Formation of Topological Defects

Tanmay Vachaspati

Physics Department, Case Western Reserve University,
Cleveland OH 44106-7079, USA.

In these lectures, I review cosmological phase transitions and the topological aspects of spontaneous symmetry breaking. I then discuss the formation of walls, strings and monopoles during phase transitions including lattice based studies of defect formation and recent attempts to go beyond the lattice. The close connection of defect formation with percolation is described. Open problems and possible future directions are mentioned.

1 Motivation and Introduction

An exciting development in cosmology has been the realization that the universe may behave very much like a condensed matter system. After all, the cosmos is the arena where very high energy particle physics is relevant and this is described by quantum field theory which is also the very tool used in condensed matter physics.

In condensed matter systems we have seen a rich array of phenomenon and we expect that the cosmos has seen an equally rich past. A routine observation in condensed matter (and daily life!) is that of symmetry breaking which is the basis of all schemes in particle physics to achieve unification of forces. Symmetry breaking would lead to phase transitions in the early universe, making their study crucial to our understanding of the cosmos. There are many cosmic theories that hinge on processes happening during phase transitions. These include a large number of inflationary models, baryogenesis and structure formation.

Many of the concepts that have gone into the explosion of cosmology in the last two decades are linked to each other. As an example, consider the birth of the inflationary idea. The success of the electroweak model led particle physicists to attempt to unify the strong force with the electroweak forces by postulating Grand Unified Theories (GUTs). In such theories, a number of phase transitions occur and, based on mathematical results on the topology of group manifolds, it is known that GUTs always contain topological defects known as magnetic monopoles. The occurrence of phase transitions in cosmology then tells us that monopoles must have formed in the early universe. In fact, standard cosmology then predicts an over-abundance of magnetic monopoles in our present universe that is clearly in conflict with observation. This head-on confrontation of particle physics and cosmology led Guth to come
up with the idea of an inflationary universe - an idea that is now deeply embedded in cosmology as it not only solves the monopole problem but also several other unrelated problems that standard (pre-inflationary) cosmology did not address.

Magnetic monopoles are one example of topological defects that can arise during phase transitions. But topological defects can occur in other varieties as well. They can be one dimensional, in which case they are called “strings”. If they are two dimensional, they are called “domain walls”. Magnetic monopoles and domain walls are two kinds of topological defects that have unpleasant cosmological consequences unless they are kept very light or are somehow eliminated at some early epoch (for example, by inflation). Cosmic strings are believed to be more benevolent and may even have been responsible for structure formation in the universe if they were formed at the GUT phase transition.

The study of topological defects is fascinating for several reasons. As in the example of magnetic monopoles and inflation above, they can provide important constraints on particle physics models and cosmology. On the other hand, if a GUT topological defect is found, it would provide a direct window on the universe at about $10^{-35}$ secs. In a manner of speaking, a part of the early universe is trapped in the interior of the defect - much like dinosaur fossils trapped in amber. GUT topological defects would also shed light on the problem of how galaxies were formed. Their discovery would give us important constraints on the symmetry structure of very high energy particle physics - this is non-perturbative information. In addition, it would give us information about phase transitions in the early universe, giving confidence in our understanding of the thermal history of the universe. Also, the topological defects would most likely have tremendous astrophysical impact since they are generally very massive and have unusual gravitational and electromagnetic properties.

In the early 80’s, people would talk of the marriage of particle physics and cosmology. I actually think there is something wrong with this picture since a third party is also involved. This is condensed matter physics. When it comes to understanding cosmological phase transitions, we are forced to consider the corresponding advances in condensed matter physics since many of the ideas are the same. Topological defects have been studied by condensed matter physicists for many, many years. If we want experimental input in our theories of the very early universe, we must go to the condensed matter laboratory. This is a growing area of research - condensed matter experiments are being inspired by cosmological questions, and, cosmologists are revising their theories based on experimental input.
In these lectures, my aim is to discuss the formation of topological defects and to bring the reader to a point where the most relevant questions are apparent. The existing work on topological defects paints a certain picture of their formation but there are some limitations. I will describe both the picture and the limitations. I will start out by describing phase transitions and the “effective potential” way of treating them in field theory. Then I will describe why topological defects come about and finally get to the work on their formation in a phase transition. It is impossible to describe every aspect of this subject within these lectures especially since the subject branches out into a large variety of different areas of research. The references provided in the bibliography should help the reader dive deeper into whichever area he/she chooses to pursue.

2 Phase Transitions

2.1 Effective Potential: Formalism

In statistical mechanics, the basic quantity one tries to find is the partition function

\[ Z = \sum e^{-\beta H} \]

where \( \beta = \frac{1}{T} \) and the sum is over all states with energy \( H \). From the partition function one can derive the Helmholtz free energy \( A \) by

\[ Z = e^{-A} \]

and the derivatives of \( A \) then lead to the thermodynamic functions.

To discuss phase transitions, one performs a Legendre transform on the Helmholtz free energy to get the Gibbs free energy \( G \). For a gas, we have

\[ G(P, T) = A + PV \]

while for an ensemble of spins,

\[ G(M, T) = A + HM \]

where \( H \) is the external magnetic field and \( M \) is the magnetization of the system.

In the magnetic system, we have

\[ \frac{\partial G}{\partial M} = H \]
and so, in the absence of an external magnetic field, the minima of $G$ describe the various phases of the system.

In field theory, these concepts carry over almost word for word. For the time being we restrict ourselves to the zero temperature case. Then, in analogy with the partition function, one defines the generating functional in the presence of an external current $J(x)$ (analogous to the external magnetic field)

$$Z[J] = \int D\phi \exp[i \int_0^T dt \int d^3x (\mathcal{L}[\phi] + J(x)\phi(x))]$$  \hspace{1cm} (1)

where, the time integration is over a large but finite interval and $\mathcal{L}$ is a suitable Lagrangian density for the system. In other words,

$$Z[J] = <\Omega| e^{-iHT} |\Omega> \equiv \exp[-E[J]]$$  \hspace{1cm} (2)

where, $|\Omega>$ is the vacuum state and $H$ is the Hamiltonian. The energy functional $E[J]$ is the analog of the Helmholtz free energy. (Note however that $E$ is not really an energy - for example, it does not have the dimensions of energy.) Now, we find,

$$\frac{\delta E[J]}{\delta J(x)} = -\int D\phi e^{i \int (\mathcal{L} + J\phi)\phi(x)}$$  \hspace{1cm}

$$= <\Omega|\phi(x)|\Omega>_{J}$$

$$= -\phi_{cl}(x)$$

which is (minus) the order parameter. Next we get the effective action, which is the analog of the Gibbs free energy, by Legendre transforming the energy functional:

$$\Gamma[\phi_{cl}] = -E[J] - \int d^4y J(y)\phi_{cl}(y) .$$

To obtain the value of the order parameter for any given external current, we need to solve:

$$\frac{\delta \Gamma[\phi_{cl}]}{\delta \phi_{cl}(x)} = -J(x)$$

and, in particular, when the external current vanishes, the phases are described by the extrema of the effective action.

So far we have taken the external current to be a function of space but, in practice, the current is taken to be a constant and $\phi_{cl}$ is also constant. Then, in this restricted case, only non-derivative terms can be present in the effective...
action and the effective action leads to an effective potential with quantum corrections:

\[ V_{\text{eff}, q}(\phi_d) \equiv -\frac{1}{VT} \Gamma[\phi_d] \]

where, \( VT \) is the spacetime volume.

These issues and, in particular, the correspondence between statistical mechanics and field theory, are described very clearly in the textbooks by Peskin and Schroeder and Rivers.

