Geometric description of the Schrödinger equation
in $3n+1$ dimensional configuration space

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We show that for non-relativistic free particles, the (bosonic) many particle equations can be rewritten in geometric fashion in terms of a classical theory of conformally stretched spacetime. We further generalize the results for the particles subject to a potential.

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I. INTRODUCTION

In general relativity (GR), gravitational interactions and phenomena are formulated and understood in terms of purely geometrical quantities. In this theory, mass-and energy-sources geometrically curve spacetime and point particles move on simple geodesics in this background. This beautiful and intuitive geometric fundament (apart from its great phenomenological success) are the reason for GRs unbroken popularity, more than 100 years after its formulation.

Quantum mechanics (QM) on the other hand deals with more abstract objects such as wave functions or probability amplitudes, which are usually less familiar than the trajectories and curved surfaces that appear in GR. It is thus natural that there are numerous attempts to reformulate quantum mechanics in a more geometrical way\cite{1,2}.

In order to cast quantum mechanics in the geometric language of GR one typically needs to define physical trajectories and a background space. Using the language of \cite{21,22}, it was shown that such trajectories naturally arise in the configuration space for the complex Klein Gordon equation. It was further found that the evolution equation for those trajectories can be cast in the form of a geodesics equation in a conformally rescaled configuration space\cite{2,3,7,9,18,19}. Thus, the relativistic Klein Gordon equation can be rewritten in a geometric language with non-trivial trajectories in configuration space.

The purpose of this article is to discuss the non-relativistic limit of those findings and to examine to which extent one can accommodate a quantum mechanical potential in this reformulation.

The paper is organized as follows: in section II we examine the non-relativistic limit of the quantum Klein-Gordon equation and for a system of free bosonic particles, which is rewritten by splitting the wave function in amplitude and phase. In section III, a geometric theory using an $n$-particle action is presented and it is shown that this geometric formulation can (by the use of suitable matching conditions) be identified with the rewritten quantum mechanics given in section II. In section IV, the results are generalized to case where the particles are not free but are subject to a potential. Section V contains some useful discussions and section VI summarizes the results.

II. NON-RELATIVISTIC LIMIT OF THE KLEIN-GORDON EQUATION

In this section it is shown how the non relativistic limit of the quantum Klein Gordon can be rewritten by splitting the wave function in amplitude and phase. The notation will largely follow \cite{12}. In the non-relativistic limit space and time are not on equal footing so here spatial and temporal derivatives are treated differently. The quantum Klein-Gordon equation. is

$$\left(\Box + \frac{M^2c^2}{\hbar^2}\right)\psi = 0. \quad (1)$$

To get non-relativistic limit of (11) we make use of the following ansatz

$$\psi(x,t) = \phi(x,t)\exp\left(-\frac{iMc^2t}{\hbar}\right). \quad (2)$$

In the non-relativistic limit the energy difference between total energy and rest mass is assumed to be small such that

$$E' = E - Mc^2,$$

with

$$E' \ll Mc^2.$$

These approximations yield the Schrödinger equation for spinless particles. The many particle Schrödinger equa-
tion can be written as
\[ \left( \sum_{j=1}^{n} \frac{\hbar}{2M_j} \partial_j^m \partial_j m + i \partial_0 \right) \psi(x_1, x_2, ..., x_n, t_1) = 0. \] (3)

Here, \( M \) is the single particle mass and \( M_j \) represents \( n \)-particle mass. The particle to be affected out of \( n \) particles is denoted by the index \( j \) and \( m \) is the space index in three flat dimensions. One factorize the wave function into amplitude and phase as just like it is done in [21, 22]
\[ \psi = Pe^{iS/\hbar}. \]

Since we are working in non-relativistic limit, so "time" is absolute, and we have only one time coordinate \( t_1 \) instead of \( n \) time coordinates \( t_j \). The quantum phase then reads
\[ S(t_j, \vec{x}_j) = -Mt_1 + \tilde{S}(t_1, \vec{x}_j). \]

The same projection onto a single time coordinate is true for Hamilton’s principle function \( S_H \), and the amplitude \( P \)
\[ P(t_j, \vec{x}_j) = P(t_1, \vec{x}_j). \]

