STRUCTURE OF SETS WITH NEARLY MAXIMAL FAVARD LENGTH

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ABSTRACT. Let $E \subset B(1) \subset \mathbb{R}^2$ be an $\mathcal{H}^1$ measurable set with $\mathcal{H}^1(E) < \infty$, and let $L \subset \mathbb{R}^2$ be a line segment with $\mathcal{H}^1(L) = \mathcal{H}^1(E)$. It is not hard to see that $\text{Fav}(E) \leq \text{Fav}(L)$. We prove that in the case of near equality, that is, $\text{Fav}(E) \approx \text{Fav}(L) - \delta$, the set $E$ can be covered by an $\epsilon$-Lipschitz graph, up to a set of length $\epsilon$. The dependence between $\epsilon$ and $\delta$ is polynomial: in fact, the conclusions hold with $\epsilon = C\delta^{1/70}$ for an absolute constant $C > 0$.

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1. INTRODUCTION

Let $E \subset \mathbb{R}^2$ be $\mathcal{H}^1$ measurable with $\mathcal{H}^1(E) < \infty$. We recall the definition of Favard length:

$$\text{Fav}(E) = \int_0^\pi \mathcal{H}^1(\pi_\theta(E)) \, d\theta.$$  

Here $\pi_\theta : \mathbb{R}^2 \rightarrow \mathbb{R}$ is the orthogonal projection $\pi_\theta(x) = x \cdot (\cos \theta, \sin \theta)$. The definition of $\text{Fav}(E)$ can be posed without the assumption $\mathcal{H}^1(E) < \infty$, but this hypothesis will be crucial for most of the statements below, and it will be assumed unless otherwise stated. A fundamental result in geometric measure theory is the Besicovitch projection theorem [2] which relates Favard length and rectifiability: $\text{Fav}(E) > 0$ if and only if $\mathcal{H}^1(E \cap \Gamma) > 0$ for some Lipschitz graph $\Gamma \subset \mathbb{R}^2$—in other words, $E$ is not purely 1-rectifiable.

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The proof of the Besicovitch projection theorem is famous for being difficult to quantify, partly because of its reliance on the Lebesgue differentiation theorem: it is hard to decipher from the argument just how large the intersection $E \cap \Gamma$ is, and what the Lipschitz constant of $\Gamma$ is. In fact, it is non-trivial to even find the right question: for example, if $E \subset B(1)$, $\mathcal{H}^1(E) = 1$, and $\text{Fav}(E) \geq \delta$ for some small but fixed constant $\delta > 0$, then it is not true that $\mathcal{H}^1(E \cap \Gamma) \geq \epsilon$ for some $\epsilon^{-1}$-Lipschitz graph $\Gamma \subset \mathbb{R}^2$, where $\epsilon = \epsilon(\delta) > 0$. We construct a relevant counterexample in Section 6.

In Theorem 1.1, we show that similar counterexamples are no longer possible if the assumption $\text{Fav}(E) \geq \delta$ is upgraded to $\text{Fav}(E) \geq 2\mathcal{H}^1(E) - \delta$ for a sufficiently small constant $\delta > 0$. The number $2$ comes from the fact that $\text{Fav}([0, 1] \times \{0\}) = 2$ and that $[0, 1] \times \{1\}$ has the maximal Favard length among sets of length unity (see (2.4)).

**Theorem 1.1.** For every $\epsilon > 0$ there exists $\delta > 0$ such that the following holds. Let $E \subset B(1)$ be an $\mathcal{H}^1$ measurable set with $\mathcal{H}^1(E) < \infty$, and assume that

$$\text{Fav}(E) \geq \text{Fav}(L) - \delta,$$

where $L \subset \mathbb{R}^2$ is a line segment with $\mathcal{H}^1(L) = \mathcal{H}^1(E)$. Then, there exists an $\epsilon$-Lipschitz graph $\Gamma \subset \mathbb{R}^2$ such that $\mathcal{H}^1(E \cap \Gamma) \geq \mathcal{H}^1(E) - \epsilon$. One can take $\delta = \epsilon^{70}/C$ for an absolute constant $C > 1$.

Thus, if $\text{Fav}(E)$ is nearly maximal, the Besicovitch projection theorem can be quantified in a very strong way, whereas the example constructed in Section 6 shows that any similar conclusion fails completely if we make the weaker assumption $\text{Fav}(E) \geq \delta$. However, it remains plausible that the assumption $\text{Fav}(E) \geq \delta$ is sufficient to guarantee a quantitative version of Besicovitch’s theorem under the additional assumption that $E$ is 1-Ahlfors regular, or satisfies other “multi-scale 1-dimensionality” hypotheses. For recent partial results, and more discussion on this question, see [8, 17, 21, 24]. The problem is closely related to Vitushkin’s conjecture [25] on the connection between analytic capacity and Favard length, see [6, 9].

We briefly mention another closely related topic: if $E \subset \mathbb{R}^2$ is self-similar and purely 1-unrectifiable, then $\text{Fav}(E) = 0$ by the Besicovitch projection theorem. It is an interesting and very popular question to attempt quantifying the (sharp) rate of decay at which $\text{Fav}(E_n) \to 0$, where $E_n$ is the “$n$th iteration” of the self-similar set. For recent developments, see [1, 3, 4, 5, 7, 16, 14, 15, 20, 22].

### 1.1. Outline of the paper

A quick outline of the article is as follows: in Section 2 we introduce Crofton’s formula and prove that line segments maximise Favard length. In Section 3 we show how to prove Theorem 1.1 using two main propositions, Proposition 3.3 and Proposition 3.10. The former allows us to cover a set with almost maximal Favard length by a bounded number of Lipschitz graphs with small constant. The latter says that, in fact, there can only be one such graph. These two propositions are then proven in Section 4 and Section 5, respectively. Section 6 contains the counterexample mentioned above Theorem 1.1. Finally, in Appendix A we give an exact formula for the measure of lines spanned by two rectifiable curves - this is used in Section 5 but it might be of independent interest.
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2. Measure theoretic preliminaries

2.1. Notation. For \( x \in \mathbb{R}^d \) and \( r > 0 \), the notation \( B(x, r) \) stands for a closed ball of radius \( r \) centred at \( x \). For \( A \subset \mathbb{R}^d \), we denote the cardinality of \( A \) by \( \# A \), and we write \( A(r) := \{ x \in \mathbb{R}^d : \text{dist}(x, A) \leq r \} \), where ”dist” is Euclidean distance. For \( f, g \geq 0 \), we write \( f \preceq g \) if there exists an absolute constant \( C > 0 \) such that \( f \leq Cg \). The notation \( f \gtrsim g \) means the same as \( g \preceq f \), and \( f \sim g \) is shorthand for \( f \preceq g \preceq f \). If the constant \( C > 0 \) is allowed to depend on some parameter ”\( p \)”, we signify this by writing \( f \gtrsim_p g \).

2.2. Integralgeometry and Crofton’s formula. One of the main tools is Crofton’s formula for rectifiable sets, which states the following. If \( E \subset \mathbb{R}^2 \) is \( \mathcal{H}^1 \)-measurable 1-rectifiable set with \( \mathcal{H}^1(E) < \infty \), then

\[
\mathcal{H}^1(E) = \frac{1}{2} \int_0^{\pi} \int_{\mathbb{R}} \#(E \cap \pi^{-1}_\theta \{ t \}) \, dt \, d\theta. \tag{2.1}
\]

The equation (2.1) is false without the rectifiability assumption, but the inequality ”\( \geq \)” remains valid in this case. This formula (and the inequality) is a special case of a more general relation between Hausdorff measure and integralgeometric measure for \( n \)-rectifiable sets in \( \mathbb{R}^d \), see Federer’s paper [11, Theorem 9.7], or [12, Theorem 3.2.26]. We next rephrase the formula (2.1) in slightly more abstract terms. We define the following measure \( \eta \) on the family \( \mathcal{A} := \mathcal{A}(2, 1) \) of all affine lines in \( \mathbb{R}^2 \):

\[
\eta(\mathcal{L}) = \int_0^{\pi} \mathcal{H}^1(\{ t \in \mathbb{R} : \pi^{-1}_\theta \{ t \} \in \mathcal{L} \}) \, d\theta, \quad \mathcal{L} \subset \mathcal{A}.
\]

With this notation, the Crofton formula (2.1) can be rewritten as

\[
\mathcal{H}^1(E) = \frac{1}{2} \int_{\mathcal{L}(E)} \#(E \cap \ell) \, d\eta(\ell), \tag{2.2}
\]

where

\[
\mathcal{L}(E) := \{ \ell \in \mathcal{A} : E \cap \ell \neq \emptyset \}.
\]

Lemma 2.3 (The line segment maximizes Favard length). If \( E \subset \mathbb{R}^2 \) is \( \mathcal{H}^1 \)-measurable, \( \mathcal{H}^1(E) < \infty \), and \( L \subset \mathbb{R}^2 \) is a line segment with \( \mathcal{H}^1(E) = \mathcal{H}^1(L) \), then

\[
\text{Fav}(E) \leq \text{Fav}(L) \tag{2.4}
\]

and

\[
\text{Fav}(L) - \text{Fav}(E) \geq \int_{\mathcal{L}(E)} \#(E \cap \ell) - 1 \, d\eta(\ell). \tag{2.5}
\]

If \( E \) is rectifiable, then equality holds in (2.5).

Proof. Suppose \( E \subset \mathbb{R}^2 \) is \( \mathcal{H}^1 \)-measurable, \( \mathcal{H}^1(E) < \infty \), and \( L \subset \mathbb{R}^2 \) is a line segment with \( \mathcal{H}^1(E) = \mathcal{H}^1(L) \). Then

\[
\text{Fav}(E) = \eta(\mathcal{L}(E)) = \int_{\mathcal{L}(E)} \, d\eta(\ell) \leq \int_{\mathcal{L}(E)} \#(E \cap \ell) \, d\eta(\ell) \leq 2 \mathcal{H}^1(E). \tag{2.6}
\]
If we replace $E$ with the line segment $L$, then equality holds in both inequalities above. Thus, $\text{Fav}(L) = 2\mathcal{H}^1(L) = 2\mathcal{H}^1(E)$, which combined with (2.6) (for $E$) proves (2.5).

Next, (2.4) follows from the fact that the right-hand side of (2.5) is nonnegative. Finally, if $E$ is rectifiable, then the second inequality in (2.6) becomes an equality, which implies that equality holds in (2.5). □

2.3. Coarea formula. We then record another tool in the proof of Theorem 1.1. It is closely related to Crofton’s formula, but only considers the intersections with lines in a fixed direction. The price to pay is that the tangent of the rectifiable set enters the formula. It is a generalization of the following standard fact: If $f : [a, b] \to \mathbb{R}$ is $\alpha$-Lipschitz, then

$$\mathcal{H}^1((\{t, f(t)\} : t \in [a, b])) = \int_a^b \sqrt{1 + f'(t)^2} \, dt \leq \sqrt{1 + \alpha^2} (b - a).$$

Lemma 2.7 (Coarea formula). Let $\alpha > 0$. Let $E \subset \mathbb{R}^2$ be a countable union of $\alpha$-Lipschitz graphs over the $x$-axis. Then,

$$\mathcal{H}^1(A) \leq \sqrt{1 + \alpha^2} \int \#(A \cap \pi_0^{-1}\{t\}) \, dt$$

for all $\mathcal{H}^1$ measurable subsets $A \subset E$. (Recall that $\pi_0 : \mathbb{R}^2 \to \mathbb{R}$ is the projection onto the $x$-axis.)

Proof. This follows from the coarea formula for rectifiable sets. (See, e.g., [12, Theorem 3.2.22] or [13, Theorem 5.4.9].) □

3. PROOF OF THEOREM 1.1 IN TWO MAIN STEPS

In this section we prove our main result using Proposition 3.3 and Proposition 3.10 introduced below. The former says that we can cover all of $E$, save for a tiny exceptional set, by a union of boundedly many Lipschitz graphs with small constant. The latter says that, in fact, there can be only one Lipschitz graph with small constant covering most of $E$, otherwise we run into contradiction with the assumption of almost maximal Favard length.

3.1. Step 1. First reductions. Let $E \subset \mathbb{R}^2$ be a Borel set with $\mathcal{H}^1(E) < \infty$. We start with the following simple lemma.

Lemma 3.1. It suffices to prove Theorem 1.1 under the additional assumption that $E$ is a finite union of disjoint $C^1$ curves.