2.2 Effective Potential: Quantum Corrections

The actual calculation of the quantum effective potential is done by using perturbation theory. The general result is as follows. For the model:

\[ L = \frac{1}{2} D_\mu \Phi D^\mu \Phi - V(\Phi) - \frac{1}{4} F^a_{\mu \nu} F^{a \mu \nu} + L_F \]

where

\[ \partial_\mu \phi = (\partial_\mu - ieA^a_\mu T^a)\phi \]

\[ F^a_{\mu \nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + e f_{abc} A^b_\mu A^c_\nu \]

\[ L_F = i \bar{\psi} \gamma^\mu D_\mu \psi - \bar{\psi} \Gamma_i \psi \phi_i \]

in conventional notation, the one loop correction to the potential is the Coleman-Weinberg correction:

\[ V_{1,q}(\phi) = \frac{1}{64\pi^2} \left[ \text{Tr}(\mu^4 \ln \frac{\mu^2}{\sigma^2}) + 3\text{Tr}(M^4 \ln \frac{M^2}{\sigma^2}) - 4\text{Tr}(m^4 \ln \frac{m^2}{\sigma^2}) \right] . \] (3)

Here \( \mu, M \) and \( m \) are the scalar, vector and spinor masses and \( \sigma \) is a renormalization scale. The factors of 1, 3 and 4 in front of the three terms are due to the spin degrees of freedom of a scalar, massive vector and spinor (fermion-antifermion) field. Also, the - sign in front of the fermionic contribution is due to Fermi statistics. The renormalization conditions used to derive this form of \( V_{1,q}(\phi) \) are the “zero momentum” conditions.

As a specific simple example, consider the Lagrangian

\[ L = \frac{1}{2} (\partial_\mu \Phi)^2 - V(\Phi) \] (4)

where, \( \Phi \) is real and

\[ V(\Phi) = -\frac{\mu^2}{2} \Phi^2 + \frac{\lambda}{4} \Phi^4 . \]
Next write
\[ \Phi(x) = \phi_{cl} + \chi(x) \]
where, \( \phi_{cl}^2 \) is a constant and equal to \( \mu^2 / \lambda \) at tree level. Then,
\[ L = \frac{1}{2} (\partial_{\mu} \chi)^2 - \frac{1}{2} (3\lambda \phi_{cl}^2 - \mu^2) \chi^2 - \lambda \phi_{cl} \chi^3 - \frac{\lambda}{4} \chi^4 + \frac{1}{2} \mu^2 \phi_{cl}^2 - \frac{\lambda}{4} \phi_{cl}^4 \]
The last two terms are the tree level effective action for \( \phi_{cl} \). We now want to find the one loop correction to the effective action. This comes about due to two ingredients: (i) the \( \chi \) fluctuations contribute to the energy of the vacuum, and, (ii) the mass of the \( \chi \) particles depends on the value of \( \phi_{cl} \). Hence the \( \chi \) vacuum fluctuations (loops in Feynman diagram language) contribute to the potential felt by \( \phi_{cl} \).

There is only one Feynman diagram that contributes at one loop - simply a \( \chi \) particle propagating in a loop. Then the contribution to the effective potential from this one loop is:
\[ V_{1,\chi}(\phi_{cl}) = \frac{1}{2(2\pi)^4} \int d^4 k \ln (k^2 + m^2(\phi_{cl})) \]
where,
\[ m^2(\phi_{cl}) = (3\lambda \phi_{cl}^2 - \mu^2) \]
A clear physical meaning can be obtained by performing the integration over \( k_0 \):
\[ V_{1,\chi}(\phi_{cl}) = \frac{1}{(2\pi)^3} \int d^3 k \sqrt{k^2 + m^2(\phi_{cl})}^{1/2} \quad (5) \]
where an infinite constant has been removed by shifting the zero energy level. The integrand now is the energy of a \( \chi \) particle with momentum \( \vec{k} \) and so the one loop correction to the effective potential is just the energy in all the modes of \( \chi \) - corresponding to the sum over \( \hbar \omega / 2 \) in field theory. In elementary applications of quantum field theory, this zero point energy is removed by normal ordering, but here it contains a non-trivial dependence on \( \phi_{cl} \) which cannot be removed and should be included in the effective potential.

The integration in eq. (5) is divergent and needs to be renormalized. This is done by choosing a set of renormalization conditions. Linde’s choice is:
\[ \frac{dV}{d\phi} \bigg|_{\phi = \mu \lambda^{-1/2}} = 0 \]
and
\[ \frac{d^2V}{d\phi^2} \bigg|_{\phi = \mu \lambda^{-1/2}} = 2\mu^2 \]
where we have dropped subscripts on the effective potential and the order parameter to simplify notation. The final result for the effective potential is:

\[ V(\phi) = -\frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 + \frac{1}{64\pi^2}(3\lambda\phi^2 - \mu^2)^2\ln\left(\frac{3\lambda\phi^2 - \mu^2}{2\mu^2}\right) + \frac{21\lambda\mu^2}{64\pi^2}\phi^2 - \frac{27\lambda^2}{128\pi^2}\phi^4. \]

This result differs from the general form in eq. (3) because of a different choice of renormalization conditions. The physical consequences are, however, independent of the renormalization conditions one chooses.

2.3 Effective Potential: Thermal Corrections

Now we look at thermal corrections to the potential. These corrections can be understood as follows: a thermal bath necessarily contains a thermal distribution of particles. The properties of any particle is influenced by its interactions with the particles in the thermal background. This leads to an effective Lagrangian in which the effects of temperature are already included.

The derivation of the temperature dependence of the effective potential is closely analogous to the derivation of the quantum corrections. The calculation is now done in Euclidean space with the Euclidean time coordinate being periodic with period \( \frac{1}{2\pi T} \). This means that the zeroth component, \( k_0 \), of the four momentum of a particle is now discrete. For bosons, periodic boundary conditions are used and so \( k_0 \) is replaced by \( n(2\pi T) \) while for fermions, anti-periodic boundary conditions are necessary and \( k_0 \) is replaced by \((n + 1/2)2\pi T\) where \( n \) is any integer. Then integrals over \( k_0 \) get transformed to a sum over \( n \):

\[
\int dk_0 \rightarrow 2\pi T \sum_{n=-\infty}^{n=+\infty}
\]

For example, in the simple model at the end of the previous section, the temperature dependent correction to the effective potential is:

\[
V_{1,T}(\phi) = \frac{T}{2(2\pi)^3} \sum_{n=-\infty}^{n=+\infty} \int d^3k \ln[(2\pi n T)^2 + \vec{k}^2 + m^2(\phi)]
\]

where, once again,

\[
m^2(\phi) = 3\lambda\phi^2 - \mu^2.
\]

The result of doing the sum and the integration and applying the renormalization conditions is:

\[
V(\phi, T) = -\frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 - \frac{\pi^2}{90}T^4 + \frac{m^2(\phi)}{24}T^2 (6)
\]
in the temperature range $T \gg m$ to first order in $\lambda$.

