The Semiparametric Cramér-Rao Bound for Complex Elliptically Symmetric Distributions

Stefano Fortunati, Member, IEEE, Fulvio Gini, Fellow, IEEE,
Maria S. Greco, Fellow, IEEE, and Abdelhak M. Zoubir, Fellow, IEEE

Abstract

This letter aims at extending the Constrained Semiparametric Cramer-Rao Bound (CSCRB) for the joint estimation of mean vector and scatter matrix of Real Elliptically Symmetric (RES) distributions to Complex Elliptically Symmetric (CES) distributions. A closed form expression for the complex CSCRB (CCSCRB) is derived by exploiting the so-called Wirtinger or CR-calculus. Finally, the CCSCRB for the estimation of the complex mean vector and scatter matrix of a set of complex t-distributed random vectors is provided as an example of application.

Index Terms

Semiparametric model, Semiparametric Cramér-Rao Bound, Complex Elliptically Symmetric distributions, scatter matrix estimation, robust estimation.

I. INTRODUCTION

Statistical analysis of complex data is a well-established field in Signal Processing (see [1], [2], [3], [4], [5], [6], and [7] just to cite a few). The use of a complex representation for the acquired data can simplify the modeling and the inference tasks in many applications such as acoustics, optics, seismology, communications and radar/sonar Signal Processing. This fact, together with the need to model the non-Gaussian, heavy-tailed statistical behavior of the noise, leads to the
introduction of the wide family of Complex Elliptical Symmetric (CES) distributions ([8], [9, Ch. 3], [10] and [11, Ch. 4]). CES distributions are the complex extension of Real Elliptically Symmetric distributions ([12], [13]) from which they inherit most of their properties.

Our recent paper [14] focuses on the particular semiparametric structure of the RES distributions family. As discussed in [15] and [16] (Sec. 4.2 and 7.2), the RES distributions can be considered as a semiparametric group model whose parametric part is given by the mean vector and by the constrained scatter matrix to be jointly estimated while the non-parametric part, nuisance, is given by the density generator (see [14] and [16] for additional details and references on this topic). Moreover, in [14], a closed form expression for the CSCRB on the joint estimation of the parametric part of the RES model has also been derived. It is worth noticing that the CSCRB for the estimation of the mean vector and of the constrained scatter matrix has been already derived in [17], [18], [19], [20] by using a more general, but more abstract, procedure based on the LeCam’s theory [21].

The aim of this letter is to generalize the results on the CSCRB, already derived in the context of RES distributions, to CES distributions. In particular, we will provide a closed form expression for the CSCRB on the Mean Square Error (MSE) of the joint estimation of the complex mean vector \( \mathbf{\mu} \) and complex constrained scatter matrix \( \Sigma \) of a set of CES distributed random vectors. This generalization relies on the Wirtinger or \( \mathbb{C} \mathbb{R} \)-calculus ([22], [23], [5], [6], [7], [24], [25]) and on its application on the derivation of standard and misspecified CRBs ([26], [27], [28], [29], [30] and [31]).

Notation: Throughout this paper, italics indicates scalar quantities (\( a \)), lower case and upper case boldface indicate column vectors (\( \mathbf{a} \)) and matrices (\( \mathbf{A} \)), respectively. Note that the word “vector” indicates both Euclidean vectors and vector-valued functions. For the sake of clarity, we indicate sometimes a vector-valued function as \( \mathbf{a} \equiv \mathbf{a}(z) \). The asterisk * indicates complex conjugation. The superscripts \( T \) and \( H \) indicate the transpose and the Hermitian operators, respectively then \( \mathbf{A}^H = (\mathbf{A}^\ast)^T \). Moreover, \( \mathbf{A}^{-T} \triangleq (\mathbf{A}^{-1})^T = (\mathbf{A}^T)^{-1}, \mathbf{A}^{-\ast} \triangleq (\mathbf{A}^{-1})^\ast = (\mathbf{A}^\ast)^{-1} \) and \( \mathbf{A}^{-H} \triangleq (\mathbf{A}^{-1})^H = (\mathbf{A}^H)^{-1} \). According to the notation introduced in [32] and [14], we indicate the true parameter vector as \( \phi_0 \) and the related true pdf as \( p_0(z) \triangleq p_Z(z|\phi_0, h_0) \), where \( h_0 \) indicates the true density generator. Moreover, \( E_0 \{ \cdot \} \) indicates the expectation operator with respect to \( \text{(w.r.t.)} \) the true pdf \( p_0(z) \). Finally, for random variables or vectors, \( =_d \) stands for ”has the same distribution as”.
II. CES DISTRIBUTIONS

This section provides a short overview of CES distributions with a specific focus on the properties that will play a crucial role in the derivation of the CCSCRB.

