SOME NEW RESULTS ON PROPER COLOURING OF EDGE-SET GRAPHS

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Abstract. In this paper, we present a foundation study for proper colouring of edge-set graphs. The authors consider that a detailed study of the colouring of edge-set graphs corresponding to the family of paths is best suitable for such foundation study. The main result is deriving the chromatic number of the edge-set graph of a path, \( P_{n+1} \), \( n \geq 1 \). It is also shown that edge-set graphs for paths are perfect graphs.

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1. INTRODUCTION

For general notation and concepts in graphs and digraphs see [1, 2, 12]. Unless mentioned otherwise, all graphs we consider in this paper are finite, simple, connected and undirected graphs.

For a set of distinct colours \( \mathcal{C} = \{ c_1, c_2, c_3, \ldots, c_\ell \} \), a vertex colouring of a graph \( G \) is an assignment \( \phi : V(G) \rightarrow \mathcal{C} \). A vertex colouring is said to be a proper vertex colouring of a graph \( G \) if no two distinct adjacent vertices have the same colour. The cardinality of a minimum set of colours in a proper vertex colouring of \( G \) is called the chromatic number of \( G \) and is denoted...
χ(G). A colouring of G with exactly χ(G) colours may be called a χ-colouring or a chromatic colouring of G.

A minimum parameter colouring of a graph G is a proper colouring of G which consists of the colours c_i; 1 ≤ i ≤ ℓ, with minimum possible values for the subscripts i. Unless stated otherwise, we consider minimum parameter colouring throughout this paper.

The set of vertices of G having the colour c_i is said to be the colour class of c_i in G and is denoted by 𝒞_i. The cardinality of the colour class 𝒞_i is said to be the weight of the colour c_i, denoted by θ(c_i). Note that ∑_{i=1}^{ℓ} θ(c_i) = ν(G).

Unless mentioned otherwise, we colour the vertices of a graph G in such a way that 𝒞_1 = I_1, the maximal independent set in G, 𝒞_2 = I_2, the maximal independent set in G_1 = G − 𝒞_1 and proceed like this until all vertices are coloured. This convention is called rainbow neighbourhood convention (see [5]. The number of vertices in G which yield rainbow neighbourhoods, denoted by rχ(G), is called the rainbow neighbourhood number of G.

In [5], the bounds on rχ(G) corresponding to of minimum proper colouring, denoted by rχ(G) and rχ(G), have been defined as the minimum value and maximum value of rχ(G) over all permissible colour allocations. If we relax connectedness, it follows that the null graph Ω_n of order n ≥ 1 has r−(Ω_n) = r+(Ω_n) = n. For bipartite graphs and complete graphs, Kn it follows that, r−(G) = r+(G) = n and r−(Kn) = r+(Kn) = n.

We observe that if it is possible to permit a chromatic colouring of any graph G of order n such that the star subgraph obtained from vertex v as center and its open neighbourhood N(v) the pendant vertices, has at least one coloured vertex from each colour for all v ∈ V(G) then rχ(G) = n. Certainly, examining this property for any given graph is complex.

Lemma 1.1. [5] For any graph G the graph G' = K_1 + G has rχ(G') = 1 + rχ(G).

2. RAINBOW NEIGHBOURHOOD NUMBER OF EDGE-SET GRAPHS

Edge-set graphs were introduced in [4]. As the notion of an edge-set graph seems to be largely unknown. Therefore, the main definition and some important observations from [4] will be presented in this section.
Let $A$ be a non-empty finite set. Let the set of all $s$-element subsets of $A$ (arranged in some order), where $1 \leq s \leq |A|$, be denoted by $S$ and the $i$-th element of $S$ by $A_{i,s}$.

