Abstract

A calcular algebra is a subalgebra of $H^\infty(\Omega)$ with norm given by $\|\phi\| = \sup \|\phi(T)\|$ as $T$ ranges over a given class of commutative $d$-tuples of operators with Taylor spectrum in $\Omega$. We discuss what algebras arise this way, and how they can be represented.

1 Introduction

Let $\Omega$ be a bounded open set in $\mathbb{C}^d$. We say that a class $\mathcal{C}$ is subordinate to $\Omega$ if:

(i) Each element $T$ of $\mathcal{C}$ is a commuting $d$-tuple of bounded operators on a Hilbert space, with its Taylor spectrum $\sigma(T)$ in $\Omega$.

(ii) For some non-zero Hilbert space $\mathcal{H}$, the set of scalars$
\{(\lambda^1, \ldots, \lambda^d) : \lambda \in \Omega\} \subseteq \mathcal{C}$,

where we think of $\lambda$ as a $d$-tuple of scalar multiples of the identity acting on $\mathcal{H}$. (Note that we use superscripts to denote the coordinates.)

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$^1$For a definition of Taylor spectrum of a commuting tuple, see [11].
Given a class $\mathcal{C}$ subordinate to $\Omega$, we define $H^\infty(\mathcal{C})$ to be those holomorphic functions on $\Omega$ for which

$$
\|\phi\|_\mathcal{C} = \sup\{\|\phi(T)\| : T \in \mathcal{C}\}
$$

is finite. It can be shown (see Prop. 2.1 below) that this algebra is always complete, so it is a Banach algebra, which by Property (ii) is always contained contractively in the algebra $H^\infty(\Omega)$ of bounded holomorphic functions on $\Omega$. (We are using $H^\infty$ in two apparently different ways, but identifying $\Omega$ with the set of scalars makes the two usages agree). Any Banach algebra of holomorphic functions arising in this way we shall call a *calcular algebra over* $\Omega$.

We shall call the closed unit ball of $H^\infty(\mathcal{C})$ the *Schur class of $\mathcal{C}$*, and denote it by $\mathcal{S}(\mathcal{C})$.

$$
\mathcal{S}(\mathcal{C}) = \{\phi \in \text{Hol}(\Omega) : \|\phi(T)\| \leq 1, \forall T \in \mathcal{C}\}.
$$

Let $\mathcal{S}(\Omega)$ denote the closed unit ball of $H^\infty(\Omega)$.

If $\mathcal{H}$ is a Hilbert space (we shall always assume that Hilbert spaces are not zero-dimensional to avoid trivialities), let $CB(\mathcal{H})^d$ denote the set of commuting $d$-tuples of elements of $B(\mathcal{H})$, the bounded linear operators on $\mathcal{H}$. Given a set $S$ of bounded holomorphic functions on $\Omega$, and a Hilbert space $\mathcal{H}$, one can form the set

$$
\mathcal{H}(S) = \{T \in CB(\mathcal{H})^d : \sigma(T) \subseteq \Omega \text{ and } \|\phi(T)\| \leq 1 \forall \phi \in S\}.
$$

If $\mathcal{H}$ is a Hilbert space, $\mathcal{C} \subseteq CB(\mathcal{H})^d$ and $S \subseteq \mathcal{S}(\Omega)$, then tautologically one has

$$
\mathcal{H}(\mathcal{S}(\mathcal{C})) \supseteq \mathcal{C} \quad (1.1)
$$

$$
\mathcal{S}(\mathcal{H}(S)) \supseteq S. \quad (1.2)
$$

Typically these inclusions will be strict. For example, let $d = 1$, and let $\Omega$ be the open unit disk $\mathbb{D}$. Let $\mathcal{H}$ be any Hilbert space, and let $\mathcal{C}$ be the set $\{\lambda I : \lambda \in \mathbb{D}\}$. Then $\mathcal{S}(\mathcal{C})$ will equal $\mathcal{S}(\mathbb{D})$, and, by von Neumann’s inequality [13], $\mathcal{H}(\mathcal{S}(\mathbb{D}))$ will consist of all contractions on $\mathcal{H}$ whose spectrum is in $\mathbb{D}$. Likewise if $S$ just contains the function $z$, then $\mathcal{H}(S)$ will be the contractions on $\mathcal{H}$ whose spectrum is in $\mathbb{D}$, and the Schur class of this set will be all of $\mathcal{S}(\mathbb{D})$. 2
Our first result is that the operations $H$ and $S$ stabilize after 3 steps, provided $H$ is infinite dimensional.

