PSU(2,2|4) Exchange Algebra
of \( \mathcal{N} = 4 \) Superconformal Multiplets

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Abstract

It is known that the unitary representation of the \( D = 4 \), \( \mathcal{N} = 4 \) superconformal multiplets and their descendants are constructed as supercoherent states of bosonic and fermionic creation oscillators which covariantly transform under SU(2,2|4). We non-linearly realize those creation oscillators on the coset superspace PSU(2,2|4)/\{SO(1,4)\otimes SO(5)\} which is reparametrized by the \( D = 10 \) supercoordinates (\( X, \Theta \)). We consider a \( D = 2 \) non-linear \( \sigma \) model on the coset superspace and set up Poisson brackets for \( X \) and \( \Theta \) on the light-like line. It is then shown that the non-linearly realized creation oscillators satisfy the classical exchange algebra with the classical r-matrix of PSU(2,2|4). We have recourse to purely algebraic quantization of the classical exchange algebra in which the r-matrix is promoted to the universal R-matrix. The quantum exchange algebra essentially characterizes correlation functions of the \( D = 4 \), \( \mathcal{N} = 4 \) superconformal multiplets and their descendants on the light-like line. It is because they are supercoherent states of the oscillators. The arguments are straightforwardly extended to the case where those quantities are endowed with the U(\( N \)) YM gauge symmetry.

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1 Introduction

The gauge/string duality between the $D = 4, \mathcal{N} = 4$ SUSY YM theory and the IIB string theory on $AdS_5 \times S^5$ is one of the subjects which have been discussed with great interest in recent years. The integrability and the superconformal symmetry $PSU(2,2|4)$ play crucial roles on both the sides of the duality.

The $D = 4, \mathcal{N} = 4$ SUSY YM theory on one side was casted to a $D = 2$ spin-chain system with the superconformal symmetry $PSU(2,2|4)$. For this system the Bethe ansatz and the R-matrix were extensively studied by assuming the integrability. However the origin of the integrability is obscure in this approach. Moreover the existence of the superconformal symmetry $PSU(2,2|4)$ is also hypothetical, since it is broken by the Bethe ansatz to two copies of the subgroup $PSU(1,1|2)$ with central charges. The appearance of central charges makes the purely algebraic construction of the universal R-matrix for a simple group unreliable. That is, the plug-in formula for the universal R-matrix works only if we concern a simple (super)group G and its Yangian generalization Y(G). This unusual feature of the R-matrix attracted a particular interest as a challenging subject.

It gives a clue to study the anomalous scaling dimension of the $\mathcal{N} = 4$ SUSY YM theory by means of the R-matrix of a $D = 2$ spin-chain.

The IIB string theory on the other side was effectively described by a $D = 2$ non-linear $\sigma$-model on the coset superspace $PSU(2,2|4)/(SO(1,4) \otimes SO(5))$. It is integrable at the classical level admitting an infinite number of conserved currents. Quantum extension of the integrability was argued by the Bethe ansatz. The Bethe ansatz is a common language to understand the gauge/string duality on both of the sides. In this approach the origin of the integrability and the superconformal symmetry $PSU(2,2|4)$ are clear because of the Poisson structure of the non-linear $\sigma$-model and the resemblance to the Green-Schwarz superstring respectively.

In this paper we pursue the approach of the string side. For the non-linear $\sigma$-model on $PSU(2,2|4)/(SO(1,4) \otimes SO(5))$ we set up Poisson brackets for the basic fields on the light-like line $x^+ = \text{const}$ instead of the equal-time line $x^0 = \text{const}$. In the consistency and the virtue for doing this were shown for the non-linear $\sigma$-model on the general bosonic coset space $G/H$. Namely, the Poisson brackets satisfy the three conditions. (i) They satisfies the Jacobi identities. (ii) The energy-momentum tensor $T_{\cdot \cdot}$ generates diffeomorphism on the light-like line by means of the Poisson brackets. (iii) At the origin of the coset space they coincide with the Poisson brackets of the free boson theory. There exists a quantity $\Upsilon$, called Killing scalar, which transforms as a linear representation vector of $G$ by the Killing vectors of $G/H$. It exists in any representation of $G$ and obeys the classical exchange algebra

$$\{ \Upsilon(x) \otimes \Upsilon(y) \} = -h r_{xy} \Upsilon(x) \otimes \Upsilon(y), \quad (1.1)$$

on the light-like line with the Poisson brackets for the basic fields. Here $r_{xy}$ is the classical r-matrix of $G$. If $G$ is a simple group, we may have recourse to the plug-in formula to promote it to the universal R-matrix $R_{xy}$, which is expressed purely in terms of the
generators of $G[4,5]$. Then (1.1) becomes the quantum exchange algebra

$$R_{xy} \Upsilon(x) \otimes \Upsilon(y) = \Upsilon(y) \otimes \Upsilon(x),$$

(1.2)

Its classical correspondence to (1.1) can be seen by

$$R_{xy} = 1 + hr_{xy} + O(h^2).$$

Correlation functions of $\Upsilon$s arrayed on the light-like line, may be obtained by using the quantum exchange algebra to braid $\Upsilon$s at adjacent positions successively.

In this paper we apply all of these arguments to the non-linear $\sigma$-model on $\text{PSU}(2,2|4)/\{\text{SO}(1,4) \otimes \text{SO}(5)\}[7]$. Now the basic fields of the coset space are the $D = 10$ supercoordinates $(X, \Theta)$. For this non-linear $\sigma$-model the exchange algebra (1.1) or (1.2) appears with the $r$- or $R$-matrix of the superconformal group $\text{PSU}(2,2|4)$. The Killing scalar $\Upsilon$ is the $\mathcal{N} = 4$ superconformal multiplet. But $\text{PSU}(2,2|4)$ is non-compact. Hence the unitary representation contains infinitely many descendants. We call them as a whole the superconformal multiplet $\mathcal{V}$. In the previous work [[11]] the exchange algebra (1.1) or (1.2) of the non-linear $\sigma$-model on $G/H$ was discussed in an arbitrary representation. But the dimension of the unitary representation was finite by assuming that $G$ is a compact group. It is awkward to simply apply the arguments in [11] to the case where the dimension of the unitary representation is necessarily infinite. It is our main concern to make a bridge over this gap.

To this end we remember that the $\mathcal{N} = 4$ superconformal multiplet $\mathcal{V}$ can constructed over a supercoherent space of bosonic and fermionic creation oscillators[12]. The oscillators form a 8-d vector, say $\psi$, transforming covariantly under the superconformal group $\text{PSU}(2,2|4)$. Let us write the covariant action on $\psi$ as an $8 \times 8$ supermatrix $e^{iM}$. Then we show that the unitary representation is given by $\hat{U} = e^{i\psi M} \psi$, which is an infinite dimensional representation of $\text{PSU}(2,2|4)$. Acting on supercoherent states it induces the group action $e^{iM}$ on the 8-d vector $\psi$, as shown by the state-operator relations (3.3) and (3.4).

Therefore the Killing scalar $\Upsilon$ which we want to let satisfy the classical exchange algebra (1.1) is not necessarily the $\mathcal{N} = 4$ superconformal multiplet $\mathcal{V}$, but may be the 8-d covariant vector $\psi$. The Killing scalar $\Upsilon$ transforming identically with $\psi$ can be readily constructed on the coset space $\text{PSU}(2,2|4)/\{\text{SO}(1,4) \otimes \text{SO}(5)\}$, following [11]. Once this is done, the whole arguments in [11] can be applied to the non-linear $\sigma$-model on this coset space as well. That is, this Killing scalar $\Upsilon$ satisfies the classical exchange algebra (1.1) with the $r$-matrix in the $8 \times 8$ matrix representation of $\text{PSU}(2,2|4)$. $\text{PSU}(2,2|4)$ is a simple group. Hence the finite-dimensional $r$-matrix can be quantized to the universal $R$-matrix by means of the plug-in formula[4,5]. Thus we get the quantum exchange algebra (1.2) for the Killing scalar $\Upsilon$ or equivalently for the covariant vector $\psi$. From this we can calculate the quantum exchange algebra for the $\mathcal{N} = 4$ superconformal multiplet $\mathcal{V}$, because $\mathcal{V}$ consists of $\psi$ as shown in table 2. The $R$-matrix for $\mathcal{V}$ is infinite-dimensional and yet algebraically the same as for the covariant vector $\psi$ owing to the operators-state relations

\footnote{The $R$-/S-matrix discussed in [2] is not the one for $\text{PSU}(2,2|4)$, but for a non-simple group such as $\text{PSU}(1,1|2) \times \mathbb{R}^3$. Further comments on this will be made at the end of this paper.}
Thus we dispense with meeting the R-matrix in an infinite-dimensional representation head-on.

The paper is organized as follows. In section 2 we explain the $\mathcal{N} = 4$ superconformal algebra of PSU(2,2|4) in terms of bosonic and fermionic oscillators forming an 8-d covariant vector $\psi$. It is done by following [2] closely. In section 3 we construct the unitary representation of the superconformal group PSU(2,2|4) over a supercoherent space of the oscillators, following [12]. In particular we focus on the $\mathcal{N} = 4$ field strength multiplet appearing as a half-BPS state in the unitary representation of PSU(2,2|4), which was discussed in [13]. Arguments on more general superconformal multiplets $\mathcal{V}$ are given in appendix A. The reader who is familiar the subjects may skip sections 2 and 3. In section 4 we discuss the $8 \times 8$ supermatrix representation of PSU(2,2|4). The operator-state relations (3.3) and (3.4) establish a one-to-one map between the unitary(oscillator) representation in section 3 and the matrix representation. In section 5 the superconformal group PSU(2,2|4) is non-linearly realized on the coset space PSU(2,2|4)/\{SO(1,4)⊗SO(5)\}, in a way independent of the representation. Embedding the subgroup SO(1,4)⊗SO(5) in PSU(2,2|4) is carefully studied. The salient feature of this coset space is that the basis fields of the coset space are the $D = 10$ supercoordinates $(X, \Theta)$. In section 6 the oscillators, forming the 8-d covariant vector $\psi$ of PSU(2,2|4), are non-linearly realized on PSU(2,2|4)/\{SO(1,4)⊗SO(5)\} as the Killing scalar $\Upsilon$. In section 7, we consider the non-linear $\sigma$-model on the coset space and impose Poisson brackets for $(X, \Theta)$, according to [11]. Then we get the classical exchange algebra for the non-linearly realized oscillators $\Upsilon$ and discuss its implication for correlation functions when the non-linear $\sigma$-model is quantized on the light-like line. Appendix A is devoted to complete the argument on the unitary(oscillator) representation of PSU(2,2|4) in section 3. Superconformal multiplets other than the field strength multiplet appear as larger BPS multiplets. Though they were argued in various works [13] [14] [15], here we straighten the arguments by unifying the notations. Finally in appendix B we explain how to calculate the Killing vectors of the general coset space $G/H$ in a way independent of the representation, i.e., by using only the Lie-algebra. The unitary(oscillator) representation of PSU(2,2|4) as well as the matrix one require central charges as shown in section 3 and 4. The algebraic calculation in appendix B dispenses us with meeting central charges. It is desirable since PSU(2,2|4) is a simple group which is free from central charges at the algebraic level and so are the Killing vectors.

2 The $\mathcal{N} = 4$ SUSY YM theory and PSU(2,2|4)

The $\mathcal{N} = 4$ SUSY YM theory is described by a set of fundamental fields

$$A_\mu, \Psi_{\alpha a}, \Psi_\alpha^a, \Phi_{[c,d]}.$$  

Our index convention is as follows: $\mu$ refers to vector indices of the Lorentz group SO(1,3), taking four values. $\alpha, \dot{\alpha}$ refer to two independent spinor indices of SU(2)⊗SU(2) ($\in$ SU(2,2)). They respectively take two values. $a, b$ refer spinor indices of the R-symmetry group.
SU(4), taking four values. [ , ] indicates anti-symmetrization of them. Complex conjugation of the spinor representation is indicated by raising or lowering indices. The $\mathcal{N}=4$ SUSY field strength multiplet is constructed out of these fundamental fields as shown in table 1. There $F$ indicates the field strength $F_{\mu\nu}$, which has been split into $F_{\{\alpha\beta\}}$ and $F_{\{\dot{\alpha}\dot{\beta}\}}$ by using the spinor indices of SU(2)$\otimes$SU(2). { , } indicates symmetrization of the indices. $\mathcal{D}$ indicates space-time derivative $\mathcal{D}_\mu$, which may be written as $\mathcal{D}_{\alpha\dot{\beta}}$. The representation of SU(2)$\otimes$SU(2) and SU(4) are indicated by the Dynkin labels for the highest weight as $[s_1,s_2]$ and $[r_1,r_2,r_3]$ respectively. The Young tableau representing the SU(4) representation is drawn in figure 1. Their dimensions are given by

\[
\dim[s_1,s_2] = \tilde{s}_1 \tilde{s}_2,
\]

\[
\dim[r_1,r_2,r_3] = \frac{1}{12} \tilde{r}_1 \tilde{r}_2 \tilde{r}_3 (\tilde{r}_1 + \tilde{r}_2)(\tilde{r}_2 + \tilde{r}_3)(\tilde{r}_1 + \tilde{r}_2 + \tilde{r}_3),
\]

with $\tilde{s}_i = s_i + 1$ and $\tilde{r}_a = r_a + 1$.

| field    | SU(2)$\otimes$SU(2) h.w. | SU(4) h.w. |
|----------|---------------------------|------------|
| $\mathcal{D}^k F$ | $[k+2,k]$ | $[0,0]$ |
| $\mathcal{D}^k \Psi$ | $[k+1,k]$ | $[1,0]$ |
| $\mathcal{D}^k \Phi$ | $[k,k]$ | $[0,1]$ |
| $\mathcal{D}^k \dot{\Psi}$ | $[k,k+1]$ | $[0,0,1]$ |
| $\mathcal{D}^k \dot{F}$ | $[k,k+2]$ | $[0,0]$ |

Table 1: $\mathcal{N}=4$ SUSY field strength multiplet.

