Stress state of a piecewise uniform layered space with doubly periodic internal cracks

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Abstract. The present paper deals with the stress state of a piecewise homogeneous plane formed by alternation junction of two distinct strips of equal height manufactured of different materials. There is a doubly periodic system of cracks on the plane. The governing system of singular integral equations of the first kind for the density of the crack dislocation is derived. The solution of the problem in the case where only one of the repeated strips contains one doubly-periodic crack is obtained by the method of mechanical quadratures.

Introduction

A large number of studies deal with periodic and doubly-periodic problems for massive homogeneous bodies with cracks. The monographs [1, 2] summarizing the basic results obtained in this field deserve our attention. In the recent years, the periodic and doubly-periodic problems for piecewise uniformly homogeneous layered bodies with interphase crack-type defects namely, foreign or absolutely rigid inclusions, are currently actual in the light of layered composites studied in [3–6]. Our review of the relevant literature showed that the research of the stress state of piecewise uniformly homogeneous layered bodies with internal doubly-periodic cracks is carried out for the first time in the present paper.

1. Statement of the problem and derivation of the governing equations

Assume that, in the Cartesian coordinates Oxy, a piecewise homogeneous elastic plane, which is formed by alternate junction of strips of thickness 2h manufactured of different materials with Young moduli $E_1$ and $E_2$ and Poisson ratios $\nu_1$ and $\nu_2$, contains the same periodic internal cracks with period 2l on the middle lines of the distinct strips $y = (2n + 1)h$ ($n \in \mathbb{Z}$) along the lines $L_1 = \bigcup_{i=1}^{N} (a_i, b_i)$ and $L_2 = \bigcup_{i=1}^{M} (c_i, d_i)$. Let us further assume that the plane is deformed under the action of identical loads, which are symmetric with respect to the lines $x = 2nl$ ($n \in \mathbb{Z}$) and are applied on the crack edges so that the lines $x = (2n + 1)l$ and $y = (2n + 1)h$ ($n \in \mathbb{Z}$) are the lines of symmetry. Taking these assumptions into account, we can define our problem as a problem for the piecewise homogeneous rectangle (basic cell) occupying the region $\Omega\{|x| \leq l; |y| \leq H\}$ on the plane with symmetry conditions on the boundaries and on the lines $y = \pm h$ beyond the cracks; the normal stresses of intensity $P^{(j)}_0(x)$ are given on the crack edges and the shear stresses are absent.
The goal in this paper is to construct the solutions of the above-defined problem and to reveal how the laws of changes in the stress intensity factors at the crack end-points and the crack opening depend on the elastic and geometrical characteristics of the problem.

The problem is equivalent to the following boundary-value problem for the piecewise homogeneous rectangle $\Omega\{ -l \leq x \leq l; -h \leq y \leq h \}$:

\[
\begin{align*}
\tau_{xy}^{(1)}(\pm l, y) &= U_1(\pm l, y) = 0, & 0 < y < h, \\
\tau_{xy}^{(2)}(\pm l, y) &= U_2(\pm l, y) = 0, & -h < y < 0, \\
\tau_{xy}^{(2)}(x, (-1)^{j+1}h) &= V_j(x, (-1)^{j+1}h) = 0, & x \notin L_j, \\
\tau_{xy}^{(1)}(x, (-1)^{j+1}h) &= 0, & x \in L_j, \ j = 1, 2, \\
\sigma_y^{(1)}(x, (-1)^{j+1}h) &= P_0^{(j)}(x), & x \in L_j, \\
\sigma_y^{(1)}(x, 0) &= \sigma_y^{(2)}(x, 0), \quad \tau_{xy}^{(1)}(x, 0) = \tau_{xy}^{(2)}(x, 0), \quad |x| \leq l.
\end{align*}
\]

The functions $P_0^{(j)}(x)$ are even.

Here $U_j(x, y)$ and $V_j(x, y)$ ($j = 1, 2$) are horizontal and vertical components of the displacement vector of the rectangle, which satisfy the Lamé equations, while $\sigma_y^{(j)}(x, y)$ and $\tau_{xy}^{(j)}(x, y)$ ($j = 1, 2$) are components of normal and shear stresses acting in the respective rectangle.

To solve the boundary-value problem, we should initially solve an auxiliary problem, which differs from problem (1) in the following relations considered instead of (1a):

\[ V_j(x, (-1)^{j+1}h) = V_j(x) \quad (j = 1, 2; x \in L). \]  

Here the functions $V_j(x)$ are the unknown displacements of the crack edges.