We now describe a more intuitive (but equivalent) way of deriving the thermal corrections to the potential. The idea is that if we write

$$\Phi = \phi + \chi$$

there is a thermal background of $\chi$ particles that contribute to the effective potential for $\phi$. This contribution can be calculated by taking the ensemble average of the microscopic (bare) potential for $\phi$. So,

$$V_{\text{eff},T}(\phi) = -\frac{\mu^2}{2} \langle (\phi + \chi)^2 \rangle + \frac{\lambda}{4} \langle (\phi + \chi)^4 \rangle$$

where $\langle \rangle$ denotes ensemble averaging. The ensemble average of odd powers of $\chi$ vanish by symmetry while the average of $\mu^2 \chi^2$ and $\chi^4$ simply shift the zero level. The only non-trivial contribution comes from

$$\frac{3\lambda}{2} \phi^2 \langle \chi^2 \rangle.$$

Replacing $\chi$ by an expansion in terms of creation and annihilation operators gives:

$$\langle \chi^2 \rangle = \frac{1}{(2\pi)^3} \int \frac{d^3 k}{2\omega_k} 2a_+^k a_k + 1$$

The operator $a_+^k a_k$ is simply the number operator and, since the number distribution of spin zero particles with momentum $\vec{k}$ in a thermal bath is given by the Bose distribution, we have

$$\langle a_+^k a_k \rangle = n(\omega_k) = (e^{\omega_k/T} - 1)^{-1}$$

where, $\omega_k = (\vec{k}^2 + m^2)^{1/2}$. After subtracting out the constant infinite piece from $\langle \chi^2 \rangle$ we find

$$V_{\text{eff},T}(\phi) = \left[ -\frac{\mu^2}{2} + \frac{3\lambda T^2}{4\pi^2} \int \frac{k^2 dk}{\sqrt{k^2 + \bar{m}^2}} \frac{1}{e^{\sqrt{k^2 + \bar{m}^2}T} - 1} \right] \phi^2 + \frac{\lambda}{4} \phi^4$$

where we have also rescaled the integration variable by $1/T$ to make it dimensionless and $\bar{m} = m/T$. In the limit of large $T$, we have $\bar{m} \to 0$ and the integral can be done in closed form. In this limit, the result reduces to eq. (6) without the $\phi$ independent terms.

A plot of the effective potential is shown in Fig. 1. At low temperatures, there are two minima and, in a rapid quench, the system would have to transit
from the minimum at $\phi = 0$ to the minimum at $\phi \neq 0$. Here the transition can take place continuously and is a second-order phase transition.

A number of different behaviours for the effective potential have been found. In certain systems, there are two phases with an energy barrier separating them. This leads to first order phase transitions. If the barrier separating the two phases is very large, the transition from one phase to another would have to be somehow activated over the barrier or quantum mechanical tunneling would eventually complete the transition. However, both processes can be very slow to occur and so the system can be trapped in the higher energy phase (false vacuum) for a long time. In other words, the system can supercool.

Note that we have been working with examples where cooling leads to spontaneous reduction in symmetry. This is indeed generic. However, many
instances are also known where cooling leads to symmetry restoration. This has been observed in Rochelle salts.

2.4 Order of the Transition: Ehrenfest Classification

The role of the thermodynamic potentials is played by the effective potential. So one should take derivatives of \( V_{\text{eff}} \) with respect to the temperature and find the order of the lowest derivative which is discontinuous at the critical temperature. This will give the order of the phase transition.

As an example, consider the simple model in eq. (4) for which the effective potential is given in eq. (6). For convenience, let us write:

\[
V(\phi; T) = -\frac{1}{2}M^2(T)\phi^2 + \frac{\lambda}{4}\phi^4
\]

where, we have defined \( M^2 \) to absorb the lengthy expressions in eq. (6). The critical temperature, \( T_c \), is defined by

\[
M^2(T_c) = 0.
\]

The stable phases of the system are defined by

\[
\frac{dV}{d\phi} = 0
\]

and with the second derivative being positive. Hence, when \( M^2(T) \) is negative \((i.e.\) high temperatures), the phase is:

\[
\phi = 0, \quad \text{Phase I}
\]

and when \( M^2(T) \) is positive \((i.e.\) low temperatures):

\[
\phi = \frac{M(T)}{\sqrt{\lambda}}, \quad \text{Phase II}
\]

Now we find the potential in both phases:

\[
V_I(T) = 0,
\]

\[
V_{II}(T) = -\frac{[M(T)]^4}{4\lambda}.
\]

From these expressions, using \( M(T_c) = 0 \), we have

\[
V_I(T_c) = 0 = V_{II}(T_c).
\]
Also,
\[ \frac{dV_I}{dT} \bigg|_{T_c} = 0 = \frac{dV_{II}}{dT} \bigg|_{T_c} \]
since \( M^2(T) \) is of the form \( \mu^2 - aT^2 \) where \( a \) is a constant. (Therefore the derivative of \( M^2(T) \) with respect to \( T \) is well behaved and so the derivative of \( M^4 \) vanishes.) However,
\[ \frac{d^2V_I}{dT^2} \bigg|_{T_c} = 0 \neq \frac{d^2V_{II}}{dT^2} \bigg|_{T_c} \]
and so we conclude that the phase transition is second order.

The above scheme (Ehrenfest classification) for defining the order of a transition in terms of discontinuities in the derivatives of the potentials can fail if any of the derivatives of the potential do not exist. Generally, one says that the phase transition is first order if there is a barrier in the effective potential that separates the two vacua. If the tunneling probability from the false vacuum to the true vacuum is very small, the phase transition is said to be strongly first order but if the tunneling rate is not too small, it is weakly first order. If there is no barrier between the different phases, the phase transition is said to be second-order. However, this terminology does not imply a definite understanding of how the phase transition proceeds. Only in the case of a strongly first order phase transition does one know that the phase transition proceeds by the growth of bubbles of the lower energy (true vacuum) phase.

2.5 Limitations of the Effective Potential

Even in thermodynamics, the Van der Waal's equation of state for a gas leads to a \( PV \) diagram in which the derivative of \( P \) with respect to \( V \) is positive. But this is unphysical since it is an unstable situation: an increase in the volume leads to an increase in the pressure, which leads to a further increase in the volume and so on. The resolution was found in the assumption used to derive the \( PV \) diagram - that the sample is entirely in one phase. More accurately, there will be regions of \( PV \) space where the system will consist of an admixture of phases. And the free energy of the coexisting phases can be lower than the free energy of just one phase.

In our discussion of phase transitions in field theory we have also treated the order parameter \( \phi_{cl} \) as being uniform in space. In reality, \( \phi_{cl} \) will vary over space and the two phases will coexist. If one applies a Maxwell construction to the effective potential, the result is a straight line joining the two vacua.

The dynamics of transiting from one phase to the other depends on the rate at which external parameters are varied, the shape and structure of the
effective potential, and other factors. Gravitational effects may be important in certain situations too. For example, if the tunneling rate is very small, the universe could start inflating and the phase transition would never complete. To add to these complications is the presence of topological defects that prevent the transition from false to true vacua from occurring globally.

3 Topological Defects

We first consider the existence of topological defects. In the context of the previous lectures, these may be viewed as obstructions to the completion of a phase transition. In this lecture, however, we will simply view them as classical solutions in a model that exist for topological reasons. I will start out by providing the simplest examples of topological defects and then later discuss their classification via homotopy groups.

3.1 Domain Walls

Consider the $Z_2$ Lagrangian in 1+1 dimensions

$$L = (\partial_\mu \phi)^2 - \frac{\lambda}{4} (\phi^2 - \eta^2)^2$$

(7)

where $\phi$ is a real scalar field - also called the order parameter. The Lagrangian is invariant under $\phi \rightarrow -\phi$ and hence possesses a $Z_2$ symmetry. For this reason, the potential has two minima: $\phi = \pm \eta$. And the “vacuum manifold” has two-fold degeneracy.

Consider the possibility that $\phi = +\eta$ at $x = +\infty$ and $\phi = -\eta$ at $x = -\infty$. In this case, the continuous function $\phi(x)$ has to go from $-\eta$ to $+\eta$ as $x$ is taken from $-\infty$ to $+\infty$ and so must necessarily pass through $\phi = 0$. But then there is energy in this field configuration since the potential is non-zero when $\phi = 0$. Also, this configuration cannot relax to either of the two vacuum configurations, say $\phi(x) = +\eta$, since that involves changing the field over an infinite volume from $-\eta$ to $+\eta$, which would cost an infinite amount of energy.

