On Beurling’s sampling theorem in $\mathbb{R}^n$

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Abstract

We present an elementary proof of the classical Beurling sampling theorem which gives a sufficient condition for sampling of multi-dimensional band-limited functions.

1 Introduction

Let $\mathcal{S} \subset \mathbb{R}^n$, $n \geq 1$, be a compact. The Bernstein space $B_{\mathcal{S}}$ consists of all bounded functions on $\mathbb{R}^n$ whose spectrum belongs to $\mathcal{S}$. The latter means that

$$\int_{\mathbb{R}^n} f(x) \hat{\varphi}(x) \, dx = 0, \quad f \in B_{\mathcal{S}},$$

for every smooth function $\varphi(x)$ whose support belongs to a ball disjoint from $\mathcal{S}$. Here $\hat{\varphi}$ denotes the Fourier transform

$$\hat{\varphi}(x) = \int_{\mathbb{R}^n} e^{-it \cdot x} \varphi(t) \, dt.$$

A set $\Lambda \subset \mathbb{R}^n$ is called a sampling set for $B_{\mathcal{S}}$, if there is a positive constant $C$ such that

$$\|f\|_{\infty} \leq C\|f|_{\Lambda}\|_{\infty}, \quad \text{for every} \quad f \in B_{\mathcal{S}},$$

where

$$\|f\|_{\infty} := \sup_{x \in \mathbb{R}^n} |f(x)|, \quad \|f|_{\Lambda}\|_{\infty} := \sup_{\lambda \in \Lambda} |f(\lambda)|.$$

It is a classical problem to determine when $\Lambda$ constitutes a sampling set for $B_{\mathcal{S}}$. Beurling discovered the importance of the lower uniform density $D^{-}(\Lambda)$ of $\Lambda$ for this problem:

$$D^{-}(\Lambda) := \lim_{r \to \infty} \min_{x \in \mathbb{R}^n} \frac{\text{Card}(\Lambda \cap (x + r\mathcal{B}))}{|r\mathcal{B}|},$$
where $\mathcal{B}$ is the unit ball in $\mathbb{R}^n$, $x + r\mathcal{B}$ is the ball of radius $r$ centered at $x$ and $|S|$ denotes the measure of a set $S$. In [2] he proved the following

**Theorem 1** Let $S = [a, b] \subset \mathbb{R}$. Then $\Lambda \subset \mathbb{R}$ is a sampling set for $B_{S}$ if and only if

$$D^{-}(\Lambda) > |S|/2\pi.$$  

(1)

Hence, when $S$ is an interval in $\mathbb{R}$, the sampling problem can be solved in terms of the density $D^{-}(\Lambda)$. Condition $D^{-}(\Lambda) \geq |S|/(2\pi)^n$ remains necessary for sampling in $B_{S}$, for every compact set $S \subset \mathbb{R}^n$. This follows from a general result of Landau [6]. On the other hand, simple examples show that in dimension one condition (1) ceases to be sufficient already when $S$ is a union of two intervals.

A new phenomenon occurs in several dimensions: Even for the simplest sets $S$ like a ball or a cube, no sufficient conditions for sampling in $B_{S}$ can be expressed in terms of $D^{-}(\Lambda)$. The reason for that is that the zeros of the multi-dimensional entire functions are not discrete. One can check that if $S \subset \mathbb{R}^n$ contains at least two points, then there are functions $f \in B_{S}$ whose zero set contains sets $\Lambda \subset \mathbb{R}^n$ with arbitrarily large $D^{-}(\Lambda)$. Clearly, if a function $f \in B_{S}$ vanishes on $\Lambda$, then $\Lambda$ is not a sampling set for $B_{S}$ (see also discussion in [8], pp. 122–123).

