DEFORMATIONS OF COMPACT HOLOMORPHIC POISSON SUBMANIFOLDS

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Abstract. In this paper, we study deformations of compact holomorphic Poisson submanifolds which extend Kodaira’s series of papers on semi-regularity (deformations of compact complex submanifolds of codimension 1) ([KS59]), deformations of compact complex submanifolds of arbitrary codimensions ([Kod62]), and stability of compact complex submanifolds ([Kod63]) in the context of holomorphic Poisson deformations. We also study simultaneous deformations of holomorphic Poisson structures and holomorphic Poisson submanifolds on a fixed underlying compact complex manifold. In appendices, we present deformations of Poisson closed subschemes in the language of functors of Artin rings which is the algebraic version of deformations of holomorphic Poisson submanifolds. We identify first-order deformations and obstructions.

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1. Introduction

In this paper, we study deformations of compact holomorphic Poisson submanifolds which extend Kodaira’s series of papers on semi-regularity (deformations of compact complex submanifolds of codimension 1) ([KS59]), deformations of compact complex submanifolds of arbitrary codimensions ([Kod62]), and stability of compact complex submanifolds ([Kod63]) in the context of holomorphic Poisson deformations. We will review deformation theory of compact complex submanifolds presented in [KS59], [Kod62], [Kod63], and explain how the theory can be extended in the context of holomorphic Poisson deformations.

Let us review deformations of compact complex submanifolds of codimension 1 of a complex manifold presented in [KS59] where Kodaira-Spencer proved the theorem of completeness of characteristic systems of semi-regular complex submanifolds of codimension 1. For the precise statement, we recall the definitions of a complex analytic family of compact complex submanifolds of codimension 1, and maximality (or completeness) of a complex analytic family:

Definition 1.0.1. Let $W$ be a complex manifold of dimension $n + 1$. We denote a point in $W$ by $w$ and a local coordinate of $w$ by $(w^1, \ldots, w^{n+1})$. By a complex analytic family of compact complex submanifolds of codimension 1 of $W$, we mean a complex submanifold $V \subset W \times M$ of codimension 1 where $M$ is a complex manifold, such that $V_t \times t := \omega^{-1}(t) = V \cap \pi^{-1}(t)$ for each point $t \in M$ is a connected compact complex submanifold of $W \times t$, where $\omega : V \to M$ is the map induced from the canonical projection $\pi : W \times M \to M$.

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In this paper, all manifolds under consideration are paracompact and connected.
and for each point $p \in V$, there is a holomorphic function $S(w, t)$ on a neighborhood $U_p$ of $p$ in $W \times M$ such that $\sum_{\alpha=1}^{n+1} \frac{\partial S(w, t)}{\partial w_\alpha} \neq 0$ at each point in $U_p \cap V$, and $U_p \cap V$ is defined by $S(w, t) = 0$.

**Definition 1.0.2.** Let $V \subset W \times M \xrightarrow{\omega} M$ be a complex analytic family of compact complex submanifolds of $W$ of codimension 1 and let $t_0$ be a point on $M$. We say that $V \xrightarrow{\omega} M$ is maximal at $t_0$ if, for any complex analytic family $V' \subset W \times M' \xrightarrow{\omega'} M'$ of compact complex submanifolds of $W$ of codimension 1 such that $\omega^{-1}(t_0) = \omega'^{-1}(t'_0)$, there exists a holomorphic map $h$ of a neighborhood $N'$ of $t'_0$ on $M'$ into $M$ which maps $t_0$ to $t_0$ such that $\omega^{-1}(t') = \omega'^{-1}(h(t'))$ for $t' \in N'$. We note that if we set a holomorphic map $h : W \times N' \rightarrow W \times M$ defined by $(w, t') \mapsto (w, h(t'))$, then the restriction map of $h$ to $V'|_{N'} = \omega^{-1}(N') \subset W \times N'$ defines a holomorphic map $V'|_{N'} \rightarrow V$ so that $V'|_{N'}$ is the family induced from $V$ by $h$, which means $V \xrightarrow{\omega} M$ is complete at $t_0$.

Given a complex analytic family $V \subset W \times M \xrightarrow{\omega} M$ of compact complex submanifolds of codimension 1, each fibre $V_t = \omega^{-1}(t)$ of $V$ for $t \in M$ defines a complex line bundle $N_t$ on $W$. Then infinitesimal deformations of $V_t$ in the family $V$ are encoded in the cohomology group $H^0(V_t, N_t|_{V_t})$, and we can define the characteristic map (see [KS59] p.479-480)

$$\rho_{d,t} : T_t M \rightarrow H^0(V_t, N_t|_{V_t})$$

In [KS59], Kodaira-Spencer defined a concept of semi-regularity, and showed that the semi-regularity is ‘the right condition’ for ‘theorem of existence’ and thus ‘theorem of completeness’ for deformations of compact complex submanifolds of codimension 1 as follows.

**Definition 1.0.3.** Let $V_0$ be a compact complex submanifold of $W$ of codimension 1 and $N_0$ be the complex line bundle over $W$ determined by $V_0$. Let $r_0 : N_0 \rightarrow N_0|_{V_0}$ be the restriction map which induces a homomorphism $r_0^* : H^1(W, N_0) \rightarrow H^1(V_0, N_0|_{V_0})$. We say that $V_0$ is semi-regular if $r_0^* H^1(W, N_0) = 0$.

**Theorem 1.0.4.** (Theorem of existence). If $V_0$ is semi-regular, then there exists a complex analytic family $V \subset W \times M \xrightarrow{\omega} M$ of compact complex submanifolds of $W$ containing $V_0$ as the fibre $\omega^{-1}(0)$ over $0 \in M$ such that the characteristic map

$$\rho_{d,0} : T_0 M \rightarrow H^0(V_0, N_0|_{V_0})$$

is an isomorphism.

**Theorem 1.0.5** (Theorem of completeness). Let $V \subset W \times M \xrightarrow{\omega} M$ be a complex analytic family of compact complex submanifolds of $W$ of codimension 1. If the characteristic map

$$\rho_{d,0} : T_0 M \rightarrow H^0(V_0, N_0|_{V_0})$$

is an isomorphism, then the family $V \xrightarrow{\omega} M$ is maximal at the point $t = 0$.

In section 2, we extend the concept of semi-regularity and prove an analogue of theorem of existence (Theorem 1.0.4) and an analogue of theorem of completeness (Theorem 1.0.5) in the context of holomorphic Poisson deformations. A holomorphic Poisson manifold $W$ is a complex manifold whose structure sheaf is a sheaf of Poisson algebras. A holomorphic Poisson structure on $W$ is encoded in a holomorphic section (a holomorphic bivector field) $\Lambda_0 \in H^0(W, \wedge^2 T_W)$ with $[\Lambda_0, \Lambda_0] = 0$, where $T_W$ is the sheaf of germs of holomorphic vector fields, and the bracket $[-, -]$ is the Schouten bracket on $W$. In the sequel a holomorphic Poisson manifold will be denoted by $(W, \Lambda_0)$. Let $V$ be a complex submanifold of a holomorphic Poisson manifold $(W, \Lambda_0)$ and let $i : V \rightarrow W$ be the embedding. Then $V$ is called a holomorphic Poisson submanifold of $(W, \Lambda_0)$ if $V$ is a holomorphic Poisson manifold and the embedding $i$ is a Poisson map with respect to the holomorphic Poisson structures. Then the holomorphic Poisson structure on $V$ is unique. Equivalently a holomorphic Poisson submanifold $V$ of $(W, \Lambda_0)$ can be characterized in the following way: let $V$ be covered by coordinate neighborhoods $W_i, i \in I$ in $W$. We choose a local coordinate $(w_i, z_i) := (w_1^i, ..., w_{n_i}^i, z_1^i, ..., z_{r_i}^i)$ on each neighborhood $W_i$ such that $w_1^i = \cdots = w_{n_i}^i = 0$ defines $V \cap W_i$. Then $V$ is a holomorphic Poisson submanifold of $(W, \Lambda_0)$ if the restriction $[\Lambda_0, w_\alpha^i]|_{V \cap U_i} : [\Lambda_0, w_\alpha^i]|_{w_\alpha^i=0} = 0, \alpha = 1, ..., r$, or $[\Lambda_0, w_\alpha^i]$ is of the form: $[\Lambda_0, w_\alpha^i] = \sum_{\beta=1}^{n_i} w_{\alpha}^i T_{\alpha \beta}^i (w_i, z_i)$ for some $T_{\alpha \beta}^i (w_i, z_i) \in \Gamma(W_i, T_{W_i})$.

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2We refer to [LPVY13] for general information on Poisson geometry.
Definition 1.0.6. Let \((W, \Lambda_0)\) be a holomorphic Poisson manifold of dimension \(n+1\). We denote a point in \(W\) by \(w\) and a local coordinate of \(w\) by \((w^1, ..., w^{n+1})\). By a Poisson analytic family of compact holomorphic Poisson submanifolds of codimension 1 of \((W, \Lambda_0)\), we mean a holomorphic Poisson submanifold \(\mathcal{V} \subset (W \times M, \Lambda_0)\) of codimension 1 where \(M\) is a complex manifold and \(\Lambda_0\) is the holomorphic Poisson structure on \(W \times M\) induced from \((W, \Lambda_0)\), such that \(V_t := \omega^{-1}(t) = \mathcal{V} \cap \pi^{-1}(t)\) for each point \(t \in M\) is a connected compact holomorphic Poisson submanifold of \((W \times t, \Lambda_0)\), where \(\omega : \mathcal{V} \to M\) is the map induced from the canonical projection \(\pi : W \times M \to M\), and for each point \(p \in \mathcal{V}\), there is a holomorphic function \(S(w, t)\) on a neighborhood \(U_p\) of \(p\) in \(W \times M\) such that \(\sum_{n+1}^{n+1} \left(\frac{\partial S(w, t)}{\partial t}\right)^2 \neq 0\) at each point in \(U_p \cap \mathcal{V}\), and \(U_p \cap \mathcal{V}\) is defined by \(S(w, t) = 0\).

Definition 1.0.7. Let \(\mathcal{V} \subset (W \times M, \Lambda_0) \xrightarrow{\omega} M\) be a Poisson analytic family of compact holomorphic Poisson submanifolds of \((W, \Lambda_0)\) of codimension 1 and let \(t_0\) be a point on \(M\). We say that \(\mathcal{V} \xrightarrow{\omega} M\) is maximal at \(t_0\) if, for any Poisson analytic family \(\mathcal{V}' \subset (W \times M', \Lambda_0) \xrightarrow{\omega'} M'\) of compact holomorphic Poisson submanifolds of \((W, \Lambda_0)\) of codimension 1 such that \(\text{dim}(t_0) = \omega'^{-1}(t_0), t'_0 \in M'\), there exists a holomorphic map \(h\) of a neighborhood \(N'\) of \(t'_0\) into \(M'\) which maps \(t_0\) to \(t_0\) such that \(\omega'^{-1}(t') = \omega^{-1}(h(t'))\) for \(t' \in N'\). We note that if we set a Poisson map \(h : (W \times N', \Lambda_0) \to (W \times M, \Lambda_0)\) defined by \((w, t') \to (w, h(t'))\), then the restriction map of \(h\) to \(\mathcal{V}'|_{N'} = \omega'^{-1}(N') \subset (W \times N', \Lambda_0)\) defines a Poisson map \(\mathcal{V}'|_{N'} \to \mathcal{V}\) so that \(\mathcal{V}'|_{N'}\) is the family induced from \(\mathcal{V}\) by \(h\), which means \(\omega \xrightarrow{\omega'} M\) is complete at \(t_0\).

Given a Poisson analytic family \(\mathcal{V} \subset (W \times M, \Lambda_0) \to M\) of compact holomorphic Poisson submanifolds of codimension 1, each fibre \(\mathcal{V}_t = \omega^{-1}(t)\) of \(\mathcal{V}\) for \(t \in M\) defines a Poisson line bundle \((N_t, \nabla_t)\) on \((W, \Lambda_0)\), where \(\nabla_t\) is the Poisson connection on \(N_t\) which defines the Poisson line bundle structure (see \([Kim14a]\)) so that we have a complex of sheaves on \(W\) (see \([Kim14a]\))

\[ N_t^* : N_t \xrightarrow{\nabla_t} N_t \otimes T_W \xrightarrow{\nabla_t} N_t \otimes \wedge^2 T_W \xrightarrow{\nabla_t} \cdots \]

We will denote the \(i\)-th hypercohomology group by \(\mathbb{H}^i(W, N_t^*)\). We note that \(N_t^*\) induces, by restriction on \(V_t\), the complex of sheaves on \(V_t\)

\[ N_t^*|_{V_t} : N_t|_{V_t} \xrightarrow{\nabla_t|_{V_t}} N_t|_{V_t} \otimes T_W|_{V_t} \xrightarrow{\nabla_t|_{V_t}} N_t|_{V_t} \otimes \wedge^2 T_W|_{V_t} \xrightarrow{\nabla_t|_{V_t}} \cdots \]

We will denote the \(i\)-th hypercohomology group by \(\mathbb{H}^i(V_t, N_t^*|_{V_t})\). Then infinitesimal deformations of \(V_t\) in the family \(\mathcal{V}\) are encoded in the cohomology group \(\mathbb{H}^0(V_t, N_t^*|_{V_t})\), and we can define the characteristic map (see subsection 2.1)

\[ \rho_{d,t} : T_t M \to \mathbb{H}^0(V_t, N_t^*|_{V_t}) \]

As in the concept of semi-regularity in \([KS59]\), we similarly define a concept of Poisson semi-regularity and show that Poisson semi-regularity (see Definition 2.4.8) implies ‘theorem of existence’ (see Theorem 2.4.2) and thus ‘theorem of completeness’ (see Theorem 2.4.2) for deformations of compact holomorphic Poisson submanifolds of codimension 1 as follows.

Definition 1.0.8. Let \(V_0\) be a compact holomorphic Poisson submanifold of a holomorphic Poisson manifold \((W, \Lambda_0)\) of codimension 1 and let \((N_0, \nabla_0)\) be the Poisson line bundle over \((W, \Lambda_0)\) determined by \(V_0\). We denote by \(r_0 : N^* \to N^*|_{V_0}\) the restriction map of the following complex of sheaves

\[ N^*_0 : N_0 \xrightarrow{\nabla_0} N_0 \otimes T_W \xrightarrow{\nabla_0} N_0 \otimes \wedge^2 T_W \xrightarrow{\nabla_0} \cdots \]

\[ N^*_0|_{V_0} : N_0|_{V_0} \xrightarrow{\nabla_0|_{V_0}} N_0|_{V_0} \otimes T_W|_{V_0} \xrightarrow{\nabla_0|_{V_0}} N_0|_{V_0} \otimes \wedge^2 T_W|_{V_0} \xrightarrow{\nabla_0|_{V_0}} \cdots \]

which induces a homomorphism \(r_0^* : \mathbb{H}^1(W, N^*_0) \to \mathbb{H}^1(V_0, N^*_0|_{V_0})\). We say that \(V_0\) is Poisson semi-regular if the image \(r_0^* \mathbb{H}^1(W, N^*_0)\) is zero.
**Theorem 1.0.9** (Theorem of existence). If $V_0$ is Poisson semi-regular, then there exists a Poisson analytic family $V \subset (W \times M, \Lambda_0) \overset{\sim}{\rightarrow} M$ of compact holomorphic Poisson submanifolds of $(W, \Lambda_0)$ containing $V_0$ as the fibre $\omega^{-1}(0)$ over $0 \in M$ such that the characteristic map

$$\rho_{d,0} : T_0 M \rightarrow \mathbb{H}^0(V_0, \mathcal{N}_{V_0}^*)$$

is an isomorphism.

**Theorem 1.0.10** (Theorem of completeness). Let $V \subset (W \times M, \Lambda_0) \overset{\sim}{\rightarrow} M$ be a Poisson analytic family of compact Poisson submanifolds of $(W, \Lambda_0)$ of codimension 1. If the characteristic map

$$\rho_{d,0} : T_0 M \rightarrow \mathbb{H}^0(V_0, \mathcal{N}_{V_0}^*)$$

is an isomorphism, then the family $V \overset{\sim}{\rightarrow} M$ is maximal at the point $t = 0$.

Next we review deformation theory of compact complex submanifolds of arbitrary codimensions presented in [Kod62] and explain how the theory can be extended to the theory of compact holomorphic Poisson submanifolds of arbitrary codimensions in terms of holomorphic Poisson deformations. In [Kod62], Kodaira showed that deformations of a compact complex submanifold $V$ of a complex manifold $W$ is controlled by the normal bundle $\mathcal{N}_{V/W}$ of $V$ in $W$ so that infinitesimal deformations are encoded in the cohomology group $H^0(V, \mathcal{N}_{V/W})$ and obstructions are encoded in the cohomology group $H^1(V, \mathcal{N}_{V/W})$. For the precise statement, we recall the following definition which generalize Definition 1.0.1 to arbitrary codimensions.

**Definition 1.0.11** ([Kod62]). Let $W$ be a complex manifold of dimension $d + r$. We denote a point in $W$ by $w$ and a local coordinate of $W$ by $(w^1, ..., w^{d + r})$. By a complex analytic family of compact complex submanifolds of dimension $d$ of $W$, we mean a complex submanifold $V \subset W \times M$ of codimension $r$, where $M$ is a complex manifold, such that

1. For each point $t \in M$, $V_t \times t := \omega^{-1}(t) = V \cap \pi^{-1}(t)$ is a connected compact complex submanifold of $W \times t$ of dimension $d$, where $\omega : V \rightarrow M$ is the map induced from the canonical projection $\pi : W \times M \rightarrow M$.
2. For each point $p \in V$, there exist $r$ holomorphic functions $f_\alpha(w, t), \alpha = 1, ..., r$ defined on a neighborhood $U_p$ of $p$ in $W \times M$ such that rank $\left\{ \frac{\partial (f_1, ..., f_r)}{\partial (w^1, ..., w^{d+r})} \right\} = r$, and $U_p \cap V$ is defined by the simultaneous equations $f_\alpha(w, t) = 0, \alpha = 1, ..., r$.

We call $V \subset W \times M$ a complex analytic family of compact complex submanifolds $V_t, t \in M$ of $W$. We also call $V \subset W \times M$ a complex analytic family of deformations of a compact complex submanifold $V_{t_0}$ of $W$ for each fixed point $t_0 \in M$.

We can define the concept of maximality (or completeness) of a complex analytic family of compact complex submanifolds of arbitrary codimensions as in Definition 1.0.2. Given a complex analytic family $V \subset W \times M \overset{\sim}{\rightarrow} M$, for each fibre $V_t = \omega^{-1}(t)$ of $V$ for $t \in M$, infinitesimal deformations of $V_t$ in the family $V$ are encoded in the cohomology group $H^0(V_t, \mathcal{N}_{V_t/W})$, and we can define the characteristic map (see [Kod62] p.147-150)

$$\sigma_t : T_t M \rightarrow H^0(V_t, \mathcal{N}_{V_t/W})$$

In [Kod62], Kodaira showed that given a compact complex submanifold $V$ of a complex manifold $W$, obstructions of deformations of $V$ in $W$ are encoded in $H^1(V, \mathcal{N}_{V/W})$ and so when obstructions vanish, we can prove ‘theorem of existence’ and ‘theorem of completeness’ as follows.

**Theorem 1.0.12** (theorem of existence). Let $V$ be a compact complex submanifold of a complex manifold $W$. If $H^1(V, \mathcal{N}_{V/W}) = 0$, then there exists a complex analytic family $V$ of compact complex submanifolds $V_t, t \in M_t$ of $W$ such that $V_0 = V$ and the characteristic map

$$\sigma_0 : T_0(M_t) \rightarrow H^0(V_t, \mathcal{N}_{V_t/W})$$

is an isomorphism.

**Theorem 1.0.13** (theorem of completeness). Let $V$ be a complex analytic family of compact holomorphic Poisson submanifolds $V_t, t \in M_t$, of $W$. If the characteristic map

$$\sigma_0 : T_0(M_t) \rightarrow H^0(V_0, \mathcal{N}_{V_0/W})$$

is an isomorphism, then the family $V$ is maximal at $t = 0$. 
In section \ref{sec:compact} we extend Definition \ref{def:compact} to define a concept of compact holomorphic Poisson submanifolds of arbitrary codimensions, and prove an analogue of theorem of existence (Theorem \ref{thm:existence}) and an analogue of theorem of completeness (Theorem \ref{thm:completeness}) in the context of holomorphic Poisson deformations. Let $V$ be a holomorphic Poisson submanifold of a holomorphic Poisson manifold $(W, \Lambda_0)$. Then we can define a complex of sheaves associated with the normal bundle $N_{V/W}$ (see subsection \ref{subsec:normalbundle} and Definition \ref{def:normalbundle})

$$N_{V/W} : N_{V/W} \to N_{V/W} \otimes T_W|_V \to N_{V/W} \otimes \wedge^2 T_W|_V \to N_{V/W} \otimes \wedge^3 T_W|_V \to \cdots$$

which is called the complex associated with the normal bundle $N_{V/W}$ of a holomorphic Poisson submanifold $V$ of a holomorphic Poisson manifold $(W, \Lambda_0)$, and denote its $i$-th hypercohomology group by $H^i(V, N_{V/W}^\bullet)$. Then holomorphic Poisson deformations of $V$ in $(W, \Lambda_0)$ is controlled by the complex of sheaves $N_{V/W}^\bullet$ so that infinitesimal deformations are encoded in the cohomology group $H^1(V, N_{V/W}^\bullet)$ and obstructions are encoded in the cohomology group $H^2(V, N_{V/W}^\bullet)$. For the precise statement, we define a concept of compact holomorphic Poisson submanifolds of an arbitrary codimension which generalize Definition \ref{def:compact} and Definition \ref{def:compact} as follows.

**Definition 1.0.14** (compare Definition \ref{def:compact}). Let $(W, \Lambda_0)$ be a holomorphic Poisson manifold of dimension $d + r$. We denote a point in $W$ by $w$ and a local coordinate of $w$ by $(w^1, \ldots, w^{n-r})$. By a Poisson analytic family of compact holomorphic Poisson submanifolds of dimension $d$ of $(W, \Lambda_0)$, we mean a holomorphic Poisson submanifold $V \subset (W \times M, \Lambda_0)$ of codimension $r$, where $M$ is a complex manifold and $\Lambda_0$ is the holomorphic Poisson structure on $W \times M$ induced from $(W, \Lambda_0)$, such that

1. for each point $t \in M$, $V_t \times t := \omega^{-1}(t) = V \cap \pi^{-1}(t)$ is a connected compact holomorphic Poisson submanifold of $(W \times t, \Lambda_0)$ of dimension $d$, where $\omega : V \to M$ is the map induced from the canonical projection $\pi : W \times M \to M$.
2. for each point $p \in V$, there exist $r$ holomorphic functions $f_\alpha(w, t), \alpha = 1, \ldots, r$ defined on a neighborhood $U_p$ of $p$ in $W \times M$ such that rank $\frac{\partial (f_1, \ldots, f_r)}{\partial (w^1, \ldots, w^{n-r})} = r$, and $U_p \cap V$ is defined by the simultaneous equations $f_\alpha(w, t) = 0, \alpha = 1, \ldots, r$.

We call $V \subset (W \times M, \Lambda_0)$ a Poisson analytic family of compact holomorphic Poisson submanifolds $V_t, t \in M$ of $(W, \Lambda_0)$. We also call $V \subset (W \times M, \Lambda_0)$ a Poisson analytic family of deformations of a compact holomorphic Poisson submanifold $V_{t_0}$ of $(W, \Lambda_0)$ for each fixed point $t_0 \in M$.

We can define the concept of maximality (or completeness) of a Poisson analytic family of compact holomorphic Poisson submanifolds of arbitrary codimensions as in Definition \ref{def:compact}. Given a Poisson analytic family $V \subset (W \times M, \Lambda_0) \to M$ of compact holomorphic Poisson submanifolds, for each fibre $V_t = \omega^{-1}(t)$ of $V$ for $t \in M$, infinitesimal deformations of $V_t$ in the family $V$ are encoded in the cohomology group $H^0(V_t, N_{V_t/W}^\bullet)$, and we can define the characteristic map (see subsection \ref{sec:characteristic})

$$\sigma_t : T_t M \to H^0(V_t, N_{V_t/W}^\bullet)$$

Given a compact holomorphic Poisson submanifold $V$ of a holomorphic Poisson manifold $(W, \Lambda_0)$, obstructions of holomorphic Poisson deformations of $V$ in $(W, \Lambda_0)$ are encoded in $H^2(V, N_{V/W}^\bullet)$ and so when obstructions vanish, we can prove ‘theorem of existence’ (see Theorem \ref{thm:existence}) and ‘theorem of completeness’ (see Theorem \ref{thm:completeness}) as follows.

**Theorem 1.0.15** (theorem of existence). Let $V$ be a compact holomorphic Poisson submanifold of a holomorphic Poisson manifold $(W, \Lambda_0)$. If $H^0(V, N_{V/W}^\bullet) = 0$, then there exists a Poisson analytic family $V$ of compact holomorphic Poisson submanifolds $V_t$, $t \in M_1$, of $(W, \Lambda_0)$ such that $V_0 = V$ and the characteristic map

$$\sigma_0 : T_{t_0} M_1 \to H^0(V_{t_0}, N_{V_{t_0}/W}^\bullet)$$

is an isomorphism.

**Theorem 1.0.16** (theorem of completeness). Let $V$ be a Poisson analytic family of compact holomorphic Poisson submanifolds $V_t$, $t \in M_1$, of $(W, \Lambda_0)$. If the characteristic map

$$\sigma_0 : T_{t_0} M_1 \to H^0(V_{t_0}, N_{V_{t_0}/W}^\bullet)$$

is an isomorphism, then the family $V$ is maximal at $t = 0$. 
In Section \textbf{I}, we study simultaneous deformations of holomorphic Poisson structures and compact holomorphic Poisson submanifolds. Let $V$ be a holomorphic Poisson submanifold of a holomorphic Poisson manifold $(W, \Lambda_0)$. As we saw above, the complex associated with the normal bundle

\begin{equation}
\mathcal{N}_{V/W} : \mathcal{N}_{V/W} \to \mathcal{N}_{V/W} \otimes T_W|_V \to \mathcal{N}_{V/W} \otimes \wedge^2 T_W|_V \to \cdots
\end{equation}

controls holomorphic Poisson deformations of $V$ of $(W, \Lambda_0)$, and the complex

\begin{equation}
\wedge^2 T_W^* : \wedge^2 T_W \to \wedge^3 T_W \to \wedge^4 T_W \to \cdots
\end{equation}

controls deformations of the holomorphic Poisson structure $\Lambda_0$ on the fixed underlying complex manifold $W$ (see Appendix [A]). By combining two complexes (1.0.17) and (1.0.18), we can define a complex of sheaves on $W$ (see subsection \textbf{II} and Definition \textbf{II.18})

\[
(\wedge^2 T_W \oplus i_\ast \mathcal{N}_{V/W})^\ast : \wedge^2 T_W \oplus i_\ast \mathcal{N}_{V/W} \to \wedge^3 T_W \oplus i_\ast (\mathcal{N}_{V/W} \otimes T_W|_V) \to \wedge^4 T_W \oplus i_\ast (\mathcal{N}_{V/W} \otimes \wedge^2 T_W|_V) \to \cdots
\]

where $i : V \to W$ is the embedding, which is called the extended complex associated with the normal bundle $\mathcal{N}_{V/W}$ of a holomorphic Poisson submanifold $V$ of a holomorphic Poisson manifold $W$, and denotes its $i$-th hypercohomology group by $\mathbb{H}^i(W, (\wedge^2 T_W \oplus i_\ast \mathcal{N}_{V/W})^\ast)$. Then $(\wedge^2 T_W \oplus i_\ast \mathcal{N}_{V/W})^\ast$ controls simultaneous deformations of $\Lambda_0$ and $V$ in $(W, \Lambda_0)$ so that infinitesimal deformations are encoded in the cohomology group $\mathbb{H}^0(W, (\wedge^2 T_W \oplus i_\ast \mathcal{N}_{V/W})^\ast)$ and obstructions are encoded in $\mathbb{H}^1(W, (\wedge^2 T_W \oplus i_\ast \mathcal{N}_{V/W})^\ast)$. For the precise statements, we extend Definition \textbf{I.0.14} of a Poisson analytic family of compact holomorphic submanifolds of a holomorphic Poisson manifold $(W, \Lambda_0)$ by deforming $\Lambda_0$ on the fixed complex manifold $W$ as well as a holomorphic Poisson submanifold $V$ of $(W, \Lambda_0)$ as follows (see Definition \textbf{I.0.11}).

**Definition 1.0.19.** Let $W$ be a complex manifold of dimension $d + r$. We denote a point in $W$ by $w$ and a local coordinate of $w$ by $(w^1, \ldots, w^{d+r})$. By an extended Poisson analytic family of compact holomorphic Poisson submanifolds of dimension $d$ of $W$, we mean a holomorphic Poisson submanifold $V \subset (W \times M, \Lambda)$ of codimension $r$, where $M$ is a complex manifold and $\Lambda$ is a holomorphic Poisson structure on $W \times M$, such that

1. The canonical projection $\pi : (W \times M, \Lambda) \to M$ is a Poisson analytic family in the sense of [Kim14b] (but we allow non-compact fibres) so that $\Lambda \in H^0(W \times M, \wedge^2 T_{W \times M/M})$ and $\pi^{-1}(t) := (W \times t, \Lambda_t)$ is a holomorphic Poisson submanifold of $(W \times M, \Lambda)$ for each point $t \in M$.
2. For each point $t \in M$, $V_t \times t := \omega^{-1}(t) = V \cap \pi^{-1}(t)$ is a connected compact holomorphic Poisson submanifold of $(W \times t, \Lambda_t)$ of dimension $d$, where $\omega : V \to M$ is the map induced from $\pi$.
3. For each point $p \in W$, there exist $r$ holomorphic functions $f_\alpha(w, t), \alpha = 1, \ldots, r$ defined on a neighborhood $U_p$ of $p$ in $W \times M$ such that $\text{rank} \frac{\partial (f_1, \ldots, f_r)}{\partial (w^1, \ldots, w^{d+r})} = r$, and $U_p \cap V$ is defined by the simultaneous equations $f_\alpha(w, t) = 0, \alpha = 1, \ldots, r$.

We call $V \subset (W \times M, \Lambda)$ an extended Poisson analytic family of compact holomorphic Poisson submanifolds $V_t, t \in M$ of $(W, \Lambda_t)$. We also call $V \subset (W \times M, \Lambda)$ an extended Poisson analytic family of simultaneous deformations of a holomorphic Poisson submanifold $V_{t_0}$ of $(W, \Lambda_{t_0})$ for each fixed point $t_0 \in M$.

As in Definition \textbf{I.0.2} we can similarly define the concept of maximality (or completeness) of an extended Poisson analytic family (see Definition \textbf{I.5.1}). Given an extended Poisson analytic family $V \subset (M \times M, \Lambda) \to M$ of compact holomorphic Poisson submanifolds, for each fibre $V_t = \omega^{-1}(t) \subset (W, \Lambda_t)$ of $V$ for $t \in M$, infinitesimal deformations of $(\Lambda_t, V_t)$ in the family $V$ are encoded in the cohomology group $\mathbb{H}^0(W, (\wedge^2 T_W \oplus i_\ast \mathcal{N}_{V/W})^\ast)$, and we can define the characteristic map (see subsection \textbf{I.2})

$$\sigma_t : T_t M \to \mathbb{H}^0(W, (\wedge^2 T_W \oplus i_\ast \mathcal{N}_{V/W})^\ast)$$

Given a compact holomorphic Poisson manifold $V$ of a compact holomorphic Poisson manifold $(W, \Lambda_0)$, obstructions of simultaneous deformations of $\Lambda_0$ and $V$ are encoded in $\mathbb{H}^1(W, (\wedge^2 T_W \oplus i_\ast \mathcal{N}_{V/W})^\ast)$ and so when obstructions vanish, we can prove ‘theorem of existence’ (see Theorem \textbf{I.3.1} and ‘theorem of completeness’ (see Theorem \textbf{I.5.2}) as follows.

**Theorem 1.0.20** (theorem of existence). Let $V$ be a holomorphic Poisson submanifold of a compact holomorphic Poisson manifold $(W, \Lambda_0)$. If $\mathbb{H}^1(W, (\wedge^2 T_W \oplus i_\ast \mathcal{N}_{W/V})^\ast) = 0$, then there exists an extended Poisson
analytic family $\mathcal{V} \subset (W \times M_1, \Lambda)$ of compact holomorphic Poisson submanifolds $V_t$, $t \in M_1$, of $(W, \Lambda_0)$ such that $V = V_0 \subset (W, \Lambda_0)$ and the characteristic map

$$\sigma_0 : T_0(M_1) \to H^0(W_1, (\wedge^2 T_W \oplus i_* N_{V/W})^\bullet)$$

is an isomorphism.

**Theorem 1.0.21** (theorem of completeness). Let $\mathcal{V} \subset (W \times M_1, \Lambda)$ be an extended Poisson analytic family of compact holomorphic Poisson submanifolds $V_t$ of $(W, \Lambda_t)$. If the characteristic map

$$\rho_0 : T_0(M_1) \to H^0(W_1, (\wedge^2 T_W \oplus i_* N_{V/W})^\bullet)$$

is an isomorphism, then the family $\mathcal{V}$ is maximal at $t = 0$.

Lastly we review stability of compact complex submanifolds presented in [Kod63] and explain how we can extend the concept of stability in the context of holomorphic Poisson deformations. In [Kod63], Kodaira defined a concept of stability of compact complex submanifolds as follows:

**Definition 1.0.22.** Let $V$ be a compact complex submanifold of a complex manifold $W$. We call $V$ a stable complex submanifold of $W$ if and only if, for any complex fibre manifold $N$ of compact holomorphic Poisson submanifolds $(W, B, p)$ with $p : W \to B$ such that $p^{-1}(0) = W$ for a point $0 \in B$, there exist a neighborhood $N$ of $0$ in $B$ and a complex fibre submanifold $V$ with compact fibres of the complex fibre manifold $W|_N$ such that $V \cap W = V$, where $W|_N$ is the restriction $(p^{-1}(N), N, p)$ of $(W, B, p)$ to $N$.

and he proved

**Theorem 1.0.23.** Let $V$ be a compact complex submanifold of a complex manifold $W$. If the first cohomology group $H^1(V, N_{V/W})$ vanishes, then $V$ is a stable complex submanifold of $W$.

In section 5 we similarly define a concept of stable compact holomorphic Poisson submanifolds as follows (see Definition 5.0.22):

**Definition 1.0.24.** Let $V$ be a compact holomorphic Poisson submanifold of a holomorphic Poisson manifold $(W, \Lambda_0)$. We call $V$ a stable holomorphic Poisson submanifold of $(W, \Lambda_0)$ if and only if, for any holomorphic Poisson fibre manifold $(W, \Lambda, B, p)$ (see Definition 5.0.1) such that $p^{-1}(0) = (W, \Lambda_0)$ for a point $0 \in B$, there exist a neighborhood $N$ of $0$ in $B$ and a holomorphic Poisson fibre submanifold $V$ with compact fibres of the holomorphic Poisson fibre manifold $(W, \Lambda)|_N$ such that $V \cap W = V$, where $(W, \Lambda)|_N$ is the restriction $(p^{-1}(N), \Lambda|_{p^{-1}(N)}, N, p)$ of $(W, \Lambda, B, p)$ to $N$.

and we prove (see Theorem 5.1.1)

**Theorem 1.0.25.** Let $V$ be a compact holomorphic Poisson submanifold of a holomorphic Poisson manifold $(W, \Lambda_0)$. If the first cohomology group $H^1(V, N_{V/W})$ vanishes, then $V$ is a stable holomorphic Poisson submanifold of $(W, \Lambda_0)$.

In appendices E and C we present deformations of Poisson closed subschemes in the language of functors of Artin rings which is the algebraic version of deformations of holomorphic Poisson submanifolds. We identify first-order deformations and obstructions (see Proposition 5.2.3 and Proposition C.2.2).

2. DEFORMATIONS OF COMPACT HOLOMORPHIC POISSON SUBMANIFOLDS OF CODIMENSION 1 AND POISSON SEMI-REGULARITY

**Definition 2.0.26.** Let $(W, \Lambda_0)$ be a holomorphic Poisson manifold. We denote a point in $W$ by $w$ and a local coordinate of $w$ by $(w^1, ..., w^{n+1})$. By a Poisson analytic family of compact holomorphic Poisson submanifolds of codimension 1 of $(W, \Lambda_0)$, we mean a holomorphic Poisson submanifold $\mathcal{V} \subset (W \times M, \Lambda_0)$ of codimension 1 where $M$ is a complex manifold and $\Lambda_0$ is the holomorphic Poisson structure on $W \times M$ induced from $(W, \Lambda_0)$, such that $V_t \times t := \omega^{-1}(t) = V \cap \pi^{-1}(t)$ for each point $t \in M$ is a connected compact holomorphic Poisson submanifold of $(W \times t, \Lambda_0)$, where $\omega : V \to M$ is the map induced from the canonical projection $\pi : W \times M \to M$, and for each point $p \in \mathcal{V}$, there is a holomorphic function $S(w, t)$ on a neighborhood $U_p$ of $p$ in $W \times M$ such that $\sum_{\alpha=1}^{n+1} \left| \frac{\partial S(w,t)}{\partial w^\alpha} \right|^2 \neq 0$ at each point in $U_p \cap \mathcal{V}$, and $U_p \cap \mathcal{V}$ is defined by $S(w, t) = 0$.

3 for the definition of a complex fibre manifold and a complex fibre submanifold, see [Kod63] p.79
2.1. Infinitesimal deformations.

Let \((W, \Lambda_0)\) be a holomorphic Poisson manifold. We denote by \(w\) a point on \((W, \Lambda_0)\) and by \((w_1, ..., w^{n+1})\) the local holomorphic coordinates (not specified) of \(w\). Consider a small spherical neighborhood \(N\) of a point on \(M\) and let \(\{U_i\}\) be a locally finite covering of \(W\) by sufficiently small coordinate neighborhoods \(U_i\).

Consider a Poisson analytic family \(\mathcal{V} \subset (W \times M, \Lambda_0)\) of compact holomorphic Poisson submanifolds of codimension 1 as in Definition 2.0.23. Let \(N\) be a small spherical neighborhood of a point on \(M\). Then the holomorphic Poisson submanifold \(\mathcal{V}\) of \((W \times M, \Lambda_0)\) is defined in each neighborhood \(U_i \times N\) by a holomorphic equation \(S_i(w, t) = 0\) where \(S_i(w, t)\) is a holomorphic function on \(U_i \times N\) such that \(\sum_{n=1}^{n+1} \left| \frac{\partial S_i(w, t)}{\partial w^n} \right|^2 \neq 0\) at each point \((w, t)\) of \(\mathcal{V} \cap (U_i \times N)\) if \(\mathcal{V} \cap (U_i \times N)\) is empty, we set \(S_i(w, t) = 1\). By letting

\[
S_i(w, t) = f_{ik}(w, t)S_k(w, t), \quad w \in U_i \cap U_k,
\]
we get a system \(\{f_{ik}(w, t)\}\) of non-vanishing holomorphic functions \(f_{ik}(w, t)\) defined, respectively, on \((U_i \times N) \cap (U_k \times N)\) satisfying

\[
f_{ik}(w, t) = f_{ij}(w, t)f_{jk}(w, t), \quad w \in U_i \cap U_j \cap U_k.
\]

On the other hand, since \(S_i(w, t) = 0\) define a holomorphic Poisson submanifold, \([\Lambda_0, S_i(w, t)]\) is of the form

\[
[\Lambda_0, T_i(w, t)] = 0, \quad w \in U_i
\]
for some \(T_i(w, t) = \sum_{n=1}^{n+1} T^n_i(w, t) \frac{\partial}{\partial w^n}, \) where \(T^n_i(w, t)\) is a holomorphic function on \(U_i \times N\). By taking \([\Lambda_0, -]\) on (2.1.3), we have \(0 = -[\Lambda_0, S_i(w, t)] \wedge T_i(w, t)] + S_i(w, t)[\Lambda_0, T_i(w, t)] = -S_i(w, t)T_i(w, t) \wedge T_i(w, t) + S_i(w, t)[\Lambda_0, T_i(w, t)] = S_i(w, t)[\Lambda_0, T_i(w, t)]\) so that

\[
[\Lambda_0, T_i(w, t)] = 0, \quad w \in U_i
\]
From (2.1.1) and (2.1.3), \(f_{ik}(w, t)S_k(w, t)T_i(w, t) = S_i(w, t)T_i(w, t) = [\Lambda_0, S_i(w, t)] = [\Lambda_0, f_{ik}(w, t)S_k(w, t)] = S_k(w, t)[\Lambda_0, f_{ik}(w, t)] + f_{ik}(w, t)[\Lambda_0, S_k(w, t)] = S_k(w, t)[\Lambda_0, f_{ik}(w, t)] + f_{ik}(w, t)S_k(w, t)T_k(w, t)\) so that

\[
f_{ik}(w, t)T_i(w, t) - f_{ik}(w, t)T_k(w, t) = [\Lambda_0, f_{ik}(w, t)], \quad w \in U_i \cap U_k
\]
Then the conditions (2.1.2), (2.1.4), (2.1.5) show that \((\{f_{ik}(w, t), \{T_i(w, t))\}\) defines the Poisson line bundle over \((W, \Lambda_0)\) for each \(t \in \mathbb{N}\). We will denote the Poisson line bundle by \((N_t, \nabla_t)\) for \(t \in \mathbb{N}\), where \(\nabla_t\) is the Poisson connection on \(N_t\) which defines the Poisson line bundle structure (see [Kim14a]). Then we have a complex of sheaves on \(W\) (see [Kim14a])

\[
N_t^* : N_t \xrightarrow{\nabla_t} N_t \otimes T_W \xrightarrow{\nabla_t} N_t \otimes \wedge^2 T_W \xrightarrow{\nabla_t} \cdots
\]
We will denote the \(i\)-th hypercohomology group by \(\mathbb{H}^i(W, N_t^*)\). We note that \(N_t^*\) induces, by restriction on \(V_t\), the complex of sheaves on \(V_t\)

\[
N_t^*[V_t] : N_t|_{V_t} \xrightarrow{\nabla_t|_{V_t}} N_t|_{V_t} \otimes T_W|_{V_t} \xrightarrow{\nabla_t|_{V_t}} N_t|_{V_t} \otimes \wedge^2 T_W|_{V_t} \xrightarrow{\nabla_t|_{V_t}} \cdots
\]
We will denote the \(i\)-th hypercohomology group by \(\mathbb{H}^i(V_t, N_t^*[V_t])\).

Denote by \((t_1, ..., t_m)\) a system of holomorphic coordinates on \(N\). For any tangent vector \(v = \sum_{r=1}^m v_r \frac{\partial}{\partial z_r}\) of \(M\) at \(t \in \mathbb{N}\), we set \(\psi_i(w, t) = -\sum_{r=1}^m v_r \frac{\partial S_i(w, t)}{\partial z_r}.\) Since \(S_i(w, t) = 0\) for \(w \in V_t\), by taking the derivative of (2.1.1) and (2.1.3) with respect to \(t\), we obtain

\[
\psi_i(w, t) = f_{ik}(w, t)\psi_k(w, t), \quad w \in V_i \cap U_i \cap U_k,
\]
\[
[\Lambda_0, \psi_i(w, t)]|_{V_t} = [\Lambda_0, \psi_i(w, t)]|_{S_i(w, t)\psi_i = 0} = \psi_i(w, t)T_i(w, t), \quad w \in V_i \cap U_i
\]
From (2.1.6) and (2.1.7), \(\{\psi_i(w, t)\}\) define an element in \(\mathbb{H}^0(V_t, N_t^*[V_t])\) so that we have a linear map

\[
\rho_{d,t} : T_t M \rightarrow \mathbb{H}^0(V_t, N_t^*[V_t])
\]
\[
\frac{\partial}{\partial t} \mapsto \frac{\partial V_t}{\partial t} := \{\psi_i(z, t)\}
\]
We call \(\rho_{d,t}\) the characteristic map.

**Definition 2.1.8.** Let \(V_0\) be a compact holomorphic Poisson submanifold of a holomorphic Poisson manifold \((W, \Lambda_0)\) of codimension 1 and let \((N_0, \nabla_0)\) be the Poisson line bundle over \((W, \Lambda_0)\) determined by \(V_0\). We denote by \(r_0 : N^* \rightarrow N^*[V_0]\) the restriction map of the following complex of sheaves.
then the system $(V \subset \text{family } \beta H)$ defined on $(U \text{ is an isomorphism.})$

identification $W$, tried to maintain notational consistency with [KS59]. Let $(\text{Theorem of existence.})$

if the image $r$ which induces a homomorphism $r^*: H^1(W, N_0^*) \to H^1(V_0, N_0^*)$. We say that $V_0$ is Poisson semi-regular if the image $r^*: H^1(W, N_0^*)$ is zero.

