DIMENSIONAL RENORMALIZATION OF YUKAWA THEORIES VIA WILSONIAN METHODS

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ABSTRACT

In the ’t Hooft-Veltman dimensional regularization scheme it is necessary to introduce finite counterterms to satisfy chiral Ward identities. It is a non-trivial task to evaluate these counterterms even at two loops.

We suggest the use of Wilsonian exact renormalization group techniques to reduce the computation of these counterterms to simple master integrals.

We illustrate this method by a detailed study of a generic Yukawa model with massless fermions at two loops.

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Introduction

The dimensional regularization scheme devised by 't Hooft and Veltman [1, 2] and later systematized by Breitenlohner and Maison [3] (BMHV) is the only dimensional regularization scheme which is known to be consistent in presence of $\gamma_5$. It gives the correct result for the axial anomaly, at the price of breaking $d$-dimensional Lorentz symmetry and chiral symmetry; while the former is easily recovered, it is a non-trivial task to satisfy the chiral Ward identities.

More popular is the naive dimensional regularization scheme (NDR) [4] which, although inconsistent, is much easier to use and can be handled safely in most practical situations. For a review on the subject, see e.g. [5].

Due to the relevance of higher loop computations in the standard model, where it is difficult to guarantee the consistency of the naive scheme, it is worthwhile to investigate thoroughly consistent renormalization schemes.

The difficulties encountered in computing systematically in the BMHV scheme the non-invariant counterterms can be divided in three categories:

i) it is a complicated (but generally solvable) algebraic problem to satisfy all the Ward identities (modulo anomaly problems), determining these counterterms as combinations of Feynman integrals;

ii) it is an analytically difficult problem to evaluate these counterterms, which in general involve Feynman integrals with several masses and/or momenta; explicit formulae with several masses at more than one loop usually range from being quite complicated (see e.g. [6]) to being as yet unknown;

iii) it is burdensome to store these counterterms (which include evanescent ones) and to compute the renormalization group beta function using them [7].

By comparison, we recall that in a vectorial theory (or in NDR, whenever possible) step (i) is trivial in the minimal subtraction (MS) scheme, since the Ward identities are automatically satisfied; step (ii) is much simpler, since the poles are much easier to compute than the finite parts of the Green functions; step (iii) is almost trivial, since in general multiplicative renormalization holds; simple formulae for the beta and gamma renormalization group functions are available in the MS scheme [8].

The short-cuts used to get the appropriate answer for the problem at hand include:

I) in a few cases, like in the case of an axial current insertion in a vectorial gauge theory, it is sufficient to equate the axial vertex to the corresponding vector vertex multiplied by $\gamma_5$ to be sure that the chiral Ward identities are satisfied [9];

II) after solving the algebraic problem, avoid carrying out step (ii); this is the philosophy of algebraic renormalization (for a review see [10]), used generally only for demonstrative purposes, but which can also be used for making explicit computations, as shown in some two-loop examples carried out recently [11].

To our knowledge the only paper in which the counterterms are determined systematically at one loop in the BMHV scheme in a chiral gauge theory is [12], in which the Bonneau-Zimmermann [13] identities are used.

There are several computations of one-loop counterterms in the BMHV scheme for particular processes in the standard model [14], but no systematic one-loop treatment has been given in such a scheme.

In this paper we describe a general method for simplifying step (ii) and partly (iii), i.e. we show that the counterterms can be reduced to zero-momentum
Feynman integrals with the same auxiliary mass in all propagators, and that the beta function can be computed easily without making a direct use of the counterterms.

The auxiliary mass technique in this form has been used in the past at two or more loops [14, 15, 16] in conjunction with MS. Being the MS scheme mass-independent [17], the auxiliary mass technique gives gauge-breaking terms that are polynomial in the auxiliary mass and then can be easily treated; however in chiral theories with BMHV the MS scheme cannot be used, and it is not easy to disentangle the gauge-breaking terms.

We will show how to do this by Wilsonian methods [18].

Wilsonian methods have been used by Polchinski [19] to simplify the proof of renormalizability in $\phi^4$; this proof has been further simplified and generalized to other cases [20, 21, 22, 23] in particular gauge theories have been renormalized using the effective Ward identities introduced in [20]. In [23] mass-independent renormalization has been studied with these Wilsonian methods; it has been also shown in this paper that the Wilsonian effective action satisfies an effective renormalization group equation, which is the analogous of the effective Ward identities.

In this paper we propose the exploitation in the BMHV scheme of the effective Ward identities and effective renormalization group equation to compute the finite counterterms and the beta function in terms of Wilsonian Green functions at zero momenta and masses, which are easy to evaluate.

As a simple testing ground for our proposal we renormalize systematically at two loops the most general Yukawa theory with massless Dirac fermions; so far only the simplest Yukawa model with pseudoscalar coupling (without chiral symmetry) has been renormalized at two loops in the minimal BMHV subtraction scheme [7].

We impose Wilsonian mass-independent renormalization conditions compatible with the rigid chiral Ward identities. Our renormalization scheme is chosen to coincide with the minimal subtraction scheme in the non-chiral Yukawa case (this choice is done to simplify step (i), but is not essential in our method); in the general case it gives the same two-loop $\beta$ and $\gamma$ functions as in the minimal naive scheme [24].

Finally we discuss the coupling with external gauge fields; we impose Wilsonian renormalization conditions compatible with the effective Ward identities at one loop. At two loop we check only the gauge two-point function Ward identity and the Adler-Bardeen non-renormalization theorem [25].

Our method is based on the use of a Wilsonian flow belonging to the class characterized by the cut-off functions $K^{(n)}(x) = (\Lambda^2 / x + \Lambda^2)^n$ with $n = 2, 3, \ldots$; these cut-off functions separate the propagators into hard and soft parts. For $n \geq 2$ the counterterms chosen at a particular $\Lambda$ renormalize the flow for any $\Lambda$.

We renormalize the theory at zero momenta and masses along the flow $n = 2$, using the $n = 3$ case as a check on computations.

The counterterms are zero-momentum integrals with the same $\Lambda$ in all the propagators, which can be reduced to a single ‘master integral’ at two loops using recursion relations (for a review see [14]). We have used Mathematica [26] to evaluate these integrals.

In the first section we show how the exact renormalization group method can be used in connection with dimensional renormalization to renormalize the
In the second section, after reviewing the recursion formula for the massive zero-momentum two-loop integrals, we give the one and two loops results for the Yukawa models.

In the third section we discuss the effective Ward identities for the Yukawa model in presence of external gauge fields.

We end with a concluding section.

In the appendix we give the complete two-loop bare Yukawa action.

1 Dimensional renormalization and the Wilsonian effective action

1.1 ’t Hooft–Veltman regularization scheme

We recall how gamma matrices are treated in the ’t Hooft and Veltman [1] dimensional regularization scheme as elaborated by Breitenlohner and Maison [3] (BMHV).

We work in Euclidean space; $\delta_{\mu\nu}$ is the Kronecker delta in $d = 4 - \epsilon$ dimensions;

$$\delta_{\mu\nu} p_\nu = p_\mu ; \quad \delta_{\mu\mu} = d$$

(1)

The gamma matrices $\gamma_\mu$ satisfy the relation

$$\{ \gamma_\mu, \gamma_\nu \} = -2 \delta_{\mu\nu} I$$

(2)

where

$$tr I = 4 ; \quad I \gamma_\mu = \gamma_\mu I = \gamma_\mu$$

(3)

In the BMHV scheme $O(d)$ invariance is broken and one introduces $O(4) \times O(d - 4)$ invariant tensors: the $(d - 4)$-dimensional Kronecker delta $\hat{\delta}_{\mu\nu}$ and the 4-dimensional antisymmetric tensor $\epsilon_{\nu\rho\sigma\tau}$, satisfying

$$\hat{\delta}_{\mu\nu} \delta_{\nu\rho} = \hat{\delta}_{\mu\rho} ; \quad \hat{\delta}_{\mu\nu} = -\epsilon ; \quad \hat{\delta}_{\mu\nu} \epsilon_{\nu\rho\sigma\tau} = 0$$

(4)

Moreover one defines the ‘evanescent’ tensors

$$\hat{p}_\mu \equiv \hat{\delta}_{\mu\nu} p_\nu ; \quad \hat{\gamma}_\mu \equiv \hat{\delta}_{\mu\nu} \gamma_\nu$$

(5)

The matrix $\gamma_5$ is defined by

$$\gamma_5 \equiv \frac{1}{4!} \epsilon_{\mu\nu\rho\sigma} \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma$$

(6)

satisfying

$$\{ \gamma_5, \gamma_\mu \} = \{ \gamma_5, \hat{\gamma}_\mu \} = 2 \hat{\gamma}_\mu \gamma_5 ; \quad \hat{\gamma}_5^2 = I$$

(7)

In Euclidean space the reflection symmetry takes the place of hermiticity. Reflection symmetry is an antilinear involution, under which

$$\Theta \gamma_\mu = -\gamma_\mu ; \quad \Theta \psi(x) = \bar{\psi}(x^\prime) \gamma_1 ; \quad \Theta \bar{\psi}(x) = \gamma_1 \psi(x^\prime)$$

(8)
where \( x'^1 = -x^1, \) \( x'^\mu = x^\mu \) for \( \mu \neq 1 \).

For a multiplet of fermionic fields, a general local marginal fermionic bilinear, involving a scalar \((A)\), pseudoscalar \((B)\), vector \((V_\mu)\) and pseudovector \((A_\mu)\), all of them real fields, can be written as

\[
\begin{align*}
&\int \bar{\psi} (H_1 \gamma_\mu + H_2 [\gamma_\mu, \gamma_5] + H_3 \gamma_\mu + i H_4 \gamma_\mu \gamma_5) \partial_\mu \psi + \\
&\int \bar{\psi} (i H_5 A + H_6 \gamma_5 B + i H_7 \gamma_\mu V_\mu + i H_8 [\gamma_\mu, \gamma_5] A_\mu + \\
&\quad i H_9 \gamma_\mu V_\mu + H_{10} \gamma_\mu \gamma_5 A_\mu) \psi
\end{align*}
\]

where \( H_i \) are matrices over flavour indices. Reflection invariance requires that \( H_i \) are hermitian.

Green’s functions in \( d \) dimensions are obtained performing the Dirac algebra in the Feynman graphs with the above rules. Actually the analytic continuation to continuous dimensions is defined on the scalar coefficients of the Green’s functions expanded on a basis for tensorial structures. For chiral theories, being \( O(d) \) broken, such a basis includes also ‘hatted’ tensors.