Definition II.1. ([8], [9], [10] and [11, Ch. 4]) Let \( z \triangleq x_R + j x_I \in \mathbb{C}^N \) be a complex random vector and let \( x_R \in \mathbb{R}^N \) and \( x_I \in \mathbb{R}^N \) be two real random vectors that represent the real and the imaginary part of \( z \), respectively. Then \( z \) is said to be CES-distributed with mean vector \( \mu \) and scatter matrix \( \Sigma \) such that (s.t.):

\[
\mu = \mu_R + j \mu_I \in \mathbb{C}^N \quad \Sigma = C_1 + j C_2 \in \mathbb{C}^{N \times N},
\]

if and only if the real random vector \( \tilde{x} \triangleq (x^T_R, x^T_I)^T \in \mathbb{R}^{2N} \) is RES-distributed with mean vector \( \tilde{\mu} = (\mu^T_R, \mu^T_I)^T \) and scatter matrix \( \tilde{\Sigma} \) that satisfies the following structure

\[
\tilde{\Sigma} = \frac{1}{2} \begin{pmatrix}
C_1 & -C_2 \\
C_2 & C_1
\end{pmatrix}.
\]

We note that, as a consequence of Definition II.1, any CES-distributed random vector \( z \) satisfies the circularity property, i.e. \( (z - \mu) =_d e^{j\theta} (z - \mu), \forall \theta \in \mathbb{R} \). Moreover, if \( \text{rank}(\Sigma) = N \), the pdf of the CES-distributed vector \( z \) can be directly obtained from the one of the RES-distributed vector \( \tilde{x} \sim RES_{2N}(\tilde{x}; \tilde{\mu}, \tilde{\Sigma}, g) \). Specifically, ([9, Sec. 3.5] and [11, Sec. 4.2.2]):

\[
RES_{2N}(\tilde{x}; \tilde{\mu}, \tilde{\Sigma}, g) \triangleq p_{\tilde{x}}(\tilde{x}; \tilde{\mu}, \tilde{\Sigma}, g) = 2^{-(2N)/2} |\tilde{\Sigma}|^{-1/2} g((\tilde{x} - \tilde{\mu})^T \tilde{\Sigma}^{-1} (\tilde{x} - \tilde{\mu}))
\]

\[
= |\Sigma|^{-1} g \left( 2(z - \mu)^H \Sigma^{-1} (z - \mu) \right)
\]

\[
= p_Z(z; \mu, \Sigma, h) \triangleq CES_N(z; \mu, \Sigma, h),
\]

where \( h(t) \triangleq g(2t) \). Note that by moving from the real to the complex representation, the functional form of the density generator remains unchanged except for the scaling factor 2 of its argument.

As for RES distributed vectors, any CES distributed vector \( z \) can be represented as ([8], [10] and [9, Sec. 3.5]):

\[
z =_d \mu + \sqrt{Q} \Sigma^{1/2} u
\]

where \( u \sim \mathcal{U}(\mathbb{C}S^N) \) is a complex random vector uniformly distributed on the unit complex \( N \)-sphere \( \mathbb{C}S^N \) and \( Q \) is the so-called 2nd-order modular variate, s. t.:

\[
Q =_d Q \triangleq (z - \mu)^H \Sigma^{-1} (z - \mu),
\]
whose pdf is given by:

\[ p_Q(q) = 2^{-1} s_N q^{N-1} h(q) = \pi^N \Gamma(N)^{-1} q^{N-1} g(2q), \]  

(6)

where \( s_N \triangleq 2\pi^N / \Gamma(N) \) is the surface area of \( \mathbb{C}S^N \).

Similarly to the real case, the 2nd-order modular variate \( Q \) and the scatter matrix \( \Sigma \) can be identified only up to a scalar factor. To avoid this scale ambiguity problem, we put the usual constraint on trace of \( \Sigma \), i.e. \( \text{tr}(\Sigma) = N \).

