ON AN ANISOTROPIC $p$-LAPLACE EQUATION WITH VARIABLE SINGULAR EXPONENT

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Abstract. In this article, we study the following anisotropic $p$-Laplace equation with variable exponent given by

$$
(P) \begin{cases}
-\Delta_{H,p} u = \frac{\lambda f(x)}{u^{q(x)}} + g(u) & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
$$

under the assumption $\Omega$ is a bounded smooth domain in $\mathbb{R}^N$ with $p, N \geq 2$, $\lambda > 0$ and $0 < q \in C(\Omega)$. For the purely singular case that is $g \equiv 0$, we proved existence and uniqueness of solution. We also demonstrate the existence of multiple solution to $(P)$ provided $f \equiv 1$ and $g(u) = u^r$ for $r \in (p-1, p^* - 1)$.

Key words: Anisotropic $p$-Laplace equation, Singular nonlinearity, Variable exponent, Approximation method, Variational approach.

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1. Introduction and Main results

In this article, we establish the existence of unique weak solution for the problem

$$
(1.1) \begin{cases}
-\Delta_{H,p} u = \frac{f(x)}{u^{q(x)}} & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
$$

and existence of at least two weak solutions for

$$
(1.2) \begin{cases}
-\Delta_{H,p} u = \frac{\lambda}{u^{q(x)}} + u^r & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
$$

where $\Omega$ is a bounded smooth domain in $\mathbb{R}^N$ with $N \geq 2$ and $0 < q \in C(\Omega)$. Our other important assumptions for $(1.1)$ are $2 \leq p < \infty$ and $f \in L^1(\Omega) \setminus \{0\}$ is a nonnegative function (see Theorem 1.4-1.5 for precise assumptions). For the problem $(1.2)$, we assume the parameter $\lambda$ to be positive and $r \in (p-1, p^* - 1)$, where $p^* = \frac{Np}{N-p}$ is the critical Sobolev exponent for $1 < p < N$ (refer Theorem 1.6-1.7). The key operator in our problems, that is the Finsler $p$-Laplacian is defined as

$$
\Delta_{H,p} u := \text{div}(H(\nabla u)^{p-1}\nabla\eta H(\nabla u)),
$$

where $\nabla\eta$ denotes the gradient operator with respect to the $\eta$ variable and $H : \mathbb{R}^N \to [0, \infty)$ is known as the Finsler-Minkowski norm satisfying the following hypothesis:

(H0) $H(x) \geq 0$, for every $x \in \mathbb{R}^N$.

(H1) $H(x) = 0$, if and only if $x = 0$. 

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(H2) $H(tx) = |t| H(x)$, for every $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$.
(H3) $H \in C^\infty(\mathbb{R}^N \setminus \{0\})$.
(H4) the Hessian matrix $\nabla^2 H(x)$ is positive definite for all $x \in \mathbb{R}^N \setminus \{0\}$.

The dual $H_0 : \mathbb{R}^N \to [0, \infty)$ of $H$ is defined by

$$H_0(\xi) := \sup_{x \in \mathbb{R}^N \setminus \{0\}} \frac{\langle x, \xi \rangle}{H(x)}.$$  

More details on this can be found in Bao-Chern-Shen [7], Xia [47] and Rockafellar [45].

Anisotropic $p$-Laplace equations has been a topic of considerable attention in the recent years. We refer to Alvino-Ferone-Trombetti-Lions [3], Belloni-Ferone-Kawohl [8], Belloni-Kawohl-Juutinen [9], Bianchini-Giulio [10], Xia [47], Cianchi-Salani [18], Ferone-Kawohl [25], Kawohl-Novaga [33] and the references therein, for detailed analysis of the operator and its important features. We present some remarks and examples now to give a little more insight on the anisotropic $p$-Laplace operator and the adjoined norm.

**Remark 1.1.** Since all norms in $\mathbb{R}^N$ are equivalent, there exist positive constants $C_1, C_2$ such that

$$C_1 |x| \leq H(x) \leq C_2 |x|, \quad \forall x \in \mathbb{R}^N.$$

We are able to define an equivalent norm on the Sobolev space $W^{1,p}_0(\Omega)$ as

$$\|u\|_X := \left( \int_{\Omega} H(\nabla u)^p \, dx \right)^{\frac{1}{p}}.$$  

**Examples:** Let $x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N$,

(i) then, for $t > 1$, we define

$$H_t(x) := \left( \sum_{i=1}^{N} |x_i|^t \right)^{\frac{1}{t}};$$

(ii) for $\lambda, \mu > 0$, we define

$$H_{\lambda,\mu}(x) := \sqrt{\lambda \sum_{i=1}^{N} x_i^4 + \mu \sum_{i=1}^{N} x_i^2}.$$  

Then, one may check that the functions $H_t, H_{\lambda,\mu} : \mathbb{R}^N \to [0, \infty)$ given by (1.5) and (1.6) satisfies all the hypothesis from $(H0) - (H4)$ or refer Mezei-Vas [36].

**Remark 1.2.** For $i = 1, 2$ if $\lambda_i, \mu_i$ are positive real numbers such that $\frac{\lambda_1}{\mu_1} \neq \frac{\lambda_2}{\mu_2}$, then $H_{\lambda_1,\mu_1}$ and $H_{\lambda_2,\mu_2}$ given by (1.6) defines two non-isometric norms in $\mathbb{R}^N$.

**Remark 1.3.** Moreover, for $H = H_t$ given by (1.5) we have

$$\Delta_{H_t} u = \begin{cases} \Delta_p u := \text{div}(\nabla u |\nabla u|^{p-2}\nabla u), \text{ (p-Laplacian)} & \text{if } t = 2, 1 < p < \infty, \\ S_p = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} (|u_i|^{p-2} u_i), & \text{if } t = p \in (1, \infty), \end{cases}$$

where $u_i := \frac{\partial u}{\partial x_i}$, for $i = 1, 2, \ldots, N$. 


This last remark puts an important touch to our problem stating that Finsler p-Laplacian generalizes the p-Laplacian. So our result are true for a general set of quasilinear operators and we strongly point out here that (1.1) and (1.2) are new even for Laplacian.

The equations (1.1) and (1.2) are singular in the sense that the nonlinearities in our consideration blow up near the origin, due to the presence of the term \( u^{-q(x)} \) where \( q \) is a positive function. The fundamental feature of our article is the singular variable exponent \( q : \Omega \to (0, \infty) \) which is a continuous function up to the boundary and it is positive. Due to the blow up phenomenon, critical point theory fails here and the standard technique of approximating the problem with a non singular set up comes to our rescue. Another striking feature is the singular exponent being a function here which covers both \( 0 < q(x) < 1 \) and \( q(x) \geq 1 \) cases together. In the \( 0 < q(x) < 1 \) for all \( x \in \Omega \), we may employ some well known techniques to derive the weak solution. But when \( q \) is not bounded by 1, under an additional assumption of restricting \( q \) in \((0, 1]\) near the boundary of \( \Omega \), we were able to show existence of unique weak solution to (1.1) i.e. purely singular case. For (1.2), fixing \( 0 < q(x) < 1 \) we show that the variational techniques using the Gateaux differentiability of corresponding energy functional can be advantageous to show existence of at least two weak solutions.

When \( q(x) = q \) is a fixed positive constant, singular problems has been investigated in many different contexts in the recent years. In particular, the p-Laplace equation

\[
-\Delta_p u = h(x, u) \quad \text{in } \Omega, \quad u > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,
\]

has been studied concerning the existence, uniqueness and multiplicity results widely for any \( 1 < p < \infty \), including both purely singular and perturbed singular nonlinearity \( h \).

In the purely singular case, we refer the reader to Crandall-Rabinowitz-Tartar [19], Boccardo-Orsina [13] for the semilinear case and De Cave [20], Canino-Sciunzi-Trombetta [15] for the quasilinear case respectively. Whereas for the perturbed singular case, one can look into Arcoya-Boccardo [4] for the semilinear case and Giacomoni-Schindler-Takac [31], Bal-Garain [5] for the quasilinear case respectively. One may also refer to Garain-Mukherjee [26,28] and Leggat-Miri [35], dos Santos-Figueiredo-Tavares [21], Bal-Garain [6,27] for the study of singular problems in the context of weighted and anisotropic p-Laplace operator respectively. Furthermore, we motivate readers to read Ghergu-Radulescu [30], Oliva-Orsina-Petitta [39–42] for an extensive study of singular elliptic problems.

As stated earlier, our main concern in this article is to investigate the existence results in the presence of a variable singular exponent \( q \). To the best of our knowledge, the first article to deal with such phenomenon was by Carmona-Aparicio [16], where the authors studied the following singular Laplace equation

\[
-\Delta u = u^{-q(x)} \quad \text{in } \Omega, \quad u > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.
\]

In addition to (1.9), for purely singular nonlinearity, Chu-Gao-Gao [17] studied a more general class of second order elliptic equation; Miri [37] studied anisotropic p-Laplace equations; Zhang [49] studied \( p(x) \)-Laplace equation accommodating variable singular exponent. Alves-Santos-Siqueira [2] obtained existence and uniqueness for a more general \( p(x) \)-Laplace equation. For the existence results of a system of \( p(x) \)-Laplace equation, we refer Alves-Moussaoui [1].
In the purturbed case, Byun-Ko [14] discussed multiplicity for a class of \( p(x) \)-Laplace equation. Papageorgiou-Scapellato [43] obtained existence results for a purely singular \((p(x), q(x))\)-Laplace equation. In the nonlocal setting, problem (1.9) has been investigated recently by Garain-Mukherjee in [29]. For an extensive literature of variable exponent problems, interested readers can go through Rădulescu-Repovš [46].

