One-shot assisted concentration of coherence

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Abstract
We find one-shot bounds for concentration of maximally coherent states in the so called assisted scenario. In this setting, Bob is restricted to performing incoherent operations on his quantum system, however he is assisted by Alice, who holds a purification of Bob’s state and can send classical data to him. We further show that in the asymptotic limit our one-shot bounds recover the previously computed rate of asymptotic assisted concentration.

Keywords: quantum coherence, assisted concentration of coherence, quantum information

1. Introduction

Resource theories have become a powerful tool for developing various subjects within quantum information theory. In general, quantum resource theories study how certain features of quantum systems behave when the physical operations and manipulations of the system are limited. For instance, quantum entanglement is a feature that emerges in multipartite quantum systems, and it is natural to consider how entanglement behaves when the spatially separated parties are restricted to local operations and classical communication (LOCC). Using resource theories, researchers are able to more precisely identify and quantify the role that certain quantum features, such as entanglement, play in the performance of different quantum computational tasks [1, 2]. Beyond entanglement, the resource-theoretic approach has found application in the study of quantum Shannon theory [3], quantum thermodynamics [4, 5] shared reference frames [6], and many others [7]. General measures such as the relative entropy of resource can be applied in different resource theories and carry analogous operational interpretations in each [8–11].

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The focus of this paper is the resource theory of quantum coherence. The fact that coherent superpositions of quantum states are valid physical states is an essential feature of quantum mechanics, and it is the first point to consider when identifying advantages of quantum computation over its classical counterpart [12]. Recently, the phenomenon of coherent superpositions has received rigorous development through the lens of a quantum resource theory [13–19]. See also [20] for a detailed review. In the resource theory of coherence, a state is considered resourceful if it is non-diagonal in a particular fixed basis. All other states are called ‘free’, and they take the form \( \delta = \sum_i \delta_i |i\rangle \langle i| \), where \( \{|i\rangle\} \) is some fixed basis known as the incoherent basis. Note that the free states essentially represent classical probability distributions encoded in some physical system, and thus the resource theory of coherence captures what is perhaps the most basic non-classical feature that quantum mechanics allows.

Like all resource theories, only certain quantum operations are permitted when characterizing the operational capabilities of coherence. Several different families of allowed, or ‘free’, operations have been proposed in the literature, and they all share in the property of being non-coherence-generating; i.e. they map the set of diagonal states onto itself. Quantifying the amount of coherence in a quantum state is then achieved through several coherence monotones, one such coherence monotone being the relative entropy of coherence \( C_r \). When transforming any state \( \rho \) to another state \( \sigma \) by the free operations, \( C_r \) is non-increasing. While this is a necessary condition of state convertibility, the exact conditions for when one state can be converted into another via free operations is an important operational question that has been answered for pure states but is still unknown for mixed states [18].

However, it is known that for any \( M \)-dimensional quantum system, a maximally coherent state \( |\Phi_M\rangle \) exists that can be transformed to any other \( M \)-dimensional quantum state \( \rho \). This conversion \( |\Phi_M\rangle \rightarrow \rho \) is often referred to as coherence dilution. The reverse transformation \( \rho \rightarrow |\Phi_M\rangle \), with \( M' < M \), is typically called coherence concentration since all the coherence in mixed-state form \( \rho \) becomes concentrated into maximal form \( |\Phi_M\rangle \). Interestingly, when the free operations are identified as the so-called class of incoherent operations [13], the relative entropy of coherence turns out to be the optimal asymptotic rate for obtaining maximally coherent states, thereby providing an operational interpretation of the measure.

The specific problem we consider here involves coherence concentration on one system under the assistance of a second party. The operational scenario is depicted in figure 1. We suppose that Alice and Bob initially share some bipartite entangled state \( |\psi\rangle^{AB} \), and the goal is to concentrate the largest amount of coherence on Bob’s side under the constraints that (a) Alice and Bob can only communicate classically, and (b) Bob can only perform incoherent operations. While this is a two-body problem as described, it generalizes a many-body system in which a large number of parties are collectively being called ‘Alice’. Our question then find application in classically-connected quantum networks where the goal is to concentrate coherence at one of the nodes in order to perform some quantum information processing task.

There is a strong similarity between the resource theories of coherence and entanglement, and some of these connections have been pointed out in [21–24]. The equivalence in structure between the coherence of assistance and the entanglement of assistance was exploited in [25] to find the asymptotic coherence of assistance. Inspired by previous work on the problem of one-shot, or single-copy, assisted entanglement concentration [26], in this paper we bound the one-shot assisted coherence concentration. In the assisted concentration scenario, Alice and Bob share a bipartite pure state \( |\psi\rangle^{AB} \) and the goal is to maximize the rate of concentration of unit maximally coherent states (MCS) \( |\Phi_2\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \) on Bob’s side, while Bob is restricted to using incoherent operations and one-way communication is allowed from Alice to Bob. The ideal assisted concentration rate in the asymptotic setting \( C_r(|\psi^{AB}\rangle) \), i.e. when Alice
and Bob share many copies of the state $|\psi\rangle^{AB}$, is known to be equal to coherence of assistance [25]. While this rate is achievable with many copies of the state, in realistic scenarios resources are limited. Thus a more practical question is the following: if we allow for some bounded error in the process, how many copies of a maximally coherent state can we generate from just a single copy of the given pure state $|\psi\rangle^{AB}$?

While this question has been answered for concentration and dilution in the unassisted setting [27–29], it has remained an open question for the one-shot assisted concentration paradigm, and it is one that we answer in this paper.

An outline of our approach is as follows: we argue that Alice can prepare any pure state ensemble consistent with Bob’s local density operator on Bob’s side using arbitrary local operations and classical communication. Alice will choose to prepare the most optimal ensemble she can on Bob’s system and now Bob is left with the task of concentrating this pure state ensemble using incoherent operations to create MCS. We derive bounds on the maximum rate at which Bob can achieve this concentration in two steps. First we derive bounds for the concentration rate $C_c(\psi, \epsilon)$ for a pure state $\psi$ using incoherent operations, where $\epsilon$ is the allowed error. While this problem has been previously solved in [27], our approach uses different techniques. Then we generalize our pure state proof to find the bounds for the concentration rate $C_c(\mathcal{E}, \epsilon)$ for an ensemble of pure states $\mathcal{E} = \{p_i, \psi_i\}_i$ with error $\epsilon$ and hence find bounds up to Alice’s initial optimization for the one-shot assisted concentration problem. We then show that our one-shot rate recovers the asymptotic rate in the appropriate limits.

The paper is organized as follows: in section 2 we clarify notation and present definitions for the quantities we use in various proofs in this paper. In section 3 we derive bounds on the one-shot (unassisted) concentration of MCS from an arbitrary pure state. In section 4 we generalize these bounds to get the average rate of concentration from an ensemble of pure states and in section 5 we show that in the asymptotic limit we recover the appropriate rate. Finally we present our conclusions in section 6.

Figure 1. The general task considered in this paper involves assisted coherence concentration. In phase (i), Alice and Bob share some entangled state $|\psi\rangle^{AB}$. In phase (ii), Alice makes a measurement on her system and communicates the measurement result to Bob. Bob then performs local incoherent operations to maximize the coherence $|\Phi_M\rangle^B$ of his system.
2. Definitions and notations

We fix a particular basis \{\ket{i}\} in a given Hilbert space \(\mathcal{H}\) as the incoherent basis and let \(I\) denotes the set of states which are represented by diagonal density matrices (incoherent states) in this basis. The maximally coherence state of rank \(M\) is defined with reference to this basis as,

\[
\ket{\Phi_M} = \sum_i \frac{1}{\sqrt{M}} \ket{i}.
\] (1)

We use the notation \(\psi\) and \(|\psi\rangle\langle\psi|\) interchangeably. We will use the fidelity measure defined as,

\[
F(\rho, \sigma) = \text{Tr} \left( \sqrt{\sqrt{\sigma} \rho \sqrt{\sigma}} \right) = \|\sqrt{\rho} \sqrt{\sigma}\|_1.
\] (2)

The following lemmas are well-known.

**Lemma 1.** For any self-adjoint operator \(A\) and \(B\) and any positive operator \(0 \leq P \leq I\),

\[
\text{Tr}(P(A - B)) \leq \text{Tr}(A - B)_+ \leq \|A - B\|_1,
\] (3)

where \((X)_+\) denotes the positive part of the operator \(X\).

