Feedback Ansatz for Adaptive-Feedback Quantum Metrology Training with Machine Learning

Yi Peng\(^1,2\) and Heng Fan\(^1,2,3,4,\ast\)

\(^1\)Institute of Physics, Chinese Academy of Sciences, Beijing 100190, China
\(^2\)School of Physical Sciences, University of Chinese Academy of Sciences, Beijing 100190, China
\(^3\)CAS Center for Excellence in Topological Quantum Computation, University of Chinese Academy of Sciences, Beijing 100190, China
\(^4\)Songshan Lake Materials Laboratory, Dongguan 523808, Guangdong, China

(Dated: October 9, 2019)

It is challenging to construct metrology schemes which harness quantum features such as entanglement and coherence to surpass the standard quantum limit. We propose an ansatz for devising adaptive-feedback quantum metrology (AFQM) strategy which reduces greatly the searching space. Combined with the Markovian feedback assumption, the computational complexity for designing AFQM would decrease from \(N^7\) to \(N^4\), for \(N\) probing systems. The feedback scheme devising via machine learning such as particle-swarm optimization and derivative evolution requires much less time and produces equally well imprecision scaling. We have thus devised an AFQM for 207-partite system. The imprecision scaling would persist steadily for \(N > 207\) when the parameter settings for 207-partite system is employed without further training.

Introduction—Given \(N\) entangled probing systems, quantum metrology promises parameter estimation with imprecision below the lowest limit \(1/\sqrt{N}\) allowed by classical theory. This is known as the standard quantum limit (SQL). The lowest imprecision permitted by quantum mechanics is \(1/N\), i.e. the so-called Heisenberg limit (HL) \([1-6]\). Such kind of quantum superiority over the classical schemes attracts much attention in both academic and industry communities. Because it has wide range of applications including spectroscopy \([7]\), accurate clock construction \([8-10]\), gravitational wave detection \([11, 12]\), fundamental biology research and medicine development \([13]\), and others \([2, 5, 6, 14]\).

To harness the advantage offered by quantum mechanics for practical metrology, there exist at least three prominent challenges. a) Both SQL and HL are asymptotic and require great amount of data to approach. It is a serious limitation in many circumstances. For instance, the gravitation detection window is very narrow \([11, 12]\) while many biological samples are too fragile to endure much photon bombardment \([13, 15]\). Thus, we need to finish the interference in a restricted time period and with limited number of prob systems (such as photons). b) Environment noise which is inevitable in practical platforms can completely demolish such quantum advantage \([16-18]\). c) Many metrology schemes proposed previously require input states or final measurements which are difficult to realize. For example, the Greenberger-Horne-Zeilinger (GHZ) state has ability to asymptotically achieve HL \([2-4, 19-21]\). Synthesizing GHZ state is well recognized as highly complicated and inefficient in many platforms \([20, 22-28]\). In typical phase estimation tasks, canonical positive-operator-valued measure (POVM) based on the so-called phase state \(|\phi\rangle\langle\phi|\) and the sine input state \((9)\) have been frequently utilized to demonstrate asymptotic HL. To our knowledge, there is no clear way to realize either of them for \(N \geq 3\) \([29-33]\).

