Frequency spectrum of nonlinear oscillations and resonance phenomena for graphene plates

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Abstract

The paper studies oscillations of graphene plates under the hypothesis that deformations are much more than the thickness of plate. In this most realistic case the oscillations are described by the system of nonlinear partial differential equations (the Föppl-von Kármán equations). This system is reduced to one nonlinear ordinary differential equation and investigated by means of the Bogoliubov-Mitropolsky asymptotic methods. As a result, we have the real frequency spectrum for rectangular graphene plates. Next we have examined the nonlinear resonance effects under forced oscillations. These outcomes can apply for variable strain-induced pseudomagnetic fields. Such fields permit better to understand the properties of flexural phonons connected with transport process. Resonance phenomena play a leading role in application of graphene plates in engineering constructions.

Keywords: graphene; frequency spectrum; nonlinear oscillations; flexural phonons; Bogoliubov-Mitropolsky asymptotic methods; resonance phenomena

1. Introduction

A major direction of solid state physics is the investigation of graphene (Kittel [7] Chap.18). There is a need to develop fundamental analysis of graphene properties on the base of quantum and classical physics. These problems are examined in the comprehensive monograph of Katsnelson [6] as well as in the books of Gorbar & Sharapov [3], Mikhailov [14], Shafraniuk [17].

In particularly, in the ninth chapter of Katsnelson [6] the deformations of graphene plates are considered in the view of the theory elasticity and de-
scribed by the *Föppl-von Kármán equations*, which are sufficiently difficult for analysis. In addition, several works are devoted to the mechanical properties of graphene plates: Booth et al. [3]; Lee et al. [11]; Los, Fasolino & Katsnelson [12].

In our paper the solution of the Föppl-von Kármán equations is proposed on the base of nonlinear oscillations theory. We shall consider the nonlinear oscillations of defined mode presented in terms of the product of sinusoidal functions. Therefore, the stress function is described in the explicit form, and the system of nonlinear partial differential equations is reduced to one nonlinear ordinary differential equation. The latter is investigated by means of the *Bogoliubov-Mitropolsky asymptotic methods* [2]. As a result, we have the real frequency spectrum for graphene plates under the situation when deformations are much more than the thickness of plate. Next we considered the nonlinear forced oscillations and resonance phenomena.

In the tenth chapter of Katsnelson [6] the strain-induced *pseudomagnetic fields* are examined. The review of Amorim et al. [1] contains the all-round description of these subjects. Our results concerning the Föppl-von Kármán equations can help to study the properties of strained graphene.

The oscillations of graphene plates lead to the concept of *flexural phonons*. The properties of flexural phonons are connected with transport processes (see Katsnelson [6] Sec.11.4). The flexural phonons are considered as well in the papers of Stauber et al. [18]; Morozov et al. [15]; Mariani & von Oppen [13]; Castro et al. [4]; Ochoa et al. [16]. These researches are based on the linear theory of oscillations.

Besides, the investigation of oscillations has the important practical meaning. Graphene plates in engineering constructions undergo perturbations and will vibrate. This can lead to violations of work or even to the destruction of plate.

### 2. Föppl-von Kármán equations

In this section we consider the equations of oscillations for graphene plates. These equations are based on the elasticity theory.\(^1\) The coordinate axes \(x, y\) of a rectangular plate are directed along sides. We shall use such denotations: \(E\) is *Young’s modulus*, \(\sigma\) is the *Poisson ratio*, \(\rho\) is the mass

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\(^1\)See Landau & Lifshitz [9].
density, $h$ is the thickness of plate,

\[ \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \]  

\[ D = \frac{Eh^3}{12(1 - \sigma^2)}. \]  