In the same context, we believe that Anisotropic \( p \)-Laplace equations with singular nonlinearities have been very less understood and considered. We refer to the recent articles by Bistriţeanu-Mohammed [11], Farkas-Winkert [24] and Farkas-Fiscella-Winkert [23] in this regard. We seek some ideas of approximation from Boccardo-Orsina [90]. On the other side, we obtain multiplicity results for the problem (1.2) using the variational approach from Arcoya-Boccardo [4].

**Notation:** We gather below all the notations that has been frequently used in our article-

- \( X := W^{1,p}_0(\Omega) \).
- For \( u \in X \), we denote by \( \|u\| \) to mean \( \|u\|_X \) which is defined by (1.4).
- For a given constant \( c \) and a set \( S \), by \( u \geq c \) in \( S \), we mean \( u \geq c \) almost everywhere in \( S \). Moreover, we write \( |S| \) to denote the Lebesgue measure of \( S \).
- \( \langle , \rangle \) denotes the standard inner product in \( \mathbb{R}^N \).
- The conjugate exponent of \( \theta > 1 \) is given by \( \theta' := \frac{\theta}{\theta - 1} \).
- For \( 1 < p < N \), we denote by \( p^* := \frac{Np}{N-p} \) to mean the critical Sobolev exponent.
- For \( a \in \mathbb{R} \), we denote by \( a^+ := \max\{a, 0\} \) and \( a_- := \min\{a, 0\} \).
- We write by \( c, C \) or \( C_i \) for \( i \in \mathbb{N} \) to mean a positive constant which may vary from line to line or even in the same line. If a constant \( C \) depends on \( r_1, r_2, \ldots \), we denote it by \( C(r_1, r_2, \ldots) \).

Now, we define the notion of weak solutions as follows.

**Weak solution:** We say that \( u \in X \) is a weak solution of the problem (1.1), if \( u > 0 \) in \( \Omega \) and for every \( \omega \Subset \Omega \), there exists a positive constant \( c_\omega \) such that \( u \geq c_\omega > 0 \) in \( \omega \), satisfying

\[
(1.10) \quad \int_\Omega H(\nabla u)^{p-1}\nabla \eta H(\nabla u) \nabla \phi \, dx = \int_\Omega \frac{f(x)}{u^q(x)} \phi \, dx, \quad \forall \phi \in C^1_c(\Omega).
\]

Analogously, we say that \( u \in X \) is a weak solution of the problem (1.2), if \( u > 0 \) in \( \Omega \) and for every \( \omega \Subset \Omega \), there exists a positive constant \( c_\omega \) such that \( u \geq c_\omega > 0 \) in \( \omega \), satisfying

\[
(1.11) \quad \int_\Omega H(\nabla u)^{p-1}\nabla \eta H(\nabla u) \nabla \phi \, dx = \int_\Omega \left( \frac{\lambda}{u^{q(x)}} + u^r \right) \phi \, dx, \quad \forall \phi \in C^1_c(\Omega).
\]

Our main results in this article reads includes the following.

**Theorem 1.4.** Let \( 2 \leq p < \infty \) and \( q \in C(\overline{\Omega}) \) be positive.

(a) **Uniqueness:** Then for any nonnegative \( f \in L^1(\Omega) \setminus \{\emptyset\} \), the problem (1.1) admits at most one weak solution in \( X \).

(b) **Existence:** Assume that there exists a constant \( \delta > 0 \) such that \( 0 < q(x) \leq 1 \) in \( \Omega_\delta \), where

\[
\Omega_\delta := \{ x \in \Omega : \text{dist}(x, \partial \Omega) < \delta \}.
\]
Then the problem \((1.1)\) admits a unique weak solution \(u \in X\), for any nonnegative \(f \in L^m(\Omega) \setminus \{0\}\), where

\[
m = \begin{cases} 
(p^*)', & \text{if } 2 \leq p < N, \\
> 1, & \text{if } p = N, \\
1, & \text{if } p > N.
\end{cases}
\]

In particular, we have the following existence and uniqueness result for any \(1 < p < \infty\).

**Theorem 1.5.** If

\[\Delta_{H,p} \equiv \Delta_p \text{ or } S_p,\]

as given by \((1.7)\), then Theorem 1.4 holds for any \(1 < p < \infty\).

**Theorem 1.6.** Let \(2 \leq p < N\) and \(q \in C(\overline{\Omega})\) be such that \(0 < q(x) < 1\) for all \(x \in \overline{\Omega}\). Then there exists \(\Lambda > 0\) such that for all \(\lambda \in (0, \Lambda)\), the problem \((1.2)\) admits at least two distinct weak solutions in \(X\), for any \(r \in (p-1, p^* - 1)\).

In particular, we have the following multiplicity result for any \(1 < p < N\).

**Theorem 1.7.** If

\[\Delta_{H,p} \equiv \Delta_p \text{ or } S_p,\]

as given by \((1.7)\), then Theorem 1.6 holds for any \(1 < p < N\).

We have segmented our paper as follows: In Section 2, we discuss some auxiliary results in our functional setting. In Section 3 and Section 4, we obtain some preliminary results for the proof of Theorem 1.4, Theorem 1.5 and Theorem 1.6, Theorem 1.7 respectively. Finally, the proof of all main results has been collected in Section 5.

### 2. Auxiliary results

This section is devoted to preliminary ideas which will be helpful to advance towards establishing our principle results. We fix our assumption to \(1 < p < \infty\), unless otherwise mentioned. We state some results below which are taken from Farkas-Winkert [24, Proposition 2.1] and Xia [47, Proposition 1.2].

**Lemma 2.1.** For every \(x \in \mathbb{R}^N \setminus \{0\}\) and \(t \in \mathbb{R} \setminus \{0\}\) we have

\(A\) \(x \cdot \nabla \eta H(x) = H(x)\).

\(B\) \(\nabla \eta H(tx) = \text{sign}(t)\nabla \eta H(x)\).

\(C\) \(|\nabla \eta H(x)| \leq C\), for some positive constant \(C\).

\(D\) \(H\) is strictly convex.

**Corollary 2.2.** On account of Lemma 2.1 and Remark 1.1, we have the following relations-

\[H(x)^{p-1}\nabla \eta H(x) \cdot x = H(x)^p \geq C_1|x|^p, \quad \forall x \in \mathbb{R}^N,\]

\[|H(x)^{p-1}\nabla \eta H(x)| \leq C_2|x|^{p-1}, \quad \forall x \in \mathbb{R}^N \text{ and}\]

\[H(tx)^{p-1}\nabla \eta H(tx) = |t|^{p-2}tH(x)^{p-1}\nabla \eta H(x), \quad \forall x \in \mathbb{R}^N \text{ and } t \in \mathbb{R} \setminus \{0\}.\]
The following result has been adapted from Ohta [38, Proposition 4.6].

**Lemma 2.3.** For every $x, y \in \mathbb{R}^N$, there exists a constant $C \geq 1$ such that
\begin{equation}
H \left( \frac{x + y}{2} \right)^2 + \frac{1}{4C^2} H(x - y)^2 \leq \frac{H(x)^2 + H(y)^2}{2}.
\end{equation}

Lemma 2.4-2.5 given below follows from arguments given in the proof of Xia [48, Lemma 3.1 – 3.2].

**Lemma 2.4.** Let $2 \leq p < \infty$. Then, for every $x, y \in \mathbb{R}^N$,
\begin{equation}
H(x)^p \geq H(y)^p + pH(y)^{p-1} \nabla_y H(y)(x - y).
\end{equation}
Moreover, there exists a positive constant $c = c(C, p)$, where $C$ is given by Lemma 2.3 such that for every $x, y \in \mathbb{R}^N$,
\begin{equation}
H(x)^p \geq H(y)^p + pH(y)^{p-1} \nabla_y H(y)(x - y) + cH(x - y)^p.
\end{equation}

**Proof.** Firstly, we observe that $H^p$ is convex. Indeed, for any $x, y \in \mathbb{R}^N$ and $s \in [0, 1]$, by the convexity of $H$ (follows from Lemma 2.1), we have
\begin{align*}
H(sx + (1-s)y)^p &\leq (sH(x) + (1-s)H(y))^p \\
&\leq sH(x)^p + (1-s)H(y)^p,
\end{align*}
where in the final step above, we have used the convexity of $| \cdot |^p$. Hence the estimate (2.5) follows. Next we prove (2.6). For any $a, b \geq 0$ and $p \geq 2$, we have the following elementary inequality (see Xia [48])
\begin{equation}
(a^p + b^p) \leq (a^2 + b^2)^{p/2}.
\end{equation}
Let $C \geq 1$ be given by Lemma 2.3. Choosing $a = H\left(\frac{x+y}{2}\right)$ and $b = H\left(\frac{x-y}{2C}\right)$ in (2.7) and using Lemma 2.3 we get
\begin{equation}
H\left(\frac{x+y}{2}\right)^p + H\left(\frac{x-y}{2C}\right)^p \leq \left(\frac{H(x+y)^2}{4} + \frac{1}{4C^2} H(x-y)^2\right)^{p/2}
\leq \left(\frac{H(x)^2 + H(y)^2}{2}\right)^{p/2} \leq \frac{H(x)^p + H(y)^p}{2},
\end{equation}
where in the last line above we have used the convexity of $| \cdot |^2$ for $p \geq 2$. Moreover, by (2.5), we have
\begin{equation}
H\left(\frac{x+y}{2}\right)^p \geq H(y)^p + \frac{p}{2} H(y)^{p-1} \nabla_y H(y)(x - y).
\end{equation}
Hence using (2.9) in (2.8) the estimate (2.6) follows. 