Notation: If $T$ is a commuting $d$-tuple of bounded operators on a Hilbert space $H$, we call $H$ the carrier of $T$, and write $H = \text{car}(T)$.

**Theorem 1.3.** Let $\Omega$ be a bounded open set in $\mathbb{C}^d$, and let $C$ be any class subordinate to $\Omega$. Let $S$ be a non-empty subset of $\mathcal{S}(\Omega)$. For any Hilbert space $H$, we have

$$H(S(H(S))) = H(S). \quad (1.4)$$

If the dimension of $H$ is either infinite, or greater than or equal to $\sup \{\dim(\text{car}(T)) : T \in C\}$, then

$$\mathcal{S}(H(S(C))) = \mathcal{S}(C) \quad (1.5)$$

**Proof:** By (1.2), we have

$$H(S(H(S))) \subseteq H(S). \quad (1.6)$$

Suppose now that $T \in H(S)$, and $\phi$ is any function in $\mathcal{S}(H(S))$. Then $\|\phi(T)\| \leq 1$, so $T$ is in $H(S(H(S)))$, proving (1.4).

By (1.2) again, with $S = \mathcal{S}(C)$, we get

$$\mathcal{S}(C) \subseteq \mathcal{S}(H(S(C))). \quad (1.7)$$

Now, assume the dimension of $H$ is as in the hypothesis. Let $\phi \in \mathcal{S}(H(S(C)))$, and let $T \in C$, with $\text{car}(T) = K$. We need to show $\|\phi(T)\| \leq 1$. If the dimension of $H$ is equal to the dimension of $K$, then $T$ is unitarily equivalent to a $d$-tuple $R$ on $H$, and $R \in H(S(C))$ since $\|\psi(R)\| = \|\psi(T)\| \leq 1$ for every $\psi$ in $\mathcal{S}(C)$. Therefore $\|\phi(T)\| = \|\phi(R)\| \leq 1$, and we are done.

If the dimension of $H$ is larger than the dimension of $K$, write $H = H_1 \oplus H_2$ where $\dim(H_1) = \dim(K)$, and let $R_1$ on $H_1$ be unitarily equivalent to $T$. Choose $\lambda = (\lambda^1, \ldots, \lambda^d)$ in $\Omega$, and let

$$R = (R_1^1 \oplus \lambda^1 I_{H_2}, \ldots, R_1^d \oplus \lambda^d I_{H_d}),$$

Then for any $\psi \in O(\Omega)$, the set of holomorphic functions on $\Omega$, we have $\psi(R) = \psi(R_1) \oplus \psi(\lambda) I_{H_2}$, so if $\psi \in \mathcal{S}(C)$, we have $\|\psi(R)\| \leq 1$, and therefore $R \in H(S(C))$. Now we get $\|\phi(T)\| \leq \|\phi(R)\| \leq 1$, and again we are done.

Finally we consider the case where $H$ is infinite dimensional, but the carriers of the elements of $C$ may have larger dimension. We can assume
without loss of generality that \( \mathcal{H} \) is separable. We need to find \( R \in \mathcal{H}(\mathcal{S}(\mathcal{C})) \) with \( \|\phi(R)\| = \|\phi(T)\| \). To do this, it is sufficient to show that there is a separable subspace \( \mathcal{K}_1 \) of \( \mathcal{K} \) that is reducing for \( f(T) \) for every \( f \in O(\Omega) \) and such that \( \|\phi(T)|_{\mathcal{K}_1}\| = \|\phi(T)\| \); for then we can choose \( R_1 \) on \( \mathcal{H} \) unitarily equivalent to \( P_{\mathcal{K}_1}T|_{\mathcal{K}_1} \), where \( P_{\mathcal{K}_1} \) is projection onto \( \mathcal{K}_1 \); the fact that \( \mathcal{K}_1 \) is reducing means that \( \phi(P_{\mathcal{K}_1}T|_{\mathcal{K}_1}) = P_{\mathcal{K}_1}\phi(T)|_{\mathcal{K}_1} \).