The adaptive-feedback quantum metrology (AFQM) is believed to be a promising candidate capable of giving good parameter estimation with limited number of measurements and thus resolve issue a). As an example, the so-called Berry-Wiseman-Breslin scheme (BWB) can provide single-shot estimation achieving imprecision below SQL. Besides, BWB employs local projective measurements which partially resolves issue c). The limitation of BWB is its employment of the sine input state which is hard to generate \([34, 35]\). BWB is a well-educated heuristic strategy. Devising AFQM is highly challenging. Considering the AFQM employing local projective measurement described in Fig. 1, the total measurement outcome combinations as well as the feedbacks would amount to \(2^N\) if \(N\) qubits are employed. It indicates plenty flexibility of this type of AFQM scheme as well as a great challenge of optimizing it. It was firstly shown by Hentschel and Sanders that we can reduce the dimension of the feedback parameter space to \(N\) assuming a Markovian feedback sequence. By employing machine learning such as particle-swarm optimization (PSO) and differential evolution (DE), one can devise promising AFQM autonomously. We call such a scheme devising procedure as the Hentschel-Sanders approach (HS) \([36, 37]\). Without noise, the achievable imprecision breaches SQL and shows superiority over BWB (cf. Table I). Given permutation symmetric input state, schemes thus devised have remarkable resilience against environment noise. They can also provide single-shot estimation. Though only the sine and product input states have been considered up to now, HS can be applied to other types of input states. Hence HS approach can solve a), b) and c) simultaneously. Generating such an AFQM would consume time \(O(N^7)\) and memory space \(O(N)\) for...
FIG. 1. (Color online) Quantum circuit of AFQM employing local projective measurement for $N = 4$. $|\psi_{in}\rangle$ is the total input state. The interference process $\hat{U}_\phi$ is controlled by $\phi$. $\phi_0$ is the initial random guess generated by a random number generator (RNG). $\phi_1$ is feedback information gathered from the first measurement, $\phi_2$ from the first and second measurements while $\phi_3$ from the first three measurements. $\phi_k$ is the final single-shot estimation of $\phi$ determined by all the measurements. Case of arbitrary $N$ is similar.

Computation. AFQM for up to $N = 100$ has thus been devised [36–38, 40, 42–46].

Here we introduce an ansatz for devising AFQM. Drawn from HL and SQL, the ansatz suggests that the adjustment of the feedback in every step should be a polynomial of the inverse powers of the step. It can reduce the feedback parameter space dimension from $N$ to a chosen constant, if we adopt HS. The memory space for parameter storing would also be a constant. With the parameter space dimension reduced to a constant, we can ensure persistent improvement for at least $N = 207$ without increasing the training time of our policy for bigger $N$. Further, we can generate an $N$-partite scheme without knowing schemes for fewer qubits which is required in the previous HS implementations [36–38, 40, 42–46]. The computation time thus scales as $O(N^4)$. We test the ansatz for devising AFQM via PSO as well as DE. Both the previously studied sine state (9) and the spin-squeezed state (SSS) are considered. It is widely believed that SSS has admirable resilience against environmental noise [47–52] and its synthesis has been realized in many labs [53–62].

The performance of AFQM thus devised is as good as the performance of the AFQM devised via previous HS. One of the most intriguing part is that when applying the parameters obtained for 207-partite system to bigger systems $N > 207$, the imprecision scaling persists for an admirable range.

TABLE I. Summary of previous results. $\alpha$ is the inverse-scaling power of the imprecision $\delta \alpha$ with respect to $N$. $N_{\text{max}}$ is maximum prob number for which AFQM can be obtained via HS machine learning approach.

| Ref. | BWB | PSO | DE |
|------|-----|-----|-----|
| $\alpha$ | 0.704 | 0.708 | 0.736 | 0.747 | 0.74 | 0.71 | 0.7198 | 0.729 |
| $N_{\text{max}}$ | $\leq 14$ | $\leq 50$ | $\leq 98$ | $\leq 100$ | $\leq 100$ | $\leq 100$ | $\leq 100$ |

Feedback ansatz for AFQM. Suppose there are $N$ spin-1/2 probes. The interference process parameterized by $\phi$ is

$$\hat{U}_\phi = e^{-i\phi \hat{J}_y}, \quad \text{with} \quad \hat{J}_y = \sum_{n=1}^{N} \hat{s}^{(n)}_y. \tag{1}$$

$\hat{J}_{x,y,z}$ denotes total angular momentum along $x$, $y$ and $z$ direction respectively while $\hat{s}^{(n)}_{x,y,z}$ are spin operators of the $k$th probe. After the probes have passed through the parameter channel $\hat{U}_\phi$, we apply feedback adjustment $\hat{U}_\phi^{(k-1)}$ to compensate $\hat{U}_\phi$ as closely as possible. The initial compensation $\phi_0$ is a random guess between $-\pi$ and $\pi$. Note that we assume $\phi \in [\pi, \pi]$. Then we locally measure $\hat{s}^{(1)}_z$, the result of which would be estimated by the next compensation $\phi_1$. The estimation-compensation-measurement procedure carries on until we obtain the final estimation $\phi_N$. The $n$th compensation $\phi_n$ can be regarded as an update of $\phi_{n-1}$ with an adjustment depending on the $n$ previous measurement outcomes $s_1, \ldots, s_n$ of $\hat{s}^{(1)}_z, \ldots, \hat{s}^{(n)}_z$:

