Charge pairing, superconducting transition and supersymmetry in high-temperature cuprate superconductors

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We propose a model for high-\(T_c\) superconductors, valid for \(0 \leq \delta \leq \delta_{\text{SC}}\), that includes both the spin fluctuations of the \(\text{Cu}^{++}\) magnetic ions and of the \(\text{O}^{--}\) doped holes. Spin-charge separation is taken into account with the charge of the doped holes being associated to quantum skyrmion excitations (holons) of the \(\text{Cu}^{++}\) spin background. The holon effective interaction potential is evaluated as a function of doping, indicating that Cooper pair formation is determined by the competition between the spin fluctuations of the \(\text{Cu}^{++}\) background and of spins of the \(\text{O}^{--}\) doped holes (spinons). The superconducting transition occurs when the spinon fluctuations dominate, thereby reversing the sign of the interaction. At this point \((\delta = \delta_{\text{SC}})\), the theory is supersymmetric at short distances and, as a consequence, the leading order results are not modified by radiative corrections. The critical doping parameter for the onset of superconductivity at \(T = 0\) is obtained and found to be a universal constant determined by the shape of the Fermi surface. Our theoretical values for \(\delta_{\text{SC}}\) are in good agreement with the experiment for both LSCO and YBCO.

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Introduction. High-temperature superconducting cuprates have a very rich and complex phase diagram whose understanding is an important issue. In the underdoped region, for instance, a wide variety of physical phenomena like Néel and metal-insulator transitions, transport (non-Fermi liquid) anomalies, the occurrence of a spin-pseudogap, absence of a sharp quasiparticle peak (spin-charge separation), etc., have inspired a large amount of theoretical and experimental work for about fifteen years [1]. In spite of that, even the nature of the ground state and of its elementary excitations has not yet been fully determined and many different pictures are available, ranging from a dimerized ground state with spin-Peierls or valence-bond order [2] until the so called staggered-flux (\(d\)-wave) phase [3].

Another fundamental point yet to be understood is the mechanism of charge pairing leading to the superconducting transition. In connection to this, we stress that it is by now well established that antiferromagnetic spin correlations play an important role in the dynamics of the system, even after the destruction of the Néel state. Indeed, different spin-fluctuation models have been successfully used to explain the observed spectral weight in ARPES data in the quantum disordered phase of high-\(T_c\) materials [4], as well as other anomalies [5]. Moreover, the idea of spin-fluctuation induced charge pairing and superconductivity has been used recurrently [6].

In this work we propose a model for high-\(T_c\) cuprates valid for dopant concentrations ranging from zero up to the superconducting transition, \(0 \leq \delta \leq \delta_{\text{SC}}\), that takes into account the spin fluctuations of the \(\text{Cu}^{++}\) magnetic ions and of the \(\text{O}^{--}\) doped holes on different footing, as suggested by the different temperature dependences of the NMR relaxation rates for the Oxygen and Copper spins [7]. Our model also incorporates spin-charge separation [8] as follows. The charge of the dopants introduced in the \(\text{CuO}_2\) planes is associated to skyrmion quantum spin excitations of the \(\text{Cu}^{++}\) background (holons) which, in the Néel phase appear as finite energy defects closely related to their classic counterparts, whereas in the quantum disordered phase are nontrivial zero energy purely quantum mechanical excitations. The spin of the doped holes, on the other hand, is represented by chargeless, massless Dirac fermion fields (spinons) [9]. We then calculate the effective interaction potential between the quantum skyrmion topological excitations, as a function of doping, in order to study charge pairing. It becomes clear that Cooper pair formation at zero temperature is controlled by the competition between two different contributions to the quantum skyrmion effective interaction energy; one coming from the spin fluctuations of \(\text{Cu}^{++}\) magnetic ions and the other from the corresponding fluctuations of the spins of the doped holes (spinons). The superconducting transition occurs when the latter dominates, thereby reversing the sign of the effective quantum skyrmion (dopant charge) interaction potential, at short distances, from repulsive to attractive. Interestingly, the model becomes supersymmetric at short distances, precisely at the onset of superconductivity, thereby making our one-loop results robust against radiative corrections, in accordance to Witten’s theorem [10]. The critical value of the doping parameter for the superconducting transition at zero temperature, \(\delta_{\text{SC}}\), is universally determined by the shape of the Fermi surface and is in good agreement with experiment for both \(\text{La}_{2-x}\text{Sr}_x\text{CuO}_4\) and \(\text{YBa}_2\text{Cu}_3\text{O}_{6+x}\) compounds. The model also correctly
predicts the zero temperature magnetization curves as a function of the critical doping in the AF ordered phase.