Observe that \( \{f(T) : f \in O(\Omega)\} \) has a countable dense subset \( \mathcal{D} \) in the norm topology of \( CB(\mathcal{K})^d \), since \( O(\Omega) \) is separable. Let \( u_j \) be a sequence of unit vectors in \( \mathcal{K} \) such that \( \|\phi(T)u_j\| \to \|\phi(T)\| \). Let \( \mathcal{K}_1 \) be the closed linear span of finite products of elements of \( \mathcal{D} \cup \mathcal{D}^* \) applied to finite linear combinations of the vectors \( u_j \). By \( \mathcal{D}^* \) we mean

\[
\mathcal{D}^* = \{((T^1)^*, \ldots, (T^d)^*) : (T^1, \ldots, T^d) \in \mathcal{D}\}.
\]

Then \( \mathcal{K}_1 \) is a separable subspace of \( \mathcal{K} \) on which \( \phi(T) \) achieves its norm and that is reducing for every \( f(T) \). \( \square \)

For a given class \( \mathcal{C} \), it is of interest to know the smallest dimension of \( \mathcal{H} \) that gives equality in (1.5).

**Example 1.8** Let \( \Omega = \mathbb{D} \), and let \( \mathcal{C} \) be all contractions with spectrum in \( \mathbb{D} \). Then we can take \( \mathcal{H} \) to be one dimensional. Similarly, if \( \Omega = \mathbb{D}^2 \), and \( \mathcal{C} \) is all pairs of commuting contractions with spectrum in \( \mathbb{D}^2 \); Andô’s inequality [4] yields that we can take \( \mathcal{H} \) to be one dimensional again.

However, if \( d \geq 3 \), we let \( \Omega = \mathbb{D}^d \), and \( \mathcal{C} \) be the class of all \( d \)-tuples of commuting contractions with spectrum contained in \( \mathbb{D}^d \), then \( \mathcal{S}(\mathcal{C}) \) is the Schur-Agler class, a proper subset of \( \mathcal{S}(\mathbb{D}^d) \) [6, 12]. If \( \mathcal{H} = \mathbb{C}^n \), then \( \mathcal{H}(\mathcal{S}(\mathcal{C})) \) will be all \( d \)-tuples of commuting contractive \( n \times n \) matrices with spectrum in \( \mathbb{D}^d \). In [9], it is shown that if \( n = 3 \), then

\[
\mathcal{S}(\mathbb{C}^3(\mathcal{S}(\mathcal{C}))) = \mathcal{S}(\mathbb{D}^d).
\]

It is unknown what the minimal dimension of \( \mathcal{H} \) must be in this case to get equality in (1.5), or even whether it must be infinite.

**Example 1.9** Let \( \mathcal{K} \) be a Hilbert function space on \( \Omega \) with reproducing kernel \( k \). The multiplier algebra \( Mult(\mathcal{K}) \) is always a calular algebra. Indeed, for each finite set \( F = \{\lambda_1, \ldots, \lambda_n\} \subset \Omega \), let \( T_F \) be the commuting \( d \)-tuple \( (T^1_F, \ldots, T^d_F) \) acting on the \( n \)-dimensional subspace of \( \mathcal{K} \) spanned by the kernel
functions \( \{k_{\lambda_j} : 1 \leq j \leq n\} \) defined by

\[
T^*_F k_{\lambda_j} = \overline{\lambda_j} k_{\lambda_j} \quad 1 \leq r \leq d, 1 \leq j \leq n.
\]

Define

\[
\mathcal{C} = \{T^*_F : F \text{ a finite subset of } \Omega\}.
\]

It is straightforward to show that \( H^\infty(\mathcal{C}) = \text{Mult}(\mathcal{K}) \). Many other examples of calcular algebras are given in [2, Chapter 9].

## 2 When is a Banach algebra a calcular algebra?

**Proposition 2.1.** Let \( \mathcal{C} \) be subordinate to \( \Omega \). Then \( H^\infty(\mathcal{C}) \) is a Banach algebra.