Proof. We may assume that $E \subset B(1)$ is rectifiable, because by the Besicovitch projection theorem, the rectifiable part of $E$ continues to satisfy all the assumptions of Theorem 1.1 (with the same constant $\delta > 0$). By this assumption, $\mathcal{H}^1$ almost all of $E$ can be covered by a countable union of $C^1$-curves. Decomposing the curves further, we may assume that they are disjoint, and for any given $\eta > 0$ we may write

$$E = \bigcup_{j=1}^{M_1} (\gamma_j \cap E) \cup S,$$

where $\mathcal{H}^1(S) \leq \eta$, and $\mathcal{H}^1(E \cap \gamma_j) \geq (1 - \eta)\mathcal{H}^1(\gamma_j)$. Now, the set $\bar{E} := \bigcup_{j=1}^{M_1} \gamma_j$ satisfies

$$\mathcal{H}^1(\bar{E}) \leq (1 - \eta)^{-1}\mathcal{H}^1(E) \quad \text{and} \quad \text{Fav}(\bar{E}) \geq \text{Fav}(E) - \eta,$$
and is additionally a finite union of disjoint $C^1$-curves. If Theorem 1.1 is already known under this additional assumption, we may now infer that $H^1(E \setminus \Gamma) \leq \epsilon$, where $\Gamma$ is an $\epsilon$-Lipschitz graph. But then also $H^1(E \setminus \Gamma) \leq H^1(E \setminus \bar{E}) + H^1(E \setminus \Gamma) \leq \eta + \epsilon$, and Theorem 1.1 follows for $E$ by choosing the parameters $\epsilon, \eta$ appropriately. □

3.2. Step 2. Minigraphs and how to merge them. By Lemma 3.1, in the sequel we may assume that $E$ is a finite union of disjoint $C^1$-curves $\gamma_1, \ldots, \gamma_M$. We further chop up each curve $\gamma_j$ into connected pieces whose tangent varies by less than $\alpha$, where $\alpha$ is a small constant depending on $\epsilon$ fixed later on (see (3.5)). At this point, we have managed to write $E$ as a finite union of disjoint $\alpha$-Lipschitz graphs $\gamma_1, \ldots, \gamma_M$, where $M_1 \leq M' < +\infty$. Each of these graphs will be called a minigraph, and their collection is denoted $E$. The main task in Theorem 1.1 is to combine the minigraphs into bigger graphs.

To begin with, each of the minigraphs is an $\alpha$-Lipschitz graph over some line of the form

$$\text{span}(\cos(k\pi/M_2), \sin(k\pi/M_2)), \quad 0 \leq k \leq M_2 \sim \alpha^{-1}.$$ 

The vector $v_k := (\cos(k\pi/M_2), \sin(k\pi/M_2))$ will be called the direction of the minigraph (if there are several suitable vectors for one minigraph, fix any one of them; we will only need to know that each minigraph is an $\alpha$-Lipschitz graph over the line spanned by its direction). Statements about the (relative) angles of minigraphs should always be interpreted as statements about the relative angles of the direction vectors $v_k$.

For $k \in \{0, \ldots, M_2\}$ fixed, we write $E_k \subset E$ for the subset of minigraphs with direction $v_k$. We suggest that the reader visualise the minigraphs as line segments $I$ with $\angle(I, \text{span}(v_k)) \leq \alpha$. It seems likely that Theorem 1.1 could be reduced to the case where $E$ is a finite union of line segments, but employing the minigraphs seems to spare us some unnecessary steps.

We write $E := \cup E_k$. Thus

$$E = E_0 \cup \ldots \cup E_{M_2}. \quad (3.2)$$

It turns out that, except for a small error, each set $E_k$ is covered by a single Lipschitz graph with constant $\sim \alpha$ over $\text{span}(v_k)$. Indeed, note that Lemma 2.3 and (1.2) together imply $\int_{\ell} \#(E \cap \ell) - 1 \, d\eta(\ell) \leq \delta$. Then we have the following proposition, whose proof will be carried out in Section 4.

**Proposition 3.3.** There exist absolute constants $C_0, \alpha_0 \in (0, 1)$ and $C_{\text{lip}} > 1$ such that the following holds. Let $\delta, \epsilon \in (0, 1)$ and $\alpha \in (0, \alpha_0)$ be such that $\delta \leq C_0 \alpha^3 \epsilon^2$. Let $E \subset B(1)$ be a set with $H^1(E) < \infty$ of the form

$$E = \bigcup_{\gamma \in \mathcal{E}} \gamma,$$

where $\mathcal{E}$ is a finite collection of disjoint $\alpha$-Lipschitz graphs over a fixed line $L \subset \mathbb{R}^2$. Assume further that $E$ satisfies

$$\int_{\ell} \#(E \cap \ell) - 1 \, d\eta(\ell) \leq \delta. \quad (3.4)$$

Then, there exists a $C_{\text{lip}} \alpha$-Lipschitz graph $\Gamma$ over $L$, such that

$$H^1(E \setminus \Gamma) \leq \epsilon.$$
3. Step 3. There can only be one graph. In Proposition 3.3 we managed to pack a majority of each set \( E_j \) (as defined in (3.2)) to a Lipschitz graph of constant \( \sim \alpha \), up to errors which tend to zero as \( \delta \to 0 \) in the main assumption (1.2). However, at this point there might be up to \( \sim \alpha^{-1} \) distinct Lipschitz graphs, and to prove Theorem 1.1, we would (roughly speaking) like to reduce their number to one. That this should be possible is not hard to believe: if \( E \) consists of several distinct Lipschitz graphs of substantial measure, which nevertheless cannot be fit into a single Lipschitz graph, then \( \text{Fav}(E) \) cannot possibly be maximal.

We turn to the details. We recall the "given" constant \( \epsilon > 0 \) from the statement of Theorem 1.1, and we set

\[
\delta := \frac{\epsilon^{70}}{C_{\text{thm}}}
\]

for a sufficiently large absolute constant \( C_{\text{thm}} > 1 \). We define also

\[
\alpha := \left( \frac{\epsilon}{C_{\text{alp}}} \right)^{10}
\]

for some universal \( C_{\text{alp}} > 1 \). The universal constant \( C_{\text{thm}} \) will depend on \( C_{\text{alp}} \), whereas \( C_{\text{alp}} \) depends only on \( C_{\text{lip}} \) and another constant \( C_{\text{sep}} \), which is introduced below. We record that

\[
\alpha^7 = C_{\text{alp}}^{-70} \epsilon^{70} = C_{\text{thm}} C_{\text{alp}}^{-70} \cdot \delta.
\]

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\[
\alpha^7 = C_{\text{alp}}^{-70} \epsilon^{70} = C_{\text{thm}} C_{\text{alp}}^{-70} \cdot \delta.
\]

Recall, once more, the decompositions \( E = E_0 \cup \ldots \cup E_{M_2} \) and \( E = E_0 \cup \ldots \cup E_{M_2} \) from the previous subsection: this decomposition depends on the parameter \( \alpha \) fixed above. In addition to the decomposition \( E = E_0 \cup \ldots \cup E_{M_2} \), we will also need another, coarser, decomposition of \( E \) in this section. Write \( \kappa := \frac{1}{10} \), fix \( M_3 \sim \alpha^{-\kappa} \), and decompose \( E = F_0 \cup \ldots \cup F_{M_3} \) in such a way that

- each \( F_k \) is a union of finitely many consecutive families \( E_j \), and
- \( F_k \) contains those minigraphs whose direction makes an angle \( \leq \alpha^\kappa \) with \( w_k = (\cos(k\pi/M_3), \sin(k\pi/M_3)) \), for \( 0 \leq k \leq M_3 \).

We write

\[
F_k := F_k, \quad 0 \leq k \leq M_3 \sim \alpha^{-\kappa}.
\]

At this point, we consider two distinct cases. Let \( C_{\text{sep}} \) be a large constant depending only on the absolute constant \( C_{\text{lip}} \) appearing in Proposition 3.3 (the letters "sep" stand for "separation"). Thus, the constant \( C_{\text{sep}} \) is also absolute, and we may (and will) assume that \( C_{\text{alp}} \) is large relative to \( C_{\text{sep}} \).

**Case 1.** Given the constant \( \epsilon > 0 \) from Theorem 1.1, the first case is that we can find consecutive sets \( F_k, F_{k+1}, \ldots, F_{k+C_{\text{sep}}} \) with the property

\[
\mathcal{H}^1(E \setminus (F_k \cup \ldots \cup F_{k+C_{\text{sep}}})) \leq \epsilon.
\]

In this case we note that \( F := F_k \cup \ldots \cup F_{k+C_{\text{sep}}} \) is a union of minigraphs whose directions are within \( \lesssim C_{\text{sep}} \alpha^\kappa \) of the fixed vector \( w_k \). In particular, \( F \) can be expressed as a union of finitely many disjoint \( \alpha_0 \)-Lipschitz graphs over the line \( \text{span}(w_k) \), with \( \alpha_0 \sim C_{\text{sep}} \alpha^\kappa \). This
will place us in a positions to use Proposition 3.3 (with $E$ replaced by $F$ and $\alpha$ replaced by $a_0$). Of course also
\[ \int_{\mathcal{L}(F)} \#(F \cap \ell) - 1 \, d\eta(\ell) \leq \int_{\mathcal{L}(E)} \#(E \cap \ell) - 1 \, d\eta(\ell) \leq \delta, \]
so the analogue of the assumption (3.4) is valid for $F$ in place of $E$. We also note that
\[ \delta = C_{\text{thm}}^{-1} \epsilon^{70} \leq C_{\text{thm}}^{-1} C_{\text{alg}}^3 \cdot (\epsilon/C_{\text{alg}})^3 \cdot \epsilon^2 = (C_{\text{thm}}^{-1} C_{\text{alg}}^3) \cdot \alpha^3 \epsilon^2 \sim (C_{\text{thm}}^{-1} C_{\text{alg}}^3 C_{\text{sep}}^{-3}) \cdot \alpha^3 \epsilon^2, \]
so if $C_{\text{thm}}$ is sufficiently large relative to $C_{\text{alg}}$, then the hypothesis in Proposition 3.3 on the relation between $\delta, a_0$, and $\epsilon$ is satisfied (the constant $C_{\text{sep}}$ is large, so it can be safely ignored here). Consequently, there exists a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ of constant $\lesssim C_{\text{lip}} C_{\text{sep}} \cdot \alpha^\epsilon = C_{\text{lip}} C_{\text{sep}} \cdot \epsilon/C_{\text{alg}}$ with the property
\[ \mathcal{H}^1(F \setminus \Gamma) \leq \epsilon, \]
and consequently $\mathcal{H}^1(E \setminus \Gamma) \leq 2\epsilon$. By choosing $C_{\text{alg}}$ sufficiently large relative to $C_{\text{sep}}$ and $C_{\text{lip}}$, we may ensure that $\Gamma$ is an $\epsilon$-Lipschitz graph, as desired.

**Case 2.** We then move to consider the other option, where $E$ cannot be exhausted, up to measure $\epsilon$, by a constant number of consecutive sets $F_k, F_{k+1}, \ldots, F_{k+C_{\text{sep}}}$. Since (3.7) fails for every $k$, we may find an index pair $k, l \in \{0, \ldots, M_3\}$ with $|k - l| \geq C_{\text{sep}}$ such that
\[ \mathcal{H}^1(F_k) \geq \alpha^{2k} \quad \text{and} \quad \mathcal{H}^1(F_l) \geq \alpha^{2k}. \]
This follows immediately from the pigeonhole principle, recalling that the cardinality of the pieces $F_k$ is $\lesssim \alpha^{-\epsilon}$, and also that $\alpha^\epsilon$ is much smaller than $\epsilon$ by (3.5).

**Remark 3.9.** Recall that the "separation" constant $C_{\text{sep}}$ above has been chosen to be large relative to the constant $C_{\text{lip}}$ in Proposition 3.3: morally, if $\Gamma_1, \Gamma_2$ are two $C_{\text{lip}} \alpha^{-\epsilon}$-Lipschitz graphs over lines $L_1, L_2$ with $\angle(L_1, L_2) \geq C_{\text{sep}} \alpha^\epsilon$, we need to know that $\Gamma_1$ and $\Gamma_2$ are still "transversal" (their tangents form angles $\geq 1/2 C_{\text{sep}} \alpha^\epsilon$ with each other).