**Proof.** See [30] \(\square\)

**Lemma 2.** For any state \(\rho\) and an operator \(0 \leq \Lambda \leq I\) such that \(\text{Tr}(\Lambda \rho) \geq 1 - \epsilon\) then,

\[
\|\rho - \sqrt{\Lambda} \rho \sqrt{\Lambda}\|_1 \leq 2\sqrt{\epsilon}.
\] (4)

**Proof.** See [31, 32]. \(\square\)

We also define the following entropic quantities: for any two operators \(\rho\) and \(\sigma\) in a Hilbert space \(\mathcal{H}\) such that \(\rho, \sigma \geq 0\) and any operator \(P\) such that \(0 \leq P \leq I\), and \(\alpha \in (0, \infty) \setminus \{1\}\),

\[
S_P^\alpha(\rho\|\sigma) = \frac{1}{\alpha - 1} \log_2 \text{Tr} \left( \sqrt{\sqrt{\rho} \alpha \sqrt{\sigma}^{1 - \alpha}} \right).
\] (5)

Notice that for \(P = I\), this reduces to the relative Rényi entropy. We will be often using the quantity,

\[
S_0^\alpha(\rho\|\sigma) = \lim_{\alpha \to 0} S_P^\alpha(\rho\|\sigma) = - \log_2 \text{Tr} \left( \sqrt{\rho} \alpha \sqrt{\sigma}^{1 - \alpha} \right),
\] (6)

where \(\Pi_\rho\) is the projector unto the support of \(\rho\) in \(\mathcal{H}\). Notice that the quantity,

\[
S_0^\alpha(\rho\|\sigma) = S_0(\rho\|\sigma) = - \log_2 (\text{Tr} \Pi_\rho \sigma)
\] (7)

is the relative Rényi entropy of order 0. The relative entropy of coherence is defined as,

\[
C_\rho := \min_{\delta \in \mathcal{I}} S_0(\rho\|\delta) = S(\Delta(\rho)) - S(\rho),
\] (8)

where \(\Delta\) is the dephasing operation which deletes off-diagonal terms in the reference basis, mathematically defined as \(\Delta(\rho) = \sum_i \ket{i} \langle i | \rho \langle i | i\rangle \langle i | \). \(S(\cdot\|\cdot) \equiv S_1(\cdot\|\cdot) \equiv S(\cdot)\) is the relative entropy and \(S(\cdot)\) is the von-Neumann entropy. We use \(S_0(\rho\|\sigma)\) to define the min-entropy of coherence as,
\[ C_{\min}(\rho) = \min_{\sigma \in \mathcal{I}} S_{0}(\rho\|\sigma) \]

where \( \mathcal{I} \) is the set of incoherent states. We also define the min-entropy as,

\[ S_{\min}(\rho) = -\log_2(\lambda_{\max}(\rho)), \]

where \( \lambda_{\max}(\rho) \) is the largest eigenvalue of \( \rho \). To define smoothed versions of these entropic quantities we define the \( \epsilon \)-close ball for any state \( \rho \) and \( \epsilon \geq 0 \) as,

\[ b(\rho, \epsilon) = \{ \sigma : \sigma \geq 0, \Tr[\sigma] = 1, F(\rho, \sigma) \geq 1 - \epsilon \}. \]

Similarly we define a \( \epsilon \)-close ball of sub-normalized pure states as,

\[ b_{\epsilon}(\rho, \epsilon) := \{ \psi : \Tr(\psi) \leq 1, F(\psi, \rho) \geq 1 - \epsilon \} \]

where \( \overline{\psi} \) are pure states. The normalized version of this \( \epsilon \)-ball is defined as

\[ b_{\epsilon}(\rho, \epsilon) := \{ \overline{\psi} : \Tr(\overline{\psi}) = 1, \overline{\psi} \in b_{\epsilon}(\rho, \epsilon) \}. \]

The optimal rate for concentration of coherence with assistance and asymptotically many copies of the state \( \psi^{AB} \) is known to be equal to the coherence of assistance \( D_{\alpha}(\rho^B) \), where \( \rho^B = \Tr(A(\psi^{AB})) \), defined as [25],

\[ D_{\alpha}(\rho^B) := \max_{\{\rho, \psi\}} \frac{\sum p_i C_i(\psi_i^B)}{\sum p_i \sigma_i}, \]

where \( C_i(\rho) \) is the relative entropy of coherence, \( S(\rho) \) is the von-Neumann entropy and \( \Delta \) is the dephasing operation in the fixed reference basis. We define the one-shot assisted coherence concentration as,

\[ C_{\alpha}^{AB,\epsilon}(|\psi\rangle^{AB}) := \max_{\Lambda \in \mathcal{O}} \{ \log_2 M : F^2(\Lambda^{AB\to B'}(|\psi\rangle^{AB}), \Phi'_M) \geq 1 - \epsilon \}, \]

where \( \mathcal{O} \) is the set of local quantum-incoherent operations with one-way classical communication (LQICC-1), \( \epsilon \geq 0 \), \( F(\rho, \sigma) \) is the fidelity and \( \Phi'_M \) is the maximally coherent state of rank \( M \) in the output Hilbert space \( H^B' \). The optimal procedure for Alice to assist Bob would be to perform some POVM \( \{P_i^A\} \), on her part of the state and communicate the result to Bob who would then apply an incoherent operation \( \Lambda_i \) depending on Alice’s outcome. Let \( \mathbb{N} \) be the set of natural numbers \( \{0, 1, \ldots\} \), then the optimal rate must be equal to the following quantity:

\[ C_{\alpha}(\rho^B, \epsilon) := \max_{\{p_i\}} \max_{M \in \mathbb{N}} \left\{ \log_2 M : \max_{\{\Lambda_i^A\}} F^2 \left( \sum_i p_i \Lambda_i^A(\rho_i^B), \Phi'_M \right) \geq 1 - \epsilon \right\}; \]

which we call the one-shot coherence of assistance, where \( p_i \rho_i^B = \Tr(A((P_i^A \otimes 1^B)|\psi^{AB})) \). The one-shot coherence of assistance can be equivalently defined as,

\[ C_{\alpha}(\rho^B, \epsilon) := \max_{\{p_i\}} \max_{M \in \mathbb{N}} \left\{ \log_2 M : \max_{\{\Lambda_i^A\}} F^2 \left( \sum_i p_i \Lambda_i^A(\psi_i^B), \Phi'_M \right) \geq 1 - \epsilon \right\}, \]

where \( \rho^B = \sum_i p_i \psi_i^B \), since without loss of generality, the maximization over POVMs \( \{P_i^A\} \), can be restricted to rank-1 POVMs and this is equivalent to preparing any ensemble on Bob’s side consistent with his reduced state \( \rho^B \) [26]. Operationally the concentration task can be split into two parts; Alice prepares an optimal pure state ensemble \( \{p_i, \psi_i^B\}_i \), by performing a suitable measurement and communicates the index \( i \) to Bob. Bob then performs an optimal
incoherent operation on this state to distil the maximally coherent state. Then our task is reduced to finding the optimal rate of distilling the optimal pure state ensemble which will be the best achievable rate on average.

3. Pure state concentration

We will now derive bounds for the one-shot pure state concentration of MCS. The one-shot coherence concentration rate for a pure state $\psi$, a set of incoherent operations $O$ and $\epsilon \geq 0$ is defined as:

$$C_c(\psi, \epsilon) := \max_{M \in \mathbb{N}} \left\{ \log_2 M : \max_{\Lambda \in O} F^2(\Lambda(\psi), \Phi_M) \geq 1 - \epsilon \right\}. \quad (18)$$

We will make use of the following lemma,

**Lemma 3.** For any two pure states $\psi$ and $\phi$ if the condition $\Delta(\psi) \succ \Delta(\phi)$ where the notation $\rho \succ \sigma$ indicates that $\rho$ majorizes $\sigma$, then there exists an incoherent operation $\Lambda$ such that $\Lambda(\phi) = \psi$.

