On the algebraic set of singular elements 
in a complex simple Lie algebra

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Abstract. Let $G$ be a complex simple Lie group and let $\mathfrak{g} = \text{Lie} G$. Let $S(\mathfrak{g})$ be the $G$-module of polynomial functions on $\mathfrak{g}$ and let Sing $\mathfrak{g}$ be the closed algebraic cone of singular elements in $\mathfrak{g}$. Let $\mathcal{L} \subset S(\mathfrak{g})$ be the (graded) ideal defining Sing $\mathfrak{g}$ and let $2r$ be the dimension of a $G$-orbit of a regular element in $\mathfrak{g}$. Then $\mathcal{L}^k = 0$ for any $k < r$. On the other hand, there exists a remarkable $G$-module $M \subset \mathcal{L}^r$ which already defines Sing $\mathfrak{g}$. The main results of this paper are a determination of the structure of $M$.

0. Introduction

0.1. Let $G$ be a complex simple Lie group and let $\mathfrak{g} = \text{Lie} G$. Let $\ell = \text{rank} \mathfrak{g}$. Then in superscript centralizer notation one has $\text{dim} \mathfrak{g}^x \geq \ell$ for any $x \in \mathfrak{g}$. An element $x \in \mathfrak{g}$ is called regular (resp. singular) if $\text{dim} \mathfrak{g}^x = \ell$ (resp. $> \ell$). Let Reg $\mathfrak{g}$ be the set of all regular elements in $\mathfrak{g}$ and let Sing $\mathfrak{g}$, its complement in $\mathfrak{g}$, be the set of all singular elements in $\mathfrak{g}$. Then one knows that Reg $\mathfrak{g}$ is a nonempty Zariski open subset of $\mathfrak{g}$ and hence Sing $\mathfrak{g}$ is a closed proper algebraic subset of $\mathfrak{g}$.

Let $S(\mathfrak{g})$ (resp. $\wedge \mathfrak{g}$) be the symmetric (resp. exterior) algebra over $\mathfrak{g}$. Both algebras are graded and are $G$-modules by extension of the adjoint representation. Let $\mathcal{B}$ be the natural extension of the Killing form to $S(\mathfrak{g})$ and $\wedge \mathfrak{g}$. The inner product it induces on $u$ and $v$ in either $S(\mathfrak{g})$ or $\wedge \mathfrak{g}$ is denoted by $(u,v)$. The use of $\mathcal{B}$ permits an identification of $S(\mathfrak{g})$ with the algebra of polynomial functions on $\mathfrak{g}$. Since Sing $\mathfrak{g}$ is clearly a cone the ideal, $\mathcal{L}$, of all $f \in S(\mathfrak{g})$ which vanish on Sing $\mathfrak{g}$ is graded. Let $n = \text{dim} \mathfrak{g}$ and let $r = (n - \ell)/2$. One knows that $n - \ell$ is even so that $r \in \mathbb{Z}_+$. It is easy to show that

$$\mathcal{L}^k = 0, \text{ for all } k < r. \quad (0.1)$$
The purpose of this paper is to define and study a rather remarkable $G$-submodule

\[ M \subset \mathcal{L}^r \quad (0.2) \]

which in fact defines $\text{Sing}\ g$. That is, if $x \in g$, then

\[ x \in \text{Sing}\ g \iff f(x) = 0, \ \forall f \in M \quad (0.3) \]

0.2. We will now give a definition of $M$. The use of $\mathcal{B}$ permits an identification of $\wedge g$ with the underlying space of the cochain complex defining the cohomology of $g$. The coboundary operator is denoted here by $d$ (and $\delta$ in [Kz]) is a (super) derivation of degree 1 of $\wedge g$ so that $dx \in \wedge^2 g$ for any $x \in g$. Since $\wedge^{\text{even}} g$ is a commutative algebra there exists a homomorphism

\[ \gamma : S(g) \to \wedge^{\text{even}} g \]

where for $x \in g$, $\gamma(x) = -dx$. One readily has that

\[ S^k(g) \subset \text{Ker} \gamma, \ \text{for all } k > r. \quad (0.4) \]

Let $\gamma_r = \gamma|S^r(g)$ so that

\[ \gamma_r : S^r(g) \to \wedge^{2r} g. \quad (0.5) \]

If $x \in g$, one readily has

\[ x^r \in \text{Ker} \gamma_r \iff x \in \text{Sing}\ g. \quad (0.6) \]

Let $\Gamma$ be the transpose of $\gamma_r$ so that one has a $G$-map

\[ \Gamma : \wedge^{2r} \to S^r(g). \quad (0.7) \]

By definition

\[ M = \text{Im} \Gamma. \quad (0.8) \]
0.3. Let \( J = S(\mathfrak{g})^G \) so that (Chevalley) \( J \) is a polynomial ring \( \mathbb{C}[p_1, \ldots, p_\ell] \) where the invariants \( p_j \) can be chosen to be homogeneous. In fact if \( m_j, j = 1, \ldots, \ell, \) are the exponents of \( \mathfrak{g} \) we can take \( \deg p_j = m_j + 1 \). For any linearly independent \( u_1, \ldots, u_\ell \in \mathfrak{g} \), let

\[
\psi(u_1, \ldots, u_\ell) = \det \partial_{u_i} p_j
\]

(0.9)

where, if \( v \in \mathfrak{g} \), \( \partial_v \) is the operator of partial derivative by \( v \) in \( S(\mathfrak{g}) \). One has

\[
\psi(u_1, \ldots, u_\ell) \in S^r(\mathfrak{g})
\]

(0.10)

since, as one knows, \( \sum_{i=1}^\ell m_i = r \).

Let \( \Sigma_{2r} \) be the permutation group of \( \{1, \ldots, 2r\} \) and let \( \Pi_r \subset \Sigma_{2r} \) be a subset (of cardinality \( (2r - 1)(2r - 3) \cdots 1 \)) with the property that \( sg \nu = 1 \) for all \( \nu \in \Pi_r \) and such that, as unordered,

\[
\{(\nu(1), \nu(2)), \ldots, (\nu(2r-1), \nu(2r)) \mid \nu \in \Pi_r\}
\]

is the set of all partitions of \( \{1, \ldots, 2r\} \) into a union of \( r \) subsets each of which has two elements. The following is one of our main theorems. Even more than explicitly determining \( \psi(u_1, \ldots, u_\ell) \) one has

**Theorem 0.1.** Let \( u_1, \ldots, u_\ell \) be any \( \ell \) linearly independent elements in \( \mathfrak{g} \) and let \( w_1, \ldots, w_{2r} \) be a basis of the \( \mathcal{B} \)-orthogonal subspace to the span of the \( u_i \). Then there exists some fixed \( \kappa \in \mathbb{C}^\times \) such that, for all \( x \in \mathfrak{g} \),

\[
\sum_{\nu \in \Pi_r} ([w_{\nu(1)}, w_{\nu(2)}], x) \cdots ([w_{\nu(2r-1)}, w_{\nu(2r)}], x) = \kappa \psi(u_1, \ldots, u_\ell)(x). \tag{0.11}
\]

Moreover \( \psi(u_1, \ldots, u_\ell) \in M \). In fact the left side of (0.11) is just \( \Gamma(w_1 \wedge \cdots \wedge w_{2r})(x) \). In addition \( M \) is the span of \( \psi(u_1, \ldots, u_\ell) \), over all \( \{u_1, \ldots, u_\ell\} \), taken from the \( \binom{n}{\ell} \) subsets of \( \ell \)-elements in any given basis of \( \mathfrak{g} \).

