like (10), this field can be formally calculated, not asking (as yet) a question about physical meaning of coordinates $x^\alpha$ of the space to act the internal symmetry group:

\[
\tilde{F}_{\alpha\beta\mu\nu} = (-2f - r^2(f^2 - g^2))\delta_{\alpha\beta\mu\nu} - (2g + rg')\varepsilon_{\alpha\beta\mu\nu} + (f^2 - g^2 - \frac{f'}{r})X_{\alpha\beta\mu\nu} - (2fg - \frac{g'}{r})X_{*\alpha\beta\mu\nu}
\]

This almost coincide with (10) (the difference is in the last term: $*X$ instead of $X*$):

\[
F_{\alpha\beta\mu\nu} = \tilde{F}_{\alpha\beta\mu\nu} + (2fg - \frac{g'}{r})(X_{\alpha\beta\mu\nu} - X_{*\alpha\beta\mu\nu})
\]

This field should obey to Bianchi’s identity, which here is of the form:
\[
\tilde{D}_\beta *F_{\alpha\beta\mu\nu} \equiv 0, \quad \tilde{D}_\beta = \partial_\beta + [\tilde{A}_\beta,]
\]

From this and from the relation between $F$ and $\tilde{F}$ it follows one more identity on the field $\tilde{F}$:
\[
\tilde{D}_\beta (*F_{\alpha\beta\mu\nu} + (\frac{g'}{r} - 2fg)(X_{\alpha\beta\mu\nu} - X_{*\alpha\beta\mu\nu})) \equiv 0
\]
and as well:
\[
\tilde{D}_\beta (*F_{\alpha\beta\mu\nu} + (\frac{g'}{r} - 2fg)(X_{*\alpha\beta\mu\nu} + X_{\alpha\beta\mu\nu})) \equiv 0
\]

Let us be back to the field $F_{\alpha\beta\mu\nu}$. Equations
\[ F_{\alpha\beta\mu\nu} = \pm *F_{\alpha\beta\mu\nu} \tag{13} \]
are important.
If the equations (13) hold, then on account of the Bianchi identity the Yang-Mills equations for the field $F_{\alpha\beta\mu\nu}$ will hold too. Therefore, on the base of (13) it is convenient to introduce [4] tensors $S_{\alpha\beta\mu\nu}$ and $R_{\alpha\beta\mu\nu}$:

$$S_{\alpha\beta\mu\nu} = \frac{1}{2}(F_{\alpha\beta\mu\nu} + \ast F_{\ast\alpha\beta\mu\nu})$$

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}(F_{\alpha\beta\mu\nu} - \ast F_{\ast\alpha\beta\mu\nu})$$

Then now (13) are equivalent to equations:

$$S_{\alpha\beta\mu\nu} = 0, \quad (14)$$

$$R_{\alpha\beta\mu\nu} = 0 \quad (15)$$

These equations are equivalent to the (anti)selfduality conditions. It is true for groups $SL(2,C)(SU(2))$ as well.

The application of these equations to gravity enabled to generalize and extend General Relativity [6]. The gravitational analog of (14) is an inhomogeneous equation and includes an energy-momentum current. It would be not bad to learn something about the structure of the Yang-Mills current taking into account the gravitational analogy and using some phenomenological arguments [4].

The tensors $S_{\alpha\beta\mu\nu}$ can be calculated with using the decomposition (10) and the properties (5)—(9):

$$2S_{\alpha\beta\mu\nu} = (-r^2 f^2 + r^2 g^2 + rf')\delta_{\alpha\beta\mu\nu} - \frac{2}{r^2}X_{\alpha\beta\mu\nu} + (2r^2 fg - rg')\varepsilon_{\alpha\beta\mu\nu} - \frac{2}{r^2}X^*_{\alpha\beta\mu\nu}$$

Then the equation (13) reduces to

$$f' = r(f^2 - g^2), \quad g' = 2rf g$$

Multiplying the second formula by imaginary unit, then plus or minus the first one, we obtain:

$$f' \pm ig' = r(f \pm ig)^2 \quad (16)$$

Similarly, (15) gives

$$2R_{\alpha\beta\mu\nu} = -(4f + rf' + r^2(f^2 - g^2))\delta_{\alpha\beta\mu\nu} - (4g + rg' + 2r^2 fg)\varepsilon_{\alpha\beta\mu\nu} = 0$$

from here

$$4f + rf' + r^2(f^2 - g^2) = 0, \quad 4g + rg' + 2r^2 fg = 0$$

Again, multiplying the second equation by $i$, then plus or minus the first formula, we have

$$4(f \pm ig) + r(f' \pm ig') + r^2(f \pm ig)^2 = 0 \quad (17)$$

So real conditions of (anti)selfduality of the gauge $O(1,3)$-theory within the frame of (2) with real functions $f$ and $g$ are equivalent to the equations (16), (17) for complex-analytic function! 