Another way to see this is to notice the presence of a conserved current:

$$j^\mu = e^{\mu \nu} \partial_\nu \phi$$

where $\mu, \nu = 0, 1$ and $e^{\mu \nu}$ is the antisymmetric symbol in 2 dimensions. Clearly $j^\mu$ is conserved and so we have a conserved charge in the model:

$$Q = \int dx j^0 = \phi(+\infty) - \phi(-\infty).$$
For the vacuum $Q = 0$ and for the configuration described above $Q = 1$. So the configuration cannot relax into the vacuum - it is in a different topological sector.

To get the field configuration with the boundary conditions $\phi(\pm \infty) = \pm \eta$, one would have to solve the field equation resulting from the Lagrangian (7). This would be a second order differential equation. Instead, one can use the clever method first derived by Bogomolnyi and obtain a first order differential equation. The method uses the energy functional:

$$ E = \int dx [\partial_t \phi]^2 + (\partial_x \phi)^2 + V(\phi) $$

$$ = \int dx [\partial_t \phi]^2 + (\partial_x \phi + \sqrt{V(\phi) \partial_x \phi}] $$

$$ = \int dx [\partial_t \phi]^2 + (\partial_x \phi + \sqrt{V(\phi) \partial_x \phi}] - 2 \int_{\phi(\pm \infty)}^{\phi(\pm \infty)} d\phi' \sqrt{V(\phi')} $$

Then, for fixed values of $\phi$ at $\pm \infty$, the energy is minimized if

$$ \partial_t \phi = 0 $$

and

$$ \partial_x \phi + \sqrt{V(\phi)} = 0. $$

Furthermore, the minimum value of the energy is:

$$ E_{\text{min}} = 2 \int_{\phi(-\infty)}^{\phi(+\infty)} d\phi' \sqrt{V(\phi')} $$

In our case,

$$ V(\phi) = \frac{\lambda}{4} (\phi^2 - \eta^2)^2 $$

which can be inserted in the above equations to get the "kink" solution:

$$ \phi = \eta \tanh \left( \frac{\sqrt{\lambda} \eta x}{2} \right) $$

for which the energy is:

$$ E_{\text{kink}} = \frac{4}{3} \sqrt{\lambda} \eta^3 $$

Note that the energy density is localized in the region where $\phi$ is not in the vacuum, i.e. in a region of thickness $2(\sqrt{\lambda} \eta)^{-1}$ around $x = 0$. 13
We can extend the model in eq. (7) to 3+1 dimensions and consider the case when \( \phi \) only depends on \( x \) but not on \( y \) and \( z \). We can still obtain the kink solution for every value of \( y \) and \( z \) and so the kink solution will describe a “domain wall” in the \( yz \)-plane.

Notice that the existence of the domain wall only depends on the fact that there were discrete vacuua in the theory.

3.2 Cosmic Strings

Consider the Lagrangian:

\[
L = |\partial_\mu \phi|^2 - \frac{\lambda}{4} (|\phi|^2 - \eta^2)^2
\]

where \( \phi \) is now taken to be a complex scalar field. The Lagrangian is invariant under

\[
\phi \rightarrow \phi' = e^{i\alpha} \phi
\]

and hence the model has a \( U(1) \) (global) symmetry. The vacuum expectation value of \( \phi \) is \( \eta e^{i\alpha} \) where \( \alpha \) can take any value. So the ground state of the model has continuous degeneracy. The degeneracy is labelled by the phase angle \( \alpha \) and hence the vacuum manifold is a circle.

Vortices are formed if we consider the model in two spatial dimensions and let \( \alpha \) be such as to wrap around the vacuum manifold. For example, we could take \( \alpha = \theta \), the polar angle. Then, since the field is single valued everywhere, there must be at least one point at which \( \phi = 0 \). The field carries energy at this point since \( \phi = 0 \) is not on the vacuum manifold. The location of this point may be defined as the location of a vortex. (An example of a vortex is universally encountered by people taking baths or washing dishes. As the water flows down the drain, it circulates. We cannot interpolate the circulating velocity field all the way to the center of the vortex since it would have to become multi-valued. Instead the fluid density in the central region of the vortex vanishes.)

If we now take the model in three spatial dimensions, the vortex becomes a line stretching in the third dimension. The vortex line is called a “string”.

The crucial element in the existence of the vortex was that \( \alpha \) could “wrap around” the vacuum manifold. In other words, vortices exist if the vacuum manifold contains incontractable closed paths.

Bogomolnyi’s method cannot be applied to construct the vortex solution in model (8). In fact, the energy of an isolated global vortex diverges. If the model (8) is gauged, then Bogomolnyi’s method does work for a particular choice of parameters in the model. Note that gauging the model makes no
difference to the vacuum manifold and so the topological arguments that show the existence of vortices still apply.

3.3 Monopoles

The model:

\[ L = |\partial_\mu \vec{\phi}|^2 - \frac{\lambda}{4} (\vec{\phi}^2 - \eta^2)^2 \]

where \( \vec{\phi} \) is a triplet of scalar fields contains global monopole solutions. To see this, note that the Lagrangian is invariant under

\[ \vec{\phi} \rightarrow \vec{\phi}' = R \vec{\phi} \]

where, \( R \) is a rotation matrix in three dimensions. Hence the model has an \( O(3) \) (global) symmetry which is broken down to \( O(2) \) once \( \vec{\phi} \) gets a vacuum expectation value. For example, if \( \vec{\phi} \propto \hat{e}_3 \), then the rotations in the unbroken group are the rotations about the \( \hat{e}_3 \) axis.

A monopole solution is obtained when

\[ \vec{\phi} \propto \hat{e}_r = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \]

where \( \theta \) and \( \phi \) are the angular spherical coordinates. For this “hedgehog” configuration of the field, there must be a point in space where \( \vec{\phi} = 0 \) and the energy density is non-vanishing. In fact, in the global symmetry case the energy of the monopole is infinite because of the slow fall off of gradient energy at infinity. If the model is gauged, the \( \vec{\phi} \) field configuration can be accompanied by a gauge field that cancels off the gradient energy at infinity. This then leads to a finite energy solution but with a non-vanishing magnetic flux at infinity - the famous “magnetic monopole” of 't Hooft and Polyakov.

For the magnetic monopole, the field on the asymptotic two sphere has to be non-trivial. So if the vacuum manifold admits incontractable two spheres, we can have mappings from spatial infinity to the vacuum manifold that cannot be smoothly deformed to the trivial mapping (in which all of space is assigned the same point on the vacuum manifold). And each such mapping would lead to a monopole solution.

3.4 Textures

A simple model with (global) texture is

\[ L = (\partial_\mu \phi^a)^2 - \frac{\lambda}{4} (\phi^a \phi^a - \eta^2)^2 \] (9)
where $a = 1, 2, 3, 4$. Now the vacuum manifold is a three sphere. If our universe is a three sphere, $\phi^a$ could wrap around it an integer number of times. Such a solution is a texture - the field does not vanish anywhere but there is still some gradient energy present.

In the cosmology literature, the term texture does not necessarily refer to the case when the universe is a three sphere. One can consider a ball in space and find the winding of the field within this ball. Such configurations can have any winding. The term texture is taken to apply to the situation where the winding is unity. Such textures are time-dependent solutions. They collapse to a point where the configuration unwinds and then the energy radiates away. (The scalar field is zero at one point in space-time.)

