NORMALIZERS OF AD-NILPOTENT IDEALS

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Let $\mathfrak{g}$ be a complex simple Lie algebra. Fix a Borel subalgebra $\mathfrak{b}$ and a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{b}$. The nilpotent radical of $\mathfrak{b}$ is denoted by $\mathfrak{u}$. The corresponding set of positive (resp. simple) roots is $\Delta^+$ (resp. $\Pi$).

An ideal of $\mathfrak{b}$ is called ad-nilpotent, if it is contained in $[\mathfrak{b}, \mathfrak{b}]$. The theory of ad-nilpotent ideals has attracted much recent attention in the work of Kostant, Cellini-Papi, Sommers, and others. The goal of this paper is to study the normalizer of an ad-nilpotent ideal.

We obtain several general descriptions of the normalizer, and present a number of more explicit results for $\mathfrak{g} = \mathfrak{sl}_n$ or $\mathfrak{sp}_{2n}$.

Let $c$ be an ad-nilpotent ideal. Being a $\mathfrak{t}$-stable subspace of $\mathfrak{u}$, it is a sum of root spaces. The sum of the corresponding roots is an integral weight, denoted $|c|$. We show that $|c|$ is a dominant weight and that the normalizer of $c$, $n_\mathfrak{g}(c)$, is completely determined by $|c|$. Since $n_\mathfrak{g}(c)$ contains $\mathfrak{b}$, it suffices to realize which root spaces $\mathfrak{g} - \alpha$ ($\alpha \in \Pi$) are contained in $n_\mathfrak{g}(c)$. We prove that $\mathfrak{g} - \alpha \subset n_\mathfrak{g}(c)$ if and only if $(|c|, \alpha) = 0$.

Another type of descriptions is based on a relationship between the ad-nilpotent ideals and certain elements of the affine Weyl group $\hat{W}$. Let $\tilde{\mathfrak{A}} = \tilde{\mathfrak{A}}(\mathfrak{g})$ denote the set of all ad-nilpotent ideals of $\mathfrak{b}$. By [3], to each $c \in \tilde{\mathfrak{A}}$ one associates an element of $\hat{W}$, which we denote by $w_{\text{min}, c}$. An ad-nilpotent ideal is called strictly positive, if it is contained in $[\mathfrak{u}, \mathfrak{u}]$. The set of strictly positive ideals is denoted by $\tilde{\mathfrak{A}}_0$. By [15], to each $c \in \tilde{\mathfrak{A}}_0$ one associates an element of $\hat{W}$, which we denote by $w_{\text{max}, c}$. The group $\hat{W}$ acts linearly on a vector space $\hat{V}$, containing affine root system, and we prove that

$$\mathfrak{g} - \alpha \subset n_\mathfrak{g}(c) \iff w_{\text{min}, c}(\alpha) \text{ is an affine simple root}.$$

If $c \in \tilde{\mathfrak{A}}_0$, then both $w_{\text{min}, c}$ and $w_{\text{max}, c}$ are defined, and we have

$$w_{\text{min}, c}(\alpha) \text{ is an affine simple root} \iff w_{\text{max}, c}(\alpha) \text{ is an affine simple root},$$

see Sections 11 and 12 for details. It is worth noting that for $c \in \tilde{\mathfrak{A}}_0$ the elements $w_{\text{min}, c}$ and $w_{\text{max}, c}$ can be different and simple roots $w_{\text{min}, c}(\alpha)$, $w_{\text{max}, c}(\alpha)$ can also be different.

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Our geometric characterization of the normalizer is connected with Shi’s bijection between the ad-nilpotent ideals and the dominant regions of the Shi arrangement. Given an ideal \( c \in \mathfrak{A}_\mathfrak{b} \), let \( R_c \) denote the corresponding region. We show that \( g_{-\alpha} \subset n_g(c) \) if and only if the wall of the dominant Weyl chamber orthogonal to \( \alpha \) is also a wall of \( R_c \).

An interesting general problem is to study the partition of \( \mathfrak{A}_\mathfrak{b} \) into the subsets \( \mathfrak{A}_\mathfrak{b}(p) \) parametrized by the standard parabolic subalgebras. Here \( \mathfrak{A}_\mathfrak{b}(p) := \{ c \in \mathfrak{A}_\mathfrak{b} \mid n_g(c) = p \} \). Obviously, the nilpotent radical of \( p \) is the unique maximal element of \( \mathfrak{A}_\mathfrak{b}(p) \). But there can be several minimal elements. It seems that the most interesting subset is \( \mathfrak{A}_\mathfrak{b}(b) \). We give a description of it in the spirit of Cellini-Papi. According to [4], there is a bijection between \( \mathfrak{A}_\mathfrak{b} \) and the points of the coroot lattice \( Q^\vee \) lying in the simplex \( D_{\min} = \{ x \in V \mid (\alpha, x) \geq -1 \forall \alpha \in \Pi \& (\theta, x) \leq 2 \} \). Here \( V = \bigoplus_{\alpha \in \Pi} \mathbb{R}_{\alpha}, \theta \) is the highest root, and \( ( \ , \ ) \) is a \( W \)-invariant inner product on \( V \). Our result says that the normalizer of the ideal corresponding to \( x \in D_{\min} \cap Q^\vee \) is \( b \) if and only if \( (x, \alpha) \neq 0 \) for all \( \alpha \in \Pi \) and \( (x, \theta) \neq 1 \).

This description allows us to interpret the number \( \#\mathfrak{A}_\mathfrak{b}(b) \) as the coefficient of \( x \) in a certain Laurent series. This series depends only on the coefficients of \( \theta \) in the basis of the simple roots. Namely, if \( \theta = \sum_{i=1}^n c_i \alpha_i, \ c_0 = 1, \) and \( f = \#\{ j \mid c_j = 1 \} \), then

\[
\#\mathfrak{A}_\mathfrak{b}(b) = \frac{1}{f}[x] \prod_{i=0}^n \left( \frac{x^{-c_i}}{1 - x^{c_i}} - 1 \right).
\]

A similar characterization is obtained for \( \mathfrak{A}_\mathfrak{b}(b) := \mathfrak{A}_\mathfrak{b}(b) \cap \mathfrak{A}_0 \). Using these results we explicitly compute the numbers \( \#\mathfrak{A}_\mathfrak{b}(b) \) and \( \#\mathfrak{A}_0(b) \) for the classical Lie algebras.

In case of \( g = sl_{n+1} \) and \( sp_{2n} \), our results are more precise. We explicitly describe the set \( \mathfrak{A}_\mathfrak{b}(p) \) for any \( p \). For \( g = sl_{n+1} \), it follows from that description that \( \#\mathfrak{A}_\mathfrak{b}(p) \) depends only on the difference \( s = n - \text{srk} \ p \). Namely, it is the \( s \)-th Motzkin number. For \( g = sp_{2n} \), the quantity \( \#\mathfrak{A}_\mathfrak{b}(p) \) depends only on the number of short simple roots, say \( s \), that are not in the Levi subalgebra of \( p \). Namely, it is the number of directed animals of size \( s + 1 \).

In these two cases, it is also shown that \( \mathfrak{A}_\mathfrak{b}(p) \) has always a unique minimal element. The interest of two series, \( sl_{n+1} \) and \( sp_{2n} \), is revealed via the fact that one can tie together the notion of duality for ad-nilpotent ideals [10], the minimax ideals [12], and the ideals whose normalizer equals \( b \). That is, we prove that \( n_g(c) = b \) if and only if the dual ideal, \( c^* \), is minimax. Also, it turns out that considering the normalizers of ad-nilpotent ideals provides a natural framework for demonstrating various identities related to Catalan, Motzkin, Riordan, and other numbers.

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1. Notation and other preliminaries

(1.1) Main notation. \( \Delta \) is the root system of \((g, t)\) and \( W \) is the usual Weyl group. For \( \mu \in \Delta \), \( g_\mu \) is the corresponding root space in \( g \).
\(\Delta^+\) is the set of positive roots, \(\theta\) is the highest root in \(\Delta^+\), and \(\rho = \frac{1}{2} \sum_{\mu \in \Delta^+} \mu\).

\(\Pi = \{\alpha_1, \ldots, \alpha_p\}\) is the set of simple roots in \(\Delta^+\) and \(\varphi_i\) is the fundamental weight corresponding to \(\alpha_i\).

We set \(V := t_\mathbb{R} = \oplus_{i=1}^p \mathbb{R} \alpha_i\) and denote by \((, )\) a \(W\)-invariant inner product on \(V\). As usual, \(\mu^\vee = 2\mu/(\mu, \mu)\) is the coroot for \(\mu \in \Delta\).

\(\mathcal{C} = \{x \in V \mid (x, \alpha) > 0 \ \forall \alpha \in \Pi\}\) is the (open) fundamental Weyl chamber.

\(\mathcal{A} = \{x \in V \mid (x, \alpha) > 0 \ \forall \alpha \in \Pi \ \& \ (x, \theta) < 1\}\) is the fundamental alcove.

\(Q^+ = \{\sum_{i=1}^p n_i \alpha_i \mid n_i = 0, 1, 2, \ldots\}\) and \(Q^\vee = \oplus_{i=1}^p \mathbb{Z} \alpha_i^\vee \subset V\) is the coroot lattice.

For \(\mu, \gamma \in \Delta^+\), write \(\mu \preceq \gamma\), if \(\gamma - \mu \in Q^+\). We regard \(\Delta^+\) as poset under \(\preceq\).

Letting \(\widehat{\mathcal{V}} = V \oplus \mathbb{R} \delta \oplus \mathbb{R} \lambda\), we extend the inner product \((, )\) on \(\widehat{\mathcal{V}}\) so that \((\delta, V) = (\lambda, V) = (\delta, \delta) = (\lambda, \lambda) = 0\) and \((\delta, \lambda) = 1\).

\(\widehat{\Delta} = \{\Delta + k\delta \mid k \in \mathbb{Z}\}\) is the set of affine real roots and \(\widehat{W}\) is the affine Weyl group.

Then \(\widehat{\Delta}^+ = \Delta^+ \cup \{\Delta + k\delta \mid k \geq 1\}\) is the set of positive affine roots and \(\widehat{\Pi} = \Pi \cup \{\alpha_0\}\) is the corresponding set of affine simple roots, where \(\alpha_0 = \delta - \theta\). The inner product \((, )\) on \(\widehat{\mathcal{V}}\) is \(\widehat{W}\)-invariant. The notation \(\beta > 0\) (resp. \(\beta < 0\)) is a shorthand for \(\beta \in \widehat{\Delta}^+\) (resp. \(\beta \in -\widehat{\Delta}^+\)).

For \(\alpha_i\) \((0 \leq i \leq p)\), we let \(s_i\) denote the corresponding simple reflection in \(\widehat{W}\). If the index of \(\alpha \in \widehat{\Pi}\) is not specified, then we merely write \(s_\alpha\). The length function on \(\widehat{W}\) with respect to \(s_0, s_1, \ldots, s_p\) is denoted by \(\ell\). For any \(w \in \widehat{W}\), we set

\[N(w) = \{\mu \in \widehat{\Delta}^+ \mid w(\mu) \in -\widehat{\Delta}^+\}.\]

It is standard that \(\#N(w) = \ell(w)\) and \(N(w)\) is bi-convex. The latter means that both \(N(w)\) and \(\widehat{\Delta}^+ \setminus N(w)\) are subsets of \(\widehat{\Delta}^+\) that are closed under addition. Furthermore, the assignment \(w \mapsto N(w)\) sets up a bijection between the elements of \(\widehat{W}\) and the finite bi-convex subsets of \(\widehat{\Delta}^+\).

\textbf{(1.2) Ideals and antichains.} Our \(\mathfrak{b}\) is the Borel subalgebra of \(\mathfrak{g}\) corresponding to \(\Delta^+\) and \(u = [\mathfrak{b}, \mathfrak{b}]\). If \(\mathfrak{c}\) is an ad-nilpotent ideal of \(\mathfrak{b}\), then \(\mathfrak{c} = \bigoplus_{\alpha \in I_\mathfrak{c}} \mathfrak{g}_\alpha\) for some \(I_\mathfrak{c} \subset \Delta^+\). The set of roots \(I_\mathfrak{c}\) \((c \in \mathfrak{X}\mathfrak{d})\) arising in this way is an upper ideal of (the poset) \(\Delta^+\). This means that \(I_\mathfrak{c}\) satisfies the following property:

if \(\gamma \in I_\mathfrak{c}, \mu \in \Delta^+, \text{ and } \gamma + \mu \in \Delta\), then \(\gamma + \mu \in I_\mathfrak{c}\).

In view of the obvious bijection between the ad-nilpotent ideals and the upper ideals of \(\Delta^+\), we will often identify them. A root \(\gamma \in I_\mathfrak{c}\) is called a generator of \(I_\mathfrak{c}\) or \(c\), if \(\gamma - \alpha \not\in I_\mathfrak{c}\) for any \(\alpha \in \Pi\). In other words, \(\gamma\) is a minimal element of \(I_\mathfrak{c}\) with respect to \(\preceq\). We write \(\Gamma(I_\mathfrak{c})\) or \(\Gamma(c)\) for the set of generators. It is easily seen that \(\Gamma(I_\mathfrak{c})\) is an antichain of \(\Delta^+\), i.e., \(\gamma_i \not\approx \gamma_j\) for any pair \((\gamma_i, \gamma_j)\) in \(\Gamma(I_\mathfrak{c})\). Conversely, if \(\Gamma \subset \Delta^+\) is an antichain, then the upper ideal \(I(\Gamma) := \{\mu \in \Delta^+ \mid \mu \succeq \gamma_i \text{ for some } \gamma_i \in \Gamma\}\) has \(\Gamma\) as the set of generators.

