Multiple-Source Multiple-Sink Maximum Flow in Directed Planar Graphs in $O(n^{1.5} \log n)$ Time

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Abstract

We give an $O(n^{1.5} \log n)$ algorithm that, given a directed planar graph with arc capacities, a set of source nodes and a set of sink nodes, finds a maximum flow from the sources to the sinks.

1 Introduction

In this paper we give an $O(n^{1.5} \log n)$-time algorithm for the maximum flow problem in directed planar graphs with multiple sources and sinks (MSMS).

- **Input:**
  - a directed planar embedded graph $G$ with non-negative arc capacities
  - a set $S$ of source nodes
  - a set $T$ of sink nodes

- **Output:** a feasible flow in $G$ that maximizes the total flow into $T$.

Several new results were recently obtained for this and related problems. A few months ago Borradaile and Wulff-Nilsen [1] and Klein and Mozes [5] independently presented two $O(n^{1.5} \log n)$-time algorithms for the maximum flow problem in directed planar graphs with multiple sources and a single sink (MSSS). Very recently Nussbaum [7] presented a recursive $O(n^{1.5} \log^2 n)$-time algorithm for MSMS. His algorithm uses, among other techniques, MSSS max flow computations.

The proposed algorithm is similar to that of Nussbaum, but uses a simpler recursive approach. It uses a technique to redistribute excess flow among a certain set of nodes called boundary nodes. This technique was used by Nussbaum for the case where the sources and sinks are separated by the boundary nodes. Here we prove that it is applicable in the general case.
2 preliminaries

2.1 Jordan Separators for Embedded Planar Graphs

Miller \cite{miller} gave a linear-time algorithm that, given a triangulated two-connected \( n \)-node planar embedded graph, finds a simple cycle separator consisting of at most \( 2\sqrt{\frac{3}{2}}n \) nodes, such that at most \( 2n/3 \) nodes are strictly enclosed by the cycle, and at most \( 2n/3 \) nodes are not enclosed.

For an \( n \)-node planar embedded graph \( G \) that is not necessarily triangulated or two-connected, we define a Jordan separator to be a Jordan curve \( C \) that intersects the embedding of the graph only at nodes such that at most \( 2n/3 \) nodes are strictly enclosed by the curve and at most \( 2n/3 \) nodes are not enclosed. The nodes intersected by the curve are called boundary nodes. To find a Jordan separator with at most \( 2\sqrt{\frac{3}{2}}n \) boundary nodes, add artificial edges with zero capacity to triangulate the graph and make it two-connected without changing the maximum flow in the graph. Now apply Miller’s algorithm.

A cycle separator \( C \) separates the graph \( G \) into two subgraphs \( G_1, G_2 \) called pieces. \( G_1 \) is the embedded subgraph consisting of the nodes and edges enclosed by \( C \), i.e. including the nodes intersected by \( C \). Similarly, \( G_2 \) is the subgraph consisting of the nodes and edges not strictly enclosed by \( C \), i.e. again including the nodes intersected by \( C \).

2.2 Flow

Let \( G \) be a directed graph with arc set \( A \), node set \( V \), source set \( S \subset V \) and sink set \( T \subseteq V - S \). For notational simplicity, we assume here and henceforth that \( G \) has no parallel arcs and no self-loops.

We associate with each arc \( a \) two darts \( d \) and \( d' \), one in the direction of \( a \) and the other in the opposite direction. We say that those two darts are reverses of each other, and write \( d = \text{rev}(d') \). Given an arc \( a \) with capacity \( c \), the capacity associated with the dart \( d \) going in the direction of \( a \) is \( c \). The capacity associated \( \text{rev}(d) \) is zero.

A flow assignment \( f(\cdot) \) is a real-valued function on darts that satisfies anti-symmetry:

\[ f(\text{rev}(d)) = -f(d) \]

A flow assignment \( f(\cdot) \) is said to respect capacities if, for every dart \( d \), \( f(d) \leq c(d) \). Such a flow assignment is also called a pseudoflow.

For a given flow assignment \( f(\cdot) \), the net inflow (or just inflow) of a node \( v \) is inflow\(_f\)(\( v \)) = \( \sum_{a \in A : \text{head}(a) = v} f(a) - \sum_{a \in A : \text{tail}(a) = v} f(a) \).

