GLOBAL SECOND ORDER SOBOLEV-REGULARITY OF $p$-HARMONIC FUNCTIONS

AKSELI HAARALA AND SAARA SARSA

ABSTRACT. We prove a global version of the classical result that $p$-harmonic functions belong to $W^{2,2}_{\text{loc}}$ for $1 < p < 3 + \frac{2}{n-2}$. The proof relies on Cordes’ matrix inequalities [7] and techniques from the work of Cianchi and Maz’ya [5,6].

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz domain and $1 < p < \infty$. For some $\varphi \in W^{1,p}(\Omega)$ consider the problem of minimizing the $p$-energy functional

$$\int_{\Omega} |Dv|^p dx$$

among all functions $v \in \varphi + W^{1,p}_0(\Omega) := \{ v \in W^{1,p}(\Omega) : v - \varphi \in W^{1,p}_0(\Omega) \}$. Here $Dv = (v_{x_1}, \ldots, v_{x_n})$ denotes the weak gradient of $v$ and $|Dv| = (v^2_{x_1} + \cdots + v^2_{x_n})^{1/2}$ its Euclidean norm. By the direct method of calculus of variations, there exists a unique minimizer $u \in \varphi + W^{1,p}_0(\Omega)$. Moreover, the minimizer $u$ is $p$-harmonic in $\Omega$. That is, $u$ solves the homogeneous $p$-Laplacian equation

$$\Delta_p u := \text{div}(|Du|^{p-2}Du) = 0$$

in the weak sense. More precisely,

$$\int_{\Omega} |Du|^{p-2} \langle Du, D\phi \rangle dx = 0$$

for all $\phi \in C^\infty_0(\Omega)$. Thus the minimizer $u$ is a weak solution to the Dirichlet problem

$$\begin{cases}
\Delta_p u = 0 & \text{in } \Omega; \\
u = \varphi & \text{on } \partial \Omega.
\end{cases}$$

For the definitions of the function spaces that are ubiquitous in this paper, see the end of this section.

The regularity theory of $p$-harmonic functions is well known. Any $p$-harmonic function belongs to $C^{1,\alpha}_{\text{loc}}$, where $\alpha = \alpha(n,p) \in (0,1]$, see [11, 23, 24] for the case $p \geq 2$ and [9, 14, 16, 22, 25] for the full range $1 < p < \infty$. The standard second order regularity result of $p$-harmonic functions states that $|Du|^{\frac{p}{2}}Du \in W^{1,2}_{\text{loc}}$, see [3] for $p \geq 2$ and [8] for $1 < p < \infty$. Manfredi and Weitsman [17, Lemma 5.1] showed that $u \in W^{2,2}_{\text{loc}}(\Omega)$ under the condition $1 < p < 3 + \frac{2}{n-2}$. The restriction to the range of $p$ arises from a so-called Cordes condition. We also mention a recent result by Dong, Peng, Zhang and Zhou [10] that $|Du|^{\beta}Du \in W^{1,2}_{\text{loc}}$.
for $\beta > -1 + \frac{(n-2)(p-1)}{2(n-1)}$, The range of $\beta$ was later improved to $\beta > -1 + \frac{(n-2)(p-1)}{2(n-1)}$ by the second author [20].

The purpose of this paper is to investigate a global version of the classical result by Manfredi and Weitsman [17, Lemma 5.1] that if $1 < p < 3 + \frac{2}{n-2}$, then $u \in W^{2,2}_{\text{loc}}(\Omega)$. We prove that if $1 < p < 3 + \frac{2}{n-2}$ and $\varphi \in W^{1,p}(\Omega) \cap W^{2,2}(\Omega)$ and the boundary $\partial \Omega$ satisfies certain additional regularity assumptions, then $u \in W^{2,2}(\Omega)$.

For the regularity assumptions on the boundary $\partial \Omega$, we refer to two recent papers by Cianchi and Maz’ya [5, 6]. See also [2]. In [5] the authors study the global second order Sobolev-regularity of the so-called generalized solutions to

\begin{equation}
-\text{div}(a(|Du|)Du) = f \quad \text{in } \Omega
\end{equation}

with both Dirichlet boundary condition $u = 0$ on $\partial \Omega$ and Neumann boundary condition $\frac{\partial u}{\partial \nu} = 0$ on $\partial \Omega$. Here $a: (0, \infty) \to (0, \infty)$ is a $C^1$-function such that

\begin{equation}
-1 < i_u := \inf_{t > 0} \frac{ta'(t)}{a(t)} \leq \sup_{t > 0} \frac{ta'(t)}{a(t)} =: s_u < \infty
\end{equation}

and $a'$ denotes the derivative of $a$. The authors prove that $a(|Du|)Du \in W^{1,2}(\Omega)$ if and only if $f \in L^2(\Omega)$, provided that the boundary $\partial \Omega$ of the Lipschitz domain $\Omega$ is sufficiently regular. For the regularity of the boundary $\partial \Omega$, firstly they assume that $\partial \Omega \in W^{2,1}$, that is, the boundary $\partial \Omega$ can be locally viewed as a graph of a twice weakly differentiable function. Secondly, they assume that the weak second fundamental form of $\partial \Omega$ satisfies certain summability assumptions. In [6] the authors prove a similar result in the case when the equation (1.3) is replaced by the $p$-Laplace system. They also relax the boundary regularity assumption compared to [5].

Our regularity assumption on the boundary $\partial \Omega$ is the same one that was used in [6]. We leave the more detailed discussion concerning the boundary regularity results from [5, 6] to Section 5. Here we merely introduce the notation that is needed to state our main theorem. Our notation is adopted from [6].

We denote the diameter of $\Omega$ by $d_\Omega$ and the Lipschitz constant of the boundary $\partial \Omega$ by $L_\Omega$. Suppose that $\partial \Omega \in W^{2,1}$. In particular, the second fundamental form $\mathcal{B}: \partial \Omega \to \mathbb{R}^{(n-1) \times (n-1)}$ exists in the weak sense. Denote its norm by $|\mathcal{B}|$.

Let us define $\mathcal{K}_\Omega: (0, 1) \to [0, \infty]$ as

\begin{equation}
\mathcal{K}_\Omega(r) := \sup_{x \in \partial \Omega} \left( \sup_{E \subset \partial \Omega \cap B_1(x)} \int_E |\mathcal{B}|d\mathcal{H}^{n-1} \cap B_1(x) \right),
\end{equation}

where $d\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure.

We can now state our main theorem.

**Theorem 1.1.** Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with diameter $d_\Omega > 0$ and Lipschitz constant $L_\Omega > 0$. Suppose in addition that $\partial \Omega \in W^{2,1}$. Assume that $1 < p < 3 + \frac{2}{n-2}$ and let $u \in W^{1,p}(\Omega)$ solve (1.2) with $\varphi \in W^{1,p}(\Omega) \cap W^{2,2}(\Omega)$. There exists a constant $\mathcal{K}_0 = \mathcal{K}_0(n, p, d_\Omega, L_\Omega) > 0$ such that if

\begin{equation}
\lim_{r \to 0} \mathcal{K}_\Omega(r) < \mathcal{K}_0,
\end{equation}

then $Du \in W^{1,2}(\Omega; \mathbb{R}^n)$ and

$$
1.1 \quad \|Du\|_{W^{1,2}(\Omega; \mathbb{R}^n)} \leq C \left( \|D\phi\|_{W^{1,2}(\Omega; \mathbb{R}^n)} + \|D\phi\|_{L^p(\Omega; \mathbb{R}^n)} \right)
$$

for some $C = C(n, p, \Omega)$.

The proof of Theorem 1.1 is based on Cordes’ matrix inequalities [7] and the aforementioned techniques taken from the work of Cianchi and Maz’ ya [5, 6].

To illustrate the idea of the proof of Theorem 1.1, consider the well-known identity

$$
1.8 \quad |D^2u|^2 = \text{div} (D^2uDu - \Delta uDu) + (\Delta u)^2
$$

that holds for any function $u \in C^3(\Omega)$. Here $D^2u = (u_{ij})_{i,j=1}^n$ denotes the Hessian matrix of $u$, $|D^2u| = (\sum_{i,j=1}^n u_{ij}^2)^{1/2}$ denotes its (Hilbert-Schmidt) norm, and $\Delta u = \sum_{i=1}^n u_{ii}$ denotes the Laplacian of $u$. The identity (1.8) appears repeatedly in the literature, see for instance [2, 10, 13, 15, 21].

The identity (1.8) has been generalized by replacing the Laplacian $\Delta u$ by a more general elliptic operator.

Let $A : \Omega \to \mathbb{R}^{n \times n}$ be a matrix-valued function, $A = (A_{ij})_{i,j=1}^n$. Denote the norm of $A$ by $|A| = (\sum_{i,j=1}^n A_{ij}^2)^{1/2}$ and the trace of $A$ by $\text{tr}(A) = \sum_{i=1}^n A_{ii}$. Talenti [21] showed that if $A$ satisfies the so-called Cordes condition

$$
1.9 \quad (n-1+\delta)|A|^2 \leq (\text{tr} A)^2 \quad \text{in } \Omega
$$

for some $0 < \delta < 1$, then for any $u \in C^3(\Omega)$

$$
1.10 \quad c|D^2u|^2 \leq \text{div} (D^2uDu - \Delta uDu) + \frac{C}{|A|^2} \langle A, D^2u \rangle^2
$$

where $c = c(n, \delta) > 0$ and $C = C(n, \delta) > 0$. In the display (1.10) the brackets $\langle \cdot, \cdot \rangle$ denote the inner product of two matrices, that is, $\langle A, D^2u \rangle = \sum_{i,j=1}^n A_{ij}u_{ij}$. If $A$ happens to be the identity matrix, one can see that indeed, (1.10) can be viewed as a generalization of (1.8).

Cianchi and Maz’ ya [5, Lemma 3.1] proved another generalization of (1.8). If $a : [0, \infty) \to (0, \infty)$ is a (sufficiently well-behaving) $C^1$-function such that the first inequality in (1.4) holds, then for any $u \in C^3(\Omega)$

$$
1.11 \quad c(a(|Du|^2))|D^2u|^2 \leq \text{div} (a(|Du|^2)(D^2uDu - \Delta uDu) + (\text{div} (a(|Du|)Du))^2
$$

where $c = c(n, i_a) > 0$. For the precise assumptions on the function $a$, we refer to the statement of Lemma 3.1 in [5]. Similarly as (1.10), the inequality (1.11) can be viewed as a generalization of (1.8).

Our central observation is that Cordes’ matrix inequalities [7] imply yet another generalization of (1.8), that contains both (1.10) and (1.11) as a special case.

