Quantum Liouville Theory On The Riemann Sphere With $n > 3$ Punctures

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ABSTRACT

We have studied the quantum Liouville theory on the Riemann sphere with $n > 3$ punctures. While considering the theory on the Riemann surfaces with $n=4$ punctures, the quantum theory near an arbitrary but fixed puncture can be obtained via canonical quantization and an extra symmetry is explored. While considering more than four distinguished punctures, we have found the exchange relations of the monodromy parameters from which we can get a reasonable quantum theory.

PACS:02.20.+b

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1. Introduction

The Liouville theory has attracted much attention for a long time. The early interests in it rests mainly on the uniformization theory of two dimensional Riemann sphere and a lot of mathematicians such as Klein, Poincare, Koebe [1] etc. have done a lot of excellent works in this domain. The recent interests in it revived mainly with the Polyakov string [2] and two dimensional quantum gravity[3] where the Liouville action plays the role of Weyl anomaly. It is inevitable for us to study the classical and quantum Liouville theory in order to study the noncritical string theory.

Despite the nice work for the Liouville theory[4-5], there still exist a lot of mysterious aspects in both the classical and quantum Liouville theory. In [6], we have mainly studied the classical Liouville theory on the Riemann sphere with \( n > 3 \) punctures and interesting results have been obtained there.

In this paper, we mainly study the quantum Liouville theory on the Riemann sphere with \( n > 3 \) punctures. Starting from the Poisson bracket relations, we express these two chiral components which are two linearly independent solutions of the the uniformization equation in terms of the monodromy parameters and the fields which depend only on the space-time coordinates. Starting from the Poisson relations of the monodromy parameters and the free field, we directly get the quantum Liouville theory in the neighborhood of a puncture via canonical quantization. In this case, we find there exists an extra symmetry which leads to a certain arbitrariness for the matrices dominating the classical and quantum exchange algebra relations. If we restrict them to satisfy the classical and quantum Yang-Baxter equations respectively, we find a family of solutions to both these equations.

When we consider the theory on the riemann sphere with arbitrary number of punctures, we will meet some monodromy parameters except the fields depending only on the space time. We wish to get a quantum theory including that of the monodromy parameters and that of the space-time dependent fields. To do this, we will adopt a postulation of the quantum exchange properties of the monodromy parameters as well as the space-time dependent fields. Starting from this postulation, we can get the quantum exchange algebra relations of the chiral components from which we can see whether our postulation is reasonable or not since the quantum Liouville theory has the \( SL(2, R) \) quantum group symmetry.

This paper is organized as follows: In section 2, we introduce concisely, for the convenience of notations, some background knowledge about the uniformization theory of the Riemann sphere with \( n > 3 \) punctures and some main results of [6]. In section 3, we study the quantum Liouville theory, via canonical quantization, near an arbitrary but fixed puncture on the Riemann sphere. In section 4, we study the quantum Liouville theory considering all the monodromy parameters of \( n > 3 \) punctures. Finally, we will give some concluding remarks.

2. Introduction to the Liouville theory on the Riemann sphere with \( n > 3 \)
punctures.

In this section, we will introduce some background knowledge about the classical Liouville theory and some main results of [7] for the convenience of notation in the context.

The Riemann sphere $X$ with $n$ punctures whose coordinates are $z_1, \ldots, z_{n-3}, 0, 1, \text{and } \infty$ without loss of generality can be realized as the quotient space $H/\Gamma$ where $H$ is the upper half plane and $\Gamma \subset \text{PSL}(2, \mathbb{R})$ is the Fuchsian group. That is to say, there exists a covering $J : H \to X$ with $J(\gamma \omega) = J(\omega)$ to arbitrary $\gamma \in \Gamma, \omega \in H$.