3.5 Hybrids

In a sequence of symmetry breakings, one can get “hybrid” defects such as walls bounded by strings, or, monopoles connected by strings. For example, in the symmetry breaking sequence:

$$SU(2) \rightarrow U(1) \rightarrow 1$$

the first symmetry breaking yields monopoles which get connected by strings in the second stage of symmetry breaking. Similarly one can get domain walls that are bounded by strings. For further discussion of hybrid defects see Vilenkin and Shellard.

3.6 Topological Criterion and Homotopy Groups

The criterion for having domain walls in a model in which the symmetry group $G$ is spontaneously broken to $H$ by the vacuum expectation value of a field can now be specified. First, if $H$ is the trivial group, any element of $G$ acting on the order parameter would yield a possibly new value of the order parameter which would still be in the minima of the potential and hence would be a degenerate vacuum. Then the manifold of vacuum states is simply given by the manifold of $G$. Next, if $H$ is not the trivial group, the non-trivial elements of $H$ leave the order parameter invariant. But since $H$ is a subgroup of $G$, there are elements of $G$ whose action on the order parameter is identical. In fact, the elements of $G$ can be ascribed to equivalence classes - the action of each element within an equivalence class on the order parameter is identical but the action of two elements belonging to two different equivalence classes can be different. These equivalence classes are nothing but the elements $gH$ of the coset space $G/H$ - elements of $gH$ acting on the order parameter give the same
result as $g$ acting on the order parameter since elements of $H$ leave the order parameter invariant. So the vacuum manifold is the manifold corresponding to $G/H$. The connectivity of a manifold is described by the zeroth homotopy group of the manifold: $\pi_0(G/H)$. Domain walls are present in the model if the vacuum manifold has disconnected components, that is, if $\pi_0(G/H)$ is not trivial.

Now we can generalize these considerations further. The next step is to consider a vacuum manifold that contains incontractable closed paths. For example, the vacuum manifold could be a one sphere as in the $U(1)$ case when $\phi$ is a complex scalar field. Then the vacuum expectation value of $\phi$ at infinity determines a path in the vacuum manifold (and vice versa) by the equation:

$$\phi_\infty(\theta) = g(\theta)\phi_\infty(0)$$

where $\theta \in [0, 2\pi]$ is the polar angle. The group elements $g(\theta)$ determine a path on the vacuum manifold parametrized by $\theta$. Then one could consider a configuration $\phi_\infty(\theta)$ for which $g(\theta)$ traces one of the incontractable closed paths. This mapping from the circle at spatial infinity to the vacuum manifold has a non-trivial topological index and hence the mapping from any two dimensional disk bounded by the circle at spatial infinity to the vacuum manifold must have a singularity. In the field theory case, the singularity is simply a location where the field $\phi$ vanishes. Now, since we can choose to look at any surface bounded by the circle at spatial infinity, there must be a curve on which $\phi$ vanishes. The form of the potential tells us that there is energy wherever $\phi = 0$ and so there is energy distributed along a one-dimensional curve - that is, a string. The conclusion is that there are strings in the model whenever there exist incontractable paths on the vacuum manifold.

In mathematical language, paths on manifolds are put in equivalence classes depending on whether they can be deformed into one another (that is, an equivalence class contains homtopically equivalent paths). Furthermore, there are combination rules for paths - two paths can be combined to give a third path. The set of homotopically equivalent paths, together with the combination rule for paths leads to a group structure which is called the first homotopy group. For the manifold $G/H$, this is denoted by $\pi_1(G/H)$. In this language, the field theoretic model contains strings whenever $\pi_1(G/H)$ is non-trivial.

The next generalization is to consider vacuum manifolds on which there exist incontractable two spheres. Here the mapping from the two sphere at infinity to the vacuum manifold can be non-trivial and this would lead to a point-like singularity which we call a “monopole”. Therefore there are monopoles in the model if $\pi_2(G/H)$ is non-trivial.
One could also consider the possibility of incontractable three spheres on vacuum manifolds. These would be given by non-trivial third homotopy groups, \( \pi_3(G/H) \), and result in “textures”. If space is a three sphere, we can have a texture wrapped around the entire three sphere and this would give a static solution in the model.

The problem of finding the types of topological defects present in a given model reduces to finding the homotopy groups for a certain symmetry breaking \( G \rightarrow H \). That is, we need to find \( \pi_n(G/H) \) \((n = 0, 1, 2, 3)\) given the groups \( G \) and \( H \). In general, this can be quite complicated but there is an immensely useful theorem which is often applicable and simplifies matters. I now describe this theorem.

Mathematicians have found that there are homomorphisms between certain homotopy groups that can be written as an “exact homotopy sequence”. An important sequence is:

\[
\ldots \rightarrow \pi_n(G) \xrightarrow{\alpha} \pi_n(G/H) \xrightarrow{\beta} \pi_{n-1}(H) \xrightarrow{\gamma} \pi_{n-1}(G) \rightarrow \ldots
\]

The sequence denotes that there exist homomorphisms \( \alpha \), \( \beta \) and \( \gamma \) such that

\[
\text{Im}(\alpha) = \text{Ker}(\beta)
\]

\[
\text{Im}(\beta) = \text{Ker}(\gamma)
\]

where, \( \text{Im}() \) stands for the image of a map and \( \text{Ker}() \) stands for the kernel of a map (i.e. the set of all elements mapped to the identity element). This relationship is depicted in Fig. 2.

In particular, if \( \pi_n(G) \) and \( \pi_{n-1}(G) \) are trivial (that is, contain the identity element only), \( \text{Im}(\alpha) \) has to be the identity of \( \pi_n(G/H) \). But then, by the exact sequence, \( \text{Ker}(\beta) \) is also the identity element. Using this fact one can prove that \( \beta \) must be one to one. Next, we show that \( \beta \) is onto. The homomorphism \( \gamma \) in the sequence maps the entire group \( \pi_{n-1}(H) \) to \( \pi_{n-1}(G) \) since \( \pi_{n-1}(G) \) is trivial. That is, \( \text{Ker}(\gamma) = \pi_{n-1}(H) \). Then \( \text{Im}(\beta) = \text{Ker}(\gamma) = \pi_{n-1}(H) \) and so \( \beta \) is both one-one and onto and hence, is an isomorphism. Therefore:

\[
\pi_n(G/H) = \pi_{n-1}(H)
\]

if \( \pi_n(G) = 1 = \pi_{n-1}(G) \).

As an example, for \( n = 2 \), we know that \( \pi_2(G) = 1 \) for any Lie group and, if \( \pi_1(G) = 1 \), then

\[
\pi_2(G/H) = \pi_1(H)
\]

\(^a\)The use of the word “texture” in condensed matter is different - it refers to any variation of the order parameter, not necessarily having non-trivial \( \pi_3 \).
In particular, if $H$ contains any $U(1)$ factors, $\pi_1(H)$ is non-trivial and the model has monopoles. In Grand Unified Theories, the Grand Unified group is usually taken to be simply connected and then, since $H$ necessarily contains the electromagnetic $U(1)$ symmetry group, $\pi_2(G/H)$ is non-trivial and hence, (magnetic) monopoles are necessarily predicted.

There is a subtlety in the homotopic classification that we have glossed over. The subtlety is that the first homotopy group classifies the paths on the vacuum manifold that pass through some arbitrary but fixed base point. It is possible that two closed paths may not be deformable to each other if we impose the constraint that they should continue to pass through the base point but, if we were to relax this constraint, they would indeed be deformable into one another. As there are no constraints on the field configurations that correspond to having a fixed base point, strings are classified by “free” homotopy. This subtlety does not play a role when the first homotopy group is abelian but it can be important when the group is non-abelian.