A flow assignment \( f(\cdot) \) is said to obey conservation if for every node \( v \notin S \cup T \), \( \text{inflow}_f(v) = 0 \). A pseudoflow that obeys conservation is called a feasible flow. The value of a feasible flow \( f(\cdot) \) is the sum of inflows at the sinks, \( \sum_{t \in T} \text{inflow}_f(t) \). A maximum flow is a feasible flow whose value is maximum.

The residual graph of \( G \) with respect to a flow assignment \( f(\cdot) \) is the graph \( G_f \) with the same arc-set, node-set, sources and sinks, and with capacity assignment \( c_f(\cdot) \) defined by \( c_f(d) = c(d) - f(d) \) for every dart \( d \).
3 The Algorithm

We present the algorithm as a recursive procedure with calls to the following subroutines:

- **MultipleSourceSingleSinkMaxFlow** \((G, S, t)\) – computes a maximum flow in \(G\) from source set \(S\) to sink node \(t\). This can be implemented in \(O(n^{1.5} \log n)\) time using the algorithm in \([5]\) or \([1]\).

- **SingleSourceMultipleSinkMaxFlow** \((G, s, T)\) – computes a maximum flow in \(G\) from source node \(s\) to sink set \(T\). This can be implemented using the multiple-source single-sink maximum flow subroutine.

- **SingleSourceSingleSinkLimitedMaxFlow** \((G, s, t, \Delta)\) – computes a flow in \(G\) from source node \(s\) to sink node \(t\), whose value is the minimum between \(\Delta\) and the value of the maximum \(s\)-to-\(t\) flow. This can be implemented in linear time when \(s\) and \(t\) are incident to the same face \([2, 3]\).

We omit discussion of the base case of the recursion (the case where the graph size is smaller than a certain constant). Each of the recursive calls operates on a subgraph of the original input graph. We assume one global flow assignment \(f(\cdot)\) for the original input graph, and one global capacity assignment \(c(\cdot)\). Whenever a subroutine is called, it takes as part of its input the current residual capacity function \(c_f(\cdot)\), computes a flow assignment \(\hat{f}(\cdot)\), and then updates the global flow assignment \(f(\cdot) := f(\cdot) + \hat{f}(\cdot)\) for every dart in the subgraph. In the pseudocode, we do not explicitly mention \(f(\cdot), c(\cdot), \sigma(\cdot), c_f(\cdot), \) or \(\sigma_f(\cdot)\). The pseudocode for the algorithm is given below.

The algorithm first finds a Jordan separator \(C\) with pieces \(G_1\) and \(G_2\). For each piece, it calls itself recursively (Line 3), so that after the call there are no \(S\)-to-\(T\) residual paths within \(G_i\). It then pushes flow from the sources in \(G_i\) to the boundary nodes using a multiple-sources single-sink max flow computation (Line 5), and similarly pushes flow from the boundary nodes to the sinks in \(G_i\) (Line 6). After the first loop terminates, there are no \(S\)-to-\(C\) residual paths, no \(C\)-to-\(T\) residual paths and no \(S\)-to-\(T\) residual paths in the entire graph. The resulting flow assignment is a pseudoflow rather than a feasible flow since it does not satisfy conservation at the boundary nodes.

The algorithm then handles the boundary nodes one by one in cyclic order. Along the iterations of the second loop, we say that a boundary node \(p_i\) is *processed* if the current value of the loop variable \(i\) is greater than \(j\), and unprocessed otherwise. If the inflow at an unprocessed node \(p_i\) is positive, the algorithm tries to resolve the violation of conservation at \(p_i\) by sending the excess flow from \(p_i\) to other unprocessed nodes. Similarly, if the inflow at \(p_i\) is negative, the algorithm tries to send flow from other unprocessed nodes to \(p_i\). This approach for resolving excess flow in the boundary nodes was very recently used in \([7]\) for the special case where all sources are in one piece and all sinks are in the other. We prove that the procedure works even in the general case where the sources and sinks are in both pieces.
Algorithm 1 MultipleSourceMultipleSinkMaxFlow(graph $G$, sources $S$, sinks $T$)