Using the above definition of amplitude and phase in non-relativistic limit we rewrite the wave function as
\[ \psi(t_1, \vec{x}_j) = P(t_1, \vec{x}_j)e^{i(S(t_1, \vec{x}_j) - Mt_1)/\hbar}. \] (4)

Thus, (3) reads
\[ \left( \sum_{j=1}^{n} \frac{\hbar}{2M_j} \partial_j^m \partial_j m + i \partial_0 \right) P(t_1, \vec{x}_j)e^{i(S(t_1, \vec{x}_j) - Mt_1)/\hbar} = 0. \]

Here, \( \partial P/\partial t_1 = 0 \), as for large \( t \) the amplitude on the average is zero. Taking the real part of the above equation after using Taylor series, one gets
\[ \sum_{j=1}^{n} \frac{\hbar}{2M_j} \partial_j^m \partial_j m P(t_1, \vec{x}_j) = \sum_{j=1}^{n} \frac{(\partial_j \tilde{S})(\partial_j \tilde{S})}{2M_j} + \tilde{S} - M, \]

where the derivative with respect to \( t_1 \) is represented by a dot. Thus, one can write
\[ Q = \sum_{j=1}^{n} \frac{(\partial_j \tilde{S})(\partial_j \tilde{S})}{2M_j} + \tilde{S} - M, \] (5)

where,
\[ Q = \sum_{j=1}^{n} \frac{\hbar}{2M_j} \partial_j^m \partial_j m P(t_1, \vec{x}_j). \]

Equation (5) can be thought of as Hamilton-Jacobi equation as in classical mechanics. The term \( Q \) here represents the quantum potential \( Q(x_1, x_2, ..., x_n, t_1) \) and \( P(x_1, x_2, ..., x_n, t_1) \) is the pilot wave [21, 22]. The defined non relativistic wave function allows to construct conserved current as
\[ \partial_0(\psi^* \psi) - \sum_{j=1}^{n} \frac{\partial_j^m}{2M_j} (\psi^* \partial_j \psi) = 0. \]

The conserved current with the defined non-relativistic wave function yields a real equation
\[ \partial_0(P^2) + \sum_{j=1}^{n} \partial_j m \left( \frac{P^2}{M_j} \partial_j^m \tilde{S} \right) = 0. \] (6)

Equation (6) is the second equation representing the conserved current. Notice that \( \partial_0(P^2) \) is not zero because here negative changes in amplitude would also become positive.

The interpretation in terms of trajectories is introduced by defining velocities and momentum by
\[ p_j^n = M_j \frac{dx_j^n}{ds} = \partial_j^m \tilde{S}. \] (7)

Now the equation of motion for \( n \) non-relativistic particles is given by
\[ \frac{dx_j^n}{ds} = \frac{\partial_j^m \tilde{S}}{M_j}. \]

Using
\[ \frac{d}{ds} = \sum_{j=1}^{n} \frac{d}{dx_j^n} \frac{dx_j^n}{ds}, \]

one finds
\[ \frac{d^2 x_j^n}{ds^2} = \sum_{i=1}^{n} \frac{(\partial_i^j \tilde{S})(\partial_i^m \partial_i \tilde{S})}{M_j^2}. \] (8)

Here, \( ds = dt_1 \) is the common time coordinate for all \( n \) particles. Please note that those equations are just a rewriting of the initial equation (3). Please note further that in addition to analogous manipulations, the de Broglie Bohm (dBB) interpretation consists in associating physical reality to the trajectories (8). Since the dBB theory is deterministic and non-local, equation (8) confirms the non-local nature of dBB theory i.e., position of one particle depends on the position of all other particles constituting the system. This can be thought of as Newton’s second law of motion and this equation shows that the motion of \( j \)th particle is affected by a ‘force’ \[ \sum_{i=1}^{n} \frac{\partial_i^j \tilde{S})(\partial_i^m \partial_i \tilde{S})}{M_j^2} \] which involves the position of all the particles [23].