The poles of the Green’s functions for \( \epsilon \to 0 \) are removed by local counterterms. Loop by loop the singular part of the counterterms must subtract exactly the pole part of the Green’s functions, including the ‘hatted’ components; the finite parts of the counterterms instead are not uniquely determined by the renormalization conditions which constrain only their 4–dimensional part. In fact the \( d \to 4 \) prescription requires, besides to take the limit \( \epsilon \to 0 \), to set the hatted tensors to zero.

In order to fix completely the fermionic counterterms, in the decomposition \( \psi \) we will choose that \( H_3, H_4, H_9, H_{10} \) have vanishing finite term in their Laurent expansion. The same convention is assumed for the coefficients \( b_{ij}, d_{ij}^a \) and \( d_{ij}^{ab} \) in the bilinear scalar counterterms, which have the form

\[
\frac{1}{2} \int \partial_\mu \phi_i \partial_\nu \phi_j (a_{ij} \delta_{\mu\nu} + b_{ij} \hat{\delta}_{\mu\nu}) + V_\mu^\alpha \phi_i \partial_\mu \phi_j c_{ij}^a + \phi_i \hat{\partial}_\mu \phi_j d_{ij}^a + V_\mu \partial_\mu V_\nu \phi_i \phi_j d_{ij}^{ab} (10)
\]

1.2 Yukawa models

Consider the most general Yukawa model with massless fermions.

In dimensional regularization the bare action is

\[
S = \int \frac{1}{2} \phi_i c_{ij} \phi_j + \bar{\psi} c_1 \psi \phi_i + \mu \epsilon/2 \bar{\psi} c_{1/2} \psi \phi_i + \frac{\mu_\epsilon}{4!} c_{ijkl} \phi_i \phi_j \phi_k \phi_l (11)
\]

and it is chosen to be reflection symmetric.

At tree level

\[
c_{ij}^{(0)}(p) = \delta_{ij} p^2 + m_{ij}^2 ; \quad c_i^{(0)}(p) = i \gamma_\mu p_\mu
\]

\[
c_{ij}^{(0)} = iy_i ; \quad c_{ijkl}^{(0)} = h_{ijkl}
\]

We define

\[
y_i = S_i I + iP_i \gamma_5 ; \quad Y_i = S_i + iP_i
\]
The constants $c, c_i$ are matrix-valued, $c = c_{iJ}$, etc., where $i,J$ are the internal indices of the fermions $\psi_i$. The matrices $S_i, P_i$ are hermitian.

We will consider a group $G$ which is not necessarily semi-simple, with structure constants $f^{abc}$. The fields transform under linear (chiral) representations of the group which are in general reducible:

$$\psi \to g \psi; \quad \bar{\psi} \to \bar{g} \psi^{-1}; \quad \phi_i \to h_{ij} \phi_j$$  \hspace{1cm} \text{(14)}

where

$$g = \exp(i \epsilon^a t^a) \quad ; \quad \bar{g} = \exp(i \epsilon^a \bar{t}^a) \quad ; \quad h = \exp(i \epsilon^a \theta^a)$$  \hspace{1cm} \text{(15)}

and

$$t^a = t^a_R P_R + t^a_L P_L = t^a_s + i t^a_5 \gamma_5 \quad ; \quad \bar{t}^a = t^a_L P_R + t^a_R P_L = t^a_s - i t^a_5 \gamma_5$$

$$P_R = \frac{1}{2} (1 + \gamma_5) \quad ; \quad P_L = \frac{1}{2} (1 - \gamma_5)$$  \hspace{1cm} \text{(16)}

$t^a_R$ and $t^a_L$ belong in general to different representations of $G$. The scalar fields are in a real representation; then $\theta^a$ are antisymmetric imaginary matrices.

The Yukawa coupling $\bar{\psi} y_i \psi_i \phi_i$ is invariant under these transformations provided the following relations hold

$$y_j \theta^a_{ji} + y_i t^a - \bar{t}^a y_i = 0 \quad ; \quad y_j \theta^a_{ji} + y_i \bar{t}^a - t^a y_i = 0$$  \hspace{1cm} \text{(17)}$$

or equivalently

$$Y_j \theta^a_{ji} + Y_i t^a_R - t^a_L Y_i = 0 \quad ; \quad Y_j \theta^a_{ji} + Y_i \bar{t}^a_R - \bar{t}^a_L Y_i = 0$$  \hspace{1cm} \text{(18)}$$

The tree-level action is invariant under these chiral transformations, apart from an evanescent fermionic kinetic term. Higher corrections in the bare action will require also non-invariant counterterms; in the next subsection we will discuss how Wilsonian methods can be applied to determine the counterterms in order to preserve the Ward identities.

### 1.3 Wilsonian effective action

The bare action $S(\Phi)$ has the form

$$S(\Phi) = \frac{1}{2} \Phi D^{-1} \Phi + S^I(\Phi)$$  \hspace{1cm} \text{(19)}$$

where we use a compact notation in which $\Phi$ is a collection of fields. $S^I$ contains the tree-level interaction terms and the counterterms.

Consider an analytic cut-off function

$$K_\Lambda(p, M) = \left( \frac{\Lambda^2}{p^2 + M^2 + \Lambda^2} \right)^n$$  \hspace{1cm} \text{(20)}$$

for $n \geq 2$; $M^2 = m^2 \geq 0$ in the scalar sector; $M^2 = 0$ in the fermionic sector.

Let us split the propagator in two parts characterized by a scale $\Lambda > 0$; defining

$$D_H = D_\Lambda = D(1 - K_\Lambda) \quad ; \quad D_S = D K_\Lambda$$  \hspace{1cm} \text{(21)}$$

5
Let us make an ‘incomplete integration’ over the hard modes

\[ Z_\Lambda[J] = \exp \frac{1}{\hbar} W_\Lambda[J] = \int \mathcal{D}\Phi e^{-\frac{1}{\hbar}(S_\Lambda(\Phi) - J\Phi)} \]  

with bare action

\[ S_\Lambda(\Phi) = \frac{1}{2} \Phi D_\Lambda^{-1} \Phi + S'(|\Phi|) \]  

The Green functions obtained from \( Z_\Lambda[J] \) are infrared finite for \( \Lambda > 0 \) even at exceptional momenta.

The flow of this functional from \( \Lambda \) to zero can be represented as

\[ Z[J] = Z_0[J] = e^{\frac{\Phi}{\hbar} \frac{D_\Lambda^{-1}}{\Lambda} K_\Lambda \frac{\Phi}{\hbar}} Z_\Lambda[J] \]  

The Wilsonian theory at scale \( \Lambda \) has a bare action differing from the one of the usual theory at \( \Lambda = 0 \) by the term

\[ S_\Lambda(\Phi) - S(\Phi) = \frac{1}{2} \Phi D_\Lambda^{-1} K_\Lambda \Phi \]  

which has ultraviolet dimension less or equal to zero, due to the above made choice of the cut-off function \( K_\Lambda \); the renormalization of the usual theory implies the renormalization of the Wilsonian theory and viceversa.

To impose the renormalization conditions it is useful to separate the Wilsonian 1-PI functional generator, obtained by making a Legendre transformation on \( W_\Lambda[J] \), into the quadratic tree-level and interacting parts:

\[ \Gamma_\Lambda[\Phi] = \frac{1}{2} \Phi D_\Lambda^{-1} \Phi + \Gamma'_\Lambda[\Phi] \]  

and to make a Taylor expansion of \( \Gamma'_\Lambda[\Phi] \) in fields, \( d \)-dimensional momenta and masses; the four-dimensional terms in this expansion form a local functional, which will be called \( S_W \).

Let us now discuss briefly why the renormalization program can be performed on the \( Z_\Lambda \) functional with some advantages:

i) since the hard propagator \( D_H(p) \) is regular at \( p = 0 \) there are no infrared problems in the evaluation of the counterterms, so that the counterterms can be chosen to be Feynman integrals at zero momenta and masses using a mass-independent Wilsonian renormalization scheme [23];

ii) for a Wilsonian flow with \( n \geq 2 \) it is simple to prove that the renormalization at a fixed \( \Lambda \) implies the finiteness of the Wilsonian effective action at every value \( \Lambda \) of the Wilsonian flow, in particular at \( \Lambda = 0 \); moreover because the propagators \( D_\Lambda \) at \( \Lambda = 0 \) do not depend on \( n \), a bare action that renormalizes a given flow will also make finite the flows for every \( n \geq 2 \).

iii) for a Wilsonian flow with \( n \geq 2 \) the Ward identities and the renormalization group equation on the functional \( Z_{\Lambda=0} \) are equivalent to the corresponding effective Ward identities [20, 22] and effective renormalization group equation [23] on \( Z_\Lambda \), so that choosing renormalization conditions compatible with the effective Ward identities the validity of the Ward identities on \( Z_{\Lambda=0} \) follows; furthermore the renormalization group beta function can be expressed in terms of Feynman integrals at zero momenta and masses.
These considerations hold for any renormalizable theory in four dimensions; we consider here only the Yukawa model, which is the simplest chiral theory. Actually its rigid Ward identities are too simple to illustrate point (iii), since $Z_{\Lambda}$ satisfies the same rigid Ward identities; however introducing external currents one has to study the local Ward identities, whose effective form will be used in a later section.

In the Yukawa model the bare constants in the action are chosen of the form

$$c_{A} = \sum_{l \geq 1} h^l N_{d}^{(l)} A_{l}(\epsilon)$$

$$N_{d} = (4\pi)^{\epsilon/2 - 2} \Gamma(1 + \frac{\epsilon}{2}) ; \quad c_{A}^{(l)}(\epsilon) = \sum_{r \geq 0} c_{A}^{(l), r} \epsilon^{-r} \quad (27)$$

We use a modified subtraction scheme [27], in which $N_{d}$ is introduced in order to avoid $\gamma_{E}$, $\ln(4\pi)$ and $\pi^{2}$ factors. We choose a mass-independent renormalization scheme, i.e. all the bare terms, apart from $c_{ij}$, are independent from $m^{2}$; the latter term depends polynomially on it.

The bare constants must be fixed by suitable renormalization conditions on the Wilsonian effective action.

In the Yukawa model the marginal part $S_{W}^{A}$ of the Wilsonian effective action is defined according to:

$$\Gamma^{A} = S_{W}^{A} + \Gamma_{irr}^{A}$$

$$S_{W}^{A} = \int \bar{\psi} \gamma_{\mu} a \partial_{\mu} \psi - \frac{1}{2} a_{ij} \phi_{i} \partial^{2} \phi_{j} + \frac{1}{2} \phi_{i} m_{rs} a_{ij}^{rs} \phi_{j} + \bar{\psi} a_{ij} \psi \phi_{i} + \frac{1}{4!} a_{ijkl} \phi_{i} \phi_{j} \phi_{k} \phi_{l} \quad (28)$$

$\Gamma_{irr}$ contains all the terms of dimension greater than four in the expansion of $\Gamma^{A}$ in the fields, momenta and masses.

