Asymptotic behavior at infinity of solutions of Monge-Ampère equations in half spaces

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Abstract

We prove that any convex viscosity solution of \( \det D^2 u = 1 \) outside a bounded domain of \( \mathbb{R}^n_+ \) tends to a quadratic polynomial at infinity with rate at least \( \frac{|x|}{|x|^n} \) if \( u \) is a quadratic polynomial on \( \{ x_n = 0 \} \) and satisfies \( \mu |x|^2 \leq u \leq \mu^{-1} |x|^2 \) as \( |x| \to \infty \) for some \( 0 < \mu \leq \frac{1}{2} \).

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1. Introduction

In this paper we investigate the asymptotic behavior at infinity of convex viscosity solution of the Monge-Ampère equation

\[
\begin{cases}
\det D^2 u = f & \text{in } \mathbb{R}^n_+,
\vspace{2mm}
u = p(x') & \text{on } \{x_n = 0\},
\end{cases}
\tag{1.1}
\]

where the space dimension \( n \geq 2 \), \( p(x') \) is a quadratic polynomial of \( n - 1 \) variables and \( f \in C^0(\mathbb{R}^n_+) \) satisfies

\[
0 < \lambda \leq \inf_{\mathbb{R}^n_+} f \leq \sup_{\mathbb{R}^n_+} f \leq \Lambda < \infty.
\tag{1.2}
\]

It is also assumed that for some \( R_0 > 0 \),

\[
\Omega_0 := \text{support}(f - 1) \subset B^+_R,
\tag{1.3}
\]

and for some \( 0 < \mu \leq \frac{1}{2} \),

\[
\mu |x|^2 \leq u(x) \leq \mu^{-1} |x|^2 \quad \text{in } \mathbb{R}^n_+ \setminus B^+_R.
\tag{1.4}
\]

Condition (1.4) implies that \( u \) is quadratically increasing in \( \mathbb{R}^n_+ \setminus B^+_R \) and that \( p(x') \) is non-degenerate and strictly convex. A counterexample will be given in Section 2 to show the necessity of (1.4).
K. Jögens \((n = 2, \text{see } [14])\), E. Calabi \((n \leq 5, \text{see } [7])\) and A. V. Pogorelov \((n \geq 2, \text{see } [16])\) proved that any classical convex solution of \(\det D^2 u = 1\) in \(\mathbb{R}^n\) is a quadratic polynomial. In \([4, 5]\), L. A. Caffarelli extended above result to viscosity solutions. The asymptotic behavior at infinity of viscosity solution of \(\det D^2 u = 1\) outside a bounded subset of \(\mathbb{R}^n\) was obtained by L. A. Caffarelli and Y. Y. Li in \([5]\). The main conclusion of \([5]\) is that for \(n \geq 3\), \(u\) tends to a quadratic polynomial at infinity with rate at least \(|x|^{2-n}\); for \(n = 2\), \(u\) tends to a quadratic polynomial plus \(d \log |x|\) at infinity with rate at least \(|x|^{-1}\), where \(d\) is a constant. When \(n = 2\), L. Ferrer, A. Martínez and F. Milán obtained the same result using complex variable methods (see \([9, 10]\)).

As for Monge-Ampère equations in half spaces, if \(u\) is the viscosity convex solution of \((1.1)\) with \(f \equiv 1\) and satisfies \((1.4)\), then \(u\) is a quadratic polynomial. This is a well known result and was mentioned in \([15, 18]\). We will give it as a corollary of our main theorem.

The aim of this paper is to investigate the case as \(f \not\equiv 1\) in the half space and our main result is:

**Theorem 1.1.** Let \(p(x')\) be a quadratic polynomial of \(n - 1\) variables and \(f \in C^0(\mathbb{R}^n_+)\) satisfy \((1.2)\) and \((1.3)\). Assume that \(u\) is a convex viscosity solution of \((1.1)\) such that \((1.4)\) holds. Then there exist some symmetric positive definite matrix \(A\) with \(\det A = 1\), vector \(b \in \mathbb{R}^n\) and constant \(c \in \mathbb{R}\) such that

\[
\left| u(x) - \left( \frac{1}{2} x^T A x + b \cdot x + c \right) \right| \leq C \frac{x_n}{|x|^n} \quad \text{in } \mathbb{R}^n_+ \setminus B_R^+, \quad (1.5)
\]

where \(x = (x', x_n)\), and \(C\) and \(R\) depend only on \(R_0, \mu\) and \(n\). Moreover, \(u \in C^\infty(\mathbb{R}^n_+ \setminus \Omega_0)\) and for any \(k \geq 1\),

\[
|x|^{n-1+k} \left| D^k \left( u(x) - \frac{1}{2} x^T A x - b \cdot x - c \right) \right| \leq C \quad \text{in } \mathbb{R}^n_+ \setminus B_R^+, \quad (1.6)
\]

where \(C\) also depends on \(k\).

**Remark 1.2.** (i) By \((1.5)\) and the boundary condition in \((1.1)\), we have the following compatibility condition

\[
p(x') = \frac{1}{2} (x', 0)^T A(x', 0) + b \cdot (x', 0) + c.
\]

(ii) Observe that in Theorem 1.1, the approximation rate of \(u\) to the quadratic polynomial at infinity is \(\frac{x_n}{|x|^n}\), which is the Poisson kernel of Laplace’s equation in \(\mathbb{R}^n_+\). Nevertheless, in \(\mathbb{R}^n (n \geq 3)\), by \([5, \text{Theorem 1.2}]\), the approximation rate is \(|x|^{2-n}\), which is the fundamental solution of Laplace’s equation in \(\mathbb{R}^n\).

(iii) Since we have the boundary condition, there is no difference in our result between \(n \geq 3\) and \(n = 2\). However, in the whole space, the results for \(n \geq 3\) and for \(n = 2\) are different (see \([5, \text{Theorem 1.2}]\)).
The proof of Theorem 1.1 borrows the idea of [5] and we separate it into two steps: nonlinear approach and linear approach. In nonlinear approach, we show that there exist some matrix $T$ and some constant $\epsilon > 0$ such that $|u(Tx) - \frac{1}{2}|x|^2| = O(|x|^{2-\epsilon})$ as $|x| \to \infty$ by Pogorelov estimates in half domain and the comparison principle, where auxiliary functions are constructed via solving scaled problems. In linear approach, we obtain some linear function $l(x)$ such that $|u(Tx) - \frac{1}{2}|x|^2| - l(x)| = O(|x^n|)$ at infinity by linearizing the equation and using asymptotic behavior of linear elliptic equations in half spaces.

The following corollary is a simple consequence of Theorem 1.1, which was also mentioned in [15, 18].

**Corollary 1.3.** Let $u$ be a convex viscosity solution of
\[
\begin{align*}
\det D^2 u &= 1 & \text{in } \mathbb{R}^n_+,
\text{det} A &= 1, \\
\det A &= 1, & v &= p(x') & \text{on } \{x_n = 0\}
\end{align*}
\]
and satisfy (1.4), where $p(x')$ is a quadratic polynomial. Then $u$ is a quadratic polynomial.

Our next theorem gives the existence of solutions of (1.1) with prescribed asymptotic behavior at infinity.

**Theorem 1.4.** Let $f \in C^0(\mathbb{R}^n_+)$ satisfy (1.2) and (1.3). Then for any symmetric positive definite matrix $A$ with $\text{det} A = 1$, vector $b \in \mathbb{R}^n$, constant $c \in \mathbb{R}$ and quadratic polynomial $p(x') = \frac{1}{2}(x',0)^T A(x',0) + b \cdot (x',0) + c$, there exists a unique convex solution $u$ of (1.1) satisfying
\[
\lim_{|x| \to \infty} u(x) = \frac{1}{2}x^T Ax + b \cdot x + c.
\]

The paper is organized as follows. In Section 2, we list some fundamental results on Monge-Ampère equations, give as a corollary estimates of derivatives of Monge-Ampère equations in half spaces, and demonstrate a counterexample to show the necessity of (1.4). In Section 3, we investigate the asymptotic behavior at infinity of solutions of a class of linear elliptic equations in half spaces, which will be used in the second step of the proof of Theorem 1.1. In Section 4, we show Theorem 1.1 by two steps as mentioned above. In Section 5, Corollary 1.3 and Theorem 1.4 are proved.

Throughout this paper, we use the following standard notations.

- For any $x \in \mathbb{R}^n$, $x = (x_1, x_2, \ldots, x_n) = (x', x_n)$, $x' \in \mathbb{R}^{n-1}$.
- $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$, $\mathbb{R}^n_{++} = \{x \in \mathbb{R}^n : x_n \geq 0\}$.
- For any $x \in \mathbb{R}^n$ and $r > 0$, $B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\}$ and $B_r^+(0) = B_r(0) \cap \{x_n > 0\}$. $B_r = B_r(0)$ and $B_r^+ = B_r^+(0)$.
- For any $r > 0$, $Q^r = \{x \in \mathbb{R}^n : |x'| < r, \ r > x_n > 0\}$.

2. Preliminary results for the Monge-Ampère equation

In this section we list some basic definitions and results on Monge-Ampère equations as follows (see [11, 13]).
Definition 2.1. Let $\Omega$ be an open subset of $\mathbb{R}^n$, $u \in C(\Omega)$ be a convex function and $f \in C(\Omega)$, $f \geq 0$. The convex function $u$ is a viscosity subsolution (supersolution) of the equation $\det D^2u = f$ in $\Omega$ if whenever convex $\phi(x) \in C^2(\Omega)$ and $x_0 \in \Omega$ are such that $(u-\phi)(x) \leq (\geq)(u-\phi)(x_0)$ for all $x$ in a neighborhood of $x_0$, then we must have

$$\det D^2\phi(x_0) \geq (\leq)f(x_0).$$

If $u$ is a viscosity subsolution and supersolution, we call it viscosity solution.

Definition 2.2. The normal mapping of $u$ at $x_0$ is the set-valued function $\partial u : \Omega \to \mathbb{R}^n$ defined by

$$\partial u(x_0) = \{l \in \mathbb{R}^n : u(x) \geq u(x_0) + l \cdot (x - x_0) \quad \forall x \in \Omega\}.$$

For any subset $E \subset \Omega$, we define $\partial u(E) = \bigcup_{x \in E} \partial u(x)$.