**Proof.** We present a proof given in [18] here for completeness. Let $\text{spec}(\Delta(\psi)) = (\psi_1, \psi_2, \ldots, \psi_d)$ and $\text{spec}(\Delta(\phi)) = (\phi_1, \phi_2, \ldots, \phi_d)$. The majorization condition implies that there exist permutations $\{\pi\}$ and a set of real numbers $\{\lambda_\pi : 0 \leq \lambda_\pi \leq 1, \sum_\pi \lambda_\pi = 1\}$, such that

$$\phi = \sum_{\pi} \lambda_\pi \psi_\pi \quad (20)$$

where $\psi_\pi$ is a vector with the components of $\psi$ permuted by $\pi$. An explicit construction of $\Lambda$ is given in terms of it’s Kraus operators as,

$$\Lambda(\phi) = \sum_{\pi} K_\pi \phi K_\pi^\dagger, \quad (21)$$

where,

$$K_\pi = \sum_i \sqrt{\lambda_\pi} \left( \frac{p_\pi(i)}{q(i)} \right) |\pi(i)\rangle \langle i| \quad (22)$$

where $p_\pi(i)$ and $q(i)$ are the $i$th components of $\psi_\pi$ and $\phi$ respectively. It can be verified that $\Lambda(\phi) = \psi$ by substituting the definition of the Kraus operators $K_\pi$ from equation (22) in equation (21), thus proving the lemma. Note that as the operation $\Lambda$ does not decrease the rank of the input state, it is classified as a strictly incoherent operation (SIO) [18].

**Theorem 1.** For any pure state $\psi$ and $\epsilon \geq 0$

$$\max_{\psi \in b^*} S_{\min}(\Delta(\psi)) - \delta \leq C_c(\psi, \epsilon) \leq \max_{\psi \in K^* \psi, s} S_{\min}(\Delta(\psi)), \quad (23)$$

where $0 \leq \delta \leq 1$ is a number which ensures the lower limit is the logarithm of an integer.

**Proof.** For any pure states $\psi$ such that, $\Delta(\Phi_M) \succ \Delta(\psi)$, then from lemma 3 there exists an incoherent operation $\Lambda$ such that $\Lambda(\psi) = \Phi_M$. Let $\text{spec}(\Delta(\Phi_M)) = \left( \frac{1}{M}, \frac{1}{M}, \ldots, \frac{1}{M} \right)$ and $\text{spec}(\Delta(\psi)) = \left( \psi_1, \psi_2, \ldots, \psi_d \right)$. Then the majorization condition implies that,
\[
\sum_{i=1}^{k} \frac{1}{M} \geq \sum_{i=1}^{k} \psi_i^+ \text{, } \forall k, d, \quad (24)
\]

where the \( \psi_i^+ \) are the \( \psi_i \) in a monotonically decreasing order. Notice that in this case \( \frac{1}{M} \geq \psi_{\text{max}} \equiv \max \psi_j \) is sufficient to imply the majorization condition in equation (24) and ensuring the existence of a \( \Lambda \) that achieves the desired transformation. This implies that \( \Lambda(\psi) = \Phi_M \) for any \( M \) such that \( S_{\text{min}}(\Delta(\psi)) = -\log \lambda_{\text{max}} \geq \log M \). In particular \( M = \lfloor 2^{S_{\text{min}}(\Delta(\psi))} \rfloor \) is always achievable. Consequently, for any pure state \( \overline{\psi} \in b_*(\psi, \epsilon) \) there exists an SIO operation \( \Lambda \) such that \( \Lambda(\overline{\psi}) = \Phi M \). Due to the monotonicity of fidelity under positive trace-preserving maps we have,

\[
1 - \epsilon \leq F(\psi, \overline{\psi}),
\]

\[
\leq F(\Lambda(\psi), \Lambda(\overline{\psi})),
\]

\[
= F(\Lambda(\psi), \Phi_M).
\]

(25)

Hence, \( C_c(\psi, \epsilon) \geq \log_2 M \) for any state \( \overline{\psi} \in b_*(\psi, \epsilon) \), or

\[
C_c(\psi, \epsilon) \geq \max_{\overline{\psi} \in b_*(\psi, \epsilon)} \log_2 \lfloor 2^{S_{\text{min}}(\Delta(\overline{\psi}))} \rfloor.
\]

(26)

For the converse, let \( M \) be the maximum of all \( \epsilon \)-achievable rates for concentration of the pure state \( \psi \), i.e. there exists an incoherent operation \( \Lambda \) such that \( F^2(\Lambda(\psi), \Phi_M) \geq 1 - \epsilon \). Note that for any incoherent state \( \gamma \in \mathcal{I} \) we

\[
\Phi_M^\Lambda(\gamma)\Phi_M = \frac{1}{M} \Phi_M,
\]

(27)

since \( \delta \in \mathcal{I} \) implies \( \Phi_M^\delta \Phi_M = \frac{1}{M} \Phi_M \). Multiplying both sides of equation (27) with \( \Lambda(\psi) \) and taking the trace gives,

\[
\text{Tr}(\Lambda(\psi)\Phi_M^\Lambda(\gamma)\Phi_M) = \frac{1}{M} \text{Tr}(\Lambda(\psi)\Phi_M),
\]

\[
\leq \frac{1}{M}.
\]

(28)

where in the second line we have used the fact that \( \Lambda(\psi) \leq I \). Continuing from equation (28) we have,

\[
\log_2 M \leq -\log_2 \text{Tr}(\Phi_M^\Lambda(\psi)\Phi_M^\Lambda(\gamma)),
\]

\[
= -\log_2 \text{Tr}(\Lambda^*(\Phi_M^\Lambda(\psi)\Phi_M^\Lambda)\gamma),
\]

(29)

where \( \Lambda^* \) is the dual map of \( \Lambda \) such that \( \text{Tr}(X\Lambda(\rho)) = \text{Tr}(\Lambda^*(X)\rho) \). Defining \( Q := \Lambda^*(\Phi_M^\Lambda(\psi)\Phi_M^\Lambda) \) we have,

\[
\log_2 M \leq -\log_2 \text{Tr}(Q\gamma),
\]

\[
\leq -\log_2 \text{Tr}(\sqrt{Q}\psi\sqrt{Q}\gamma),
\]

\[
\leq -\log_2 \text{Tr}(\psi\gamma),
\]

(30)
where we use the fact that $\sqrt{Q}\psi \leq Q$ and we have introduced the sub-normalized state $|\tilde{\psi}\rangle \equiv \sqrt{Q}|\psi\rangle$. Since $\gamma$ is an arbitrary incoherent state, we thus have

$$\log_2 M \leq \min_{\gamma \in \mathcal{Z}} \left\{ -\log_2 \text{Tr}(\tilde{\psi}^{\frac{1}{2}}) \right\},$$

$$= -\log_2 (\lambda_{\text{max}}(\Delta(\tilde{\psi}))),$$

$$= S_{\text{min}}(\Delta(\tilde{\psi})). \quad (31)$$

We will now show that $\tilde{\psi} \in b^*_c(\psi, 2\epsilon)$. Note that,

$$\text{Tr}(Q\psi) = \text{Tr}(\Phi_M \Lambda(\psi) \Phi_M),$$

$$= \left\langle \Phi_M | \Lambda(\psi) | \Phi_M \right\rangle^2,$$

$$\geq \left( F^2(\Lambda(\psi),\Phi_M) \right)^2 \geq 1 - 2\epsilon \quad (32)$$

where for the last inequality we use the fact that $F^2(\Lambda(\psi),\Phi_M) \geq 1 - \epsilon$. Now we can see that,

$$F(\psi, \tilde{\psi}) = \left\langle \psi | \sqrt{Q} | \psi \right\rangle,$$

$$\geq \left\langle \psi | Q | \psi \right\rangle = \text{Tr}(Q\psi),$$

$$\geq 1 - 2\epsilon, \quad (33)$$

where the last inequality follows from equation (32). This implies that $\tilde{\psi} \in b^*_c(\psi, 2\epsilon)$. From equation (31) we can write,

$$\log_2 M \leq S_{\text{min}}(\Delta(\tilde{\psi})),$$

$$\leq \max_{\psi \in b^*_c(\psi, 2\epsilon)} S_{\text{min}}(\Delta(\tilde{\psi})), \quad (34)$$

thus proving the theorem.