Figure 1: The SU(4) Young tableau for the representation with the Dynkin label $[r_1,r_2,r_3]$.

The $\mathcal{N}=4$ SUSY YM theory has the superconformal symmetry defined by the supergroup PSU(2,2$|\hspace{1pt}4$). It is represented as a subgroup of the slightly enlarged supergroup U(2,2$|\hspace{1pt}4$). The Lie-algebra of U(2,2$|\hspace{1pt}4$) is decomposed as

\[ T^- \oplus T^0 \oplus T^+, \]

where $T^0$ represents generators of the compact subgroup U(2,2$|\hspace{1pt}4$)$\otimes$U(4)$\otimes$U(1) and $T^- \oplus T^+$ represents non-compact ones such that

\[ [T^0, T^\pm] = T^\pm, \quad [T^\pm, T^\pm] = T^\pm. \]
Here the bracket $[\cdot,\cdot]$ is a graded commutator understood as an anti-commutator between fermionic generators, and as a commutator otherwise. We introduce two set of bosonic oscillators $(a_\alpha, a^{\dagger}_\alpha)$ and $(b_\beta, b^{\dagger}_\beta)$ and one set of fermionic ones $(c_a, c^{\dagger}_a)$ to realize these generators. The non-trivial commutation relations are

$$[a^\alpha, a^{\dagger}_\beta] = \delta^\alpha_\beta, \quad [b^{\dagger}_\beta, b^\alpha] = \delta^\beta_\alpha, \quad \{c^a, c^{\dagger}_b\} = \delta^a_b.$$

(2.1)

To be explicit, $T^0$ consists of the generators

$$L^\alpha_\beta = a^{\dagger}_\beta a^\alpha - \frac{1}{2} \delta^\alpha_\beta a^{\dagger}_\gamma a^\gamma, \quad L^{\dagger}_\beta = b^{\dagger}_\beta b^\alpha - \frac{1}{2} \delta^\beta_\alpha b^{\dagger}_\gamma b^\gamma, \quad R^a_b = c^{\dagger}_b c^a - \frac{1}{4} \delta^a_b c^c c^c,$$

(2.2)

and the three $U(1)$ generators

$$D = 1 + \frac{1}{2} a^{\dagger}_\gamma a^\gamma + \frac{1}{2} b^{\dagger}_\gamma b^\gamma = \frac{1}{2} a^{\dagger}_\gamma a^\gamma + \frac{1}{2} b^{\dagger}_\gamma b^\gamma,$$

$$C = 1 - \frac{1}{2} a^{\dagger}_\gamma a^\gamma + \frac{1}{2} b^{\dagger}_\gamma b^\gamma - \frac{1}{2} c^{\dagger}_c c^c = - \frac{1}{2} a^{\dagger}_\gamma a^\gamma + \frac{1}{2} b^{\dagger}_\gamma b^\gamma - \frac{1}{2} c^{\dagger}_c c^c,$$

$$B = -1 + \frac{1}{2} a^{\dagger}_\gamma a^\gamma - \frac{1}{2} b^{\dagger}_\gamma b^\gamma = \frac{1}{2} a^{\dagger}_\gamma a^\gamma - \frac{1}{2} b^{\dagger}_\gamma b^\gamma.$$

(2.3)

The generators in $T^+$ are given by

$$Q^a_\alpha = a^{\dagger}_\alpha c^a, \quad \dot{Q}^{\dagger}_\alpha = b^{\dagger}_\alpha c^{\dagger}_a, \quad P^{\dagger}_{\alpha\beta} = b^{\dagger}_\alpha a^{\dagger}_\beta,$$

while those in $T^-$ by

$$S^a_\alpha = c^{\dagger}_a a^\alpha, \quad \dot{S}^{\dagger}_\alpha = b^{\dagger}_a c^{\dagger}_a, \quad K^{\alpha\beta} = a^\alpha b^{\dagger}_\beta.$$

Then the generators in (2.2) form the subalgebra $SU(2) \otimes SU(2) \otimes SU(4)$ of $U(2,2|4)$

$$[L^\alpha_\beta, L^\gamma_\delta] = -\delta^\gamma_\delta L^\alpha_\beta + \delta^\alpha_\delta L^\gamma_\beta, \quad [L^{\dagger}_\beta, L^{\dagger}_\delta] = -\delta^{\dagger}_\delta L^{\dagger}_\beta + \delta^{\dagger}_\alpha L^{\dagger}_\gamma, \quad [R^a_b, R^c_d] = -\delta^c_d R^a_b + \delta^a_d R^c_b.$$

(2.4)

The algebra $[T^+, T^-] = T^\pm$ is nilpotent in the sense that $[T^+[T^+, T^-]] = 0$, and is given by

$$\{\dot{Q}^{\dagger}_{\alpha\beta}, Q^a_\alpha\} = \delta^a_b P^{\dagger}_{\alpha\beta}, \quad \{\dot{S}^{\dagger}_\alpha, S^a_\beta\} = \delta^a_b K^{\alpha\beta},$$

(2.5)

If $Q^a_\alpha = a^{\dagger}_a c^a, S^a_\alpha = c^{\dagger}_a a^\alpha$ are replaced by $Q_{a\alpha} = a^{\dagger}_{a\alpha}, S^{a\alpha} = c^a c^\alpha$, they form the algebra

$$[T^0, T^\pm] = T^\pm, \quad [T^+, T^-] = T^0, \quad \{T^\pm, T^\pm\} = 0.$$

This form of the Lie-algebra $U(2,2|4)$ was used to discuss the unitary representation in refs [16]
while the algebra \([T^+, T^-] = T^0\) by

\[
[K^{\alpha\beta}, P_{\gamma\delta}] = \delta^{\delta}_{\gamma} L^{\alpha}_{\beta} + \delta^{\alpha}_{\delta} \hat{L}^{\beta}_{\gamma} + \delta^{\beta}_{\gamma} \delta^{\alpha}_{\delta} D,
\]

\[
\{S^{\alpha}_{b}, Q_{\alpha}^{a}\} = \delta^{a}_{b} L^{\alpha}_{\beta} + \delta^{\alpha}_{b} R^{\beta}_{a} + \frac{1}{2} \delta^{\alpha}_{b} \delta^{\alpha}_{\beta} (D - C),
\]

\[
\{\hat{S}^{\alpha\beta}, \hat{Q}_{\alpha\beta}\} = \delta^{\alpha}_{b} \hat{L}^{\beta}_{\alpha} - \delta^{\beta}_{\alpha} \hat{R}^{\alpha}_{b} + \frac{1}{2} \delta^{\alpha}_{b} \delta^{\alpha}_{\beta} (D + C).
\]

Finally the algebra \([T^+, T^-], which does not close into \(T^0\), is given by

\[
[S^{\alpha}_{b}, P_{\alpha\beta}] = \delta^{\alpha}_{\beta} \hat{Q}_{\alpha\beta}, \quad [K^{\alpha\beta}, Q_{\alpha}^{a}] = \delta^{\alpha}_{a} S^{\beta}_{b},
\]

\[
[\hat{S}^{\alpha\beta}, P_{\alpha\beta}] = \delta^{\beta}_{a} Q^{\alpha}_{b}, \quad [K^{\alpha\beta}, Q_{\alpha}^{a}] = \delta^{\alpha}_{a} \hat{S}^{\alpha\beta}.
\]

We omit the algebra \([T^0, T^\pm] = T^\pm\), which can be easily written down. Altogether the algebras \((2.4) - (2.7)\) define the Lie-algebra of \(U(2,2|4)\) [2].

It is instructive to put the generators in a tensor product form of the row and column vectors

\[
\psi^\dagger = (a_{\alpha}^\dagger, c_{\alpha}^\dagger, b_{\beta}^\dagger), \quad \psi = \begin{pmatrix} a_{\alpha}^\dagger \\ c_{\alpha} \\ b_{\beta} \end{pmatrix},
\]

as

\[
\psi^\dagger \otimes \psi =
\begin{pmatrix}
L^{\alpha}_{\beta} & S^{\alpha}_{b} & K^{\alpha\beta} \\
Q^{\alpha}_{b} & R^{\beta}_{a} & \hat{S}^{\alpha\beta} \\
P_{\alpha\beta} & \hat{Q}_{\alpha\beta} & \hat{L}^{\beta}_{\alpha}
\end{pmatrix}
+ \begin{pmatrix}
\frac{1}{2} \delta^{\alpha}_{\beta} (D + B) & 0 & 0 \\
0 & -\frac{1}{4} (C + B) & 0 \\
0 & 0 & \frac{1}{2} \delta^{\alpha}_{\beta} (D - B)
\end{pmatrix}.
\]

Here use was made of \((2.3)\). \(P_{\alpha\beta}, Q^{\alpha}_{b}, \hat{Q}_{\alpha\beta}\) in the lower-left blocks are generators of (super) translation while \(K^{\alpha\beta}, S^{\alpha}_{b}, \hat{S}^{\alpha\beta}\) in the upper-right blocks are generators of (super) boost. In the diagonal blocks \(L^{\alpha}_{\beta}, \hat{L}^{\beta}_{\alpha}, R^{\beta}_{a}\) are generators of the Lorentz subsymmetry \(SU(2) \otimes SU(2) (\subset SU(2,2))\) and the R-symmetry \(SU(4)\), and \(D, C, B\) are three \(U(1)\) charges. \(D\) is the dilatation. \(B\) never appears in the above superalgebrae of \(U(2,2|4)\), \((2.4) - (2.7)\). All the generators commute with \(C\). Hence \(C\) is a central charge.

Finally we get the quadratic Casimir in the form

\[
L^{\alpha}_{\beta} L^{\beta}_{\alpha} - R^{\beta}_{a} R^{\beta}_{a} + L^{\beta}_{\alpha} L^{\beta}_{\alpha} + D^2 - \{P_{\alpha\beta}, K^{\alpha\beta}\} - \{Q^{\alpha}_{b}, S^{\alpha}_{b}\} - \{\hat{Q}_{\alpha\beta}, \hat{S}^{\alpha\beta}\},
\]

as can be checked by a direct calculation.

### 3 Unitary(oscillator) representation of \(PSU(2,2|4)\)

The superconformal transformations act on the \(\mathcal{N} = 4\) SUSY field strength multiplet given in table 1. In quantum field theory they are represented as unitary linear transformations in the Hilbert space. Hence the unitary representation of the superconformal group
PSU(2,2|4) is the primary concern for quantization of the $\mathcal{N}=4$ SUSY YM theory. Since PSU(2,2|4) is non-compact, the unitary representation is necessarily infinite-dimensional. The $\mathcal{N}=4$ SUSY field strength multiplet is one of infinitely many multiplets in the unitary representation of PSU(2,2|4). Other multiplets, generally called $\mathcal{N}=4$ superconformal multiplets, are known by a systematic analysis of the unitary representation[13, 14, 15]. They are given in appendix A.

A unitary operator $\hat{U}$ representing U(2,2|4) may be given by

$$\hat{U} = e^{i\bar{\psi}M\psi},$$

with $\bar{\psi} = \psi^\dagger\gamma[12]$. Here $M$ and $\gamma$ are $8 \times 8$ supermatrices of the block form

$$M = \begin{pmatrix} V & \theta & X \\ \theta^\dagger & W & \epsilon \\ -X^\dagger & -\epsilon^\dagger & Z \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

in which $V, W, Z$ are Hermitian matrices, $X$ is a complex matrix, but $\theta$ (or $\epsilon$) is a $2 \otimes 4$ (or $4 \otimes 2$) matrix of which elements are Grassmannian numbers. The unitarity of $\hat{U}$ follows from the Hermiticity of $\bar{\psi}M\psi$, i.e., $(\bar{\psi}M\psi)^\dagger = \bar{\psi}M\psi$. The vector $\psi$ transforms covariantly by the action of U(2,2|4) as

$$\hat{U}^\dagger \psi \hat{U} = e^{iM}\psi,$$

and $\bar{\psi}$ contravariantly as

$$\hat{U}^\dagger \bar{\psi} \hat{U} = \bar{\psi}e^{-iM}.$$

The *minus* sign in $M$ is a hallmark of non-compactness of U(2,2|4). It comes from the fact that we have chosen $\psi$ in (2.8) as having creation and annihilation oscillators mixed. For representing the compact supergroup U(4|4), it suffices to define $\psi$ by annihilation oscillators alone. Consequently the *minus* sign is not needed for the block matrices $X^\dagger$ and $\epsilon^\dagger$ in $M$. Then $M$ is Hermitian in itself and $\gamma$ is not needed either.

We explain this point of the unitary(oscillator) representation by taking much simpler groups SU(1,1) and SU(2) as examples. Both Lie-algebrae are realized by using two pairs of oscillators $(a, a^\dagger)$, and $(b, b^\dagger)$. The non-trivial commutation relations are

$$[a, a^\dagger] = 1, \quad [b, b^\dagger] = 1.$$

Then SU(1,1) is realized by the unitary operator (3.1) with

$$\psi^\dagger = (a^\dagger, b), \quad \psi = \begin{pmatrix} a \\ b^\dagger \end{pmatrix}, \quad M = \begin{pmatrix} V & \phi \\ -\phi^* & -V \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

while SU(2) by the unitary operator $e^{i\psi_U^\dagger M_U\psi_U}$ with

$$\psi_U^\dagger = (a^\dagger, b^\dagger), \quad \psi_U = \begin{pmatrix} a \\ b \end{pmatrix}, \quad M_U = \begin{pmatrix} V & \phi \\ \phi^* & -V \end{pmatrix}. $$

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From $\bar{\psi} M \psi$ and $\psi_U^\dagger M_U \psi_U$ we read the generators of the respective group as

$$\begin{align*}
SU(1, 1) : & \quad T^+ = a^\dagger b^\dagger, \quad T^- = ab, \quad T^0 = \frac{1}{2}(a^\dagger a + b^\dagger b), \\
SU(2) : & \quad T_U^+ = a^\dagger b, \quad T_U^- = ab^\dagger, \quad T_U^0 = \frac{1}{2}(a^\dagger a - b^\dagger b),
\end{align*}$$

which satisfy the algebras

$$\begin{align*}
[T^+, T^-] &= -2T^0, \quad [T^0, T^\pm] = \pm T^\pm, \\
[T_U^+, T_U^-] &= 2T_U^0, \quad [T_U^0, T_U^\pm] = \pm T_U^\pm.
\end{align*}$$

Let $|0\rangle$ to be the vacuum of the Fock space. Then we have

$$(T^+)^n a^\dagger |0\rangle \neq 0, \quad (T_U^+)^n a^\dagger |0\rangle = 0,$$

for a positive integer $n$. Thus by means of the unitary operator (3.1) we can realize the non-compact group $U(1,1)$ in an infinite dimensional representation.