Just as in [3, 4], we use biharmonic functions of stresses (Airy functions) to construct solutions of the auxiliary problem. For the heterogeneous rectangles, these functions are represented as the series

\[
F_j(x, y) = \sum_{k=1}^{\infty} \{A_k^{(j)} \cosh(\alpha_k y) + B_k^{(j)} \sinh(\alpha_k y) + \alpha_k y[ C_k^{(j)} \cosh(\alpha_k y) + D_k^{(j)} \sinh(\alpha_k y)] \} \cos(\alpha_k x) + a_j x^2 + b_j y^2 \ (j = 1, 2),
\]

where $\alpha_k = \pi k/l \ (-l \leq x \leq l; 0 \leq y \leq h)$ for $j = 1$ and $(-l \leq x \leq l; -h \leq y \leq h)$ for $j = 2$. The coefficients $A_k^{(j)}, B_k^{(j)}, C_k^{(j)}, D_k^{(j)}, a_j$ and $b_j \ (j = 1, 2)$ are unknown constants to be determined.

We take into account the relations

\[
\begin{align*}
\sigma_x(x, y) &= \frac{\partial^2 F_j(x, y)}{\partial y^2}, & \sigma_y(x, y) &= \frac{\partial^2 F_j(x, y)}{\partial x^2}, & \tau_{xy}(x, y) &= -\frac{\partial^2 F_j(x, y)}{\partial x \partial y}, \\
\frac{E_j}{1 - \nu_j} U_j(x, y) &= \int \frac{\partial^2 F_j(x, y)}{\partial y^2} \ dx - \nu_j \frac{\partial F_j(x, y)}{\partial x} + a_0^{(j)} y + c_0^{(j)}, \\
\frac{E_j}{1 - \nu_j} V_j(x, y) &= \int \frac{\partial^2 F_j(x, y)}{\partial x^2} \ dy - \nu_j \frac{\partial F_j(x, y)}{\partial y} - a_0^{(j)} x + b_0^{(j)}
\end{align*}
\]

connecting the stress and displacement components with the stress function, and satisfy conditions (1), besides the last relations in (1a), and relations (2) to obtain the unknown coefficients in terms of the Fourier coefficients $V_j(x) \ (j = 1, 2)$. 

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The following is obtained:

\[
C_k^{(1)} = -\tanh \beta_k D_k^{(1)} + \frac{\nu_k^{(1)}}{\cosh \beta_k}, \quad C_k^{(2)} = \tanh \mu_k D_k^{(2)} + \frac{\nu_k^{(2)}}{\cosh \beta_k},
\]

\[
D_k^{(1)} = \frac{\nu_k^{(1)}(\theta_2 + \theta_1(1 - 2\nu_1)) - 2\theta_1 \nu_k^{(2)}(1 - \nu_2) + (\theta_2 - \theta_1) \nu_k^{(3)}}{\Delta_k^{(1)} \sinh \beta_k},
\]

\[
D_k^{(2)} = \frac{-2\theta_1 \nu_k^{(1)}(1 - \nu_1) + \nu_k^{(2)}(\theta_1 + \theta_2(1 - 2\nu_2)) + (\theta_2 - \theta_1) \nu_k^{(4)}}{\Delta_k^{(2)} \sinh \beta_k},
\]

\[
A_k^{(1)} = A_k^{(2)} = -\left[ \frac{\beta_k}{\sinh(2\beta_k)} + \frac{1}{2} \right] (D_k^{(1)} + D_k^{(2)}) + \frac{\nu_k^{(1)} - \nu_k^{(2)} + \nu_k^{(3)} + \nu_k^{(4)}}{2 \sinh \beta_k},
\]

\[
B_k^{(1)} = -A_k^{(1)} \tanh \beta_k \frac{\beta_k D_k^{(1)}}{\cosh^2 \beta_k} + \frac{\nu_k^{(3)}}{\cosh \beta_k}, \quad B_k^{(2)} = A_k^{(2)} \tanh \beta_k + \frac{\beta_k D_k^{(2)}}{\cosh^2 \beta_k} - \frac{\nu_k^{(4)}}{\cosh \beta_k},
\]

\[
b_0^{(j)} = 2h(-1)^j \frac{(1 - 2\nu_j)}{(1 - \nu_j)^2} a_j + \frac{E_j}{(1 - \nu_j^2)}, \quad a_1 = a_2 = \frac{(V_0^{(1)} - V_0^{(2)})}{2h(m_1 + m_2)}, \quad b_j = \frac{\nu_j a_j}{1 - \nu_j},
\]