Definition II.1 and the equality chain in (3) suggest the existence of a one-to-one mapping between the subset of the RES distributions satisfying the covariance structure specified in (2) and the family of CES distributions. In other words, the CES “framework” is just a convenient and compact representation of a subset of RES distributions. This implies that the theory already developed for RES class holds true for CES class. In particular, by relying on the approach proposed in [9, Sec. 3.5], CES distributions can be interpreted as the semiparametric group model generated from the set of Complex Spherically Symmetric (CSS) distributions through the action of the group of affine transformations.

III. THE CSCRB FOR COMPLEX PARAMETER ESTIMATION IN CES DISTRIBUTIONS

The derivation of the CSCRB for the joint estimation of the complex mean vector \( \mu \) and of the complex constrained scatter matrix \( \Sigma \) for CES-distributed vectors strictly follows the one described in [14] for the real case. However, in the complex case, the derivatives have to be considered as Wirtinger derivatives. More precisely, following Theorem IV.1 in [14], the steps to be covered are:

A. Define the complex constrained parameter space \( \bar{\Omega}_c \).

B. Evaluate the semiparametric efficient score vector \( \bar{\mathbf{s}}_0(z) \) using the Wirtinger derivatives.

C. Derive the semiparametric Fisher Information Matrix (SFIM) for the joint estimation of \( \mu \) and \( \Sigma \).

D. Obtain a closed form expression for the complex CSCRB.

A. The complex constrained parameter space \( \bar{\Omega}_c \)

As mentioned before, the parametric part of the semiparametric CES model is given by the mean vector \( \mu \) and by the Hermitian scatter matrix \( \Sigma \). According to the rules of the Wirtinger calculus, to define a complex parameter space, we have to take into account the parameters to
be estimated together with their complex conjugates ([26], [27], [28], [29], [30], [31]). To this end, we note that, while $\mu$ is composed of $N$ complex free parameters, i.e. all its $N$ entries, the Hermitian scatter matrix $\Sigma$ can be parametrized by means of its $N$ real diagonal entries and of its $N(N-1)/2$ complex entries that are positioned strictly below the main diagonal [33]. More formally, and following the notation in [29] and [30], the parametric part of the CES distributions model can be described by the parameter vector $\theta = (\theta_c^T, \theta_H^T, \theta_r^T)^T$, where:

$$\theta_c = (\mu^T, \text{vec}(\Sigma)^T)^T, \quad \theta_r = \text{diag}(\Sigma),$$

(7)

the operator $\text{vec}_l(\cdot)$ selects all the entries strictly below the main diagonal of $\Sigma$ taken in the same column-wise order as the ordinary $\text{vec}(\cdot)$ operator [33, Sec. 2.4] while $\text{diag}(\Sigma)$ is a column vector collecting the diagonal elements of $\Sigma$.

For ease of calculation, we decide to express the parameter vector $\theta$ with respect to a different basis. In particular, let us introduce a permutation matrix $P$ s. t.:

$$\phi \equiv (\mu^T, \mu^H, \text{vec}(\Sigma)^T)^T = P\theta.$$

(8)

It is worth stressing here that the previous two characterizations of the augmented complex parameter vectors $\theta$ and $\phi$ given in (7) and (8), respectively, are equivalent, since the scatter matrix $\Sigma$ is an Hermitian matrix (see [33, Sec. 6.5.5]) and the permutation matrix $P$ only represents an orthogonal change of basis. Consequently, let us define the “augmented” complex parameter space $\Omega_C \subseteq \mathbb{C}^q$ of dimension $q = N(N+2)$ as:

$$\Omega_C = \{ \phi \in \mathbb{C}^q | \phi \text{ is as in (8); } \mu \in \mathbb{C}^N, \Sigma \in M_C^N \},$$

(9)

where $M_C^N$ is the set of all the Hermitian, positive-definite matrices of dimension $N \times N$. As previously discussed, in order to avoid the scale ambiguity between the scatter matrix and the density generator of a CES distribution, we choose to impose a constraint on the trace of $\Sigma$.