We now deal with the \( G \)-module structure of \( M \). For any subspace \( \mathfrak{s} \) of \( \mathfrak{g} \), say of dimension \( k \), let \( [\mathfrak{s}] = \mathbb{C}v_1 \wedge \cdots \wedge v_k \subset \wedge^k \mathfrak{g} \) where the \( v_i \) are a basis of \( \mathfrak{s} \). Let
Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and let $\Delta$ be the set of roots for the pair $(\mathfrak{h}, \mathfrak{g})$. For any $\varphi \in \Delta$ let $e_\varphi \in \mathfrak{g}$ be a corresponding root vector. Let $\Delta_+ \subset \Delta$ be a choice of a set of positive roots and let $\mathfrak{b}$ be the Borel subalgebra spanned by $\mathfrak{h}$ and all $e_\varphi$ for $\varphi \in \Delta_+$. For any subset $\Phi \subset \Delta$ let $a_\Phi \subset \mathfrak{g}$ be the span of $e_\varphi$ for $\varphi \in \Phi$. Also let $\langle \Phi \rangle = \sum_{\varphi \in \Phi} \varphi$ so that $[a_\Phi]$ is an $\mathfrak{h}$-weight space for the $\mathfrak{h}$-weight $\langle \Phi \rangle$.

A subset $\Phi \in \Delta_+$ will be said to be an ideal in $\Delta_+$ if $a_\Phi$ is an ideal of $\mathfrak{b}$. In such a case, if $\text{card} \Phi = k$, then the span $V_\Phi$ of $G \cdot [a_\Phi]$ is an irreducible $G$-submodule of $\wedge^k \mathfrak{g}$ having $[a_\Phi]$ as highest weight space and $\langle \Phi \rangle$ as highest weight. Let $\mathcal{I}$ be the set of all ideals $\Phi$ in $\Delta_+$ of cardinality $\ell$. It is shown in [KW] that all ideals in $\mathfrak{b}$ of dimension $\ell$ are abelian and hence are of the form $a_\Phi$ for a unique $\Phi \in \mathcal{I}$. Specializing $k$ in [K3] to $\ell$ one has that, by definition, $A_\ell \subset \wedge^\ell \mathfrak{g}$ is the span of $[s]$ over all abelian subalgebras $s \subset \mathfrak{g}$ of dimension $\ell$. Using results in [K3] and that in [KW] above, one also has that $A_\ell$ is a multiplicity one $G$-module with the complete reduction

$$A_\ell = \bigoplus_{\Phi \in \mathcal{I}} V_\Phi$$

so that there are exactly $\text{card} \mathcal{I}$ irreducible components. In addition it has been shown in [K3] that $\ell$ is the maximal eigenvalue of the $(\mathcal{B}$ normalized) Casimir operator, $\text{Cas}$, in $\wedge^\ell \mathfrak{g}$ and $A_\ell$ is the corresponding eigenspace. In the present paper the $G$-module structure of $M$ is given in

**Theorem 0.2.** As $G$-modules one has an equivalence

$$M \cong A_\ell$$

so that $M$ is a multiplicity one module with $\text{card} \mathcal{I}$ irreducible components. Moreover the components can be parameterized by $\mathcal{I}$ in such a way that the component corresponding to $\Phi \in \mathcal{I}$ has highest weight $\langle \Phi \rangle$. In addition $\text{Cas}$ takes the value $\ell$ on each and every irreducible component of $M$. 

4
1. Preliminaries

1.1. Let \( g \) be a complex semisimple Lie algebra and let \( G \) be a Lie group such that \( g = \text{Lie} \ G \). Let \( \mathfrak{h} \subset \mathfrak{g} \) be a Cartan subalgebra of \( \mathfrak{g} \) and let \( \ell \) be the rank of \( \mathfrak{g} \) so that \( \ell = \dim \mathfrak{h} \). Let \( \Delta \) be the set of roots for the pair \( (\mathfrak{h}, \mathfrak{g}) \) and let \( \Delta_+ \subset \Delta \) be a choice of a set of positive roots. Let \( r = \text{card} \Delta_+ \) so that

\[
n = \ell + 2r
\]

(1.1)

where we let \( n = \dim \mathfrak{g} \). Let \( \mathcal{B} \) be Killing form \((x, y)\) on \( \mathfrak{g} \). For notational economy we identify \( \mathfrak{g} \) with its dual \( \mathfrak{g}^* \) using \( \mathcal{B} \). The bilinear form \( \mathcal{B} \) extends to an inner product \((p, q)\), still denoted by \( \mathcal{B} \), on the two graded algebras, the symmetric algebra \( S(\mathfrak{g}) \) of \( \mathfrak{g} \) and the exterior algebra \( \land \mathfrak{g} \) of \( \mathfrak{g} \). If \( x_i, y_j \in \mathfrak{g}, i = 1, \ldots, k, j = 1, \ldots, m \), then the product of \( x_i \) is orthogonal to the product of \( y_j \) in both \( S(\mathfrak{g}) \) and \( \land \mathfrak{g} \) if \( k \neq m \), whereas if \( k = m \),

\[
(x_1 \cdots x_k, y_1 \cdots y_k) = \sum_{\sigma \in \Sigma_k} (x_1, y_{\sigma(1)}) \cdots (x_k, y_{\sigma(k)}) \quad \text{in } S(\mathfrak{g})
\]

(1.2)

\[
(x_1 \land \cdots \land x_k, y_1 \land \cdots \land y_k) = \sum_{\sigma \in \Sigma_k} s_\sigma(\sigma)(x_1, y_{\sigma(1)}) \cdots (x_k, y_{\sigma(k)}) \quad \text{in } \land \mathfrak{g}.
\]

Here \( \Sigma_k \) is the permutation group on \( \{1, \ldots, k\} \) and \( s_\sigma \) abbreviates the signum character on \( \Sigma_k \).