However, the purpose of this article is not to assign physical reality to the laws of quantum motion. Instead the purpose is to show that quantum mechanics can be rewritten in a geometric language, independent of the
question of physical reality of the appearing trajectories. In the equations [4, 5], the amplitude $P$, and the phase $S$, and $Q$ depend on $1 + 3n$ coordinates, $3n$ of space and 1 of time. In order to abbreviate notation one can define

$$x^L = (t_1, \tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_n),$$

such that $\partial_j^m \to \partial^L$ and $\partial_{jm} \to \partial_L$; with

$$Q = \frac{\hbar^2}{2M_j} \partial^L \partial_L P(t_1, \tilde{x}_j).$$

As mentioned above, we will take the temporal derivative of amplitude as zero as on the average $\partial P/\partial t_1 = 0$. So there remain only spatial derivatives.

Thus the non-relativistic limit of [6] is equivalently given by the following set of equations

$$Q = \frac{(\partial^L \tilde{S})(\partial_L \tilde{S})}{2M_j} + \tilde{S} - M, \quad (9)$$

$$\partial_L \left( \frac{P^2}{M_j} \partial^L \tilde{S} \right) + \partial_0 (P^2) = 0, \quad (10)$$

$$p^L = M_j \frac{dx^L}{ds} = \partial^L \tilde{S}, \quad (11)$$

$$\frac{d^2 x^L}{ds^2} = \frac{(\partial^N \tilde{S})(\partial_L \partial_N \tilde{S})}{M_j^2}. \quad (12)$$

It is the purpose of the following section to show that this set of equations can be obtained from a purely geometrical formulation in configuration space.

III. GEOMETRY OF CONFIGURATION SPACE

We consider a $1 + 3n$ dimensional configuration space of $n$-particles with one single time coordinate. The coordinates are denoted by

$$\tilde{x}^L = (\tilde{t}_1, \tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_n).$$

The model specifying the curvature of such space uses a single $1 + 3n$ dimensional scalar equation given by

$$P_s \left( \tilde{R} + k \tilde{L}_M \right) = \tilde{R} + k \tilde{L}_M. \quad (13)$$

Here $P_s$ is symmetrization operator between different particles $x^L_i$ and $x^L_j$, $\tilde{R}$ is Ricci scalar, $\tilde{L}_M$ is matter lagrangian and $k$ is coupling constant representing the extent of interaction between particles and field.

With the symmetrization condition [13] the particle action reads

$$S (\tilde{g}_{\Delta}) = \int dt \int dx^3n \sqrt{|g|} \left( \tilde{R} + k \tilde{L}_M \right) \quad (14)$$

The local conformal part of the theory is described separately by splitting the metric $\tilde{g}$ into a conformal function $\phi(\tilde{x}, t_1)$ and a flat part $\eta$ [19]. The conformal transformation here is given by

$$\tilde{g}_{\Lambda \Gamma} = \phi^{\frac{-\Lambda}{M}} \frac{\eta_{\Lambda} \eta_{\Gamma}}{\eta_{\Delta} \Lambda_{\Delta}}. \quad (15)$$

The inverse of the metric is given by

$$\tilde{g}_{\Lambda \Gamma} = \phi^{\frac{-\Lambda}{M}} \eta^L \eta^R. \quad (16)$$

The lower Greek and lower Roman index are identified as $\delta_{\Lambda} = \partial_L$ so that the adjoint derivatives are different in each notation i.e.,

$$\delta^\Lambda = g^{\Lambda \Sigma} \partial^\Sigma = \phi^{\frac{-\Lambda}{M}} \eta^L \partial^S$$

$$\delta^\Lambda = \phi^{\frac{-\Lambda}{M}} \partial_L$$

$$\delta^\Lambda = \phi^{\frac{-\Lambda}{M}} \partial_L.$$

A. Geometric Dual of the 1st Equation:

The particle action in terms of $\phi$ and $g_{LD}$ is [19]

$$S (\phi, g_{LD}) = \int dt \int dx^3n \sqrt{|g|} \left( \frac{12n}{1 + 3n} \partial^L \phi (\partial_L \phi) + \phi^2 (R + kL_M) \right)$$

