Asymptotic upper bound on prime gaps

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Abstract

The Cramér-Granville conjecture is an upper bound on prime gaps, \( g_n = p_{n+1} - p_n < c \log^2 p_n \) for some constant \( c \geq 1 \). Using a formula of Selberg, we first prove the weaker summed version:

\[ \sum_{n=1}^{N} g_n < \sum_{n=1}^{N} \log^2 p_n. \]

In the remainder of the paper we investigate which properties of the fluctuations \( f(x) = \pi(x) - \text{Li}(x) \) would imply the Cramér-Granville conjecture is true and present two such properties, one of which assumes the Riemann Hypothesis; however we are unable to prove these properties are indeed satisfied. We argue that the conjecture is related to the enormity of the Skewes number.

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Let \( p_n \) denote the \( n \)-th prime number with \( p_1 = 2 \), and define the gaps

\[
g_n = p_{n+1} - p_n
\]

The lowest gap is obviously equal to 2, however an interesting question is how often this minimal gap occurs, and the twin primes conjecture says an infinite number of times. There has recently been progress on this problem by Yitang Zhang \[1\]. In the other direction, upper bounds on gaps are also of interest. In practice the latter are more important since, given knowledge of a prime, they can aid in the location of the next prime.

The Prime Number Theorem leads to \( p_n \approx n \log n \), which implies the average gap \( g_n \) is \( \log n \). However it is known that the maximal gaps grow faster than this \[2\]:

\[
\limsup_{n \to \infty} \frac{p_{n+1} - p_n}{\log p_n} = \infty
\]

The Cramér-Granville conjecture \[3, 4\] is the statement

\[
g_n < c \log^2 p_n, \quad \forall n
\]

for some constant \( c \geq 1 \). Cramér’s model supports the conjecture with \( c = 1 \), whereas Granville proposed that \( c > 1 \) and suggested \( c \geq 2e^{-\gamma} \approx 1.1229 \ldots \). There is extensive numerical evidence for the conjecture \[3\]. Furthermore, thus far this evidence supports the value \( c = 1 \) at least for \( n > 4 \): the greatest known value of the ratio \( g_n / \log^2 p_n \) is 0.9206386... for the prime 1693182318746371, which is somewhere around the \( n = 5 \times 10^{13} \)-th prime.

As usual let \( \pi(x) \) denote the number of primes less than or equal to \( x \). Riemann derived an explicit formula for \( \pi(x) \) in terms of an infinite sum over zeros of \( \zeta(s) \) inside the critical strip \( 0 \leq \Re(s) \leq 1 \). The Prime Number Theorem (PNT) is the result that the leading term is

\[
\pi(x) = \sum_{p \leq x} 1 \approx \int_2^x \frac{dt}{\log t} = \text{Li}(x)
\]

It was proven independently by Hadamard and de la Vallée Poussin using Riemann’s formula for \( \pi(x) \) and showing that there are no zeros with \( \Re(s) = 1 \).

Cramér’s original conjecture is essentially based on the PNT. The Cramér model is a probabilistic model of the primes, in which one assumes that the probability of a natural number of size \( x \) being prime is \( 1/\log x \). Cramér proved that in this model, the conjecture holds true with probability one. Since there is a great deal of numerical evidence the conjecture is correct, this suggests that a proof of it may only require something slightly stronger.
than the PNT, and this will what we will investigate here. A result of this kind is due to Selberg \[6\]. The latter led to an independent, so-called elementary proof of the PNT, in that it does not rely on Riemann’s formula for $\pi(x)$. The starting point for this article is a formula of Selberg, namely Theorem 1 below.

The Cramér-Granville conjecture may be difficult, or even impossible, to prove for a single given $n$. On the other hand, if it is known to be true by direct computation for all $n < N_0$ for some high enough $N_0$, one can attempt to prove the conjecture with a bootstrap principle that extrapolates to infinity using some asymptotic formulas valid in the limit of large $N$. The asymptotic formulas must be such that the relative fluctuations decrease fast enough not to spoil the validity of the conjecture. This is in analogy to statistical physics of a system of $N$ particles, where the relative fluctuations are typically of order $1/\sqrt{N}$ so that results just get better and better as one increases $N$, i.e. in the so-called thermodynamic limit.

In the sequel, relative fluctuations will be quantified as follows. Since $\text{Li}(x)$ is the leading term,

$$\pi(x) = \text{Li}(x) + \mathcal{f}(x)$$

where the fluctuating term $\mathcal{f}(x)$ grows more slowly than the smooth part $\text{Li}(x) \approx x/\log x$, i.e. $\lim_{x \to \infty} \mathcal{f}(x)/\text{Li}(x) = 0$. We will consider different bounds on $\mathcal{f}(x)$ depending on whether one assumes the Riemann Hypothesis or not.