Our next Lemma is a well know identity for the $p$-Laplacian operators, we prove it for Finsler $p$-Laplacian (a general case).

**Lemma 2.5.** Let $2 \leq p < \infty$. Then, for every $x, y \in \mathbb{R}^N$, there exists a positive constant $c = c(C, p)$, where $C$ is given by Lemma 2.3 such that
\begin{equation}
\langle H(x)^{p-1} \nabla_y H(x) - H(y)^{p-1} \nabla_y H(y), x - y \rangle \geq cH(x - y)^p.
\end{equation}
Lemma 2.8. (D1) Cauchy-Schwartz inequality:

\begin{equation}
H(x)^p \geq H(y)^p + pH(y)^{p-1} \nabla y H(y)(x - y) + cH(x - y)^p,
\end{equation}

and

\begin{equation}
H(y)^p \geq H(x)^p + p H(x)^{p-1} \nabla y H(x)(y - x) + cH(y - x)^p.
\end{equation}

Adding (2.11), (2.12) and using (H2) the estimate (2.10) follows. \hfill \Box

Remark 2.6. When 2 \leq p < \infty and H(x) = H_2(x) = |x| as given by (1.5), Lemma 2.5 coincides with the well-known algebraic inequality from Lemma 2.12.

Corollary 2.7. As a consequence of Lemma 2.5, we have

\begin{equation}
\langle H(x)^{p-1} \nabla y H(x) - H(y)^{p-1} \nabla y H(y), x - y \rangle > 0, \quad \forall x \neq y \in \mathbb{R}^N.
\end{equation}

The following inequalities has been borrowed from [47, Proposition 1.3 and (1.2)].

Lemma 2.8. (D1) Cauchy-Schwartz inequality:

\begin{equation}
\langle x, \xi \rangle \leq H(x)H_0(\xi), \quad \forall x, \xi \in \mathbb{R}^N.
\end{equation}

(D2) For all \(x \in \mathbb{R}^N \setminus \{0\}\), it holds

\begin{equation}
H_0(\nabla H(x)) = 1.
\end{equation}

Below is an important output of above inequalities.

Corollary 2.9. For any \(x, y \in \mathbb{R}^N\), it holds that

\begin{equation}
\langle H(x)^{p-1} \nabla y H(x) - H(y)^{p-1} \nabla y H(y), x - y \rangle \geq \left( H(x)^{p-1} - H(y)^{p-1} \right) \left( H(x) - H(y) \right).
\end{equation}

Proof. Using Lemma 2.8 we observe that

\begin{equation}
H(x)^{p-1} \langle \nabla y H(x), y \rangle \leq H(x)^{p-1} H_0(\nabla y H(x)) H(y) = H(x)^{p-1} H(y)
\end{equation}

and

\begin{equation}
H(y)^{p-1} \langle \nabla y H(y), x \rangle \leq H(y)^{p-1} H_0(\nabla y H(y)) H(x) = H(y)^{p-1} H(x),
\end{equation}

holds for every \(x, y \in \mathbb{R}^N\). Hence using (2.17) and (2.18), for all \(x, y \in \mathbb{R}^N\) we obtain

\[
\langle H(x)^{p-1} \nabla y H(x) - H(y)^{p-1} \nabla y H(y), x - y \rangle \\
= H(x)^p + H(y)^p - H(x)^{p-1} \langle \nabla y H(x), y \rangle - H(y)^{p-1} \langle \nabla y H(y), x \rangle \\
\geq H(x)^{p-1} + H(y)^{p-1} - H(x)^{p-1} H(y) - H(y)^{p-1} H(x) \\
= \left( H(x)^{p-1} - H(y)^{p-1} \right) \left( H(x) - H(y) \right).
\]

The following result follows from Belloni-Ferone-Kawohl [8, Theorem 3.1].

Lemma 2.10. There exists a positive eigenfunction \(e_1 \in X \cap L^\infty(\Omega)\) such that \(\|e_1\|_{L^\infty(\Omega)} = 1\) corresponding to the first eigenvalue \(\lambda_1 > 0\) satisfying the equation

\begin{equation}
-\Delta_{H,p} v = \lambda_1 |v|^{p-2} v \text{ in } \Omega, \quad v = 0 \text{ in } \partial \Omega.
\end{equation}
For the following Sobolev embedding, see Evans [22].

**Lemma 2.11.** The inclusion map

\[
W_0^{1,p}(\Omega) \hookrightarrow \begin{cases} 
L^t(\Omega), & \text{for } t \in [1,p^*], \text{ if } 1 < p < N, \\
L^1(\Omega), & \text{for } t \in [1,\infty), \text{ if } p = N, \\
C(\overline{\Omega}), & \text{if } p > N,
\end{cases}
\]

is continuous. Moreover, the above mapping is compact except for \( r = p^* \), when \( 1 < p < N \).

Next, we state the algebraic inequality from Peral [44, Lemma A.0.5].

**Lemma 2.12.** For any \( a, b \in \mathbb{R}^N \), there exists a constant \( C = C(p) > 0 \), such that

\[
\langle |a|^{p-2}a - |b|^{p-2}b, a - b \rangle \geq \begin{cases} 
C|a - b|^p, & \text{if } 2 \leq p < \infty, \\
C\frac{|a - b|^2}{(|a| + |b|)^{p-2}}, & \text{if } 1 < p < 2.
\end{cases}
\]

Our next result ensures that test functions in (1.10) and (1.11) can be chosen from the space \( X \) itself.

**Lemma 2.13.** Let \( u \in X \) be a weak solution of the problem (1.1) or (1.2), then (1.10) or (1.11) holds, for every \( \phi \in X \) respectively.

**Proof.** Let \( u \in X \) solves the problem (1.1) or (1.2). Suppose

\[
g(x,u) = f(x)u^{-q(x)} \text{ or } \lambda u^{-q(x)} + u^r.
\]

Therefore, for every \( \phi \in C^1_c(\Omega) \), from (1.10) and (1.11) we have

\[
\int_{\Omega} H(\nabla u)^{p-1}\nabla \eta H(\nabla u)\nabla \phi \, dx = \int_{\Omega} g(x,u)\phi \, dx.
\]

Now for every \( \psi \in X \), there exists a sequence of function \( 0 \leq \psi_n \in C^1_c(\Omega) \rightarrow |\psi| \) strongly in \( X \) as \( n \rightarrow \infty \) and pointwise almost everywhere in \( \Omega \). We observe that

\[
\left| \int_{\Omega} g(x,u)\psi \, dx \right| \leq \int_{\Omega} g(x,u)|\psi| \, dx \leq \liminf_{n \rightarrow \infty} \int_{\Omega} g(x,u)\psi_n \, dx
\]

\[
= \liminf_{n \rightarrow \infty} < -\Delta_{H,p}u, \psi_n > \leq C\|u\|^{p-1} \liminf_{n \rightarrow \infty} \|\psi_n\|
\]

\[
\leq C\|u\|^{p-1}\|\psi\| \leq C\|u\|^{p-1}\|\psi\|
\]

for some positive constant \( C \). Let \( \phi \in X \), then there exists a sequence \( \{\phi_n\} \subset C^1_c(\Omega) \) which converges to \( \phi \) strongly in \( X \). We claim that

\[
\lim_{n \rightarrow \infty} \int_{\Omega} g(x,u)\phi_n \, dx = \int_{\Omega} g(x,u)\phi \, dx.
\]

Indeed, using \( \psi = \phi_n - \phi \) in (2.22), we obtain

\[
\lim_{n \rightarrow \infty} \left| \int_{\Omega} g(x,u)(\phi_n - \phi) \, dx \right| \leq C\|u\|^{p-1} \lim_{n \rightarrow \infty} \|\phi_n - \phi\| = 0.
\]

Again, since \( \phi_n \rightarrow \phi \) strongly in \( X \) as \( n \rightarrow \infty \), we have

\[
\lim_{n \rightarrow \infty} H(\nabla u)^{p-1}\nabla \eta H(\nabla u)\nabla (\phi_n - \phi) \, dx = 0.
\]

Hence, using (2.23) and (2.24) in (2.21) the result follows. \( \square \)
3. Preliminaries for the proof of Theorem 1.4 and Theorem 1.5

We present proof of our first main result in this section. Here again, we assume $1 < p < \infty$, unless otherwise mentioned. For $n \in \mathbb{N}$ and nonnegative $f \in L^1(\Omega) \setminus \{0\}$, let us define $f_n(x) := \min\{f(x), n\}$ and consider the following approximated problem

\[
\begin{aligned}
-\Delta_{H,p} u &= \frac{f_n(x)}{(u^+ + \frac{1}{n})q(x)} \quad \text{in } \Omega, \\
 u &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

(3.1)

First, we prove the following useful result.

Lemma 3.1. Let $g \in L^\infty(\Omega) \setminus \{0\}$ be a nonnegative function in $\Omega$. Then there exists a unique solution $u \in X \cap L^\infty(\Omega)$ of the problem

\[
-\Delta_{H,p} u = g \quad \text{in } \Omega, \quad u > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega,
\]

(3.2)

such that for every $\omega \subset \Omega$, there exists a constant $c_\omega$ satisfying $u \geq c_\omega > 0$ in $\omega$.