It is interesting to note that in a truly gravitational theory a similar factorization of the conformal factor can be used to mimic the effects of dark matter [21]. However, in this toy model in configuration space we are only interested in studying flat “Minkowski” background space so $g_{LG} = \eta_{LG}$ and $|g| = 1$ and $R = 0$. So the action simplifies to

$$S (\phi) = \int dt \int dx^3n \left( \frac{12n}{1 + 3n} \partial^L \phi (\partial_L \phi) + \phi^2 (kL_M) \right) \quad (18)$$

The equation of motion for $\phi$ is

$$\frac{12n}{1 + 3n} \partial^L \partial_L \phi = kL_M. \quad (17)$$

The matter Lagrangian $L_M$ is given by

$$L_M = \frac{2(\partial^L \tilde{S}_H)(\partial_L \tilde{S}_H)}{2M_G} + \frac{\tilde{S}_H}{\partial t_1} - \dot{M},$$

which also contains square of first order temporal derivatives. The equation of motion for $\phi$ reads

$$\frac{12n}{k(1 + 3n)} \partial^L \partial_L \phi = \frac{2(\partial^L \tilde{S}_H)(\partial_L \tilde{S}_H)}{2M_G} + \dot{\tilde{S}_H} - \dot{M}. \quad (18)$$

With the following matching conditions

$$k = \frac{12n}{1 + 3n} \frac{2M_G}{\hbar^2}.$$
we find that the geometrical equation (18) is identical to the quantum equation (19). Please note that in relativistic case the dual theory was developed in $4n$ dimensions while here equation (18) is obtained from the geometric theory developed in $(1+3)$ dimensions. This clearly reflects the fact that here space and time are treated differently. In contrast to the relativistic case, the non-relativistic case in addition to spatial changes also incorporates temporal changes of the pilot wave.

\[ \phi(x_j,t_1) = P(x_j,t_1) \]

\[ \tilde{S}_H(x_j,t_1) = \tilde{S}(x_j,t_1) \]

\[ M_j = \tilde{M}_G \]

B. Geometric Dual of the 2nd Equation

The stress energy tensor is of the matter part given by

\[ T^{\Lambda\Delta} = \frac{2(\partial^\Lambda \tilde{S}_H)(\partial^\Delta \tilde{S}_H)}{M_G} + g^{\Lambda\Delta} \left( \frac{\partial^\Lambda \tilde{S}_H(\partial^\Delta \tilde{S}_H)}{M_G} + \partial_0 \tilde{S}_H - \tilde{M} \right), \]

where $\partial_0 = \partial/\partial t_1$.

Since the stress energy tensor is covariantly conserved, so

\[ \nabla_\Lambda T^{\Lambda\Delta} = 0 \]

From this, we can write

\[ \frac{(\partial^\Lambda \tilde{S}_H)\nabla_\Lambda (\partial^\Delta \tilde{S}_H)}{M_G} = 0, \tag{19} \]

\[ \frac{(\partial^\Lambda \tilde{S}_H)\nabla^\Delta (\partial^\Lambda \tilde{S}_H)}{M_G} = 0, \tag{20} \]

\[ \frac{(\partial_0 \tilde{S}_H)\nabla_\Lambda (\partial^\Lambda \tilde{S}_H)}{M_G} + \nabla_\Lambda (\partial_0 \tilde{S}_H) = 0. \tag{21} \]

The Levi-Civita connection is given by

\[ \Gamma^{\Sigma\Delta}_{\Lambda\Delta} = \frac{1}{2} g^{\Sigma\Xi} \left( \partial_\Lambda g_{\Xi\Xi} + \partial_\Xi g_{\Xi\Lambda} - \partial_\Xi g_{\Xi\Lambda} \right), \]

\[ \Gamma^{\Sigma\Delta}_{\Lambda\Delta} = \frac{1}{2} \phi^{\frac{\Lambda}{3n-1}} \left[ (\partial_L \phi^{\frac{4}{3n-1}}) \delta^S_D + (\partial_D \phi^{\frac{4}{3n-1}}) \delta^S_L - (\partial^S \phi^{\frac{4}{3n-1}}) \eta_{L,D} \right]. \]