The uniformization problem of the Riemann sphere is connected with the differential equation of $J^{-1}(\omega)$. It has been shown [1] that $J^{-1}$ satisfy

$$Q_X = \frac{\partial^3 J^{-1}}{\partial \omega^3} - 3 \left( \frac{\partial^2 J^{-1}}{\partial \omega^2} \right)^2 = \sum_{i=1}^{n-1} \left( \frac{1}{2(\omega - \omega_i)^2} + \frac{c_i}{\omega - \omega_i} \right)$$

(1)

where $c_i$ are the accessory parameters and satisfies

$$\sum_{i=1}^{n-1} c_i = 0, \quad \sum_{i=1}^{n-1} c_i \omega_i = 1 - \frac{n}{2}, \quad \sum_{i=1}^{n-1} \omega_i (1 + c_i \omega_i) = c_n$$

(2)

which lead to the asymptotic behavior of $Q_X$

$$Q_X = \frac{1}{2z^2} + \frac{c_n}{z^3}, \quad z \to \infty$$

The projection monodromy group here is just the Fuchsian uniformization group. The elements of this group are parabolic if we only consider the punctured Riemann surfaces. We can choose in this group a standard system of parabolic generators $M_1, \ldots, M_n$ satisfying the single relation $M_1 \cdots M_n = 1, \text{Tr} M_\lambda = 2$ and $M_\lambda$ has one fixed point $z_\lambda$, the matrices satisfying these conditions can be represented as

$$M_\lambda = \begin{pmatrix} 1 + \alpha_\lambda z_\lambda & -\alpha_\lambda z_\lambda^2 \\ \alpha_\lambda & 1 - \alpha_\lambda z_\lambda \end{pmatrix}$$

(3)

and

$$M_n = \begin{pmatrix} 1 & \alpha_n \\ 0 & 1 \end{pmatrix}$$

(4)

where we take $z_n = \infty$.

Equation (1) is related to a second order linear differential equation

$$\frac{d^2 \eta}{dz^2} + \frac{1}{2} Q_X(z) \eta = 0$$

(5)

by that $J^{-1}$ can be represented as the quotient of the two linearly independent solutions of (5).
The poisson bracket relations between these two solutions of equation (5) is dominated by
\[ \{ \eta_i(z), \eta_j(z') \} = S_{ij}^{kl} \eta_k(z) \eta_l(z') \]
where \( i, j, k, l = 1, 2 \) and
\[ S_{ij}^{kl} = -\frac{1}{16} [r_+ \theta(|z| - |z'|) + r_- \theta(|z'| - |z|)]_{ij}^{kl} \]
and \( \theta(z) \) is the step function. The \( 4 \times 4 \) matrices \( r_{\pm} \), called the classical \( r \) matrices, are the solutions of the classical Yang-Baxter equation:
\[ [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0, \]
and can be expressed by the generators of the Lie algebra \( sl(2, R) \):
\[ r_{\pm} = \pm H \otimes H \pm 4E_{\pm} \otimes E_{\mp} \]
\[ [H, E_{\pm}] = \pm 2E_{\pm}, \quad [E_+, E_-] = H \]

When we circle around the \( \lambda \)-th puncture, \( \eta_1, \eta_2 \) transform according to
\[ \begin{pmatrix} \eta_{1\lambda}^\lambda \\ \eta_{2\lambda}^\lambda \end{pmatrix} = (M_\lambda) \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \]
Because the Liouville field \( \phi \) is related to \( \eta_i \) by
\[ e^{-\frac{1}{2} \phi} = \frac{Im(\eta_1 \bar{\eta}_2)}{|\eta_1 \bar{\eta}_2 - \eta_2 \bar{\eta}_1|} \]
so we can find that \( \phi(P) \) is periodic for a closed path \( \Gamma_\lambda \) around any puncture on \( z \) when the analytic continuation of a pair of solutions \( \eta_1(P) \) and \( \eta_2(P) \) along \( \Gamma_\lambda \) results in a pair of new solutions \( \eta_1^\lambda(P) \) and \( \eta_2^\lambda(P) \).

That \( \phi(z) \) is single valued around puncture on \( \Omega \) enables us to make the following proposition:
\[ \{ \phi(P), \pi(P') \} = \{ \phi(P + \Gamma_\lambda), \pi(P' + \Gamma_\lambda) \} = \Delta(P - P') \]
where \( P, P' \in C_\tau \) and \( C_\tau \) is the level curve, \( \Delta(P - P') \) is the delta function on \( C_\tau \). We define \( C_\tau \) by
\[ C_\tau = \{Q | Re \int_{Q_0}^Q \omega = \tau. \} \]
where \( e^{\phi(z)} = \omega(z) \bar{\omega}(\bar{z}) \) near the puncture.