The next key proposition will imply that Case 2 cannot happen:

**Proposition 3.10.** Suppose that $C_{\text{sep}} > 0$ is sufficiently large, and suppose that there are $k, l \in \{0, \ldots, M_3\}$ with $|k - l| \geq C_{\text{sep}}$ such that
\[ \mathcal{H}^1(F_k) \geq \alpha^{2k} \quad \text{and} \quad \mathcal{H}^1(F_l) \geq \alpha^{2k}. \]
Then
\[ \int_{\mathcal{L}(E)} \#(E \cap \ell) - 1 \, d\eta(\ell) \gtrsim \alpha^7. \quad (3.11) \]

As we recorded in (3.6), we have $\alpha^7 = C_{\text{thm}} C_{\text{alg}}^{-70} \delta$. Thus, if $C_{\text{thm}}$ is chosen sufficiently large relative to $C_{\text{alg}}$ and the implicit absolute constants in (3.11), then (3.11) would lead to the contradiction
\[ \delta \gtrsim \int_{\mathcal{L}(E)} \#(E \cap \ell) - 1 \, d\eta(\ell) > \delta. \]
(For the first inequality, recall (2.5) and our main assumption (1.2).) Thus, with the choices of constants specified in this section, Case 2 cannot occur. This concludes the proof of Theorem 1.1.

In the next two sections we prove the two key results used above, Propositions 3.3 and 3.10.
4. Proof of Proposition 3.3

Let $E \subset \mathbb{R}^2$ be as in the proposition. With no loss of generality, we may assume that $L$ is the $x$-axis, so the minigraphs in $E$ are roughly horizontal. We introduce further notation. We write

$$C_\beta := \{(x, y) \in \mathbb{R}^2 : |y| \geq \beta |x|\}, \quad \beta > 0.$$ 

Thus, the smaller the $\beta$, the wider the cone. We also write

$$C_\beta(x) := x + C_\beta \quad \text{and} \quad C_\beta(x, r) := C_\beta(x) \cap B(x, r).$$

With this notation, if a set $\Gamma \subset \mathbb{R}^2$ satisfies $\Gamma \cap C_\beta(x) = \{x\}$ for all $x \in \Gamma$, then $\Gamma$ is (a subset of) a $\beta$-Lipschitz graph. Thus, in view of Proposition 3.3, it would be desirable to show that $E \cap C_{\text{lip}}(x) = \{x\}$ for all $x \in E$. In reality, we will prove a similar statement about a subset of $E$ (of nearly full length). It is worth noting that a toy version of these statements is already present in our hypotheses: each minigraph $\gamma \in E$ is an $\alpha$-Lipschitz graph over the $x$-axis.

Define the maximal conical density

$$\Theta^+_E(x, \beta) = \sup_{r > 0} \frac{\mathcal{H}^1(C_\beta(x, r) \cap E)}{r}.$$ 

Lemma 4.1 says that points of high conical density are negligible, whereas Lemma 4.18 says that points of low conical density can be mostly contained in a Lipschitz graph.

**Lemma 4.1 (High conical density points are negligible).** Let $E \subset B(1)$, $\alpha \in (0, \alpha_0)$ and $\delta \in (0, 1)$ be as in Proposition 3.3, so that in particular (3.4) holds. Let $\varepsilon > 0$. If the absolute constant $C_{\text{lip}} > 0$ is chosen sufficiently large, then

$$\mathcal{H}^1(\{x \in E : \Theta^+_E(x, \beta) \geq \varepsilon\}) \lesssim \frac{\delta}{\varepsilon \alpha^2}, \quad (4.2)$$

where $\alpha' := C_{\text{lip}} \alpha/2$.

Write $\ell_{x, \theta} := \pi_{\theta}^{-1}\{\pi_{\theta}(x)\}$ for $\theta \in [0, \pi)$, so that $\ell_{0, \theta} = \text{span}(\cos \theta, \sin \theta)^\perp$. Let $J(\beta) \subset [0, \pi)$ be the set of directions in the cone $C_\beta$, i.e.,

$$J(\beta) = \{\theta \in [0, \pi) : \ell_{0, \theta} \subset C_\beta\} = \{\theta \in [0, \pi) : \text{span}(\cos \theta, \sin \theta)^\perp \subset C_\beta\}.$$

If $\ell$ is a line, we let $\ell(w)$ denote the tube that is the $w$-neighborhood of $\ell$. For a tube $T = \ell(w)$, we denote $w(T) = w$.

To prove Lemma 4.1, we use the Besicovitch alternative:

**Lemma 4.3 (The Besicovitch alternative).** Let $E \subset \mathbb{R}^2$ and $\beta \leq 1$. Then for all $x \in E$ and $H \geq 1$, at least one of the following two alternatives holds:

(A1) There exists a set $I_x \subset J(\beta)$ of measure $\mathcal{H}^1(I_x) \geq H^{-1}$ such that

$$\#(E \cap \ell_{x, \theta}) \geq 2, \quad \theta \in I_x.$$

(A2) There exists a set $J_x \subset J(\beta)$ of measure $\mathcal{H}^1(J_x) \gtrsim H^{-1}$ and the following property: for every $\theta \in J_x$, there is a tube $T = T_{x, \theta} = \ell_{x, \theta}(w(T))$ centred around $\ell_{x, \theta}$ such that

$$\mathcal{H}^1(E \cap T) \gtrsim \Theta^+_E(x) \cdot H \cdot w(T).$$

This alternative is part of Besicovitch’s original argument [2] for the Besicovitch projection theorem. For a more recent presentation, see [10, p. 86-87]. We include the details for completeness.
Proof of Lemma 4.3. Let $E, x, \beta, H$ be as in the statement of the lemma. Let $\varepsilon := \Theta_{E, \beta}^*(x)$, so that there exists an $r > 0$ such that $\mathcal{H}^1(C_\beta(x, r) \cap E) \geq \varepsilon r$. We set also $J := J(\beta)$.

If the alternative (A1) fails, then
$$\mathcal{H}^1(\{\theta \in J : \#(C_\beta(x, r) \cap E \cap \ell_{x, \theta}) \geq 2\}) \leq H^{-1}.$$ 
Since evidently $x \in C_\beta(x, r) \cap E \cap \ell_{x, \beta}$, this implies that most of the lines $\ell_{x, \theta}$ do not intersect the set $C_\beta(x, r) \cap E$ outside $x$. Consequently, $C_\beta(x, r) \cap E$ is contained in a union of narrow cones $C_1, C_2, \ldots$ which are centred around certain lines $\ell_{x, \theta_j}$ with $\theta_j \in J$, and whose opening angles $\beta_1, \beta_2, \ldots$ satisfy $\sum \beta_j \leq 2H^{-1}$. We may arrange that the cones have the form
$$C_j := C(I_j) := \cup \{\ell_{x, \theta} : \theta \in I_j\},$$
where $I_j \subset J$ is a dyadic interval, $|I_j| = \beta_j$, and $\theta_j \in J$ is the midpoint of $I_j$. We may also assume that the dyadic intervals $I_j$ are disjoint, so the sets $C_j \setminus \{x\}$ are disjoint.

To use these cones to arrive at alternative (A2), recall that $E, \beta$ also satisfied an upper bound roughly matching the lower bound in (4.4). If we knew this, then we could estimate $\mathcal{H}^1(\{\theta \in J : \#(C_\beta(x, r) \cap E \cap \ell_{x, \theta}) \geq 2\}) \leq H^{-1}$. This would be easy if the heavy cones also satisfied an upper bound roughly matching the lower bound in (4.4). If we knew this, then we could estimate
$$\sum_{j \in \mathbb{N}} |I_j| \geq (\varepsilon H r)^{-1} \sum_{j \in \mathbb{N}} \mathcal{H}^1(C_j \cap B(x, r) \cap E) \geq H^{-1}. \tag{4.5}$$
This desired upper bound in (4.4) need not be true to begin with, but can be easily arranged. Fix a heavy cone $C(I_j)$, and perform the following stopping time argument: the dyadic interval $I_j$ is successively replaced by its parent “$\hat{I}_j$” until either the upper bound
$$\mathcal{H}^1(C(\hat{I}_j) \cap B(x, r) \cap E) \leq \varepsilon H |\hat{I}_j| \cdot r \tag{4.6}$$
holds, or then $\hat{I}_j = J$. This procedure gives rise to a new collection of cones $C(\hat{I}_j)$ which are evidently still heavy, and whose union covers the union of the initial heavy cones. Since the intervals $\hat{I}_j$ are dyadic, we may arrange that the new heavy cones are disjoint outside $\{x\}$ without violating the previous two properties.

At this point, either $\hat{I}_j = J$ for some index $j$, in which case (4.5) is trivially true (using $|J| \sim 1$), or then the upper bound (4.6) holds for all the heavy cones. In this case the lower bound (4.5) holds by the very calculation shown in (4.5).

We are now fully equipped to establish alternative (A2). Consider a line $\ell_{x, \theta}$ contained in the union of the heavy cones. According to (4.5), the set of angles $\theta \in J$ of such lines has length $\gtrsim H^{-1}$. This set of angles is the set $J_x \subset J$ whose existence is claimed in (A2). It remains to associate the tube $T_{x, \theta}$ to each line $\ell_{x, \theta}$ with $\theta \in J_x$. Let $C(I_j) = C_j \supset \ell_{x, \theta}$ be
the (unique) heavy cone containing $\ell_{x,\theta}$. The opening angle of $C_j$ is $|I_j| \in (0, |J|]$, and it follows by elementary geometry that
\[ C_j \cap B(x,r) \subset \ell_{x,\theta}(2|I_j|r) =: T_{x,\theta}. \]
Finally,
\[ \mathcal{H}^1(E \cap T_{x,\theta}) \geq \mathcal{H}^1(C_j \cap B(x,r) \cap E) \geq \varepsilon H_{\beta_j} \cdot r \sim \varepsilon H \cdot w(T), \]
as claimed in alternative (A2).

**Proof of Lemma 4.1.** The main geometric observation is the following: every minigraph in $\mathcal{E}$ is an $\alpha^{-1}$-Lipschitz graph over every line $L_\theta := \text{span}(\cos \theta, \sin \theta) = \ell_{0,\theta}^\perp$ with $\theta \in J(\alpha')$ (recall that $\alpha' = C_{\alpha'} \alpha/2$). This is simply because the minigraphs in $\mathcal{E}$ are $\alpha$-Lipschitz graphs over the $x$-axis, but for all $\theta \in J(\alpha')$, the lines $L_\theta$ form an angle $\geq \alpha$ with the $y$-axis. Thus, $E$ is a union of finitely many $\alpha^{-1}$-Lipschitz graphs over $L_\theta$, for every $\theta \in J(\alpha')$. This places us in a position to use the area formula (2.8): for every $\theta \in J(\alpha')$ and every $\mathcal{H}^1$ measurable subset $E' \subset E$ we have
\[ \int_{\pi_\theta(E')} \#(E' \cap \pi_\theta^{-1}\{t\}) \, dt \geq \alpha \mathcal{H}^1(E'). \tag{4.7} \]
Let
\[ R = \{ x \in E : \Theta_{E,\alpha'}^+(x) \geq \varepsilon \}. \]
Fix $H \geq 1$. (We will eventually choose $H \sim 1/(\alpha \varepsilon)$; see (4.16) below.) By Lemma 4.3 (with $\beta = \alpha'$), we can write $R = R_1 \cup R_2$, where alternative (A1) holds on $R_1$ and (A2) holds on $R_2$. To prove (4.2), it suffices to show
\[ \mathcal{H}^1(R_i) \lesssim \frac{\delta}{\varepsilon \alpha^2} \quad \text{for } i = 1, 2. \tag{4.8} \]

We first consider $R_1$. Recall the sets $I_x \subset J(\alpha')$ defined in (A1). Since $E$ is a union of finitely many compact Lipschitz graphs, there are no measurability issues, and we may freely use Fubini’s theorem:
\[ H^{-1} \mathcal{H}^1(R_1) \leq \int_{R_1} \mathcal{H}^1(I_x) \, d\mathcal{H}^1(x) = \int_{J(\alpha')} \mathcal{H}^1(\{x \in R_1 : \theta \in I_x\}) \, d\theta. \tag{4.9} \]
For $\theta \in J(\alpha')$ fixed, abbreviate $R'_\theta := \{ x \in R_1 : \theta \in I_x \}$. Write also
\[ E'_\theta := \bigcup_{t \in \pi_\theta(R'_\theta)} (E \cap \pi_\theta^{-1}\{t\}), \]
so certainly $R'_\theta \subset E'_\theta$. Note that if $t \in \pi_\theta(E'_\theta)$, then $t = \pi_\theta(x)$ for some $x \in R'_\theta$. Thus $\theta \in I_x$ by definition, so
\[ \#(E'_\theta \cap \pi_\theta^{-1}\{t\}) = \#(E \cap \ell_{x,\theta}) \geq 2. \]

Therefore
\[ \#(E'_\theta \cap \pi_\theta^{-1}\{t\}) - 1 \sim \#(E'_\theta \cap \pi_\theta^{-1}\{t\}), \quad t \in \pi_\theta(E'_\theta). \tag{4.10} \]

We now may deduce from (4.7) applied to $E' := E'_\theta$, and (4.10), that
\[ \int_{\pi_\theta(E'_\theta)} \#(E'_\theta \cap \pi_\theta^{-1}\{t\}) \, dt \sim \int_{\pi_\theta(E'_\theta)} \#(E'_\theta \cap \pi_\theta^{-1}\{t\}) \, dt \gtrsim \alpha \mathcal{H}^1(E'_\theta) \gtrsim \alpha \mathcal{H}^1(R'_\theta), \]
and finally
\[
\int_{\mathcal{L}(E)} \#(E \cap \ell) - 1 \, d\eta(\ell) \geq \int_{J(\alpha')} \#(E^\alpha_\theta \cap \pi^{-1}_\theta \{t\}) - 1 \, dt \, d\theta \quad (4.9)
\]
By (3.4) the left hand side is bounded from above by \(\delta\), so
\[
H^1(R_1) \lesssim \frac{\delta H}{\alpha}. \quad (4.11)
\]
Recalling that we promised to choose \(H \sim 1/(\alpha \varepsilon)\) in the end, the bound above implies (4.8) for \(R_1\).