Proof. **Existence:** We define the energy functional $J : X \to \mathbb{R}$ by

\[
J(u) := \frac{1}{p} \int_\Omega H(\nabla u)^p \, dx - \int_\Omega gu \, dx.
\]

Noting Lemma 2.1 and Lemma 2.11, it can be easily seen that $J$ is coercive, weakly lower semicontinuous and a strictly convex functional. Therefore, $J$ has a unique minimizer $u$ in $X$. Since $J \in C^1(X)$, so $u \in X$ is the unique solution of the equation

\[
-\Delta_{H,p} u = g \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega.
\]

(3.3)

Noting $g \geq 0$ and choosing $u_- := \min\{u, 0\}$ as a test function in (3.3), by (2.1) we obtain

\[
C_1 \int_\Omega |\nabla u_-|^p \, dx \leq \int_\Omega H(\nabla u_-)^p \, dx = \int_\Omega H(\nabla u)^p \nabla u_\eta H(\nabla u) \nabla u_- \, dx = \int_\Omega gu_- \, dx \leq 0,
\]

which gives, $u \geq 0$ in $\Omega$.

**Boundedness:** For any $k \geq 1$, we define a subset of $\Omega$ given by

\[
A(k) := \{x \in \Omega : u(x) \geq k \text{ in } \Omega\}.
\]

Choosing $\phi_k := (u - k)^+ = \max\{u - k, 0\}$ as a test function in (3.3) and using the continuity of the mapping $X \hookrightarrow L^l(\Omega)$ for some $l > p$ from Lemma 2.11 along with (2.1) we obtain

\[
\int_\Omega H(\nabla \phi_k)^p \, dx = \int_\Omega g\phi_k \, dx \leq \|g\|_{L^\infty(\Omega)} \int_{A(k)} (u - k) \, dx \leq C_0\|g\|_{L^\infty(\Omega)} |A(k)|^{\frac{l}{l-1}} \|\phi_k\|,
\]

where $C_0$ is the Sobolev constant. Hence, we have

\[
\|\phi_k\|^{p-1} \leq C |A(k)|^{\frac{l}{l-1}},
\]

(3.4)
for some positive constant $C$ depending on $\|g\|_{L^\infty(\Omega)}$. We choose $h$ such that $1 < k < h$ then using (3.4), $(u(x) - k)^p \geq (h - k)^p$ on $A(h)$ and $A(h) \subset A(k)$ we get

$$
(h - k)^p |A(h)|^{\frac{p}{p-1}} \leq \left( \int_{A(h)} (u(x) - k)^l \, dx \right)^{\frac{p}{p-1}} \leq \left( \int_{A(k)} (u(x) - k)^l \, dx \right)^{\frac{p}{p-1}} \leq \|\phi_k\|^p \leq C|A(k)|^{\frac{p(l-1)}{l(p-1)}}.
$$

Hence, we obtain

$$
|A(h)| \leq \frac{C}{(h-k)^l} |A(k)|^{\frac{l-1}{p-1}}.
$$

Therefore, observing that $\frac{l-1}{p-1} > 1$, using Kinderlehrer-Stampacchia [34, Lemma B.1] we have

$$
\|u\|_{L^\infty(\Omega)} \leq C,
$$

for some positive constant $C$ depending on $\|g\|_{L^\infty(\Omega)}$.

**Positivity:** Moreover since $g \neq 0$, we have $u \neq 0$ in $\Omega$. Noting Corollary 2.2 and Corollary 2.7, we can apply Heinonen-Kilpeläine-Martio [32, Theorem 3.59] so that for every $\omega \in \Omega$, there exists a constant $c_\omega > 0$ such that $u \geq c_\omega > 0$ in $\omega$. Thus $u > 0$ in $\omega$.

Then, we have the following result concerning the problem (3.1).

**Lemma 3.2.** Let $2 \leq p < \infty$. Then for every $n \in \mathbb{N}$, there exists a unique positive solution $u_n \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$ to the problem (3.1) such that $u_{n+1} \geq u_n$ in $\Omega$, for every $n \in \mathbb{N}$. Moreover for every $\omega \subset \subset \Omega$, there exists a constant $c_\omega > 0$ (independent of $n$) such that $u_n \geq c_\omega > 0$ in $\omega$.

**Proof.** Step 1. (Existence) We observe that for every $n \in \mathbb{N}$ and $v \in L^p(\Omega)$, the function

$$
g_n(x) := \frac{f_n(x)}{(v^+ + \frac{1}{n})q(x)} \in L^\infty(\Omega) \setminus \{0\}
$$

is nonnegative. Therefore, by Lemma 3.1 we can define the map $T : L^p(\Omega) \to L^p(\Omega)$ by

$$
T(v) = w_n,
$$

where $w_n \in X \cap L^\infty(\Omega)$ is the unique solution to the problem

$$
(3.5) \begin{cases} -\Delta_{H,p}w_n = \frac{f_n(x)}{(v^+ + \frac{1}{n})q(x)} & \text{in } \Omega, \\ w_n > 0 \text{ in } \Omega, w_n = 0 \text{ on } \partial \Omega, \end{cases}
$$

such that for every $\omega \subset \subset \Omega$ there exists a positive constant $c_\omega$ satisfying $w_n \geq c_\omega > 0$ in $\omega$. Now, using $w_n$ as a test function in the problem (3.5) and the fact that $q \in C(\overline{\Omega})$, we obtain

$$
(3.6) \quad \|w_n\| \leq Cn \frac{\|g\|_{L^\infty(\Omega)}^{l+1}}{l^{l-1}},
$$

for some constant $C = C(p, N, \Omega)$. We define

$$
S := \{v \in L^p(\Omega) : \lambda T(v) = v \text{ for } 0 \leq \lambda \leq 1\}.
$$
Indeed, let \( v_1, v_2 \in S \). Then using (3.6) and Lemma 2.11
\[
\|v_1 - v_2\|_{L^p(\Omega)} = \lambda \|T(v_1) - T(v_2)\|_{L^p(\Omega)} \leq 2\lambda C_1 \|u\|_{L^{\infty}(\Omega)}^{p+1},
\]
where \( C = C(p, N, \Omega) \) is given by (3.6) and \( C_1 \) is the Sobolev constant. Therefore, the set \( S \) is bounded in \( L^p(\Omega) \). To prove the continuity of \( T \), let \( v_k \to v \) strongly in \( L^p(\Omega) \) and \( T(v_k) = w_{n,k} \). Then, we have
\[
\int_{\Omega} H(\nabla w_{n,k})^{p-1} \nabla \eta_{n}(\nabla w_{n,k}) \nabla (w_{n,k} - u_n) \, dx = \int_{\Omega} \frac{f_n(x)}{v_k + \frac{1}{n}q(x)} (w_{n,k} - u_n) \, dx,
\]
and
\[
\int_{\Omega} H(\nabla w_{n,k})^{p-1} \nabla \eta_{n}(\nabla w_{n,k}) \nabla (w_{n,k} - w_n) \, dx = \int_{\Omega} \frac{f_n(x)}{v^+ + \frac{1}{n}q(x)} (w_{n,k} - w_n) \, dx.
\]
Subtracting (3.8) from (3.7), since \( 2 \leq p < \infty \), we can use Lemma 2.5 to obtain
\[
\int_{\Omega} f_n(x) \left\{ \left( v_k^+ + \frac{1}{n} \right)^{-q(x)} - \left( v^+ + \frac{1}{n} \right)^{-q(x)} \right\} (w_{n,k} - w_n) \, dx \leq c \int_{\Omega} H(\nabla (w_{n,k} - w_n))^p \, dx,
\]
for some positive constant \( c \). From Hölder’s inequality, we obtain
\[
\int_{\Omega} f_n(x) \left\{ \left( v_k^+ + \frac{1}{n} \right)^{-q(x)} - \left( v^+ + \frac{1}{n} \right)^{-q(x)} \right\} (w_{n,k} - w_n) \, dx \leq n \left( \int_{\Omega} \left[ \left( v_k^+ + \frac{1}{n} \right)^{-q(x)} - \left( v^+ + \frac{1}{n} \right)^{-q(x)} \right]^{-\frac{p}{p-1}} \, dx \right)^{-\frac{p-1}{p}} \|w_{n,k} - w_n\|_{L^p(\Omega)}.
\]
Since \( v_k \to v \) strongly in \( L^p(\Omega) \), it follows that up to a subsequence \( v_k \to v \) almost everywhere in \( \Omega \). Also, it is easy to observe that
\[
\left| \left( v_k^+ + \frac{1}{n} \right)^{-q(x)} - \left( v^+ + \frac{1}{n} \right)^{-q(x)} \right| \leq 2n \|q\|_{L^{\infty}(\Omega)}.
\]
Since the limit is independent of the choice of the subsequence, applying the Lebesgue dominated convergence theorem, we obtain
\[
\lim_{k \to \infty} \int_{\Omega} \left( v_k^+ + \frac{1}{n} \right)^{-q(x)} - \left( v^+ + \frac{1}{n} \right)^{-q(x)} \left| \frac{1}{p-1} \right| \, dx = 0.
\]
Thus using (3.9) along with (3.10) and (3.11) it follows that \( w_{n,k} \to w_n \) strongly in \( X \). As a consequence, the mapping \( T \) is continuous. The compactness of \( T \) follows from the fact (3.6) and the compactness of the mapping \( X \hookrightarrow L^p(\Omega) \) which follows from Lemma 2.11. Thus by the Schauder fixed point theorem, \( T \) has a fixed point, say \( u_n \in X \) which solves the problem (3.1). Note that \( u_n \geq 0 \) in \( \Omega \).
Step 2. (Monotonicity) Choosing \((u_n - u_{n+1})^+ := \max\{u_n - u_{n+1}, 0\}\) as a test function in (3.1) we have

\[
\int_{\Omega} H(\nabla u_n)^{p-1} \nabla \eta H(\nabla u_n) \nabla (u_n - u_{n+1})^+ \, dx = \int_{\Omega} \frac{f_n(x)}{(u_n + \frac{1}{n})^q(x)} (u_n - u_{n+1})^+ \, dx,
\]

and

\[
\int_{\Omega} H(\nabla u_{n+1})^{p-1} \nabla \eta H(\nabla u_{n+1}) \nabla (u_n - u_{n+1})^+ \, dx = \int_{\Omega} \frac{f_{n+1}(x)}{(u_{n+1} + \frac{1}{n+1})^q(x)} (u_n - u_{n+1})^+ \, dx.
\]