In [1] Beurling obtained the following sufficient condition for sampling in $B_{S}$:

**Theorem 2** Assume $\Lambda \subset \mathbb{R}^n$, $n \geq 1$, and $\rho < \frac{n}{2}$ satisfy

$$\Lambda + \rho \mathcal{B} = \mathbb{R}^n.$$  

Then

$$\|f\|_{\infty} \leq \frac{1}{1 - \sin \rho} \|f|_{\Lambda}\|_{\infty}, \text{ for every } f \in B_{\mathcal{B}},$$  

(2)

and so $\Lambda$ is a sampling set for $B_{\mathcal{B}}$.

In fact, Beurling in [1] proves a result on balayage of Fourier–Stieltjes transforms which is equivalent to Theorem 2: For every Dirac’s measure $\delta_{\xi}$, there exists a finite measure with masses on $\Lambda$ such that the values of their Fourier–Stieltjes transforms agree in the ball $\mathcal{B}$. We use a completely different elementary approach which allows us to get a more general result, see Theorem 3 below. We
shall see that unlike the case of interpolation in several dimensions (see [7]), the ”Beurling-type” sampling is in fact a one-dimensional phenomenon.

Observe that condition $\Lambda + \rho \mathcal{B} = \mathbb{R}^n$ in Theorem 2 means that $\Lambda$ is a $\rho$-net, i.e. for every $x \in \mathbb{R}^n$ there exists $\lambda \in \Lambda$ with $|x - \lambda| \leq \rho$. Hence, every $\rho$-net with $\rho < \pi/2$ is a sampling set for $B_\mathcal{B}$. This is sharp: Beurling shows that the theorem ceases to be true for $\pi/2$–nets.

Let us in what follows denote by $\mathcal{K}$ a closed convex central-symmetric body with positive measure. Then

$$
\mathcal{K}^o := \{x \in \mathbb{R}^n : x \cdot t \leq 1 \text{ for all } t \in \mathcal{K}\}
$$

denotes the polar body of $\mathcal{K}$. In particular, we have $\mathcal{B}^o = \mathcal{B}$.

The following propositions are formulated in [1] without proof:

(i) Estimate (2) in Theorem 2 can be replaced with a better one:

$$
\|f\|_\infty \leq \frac{1}{\cos \rho} \|f|_\Lambda\|_\infty.
$$

(ii) Every set $\Lambda$ satisfying $\Lambda + \rho \mathcal{K}^o = \mathbb{R}^n$ with some $\rho < \pi/2$ is a sampling set for $B_\mathcal{K}$.

We show that estimate (3) holds for every convex central-symmetric body $\mathcal{K}$:

**Theorem 3** Assume $\Lambda \subset \mathbb{R}^n$ and $\rho < \frac{\pi}{2}$ satisfy

$$
\Lambda + \rho \mathcal{K}^o = \mathbb{R}^n.
$$

Then (3) is true, and so $\Lambda$ is a sampling set for $B_\mathcal{K}$.

Clearly, condition (4) means that for every $x \in \mathbb{R}^n$ there exists $\lambda \in \Lambda$ such that $\|x - \lambda\|_{\mathcal{K}^o} \leq \rho$, where $\|x\|_{\mathcal{K}^o} := \inf_{a > 0} \{x \in a\mathcal{K}^o\}$. Hence, every $\rho$-net in the norm $\|\cdot\|_{\mathcal{K}^o}$ is a sampling set for $B_\mathcal{K}$ provided $\rho < \pi/2$. This is sharp:

**Proposition 1.** Suppose a closed convex central-symmetric body $\mathcal{S}$ contains a point $x_0$ with $\|x_0\|_{\mathcal{K}^o} = \pi/2$. Then there exists $\Lambda \subset \mathbb{R}^n$ with $\Lambda + \mathcal{S} = \mathbb{R}^n$ and a function $f \in B_\mathcal{K}$ such that $f(\lambda) = 0$, $\lambda \in \Lambda$.