We begin with:

**Theorem 1.** (Selberg)

$$2x \log x + O(x) = \sum_{p \leq x} \log^2 p + \sum_{p, q \text{ with } pq \leq x} \log p \log q$$

where $p, q$ are primes.

Let us first make some simple observations based on the above theorem. Let $S_1(x)$ denote the first sum on the RHS of the above equation and $S_2(x)$ the second sum. $S_1$ includes terms up to $\log^2 x$, whereas $S_2$ contains terms with $p = q$ up to $\log^2 \sqrt{x}$, which are much smaller. There are additional terms in $S_2$ compared to $S_1$ for $p > q$, where the largest prime is
approximately $x/2$ corresponding to the term $\log(x/2)\log 2$, and they are also considerably smaller. This strongly suggests that $S_1(x) > S_2(x)$. One can easily check numerically that for all $x < p_{10^4} = 104729$, $S_1(x)$ is significantly larger than $S_2(x)$ and the difference increases with $x$. For instance, for $x = p_{10^4}$, $S_1(x) - S_2(x) = 686787.25...$ For larger $x$, the difference $S_1 - S_2$ only continues to grow. These considerations lead us to formulate the following lemma:

**Lemma 1.**

\[ \sum_{p, q \text{ with } pq \leq x} \log p \log q < \sum_{p \leq x} \log^2 p \quad (6) \]

**Proof.** For $x < p_{N_0}$ where $N_0 = 10^4$ for instance or even much smaller, the above inequality is easily verified by direct computation. For higher $x$, the relative fluctuations of $S_1$ and $S_2$ are very small compared to the difference $S_1 - S_2$. Thus, to go to higher $x$, one can use the following asymptotic formulas:

\[ \sum_{p \leq x} \log^2 p \approx \int_2^x \frac{dt}{\log t} \left[ \log^2 t \right] = x \log x - x - 2 \log 2 + 2 \quad (7) \]

\[ \sum_{p, q \text{ with } pq \leq x} \log p \log q \approx \int_2^x \frac{dt}{\log t} \int_2^{x/t} \frac{du}{\log u} \left[ \log t \log u \right] = x \log x - (2 + \log 2)x + 4 \]

Therefore $S_1(x) - S_2(x) \approx (1 + \log 2)x$ for large $x$, and this linear growth in $x$ is much larger than the fluctuations for large enough $x$. \( \square \)

**Theorem 2.** For large enough $N > N_0$ for some finite $N_0$, one has

\[ \sum_{n=1}^{N} g_n < \sum_{n=1}^{N} \log^2 p_n \quad (8) \]

**Proof.** From Lemma 1 and Theorem 1, one has

\[ x \log x + O(x) < \sum_{p \leq x} \log^2 p \quad (9) \]

Let $x = p_{N+1} - \epsilon$ with $\epsilon$ small and positive, and consider the limit $\epsilon \to 0$. Then (9) can be expressed as

\[ p_{N+1} \log p_{N+1} + O(p_{N+1}) < \sum_{n=1}^{N} \log^2 p_n \quad (10) \]
It should be kept in mind that the $O(p_{N+1})$ term can be negative here. Nevertheless, for large enough $N$, $p_{N+1} < p_{N+1} \log p_{N+1} + O(p_{N+1})$. Next, noting that $p_{N+1} = \sum_{n=1}^{N} g_n + 2$ proves the theorem.

Theorem 2 is consistent with the Cramér-Granville conjecture, and suggests $c = 1$, although it is also consistent with $c > 1$ since the fluctuations in the sum could just average out to give $c = 1$. Furthermore it is clearly not enough to establish it since it does not imply that each individual term in the sum satisfies the inequality. For the remainder of this article, we propose conditions on the fluctuations which would imply the Cramér-Granville conjecture, however we are unable to prove these conditions are true.

**Proposition 1.** Define

$$\Delta(x) = \sum_{p < x} \left( \log^2(p) - \frac{g(p)}{c} \right)$$

where $g(p_n) = g_n$. Then if $\Delta(x)$ with $c > 1$ is a monotonically increasing function of $x$ in a region $x_1 < x < x_2$ then the Cramér-Granville conjecture is true for all primes $p$ in the region $x_1 < p < x_2$.