$$\phi_n = \phi_{n-1} - \Delta_n(s_1, \ldots, s_n). \tag{2}$$

We want $\Delta_n(s_1, \ldots, s_n)$ to bring $\phi_n$ closer to $\phi$ in each step and $|\phi_n - \phi|$ increases with respect to $n$. One can regard $\phi_1, \ldots, \phi_N$ as a serial of estimations of $\phi$. We would expect $|\phi_n - \phi|$ to be of the order of $1/n^\alpha$ with $\alpha$ being some positive constant between 1/2 and 1. $\alpha = 1/2$ correspond to SQL while $\alpha = 1$ to HL. We cannot allow $\Delta_n(s_1, \ldots, s_n)$ being too big compared with $|\phi_{n-1} - \phi|$. $\Delta_n(s_1, \ldots, s_n)$ cannot be too small either. Consider the case when $\Delta_n(s_1, \ldots, s_n) = s_n/2^{n-1}$. If the measurement result $s_n$ makes $\Delta_n(s_1, \ldots, s_n)$ move $\phi_n$ away from $\phi$, the best of all the later adjustments $\Delta_n'(s_1, \ldots, s_n)$ with $n' > n$ can achieve is to neutralize the detrimental effect of $\Delta_n(s_1, \ldots, s_n)$. In such circumstances, the final estimation $\phi_N$ would be worse than $\phi_n$. It seems setting $\Delta_n(s_1, \ldots, s_n)$ around the order of $|\phi_n - \phi|$ would be reasonable. Another fact should be noted is that $|\phi_n - \phi|$ and $|\phi_{n+1} - \phi|$ are about the same order. Thus $\Delta_n(s_1, \ldots, s_n)$ should be smaller than $1/|\phi_n - \phi|$. Now comes the feedback ansatz

$$\Delta_n(s_1, \ldots, s_n) \propto 1/(n+1)^{\rho_n(s_1, \ldots, s_n)}. \tag{3}$$

where $\rho_n(s_1, \ldots, s_n)$ can be out of the range $[1/2, 1]$ bounded by SQL and HL. We used $1/(n+1)^{\rho_n(s_1, \ldots, s_n)}$ instead of $1/n^{\rho_n(s_1, \ldots, s_n)}$ to ensure that the variation of $\rho_n(s_1, \ldots, s_n)$ matters. So far the ansatz (3) can only reduce the volume of the AFQM parameter space. Combined with other assumptions, it can reduce the parameter space drastically as shown in the following.

Hentschel-Sanders AFQM parameter space reduced by feedback ansatz. Here we provide an example of application of the feedback ansatz (3) in the HS AFQM devising approach. HS indicates that the adjustment of $\phi_n$
from the immediate former compensation \( \phi_{n-1} \) depends only on the measurement result \( s_n \) of \( s_n^{(\ell)} \).

\[
\phi_n = \phi_{n-1} - 2s_n \Delta_n. \tag{4}
\]

\( \Delta_1, \ldots, \Delta_N \) thus constitutes the AFQM parameter search space [36–38, 40, 43–46]. By invoking ansatz (3), we would have

\[
\phi_n = \phi_{n-1} - 2s_n/(n+1)^\alpha. \tag{5}
\]

The parameter space becomes that of \( \varphi_1, \ldots, \text{and } \varphi_N \). If \( |\phi_n - \phi| \) is about the scale of \( 1/n^\alpha \) and \( \alpha \) is nearly the same for almost every \( \phi_n \), then we may expect \( \Delta_n \) to be of the order \( 1/(n+1)^\alpha \) with \( \varphi \) being almost the same for every \( \Delta_n \), \( \varphi_1, \ldots, \) and \( \varphi_N \) are located in an small zone around \( \varphi \). Generally, we expect the adjustment to be a polynomial of the inverse power of \( n \)

\[
\Delta_n = \sum_{\ell=0}^{N_s-1} c_{\ell} \pi \left( \frac{n}{n+1} \right)^{\frac{\ell}{2}}, \tag{6}
\]