Subtracting (3.13) from (3.12), since \(2 \leq p < \infty\), using Lemma 2.5 we obtain

\[
\int_{\Omega} \left\{ \frac{f_n(x)}{(u_n + \frac{1}{n})^q(x)} - \frac{f_{n+1}(x)}{(u_{n+1} + \frac{1}{n+1})^q(x)} \right\} (u_n - u_{n+1})^+ \, dx
\]

\[
= \int_{\Omega} \{ H(\nabla u_n)^{p-1} \nabla \eta H(\nabla u_n) - H(\nabla u_{n+1})^{p-1} \nabla \eta H(\nabla u_{n+1}) \} \nabla (u_n - u_{n+1})^+ \, dx
\]

\[
\geq c \int_{\Omega} H \left( \nabla (u_n - u_{n+1})^+ \right)^p \, dx,
\]

for some positive constant \(c\). Since \(f_n(x) \leq f_{n+1}(x)\) for almost every \(x \in \Omega\), we get

\[
\int_{\Omega} \left\{ \frac{f_n(x)}{(u_n + \frac{1}{n})^q(x)} - \frac{f_{n+1}(x)}{(u_{n+1} + \frac{1}{n+1})^q(x)} \right\} (u_n - u_{n+1})^+ \, dx
\]

\[
\leq \int_{\Omega} f_{n+1}(x) \left( u_{n+1} + \frac{1}{n+1} \right) q(x) \left( u_{n+1} + \frac{1}{n+1} \right) q(x) (u_n - u_{n+1})^+ \, dx \leq 0.
\]

Hence from (3.14) and (3.15), we have \(u_{n+1} \geq u_n\) in \(\Omega\).

Step 3. (Uniqueness) Let \(n \in \mathbb{N}\) and \(u_n, v_n \in X \cap L^\infty(\Omega)\) solves the problem (3.1). Then choosing \((u_n - v_n)^+ := \max\{u_n - v_n, 0\}\) as a test function in (3.1) and proceeding similarly as the proof of Step 2 above, we get \(u_n \geq v_n\) in \(\Omega\). Choosing \((v_n - u_n)^+ := \max\{v_n - u_n, 0\}\) as a test function in (3.1), we get \(v_n \geq u_n\) in \(\Omega\). This gives that \(u_n = v_n\) in \(\Omega\).

Step 4. (Positivity) From Step 1, we infer that there exists a constant \(c_\omega > 0\) such that \(u_1 \geq c_\omega > 0\), for every \(\omega \in \Omega\). Hence taking into account Step 2, we have \(u_n \geq c_\omega > 0\) in \(\omega\) for every \(\omega \in \Omega\), where \(c_\omega > 0\) is independent of \(n\). Thus \(u_n > 0\) in \(\Omega\) for all \(n \in \mathbb{N}\), from which the result follows.

Remark 3.3. We emphasize that if \(\Delta_{\mu,p} = \Delta_p\) or \(S_p\) as given by (1.7), then noting Lemma 2.12, the results in Lemma 3.2 will hold for any \(1 < p < \infty\).

4. Preliminaries for the proof of Theorem 1.6 and Theorem 1.7

Throughout this section, we assume \(1 < p < N\), unless otherwise mentioned. Here we prove some preliminary results required to prove Theorem 1.6-1.7.
Let us denote the energy functional $I_\lambda : X \to \mathbb{R} \cup \{\pm \infty\}$ corresponding to the problem (1.2) by

$$I_\lambda(u) := \frac{1}{p} \int_\Omega H(\nabla u)^p \, dx - \lambda \int_\Omega \frac{(u^+)^{-q(x)}}{1-q(x)} \, dx - \frac{1}{r+1} \int_\Omega (u^+)^{r+1} \, dx.$$  

Now for $\epsilon > 0$, we consider the following approximated problem

$$I_{\lambda, \epsilon}(u) = \frac{1}{p} \int_\Omega H(\nabla u)^p \, dx - \lambda \int_\Omega \frac{[(u^+ + \epsilon)^{-q(x)} - \epsilon^{-q(x)}]}{1-q(x)} \, dx - \frac{1}{r+1} \int_\Omega (u^+)^{r+1} \, dx.$$

for which the corresponding energy functional is given by

$$I_{\lambda, \epsilon}(u) = \frac{1}{p} \int_\Omega H(\nabla u)^p \, dx - \lambda \int_\Omega \frac{[(u^+ + \epsilon)^{-q(x)} - \epsilon^{-q(x)}/(1-q(x))]}{1-q(x)} \, dx - \frac{1}{r+1} \int_\Omega (u^+)^{r+1} \, dx.$$

It is easy to verify that $I_{\lambda, \epsilon} \in C^1(X, \mathbb{R})$, $I_{\lambda, \epsilon}(0) = 0$ and $I_{\lambda, \epsilon}(v) \leq I_{0, \epsilon}(v)$, for all $v \in X$.

Our next Lemma states that $I_{\lambda, \epsilon}$ satisfies the Mountain Pass Geometry.

**Lemma 4.1.** There exists $R > 0$, $\rho > 0$ and $\Lambda > 0$ depending on $R$ such that

$$\inf_{\|v\| \leq R} I_{\lambda, \epsilon}(v) < 0 \quad \text{and} \quad \inf_{\|v\| = R} I_{\lambda, \epsilon}(v) \geq \rho, \quad \text{for} \quad \lambda \in (0, \Lambda).$$

Moreover, there exists $T > R$ such that $I_{\lambda, \epsilon}(Te_1) < -1$ for $\lambda \in (0, \Lambda)$, where $e_1$ is given by Lemma 2.10.

**Proof.** We fix $l = |\Omega|^{\frac{1}{(p-1)}}$. Then using Hölder’s inequality and Lemma 2.11, for any $v \in X$ we get

$$\int_\Omega (v^+)^{r+1} \, dx \leq \left( \int_\Omega |v|^p \right)^{\frac{r+1}{p}} |\Omega|^{\frac{1}{p-1}} \|v\|^{r+1} \leq Cl\|v\|^{r+1},$$

for some positive constant $C$ independent of $v$. Observing

$$\lim_{t \to 0} \frac{I_{\lambda, \epsilon}(te_1)}{t} = -\lambda \int_\Omega \epsilon^{-q(x)} e_1 \, dx < 0,$$

we can choose $k \in (0, 1)$ sufficiently small and set $\|v\| = R := k\left(\frac{R+1}{pCl}\right)^{\frac{1}{r+1}}$ such that

$$\inf_{\|v\| \leq R} I_{\lambda, \epsilon}(v) < 0.$$

Moreover, since $R < \left(\frac{R+1}{pCl}\right)^{\frac{1}{r+1}}$, we obtain

$$I_{0, \epsilon}(v) \geq \frac{Rp}{p} - \frac{ClR^{r+1}}{r+1} := 2\rho \quad \text{(say)} > 0.$$

We define

$$\Lambda := \sup_{\|v\| = R} \left(\frac{1}{1-q(x)} \int_\Omega |v|^{1-q(x)} \, dx\right),$$

which is a positive constant and since $\rho, R$ depends on $k, r, p, |\Omega|, C$ so does $\Lambda$. We know that

$$(v^+ + \epsilon)^{1-q(x)} - \epsilon^{1-q(x)} \leq (v^+)^{1-q(x)},$$

and

$$(v^+ + \epsilon)^{1-q(x)} - \epsilon^{1-q(x)} \leq (v^+)^{1-q(x)}.$$
which gives
\[ I_{λ,ε}(v) \geq \frac{1}{p} \int_{Ω} H(∇v)^p \, dx - \frac{1}{r + 1} \int_{Ω} (v^+)^{r+1} \, dx - \frac{λ}{1 - q(x)} \int_{Ω} (v^+)^{1-q(x)} \, dx \]
\[ = I_{0,ε}(v) - \frac{λ}{1 - q(x)} \int_{Ω} (v^+)^{1-q(x)} \, dx. \]

Therefore
\[ \inf_{\|v\|=R} I_{λ,ε}(v) \geq \inf_{\|v\|=R} I_{0,ε}(v) - λ \sup_{\|v\|=R} \left( \frac{1}{1 - q(x)} \int_{Ω} |v|^{1-q(x)} \, dx \right) \]
\[ \geq 2ρ - λ \sup_{\|v\|=R} \left( \frac{1}{1 - q(x)} \int_{Ω} |v|^{1-q(x)} \, dx \right) \geq ρ, \]
if \( λ \in (0, Λ) \). Lastly, it is easy to see that \( I_{0,ε}(tε_1) \to -∞ \), as \( t \to +∞ \) which implies that we can choose \( T > R \) such that \( I_{0,ε}(Tε_1) < -1 \). Hence
\[ I_{λ,ε}(Tε_1) \leq I_{0,ε}(Tε_1) < -1 \]
which completes the proof. \( \square \)

As a consequence of Lemma 4.1, we have
\[ \inf_{\|v\|=R} I_{λ,ε}(v) \geq ρ \max\{I_{λ,ε}(Tε_1), I_{λ,ε}(0)\} = 0. \]

Our next Lemma ensures that \( I_{λ,ε} \) satisfies the Palais Smale (PS)_c condition.

