A robust data completion method for 2D Laplacian Cauchy problems

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Abstract. The purpose is to propose an improved regularization method for data completion problems. This method is presented on the Cauchy problem for the Laplace equation in 2D situations. Many numerical simulations using finite element method highlight the efficiency of this new approach. In particular, it gives reconstructions with an increased accuracy, it is stable with respect to strong perturbations on the data and is able to deblur noisy data.

1. Introduction

The purpose is to propose an improved method to solve Cauchy inverse problems. These problems are called data completion problems. They arise in many engineering fields like thermal, electrostatic or elastostatic problems [1] and find applications in non destructive testing or in medical investigations. They consist in recovering data on a part of the boundary given the partial differential equation within the domain and overspecified data available on the remaining part of the boundary. Many methods have been introduced [2, 3, 4, 5, 6, 7, 8, 9] to solve this problem.

In order to improve the reconstruction of the normal derivative, we introduced a first order method [10]. It connects the determination of the solution of the Cauchy problem with the determination of two auxiliary functions, which are the partial derivatives of the solution, when the data are compatible. The first order problem needs boundary additional conditions which have to be derived from the given boundary conditions of the Cauchy problem. The evaluation of these informations requires to tangentially differentiate the boundary conditions. These tangential derivatives have to be numerically evaluated.

We also proposed a resolution method [11] for the Cauchy problem based on the same principle as the first order method but requiring no numerical evaluations of additional data. This method relies on a system of two boundary integral equations which connect the two partial derivatives of a harmonic function and not their normal derivatives. The purpose of this paper is to extend this inverse method to the finite element method.
2. Presentation of the method

2.1. The Cauchy problem for the Laplace equation

Here we are concerned with the Cauchy problem for the Laplace equation in 2D situations (1):

\[
\begin{align*}
\Delta u &= 0 \quad \text{in } \Omega \subset \mathbb{R}^2 \\
\quad u &= \phi_d \quad \text{on } \Gamma_d \\
\quad u' &= \psi_d \quad \text{on } \Gamma_d
\end{align*}
\]

On the part \( \Gamma_d \) of the boundary \( \Gamma = \partial \Omega \), overspecified boundary conditions are given. The aim is to calculate \( u \) and its normal derivative denoted by \( u' \) on \( \Gamma \setminus \Gamma_d \). This problem (1) admits a unique solution when the data \( \phi_d \) and \( \psi_d \) are compatible. But its solution is very sensitive to small data perturbations, since Cauchy problems are ill-posed problems [5, 9].

2.2. Weak formulation of a harmonic function

Let \( u \) be a harmonic function. We have the following weak formulation:

\[
\langle r(u, u'), v \rangle \equiv \int_{\Omega} \nabla u \nabla v \, d\Omega - \int_{\Gamma} u' v \, ds = 0 \quad \forall v \in H^1(\Omega)
\]

2.3. Properties of harmonic function partial derivatives

Under the regularity assumption \( u \in C^2(\Omega) \cap C^3(\Omega) \), we first obtain that \( \frac{\partial u}{\partial x_i} \) is harmonic \( \forall i = 1, 2 \)

\[
\Delta \left( \frac{\partial u}{\partial x_i} \right) = \frac{\partial (\Delta u)}{\partial x_i} = 0 \quad \text{in } \Omega, \forall i = 1, 2
\]

Next, using \( \Delta u = 0 \) on the boundary \( \Gamma \), we obtain:

\[
\frac{\partial}{\partial n} \frac{\partial u}{\partial x_1} = \frac{\partial}{\partial s} \frac{\partial u}{\partial x_1}, \quad \text{and} \quad \frac{\partial}{\partial n} \frac{\partial u}{\partial x_2} = - \frac{\partial}{\partial s} \frac{\partial u}{\partial x_2} \quad \forall x \in \Gamma
\]

The relation (2) can be written for \( \frac{\partial u}{\partial x_1} \) and \( \frac{\partial u}{\partial x_2} \). After integration by parts, we obtain the two equations (5-6):

\[
\begin{align*}
\langle R_1(u_1, u_2), v \rangle &\equiv \int_{\Omega} \nabla (\frac{\partial u}{\partial x_1}) \nabla v \, d\Omega - \int_{\Gamma} \frac{\partial u}{\partial x_2} \frac{\partial v}{\partial s} \, ds = 0 \quad \forall v \in H^1(\Omega) \\
\langle R_2(u_1, u_2), v \rangle &\equiv \int_{\Omega} \nabla (\frac{\partial u}{\partial x_2}) \nabla v \, d\Omega + \int_{\Gamma} \frac{\partial u}{\partial x_1} \frac{\partial v}{\partial s} \, ds = 0 \quad \forall v \in H^1(\Omega)
\end{align*}
\]

2.4. An iterative Tikhonov type algorithm with penalization

We introduce the following problem (7):

\[
\begin{align*}
r(u, n_1u_1 + n_2u_2) &= 0 \quad (a) \\
u &= \phi_d \quad \text{on } \Gamma_d \quad (b) \\
R_1(u_1, u_2) &= 0 \quad (d)
\end{align*}
\]