This subtlety also applies in a slightly varied form to the classification of magnetic monopoles. It can be important in the cases where both strings and monopoles are present in the model\[4\].

Figure 2: An illustration of a part of the homotopy sequence. The largest ellipse denotes the homotopy group specified below the ellipse, the middle shaded ellipse denotes the kernel of the map which is also the image of the previous map, and the innermost blackened ellipse denotes the identity element.
3.7 Exceptions: Semilocal Strings

The classification of defects by homotopy groups is purely topological and does not take any dynamical effects into consideration. For example, the first homotopy group tells us that there are non-trivial paths on the vacuum manifold but does not tell us which of these paths will be preferred when we solve the field theoretic equations of motion. For the same reason, there are a number of solitons that cannot be detected by the homotopy classification. These include non-topological solitons and semilocal strings. I will discuss the latter here.

Consider the Lagrangian

\[ L = |D_\mu \Phi|^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\lambda}{4} (\Phi^\dagger \Phi - \eta^2)^2 \]

where \( \Phi \) is an \( SU(2) \) doublet, and,

\[ D_\mu = \partial_\mu - ieA_\mu \]

where, \( A_\mu \) is an Abelian gauge potential.

Note that the model has an \( [SU(2) \times U(1)]/\mathbb{Z}_2 \) symmetry where the \( SU(2) \) is a global symmetry and the \( U(1) \) is a local (gauged) symmetry. (The \( \mathbb{Z}_2 \) in the denominator is present because the center of the \( SU(2) \) factor is also contained in the \( U(1) \) factor. Without modding out by \( \mathbb{Z}_2 \), we would be including these elements twice.) Once \( \Phi \) acquires a vacuum expectation value, the symmetry is broken to \( U(1)' \) which is a global symmetry group. This model is exactly the bosonic sector of the standard model of the electroweak interactions with weak mixing angle \( \theta_w = \pi/2 \).

The vacuum manifold of the standard model is known to be a three sphere. The simplest way to see it is to consider the minima of the potential:

\[ \Phi^\dagger \Phi = \eta^2 \]

and this is a three sphere. The homotopy classification now tells us that the three sphere is simply connected and so there are no strings in this model. But this conclusion is wrong.

A way to understand why there may be strings in the model is that if one only considers the gauged symmetries, then the breaking pattern is \( U(1) \to 1 \). And we know that this symmetry breaking yields strings. However, since the string is not topological, there are ways to deform the string so that it is equivalent to the vacuum. What might prevent this from happening is not topology but an energy barrier. A stability analysis for semilocal strings tells us that they are stable only if \( \lambda/2e^2 < 1 \).
4 Formation in Cosmology

Now we are ready to discuss the cosmological formation of topological defects.

4.1 General Discussion

In a symmetry breaking, the order parameter $\phi$ can have a vacuum expectation value anywhere on the vacuum manifold. For example, in the case of a $U(1)$ symmetry breaking, at some time after the symmetry breaking, $\phi = \eta \exp(i\alpha)$ where $\alpha$ can be any space dependent function. We expect that the thermal nature of the symmetry breaking would lead to a domain structure of $\alpha$ where the correlation of $\alpha$ at two spatial points decays with the distance between the two points. In addition, large variations of $\alpha$ should be suppressed by the terms in the effective action that contain the gradient of $\alpha$. In other words, we expect

$$|\nabla \alpha| \sim T$$

that is, $\alpha$ changes by order 1 as a distance of $\sim T^{-1}$ is traversed.

While the above argument is reasonable for the breaking of global symmetries where $\alpha$ is a physical variable, it breaks down when the symmetries are gauged. The point here is that the gradient of the phase of $\phi$ has no gauge invariant meaning. The gradient of $\alpha$ must now be replaced by the covariant gradient which also contains the gauge fields in it. Now we have

$$|\nabla \alpha - eA| \sim T.$$ 

This means that we can transfer some of the variation in $\alpha$ to the gauge field. But the defect depends on

$$\int d\alpha$$

where the integral is taken along a closed path. This closed line integral is gauge invariant but a relation with the covariant derivative - which is the physical one - cannot be made without first specifying the gauge field.

The bottom line is that one cannot say that $\alpha$ has a certain value at a spatial point unless one fixes the gauge. In the lattice simulations that I will describe, this difficulty is ignored and it is assumed that $\alpha$ can be treated as being a physical field just as in the global symmetry case. In more recent work, this assumption has been questioned but further work has provided some justification for working with $\alpha$ as if it was physical.

\footnote{More rigorously one should find the correlation length at the phase transition and use that to find the $\alpha$ domain size. This will be discussed in the lectures by Professor Kibble.}
Table 1: Size distribution of + clusters found by simulations on a cubic lattice.

| Cluster size | 1  | 2  | 3  | 4  | 6  | 10 | 31082 |
|--------------|----|----|----|----|----|----|-------|
| Number       | 462| 84 | 14 | 13 | 1  | 1  | 1     |

5 Lattice Simulations

5.1 Domain Walls

The simplest domain walls are formed when the vacuum is two-fold degenerate. Let us call these vacuua + and -. At the phase transition, there are domains in which the order parameter chooses the + vacuum and other domains in which it chooses the - domain. Since these are degenerate, the probability for choosing + or - is 1/2.

What is the size distribution of walls formed in this phase transition?

This problem is closely related to the problem studied in percolation theory. There the problem is to assign + with probability $p$ and - with probability $1 - p$ to every domain and then to find out the cluster size distribution of + domains. The result is that, if the domains are taken to form a cubic lattice, there is one infinite size, + domain cluster if $p > 0.31$. (That is, the + domains percolate if $p > 0.31$.) In our problem, $p = 0.5$ and so both + and - will percolate. The boundary of + and - domains is the location of domain walls and so we will get one infinite domain wall. In addition, we might find a few smaller domain walls.

The cluster size distribution is easy to find by doing a simulation. In Table 1, I give the results from a paper by Vilenkin and me.

5.2 Strings

The formation of strings can be studied numerically by assigning the phase $\alpha$ randomly on lattice sites - say of a cubic lattice. Then one can go along the edges of each plaquette of the lattice and evaluate differences in $\alpha$. To do this, it is necessary to interpolate between the values of $\alpha$ at two sites. Then one finds the integral in eq. 10 around a plaquette. If this is non-zero, it indicates that there is a string or anti-string passing through the plaquette. In this way, all the strings are found. Then they are connected and information about the distribution of string is stored.

A subtle issue in calculating the integral above is the interpolation as we
Figure 3: The triangle is a plaquette in space and the circle denotes the vacuum manifold. At each vertex of the plaquette a phase is assigned at random. In traversing from $A$ to $B$ on the triangle, the phase must change from $\alpha_A$ to $\alpha_B$. However, there is an infinite degeneracy in the path from $\alpha_A$ to $\alpha_B$ on the vacuum manifold since the path can wrap around the entire circle any number of times.

go from one lattice site to another. Consider a $U(1)$ string simulation as shown in Fig. 3. As we traverse the triangle ABC in space, the phase varies from $\alpha_A$ to $\alpha_B$ to $\alpha_C$ and then back to $\alpha_A$. These are simply points on a circle and we know that there are infinitely many paths joining any two points on a circle. (The paths can go around the circle infinitely many times.) So at every stage of the construction, we need to interpolate between the phases and there is an infinite-fold ambiguity in this interpolation. How do we resolve this ambiguity?