2. The weight and normalizer of an ad-nilpotent ideal

A parabolic subalgebra of \(\mathfrak{g}\) is called standard, if it contains \(\mathfrak{b}\). If \(\Pi' \subset \Pi\), then \(p(\Pi')\) stands for the standard parabolic subalgebra which is generated by \(\mathfrak{b}\) and the spaces \(\mathfrak{g}_{-\alpha},\)
\( \alpha \in \Pi' \). For instance, \( p(\emptyset) = b \) and \( p(\Pi) = g \). The maximal (proper) parabolic subalgebra \( p(\Pi \setminus \{ \alpha \}) \) is also denoted by \( p(\{ \alpha \}) \), and we write \( p(\alpha) \) in place of \( p(\{ \alpha \}) \). The standard Levi subalgebra of \( p(\Pi') \), denoted \( l(\Pi') \), is generated by \( t \) and the subspaces \( g_{\alpha}, \alpha \in \pm \Pi' \). Write \( \srk p(\Pi') \) for \( \srk [l(\Pi'), l(\Pi')] \), the semisimple rank of \( p(\Pi') \). We have \( \srk p(\Pi') = \# \Pi' \).

Let \( n_g(\mathfrak{c}) \) be the normalizer in \( g \) of an ad-nilpotent ideal \( \mathfrak{c} \). It is a standard parabolic subalgebra of \( g \). Therefore, to describe \( n_g(\mathfrak{c}) \) explicitly, one has to only realize when \( g_{-\alpha} \) is contained in \( n_g(\mathfrak{c}) \) for an \( \alpha \in \Pi \). A description of \( n_g(\mathfrak{c}) \) in terms of \( \Gamma(\mathfrak{c}) \) is given in [13 Theorem 3.2]:

**2.1 Theorem.** \( g_{-\alpha} \not\subset n_g(\mathfrak{c}) \) if and only if \( \gamma = \alpha \in \Delta^+ \cup \{0\} \) for some \( \gamma \in \Gamma(\mathfrak{c}) \).

The aim of this section is to give some other descriptions of \( n_g(\mathfrak{c}) \) associated with the combinatorial theory of ad-nilpotent ideals.

Recall some basic results concerning a connection between the ad-nilpotent ideals and certain elements in the affine Weyl group. Given \( \mathfrak{c} \in \mathfrak{A}_0 \) with the corresponding upper ideal \( I_{\mathfrak{c}} \subset \Delta^+ \), there is a unique element \( w = w_{\min, c} \in \hat{W} \) satisfying the following properties (see [3]):

\[
\begin{align*}
(\diamond) & \quad \text{For } \gamma \in \Delta^+, \text{ we have } \gamma \in I_{\mathfrak{c}} \text{ if and only if } w(\delta - \gamma) < 0; \\
(\text{dom}) & \quad w(\alpha) > 0 \text{ for all } \alpha \in \Pi; \\
(\text{min}) & \quad \text{if } \alpha \in \hat{\Pi} \text{ and } w^{-1}(\alpha) = k\delta + \mu \text{ for some } \mu \in \Delta, \text{ then } k \geq -1.
\end{align*}
\]

This element \( w \) is said to be the **minimal element of** \( \mathfrak{c} \). The elements of \( \hat{W} \) satisfying property (dom) are called **dominant**. The elements of \( \hat{W} \) satisfying the last two properties are called **minimal**. The minimal element of \( \mathfrak{c} \) can also be characterized as the unique element of \( \hat{W} \) satisfying properties (\( \diamond \)), (dom), and having the minimal possible length. This explains the term.

An ad-nilpotent ideal \( \mathfrak{c} \) is called **strictly positive**, if it is contained in \([u, u]\) (i.e., \( I_{\mathfrak{c}} \) contains no simple roots). The set of strictly positive ideals is denoted by \( \mathfrak{A}_0 \). If \( \mathfrak{c} \in \mathfrak{A}_0 \), then there is a unique element \( w = w_{\max, \mathfrak{c}} \in \hat{W} \) satisfying properties (\( \diamond \)) and (dom), as above, and also the property

\[
(\text{max}) \quad \text{if } \alpha \in \hat{\Pi} \text{ and } w^{-1}(\alpha) = k\delta + \mu \text{ for some } \mu \in \Delta, \text{ then } k \leq 1,
\]

(see [15]). This element is said to be the **maximal element of** \( \mathfrak{c} \). The elements of \( \hat{W} \) satisfying properties (dom) and (max) are called **maximal**. The maximal element of a strictly positive ideal can also be characterized as the unique element of \( \hat{W} \) satisfying properties (\( \diamond \)), (dom), and having the maximal possible length. This explains the term.

Usually, we have \( w_{\min, \mathfrak{c}} \neq w_{\max, \mathfrak{c}} \). The case of coincidence of these two elements is studied in [12]. The respective ideals are called **minimax**.

For any finite subset \( M \subset \hat{\Delta}^+ \), we set \( |M| := \sum_{\gamma \in M} \gamma \). If \( \mathfrak{c} \in \mathfrak{A}_0 \) and \( I_{\mathfrak{c}} \) is the corresponding upper ideal, then put \( |\mathfrak{c}| := |I_{\mathfrak{c}}| \). We say that \( |\mathfrak{c}| \) is the **weight** of the ad-nilpotent ideal \( \mathfrak{c} \). Our first aim is to look at the weights occurring in this way. The following result
is due to Kostant [8, Theorem 7]. For the sake of completeness, we give a proof, which demonstrates the role of minimal elements.

2.2 Proposition. Suppose $c_1, c_2 \in \mathfrak{A} \otimes$ and $|c_1| = |c_2|$. Then $c_1 = c_2$.

Proof. Let $I_1, I_2$ be the corresponding upper ideals. Assume $I_1 \neq I_2$. Then both sets $I_1 \setminus I_2$ and $I_2 \setminus I_1$ are non-empty and we have $|I_1 \setminus I_2| = |I_2 \setminus I_1|$. Let us rewrite this equality in the form:

$$(2.3) \quad \sum_{\gamma \in I_1 \setminus I_2} (\delta - \gamma) - c\delta = \sum_{\gamma \in I_2 \setminus I_1} (\delta - \gamma),$$

where $c = \#I_1 - \#I_2$. Without loss of generality, we may assume that $\dim \ c_1 \geq \dim \ c_2$, i.e., $c \geq 0$. Let $w_1 \in \hat{\mathcal{W}}$ be the minimal element of $c_1$. Applying $w_1$ to Eq. (2.3) and using property $(\diamond)$, we see that $w_1(\text{LHS})$ (resp. $w_1(\text{RHS})$) is a sum of negative (resp. positive) roots. A contradiction! \hfill $\Box$

For $\mathfrak{c} \in \mathfrak{A} \otimes$, we set $c^1 = c$ and $c^k = [c^{k-1}, c]$ for $k \geq 2$. Then $c^m = 0$ for $m \gg 0$.

2.4 Theorem. Let $\mathfrak{c}$ be an ad-nilpotent ideal of $\mathfrak{b}$ and $\alpha \in \Pi$. Then

(i) $(|\mathfrak{c}|, \alpha) \geq 0$;

(ii) $\mathfrak{g}_{-\alpha} \subseteq \mathfrak{n}_\mathfrak{g}(\mathfrak{c})$ $\iff$ $(|\mathfrak{c}|, \alpha) = 0$;

(iii) $(|\mathfrak{c}|, \alpha) = 0 \iff (\sum_{k \geq 1} |\mathfrak{c}|, \alpha) = 0$.

Proof. (i), (ii) For $\alpha \in \Pi$, let $\mathfrak{sl}_2(\alpha)$ be the simple three-dimensional subalgebra of $\mathfrak{g}$ generated by $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$. Let $\{x_\alpha, h_\alpha, y_\alpha\}$ be a basis for $\mathfrak{sl}_2(\alpha)$, where $x_\alpha \in \mathfrak{g}_\alpha$, $y_\alpha \in \mathfrak{g}_{-\alpha}$, and $h_\alpha = [x_\alpha, y_\alpha]$. Obviously, $\mathfrak{c}$ is a $\langle x_\alpha, h_\alpha \rangle$-module. Since $\mathfrak{c}$ is a subspace of an $\mathfrak{sl}_2(\alpha)$-module, we conclude that $(|\mathfrak{c}|, \alpha) \geq 0$. This proves part (i). Furthermore, $(|\mathfrak{c}|, \alpha) = 0$ if and only if $\mathfrak{c}$ is an $\mathfrak{sl}_2(\alpha)$-module, i.e., $y_{-\alpha} \in \mathfrak{n}_\mathfrak{g}(\mathfrak{c})$.

(iii) Since $c^k$ is an ad-nilpotent ideal for all $k \geq 1$, we have $(|c^k|, \alpha) \geq 0$ by part (i). This gives the implication "$\Leftarrow". On the other hand, if $\mathfrak{g}_{-\alpha} \subseteq \mathfrak{n}_\mathfrak{g}(\mathfrak{c})$, then $\mathfrak{g}_{-\alpha} \subseteq \mathfrak{n}_\mathfrak{g}(c^k)$ as well, and one may apply part (ii) to $c^k$. \hfill $\Box$

Thus, the weight of any ideal is dominant, different ideals have different weights, and the normalizer of an ad-nilpotent ideal is completely determined by its weight.

2.5 Remarks. 1. If $c_1$ and $c_2$ are two ad-nilpotent ideals, then $c_1 \cap c_2$ and $c_1 + c_2$ are ad-nilpotent ideals as well. Also, $|c_1 + c_2| + |c_1 \cap c_2| = |c_1| + |c_2|$. Clearly,

$$\mathfrak{n}_\mathfrak{g}(c_1 + c_2) \supseteq \mathfrak{n}_\mathfrak{g}(c_1) \cap \mathfrak{n}_\mathfrak{g}(c_2) \text{ and } \mathfrak{n}_\mathfrak{g}(c_1 \cap c_2) \supseteq \mathfrak{n}_\mathfrak{g}(c_1) \cap \mathfrak{n}_\mathfrak{g}(c_2).$$

But both these containments can be strict even if $\mathfrak{n}_\mathfrak{g}(c_1) = \mathfrak{n}_\mathfrak{g}(c_2)$, see Example 2.12(2) below.

2. It is an interesting open problem to characterize abstractly the set of weights $\{|\mathfrak{c}| \mid \mathfrak{c} \in \mathfrak{A} \otimes\}$. For instance, if $\mathfrak{g} = \mathfrak{G}_2$, then this set is equal to $\{0, \varphi_2, 3\varphi_1, 4\varphi_1, 3\varphi_1 + \varphi_2, 5\varphi_1, 3\varphi_2, 2\varphi_1 + 2\varphi_2\}$.
We wish also to obtain a description of \( n_{\vartheta}(c) \) in terms of \( w_{\min, c} \) (and \( w_{\max, c} \) if \( c \in \mathfrak{A}_0 \)). To this end, consider
\[
\hat{\rho} = \rho + \frac{(\theta, \theta)}{2}(1 + (\rho, \theta'))\lambda \in \hat{V}.
\]
Since \((\rho, \rho') = 1\) for any \( \alpha \in \Pi \), it readily follows that \( \hat{\rho} \) is the unique element of \( V \oplus \mathbb{R}\lambda \) having the property that \((\hat{\rho}, \rho') = 1\) for all \( \alpha \in \hat{\Pi} \).

2.6 Proposition. For any \( w \in \hat{W} \), we have \( \hat{\rho} - w^{-1}(\hat{\rho}) = |N(w)| \).

Proof. We argue by induction on \( \ell(w) \). If \( w = s_{\alpha}, \alpha \in \hat{\Pi} \), then \( N(s_{\alpha}) = \{ \alpha \} \) and the claim follows from the definition of \( \hat{\rho} \). Suppose \( \ell(w) > 1 \) and \( w = s_{\alpha}w_{\tilde{w}} \), where \( \ell(w) = \ell(w_{\tilde{w}}) + 1 \). Then \( N(w) = N(w_{\tilde{w}}) \cup \{ w_{\tilde{w}}^{-1}(\alpha) \} \). Assume that the claim holds for \( \tilde{w} \). Then \( \hat{\rho} - w^{-1}(\hat{\rho}) = \hat{\rho} - w^{-1}(\hat{\rho}) + w_{\tilde{w}}^{-1}(\hat{\rho}) - w_{\tilde{w}}^{-1}(\hat{\rho}) = |N(w_{\tilde{w}})| + w_{\tilde{w}}^{-1}(\hat{\rho} - s_{\alpha}\hat{\rho}) = |N(w_{\tilde{w}})| + w_{\tilde{w}}^{-1}(\hat{\rho}) = |N(w)| \). \( \square \)

2.7 Lemma. Let \( w \in \hat{W} \) be a dominant element. Then
\[
(i) \quad (|N(w)|, \alpha) \leq 0 \text{ for any } \alpha \in \Pi;
(ii) \quad (|N(w)|, \alpha) = 0 \text{ if and only if } w(\alpha) \in \hat{\Pi};
\]

Proof. By Proposition 2.6, \( \hat{\rho} - w^{-1}(\hat{\rho}) = |N(w)| \). It follows that \((|N(w)|, \rho') = 1 - (\hat{\rho}, w(\alpha)\rho') \). By property (dom), we have \( w(\alpha) \) is positive. This yields all the assertions. \( \square \)

The following is our main result for minimal elements.

2.8 Theorem. Let \( c \) be an arbitrary ad-nilpotent ideal of \( b \) and \( \alpha \in \Pi \). Then \( g_{-\alpha} \subset n_{\vartheta}(c) \) if and only if \( w_{\min, c}(\alpha) \in \hat{\Pi} \).