2.2. Theorem of existence.

We extend the argument in [KS59] in the context of Poisson deformations (see [KS59] p.484-493). We tried to maintain notational consistency with [KS59]. Let $(W, \Lambda_0)$ be a holomorphic Poisson manifold of dimension $n + 1 \geq 2$ and let $V_0$ be a compact holomorphic Poisson submanifold of $(W, \Lambda_0)$ of dimension $n$. In what follows we denote by $p$ a point on $W$ and by $(w^1(p), \ldots, w^{n+1}(p))$ the coordinate of $p$ with respect to a system of local holomorphic coordinate $(w^1, ..., w^{n+1})$ on $W$. We choose a locally finite covering $\mathcal{U} = \{U_i\}$ of $W$ such that

1. each neighborhood $U_i$ is a poly cylinder: $U_i = \{p|\text{image } w^1(p) < 1, ..., |w^{n+1}(p)| < 1\}$ where $(w^1, ..., w^{n+1})$

2. $V_0 \cap U_i$ coincides with the coordinate plane $w^{n+1} = 0$ if $0 \cap U_i$ is not empty,

3. $V_0 \cap U_i \cap U_k$ is not empty if $V_0 \cap U_i, V_0 \cap U_k$ and $U_i \cap U_k$ are not empty.

4. if $V_0 \cap U_i$ is not empty, we write $w_i = w_i^{n+1}$, $z_i^{1} = w_i^{1}, ..., z_i^{n} = w_i^{n}$.

5. $\{U_i\}$ covers $W$, where $U_i = \{p|\text{image } w_i(p) < 1 - \delta, ..., |z_i^{1}(p)| < 1 - \delta, ..., |z_i^{n}(p)| < 1 - \delta, |w_i(p)| < 1\}$ for a sufficiently small $\delta$.

Let $S_{i|0}(p) = w_i(p)$ if $V_0 \cap U_i \neq \emptyset$ and $S_{i|0}(p) = 1$ if $V_0 \cap U_i = \emptyset$ if $p \in U_i$. If we let

$$f_{ik|0}(p) := \frac{S_{i|0}(p)}{S_{k|0}(p)}, \text{ for } p \in U_i \cap U_k,$$

$$T_{i|0}(p) := [\Lambda_{0}, \log S_{i|0}(p)], \text{ i.e. } [\Lambda_{0}, S_{i|0}(p)] = S_{i|0}(p)T_{i|0}(p), \text{ for } p \in U_i,$$

then the system $(\{f_{ik|0}(p)\}, \{T_{i|0}(p)\}))$ defines the Poisson line bundle $\mathcal{N}_0$. Let $\{\beta_1, ..., \beta_m\}$ be a basis of $H^0(V_0, N_0^*)$. Each $\beta_i$ is a holomorphic section of $\mathcal{N}_0$ over $V_0$ and $\beta_i$ is written in the form via the identification $\mathcal{N}_0|_{U_i} \cong U_i \times \mathbb{C},$

$$\beta_i : p \mapsto (p, \beta_i(p))$$

where $\beta_i(p)$ are holomorphic functions on $V_0 \cap U_i$ satisfying

$$(2.2.1) \quad \beta_i(p) = f_{ik|0}(p)\beta_k(p), \text{ for } p \in V_0 \cap U_i \cap U_k.$$

$$(2.2.2) \quad -[\Lambda_{0}, \beta_i(p)]|_{V_0} + \beta_i(p)T_{i|0}(p) = 0, \quad \text{for } p \in V_0 \cap U_i$$

With this preparation, we prove

Theorem 2.2.3 (Theorem of existence). If $V_0$ is Poisson semi-regular, then there exists a Poisson analytic family $V \subset (W \times M, \Lambda_0) \xrightarrow{\tilde{\omega}^{-1}} M$ of compact holomorphic Poisson submanifolds of $(W, \Lambda_0)$ containing $V_0$ as the fibre $\omega^{-1}(0)$ over $0 \in M$ such that the characteristic map

$$\rho_{d,0} : T_0M \to \mathbb{H}^0(V_0, N_0^*)$$

$$\frac{\partial}{\partial t} \mapsto \left(\frac{\partial V_i}{\partial t}\right)_{t=0}$$

is an isomorphism.

Proof. Let $N$ be a spherical neighborhood of 0 on the space of $m$ complex variables $t_1, ..., t_m$, where $m = \dim \mathbb{H}^0(V_0, N_0^*)$. In order to prove Theorem 2.2.3, it suffices to construct a system $\{S_i(p, t)\}$ of holomorphic functions $S_i(p, t)$ defined on $U_i \times N$ and a system $\{f_{ik}(p, t)\}$ of non-vanishing holomorphic functions $f_{ik}(p, t)$ defined on $(U_i \cap U_k) \times N$ and a system $\{T_i(p, t)\}$ of holomorphic vector fields $T_i(p, t)$ defined on $U_i \times N$ such
that

\( S_i(p, t) = f_{ik}(p, t)S_k(p, t), \) for \( p \in U_i \cap U_k, \)

\( [A_0, S_i(p, t)] = S_i(p, t)T_i(p, t), \) for \( p \in U_i, \)

\( S_i(0, p) = S_{i|0}(p), \ f_{ik}(0, p) = f_{ik|0}(p), \ T_i(0, p) = T_{i|0}(p) \)

\( S_i(p, t) \neq 0, \) if \( V_0 \cap U_i = \emptyset \)

\( \frac{\partial S_i(p, t)}{\partial r_i} = \beta_{r_i}(p), \ r = 1, \ldots, m. \)

We write \( S_i(p, t), f_{ik}(p, t) \) and \( T_i(p, t) \) in the forms

\[
S_i(p, t) = S_{i|0}(p) + \sum_{\mu=1}^{\infty} S_{i|\mu}(p, t), \ f_{ik}(p, t) = f_{ik|0}(p) + \sum_{\mu=1}^{\infty} f_{ik|\mu}(p, t), \ T_i(p, t) = T_{i|0}(p) + \sum_{\mu=1}^{\infty} T_{i|\mu}(p, t)
\]

where \( S_{i|\mu}(p, t), f_{ik|\mu}(p, t) \) and \( T_{i|\mu}(p, t) \) are homogenous polynomials in \( t = (t_1, \ldots, t_m) \) of degree \( \mu \) whose coefficients are holomorphic functions on \( U_i, \) on \( U_i \cap U_k, \) and holomorphic vector fields on \( U_i, \) respectively.

Let

\[
S_i^\mu(p, t) = S_{i|0}(p) + \sum_{\lambda=1}^{\mu} S_{i|\lambda}(p, t)
\]

\[
f_{ik}^\mu(p, t) = f_{ik|0}(p) + \sum_{\lambda=1}^{\mu} f_{ik|\lambda}(p, t)
\]

\[
T_i^\mu(p, t) = T_{i|0}(p) + \sum_{\lambda=1}^{\mu} T_{i|\lambda}(p, t)
\]

**Notation 1.** We write the power series expansion of a holomorphic function \( P(t) \) in \( t_1, \ldots, t_m \) defined on a neighborhood of the origin 0 in the form:

\[
P(t) = P(0) + P_1(t) + \cdots + P_m(t) + \cdots,
\]

where each \( P_\mu(t) \) denotes a homogenous polynomial of degree \( \mu \) in \( t_1, \ldots, t_m. \) We set \( P^\mu(t) := P(0) + P_1(t) + \cdots + P_\mu(t). \) We write \( [P(t)]_\mu \) for \( P_\mu(t) \) when we substitute a complicated expression for \( P(t). \) For any power series \( P(t), \) and \( Q(t) \) in \( t = (t_1, \ldots, t_m), \) we indicate \( P(t) \equiv Q(t) \) that \( P(t) - Q(t) \) contains no term of degree \( \leq \mu \) in \( t. \)

By Notation 1 (2.2.4) and (2.2.5) are equivalent to the system of congruences

\[
S_i^\mu(p, t) \equiv f_{ik}^\mu(p, t)S_k^\mu(p, t)
\]

\[
[A_0, S_i^\mu(p, t)] \equiv S_{i|0}^\mu(p, t)T_i^\mu(p, t), \ \mu = 1, 2, 3, \ldots
\]

We will construct \( S_{i|\mu}(p, t), f_{ik|\mu}(p, t) \) by induction on \( \mu \) satisfying (2.2.12) and (2.2.13). We assume the following special forms for \( S_{i|\mu}(p, t), \mu \geq 1: \)

\[
S_{i|\mu}(p, t) = \begin{cases} \psi_{i|\mu}(z_i(p), t), & \text{if } V_0 \cap U_i \neq \emptyset \\ 0, & \text{if } V_0 \cap U_i = \emptyset \end{cases}
\]

where \( z_i(p) = (z^1_i(p), \ldots, z^n_i(p)) \) and \( \psi_{i|\mu}(z_i, t) \) is a homogeneous polynomial of degree \( \mu \) in \( t \) whose coefficients are holomorphic functions of \( z_i = (z^1_i, \ldots, z^n_i) \) defined on the polycylinder: \( |z^1_i| < 1, \ldots, |z^n_i| < 1. \)

We define \( \psi_{i|1}(z_i, t) \) by

\[
\psi_{i|1}(z_i(p), t) = \sum_{r=1}^{m} t_r \beta_{r_i}(p), \ p \in V_0 \cap U_i.
\]

and determine \( S_{i|0}(p, t), S_{i|1}(p, t) \) by (2.2.11) and (2.2.13). By letting \( f_{ik|1}(p, t) = \frac{S_{i|1}(p, t) - f_{ik|0}(p)S_{i|1}(p, t)}{S_{i|0}(p)}, \)

\( f_{ik}(p, t) = f_{ik|0}(p) + f_{ik|1}(p, t) \) is holomorphic in \( p \) and satisfies (2.2.12) (for the detail, see [KS59] p.486).

On the other hand, we note that \( [A_0, S_{i|1}(p, t)] \equiv 1 \ S_{i|1}(p, t)T_{i|1}(p, t) \) is equivalent to \( [A_0, S_{i|1}(p, t)] = S_{i|1}(p, t)T_{i|0}(p) + S_{i|0}(p)T_{i|1}(p, t). \) By letting

\[
T_{i|1}(p, t) = \frac{[A_0, S_{i|1}(p, t)] - S_{i|1}(p, t)T_{i|0}(p)}{S_{i|0}(p)}
\]
Hence we obtain from (2.2.18). Then we have, by restricting to $V_\equiv$, we define homogenous polynomials $f_{ik}(p, t)$, $f_{ik}^\mu(p, t)$, $T^\mu_i(p, t)$ satisfying (2.2.12) and (2.2.13), which imply that

\begin{align*}
(2.2.16) & \quad f_{ik}^\mu(p, t) \equiv \mu f_{ij}^\mu(p, t) f_{jk}^\mu(p, t) \\
(2.2.17) & \quad [\Lambda_0, T^\mu_i(p, t)] = 0 \\
(2.2.18) & \quad [\Lambda_0, f_{ik}^\mu(p, t)] + f_{ik}^\mu(p, t) T^\mu_i(p, t) - f_{ik}^\mu(p, t) T^\mu_i(p, t) \equiv \mu 0
\end{align*}

We define homogenous polynomials $\psi_{ik|\mu+1}(p, t)$ in $t$ of degree $\mu + 1$ whose coefficients are in $\Gamma(V_0 \cap U_1 \cap U_k, \mathcal{O}_V_0)$ and homogenous polynomials $W_{ij|\mu+1}(p, t)$ of degree $\mu + 1$ whose coefficient are in $\Gamma(U_i \cap V_0, T_W|_{V_0})$ by

\begin{align*}
(2.2.19) & \quad \psi_{ik|\mu+1}(p, t) \equiv \mu+1 f_{ik}^\mu(p, t) S^\mu_k(p, t) - S^\mu_i(p, t) \quad \text{for } p \in V_0 \cap U_i \cap U_k \\
(2.2.20) & \quad W_{ij|\mu+1}(p, t) \equiv \mu+1 [\Lambda_0, S^\mu_i(p, t)]|_{V_0} - S^\mu_i(p, t) T_i^\mu(p, t), \quad \text{for } p \in V_0 \cap U_i
\end{align*}

Then we have (for the detail, see [KS89] p.487)

\begin{align*}
(2.2.21) & \quad \psi_{ik|\mu+1}(p, t) = \psi_{ij|\mu+1}(p, t) + f_{ij|0}(p) \psi_{jk|\mu+1}(p, t), \quad \text{for } p \in V_0 \cup U_i \cup U_j \cup U_k \\
(2.2.22) & \quad -[W_{ij|\mu+1}(p, t), \Lambda_0]|_{V_0} + (\Lambda_0, f_{ik}^\mu(p, t))- [\Lambda_0, f_{ik}^\mu(p, t)]. \quad \text{We note that } G_{ik|\mu+1}(p, t) \equiv \mu 0 \text{ from (2.2.18)}
\end{align*}

Lastly, let $\tilde{W}_{ij|\mu+1}(p, t) \equiv \mu+1 f_{ik}^\mu(p, t)(T_i^\mu(p, t) - T_k^\mu(p, t)) - [\Lambda_0, f_{ik}^\mu(p, t)]$. Then from (2.2.18), we obtain

\begin{align*}
(2.2.22) & \quad -[W_{ij|\mu+1}(p, t), \Lambda_0]|_{V_0} + (\Lambda_0, f_{ik}^\mu(p, t))- [\Lambda_0, f_{ik}^\mu(p, t)] = 0, \\
& \quad \equiv \mu+1 f_{ik}^\mu(p, t)|[\Lambda_0, S^\mu_k(p, t)]|_{V_0} - S^\mu_i(p, t) T^\mu_i(p, t) - [\Lambda_0, S^\mu_i(p, t)]|_{V_0} \\
& \quad \equiv \mu+1 f_{ik}^\mu(p, t)|W_{ik|\mu+1}(p, t) + f_{ik}^\mu(p, t) S^\mu_k(p, t) T_i^\mu(p, t) + S^\mu_i(p, t) [\Lambda_0, f_{ik}^\mu(p, t)]|_{V_0} - W_{ik|\mu+1}(p, t) - S^\mu_i(p, t) T^\mu_i(p, t) - S^\mu_i(p, t) [\Lambda_0, T^\mu_i(p, t)] \\
& \quad \equiv \mu+1 f_{ik}^\mu(p, t)|W_{ik|\mu+1}(p, t) - W_{ik|\mu+1}(p, t) + W_{ik|\mu+1}(p, t) + \psi_{ik|\mu+1}(p, t) T_i^\mu(p, t) - W_{ik|\mu+1}(p, t)
\end{align*}

Hence we obtain

\begin{align*}
(2.2.23) & \quad -[\psi_{ik|\mu+1}(p, t), \Lambda_0]|_{V_0} + \psi_{ik|\mu+1}(p, t) T_i|_{V_0} + f_{ik|0}(p) W_{ik|\mu+1}(p, t) - W_{ik|\mu+1}(p, t) = 0
\end{align*}

Hence from (2.2.21), (2.2.22), (2.2.23), $(\psi_{\mu+1}(t), W_{\mu+1}(t)) := \{(\psi_{ik|\mu+1}(p, t), W_{ik|\mu+1}(p, t))\}$ defines a 1-cocycle in the following Cech resolution of $\mathcal{N}_0|_{V_0}$:

\begin{align*}
C^0(U \cap V_0, \mathcal{N}_0|_{V_0}) \oplus T^2 W|_{V_0} & \quad \nabla_0|_{V_0} \\
C^0(U \cap V_0, \mathcal{N}_0|_{V_0} \otimes T W|_{V_0}) & \quad \delta \rightarrow C^1(U \cap V_0, \mathcal{N}_0|_{V_0} \otimes T W|_{V_0}) \nabla_0|_{V_0} \\
C^0(U \cap V_0, \mathcal{N}_0|_{V_0}) & \quad \delta \rightarrow C^2(U \cap V_0, \mathcal{N}_0|_{V_0})
\end{align*}
Next we show that if \((\psi_{\mu+1}(t), W_{\mu+1}(t))\) vanishes identically, we can construct \(S_{\mu+1}^i(p, t), f_{ik}^{\mu+1}(p, t)\) and \(T_{\mu+1}^i(p, t)\) satisfying (2.2.12) and (2.2.13). Indeed, let us assume that \((\psi_{\mu+1}(t), W_{\mu+1}(t))\) vanishes identically. Then there exists homogenous polynomials \(\phi_{i|\mu+1}(p, t)\) in \(t\) of degree \(\mu + 1\) whose coefficients are holomorphic functions on \(V_0 \cap U_i\) such that
\[
(2.2.24) \quad \psi_{ik|\mu+1}(p, t) = \phi_{ik|\mu+1}(p, t) - f_{ik|0}(p)\phi_{ik|\mu+1}(p, t), \quad p \in V_0 \cap U_i \cap U_k,
\]
\[
(2.2.25) \quad W_{i|\mu+1}(p, t) = -[\Lambda_0, \phi_{i|\mu+1}(p, t)]|_0 + \phi_{i|\mu+1}(p, t)T_{i|0}(p), \quad p \in V_0 \cap U_i.
\]
We define \(\psi_{i|\mu+1}(z, t) := \psi_{i|\mu+1}(z(p), t) = \phi_{i|\mu+1}(p, t), p \in V_0 \cap U_i\) and determine \(S_{i|\mu+1}(p, t)\) by (2.2.14), and \(S_{i|\mu+1}^i(p, t)\) by (2.2.15). We define \(f_{ik|\mu+1}(p, t) \equiv_{\mu+1} S_{ik}^p(p, t) - f_{ik|0}(p)S_{ik}^p(p, t)\) and determine \(f_{ik}^{\mu+1}(p, t)\) by (2.2.10). Then \(S_{i|\mu+1}^i(p, t)\) and \(f_{ik}^{\mu+1}(p, t)\) satisfy (2.2.12) for the detail, see [KS59] p.488.

On the other hand, we note that from (2.2.20), we have
\[
(2.2.26) \quad W_{i|\mu+1}(p, t) = -[\Lambda_0, S_{i|\mu+1}(p, t)]|_0 + S_{i|\mu+1}(p, t)T_{i|0}(p), \quad p \in V_0 \cap U_i.
\]
We set
\[
(2.2.27) \quad T_{i|\mu+1}(p, t) :=_{\mu+1} \frac{\Phi^p(p, t)}{S_{i|0}(p)} = \frac{[\Lambda_0, S_i^p(p, t)] - S_i^p(p, t)T_i^p(p, t) + [\Lambda_0, S_{i|\mu+1}(p, t)] - S_{i|\mu+1}(p, t)T_{i|0}(p)}{S_{i|0}(p)}
\]
Then \(\Phi^p(p, t) \equiv_{\mu+1} 0\) for \(p \in V_0 \cap U_i\) from (2.2.20) and (2.2.26) so that \(T_{i|\mu+1}(p, t)\) are holomorphic in \(p\) and we have, from (2.2.27),
\[
[\Lambda_0, S_i^p(p, t) + S_{i|\mu+1}(p, t)] \equiv_{\mu+1} S_i^p(p, t)T_i^p(p, t) + S_{i|\mu+1}(p, t)T_{i|0}(p) + S_{i|0}(p)T_{i|\mu+1}(p, t)
\]
\[
\equiv_{\mu+1} (S_i^p(p, t) + S_{i|\mu+1}(p, t))T_i^p(p, t) + T_{i|\mu+1}(p, t)
\]
so that \(S_{i|\mu+1}^i(p, t)\) and \(T_{i|\mu+1}^i(p, t)\) satisfy (2.2.13).

Now we prove that \((\psi_{\mu+1}(t), W_{\mu+1}(t))\) vanishes identically if \(V_0\) is Poisson semi-regular. For this purposes it suffices to construct a polynomial \((\eta_{ik}(t), \omega_i(t))\) in \(t\) of degree \(\mu + 1\) with coefficients in \(H^1(W, N^*_0)\) such that
\[
(2.2.28) \quad \psi_{ik|\mu+1}(p, t) = \eta_{ik}(p, t) \quad \text{for } p \in V_0 \cap U_i \cap U_k,
\]
\[
(2.2.29) \quad W_{i|\mu+1}(p, t) = \omega_i(p, t) \quad \text{for } p \in V_0 \cap U_i.
\]

In fact, \((\eta_{ik}(p, t), \omega_i(p, t))\) represents a polynomial \((\eta(t), \omega(t))\) in \(t\) with coefficients in \(H^1(W, N^*_0)\), and (2.2.28) and (2.2.29) imply that \((\psi_{\mu+1}(t), W_{\mu+1}(t)) = 0\) \((\eta(t), \omega(t))\). Hence we obtain \((\psi_{\mu+1}(t), W_{\mu+1}(t))\) vanishes if \(V_0\) is Poisson semi-regular.

**Lemma 2.2.30** (see [KS59] Lemma 1 p.488). **For each integer \(\lambda \leq \mu\), there exist polynomials \(g_{ik} := g_{ik}^\lambda(p, t)\) in \(t\) of degree \(\lambda\) whose coefficients are holomorphic functions in \(p\) defined on \(U_i \cap U_k\) such that**
\[
(2.2.31) \quad f_{ik|0}(p) \exp g_{ik}^\lambda(p, t) \equiv_{\lambda} f_{ik}^\lambda(p, t)
\]
We define polynomials \(f_{ik}^{\mu+1} := \hat{f}_{ik}^{\mu+1}(p, t) \equiv_{\mu+1} f_{ik|0}(p) \exp g_{ik}^\mu(p, t)\) in \(t\) of degree \(\mu + 1\). Then \(f_{ik}^{\mu+1}(p, t) \equiv_{\mu} f_{ik}(p, t) + \hat{f}_{ik}^{\mu+1}(p, t)S_{ik}^p(p, t)\). By letting \(\eta_{ik}(p, t) \equiv_{\mu+1} \hat{f}_{ik}^{\mu+1}(p, t)S_{ik}^p(p, t) - S_i^p(p, t)\), we get (for the detail, see [KS59] p.489)
\[
(2.2.32) \quad \eta_{ik}(p, t) = \eta_{ik}(p, t) + f_{ik|0}(p)\eta_{ik}(p, t)
\]

On the other hand, we denote \(f_{ik} := f_{ik}(p, t)\) and \(T_i := T_i(p, t)\), we note that from (2.2.18), we have
\[
[\Lambda_0, f_{ik}^\mu + f_{ik}(T_k^\mu - T_i^\mu)] \equiv_{\mu} 0.
\]
Since \(f_{ik|0}(p) \exp(\hat{g}_{ik}^\mu) \equiv_{\lambda} f_{ik}^\lambda(p, t)\) by (2.2.31), we have
\[
[\Lambda_0, f_{ik|0} \exp g_{ik}^\mu] + f_{ik|0} \exp(g_{ik}^\mu)(T_k^\mu - T_i^\mu) \equiv_{\mu} 0
\]
\[
\iff [\Lambda_0, f_{ik|0}] + [\Lambda_0, g_{ik}^\mu] + (T_k^\mu - T_i^\mu) = 0
\]
\[
(2.2.33) \quad \iff \{\Lambda_0, \hat{f}_{ik}^{\mu+1}\} + \hat{f}_{ik}^{\mu+1}(T_k^\mu - T_i^\mu) = 0
\]
From (2.2.12), we have \([A_0, S_{p}^{\mu}(p, t)] \equiv_{\mu} S_{p}^{\mu}(p, t) T_{i}^{\mu}(p, t)\). We define \(\omega_1 := \omega_1(p, t) \equiv_{\mu+1} [A_0, S_{p}^{\mu}(p, t)] - S_{p}^{\mu}(p, t) T_{i}^{\mu}(p, t)\). Then we have from (2.2.33)

\[
[A_0, \eta_{ik}] \equiv_{\mu+1} [A_0, \tilde{f}_{ik}^{\mu+1} S_{k}^{\mu}] - [A_0, S_{i}^{\mu}] \equiv_{\mu+1} S_{i}^{\mu}[A_0, \tilde{f}_{ik}^{\mu+1}] + \tilde{f}_{ik}^{\mu+1}[A_0, S_{k}^{\mu}] - [A_0, S_{i}^{\mu}]
\]

\[
\equiv_{\mu+1} S_{i}^{\mu}[A_0, \tilde{f}_{ik}^{\mu+1}] + \tilde{f}_{ik}^{\mu+1} S_{k}^{\mu} T_{k} + f_{ik[0]} \omega_{k} - S_{i}^{\mu} T_{i} - \omega_{i}
\]

\[
\equiv_{\mu+1} S_{i}^{\mu}[A_0, \tilde{f}_{ik}^{\mu+1}] + \tilde{f}_{ik}^{\mu+1} S_{k}^{\mu} T_{k} + f_{ik[0]} \omega_{k} + (\eta_{ik} - \tilde{f}_{ik}^{\mu+1} S_{k}^{\mu}) T_{i} - \omega_{i}
\]

\[
\equiv_{\mu+1} S_{i}^{\mu}[A_0, \tilde{f}_{ik}^{\mu+1}] + f_{ik[0]} \omega_{k} - \omega_{i}
\]

so that we obtain the equality

(2.2.34) \[-[\eta_{ik}(p, t), A_0] + \eta_{ik}(p, t) T_{i[0]}(p) + f_{ik[0]}(p) \omega_{k}(p) - \omega_{i}(p) = 0, \ p \in U_i \cap U_k.\]

Lastly, we have from (2.2.17),

\[
[A_0, \omega_i] \equiv_{\mu+1} -[A_0, S_{p}^{\mu} T_{i}] \equiv_{\mu+1} [A_0, S_{i}^{\mu}] T_{i} - S_{i}^{\mu} [A_0, T_{i}] \equiv_{\mu+1} \omega_i T_{i} + S_{i}^{\mu} T_{i}^{\mu} - S_{i}^{\mu} [A_0, T_{i}] \equiv_{\mu+1} \omega_i T_{i[0]}
\]

so that we obtain the equality

(2.2.35) \[-[\omega_i(p, t), A_0] + (-1)^{1} \omega_i(p, t) \cap T_{i[0]}(p) = 0, \ p \in U_i,\]

Form (2.2.12), (2.2.1), and (2.2.13), \((\{\eta_{ik}(p, t), \{\omega_i(p, t)\}\})\) is a polynomial in\(t\) whose coefficients in \(\mathbb{C}^{1}(W, N_0)\). Then since \(S_{k}^{\mu}(p, t) \equiv 0 \) for \(p \in V_0 \cap U_k\), and \(f_{ik}^{\mu+1}(p, t) \equiv_{\mu} f_{ik}^{\mu}(p, t)\), we obtain

\[
\eta_{ik}(p, t) \equiv_{\mu+1} f_{ik}^{\mu}(p, t) S_{k}^{\mu}(p, t) - S_{i}^{\mu}(p, t) \equiv_{\mu+1} \psi_{ik[\mu+1]}(p, t), \ p \in V_0 \cap U_k,
\]

\[
\omega_i(p, t) \equiv_{\mu+1} [A_0, S_{p}^{\mu}(p, t)]|_{V_0} - S_{i}^{\mu}(p, t) T_{i}(p, t) \equiv_{\mu+1} W_{i[\mu+1]}(p, t), \ p \in V_0 \cap U_i.
\]

Hence when \(V_0\) is Poisson semi-regular, we can construct \(S_{p}^{\mu}(p, t), f_{ik}(p, t)\) and \(T_{i}(p, t)\) satisfying (2.2.12) and (2.2.13), by induction on \(\mu\), and therefore we obtain formal power series \(S_{i}(p, t), f_{ik}(p, t)\) and \(T_{i}(p, t)\), satisfying (2.2.4), (2.2.5), (2.2.6), (2.2.7), and (2.2.8).

2.3. Proof of convergence.

Notation 2. Consider a formal power series \(f(t) = f(p, t) = \sum f_{h_1, h_2, \ldots, h_m}(p)(t_1)^{h_1}(t_2)^{h_2} \cdots (t_m)^{h_m}\) whose coefficients \(f_{h_1, h_2, \ldots, h_m}(p)\) are vector-valued holomorphic functions in \(p\) defined on a domain and a power series \(a(t) = \sum a_{h_1, h_2, \ldots, h_m}(t_1)^{h_1}(t_2)^{h_2} \cdots (t_m)^{h_m}, a_{h_1, h_2, \ldots, h_m} \geq 0.\) We indicate by \(f(p, t) \ll a(t)\) that \(|f_{h_1, h_2, \ldots, h_m}(p)| < a_{h_1, h_2, \ldots, h_m}|.\) Let

\[
A(t) = \frac{b}{64c} \sum_{\mu=1}^{\infty} \left(\frac{c^\mu}{\mu^2}\right)^v A(t), \ v = 2, 3, \ldots.
\]

where \(b, c\) are positive constants. Then we have

(2.3.1) \[A(t)^v \ll \left(\frac{b}{c}\right)^{v-1} A(t), \ v = 2, 3, \ldots.\]

We will show that the formal power series \(S_{i}(t) := S_{i}(p, t), f_{ik}(t) := f_{ik}(p, t)\) and \(T_{i}(t) := T_{i}(p, t)\) constructed in the previous subsection satisfy

(2.3.2) \[S_{i}(t) - S_{i[0]} \ll A(t), \ p \in U_i \cap U_k,\]

(2.3.3) \[f_{ik}(t) - f_{ik[0]} \ll c_{1} A(t), \ p \in U_i,\]

(2.3.4) \[T_{i}(t) - T_{i[0]} \ll d_{1} A(t), \ p = (z_{l}(p), w_{l}(p)) \in U_{i} \text{ with } |w_{l}(p)| < 1, |z_{l}(p)| < 1 - \delta \iff p \in U_{i}^\delta,\]

for some constants \(b, c, c_{1}, d_{1}\) from Notation 2 and a sufficiently small number \(\delta > 0\) in the beginning of subsection 2.2. Here we write \(T_{i}(t) = T_{i}(p, t)\) by the form \(T_{i}(p, t) = T_{i}^{\mu}(p, t) \frac{d}{d z_{l}^{\mu}} + \cdots + T_{i}^{m}(p, t) \frac{d}{d z_{l}^{m}} + T_{i}^{n+1}(p, t) \frac{d}{d w_{l}}\)

by which we consider \(T_{i}(p, t)\) a power series in \(t\) whose coefficients are vector-valued holomorphic functions on \(U_{i}\).

We may assume that \(|f_{ik}(p)| < c_{2}, p \in U_{i} \cap U_{k}\) for some constant \(c_{2} > 0.\) Then \(S_{i}(t) - S_{i[0]} \ll A(t)\) if \(b\) is sufficiently large.
Suppose that
\[(2.3.5)\quad f^{\mu \leftarrow 1}_{ik}(t) - f_{ik|0} \ll c_1 A(t), \quad p \in U_i \cap U_k\]
\[(2.3.6)\quad S^{\mu}_i(t) - S_{i|0} \ll A(t), \quad p \in U_i\]
\[(2.3.7)\quad T^{\mu \leftarrow 1}_i(t) - T_{i|0} \ll d_1 A(t), \quad p = (z_i(p), w_i(p)) \in U_i \text{ with } |w_i(p)| < 1, |z_i(p)| < 1 - \delta \iff p \in U_i^\delta\]

First we show that 
\[f^{\mu}_{ik}(t) - f_{ik|0} \ll c_1 A(t), p \in U_i \cap U_k \text{ for some constant } c_1 > 0.\]
We briefly summarize Kodaira’s result in the following (see [KS59] p.491-492): by setting 
\[c = \frac{2(\alpha + 2)(\alpha + \epsilon)}{\epsilon} \quad \text{for some sufficiently small constant } 0 < \epsilon < 1, \text{ and assuming}\]
\[(2.3.8)\quad c > 2bc_1(1 + c_2^2),\]
we get 
\[f^{\mu}_{ik}(t) - f_{ik|0} \ll c_1 A(t), p \in U_i \cap U_k.\]
Next we show that
\[(2.3.9)\quad T^{\mu}_i(t) - T_{i|0} \ll d_1 A(t), \quad p \in U_i^\delta\]
for some constant \(d_1 > 0\). We may assume that
\[(2.3.10)\quad |T_{i|0}(p)| < d_2, \quad p \in U_i\]
for some constant \(d_2 > 0\). We recall from \((2.2.27)\) that
\[(2.3.11)\quad T_{i|\mu}(p, t) = \frac{[A_0, S^{\mu - 1}_i]}{S_{i|0}} - T^{\mu \leftarrow 1}_i S^{\mu - 1}_i + \frac{[A_0, S_{i|0}]}{S_{i|0}} - T^{i \leftarrow 0}_i S^{i}_i = \frac{[A_0, S^{\mu}_i]}{S_{i|0}} - T^{\mu \leftarrow 1}_i S^{\mu}_i\]
We estimate \([A_0, S^{\mu}_i] - T^{\mu \leftarrow 1}_i S^{\mu}_i\). We note that
\[(2.3.12)\quad [A_0, S^{\mu}_i] - T^{\mu \leftarrow 1}_i S^{\mu}_i = \frac{[A_0, S^{\mu}_i - S_{i|0}]}{S_{i|0}} - (T^{\mu \leftarrow 1}_i - T_{i|0})(S^{\mu}_i - S_{i|0}) + [A_0, S_{i|0} - T_{i|0}](S^{\mu}_i - S_{i|0}) - (T^{\mu \leftarrow 1}_i - T_{i|0})S_{i|0} - T_{i|0}S_{i|0}\]
We note that since \([A_0, S^{\mu}_i] - T^{\mu \leftarrow 1}_i S^{\mu}_i \ll_{\mu} 0, (T^{\mu \leftarrow 1}_i - T_{i|0})S_{i|0} \text{ contributes nothing to } [A_0, S^{\mu}_i] - S^{\mu}_i T^{\mu \leftarrow 1}_i\). Let us estimate \([A_0, S^{\mu}_i - S_{i|0}]\) in \((2.3.13)\). Let \(A_0 = \sum_{\alpha, \beta = 1}^{n+1} \lambda_i \alpha \beta \frac{\partial S_{\alpha \beta}}{\partial x_i} \lambda_i \nu \alpha \beta \frac{\partial S_{\nu \beta}}{\partial x_i}\) with \(A^{\alpha \beta}_i = -A^{\nu \alpha}_i\) for some positive constant \(M > 0\).
\[(2.3.14)\quad |A_0, S^{\mu}_i - S_{i|0}| = \sum_{\alpha, \beta = 1}^{n+1} 2\lambda_i \alpha \beta \frac{\partial S_{\alpha \beta}}{\partial x_i} \lambda_i \nu \alpha \beta \frac{\partial S_{\nu \beta}}{\partial x_i} \ll 2(n + 1)^2 M A(t)\]
Hence we have, from \((2.3.12), (2.3.7), (2.3.13), (2.3.10)\) and \((2.3.14)\),
\[(2.3.15)\quad T_{i|\mu}(t) \ll d_3 \quad \text{for some constant } d_3 > 0.\]
We claim that
\[(2.3.16)\quad T_{i|\mu}(t) \ll \frac{d_3}{\epsilon} A(t), \quad p \in U_i^\delta\]
Indeed, from \(S_{i|0}(p) = w_i(p), (2.3.11)\) and \((2.3.14)\), if \(|w_i(p)| = \epsilon, T_{i|\mu}(t) \ll \frac{d_3}{\epsilon} A(t).\) If \(|w_i(p)| < \epsilon, we get\]
\[(2.3.17)\quad T_{i|\mu}(t) \ll \frac{d_4}{\epsilon} A(t) \text{ by the maximum principle.}\]
On the other hand, from \((2.3.15)\), we have \(\frac{d_4}{\epsilon} = \frac{d_4}{\epsilon} + \frac{d_4}{\epsilon} c\). Now we set \(d_1 = \frac{2d_4}{\epsilon}\). If we take
\[(2.3.18)\quad c > \frac{2d_4}{\epsilon},\]
then we get \(\frac{d_4}{\epsilon} < \frac{d_4}{\epsilon} + \frac{d_4}{\epsilon} = d_1\) so that we obtain \((2.3.9)\) from \((2.3.7)\) and \((2.3.10)\).
Lastly we show that
\begin{equation}
S_{i}^{µ+1}(t) - S_{i0} \ll A(t), \quad p \in U_i
\end{equation}
We note that (for the detail, see [KS59] p.493)
\begin{equation}
\psi_{ik|µ+1}(t) \ll \frac{bc_i}{c} A(t), \quad p \in V_0 \cap U_i \cap U_k
\end{equation}
Recall from \((2.2.22)\) that \(W_{i|µ+1}(p, t) \equiv_{µ+1} [A_0, S_{i|0}^µ(p, t)]V_0 - S_{i|0}^µ(p, t)T_{i|0}^µ(p, t)\). Since \([A_0, S_{i|0}^µ] - S_{i|0}^µT_{i|0}^µ \equiv_µ 0\), we get \([A_0, S_{i|0}^µ] - S_{i|0}^µ(T_{i|0} - T_{i|0}) = S_{i|0}^µT_{i|0}^µ - S_{i|0}^µ(T_{i|0} - T_{i|0})\). Since \([A_0, S_{i|0}^µ] - S_{i|0}^µT_{i|0}^µ \equiv_µ 0\), we obtain, from \((2.3.30)\) and \((2.3.33)\).
\begin{equation}
W_{i|µ+1}(p, t) \ll \frac{bd_i}{c} A(t), \quad p \in V_0 \cap U_i^δ
\end{equation}

**Lemma 2.3.21** (compare [KS59] p.499). We can choose \(\phi_{i|µ+1}(p, t)\) satisfying
\begin{align*}
&\psi_{ik|µ+1}(p, t) = \phi_{i|µ+1}(p, t) - f_{ik|0}(p)\phi_{k|µ+1}(p, t) \\
&W_{i|µ+1}(p, t) = - [A_0, \phi_{i|µ+1}(p, t)]V_0 + \phi_{i|µ+1}T_{i|0}(p)
\end{align*}
such that \(\phi_{i|µ+1} \equiv c_4 \left( \frac{bc_i}{c} + \frac{bd_i}{c} \right) A(t)\), where the constant \(c_4\) is independent of \(µ\).

**Proof.** For simplicity, we write \(U_i\) for \(V_0 \cap U_i, U_i^δ\) for \(V_0 \cap U_i^δ\), and let \(U = \{U_i\}\) be the covering of \(V_0\). For any 0-cochain \(\phi = \{\phi_i(p)\}\), 1-cochain \(\psi = \{\psi_{ik}(p), W_i(p)\}\) on \(U\), we define the norms of \(\phi, (ψ, W)\) by
\begin{align*}
||\phi|| & := \max \sup_{i} |\phi_i(p)|, \\
||\psi, W|| & := \max \sup_{i, k} |\psi_{ik}(p)| + \max \sup_{i} |W_i(p)|
\end{align*}
The coboundary \(\phi\) is defined by
\begin{equation}
f_{ik}(p)\phi_k(p) - \phi_i(p), \quad p \in U_i \cap U_k, \quad -[\phi_i(p), A_0]V_0 + \phi_i(p)T_{i|0}(p), \quad p \in U_i
\end{equation}
For any \((ψ, W)\), we define
\begin{equation}
\iota(ψ, W) = \inf_{δ(ψ) = (ψ, W)} ||ψ||
\end{equation}
It suffices to prove the existence of constant \(c\) such that \(\iota(ψ, W) \leq c||ψ, W||\). Assume that such a constant \(c\) does not exist. Then we can find a sequence \((ψ', W'), (ψ'', W''), \ldots, (ψ^{(µ)}, W^{(µ)}), \ldots\) such that
\begin{equation}
\iota(ψ^{(µ)}, W^{(µ)}) = 1, \quad ||ψ^{(µ)}, W^{(µ)}|| < \frac{1}{µ}
\end{equation}
We take a covering \(\{U_i^δ\}\) of \(V_0\). Since \(φ_i^δ(p) < 2\) for \(p \in U_i\), there exists a subsequence \(φ_1^{(µ_1)}, φ_2^{(µ_2)}, \ldots, φ_ν^{(µ_ν)}\) of \(φ^{(µ)}\) converging absolutely and uniformly on \(U_i^δ\) for each \(η\). Since \(V_0\) is compact, we can choose a subsequence that works for all \(η\). On the other hand, since \(||ψ, W|| < \frac{1}{µ}\), we have in particular
\begin{equation}
|f_{ik|0}(p)φ_i^{(ν)}(p) - φ_i^{(µ)}(p)| < \frac{1}{µ}, \quad p \in U_i \cap U_k, \quad -[φ_i^{(µ)}(p), A_0]V_0 + φ_i^{(µ)}(p)T_{i|0}(p) < \frac{1}{µ}, \quad p \in U_i^δ
\end{equation}
Then \(φ_i^{(µ_ν)}\) converges absolutely and uniformly on the whole \(U_i\). Let \(φ_i(p) = \lim_n φ_i^{(µ_n)}(p)\) and let \(φ = \{φ_i(p)\}\). Then we have \(||φ^{(µ_n)} - φ|| \to 0\) as \(n \to ∞\). On the other hand, from \((2.3.22)\), we have \(δ(φ) = (0, W_φ)\), where \(W_φ(p) = 0\) for \(p \in U_i^δ\). By identity theorem, \(W_φ(p) = 0\) for \(p \in U_i\). Hence we have \(δ(φ^{(µ_ν)} - φ) = (ψ^{(µ_ν)}, W^{(µ_ν)})\) which contradicts to \(\iota(ψ^{(µ_ν)}, W^{(µ_ν)}) = 1\). □

By Lemma \((2.3.21)\), we can choose \(S_{i|µ+1}(t) \ll c_4 \left( \frac{bc_i}{c} + \frac{bd_i}{c} \right) A(t)\). We note \(\frac{bd_i}{c} A(t)\) and \(\frac{bc_i}{c} A(t)\). By setting \(c > \max\{2b_1c_1(1 + e^2), \frac{2b}{c}, c_4bc_1 + c_4bd_1\}\), we get \(\frac{bd_i}{c} A(t)\). Since the constants \(b, c, c_1\) are independent of \(µ\), we have \(\frac{bd_i}{c} A(t)\), \(\frac{bc_i}{c} A(t)\), and \(\frac{bd_i}{c} A(t)\). By letting \(N = \{t| \sum_{i=1}^n |t_i|^2 < \frac{δ}{n^2}\}\), the power series \(S_i(p, t), f_{ik}(p, t)\) and \(T_i(p, t)\) converges absolutely and uniformly for \(t \in N\) so that \(S_i(p, t), T_i(p, t)\) and \(f_{ik}(p, t)\) are holomorphic on \(U_i^δ \times N\) and \(U_i^δ \cap U_i^δ \times N\), respectively, and satisfy \(\frac{bd_i}{c} A(t)\), \(\frac{bc_i}{c} A(t)\), \(\frac{bd_i}{c} A(t)\), \(\frac{bc_i}{c} A(t)\), \(\frac{bd_i}{c} A(t)\), and \(\frac{bc_i}{c} A(t)\) by replacing \(U_i\) by \(U_i^δ\). This completes the proof of Theorem \((2.2.23)\). □
2.4. Maximal families: Theorem of completeness.

**Definition 2.4.1.** Let \( \mathcal{V} \subset (W \times M, \Lambda_0) \) be a Poisson analytic family of compact holomorphic Poisson submanifolds of \((W, \Lambda)\) of codimension 1 and let \( t_0 \) be a point on \( M \). We say that \( \mathcal{V} \) is maximal at \( t_0 \) if, for any Poisson analytic family \( \mathcal{V}' \subset (W \times M', \Lambda) \) of compact holomorphic Poisson submanifolds of \((W, \Lambda_0)\) of codimension 1 such that \( \omega^{-1}(t_0) = \omega^{-1}(t_0') = M' \), there exists a holomorphic map \( h \) of a neighborhood \( N' \) of \( t_0' \) on \( M' \) into \( M \) which maps \( t_0' \) to \( t_0 \) such that \( \omega^{-1}(t') = \omega^{-1}(h(t')) \) for \( t' \in N' \). We note that if we set a Poisson map \( h : (W \times N', \Lambda_0) \rightarrow (W \times M, \Lambda_0) \) defined by \((w, t') \rightarrow (w, h(t'))\), then the restriction map of \( h \) to \( \mathcal{V}'|_{N'} = \omega^{-1}(N') \subset (W \times N', \Lambda) \) defines a Poisson map \( \mathcal{V}'|_{N'} \rightarrow \mathcal{V} \) so that \( \mathcal{V}'|_{N'} \) is the family induced from \( \mathcal{V} \) by \( h \), which means \( \mathcal{V} \) is maximal at \( t_0 \).