The renormalization conditions fix the limit for $\epsilon \to 0$ of the functions $g_{A} = \{a, a_{ij}, a_{ij}^{rs}, a_{ij}, a_{ijkl}\}$ at $\Lambda = \mu$ and perturbatively determine uniquely the bare action, as discussed in the first section.

From the Feynman graphs rules the terms $g_{A}$ at order $l$ in the loop expansion are

$$g_{A}^{(l)} = \sum_{r=0}^{l} \left( \frac{\mu}{\Lambda} \right)^{r} g_{A}^{(r,l-r)} \quad (29)$$

in which the dependence on $\Lambda$ and $\mu$ is made explicit; $g_{A}^{(l,0)}$ is the $l$-loop graph contribution, while $g_{A}^{(r,l-r)}$, $r = 0, ..., l - 1$ is the contribution due to the $r$-loop graphs with counterterms of overall loop order $l - r$; $g_{A}^{(r,l-r)}$ are independent from $\mu$ and $\Lambda$.

Being the theory renormalized, one gets, for $\epsilon \to 0$

$$\mu \frac{\partial}{\partial \mu} g_{A}^{(l)} = -\Lambda \frac{\partial}{\partial \Lambda} g_{A}^{(l)} \to \sum_{r=1}^{l} r g_{A}^{(r,l-r)} \quad (30)$$

---

2In a Wilsonian mass-independent scheme, as discussed in [23], the renormalization of the mass parameter is treated in a way very similar to the kinetic term and therefore is included in the marginal part of the action.
Taking \( s \) derivatives with respect to \( \Lambda \) in (29) one has still a finite expression for \( \epsilon \to 0 \), so that one obtains the consistency conditions

\[
\sum_{r=1}^{l} r^s g_{\Lambda, -n}^{(r, l-r)} = 0
\]

(31)

for \( n = 2, \ldots, l \) and \( s = 1, \ldots, n-1 \).

For the two-loop case the consistency condition (31) for \( n = 2 \) and \( s = 1 \) is known to hold for each Feynman graph and its counterdiagrams [28], providing a useful check.

To renormalize the theory we assign finite values to the constants \( g_A \) at \( \Lambda = \mu \)

\[
g_A = r_A + O(\epsilon)
\]

(32)

Each coefficient in the Laurent expansion in \( \epsilon \) must be determined to establish completely the renormalization scheme. In a subtraction scheme in which the bare constants are fixed to have all positive powers of \( \epsilon \) equal to zero (see eq. (27)) and the renormalized quantities are fixed to be non singular and with determined \( \epsilon^0 \) coefficient, the renormalization scheme is completely fixed. The \( O(\epsilon) \) term in eq. (32) refers to the fact that one cannot fix the positive powers of \( \epsilon \) both in the renormalization condition and in the bare action.

In order to respect the rigid Ward identities one must choose for \( r_A \) group-covariant quantities under the chiral transformations. They can be constructed out of the tree coupling constants \( y \) and \( h_{ijkl} \), taking into account the invariance properties

\[
\bar{g}^{-1} y g = h_{ij} y_j \quad ; \quad g^{-1} y_i^\dagger \bar{g} = h_{ij} y_j^\dagger
\]

(33)

Furthermore one can examine which choice for \( r_A \) gives the simplest expression for the counterterms.

For the non-chiral theory, in which the couplings \( P_i \) of eq. (13) are vanishing and \( g = \bar{g} = g_s \), the simplest counterterms are those determined by the MS scheme, corresponding to renormalization conditions \( r_A(S) \) that can be explicitly computed.

For the chiral theory we will define the counterterms as suitable functions \( c_A(S, P) \) which, for \( P = 0 \), coincide with the corresponding functions of the non-chiral case. To this aim loop by loop we will choose \( r_A \) applying a covariantization formula to the corresponding \( r_A(S) \). A comparison of eq. (33) with the analogous equation in the non-chiral case with the same group \( G \):

\[
g_s^{-1} S \rightarrow h_{ij} S_j
\]

(34)

suggests the recipe for the Green’s functions of interest :

\[
S_{i_1} S_{i_2} \ldots S_{i_{2n+1}} I \rightarrow y_{i_1} y_{i_2}^\dagger \ldots y_{i_{2n+1}}
\]
\[
\gamma_\mu S_{i_1} S_{i_2} \ldots S_{i_{2n}} \rightarrow \gamma_\mu y_{i_1} y_{i_2}^\dagger \ldots y_{i_{2n}}
\]
\[
Tr \left[ S_{i_1} S_{i_2} \ldots S_{i_{2n}} \right] \rightarrow \frac{1}{4} Tr tr \left[ y_{i_1} y_{i_2}^\dagger \ldots y_{i_{2n}} \right] =
\]
\[
\frac{1}{2} Tr \left( Y_{i_1} Y_{i_2}^\dagger Y_{i_2n-1} Y_{i_{2n}} + Y_{i_1} Y_{i_2}^\dagger Y_{i_{2n-1}} Y_{i_{2n}} \right)
\]

(35)
where \( Tr \) denotes the trace over the internal fermionic indices.

As an example let us consider the renormalization condition to be imposed on the fermionic self-energy. In the non-chiral theory, using the minimal subtraction one computes:

\[
\Sigma(p)|_{\text{marg}} = i \not{p} r(S)
\]  

Using the covariantization procedure \( i \not{p} r(S) \) must be replaced by \( i \not{p} r(y, y^\dagger) \). In order to be compared with the standard form of eq.(36) \( \Sigma \) can be written as

\[
\Sigma(p)|_{\text{marg}} = i \left\{ \not{p} \left[ r(Y, Y^\dagger) + r(Y^\dagger, Y) \right] + \frac{i}{2} \not{[p, \gamma_5]} \left[ r(Y, Y^\dagger) - r(Y^\dagger, Y) \right] \right\}
\]

which differs from \( i \not{p} r(y, y^\dagger) \) by evanescent terms. Eq.(37) gives the renormalization condition for the chiral theory. Observe that, if the theory is reflection symmetric, \( r(y, y^\dagger) \) is hermitian.

The counterterms \( c_A \) are completely determined by the pole part and by the constant part of the corresponding vertices. They can be decomposed into two parts

\[
c_A = c_A^{NDR} + \Delta c_A
\]

where \( c_A^{NDR} \) has the same group structure as \( r_A \) and, due to the choice of renormalization conditions, has only pole part in \( \epsilon \) (see comment after eq.(32)).

The remaining part \( \Delta c_A \) vanishes in the non-chiral case \( (P_i = 0) \).

The non-marginal relevant terms satisfy

\[
\Gamma^{A=0}_{i,j}|_{p=m=0} = 0 \quad ; \quad \Gamma^{A=0}_{IJ}|_{p=m=0} = 0
\]

In dimensional regularization, being the massless tadpoles equal to zero, eq.(39) corresponds to have vanishing bare relevant counterterms.

2 Explicit computations

2.1 Master integrals

Using the cut-off function (20) the hard propagator has the form

\[
D_A(p, M) = D(p, M) \frac{p^2 + M^2}{\Lambda^2} \sum_{r=1}^{n} \left( \frac{\Lambda^2}{p^2 + M^2 + \Lambda^2} \right)^r
\]

The counterterms are computed in terms of Wilsonian Green functions at zero momenta and masses \( M = 0 \); the corresponding Wilsonian Feynman graphs with \( I \) internal lines have \( I \) sums \( \sum_{r_1=1}^{n} \ldots \sum_{r_I=1}^{n} \).

Since \( D(p, M)(p^2 + M^2) \) is polynomial in \( p \) the counterterms are expressed in terms of massive zero momentum tensor integrals, in which all the propagators have ‘mass’ \( \Lambda \). Taking traces one can reduce these tensor integrals in terms of scalar integrals.
At one loop the scalar integrals are
\[ I_a = \int \frac{d^d q_1}{N_d \pi^{d/2}} \frac{1}{(q_1^2 + 1)^a} \] (41)

At two loops the scalar integrals have the form
\[ I_{a,b,c} = \int \frac{d^d q_1}{N_d \pi^{d/2}} \int \frac{d^d q_2}{N_d \pi^{d/2}} \frac{1}{(q_1^2 + 1)^a(q_2^2 + 1)^b((q_1 + q_2)^2 + 1)^c} \] (42)
and so on at higher loops.

At one loop one has the well-known explicit expression
\[ I_a = \frac{\Gamma(a - d/2)}{N_d \Gamma(a)} \] (43)

At higher loops an exact expression for these integrals is not known; they can be reduced, using recursion relations, to a small number of master integrals which can be expanded in \( \epsilon \); to renormalize at \( l \)-loop one must know the \( I_{l-1} \) up to the \( O(1) \) term, the \( I_l \) up to the \( O(1/\epsilon) \) term and so on (in minimal subtraction it is sufficient to know the \( I_{l-1} \) up to the \( O(1/\epsilon) \) term, the \( I_l \) up to the \( O(1) \) term and so on; however in presence of chiral symmetries MS is not sufficient).

At two loops there is only one master integral (for a review see e.g. [15]); it is known how to compute the finite parts of the three-loop master integral [16].