Proof. By [3] Section 2, we have \( N(w_{\min, c}) = \bigcup_{k \geq 1} (k\delta - I_c) \). Hence,
\[
(|N(w_{\min, c})|, \alpha) = -(\sum_{k \geq 1} |k\delta - I_c|, \alpha).
\]
Therefore, combining Theorem 2.4 and Lemma 2.7, we obtain:

\( g_{-\alpha} \subset n_{\vartheta}(c) \) if and only if \((|N(w_{\min, c})|, \alpha) = 0 \) if and only if \( w_{\min, c}(\alpha) \in \hat{\Pi} \). \( \square \)

Next, we show that, for a strictly positive ideal \( c \), the similar claim holds with \( w_{\max, c} \).

2.9 Theorem. Suppose \( c \in \mathfrak{A}_0 \) and \( \alpha \in \Pi \). Then \( w_{\max, c}(\alpha) \in \hat{\Pi} \) if and only if \( w_{\min, c}(\alpha) \in \hat{\Pi} \).

Proof. We have already proved that the two conditions:

- \( w_{\min, c}(\alpha) \in \hat{\Pi} \)
- \( c \) is an \( sl_2(\alpha) \)-module

are equivalent. Therefore, by Lemma 2.7, it suffices to prove that \((|N(w_{\max, c})|, \alpha) = 0 \) if and only if \( c \) is an \( sl_2(\alpha) \)-module. A description of \( N(w_{\max, c}) \) is due to Sommers [15], see also [11] 2.11. We state it in a form convenient for our purposes. Let \( m \) be the (unique) t-stable complement of \( c \) in \( u \). Set \( m^1 = m \) and \( m^k = [m^{k-1}, m] \) for \( k \geq 2 \). Let \( \xi^k \) be the
t-stable complement of $m + m^2 + \ldots + m^k$ in $u$. Then $c = \tilde{c} \supset \tilde{c}^2 \supset \ldots$ and $\tilde{c}^m = 0$ for $m \gg 0$. By [15], we have $\tilde{c}^k \subset \tilde{c}^k$ for any $k$ and $N(w_{max, c}) = \bigcup_{k \geq 1} (k\delta - I_{\tilde{c}^k})$. Hence

$$
(\mid N(w_{max, c})\mid, \alpha) = -\sum_k (|\tilde{c}^k|, \alpha).
$$

(2.10)

Notice that $g_\alpha \subset m$. Let $m_\alpha$ be the t-stable complement of $g_\alpha$ in $m$. Then $m_\alpha \oplus c = p(\alpha)^{nil}$, the nilpotent radical of the minimal parabolic subalgebra $p(\alpha)$. Since $p(\alpha)$ is an $sl_2(\alpha)$-module and $c$ is an $\langle x_\alpha, h_\alpha \rangle$-module, $m_\alpha$ is an $\langle y_\alpha, h_\alpha \rangle$-module. Furthermore, $m_\alpha$ is an $sl_2(\alpha)$-module if and only if $c$ is. Next, we have $m + m^2 + \ldots + m^k = g_\alpha \oplus (m_\alpha + m^2 + \ldots + m^k)$. Set $m_\alpha^{(k)} := (m_\alpha + m^2 + \ldots + m^k)$ for $k \geq 2$. Then $m_\alpha^{(k)} \oplus \tilde{c}^k = p(\alpha)^{nil}$. By induction, it is easily seen that each $m_\alpha^{(k)}$ is an $\langle y_\alpha, h_\alpha \rangle$-module. Therefore $\tilde{c}^k$ is an $\langle x_\alpha, h_\alpha \rangle$-module and hence $(|\tilde{c}^k|, \alpha) \geq 0$ for all $k \geq 1$. If $(\mid N(w_{max, c})\mid, \alpha) = 0$, then it follows from Eq. (2.10) that $(|\tilde{c}^k|, \alpha) = 0$ for all $k$. In particular, $\tilde{c} = c$ is an $sl_2(\alpha)$-module. Conversely, if $c$ is an $sl_2(\alpha)$-module, then $m_\alpha$ is. This easily implies that $m_\alpha^{(k)}$ is an $sl_2(\alpha)$-module for all $k \geq 2$. Hence all $\tilde{c}^k$, $k \geq 1$, are $sl_2(\alpha)$-modules and we conclude from Eq. (2.10) that $(\mid N(w_{max, c})\mid, \alpha) = 0$.

\[ \square \]

It is not however true in general that two simple roots $w_{max,c}(\alpha) \in \tilde{\Pi}$ and $w_{min,c}(\alpha) \in \tilde{\Pi}$ are equal, see Example 2.12(4) below.

2.11 Corollary. For $c \in \mathfrak{sl}_9$ and $\alpha \in \Pi$, we have $g_{-\alpha} \subset n_\theta(c)$ if and only if $(\sum_k |\tilde{c}^k|, \alpha) = 0$ if and only if $w_{max, c}(\alpha) \in \tilde{\Pi}$.

2.12 Examples. Here we give some illustrations to previous results. The numbering of the simple roots and fundamental weights is the same as in [17].

(1) Let $c$ be the ad-nilpotent ideal for $g = sl_5$ with $\Gamma(c) = \{\alpha_1 + \alpha_2, \alpha_2 + \alpha_3 + \alpha_4\}$. It is an Abelian ideal, i.e., $c^2 = 0$. Here $|c| = 2\varphi_1 + 2\varphi_2 + \varphi_4$. Using either Theorem 2.1 or Theorem 2.4 we obtain $n_\theta(c) = p(\alpha_3)$. Using the algorithm given in [9], one finds that $w_{min,c} = s_1 s_4 s_5 s_0$. Therefore the action of $w_{min,c}$ on $\Pi$ is given by

$$
w_{min,c} : \begin{cases} 
\alpha_1 &\mapsto \delta - \alpha_1 - \alpha_2 \\
\alpha_2 &\mapsto \alpha_1 + \alpha_2 + \alpha_3 \\
\alpha_3 &\mapsto \alpha_4 \\
\alpha_4 &\mapsto \delta - \alpha_2 - \alpha_3 - \alpha_4
\end{cases}
$$

Then using Theorem 2.8 we obtain again the same description of $n_\theta(c)$. In this case $c^2 = 0$, but $\tilde{c}^2 = g_\theta$ and $\tilde{c}^3 = 0$. Therefore $\sum_k |\tilde{c}^k| = |c| + \theta = 3\varphi_1 + 2\varphi_2 + 2\varphi_4$, which yields an illustration to Corollary 2.11.

Finally, one can compute that $w_{max,c} = s_2 w_{min,c}$.

(2) Take $g = sl_7$ and the ideals $c_1, c_2$ with $\Gamma(c_1) = \{\alpha_1 + \alpha_2, \alpha_3 + \alpha_4 + \alpha_5, \alpha_4 + \alpha_5 + \alpha_6\}$, $\Gamma(c_2) = \{\alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3 + \alpha_4, \alpha_4 + \alpha_5 + \alpha_6\}$. Then one easily computes that $n_\theta(c_1) = n_\theta(c_2) = b$, whereas $n_\theta(c_1 \cap c_2) = p(\alpha_2)$ and $n_\theta(c_1 + c_2) = p(\alpha_3)$.

(3) $g = F_4$.

Write $[n_1, n_2, n_3, n_4]$ for $\sum_i n_i \alpha_i$. Consider the ideal $c$ with $\Gamma(c) = \{[0, 2, 2, 1], [2, 2, 1, 0]\}$. 

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Here \(|c| = [16, 28, 20, 10] = 4\varphi_1 + 2\varphi_3\). Next, \(c^2 = \tilde{c}^2 = g_{[2, 4, 3, 1]} \oplus g_{[2, 4, 3, 2]}\) and \(\tilde{c}^3 = 0\). Therefore \(\sum_k |c^k| = \sum_k |\tilde{c}^k| = [20, 36, 26, 13] = 4\varphi_1 + 3\varphi_3\). Thus, \(n_\mathfrak{g}(c) = p(\{\alpha_2, \alpha_4\})\). In this case, we have \(w_{\min, \mathfrak{c}} = w_{\max, \mathfrak{c}} = s_0s_4s_3s_2s_0s_4s_3s_1s_2s_3s_4s_0\).

(4) \(\mathfrak{g} = \mathfrak{G}_2\).

Consider the Abelian ideal \(\mathfrak{c}\) with \(\Gamma(\mathfrak{c}) = \{2\alpha_1 + \alpha_2\}\). Here \(w_{\min, \mathfrak{c}} = s_1s_2s_0\) and \(w_{\max, \mathfrak{c}} = s_0s_2s_1s_2s_0\). Therefore

\[
\begin{align*}
    w_{\min, \mathfrak{c}} : & \quad \alpha_1 \mapsto 2\alpha_1 + \alpha_2, \quad \alpha_2 \mapsto \delta - 3\alpha_1 - 2\alpha_2 = \alpha_0, \\
    w_{\max, \mathfrak{c}} : & \quad \alpha_1 \mapsto \delta - \alpha_1 - \alpha_2, \quad \alpha_2 \mapsto \alpha_2.
\end{align*}
\]

Thus, we have \(n_\mathfrak{g}(\mathfrak{c}) = p(\alpha_2), \) but \(w_{\min, \mathfrak{c}}(\alpha_2) \neq w_{\max, \mathfrak{c}}(\alpha_2)\).

3. A GEOMETRIC DESCRIPTION OF THE NORMALIZER

For \(\mu \in \Delta^+\) and \(k \in \mathbb{Z}\), define the hyperplane \(\mathcal{H}_{\mu, k}\) in \(V\) as \(\{x \in V \mid (x, \mu) = k\}\). The collection of all these hyperplanes is called the affine arrangement in \(V\). The regions of an arrangement are the connected components of the complement in \(V\) of the union of all its hyperplanes. As is well-known (see e.g. [7]), the regions of the affine arrangement are simplices (alcoves), and \(\mathcal{A}\) is one of them.

Recall a bijection between the ad-nilpotent ideals (or upper ideals of \(\Delta^+\)) and the dominant regions of the Shi arrangement, which is due to J.-Y. Shi [14, Theorem 1.4]. The Shi arrangement, \(\text{Shi}(\Delta)\), is the sub-arrangement of the affine arrangement consisting of all hyperplanes \(\mathcal{H}_{\mu, k}\) with \(k = 0, 1\). Any region of \(\text{Shi}(\Delta)\) lying in \(\mathcal{C}\) is said to be dominant.

The Shi bijection takes an ad-nilpotent ideal \(\mathfrak{c}\) with corresponding upper ideal \(I_\mathfrak{c} \subset \Delta^+\) to the dominant region

\[
R_\mathfrak{c} = \{x \in \mathcal{C} \mid (x, \gamma) > 1, \text{ if } \gamma \in I_\mathfrak{c} \text{ and } (x, \gamma) < 1, \text{ if } \gamma \notin I_\mathfrak{c}\}.
\]

Our goal is to describe \(n_\mathfrak{g}(\mathfrak{c})\) in terms of \(R_\mathfrak{c}\). To this end, we use relations between the two actions of \(\hat{W}\): the linear action on \(\hat{V}\) and the affine action on \(V\). We use ‘∗’ to denote the affine action: \((w, x) \mapsto w \ast x, w \in \hat{W}, x \in V\). For any \(\alpha \in \hat{\Pi}\), let \(H_\alpha\) denote the corresponding wall of \(\mathcal{A}\). That is, \(H_\alpha = \left\{ \begin{array}{ll} \mathcal{H}_{\alpha, 0}, & \text{if } \alpha \in \Pi, \\ \mathcal{H}_{\alpha, 1}, & \text{if } \alpha = \alpha_0. \end{array} \right. \)

The generator \(s_\alpha \in \hat{W}\) acts on \(V\) as (affine) reflection relative to \(H_\alpha\). Our next arguments will be based on comparing properties of these two actions. The following is Eq. (1.1) in [3]. Suppose \(\mu \in \Delta^+, k > 0,\) and \(h \geq 0\). Then

\[
\begin{align*}
    w(k\delta - \mu) < 0 & \quad \text{if and only if } \mathcal{H}_{\mu, k} \text{ separates } \mathcal{A} \text{ and } w^{-1} \ast \mathcal{A}, \\
    w(h\delta + \mu) < 0 & \quad \text{if and only if } \mathcal{H}_{\mu, -h} \text{ separates } \mathcal{A} \text{ and } w^{-1} \ast \mathcal{A}.
\end{align*}
\]

It follows from these equations that \(w \in \hat{W}\) is dominant if and only if \(w^{-1} \ast \mathcal{A} \subset \mathcal{C}\). Another useful relation is

\[
w \ast \mathcal{H}_{\mu_1, k_1} = \mathcal{H}_{\mu_2, k_2} \quad \text{if and only if } w(k_1\delta - \mu_1) = \pm (k_2\delta - \mu_2),
\]

where \(\mu_i \in \Delta^+\) and \(k_i \in \mathbb{Z}\). It suffices to verify this only for the simple reflections, the case of \(w = s_i (i > 0)\) being trivial. Some calculations are only needed for \(w = s_0\).
Notice that if $R$ is a dominant region, then its walls belong to the set $\mathcal{H}_{\gamma,1}$, $\gamma \in \Delta^+$, and $\mathcal{H}_{\alpha,0}$, $\alpha \in \Pi$.

3.4 Theorem. Suppose $c \in \mathfrak{A}^0$ and $\alpha \in \Pi$. Then $\mathfrak{g}_{-\alpha} \subset \mathfrak{n}_g(c)$ if and only if $\mathcal{H}_{\alpha,0}$ is a wall of $R_c$.

Proof. For any $w \in \hat{W}$, let $\mathcal{L}(w)$ denote the set of all hyperplanes $\mathcal{H}_{\gamma,k}$ separating $\mathcal{A}$ and $w \ast \mathcal{A}$.