**Definition 2.1.** [4] Let $G(V,E)$ be a non-empty finite graph with $|E| = \varepsilon \geq 1$ and $E = \mathcal{P}(E) - \{\emptyset\}$, where $\mathcal{P}(E)$ is the power set of the edge set $E(G)$. For $1 \leq s \leq \varepsilon$, let $S$ be the collection of all $s$-element subsets of $E(G)$ and $E_{s,i}$ be the $i$-th element of $S$. Then, the edge-set graph corresponding to $G$, denoted by $G'$, is the graph with the following properties.

(i) $|V(G')| = 2^\varepsilon - 1$ so that there exists a one to one correspondence between $V(G')$ and $E$;

(ii) Two vertices, say $v_{s,i}$ and $v_{t,j}$, in $G'$ are adjacent if some elements (edges of $G$) in $E_{s,i}$ is adjacent to some elements of $E_{t,j}$ in $G$.

From the above definition, it can be seen that the edge-set graph $G'$ of a given graph $G$ is dependent not only on the number of edges $\varepsilon$, but the structure of $G$ also. Note that it was erroneously remarked in [4] that non-isomorphic graphs of the same size have distinct edge-set graphs. Figure 2 illustrates one contradictory case.

Note that an edge-set graph $G'$ has an odd number of vertices. If $G$ is a trivial graph, then $G'$ is an empty graph (since $\varepsilon = 0$). Also, $G_{P_2} = K_1$ and $G_{P_3} = C_3$. In [4] the following conventions were used.

(i) If an edge $e_j$ is incident with vertex $v_k$, then we write it as $(e_j \to v_k)$.

(ii) If the edges $e_i$ and $e_j$ of a graph $G$ are adjacent, then we write it as $e_i \sim e_j$.

(iii) The $n$ vertices of the path $P_n$ are positioned horizontally and the vertices and edges are labeled from left to right as $v_1,v_2,v_3,\ldots,v_n$ and $e_1,e_2,e_3,\ldots,e_{n-1}$, respectively.

(iv) The $n$ vertices of the cycle $C_n$ are seated on the circumference of a circle and the vertices and edges are labeled clockwise as $v_1,v_2,v_3,\ldots,v_n$ and $e_1,e_2,e_3,\ldots,e_n$, respectively such that $e_i = v_iv_{i+1}$, in the sense that $v_{n+1} = v_1$.

Invoking the definition and observations given above, it is noticed that both $d_{G(e)}(G)$ and $d_{G(e)}(v_i)$ are single values, while $d_{G(v_k)}(e_j) \leq d_{G(v_m)}(e_j), (e_j \to v_k), (e_j \to v_m)$. The graphs having three edges $e_1,e_2,e_3$ are graphs $P_4,C_3$, and $K_{1,3}$. The corresponding edge-set graphs
on the vertices \( v_{1,1} = \{e_1\}, v_{1,2} = \{e_2\}, v_{1,3} = \{e_3\}, v_{2,1} = \{e_1,e_2\}, v_{2,2} = \{e_1,e_3\}, v_{2,3} = \{e_2,e_3\}, v_{3,1} = \{e_1,e_2,e_3\} \) are depicted below.

Figure 1 depicts the edge-set graph \( G_{P_4} \).

![Figure 1. Edge-set graph \( G_{P_4} \).](image1)

Figure 2 depicts the edge-set graph \( G_{C_3} = G_{K_{1,3}} = K_7 \).

![Figure 2. Edge-set graph \( G_{C_3} = G_{K_{1,3}} = K_7 \).](image2)

Notice that both \( G_{C_3} \) and \( G_{K_{1,3}} \) are complete graphs.

3. **Proper Colouring of the Edge-set Graphs of Paths**

   It is known that for a given size \( \varepsilon \geq 1 \) a graph of maximum order \( v \), is a tree. Hence, for a given size the graphs with maximum structor index \( si(G) \) are the corresponding trees,
It easily follows that for $\varepsilon(T) \geq 3$ only the star graphs have $G_{\varepsilon+1}$, complete. Put another way, a tree $T$ has $G_{\varepsilon}$ complete if and only if $\text{diam}(T) \leq 2$. From the family of trees, a path corresponding to a given $\varepsilon$, denoted by $P_{\varepsilon}$, has largest diameter. These observations motivate a detailed study of the proper colouring and associated colour parameters of edge-set graphs of paths to lay the foundation for studying more complex graph classes.