The coefficients \( c_0, \ldots, \text{and } c_{N_s-1} \) gives us enough flexibility in AFQM generation. \( N_s \) is our choice of number of terms in the expansion. (6) can be seen as a derivative of our feedback ansatz (3). Including \( \varphi \), there are \( N_s + 1 \) control parameters, the combination of which we call an inverse-scaling policy \( \mathcal{P} \). This reduces the search space dimension to \( N_s \) independent of \( N \). It enables a reduction of computation complexity. The memory space required to store \( \mathcal{P} \) is also constant.

Devising AFQM via machine learning: cost function and complexity.—Following the HS approach [36–38, 40, 43–45], we implement machine learning algorithm such as PSO and DE to generate AFQM under the guidance of our ansatz derivative (6). Employing machine learning to optimize the AFQM with the feedback ansatz is very much like a treasure hunting under the guidance of SQL and HL.

Cost function. We employ Holevo variance to quantify the imprecision \( \delta \phi \) of the final estimation \( \phi_N \) as before [34–38, 40, 43–46, 63, 64]

\[
V_\phi = (\delta \phi)^2 = \frac{1}{N_s^2} - 1, \text{ with } S = \left| \int_{-\pi}^{\pi} d\phi P(\phi)e^{i(\phi - \phi_N)} \right|. \tag{7}
\]

Note that \( V_\phi \) is an good approximation of the traditional variance in statistics when \( \phi_N \) is very close to \( \phi \) [64]. One can simulate \( K = 10N_s^2 \) trials of experiment and obtain thus many estimations \( \phi_N^{(k)} \). \( S \) is the so-called sharpness and can be estimated via Monte-Carlo method as [34–38, 40, 43, 44, 46]

\[
S = \left| \frac{1}{K} \sum_{k=1}^{K} e^{i(\phi - \phi_N^{(k)})} \right|. \tag{8}
\]

Complexity. Both PSO and DE employ a group of searching agents and record the best strategy found by the agents throughout their evolving [65, 66]. With greater number of searching agents, one can find better outcome at the cost of adding computation complexity. By our ansatz derivative (6), we need \( \Xi = 20(N_s + 1) \) searching agents instead of \( 20N \) [36–38, 40, 43–45] when the input state is the so-called sine state

\[
|\psi_{\sin}\rangle = \frac{1}{\sqrt{J+1}} \sum_{\mu=-J}^{J} \sin \left( \frac{\mu + J}{2} \pi \right) |j\mu\rangle, \tag{9}
\]

\( j = N/2 \) and \( |j\mu\rangle \) is the eigenstate of \( \hat{J}_x, \hat{J}_y, \hat{J}_z \) respectively, belonging to eigenvalue \( \mu \). We have also considered feeding the spin-squeezed state [67]

\[
|\psi_{\sss}\rangle = e^{i J_s \delta_{\adj}} e^{-i J^2 S} |\mu\rangle \text{ with } \delta_{\adj} = \frac{1}{2} \arctan \frac{B}{A}, \frac{1}{10}
\]

where \( A = 1 - (cos 2T_s)^N - 1 \) and \( B = 4 \sin T_s (cos T_s)^{(N-2)} \). Adding \( T_s \), the parameter space dimension would be \( N_s + 2 \) and thus we dispatch \( \Xi = 20(N_s + 2) \) agents to search if SSS has been fed to the interferometer. Given \( N_s = 4 \), the search space boundaries has been chosen according to Table II. We iterate both PSO and DE for

| \( \delta \) | \( c_\ell \) | \( T_s \) |
|---|---|---|
| \( [0,5] \) | \( [-5,5] \) | \( [0,2/\sqrt{N}] \) |

\( N_l = 300 \) times as has been done in Ref. [36–38, 40, 43–45]. The \( N \)-partite inverse-scaling policy can be generated directly, without knowing any \((N-k)\)-partite policy for \( 1 \leq k \leq N - 1 \). To generate a \( N \)-partite inverse-scaling policy we need first time of \( O(KN^2) = O(N^4) \). Recall that the number \( \Xi \) of searching agents is constant independent of \( N \) while each simulation of the adaptive feedback metrology progress consumes time of \( N^2 \) [42].