The identification of \( \mathfrak{g} \) with its dual has the effect of identifying \( S(\mathfrak{g}) \) with the algebra of polynomial functions \( f(y) \) on \( \mathfrak{g} \). Thus if \( x, y \in \mathfrak{g} \), then \( x(y) = (x, y) \) and if \( x_i \in \mathfrak{g}, i = 1, \ldots, k \), then

\[
(x_1 \cdots x_k)(y) = \prod_{i=1}^k (x_i, y)
\]

(1.3)

\[
= (x_1 \cdots x_k, \frac{1}{k!} y^k).
\]

The identification of \( \mathfrak{g} \) with its dual also has the effect of identifying the (supercommutative) algebra \( \land \mathfrak{g} \) with the underlying space of the standard cochain complex
defining the cohomology of $\mathfrak{g}$. Let $d$ be the (super) derivation of degree 1 of $\wedge \mathfrak{g}$, defined by putting

$$d = \frac{1}{2} \sum_{i=1}^{n} \varepsilon(w_i)\theta(z_i). \quad (1.4)$$

Here $\varepsilon(u)$, for any $u \in \wedge \mathfrak{g}$, is left exterior multiplication by $u$ so that $\varepsilon(u) v = u \wedge v$ for any $v \in \wedge \mathfrak{g}$. Also $w_i, i = 1, \ldots, n$, is any basis of $\mathfrak{g}$ and $z_i \in \mathfrak{g}$, $i = 1, \ldots, n$, is the $\mathcal{B}$ dual basis. $\theta(x)$, for $x \in \mathfrak{g}$, is the derivation of $\wedge \mathfrak{g}$, of degree 0, defined so that $\theta(x)y = [x, y]$ for any $y \in \mathfrak{g}$. One readily notes that (1.4) is independent of the choice of the basis $w_i$. Thus if $x \in \mathfrak{g}$, then $dx \in \wedge^2 \mathfrak{g}$ is given by

$$dx = \frac{1}{2} \sum_{i=1}^{n} w_i \wedge [z_i, x]. \quad (1.5)$$

Any element $\omega \in \wedge^2 \mathfrak{g}$ defines an alternating bilinear form on $\mathfrak{g}$. Its value $\omega(y, z)$ on $y, z \in \mathfrak{g}$ may be given in terms of $\mathcal{B}$ by

$$\omega(y, z) = (\omega, y \wedge z). \quad (1.6)$$

The rank of $\omega$ is necessarily even. In fact if rank $\omega = 2k$, then there exist $2k$ linearly independent elements $v_i \in \mathfrak{g}$, $i = 1, \ldots, 2k$, such that

$$\omega = v_1 \wedge v_2 + \cdots + v_{2k-1} \wedge v_{2k}. \quad (1.7)$$

The radical of $\omega$, denoted by $\text{Rad} \omega$, is the space of all $y \in \mathfrak{g}$ such that $\omega(y, z) = 0$ for all $z \in \mathfrak{g}$. For $u \in \wedge \mathfrak{g}$, let $\iota(u)$ be the transpose of $\varepsilon(u)$ with respect to $\mathcal{B}$ on $\wedge \mathfrak{g}$. If $u = y \in \mathfrak{g}$, then one knows that $\iota(y)$ is the (super) derivation of degree minus 1 defined so that if $z \in \mathfrak{g}$, then $\iota(y)z = (y, z)$. (See p. 8 in [Kz]). From (1.6) one has

$$\text{Rad} \omega = \{ y \in \mathfrak{g} \mid \iota(y)\omega = 0 \}. \quad (1.8)$$

If $\mathfrak{s}$ is any subspace of $\mathfrak{g}$, let $\mathfrak{s}^\perp$ be the $\mathcal{B}$ orthogonal subspace to $\mathfrak{s}$. From (1.7) one then has that

$$\{ v_i \}, \ i = 1, \ldots, 2k, \text{ is a basis of } \text{Rad} \omega^\perp. \quad (1.9)$$
If \( s \subset g \) is any subspace, say of dimension \( m \), let \([s] \in \wedge^m g\) be the \( \mathbb{C} \) span of the decomposable element \( u_1 \wedge \cdots \wedge u_m \) where \( \{u_i, i = 1, \ldots, m\} \) is a basis of \( s \). One notes that if \( \omega \in \wedge^2 g \) is given as in (1.7), then

\[
\omega^k = k! \, v_1 \wedge \cdots \wedge v_{2k}
\]

so that

\[
\omega^j \neq 0 \iff j \leq k \text{ and } \omega^k \in [\text{Rad } \omega^\perp].
\]

Let \( \{w_j, j = 1, \ldots, n\} \) be a \( \mathcal{B} \) orthonormal basis of \( g \). Put \( \mu = w_1 \wedge \cdots \wedge w_n \) so that

\[
(\mu, \mu) = 1
\]

so that \( \mu \) is unique up to sign and \( \wedge^n g = \mathbb{C} \mu \). For any \( v \in \wedge g \) let \( v^* = \iota(v)\mu \). We recall the more or less well known.

**Proposition 1.1.** If \( s \subset g \) is any subspace and \( 0 \neq u \in [s] \), then

\[
0 \neq u^* \in [s^\perp].
\]

Moreover if \( s, t \in \wedge g \), one has

\[
(s, t) = (s^*, t^*).
\]

**Proof.** Let \( \{y_i, i = 1, \ldots, m\} \) be a basis of \( s \) chosen so that \( u = y_1 \wedge \cdots \wedge y_m \) and let \( \{z_j, j = 1, \ldots, n - m\} \) be a basis of \( s^\perp \). Then if \( y'_k, k = 1, \ldots, m \), are chosen in \( g \) such that \( (y_i, y'_k) = \delta_{ik} \), it is immediate that the \( y'_k \) together with the \( z_j \) form a basis of \( g \) so that for some \( \lambda \in \mathbb{C}^\times \) one has

\[
\lambda y'_1 \wedge \cdots \wedge y'_m \wedge z_1 \wedge \cdots \wedge z_{n-m} = \mu.
\]

But since interior product is the transpose of exterior product one has

\[
\iota(q) \iota(p) = \iota(p \wedge q)
\]
for any \( p,q \in \wedge^g \). Thus by (1.15) one has

\[ u^* = \lambda z_1 \wedge \cdots \wedge z_{n-m} \]

establishing (1.13). To prove (1.14) it suffices by linearity to assume that both \( s \) and \( t \) are decomposable of some degree \( m \). Thus we can assume \( s = y_1 \wedge \cdots \wedge y_m \) and \( t = z_1 \wedge \cdots \wedge z_m \) for \( y_i, z_j \in g \). But now, as one knows, and readily establishes,

\[ \varepsilon(y) \iota(z) + \iota(z)\varepsilon(y) = (y, z)\text{Id}_g \quad (1.17) \]

for \( y, z \in g \). Thus

\[ (s^*, t^*) = (\iota(s)\mu, \iota(t)\mu) = (\mu, \varepsilon(s)\iota(t)\mu). \quad (1.18) \]

But then using (1.17) and the fact that \( \varepsilon(y)\mu = 0 \) for any \( y \in g \), one has

\[ (\mu, \varepsilon(s)\iota(t)\mu) = \sum_{j=0}^{m-1} (-1)^j (y_m, z_{m-j})(\mu, \varepsilon(y_1) \cdots \varepsilon(y_{m-1}) \iota(z_m) \cdots \iota(z_{m-j}) \cdots \iota(z_1)\mu). \]

But then by induction and the expansion of the determinant defined by the last row one has

\[ (\mu, \varepsilon(s)\iota(t)\mu) = \det(y_i, z_j)(\mu, \mu) = (s, t) \]

proving (1.14). QED

1.2. The algebra \( S(g) \) is a \( G \)-module extending the adjoint representation. Let \( J = S(g)^G \) be the subalgebra of \( g \)-invariants. Let \( H \subset S(g) \) be the graded \( g \)-submodule of harmonic elements in \( S(g) \) (See §1.4 in [K2] for definitions). Then one knows

\[ S(g) = J \otimes H. \quad (1.19) \]

See (1.4.3) in [K2].