For this section paths of the form $P_{n+1} = v_1e_1v_2e_2v_3\cdots e_nv_{n+1}$, will be considered. Such graph will be abbreviated to $P_{n+1} = v_i\, e_i \, v_{i+1}$, $1 \leq i \leq n$. To easily relate the results with Definition 2.1, note that $\varepsilon(P_{n+1}) = n$. It can be easily verified that $G_{P_2} = K_1$. Hence, $\chi(G_{P_2}) = 1$. Also, $G_{P_3} = K_3$ and hence, $\chi(G_{P_3}) = 3$. These observations bring the main results. First, we state an important lemma.

**Lemma 3.1.** Let $G(V,E)$ be a non-empty finite graph with $|E| = \varepsilon \geq 1$ and $\mathcal{E} = \mathcal{P}(E) - \{\emptyset\}$, where $\mathcal{P}(E)$ is the power set of the edge set $E(G)$. Then each edge $e_i$ is in exactly $2^{\varepsilon-1}$ subsets of $\mathcal{E}$.

**Proof.** The result follows directly from the well-definedness and well-ordering of the power set, $\mathcal{P}(E)$. \hfill \Box

It is observed that if the number of subsets which has say, $e_i$ as element is $t$, then within the corresponding $t$ subsets the edge $e_j, j \neq i$ will be in $\frac{t}{2} = 2^{\varepsilon-2}$ of those subsets.

**Theorem 3.2.** The edge-set graph $G_{P_{n+1}}$, $n \geq 1$ has

$$
\chi(G_{P_{n+1}}) = \begin{cases} 
1 \text{ or } 3, & \text{if } P_2 \text{ or } P_3 \text{ respectively,} \\
5, & \text{if } P_4, \\
2^{n-1} + 2^{n-2} - 2, & \text{for } P_{n+1}, \, n \geq 4.
\end{cases}
$$

**Proof. Part 1:** Trivial is the observation that $G_{P_2} = K_1$ and that result in equality. It has been observed that $G_{P_3} = K_3$ and hence $\chi(G_{P_3}) = 3$.

**Part 2:** In constructing $G_{P_4}$ begin with $G_{P_3}$ which has vertices $\{e_1\}, \{e_2\}, \{e_1, e_2\}$. Add a disjoint copy of $G_{P_3}$ and relabel the vertices of this copy to be $\{e_1, e_3\}, \{e_2, e_3\}, \{e_1, e_2, e_3\}$ to obtain, $G_{P_4}$. Clearly, $G_{P_4}$ complies with Definition 2.1.
Consider $H = \mathcal{G}_3 \cup \mathcal{G}_3$ and add the cut edges, $\{e_2\}\{e_1, e_3\}$, $\{e_2\}\{e_2, e_3\}$, $\{e_2\}\{e_1, e_2, e_3\}$, $\{e_1, e_2\}\{e_1, e_3\}$, $\{e_1, e_2\}\{e_2, e_3\}$, $\{e_1, e_2\}\{e_1, e_2, e_3\}$. Clearly, the induced subgraph, $\langle \{e_2\}, \{e_1, e_2\}, \{e_2, e_3\}, \{e_1, e_2, e_3\} \rangle = K_5$. Now add all additional bridges in accordance with Definition 2.1 to obtain graph $H'$. Due to symmetry considerations between edges $e_1$ and $e_2$ in $P_3$, exactly two maximum cliques $K_5$ come into existence hence, $\omega(H') = 5$. Finally, by adding vertex $\{e_3\}$ and the corresponding edges in accordance with Definition 2.1 and by symmetry considerations between edges $e_1$ and $e_3$ in $P_4$, the edge-set graph $\mathcal{G}_4$ has exactly four maximum cliques $K_5$. Therefore, $\chi(\mathcal{G}_4) \geq 5$.