In this paper we will restrict to two-loop computations; the recursion relation is obtained from the identity
\[ \int \frac{d^d q_1}{N_d \pi^{d/2}} \int \frac{d^d q_2}{N_d \pi^{d/2}} \frac{\partial}{\partial q_1^\mu} q_1^\mu (q_1^2 + 1)^a(q_2^2 + 1)^b((q_1 + q_2)^2 + 1)^c = 0 \] (44)
which implies
\[ (d - 2b - c)I_{a+1,b,c} - c(I_{a,b-1,c+1} - I_{a-1,b,c+1}) + \]
\[ 2bI_{a+1,b,c} + cI_{a,b,c+1} = 0 \] (45)

From this relation and the fact that \( I_{a,b,c} \) is totally symmetric in its indices it follows the recursion relation
\[ 3a \ I_{a+1,b,c} = c \ (I_{a-1,b,c+1} - I_{a,b-1,c+1}) + \]
\[ b \ (I_{a-1,b+1,c} - I_{a,b+1,c-1}) + (3a - d)I_{a,b,c} \] (46)

Using this recursion relation one can express all \( I_{a,b,c} \), for \( a, b, c > 0 \) in terms of the master integral \( I_{1,1,1} \) and in terms of \( I_{a',b',c'} \), with one of the indices vanishing; in the latter case the two-loop integral reduces to the product of two one-loop integrals:
\[ I_{a,b,0} = I_a I_b \] (47)

One has
\[ I_{1,1,1} = -\frac{6}{c^2} - \frac{9}{c} + \frac{3}{2} (v - 5) + O(\epsilon) \]
\[ v = 3 \lim_{\epsilon \to 0} I_{2,2,2} = -2 - \frac{4}{\sqrt{3}} \int_0^{\pi/3} dx \ln \left(2 \sin \frac{x}{2}\right) \simeq 0.34391 \] (48)

These recursion relations can be solved very fast by computer in the cases of interest. Recently all the Laurent series of \( I_{1,1,1} \) has been computed in [29].
2.2 One-loop results

At one loop the renormalization conditions \([32]\) on the Wilsonian vertices are

\[
\begin{align*}
    r_{ij}^{(1)} &= r_{ij}^{(1)\phi} Y_{ij} \quad ; \quad r_{ij}^{(1)rs} = i r_{ij}^{m^2} h_{ijrs} \\
    r_{i}^{(1)} &= r_{i}^{(1)\psi} y_{i} \quad ; \quad r_{i}^{(1)} = i r_{i}^{v_3} h_{ij} y_{j} \\
    r_{ijkl}^{(1)} &= r_{ijkl}^{(1)Y} + r_{ijkl}^{(1)h} h_{ijkl}^{rs}
\end{align*}
\]

where the symmetrizations \((\ldots)\) are with weight one; e.g. \(h_{ijkl}^{rs} = h_{ijkl}\); we defined

\[
Y_{1i_2...i_{2n-1}i_{2n}} = \frac{1}{2} Tr \left( Y_{i_1} Y_{i_2}^\dagger ... Y_{i_{2n-1}} Y_{i_{2n}}^\dagger + Y_{i_1}^\dagger Y_{i_2} ... Y_{i_{2n-1}}^\dagger Y_{i_{2n}} \right)
\]

(50)

The quantities \(r_{A}\) depend in general on the choice of the Wilsonian flow; in Table 1 we give the values of \(r_{A}\) and \(c_{A}^{NDR}\) for the flow \(n = 2\), figure 1 gives the corresponding graphs. To compute the coefficients in Table 1 one has to make an expansion in masses and momenta of graphs in fig.1 as mentioned after eq.(28).

The remaining part of the one-loop bare constants is

\[
\begin{align*}
    \Delta c_{ij}^{(1)} &= \frac{4}{3} \left( -p^2 + \frac{2}{e} p^2 \right) Tr \left[ P_i P_j \right] \\
    \Delta c_{i}^{(1)} &= \frac{i}{e} \gamma \left( 2 P_i P_i - i \gamma S_i \right) \\
    \Delta c_{i}^{(1)} &= y_j \gamma^5 P_j y_j \\
    \Delta c_{ijkl}^{(1)} &= -96 Tr \left[ S_i S_j P_k P_l + \frac{2}{3} P_i P_j P_k P_l \right]
\end{align*}
\]

(51)

| structure | \(c_{A}^{NDR}\) | \(r_{A}^{#}\) | \(\mu\partial_{\mu}g\) | graphs |
|-----------|----------------|-------------|----------------|--------|
| \(Y_{ij}\) | \(-\frac{4}{e}\) | \(\frac{26}{19}\) | 4 | Fig.1(a) |
| \(h_{ijkl}\) | \(\frac{1}{e}\) | \(-\frac{7}{15}\) | -1 | Fig.1(b) |
| \(y_{i} y_{i}\) | \(-\frac{1}{e}\) | \(\frac{7}{12}\) | 1 | Fig.1(c) |
| \(i y_{j} y_{i} y_{j}\) | \(\frac{2}{e}\) | \(-\frac{42}{60}\) | -2 | Fig.1(d) |
| \(Y_{(ijkl)}\) | \(-\frac{48}{e}\) | \(\frac{454}{35}\) | 48 | Fig.1(e) |
| \(h_{ijkl}^{rs} h_{kl}^{rs}\) | \(\frac{2}{e}\) | \(-\frac{7}{4}\) | -3 | Fig.1(f) |

Table 1: One-loop coefficients
In [7] MS is applied in BMHV in the simplest Yukawa model with a pseudoscalar, where there is no chiral symmetry to be maintained. Here we introduced finite counterterms, which are exactly those needed to obtain the same renormalized Green functions as in the MS NDR scheme. In [7] it was shown that the beta function at two loop in the MS BMHV scheme differs from the one in MS NDR scheme by a renormalization group transformation involving finite one-loop counterterms, which are in agreement with eq. (51).

2.3 Two-loop results

At two loops the renormalization conditions on the Wilsonian effective action read

\[
\begin{align*}
\gamma_{ij} &= \gamma^p_{ij} p^2 + \gamma^m_{ij} m^2 \\
\gamma^p_{ij} &= \gamma^h_{ikmn} h_{jkmn} + \gamma^i_{ij} + \gamma^y_{ijkl} y_{ij}
\end{align*}
\]

The two-point fermionic term is

\[
r = y_j [\gamma^p_{ij} y_i y_j + \gamma^m_{ij} m^2]
\]

The two-loop fermion-fermion-scalar term is

\[
r_i = \gamma^h_{ij} y_i y_j
\]

The two-loop quartic scalar term is

\[
r_{ijkl} = \gamma^h_{ijkl} + \gamma^h_{ij} y_i y_j
\]

where symmetrization over the indices \(i, j, k, l\) is understood.
Table 2: Two-loop coefficients for the two-point bosonic functions

| structure | $c^{NDR}$ | $r^\phi$, $r^{\mu^2}$ | $\mu \partial \mu g$ | $\tau$ | graphs |
|-----------|-----------|------------------------|--------------------|--------|--------|
| $h_{ikmn}h_{jkmn}p^2$ | $-\frac{1}{12\epsilon}$ | $\frac{7}{143} + \frac{5v}{81}$ | $\frac{1}{6}$ | 0 | Fig. 2(a) |
| $Y_{ikjk}p^2$ | $-\frac{8}{\epsilon^2} + \frac{2}{\epsilon}$ | $-\frac{25}{9} + \frac{178v}{81}$ | $-\frac{164}{15}$ | 0 | Fig. 2(b) |
| $Y_{ijkk}p^2$ | $-\frac{4}{\epsilon^2} + \frac{3}{\epsilon}$ | $-\frac{16621}{8505} + \frac{49v}{729}$ | $-\frac{279}{35}$ | $-\frac{157}{105}$ | Fig. 2(c) |
| $h_{ikmn}h_{jlmn}m_{kl}^2$ | $\frac{1}{\epsilon^2} - \frac{1}{2\epsilon}$ | $\frac{67}{108} + \frac{29v}{324}$ | | | Fig. 2(a) |
| $h_{ijmn}h_{mkln}m_{kl}^2$ | $\frac{1}{\epsilon^2}$ | $\frac{49}{144}$ | | | Fig. 2(d) |
| $Y_{ikjl}m_{kl}^2$ | $-\frac{8}{\epsilon^2} + \frac{4}{\epsilon}$ | $-\frac{1642}{729} + \frac{5078v}{2187}$ | | | Fig. 2(b) |
| $Y_{ijkl}m_{kl}^2$ | $-\frac{16}{\epsilon^2}$ | $-\frac{146}{81} + \frac{4v}{243}$ | | | Fig. 2(c) |
| $h_{ijkm}Y_{lm}m_{kl}^2$ | $\frac{4}{\epsilon^2} - \frac{2}{\epsilon}$ | $\frac{4421}{545} + \frac{121v}{27}$ | | | Fig. 2(c) |

Figure 2: Two-loop graphs for the two-point bosonic function

Table 3: Two-loop coefficients for the fermionic two-point functions

| structure | $c^{NDR}$ | $r^\psi$ | $\mu \partial \mu g$ | $\tau$ | graphs |
|-----------|-----------|-----------|--------------------|--------|--------|
| $y_j^\dagger y_j^\dagger y_j$ | $-\frac{1}{2\epsilon^2} + \frac{1}{8\epsilon}$ | $-\frac{2359}{9720} + \frac{289v}{5832}$ | $-\frac{29}{30}$ | $\frac{13}{120}$ | Fig. 3(a) |
| $y_j^\dagger y_j^\dagger y_j$ | $-\frac{2}{\epsilon^2}$ | $-\frac{229}{324} + \frac{419v}{486}$ | $-\frac{7}{9}$ | 0 | Fig. 3(b) |
| $y_j^\dagger y_i Y_{ij}$ | $-\frac{2}{\epsilon^2} + \frac{3}{2\epsilon}$ | $-\frac{2375}{486} + \frac{6197v}{1458}$ | $\frac{127}{30}$ | $\frac{11}{10}$ | Fig. 3(c) |
| $h_{ijkk}y_j^\dagger y_i$ | $0$ | $\frac{13}{30}$ | 0 | 0 | Fig. 3(d) |
Figure 3: Two-loop graphs for the two-point fermionic function

\[
\begin{align*}
Y_{jk} y_k y_j^\dagger y_j & : \frac{4}{\eps^2} - \frac{2}{\eps} + \frac{19384}{2835} + \frac{1976}{243} + \frac{647}{105} - \frac{34}{35} \\
y_k y_j^\dagger y_i y_k^\dagger y_k & : \frac{2}{\eps^2} - \frac{1}{\eps} + \frac{47}{81} + \frac{13v}{486} + \frac{107}{30} + 0 \\
y_k (y_i^\dagger y_j y_j^\dagger y_k + y_i^\dagger y_j y_j^\dagger y_k^\dagger) y_k & : \frac{1}{\eps^2} - \frac{1}{2\eps} + \frac{10009}{22050} + \frac{23v}{972} + \frac{647}{420} - \frac{17}{70} \\
y_k (y_i^\dagger y_j^\dagger y_k + y_i^\dagger y_k y_k^\dagger) y_j & : \frac{2}{\eps^2} + \frac{415}{972} + \frac{169v}{1458} + \frac{47}{30} + 0 \\
y_k y_j y_i y_k^\dagger y_j & : \frac{1}{\eps} + \frac{76}{279} + \frac{920v}{2187} - 2 + 0 \\
h_{ijk} y_k y_i y_j & : 0 - \frac{33}{70} + 0 + 0 \\
h_{ijk} y_k y_j^\dagger y_i & : -\frac{1}{\eps} + \frac{7}{81} + \frac{160v}{243} + 2 + 0
\end{align*}
\]