Results and analysis.— As has been mentioned previous, we consider both the sine state and SSS as input state. We consider the sine state for comparison with previous results. SSS is considered due to its well recognized noise-resisting ability and proven synthesis procedure in labs [53–62]. We generate AFQM for both kinds of input states via PSO as well as DE. There are four groups of data which we analyze and present in the following: PSO-SSS, PSO-Sine, DE-SSS and DE-Sine.

We summarized eight main conclusions drawn from our numerical data. i) AFQM generated with our feed-

- **TABLE II. Boundaries of inverse-scaling policy parameters.**
- Since we suspect \( \delta \) to be very close to the region between 1/2 (SQL) and 1 (HL), we choose the search zone that covers the region between SQL and HL and 10 times bigger. The boundaries for \( c_0, \ldots, \) and \( c_{N_s-1} \) are empirical which provides good results but not guaranteed to be optimal. We choose the upper bound 2/\( \sqrt{N} \) for spin squeezing time \( T_s \) since 1/\( \sqrt{N} \) is the minimum time needed to ensure maximal quantum Fisher information for |\( \psi_{\sss}\rangle \) [68]. Recall that quantum Fisher information quantities the metrology prowess of |\( \psi_{\sss}\rangle \) [1].
back ansatz is equally good as the previous AFQM generated via HS. Given sine input state, this is clear from Fig. 2(a,c) and Table II. ii) We can obtain inverse-scaling policy for bigger $N$ in shorter time. For example, we generated the 207-partite inverse-scaling policy for the sine state via PSO at the cost of approximately 200 hours running of 120 CPUs at 2.6 GHz. iii) Since the parameter space having a much small dimension $N$, PSO and DE produces almost equally good AFQM. There is no breakdown of PSO up to $N = 207$. iv) By optimizing the squeezing time as well, feeding SSS state to the interferometer can outperform AFQM with sine input state. v) The inverse-scaling policy trained for $N = 207$ can also sever as a good policy for AFQM with bigger $N$. As shown in Fig. 2, the power-law scaling of imprecision $\delta \varphi$ does not breakdown immediately for $N > 207$ if the inverse-scaling policy of $N = 207$ is applied without further training. In the case of sine state, the imprecision scaling of the 207-partite policy for $N > 207$ would gradually broke but would stay below SQL for a considerable range (cf. Fig. 2 (a,c)). Given SSS, the performance scaling would endure for much bigger $N$ when 207-partite policy is employed without further training (cf. Fig. 2 (b,d)). vi) As a matter of fact, we can see a general trend of the leading inverse-scaling exponentiation $\varphi$ (cf. Fig. 3(a)). From the fairly success of the 207-partite inverse-policy applying to bigger prob ensemble as well as the general trend of $\varphi$, one can see the validity and merit of our ansatz (3) and its derivative (6). vii) In fact we also see a general trend the optimal spin-squeezing time $T_s$ (cf. Fig. 3(b)). For $N$ big enough ($N \gtrsim 100$), our numerical result suggests that the optimal spin-squeezing time should be approximately $0.6/N^{2/3}$. In fact, we have been optimizing $c_s = T_s N^{2/3}$ in our simulation. In applying the 207-partite inverse-policy for SSS with $N > 207$, it is $c_s$ that has been inherited instead of $T_s$. This hints that the optimal squeezing time for employing SSS in AFQM should scale as $1/N^{2/3}$. viii) As long as the inverse-policy has been trained employing either PSO or DE, the scaling of the imprecision $\delta \varphi$ would not break up to at least $N = 207$. This upper limit for $N$ would be much bigger, since we can see the scaling persistence in a moderate range when the 207-partite inverse-policy has been applied without training for $N > 207$.