Invoking Definition 2.1, consider the following colouring of $\mathcal{G}_4$. Let $c(v_{1,1}) = c_1$, $c(v_{1,3}) = c_1$, $c(v_{2,2}) = c_1$, $c(v_{1,2}) = c_2$, $c(v_{2,1}) = c_3$, $c(v_{2,3}) = c_4$, $c(v_{3,1}) = c_5$. Clearly, the colouring is proper and hence $\chi(\mathcal{G}_4) \leq 5$. Hence we have $\chi(\mathcal{G}_4) = 5$.

**Part 3**: For $n \geq 4$, and the path path $P_{n+1}$ the edge-set graph $\mathcal{G}_{P_{(n-1)+1}}$ of the preceding path hence, the $(n - 1)$-edge path $P_{(n-1)+1}$, is incomplete. In accordance with the procedure described in Part 2, consider $\mathcal{G}_{P_{(n-1)+1}}$ and $\mathcal{G}_{P_{(n-1)+1}}'$. Since in $\mathcal{G}_{P_{(n-1)+1}}'$ the edge $e_n$ has been added to each vertex corresponding to the vertices $v_{i,j} \in V(\mathcal{G}_{P_{(n-1)+1}})$, the new edges in accordance with Definition 2.1 are those between all pairs of vertices for which at least one vertex has $e_{n-1} \in v'_{i,j}$. From Lemma 3.1, it follows that at least one complete induced subgraph, $K_{2n-2}$ exists in $\mathcal{G}_{P_{(n-1)+1}}'$. All pairs of vertices which has both $e_{n-2}, e_{n-1} \in v'_{i,j}$ is an edge in $\mathcal{G}_{P_{(n-1)+1}}'$ so least one complete induced subgraph, $K_{2n-2+1}$ exists in $\mathcal{G}_{P_{(n-1)+1}}'$. Proceeding to vertices for which edge $e_{n-3} \in v'_{i,j}$ and so on until the edge $e_1$ has been accounted for results in $\mathcal{G}_{P_{(n-1)+1}}'$ being complete. Hence, $\chi(\mathcal{G}_{P_{(n-1)+1}}') = 2^{n-1} - 1$.

Finally, by adding the bridges between $\mathcal{G}_{P_{(n-1)+1}}$ and $\mathcal{G}_{P_{(n-1)+1}}'$ and through similar arguments in respect of edges $e_{n-2}, e_{n-1} \in v_{i,j} \in V(\mathcal{G}_{P_{(n-1)+1}})$ and so on, it follows that at least one maximum induced clique, of order $2^{n-2} - 1 + \chi(\mathcal{G}_{P_{(n-1)+1}})$, exists in $\mathcal{G}_{P_{n+1}}$. Therefore, $\chi(\mathcal{G}_{P_{n+1}}) \geq 2^{n-1} + 2^{n-2} - 2$. By allocating colours similar to the procedure described in Part-2, it follows that $2^{n-1} + 2^{n-2} - 2 \leq \chi(\mathcal{G}_{P_{n+1}}) \leq 2^{n-1} + 2^{n-2} - 2 \iff \chi(\mathcal{G}_{P_{n+1}}) = 2^{n-1} + 2^{n-2} - 2$. Therefore, by immediate induction, the result follows for all $n \geq 4$.

**Corollary 3.3.** (a) Each vertex in an edge-set graph $\mathcal{G}_{P_{n+1}}$, $n \geq 2$ belongs to some maximum clique in $\mathcal{G}_{P_{n+1}}$. 
(b) The edge-set graphs $G_{P_n+1}$, $n \geq 1$ has clique number, $\omega(G_{P_n+1}) = 2^{n-1} + 2^{n-2} - 2$.