Suppose $\mathfrak{g}_{-\alpha} \subset \mathfrak{n}_g(c)$. Then $w_{\text{min},c}(\alpha) =: \nu \in \hat{H}$. Then $N(s_{\nu}w_{\text{min},c}) = N(w_{\text{min},c}) \cup \{\alpha\}$. This already means that $\hat{w} := s_{\nu}w_{\text{min},c}$ is not dominant. Furthermore, by Theorem 4.5 in [7], we have $\mathcal{L}((\hat{w})^{-1}) = \mathcal{L}(w_0^{-1}) \cup \{w_0^{-1} \ast H_{\nu}\}$ and by Eq. (3.3), $w_0^{-1} \ast H_{\nu} = \mathcal{H}_{\alpha,0}$. That is, the hyperplane $\mathcal{H}_{\alpha,0}$ separates the alcoves $\hat{w}^{-1} \ast \mathcal{A}$ and $w_0^{-1} \ast \mathcal{A}$. Since $w_0^{-1} \ast \mathcal{A} \subset R_c$ [4], we conclude that $\mathcal{H}_{\alpha,0}$ is a wall of $R_c$.

Conversely, suppose $\mathcal{H}_{\alpha,0}$ is a wall of $R_c$. This means that there is a $w \in \hat{W}$ such that $w^{-1} \ast \mathcal{A} \subset R_c$ (hence $w$ is dominant!) and $\mathcal{H}_{\alpha,0}$ is a wall of the alcove $w^{-1} \ast \mathcal{A}$. Equivalently, $w \ast \mathcal{H}_{\alpha,0}$ is a wall of $A$. Then $w \ast \mathcal{H}_{\alpha,0} = H_{\nu}$ for some $\nu \in \hat{H}$ and hence $w(\alpha) = \pm \nu$, by Eq. (3.3). Since $w$ is dominant, we actually have $w(\alpha) = \nu$. Next, it follows from the dominance of $w$ that $N(w) = \bigcup_{k \geq 1} (k \delta - I_k)$, where each $I_k$ is an upper ideal. Furthermore, in view of Eq. (3.2), the condition $w^{-1} \ast \mathcal{A} \subset R_c$ precisely means that $\delta - \gamma \in N(w)$ if and only if $\gamma \in I_1$, i.e., $I_1 = I_c$. Since $w(\alpha) \in \hat{H}$, it follows from Lemma 2.7 that $(|N(w)|, \alpha) = 0$ and hence $(|I_1|, \alpha) = 0$ for all $k$. In particular, $(|I_1|, \alpha) = (|\alpha|, \alpha) = 0$, i.e., $\mathfrak{g}_{-\alpha} \subset \mathfrak{n}_g(c)$.

Remark. For $c \in \mathfrak{A}_0$, the previous result and Theorem 2.9 show that the alcoves $w_{\text{min},c}^{-1} \ast \mathcal{A}$ and $w_{\text{max},c}^{-1} \ast \mathcal{A}$ have the same walls of the form $\mathcal{H}_{\alpha,0}$. Furthermore, the following theorem shows that if $\mathcal{A}$ is an arbitrary alcove in $R_c$, where $c$ is not necessarily in $\mathfrak{A}_0$, then any of its walls of the form $\mathcal{H}_{\alpha,0}$ is also a wall of $w_{\text{min},c}^{-1} \ast \mathcal{A}$.

3.5 Theorem. Suppose $w \in \hat{W}$ is dominant, and let $c$ be the first layer ideal of $w$ (i.e., $c = \{\gamma \in \Delta^+ \mid w(\delta - \gamma) < 0\}$). If $w(\nu) \in \hat{H}$ for $\nu \in \Pi$, then $\mathfrak{g}_{-\nu} \subset \mathfrak{n}_g(c)$.

Proof. Assume $\mathfrak{g}_{-\nu} \not\subset \mathfrak{n}_g(c)$. Then there is a $\gamma \in I_c$ such that either $\gamma - \nu \in \Delta^+ \setminus A$ or $\gamma = \nu$. In the first case we have $w(\delta - \gamma) < 0$ and $w(\delta - \gamma + \nu) > 0$. This clearly implies that $\text{ht } w(\nu) \geq 2$, i.e., this root is not simple. If $\gamma = \nu$, then $\kappa := w(\delta - \gamma) < 0$. Hence $w(\nu) = \delta - \kappa$. This root also cannot be simple.\hfill $\square$

Theorem 3.4 says that $\mathfrak{n}_g(c) \supset \mathfrak{p}(\alpha)$ if and only if $\mathcal{H}_{\alpha,0}$ is a wall of $R_c$. This can also be restated in the following form. Consider the restricted arrangement

$$\text{Shi}(\Delta)_\alpha = \{H \cap \mathcal{H}_{\alpha,0} \mid H \in \text{Shi}(\Delta) \setminus \{\mathcal{H}_{\alpha,0} \cup \mathcal{H}_{\alpha,1}\}\}.$$  

Let us say that the region of $\text{Shi}(\Delta)_\alpha$ is dominant, if it belongs to $\mathfrak{C} \cap \mathcal{H}_{\alpha,0}$. Hence the ideals $c$ with $\mathfrak{n}_g(c) \supset \mathfrak{p}(\alpha)$ are in bijection with the dominant regions of $\text{Shi}(\Delta)_\alpha$. Notice also that $\mathcal{H}_{\gamma,1} \cap \mathcal{H}_{\alpha,0} = H_{\gamma,1} \cap \mathcal{H}_{\alpha,0}$ if and only if $\gamma - \hat{\gamma} = k\alpha$ for some $k \in \mathbb{Z}$. This means the hyperplanes of the restricted arrangement that dissect $\mathfrak{C} \cap \mathcal{H}_{\alpha,0}$ are in a bijection with $I(\alpha)$-submodules of $\mathfrak{p}(\alpha)^{ad}$. (Here $I(\alpha) = \mathfrak{sl}_2(\alpha) + t$ is the standard Levi subalgebra of $\mathfrak{p}(\alpha)$.)
On the other hand, any ad-nilpotent ideal whose normalizer contains \( p(\alpha) \) lies in \( p(\alpha)^{nil} \) and is a sum of \( l(\alpha) \)-modules. In the general case, the condition that a \( t \)-stable subspace of \( u = b^{nil} \) is actually \( b \)-stable led us to the notion of an upper ideal of \( \Delta^+ \). Accordingly, in this situation, the condition that an \( l(\alpha) \)-stable subspace of \( p(\alpha)^{nil} \) is actually \( p(\alpha) \)-stable lead us to the notion of an upper ideal of the poset of \( l(\alpha) \)-modules in \( p(\alpha)^{nil} \). The latter can be defined as the quotient \( \Delta^+_\alpha := (\Delta^+ \setminus \{\alpha\})/\sim \), where the equivalence \( \sim \) is defined as \( \gamma \sim \tilde{\gamma} \) if and only if \( \gamma - \tilde{\gamma} = k\alpha \) for some \( k \in \mathbb{Z} \). It is easily seen that "\( \leq \)" induces a well-defined partial order in \( \Delta^+_\alpha \). Thus, we obtain a "restricted" version of the Shi correspondence:

\[
\text{(3.6)} \quad \text{There is a bijection between the upper ideals of } (\Delta^+_\alpha, \leq) \text{ and the dominant regions of the restricted arrangement } \text{Shi}(\Delta)_{\alpha}.
\]

Clearly, one can proceed further, and consider arbitrary parabolic subalgebras (i.e., not necessarily minimal ones) and the restricted Shi arrangement determined by the respective face of the dominant Weyl chamber. We leave it to the interested reader to give an accurate statement. It would be interesting to find a closed formula for the number of such dominant regions.

Let \( \mathcal{P} = \mathcal{P}(g) \) denote the set of all standard parabolic subalgebras of \( g \). We have a natural mapping \( \psi : \mathcal{A} \to \mathcal{P} \), which takes an ad-nilpotent ideal to its normalizer. It is interesting to study the fibres of \( \psi \). Write \( \mathcal{A}(p) \) for \( \psi^{-1}(p) \), the set of all ideals whose normalizer equals \( p \). Whenever we wish to make the dependence on \( g \) explicit, we write \( \mathcal{A}(g)(p) \). Each \( \mathcal{A}(p) \), as well as the whole of \( \mathcal{A} \), is regarded as poset under the usual containment of subspaces of \( u \). The following is obvious.

**3.7 Lemma.** The unique maximal element of \( \mathcal{A}(p) \) is \( p^{nil} \), the nilpotent radical of \( p \). In particular, \( \psi \) is onto.

It is not however true that \( \mathcal{A}(p) \) always has a unique minimal element.

**Example.** Take \( g = so_8 \) and \( p = b \). Then \( \mathcal{A}(so_8)(b) \) has three minimal elements (ideals). One of them has the generators \( \alpha_1 + \alpha_2, \alpha_2 + \alpha_3 + \alpha_4 \). The other two correspond to the cyclic permutations of \( \{1,3,4\} \).

Below, we show that if \( g = sl_n \) or \( sp_{2n} \), then \( \mathcal{A}(p) \) has a unique minimal element for any \( p \in \mathcal{P} \).

Another easy observation is connected with the maximal parabolic subalgebras.

**3.8 Lemma.** The poset \( \mathcal{A}(p(i)) \) is a chain and \( \# \mathcal{A}(p(i)) = n_i \), where \( \theta = \sum n_i \alpha_i \).

**Proof.** For any \( \mu \in Q \), let \( [\mu : \alpha_i] \) denote the coefficient of \( \alpha_i \) in the expansion of \( \mu \) via the basis \( \Pi \). Set \( I(\alpha_i)_j = \{ \mu \in \Delta^+ | [\mu : \alpha_i] \geq j \} \). It is an upper ideal, and it is easily seen that the corresponding ad-nilpotent ideals with \( j = 1, \ldots, n_i \) exhaust the fibre \( \mathcal{A}(p(i)) \). \( \square \)

In the rest of the section, we give a geometric description of the set \( \mathcal{A}(b) \). Recall that \( \hat{W} \) is isomorphic to a semi-direct product of \( W \) and \( Q' \). Given \( w \in \hat{W} \), there is a unique
factorization
\[ w = v \cdot t_r , \]
where \( v \in W \) and \( t_r \) is the translation corresponding to \( r \in Q^\vee \). Then \( w^{-1} = v^{-1} \cdot t_{-v(r)} \).

In terms of this factorization for \( w \), the linear action of \( \hat{W} \) on \( V \oplus \mathbb{R} \delta \subset \hat{V} \) is given by
\[ w^{-1}(x) = v^{-1}(x) + (x, v(r))\delta \quad \text{for any } x \in V \oplus \mathbb{R} \delta. \]

In particular,
\[
\begin{align*}
(3.9) \quad w^{-1}(\alpha_i) &= v^{-1}(\alpha_i) + (\alpha_i, v(r))\delta, \quad i \geq 1, \\
(3.10) \quad w^{-1}(\alpha_0) &= -v^{-1}(\theta) + (1 - (\theta, v(r)))\delta.
\end{align*}
\]

Given \( c \in A_0 \), consider \( w_{\min, c} \) and the corresponding factorization (3.9) for it. (To simplify notation, we do not endow the components \( v \) and \( r \) with subscripts.) Form the element \( z_c := v(r) \in Q^\vee \subset V \). The following fundamental result is due to Cellini and Papi [4].

3.11 Theorem.
(i) \( z_c \in D_{\min} := \{ x \in V \mid (x, \alpha) \geq -1 \ \forall \alpha \in \Pi \ \& \ (x, \theta) \leq 2 \} \);
(ii) The mapping \( A_0 \rightarrow D_{\min} \cap Q^\vee, c \mapsto z_c \) is a bijection.

Our next description of \( A_0\{b\} \) says which points of \( D_{\min} \cap Q^\vee \) correspond to the ideals whose normalizer is equal to \( b \).

3.12 Theorem. For \( c \in A_0 \), we have
(i) \( n_0(c) = b \) if and only if \( (z_c, \alpha) \neq 0 \ \forall \alpha \in \Pi \) and \( (z_c, \theta) \neq 1 \). In other words, there is a one-to-one correspondence
\[
A_0\{b\} \leftrightarrow \{ x \in D_{\min} \cap Q^\vee \mid x \not\in H_\alpha \ \forall \alpha \in \Pi \}.
\]
(ii) The semisimple rank of \( n_0(c) \) is equal to the number of hyperplanes \( H_\alpha (\alpha \in \hat{\Pi}) \) to which \( z_c \) belongs.

Proof. (i) By Theorem 2.8, we have
\[
c \in A_0\{b\} \iff w_{\min, c}(\alpha) \not\in \hat{\Pi} \ \forall \alpha \in \Pi \iff w^{-1}_{\min, c}(\hat{\Pi}) \cap \Pi = \emptyset.\]

Actually, we will prove a bit more precise statement that

for \( \alpha \in \Pi \), \( w^{-1}_{\min, c}(\alpha) \in \Pi \) if and only if \((z_c, \alpha) = 0\),

for \( \alpha = \alpha_0 \), \( w^{-1}_{\min, c}(\alpha_0) \in \Pi \) if and only if \((z_c, \theta) = 1\),

which implies the assertion. Indeed, if \( \alpha \in \Pi \) and \( w^{-1}_{\min, c}(\alpha) \in \Pi \), then it follows from the first row in Eq. (3.10), with \( z_c = v(r) \), that \( v^{-1}(\alpha) \in \Pi \) and \( (z_c, \alpha) = 0 \). Conversely, if \( (z_c, \alpha) = 0 \), then \( w^{-1}_{\min, c}(\alpha) = v^{-1}(\alpha) =: \gamma \in \Delta \). Since \( w_{\min, c} \) is dominant, \( \gamma \) must be positive. Assuming \( \gamma \not\in \Pi \) and hence \( \gamma = \gamma_1 + \gamma_2 \ (\gamma_i \in \Delta^+) \), we obtain \( w_{\min, c}(\gamma_1) + w_{\min, c}(\gamma_2) = \alpha \).