**Theorem 2.4.2** (Theorem of completeness). Let \( \mathcal{V} \subset (W \times M, \Lambda_0) \) be a Poisson analytic family of compact holomorphic Poisson submanifolds of \((W, \Lambda_0)\) of codimension 1. If the characteristic map

\[ \rho_{d,0} : T_0 M \rightarrow \mathbb{H}^0(V_0, N_0^*|_{V_0}) \]

is an isomorphism, then the family \( \mathcal{V} \) is maximal at the point \( t = 0 \).

**Proof.** We extend the arguments in \([\text{KS}59]\) p.494-496 in the context of holomorphic Poisson deformations. We tried to maintain notational consistency with \([\text{KS}59]\).

Suppose that \( M = \{ t \mid \sum_{r=1}^s |t_r|^2 < 1 \} \) and that \( \rho_{d,0} : T_0 M \rightarrow \mathbb{H}^0(V_0, N_0^*|_{V_0}) \) is an isomorphism. Let \( \mathcal{V}' \subset (W \times M, \Lambda_0) \) be an arbitrary Poisson analytic family of holomorphic Poisson submanifolds of \((W, \Lambda)\) of codimension 1 such that \( \omega^{-1}(0) = V_0 \), where \( M' = \{ s \mid \sum_{r=1}^s |s_r|^2 < 1 \} \). We will construct a holomorphic map \( h : s \rightarrow t = h(s) \) of \( N' \) into \( M \) with \( h(0) = 0 \) such that \( \omega^{-1}(s) = \omega^{-1}(h(s)) \) where \( N' = \{ s \mid \sum_{r=1}^s |s_r|^2 < \delta \} \subset M' \) for a sufficiently small number \( \delta > 0 \).

We keep the notations of subsection 2.2 so that \( \{ S_i(p, t) \}, \{ f_i(p, t) \}, \) and \( \{ T_i(p, t) \} \) determine the Poisson analytics family \( \mathcal{V} \). Let \( \{ R_i(p, s) \}, \{ e_i(p, s) \} \) and \( \{ Q_i(p, t) \} \) be the corresponding system defining \( \mathcal{V}' \subset (W \times M', \Lambda_0) \) so that we have

\[ S_i(p, t) = f_i(p, t)S_k(p, t), \quad [\Lambda_0, S_i(p, t)] = S_i(p, t)T_i(p, t) \]

\[ R_i(p, s) = e_i(p, s)R_k(p, s), \quad [\Lambda_0, R_i(p, s)] = R_i(p, s)Q_i(p, s) \]

We may assume that

\[ S_i(p, 0) = R_i(p, 0) = w_i(p), \quad f_i(p, 0) = e_i(p, 0) = f_{ik0}(p), \quad T_i(p, 0) = R_i(p, 0) = T_{ik0}(p). \]

We expand \( S_i(p, t) = w_i(p) + S_{i0}(p,t) + S_{i2}(p,t) + \cdots \) and let \( S_{i1}(p, t) = \sum_{r=1}^s B_r(p) t_r \). Then the restriction \( \beta_r(p) \) of \( B_r(p) \) to \( V_0 \) satisfy

\[ \beta_r(p) = f_{ik0}(p) \beta_r(p), \quad p \in V_0 \cap U_i \cap U_k, \]

\[ -[\beta_r(p), \Lambda_0]|_{V_0} + \beta_r(p) T_{ik0}(p) = 0, \quad p \in V_0 \cap U_i \]

and \( \{ \beta_1, \ldots, \beta_m \} \) forms a basis of \( \mathbb{H}^0(V_0, N_0^*|_{V_0}) \) by the hypothesis.

If there exist non-vanishing holomorphic functions \( f_i(p, s) \) defined on \( U_i \times N' \) satisfying

\[ f_i(p, s) R_i(p, s) = S_i(p, h(s)), \]

we get \( \omega^{-1}(s) = \omega^{-1}(h(s)) \). Recall Notation 1 and let us write \( h(s) \) and \( f_i(p, s) \) in the following form:

\[ h(s) = (h_1(s) = \sum_{\mu=1}^\infty h_{r1\mu}(s), \ldots, h_m(s) = \sum_{\mu=1}^\infty h_{r1\mu}(s)), \quad f_i(p, s) = 1 + \sum_{\mu=1}^\infty f_{i\mu}(p, s), \]

We will construct such \( f_i(p, s) \) and \( h(s) \) satisfying (2.4.8) by solving the system of congruences by induction on \( \mu \)

\[ f_{i\mu}(p, s) R_i(p, s) \equiv \mu_\mu S_i(p, h^\mu(s)), \quad \mu = 0, 1, 2, \cdots . \]

(2.4.10) follows from (2.4.15). Now assume that \( h^\mu(s) \) and \( f_i^{\mu-1}(p, s) \) satisfying (2.4.9) are already determined. We will find \( h_{\nu\mu}(s) \) and \( f_{i\nu\mu}(p, s) \) such that \( h^\mu = h^{\mu-1}(s) + h^\mu(s) \) and \( f_i^{\mu-1}(p, s) + f_{i\mu}(p, s) \) satisfy (2.4.10). We can define homogenous polynomials \( \Gamma_{i\mu}(p, s) \) of degree \( \mu \) in \( s \) by

\[ \Gamma_{i\mu}(p, s) \equiv \mu \int f_{i\mu}(p, s) R_i(p, s) - S_i(p, h^{\mu-1}(s)), \quad p \in U_i \]
Then we claim that

\[(2.4.11)\quad \Gamma_{i|\mu}(p, s) = f_{ik|0}(p)\Gamma_{k|\mu}(p, s), \quad p \in V_0 \cap U_i \cap U_k\]

\[(2.4.12)\quad [\Lambda_0, \Gamma_{i|\mu}(p, s)]|_{w_i(p)=0} = \Gamma_{i|\mu}(p, s)T_i(p), \quad p \in V_0 \cap U_i\]

Indeed, \[(2.4.11)\] follows from \[KS59\] p.496. On the other hand, to prove \[(2.4.12)\], we remark that

\[(2.4.13)\quad [\Lambda_0, f_i|^{\mu-1}(p, s)] + Q_i(p, s)f_i|^{\mu-1}(p, s) - f_i|^{\mu-1}(p, s)T_i(p, h|^{\mu-1}(p, s)) \equiv_{\mu-1} 0\]

Indeed, by applying \([\Lambda_0, -]\) on \(f_i|^{\mu-1}(p, s)R_i(p, s) \equiv_{\mu-1} S_i(p, h|^{\mu-1}(s))\) in \[(2.4.11)\] for \(\mu-1\), we get, from \[(2.4.13)\], \[(2.4.14)\], and \[(2.4.19)\],

\[(2.4.14)\quad [\Lambda_0, \Gamma_{i|\mu}(p, s)] \equiv_{\mu} [\Lambda_0, f_i|^{\mu-1}(p, s)R_i(p, s)] - [\Lambda_0, S_i(p, h|^{\mu-1}(s))]\]

\[=\mu [\Lambda_0, f_i|^{\mu-1}(p, s)]R_i(p, s) + [\Lambda_0, R_i(p, s)]f_i|^{\mu-1}(p, s) - [\Lambda_0, S_i(p, h|^{\mu-1}(s))]\]

\[=\mu [\Lambda_0, f_i|^{\mu-1}(p, s)]R_i(p, s) + R_i(p, s)Q_i(p, s)f_i|^{\mu-1}(p, s) - S_i(p, h|^{\mu-1}(s))T_i(p, h|^{\mu-1}(s))\]

\[=\mu [\Lambda_0, f_i|^{\mu-1}(p, s)]R_i(p, s) + R_i(p, s)Q_i(p, s)f_i|^{\mu-1}(p, s) - \Gamma_{i|\mu}(p, s)T_i(p, h|^{\mu-1}(s)) - f_i|^{\mu-1}(p, s)R_i(p, s)T_i(p, h|^{\mu-1}(s))\]

\[=\mu (\Lambda_0, f_i|^{\mu-1}(p, s)] + Q_i(p, s)f_i|^{\mu-1}(p, s) - f_i|^{\mu-1}(p, s)T_i(p, h|^{\mu-1}(p, s)) + \Gamma_{i|\mu}(p, s)T_i(p, h|^{\mu-1}(s)))R_i(p, s) + \Gamma_{i|\mu}(p, s)T_i(p, h|^{\mu-1}(s))\]

By restricting \[(2.4.14)\] to \(V_0\) (i.e by setting \(S_i(p, 0) = R_i(p, 0) = w_i(p)\)), we obtain

\[(2.4.15)\quad [\Lambda_0, \Gamma_{i|\mu}(p, s)]|_{w_i(p)=0} = \mu \Gamma_{i|\mu}(p, s)T_i(p, h|^{\mu-1}(s)), \quad p \in V_0 \cap U_i\]

This proves \[(2.4.12)\]. From \[(2.4.11)\] and \[(2.4.12)\], there exist homogenous polynomials \(h_{t|\mu}\) of degree \(\mu\) in \(s\) such that

\[(2.4.16)\quad \sum_{r=1}^{m} \beta_{t|\mu}(s)h_{t|\mu}(s) = \Gamma_{t|\mu}(p, s), \quad p \in V_0 \cap U_i\]

From \(h_{t|\mu}(s) = h_{t|\mu}(s) + h_{t|\mu}(s)\), the congruence \[(2.4.16)\] is equivalent to (for the detail, see \[KS59\] p.496)

\[(2.4.17)\quad \sum_{r=1}^{m} B_{t|\mu}(p)h_{t|\mu}(s) = w_i(p)f_{i|\mu}(p, s) + \Gamma_{i|\mu}(p, s)\]

By setting \(f_{i|\mu}(p, s) := \sum_{r=1}^{m} B_{t|\mu}(p)h_{t|\mu}(s) - \Gamma_{i|\mu}(p, s)\), we get \[(2.4.17)\], which completes the inductive construction of \(h^\mu(s)\) and \(f_i|^{\mu}(p, s)\) satisfying \[(2.4.19)\].

\[\Box\]

2.5. Proof of convergence.

The convergence of \(h_{t|\mu}(s)\), \(f_i(p, s)\) follows from the same arguments in \[KS59\] p.497-498, which completes Theorem \[(2.4.2)\].

3. Deformations of compact holomorphic Poisson submanifolds of arbitrary codimensions

We extend Definition \[(2.0.26)\] to arbitrary codimensions.

**Definition 3.0.1** (compare \[Ko62\]). Let \((W, \Lambda_0)\) be a holomorphic Poisson manifold of dimension \(d + r\). We denote a point in \(W\) by \(w\) and a local coordinate of \(w\) by \((w^1, ..., w^{d+r})\). By a Poisson analytic family of compact holomorphic Poisson submanifolds of \((W, \Lambda_0)\) we mean a holomorphic Poisson submanifold \(V \subset (W \times M, \Lambda_0)\) of dimension \(r\), where \(M\) is a complex manifold and \(\Lambda_0\) is the holomorphic Poisson structure on \(W \times M\) induced from \((W, \Lambda_0)\), such that

1. for each point \(t \in M, V_t \times t := \omega^{-1}(t) = V \cap \pi^{-1}(t)\) is a connected compact holomorphic Poisson submanifold of \((W \times t, \Lambda_0)\) of dimension \(d\), where \(\omega : V \rightarrow M\) is the map induced from the canonical projection \(\pi : W \times M \rightarrow M\).
(2) for each point \( p \in V \), there exist \( r \) holomorphic functions \( f_\alpha(w, t), \alpha = 1, \ldots, r \) defined on a neighborhood \( U_p \) of \( p \) in \( W \times M \) such that \( \text{rank} \left( \frac{\partial (f_1, \ldots, f_r)}{\partial (w_1, \ldots, w_r)} \right) = r \), and \( U_p \cap V \) is defined by the simultaneous equations \( f_\alpha(w, t) = 0, \alpha = 1, \ldots, r \).

We call \( V \subset (W \times M, \Lambda_0) \) a Poisson analytic family of compact holomorphic Poisson submanifolds \( V_t, t \in M \) of \((W, \Lambda_0)\). We also call \( V \subset (W \times M, \Lambda_0) \) a Poisson analytic family of deformations of a compact holomorphic Poisson submanifold \( V_{t_0} \) of \((W, \Lambda_0)\) for each fixed point \( t_0 \in M \).

### 3.1. The complex associated with the normal bundle of a holomorphic Poisson submanifold in a holomorphic Poisson manifold.

Let \((W, \Lambda_0)\) be a holomorphic Poisson manifold and \( V \) be a holomorphic Poisson submanifold of \((W, \Lambda_0)\).

Let \( \mathcal{U} = \{W_i\} \) be an open covering such that \( W_i \) is a polycylinder with a local coordinate \((w_i, z_i) = (w_i^1, \ldots, w_i^r, z_i^1, \ldots, z_i^d)\) such that \( W_i = \{(w_i, z_i) ||w_i|| < 1, |z_i| < 1\} \), where \( |w_i| = \max_\alpha |w_i^\alpha|, |z_i| = \max_\gamma |z_i^\gamma| \), the local coordinate \((w_i, z_i)\) can be extended to a point containing the closure of \( W_i \) and on each neighborhood \( W_i, V \cap W_i \) coincides with the subspace of \( W_i \) determined by \( w_i^1 = \cdots = w_i^r = 0 \). On the intersection \( W_i \cap W_k, \) the coordinates \( w_i^1, \ldots, w_i^r, z_i^1, \ldots, z_i^d \) are holomorphic functions of \( w_k \) and \( z_k \): \( w_i^\alpha = f_{ik}^\alpha(w_k, z_k), \alpha = 1, \ldots, r, z_i^\gamma = g_{ik}^\gamma(w_k, z_k), \gamma = 1, \ldots, d \). We set \( f_{ik}(w_k, z_k) := (f_{ik}^1(w_k, z_k), \ldots, f_{ik}^r(w_k, z_k)) \) and \( g_{ik}(w_k, z_k) = (g_{ik}^1(w_k, z_k), \ldots, g_{ik}^d(w_k, z_k)) \) so that we write the formula in the form \( w_i = f_{ik}(w_k, z_k), z_i = g_{ik}(w_k, z_k) \).

Then we have \( f_{ik}(0, z_k) = 0 \) and so \( w_i^\alpha = f_{ik}^\alpha(w_k, z_k) \) has the following form:

\[
i^\alpha = f_{ik}^\alpha(w_k, z_k) = \sum_{\beta=1}^r w_k^\beta F_{ik\beta}^\alpha(w_k, z_k)
\]

We set \( U_i = V \cap W_i = \{(0, z)| ||z|| < 1\} \). We denote a point of \( V \) by \( z = (0, z_k) \in U_i \), we consider \( z_i = (z_i^1, \ldots, z_i^d) \) as the coordinate of \( z \) on \( U_i \). We indicate by writing \( z = (0, z_k) \in U_k \cap U_i \) that \( z \) is a point in \( U_k \cap U_i \). Then we denote a point of \( U_k \) by \( z_k \). We note that

\[
F_{ik\beta}^\alpha(0, z_k) = \left. \frac{\partial f_{ik}^\alpha(w_k, z_k)}{\partial w_k^\beta}\right|_{w_k=0}, \quad z = (0, z_k) \in U_i \cap U_k,
\]

and let \( F_{ik}(z) := (F_{ik\beta}^\alpha(0, z))_{\alpha, \beta=1, \ldots, r} \). Then the matrix valued functions \( F_{ik}(z) \) satisfy \( F_{ik}(z) = F_{ij}(z) F_{jk}(z) \) for \( z \in U_i \cap U_j \cap U_k \). Therefore they define the normal bundle \( N_{V/W} \).

On the other hand, since \( V \) is a holomorphic Poisson submanifold of \((W, \Lambda_0)\), \([A_0, w_i^\alpha] \) is of the form

\[
[A_0, w_i^\alpha] = \sum_{\beta=1}^r w_i^\beta T_{ia}^\alpha(w_i, z_i)
\]

where \( T_{ia}^\beta(w_i, z_i) = \sum_{\gamma=1}^d P_{ia,\gamma}^\beta(w_i, z_i) \frac{\partial}{\partial w_i^\gamma} + \cdots + P_{ia,\beta}^1(w_i, z_i) \frac{\partial}{\partial z_i^\gamma} + \cdots + P_{ia,\beta}^d(w_i, z_i) \frac{\partial}{\partial z_i^\gamma} \in \Gamma(W_i, T_W) \) by which we consider \( T_{ia}^\beta(w_i, z_i) \) a vector-valued holomorphic function on \( W_i \). Then on \( W_i \cap W_k \), we have

\[
[A_0, w_i^\alpha] = \sum_{\beta=1}^r w_k^\beta T_{ia}^\alpha(w_k, z_k) = \sum_{\beta=1}^r f_{ik}^\beta(w_k, z_k) T_{ia}^\beta(w_k, z_k)
\]

On the other hand, from (3.1.1) and (3.1.3),

\[
[A_0, w_i^\alpha] = \sum_{\beta=1}^r w_k^\beta [A_0, F_{ik\beta}^\alpha(w_k, z_k)] + \sum_{\beta=1}^r F_{ik\beta}^\alpha(w_k, z_k) [A_0, w_k^\beta]
\]

\[
= \sum_{\beta=1}^r w_k^\beta [A_0, F_{ik\beta}^\alpha(w_k, z_k)] + \sum_{\beta, \gamma=1}^r F_{ik\beta}^\alpha(w_k, z_k) w_k^\gamma T_{ik\beta}^\gamma(w_k, z_k)
\]

By taking the derivative of (3.1.4) and (3.1.5) with respect to \( w_k^\gamma \) and setting \( w_k = 0 \), we get from (3.1.2), on \( \Gamma(U_i \cap U_k, T_W|V) \),

\[
\sum_{\beta=1}^r F_{ik\gamma}^\alpha(0, z_k) T_{ia}^\beta(0, z_i) = [A_0, F_{ik\gamma}^\alpha(0, z_k) |w_k=0] + \sum_{\beta=1}^r F_{ik\beta}^\alpha(0, z_k) T_{ik\beta}^\gamma(0, z_k)
\]
On the other hand, from (3.1.3), we have

\begin{align*}
0 &= [\Lambda_0, [\Lambda_0, w_i^\beta]] = \sum_{\beta=1}^r [\Lambda_0, w_i^\beta T_{i\alpha}^\beta(w_i, z_i)] = \sum_{\beta=1}^r w_i^\beta [\Lambda_0, T_{i\alpha}^\beta(w_i, z_i)] - [\Lambda_0, w_i^\beta] \cdot T_{i\alpha}^\beta(w_i, z_i) \\
&= \sum_{\beta=1}^r w_i^\beta [\Lambda_0, T_{i\alpha}^\beta(w_i, z_i)] - \sum_{\beta=1}^r w_i^\gamma T_{i\alpha}^\gamma(w_i, z_i) \cdot T_{i\alpha}^\beta(w_i, z_i) \\
(3.1.7) &= \sum_{\beta=1}^r w_i^\beta \left( [\Lambda_0, T_{i\alpha}^\beta(w_i, z_i)] - \sum_{\gamma=1}^r T_{i\alpha}^\gamma(w_i, z_i) \cdot T_{i\alpha}^\beta(w_i, z_i) \right)
\end{align*}

By taking the derivative of (3.1.7) with respect to $w_i^\beta$ and setting $w_i = 0$, we get, on $\Gamma(U_i, T_W|_\mathcal{V})$,

\begin{align*}
(3.1.8) \quad [\Lambda_0, T_{i\alpha}^\beta(0, z_i)]|_{w_i=0} - \sum_{\gamma=1}^r T_{i\alpha}^\gamma(0, z_i) \cdot T_{i\alpha}^\beta(0, z_i) = 0
\end{align*}

Now we define a complex of sheaves associated with the normal bundle $\mathcal{N}_{V/W}$:

\begin{align*}
\mathcal{N}_{V/W} \quad \rightarrow \mathcal{N}_{V/W} \otimes T_W|_\mathcal{V} \quad \rightarrow \mathcal{N}_{V/W} \otimes \wedge^2 T_W|_\mathcal{V} \quad \rightarrow \mathcal{N}_{V/W} \otimes \wedge^3 T_W|_\mathcal{V} \quad \rightarrow \cdots
\end{align*}

First we define $\nabla: \mathcal{N}_{V/W} \rightarrow \mathcal{N}_{V/W} \otimes T_W|_\mathcal{V}$ and then extend to $\nabla: \mathcal{N}_{V/W} \otimes \wedge^p T_W|_\mathcal{V} \rightarrow \mathcal{N}_{V/W} \otimes \wedge^{p+1} T_W|_\mathcal{V}$ in the following. We note that $\Gamma(U_i, \mathcal{N}_{V/W}) \cong \oplus \Gamma(U_i, \mathcal{O}_\mathcal{V})$ and $\Gamma(U_i, \mathcal{N}_{V/W} \otimes T_W|_\mathcal{V}) \cong \oplus \Gamma(U_i, T_W|_\mathcal{V})$. Using these isomorphism, we define $\nabla$ on $\mathcal{N}_{V/W}$ by the rule

\begin{align*}
\nabla(e_\alpha^i) := \sum_{\beta=1}^r T_{i\beta}^\alpha(0, z_i) e_\beta^i
\end{align*}

where $e_\alpha^i = (0, \ldots, 1_{\alpha-th}, \ldots, 0) \in \oplus_{\alpha=1}^r \Gamma(U_i, \mathcal{O}_\mathcal{V})$. In general, we define

\begin{align*}
\nabla: \oplus \Gamma(U_i, \mathcal{O}_\mathcal{V}) \rightarrow \oplus \Gamma(U_i, T_W|_\mathcal{V})
\end{align*}

\begin{align*}
\sum_{\alpha=1}^r g_{i\alpha} e_\alpha^i \rightarrow \sum_{\alpha=1}^r -[g_{i\alpha}, \Lambda_0]|_{w_i=0} \cdot e_\alpha^i + \sum_{\alpha=1}^r g_{i\alpha} \nabla(e_\alpha^i) = \sum_{\alpha=1}^r \left( -[g_{i\alpha}, \Lambda_0]|_{w_i=0} + \sum_{\beta=1}^r g_{i\alpha} T_{i\beta}^\alpha(0, z_i) \right) e_\alpha^i
\end{align*}

where $g_{i\alpha}^\alpha \in \Gamma(U_i, \mathcal{O}_\mathcal{V})$.

We extend $\nabla$ on $\mathcal{N}_{V/W} \otimes \wedge^p T_W|_\mathcal{V}$. $\mathcal{N}_{V/W} \otimes \wedge^p T_W|_\mathcal{V} \rightarrow \mathcal{N}_{V/W} \otimes \wedge^{p+1} T_W|_\mathcal{V}$ is locally defined in the following way: we note that $\Gamma(U_i, \mathcal{N}_{V/W} \otimes \wedge^p T_W|_\mathcal{V}) \cong \oplus \Gamma(U_i, \wedge^p T_W|_\mathcal{V})$ and $\Gamma(U_i, \mathcal{N}_{V/W} \otimes \wedge^{p+1} T_W|_\mathcal{V}) \cong \oplus \Gamma(U_i, \wedge^{p+1} T_W|_\mathcal{V})$. From these isomorphism, we define $\nabla$ by the rule

\begin{align*}
\nabla: \oplus \Gamma(U_i, \wedge^p T_W|_\mathcal{V}) \rightarrow \oplus \Gamma(U_i, \wedge^{p+1} T_W|_\mathcal{V})
\end{align*}

\begin{align*}
\sum_{\alpha=1}^r g_{i\alpha} e_\alpha^i \rightarrow \sum_{\alpha=1}^r -[g_{i\alpha}, \Lambda_0]|_{w_i=0} \cdot e_\alpha^i + (-1)^p \sum_{\alpha=1}^r g_{i\alpha} \wedge \nabla(e_\alpha^i) \\
= \sum_{\alpha=1}^r \left( -[g_{i\alpha}, \Lambda_0]|_{w_i=0} + (-1)^p \sum_{\beta=1}^r g_{i\alpha} T_{i\beta}^\alpha(0, z_i) \right) e_\alpha^i
\end{align*}

where $g_{i\alpha}^\alpha \in \Gamma(U_i, \wedge^p T_W|_\mathcal{V})$. 
First we show that $\nabla$ defines a complex, i.e. $\nabla \circ \nabla = 0$. Indeed,

$$\nabla \circ \nabla \left( \sum_{\alpha=1}^{r} g_{\alpha}^{\alpha} e_{\alpha}^{\alpha} \right) = \nabla \left( \sum_{\alpha=1}^{r} \left( -[g_{\alpha}^{\alpha}, A_{0}]_{w_{\alpha}=0} + (-1)^{p} \sum_{\beta=1}^{r} g_{\alpha}^{\beta} \wedge T_{\alpha}^{\beta}(0, z_{\alpha}) \right) e_{\alpha}^{\alpha} \right)$$

$$= -(-1)^{p} \sum_{\beta=1}^{r} g_{\alpha}^{\beta} \wedge T_{\alpha}^{\beta}(0, z_{\alpha}) W/V_{\alpha} e_{\alpha}^{\alpha} + (-1)^{p+1} \sum_{\alpha=1}^{r} \left( -[g_{\alpha}^{\alpha}, A_{0}]_{w_{\alpha}=0} + (-1)^{p} \sum_{\beta=1}^{r} g_{\alpha}^{\beta} \wedge T_{\alpha}^{\beta}(0, z_{\alpha}) \right) \nabla e_{\alpha}^{\alpha}$$

$$= (-1)^{p+1} \sum_{\alpha, \beta=1}^{r} [g_{\alpha}^{\beta} \wedge T_{\alpha}^{\beta}(0, z_{\alpha}), A_{0}]_{w_{\alpha}=0} \wedge T_{\alpha}^{\beta}(0, z_{\alpha}) e_{\alpha}^{\alpha} - \sum_{\alpha, \beta, \gamma=1}^{r} g_{\alpha}^{\beta} \wedge T_{\gamma}^{\beta}(0, z_{\gamma}) \wedge T_{\gamma}^{\alpha}(0, z_{\gamma}) e_{\alpha}^{\alpha}$$

Hence in order to show $\nabla \circ \nabla = 0$, we have to show that

$$(3.1.9)$$

$$(-1)^{p+1} \sum_{\beta=1}^{r} [g_{\alpha}^{\beta} \wedge T_{\alpha}^{\beta}(0, z_{\alpha}), A_{0}]_{w_{\alpha}=0} \wedge T_{\alpha}^{\beta}(0, z_{\alpha}) - \sum_{\beta, \gamma=1}^{r} g_{\alpha}^{\beta} \wedge T_{\gamma}^{\beta}(0, z_{\gamma}) \wedge T_{\gamma}^{\alpha}(0, z_{\gamma}) = 0$$

Indeed, from \textbf{3.1.3}, \textbf{3.1.4} becomes

$$(-1)^{p+1} \sum_{\beta=1}^{r} [g_{\alpha}^{\beta} \wedge T_{\alpha}^{\beta}(0, z_{\alpha}), A_{0}]_{w_{\alpha}=0} \wedge T_{\alpha}^{\beta}(0, z_{\alpha}) + (-1)^{p+1} \sum_{\beta=1}^{r} g_{\alpha}^{\beta} \wedge [T_{\alpha}^{\beta}(0, z_{\alpha}), A_{0}]_{w_{\alpha}=0}$$

$$+ (-1)^{p} \sum_{\beta=1}^{r} [g_{\alpha}^{\beta} \wedge T_{\alpha}^{\beta}(0, z_{\alpha}), A_{0}]_{w_{\alpha}=0} \wedge T_{\alpha}^{\beta}(0, z_{\alpha}) - \sum_{\beta, \gamma=1}^{r} g_{\alpha}^{\beta} \wedge T_{\gamma}^{\beta}(0, z_{\gamma}) \wedge T_{\gamma}^{\alpha}(0, z_{\gamma})$$

$$= \sum_{\beta=1}^{r} g_{\alpha}^{\beta} \wedge [A_{0}, T_{\alpha}^{\beta}(0, z_{\alpha})]_{w_{\alpha}=0} - \sum_{\beta, \gamma=1}^{r} g_{\alpha}^{\beta} \wedge T_{\gamma}^{\beta}(0, z_{\gamma}) \wedge T_{\gamma}^{\alpha}(0, z_{\gamma})$$

$$= \sum_{\beta=1}^{r} g_{\alpha}^{\beta} \left( [A_{0}, T_{\alpha}^{\beta}(0, z_{\alpha})]_{w_{\alpha}=0} - \sum_{\gamma=1}^{T_{\alpha}^{\gamma}(0, z_{\gamma}) \wedge T_{\alpha}^{\gamma}(0, z_{\gamma}) \wedge T_{\gamma}^{\alpha}(0, z_{\gamma}) \right) = 0$$

Hence $\nabla \circ \nabla = 0$.

Next we show $\nabla$ is well-defined. In other words, on $U_{i} \cap U_{k}$, the following diagram commutes

$$(3.1.10)$$

$$\Gamma(U_{k}, N_{W/V} \otimes \wedge^{p} T_{W/V}) \cong \oplus \Gamma(U_{k}, \wedge^{p} T_{W/V}) \hspace{1cm} \text{on } U_{i} \cap U_{k} \cong \Gamma(U_{i}, N_{W/V} \otimes \wedge^{p} T_{W/V}) \cong \oplus \Gamma(U_{i}, \wedge^{p} T_{W/V})$$

$$\nabla \downarrow \hspace{1cm} \nabla \downarrow$$

$$\Gamma(U_{k}, N_{W/V} \otimes \wedge^{p+1} T_{W/V}) \cong \oplus \Gamma(U_{k}, \wedge^{p+1} T_{W/V}) \hspace{1cm} \text{on } U_{i} \cap U_{k} \cong \Gamma(U_{i}, N_{W/V} \otimes \wedge^{p+1} T_{W/V}) \cong \oplus \Gamma(U_{i}, \wedge^{p+1} T_{W/V})$$

Let $\sum_{\alpha=1}^{r} g_{\alpha}^{\alpha} e_{\alpha}^{\alpha} \in \oplus \Gamma(U_{k}, \wedge^{p} T_{W/V})$. Then $\nabla(\sum_{\alpha=1}^{r} g_{\alpha}^{\alpha} e_{\alpha}^{\alpha}) = \sum_{\alpha=1}^{r}(-[g_{\alpha}^{\alpha}, A_{0}]_{w_{\alpha}=0} + (-1)^{p} \sum_{\beta=1}^{r} g_{\alpha}^{\beta} \wedge T_{\alpha}^{\beta}(0, z_{\alpha})) e_{\alpha}^{\alpha}$ is identified on $U_{i} \cap U_{k}$ with

$$(3.1.11)$$

$$\sum_{\gamma=1}^{r} \left( \sum_{\alpha=1}^{r} -F_{\gamma \alpha}(0, z_{\alpha}) [g_{\alpha}^{\alpha}, A_{0}]_{w_{\alpha}=0} + (-1)^{p} \sum_{\alpha, \beta=1}^{r} F_{\gamma \alpha}(0, z_{\alpha}) g_{\alpha}^{\beta} \wedge T_{\alpha}^{\beta}(0, z_{\alpha}) \right) e_{\gamma}^{\alpha}$$

On the other hand, $\sum_{\alpha=1}^{r} g_{\alpha}^{\alpha} e_{\alpha}^{\alpha}$ is identified on $U_{i} \cap U_{k}$ with $\sum_{\alpha, \gamma=1}^{r} (F_{\gamma \alpha}(0, z_{\alpha}) g_{\alpha}^{\alpha} e_{\gamma}^{\alpha}) \in \oplus \Gamma(U_{i}, \wedge^{p} T_{W/V})$ and

$$(3.1.12)$$

$$\nabla(\sum_{\alpha, \gamma=1}^{r} (F_{\gamma \alpha}(0, z_{\alpha}) g_{\alpha}^{\alpha} e_{\gamma}^{\alpha}) = \sum_{\gamma=1}^{r} \left( \sum_{\alpha=1}^{r} -[F_{\gamma \alpha}(0, z_{\alpha}) g_{\alpha}^{\alpha}, A_{0}]_{w_{\alpha}=0} + (-1)^{p} \sum_{\alpha, \beta=1}^{r} F_{\gamma \alpha}(0, z_{\alpha}) g_{\alpha}^{\beta} \wedge T_{\gamma}^{\beta}(0, z_{\gamma}) \right) e_{\gamma}^{\alpha}$$
Hence in order for the diagram \(3.1.10\) to commute, we have to show that \(3.1.11\) coincides with \(3.1.12\):

\[
(-1)^p \sum_{\alpha, \beta = 1}^r F^\gamma_{ik\alpha}(0, z_k) g^\beta_k \wedge T^\beta_{\alpha}(0, z_k) = \sum_{\alpha = 1}^r -[F^\gamma_{ik\alpha}(0, z_k), A_0]|_{w_k = 0} \wedge g^\alpha_k + (-1)^p \sum_{\alpha, \beta = 1}^r F^\beta_{ik\alpha}(0, z_k) g^\alpha_k \wedge T^\gamma_{\beta}(0, z_l) = 0
\]

\[
\Longleftrightarrow \sum_{\alpha, \beta = 1}^r F^\gamma_{ik\beta}(0, z_k) T^\alpha_{\beta}(0, z_k) \wedge g^\alpha_k = \sum_{\alpha = 1}^r -[F^\gamma_{ik\alpha}(0, z_k), A_0]|_{w_k = 0} \wedge g^\alpha_k + \sum_{\alpha, \beta = 1}^r F^\beta_{ik\alpha}(0, z_k) T^\gamma_{\beta}(0, z_l) \wedge g^\alpha_k = 0
\]

which follows from \(3.1.9\). Therefore \(\nabla\) is well-defined.

**Definition 3.1.13.** We call the complex defined as above

\[
\mathcal{N}_{V/W}^\bullet := \mathcal{N}_{V/W} \otimes \mathcal{N}_{V/W} \otimes T_W | V \mathcal{N}_{V/W} \otimes \wedge^2 T_W | V \mathcal{N}_{V/W} \otimes \wedge^3 T_W | V \mathcal{N}_{V/W} \otimes \cdots
\]

the complex associated with the normal bundle \(\mathcal{N}_{V/W}\) of a holomorphic Poisson submanifold \(V\) of a holomorphic Poisson manifold \(W\) and denote its i-th hypercohomology group by \(\mathcal{H}^i(V, \mathcal{N}_{V/W})\).

**Example 1.** On \(W = \mathbb{C}^3\), let \((z, w_1, w_2)\) be a coordinate and \(A_0 = w_1 z \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial t}\) be a holomorphic Poisson structure on \(\mathbb{C}^3\). Then \(w_1 = w_2 = 0\) is a holomorphic Poisson submanifold \(V \subset W\) with coordinate \(z\) so that the normal bundle is \(\mathcal{N}_{V/W} \cong \mathcal{O} \oplus \mathcal{O}\). Then \([A_0, w_1] = w_1 z \frac{\partial}{\partial w_1}\) and \([A_0, w_2] = w_1(-z \frac{\partial}{\partial w_2})\) so that we get \(T^1_1(z) = z \frac{\partial}{\partial w_1}, T^1_2(z) = 0\) and \(T^2_2(z) = -z \frac{\partial}{\partial w_2}\); \(T^1_2(z) = 0\). Since \([A_0, f(z)]|_{w_1 = w_2 = 0} = 0\) for entire functions \(f_i(z), i = 1, 2\), we have \(\nabla(f_1 e_1 + f_2 e_2) = (f_1 T^1_1(z) + f_2 T^2_2(z)) e_1 + (f_1 T^1_2(z) + f_2 T^2_2(z)) e_2 = (f_1 z \frac{\partial}{\partial w_1}, -f_2 z \frac{\partial}{\partial w_2})\) so that \(\mathcal{H}^0(V, \mathcal{N}_{V/W}) = \{(0, f_2(z))|_{f_2(z)}\) is an entire function\}.

### 3.2. Infinitesimal deformations.

Let \(M_1 = \{t = (t_1, \ldots, t_t) \in \mathbb{C}^t| |t| < 1\}\). Consider a Poisson analytic family \(V \subset (W \times M_1, A_0)\) of compact holomorphic Poisson submanifolds \(V_t \subset M_1\) of \((W, A_0)\) and let \(V = V_0\) as in Definition 3.0.1. We keep the notations in subsection 3.1. Let \(\epsilon\) be a sufficiently small positive number. Then for \(|t| < \epsilon\), the holomorphic Poisson submanifold \(V_t\) of \(W\) is defined in each neighborhood \(W_t\) by simultaneous equations of the form \(w^\lambda_{t} = \phi^\lambda_{t}(z_i, t), \lambda = 1, \ldots, r\) where the \(\phi^\lambda_{t}(z_i, t)\) are holomorphic functions of \(z_i, |z_i| < 1\), depending holomorphically on \(t, |t| < \epsilon\), and satisfying the boundary conditions \(\phi^\lambda_{t}(z_i, 0) = \lambda = 1, \ldots, r\). By setting \(\phi^\lambda_{t}(z_i, t) = (\phi^\lambda_{t}(z_i, t), \ldots, \phi^\lambda_{t}(z_i, t))\), we write the simultaneous equation as \(w_{t} = \phi^\lambda_{t}(z_i, t)\). Then we have \(\phi^\lambda_{t}(\varphi_{ik}(z_i, t), z_k, t) = f_{ik}(\varphi_{ik}(z_i, t), z_k)\). For each \(t, |t| < \epsilon\), we set \(w^\lambda_{t} = w^\lambda_{t} - \phi^\lambda_{t}(z_i, t)\), \(\lambda = 1, \ldots, r\) so that \((w^1_{t}, \ldots, w^r_{t}, z^1, \ldots, z^d)\) form a local coordinate defined on \(V_t\). We define \(F_{ik}(z_i) := \left(\frac{\partial \phi^\lambda_{t}(z_i, t)}{\partial z_{ik}}\right)_{\lambda, \mu = 1, \ldots, r}\) for \(z_i \in V_t \cap W_t \cap W_k\), where we denote by \(\frac{\partial \phi^\lambda_{t}(z_i, t)}{\partial z_{ik}}\) the value of the partial derivatives at a point \(z_i\) on \(V_t\). Then the collection \(\{F_{ik}(z_i)\}\) of \(F_{ik}(z_i)\) forms a system of transition matrices defining the normal bundle \(F_t\) of \(V_t\) in \(W_t\). Note that \(F_{ik}(z) = F_{ik}(z)\) for \(z \in U_{t} \cap U_{t'}\) from \(3.1.2\).

Take an arbitrary tangent vector \(\frac{\partial}{\partial t}\) of \(M_1\) at \(t, |t| < \epsilon\), and let \(\psi_{t}(z_i, t) = \frac{\partial \phi^\lambda_{t}(z_i, t)}{\partial t}\) for \(z_i = (\varphi_{t}(z_i, t), z_i)\). Then we obtain the equality

\[
(3.2.1) \quad \psi_{t}(z_i, t) = F_{ik}(z_i) \cdot \psi_{k}(z_i, t), \quad \text{for } z_i \in V_t \cap W_t \cap W_k.
\]

On the other hand, \(w_i - \varphi_{t}(z_i, t) = 0\) define a holomorphic Poisson submanifold, we have

\[
(3.2.2) \quad [A_0, w^\lambda_{t} - \varphi^\lambda_{t}(z_i, t)] = \sum_{\mu = 1}^r (w^\mu_{t} - \varphi^\mu_{t}(z_i, t)) T^\mu_{\lambda}(w, z_i, t)
\]

for some \(T^\mu_{\lambda}(w, z_i, t)\) which is of the form

\[
T^\mu_{\lambda}(w, z_i, t) = P^\mu_{11}(w, z_i, t) \frac{\partial}{\partial w_{1}} + \cdots + P^\mu_{ir}(w, z_i, t) \frac{\partial}{\partial w_{r}} + Q^\mu_{11}(w, z_i, t) \frac{\partial}{\partial z_{1}} + \cdots + Q^\mu_{rd}(w, z_i, t) \frac{\partial}{\partial z_{d}}
\]

by which we consider \(T^\mu_{\lambda}(w, z_i, t)\) as a vector valued holomorphic function of \((w, z_i, t)\).

By taking the derivative of \(3.2.2\) with respect to \(t\), we get

\[
[A_{0}, - \frac{\partial \varphi^\lambda_{t}(z_i, t)}{\partial t}] = \sum_{\mu = 1}^r - \frac{\partial \varphi^\mu_{t}(z_i, t)}{\partial t} T^\mu_{\lambda}(w, z_i, t) + \sum_{\mu = 1}^r (w^\lambda_{t} - \varphi^\mu_{t}(z_i, t)) \frac{\partial \varphi^\mu_{t}(z_i, t)}{\partial t}.
\]

By restricting to \(V_t\), equivalently setting \(w_i - \varphi_{t}(z_i, t) = 0\), we get on
Let \( \sigma_t \) be a small positive number. In order to prove Theorem 3.3.1, it suffices to construct vector-valued holomorphic functions

\[
\varphi_i(z, t) = (\varphi_1^i(z, t), \ldots, \varphi_r^i(z, t))
\]

in \( z_i \), and \( t \) with \( |z_i| < 1, |t| < \epsilon \), with \( |\varphi_i(z_i, t)| < 1 \) satisfying the boundary condition

\[
\varphi_i(z_i, 0) = 0, \quad \frac{\partial \varphi_i(z_i, t)}{\partial t} |_{t=0} = \gamma_{ri}(z)
\]

such that

\[
\varphi_i(g_i(z_k, t), z_k, t) = f_i(z_k, t), \quad (\varphi_i(z_k, t), z_k) \in W_k \cap W_i,
\]

\[
[\Lambda_0, w_i^\alpha - \varphi_i(z_i, t)]|_{w_i=\varphi_i(z_i, t)} = 0, \quad \alpha = 1, \ldots, r
\]
Recall Notation 1. Then the equalities (3.3.4) and (3.3.5) are equivalent to the system of congruences
\[(3.3.6)\quad \varphi_i^m(g_k(\varphi_k^m(z_k, t), z_k, t)) \equiv_m f_k(\varphi_k^m(z_k, t), z_k), \quad m = 1, 2, 3, \ldots\]
\[(3.3.7)\quad [A_0, w_\alpha^m - \varphi^m_\alpha(z_i, t)]|_{w_\alpha=\varphi^m_\alpha(z_i, t)} \equiv_0 0, \quad m = 1, 2, 3, \ldots, \quad \alpha = 1, \ldots, r.

We will construct the formal power series \(\varphi_i^m(z_i, t)\) satisfying (3.3.6) and (3.3.7) by induction on \(m\).

We define \(\varphi_i^m(z_i, t) = \sum_{\beta=1}^{r} \gamma_{\beta}^m(z_i), \alpha = 1, \ldots, r\). Then from (3.3.2), \(\varphi_i^m(z_i, t)\) satisfies (3.3.6) and (3.3.7). On the other hand, since \([A_0, \gamma_{\beta}^m(z_i)]|_{w_\beta=\gamma_{\beta}^m(z_i)} = 0\), \(\alpha = 1, \ldots, r\) from (3.3.3), we get
\[[A_0, \varphi_i^m(z_i, t)]|_{w_\alpha=\varphi_i^m(z_i, t)} = [A_0, \sum_{\rho=1}^{l} t_{\rho}^{m} \gamma_{\rho}^m(z_i)]|_{w_{\rho}=\gamma_{\rho}^m(z_i)} = \sum_{\beta=1}^{r} \sum_{\rho=1}^{l} t_{\rho}^{m} \gamma_{\beta}^m(z_i) T_{\beta}^{m}(z_i, 0) = \sum_{\beta=1}^{r} \varphi_i^m(z_i, t) T_{\beta}^{m}(z_i, 0)\]

Then we have \([A_0, \varphi_i^m(z_i, t)] = \sum_{\beta=1}^{r} \varphi_i^m(z_i, t) T_{\beta}^{m}(w_i, z_i) - \sum_{\beta=1}^{r} w_i^{m} P_{\beta}^{m}(w_i, z_i, t)\) for some \(P_{\beta}^{m}(w_i, z_i, t)\) which are homogenous polynomials in \(t_1, \ldots, t_l\) of degree 1 with coefficients in \(\Gamma(W, T_W)\). Hence from (3.1.3), we have
\[[A_0, w_\alpha^m - \varphi_i^m(z_i, t)] = \sum_{\beta=1}^{r} (w_i^{m} - \varphi_i^{m}(z_i, t)) T_{\beta}^{m}(w_i, z_i) + \sum_{\beta=1}^{r} w_i^{m} P_{\beta}^{m}(w_i, z_i, t)\]
so that we obtain \([A_0, w_\alpha^m - \varphi_i^m(z_i, t)]|_{w_{\beta}=\varphi_i^m(z_i, t)} \equiv 0\) and so \(\varphi_i^m(z_i, t)\) satisfies (3.3.7). Hence the induction holds for \(m = 1\).