Namely, let $a, b : [0, \infty) \to (0, \infty)$ be $C^1$-functions and denote

$$
\bar{a}(t) := \frac{ta'(t)}{a(t)} \quad \text{and} \quad \bar{b}(t) := \frac{tb'(t)}{b(t)} \quad \text{for all } t \geq 0.
$$

For any $u \in C^3(\Omega)$, consider the vector fields

$$
V_a := a(|Du|)Du \quad \text{and} \quad V_b := b(|Du|)Du.
$$

We show that if $a$ and $b$ satisfy the growth conditions

$$
-1 < \inf_{t \geq 0} \bar{a}(t) \leq \sup_{t \geq 0} \bar{a}(t) < \infty \quad \text{and} \quad -1 < \inf_{t \geq 0} \bar{b}(t) \leq \sup_{t \geq 0} \bar{b}(t) < \infty,
$$
and if they are "sufficiently close to each other", that is, if

\[
0 < \inf_{t \geq 0} \frac{1 + \vartheta_a(t)}{1 + \vartheta_b(t)} \leq \sup_{t \geq 0} \frac{1 + \vartheta_a(t)}{1 + \vartheta_b(t)} < \frac{2(n-1)}{n-2},
\]

then

\[
c|D V_b|^2 \leq \text{div} \left( (D V_b - \text{tr}(D V_b) J) V_b \right) + C \left( \frac{b(|Du|)}{a(|Du|)} \right)^2 \left( \text{div}(V_a) \right)^2,
\]

for some positive constants \( c > 0 \) and \( C > 0 \). Our proof of (1.13) is based on Cordes' matrix inequalities which we discuss in Section 2. The proof of inequality (1.13) is given in Section 3, see Lemma 3.2.

The inequality (1.13) is the key tool of this paper. It is a generalization of the aforementioned inequalities (1.10) and (1.11). To recover Talenti’s inequality (1.10), put \( b \equiv 1 \) in (1.13). On the other hand, to recover Cianchi and Mazy’a’s inequality (1.11), put \( b = a \) in (1.13).

In fact, our proof of (1.13) with \( b = a \) is simpler and shorter than that of (1.11) in [5]. See Remark 3.2 for details. In Remark 3.2 we also explain why in the case \( b = a \) we can choose \( C = 1 \) in (1.13).

To prove the estimate (1.7) in Theorem 1.1, we use a further generalization of (1.13). Namely, if (1.12) holds and \( W \in C^{2}(\Omega; \mathbb{R}^n) \) is an arbitrary vector field, then

\[
c|D V_b|^2 \leq \text{div} \left( (D V_b - W - \text{tr}(D V_b - W) J) (V_b - W) \right)
+ C \left( |D W|^2 + \left( \frac{b(|Du|)}{a(|Du|)} \right)^2 \left( \text{div}(V_a) \right)^2 \right).
\]

See Corollary 3.3 for details.

Observe that the inequality (1.14) can be used to easily derive local \( L^2 \)-estimates for \( D V_b \) in terms of \( L^2 \)-oscillation of \( V_b \), see Remark 6.1.

Let us briefly explain the outline of the proof of Theorem 1.1. Once the inequality (1.14) is established in Sections 2 and 3, we follow the idea of the proof of Theorem 2.4 from [5]. We first derive the estimate (1.7) under additional regularity assumptions on the domain \( \Omega \), the boundary data \( \varphi \) and the operator \( \Delta_p \). For \( \varepsilon > 0 \) small, we set \( a(t) = (t^2 + \varepsilon)^{\frac{n-2}{2}} \) and \( b \equiv 1 \) in (1.14). In this case the condition (1.12) is satisfied precisely when

\[
1 < p < 3 + \frac{2}{n-2}.
\]

The key idea is to set \( W = D \varphi \) in (1.14). Since \( u = \varphi \) on \( \partial \Omega \), we have \( D_T u = D_T \varphi \) on \( \partial \Omega \). Here \( D_T \) refers to the tangential gradient on the boundary, see Section 4 for details. In other words, the vector field \( Du - D \varphi \) is always normal to the boundary manifold \( \partial \Omega \). The flow of such vector field across the boundary \( \partial \Omega \) can be estimated in terms of the second fundamental form of the boundary manifold. Consequently, we may derive appropriate boundary estimates similarly as in [5,6].

In Section 5 we state a weighted trace inequality from [6] that explains how the quantity \( \mathcal{X}_\Omega \) comes into play. In Section 5 we also make some remarks about the meaning of the condition (1.6).

In Section 6 we put the tools from the previous sections into use to derive a regularized version of the estimate (1.7), see Proposition 6.3. Once the regularized version of (1.7) is established, we carefully pass to the limit to conclude the proof of Theorem 1.1. See Section 7 for details.

Throughout this paper, \( W^{1,p}(\Omega) \) denotes the Sobolev space of weakly differentiable functions \( v: \Omega \to \mathbb{R} \) such that \( v, v_{x_i} \in L^p(\Omega) \) for all \( i = 1, \ldots, n \). \( W^{2,2}(\Omega) \) denotes the
Sobolev space of twice weakly differentiable functions \( v: \Omega \to \mathbb{R} \) such that \( v, v_x, v_{x_i} \in L^2(\Omega) \) for all \( i, j = 1, \ldots, n \). \( C^\infty_0(\Omega) \) denotes the space of smooth functions that are compactly supported on \( \Omega \). \( W^{1,p}_0(\Omega) \) denotes the closure of \( C^\infty_0(\Omega) \) with respect to the Sobolev norm

\[
\|v\|_{W^{1,p}(\Omega)} = \left( \int_{\Omega} |v|^p dx + \sum_{i=1}^n \int_{\Omega} |v_{x_i}|^p dx \right)^{1/p}.
\]

2. Cordes’ matrix inequalities

All the matrix inequalities in this section are due to Cordes \cite{Cordes}. See also \cite[Lemma 3]{Zhang}. We provide detailed proofs for the convenience of the reader.

Consider the space of real square matrices \( \mathbb{R}^{n \times n} \). The space \( \mathbb{R}^{n \times n} \) equipped with the usual Hilbert-Schmidt inner product

\[
\langle M, N \rangle := \sum_{i,j=1}^n M_{ij}N_{ij}, \quad \text{for all } M = (M_{ij})_{i,j=1}^n \in \mathbb{R}^{n \times n} \text{ and } N = (N_{ij})_{i,j=1}^n \in \mathbb{R}^{n \times n},
\]

is a Hilbert space. Given \( M = (M_{ij})_{i,j=1}^n \in \mathbb{R}^{n \times n} \), we denote its Hilbert-Schmidt norm by \( |M| := \langle M, M \rangle^{1/2} \), transpose by \( M^\top := (M_{ij})_{i,j=1}^n \) and trace by \( \text{tr} \langle M \rangle := \sum_{i=1}^n M_{ii} \). If \( M \) is invertible, we denote its inverse by \( M^{-1} \). The identity matrix is denoted by \( I := \text{diag}(1, \ldots, 1) \).

**Definition 1.** A symmetric matrix \( A \in \mathbb{R}^{n \times n} \) satisfies Cordes condition with respect to a symmetric, positive definite matrix \( B \in \mathbb{R}^{n \times n} \) if \( A \neq 0 \) and

\[
(n - 1 + \delta)\langle B^{-1}A, (B^{-1}A)^\top \rangle \leq (\text{tr} (B^{-1}A))^2
\]

for some \( 0 < \delta < 1 \).

**Remark 2.1.** Any symmetric, positive definite matrix satisfies Cordes condition with respect to itself.

**Lemma 2.1.** Suppose that \( A \in \mathbb{R}^{n \times n} \) is a symmetric matrix that satisfies Cordes condition (2.1) with respect to the identity matrix \( I \in \mathbb{R}^{n \times n} \) for some \( 0 < \delta < 1 \). Then there exist constants \( c = c(n, \delta) > 0 \) and \( C = C(n, \delta) > 0 \) such that

\[
|A| \leq |A_\parallel| + \frac{C}{|A_\parallel|^2} \langle A, M \rangle^2
\]

for any symmetric matrix \( M \in \mathbb{R}^{n \times n} \).

**Proof.** Write

\[
A = A_\parallel + A_\perp
\]

where

\[
A_\parallel := \frac{\text{tr} (A)}{n} I
\]

is parallel to the identity matrix and

\[
A_\perp := A - A_\parallel
\]

is perpendicular to the identity matrix. Then the Cordes condition (2.1) with respect to the identity matrix can be written as

\[
\delta |A_\parallel|^2 \leq |A_\parallel|^2 - (n - 1)|A_\perp|^2
\]
for some $0 < \delta < 1$. Notice that it suffices to study the case $A_\parallel \neq 0$ and $A_\perp \neq 0$. Namely, if $A_\parallel = 0$, then (2.3) cannot hold. On the other hand, if $A_\perp = 0$, then the claim is trivially true for $c = 1$ and $C = n$.

Let $M \in \mathbb{R}^{n \times n}$ be any symmetric matrix. We write

$$M = m_\parallel A_\parallel + m_\perp A_\perp + \tilde{M}$$

where

$$m_\parallel := \frac{A_\parallel}{|A_\parallel|} \text{ and } m_\perp := \frac{A_\perp}{|A_\perp|}$$

are the coordinates of $M$ with respect to $\text{span}\{A_\parallel\}$ and $\text{span}\{A_\perp\}$ respectively, and $\tilde{M}$ denotes the part of $M$ that is perpendicular to both $I$ and $A$. Then

$$|M|^2 = m_\parallel^2 + m_\perp^2 + |	ilde{M}|^2$$

and

$$(\text{tr}(M))^2 = nm_\parallel^2$$

and

$$\langle A, M \rangle^2 = \left( \langle A_\parallel, M \rangle + \langle A_\perp, M \rangle \right)^2$$

$$= \left( |A_\parallel| m_\parallel + |A_\perp| m_\perp \right)^2$$

$$= |A_\parallel|^2 m_\parallel^2 + 2|A_\parallel||A_\perp|m_\parallel m_\perp + |A_\perp|^2 m_\perp^2.$$  

Now the right hand side of (2.2) can be written as

$$|M|^2 - (\text{tr}(M))^2 + \frac{C}{|A|^2} \langle A, M \rangle^2$$

$$= (m_\parallel^2 + m_\perp^2 + |	ilde{M}|^2) - nm_\parallel^2 + \frac{C}{|A|^2} \left( |A_\parallel|^2 m_\parallel^2 + 2|A_\parallel||A_\perp|m_\parallel m_\perp + |A_\perp|^2 m_\perp^2 \right)$$

$$= Q + |	ilde{M}|^2$$

where

$$Q := \left( 1 - n + \frac{C|A_\parallel|^2}{|A|^2} \right) m_\parallel^2 + \left( 1 + \frac{C|A_\perp|^2}{|A|^2} \right) m_\perp^2 + \frac{2C|A_\parallel||A_\perp|}{|A|^2} m_\parallel m_\perp$$

is a quadratic form in $m_\parallel$ and $m_\perp$. By the Cordes condition (2.3), the determinant of $Q$ has a lower bound

$$\det Q = 1 - n + \frac{C}{|A|^2} \left( |A_\parallel|^2 - (n - 1)|A_\perp|^2 \right)$$

$$\geq 1 - n + C \delta.$$  

Consequently, we can select $C = C(n, \delta) > 0$ such that $\det Q > 0$. We can then fix $c = c(n, \delta) > 0$ such that

$$Q \geq c(m_\parallel^2 + m_\perp^2)$$

and thus (2.2) holds.