Next, we tackle \(R_2\). This time we define \(R'_\theta := \{x \in R_2 : \theta \in J_x\} \subset E\), and we deduce exactly as in (4.9) that
\[
H^{-1}H^1(R_2) \lesssim \int_{J(\alpha')} H^1(R'_\theta) \, d\theta. \quad (4.12)
\]
Fix \(\theta \in J(\alpha')\) with \(R'_\theta \neq \emptyset\). For each \(x \in R'_\theta\), by definition, there exists a tube \(T = T_{x,\theta}\) centred around \(\ell_{x,\theta}\) with the property
\[
H^1(E \cap T) \gtrsim \varepsilon H \cdot w(T). \quad (4.13)
\]
The tubes \(\{T_{x,\theta} : x \in R'_\theta\}\) may overlap, but they are all parallel. It follows from an application of the Besicovitch covering theorem (to the projections \(I_{x,\theta} := \pi_\theta(T_{x,\theta}) \subset \mathbb{R}\)) that there exists a countable sub-collection \(T_0 \subset \{T_{x,\theta} : x \in R'_\theta\}\), with the properties
\[
R'_\theta \subset \bigcup_{x \in R'_\theta} T_{x,\theta} \subset \bigcup_{T \in T_0} T \quad \text{and} \quad \sum_{T \in T_0} 1_T \leq 1. \quad (4.14)
\]
Fix \(T \in T_0\), and let \(\text{Bad}(E \cap T) \subset E \cap T\) consist of those points \(x \in E \cap T\) with \(\#(\ell_{x,\theta} \cap E) = 1\). We apply the coarea formula (2.8) to the set \(A := \text{Bad}(E \cap T) \subset E\). Recalling that for every \(\theta \in J(\alpha')\) the set \(E\) is a union of finitely many \(\alpha^{-1}\)-Lipschitz graphs over \(L_\theta\) (see remark above (4.7)) we get that
\[
H^1(\text{Bad}(E \cap T)) \leq \frac{1}{\alpha} \int_{\pi_\theta(T)} 1 \, dt = \frac{w(T)}{\alpha}. \quad (4.15)
\]
Now, for a suitable choice \(H \sim 1/(\alpha \varepsilon)\), a combination of (4.13) and (4.15) shows that
\[
H^1((E \cap T) \setminus \text{Bad}(E \cap T)) \geq \frac{1}{2} H^1(E \cap T). \quad (4.16)
\]
At this point, we simplify notation by setting
\[
E_\theta := \bigcup_{T \in T_0} (E \cap T) \setminus \text{Bad}(E \cap T) \subset E.
\]
By the definition of the sets \(\text{Bad}(E \cap T)\), if \(x \in E_\theta\), then \(\#(E \cap \ell_{x,\theta}) \geq 2\), and therefore
\[
\#(E \cap \pi^{-1}_\theta \{t\}) - 1 \sim \#(E \cap \pi^{-1}_\theta \{t\}) \geq \#(E_\theta \cap \pi^{-1}_\theta \{t\}), \quad t \in \pi_\theta(E_\theta). \quad (4.17)
\]
It follows that
\[
\int_{E(E)} \#(E \cap \ell) - 1 \, d\eta(\ell) \geq \int_{J(\alpha')} \int_{\pi_\alpha(E_\theta)} \#(E_\theta \cap \pi_\theta^{-1}\{t\}) - 1 \, dt \, d\theta
\]
(4.17)
\[
\geq \int_{J(\alpha')} \sum_{T \in T_\theta} \int_{\pi_\theta(E_\theta \cap T)} \#(E_\theta \cap \pi_\theta^{-1}\{t\}) \, dt \, d\theta
\]
(4.14)
\[
\geq \alpha \int_{J(\alpha')} \mathcal{H}^1(E_\theta \cap T) \, d\theta
\]
(4.16)
\[
\geq \frac{\alpha}{2} \int_{J(\alpha')} \sum_{T \in T_\theta} \mathcal{H}^1(E \cap T) \, d\theta
\]
(4.14)
\[
\geq \alpha \int_{J(\alpha')} \mathcal{H}^1(R_\theta) \, d\theta \quad (4.12)
\]
(4.14)
\[
\geq \alpha \frac{1}{2} \mathcal{H}^1(R_\theta) \geq \frac{\alpha}{H} \cdot \mathcal{H}^1(R_2).
\]
(4.16)

Recalling once again from (3.4) that the left hand side above is \(\leq \delta\), we deduce that
\[
\mathcal{H}^1(R_2) \leq \frac{\delta H}{\alpha} \sim \frac{\delta}{\varepsilon \alpha^2},
\]
which is (4.8) for \(R_2\). The proof of Lemma 4.1 is complete. \(\square\)

Next, repeating the classical "two cones" argument of Besicovitch, we show that we can pack most of points of low conical density into a single Lipschitz graph.

**Lemma 4.18** (Most low conical density points fit into a Lipschitz graph). Let \(E \subset B(1) \subset \mathbb{R}^2\) and let \(\varepsilon \in (0, 1)\), \(\beta \in (0, \frac{1}{2})\). Then, there exists a \(2\beta\)-Lipschitz graph \(\Gamma \subset \mathbb{R}^2\) over the \(x\)-axis such that
\[
\mathcal{H}^1(\{x \in E : \Theta_{E,\beta}^*(x) \leq \varepsilon\} \setminus \Gamma) \lesssim \varepsilon / \beta.
\]

**Proof.** Let \(G = \{x \in E : \Theta_{E,\beta}^*(x) \leq \varepsilon\}\). Our task is to find a subset \(\Gamma \subset G\) with \(\mathcal{H}^1(G \setminus \Gamma) \lesssim \varepsilon / \beta\) and the property \(C_{2\beta}(x) \cap \Gamma = \{x\}\) for all \(x \in \Gamma\). Then \(\Gamma\) extends to a \(2\beta\)-Lipschitz graph, as desired.

Let \(B\) be the set of points \(x \in G\) with the "bad" property that there exists a point \(y \in G \cap C_{2\beta}(x)\) with \(y \neq x\). The goal is to show that \(\mathcal{H}^1(B) \lesssim \varepsilon / \beta\). For each \(x \in B\), let \(r(x) = \sup\{|x - y| : y \in G \cap C_{2\beta}(x)\}\), so
\[
B \cap C_{2\beta}(x) \subset B(x, r(x)), \quad x \in B.
\]
(4.19)

See Figure 1 for an illustration.

Let \(T_x\) be the tube around the vertical line passing through \(x\) with \(w(T_x) := \frac{1}{10} \beta r(x)\). Then
\[
T_x \setminus B(x, \frac{1}{2} \beta r(x)) \subset C_1(x) \subset C_{2\beta}(x) \subset C_\beta(x).
\]
(4.20)
(Recall that $2\beta \leq 1$.) In particular, (4.20) implies $T_x \setminus B(x, r(x)) \subset C_{2\beta}(x)$. Using this corollary, we observe that

$$B \cap T_x \subset B(x, r(x)) \cup [(B \cap T_x) \setminus B(x, r(x))]$$

$$= B(x, r(x)) \cup [B \cap (T_x \setminus B(x, r(x)))]$$

$$\subset B(x, r(x)) \cup [B \cap C_{2\beta}(x)] \subset B(x, r(x)). \quad (4.21)$$

Choose a point $y(x) \in G \cap C_{2\beta}(x)$ such that $|x - y(x)| \geq \frac{9}{10} r(x)$. A slightly more delicate geometric fact is that $T_x \subset C_{\beta}(x) \cup C_{\beta}(y(x))$. This is an exercise in elementary geometry, see Figure 1 (or the proof in [18, Lemma 15.14] for a more formal argument): the disc $B(\frac{1}{2} \beta r(x))$, and in particular the intersection $T_x \cap B(\frac{1}{2} \beta r(x))$, is contained in the cone $C_{\beta}(y(x))$, whereas the rest of $T_x$ is contained in $C_{\beta}(x)$, as already noted in (4.20). Consequently, using (4.21), the trivial inclusion $B(x, r(x)) \subset B(y(x), 2r(x))$, and $x, y(x) \in G$, we have

$$\mathcal{H}^1(B \cap T_x) \leq \mathcal{H}^1(C_{\beta}(y(x), 2r(x)) \cap E) + \mathcal{H}^1(C_{\beta}(x, r(x)) \cap E)$$

$$\leq 2r(x) + \varepsilon r(x) \leq 30(\varepsilon/\beta) \cdot w(T_x).$$

We have now shown that every point $x \in B$ is contained on the central line of a vertical tube $T_x$ satisfying the estimate above. By the Besicovitch covering theorem, as in the proof of Lemma 4.1, we may then find a countable, boundedly overlapping sub-family $T$ of these tubes which still cover $B$. All the tubes intersect $B(1) \supset B$, so $\sum_{T \in T} w(T) \lesssim 1$. It follows that

$$\mathcal{H}^1(B) \leq \sum_{T \in T} \mathcal{H}^1(B \cap T) \leq \frac{30\varepsilon}{\beta} \sum_{T \in T} w(T) \lesssim \frac{\varepsilon}{\beta}.$$ 

This completes the proof of Lemma 4.18. \qed

We are then ready to prove Proposition 3.3:

**Proof of Proposition 3.3.** Fix $\varepsilon > 0$ as in the statement of the proposition, and set $\alpha' = C_{\text{lip}}\alpha/2$. Define $\varepsilon_1 := \alpha \varepsilon/C$ for a suitable absolute constant $C > 0$. By Lemma 4.1 applied to $\varepsilon = \varepsilon_1$, we know that the set $R \subset E$ of bad points $x \in E$ with

$$\Theta_{E, \alpha'}^\varepsilon(x) \geq \varepsilon_1$$
satisfies
\[ H^1(R) \lesssim \delta \cdot \epsilon_1^{-1} \alpha^{-2} = C \delta \cdot \epsilon_1^{-1} \alpha^{-3}. \]
Since \( \delta \leq C_0 \epsilon^2 \alpha^3 \), taking \( C_0 = C^{-2} \) gives \( H^1(R) \leq \epsilon/2 \) (assuming that \( C > 0 \) was large enough).

The set \( G := E \setminus R \) satisfies the hypotheses of Lemma 4.18 (with \( \beta = \alpha' = C_{\text{lip}} \alpha/2 \) and \( \epsilon = \epsilon_1 \)), so there exists a \( C_{\text{lip}} \alpha \)-Lipschitz graph \( \Gamma \subset \mathbb{R} \) over the \( x \)-axis such that \( H^1(G \setminus \Gamma) \leq \epsilon_1/\alpha = \epsilon/C \). If the constant \( C > 0 \) was chosen large enough, we see that
\[ H^1(E \setminus \Gamma) \leq H^1(R) + H^1(G \setminus \Gamma) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \]
This concludes the proof of Proposition 3.3.