Now we assume that we have already constructed \(\varphi_i^m(z_i, t) = (\varphi_i^1(z_i, t), \ldots, \varphi_i^m(z_i, t), \ldots, \varphi_i^m(z_i, t))\) satisfying (3.3.6) and (3.3.7) such that \([A_0, w_\alpha^m - \varphi_i^m(z_i, t)]\) is of the form
\[(3.3.8)\quad [A_0, w_\alpha^m - \varphi_i^m(z_i, t)] = \sum_{\beta=1}^{r} (w_i^{m} - \varphi_i^{m}(z_i, t)) T_{\beta}^{m}(w_i, z_i) + q_i^{m}(z_i, t) + \sum_{\beta=1}^{r} w_i^{m} P_{\beta}^{m}(w_i, z_i, t)\]
such that the degree of \(P_{\beta}^{m}(w_i, z_i, t)\) is at least 1 in \(t_1, \ldots, t_l\). We note that we can rewrite (3.3.8) in the following way.
\[
\sum_{\beta=1}^{r} (w_i^{m} - \varphi_i^{m}(z_i, t)) T_{\beta}^{m}(w_i, z_i) + Q_i^{m}(z_i, t) + \sum_{\beta=1}^{r} q_i^{m}(z_i, t) P_{\beta}^{m}(\varphi_i^m(z_i, t), z_i, t) + \sum_{\beta=1}^{r} (w_i^{m} - \varphi_i^{m}(z_i, t)) L_{\beta}^{m}(w_i, z_i, t)
\]
such that the degree of \(L_{\beta}^{m}(w_i, z_i, t)\) in \(t_1, \ldots, t_l\) is at least 1 so that (3.3.8) becomes the following form:
\[(3.3.9)\quad [A_0, w_\alpha^m - \varphi_i^m(z_i, t)] = \sum_{\beta=1}^{r} (w_i^{m} - \varphi_i^{m}(z_i, t)) T_{\beta}^{m}(w_i, z_i) + K_i^{m}(z_i, t) + \sum_{\beta=1}^{r} (w_i^{m} - \varphi_i^{m}(z_i, t)) L_{\beta}^{m}(w_i, z_i, t)\]
where \(K_i^{m}(z_i, t) := Q_i^{m}(z_i, t) + \sum_{\beta=1}^{r} q_i^{m}(z_i, t) P_{\beta}^{m}(\varphi_i^{m}(z_i, t), z_i, t)\).

We set
\[(3.3.10)\quad \psi_k(g_k, z_k, t) := [\varphi_k^{m}(g_k(\varphi_k^{m}(z_k, t), z_k, t)) - f_k(g_k(\varphi_k^{m}(z_k, t), z_k))]_{m+1}
\]
\[(3.3.11)\quad G_i^{a}(z_i, t) := [[A_0, w_\alpha^m - \varphi_i^m(z_i, t)]|_{w_\alpha=\varphi_i^m(z_i, t)}]_{m+1}, \quad \alpha = 1, \ldots, r.

We claim that \(\{\psi_k(g_k(z_k, t), \ldots, \psi_r(g_r(z_r, t))\} \oplus \{-G_i^{a}(z_i, t), \ldots, -G_i^{a}(z_i, t)\}\) defines a 1-cocycle in the following Čech resolution of \(N_i^{\bullet}/V\):
\[
C^0(\Omega \cap V, N_{W/V} \otimes \Lambda^2 T_W|V) \xrightarrow{\delta} C^1(\Omega \cap V, N_{W/V} \otimes T_W|V)
\]
\[
C^0(\Omega \cap V, N_{W/V}) \xrightarrow{\delta} C^1(\Omega \cap V, N_{W/V}) \xrightarrow{\delta} C^2(\Omega \cap V, N_{W/V})
\]
By defining $\psi_{ik}(z, t) = \psi_{ik}(z_k, t)$ for $(0, z_k) \in U_k \cap U_i$, we have the equality (see Kod62 p.152-153)

\[(3.3.12) \quad \psi_{ik}(z, t) = \psi_{ij}(z, t) + F_{ij} \cdot \psi_{jk}(z, t), \quad \text{for } z \in U_i \cap U_j \cup U_k\]

On the other hand, by applying $[\Lambda_0, -]$ on (3.3.9), we have

\[(3.3.13) \quad 0 = [\Lambda_0, [\Lambda_0, w^\alpha_i - \varphi_i^m]] = \sum_{\beta=1}^r -[\Lambda_0, w^\beta_i - \varphi_i^m(z_i, t)] \cap T_{i\alpha}^\beta(w_i, z_i) + \sum_{\beta=1}^r (w^\beta_i - \varphi_i^m(z_i, t))[\Lambda_0, T_{i\alpha}^\beta(w_i, z_i)] \]

\[+ [\Lambda_0, K_i^m(z_i, t)] + \sum_{\beta=1}^r -[\Lambda_0, w^\beta_i - \varphi_i^m(z_i, t)] \cap L_{i\alpha}^\beta(w_i, z_i, t) + \sum_{\beta=1}^r (w^\beta_i - \varphi_i^m(z_i, t))[\Lambda_0, L_{i\alpha}^\beta(w_i, z_i, t)]\]

By restricting (3.3.13) to $w_i = \varphi_i^m(z_i, t)$, since $G_0^\alpha(z_i, t) \equiv 0, \alpha = 1, ..., r$, we get

\[0 \equiv_{m+1} \sum_{\beta=1}^r -G_i^\beta(z_i, t) \cap T_{i\alpha}^\beta(0, z_i) + [\Lambda_0, K_i^m(z_i, t)] |_{w_i=\varphi_i^m(z_i, t)} + \sum_{\beta=1}^r -G_i^\beta(z_i, t) \cap L_{i\alpha}^\beta(\varphi_i^m(z_i, t), z_i, t)\]

Since the degree $L(w_i, z_i, t)$ is at least 1 in $t_1, ..., t_l$ and we have, from (3.3.9),

\[(3.3.14) \quad G_i^\alpha(z_i, t) \equiv_{m+1} K_i^m(z_i, t) = Q_k^\alpha(z_i, t) + \sum_{\beta=1}^r \phi_i^m(z_i, t)P_{i\alpha}^\beta(\varphi_i^m(z_i, t), z_i, t),\]

we obtain

\[(3.3.15) \quad 0 \equiv_{m+1} \sum_{\beta=1}^r -G_i^\beta(z_i, t) \cap T_{i\alpha}^\beta(0, z_i) + [\Lambda_0, G_i^\alpha(z_i, t)]|_{w_i=0}\]

Next, since $f_{ik}^m(w_k, z_k) - \varphi_i^m(g_k(w_k, z_k), t) - [f_{ik}^m(\varphi_i^m(z_k, t), z_k)] - \varphi_i^m(g_k[\varphi_i^m(z_k, t), z_k], t)] = \sum_{\beta=1}^r (w^\beta_k - \varphi_k^m(z_k, t)) \cdot S_{k\alpha}^\beta(w_k, z_k, t)$ for some $S_{k\alpha}^\beta(w_k, z_k, t)$. By setting $t = 0$, we get $f_{ik}^m(w_k, z_k) - f_{ik}^m(0, z_k) = \sum_{\beta=1}^r w_k^\beta \cdot S_{k\alpha}^\beta(w_k, z_k, 0)$, and then by taking the derivative with respect to $w_k$, and setting $w_k = 0$, we obtain

\[(3.3.16) \quad \frac{\partial f_{ik}^m(w_k, z_k)}{\partial w_k}|_{w_k=0} = S_{k\alpha}^\beta(0, z_k, 0).\]

Then we have

\[\begin{align*}
[\Lambda_0, f_{ik}^m(w_k, z_k) - \varphi_i^m(g_k(w_k, z_k), t)]|_{w_k=\varphi_i^m(z_k, t)} + [\Lambda_0, \psi_{ik}^\alpha(z_k, t)]|_{w_k=0} \\
\equiv_{m+1} [\Lambda_0, f_{ik}^m(w_k, z_k) - \varphi_i^m(g_k(w_k, z_k), t)]|_{w_k=\varphi_i^m(z_k, t)} + [\Lambda_0, \psi_{ik}^\alpha(z_k, t)]|_{w_k=0} \\
\equiv_{m+1} [\Lambda_0, f_{ik}^m(w_k, z_k) - \varphi_i^m(g_k(w_k, z_k), t)]|_{w_k=\varphi_i^m(z_k, t)} + [\Lambda_0, \psi_{ik}^\alpha(z_k, t)]|_{w_k=0} \\
\equiv_{m+1} [\Lambda_0, \sum_{\beta=1}^r (w_k^\beta - \varphi_k^m(z_k, t))S_{k\alpha}^\beta(w_k, z_k, t)]|_{w_k=\varphi_i^m(z_k, t)} \\
\equiv_{m+1} \sum_{\beta=1}^r [\Lambda_0, w_k^\beta - \varphi_k^m(z_k, t)]|_{w_k=\varphi_i^m(z_k, t)} \cdot S_{k\alpha}^\beta(\varphi_i^m(z_k, t), z_k, t) \\
\equiv_{m+1} \sum_{\beta=1}^r G_k^\beta(z_k, t) \cdot S_{k\alpha}^\beta(0, z_k, 0) \equiv_{m+1} \sum_{\beta=1}^r G_k^\beta(z_k, t) \cdot \frac{\partial f_{ik}^m(w_k, z_k)}{\partial w_k}|_{w_k=0}
\end{align*}\]

Hence we obtain the equality

\[(3.3.17) \quad [\Lambda_0, f_{ik}^m(w_k, z_k) - \varphi_i^m(g_k(w_k, z_k), t)]|_{w_k=\varphi_i^m(z_k, t)} + [\Lambda_0, \psi_{ik}^\alpha(z_k, t)]|_{w_k=0} \equiv_{m+1} \sum_{\beta=1}^r G_k^\beta(z_k, t) \cdot S_{k\alpha}^\beta(0, z_k, 0) = \sum_{\beta=1}^r G_k^\beta(z_k, t) \cdot \frac{\partial f_{ik}^m(w_k, z_k)}{\partial w_k}|_{w_k=0}\]
On the other hand, from (3.3.9), we have

\[(3.3.18)\quad \Lambda_0, f_{ik}^\beta (w_k, z_k) - \varphi_i^{am}(g_{ik}(w_k, z_k), t) = \sum_{\beta=1}^r (f_{ik}^\beta (w_k, z_k) - \varphi_i^{am}(g_{ik}(w_k, z_k), t)) T_{\alpha a}^\beta (w_i, z_i) \]

\[+ K_i^{am}(z_i, t) + \sum_{\beta=1}^r (f_{ik}^\beta (w_k, z_k) - \varphi_i^{am}(g_{ik}(w_k, z_k), t)) L_{\alpha a}^\beta (w_i, z_i, t) \]

By restricting (3.3.18) to \( w_k = \varphi_k^m(z_k, t) \), we get, from (3.3.14),

\[(3.3.19)\quad \Lambda_0, f_{ik}^\beta (w_k, z_k) - \varphi_i^{am}(g_{ik}(w_k, z_k), t) \rvert_{w_k = \varphi_k^m(z_k, t)} = m+1 \sum_{\beta=1}^r -\psi_{ik}^\beta(z_k, t) T_{\alpha a}^\beta(0, z_i) + G_i^a(z_i, t) \]

Hence from (3.3.17), (3.3.19) and (3.3.16), we obtain

\[(3.3.20)\quad \sum_{\beta=1}^r -\psi_{ik}^\beta(z_k, t) T_{\alpha a}^\beta(0, z_i) + G_i^a(z_i, t) = \sum_{\beta=1}^r G_i^a(z_i, t) \cdot \frac{\partial f_{ik}^\beta(w_k, z_k)}{\partial w_k^\alpha} \rvert_{w_k=0} \]

Hence from (3.3.12), (3.3.17), (3.3.20), \( \{(\psi_{ik}^\beta(z_k, t), ..., \psi_{ik}^\beta(z_k, t)) \} \oplus \{(G_i^1(z_i, t), ..., G_i^1(z_i, t))\} \) defines a 1-cocycle in the above complex so that we get the claim. We call \( \psi_{m+1}(t) := \{(\psi_{ik}^\beta(z_k, t), ..., \psi_{ik}^\beta(z_k, t))\} \) and \( G_{m+1}(t) := \{(G_i^1(z_i, t), ..., G_i^1(z_i, t))\} \) the m-th obstruction so that the coefficients of \( (\psi_{m+1}(t), G_{m+1}(t)) \) in \( t_1, ..., t_r \) lies in \( \mathbb{H}^1(V, \mathcal{N}^{\bullet}/W) \).

On the other hand, by hypothesis, the cohomology group \( \mathbb{H}^1(V, \mathcal{N}^{\bullet}/W) \) vanishes. Therefore there exists \( \varphi_{i[m+1]}(z_i, t) \) such that \( \psi_{ik}(z, t) = F_i(z) \varphi_{i[m+1]}(z, t) - \varphi_{i[m+1]}(z, t) \) and \( \sum_{\beta=1}^r \varphi_{i[m+1]}^\beta(z, t) T_{\alpha a}^\beta(0, z_i) - [\Lambda_0, \varphi_{i[m+1]}^\alpha(z_i, t)] \rvert_{w_k=0} = -G_i^a(z_i, t) \). Then we can show (3.3.9) \( m+1 \) (for the detail, see [Kod62] p.154). On the other hand,

\[-[\Lambda_0, \varphi_{i[m+1]}^\alpha(z_i, t)] = -\sum_{\beta=1}^r \varphi_{i[m+1]}^\beta(z_i, t) T_{\alpha a}^\beta(w_i, z_i) - G_i^a(z_i, t) + \sum_{\beta=1}^r w_i^\beta R_{\alpha a}^\beta(w_i, z_i, t) \]

where the degree of \( R_{\alpha a}^\beta(w_i, z_i, t) \) is \( m+1 \) in \( t_1, ..., t_r \). Then from (3.3.8), we have

\[(3.3.21)\quad [\Lambda_0, w_i^\alpha - \varphi_i^{am}(z_i, t) - \varphi_i^{am}(z_i, t)] = [\Lambda_0, w_i^\alpha - \varphi_i^{am}(z_i, t)] + [\Lambda_0, \varphi_i^{am}(z_i, t)] \]

\[= \sum_{\beta=1}^r (w_i^\beta - \varphi_i^{am}(z_i, t)) T_{\alpha a}^\beta(w_i, z_i) + Q_i^a(\varphi_i^{am}(z_i, t) - G_i^a(z_i, t) + \sum_{\beta=1}^r w_i^\beta P_{\alpha a}^{bm}(w_i, z_i, t) + R_{\alpha a}^\beta(w_i, z_i, t) \]

By setting \( \varphi_i^{(m+1)}(z_i, t) := \varphi_i^{am}(z_i, t) + \varphi_i^{am}(z_i, t) \), we show that \( [\Lambda_0, w_i^\alpha - \varphi_i^{am}(z_i, t)] \rvert_{w_i = \varphi_i^{am}(z_i, t)} = m+1 \)

0. Indeed, from (3.3.21) and (3.3.14),

\[= \sum_{\beta=1}^r (\varphi_i^{am}(z_i, t) + \varphi_i^{am}(z_i, t)) P_{\alpha a}^{bm}(\varphi_i^{am}(z_i, t) + \varphi_i^{am}(z_i, t), z_i, t) \]

\[= \sum_{\beta=1}^r Q_i^a(\varphi_i^{am}(z_i, t) - G_i^a(z_i, t) + \sum_{\beta=1}^r w_i^\beta P_{\alpha a}^{bm}(\varphi_i^{am}(z_i, t), z_i, t) \equiv m+1 K_i^{am}(z_i, t) \equiv m+1 0 \]

which shows (3.3.7) \( m+1 \). This completes the inductive construction of the polynomials \( \varphi_i^{am}(z_i, t), i \in I \).

3.4. Proof of convergence.

We will show that we can choose \( \varphi_i^{am}(z_i, t) \) in each inductive step so that the formal power series \( \varphi_i(z_i, t), i \in I \) constructed in the previous subsection, converges absolutely for \( |t| < \epsilon \) for a sufficiently small number \( \epsilon > 0 \).
Notation 3. Recall Notation 3. We write

\[(3.4.1) \quad A(t) = \frac{a}{16b} \sum_{n=1}^{\infty} \frac{b^n(t_1 + \cdots + t_i)^n}{n^2} \]

instead $A(t)$ in Notation 3 to keep the notational consistency with \text{Kod62}. We also note that

\[(3.4.2) \quad A(t)^v \ll \left(\frac{a}{b}\right)^{v-1} A(t) \quad \text{for} \quad v = 2, 3, \ldots \]

We may assume that $|P_{ik}(0, z)| < c_0$ with $c_0 > 1$. Then $\varphi_{i1}(z_i, t) \ll A(t)$ if $b$ is sufficiently large.

We assume that

\[(3.4.3) \quad \varphi_i^m(z_i, t) \ll A(t) \]

for an integer $m \geq 1$, we shall estimate the coefficients of the homogenous polynomials $\psi_{ik}(z, t)$ and $G_{\alpha}(z, t)$ from (3.3.10) and (3.3.11).

Let $W_{i}^{\delta}$ be the subdomain of $W_i$ consisting of all points $(w_i, z_i)$, $|w_i| < 1 - \delta, |z_i| < 1 - \delta$ for a sufficiently small number $\delta > 0$ such that \{\$W_{i}^{\delta}$i \in I\} forms a covering of $W_i$, and $\{U_i = W_{i}^{\delta} \cap V| i \in I\}$ forms a covering of $V$.

First we estimate the coefficients of the homogenous polynomials $\psi_{ik}(z, t)$. We briefly summarize Kodaira’s result in the following: we expand $f_{ik}(w_k) = f_{ik}(w_k, z_k)$ and $g_{ik}(w_k) = g_{ik}(w_k, z_k)$ into power series in $w_{k1}, \ldots, w_{kr}$, whose coefficients are vector-valued holomorphic functions of $z = (0, z_k)$ defined on $U_k \cap U_i$.

We assume that $f_{ik}(w_k) \ll \sum_{n=1}^{\infty} e_{i1}^n(w_{k1} + \cdots + w_{kr})^n$ and $g_{ik}(w_k) = \sum_{n=0}^{\infty} c_{i1}^n(w_{k1} + \cdots + w_{kr})^n$. Then we can estimate

\[(3.4.4) \quad \psi_{ik}(z_k, t) \ll c_{3} A(t), \; z_k \in U_k \cap U_i, \]

where $c_3 = 2cr_0 \frac{4cr_0}{3} + r_{c1}$ with

\[(3.4.5) \quad b > \max\{2cr_0, \frac{4cr_0}{3}\}. \]

Second we estimate the coefficients of the homogenous polynomials $G_{\alpha}(z, t), \alpha = 1, \ldots, r$. Let $\Lambda_0 = \Lambda_0(w_i, z_i) = \sum_{p,q=1}^{r} \Lambda_{pq}^{i}(w_i, z_i) \frac{\partial}{\partial x^p_i} \wedge \frac{\partial}{\partial x^q_i}$ with $\Lambda_{pq}^{i}(w_i, z_i) = -\Lambda_{qp}^{i}(w_i, z_i)$, where $x_i = (w_i, z_i)$ on $W_i$. By considering coefficients $\Lambda_{pq}^{i}(w_i, z_i)$ of $\frac{\partial}{\partial x^p_i} \wedge \frac{\partial}{\partial x^q_i}$, we can consider $\Lambda_i(w_i, z_i)$ a vector-valued holomorphic function on $W_i$. We expand $\Lambda_i(w_i) = \Lambda_i(w_i, z_i)$ into power series in $w_{i1}, \ldots, w_{ir}$, whose coefficients are vector valued holomorphic functions of $z = (0, z_i)$ defined on $U_i$ and we may assume, for any $p, q$, $\Lambda^{i}_{pq}(w_i, z_i) \ll \sum_{n=0}^{\infty} e_{i1}^n(w_{i1} + \cdots + w_{ir})^n$

for some constant $c_1 > 0$. Now we estimate

\[(3.4.6) \quad G_{\alpha}^{i}(z_i, t) = ([\Lambda_0, w_{i\alpha} - \varphi_{i\alpha}^{m}(z_i, t)]|_{w_i = \varphi_{i\alpha}^m(z_i, t)})_{m+1} \quad \text{for} \quad z_i \in U_i^{\delta}. \]

First we estimate $[[\Lambda_0, w_{i\alpha}]|_{w_i = \varphi_{i\alpha}^m(z_i, t)}]_{m+1}$ in (3.4.7). We note that

\[(3.4.8) \quad [\Lambda_0, w_{i\alpha}]|_{w_i = \varphi_{i\alpha}^m(z_i, t)} = \sum_{p,q=1}^{d+r} 2\Lambda_{pq}^{i}(\varphi_{i\alpha}^m(z_i, t), z_i) \frac{\partial w_{i\alpha}^p}{\partial x^p_i} \frac{\partial}{\partial x^q_i} \]

Since constant terms and linear terms of $\Lambda_{pq}^{i}(\varphi_{i\alpha}^m(z_i, t), z_i)$ does not contribute to $[\Lambda_{pq}^{i}(\varphi_{i\alpha}^m(z_i, t), z_i)]_{m+1}$, we get, from (3.4.6) and (3.4.2),

\[(3.4.9) \quad [\Lambda_{pq}^{i}(\varphi_{i\alpha}^m(z_i, t), z_i)]_{m+1} \ll \sum_{n=2}^{\infty} e_{i1}^{m\alpha} r^n A(t)^n = e_{1r}^{m\alpha} \sum_{n=1}^{\infty} e_{i1}^{m\alpha} r^n A(t)^{n+1} \ll e_{1r} A(t) \sum_{n=1}^{\infty} \left(\frac{e_{1r} a}{b}\right)^n \]

Assuming that

\[(3.4.10) \quad b > 2e_{1r} a, \]

\[\sum_{n=0}^{\infty} \left(\frac{e_{1r} a}{b}\right)^n \]

\[\sum_{n=0}^{\infty} \left(\frac{e_{1r} a}{b}\right)^n \]
we obtain, from (3.4.8) and (3.4.9)

\[(3.4.11) \quad \|[\Lambda_0, w^n_i]\|_{w_i=\varphi^{\alpha_m}(z_i,t)}_{m+1} \ll 2(d+r)^2 \frac{e_1 r^2 a}{b} A(t)\]

Second, we estimate \([\Lambda_0, \varphi_i^{\alpha_m}(z_i,t)]\|_{w_i=\varphi^{\alpha_m}(z_i,t)}_{m+1}\) in (3.4.7). We note that by Cauchy’s integral formula and (3.4.13),

\[(3.4.12) \quad \frac{\partial \varphi_i^{\alpha_m}(z_i,t)}{\partial w_i^\gamma} = 0, \quad \frac{\partial \varphi_i^{\alpha_m}(z_i,t)}{\partial z_i^\gamma} = \frac{1}{2\pi i} \int_{|\xi-z_i^{\gamma}|=\delta} \frac{\varphi_i^{\alpha_m}(z_i^1, \ldots, \xi^i, \ldots, z_i^d, t)}{(\xi-z_i^\gamma)^2} d\xi \ll \frac{A(t)}{\delta} \text{ for } |z_i| < 1 - \delta.
\]

Since constant term of \(\Lambda_{pq}(w_i, z_i)\) with respect to \(w_i^1, \ldots, w_i^r\) does not contribute to \([\Lambda_0, \varphi_i^{\alpha_m}(z_i,t)]\|_{w_i=\varphi^{\alpha_m}(z_i,t)}_{m+1}\), we get, from (3.4.6), (3.4.2) and (3.4.12),

\[(3.4.13) \quad \|[\Lambda_0, \varphi_i^{\alpha_m}(z_i,t)]\|_{w_i=\varphi^{\alpha_m}(z_i,t)}_{m+1} = \sum_{p,q=1}^{r+d} 2\Lambda_{pq}^{ij}(\varphi_i^{\alpha_m}(z_i,t), z_i) \frac{\partial \varphi_i^{\alpha_m}(z_i,t)}{\partial x_i^p} \frac{\partial \varphi_i^{\alpha_m}(z_i,t)}{\partial x_i^q} \ll 2(r + d)^2 \frac{A(t)}{\delta} \sum_{n=1}^{\infty} e_1^n r^n A(t)^n, \quad z_i \in U_i^\delta
\]

\[
\ll 2(r + d)^2 \frac{1}{\delta} \sum_{n=1}^{\infty} e_1^n r^n A(t)^{n+1} \ll \frac{2(r + d)^2}{\delta} \sum_{n=1}^{\infty} \left(\frac{e_1 r a}{b}\right)^n A(t)
\]

Assuming that \(b > 2e_1 r a\), we get

\[(3.4.14) \quad \|[\Lambda_0, \varphi_i^{\alpha_m}(z_i,t)]\|_{w_i=\varphi^{\alpha_m}(z_i,t)}_{m+1} \ll \frac{4(r + d)^2 e_1 r a}{b} A(t), \quad z_i \in U_i^\delta.
\]

Hence from (3.4.11), and (3.4.14), we obtain

\[(3.4.15) \quad G_i^\alpha(z_i,t) = \|[\Lambda_0, w_i^\alpha - \varphi_i^{\alpha_m}(z_i,t)]\|_{w_i=\varphi^{\alpha_m}(z_i,t)}_{m+1} \ll e_2 A(t), \quad z_i \in U_i^\delta,
\]

where

\[(3.4.16) \quad e_2 = \frac{4(d+r)^2 e_1^2 a}{b} + \frac{4(r + d)^2 e_1 r a}{b}.
\]

**Lemma 3.4.17.** We can choose the homogenous polynomials \(\varphi_i^{(m+1)}(z_i,t), i \in I\) satisfying

\[
\psi_k(z,k) = F_k(z)\varphi_k^{(m+1)}(z,t) - \varphi_i^{(m+1)}(z,t)
\]

\[-G_i^\alpha(z,t) = -[\varphi_i^{(m+1)}(z,t), \Lambda_0]_{w_i=0} + \sum_{\beta=1}^{r} \varphi_i^{\beta}(z,t)T_{\beta\alpha}^i(0, z)
\]

in such a way that \(\varphi_i^{(m+1)}(z,t) \ll c_4(e_2 + c_3) A(t)\), where \(c_4\) is independent of \(m\).

**Proof.** For any 0-cochain \(\varphi = \{\varphi_i(z)\}\), 1-cochain \((\psi, G) = \{\psi_k(z)\}, \{\sum_{\alpha=1}^{r} G_i^\alpha(z_i)e_\alpha^\alpha\}\), we define the norms of \(\varphi\) and \((\psi, G)\) by

\[
||\varphi|| := \max_{i} \sup_{z \in U_i} |\varphi_i(z)|,
\]

\[
||(\psi, G)|| := \max_{i,k} \sup_{z \in U_i \cap U_k} |\psi_i(z)| + \max_{i,\alpha} \sup_{z \in U_i^\alpha} |G_i^\alpha(z_i)|
\]

The coboundary \(\varphi\) is defined by

\[
\delta(\varphi) := (-F_k(z)\varphi_k(z,t) + \varphi_i(z), -[\varphi_i^{\alpha}(z_i), \Lambda_0]_{w_i=0} + \sum_{\beta=1}^{r} \varphi_i^{\beta}(z)T_{\beta\alpha}^i(0, z_i))
\]

For any coboundary \((\psi, G)\), we define

\[
\iota(\psi) = \inf_{\delta \varphi = (\psi, G)} ||\varphi||.
\]
To prove Lemma 3.4.17 it suffices to show the existence of a constant \( c \) such that \( \iota(\psi, G) \leq c \| (\psi, G) \| \). Assume that such a constant \( c \) does not exist. Then we can find a sequence \((\psi^n, G^n), (\psi'^n, G'^n), \ldots, (\psi^{(n)}(\mu), G^{(n)}(\mu)), \ldots\) such that there exists \( \varphi^{(\mu)} \) with \( \delta^{(\mu)} = (\psi^{(\mu)}, G^{(\mu)}) \) satisfying \( \| \varphi^{(\mu)} \| < 2 \). Then we can show that there is a subsequence \( \varphi^{(\mu)}_i, \varphi^{(\mu)}_j, \ldots \) such that \( \varphi^{(\mu)}_i(z_i) \) converges absolutely and uniformly on \( U_i \). Let \( \varphi_i(z_i) = \lim \varphi^{(\mu)}_i(z_i) \) and let \( \varphi = \{ \varphi_i(z_i) \} \). Then we have \( \| \varphi^{(\mu)} - \varphi \| \to 0 \). On the other hand \( \delta(\varphi) = (0, G_\varphi) \), where \( G_\varphi(z) = 0 \) for \( z \in U_\varphi \). By identity theorem \( G_\varphi(z) = 0 \) for \( z \in U_\varphi \) so that \( \delta(\varphi) = (0, 0) \). Therefore \( \delta(\varphi^{(\mu)} - \varphi) = (\varphi^{(\mu)}, G^{(\mu)}(\mu)) \) which contradict to \( \iota(\varphi^{(\mu)} = 1) \). □

From 3.4.4 and 3.4.16, we have

\[
(3.4.18) \quad c_1(c_3 + e_2) = c_4c_3 + c_4e_2 = \frac{8c_4c_0c_1r^2a}{b} \left( \frac{2d}{b} + rc_1 \right) + c_4 \left( \frac{4(d + r)^2c_1^2r^2a}{\delta} + \frac{4(r + d)c_1ra}{\delta} \right)
\]

From 3.4.10, (3.4.18) and Lemma 3.4.17 by assuming

\[
b > \max \{8c_4c_0c_1r^2a \left( \frac{2d}{b} + rc_1 \right) + c_4 \left( \frac{4(d + r)^2c_1^2r^2a}{\delta} + \frac{4(r + d)c_1ra}{\delta} \right) \}
\]

we can choose \( \varphi_{1,|\mu|+1}(z_i) \ll A(t) \) and so \( \varphi(z_i, t) \ll A(t) \) so that the power series \( \varphi(z_i, t) \) converges for \( |t| < \frac{1}{\sqrt{b}} \).

Then by the argument of [Kod62] p.158, we obtain the equality

\[
\varphi_i(g_k(\varphi_k(z_k, t), z_k), t) = f_k(\varphi_k(z_k, t), z_k), \quad \text{for} \quad |t| < \epsilon, \quad (\varphi_k(z_k, t), z_k) \in W_1^\delta \cap W_2^\delta
\]

for a sufficiently small number \( \epsilon > 0 \), which proves Theorem 3.3.1 □

In the case \( H^1(V, N^*_V/W) \neq 0 \), our proof of Theorem 3.3.1 proves the following:

**Theorem 3.4.19.** If the obstruction \( (\psi_{m+1}(t), G_{m+1}(t)) \) vanishes for each integer \( m \geq 1 \), then there exists a Poisson analytic family \( V \) of compact holomorphic Poisson submanifolds \( V_t, t \in M_1, \) of \( (W, A_0) \) such that \( V_0 = V \) and the characteristic map

\[
\sigma_0 : T_0(M_1) \to H^0(V, N^*_V/W)
\]

\[
\frac{\partial}{\partial t} \mapsto \left( \frac{\partial V_t}{\partial t} \right)_{t=0}
\]

is an isomorphism.

### 3.5. Maximal families: Theorem of completeness.

We note that Definition 2.4.1 can be extended to arbitrary codimensions.

**Theorem 3.5.1** (theorem of completeness). Let \( V \) be a Poisson analytic family of compact holomorphic Poisson submanifolds \( V_t, t \in M_1, \) of \( (W, A_0) \). If the characteristic map

\[
\sigma_0 : T_0(M_1) \to H^0(V, N^*_V/W)
\]

\[
\frac{\partial}{\partial t} \mapsto \left( \frac{\partial V_t}{\partial t} \right)_{t=0}
\]

is an isomorphism, then the family \( V \) is maximal at \( t = 0 \).

**Proof.** We extend the arguments in [Kod62] p.158-160 in the context of holomorphic Poisson deformations.

Consider an arbitrary Poisson analytic family \( V' \) of compact holomorphic Poisson submanifolds \( V'_s, s \in M' \) of \( (W, A_0) \) such that \( V'_0 = V_0 \), where \( M' = \{ s = (s_1, \ldots, s_q) \in \mathbb{C}^q ||s|| < 1 \} \). We shall construct a holomorphic map \( h : s \to t = h(s) \) of a neighborhood \( N' \) of \( 0 \) into \( M_1 \) such that \( h(0) = 0 \) and \( V'_s = V_{h(s)} \).

We keep the notations in 3.2 so that the holomorphic Poisson submanifold \( V_t \) is defined in each domain \( W_i, i \in I \) by \( w_i = \varphi_i(z_i, t) \) and satisfy

\[
(3.5.1) \quad [\Lambda_0, w_i^\beta - \varphi_i^\beta(z_i, t)] = \sum_{\beta=1}^{r} (w_i^\beta - \varphi_i^\beta(z_i, t)T_{\alpha\beta}(w_i, z_i, t).
\]
We may assume that $V'_3$ is defined in each domain $W_i$, $i \in I$, by $w_i = \theta_i(z_i, t)$, where $\theta_i(z_i, t)$ is a vector-valued holomorphic function of $z_i, s$ with $|z_i| < 1, |s| < 1$, and satisfy

\[(3.5.3) \quad [A_0, w_i^\alpha - \theta_i^\alpha(z_i, s)] = \sum_{\beta=1}^r (w_i^\beta - \theta_i^\beta(z_i, s)) P_{i\alpha}^\beta(w_i, z_i, s)\]

for some $P_{i\alpha}^\beta(w_i, z_i, s)$ which are power series in $s$ with coefficients in $\Gamma(W_i, T_W)$ and $P_{i\alpha}^\beta(0, z_i, 0) = T_{i\alpha}^\beta(0, z_i)$. Then $V'_s = V_h(s)$ is equivalent to

\[(3.5.4) \quad \theta_i(z_i, s) = \varphi_i(z_i, h(s))\]

Recall Notation 1 and let us write

\[(3.5.6) \quad \text{Proof of convergence.} \]

Let $\omega$ which completes the inductive construction of $h$ for some $\omega$. We may assume that $\omega$. We claim that $\omega$. We have already constructed $h(m)(s)$ satisfying (3.5.5) and $h_{m+1}(s)$ satisfy (3.5.5). Let $\omega_i(z_i, s) = [\theta_i(z_i, s) - \varphi_i(z_i, h^m(s))]_{m+1}$. We claim that

\[(3.5.6) \quad \omega_i(z_i, s) = F_{ik}(z_i, s)\]

\[(3.5.7) \quad \omega_i(z_i, s) = F_{ik}(z_i, s), \quad z \in U_i \cap U_k\]

For the proof of (3.5.6), see [Kod62] p.160. Let us show (3.5.7). From (3.5.2) and (3.5.3), we have

\[\begin{align*}
[A_0, w_i^\alpha, \theta_i(z_i, s)]_{w_i=0} & \equiv 1 \quad \text{[A_0, w_i^\alpha(z_i, s)]}_{w_i=0} = [\theta_i(z_i, s) - w_i^\alpha + w_i^\alpha - \varphi_i^\alpha(z_i, h^m(s))]_{w_i=\theta_i(z_i, t)} \\
\equiv m+1 \quad [A_0, w_i^\alpha - \theta_i^\alpha(z_i, s)]_{w_i=\theta_i(z_i, t)} & = [\theta_i(z_i, s) - w_i^\alpha + w_i^\alpha - \varphi_i^\alpha(z_i, h^m(s))]_{w_i=\theta_i(z_i, t)} \\
\equiv m+1 \quad \sum_{\beta=1}^r (\theta_i^\beta(z_i, s) - \varphi_i^\beta(z_i, h(s))) T_{i\alpha}^\beta(w_i, z_i, h(s))_{w_i=0} & = m+1 \quad \sum_{\beta=1}^r \omega_i^\beta(z_i, s) T_{i\alpha}^\beta(0, z_i) \\
\end{align*}\]

This proves (3.5.7). From (3.5.6) and (3.5.7), $\omega_i(z_i, s)$ is a homogenous polynomial of degree $m+1$ in $s$ with coefficients in $\mathbb{H}^0(V, N_{V/\mathbb{P}_3}^\bullet)$ so that there exists a homogenous polynomial $h_{m+1}(s)$ of degree 1 in $s$ such that $\omega_i(z_i, s) = \varphi_i(z_i, h_{m+1}(s))$. Therefore we have $\varphi_i(z_i, h_{m+1}(s)) = m+1 \varphi_i(z_i, h^m(s)) + \omega_i(z_i, s) \equiv m+1 \theta_i(z_i, s)$ which completes the inductive construction of $h^m(s)$ satisfying (3.5.5)_{m+1}.

### 3.6. Proof of convergence.

The convergence of the power series $h(s)$ follows from the same arguments in [Kod62] p.160-161. This completes the proof of Theorem 3.5.1.

---

**Example 3.** Let $[\xi_0, \xi_1, \xi_2, \xi_3]$ be the homogenous coordinate on $\mathbb{P}_3^3$ and a hyperplane $V$ defined by $\xi_3 = 0$ so that $N_{V/\mathbb{P}_3^3} \cong \mathcal{O}_V(1)$. Let $[1, z_1, z_2, z_3] = [1, \xi_0, \xi_1, \xi_2, \xi_3]$. Consider a Poisson structure $\Lambda_0 = z_1 \frac{\partial}{\partial \xi_1} \wedge \frac{\partial}{\partial \xi_2}$ on $\mathbb{P}_3^3$. Then $V \cong \mathbb{P}_3^3$ is a holomorphic Poisson submanifold. We compute $\mathbb{H}^0(V, N_{V/\mathbb{P}_3^3})$ which is the kernel $\nabla : H^0(V, \mathcal{O}_V(1)) \to H^0(V, T_{\mathbb{P}_3^3}[V(1)])$. Since $[\Lambda_0, z_3] = 0$,

$$\nabla(a z_1 + b z_2 + c) = -[\Lambda_0, a z_1 + b z_2 + c]_{z_3=0} = -a z_1 \frac{\partial}{\partial z_2} + b z_1 \frac{\partial}{\partial z_1} = 0 \iff a = b = 0$$
such that $d$ is a local coordinate of $w$ by Definition 3.0.1 by deforming holomorphic Poisson structures as well on a fixed complex manifold.

We extend the definition of a Poisson analytic family of compact holomorphic Poisson submanifolds in Definition 3.0.1 by deforming holomorphic Poisson structures as well on a fixed complex manifold $W$.

Definition 4.0.1. Let $W$ be a complex manifold of dimension $d + r$. We denote a point in $W$ by $w$ and a local coordinate of $w$ by $(w^1, ..., w^{r+d})$. By an extended Poisson analytic family of compact holomorphic Poisson submanifolds of dimension $d$ of $W$, we mean a holomorphic Poisson submanifold $V \subset (W \times M, \Lambda)$ of codimension $r$, where $M$ is a complex manifold and $\Lambda$ is a holomorphic Poisson structure on $W \times M$, such that...
(1) the canonical projection \( \pi : (W \times M, \Lambda) \to M \) is a holomorphic Poisson fibre manifold as in Definition 5.0.1, so that \( \Lambda \in H^0(W \times M, \Lambda^2 T_{W \times M/M}) \) and \( \pi^{-1}(t) := (W, \Lambda_t) \) is a holomorphic Poisson submanifold of \((W \times M, \Lambda)\) for each point \( t \in M \).

(2) for each point \( t \in M \), \( V_t \times t := \omega^{-1}(t) = V \cap \pi^{-1}(t) \) is a connected compact holomorphic Poisson submanifold of \((W, \Lambda_t)\) of dimension \( d \), where \( \omega : V \to M \) is the map induced from \( \pi \).

(3) for each point \( p \in V \), there exist \( r \) holomorphic functions \( f_\alpha(w, t), \alpha = 1, ..., r \) defined on a neighborhood \( \mathcal{U}_p \) of \( p \) in \( W \times M \) such that \( \text{rank} \frac{\partial (f_1, ..., f_r)}{\partial (w, w^t, ..., w^r)} = r \), and \( \mathcal{U}_p \cap V \) is defined by the simultaneous equations \( f_\alpha(w, t) = 0, \alpha = 1, ..., r \).

We call \( V \subset (W \times M, \Lambda) \) an extended Poisson analytic family of compact holomorphic Poisson submanifolds \( V_t, t \in M \) of \((W, \Lambda_t)\). We also call \( V \subset (W \times M, \Lambda) \) an extended Poisson analytic family of simultaneous deformations of a holomorphic Poisson submanifold \( V_0 \) of \((W, \Lambda_0)\) for each fixed point \( t_0 \in M \).

4.1. The extended complex associated with the normal bundle of a holomorphic Poisson submanifold of a holomorphic Poisson manifold.

Let \( V \) be a holomorphic Poisson submanifold of a holomorphic Poisson manifold \((W, \Lambda_0)\). We will describe a complex of sheaves to control simultaneous deformations of holomorphic Poisson structures and holomorphic Poisson submanifolds. We recall that the complex associated with the normal bundle (see Definition 3.1.3).

\[
\mathcal{N}_{V/W} : N_{V/W} \xrightarrow{-} N_{V/W} \otimes T_W|_V \xrightarrow{-} N_{V/W} \otimes \wedge^2 T_W|_V \xrightarrow{-} \cdots
\]

controls holomorphic Poisson deformations of \( V \) in \((W, \Lambda_0)\), and the complex

\[
\wedge^2 T_W : \wedge^2 T_W \xrightarrow{-} \wedge^3 T_W \xrightarrow{-} \wedge^4 T_W \xrightarrow{-} \cdots
\]

controls deformations of the holomorphic Poisson structure \( \Lambda_0 \) on the fixed underlying complex manifold \( W \) (see Appendix A). By combining two complexes \([4.1.1]\) and \([4.1.2]\), we shall define a complex of sheaves on \( W \):

\[
(\wedge^2 T_W \oplus i_*N_{V/W})^* : \wedge^2 T_W \oplus i_*N_{V/W} \xrightarrow{-} \wedge^3 T_W \oplus i_* (N_{V/W} \otimes T_W|_V) \xrightarrow{-} \wedge^4 T_W \oplus i_* (N_{V/W} \otimes \wedge^2 T_W|_V) \xrightarrow{-} \cdots
\]

which controls simultaneous deformations of the holomorphic Poisson structure \( \Lambda_0 \) and the holomorphic Poisson submanifold \( V \) of \((W, \Lambda_0)\), where \( i : V \hookrightarrow W \) is the embedding. We keep the notations in subsection 3.1.