Let \( r \) be as in (1.1). For the convenience of the reader we repeat a paragraph in §1.2 of [K4]. Let \( \Sigma_{2r,2} \) be the subgroup of all \( \sigma \in \Sigma_{2r} \) such that \( \sigma \) permutes the set of
unordered pairs \{((1, 2), (3, 4), \ldots, (2r - 1, 2r))\}. It is clear that \(\Sigma_{2r, 2}\) has order \(r! 2^r\).

Now let \(\Pi_r\) be a cross-section of the set of left cosets of \(\Sigma_{2r, 2}\) in \(\Sigma_{2r}\). Thus one has a disjoint

\[
\Sigma_{2r} = \bigcup_{\nu \in \Pi_r} \nu \Sigma_{2r, 2}. \tag{1.20}
\]

One notes that the cardinality of \(\Pi_r\) is \((2r - 1)(2r - 3) \cdots 1\) (the index of \(\Sigma_{2r, 2}\) in \(\Sigma_{2r}\)) and the correspondence

\[
\nu \mapsto ((\nu(1), \nu(2)), (\nu(3), \nu(4)), \ldots, (\nu(2r - 1), \nu(2r))) \tag{1.21}
\]

sets up a bijection of \(\Pi_r\) with the set of all partitions of \((1, 2, \ldots, 2r)\) into a union of subsets, each of which has two elements. Furthermore, since the signum character restricted to \(\Sigma_{2r, 2}\) is nontrivial we may choose \(\Pi_r\) so that

\[
sg(\nu) = 1
\]

for all \(\nu \in \Pi_r\).

In [K4] we defined a map \(\Gamma : \wedge^{2r} \mathfrak{g} \rightarrow S(\mathfrak{g})\); (Its significance will become apparent later). Here, using Proposition 1.2 in [K4] we will give a simpler definition of \(\Gamma\). By Proposition 1.2 in [K4] one has

**Proposition 1.2.** There exists a map

\[
\Gamma : \wedge^{2r} \mathfrak{g} \rightarrow S^r(\mathfrak{g}) \tag{1.21a}
\]

such that for any \(w_i \in \mathfrak{g}, i = 1, \ldots, 2r\), one has

\[
\Gamma(w_1 \wedge \cdots \wedge w_{2r}) = \sum_{\nu \in \Pi_r} [w_{\nu(1)}, w_{\nu(2)}] \cdots [w_{\nu(2r-1)}, w_{\nu(2r)}]. \tag{1.22}
\]

As a polynomial function of degree \(r\) on \(\mathfrak{g}\), one notes that

\[
\Gamma(w_1 \wedge \cdots \wedge w_{2r})(x) = \sum_{\nu \in \Pi_r} ([w_{\nu(1)}, w_{\nu(2)}], x) \cdots ([w_{\nu(2r-1)}, w_{\nu(2r)}], x). \tag{1.23}
\]
This clear from (1.1.7) in [K4] and (1.3) here.

The algebra $\land g$ is a natural $G$-module by extension of the adjoint representation. It is clear that $\Gamma$ is a $G$-map. Let $M \subset S^r(g)$ be the image of $\Gamma$. The following is proved as Corollary 3.3 in [K4].

**Theorem 1.3.** One has $M \subset H^r$ so that $M$ is a $G$-module of harmonic polynomials of degree $r$ on $g$.

Giving properties of $M$ and determining its rather striking $g$-module structure is the main goal of this paper.

For any $y \in g$ one has the familiar supercommutation formula $\iota(y) d + d\iota(y) = \theta(y)$. See e.g., (92) in [K5]. Now let $x, y \in g$. Since $d\iota(y)(x) = 0$ one has $\iota(y)dx = [y, x]$. Thus, by (1.8), using superscript notation for centralizers one has

$$\text{Rad } dx = g^x. \quad (1.24)$$

Clearly $[x, g]$ is the $B$ orthogonal subspace in $g$ to $g^x$ so that

$$[x, g] = (\text{Rad } dx)^\perp \quad (1.25)$$

for any $x \in g$.

For any $x \in g$ one knows $\dim g^x \geq \ell$. Recall that an element $x \in g$ is called regular if $\dim g^x = \ell$. The set $\text{Reg } g$ of regular elements is nonempty and Zariski open. Its complement, $\text{Sing } g$, is the Zariski closed set of singular elements. One notes, by (1.11), that

$$\text{Sing } g = \{x \in g \mid (dx)^r = 0\}. \quad (1.26)$$

Now $\land^{\text{even}} g$ is a commutative algebra and hence there exists a homomorphism

$$\gamma : S(g) \to \land^{\text{even}} g \quad (1.27)$$
such that for $x \in \mathfrak{g}$,

$$\gamma(x) = -dx.$$ 

Let $\gamma_r$ be the restriction of $\gamma$ to $S^r(\mathfrak{g})$. The following result, established as Theorem 1.4 in [K4], asserts that $\Gamma$ is the transpose of $\gamma_r$.

**Theorem 1.4.** Let $y_1, \ldots, y_r \in \mathfrak{g}$ and let $\zeta \in \wedge^{2r}(\mathfrak{g})$. Then

$$(y_1 \cdots y_r, \Gamma(\zeta)) = (-1)^r(dy_1 \wedge \cdots \wedge dy_r, \zeta). \quad (1.28)$$

Now one knows that $S^r(\mathfrak{g})$ is (polarization) spanned by all powers $x^r$ for $x \in \mathfrak{g}$. Using (1.3), (1.26) and Theorem 1.4 we recover Proposition 3.2 in [K4]. The key point is that $M$ defines the variety Sing $\mathfrak{g}$.

**Theorem 1.5.** Let $x \in \mathfrak{g}$ and $\zeta \in \wedge^{2r}(\mathfrak{g})$. Then

$$\Gamma(\zeta)(x) = \frac{(-1)^r}{r!}((dx)^r, \zeta). \quad (1.29)$$

In particular

$$f(x) = 0, \ \forall f \in M \iff x \in \text{Sing}(\mathfrak{g}). \quad (1.30)$$

If $\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$, then one knows that $\mathfrak{a} \cap \text{Sing} \mathfrak{g}$ is a union of the root hyperplanes in $\mathfrak{a}$. Hence as a corollary of Theorem 1.5 one has

**Theorem 1.6.** Let $\mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{g}$. Let $\Delta_+(\mathfrak{a})$ be a choice of positive roots for the pair $(\mathfrak{a}, \mathfrak{g})$. Then for any $f \in M$ one has

$$f|_{\mathfrak{a}} \in \mathbb{C} \prod_{\beta \in \Delta_+(\mathfrak{a})} \beta. \quad (1.31)$$