**Proof.** Define the discrete function of $N$

$$D(N) \equiv \sum_{n=1}^{N} \left( \log^2 p_n - \frac{g_n}{c} \right)$$

Now $D(N)$ may be viewed as the above function $\Delta$ of $p_{N+1}$, i.e. $D(N) = \Delta(p_{N+1})$. Since $p_{N+1}$ is a monotonically increasing function of $N$, then if $\Delta(p)$ is a monotonically increasing function of $p$ for large enough $p$, then $D(N)$ is a monotonically increasing function of $N$ for large enough $N > N_0$. This in turn implies that for $N > N_0$, at each step in the sum, $N - 1 \to N$, one has $c \log^2 p_N > g_N$.

**Proposition 2.**

$$\Delta(x) = x \log x - \frac{(c + 1)}{c} x + \hat{h}(x) + O(1)$$

where

$$-\frac{Bx}{\log x} < \hat{h}(x) < \frac{Bx}{\log x}$$

for some constant $B \approx 5$. 

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Proof. One has the exact formula

$$\sum_{p < x} \log^2 p = \int_{2}^{x} d\pi(t) \log^2 t$$  \hspace{1cm} (15)$$

It is known that \[7, 8\]

$$\frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{1.8x}{\log^3 x} < \pi(x) < \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2.51x}{\log^3 x}$$  \hspace{1cm} (16)$$

for large enough \(x\); the first inequality requires \(x \geq 32299\) and the second \(x \geq 355991\). Thus we can simply write this as

$$\pi(x) = \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2x}{\log^3 x} + O\left(\frac{x}{\log^3 x}\right)$$  \hspace{1cm} (17)$$

where the first three terms come from the expansion of \(\text{Li}(x)\). The bound (17) is only valid for large enough \(x\). However by changing the constants one can obtain a bound valid for all \(x\):

$$\pi(x) = \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2x}{\log^3 x} + \hat{f}(x)$$  \hspace{1cm} (18)$$

where

$$-\frac{Bx}{\log^3 x} < \hat{f}(x) < \frac{Bx}{\log^3 x}$$  \hspace{1cm} (19)$$

for some constant \(B \approx 5\). We determined this value of \(B\) by verifying (19) is valid for \(x\) below the value where (17) becomes valid. Integrating (15) by parts, this implies (14).

Define the function \(b(x)\) such that

$$\hat{h}(x) = \frac{b(x)x}{\log x}$$  \hspace{1cm} (20)$$

where \(-B < b(x) < B\). One has

$$\frac{d\Delta}{dx} = \log x - \frac{1}{c} + \frac{b(x)}{\log x} \left(1 - \frac{1}{\log x}\right) + \frac{b'(x)x}{\log x}$$  \hspace{1cm} (21)$$

where \(b'(x)\) is formally \(db(x)/dx\). Due to the jump discontinuities of \(b(x)\) where the derivative is not defined, one needs to formally define \(b'(x)\). On a prime \(p_n\) let us define it as follows:

$$b'(p_n) = \frac{b(p_{n+1}) - b(p_n)}{p_{n+1} - p_n}$$  \hspace{1cm} (22)$$

We then define \(b'(x)\) for other \(x\) as a linear interpolation between \(b'(p_n), n = 1, 2, 3\ldots\)

Recall that based on Proposition 1 we wish to show that \(d\Delta(x)/dx > 0\) for \(x > x_0\) for some finite \(x_0\). Let us first provide some evidence that the \(b'\) term in (21) can be neglected,
i.e. does not spoil the monotonicity of $\Delta(x)$ at high $x$. To this end, let us first assume $b' = 0$ so that $b$ is constant. Then as far as monotonicity goes, in the worse case $b(x) = -B$, and $\Delta(x)$ would be monotonically increasing for

$$x \geq e^{1/2c+\sqrt{1/4c^2+B}}$$  \hspace{1cm} (23)

Note that for a given $B$, the above equation does not single out any particularly special value of $c$. However, if, for instance, one wishes the Cramér-Granville conjecture to be true beyond the first 5 primes, then $c \approx 1$, since for $B = 5$, \[(23)\] gives $x > 16.3 > p_6$. Interestingly, the original Cramér conjecture with $c = 1$ is only known to fail for the first 4 primes, and beyond this has been verified numerically to very high $N > 10^{10}$ or so. If we chose a smaller value of $c$, then $\Delta(x)$ simply becomes monotonically increasing beyond a higher value of $x$. The fact that this analysis led to the very likely correct prediction that the Cramér-Granville conjecture is valid for primes $p > p_5$, suggests that the $b'(x)$ must indeed be very small. As we discuss below, this appears to be related to the enormous value of the Skewes number.

This analysis also suggests the value of $c$ is fixed by the gaps in the low primes.