Let $\chi(x, y)$ be the stress function, $\zeta(x, y)$ be the out-of-plane deflection of the plate. Then we get the next equations to describe the nonlinear oscillations of graphene plates:\[^2\]

\[ D\Delta^2\zeta - hL(\zeta, \chi) + h\rho\frac{\partial^2 \zeta}{\partial t^2} = 0, \]

\[ \Delta^2 \chi + \frac{E}{2}L(\zeta, \zeta) = 0, \]  

where

\[ L(\zeta, \chi) = \frac{\partial^2 \zeta}{\partial x^2} \frac{\partial^2 \chi}{\partial y^2} + \frac{\partial^2 \zeta}{\partial y^2} \frac{\partial^2 \chi}{\partial x^2} - 2 \frac{\partial^2 \zeta}{\partial x \partial y} \frac{\partial^2 \chi}{\partial x \partial y}. \]  

To avoid cumbersome calculations, we shall assume in the following that plate edges rest on a fixed support (Landau & Lifshitz §12).

3. Linear oscillations

Preparatory to analysing nonlinear oscillations, first we shall consider linear oscillations. Such vibrations will be under the hypothesis of small enough deformations. Then we derive the linear equation:\[^3\]

\[ D\Delta^2\zeta + h\rho\frac{\partial^2 \zeta}{\partial t^2} = 0. \]

Taking into account the boundary conditions, we present the solution of Eq.(6) in terms of the series

\[ \zeta(x, y, t) = \sum_{n,m} A_{mn}(t) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right), \]  

\[^2\]See Landau & Lifshitz §14.  

\[^3\]See Landau & Lifshitz §25.
where $a$ is the length of plate and $b$ is the width of plate, $m$ and $n$ are positive integers. Each member of the series corresponds to the defined mode of oscillations. Substituting the series (7) in Eq.(6), multiplying by $\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$ and integrating over all area of plate, we obtain the equation

$$\frac{d^2 A_{mn}}{dt^2} + \omega_{0,mn}^2 A_{mn} = 0,$$

where $\omega_{0,mn}$ are the frequencies of linear oscillations that are computed using the expression

$$\omega_{0,mn}^2 = \frac{Eh^2\pi^4}{12\rho(1 - \sigma^2)} \left[ \left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 \right]^2.$$

Consequently, we have found the frequency spectrum of linear oscillations for graphene plates.

Consider now a plane wave as the solution of Eq.(6). Substituting

$$\zeta = c \cdot \exp\left[ i \left( k_x x + k_y y - \omega_{0,mn} t \right) \right]$$

in Eq.(6), one obtains

$$-\rho \omega_{0,mn}^2 + \frac{D}{h^4} k^4 = 0.$$ (11)

Then

$$k^2 = \pi^2 \left[ \left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 \right].$$ (12)

Since arbitrary oscillations are described by the superposition of plane waves (10), the method of secondary quantization leads to the concept of flexural phonons with the energy $E = \hbar \omega_{0,mn}$ and the quasi-momentum $\vec{p} = \hbar \vec{k}$ (see Kosevich [8] Sec.6.6.; Landau & Lifshitz [10] §§71-72).

4. Nonlinear oscillations of presented mode

Let us consider the nonlinear oscillations described by Eqs.(3) and (4). We shall study the nonlinear oscillations of defined mode (for the fixed values $m$ and $n$):
\[ \zeta = f_{mn}(t)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right). \]  

(13)

Substituting this expression in

\[ L(\zeta, \zeta) = 2 \frac{\partial^2 \zeta}{\partial x^2} \frac{\partial^2 \zeta}{\partial y^2} - 2\left(\frac{\partial^2 \zeta}{\partial x \partial y}\right)^2, \]

(14)

one obtains

\[ L(\zeta, \zeta) = 2f_{mn}^2 \pi^4 \frac{m^2 n^2}{a^2 b^2} \left[ \sin^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi y}{b}\right) - \cos^2\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi y}{b}\right) \right] \]

(15)

As a result, we have the equation

\[ \Delta^2 \chi = \frac{1}{2} Ef_{mn}^2 \pi^4 \frac{m^2 n^2}{a^2 b^2} \left[ \cos^2\left(\frac{2m\pi x}{a}\right) + \cos^2\left(\frac{2n\pi y}{b}\right) \right]. \]