**Corollary 1.** Suppose a closed convex central-symmetric body $\mathcal{S}$ has the property that every set $\Lambda \subset \mathbb{R}^n$ satisfying $\Lambda + \mathcal{S} = \mathbb{R}^n$ is a sampling set for $B_\mathcal{K}$. Then $\mathcal{S} \subset \rho \mathcal{K}^o$ for some $\rho < \pi/2$. 

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2. Proofs

1. Proof of Proposition 1. By assumption, there exist $x_0 \in S$ and $t_0 \in K$ such that $x_0 \cdot t_0 = \pi/2$. The spectrum of the function $\sin(x \cdot t_0)$ consists of two points $\pm t_0 \in K$, and so $\sin(x \cdot t_0) \in B_K$.

Let $\Lambda := \{x \in \mathbb{R}^n : x \cdot t_0 \in \pi \mathbb{Z}\}$ be the zero set of $\sin(x \cdot t_0)$. Denote by $I = \{-1 \leq \tau \leq 1\} \subseteq S$ the interval from $-x_0$ to $x_0$. Clearly, for every point $y \in \mathbb{R}^n$ there exist $n \in \mathbb{Z}$ and $-1 \leq \tau \leq 1$ such that $y \cdot t_0 = \pi n - \pi \tau/2$. Hence, $y - \tau x_0 \in \Lambda$, which implies $\Lambda + I = \mathbb{R}^n$. □

2. Proof of Theorem 3. We shall deduce Theorem 3 from the following

**Lemma 1** Suppose a function $g \in B_{[-\tau, \tau]}$ satisfies $|g(0)| = \|g\|_{\infty}$. Then

$$|g(u)| \geq |g(0)| \cos(\tau u), \quad |u| < \pi/2\tau. \quad (5)$$

This lemma is proved in [3, proof of Theorem 4]. For completeness of presentation, we sketch the proof below.

Let us now prove Theorem 3. Take any function $f \in B_K$. Assume first that $|f|$ attains maximum on $\mathbb{R}^n$, i.e. $|f(x_0)| = \|f\|_{\infty}$ for some $x_0 \in \mathbb{R}^n$. By (4), there exists $\lambda_0 \in \Lambda$ with $\|\lambda_0 - x_0\|_{\mathcal{K}^0} \leq \rho$. Consider the function of one variable $g(u) := f(x_0 + u(\lambda_0 - x_0)), u \in \mathbb{R}$. One may check that $g \in B_{[-\tau, \tau]}$ with $\tau = \|\lambda_0 - x_0\|_{\mathcal{K}^0}$. Also, clearly $|g(0)| = \|g\|_{\infty}$ and $g(1) = f(\lambda_0)$. Since $\tau \leq \rho < \pi/2$, we may use inequality (5) with $u = 1$:

$$\|f\|_{\infty} = |f(x_0)| = |g(0)| \leq \frac{g(1)}{\cos \tau} \leq \frac{|f(\lambda_0)|}{\cos \rho} \leq \frac{1}{\cos \rho} \|f|_{\lambda}\|_{\infty}. \quad (6)$$

If $|f|$ does not attain maximum on $\mathbb{R}^n$, we consider the function $f_\epsilon(x) := f(x)\varphi(\epsilon x)$, where $\varphi \in B_{K^0}$ is any function satisfying $\varphi(0) = 1$ and $\varphi(x) \to 0$ as $|x| \to \infty$. It is clear that $f_\epsilon \in B_{K^0+\mathcal{B}}$ and that $f_\epsilon$ attains maximum on $\mathbb{R}^n$. Set $g_\epsilon(u) := f_\epsilon(x_0 + u(\lambda_0 - x_0)), u \in \mathbb{R}$, where $x_0$ and $\lambda_0$ are chosen so that $|g_\epsilon(0)| = \|f_\epsilon\|_{\infty}$ and $\|\lambda_0 - x_0\|_{\mathcal{K}^0} \leq \rho$. We have $g \in B_{[-\tau - \delta, \tau + \delta]}$, where $\tau = \|\lambda_0 - x_0\|_{\mathcal{K}^0} \leq \rho < \pi/2$ and $\delta = \delta(\epsilon) \to 0$ as $\epsilon \to 0$. So, if $\epsilon$ is so small that $\tau + \epsilon < \pi/2$, we may repeat the argument above to obtain $\|f_\epsilon\|_{\infty} \leq \|f_\epsilon|_{\Lambda}\|_{\infty}/\cos(\rho + \delta)$. By letting $\epsilon \to 0$, we obtain (3). □
3. Proof of Lemma 1