Specifically, let us define the scalar, real-valued, constraint function as:

$$c(\Sigma) \equiv \text{tr}(\Sigma) - N = \sum_{i \in I} [\text{vec}(\Sigma)]_i - N = 0,$$

(10)

and $I \equiv \{ i | i = j + (j-1)N, j = 1, \ldots, N \}$. Then, the function $c(\Sigma)$ constrains the parameter vector $\phi$ in a smooth sub-manifold of $\Omega_C$ defined as:

$$\bar{\Omega}_C = \{ \phi \in \Omega_C | c(\Sigma) = 0 \},$$

(11)

of dimension $\bar{q} = q - 1$. From now on, $\bar{\Omega}_C$ will be considered as the reference parameter space.
B. The complex semiparametric efficient score vector $\bar{s}_0(z)$

This subsection provides a closed form expression for the semiparametric efficient score vector $\bar{s}_0(z)$, evaluated at the true parameter vector $\phi_0 \in \Omega_C$. The complex extension of the semiparametric efficient score vector given in Theorem IV.1 in [14] can be defined as:

$$\bar{s}_0 = [s^T_\mu_0, s^T_{\mu_0^*}, s^T_{\text{vec}(\Sigma_0)}]^T = s_{\phi_0} - \Pi(s_{\phi_0}|T_{h_0}),$$

(12)

where $s_{\phi_0}$ is the score vector w.r.t. $\phi_0$ and $\Pi(s_{\phi_0}|T_{h_0})$ is the orthogonal projection of $s_{\phi_0}$ on the nuisance tangent space evaluated at the true density generator $h_0$.

The score vector w.r.t. $\phi_0$ can be expressed as:

$$s_{\phi_0} \triangleq \nabla_{\phi} \ln p_Z(z; \phi_0, h_0) = [s^T_{\mu_0}, s^T_{\mu_0^*}, s^T_{\text{vec}(\Sigma_0)}]^T$$

(13)

where, following the approach detailed in [29], the complex gradient operator of a scalar, real-valued, function $f(\phi)$, evaluated in $\phi_0$, is defined as:

$$[\nabla_{\phi} f(\phi_0)]_i = \partial f(\phi)/\partial {\phi_i^*}|_{\phi = \phi_0}, \ i = 1, \ldots, q.$$

(14)

The closed form expression for $s_{\mu_0}, s_{\mu_0^*}$ and $s^T_{\text{vec}(\Sigma_0)}$ can be obtained by applying the standard rules of the Wirtinger matrix calculus. For an excellent and comprehensive book about this topic, we refer the reader to [33]. Here, to not clutter the presentation with too many technicalities, we will provide only the final outcomes without reporting all the steps.

The complex gradient w.r.t. $\mu$ of $\ln p_Z(z; \phi_0, h_0)$ can be obtained by applying the rules listed in Table 4.2 of [33] as:

$$s_{\mu_0}(z) = -\psi_0(Q_0)\Sigma_0^{-1}(z - \mu_0) = -\sqrt{Q}\psi_0(Q)\Sigma_0^{-1/2}u.$$  

(15)

Consequently, we have that:

$$s_{\mu_0^*}(z) = s_{\mu_0^*}(z) = -\sqrt{Q}\psi_0(Q)(\Sigma_0^*)^{-1/2}u^*,$$

(16)

where $\psi_0(t) \triangleq d\ln h_0(t)/dt$. Moreover, by applying the derivative rules listed in Table 4.3 and the equality in [33, eq. 6.199], we get:

$$s_{\text{vec}(\Sigma_0)}(z) = -\text{vec}(\Sigma_0^{-1}) - \psi_0(Q_0)\Sigma_0^{-*} \otimes \Sigma_0^{-1}\text{vec}((z - \mu_0)(z - \mu_0)^H)$$

$$= -\text{vec}(\Sigma_0^{-1}) - Q\psi_0(Q)((\Sigma_0^*)^{-1/2} \otimes \Sigma_0^{-1/2})\text{vec}(uu^H).$$

(17)

The next step is the derivation of the orthogonal projection of the score vector $s_{\phi_0}$ on the nuisance tangent space of the CES semiparametric group model evaluated at the true density
generator \( h_0 \). The procedure to obtain a closed form expression for \( \Pi(s_{\phi_0}|T_{h_0}) \) is exactly as the one described in [14, Sec. IV.B] for the real case. Specifically, the properties of the semiparametric group models collected in Proposition II.1 of [14] can be applied straight to derive \( \Pi(s_{\phi_0}|T_{h_0}) \). Then, by replicating step-by-step the procedure discussed in [14, Sec. IV.B], we obtain:

\[
\Pi(s_{\mu_0}|T_{h_0}) = \Pi(s_{\mu_0}^*|T_{h_0}) = 0_N, \tag{18}
\]

\[
\Pi(s_{\text{vec}(\Sigma_0)}|T_{h_0}) = -(1 + N^{-1}Q\psi_0(\mathcal{Q}))\text{vec}(\Sigma_0^{-1}). \tag{19}
\]

Note that, as for the real case, \( s_{\mu_0} \) and \( s_{\mu_0}^* \) are orthogonal to the nuisance tangent space \( T_{h_0} \). This implies that we will get the same (asymptotic) performance in the estimation of \( \mu_0 \) by knowing or not knowing the true density generator \( h_0 \).