Definition 2.3. Let $\Omega$ be an open and convex subset of $\mathbb{R}^n$ and $\nu$ be a finite Borel measure in $\Omega$. The convex function $u \in C(\Omega)$ is called a generalized solution of the Monge-Ampère equation

$$\det D^2u = \nu$$

if for any Borel set $E \subset \Omega$, there is $|\partial u(E)| = \nu(E)$, where $|\partial u(E)|$ is the Lebesgue measure of the normal mapping set $\partial u(E)$.

In particular, if $\nu = fdx$ for integrable $f \geq 0$ in $\Omega$, we denote $\det D^2u = f$.

Definition 2.4. For any convex function $u$ defined on a convex domain $\overline{\Omega}$, if $L(x) = u(x_0) + l \cdot (x - x_0)$ is a supporting plane to $u$ at $(x_0, u(x_0))$, we denote by $S_h(u, l, x_0)$ the cross section centered at $x_0$ and height $h > 0$

$$S_h(u, l, x_0) = \{x \in \overline{\Omega} : u(x) < L(x) + h\}.$$

If $u$ is of $C^1$, $l$ is unique and equals $Du(x_0)$. For simplicity, we denote $S_h(u, l, x_0)$ by $S_h(u)$ if there is no confusion of $l$ and $x_0$.

Now we give the existence and uniqueness of generalized solutions to the Dirichlet problem for Monge-Ampère equations, which was proved by A. D. Aleksandrov [1] and I. J. Bakel’man [2] (see also [19, Theorem 2.1]).

Theorem 2.5. Let $\Omega$ be a bounded open convex domain in $\mathbb{R}^n$, $\nu$ be a finite nonnegative measure and $\varphi \in C(\overline{\Omega})$ be convex in $\Omega$. Then there is a unique generalized solution $u \in C(\overline{\Omega})$ of

$$\begin{cases} 
\det D^2u = \nu & \text{in } \Omega, \\
u = \varphi & \text{on } \partial \Omega. 
\end{cases} \quad (2.1)$$

Remark 2.6. If $\Omega$ is strictly convex, we can replace convex $\varphi \in C(\overline{\Omega})$ by $\varphi \in C^0(\partial \Omega)$ (see [11, 13]).

Theorem 2.7 (Comparison Principle). Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ and $u, v \in C(\overline{\Omega})$ be locally convex functions. Assume that for any Borel set $E \subset \Omega$, $|\partial u(E)| \geq |\partial v(E)|$ and $u \leq v$ on $\partial \Omega$. Then $u \leq v$ in $\Omega$. 

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The following theorem shows the equivalence of generalized solutions and viscosity solutions if \( f \in C(\overline{\Omega}) \) is positive (see [13]).

**Theorem 2.8.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \), \( f \in C(\overline{\Omega}) \) with \( f \geq 0 \) in \( \Omega \). Then any generalized solution of \( \det D^2 u = f \) is a viscosity solution of \( \det D^2 u = f \). Furthermore, if \( f > 0 \) in \( \Omega \), any viscosity solution of \( \det D^2 u = f \) is a generalized solution of \( \det D^2 u = f \).

**Remark 2.9.** Theorem 2.8 allows us to apply Theorem 2.5 and Theorem 2.7 to viscosity solutions of \( \det D^2 u = f \) for any continuous and positive \( f \).

If \( f \) is smooth, the generalized solution will be smooth as the following theorem shows (see [8]).

**Theorem 2.10.** Let \( \Omega \) be a bounded open convex domain in \( \mathbb{R}^n \), \( f \in C^0(\overline{\Omega}) \cap C^\infty(\Omega) \) be a positive function. Then there exists a unique convex solution \( w \in C^0(\overline{\Omega}) \cap C^\infty(\Omega) \) of

\[
\begin{aligned}
\det D^2 w &= f \quad \text{in } \Omega, \\
w &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

**Remark 2.11.** If \( \Omega \) is strictly convex with boundary \( \partial \Omega \in C^\infty \) and \( f \in C^\infty(\overline{\Omega}) \), then \( u \) belongs to \( C^\infty(\overline{\Omega}) \) (see [6]).

Next we introduce the Pogorelov estimate in half domain, which was obtained by O. Savin and played an important role in establishing the boundary pointwise \( C^{2,\alpha} \) estimates when the domain is not strictly convex (see [17, Proposition 6.1, Remark 6.3 and Theorem 6.4]). It is also crucial to establish our main results.

**Theorem 2.12** (Pogorelov estimate in half domain). Let \( u \in C(\overline{\Omega}) \) be a convex viscosity solution of

\[
\det D^2 u = 1 \quad \text{in } \Omega.
\]

Assume that for some constants \( \rho_1, \rho_2, \rho_3 > 0 \),

\[
B^+_{\rho_1} \subset \Omega \subset B^+_{\rho_1^{-1}}
\]

and

\[
\begin{aligned}
u = p(x') & \quad \text{on } \{x_n = 0\} \cap \partial \Omega, \\
\rho_2 \leq u \leq \rho_2^{-1} & \quad \text{on } \{x_n > 0\} \cap \partial \Omega,
\end{aligned}
\]

where \( p(x') \) is a quadratic polynomial that satisfies

\[
\rho_3 |x'|^2 \leq p(x') \leq \rho_3^{-1} |x'|^2.
\]

Then there exists \( c_0 > 0 \) depending only on \( \rho_1, \rho_2, \rho_3 \) and \( n \) such that

\[
||u||_{C^{3,1}(\overline{B_{\rho_0}^+})} \leq c_0^{-1}.
\]

(2.2)
Corollary 2.13. Let \( w \in C^\infty(\mathbb{R}_+^n \setminus B_1^+) \) satisfy
\[
[I_n + D^2 w] > 0, \quad \det(I_n + D^2 w) = 1 \quad \text{in } \mathbb{R}_+^n \setminus B_1^+, \\
w(x) = 0 \quad \text{on } \{ x : |x'| \geq 1, x_n = 0 \}
\] (2.3)
and for some constants \( \beta > 0 \) and \( \gamma > -2 \),
\[
|w(x)| \leq \frac{\beta}{|x|^\gamma} \quad \text{in } \mathbb{R}_+^n \setminus B_1^+.
\] (2.4)

Then there exists \( R_0 \geq 1 \) depending only on \( n, \beta \) and \( \gamma \) such that for any \( k \geq 1 \),
\[
|D^k w(x)| \leq \frac{C}{|x|^\gamma+k} \quad \text{in } \mathbb{R}_+^n \setminus B_{R_0}^+,
\]
where \( C \) depends only on \( \beta, \gamma, k \) and \( n \).

Proof. We prove this corollary by two steps.

Step 1. For any \( x_0 \in \{ |x| = R \geq R_0, x_n = 0 \} \) and any \( k \geq 1 \),
\[
|D^k w(x)| \leq \frac{C}{|x|^\gamma+k} \quad \text{in } \overline{B}_{R_0}^+(x_0),
\] (2.5)
where \( \theta > 0 \) and \( R_0 \geq 1 \) depend only on \( \beta, \gamma \) and \( n \), and \( C \) depends only on \( \beta, \gamma, k \) and \( n \).

Indeed, for any \( x_0 \in \{ |x| = R \geq 3, x_n = 0 \} \), let
\[
\eta(x) = w(x) + \frac{1}{2}|x - x_0|^2, \quad x \in \mathbb{R}_+^n,
\]
\[
\eta_R(y) = \left( \frac{4}{R} \right)^2 \eta \left( x_0 + \frac{4}{R}y \right), \quad y \in B_2^+
\]
and
\[
w_R(y) = \left( \frac{4}{R} \right)^2 w \left( x_0 + \frac{4}{R}y \right) = \eta_R(y) - \frac{1}{2}|y|^2, \quad y \in B_2^+.
\]

In view of (2.3), we have
\[
[D^2 \eta_R(y)] > 0, \quad \det D^2 \eta_R(y) = 1 \quad \text{in } B_2^+,
\]
\[
\eta_R(y) = \frac{1}{2}|y|^2 \quad \text{on } \partial B_2^+ \cap \{ y_n = 0 \}.
\]

By (2.4), there exists \( R_0 \geq 1 \) depending only on \( \beta \) and \( \gamma \) such that for any \( R \geq R_0 \) and any \( y \in \partial B_2 \cap \{ y_n \geq 0 \} \),
\[
1 \leq \frac{1}{2}|y|^2 - \frac{16\beta}{R^2} \left| x_0 + \frac{R}{4}y \right|^{-\gamma} \leq \eta_R(y) \leq \frac{1}{2}|y|^2 + \frac{16\beta}{R^2} \left| x_0 + \frac{R}{4}y \right|^{-\gamma} \leq 3.
\] (2.6)
Then by Theorem 2.12 there exists $c_0 > 0$ depending only on $n$ such that

$$\|\eta_R(y)\|_{C^{\beta,\alpha}(\overline{B}_{c_0})} \leq c_0^{-1}. $$

In particular, combining with $\det D^2\eta_R(y) = 1$, it implies that

$$C^{-1}I_n \leq [D^2\eta_R(y)] \leq CI_n \quad \text{in } y \in \overline{B}_{c_0},$$

where $C$ depending only on $n$.