(c) The edge-set graphs $G_{P_n+1}$, $n \geq 1$ are perfect graphs.

(d) The edge-set graph $G_{P_n+1}$ has, $r^-_{chi}(G_{P_n+1}) = r^+_{chi}(G_{P_n+1}) = 2^n - 1$.

Proof. The results are a direct consequence from the proof of Theorem 3.2. □

Theorem 3.4. An edge-set graph $G_{P_{n+1}}$, $n \geq 1$ is a perfect graph.

Proof. For $P_1, P_2$ the result is trivial. From Theorem 3.2 and Corollary 3.3(b) we have, $n \geq 2$ and hence it follows that $\omega(G_{P_{n+1}}) = 2^{n-1} + 2^{n-2} - 2 = \chi(G_{P_{n+1}})$. Hence, an edge-set graph is weakly perfect. From Definition 2.1, it follows that an edge-set graph has a unique maximum independent set $X$. Furthermore, $\langle X \rangle$ is a null graph hence, any subgraph thereof is perfect.

Also, from Corollary 3.3(a), each vertex in $V(G_{P_{n+1}})$ is in some induced maximum clique. It then follows that $\omega(H) = \chi(H)$, $\forall H \subseteq G_{P_{n+1}}$, $n \geq 1$. Hence the result. □

Conjecture 1. The edge-set graphs of acyclic graphs are perfect graphs.

4. Conclusion

Research problem: The notion of a chromatic core subgraph of a graph $G$ was introduced in [9]. We recall that, for a graph $G$ its structural size is measured by its structor index denoted and defined as, $si(G) = v(G) + e(G)$. We say that the smaller of graphs $G$ and $H$ is the graph satisfying the condition, $\text{min}\{si(G), si(H)\}$. If $si(G) = si(H)$ the graphs are of equal structural size but not necessarily isomorphic. A straightforward example is the path, $P_4$ and the star graph, $S_3$.

Definition 4.1. For a finite, undirected simple graph $G$ of order $v(G) = n \geq 1$ a chromatic core subgraph $H$ is a smallest induced subgraph $H$ (smallest in respect of $si(H)$) such that, $\chi(H) = \chi(G)$.

From the construction used in the proof of Theorem 3.2 it follows that a finite number of distinct maximum cliques can be associated with a given edge-set graph $G_{P_{n+1}}$. As an application, the largest number of vertices common to the maximum number of chromatic core subgraphs can be considered the most strategic vertices for protection from a disaster management and
recovery plan in the event of graph destruction. The aforesaid observation motivates us to introduce a new graph parameter called the *chromatic cluster number* of a graph $G$. It is denoted by $\mathcal{C}(G)$. From Theorem 3.2 it follows that $\mathcal{C}(\mathcal{G}_{P_2}) = \mathcal{C}(\mathcal{G}_{P_3}) = 1$ and $\mathcal{C}(\mathcal{G}_{P_4}) = 4$. Note that the vertices $v_{1,1} = \{e_1\}$, $v_{1,3} = \{e_3\}$, $v_{2,2} = \{e_1, e_3\}$ and $v_{1,3} = \{e_1, e_2, e_3\}$ corresponds to $\mathcal{C}(\mathcal{G}_{P_4})$.

**Problem 1.** For the edge-set graph $\mathcal{G}_{P_{n+1}}$, $n \geq 4$, determine $\mathcal{C}(\mathcal{G}_{P_{n+1}})$.

The research on set-graphs (see [3]) and edge-set graphs naturally leads to new concepts such as vertex degree sequence set-graphs and colour set-graphs and colour-string set-graphs. Preliminary definitions are provided below.

