MULTI-INTERVAL DISSIPATIVE STURM–LIOUVILLE BOUNDARY-VALUE PROBLEMS WITH DISTRIBUTIONAL COEFFICIENTS

ANDRII GORIUNOV

Abstract. The paper investigates spectral properties of multi-interval Sturm-Liouville operators with distributional coefficients. Constructive descriptions of all self-adjoint and maximal dissipative/accumulative extensions in terms of boundary conditions are given. Sufficient conditions for the resolvents of these operators to be operators of the trace class and for the systems of root functions to be complete are found. Results of paper are new for one-interval boundary value problems as well.

1. Introduction.
Differential operators, generated by the Sturm-Liouville expression

$$l(y) = -(py')' + qy,$$

arise in numerous problems of analysis and its applications. The classic assumptions on its coefficients are the following:

$$q \in C([a, b]; \mathbb{R}), \quad 0 < p \in C^1([a, b]; \mathbb{R}).$$

Principal statements of theory of such operators remain true under more general assumptions

$$q, 1/p \in L_1([a, b], \mathbb{C}).$$

However, many problems of mathematical physics require study of differential operators with complex coefficients which are Radon measures or even more singular distributions. In papers [1–4] a new approach to investigation of such operators was proposed based on definition of these operators as quasi-differential, which allows also to consider differential operators of higher order [3, 5].

The purpose of this paper is to develop a spectral theory of non-self-adjoint Sturm-Liouville operators, given on the finite system of bounded intervals under minimal conditions for the regularity of the coefficients.

Multi-interval differential and quasi-differential operators were investigated, particularly, in the papers [6–9].

2. Preliminary results.
Let $[a, b] \subset \mathbb{C}$ be a compact interval, $m \in \mathbb{N}$, and let $a = a_0 < a_1 < \cdots < a_m = b$ be a partition of interval $[a, b]$ into $m$ parts. Let us consider the space $L_2([a, b], \mathbb{C})$ as a direct sum $\oplus_{k=1}^m L_2([a_{k-1}, a_k], \mathbb{C})$ which consists of vector functions $f = \oplus_{k=1}^m f_k$ such that $f_k \in L_2([a_{k-1}, a_k], \mathbb{C})$.

Let on every interval $(a_{k-1}, a_k), k \in \{1, \ldots, m\}$ the formal Sturm-Liouville differential expression

$$l_k(y) = -(p_k(t)y')' + q_k(t)y + i((r_k(t)y')' + r_k(t)y'),$$

2010 Mathematics Subject Classification. 34L40, 34B45.

Key words and phrases. Sturm-Liouville operator; multi-interval boundary value problems; distributional coefficients; maximal dissipative extension; completeness of root functions.
be given with coefficients \( p_k, q_k \) and \( r_k \) which satisfy the conditions:

\[
q_k = Q_k', \quad \frac{1}{\sqrt{|p_k|}}, \quad \frac{Q_k}{\sqrt{|p_k|}}, \quad \frac{r_k}{\sqrt{|p_k|}} \in L_2([a_{k-1}, a_k]; \mathbb{C}),
\]

where the derivatives \( Q_k' \) are understood in the sense of distributions.

Similarly to [3] (see also [11]) we introduce by the coefficients of the expression (1) on every interval \([a_{k-1}, a_k]\) the quasi-derivatives in the following way:

\[
D_k^0 y := y; \quad D_k^1 y := p_k y' - (Q_k + ir_k) y; \quad D_k^2 y := (D_k^1 y)' + \frac{Q_k - ir_k}{p_k} D_k^1 y + \frac{Q_k^2 + r_k^2}{p_k} y.
\]

Also denote for all \( t \in [a_{k-1}, a_k] \)
\[
\hat{y}_k(t) = (D_k^0 y(t), D_k^1 y(t)) \in \mathbb{C}^2.
\]

Under assumptions (2) these expressions are Shin-Zettl quasi-derivatives (see [10][11]). One can easily verify that for the smooth coefficients \( p_k, q_k \) and \( r_k \) the equality \( l_k(y) = -D_k^2 y \) holds.