It can be shown that for compatible data, \( u_1 \) and \( u_2 \) are respectively the partial derivatives of \( u \). As in [4], we distinguish the reliable information (equilibrium equation (7a)) and the uncertain
information (boundary conditions (7b and 7c)). We introduce the iterative Tikhonov type algorithm where the equations (7d and 7e) are taken into account through penalization terms:

\[
\begin{align*}
\text{Let consider } & c > 0, \alpha > 0 \text{ and } U^0 = 0 \\
\text{Find } U^{k+1} & = (u^{k+1}, u_1^{k+1}, u_2^{k+1}) \in H(\Gamma) \text{ such that } \\
J_k(U^{k+1}) & \leq J_k(W) \quad \forall W \in H(\Gamma) \text{ with:} \\
J_k(W) & = ||w - \phi_d||_D^2 + ||(w_1n_1 + w_2n_2) - \psi_d||_D^2 + c||w - u||_H^2 \\
& + \alpha (||R_1(w_1, w_2)||_F^2 + ||R_2(w_1, w_2)||_F^2)
\end{align*}
\]

where \( H(\Gamma) \) is a space of functions traces satisfying equation (7a).

3. FEM implementation

In practice the equation (2) is numerically approximated using the simplest finite element method. The boundary \( \Gamma \) is approximated using \( N \) finite elements. After condensation on the boundary, we obtained a system of \( N \) linear equations:

\[
AU + B_1U_1 + B_2U_2 = 0
\]

The matrices \( A, B_1 \) and \( B_2 \) only depend on the domain mesh. The vectors \( U, U_1 \) and \( U_2 \) are respectively the discretized values of the function and of its partial derivatives at each node on the boundary. The linear system (9) of \( N \) equations for \( 3N \) unknowns define the discretization of the space \( H(\Gamma) \). Considering the same discretization, the system of \( N \) equations (10) (respectively the system (11)) represents the discrete formulation of equation (5) ((respectively of equation (6)):

\[ AU_1 - CU_2 = 0 \]
\[ CU_1 + AU_2 = 0 \]

For \( u = x_1 + x_2 \), one gets \( U_1 = (1, ..., 1) \) and \( U_2 = (1, ..., 1) \). Obviously these two vectors satisfy the equations (10)(11). So these equations are not independent.

A discrete formulation of the continuous iterative process (8) leads at each step to an optimization problem under equality constraints:

\[
\begin{align*}
\text{Find } \Psi^{k+1} & = (U^{k+1}, U_1^{k+1}, U_2^{k+1}) \in \mathbb{R}^{3N} \text{ such that } \\
J_k(\Psi^{k+1}) & \leq J_k(\Psi) \quad \forall \Psi \in \mathbb{R}^{3N} \\
\text{under the equality constraints } \\
AU + B_1U_1 + B_2U_2 & = 0
\end{align*}
\]

We introduce a \( N \) vector of Lagrange multipliers to take into account the \( N \) equality constraints. Each iteration in the algorithm needs to solve a system of \( 4N \) linear equations with \( 4N \) unknowns. The corresponding system for the first order method [10] involved a system of \( 9N + 1 \) equations for \( 9N + 1 \) unknowns and more over needed to numerically differentiate twice \( \phi_d \) and once \( \psi_d \).

4. Numerical simulations

Numerical simulations are performed on a square domain \((\Omega = [0, 1] \times [0, 1])\). The \( \Gamma_d \) boundary part is composed of two sides \((y = 0 \text{ and } x = 1)\) (Figure 1). The function \( u \) to be recovered is defined by:

\[
u(x, y) = \cos(x) \cosh(y) + \sin(x) \sinh(y).
\]
4.1. Behaviour with respect to $c$ parameter

The boundary mesh is a regular one with 80 nodes on each side. Figure 2 displays $L^2(\Gamma)$ errors on $u$ and on its normal derivative $u'$. As expected this error does not depend on the parameter $c$ value. The parameter $c$ only acts on the convergence rate. Figure 3 shows the evolution of the number of iterations $n$ required to converge with respect to the parameter $c$.

4.2. Behaviour with respect to boundary mesh

Figure 4 displays $L^2(\Gamma)$ errors with respect to the number of nodes on the boundary. We can notice that the inverse method is stable when the number of nodes increases.

4.3. Behaviour with respect to $\alpha$ parameter

The parameter $\alpha$ is a penalization parameter and the obtained solution depends on its choice. Figure 5 shows the evolution of $L^2(\Gamma)$ errors with respect to $-\log\alpha$. We can notice that the $\alpha$ value can be choosen in a large range ($[10^{-7} : 10^{-4}]$). But this choice must be made independently from the solution on the whole boundary. We plot $\|u - \phi_d\|_{L^2_d}$ and $\|u' - \psi_d\|_{L^2_d}$ (called $L^2(\Gamma_d)$ norms) with respect to the $\alpha$ value (figure 6). We obtain classical L-curves [12] which allow us to determine the optimal $\alpha$ value (roughly $10^{-6}$).
4.4. Behaviour with respect to noisy data

In this section, we discuss the numerical stability of the algorithm to noisy data. For each case, the noise is a random one whose amplitude is equal to a percentage of the data maximum value on the boundary. Firstly, we look at the algorithm stability when only the data $\phi_d$ is noisy. Figure 7 gives the evolution of $L^2(\Gamma)$ errors with respect to the noise level. Figures 9 and 10 give the function and its normal derivative reconstructions when the noise level is equal to 10%. We can notice that the algorithm is able to deblur the noisy data $\phi_d$.