**Proof:** We need to prove completeness. Consider a Cauchy sequence \( \{\phi_n\} \) in \( H^\infty(\mathcal{C}) \). Since \( \mathcal{C} \) is subordinate to \( \Omega \), \( \{\phi_n\} \) is a Cauchy sequence in \( H^\infty(\Omega) \). Therefore, as \( H^\infty(\Omega) \) is complete, there exists \( \phi \in H^\infty(\Omega) \) such that

\[
\sup_{\lambda \in \Omega} |\phi_n(\lambda) - \phi(\lambda)| \to 0 \text{ as } n \to \infty. \tag{2.2}
\]

We claim that

\[
\phi \in H^\infty(\mathcal{C}) \tag{2.3}
\]

and

\[
\phi_n \to \phi \text{ in } H^\infty(\mathcal{C}). \tag{2.4}
\]

To prove statement (2.3), note that for each \( T \in \mathcal{C} \), we have \( \Omega \) is a neighborhood of \( \sigma(T) \). Consequently, continuity of the functional calculus implies that

\[
\phi_n(T) \to \phi(T) \quad \forall T \in \mathcal{C}. \tag{2.5}
\]

Also, as \( \{\phi_n\} \) is a Cauchy sequence in \( H^\infty(\mathcal{C}) \), there exists a constant \( M \) such that

\[
\|\phi_n\|_{\mathcal{C}} \leq M \quad \forall n.
\]

Therefore, if \( T \in \mathcal{C} \),

\[
\|\phi(T)\|_{\mathcal{C}} = \lim_{n \to \infty} \|\phi_n(T)\| \leq \limsup_{n \to \infty} \|\phi_n\|_{\mathcal{C}} \leq M.
\]
Hence,
\[ \| \phi \|_{C} = \sup_{T \in C} \| \phi(T) \|_{C} \leq M, \]
which proves the membership (2.3).

To prove the limiting relation (2.4), let \( \varepsilon > 0 \). Choose \( N \) such that
\[ m, n \geq N \implies \| \phi_{n} - \phi_{m} \|_{C} < \varepsilon. \]

By definition of the norm, this means
\[ m, n \geq N \implies \| \phi_{n}(T) - \phi_{m}(T) \| < \varepsilon \quad \forall T \in C. \]

Letting \( m \to \infty \) and using statement (2.5) we deduce that
\[ n \geq N \implies \| \phi_{n}(T) - \phi(T) \| < \varepsilon \quad \forall T \in C. \]

Hence, since \( \| \phi_{n} - \phi \|_{C} = \sup_{T \in C} \| \phi_{n}(T) - \phi(T) \| \),
\[ n \geq N \implies \| \phi_{n} - \phi \|_{C} \leq \varepsilon. \]
\[ \square \]

Let \( \mathcal{A} \) be a unital Banach algebra contractively contained in \( H_{\infty}(\Omega) \). When can it be realized as a calcular algebra? Let \( S \) be its unit ball. By Theorem 1.3, \( \mathcal{A} \) is a calcular algebra if and only if \( \mathcal{S}(\mathcal{H}(S)) = S \), where \( \mathcal{H} \)

is an infinite dimensional Hilbert space.

This imposes a constraint on \( \mathcal{A} \). In particular, there must be an isometric homomorphism from \( \mathcal{A} \) into \( B(\mathcal{K}) \) for some Hilbert space \( \mathcal{K} \). There is another constraint which stems from the requirement that all the operators in the class have spectrum in the open set \( \Omega \).

**Proposition 2.6.** If \( \mathcal{A} \) is a calcular algebra, then:

(i) There is an isometric homomorphism into \( B(\mathcal{K}) \) for some Hilbert space \( \mathcal{K} \).

(ii) If \( \phi_{n} \) is a bounded sequence in \( \mathcal{A} \) that converges uniformly on compact subsets of \( \Omega \) to a function \( \psi \), then \( \psi \in \mathcal{A} \), and \( \| \psi \| \leq \lim inf \| \phi_{n} \| \).

**Proof:** (i) Suppose \( \mathcal{A} \) is \( H_{\infty}(\mathcal{C}) \) for some class \( \mathcal{C} \) subordinate to an open set \( \Omega \). Let \( \mathcal{H} \) be any infinite dimensional Hilbert space, and let \( \mathcal{C}_{1} = \mathcal{H}(\mathcal{S}(\mathcal{C})) \). By Theorem 1.3, we have
\[ \mathcal{A} = H_{\infty}(\mathcal{C}_{1}). \quad (2.7) \]
Let $\mathcal{K}$ be the direct sum of cardinality($C_1$) copies of $\mathcal{H}$, with the sum indexed by $C_1$. Define a map $\pi : \mathcal{A} \to B(\mathcal{K})$ by

$$\pi(\phi) = \bigoplus_{T \in C_1} \phi(T).$$

Then $\pi$ is a homomorphism, and by (2.7) it is isometric.