Going to the opposite extreme we recall that a nilpotent element $e$ is called principal if it is regular. Let $e$ be a principal nilpotent element. Then by Corollary 5.6
in [K1] there exists a unique nilpotent radical \( n \) of a Borel subalgebra such that \( e \in n \). Furthermore \( g^e \cap [n, n] \) is a linear hyperplane in \( g^e \) and \( g^e \cap [n, n] = (\text{Sing } g) \cap g^e \) by Theorem 5.3 and Theorem 6.7 in [K1]. Thus there exists a nonzero linear functional \( \xi \) on \( g^e \) such that

\[
\text{Ker } \xi = (\text{Sing } g) \cap g^e.
\] (1.32)

This establishes

**Theorem 1.7.** Let \( e \in g \) be principal nilpotent. Let \( f \in M \). Then using the notation of (1.32) one has

\[
f|g^e \in \mathbb{C} \xi^r.
\] (1.33)

Since \( \text{Sing } g \) is clearly a cone it follows that the ideal \( \mathcal{L} \) of \( f \in S(g) \) which vanishes on \( \text{Sing } g \) is graded. One of course has that \( M \subset \mathcal{L}^r \). We now observe that \( r \) is the minimal value of \( k \) such that \( \mathcal{L}^k \neq 0 \)

**Proposition 1.8.** Assume that \( 0 \neq f \in \mathcal{L}^k \). Then \( k \geq r \).

**Proof.** Since \( f \neq 0 \) there clearly exists a Cartan subalgebra \( a \) of \( g \) such that \( f|a \neq 0 \). But then using the notation of Theorem 1.6 it follows from the prime decomposition that \( \beta \) divides \( f|a \) for all \( \beta \in \Delta_+(a) \). Thus \( k \geq r \). QED

**2. The structure of \( M \) in terms of minors and as a \( G \)-module**

**2.1.** For any \( z \in g \) let \( \partial_z \) be the partial derivative of \( S(g) \) defined by \( z \). Let \( W(g) = S(g) \otimes \wedge g \) so that \( W(g) \) can be regarded as the supercommutative algebra of all differential forms on \( g \) with polynomial coefficients. To avoid confusion with the already defined \( d \), let \( d_W \) be the operator of exterior differentiation on \( W(g) \). That is, \( d_W \) is a derivation of degree 1 defined so that if \( \{z_i, w_j\}, i, j = 1, \ldots, n \), are dual \( B \)
bases of $\mathfrak{g}$, then
\[
d_W(f \otimes u) = \sum_i^n \partial z_i f \otimes \varepsilon(w_i) u \tag{2.1}
\]
where $f \in S(\mathfrak{g})$ and $u \in \wedge \mathfrak{g}$. Of course $d_W$ is independent of the choice of bases. In particular $d_W f$ is a differential form of degree 1 on $\mathfrak{g}$.

For any $x \in \mathfrak{g}$ one has a homomorphism
\[
W(\mathfrak{g}) \rightarrow \wedge \mathfrak{g}, \quad \varphi \mapsto \varphi(x) \tag{2.2}
\]
defined so that if $\varphi = f \otimes u$, using the notation of (2.1), then $\varphi(x) = f(x)u$. Next one notes that the $G$-module structures on $S(\mathfrak{g})$ and $\wedge \mathfrak{g}$ define, by tensor product, a $G$-module structure on $W(\mathfrak{g})$. Clearly $d_W$ is a $G$ map. If $a \in G$ and $\varphi \in W(\mathfrak{g})$, the action of $a$ on $\varphi$ will simply be denoted by $a \cdot \varphi$. If $x \in \mathfrak{g}$ one readily has
\[
a \cdot (\varphi(x)) = a \cdot \varphi(a \cdot x). \tag{2.3}
\]

One knows (Chevalley) that $J$ is a polynomial ring $\mathbb{C}[p_1, \ldots, p_\ell]$ where the $p_j$ are homogeneous polynomials. If $d_j = \deg p_j$, for $j = 1, \ldots, \ell$, and $m_j = d_j - 1$, then the $m_j$ are exponents of $\mathfrak{g}$ so that
\[
\sum_{j=1}^{\ell} m_j = r. \tag{2.4}
\]
Moreover we can choose the $p_j$ so that $\partial_y p_j \in H$ for any $y \in \mathfrak{g}$ (see Theorem 67 in [K5]). In fact, if $H_{ad}$ is the primary component of $H$ corresponding to the adjoint representation, then the multiplicity of the adjoint representation in $H_{ad}$ is equal to $\ell$ and $\tau_j$, $j = 1, \ldots, \ell$, is a basis of $\text{Hom}_G(\mathfrak{g}, H_{ad})$ where
\[
\tau_j(y) = \partial_y p_j \tag{2.5}
\]
for any $y \in \mathfrak{g}$. Again see Theorem 67 in [K5].

**Remark 2.2.** Using the notation of (2.1) note that
\[
\{w_{i_1} \wedge \cdots \wedge w_{i_\ell} \mid 1 \leq i_1 < \cdots < i_\ell \leq n\}
\]
is a basis of $\wedge^\ell g$. Furthermore

$$\{z_{j_1} \wedge \cdots \wedge z_{j_\ell} \mid 1 \leq j_1 < \cdots < j_\ell \leq n\}$$

is the dual basis since clearly

$$(w_{i_1} \wedge \cdots \wedge w_{i_\ell}, z_{j_1} \wedge \cdots \wedge z_{j_\ell}) = \prod_{k=1}^{n} \delta_{i_k,j_k}. \quad (2.6)$$

In addition if the $w_i$ are a $B$-orthonormal basis of $g$, then $w_i = z_i$, $i = 1, \ldots, n$, and hence (2.6) implies that $\{w_{i_1} \wedge \cdots \wedge w_{i_\ell} \mid 1 \leq i_1 < \cdots < i_\ell \leq n\}$ is a $B$ orthonormal basis of $\wedge^\ell g$.

Now for any $y_i \in g$, $i = 1, \ldots, \ell$, let $\psi(y_1, \ldots, y_\ell) = \det \partial_{y_i} p_j$ so that

$$\psi(y_1, \ldots, y_\ell) \in S^r(g) \quad (2.7)$$

by (2.4). But now $d_WP_j$ is an invariant 1-form on $g$. If $x \in g$, then $d_WP_j(x) \in \wedge^1 g$. Explicitly, using the notation in (2.1), one has

$$d_WP_j(x) = \sum_{i=1}^{n} \partial_{z_i} p_j(x) w_i. \quad (2.8)$$

One notes that $\partial_{z_i} p_j$ is an $n \times \ell$ matrix of polynomial functions. There are $\binom{n}{\ell} \ell \times \ell$ minors for this matrix. The determinants of these minors all lie in $S^r(g)$ and appear in the following expansion.