(16)

We present the solution of this equation in terms of the expression

\[ \chi = A \cdot \cos\frac{2m\pi x}{a} + B \cdot \cos\frac{2n\pi y}{b}. \]

(17)

Then we shall find that

\[ A = \frac{Ef_{mn}^2 n^2 a^2}{32 m^2 b^2}, \quad B = \frac{Ef_{mn}^2 m^2 b^2}{32 n^2 a^2}. \]

(18)

Substituting the expressions (13) and (17) in Eq.(3), one derives

\[ \frac{D^2}{\partial t^2} \zeta - \left(\frac{\partial^2 \zeta}{\partial y^2} \frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial x^2} \frac{\partial^2 \zeta}{\partial y^2} - 2\frac{\partial^2 \zeta}{\partial x \partial y} \frac{\partial^2 \zeta}{\partial x \partial y}\right) + \rho \frac{\partial^2 \zeta}{\partial t^2} = \]

\[ = \left[ f_{mn} \frac{D^2}{\partial t^2} \pi^4 \left(\frac{m^2}{a^2} + \frac{n^4}{b^4} + 2\frac{m^2 n^2}{a^2 b^2}\right) - B\left(\frac{2\pi n}{b}\right)^2 f_{mn} \frac{m^2 x^2}{a^2} \cos\frac{2\pi n y}{b} - \right. \]

\[ - A\left(\frac{2\pi m}{a}\right)^2 f_{mn} \frac{n^2 y^2}{b^2} \cos\frac{2\pi n y}{a} + \rho \frac{d^2 f_{mn}}{dt^2} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} = 0. \]

(19)

Using the values of A and B from the relations (18), Eq.(19) is transformed to

\[ \left[ f_{mn} \frac{D^2}{\partial t^2} \pi^4 \left(\frac{m^2}{a^2} + \frac{n^4}{b^2}\right) - Ef_{mn}^3 \pi^4 \left(\frac{m^4}{a^4} \cos\frac{2\pi n y}{b} + \frac{n^4}{b^4} \cos\frac{2\pi m x}{a}\right) + \right. \]

\[ + \rho \frac{d^2 f_{mn}}{dt^2} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} = 0. \]

(20)
Multiplying by $\sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$ and integrating over all area of plate, we shall find that

$$f_{mn} \frac{D}{h} \pi^4 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right) + \frac{Ef_{mn}^3}{16} \pi^4 \left( \frac{m^4}{a^4} + \frac{n^4}{b^4} \right) + \rho \frac{d^2 f_{mn}}{dt^2} = 0, \quad (21)$$

since

$$\int_0^a \sin^2\left(\frac{m\pi x}{a}\right) dx \int_0^b \sin^2\left(\frac{n\pi y}{b}\right) dy = \frac{ab}{4},$$

$$\int_0^a \sin^2\left(\frac{m\pi x}{a}\right) \cos\left(\frac{2\pi mx}{a}\right) dx \int_0^b \sin^2\left(\frac{n\pi y}{b}\right) dy = -\frac{ab}{8},$$

$$\int_0^a \sin^2\left(\frac{m\pi x}{a}\right) dx \int_0^b \sin^2\left(\frac{n\pi y}{b}\right) \cos\left(\frac{2\pi ny}{b}\right) dy = -\frac{ab}{8}. \quad (22)$$

We shall rewrite Eq.(21) as

$$\frac{d^2 f_{mn}}{dt^2} + \omega_{0,mn}^2 f_{mn} + K f_{mn}^3 = 0, \quad (23)$$

where

$$K = \frac{E\pi^4}{16\rho} \left( \frac{m^4}{a^4} + \frac{n^4}{b^4} \right). \quad (24)$$

Thus, we have obtained the nonlinear ordinary differential equation.