**Corollary 2.2.** Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix that satisfies the Cordes condition (2.1) with respect to a symmetric, positive definite matrix $B \in \mathbb{R}^{n \times n}$ for some $0 < \delta < 1$. Then there exist constants $c = c(n, \delta) > 0$ and $C = C(n, \delta) > 0$ such that

$$C \langle BM, (BM)^T \rangle \leq \langle BM, (BM)^T \rangle - (\text{tr}(BM))^2 + \frac{C}{(B^{-1}A, (B^{-1}A)^T)} \langle A, M \rangle^2$$
for any symmetric matrix $M \in \mathbb{R}^{n \times n}$.

Proof. Since $B$ is symmetric and positive definite, there exists an invertible matrix $\Phi \in \mathbb{R}^{n \times n}$ such that $\Phi^T B \Phi = I$. Consequently $\Phi^T A \Phi$ is a symmetric matrix that satisfies Cordes condition (2.1) with respect to the identity matrix. We apply Lemma 2.1 with $\Phi^T A \Phi$ in place of $A$ and $\Phi^{-1} M (\Phi^{-1})^T$ in place of $M$ to obtain the desired result. \hfill \Box

Lemma 2.3. Let $B \in \mathbb{R}^{n \times n}$ be a positive definite symmetric matrix with the smallest eigenvalue $\lambda_{\min}$ and the largest eigenvalue $\lambda_{\max}$. Then

$$|BM|^2 \leq \left( \frac{\lambda_{\max}}{\lambda_{\min}} \right)^2 \langle BM, (BM)^T \rangle$$

for any symmetric matrix $M \in \mathbb{R}^{n \times n}$.

Proof. Without loss of generality we may assume that $B$ is a diagonal matrix, that is, $B = \text{diag}(\lambda_1, \ldots, \lambda_n)$ for some $\lambda_1, \ldots, \lambda_n \in (0, \infty)$. Then

\begin{equation}
|BM|^2 = \sum_{i,j=1}^{n} (\lambda_i M_{ij})^2 \leq \lambda_{\max}^2 |M|^2 \tag{2.4}
\end{equation}

and

\begin{equation}
\langle BM, (BM)^T \rangle = \sum_{i,j=1}^{n} \lambda_i M_{ij} \lambda_j M_{ji} = \sum_{i,j=1}^{n} \lambda_i \lambda_j M_{ij}^2 \geq \lambda_{\min}^2 |M|^2. \tag{2.5}
\end{equation}

The estimates (2.4) and (2.5) together imply the desired result. \hfill \Box

3. Differential Inequality

In this section we prove our key inequalities (1.13) and (1.14). Let $\Omega \subset \mathbb{R}^n$ be a domain and $u \in C^3(\Omega)$. Consider the two vector fields

$$V_a := a(|Du|) Du \quad \text{and} \quad V_b := b(|Du|) Du,$$

where $a: [0, \infty) \to (0, \infty)$ and $b: [0, \infty) \to (0, \infty)$ are $C^1$-functions. We introduce some notation, similar to [5]. Let

$$\theta_a(t) := \frac{t a'(t)}{a(t)} \quad \text{and} \quad \theta_b(t) := \frac{t b'(t)}{b(t)} \quad \text{for all } t \geq 0,$$

and

$$i_a := \inf_{t \geq 0} \theta_a(t) \quad \text{and} \quad s_a := \sup_{t \geq 0} \theta_a(t),$$

and

$$i_b := \inf_{t \geq 0} \theta_b(t) \quad \text{and} \quad s_b := \sup_{t \geq 0} \theta_b(t).$$

Our standing assumption on $a$ and $b$ is that

\begin{equation}
-1 < i_a \leq s_a < \infty \quad \text{and} \quad -1 < i_b \leq s_b < \infty. \tag{3.1}
\end{equation}

Observe that

$$DV_a = \left(\frac{(a(|Du|)u_{x_i})}{|Du|^2}\right)_{i,j=1}^{n} = a(|Du|) AD^2 u$$

and

$$DV_b = \left(\frac{(b(|Du|)u_{x_i})}{|Du|^2}\right)_{i,j=1}^{n} = b(|Du|) BD^2 u$$

where

\begin{equation}
A := I + \theta_a(|Du|) \frac{Du \otimes Du}{|Du|^2} \quad \text{and} \quad B := I + \theta_b(|Du|) \frac{Du \otimes Du}{|Du|^2}. \tag{3.2}
\end{equation}
are symmetric $n \times n$ matrices. In the above display $I$ stands for the identity matrix and $Du \otimes Du$ stands for the tensor product of the vector $Du$ with itself.

We record certain elementary facts on the matrices $A$ and $B$. If $Du \neq 0$, the eigenvectors of both $A$ and $B$ are

$$e_1, \ldots, e_{n-1}, \frac{Du}{|Du|}$$

where $\{e_1, \ldots, e_{n-1}\}$ is an orthogonal basis of the orthogonal complement of $\text{span}\{Du\}$. If $Du = 0$, then $A$ and $B$ are well defined and equal to $I$, because $\vartheta_a(0) = 0 = \vartheta_b(0)$. We conclude that the eigenvalues of $A$ and $B$ are

$$1, \ldots, 1, 1 + \vartheta_a(|Du|) \quad \text{and} \quad 1, \ldots, 1, 1 + \vartheta_b(|Du|),$$

respectively. The assumption (3.1) guarantees that $A$ and $B$ are uniformly elliptic in $\Omega$.

The following Lemma is elementary yet useful to state for record. We noticed that it appears implicitly in [13, Theorem 3.1.1.1] and [19], and undoubtedly in many other instances in the literature.

**Lemma 3.1.** If $X \in C^2(\Omega; \mathbb{R}^n)$, then

$$\text{div} ((DX - \text{tr} (DX) I) X) = \langle DX, DX^\top \rangle - (\text{tr} (DX))^2$$

everywhere in $\Omega$.

**Proof.** The proof is a direct calculation. Indeed, notice that if $M \in C^1(\Omega; \mathbb{R}^{n \times n})$ and $V \in C^1(\Omega; \mathbb{R}^n)$, then

$$\text{div} (MV) = \sum_{i=1}^n \frac{\partial}{\partial x_i} \left( \sum_{j=1}^n M_{ij} V_j \right)$$

$$= \sum_{i,j=1}^n \left( \frac{\partial M_{ij}}{\partial x_i} V_j + M_{ij} \frac{\partial V_j}{\partial x_i} \right)$$

$$= \langle \text{div} (M^\top), V \rangle + \langle M^\top, DV \rangle.$$  

In the above display we use the convention that the divergence of the matrix $M^\top$ is a vector whose components are divergences of the rows of $M^\top$.

In our case $M = DX - \text{tr} (DX) I$ and $V = X$. Then

$$\text{div} ((DX - \text{tr} (DX) I) X) = \langle \text{div} (DX^\top - \text{tr} (DX) I), X \rangle + \langle DX^\top, DX \rangle - (\text{tr} (DX))^2$$

$$= \langle DX^\top, DX \rangle - (\text{tr} (DX))^2$$

because for each $i = 1, \ldots, n$

$$(\text{div} (DX^\top - \text{tr} (DX) I))_i = \sum_{j=1}^n \frac{\partial (DX^\top}_{x_j} - \sum_{j=1}^n \frac{\partial (\text{tr} (DX) \delta_{ij})}{\partial x_j}$$

$$= \sum_{j=1}^n \frac{\partial X_j}{\partial x_j} - \frac{\partial}{\partial x_i} \left( \sum_{k=1}^n \frac{\partial X_k}{\partial x_k} \right)$$

$$= 0.$$

The proof is finished.  

**Remark 3.1.** It is easy to verify that Lemma 3.1 holds even more generally. Namely, if $X \in C^2(\Omega; \mathbb{R}^n)$ and $Y \in C^1(\Omega; \mathbb{R}^n)$ then

$$\text{div} ((DX - \text{tr} (DX) I) Y) = \langle DX, DY^\top \rangle - \text{tr} (DX) \text{tr} (DY)$$

everywhere in $\Omega$.  

Remark 3.2. If we set $X = V_a$ in Lemma 3.1, we recover the inequality (1.11) of Cianchi and Maz’ya. Indeed,

$$\text{div} \left( (DV_a - \text{tr}(DV_a) I)V_a \right)$$

$$= \text{div} \left( (a(|Du|))^2 \left( I + \vartheta_a(|Du|) \frac{Du \otimes Du}{|Du|^2} \right) D^2 u Du \right)$$

$$- \text{div} \left( (a(|Du|))^2 \text{tr} \left( \left( I + \vartheta_a(|Du|) \frac{Du \otimes Du}{|Du|^2} \right) D^2 u \right) Du \right)$$

$$= \text{div} \left( (a(|Du|))^2 (D^2 u Du - \Delta u Du) \right)$$

$$+ \text{div} \left( (a(|Du|))^2 \vartheta_a(|Du|) (\{ Du \otimes Du \}) D^2 u Du - \text{tr} (\{ Du \otimes Du \} D^2 u) Du \right)$$

$$= \text{div} \left( (a(|Du|))^2 (D^2 u Du - \Delta u Du) \right).$$

In the last equality of the above display we used the fact that

$$(Du \otimes Du) D^2 u Du = \Delta u Du = \text{tr} ((Du \otimes Du) D^2 u) Du$$

for any sufficiently smooth function $u$. Moreover, the estimate (2.5) in the proof of Lemma 2.3 implies that

$$(3.4) \langle DV_a, DV_a^\top \rangle = (a(|Du|))^2 (A D^2 u, (A D^2 u)^\top) \geq \left( \min \{ 1, 1 + i_a \} \right)^2 (a(|Du|))^2 |D^2 u|^2.$$ 

Finally notice that

$$(3.5) \text{tr} (DV_a) = \text{div} (V_a) = \text{div} (a(|Du|) Du).$$

As we combine (3.3), (3.4) and (3.5) with Lemma 3.1, we get an elementary proof of (1.11). Moreover, notice that in the special case when $a \equiv 1$ the inequality (3.4) improves to an equality and we recover the basic identity (1.8).