Fig. 4: Two-loop graphs for the cubic vertex

Table 4: Two-loop coefficients for the cubic vertex

Figure 4: Two-loop graphs for the cubic vertex
| structure       | $c^{NDR}$               | $\rho^{v4}$          | $\mu \partial_{\mu} g$ | $\tau$ | graphs  |
|-----------------|-------------------------|-----------------------|-------------------------|--------|---------|
| $Y_{niijkl}$    | $\frac{192}{\epsilon^2} + \frac{96}{\epsilon}$ | $-\frac{3782672}{6561}$ + $\frac{56416u}{5961}$ | $-\frac{10352}{35}$ | 0      | Fig. 5(a) |
| $Y_{niijnkl}$   | $\frac{48}{\epsilon}$  | $-\frac{15088}{18225}$ + $\frac{24272u}{2187}$ | $-96$ | 0      | Fig. 5(b) |
| $Y_{niijln}$    | $\frac{96}{\epsilon^2} + \frac{48}{\epsilon}$ | $-\frac{5690407}{8164}$ + $\frac{155224u}{19683}$ | $-\frac{13756}{405}$ | $\frac{1772}{105}$ | Fig. 5(c) |
| $h_{mnij}Y_{mkin}$ | $\frac{96}{\epsilon^2}$ | $-\frac{292}{27} + \frac{8u}{81}$ | $-112$ | 0      | Fig. 5(d) |
| $h_{mnij}Y_{mknl}$ | $\frac{48}{\epsilon^2} + \frac{24}{\epsilon}$ | $-\frac{4766}{224} + \frac{10156u}{729}$ | $-104$ | 0      | Fig. 5(e) |
| $h_{mnij}Y_{mp}h_{pijkl}$ | $\frac{12}{\epsilon^2} - \frac{6}{\epsilon}$ | $\frac{949}{288} + \frac{121u}{2}$ | $\frac{107}{9}$ | $-\frac{23}{3}$ | Fig. 5(f) |
| $h_{mnij}h_{mpq}h_{pqkl}$ | $\frac{3}{\epsilon^2}$ | $\frac{61}{45}$ | $\frac{7}{2}$ | 0      | Fig. 5(g) |
| $h_{mnij}h_{mpkl}h_{npqq}$ | 0 | $-\frac{39}{20}$ | 0 | 0      | Fig. 5(h) |
| $h_{mnij}h_{npq}h_{mpql}$ | $\frac{6}{\epsilon^2} - \frac{3}{\epsilon}$ | $\frac{38}{9} + \frac{29u}{54}$ | 13 | 0      | Fig. 5(i) |

Table 5: Two-loop coefficients for the quartic vertex

![Figure 5: Two-loop graphs for the quartic vertex](image-url)
The coefficients in eqs. (52), (53), (54), (55) and the corresponding ones for the two-loop $c^{NDR}_N$ in the case of the Wilsonian flow $n = 2$ are given respectively in the Tables 2, 3, 4, 5. Figures 2, 3, 4, 5 show the graphs contributing to these coefficients; for each diagram shown there are in general corresponding graphs with one-loop counterterms that are not represented.

The naive part $c^{NDR}_A$ of the two-loop bare constants agree with (24). In order to make the comparison with (24), where a two-component spinor formalism is used, one must substitute the traces $k \text{Tr} \left[ Y_{i_1} Y_{i_2}^\dagger ... Y_{i_{2n-1}} Y_{i_{2n}}^\dagger \right]$ in (24) with $Y_{i_1 i_2 ... i_{2n-1} i_{2n}}$.

The remaining part of the bare counterterms is given in the Appendix.

With the bare action determined using the flow $n = 2$, according to the lines of sect. (1.3) one can define a new Wilsonian action $\Gamma^{(n=3)}$. For $n = 3$ (or bigger) the new effective action is finite and satisfies the Ward identities. The graphs of $\Gamma^{(n=3)}$ now have $n = 3$ propagators $D_{\Lambda}$ but contain the $n = 2$ counterterms. This suggests the following consistency check: consider the contributions of $n = 3$ bare (marginal) graph, of its relative countergraph and two loop counterterm; we have verified that although the two first quantities are divergent and separately not invariant, the total sum is finite and invariant. Furthermore this total sum, being computed on a different flow is different from the $n = 2$ case.

### 2.4 Renormalization group equation

Let us compute the renormalization group beta and gamma functions in the BMHV scheme at two loops. In the case of the simplest Yukawa model this has been done in [7] using the minimal subtraction formulas [2] and taking into account the one-loop evanescent tensors in the bare action; after obtaining a renormalization group equation involving also insertions of evanescent tensors, these are solved in terms of relevant couplings obtaining the usual renormalization group equation.

To avoid making a similar subtle analysis of the bare couplings in our case, we will obtain the beta and gamma functions working directly with the Wilsonian effective action, which satisfies the effective renormalization group equation (23). We will restrict to the massless case; in (23) it is considered also the massive case.

In the compact notation of subsection 1.3 the Gell-Mann and Low renormalization group equation reads

$$
(\mu \frac{\partial}{\partial \mu} + \beta \cdot \frac{\partial}{\partial g} + J\gamma^T \delta \frac{\delta}{\delta J})Z[J] = \mathcal{E}[J] 
$$

where $Z$ is the renormalized functional in which the limit $\epsilon \to 0$ has not yet been taken; $\mathcal{E}$ is an evanescent functional, which in general is not vanishing for finite value of $\epsilon$ since the renormalization of the theory is not strictly multiplicative, and

$$
\beta \cdot \frac{\partial}{\partial g} = \beta_{ijkl} \frac{\partial}{\partial h_{ijkl}} + Tr(\beta_{Y_i} \frac{\partial}{\partial Y_i} + \beta_{Y_i}^\dagger \frac{\partial}{\partial Y_i^\dagger}) 
$$

From eq. (24) it follows that $Z_{\Lambda}[J]$ satisfies the ‘effective renormalization
group equation’

\[
(\mu \frac{\partial}{\partial \mu} + \beta \cdot \frac{\partial}{\partial g} + J \gamma^T_\phi \delta - \hbar \delta J_\phi D^{-1}_\Lambda K_\Lambda \delta J) Z_\Lambda[J] = \mathcal{E}_\Lambda[J]
\]  

(58)

Define the functional \( Z_\Lambda[J, \chi] \) as in (22), but with

\[
S_\Lambda(\Phi, \chi) = S_\Lambda(\Phi) - \chi \Delta_\gamma(\Phi) ; \quad \Delta_\gamma(\Phi) = \Phi \gamma^T_\phi D^{-1}_\Lambda K_\Lambda \Phi
\]

(59)

The effective renormalization group equation can be written as

\[
(\mu \frac{\partial}{\partial \mu} + \beta \cdot \frac{\partial}{\partial g}) |_{\chi=0} Z_\Lambda[J, \chi] = \mathcal{E}_\Lambda[J]
\]

(60)

which in terms of the 1-PI functional generator reads

\[
(\mu \frac{\partial}{\partial \mu} + \beta \cdot \frac{\partial}{\partial g} - \Phi \gamma^T_\phi \delta \delta \Phi |_{\chi=0}) \Gamma_\Lambda[\Phi, \chi] = \mathcal{E}_\Lambda[\Phi]
\]

(61)

or equivalently

\[
(\mu \frac{\partial}{\partial \mu} + \beta \cdot \frac{\partial}{\partial g} - \Phi \gamma^T_\phi \delta \delta \Phi) \Gamma_\Lambda[\Phi] = \mathcal{E}_\Lambda[\Phi]
\]

(62)

where \( \mathcal{T}_\Lambda[\Phi] \) represents the insertion of the non local operator \( \Delta_\gamma(\Phi) \) on the functional \( \Gamma_\Lambda[\Phi] \); its Feynman graphs contain the new vertices

\[
\begin{align*}
\text{between two bosonic or fermionic lines; due to its nonlocality } \Delta_\gamma(\Phi) \text{ does not require renormalization.}
\end{align*}
\]

Let us finally rewrite in a suitable way the gradient terms in (62); using (30) one gets

\[
\mu \frac{\partial g_A^{(1)}}{\partial \mu} = g_A^{(1,0)} = -e_A^{(1)} ; \quad \mu \frac{\partial g_A^{(2)}}{\partial \mu} = 2 g_{A,1}^{(2,0)} + g_{A,1}^{(1,1)}
\]

(64)

These gradients have the same group structure as the renormalization constants given in eqs.(59), (52); the coefficients of these group structures are given in the tables 3, 4, 5, 6, 7 in the case of the \( n = 2 \) flow. Observe that, while \( g_{A,1}^{(2,0)} \) and \( g_{A,1}^{(1,1)} \) are not separately chiral group-covariant, the gradient term \( 2 g_{A,1}^{(2,0)} + g_{A,1}^{(1,1)} \) must be covariant. This provides a check on the computations, besides the double pole rule mentioned after (31), that we made in a systematic way.

At one loop the Wilsonian gradients can be expressed in terms of the pole of the bare couplings, so that the formulae for the beta and gamma functions are equivalent to the standard formulae \[2\]; using \( \beta_A = \sum \frac{\lambda^l}{(4\pi)^l} \beta_A^{(l)} \) and defining \( \beta_i = \beta_Y \cdot P_R + \beta_{\gamma} \cdot P_L \) we get

\[
\begin{align*}
\gamma_{ij}^{(1)} &= 2 Y_{ij} ; \quad \bar{\gamma}^{(1)} = \frac{1}{2} y_i y_j^\dagger ; \quad \bar{\gamma}^{(1)} = \frac{1}{2} y_i \dagger y_i \\
\beta_i^{(1)} &= 2 y_i y_i^\dagger y_j + \frac{1}{2}(y_i y_j^\dagger y_j + y_j y_i^\dagger y_i) + 2 Y_{ij} y_j \\
\beta_{ij}^{(1)} &= -48 Y_{(ijkl)} + 3 h_{ij}^r h_{ijkl} + 8 Y_{m(i} h_{jkl)}
\end{align*}
\]

(65)
At two loops the Wilsonian gradients are not given by the simple pole of the bare counterterms (64); the beta and gamma functions are given in terms of these Wilsonian gradients, the $T$ insertions, given in the tables 2, 3, 4, 5, and the one-loop renormalization conditions (49).

We get

\begin{align*}
\gamma^{(2)}_{ij} &= \frac{1}{12} h_{ikmn} h_{jkmn} - 2 Y_{ikjk} - 3 Y_{ijjk} \\
\gamma^{\psi(2)} &= -\frac{1}{8} y_j y_i y_j y_i - \frac{3}{2} Y_{ij} y_i y_i \\
\gamma^{\bar{\psi}(2)} &= -\frac{1}{8} y_j y_i y_j y_i - \frac{3}{2} Y_{ij} y_j y_j \\
\beta^{(2)}_i &= 2 y_k y_i y_i (-y_j y_k + y_j y_j) - 2 h_{ijkl} y_j y_k y_i y_i - 4 Y_{ijkk} y_i y_i y_i - y_i (y_i y_i y_i + y_i y_j y_j) + y_i \gamma^{\psi(2)} + \gamma^{\bar{\psi}(2)} y_i + \gamma^{(2)}_{ij} y_j \\
\beta^{(2)}_{ijkl} &= 96 (2 Y_{njmkl} + Y_{nijml} + Y_{nikml}) + 6 h_{mijn}(8 Y_{mknl} - 2 Y_{mpnk} h_{mpnk} - h_{mpnk} h_{mpnk} + 4 \gamma^{(2)}_{lm} h_{mijkl})
\end{align*}

where in the last equation the indices $i, j, k, l$ must be totally symmetrized.