From (6) and (11), we can assume
\[ \{ \eta_1^\lambda(P), \otimes \eta_2^\lambda(P') \} = S_{ij}^{kl} \eta_k^\lambda(P) \eta_l^\lambda(P') \]
from which the following Poisson brackets can be uniquely determined

\[
\{\eta_1, \alpha_\lambda\} = -\frac{1}{4} \alpha_\lambda z_\lambda \eta_2, \quad \{\eta_1, z_\lambda\} = -\frac{1}{8} z_\lambda^2 \eta_2,
\]

\[
\{\eta_2, \alpha_\lambda\} = 0, \quad \{\eta_2, z_\lambda\} = -\frac{1}{8} \eta_1
\]

\[
\{\alpha_\lambda, \alpha_\rho\} = 0, \quad \{z_\lambda, z_\rho\} = \frac{1}{8}(z_\rho^2 - z_\lambda^2)
\]

\[
\{\alpha_\lambda, z_\rho\} = \frac{1}{4} \alpha_\lambda z_\lambda
\]

(14)

where \(\lambda = 1, \cdots, n-3\) and \(n > 3\) is the number of punctures. Thus the elements of the monodromy matrices are dynamical variables in our case.

3. Liouville theory near one puncture on the Riemann sphere with \(n = 4\) punctures.

Now we are ready to study the simplest case of our theory i.e. the Liouville theory on the Riemann surfaces with \(n = 4\) punctures. Without loss of generality, we may pick the coordinates of three of these punctures as 0, 1, \(\infty\) and the another as \(z_\lambda\). The monodromy matrices corresponding to the \(\lambda\)-th puncture is as shown in equation (3).

In this case, the fundamental relations in (14) can be rewritten in a more compact form as

\[
\{q_\lambda, p_\lambda\} = 1, \quad \{q_\lambda, q_\lambda\} = \{p_\lambda, p_\lambda\} = 0,
\]

\[
\{\eta_1, q_\lambda\} = -\frac{1}{2} e^{\frac{i}{4}(2p_\lambda - q_\lambda)} \eta_2, \quad \{\eta_2, q_\lambda\} = 0,
\]

\[
\{\eta_1, p_\lambda\} = 0, \quad \{\eta_2, p_\lambda\} = \frac{1}{4} e^{-\frac{i}{4}(2p_\lambda - q_\lambda)} \eta_2,
\]

where

\[
q_\lambda = 2 \ln \alpha_\lambda, \quad p_\lambda = \ln \alpha_\lambda + 2 \ln z_\lambda.
\]

(15)

(16)

From these relations, it is natural for us to define the Poisson bracket relation in the monodromy parameter space in terms of \(p_\lambda\) and \(q_\lambda\) as

\[
\{A(q_\lambda, p_\lambda), B(q_\lambda, p_\lambda)\}_P = \frac{\delta A}{\delta q_\lambda} \frac{\delta B}{\delta p_\lambda} - \frac{\delta A}{\delta p_\lambda} \frac{\delta B}{\delta q_\lambda}
\]

(17)

where the index \(P\) represents that the Poisson bracket is defined in the monodromy parameter space. With this explicit formulation, it is easy for us to get from (15) and (17) these derivative equations satisfied by \(\eta_1\) and \(\eta_2\)

\[
\frac{\delta m_1}{\delta p_\lambda} = \frac{1}{2} e^{\frac{i}{4}(2p_\lambda - q_\lambda)} \eta_2, \quad \frac{\delta m_2}{\delta p_\lambda} = 0,
\]

\[
\frac{\delta m}{\delta q_\lambda} = 0, \quad \frac{\delta m_2}{\delta q_\lambda} = \frac{1}{4} e^{-\frac{i}{4}(2p_\lambda - q_\lambda)} \eta_1.
\]

(18)

By solving these equations, we get

\[
\eta_1 = e^{\frac{i}{4}p_\lambda} f(z), \quad \eta_2 = e^{\frac{i}{4}q_\lambda} f(z)
\]

(19)
where \( f(z) \) is a function depending only on the space-time coordinates.