5. PROOF OF PROPOSITION 3.10

In this section we prove Proposition 3.10. Recall that we are assuming to be in "Case 2"; that is, \( E \) cannot be exhausted, up to measure \( \epsilon \), by a a constant number of consecutive sets \( F_k, F_{k+1}, \ldots, F_{k+C_{\text{sep}}} \) (recall this notation from Subsection 3.3). More precisely, this meant that
\[ H^1(E \setminus (F_k \cup \ldots \cup F_{k+C_{\text{sep}}})) \leq \epsilon. \] (5.1)
fails for every \( k \); thus we found an index pair \( k, l \) with \( |k - l| \geq C_{\text{sep}} \) such that
\[ H^1(F_k) \geq \alpha^{2\kappa} \quad \text{and} \quad H^1(F_l) \geq \alpha^{2\kappa}. \] (5.2)
Recall that all the minigraphs in \( F_k \) make an angle \( \leq \alpha^\kappa \) with \( L_k := \text{span}(w_k) = \text{span}(\cos(k\pi/M_3), \sin(k\pi/M_3)) \)
and similarly all the minigraphs in \( F_l \) make an angle \( \leq \alpha^\kappa \) with \( L_l = \text{span}(w_l) \).

![Figure 2. A configuration where positively many lines hit E twice.](image)

The existence of \( F_k \) and \( F_l \) will imply a configuration such as the one depicted in Figure 2. A more precise definition is given in the lemma below.

**Lemma 5.3.** If (5.2) holds, then there exists an absolute constant \( C \sim C_{\text{lip}} \) (the constant from Proposition 3.3) such that the following objects exist:

1. Affine lines \( \ell_k \) and \( \ell_l \) with \( \angle(\ell_k, L_k) \leq \alpha^\kappa \) and \( \angle(\ell_l, L_l) \leq \alpha^\kappa \).
2. Tubes \( T_k := \ell_k(C\alpha) \) and \( T_k := \ell_k(\alpha^{1/2}) \).
3. Tubes \( T_l := \ell_l(C\alpha) \) and \( T_l := \ell_l(\alpha^{1/2}) \).
4. \( C_{\text{lip}} \alpha \)-Lipschitz graphs \( \gamma_k, \gamma_l \) over the lines \( \ell_k, \ell_l \), respectively such that
   \[ \gamma_k \cap B(1) \subset T_k' \quad \text{and} \quad \gamma_l \cap B(1) \subset T_l'. \]
Indeed, if $t$ with this notation, we claim that $G \subset \mathbb{R}^2$ and $G_i \subset \mathbb{R}^2$ for all $i \in \mathcal{L}(G_k, G_i)$.

The key geometric observation is the following: if $\ell \cap G_k \neq \emptyset$ and $\ell \cap G_i = \emptyset$ for all $\ell \in \mathcal{L}(G_k, G_i)$.

Proof. $\ell \cap G_k \neq \emptyset$ and $\ell \cap G_i = \emptyset$ for all $\ell \in \mathcal{L}(G_k, G_i)$.

Lemma 5.5. There exists a set of lines $\mathcal{L}(G_k, G_i)$ of measure $\eta(\mathcal{L}(G_k, G_i)) \geq \alpha^7$ such that $\ell \cap G_k \neq \emptyset$ and $\ell \cap G_i = \emptyset$ for all $\ell \in \mathcal{L}(G_k, G_i)$.

Proposition 3.10 follows immediately by Lemma 5.5. We will next derive Lemma 5.5 from Lemma 5.3. (See Remark 5.10 and Appendix A for an alternative proof of Lemma 5.5.)

Proof. The key geometric observation is the following: if $\ell \subset \mathbb{R}^2$ is any line with $G_k \cap \ell \neq \emptyset \neq G_i \cap \ell$, then $\ell$ must make an angle $\alpha^{1/2}$ with both $\ell_k$ and $\ell_i$, see Figure 2: indeed, if for example $\angle(\ell, \ell_i) \ll \alpha^{1/2}$ and $\ell \cap G_i \neq \emptyset$, then $\ell \cap B(1) \subset T_i$, and hence $\ell \cap G_k = \emptyset$ by (5.4). It follows that both $\ell_k, \ell_i$ are $C\alpha^{-1/2}$-graphs over $\ell \perp$, for any line $\ell$ connecting $G_k$ and $G_i$.

But since $\gamma_k, \gamma_i$ were by definition $C_{\text{lip}}$-Lipschitz graphs over $\ell_k, \ell_i$, it follows that also $\gamma_k, \gamma_i$ are $C\alpha^{-1/2}$-Lipschitz graphs over $\ell \perp$ (assuming that $\alpha > 0$ is small enough).

To prove the lower bound (5.6), start by fixing $x \in G_i \subset \gamma_i$, recall that $\ell_{x, \theta} := \pi_{\theta}^{-1}(\pi_{\theta}(x))$, and consider the set of directions

$\Theta(x, G_k) := \{ \theta \in [0, \pi) : \ell_{x, \theta} \cap G_k \neq \emptyset \}$.

With this notation, we claim that

$$\mathcal{H}^1(\Theta(x, G_k)) \geq \alpha^{1/2}\mathcal{H}^1(G_k), \quad x \in G_i.$$  \hfill (5.7)

Indeed, if $\{B(\theta_j, r_j)\}_{j \in \mathbb{N}}$ is an arbitrary cover of $\Theta(x, G_k)$, then the tubes $\ell_{x, \theta_j}(C_j)$ cover $G_k$, where $C > 0$ is an absolute constant. This is because $G_k$ is covered by the cones $C_j := \bigcup(\ell_{x, \theta} : \theta \in B(\theta_j, r_j))$ by definition, and each intersection $G_k \cap C_j \subset B(1) \cap C_j$ is further covered by a tube of the form $\ell_{x, \theta_j}(C_j)$. Now recall that $\gamma_k \supset G_k$ is an $\alpha^{-1/2}$-Lipschitz graph over each line $\ell_{x, \theta_j}^\perp$; this gives

$$\alpha^{-1/2} \sum_{j \in \mathbb{N}} r_j \geq \sum_{j \in \mathbb{N}} \mathcal{H}^1(G_k \cap \ell_{x, \theta_j}(r_j)) \geq \mathcal{H}^1(G_k),$$

which implies (5.7).

We now infer from (5.7) and Fubini’s theorem that

$$\int_0^\pi \mathcal{H}^1([x \in G_i : \theta \in \Theta(x, G_k)]) d\theta$$

$$= \int_{G_i} \mathcal{H}^1(\Theta(x, G_k)) d\mathcal{H}^1(x) \geq \alpha^{1/2}\mathcal{H}^1(G_k)\mathcal{H}^1(G_i).$$  \hfill (5.8)
To proceed, write \( G_t(\theta) := \{ x \in G_t : \theta \in \Theta(x, G_t) \} \). We claim that
\[
\mathcal{H}^1(G_t(\theta)) \neq 0 \quad \Rightarrow \quad \mathcal{H}^1(\pi \Theta(G_t(\theta))) \gtrsim \alpha^{1/2} \mathcal{H}^1(G_t(\theta)), \quad \theta \in [0, \pi). \quad (5.9)
\]
This will complete the proof of the corollary, because (5.8) then implies
\[
\int_0^\pi \mathcal{H}^1(\pi \Theta(G_t(\theta))) d\theta \overset{(5.8)}{=} \alpha \mathcal{H}^1(G_k) \mathcal{H}^1(G_t) \overset{L.5,3}{\gtrsim} \alpha^7,
\]
and the left hand side above is a lower bound for \( \eta(\mathcal{L}(G_k, G_t)) \).

Finally, let us prove (5.9). If \( \mathcal{H}^1(G_t(\theta)) \neq 0 \), then \( \theta \in \Theta(x, \gamma_k) \) for at least one \( x \in G_t \), which means that \( \ell_{x, \theta} = \pi \theta^{-1}(\{ \pi \theta(x) \} \) intersects both \( G_k \) and \( G_t \). Thus, \( \gamma_k \) is a \( C \alpha^{-1/2} \)-Lipschitz graph over the line \( \ell_{x, \theta} \). Consequently, the relation \( \mathcal{H}^1(\pi \Theta(H)) \gtrsim \alpha^{1/2} \mathcal{H}^1(H) \) holds for all \( \mathcal{H}^1 \) measurable subsets \( H \subset \gamma_k \), in particular for \( H := G_t(\theta) \).

\[\square\]

Remark 5.10. In fact, we have an exact expression for \( \eta(\mathcal{L}(G_k, G_t)) \): \[
\eta(\mathcal{L}(G_k, G_t)) = \int_{G_k \times G_t} \frac{|\pi \theta(x_k, x_l)(\tau_k(x_k))| |\pi \theta(x_k, x_l)(\tau_l(x_l))|}{|x_k - x_l|} d(\mathcal{H}^1 \times \mathcal{H}^1)(x_k, x_l). \quad (5.11)
\]
In (5.11), \( \tau_k(x) \) denotes the unit tangent vector to \( \gamma_k \) at \( x \in \gamma_k \), and \( \tau_l(x) \) is defined similarly. For distinct \( x, x' \in \mathbb{R}^2 \), \( \theta(x, x') \) denotes the angle \( \theta \) such that \( \pi \theta(x) = \pi \theta(x') \).

Now we show how (5.11) implies Lemma 5.5. By the key geometric observation in the first paragraph of the proof of Lemma 5.5 and the fact that \( G_k, G_t \subset B(1) \), the integrand in (5.11) is \( \gtrsim \frac{\alpha^{1/2} \alpha^{1/2}}{1} = \alpha \). Thus, \( \eta(\mathcal{L}(G_k, G_t)) \gtrsim \alpha \mathcal{H}^1(G_k) \mathcal{H}^1(G_t) \gtrsim \alpha^7 \).

We state and prove a more general form of (5.11) in Appendix A.

The remainder of this section is devoted to constructing the objects listed in Lemma 5.3. This is based on the assumption (3.8), that is, \( \mathcal{H}^1(F_k) \gtrsim \alpha^{2k} \) and \( \mathcal{H}^1(F_l) \gtrsim \alpha^{2k} \). Recall also that \( F_k, F_l \) were the unions of the minigraphs in \( F_k \) and \( F_l \). The minigraphs in \( F_k \) make an angle \( \leq \alpha^k \) with \( L_k \), while the minigraphs in \( F_l \) make an angle \( \leq \alpha^k \) with \( L_l \). Furthermore, \( \angle(L_k, L_l) \geq C_{\text{sep}} \alpha^k \), so the minigraphs from \( F_k \) and \( F_l \) point in quantitatively different directions. We also recall that \( F_k \) (respectively \( F_l \)) can be expressed as a union of certain consecutive families \( \mathcal{E}_i \):
\[
F_k = \mathcal{E}_0 \cup \mathcal{E}_{s+1} \cup \ldots \cup \mathcal{E}_{s+m} \quad \text{and} \quad F_l = \mathcal{E}_t \cup \ldots \cup \mathcal{E}_{t+m}.
\]
(5.12)
Some of these families may be empty, but not all, according to (5.2). Of course
\[
m \gtrsim \alpha^{-1}, \quad (5.13)
\]
since there were no more than \( \alpha^{-1} \) of the families \( \mathcal{E}_j \) altogether.

5.1. Sketch of the proof. We now explain the proof strategy with a picture. In Figure 3, we have depicted the sets \( F_k \) and \( F_l \), which are roughly speaking \( \alpha^k \)-Lipschitz graphs over the lines \( L_k, L_l \) by Proposition 3.3 (details will follow). Both \( F_k \) and \( F_l \) are, moreover, tiled by \( \gtrsim \alpha^{-1} \) of the sets \( E_j \). Most of sets \( E_j \) are (individually) contained on \( \alpha \)-Lipschitz graphs \( \gamma_j \), by another application of Proposition 3.3. The red sets shown in Figure 3 illustrate sets of the form
\[
G_j = E_j \cap \gamma_j \cap B_j,
\]
where \( B_j \) is some ball of radius \( \alpha \) with the property that \( \mathcal{H}^1(G_j) \sim \alpha \mathcal{H}^1(E_j) \). Each \( G_j \) is contained in a tube \( T_j \) of width \( \alpha^{1/2} \) (or even a tube of width \( \alpha \), which was also required
in Lemma 5.3). So, picking \( G_k \subset F_k \) and \( G_l \subset F_l \) arbitrarily, we would satisfy all the points (1)-(5) in Lemma 5.3, except for the inclusions (5.4).