To study Eq.(23), we shall use the asymptotic methods proposed by Bogoliubov-Mitropolsky [2]. These methods are explained briefly in Appendix A. Applying general formulas from Appendix A for Eq.(23), we have $u = f_{mn}$, $f_{mn} = \alpha \cos \psi$,

$$\varepsilon \phi(u) = -K f_{mn}^3, \quad \omega^2 = \omega_{0,mn}^2, \quad (25)$$

$$C_1 = -\frac{K \alpha^3}{\pi \varepsilon} \int_0^{2\pi} \cos^4 \psi d\psi = -\frac{3K \alpha^3}{4\varepsilon}. \quad (26)$$

For the corrected frequency we have the formula

$$\omega_{1,mn}(\alpha) = \omega_{0,mn} + \frac{3K \alpha^2}{8\omega_{0,mn}}. \quad (27)$$

It should be noted that we have found the frequency spectrum for the defined mode of nonlinear oscillations. There are as well anharmonic effects.
connected with combination frequencies (Landau & Lifshitz §26). However, it is difficult enough to realize corresponding calculations.

Since in the first approximation the oscillations of plates are harmonic, the plane wave
\[ \zeta = c \cdot \exp \left[ i (k_x x + k_y y - \omega_{I,mn}(\alpha) t) \right] \] (28)
describes the nonlinear oscillations of defined mode. Then one can speak about the flexural phonons with the energy \( E = \hbar \omega_{I,mn}(\alpha) \) and the quasi-momentum \( \vec{p} = \hbar \vec{k} \) in assuming that
\[ k^2 = \sqrt{\frac{\hbar \rho}{D}} \omega_{I,mn}(\alpha). \] (29)

5. Linear forced oscillations

Consider the equation
\[ D \Delta^2 \zeta + h \rho \frac{\partial^2 \zeta}{\partial t^2} = q(x, y) \sin \Omega t - 2 \rho h^2 \beta \frac{\partial \zeta}{\partial t}, \] (30)
where \( q(x, y) \) is the intensity of transverse loading, the latter member describes the dissipation force. Substituting the expression (7) in Eq.(57), multiplying by \( \sin \left( \frac{m \pi x}{a} \right) \sin \left( \frac{n \pi y}{b} \right) \) and integrating over all area of plate, we obtain the equation
\[ \frac{d^2 A_{mn}}{dt^2} + 2 \beta \frac{dA_{mn}}{dt} + \omega_{0,mn}^2 A_{mn} = Q_{mn} \sin \Omega t, \] (31)
where
\[ Q_{mn} = \frac{4}{abh\rho} \int_0^a \int_0^b q(x, y) \sin \left( \frac{m \pi x}{a} \right) \sin \left( \frac{n \pi y}{b} \right) dx \, dy \] (32)
Presenting the solution of Eq.(58) in terms of
\[ A_{mn} = B_{mn} \cos (\Omega t + \theta), \] (33)
we find that
\[ B_{mn} = \frac{Q_{mn}}{\sqrt{\left( \omega_{0,mn}^2 - \Omega^2 \right)^2 + 4 \beta^2 \Omega^2}}. \] (34)
In the case of the resonance $\Omega = \omega_{0,mn}$ we have

$$B_{mn} = \frac{Q_{mn}}{2\beta \Omega} \quad (35)$$

It should be borne in mind that under the resonance $\Omega = \omega_{0,mn}$ the only oscillations mode $\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$ is selected in the expression (7) since $B_{mn} >> B_{ij}$ under the small enough $\beta$.

6. Nonlinear resonance phenomena

In the case of forced oscillations we get the equation

$$D\Delta^2 \zeta - hL(\zeta, \chi) + h\rho \frac{\partial^2 \zeta}{\partial t^2} = q(x, y)\sin \Omega t - 2\rho h \beta \frac{\partial \zeta}{\partial t} \quad (36)$$

instead of Eq.(3). Repeating the above reasoning from the section 4, we obtain the equation

$$\frac{d^2 f_{mn}}{dt^2} + 2\beta \frac{df_{mn}}{dt} + \omega_{0,mn}^2 f_{mn} + K f_{mn}^3 = Q_{mn}\sin \Omega t \quad (37)$$

To study Eq.(37), we use the equivalent linearization that is considered in Appendix B.