(ii) Let $\phi_n$ be a bounded sequence in $H^\infty(\mathcal{C})$ converging to $\psi$ locally uniformly on $\Omega$. Without loss of generality, we may assume that each $\phi_n$ is in $\mathcal{S}(\mathcal{C})$. For each $T$ in $\mathcal{C}$, since $\sigma(T) \subseteq \Omega$, it follows from the continuity of the functional calculus that $\psi(T)$ is the limit in norm of $\phi_n(T)$, so $\psi$ is in $\mathcal{S}(\mathcal{C})$. Replacing $\phi_n$ by a subsequence whose norms converge to $\lim \inf \|\phi_n\|$ gives the last inequality. $\square$

Remark: If one defines $S = \bigoplus_{T \in C_1}(T)$, then one can interpret $\phi(S)$ as $\pi(\phi)$. However, the spectrum of $S$ will be $\overline{\Omega}$, so $S$ is not contained in any class subordinate to $\Omega$. The Taylor functional calculus is defined only for functions holomorphic on a neighborhood of the Taylor spectrum of the $d$-tuple.

A necessary condition for a Banach algebra to be isometrically isomorphic to an algebra of operators on a Hilbert space is that it satisfies von Neumann’s inequality: $\|p(x)\| \leq \|p\|_{H^\infty(D)}$ for any $x$ in the unit ball of the Banach algebra, and any polynomial $p$. It is not known whether this condition is sufficient.

Calculus algebras come with a sequence of matrix norms. If $[\phi_{ij}]$ is an $n$-by-$n$ matrix of elements of $H^\infty(\mathcal{C})$, one can define

$$\|\phi_{ij}\|_n = \sup\{\|\phi_{ij}(T)\| : T \in \mathcal{C}\},$$

where the norm on the right-hand side is the operator norm on $\text{car}(T) \otimes \mathbb{C}^n$. By a similar argument to Proposition 2.6, one can show that calculus algebras have completely isometric homomorphic embeddings into some $B(\mathcal{K})$.

Algebras that can be completely isometrically realized in this way are characterized by the Blecher-Ruan-Sinclair theorem [5], [10, Cor. 16.7]. This says that the algebra $\mathcal{A}$ must satisfy the Ruan axioms:

$$\forall n \in \mathbb{N}, \forall a \in M_n(\mathcal{A}), \forall X, Y \in M_n(\mathcal{C}), \quad \|XaY\|_n \leq \|X\|\|a\|_n\|Y\|$$

$$\forall m, n \in \mathbb{N}, \forall a \in M_m(\mathcal{A}), b \in M_n(\mathcal{A}), \quad \|a \oplus b\|_{m+n} = \max\|a\|_m, \|b\|_n,$$

and hence be isometrically realizable as an operator space; and the matrix multiplication at each level $n$ must be contractive, i.e. if $a = [a_{ij}]$ and $b = [b_{ij}]$
are in $M_n(\mathcal{A})$, then
\[ \| [\sum_{k=1}^n a_{ik} b_{kj}]\|_n \leq \| [a_{ij}]\|_n \| [b_{ij}]\|_n. \]

It is straightforward to check that a calcular algebra satisfies the hypotheses of the Blecher-Ruan-Sinclair theorem.

We do not know in general what intrinsic necessary and sufficient conditions on a sub-algebra of $H^\infty(\Omega)$ make it a calcular algebra; we can say something with strong convexity assumptions. Let $\mathcal{P} = \mathbb{C}[z_1, \ldots, z_d]$ denote the polynomials. If $f$ is a function and $r > 0$, define $f_r$ by $f_r(z) = f(rz)$.