The efficient score vector \( \bar{s}_0 \) in (12) can now be easily derived by collecting the previous results. In particular, we have that \( \bar{s}_{\mu_0} \equiv s_{\mu_0} \) and \( \bar{s}_{\mu_0}^* \equiv s_{\mu_0}^* \) since, as reported in (18), the projection is nil, and

\[
\bar{s}_{\text{vec}(\Sigma_0)} = d \mathcal{Q}\psi_0(\mathcal{Q})(\Sigma_0^*)^{-1/2} \otimes \Sigma_0^{-1/2} \text{vec}(\mathcal{Q}^H) - N^{-1} \text{vec}(\Sigma_0^{-1}). \tag{20}
\]

C. The SFIM \( \bar{I}(\phi_0|h_0) \)

The SFIM can be expressed as the following block matrix:

\[
\bar{I}(\phi_0|h_0) = \begin{pmatrix}
\bar{I}(\mu_0|h_0) & 0_{2N \times N^2} \\
0_{2N \times N^2} & C_0(\bar{s}_{\text{vec}(\Sigma_0)})
\end{pmatrix}, \tag{21}
\]

where \( C_0(1) \triangleq E_0\{1^H\} \), for every random function \( l(z) \) and the off-diagonal block matrices vanish because all the third-order moments of \( u \) vanish [10, Lemma 1] and

\[
\bar{I}(\mu_0|h_0) = \begin{pmatrix}
C_0(\bar{s}_{\mu_0}) & 0_{N \times N} \\
0_{N \times N} & C_0'(\bar{s}_{\mu_0})
\end{pmatrix}, \tag{22}
\]

\[
C_0(\bar{s}_{\mu_0}) = N^{-1} E\{Q\psi_0(\mathcal{Q})^2\} \Sigma_0^{-1}. \tag{23}
\]

Note that the off-diagonal matrices in (22) vanish due to the circularity of \( u \), while to derive (23), we used the fact that \( E\{uu^H\} = N^{-1} I \) ([10], Lemma 1). Moreover, by relying on the procedure given in [34] and [35], after some standard complex matrix manipulations, we get:

\[
C_0(\bar{s}_{\text{vec}(\Sigma_0)}) = \frac{E\{Q^2\psi_0(\mathcal{Q})^2\}}{N(N + 1)} (\Sigma_0^T \otimes \Sigma_0^{-1} - N^{-1} \text{vec}(\Sigma_0^{-1})\text{vec}(\Sigma_0^{-1})^H). \tag{24}
\]

It is worth noticing that the constraint on the trace of the scatter matrix \( \Sigma_0 \) has not been imposed yet.
D. The complex CSCRB: CCSCRB(φ₀|h₀)

We are now ready to derive a closed form expression of the CSCRB for the constrained estimation of the complex parameter vector φ₀ ∈ ℂC, i.e. CCSCRB(φ₀|h₀). As showed in Theorem IV.1 in [14] for the real case, the first step to obtain CCSCRB(φ₀|h₀) is the derivation of matrix U whose columns form an orthonormal basis for the null space of the Jacobian matrix of the constraint function c(Σ₀) in (10). Since, in our case, c(Σ₀) involves only the real diagonal elements of the Hermitian matrix Σ₀, U ∈ ℝ^{N²×(N²−1)} is the matrix that satisfies the following two conditions:

\[ \nabla^T_{\text{vec}(Σ)} c(Σ₀) U = 0, \quad U^T U = I_{N²−1}. \]  

Through direct calculation, we have that:

\[ \nabla^T_{\text{vec}(Σ)} c(Σ₀) = \nabla^T_{\text{vec}(Σ)} \sum_{i \in I} [\text{vec}(Σ₀)]_i = \text{vec}(I_{N²})^T. \]  

Then, matrix U can be obtained numerically by evaluating the N²−1 orthonormal eigenvectors associated with the zero eigenvalue of Iₙ through SVD.