\[
a_0^{(j)} = c_0^{(j)} = 0 \quad (j = 1, 2),
\]

where

\[
\Delta_k^{(1)} = \frac{2(\theta_2 - \theta_1)\beta_k}{\sinh(2\beta_k)} + \Delta_1, \quad \Delta_k^{(2)} = \frac{2(\theta_2 - \theta_1)\beta_k}{\sinh(2\beta_k)} - \Delta_2, \quad \Delta_1 = \theta_2 + \theta_1 \nu_1, \quad \Delta_2 = \theta_1 + \theta_2 \nu_2,
\]

\[
\nu_k^{(j)} = \frac{V_k^{(j)}}{2\theta_j \alpha_k(1 - \nu_j)} \quad (j = 1, 2), \quad \nu_k^{(j)} = (-1)^j(\beta_k \tanh \beta_k + 1)\nu_k^{(j-2)} \quad (j = 3, 4),
\]

\[
m_j = \frac{\theta_j(1 - 2\nu_j)}{(1 - \nu_j)}, \quad \beta_k = \frac{\pi k}{l}, \quad \theta_j = \frac{1 + \nu_j}{E_j}, \quad \nu_j = 3 - 4\nu_j, \quad V_0^{(j)} = \frac{1}{2l} \int_{L_j} V_j(x) \, dx,
\]

\[
V_k^{(j)} = \frac{1}{l} \int_{L_j} V_j(x) \cos(\alpha_k x) \, dx \quad (j = 1, 2; k = 1, 2, \ldots).
\]

Then, substituting the obtained expressions for the coefficients into (3) and taking (4) into account, we can express the stress function, and hence all values characterizing the stress state in the piecewise homogeneous rectangle in terms of the functions \(V_j(x)\) \((j = 1, 2)\).

In particular, the following expressions are obtained for the components of stresses acting on the crack edges:

\[
\sigma_y^{(1)}(x, h) = -\frac{1}{2l(1 - \nu_1) \theta_1} \left[ \int_{L_1} \cot \frac{\pi(s - x)}{2l} V_1'(s) \, ds + \int_{L_1} K_{11}(s - x) V_1'(s) \, ds \right]
\]

\[
+ \frac{1}{2l(1 - \nu_2) \theta_2} \int_{L_2} K_{12}(s - x) V_2'(s) \, ds + 2a_1 \quad (-l < x < l),
\]

\[
\sigma_y^{(2)}(x, -h) = \frac{1}{2l(1 - \nu_2) \theta_2} \left[ \int_{L_2} \cot \frac{\pi(s - x)}{2l} V_2'(s) \, ds + \int_{L_2} K_{22}(s - x) V_2'(s) \, ds \right]
\]

\[
+ \frac{1}{2l(1 - \nu_1) \theta_1} \int_{L_1} K_{21}(s - x) V_1'(s) \, ds + 2a_1 \quad (-l < x < l),
\]

\[
K_{ij}(x) = \frac{1}{l} \sum_{k=1}^{\infty} A_k^{(i,j)} \sin(\alpha_k x) \quad (i, j = 1, 2),
\]
\[
\begin{align*}
A_{k}^{(1,1)} &= \left\{ \left( \beta_k \tanh \beta_k + \beta_k \coth \beta_k + 1 \right) \left[ 2\vartheta_1 (1 - \nu_1) - (\vartheta_2 - \vartheta_1) \tanh \beta_k \right] + \frac{2(1 - \nu_1) \vartheta_1}{\Delta_k^{(1)} \sinh(2\beta_k)} \left[ \frac{2\beta_k}{\sinh(2\beta_k)} - \frac{\beta_k}{2 \cosh^2 \beta_k} + \tanh \beta_k - 1 \right] \right\}, \\
A_{k}^{(1,2)} &= \left\{ \left( 2\vartheta_2 (1 - \nu_2) (\beta_k \tanh \beta_k + \beta_k \coth \beta_k + 1) \right) \Delta_k^{(1)} \sinh(2\beta_k) \\
&\quad - \frac{2(1 - \nu_2) \vartheta_2 (\vartheta_2 - \vartheta_1) \beta_k \tanh \beta_k}{\sinh(2\beta_k)} \left[ \frac{2\beta_k}{\sinh(2\beta_k)} + 1 \right] - \frac{\beta_k}{2 \cosh^2 \beta_k} + \tanh \beta_k - 1 \right\}; \\
A_{k}^{(2,2)} &= \left\{ - \left( \beta_k \tanh \beta_k + \beta_k \coth \beta_k + 1 \right) \left[ 2\vartheta_2 (1 - \nu_2) + (\vartheta_2 - \vartheta_1) \tanh \beta_k \right] \Delta_k^{(2)} \sinh(2\beta_k) \\
&+ \frac{2(1 - \nu_2) \vartheta_2}{\Delta_k^{(1)} \sinh(2\beta_k)} \left[ \frac{2\beta_k}{\sinh(2\beta_k)} + 1 \right] - \frac{\beta_k}{2 \cosh^2 \beta_k} + \tanh \beta_k - 1 \right\}; \\
A_{k}^{(2,1)} &= \left\{ \left( 2\vartheta_1 (1 - \nu_1) (\beta_k \tanh \beta_k + \beta_k \coth \beta_k + 1) \right) \Delta_k^{(2)} \sinh(2\beta_k) \\
&- \frac{2(1 - \nu_1) \vartheta_1 (\vartheta_2 - \vartheta_1) \beta_k \tanh \beta_k}{\sinh(2\beta_k)} \left[ \frac{2\beta_k}{\sinh(2\beta_k)} + 1 \right] - \frac{\beta_k}{2 \cosh^2 \beta_k} \right\}.
\end{align*}
\]