Conclusion and discussion. — We have proposed the feedback ansatz (3) for devising AFQM. When combined with the previous HS scheme, we demonstrated the prowess of our ansatz via numerical simulation. Though only the noise-less situation has been tested here, we can expect similar application in devising AFQM infested by environment noise. It may also be useful in devising multi-parameter estimation schemes. It is interesting to see that HL and SQL can be used as guidelines for AFQM designing instead of being mere metrology performance borderlines. Our method may provide more insight on this direction of research.

We thank B.C. Sanders for stimulating discussions. This work was supported National Key R & D Program of China (Grant Nos. 2016YFA0302104 and 2016YFA0300600), National Natural Science Foundation of China (Grant Nos. 11774406 and 11934018), Strategic Priority Research Program of Chinese Academy of Sci-

![FIG. 2. (Color online) Imprecision $\delta \varphi$ of AFQM generated via machine learning. The dots are numerical data while the solid line are generated by least-squares fitting the data represented by solid dots. Every solid dot represents AFQM employing inverse-scaling policy for $N = 207$ without further training.](image)

![FIG. 3. (Color online) General Trending of inverse-scaling policy. (a) Leading inverse-scaling exponentiate $\varphi$ of the adjustment $\Delta_\alpha$ of feedback and (b) optimal squeezing time for AFQM.](image)
ences (Grant No. XDB28000000), and Beijing Academy of Quantum Information Science (Grant No. Y18G07).

\[ \text{References (Grant No. XDB28000000), and Beijing Academy of Quantum Information Science (Grant No. Y18G07).} \]
(2013).

[52] L. Pezzé and A. Smerzi, Phys. Rev. Lett. 110, 163604 (2013).

[53] J. Hald, J. L. Sørensen, C. Schori, and E. S. Polzik, Phys. Rev. Lett. 83, 1319 (1999).

[54] T. Fernholz, H. Krauter, K. Jensen, J. F. Sherson, A. S. Sørensen, and E. S. Polzik, Phys. Rev. Lett. 101, 073601 (2008).

[55] T. Takano, M. Fuyama, R. Namiki, and Y. Takahashi, Phys. Rev. Lett. 102, 033601 (2009).

[56] C. Gross, T. Zibold, E. Nicklas, J. Estève, and M. K. Oberthaler, Nature 464, 1165 (2010).

[57] I. D. Leroux, M. H. Schleier-Smith, and V. Vuletić, Phys. Rev. Lett. 104, 073602 (2010).

[58] C. D. Hamley, C. S. Gerving, T. M. Hoang, E. M. Bookjans, and M. S. Chapman, Nat. Phys. 8, 305 (2012).

[59] R. J. Sewell, M. Koschorreck, M. Napolitano, B. Dubost, N. Behbood, and M. W. Mitchell, Phys. Rev. Lett. 109, 253605 (2012).

[60] W. Muessel, H. Strobel, D. Linnemann, D. B. Hume, and M. K. Oberthaler, Phys. Rev. Lett. 113, 1 (2014).

[61] O. Hosten, N. J. Engelsen, R. Krishnakumar, and M. A. Kasevich, Nature 529, 505 (2016).

[62] Y.-Q. Zou, L.-N. Wu, Q. Liu, X.-Y. Luo, S.-F. Guo, J.-H. Cao, M. K. Tey, and L. You, Proc. Natl. Acad. Sci. 115, 6381 (2018).

[63] A. S. Holevo, in Quantum Probability and Applications to the Quantum Theory of Irreversible Processes, edited by L. Accardi, A. Frigerio, and V. Gorini (Springer Berlin Heidelberg, Berlin, Heidelberg, 1984) pp. 153–172.

[64] D. W. Berry, Adaptive Phase Measurements, Ph.D. thesis, University of Queensland (2001).

[65] J. Kennedy and R. Eberhart, in Proc. ICNN’95 - Int. Conf. Neural Networks, Vol. 4 (IEEE, 1995) pp. 1942–1948.

[66] R. Storn, in Bienn. Conf. North Am. Fuzzy Inf. Process. Soc. - NAFIPS (IEEE, 1996) pp. 519–523.

[67] M. Kitagawa and M. Ueda, Phys. Rev. A 47, 5138 (1993).

[68] L. Pezzé and A. Smerzi, Phys. Rev. Lett. 102, 100401 (2009).