Differentiating $\ln(\det D^2\eta_R) = 0$ with respect to $y_k$, we have

$$a_{ij}(y)D_{ij}(\eta_R)_k(y) = 0 \quad \text{in } B_{c_0}^+, $$

where $a_{ij}(y) = \{[D^2\eta_R]^{-1}\}_{ij}(y)$. By Schauder estimates, we have for any $k \geq 1$,

$$\|\eta_R(y)\|_{C^k(\overline{B}_{c_0/2})} \leq C, $$

where $C$ depends only on $n$ and $k$. Combining with (2.7), it follows that

$$\|w_R(y)\|_{C^k(\overline{B}_{c_0/2})} \leq C, \quad C^{-1}I_n \leq (I_n + D^2w_R) \leq CI_n \quad \text{on } \overline{B}_{c_0/2}. $$

(2.8) follows that for any $k \geq 1$,

$$\|\tilde{a}_{ij}\|_{C^k(\overline{B}_{c_0/2})} \leq C, \quad C^{-1}I_n \leq [\tilde{a}_{ij}] \leq CI_n \quad \text{on } \overline{B}_{c_0/2}. $$

By Schauder estimates, we have for any $k \geq 1$, in $\overline{B}_{c_0/4},$

$$|D^k w_R(y)| \leq C\|w_R(y)\|_{L^{\infty}(\overline{B}_{c_0/4})} \leq CR^{-\gamma-2} \quad \text{(by (2.4))},$$

where $C$ depends only on $\beta, \gamma, k$ and $n$. It yields that

$$|D^k w(x)| \leq \frac{C}{|x|^{\gamma+k}} \quad \text{in } \overline{B}_{\theta R}(x_0)$$

for any $k \geq 1$ and $\theta = \frac{1}{16}c_0$, where $C$ depends only on $\beta, \gamma, k$ and $n$.

**Step 2.** For any $x_0 \in \{|x| = R \geq R_0, x_n \geq \frac{1}{2}\theta R\}$ and any $k \geq 1$,

$$|D^k w(x)| \leq \frac{C}{|x|^{\gamma+k}} \quad \text{in } B_{\frac{1}{16}\theta R}(x_0),$$

where $\theta$ and $R_0$ is given by Step 1 and $C$ depends only on $\beta, \gamma, k$ and $n$.

In fact, since the proof is similar to that of [3, Lemma 3.5], we omit it. \qed
Example 2.14. We give a counterexample to demonstrate the necessity of (1.4), which was mentioned in [15, 18].

For $n = 2$, we consider solutions of the following Monge-Ampère equation

$$
\begin{aligned}
\begin{cases}
\det D^2u = 1 & \text{in } \mathbb{R}^2_+,
\quad \\
u = \frac{1}{2}x^2_1 & \text{on } \{x_2 = 0\}.
\end{cases}
\end{aligned}
$$

(2.9)

Suppose $u(x)$ satisfies

$$u(x_1, x_2) = f(x_1)g(x_2) + w(x_2)$$

with some one variable functions $f(x_1)$, $g(x_2)$ and $w(x_2)$. On $\{x_2 = 0\}$,

$$u(x_1, 0) = \frac{1}{2}x^2_1 = f(x_1)g(0) + w(0).$$

Assume that $g(0) = 1$. Then $f(x_1) = \frac{1}{2}x^2_1$ and $w(0) = 0$.

By $\det D^2u = 1$, we have

$$\frac{1}{2}x^2_1(g''(x_2)(x_2) - 2(g'(x_2))^2) + g(x_2)w''(x_2) = 1. \quad (2.10)$$

Since $x_1$ in (2.10) is arbitrary, we have

$$g(x_2)g''(x_2) - 2(g'(x_2))^2 = 0, \quad g(x_2)w''(x_2) = 1.$$  

By a simple calculation, we get

$$g(x_2) = \frac{1}{x_2 + 1}, \quad w(x_2) = \frac{1}{6}x_2^3 + \frac{1}{2}x_2^2.$$  

Then

$$u(x_1, x_2) = \frac{x^2_1}{2(x_2 + 1)} + \frac{x_2^3 + 3x_2^2}{6}$$

solves (2.9) and is convex. However, $u$ is not a quadratic polynomial.

For the higher dimensional space,

$$u(x', x_n) = \frac{|x'|^2}{2(x_n + 1)} + \frac{x_n^{n+1} + (n + 1)x_n^n}{n(n + 1)}$$

is a counterexample.

3. Asymptotic behavior of linear elliptic equation in half space

In this section we study the asymptotic behavior at infinity of solutions of a class of linear elliptic equations outside a bounded domain of $\mathbb{R}^n_+$, which will used in the second step of the proof of Theorem [11]. Related results outside a bounded domain of $\mathbb{R}^n$ were given by D. Gilbarg and J. Serrin (see [12]).

We first show two auxiliary lemmas by Harnack inequality and maximum principle.
Lemma 3.1. Let \( u(x) \) solve the following Dirichlet problem

\[
\begin{aligned}
  a_{ij}(x)D_{ij}u(x) &= 0 & \text{in } B_{4R}^+ \backslash \overline{B}_R^+, \\
  u(x) &\leq 1 & \text{on } \partial(B_{4R}^+ \backslash \overline{B}_R^+) \cap \{x_n > 0\}, \\
  u(x) &\leq \frac{1}{2} & \text{on } \partial(B_{4R}^+ \backslash \overline{B}_R^+) \cap \{x_n = 0\},
\end{aligned}
\]  

(3.1)

where \( a_{ij}(x) \in C^0(B_{4R}^+ \backslash \overline{B}_R^+) \) and \( \lambda I \leq [a_{ij}(x)] \leq \Lambda I \) in \( B_{4R}^+ \backslash \overline{B}_R^+ \) for some \( R > 0 \) and \( 0 < \lambda \leq \Lambda < \infty \). Then there exists \( \varepsilon_0 > 0 \) depending only on \( \lambda, \Lambda \) and \( n \) such that

\[
u(x', x_n) \leq 1 - \varepsilon_0 \quad \text{on } \partial B_{2R} \cap \{x_n \geq 0\}.
\]

Proof. Without loss of generality, we assume that \( R = 1 \).

By the third inequality in (3.1) and classical Hölder continuity theory up to the boundary, there exists constant \( 0 < \delta \leq 1 \) depending only on \( \lambda, \Lambda \) and \( n \) such that

\[
u(x) \leq \frac{2}{3} \quad \text{on } \partial B_2 \cap \{0 \leq x_n \leq \delta\}. \tag{3.2}
\]

Applying interior Harnack inequality to \( 1 - u \), there exists a positive constant \( C \geq 1 \) depending only on \( \lambda, \Lambda \) and \( n \) such that

\[
C \inf_{\partial B_2 \cap \{x_n \geq \delta\}} (1 - u) \geq \sup_{\partial B_2 \cap \{x_n \geq \delta\}} (1 - u) \geq \sup_{\partial B_2 \cap \{x_n = \delta\}} (1 - u) \geq \frac{1}{3}.
\]

It yields that

\[
u(x) \leq 1 - \frac{1}{3C} \quad \text{on } \partial B_2 \cap \{x_n \geq \delta\}. \tag{3.3}
\]

By (3.2) and (3.3), the proof is completed with \( \varepsilon_0 = \frac{1}{3C} \).

Lemma 3.2. Let \( u(x) \in C^2(\mathbb{R}^n_+ \backslash B_{R_0}^+) \) be a solution of

\[
a_{ij}(x)D_{ij}u(x) = 0 \quad \text{in } \mathbb{R}^n_+ \backslash B_{R_0}^+, \tag{3.1}
\]

where \( \lambda I \leq [a_{ij}(x)] \leq \Lambda I \) in \( \mathbb{R}^n_+ \backslash B_{R_0}^+ \) for some \( R_0 > 0 \) and \( 0 < \lambda \leq \Lambda < \infty \). Assume that \( |u(x)| \leq 1 \) on \( \partial B_{R_0} \cap \{x_n > 0\} \cup \{x_n = 0, |x| \geq R_0\} \), \( u(x', 0) \to 0 \) as \( |x'| \to \infty \) and \( |Du(x)| \to 0 \) as \( |x| \to \infty \). Then \( |u| \leq 1 \) in \( \mathbb{R}^n_+ \backslash B_{R_0}^+ \).

Proof. For any \( \varepsilon > 0 \), by \(|Du| \to 0\) as \(|x| \to \infty\), there exists \( R_\varepsilon \geq R_0 \) such that

\[
|Du| \leq \varepsilon \quad \text{in } \mathbb{R}^n_+ \backslash Q_{R_\varepsilon}^+, \tag{3.4}
\]

where \( Q_{R_\varepsilon}^+ = \{(x', x_n) : |x'| < R_\varepsilon, 0 < x_n < R_\varepsilon\} \) is a cylinder.

Since \(|u| \leq 1\) on \( \{x_n = 0, |x| \geq R_0\} \), by (3.4) and Newton-Leibniz formula, we get

\[
|u(x)| \leq 1 + 2\varepsilon x_n \quad \text{on } \partial Q_{R_\varepsilon}^+ \cap \{x_n > 0\}.
\]

Combining with \(|u| \leq 1\) on \( \partial B_{R_0} \cap \{x_n > 0\} \cup \{x_n = 0, |x| \geq R_0\} \), it implies that

\[
|u(x)| \leq 1 + 2\varepsilon x_n \quad \text{on } \partial (Q_{R_\varepsilon}^+ \backslash B_{R_0}^+).
\]
By the comparison principle, we have
\[ |u(x)| \leq 1 + 2 \varepsilon x_n \quad \text{in} \quad Q_{R_{0}}^{0} \setminus B_{R_{0}}^{0}.\]
Letting \( \varepsilon \to 0 \), we have \( |u(x)| \leq 1 \) in \( \mathbb{R}^{n} \setminus B_{R_{0}}^{0} \).

The following two lemmas are our main results in this section.

**Lemma 3.3.** Let \( u(x) \in C^{2}(\overline{\mathbb{R}^{n}} \setminus B_{R_{0}}^{0}) \) be a solution of
\[ a_{ij}(x)D_{ij}u(x) = 0 \quad \text{in} \quad \mathbb{R}^{n} \setminus B_{R_{0}}^{0}, \]
where \( a_{ij}(x) \in C^{0}(\overline{\mathbb{R}^{n}} \setminus B_{R_{0}}^{0}) \) and \( \lambda I \leq [a_{ij}(x)] \leq \Lambda I \) in \( \overline{\mathbb{R}^{n}} \setminus B_{R_{0}}^{0} \) for some \( R_{0} > 0 \) and \( 0 < \lambda \leq \Lambda < \infty \). Assume that \( |u| \leq 1 \) on \( \partial B_{R_{0}} \cap \{ x_{n} = 0 \} \cup \{ x_{n} = 0, |x| \geq R_{0} \} \), \( u(x', 0) \to \beta \) as \( |x'| \to \infty \) and \( Du(x) \to 0 \) as \( |x| \to \infty \). Then \( u(x) \to \beta \) as \( |x| \to \infty \).