Therefore one may correctly define expressions (1) under assumptions (2) as Shin-Zettl quasi-differential expressions:

\[
l_k[y] := -D_k^2 y.
\]

The corresponding Shin-Zettl matrices (see [10][11]) have the form

\[
A_k = \begin{pmatrix}
\frac{Q + ir_k}{p} & \frac{1}{p} \\
-\frac{Q^2 + r_k^2}{p} & -\frac{Q - ir_k}{p}
\end{pmatrix} \in L_1([a, b]; \mathbb{C}^{2 \times 2}).
\]

Then on the Hilbert spaces \( L_2([a_{k-1}, a_k], \mathbb{C}) \) minimal and maximal differential operators are defined, which are generated by the quasi-differential expressions \( l_k[y] \) (see [10][11]):

\[
L_{k,1} : y \rightarrow l_k[y], \quad \text{Dom}(L_{k,1}) := \left\{ y \in L_2 \left| y, D_k^1 y \in AC([a_{k-1}, a_k], \mathbb{C}), D_k^2 y \in L_2 \right. \right\},
\]

\[
L_{k,0} : y \rightarrow l_k[y], \quad \text{Dom}(L_{k,0}) := \{ y \in \text{Dom}(L_{k,1}) \mid \hat{y}_k(a_{k-1}) = \hat{y}_k(a_k) = 0. \}.
\]

Results of [10][11] for general Shin-Zettl quasi-differential operators together with formula (3) imply that operators \( L_{k,1}, L_{k,0} \) are closed and densely defined on the space \( L_2([a_{k-1}, a_k], \mathbb{C}) \).

In the case where \( p_k, q_k \) and \( r_k \) are real-valued, the operator \( L_{k,0} \) is symmetric with the deficiency index (2, 2) and

\[
L_{k,0}^* = L_{k,1}, \quad L_{k,1}^* = L_{k,0}.
\]

3. Dissipative boundary-value problems.

We consider the space \( L_2([a, b], \mathbb{C}) \) as a direct sum \( \oplus_{k=1}^m L_2([a_{k-1}, a_k], \mathbb{C}) \) which consists of vector functions \( f = \oplus_{i=1}^m f_i \) such that \( f_i \in L_2([a_{i-1}, a_i], \mathbb{C}) \). In this space \( L_2([a, b], \mathbb{C}) \) we consider maximal and minimal operators \( L_{\max} = \oplus_{i=1}^m L_{i,1} \) and \( L_{\min} = \oplus_{i=1}^m L_{i,0} \).

It is easy to see that operators \( L_{\max}, L_{\min} \) are closed and densely defined on the space \( L_2([a, b], \mathbb{C}) \).

Throughout the rest of the paper we assume functions \( p_k, q_k \) and \( r_k \) to be \textit{real-valued} for all \( k \) and therefore operators \( L_{k,0} \) to be symmetric with the deficiency indices (2, 2). Then the operator \( L_{\min} \) is symmetric with the deficiency index \( (2m, 2m) \) and

\[
L_{\min}^* = L_{\max}, \quad L_{\max}^* = L_{\min}.
\]
Then naturally arises the problem of describing all its self-adjoint, maximal dissipative and maximal accumulative extensions in terms of homogeneous boundary conditions of the canonical form. For this purpose it is convenient to apply the approach based on the concept of boundary triplets. It was developed in the papers by Kochubei \[12\], see also the monograph \[13\] and the references therein.

Note that the minimal operator \(L_{\text{min}}\) may be not semi-bounded even in the case of a single-interval boundary-value problem since the function \(p\) may reverse sign.

Recall that a boundary triplet of a closed densely defined symmetric operator \(T\) with equal (finite or infinite) deficiency indices is called a triplet \((H, \Gamma_1, \Gamma_2)\) where \(H\) is an auxiliary Hilbert space and \(\Gamma_1, \Gamma_2\) are the linear maps from \(\text{Dom}(T^*)\) into \(H\) such that:

1. for any \(f, g \in \text{Dom}(T^*)\) there holds \((T^* f, g)_H - (f, T^* g)_H = (\Gamma_1 f, \Gamma_2 g)_H - (\Gamma_2 f, \Gamma_1 g)_H\),

2. for any \(g_1, g_2 \in H\) there is a vector \(f \in \text{Dom}(T^*)\) such that \(\Gamma_1 f = g_1\) and \(\Gamma_2 f = g_2\).