It is worth noticing that \( f(z) \) is a local solution of equation (5). Since \( J^{-1} \) can be represented as the quotient of two linearly independent solutions of the Fuchsian equation (5), we find locally

\[
\omega(z) = J^{-1}(z) = \frac{\eta_1}{\eta_2} = z_i
\]  

according to equation (19). It should be noticed that \( J^{-1} \) in general is a multivalued analytic function on \( X \). Suppose \( (\eta'_1, \eta'_2) \) is a pair of new solutions related to the original pair of solution \( (\eta_1, \eta_2) \) by an arbitrary Mobius transformation \( \gamma \in \Gamma \). If \( \gamma \neq M_\lambda, J^{-1}(z) = \frac{\eta'_1}{\eta'_2} \) has a branch different from that of \( J^{-1}(z) \) in equation (20). On the other hand, due to \( J(\gamma \omega) = J(\omega), \omega \in H \), for any \( \gamma \in \Gamma \), we find that \( z_i \) corresponds to \( \omega_i \) and \( \gamma \omega_i \) on \( H \) for any \( \gamma \in \Gamma \). On the other hand, \( J^{-1} \) is understood to depend on both the puncture’s position and the space-time coordinates. The expression (20) is a local one which holds in the neighbourhood of the puncture \( z_\lambda \).

From (19), we may easily verify that \( \eta_1, \eta_2 \) are invariant under the action of the monodromy group with generator (3). That is

\[
\eta_1^\lambda = (1 + \alpha_\lambda z_\lambda)\eta_1 - \alpha_\lambda z_\lambda^2 \eta_2 = \eta_1
\]

\[
\eta_2^\lambda = \alpha_\lambda \eta_1 + (1 - \alpha_\lambda z_\lambda)\eta_2 = \eta_2
\]

Therefore \( f(z) \) must be a local single valued function.

Due to equations (6) and (17), the definition of the Poisson bracket relation in (19) must be modified in the whole space \( W \) spanned by \( p_\lambda, q_\lambda \) and \( f \). Since the Poisson bracket between the monodromy parameters and \( f \) equals zero, \( W \) can be regarded as the direct sum of two orthogonal subspaces spanned by \( p_\lambda, q_\lambda \) and by \( f \) respectively. The Poisson bracket relation between \( f \)s at different positions can be obtained from

\[
\{\eta_i(z), \eta_i(z')\} = -\frac{1}{16} \epsilon(|z| - |z'|)\eta_i(z)\eta_i(z'), \quad i = 1, 2
\]  

in (6) and (15) to be

\[
\{f(z), f(z')\} = -\frac{1}{16} \epsilon(|z| - |z'|)f(z)f(z')
\]  

If we denote \( \varphi(z) = 4 \ln f(z) \), equation (22) turns to be

\[
\{\varphi(z), \varphi(z')\} = -\epsilon(|z| - |z'|).
\]  

So we can see that \( f \) is similar to a vertex constructing from a free field \( \varphi \).

Hence the Poisson bracket relation in \( W \) in terms of the monodromy parameters and \( \varphi \) can be expressed as

\[
\{A, B\} = \frac{\delta A}{\delta p_\lambda} \frac{\delta B}{\delta q_\lambda} - \frac{\delta A}{\delta q_\lambda} \frac{\delta B}{\delta p_\lambda} - \left( \frac{\delta A}{\delta \varphi(z)} \frac{\delta B}{\delta \varphi(z')} - \frac{\delta A}{\delta \varphi(z')} \frac{\delta B}{\delta \varphi(z)} \right) \epsilon(|z| - |z'|)
\]  

(24)
where $A$ and $B$ are arbitrary functions smoothly depending on the monodromy parameters and function $f$.

From (19), we can find that there exists an extra symmetry in our theory, that is
\begin{equation}
\eta_1(z)\eta_2(z') = \eta_2(z)\eta_1(z').
\end{equation}
This extra symmetry results from the monodromy symmetry around the puncture of the theory and leads us to a modified classical exchange algebra relation.

\begin{equation}
\{\eta_i(z), \eta_j(z')\} = \bar{S}_{ij}^{kl} \eta_k(z)\eta_l(z')
\end{equation}

where $i, j, k, l = 1, 2$ and $\bar{S}$, in general, can be expressed as
\begin{equation}
\bar{S} = -\frac{1}{16} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 + l_1 & 4 - l_1 & 0 \\
0 & l_2 & -1 - l_2 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\end{equation}
where we have assumed $|z| > |z'|$ for simplicity without loss of generality and $l_1, l_2$ are some arbitrary constants.