The model. In previous works [1], we have proposed a model for doping quantum Heisenberg antiferromagnets that successfully described the magnetization curves and the AF part of the phase diagrams of the two best studied high-$T_c$ compounds, namely LSCO and YBCO. One of the important consequences of that model is the observation that each hole added to the CuO$_2$ planes creates a skyrmion topological defect, as has been proposed earlier [2]. The dopant charge, in particular, is attached to the skyrmion charge and its dynamics becomes totally determined by the quantum skyrmion correlation functions. In the Néel ordered Mott insulating phase, the skyrmions have a finite excitation energy and this reflects the existence of a gap for charge that can be associated to the breakdown of translational invariance of the lattice. The model proposed in [11], however, is restricted to the antiferromagnetic part of the phase diagram, where $\delta \leq \delta_{AF}$. Nevertheless, we shall pursue the picture in which skyrmions are in general the charge carriers of the doped holes. In particular, we shall exploit this idea in the quantum disordered phase, $\delta_{AF} \lesssim \delta \lesssim \delta_{SC}$, where the skyrmions are purely quantum mechanical and have zero energy. This is again consistent with charge response experiments, like in-plane optical conductivity [3], which indicates the absence of a gap for charge excitations in the quantum disordered metallic phase.

Let us consider the zero temperature Euclidean partition function

$$Z = \int \mathcal{D}z \mathcal{D}\bar{z} \mathcal{D}\bar{\psi} \mathcal{D}\psi \mathcal{D}A_\mu \delta[\bar{z}z - 1]e^{-S(\bar{z}, z, \bar{\psi}, \psi, A_\mu)}$$

where

$$S(\bar{z}, z, \bar{\psi}, \psi, A_\mu) = \int d^3x \left\{ \frac{1}{g_0} \left| (\partial_\mu - iA_\mu) z_1 \right|^2 + \bar{\psi} \gamma_\mu (i\partial^\mu + qA^\mu) \psi_1 \right\}$$

and we use units in which $\hbar = c = 1$. In the above expression $z_1^i, z_i, i = 1, 2,$ are Schwinger boson fields related to the local spin density of Cu$^{++}$ ions through $S = z_1^i \sigma_j z_j$, and $\psi_1^i, \psi_1^a, a = \uparrow, \downarrow$, are chargeless 2-component Dirac spinor fields (spinons) describing the local spin density of the doped O$^{2-}$ holes through $S_h = \psi_1^i \sigma_{ab} \psi_2$. As usual, $A_\mu$ is the Hubbard-Stratonovich field and $g_0$ the bare coupling constant of the CP$^1$ model. The constant $q$ measures the strength of the coupling of spinons.

Spin-charge separation is manifested in our model through the fact that the massless Dirac fermions carry the spin of the doped holes, whereas all information about their charge is carried by the quantum skyrmion excitations (holons) created out of the Schwinger boson background [11]. The description of spinons as massless Dirac fields arises naturally in the continuous limit of microscopic models [1]. The full treatment of the quantum skyrmions of the theory described by [2], on the other hand, has been carried out in [11].

