The cosmological constant effect on the quantum entanglement

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Abstract. In Schwarzschild-de Sitter space-time, the effect of a gravitational field near a massive body black hole on the spin entanglement has been studied for two-qubits in the case of triplet and singlet states of a two particles system in a circular geodesic motion. It is found that the effect of the cosmological constant on the robustness of the Wooters concurrence is more important in the triplet state than in the singlet one even in the extended region around the black hole

1. Introduction

Quantum entanglement is a physical phenomenon observed in quantum mechanics and it plays an important role in the field of quantum information processing such as quantum teleportation [1] and cryptography [2]; meanwhile, it is at the heart of philosophical discussions on the interpretation of quantum mechanics. The first idea appeared with a famous article prepared by Einstein, Podolsky and Rosen (EPR) [3] generally referred to EPR paradox. This phenomenon occurs when pairs or a set of particles interact in a such way that the quantum state of each particle cannot be described independently of the others even when the particles are separated by a very large distance. Measurements of some physical properties such as position, momentum, spin etc., performed on entangled particles are found to be correlated.

Recently, there are a number of articles that treat the effect of gravitational field on the quantum information by introducing the idea of local inertial frames namely the work of Terashima and Ueda who studied the spin rotation caused by the space time curvature for spin 1/2 particles moving in a gravitational field [4] and the gravitational spin entropy production for particles with arbitrary spin [5]. Many other issues on the problem were widely discussed in the literature [5,6,7]. On the other hand, cosmology is currently confronted with two unknown components: dark matter and energy. Dark matter is introduced to obtain the gravitational field needed to describe observations like the galactic rotation curves, gravitational lensing or the structure of the cosmic microwave background while dark energy is needed to explain the observed accelerated expansion of the universe. The repulsive force necessary to obtain an accelerated expansion of the universe can be provided by the inclusion of a vacuum energy. This corresponds to the well-known modification of the Einstein equations consisting of the addition of a cosmological term $\Lambda g_{\mu\nu}$. The Schwarzschild-de Sitter space-time describes the static gravitational field of a spherically symmetric mass in a universe with a cosmological constant $\Lambda$. 
In fact, de Sitter space with a positive cosmological constant is spherically symmetric and has a cosmological horizon surrounding any observer. The Schwarzschild solution is a spherically symmetric solution of the Einstein equations with zero cosmological constant and it describes a black hole event horizon. Then, Schwarzschild-de Sitter space-time is a combination of the two [8].

In this paper we discuss the effect of the gravitational field of a massive body (black hole) on the spin entanglement of a two particles system in a circular geodesic motion in a Schwarchild-de Sitter space-time and the role of the cosmological constant and its influence on the entanglement of the singlet and triplet states. In section 2, we present the mathematical formalism. In section 3, we deduce the form of the corresponding Wigner rotational matrix. In section 4, we calculate and discuss the Wooters concurrence to quantify the spin entanglement of a mixed state bipartite singlet and triplet states and finally in section 5, we draw our conclusions.

2. Mathematical formulation

On a curved space-time, the spin of the particle is not well defined. Thus, one has to define it locally. For this, one introduces a local inertial frame at each point by using a Vierbein (or tetrad) \( e_\mu^a \) defined by:

\[
g_{\mu\nu}e_\mu^a e_\nu^b = \eta_{ab} \tag{1}\]

where \( g_{\mu\nu} \) (resp. \( \eta_{ab} \)) is the metric of the curved (resp. Minkowski) space-time. If \( p^\mu \) is the four energy-momentum tensor of the particle, the corresponding spin quantum state is denoted by \( |\sigma, \sigma\rangle \) where \( \sigma (=\uparrow, \downarrow) \). If this particle moves to another point of the space-time, its state in the local frame becomes \([9, 10]\)

\[
\sum_{\sigma} D_{\sigma'\sigma} (\Lambda, p) |\sigma, \sigma\rangle \tag{2}
\]

where \( \Lambda \) is the Lorentz transformation matrix and \( D_{\sigma'\sigma} \) a 2x2 Wigner rotation matrix elements corresponding to a momentum dependent change of the spin state of a relativistic particle with a change of the referential frame\([11, 12, 13]\).