Finally, the CCSCRB for the estimation of φ₀ ∈ ℂC in (11) can be expressed as:

\[ \text{CCSCRB}(φ₀|h₀) = \begin{pmatrix} \bar{I}(μ₀|h₀)^{-1} & 0_{N²×N²} \\ 0_{N²×N²} & \bar{I}(Σ₀|h₀)^{-1} \end{pmatrix}, \]  

where the two block-diagonal matrices are the inverse of the SFIMs for the estimation of the mean vector μ₀ and of the constrained scatter matrix Σ₀ that can be expressed as:

\[ \bar{I}(μ₀|h₀)^{-1} = \frac{N}{E\{Qψ₀(Q)^2\}} \begin{pmatrix} Σ₀ & 0_{N×N} \\ 0_{N×N} & Σ₀^* \end{pmatrix}, \]  

\[ \bar{I}(Σ₀|h₀)^{-1} = U (U^T C₀(s_{\text{vec}}(Σ₀)) U)^{-1} U^T. \]  

Note that, as for the real case, the block-diagonal structure of CCSCRB(φ₀|h₀) implies that not knowing the mean vector μ₀ have no impact on the optimal asymptotic performance in the estimation of the scatter matrix Σ₀.

IV. A NUMERICAL EXAMPLE

In this section, we calculate the CCSCRB for the constrained estimation of the scatter matrix of complex t-distributed random vectors. The pdf related to the complex t-distribution can be obtained from the real t-distribution by applying the equality chain in (3). Specifically, the
relevant density generator $h_0$ can be obtained from the one given in eq. (75) in [14] through a
change of variables $N \to 2N$, $\lambda \to \lambda/2$ as:

$$h_0(t) = (\pi^N \Gamma(\lambda))^{-1} \Gamma(\lambda + N)(\lambda/\eta)\lambda(\lambda/\eta + t)^{-(\lambda+N)}$$

and then $\psi_0(t) = -(\lambda + N)(\lambda/\eta + t)^{-1}$. From (6), we have that:

$$p_\mathcal{Q}(q) = \frac{\Gamma(\lambda + N)}{\Gamma(N)\Gamma(\lambda)} \frac{(\lambda/\eta)^\lambda}{q^{N-1}} \left(\frac{\lambda}{\eta} + q\right)^{-(\lambda+N)}.$$  

Using the integral in [36, pp. 315, n. 3.194 (3)], we get:

$$E\{Q\psi(Q)^2\} = \frac{\eta N(\lambda + N)}{N + \lambda + 1},$$

$$E\{Q^2\psi(Q)^2\} = \frac{N(N + 1)(\lambda + N)}{(N + \lambda + 1)}.$$  

Finally, by replacing the two previously derived expectations in (23) and (24), we obtain closed
form expressions for the matrices $C_0(\hat{s}_\mu_0)$ and $C_0(\hat{s}_{vec(\Sigma_0)})$ and consequently the CCSCRB in
(27).

In Fig. 1, the performance of the constrained Sample Covariance Matrix (CSCM) estimator
and the Constrained Tyler’s (C-Tyler) estimator are compared against the CCSCRB. The explicit
expressions of these two estimators can be obtained form the ones provided in [14] by replacing
the transpose with the Hermitian operator. The simulation parameters are:

- $\Sigma_0$ is a Toeplitz Hermitian matrix whose first row is given by $[1, \rho^*, \ldots, (\rho^*)^{N-1}]$, where
  $\rho = 0.8e^{i2\pi/5}$ and $N = 8$.
- The data power is chosen to be $\sigma_X^2 = E\{Q\}/N = 4$.
- The data mean vector is assumed to be perfectly known, i.e. we will not estimate it, and
  $\mu_0 = 0_N$.
- The number of the available i.i.d. data vectors is $M = 3N = 24$. Since we assume to have
  $M$ i.i.d. data vectors, the CCSCRB in (27) has to be divided by $M$.
- The number of independent Monte Carlo runs is $10^6$.

As MSE indices and bound, in Fig. 1 we use:

$$\varepsilon_\alpha \triangleq \|E\{(\text{vec}(\hat{\Sigma}_\alpha) - \text{vec}(\Sigma_0))(\text{vec}(\hat{\Sigma}_\alpha) - \text{vec}(\Sigma_0))^H\}\|_F,$$

where $\alpha = \{CSCM, C - Tyler\}$ and

$$\varepsilon_{CCSCRB, \Sigma_0} \triangleq \||\text{CCSCRB}(\phi_0, h_0)]\Sigma_0\|_F.$$(35)
Fig. 1: MSE indices for the C SCM and C-Tyler’s estimators and the related CCSCRB as functions of the shape parameter $\lambda$ for complex $t$-distributed data ($M = 3N$).