Secondly, we look at the algorithm stability when only the data $\psi_d$ is noisy. Figure 8 gives the evolution of $L^2(\Gamma)$ errors with respect to the noise level. Figures 11 and 12 give the function and its normal derivative reconstructions when the noise level is equal to 10%. We can notice that the algorithm, contrary to the algorithm presented in [4], is able to deblur the noisy data $\psi_d$. 

Figure 5. Influence of $\alpha$ on $L^2(\Gamma)$ errors

Figure 6. Influence of $\alpha$ on $L^2(\Gamma_d)$ norms

Figure 7. Influence of $u$ noise level (in %) on $L^2(\Gamma)$ errors

Figure 8. Influence of $u'$ noise level (in %) on $L^2(\Gamma)$ errors
4.5. **Comparison with other methods**

In this section, we compare the reconstructions obtained with this new method with ones obtained by other methods.

In the first case, the data are not noisy and are prescribed on half of the boundary. It was studied in [13] using a Tikhonov like method, in [4] using a fading regularization method and in [8] using an iterative method. All these methods gave similar results. This example was also studied using a first order method [10]. Figures 13 and 14 display all these reconstructions. The new method and the method introduced in [10] give more accurate normal derivative reconstructions.

Using [4], it has been observed [14] that FEM and BEM give identical reconstructions for $u$ and $u'$. So the accuracy of the $u'$ reconstruction seems not to depend on the chosen discretization method. The second case, where the $\phi_d$ and $\psi_d$ data are both noisy, was studied in [10]. This method needs to differentiate numerically the boundary data $\phi_d$ and $\psi_d$. Figures 15 and 16 display the reconstructions and show that the present method is more accurate.
5. Concluding remarks
A new method has been introduced for 2D Cauchy problems for the Laplace equation. It has introduced a first order penalization term. The numerical simulations prove the efficiency and the robustness of this approach. In particular, it gives $u'$ with an increased accuracy, it is stable with respect to strong perturbations on the data and is able to deblur the two boundary data $\phi_d$ and $\psi_d$ when they are noisy.

6. References
[1] Bui H D 1993 *Introduction aux problèmes inverses en mécanique des matériaux* (Paris: Eyrolles)
[2] Andrieux S, Baranger T N and Ben Abda A 2006 Solving Cauchy problems by minimizing an energy-like functional *Inv. Probl.* 22 115-33
[3] Bourgeois L 2005 A mixed formulation of quasi-reversibility to solve the Cauchy problem for Laplace’s equation *Inv. Probl.* 21 1087-1104
[4] Cimetière A, Delvare F, Jaoua M and Pons F 2001 Solution of the Cauchy problem using iterated Tikhonov regularization *Inv. Probl.* 17 555-70
[5] Engl H W, Hanke M and Neubauer A 2000 *Regularisation of inverse problems* (Dordrecht:Kluwer)
[6] Lattès R and Lions J L 1967 *Méthode de quasi-reversibilité et applications* (Paris:Dunod)
[7] Kozlov V A, Maz'ya V G and Fomin A F 1991 An iterative method for solving the Cauchy problem for elliptic equations *Comput. Math. Phys.* 31 45-52
[8] Lesnic D, Elliott L and Ingham D B 1997 An iterative boundary element method for solving the Cauchy problem for the Laplace equation *Eng. Anal. Bound. Elem.* 20 123-33
[9] Tikhonov A and Arsenine V 1977 *Méthode de résolution de problèmes mal posés* (Moscow:Mir)
[10] Delvare F and Cimetière A 2008 A first order method for the Cauchy problem for the Laplace equation using BEM, *Comput. Mech* 41 789-796
[11] Cimetière A, Delvare F and Pons F 2007 Système d’équations intégrales de frontière pour les dérivées partielles d’une fonction harmonique dans un ouvert du plan. Application à un problème inverse de Cauchy *Sème coll. nat. cal. struct.* Vol 2, 481-6
[12] Hansen C 1992 Analysis of Discrete Ill-Posed Problems by Means of the L-Curve *SIAM Review*, 34 561-80
[13] Hayashi K, Ohura Y and Onishi K 1992 Direct method of solution for general boundary value problem of the Laplace equation *Eng. Anal. Bound. Elem.* 26 763-71
[14] Delvare F and Cimetière A and Pons F 2008 An iterative boundary element method for Cauchy inverse problems *Comput. Mech* 28 291-302