Using the obtained expressions we satisfy the last relations in (1a). After some simplifications, the following system of singular integral equations of the first kind is obtained to determine the unknown derivatives of the functions \( V_j(x) \) \((j = 1, 2)\):

\[
\begin{align*}
\frac{1}{\pi} \int_{L_1} \frac{V_1'(s) \, ds}{s - x} + \int_{L_2} R_{11}(s - x) V_1'(s) \, ds &+ \int_{L_1} R_{12}(s - x) V_2'(s) \, ds = f_1(x) \quad (x \in L_1), \\
\frac{1}{\pi} \int_{L_2} \frac{V_2'(s) \, ds}{s - x} + \int_{L_1} R_{21}(s - x) V_1'(s) \, ds &+ \int_{L_2} R_{22}(s - x) V_2'(s) \, ds = f_2(x) \quad (x \in L_2).
\end{align*}
\]

This system should be considered under the continuity conditions for the displacements at the crack end-points

\[
\int_{a_j}^{b_j} V_1'(s) \, ds = 0, \quad \int_{c_j}^{d_j} V_2'(s) \, ds = 0 \quad (j = 1, N; i = 1, M).
\]

Here

\[
\begin{align*}
f_j(x) &= (1 - \nu_j) \vartheta_j P_j^{(j)}(x) \quad (j = 1, 2), \\
R_{ij}(t) &= \frac{1}{2l} \cot \frac{\pi t}{2l} - \frac{1}{\pi t} + \frac{1}{2l} K_{ij}(t) + \frac{2(1 - \nu_j) \vartheta_j t}{hl(m_1 + m_2)} \quad (j = 1, 2), \\
R_{ij}(t) &= (-1)^j \frac{m_{12}}{2l} K_{ij}(t) - (-1)^j \frac{2(1 - \nu_i) \vartheta_i t}{hl(m_1 + m_2)} \quad (i, j = 1, 2; i \neq j).
\end{align*}
\]

Note that, in the general case, system (7) with conditions (8) can be represented as a system of \( N + M \) equations for the displacement dislocations on each of the intervals \((a_j, b_j) \quad (j = 1, M)\) and \((c_j, d_j) \quad (j = 1, N)\). This system can be solved either by using the apparatus of Chebyshev orthogonal polynomials reducing it to a quasi-complete regular system of infinite algebraic equations [7] or by the method of mechanical quadratures [8].
2. A piecewise homogeneous plane with a doubly-periodic system of internal cracks located in one of the repeated strips

As an example, we consider the case where there is a periodically repeated crack of length $2a$ occupying the interval $(a_1, b_1) = (-a, a)$ on the boundary $y = h$ of the basic cell on whose edges the symmetric normal stresses of intensity $P_0(x)$ are acting, i.e., $M = 1, N = 0$ and $P_0^{(1)}(x) = P_0(x)$.