In this case, we will find that $\bar{S}$, in general, satisfies the inhomogeneous Yang-Baxter equation:
\begin{equation}
[S_{12}, S_{13}] + [S_{12}, S_{23}] + [S_{13}, S_{23}] = \frac{1}{16^2} l_2(1 - l_1)\epsilon_{ijk} A_i \otimes A_j \otimes A_k
\end{equation}
where $A_1 = E_+$, $A_2 = H$, $A_3 = E_-$ and $\epsilon_{ijk} = 1$ for even permutations of $123$ and $-1$ for odd permutations.

In two special cases, i.e.

(i): $l_1 = 1$, and $l_2$ is some arbitrary constant,

(ii): $l_2 = 0$, and $l_1$ is some arbitrary constant,

we will get the ordinary classical Yang-Baxter equation. Hence, we have got two classes of solutions of the classical Yang-Baxter equation by substituting (i) and (ii) into the general expression (27):
\begin{equation}
\bar{S} = -\frac{1}{16} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 \\
0 & l_2 & -1 - l_2 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\end{equation}
\begin{equation}
\bar{S} = -\frac{1}{16} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 + l_1 & 4 - l_1 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\end{equation}

When $l_1 = l_2 = 0$, $\bar{S}$ in equation (27) is just the ordinary $SL(2,R)$ solution $r_+$ in (8) to the classical Yang-Baxter equation.

Now we are ready to study the quantum Liouville theory on the Riemann sphere with four punctures. The method to the quantization is the usual canonical quantization, i.e. we first regard all the classical functions such as $\eta_1, \eta_2, p_\lambda, q_\lambda, \varphi$ in (15),
(23) and etc. as operators, then we replace the canonical Poisson brackets of \( q_\lambda \) and \( p_\lambda \) in (15) and that of \( \varphi \) in (23) by commutators according to the quantization principle, i.e. in the quantum case, we have these commutation relations:

\[
[q_\lambda, p_\lambda] = i\hbar, \quad [p_\lambda, \varphi(z)] = [q_\lambda, \varphi(z)] = 0
\]

\[
[\varphi(z), \varphi(z')] = -i\hbar \epsilon(|z| - |z'|)
\]

In the quantum case, \( \eta_1 \) and \( \eta_2 \) can be defined as

\[
\eta_1 = :e^{\frac{i}{\hbar}p_\lambda} :: e^{\frac{i}{\hbar}\varphi} :, \quad \eta_2 = :e^{\frac{i}{\hbar}q_\lambda} :: e^{\frac{i}{\hbar}\varphi} :
\]

where : : denotes normal ordering. With this definition, we can find, corresponding to the classical case, there exists in the quantum case an extra symmetry:

\[
\eta_1(z)\eta_2(z') = e^{-\frac{i}{\hbar}h} \eta_2(z)\eta_1(z')
\]

which leads to these exchange algebra relation:

\[
\eta_1(z)\eta_2(z') = R_{ij}^{mn} \eta_m(z')\eta_n(z)
\]

where

\[
R = e^{-i\hbar} e^{\frac{i}{\hbar}h} - k_1 e^{\frac{i}{\hbar}h} - k_1 e^{\frac{i}{\hbar}h} 0
0 e^{\frac{i}{\hbar}h} - k_2 0
0 0 0 1
\]

and \( k_1, k_2 \) are some quantities to be determined.

Before we set to find the equation satisfied by (33), we first show that if \( k_1 = \frac{1}{16}i\hbar + O(h^2) \) and \( k_2 = \frac{1}{16}i\hbar + O(h^2) \) when \( h \to 0 \), then we have

\[
R = P_{12} I \otimes I + i\hbar \mathcal{S} + O(h^2)
\]

where \( P_{12} \) is the permutation operator of 1 and 2 and \( \mathcal{S} \) is as shown in equation (27). Hence we may say that (33) can be regarded as the quantum counterpart of the \( \mathcal{S} \) matrix in (27).