**Proof.** Suppose that \( \beta = 0 \). Otherwise, we consider
\[ (u(x) - \beta)/(1 + |\beta|).\]
By Lemma 3.2, we have \( |u(x)| \leq 1 \) in \( \mathbb{R}^{n} \setminus B_{R_{0}}^{0} \). Then \( u \) has finite superior limit \( \overline{u} \) and inferior limit \( \underline{u} \) at infinity. By \( u(x', 0) \to 0 \) as \( |x'| \to \infty \), we have \( \overline{u} \geq 0 \geq \underline{u} \).

Now we argue by contradiction. If this lemma is not true, then \( \overline{u} > 0 \) or \( \underline{u} < 0 \). We may assume \( \overline{u} > 0 \). Otherwise, we consider \( -u \).

Let \( \varepsilon_{0} \) be given by Lemma 3.1. By the definition of \( \overline{u} \), there exists large \( R_{1} \geq R_{0} \) such that for all \( R \geq R_{1} \),
\[ u(x) \leq (1 + \frac{\varepsilon_{0}}{2})\overline{u} \quad \text{in} \quad \mathbb{R}^{n} \setminus B_{R}^{0} \]
and
\[ u(x', 0) \leq \frac{1}{2}(1 + \frac{\varepsilon_{0}}{2})\overline{u} \quad \text{on} \quad \{ x_{n} = 0, |x'| > R \}.\]
Applying Lemma 3.1 to \( \frac{u(x)}{(1 + \varepsilon_{0}/2)^{n}} \) in \( B_{1/4R}^{0} \setminus B_{R}^{0} \), we obtain that
\[ u(x) \leq (1 - \varepsilon_{0})(1 + \frac{\varepsilon_{0}}{2})\overline{u} \leq (1 - \frac{\varepsilon_{0}}{2})\overline{u} \quad \text{on} \quad \partial B_{2R} \cap \{ x_{n} = 0 \}.\]
However, by the arbitrariness of \( R \geq R_{1} \), we have
\[ u(x) \leq (1 - \frac{\varepsilon_{0}}{2})\overline{u} \quad \text{in} \quad \overline{\mathbb{R}^{n}} \setminus B_{2R_{1}}^{0}, \]
which contradicts the definition of \( \overline{u} \).

**Lemma 3.4.** Let \( u(x) \in C^{2}(\overline{\mathbb{R}^{n}} \setminus B_{R_{0}}^{0}) \) be a solution of
\[
\begin{cases}
  a_{ij}(x)D_{ij}u(x) = 0 & \text{in} \quad \mathbb{R}^{n} \setminus B_{R_{0}}^{0}, \\
  u(x) = 0 & \text{on} \quad \{ x_{n} = 0, |x| \geq R_{0} \},
\end{cases}
\]
(3.5)
where $a_{ij}(x) \in C^0(\mathbb{R}^n_+ \setminus B^+_R)$, $\lambda I \leq [a_{ij}(x)] \leq \Lambda I$ and $|a_{ij}(x) - \delta_{ij}| \leq C|x|^{-s}$ in $\mathbb{R}^n_+ \setminus B^+_R$ for some $R_0 > 0$, $s > 0$ and $0 < \lambda \leq \Lambda < \infty$. Assume that $|u| \leq 1$ on $\partial B_{R_0} \cap \{x_n > 0\}$, $|Du(x)| \leq 1$ in $\mathbb{R}^n_+ \setminus B^+_R$ and $|Du(x)| \to 0$ as $|x| \to \infty$. Then

$$|u(x)| \leq C \frac{x_n}{|x|^n} \text{ in } \mathbb{R}^n_+ \setminus B^+_R,$$

(3.6)

where $C$ and $R \geq R_0$ are positive constants depending only on $C_1$, $R_0$, $s$ and $n$.

Proof. By Lemma 3.3 we have

$$u(x) \to 0 \quad \text{as } |x| \to \infty.$$

Fix $0 < \delta < \min\{1, \frac{s}{n-1}\}$ and let

$$w(x) = \frac{x_n}{|x|^n} \left( \frac{x_n}{|x|^n} \right)^{1+\delta}.$$

Then for all $1 \leq i, j \leq n$,

$$D_i w = \left( 1 - (1 + \delta) \left( \frac{x_n}{|x|^n} \right)^\delta \right) \left( \frac{\delta_n^i}{|x|^n} - \frac{n x_n x_i}{|x|^{n+2}} \right).$$

and

$$D_{ij} w = \left( 1 - (1 + \delta) \left( \frac{x_n}{|x|^n} \right)^\delta \right) \left( -n(\delta_n^i x_j + \delta_n^j x_i + \delta_j^i x_n) \frac{1}{|x|^{n+2}} \right) + \frac{n(n+2)x_n x_i x_j}{|x|^{n+4}}$$

$$+ \left( -\delta(1+\delta) \left( \frac{x_n}{|x|^n} \right)^{\delta-1} \right) \left( \frac{\delta_n^i}{|x|^n} - \frac{n x_n x_i}{|x|^{n+2}} \right) \left( \frac{\delta_n^j}{|x|^n} - \frac{n x_n x_j}{|x|^{n+2}} \right).$$

Consequently

$$\Delta w = -\delta(1+\delta) \left( \frac{x_n}{|x|^n} \right)^{\delta-1} \left( \frac{1}{|x|^{2n}} + \frac{(n^2 - 2n) x_n^2}{|x|^{2n+2}} \right)$$

$$\leq -\delta(1+\delta) \frac{1}{|x|^{2n}} \left( \frac{x_n}{|x|^n} \right)^{\delta-1}$$

(3.7)

and there exists $C$ depending only on $n$ and $\delta$ such that for all $1 \leq i, j \leq n$,

$$|D_{ij} w| \leq C \left( \frac{1}{|x|^{n+1}} + \frac{1}{|x|^{n+1}} \left( \frac{x_n}{|x|^n} \right)^\delta + \frac{1}{|x|^{2n}} \left( \frac{x_n}{|x|^n} \right)^{\delta-1} \right)$$

$$\leq C \left( \frac{1}{|x|^{n+1}} + \frac{1}{|x|^{2n}} \left( \frac{x_n}{|x|^n} \right)^{\delta-1} \right).$$

(3.8)
By (3.7), (3.8) and \( |a_{ij}(x) - \delta_{ij}| \leq C_1|x|^{-s} \), there exists \( C \) depending only on \( C_1, \delta, s \) and \( n \) such that

\[
a_{ij}(x) D_{ij} w(x) = \delta_{ij} D_{ij} w(x) + (a_{ij}(x) - \delta_{ij}) D_{ij} w(x) \\
\leq -\delta (1 + \delta) \frac{1}{|x|^{2n}} \left( \frac{x_n}{|x|^n} \right)^{\delta - 1} \\
+ C|x|^{-s} \left( \frac{1}{|x|^{n+1}} + \frac{1}{|x|^{2n}} \left( \frac{x_n}{|x|^n} \right)^{\delta - 1} \right) \\
\leq (-\delta (1 + \delta) + C|x|^{-s}) |x|^{-2n} \left( \frac{x_n}{|x|^n} \right)^{\delta - 1} + C|x|^{-s-n-1}.
\]

(3.9)

From \( 0 < \delta < \min\{1, \frac{s}{n-1}\} \) and

\[
\left( \frac{x_n}{|x|^n} \right)^{\delta - 1} \geq |x|^{-(n-1)(\delta - 1)},
\]

(3.9) yields that there exists some \( R \geq 2R_0 \) large enough (depending only on \( C_1, \delta, s \) and \( n \)) such that

\[
a_{ij}(x) D_{ij} w(x) \leq 0 \quad \text{in} \quad \mathbb{R}^n_+ \setminus \overline{B}_R.
\]

(3.10)

Since \( |Du(x)| \leq 1 \) in \( \mathbb{R}^n_+ \setminus B_{R_0}^+ \), by Newton-Leibniz formula, we have

\[
|u(x)| \leq 3x_n \quad \text{on} \quad \partial B_R \cap \{x_n \geq 0\}.
\]

On the other hand, on \( \partial B_R \cap \{x_n \geq 0\} \),

\[
w(x) = \frac{x_n}{|x|^n} \left( 1 - \left( \frac{x_n}{|x|^n} \right)^{\delta} \right) \geq \frac{x_n}{|x|^n} \left( 1 - \left( \frac{1}{|x|^{n-1}} \right)^{\delta} \right) = \frac{x_n}{R^n} \left( 1 - R^{(1-n)\delta} \right).
\]

It follows that for some \( C \) depending only on \( R_0, C_1, s, \delta \) and \( n \), we have

\[
|u(x)| \leq Cw(x), \quad \text{on} \quad \partial B_R \cap \{x_n \geq 0\}
\]

(3.11)

For any \( \epsilon > 0 \), by \( u(x) \to 0 \) as \( |x| \to \infty \), there exists \( R_\epsilon > R \) such that

\[
|u(x)| \leq \epsilon, \quad x \in \partial B_{R_\epsilon} \cap \{x_n \geq 0\}.
\]

(3.12)

By (3.11), (3.12) and \( u(x) = 0 \) on \( (B_{R_\epsilon} \setminus \overline{B}_R) \cap \{x_n = 0\} \) we have

\[
|u(x)| \leq Cw(x) + \epsilon \quad \text{on} \quad \partial (B_{R_\epsilon}^+ \setminus \overline{B}_R),
\]

By the comparison principle, it follows that

\[
|u(x)| \leq Cw(x) + \epsilon \quad \text{in} \quad B_{R_\epsilon}^+ \setminus \overline{B}_R.
\]

Taking \( \epsilon \to 0 \), we have (3.6). \( \square \)
4. Proof of Theorem 1.1

In this section we prove Theorem 1.1, which is equivalent to the following Theorem 4.1.