We return to the main arguments on $PSU(2,2|4)$. The unitary operator (3.1) for $PSU(2,2|4)$ acts on a Fock space given by all possible oscillator excitations

$$\prod_{a=1}^{2} \prod_{n_{a\alpha} = 1}^{\infty} (a_\alpha^\dagger)^{n_{a\alpha}} \prod_{a=1}^{2} \prod_{n_{b\dot{\alpha}} = 1}^{\infty} (b_{\dot{\alpha}}^\dagger)^{n_{b\dot{\alpha}}} \prod_{a=1}^{4} \prod_{n_{c\alpha} = 1}^{\infty} (c_{\alpha}^\dagger)^{n_{c\alpha}} |0\rangle.$$

Thus it is the unitary representation of $U(2,2|4)$. $PSU(2,2|4)$ is represented in a subsector of the Fock space constrained by

$$C = 1 - \frac{1}{2} \sum_{\alpha}^{2} n_{a\alpha} + \frac{1}{2} \sum_{\dot{\alpha}}^{2} n_{b\dot{\alpha}} - \frac{1}{2} \sum_{\alpha}^{4} n_{c\alpha} = 0,$$

with $C$ given in (2.3). If we have

$$a^\alpha |0\rangle = 0, \quad b_{\dot{\alpha}}^\dagger |0\rangle = 0, \quad c^\alpha |0\rangle = 0,$$

the vacuum $|0\rangle$ is not in this subsector because $C = 1$. Hence we define a new physical vacuum $Z$ which has $C = 0$. It may be realized by

$$Z = c_3^\dagger c_4^\dagger |0\rangle.$$

It is convenient to rename the whole fermionic oscillators $c^\alpha, c_{\alpha}^\dagger, a = 1, 2, 3, 4$ as

$$\begin{align*}
(c_1^\dagger, c_2^\dagger) &\equiv c_3^\dagger, \quad (c_3^\dagger, c_4^\dagger) = (d_3^\dagger, d_4^\dagger) \equiv d_3^\dagger, \\
(c_1^\dagger, c_2^\dagger) &\equiv c_3^\dagger, \quad (c_3^\dagger, c_4^\dagger) = (d_3^\dagger, d_4^\dagger) \equiv d_3^\dagger.
\end{align*}$$
Then $Z$ satisfies
\[ a^\alpha |0> = 0, \quad b^\hat{a} |0> = 0, \quad c^\hat{a} |0> = 0, \quad d^\hat{a} |0> = 0. \]

The physical Fock space is built up on this $Z$ as
\[ \prod_{\alpha=1}^{2} (a^\dagger_\alpha)^{n_{a\alpha}} \prod_{\hat{a}=1}^{2} (b^\dagger_{\hat{a}})^{n_{b\hat{a}}} \prod_{\hat{a}=1}^{2} (c^\dagger_{\hat{a}})^{n_{c\hat{a}}} \prod_{\hat{a}=3}^{4} (d^\dagger_{\hat{a}})^{n_{d\hat{a}}} Z. \tag{3.8} \]

The constraint (3.5) becomes
\[ C = 2 \sum_{\alpha=1}^{2} n_{a\alpha} - \sum_{\hat{a}=1}^{2} n_{b\hat{a}} + 2 \sum_{\hat{a}=1}^{2} n_{c\hat{a}} - 4 \sum_{\hat{a}=3}^{4} n_{d\hat{a}} = 0. \]

According to this redefinition all the generators representing $U(2,2\mid 4)$ in (2.9) get the central charge $C = 0$. Among them the following generators non-trivially act on $Z$,
\[ Q^{\hat{a}}_\beta, \quad \hat{Q}_{\hat{a}}^{\hat{b}}, \quad P_{\alpha\beta}, \quad R^{\hat{a}}_{\hat{b}}, \]
with the renamed indices by (3.7). To be explicit, they are
\[ a_{\hat{a}}^{\dagger} b_{\hat{a}}^{\dagger}, \quad b_{\hat{a}}^{\dagger} c_{\hat{a}}^{\dagger}, \quad a_{\beta}^{\dagger} b_{\hat{a}}^{\dagger}, \quad c_{\hat{a}}^{\dagger} d_{\hat{a}}^{\dagger}. \]

Acting on $Z$ the fermionic generators create the states as shown in table 2\[13\]. They exactly correspond to the fundamental fields of the $\mathcal{N}=4$ SUSY field strength multiplet in table 1. Furthermore acting on those states $P_{\alpha\beta} = a_{\alpha}^{\dagger} b_{\beta}^{\dagger}$ and $R^{\hat{a}}_{\hat{b}} = c_{\hat{a}}^{\dagger} d_{\hat{b}}^{\dagger}$ create $SU(2)\otimes SU(2)$ excited states with the Dynkin label $[k,k]$ respectively. The former excitation implies space-time derivative $\mathcal{D}$ of the $\mathcal{N} = 4$ SUSY field strength multiplet. (See table 1.) The latter excitation occurs in the representation space of the R-symmetry $SU(4)$. All of these states have the central charge $C = 0$, so that they are indeed in the infinite-dimensional unitary representation of $PSU(2,2\mid 4)$. The remaining generators annihilate $Z$. In particular the fermionic ones are given by
\[ Q^{\hat{a}}_\beta, \quad \hat{Q}_{\hat{a}}^{\hat{b}}, \tag{3.9} \]
which are
\[ a_{\beta}^{\dagger} c^{\hat{a}}, \quad b_{\hat{a}}^{\dagger} d^{\hat{b}}. \]

They are half of the 16 supercharges. Thus the states in table 2 form a half-multiplet\[13\]. They are the smallest BPS multiplet. The vacuum $Z$ is the highest weight vector of the multiplet, which is denoted by the $SU(2,2)$ Dynkin label $[0,1,0]$.

Larger BPS multiplets for the $\mathcal{N} = 4$ SUSY theory, i.e., other $\mathcal{N} = 4$ superconformal multiplets, can be also constructed by generalizing the above construction. It will be done in appendix A to complete the argument.
Table 2: Oscillator representation of the $\mathcal{N} = 4$ SUSY field strength multiplet.

4 Matrix representation of $\text{PSU}(2,2|4)$

So far we have considered the unitary(oscillator) representation of $U(2,2|4)$ taking a base obtained by the tensor product $\psi^\dagger \otimes \psi$. In this section we discuss a matrix representation of $U(2,2|4)$ which is induced from the unitary(oscillator) representation by (3.3) and (3.4). To this end we put 64 generators in a base which manifests $U(2,2|4)$ more faithfully than (2.9), i.e.,

$$
\begin{bmatrix}
T^\alpha_{\beta} & T^\alpha_{\bar{\beta}} & T^\alpha_{\bar{\beta}} \\
T^\alpha_{\bar{\beta}} & T^\alpha_{\bar{\beta}} & T^\alpha_{\bar{\beta}} \\
T^{\dot{\alpha}}_{\dot{\beta}} & T^{\dot{\alpha}}_{\dot{\beta}} & T^{\dot{\alpha}}_{\dot{\beta}}
\end{bmatrix}.
$$

Using an $8 \times 8$ supermatrix with the index convention

$$
\begin{pmatrix}
t^\gamma_{\delta} & t^\gamma_{\delta} & t^\gamma_{\delta} \\
t^\gamma_{\delta} & t^\gamma_{\delta} & t^\gamma_{\delta} \\
t^\gamma_{\delta} & t^\gamma_{\delta} & t^\gamma_{\delta}
\end{pmatrix},
$$

we write the generators as

$$
T^\alpha_{\beta} = \begin{pmatrix}
\delta^\alpha_{\delta} \delta^\beta_{\delta} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad T^\alpha_{\bar{\beta}} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad T^\alpha_{\bar{\beta}} = \begin{pmatrix}
0 & 0 & \delta^\alpha_{\bar{\beta}} \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
$$

$$
T^\alpha_{\bar{\beta}} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad T^\alpha_{\bar{\beta}} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad T^\alpha_{\bar{\beta}} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
$$
Here keep in mind the *minus* sign in the last line which accounts for non-compactness of $U(2,2|4)$. Bosonic generators in the diagonal blocks of (4.1) form the Lie-algebra of $U(2,2) \otimes U(4)$

$$
[T^\alpha_\beta, T^\gamma_\delta] = -\delta_\beta^\delta T^\alpha_\gamma + \delta_\delta^\gamma T^\alpha_\beta, \quad [T^{\dot{\alpha}}_\dot{\beta}, T^{\dot{\gamma}}_\dot{\delta}] = \delta^{\dot{\gamma}}_\dot{\beta} T^{\dot{\alpha}}_\dot{\gamma} - \delta^{\dot{\alpha}}_\dot{\delta} T^{\dot{\beta}}_\dot{\gamma},
$$

$$
[T^\alpha_\beta, T^c_d] = -\delta^c_\beta T^\alpha_d + \delta^c_d T^\alpha_\beta, \quad [T^{\dot{\alpha}}_\dot{\beta}, T^{\dot{c}}_\dot{d}] = \delta^{\dot{c}}_\dot{d} T^{\dot{\alpha}}_\dot{c} - \delta^{\dot{\alpha}}_\dot{c} T^{\dot{\beta}}_\dot{d},
$$

\[4.3\]

Anti-commuting fermionic generators in the off-diagonal blocks with each other yields

$$
\{T^a_b, T^a_\beta\} = \delta^a_\beta T^a_b, \quad \{T^{\dot{a}}_b, T^{\dot{a}}_\beta\} = -\delta^{\dot{a}}_\beta T^{\dot{a}}_b, \quad \{T^a_\beta, T^{\dot{a}}_b\} = \delta^a_\beta T^{\dot{a}}_b, \quad \{T^{\dot{a}}_\beta, T^a_b\} = \delta^{\dot{a}}_\beta T^a_b.
$$

\[4.4\]

Commuting these fermionic generators with bosonic generators yields

$$
[T^a_\beta, T^c_d] = \delta^c_\beta T^a_d, \quad [T^a_\beta, T^{\dot{c}}_\dot{d}] = -\delta^{\dot{c}}_\dot{d} T^a_\beta, \quad [T^a_\beta, T^{\dot{a}}_\delta] = 0, \quad [T^a_\beta, T^c_d] = 0,
$$

\[4.5\]

while commuting them with bosonic generators

$$
[T^a_\beta, T^{\dot{a}}_\delta] = 0, \quad [T^{\dot{a}}_\beta, T^c_d] = \delta^{\dot{a}}_\delta T^c_d, \quad [T^{\dot{a}}_\beta, T^{\dot{a}}_\delta] = 0,
$$

\[4.6\]

All other (anti-)commutation relations are vanishing. The diagonal blocks contain the generators of the subgroup $SU(2) \otimes SU(2) \otimes SU(4)$ given by

$$
\mathcal{L}^\alpha_\beta = T^\alpha_\beta - \frac{1}{2} \delta^\alpha_\beta T^\gamma_\gamma, \quad \mathcal{L}^{\dot{\alpha}}_\dot{\beta} = T^{\dot{\alpha}}_\dot{\beta} - \frac{1}{2} \delta^{\dot{\alpha}}_\dot{\beta} T^{\dot{\gamma}}_\dot{\gamma}, \quad \mathcal{R}^a_b = T^a_b - \frac{1}{4} \delta^a_b T^c_c,
$$

\[4.7\]

and three $U(1)$ generators defined by

$$
D = \frac{1}{2} \left( \begin{array}{ccc} 
\delta^\gamma_\delta & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -\delta^\gamma_\delta 
\end{array} \right) = \frac{1}{2} (T^\alpha_\alpha + T^{\dot{\alpha}}_{\dot{\alpha}}),
$$

\[4.8\]

$$
-C = \frac{1}{2} \left( \begin{array}{ccc} 
\delta^\gamma_\delta & 0 & 0 \\
0 & \delta^c_d & 0 \\
0 & 0 & \delta^\gamma_\delta 
\end{array} \right) = \frac{1}{2} (T^\alpha_\alpha - T^{\dot{\alpha}}_{\dot{\alpha}} + T^a_a),
$$

\[4.8\]
The quadratic Casimir is given by

\[ B = \frac{1}{2} \left( \begin{array}{ccc} \delta^\gamma_\delta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \delta^\gamma_\delta \end{array} \right) = \frac{1}{2} (T^\alpha_\alpha - T^{\dot{\alpha}}_{\dot{\alpha}}). \]

They are identical to the one given in the unitary(oscillator) representation (2.3). U(2,2|4) becomes SU(2,2|4) or PU(2,2|4) when the U(1) generators are constrained by \( B = 0 \) or \( C = 0 \) respectively. When imposed both constraints, it becomes PSU(2,2|4).