Now, let us consider a system consisting of two spin 1/2 non interacting particles (separable state) where the center (centroid) of this system is described by a wave packet in the local frame where the initial state is written as:

\[
|\psi^i\rangle = \sum_{\sigma_1, \sigma_2} \int dp_1 dp_2 g_{\sigma_1, \sigma_2}(p_1, p_2) |p_1, \sigma_1; p_2, \sigma_2\rangle \tag{3}
\]

where:

\[
\sum_{\sigma_1, \sigma_2} \int dp_1 dp_2 \left| g_{\sigma_1, \sigma_2}(p_1, p_2) \right|^2 = 1 \tag{4}
\]

with \( p_1 \) and \( p_2 \) are the four-momenta of the two particles. When the system reaches a final point, the wave packet \( |\psi^f\rangle \) in the local inertial frame becomes:

\[
|\psi^f\rangle = \sum_{\sigma_1, \sigma_2, \sigma_1', \sigma_2'} \int dp_1 dp_2 \sqrt{ \frac{(Ap_1)(Ap_2)}{p_1^0 p_2^0} } g_{\sigma_1, \sigma_2}(p_1, p_2) \times D_{\sigma_1'\sigma_1}(A, p_1) D_{\sigma_2'\sigma_2}(A, p_2) |Ap_1, \sigma_1'; Ap_2, \sigma_2\rangle \tag{5}
\]

On the other hand, the change in the local inertial frame is given by:

\[
\delta e_\mu^a(x) = u^a(x) d\tau \nabla_\nu e_\mu^a(x) = -u^a(x) \omega_{0b} e_\mu^b(x) d\tau = \chi_{0b}^a(x) e_\mu^b(x) d\tau \tag{6}
\]
where
\[ w_{\nu b}^a = -e_b^\mu(x)\nabla_\nu e_\mu^a(x) = e_\mu^a(x)\nabla_\nu e_b^\mu(x) \] (7)

and
\[ \chi_b^a(x) = -u^\nu(x)w_{\nu b}^a \] (8)

where \( w_{\nu b}^a \) is the spin connection elements and \( \chi_b^a(x) \) denotes the change in the local inertial frame along \( u^\nu(x) \). The change \( \delta p^\mu \) in the momentum has the form:
\[ \delta p^\mu(x) = u^\nu(x)d\tau\nabla_\nu p^\mu(x) = ma^\mu(x)d\tau \] (9)

where \( m \) is the rest mass of the particle and \( a^\mu(x) = u^\nu(x)\nabla_\nu u^\mu(x) \) (10)

is the acceleration due to an external force and \( \nabla_\nu \) is the covariant derivative. Thus, we can rewrite eq.(9) as:
\[ \delta p^\mu(x) = -\frac{1}{mc^2}[a^\mu(x)p_\nu(x) - p^\mu(x)a_\nu(x)]p^\nu(x)d\tau \] (11)

with \( c \) the velocity of light and \( p^\mu(x)p_\mu(x) = -m^2c^2; \quad p^\mu(x)a_\mu(x) = 0 \) (12)

We deduce that in the local frame:
\[ \delta p^a(x) = \lambda_a^b(x)p^b(x)d\tau \] (13)

where
\[ \lambda_a^b(x) = -\frac{1}{mc^2}[a^a(x)p_b(x) - p^a(x)a_b(x)] + \chi_b^a(x) \] (14)

3. Wigner rotational matrix in the Schwarzschild-de Sitter space-time

Consider a system of two particles (bipartite state) moving in the gravitational field of the Schwarzschild-de Sitter space-time where the metric is given by:
\[ ds^2 = -c^2\left(f(r) - \frac{\Lambda r^2}{3}\right)dt^2 + \left(f(r) - \frac{\Lambda r^2}{3}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2) \] (15)