**Theorem 2.8.** Let $\mathcal{A}$ be a unital Banach algebra that is contractively contained in $H^\infty(\Omega)$, for some bounded open convex set $\Omega$ in $\mathbb{C}^d$ that contains 0. Suppose that $\mathcal{P}$ is contained in $\mathcal{A}$ and that for every function $\phi \in \mathcal{A}$, there is a sequence in $\mathcal{P}$ that is bounded in norm by $\|\phi\|$ and converges to $\phi$ locally uniformly on $\Omega$. Suppose moreover that for every polynomial $p \in \mathcal{P}$, we have $\|p_r\| \leq \|p\|$ for $0 < r < 1$.

Then $\mathcal{A}$ is a calcular algebra over $\Omega$ if and only if the necessary conditions of Proposition 2.6 hold.

**Proof:** Suppose both conditions hold, and $\pi$ embeds $\mathcal{A}$ isometrically in $B(K)$. For each of the coordinate functions $z^j, 1 \leq j \leq d$, define $T^j = \pi(z^j)$. Let $T \in CB(K)^d$ be the tuple $(T^1, \ldots, T^d)$. Then for any polynomial $p \in \mathcal{P}$ we have $\pi(p) = p(T)$. Moreover, if $p$ has no zeroes on $\overline{\Omega}$, then
\[ \pi(p \frac{1}{p}) = 1_K = p(T)\pi\left( \frac{1}{p} \right), \]
so
\[ \pi\left( \frac{1}{p} \right) = p(T)^{-1}. \]

As $p$ ranges over affine functions whose zero sets are hyperplanes not intersecting $\overline{\Omega}$, we see that $\sigma(T)$ must be contained in $\overline{\Omega}$.

We want the elements of $\mathcal{C}$ to have spectrum in $\Omega$. Let $\mathcal{C} = \{rT : 0 \leq r < 1\}$. 
For any polynomial $p$ and any sequence $r_n \uparrow 1$ we have

$$
\|p\|_A = \|\pi(p)\| = \|p(T)\|
$$

$$
= \lim_{n \to \infty} \|p(r_n T)\|
$$

$$
\leq \|p\|_C
$$

$$
= \sup_{0<r<1} \|p_r(T)\|
$$

$$
= \sup_{0<r<1} \|\pi(p_r)\|
$$

$$
\leq \|p\|_A.
$$

So $\mathcal{A}$ and $H^\infty(C)$ assign the same norm to polynomials.

Let $\psi$ be in $\mathcal{A}$ of norm 1. By hypothesis, there is a sequence of polynomials $q_n$ that converges locally uniformly to $\psi$, with $\|q_n\|_A \leq 1$. Therefore for each $0 \leq r < 1$,

$$
\|\psi(rT)\| = \lim_{n \to \infty} \|q_n(rT)\| \leq 1.
$$

Therefore $\psi$ is in the unit ball of $H^\infty(C)$, and hence $\mathcal{A}$ is contractively contained in $H^\infty(C)$.

Conversely, let $\phi \in \mathscr{S}(C)$. Since $\Omega$ is convex, $\phi_r$ will converge to $\phi$ locally uniformly on $\Omega$ as $r \uparrow 1$. Fix $r < 1$. There is a sequence $q_n$ of polynomials that converges uniformly to $\phi_r$ on a neighborhood of $\overline{\Omega}$. Therefore $\lim_{n \to \infty} q_n(T) = \phi_r(T)$ is a contraction, and so by Property (ii) we have

$$
\|\phi_r\|_A \leq 1.
$$

By a diagonalization argument, we can modify this construction to find polynomials $q_n$ in the unit ball of $\mathcal{A}$ that converge locally uniformly to $\phi$, and hence

$$
\|\phi\|_A \leq 1.
$$

So $H^\infty(C)$ is contractively contained in $\mathcal{A}$, and hence the two algebras are isometrically isomorphic.

\begin{example}
\end{example}

The disk algebra $A(\mathbb{D})$ cannot be a calcular algebra, since it fails (ii). However, there are subalgebras of the disk algebra that are the multiplier algebra of some Hilbert function spaces on the disk, e.g. the space
with reproducing kernel

\[ k(w, z) = \sum_{n=0}^{\infty} (n+1)^2 z^n \bar{w}^n. \]

Multiplier algebras for spaces of holomorphic functions are always calcular, as shown in Example 1.9.

**Problem 2.10** Find necessary and sufficient conditions for a subalgebra of \( H^\infty(\Omega) \) to be a calcular algebra.