Fig. 1 shows the MSE indices of the C SCM and of C-Tyler’s estimator compared with the CCSCRB as function of the shape parameter $\lambda$. When $\lambda \to \infty$, i.e. when the data tends to be Gaussian distributed, the C SCM reaches the C SCRB. On the other hand, Fig. 1 shows that C-Tyler’s estimator is not an efficient estimator w. r. t. the CCSCRB.

V. Conclusion

In this letter, we derived the CCSCRB for the constrained estimation of the complex mean vector and scatter matrix of CES-distributed random vectors. This result generalizes the one derived in [14] for RES distributions. The proposed CCSCRB is a lower bound on the estimation accuracy of any robust $M$-estimator of $\mu$ and $\Sigma$. Current research efforts are devoted to investigate the trade-off between estimator efficiency (w.r.t. the CCSCRB) and robustness.

Acknowledgment

The work of Stefano Fortunati has been partially supported by the Air Force Office of Scientific Research under award number FA9550-17-1-0065.

References

[1] D. H. Brandwood, “A complex gradient operator and its application in adaptive array theory,” Communications, Radar and Signal Processing, IEE Proceedings F, vol. 130, no. 1, pp. 11–16, February 1983.

[2] B. Picinbono, “On circularity,” IEEE Transactions on Signal Processing, vol. 42, no. 12, pp. 3473–3482, 1994.

[3] ——, “Second-order complex random vectors and normal distributions,” IEEE Transactions on Signal Processing, vol. 44, no. 10, pp. 2637–2640, Oct 1996.
11

[4] S. C. Olhede, “On probability density functions for complex variables,” *IEEE Transactions on Information Theory*, vol. 52, no. 3, pp. 1212–1217, March 2006.

[5] P. J. Schreier and L. L. Scharf, *Statistical Signal Processing of Complex-Valued Data: the Theory of Improper and Noncircular Signals*. Cambridge UK: Cambridge Univ. Press, 2010.

[6] J. Eriksson, E. Ollila, and V. Koivunen, “Essential statistics and tools for complex random variables,” *IEEE Transactions on Signal Processing*, vol. 58, no. 10, pp. 5400–5408, Oct 2010.

[7] T. Adali, P. J. Schreier, and L. L. Scharf, “Complex-valued signal processing: The proper way to deal with impropriety,” *IEEE Transactions on Signal Processing*, vol. 59, no. 11, pp. 5101–5125, Nov 2011.

[8] P. Krishnaiah and J. Lin, “Complex elliptically symmetric distributions,” *Communications in Statistics - Theory and Methods*, vol. 15, no. 12, pp. 3693–3718, 1986.

[9] C. D. Richmond, “Adaptive array signal processing and performance analysis in non-Gaussian environments,” Ph.D. dissertation, Massachusetts Institute of Technology, 1996. [Online]. Available: https://dspace.mit.edu/handle/1721.1/11005

[10] E. Ollila, D. E. Tyler, V. Koivunen, and H. V. Poor, “Complex elliptically symmetric distributions: Survey, new results and applications,” *IEEE Transactions on Signal Processing*, vol. 60, no. 11, pp. 5597–5625, 2012.

[11] A. M. Zoubir, V. Koivunen, E. Ollila, and M. Muma, *Robust Statistics for Signal Processing*. Cambridge University Press, 2018.

[12] S. Cambanis, S. Huang, and G. Simons, “On the theory of elliptically contoured distributions,” *Journal of Multivariate Analysis*, vol. 11, no. 3, pp. 368 – 385, 1981.

[13] K.-T. Fang, S. Kotz, and K. W. Ng, *Symmetric Multivariate and Related Distributions*. Monographs on Statistics and Applied Probability, Springer US, 1990.

[14] S. Fortunati, F. Gini, M. S. Greco, and A. M. Zoubir, “Semiparametric inference and lower bounds for real elliptically symmetric distributions,” *Submitted IEEE Transactions on Signal Processing*, 2018. [Online]. Available: http://arxiv.org/abs/1807.07811

[15] P. J. Bickel, “On adaptive estimation,” *The Annals of Statistics*, vol. 10, no. 3, pp. 647–671, 1982.