One can state something more precise as follows. Only if $b'(x) < 0$ can it spoil the monotonicity. If the following condition holds

$$b'(p) > -\frac{\log^2 p}{p} \left(1 - \frac{1}{c \log p}\right)$$  \hspace{1cm} (24)

where $p$ is prime with $p > x_0$ for some $x_0$, then the Cramér-Granville conjecture is true for $p > x_0$. Numerically, we checked that the above condition is valid for $5 < p < 10^{12}$ with $c = 1$, however we cannot prove that it is valid beyond this.

Stronger bounds on the fluctuations do not significantly improve the analysis. Since it is believed that the strongest bounds come from assuming the Riemann Hypothesis, let us do so. Schoenfeld showed \[9\] that

$$\left|f(x)\right| = |\pi(x) - \text{Li}(x)| < K \sqrt{x} \log x$$  \hspace{1cm} (25)

for $x > 2657$ where $K = 1/8\pi$. By increasing $K$, one can make the above bound valid for all $x$; we found $K \approx 1/3$. Repeating the above steps leads to

$$\Delta(x) = x \log x - \frac{(c + 1)}{c} x + h(x) + O(1)$$  \hspace{1cm} (26)

where

$$h(x) = k(x) \sqrt{x} \log^3 x$$  \hspace{1cm} (27)
From (25) one has
\[-K < k(x) < K\] (28)

In (27) we have dropped a few terms with lower powers of \(\log x\), such as \(\sqrt{x} \log^2 x\) since they will not significantly change our conclusions.

Define the formal derivative \(k'(x)\) of \(k(x)\) as in (22) with \(b \to k\). In Figure 1 we plot some values. We then have

\[
\frac{d\Delta(x)}{dx} = \log x - \frac{1}{c} + \frac{k(x) \log x}{\sqrt{x}} + k'(x)\sqrt{x} \log^3 x
\] (29)

As before, if one first assumes the \(k'\) term is negligibly small, then the worse case is \(k = -K\). One finds that for \(c = 1\) and \(K = 1/3\), \(d\Delta(x)/dx > 0\) for \(x > p_2\). Thus, once again we find that neglecting the \(k'\) term apparently leads to the right conclusion, i.e. that the Cramér-Granville conjecture is true with \(c = 1\) for all primes \(p > p_4\). In particular, if the following condition holds

\[
k'(p) > -\frac{1}{\sqrt{p \log^2 p}} \left(1 - \frac{1}{c \log p}\right)
\] (30)

for \(p > x_0\) then the Cramér-Granville conjecture holds for primes \(p > x_0\). We checked numerically that the above condition with \(c = 1\) is satisfied for all primes \(p > 3\) up to \(10^{12}\), however again we cannot prove this result. See Figure 1.

Additional evidence for \(k'(x)\) being very small comes from the largeness of the Skewes number. Up to values of \(x\) that are within reach numerically, \(\text{Li}(x) > \pi(x)\), and for a long time it was believed that this persists to infinity. Let \(\text{Sk}_1\) denote the Skewes number, which is the first \(x\) where the crossover \(\text{Li}(x) < \pi(x)\) occurs. Littlewood proved \(\text{Sk}_1\) exists, i.e. 
isn’t infinite, and Skewes first estimated it as Sk\textsubscript{1} < 10^{10^{10^{34}}}. This has since been reduced to Sk\textsubscript{1} < 10^{316} [10]. Numerically it is also known that Sk\textsubscript{1} > 10^{14}. Now k(x) starts out negative with k(x) > −1/8\pi for low x, then very slowly increases on average until it finally changes sign at Sk\textsubscript{1}. Thus the average of k′(x) in the region 2 < x < Sk\textsubscript{1} is approximately (8\pi Sk\textsubscript{1})^{-1}, which is exceedingly small.

It is interesting to try and determine the Skewes number from the smallness of k′(x). Numerically we find that k(x) = O(\log \log \log x), so that k′(x) is indeed small at large x. In particular, For x < 10^{10} a reasonably good fit is that on average

\[ k(x) \approx -A(\alpha - \log \log \log x) \quad (31) \]

with \( \alpha \approx 1.3 \). If this approximation persists, then Sk\textsubscript{1} = e^{e^{e^\alpha}}. This is obviously very sensitive to \( \alpha \). For \( \alpha = 1.3 \), Sk\textsubscript{1} = 10^{17}, however a small increase to \( \alpha = 1.5 \) would already give Sk\textsubscript{1} = 10^{38}. Any values \( \alpha > 2 \) are already ruled out since for \( \alpha = 2 \), Sk\textsubscript{1} > 10^{702}.

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