**Proposition 4.2.** Let \( 2 \leq p < N \), then \( I_{λ,ε} \) satisfies the (PS)_c condition, for any \( c \in \mathbb{R} \) that is if \( \{u_k\} \subset X \) is a sequence satisfying
\[ (4.2) \quad I_{λ,ε}(u_k) \to c \text{ and } I'_{λ,ε}(u_k) \to 0 \]
as \( k \to ∞ \), then \( \{u_k\} \) contains a strongly convergent subsequence in \( X \).

**Proof.** Let \( \{u_k\} \subset X \) satisfies (4.2) then we claim that \( \{u_k\} \) must be bounded in \( X \). To see this using (4.1), we obtain
\[ (4.3) \quad I_{λ,ε}(u_k) - \frac{1}{r + 1} I'_{λ,ε}(u_k) u_k = \left( \frac{1}{p} - \frac{1}{r + 1} \right) \int_{Ω} H(∇u_k)^p \, dx - λ \int_{Ω} \frac{(u_k^+ + ε)^{1-q(x)} - ε^{1-q(x)}}{1 - q(x)} \, dx \]
\[ + \frac{λ}{r + 1} \int_{Ω} (u_k^+ + ε)^{-q(x)} u_k \, dx \]
\[ \geq \left( \frac{1}{p} - \frac{1}{r + 1} \right) \int_{Ω} H(∇u_k)^p \, dx - λ \int_{Ω} \frac{(u_k^+)^{1-q(x)}}{1 - q(x)} \, dx \]
\[ + \frac{λ}{r + 1} \int_{Ω} (u_k^+ + ε)^{-q(x)} u_k \, dx \]
\[ \geq \left( \frac{1}{p} - \frac{1}{r + 1} \right) \int_{Ω} H(∇u_k)^p \, dx - λ \int_{Ω} \frac{(u_k^+)^{1-q(x)}}{1 - q(x)} \, dx - \frac{λC}{ε(r + 1) \|u_k\|}, \]
for some positive constant $C$ (independent of $k$) where we have used Lemma 2.11 and the fact $0 < q(x) < 1$ in $\Omega$. Due to the same reasoning, we obtain

\begin{equation}
- \int_\Omega \frac{(u_k^+)^{1-q(x)}}{1-q(x)} \, dx \geq - \int_\Omega \frac{|u_k|^{1-q(x)}}{1-q(x)} \, dx
\end{equation}

\begin{equation}
\geq \frac{-1}{1 - \|q\|_{L^\infty(\Omega)}} \left( \int_{\Omega \cap \{|u_k| \geq 1\}} |u_k|^{1-q(x)} \, dx + \int_{\Omega \cap \{|u_k| < 1\}} |u_k|^{1-q(x)} \, dx \right)
\end{equation}

\begin{equation}
\geq \frac{-1}{1 - \|q\|_{L^\infty(\Omega)}} \left( \int_{\Omega \cap \{|u_k| \geq 1\}} |u_k| \, dx + \int_{\Omega \cap \{|u_k| < 1\}} |u_k|^{1-\|q\|_{L^\infty(\Omega)}} \, dx \right)
\end{equation}

\begin{equation}
\geq -C \left( \|u_k\| + \|u_k\|^{1-\|q\|_{L^\infty(\Omega)}} \right),
\end{equation}

for some positive constant $C$ independent of $k$. Thus inserting (4.4) into (4.3) and using the fact $r + 1 > p$ along with Remark 1.1, we get

\begin{equation}
I_{\lambda,\epsilon}(u_k) - \frac{1}{r+1} I'_{\lambda,\epsilon}(u_k) u_k \geq C_1 \|u_k\|^p - C \left( \|u_k\| + \|u_k\|^{1-\|q\|_{L^\infty(\Omega)}} \right),
\end{equation}

for some positive constant $C_1$ (independent of $k$). Also from (4.2) it follows that for $k$ large enough

\begin{equation}
\left| I_{\lambda,\epsilon}(u_k) - \frac{1}{r+1} I'_{\lambda,\epsilon}(u_k) u_k \right| \leq C + o(\|u_k\|),
\end{equation}

for some positive constant $C$ (independent of $k$). Combining (4.5) and (4.6), our claim follows since $p > 1$. By reflexivity of $X$, there exists $u_0 \in X$ such that up to a subsequence, $u_k \rightharpoonup u_0$ weakly in $X$ as $k \to \infty$.

Claim: $u_k \to u_0$ strongly in $X$ as $k \to \infty$.

By (4.2), we have that

\begin{equation}
\lim_{k \to \infty} \left( \int_\Omega H(\nabla u_k)^p \nabla H(\nabla u_k) \nabla u_0 \, dx - \lambda \int_\Omega (u_k^+ + \epsilon)^{-q(x)} u_0 \, dx - \int_\Omega (u_k^+)^r u_0 \, dx \right) = 0
\end{equation}

and

\begin{equation}
\lim_{k \to \infty} \left( \int_\Omega H(\nabla u_k)^p \nabla H(\nabla u_k) \nabla u_k \, dx - \lambda \int_\Omega (u_k^+ + \epsilon)^{-q(x)} u_k \, dx - \int_\Omega (u_k^+)^r u_k \, dx \right) = 0.
\end{equation}

Now

\begin{equation}
\lim_{k \to \infty} \left( \int_\Omega H(\nabla u_k)^p \nabla H(\nabla u_k) - H(\nabla u_0)^p \nabla H(\nabla u_0) \right) (\nabla (u_k - u_0) \, dx
\end{equation}

\begin{equation}
= \lim_{k \to \infty} \left( \lambda \int_\Omega (u_k^+ + \epsilon)^{-q} u_k \, dx + \int_\Omega (u_k^+)^r u_k \, dx - \lambda \int_\Omega (u_k^+ + \epsilon)^{-q} u_0 \, dx - \int_\Omega (u_k^+)^r u_0 \, dx \right)
\end{equation}

\begin{equation}
- \lim_{k \to \infty} \left( \int_\Omega H(\nabla u_0)^p \nabla H(\nabla u_0) \nabla u_k \, dx - \int_\Omega H(\nabla u_0)^p \, dx \right).
\end{equation}

From weak convergence of $\{u_k\}$ we get

\begin{equation}
\lim_{k \to \infty} \left( \int_\Omega H(\nabla u_0)^p \nabla H(\nabla u_0) \nabla u_k \, dx - \int_\Omega H(\nabla u_0)^p \, dx \right) = 0.
\end{equation}
Also, we have
\[ \left| (u_k^+ + \epsilon)^{-q(x)}u_0 \right| \leq \epsilon^{-q(x)}u_0 \text{ and } \int_{\Omega} \left| e^{-q(x)}u_0 \right| \, dx \leq \|e^{-q(x)}\|_{L^\infty(\Omega)} \int_{\Omega} |u_0| \, dx < +\infty. \]

Thus Lebesgue Dominated convergence theorem gives that
\[ (4.9) \lim_{k \to \infty} \int_{\Omega} (u_k^+ + \epsilon)^{-q(x)}u_0 \, dx = \int_{\Omega} (u_0^+ + \epsilon)^{-q(x)}u_0 \, dx. \]

Since \( u_k \to u_0 \) pointwise in \( \Omega \), for any measurable subset \( E \) of \( \Omega \) we have
\[ \int_E \left| (u_k^+ + \epsilon)^{-q(x)}u_k \right| \, dx \leq \int_E e^{-q(x)}u_k \, dx \leq \|e^{-q(x)}\|_{L^\infty(\Omega)} \|u_k\|_{L^{p^*}(\Omega)} |E|^\frac{p^* - q}{p^*} \leq C(\epsilon)|E|^\frac{p^* - q}{p^*}, \]
so from Vitali convergence theorem, it follows that
\[ (4.10) \lim_{k \to \infty} \lambda \int_{\Omega} (u_k^+ + \epsilon)^{-q(x)}u_k \, dx = \lambda \int_{\Omega} (u_0^+ + \epsilon)^{-q(x)}u_0 \, dx. \]

Similarly, due to \( r + 1 < p^* \), we have
\[ \int_E \left| (u_k^+)^ru_0 \right| \, dx \leq \|u_0\|_{L^{p^*}(\Omega)} \left( \int_E (u_k^+)^{rp^*} \, dx \right)^{\frac{1}{p^*}} \leq C_3 |E|^{\alpha} \]
and
\[ \int_E \left| (u_k^+)^ru_k \right| \, dx \leq \|u_k\|_{L^{p^*}(\Omega)} \left( \int_E (u_k^+)^{rp^*} \, dx \right)^{\frac{1}{p^*}} \leq C_4 |E|^{\beta} \]
for some positive constants \( C_3, C_4, \alpha \) and \( \beta \). Therefore Vitali convergence theorem gives
\[ (4.11) \lim_{k \to \infty} \int_{\Omega} (u_k^+)^ru_0 \, dx = \int_{\Omega} (u_0^+)^ru_0 \, dx, \]
and
\[ (4.12) \lim_{k \to \infty} \int_{\Omega} (u_k^+)^ru_k \, dx = \int_{\Omega} (u_0^+)^ru_0 \, dx. \]

Using (4.8), (4.9), (4.10), (4.11) and (4.12) in (4.7), we obtain
\[ \lim_{k \to \infty} \int_{\Omega} \left\{ H(\nabla u_k)^{p-1}\nabla \eta H(\nabla u_k) - H(\nabla u_0)^{p-1}\nabla \eta H(\nabla u_0) \right\} \nabla (u_k - u_0) \, dx = 0. \]