Before we proceed, it is important to determine how the doping dependence will be introduced in our theory. As we explained above, we may identify in principle, at least two phases in the model given by [11], at $T = 0$. An ordered Néel phase, $g_0 < g_c$, for which there is a nonzero spin stiffness $\rho_s = 1/g_0 - 1/g_c > 0$ (where $g_c$ is the quantum critical coupling) and a quantum disordered phase, $g_0 > g_c$, in which the Schwinger bosons $z_i$ acquire a mass $m^2 \propto \left[1/g_0 - 1/g_c\right]$, and $\rho_s = 0$. As we explained in [11], the whole dynamics of the in-plane dopant charge is identical the quantum dynamics of skyrmions. The quantum skyrmion correlation function corresponding to [3] has been evaluated in [13], in the ordered phase, giving

$$\langle \mu(x) \mu^\dagger(y) \rangle = \frac{e^{-2\pi \rho_s |x - y|}}{|x - y|^{\pi/2}},$$

where $\mu^\dagger$ is the quantum skyrmion creation operator. Conversely, for the theory studied in [11] the corresponding correlator was found to be

$$\langle \mu(x) \mu^\dagger(y) \rangle = \frac{e^{-2\pi \rho_s |x - y|}}{|x - y|^{\alpha(|\delta|)}},$$

where the expressions for $\rho_s(\delta)$ and $\alpha(\delta)$ have been determined in [11]. In particular,

$$\alpha(\delta) = \frac{64}{\pi^2 + 16} \frac{\alpha_{EM}}{4\pi^2} (n\delta)^2$$

with $n = 1$ for YBCO and $n = 4$ for LSCO, the factor of four being a consequence of the existence of four branches in the Fermi surface for this compound, see discussion in [11]. The $\rho_s(\delta)$ function is given by $\rho_s(\delta) = \rho(0)\left[1 - A\delta^2\right]$, for YBCO and $\rho_s(\delta) = \rho(0)\left[1 - B\delta - C\delta^2\right]^{1/2}$, for LSCO, and again the different behavior being ascribed to the form of the Fermi surface in each case [11]. The constants $A, B$ and $C$ have been evaluated from first principles in [11]. In [3], $\alpha_{EM}$ is the electromagnetic fine structure constant and accounts for the contribution of the electromagnetic interaction of the doped holes to the skyrmion correlation function. Examining [3] we see that actually this term can be neglected when compared to the first one. In order to obtain the $\delta$-dependence of the spin stiffness $\rho_s$ and of the spinon coupling $q$ in our model (2), we now match the two correlation functions in [3] and [11] (ordered phase), obtaining $\rho_s = \rho(\delta)$ and $q = [256/(\pi^2 + 16)]^{1/2}(n\delta)$, where we have already neglected the electromagnetic part. We immediately conclude that the sublattice magnetization in the ordered phase is given by $M(\delta) = \sqrt{\rho(\delta)}$. From this we can readily obtain $\delta_{AF}$ from $\rho(\delta_{AF}) = 0$, see also [11]. For $\delta > \delta_{AF}$, on the other hand, where $\rho_s = 0$, we assume that the expression for $q(\delta)$ still holds.

Cooper pair formation. Let us now investigate the conditions for skyrmion pairing and consequent formation of
Cooper pairs, by analyzing the effective interaction potential between quantum skyrmions in the quantum disordered underdoped phase. For this purpose, we introduce the skyrmion current $J^\mu = \frac{1}{\pi^2} \epsilon^{\mu\nu\alpha\beta} \partial_\alpha A_\beta$ through the identity

$$Z = \int \mathcal{D}J_\mu \mathcal{D}A_\mu \mathcal{D}z \mathcal{D}\psi \mathcal{D}\bar{\psi} \delta[J_\mu - \frac{1}{2\pi} \epsilon^{\mu\nu\alpha\beta} \partial_\alpha A_\beta] \times e^{-S[\hat{z},\hat{\psi},\bar{\psi},A_\mu]},$$  

(6)

Integrating over $\hat{z}, \hat{\psi}$ and $\bar{\psi}, A_\mu$, we obtain, at one-loop level, the effective action

$$S_{\text{eff}}[A_\mu] = \int d^3x \int d^3y \left\{ \frac{1}{4} F_{\mu\nu}(x) \Sigma(x - y) F_{\mu\nu}(y) \right\}.$$