The definition of the boundary triplet implies that \(f \in \text{Dom}(T)\) if and only if \(\Gamma_1 f = \Gamma_2 f = 0\). A boundary triplet \((H, \Gamma_1, \Gamma_2)\) with \(\dim H = n\) exists for any symmetric operator \(T\) with equal non-zero deficiency indices \((n, n)\) \((n \leq \infty)\), but it is not unique.

For the minimal quasi-differential operators \(L_{k,0}\) the boundary triplet is explicitly given by the following theorem which follows from the results of \[2\].

**Theorem 1.** For every \(k = 1, \ldots, m\) the triplet \((C^2, \Gamma_{1,k}, \Gamma_{2,k})\), where \(\Gamma_{1,k}, \Gamma_{2,k}\) are linear maps
\[
\Gamma_{1,k} := \left(D_k^{[1]} y(a_{k-1}^+), -D_k^{[1]} y(a_{k-1}^-)\right), \quad \Gamma_{2,k} y := (y(a_{k-1}^+), y(a_{k-1}^-)),
\]
from \(\text{Dom}(L_{k,1})\) onto \(C^2\) is a boundary triplet for the operator \(L_{k,0}\).

For the minimal operator \(L_{\text{min}}\) in the space \(L_2([a, b], C)\) the boundary triplet is explicitly given by the following theorem.

**Theorem 2.** The triplet \((C^{2m}, \Gamma_1, \Gamma_2)\), where \(\Gamma_1, \Gamma_2\) are linear maps
\[
(4) \quad \Gamma_1 y := (\Gamma_{1,1} y, \Gamma_{1,2} y, \ldots, \Gamma_{1,m} y), \quad \Gamma_2 y := (\Gamma_{2,1} y, \Gamma_{2,2} y, \ldots, \Gamma_{2,m} y),
\]
from \(\text{Dom}(L_{\text{max}})\) onto \(C^{2m}\) is a boundary triplet for the operator \(L_{\text{min}}\).

Denote by \(L_K\) the restriction of operator \(L_{\text{max}}\) onto the set of functions \(y \in \text{Dom}(L_{\text{max}})\) satisfying the homogeneous boundary condition
\[
(5) \quad (K - I) \Gamma_1 y + i (K + I) \Gamma_2 y = 0,
\]
where \(K\) is an arbitrary bounded operator on the space \(C^{2m}\).

Similarly, denote by \(L^K\) the restriction of \(L_{\text{max}}\) onto the set of functions \(y \in \text{Dom}(L_{\text{max}})\) satisfying the homogeneous boundary condition
\[
(6) \quad (K - I) \Gamma_1 y - i (K + I) \Gamma_2 y = 0,
\]
where \(K\) is an arbitrary bounded operator on the space \(C^{2m}\).

Theorem \[1\] together with \[13\], Th. 1.6 lead to the following description of all self-adjoint, maximal dissipative and maximal accumulative extensions of operator \(L_{\text{max}}\).
Theorem 3. Every $L_K$ with $K$ being a contracting operator in the space $\mathbb{C}^{2m}$, is a maximal dissipative extension of operator $L_{\text{min}}$. Similarly every $L^K$ with $K$ being a contracting operator in $\mathbb{C}^{2m}$, is a maximal accumulative extension of the operator $L_{\text{min}}$.

Conversely, for any maximal dissipative (respectively, maximal accumulative) extension $\tilde{L}$ of the operator $L_{\text{min}}$ there exists the unique contracting operator $K$ such that $\tilde{L} = L_K$ (respectively, $\tilde{L} = L^K$).

The extensions $L_K$ and $L^K$ are self-adjoint if and only if $K$ is a unitary operator on $\mathbb{C}^{2m}$.

The mappings $K \rightarrow L_K$ and $K \rightarrow L^K$ are injective.

All functions from $\text{Dom}(L_{\text{max}})$ together with their first quasi-derivatives belong to $\oplus_{k=1}^m AC([a_{k-1}, a_k], \mathbb{C})$. This implies that following definition is correct.