With this, the relation (21) reads for the first term

\[ \nabla_\Lambda (\partial^\Lambda \tilde{S}_H) = \partial_\Lambda (\partial^\Lambda \tilde{S}_H) \]

\[ + \frac{1}{2} g^{\Sigma\Xi} [\partial_\Lambda g_{\Xi\Xi} + \partial_\Xi g_{\Xi\Lambda} - \partial_\Xi g_{\Xi\Lambda}] \times [\partial^\Lambda \tilde{S}_H] = 0, \]

or

\[ \phi^{\frac{2-6n}{2n}} \partial_L (\phi^2 (\partial^L \tilde{S}_H)) = 0 \tag{22} \]

For the second term in (21) we used

\[ \nabla_\Lambda (\partial_0 \tilde{S}_H) = \partial_\Lambda (\partial_0 \tilde{S}_H) \]

and

\[ \nabla_\Lambda (\partial_0 \tilde{S}_H) = \partial_0 (\partial_L \tilde{S}_H). \tag{23} \]

From (21), with (22) and (23), one gets

\[ \left( \frac{\partial_L \left[ \phi^2 (\partial^L \tilde{S}_H) \right]}{M_G} + \partial_0 (\phi^2) \right) = 0. \tag{24} \]

With the given matching conditions, one confirms that equation (24) is identical to (10).

In non-relativistic limit $\psi^* \psi$ can be interpreted as probability density, but it is not possible to provide an interpretation for the probability of Klein-Gordon equation with $\psi^* \psi$. The probability interpretation of Klein-Gordon equation is given in terms of Klein-Gordon current, that is conserved with respect to time. Here $\phi(t_1, \bar{x}_j)$ enters two ways, $\phi(t_1, \bar{x}_j)$ represents the interaction of matter with the conformal field and its square $\phi^2(t_1, \bar{x}_j)$ represents the probability density.

C. Geometric Dual of the 3rd Equation:

The momentum is defined by the derivative of the Hamilton principle function $\tilde{S}_H$ as suggested by Hamilton-Jacobi formalism

\[ \tilde{p}^\Lambda = (\partial^\Lambda \tilde{S}_H). \tag{25} \]

This is identical to the third equation (11).

D. Trajectory Equation of Motion

It remains to obtain the geometric dual of the relation (12). One notes that the total derivative is

\[ \frac{d}{ds} = \frac{d\bar{x}^\Lambda}{ds} \partial_\Lambda \]

\[ \frac{d}{ds} = \phi^{\frac{4}{3n-1}} \frac{dx^L}{ds} \partial_L \]

\[ \frac{d}{ds} = \phi^{\frac{4}{3n-1}} \frac{d}{ds} \]
Applying this relation to momenta gives

$$\hat{M}_G \left( \frac{d\hat{x}^\Lambda}{ds} \right) = (\hat{\partial}^\Lambda \hat{S}_H),$$

and

$$\frac{d^2\hat{x}^\Lambda}{ds^2} = \frac{d}{ds} \frac{(\hat{\partial}^\Lambda \hat{S}_H)}{M_G}.$$ 

Using the identity

$$\frac{d}{ds} = \hat{\partial}_s \frac{d\hat{x}^\Lambda}{ds},$$

one finds

$$\frac{d^2\hat{x}^\Lambda}{ds^2} = \frac{(\hat{\partial}^\Lambda \hat{S}_H)}{M_G} \hat{\partial}_s \left( \frac{\hat{\partial}^\Lambda \hat{S}_H}{M_G} \right).$$ \hspace{1cm} (26)

This is the equation of motion for non-relativistic particles. We can relate it with another equation, the "Geodesic equation of motion" in non-relativistic limit.

$$\frac{d^2\hat{x}^\Lambda}{ds^2} = -\Gamma^\Lambda_{\Phi\Phi}$$ \hspace{1cm} (27)

$$\frac{d^2\hat{x}^\Lambda}{ds^2} = \frac{1}{2} \delta^\Lambda_{ij} \hat{\partial}_{\Phi_{ij}}.$$

Applying a conformal transformation it leads to

$$M \frac{d^2\hat{x}^\Lambda}{ds^2} = -\frac{1}{2} M (\nabla \phi)^{\pi_{\Lambda-\Theta}}$$

which is just like the equation of motion in "Newtonian gravity". With the given matching conditions, equation [20] is identical to [12].