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1Cosmas Zachos pointed out that expressing covariant derivative of the curvature tensor by the energy-momentum current can be obtained without references to duality just by contracting the Bianchi identity, then taking into account Einstein’s equation.
This result can be easier obtained by going over to $SL(2, C) \times SL(2, C)$-strength:

$$\frac{1}{2} (F_{\alpha \beta \mu \nu} \pm i*F_{\alpha \beta \mu \nu}) = (- (f \mp ig) - r^2 (f \mp ig)^2) (\delta_{\alpha \beta \mu \nu} \pm i\varepsilon_{\alpha \beta \mu \nu}) +$$

$$\frac{1}{2} \left( (f \mp ig)^2 - \frac{1}{r} (f' \mp ig') \right) (X_{\alpha \beta \mu \nu} \pm i* X_{\alpha \beta \mu \nu})$$

And most direct way to obtain an equation for function $f \pm ig$ as a selfduality condition is from the very beginning, on the base $sl(2, C)$-potential:

$$a_\mu = 1 \frac{1}{2} a_{\alpha \beta \mu}^{\mu \beta} = 1 \frac{1}{2} u(\zeta)(\delta_{\alpha \beta \mu \lambda} + i\varepsilon_{\alpha \beta \mu \lambda})x^\lambda t^\alpha$$

where $r \to \zeta$ (and $x^\mu$) — are complex, and

$$t^\alpha \beta = \frac{1}{2} (T^{\alpha \beta} + i4\Gamma^{\alpha \beta}), \ [t^{\alpha \beta}, t^{\gamma \delta}] = -\delta^{\alpha \gamma} t^{\beta \delta} - \delta^{\beta \gamma} t^{\alpha \delta} + \delta^{\beta \delta} t^{\alpha \gamma}$$

To calculate this algebra is helpful to apply a formula:

$$\varepsilon^\alpha \beta \gamma \lambda T^\delta_\lambda = \delta^\delta \alpha T^\beta \gamma + \delta^\delta \beta T^\gamma \alpha + \delta^\delta \gamma T^\alpha \beta$$

Now the strengths are easily calculated:

$$f_{\mu \nu} = \partial_\mu a_\nu - \partial_\nu a_\mu + [a_\mu, a_\nu] = \frac{1}{2} t^{\mu \beta} \left( (\delta_{\alpha \beta \mu \nu} + i\varepsilon_{\alpha \beta \mu \nu})(-u - \frac{1}{2} \zeta^2 u^2) + 

(X_{\alpha \beta \mu \nu} + i* X_{\alpha \beta \mu \nu}) \frac{1}{2}(- \frac{u'}{\zeta} + u^2) \right)$$

Thus the field $f_{\mu \nu}$ is selfdual and complex-analytic manifold, if:

$$\frac{1}{r} u'(\zeta) - u^2(\zeta) = 0.$$  

### 3 Other ansätze

Complexifying by means of introduction into ansatz the Levi-Civita tensor can be carried out for other substitutions as well. We demonstrate it for Corrigan-Fairlie-t’Hooft-Wilczek ansatz \[ (O(1, 3))-gauge group) : \]

$$A_\mu = \frac{1}{2} A_{\alpha \beta \mu} T^{\alpha \beta}, \ A_{\alpha \beta \mu} = \partial_{\lambda} \ln \phi(x) \delta_{\alpha \beta \mu \lambda}. \quad (19)$$

It leads to the field

$$F_{\alpha \beta \mu \nu} = \frac{2}{\phi^2} \phi_{\alpha \beta \mu \nu} - \frac{1}{\phi^2} \phi_{\alpha \beta \mu \nu} - \phi_{\lambda} \phi_{\lambda} \delta_{\alpha \beta \mu \nu} \quad (20)$$
Here
\[ \phi_\lambda = \partial \phi, \quad \phi_{\alpha \beta} = \partial^2 \phi_{\alpha \beta}, \quad \phi_{\alpha \beta \mu \nu} = \phi_{\alpha \phi_{\mu \delta \beta \mu}} + \phi_{\beta \phi_{\delta \alpha \mu}} - \phi_{\alpha \phi_{\delta \beta \mu}} - \phi_{\beta \phi_{\delta \alpha \mu}}, \]
\[ \varphi_{\alpha \beta \mu \nu} = \phi_{\alpha \mu \delta \beta \mu} + \phi_{\beta \delta \alpha \mu} - \phi_{\alpha \phi_{\delta \beta \mu}} - \phi_{\beta \delta \alpha \mu}. \]

Conditions of selfduality [14] and antiselfduality [15] are, respectively, of the form:
\[ S_{\alpha \beta \mu \nu} = \left( \frac{1}{2\phi} \partial^\rho \phi - \frac{\phi_\lambda \phi^\lambda}{\phi^2} \right) \delta_{\alpha \beta \mu \nu} + \frac{2}{\phi^2} \varphi_{\alpha \beta \mu \nu} - \frac{1}{\phi} \varphi_{\alpha \beta \mu \nu} = 0 \quad (21) \]
\[ R_{\alpha \beta \mu \nu} = -\frac{1}{2} \partial^\rho \phi \delta_{\alpha \beta \mu \nu} = 0, \quad (22) \]