For the statement of the next lemma we set

$$\theta(t) := \frac{1 + \vartheta_a(t)}{1 + \vartheta_b(t)}.$$ 

Note that $\theta$ depends on both $a$ and $b$. In addition we denote

$$i_\theta := \inf_{t \geq 0} \theta(t) \quad \text{and} \quad s_\theta := \sup_{t \geq 0} \theta(t).$$

Lemma 3.2. Suppose that $a, b : [0, \infty) \to (0, \infty)$ are functions as described above and

$$(3.6) \quad 0 < i_\theta \leq s_\theta < \frac{2(n - 1)}{n - 2}.$$ 

Then there exist constants $c = c(n, i_\theta, s_\theta, i_b, s_b) > 0$ and $C = C(n, i_\theta, s_\theta) > 0$ such that

$$(3.7) \quad c |DV_b|^2 \leq \text{div} ((DV_b - \text{tr}(DV_b) I)V_b) + C \left( \frac{b(|Du|)}{a(|Du|)} \right)^2 (\text{div}(V_a))^2.$$ 

Proof. Let $A$ and $B$ be as in (3.2). We compute that

$$B^{-1} A = I + \frac{\vartheta_a(|Du|) - \vartheta_b(|Du|)}{1 + \vartheta_b(|Du|)} \frac{Du \otimes Du}{|Du|^2}.$$ 

Then

$$\langle B^{-1} A, (B^{-1} A)^\top \rangle = n - 1 + (\theta(|Du|))^2.$$
and
\[
(\text{tr}(B^{-1}A))^2 = \left(n - 1 + \theta(|Du|)\right)^2.
\]
We conclude that \( A \) satisfies Cordes condition with respect to \( B \) if and only if
\[
(n - 1 + \delta)(n - 1 + (\theta(|Du|))) \leq (n - 1 + \theta(|Du|))^2
\]
holds for some \( 0 < \delta < 1 \). In other words, we need that
\[
\delta \leq \frac{2(n - 1) - (n - 2)\theta(|Du|)\theta(|Du|)}{n - 1 + (\theta(|Du|))^2}.
\]
Indeed, under the condition (3.6), the right hand side of (3.9) is uniformly positive and we may find such \( \delta = \delta(n, i_\theta, s_\theta) > 0 \).

We apply Corollary 2.2 with \( M = D^2u \) to obtain
\[
c\langle BD^2u, BD^2u \rangle \leq \langle BD^2u, BD^2u \rangle - (\text{tr}(BD^2u))^2 + C\langle A, D^2u \rangle
\]
for some \( c = c(n, i_\theta, s_\theta) > 0 \) and \( C = C(n, i_\theta, s_\theta) > 0 \).

By Lemma 2.3
\[
\langle BD^2u, BD^2u \rangle \geq \left( \frac{\min\{1, 1 + i_b\}}{\max\{1, 1 + s_b\}} \right)^2 |BD^2u|^2.
\]
The estimates (3.10) and (3.11) yield that
\[
c|BD^2u|^2 \leq \langle BD^2u, BD^2u \rangle - (\text{tr}(BD^2u))^2 + C\langle A, D^2u \rangle
\]
where \( c = c(n, i_\theta, s_\theta, i_b, s_b) > 0 \) and \( C = C(n, i_\theta, s_\theta) > 0 \). Multiplying the estimate (3.12) by \( |BD^2u|^2 \) yields that
\[
c|D(b(|Du|)Du)|^2 \leq \langle D(b(|Du|)Du), D(b(|Du|)Du) \rangle - (\text{tr}(D(b(|Du|)Du)))^2
\]
\[+ C(b(|Du|)\langle A, D^2u \rangle)^2.
\]
As we employ Lemma 3.1 and note that
\[
\text{div}(a(|Du|)Du) = a(|Du|)\langle A, D^2u \rangle,
\]
the desired estimate follows immediately. \( \square \)

**Corollary 3.3.** Let \( W \in C^2(\Omega; \mathbb{R}^n) \). Under the assumptions of Lemma 3.2,
\[
c|DV_b|^2 \leq \text{div} \left( (D(V_b - W) - \text{tr}(D(V_b - W))I)(V_b - W) \right)
\]
\[+ C\left( |DW|^2 + \frac{b(|Du|)}{a(|Du|)} \right)^2 (\text{div}(V_a))^2 \]
where \( c = c(n, i_\theta, s_\theta, i_b, s_b) > 0 \) and \( C = C(n, i_\theta, s_\theta) > 0 \).

Note that Corollary 3.3 reduces to Lemma 3.2 if \( W = 0 \).

**Proof of Corollary 3.3.** Let us fix \( W \in C^2(\Omega; \mathbb{R}^n) \). Write \( V_b = X + W \), where \( X := V_b - W \).

By Lemma 3.2 and Lemma 3.1
\[
c|DV_b|^2 \leq \langle DV_b, DV_b \rangle - (\text{tr}(DV_b))^2 + C\left( \frac{b(|Du|)}{a(|Du|)} \right)^2 (\text{div}(V_a))^2
\]
\[= \text{div} \left( (DX - \text{tr}(DX)I)(X) \right)
\]
\[+ 2\langle DX, (DW) \rangle - \text{tr}(DX)\text{tr}(DW) + \langle DW, (DW) \rangle - (\text{tr}(DW))^2
\]
\[+ C\left( \frac{b(|Du|)}{a(|Du|)} \right)^2 (\text{div}(V_a))^2.
\]
We apply Young’s inequality to estimate
\[
2\left(\langle DX, (DW)^T \rangle - \text{tr} (DX) \text{tr} (DW) + \langle DW, (DW)^T \rangle - (\text{tr} (DW))^2\right)
\]
\[
= \langle DV_b, (DW)^T \rangle - \text{tr} (DV_b) \text{tr} (DW) - \langle (DW), (DW)^T \rangle + (\text{tr} (DW))^2,
\]
\[
\leq \frac{c}{2}|DV_b|^2 + C|DW|^2,
\]
where \(C = C(n, c) > 0\). The desired inequality now follows from (3.14) and (3.15).

4. A BOUNDARY IDENTITY AND ITS COROLLARIES

This section is based on [13, Section 3.1.1]. Let \(\Omega \subset \mathbb{R}^n\) be a bounded domain such that the boundary \(\partial \Omega\) is smooth. We consider \(\partial \Omega\) as an \((n - 1)\)-dimensional smooth submanifold of \(\mathbb{R}^n\). For each boundary point \(x \in \partial \Omega\), let \(\tau_1, \ldots, \tau_{n-1}\) be unit vectors that form an orthonormal basis of the tangent space \(T_x \partial \Omega\). Let \(v \in \mathbb{R}^n\) denote the outward unit normal of \(\partial \Omega\). Then \(\{\tau_1, \ldots, \tau_{n-1}, v\}\) forms an orthonormal basis of \(\mathbb{R}^n\) at each boundary point. Due to the smoothness of \(\partial \Omega\), we can assume that these basis vectors are smooth functions \(\partial \Omega \to \mathbb{R}^n\).

Let \(X \in C^1(\partial \Omega; \mathbb{R}^n)\) be a vector field. We write
\[
X = X_T + \langle X, v \rangle v,
\]
where
\[
X_T := X - \langle X, v \rangle v = \sum_{i=1}^{n-1} \langle X, \tau_i \rangle \tau_i
\]
denotes the part of \(X\) that is tangential to the boundary \(\partial \Omega\). The tangential divergence of \(X\) on the boundary \(\partial \Omega\) is given by
\[
\text{div}_T(X) = \sum_{i=1}^{n-1} \frac{\partial X}{\partial \nu_i} \cdot \tau_i
\]
where \(\frac{\partial}{\partial \nu_i}\) denotes the directional derivative with respect to \(\tau_i\). Let \(f \in C^1(\partial \Omega)\) be a function. Similarly to (4.1), the tangential gradient of \(f\) on the boundary \(\partial \Omega\) is given by
\[
D_T f = \sum_{i=1}^{n-1} \frac{\partial f}{\partial S_i} \tau_i.
\]

**Lemma 4.1** ([13, Equation (3.1.1.8)]). Let \(\Omega \subset \mathbb{R}^n\) be a bounded domain with a smooth boundary \(\partial \Omega\). If \(X \in C^1(\partial \Omega; \mathbb{R}^n)\), then
\[
\langle DX \cdot X - \text{div} (X) X, v \rangle = \langle X_T, D_T \langle X, v \rangle \rangle - \langle X, v \rangle \text{div}_T (X_T)
\]
\[
+ \mathcal{B}(X_T, X_T) + \langle X, v \rangle^2 \text{tr} (\mathcal{B})
\]
on the boundary \(\partial \Omega\). Here \(\mathcal{B}\) denotes the second fundamental form of \(\partial \Omega\).

**Corollary 4.2.** Under the assumptions of Lemma 4.1, if in addition \(X_T = 0\) on the boundary \(\partial \Omega\), then
\[
|\langle DX \cdot X - \text{div} (X) X, v \rangle| \leq C|\mathcal{B}| |X|^2
\]
on the boundary \(\partial \Omega\), where \(C = C(n) > 0\).
Corollary 4.3. Under the assumptions of Lemma 4.1, if in addition \( \Omega \) is convex and \( X_T = 0 \) on the boundary \( \partial \Omega \), then

\[
\langle DX \cdot X - \text{div}(X)X, v \rangle \leq 0
\]
on the boundary \( \partial \Omega \).

5. Weighted trace inequality

In this section we state a weighted trace inequality from [6] which is a crucial part of the proof of Theorem 1.1. We also discuss about the meaning of the boundary regularity assumption (1.6). All the results in this section are from [5, 6].

We assume that \( \Omega \subset \mathbb{R}^n \) is a bounded Lipschitz domain with diameter \( d_\Omega > 0 \) and Lipschitz constant \( L_\Omega > 0 \). Suppose in addition that \( \partial \Omega \in W^{2,1} \). Recall that \( \mathcal{K}_\Omega : (0,1) \to [0,\infty] \) is defined by

\[
(5.1) \quad \mathcal{K}_\Omega(r) := \sup_{x \in \partial \Omega} \sup_{E \subset \partial \Omega \cap B_r(x)} \frac{\int_E |\mathcal{B}| d\mathcal{H}^{n-1}}{\text{cap}_{B_1(x)}(E)}
\]

where \( \mathcal{B} \) denotes the weak second fundamental form of \( \partial \Omega \) and \( \text{cap}_{B_1(x)}(E) \) denotes the capacity of a set \( E \) relative to a ball \( B_1(x) \).