We note that our theorem 1 is essentially equivalent to the result given in [27]. Using the theory of distillation norms, the authors of [27] have shown the one-shot pure state concentration of coherence to be

$$C_c(\psi, \epsilon) = \min_{\sigma \in \mathcal{Z}} D_{HF}(\psi||\sigma) - \delta, \quad (35)$$

where $D_{HF}(\psi||\sigma)$ is the smoothed hypothesis testing relative entropy. That is, $D_{HF}(\psi||\sigma) = -\log \min_{\{M \leq 1 \}} \{ \text{Tr} [\sigma M] : 0 \leq M \leq 1, F(\psi, M) > 1 - \epsilon \}$. By applying Sion’s minimax theorem [33], we see that equation (35) reduces to

$$C_c(\psi, \epsilon) = \max_{M \in B^*(\psi, \epsilon)} S_{\text{min}}(\Delta(M)), \quad (36)$$

where $B^*(\psi, \epsilon) = \{ M : 0 \leq M \leq 1, F(\psi, M) > 1 - \epsilon \}$ is the so-called operator ball around $\psi$. Note that $B^*(\psi, \epsilon) \supset b^*(\psi, \epsilon) \supset b^*_c(\psi, \epsilon) \supset b^*_c(\psi, \epsilon)$. Our lower bound in theorem 1 therefore implies that the maximum in equation (36) is attained by a pure state $M$.

4. Coherence concentration for an ensemble of pure states

For any given pure state ensemble $\mathcal{E} = \{ p_i, \psi_i \}$, we define the coherence concentration for $\mathcal{E}$ as :

$$C_c(\mathcal{E}, \epsilon) := \max_{M \in \mathbb{N}} \left\{ \log_2 M : \max_{\{ \Lambda_i \}} F^2 \left( \sum_i p_i \Lambda_i(\psi_i), \Phi_M \right) \geq 1 - \epsilon \right\}, \quad (37)$$

where $\mathbb{N}$ is the set of natural numbers.
where $\Lambda_i$ are incoherent operators. The one-shot coherence of assistance is then given by,

$$C_a(\rho, \epsilon) = \max_{\mathcal{E}} C_c(\mathcal{E}, \epsilon),$$

where $\mathcal{E}$ are all possible pure state ensemble decompositions of $\rho$. We will now define for any ensemble $\mathcal{E} = \{p_i, \psi_i\}$, the following quantity:

$$F^A_{\text{min}}(\mathcal{E}) := \min_i S_{\text{min}}(\Delta(\psi_i)).$$

This is an estimate of the minimum coherence that can be distilled from the ensemble $\mathcal{E}$. Also for any ensemble $\mathcal{E}$ and $\epsilon \geq 0$ we define the set:

$$b'(\mathcal{E}, \epsilon) := \left\{ \mathcal{T} = \{p_i, \psi_i\}_i : \text{Tr}(\psi_i) \leq 1, \sum_i p_i F(\psi_i, \psi_i) \geq 1 - \epsilon \right\}$$

and also the subset of $b(\mathcal{E}, \epsilon)$ with normalized pure states as,

$$b(\mathcal{E}, \epsilon) := \left\{ \mathcal{T} = \{p_i, \psi_i\}_i \in b'(\mathcal{E}, \epsilon) : \text{Tr}(\psi_i) = 1 \right\}.$$

Now we state our main result:

**Theorem 2.** For any given ensemble $\mathcal{E} = \{p_i, \psi_i\}_i$ of pure states, and any $\epsilon \geq 0$,

$$\max_{\mathcal{T} \in b(\mathcal{E}, \epsilon)} F^A_{\text{min}}(\mathcal{T}) - \delta \leq C_a(\mathcal{E}, \epsilon) \leq \max_{\mathcal{T} \in b'(\mathcal{E}, 2\epsilon)} F^A_{\text{min}}(\mathcal{T}),$$

where $0 \leq \delta \leq 1$ is a number to ensure that the lower limit is the logarithm of an integer.

**Proof.** Our proof of theorem 2 follows in parallel to the proof of theorem 1. For the lower bound, let $\mathcal{T} = \{p_i, \psi_i\}_i$ be any ensemble such that $\mathcal{T} \in b(\mathcal{E}, \epsilon)$, i.e. $\sum_i p_i F(\psi_i, \psi_i) \geq 1 - \epsilon$. As in the proof of theorem 1, we know that for each pure state $\psi_i$ Bob can distill a maximally coherent state of length $\log_2 \left[ 2^{S_{\text{min}}(\Delta(\psi_i))} \right]$ without error. Then there exists a set of incoherent operations $\{\Lambda_i\}$ such that $\Lambda_i(\psi_i) = \Phi M(\mathcal{T})$, where $M(\mathcal{T}) = \min_i \left[ 2^{S_{\text{min}}(\Delta(\psi_i))} \right]$. This is because each $\psi_i \in \mathcal{T}$ can attain a maximally coherent state of at least length $M(\mathcal{T})$ using incoherent operations. Then,

$$1 - \epsilon \leq \sum_i p_i F(\psi_i, \psi_i),$$

$$\leq \sum_i p_i F(\Lambda_i(\psi_i), \Lambda_i(\psi_i)), $$

$$= \sum_i p_i F(\Lambda_i(\psi_i), \Phi M(\mathcal{T})), $$

$$= F\left( \sum_i p_i \Lambda_i(\psi_i), \Phi M(\mathcal{T}) \right),$$

where the second line follows from the monotonicity of fidelity under CP maps. Since this holds for any $\mathcal{T} \in b(\mathcal{E}, \epsilon)$, we conclude that
\[ C_c(\mathcal{E}, \epsilon) \geq \max_{\mathcal{E} \in b' (\mathcal{E}, \epsilon)} \min_i S_{\text{min}}(\Delta(\tilde{\psi}_i)) - \delta, \]
\[ = \max_{\mathcal{E} \in b' (\mathcal{E}, \epsilon)} F_{\text{min}}^\Delta(\mathcal{E}) - \delta, \quad (44) \]

thus proving the direct part of the theorem.

For the converse part, suppose that \( C_c(\mathcal{E}, \epsilon) = \log_2 M \). Then there exists a family of incoherent maps \( \{ \Lambda_i \} \) such that
\[ 1 - \epsilon \leq F^2 \left( \sum_i p_i \Lambda_i(\psi_i), \Phi_M \right) = \sum_i p_i \langle \Phi_M | \Lambda_i(\psi_i) | \Phi_M \rangle. \quad (45) \]
Since each \( \Lambda_i \) is incoherent, for any \( \gamma \in I \) we have that
\[ \Phi_M \Lambda_i(\gamma) | \Phi_M \leq \frac{1}{M} | \Phi_M. \quad (46) \]
With \( \Lambda_i(\psi_i) \leq 1 \), we can multiply both sides of the previous equation by \( \Lambda_i(\psi_i) \) and take the trace to obtain
\[ \log_2 M \leq - \log \text{Tr} [ \Phi_M \Lambda_i(\psi_i) | \Phi_M \Lambda_i(\gamma) ] = - \log \text{Tr} [ \Lambda_i^* (\Phi_M | \Lambda_i(\psi_i) | \Phi_M) \gamma ] \]
\[ \leq - \log \text{Tr} \left( \sqrt{Q} \psi_i \sqrt{Q} \gamma \right) \]
\[ \leq - \log \text{Tr} [\tilde{\psi}_i \gamma], \quad (47) \]
where we have used the fact that \( \sqrt{Q} \psi_i \sqrt{Q} \leq Q \) and we have introduced the sub-normalized states \( | \tilde{\psi}_i \rangle \equiv \sqrt{Q} | \psi_i \rangle \). Define the pure state ensemble \( \tilde{\mathcal{E}} \equiv \{ p_i, \tilde{\psi}_i \} \). Returning to equation (47), we can choose the incoherent state \( \gamma \) to be an eigenvector associated with the largest eigenvalue of \( \Delta(\tilde{\psi}_i) \). Using this inequality on every \( | \tilde{\psi}_i \rangle \in \tilde{\mathcal{E}} \), we obtain
\[ \log_2 M \leq \min_i S_{\text{min}}(\Delta(\tilde{\psi}_i)) = F_{\text{min}}^\Delta(\tilde{\mathcal{E}}). \quad (48) \]
It remains to show that \( \tilde{\mathcal{E}} \in b' (\mathcal{E}, 2\epsilon) \). Using the inequality in equation (33), we have
\[ \sqrt{\sum_i p_i F(\psi_i, \tilde{\psi}_i)} \geq \sqrt{\sum_j p_j \text{Tr} [Q \tilde{\psi}_j]} \]
\[ = \sqrt{\sum_i p_i \langle \Phi_M | \Lambda_i(\psi_i) | \Phi_M \rangle^2} \]
\[ \geq \sum_i p_i \langle \Phi_M | \Lambda_i(\psi_i) | \Phi_M \rangle \geq 1 - \epsilon, \quad (49) \]
where the second inequality follows from the concavity of the function \( f(x) = \sqrt{x} \). Hence \( \sum_i p_i F(\psi_i, \tilde{\psi}_i) \geq (1 - \epsilon)^2 \geq 1 - 2\epsilon \). So we have,
\[ \log_2 M \leq F_{\text{min}}^\Delta(\tilde{\mathcal{E}}) \leq \max_{\mathcal{E} \in b' (\mathcal{E}, 2\epsilon)} F_{\text{min}}^\Delta(\mathcal{E}). \quad (50) \]
5. Asymptotic coherence of assistance