**Proposition 2.1.** Let the notation be as in (2.1). Let $x \in g$. Then in $\wedge^\ell g$ one has

$$(d_WP_1(x) \wedge \cdots \wedge d_WP_\ell(x) = \sum_{1 \leq i_1 < \cdots < i_\ell \leq n} \psi(z_{i_1}, \ldots, z_{i_\ell})(x) w_{i_1} \wedge \cdots \wedge w_{i_\ell}. \quad (2.9)$$

**Proof.** This is just standard exterior algebra calculus using (2.8). QED
Theorem 2.2. Let $v_i, i = 1, \ldots, n$, be a $B$ orthonormal basis of $\mathfrak{g}$ chosen and ordered so that $v_i, i = 1, \ldots, \ell$, is a basis of $\mathfrak{h}$. Then there exists a scalar $\kappa \in \mathbb{C}^\times$ such that, for any $y \in \mathfrak{h}$,

$$d_W p_1(y) \wedge \cdots \wedge d_W p_\ell(y) = \kappa \left( \prod_{\varphi \in \Delta_+} \varphi(y) \right) v_1 \wedge \cdots \wedge v_\ell. \quad (2.10)$$

Proof. If $a \in G$, $x \in \mathfrak{g}$ and $j = 1, \ldots, \ell$, then since $d_W p_j$ is $G$-invariant one has

$$a \cdot d_W p_j(x) = d_W p_j(a \cdot x). \quad (2.11)$$

But this implies that

$$d_W p_j(x) \in \text{cent } \mathfrak{g}^x \quad (2.12)$$

since if we choose $a \in G^x$ in (2.11) it follows from (2.11) that $d_W p_j(x)$ commutes with $\mathfrak{g}^x$. But $x \in \mathfrak{g}^x$ so that $d_W p_j(x) \in \mathfrak{g}^x$. This establishes (2.12).

Now by Theorem 9, p. 382 in [K2] one has that if $x \in \mathfrak{g}$, then

$$\{d_W p_1(x), \ldots, d_W p_\ell(x)\} \text{ are linearly independent } \iff \ x \in \text{Reg } \mathfrak{g}. \quad (2.12a)$$

Thus the left side of (2.10) vanishes if and only if $y \in \text{Sing } \mathfrak{g} \cap \mathfrak{h}$. In particular, choosing the $z_i$ in (2.9) so that $v_j = z_j$ for $j = 1, \ldots, \ell$, one has $\psi(v_1, \ldots, v_\ell)(y) = 0$ if $y$ is singular by the expansion (2.9). One the other hand, if $y \in \mathfrak{h}$ is regular then, by (2.12), one must have that

$$\{d_W p_j(y), j = 1, \ldots, \ell\} \text{ is a basis of } \mathfrak{h}. \quad (2.13)$$

Thus if $y$ is regular, the left side of (2.10) equals $\nu v_1 \wedge \cdots \wedge v_\ell$ for some $\nu \in \mathbb{C}^\times$. Comparing with the expansion (2.9) one must have $\nu = \psi(v_1, \ldots, v_\ell)(y)$. But then $\psi(v_1, \ldots, v_\ell)|\mathfrak{h}$ is a polynomial of of degree $r$ which vanishes on $y \in \mathfrak{h}$ if and only if $y \in \mathfrak{h}$ is singular. Thus

$$\psi(v_1, \ldots, v_\ell)|\mathfrak{h} = \kappa \prod_{\varphi \in \Delta_+} \varphi$$
for some nonzero constant \( \kappa \). This proves (2.10). QED

### 2.2. For any root \( \varphi \in \Delta \) let \( e_\varphi \in \mathfrak{g} \) be a corresponding root vector. We will make choices so that

\[
(e_\varphi, e_{-\varphi}) = 1. \tag{2.14}
\]

For any \( x \in \mathfrak{h} \), one then has

\[
dx = \sum_{\varphi \in \Delta_+} \varphi(x) e_\varphi \wedge e_{-\varphi}. \tag{2.15}
\]

See Proposition 37, p. 311 in [K5], noting (106), p. 302 and (142), p. 309 in [K5]. But then recalling (1.27) one has

\[
\gamma_r(x^r) = r! (-1)^r \prod_{\varphi \in \Delta_+} \varphi(x) e_\varphi \wedge e_{-\varphi}. \tag{2.16}
\]

But since \((e_\varphi \wedge e_{-\varphi}, e_\varphi \wedge e_{-\varphi}) = -1\), by (2.14), for any \( \varphi \in \Delta_+ \) one has that

\[
(\prod_{\varphi \in \Delta_+} e_\varphi \wedge e_{-\varphi}, \prod_{\varphi \in \Delta_+} e_\varphi \wedge e_{-\varphi}) = (-1)^r. \tag{2.17}
\]

But then if \( \{v_i \mid i = 1, \ldots, \ell\} \) is an orthonormal basis of \( \mathfrak{h} \), one has

\[
(v_1 \wedge \cdots \wedge v_\ell \wedge \prod_{\varphi \in \Delta_+} e_\varphi \wedge e_{-\varphi}, v_1 \wedge \cdots \wedge v_\ell \wedge \prod_{\varphi \in \Delta_+} e_\varphi \wedge e_{-\varphi}) = (-1)^r. \tag{2.18}
\]

But then we may choose an ordering of the \( v_i \) such that

\[
\mu = i^r v_1 \wedge \cdots \wedge v_\ell \wedge \prod_{\varphi \in \Delta_+} e_\varphi \wedge e_{-\varphi} \tag{2.19}
\]

so that

\[
(v_1 \wedge \cdots \wedge v_\ell)^* = i^r \prod_{\varphi \in \Delta_+} e_\varphi \wedge e_{-\varphi}. \tag{2.20}
\]

But then one has

**Theorem 2.3** There exists \( \kappa_o \in \mathbb{C}^\times \) such that for any \( x \in \mathfrak{g} \),

\[
(d_W p_1(x) \wedge \cdots \wedge d_W p_\ell(x))^* = \kappa_o \frac{(-dx)^r}{r!} \tag{2.21}
\]

\[= \kappa_o \gamma_r \left( \frac{x^r}{r!} \right). \]
Proof. If \( y \in \mathfrak{h} \) is regular, then (2.21), for \( y = x \), follows from (2.16),(2.20) and Theorem 2.2. That is
\[
(dW p_1(y) \wedge \cdots \wedge dW p_\ell(y))^* = \kappa_o \frac{(-dy)^r}{r!} = \kappa_o \gamma_r \frac{y^r}{r!}.
\] (2.22)

But now if \( x \in \mathfrak{g} \) regular and semisimple there exist \( a \in G \) and a regular \( y \in \mathfrak{h} \) such that \( a \cdot y = x \). But now since \( * \) and \( \gamma_r \) are clearly \( G \)-maps one has (2.21) by applying the action of \( a \) to both sides of (2.22). However the set of regular semisimple elements in \( \mathfrak{g} \) is dense (this nonempty set is Zariski open) one has (2.21) for all \( x \in \mathfrak{g} \) by continuity. QED

Returning to our module \( M \) of harmonic polynomials on \( \mathfrak{g} \) of degree \( r \) it is obvious, by definition, that \( M \) is spanned by all \( f \in S^r \) of the form \( f = \Gamma(w_1 \wedge \cdots \wedge w_{2r}) \) where the \( w_i \in \mathfrak{g} \) are linearly independent. Explicitly \( \Gamma(w_1 \wedge \cdots \wedge w_{2r}) \) is given by (1.22). We now show that \( \Gamma(w_1 \wedge \cdots \wedge w_{2r}) \) may also be given as the determinant of one of the \( \ell \times \ell \) minors in the expansion (2.9).