These formulae are in agreement with [24]. No renormalization group transformation is needed, because the renormalization conditions are the same.

As an example, we give the separate contributions to the term $Y_{njmkl}$ of $\beta^{(2)}_{ijkl}$. In (63) are indicated the Feynman rules for the insertions $T^{\gamma}_{jkl}$.

$-\mu \frac{\partial g^{(2)}_{ij}}{\partial \mu}$ (see table 3) contributes $\frac{12756}{105}$.

$-\beta^{(1)}(1) \frac{\partial g^{(1)}_{ij}}{\partial y_j}$ gets the contribution $-\frac{1816}{35}$ due to the term $\frac{454}{35} Y^{(1)}_{ijkl}$ of $g^{(1)}_{ijkl}$ (see table 1) and the term $\frac{1}{2} h_{ijkl}$ of $\beta^{(1)}_i$.

Finally $T^{\gamma}_{jkl}$ contributes to $Y_{njmkl}$ through the graph in fig. 5 and gives the coefficient $\frac{1772}{105}$.

These three contributions add up to the expected value 96 $Y_{njmkl}$ for $\beta^{(2)}_{ijkl}$ in equation (66).

3 External currents and local Ward identities

3.1 External currents

In order to define the currents associated to the symmetry transformations (14) we will add in the action (11) an external gauge field $A^a_{\mu}$.

At tree level the action is now

\begin{equation}
S^{(0)} = \int \bar{\psi} \gamma_{\mu} D_{\mu} \psi + \frac{1}{2} (D_{\mu} \phi)^2 + i \bar{\psi} y_{i} \psi \phi_{i} + \frac{1}{4} h_{ijkl} \phi_{i} \phi_{j} \phi_{k} \phi_{l} \tag{67}
\end{equation}
\[ D_\mu \psi = (\partial_\mu + iA_\mu^a t^a) \psi \quad ; \quad D_\mu \phi_i = (\partial_\mu \delta_{ij} + iA_\mu^a \theta_a^i) \phi_j \] (68)

We can choose for the external gauge field the condition \( \hat{A}_a^\mu = 0 \), so that \( \gamma_\mu t^a \) is equivalent to \( \bar{t}^a \gamma_\mu \) and the fermionic gauge coupling \( i\gamma_\mu t^a \) is equal to the reflection-symmetric expression \( \frac{i}{2} (\gamma_\mu t^a + \bar{t}^a \gamma_\mu) \).

We set the \( \mu \) factors to one; they can be easily reintroduced by dimensional analysis.

The local infinitesimal version of transformations (14) is now

\[ \delta A^a_\mu = -\partial_\mu \epsilon^a - f^{abc} A^b_\mu \epsilon^c = -D^a_\mu \epsilon^b \]

where \( \hat{\partial}_\mu \epsilon^a = 0 \). The tree-level action is invariant under these transformations apart from an evanescent fermionic kinetic term.

To renormalize arbitrary products of currents we will consider in the bare action local monomials in \( A^a_\mu \) up to fourth order.

The bare action (11) gets additional terms:

\[ S(A) = \int \left[ \frac{1}{2} \epsilon^{\rho\sigma} A^a_\mu A^a_\nu + \frac{1}{3!} \epsilon^{abc} A^a_\mu A^b_\nu A^c_\rho + \frac{1}{4!} \epsilon^{abcd} A^a_\mu A^b_\nu A^c_\rho A^d_\sigma + A^a_\mu \bar{\psi} c^a_\mu \psi + \frac{1}{2} \epsilon^{a\mu ij} A^a_\mu \phi_i \phi_j + \frac{1}{4} \epsilon^{abc} A^a_\mu A^b_\nu \phi_i \phi_j \right] \] (70)

The marginal part \( S_W \) of the Wilsonian effective action will have a similar dependence on \( A^a_\mu \).

The renormalization conditions on \( S_W \) analogous to (32), which include the new vertices, have to be chosen in such a way that the local chiral Ward identities hold; to this aim the renormalization conditions are fixed compatibly with the effective Ward identities. These identities have been introduced in [20] and further studied in [22].

An analysis similar to the one in sect.2.4 shows that the extra term of the effective Ward identity can be represented as the insertion of non-local vertices. Under transformation (69) one gets:

\[ \delta \Gamma^\Lambda = \mathcal{T}^\Lambda + \mathcal{O}^\Lambda \quad ; \quad \mathcal{T}^\Lambda = \left[ \delta \left( \frac{1}{2} \Phi D^{-1}_\Lambda K_\Lambda \Phi \right) \right] \cdot \Gamma^\Lambda \] (71)

where \( \mathcal{T}^\Lambda \) is the generator of the 1-PI Green functions with one insertion of the operator \( \delta \left( \frac{1}{2} \Phi D^{-1}_\Lambda K_\Lambda \Phi \right) \). Its Feynman rules are

\[ -i\epsilon^a(p) [S^{-1}_H(p + q) K_\Lambda(p + q) t^a - \bar{t}^a S^{-1}_H(q) K_\Lambda(q)] \] (72)

for insertion on the fermionic lines and

\[ -i\epsilon^a(p) [D^{-1}_H(p + q) K_\Lambda(p + q) \theta^a - \theta^a D^{-1}_H(q) K_\Lambda(q)] \] (73)

for insertion on the scalar lines.

If one can choose the renormalization conditions on \( \Gamma^\Lambda \) such that \( \mathcal{O}^\Lambda \) is evanescent, the effective Ward identities are not anomalous and the usual Ward
identities are recovered at $\Lambda = 0$; otherwise it is anyway possible to separate the functional $O^\Lambda$ in a part corresponding to the local anomaly operator plus a part representing an evanescent operator:

$$O^\Lambda = A^\Lambda + E^\Lambda$$  \hspace{1cm} (74)$$

Equation (71) is the effective Ward identity \cite{20, 22}, cast in a form which we find convenient for explicit computations.

The anomaly operator is defined by its renormalization conditions at the renormalization scale $\Lambda = \mu$ therefore we have to consider the marginal projection of eq.(71) in the limit $\epsilon \to 0$:

$$\delta S_W = T + A$$  \hspace{1cm} (75)$$

where $T$ and $A$ are the marginal parts of $T^\Lambda$ and $A^\Lambda$.

The additional renormalization conditions to which we refer in the following sections, have been chosen through a covariantization procedure of the renormalization conditions of the corresponding vector theory with minimal subtraction. For example to the tensor $Tr[t^a_i t^b_j t^c_k S]$ we associate in the chiral theory the tensor $Tr[t^a_i y^b_j y^c_k]$, which has the same transformation property. According to this rule by construction $S_W$ does not contain monomials in the gauge fields with the Levi-Civita tensor. This leads to the definition of the anomaly in the left-right symmetric form.

The Bardeen anomaly can be obtained through different renormalization condition, allowing in $S_W$ terms proportional to the Levi-Civita tensor.

3.2 One loop

At one loop using the tree level action (67) and the rules (72,73) for the insertions one computes:

$$T = \int \partial_\mu \epsilon^e \left[ b^{ca}_e D^{ab}_\nu \partial_\nu A^b_\mu + b^{ac}_e f^{cab} A^b_\nu (\partial_\mu A^b_\nu - \partial_\nu A^b_\mu) + \right. \hspace{1cm} \text{(76)}$$

$$\left. (f^{cde} b^{ey}_6 f^{pac} + \frac{1}{3!} b^{abc}) A^a_\mu A^b_\nu A^c_\sigma \right. \hspace{1cm} \text{where } A^a_\mu \equiv i t^a_R A^a_{R\mu}, A^a_L \equiv i t^a_L A^a_{L\mu} \hspace{1cm} \text{with the notations:}$$

\begin{align*}
S_2(F)^{ab} &= \frac{1}{2} Tr[t^a_i t^b_j \partial_\mu A^\mu + t^a_i t^b_i] ; \quad S_2(S)^{ab} = \theta^a_i \theta^b_j \\
S_4(F)^{abcd} &= \frac{1}{2} Tr[t^a_i t^b_j t^c_k t^d_l + t^a_i t^d_i t^c_k t^d_l] \\
S_4(S)^{abcd} &= \theta^a_i \theta^b_j \theta^c_k \theta^d_l + \theta^a_i \theta^d_i \theta^c_k \theta^d_l + \theta^a_i \theta^b_j \theta^c_k \theta^d_l + \theta^a_i \theta^c_k \theta^b_j \theta^d_l \hspace{1cm} (78)
\end{align*}
the values of the coefficients in eq. (76) are:

\[ b_1^{ab} = -\frac{2}{3}S_2(F)^{ab} + \frac{1}{60}S_2(S)^{ab} \]
\[ b_4^{ab} = -\frac{5}{9}S_2(F)^{ab} + \frac{1}{72}S_2(S)^{ab} \]
\[ b_6^{ab} = -\frac{34}{105}S_2(F)^{ab} - \frac{7}{360}S_2(S)^{ab} \]
\[ b^{abcd} = 4S_4(F)^{abcd} - \frac{11}{210}S_4(S)^{abcd} \]
\[ b^a = -\frac{53}{120}y^a_i p^b_i y^b_i + \frac{7}{120}y^a_i y^b_i \theta^a_{ij} \]
\[ b_{ij}^a = -\frac{121}{210} (Tr[t^a_i t^b_j] Y^i_L Y^j_L + (t^a_i t^b_j) Y^i_L Y^j_R] - (i \leftrightarrow j)) \]
\[ b_{ij}^{ab} = \frac{16}{105} (Tr[t^a_i t^b_j Y^i_R Y^j_L + t^a_i t^b_j Y^i_L Y^j_R] + (i \leftrightarrow j)) \]

To determine the Wilsonian renormalization conditions one must find \( S_W \); choose them of the form

\[
S_W^{(A)} = \int \frac{1}{2} \psi_\mu \Gamma^\mu \psi_\mu + \frac{1}{2} F_{\mu \nu} F^{\mu \nu} + a_3 A_\mu \partial_\mu A_\nu + a_4 f^{abc} \partial_\mu A_\mu^a A_\nu^b A_\lambda^c + \frac{1}{4} (f^{abc} f^{bcd} + 6 \theta^{abcd}) A_\mu^a A_\nu^b A_\lambda^c A_\sigma^d + i A_\mu^a \bar{\psi} \gamma_\mu a^a \psi + A_\mu^a \phi_i a_{ij}^a \phi_j + \frac{1}{4} A_\mu^a A_\nu^b \phi_i \phi_j a_{ij}^{ab} \]  
(80)