The following theorem is a consequence of the Adams’ potential embedding theorem, see [1, Theorem 7.2.1]. Here and in similar occurrences in what follows, the dependence of a constant on \( L_\Omega \) and \( d_\Omega \) is understood just via an upper bound for them.

Lemma 5.1 ([6, Lemma 3.5]). Suppose that \( x_0 \in \partial \Omega \) and let \( r > 0 \) be small. There exists a constant \( C = C(n, L_\Omega, d_\Omega) > 0 \) such that

\[
(5.2) \quad \int_{\partial \Omega \cap B_r(x_0)} |v|^2 |\mathcal{B}| d\mathcal{H}^{n-1} \leq C \mathcal{K}_\Omega(r) \int_{\Omega \cap B_r(x_0)} |Dv|^2 dx
\]

holds for any \( v \in C^1_0(\overline{\Omega} \cap B_r(x_0)) \).

Note that if \( \mathcal{K}_\Omega(r) = \infty \), then the inequality (5.2) is trivially true. Obviously we need \( \mathcal{K}_\Omega(r) < \infty \). Moreover, we will need that \( \mathcal{K}_\Omega(r) \) is sufficiently small when \( r \to 0 \), which is the condition (1.6).

By [5], in order to guarantee that \( \mathcal{K}_\Omega(r) \) is finite, it suffices to assume that the weak second fundamental form \( \mathcal{B} \) belongs to a weak Lebesgue space (for \( n \geq 3 \)) or to a weak Zygmund space (for \( n = 2 \)).

For the definition of the weak Lebesgue space and weak Zygmund space, we need to introduce some preliminary concepts. Denote \( \mu := \mathcal{H}^{n-1} |_{\partial \Omega} \). That is, \( \mu \) is the restriction of the \((n-1)\)-dimensional Hausdorff measure to the boundary \( \partial \Omega \), so that \( \mu \) is a natural measure on \( \partial \Omega \).

The distribution function of \((\mu\text{-measurable})\) function \( \psi : \partial \Omega \to \mathbb{R} \) is

\[
(5.3) \quad \mu_\psi(\lambda) = \mu(\{x \in \partial \Omega : |\psi(x)| > \lambda \}) \quad \text{for all } \lambda > 0.
\]

Given the distribution function \( \mu_\psi \), we define the decreasing rearrangement of \( \psi \), denoted by \( \psi^* \), as

\[
\psi^*(t) := \sup\{\lambda > 0 : \mu_\psi(\lambda) > t\} \quad \text{for all } t > 0.
\]

It holds that

\[
(5.4) \quad \int_{\partial \Omega} |\psi| d\mu = \int_0^{\mu(\partial \Omega)} \psi^*(s) ds.
\]
Let $q > 1$. The weak Lebesgue space $L^{q,\infty}(\partial \Omega)$ is the space of all $\mu$-measurable functions $\psi$ such that
\begin{equation}
\|\psi\|_{L^{q,\infty}(\partial \Omega)} := \sup_{0 < s < \mu(\partial \Omega)} s^{\frac{1}{q} - 1} \int_0^s |\psi|^q(t)\,dt < \infty.
\end{equation}
The weak Zygmund space $L^{1,\infty}\log L(\partial \Omega)$ is the space of all $\mu$-measurable functions $\psi$ such that
\begin{equation}
\|\psi\|_{L^{1,\infty}\log L(\partial \Omega)} := \sup_{0 < s < \mu(\partial \Omega)} \log \left(1 + \frac{\psi}{s}\right) \int_0^s |\psi|^q(t)\,dt < \infty.
\end{equation}

By [5, Proof of Theorem 2.4], see also Lemmas 3.5 and 3.7 in [6],
\begin{equation}
\mathcal{K}_\Omega(r) \lesssim \begin{cases}
\sup_{x \in \partial \Omega} \|\mathcal{B}\|_{L^{r-1,\infty}(\partial \Omega; B_r(x))} & \text{if } n \geq 3,
\sup_{x \in \partial \Omega} \|\mathcal{B}\|_{L^{1,\infty}\log L(\partial \Omega; B_r(x))} & \text{if } n = 2,
\end{cases}
\end{equation}
up to some positive constant depending on $n$, $d_\Omega$ and $L_\Omega$. In particular, if $\mathcal{B} \in L^{r-1,\infty}(\partial \Omega)$ for $n \geq 3$ or $\mathcal{B} \in L^{1,\infty}\log L(\partial \Omega)$ for $n = 2$, then $\mathcal{K}_\Omega(r) < \infty$ for all $r \in (0, 1)$ sufficiently small. The smallness assumption (1.6) for $\mathcal{K}_\Omega$ is then certainly satisfied if
\begin{equation}
\lim_{r \to 0} \sup_{x \in \partial \Omega} \|\mathcal{B}\|_{L^{r-1,\infty}(\partial \Omega; B_r(x))} < \mathcal{K}_0' & \text{if } n \geq 3,
\end{equation}
or
\begin{equation}
\lim_{r \to 0} \left( \sup_{x \in \partial \Omega} \|\mathcal{B}\|_{L^{1,\infty}\log L(\partial \Omega; B_r(x))} \right) < \mathcal{K}_0' & \text{if } n = 2,
\end{equation}
with a suitable $\mathcal{K}_0' = \mathcal{K}_0'(n, p, d_\Omega, L_\Omega) > 0$. The assumptions (5.8) and (5.9) were the main boundary regularity assumptions used in [5, Theorem 2.4]. The quantity $\mathcal{K}_\Omega$ was introduced in the later work [6]. By [6, Remark 2.5], if $\partial \Omega \in C^2$, then
\begin{equation}
\lim_{r \to 0} \mathcal{K}_\Omega(r) = 0,
\end{equation}
and thus (1.6) holds trivially.

6. Regularized Case

In this section we prove a regularized version of our main theorem. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial \Omega$. We denote the diameter of $\Omega$ by $d_\Omega$ and the Lipschitz constant of the boundary $\partial \Omega$ by $L_\Omega$.

Let $\varphi \in C^\infty(\Omega)$. For $1 < p < \infty$ and $0 < \varepsilon < 1$, consider the problem of minimizing the regularized $p$-energy functional
\begin{equation}
\int_\Omega (|Dv|^2 + \varepsilon)^{p/2} dx
\end{equation}
among $v \in \varphi + W^{1,p}_0(\Omega)$. By the standard methods of calculus of variations, there exists a unique minimizer $u \in \varphi + W^{1,p}_0(\Omega)$. Moreover, the minimizer $u$ is a weak solution to the Dirichlet problem
\begin{equation}
\begin{cases}
\text{div}(a(|Du|)Du) = 0 & \text{in } \Omega;
\end{cases}
\end{equation}
\begin{equation}
\begin{cases}
u = \varphi & \text{on } \partial \Omega,
\end{cases}
\end{equation}
where $a : [0, \infty) \to (0, \infty)$ is given by
\begin{equation}
a(t) := (t^2 + \varepsilon)^{p/2} \quad \text{for all } t \geq 0.
\end{equation}
By the classical regularity theory [12], we have \( u \in C^\infty(\overline{\Omega}) \).

Denote \( V_\theta := a(\lvert Du \rvert)Du \). Then \( V_\theta \) is of divergence free by the PDE in \((6.1)\). Let \( \beta > -1 \) and consider the vector field \( V_b = b(|Du|)Du \) where

\[
    b(t) := (t^2 + \epsilon)^\frac{\beta}{2}
\]

for all \( t \geq 0 \).

The functions \( a \) and \( b \) satisfy the assumption \((3.1)\) with

\[
    i_a = \min\{p-2, 0\} \quad \text{and} \quad s_a = \max\{p-2, 0\},
\]

and

\[
    i_b = \min\{\beta, 0\} \quad \text{and} \quad s_b = \max\{\beta, 0\}.
\]

Moreover,

\[
    \theta(t) = \frac{1 + \theta_a(t)}{1 + \theta_b(t)} = \frac{(p-1)t^2 + \epsilon}{(p+1)t^2 + \epsilon}
\]

and

\[
    i_\theta = \min\left\{ \frac{p-1}{p+1}, 1 \right\} \quad \text{and} \quad s_\theta = \max\left\{ \frac{p-1}{p+1}, 1 \right\}.
\]

In particular \((3.6)\) is satisfied if and only if

\[
    \frac{p-1}{\beta+1} < \frac{2(n-1)}{n-2},
\]

or equivalently

\[
    (6.2) \quad \beta > -1 + \frac{(n-2)(p-1)}{2(n-1)}.
\]

The following auxiliary lemma can be applied to derive either local or global estimates.

**Lemma 6.1.** Let \( 1 < p < \infty \) and let \( u \in C^\infty(\overline{\Omega}) \) be a solution to \((6.1)\). If

\[
    \beta > -1 + \frac{(n-2)(p-1)}{2(n-1)},
\]

then there exists a constant \( C = C(n, p, \beta) > 0 \) such that

\[
    (6.3) \quad \int_{\Omega} |DV_b|^2 \eta^2 dx \leq C \int_{\partial\Omega} ((D(V_b - W) - \text{tr}(D(V_b - W))I)(V_b - W), \nu) \eta^2 d\mathcal{H}^{n-1} \]

\[
    + C \int_{\Omega} |V_b - W|^2 |D\eta|^2 dx + C \int_{\Omega} |DW|^2 \eta^2 dx
\]

for any \( \eta \in C^\infty(\mathbb{R}^n) \) and for any \( W \in C^2(\Omega; \mathbb{R}^n) \).