For general \( k_1 \) and \( k_2 \), \( R \) satisfies the following inhomogeneous quantum Yang-Baxter equation:

\[
R_{12} R_{13} R_{23} - R_{23} R_{13} R_{12} = k_1 k_2 \left( \frac{1}{2} I \otimes (A + D) + \frac{1}{2} H \otimes (A - D) + E_\xi \otimes B + E_\eta \otimes C \right)
\]

where

\[
A = (a^3 - \frac{k_1}{a^4})(E_- \otimes E_+ - E_+ \otimes E_-) + \frac{1}{4}(\frac{k_1}{a^4} - a)(I - H) \otimes (I + H)
\]

\[
B = a(a^2 k_1 - 1)H \otimes E_+ + \frac{1}{2a}(1 - a^2 k_1 - a^4 + k_2)E_+ \otimes I + \frac{1}{2a}(1 - a^2 k_1 + a^4 - k_2)E_+ \otimes H
\]

\[
C = (a - \frac{k_1}{a^4})H \otimes E_- - \frac{1}{2a}(1 - a^2 k_1 - a^4 + k_2)E_- \otimes I - \frac{1}{2a}(1 - a^2 k_1 + a^4 - k_2)E_- \otimes H
\]

\[
D = \frac{1}{4}(a^3 k_1 - \frac{k_1}{a^4})(I + H) \otimes (I - H) + \frac{1}{4}(a^3 k_1 - 1)(E_- \otimes E_+ - E_+ \otimes E_-)
\]
and $a = e^{-\frac{i}{\hbar}h}$.

In these special cases

(i): $k_1 = 0$, for arbitrary $k_2$.

(ii): $k_2 = 0$, for arbitrary $k_1$.

(iii): $k_2 = a^4$, $k_1 = a^{-2}$.

equation (37) reduces to the ordinary Quantum Yang-Baxter equation. So three kinds of matrices

\[
R = e^{-\frac{i}{\hbar}h} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & e^{-\frac{i}{\hbar}h} & 0 \\ 0 & e^{\frac{i}{\hbar}h} - k_2 & k_2 e^{\frac{i}{\hbar}h} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

and

\[
R = e^{-\frac{i}{\hbar}h} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & k_1 e^{\frac{i}{\hbar}h} & e^{-\frac{i}{\hbar}h} - k_1 e^{\frac{i}{\hbar}h} & 0 \\ 0 & e^{\frac{i}{\hbar}h} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

in our case satisfy the quantum Yang-Baxter equation:

\[R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}\]

4. General Case.

In this section, we consider the Liouville theory considering arbitrary number of punctures.

Let us first reparametrize the monodromy parameters $\alpha_\lambda$, $z_\lambda$ as

\[
\alpha_\lambda^\frac{1}{2} = Q_\lambda, \quad z_\lambda \alpha_\lambda^\frac{1}{2} = P_\lambda.
\]

Then the Poisson bracket relation (14) can be rewritten as

\[
\{Q_\lambda, P_\rho\} = \frac{1}{8} P_\lambda Q_\rho, \quad \{Q_\lambda, Q_\rho\} = 0
\]

\[
\{\eta_1, Q_\lambda\} = -\frac{1}{8} P_\lambda \eta_2, \quad \{P_\lambda, P_\rho\} = 0
\]

\[
\{\eta_1, P_\lambda\} = 0, \quad \{\eta_2, Q_\lambda\} = 0
\]

\[
\{\eta_2, P_\lambda\} = \frac{1}{8} Q_\lambda \eta_1
\]

Comparing with the notation of the last section in the neighbourhood of the $\lambda$-th puncture, we have

\[
P_\lambda = e^{\frac{i}{\hbar}P_\lambda}, \quad Q_\lambda = e^{\frac{i}{\hbar}Q_\lambda}
\]

Generally, if we consider the Poisson bracket relation between the functions $F = F(Q_i, P_i)$ and $G = G(Q_i, P_i)$, we can get from equation the property of the Poisson bracket and (36),

\[
\{F, G\} = \frac{\partial F}{\partial Q_i} \frac{\partial G}{\partial Q_j} \{Q_i, Q_j\} + \frac{\partial F}{\partial Q_i} \frac{\partial G}{\partial P_j} \{Q_i, P_j\} + \frac{\partial F}{\partial P_i} \frac{\partial G}{\partial Q_j} \{P_i, Q_j\} + \frac{\partial F}{\partial P_i} \frac{\partial G}{\partial P_j} \{P_i, P_j\}
\]

\[= \frac{1}{8}[\nabla_Q F \cdot \nabla_P G - \nabla_P F \cdot \nabla_Q G]
\]

(37)
\( \nabla Q = \sum_i P_i \frac{\partial}{\partial Q_i} \), \( \nabla Q = \sum_i Q_i \frac{\partial}{\partial P_i} \).