Using the generators defined by (4.7) we rewrite the algebrae (4.3)\sim(4.5). The first three algebrae in (4.3) remain in the same form

\[ [\mathcal{L}_\beta^\alpha, \mathcal{L}_\delta^\gamma] = -\delta^\gamma_\delta \mathcal{L}_\beta^\alpha + \delta^\alpha_\delta \mathcal{L}_\beta^\gamma, \quad [\mathcal{L}_\beta^\gamma, \mathcal{L}_\delta^\gamma] = \delta^\gamma_\delta \mathcal{L}_\beta^\gamma - \delta^\delta_\beta \mathcal{L}_\beta^\gamma, \]

\[ [R_a^b, R_c^d] = -\delta^c_d R_a^b + \delta^d_c R_a^b. \] (4.9)

Other algebrae in (4.3)\sim(4.5) also do not change significantly the forms, except for the last algebra in (4.3) and the first two in (4.4). Those are found to be

\[ [T_\beta^\alpha, T_\gamma^\delta] = \delta^\gamma_\delta T_\beta^\alpha + \delta^\alpha_\delta T_\beta^\gamma + \delta^\delta_\beta T_\beta^\delta D, \]

\[ \{T_\beta^\alpha, T_\gamma^\delta\} = \delta^\alpha_\delta T_\beta^\gamma + \delta^\gamma_\delta T_\beta^\delta + \frac{1}{2} \delta^\delta_\beta \delta^\delta_\beta (D - C), \] (4.10)

\[ \{T_\beta^\alpha, T_\delta^\beta\} = \delta^\beta_\alpha T_\beta^\delta - \delta^\delta_\beta T_\beta^\delta + \frac{1}{2} \delta^\delta_\beta \delta^\delta_\beta (D + C). \]

The quadratic Casimir is given by

\[ \mathcal{L}_\beta^\alpha \mathcal{L}_\alpha^\beta - R_a^b R_a^b + \mathcal{L}_\beta^\alpha \mathcal{L}_\alpha^\beta + D^2 - \{T_\beta^\alpha, T_\gamma^\delta\} - [T_\beta^\alpha, T_\gamma^\beta] - [T_\beta^\delta, T_\alpha^\beta]. \] (4.11)

Now we compare the algebrae (4.3)\sim(4.6) with (2.4)\sim(2.7) in the unitary(oscillator) representation. We find them to be equivalent by redefining the generators as

\[ \epsilon^{\alpha\dot{\alpha}} \epsilon^{\beta\dot{\beta}} \mathcal{L}_\delta^\gamma = \hat{L}_\beta^\alpha, \quad -\hat{L}_\beta^\alpha = -\mathcal{L}_\beta^\alpha, \quad \epsilon^{\beta\dot{\beta}} T_\delta^\alpha = K^{\alpha\beta}, \quad \epsilon^{\alpha\gamma} T_\beta^\gamma = P_{\alpha\beta}, \]

\[ \epsilon^{\beta\dot{\beta}} T_\delta^\alpha = \hat{S}_\alpha^\beta, \quad \epsilon^{\alpha\gamma} T_\beta^\gamma = \hat{Q}_{\alpha\beta}, \]

\[ \mathcal{L}_\beta^\alpha = L_\beta^\alpha, \quad R_a^b = R_a^b, \quad S_a^b = T_\beta^\alpha, \quad T_\beta^\alpha = Q_{\alpha\beta}, \quad D = D. \]

The redefinition does not change the form of the quadratic Casimir (4.11). It coincides with the quadratic Casimir (2.10), given in the unitary(oscillator) representation. But the redefinition changes the sign of the algebrae linearly containing \( \hat{L}_\beta^\alpha \) in (4.3)\sim(4.6). For instance, the second one in (4.9) becomes that of (2.4).

We compare also the algebrae (4.3)\sim(4.6) with those of U(4|4) and U(8). If the matrices (4.2) get all entries with plus sign, i.e., \( (\mathcal{T}^\alpha_A)^\beta = \delta^\alpha_\beta \delta^\alpha_\beta \), they become the generators of U(4|4). They satisfy the Lie-algebrae (4.3)\sim(4.6) where \( T_\beta^\gamma, T_\beta^\gamma, T_\beta^\delta \) get the sign changed. Accordingly the quadratic Casimir (4.11) changes the form as

\[ \sum_{B=\beta,\beta,b} T_\beta^\alpha T_B^\alpha + \sum_{B=\beta,\beta,b} T_\beta^\alpha T_B^\alpha - \sum_{B=\beta,\beta,b} T_\beta^\alpha T_B^\alpha = (-1)^{g(A)} T^A_B T^B_A. \]
Here we have assigned the grading \( g(A) \) to the index \( A \) in such a way \( g(A) = 1 \) for a fermionic index and otherwise \( g(A) = 0 \). So \( T^A_B \) has the grading \( g(A)g(B) \). If we do not assign the grading, \( T^A_B \) satisfies the Lie-algebra of U(8)

\[
[T^A_B, T^C_D] = -\delta^C_B T^A_D + \delta^A_B T^C_D.
\]

being defined as \( (T^A_B)^C_D = \delta^A_B \delta^D_C \). The quadratic Casimir of U(8) is simply

\[
\sum_{B=\beta,\dot{\beta},b} T^\alpha_B T^B_\alpha + \sum_{B=\beta,\dot{\beta},b} T^\dot{\alpha}_B T^B_{\dot{\alpha}} + \sum_{B=\beta,\dot{\beta},b} T^a_B T^B_a = T^A_B T^B_A.
\]

Or we had better formulate the superalgebras of U(2,2|4) and U(4|4) in a converse way, i.e., starting with this form of the algebra of U(8) instead of the graded form of (4.3)~(4.6).

5 Non-linear realization of PSU(2,2|4)

Both the unitary(oscillator) representation and the matrix one allow linear realization of PSU(2,2|4) only as a subgroup of its centrally extended group SU(2,2|4). It can been seen from the respective algebrae (2.6) and (4.10). PSU(2,2|4) is a simple group so that we do not need the central extension at the algebraic level. In this section we want to discuss a purely algebraic method to non-linearly realize PSU(2,2|4), which does not rely on the explicit representations and is consequently free from the central charge of SU(2,2|4). To this end we begin by writing the Lie-algebra of PSU(2,2|4) in a common form by which we can freely change the unitary(oscillator) representation to the matrix one and \textit{vice versa}. Then using that algebra we give general accounts of non-linear realization of PSU(2,2|4) on the coset space PSU(2,2|4)/H, without being bothered by the specifics of a chosen subgroup H. We discuss afterwards the case where H is SO(1,4)⊗SO(5), which is the main concern in this paper.

5.1 Algebraic method of non-linear realization

Let us put the generators of PU(2,2|4) in a row and denote them by \( \{T^\Xi\} \). That is, 62 generators in the unitary(oscillator) representation, discussed in section 3, are denoted by

\[
\{T^\Xi\} = \{L^\alpha_\beta, \dot{L}^{\dot{\alpha}}_{\dot{\beta}}, R^a_b, D, P_{\alpha\beta}, K_{\beta\dot{\alpha}}, S^\alpha_b, Q^a_\alpha, \dot{S}^{\dot{a}}_{\dot{\alpha}}, \dot{Q}_{\dot{a}a}\}, \tag{5.1}
\]

while the corresponding generators in the matrix representation, discussed in section 4, by

\[
\{T^\Xi\} = \{L^\alpha_\beta, \dot{L}^{\dot{\alpha}}_{\dot{\beta}}, R^a_b, D, P_{\alpha\beta}, K_{\beta\dot{\alpha}}, S^\alpha_b, Q^a_\alpha, \dot{S}^{\dot{a}}_{\dot{\alpha}}, \dot{Q}_{\dot{a}a}\}. \tag{5.2}
\]

Using either set of these 62 generators we represent PSU(2,2|4) in a common form as

\[
e^{i\mathcal{M}^\Xi T^\Xi} \in \text{PSU}(2,2|4), \tag{5.3}
\]
in which $M^\Xi$ are 62 elements of the supermatrix $M$ given in (3.2)

\[
\{M^\Xi\} = \left\{ V_\beta^\alpha - \frac{1}{2} \delta_\beta^\alpha, Z_\beta^\hat{\alpha} - \frac{1}{2} \delta_\beta^\hat{\alpha}, W_b^a - \frac{1}{4} \delta_b^a, D, -X_\alpha^1, X_\beta^\hat{\alpha}, \theta_\beta^\alpha, \theta_\alpha^1, \epsilon^\alpha, -\epsilon_\alpha^1 \right\}.
\]

We find explicit forms of $M^\Xi T^\Xi$ for the respective representations, expanding $\bar{\psi} M \psi$ and $M$ in terms of the generators (5.1) and (5.2). The expansion of the former reads

\[
\bar{\psi} M \psi \equiv \sum_{B=\beta,\hat{\beta},b} \psi^\dagger_B (\gamma M)_{B}^\alpha \psi^B + \sum_{B=\beta,\hat{\beta},b} \psi^\dagger_M (\gamma M)_{B}^\alpha \psi^B
\]

\[
= \left[ V^\alpha_\beta L^\beta_\alpha + X^\alpha\hat{\beta} P^\beta_\alpha + \theta^\alpha b Q^a_\alpha \right] + \left[ (-Z_\beta^\hat{\alpha} L^{\hat{\alpha}}_\beta + Z^\gamma_\hat{\gamma}) + X^{\hat{\alpha}}_{\alpha \beta} K^{\beta \hat{\alpha}} + \epsilon^{\alpha \hat{\alpha}} \hat{\epsilon}^{\alpha \hat{\alpha}} \right]
\]

\[
+ \left[ W^a_b R^b_a - \theta^b_a S^a_\beta - \epsilon^{a\hat{\alpha}} \hat{\epsilon}^{a\hat{\alpha}} \right] + \frac{1}{2} (V^\gamma_\gamma - Z^\gamma_\gamma) D
\]

\[
+ \frac{1}{2} (V^\gamma_\gamma + Z^\gamma_\gamma) B - \frac{1}{2} W^c(B + C)
\]

\[
\equiv M^\Xi T^\Xi + \frac{1}{2} (V^\gamma_\gamma + Z^\gamma_\gamma) B - \frac{1}{2} W^c(B + C),
\]

(5.4)

by using the commutation relations (2.1) and the generators defined by (2.2) and (2.3). On the other hand the expansion of the latter reads

\[
M = \sum_{B=\beta,\hat{\beta},b} M^a_B \cdot \gamma T^B_a + \sum_{B=\beta,\hat{\beta},b} M^\hat{\alpha}_B \cdot \gamma T^B_{\hat{\alpha}} + \sum_{B=\beta,\hat{\beta},b} M^\alpha_B \cdot \gamma T^B_\alpha
\]

\[
= \left[ V^\alpha_\beta L^\beta_\alpha + X^\alpha\hat{\beta} P^\beta_\alpha + \theta^\alpha b Q^a_\alpha \right] + \left[ (-Z_\beta^\hat{\alpha} L^{\hat{\alpha}}_\beta + Z^\gamma_\hat{\gamma}) + X^{\hat{\alpha}}_{\alpha \beta} K^{\beta \hat{\alpha}} - \epsilon^{\alpha \hat{\alpha}} \hat{\epsilon}^{\alpha \hat{\alpha}} \right]
\]

\[
+ \left[ W^a_b R^b_a + \theta^b_a S^a_\beta + \epsilon^{a\hat{\alpha}} \hat{\epsilon}^{a\hat{\alpha}} \right] + \frac{1}{2} (V^\gamma_\gamma - Z^\gamma_\gamma) D
\]

\[
+ \frac{1}{2} (V^\gamma_\gamma + Z^\gamma_\gamma) B - \frac{1}{2} W^c(B + C)
\]

\[
\equiv M^\Xi T^\Xi + \frac{1}{2} (V^\gamma_\gamma + Z^\gamma_\gamma) B - \frac{1}{2} W^c(B + C),
\]

(5.5)

by using the generators defined by (4.7) and (4.8) and noting $(\gamma T^A_B)_{D} = \delta^A_D \delta_D^B$. A sign difference in the third square brackets $[\cdots]$ of the the respective expansions (5.4) and (5.5) does not indicate anything wrong. This is due to the different prescription in grading $T^\Xi$ in both representations. In (5.4) we have employed the prescription

\[
T^\Xi M^\Phi = (-1)^{\varphi(\Xi) \varphi(\Phi)} M^\Phi T^\Xi.
\]

(5.6)

Here the grading of $T^\Xi$ is the same as $M^\Xi$, i.e., $g(\Xi) = g(A)g(B)$ when $\{T^\Xi\}$ is put in the tensor form $\{\psi^A_b \psi^B\}$ as (2.9). Hence $\bar{\psi} M \psi$ is a bosonic operator acting on the Fock space (3.8). On the other hand, in (5.5) we have employed the prescription

\[
T^\Xi M^\Phi = M^\Phi T^\Xi,
\]

(5.7)
assigning no grading to $T^\Xi$. This is also reasonable because the generators (5.2) consist of bosonic elements as (3.2) and commute any element of $M$. The reader may see appendix B for more arguments on these prescriptions.