Then $u = f_{mn}$, $f_{mn} = \alpha \cos \psi$,

$$\varepsilon\phi(u, \frac{du}{dt}) = -K f_{mn}^3 - 2\beta \frac{df_{mn}}{dt}$, \quad \omega^2 = \omega_{0,mn}^2, R = Q_{mn}. \quad (38)$$

Now we can compute $\lambda_e(\alpha)$ and $k_e(\alpha)$:

$$\lambda_e(\alpha) = \frac{1}{\pi \alpha \omega_{0,mn}} \int_0^{2\pi} (-K\alpha^3 \cos^3 \psi + 2\alpha \beta \Omega \sin \psi) \sin \psi \, d\psi = 2\beta, \quad (39)$$

$$k_e(\alpha) = \omega_{0,mn}^2 - \frac{1}{\pi \alpha} \int_0^{2\pi} (-K\alpha^3 \cos^3 \psi + 2\alpha \beta \Omega \sin \psi) \cos \psi \, d\psi = \omega_{0,mn}^2 + \frac{3K\alpha^2}{4}. \quad (40)$$

Then $\delta_e(\alpha) = \beta$, and

$$\omega_e^2 = \omega_{0,mn}^2 + \frac{3K\alpha^2}{4}. \quad (41)$$
The relation between the amplitude of stationary oscillations $\alpha$ and the frequency of external force $\Omega$ gets the mode

$$\alpha^2 = \frac{Q_{mn}^2}{(\omega_{0,nn}^2 + \frac{3K\alpha^2}{4} - \Omega^2)^2 + 4\beta^2\Omega^2}. \quad (42)$$

7. Conclusion

We have considered the Föppl-von Kármán equations to describe the nonlinear oscillations of graphene plates. Using the presentation of oscillation mode as the product of sinusoidal functions, we have reduced these equations to one nonlinear ordinary differential equation. As a result we have found the real frequency spectrum for graphene plates. Next we have studied the nonlinear resonance phenomena in the case of forced oscillations.

An important outcome of our analysis is that the investigations of nonlinear oscillations of graphene plates can be applied for variable strain-induced pseudomagnetic fields. Such fields permit better to understand the properties of flexural phonons connected with transport processes.

Appendix A. Asymptotic methods

Consider the differential equation with small parameter $\varepsilon$:

$$\frac{d^2 u}{dt^2} + \omega^2 u = \varepsilon \phi(u, \frac{du}{dt}). \quad (A.1)$$

We shall present the solution of this equation in terms of the sum $^3$:

$$u = \alpha \cos \psi + \varepsilon u_1(\alpha, \psi) + \varepsilon^2 u_2(\alpha, \psi) + ... \quad (A.2)$$

where $u_1(\alpha, \psi), u_2(\alpha, \psi), ...$ are the periodic functions of the angle $\psi$ with the period $2\pi$, $\alpha$ and $\psi$ are the time functions determined as

$$\frac{d\alpha}{dt} = \varepsilon A_1(\alpha) + \varepsilon^2 A_2(\alpha) + ..., \quad (A.3)$$

$$\frac{d\psi}{dt} = \omega + \varepsilon B_1(\alpha) + \varepsilon^2 B_2(\alpha) + ... \quad (A.4)$$