The problem is that if we pick \( G_k \subset F_k \) and \( G_l \subset F_l \) arbitrarily, the tube \( T_k \) associated with \( G_k \) might intersect \( G_l \), or vice versa, violating (5.4). To satisfy (5.4), we need to pick \( G_k, G_l \) in such a way that the \( G_k \)-tube avoids \( G_l \) and the \( G_l \)-tube avoids \( G_k \). To achieve this, we roughly choose 3 well-separated sets \( G_{l1}, G_{l2}, G_{l3} \subset F_l \) and 2 further well-separated sets \( G_{k1}, G_{k2} \subset F_k \).

Then, we use the "transversality" of the graphs \( F_k, F_l \) to deduce the following: each \( G_{k1} \)-tube can intersect at most one of the sets \( G_{lj} \), and vice versa. At this point, we may deduce from the pigeonhole principle that there must exists a pair \( (G_{k1}, G_{lj}) \) such that the \( G_{k1} \)-tube does not intersect \( G_{lj} \), and the \( G_{lj} \)-tube does not intersect \( G_{k1} \). Indeed, there are six pairs \( (G_{k1}, G_{lj}) \), but only five tubes. This will complete the proof.

5.2. Proof. We turn to the details. First, we apply Proposition 3.3 to the sets \( F_k, F_l \), each of which can be written as a finite union of \( \alpha\kappa \)-Lipschitz minigraphs over the lines \( L_k, L_l \), respectively. It follows from the choice of constants \( \delta = \epsilon^{10}/C_{\text{thm}} \) and \( \alpha = (\epsilon/C_{\text{alp}})^{10} \) made in Section 3.3 that \( \delta \ll \alpha^{5\kappa} \), assuming that \( C_{\text{thm}} \) is chosen sufficiently small compared to the absolute constant \( C_{\text{alp}} \). Writing \( \alpha^{5\kappa} = (\alpha^\kappa)^3 \alpha^{2\kappa} \), this means that the main hypothesis of Proposition 3.3 is valid with constants \( \delta \ll \alpha^{5\kappa} \) in place of \( \alpha \) and \( \epsilon \). It follows that there exist \( C_{\text{lip}}^\alpha \)-Lipschitz graphs \( \Gamma_k, \Gamma_l \) over \( L_k, L_l \), respectively, which cover most of \( F_k \) and \( F_l \) in the sense

\[
\mathcal{H}^1(F_k \setminus \Gamma_k) \leq \frac{1}{2} \alpha^{2\kappa} \leq \frac{1}{2} \mathcal{H}^1(F_k) \quad \text{and} \quad \mathcal{H}^1(F_l \setminus \Gamma_l) \leq \frac{1}{2} \mathcal{H}^1(F_l).
\]

We write \( F_k' := F_k \cap \Gamma_k \) and \( F_l' := F_l \cap \Gamma_l \). Next, recall from (5.12) that

\[
F_k = E_s \cup \ldots \cup E_{s+m} \quad \text{and} \quad F_l = E_t \cup \ldots \cup E_{t+m},
\]

and each \( E_j \) is a finite union of \( \alpha \)-Lipschitz minigraphs \( E_j \) over a certain line (which makes an angle \( \leq \alpha^\kappa \) with \( L_k \)). Applying Proposition 3.3 again, for each \( E_j \) with either \( j \in \{s, \ldots, s+m\} \) or \( j \in \{t, \ldots, t+m\} \), we find Lipschitz graphs \( \gamma_j \) with constant \( \leq C_{\text{lip}}^\alpha \).
and the property
\[ \mathcal{H}^1(E_j \setminus \gamma_j) \lesssim \alpha^2, \quad s \leq j \leq s + m \text{ or } t \leq j \leq t + m. \]

For this application of Proposition 3.3 to be legitimate, we need \( \delta \ll \alpha^3(\alpha^2)^2 = \alpha^7 \), which also follows from our choice of constants recalled above, taking \( C_{\text{lim}} \gg C_{\text{lip}}^{10} \). We write \( E_j' := E_j \cap \gamma_j \). With these choices, a major part of \( E_j' \) is covered by the union of the graphs \( \gamma_j \): indeed since \( F_k' \subset F_k \subset (E_s \cup \ldots \cup E_{s+m}) \), we have
\[ \mathcal{H}^1 \left( F_k' \setminus \bigcup_{j=1}^m E_{s+j}' \right) \leq \sum_{j=1}^m \mathcal{H}^1(E_{s+j} \setminus \gamma_{s+j}) \lesssim \sum_{j=1}^m \alpha^2 \lesssim \alpha. \]

Since \( \mathcal{H}^1(F_k') \gtrsim \mathcal{H}^1(F_k) \gtrsim \alpha^{2k} \), and \( \kappa = \frac{1}{10} \), we infer that at least half of \( F_k' \) is covered by the (subsets of) \( \alpha \)-Lipschitz graphs \( E_j' \) with \( s \leq j \leq s + m \). The same conclusion \textit{mutatis mutandis} holds for \( F_k' \) and the sets \( E_j' \) with \( t \leq j \leq t + m \). We finally redefine \( F_k := F_k' \cap \bigcup_{j=1}^m E_{s+j}' \) and \( F_i := F_i' \cap \bigcup_{j=1}^m E_{t+j}' \).

This should cause no confusion, since the original sets \( F_k, F_i \) will no longer be used. We list all the properties of \( F_k, F_i \) we will need in the sequel:
- \( F_k, F_i \subset E \) and \( \mathcal{H}^1(F_k) \gtrsim \alpha^{2k} \) and \( \mathcal{H}^1(F_i) \gtrsim \alpha^{2k} \) (compare with (3.8)),
- \( F_k \) is covered by the Lipschitz graph \( \Gamma_k \) over \( L_k \) with constant \( \leq C_{\text{lip}} \alpha^k \),
- \( F_i \) is covered by the Lipschitz graph \( \Gamma_i \) over \( L_i \) with constant \( \leq C_{\text{lip}} \alpha^k \),
- \( F_k \) is covered by the union of \( \lesssim \alpha^{-1} \) Lipschitz graphs \( \gamma_s, \ldots, \gamma_{s+m} \) with constant \( \leq C_{\text{lip}} \alpha \) over certain lines \( \ell_{s+j} \) making an angle \( \lesssim \alpha^k \) with \( L_k \),
- \( F_i \) is covered by the union of \( \lesssim \alpha^{-1} \) Lipschitz graphs \( \gamma_t, \ldots, \gamma_{t+m} \) with constant \( \leq C_{\text{lip}} \alpha \) over certain lines \( \ell_{t+j} \) making an angle \( \lesssim \alpha^k \) with \( L_i \).

We have now defined carefully the objects \( F_k, F_i \) in Figure 3. In defining the objects \( E_k \) and \( E_\beta \) in the same picture, there is the technical problem that the "initial" sets \( E_j \) need not be localised, as the picture suggests. This will be easily fixed by intersecting the initial sets \( E_j \) with balls. First, using that \( \mathcal{H}^1(F_k) \gtrsim \alpha^{2k} \), we choose two special points \( x_1, x_2 \in \tilde{F}_k \) with the properties
\[ |x_1 - x_2| \gtrsim \alpha^{2k} \quad \text{and} \quad \mathcal{H}^1(F_k \cap B(x_j, \alpha)) \gtrsim \alpha^2 \quad \text{for } j \in \{1, 2\}. \]

This can be arranged, because the set of points \( x \in F_k \) with \( \mathcal{H}^1(F_k \cap B(x, \alpha)) \lesssim \alpha^2 \) has total length at most \( \lesssim \alpha \ll \mathcal{H}^1(F_k) \). Thus, the admissible points for the second condition in (5.14) have total length \( \gtrsim \frac{1}{2} \mathcal{H}^1(F_k) \gtrsim \alpha^{2k} \). Then, to finish the selection, it remains to pick two of these points with separation \( \alpha^{2k} \): this is possible because \( F_k \) lies on a Lipschitz graph with constant \( \leq 1 \), so in particular \( \mathcal{H}^1(F_k \cap B(x, r)) \lesssim r \) for all \( r > 0 \).

Next, we move attention from \( F_k \) to \( F_i \). This time we pick 3 special points \( y_1, y_2, y_3 \in \tilde{F}_i \) with properties similar to those in (5.14):
\[ |y_i - y_j| \gtrsim \alpha^{2k} \quad \text{for } i \neq j \quad \text{and} \quad \mathcal{H}^1(F_i \cap B(y_j, \alpha)) \gtrsim \alpha^2 \quad \text{for } j \in \{1, 2, 3\}. \]

The details of the selection are the same as we have seen above.

Next, recall that both \( F_k \) and \( F_i \) can be written as a finite union of (subsets of) \( C_{\text{lip}} \alpha \)-Lipschitz graphs: the covering graphs for \( F_k \) were denoted \( \gamma_s, \ldots, \gamma_{s+m} \) and the covering graphs for \( F_i \) were denoted \( \gamma_t, \ldots, \gamma_{t+m} \), where \( m \lesssim \alpha^{-1} \). Since \( \mathcal{H}^1(F_k \cap B(x_1, \alpha)) \gtrsim \alpha^2 \),
at least one of the graphs $\gamma_s, \ldots, \gamma_{s+m}$ must have large intersection with $F_k \cap B(x_1, \alpha)$. We denote this graph by $\gamma_k^j$; then we have

$$\mathcal{H}^1(F_k \cap \gamma_k^j \cap B(x_1, \alpha)) \gtrsim \alpha^3. \quad (5.16)$$

We find similarly a graph $\gamma_k^j \in \{\gamma_s, \ldots, \gamma_{s+m}\}$ such that $\mathcal{H}^1(F_k \cap \gamma_k^j \cap B(x_2, \alpha)) \gtrsim \alpha^3$. Then, we also repeat the argument for the three balls $B(y_j, \alpha)$: we find three graphs $\gamma_1^j, \gamma_2^j, \gamma_3^j \in \{\gamma_t, \ldots, \gamma_{t+m}\}$ with the property

$$\mathcal{H}^1(F_i \cap B(y_j, \alpha) \cap \gamma_i^j) \gtrsim \alpha^3, \quad 1 \leq j \leq 3. \quad (5.17)$$

The sets

$$G_i^k := F_k \cap \gamma_i^k \cap B(x_1, \alpha), \quad i = 1, 2, \quad \text{and}$$

$$G_j^l := F_l \cap \gamma_j^l \cap B(y_j, \alpha), \quad j = 1, 2, 3$$

are the ones we informally discussed below Figure 3.

Next, we associate the lines and tubes (required by Lemma 5.3) to the sets $G_i^k, G_j^l$. We associate to each graph $\gamma_i^k$ or $\gamma_j^l$ an affine line $\ell_i^k$ or $\ell_j^l$ with the following properties:

- $\gamma_i^k$ is a $C_{\text{lip}}$-Lipschitz graph over $\ell_i^k$ for $i \in \{1, 2\}$,
- $\gamma_j^l$ is a $C_{\text{lip}}$-Lipschitz graph over $\ell_j^l$ for $j \in \{1, 2, 3\}$,
- The lines are chosen so that $G_i^k \subset \ell_i^k(C\alpha)$ for $i \in \{1, 2\}$ and $G_j^l \subset \ell_j^l(C\alpha)$ for $j \in \{1, 2, 3\}$,

where $C \sim C_{\text{lip}}$. We now define

$$(T_i^k)' := \ell_i^k(C\alpha) \quad \text{and} \quad T_i^k := \ell_i^k(\alpha^{1/2})$$

for $i \in \{1, 2\}$, and similarly

$$(T_j^l)' := \ell_j^l(C\alpha) \quad \text{and} \quad T_j^l := \ell_j^l(\alpha^{1/2})$$

for $j \in \{1, 2, 3\}$. Thus, $G_i^k \subset (T_i^k)' \subset T_i^k$ and $G_j^l \subset (T_j^l)' \subset T_j^l$. Since moreover $\mathcal{H}^1(G_i^k) \gtrsim \alpha^3$ and $\mathcal{H}^1(G_j^l) \gtrsim \alpha^3$ by (5.16)-(5.17), any pair $(G_i^k, G_j^l)$ (with associated lines and tubes) would now satisfy all the requirements of Lemma 5.3, except perhaps the inclusions (5.4).