Here \( f(r) = 1 - \frac{r_\text{S}}{r} \) and \( r_\text{S} = 2GM \) is the Schwarzschild radius and \( r > r_\text{S} \). Let us introduce a local inertial frame by choosing the tetrad:
\[ e_0^\ell = \frac{1}{c\sqrt{f(r) - \frac{\Lambda r^2}{3}}}, \quad e_1^r = \sqrt{f(r) - \frac{\Lambda r^2}{3}}, \quad e_2^\theta = \frac{1}{r}, \quad e_3^\phi = \frac{1}{r\sin\theta} \] (16)

Now, for a particle in a circular motion on the equatorial plane where \( \theta = \frac{\pi}{2} \), the four-velocity of the centroid is given by:
\[ u^\ell(x) = \frac{\cosh\xi}{\sqrt{f(r) - \frac{\Lambda r^2}{3}}} \quad \text{and} \quad u^\phi(x) = \frac{\sinh\xi}{r} \] (17)
Where $\xi$ is the rapidity in the local inertial frame defined by $\tanh \xi = \frac{V}{c}$ and $V$ denotes the velocity. Notice that the condition

$$\left(1 - \frac{r_s}{r} - \frac{\Lambda r_s^2}{3}\right) > 0$$

(18)

Implies $\Lambda r_s^2 < \frac{4}{9}$. In what follows, we denote by:

$$z = \frac{r}{r_s}; \quad w = \Lambda r_s^2; \quad \Sigma = \frac{ct}{r_s}$$

(19)

In this case, $\Theta$ takes the form:

$$\Theta = q\sqrt{q^2 + 1}\left[\sqrt{q^2 + 1} - \frac{q}{\sqrt{p^2+1+1}} \right] \frac{z^{1/2}}{\left(z - 1 - \frac{z^2 w}{3}\right)} (z - 3)$$

(20)

and the two dimensional representation of the Wigner rotation matrix $D(\Theta)$ reads:

$$D(\Theta) = e^{-ijz\theta} = \begin{pmatrix} \cos \frac{\theta}{2} & -\sin \frac{\theta}{2} \\ \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix}$$

(21)

### 4. Measure of entanglement

The most widely used measures to quantify the entanglement for a mixed state is the so called Wooters concurrence defined as [14,15,16,17]:

$$C(\rho) = \max (0, \sqrt{\lambda_1}, -\sqrt{\lambda_2}, -\sqrt{\lambda_3}, -\sqrt{\lambda_4})$$

(22)

Where $\rho$ is the density matrix of the state and $\sqrt{\lambda_i}$'s are the eigenvalues of $\rho \tilde{\rho}$ with

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y)$$

(23)

and $\sigma_y$ is the Pauli matrix.

#### 4.1. The singlet entangled state

the wave packet in the momentum representation for a system of two spin 1/2 non-interacting particles observed in a local frame and located at an initial point can be written as:

$$|\psi_i\rangle = \sum_{\sigma_1, \sigma_2} \int dp_1 dp_2 g_{\sigma_1, \sigma_2}(p_1, p_2) |p_1, \sigma_1; p_2, \sigma_2\rangle$$

(24)

where

$$\sum_{\sigma_1, \sigma_2} \int dp_1 dp_2 |g_{\sigma_1, \sigma_2}(p_1, p_2)|^2 = 1$$

(25)

In the singlet state, $g_{\sigma_1, \sigma_2}(p_1, p_2)$ is given by

$$g_{\sigma_1, \sigma_2}(p_1, p_2) = \frac{1}{\sqrt{2}} (\delta_{\sigma_1, \sigma_2} - \delta_{\sigma_1, \sigma_2}) f(p_1) f(p_2)$$

(26)

and the reduced matrix density for the initial state has the form [11,12]:
For the final state, the reduced matrix density elements are:

\[
\rho_f = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & +1 & -1 & 0 \\
0 & -1 & +1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]  

(27)