For a mixed state $\rho \equiv \rho^B$, its one-shot coherence of assistance is given by

$$C_a(\rho, \epsilon) = \max_{E} C_c(E, \epsilon),$$

where the maximization is over all ensemble decompositions $E$ of $\rho$. The coherence of assistance for $\rho$ is defined by

$$D_a(\rho) = \max_{E} = \{p_i, \psi_i\} \sum_i p_i S(\Delta(\psi_i)),$$

with its regularized version being $D^\infty_a(\rho) = \lim_{n \to \infty} \frac{1}{n} D_a(\rho^\otimes n)$. The asymptotic assisted coherence concentration for Alice and Bob sharing a pure state $|\psi\rangle^{AB}$ is given by [25],

$$D_{A|B}^A(|\psi\rangle^{AB}) = D^\infty_a(\rho^B) = S(\Delta(\rho^B)),$$

where $\rho^B = \text{Tr}_A(|\psi\rangle^{AB})$. We define the asymptotic limit of the one-shot coherence of assistance as,

$$C^\infty_a(\rho) = \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} C_a(\rho^\otimes n, \epsilon).$$

We will now show that under this limit we recover the asymptotic expression.

Theorem 3. For any state $\rho$,

$$C^\infty_a(\rho) = D^\infty_a(\rho).$$

Lemma 4. For any state $\rho$,

$$C^\infty_a(\rho) \leq D^\infty_a(\rho).$$

Proof. Suppose $\rho$ has support on a $d$-dimensional Hilbert space. From theorem 2, we have

$$C_a(\rho^\otimes n, \epsilon) \leq \max_E \max_{\varphi \in \mathcal{E}^\otimes (\epsilon, 2\epsilon)} F_{\text{min}}^\Delta(\varphi) \leq \max_E \max_{\varphi \in \mathcal{E}^\otimes (\epsilon, 2\epsilon)} \sum_i p_i S(\Delta(\psi_i)) \leq \max_E \max_{\varphi \in \mathcal{E}^\otimes (\epsilon, 2\epsilon)} \sum_i p_i S(\Delta(\psi_i)),$$

where the first maximization is taken over all ensembles $E$ generating $\rho^\otimes n$. To bound the last term introduce the QC states $\sigma^{BX} = \sum_i p_i |\psi_i\rangle \otimes |i\rangle\langle i|$, $\sigma^{BX} = \sum_i p_i |\psi_i\rangle \otimes |i\rangle\langle i|$, and note

$$\|\sigma^{BX} - \sigma^{BX}\|_1 = \sum_i p_i \|\psi_i - \overline{\psi}_i\|_1 = 2 \sum_i p_i T(\psi_i, \overline{\psi}_i) \leq 8 \epsilon + 8\sqrt{\epsilon},$$

where $T(\psi_i, \overline{\psi}_i)$ is the trace norm and the proof of the last inequality can be found in appendix B. If we let $\Delta^B$ denote the dephasing map on system $B$ then we further have $\delta := \|\Delta^B(\sigma^{BX}) - \Delta^B(\sigma^{BX})\|_1 \leq 8 \epsilon + 8\sqrt{\epsilon}$. An application of the Alicki–Fannes inequality [34] to the (classical) states $\Delta^B(\sigma^{BX})$ and $\Delta^B(\sigma^{BX})$ yields

$$\left| \sum_i p_i S(\Delta(\psi_i)) - \sum_i p_i S(\Delta(\psi_i)) \right| \leq 4\delta n \log(d) + h(\delta),$$

where $h(\delta)$ is a function of $\delta$. This completes the proof.
where \( h(\delta) := -\delta \log_2(\delta) - (1 - \delta) \log_2(1 - \delta) \), is the binary entropy function. Hence
\[
C_\epsilon(\rho^{\otimes n}, \epsilon) \leq \max_j \sum_i p_i s(\Delta(\psi_i)) + 4\delta n \log(d) + h(\delta) = D_\epsilon(\rho^{\otimes n}) + 4\delta n \log(d) + h(\delta).
\] (60)

Dividing both sides by \( n \) and taking the limits \( n \to \infty, \epsilon \to 0 \) yields the desired result. \( \square \)

**Definition 1.** We define the quantum-classical state corresponding to any pure state ensemble \( \mathcal{E} = \{p_i, \psi_i^B\} \) as,
\[
\sigma^{BZ}_\mathcal{E} := \sum_i p_i \psi_i^B \otimes \pi_i^Z
\] (61)
where \( \pi_i^Z \) are orthogonal rank one projectors \( |i\rangle \langle i|^Z \).

We define the function \( \overline{\mathcal{C}}_{\text{min}} : \mathcal{D}(\mathcal{H}^B \otimes \mathcal{H}^Z) \to \mathbb{R} \) which is a smoothed version of \( C_{\text{min}}(\cdot) \) introduced in equation (9) but defined for Q.C. states;
\[
\overline{\mathcal{C}}_{\text{min}}(\sigma^{BZ}_\mathcal{E}) := \max_{\mathcal{E} \in \mathcal{B}(\mathcal{E}, \epsilon)} \min_{\nu^Z \in \mathcal{I}} \mathcal{S}_0(\nu^{BZ} | \nu^{BZ}).
\] (62)
Further we will make use of the following lemmas,

**Lemma 5.** For any state \( \rho^B \) and any \( \epsilon \geq 0 \),
\[
\max_{\mathcal{E}} \overline{\mathcal{C}}_{\text{min}}(\sigma^{BZ}_\mathcal{E}) - \delta \leq C_\epsilon(\rho^B, \epsilon),
\] (63)
where the maximization is taken over all ensembles \( \mathcal{E} = \{p_i, \psi_i\} \), such that \( \rho^B = \sum_i p_i \psi_i^B, \)
\[
\sigma^{BZ}_\mathcal{E} := \sum_i p_i \psi_i \otimes \pi_i \text{ and } 0 \leq \delta \leq 1 \text{ ensures the lower limit is the logarithm of a positive integer.}
\]

**Proof.** Notice that,
\[
\overline{\mathcal{C}}_{\text{min}}(\sigma^{BZ}_\mathcal{E}) := \max_{\mathcal{E} \in \mathcal{B}(\mathcal{E}, \epsilon)} \min_{\nu^Z \in \mathcal{I}} \left\{ -\log_2 \mathcal{F}_{\text{min}}(\nu^{BZ}) \right\},
\]
\[
= \max_{\{p_i, \psi_i\} \in \mathcal{B}(\mathcal{E}, \epsilon)} \min_i \left\{ -\log_2 \mathcal{F}_{\text{min}}(\nu^{BZ}_i) \right\},
\]
\[
= \max_{\{p_i, \psi_i\} \in \mathcal{B}(\mathcal{E}, \epsilon)} \min_i \left\{ -\log_2 \lambda_{\text{max}}(\Delta(\psi_i)) \right\},
\]
\[
= \max_{\{p_i, \psi_i\} \in \mathcal{B}(\mathcal{E}, \epsilon)} \min_i \mathcal{S}_{\text{min}}(\Delta(\psi_i)),
\]
\[
= \max_{\mathcal{E} \in \mathcal{B}(\mathcal{E}, \epsilon)} F_{\Delta}(\mathcal{E}),
\]
\[
\leq C_\epsilon(\mathcal{E}, \epsilon),
\] (64)
where the inequality comes from theorem 2. Maximizing over \( \mathcal{E} \) proves the lemma. \( \square \)