The effective Ward identities give the following relations between the \( S_W^{(A)} \) coefficients and those for \( T \):

\[
(a_1 + a_2 + b_1)^{ab} = 0 \quad \quad (a_1 + a_3 - b_4)^{ab} = 0 \\
(a_3 - a_4 - b_6)^{ab} = 0 \quad \quad (a + b)^{abcd} = 0 \\
t^a a^a + b^a = 0 \quad \quad [t^a, a^a] = 0 \\
(a^a + b^a - i\theta^a)_{ij} = 0 \quad \quad i a_{ik}^a \theta^k_{ij} + i a_{jk}^a \theta^k_{ij} - a_{ij}^{ab} - b_{ij}^{ba} = 0 \]  
(81)

Using \( \{a_i\} \), a set of independent parameters of \( S_W^{(A)} \) is \( a, a_{ij} \) and \( a_{ij}^{ab} \). The former two parameters have been fixed in the previous sections, to be compatible with minimal subtraction in the non-chiral case; making the analogous choice for the new parameter we get

\[
a_{ij}^{ab} = \frac{8}{5} S_2(F)^{ab} + \frac{11}{60} S_2(S)^{ab} + z^{ab} \]  
(82)

with \( z^{ab} = 0 \) for the minimal subtraction choice. We have checked all the relations \( \{a_i\} \) by explicit computation of the marginal Wilsonian Green functions and the corresponding \( T \) insertions.

Having determined the renormalization conditions, we can now compute the one-loop bare action:

\[
S_{(1)} = \int \frac{1}{4} F_{\mu \nu}^a [-8 \frac{1}{3 \epsilon} S_2(F)^{ab} + \frac{1}{3 \epsilon} S_2(S)^{ab} + z^{ab}] F_{\mu \nu}^b + S_{(1)}^{\psi NDR} + S_{(1)}^{\phi NDR} + \Delta S_{(1)} \]  
(83)
where $F_{\mu\nu}^a \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - f^{abc} A^b_\mu A^c_\nu$, $S_{(1)}^{\phi NDR}$ and $S_{(1)}^{\psi NDR}$ are the covariantization of the kinetic terms for fermions and scalars in absence of the gauge field; $\Delta S_{(1)}$ is the non-naive part of $S_{(1)}$.

$A$ in (77) is the Adler-Bardeen anomaly [25]; for a previous derivation using Wilsonian methods, see [30].

In order to compute the Bardeen anomaly in the next subsection we write $\Delta S_{(1)}^{(A)}$, the gauge-dependent part of $\Delta S_{(1)}$, with the variables $A_\mu$ and $V_\mu$ so defined:

$$A_\mu = V_\mu + \gamma_5 A_\mu ; \quad V_\mu = i t^a A^a_\mu ; \quad A_\mu = i t^a A^a_\mu$$ (84)

As a consequence:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = V_{\mu\nu} + \gamma_5 A_{\mu\nu}$$
$$V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu + [V_\mu, V_\nu] + [A_\mu, A_\nu]$$
$$A_{\mu\nu} = D^V_{\mu} A_\nu - D^V_{\nu} A_\mu$$ (85)

where we define $D^V_{\mu} f = \partial_\mu f + [V_\mu, f]$.

A similar decomposition for the scalars fields is introduced:

$$\sigma \equiv S_1 \phi_i ; \quad \pi \equiv P_1 \phi_i$$ (86)

$\Delta S_{(1)}^{(A)}$ is then cast in the form:

$$\Delta S_{(1)}^{(A)} = 4 \int Tr \left[ \frac{1}{6} (D^V_{\mu} A_\nu)^2 + V_{\mu\nu} A_\mu A_\nu - \frac{1}{3} [A_\mu, A_\nu]^2 - \frac{1}{3} A^2_{\mu} A^2_{\nu} + \frac{1}{3} \epsilon^{\mu\rho\sigma} (\partial_\mu V_\rho A_\sigma + V_\mu \partial_\rho V_\sigma A_\rho + \frac{3}{2} V_\mu V_\nu V_\rho A_\sigma + \frac{1}{2} V_\mu A_\nu A_\rho A_\sigma) - \frac{1}{6} (D^V_{\mu} \pi)^2 + i A_\mu (\pi, D^V_{\mu} \pi) + A^2_\mu (\pi^2 + \pi^2) + \frac{1}{2} [A_\mu, \pi]^2 \right] + 
\int \bar{\psi} y \gamma_\mu \gamma_5 A_\mu y \psi$$ (87)

### 3.3 Bardeen anomaly

The theory we considered up to now is very general, so for instance one should be able to compute the Bardeen anomaly [31]. This requires a suitable modification of the non naive counterterm part $\Delta S_{(1)}^{(A)}$. Let’s consider the case in which $G = \tilde{G}_R \times \tilde{G}_L$. In our model this is made by choosing that half of the generators $t^a_R$ and half of the generators $t^a_L$ vanish in such a way that $t^a_R t^b_R = t^a_L t^b_L = 0$.

Let’s consider moreover the case in which the left and the right representations of $\tilde{G}$ are the same so that the vectorial representation of $\tilde{G}$ turns out to be defined.

Decompose the gauge parameter as

$$\epsilon \equiv i t^a \epsilon^a = \alpha + \gamma_5 \beta$$ (88)

Now $V_\mu$ and $A_\mu$ as well as $\alpha$ and $\beta$, which are respectively the vectorial and axial gauge parameters, can be considered as independent quantities. The gauge transformation

$$\delta A_\mu = - \partial_\mu \epsilon + [\epsilon, A_\mu]$$ (89)
decomposes into vectorial and axial gauge transformations.

Under vectorial gauge transformations one has
\[ \delta_\alpha V_\mu = -D_\mu^V \alpha ; \quad \delta_\alpha A_\mu = [\alpha, A_\mu] \] (90)
and the scalar fields introduced in eq.(86) transform homogeneously.

Under axial gauge transformations one has
\[ \delta_\beta V_\mu = [\beta, A_\mu] ; \quad \delta_\beta A_\mu = -D_\mu^V \beta \] (91)

Under a vectorial gauge transformation one gets
\[ \delta_\alpha S_{\text{bare}} = A(\alpha) \] (92)
where \( A(\alpha) \) is the vectorial part of the anomaly (77) which indeed can be decomposed as
\[ A = A(\alpha) + A(\beta) \] (93)

where
\[
A(\xi) = -\frac{2}{3} \int \epsilon^{\mu\nu\rho\sigma} Tr \left[ \partial_\nu \xi \left( \partial_\rho A_{R\rho} A_{R\sigma} + \frac{1}{2} A_{R\rho} A_{R\rho} A_{R\sigma} - \frac{1}{2} A_{L\rho} A_{L\rho} A_{L\sigma} \right) \right]
\] (94)

On the other hand under an axial gauge transformation one gets
\[ \delta_\beta S_{(\epsilon-\text{part})} = A(\beta) - A_{\text{Bardeen}} \]
\[ A_{\text{Bardeen}} = 4 \int \epsilon^{\mu\nu\rho\sigma} Tr \left\{ \beta \left[ \frac{1}{4} V_{\mu\nu} V_{\rho\sigma} + \frac{1}{12} A_{\mu\nu} A_{\rho\sigma} - \frac{2}{3} (A_\mu V_{\nu\rho} A_{\sigma} + A_\mu A_{\nu\rho} A_{\sigma} + V_{\mu\nu} A_{\rho\sigma}) + \frac{8}{3} A_\mu A_\nu A_\rho A_\sigma \right] \right\} \] (95)

One can then modify the renormalization of the product of the currents by subtracting out all the terms depending on the Levi-Civita tensor in the finite part of the bare action (87) and correspondingly by introducing their gauge variation in eq.(75). Being \( T \) unchanged the effect of subtracting eq.(92) is to recover the vectorial gauge invariance, the subtraction of eq.(95) put the anomaly in the Bardeen form (31).

Observe that the \( \epsilon \)-dependent part of (87) is not specific of the BMHV regularization, being fixed its gauge variations (92, 93).

The \( \epsilon \)-independent part of (87) is specific of the BMHV regularization; for instance it takes a different expression in [32], where a different renormalization scheme is used. Using (87) with \( V_\mu = \frac{1}{2} A_\mu (t_R + t_L) \) and \( A_\mu = \frac{1}{2} A_\mu (t_R - t_L) \) and making the non-minimal choice \( z^{ab} = \frac{4}{7} S_2(F)^{ab} \) we find agreement with the finite counterterm computed in [11] using the Bonneau identities [12].

### 3.4 Two loop

At two loops similar computations could be performed. We have computed the marginal Wilsonian action in the two gauge field sector, that is the coefficients
Figure 7: Graphs of the two-point gauge field Green function.

Table 6: Contributions to the two-point gauge Green function: the numbers include the contributions of the two-loop graphs with the corresponding one-loop subtraction and sum over external legs permutation

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
 & $a_{1}^{ab}$ & & $a_{2}^{ab}$ & graphs \\
\hline
$K_{1}^{ab}$ & $K_{2}^{ab}$ & $K_{1}^{ab}$ & $K_{2}^{ab}$ & \\
\hline
$-\frac{593}{3649} + \frac{674}{2187} v$ & 0 & $\frac{208}{725} - \frac{2066}{2187} v$ & 0 & Fig.7(a) \\
0 & $\frac{57469}{51030} - \frac{98}{2187} v$ & 0 & $\frac{38657}{25515} + \frac{98}{2187} v$ & Fig.7(b) \\
$\frac{13178}{3645} - \frac{9956}{2187} v$ & $-\frac{13178}{3645} + \frac{9956}{2187} v$ & $-\frac{9674}{2187} v$ & $\frac{9674}{2187} v$ & Fig.7(c) \\
$\frac{80042}{25515} - \frac{1411}{2187} v$ & $-\frac{80042}{25515} + \frac{1411}{2187} v$ & $-\frac{14111}{2187} v$ & $\frac{13858}{2187} v$ & Fig.7(d) \\
\hline
\end{tabular}
\end{table}

$a_{1}^{ab}$ and $a_{2}^{ab}$ of eq. (80) at two loop. The graphs involved are shown in fig. 7; their contributions to $a_{1}^{ab}$ and $a_{2}^{ab}$ are expanded on the basis of invariant symmetric tensors:

\begin{align}
K_{1}^{ab} &= \frac{1}{2} \left( \text{tr} \left[ t_{L}^{a} Y_{i} t_{R}^{b} Y_{i}^{\dagger} \right] + \text{tr} \left[ t_{R}^{a} Y_{i}^{\dagger} t_{L}^{b} Y_{i} \right] \right) \\
K_{2}^{ab} &= \frac{1}{2} \left( \text{tr} \left[ t_{L}^{a} t_{R}^{b} Y_{i} Y_{i}^{\dagger} \right] + \text{tr} \left[ t_{R}^{a} t_{L}^{b} Y_{i}^{\dagger} Y_{i} \right] \right)
\end{align}

and are shown in table 6. In order to check the first Ward identity relation in (81) we have also computed the graphs shown in fig. 8 and the results are collected in table 7. Actually we did not consider in figs. 7, 8 the graph with only scalar internal lines and quartic scalar vertex: it is finite and rigid invariant, therefore it does not contribute to the bare action and it has to match separately its own $T$ insertion.