**Theorem 4.1.** Let $f \in C^0(\mathbb{R}^n_+)$ satisfy (1.2) and

$$\Omega_0 = \text{support}(f - 1) \subset B_1^+$$

(4.1)

Assume that $u$ is a convex viscosity solution of

$$\begin{cases}
\text{det } D^2 u = f & \text{in } \mathbb{R}^n_+,
\end{cases}$$

and satisfies

$$\mu |x|^2 \leq u \leq \mu^{-1} |x|^2 \quad \text{in } \mathbb{R}^n_+ \setminus B_1^+.$$  

Then there exist some symmetric positive definite matrix $A$ with $\det A = 1$ and constant $b_n \in \mathbb{R}$ such that

$$\left| u(x) - \left( \frac{1}{2} x^T A x + b_n x_n \right) \right| \leq C \frac{x_n}{|x|^n} \quad \text{in } \mathbb{R}^n_+ \setminus B_1^+,$$

(4.4)

where $C$ and $R$ depend only on $\mu$ and $n$. Moreover, $u \in C^\infty(\mathbb{R}^n_+ \setminus \Omega_0)$ and

$$|x|^{n-1+k} \left| D^k \left( u(x) - \frac{1}{2} x^T A x - b_n x_n \right) \right| \leq C \quad \text{in } \mathbb{R}^n_+ \setminus B_1^+,$$

(4.5)

where $C$ also depends on $k$.

**Remark 4.2.** In fact, by the boundary condition, $\frac{1}{2} (x',0)^T A (x',0) = \frac{1}{2} |x'|^2$, and by $\det A = 1$ and (1.4), there exist bounded constants $\nu_i$, $1 \leq i \leq n - 1$, such that

$$A = \begin{bmatrix}
1 & 0 & \cdots & \nu_1 \\
0 & 1 & \cdots & \nu_2 \\
\vdots & \vdots & \ddots & \vdots \\
\nu_1 & \nu_2 & \cdots & 1 + \sum_{i=1}^{n-1} \nu_i^2
\end{bmatrix}.$$

From Theorem 4.1, we can show Theorem 1.1 by the following way. Indeed, after subtracting a linear function, $p(x')$ is homogeneous of degree 2, that is, $p(x') = \frac{1}{2} x'^T P x'$ for some $(n - 1) \times (n - 1)$ symmetric positive definite matrix $P$. Let

$$\tilde{P} = \begin{bmatrix}
P & 0 \\
0 & \frac{1}{\det P}
\end{bmatrix}.$$
There exists a rotation $R$ (with $\det R = 1$), which leaves the $x_n$ coordinate invariant, such that $R^T \bar{P} R$ is diagonal. Then there exists a dilation $Q$ such that $Q^T R^T \bar{P} R Q = I_n$, where $I_n$ is the $n \times n$ unit matrix.

Let $\tilde{u}(x) = u(RQx)$, $x \in \mathbb{R}^n_+$. Hence $\tilde{u}(x) = \frac{1}{2}|x'|^2$ on $\{x_n = 0\}$. By $\det R = 1$, $\det \bar{P} = 1$ and $Q^T R^T \bar{P} R Q = I_n$, we have $\det Q = 1$. Then in the viscosity sense,

$$\det D^2 \tilde{u}(x) = (\det R)^2 (\det Q)^2 \det D^2 u(RQx) = f(RQx) := \tilde{f}(x).$$

Obviously, $\det D^2 \tilde{u}(x) \equiv 1$ in $\mathbb{R}^n_+ \setminus (RQ)^{-1} \Omega_0$ and $\tilde{f}$ satisfies (1.2). By (1.4), we have

$$2\mu I_{n-1} \leq P \leq 2\mu^{-1} I_{n-1},$$

and then

$$(2\mu)^{n-1} I_n \leq \bar{P} \leq (2\mu)^{-n} I_n.$$ Combining with $Q^T R^T \bar{P} R Q = I_n$, it yields that

$$(2\mu)^{\frac{n+1}{2}} |x| \leq |(RQ)^{-1} x| \leq (2\mu)^{\frac{n-1}{2}} |x| \quad \forall x \in \mathbb{R}^n.$$ That is, $(RQ)^{-1} \Omega_0$ is bounded. Then there exists large enough $M$ (depending only on $\mu$, $R_0$ and $n$) such that

$$\hat{u}(x) := \frac{\tilde{u}(Mx)}{M^2}$$
solves $\det D^2 \hat{u}(x) = \tilde{f}(Mx) := \hat{f}(x)$ with $\hat{f}$ satisfying (4.1) and $\hat{u}$ satisfying (4.3).

Hence, we prove Theorem 4.1 instead and first show the smoothness of $u$.

**Lemma 4.3.** Let $u$ be as in Theorem 4.1. Then $u \in C^\infty(\mathbb{R}^n_+ \setminus \Omega_0)$.

**Proof.** For any $x_0 \in \{x_n = 0\} \setminus \Omega_0$, let $d = \text{dist}(x_0, \Omega_0)$ and

$$\tilde{u}(x) = u(x) - u(x_0) - x_0 \cdot (x - x_0) + C x_n,$$

where $C$ is chosen such that $\tilde{u}$ is positive on $\partial B_d(x_0) \cap \{x_n \geq 0\}$. Actually, since $\tilde{u}(x', 0) = \frac{1}{2}|x' - x_0|^2$, there exists $\delta > 0$ such that $\tilde{u}(x) \geq d^2/4$ on $\partial B_d(x_0) \cap \{0 \leq x_n \leq \delta\}$ and then we have the existence of $C$. By Theorem 2.12, $\tilde{u} \in C^{3,1}(\overline{B_c^+ (x_0)})$ for some $c \leq d$. Linearizing the equation and by Schauder estimates, we have $u \in C^\infty(\overline{B_c^+ (x_0)})$.

For any $x_0 \in \{x_n > 0\} \setminus \Omega_0$, let $d = \text{dist}(x_0, \partial(\mathbb{R}^n_+ \setminus \Omega_0))$ and $l_{x_0}(x)$ be a support plane of $u$ at $x_0$. Now we show that

$$\alpha := \min_{\partial B_{d/2}(x_0)} (u - l_{x_0})(x) > 0. \quad (4.6)$$

Indeed, if $\alpha = 0$, then by [3, Theorem 1], there exists an endless line $L \subset \{u - l_{x_0} = 0\}$ and this contradicts (4.3). By (4.6), we have $S_\beta(u) \subset B_{d/2}(x_0)$ for any $0 < \beta < \alpha$. By Theorem 2.10, we have $u \in C^\infty(B_{d/2}(x_0))$. \hfill \Box

We divide the remaining proof of Theorem 4.1 into two steps: nonlinear approach and linear approach.
4.1. Nonlinear approach

By (4.3), for any $M \geq \mu^{-1}$,
\[ M^{1/2} \mu^{1/2} B_1^+ \subset S_M(u) \subset M^{1/2} \mu^{-1/2} B_1^+. \]  
(4.7)

For any $M \geq \mu^{-1}$, let
\[ \hat{u}(x) = \frac{1}{M} u(M^{1/2} x), \quad x \in \mathcal{O} := \frac{1}{M^{1/2}} S_M(u). \]  
(4.8)

In view of (4.7), we have
\[ \mu^{1/2} B_1^+ \subset \mathcal{O} \subset \mu^{-1/2} B_1^+. \]  
(4.9)

Clearly, $\hat{u}(x)$ solves
\[
\begin{cases}
\det D^2 \hat{u} = f(M^{1/2} x) & \text{in } \mathcal{O}, \\
\hat{u} = \frac{1}{2} |x'|^2 & \text{on } \partial \mathcal{O} \cap \{x_n = 0\}, \\
\hat{u} = 1 & \text{on } \partial \mathcal{O} \cap \{x_n > 0\}.
\end{cases}
\]  
(4.10)

By (4.3) and (4.8), we have
\[ \mu M^{-1} \leq \hat{u} \leq \mu^{-1} M^{-1} \quad \text{on } \partial B_{M^{-1/2}}^+ \cap \{x_n > 0\}. \]  
(4.11)

Now we consider the following Dirichlet problem
\[
\begin{cases}
\det D^2 \xi = 1 & \text{in } \mathcal{O}, \\
\xi = \frac{1}{2} |x'|^2 & \text{on } \partial \mathcal{O} \cap \{x_n = 0\}, \\
\xi = 1 & \text{on } \partial \mathcal{O} \cap \{x_n > 0\}.
\end{cases}
\]  
(4.12)

The existence of $\xi$ can be obtained by Theorem 2.5 where we need extend the boundary value of (4.12) to a convex function on $\overline{\mathcal{O}}$ which can be defined by
\[ \sup \{l(x) : l \text{ is a linear function and } l \leq \xi|_{\partial \mathcal{O}} \text{ on } \partial \mathcal{O}\}, \quad \forall x \in \mathcal{O}. \]

For any $M \geq \mu^{-1}$, applying Theorem 2.12 to $\xi$ in $\mathcal{O}$, there exists $c_0 > 0$ depending only on $\mu$ and $n$ such that
\[ |D\xi(x)| \leq c_0^{-1}, \quad c_0 I \leq [D^2 \xi(x)] \leq c_0^{-1} I, \quad |D^3 \xi(x)| \leq c_0^{-1} \quad \text{in } B_{c_0}^+. \]  
(4.13)

By Newton-Leibniz formula, it follows that for any $M \geq \{\mu^{-1}, c_0^2\}$,
\[ |\xi(x)| \leq c_0^{-1} M^{-1/2} \quad \text{in } B_{M^{-1/2}}^+. \]  
(4.14)

Denote $\mathcal{Q} = \mathcal{O} \setminus B_{M^{-1/2}}^+$. Observe that $f \equiv 1$ in $\mathcal{Q}$. 

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Lemma 4.4. For any $M \geq \{\mu^{-1}, c_0^{-2}\}$,
\[ |\hat{u} - \xi| \leq CM^{-1/2} \quad \text{in } \overline{Q}, \]
where $C$ depends only on $\mu$ and $n$.