**Lemma 6.** Given a quantum classical state \( (\sigma^{BZ})^{\otimes n} \) and any general pure state ensemble \( \mathcal{E}_n = \{p_i^{(n)}, \psi_i^{(n)}\} \), such that \( (\sigma^{BZ})^{\otimes n} = \sum_i p_i^{(n)} \psi_i^{(n)} \), we have
\[
\lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \max_{\mathcal{E}_n} \overline{\mathcal{C}}_{\text{min}}(\sigma^{BZ}_\mathcal{E}_n) \geq \max_{\mathcal{E}} C_\epsilon(\sigma^{BZ}_\mathcal{E}),
\] (65)
where \( C_\epsilon(\sigma) \) is relative entropy of coherence.
Proof. We need to use some results from the quantum information spectrum approach. □

Definition 2. Given a sequence of states $\hat{\rho} = \{\rho^n\}_{n=1}^{\infty}$ with $\rho^n \in D(\mathcal{H}_n)$ (set of density operators in $\mathcal{H}_n$) and positive operators $\sigma = \{\sigma^n\}_{n=1}^{\infty}$ with $\sigma^n \in \mathcal{B}(\mathcal{H}_n)$ (set of positive operators acting on $\mathcal{H}_n$), and defining $\Delta^n(\gamma) := \rho^n - 2^{\gamma} \sigma^n$, the quantum spectral inf-divergence rate is defined as,

$$D(\hat{\rho}||\hat{\sigma}) := \sup \left\{ \gamma : \lim_{n \to \infty} \inf \{ \Delta^n \neq 0 \} : \Delta^n = 1 \right\},$$

(66)

where $\{X \geq 0\}$ for a self-adjoint operator $X$ denotes the projector unto the non-negative eigenspace of $X$.

Lemma 7. Given a state $\rho_n$ and a self-adjoint operator $\omega_n$, for any real $\gamma$, we have,

$$\text{Tr} \left\{ (\rho_n - 2^{\gamma} \omega_n) \omega_n \right\} \leq 2^{-\gamma\gamma}.$$

(67)

Proof. See [35]. □

Lemma 8. For any given state $\rho^n$, let $\mathcal{E} = \{p, \psi\}$ denote a pure state decomposition and $\mathcal{E}_n = \{p_n, \psi_n\}$ denote a pure state decomposition of the state $(\rho^n)^{\otimes n}$, then we have,

$$\lim_{\epsilon \to 0} \frac{1}{n} \max_{e,x} \sum_{\epsilon, (\sigma_e^{BZ})^{\otimes n}} \min_{\epsilon, (\nu^{BZ})^{\otimes n}} D(\sigma_e^{BZ} || \nu^{BZ}),$$

(68)

where $\sigma_e^{BZ} = \{(\sigma_e^{BZ})^{\otimes n}\}_{n=1}^{\infty}$ and $\nu^{BZ} = \{(\nu^{BZ})^{\otimes n}\}_{n=1}^{\infty}$.

Proof. Let $\mathcal{E}^*$ be an ensemble such that it achieves the maximum in equation (68). By definition we have,

$$\max_{\mathcal{E}^*} \sum_{\epsilon, (\sigma_e^{BZ})^{\otimes n}} \min_{\epsilon, (\nu^{BZ})^{\otimes n}} S_0(\sigma_e^{BZ} || \nu^{BZ}) \geq \max_{\mathcal{E}^*} \sum_{\epsilon, (\sigma_e^{BZ})^{\otimes n}} \min_{\epsilon, (\nu^{BZ})^{\otimes n}} S_0(\sigma_e^{BZ} || \nu^{BZ}) \geq \max_{\mathcal{E}^*} \sum_{\epsilon, (\sigma_e^{BZ})^{\otimes n}} \min_{\epsilon, (\nu^{BZ})^{\otimes n}} S_0(\sigma_e^{BZ} || \nu^{BZ}),$$

(69)

where $\mathcal{E}^{\otimes n}$ is the product pure state ensemble $\{p_n, \psi_n\}^{\otimes n}$. For each $\nu^{BZ}$ and any $\gamma \in \mathbb{R}$ we define the projector,

$$P^n_{\gamma} \equiv P^n_{\gamma}(\nu^{BZ}) := \{(\sigma_e^{BZ})^{\otimes n} - 2^{\gamma\gamma} \nu^{BZ} \geq 0\}.$$

(70)

Since $\nu^{BZ}$ are incoherent states, the projector $P^n_{\gamma}$ also has a Q.C. structure. Let $\hat{\sigma}_e^{BZ}$ be the i.i.d. sequence of states $\{(\sigma_e^{BZ})^{\otimes n}\}_{n=1}^{\infty}$. For a sequence $\nu^{BZ} := \{\nu_n^{BZ}\}_{n=1}^{\infty}$, fix $\delta > 0$ and choose $\gamma \equiv \gamma(\nu^{BZ}) := D(\hat{\sigma}_e^{BZ} || \nu^{BZ}) - \delta$. Then from the definition of the quantum inf-divergence rate in equation (66), there exists an $n$ large enough such that,

$$\text{Tr} \left\{ P^n_{\gamma}(\sigma_e^{BZ})^{\otimes n} \right\} \geq 1 - \epsilon,$$

(71)

for any $\epsilon > 0$. Here we have used the fact that the quantum inf-divergence rate can be alternatively defined as (see proposition 2 in [30]).
\[
D(\hat{\rho} \parallel \hat{\sigma}) := \sup \left\{ \gamma : \liminf_{n \to \infty} \Tr (\{ \Delta^n \geq 0 \} \rho^n) = 1 \right\}.
\]

(72)

Now we define,

\[
P_n^\gamma (\sigma_B^Z \otimes_P \sigma_E^Z \otimes_P n) = \sum_i p_{i,n} \psi_i^n \otimes \pi_{i,n} \]

\[
\Tr (P_n^\gamma (\sigma_B^Z \otimes_P \sigma_E^Z \otimes_P n)) = \omega_n^B Z_n \gamma (\nu_n^B Z_n);
\]

(73)

where \( \pi_{i,n} = |i\rangle \langle i| \) and \( \mathcal{E}_n^\gamma \) is the pure state ensemble \( \{p_{i,n}, \sum_{i} \Tr (\sigma_{i,n} \otimes \pi_{i,n}) \} \). We will now show that \( \mathcal{E}_n^\gamma \in \mathcal{S}_{\infty}((\mathcal{E}^*)^\otimes n, \epsilon) \). Since \( P_n^\gamma \) has a Q.C. structure, we can write it as \( P_n^\gamma = \sum_i \Pi_i^\gamma \otimes \pi_i^n \). Where \( \Pi_i^\gamma \) are projectors acting on the Hilbert space \( (H_B)^\otimes n \). Now we have,

\[
1 - \epsilon \leq \Tr \left( P_n^\gamma (\sigma_B^Z \otimes_P \sigma_E^Z \otimes_P n) \right),
\]

(74)

but note that,

\[
F \left( \frac{\bar{\psi}_i^n}{\Tr (P_n^\gamma (\sigma_B^Z \otimes_P \sigma_E^Z \otimes_P n))}, \psi_i^n \right) = \frac{1}{\sqrt{\sum_j p_{j,n} \Tr (\Pi_j^\gamma \psi_j^n)}} \Tr \left( \sqrt{\psi_i^n \Pi_j^\gamma \psi_j^n} \right),
\]

(75)

Hence we have,

\[
\sum_i p_{i,n} F \left( \frac{\bar{\psi}_i^n}{\Tr (P_n^\gamma (\sigma_B^Z \otimes_P \sigma_E^Z \otimes_P n))}, \psi_i^n \right) = \sqrt{\sum_j p_{j,n} \Tr (\Pi_j^\gamma \psi_j^n)} \geq \sum_j p_{j,n} \Tr (\Pi_j^\gamma \psi_j^n) \geq 1 - \epsilon,
\]

(76)

where the last inequality follows from equation (74). Equation (76) implies that \( \mathcal{E}_n^\gamma \in \mathcal{S}_{\infty}((\mathcal{E}^*)^\otimes n, \epsilon) \). Proceeding from equation (69) we have
\[
\lim_{n \to \infty} \frac{1}{n} \left( \max_{\mathcal{E} \in \mathcal{B}(k(\mathcal{E}))} \min_{\nu^{RZ} \in \mathcal{I}} S_0(\sigma^{E^* Z} \| \nu^{RZ}) \right).
\]

For the second inequality, we have used the fact that \( \Pi_{\mathcal{E}} \leq \mathcal{P} \) and the third inequality follows from lemma 7. As this holds for arbitrary \( \delta \geq 0 \) we recover the statement of lemma 8 in the limit \( \epsilon \to 0 \).

\[\tag{77}\]

**Lemma 9.** For any sequence of states \( \hat{\rho} = \{\rho^{\otimes n}\}_{n \geq 1} \),

\[
\min_{\hat{\sigma}} D(\hat{\rho} \| \hat{\sigma}) = C_\epsilon(\rho),
\]

where \( \hat{\sigma} = \{\sigma^n\}_{n \geq 1} \) with \( \sigma^n \in \mathcal{I} \) and \( C_\epsilon(\rho) = \min_{\delta \in \mathcal{I}} S(\rho \| \delta) \) is the relative entropy of coherence.