It is the fact that $\exp(i\bar{\psi}M\psi)$ and $\exp(iM)$ with (5.4) and (5.5) are related by the operator-state relations (3.3) and (3.4). Note that owing to these relations the multiplication $\exp(i\bar{\psi}M_1\psi) \times \exp(i\bar{\psi}M_2\psi)$ in the Fock space induces that of supermatrices as $M_1M_2$. Thus we are now in a position to discuss the coset space $\text{PSU}(2,2|4)/H$ in either of the representations. By using the common form of the representation (5.3) we can freely change one representation to another in the following discussion. Decompose the generators of $\text{PSU}(2,2|4)$, given by either (5.1) or (5.2), under a subgroup $H$ as

$$\{T^\Xi\} = \{T^i, H^I\},$$

in which $H^I$ are generators of $H$, while $T^i$ coset ones. Then we consider a coset element

$$e^{i\phi^1 T^1 + i\phi^2 T^2 + \cdots} \equiv e^{i\phi T}.$$  \hspace{1cm} (5.9)

Here $\phi^1, \phi^2, \cdots$, are coordinates reparametrizing the coset space, denoted by $\phi^i$. (We keep the index $i$ for indicating a vector component in the tangent frame as (5.8).) Of course they have the same grading as the coset part of $M^\Xi$, i.e., $g(\bar{i}) = g(\Xi)|_{\Xi=i}$. For left multiplication of an element $e^{iM^\Xi T^\Xi} \in G$, (5.3), the coset element changes as

$$e^{iM^\Xi T^\Xi} e^{\phi T} = e^{\phi'(\phi) T} e^{i\rho(\phi, M)},$$

with an appropriate compensator $e^{i\rho(\phi, M)}$. This defines a transformation of the coordinates $\phi^i \to \phi'^i(\phi)$. When $M^\Xi$ are infinitesimally small, this relation defines the Killing vectors $R^{\Xi\bar{i}}$ as

$$\delta \phi^i = \phi'^i(\phi) - \phi^i \equiv M^\Xi R^{\Xi\bar{i}} \equiv M^\Xi \delta^\Xi \phi^i.$$ \hspace{1cm} (5.11)

They satisfy the Lie-algebra of $\text{PSU}(2,2|4)$

$$R^{\Xi\bar{i}} \frac{\partial}{\partial \phi^i} R^{\bar{\phi}^j} - (-1)^{g(\Xi)g(\bar{\phi})} R^{\bar{\phi}^i} \frac{\partial}{\partial \phi^i} R^{\Xi\bar{j}} = f^{\Xi\Phi\Sigma} R^{\Sigma\bar{j}},$$

with the structure constants $f^{\Xi\Phi\Sigma}$ of $\text{PSU}(2,2|4)$.

We would like to make important comments on the above algebraic construction. First of all, the construction does not need any representation at all, although we have proceeded the arguments having the unitary(oscillator) representation or the matrix representation in mind. That is, the above machinery to construct the Killing vectors $R^{\Xi\bar{i}}$ works at the algebraic level, once given the Lie-algebra of the generators $T^\Xi$. We give a demonstration for this in appendix B. Hence the forms of the Killing vectors $R^{\Xi\bar{i}}$ are the same if two representations take the same form of the Lie-algebra, like the unitary(oscillator) representation (5.1) and the matrix one (5.2). Moreover the Killing vectors $R^{\Xi\bar{i}}$ are free from any extra $U(1)$ factor of the central charge $C$, since the calculation is purely algebraic. On
the contrary, if the construction is done by using the unitary(oscillator) representation or the matrix one, in (5.10) the compensator acquires an extra U(1) factor as

$$e^{\rho(\phi, M)} = e^{\rho(\phi, M)^I H^I + c(\phi, M)C} \in H \otimes U(1),$$

even though $$e^{iM_{\Xi T}^\Xi T}$$ does not have it. Here $$\rho(\phi, M)^I$$ and $$c(\phi, M)$$ are appropriate functions of $$\phi^i$$. This is due to the fact that the Lie-algebra of PSU(2,2|4) is merely realized by embedding it in SU(2,2|4). We would like to emphasize that the Killing vectors $$R^{\Xi a}$$ realize the Lie-algebra of PSU(2,2|4), given by (5.12), without the central charge. This is an advantage of the non-linear realization over the other two representations.

5.2 PSU(2,2|4)/{SO(1,4)⊗SO(5)}

So far non-linear realization of PSU(2,2|4) has been discussed on the coset space PSU(2,2|4)/H without specifying a subgroup H. Now we take H to be SO(1,4)⊗SO(6) to proceed with our discussions. First of all we note that

$$\text{SU}(2,2) \otimes \text{SU}(4) \cong \text{SO}(2,4) \otimes \text{SO}(6) \supset \text{SO}(1,4) \otimes \text{SO}(5).$$

The matrix representation of SU(2,2)⊗SU(4) so far discussed can be identified with the chiral spinor representation of SO(2,4)⊗SO(6). The Dirac algebrae of SO(2,4) and SO(5) respectively read

$$\{\Gamma_p, \Gamma_q\} = 2\eta_{pq} = 2(-1, 1, 1, 1, 1, -1), \quad p, q = 0, 1, \ldots, 5,$$
$$\{\hat{\Gamma}_{\hat{p}}, \hat{\Gamma}_{\hat{q}}\} = 2\hat{\eta}_{\hat{p}\hat{q}} = 2(1, 1, 1, 1, 1, 1), \quad \hat{p}, \hat{q} = 0, 1, \ldots, 5.$$

In the chiral spinor representation the Dirac matrices of SO(2,4) are, for example, given by

$$\Gamma_5 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Gamma_m = \begin{pmatrix} 0 & \gamma_m \\ \gamma_m & 0 \end{pmatrix}, \quad (5.13)$$

with 4×4 $$\gamma$$ matrices satisfying

$$\{\gamma_m, \gamma_n\} = 2\eta_{mn} = 2(-1, 1, 1, 1, 1, 1), \quad m, n = 0, 1, 2, \ldots, 4.$$

On the other hand the Dirac matrices of SO(6) are given by

$$\hat{\Gamma}_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\Gamma}_{\hat{m}} = \begin{pmatrix} 0 & i\hat{\gamma}_{\hat{m}} \\ -i\hat{\gamma}_{\hat{m}} & 0 \end{pmatrix}, \quad (5.14)$$

with 4×4 $$\hat{\gamma}$$ matrices satisfying

$$\{\hat{\gamma}_{\hat{m}}, \hat{\gamma}_{\hat{n}}\} = 2\hat{\eta}_{\hat{m}\hat{n}} = 2(1, 1, 1, 1, 1, 1), \quad \hat{m}, \hat{n} = 1, 2, \ldots, 5.$$
By using these Dirac matrices the generators of SO(2,4) and SO(6) are given by

\[ \Gamma_{pq} = \frac{1}{4} \{ \Gamma_p, \Gamma_q \} P_+, \quad \hat{\Gamma}_{\hat{p}\hat{q}} = \frac{1}{4} \{ \hat{\Gamma}_{\hat{p}}, \hat{\Gamma}_{\hat{q}} \} \hat{P}_+. \]

The chiral projectors take the diagonalized forms

\[ P_+ = \frac{1}{2} (1 + \Gamma_7) = \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right), \quad \hat{P}_+ = \frac{1}{2} (1 + \hat{\Gamma}_7) = \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right), \]

when we choose the Weyl representation

\[ \gamma_0 = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right), \quad \gamma_1 = \left( \begin{array}{cc} 0 & \sigma_1 \\ \sigma_1 & 0 \end{array} \right), \quad \gamma_2 = \left( \begin{array}{cc} 0 & \sigma_2 \\ \sigma_2 & 0 \end{array} \right), \]

\[ \gamma_3 = \left( \begin{array}{cc} 0 & \sigma_3 \\ \sigma_3 & 0 \end{array} \right), \quad \gamma_4 = i \gamma_0 \gamma_1 \gamma_2 \gamma_3, \]

for \( \gamma_m \) in (5.13) and a similar representation for \( \gamma_{\hat{m}} \) in (5.14). We have one to one correspondence between the generators of SU(2,2) \( \otimes \) SU(4) in (5.2) and those of SO(2,4) \( \otimes \) O(6) as

\[ \{ \mathcal{L}^a_{\beta} \} = \{ \Gamma_{12}, \Gamma_{23}, \Gamma_{31} \}, \quad \{ P_{\alpha\beta} \} = \{ \Gamma_{\mu 4} + \Gamma_{\mu 5} \}, \]

\[ \{ \hat{\mathcal{L}}_{\hat{a}\hat{\beta}} \} = \{ \Gamma_{01}, \Gamma_{02}, \Gamma_{03} \}, \quad \{ K_{\alpha\beta} \} = \{ \Gamma_{\mu 4} - \Gamma_{\mu 5} \}, \]

\[ \mathcal{D} = \Gamma_{45}, \quad \{ R_{ab} \} = \{ \hat{\Gamma}_{\hat{p}\hat{q}} \} \]

with \( \mu = 0, 1, 2, 3 \).

We further decompose the generators of SO(2,4) \( \otimes \) SO(6) under SO(1,4) \( \otimes \) SO(5) as

\[ \Gamma_{pq} = \begin{bmatrix} \Gamma_{mn} & \Gamma_{m5} \\ \Gamma_{5n} & 0 \end{bmatrix}, \quad \hat{\Gamma}_{\hat{p}\hat{q}} = \begin{bmatrix} \hat{\Gamma}_{\hat{m}\hat{n}} & \hat{\Gamma}_{\hat{m}5} \\ \hat{\Gamma}_{\hat{5}\hat{n}} & 0 \end{bmatrix}, \]

\[ m, n = 0, 1, \ldots, 4, \quad \hat{m}, \hat{n} = 0, 1, \ldots, 4. \]

Using this basis we rewrite the generators of PSU(2,2|4) in the matrix representation, given by (5.2), as

\[ \{ T^\Xi \} = \{ \Gamma_{mn}, \Gamma_{m5}, \hat{\Gamma}_{\hat{m}\hat{n}}, \hat{\Gamma}_{\hat{m}5}, S_{ab}^{\alpha}, Q_{\alpha a}, \hat{S}_{\alpha\hat{a}}, \hat{Q}_{\alpha\hat{a}} \}. \]

Now we are in a position to construct the coset superspace PSU(2,2|4)/SO(1,4) \( \otimes \) SO(5), following the general method given previously. The coset element \( e^{i\phi T} \), given by (5.9), takes an explicit form with

\[ \{ T^i \} = \{ \Gamma_{m5}, \hat{\Gamma}_{\hat{m}5}, S_{ab}^{\alpha}, Q_{\alpha a}, \hat{S}_{\alpha\hat{a}}, \hat{Q}_{\alpha\hat{a}} \}. \]

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These coset generators act on $\text{PSU}(2,2|4)/\text{SO}(1,4) \otimes \text{SO}(5)$ transitively. They are identified with the corresponding generators of the $D = 10$ Poincaré superalgebra at the origin of the coset superspace. After this identification it is natural to rename the generators of $\{T^i\}$ as
\[
\{\Gamma_m, \hat{\Gamma}_{\bar{m}}\} = \{\Gamma^M\}, \quad \{S^a_{\alpha}, Q^a_{\alpha}, \hat{S}^{\dot{a} \dot{a}}, \hat{Q}_{\dot{a} \dot{a}}\} = \{Q^{\mathfrak{M}}\},
\] (5.16)
with $M = 0, 1, 2, \cdots, 9$ and $\mathfrak{M} = 1, 2, \cdots, 32$. Correspondingly the coordinates $\phi^\bar{i}$ reparametrizing $\text{PSU}(2,2|4)/\text{SO}(1,4) \otimes \text{SO}(5)$ are renamed as
\[
\{\phi^\bar{i}\} = \{X^M, \Theta^{\mathfrak{M}}\}. \tag{5.17}
\]
They are identified with supercoordinates in the $D = 10$ curved spacetime. By using them we may write the coset element (5.9) in a form looking like a vertex operator of the Green-Schwarz string theory as
\[
e^{i\phi^T} = e^{i(X \cdot \Gamma + \Theta \cdot Q)}. \tag{5.18}
\]

The Killing vectors defined by (5.11) are found as functions of the $D = 10$ supercoordinates,
\[
\delta^X \phi^\bar{i} = R^{X \bar{i}}(\phi) = \left( R^{X M}(X, \Theta), R^{\mathfrak{M} \bar{M}}(X, \Theta) \right). \tag{5.19}
\]

6 Non-linear realization of Oscillators

In the previous section we have discussed that the coset element (5.18) looks like a vertex operator and it transforms according to (5.10), i.e.,
\[
e^{i(X \cdot \Gamma + \Theta \cdot Q)} \longrightarrow e^{iX'(X, \Theta) \cdot \Gamma + \Theta'(X, \Theta) \cdot Q} = e^{iM^{\bar{X} \bar{M}} e^{i(X \cdot \Gamma + \Theta \cdot Q)} e^{-i\rho(X, \Theta, M)}, \tag{6.1}
\] in which the non-linear transformations $X'(X, \Theta)$ and $\Theta'(X, \Theta)$ are generated by the Killing vectors (5.19). The arguments have been given in an algebraic way which does not rely on either of the unitary(oscillator) representation and the matrix one. However let us now choose the matrix representation. Then the transformation (6.1) is written by an $8 \times 8$ supermatrix. If there also exists an 8-d column vector $\eta(X, \Theta)$ transforming as
\[
\eta(X, \Theta) \longrightarrow \eta'(X', \Theta') = e^{i\rho(X, \Theta, M)} \eta(X, \Theta), \tag{6.2}
\] by the non-linear transformations $X'(X, \Theta)$ and $\Theta'(X, \Theta)$, then (6.1) becomes
\[
e^{i(X \cdot \Gamma + \Theta \cdot Q)} \eta(X, \Theta) \longrightarrow e^{iM^{\bar{X} \bar{M}} e^{i(X \cdot \Gamma + \Theta \cdot Q)} \eta(X, \Theta). \tag{6.3}
\] It implies that $e^{i(X \cdot P + \Theta \cdot Q)} \eta(X, \Theta)$ is a covariant vector under $\text{PSU}(2,2|4)$. In [11] such a quantity is called Killing scalar $\Upsilon$, i.e.,
\[
\Upsilon(X, \Theta) = e^{i(X \cdot \Gamma + \Theta \cdot Q)} \eta(X, \Theta)). \tag{6.4}
\]
The transformation is exactly the same as for the 8-d column vector

$$\psi = \begin{pmatrix}
\alpha \\
c \\
a \\
b^\dagger \\
\dot{\alpha}
\end{pmatrix}, \quad (6.5)$$

which was defined by (2.8). Making the identification

$$\Upsilon(X, \Theta) = \psi, \quad (6.6)$$

we claim that this is a non-linear realization of the oscillators.