### 3 Realization formulas

In [7] and [8], Dritschel, Marcantognini, and McCullough proved a very general realization formula, building on work of Ambrozie and Timotin in [3], which can be adapted to our current setting.

Let \( S \) be a set of functions from a set \( X \) to the unit disk \( \mathbb{D} \). In this section, we shall make the standing assumption that \( S \) restricted to any finite set \( F \) generates, as an algebra, all the complex-valued functions on \( F \).

We define \( K_S \) to be the set of kernels on \( X \) that satisfy

\[ K_S = \{ k \mid (1 - \psi(z)\bar{\psi}(w))k(z, w) \geq 0 \quad \forall \psi \in S \}. \]

We define \( A^\infty(K_S) \) to be

\[ A^\infty(K_S) = \{ \phi : X \to \mathbb{C} \mid \exists M \geq 0 \text{ s.t. } (M^2 - \phi(z)\bar{\phi}(w))k(z, w) \geq 0 \forall k \in K_S \}, \]

and define \( \| \phi \| \) to be the smallest \( M \) that works.

Endow \( S \) with the topology of pointwise convergence. Let \( C_b(S) \) denote the continuous bounded functions on \( S \), which we think of as a C*-algebra. Let \( E : X \to C_b(S) \) be the evaluation map \( E(z)(\psi) = \psi(z) \), and let \( E(w)^* \) mean the complex conjugate of this, the adjoint in the C*-algebra, \( E(w)^*(\psi) = \overline{\psi(w)} \).

If \( \psi \) is a function from \( X \) to \( \mathbb{C} \), we say it has a network realization formula if there exists a Hilbert space \( \mathcal{M} \), a unital *-representation \( \rho : C_b(S) \to B(\mathcal{M}) \), and a unitary \( U : \mathbb{C} \oplus \mathcal{M} \to \mathbb{C} \oplus \mathcal{M} \) that in block matrix form is

\[
U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}
\]
so that
\[ \psi(z) = A + B\rho(E(z))(I - D\rho(E(z)))^{-1}C. \] (3.1)

If $B$ is a C*-algebra, a positive kernel on a set $X$ with values in $B^*$, the dual of $B$, is a function $\Gamma : X \times X \to B^*$ such that for every finite set $F \subseteq X$, and every $f : F \to B$ we have
\[ \sum_{z,w \in F} \Gamma(z,w)(f(w)^*f(z)) \geq 0. \]

Here is the Dritschel, Marcantognini, and McCullough theorem.

**Theorem 3.2.** Let $S$ be a set of functions from $X$ to $\mathbb{D}$, and let $\phi : X \to \mathbb{D}$.

The following are equivalent:

(i) $\phi \in A^\infty(K_S)$ and $\|\phi\|_{A^\infty(K_S)} \leq 1$.

(ii) For each finite set $F \subseteq X$ there exists a positive kernel $\Gamma : F \times F \to C_b(S)^*$ so that, for all $z, w \in F$,
\[ 1 - \phi(z)\overline{\phi(w)} = \Gamma(z,w)(1 - E(z)E(w)^*). \] (3.3)

(iii) $\phi$ has a network realization formula.

Now let us assume that the functions in $S$ are all holomorphic functions on the open set $\Omega$ in $\mathbb{C}^d$. By definition, we always have $S$ is contained in the unit ball of $A^\infty(K_S)$, so when $\mathcal{H}$ is infinite dimensional we have $H^\infty(\mathcal{H}(S))$ is contractively contained in $A^\infty(K_S)$ by Theorem 1.3. We shall show in Theorem 3.7 and Proposition 3.5 that the converse holds if $S$ is finite, or if a certain generic assumption holds.

We shall say that $T$ is a **generic matrix $d$-tuple on $\Omega$** if, for some $n \in \mathbb{N}$, we have that $T$ is a $d$-tuple of commuting $n$-by-$n$ matrices that have a common set of $n$ linearly independent eigenvectors with distinct joint eigenvalues, which means there are $n$ linearly independent eigenvectors $v_j$ in $\mathbb{C}^n$ so that
\[ T^r v_j = \lambda_j^r v_j, \quad 1 \leq r \leq d, \ 1 \leq j \leq n, \] (3.4)
and the $n$ points $\lambda_j = (\lambda_j^1, \ldots, \lambda_j^d)$ are distinct points in $\Omega$. The advantages of working with generic $d$-tuples were pointed out in [1].