(1) If the degree sequence of a graph $G$ of order $n \geq 1$ is $(d_1 \leq d_2 \leq d_3 \leq \cdots \leq d_n)$, then for a subsequence $(d_{t+1} = d_{t+2} = \cdots = d_{t+\ell} = m_i)$, $t \geq 0$, $1 \leq \ell \leq n$, label the corresponding vertices to be $m_{i,1}, m_{i,2}, m_{i,3}, \ldots, m_{i,\ell}$. Consider the set $\mathcal{V}(G) = \mathcal{P}(V) - \emptyset$ where, $\mathcal{P}(V)$ is the power set of $V(G)$.

**Definition 4.2.** The degree sequence set-graph corresponding to $G$, denoted by $\mathcal{S}_{\mathcal{V}}(G)$, is the graph with the following properties.

(i) $|\mathcal{S}_{\mathcal{V}}(G)| = 2^v - 1$ so that there exists a one to one correspondence between $\mathcal{V}(\mathcal{S}_{\mathcal{V}}(G))$ and $\mathcal{V}(G)$.

(ii) Two vertices, say $v_{s,i}$ and $v_{t,j}$, in $\mathcal{S}_{\mathcal{V}}(G)$ are adjacent if some element(s) (specific vertex degree(s) of $G$) in $v_{s,i}$ is adjacent to some element(s) of $v_{t,j}$ in $G$.

It follows easily that for a complete graph $K_n$, $n \geq 1$ has its corresponding degree sequence set-graph, a complete graph.

**Problem 2.** Discuss the properties of the degree sequence set-graph corresponding to graph $G$.

(2) Let the minimum colour set $\mathcal{C} = \{c_1, c_2, c_3, \ldots, c_\chi\}$ permit a chromatic colouring of $G$ in accordance with the rainbow neighbourhood convention. Let $\mathcal{C}^{(1)}(G) = \mathcal{P}(\mathcal{C}) - \emptyset$ where, $\mathcal{P}(\mathcal{C})$ is the power set of $\mathcal{C}$.

**Definition 4.3.** The colour set-graph corresponding to $G$, denoted by $\mathcal{S}_{\mathcal{C}^{(1)}}(G)$, is the graph with the following properties.
(i) $|\mathcal{G}_{\mathcal{C}}(G)| = 2^\chi - 1$ so that there exists a one to one correspondence between $V(\mathcal{G}_{\mathcal{C}}(G))$ and $\mathcal{C}\{\chi\}(G)$.

(ii) Two vertices, say $v_{s,i}$ and $v_{t,j}$, in $\mathcal{G}_{\mathcal{C}}(G)$ are adjacent if some element(s) (specific vertex degree(s) of $G$) in $v_{s,i}$ is adjacent to some element(s) of $v_{t,j}$ in $G$.

Clearly, for all graphs $G$ with $\chi(G) = 2$ the colour set-graph is $K_3$.

**Problem 3.** Discuss the properties of the colour set-graph corresponding to a chromatic colouring of a graph $G$.

This problem is similar to (1). For a minimum colour set $\mathcal{C} = \{c_1, c_2, c_3, \ldots, c_\chi\}$ the corresponding colour weight sequence is $(\frac{c_1}{\theta(c_1)\text{ entries}}, \frac{c_2}{\theta(c_2)\text{ entries}}, \ldots, \frac{c_\chi}{\theta(c_\chi)\text{ entries}})$.

Let $\mathcal{C}^\circ(G) = \{c_{1,1}, c_{1,2}, c_{1,3}, \ldots, c_{\chi,1}, c_{\chi,2}, c_{\chi,3}, \ldots, c_{\chi,\theta(c_\chi)}\}$. We can define the colour-string set-graph, $\mathcal{G}_{\mathcal{C}^\circ}(G)$ similar to Definition 4.2.

**Problem 4.** Research the properties of the colour-string set-graph corresponding to a chromatic colouring of a graph $G$.

**CONFLICT OF INTERESTS**

The author(s) declare that there is no conflict of interests.

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