1: find a simple cycle separator $C$ in $G$ with pieces $G_1$ and $G_2$
2: for $i = 1, 2$ do
3: MultipleSourceMultipleSinkMaxFlow($G_i, S \cap G_i, T \cap G_i$)
4: add to $G_i$ artificial bi-directional arcs with infinite capacity between the boundary nodes and an artificial node $v^*$ embedded in the face of $G_i$ resulting from the deletion of arcs of $G$ not in $G_i$.
5: MultipleSourceSingleSinkMaxFlow($G_i, S \cap G_i, v^*$)
6: SingleSourceMultipleSinkMaxFlow($G_i, v^*, T \cap G_i$)
7: remove $v^*$ and the artificial arcs from $G_i$
8: let the nodes of $C$ be $p_1, p_2, \ldots, p_k$
9: for $i = 1, 2, \ldots, k$ do
10: Add infinite capacity artificial bi-directional arcs $p_i+1p_{i+2}, p_{i+2}p_i+3, \ldots, p_k^{-1}p_k$
11: if $\text{inflow}(p_i) > 0$ then
12: SingleSourceSingleSinkLimitedMaxFlow($G, p_i, p_i+1, \text{inflow}(p_i)$)
13: else
14: SingleSourceSingleSinkLimitedMaxFlow($G, p_i+1, p_i, |\text{inflow}(p_i)|$)
15: remove artificial arcs
16: push flow from boundary nodes with positive inflow back to sources and to boundary nodes with negative inflow from sinks

After all boundary nodes are processed, the algorithm converts the pseudoflow to a maximum feasible flow by sending any remaining excess flow from the boundary back to the sources and filling flow deficiencies by sending flow to the boundary from the sinks. This can be done in linear time by first canceling flow cycles using the technique of Kaplan and Nussbaum [4], and then pushing the flow in topological sort order (cf. [5]).

3.1 Correctness

We will use the following two lemmas in the proof of correctness:

Lemma 3.1 (suffix lemma) Let $f$ be a flow with source set $X$. Let $A, B$ be two disjoint sets of nodes. If there are no $A$-to-$B$ residual paths and no $X$-to-$B$ residual paths before $f$ is pushed, then there are no $A$-to-$B$ residual paths after $f$ is pushed.

Proof: $f$ may be decomposed into a cyclic component (a circulation) and an acyclic component. Note that pushing a circulation does not change the amount of flow crossing any cut. This implies that if there were no $A$-to-$B$ residual paths before $f$ was pushed, then there are none after just the cyclic component of $f$ is pushed. It therefore suffices to show the lemma for an acyclic flow $f$. 
Suppose, for the sake of contradiction, that there exists a residual $a$-to-$b$ path $P$ after $f$ is pushed for some $a \in A$ and $b \in B$. Let $P'$ be the maximal suffix of $P$ that was residual before the push. That is, the arc $e$ of $P$ whose head is $\text{start}(P')$ was non-residual before the push, and $f(\text{rev}(e)) > 0$. The fact that $f(\text{rev}(e)) > 0$ implies that before $f$ was pushed there was a residual path $Q$ from some node $x \in X$ to $\text{head}(e)$. Therefore, the concatenation of $Q$ and $P'$ was a residual $x$-to-$b$ residual path before the push, a contradiction. QED

Lemma 3.2 (prefix lemma) Let $f$ be a flow with sink set $X$. Let $A, B$ be two disjoint sets of nodes. If there are no $A$-to-$B$ residual paths and no $A$-to-$X$ residual paths before $f$ is pushed, then there are no $A$-to-$B$ residual paths after $f$ is pushed.

Proof: Similarly to the proof of Lemma 3.1, it suffices to prove the lemma for an acyclic flow $f$. Suppose, for the sake of contradiction, that there exists a residual $a$-to-$b$ path $P$ after $f$ is pushed for some $a \in A$ and $b \in B$. Let $P'$ be the maximal prefix of $P$ that was residual before the push. That is, the arc $e$ of $P$ whose tail is $\text{end}(P')$ was non-residual before the push, and $f(\text{rev}(e)) > 0$. The fact that $f(\text{rev}(e)) > 0$ implies that before $f$ was pushed there was a residual path $Q$ from $\text{tail}(e)$ to some node $x \in X$. Therefore, the concatenation of $P'$ and $Q$ was a residual $a$-to-$X$ residual path before the push, a contradiction. QED

As Nussbaum points out [7], we may assume, without loss of generality, that no sources or sinks belong to $C$. Otherwise we may replace each such terminal (i.e., either a source or a sink) $v$ with a new terminal $v'$ embedded in a face to which $v$ is adjacent, connect $v'$ to $v$ and designate $v'$ as the terminal instead of $v$.