Here both summands in the left-hand side are positive roots, which contradicts the simplicity of \( \alpha \). Hence \( \gamma \) must be a simple root.

The argument for \( \alpha_0 \) is similar, taking into account the second row in Eq. (3.10).

(ii) This follows from the proof of part (i). If \( z_c \in H_\alpha \ (\alpha \in \hat{\Pi}) \), then the corresponding root of \( \Pi \) occurring in the standard Levi subalgebra of \( n_0(c) \) is \( w_{\min, c}(\alpha) \). \( \square \)
It would be interesting to be able to extract all information on the normalizer of $c$ directly from the indication of $z_c$, without appealing to $w_{min,c}$. So that Theorem 3.12 does not give a complete answer to this problem.

If $c \in \mathfrak{A}_0$, then similar results are valid for $w_{max,c}$ and the coroot lattice points of another simplex. For $w_{max,c} = v \cdot t$, we set $y_c = v(r)$. The following fundamental result is due to Sommers [15].

3.13 Theorem.\(\)
\begin{enumerate}[m(i)]
\item $y_c \in D_{\max} := \{ x \in V \mid (x, \alpha) \leq 1 \ \forall \alpha \in \Pi \ \& \ (x, \theta) \geq 0 \};$
\item The mapping $\mathfrak{A}_0 \rightarrow D_{\max} \cap Q^\vee, \ c \mapsto y_c$, is a bijection.
\end{enumerate}

Our next description of $\mathfrak{A}_0\{b\}$ says which points of $D_{\max} \cap Q^\vee$ correspond to the ideals whose normalizer is equal to $b$. Since the proof is identical to the proof of Theorem 3.12, it is omitted.

3.14 Theorem. For $c \in \mathfrak{A}_0$, we have
\begin{enumerate}[m(i)]
\item $n_g(c) = b$ if and only if $(y_c, \alpha) \neq 0 \ \forall \alpha \in \Pi \ \& \ (y_c, \theta) \neq 1$. In other words, there is a one-to-one correspondence $\mathfrak{A}_0\{b\} \leftrightarrow \{ x \in D_{\max} \cap Q^\vee \mid x \notin H_\alpha \ \forall \alpha \in \hat{\Pi} \}.$
\item The semisimple rank of $n_g(c)$ is equal to the number of hyperplanes $H_\alpha (\alpha \in \hat{\Pi})$ to which $y_c$ belongs.
\end{enumerate}

4. ON IDEALS WHOSE NORMALIZER IS EQUAL TO $b$

In this section, we present a practical method for counting the number of all and strictly positive ad-nilpotent ideals, respectively, whose normalizer equals $b$. It turns out that for the classical series of simple Lie algebras we meet several famous integer sequences: the Motzkin and Riordan numbers, the number of directed animals of size $n$, and trinomial coefficients. Our exposition is quite similar to that in Section 5 in [12], where an analogous problem was considered for minimax ideals.

By Theorems 3.12 and 3.14 we have to count the points in $Q^\vee$ satisfying certain constraints. However, for practical computations, it is easier to deal with points of the coweight lattice in $V$, denoted $P^\vee$. Let $\{\omega_i\}_{i=1}^n$ be the basis for $V$ that is dual to $\{\alpha_i\}_{i=1}^n$. Then the lattice generated by the $\omega_i$’s is $P^\vee$. If $y = \sum_i y_i \omega_i \in P^\vee$, then $y \in Q^\vee$ if and only if a certain congruence condition (depending on $g$) is satisfied for $(y_1, \ldots, y_n) \in \mathbb{Z}^n$. Our primary goal is to compare the numbers $\# \{ y \in D_{\min} \cap Q^\vee \mid y \notin H_\alpha \ \forall \alpha \in \hat{\Pi} \}$ and $\# \{ y \in D_{\min} \cap P^\vee \mid y \notin H_\alpha \ \forall \alpha \in \hat{\Pi} \}$; and likewise for $D_{\max}$. Define the integers $c_i \in \mathbb{Z}, i = 1, 2, \ldots, n$, by the formula $\theta = \sum_{i=1}^n c_i \alpha_i$.\(\)
It is clear that \( \# \{ y \in D_{\min} \cap P^\vee \mid y \not\in H_\alpha \ \forall \alpha \in \hat{\Pi} \} \) is equal to the number of solutions of the system of equations

\[
\begin{cases}
y_i \in \{-1, 1, 2, \ldots\} \ (i = 1, 2, \ldots, n) \\
c_1y_1 + \ldots + c_ny_n \leq 2 \\
c_1y_1 + \ldots + c_ny_n \neq 1 
\end{cases}
\]

It is convenient to set \( y_0 = 1 - (c_1y_1 + \ldots + c_ny_n) \). The new variable \( y_0 \) also ranges over \( \{-1, 1, 2, \ldots\} \), so that, letting \( c_0 = 1 \), the above system takes a more symmetric form

\[
\begin{cases}
y_i \in \{-1, 1, 2, \ldots\} \ (i = 0, 1, \ldots, n) \\
c_0y_0 + c_1y_1 + \ldots + c_ny_n = 1 
\end{cases}
\]

(4.1)

In a sense, this procedure corresponds to taking the extended Dynkin diagram of \( g \). For this reason, system (4.1) will be referred to as the \textit{min-extended} system.

Similarly, \( \# \{ y \in D_{\max} \cap P^\vee \mid y \not\in H_\alpha \ \forall \alpha \in \hat{\Pi} \} \) is equal to the number of solutions of the system of equations

\[
\begin{cases}
y_i \in \{1, -1, -2, \ldots\} \ (i = 1, 2, \ldots, n) \\
c_1y_1 + \ldots + c_ny_n \geq 0 \\
c_1y_1 + \ldots + c_ny_n \neq 1 
\end{cases}
\]

Letting \( y_0 = 1 - (c_1y_1 + \ldots + c_ny_n) \) as above, we obtain the \textit{max-extended} system

\[
\begin{cases}
y_i \in \{1, -1, -2, \ldots\} \ (i = 0, 1, \ldots, n) \\
c_0y_0 + c_1y_1 + \ldots + c_ny_n = 1 
\end{cases}
\]

(4.2)

Replacing each \( y_i \) with \( -y_i \) yields another system, which is sometimes more convenient to deal with:

\[
\begin{cases}
y_i \in \{-1, 1, 2, \ldots\} \ (i = 0, 1, \ldots, n) \\
c_0y_0 + c_1y_1 + \ldots + c_ny_n = -1 
\end{cases}
\]

(4.3)

The following theorem shows that counting points in \( P^\vee \) in place of \( Q^\vee \) does not lead us far away from our purpose. The number \( f = |P^\vee : Q^\vee| \) is called the \textit{index of connection} of \( \Delta \). It is also equal to \( \# \{ j \mid c_j = 1 \} \).

\[ \text{4.4 Theorem.} \]

\begin{itemize}
  \item[(i)] \( \# \{ x \in D_{\min} \cap Q^\vee \mid x \not\in H_\alpha \ \forall \alpha \in \hat{\Pi} \} = \frac{1}{f} \# \{ x \in D_{\min} \cap P^\vee \mid x \not\in H_\alpha \ \forall \alpha \in \hat{\Pi} \} \); \\
  \item[(ii)] \( \# \{ x \in D_{\max} \cap Q^\vee \mid x \not\in H_\alpha \ \forall \alpha \in \hat{\Pi} \} = \frac{1}{f} \# \{ x \in D_{\max} \cap P^\vee \mid x \not\in H_\alpha \ \forall \alpha \in \hat{\Pi} \} \).
\end{itemize}

\textit{Proof.} The argument amounts to a direct case-by-case verification. For each simple Lie algebra \( g \), we look at the effect of the additional congruence condition imposed on systems (4.1) and (4.2). One can define an action of the cyclic group \( \mathbb{Z}_f \) on the set of solutions of (4.1) and (4.2) such that each \( \mathbb{Z}_f \)-orbit has cardinality \( f \) and contains a unique representative from \( Q^\vee \). Since the technical details are completely the same as in the proof of Theorem 5.5 in \cite{12}, we omit them. \( \square \)
Obviously, the number of solutions of \((4.1)\) is equal to the coefficient of \(x\) in the Laurent series
\[
\prod_{i=0}^{n} (x^{-c_i} + x^{c_i} + x^{2c_i} + \ldots) = \prod_{i=0}^{n} \left( \frac{x^{-c_i}}{1 - x^{c_i}} - 1 \right).
\]
We use the standard notation that \([x^n] F(x)\) denotes the coefficient of \(x^n\) in the Laurent series \(F(x)\). Therefore, combining Theorems 3.12 and 4.4(i) we obtain
\[
\#\mathfrak{W}(b) = \frac{1}{f}[x] \prod_{i=0}^{n} \left( \frac{x^{-c_i}}{1 - x^{c_i}} - 1 \right).
\]
Similarly, starting with \((4.3)\) and combining Theorems 3.14 and 4.4(ii), we obtain
\[
\#\mathfrak{W}_0(b) = \frac{1}{f}[x^{-1}] \prod_{i=0}^{n} \left( \frac{x^{-c_i}}{1 - x^{c_i}} - 1 \right).
\]
The Equations \((4.5)\) and \((4.6)\) show that the cardinalities in question depend only on the multiset of the coefficients of the highest root. Therefore, we can already conclude that these cardinalities are equal for \(\mathfrak{sp}_{2n}\) and \(\mathfrak{so}_{2n+1}\).

Now, we are ready to consider the case of classical Lie algebras. For the future use, we introduce some notation for trinomial coefficients. The coefficient \([x^0](x^{-1} + 1 + x)^n\) is called the \(\text{central trinomial}\), denoted \(\text{ct}_n\) and \([x](x^{-1} + 1 + x)^n\) is called the \(\text{next-to-central trinomial}\), denoted \(\text{nct}_n\). In an explicit form, we have
\[
\text{ct}_n = \sum_{k \geq 0} \frac{n!}{k!k!(n-2k)!} \quad \text{and} \quad \text{nct}_n = \sum_{k \geq 0} \frac{n!}{k!(k+1)!(n-2k-1)!}.
\]
1) \(g = \mathfrak{sl}_{n+1}\). Here all \(c_i = 1\) and \(f = n + 1\). Therefore
\[
\#\mathfrak{W} (\mathfrak{sl}_{n+1}) \{ b \} = \frac{1}{n+1} [x](x^{-1} \frac{1}{1 - x} - 1)^{n+1} = \frac{1}{n+1} [x] \sum_{i=1}^{n+1} (-1)^{n+1-i} \binom{n+1}{i} \frac{x^{-i}}{(1-x)^i} = \frac{1}{n+1} \sum_{i=1}^{n+1} (-1)^{n+1-i} \binom{n+1}{i} [x^{i+1}] \frac{1}{(1-x)^i} = \frac{1}{n+1} \sum_{i=1}^{n+1} (-1)^{n+1-i} \binom{n+1}{i} (\frac{2i}{i+1}) = \sum_{i=1}^{n+1} (-1)^{n+1-i} \binom{n}{i} C_i = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} C_{j+1},
\]
where \(C_i := \frac{1}{i+1} \binom{2i}{i}\) is the \(i\)-th Catalan number. It is well-known that the last expression gives \(M_n\), the \(n\)-th Motzkin number, see e.g. [2] p.99. There is a rich literature devoted to Motzkin numbers, where the reader may find various definitions/interpretations of these numbers, see e.g. [1], [2], [5], [16] Ex. 6.37,6.38]. In [12], it was shown that \(M_n\) gives
the number of minimax ideals in $\mathfrak{A}\mathfrak{d}(s\mathfrak{t}_{n+1})$.

Similarly,

\begin{equation}
\#\mathfrak{A}\mathfrak{d}_0(s\mathfrak{t}_{n+1})\{b\} = \frac{1}{n+1}[x^{-1}](\frac{x^{-1}}{1-x} - 1)^{n+1} = \ldots
\end{equation}

\[
\frac{1}{n+1} \sum_{i=1}^{n+1} (-1)^{n+1-i} \binom{n+1}{i} [x^{i-1}] \frac{1}{(1-x)^i} = \ldots
\]

\[
\sum_{i=1}^{n+1} (-1)^{n+1-i} \binom{n+1}{i} C_{i-1} = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} C_j .
\]

This time, the last expression yields $R_n$, the $n$-th Riordan number, see [2] p.99. Similarity of the expressions for $\mathfrak{A}\mathfrak{d}(s\mathfrak{t}_{n+1})\{b\}$ and $\mathfrak{A}\mathfrak{d}_0(s\mathfrak{t}_{n+1})\{b\}$ reflects the well-known relation $M_n = R_n + R_{n+1}$. We refer to [2] for a discussion of Catalan, Motzkin, and Riordan numbers.