In the case of global strings, it is assumed that the shortest of the infinitely many paths is the correct one. The rationale for this choice is that the free energy density gets contributions from a term $|\nabla \alpha|^2$ and this is least for the shortest path. The rule of choosing the shortest path to interpolate between two points on the vacuum manifold is known as the “geodesic rule”.

In the case of gauge strings, the rationale for the geodesic rule breaks down since the contribution to the free energy involves the covariant derivative of $\alpha$ and not the ordinary derivative. Now which path should be chosen? Following the logic of the global case, it should be the path that minimizes $|\nabla \alpha - eA|^2$. In the simulation this would mean that we should not only keep track of $\alpha$ but also the gauge field $A$. We will discuss a possible way to circumvent this problem in the next subsection while here we will assume the geodesic rule to be valid.

The surprising result that emerges from numerical simulations is that most of the energy in the string network is in infinite strings. Furthermore, the strings are Brownian on large scales and the loop distribution is scale invariant.
Let us explain these results in more detail:

- **Brownian strings**: This means that the length $l$ of a string is related to the end-to-end distance $d$ by

  \[ l = \frac{d^2}{\xi} \]  

  where $\xi$ is a length scale also called the step length which would be roughly given by the lattice spacing. This result is valid for large $l$. For smaller lengths, the walk is not Brownian and lattice effects are also present.

- **Loop distribution**: Scale invariance means that there is no preferred length scale in the problem apart from the lattice cut-off and loops of all sizes are present. So the number density of loops having size between $R$ and $R + dR$ is given by dimensional analysis:

  \[ dn(R) = c \frac{dR}{R^4} \]

  where $c \sim 6$. Using eq. (11), this may be written as:

  \[ dn(l) = \frac{c}{2\xi^{3/2}} \frac{dl}{l^{5/2}} \]

  Note that the scale invariance is in the size of the loops and not in their length.

- **Infinite strings**: With the implementation of the geodesic rule, the density in infinite strings was estimated to be about 80% of the total density in strings. The way this estimate was made was to do the simulation in bigger and bigger lattices and keep track of the length in the strings that were longer than a large (compared to the lattice size) critical length $20$. As the lattice was made bigger, the fraction of string in long strings tended to stabilize around 80%. Simulations on other lattices and with periodic boundary conditions also yield infinite strings but the estimated fraction can vary upward from about 74%. Analytic estimates of the fraction which assume that the strings are random walks on a lattice, are consistent with these estimates.

Can one analytically see the presence of infinite strings? This is an open question. Some progress can be made if one assumes that strings perform a
Brownian walk. It is known that random walks do not close in 3 dimensions and this tells us that infinite strings will be present. Furthermore, estimates can be obtained for the fraction of length in infinite strings and the result is similar (though not identical) to the one obtained by simulations.

For cosmological applications such as the formation of large-scale structure, the existence of infinite strings is vital. The reason is that the small closed loops can decay by emitting gravitational and other forms of radiation but the infinite strings are destined to live forever because of their topological character. So only the infinite strings (and their off-spring loops) could live to influence late time cosmology and also to tell us the story of the Grand Unified epoch.

5.3 Relaxing the Geodesic Rule

A possible cure for the ambiguity in choosing the path on the vacuum manifold (discussed above) is to relax the geodesic rule and assume that the phase difference between two lattice sites is given by a probability distribution. So, if the values of the phases at lattice sites 1 and 2 are $\theta_1$ and $\theta_2$, the phase difference will be

$$\Delta \theta = \theta_2 - \theta_1 + 2\pi n = \delta \theta + 2\pi n$$

where $n$ is a random integer drawn from some distribution. A convenient choice for the distribution is

$$P_n = \int_{n-0.5}^{n+0.5} dm 2\sqrt{\pi} e^{-\beta (\delta \theta + 2\pi m)^2} . \quad (12)$$

with $\beta > 0$ being a parameter. This probability distribution is consistent with the idea that longer paths on the circle should be suppressed but the amount of suppression depends on the choice of $\beta$. Note that $\beta$ plays the role of inverse temperature since lower values of $\beta$ (that is, higher temperatures) allow for larger values of $n$ while larger values of $\beta$ reduce the algorithm to the geodesic rule.

In simulations that relax the geodesic rule, it is found that the fraction of infinite strings gets larger with smaller values of the parameter $\beta$. And hence the case for having infinite strings in a realistic phase transition is strengthened (see Fig. 4).

---

\[c\] The two ways in which they could decay are: a) a string meets an antistring and annihilates, and, b) a string snaps leading to a gravitational singularity. Neither process is expected to occur at a rate that would be cosmologically interesting.

\[d\] Professor Kibble has raised the interesting possibility that a population of long loops might be able to play the role of infinite strings in cosmology.
Figure 4: A plot of the infinite string density fraction versus the total string density. The total string density increases as the parameter $\beta$ is lowered. The geodesic rule is recovered in the limit that $\beta$ becomes very large.

5.4 Monopoles

Lattice simulations of monopoles largely follow the strategy adopted for strings. Here too one throws down random phases (corresponding to points on a two sphere) on lattice points, adopts the geodesic rule, and then finds the winding of the configuration. The calculations are a little more involved since one needs to calculate windings of a two sphere on a two sphere. Also, a cubic lattice leads to ambiguities and it is better to work on a lattice where the cells are (irregular) tetrahedra.

The results of these simulations is a distribution of monopoles which is correlated on small scales - for example, the cells sharing a plaquette with a cell containing a monopole can only contain an antimonopole - but these correlations decay with distance. The distance between closest monopoles is not too much larger than the distance between closest monopole and antimonopole. As a result, if the system is left to evolve further under the mutual forces of the monopoles, some of the close-by monopoles and antimonopoles annihilate but very often a monopole is left as an “odd man out”. The antimonopole it could annihilate with is located far away and was also the “odd man out” in a set of local annihilations. As a result, the correlations between monopoles and
antimonopoles are quickly washed out and the end result is a scaling density of poles.

5.5 Biased Topological Defects

So far we have assumed that the vacuum manifold is completely degenerate. But in many circumstances, the degeneracy may be broken by explicit symmetry violations. For example, in the domain wall case, the + vacuum might be slightly preferred over the - vacuum. This is likely to be relevant in condensed matter physics where there are external forces present (e.g., earth’s magnetic field, rotation of the container etc.) that could “bias” the symmetry. What happens to the statistics of the defects in such cases?

In the case of domain walls, as already discussed, the result is known from percolation theory. At a critical bias, the probability of a domain having + falls below 0.31 (on a cubic lattice) and the + domains do not percolate. Then all the domain walls are finite.

In the case of strings, we do not have results from percolation theory but a few simulations have been done. It is found that the infinite strings break down into smaller loops at some critical value of the bias and the resulting distribution of loops is not scale invariant. Instead it is well fitted by:

\[
dn(l) = \frac{c}{2\xi^2} \frac{dl}{l^a} \exp[-bl]
\]

where \(a, b\) and \(c\) are constants that depend on the value of the bias. In particular, note that the exponent \(a\) need not be the scale invariant value of \(5/2\). Instead two regimes seem to be indicated - for large bias \(a = 2\) while for small bias \(a = 5/2\).