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^3See Bogoliubov & Mitropolsky [2] §1.
At the beginning, we consider the first approximation:

\begin{align}
  u &= \alpha \cos \psi, \\
  \frac{d\alpha}{dt} &= \varepsilon A_1(\alpha), \\
  \frac{d\psi}{dt} &= \omega + \varepsilon B_1(\alpha),
\end{align}

where

\begin{align}
  A_1(\alpha) &= -\frac{1}{2\pi \omega} \int_0^{2\pi} \phi(\alpha \cos \psi, -\alpha \omega \sin \psi) \sin \psi d\psi, \\
  B_1(\alpha) &= -\frac{1}{2\pi \alpha \omega} \int_0^{2\pi} \phi(\alpha \cos \psi, -\alpha \omega \sin \psi) \cos \psi d\psi.
\end{align}

In particularly, consider the equation of specific interest for the free non-linear oscillations of plates:

\begin{equation}
  \frac{d^2u}{dt^2} + \omega^2 u = \varepsilon \phi(u). 
\end{equation}

Since \(\phi(\alpha \cos \psi)\) is the even function, we have the expansion

\begin{equation}
  \phi(\alpha \cos \psi) = \sum_{n=0}^{\infty} C_n \cos n\psi, 
\end{equation}

where

\begin{equation}
  C_n = \frac{1}{\pi} \int_0^{2\pi} \phi(\alpha \cos \psi) \cos n\psi d\psi.
\end{equation}

Then

\begin{equation}
  A_1(\alpha) = 0, \quad B_1(\alpha) = -\frac{C_1(\alpha)}{2\omega \alpha}.
\end{equation}

Thus, in the first approximation:

\begin{equation}
  u_I = \alpha \cos \psi,
\end{equation}

\footnote{See Bogoliubov & Mitropolsky \[2\] §2.}
\[
\frac{d\alpha}{dt} = 0,
\] (A.15)

\[
\frac{d\psi}{dt} = \omega - \frac{\varepsilon C_1(\alpha)}{2\omega \alpha} = \omega_I(\alpha).
\] (A.16)

Consequently, the amplitude \(\alpha\) does not depend on time and preserves an initial value. The phase \(\psi\) is equal to:

\[
\psi = \omega_I(\alpha)t + \theta.
\] (A.17)

Then in the first approximation the oscillations are harmonic with the frequency \(\omega_I(\alpha)\).

**Appendix B. Equivalent linearization**

Consider the equation

\[
\frac{d^2u}{dt^2} + \omega^2u = \varepsilon \phi(u, \frac{du}{dt}) + R \sin \Omega t.
\] (B.1)

Next we shall analyze the resonance \(\Omega \approx \omega\). The first approximation has the form \(u = \alpha \cos(\Omega t + \theta)\). The functions \(\alpha(t)\) and \(\theta(t)\) satisfy the relations (see Bogoliubov & Mitropolsky [2] §15):

\[
\frac{d\alpha}{dt} = -\delta_e(\alpha)\alpha - \frac{R \cos \theta}{\omega + \Omega},
\] (B.2)

\[
\frac{d\theta}{dt} = \omega_e(\alpha) - \Omega + \frac{R \sin \theta}{\alpha(\omega + \Omega)},
\] (B.3)

where

\[
\delta_e(\alpha) = \frac{1}{2} \lambda_e(\alpha), \quad \lambda_e(\alpha) = \frac{\varepsilon}{\pi \alpha \omega} \int_0^{2\pi} \phi(\alpha \cos \psi, -\alpha \Omega \sin \psi) \sin \psi d\psi,
\] (B.4)

\[
\omega_e(\alpha) = \sqrt{k_e(\alpha)}, \quad k_e(\alpha) = \omega^2 - \frac{\varepsilon}{\pi \alpha} \int_0^{2\pi} \phi(\alpha \cos \psi, -\alpha \Omega \sin \psi) \cos \psi d\psi,
\] (B.5)

and \(\psi = \Omega t + \theta\).
The relation between the amplitude of stationary oscillations $\alpha$ and the frequency of external force $\Omega$ is determined as

$$\alpha^2 = \frac{R^2}{(\omega_c^2 - \Omega^2)^2 + 4\delta_c^2 \Omega^2}. \quad (B.6)$$

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