Equation (37) may also be regarded as the defining relation for the Poisson bracket in the parameter space spanned by \( P_i, Q_i \).

From equation (14) and the defining relation (37), we can easily get the differential equations satisfied by \( \eta_1 \) and \( \eta_2 \) in this parameter space:

\[
\sum_{i=1}^{n-3} Q_i \frac{\partial}{\partial P_i} = \eta_2, \quad \sum_{i=1}^{n-3} P_i \frac{\partial}{\partial Q_i} = \eta_1, \\
\sum_{i=1}^{n-3} P_i \frac{\partial}{\partial Q_i} = 0, \quad \sum_{i=1}^{n-3} Q_i \frac{\partial}{\partial P_i} = 0.
\]

The general solution to this equation can be found to be [7]

\[
\eta_1 = \sum_{i=1}^{n-3} f_i(z) P_i, \quad \eta_2 = \sum_{i=1}^{n-3} f_i(z) Q_i
\]

where \( f_i(z) \) depends only on the space-time coordinates and must satisfy the uniformization equation (1).

If we consider two functions \( F = F(P_i, Q_i, f_i, f'_i) \) and \( G = G(P_i, Q_i, f_i, f'_i) \) in which \( f_i = f_i(z), \ f'_i = f_i(z') \), their Poisson bracket can be found to be

\[
\{ F, G \} = \{ F, G \}_{(Q_i, P_j)} \cdot \{ Q_i, P_j \} + \{ F, G \}_{(f_i, f'_j)} \cdot \{ f_i, f'_j \}
\]

where

\[
\{ F, G \}_{(Q_i, P_j)} = \frac{\partial F}{\partial Q_i} \cdot \frac{\partial G}{\partial P_j} - \frac{\partial F}{\partial P_j} \cdot \frac{\partial G}{\partial Q_i}, \\
\{ F, G \}_{(f_i, f'_j)} = \frac{\partial F}{\partial f_i} \cdot \frac{\partial G}{\partial f'_j} - \frac{\partial F}{\partial f'_j} \cdot \frac{\partial G}{\partial f_i}.
\]

From (6) and (39), we can get the Poisson bracket relations between \( f_i \) and \( f'_i \) as

\[
\{ f_i, f'_j \} = \frac{1}{16} \epsilon(|z| - |z'|)(f_i f'_j - 2f_j f'_i)
\]

When we consider the theory on the Riemann surface with more than four punctures, one may notice that neither the parameters \( P_i, Q_i \) nor the space-time dependent fields \( f, f' \) in (39) can be considered as free fields.

However, in the canonical quantization, when we try to get the quantum theory corresponding to the classical theory, we have made an important assumption– the principle of quantization. This assumption corresponding to the canonical Poisson bracket relation between the canonical variables is the bridge between the classical and quantum theory.

What we face now is: we have not the canonical conjugate variables at hands. So the first thing we should do is to find a principle of quantization corresponding to our
Poisson bracket relations. We assume that all the functions such as \( \eta_1, \eta_2, f(z) \) and all the monodromy parameters such as \( Q_i, P_i \) will be regarded as operators in the quantum case. The expression of \( \eta_1, \eta_2 \) in terms of \( f(z), Q_i, P_i \) is the same as its classical case in equation (39).

Furthermore, we make the following principle of quantization:

\[
P_k Q_l = Q_l P_k + (1 - A) P_l Q_k
\]

\[
f_l f'_k = (2 - B)f'_k f_l - 2(1 - B)f_l f_k
\]

where \( A \) and \( B \) are some coefficients to be determined (See [7] for details).