Then using (2.16) from Corollary 2.9, we obtain \( u_k \to u_0 \) strongly in \( X \) as \( k \to \infty \) which proves our claim. \( \square \)

**Remark 4.3.** From Lemma 4.1, Proposition 4.2 and Mountain Pass Lemma, for every \( \lambda \in (0, \Lambda) \), there exists a \( \zeta_\epsilon \in X \) such that \( I_{\lambda,\epsilon}'(\zeta_\epsilon) = 0 \) and
\[ I_{\lambda,\epsilon} \zeta_\epsilon = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_{\lambda,\epsilon} (\gamma(t)) \geq \rho > 0, \]
where
\[ \Gamma = \{ \gamma \in C([0,1],X) : \gamma(0) = 0, \gamma(1) = T_\epsilon e_1 \}. \]

Using (4.4) together with Vitali convergence theorem, if \( u_k \rightharpoonup u_0 \) weakly in \( X \), then we have
\[ \lim_{k \to \infty} \int_{\Omega} (u_k + \epsilon)^{1-q(x)} - \epsilon^{1-q(x)} \frac{dx}{1- q(x)} = \int_{\Omega} (u_0 + \epsilon)^{1-q(x)} - \epsilon^{1-q(x)} \frac{dx}{1- q(x)}. \]
Therefore, $I_{\lambda, \varepsilon}$ is weakly lower semicontinuous. Furthermore, as a consequence of Lemma 4.1, since for every $\lambda \in (0, \Lambda)$ we have $\inf_{\|v\| \leq R} I_{\lambda, \varepsilon}(v) < 0$, there exists a nonzero $\nu_{\varepsilon} \in X$ such that $\|\nu_{\varepsilon}\| \leq R$ and

$$(4.13) \quad \inf_{\|v\| \leq R} I_{\lambda, \varepsilon}(v) = I_{\lambda, \varepsilon}(\nu_{\varepsilon}) < 0 < \rho \leq I_{\lambda, \varepsilon}(\zeta_{\varepsilon}).$$

Thus, $\zeta_{\varepsilon}$ and $\nu_{\varepsilon}$ are two different nontrivial critical points of $I_{\lambda, \varepsilon}$, provided $\lambda \in (0, \Lambda)$.

**Lemma 4.4.** Let $2 \leq p < N$, then the critical points $\zeta_{\varepsilon}$ and $\nu_{\varepsilon}$ of $I_{\lambda, \varepsilon}$ are nonnegative in $\Omega$.

**Proof.** Testing $(P_{\lambda, \varepsilon})$ with $\min\{\zeta_{\varepsilon}, 0\}$ and $\min\{\nu_{\varepsilon}, 0\}$, proceeding similarly as in the proof of Lemma 3.1, we have $\zeta_{\varepsilon}, \nu_{\varepsilon} \geq 0$ in $\Omega$. □

**Remark 4.5.** As in Remark 3.3, if $\Delta_{H,p} = \Delta_p$ or $S_p$ as given by (1.7), then noting Lemma 2.12, we have Proposition 4.2 and Lemma 4.4 valid for any $1 < p < N$.

**Lemma 4.6.** There exists a constant $\Theta > 0$ (independent of $\varepsilon$) such that $\|v_{\varepsilon}\| \leq \Theta$, where $v_{\varepsilon} = \zeta_{\varepsilon}$ or $\nu_{\varepsilon}$.

**Proof.** The result trivially holds if $v_{\varepsilon} = \nu_{\varepsilon}$ so we deal with the case $v_{\varepsilon} = \zeta_{\varepsilon}$. Recalling the terms from Lemma 4.1, we define $A = \max_{t \in [0,1]} I_{0, \varepsilon}(tT_{e_1})$ then

$$A \geq \max_{t \in [0,1]} I_{\lambda, \varepsilon}(tT_{e_1}) \geq \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_{\lambda, \varepsilon}(\gamma(t)) = I_{\lambda, \varepsilon}(\zeta_{\varepsilon}) \geq \rho > 0 > I_{\lambda, \varepsilon}(\nu_{\varepsilon}).$$

Therefore

$$(4.14) \quad \frac{1}{p} \int_{\Omega} H(\nabla \zeta_{\varepsilon})^p \, dx - \lambda \int_{\Omega} \frac{(\zeta_{\varepsilon} + \varepsilon)^{1-q(x)} - \varepsilon^{1-q(x)}}{1-q(x)} \, dx - \frac{1}{r+1} \int_{\Omega} \zeta_{\varepsilon}^{r+1} \, dx \leq A.$$

Choosing $\phi = -\frac{\zeta_{\varepsilon}}{p+1}$ as a test function in $(P_{\lambda, \varepsilon})$ we obtain

$$(4.15) \quad -\frac{1}{r+1} \int_{\Omega} H(\nabla \zeta_{\varepsilon})^p \, dx + \frac{\lambda}{r+1} \int_{\Omega} \frac{\zeta_{\varepsilon}}{(\zeta_{\varepsilon} + \varepsilon)^q(x)} \, dx + \frac{1}{r+1} \int_{\Omega} \zeta_{\varepsilon}^{r+1} \, dx = 0.$$

Adding (4.14) and (4.15) we get

$$\left(\frac{1}{p} - \frac{1}{r+1}\right) \int_{\Omega} H(\nabla \zeta_{\varepsilon})^p \, dx \leq \lambda \int_{\Omega} \frac{(\zeta_{\varepsilon} + \varepsilon)^{1-q(x)} - \varepsilon^{1-q(x)}}{1-q(x)} \, dx - \frac{\lambda}{r+1} \int_{\Omega} \frac{\zeta_{\varepsilon}}{(\zeta_{\varepsilon} + \varepsilon)^q(x)} \, dx + A$$

$$\leq \lambda \int_{\Omega} \frac{(\zeta_{\varepsilon} + \varepsilon)^{1-q(x)} - \varepsilon^{1-q(x)}}{1-q(x)} \, dx + A$$

$$\leq C(\|\zeta_{\varepsilon}\| + \|\zeta_{\varepsilon}\|^{1-\|\varepsilon\|_{L\infty(\Omega)}}) + A,$$

for some positive constant $C$ being independent of $\varepsilon$, where the last inequality is deduced using the estimate (4.4), Hölder inequality along Lemma 2.11. Therefore, using $r+1 > p$, the sequence $\{\zeta_{\varepsilon}\}$ is uniformly bounded in $X$ with respect to $\varepsilon$. This completes the proof. □
5. Proof of the main results

Proof of Theorem 1.4:

(a) Uniqueness: Let \( u, v \in X \) be two distinct weak solutions of (1.1). Then by Lemma 2.13 we can choose \( \phi = (u - v)^+ \) as a test function in (1.10) to get

\[
\langle -\Delta_{H,p}u + \Delta_{H,p}v, (u - v)^+ \rangle = \int_{\Omega} f \left( \frac{1}{u^q(x)} - \frac{1}{v^q(x)} \right) (u - v)^+ \, dx \leq 0.
\]

Using Lemma 2.5 we obtain \( u \leq v \in \Omega \). Similarly, choosing \( \phi = (v - u)^+ \) as a test function in (1.10) we get \( v \leq u \in \Omega \). Hence the uniqueness follows.

(b) Existence: Let \( u_n \in X \) denote the weak solution of (3.1) then choosing \( u_n \) as a test function in (3.1) and using Remark 1.1, we get

\[
\int_{\Omega} H(\nabla u_n)^p \, dx \leq \int_{\Omega} \frac{f_n(x)u_n}{(u_n + \frac{1}{n})^q(x)} \, dx.
\]

Now denote by \( \omega_\delta = \Omega \setminus \overline{\Omega}_\delta \) then by Lemma 3.2, we get \( u_n \geq c_{\omega_\delta} > 0 \) in \( \omega_\delta \). We observe that

\[
\int_{\Omega} \frac{f_n(x)u_n}{(u_n + \frac{1}{n})^q(x)} \, dx = \int_{\overline{\Omega}_\delta} \frac{f_n(x)u_n}{(u_n + \frac{1}{n})^q(x)} \, dx + \int_{\omega_\delta} \frac{f_n(x)u_n}{(u_n + \frac{1}{n})^q(x)} \, dx
\]

\[
\leq \int_{\overline{\Omega}_\delta} f(x) u_n^{1-q(x)} \, dx + \int_{\omega_\delta} \frac{f(x)}{u_n^{q(x)}} u_n \, dx
\]

\[
\leq \int_{\overline{\Omega}_\delta \cap \{u_n \leq 1\}} f(x) \, dx + \int_{\omega_\delta \cap \{u_n \geq 1\}} f(x) u_n \, dx + \int_{\omega_\delta} \frac{f(x)}{u_n^{q(x)}} u_n \, dx
\]

\[
\leq \|f\|_{L^1(\Omega)} + (1 + \|c_{\omega_\delta}^{-q(x)}\|_{L^\infty(\Omega)}) \int_{\Omega} f(x) u_n \, dx.
\]

Let \( 2 \leq p < N \). Then, using Hölder’s inequality and Lemma 2.11 we obtain for \( f \in L^m(\Omega) \) with \( m = (p^*)' \)

\[
\|u_n\|^p \leq \|f\|_{L^1(\Omega)} + C\|f\|_{L^m(\Omega)} \|u_n\|,
\]

which implies that the sequence \( \{u_n\} \) is uniformly bounded in \( X \). Thus up to a subsequence, by Lemma 2.11 for \( 1 \leq t < p^* \), we get

\[
u_n \rightharpoonup u \ \text{weakly in} \ \ X,
\]

\[
 u_n \to u \ \text{strongly in} \ \ L^t(\Omega), \ \text{and}
\]

\[
 u_n \to u \ \text{pointwise in} \ \Omega \ \text{as} \ n \to \infty.
\]