(7)

where the kernel $\Sigma(x - y)$ has Fourier transform given by

$$\Pi_B(p) = \frac{1}{2\pi} \left[ \frac{m}{p^2} - \frac{1}{2p} \arctan \left( \frac{p}{m} \right) - \frac{2m^2}{p^2} \arctan \left( \frac{p}{2m} \right) \right],$$

(8)

and

$$\Pi_F(p) = \frac{1}{2\pi} \left[ \frac{q^2}{p^2} \right].$$

(9)

These two terms are, respectively, the contributions to the vacuum polarization coming from the complex scalar fields $z_i$ (Schwinger bosons) and fermions $\psi$ (spinons). In (8), $m$ is the mass of the $z_i$-fields (spin-gap) in the quantum disordered phase, where $\delta > \delta_{AF} (g_0 > g_c)$.

In order to obtain the effective skyrmion action, we use an exponential representation for the $\delta$-function in (8), and integrate over the corresponding Lagrange multiplier field $A_\mu$. The result is

$$Z = \int \mathcal{D}J_\mu \mathcal{D}A_\mu \left\{ \frac{e^{-i \mu^2}}{2\pi} \int d^3x \int d^3y \mathcal{J}_\mu(x) \Sigma_{\mu\nu}(x - y) \mathcal{J}_\nu(y) \right\},$$

(10)

where the Fourier transform of the kernel is given by

$$\Sigma_{\mu\nu}(p) = \frac{\Sigma(p)}{p^2} \left( p^2 \delta_{\mu\nu} - p^\mu p^\nu \right).$$

(11)

From expression (11), we can readily obtain the effective interaction energy between static skyrmions. This is given in real time, by

$$\mathcal{H}_I = -\frac{1}{(2\pi)^2} \int d^2x \int d^2y \rho(x) K(x - y) \rho(y),$$

(12)

where $\rho(x) = \mathcal{J}_0(x,0)$ is the dopant charge density and

$$K(x - y) = \int d^2p / (2\pi)^2 \, e^{i p \cdot (x - y)} \Sigma(p,0).$$

For two charges at positions $x_1$ and $x_2$, we have $\rho(x) = \delta^2(x - x_1) + \delta^2(x - x_2)$. After discarding self-interactions, we obtain the effective interaction potential for static charges, namely

$$V(x_1 - x_2) = \int d^2p \, \Sigma(p,0) \, e^{i p \cdot (x_1 - x_2)},$$

(13)

where

$$\Sigma(p,0) = -\frac{q^2}{8|p|} + \frac{1}{2\pi} \left[ \frac{1}{2|p|} \arctan \left( \frac{|p|}{m} \right) - \frac{m}{|p|^2} \right]$$

$$+ \frac{2m^2}{|p|^3} \arctan \left( \frac{|p|}{2m} \right).$$

(14)

It is well known that in high-T$_c$ superconducting cuprates, Cooper pairs form in such a way that the two charges are localized in space. Only the short distance behavior of the interaction potential, therefore, is relevant for Cooper pair formation. In this limit (large $|p|$) we have

$$V(x_1 - x_2) \to \int d^2p \left[ \frac{1}{8|p|} - \frac{q^2}{8|p|^2} \right] e^{i p \cdot (x_1 - x_2)}.$$

(15)

The first contribution inside the square brackets in the above expression corresponds to the Cu$^{++}$ spin fluctuations (Schwinger bosons) while the second corresponds to fluctuations from O$^{++}$ spins (spinons). We see that for small doping, $q^2 < 1$, the potential is always repulsive and there is no charge pairing. For $q^2 > 1$, on the other hand, the interaction potential becomes attractive and charge (skyrmion) pairing occurs. Consequently, we conclude that the critical doping for the onset of superconductivity is determined by the condition

$$q^2(\delta_{SC}) = 1.$$  

(16)

Let us remark that if we had considered a system of spinons solely, without including the Cu$^{++}$ background, we would have obtained that the interaction potential (13) would always be attractive for any $q \neq 0$, at zero temperature, and $\delta_{SC} = 0$. This is what happens in the mean field phase diagram of Kotliar and Liu for the $t - J$ model [15]. We see that the primer effect of considering the Cu$^{++}$ spin background is to shift the value of $\delta_{SC}$ to the right of the phase diagram, which is actually what is observed experimentally.