**Proof.** Consider the family of sets \( \mathcal{M} := \{\mathcal{M}_n\}_{n \geq 1} \)

\[
\mathcal{M}_n := \{\delta_n \in \mathcal{I}\}_{n \geq 1}
\]

(79)

where \( \mathcal{I}_n \) is the set of incoherent states in \( \mathcal{H}^{\otimes n} \).

**Proposition 1.** The family of sets \( \mathcal{M} \) satisfies the conditions required to apply the generalized Stein’s lemma (proposition III.1 in [37]).

**Proof.** See appendix A.

From proposition 1 we have for a given state \( \rho \),

\[
S_{\mathcal{M}}^\infty := \frac{1}{n} S_{\mathcal{M}_n}(\rho^{\otimes n}),
\]

(80)

with \( S_{\mathcal{M}_n}(\rho^{\otimes n}) := \min_{\delta_n \in \mathcal{M}_n} S(\rho^{\otimes n} \| \delta_n) \). Let \( \Delta_n(\gamma) = \rho^{\otimes n} - 2^{\gamma} \delta_n \). Then from the generalized Stein’s lemma in [37] it follows that for \( \gamma > S_{\mathcal{M}}^\infty(\rho) \),

\[
\lim_{n \to \infty} \min_{\delta_n \in \mathcal{M}_n} \text{Tr} (\{\Delta_n(\gamma) \geq 0\} \Delta_n) = 0.
\]

(81)

This implies that \( \min_{\delta_n \in \mathcal{M}_n} D(\hat{\rho} \| \hat{\sigma}) \leq S_{\mathcal{M}}^\infty(\rho) \). Conversely, for \( \gamma < S_{\mathcal{M}}^\infty(\rho) \),

\[
\lim_{n \to \infty} \min_{\delta_n \in \mathcal{M}_n} \text{Tr} (\{\Delta_n(\gamma) \geq 0\} \Delta_n) = 1,
\]

(82)
which implies that \( \min D(\hat{\rho} \parallel \hat{\sigma}) \geq S_{M}^{\infty}(\rho) \). Thus we have,
\[
D(\hat{\rho} \parallel \hat{\sigma}) = S_{M}^{\infty}(\rho).
\] (83)
But by definition \( S_{M}^{\infty}(\rho) \equiv C_{r}^{\infty}(\rho) = \lim_{n \to \infty} \min_{\delta_{n} \in I} S(\rho^{\otimes n} \parallel \delta_{n}) = C_{r}(\rho) \) because of the additivity of the relative entropy of coherence [18], thus proving lemma 9. Lemmas 8 and 9 together prove lemma 6.

**Lemma 10.** For any bipartite state \( \rho^{B} \),
\[
C_{a}^{\infty}(\rho^{B}) \geq \lim_{n \to \infty} \frac{1}{n} D_{a}( (\rho^{B})^{\otimes n} ) = D_{a}^{\infty}(\rho).
\] (84)
**Proof.** Let \( \mathcal{E} = \{ p_{i}, \psi_{i} \} \) be a pure state ensemble decomposition of \( \rho \) and \( \mathcal{E}_{n} = \{ p^{n}_{\rho}, \psi_{\rho}^{n} \} \) be such a decomposition of \( (\rho^{B})^{\otimes n} \). As before we define the q.c. state,
\[
\sigma^{BZ}_{\mathcal{E}_{n}} = \sum_{i} p_{\rho}^{n} \sigma_{\rho}^{n} \otimes \pi_{\rho}^{Z}.
\] (85)
where \( \pi_{\rho}^{Z} = |\rho\rangle\langle\rho| \) is the incoherent basis in \( \mathcal{H}_{Z}^{\otimes n} \). From lemma 5 we know that,
\[
C_{a}( (\rho^{B})^{\otimes n}, \epsilon ) \geq \max_{\mathcal{E}_{n}} C_{r}^{\mathcal{E}_{n}}(\sigma^{BZ}_{\mathcal{E}_{n}}) - \delta_{n},
\] (86)
where \( 0 \leq \delta_{n} \leq 1 \). So we have,
\[
C_{a}^{\infty}(\rho^{B}) = \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} C_{a}( (\rho^{B})^{\otimes n}, \epsilon ),
\]
\[
\geq \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \max_{\mathcal{E}_{n}} C_{r}^{\mathcal{E}_{n}}(\sigma^{BZ}_{\mathcal{E}_{n}}),
\]
\[
\geq \max_{\mathcal{E}} C_{r}(\sigma^{BZ}_{\mathcal{E}}),
\] (87)
where we have used lemma 5 for the first inequality and lemma 6 for the last inequality.

**Lemma 11.** For any quantum-classical state \( \sigma^{BZ} = \sum_{i} p_{i} \sigma_{i}^{B} \otimes \pi_{i}^{Z} \), where \( \pi_{i}^{Z} = |i\rangle\langle i|^{Z} \) are projectors onto the incoherent basis elements and \( \sigma_{i} \) are arbitrary density operators, the relative entropy of coherence of \( \sigma^{BZ} \) is given by,
\[
C_{r}(\sigma) = \sum_{i} p_{i} C_{r}(\sigma_{i}).
\] (88)
**Proof.**
\[
C_{r}(\sigma^{BZ}) = C_{r}( \sum_{i} p_{i} \sigma_{i}^{B} \otimes \pi_{i}^{Z} ) = S\left( \Delta \left( \sum_{i} p_{i} \sigma_{i}^{B} \otimes \pi_{i}^{Z} \right) \right) - S\left( \sum_{i} p_{i} \sigma_{i}^{B} \otimes \pi_{i}^{Z} \right)
\]
\[
= S\left( \sum_{i} p_{i} \Delta(\sigma_{i}^{B}) \otimes \pi_{i}^{Z} \right) - S\left( \sum_{i} p_{i} \sigma_{i}^{B} \otimes \pi_{i}^{Z} \right).
\] (89)
We have,
\[ S(\sigma_{BZ}) = -\text{Tr} \sigma_{BZ} \ln \sigma_{BZ} \]
\[ = -\text{Tr} \left( \left( \sum_i p_i \sigma_i^B \otimes \pi_i^Z \right) \ln \left( \sum_j p_j \sigma_j^B \otimes \pi_j^Z \right) \right) \]
\[ = -\text{Tr} \left( \left( \sum_{i,k} p_i \lambda_i^B |\lambda_i^B \rangle \langle \lambda_i^B | \otimes \pi_i^Z \right) \ln \left( \sum_{j,l} p_j \lambda_j^B |\lambda_j^B \rangle \langle \lambda_j^B | \otimes \pi_j^Z \right) \right) \]
\[ = -\sum_{i,k} p_i \lambda_i^B \ln(p_i \lambda_i^B) \]
\[ = -\sum_{i,k} p_i \lambda_i^B \ln(p_i) - \sum_{i,k} p_i \lambda_i^B \ln(\lambda_i^B) \]
\[ = -\sum_i p_i \ln(p_i) + \sum_i p_i S(\sigma_i) \] \hspace{1cm} (90)

where in the third equality we have used the spectral decomposition of \( \sigma_i^B = \sum_k p_i \lambda_i^B |\lambda_i^B \rangle \langle \lambda_i^B | \otimes \pi_i^Z \), thus proving the lemma.