The remaining question is whether the quantity \(\eta\) with the transformation property (6.2) really exists. In [11] the existence was shown for the general bosonic coset space \(G/H\) in an arbitrary, but finite representation of the coset element \(e^{i\phi T}\). We have chosen the matrix representation to discuss the coset space \(\text{PSU}(2,2|4)/\{\text{SO}(1,4) \otimes \text{SO}(5)\}\). Therefore \(\eta\) exists for this case similarly. Here we recall only of the point of the arguments and explain the quantity more explicitly for the coset space \(\text{PSU}(2,2|4)/\{\text{SO}(1,4) \otimes \text{SO}(5)\}\).

First of all we consider the Cartan-Maurer 1-form

$$g^{-1} dg = i(e^i J^i + \omega^I J^I) d\phi^j, \quad (6.7)$$

denoting the coset element (5.18) as \(g\) and using the index notation (5.17). This defines the vielbein \(e^i_j\) and the connection \(\omega^I_j\) in the tangent frame of the coset space. Under the transformation (6.1) they transform as

$$e^i_j d\phi^j \rightarrow [e^{i(\phi,M)}]^{ik} e^k_j d\phi^j,$$

$$\omega^I_j J^I d\phi^j \rightarrow e^{-i\phi(M)} [d - i\omega^I J^I d\phi^j] e^{i(\phi,M)}.$$

Then we have the Wilson line-operator

$$W(\phi, \phi_0) = P \exp i \int_{\phi_0}^{\phi} \omega^I J^I d\phi^j,$$

which transforms as

$$W(\phi, \phi_0) \rightarrow e^{i\phi(M)} W(\phi, \phi_0) e^{-i\phi(M)}.$$

The compensator \(e^{i\phi(M)}\) becomes a constant element at the origin \(\phi = 0\) of the coset space, i.e.,

$$e^{i\phi(M)} \Big|_{\phi=0} = e^{iM^I J^I} \in H = \text{SO}(1,4) \otimes \text{SO}(5). \quad (6.8)$$

Let \(e^{iM^I_0 J^I}\eta_0\) to be a linear representation vector with \(M^I_0\) parametrizing the subgroup \(H\). Then it transforms by the compensator (6.8) at the origin as

$$e^{iM^I_0 J^I}\eta_0 \rightarrow e^{iM^I J^I} e^{M^I_0 J^I} \eta_0.$$
Here $\eta_0$ is a constant vector fixed in the representation space of $H$. To be concrete for the case of $\text{PSU}(2,2|4)/\{\text{SO}(1,4) \otimes \text{SO}(5)\}$, we have

$$e^{iM^I_{\phi} H^I} \eta_0 = e^{i(M^m_{\phi} \Gamma_m + M^\alpha_{\phi} \tilde{\Gamma}_m)} \eta_0,$$

by using the generators in (5.15). Hence $\eta_0$ is now a constant chiral spinor of $\text{SO}(1,4) \otimes \text{SO}(5)$. As the result we find the quantity

$$\eta(X, \Theta) = \eta(\phi) = W(\phi, 0) e^{iM^I_{\phi} H^I} \eta_0,$$

with $\{\phi^i\} = \{X^\overline{M}, \Theta^\overline{M}\}$, which has the transformation property (6.2). Thus we have justified the identification (6.4) with $\eta(X, \Theta)$ of this form.

### 7 Exchange algebra

In the previous section we have identified the Killing scalar $\Upsilon(X, \Theta)$ of the coset space $\text{PSU}(2,2|4)/\{\text{SO}(1,4) \otimes \text{SO}(5)\}$ with the 8-d column vector $\psi$ given by (6.5). In [11] the general accounts for the Killing scalar were given for the ordinary coset space $G/H$, i.e., $G$ is not a supergroup. It was shown that it satisfies the classical exchange algebra of $G$ in the non-linear $\sigma$-model on $G/H$ with the Poisson brackets set up on the light-like line. For this it was essential to have the linear transformation property (6.3), i.e.,

$$\delta^\Xi \Upsilon = T^\Xi \Upsilon,$$  \hspace{1cm} (7.1)

by the Killing vectors (5.19). In this section we show that this is also true for the Killing scalar (6.1) of the non-linear $\sigma$-model on $\text{PSU}(2,2|4)/\{\text{SO}(1,4) \otimes \text{SO}(5)\}$. The identification (6.6) implies that the 8-d covariant vector $\psi$ given by (6.5) satisfies the classical exchange algebra of $\text{PSU}(2,2|4)$. Then the arguments go in the same way for the most part even for the coset superspace. We shall here explain them taking a care of the points for the supersymmetric generalization. First of all we write the action of the non-linear $\sigma$-model on $\text{PSU}(2,2|4)/\{\text{SO}(1,4) \otimes \text{SO}(5)\}$

$$S = \frac{1}{2} \int d^2 x \eta^+ (e^i_j \partial_+ \phi^j)(e^j_k \partial_- \phi^k)(-1)^{\vartheta(i)},$$  \hspace{1cm} (7.2)

with the vielbein $e^i_j$ defined by (6.7) and the supercoordinates $\phi^i$ given by (5.17). Here we have the graded summation for the index $i$ according the quadratic Casimir (4.11). We set up the Poisson brackets on the light-like line $x^+ = y^+$

$$\{\phi^i(x) \otimes \phi^j(y)\} = -\frac{1}{4} \left[ \theta(x-y) t^+_\phi \delta^\Xi \phi^i(x) \otimes \delta^\Phi \phi^j(y) - \theta(y-x) t^+_\phi \delta^\Xi \phi^j(y) \otimes \delta^\Phi \phi^i(x)(-1)^{\vartheta(i)\vartheta(j)}(-1)^{\vartheta(\Xi)\vartheta(\Phi)} \right].$$  \hspace{1cm} (7.3)

The notation is as follows. $\theta(x)$ is the step function. $\delta^\Xi \phi^i(x)$ are the Killing vectors defined by (5.11). More correctly they should be written as $\delta^\Xi \phi^i(\phi(x))$, but the dependence of
\( \phi^i(x) \) was omitted to avoid an unnecessary complication. The quantity \( t^+_\Xi \) is the most crucial in our arguments. It is a modified Killing metric of \( t_\Xi \). By means of it we define the classical r-matrix satisfying the classical Yang-Baxter equation. To explain this quantity let us remember the definition of the classical r-matrices for the ordinary group

\[
    r^\pm = \sum_{\alpha \in R} \text{sgn} \, \alpha E_\alpha \otimes E_{-\alpha} \pm \sum_{A,B} t_{AB} T^A \otimes T^B \equiv t^\pm_{AB} T^A \otimes T^B. \tag{7.4}
\]

Here \( T^A \) denote the generators of the group \( G \) with \( t_{AB} \) the Killing metric. They are given in the Cartan-Weyl basis as \( \{ E^\pm_\alpha, H_\mu \} \) with \( \text{sgn} \, \alpha = \pm \) according as the roots are positive or negative. Note the relation \( t^\pm_{AB} = -t^-_{BA} \). The r-matrix satisfies the classical Yang-Baxter equation

\[
    [r_{xy}, r_{xz}] + [r_{xy}, r_{yz}] + [r_{xz}, r_{yz}] = 0. \tag{7.5}
\]

Here the r-matrix acts at on a tensor product of the Killing scalars \( \Upsilon(\phi(x)) \otimes \Upsilon(\phi(y)) \otimes \Upsilon(\phi(z)) \) but only at the designated positions\([11, 18]\). For the supergroup \( \text{PSU}(2,2|4) \) the r-matrix is generalized as follows. With the generators written as \( (5.2) \) we have the quadratic Casimir \( (4.11) \), i.e.,

\[
    \sum_{\Xi,\Phi} t_{\Xi \Phi} \Xi \Phi \equiv \mathcal{L}_\beta \mathcal{L}_\alpha - \mathcal{R}_b^a \mathcal{R}_a^b + \hat{\mathcal{L}}_\hat{\beta} \hat{\mathcal{L}}_\hat{\alpha} + \mathcal{D}^2 - \{ T_\alpha^\beta, T_\beta^\alpha \} - [T_\alpha^\beta, T_\beta^\alpha] - [T_\alpha^\beta, T^\beta_a].
\]

Correspondingly to this expression the r-matrix of \( \text{PSU}(2,2|4) \) is given by

\[
    r^+ = \sum_{\Xi,\Phi} t^+_{\Xi \Phi} \Xi \Phi
    = \sum_{\alpha > \beta} \mathcal{L}_\beta \mathcal{L}_\alpha - \sum_{a > b} \mathcal{R}_b^a \mathcal{R}_a^b + \sum_{\hat{\alpha} > \hat{\beta}} \hat{\mathcal{L}}_\hat{\beta} \hat{\mathcal{L}}_\hat{\alpha} - \sum_{a, \hat{\beta}} T_\alpha^\beta T^\beta_a - \sum_{a, \beta} T_\alpha^\beta T_\beta^a - \sum_{a, \beta} T_\alpha^\beta T^\beta_a + \sum_{\Xi,\Phi} t_{\Xi \Phi} \Xi \Phi.
\]

That is, \( t^+_\Xi \) in \((7.3)\) is a simple generalization of the quantity in \((7.4)\) for the case of \( \text{PSU}(2,2|4) \). It is straightforward to show that the r-matrix generalized in this way satisfies the classical Yang-Baxter equation

\[
    [r_{xy}, r_{xz}] + [r_{xy}, r_{yz}] + [r_{xz}, r_{yz}] = 0,
\]

with the graded commutator \([ , ]\). Note that now we have

\[
    t_{\Xi \Phi} = (-1)^{g(\Xi)g(\Phi)} t_{\Phi \Xi}, \quad t^+_{\Xi \Phi} = -(-1)^{g(\Xi)g(\Phi)} t_{\Phi \Xi}.
\]

Then it follows that

\[
    \{ \phi^i(x) \otimes \phi^j(y) \} = (-1)^{g(i)g(j)} \{ \phi^i(y) \otimes \phi^j(x) \},
\]
for the Poisson brackets given by (7.3). All the arguments here on the classical Yang-Baxter equation were done for the supergroup SL(1|2) and OSP(2|2) in [19, 20]. There the r-matrix of the respective supergroup appeared as showing integrability of the $D = 2$, $(1,0)$ and $(2,0)$ effective gravity.

Finally we can show the consistency of the Poisson brackets (7.3). First of all it satisfies the Jacobi identities owing to the classical Yang-Baxter equation for the r-matrix. Secondly the energy-momentum tensor of the non-linear $\sigma$-model (7.2) reproduces the diffeomorphism

\[ \delta_{\text{diff}} \phi^i(x^+, x^-) \equiv \epsilon(x^-) \partial_- \phi^i(x^+, x^-) = \int dy^- \epsilon(y^-) \{\phi^i(x), T_-(\phi(y))\} \bigg|_{x^+ = y^+}, \]

Thirdly the Poisson brackets tend to those of the free boson and fermion theory as

\[ \{\phi^i(x) \otimes \phi^j(y)\} = -\frac{1}{4} [\theta(x - y)\delta^{ij} - \theta(y - x)\delta^{ji} (-1)^{\bar{i}\bar{j}}]. \]

These statements can be verified in the same way as for the ordinary non-linear $\sigma$-model.

With the Poisson brackets (7.3) let us calculate $\{\Upsilon(x) \otimes \Upsilon(y)\}$ for the Killing scalar $\Upsilon$ using the property

\[ \{\tilde{\phi}^i(x) \otimes \Upsilon(y)\} = \{\tilde{\phi}^i(x) \otimes \phi^j(y)\} \frac{\delta \Upsilon(y)}{\delta \phi^j(y)}, \]

together with (7.1). We then get the classical exchange algebra in the form

\[ \{\Upsilon(x) \otimes \Upsilon(y)\} = -\frac{1}{4} [\theta(x - y)r^+ + \theta(y - x)r^-] \Upsilon(x) \otimes \Upsilon(y), \quad (7.6) \]

on the light-like plane $x^+ = y^+$. Here $\Upsilon(x)$ should be understood with an abbreviated notation for $\Upsilon(\phi(x))$. It is a non-linear realization of the oscillators by the identification (6.4). Thus the oscillators obey the classical exchange algebra (7.6).

The supergroup $\text{PSU}(2,2|4)$ is a simple group. Hence we may use the plug-in formula to promote the r-matrix to the universal R-matrix $R_{xy}$. It is expressed purely in terms of generators of G. Then (7.6) becomes

\[ \Upsilon(x) \otimes \Upsilon(y) = \theta(x - y)R_{xy}^+ \Upsilon(y) \otimes \Upsilon(x) + \theta(y - x)R_{xy}^- \Upsilon(y) \otimes \Upsilon(x). \quad (7.7) \]

Here the universal R-matrix satisfies the quantum Yang-Baxter equation

\[ R_{xy} R_{xz} R_{yz} = R_{yz} R_{xz} R_{xy}. \quad (7.8) \]

(7.6) and (7.3) are the respective classical correspondents of (7.7) and (7.8) obtained by

\[ R_{xy}^\pm = 1 + h r_{xy}^\pm + O(h^2), \]
with \( h = -\frac{1}{4} \). For \( G = \text{SU}(2) \) or \( \text{SU}(1,1) \) the plug-in formula of the universal R-matrix is given in a rather simple form as

\[
R = q^{\frac{1}{2} H \otimes H} \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2} n(n-1)}(q - q^{-1})^n}{[n]_q!} (E^+)^n \otimes (E^-)^n = \exp_q[(q - q^{-1})E^+ \otimes E^-], \tag{7.9}
\]

by using the notation (7.4) for the Lie-algebra. Here the \( q \)-exponential is defined by

\[
\exp_q(x) = \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2} n(n-1)}x^n}{[n]_q!},
\]

with \( q = e^h \) and

\[
[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.
\]

The formula (7.9) can be generalized for the general simple group. The generalized plug-in formula for the ordinary group can be found in [4]. That for supergroups was given in [5].