Denote by $f(t-)$ the left germ and by $f(t+)$ the right germ of the continuous function $f$ at point $t$. Similarly to the paper [2] we say that boundary conditions which define the operator $L \subset L_{\text{max}}$ are called local, if for any functions $y \in \text{Dom}(L)$ and for any $y_1, \ldots, y_m \in \text{Dom}(L_{\text{max}})$ equalities $y_j(a_j-) = y(a_j-)$, $y_j(a_j+) = y(a_j+)$ and $y_j(a_k-) = y_j(a_k+) = 0$, $k \neq j$ imply that $y_j \in \text{Dom}(L)$. For $j = 0$ and $j = m$ we consider only the right and left germs respectively.

The following statement gives a description of extensions $L_K$ and $L^K$ which are given by local boundary conditions.

Theorem 4. The boundary conditions [5] and [6] defining extensions $L_K$ and $L^K$ respectively are local if and only if the matrix $K$ has the block form

$$
(7) \quad K = \begin{pmatrix}
K_{a_0} & 0 & \ldots & 0 \\
0 & K_{a_1} & \ldots & 0 \\
0 & 0 & \ldots & K_{a_n}
\end{pmatrix},
$$

where $K_{a_0}$ and $K_{a_n} \in \mathbb{C}$ and other $K_{a_j} \in \mathbb{C}^{2 \times 2}$.

4. Generalized resolvents.

Let us recall that a generalized resolvent of a closed symmetric operator $L$ in a Hilbert space $\mathcal{H}$ is an operator-valued function $\lambda \mapsto R_\lambda$, defined on $\mathbb{C} \setminus \mathbb{R}$ which can be represented as

$$
R_\lambda f = P^+ (L^+ - \lambda I^+)^{-1} f, \quad f \in \mathcal{H},
$$

where $L^+$ is a self-adjoint extension of operator $L$ which acts in a certain Hilbert space $\mathcal{H}^+ \supset \mathcal{H}$, $I^+$ is the identity operator on $\mathcal{H}^+$, and $P^+$ is the orthogonal projection operator from $\mathcal{H}^+$ onto $\mathcal{H}$. It is known that an operator-valued function $R_\lambda$ is a generalized resolvent of a symmetric operator $L$ if and only if it can be represented as

$$
(R_\lambda f, g)_\mathcal{H} = \int_{-\infty}^{+\infty} \frac{d(F_\mu f, g)}{\mu - \lambda}, \quad f, g \in \mathcal{H},
$$

where $F_\mu$ is a generalized spectral function of the operator $L$. This implies that the operator-valued function $F_\mu$ has the following properties:

1. For $\mu_2 > \mu_1$ the difference $F_{\mu_2} - F_{\mu_1}$ is a bounded non-negative operator.
2. $F_{\mu_1} = F_{\mu}$ for any real $\mu$.
3. For any $x \in \mathcal{H}$ following equalities hold:

$$
\lim_{\mu \to -\infty} ||F_\mu x||_\mathcal{H} = 0, \quad \lim_{\mu \to +\infty} ||F_\mu x - x||_\mathcal{H} = 0.
$$

The following theorem provides a parametric description of all generalized resolvents of symmetric operator $L_{\text{min}}$ (see also [14]).
Theorem 5. 1) Every generalized resolvent $R_\lambda$ of the operator $L_{\min}$ in the half-plane $\text{Im} \lambda < 0$ acts by the rule $R_\lambda h = y$, where $y$ is a solution of the boundary-value problem

$$l(y) = \lambda y + h,$$

$$(K(\lambda) - I) \Gamma_1 f + i (K(\lambda) + I) \Gamma_2 f = 0.$$

Here $h(x) \in L_2([a, b], \mathbb{C})$ and $K(\lambda)$ is a $2m \times 2m$ matrix-valued function which is holomorphic in the lower half-plane and such that $|K(\lambda)| \leq 1$.

2) In the half-plane $\text{Im} \lambda > 0$ every generalized resolvent of operator $L_{\min}$ acts by $R_\lambda h = y$, where $y$ is a solution of the boundary-value problem

$$l(y) = \lambda y + h,$$

$$(K(\lambda) - I) \Gamma_1 f - i (K(\lambda) + I) \Gamma_2 f = 0.$$

Here $h(x) \in L_2([a, b], \mathbb{C})$ and $K(\lambda)$ is a $2m \times 2m$ matrix-valued function which is holomorphic in the upper half-plane and satisfies $|K(\lambda)| \leq 1$.