**Theorem 2.4.** Let \( w_k \in \mathfrak{g}, k = 1, \ldots, 2r, \) be linearly independent and let \( \mathfrak{s} \subset \mathfrak{g} \) be the span of the \( w_k \) and let \( u_i \in \mathfrak{g}, = 1, \ldots, \ell, \) be a basis of \( \mathfrak{s}^\perp \). Then there exists a constant \( \kappa_1 \in \mathbb{C}^\times \) such that
\[
\Gamma(w_1 \wedge \cdots \wedge w_{2r}) = \kappa_1 \psi(u_1, \ldots, u_\ell)
\]
(2.23)

\[
= \kappa_1 \det \partial_{u_i} p_j.
\]

Furthermore \( M \) is the span of all \( \ell \times \ell \) determinant minors \( \psi(v_1, \ldots, v_\ell) \) where \( v_i \in \mathfrak{g}, i = 1, \ldots, \ell, \) are linearly independent.

**Proof.** Clearly we may choose the two dual bases in (2.1) so that the given \( w_k \) are the first \( 2r \)-elements of the \( w \) basis and the \( u_i \) are the last \( \ell \) elements of the \( z \) basis. Thus there exists \( \kappa_2 \in \mathbb{C}^\times \) such that
\[
(u_1 \wedge \cdots \wedge u_\ell)^* = \kappa_2 w_1 \wedge \cdots \wedge w_{2r}.
\] (2.24)
Now let \( x \in \mathfrak{g} \). Then by the expansion (2.9) one has

\[
(dW p_1(x) \wedge \cdots \wedge dW p_\ell(x), u_1 \wedge \cdots \wedge u_\ell) = \psi(u_1, \ldots, u_\ell)(x).
\] (2.25)

But then by (1.3), (1.14), (1.29) and (2.21) one has

\[
\psi(u_1, \ldots, u_\ell)(x) = ((dW p_1(x) \wedge \cdots \wedge dW p_\ell(x))^*, (u_1 \wedge \cdots \wedge u_\ell)^*)
\]

\[
= \kappa_0 \kappa_2 \left( \gamma_r \left( \frac{x^r}{r!} \right), w_1 \wedge \cdots \wedge w_{2r} \right)
\]

\[
= \kappa_1^{-1} \Gamma(w_1 \wedge \cdots \wedge w_{2r})(x)
\] (2.26)

where \( \kappa_1^{-1} = \kappa_0 \kappa_2 \). The last statement in the theorem is obvious since clearly \( u_i, i = 1, \ldots, \ell \), is an arbitrary set of \( \ell \)-independent elements in \( \mathfrak{g} \). QED

**2.3.** Let \( \{z_i, w_j\} \) be the arbitrary dual bases of \( \mathfrak{g} \) as in (1.4). Then, independent of the choice of bases, the Casimir operator \( \text{Cas} \) on \( \wedge \mathfrak{g} \) is given by

\[
\text{Cas} = \sum_{i=1}^{n} \theta(z_i)\theta(w_i).
\]

We recall special cases of some results in [K3]. Let \( A_\ell \subset \wedge^\ell \mathfrak{g} \) be the span in \( \wedge^\ell \mathfrak{g} \) of all \( [c] \) where \( c \in \mathfrak{g} \) is a commutative Lie subalgebra of dimension \( \ell \). Since the set of such subalgebras includes, for example, Cartan subalgebras it is obvious that \( A_\ell \neq 0 \). In fact note that

\[
[\mathfrak{g}^y] \subset A_\ell
\] (2.27)

for any \( y \in \text{Reg} \mathfrak{g} \) since, as one knows, \( \mathfrak{g}^y \) is abelian if \( y \) is regular. Clearly \( A_\ell \) is a \( G \) submodule of \( \wedge^{\ell} \mathfrak{g} \). On the other hand, let \( m_\ell \) be the maximal value of \( \text{Cas} \) on \( \wedge^\ell \mathfrak{g} \) and let \( M_\ell \) be the corresponding Cas eigenspace. Again, clearly \( M_\ell \) is a \( G \)-submodule of \( \wedge^\ell \mathfrak{g} \). From the definition of \( M_\ell \) it is obvious that \( \text{Hom}_G(M_\ell, \wedge^\ell \mathfrak{g}/M_\ell) = 0 \). Since \( B| \wedge^\ell \mathfrak{g} \) is nonsingular it follows that

\[
\mathcal{B}|M_\ell \text{ is nonsingular}
\] (2.28)
and hence $M_\ell$ is self-contragredient. Noting the $1/2$ in (2.1.7) of [K3] the following result is a special case of Theorem (5), p. 156 in [K3].

**Theorem 2.5.** One has

$$A_\ell = M_\ell$$

and in addition

$$m_\ell = \ell.$$  \hspace{1cm} (2.30)

For any ordered subset $\Phi \subset \Delta$, $\Phi = \{\varphi_1, \ldots, \varphi_k\}$, let $e_\Phi = e_{\varphi_1} \wedge \cdots \wedge e_{\varphi_k}$ and put $\langle \Phi \rangle = \sum_{\varphi \in \Phi} \varphi$ so that with respect to $h$,

$$e_\Phi \in \land^k g$$

is a weight vector of weight $\langle \Phi \rangle$.  \hspace{1cm} (2.31)

Let $b \subset g$ be the Borel subalgebra of $g$ spanned by $h$ and $\{e_\varphi\}$, for $\varphi \in \Delta_+$, and put $n = [b, b]$. Any ideal $a$ of $b$ where $a \subset n$ is necessarily spanned by root vectors. We will say that $\Phi$, as above, is an ideal of $\Delta_+$ if $\Phi \subset \Delta_+$ and $a_\Phi = \sum_{i=1}^k C e_{\varphi_i}$ is an ideal in $b$.

**Remark 2.6.** One notes that if $\Phi$ is an ideal of $\Delta_+$ and $V_\Phi \subset \land^k g$ is the $G$-module spanned by $G \cdot e_\Phi$, then $V_\Phi$ is irreducible having $e_\Phi$ as highest weight vector and $\langle \Phi \rangle$ as highest weight.

As already noted in [K3] (see bottom of p. 158) it is immediate that if $a$ is any abelian ideal in $b$, then $a \subset n$ so that $a = a_\Phi$ for an ideal $\Phi \subset \Delta_+$. Much more subtly it has been established in [KW] (see Lemma 12, p. 113 in [KW]) that any ideal $a$ of $b$ having dimension $\ell$ is in fact abelian. Let $\mathcal{I}$ be the (obviously finite) set of all ideals $\Phi$ in $\Delta_+$ which have cardinality $\ell$. If $\Phi_1, \Phi_2 \in \mathcal{I}$ are distinct, then $\langle \Phi_1 \rangle \neq \langle \Phi_2 \rangle$ by Theorem (7), p. 158 in [K3] so that $V_{\Phi_1}$ are inequivalent $g$ and $G$ modules. Then Theorem (8), p. 159 in [K3] implies
**Theorem 2.7.** $M_\ell$ is a multiplicity one $G$-module. In fact

$$M_\ell = \bigoplus_{\Phi \in \mathcal{I}} V_\Phi$$  \hspace{1cm} (2.32)

so the number of irreducible components in $M_\ell$ is the cardinality of $\mathcal{I}$.