We consider a correlation function

\[
< \mathcal{V}_1(x_1) \otimes \mathcal{V}_2(x_2) \otimes \cdots \otimes \mathcal{V}_i(x_i) \otimes \mathcal{V}_{i+1}(x_{i+1}) \otimes \cdots \otimes \mathcal{V}_N(x_N) > . \tag{7.10}
\]

Here \( \mathcal{V}_i(x_i) \) are the \( \mathcal{N} = 4 \) SUSY field strength multiplet and their descendants in table 1, i.e.,

\[
\{ \mathcal{V}(x) \} = \{ \mathcal{D}^k F(x), \mathcal{D}^k \psi(x), \mathcal{D}^k \Phi(x), \mathcal{D}^k \dot{\psi}(x), \mathcal{D}^k \dot{\Phi}(x) \}, \quad k = 0, 1, 2, \ldots .
\]

They are arrayed on the light-like line \( x_1^+ = x_2^+ = \cdots = x_N^+ \) as shown in figure 2.

![Figure 2: Arrows stand for SU(2)-spins of the oscillators \( a^\alpha, b^\dot{\alpha}, c^\alpha, d^\dot{\alpha} \). Encircling them indicates a component of the \( \mathcal{N} = 4 \) field strength multiplet \( \mathcal{V}_i(x_i) \).](image)

Each of the multiplets is given in terms of the oscillators as in table 2. Let \( \mathcal{V}_i(x_i) \) and \( \mathcal{V}_{i+1}(x_{i+1}) \) at the adjacent positions to be one of the components belonging to \( \mathcal{D}_{\alpha\dot{\beta}} \Phi \) and \( \Psi \) respectively. For instance, we have

\[
\mathcal{V}_i(x_i) = (a^\dagger_\alpha(x_i) b^\dagger_\dot{\alpha}(x_i))(c^\dagger_\alpha(x_i) d^\dagger_\dot{\alpha}(x_i)), \\
\mathcal{V}_{i+1}(x_{i+1}) = a^\dagger_\alpha(x_{i+1}) c^\dagger_\alpha(x_{i+1}) d^\dagger_\dagger(x_{i+1}) d^\dagger_\dot{\alpha}(x_{i+1}).
\]
\[ D_{\alpha \beta} = P_{\alpha \beta} = a_{\alpha}^\dagger b_{\beta}^\dagger. \]

The quantum exchange algebra for the field strength multiplet \( \mathcal{V}_i(x_i) \) follows from (7.7), because the oscillators are identified with the Killing scalar \( \Upsilon \) as (6.6). That is, the quantum exchange algebra is obtained by braiding the respective oscillators in \( \mathcal{V}_i(x_i) \) and \( \mathcal{V}_j(x_j) \) one by one. The R-matrix is now in an infinite-dimensional representation.

The \( \mathcal{N} = 4 \) SUSY field strength multiplet \( \mathcal{V} \) is the simplest one. The arguments can be similarly applied to other multiplets than \( \mathcal{N} = 4 \) SUSY field strength multiplet \( \mathcal{V} \). A summary of the general superconformal multiplets is given in appendix A.

8 Conclusions

Thus we are led to conclude that the supercoherent space of the unitary(oscillator) representation of \( \text{PSU}(2,2|4) \) becomes non-commutative. To show this, we have constructed the oscillators as the Killing scalar \( \Upsilon \) in the non-linear \( \sigma \)-model on \( \text{PSU}(2,2|4)/\{\text{SO}(1,4) \otimes \text{SO}(5)\} \).

It was argued that they satisfy the exchange algebra when the model is quantized on the light-like line. They took a suggestive form of the vertex operator of the Green-Schwarz superstring as given by (6.4), i.e.,

\[ \Upsilon = e^{i(X \cdot P + \Theta \cdot Q)} \eta(X, \Theta). \]

We comment on the \( D = 10 \) flat space-time limit of the non-linear \( \sigma \)-model on \( \text{PSU}(2,2|4)/\{\text{SO}(1,4) \otimes \text{SO}(5)\} \). Taking a naive limit where the \( \text{AdS}_5 \otimes S^5 \) radius tends to \( \infty \) does not give the desired \( D = 10 \) super-Poincaré invariance to the non-linear \( \sigma \)-model. According to [10] a correct way to go to the flat space-time limit is to rescale the structure constants in the algebras (2.6) as

\[
\{ \mathcal{S}^a_{\beta b}, Q^a_{\gamma b} \} = \frac{1}{r} \delta^a_{\beta b} L^\beta_{\gamma b} + \frac{1}{r} \delta^a_{\beta b} R^\beta_{\gamma b} + \frac{1}{2} \delta^a_{\beta b} \delta^\beta_{\gamma b} (D - C),
\]

\[
\{ \mathcal{\dot{S}}^{\dot{a}}_{\dot{b} \dot{\alpha}}, \dot{Q}^{\dot{a}}_{\dot{b} \dot{\beta}} \} = \frac{1}{r} \delta^{\dot{a}}_{\dot{b} \dot{\alpha}} \dot{L}^{\dot{\beta}}_{\dot{\alpha} \dot{b}} - \frac{1}{r} \delta^{\dot{a}}_{\dot{b} \dot{\alpha}} \dot{R}^{\dot{\beta}}_{\dot{\alpha} \dot{b}} + \frac{1}{2} \delta^{\dot{a}}_{\dot{b} \dot{\alpha}} \delta^{\dot{\beta}}_{\dot{\alpha} \dot{b}} (D + C).
\]

Then in the limit \( r \to \infty \) the vielbein defined by (6.7) tends to

\[ e^M_\dot{J} d\theta^\dot{J} = dX^M - \Theta \cdot \Gamma^M d\Theta, \quad e^M_{\dot{J}} \theta^\dot{J} = d\Theta^M, \quad (8.1) \]

by using (5.16) and (5.17). We no longer need distinguish the coordinates of the coset space and those of the tangent space, so that

\[ X^M = X^M, \quad \Theta^{\Theta} = \Theta^{\Theta}. \]

The vielbeins in (8.1) are invariant under the global supertransformation

\[ \delta X^M = \alpha \cdot \Gamma^M \Theta, \quad \delta \Theta^{\Theta} = \alpha^{\Theta}. \]

We also comment on the fact that our correlation function (7.10) is independent of the position \( x_i \) of the observables \( \mathcal{V}_i(x_i) \) on the light-like line. It might be considered as
a correlation function of the similar kind to the one in the topological field theory. The exchange algebra of OSP(2|2) for the (2,0) topological gravity was discussed in such a context in [20]. We may think of position-dependence for the correlation function such as

\[ e^{ip_1 x_1} e^{ip_2 x_2} \cdots e^{ip_N x_N} < \mathcal{V}_1(x_1) \otimes \mathcal{V}_2(x_2) \otimes \cdots \otimes \mathcal{V}_N(x_N) >, \tag{8.2} \]

in which \( p_1, p_2, \ldots, p_N \) are momenta excited along the light-like line. But PSU(2,2|4) is too restrictive to allow for such a dependence. Therefore we think of breaking the symmetry of PSU(2,2|4) to a subgroup symmetry which contains a certain number of central charges as factor groups. In [2] they took such a subgroup to be PSU(1,1|2) \( \ltimes \mathbb{R}^3 \). The generators \( D, P, K \) in (2.9) are reduced to three U(1) charges acting on the correlation function (8.2). The position-dependent R-matrix acting on the correlation function (8.2) was given for this residual subgroup in [2]. It played a crucial role in discussing the duality between the \( D = 2 \) spin-chain and the \( D = 4, \mathcal{N} = 4 \) SUSY YM theory. It is interesting to investigate how such a position-dependent R-matrix occurs as symmetry breaking of the universal R-matrix of PSU(2,2|4) in the non-linear \( \sigma \)-model. The issue will be discussed in a forthcoming publication [21].

A \( \frac{1}{4} \) and \( \frac{1}{8} \)-BPS multiplets

In section 3 we have argued that the \( \mathcal{N} = 4 \) SUSY field strength multiplet is the smallest BPS multiplet of the unitary (oscillator) representation of PSU(2,2|4). In this appendix we continue the arguments [13, 14, 15] to give larger BPS multiplets representing other \( \mathcal{N} = 4 \) superconformal multiplets. Our arguments on the exchange algebra in this paper can be straightforwardly applied to those multiplets as well.

In order to enlarge the BPS multiplet discussed in section 3, we retain only half of the supercharges in (3.9)

\[ Q_\beta^1, \quad \dot{Q}_{\dot{\alpha}4}, \]

as fermionic generators annihilating the vacuum. They break the R-symmetry SU(2) \( \otimes \) SU(2) of the physical vacuum of the \( \frac{1}{2} \)-BPS multiplet to U(1) \( \otimes \) U(1). To be explicit, they are

\[ a_\beta^i c^1, \quad b_\beta^i d^4. \]

Then the following states

\[ Z, \quad Y = c^1 d^i Z, \]
Table 3: The BPS multiplets. \( \#Q \) and \( \hat{\#Q} \) are the numbers of supercharges annihilating the highest weight state. The third column indicates the highest weight state of \( SU(2) \otimes SU(2) \), excited from the highest state in the fourth column by applying the remaining supercharges. Further excitation is possible by applying the space-time derivative \( P_{\alpha\dot{\beta}} \). \( \Delta \) in the last column is the conformal dimension defined by the dilatation in (2.3).

\[
\begin{array}{|c|c|c|c|}
\hline
\text{type} & (\#Q, \hat{\#Q}) & \text{SU(2)} \otimes \text{SU(2)} & \text{SU(2)} \otimes \text{SU(2)} \otimes \text{SU(4)} \\
\hline
\frac{1}{4}-\text{BPS} & (4,4) & [j+2, j+2] & [j,j][0,k,0], \ k > 0 \\
\frac{1}{2}-\text{BPS} & (2,2) & [j+3, j+3] & [j,j][l,k,l], \ l > 0 \\
\frac{1}{8}-\text{BPS} & (2,0) & [3, j+4] & [0,j][l,k,l+2m], \ m > 0 \\
\hline
\end{array}
\]

are annihilated by them and satisfies the constraint \( C = 0 \). This defines the physical vacuum of the \( \frac{1}{8} \)-BPS multiplet. If we further halve the number of the annihilating supercharges and retain only

\[ \hat{Q}_{\dot{\alpha}4} \]

the R-symmetry gets enlarged as \( SU(3) \otimes U(1) \). The states which are annihilated by them and satisfy \( C = 0 \) are

\[
Z, \quad Y = c_2^\dagger d_3^\dagger Z, \quad X = c_1^\dagger d_4^\dagger Z. \tag{A.1}
\]

This defines the vacuum of the \( \frac{1}{8} \)-BPS multiplet.

Let us denote the fermionic oscillators belonging to the representation \( 4^* \) of the R-symmetry \( SU(4) \) by the Dynkin label as

\[
c_1^\dagger = [-1, 0, 0], \quad c_2^\dagger = [1, -1, 0], \quad d_3^\dagger = [0, 1, -1], \quad d_4^\dagger = [0, 0, 1].
\]

Then the states \( X, Y, Z \) in (A.1) are denoted by the Dynkin label as

\[
Z = [0, 1, 0], \quad Y = [1, -1, 1], \quad X = [-1, 0, 1],
\]

which form the representation \( 3^* \) of the R-symmetry \( SU(3) \). The states of the representation \( 3 \) are

\[
Z^* = [0, -1, 0], \quad Y^* = [-1, 1, 1], \quad X^* = [1, 0, 1].
\]

In terms of the oscillators they are given by

\[
Z^* = c_1^\dagger c_2^\dagger d_3^\dagger d_4^\dagger Z, \quad Y^* = c_1^\dagger d_4^\dagger Z, \quad X^* = c_2^\dagger d_4^\dagger Z.
\]

Note that these six states were given by the field \( \Phi \) in table 2.

By a systematic analysis of the unitary representation of \( PSU(2,2|4) \) \cite{13,14,15} the BPS multiplets are known as given in table 3\cite{14}. The physical vacua of the BPS multiplets
are the highest weight states of the unitary representation of \text{PSU}(2,2|4). They are given by the following tensor products

\begin{align*}
Z^k & \quad \text{for } \frac{1}{2}\text{-BPS}, \\
Y^l Z^{k+l} & \quad \text{for } \frac{1}{4}\text{-BPS}, \\
X^m Y^{l+m} Z^{l+m+k} & \quad \text{for } \frac{1}{8}\text{-BPS},
\end{align*}

(A.2)

with (anti-)symmetrization indicated by the SO(6) Young tableau in figure 3. Here we have to take \(k\) copies of the oscillators \(a^\alpha, b^\dot{\alpha}, c^a\) and \(a^\dagger_\alpha, b^\dagger_\dot{\alpha}, c^\dagger_a\). Correspondingly the Fock space which they act on is the tensor product of \(k\) copies of the Fock space (3.8). For instance, we understand \(Z\), given by (3.6), as \(\sum_{s=1}^k c_3^{(s)\dagger} c_4^{(s)}\). It is easy to verify that the physical vacua in (A.2) are indeed the SU(4) highest weight states and the conformal dimension \(\Delta\), respectively given in table 3.