This parametrization of the generalized resolvents by the matrix-valued functions $K(\lambda)$ is bijective.

5. Completeness of system of root vectors.

Results of the paper [15] imply that in the single-interval case under the assumptions made and additionally for $r_k = r \equiv 0$ the resolvents of the operators $L_K$ and $L^K$ are Hilbert-Schmidt operators. This result is amplified and refined by the following theorem.

Theorem 6. 1) The resolvents of the maximal dissipative (maximal accumulative) operators $L_K$ and $L^K$ are Hilbert-Schmidt operators.

2) Let $\delta > 0$ exist such that for any $k \in \{1, 2, \ldots, m\}$

$$\left\{ \frac{1}{p_k}, \frac{Q_k + ir_k}{p_k} \right\} \subset W^\delta_2([a_{k-1}, a_k], \mathbb{C}).$$

Then the resolvent of the maximal dissipative (maximal accumulative) operator $L_K$ ($L^K$) is an operator from the trace class, and its system of root functions is complete in the Hilbert space $L_2([a, b], \mathbb{C})$.

References

[1] Goriunov A. S., Mikhailets V. A. Regularization of singular Sturm-Liouville equations // Meth. Funct. Anal. Topol. — 2010. — 16, № 2. — P. 120–130.

[2] Goriunov A. S., Mikhailets V. A., Pankrashkin K. Formally self-adjoint quasi-differential operators and boundary-value problems // Electron. J. Diff. Equ. — 2013. — №101. — P. 1–16.

[3] Mirzoev K. A., Shkalikov A. A. Differential operators of even order with distribution coefficients // Math. Notes. — 2016. — 99, № 5. — P. 779–784.

[4] Eckhardt J., Gesztesy F., Nichols R., Teschl G. Weyl-Titchmarsh Theory for Sturm-Liouville Operators with Distributional Coefficients // Opuscula Mathematica. — 2013. — 33, № 3. — P. 467–563.

[5] Goriunov A. S., Mikhailets V. A. Regularization of two-term differential equations with singular coefficients by quasiderivatives // Ukrainian Math. J. — 2012. — 63, № 9. — P. 1361–1378.

[6] Everitt W. N., Zettl A. Sturm–Liouville differential operators in direct sum spaces // Rocky Mountain J. Math. — 1986. — 16, № 3. — P. 497–516.

[7] Everitt W. N., Zettl A. Quasi-differential operators generated by a countable number of expressions on the real line // Proc. London Math. Soc. — 1992. — 64, № 3. — P. 524–544.
[8] Sokolov M. S. Representation results for operators generated by a quasi-differential multi-interval system in a Hilbert direct sum space // Rocky Mountain J. Math. — 2006. — № 2. — P. 721–739.

[9] Goriunov A. S. Multi-interval Sturm–Liouville boundary-value problems with distributional potentials // Dopov. Nats. Acad. Nauk. Ukr. — 2014. — № 7. — P. 43–47.

[10] Zettl A. Formally self-adjoint quasi-differential operators // Rocky Mountain J. Math. — 1975. — 5, № 3. — P. 453–474.

[11] Everitt W. N., Markus L. Boundary Value Problems and Symplectic Algebra for Ordinary Differential and Quasi–differential Operators. — Providence, American Mathematical Society, 1999. — xii+187 p.

[12] A. N. Kochubei. Extensions of symmetric operators and of symmetric binary relations. // Math. Notes. — 1975. — 17, no. 1. — P. 25–28.

[13] V. I. Gorbachuk, M. L. Gorbachuk. Boundary value problems for operator differential equations. Kluwer Academic Publishers Group, Dordrecht, 1991.

[14] Bruk V. M. A certain class of boundary value problems with a spectral parameter in the boundary condition // Mat. Sb. (N.S.). — 1976. — 100 (142), № 2(6). — P. 210–216 (Russian).

[15] Goriunov A. S. Convergence and approximation of the Sturm–Liouville operators with potentials-distributions // Ukrainian Math. J. — 2015. — 67, № 5. — P. 680–689.

Institute of Mathematics of National Academy of Sciences of Ukraine, Kyiv, Ukraine
E-mail address: goriunov@imath.kiev.ua