**Remark 2.8.** In the general case we do not have a formula for $\text{card} \, \mathcal{I}$ although computing this number in any given case does not seem to be too difficult. In the special case where $g \cong \text{Lie} \, \text{Sl}(n, \mathbb{C})$ one easily has a bijective correspondence of $\mathcal{I}$ with the set of all Young tableaux of size $n-1$ so that in this case

$$\text{card} \, \mathcal{I} = p(n - 1)$$  \hspace{1cm} (2.33)

where $p$ here is the classical partition function.

Let

$$\tau : \wedge^\ell g \to \wedge^{2r} g$$  \hspace{1cm} (2.34)

be the $G$-ismorphism defined by putting $\tau(u) = u^*$ recalling that $u^* = \iota(u) \mu$. Let $M_{2r} = \tau(M_\ell)$.

**Theorem 2.9.** $\tau$ is a $B$-isomorphism so that $B|_{M_{2r}}$ is nonsingular. Furthermore $\ell$ is the maximal eigenvalue of $\text{Cas}_{\text{on}} \wedge^{2r} g$ and $M_{2r}$ is the corresponding eigenspace. As $G$ modules one has

$$M_\ell \cong M_{2r}$$  \hspace{1cm} (2.35)

so that $M_{2r}$ is a multiplicity 1 module where in fact

$$M_{2r} \cong \bigoplus_{\Phi \in \mathcal{I}} V_\Phi.$$  \hspace{1cm} (2.36)

We recall the $V_\Phi$ is an irreducible $G$-module with highest weight $\langle \Phi \rangle$. See (2.31).

**Proof.** The first statement follows from Proposition 1.1. The remaining statements are immediate from Theorem 2.7 since $\tau$ is a $G$-isomorphism. QED
In light of equality $M_\ell = A_\ell$ (see (2.9)) Ranee Brylinski in her thesis (see [RB]) proved that $M_\ell$ is the span of $G \cdot [h]$. The thesis however has not been published. A stronger theorem (motivated by her result) appears in [KW]. The following result is just Corollary 2, p. 105 in [KW].

**Theorem 2.10.** $M_\ell$ is the span of $G \cdot [g^x]$ for any $x \in \text{Reg } g$.

Now by (2.12) and (2.12a) one has

$$\mathbb{C} dW p_1(x) \wedge \cdots \wedge dW p_\ell(x) = [g^x]$$

(2.37)

for any $x \in \text{Reg } g$. Using Theorem 2.3 we can now transfer Theorem 2.10 to $M_{2r}$ where it will have consequences for the structure of the space of functions $M \subset H^r$.

**Theorem 2.11.** $M_{2r}$ is the span of $G \cdot (\gamma_r(\frac{x^r}{r!}))$ for any $x \in \text{Reg } g$.

Proof. This is immediate from Theorem 2.3, Theorem 2.10, (2.37) and the fact that $\tau$ is a $G$-isomorphism. QED.

Let $N_{2r}$ be the $\mathcal{B}$ orthogonal subspace to $M_{2r}$ in $\wedge^{2r} g$. By the first statement in Theorem 2.9 one has a $\mathcal{B}$ orthogonal $G$-module decomposition $\wedge^{2r} g$,

$$\wedge^{2r} g = N_{2r} \oplus M_{2r}.$$  

(2.38)

**Remark 2.12.** Note that by Theorem 2.9 any eigenvalue of Cas in $N_{2r}$ is less than $\ell$.

We return now to our $G$-space $M$ of homogeneous harmonic polynomials on $g$ of degree $r$ which define $\text{Sing } g$. We recapitulate some of the properties of $M = \Gamma(\wedge^{2r} g)$ already established in this paper. Let $w_k \in g$, $k = 1, \ldots, 2r$, be linearly independent and let $z_i \in g$, $i = 1, \ldots, \ell$, be linearly independent and $\mathcal{B}$ orthogonal to the $w_k$. Then
for suitable generators $p_j, j = 1, \ldots, \ell$, of $J = S(g)^G$, we have

(1) $\Gamma(w_i \wedge \cdots \wedge w_{2r})$ is explicitly given by (1.23)

(2) $\Gamma(w_i \wedge \cdots \wedge w_{2r})$ is given as (up to scalar multiplication) $det \partial_{z_i} p_j$. See Theorem 2.4.

(3) If $f \in M$, then $f|_a$, where $a$ is any Cartan subalgebra or

$$a = g^e$$

for $e$ principal nilpotent, is given in Theorems 1.6 and 1.7.

We now determine the $G$-module structure of $M$,

**Theorem 2.13.** $N_{2r} = \text{Ker} \Gamma$ and

$$\Gamma : M_{2r} \to M$$

(2.39)

is a $G$-isomorphism so that as $G$-modules

$$M \cong M_{2r} \cong M_\ell = A_\ell$$

(2.40)

where we recall $A_\ell \subset \wedge^\ell g$ has been defined in [K3] as the span of $[s]$ over all abelian subalgebras $s \subset g$ of dimension $\ell$.

Furthermore we have defined $\mathcal{I}$ as the set of all ideals $\Phi$ in $\Delta_+$ of cardinality $\ell$, parameterizing with the notation $a_\Phi$, the set of all ideals $a$ of $b$ having dimension $\ell$. See Remark 2.6.

Moreover $M$ is a multiplicity one $G$-module with $\text{card}\mathcal{I}$ irreducible components. In addition $\mathcal{I}$ parameterizes these components in the sense that the component corresponding to $\Phi \in \mathcal{I}$ is equivalent to $V_\Phi$, using the notation of Remark 2.6, and hence has highest weight $\langle \Phi \rangle$. Finally Cas takes the value $\ell$ on each and every irreducible component of $M$.

**Proof.** By (1.27) and (1.29) one has

$$\Gamma(\zeta)(x) = (\zeta, \gamma_r(x^r) \frac{x^r}{r!})$$

(2.41)
for any $x \in \mathfrak{g}$ and any $\zeta \in \wedge^{2r}\mathfrak{g}$. Of course $\gamma_r(\frac{x^r}{r!}) = 0$ for any $x \in \text{Sing} \mathfrak{g}$ (see (2.12a) and Theorem 2.3). However $M_{2r}$ is the span of $G \cdot \gamma_r(\frac{x^r}{r!})$ for any $x \in \text{Reg} \mathfrak{g}$ by Theorem 2.11. Thus not only does (2.41) imply that $N_{2r} \subset \text{Ker} \Gamma$ but $N_{2r} = \text{Ker} \Gamma$ since if $\zeta \in M_{2r}$ and $x \in \text{Reg} \mathfrak{g}$ there exists $a \in G$ such that if $y = a \cdot x$, then $\Gamma(\zeta)(y) \neq 0$ by Theorem 2.11 and the nonsingularity of $\mathcal{B}|M_{2r}$, as asserted in Theorem 2.9. Since $\Gamma$ is a $G$-map one has the isomorphism (2.39). The remaining statements follow from Theorem 2.5 and Theorem 2.9. QED

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