Proof. By (4.11) and (4.14), there exists some $C$ depending only on $\mu$ and $n$ such that
\[ |\hat{u} - \xi| \leq CM^{-1/2} \quad \text{in } \partial B_{M-1/2}^+ \cap \{x_n > 0\}. \]
Combining with the boundary values in (4.10) and (4.12), we have
\[ |\hat{u} - \xi| \leq CM^{-1/2} \quad \text{on } \partial Q. \]
By the comparison principle, we see the conclusion. \qed

Lemma 4.5. For any $M \geq \{\mu^{-4}, c_0^{-2}\}$,
\[ |D\xi(0)| \leq CM^{-1/4}, \]
where $C$ depends only on $\mu$ and $n$.

Proof. By $\xi = \frac{1}{2}|x'|^2$ on $\{x_n = 0\}$, we have $D_i\xi(0) = 0, i = 1, \ldots, n - 1$. Next we show $|D_n\xi(0)| \leq CM^{-1/4}$, where $C$ depends only on $\mu$ and $n$.

For any $M \geq \mu^{-2}$, by (4.17) and the definition of $\hat{u}$, we have
\[ M^{-1/4} \mu^{1/2} \overline{B_1} \subset \{\hat{u} < M^{-1/2}\} \subset M^{-1/4} \mu^{-1/2} \overline{B_1}. \quad (4.15) \]
Denote $\tilde{x} = \{\hat{u} = M^{-1/2}\} \cap \{x = (x', x_n): x' = 0, x_n > 0\}$. By (4.15), for any $M \geq \mu^{-4}$,
\[ M^{-1/4} \mu^{1/2} \leq |\tilde{x}| \leq M^{-1/4} \mu^{-1/2}. \quad (4.16) \]
According to (4.16) and the definition of $Q$, we have $\tilde{x} \in \overline{Q}$.

By the convexity of $\xi$, Lemma 4.4 and (4.16), we deduce that
\[ D_n\xi(0) \leq \frac{\xi(\tilde{x}) - \xi(0)}{|\tilde{x}|} \leq \frac{\hat{u}(\tilde{x}) + CM^{-1/2}}{|\tilde{x}|} \leq \frac{M^{-1/2} + CM^{-1/2}}{M^{1/4} \mu^{1/2}} \leq CM^{-1/4}. \quad (4.17) \]

On the other hand, by the convexity of $\xi$, Lemma 4.4 and the definition of $\tilde{x}$,
\[ D_n\xi(\tilde{x}) \geq \frac{\xi(\tilde{x}) - \xi(0)}{|\tilde{x}|} \geq \frac{\hat{u}(\tilde{x}) - CM^{-1/2}}{|\tilde{x}|} \geq \frac{M^{-1/2} - CM^{-1/2}}{M^{1/4} \mu^{1/2}} \geq -CM^{-1/4}. \]
Combining with (4.13) and (4.16), it follows that
\[ D_n\xi(0) = D_n\xi(\tilde{x}) - \int_0^{\tilde{x}} D_{nn}\xi dx \geq D_n\xi(\tilde{x}) - c_0^{-1}|\tilde{x}| \geq -CM^{-1/4}. \quad (4.18) \]
By (4.17) and (4.18), we have $|D_n\xi(0)| \leq CM^{-1/4}$. \qed
Let
\[ E_M = \{ x \in \mathbb{R}^n_+ : x^T D^2 \xi(0) x \leq 1 \} . \] (4.19)

Lemma 4.6. There exist \( k_0 \) and \( \tilde{C} \) depending only on \( \mu \) and \( n \). For all \( k \geq k_0 \), \( M = 2^{(1+\tau)k} \) and \( M' \in [2^{k-1}, 2^k] \),
\[ \left( \frac{2M'}{M} - \tilde{C}2^{-\frac{3}{2}\tau k} \right)^{1/2} E_M \subset \frac{S_{M'}(u)}{M^{1/2}} \subset \left( \frac{2M'}{M} + \tilde{C}2^{-\frac{3}{2}\tau k} \right)^{1/2} E_M . \] (4.20)

Proof. Recall that
\[ \hat{u}(x) = \frac{1}{M} u(M^{1/2}x), \quad x \in \mathcal{O} = \frac{1}{M^{1/2}} S_M(u), \]
and then we have
\[ \left\{ \frac{\hat{u}}{M} < \frac{M'}{M} \right\} = \frac{1}{M^{1/2}} S_{M'}(u) . \]
By Lemma 4.4, it follows that
\[ \left\{ \xi < \frac{M'}{M} - \frac{C}{M^{1/2}} \right\} \subset \left\{ \frac{\hat{u}}{M} < \frac{M'}{M} \right\} \subset \left\{ \xi < \frac{M'}{M} + \frac{C}{M^{1/2}} \right\} . \] (4.21)
For any \( x \in \overline{E}_{c_0} \), (4.13) follows that
\[ |\xi(x) - \xi(0) - D\xi(0) \cdot x - \frac{1}{2} x^T D^2 \xi(0) x| \leq c_0^{-1} |x|^3 . \] (4.22)
Now we prove the first inclusion of (4.20).
For any \( x \in \left( \frac{2M'}{M} - \tilde{C}2^{-\frac{3}{2}\tau k} \right)^{1/2} E_M \), (4.19) implies that
\[ \frac{1}{2} x^T D^2 \xi(0) x \leq \frac{M'}{M} - \tilde{C}2^{-\frac{3}{2}\tau k} . \] (4.23)
By (4.13), it follows that
\[ |x| \leq C \left( \frac{M'}{M} \right)^{1/2} , \] (4.24)
where \( C \) depends only on \( \mu \) and \( n \).
In view of Lemma 4.4, (4.22), (4.23) and (4.24), we obtain that
\[ \xi(x) \leq \xi(0) + D\xi(0) \cdot x + \frac{1}{2} x^T D^2 \xi(0) x + 2c_0^{-1} |x|^3 \]
\[ \leq CM^{-1/4} \left( \frac{M'}{M} \right)^{1/2} + \frac{M'}{M} - \tilde{C}2^{-\frac{3}{2}\tau k} + C \left( \frac{M'}{M} \right)^{3/2} . \]
There exist \( k_0 \) and \( \tilde{C} \) large enough (depending only on \( \mu \) and \( n \)) such that for any \( k \geq k_0 \),
\[ CM^{-1/4} \left( \frac{M'}{M} \right)^{1/2} + \frac{M'}{M} - \tilde{C}2^{-\frac{3}{2}\tau k} + C \left( \frac{M'}{M} \right)^{3/2} < \frac{M'}{M} - \frac{C}{M^{1/2}} . \]
That is, \( \xi < \frac{M'}{M} - \frac{C}{M^{1/2}} \).

Then by (4.21), we have the first inclusion of (4.20).

Next we turn to show the second inclusion of (4.20).

For any \( x \in \frac{1}{M^{1/2}} S_{M'}(u) \), by (4.7),

\[
S_{M'}(u) \subset (M')^{1/2} \mu^{-1/2} B_1^+,
\]

and then

\[
|x| \leq \mu^{-1/2} \left( \frac{M'}{M} \right)^{1/2}.
\]

Hence, by Lemma 4.5, (4.21), (4.22) and (4.25), we have

\[
\frac{1}{2} x^T D^2 \xi(0) x \leq \xi(x) - \xi(0) + D \xi(0) \cdot x + 2c_0^{-1} |x|^3
\]

\[
\leq \frac{M'}{M} + CM^{-1/2} + CM^{-1/4} \left( \frac{M'}{M} \right)^{1/2} + C \left( \frac{M'}{M} \right)^{3/2}.
\]

Choosing larger \( k_0 \) and \( \tilde{C} \) depending only on \( \mu \) and \( n \), we have for any \( k \geq k_0 \),

\[
\frac{1}{2} x^T D^2 \xi(0) x \leq \frac{2M'}{M} + \tilde{C} 2^{-\frac{3}{2} \tau k}.
\]

Then by (4.19), we have the second inclusion of (4.20). \( \square \)

**Lemma 4.7.** Let \( k_0 \) and \( \tau \) be given by Lemma 4.6. Then there exists a real invertible bounded upper-triangular matrix \( T \) such that det \( T = 1 \) and

\[
\left( 1 - C M'^{-\frac{1}{2} \tau} \right) \sqrt{2M'B_1^+} \subset TS_{M'}(u) \subset \left( 1 + C M'^{-\frac{1}{2} \tau} \right) \sqrt{2M'B_1^+}
\]

for all \( M' \geq 2^{k_0} \), where \( C \) depends only on \( \mu \) and \( n \).

**Proof.** For any \( k \geq k_0 \), let \( M = 2^{(1+\tau)k} \) and \( M' \in [2^{k-1}, 2^k] \).

By LU decomposition for symmetric positive definite matrices, there exists a unique upper-triangular matrix \( T_k \) with real positive diagonal entries such that \( [D^2 \xi(0)] = T_k^T T_k \). Obviously, det \( T_k = 1 \).