The BPS multiplets with \(\Delta \geq 2\), given in table 3, are other \(\mathcal{N} = 4\) SUSY superconformal multiplets than the field strength multiplet in table 1. So far our arguments have been done by assuming the gauge group to be Abelian. All the fields of \(\mathcal{N} = 4\) superconformal multiplet are in the adjoint representation of a gauge group. Let them to be Lie-valued in the gauge algebra. Then the physical vacua in (A.2) are given as the gauge singlets

\begin{align*}
\text{tr}[Z^k] & \quad \text{for } \frac{1}{2}\text{-BPS}, \\
\text{tr}[Y^l]\text{tr}[Z^{k+l}] & \quad \text{for } \frac{1}{4}\text{-BPS}, \\
\text{tr}[X^m]\text{tr}[Y^{l+m}]\text{tr}[Z^{l+m+k}] & \quad \text{for } \frac{1}{8}\text{-BPS}.
\end{align*}

(A.3)

Here \(\text{tr}\) denotes trace over the gauge algebra. Therefore we have \(k,l,m \geq 2\) for a simple

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{The SO(6) Young tableaux representing symmetrization and anti-symmetrization for the \(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}\) BPS states.}
\end{figure}
gauge group. We may consider other multitrace states obtained by replacements such as
\[ tr[Z^k] \rightarrow tr[Z^{k-l}]tr[Z^l], \quad tr[Y^k]tr[Z^l] \rightarrow tr[Y^kZ^l], \quad etc \] (A.4)
since they do not change the dimension \( \Delta \). However we have
\[ tr[Z^k] = tr[Z^{k-l}]tr[Z^l], \]
by the symmetrization indicated by the SO(6) Young tableau in figure 3. Therefore the former replacement gives nothing new. On the other hand the anti-symmetrization by the SO(6) Young tableau implies that
\[ tr[Y^l]tr[Z^k] \neq tr[Y^lZ^k]. \]
That is, the latter replacement of (A.4) changes the physical vacuum into a non-BPS vacuum, Thus the multitrace states with multiplicity greater than 3 are irrelevant for the BPS multiplets in table 3.

Finally it is interesting to understand the vacua of the BPS multiplets so far discussed in a way which manifests the SO(6) symmetry of the Young tableau[15]. We consider the generators \( Q^a_\alpha \) and \( \tilde{Q}_{\dot{a}\dot{\alpha}} \) and make three sets as
\[ \{ Q^1_\alpha, Q^2_\alpha, \tilde{Q}_{\dot{a}3}, \tilde{Q}_{\dot{a}4} \}, \quad \{ Q^1_\alpha, Q^3_\alpha, \tilde{Q}_{\dot{a}2}, \tilde{Q}_{\dot{a}4} \}, \quad \{ Q^2_\alpha, Q^3_\alpha, \tilde{Q}_{\dot{a}1}, \tilde{Q}_{\dot{a}4} \}, \]
in each of which all the generators are anti-commuting. Then the states \( X, Y, Z \) in (A.1) satisfy constraints as
\[ Q^1_\alpha Z = Q^2_\alpha Z = \tilde{Q}_{\dot{a}3} Z = \tilde{Q}_{\dot{a}4} Z = 0, \]
\[ Q^1_\alpha Y = Q^3_\alpha Y = \tilde{Q}_{\dot{a}2} Y = \tilde{Q}_{\dot{a}4} Y = 0, \]
\[ Q^2_\alpha X = Q^3_\alpha X = \tilde{Q}_{\dot{a}1} X = \tilde{Q}_{\dot{a}4} X = 0. \]
Consequently the physical vacua (A.3) to be constrained by
\[ Q^1_\alpha tr[Z^k] = Q^2_\alpha tr[Z^k] = \tilde{Q}_{\dot{a}3} tr[Z^k] = \tilde{Q}_{\dot{a}4} tr[Z^k] = 0 \quad \text{for} \quad \frac{1}{2}\text{-BPS}, \]
\[ Q^1_\alpha \left( tr[Y^l]tr[Z^{k+l}] \right) = \tilde{Q}_{\dot{a}4} \left( tr[Y^l]tr[Z^{k+l}] \right) = 0 \quad \text{for} \quad \frac{1}{4}\text{-BPS}, \]
\[ \tilde{Q}_{\dot{a}4} \left( tr[X^m]tr[Y^{l+m}]tr[Z^{l+m+k}] \right) = 0 \quad \text{for} \quad \frac{1}{8}\text{-BPS}. \]
They are nothing but the constraints we have so far discussed.

### B Algebraic calculation of the Killing vectors

The Killing vectors of the coset space G/H are defined when G is simple or most generally speaking semi-simple. The supergroup PSU(2,2|4) is simple. But either of the
When the parameters $M$ are infinitesimally small, we may write the transformation (5.10) as

$$
e^{i[\phi T + M^2 R^2 T + O(M^2)]} = e^{iM^2 T^2} e^{i[M^2 \hat{\rho}^2 H]} e^{-i[M^2 \hat{\rho}^2 H]},$$  \hspace{1cm} (B.1)$$

Here use was made of the definition of the Killing vectors (5.11) and the notation

$$R^2(\phi) \cdot T \equiv R^{2\phi}(\phi) T^i, \quad \rho^2(\phi) \cdot T \equiv \rho^{2\phi}(\phi, M) H^i.$$  \hspace{1cm} (B.1)

B.1 becomes

$$e^{i[\phi T + M^2 R^2 T + O(M^2)]} = e^{i[\phi T + \sum_{n=0}^{\infty} \alpha_n (ad i\phi T)^n (M^2 T^2) - \sum_{n=0}^{\infty} (-1)^n \alpha_n (ad i\phi T)^n (M^2 \rho^2 H) + O(\varepsilon^2)]},$$  \hspace{1cm} (B.2)

by using the following formulae: for matrices $E$ and $X$

$$\exp E \exp X = \exp \left( X + \sum_{n=0}^{\infty} \alpha_n (ad X)^n E + O(\varepsilon^2) \right),$$  \hspace{1cm} (B.3)

$$\exp X \exp E = \exp \left( X + \sum_{n=0}^{\infty} (-1)^n \alpha_n (ad X)^n E + O(\varepsilon^2) \right),$$  \hspace{1cm} (B.4)

if $\varepsilon \ll 1$. Here $\alpha_n$ are the constants recursively determined by

$$\alpha_n = \frac{\alpha_{n-1}}{2!} + \cdots + \frac{\alpha_0}{(n+1)!} = 0, \quad \text{for } n = 1, 2, \cdots,$$  \hspace{1cm} (B.5)

as

$$\alpha_0 = 1, \quad \alpha_1 = -\frac{1}{2}, \quad \alpha_2 = \frac{1}{12}, \quad \alpha_3 = 0, \quad \alpha_4 = -\frac{1}{720}, \quad \ldots.$$  \hspace{1cm} (B.5)

(B.3)–(B.5) will be proved at the end of this appendix. In (B.2) $(ad i\phi \cdot T)^n$ is a mapping defined by the $n$-ple commutator

$$(ad i\phi \cdot T)^n O = [i\phi \cdot T, \cdots, [i\phi \cdot T, [i\phi \cdot T, O]] \cdots].$$  \hspace{1cm} (B.6)

As has been discussed in subsection 5.1, the grading of the commutator may differ depending on the representation. We employed the prescription (5.6) for the unitary(oscillator) representation, but the one (5.7) for the matrix representation. For the respective case (B.6) reads

$$i^n M^2 \phi^{i_1} \phi^{i_2} \phi^{i_3} \cdots \phi^{i_n} \{T^{i_1}, \cdots, [T^{i_n}, T^2] \} \cdots.$$  \hspace{1cm} (B.6)
or

\[ i^n \phi^{i_1} \phi^{i_2} \cdots \phi^{i_n} M^{\Xi}[T^{i_1}, \ldots, [T^{i_n}, T^{\Xi}]} \cdots \].

Here \([\cdot, \cdot]\) can be automatically read as a commutator or anti-commutator from the grading of \(\phi^{i_1}, \phi^{i_2}, \cdots\). Note that the difference of the two prescriptions reflects merely on the ordering of \(\phi^i\)'s. For a simple case where \(n = 1\) and \(M^I = 0\) it reads

\[
(ad \phi \cdot T)(T^{\Xi} M^{\Xi}) \bigg|_{M^I = 0} = X^{M_1} M^{M_2}[T^{M_1}, T^{M_2}] + X^{M_1} M^{M_2}[T^{M_1}, T^{M_2}] + \Theta^{M_1} M^{M_2}[T^{M_1}, T^{M_2}] + \Theta^{M_1} M^{M_2}[T^{M_1}, T^{M_2}],
\]

with the sign taken to be + for the unitary (oscillator) representation, but - for the matrix representation. Here use was made of the index notation (5.17) for \(\phi^i\).

We expand \(R^{\Xi}(\phi)\) and \(\rho^{\Xi}(\phi, M)\) in series of \(\phi^i\):

\[
R^{\Xi}(\phi) = R^{\Xi}_0(\phi) + R^{\Xi}_1(\phi) + \cdots + R^{\Xi}_n(\phi) + \cdots,
\]

\[
\rho^{\Xi}(\phi, M) = \rho^{\Xi}_0(\phi, M) + \rho^{\Xi}_1(\phi, M) + \cdots + \rho^{\Xi}_n(\phi, M) + \cdots.
\]

The \(n\)-th order terms \(R^{\Xi}_n(\phi)\) and \(\rho^{\Xi}_n(\phi, M)\) of the expansion obey the recursive relations obtained by comparing the powers of both sides of (B.2) order by order. We find to the 0-th order of \(\phi\)

\[
R^{\Xi}_0(z) \cdot T + \alpha_0 \left( \rho^{\Xi}_0(\phi, M) \cdot H \right) = i\alpha_0 T^{\Xi},
\]

to the first order of \(\phi\)

\[
R^{\Xi}_1(\phi) \cdot T + \alpha_0 \left( \rho^{\Xi}_1(\phi, M) \cdot H \right) = i\alpha_1 \left[ \phi \cdot T, T^{\Xi} \right] - \alpha_1 \left[ \phi \cdot T, \rho^{\Xi}_0(\phi, M) \cdot H \right],
\]

to the second order in \(z\)

\[
R^{\Xi}_2(\phi) \cdot T + \alpha_0 \left( \rho^{\Xi}_2(\phi, M) \cdot H \right) = i\alpha_2 \left[ \phi \cdot T, \left[ \phi \cdot T, T^{\Xi} \right] \right] - \alpha_1 \left[ \phi \cdot T, \rho^{\Xi}_1(\phi, M) \cdot H \right],
\]

and so on. These recursion relations are solved for \(R^{\Xi}_n(\phi)\) and \(\rho^{\Xi}_n(\phi, M)\) order by order, by setting \(R^{\Xi}_0(\phi) = i\alpha_0 \delta^{\Xi I}\) as an initial condition. Thus the Killing vectors are determined once the algebra of \(\{T^{\Xi}\}\) given with the decomposition (5.8).

Finally we will prove (B.3). (B.4) can be proved similarly. First of all, from the Hausdorff formula

\[
e^X e^Y = \exp \left[ \sum_m \sum_{p_i \geq 0, q_j \geq 0} \frac{(-1)^n}{m} (adX)^{p_1} (adY)^{q_1} \cdots (adX)^{p_m} (adY)^{q_m - 1} \right],
\]

we know that (B.3) holds with some coefficients \(\alpha_n, n = 0, 1, 2, \cdots\) if \(E \ll 1\). To determine the coefficients we expand the exponents of both sides. The expansion of the l.h.s. is simply

\[
l.h.s. = (1 + E) \sum_{n=0}^{\infty} \frac{1}{n!} X^n.
\]
But we expand the r.h.s. in $X$ and $\mathcal{E}$, retaining the terms of $O(\mathcal{E})$:

\[
\text{r.h.s.} = \sum_{n=0}^{\infty} \frac{1}{n!} (X + \sum_{n=0}^{\infty} \alpha_n (ad X)^n \mathcal{E})^n + O(\mathcal{E}^2)
\]

\[
= 1 + \left( X + \sum_{n=0}^{\infty} \alpha_n (ad X)^n \mathcal{E} \right)
\]

\[
+ \frac{1}{2!} \left( X^2 + X \sum_{n=0}^{\infty} \alpha_n (ad X)^n \mathcal{E} + \sum_{n=0}^{\infty} \alpha_n (ad X)^n \mathcal{E} \cdot X \right)
\]

\[
+ \frac{1}{3!} \left( X^3 + X^2 \sum_{n=0}^{\infty} \alpha_n (ad X)^n \mathcal{E} + X \sum_{n=0}^{\infty} \alpha_n (ad X)^n \mathcal{E} \cdot X + \sum_{n=0}^{\infty} \alpha_n (ad X)^n \mathcal{E} \cdot X^2 \right)
\]

\[+ O(\mathcal{E}^2)\]

Expanding this in $X$ furthermore we find

to the 0-th order in $X$: \[1 + \alpha_0 \mathcal{E},\]
to the first order in $X$:

\[(1 + \alpha_0 \mathcal{E})X + (\alpha_1 + \frac{\alpha_0}{2!})(ad X),\]
to the second order in $X$:

\[\frac{1}{2!} (1 + \alpha_0 \mathcal{E})X^2 + (\alpha_2 + \frac{\alpha_1}{2!} + \frac{\alpha_0}{3!})(ad X)^2 \mathcal{E} + (\alpha_1 + \frac{\alpha_0}{2!})(ad X) \mathcal{E} \cdot X,\]
to the third order in $X$:

\[\frac{1}{3!} (1 + \alpha_0 \mathcal{E})X^3 + (\alpha_3 + \frac{\alpha_2}{2!} + \frac{\alpha_1}{3!} + \frac{\alpha_0}{4!})(ad X)^3 \mathcal{E}
\]

\[+ (\alpha_2 + \frac{\alpha_1}{2!} + \frac{\alpha_0}{3!})(ad X)^2 \mathcal{E} \cdot X + \frac{1}{2!} (\alpha_1 + \frac{\alpha_0}{2!})(ad X) \mathcal{E} \cdot X^2.\]

If $\alpha_0 = 1$ and the recursion relation (B.5) holds up to $n = 3$, then both sides of (B.3) are equal up to the third order of $X$. The calculation may be inductively generalized to any order of $X$. Thus (B.3) was proved.

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