**Proof.** Fix \( \eta \in C^\infty(\mathbb{R}^n) \) and \( W \in C^2(\Omega; \mathbb{R}^n) \). Denote \( X := V_b - W \). By Corollary 3.3

\[
    c \int_{\Omega} |DV_b|^2 \eta^2 dx \leq \int_{\Omega} \left( \text{div} \left( (DX - \text{tr}(DX)I)X \right) \right) \eta^2 dx
\]

\[
    + C \int_{\Omega} |DW|^2 \eta^2 dx + C \int_{\Omega} \left( \frac{b(|Du|)}{a(|Du|)} \right)^2 (\text{div}(V_a))^2 \eta^2 dx.
\]

where

\[
    (6.5) \quad \text{div}(V_a) = \text{div}(a(|Du|)Du) = 0
\]

by the PDE in \((6.1)\).
By Gauss divergence theorem

\[
\int_{\Omega} \left( \text{div} \left( (DX - \text{tr} (DX) I)X \right) \right) \eta^2 \, dx
\]

(6.6)

\[
= \int_{\hat{\partial}\Omega} \langle (DX - \text{tr} (DX) I)X, v \rangle \eta^2 \, d\mathcal{H}^{n-1} - 2 \int_{\Omega} \langle (DX - \text{tr} (DX) I)X, D\eta \rangle \eta \, dx
\]

\[
\leq \int_{\hat{\partial}\Omega} \langle (DX - \text{tr} (DX) I)X, v \rangle \eta^2 \, d\mathcal{H}^{n-1} + C \int_{\Omega} |DX||X||D\eta||\eta| \, dx.
\]

We combine (6.4), (6.5) and (6.6) to get

\[
c \int_{\Omega} |DV_b|\eta^2 \, dx \leq \int_{\Omega} \langle (DX - \text{tr} (DX) I)X, v \rangle \eta^2 \, d\mathcal{H}^{n-1}
\]

(6.7)

\[+ C \int_{\Omega} |DX||X||D\eta||\eta| \, dx
\]

\[+ C \int_{\Omega} |DW|\eta^2 \, dx + C \int_{\Omega} \left( \frac{b(|Du|)}{a(|Du|)} \right)^2 (\text{div} (V_a))^2 \eta^2 \, dx.
\]

The claim follows from (6.7) by using triangle inequality and Young’s inequality. \( \square \)

Remark 6.1. If the support of \( \eta \) lies inside \( \Omega \), then the boundary integral in (6.3) vanishes and we can easily derive local estimates. More precisely, given some concentric balls \( B_r \subset B_{2r} \subset \Omega \), select a cutoff function \( \eta \in C_0^\infty (\mathbb{R}^n) \) such that

\( \text{spt} (\eta) \subset B_{2r}, \quad \eta \equiv 1 \text{ in } B_r \quad \text{and} \quad |D\eta| \leq \frac{10}{r}, \)

and set

\( W \equiv (V_b)_{B_{2r}} = \int_{B_{2r}} V_b \, dx \)

in Lemma 6.1. Then it follows easily from (6.3) that

(6.8)

\[
\int_{B_r} |DV_b|^2 \, dx \leq C \int_{B_{2r}} |V_b - (V_b)_{B_{2r}}|^2 \, dx.
\]

for some \( C = C(n, p, \beta) \). By letting \( \varepsilon \to 0 \) we recover Theorem 1.1 from [20].

Remark 6.2. Suppose that \( \Omega \subset \mathbb{R}^n \) is convex. If \( 1 < p < 3 + \frac{2}{n-2} \), then Lemma 6.1 is applicable. We select \( \eta \equiv 1 \) and \( W = D\phi \) in Lemma 6.1 to find

\[
\int_{\Omega} |D^2 u|^2 \, dx \leq C \int_{\partial\Omega} \langle (D^2 u - D^2 \phi) - \text{tr} (D^2 u - D^2 \phi) I \rangle (Du - D\phi), v \rangle \, d\mathcal{H}^{n-1}
\]

(6.9)

\[+ C \int_{\Omega} |D^2 \phi|^2 \, dx,
\]

for some \( C = C(n, p) > 0 \). Since \( D_T u = D_T \phi \) on the boundary \( \partial \Omega \), we can apply Corollary 4.3 to find that the boundary integral on the right hand side of (6.9) is always nonpositive. Consequently

(6.10)

\[
\int_{\Omega} |D^2 u|^2 \, dx \leq C \int_{\Omega} |D^2 \phi|^2 \, dx,
\]

for some \( C = C(n, p) > 0 \). The estimate (6.10) can be derived by using the tools presented in [17, Section 2]. See also [21].

For the proof of the following version of Sobolev’s inequality, see [18, Proof of Theorem 1.4.6/1]. See also [6, Lemma 3.4].
Lemma 6.2. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with diameter $d_\Omega > 0$ and Lipschitz constant $L_\Omega > 0$. Then for any $\sigma > 0$ we can find $C = C(\sigma, n, d_\Omega, L_\Omega) > 0$ such that

$$\int_{\Omega} |v|^2 \, dx \leq \sigma \int_{\Omega} |Du|^2 \, dx + C \left( \int_{\Omega} |v| \, dx \right)^2$$

for all $v \in W^{1,2}(\Omega)$.

Proposition 6.3. Let $\Omega \subset \mathbb{R}^n$ be a bounded, smooth domain with diameter $d_\Omega > 0$ and Lipschitz constant $L_\Omega > 0$. Suppose that $1 < p < 3 + \frac{2}{n-2}$ and let $u \in C^0(\overline{\Omega})$ solve (6.1). There exists a constant $K = K(n, p, d_\Omega, L_\Omega) > 0$ such that

$$\int_{\Omega} |v|^2 \, dx \leq \sigma \int_{\Omega} |Du|^2 \, dx + C \left( \int_{\Omega} |v| \, dx \right)^2$$

for all $v \in W^{1,2}(\Omega)$.

Proof. Let $\{B_i = B(x_i, r_i)\}_{i=1}^N$ be a covering of $\Omega$ such that either $x_i \in \partial \Omega$ or $B_i \subset \subset \Omega$. Such covering can be selected so that the multiplicity of the overlapping balls of $\{B_i\}_{i=1}^N$ only depends on $n$. Let $\eta_i \in C^\infty_0(B_i)$ be such that $|\nabla \eta_i| \leq \frac{C}{r_i}$ for some absolute constant $C > 0$ and $\{\eta_i^2\}_{i=1}^N$ forms a partition of unity subordinate to $\{B_i\}_{i=1}^N$.

Fix $\eta_i$ for some $i = 1, \ldots, N$. We consider two cases separately: either $B_i \subset \subset \Omega$ or $x_i \in \partial \Omega$.

For $B_i \subset \subset \Omega$, we select $\eta = \eta_i$ and $W = 0$ in (6.3) to get

$$\int_{\Omega} |Du|^2 \eta_i^2 \, dx \leq C \int_{\Omega} |Du|^2 \eta_i^2 \, dx,$$

for some $C = C(n, p) > 0$.

For $x_i \in \partial \Omega$, we select $\eta = \eta_i$ and $W = D\eta$ in (6.3) to get

$$\int_{\Omega} |D^2u|^2 \eta_i^2 \, dx \leq C \int_{\Omega} |Du|^2 \eta_i^2 \, dx,$$

(6.14)

$$\leq C \int_{\partial \Omega} \langle (D^2 u - D^2 \eta)(u - D \eta), v \rangle \eta_i^2 \, d\mathcal{H}^{n-1}$$

(6.15)

$$+ C \int_{\Omega} |Du - D \eta|^2 |D\eta|_i^2 \, dx + C \int_{\Omega} |D^2 \eta|^2 \eta_i^2 \, dx$$

for some $C = C(n, p) > 0$. Since $D_T u = D_T \varphi$ on the boundary $\partial \Omega$, we can apply Corollary 4.2 to estimate the boundary integral on the right hand side of (6.15). We obtain the estimate

$$\int_{\Omega} |D^2u|^2 \eta_i^2 \, dx \leq C \int_{\partial \Omega} \varphi |Du - D \varphi|^2 \eta_i^2 \, d\mathcal{H}^{n-1} + C \int_{\Omega} |Du - D \eta|^2 |D\eta|_i^2 \, dx$$

(6.16)

$$+ C \int_{\Omega} |D^2 \varphi|^2 \eta_i^2 \, dx$$
for some $C = C(n, p) > 0$. By Lemma 5.1, for small $r_i > 0$

$$\int_{\Omega} |\mathcal{A}| |Du - D\varphi|^2 \eta_i^2 \, d\mathcal{H}^{n-1}$$

(6.17) 

$$\leq C \mathcal{X}(r_i) \int_{\Omega} |D((Du - D\varphi)\eta_i)|^2 \, dx$$

$$\leq C \mathcal{X}(r_i) \left( \int_{\Omega} |D^2 u|^2 \eta_i^2 \, dx + \int_{\Omega} |D\varphi|^2 \eta_i^2 \, dx + \int_{\Omega} |Du - D\varphi|^2 |D\eta_i|^2 \, dx \right)$$

where $C = C(n, p, d_{\Omega}, L_{\Omega}) > 0$. In the last inequality of the above display (6.17) we employed the assumption (6.11).

Sum the interior estimates (6.14) and the boundary estimates (6.17) over $i = 1, \ldots, N$ to obtain

$$\int_{\Omega} |D^2 u|^2 \, dx \leq C \left( \max_i \mathcal{X}(r_i) \int_{\Omega} |D^2 u|^2 \, dx \right.$$ 

$$\quad + (1 + \max_i \mathcal{X}(r_i)) \int_{\Omega} |D^2 \varphi|^2 \, dx$$

$$\quad + \frac{(1 + \max_i \mathcal{X}(r_i))}{(\min_i r_i)^2} \left( \int_{\Omega} |Du|^2 + |D\varphi|^2 \, dx \right) \right)$$

for some $C = C(n, p, d_{\Omega}, L_{\Omega}) > 0$. By the convergence property (6.12) of the function $\mathcal{X}$, we can find $r' = r'(n, p, d_{\Omega}, L_{\Omega}, \mathcal{X}) > 0$ such that $0 < r_i < r'$ implies that

$$\mathcal{X}(r_i) \leq \mathcal{X}_0.$$ 

Similarly as in the proof of Theorem 3.1 in [6] we can choose the covering $\{B_i\}_{i=1}^N$ such that each $r_i$ is not only bounded from above by $r'$, but also bounded from below by $r'' = r''(n, p, d_{\Omega}, L_{\Omega}, \mathcal{X}) > 0$. That is, $r'' < r_i < r'$ for all $i = 1, \ldots, N$. With such covering $\{B_i\}_{i=1}^N$ we obtain

$$(1 - C \mathcal{X}_0) \int_{\Omega} |D^2 u|^2 \, dx \leq \frac{C(1 + \mathcal{X}_0)}{(r'')^2} \left( \int_{\Omega} |D^2 \varphi|^2 + |D\varphi|^2 \, dx + \int_{\Omega} |Du|^2 \, dx \right)$$

for some $C = C(n, p, d_{\Omega}, L_{\Omega}) > 0$. If $\mathcal{X}_0 \leq \frac{1}{\mathcal{X}_0}$, then we can conclude that

$$\int_{\Omega} |D^2 u|^2 \, dx \leq C \left( \int_{\Omega} |D^2 \varphi|^2 + |D\varphi|^2 \, dx + \int_{\Omega} |Du|^2 \, dx \right).$$

for some $C = C(n, p, d_{\Omega}, L_{\Omega}, \mathcal{X}) > 0$.