We shall define an algebra $H^\infty_{\text{gen}}(S)$ to be the holomorphic functions on $\Omega$ for which the norm
\[ \|\phi\|_{H^\infty_{\text{gen}}(S)} := \sup\{\|\phi(T)\| : T \text{ is a generic matrix } d\text{-tuple on } \Omega, \text{ and } \|\psi(T)\| \leq 1 \ \forall \psi \in S\}. \]
Proposition 3.5. Let $S$ be a set of holomorphic functions from $\Omega$ to $\mathbb{D}$. Then $H^\infty_{\text{gen}}(S) = A^\infty(K_S)$ isometrically.

**Proof:** Let $\phi$ be in the closed unit ball of $A^\infty(K_S)$. Let $T$ be a generic matrix tuple on $\Omega$, with eigenvectors as in (3.4), and assume that $\|\psi(T)\| \leq 1$ for all $\psi$ in $S$. Let $F = \{\lambda_1, \ldots, \lambda_n\}$. Define a kernel $k(z, w)$ on $\Omega$ by setting it to zero unless both $z$ and $w$ are in $F$, and on $F$ define

$$k(\lambda_i, \lambda_j) = \langle v_i, v_j \rangle.$$ 

Then $k \in K_S$, so

$$(1 - \phi(\lambda_i)\overline{\phi(\lambda_j)})\langle v_i, v_j \rangle \geq 0. \quad (3.6)$$

Then (3.6) says that $\|\phi(T)\| \leq 1$, so $\phi$ is in the closed unit ball of $H^\infty_{\text{gen}}(S)$.

Conversely, if $\phi$ is in the closed unit ball of $H^\infty_{\text{gen}}(S)$, then for every finite set $F \subset \Omega$, by Theorem 3.2 applied to $F$, we have that (3.3) holds on $F$. Hence by the Theorem again, we have $\phi$ is in the closed unit ball of $A^\infty(K_S)$.

$\blacksquare$

Theorem 3.7. Let $S$ be a set of holomorphic functions from $\Omega$ to $\mathbb{D}$. Let $\mathcal{H}$ be an infinite dimensional Hilbert space. If $S$ is finite, then $H^\infty(\mathcal{H}(S)) = A^\infty(K_S)$.

**Proof:** By Theorem 1.3, we have $H^\infty(\mathcal{H}(S))$ is contractively contained in $A^\infty(K_S)$. For the converse, let $\phi$ be in the closed unit ball of $A^\infty(K_S)$, with a network realization formula as above. Let $S = \{\psi_1, \ldots, \psi_n\}$.

Let $\Lambda_j$ be the elements of $C_b(S)$ defined by $\Lambda_j(\psi_i) = \delta_{ij}$. Since each $\Lambda_j$ is a projection, we get that $\rho(\Lambda_j) = P_j$ gives $n$ mutually orthogonal projections that sum to the identity on $\mathcal{M}$. Then $\rho(E(z)) = \sum_{j=1}^n \psi_j(z)P_j$.

Expanding (3.1) as a Neumann series in $D\rho(E(z))$, the partial sums $\phi_n$ will converge locally uniformly on $\Omega$. Therefore if $T$ is in $\mathcal{H}(S)$, since its spectrum is a compact subset of $\Omega$, we get that $\phi(T) = \lim_n \phi_n(T)$. We have $\rho(E(T)) = \sum_{j=1}^n \psi_j(T) \otimes P_j$, and (3.1) extends to

$$\phi(T) = I_\mathcal{H} \otimes A + (I_\mathcal{H} \otimes B)\rho(E(T))(I - (I_\mathcal{H} \otimes D)\rho(E(T)))^{-1}I_\mathcal{H} \otimes C. \quad (3.8)$$

A calculation with (3.8) shows that $I_\mathcal{H} - \phi(T)^*\phi(T) \geq 0$, so we conclude $\phi \in \mathcal{S}(\mathcal{H}(S))$. $\blacksquare$

Problem 3.9 Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Do $H^\infty(\mathcal{H}(S))$ and $A^\infty(K_S)$ coincide for all non-empty sets $S$ of holomorphic functions from $\Omega$ to $\mathbb{D}$?
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