Furthermore, we can get from (39) and (42) these exchange relations

\[
\begin{align*}
\eta_1 \eta'_1 &= B \eta'_1 \eta_1, \\
\eta_1 \eta'_2 &= (2 - B) \eta'_2 \eta_1 + (B - 2A + AB) \eta'_1 \eta_2, \\
\eta_2 \eta'_2 &= B \eta'_2 \eta_2, \\
\eta_2 \eta'_1 &= (2 - B) A(2 - A) \eta'_1 \eta_2 - (4 - 3B - 2A + AB) \eta'_2 \eta_1.
\end{align*}
\]

They can be rewritten compactly as

\[
\eta_i \eta'_j = R^{mn}_{ij} \eta_m \eta'_n
\]

where

\[
R = \begin{pmatrix}
B & 0 & 0 & 0 \\
0 & 2 - B & B - 2A + AB & 0 \\
0 & -4 + 3B + 2A - AB & (2 - B) A(2 - A) & 0 \\
0 & 0 & 0 & B
\end{pmatrix}
\]

(45)

If \( A = \frac{4 - 3B}{2 - B} \), we will get

\[
R_+ = \begin{pmatrix}
B & 0 & 0 & 0 \\
0 & 2 - B & -4 + 4B & 0 \\
0 & 0 & \frac{B(4 - 3B)}{2 - B} & 0 \\
0 & 0 & 0 & B
\end{pmatrix}
\]

(46)

If \( A = \frac{B}{2 - B} \), we will get

\[
R_- = \begin{pmatrix}
B & 0 & 0 & 0 \\
0 & 2 - B & 0 & 0 \\
0 & -4 + 4B & 0 & 0 \\
0 & 0 & \frac{B(4 - 3B)}{2 - B} & 0
\end{pmatrix}
\]

(47)

It is easy to show that both \( R_+ \) and \( R_- \) satisfies the quantum Yang-Baxter equation. If we let \( B = \exp(\frac{i \hbar}{16}) \), we find

\[
R_+|_{\hbar \rightarrow 0} = I + \frac{i \hbar}{16} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & -4 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} + o(\hbar^2) = I + \frac{i \hbar}{16} r_+ + o(\hbar^2).
\]

(48)
If we let \( B = \exp(-\frac{i \bar{h}}{16}) \), we find

\[
R_-|_{\bar{h} \to 0} = I - \frac{i \bar{h}}{16} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 4 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} + o(h^2) = I - \frac{i \bar{h}}{16} r_- + o(h^2).
\] (49)

5. Concluding Remarks.

At this stage, we have got the quantum Liouville theory on the Riemann sphere with \( n > 3 \) punctures. To make the problems clearer, let us present the logic as follows: we are studying the Liouville theory on the Riemann sphere with \( n > 3 \) punctures, the chiral components which are two independent solutions of the uniformization equation depend in general on the space-time coordinates and the monodromy parameters, its classical exchange algebra relations is dominated by the classical \( r \)-matrix and its quantum exchange algebra relation, in principle, can be assumed to be dominated by the quantum \( R \)-matrices. This is in fact the quantum integrability condition. When we only consider the quantum theory in the neighbour of an arbitrary but fixed puncture, we can find the canonical variables in the monodromy parameter space and space dependent field space. We can get the quantum theory from the classical one via canonical quantization. When we consider more then four distinguished punctures, we find that the usual canonical quantization procedure does not do. However, we want to get the quantum theory of all the dynamical variables which include the monodromy parameters. For this reason, we adopt the assumption of the quantum exchange relations of the monodromy parameters and that of the space-dependent fields, i.e. equation (42). Then we try to get the quantum exchange algebra relations between the chiral components according to this assumption. If this assumption can lead to the quantum integrability condition, we can say that this assumption is reasonable and hence we have got the quantum exchange relations between all the dynamical variables.

Acknowledgement

The authors are obliged to Prof. L. Bonora, H. Y. Guo, A. Verjovsky and R. Wang for their stimulating discussions and encouragement in the course of this work. J. M. Shen would like to thank Prof. Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste, Italy. Z. H. Wang is obliged to Profs. D. Amati, L. Bonora and R. Iengo for their hospitalities during his works in SISSA sponsored by the World Laboratory.

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