In this special case, the following integral equation is obtained from system (7):

$$
\frac{1}{\pi} \int_{-a}^{a} \frac{V'_1(s)}{s-x} ds + \int_{-a}^{a} R_{11}(s-x)V'_1(s) ds = f_1(x) \quad (|x| < a)
$$

under the condition

$$
\int_{-a}^{a} V'_1(s) ds = 0.
$$

The solution of the equation is constructed by the method of mechanical quadratures [8]. For this, after the change $s = a\xi$ and $x = a\eta$, the equation is considered on the interval $(-1, 1)$. We use the notation

$$
V'_1(\eta a) = \varphi(\eta), \quad aR_{11}(a\eta) = R(\eta), \quad f_1(\eta a) = f_*(\eta)
$$

to write the equation in the form

$$
\frac{1}{\pi} \int_{-1}^{1} \frac{\varphi(\xi)}{\xi - \eta} d\xi + \int_{-1}^{1} R(\xi - \eta)\varphi(\xi) d\xi = f_*(\eta).
$$

Condition (10) becomes

$$
\int_{-1}^{1} \varphi(\xi) d\xi = 0.
$$

It is not difficult to state that the unknown function at the end-points of the interval of integration has a root singularity, which can be represented as

$$
\varphi(\eta) = \frac{\varphi^*(\eta)}{\sqrt{1 - \eta}(1 + \eta)}.
$$

Here $\varphi^*(\eta)$ is a continuous smooth function bounded up to the ends of the interval $[-1, 1]$.

Substituting the function $\varphi(\eta)$ from (13) into (11), (12) and using relations from [10], we standardly obtain a system of algebraic equations for the values $\varphi^*(\xi_i) (i = 1, n)$, where $\xi_i$ are roots of the Chebyshev polynomial of the first kind $T_n(\xi_i) = 0$.

Using the Lagrange interpolation polynomial, we can easily reconstruct the function $\varphi^*(\eta)$ $(-1 < \eta < 1)$ and determine all necessary values characterizing the stress state in the two-component cell. In particular, the stress intensity factor at the crack end-points is determined by formula (5), which has the following form for $|\eta| > 1$ in the new notation:

$$
\sigma_y^{(1)}(a\eta, h) = \frac{1}{\vartheta_1(1 - \nu_1)} \left[ \frac{1}{\pi} \int_{-1}^{1} \frac{\varphi(\xi)}{\xi - \eta} d\xi + \int_{-1}^{1} R(\eta - \xi)\varphi(\xi) d\xi \right] \quad (|\eta| > 1).
$$

Substituting $\varphi(\eta)$ from (13) and using the relation [9]

$$
\frac{1}{\pi} \int_{-a}^{a} \frac{ds}{\sqrt{a^2 - s^2}(s-x)} = -\frac{\text{sign}(x)}{\sqrt{x^2 - a^2}} \quad (x < a; x > b)
$$
for the dimensionless stress factor at the crack end-points, we obtain

\[ K_1^*(\pm a) = \frac{K_1(\pm a)}{E_1} = \sqrt{\frac{2\pi}{\eta \pm 1 \pm 0}} \sqrt{\left| \frac{\sigma_y^{(1)}(a\eta,0)}{E_1} \right|} \pm \sqrt{\frac{\pi}{\nu_1^2(1)}}. \]

The following formula is obtained to determine the dimensionless crack opening:

\[ V_*(\eta) = \frac{V_1(a\eta)}{a} = \int_{-1}^\eta \varphi(\xi) \, d\xi. \]

3. Numerical analysis and conclusions
The numerical analysis of above-stated problem was carried out for \( P_0(x) = P_0 = \text{const} \). The laws of changes in the dimensionless stress intensity factor at the crack end-points and the crack opening depending on the Young moduli ratio \( \mu = E_2/E_1 \) of a separate strip were studied. The following values of Poisson’s ratios and geometrical characteristics of the strips were taken: \( \nu_1 = 0.28, \nu_2 = 0.3, \lambda_1 = a/l = 0.2, \lambda_2 = h/l = 0.5, \) and \( E_0 = P_0/E_1 = 0.1 \).

Figures 1 and 2 show the graphs of the crack opening \( V_*(\eta) \) and the dimensionless intensity stress factor \( K_1^*(a) \) versus the parameter \( \mu \).

It follows from these graphs that both the crack opening and the stress intensity factor at the crack end-points decrease tending to a certain limit as the parameter \( \mu \) increases.
Conclusions
The governing system of equations of elasticity for uniform piecewise homogeneous plane consisting of two periodically repeated distinct strips with doubly periodic internal cracks is obtained. In the general case, the ways of solving the problem are indicated. Numerical-analytical solutions of the equations are obtained in the special case where only one of the strips contains a periodically repeated crack of length \(2a\). It is shown that as the rigidities of the strips without cracks increase, while the other parameters remain unchanged, the crack opening and the stress intensity factors at the crack end-points decrease and tend to the values corresponding to the case where the strips with no cracks are absolutely rigid.

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