Lemma 3.3 After Line 3 is executed for piece $G_i$, there is no $S$-to-$T$ residual path in $G_i$.

Proof: By maximality of the flow pushed in Line 3 QED

Lemma 3.4 After Line 5 is executed for piece $G_i$, there is no:

1. $S$-to-$T$ residual path in $G_i$

2. $S$-to-$C$ residual path in $G_i$

Proof: Item 2 follows from the maximality of the flow pushed in Line 5. Item 1 follows by applying the suffix lemma with $A = S \cap G_i$, $B = T \cap G_i$, $X = S \cap G_i$, and $f$ the flow pushed in Line 5 QED

5
Lemma 3.5 After Line 6 is executed for piece $G_i$, there is no:

1. $S$-to-$T$ residual path in $G_i$
2. $S$-to-$C$ residual path in $G_i$
3. $C$-to-$T$ residual path in $G_i$

Proof: Item 3 is immediate from the maximality of the flow pushed in Line 6. Item 1 follows by applying the prefix lemma with $A = S \cap G_i$, $B = T \cap G_i$, $X = T \cap G_i$, and $f$ the flow pushed in Line 6. Item 2 follows by applying the prefix lemma with $A = S \cap G_i$, $B = C$, $X = T \cap G_i$, and $f$ the flow pushed in Line 6. QED

An immediate corollary of Lemma 3.5 is

Corollary 3.6 Immediately after the first loop terminates there is no:

1. $S$-to-$T$ residual path in $G$
2. $S$-to-$C$ residual path in $G$
3. $C$-to-$T$ residual path in $G$

Lemma 3.7 The following invariants are preserved throughout the execution of the second loop

1. There is no $S$-to-$T$ residual paths in $G$.
2. There is no residual $S$-to-$C$ path nor residual $C$-to-$T$ path in $G$.
3. If a processed node $p_j$ has positive inflow, there is no residual path from $p_j$ to the as-yet-unprocessed nodes. If $p_j$ has negative inflow, there is no residual path to it from the as-yet-unprocessed nodes.
4. There is no residual path from a processed node with positive inflow to a processed node with negative inflow.

Proof: By induction on the number of iterations $i$ of the loop (i.e., the number of processed nodes). By corollary 3.6 the first two invariants are satisfied immediately before the second loop is executed. The last two invariants are trivially satisfied since at that time there are no processed nodes.

Assume the invariants hold up until the beginning of the $i^{th}$ iteration. Suppose that $p_i$ has positive inflow at the beginning of the iteration (the case of negative inflow is similar). At the end of the iteration $p_i$ is processed and we need to show that the invariants still hold.

1. Invariant 1 holds by invoking the suffix lemma with $A = S$, $B = T$, $X = \{p_i\}$, and $f$ the flow pushed from $p_i$ in Line 12.
2. There are no residual $S$-to-$C$ paths by invoking the prefix lemma with $A = S$, $B = C$, $X = \{p_j : j > i\}$ and $f$ the flow pushed from $p_i$ in Line 12. There are no residual $C$-to-$T$ paths by invoking the suffix lemma with $A = C$, $B = T$, $X = \{p_i\}$ and $f$ the flow pushed from $p_i$ in Line 12.

3. Since $p_i$ had positive inflow at the beginning of the iteration, and the flow pushed in Line 12 is limited, if $p_i$ has non-zero inflow at the end of the iteration it must be positive, and the flow pushed was in fact a maximum flow from $p_i$ to $\{p_j : j > i\}$. Invariant 4 holds for $p_i$ by maximality of the flow pushed in Line 12. The invariant holds for processed nodes $p_j$ with $j < i$ by invoking the prefix lemma with $A = \{p_j : j < i\}$, $B = \{p_j : j > i\}$, $X = \{p_j : j > i\}$, and $f$ the flow pushed from $p_i$ in Line 12.