1. The proof in [3] is based on the following result from [4] (for some extension see [5]): Let \( f \in B_{[-\tau, \tau]} \) be a real function satisfying \(-1 \leq f(x) \leq 1\) for all \( x \in \mathbb{R} \). Then for every real \( a \) the function \( \cos(\tau z + a) - f(z) \) vanishes identically or else it has only real zeros. Moreover it has a zero in every interval where \( \cos(\tau z + a) \) varies between -1 and 1 and all the zeros are simple, except perhaps at points on the real axis where \( f(x) = \pm 1 \).

Sketch of proof. We may assume \( a = 0 \) and \( \tau = 1 \). Consider the function

\[ f_\epsilon(z) := (1 - \epsilon) \sin(\epsilon z) f((1 - \epsilon)z). \]

One may check that \( f_\epsilon \in B_{[-1,1]} \), \(-1 < f(t) < 1, t \in \mathbb{R}, \) and that the estimate holds

\[ |f_\epsilon(z)| \leq \frac{e|y|}{\epsilon|z|}, z = x + iy \in \mathbb{C}. \]

This shows that \( |f_\epsilon(z)| < |\cos z| \) when \( z \) lies on a rectangular contour \( \gamma \) consisting of segments of the lines \( x = \pm N\pi, y = \pm N, \) where \( N \) is every large enough integer. By Rouché’s theorem, the function \( \cos z - f_\epsilon(z) \) has the same number of zeros in \( \gamma \) as \( \cos z \), that is, \( 2N \) zeros. On the real axis \( |f_\epsilon| \leq 1 - \epsilon \). Hence, \( \cos z - f_\epsilon(z) \) is alternately plus and minus at the \( 2N + 1 \) points \( k\pi, |k| \leq N, \) so it has \( 2N \) real zeros inside \( \gamma \). Taking larger values of \( N \) we see that \( \cos z - f_\epsilon(z) \) has exclusively real and simple zeros, which lie in the intervals \( (k\pi, (k + 1)\pi) \).

The zeros of \( \cos z - f(z) \) are limit points of the zeros of \( \cos z - f_\epsilon(z) \) as \( \epsilon \to 0 \). Thus \( \cos z - f(z) \) cannot have non-real zeros. Moreover, it has an infinite number of real zeros which are all simple, except those at the points \( k\pi \) iff \( f(k\pi) = (-1)^k \). Every interval \( k\pi < z < (k + 1)\pi \) at the endpoints of which \( |f(t)| < 1 \) contains exactly one zero. If \( f(k\pi) = (-1)^k \), we have a double zero at \( k\pi \) but no further zeros in the interior or at the endpoints of the interval \( ((k - 1)\pi, (k + 1)\pi) \).

2. It suffices to prove Lemma 1 for real functions \( f \in B_{[-\pi, \pi]} \). Since \( f \) has a local maximum at \( t = 0 \), the function \( f(t) - \cos \tau t \) has a repeated zero at \( t = 0 \). By the discussion above we see that either \( f(t) \) is identically equal to \( \cos \tau t \) or \( f(t) - \cos \tau t \) does not vanish on \( [-\pi/\tau, 0) \cup (0, \pi/\tau] \). Since \( |f(\pi)| \leq 1 \), it follows that \( f(t) > \cos \tau t \) on each of the intervals \( [-\pi/\tau, 0) \) and \( (0, \pi/\tau] \). □
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