In view of (4.19), we have \( E_M = T_k^{-1} B_1^+ \). Then (4.20) yields that for any \( M' \in [2^{k-1}, 2^k] \),

\[
\left( \frac{2M'}{M} - C 2^{-\frac{3}{2} \tau k} \right)^{1/2} T_k^{-1} B_1^+ \subset \frac{S_{M'}(u)}{M^{1/2}} \subset \left( \frac{2M'}{M} + C 2^{-\frac{3}{2} \tau k} \right)^{1/2} T_k^{-1} B_1^+
\]

or

\[
\left( 1 - C 2^{-\frac{1}{4} \tau k} \right) \sqrt{2M'B_1^+} \subset T_k S_{M'}(u) \subset \left( 1 + C 2^{-\frac{1}{4} \tau k} \right) \sqrt{2M'B_1^+}.
\]
It follows that
\[
\left(1 - C^2\right)^{-1/2} \sqrt{2^k B^+_{1}} \subset T_{k} S_{2^{k-1}}(u) \subset \left(1 + C^2\right)^{-1/2} \sqrt{2^k B^+_{1}} \tag{4.27}
\]
and
\[
\left(1 - C^2\right)^{-1/2} \tau_{k} \sqrt{2^k B^+_{1}} \subset T_{k-1} S_{2^{k-1}}(u) \subset \left(1 + C^2\right)^{-1/2} \tau_{k} \sqrt{2^k B^+_{1}}. \tag{4.28}
\]

By (4.27) and (4.28), there exists some larger C such that
\[
\left(1 - C^2\right)^{-1/2} \tau_{k} \sqrt{2^k B^+_{1}} \subset T_{k-1} S_{2^{k-1}}(u) \subset \left(1 + C^2\right)^{-1/2} \tau_{k} \sqrt{2^k B^+_{1}}. \tag{4.29}
\]

Denote \(U = T_{k} T_{k-1}^{-1}\). Then \(U\) is an upper-triangular, \(\det U = 1\) and the inverse matrix \(U^{-1}\) of \(U\) satisfies
\[
\frac{1}{(1 + C^2)^{-1/2} \tau_{k}} \sqrt{2^k B^+_{1}} \subset U^{-1} B^+_{1} \subset \frac{1}{(1 - C^2)^{-1/2} \tau_{k}} \sqrt{2^k B^+_{1}}. \tag{4.30}
\]

For \(1 \leq j \leq n\), let \(e_j\) denote the unit vector with the \(j\)th component equals 1 and all the other components equal zero. Using (4.29),
\[
||U e_j|| = \sqrt{\sum_{i=1}^{j} U_{ij}^2} \leq 1 + C^2\tau_{k}. \tag{4.31}
\]

In particular,
\[
U_{jj} \leq 1 + C^2\tau_{k}, \quad 1 \leq j \leq n.
\]

Similarly, by (4.30), we have
\[
\frac{1}{U_{jj}} = U_{jj} \leq \frac{1}{1 - C^2\tau_{k}}, \quad 1 \leq j \leq n,
\]
where \(U_{jj} = [U^{-1}]_{jj}\). We deduce from above two inequalities that
\[
1 - C^2\tau_{k} \leq U_{jj} \leq 1 + C^2\tau_{k}, \quad 1 \leq j \leq n.
\]

This estimate and (4.31) imply that
\[
\sum_{i<j} U_{ij}^2 \leq (1 + C^2\tau_{k})^2 - (1 - C^2\tau_{k})^2 \leq C^2\tau_{k}, \quad 1 \leq i, j \leq n.
\]

It follows that
\[
||(U - I) e_j|| = \sqrt{\sum_{i<j} U_{ij}^2 + (U_{jj} - 1)^2} \leq C^2\tau_{k}, \quad 1 \leq j \leq n.
\]

That is,
\[
||U - I|| \leq C^2\tau_{k},
\]
and then

\[ ||T_k - T_{k-1}|| \leq C2^{-\frac{1}{2}\tau_k}||T_{k-1}||.\]

By (4.13), \( T_k \) is uniformly bounded, and then there exists a unique bounded invertible upper-triangular matrix \( T \) such that \( \det T = 1 \) and

\[ ||T_k - T|| \to 0 \quad \text{as} \quad k \to \infty.\]

It yields that

\[ \left( 1 - C2^{-\frac{1}{2}\tau_k} \right) \sqrt{2M'B_1^+} \subset TSM(u) \subset \left( 1 + C2^{-\frac{1}{2}\tau_k} \right) \sqrt{2M'B_1^+} \]

for all \( M' \in [2^{k-1}, 2^k] \) and all \( k \geq k_0 \). It follows (4.26).

**Lemma 4.8.** Let \( v(x) = u(y) \) and \( y = T^{-1}x \), where \( x \in \mathbb{R}^n_+ \). Then \( v \) solves

\[ \begin{cases} 
\det D^2v = 1 & \text{in} \; \mathbb{R}^n_+ \setminus T\Omega_0, \\
v(x) = \frac{1}{2}|x'|^2 & \text{on} \; \{x_n = 0\} 
\end{cases} \] (4.32)

and for some \( C \) depending only on \( \mu \) and \( n \),

\[ \left| v(x) - \frac{1}{2}|x|^2 \right| \leq C|x|^{2-\tau} \; \text{in} \; \{|x| \geq 2^{k_0}\} \cap \mathbb{R}^n_+, \] (4.33)

where \( \tau = \frac{1}{10} \), \( k_0 \) and \( T \) are given by Lemma 4.7.

**Proof.** By \( \det D^2u(y) = 1 \) in \( \mathbb{R}^n_+ \setminus \Omega_0 \) and \( \det T = 1 \), \( v(x) \) solves

\[ \det D^2v(x) = 1 \; \text{in} \; \mathbb{R}^n_+ \setminus T\Omega_0. \]

By (4.26) and the definition of \( v \), we obtain that for any \( M \geq 2^{k_0} \),

\[ \left( 1 - CM^{-\frac{1}{2}\tau} \right) \sqrt{2MB_1^+} \subset S_M(v) \subset \left( 1 + CM^{-\frac{1}{2}\tau} \right) \sqrt{2MB_1^+}. \]

As a consequence, we get (4.33).

Since \( u(y) = \frac{1}{2}|y|^2 \) on \( \{y_n = 0\} \) and \( T \) is upper-triangular, we have

\[ v(x) = \frac{1}{2}|T^{-1}x|^2 \; \text{on} \; \{x_n = 0\}. \] (4.34)

In view of (4.33) and (4.34), we get

\[ \left| \frac{1}{2}|T^{-1}Mx|^2 - \frac{1}{2}|Mx|^2 \right| \leq C|Mx|^{2-\tau} \; \text{on} \; \{x_n = 0\} \]

for all \( M \geq 2^{k_0} \). Let \( M \to \infty \) and it yields that

\[ |T^{-1}x|^2 = |x|^2 \; \text{on} \; \{x_n = 0\}, \]

which follows from (4.34) that \( v(x', 0) = \frac{1}{2}|x'|^2. \) 

\[ \square \]
4.2. Linear approach

In this subsection we prove that \(|v(x) - \frac{1}{2}|x|^2 - bx_n| = O \left( \frac{x_n}{|x|^n} \right) \) at infinity for some constant \(b\), where \(v\) is given by Lemma 4.8.

**Lemma 4.9.** Let \(v\) be given by Lemma 4.8. Then there exists a constant \(b_n\) such that

\[
\left| v(x) - \frac{1}{2}|x|^2 - b_n x_n \right| \leq C \frac{x_n}{|x|^n} \quad \text{in} \quad \mathbb{R}_+^n \setminus B_{R_1}^+,
\]

where \(C\) and \(R\) depend only on \(\mu\) and \(n\). Furthermore, for any \(k \geq 1\),

\[
|x|^{n-1-k} \left| D^k \left( v(x) - \frac{1}{2}|x|^2 - b_n x_n \right) \right| \leq C \quad \text{in} \quad \mathbb{R}_+^n \setminus B_{R_1}^+,
\]

where \(C\) also depends on \(k\).

**Proof.** By (4.1) and (4.26), there exists \(R_1 > 0\) depending only on \(\mu\) and \(n\) such that \(T\Omega_0 \subset B_{R_1}^+\). Combining with (4.32), it yields that \(\det D^2v = 1 \quad \text{in} \quad \mathbb{R}_+^n \setminus B_{R_1}^+\).

Let \(V(x) = v(x) - \frac{1}{2}|x|^2\). By (4.33) and Corollary 2.13 we have

\[
|DV(x)| \leq C|x|^{1-\tau} \quad \text{and} \quad |D^2V(x)| \leq C|x|^{-\tau} \quad \text{in} \quad \mathbb{R}_+^n \setminus B_{R_1}^+,
\]

where \(\tau = \frac{1}{10}\) and \(C\) depends only on \(\mu\) and \(n\).

By \(\ln \det(I_n + D^2V) = \ln \det I_n = 0\), we have

\[
\tilde{a}_{ij}(x)D_{ij}V(x) = 0 \quad \text{in} \quad \mathbb{R}_+^n \setminus B_{R_1}^+,
\]

where \(\tilde{a}_{ij}(x) = \int_0^1 [sD^2V + (1-s)I_n]_{ij} ds\).

Differentiating \(\ln \det(I_n + D^2V) = 0\) with respect to \(x_k\), \(k = 1, \cdots, n\), then

\[
a_{ij}(x)D_{ij}V_k(x) = 0 \quad \text{in} \quad \mathbb{R}_+^n \setminus B_{R_1}^+,
\]

where \(a_{ij}(x) = [D^2V + I_n]_{ij}(x)\) and \(V_k = D_k V\).

In view of (4.37), we obtain that

\[
|\tilde{a}_{ij}(x) - \delta_{ij}| + |a_{ij}(x) - \delta_{ij}| \leq C|x|^{-\tau} \quad \text{in} \quad \mathbb{R}_+^n \setminus B_{R_1}^+ \quad (4.40)
\]

and that for any \(k = 1, \cdots, n\),

\[
|DV_k(x)| \rightarrow 0 \quad \text{as} \quad |x| \rightarrow \infty. \quad (4.41)
\]

By (4.39), (4.40), (4.41) and Lemma 3.4 we have for any \(k = 1, \cdots, n - 1\),

\[
|V_k(x)| \leq C \frac{x_n}{|x|^n} \quad \text{in} \quad \mathbb{R}_+^n \setminus B_{R_1}^+ \quad (4.42)
\]
where $C$ depends only on $\mu$ and $n$. Since $V_k(x',0) = 0$, (4.42) follows that for any $k = 1, \ldots, n - 1$,

$$|V_{kn}(x',0)| \leq \frac{C}{|x'|^{n}}, \quad x \in \{|x'| \geq R_1, x_n = 0\}.$$ 

By Newton-Leibniz formula and $n \geq 2$, there exists some $b_n$ such that $V_n(x',0) \to b_n$ as $|x'| \to \infty$. (4.43)

By (4.39), (4.40), (4.37), (4.43) and Lemma 3.3, we obtain that $V_n(x) \to b_n$ as $|x| \to \infty$. (4.44)

By (4.38) and the second equation of (4.32),

\[
\begin{aligned}
\begin{cases}
\tilde{a}_{ij}(x)D_{ij}(V - b_n x_n) = 0 & \text{in } \mathbb{R}^n_+ \setminus B_{R_1}^+,

V - b_n x_n = 0 & \text{on } \{x_n = 0\}.
\end{cases}
\end{aligned}
\]

In view of (4.42) and (4.44), we have

$$|D(V(x) - b_n x_n)| \to 0 \quad \text{as } |x| \to \infty.$$ 

Then by (4.40) and Lemma 3.4 there exist $R \geq R_1$ and $C$ depending only on $\mu$ and $n$ such that (4.35) holds. And then applying Corollary 2.13 with $w = V - b_n x_n$ and $\gamma = n - 1$, we obtain (4.36). □

Finally, Theorem 4.1 follows from Lemma 4.9 immediately.