2) $\mathfrak{g} = \mathfrak{sp}_{2n}$ or $\mathfrak{so}_{2n+1}$. Here $c_0 = c_1 = 1$, $c_2 = \ldots = c_n = 2$, and $f = 2$. Therefore we have to compute the coefficients

\[
\frac{1}{2}[x^{±1}](\frac{x^{-1}}{1-x} - 1)^2(\frac{x^{-2}}{1-x^2} - 1)^{n-1} =
\]

\[
= \frac{1}{2}[x^{±1}](\frac{x^{-1} - 1 + x}{1-x})^2(\frac{x^{-2}}{1-x^2} - 1)^{n-1} =
\]

\[
= \frac{1}{2}[x^{±1}](\frac{x^{-1} + x^2}{1-x^2})^2(\frac{x^{-2}}{1-x^2} - 1)^{n-1} =
\]

\[
= \frac{1}{2}[x^{±1}](x^{-2} + 2x + x^4)\sum_{i=0}^{n-1} (-1)^{n-1-i} \binom{n-1}{i} \frac{x^{-2i}}{(1-x^2)^{i+2}}.
\]

For the parity reasons for degrees, we need only the summand $2x$ in the first factor. That is, the last expression equals

\[
[x^{±1}]\sum_{i=0}^{n-1} (-1)^{n-1-i} \binom{n-1}{i} \frac{x^{-2i+1}}{(1-x^2)^{i+2}}.
\]

For the coefficient of $x^{-1}$, we obtain

\[
\#\mathfrak{A}\mathfrak{d}_0(\mathfrak{so}_{2n+1}) = \#\mathfrak{A}\mathfrak{d}_0(\mathfrak{sp}_{2n}) = \sum_{i=0}^{n-1} (-1)^{n-1-i} \binom{n-1}{i} \frac{1}{[x^{2i-2}]^i(1-x^2)^{i+2}} =
\]

\[
\sum_{i=1}^{n-1} (-1)^{n-1-i} \binom{n-1}{i} \frac{1}{[x^{i-1}]^{i+2}} = \sum_{i=1}^{n-1} (-1)^{n-1-i} \binom{n-1}{i} \binom{2i}{i-1} =
\]

\[
(n-1)\sum_{i=1}^{n-1} (-1)^{n-1-i} \binom{n-2}{i-1} C_i = (n-1)M_{n-2} = nct_{n-1} .
\]
The last equality is explained e.g. in [12, Section 5]. For the coefficient of $x$, we obtain

$$\#\mathcal{A}(so_{2n+1}) = \#\mathcal{A}(sp_{2n}) = \sum_{i=0}^{n-1} (-1)^{n-1-i} \binom{n-1}{i} \frac{1}{(1-x^2)^{i+2}} = \ldots$$

$$= \sum_{i=0}^{n-1} (-1)^{n-1-i} \binom{n-1}{i} (2i+1).$$

Write temporarily $X_n$ for the last expression. The binomial transform of $\{X_n\}$ yields

$$\binom{2n-1}{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} X_k.$$ Comparing with Eq. (6.7), which is proved below, we see that $X_n = \text{dir}_n$, the number of directed animals of size $n$.

Some other expressions for $\text{dir}_n$ are

$$\text{dir}_n = c t_{n-1} + n c t_{n-2} = \sum_{q \geq 0} \left( \frac{q}{[q/2]} \right) \binom{n-1}{q}.$$ (See [6, Eq. (27)] and [12, 5.16].) It is also not hard to prove that $\#\mathcal{A}(so_{2n+1})\{b\} - \#\mathcal{A}_0(so_{2n})\{b\} = c t_{n-1}$. This leads to a simple expression of the Riordan numbers via trinomial coefficients:

$$R_n = c t_n - n c t_n.$$ 3) $g = so_{2n}, n \geq 4$. Here $c_0 = c_1 = c_2 = c_3 = 1, c_4 = \ldots = c_n = 2, and f = 4$. Therefore we have to compute the coefficients

$$\frac{1}{4}[x^1] \left( \frac{x^{-1}}{1-x} - 1 \right)^4 \left( \frac{x^{-2}}{1-x^2} - 1 \right)^{n-3} = \frac{1}{4}[x^1] \left( \frac{x^{-1} + x^2}{1-x^2} \right)^4 \left( \frac{x^{-2}}{1-x^2} - 1 \right)^{n-3} = \frac{1}{4}[x^1] \left( x^{-4} + 4x^{-1} + 6x^2 + 4x^5 + x^8 \right) \sum_{i=0}^{n-3} (-1)^{n-3-i} \binom{n-3}{i} \frac{x^{-2i}}{(1-x^2)^{i+4}}.$$ For the parity reasons for degrees, we need only the summand $4x^5 + 4x^{-1}$ in the first factor. That is, the last expression equals

$$[x^1] \sum_{i=0}^{n-3} (-1)^{n-3-i} \binom{n-3}{i} \frac{x^{-2i+5} + x^{-2i-1}}{(1-x^2)^{i+4}}.$$ For the coefficient of $x^{-1}$ we obtain

$$\#\mathcal{A}_0(so_{2n})\{b\} = \sum_{i=0}^{n-3} (-1)^{n-3-i} \binom{n-3}{i} \binom{2i}{i-3} + \binom{2i+3}{i}.$$
For the coefficient of \( x \) we finally obtain
\[
\#\mathfrak{A}(\mathfrak{so}_{2n})\{b\} = \sum_{i=0}^{n-3} (-1)^{n-3-i} \binom{n-3}{i} \left[ \binom{2i+1}{i} + \binom{2i+4}{i+1} \right].
\]

As in the previous case, we have the relation
\[
\#\mathfrak{A}(\mathfrak{so}_{2n})\{b\} - \#\mathfrak{A}(\mathfrak{so}_{2n})\{c\} = |\mathfrak{ct}_{n-1}|.
\]

There is also a connection between the values for \( \mathfrak{so}_{2n} \) and \( \mathfrak{so}_{2n+1} \). Namely,
\[
\#\mathfrak{A}(\mathfrak{so}_{2n+1})\{b\} - \#\mathfrak{A}(\mathfrak{so}_{2n})\{b\} = M_{n-2}.
\]

4) \( \mathfrak{g} = \mathfrak{F}_4 \). Here a straightforward calculation shows \( \#\mathfrak{A}(\mathfrak{b}) = 19, \#\mathfrak{A}(\mathfrak{c}) = 11 \).

For \( \mathfrak{g} = \mathfrak{E}_6 \), calculations based on Eq. (4.5) and (4.6) show that \( \#\mathfrak{A}(\mathfrak{b}) = 111 \) and \( \#\mathfrak{A}(\mathfrak{c}) = 53 \).

The case of \( \mathfrak{G}_2 \) is easy, and the cases of \( \mathfrak{E}_7 \) and \( \mathfrak{E}_8 \) are too difficult to do them by hand.

5. The Case of \( \mathfrak{g} = \mathfrak{s}_l_{n+1} \)

In this section, \( \mathfrak{g} = \mathfrak{s}_l_{n+1} \). We will explicitly describe \( \mathfrak{A}(\mathfrak{p}) \) for every \( \mathfrak{p} \in \mathfrak{Par}(\mathfrak{s}_l_{n+1}) \). It will be shown that \( \mathfrak{A}(\mathfrak{p}) \) has a unique minimal element and \( \#\mathfrak{A}(\mathfrak{p}) \) depends only on the difference \( n - \text{srk}\mathfrak{p} \). Using the duality construction from [10, Section 4], we produce a bijection between the minimax ad-nilpotent ideals and the ideals in \( \mathfrak{A}(\mathfrak{b}) \).

We choose \( \mathfrak{b} \) (resp. \( \mathfrak{t} \)) to be the space of upper-triangular (resp. diagonal) matrices. With the usual numbering of rows and columns of matrices, the positive roots are identified with the pairs \((i, j)\), where \(1 \leq i < j \leq n + 1\). For instance, \( \alpha_i = (i, i+1) \) and therefore \((i, j) = \alpha_i + \cdots + \alpha_{j-1}\). An ad-nilpotent ideal of \( \mathfrak{b} \) is represented by a right-justified Ferrers diagram with at most \( n \) rows, where the length of \( i \)-th row is at most \( n - i + 1 \). If a box of a Ferrers diagram corresponds to a positive root \((i, j)\), then we say that this box has the coordinates \((i, j)\). The box containing the unique northeast corner of the diagram corresponds to \( \theta \), and the southwest corners give rise to the generators of the corresponding ideal, see Figure 1.

An ideal \( \mathfrak{c} \) (Ferrers diagram) is completely determined by the coordinates of boxes that contain the southwest corners of the diagram, say \( (i_1, j_1), \ldots, (i_k, j_k) \). Then we obviously have
\[
\Gamma(\mathfrak{c}) = \{(i_1, j_1), \ldots, (i_k, j_k)\}, \text{ where } 1 \leq i_1 < \cdots < i_k \leq n, \ 2 \leq j_1 < \cdots < j_k \leq n+1.
\]

Write \([n]\) for \( \{1, 2, \ldots, n\} \). It is convenient to describe standard parabolic subalgebras of \( \mathfrak{s}_l_{n+1} \) by indicating the simple roots that are not in the standard Levi subalgebra. That is, if \( E = \{l_1, l_2, \ldots, l_s\} \) is a subset of \([n]\), then
\[
\mathfrak{p}(E) = \mathfrak{p}(l_1, l_2, \ldots, l_s) := \mathfrak{p}(\Pi \setminus \{\alpha_{l_1}, \ldots, \alpha_{l_s}\})\).
\]

We always assume that \( l_1 < \ldots < l_s \). Therefore the consecutive diagonal blocks of the standard Levi subalgebra have sizes \( l_1, l_2-l_1, \ldots, l_s-l_{s-1}, n+1-l_s \).
Recall that $\# \mathcal{A}(\mathfrak{sl}_{n+1}) = \frac{1}{n+2}(\binom{2n+2}{n+1}) = C_{n+1}$, the $(n+1)$-th Catalan number, see e.g. [3]. There is a host of combinatorial objects counted by Catalan numbers, see [16, ch. 6, Ex. 6.19] and the “Catalan addendum” at www-math.mit.edu/~rstan/ec. One of the main interpretations is that $C_n$ is equal to the number of lattice paths from $(0,0)$ to $(n,n)$ with steps $(1,0)$ and $(0,1)$, always staying in the domain $x \leq y$, i.e., the number of Dyck paths of semilength $n$. The Dyck path corresponding to a $c \in \mathcal{A}(\mathfrak{sl}_{n+1})$ is the double path in Figure 1.

Recall that $M_s$ is the $s$-th Motzkin number. One of the possible explicit expressions for them is

\[
M_s := \sum_{r \geq 0} \binom{s}{2r} C_r .
\]

5.3 Theorem.

1. If $\Gamma(c) = \{(i_1, j_1), \ldots, (i_k, j_k)\}$, then $n_g(c) = p(\{i_1, \ldots, i_k\} \cup \{j_1-1, \ldots, j_k-1\})$;

2. given a subset $E = \{l_1, l_2, \ldots, l_s\} \subset [n]$, the ideals in $\mathcal{A}\{p(E)\}$ are in a bijection with the pairs $1 \leq a_1 < \ldots < a_k \leq n$, $1 \leq b_1 < \ldots < b_k \leq n$ of integer sequences such that $a_i \leq b_i$ and $\{a_1, \ldots, a_k\} \cup \{b_1, \ldots, b_k\} = E$. (Here $k$ is not fixed, but it obviously satisfies the constraints $s/2 \leq k \leq s$.) The ideal corresponding to such a pair of sequences has the generators $\{(a_i, b_i + 1) \mid 1 \leq i \leq k\}$.

3. If $\# E = s$, then $\# \mathcal{A}\{p(E)\} = M_s$. That is, this cardinality depends only on $s = \text{rk}(\mathfrak{sl}_{n+1}) - \text{srk} p(E)$.

Proof. 1. The simple roots that can be subtracted from the generators have the numbers $i_1, \ldots, i_k, j_1-1, \ldots, j_k-1$ (repetitions are allowed!). Then the description of $n_g(c)$ follows from Theorem 2.1.

2. This readily follows from part (1).

3. Clearly, the number of pairs of sequences described in part (2) depends only on $s$ and not on the explicit form of $E$. For instance, we may assume that $E = \{1, 2, \ldots, s\}$. Then the
assertion on the number of the above pairs of sequences is precisely the characterization of Motzkin numbers given in [16 Ex. 6.38(e)].

Another (more "honest") way to see the connection with Motzkin numbers is as follows. Let us temporarily write $M_s$ for the cardinality of $\#\mathfrak{ad}\{p(E)\}$. Since the number of standard parabolic subalgebras of $\mathfrak{sl}_{n+1}$ with semisimple corank $r$ equals $\binom{n}{r}$, $\mathfrak{ad}(\mathfrak{sl}_{n+1}) = \bigsqcup_p \mathfrak{ad}(\mathfrak{sl}_{n+1})\{p\}$, and the cardinality of $\mathfrak{ad}(\mathfrak{sl}_{n+1})$ is known, we obtain for each $n \in \mathbb{N}$ the identity

$$C_{n+1} = \sum_{r=0}^{n} \binom{n}{r} M_r.$$

According to [5], there is an explicit relation between the Catalan and Motzkin numbers which is exactly of such form. Hence $M_r = M_r$ for all $r$. □

5.4 Theorem. As above, $E = \{l_1, l_2, \ldots, l_\alpha\}$. The poset $\mathfrak{ad}\{p(E)\}$ has a unique maximal and unique minimal ad-nilpotent ideal. More precisely,

- $c_{\max}(E) = p(E)^{nil}$ and
  $$\Gamma(c_{\max}(E)) = \{(l_i, l_i + 1) \mid 1 \leq i \leq s\};$$
- $\Gamma(c_{\min}(E)) = \{(l_i, l_{[s/2]+i} + 1) \mid 1 \leq i \leq \lfloor(s+1)/2\rfloor\}$.