It remains to derive an estimate for $\int_{\Omega} |Du|^2 \, dx$. If $2 \leq p < 3 + \frac{2}{n-2}$ we can simply use Hölder’s inequality to obtain

$$\int_{\Omega} |Du|^2 \, dx \leq |\Omega|^\frac{p-2}{p} \left( \int_{\Omega} |Du|^p \, dx \right)^{2/p}.$$ 

If $1 < p < 2$, we can use Lemma 6.2 to obtain

$$\int_{\Omega} |Du|^2 \, dx \leq \sigma \int_{\Omega} |D^2 u|^2 \, dx + C \left( \int_{\Omega} |Du|^2 \, dx \right)^2$$

$$\leq \sigma \int_{\Omega} |D^2 u|^2 \, dx + C |\Omega|^\frac{p-2}{p} \left( \int_{\Omega} |Du|^p \, dx \right)^{2/p}$$

for any $\sigma > 0$ and for some $C = C(n, d_{\Omega}, L_{\Omega}, \sigma) > 0$.  

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In both cases we conclude the estimate
\[(6.18) \quad \int_\Omega |D^2 u|^2 dx + \int_\Omega |Du|^2 dx \leq C \left( \left( \int_\Omega |Du|^p dx \right)^{2/p} + \int_\Omega |D^2 \varphi|^2 + |D\varphi|^2 dx \right) \]
for some \( C = C(n, p, d_\Omega, L_\Omega, \mathcal{K}) > 0. \) Since \( u \) minimizes the regularized \( p \)-energy, \n\[(6.19) \quad \int_\Omega |Du|^p dx \leq \int_\Omega (|Du|^2 + \varepsilon)^{p/2} dx \leq \int_\Omega (|D\varphi|^2 + \varepsilon)^{p/2} dx. \]
The desired estimate follows now from (6.18) and (6.19).

7. Proof of Theorem 1.1

In this section we remove the additional regularity assumptions that were imposed in the previous section.

Let \( \Omega \subset \mathbb{R}^n \) be a bounded Lipschitz domain such that \( \partial \Omega \in W^{2,1} \). We denote the diameter of \( \Omega \) by \( d_\Omega \) and the Lipschitz constant of the boundary \( \partial \Omega \) by \( L_\Omega \). Recall that the function \( \mathcal{KH} : (0, 1) \to [0, \infty] \), given by
\[(7.1) \quad \mathcal{KH}(r) = \sup_{x \in \partial \Omega} \sup_{E \in \partial \Omega \cap B_r(x)} \frac{\int_E |B_d \mathcal{H}^{n-1}|}{\text{cap}_{B_r(x)}(E)}, \]
is assumed to satisfy the smallness condition (1.6). That is, \n\[\lim_{r \to 0} \mathcal{KH}(r) < \mathcal{KH}_0\]
for a suitable upper bound \( \mathcal{KH}_0 = \mathcal{KH}_0(n, p, d_\Omega, L_\Omega) > 0. \)

Let \( u \in W^{1,p}(\Omega) \) be a weak solution to the Dirichlet problem
\[(7.2) \quad \begin{cases} \Delta_p u = 0 & \text{in } \Omega; \\ u = \varphi & \text{on } \partial \Omega, \end{cases} \]
where \( \varphi \in W^{1,p}(\Omega) \cap W^{2,2}(\Omega). \)

For the proof of Theorem 1.1, we approximate the operator \( \Delta_p \) and the domain \( \Omega \) similarly as in [5, 6]. For the approximation of the boundary data \( \varphi \), we employ the Sobolev extension theorem.

**Lemma 7.1.** If \( \Omega \subset \mathbb{R}^n \) is a bounded Lipschitz domain, then the extension operator
\[E : W^{1,p}(\Omega) \cap W^{2,2}(\Omega) \to W^{1,p}(\mathbb{R}^n) \cap W^{2,2}(\mathbb{R}^n)\]
is a well-defined bounded linear operator.

**Proof.** The proof follows from [4, Theorem 12].

Before we give the proof of Theorem 1.1, let us explain our approximation procedure in detail. For \( \varepsilon > 0 \) small, let
\[a^\varepsilon(t) := (t^2 + \varepsilon)^{\frac{n-2}{2}} \quad \text{for all } t \geq 0.\]
Let \( E\varphi : \mathbb{R}^n \to \mathbb{R} \) denote the Sobolev extension of \( \varphi \) given by Lemma 7.1. Take \( \{ \varphi_k \}_{k=1}^\infty \subset C^\infty_0(\mathbb{R}^n) \) such that
\[(7.3) \quad \varphi_k \xrightarrow{k \to \infty} E\varphi \quad \text{in } W^{2,2}(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n). \]

Finally, by [6, Lemma 5.2], we can take a sequence of approximation domains \( \{ \Omega_m \}_{m=1}^\infty \) such that
\[
\begin{aligned}
&(a) \quad \Omega \subset \Omega_m \\
&(b) \quad \partial \Omega_m \subset C^\infty
\end{aligned}
\]
(c) \( |\Omega_m \setminus \Omega| \xrightarrow{m \to \infty} 0 \)

(d) Hausdorff distance of \( \Omega \) and \( \Omega_m \) tends to 0 as \( m \to \infty \).

(e) \( d_{\Omega_m} \leq C d_{\Omega} \), where \( d_{\Omega_m} \) denotes the diameter of \( \Omega_m \) and \( C > 0 \) is a positive constant.

(f) \( L_{\Omega_m} \leq C L_{\Omega} \), where \( L_{\Omega_m} \) denotes Lipschitz constant of the boundary \( \partial \Omega_m \) and \( C > 0 \) is a positive constant.

(g) \( \mathcal{H}_{\Omega_m}(r) \leq C \mathcal{H}(r) \) for all \( r \in (0, r_0) \), where

\[
\mathcal{H}_{\Omega_m}(r) = \sup_{x \in \partial \Omega_m} \sup_{E \subset \partial \Omega_m \cap B_r(x)} \frac{\int_E |\mathcal{B}_m| d\mathcal{H}^{n-1}}{\text{cap}_{B_1(x)}(E)},
\]

\( \mathcal{B}_m \) denotes the second fundamental form of \( \partial \Omega_m \), and \( C > 0 \) and \( r_0 > 0 \) are positive constants.

In fact, note that the convergence (7.3) implies

\[
\phi_k \xrightarrow{k \to \infty} E \phi \quad \text{in} \ W^{2,2}(\Omega_m) \cap W^{1, p}(\Omega_m)
\]

for each \( m = 1, 2, \ldots \).

Consider the problem of minimizing of the regularized \( p \)-energy functional

\[
\int_{\Omega_m} (|Du|^2 + \varepsilon)|^p/2 \ dx
\]

among \( v \in \phi_k + W^{1, p}_0(\Omega_m) \). By direct method of calculus of variations, there exists a unique minimizer \( u^{\varepsilon, k, m} \in W^{1, p}(\Omega_m) \) that solves the Dirichlet problem

\[
\begin{aligned}
\text{div} (a^\varepsilon(|Du^{\varepsilon, k, m}|)Du^{\varepsilon, k, m}) &= 0 & \text{in} \ \Omega_m; \\
u^{\varepsilon, k, m} &= \phi_k & \text{on} \ \partial \Omega_m
\end{aligned}
\]

in the weak sense.

Standard elliptic regularity theory implies that \( u^{\varepsilon, k, m} \in C^\infty(\overline{\Omega_m}) \). Therefore, Proposition 6.3 is applicable to \( u^{\varepsilon, k, m} \), provided that \( 1 < p < 3 + \frac{2}{n-2} \).

Step 1: Let \( \varepsilon \to 0 \). Fix \( k, m = 1, 2, \ldots \). Consider the problem of minimizing the \( p \)-energy functional

\[
\int_{\Omega_m} |Dv|^p \ dx
\]

among \( v \in \phi_k + W^{1, p}_0(\Omega_m) \). By direct method in calculus of variations, there exists a unique minimizer \( u^{k, m} \in W^{1, p}(\Omega_m) \) that solves the Dirichlet problem

\[
\begin{aligned}
\Delta_p u^{k, m} &= 0 & \text{in} \ \Omega_m; \\
u^{k, m} &= \phi_k & \text{on} \ \partial \Omega_m
\end{aligned}
\]

in the weak sense. Moreover,

\[
u^{k, m} \xrightarrow{\varepsilon \to 0} u^{\varepsilon, k, m} \quad \text{in} \ W^{1, p}(\Omega_m).
\]

Step 2: Let \( k \to \infty \). Fix \( m = 1, 2, \ldots \). Consider the problem of minimizing the \( p \)-energy functional

\[
\int_{\Omega_m} |Dv|^p \ dx
\]

among \( v \in E \phi + W^{1, p}_0(\Omega_m) \). By direct method in calculus of variations, there exists a unique minimizer \( u^m \in W^{1, p}(\Omega_m) \) that solves the Dirichlet problem

\[
\begin{aligned}
\Delta_p u^m &= 0 & \text{in} \ \Omega_m; \\
u^m &= E \phi & \text{on} \ \partial \Omega_m
\end{aligned}
\]
in the weak sense. Moreover,

\[(7.9)\quad u^{k,m} \xrightarrow{k \to \infty} u^m \quad \text{in} \quad W^{1,p}(\Omega_m).\]

Step 3: Let \( m \to \infty \). For the final step, we have, by standard methods in calculus of variations that

\[(7.10)\quad u^m \xrightarrow{k \to \infty} u \quad \text{in} \quad W^{1,p}(\Omega),\]

where \( u \) is the solution of \((7.2)\).

We conclude that we can reach the \( p \)-harmonic function \( u \) from the regularized version \( u^{E,k,m} \) via the three steps described above. Now we are ready to prove our main theorem.