We note that \( \Gamma(W_i, \wedge^{p+2} T_W \oplus i_* (N_{V/W} \otimes \wedge^p T_W|_V)) = \Gamma(W_i, \wedge^{p+2} T_W) \oplus \Gamma(U_i, N_{V/W} \otimes \wedge^p T_W|_V) \equiv \Gamma(W_i, \wedge^{p+2} T_W) \oplus (\oplus \Gamma(U_i, \wedge^{p+1} T_W|_V)) \). From these isomorphisms, we define

\[
\tilde{\nabla} : \wedge^{p+2} T_W \oplus i_* (N_{V/W} \otimes \wedge^p T_W|_V) \to \wedge^{p+3} T_W \oplus i_* (N_{V/W} \otimes \wedge^{p+1} T_W|_V)
\]

locally in the following way:

\[
\Gamma(W_i, \wedge^{p+2} T_W) \oplus (\oplus \Gamma(U_i, \wedge^{p+1} T_W|_V)) \xrightarrow{\tilde{\nabla}} \Gamma(W_i, \wedge^{p+3} T_W) \oplus (\oplus \Gamma(U_i, \wedge^{p+1} T_W|_V))
\]

In other words,

\[
(\Pi, (g_\alpha^0 e_\alpha^0, ..., g_\omega^0 e_\omega^0)) \mapsto (- [\Pi, \Lambda_0], \sum_{\alpha=1}^r [\Pi, w_\alpha^0]|_{w_\alpha=0} e_\alpha^0 + \nabla(\sum_{\alpha=1}^r g_\alpha^0 e_\alpha^0))
\]

In words,

\[
(\Pi, (g_\alpha^0, ..., g_\omega^0)) \mapsto (- [\Pi, \Lambda_0], (\sum_{\alpha}^r [\Pi, w_\alpha^0]|_{w_\alpha=0} e_\alpha^0 + \nabla(\sum_{\alpha=1}^r [\Pi, w_\alpha^0]|_{w_\alpha=0} e_\alpha^0))
\]

First we show that \( \tilde{\nabla} \) defines a complex, i.e. \( \tilde{\nabla} \circ \tilde{\nabla} = 0 \). Since \( \nabla \circ \nabla = 0 \), we have

\[
\tilde{\nabla}(\tilde{\nabla}(\Pi, \sum_{\alpha=1}^r g_\alpha^0 e_\alpha^0)) = (0, \sum_{\alpha=1}^r -[\Pi, \Lambda_0], w_\alpha^0]|_{w_\alpha=0} e_\alpha^0 + \nabla(\sum_{\alpha=1}^r [\Pi, w_\alpha^0]|_{w_\alpha=0} e_\alpha^0))
\]

\[
= (0, \sum_{\alpha=1}^r -[\Pi, \Lambda_0], w_\alpha^0]|_{w_\alpha=0} e_\alpha^0 + \sum_{\alpha=1}^r -[\Pi, w_\alpha^0]|_{w_\alpha=0} e_\alpha^0 + (-1)^p \sum_{\alpha, \beta=1}^r [\Pi, w_\alpha^0]|_{w_\alpha=0} \wedge T_{i, \beta}^0 (0, z_i) e_\beta^0
\]
Hence $\nabla \circ \nabla = 0$ is equivalent to

$$-(\Pi_i, [\Lambda_0, w_i^0])|_{w_i=0} - \left(\Pi_i, w_i^0, \Lambda_0\right)|_{w_i=0} + (-1)^{p+1} \sum_{\beta=1}^r \left[\Pi_i, u_{i\beta}^0\right]|_{w_i=0} \wedge T_{i\alpha}^\beta(0, z_i) = 0$$

Let us show (4.1.3). We note that from (3.1.3),

$$[\Pi_i, [\Lambda_0, w_i^0]] = \sum_{\beta=1}^r \left[\Pi_i, w_i^0 T_{i\alpha}^\beta(w_i, z_i)\right] = \sum_{\beta=1}^r w_i^0 \left[\Pi_i, T_{i\alpha}^\beta(w_i, z_i)\right] + (-1)^{p+1} \sum_{\beta=1}^r [\Pi_i, w_i^0 ] \wedge T_{i\alpha}^\beta(w_i, z_i)$$

$$\Rightarrow [\Pi_i, [\Lambda_0, w_i^0]]|_{w_i=0} = (-1)^{p+1} \sum_{\beta=1}^r [\Pi_i, w_i^0]|_{w_i=0} \wedge T_{i\alpha}^\beta(0, z_i)$$

Then (4.1.3) is equivalent to

$$-(\Pi_i, [\Lambda_0, w_i^0])|_{w_i=0} - \left(\Pi_i, w_i^0, \Lambda_0\right)|_{w_i=0} + [\Pi_i, [\Lambda_0, w_i^0]] = 0$$

which follows from $-(\Pi_i, [\Lambda_0, w_i^0]) - [\Pi_i, w_i^0, \Lambda_0] + [\Pi_i, [\Lambda_0, w_i^0]] = 0$ by the graded Jacobi identity. This proves $\nabla \circ \nabla = 0$.

Next we show that $\nabla$ is well-defined. In other words, on $U_i \cap U_k$, the following diagram commutes

$$\Gamma(W_k, \wedge^{p+2} T_W) \oplus (\oplus \Gamma(U_k, \wedge^p T_W|_V)) \xrightarrow{\text{on } U_i \cap U_k} \Gamma(W_i, \wedge^{p+2} T_W) \oplus (\oplus \Gamma(U_i, \wedge^p T_W|_V))$$

(4.1.5)

$$\nabla \downarrow$$

$$\Gamma(W_k, \wedge^{p+3} T_W) \oplus (\oplus \Gamma(U_k, \wedge^{p+1} T_W|_V)) \xrightarrow{\text{on } U_i \cap U_k} \Gamma(W_i, \wedge^{p+3} T_W) \oplus (\oplus \Gamma(U_i, \wedge^{p+1} T_W|_V))$$

Let $\Pi, \sum_{\alpha=1}^r g_{k\alpha}^\alpha e_k^\alpha \in \Gamma(W_k, \wedge^{p+2} T_W) \oplus (\oplus \Gamma(U_k, \wedge^p T_W|_V))$ and let $\nabla((\Pi, \sum_{\alpha=1}^r g_{k\alpha}^\alpha e_k^\alpha)) = \sum_{\alpha=1}^r G_k^\alpha e_k^\alpha$, where

$$G_k^\alpha = -[g_{k\alpha}^\alpha, [\Lambda_0, 0]|_{w_k=0} + (-1)^p \sum_{\beta=1}^r g_{k\beta}^\beta T_{k\alpha}^\beta(0, z_k) \in \Gamma(U_k, \wedge^{p+1} T_W|_V).$$

Then $\nabla((\Pi, \sum_{\alpha=1}^r g_{k\alpha}^\alpha e_k^\alpha)) = (\Pi, \sum_{\alpha=1}^r [\Pi, g_{k\alpha}^\alpha]|_{w_k=0} e_k^\alpha + \sum_{\alpha=1}^r G_k^\alpha e_k^\alpha)$ is identified on $U_i \cap U_k$ with

$$\Pi, \sum_{\alpha=1}^r [\Pi, g_{k\alpha}^\alpha]|_{w_k=0} e_k^\alpha = \Pi, \sum_{\alpha=1}^r [\Pi, w_i^0]|_{w_k=0} e_k^\alpha$$

On the other hand, $(\Pi, \sum_{\alpha=1}^r g_{k\alpha}^\alpha e_k^\alpha)$ is identified on $U_i \cap U_k$ with $(\Pi, \sum_{\beta=1}^r \left(\sum_{\alpha=1}^r F_{i\beta k\alpha}^\alpha (0, z_k) g_{k\alpha}^\alpha\right) e_i^\beta) \in \Gamma(W_i, \wedge^{p+2} T_W) \oplus (\oplus \Gamma(U_i, \wedge^p T_W|_V))$. Then we have

$$\Pi, \sum_{\beta=1}^r \left(\sum_{\alpha=1}^r F_{i\beta k\alpha}^\alpha (0, z_k) g_{k\alpha}^\alpha\right) e_i^\beta = (\Pi, [\Lambda_0, \sum_{\beta=1}^r [\Pi, w_i^0]|_{w_k=0} e_i^\beta \nabla(\sum_{\alpha=1}^r F_{i\beta k\alpha}^\alpha (0, z_k) g_{k\alpha}^\alpha e_i^\beta))$$

Hence in order for the diagram (4.1.5) to commute, we have to show that (4.1.5) coincides with (4.1.7). By the equality of (3.1.11) and (3.1.12), it is enough to show $\Pi, w_i^0|_{w_k=0} = \sum_{\alpha=1}^r F_{i\alpha k\beta} (0, z_k)[\Pi, w_i^0]|_{w_k=0}$ which comes from (3.1.1), and

$$w_i^0 = \sum_{\alpha=1}^r F_{i\alpha k\beta} (w_k, z_k) w_k^\alpha \Rightarrow [\Pi, w_i^0] = \sum_{\alpha=1}^r (F_{i\alpha k\beta} (w_k, z_k)[\Pi, w_i^0]) + w_k^\alpha [\Pi, F_{i\alpha k\beta}].$$

Hence $\nabla$ is well-defined.

**Definition 4.1.8.** We call the complex defined as above

$$(\wedge^2 T_W \oplus i_* N_{V/W})^\bullet : \wedge^2 T_W \oplus i_* N_{V/W} \xrightarrow{\nabla} \wedge^3 T_W \oplus i_* (N_{V/W} \otimes T_W|_V) \rightarrow \wedge^4 T_W \oplus i_* (N_{V/W} \otimes \wedge^2 T_W|_V) \rightarrow \cdots$$

the extended complex associated with the normal bundle $N_{V/W}$ of a holomorphic Poisson submanifold $V$ of a holomorphic Poisson manifold $W$ and denote its $i$-th hypercohomology group by $\mathbb{H}^i(W, (\wedge^2 T_W \oplus i_* N_{V/W})^\bullet)$. 
4.2. Infinitesimal deformations.

Let $M = \{ t = (t_1, ..., t_r) \in \mathbb{C} \mid |t| < 1 \}$. Consider an extended Poisson analytic family $\mathcal{V} \subset (W \times M, \Lambda)$ of compact holomorphic Poisson submanifolds $V_t, t \in M_1$ of $(W, \Lambda_t)$ and let $V = V_0$ as in Definition 4.2. We keep the notations in subsection 4.2 and subsection 4.1 so that for $|t| < \epsilon$ for a sufficiently small number $\epsilon > 0$, $V_t$ is defined by $w_i = \phi_i(z, t)$ on each neighborhood $W_t$ and, by setting $w_i^\lambda = w_i^\lambda - \phi_i^\lambda(z, t), \lambda = 1, ..., r$, $F_{i\lambda}(z_t) := \left( \frac{\partial w_i^\lambda}{\partial w_i^\mu}(z_t) \right)_{\lambda, \mu = 1, ..., r}$ for $z_t \in V_t \cap W_t \cap W_k$ defines the normal bundle $N_{V_t/W}$ of $V_t$ in $W$. For an arbitrary tangent vector $\frac{\partial}{\partial T}$ at $z = (\phi_1(z, t), z_t)$, we get, on $\Gamma(W_t, T_{W_t}/V_t)$, we get, on $\Gamma(W_t, T_{W_t}/V_t)$, we get

\[ [\Lambda_i(w_t, z, t), \Lambda_i(w_t, z, t)] = 0 \]

and $\Lambda_i(w_t, z, t)$ is of the form

\[ \Lambda_i(w_t, z, t) := \Lambda_i(x_t, t) = \sum_{\alpha, \beta = 1}^{r+d} \Lambda_{i, \alpha, \beta}(x_t, t) \frac{\partial}{\partial x_t^\alpha} \wedge \frac{\partial}{\partial x_t^\beta}, \text{ with } \Lambda_{i, \alpha, \beta}(x_t, t) = -\Lambda_{\beta, \alpha}(x_t, t), x_t = (w_t, z_t), \]

by which we consider $\Lambda_i(w_t, z, t)$ as a vector-valued holomorphic function of $(w_t, z_t)$. Let $\pi_i(w_t, z, t) = \frac{\partial \Lambda_i(w_t, z, t)}{\partial t}$. Then

\[ \{ \pi_i(w_t, z, t) \} \in H^0(W, \wedge^2 T_W) \]

By taking the derivative of (4.2.2) with respect to $t$, we get

\[ [\Lambda_i(w_t, z, t), \frac{\partial \Lambda_i(w_t, z, t)}{\partial t}] = 0 \iff [\pi_i(w_t, z, t), \Lambda_i(w_t, z, t)] = 0 \]

Lastly, since $w_i^\lambda - \phi_i^\lambda(z, t) = 0, \lambda = 1, ..., r$ define a holomorphic Poisson submanifold, we have

\[ [\Lambda_i(w_t, z, t), w_i^\lambda - \phi_i^\lambda(z, t)] = \sum_{\mu=1}^{r} (w_i^\mu - \phi_i^\mu(z, t)) T_{i\lambda}^\mu(w_t, z, t) \]

for some $T_{i\lambda}^\mu(w_t, z, t)$ which is of the form

\[ T_{i\lambda}^\mu(w_t, z, t) = P_{i\mu}^\lambda(w_t, z, t) \frac{\partial}{\partial w_t^\lambda} + \cdots + P_{i\mu}^\lambda(w_t, z, t) \frac{\partial}{\partial w_t^\lambda} + Q_{i1}^\mu(w_t, z, t) \frac{\partial}{\partial z_t^1} + \cdots + Q_{id}^\mu(w_t, z, t) \frac{\partial}{\partial z_t^d} \]

by which we consider $T_{i\lambda}^\mu(w_t, z, t)$ as a vector valued holomorphic function of $(w_t, z_t)$.

By taking the derivative of (4.2.5) with respect to $t$, we get \( \left[ \frac{\partial \Lambda_i(w_t, z, t)}{\partial t}, w_i^\lambda - \phi_i^\lambda(z, t) \right] + \left[ \Lambda_i(w_t, z, t), -\frac{\partial \phi_i^\lambda(z, t)}{\partial t} \right] = \sum_{\mu=1}^{r} -\frac{\partial \phi_i^\mu(z, t)}{\partial t} T_{i\lambda}^\mu(w_t, z, t) + \sum_{\mu=1}^{r} (w_i^\lambda - \phi_i^\lambda(z, t)) \frac{\partial T_{i\lambda}^\mu(w_t, z, t)}{\partial t} \). By restricting to $V_t$, equivalently, by setting $w_t = \phi_t(z_t)$, we get, on $\Gamma(W_t \cap V_t, T_{W_t} | V_t)$,

\[ \left[ \frac{\partial \Lambda_i(w_t, z, t)}{\partial t}, w_i^\lambda - \phi_i^\lambda(z, t) \right]|_{V_t} - \left[ \Lambda_i(w_t, z, t), -\frac{\partial \phi_i^\lambda(z, t)}{\partial t} \right]|_{V_t} = \sum_{\mu=1}^{r} -\frac{\partial \phi_i^\mu(z, t)}{\partial t} T_{i\lambda}^\mu(\phi_t(z_t), z, t) + \sum_{\mu=1}^{r} (w_i^\lambda - \phi_i^\lambda(z, t)) T_{i\lambda}^\mu(\phi_t(z_t), z, t)|_{V_t} = 0 \]

Hence from (4.2.1), (4.2.3), (4.2.4) and (4.2.6), \( \{ \Lambda_i(w_t, z, t), \phi_t(z_t) \} \) defines an element of $H^0(W, (\wedge^2 T_W + i_{*} N_{V_t/W}^*)^*)$ so that we have a linear map

\[ \sigma_t : T_t(M_1) \to H^0(W, (\wedge^2 T_W + i_{*} N_{V_t/W}^*)) \]

\[ \frac{\partial}{\partial t} \mapsto \frac{\partial (\Lambda_i, V_t)}{\partial t} := \{ \pi_i(w_t, z, t) \}, \phi_t(z_t) \}

We call $\sigma_t$ the characteristic map.
Example 6. Let \([\xi_0, \xi_1, \xi_2]\) be the homogeneous coordinate on \(\mathbb{P}^3_C\). Let \([1, z_1, z_2] = [1, \frac{z}{z_1}, \frac{z_2}{z_1}]\). Then \(\mathcal{V} \subset (\mathbb{P}^2_C \times \mathbb{C}^2, (z_1 + t_1 z_2 + t_2) \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2})\) defined by \(\xi_1 + t_1 \xi_2 + t_2 \xi_0 = 0\) is an extended Poisson analytic family of simultaneous deformations of a holomorphic Poisson submanifold \(\xi_1 = 0\) of \((\mathbb{P}^2_C, \Lambda_0 = z_1 \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2})\) and we have the characteristic map

\[
T_0 \mathbb{C}^2 \to \mathbb{H}^0(\mathbb{P}^2_C, (\wedge^2 T_{\mathbb{P}^2_C} \oplus i_* \mathcal{O}_{\mathbb{P}^2_C}(1))^*)
\]

\[
(a, b) \mapsto a \left( z_2 \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2}, -z_2 \right) + b \left( \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2}, -1 \right)
\]

Example 7. Let \([\xi_0, \xi_1, \xi_2, \xi_3]\) be the homogeneous coordinate on \(\mathbb{P}^3_C\). Let \([1, z_1, z_2, z_3] = [1, \frac{z}{z_0}, \frac{z_1}{z_0}, \frac{z_2}{z_0}]\). Then \(\mathcal{V} \subset (\mathbb{P}^3_C \times \mathbb{C}^2, (z_1 + t_1) \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2})\) defined by \(\xi_1 + t_1 \xi_0 = \xi_3 + t_2 \xi_0 = 0\) is an extended Poisson analytic family of simultaneous deformations of a holomorphic Poisson submanifold \(\xi_1 = \xi_3 = 0\) of \((\mathbb{P}^3_C, \Lambda_0 = z_1 \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2})\) and we have the characteristic map

\[
T_0 \mathbb{C}^2 \to \mathbb{H}^0(\mathbb{P}^3_C, (\wedge^2 T_{\mathbb{P}^3_C} \oplus i_* (\mathcal{O}_{\mathbb{P}^3_C}(1) \oplus \mathcal{O}_{\mathbb{P}^2_C}(1))^*))
\]

\[
(a, b) \mapsto a \left( \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2}, (-1, 0) \right) + b(0, 0, -1)
\]

Example 8. We construct an extended Poisson analytic family of simultaneous deformations of holomorphic Poisson submanifolds of a stable elliptic surface. As in [BPVY04], let \(z(s)\) be an arbitrary holomorphic function on the unit disk \(\Delta = \{ s \in \mathbb{C} | |s| < 1 \}\) with \(\text{Im}(z(s)) > 0\). Let \(G = \mathbb{Z} \times \mathbb{Z}\) act on \(\mathbb{C} \times \Delta\) by \((m, n)(c, s) = (c + m + n z(s), s)\). The quotient \(X = (\mathbb{C} \times \Delta)/(\mathbb{Z} \times \mathbb{Z})\) is a nonsingular surface fibered over \(\Delta\) such that \(X_s\) is an elliptic curve with period \(1, z(s)\). Let \(c' = c + m + n z(s), s' = s\). Then we have \(\frac{\partial}{\partial c'} = \frac{\partial}{\partial c}, \frac{\partial}{\partial s'} = \frac{\partial}{\partial s}, \frac{\partial}{\partial t'} = \frac{\partial}{\partial t}\) so that we get \(\frac{\partial}{\partial c'} \wedge \frac{\partial}{\partial s'} = \frac{\partial}{\partial c} \wedge \frac{\partial}{\partial s}\) and so \(\Lambda = (s - t) \frac{\partial}{\partial c'} \wedge \frac{\partial}{\partial s'}\) is a \(G\)-invariant bivector field on \(X \times \Delta\) which defines a holomorphic Poisson structure \(\Lambda_t\) on \(X\) for each \(t \in \Delta\), and \(X_t = s = t\) is a holomorphic Poisson submanifold of \((X, \Lambda)\) since the holomorphic Poisson structure \(\Lambda_t\) degenerates along \(s = t\). Then \(\mathcal{V} \subset (X \times \Delta, \Lambda = (s - t) \frac{\partial}{\partial c} \wedge \frac{\partial}{\partial s})\) defined by \(s = t\) is an extended Poisson analytic family of simultaneous deformations of the holomorphic Poisson submanifold \(X_0 = s = 0\) of \((X, s \frac{\partial}{\partial c} \wedge \frac{\partial}{\partial s})\), and we have the characteristic map

\[
T_0 \mathbb{C} \to \mathbb{H}^0(X, (\wedge^2 T_X \oplus i_* N_{X_0/X})^*)
\]

\[
a \mapsto (-a \frac{\partial}{\partial c} \wedge \frac{\partial}{\partial s}, a)
\]

4.3. Theorem of existence.

Theorem 4.3.1 (theorem of existence). Let \(V\) be a holomorphic Poisson submanifold of a compact holomorphic Poisson manifold \((W, \Lambda_0)\). If \(\mathbb{H}^1(W, (\wedge^2 T_W \oplus i_* N_{W/V})^*) = 0\), then there exists an extended Poisson analytic family \(V \subset (W \times M_1, \Lambda)\) of compact holomorphic Poisson submanifolds \(V_t, t \in M_1\), of \((W, \Lambda_t)\) such that \(V = V_0 \subset (W, \Lambda_0)\) and the characteristic map

\[
\sigma_0 : T_0 (M_1) \to \mathbb{H}^0(W, (\wedge^2 T_W \oplus i_* N_{W/V})^*)
\]

\[
\frac{\partial}{\partial t} \mapsto \left( \frac{\partial (\Lambda_t, V_t)}{\partial t} \right)_{t=0}
\]

is an isomorphism.

Proof. We extend the argument in the proof of Theorem 3.3.1 in the context of simultaneous deformations. We keep the notations in subsection 4.3.

Let \(\{\eta_1, ..., \eta_p, ..., \eta_l\}\) be a basis of \(\mathbb{H}^0(W, (\wedge^2 T_W \oplus i_* N_{W/V})^*)\). On each neighborhood \(W_i\) (we recall \(U_i = W_i \cap V\)), \(\eta_\rho\) is represented as

\[
\eta_\rho = (\lambda_i^0(z_i, z)) \oplus (\gamma_\rho^1(z_1), ..., \gamma_\rho^p(z_i)) \in \Gamma(W_i, \wedge^2 T_W) \oplus (\oplus \Gamma(U_i, \mathcal{O}_V))
\]
such that

\[(4.3.2) \quad \lambda^0_i(w_i, z_i) = \lambda^0_j(w_j, z_j)\]
\[(4.3.3) \quad -[\lambda^0_i(w_i, z_i), \Lambda_0] = 0\]
\[(4.3.4) \quad \gamma_{\rho i}(z) = F_{ik}(z) \cdot \gamma_{\rho k}(z), \quad z \in U_i \cap U_k\]
\[(4.3.5) \quad [\lambda^0_i(w_i, z_i), w_i^0]|_{w_i = 0} = -[\Lambda_0, \gamma^0_{\rho i}(z_i)]|_{w_i = 0} + \sum_{\beta=1}^{r} \gamma^\beta_{\rho i} T^\beta_{i\alpha}(0, z_i) = 0, \quad z_i \in U_i, \quad \alpha = 1, ..., r.\]

We note that (4.3.2) implies \(\lambda^0 := \{\lambda^0_i(w_i, z_i)\} \in H^0(W, \wedge^2 T_W).\)

Let \(\epsilon\) be a small positive number. In order to prove Theorem 4.3.1, it suffices to construct vector-valued holomorphic functions \(\varphi_i(z_i, t) = (\varphi^1_i(z_i, t), ..., \varphi^m_i(z_i, t))\) in \(z_i\) and \(t\) with \(|z_i| < 1, |t| < \epsilon\) with \(|\varphi_i(z_i, t)| < 1,\) and \(\Lambda(t)\) which is a convergent power series in \(t\) with coefficients in \(H^0(W, \wedge^2 T_W)\) satisfying the boundary condition

\[
\varphi_i(z_i, 0) = 0, \\
\frac{\partial \varphi_i(z_i, t)}{\partial t^\rho}|_{t=0} = \gamma_{\rho i}(z) \\
\Lambda(0) = \Lambda_0 \\
\frac{\partial \Lambda(t)}{\partial t^\rho}|_{t=0} = \lambda^0
\]

such that

\[(4.3.6) \quad \varphi_i(g_{ik}(\varphi_k(z_k, t), z_k), t) = f_{ik}(\varphi(z_k, t), z_k), \quad (\varphi_k(z_k, t), z_k) \in W_k \cap W_i,\]
\[(4.3.7) \quad [\Lambda(t), w_i^0 - \varphi^0_i(z_i, t)]|_{w_i = \varphi_i(z_i, t)} = 0, \quad \alpha = 1, ..., r.\]
\[(4.3.8) \quad [\Lambda(t), \Lambda(t)] = 0\]

Recall Notation \(\ref{notation}.\) Then the equalities (4.3.6), (4.3.7), and (4.3.8) are equivalent to the system of congruences

\[(4.3.9) \quad \varphi_i^m(g_{ik}(\varphi_k^m(z_k, t), z_k), t) \equiv_m f_{ik}(\varphi_k^m(z_k, t), z_k), \quad m = 1, 2, 3, \cdots;\]
\[(4.3.10) \quad [\Lambda^m(t), w_i^0 - \varphi^0_i(z_i, t)]|_{w_i = \varphi_i(z_i, t)} \equiv_m 0, \quad m = 1, 2, 3, \cdots, \alpha = 1, ..., r,\]
\[(4.3.11) \quad [\Lambda^m(t), \Lambda^m(t)] \equiv_m 0, \quad m = 1, 2, 3, \cdots.\]

We will construct the formal power series \(\varphi_i^m(z_i, t)\) and \(\Lambda^m(t)\) satisfying (4.3.9) \(m,\) (4.3.11) \(m,\) and (4.3.11) \(m\) by induction on \(m.\)

We define \(\varphi_{i1}^m(z_i, t) = \sum_{\rho=1}^{l} t^\rho \gamma_{\rho i}^m(z_i),\) and \(\Lambda_1(t) = \sum_{\rho=1}^{l} t^\rho \lambda^0.\) Then from (4.3.11), \(\varphi_{i1}(z_i, t)\) satisfies (4.3.9) \(1.\) On the other hand, from (4.3.3), \([\Lambda_1(t), w_i^0]|_{w_i = 0} = [\Lambda_0, \varphi_{i1}^m(z_i, t)]|_{w_i = 0} + \sum_{\beta=1}^{r} \varphi_{i1}^\beta(z_i, t) T^\beta_{i\alpha}(0, z_i) = 0.\) Then we get, from (4.3.3),

\[
[\Lambda_0 + \Lambda_1(t), w_i^0 - \varphi_{i1}^m(z_i, t)] = \sum_{\beta=1}^{r} (w_i^\beta - \varphi_{i1}^\beta(z_i, t)) T^\beta_{i\alpha}(w_i, z_i) - [\Lambda_1(t), \varphi_{i1}^m(z_i, t)] + \sum_{\beta=1}^{r} w_i^\beta P^\beta_{i\alpha}(w_i, z_i, t)
\]

for some \(P^\beta_{i\alpha}(w_i, z_i, t)\) which are homogeneous polynomials of degree 1 in \(t_1, ..., t_l\) with coefficients in \(\Gamma(U_i, T_W)\) so that we obtain \(\Lambda^1(t), w_i^0 - \varphi_{i1}^m(z_i, t)]|_{w_i = \varphi_{i1}^m(z_i, t)} \equiv 1,\) which implies (4.3.10) \(1.\) Lastly from (4.3.3), we have \(-[\Lambda_1(t), \Lambda_0] = 0\) so that \([\Lambda_0 + \Lambda_1(t), \Lambda_0 + \Lambda_1(t)] \equiv 1,\) which implies (4.3.11) \(1.\) Hence the induction holds for \(m = 1.\)

Now we assume that we have already constructed \(\varphi_i^m(z_i, t) = (\varphi_i^1(z_i, t), \cdots, \varphi_i^m(z_i, t), \cdots, \varphi_i^m(z_i, t))\) satisfying (4.3.9) \(m,\) (4.3.10) \(m\) and (4.3.11) \(m\) such that for \(\alpha = 1, ..., r,\)

\[(4.3.12) \quad [\Lambda^m(t), w_i^0 - \varphi_{i1}^m(z_i, t)] = \sum_{\beta=1}^{r} (w_i^\beta - \varphi_{i1}^\beta(z_i, t)) T^\beta_{i\alpha}(w_i, z_i) + Q^m_{i\alpha}(z_i, t) + \sum_{\beta=1}^{r} w_i^\beta P^\beta_{i\alpha}(w_i, z_i, t)
\]
such that the degree of $P_{\alpha \beta}^{\xi m}(w_i, z_i, t)$ is at least 1 in $t_1, ..., t_i$. We note that we can rewrite (4.3.12) in the following way.

$$\sum_{\beta=1}^{r}(w_i^{- \phi_i^{\beta m}(z_i, t)}\lambda_{\alpha \beta}^{\beta m}(w_i, z_i) + \sum_{\beta=1}^{r}\phi_i^{\beta m}(z_i, t)P_{\alpha \beta}^{\xi m}(\varphi_i^{\beta m}(z_i, t), z_i, t) + \sum_{\beta=1}^{r}(w_i^{- \phi_i^{\beta m}(z_i, t)})L_{\alpha \beta}^{\beta m}(w_i, z_i, t)$$

such that the degree of $L_{\alpha \beta}^{\beta m}(w_i, z_i, t)$ in $t_1, ..., t_i$ is at least 1 so that (4.3.12) becomes the following form:

(4.3.13)

$$[\Lambda^m(t), w_i^{- \varphi_i^{\alpha m}(z_i, t)}] = \sum_{\beta=1}^{r}(w_i^{- \phi_i^{\beta m}(z_i, t)})\lambda_{\alpha \beta}^{\beta m}(w_i, z_i) + K_i^{\alpha m}(z_i, t) + \sum_{\beta=1}^{r}(w_i^{- \phi_i^{\beta m}(z_i, t)})L_{\alpha \beta}^{\beta m}(w_i, z_i, t)$$

where $K_i^{\alpha m}(z_i, t) := Q_i^{\alpha m}(z_i, t) + \sum_{\beta=1}^{r}\phi_i^{\beta m}(z_i, t)P_{\alpha \beta}^{\xi m}(\varphi_i^{\beta m}(z_i, t), z_i, t)$.

We set

(4.3.14)

$$\psi_{ik}(z_k, t) := [\phi_i^{\xi m}(g_{ik}(\varphi_i^{\xi m}(z_k, t), t)) - f_{ik}(\varphi_i^{\xi m}(z_k, t), z_k)]_{m+1}$$

(4.3.15)

$$G_i^\alpha(z_i, t) := [\Lambda^m(t), w_i^{- \varphi_i^{\alpha m}(z_i, t)}]_{w_i=\varphi_i^{\alpha m}(z_i, t)}^{m+1}, \quad \alpha = 1, ..., r.$$ 

(4.3.16)

$$\Pi(t) := ([\Lambda^m(t), \Lambda^m(t)])_{m+1}$$

We claim that $((0), \{\psi^\alpha_{ik}(z_k, t), ..., \psi^\alpha_{ik}(z_k, t)\}) + (-\frac{1}{2}\Pi(t), \{(G_i^\alpha(z_i, t), ..., G_i^\alpha(z_i, t))\})$ defines a 1-cocycle in the following Čech resolution of $(\wedge^2 T_W \oplus i_* N_{V/W})^\bullet$

$$C^0(U, \wedge^3 T_W \oplus i_*(N_{W/V} \otimes \wedge^2 T_W|_V))$$

\[\Phi\] $$C^0(U, \wedge^3 T_W \oplus i_*(N_{W/V} \otimes T_W|_V)) \xrightarrow{\delta} C^1(U, \wedge^4 T_W \oplus i_*(N_{W/V} \otimes T_W|_V))$$

\[\Phi\] $$C^0(U, \wedge^2 T_W \oplus i_* N_{W/V}) \xrightarrow{\delta} C^1(U, \wedge^2 T_W \oplus i_*(N_{W/V}) \xrightarrow{\delta} C^2(U, \wedge^4 T_W \oplus i_* N_{W/V})$$

By defining $\psi_{ik}(z, t) = \psi_{ik}(z_k, t)$ for $(0, z_k) \in U_k \cap U_\ell$, we have the equality (see Kodaira p.152-153)

(4.3.17)

$$\psi_{ik}(z, t) = \psi_{ij}(z, t) + F_{ijk} \psi_{jk}(z, t), \quad \text{for } z \in U_i \cap U_j \cap U_k$$

On the other hand, by applying $[\Lambda^m(t), -]$ on (4.3.13), we have

(4.3.18)

$$\frac{1}{2}[\Lambda^m(t), \Lambda^m(t), w_i^{\alpha m} - \varphi_i^{\alpha m}(z_i, t)] = \sum_{\beta=1}^{r}[-[\Lambda^m(t), w_i^{\beta m} - \varphi_i^{\beta m}(z_i, t)] \wedge T_{\alpha \beta m}(w_i, z_i) + \sum_{\beta=1}^{r}(w_i^{\beta m} - \varphi_i^{\beta m}(z_i, t))\Lambda^m(t), T_{\alpha \beta}^{\beta m}(w_i, z_i)]$$

$$+ [\Lambda^m(t), K_i^{\alpha m}(z_i, t)] + \sum_{\beta=1}^{r}[-[\Lambda^m(t), w_i^{\beta m} - \varphi_i^{\beta m}(z_i, t)] \wedge L_{\alpha \beta}^{\beta m}(w_i, z_i, t) + \sum_{\beta=1}^{r}(w_i^{\beta m} - \varphi_i^{\beta m}(z_i, t))\Lambda^m(t), L_{\alpha \beta}^{\beta m}(w_i, z_i, t)]$$

By restricting (4.3.18) to $w_i = \varphi_i^{\alpha m}(z_i, t)$, since $G_i^{\alpha m}(z_i, t) \equiv_m 0, \alpha = 1, ..., r, \Pi(t) \equiv_m 0$, the degree $L_{\alpha \beta}^{\beta m}(w_i, z_i, t)$ is at least 1 in $t_1, ..., t_i$ and we have, from (4.3.13),

(4.3.19)

$$G_i^{\alpha m}(z_i, t) \equiv_m K_i^{\alpha m}(z_i, t) = Q_i^{\alpha m}(z_i, t) + \sum_{\beta=1}^{r}\phi_i^{\beta m}(z_i, t)P_{\alpha \beta}^{\beta m}(\varphi_i^{\beta m}(z_i, t), z_i, t),$$

we obtain

(4.3.20)

$$[\frac{1}{2}\Pi(t), w_i^{\alpha m}]_{w_i=0} = \sum_{\beta=1}^{r}[-G_i^{\alpha m}(z_i, t) \wedge T_{\alpha \beta}^{\beta m}(0, z_i) + [\Lambda_0, G_i^{\alpha m}(z_i, t)]}_{w_i=0}$$

Next, since $f_{ik}^{\alpha m}(w_k, z_k) - \varphi_i^{\alpha m}(g_{ik}(w_k, z_k), t) - (f_{ik}^{\alpha m}(\varphi_i^{\alpha m}(z_k, t), z_k) - \varphi_i^{\alpha m}(g_{ik}(\varphi_i^{\alpha m}(z_k, t), z_k), t) = \sum_{\beta=1}^{r}(w_k^{\beta m} - \varphi_k^{\beta m}(z_k, t)) \cdot S_{k\alpha}^{\beta m}(w_k, z_k, t)$ for some $S_{k\alpha}^{\beta}(w_k, z_k, t)$. By setting $t = 0$, we get $f_{ik}^{\alpha m}(w_k, z_k) - f_{ik}^{\alpha m}(0, z_k) =$
Hence from (4.3.22) to \( w_k = \varphi_k^m(z_k, t) \), and then by taking the derivative with respect to \( w_k \) and setting \( w_k = 0 \), we obtain
\[
\frac{\partial f^\alpha_k(w_k, z_k)}{\partial w_k} \bigg|_{w_k=0} = S^\alpha_k(0, z_k, 0).
\]
Then we have
\[
[A^m(t), f^\alpha_k(w_k, z_k)] = [\varphi_k^m(g_k(w_k, z_k), t)] \bigg|_{w_k=\varphi_k^m(z_k, t)} + [A_0, \psi_k^m(z_k, t)] \bigg|_{w_k=0} = \sum_{\beta=1}^r G^\beta_k(z_k, t) \cdot \frac{\partial f^\alpha_k(w_k, z_k)}{\partial w_k} \bigg|_{w_k=0}
\]
Hence we obtain the equality
\[
(4.3.21)
\]
\[
[A^m(t), f^\alpha_k(w_k, z_k) - \varphi_k^m(g_k(w_k, z_k), t)] \bigg|_{w_k=\varphi_k^m(z_k, t)} + [A_0, \psi_k^m(z_k, t)] \bigg|_{w_k=0} = \sum_{\beta=1}^r G^\beta_k(z_k, t) \cdot \frac{\partial f^\alpha_k(w_k, z_k)}{\partial w_k} \bigg|_{w_k=0}
\]
On the other hand, by hypothesis, the cohomology group
\[
H^\alpha(\Lambda \oplus \Lambda^*) = 0
\]
Lastly, \([\Lambda_0, \Pi(t)] \equiv_{m+1} [A_0, [A^m(t), A^m(t)]] \equiv_{m+1} [A^m(t), [A^m(t), A^m(t)]] = 0 \) so that we get
\[
(4.3.24)
\]
Hence from (4.3.17), (4.3.20), (4.3.21) and (4.3.24), we get
\[
\left\langle \{(0), (\psi^1_k(z_k, t), ..., \psi^r_k(z_k, t))\} \right\rangle \oplus \left\langle -\frac{1}{2} \Pi(t), \{G^\gamma_1(z_1, t), ..., G^\gamma_r(z_r, t)\} \right\rangle
\]
defines a 1-cocycle in the above complex so that we get the claim. We call \([\psi_{m+1}(t) := \{\psi^1_k(z_k, t), ..., \psi^r_k(z_k, t)\}, G_{m+1}(t) := \{(G^1_1(z_1, t), ..., G^1_r(z_r, t)), \Pi_{m+1}(t) := -\frac{1}{2} \Pi(t)\) the m-th obstruction so that the coefficients of \((0, \psi_{m+1}(t)) \oplus (\frac{1}{2} \Pi_{m+1}(t), G_{m+1}(t))\) in \( t_1, ..., t_r \) falls in \( \mathbb{H}^1(W, \wedge^m T_W \oplus \iota^* N_W/W) \).
Then we can show \(4.3.9\) \(m+1\) (for the detail, see [Kod62] p.154). On the other hand, 
\[
(4.3.25)
\]
\[
[A_{m+1}(t), w_i^\alpha] - [\Lambda_0, \varphi_{i|m+1}^\alpha(z_i, t)] = - \sum_{\beta=1}^{r} \varphi_{i|m+1}^\beta(z_i, t) T_{ia}^\beta (w_i, z_i) - G_i^\alpha(z_i, t) + \sum_{\beta=1}^{r} w_i^\beta R_{ia}^\beta (w_i, z_i, t)
\]
where the degree of \(R_{ia}^\beta(w_i, z_i, t)\) is \(m+1\) in \(t\). Let 
\[
[\Lambda^m(t) - \Lambda_0, -\varphi_{i|m+1}^\alpha(z_i, t)] + [\Lambda_{m+1}(t), -\varphi_{i|m+1}^\alpha(z_i, t) - \varphi_{i|m+1}^\alpha(z_i, t)] = H_i^\alpha(z_i, t) + \sum_{\beta=1}^{r} w_i^\beta M_{ia}^\beta (w_i, z_i, t),
\]
where the degree of \(H_i^\alpha(z_i, t)\) and \(M_{ia}^\beta (w_i, z_i, t)\) is at least \(m+2\). Then from \(4.3.12\) and \(4.3.26\), we have
\[
(4.3.26)
\]
\[
[A^m(t) + \Lambda_{m+1}(t), w_i^\alpha - \varphi_{i|m+1}^\alpha(z_i, t) - \varphi_{i|m+1}^\alpha(z_i, t)]
\]
\[
= [A^m(t), w_i^\alpha - \varphi_{i}^\alpha(z_i, t)] + [\Lambda^m(t) - \Lambda_0, -\varphi_{i|m+1}^\alpha(z_i, t)] + [\Lambda_0, -\varphi_{i|m+1}^\alpha(z_i, t)]
\]
\[
= \sum_{\beta=1}^{r} (w_i^\beta - \varphi_{i}^\beta(z_i, t)) T_{ia}^\beta (w_i, z_i) + Q_i^\alpha(z_i, t) - G_i^\alpha(z_i, t) + H_i^\alpha(z_i, t)
\]
\[
+ \sum_{\beta=1}^{r} w_i^\beta (P_{ia}^\beta (w_i, z_i, t) + R_{ia}^\beta (w_i, z_i, t) + M_{ia}^\beta (w_i, z_i, t))
\]
We show that \(A^m(t), w_i^\alpha - \varphi_{i|m+1}^\alpha(z_i, t)\) to \(w_i = \varphi_{i|m+1}^\alpha(z_i, t)\), we get from \(4.3.19\).
\[
[A_{m+1}^m(t), w_i^\alpha - \varphi_{i|m+1}^\alpha(z_i, t)]_{w_i = \varphi_{i|m+1}^\alpha(z_i, t)} \equiv m+1 0.
\]
Indeed, by restricting \(4.3.26\) to \(w_i = \varphi_{i|m+1}^\alpha(z_i, t)\), we obtain
\[
[A_{m+1}^m(t), w_i^\alpha - \varphi_{i|m+1}^\alpha(z_i, t)]_{w_i = \varphi_{i|m+1}^\alpha(z_i, t)} \equiv m+1 0.
\]
Lastly \(A^m(t) + A_{m+1}(t), A^m(t) + A_{m+1}(t) \equiv m+1 [A^m(t), A^m(t)] + 2[\Lambda_0, \Lambda_{m+1}(t)] \equiv m+1 [A^m(t), A^m(t)] - \Pi(t) \equiv m+1 0\) which shows \(4.3.10\) \(m+1\). This completes the inductive constructions of \(\varphi_{i}^\alpha(z_i, t), i \in I\), and \(A^m(t)\).

4.4. Proof of convergence.

We will show that we can choose \(\varphi_{i|m}^\alpha(z_i, t)\) and \(A^m(t)\) in each inductive step so that the formal power series \(\varphi_{i}^\alpha(z_i, t), i \in I\) and \(A^m(t)\) constructed in the previous subsection, converges absolutely for \(|t| < \epsilon\) for a sufficiently small number \(\epsilon > 0\).

We keep the notations in subsection \(3.4\) For instance, \(A_0 = A_i(w_i, z_i) = \sum_{p,q=1}^{r+d} A_{pq}^i(w_i, z_i) \frac{\partial}{\partial x_p^i} \wedge \frac{\partial}{\partial x_q^i}\) with \(A_{pq}^i(w_i, z_i) = -A_{pq}^i(w_i, z_i)\), where \(x_i = (w_i, z_i)\) on \(W_i\), and \(W_i^\delta\) is the subdomain of \(W_i\) consisting of all points \((w_i, z_i)\), \(|w_i| < 1 - \delta, |z_i| < 1 - \delta\) for a sufficiently small number \(\delta > 0\) such that \(\{W_i^\delta|i \in I\}\) forms a covering of \(W\), and \(\{U_i^\delta = W_i^\delta \cap V| i \in I\}\) forms a covering of \(V\). Recall Notation 3. We denote \(A^m(t)\) on \(W_i\) by \(\Lambda^m(w_i, z_i, t)\) and \(\Pi^m(t)\) on \(W_i\) by \(\Pi^m(w_i, z_i, t)\). Then \(\Lambda^m(w_i, z_i, t)\) is of the form
\[
(4.4.1)
\]
\[
\Lambda_i^m(w_i, z_i, t) = \sum_{p,q=1}^{r+d} \Lambda_{pq}^m(w_i, z_i) \frac{\partial}{\partial x_p^i} \wedge \frac{\partial}{\partial x_q^i}, \quad \Lambda_{pq}^m(w_i, z_i, t) = -\Lambda_{pq}^m(w_i, z_i, t)
\]
such that \(\Lambda_{pq}^m(w_i, z_i, t) = \Lambda_{pq}^m(w_i, z_i)\). We may assume that \(|\Phi_{ik}(0, z)| < c_0\) with \(c_0 > 1\). Then \(\varphi_{i|m}(z_i, t) \ll A(t)\) and \(\Lambda_{i|m}(w_i, z_i, t) \ll A(t)\) if \(b\) is sufficiently large. Now, assuming the inequalities
\[
(4.4.2)
\]
\[
\varphi_{i}^m(z_i, t) \ll A(t), \quad \Lambda_i^m(w_i, z_i, t) - \Lambda_i(w_i, z_i) \ll A(t), \quad (w_i, z_i) \in W_i^\delta
\]
for an integer \(m \geq 1\), we will estimate the coefficients of the homogenous polynomials \(\psi_{ik}(z, t), \Pi_i(w_i, z_i, t),\) and \(G_i^\alpha(z_i, t)\) from \(4.3.14\), \(4.3.15\), and \(4.3.16\).