We will now use the pigeonhole principle to show that at least one of the pairs $(G_i^k, G_j^l)$ also satisfies the inclusions (5.4). The main geometric observation is the following:

$$\text{diam}(T_i^k \cap \Gamma_i) \lesssim \alpha^{1/2-\kappa} \quad \text{and} \quad \text{diam}(T_j^l \cap \Gamma_k) \lesssim \alpha^{1/2-\kappa}. \quad (5.19)$$

The first inequality holds for $i \in \{1, 2\}$, the second for $j \in \{1, 2, 3\}$. The proof of (5.19)
is contained in Figure 4. Recall that \( T^k_i \) is an \( \alpha^{1/2} \)-tube around a certain line \( \ell^k_i \) with \( \angle(\ell^k_i, L_k) \leq \alpha^k \). On the other hand, \( \angle(L_k, L_t) \geq C_{sep} \alpha^k \), so also \( \angle(\ell^k_i, L_t) \geq (C_{sep} - 1) \alpha^k \).

Finally, \( \Gamma_t \) is a \( C_{lip} \alpha^k \)-Lipschitz graph over \( L_t \), so every tangent of \( \Gamma_t \) makes an angle \( \geq C_{sep} \alpha^k \) with \( \ell^k_i \), since we chose \( C_{sep} \) much larger than \( C_{lip} \) in Section 3.3. Thus \( \Gamma_t \) is an \( \alpha^{-k} \)-Lipschitz graph over \( (\ell^k_i)^\perp \). It follows that

\[
\text{diam}(T^k_i \cap \Gamma_t) \leq \mathcal{H}^1(T^k_i \cap \Gamma_t) \lesssim \alpha^{1/2 - \kappa}.
\]

Now that we have proved (5.19), recall from (5.15) the three balls \( B(y_j, \alpha) \), all of which were centred at \( y_j \in F_t \subset \Gamma_t \), and whose centres \( y_j \) had pairwise separation \( \gtrsim \alpha^{2k} \). Since \( \kappa = \frac{1}{15p} \), we have \( \alpha^{1/2 - \kappa} \lesssim \alpha^{2k} \) for \( \alpha > 0 \) small enough (or in other words assuming that the constant \( C_{lip} > 0 \) is chosen large enough), and therefore (5.19) implies that

\[
\#\{j \in \{1, 2, 3\} : T^k_i \cap B(y_j, \alpha) \neq \emptyset \} \leq 1, \quad i \in \{1, 2\}.
\]

(5.20)

By a similar argument,

\[
\#\{i \in \{1, 2\} : T^k_i \cap B(x_i, \alpha) \neq \emptyset \} \leq 1, \quad j \in \{1, 2, 3\}.
\]

(5.21)

We finally claim, as a consequence of (5.20)-(5.21) and the pigeonhole principle, that there exists a pair of balls \( (B(x_{i_0}, \alpha), B(y_{j_0}, \alpha)) \), for some \( i_0 \in \{1, 2\} \) and \( j_0 \in \{1, 2, 3\} \) with the property

\[
T^k_{i_0} \cap B(y_{j_0}, \alpha) = \emptyset \quad \text{and} \quad T^k_{j_0} \cap B(x_{i_0}, \alpha) = \emptyset.
\]

(5.22)

This, by definition, yields

\[
G^k_{i_0} \subset \mathcal{H}^1(B(x_{i_0}, \alpha) \setminus T^k_{j_0}) \quad \text{and} \quad G^k_{j_0} \subset \mathcal{H}^1(B(y_{j_0}, \alpha) \setminus T^k_{i_0}),
\]

which (combined with (5.18)) completes the proof of the inclusions (5.4), and Lemma 5.3.

To prove (5.22), consider the bi-partite graph with 5 vertices \( \{v_1, v_2\} \cup \{w_1, w_2, w_3\} \) and the following edge set.

- For \( i \in \{1, 2\} \) and \( j \in \{1, 2, 3\} \), the edge \( (v_i, w_j) \) is included if \( T^k_i \cap B(y_j, \alpha) \neq \emptyset \).
- For \( j \in \{1, 2, 3\} \) and \( i \in \{1, 2\} \), the edge \( (w_j, v_i) \) is included if \( T^k_j \cap B(x_i, \alpha) \neq \emptyset \).

Now, (5.20)-(5.21) can be restated as follows: for \( v_i \) fixed, there can be at most one edge \( (v_i, w_j) \), and for \( w_i \) fixed, there can be at most one edge \( (w_i, v_j) \). Thus, the edge set contains at most 5 edges. On the other hand, the product set \( \{v_1, v_2\} \times \{w_1, w_2, w_3\} \) contains 6 elements, so there must be a pair \( \{v_i, w_j\} \) so that neither \( (v_i, w_j) \) nor \( (w_j, v_i) \) lies in the edge set. This is equivalent to (5.22). This completes the proof of Lemma 5.3.

6. THE GRID EXAMPLE

In this section we provide an example showing that Theorem 1.1 is optimal in the sense that the assumption \( \text{Fav}(E) \geq \text{Fav}(L) - \delta \) cannot be relaxed to \( \text{Fav}(E) \geq \delta \).

Proposition 6.1. There exists an absolute constant \( \delta > 0 \) and a sequence of compact rectifiable sets \( E_n \subset [0,1]^2 \subset \mathbb{R}^2 \) such that:

1. \( \mathcal{H}^1(E_n) = 1 \),
2. \( \text{Fav}(E_n) \geq \delta \),
3. for any \( \alpha \in [2n^{-2}, 1) \) and any curve \( \Gamma \) with \( \mathcal{H}^1(\Gamma \cap E_n) \geq \alpha \) we have \( \mathcal{H}^1(\Gamma) \geq \alpha n \).

In particular, property (3) implies that if \( M \geq 1 \), then for any \( M \)-Lipschitz graph \( \Gamma \) we have \( \mathcal{H}^1(\Gamma \cap E_n) \lesssim M n^{-1} \).
We begin the construction. Fix an integer $n \geq 2$, and let $[n] := \{1, \ldots, n\}$. For any $j = (k, l) \in [n]^2$ set
\[ x_j = \left( \frac{k}{n + 1}, \frac{l}{n + 1} \right) \tag{6.2} \]
and
\[ B_j = B \left( x_j, \frac{1}{2\pi n^2} \right). \]
Note that $B_j \subset [0,1]^2$ and if $i, j \in [n]^2$, $i \neq j$, then
\[ \text{dist}(B_i, B_j) \geq \frac{1}{n + 1} - \frac{2}{2\pi n^2} \geq \frac{1}{2n}. \tag{6.3} \]
Define $S_j = \partial B_j$, and observe that $\mathcal{H}^1(S_j) = n^{-2}$.

We define the set $E_n$ as
\[ E_n := \bigcup_{j \in [n]^2} S_j. \]
Since $\mathcal{H}^1(S_j) = n^{-2}$, we have $\mathcal{H}^1(E_n) = 1$. This verifies property (1) for $E_n$. It is also clear that $E_n$ is compact and rectifiable.

Now we check property (3). We will use the following result.

**Lemma 6.4** (Lemma 3.7 from [23]). Any compact connected set $\Gamma \subset \mathbb{R}^2$ with $\mathcal{H}^1(\Gamma) < \infty$ can be parametrized with $\gamma : [0,1] \to \mathbb{R}^2$ such that $\gamma([0,1]) = \Gamma$ and $\text{Lip}(\gamma) \leq 32 \mathcal{H}^1(\Gamma)$.

**Lemma 6.5.** For any $\alpha \in [2n^{-2},1)$ and any curve $\Gamma$ with $\mathcal{H}^1(\Gamma \cap E_n) \geq \alpha$ we have $\mathcal{H}^1(\Gamma) \geq \alpha n$.

**Proof.** Suppose that $\alpha \in [2n^{-2},1)$ and let $\Gamma$ be a curve with $\mathcal{H}^1(\Gamma \cap E_n) \geq \alpha$. Since each circle $S_j$ comprising $E_n$ has length $n^{-2}$, we get that $\Gamma$ intersects at least $\alpha n^2$ different circles. Let $J_0 \subset [n]^2$ be the set of indices such that for $j \in J_0$ we have $\Gamma \cap S_j \neq \emptyset$, so that
\[ N := \#J_0 \geq \alpha n^2. \tag{6.6} \]
To estimate $\mathcal{H}^1(\Gamma)$, we are going to use (6.6) together with the fact that the circles $S_j$ are centered on a well-separated grid (6.2), (6.3). We provide the details below.

Let $\gamma$ be the parametrisation of the curve $\Gamma$ given by Lemma 6.4. Without loss of generality, we may assume that the curve $\Gamma$ begins and ends on $E_n$, i.e., $\gamma(0), \gamma(1) \in \Gamma \cap E_n$. For all $j \in J_0$ we choose a point $y_j \in \Gamma \cap S_j$, and let $t_j \in [0,1]$ be such that $\gamma(t_j) = y_j$ ($\gamma$ might be non-injective, in which case $t_j$ is non-unique, but in this case we pick $t_j$ arbitrarily among the admissible options). The only constraint we make on our choice of $\{y_j\}_{j \in J_0}$ is so that $\gamma(0), \gamma(1) \in \{y_j\}_{j \in J_0}$. For convenience, we relabel the points $t_j$ in “ascending order”: for all $i \in \{1, \ldots, N\}$ we set $t_i := t_j$ for some $j \in J_0$, in such a way that $t_1 < t_2 < \cdots < t_N$. We relabel in a similar way $y_j$ and $S_j$.

Recalling that the circles $S_j$ are centered on a grid (6.2), it follows from the separation property (6.3) that for any $i \in \{1, \ldots, N\}$
\[ \frac{1}{2n} \leq |y_{i+1} - y_i| = |\gamma(t_{i+1}) - \gamma(t_i)| \leq \text{Lip}(\gamma) \cdot |t_{i+1} - t_i| = \text{Lip}(\gamma) \cdot (t_{i+1} - t_i). \]

Summing over $i \in \{1, \ldots, N-1\}$ we get
\[ \frac{N-1}{2n} \leq \text{Lip}(\gamma) \cdot (t_N - t_1) \leq 32 \mathcal{H}^1(\Gamma) \cdot (t_N - t_1). \]
Since we we assumed $\gamma(0), \gamma(1) \in \{y_j\}_{j \in J_0}$, we get that $t_N = 1$ and $t_1 = 0$. Thus,
\[
32 \mathcal{H}^1(\Gamma) \geq \frac{N - 1}{2n} \geq \frac{\alpha n^2 - 1}{2n} \geq \frac{\alpha n}{4}.
\]
This completes the proof of the lemma. \hfill \Box

It remains to prove the property (2), that is, $\text{Fav}(E_n) \geq \delta$. Let
\[
G_n = \bigcup_{j \in [n]^2} B_j,
\]
so that $E_n = \partial G_n$. Note that $\text{Fav}(E_n) = \text{Fav}(G_n)$. We define an auxiliary measure
\[
\mu = \mu_n = \frac{1}{\mathcal{L}^2(G_n)} \mathcal{L}^2_{|G_n|}.
\]
Recall that the 1-energy of $\mu$ is defined as
\[
I_1(\mu) = \iint \frac{1}{|x-y|} d\mu(x) d\mu(y).
\]

**Lemma 6.7.** We have
\[
I_1(\mu) \lesssim 1.
\]
As a consequence,
\[
\text{Fav}(E_n) = \text{Fav}(G_n) \gtrsim 1. \tag{6.8}
\]