4. Invariant 4 holds for $\{p_j : j < i\}$ by invoking the prefix lemma with $A = \{p_j : j < i, \text{inflow}(p_j) > 0\}$, $B = \{p_j : j < i, \text{inflow}(p_j) < 0\}$, $X = \{p_j : j > i\}$, and $f$ the flow pushed from $p_i$ in Line 12. The invariant holds for $p_i$ by invoking the suffix lemma with $A = \{p_i\}$, $B = \{p_j : j < i, \text{inflow}(p_j) < 0\}$, $X = \{p_i\}$ and $f$ the flow pushed from $p_i$ in Line 12.

QED

**Theorem 3.8** The flow computed by the algorithm is a maximum feasible flow.

**Proof:** The flow pushed by the algorithm originates only at sources and boundary nodes and terminates only at sinks and boundary nodes. Therefore, sources, sinks and boundary nodes are the only nodes whose inflow might be non-zero. Since line 16 makes the inflow at all boundary nodes zero, the flow assignment upon termination is a feasible flow. It remains to show that upon termination there is no residual $S$-to-$T$ path. Let $C_+$ ($C_-$) be the set of nodes with positive (negative) inflow just before Line 16 is executed. Let $f_+$ ($f_-$) be the flow pushed back from $C_+$ to $S$ (from $T$ to $C_-$) in Line 16. Consider first pushing back $f_+$. By Lemma 3.7, we may invoke the suffix lemma with $A = S \cup C_-$, $B = T$, $X = C_+$ and $f = f_+$ to show there are no $S$-to-$T$ residual paths nor $C_+$-to-$T$ residual paths after $f_+$ is pushed. Similarly, invoking the suffix lemma with $A = S$, $B = C_-$, $X = C_+$ and $f = f_+$ shows there is no $S$-to-$C_-$ residual path after $f_+$ is pushed. Next, consider pushing $f_-$. Invoking the prefix lemma with $A = S$, $B = T$, $X = C_-$ and $f = f_-$ shows there are no residual $S$-to-$T$ paths after Line 16 is executed. QED

### 3.2 Running Time

The algorithm performs one recursive call per piece. In addition, it performs two multiple-source single-sink max flow computations per piece, which take $O(|G_i|^{1.5} \log n)$ time. Processing the boundary nodes takes $O(n)$ per node since the maximum flow computed is between a source and a sink on the same face.
Therefore, the non-recursive part takes $O(n^{1.5} \log n)$. Since the size of the two pieces (including boundary nodes) are at most $\theta n + 2\sqrt{2} \sqrt{n}$ and $(1 - \theta)n + 2\sqrt{2} \sqrt{n}$, for some $1/3 \leq \theta \leq 2/3$, the total running time is thus bounded by

$$T(n) \leq c_1 n^{1.5} \log n + \max_{1/3 \leq \theta \leq 2/3} \left\{ T(\theta n + \sqrt{8n}) + T((1 - \theta)n + \sqrt{8n}) \right\}$$

for some constant $c_1$.

Lemma 3.9 $T(n) = O(n^{1.5} \log n)$

Proof: We prove, by induction on $n$, that there exists a constant $c$ such that for sufficiently large $n$, $T(n) < cn^{1.5} \log n$. We choose $c$ sufficiently large so that the base of the induction, where $n$ is some constant, holds. For the inductive step, using the inductive hypothesis in Eq. 1 we get

$$T(n) \leq n^{1.5} \log n \left[ c_1 + \max_{1/3 \leq \theta \leq 2/3} \left\{ \left( \theta + \sqrt{8/n} \right)^{1.5} + \left( 1 - \theta + \sqrt{8/n} \right)^{1.5} \right\} \right] \cdot c$$

By convexity, the maximum is attained at the extreme values.

Since $(1/3)^{1.5} + (2/3)^{1.5} = 0.7376 \ldots$, for sufficiently large $n$, $\left( \theta + \sqrt{8/n} \right)^{1.5} + \left( 1 - \theta + \sqrt{8/n} \right)^{1.5} < 0.74$. Therefore, $c$ can be chosen sufficiently large so that $c_1 + 0.74c < c$, proving the lemma. QED

We note that the running-time bottleneck are the multiple-source single-sink max flow computations. An $O(n^{1.5})$ bound on MSSS would imply, by the above proof an $O(n^{1.5})$ bound for multiple-sources multiple-sinks maximum flow as well.

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