First we estimate \(\psi_{ik}(z, t)\). As in \(4.4.3\), we have
\[
(4.4.3)
\]
\[
\psi_{ik}(z, t) \ll c_3 A(t), \quad z_k \in U_k \cap U_i,
\]
where \( c_3 = 2r c_0 \frac{4c_1 r a}{\delta} \left( \frac{2^d}{\delta} + r c_1 \right) \)

\[ b > \max\{2c_1 r a, \frac{4c_1 r a}{\delta}\} \]

Next we estimate \( G_m^\alpha(z_i, t) \). We note that

\[ G_m^\alpha(z_i, t) \equiv m + 1 \left[ \Lambda_i^m(w_i, z_i, t) - \Lambda_i^m(w_i, z_i, t) \right] \big|_{w_i = \varphi^m_i(z_i, t)} \]

\[ = m + 1 \left[ \Lambda_i^m(w_i, z_i, t) - \Lambda_i^m(w_i, z_i, t) \right] \big|_{w_i = \varphi^m_i(z_i, t)} - [\Lambda_i^m(w_i, z_i, t) - \Lambda_i^m(w_i, z_i, t), \varphi^m_i(z_i, t)] \big|_{w_i = \varphi^m_i(z_i, t)} \]

\[ + \left[ \Lambda_i^m(w_i, z_i, t) - \Lambda_i^m(w_i, z_i, t) \right] \big|_{w_i = \varphi^m_i(z_i, t)} \]

We estimate each term in (4.4.5). First we estimate \( [\Lambda_i^m(w_i, z_i, t) - \Lambda_i^m(w_i, z_i, t)] \big|_{w_i = \varphi^m_i(z_i, t)} \) in (4.4.6). We note that

\[ \sum_{p,q=1}^{d+r} \left( \Lambda_{pq}^m(w_i, z_i, t) - \Lambda_{pq}^m(w_i, z_i, t) \right) \frac{\partial}{\partial x_i} \Lambda_{pq}^m(w_i, z_i, t) \big|_{w_i = \varphi^m_i(z_i, t)} \]

On the other hand, \( \Phi_{pq}^m(w_i, z_i, t) := \Lambda_{pq}^m(w_i, z_i, t) - \Lambda_{pq}^m(w_i, z_i, t) \ll A(t) \) from (4.4.2) and has the degree \( m \) in \( t \). For any \( p, q \), we expand \( \Phi_{pq}^m(w_i, z_i, t) \) into power series in \( w_i^1, \ldots, w_i^r \) whose coefficients are holomorphic functions of \( z = (0, z_i) \) defined on \( U \):

\[ \Phi_{pq}^m(w_i, z_i, t) = \sum_{\mu_1, \ldots, \mu_r \geq 0}^{d+r} \Phi_{pq, \mu_1, \ldots, \mu_r}^m(z_i, t) w_i^1 \mu_1 \cdots w_i^r \mu_r \]

If \( (w_i, z_i) \in W^\delta_i \), we have, by Cauchy’s integral formula,

\[ \Phi_{pq, \mu_1, \ldots, \mu_r}^m(z_i, t) = \frac{1}{2\pi i} \int_{|\xi - w_i^1| = \delta} \cdots \int_{|\xi - w_i^r| = \delta} \frac{\Phi_{pq}^m(w_i, z_i, t)}{\prod_{1 \leq i < j \leq r} (\xi_i - w_i_j)^{\mu_i + 1}} d\xi_1 \cdots d\xi_r \]

so that we get

\[ \Phi_{pq}^m(z_i, t) \ll A(t) \frac{1}{\delta^{\mu_1 + \cdots + \mu_r}} \]

Since constant terms of \( \Phi_{pq}^m(w_i, z_i, t) \) with respect to \( w_i^1, \ldots, w_i^r \) does not contribute to \( [\Lambda_i^m(w_i, z_i, t) - \Lambda_i^m(w_i, z_i, t)] \big|_{w_i = \varphi^m_i(z_i, t)} \), from (4.4.7) and (4.4.8), we have, assuming \( \frac{\delta}{\delta^i} < \frac{1}{2} \), (for the detail, see Kod[05] p.300)

\[ \Phi_{pq}^m(\varphi^m_i(z_i, t), t) \ll A(t) \sum_{\mu_1 + \cdots + \mu_r \geq 1} \left( \frac{A(t)}{\delta} \right)^{\mu_1 + \cdots + \mu_r} \ll 2^{r+1} A(t)^2 \ll \frac{2^{r+1} A(t)}{b^d} A(t) \]

Then from (4.4.6) and (4.4.9), we obtain

\[ [\Lambda_i^m(w_i, z_i, t) - \Lambda_i^m(w_i, z_i, t)] \big|_{w_i = \varphi^m_i(z_i, t)} \ll 2(d + r)^2 \frac{2^{r+1} A(t)}{b^d} A(t) \]

Next we estimate \( [\Lambda_i^m(w_i, z_i, t) - \Lambda_i^m(w_i, z_i, t)] \big|_{w_i = \varphi^m_i(z_i, t)} \) in (4.4.5). From (4.4.2) and (3.4.12), we have

\[ \sum_{p,q=1}^{d+r} \left( \Lambda_{pq}^m(w_i, z_i, t) - \Lambda_{pq}^m(w_i, z_i, t) \right) \frac{\partial}{\partial x_i} \Lambda_{pq}^m(w_i, z_i, t) \big|_{w_i = \varphi^m_i(z_i, t)} \]

\[ = \sum_{p,q=1}^{d+r} 2(\Lambda_{pq}^m(\varphi^m_i(z_i, t), t) - \Lambda_{pq}^m(\varphi^m_i(z_i, t), t)) \frac{\partial \varphi^m_i(z_i, t)}{\partial x_i} \big|_{w_i = \varphi^m_i(z_i, t)} \ll 2(d + r)^2 A(t) \frac{A(t)}{\delta} \ll 2(d + r)^2 \frac{a}{b^d} A(t) \]

Hence we get

\[ [\Lambda_i^m(w_i, z_i, t) - \Lambda_i^m(w_i, z_i, t)] \big|_{w_i = \varphi^m_i(z_i, t)} \ll 2(d + r)^2 \frac{a}{b^d} A(t) \]
We estimate $[\Lambda_i(w_i, z_i), w_i^{m} - \varphi_i^{m}(z_i, t)]_{w_i=\varphi_i^{m}(z_i, t)}$ in (4.4.5). This comes from (3.4.15):

(4.4.13) \[ [\Lambda_i(w_i, z_i), w_i^{m} - \varphi_i^{m}(z_i, t)]_{w_i=\varphi_i^{m}(z_i, t)}]_{m+1} \ll e_2 A(t), \quad z \in U_i^\delta \]

where $e_2 = \frac{4(d+r)^2}{b_0^2} + \frac{4(r+d)\epsilon_1 r a}$ with $b > 2e_1 r a$.

Hence from (4.4.10), (4.4.12), (4.4.13), we get

(4.4.14) \[ G_i^a(z_i, t) = [[\Lambda_i^m(w_i, z_i, t), w_i^{o} - \varphi_i^{m}(z_i, t)]_{w_i=\varphi_i^{m}(z_i, t)}]_{m+1} \ll e_3 A(t) \]

where $e_3 = \frac{(d+r)^2 r^2 a}{b_0^2} + \frac{2(d+r)^2 a}{b_0} + e_2$ with

(4.4.15) \[ b > \max\left\{\frac{2a}{b}, 2e_1 r a\right\} \]

Next we estimate $\Pi_i(w_i, z_i, t) = [[\Lambda_i^m(w_i, z_i, t), \Lambda_i^m(w_i, z_i, t)]_{m+1}$. Since $\Pi_i(w_i, z_i, t) \equiv_m 0$ and

$\Pi_i(w_i, z_i, t) \equiv_m \begin{cases} \Lambda_i^m(w_i, z_i, t), & \text{if} \quad \Pi_i(w_i, z_i, t) \\ \Lambda_i^m(w_i, z_i, t) - \Lambda_i(w_i, z_i), & \text{otherwise} \end{cases}$

we have

(4.4.16) \[ \Pi_i(w_i, z_i, t) = [[\Lambda_i^m(w_i, z_i, t) - \Lambda_i(w_i, z_i), \Lambda_i^m(w_i, z_i, t) - \Lambda_i(w_i, z_i)]_{m+1} \]

We note the following two remarks.

**Remark 4.4.17.** Let $\sigma = \sum_{p, q} \sigma_{pq}^l \frac{\partial}{\partial x_p} \wedge \frac{\partial}{\partial \sigma_q}$ and $\phi = \sum_{l, k} \phi_{lk} \frac{\partial}{\partial x_l} \wedge \frac{\partial}{\partial \phi_k}$. Then $[\sigma, \phi] = \sum_{l, k} \phi_{lk} \frac{\partial}{\partial x_l} \wedge \frac{\partial}{\partial \phi_k} - \sum_{l, k} \phi_{lk} \frac{\partial}{\partial \phi_k} \wedge \frac{\partial}{\partial x_l}$.

**Remark 4.4.18.**

(4.4.19) \[ \frac{\partial}{\partial x_l} A(t) = \begin{cases} A(t), & \text{if} \quad \frac{\partial}{\partial x_l} A(t) \\ 0, & \text{otherwise} \end{cases} \]

By Remark 4.4.17, Remark 4.4.18 and (4.4.2), we have

(4.4.20) \[ \frac{1}{2} \Pi_i(w_i, z_i, t) \ll 4(d + r)^4 A(t) \cdot \frac{A(t)}{\delta} \ll \frac{4(d + r)^4 a}{\delta b} A(t) \]

Hence from (4.4.16) and (4.4.19), we obtain

(4.4.21) \[ \frac{1}{2} \Pi_i(w_i, z_i, t) \ll 4(d + r)^4 A(t) \cdot \frac{A(t)}{\delta} \ll \frac{2(d + r)^4 a}{\delta b} A(t) = e_4 A(t) \]

where $e_4 = \frac{2(d + r)^4 a}{\delta b}$.

**Notation 4.** We consider $P_k = \sum_{\alpha, \beta} P_{\alpha, \beta}^{k}(x_k) \frac{\partial}{\partial x_k} \wedge \frac{\partial}{\partial \sigma_k} \in \Gamma(W_i, \wedge^2 T_W)$ to be a vector-valued holomorphic function $P_k(x) = (P_{\alpha, \beta}^{k}(x_k))_{\alpha, \beta = 1, \ldots, d+r}$ on $W_k$. On $W_i \cap W_k$, $P_k$ is translated to $\sum_{\alpha, \beta, p, q} P_{\alpha, \beta}^{k}(x_k) \frac{\partial}{\partial x_k} \wedge \frac{\partial}{\partial \sigma_k}$ which corresponds to a vector-valued holomorphic function $\left(\sum_{\alpha, \beta, p, q} P_{\alpha, \beta}^{k}(x_k) \frac{\partial}{\partial x_k} \wedge \frac{\partial}{\partial \sigma_k}\right)_{p, q = 1, \ldots, d + r}$ on $W_i$ denoted by $H_k(x) P_k(x)$. 

We can choose the homogenous polynomials \( \varphi_{i|m+1}(z, t), \Lambda_{i|m+1}(w, z, t), i \in I \), satisfying
\[
\psi_{ik}(z, t) = F_{ik}(z) \varphi_{k|m+1}(z, t) - \varphi_{i|m+1}(z, t)
\]
\[
-G_i^\alpha(z, t) = [\Lambda_{i|m+1}(w, z, t), w_i^\alpha]|_{w_i=0} - [\varphi_{i|m+1}(z, t), \Lambda_0]|_{w_i=0} + \sum_{\beta=1}^r \varphi_{i|m+1}(z, t) T_i^\beta (0, z)
\]
\[
\frac{1}{2} \Pi_i(w_i, z_i, t) = -[\Lambda_{i|m+1}(w, z, t), \Lambda_0]
\]
\[
(\Theta_{ik}(w, z, t) = H_{ik}(w, z) \Lambda_{i|m+1}(w, z, t) - \Lambda_{i|m+1}(w, z, t) = 0)
\]
in such a way that \( \varphi_{i|m+1}(z, t) \ll c_4 (c_3 + c_3 + c_4) A(t) \) and \( \Lambda_{i|m+1}(w, z, t) \ll c_4 (c_3 + c_3 + c_4) A(t) \) for \( (w, z) \in W_i^\delta \), where \( c_4 \) is independent of \( m \).

**Proof.** For any 0-cochain \((\pi, \varphi) = (\{\pi_i\}, \{\varphi_i\})\), and 1-cochain \((\Theta, \psi) \oplus (\Pi, G) = (\{\Theta_{ik}(w, z)\}, \{\psi_{ik}(z)\}) \oplus (\{\Pi_i(w, z)\}, \{\sum_{\alpha=1}^r G_i^\alpha(z) e_i^\alpha\})\), we define the norms of \((\pi, \varphi)\) and \((\Theta, \psi) \oplus (\Pi, G)\) by
\[
|||((\pi, \varphi))|| := \max \sup_{i, k} \left|\pi_i(w, z)\right| + \max \sup_{i, k} \left|\varphi_i(z)\right|
\]
\[
|||((\Theta, \psi) \oplus (\Pi, G))|| := \max \sup_{i, k} \left|\Theta_{ik}(w, z)\right| + \max \sup_{i, k} \left|\Pi_i(w, z)\right| + \max \sup_{i, k} \left|\psi_{ik}(z)\right| + \max \sup_{i, k} \left|G_i^\alpha(z)\right|
\]

The coboundary of \((\pi, \varphi)\) is defined by
\[
\delta(\pi, \varphi) := \{-H_{ik}(x) \pi_i(x) + \pi_i(x), \{-F_{ik}(z) \varphi_k(z) + \varphi_i(z)\}
\]
\[
+ \{\{-[\pi_i(x), \Lambda_0]\}, \{\sup_{\alpha=1}^r \left|\pi_i(x), w_i^\alpha\right| - \left|\varphi_i^\alpha(z), \Lambda_0\right| + \sum_{\beta=1}^r \sup_{\alpha=1}^r G_i^{\beta \alpha}(z) T_i^{\beta \alpha} (0, z) + e_i^\alpha\} \}
\]

For any coboundary of the form \((0, \psi) \oplus (\Pi, G)\), we define
\[
\iota((0, \psi) \oplus (\Pi, G)) = \inf_{\delta(\pi, \varphi) = (0, \psi) \oplus (\Pi, G)} |||((\pi, \varphi))||
\]

To prove Lemma 4.4.21 it suffices to show the existence of a constant \( c \) such that \( \iota((0, \psi) \oplus (\Pi, G)) \leq c ||((0, \psi) \oplus (\Pi, G))|| \). Assume that such a constant does not exist. Then we can find a sequence \((0, \psi^\mu) \oplus (\Pi^\mu, G^\mu)\) such that
\[
\iota((0, \psi^\mu) \oplus (\Pi^\mu, G^\mu)) = 1, \quad ||((0, \psi^\mu) \oplus (\Pi^\mu, G^\mu))|| < \frac{1}{\mu}
\]
Then there exists \((\pi^\mu, \varphi^\mu)\) with \(\delta(\pi^\mu, \varphi^\mu) = (0, \psi^\mu) \oplus (\Pi^\mu, G^\mu)\) satisfying \(||\pi^\mu|| < 2\) and \(||\varphi^\mu|| < 2\). We note that \(\pi^\mu\) is a global bivector field in \(H^0(W, \Lambda^2 T_W)\). We take a covering \(\{W_i^\delta\}\) of \(W\) and a covering \(\{U_i^\delta\} \cap U_i^\delta\) of \(V\). Since \(\sup_{x} (|\varphi^\mu_k(x)| < 2\) for \(x \in W_k\) and \(|\phi^\mu_k(z)| < 2\) for \(z \in U_k = W_k \cap V\), there exists a subsequence \((\pi^\mu_{i_1}, \varphi^\mu_{i_1}), (\pi^\mu_{i_2}, \varphi^\mu_{i_2}), \ldots, (\pi^\mu_{i_n}, \varphi^\mu_{i_n}), \ldots\) of \((\pi^\mu, \varphi^\mu)\) such that \(\pi^\mu_{i_k}\) converges absolutely and uniformly on \(W_k^\delta\) for each \(k\) and \(\varphi^\mu_{i_k}\) converges absolutely and uniformly on \(U_k^\delta\). Since \(W\) is compact, we can choose a subsequence that works for all \(k\). On the other hand, since \(||((0, \psi^\mu) \oplus (\Pi^\mu, G^\mu))|| < \frac{1}{\mu}\), we have
\[
(4.4.22) \quad H_{ik}(x) \pi^\mu_{i_k}(x) = \pi^\mu_{i_k}(x), \quad x \in W_i \cap W_k, \quad F_{ik}(z) \varphi^\mu_{i_k}(z) - \varphi^\mu_{i_k}(z) < \frac{1}{\mu}, \quad z \in U_i \cap U_k
\]
\[
\left|\sup_{\alpha=1}^r \left|\pi^\mu_{i_k}(x), w_i^\alpha\right| - \left|\varphi^\mu_{i_k}(x), \Lambda_0\right| + \sum_{\beta=1}^r \sup_{\alpha=1}^r G_i^{\beta \alpha}(z) T_i^{\beta \alpha} (0, z) \right| < \frac{1}{\mu}, \quad z \in U_i^\delta
\]
Let \(\pi_i(x) = \lim_{n \to \infty} \pi^\mu_{i_k}(x)\) and \(\varphi_k(z) = \lim_{n \to \infty} \varphi^\mu_{i_k}(z)\). Since \(\pi^\mu_{i_k}\) converges absolutely and uniformly on \(W_i^\delta\), \(\pi_i\) is holomorphic on \(W_i^\delta\). Since \(\{W_i^\delta\}\) covers \(W\), and \(\pi_i(x) = H_{ik}(x) \pi_k(x)\) for \(x \in W_i \cap W_k\), we get \(\{\pi_i(x)\} \in H^0(W, \Lambda^2 T_W)\). On the other hand, \(\varphi^\mu_{i_k}(z)\) converges absolutely and uniformly on \(U_i^\delta\). Let \(\pi := \{\pi_i(x)\} and \varphi := \{\varphi_k(z)\}. Then we have ||\(\pi^\mu_{i_k}(x) - \pi, \varphi^\mu_{i_k}(x) - \varphi|| \to 0 as n \to \infty. On the other hand, by \(4.4.22\), \(\delta(\pi, \varphi) = (0, 0) \oplus (\{-[\pi, \Lambda_0], \{G_i, \pi, \varphi\}\})\), where \(-[\pi, \Lambda_0](x) = 0 for x \in W_i^\delta\) (hence \(-[\pi, \Lambda_0] = 0\) and \(G_i, \pi, \varphi(z) = 0 for z \in U_i^\delta\) (hence \(G_i, \pi, \varphi = 0 by identity theorem) so that \(\delta(\pi, \varphi) = (0, 0) \oplus (0, 0)\).
Hence we have \( \tilde{\delta}(\pi(\mu), \varphi(\mu)) = \tilde{\delta}(\pi(\mu) - \pi, \varphi(\mu) - \phi) = (0, \psi(\mu)) \oplus (\Pi(\mu), G(\mu)) \) which contradicts to \\
\( \iota((0, \psi(\mu)) \oplus (\Pi(\mu), G(\mu))) = 1. \)

From [4.4.3], [4.4.11] and [4.4.20], we have

\[
(4.4.23)
\]

\[
c_4(c_3 + c_3 + c_4) = c_4 c_3 + c_4 c_3 + c_4 c_4
\]

\[
= \frac{8c_3 c_0 c_1 r^2 a}{b} \left( \frac{2d}{\delta} + r c_1 \right) + c_4 \left( \frac{(d + r)^2 r^2 a}{\delta} + \frac{2(d + r)^2 a}{\delta} \right) + c_4 \left( \frac{4(r + d)c_1 r a}{\delta} \right) + c_4 \left( \frac{2(d + r)^4 a}{\delta} \right).
\]

From [4.4.3], [4.4.5], [4.4.23] and Lemma 4.4.21 by assuming

\[
b > 8c_4 c_0 c_1 r^2 a \left( \frac{2d}{\delta} + r c_1 \right) + c_4 \left( \frac{(d + r)^2 r^2 a}{\delta} + \frac{2(d + r)^2 a}{\delta} \right) + c_4 \left( \frac{4(r + d)c_1 r a}{\delta} \right) + c_4 \left( \frac{2(d + r)^4 a}{\delta} \right)
\]

\[
+ \frac{4c_1 r a}{\delta}, 2a, 2c_1 r a,
\]

we can choose \( \varphi_{i|m+1}(z_i, t) \ll A(t) \) and \( \Lambda_{i|m+1}(w_i, z_i, t) \ll A(t) \) so that \( \varphi_i(z_i, t) \ll A(t) \), and \( \Lambda_i(w_i, z_i, t) \ll A(t) \) so that \( \varphi_i(z_i, t) \ll A(t) \), and \( \Lambda_i(w_i, z_i, t) \ll A(t) \) for \( |t| < \frac{\epsilon}{10} \). Then we obtain the equality

\[
\varphi_i(g_k(\varphi_k(z_k, t), z_k), t) = f_k(\varphi_k(z_k, t), z_k), \quad \text{for } |t| < \epsilon, (\varphi_k(z_k, t), z_k) \in W_1^T \cap W_1^T \]

\[
\Lambda_i(w_i, z_i, t), w_i = \varphi_i(z_i, t) \]

\[
\Lambda_i(w_i, z_i, t), w_i = \varphi_i(z_i, t) \]

\[
\Lambda_i(w_i, z_i, t), w_i = \varphi_i(z_i, t) \]

for a sufficiently small number \( \epsilon > 0 \). This completes the proof of Theorem 4.3.1

In the case \( \mathbb{H}^1(W, (\wedge^2 T_W \oplus i_* N_{V/W})^*) \neq 0 \), our proof of Theorem 4.3.1 also proves the following:

**Theorem 4.4.24.** If the obstruction \( (0, \psi_m(t)) \oplus \frac{1}{2} \Pi_{m+1}(t), G_{m+1}(t) \) vanishes for each integer \( m \geq 1 \), then there exists an extended Poisson analytic family \( V \) of compact holomorphic Poisson submanifolds \( V_t, t \in M_1 \), of \( (W, \Lambda_t) \) such that \( V_0 = V \subset (W, \Lambda_0) \) and the characteristic map

\[
\sigma_0 : T_0(M_1) \to \mathbb{H}^0(W, (\wedge^2 T_W \oplus i_* N_{V/W})^*)
\]

\[
\frac{\partial}{\partial t} \mapsto \left( \frac{\partial(\Lambda_t, V_t)}{\partial t} \right)_{t=0}
\]

is an isomorphism.

**4.5. Maximal families: Theorem of completeness.**

**Definition 4.5.1.** Let \( V \subset (W \times M, \Lambda) \overset{w}{\rightarrow} M \) be an extended Poisson analytic family of compact holomorphic Poisson submanifolds of \( W \) so that \( \omega^{-1}(t) = V_t \) is a compact holomorphic Poisson submanifold of \( (W, \Lambda_t), t \in M \) and let \( t_0 \) be a point on \( M \). We say that \( V \overset{w}{\rightarrow} M \) is maximal at \( t_0 \) if, for any extended Poisson analytic family \( V' \subset (W \times M', \Lambda') \overset{w'}{\rightarrow} M' \) of compact holomorphic Poisson submanifolds of \( W \) such that \( \Lambda_t = \Lambda_t' \) and \( \omega^{-1}(t_0) = \omega'^{-1}(t_0') \) \( t_0', t_0 \) \( M' \), there exists a holomorphic map \( \hat{h} \) of a neighborhood \( N' \) of \( t_0' \) in \( M' \) into \( M \) which maps \( t_0' \) to \( t_0 \) such that \( \omega'^{-1}(t') = \omega^{-1}(h(t')) \) and \( \Lambda_{t'} = \Lambda_{h(t')} \) for \( t' \in N' \). We note that if we set a holomorphic map \( \tilde{h} : W \times N' \to W \times M \) defined by \( (w, t') \to (w, h(t')) \), then \( \tilde{h} \) is a Poisson map \( (W \times N', \Lambda') \to (W \times M, \Lambda) \) and the restriction map of \( \tilde{h} \) to \( V'|_{N'} = \omega'^{-1}(N') \subset (W \times N', \Lambda') \) defines a Poisson map \( V'|_{N'} \to V \) so that \( V'|_{N'} \) is the family induced from \( V \) by \( \tilde{h} \), which means \( V \overset{w}{\rightarrow} M \) is complete at \( t_0 \).

**Theorem 4.5.2** (theorem of completeness). Let \( V \subset (W \times M, \Lambda) \) be an extended Poisson analytic family of compact holomorphic Poisson submanifolds \( V_t \) of \( (W, \Lambda_t) \). If the characteristic map

\[
\rho_0 : T_0(M_1) \to \mathbb{H}^0(W, (\wedge^2 T_W \oplus i_* N_{V/W})^*)
\]

\[
\frac{\partial}{\partial t} \mapsto \left( \frac{\partial(\Lambda_t, V_t)}{\partial t} \right)_{t=0}
\]

is bijective, then the family \( V \) is maximal at \( t = 0 \).
Proof. Consider an arbitrary extended Poisson analytic family $\mathcal{V}' \subset (W \times M, \Lambda')$ of compact holomorphic Poisson submanifolds $V'_s, s \in M'$ of $(W, \Lambda'_s)$, where $M' = \{ s = (s_1, ..., s_q) \in \mathbb{C}^q \mid |s| < 1 \}$. We will construct a holomorphic map $h : s \mapsto t = h(s)$ of a neighborhood $N'$ of 0 in $M'$ into $M_1$ with $h(0) = 0$, $V'_s = V_{h(s)}$ and $\Lambda'_s = \Lambda_{h(s)}$.

We keep the notations in subsection 4.2 so that the holomorphic Poisson submanifold $V_i$ of $(W, \Lambda_t)$ is defined on each domain $W_i, i \in I$ by the equation $w_i = \varphi_i(z_i, t)$ and satisfy

\begin{equation}
[\Lambda_i(w_i, z_i, t), w_i^\alpha - \varphi_i^\alpha(z_i, t)] = \sum_{\beta=1}^{r} (w_i^\beta - \varphi_i^\beta(z_i, t)) T_{i\alpha}^\beta(w_i, z_i, t)
\end{equation}

for some $P_{i\alpha}^\beta(w_i, z_i, s)$ which are power series in $s$ with coefficients in $\Gamma(W, T_W)$ and $P_{i\alpha}^\beta(0, z_i, 0) = T_{i\alpha}^\beta(0, z_i)$. Then $V'_s = V_{h(s)}$ and $\Lambda'_s = \Lambda_{h(s)}$ are equivalent to the simultaneous equations

\begin{equation}
[\Lambda'_i(w_i, z_i, s), w_i^\alpha - \theta_i^\alpha(z_i, s)] = \sum_{\beta=1}^{r} (w_i^\beta - \theta_i^\beta(z_i, s)) P_{i\alpha}^\beta(w_i, z_i, s)
\end{equation}

Recall Notation 1 and let us write $h(s) = h_1(s) + h_2(s) + \cdots, \varphi_i(z_i, t) = \varphi_{i|1}(z_i, t) + \varphi_{i|2}(z_i, t) + \cdots, \theta_i(z_i, s) = \theta_{i|1}(s) + \theta_{i|2}(z_i, s) + \cdots, \Lambda_i(w_i, z_i, t) = \Lambda_{i|1}(w_i, z_i) + \Lambda_{i|2}(w_i, z_i, t) + \cdots$, and $\Lambda'_i(w_i, z_i, s) = \Lambda'_{i|1}(w_i, z_i) + \Lambda'_{i|2}(w_i, z_i, s) + \cdots$. We will construct $h(s)$ satisfying (4.5.5) by solving the system of congruences by induction on $m$

\begin{equation}
\theta_i(z_i, s) \equiv_m \varphi_i(z_i, h^m(s)), \quad \Lambda_i(w_i, z_i, s) \equiv_m \Lambda_i(w_i, z_i, h^m(s)), \quad i \in I, \ m = 1, 2, 3, \ldots
\end{equation}

Since $\sigma_0 : T_0(M_1) \rightarrow H^0(W, (\wedge^2 T_W \oplus i_s N_{\mathbb{C}^q}/W)\star)$ is an isomorphism by the hypothesis, any element $\{(B_i(w_i, z_i)), \{\omega_i(z_i)\}) \in H^0(W, (\wedge^2 T_W \oplus i_s N_{\mathbb{C}^q}/W)\star)$ can be written uniquely in the form

\begin{equation}
\omega_i(z) = \varphi_{i|1}(z_i, u) = \sum_{\alpha=1}^{l} \frac{\partial \varphi_{i|1}(z_i, t)}{\partial t_{\alpha}} |_{t=0} u^\alpha, \quad B_i(w_i, z_i) = \Lambda_{i|1}(w_i, z_i) = \sum_{\alpha=1}^{l} \frac{\partial \Lambda_i(z_i, s)}{\partial t_{\alpha}} |_{t=0} u^\alpha
\end{equation}

for some constant $u = (u^1, ..., u^l)$. Hence since $\{(\Lambda'_{i|1}(w_i, z_i, s)), \{\theta_{i|1}(z_i, s)\}), i \in I$ represents a linear form in $s$ whose coefficients are in $H^0(W, (\wedge^2 T_W \oplus i_s N_{\mathbb{C}^q}/W)\star)$, there exists a linear vector-valued function $h^m(s)$ of $s$ such that $\theta_{i|1}(z_i, s) = \varphi_{i|1}(z_i, h^m(s))$, and $\Lambda_{i|1}(w_i, z_i, s) = \Lambda_{i|1}(w_i, z_i, h^m(s))$. This shows (4.5.5)

Now suppose that we have already constructed $h^m(s)$ satisfying (4.5.5). We will find $h^{m+1}(s)$ such that $h^{m+1}(s) = h^m(s) + h_{m+1}(s)$ satisfy (4.5.5). Let $\omega_i(z_i, s) = [\theta_i(z_i, s) - \varphi_i(z_i, h^m(s))]_{m+1}$, and $B_i(w_i, z_i, s) = [\Lambda'_i(w_i, z_i, s) - \Lambda_i(w_i, z_i, h^m(s))]_{m+1}$.

We claim that

\begin{equation}
\omega_i(z_i, s) = F_{ik}(z) \cdot \omega_k(z_k, s)
\end{equation}

\begin{equation}
[B_i(w_i, z_i, s), w_i^\alpha]_{w_i=0} - [\omega_i^\alpha(z_i, s), \Lambda_0]_{w_i=0} + \sum_{\beta=1}^{r} \omega_i^\beta(z_i, s) T_{i\alpha}^\beta(z_i) = 0
\end{equation}

\begin{equation}
B_i(w_i, z_i, s) - B_j(w_j, z_j, s) = 0
\end{equation}

\begin{equation}
B_i(w_i, z_i, s), \Lambda_0 = 0
\end{equation}

(4.5.6) follows from [Kod82] p.160. Since $\Lambda_i(w_i, z_i, s) = \Lambda'_i(w_j, z_j, s)$ and $\Lambda_i(w_i, z_i, t) = \Lambda_j(w_j, z_j, t)$, we get (4.5.6). Since $2[\Lambda_0, B_i(w_i, z_i, s)] = [\Lambda'_i(w_i, z_i, s) + \Lambda_i(w_i, z_i, h^m(s))] - [\Lambda_i(w_i, z_i, s) - \Lambda_i(w_i, z_i, h^m(s))] = 0$, we get (4.5.6).
we get \((4.5.10)\). It remains to show \((4.5.8)\). From \((4.5.3)\) and \((4.5.4)\), we have

\[
\begin{align*}
& [\Lambda_0, \omega_0^\alpha(z_i, s)]|_{w_i = 0} = \equiv m + 1 \left[ \Lambda_i^\alpha(w_i, z_i, s), \omega_0^\alpha(z_i, s) \right]|_{w_i = \theta_i(z_i, s)} \\
& \equiv m + 1 \left[ \Lambda_i^\alpha(w_i, z_i, s), \theta_i^\alpha(z_i, s) - w_0^\alpha + w_i^\alpha - \varphi_i^\alpha(z_i, h^m(s)) \right]|_{w_i = \theta_i(z_i, s)} \\
& \equiv m + 1 \left[ \Lambda_i^\alpha(w_i, z_i, s), \theta_i^\alpha(z_i, s) - \varphi_i^\alpha(z_i, h^m(s)) \right]|_{w_i = \theta_i(z_i, s)} + [\Lambda_i(w_i, z_i, h^m(s)), \omega_i^\alpha - \varphi_i^\alpha(z_i, h^m(s))]|_{w_i = \theta_i(z_i, s)} \\
& \equiv m + 1 \left[ B_i(w_i, z_i, s), w_i^\alpha - \varphi_i^\alpha(z_i, h^m(s)) \right]|_{w_i = \theta_i(z_i, s)} + \Lambda_i(w_i, z_i, h^m(s), s), \omega_i^\alpha - \varphi_i^\alpha(z_i, h^m(s))]|_{w_i = \theta_i(z_i, s)} \\
& \equiv m + 1 \left[ B_i(w_i, z_i, s), w_i^\alpha \right]|_{w_i = 0} + \sum_{\beta = 1}^{r} [\theta_i^\beta(z_i, h(s)))|T_{j a}^\beta(\theta_i(z_i, s), z_i, h(s)) \\
& \equiv m + 1 \left[ B_i(w_i, z_i, s), w_i^\alpha \right]|_{w_i = 0} + \sum_{\beta = 1}^{r} \omega_i^\beta(z_i, s)|T_{j a}^\beta(0, z_i)
\end{align*}
\]

This proves \((4.5.8)\). From \((4.5.7)\), \((4.5.8)\), \((4.5.9)\), and \((4.5.10)\), \(\{B_i(w_i, z_i, z_i), \omega_i(z_i, s)\}\) is a homogenous polynomial of degree \(m + 1\) in \(s\) with coefficients in \(H^0(W, (\Lambda^2T_W \oplus i_*\mathcal{N}_{V_0/W})^*)\) so that there exists a homogenous polynomial \(h_{m+1}(s)\) of degree \(m + 1\) in \(s\) such that \(\omega_i(z_i, s) = \varphi_i h_{m+1}(z_i, h^m(s))\), and \(B_i(w_i, z_i, s) = \Lambda_i(w_i, z_i, h_{m+1}(s))\) so that we have \(\varphi_i(z_i, h_{m+1}(s)) \equiv m + 1 \varphi_i(z_i, s) \equiv m + 1 \theta_i(z_i, s)\), \(\varphi_i(z_i, h_{m+1}(s)) \equiv m + 1 \Lambda_i(w_i, z_i, h^m(s))\), which completes the inductive construction of \(h_{m+1}(z)\) satisfying \((4.5.6)\).

4.6. Proof of convergence.

The convergence of the power series \(h(s)\) follows from the same arguments in [Kod62]. This completes the proof of Theorem \((4.5.5)\).

Example 9. We describe holomorphic Poisson structures on rational ruled surfaces \(F_m = \mathbb{P} (\mathcal{O}_{\mathbb{P}^1}(m) \oplus \mathcal{O}_{\mathbb{P}^1})\), \(m \geq 0\) explicitly. \(F_m\) can be represented in the following way. Take two copies of \(U_i \times \mathbb{P}^1, i = 1, 2\), where \(U_i = \mathbb{C}\) and write the coordinates as \((z, [\xi_0, \xi_1])\) and \((z', [\xi'_0, \xi'_1])\). Patch \(U_i \times \mathbb{P}^1, i = 1, 2\) by the relation \(z' = \frac{1}{z}\) and \([\xi'_0, \xi'_1] = [\xi_0, z^{-m}\xi_1]\). We set \(\xi = \frac{\xi_0}{\xi_1}\) and \(\xi' = \frac{\xi'_0}{\xi'_1}\). Then we have \(\frac{\partial}{\partial z'} = -z^{-m} \frac{\partial}{\partial z} + m z \frac{\partial}{\partial \xi}\) and \(\frac{\partial}{\partial \xi} = z^{-m} \frac{\partial}{\partial z'}\) so that \(\frac{\partial}{\partial z'} \wedge \frac{\partial}{\partial z} = -z^{-m+2} \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial \xi}\). We note that a holomorphic bivector field on \(U_1 \times \mathbb{P}^1\) is of the form \((dz + e(z)\xi + f(z)\xi^2)\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial z'}\), and a holomorphic bivector field on \(U_2 \times \mathbb{P}^1\) is of the form \((p(z') + q(z')\xi') + r(z')\xi'^2)\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial z'}\), where \(d(z), e(z), f(z)\) are entire functions of \(z\) and \(p(z'), q(z'), r(z')\) are entire functions of \(z'\). For a holomorphic bivector field on \(F_m\) which has the form on each \(U_i \times \mathbb{P}^1\), we must have \(d(z) + e(z)\xi + f(z)\xi^2 = -(p(z') + q(z')\xi') + r(z')\xi'^2\) so that

\[
d(z) = -p \left( \frac{1}{z} \right) z^{-m+2}, \quad e(z) = -q \left( \frac{1}{z} \right) z^2, \quad f(z) = -r \left( \frac{1}{z} \right) z^m + 2\xi^2\]

(1) In the case of \(m = 0\), we have \(d(z) = a_0 + a_1 z + a_2 z^2, e(z) = b_0 + b_1 z + b_2 z^2, f(z) = c_0 + c_1 z + c_2 z^2\) so that \(H^0(F_0, \Lambda^2 T_{F_0}) \cong \mathbb{C}^9\).

(2) In the case of \(m = 1\), we have \(d(z) = a_0 + a_1 z, e(z) = b_0 + b_1 z + b_2 z^2, f(z) = c_0 + c_1 z + c_2 z^2 + c_3 z^3\) so that \(H^0(F_1, \Lambda^2 T_{F_1}) \cong \mathbb{C}^9\).

(3) In the case of \(m = 2\), we have \(d(z) = a_0, e(z) = b_0 + b_1 z + b_2 z^2, f(z) = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4\) so that \(H^0(F_2, \Lambda^2 T_{F_2}) \cong \mathbb{C}^9\).

(4) In the case of \(m \geq 3\), we have \(d(z) = 0, e(z) = b_0 + b_1 z + b_2 z^2, f(z) = c_0 + c_1 z + \cdots c_{m+2} z^{m+2}\) so that \(H^0(F_m, \Lambda^2 T_{F_m}) \cong \mathbb{C}^{m+6}\).

In the sequel, we keep the notations in Example 9.

Example 10. Let us consider a rational ruled surface \(F_0 = \mathbb{P}^1 \times \mathbb{P}^1\). Let us consider the Poisson structure \(\Lambda_0 = \xi \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial \xi}\). Then \(\xi = 0\) defines a holomorphic Poisson submanifold on \(F_0\) which is a nonsingular rational curve \(\cong \mathbb{P}^1\) and the normal bundle is \(\mathcal{N}_{\mathbb{P}^1/F_0} \cong \mathcal{O}_{\mathbb{P}^1}\). We compute \(H^0(F_0, (\Lambda^2 T_{F_0} \oplus i_*\mathcal{N}_{\mathbb{P}^1/F_0})^*)\) which is the
kernel of $\nabla : H^0(F_0, \wedge^2 T_{F_0}) \oplus \mathbb{C} \to H^0(\mathbb{P}^1_C, T_{F_0}|_{\mathbb{P}^1_C})$. Since $[\Lambda_0, \xi] = -\xi \frac{\partial}{\partial z}$, we have as the image of $\nabla$,

$$\begin{align*}
[(a_0 + a_1 z + a_2 z^2 + (b_0 + b_1 z + b_2 z^2)\xi + (c_0 + c_1 z + c_2 z^2)\xi^2) \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial \xi}, \xi]|_{\xi=0} &= -(a_0 + a_1 z + a_2 z^2) \frac{\partial}{\partial z} + d \frac{\partial}{\partial z} = -(a_0 + d + a_1 z + a_2 z^2) \frac{\partial}{\partial z}
\end{align*}$$

where $d$ is a constant. Hence $\dim \mathbb{H}^0(F_0, (\wedge^2 T_{F_0} \oplus i_*(N_{\mathbb{P}^1_C/F_0})) = 7$.

$$\begin{align*}
&\begin{pmatrix}
(-d + (1 + b_0 + b_1 z + b_2 z^2)\xi + (c_0 + c_1 z + c_2 z^2)\xi^2) \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial \xi}, \xi - d \end{pmatrix}|_{\xi=d} = \\
= -(b_0 d + c_0 d^2 + (b_1 d + c_1 d^2)z + (b_2 d + c_2 d^2)z^2) \frac{\partial}{\partial z} = \nabla((b_0 d + c_0 d^2 + (b_1 d + c_1 d^2)z + (b_2 d + c_2 d^2)z^2) \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial \xi}, 0))
\end{align*}$$

so that obstruction vanishes and an extended Poisson analytic family

$$\begin{align*}
\mathcal{V} &\subset (F_0 \times \mathbb{C}^7, (-d -(b_0 d + c_0 d^2) - (b_1 d + c_1 d^2)z - (b_2 d + c_2 d^2)z^2 + (1 + b_0 + b_1 z + b_2 z^2)\xi + (c_0 + c_1 z + c_2 z^2)\xi^2) \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial \xi}, a_0)
\end{align*}$$

defined by $\xi = d$ has the characteristic map

$$\begin{align*}
T_0 \mathbb{C}^7 &\to \mathbb{H}^0(F_1, (\wedge^2 T_{F_1} \oplus i_*(N_{\mathbb{P}^1_C/F_1}))
\end{align*}$$

which is an isomorphism so that $\mathcal{V}$ is complete.

**Example 11.** Let us consider a rational ruled surface $F_1$. Let us consider the Poisson structure $\Lambda_0 = \xi \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial \xi}$. Then $\xi = 0$ defines a holomorphic Poisson submanifold on $F_1$ which is a nonsingular rational curve $\cong \mathbb{P}^1_C$ and the normal bundle is $N_{\mathbb{P}^1_C/F_1} \cong \mathcal{O}_{\mathbb{P}^1_C}(-1)$ so that $H^0(\mathbb{P}^1_C, N_{\mathbb{P}^1_C/F_1}) = 0$. We compute $\mathbb{H}^0(F_1, (\wedge^2 T_{F_1} \oplus i_*(N_{\mathbb{P}^1_C/F_1}))$ which is the kernel of $\nabla : H^0(F_1, \wedge^2 T_{F_1}) \to H^0(\mathbb{P}^1_C, T_{F_1}|_{\mathbb{P}^1_C}(-1))$. Since $[\Lambda_0, \xi] = -\xi \frac{\partial}{\partial z}$, we have as the image of $\nabla$,

$$\begin{align*}
[(a_0 + a_1 z + (b_0 + b_1 z + b_2 z^2)\xi + (c_0 + c_1 z + c_2 z^2 + c_3 z^3)\xi^2) \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial \xi}, \xi]|_{\xi=0} = -(a_0 + a_1 z) \frac{\partial}{\partial z}
\end{align*}$$

so that $\mathbb{H}^0(F_1, (\wedge^2 T_{F_1} \oplus i_*(N_{\mathbb{P}^1_C/F_1})) = 7$ and an extended Poisson analytic family

$$\begin{align*}
\mathcal{V} &\subset (F_0 \times \mathbb{C}^7, (1 + b_0 + b_1 z + b_2 z^2)\xi + (c_0 + c_1 z + c_2 z^2 + c_3 z^3)\xi^2) \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial \xi}
\end{align*}$$

defined by $\xi = d$ has the characteristic map

$$\begin{align*}
T_0 \mathbb{C}^7 &\to \mathbb{H}^0(F_1, (\wedge^2 T_{F_1} \oplus i_*(N_{\mathbb{P}^1_C/F_1}))
\end{align*}$$

which is an isomorphism so that $\mathcal{V}$ is complete.

**Example 12.** Let us consider a rational ruled surface $F_m, m \geq 3$. Let us consider the trivial Poisson structure $\Lambda_0 = 0$ on $F_m$. Then $\xi = 0$ defines a holomorphic Poisson submanifold which is a nonsingular rational curve $\cong \mathbb{P}^1_C$ and the normal bundle is $N_{\mathbb{P}^1_C/F_m} \cong \mathcal{O}_{\mathbb{P}^1_C}(-m)$ so that $H^0(\mathbb{P}^1_C, N_{\mathbb{P}^1_C/F_m}) = 0$. Hence $\mathbb{H}^0(F_m, (\wedge^2 T_{F_m} \oplus i_*(N_{\mathbb{P}^1_C/F_m})) = H^0(F_m, \wedge^2 T_{F_m}) = k^{m+6}$. Let us consider an extended Poisson analytic family $\mathcal{V} \subset (F_m \times \mathbb{C}^{m+6}, \Lambda(t) = ((t_0 + t_1 z + t_2 z^2)\xi + (t_3 + t_4 z + \cdots + t_{m+5} z^{m+2})\xi^2) \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial \xi})$ defined by $\xi = 0$. Then the characteristic map

$$\begin{align*}
T_0 \mathbb{C}^{m+6} &\to \mathbb{H}^0(F_m, (\wedge^2 T_{F_m} \oplus i_*(N_{\mathbb{P}^1_C/F_m}))
\end{align*}$$

which is an isomorphism so that $\mathcal{V}$ is complete at 0.
5. Stability of compact holomorphic Poisson submanifolds

We extend the definition of a fibre manifold in [Kod63] in the context of the holomorphic Poisson category.

Definition 5.0.1. By a holomorphic Poisson fibre manifold, we shall mean a holomorphic Poisson manifold \((W, \Lambda)\) together with a holomorphic map \(p\) of \(W\) onto a complex manifold \(B\) such that the rank of the Jacobian of \(p\) at each point of \(W\) is equal to the dimension of \(B\) and \(\Lambda \in H^0(W, \wedge^2 T_{W/B})\) with \(\Lambda, \Lambda = 0\).