Let \( \phi \in C_c^\infty(\Omega) \) be such that \( \text{supp} \ \phi = \omega \) where \( \omega \subseteq \Omega \). Then by Lemma 3.2, there exists a constant \( c_\omega > 0 \) independent of \( n \) such that \( u_n \geq c_\omega > 0 \) in \( \omega \). Hence, we get

\[
\frac{f_n}{(u_n + \frac{1}{n})^q(x)} \phi \leq c_\omega q(x) f(x) \phi(x) \in L^1(\Omega).
\]

Thus we can apply Boccardo-Murat [12, Theorem 2.1] to obtain that up to a subsequence still denoted by \( \{u_n\} \), we have

\[
\lim_{n \to \infty} \nabla u_n(x) = \nabla u(x) \ for \ x \ in \ \Omega.
\]
Hence, we have
\[
\lim_{n \to \infty} H(\nabla u_n(x))^{p-1} \nabla \eta H(\nabla u_n(x)) = H(\nabla u(x))^{p-1} \nabla \eta H(\nabla u(x)) \text{ for } x \in \Omega.
\]
Moreover, by Corollary 2.2 we observe that
\[
\| H(\nabla u_n(x))^{p-1} \nabla \eta H(\nabla u_n(x)) \|_{L^{p-1}(\Omega)} \leq \| u_n \|_{p} \leq C,
\]
for some positive constant $C$ which is independent of $n$. As a consequence, the sequence
\[
H(\nabla u_n)^{p-1} \nabla \eta H(\nabla u_n) \rightharpoonup H(\nabla u)^{p-1} \nabla \eta H(\nabla u)
\]
weakly in $X$ as $n \to \infty$.

Indeed, since the weak limit is independent of the subsequence, the above fact holds for every $n$. Therefore, we have
\[
\text{(5.1)} \quad \lim_{n \to \infty} \int_{\Omega} H(\nabla u_n(x))^{p-1} \nabla \eta H(\nabla u_n(x)) \nabla \phi \, dx = \int_{\Omega} H(\nabla u)^{p-1} \nabla \eta H(\nabla u) \nabla \phi \, dx,
\]
for every $\phi \in C^{\infty}_c(\Omega)$. Moreover by the Lebesgue dominated convergence theorem, we have
\[
\text{(5.2)} \quad \lim_{n \to \infty} \int_{\Omega} \frac{f_n}{u_n + \frac{1}{n}} q(x) \phi \, dx = \int_{\Omega} \frac{f}{u q(x)} \phi \, dx \text{ for all } \phi \in C^{\infty}_c(\Omega).
\]

Now, we define by
\[
u := \lim_{n \to \infty} u_n.
\]
Then, due to the monotonicity of $\{u_n\}$ we get $u \geq c_\omega > 0$ for every $\omega \in \Omega$. Hence from (5.1) and (5.2), we get $u$ is our required solution and the result follows.

Noting Lemma 2.11, for $p \geq N$ the result follows similarly.

**Proof of Theorem 1.5:** Noting Lemma 2.12 and Remark 3.3, proceeding similarly as in the proof of Theorem 1.4, the result follows.

**Proof of Theorem 1.6:** As a resultant of Lemma 4.4 and Lemma 4.6, upto a subsequence we get that $\zeta_\epsilon \rightharpoonup \zeta_0$ and $\nu_\epsilon \rightharpoonup \nu_0$ weakly in $X$ as $\epsilon \to 0^+$, for some non negative $\zeta_0, \nu_0 \in X$.

In the sequel, we establish that $\zeta_0 \neq \nu_0$ and forms a weak solution to our problem (1.2). For convenience, we denote by $v_0$ either $\zeta_0$ or $\nu_0$. We prove the result in the following steps.

**Step 1.** Here, we prove that $v_0 \in X$ is a weak solution to the problem (1.2).

We observe that for any $\epsilon \in (0,1)$ and $t \geq 0$,
\[
\frac{\lambda}{(t + \epsilon) q(x)} + t^r \geq \frac{\lambda}{(t + 1) q(x)} + t^r \geq \min \left\{ 1, \frac{\lambda}{2} \right\} := C > 0, \text{ say.}
\]

As a consequence we get
\[
- \Delta_{H,p} v_\epsilon = \frac{\lambda}{(v_\epsilon + \epsilon) q(x)} + v_\epsilon^r \geq C > 0.
\]

By Lemma 3.1, let $\xi \in X$ satisfies
\[
- \Delta_{H,p} \xi = C \text{ in } \Omega, \xi > 0 \text{ in } \Omega,
\]
such that for every $\omega \in \Omega$, there exists a constant $c_\omega > 0$ satisfying $\xi \geq c_\omega > 0$ in $\Omega$. Then, for every nonnegative $\phi \in X$, we have

$$
\int_{\Omega} H(v_\epsilon)^{p-1} \nabla H(v_\epsilon) \nabla \phi \, dx = \int_{\Omega} \left( \frac{\lambda}{(v_\epsilon + \epsilon)^q(x)} + v_\epsilon^r \right) \phi \, dx
$$

$$
\geq \int_{\Omega} C_\phi \, dx = \int_{\Omega} H(\nabla \xi)^{p-1} \nabla H(\nabla \xi) \nabla \phi \, dx.
$$

Now choosing $\phi = (\xi - v_\epsilon)^+$, we obtain

$$
\int_{\Omega} \left\{ H(\nabla \xi)^{p-1} \nabla H(\nabla \xi) - H(\nabla v_\epsilon)^{p-1} \nabla H(\nabla v_\epsilon) \right\} \nabla (\xi - v_\epsilon)^+ \, dx \leq 0.
$$

Now applying Lemma 2.5, we obtain $v_\epsilon \geq \xi$ in $\Omega$. Hence, we get the existence of a constant $c_\omega > 0$ (independent of $\epsilon$) such that

(5.3) \hspace{1cm} v_\epsilon \geq c_\omega > 0, \text{ for every } \omega \in \Omega.

Therefore using $r + 1 < p^*$ along with Lemma 4.6 and the fact (5.3) as in the proof of Theorem 1.4 we can apply Boccardo-Murat [12, Theorem 2.1] to obtain

$$
\int_{\Omega} H(\nabla v_0)^{p-1} \nabla H(\nabla v_0) \nabla \phi \, dx = \lambda \int_{\Omega} \frac{\phi}{v_0^q(x)} \, dx + \int_{\Omega} v_0^\phi \, dx.
$$

Hence the claim follows.

**Step 2.** Now we are going to prove that $\zeta_0 \neq v_0$.

Choosing $\phi = v_\epsilon \in X$ as a test function in $(P_{\lambda, \epsilon})$ we get

$$
\int_{\Omega} H(\nabla v_\epsilon)^{p-1} \nabla H(\nabla v_\epsilon) \nabla \phi \, dx = \lambda \int_{\Omega} \frac{v_\epsilon}{(v_\epsilon + \epsilon)^q(x)} \, dx + \int_{\Omega} v_\epsilon v_\epsilon^+ \, dx.
$$

Since $r + 1 < p^*$, using Lemma 2.11 we obtain

$$
\lim_{\epsilon \to 0^+} \int_{\Omega} (v_\epsilon)^{r+1} \, dx = \int_{\Omega} v_0^{r+1} \, dx.
$$

Moreover, since

$$
0 \leq \frac{v_\epsilon}{(v_\epsilon + \epsilon)^q(x)} \leq v_\epsilon^{1-q(x)},
$$

using (4.4) together with Vitali convergence theorem, we get

$$
\lambda \lim_{\epsilon \to 0^+} \int_{\Omega} \frac{v_\epsilon}{(v_\epsilon + \epsilon)^q(x)} \, dx = \lambda \int_{\Omega} v_0^{1-q(x)} \, dx.
$$

Therefore for every $\phi \in X$, we have

$$
\lim_{\epsilon \to 0^+} \int_{\Omega} H(\nabla v_\epsilon)^{p-1} \nabla H(\nabla v_\epsilon) \nabla \phi \, dx = \lambda \int_{\Omega} v_0^{1-q(x)} \, dx + \int_{\Omega} v_0^{r+1} \, dx.
$$

Using Lemma 2.13 we can choose $\phi = v_0$ as a test function in (1.11) to deduce that

$$
\int_{\Omega} H(\nabla v_0)^p \, dx = \lambda \int_{\Omega} v_0^{1-q(x)} \, dx + \int_{\Omega} v_0^{r+1} \, dx.
$$

Hence we obtain

$$
\lim_{\epsilon \to 0} \int_{\Omega} H(\nabla v_\epsilon)^p \, dx = \int_{\Omega} H(\nabla v_0)^p \, dx.
$$
By the Vitali convergence theorem, we get
\[ \lim_{\epsilon \to 0} \int_{\Omega} [(v_\epsilon + \epsilon)^{1-q(x)} - \epsilon^{1-q(x)}] \, dx = \int_{\Omega} v_0^{1-q(x)} \, dx, \]
which together with the strong convergence of \( v_\epsilon \) in \( X \) implies \( \lim_{\epsilon \to 0} I_{\lambda, \epsilon}(v_\epsilon) = I_\lambda(v_0) \). Hence from (4.13) we get \( \zeta_0 \neq \nu_0 \).

**Proof of Theorem 1.7:** Noting Lemma 2.12 and Remark 4.5, proceeding similarly as in the proof of Theorem 1.6, the result follows.

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