The feature that I found amusing in the simulation of biased defects was that the transition from percolating strings to non-percolating strings appears to be rather sudden. In other words, there seems to be a critical minimum amount of bias that prevents infinite strings from forming.

\[\text{These results on the evolution of monopoles were derived by Preskill and have been confirmed numerically.}\]

\[\text{I sometimes wonder if this work can have application in the real world. Suppose that a manufacturer produces metal sheets by cooling molten material but is worried about the line defects that would form during cooling, run across the sheets and weaken them. One way to get rid of the line defects would be to use some sort of bias while cooling. But the question is if there is an optimal amount of bias that will be enough to clean the sheets from (long) line defects or whether it is simply that the more the bias, the fewer defects are produced. What we find here is that an optimal bias indeed exists at which the infinite line defects that run across the sample disappear even though the exact value of the bias will depend on the type of defect, type of bias and other details.}\]
Bias can also be added to simulations of monopoles. The relevant quantities to study here are the distance between closest monopoles and the distance between closest monopole ($m$) and antimonopole ($\bar{m}$), as functions of the bias. In the simulations it is found that the $mm$-distance grows with bias but the $m\bar{m}$-distance stays roughly constant. At some stage the monopole distribution is in monopole-antimonopole pairs with different pairs being widely separated. In this case, further evolution would cause the paired monopoles and the antimonopoles to annihilate and their would be no confusion about which monopole should annihilate which monopole. All the monopoles would then disappear.

5.6 Problems with Lattice Based Simulations

The simplest way to see that lattice based simulations might be suspect is to realize that the critical percolation probability depends on the lattice that is used.

Consider the case of domain walls in which the probability of laying down a $+$ is $p$ and when the critical percolation probability, $p_c$, is less than 0.5. Then there are three phases:

- $p < p_c$: the $+$ domains are islands in a sea of $-$,
- $p_c < p < 1 - p_c$: the $+$ and $-$ both form seas, and,
- $1 - p_c < p$: the $+$ form the sea, the $-$ form the islands.

In the unbiased case, we have $p = 0.5$ and so we would always get seas of $+$ and $-$, and the boundary between the $+$ and $-$ regions would also be infinite. That is, the domain walls are infinite in size.

If $p_c > 0.5$, the picture is quite different. Now we have:

- $p < 1 - p_c$: the $+$ domains are islands in a sea of $-$,
- $1 - p_c < p < p_c$: the $+$ and $-$ both form islands, and,
- $p_c < p$: the $-$ form islands in a sea of $+$.

Again, in the unbiased case, $p = 0.5$ and so both the $+$ and the $-$ form islands. The interface between the islands are finite in extent and so there are no infinite domain walls.

So one sees that the value of $p_c$ as compared to $p$ is very important for finding out if infinite domain walls are formed. What is even more interesting is that, in two spatial dimensions, $p_c = 0.5$ for a triangular lattice and $p_c = 0.59$ for a square lattice. So the domain walls in two dimensions with $p = 0.5$ are
Figure 5: The black squares denote + domains and the white squares denote - domains in a simulation with \( p = 0.5 \) on a two-dimensional square lattice. Neither the + nor the - domains percolate in this case and the domain walls, which are the boundaries between black and white squares are all finite.

(marginally) infinite on a triangular lattice and are all finite on a square lattice (see Fig. 5). Which lattice is the correct one to use to study phase transitions?

One expects the same problems to arise in the lattice based study of strings and monopoles. In fact, the study of domain walls is fundamental to understanding strings and monopoles since strings, for example, may be viewed as the intersection of two types of domain walls - one on which the real part of a complex scalar field vanishes and the other on which the imaginary part vanishes. If the two types of domain walls are all finite, the strings will also be finite. Hence it is suitable to first understand the percolation of domain walls.

6 Lattice-Free Simulations

It is easier to think about first order phase transitions since here we know that bubbles of the new phase nucleate, collide, coalesce and eventually fill space with the new phase. At every bubble collision there is a possibility that a defect will be produced. So the first step is to study the processes that can occur when two or more bubbles collide. This has been treated analytically by Kibble and Vilenkin and numerically by several groups mainly in the context of string formation. It is found that bubble collision indeed leads to the formation of strings. This can happen when two or more bubbles collide. In the case of gauge strings, when two bubbles collide, a magnetic flux tube is
formed in the shape of a closed ring at the location of the junction of the two bubbles. When several bubbles collide, these magnetic flux tubes can coalesce to form strings.

Numerical simulations have been performed to study the distribution of vortices in two spatial dimensions and strings in three spatial dimensions. The three dimensional simulation indicates that there may be an absence of infinite strings though the numerical limitations do not permit any firm conclusion. For this reason, I will now describe the easier task of studying domain wall formation in a first order phase transition in two spatial dimensions. The extension to three spatial dimensions is conceptually straightforward but has not yet been done.

7 Percolation in First Order Phase Transitions

Imagine that a phase transition occurs in two spatial dimensions and proceeds by bubble nucleation. Assume:

- The bubble nucleation probability is constant per unit time per unit volume of false vacuum.
- Once a bubble nucleates, its radius grows with constant velocity.
- When bubbles collide they continue to grow as if there was no collision.

Already there are several interesting questions one can ask. For example, what is the rate of bubble nucleation? How many collisions does an average bubble have? These questions are important if one wants to connect percolation in phase transitions to the previously studied percolation on a lattice. Here I will not go into details but only mention that the average number of collisions of a bubble is about 6 and hence is very close to the number of neighbors of a lattice site on a triangular lattice (see Fig. 6).

Percolation in the phase transition is studied by assigning a + or a - to each bubble. If a + bubble collides with a - bubble, the two bubbles are separated by a domain wall. We want to know the typical size of a + cluster for different probabilities, $p$, for assigning a + to a bubble. The simulation results show that this probability is about 0.50 which is the same as the result for a triangular lattice (Fig. 7). Hence we conclude that domain walls will (marginally) percolate in first order phase transitions in two spatial dimensions.

These results suggest what might happen in three dimensions and for other defects. If the critical percolation probability for bubbles is the same as that for tetrahedral lattices and walls percolate on a tetrahedral lattice, we expect domain walls to percolate in first order phase transitions in three dimensions.
Figure 6: The crosses denote bubble centers that are in the + phase and the filled squares denote bubble centers that are in the - phase. If two bubbles collide, their centers are joined by straight lines. The figure then shows the “bubble lattice” expected in a first order phase transition in two spatial dimensions.

Figure 7: The graphs show the moments of the + cluster size distribution excluding the largest cluster for a range of probabilities, $p$. The top curve is the zeroth moment $\sum n(s)$ where $n(s)$ is the number density of clusters having size $s$. The middle curve is $\sum s n(s)$ and the bottom curve is $\sum s^2 n(s)$. From the bottom curve, it is clear that the second moment turns over around $p = 0.5$. This turnover marks the appearance of a very large cluster which is not included in the calculation of the moments. Hence the critical percolation probability is 0.5.
as well. Then it is very likely that the intersections of two kinds of domain walls will also percolate - that is, infinite strings will be present. It would be reassuring to confirm these suggestions in actual numerical simulations.

While the above discussion lends confidence to the usual picture of defect formation, it does not provide quantitative results into the cluster size distribution and other features of interest. For this an analytic treatment would be invaluable. However, at the moment the subject is completely open and, to my knowledge, it is not even known how to start to attack the problem.

8 Directions

As discussed above, lattice-based numerical simulations have been extensively used to study defect formation. These give a picture that is well-defined and seems to hold up in general characteristics even when modifications are made to make the simulations more realistic.

Lattice-free numerical simulations have just started and as far as we can tell, have not changed the picture emerging from lattice-based simulations in any dramatic way.

Field theoretic numerical studies have also just started and we have yet to see the picture that will emerge. These studies are very computer intensive but, as computer resources improve, could become valuable for studying the formation of defects. They would also contribute to our understanding of the formation of non-topological defects such as semilocal and electroweak strings.

The most notable shortage is in an analytical understanding of defect formation. Here there is a lack of implementable techniques and there is plenty of scope for improvement. It is also likely that certain issues have already been addressed in condensed matter physics and a search of the literature may be a good starting point.

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