For any point \(u \in B\), the inverse image \(p^{-1}(u) = (W_u, \Lambda_u)\) is a holomorphic Poisson submanifold of \((W, \Lambda)\) and call it the fibre of \((W, \Lambda)\) over \(u\). We will denote the holomorphic Poisson fibre manifold \((W, \Lambda)\) by the quadruple \((W, \Lambda, B, p)\). We note that a holomorphic Poisson fibre manifold is a Poisson analytic family in the sense of [Kim14b] when fibres are compact. For any subdomain \(N\) of \(B\), we call the holomorphic Poisson fibre submanifold \((W, \Lambda)_{|N}\) if and only if \((W, \Lambda, N, p)\) forms a holomorphic Poisson fibre manifold. If, moreover, each fibre \(V_u = V \cap W_u, u \in N\), of \(V\) is compact, we call \(V\) a holomorphic Poisson fibre submanifold with compact fibres of the holomorphic Poisson fibre manifold \((W, \Lambda)_{|N}\).

We extend the definition of stability in [Kod63] in the context of the holomorphic Poisson category.

Definition 5.0.2. Let \(V\) be a compact holomorphic Poisson submanifold of a holomorphic Poisson manifold \((W, \Lambda_0)\). We call \(V\) a stable compact holomorphic Poisson submanifold of \((W, \Lambda_0)\) if and only if, for any holomorphic Poisson fibre manifold \((W, \Lambda, B, p)\) such that \(p^{-1}(0) = (W, \Lambda_0)\) for a point \(0 \in B\), there exist a neighborhood \(N\) of \(0\) in \(B\) and a holomorphic Poisson fibre submanifold \(V\) with compact fibres of the holomorphic Poisson fibre manifold \((W, \Lambda)_{|N}\) such that \(V \cap W = V\).

5.1. Stability of compact holomorphic Poisson submanifolds.

Theorem 5.1.1 (compare [Kod63] Theorem 1). Let \(V\) be a compact holomorphic Poisson submanifold of a holomorphic Poisson manifold \((W, \Lambda_0)\). Let \(N^\bullet_{V/W}\) be the complex associated with the normal bundle \(N_{V/W}\) as in Definition 5.1.13. If the first cohomology group \(H^1(V, N_{V/W}^\bullet)\) vanishes, then \(V\) is a stable holomorphic Poisson submanifold of \((W, \Lambda_0)\).

To prove Theorem 5.1.1 we extend the argument in [Kod63] p.80-85 in the context of holomorphic Poisson deformations. We tried to maintain notational consistency with [Kod63].

Let \((W, \Lambda, B, p)\) be a holomorphic Poisson fibre manifold such that \(p^{-1}(0) = (W, \Lambda_0)\) for a point \(0 \in B\) and let \((u^1, \ldots, u^d)\) denote a local coordinate on \(B\) with the center 0. Considering \(V \subset W \subset W\) as a submanifold of \((W, \Lambda_0)\), we cover \(V\) by a finite number of coordinate neighborhoods \(U_k\) in \(W\) and choose a local coordinate \((z_i, w_i, u) = (z_1, \ldots, z_d, w_1, \ldots, w_r, u^1, \ldots, u^q)\) such that the simultaneous equations \(w_1 = \cdots = w_r = u^1 = \cdots = u^q = 0\) define \(V\). We assume that each neighborhood \(U_k\) is a polycylinder consisting of all points \((z_i, w_i, u), |z_i| < 1, |w_i| < 1, |u| < 1\). On the intersection \(U_k \cap U_{k'}\) the local coordinates \(z_i^\nu, w_i\) are holomorphic functions of \(z_k, w_k, u\): \(z_i^\nu = g_i^\nu(z_k, w_k, u), \alpha = 1, \ldots, d, w_i = f_i(z_k, w_k, u), \lambda = 1, \ldots, r, \beta = 1, \ldots, q\). Note that \(f_i(z_k, 0, 0) = 0\). We set \(U_{k'} = V \cap U_{k'}\). We denote a point on \(V\) by \(z\) and, if \(z = (z_k, 0, 0) \in U_k\), we call \(z_k = (z_k^1, \ldots, z_k^d)\) the local coordinate of \(z\) on \(U_k\). We define \(a_{ik\mu}^\lambda(z) = \frac{\partial f_i^\lambda(z_k, w_k, u)}{\partial w_k}\bigg|_{w_k=0}\). Then the normal bundle of \(V\) in \(W\) is defined by the system of transition matrices

\[
\begin{pmatrix}
(a_{ik}(z) & b_{ik}(z) \\
0 & 1
\end{pmatrix}
\]

so that we have the exact sequence \(0 \rightarrow N_{V/W} \rightarrow N_{V/W} \rightarrow \oplus \mathcal{O}_V \rightarrow 0\).

The other hand, since \(w_i^1 = \cdots = w_i^r = u^1 = \cdots = u^q = 0\) defines a holomorphic Poisson submanifold, \([A, w_i^\mu] = \sum_{\beta=1}^r w_i^\beta T_{i\alpha}^\beta(z_i, w_i, u)\) for some \(T_{i\alpha}^\beta(z_i, w_i, u) \in \Gamma(U_k, T_W)\), \(\gamma = 1, \ldots, r, q, \alpha = 0\). We note that \(T_{i\alpha}^\beta(z_i, 0, 0) \in \Gamma(U_k, T_W|V)\). Setting \(T_{i\alpha}^\beta(z_i, w_i, u) = 0\) for \(\rho = 1, \ldots, q, \gamma = 1, \ldots, r + q\), we can write \([A, w_i^\mu] = \sum_{\alpha=1}^r w_i^\alpha T_{i\alpha}^\beta(z_i, w_i, u)\)

We note that we can extend \(0 \rightarrow N_{V/W} \rightarrow N_{V/W} \rightarrow \oplus \mathcal{O}_V \rightarrow 0\) to obtain an exact sequence of complex sheaves

\[
(5.1.2)
0 \rightarrow N_{V/W}^\bullet \rightarrow R^\bullet \rightarrow Q^\bullet \rightarrow 0.
\]
where the first vertical complex $N^\bullet_{V/W}$ is the complex associated with the normal bundle $N_{V/W}$ and the second vertical complex $R^\bullet$ is the subcomplex of the complex $N^\bullet_{V/W}$ associated with the normal bundle $N_{V/W}$ as in Definition 5.1.3. First we show that the second vertical complex is well-defined. Indeed, we simply note that $T^\beta(z_i,0,0) = 0, \rho = 1, \ldots, q, \beta = 1, \ldots, r + q, T^\beta(z_i,0,0) \in \Gamma(U_i,W_T|V), \alpha = 1, \ldots, \rho, \beta = 1, \ldots, r + q,$ and $-\{g,\Lambda\}|V \in \Gamma(U_i,\wedge^p T^p W_T|V)$ for $g \in \Gamma(U_i,\wedge^p T^p W_T|V)$.

Next we show that the sequence of complex of sheaves (5.1.2) is well-defined. In other words, the above diagram commutes. The commutativity of the first two complexes follows from the following local commutativity:

\[
\begin{array}{c}
(\{i_0^1, \ldots, i^k, A\} + (-1)^k \sum_{\beta=1}^r \beta^{i_0^1}_{i^1} \beta^{i_1}_{i^2} \cdots \beta^{i_{k-1}}_{i^k}A) \cdots \{i_0^1, A\} + (-1)^k \sum_{\beta=1}^r \beta^{i_0^1}_{i^1} \beta^{i_1}_{i^2} \cdots \beta^{i_{k-1}}_{i^k}A) \\
(\{i_0^1, \ldots, i^k, A\} + (-1)^k \sum_{\beta=1}^r \beta^{i_0^1}_{i^1} \beta^{i_1}_{i^2} \cdots \beta^{i_{k-1}}_{i^k}A) \cdots \{i_0^1, A\}
\end{array}
\]

where $(f_0^1, \ldots, f_r^k) \in \oplus U(T(U_i,\wedge^p T^p W_T|V))$. On the other hand, the commutativity of the last two complexes follows from the following local commutativity:

\[
\begin{array}{c}
(\{i_0^1, A\} + (-1)^k \sum_{\beta=1}^r \beta^{i_0^1}_{i^1} \beta^{i_1}_{i^2} \cdots \beta^{i_{k-1}}_{i^k}A) \cdots \{i_0^1, A\} \cdots \{i_0^1, A\}
\end{array}
\]

where $(f_0^1, \ldots, f_r^k, g_0^1, \ldots, g_r^k) \in \oplus \wedge^k T(U_i,\wedge^p T^p W_T|V)$.

Since $V$ is compact, $\mathbb{H}^0(V, Q^\bullet) = H^0(V, Q^\bullet_V) \cong \mathbb{C}^q$ so that from the hypothesis $\mathbb{H}^1(V, N_{V/W}) = 0$ and the exact sequence $0 \to N_{V/W}^\bullet \to R^\bullet \to Q^\bullet \to 0$, we obtain an exact sequence

\[
\begin{array}{c}
0 \to \mathbb{H}^0(V, N_{V/W}^\bullet) \to \mathbb{H}^0(V, R^\bullet) \cong \mathbb{C}^q \to 0
\end{array}
\]

Note that any element of $\mathbb{H}^0(V, R^\bullet)$ is a collection $\{\psi_i(z)\}$ of vector-valued holomorphic functions

\[
\begin{array}{c}
\psi_i(z) = (\psi_i^1(z), \ldots, \psi_i^r(z), \psi^{r+1}, \ldots, \psi^{r+q})
\end{array}
\]

defined respectively on $U_i$ satisfying $\psi_i^\beta(z) = \sum_{k=1}^r a_{ik}\mu_k \psi_k^\beta(z) + \sum_{\rho=1}^q b_{ik}\rho(z) \psi^{r+\rho} + \{\psi_i^\beta(z), A\}|V + \sum_{\beta=1}^r \beta^{i_0^1}_{i^1} \beta^{i_1}_{i^2} \cdots \beta^{i_{k-1}}_{i^k}T^\beta(z_i,0,0) + \sum_{\beta=1}^r \beta^{i_0^1}_{i^1} \beta^{i_1}_{i^2} \cdots \beta^{i_{k-1}}_{i^k}T^\beta(z_i,0,0) = 0, \lambda = 1, \ldots, r$ and we have $\kappa \psi = (\psi^{r+1}, \ldots, \psi^{r+q})$.

Let $M_e = \{ t = (t_1, \ldots, t_n) \in \mathbb{C}^n ||t|| < \epsilon \}$, where $\epsilon$ is a small positive number. Consider a Poisson analytic family $F$ of compact holomorphic Poisson submanifolds $V_t, t \in M_e, (W, A)$ such that $V_0 = V$ (see Definition 3.0.1). Then $F$ is a holomorphic Poisson submanifold of $(W \times M_e, \Lambda)$ such that $F \cap (W \times t) = V_t \times t$. We assume that $F$ is covered by the neighborhoods $U_i \times M_e$. On each neighborhood $U_i \times M_e$, the Poisson submanifold $F$ is defined by simultaneous holomorphic equations of the form $w^\lambda_t = \theta^\lambda_t(z_t, t), \lambda = 1, \ldots, r, w^\rho_t = \theta^{r+\rho}(t), \rho = 1, \ldots, q$. For any tangent vector $\frac{\partial}{\partial z} = \sum_{c} c_i \frac{\partial}{\partial z_i}$ of $M_e$ at $t = 0$, we set $\psi_i(z) = (\frac{\partial \psi^\lambda_i(z, t)}{\partial t})_{t=0}, \lambda = 1, \ldots, r, \psi^{r+\rho} = (\frac{\partial \psi^{r+\rho}(t)}{\partial t})_{t=0}, \rho = 1, \ldots, q$ and let $\psi_i(z) = (\psi_i^1(z), \ldots, \psi_i^r(z), \psi^{r+1}, \ldots, \psi^{r+q})$. Then the collection $\{\psi_i(z)\}$ of $\psi_i(z)$ represents an element of $\mathbb{H}^0(V, N_{V/W}) = \mathbb{H}^0(V, R^\bullet)$ (see subsection 3.2). With this preparation, we prove

**Theorem 5.1.4** (compare Kod63 Theorem 2). There exists a Poisson analytic family $F$ of compact holomorphic Poisson submanifolds $V_t, t \in M_e, \epsilon > 0$, of $(W, A)$ such that $V_0 = V$ and the characteristic map: $\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} |_{t=0}$ maps the tangent space $T_0(M_e)$ isomorphically onto $\mathbb{H}^0(V, N_{V/W}) = \mathbb{H}^0(V, R^\bullet)$ provided that cohomology group $\mathbb{H}^1(V, N_{V/W})$ vanishes.
Proof. Let $n = \mathbb{H}^0(V, R^*)$. We can choose a base $\{\beta_1, \ldots, \beta_n\}$ of $\mathbb{H}^0(V, R^*)$ such that $\beta_v^{r+\rho} = 1$ if $v = n - q + \rho$ or $\beta_v^{r+\rho} = 0$ otherwise for $\rho = 1, \ldots, q$.

We shall construct on each neighborhood $U_t \times M_t$ a vector-valued holomorphic function of the form

$$\phi_t(z_i, t) = (\theta_t^i(z_i, t), \ldots, \theta_t^n(z_i, t), t_{n-q+1}, \ldots, t_n), \quad \text{where } t = (t_1, \ldots, t_n)$$

satisfying the boundary conditions

$$\phi_t(z_i, 0) = 0,$$

$$\left( \frac{\partial \phi_t(z_i, t)}{\partial t_v} \right)_{t=0} = \beta_v z_i, \quad v = 1, \ldots, n.$$ 

such that

$$(5.1.5) \quad \phi_t(g_k z_i, \phi_k(z_k, t), t) = f_k(z_k, \phi_k(z_k, t)), \quad \phi_t^m(g_k z_i, \phi_k^m(z_k, t), t) = f_k z_i, \phi_k^m(z_k, t), \quad m = 1, 2, 3, \ldots$$

$$(5.1.6) \quad [\Lambda, w_{\alpha}^0 - \theta_1^m(z_i, t)]|_{w_{\alpha}=\theta_1^m(z_i, t), w_{\alpha}=t_{n-q+\rho}} = 0, \quad \alpha = 1, \ldots, r.$$  

(Note that we have $[\Lambda, w_{\alpha}^0 - t_{n-q+\rho}] = 0, \rho = 1, \ldots, q$ so that we do not need to consider them.)

Recall Notation 1 Then (5.1.5) and (5.1.6) are equivalent to the system of congruences

$$(5.1.7) \quad [\Lambda, w_{\alpha}^0 - \theta_1^m(z_i, t)]|_{w_{\alpha}=\theta_1^m(z_i, t), t_{\alpha}=t_{n-q+\rho}} = 0, \quad m = 1, 2, 3, \ldots, \alpha = 1, \ldots, r.$$  

As in the proof of Theorem 3.3.1, we will construct the formal power series

$$\phi_t(z_i, t) = (\theta_1^m(z_i, t), \ldots, \theta_1^m(z_i, t), n-q+1, \ldots, n)$$

satisfying (5.1.7) by induction on $m$.

We define $\phi_t(z_i, t) = \sum \beta_v(z_i, t) t_v$. Then (5.1.7) holds. On the other hand, since $[\Lambda, \beta_v(z_i)]|_{w_{\alpha}=u=0} = \sum \beta_v(z_i) T^{\beta}_{i=0}(z_i, 0, 0), \alpha = 1, \ldots, r, [\Lambda, w_{\alpha}^0 - \theta_1^m(z_i, t)]$ is of the form

$$(5.1.9) \quad [\Lambda, w_{\alpha}^0 - \theta_1^m(z_i, t)] = \sum \beta_v(z_i, t) T^{\beta}_{i=0}(z_i, w_{\alpha}, u) + \sum (u_{\alpha} - t_{n-q+\rho}) T^{\beta}_{i=0}(z_i, w_{\alpha}, u) + \sum w_{\alpha}^0 P_{i=0}^{\beta}(z_i, w_{\alpha}, u, t) + \sum u_{\alpha} P_{i=0}^{\beta}(z_i, w_{\alpha}, u, t),$$

where the degree of $P_{i=0}^\gamma(z_i, w_{\alpha}, u, t), \gamma = 1, \ldots, r + q$ in $t$ is 1 so that (5.1.9) holds. Now we assume that we have already constructed $\phi_t^m(z_i, t)$ satisfying (5.1.7) and (5.1.8) such that $[\Lambda, w_{\alpha}^0 - \theta_1^m(z_i, t)]$ is of the form (as in (3.3.8))

Such that the degree of $P_{i=0}^\gamma(z_i, w_{\alpha}, u, t), \gamma = 1, \ldots, r + q$ is at least 1 in $t$. Let

$$\psi_{ik}(z, t) \equiv_{m+1} \phi_t^m(g_k z_i, \phi_t^m(z_k, t), t) - f_k z_i, \phi_t^m(z_k, t))$$

$$(5.1.10) \quad [G_{i=0}^m(z_i, t)]|_{w_{\alpha}=\theta_1^m(z_i, t), t_{\alpha}=t_{n-q+\rho}}, \gamma = 1, \ldots, r + q$$

As in the proof of Theorem 3.3.1, we can show that the collection $\{\pi_{ik}(z, t)\} \in C^1(U \cap V, N_{V/W}^\gamma)$ and $\{(G_1^m(z_i, t), \ldots, G_n^m(z_i, t)) \in C^0(U \cap V, N_{V/W}^\gamma \otimes T_{V/W})\}$ define a 1-cocycle in the Čech resolution of $N_{V/W}^\gamma$. We note that $\{\psi_{ik}(z, t)\}$ are in $C^1(U \cap V, N_{V/W})$ and $\{(G_1^m(z_i, t), \ldots, G_n^m(z_i, t))\}$ are in $C^0(U \cap V, N_{V/W}^\gamma \otimes T_{V/W})$ so that $\{(\pi_{ik}(z, t), \{G_1^m(z_i, t), \ldots, G_n^m(z_i, t)\})\}$ define an element in $\mathbb{H}^1(V, N_{V/W}^\gamma)$. Since $\mathbb{H}^1(V, N_{V/W}^\gamma) = 0$, there exists $\{\chi_i(z, t) = (\chi_1^m(z_i, t), \ldots, \chi_n^m(z_i, t))\}$ which are homogenous polynomials of degree $m+1$ in $t_1, \ldots, t_n$. 


whose coefficients are in $C^0(\mathcal{U} \cap V, \mathcal{N}_{V/W})$ such that

$$
\psi_{ik}^\lambda(z, t) = \sum_{\mu=1}^{r} a_{ik}^\lambda(z) \chi_{ik}^\mu(z, t) - \lambda_i^\mu(z, t), \quad \lambda = 1, ..., r.
$$

$$
-G_i^\alpha(z_i, t) = -[\chi_i^\alpha(z_i, t), \Lambda_0]|_{w_i=0} + \sum_{\beta=1}^{r} \chi_i^\beta(z_i, t) T_{i\alpha}^{\beta}(z_i, 0, 0), \quad \alpha = 1, ..., r.
$$

where $\chi_i^\alpha(z_i, t) = (\chi_i^1(z_i, t), ..., \chi_i^r(z_i, t), 0, ..., 0)$. Then $\phi_i^m(z_i, t) = \phi_i^m(z_i, t) + \phi_i|_{m+1}(z_i, t)$ satisfy

$$
\phi_i(m+1) = \sum_{\beta=1}^{r} \chi_i^\beta(z_i, t) T_{i\alpha}^{\beta}(z_i, 0, 0)
$$

and the degree of $R^\alpha_{i\alpha}(z_i, w_i, u, t)$ is $m+1$ in $t$. Therefore, we have, from

$$
\Lambda, w_i^\alpha - \theta_i^m(z_i, t) - \chi_i^\alpha(z_i, t) = \sum_{\beta=1}^{r} \chi_i^\beta(z_i, t) T_{i\alpha}^{\beta}(z_i, w_i, u, t) + \sum_{\rho=1}^{q} u^{-t-q+\rho} R_{i\alpha}^{\rho+(n-q)\Lambda}(z_i, w_i, u, t)
$$

$\Lambda, w_i^\alpha - \theta_i^m(z_i, t) - \chi_i^\alpha(z_i, t) = \sum_{\beta=1}^{r} \chi_i^\beta(z_i, t) T_{i\alpha}^{\beta}(z_i, w_i, u, t) + \sum_{\rho=1}^{q} u^{-t-q+\rho} R_{i\alpha}^{\rho+(n-q)\Lambda}(z_i, w_i, u, t)$

Lastly, we note that from (5.1.11), we have

$$
G_i^\alpha(z_i, t) = \sum_{\beta=1}^{r} \theta_i^m(z_i, t) P_{i\alpha}^{\beta}(z_i, \phi_i^m(z_i, t), t) + \sum_{\rho=1}^{q} t_{n-q+\rho} P_{i\alpha}^{n+q+\rho}(z_i, \phi_i^m(z_i, t), t)
$$

so that we obtain, from (5.1.9) and (5.1.11),

$$
[\Lambda, w_i^\alpha - \theta_i^m(z_i, t) - \chi_i^\alpha(z_i, t)]|_{w_i=0} = \theta_i^m(z_i, t) + \chi_i^\alpha(z_i, t), w_i^\alpha = t_{n-q+\rho}
$$

for $\rho = 1, ..., q$. Hence (5.1.13) holds. By subsection 5.3 the formal power series $\phi_i(z_i, t)$ converges for $|t| < \epsilon$ for sufficiently small positive number $\epsilon$. Let $M_{\alpha} = \{t = (t_1, ..., t_n) \in \mathbb{C}^n | |t| < \epsilon\}$. Then on each neighborhood $\mathcal{U}_i \times M_e$ of $\mathcal{W} \times M_e$, the simultaneous equation $w_i^\alpha - \theta_i^m(z_i, t) = w^\alpha - t_{n-q+\rho} = 0, \alpha = 1, ..., r, \rho = 1, ..., q$ defines the desired Poisson analytic family $\mathcal{F}$ of compact holomorphic Poisson submanifolds of $V_t, t \in M_e$ of $(W, \Lambda)$ such that $V_0 = V$. This completes the proof of Theorem 5.1.4.

Proof of Theorem 5.1.1. Let $N_e = \{u = (u^1, ..., u^q) \in B | |u| < \epsilon\}$, where $\epsilon$ is a small positive number. Let $\mathcal{F} \subset (W \times M_e, \Lambda)$ be the Poisson analytic family of compact holomorphic Poisson submanifolds $V_t, t \in M_e$ of $(W, \Lambda)$ defined by $w_i^\alpha = \theta_i^m(z_i, t), \lambda = 1, 2, ..., r, w^\alpha = t_{n-q+\rho}, \rho = 1, 2, ..., q$, and $\Lambda$ on $\mathcal{U}_i \times M_e$ as in the proof of Theorem 5.1.4. If we define a linear map $u \rightarrow t(u) = (0, ..., 0, u^1, ..., u^q)$ of $N_e$ into $M_e$, then the union $\mathcal{V} = \bigcup_{u} V_t(u)(w)$ of the compact holomorphic Poisson submanifolds $V_t(u), u \in N_e$ of $(W, \Lambda)$ which is defined by $w_i^\alpha = \theta_i^m(z_i, 0, ..., 0, u^1, ..., u^q)$ on $\mathcal{U}_i$ forms a holomorphic Poisson fibre submanifold with compact fibres of the holomorphic Poisson fibre manifold $(W, \Lambda)|_{N_e}$ with $\mathcal{V} \cap \mathcal{W} = V$ so that $V$ is a stable holomorphic Poisson submanifold of $(W, \Lambda_0)$.

As in Theorem 3 in [Kod63], by combining the exact sequence (5.1.3) and Theorem 5.1.1, we can show

**Theorem 5.1.12.** Let $(W, \Lambda, B, p)$ be a holomorphic Poisson fibre manifold such that $(W, \Lambda_0) = p^{-1}(0)$ is the fibre of $(W, \Lambda)$ over a point $0 \in B$. Let $V$ be a compact holomorphic Poisson submanifold of $(W, \Lambda_0)$. Assume that $H^1(V, \mathcal{N}_{W/V}^\bullet) = H^0(V, \mathcal{N}_{W/V}^\bullet) = 0$. Then, for a sufficiently small neighborhood $N$ of $0 \in B,$
there exists the unique holomorphic Poisson fibre submanifold $V$ with compact fibers of the holomorphic Poisson fibre manifold $(W, \Lambda)|_N$ such that $V \cap W = V$.

**Example 13.** We keep the notations in Example [9]. Let us consider a Poisson rational ruled surface $(F_2, z^2 \xi \frac{\partial}{\partial z} + \frac{\partial}{\partial t})$. We show that the holomorphic Poisson submanifold $\xi = \xi' = 0$ of $(F_2, z^2 \xi \frac{\partial}{\partial z} + \frac{\partial}{\partial t})$ is unstable. Take two copies of $U_i \times \mathbb{P}_C^1 \times \mathbb{C}$, where $U_i = \mathbb{C}$ and write the coordinates as $(z, [\xi_0, \xi_1], t)$ and $(\xi', [\xi_0, \xi_1]', t')$. Patch $U_i \times \mathbb{P}_C^1 \times \mathbb{C}, i = 1, 2$ by the relation $\xi' = \frac{1}{2}z, t = t'$ and $[\xi_0, \xi_1] = [\xi_0, z^2 \zeta_1 + tz \zeta_0]$ and denote it by $X$. Then the projection $\pi: X \to \mathbb{C}$ defines a complex analytic family of deformations of $F_2$. We give a holomorphic Poisson structure on $X$ to make a Poisson analytic family. We set $\xi = \frac{\xi}{\xi}$ and $\xi' = \frac{\xi'}{\xi}$. Since $-\xi' \frac{\partial}{\partial z} + \frac{\partial}{\partial t} = (z^2 \xi + tz) \frac{\partial}{\partial z} + \frac{\partial}{\partial t}$, $\pi: (X, \Lambda = (z^2 \xi + tz) \frac{\partial}{\partial z} + \frac{\partial}{\partial t}) \to \mathbb{C}$ defines a Poisson analytic family. Since $\xi = \xi' = 0$ cannot be extended to a complex analytic family as in [Kod63] p.86, it cannot be extended to a Poisson analytic family as a holomorphic Poisson fibre submanifold of $(X, \Lambda)$ so that it is not stable.

**Example 14.** Let us consider $F_0 \cong \mathbb{P}_C^1 \times \mathbb{P}_C^1$ and a Poisson structure $\Lambda_0 = \xi \frac{\partial}{\partial z} + \frac{\partial}{\partial t}$ on $F_0$. We keep the notations in Example [9]. We show that the holomorphic Poisson submanifold $V : \xi = 0$ is unstable. Let us consider a Poisson analytic family $(F_0 \times \mathbb{C}, \Lambda = (z-tz) \frac{\partial}{\partial z} + \frac{\partial}{\partial t})$. Assume that there is a holomorphic Poisson fibre manifold of $V \subset (F_0 \times B, \Lambda)$, where $B = \{t \in \mathbb{C} | |t| < \epsilon \}$ for a sufficiently small number $\epsilon > 0$ such that $V|_{t=0}$ is $\xi = \xi' = 0$. We may assume that $V$ is defined by $\xi - \varphi_1(z, t) = 0$ on $U_1 \times \mathbb{P}_C^1 \times \mathbb{C}$ and $\xi' - \varphi_2(z', t) = 0$ on $U_2 \times \mathbb{P}_C^1 \times \mathbb{C}$, where $U_i = \mathbb{C}, i = 1, 2$ so that we have a relation $\varphi_1(z, t) - \varphi_2(z', t) = 0$. Hence $\varphi_1(z, t)$ is of the from $\varphi_1(z, t) = f(t)$. On the other hand, since $\xi - \varphi_1(z, t)$ defines a holomorphic Poisson submanifold of $F_0 \times \mathbb{C}$, $[\Lambda, \xi - \varphi_1(z, t)]|_{\xi = \varphi_1(z, t)} = 0$ so that $-\xi - tz \frac{\partial}{\partial t}|_{\xi = \varphi_1(z, t)} = 0 \iff \varphi_1(z, t) = tz$ which contradicts to $\varphi_1(z, t) = f(t)$. Hence $V : \xi = 0$ is not stable as a holomorphic Poisson submanifold while it is stable as a complex submanifold since $H^1(V, N_{V/F_0}) \cong H^1(\mathbb{P}_C^1, \mathcal{O}_{\mathbb{P}_C^1}) = 0$.

**Appendix A. Deformations of Poisson structures**

We denote by $\text{Art}$ the category of local artinian $k$-algebras with residue field $k$, where $k$ is an algebraically closed field with characteristic $0$, and by $k[e]$ by the ring of dual numbers.

**Definition A.0.13.** Let $(Y, \Lambda_0)$ be a nonsingular Poisson variety. An infinitesimal deformation of $\Lambda_0$ over $A \in \text{Art}$ is an algebraic Poisson scheme $(Y \times \text{Spec}(k)[e], \Lambda)$ which induces $(Y, \Lambda_0)$, where $\Lambda \in H^0(Y, \wedge^2 T_Y) \otimes A$. Then for each $A \in \text{Art}$, we can define a functor of Artin rings

$$\text{Def}_{\Lambda_0} : \text{Art} \to \text{(sets)}$$

$$A \mapsto \{\text{infinitesimal deformations of } \Lambda_0 \text{ over } A\}$$

We will denote by $\mathbb{H}^i(Y, \wedge^2 T^*_{Y^{-1}})$ the $i$-th hypercohomology group of the following complex of sheaves

$$\wedge^2 T_{Y^{-1}} : \wedge^2 T_Y \xrightarrow{[-, \cdot]_{\Lambda_0}} \wedge^3 T_Y \xrightarrow{[-, \cdot]_{\Lambda_0}} \wedge^4 T_Y \xrightarrow{[-, \cdot]_{\Lambda_0}} \cdots$$

**Proposition A.0.15.** Let $(Y, \Lambda_0)$ be a nonsingular Poisson variety. Then

1. There is a natural identification $\text{Def}_{\Lambda_0}(k[e]) \cong H^0(Y, \wedge^2 T^*_{Y^{-1}})$.
2. Given an infinitesimal deformation $\eta$ of $\Lambda_0$ over $A \in \text{Art}$ and a small extension $0 \to \eta \to \hat{A} \to A \to 0$, we can associate an element $a_0(\eta) \in \mathbb{H}^1(Y, \wedge^2 T^*_{Y^{-1}})$, which is zero if and only if there is a lifting of $\eta$ to $\hat{A}$.

**Proof.** Let $\Lambda \in H^0(Y, \wedge^2 T_Y) \otimes k[e]$ be an infinitesimal deformation of $\Lambda_0$ over $\text{Spec}(k[e])$ so that $\Lambda = \Lambda_0 + e\Lambda'$ for some $\Lambda' \in H^0(Y, \wedge^2 T_Y)$. Since $[\Lambda_0 + e\Lambda', \Lambda_0 + e\Lambda'] = 0$ so that $-\Lambda', \Lambda_0] = 0$. Hence $\Lambda' \in \mathbb{H}^1(Y, \wedge^2 T_Y)$.

Now we identify obstructions. Consider a small extension $e : 0 \to \eta \to \hat{A} \to A \to 0$. Let $\hat{A}$ be an arbitrary lifting of $\Lambda_0$ to $\hat{A}$ so that
Now we choose another arbitrary lifting $\tilde{\Lambda}$ class which induces (A.0.24) Remark A.0.21.

We note that $\tilde{\Lambda} = \Lambda + tD_i$ for some $D_i \in \Gamma(U_i, \Lambda^2 T_Y)$. Then

(A.0.16) $\quad t[\Lambda_0, \Pi_i] = [\tilde{\Lambda}_i, [\tilde{\Lambda}_i, \tilde{\Lambda}_i]] = 0 \iff -[\frac{1}{2} \Pi_i, \Lambda_0] = 0$

(A.0.17) $\quad \frac{1}{2} \Pi_i - \frac{1}{2} \Pi_j = \frac{1}{2} [\tilde{\Lambda}_i, \tilde{\Lambda}_j] - \frac{1}{2} [\Lambda_i, \Lambda_j] = t[\Lambda_0, \Lambda'_{ij}] \iff \delta(\frac{1}{2} \Pi_i) - [-\Lambda'_{ij}, \Lambda_0] = 0$

(A.0.18) $\quad t(\Lambda'_{ij} - \Lambda'_{ik} + \Lambda'_{jk}) = \tilde{\Lambda}_j - \tilde{\Lambda}_k - \tilde{\Lambda}_i + \Lambda_k + \Lambda_i - \Lambda_j = 0 \iff -\delta(\Lambda'_{ij}) = 0$.

Hence $\{\frac{1}{2} \Pi_i, [-\Lambda'_{ij}]\} \in C^0(\mathcal{U}, \Lambda^3 T_Y) \oplus C^1(\mathcal{U}, \Lambda^2 T_Y)$ define a 1-cocycle in the following Čech resolution of $\Lambda^2 T_X^{-1}$:

\[
\begin{array}{c}
C^0(\mathcal{U}, \Lambda^3 T_Y) \\
[-\cdot, \Lambda_0] \\
\downarrow \\
C^0(\mathcal{U}, \Lambda^3 T_Y) \xrightarrow{\delta} C^1(\mathcal{U}, \Lambda^3 T_Y) \\
[-\cdot, \Lambda_0] \\
\downarrow \\
\cdots \\
\end{array}
\]

Now we choose another arbitrary lifting $\tilde{\Lambda}' \in \Gamma(U_i, \Lambda^2 T_Y) \otimes \tilde{\Lambda}$ of $\Lambda_i$. We show that the associated cohomology class $b := \{\frac{1}{2} \Pi_i, [-\Lambda''_{ij}]\}$ is cohomologous to $a := \{\frac{1}{2} \Pi_i, [-\Lambda'_{ij}]\}$. We note that $\tilde{\Lambda}' = \tilde{\Lambda} + tD_i$ for some $D_i \in \Gamma(U_i, \Lambda^2 T_Y)$. Then

(A.0.19) $\quad \frac{1}{2} \Pi_i' - \frac{1}{2} \Pi_i = \frac{1}{2} [\tilde{\Lambda}', \tilde{\Lambda}'] - \frac{1}{2} [\tilde{\Lambda}_i, \tilde{\Lambda}_i] = t[D_i, \Lambda_0] \iff \frac{1}{2} \Pi_i - \frac{1}{2} \Pi_i' = -[D_i, \Lambda_0]$

(A.0.20) $\quad t\Lambda''_{ij} - t\Lambda'_{ij} = \tilde{\Lambda}_j - \tilde{\Lambda}_i + \tilde{\Lambda}_i + \tilde{\Lambda}_j = t(D_i - D_j) \iff -\Lambda'_{ij} - (-\Lambda''_{ij}) = -\delta(D_i)$

Hence $\{D_i\} \in C^0(\mathcal{U}, \Lambda^2 T_Y)$ is mapped to $a - b$ so that $a$ is cohomologous to $b$. So given a small extension $e : 0 \to (t) \to \tilde{\Lambda} \to A \to 0$, we can associate an element $o_e(e) := \text{the cohomology class of } a \in \mathbb{H}^1(Y, \Lambda^2 T_Y^{-1})$. We note that $o_e(e) = 0$ if and only if there exists a collection $\{\tilde{\Lambda}_i\}$ such that $\Pi_i = 0$ (which means $[\tilde{\Lambda}_i, \tilde{\Lambda}_i] = 0$) and $\Lambda'_{ij} = 0$ (which means $\{\tilde{\Lambda}_i\}$ glues together to define a Poisson structure on $Y \times \text{Spec}(\tilde{A})$) if and only if there is a lifting of $\eta$ to $\tilde{A}$.

\[\square\]

**Remark A.0.21.** We have an exact sequence of complex of sheaves $0 \to \Lambda^2 T_X^{-1} \to T_X \to T_X \to 0$:

\[
\begin{array}{cccccccc}
0 & \longrightarrow & \Lambda^2 T_X & \longrightarrow & \Lambda^3 T_X & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Lambda^2 T_X & \longrightarrow & \Lambda^3 T_X & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & T_X & \longrightarrow & T_X & \longrightarrow & 0
\end{array}
\]

which induces

(A.0.22) $\mathbb{H}^0(X, \Lambda^2 T_X^{-1}) \to \mathbb{H}^1(X, T_X^\bullet) \to H^1(X, T_X)$

(A.0.23) $\mathbb{H}^1(X, \Lambda^2 T_X^{-1}) \to \mathbb{H}^2(X, T_X^\bullet) \to \mathbb{H}^2(X, T_X)$

We also have morphisms of deformation functors

(A.0.24) $\text{Def}_{\Lambda_0} \to \text{Def}_{(X, \Lambda_0)} \to \text{Def}_X$

where $\text{Def}_{(X, \Lambda_0)}$ is the functor of flat Poisson deformations of $(X, \Lambda_0)$ (see [Kim14]) and $\text{Def}_X$ is the functor of flat deformations of $X$ (see [Ser06] p.64). Then (A.0.22) represents the morphisms of tangent
spaces for \([A.0.24]\) : \(\text{Def}_{\Lambda_0}(k[\epsilon]) \to \text{Def}_{(X,\Lambda_0)}(k[\epsilon]) \to \text{Def}_X(k[\epsilon])\), and \([A.0.23]\) represents the obstruction maps for \([A.0.24]\).

### Appendix B. Deformations of Poisson closed subschemes

#### B.1. The local Poisson Hilbert functor.

Let \(X \subset (Y, \Lambda_0)\) be a closed embedding of algebraic Poisson schemes, where \((Y, \Lambda_0)\) is a nonsingular Poisson variety. An infinitesimal deformation of \(X\) in \((Y, \Lambda_0)\) over \(A \in \text{Art}\) is a cartesian diagram of morphisms of schemes

\[
\begin{array}{ccc}
X & \longrightarrow & X \subset (Y \times \text{Spec}(A), \Lambda_0) \\
\downarrow & & \downarrow \\
\text{Spec}(k) & \longrightarrow & \text{Spec}(A)
\end{array}
\]

where \(\pi\) is flat and induced by a projection from \(Y \times S\) to \(S\), and \(X\) is a Poisson closed subscheme of \((Y \times S, \Lambda_0)\). Then we can define a functor of Artin rings (called the local Poisson Hilbert functor of \(X\) in \((Y, \Lambda_0)\))

\[
H^{(Y, \Lambda_0)}_X : \text{Art} \to (\text{Sets})
\]

\[
A \mapsto \{ \text{infinitesimal deformations of } X \text{ in } (Y, \Lambda_0) \text{ over } A \}
\]

#### B.2. The complex associated with the normal bundle of a Poisson closed subscheme of a nonsingular Poisson variety.

Let \((Y, \Lambda_0)\) be a nonsingular Poisson variety and \(X\) be a Poisson closed subscheme of \((Y, \Lambda)\) defined by a Poisson ideal sheaf \(\mathcal{I}\). Assume that \(i : X \hookrightarrow Y\) be a regular embedding. Let \(U_i\) be an affine open cover of \(Y\) such that \(I_i = (f_1^i, \ldots, f_n^i)\) be a Poisson ideal of \(\Gamma(U_i, \mathcal{O}_Y)\) defining \(X \cap U_i\) and \((f_1^i, \ldots, f_n^i)\) is a regular sequence. Since \((f_1^i, \ldots, f_n^i) = (f_1^i, \ldots, f_N^i)\), \(f_0^i = \sum_{\beta=1}^N r_{ij}^\alpha f_j^\beta\) for some \(r_{ij}^\alpha \in \Gamma(U_i \cap U_j, \mathcal{O}_Y)\). Since \(I_i/I_j^2\) is free \(\Gamma(U_i \cap U_j, \mathcal{O}_X)\)-module and generated by \((f_0^\alpha + I_j^2, \alpha = 1, \ldots, N)\), \(r_{ij}^\alpha\) is uniquely determined, where \(r_{ij}^\alpha\) is the restriction of \(r_{ij}^\beta\) to \(\Gamma(U_i \cap U_j \cap X, \mathcal{O}_X)\).

Then the normal sheaf \(N_{X/Y} := \mathcal{H}om_{\mathcal{O}_X} (\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X)\) is locally described in the following way: \(\text{Hom}_{\mathcal{O}_X}( \mathcal{I}/\mathcal{I}^2, \mathcal{O}_X) \cong \oplus \Gamma(U_i \cap X, \mathcal{O}_X)\), \(\phi \mapsto (\phi(f_1^i), \ldots, \phi(f_N^i))\), where \(f_0^i\) is the image of \(f_0^\alpha\) in \(I_i/I_j^2\) and on \(U_i \cap U_j\), \((g_1^i, \ldots, g_N^i) \in \oplus \Gamma(U_j \cap X, \mathcal{O}_X)\) is identified with \(\sum_{\beta=1}^N r_{ij}^\alpha g_\beta = \sum_{\beta=1}^N r_{ij}^\alpha g_\beta \in \oplus \Gamma(U_i \cap X, \mathcal{O}_X)\).

On the other hand, \((f_1^i, \ldots, f_n^i)\) is a Poisson ideal, in other words, \((f_1^i, \ldots, f_n^i, O_Y) \subset (f_1^i, \ldots, f_n^i, O_Y), \alpha = 1, \ldots, N\) so that \([A_0, f_0^i] = \sum_{\beta=1}^N f_0^\alpha T_{i\alpha}\) for some \(T_{i\alpha} \in \Gamma(U_i \cap X, T_Y|_X)\). Let \(T_{i\alpha}^3\) be the image of \(T_{i\alpha}^3\) in \(\Gamma(U_i \cap X, T_Y|_X)\).

Then we have

\[(1) \quad \text{We note that } \sum_{\beta=1}^N r_{ij}^\alpha f_j^\beta = \sum_{\beta=1}^N f_0^\alpha T_{i\alpha}, \quad \text{where } \sum_{\beta=1}^N r_{ij}^\alpha f_j^\beta = \sum_{\beta=1}^N [A_0, f_0^\alpha] = [A_0, \sum_{\beta=1}^N r_{ij}^\alpha f_j^\beta] = \sum_{\beta=1}^N [A_0, f_0^\alpha] T_{i\alpha}^3 + \sum_{\beta=1}^N r_{ij}^\alpha f_j^\beta T_{i\alpha}^3. \]

\[(2) \quad \text{Then } \sum_{\gamma=1}^N f_j^\gamma [\sum_{\beta=1}^N r_{ij}^\alpha T_{i\alpha}^3] = \sum_{\beta=1}^N f_0^\alpha T_{i\alpha}^3, \quad \text{where } \sum_{\beta=1}^N r_{ij}^\alpha f_j^\beta T_{i\alpha}^3 = 0 \text{ so that we get}
\]

\[
\sum_{\beta=1}^N T_{i\alpha}^3 \cdot T_{i\alpha}^3 = \sum_{\beta=1}^N T_{i\alpha}^3 \cdot T_{i\alpha}^3 = 0.
\]

Now we define the complex \(\mathcal{N}_{X/Y}^*\) associated with the normal bundle \(\mathcal{N}_{X/Y}\):

\[
\mathcal{N}_{X/Y}^* : \mathcal{N}_{X/Y} \xhookrightarrow{} \mathcal{N}_{X/Y} \otimes T_Y|_X \xhookrightarrow{} \mathcal{N}_{X/Y} \otimes \wedge^2 T_Y|_X \xhookrightarrow{} \cdots
\]
Proof.

Let $\mathcal{U} = \text{bedding of algebraic Poisson schemes}$.

The complex is defined locally in the following way:

$$
\nabla : \oplus_i \Gamma(U_i \cap X, \wedge^p T_Y|_X) \rightarrow \oplus_i \Gamma(U_i \cap X, \wedge^{p+1} T_Y|_X)
$$

$$(g_1^i, ..., g_N^i) \mapsto (-[g_1^i, \Lambda_0] + (-1)^p \sum_{\beta=1}^N g_\beta^i \wedge T_{\beta_1}, ..., -[g_N^i, \Lambda_0] + (-1)^p \sum_{\beta=1}^N g_\beta^i \wedge T_{\beta_N})$$

We denote the $i$-th hypercohomology group of $\mathcal{N}_{X/Y}^*$ by $\mathbb{H}^i(X, \mathcal{N}_{X/Y}^*)$.

**Proposition B.2.4** (compare [Ser06] Proposition 3.2.1 and Proposition 3.2.6). *Given a regular closed embedding of algebraic Poisson schemes $i : X \rightarrow (Y, \Lambda_0)$, where $(Y, \Lambda_0)$ is a nonsingular Poisson variety, then*

1. There is a natural identification

$$
H_X^{(Y, \Lambda_0)}(k[\epsilon]) \cong \mathbb{H}^0(X, \mathcal{N}_{X/Y}^*)
$$

2. Given an infinitesimal deformation $\eta$ of $X$ in $(Y, \Lambda_0)$ over $A \in \text{Art}$ and a small extension $e : 0 \rightarrow (t) \rightarrow \hat{A} \rightarrow A \rightarrow 0$, we can associate an element $o_0(e) \in \mathbb{H}^1(X, \mathcal{N}_{X/Y}^*)$, which is zero if and only if there is a lifting of $\eta$ to $\hat{A}$.