For the final state, the reduced matrix density elements are:

\[
\rho_{\sigma_1 \sigma_2; \sigma_1 \sigma_2}^f = \frac{1}{2} \iint dp_1 dp_2 |f(p_1)|^2 |f(p_2)|^2 [D_{\sigma_1;1}(\Theta_1)D_{\sigma_2;2}(\Theta_2) - D_{\sigma_1;2}(\Theta_1)D_{\sigma_2;1}(\Theta_2) - D_{\sigma_1;1}(\Theta_2)D_{\sigma_2;2}(\Theta_1) + D_{\sigma_1;2}(\Theta_2)D_{\sigma_2;1}(\Theta_1)]
\]

(28)

Then, straightforward calculations show that the wooters concurrence takes the following form:

\[
C(\rho_f) = \langle \cos \Theta \rangle^2 + \langle \sin \Theta \rangle^2
\]  

(29)

where \( \langle X \rangle = \int dp |f(p)|^2 X \). We note that the expectation values values in eq.(29) have to be solved numerically using a Gaussian distribution. Fig.1 shows the wooters concurrence \( C(\rho_f) \) as a function of \( z \) for fixed values of \( q, \Sigma \) and \( w \).

![Figure 1. \( C(\rho_f) \) as a function of \( z \) for fixed \( q, \Sigma \) and \( w \)](image_url)

Notice that as \( z \) (equivalently the distance \( r \)) increases i.e. going far away from the horizon of the black hole, the entanglement increases and becomes more robust. This is due to the fact that the gravitational field decreases. Fig.2, displays the variation of the concurrence \( C(\rho_f) \) as a function of the centroid momentum \( q \) for fixed values of \( z, \Sigma \) and \( w \). Notice the damping periodic oscillatory behavior due to the sine and cosine functions in the Wigner matrix. Furthermore, as \( q \) increases, the maximum of the oscillations decreases due to the fact that as the velocity increases the spin.
decoherence phenomenon increases. Fig.3, shows the variation of the concurrence $C(\rho^f)$ as a function of $w$ for fixed values of $z$, $\Sigma$ and $q$. Notice that if the cosmological constant $\Lambda$ increases, we will approach more the cosmological horizon where the cosmological curvature becomes stronger and therefore the entanglement decreases. According to the Hawking-Unruh effect, an accelerating particle
will radiate and lose information and entanglement decreases. Fig.4 displays a 3D plot representing the Wooters concurrence as a function of $z$ and $q$ for fixed $\Sigma$ and $w$. Fig.5 displays a 3D plot representing the Wooters concurrence as a function of $z$ and $w$ for fixed values of $\Sigma$ and $q$.

\[ \text{Figure 5. } 3D \text{ plot of Wooters concurrence as a function of } z \text{ and } w \text{ for fixed } \Sigma \text{ and } q \]

4.2. The triplet entangled state

In the triplet state, $g_{\sigma_1,\sigma_2}(p_1, p_2)$ takes the form:

\[ g_{\sigma_1,\sigma_2}(p_1, p_2) = \frac{1}{\sqrt{2}} \left( \delta_{\sigma_1,\delta_{\sigma_2}} + \delta_{\sigma_1,\delta_{\sigma_2}} f(p_1) f(p_2) \right) \]

(30)

In this case, one can show that after straightforward but tedious calculations, the Wooters concurrence has the following expression:

\[ C(\rho) = \sqrt{\langle \cos 2\Theta \rangle^2 + \langle \sin 2\Theta \rangle^2} \]

(31)

Fig.6, fig.7 and fig.8 are the same as fig.2, fig.1 and fig.3 respectively but with a triplet entangled state.

\[ \text{Figure 6. } \text{ same as fig.2 but with a triplet entangled state.} \]

Notice that contrary to ref. [8], the concurrence in the quantum entangled triplet state is more robust than in the singlet case. Moreover, the effect of the cosmological constant is very remarkable even in extended regions around the black hole.
5. Conclusion
In this work, which falls within the framework of quantum information in a curved static space-time, Wooters concurrence behavior of two spin 1/2 entangled particles moving in a circular motion is studied in the context of the Schwarzschild-de Sitter metric. We have found that the spin entanglement in the triplet quantum state is more important than in the singlet case. Furthermore, the cosmological constant plays an important role in the neighborhood region around the black hole (more study is under investigation).

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