5. Proofs of Corollary 1.3 and Theorem 1.4

Proof of Corollary 1.3

By Theorem 1.1, $u \in C^\infty(\mathbb{R}^n_+)$ and there exist some symmetric positive definite matrix $A$ with $\det A = 1$, vector $b \in \mathbb{R}^n$ and constant $c \in \mathbb{R}$ such that

$$u(x) - \frac{1}{2}x^T A x - b \cdot x - c \to 0 \quad \text{as } |x| \to \infty.$$ 

Denote $E(x) = u(x) - \frac{1}{2}x^T A x - b \cdot x - c$. Since $p(x')$ is a quadratic polynomial,

$$E = 0 \quad \text{on } \{x_n = 0\}.$$ 

Furthermore, $\det(A + D^2E) - \det A = \det D^2u - 1 = 0$ and $[D^2u] = [A + D^2E]$ is positive definite. Thus, $E$ solves

$$a_{ij}D_{ij}E = 0 \quad \text{in } \mathbb{R}^n_+,$$

where $a_{ij}(x) = \int_0^1 [sD^2u + (1 - s)A]^{ij}(x) ds$.

By the maximum principle, $E(x) \equiv 0$, i.e., $u(x) = \frac{1}{2}x^T A x + b \cdot x + c$. □

Before proving Theorem 1.4, we need construct two barrier functions.
Lemma 5.1. For any $x \in \mathbb{R}^n_+$, let

$$u_+(x) = u_+(|x|) = \begin{cases} 
\int_1^{|x|} \left( s^{n+1} + 1 \right)^{\frac{1}{n+1}} ds, & |x| \geq 1, \\
\lambda^\frac{n}{2} \left( \frac{1}{2} |x|^2 + a|x| - a - \frac{1}{2} \right), & 0 \leq |x| < 1
\end{cases}$$

and

$$u_-(x) = u_-(|x|) = \begin{cases} 
\int_1^{|x|} \left( s^{n+1} - \frac{1}{2} \right)^{\frac{1}{n+1}} ds, & |x| \geq 1, \\
\Lambda^\frac{n}{2} \left( \frac{1}{2} |x|^2 - \tilde{a}|x| + \tilde{a} - \frac{1}{2} \right), & 0 \leq |x| < 1
\end{cases}$$

where $a > 2^{\frac{1}{n+1}} \lambda^{-\frac{1}{n+1}} - 1$ and $1 - 2^{\frac{1}{n+1}} \Lambda^{-\frac{1}{n+1}} < \tilde{a} < 1$. Then $u_\pm \in C^0(\mathbb{R}^n_+) \cup C^\infty(B^+_1) \cup C^\infty(\mathbb{R}^n_+ \setminus B^+_1)$ satisfy

(i) $\det D^2 u_+ = \begin{cases} 
\left( 1 + \frac{1}{|x|^{n+1}} \right)^{-\frac{1}{n+1}} & \text{in } \mathbb{R}^n_+ \setminus B^+_1, \\
\lambda & \text{in } B^+_1
\end{cases}$ \hspace{1cm} (5.1)

and

$$\lim_{r \to 1^-} D_r u_+(x) > \lim_{r \to 1^+} D_r u_+(x);$$ \hspace{1cm} (5.2)

(ii) $\det D^2 u_- = \begin{cases} 
\left( 1 - \frac{1}{2|x|^{n+1}} \right)^{-\frac{1}{n+1}} & \text{in } \mathbb{R}^n_+ \setminus B^+_1, \\
\Lambda & \text{in } B^+_1
\end{cases}$ \hspace{1cm} (5.3)

and

$$\lim_{r \to 1^-} D_r u_-(x) < \lim_{r \to 1^+} D_r u_-(x);$$ \hspace{1cm} (5.4)

(iii) there exists some constant $C$ depending only on $a$, $\tilde{a}$, $\lambda$, $\Lambda$ and $n$ such that

$$\sup_{\mathbb{R}^n_+} \left| u_\pm(x) - \frac{1}{2} |x|^2 \right| \leq C.$$ \hspace{1cm} (5.5)

Proof. Since $u_\pm$ are radial, (5.1)-(5.4) are clear. Observe $\det D^2 u_\pm = u''_\pm \left( \frac{u'}{r} \right)^{n-1}$. Since for any $s \geq 1$,

$$s \leq (s^{n+1} + 1)^{\frac{1}{n+1}} = s \left( 1 + s^{-n-1} \right)^{\frac{1}{n+1}} \leq s \left( 1 + \frac{s^{-n-1}}{n+1} \right)$$

and

$$s \geq \left( s^{n+1} - \frac{1}{2} \right)^{\frac{1}{n+1}} = s \left( 1 - \frac{1}{2} s^{-n-1} \right)^{\frac{1}{n+1}} \geq s \left( 1 - \frac{1}{2} s^{-n-1} \right),$$

we deduce (5.5). \qed
Remark 5.2. Here we modified the barrier functions in [5], which are applicable for all $n \geq 2$.

Proof of Theorem 1.4. The uniqueness of solutions in Theorem 1.4 can be deduced from the comparison principle. As for the existence part, we only need to show it under additional hypothesis $\Omega_0 \subset B^+_1$, $A = I_n$, $b = 0$ and $c = 0$. In fact, by LU decomposition for symmetric positive definite matrices, there exists a unique upper-triangular matrix $Q$ with real positive diagonal entries such that $Q^T Q = A$ and $\det Q = 1$. Then the existence of $u$ satisfying (1.1), (1.8) and (1.3) is equivalent to the existence of $w$ satisfying

$$\begin{cases}
\det D^2 w = \tilde{f} & \text{in } \mathbb{R}^n, \\
w = \frac{1}{2} |x|^2 & \text{on } \{x_n = 0\}
\end{cases}$$

and

$$\lim_{|x| \to \infty} w(x) = \frac{1}{2} |x|^2, \quad \text{support}(\tilde{f} - 1) \subset B^+_1$$

by setting

$$u(x) = L^2 w \left( \frac{Qx}{L} \right) + b(x) + c, \quad f(x) = \tilde{f} \left( \frac{Qx}{L} \right)$$

for $L$ large enough.

Next we prove the existence of Theorem 1.4 under additional hypothesis $\Omega_0 \subset B^+_1$, $A = I_n$, $b = 0$ and $c = 0$.

For any $R > 1$, let $u_R(x)$ be the unique convex viscosity solution of

$$\begin{cases}
\det D^2 u_R = f & \text{in } B^+_R, \\
u_R = \frac{1}{2} |x|^2 & \text{on } \partial B^+_R.
\end{cases} \quad (5.6)$$

By Lemma 5.1, we can obtain that

$$u_-(x) - C \leq u_R(x) \leq u_+(x) + C \quad \forall x \in B^-_R, \quad (5.7)$$

where $C$ is given by (5.5). Actually, the second inequality in (5.7) can be showed by the following way. If it dose not hold, then there exist $M > C$ and $x_0 \in \overline{B}^-_R$ such that

$$u_R(x) \leq u_+(x) + M, \quad u_R(x_0) = u_+(x_0) + M.$$

By the definition of $u_R$, $u_+$ and (5.5), we have $x_0 \notin \partial B^+_R$. By (5.4) and the smoothness of $u_R$ on $|x| = 1$, we get $|x_0| \neq 1$. Then by the comparison principle, $x_0 \notin B^+_R \setminus B^+_1$ and $x_0 \notin B^+_1$. It contradicts $x_0 \in \overline{B}^-_R$. Similarly, we can show the first inequality in (5.7).

By (5.5) and (5.7), we have for any $R \geq 1$,

$$\sup_{\overline{B}^-_R} \left| u_R(x) - \frac{1}{2} |x|^2 \right| \leq C, \quad (5.8)$$

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where $C$ depends only on $\lambda$, $\Lambda$ and $n$.

Fix large $R$. By (5.8) and similar arguments in Corollary 2.13, we have for any $R \geq 2\overline{R}$,

$$||u_R||_{C^{3,1}(B_{\overline{R}+1})} \leq C,$$

where $C$ depends only on $\overline{R}$ and $n$. It follows that $||u_R||_{C^{3,1}(\partial B_{\overline{R}}^+)} \leq C$. And then by [19, Lemma 3.3], $||u_R||_{C^0(\overline{B}_{\overline{R}}^+)} \leq C$ for some $\alpha \in (0, 1)$ and some $C > 0$ depending only on $\overline{R}$, $\lambda$, $\Lambda$ and $n$. Therefore there exists a sequence $\{u_{R_j}\}$ such that

$$u_{R_j} \to u \quad \text{in } C^0(\overline{B}_{\overline{R}}^+) \quad \text{as } R_j \to \infty \quad (j \to \infty).$$

Moreover, $u$ is a convex viscosity solution of

$$\begin{cases}
\det D^2 u = f & \text{in } B_{\overline{R}}^+,

u = \frac{1}{2}|x|^2 & \text{on } \{x_n = 0\}
\end{cases}$$

and satisfies

$$\sup_{\overline{B}_{\overline{R}}^+} \left| u(x) - \frac{1}{2}|x|^2 \right| \leq C.$$

Then by standard diagonal arguments, $u$ can be defined on $\mathbb{R}_+^n$ such that it is a convex viscosity solution of (1.1). By (5.8) and Theorem 1.1, (1.8) holds with $A = I$, $b = 0$ and $c = 0$. \hfill \square

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