**Proof.** Let $\mathcal{U} = \{U_i\}$ be an affine open covering of $Y$ and let $I_i = (f_1^i, ..., f_N^i)$ be a Poisson ideal of $\Gamma(U_i, \mathcal{O}_Y)$ defining $U_i \cap X$ such that $(f_1^i, ..., f_N^i)$ is a regular sequence. We keep the notations in subsection B.2.

A first-order deformation of $X$ in $(Y, \Lambda_0)$ is a flat family

$$
X \longrightarrow \mathcal{X} \subset (Y \times \text{Spec}(k[\epsilon]), \Lambda_0)
$$

so that $\mathcal{X}$ is determined by a Poisson ideal sheaf $\mathcal{I}$ generated by $\{f_\alpha^i + \epsilon g_\alpha^i\}$, $\alpha = 1, ..., N$ for some $g_\alpha^i \in \Gamma(U_i, \mathcal{O}_Y)$. Let $(\bar{g}_1^i, ..., \bar{g}_N^i) \in \oplus_i \Gamma(X \cap U_i, \mathcal{O}_X)$ be the image of $(g_1^i, ..., g_N^i) \in \oplus_i \Gamma(U_i, \mathcal{O}_Y)$. Since $(f_1^i + \epsilon g_1^i, ..., f_N^i + \epsilon g_N^i) = (f_1^i + \epsilon g_1^i, ..., f_N^i + \epsilon g_N^i)$, $f_\alpha^i + \epsilon g_\alpha^i = \sum_{\beta=1}^N (r_{i\beta}^\alpha + \epsilon h_{i\beta}^\alpha)(f_\beta^\alpha + \epsilon g_\beta^\alpha)$ for some $h_{i\beta}^\alpha \in \Gamma(U_i \cap U_j, \mathcal{O}_Y)$ so that we have $g_\alpha^i = \sum_{\beta-1}^N r_{i\beta}^\alpha g_\beta^i$. Hence $\{\bar{g}_1^i, ..., \bar{g}_N^i\}$ define a global section in $\mathbb{H}^0(X, \mathcal{N}_{X/Y})$.

On the other hand, since $(f_1^i + \epsilon g_1^i, ..., f_N^i + \epsilon g_N^i)$ is a Poisson ideal, we have $[\Lambda_0, f_\alpha^i + \epsilon g_\alpha^i] = \sum_{\beta=1}^N (f_\beta^i + \epsilon g_\beta^i)(T_{i\beta}^\alpha + \epsilon W_{i\beta}^\alpha)$ for some $W_{i\beta}^\alpha \in \Gamma(U_i, \mathcal{T}_Y)$. Then $[\Lambda_0, \bar{g}_\alpha^i] = \sum_{\beta=1}^N g_{i\beta}^\alpha T_{i\beta}^\alpha$ so that $\nabla(\{\bar{g}_1^i, ..., \bar{g}_N^i\}) = 0$. Hence $\{(\bar{g}_1^i, ..., \bar{g}_N^i)\} \in \mathbb{H}^0(X, \mathcal{N}_{X/Y})$.

Now we identify obstructions. Consider a small extension $e : 0 \rightarrow \hat{A} \rightarrow A \rightarrow 0$. Let $\eta := (\mathcal{X} \subset (Y \times \text{Spec}(\hat{A}), \Lambda_0))$ be an infinitesimal deformation of $X$ in $(Y, \Lambda_0)$ over $A$. Then $X$ is determined by a Poisson ideal sheaf $\mathcal{I}_A$ generated by $\{f_\alpha^i + \epsilon g_\alpha^i\}$, $\alpha = 1, ..., N$ for some $g_\alpha^i \in \Gamma(U_i, \mathcal{O}_Y)$ such that $F^i_\alpha \equiv f_\alpha^i \mod (m_A)$ and $\{F^i_1, ..., F^i_N\}$ is a regular sequence. Since $(F^i_1, ..., F^i_N) = (F^i_1, ..., F^i_N)$, we have $F^i_\alpha = \sum_{\beta=1}^N r_{i\beta}^\alpha T_{i\beta}^\alpha$ for some $R_{i\beta}^\alpha \in \Gamma(U_i, \mathcal{O}_Y)$.

On the other hand, since $(F^i_1, ..., F^i_N)$ is a Poisson ideal, we have $[\Lambda_0, F^i_\alpha] = \sum_{\beta=1}^N F^i_\beta W_{i\beta}^\alpha$ for some $W_{i\beta}^\alpha \in \Gamma(U_i, \mathcal{T}_Y) \otimes A$. Let $\bar{F}^i_\alpha \in \Gamma(U_i, \mathcal{O}_Y) \otimes A$ be an arbitrary lifting of $F^i_\alpha$, $\bar{R}^i_{i\beta} \in \Gamma(U_i, \mathcal{O}_Y) \otimes A$ be an arbitrary lifting of $W_{i\beta}^\alpha$, and $\bar{h}_{i\beta}^\alpha \in \Gamma(U_i \cap U_j, \mathcal{O}_Y)$ be an arbitrary lifting of $R_{i\beta}^\alpha$. Then $[\Lambda_0, \bar{F}^i_\alpha] - \sum_{\beta=1}^N \bar{F}^i_\beta \bar{R}^i_{i\beta} \equiv tG_i^\alpha$ for some $G_i^\alpha \in \Gamma(U_i, \mathcal{T}_Y)$, $\bar{F}^i_\alpha - \sum_{\beta=1}^N \bar{R}^i_{i\beta} \bar{h}_{i\beta}^\alpha \equiv tP_{i\beta}^\alpha$ for some $P_{i\beta}^\alpha \in \Gamma(U_i \cap U_j \cap U_k, \mathcal{O}_Y)$. We will show that $\{(\bar{G}_1^\alpha, ..., \bar{G}_N^\alpha) \oplus \{\bar{h}_{i\beta}^\alpha \}_1 \in C^0(\mathcal{U}, \mathcal{N}_{X/Y} \otimes \mathcal{T}_Y|_X) \oplus C^1(\mathcal{U}, \mathcal{N}_{X/Y})$ define a 1-cocycle in the following Čech resolution of $\mathcal{N}_{X/Y}^*$.
By taking
\[ \sum_{\beta=1}^{\gamma=1} \alpha \cap \gamma = 0 \]
Next we show that
\[ \delta \sum_{\beta=1}^{\gamma=1} \alpha \cap \gamma = 0 \]
Then we have
\[ (B.2.5) \]
\[ t[A_0, G_i^\alpha] = -\sum_{\gamma=1}^{N} \sum_{\beta=1}^{N} t[F_i^\beta \cap \gamma = 0 \]\nBy taking \(- \) on \(B.2.5\), we obtain
\[ (B.2.6) \]
\[ t[A_0, G_i^\alpha] = \sum_{\gamma=1}^{N} \sum_{\beta=1}^{N} t[F_i^\beta \cap \gamma = 0 \]
Next we show that \(\delta \{(\tilde{G}_i^1, ..., \tilde{G}_i^N)\} + \nabla(\{\tilde{h}_{ij}, ..., \tilde{h}_{ij}^N\}) = 0\). We have
\[ (B.2.7) \]
On the other hand, we have
\[ (B.2.8) \]
As in \(B.2.1\), we can show
\[ (\sum_{\beta=1}^{\gamma=1} \sum_{\beta=1}^{N} R_{ij}^\gamma W_{i\alpha} - [A_0, R_{ij}^\gamma W_{i\alpha}] - \sum_{\beta=1}^{N} R_{ij}^\gamma W_{i\alpha} = 0 \]
Then from \(B.2.7\) and \(B.2.8\), we obtain
\[ (B.2.9) \]
By taking \(- \) on \(B.2.9\), we get
\[ (B.2.10) \]
Lastly, we show that $\delta((\tilde{h}^1_{ij}, ..., \tilde{h}^N_{ij})) = 0$.

(B.2.11) $t(\sum_{\beta=1}^{N} \tilde{r}^\alpha_{ij\beta} h_{jk} - \tilde{h}^\alpha_{ik} + h_{ij}) = \sum_{\beta=1}^{N} \tilde{R}^\alpha_{ij\beta} \tilde{F}_j - \sum_{\beta, \gamma=1}^{N} \tilde{R}^\alpha_{ij\beta} \tilde{R}^\beta_{jk\gamma} \tilde{F}_\gamma - \tilde{F}_i^\alpha + \sum_{\beta=1}^{N} \tilde{R}^\alpha_{ik\beta} \tilde{F}_k + \tilde{F}_i^\alpha - \sum_{\beta=1}^{N} \tilde{R}^\alpha_{ijk\beta} \tilde{F}_j = - \sum_{\beta, \gamma=1}^{N} \tilde{R}^\alpha_{ij\beta} \tilde{R}^\beta_{jk\gamma} \tilde{F}_\gamma + \sum_{\beta=1}^{N} \tilde{R}^\alpha_{ik\beta} \tilde{F}_k = \sum_{\gamma=1}^{N} \tilde{t}^\alpha_{ijk\gamma} \tilde{F}_\gamma = \sum_{\gamma=1}^{N} \tilde{t}^\alpha_{ijk\gamma} \tilde{F}_\gamma$.

By taking $-\text{on (B.2.11)},$ we get

(B.2.12) $t(\sum_{\beta=1}^{N} \tilde{r}^\alpha_{ij\beta} \tilde{h}_{jk} - \tilde{h}^\alpha_{ik} + \tilde{h}^\alpha_{ij}) = \sum_{\gamma=1}^{N} \tilde{t}^\alpha_{ijk\gamma} \tilde{f}_k = 0 \iff \sum_{\beta=1}^{N} \tilde{r}^\alpha_{ij\beta} \tilde{h}_{jk} - \tilde{h}^\alpha_{ik} + \tilde{h}^\alpha_{ij} = 0$.

Hence from (B.2.13), (B.2.10) and (B.2.12), $\{(-\tilde{G}^1_{ij}, ..., -\tilde{G}^N_{ij})\} \oplus \{(h^1_{ij}, ..., \tilde{h}^N_{ij})\} \in C^0(U \cap X, \mathbb{N}_{X/Y} \otimes T_Y|_X) \oplus C^1(U \cap X, \mathbb{N}_{X/Y})$ define a 1-cocycle in the above Čech resolution.

Now we choose another arbitrary lifting $F^\alpha_i \in \Gamma(U_i, \mathcal{O}_Y) \otimes \hat{A}$ of $F^\alpha_i$, another arbitrary lifting $\hat{T}_{i\alpha}^\beta \in \Gamma(U_i, T_Y) \otimes \hat{A}$ of $W_{i\alpha}^\beta$ and another arbitrary lifting $R^\alpha_{ij\beta} \in \Gamma(U_i \cap U_j, \mathcal{O}_Y) \otimes \hat{A}$. We show that the associated cohomology class $b := \{(-\tilde{G}^1_{ij}, ..., -\tilde{G}^N_{ij})\} \oplus \{(h^1_{ij}, ..., \tilde{h}^N_{ij})\}$ is cohomologous to $a := \{(-\tilde{G}^1_{ij}, ..., -\tilde{G}^N_{ij})\} \oplus \{(\tilde{h}^1_{ij}, ..., \tilde{h}^N_{ij})\}$. We note that $\tilde{F}^\alpha_i = \tilde{F}^\alpha_i + tA^\alpha_i$ for some $A^\alpha_i \in \Gamma(U_i, \mathcal{O}_Y)$, $\tilde{T}_{i\alpha}^\beta = \tilde{T}_{i\alpha}^\beta + tB_{i\alpha}^\beta$ for some $B_{i\alpha}^\beta \in \Gamma(U_i, T_Y)$ and $\tilde{R}^\alpha_{ij\beta} = \tilde{R}^\alpha_{ij\beta} + tC^\alpha_{ij\beta}$ for some $C^\alpha_{ij\beta} \in \Gamma(U_i \cap U_j, \mathcal{O}_Y)$.

(B.2.13) $t(G^\alpha_i - G^\alpha_{ij}) = \{A_0, \tilde{F}^\alpha_i\} - \sum_{\beta=1}^{N} \tilde{F}_i^\beta \tilde{T}_{i\alpha}^\beta = \{A_0, \tilde{F}^\alpha_i\} + \sum_{\beta=1}^{N} \tilde{F}_i^\beta \tilde{T}_{i\alpha}^\beta = \{A_0, tA^\alpha_i\} - \sum_{\beta=1}^{N} tA^\beta_i \tilde{T}_{i\alpha}^\beta = \sum_{\beta=1}^{N} tB_{i\alpha}^\beta$.

By taking $-\text{on (B.2.13)},$ we get $-G^\alpha_i - (-G^\alpha_{ij}) = [-A^\alpha_i, A_0] + \sum_{\beta=1}^{N} \tilde{A}^\beta_i \tilde{T}_{i\alpha}^\beta$.

On the other hand,

(B.2.14) $t(h^\alpha_{ij} - h^\alpha_{ij}) = \tilde{F}^\alpha_i - \sum_{\beta=1}^{N} \tilde{R}^\alpha_{ij\beta} \tilde{F}_j = \tilde{F}^\alpha_i + \sum_{\beta=1}^{N} \tilde{R}^\alpha_{ij\beta} \tilde{F}_j = tA^\alpha_i - \sum_{\beta=1}^{N} \tilde{r}^\alpha_{ij\beta} tA^\beta_i - \sum_{\beta=1}^{N} tC^\alpha_{ij\beta} f_{ij}^\beta$.

By taking $-\text{on (B.2.14)},$ we get $h^\alpha_{ij} - h^\alpha_{ij} = \tilde{A}^\alpha_i - \sum_{\beta=1}^{N} \tilde{r}^\alpha_{ij\beta} \tilde{A}^\beta_i$. Hence $\{(\tilde{A}^1_i, ..., \tilde{A}^N_i)\}$ is mapped to $b - a$ so that $a$ is cohomologous to $a$. So given a small extension $e: 0 \rightarrow (t) \rightarrow \hat{A} \rightarrow A \rightarrow 0$, we can associate an element $o_\theta(e)$ := the cohomology class $a \in \mathbb{H}^1(X, \mathbb{N}_{X/Y})$. We note that $o_\theta(e) = 0$ if and only if there exists a collection $\{\tilde{F}^\alpha_i\}, \{\tilde{T}_{i\alpha}^\beta\}$ and $\{\tilde{R}^\alpha_{ij\beta}\}$ such that $\tilde{h}^\alpha_{ij} = 0$ and $\tilde{G}^\alpha_i = 0, a = 1, ..., N$:

1. If $\tilde{h}^\alpha_{ij} = 0$, then $\tilde{h}^\alpha_{ij} = f_{ij}^1 L_1^\beta + ... + f_{ij}^N L_N^\beta$ for some $L_1^\beta \in \Gamma(U_i \cup U_j, \mathcal{O}_Y)$. Then $\tilde{F}^\alpha_i = \sum_{\beta=1}^{N} \tilde{R}^\alpha_{ij\beta} \tilde{F}_j = t(\sum_{\beta=1}^{N} \tilde{F}^\beta_j L_1^\beta)$. Hence $\tilde{F}^\alpha_i = \sum_{\beta=1}^{N} (\tilde{r}^\alpha_{ij\beta} + tL_1^\beta) \tilde{F}_j = 0$ so that $\tilde{F}^\alpha_i = (\tilde{F}^1_j, ..., \tilde{F}^N_j)$. Hence $\{(\tilde{F}^1_i, ..., \tilde{F}^N_i)\}$ is an ideal sheaf on $Y \times \text{Spec}(\hat{A})$.

2. If $\tilde{G}^\alpha_i = 0$, then $\tilde{G}^\alpha_i = f_{ij}^1 P_1^\beta + ... + f_{ij}^N P_N^\beta$ for some $P_1^\beta \in \Gamma(U_i, T_Y)$, then $\tilde{A}^\alpha_i = \sum_{\beta=1}^{N} \tilde{T}^\alpha_i T_{i\alpha}^\beta = t \sum_{\beta=1}^{N} \tilde{T}^\alpha_i \tilde{F}_j^\beta = t \sum_{\beta=1}^{N} \tilde{T}^\alpha_i T_{i\alpha}^\beta$ so that $\{(\tilde{F}^1_i, ..., \tilde{F}^N_i)\}$ is a Poisson ideal.

Hence $\theta_\theta(e) = 0$ if and only if there is a lifting of $\eta$ to $\hat{A}$.

\[ \square \]

B.3. Deformations of Poisson closed subschemes of codimension 1 and Poisson semi-regularity.

(Compare, e.g., p.143-144)

Let $(L, \nabla)$ be a Poisson invertible sheaf on a nonsingular Poisson variety $(X, \Lambda_0)$, where $\nabla$ is a Poisson connection on $L$, and we denote by $\mathbb{H}(X, L^\bullet)$ the $i$-th hypercohomology group of the following complex of sheaves (see [Kim14a]):

\[ L^\bullet : L \xrightarrow{\nabla} L \otimes T_X \xrightarrow{\nabla} L \otimes \wedge^2 T_X \xrightarrow{\nabla} L \otimes \wedge^2 T_X \xrightarrow{\nabla} \cdots \]
Let $s \in \mathbb{H}^0(X, L^\bullet)$ and $D = \text{div}(s) \subset (X, \Lambda_0)$ be the Poisson divisor associated with $s$. Then we have a morphism of functors of Artin rings:

\[(B.3.2) \quad H^i_D(X, \Lambda_0) \to \text{Def}_{f(L, \nabla)}\]

where $\text{Def}_{f(L, \nabla)}$ is the functor of deformations of $(L, \nabla)$ (see [Kim14a]). Let us consider the following exact sequence $0 \to \mathcal{O}_X \xrightarrow{s} L^\bullet \to L_D^\bullet \to 0$:

\[
\begin{array}{cccc}
\cdots & \cdots & \cdots \\
\text{[-1,0]} & \Rightarrow & \Rightarrow & \\
0 & \to & \wedge^2 T_X & s \\
\text{[-1,0]} & \Rightarrow & \Rightarrow & \\
0 & \to & T_X & s \\
\text{[-1,0]} & \Rightarrow & \Rightarrow & \\
0 & \to & \mathcal{O}_X & s \\
\end{array}
\]

so that we have a long exact sequence

\[0 \to \mathbb{H}^0(X, \mathcal{O}_X^\bullet) \to \mathbb{H}^0(X, L^\bullet) \to \mathbb{H}^0(D, L_D^\bullet) \xrightarrow{\delta_0} \mathbb{H}^1(X, \mathcal{O}_X^\bullet) \to \mathbb{H}^1(D, L_D^\bullet) \xrightarrow{\delta_1} \mathbb{H}^2(X, \mathcal{O}_X^\bullet) \to \cdots\]

Then $\delta_0$ represents the morphism of tangent spaces for $H^1_D(X, \Lambda_0)(k[\epsilon]) \to \text{Def}_{f(L, \nabla)}(k[\epsilon])$ and $\delta_1$ represents the obstruction map for $B.3.2$.

**Remark B.3.3** (Poisson semi-regularity). Let $(X, \Lambda_0)$ be a nonsingular Poisson projective variety. A Poisson Cartier divisor $D$ on $X$ is called Poisson semi-regular if the natural map

\[\mathbb{H}^1(X, \mathcal{O}_X(D)^\bullet) \to \mathbb{H}^1(D, \mathcal{O}_D(D)^\bullet)\]

is zero. If $D \subset (X, \Lambda_0)$ is Poisson semi-regular and $\text{Def}_{f(L, \nabla)}$ is smooth, then $H^1_D(X, \Lambda_0)$ is smooth so that $D \subset (X, \Lambda_0)$ is unobstructed.

### Appendix C. Simultaneous deformations of Poisson structures and Poisson closed subschemes

**C.1. The local extended Poisson Hilbert functor.**

Let $X \subset (Y, \Lambda_0)$ be a closed embedding of algebraic Poisson schemes where $(Y, \Lambda_0)$ is a nonsingular Poisson variety, and $A \in \textbf{Art}$. An infinitesimal simultaneous deformation of $X$ in $(Y, \Lambda_0)$ over $A \in \textbf{Art}$ is a cartesian diagram of morphisms of schemes

\[
\begin{array}{ccc}
X & \longrightarrow & X \subset (Y \times \text{Spec}(A), \Lambda) \\
\eta : & & \downarrow \pi \\
\text{Spec}(k) & \longrightarrow & S = \text{Spec}(A)
\end{array}
\]

where $\pi$ is flat, and it is induced by the projection from $Y \times S$ to $S$, $\Lambda \in \Gamma(Y \times S, \mathcal{H}om(\wedge^2 \Omega^1_{Y \times S}, \mathcal{O}_{Y \times S}))$ defines a Poisson structure on $Y \times S$ over $S$, $X$ is a Poisson closed subscheme of $(Y \times S, \Lambda)$, and $(Y \times S, \Lambda)$ induces $(Y, \Lambda_0)$. Then we can define a functor of Artin rings (called the local extended Poisson Hilbert functor of $X$ in $(Y, \Lambda_0)$)

\[EH^Y_X(Y, \Lambda_0) : \textbf{Art} \to (\text{Sets})\]

\[A \mapsto \{\text{infinitesimal simultaneous deformations of } X \text{ in } (Y, \Lambda_0)\}\]

**C.2. The extended complex associated with the normal bundle $N_{X/Y}$ of a Poisson closed subschemes of a nonsingular Poisson variety.**

Let $(Y, \Lambda_0)$ be a nonsingular Poisson variety and $X$ be a Poisson closed subscheme of $(Y, \Lambda_0)$ defined by a Poisson ideal sheaf $\mathcal{I}$. Assume that $i : X \hookrightarrow Y$ be a regular embedding. We keep the notations in subsection B.2.
We define the extended complex $(\wedge^2 T_Y \oplus i_* N_{X/Y})^\bullet$ associated with the normal bundle $N_{X/Y}$:

$$(\wedge^2 T_Y \oplus i_* N_{X/Y})^\bullet : \wedge^2 T_Y \oplus i_* N_{X/Y} \xrightarrow{\nabla} \wedge^3 T_Y \oplus (i_* N_{X/Y} \otimes T_Y |_X) \xrightarrow{\nabla} \wedge^4 T_Y \oplus (i_* N_{X/Y} \otimes \wedge^2 T_Y |_X) \xrightarrow{\nabla} \cdots$$

The complex is defined locally in the following way:

$$\nabla : \Gamma(U_i, \wedge^{p+2}T_Y) \oplus (\oplus \Gamma(U_i \cap X, \wedge^p T_Y |_X) \rightarrow \Gamma(U_i, \wedge^{p+3}T_Y) \oplus (\oplus \Gamma(U_i \cap X, \wedge^{p+1} T_Y |_X)$$

$$(\Pi_i, (g_i, \ldots, g_i^N)) \mapsto (-[\Pi_i, \Lambda_0], [\Pi_i, f_i^1] - [g_i^1, \Lambda_0] + (-1)^p \sum_{\beta=1}^N g_i^\beta \wedge \tilde{T}_{i1}^\beta, \ldots, [\Pi_i, f_i^N] - [g_i^N, \Lambda_0] + (-1)^p \sum_{\beta=1}^N g_i^\beta \wedge \tilde{T}_{iN}^\beta)$$

**Proposition C.2.2.** Given a regular closed embedding of algebraic Poisson schemes $i : X \hookrightarrow (Y, \Lambda_0)$, where $(Y, \Lambda_0)$ is a nonsingular Poisson variety,

1. There is a natural identification $\text{EH}_X^{(Y, \Lambda_0)}(k[e]) \cong \mathbb{H}^0(Y, (\wedge^2 T_Y \oplus i_* N_{X/Y})^\bullet)$

2. Given an infinitesimal simultaneous deformation $\eta$ of $X$ in $(Y, \Lambda_0)$ over $A \in \text{Art}$ and a small extension $e : 0 \rightarrow (t) \rightarrow \bar{A} \rightarrow A \rightarrow 0$, we can associate an element $o_\eta(e) \in \mathbb{H}^1(Y, (\wedge^2 T_Y \oplus i_* N_{X/Y})^\bullet)$, which is zero if and only if $\eta$ is a lifting of $\eta$ to $\bar{A}$.

**Proof.** Let $U = \{U_i\}$ be an affine open covering of $Y$ and let $I_i = (f_i^1, \ldots, f_i^N)$ be a Poisson ideal of $\Gamma(U_i, O_Y)$ defining $U_i \cap X$ such that $(f_i^1, \ldots, f_i^N)$ is a regular sequence. We keep the notations in subsection [B.2.4]

A first-order simultaneous deformation of $X$ in $(Y, \Lambda_0)$ is a flat family

$$X \longrightarrow X \subset (Y \times \text{Spec}(k[e]), \Lambda_0 + \epsilon \Lambda')$$

Since $\Lambda_0 + \epsilon \Lambda'$ with $\Lambda' \in H^0(Y, \wedge^2 T_Y)$ define a Poisson structure on $Y \times \text{Spec}(k[e])$, we have $[\Lambda_0, \Lambda'] = 0$. Assume that $\Lambda$ is determined by a Poisson ideal sheaf $\mathcal{I}_A$ generated by $\{f_i^\alpha + \epsilon g_i^\alpha, \alpha = 1, \ldots, N$ for some $g_i^\alpha \in \Gamma(U_i, O_Y)$. Then $(\tilde{g}_i^1, \ldots, \tilde{g}_i^N)$ define a global section in $H^0(X, N_{X/Y})$ as in the proof of Proposition [B.2.4]. On the other hand, $(f_i^1 + \epsilon g_i^1, \ldots, f_i^N + \epsilon g_i^N)$ is a Poisson ideal, we have $[\Lambda_0 + \epsilon \Lambda', f_i^\alpha + \epsilon g_i^\alpha] = \sum_{\beta=1}^N (f_i^\beta + \epsilon g_i^\beta) \wedge (T_{i\alpha}^\beta + \epsilon W_{i\alpha})$ for some $W_{i\alpha} \in \Gamma(U_i, T_Y)$. Then $[\Lambda_0, g_i^1 + \epsilon \Lambda', f_i^\alpha + \epsilon g_i^\alpha] = \sum_{\beta=1}^N g_i^\beta T_{i\alpha}^\beta W_{i\alpha}$ so that $\nabla(-\Lambda', (\tilde{g}_i^1, \ldots, \tilde{g}_i^N)) = 0$. Hence $(-\Lambda', (\tilde{g}_i^1, \ldots, \tilde{g}_i^N)) \in \mathbb{H}^0(Y, (\wedge^2 T_Y \oplus i_* N_{X/Y})^\bullet)$.

Next we identify obstructions. Consider a small extension $e : 0 \rightarrow (t) \rightarrow \bar{A} \rightarrow A \rightarrow 0$. Let $\eta := (X \subset (Y \times \text{Spec}(A), \Lambda))$ be an infinitesimal simultaneous deformation of $X$ in $(Y, \Lambda_0)$ over $A$. Then $\Lambda$ is determined by a Poisson ideal sheaf $\mathcal{I}_A$ generated by $(F_i^1, \ldots, F_i^N)$ in $\Gamma(U_i, O_Y) \otimes A$ such that $F_i^\alpha \equiv f_i^\alpha \mod A$, and $(F_i^1, \ldots, F_i^N)$ is a regular sequence. Let $\Lambda_1 \in \Gamma(U_i, \wedge^2 T_Y) \otimes A$ be the restriction of $\Lambda$ on $U_i$. Since $(F_i^1, \ldots, F_i^N) = (f_i^1, \ldots, F_i^N)$, we have $F_i^\alpha = \sum_{\beta=1}^N R_{i\beta} F_i^\beta$ for some $R_{i\beta} \in \Gamma(U_i \cap U_j, O_Y) \otimes A$. Since $(F_i^1, \ldots, F_i^N)$ is a Poisson ideal, we have $[\Lambda_1, F_i^\alpha] = \sum_{\beta=1}^N F_i^\beta R_{i\beta}$ for some $W_{i\alpha} \in \Gamma(U_i, T_Y) \otimes A$. Let $\tilde{\Lambda}_1 \in \Gamma(U_i, \wedge^2 T_Y) \otimes A$ be an arbitrary lifting of $\Lambda_1$, $\tilde{F}_i^\alpha \in \Gamma(U_i, O_Y)$ be an arbitrary lifting of $F_i^\alpha$, $\tilde{T}_{i\alpha} \in \Gamma(U_i, T_Y) \otimes A$ be an arbitrary lifting of $W_{i\alpha}$, and $\tilde{R}_{i\beta} \in \Gamma(U_i \cap U_j, \wedge^2 T_Y) \otimes A$ be an arbitrary lifting of $R_{i\beta}$. Then $[\tilde{\Lambda}_1, \tilde{F}_i^\alpha] = \tilde{g}_i^\alpha$ for some $g_i^\alpha \in \Gamma(U_i, O_Y)$, $\tilde{\Lambda}_1 - \bar{\Lambda}_1 = t\tilde{\Lambda}_1$ for some $\tau \in \Gamma(U_i \cap U_j, \wedge^2 T_Y)$, $[\tilde{\Lambda}_1, \tilde{F}_i^\alpha] - \sum_{\beta=1}^N \tilde{T}_{i\beta}^\alpha R_{i\beta} = tG_i^\alpha$ for some $G_i^\alpha \in \Gamma(U_i, T_Y)$, $\tilde{F}_i^\alpha - \sum_{\beta=1}^N \tilde{R}_{i\beta} \tilde{T}_{i\beta}^\alpha = th_i^\alpha$ for some $h_i^\alpha \in \Gamma(U_i \cap U_j, O_Y)$, and $\tilde{R}_{i\beta} - \sum_{\beta=1}^N \tilde{R}_{i\beta} \tilde{T}_{i\beta}^\alpha \tilde{R}_{j\gamma} = tP_{ijk}$ for some $P_{ijk} \in \Gamma(U_i \cap U_j \cap U_k, O_Y)$. We will show that $((\tilde{\Lambda}_1, \tilde{F}_i^\alpha, \tilde{G}_i^\alpha, \tilde{h}_i^\alpha)) \in \mathbb{H}^0(U, \wedge^3 T_Y \oplus i_* (N_{X/Y} \otimes T_Y |_X)) \Rightarrow C^1(U, \wedge^2 T_Y \oplus i_* N_{X/Y})$ define a 1-cocycle in the following Čech resolution of $(\wedge^2 T_Y \oplus i_* N_{X/Y})^\bullet$:
First we show that $\hat{\nabla}((\{\frac{1}{2}\Pi_i\}, \{-\hat{G}_i^1, ..., -\hat{G}_i^N\})) = 0$. From (A.0.16), we have

(C.2.3) $-\frac{1}{2} \Pi_i, \Lambda_0 = 0$

On the other hand, as in (15.2.2), we can show $\sum_{\gamma=1}^N \bar{F}^\gamma_i (\bar{A}_i, \bar{T}^\alpha_{i\alpha}) = \sum_{\beta=1}^N \bar{F}^\beta_i t Q^\gamma_{i\alpha}$ for some $\hat{Q}_{i\alpha} \in \Gamma(U, \wedge^2 T_y)$. Then we have

(C.2.4) $t[\Lambda_0, G^\alpha_i] = \left[\bar{A}_i, \bar{F}^\alpha_i\right] - \sum_{\beta=1}^N \bar{F}^\beta_i \bar{T}^\beta_{i\alpha} = t[\frac{1}{2} \Pi_i, f^{\alpha_i}] - \sum_{\beta=1}^N \bar{F}^\beta_i \bar{A}_i, \bar{T}^\beta_{i\alpha} + \sum_{\beta=1}^N \bar{F}^\beta_i T^\alpha_{i\alpha}$

By taking $-\frac{1}{2} \Pi_i, f^{\alpha_i}$, we obtain

(C.2.5) $\iff \left[\frac{1}{2} \Pi_i, f^{\alpha_i}\right] - \left[-G^\alpha_i, \Lambda_0\right] = (-1)^1 \sum_{\beta=1}^N \bar{G}^\alpha_i \wedge \bar{T}^\beta_{i\alpha} = 0$

Next we show that $\delta((\{\frac{1}{2}\Pi_i\}, \{-\Lambda_i^1, ..., -\Lambda_i^N\})) + \hat{\nabla}((\{\Lambda_i^1\}, \{\hat{h}_{ij}^1, ..., \hat{h}_{ij}^N\})) = 0$. From (A.0.17), we have

(C.2.6) $\delta(\frac{1}{2} \Pi_i) = \left[-\Lambda_i, \Lambda_0\right] = 0$

On the other hand, we have

(C.2.7) $t(G^\alpha_i - \sum_{\beta=1}^N \bar{F}_i^{\alpha_\beta} G^\beta_j) = \left[\bar{A}_i, \bar{F}^\alpha_i\right] - \sum_{\beta=1}^N \bar{F}^\beta_i \bar{T}^\beta_{i\alpha} - \sum_{\beta=1}^N \bar{R}^\alpha_{i\beta} \bar{A}_i, \bar{T}^\beta_{i\beta} + \sum_{\beta, \gamma=1}^N \bar{R}^\alpha_{i\beta} \bar{F}^\gamma_{i\beta}$

(C.2.8) $\left[\Lambda_0, h^\alpha_{ij}\right] + t \sum_{\beta=1}^N h^\beta_{ij} T^\alpha_{\beta\alpha} = \left[\bar{A}_i, \bar{F}^\alpha_i\right] - \sum_{\beta=1}^N \bar{F}^\beta_i \bar{A}_i, \bar{T}^\beta_{ij} - \sum_{\beta=1}^N \bar{R}^\alpha_{ij\beta} \bar{F}^\beta_{ij} + \sum_{\beta, \gamma=1}^N \bar{R}^\alpha_{ij\beta} \bar{T}^\beta_{ij\beta}$

As in (15.2.1), we can show $\sum_{\gamma=1}^N \bar{F}^\gamma_{ij} (\sum_{\beta=1}^N \bar{R}^\alpha_{ij\beta} W^\gamma_{i\beta} - \bar{A}_i, \bar{R}^\alpha_{ij\gamma}) - \sum_{\beta=1}^N \bar{R}^\alpha_{ij\beta} W^\gamma_{ij\beta}) = 0$ so that we have $\sum_{\gamma=1}^N \bar{F}^\gamma_{ij} (\sum_{\beta=1}^N \bar{R}^\alpha_{ij\beta} T^\gamma_{i\beta} - \bar{A}_i, \bar{R}^\alpha_{ij\gamma}) - \sum_{\beta=1}^N \bar{R}^\alpha_{ij\beta} T^\gamma_{ij\beta}) = \sum_{\gamma=1}^N \bar{F}^\gamma_{ij} S^\alpha_{ij\gamma}$ for some $S^\alpha_{ij\gamma} \in \Gamma(U_i \cap U_j, T_y)$. Then
from (C.2.7) and (C.2.8), we get

\[ (C.2.9) \quad t(G^\alpha_i - \sum_{\beta=1}^N r^\alpha_{ij\beta} G^\beta_j) - t[A_0, h^\alpha_i] + t \sum_{\beta=1}^N h^\beta_i T^\beta_\alpha = \sum_{\beta=1}^N \tilde{R}^\alpha_{ij\beta} \tilde{F}^\beta_j \tilde{T}^\gamma_j \gamma + \sum_{\gamma=1}^N \tilde{F}^\alpha_j [\tilde{A}_i, \tilde{R}^\alpha_{j\gamma}] + \sum_{\beta=1}^N r^\alpha_{ij\beta} [tA'_{ij}, f^\beta_j] - \sum_{\beta=1}^N \tilde{R}^\alpha_{ij\gamma} \tilde{F}^\gamma_j \tilde{T}^\beta_\alpha \]

By taking \(-\) on (C.2.9), we get

\[ (C.2.10) \quad \iff \sum_{\beta=1}^N \tilde{F}^\alpha_{ij\beta} \cdot (-\tilde{G}^\beta_j) + [\bar{A}'_{ij}, f^\beta_j] - [h^\alpha_i, A_0] + \sum_{\beta=1}^N \tilde{h}^\beta_i \tilde{T}^\beta_\alpha = 0 \]

Lastly, from (A.0.18) and (B.2.12), we have

\[ (C.2.11) \quad \delta(\{-\Lambda'_{ij}\}, \{\tilde{h}^1_{ij}, ..., \tilde{h}^N_{ij}\}) = 0. \]

Hence from (C.2.3), (C.2.5), (C.2.6), (C.2.10), and (C.2.11), \(((\frac{1}{2} \Pi_i), \{(-G^1_i, ..., \tilde{G}^N_i)\}) \oplus \{(-\Lambda'_{ij}), \{(\tilde{h}^1_{ij}, ..., \tilde{h}^N_{ij})\}\} \in C^0(\mathcal{U}, \psi^2 T_Y \oplus i_* \mathcal{N}_{X/Y} \otimes T_X) \oplus C^1(\mathcal{U}, \psi^2 T_Y \oplus i_* \mathcal{N}_{X/Y})\) define a 1-cocycle in the above Čech resolution.

Now we choose another arbitrary lifting \(F'^\alpha_i \in \Gamma(U_i, \mathcal{O}_Y) \otimes \tilde{A}\) of \(F^\alpha_i\), another arbitrary lifting \(\tilde{T}'^\beta_\alpha \in \Gamma(U_i, T_Y) \otimes \tilde{A}\) of \(T^\beta_\alpha\) and another arbitrary lifting \(\bar{A}'_i \in \Gamma(U_i, \lambda^2 T_Y) \otimes \tilde{A}\) of \(A'_i\). We show that the associated cohomology class \(b := (\frac{1}{2} \Pi_i), \{(-\tilde{G}^1_i, ..., \tilde{G}^N_i)\}) \oplus \{(-\Lambda'_ij), \{(\tilde{h}^1_{ij}, ..., \tilde{h}^N_{ij})\}\} \) is cohomologous to \(a := (\frac{1}{2} \Pi_i), \{(-\tilde{G}^1_i, ..., \tilde{G}^N_i)\}) \oplus \{(-\Lambda'_{ij}), \{(\tilde{h}^1_{ij}, ..., \tilde{h}^N_{ij})\}\} \). We note that \(\tilde{F}'^\alpha_i = \tilde{F}^\alpha_i + tA'_i\) for some \(A'_i \in \Gamma(U_i, \mathcal{O}_Y)\), \(\tilde{T}'_\alpha = \tilde{T}_\alpha + tB_\alpha\) for some \(B_\alpha \in \Gamma(U_i, T_Y)\), \(\bar{F}'_{ij\beta} = \bar{F}_{ij\beta} + tC^\beta_{ij\beta}\) for some \(C^\beta_{ij\beta} \in \Gamma(U_i \cap U_j, \mathcal{O}_Y)\) and \(\bar{A}'_{ij} = \bar{A}_{ij} + tD_i\) for some \(D_i \in \Gamma(U_i, \lambda^2 T_Y)\).

\[ (C.2.12) \quad t(G'^\alpha_i - G^\alpha_i) = [\bar{A}', \tilde{F}'^\alpha_i] - \sum_{\beta=1}^N \tilde{F}^\beta_{ij\beta} \tilde{T}^\beta_\alpha = [\bar{A}, \tilde{F}^\alpha_i] + \sum_{\beta=1}^N \tilde{F}^\beta_{ij\beta} \tilde{T}^\beta_\alpha = [tD_i, f^\alpha_i] + [A_0, tA'_i] - \sum_{\beta=1}^N tA^\beta_i T^\beta_\alpha - \sum_{\beta=1}^N \tilde{f}^\beta_i tB^\beta_\alpha \]

By taking \(-\) on (C.2.12), we get

\[ (C.2.13) \quad \tilde{G}'^\alpha_i - G^\alpha_i = [D_i, f^\alpha_i] + [A_0, A'_i] - \sum_{\beta=1}^N \tilde{A}^\beta_i \tilde{T}^\beta_\alpha \iff -\tilde{G}_i^\alpha = (-\tilde{G}'_i^\alpha) = [D_i, f^\alpha_i] - [A_0, -A'_i] + \sum_{\beta=1}^N -\tilde{A}^\beta_i \tilde{T}^\beta_\alpha \]

and from (B.2.14), we have \(\tilde{h}^\alpha_{ij} - \bar{h}^\alpha_{ij} = \tilde{A}^\alpha_i - \sum_{\beta=1}^N \tilde{h}^\beta_{ij} \tilde{A}^\beta_j\) so that \(\{\tilde{h}^\alpha_{ij} - \bar{h}^\alpha_{ij}\} = -\delta(\{-A^\alpha_i\})\). On the other hand, from (A.0.19) and (A.0.20), we have \(\frac{1}{2} \Pi_i - \frac{1}{2} \Pi'_i = [D_i, A_0]\) and \(-\Lambda'_{ij} - (-\Lambda^0_{ij}) = -\delta(D_i)\). Hence \((\{D_i\} \oplus \{-A^0_{ij}\}) \) is mapped to \(a\) so that \(a\) is cohomologous to \(b\). So given a small extension \(e: 0 \to (t) \to \tilde{A} \to A \to 0\), we can associate an element \(o_\eta(e):=\) the cohomology class \(a \in \mathbb{H}^1(Y, (\lambda^2 T_Y \oplus i_* \mathcal{N}_{X/Y})*).\) We note that \(o_\eta(e) = 0\) if and only if there exists collections \(\{\tilde{F}_i^\alpha\}, \{\tilde{T}_\alpha\}, \{\bar{F}_{ij\beta}\}\) and \(\{\bar{A}\}\) such that \(\tilde{h}^\alpha_{ij} = 0, \tilde{G}_i^\alpha = 0, \alpha = 1, ..., N, \Pi_i = 0, \) and \(\bar{A}'_{ij} = 0\):

1. If \(\tilde{h}^\alpha_{ij} = 0\), then \((\tilde{F}_1^i, ..., \tilde{F}_N^i) = (\tilde{F}_1^i, ..., \tilde{F}_N^i)\) so that \(\{(\tilde{F}_1^i, ..., \tilde{F}_N^i)\}\) define an ideal sheaf on \(Y \times \text{Spec}(\tilde{A})\).
2. If \(\Pi_i = 0\) and \(\Lambda'_{ij} = 0\), then \([\bar{A}, \bar{A}_{ij}] = 0\), and \(\{\bar{A}\}\) glues together to define a Poisson structure on \(Y \times \text{Spec}(\tilde{A})\).
3. If \(G^\alpha_i = 0\), then \(G^\alpha_i = \sum_{\beta=1}^N f^\beta_i \tilde{P}_i^\beta\) for some \(P_i^\beta \in \Gamma(U_i, T_Y)\). Then \(\bar{A}, \bar{F}_1^\alpha = \sum_{\beta=1}^N \tilde{f}^\beta_i + tP_i^\beta \tilde{F}_1^\beta\) so that \((\tilde{F}_1^i, ..., \tilde{F}_N^i)\) defines a Poisson ideal.

Hence \(o_\eta(e) = 0\) if and only if there is a lifting of \(\eta\) to \(\tilde{A}\). \(\square\)
REFERENCES

[BHPVdV04] Wolf P. Barth, Klaus Hulek, Chris A. M. Peters, and Antonius Van de Ven, Compact complex surfaces, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 4, Springer-Verlag, Berlin, 2004. MR 2030225 (2004m:14070)

[Kim14a] Chunghoon Kim, Deformations of nonsingular Poisson varieties and Poisson invertible sheaves, preprint (2014).

[Kim14b] Chunghoon Kim, Deformations of nonsingular Poisson varieties and Poisson invertible sheaves, preprint (2014).

[Kod62] K. Kodaira, A theorem of completeness of characteristic systems for analytic families of compact submanifolds of complex manifolds, Ann. of Math. (2) 75 (1962), 146–162. MR 0153033 (27 #3002)

[Kod63] K. Kodaira, On stability of compact submanifolds of complex manifolds, Amer. J. Math. 85 (1963), 79–94. MR 0153033 (27 #3002)

[Kod05] Kunihiko Kodaira, Complex manifolds and deformation of complex structures, english ed., Classics in Mathematics, Springer-Verlag, Berlin, 2005, Translated from the 1981 Japanese original by Kazuo Akao. MR 2109686 (2005h:32030)

[KS59] K. Kodaira and D. C. Spencer, A theorem of completeness of characteristic systems of complete continuous systems, Amer. J. Math. 81 (1959), 477–500. MR 0112156 (22 #3011)

[LGPV13] Camille Laurent-Gengoux, Anne Pichereau, and Pol Vanhaecke, Poisson structures, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 347, Springer, Heidelberg, 2013. MR 2906391

[Ser06] Edoardo Sernesi, Deformations of algebraic schemes, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 334, Springer-Verlag, Berlin, 2006. MR 2247603 (2008e:14011)

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