**Proof of Theorem 1.1.** Suppose that \( 1 < p < 3 + \frac{2}{n-2} \). Let \( u \in W^{1,p}(\Omega) \) solve \((7.2)\) and \( u^{E,k,m} \in C^\infty(\bar{\Omega}_m) \) solve \((7.5)\), as explained above. By Proposition 6.3, together with the properties (e) – (g) of the approximating domain \( \Omega_m \), there exists a constant \( \mathcal{K}_0 = \mathcal{K}_0(n,p,d_\Omega,L_\Omega) > 0 \) such that if

\[
\lim_{r \to 0} \mathcal{K}_\Omega(r) < \mathcal{K}_0
\]

then

\[(7.11)\quad \|D u^{E,k,m}\|_{W^{1,2}(\Omega_m;\mathbb{R}^n)} \leq C \left( \|D \phi_k\|_{W^{1,2}(\Omega_m;\mathbb{R}^n)} + \|D \phi_k\|_{L^p(\Omega_m;\mathbb{R}^n)} + \epsilon \right).\]

for all \( \epsilon > 0 \) and \( k,m = 1,2,\ldots \) and for some constant \( C = C(n,p,d_\Omega,L_\Omega,\mathcal{K}_\Omega) > 0 \). In particular, notice that the constant \( C = C(n,p,d_\Omega,L_\Omega,\mathcal{K}_\Omega) \) is uniform with respect to all the regularization parameters \( \epsilon, k \) and \( m \). In the following, \( C \) denotes a positive constant that is allowed to depend on \( n, p, d_\Omega, L_\Omega \) and \( \mathcal{K}_\Omega \).

Step 1: Let \( \epsilon \to 0 \). For \( k,m = 1,2,\ldots \) fixed, consider the family \( \{D u^{E,k,m}\}_{0 < \epsilon < 1} \). By \((7.11)\),

\[(7.12)\quad \|D u^{E,k,m}\|_{W^{1,2}(\Omega_m;\mathbb{R}^n)} \leq C \left( \|D \phi_k\|_{W^{1,2}(\Omega_m;\mathbb{R}^n)} + \|D \phi_k\|_{L^p(\Omega_m;\mathbb{R}^n)} + 1 \right).\]

The right hand side of \((7.12)\) is independent of \( \epsilon \), which means that the family \( \{D u^{E,k,m}\}_{0 < \epsilon < 1} \) is uniformly bounded in \( W^{1,2}(\Omega_m;\mathbb{R}^n) \). By weak compactness, we can select a subsequence \( \{D u^{E_j,k,m}\}_{j=1}^\infty \) such that we have the weak convergence

\[(7.13)\quad D u^{E_j,k,m} \xrightarrow{j \to \infty} U \quad \text{in} \quad W^{1,2}(\Omega_m;\mathbb{R}^n)\]

for some \( U \in W^{1,2}(\Omega_m;\mathbb{R}^n) \). We claim that \( U = D u^{k,m} \), where \( u^{k,m} \) solves \((7.6)\). This follows easily from the convergences \((7.7)\) and \((7.13)\) and the uniqueness of weak limit.

Indeed, if \( 2 \leq p < 3 + \frac{2}{n-2} \), then \((7.7)\) implies that

\[(7.14)\quad D u^{E_j,k,m} \xrightarrow{j \to \infty} D u^{k,m} \quad \text{in} \quad L^2(\Omega_m;\mathbb{R}^n)\]

and in particular,

\[(7.15)\quad D u^{E_j,k,m} \xrightarrow{j \to \infty} D u^{k,m} \quad \text{in} \quad L^2(\Omega_m;\mathbb{R}^n)\]

Since weak limit is unique, \((7.13)\) and \((7.14)\) imply that \( U = D u^{k,m} \). On the other hand, if \( 1 < p < 2 \), then \((7.13)\) implies that

\[(7.16)\quad D u^{E_j,k,m} \xrightarrow{j \to \infty} U \quad \text{in} \quad L^p(\Omega_m;\mathbb{R}^n)\]

Trivially \((7.7)\) implies that

\[(7.17)\quad D u^{E_j,k,m} \xrightarrow{j \to \infty} D u^{k,m} \quad \text{in} \quad L^p(\Omega_m;\mathbb{R}^n).\]
Since weak limit is unique, (7.15) and (7.16) imply that $U = Du^m$.

We conclude that $Du^{k,m} \in W^{1,2}(\Omega_{m}; \mathbb{R}^n)$ and $Du^{j,k,m} \xrightarrow{j \to \infty} Du^{k,m}$ in $W^{1,2}(\Omega_{m}; \mathbb{R}^n)$. We let $j \to \infty$ in (7.11) to conclude that
\[
\|Du^{k,m}\|_{W^{1,2}(\Omega_{m}; \mathbb{R}^n)} \leq \liminf_{j \to \infty} \|Du^{j,k,m}\|_{W^{1,2}(\Omega_{m}; \mathbb{R}^n)}
\leq \liminf_{j \to \infty} C \left( \|D\phi_k\|_{W^{1,2}(\Omega_{m}; \mathbb{R}^n)} + \|D\phi_k\|_{L^p(\Omega_{m}; \mathbb{R}^n)} + \varepsilon \right)
= C \left( \|D\phi_k\|_{W^{1,2}(\Omega_{m}; \mathbb{R}^n)} + \|D\phi_k\|_{L^p(\Omega_{m}; \mathbb{R}^n)} \right).
\]
So the final conclusion of Step 1 is the estimate
\[(7.17) \quad \|Du^{k,m}\|_{W^{1,2}(\Omega_{m}; \mathbb{R}^n)} \leq C \left( \|D\phi_k\|_{W^{1,2}(\Omega_{m}; \mathbb{R}^n)} + \|D\phi_k\|_{L^p(\Omega_{m}; \mathbb{R}^n)} \right).\]

**Step 2:** Let $k \to \infty$. For $m = 1, 2, \ldots$ fixed, consider the sequence $\{Du^{k,m}\}_{k=1}^\infty$. By the estimate (7.17) and the convergence (7.4),
\[(7.18) \quad \|Du^{k,m}\|_{W^{1,2}(\Omega_{m}; \mathbb{R}^n)} \leq C \left( \|D(E\phi)\|_{W^{1,2}(\Omega_{m}; \mathbb{R}^n)} + \|D(E\phi)\|_{L^p(\Omega_{m}; \mathbb{R}^n)} + 1 \right).\]
The right hand side of (7.18) is independent of $k$, which means that the sequence $\{Du^{k,m}\}_{k=1}^\infty$ is uniformly bounded in $W^{1,2}(\Omega_{m}; \mathbb{R}^n)$. By weak compactness, we can select a subsequence $\{Du^{j,m}\}_{j=1}^\infty$ such that we have the weak convergence
\[(7.19) \quad Du^{j,m} \xrightarrow{j \to \infty} U \quad \text{in} \quad W^{1,2}(\Omega_{m}; \mathbb{R}^n)\]for some $U \in W^{1,2}(\Omega_{m}; \mathbb{R}^n)$. By a similar argument as in Step 1, the convergences (7.9) and (7.19), together with the uniqueness of weak limit, imply that $U = Du^m$, where $u^m$ solves (7.8).

We conclude that $Du^m \in W^{1,2}(\Omega_{m}; \mathbb{R}^n)$ and $Du^{j,m} \xrightarrow{j \to \infty} Du^m$ in $W^{1,2}(\Omega_{m}; \mathbb{R}^n)$. We let $j \to \infty$ in (7.17) to conclude that
\[
\|Du^m\|_{W^{1,2}(\Omega_{m}; \mathbb{R}^n)} \leq \liminf_{j \to \infty} \|Du^{j,m}\|_{W^{1,2}(\Omega_{m}; \mathbb{R}^n)}
\leq \liminf_{j \to \infty} C \left( \|D\phi_j\|_{W^{1,2}(\Omega_{m}; \mathbb{R}^n)} + \|D\phi_j\|_{L^p(\Omega_{m}; \mathbb{R}^n)} \right)
= C \left( \|D(E\phi)\|_{W^{1,2}(\Omega_{m}; \mathbb{R}^n)} + \|D(E\phi)\|_{L^p(\Omega_{m}; \mathbb{R}^n)} \right).
\]
Here we also employed the convergence (7.4). The final conclusion of Step 2 is the estimate
\[(7.20) \quad \|Du^m\|_{W^{1,2}(\Omega; \mathbb{R}^n)} \leq C \left( \|D(E\phi)\|_{W^{1,2}(\Omega_{m}; \mathbb{R}^n)} + \|D(E\phi)\|_{L^p(\Omega_{m}; \mathbb{R}^n)} \right).\]

**Step 3:** Let $m \to \infty$. Consider the sequence $\{Du^m\}_{m=1}^\infty$. By the estimate (7.20)
\[(7.21) \quad \|Du^m\|_{W^{1,2}(\Omega; \mathbb{R}^n)} \leq C \left( \|D(E\phi)\|_{W^{1,2}(\Omega; \mathbb{R}^n)} + \|D(E\phi)\|_{L^p(\Omega; \mathbb{R}^n)} \right).\]
The right hand side of (7.21) is independent of $m$, which means that the sequence $\{Du^m\}_{m=1}^\infty$ is uniformly bounded in $W^{1,2}(\Omega; \mathbb{R}^n)$. By weak compactness, we can select a subsequence $\{Du^{j,m}\}_{j=1}^\infty$ such that we have the weak convergence
\[(7.22) \quad Du^{j,m} \xrightarrow{j \to \infty} U \quad \text{in} \quad W^{1,2}(\Omega; \mathbb{R}^n)\]By a similar argument as in Step 1 and Step 2, the convergences (7.10) and (7.22), together with the uniqueness of weak limit, imply that $U = Du$, where $u$ solves (7.2).
We conclude that $Du \in W^{1,2}(\Omega; \mathbb{R}^n)$ and $Du^{m_j} \xrightarrow{j \to \infty} Du$ in $W^{1,2}(\Omega; \mathbb{R}^n)$. We let $j \to \infty$ in (7.20) to conclude that
\[
\|Du\|_{W^{1,2}(\Omega; \mathbb{R}^n)} \leq \liminf_{j \to \infty} \|Du^{m_j}\|_{W^{1,2}(\Omega; \mathbb{R}^n)}
\leq \liminf_{j \to \infty} C\left(\|D(E\phi)\|_{W^{1,2}(\Omega; \mathbb{R}^n)} + \|D(E\phi)\|_{L^p(\Omega; \mathbb{R}^n)}\right)
= C\left(\|D\phi\|_{W^{1,2}(\Omega; \mathbb{R}^n)} + \|D\phi\|_{L^p(\Omega; \mathbb{R}^n)}\right)
\]
which is the desired estimate. The proof is finished. □

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Mathematiikan ja Tilastotieteen laitos, Helsingin yliopisto, PL 68 (Pietari Kalmin Katu 5) 00014 Helsingin yliopisto, Helsinki, Finland

*Email address*, Akseli Haarala: akseli.haarala@helsinki.fi

Mathematiikan ja Tilastotieteen laitos, Helsingin yliopisto, PL 68 (Pietari Kalmin Katu 5) 00014 Helsingin yliopisto, Helsinki, Finland

*Email address*, Saara Sarsa: saara.sarsa@helsinki.fi