We extend this ansatz:
\[ A_{\alpha \beta \mu} = \partial^\lambda \left[ \ln \phi(x) \delta_{\alpha \beta \mu \lambda} + \ln \psi(x) \varepsilon_{\alpha \beta \mu \lambda} \right] \quad (23) \]

sl(2, c) × sl(2, c)-potential is
\[ A_{\alpha \beta \mu} + i A_{\alpha \beta \mu} = \partial^\lambda \left[ \ln \phi - i \ln \psi \right] (\delta_{\alpha \beta \mu \lambda} + i \varepsilon_{\alpha \beta \mu \lambda}) \]

Now we can see that e.g. antiselfduality condition (22) is determined by complex function \( \Phi(x) \)
\[ \ln \Phi = \ln \phi - i \ln \psi = \ln(\phi \exp(-i \ln \psi)), \quad \Phi = \phi \exp(-i \ln \psi), \]

which obeyed d’Alambert equation:
\[ \partial^\rho \phi \Phi = 0 \quad (24) \]

One can continue holomorphically the function \( \Phi \) in variables \( x^\mu \), then take the complex t’Hooft [8] solution to equation (24) with real \( x^\mu \):
\[ \phi \exp(-i \ln \psi) = 1 + \frac{\lambda_1 + i \kappa_1}{(x - x_1)^2} + \frac{\lambda_2 + \kappa_2}{(x - x_2)^2} + \ldots \]

Separating here real and imaginary parts, we find real 6q-parametric generalization of the 6q-instantonic t’Hooft solution.

4 Conclusion

So in some cases, we managed to find some new solutions and continue analytically the instantonic one by means of extension well-known standard substitutions (ansätze). In this connection it is possible a reasoning as follows. All complex-analytic solutions contain real Euclidean ones. On the other hands, complex-analytic solution is analytic continuation of the Euclidean solution carrying topological numbers. Can we assert taking into account holomorphy and
Other models with internal and gauge symmetries could be treated within the approach described. For example, Euclidean $O(2) \times O(2)$-theory which is widely adopted in different branches of physics, particularly in the condensed matter physics [9]. There are instantons in the model to be found by means of $O(2)$-symmetric ansatz. Also the model is quite rich in duality properties admitting complex structure introduction.

The same is right for $O(3)$-models with curls, monopoles etc. But of special interest is the use of these ideas in Gravitation. Unfortunately, direct attempts to do something like that do not pass through. Gravitational gauge potentials in the standard Einsteinian theory (Cristoffel symbols, riemannian connection) $\Gamma_{\alpha\beta\mu}$ in coordinate (holonomic) basis are not antisymmetric in $\alpha, \beta$:

$$\Gamma_{\alpha\beta\mu} = \frac{1}{2}(\partial_\mu g_{\alpha\beta} + \partial_\beta g_{\alpha\mu} - \partial_\alpha g_{\beta\mu}) = \frac{1}{2}(\partial_{\mu} g_{\alpha\beta} + \partial_\nu (g_{\alpha\beta\mu}^{\nu})),$$

what immediately spoils complexifying, i.e. going over group $SL(2, C)$. Further, incorporation of the Levi-Civita tensor into the connection changes geometrical sense of the theory radically. For example:

$$\tilde{\Gamma}_{\alpha\beta\mu} = \Gamma_{\alpha\beta\mu} + \frac{1}{2}\partial_\nu (E_{\alpha\beta\mu}^{\nu})$$

Generally speaking, definition of parallel transport (covariant derivative) and the curvature tensor form are changed. Unclear, whether all that can be done in the consistent way.

Perhaps it would be more logically to treat gravity as a gauge theory. However there are many other problems there.

Utiyama [10] and Kibble [10] were the pioneers in the development of the approach. Their theories did not coincide with General Relativity. In order to obtain the latter one had to attract ones or others additional considerations. Still so far it does not succeed to obtain General Relativity satisfactorily within the gauge approach [12]. One of the problem here is presence of tetrad in the formalism together with other field functions to describe gravity. The tetrad formalism in the pure gravity leads to unnecessary dualism in description.

One more argument for pseudoeuclidean approach rather than Euclidean one. In the [13] method of functional saddle point (functional steepest descent method) had been proposed. It is shown that within the framework of some nonperturbation approximation without any going over "Euclidean" case the transition amplitude $\int \exp(iS)$ is obtained in the form of the (nongaussian) path integral of damping exponential.

By this we want to stress that for any arguments pro 'Euclidean regime' in Physics one finds their contra.

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2There exist different viewpoints to the question
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