Comparison with experiment. From the expression of $q$ in terms of $\delta$ (see [10]), we may infer that $\delta_{SC}$ is an universal constant, only determined by the shape of the Fermi surface. We see, in particular, that $\delta_{SC}^{BOCO} = 4\delta_{SC}^{YBCO}$, a relation that is verified by experiments, if we take in account the relation between $\delta$

\[3\]
and the stoichiometric doping parameter \( x \), namely \( \delta = x \) for LSCO and \( \delta = x - 0.20 \) for YBCO. Another prediction of our model is that compounds with similar Fermi surfaces should have the same superconducting critical doping \( \delta_{SC} \). From (4) and (5), we calculate \( \delta_{YBCO}^{SC} = 0.318 \) and \( \delta_{LSCO}^{SC} = 0.079 \), which have a fairly good agreement with experiment. We show below that taking in account the presence of disorder, we can obtain better values for these critical doping parameters.

**Disorder.** Disorder may be modelled in the ordered Néel phase of a doped antiferromagnet by considering a continuous random distribution of spin stiffnesses \( \psi_i \). If we introduce a Gaussian \( |\rho - \rho_i|^{-\nu} \) distribution, with exponentially suppressed magnetic dilution, in the original model \( \psi_i \) used to describe the antiferromagnetic phase, we obtain a correction for (4), namely \( \alpha'(\delta) = \alpha(\delta) + \nu \). Choosing \( \nu = \frac{1}{4} \) for both compounds, we get

\[
\delta_{SC} = \frac{1}{n} \sqrt{\pi^2 + 16} \frac{512}{n^2}
\]

(17)

Now, the critical doping parameters at \( T = 0 \) become \( \delta_{YBCO}^{SC} = 0.225 \) and \( \delta_{LSCO}^{SC} = 0.056 \), corresponding to \( x_{YBCO} = 0.425 \) and \( x_{LSCO} = 0.056 \), which are in good agreement with experiment. Notice that with a single choice for the disorder distribution we correctly obtained the critical dopings for both YBCO and LSCO.

**Supersymmetry.** Let us observe now a remarkable fact. At short distances, when the mass of the Schwinger bosons, \( m \), may be neglected, our model becomes supersymmetric precisely at the point \( q(\delta_{SC}) = 1 \), where the superconducting transition occurs. Supersymmetry relates Schwinger bosons \( z_i \) and spinons \( \psi_a \) and that is why the contributions of both to the holon (skyrmion) interaction potential are identical but with opposite signs at this point. An important consequence is that our one-loop derivation of the holon effective interaction potential and critical dopings are unchanged by radiative corrections. Indeed, we have actually checked that the contribution of these corrections to the short distance behavior of the effective holon interaction potential (15) is subdominant and can be neglected. This can be understood on general grounds, as a result of Witten’s theorem [10], which states that supersymmetry cannot be broken perturbatively.

**Conclusions.** We have calculated the effective interaction potential between holons in a spin-charge separated, spin fluctuation model for high-Tc cuprates. We have shown that Cooper pair formation and superconductivity is determined by the competition between the spin fluctuations of the Cu\(^{++}\) antiferromagnetic background and the spin fluctuations of the doped O\(^{--}\) holes, in the underdoped regime. Our prediction of the critical doping for the onset of superconductivity at zero temperature, \( \delta_{SC} \), is in good agreement with experiment for either LSCO and YBCO compounds. We stress that the pairing must be between skyrmions and not between skyrmion and anti-skyrmion, so that the total electric charge of the pair is nonzero.

At finite temperatures, the pairing shall no longer occur for \( q^2(\delta) = 1 \). There will be finite temperature corrections for both \( \Pi_B \) and \( \Pi_F \) and, since supersymmetry is broken at any finite temperature, the fermionic and bosonic contributions for the interaction potential should cancel at \( q \neq 1 (\delta \neq \delta_{SC}) \), in agreement with experimental results. We are presently investigating this point.

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