Since \( \sigma_{BZ} \) from equation (87) is a quantum classical state, we use lemma 11 to get,
\[ C_r \left( \sum_i p_i \phi_i^B \otimes \pi_i^Z \right) = \sum_i p_i C_r(\phi_i) \] \hspace{1cm} (91)

Hence we have,
\[ C^\infty_a(\rho^B) := \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} C_a((\rho^B)^{\otimes n}, \epsilon) \geq \max_{(p,\phi^B)} \sum_i p_i C_r(\phi_i^B) = D_a(\rho^B). \] \hspace{1cm} (92)

Lemmas 4 and 10 proves theorem 3.

6. Conclusions

We have derived bounds for the one-shot concentration of maximally coherent states for pure states and average rate for an ensemble of pure states. Using this we have given bounds on the one-shot coherence of assistance and hence the assisted coherence concentration. We further show that asymptotically the one-shot quantity reduces to the correct known result. Finding the one-shot concentration rate for a more general scenario than assistance where communication is not restricted to being one-way and with multiple parties helping Bob, the so-called collaboration scenario, remains an open question. Our results highlight how techniques used in the resource theory of entanglement can find ready application to the resource theory of coherence and we hope it will help deepen understanding of the relationship between the two.
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Note added in proof. We would like to remark that during the preparation of this manuscript an independent study of one-shot assisted concentration was reported using different techniques [36].

Appendix A. \( \mathcal{M} \) satisfies generalized Stein’s lemma

For the generalized Stein’s lemma to hold for a family of sets \( \mathcal{M} \) the following conditions need to be met [37]

1. Each \( \mathcal{M}_n \) must be closed and convex.
2. Each \( \mathcal{M}_n \) contains \( \sigma^{\otimes n} \) for a full rank state \( \sigma \in \mathcal{D}(\mathcal{H}) \).
3. If \( \rho \in \mathcal{M}_n \) and \( \nu \in \mathcal{M}_m \), then \( \rho \otimes \nu \in \mathcal{M}_{n+m} \).
4. If \( \rho \in \mathcal{M}_n \) then \( P_\pi \rho P_\pi \in \mathcal{M}_n \) for every \( \pi \in S_n \), where \( P_\pi \) is the representation of a permutation \( \pi \) in \( \mathcal{H}^{\otimes n} \) and \( S_n \) is symmetric group of order \( n \).

The set of incoherent states in \( \mathcal{H}^{\otimes n} \) will be convex and closed satisfying the first condition. \( \delta^{\otimes n} \in I_n \) satisfying condition 2. \( \text{Tr}_k(\delta_{n+1}) \in I_n \) where \( \delta_{n+1} \in I_{n+1} \) for any \( k \in \{1, \ldots, n+1\} \) satisfying condition 3. \( \delta_n \otimes \nu_m \in I_{m+n} \) when \( \delta_n \in I_n \) and \( \nu_m \in I_m \) hence condition 4. is satisfied. Finally the permutation operation is just a relabelling of the incoherent basis hence the set of incoherent states will be closed under such a permutation and condition 5. is satisfied.

Appendix B. Upper bound for trace distance

Given that \( \sum_i p_i F(\psi_i, \overline{\psi}_i) \geq 1 - 2\epsilon \), and \( \text{Tr}(\psi_i) = 1, \text{Tr}(\overline{\psi}_i) \leq 1 \) we want to find an upper bound in terms of \( \epsilon \) for the expression,

\[
\sum_i p_i T(\psi_i, \overline{\psi}_i),
\]

where \( T \) is the trace distance. Let \( \overline{\psi}_i = a|\psi_i\rangle + b|\psi_i^\perp\rangle \), where \( \langle \psi_i|\psi_i^\perp\rangle = 0 \) and \( |a|^2 + |b|^2 = \delta_i \leq 1 \). We have,

\[
T(\psi_i, \overline{\psi}_i) := \frac{1}{2} ||\psi_i - \overline{\psi}_i|| = \sum_j |\lambda_j^i|,
\]

where \( \{\lambda_j^i\} \) are the eigenvalues of the operator \( \psi_i - \overline{\psi}_i \). Expressing this operator in the basis \( \{|\psi_i\rangle, |\psi_i^\perp\rangle\} \) we get

\[
\psi_i - \overline{\psi}_i = (1 - |a|^2)|\psi_i\rangle \langle \psi_i| - |b|^2|\psi_i^\perp\rangle \langle \psi_i^\perp| - ab^*|\psi_i\rangle \langle \psi_i^\perp| - a^*b|\psi_i^\perp\rangle \langle \psi_i|,
\]

or in matrix form,

\[
\begin{pmatrix}
1 - |a|^2 & -ab^* \\
-a^*b & |b|^2
\end{pmatrix}.
\]
If we calculate the eigenvalues of the above matrix we get,

$$\lambda_i^\pm = \frac{1 - \delta_i}{2} \pm \frac{1}{2} \sqrt{(1 + \delta_i)^2 - 4|a_i|^2}.$$  \hfill (B.5)

Using equation (B.2) we have,

$$T(\psi_i, \bar{\psi}_i) = \left| \frac{1 - \delta_i}{2} + \frac{1}{2} \sqrt{(1 + \delta_i)^2 - 4|a_i|^2} \right| + \left| \frac{1 - \delta_i}{2} - \frac{1}{2} \sqrt{(1 + \delta_i)^2 - 4|a_i|^2} \right| \hfill (B.6)$$

$$\leq |1 - \delta_i| + \sqrt{(1 + \delta_i)^2 - 4|a_i|^2}. \hfill (B.7)$$

Now we have,

$$\sum_i p_i T(\psi_i, \bar{\psi}_i) \leq \sum_i p_i |1 - \delta_i| + \sum_i p_i \left| \sqrt{(1 + \delta_i)^2 - 4|a_i|^2} \right|. \hfill (B.8)$$

We know that $\delta_i \leq 1$ so we can find an upper bound by substituting $\delta_i = 1$ in the second term in the above equation, i.e.

$$\sum_i p_i T(\psi_i, \bar{\psi}_i) \leq \sum_i p_i |1 - \delta_i| + \sum_i p_i \sqrt{4 - 4|a_i|^2}. \hfill (B.9)$$

$$\leq \sum_i p_i (1 - \delta_i) + 2 \sum_i p_i \sqrt{1 - |a_i|^2}. \hfill (B.10)$$

$$\leq \sum_i p_i (1 - \delta_i) + 2 \left( 1 - \sum_i p_i |a_i|^2 \right). \hfill (B.11)$$

Notice that

$$F^2(\psi_i, \bar{\psi}_i) = |\langle \psi_i | \bar{\psi}_i \rangle|^2 = |a_i|^2 \leq \delta_i. \hfill (B.12)$$

So,

$$\sum_i p_i T(\psi_i, \bar{\psi}_i) \leq \sum_i p_i (1 - \delta_i) + 2 \left( 1 - \sum_i p_i F^2(\psi_i, \bar{\psi}_i) \right). \hfill (B.13)$$

Consider the following inequality,

$$1 - 2\epsilon \leq \sum_i p_i F(\psi_i, \bar{\psi}_i) \leq \sqrt{\sum_i p_i F^2(\psi_i, \bar{\psi}_i)}. \hfill (B.14)$$

So we have,

$$\sum_i p_i F^2(\psi_i, \bar{\psi}_i) \geq 1 - 4\epsilon. \hfill (B.15)$$

Since $\delta_i \geq F^2(\psi_i, \bar{\psi}_i)$, we also have,

$$\sum_i p_i \delta_i \geq 1 - 4\epsilon. \hfill (B.16)$$

An upper bound for equation (B.13) can be found by substituting the values $\sum_i p_i \delta_i = 1 - 4\epsilon = \sum_i p_i F^2(\psi_i, \bar{\psi}_i)$, so we get,
\[ \sum_i p_i T(\psi_i, \bar{\psi}_i) \leq (1 - (1 - 4\epsilon)) + 2\sqrt{1 - (1 - 4\epsilon)}, \quad (B.17) \]
\[ = 4\epsilon + 4\sqrt{\epsilon}. \quad (B.18) \]

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