**Proof.** We write
\[
I_1(\mu) = \iint \frac{1}{|x-y|} d\mu(x) d\mu(y) = \sum_{i,j \in [n]^2} \int_{B_i} \int_{B_j} \frac{1}{|x-y|} d\mu(x) d\mu(y)
\]
\[
+ \sum_{i \in [n]^2} \int_{B_i} \int_{B_j} \frac{1}{|x-y|} d\mu(x) d\mu(y) + \sum_{i,j \in [n]^2, i \neq j} \int_{B_i} \int_{B_j} \frac{1}{|x-y|} d\mu(x) d\mu(y) = A_1 + A_2.
\]
To estimate $A_1$ we note that for any $i \in [n]^2$ and any fixed $x \in B_i$
\[
\int_{B_i} \frac{1}{|x-y|} d\mu(y) \leq \sum_{k = \lfloor \log_2 n^2 \rfloor}^{\infty} \int_{B(x, 2^{-k}) \setminus B(x, 2^{-k-1})} \frac{1}{|x-y|} d\mu(y)
\]
\[
\sim \sum_{k = \lfloor \log_2 n^2 \rfloor}^{\infty} 2^k \mu(B(x, 2^{-k}) \setminus B(x, 2^{-k-1})) \lesssim \frac{1}{\mathcal{L}^2(G_n)} \sum_{k = \lfloor \log_2 n^2 \rfloor}^{\infty} 2^k \mathcal{L}^2(B(x, 2^{-k}))
\]
\[
\sim n^2 \sum_{k = \lfloor \log_2 n^2 \rfloor}^{\infty} 2^k \cdot 2^{-2k} \sim 1.
\]
Hence,
\[
A_1 = \sum_{i \in [n]^2} \int_{B_i} \int_{B_i} \frac{1}{|x-y|} d\mu(x) d\mu(y) \lesssim \sum_{i \in [n]^2} \mu(B_i) = 1.
\]

We move on to estimating $A_2$. Let $Q_j$ denote the square centered at $x_j$ with sidelength $1/(n + 1)$. Note that $B_j \subset Q_j$, and the squares $Q_j, j \in [n]^2$ are pairwise disjoint. If $x \in B_i$
and $y \in B_j$, with $i \neq j$, then $|x - y| \sim \text{dist}(B_i, B_j) \sim |x - z|$ for any $z \in Q_j$. It follows that for a fixed $x \in B_i$,

$$
\int_{B_j} \frac{1}{|x - y|} \, d\mu(y) \sim \text{dist}(B_i, B_j)^{-1} \, \mu(B_j) \sim \text{dist}(B_i, B_j)^{-1} \, L^2(Q_j) \sim \int_{Q_j} \frac{1}{|x - z|} \, dL^2(z)
$$

Summing over $j \in [n]^2 \setminus \{i\}$ yields

$$
\sum_{j \in [n]^2 \setminus \{i\}} \int_{B_j} \frac{1}{|x - y|} \, d\mu(y) \sim \sum_{j \in [n]^2 \setminus \{i\}} \int_{Q_j} \frac{1}{|x - z|} \, dL^2(z) \leq \sum_{j = 1,2} \int_{|x - z| \leq 1} \frac{1}{|x - z|} \, dL^2(z)
$$

Thus,

$$
A_2 = \sum_{i \in [n]_2} \int_{B_i} \left( \sum_{j \in [n]^2 \setminus \{i\}} \int_{B_j} \frac{1}{|x - y|} \, d\mu(y) \right) \, d\mu(x) \leq \sum_{i \in [n]_2} \mu(B_i) = 1.
$$

It follows that $I_1(\mu) \lesssim 1$.

To see (6.8), we use Theorem 4.3 from [19] to conclude that

$$
\text{Fav}(E_n) = \text{Fav}(G_n) \gtrsim \frac{1}{I_1(\mu)} \gtrsim 1.
$$

This concludes the proof of Proposition 6.1. \qed

### APPENDIX A. LINES SPANNED BY RECTIFIABLE CURVES

We state and prove a generalization of (5.11), which was mentioned in Remark 5.10.

**Lemma A.1.** Let $\gamma_1, \gamma_2 \subset \mathbb{R}^2$ be rectifiable curves. For $\mathcal{H}^1$ almost every $x \in \gamma_i$, let $\tau_i(x)$ denote the unit tangent vector to $\gamma_i$ at $x$. (The choice of direction is irrelevant.) Then for any $G_1 \subset \gamma_1$ and $G_2 \subset \gamma_2$, we have

$$
\int_A \# \{(x_1, x_2) \in G_1 \times G_2 : x_1 \neq x_2 \text{ and } x_1, x_2 \in \ell \} \, d\eta(\ell) = \iint_{G_1 \times G_2} \frac{|\pi_\theta(x_1, x_2)(\tau_1(x_1))| |\pi_\theta(x_1, x_2)(\tau_2(x_2))|}{|x_1 - x_2|} \, d(\mathcal{H}^1 \times \mathcal{H}^1)(x_1, x_2)
$$

where $\theta(x_1, x_2)$ denotes the angle $\theta$ such that $\pi_\theta(x_1) = \pi_\theta(x_2)$.

**Proof.** Let $\phi_\ell(s)$ be a parametrization of $\gamma_i$ by arclength. Consider the map $\Psi : (s_1, s_2) \mapsto (\theta, t)$ defined implicitly by

$$
\pi_\theta(\phi_1(s_1)) = \pi_\theta(\phi_2(s_2)) = t. \tag{A.2}
$$

By the change of variables formula,

$$
\int_A \# \{(x_1, x_2) \in G_1 \times G_2 : x_1 \neq x_2 \text{ and } x_1, x_2 \in \ell \} \, d\eta(\ell) = \int_{[0,\pi] \times \mathbb{R}} \# \{(x_1, x_2) \in G_1 \times G_2 : x_1 \neq x_2 \text{ and } x_1, x_2 \in \pi_\theta^{-1}(t) \} \, d\mathcal{H}^2(\theta, t)
$$

$$
= \iint_{s_1 \in \phi_1^{-1}(G_1), s_2 \in \phi_2^{-1}(G_2)} J\Psi(s_1, s_2) \, ds_1 \, ds_2,
$$

where $J\Psi(s_1, s_2)$ is the Jacobian of the change of variables $\Psi$. The proof is completed by the formula for the Jacobian of the change of variables.
where $J\Psi$ denotes the Jacobian determinant of $\Psi$. (Note that the set \{(s_1, s_2) : \phi_1(s_1) = \phi_2(s_2)\} has $H^2$-measure zero.)

We now prove that
\[
J\Psi(s_1, s_2) := \left| \frac{\partial^2 \phi_2(s_1)}{\partial s_1 \partial s_2} - \frac{\partial^2 \phi_1(s_1)}{\partial s_1 \partial s_2} \right| = \frac{\left| \pi_{\theta(s_1, s_2)}(\gamma_1'(s_1)) \right|}{\left| \gamma_1(s_1) - \gamma_2(s_2) \right|}. \tag{A.3}
\]

Note that this would finish the proof of the lemma. To show (A.3), define $e_0 = (\cos \theta, \sin \theta)$ and $e_0^\perp = -e_0 = (-\sin \theta, \cos \theta)$. By differentiating (A.2) with respect to $s_1$ and $s_2$, we obtain
\[
e_0 \cdot \phi_1'(s_1) + e_0^\perp \cdot \phi_1(s_1) \hat{c}_{s_1} \theta = e_0 \cdot \phi_2(s_2) \hat{c}_{s_1} \theta = \hat{c}_{s_1} t,
\]
\[
e_0 \cdot \phi_2'(s_2) + e_0^\perp \cdot \phi_2(s_2) \hat{c}_{s_2} \theta = e_0 \cdot \phi_1(s_1) \hat{c}_{s_2} \theta = \hat{c}_{s_2} t.
\]
The two equalities on the left give
\[
|\hat{c}_{s_1} \theta| = \frac{|e_0 \cdot \phi_1'(s_1)|}{|e_0^\perp \cdot (\phi_1(s_1) - \phi_2(s_2))|} \quad \text{for } i = 1, 2
\]
which, when combined with the two equalities on the right, give
\[
J\Psi(s_1, s_2) = |\hat{c}_{s_1} \theta| |\hat{c}_{s_2} \theta| |\frac{e_0^\perp \cdot (\phi_1(s_1) - \phi_2(s_2))}{e_0 \cdot (\phi_1(s_1) - \phi_2(s_2))}| = \frac{|e_0 \cdot \phi_1'(s_1)| |e_0 \cdot \phi_2'(s_2)|}{|e_0^\perp \cdot (\phi_1(s_1) - \phi_2(s_2))|}.
\]

Finally, observe that $e_0 \cdot (\phi_1(s_1) - \phi_2(s_2)) = 0$ by the definition of $\Psi$, which implies $|e_0^\perp \cdot (\phi_1(s_1) - \phi_2(s_2))| = |\phi_1(s_1) - \phi_2(s_2)|$. This completes the proof of (A.3). \qed

References

[1] Michael Bateman and Alexander Volberg. An estimate from below for the Buffon needle probability of the four-corner Cantor set. Math. Res. Lett., 17(5):959–967, 2010.
[2] Abram S. Besicovitch. On the fundamental geometrical properties of linearly measurable plane sets of points (III). Math. Ann., 116(1):349–357, 1939.
[3] Matthew Bond, Izabella Laba, and Alexander Volberg. Buffon’s needle estimates for rational product Cantor sets. Amer. J. Math., 136(2):357–391, 2014.
[4] Matthew Bond and Alexander Volberg. Buffon needle lands in $c$-neighborhood of a 1-dimensional Sierpinski gasket with probability at most $|\log c|^{−c}$. C. R. Math. Acad. Sci. Paris, 348(11-12):653–656, 2010.
[5] Matthew Bond and Alexander Volberg. Buffon’s needle landing near Besicovitch irregular self-similar sets. Indiana Univ. Math. J., 61(6):2085–2109, 2012.
[6] Alan Chang and Xavier Tolsa. Analytic capacity and projections. J. Eur. Math. Soc. (JEMS), 22(12):4121–4159, 2020.
[7] Laura Cladek, Blair Davey, and Krystal Taylor. Upper and lower bounds on the rate of decay of the Favard curve length for the four-corner Cantor set. arXiv e-prints, page arXiv:2003.03620, March 2020.
[8] Blair Davey and Krystal Taylor. A Quantification of a Besicovitch Nonlinear Projection Theorem via Multiscale Analysis. arXiv e-prints, page arXiv:2104.00826, April 2021.
[9] Damian Dabrowski and Michele Villa. Analytic capacity and dimension of sets with plenty of big projections. In preparation.
[10] Kenneth J. Falconer. The geometry of fractal sets, volume 85 of Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 1986.
[11] Herbert Federer. The $(\varphi, k)$ rectifiable subsets of $n$-space. Trans. Amer. Math. Soc., 62:114–192, 1947.
[12] Herbert Federer. Geometric measure theory. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.
[13] Steven G. Krantz and Harold R. Parks. Geometric integration theory. Cornerstones. Birkhäuser Boston, Inc., Boston, MA, 2008.
[14] Izabella Laba. Recent progress on Favard length estimates for planar Cantor sets. In Operator-related function theory and time-frequency analysis, volume 9 of Abel Symp., pages 117–145. Springer, Cham, 2015.
[15] Izabella Łaba and Caleb Marshall. Vanishing sums of roots of unity and the Favard length of self-similar product sets. *arXiv e-prints*, page arXiv:2202.07555, February 2022.

[16] Izabella Łaba and Kelan Zhai. The Favard length of product Cantor sets. *Bull. Lond. Math. Soc.*, 42(6):997–1009, 2010.

[17] Henri Martikainen and Tuomas Orponen. Characterising the big pieces of Lipschitz graphs property using projections. *J. Eur. Math. Soc. (JEMS)*, 20(5):1055–1073, 2018.

[18] Pertti Mattila. *Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability*. 1st paperback ed. Cambridge: Cambridge University Press, 1st paperback ed. edition, 1999.

[19] Pertti Mattila. *Fourier Analysis and Hausdorff Dimension*. Cambridge University Press, Cambridge, England, UK, 2015.

[20] Fedor Nazarov, Yuval Peres, and Alexander Volberg. The power law for the Buffon needle probability of the four-corner Cantor set. *Algebra i Analiz*, 22(1):82–97, 2010.

[21] Tuomas Orponen. Plenty of big projections imply big pieces of Lipschitz graphs. *Invent. Math.*, 226(2):653–709, 2021.

[22] Yuval Peres and Boris Solomyak. How likely is Buffon’s needle to fall near a planar Cantor set? *Pacific J. Math.*, 204(2):473–496, 2002.

[23] Raanan Schul. Subsets of rectifiable curves in Hilbert space—the analyst’S TSP. *J. Anal. Math.*, 103(1):331–375, 2007.

[24] Terence Tao. A quantitative version of the Besicovitch projection theorem via multiscale analysis. *Proc. Lond. Math. Soc. (3)*, 98(3):559–584, 2009.

[25] Anatoli G. Vitushkin. Analytic capacity of sets in problems of approximation theory. *Uspehi Mat. Nauk*, 22(6 (138)):141–199, 1967.