Proof. The first claim is a particular case of Lemma 5.7. As for the second claim, it is clear that the ideal with given generators lies in $\mathfrak{ad}\{p(E)\}$. Suppose $(a_1, \ldots, a_\alpha)$, $(b_1, \ldots, b_\beta)$ is an arbitrary pair of sequences as in Theorem 5.3(2). Since each $l_j$ must appear among the $a_i$’s and $b_i$’s, we have $k \geq \lfloor(s+1)/2\rfloor$. Also, $a_i \geq l_i$ and $b_i \leq l_{s-k+i} \leq l_{s-\lfloor(s+1)/2\rfloor+i} = l_{[s/2]+i}$. These inequalities mean that each root $(a_i, b_i + 1)$ lies in the ideal with the generators $(l_j, l_{[s/2]+j} + 1, 1 \leq j \leq \lfloor(s+1)/2\rfloor)$, which completes the proof. □

5.5 Remarks. Theorems 5.3, 5.4 have a number of interesting consequences.

1. Notice that $c_{\min}(E)$ is always an Abelian ideal. This reflects the fact that, for $\mathfrak{sl}_{n+1}$ (and $\mathfrak{sp}_{2n}$), the mapping $e \mapsto n_{\mathfrak{g}}(e)$ sets up a bijection between $\mathfrak{bar}$ and the subset of Abelian ideals in $\mathfrak{ad}$, see [13 Section 3]. Thus, $c_{\min}(E)$ is the unique Abelian ideal with normalizer $p(E)$. The description of the minimal ideal in $\mathfrak{ad}\{p\}$ can also be stated in a "coordinate-free" form. If $\text{srk } p = n - s$, then the minimal ideal in $\mathfrak{ad}\{p\}$ is $(p^{nil})^{\lfloor(s+1)/2\rfloor}$.

2. It is easily seen that the poset $\mathfrak{ad}(\mathfrak{sl}_{n+1})\{p(l_1, l_2, \ldots, l_\alpha)\}$ is naturally isomorphic to the poset $\mathfrak{ad}(\mathfrak{sl}_{n+1})\{b\}$. In other words, the structure of $\mathfrak{ad}\{p(l_1, l_2, \ldots, l_\alpha)\}$ depends only on the number of diagonal blocks of a Levi subalgebra, but not on the sizes of blocks. Therefore, in a sense, it suffices to consider only the ad-nilpotent ideals whose normalizer equals $b$.

3. The decomposition $\mathfrak{ad}(\mathfrak{sl}_{n+1}) = \bigsqcup_p \mathfrak{ad}(\mathfrak{sl}_{n+1})\{p\}$ yields a “materialization” of the identity $C_{n+1} = \sum_{r=0}^{n} \binom{n}{r} M_r$. 

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As in [12], we write \( \mathfrak{A}_m = \mathfrak{A}_m(g) \) for the subset of minimax ideals in \( \mathfrak{A}(g) \). Recall that \( c \) is called minimax, if \( w_{\text{min},c} = w_{\text{max},c} \), which means in particular that \( \mathfrak{A}_m \subset \mathfrak{A}_0 \).

The geometric characterization is that \( c \) is minimax if and only if \( R_c \) is a single alcove. In [12 Sect. 6], we obtained a description of the minimax ideals for \( \mathfrak{sl}_{n+1} \) and \( \mathfrak{sp}_{2n} \). In particular, \( \#\mathfrak{A}_m(\mathfrak{sl}_{n+1}) = M_n \).

Now, we show that the equality \( \#\mathfrak{A}(\mathfrak{sl}_{n+1})\{b\} = \#\mathfrak{A}_m(\mathfrak{sl}_{n+1}) \) is not a mere coincidence. To this end, recall the notion of the dual ideal for an ideal \( c \in \mathfrak{A}(\mathfrak{sl}_{n+1}) \). If \( \Gamma(c) = \{(i_1, j_1), \ldots, (i_k, j_k)\} \), where \( 1 \leq i_1 < \cdots < i_k \leq n \) and \( 2 \leq j_1 < \cdots < j_k \leq n + 1 \), then put

\[
X(c) = \{i_1, \ldots, i_k\} \text{ and } \tilde{Y}(c) = \{j_1-1, \ldots, j_k-1\}.
\]

By definition, the dual ideal for \( c \), denoted \( c^* \), is the ideal determined by the equalities \( X(c^*) = [n] \setminus \tilde{Y}(c) \) and \( \tilde{Y}(c^*) = [n] \setminus X(c) \). The operation \( c \mapsto c^* \) is well-defined, and \( (c^*)^* = c \). It has also a number of other nice properties, see [10 Section 4] for more details.

**5.6 Theorem.** For \( g = \mathfrak{sl}_{n+1} \), we have \( c \in \mathfrak{A}_m \) if and only if \( n_g(c^*) = b \).

**Proof.** Let \( c \in \mathfrak{A}(\mathfrak{sl}_{n+1}) \) be an arbitrary ideal with \( X(c) = \{a_1, \ldots, a_k\} \) and \( \tilde{Y}(c) = \{b_1, \ldots, b_k\} \). Then we have the following two characterizations:

- \( c \) is minimax if and only if \( X(c) \cap \tilde{Y}(c) = \emptyset \). (See [12 Corollary 6.5]);
- \( n_g(c) = b \) if and only if \( X(c) \cup \tilde{Y}(c) = [n] \). (See Theorem 5.3 (2).)

The very definition of \( c^* \) shows that \( X(c) \cap \tilde{Y}(c) = \emptyset \) if and only if \( X(c^*) \cup \tilde{Y}(c^*) = [n] \). \( \Box \)

As a consequence, we immediately obtain

**5.7 Corollary.** An ideal \( c \) is self-dual (i.e., \( c = c^* \)) if and only if \( c \) is minimax and \( n_g(c) = b \).

It was shown in [10 Theorem 4.6] that the number of self-dual ideals in \( \mathfrak{A}(\mathfrak{sl}_{2m+1}) \) is equal to \( C_m \) (while it is obviously 0 for \( \mathfrak{sl}_{2n} \)).

**5.8 Remark.** Considering the normalizers of minimax ideals yields a materialization of Eq. (5.2). Since \( c \) is minimax if and only if \( X(c) \cap \tilde{Y}(c) = \emptyset \), we see that \( c \in \mathfrak{A}_m \) has \( k \) generators if and only if \( n_g(c) \) has semisimple corank \( 2k \). (That is, unlike the general case, there is a strong correlation between \( \text{srk} n_g(c) \) and \( \#\Gamma(c) \).) Obviously, any \( p \in \mathfrak{P}(\mathfrak{sl}_{n+1}) \) having an even semisimple corank appears in this way. On the other hand, if \( p = p(l_1, \ldots, l_{2k}) \), then the minimax ideals in \( \mathfrak{A}\{p\} \) are in a bijection with the disjoint partitions

\[
\{l_1, \ldots, l_{2k}\} = \{a_1, \ldots, a_k\} \cup \{b_1, \ldots, b_k\}
\]

such that \( a_1 < \cdots < a_k, b_1 < \cdots < b_k, \) and \( a_i < b_i \). It is well-known and easy to prove that the number of such partitions equals \( C_k \). Thus, if \( n - \text{srk} p = 2k \), then \( \#\{c \in \mathfrak{A}_m | n_g(c) = p\} = C_k \). This yields Eq. (5.2).
6. The Case of $\mathfrak{g} = \mathfrak{sp}_{2n}$

Roughly speaking, the results for $\mathfrak{sp}_{2n}$ are similar to those for $\mathfrak{sl}_{n+1}$. One of the notable distinctions is that the cardinalities of posets $\mathfrak{A}(\mathfrak{sp}_{2n})\{p\}$ are now expressed in terms of numbers of directed animals in place of Motzkin numbers.

We use a standard matrix model of $\mathfrak{sp}_{2n}$ corresponding to a Witt basis for alternating bilinear form. For this basis of $\mathbb{C}^{2n}$, the algebra $\mathfrak{sp}_{2n}$ has the following block form:

$$\mathfrak{sp}_{2n} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid B = \hat{B}, \ C = \hat{C}, \ D = -\hat{A} \right\},$$

where $A, B, C, D$ are $n \times n$ matrices and $A \mapsto \hat{A}$ is the transpose relative to the antidiagonal. If $\mathfrak{b}$ is the standard Borel subalgebra of $\mathfrak{sl}_{2n}$, then $\mathfrak{b} := \mathfrak{b} \cap \mathfrak{sp}_{2n}$ is a Borel subalgebra of $\mathfrak{sp}_{2n}$. (See also [10, 5.1].) We identify the positive roots of $\mathfrak{sp}_{2n}$ with the set \[\{(i, j) \mid i < j, \ i + j \leq 2n + 1\}\]. Here the simple roots are $\alpha_i = (i, i + 1), 1 \leq i \leq n$, and therefore:

$$(i, j) = \begin{cases} \alpha_i + \ldots + \alpha_{j-1}, & \text{if } j \leq n+1, \\ \alpha_i + \ldots + \alpha_{2n-j} + 2(\alpha_{2n-j+1} + \ldots + \alpha_{n-1}) + \alpha_n, & \text{if } j > n+1. \end{cases}$$

The root $(i, j)$ is long if and only if $i + j = 2n + 1$, i.e., the corresponding matrix entry lies on the anti-diagonal. The ideals for $\mathfrak{sp}_{2n}$ can be identified with the ideals (Ferrers diagrams) for $\mathfrak{sl}_{2n}$ that are symmetric with respect to the antidiagonal (= self-conjugate). In other words, there is a natural bijection between $\mathfrak{A}(\mathfrak{sp}_{2n})$ and the self-conjugate ideals in $\mathfrak{A}(\mathfrak{sl}_{2n})$. More precisely, suppose $\bar{c} \in \mathfrak{A}(\mathfrak{sl}_{2n})$ and $\Gamma(\bar{c}) = \{(i_1, j_1), \ldots, (i_k, j_k)\}$ with $i_1 < \ldots < i_k$, where we use our convention on the roots of $\mathfrak{sl}_{2n}$. By definition, $\bar{c}$ is self-conjugate if and only if $i_m + j_{k+1-m} = 2n + 1$ for all $m$. Then the corresponding ideal $c \in \mathfrak{A}(\mathfrak{sp}_{2n})$ has the generators $\Gamma(c) = \{(i_m, j_m) \mid m \leq \lceil (k + 1)/2 \rceil\}$. Conversely, given $c \in \mathfrak{A}(\mathfrak{sp}_{2n})$, we obtain $\Gamma(\bar{c})$ by replacing each $(i, j) \in \Gamma(c)$ with $(i, j)$ and $(2n+1-j, 2n+1-j)$. If $\# \Gamma(c) = s$, then

$$\# \Gamma(\bar{c}) = \begin{cases} 2s, & \text{if all roots in } \Gamma(c) \text{ are short} \\ 2s - 1, & \text{if } \Gamma(c) \text{ contains a (unique!) long root.} \end{cases}$$

We shall say that $\bar{c} \in \mathfrak{A}(\mathfrak{sl}_{2n})$ is the symmetrization of $c \in \mathfrak{A}(\mathfrak{sp}_{2n})$. It is also clear that $n_{\mathfrak{sl}_{2n}}(\bar{c})$ is a self-conjugate (in the above sense) standard parabolic subalgebra, and that $n_{\mathfrak{sp}_{2n}}(\bar{c})$ is the symmetrization of $n_{\mathfrak{sp}_{2n}}(c)$.

Our general line of reasoning is as follows. Given an ideal $c \in \mathfrak{A}(\mathfrak{sp}_{2n})$, we take its symmetrization $\bar{c} \in \mathfrak{A}(\mathfrak{sl}_{2n})$ and then work with the generators of $\bar{c}$. If $\Gamma(\bar{c}) = \{(i_1, j_1), \ldots, (i_k, j_k)\}$, then put $E_c := \{(i_1, \ldots, i_k) \cup \{j_1-1, \ldots, j_k-1\}\} \cap [n] = (X(\bar{c}) \cup \hat{Y}(\bar{c})) \cap [n]$. We use the same notation for parabolic subalgebras as for $\mathfrak{sl}_{n+1}$, see Eq. (5).

6.1 Proposition.

(i) For any $c \in \mathfrak{A}(\mathfrak{sp}_{2n})$, we have $n_g(c) = p(E_c)$.

(ii) Given $E \subseteq [n]$, there is a bijection between the ideals in $\mathfrak{A}(\mathfrak{p}(E))$ and the the pairs $1 \leq a_1 < \ldots < a_k \leq 2n-1, 1 \leq b_1 < \ldots < b_k \leq 2n-1$ of integer sequences such
that \( a_i \leq b_i, a_i + b_{k+1-i} = 2n \) for any \( i \), and \( \{a_1, \ldots, a_k\} \cup \{b_1, \ldots, b_k\} \cap [n] = D \). (Here \( k \) is not fixed.) The ideal corresponding to such a pair of sequences has the generators \( \{(a_i, b_i) \mid 1 \leq i \leq [(k + 1)/2]\} \).

**Proof.** (i) By Theorem 2.1, it suffices to realize which simple roots can be subtracted from the generators of \( c \). For \( (i, j) \) with \( j \leq n + 1 \) these are \( \alpha_i \) and \( \alpha_{j-1} \), while for \( (i, j) \) with \( j > n + 1 \) these are \( \alpha_i \) and \( \alpha_{2n-j+1} \). Thus, in view of previous definitions, the numbers of these roots form exactly the set \( E_c \).

(ii) This is just a reformulation of the above formulae expressing \( E_c \) in terms of the generators of the symmetrization of \( c \).

Obviously, two strongly increasing sequences \( (a_i), (b_i) \), as in Theorem 6.1(ii), are determined by the intersections \( \{a_1, \ldots, a_k\} \cap [n] \) and \( \{b_1, \ldots, b_k\} \cap [n] \), cf. the next proof.

Now, we show that the long simple root, \( \alpha_n \), plays a special role in the symplectic case.

**6.2 Proposition.** Let \( E \) be a subset of \([n−1]\). Then there is a natural bijection between \( \mathcal{A}(p(E)) \) and \( \mathcal{A}(p(E \cup \{n\})) \).