[16] P. Bickel, C. Klaassen, Y. Ritov, and J. Wellner, *Efficient and Adaptive Estimation for Semiparametric Models*. Johns Hopkins University Press, 1993.

[17] M. Hallin and D. Paindaveine, “Semiparametrically efficient rank-based inference for shape I. optimal rank-based tests for sphericity,” *The Annals of Statistics*, vol. 34, no. 6, pp. 2707–2756, 2006.

[18] M. Hallin, H. Oja, and D. Paindaveine, “Semiparametrically efficient rank-based inference for shape II. optimal r-estimation of shape,” *The Annals of Statistics*, vol. 34, no. 6, pp. 2757–2789, 2006.

[19] M. Hallin and D. Paindaveine, “Parametric and semiparametric inference for shape: the role of the scale functional,” *Statistics & Decisions*, vol. 24, no. 3, pp. 327–350, 2009.

[20] D. Paindaveine, “A canonical definition of shape,” *Statistics & Probability Letters*, vol. 78, no. 14, pp. 2240 – 2247, 2008.

[21] L. LeCam and G. L. Yang, *Asymptotics in Statistics: Some Basic Concepts (second edition)*. Springer series in statistics, 2000.

[22] A. van den Bos, “Complex gradient and hessian,” *IEE Proceedings - Vision, Image and Signal Processing*, vol. 141, no. 6, pp. 380–383, Dec 1994.

[23] R. Remmert, *Theory of Complex Functions*. New York: Springer, 1991.

[24] H. Li and T. Adali, “Complex-valued adaptive signal processing using nonlinear functions,” *EURASIP Journal on Advances in Signal Processing*, vol. 2008, no. 1, p. 765615, Feb 2008.

[25] K. Kreutz-Delgado, “The complex gradient operator and the CR-calculus,” in *ISI World Statistics Congress 2017 (ISI2017)*, 2017. [Online]. Available: https://arxiv.org/abs/0906.4835
[26] A. van den Bos, “A cramer-rao lower bound for complex parameters,” *IEEE Transactions on Signal Processing*, vol. 42, no. 10, p. 2859, Oct 1994.

[27] A. K. Jagannatham and B. D. Rao, “Cramer-rao lower bound for constrained complex parameters,” *IEEE Signal Processing Letters*, vol. 11, no. 11, pp. 875–878, Nov 2004.

[28] E. Ollila, V. Koivunen, and J. Eriksson, “On the Cramér-Rao bound for the constrained and unconstrained complex parameters,” in *2008 5th IEEE Sensor Array and Multichannel Signal Processing Workshop*, July 2008, pp. 414–418.

[29] T. Menni, E. Chaumette, P. Larzabal, and J. P. Barbot, “New results on deterministic Cramér-Rao Bounds for real and complex parameters,” *IEEE Transactions on Signal Processing*, vol. 60, no. 3, pp. 1032–1049, March 2012.

[30] C. D. Richmond and L. L. Horowitz, “Parameter bounds on estimation accuracy under model misspecification,” *IEEE Transactions on Signal Processing*, vol. 63, no. 9, pp. 2263–2278, 2015.

[31] S. Fortunati, “Misspecified Cramér-Rao bounds for complex unconstrained and constrained parameters,” in *2017 25th European Signal Processing Conference (EUSIPCO)*, 2017.

[32] S. Fortunati, F. Gini, M. S. Greco, A. M. Zoubir, and M. Rangaswamy, “A fresh look at the semiparametric Cramér-Rao bound,” in *26th European Signal Processing Conference (EUSIPCO)*, 2018. [Online]. Available: http://arxiv.org/abs/1803.00267

[33] A. Hjørungnes, *Complex-Valued Matrix Derivatives With Applications in Signal Processing and Communications*. Cambridge University Press, 2011.

[34] M. Greco and F. Gini, “Cramér-Rao lower bounds on covariance matrix estimation for complex elliptically symmetric distributions,” *IEEE Transactions on Signal Processing*, vol. 61, no. 24, pp. 6401–6409, 2013.

[35] M. Greco, S. Fortunati, and F. Gini, “Maximum likelihood covariance matrix estimation for complex elliptically symmetric distributions under mismatched conditions,” *Signal Processing*, vol. 104, pp. 381–386, 2014.

[36] I. S. Gradshteyn and M. Ryzhik, *Tables of Integrals, Series, and Products* (7th edition). Academic Press, Orlando, Florida, 2007.