IV. INCLUDING A POTENTIAL

The Schrödinger equation in presence of potential is

$$\left( -\hbar^2 \frac{\nabla^2}{2M} + V \right) \psi = i\hbar \frac{\partial \psi}{\partial t}$$

It is straight forward to see that this modifies the previous discussion only by an additional term in

$$Q = \sum_{j=1}^{n} \left( \frac{\partial_j \hat{S}}{2M_j} \right) + \hat{S} + V - M.$$ \hspace{1cm} (28)

A. Geometric Dual

The particle action in terms of $\phi$ and $g_{LD}$ is

$$S(\phi, g_{LD}) = \int dt \int dx^{3n} \sqrt{|g|} \left[ \frac{12n}{1-3n} (\partial^L \phi)(\partial_L \phi) + \phi^2 (R + kV + kL_M) \right]$$

We are interested in studying flat Minkowski background space so $g_{LG} = \eta_{LG}$ and $|g| = 1$ and $R = 0$. So the geometric action is again

$$S(\phi) = \int dt \int dx^{3n} \left( \frac{12n}{1-3n} (\partial^L \phi)(\partial_L \phi) + \phi^2 (kV + kL_M) \right),$$

where the appearance of the potential $V$ was imposed externally.

From this action, the equation of motion for $\phi$ is

$$\frac{12n}{1-3n} \frac{\partial^L \partial_L \phi}{\phi} = k(V + L_M)$$

with

$$L_M = \frac{2(\partial^L \hat{S}_H)(\partial_L \hat{S}_H)}{2M_G} + \frac{\partial \hat{S}_H}{\partial t} - \hat{M}$$

and thus

$$\frac{12n}{k(1-3n)} \frac{\partial^L \partial_L \phi}{\phi} = \frac{2(\partial^L \hat{S}_H)(\partial_L \hat{S}_H)}{2M_G} + \frac{\partial \hat{S}_H}{\partial t} + V - \hat{M}. $$ \hspace{1cm} (29)

With the defined matching conditions, [20] is identical to [28].

V. DISCUSSIONS

In context with the above relation one should mention a few points

- Locality:

  Any geometric description in analogy to the language of GR is supposed to be local, where the rewritten quantum equations [9] are clearly non-local. “How, could such descriptions be equivalent?”

  The answer to this question comes from the fact that the coordinates $x^L$ of the (local) geometric description are defined in the higher dimensional configuration space of the system, while the coordinates $\hat{x}$ in the (non-local) quantum description are initially defined in three dimensional space plus a universal time.

- Is the presented geometric description “gravity”? Even thought, the geometric description uses a covariant metric formalism, it is not gravity, it is just a geometrical rewriting of the same quantum mechanical theory. The distinction from gravity comes
due to several reasons. First, it is defined in configuration space and not in 4-D spacetime. Second, only the covariant derivatives of the conformal factor $\phi$ on a flat background is contemplated and off-diagonal contributions are neglected, like it would be done in a perturbation theory ansatz. Third, the matter Lagrangian and its coupling constant differ from the usual coupling between gravity and matter.

- Matching conditions:
  A set of matching conditions is defined to connect non-relativistic quantum mechanics with the classical geometric theory. These matching conditions are not unique but are chosen depending upon the situation, so as to provide a suitable connecting link between the two theories. These conditions connect the quantum phase $\tilde{S}$ with the Hamilton principle function $\tilde{S}_H$, the amplitude of pilot wave $P$ with the conformal function of the metric $\phi$, and the mass $M_j$ with the mass $\hat{M}_G$. The coupling constant in the geometrical reformulation is given by

$$k = \frac{12n}{1-3n} \frac{2\hat{M}_G}{\hbar^2},$$

which depends on the number of particles e.g. for $n = 1, k = -\frac{12\hat{M}_G}{\hbar^2}$.

VI. SUMMARY

It has been shown that, for spinless particles, the $n$-particle equations of non-relativistic quantum mechanics can be rewritten in the language of a geometric theory in the $1 + 3n$ dimensional configuration space. A set of matching conditions is defined to translate one formulation to the other. We further generalized the study for bosons subjected to a potential $V$.

It is hoped that this work on the many particles Schrödinger equation allows the interested reader to add a different, a geometric, perspective on the laws of quantum mechanics.

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