**Proof.** Suppose \( n_0(e) = p(E) \) and let \( (a_1, \ldots, a_k), (b_1, \ldots, b_k) \) be the corresponding integer sequences as described in Proposition 6.1(ii). Then \( n \) does not appear in both of them. Let us insert \( n \) in the appropriate place of each sequence. It is easily seen that the two sequences obtained satisfy the conditions of Proposition 6.1(ii) and thereby determine an ideal in \( \mathcal{A}(p(E \cup \{n\})) \). Indeed, we have

\[
\begin{align*}
    a_1 < \ldots < a_t < n < a_{t+1} < \ldots < a_k, \\
    b_1 < \ldots < b_{k-t} < n < b_{k-t+1} < \ldots < b_k,
\end{align*}
\]

for some \( t \) in view of the conditions \( a_i + b_{k+1-i} = 2n \). Furthermore, \( t \geq k−t \). For, otherwise we would have \( a_{t+1} > n > b_{t+1} \).

Obviously, this procedure can be reversed. With this notation, we also have \( E = \{a_1, \ldots, a_t\} \cup \{b_1, \ldots, b_{k-t}\} \), where the union is not necessarily disjoint.

Thanks to this result, we can restrict ourselves to considering only parabolic subalgebras \( p(E) \) with \( E \subset [n−1] \).

**6.4 Theorem.** Suppose \( E = \{l_1, \ldots, l_s\} \subset [n−1] \). Then there is a bijection between the ideals in \( \mathcal{A}(p(E)) \) and the words \( (v_1 \ldots v_s) \) in the alphabet \( \{-1, 0, 1\} \) such that all partial sums \( \sum_{i \leq m} v_i \) are non-negative.

**Proof.** As we already know, an ideal in \( \mathcal{A}(p(E)) \) is determined by two sequences \( (6.3) \) such that \( a_i \leq b_i, a_i + b_{k+1-i} = 2n \), and \( \{a_1, \ldots, a_t\} \cup \{b_1, \ldots, b_{k-t}\} = E \). The word \( (v_1 \ldots v_s) \) is defined by the following rule:

\[
v_i = \begin{cases} 
    1, & \text{if } l_i \in \{a_1, \ldots, a_t\} \setminus \{b_1, \ldots, b_{k-t}\} \\
    0, & \text{if } l_i \in \{a_1, \ldots, a_t\} \cap \{b_1, \ldots, b_{k-t}\} \\
    -1, & \text{if } l_i \in \{b_1, \ldots, b_{k-t}\} \setminus \{a_1, \ldots, b_t\}.
\end{cases}
\]
It is easily seen that non-negativity of all partial sums is guaranteed by the condition \( a_i \leq b_i \).

Conversely, given a word \((v_1 \ldots v_s)\), we can restore the presentation of \( E \) as a not necessarily disjoint union \( \{a_1, \ldots, a_t\} \cup \{b_1, \ldots, b_{k-t}\} \). The non-negativity of partial sums implies that \( t > k - t \) and \( a_i \leq b_i \). Therefore these “truncated” \( a \)- and \( b \)-sequences can be extended to the whole sequences \((6.3)\), using the conditions \( a_i + b_{k+1-i} = 2n \). \( \square \)

The number of directed animals of size \( n \), denoted \( \text{dir}_n \), is defined as the number of certain \( n \)-element subsets of a two-dimensional lattice. An explicit expression is

\[
\text{dir}_n = \sum_{q \geq 0} \left( \begin{array}{c} q \\ \lfloor q/2 \rfloor \end{array} \right) \left( \begin{array}{c} n-1 \\ q \end{array} \right),
\]

see [6, Eq. (27)]. According to a beautiful result of D. Gouyou-Beauchamps and G. Viennot [6, Theorem 1], the number of directed animals of size \( s + 1 \) equals the number of words \((v_1 \ldots v_s)\), as above. Thus, we obtain the following

**6.6 Corollary.** If \( E \subset [n-1] \) and \( \#(E) = s \), then

\[
\# \mathfrak{Ad}(\mathfrak{sp}_{2n})\{p(E)\} = \# \mathfrak{Ad}(\mathfrak{sp}_{2n})\{p(E) \cup \{n\}\} = \text{dir}_{s+1}.
\]

In particular, \( \# \mathfrak{Ad}(\mathfrak{sp}_{2n})\{b\} = \# \mathfrak{Ad}(\mathfrak{sp}_{2n})\{p(\alpha_n)\} = \text{dir}_n \).

Using this corollary and the equality \( \# \mathfrak{Ad}(\mathfrak{sp}_{2n}) = \binom{2n}{n} \) [3], the decomposition \( \mathfrak{Ad}(\mathfrak{sp}_{2n}) = \bigsqcup_p \mathfrak{Ad}(\mathfrak{sp}_{2n})\{p\} \) is being translated to the identity

\[
\binom{2n}{n} = 2 \sum_{k=0}^{n-1} \binom{n-1}{k} \text{dir}_{k+1}
\]

or

\[
\binom{2n-1}{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} \text{dir}_{k+1},
\]

which is seem to be new. All explicit results on \( \text{ad} \)-nilpotent ideals for \( \mathfrak{sp}_{2n} \) are based on the relation between an ideal \( c \in \mathfrak{Ad}(\mathfrak{sp}_{2n}) \) and its symmetrization \( \bar{c} \in \mathfrak{Ad}(\mathfrak{sl}_{2n}) \). There is an analogue of Proposition 5.4:

**6.8 Proposition.** For any \( p \in \mathfrak{Par}(\mathfrak{sp}_{2n}) \), the poset \( \mathfrak{Ad}\{p\} \) has a unique minimal element (ideal), which is Abelian.

**Proof.** Take the self-conjugate parabolic subalgebra \( \bar{p} \in \mathfrak{Par}(\mathfrak{sl}_{2n}) \) corresponding to \( p \). It follows from Theorem 5.4 that the minimal ideal in \( \mathfrak{Ad}(\mathfrak{sl}_{2n})\{\bar{p}\} \) is self-conjugate and Abelian, and therefore it determines a (necessarily minimal and Abelian) ideal in \( \mathfrak{Ad}(\mathfrak{sp}_{2n})\{p\} \). \( \square \)

Recall the construction of the dual ideal in the symplectic setting. Given \( c \in \mathfrak{Ad}(\mathfrak{sp}_{2n}) \), one defines \( c^* \) in a round-about way via the symmetrization. That is, \( c^* \) is the ideal whose symmetrization is \( (\bar{c})^* \in \mathfrak{Ad}(\mathfrak{sl}_{2n}) \), see [10 Sect. 5].
6.9 Theorem. For \( g = \text{sp}_{2n} \), we have \( e \in \mathcal{A}_mm \) if and only if \( n_g(e) = b \).

Proof. It follows from Proposition 6.1(ii) that \( n_g(e) = b \) if and only if \( n_{\text{sl}_{2n}}(e) = \bar{b} \). On the other hand, the description of the minimax ideals in \( \mathcal{A}(\text{sp}_{2n}) \) [12 Cor. 6.8] essentially says that \( e \in \mathcal{A}_mm(\text{sp}_{2n}) \) if and only if \( \bar{e} \in \mathcal{A}_mm(\text{sl}_{2n}) \). Thus, the result follows from Theorem 5.6. \( \square \)

Finally, we consider the normalizers of minimax ideals and characterize the sets \( \mathcal{A}(\{p\}) \cap \mathcal{A}_mm \).

6.10 Theorem.

(i) The algebra \( p(E) \in \mathcal{P}(\text{sp}_{2n}) \) is the normalizer of a minimax ideal if and only if \( E \subset [n-1] \).

(ii) For \( E \subset [n-1] \) with \( \# E = s \), we have \( \#\{e \in \mathcal{A}_mm \mid n_g(e) = p(E)\} = \binom{s}{\lfloor s/2 \rfloor} \).

Proof. (i) If \( \bar{e} \in \mathcal{A}_mm(\text{sl}_{2n}) \) is the symmetrization of a minimax ideal \( e \), then the condition \( X(\bar{e}) \cap \bar{Y}(\bar{e}) = \emptyset \), which characterizes the minimax ideals, readily implies that \( n \notin X(\bar{e}) \cup \bar{Y}(\bar{e}) \).

(ii) If \( E = \{l_1, \ldots, l_s\} \), then the minimax ideals in \( \mathcal{A}(p(E)) \) are in a bijection with the disjoint partitions

\[ \{a_1, \ldots, a_t\} \cup \{b_1, \ldots, b_{s-t}\} = E \]

such that \( a_1 < \ldots < a_t, b_1 < \ldots < b_{s-t} \), and \( a_i \leq b_i \). Using the rule from the proof of Theorem 6.4, one sees that the number of such partitions is equal to the number of words \( (v_1 \ldots v_s) \) in the alphabet \( \{-1, 1\} \) such that all partial sums \( \sum_{i \leq m} v_i \) are nonnegative. A direct calculation shows that the latter is equal to \( \binom{s}{\lfloor s/2 \rfloor} \), see Lemma 6.11. \( \square \)

Notice that this theorem and the partition \( \mathcal{A}_mm(\text{sp}_{2n}) = \cup \mathcal{A}_mm(\text{sp}_{2n})\{p\} \) yield a materialization of the identity Eq. (6.5).

For convenience of the reader, we give a proof of the following result, which was used in the proof of Theorem 6.10.

6.11 Lemma. The number of words \( (v_1 \ldots v_s) \) in the alphabet \( \{-1, 1\} \) such that all partial sums \( \sum_{i \leq m} v_i \) are nonnegative is equal to \( \binom{s}{\lfloor s/2 \rfloor} \).

Proof. Consider all words \( (v_1 \ldots v_s) \) such that the total sum \( \sum_{i \leq s} v_i \) is nonnegative. Clearly, the number of such words is

\[ \binom{s}{\lfloor s/2 \rfloor} + \binom{s}{\lfloor s/2 \rfloor - 1} + \binom{s}{\lfloor s/2 \rfloor - 2} + \ldots . \]

(The number of \(-1\)'s can be any integer \( \leq \lfloor s/2 \rfloor \).) Let us say that a word is \textit{bad} if at least one partial sum is negative. Suppose \( (v_1 \ldots v_s) \) is bad, and let \( \sum_{i \leq t} v_i = -1 \) be the first negative partial sum. Consider the word \( (w_1 \ldots w_s) \), where \( w_i = \begin{cases} -v_i, & \text{if } i \leq t, \\ v_i, & \text{if } i > t. \end{cases} \) Then \( \sum_{i \leq s} w_i = \sum_{i \leq s} v_i + 2 \geq 2 \), and it is easily seen that this procedure yields a bijection
between the bad words and all the words with the total sum ≥ 2. The number of the latter is
\[ \binom{s}{\lfloor s/2 \rfloor - 1} + \binom{s}{\lfloor s/2 \rfloor - 2} + \ldots . \]
Whence the number of non-bad words is equal to \( \binom{s}{\lfloor s/2 \rfloor} \). \qed

The reader may recognize that we have used in the proof the reflection principle for lattice paths in an algebraic form.

7. SOME REMARKS ON sl_{n+1}, sp_{2n}, AND OTHER SIMPLE LIE ALGEBRAS

It seems that sl_{n+1} and sp_{2n} are the most attractive simple Lie algebras from the point of view of the theory of ad-nilpotent ideals. Let us list some relevant nice properties of these two series that do not hold in general:

- There is a natural procedure of constructing the dual ad-nilpotent ideal, see [10, Sect. 4 & 5].
- Taking the dual ideal yields a bijection between \( \mathcal{A}d_{mm} \) and \( \mathcal{A}d \{b\} \), see Theorems 5.6 and 6.9.
- There is an explicit description of the minimax ideals, see [12, Sect. 6].
- Write \( \mathcal{A}b = \mathcal{A}b(g) \) for the set of all Abelian ideals. It is shown in [13] that the assignment \( a \mapsto n_g(a), a \in \mathcal{A}b(g) \), yields a bijection between \( \mathcal{A}b(g) \) and \( \mathcal{P}ar(g) \) if and only if \( g = sl_n \) or \( sp_{2n} \).

As we have seen in Sections 5 and 6, considering various classes of ideals for sl_{n+1} and sp_{2n} provides a tool for demonstrating some identities related to Catalan and Motzkin numbers and numbers of directed animals. Let us present one more speculation of this kind. Consider the generating function for the number of minimax ideals in \( g \) with a given semisimple corank of the normalizer:

\[ F_{n-mm}(g, t) := \sum_s \#\{ \epsilon \in \mathcal{A}d_{mm} \mid \text{rk } g - \text{srk } n_g(\epsilon) = s \} \cdot t^s , \]

Our computation in Theorem 6.10 shows that

\[ F_{n-mm}(sp_2, t) = \sum_{s=0}^{n-1} \binom{n-1}{s} \binom{s}{\lfloor s/2 \rfloor} t^s , \]

and we know that \( F_{n-mm}(sp_2, 1) = \text{dir}_n \). It is curious to observe that \( F_{n-mm}(sp_2, -1) = R_n \), the \( n \)-th Riordan number.

In case of \( so_{2n+1} \), there is an ersatz construction of duality that is based on the similarity of shifted Ferrers diagrams representing ad-nilpotent ideals in sp_{2n} and so_{2n+1} [10, Sect. 5]. However, this construction does not yield a bijection between \( \mathcal{A}d_{mm}(so_{2n+1}) \) and \( \mathcal{A}d(so_{2n+1}) \{b\} \), which is already seen for \( n = 3 \). Also, no explicit description of minimax ideals for so_{2n+1} is known.
Furthermore, the cardinalities of the sets $\mathfrak{A}_nm$ and $\mathfrak{A}\{b\}$ are not always equal. The following table, which presents results of our explicit calculations, shows various possibilities:

|       | $\mathfrak{so}_8$ | $\mathfrak{so}_{10}$ | $E_6$ | $F_4$ | $G_2$ |
|-------|-------------------|------------------------|-------|-------|-------|
| $\#\mathfrak{A}_{nm}$ | 9                 | 23                     | 67    | 17    | 3     |
| $\#\mathfrak{A}\{b\}$  | 11                | 31                     | 111   | 19    | 2     |

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