From the pole part of graphs in fig. 7 we computed the gauge invariant part of $S^{(A)}$ at two loops:

\begin{align}
S_{(2)}^{(A)} &= \int \frac{1}{4} F_{\mu \nu}^{a} \left[ -\frac{2}{\epsilon} K_{2}^{ab} \right] F_{\mu \nu}^{b} \\
\text{finding agreement with } & [24].
\end{align}

From the non-invariant part we derived the following contribution to the
Figure 8: Tau-gauge graphs

Table 7: Contributions to the one-point $\mathcal{T}$ Green function.

| $b_1^{ab}$ | $K_1^{ab}$ | $K_2^{ab}$ | graphs |
|------------|-------------|-------------|--------|
|            | $\frac{262}{1215} + \frac{748}{279}v$ | 0 | Fig. 8(a) |
|            | 0           | $\frac{1171}{2410} - \frac{248}{729}v$ | Fig. 8(b) |
|            | $\frac{253}{243} - \frac{1508}{729}v$ | $-\frac{253}{241} + \frac{1508}{729}v$ | Fig. 8(c) |
|            | $-\frac{157}{405} + \frac{268}{243}v$ | $\frac{157}{135} - \frac{268}{81}v$ | Fig. 8(d) |
|            | $\frac{1751}{1890} - \frac{124}{81}v$ | $-\frac{1751}{1890} + \frac{124}{81}v$ | Fig. 8(e) |
bare action:

\[
\Delta S_{(2)}^{(A)} = \frac{1}{2} \int Tr \left[ \left( \frac{2}{3c} - \frac{53}{36} \right) \{A_\mu, P_1\}(D^\nu_\nu)^2\{A_\mu, P_1\} + 2A_\mu(D^\nu_\nu)^2A_\mu P_1P_1 - \left( \frac{2}{3c} + \frac{13}{36} \right) \{A_\mu, S_1\}(D^\nu_\nu)^2\{A_\mu, S_1\} + \frac{5}{3} A_\mu(D^\nu_\nu)^2A_\mu S_1S_1 \right] + \\
\frac{1}{4} \int Tr \left[ \left( \frac{8}{3c} - \frac{13}{9} \right) (A_{\mu\nu}S_1)^2 - \frac{5}{9} (A_{\mu\nu})^2(S_1)^2 + \left( \frac{4}{3c} - \frac{22}{9} \right) (A_{\mu\nu}P_1)^2 + \left( \frac{4}{3c} - \frac{4}{9} \right) (A_{\mu\nu})^2(P_1)^2 - \left( \frac{4}{9c} + \frac{31}{27} \right) (V_{\mu\nu}P_1)^2 + \left( \frac{28}{9c} - \frac{29}{27} \right) (V_{\mu\nu})^2(P_1)^2 + i \left( \frac{8}{3c} - \frac{13}{9} \right) V_{\mu\nu}S_1 A_{\mu\nu} P_1 \right. \\
\left. + i \left( -\frac{16}{9c} + \frac{35}{27} \right) V_{\mu\nu} P_1 A_{\mu\nu} S_1 + i \left( \frac{4}{3c} - \frac{1}{9} \right) V_{\mu\nu} A_{\mu\nu} P_1 S_1 - \frac{5}{9} i V_{\mu\nu} A_{\mu\nu} S_1 P_1 + i \left( -\frac{4}{9c} + \frac{14}{27} \right) V_{\mu\nu} P_1 S_1 A_{\mu\nu} \right]
\]

(98)

We used the notation of eqs. \[84\]; the r.h.s. of eq. \[88\] is not simply quadratic in the gauge fields because we completed it in order to have an expression gauge-invariant under vectorial transformation. This fact has been possible because the derivatives \(\partial_\mu V_\nu\) of the gauge fields \(V_\mu\) appear only in the combination \(\partial_\mu V_\nu\) as expected, since dimensional regularization and ours renormalization condition respect vectorial symmetry.

We checked the non renormalization theorem of the anomaly only in the two-gluon sector of eq. \[75\]. We have computed the part of \(\mathcal{T}\) from which the anomaly might arise:

\[
\mathcal{T} = \int \partial_\mu \epsilon^a(x) \partial_\nu A^b_\mu A^c_\nu \epsilon^{\mu\nu\rho\sigma} T_{abc}
\]

(99)

where \(T_{abc}\) is an invariant tensor symmetric in the last two indices. The non renormalization theorem requires that the completely symmetric part \(T_{(abc)}\), which cannot be eliminated by a local counterterm, must vanish.

Fig. 2 shows the graphs which give contributions to \(T_{(abc)}\). Each contribution is decomposed on a suitable basis of symmetric tensor, we have chosen a basis in which \(\theta_{ij}^a\) never appears explicitly (symmetrization in the indices \(a, b, c\) is understood):

\[
T_{abc}^{(1)} = Tr[Y_k^a t^a_k t^b_R t^c_R] - Tr[Y_k^a t^a_k t^c_R t^b_R] \\
T_{abc}^{(2)} = Tr[Y_k^a t^a_k t^b_R t^c_R] - Tr[Y_k^a t^b_R t^a_k t^c_R]
\]

(100)

The results of our calculations are summarized in table 8 as expected the sum of the contributions of every column vanishes, so that the Adler-Bardeen theorem is verified.

One could address the question whether \(T_{abc}\) and not only its symmetric part is actually vanishing. For \(G = SU(N)\), using the Young tableaux one can easily prove that invariant tensors \(X_{abc}\) with mixed symmetry do not exist; \[3\] for more general groups see \[33\].

\[\text{We thank C. Destri for explaining this point to us.}\]
Figure 9: Two-loop graphs contributing to the anomaly

\[
\begin{array}{c|c|c}
\tau_{\alpha \beta \gamma}^{(1)} & \tau_{\alpha \beta \gamma}^{(2)} & \text{graphs} \\
\hline
\frac{-6797}{21870}i - \frac{4688}{6561}iv & 0 & \text{Fig. 9(a)} \\
\frac{18119}{21870}i - \frac{12880}{6561}iv & 0 & \text{Fig. 9(b)} \\
0 & -\frac{55441}{204129}i - \frac{76}{2187}iv & \text{Fig. 9(c)} \\
0 & -\frac{10931}{204129}i - \frac{308}{2187}iv & \text{Fig. 9(d)} \\
-\frac{254}{243}i + \frac{2080}{729}iv & 0 & \text{Fig. 9(e)} \\
\frac{98}{135}i - \frac{440}{243}iv & -\frac{98}{135}i + \frac{440}{243}iv & \text{Fig. 9(f)} \\
\frac{4}{3}i & 0 & \text{Fig. 9(g)} \\
-\frac{1861}{1215}i + \frac{1192}{729}iv & \frac{1861}{1215}i - \frac{1192}{729}iv & \text{Fig. 9(h)} \\
\end{array}
\]

Table 8: Two-loop coefficients for the anomaly
4 Concluding remarks

We have shown that Wilsonian methods are useful in the BMHV dimensional renormalization of theories with chiral symmetries; the two-loop renormalization of the most general Yukawa theory becomes straightforward in this scheme, while in a more conventional approach, based on the verification of usual Ward identities, it is a non trivial task.

In our approach one renormalizes the effective Wilsonian action along a flow on which the subtraction integrals are easily computed using standard techniques.

There is some arbitrariness in the choice of this flow; we worked with the $n = 2$ flow, which is the simplest one from a computational viewpoint: it is quite obvious that calculations with the $n \geq 3$ flows are more complicated, although the bare action is the same. On the other hand the $n = 1$ flow, which coincides with the auxiliary mass method, does not respect renormalizability so the procedure described in this paper fails down. It is a convenient method in nonchiral theories where minimal subtraction respects the Ward identities.

In the future we intend to apply our formalism to chiral gauge theories.
Appendix. Two-loop bare Yukawa action

We have written the naive part of the counterterms in the Tables; here we report the remaining counterterms, necessary to recover the rigid chiral symmetry in the BMHV scheme.

The two-loop scalar quadratic counterterm is

\[
\Delta \sigma_{ij} (p) = c_{ij} \tilde{p}^2 + c_{ij}^{m^2,kl} (n)m_{kl}^2 + \tilde{p}^2 \text{terms}
\]

\[
c_{ij}^{\phi} = Tr \left[ 2S_kP_kS_jP_j + \left( -\frac{8}{9} + \frac{20}{3\epsilon} \right) P_iS_kP_jS_k + \left( \frac{23}{9} - \frac{20}{3\epsilon} \right) (S_iS_kP_jP_k + S_jS_kP_iP_i) \right] -
\]

\[
\left( \frac{7}{9} - \frac{4}{3\epsilon} \right) (S_iS_kP_jP_j + S_jP_jS_iS_i) + \left( \frac{13}{9} - \frac{16}{3\epsilon} \right) P_iP_jP_kP_k \]

\[
c^{m^2,kl}_{ij} = Tr \left[ 3 - \frac{4}{\epsilon} \left( S_jS_kP_iP_i + S_jP_iP_kS_i + P_jS_iS_iP_i + P_jS_kP_iS_i - 8P_jS_kP_iS_i + \left( \frac{64}{9} - \frac{32}{3\epsilon} \right) P_jP_kP_iP_i + \left( \frac{80}{9} - \frac{64}{3\epsilon} \right) P_jP_kP_iP_i + (2 - \frac{8}{\epsilon}) (S_jS_kP_iP_i + S_jP_iP_iS_i + P_jS_kS_iS_i + P_jP_kS_iS_i) + \left( \frac{2}{\epsilon} - \frac{7}{9} \right) P_mP_l \right] h_{ijk}\]

and the symmetrization in \( i, j \) is understood.

The two-loop fermionic quadratic counterterm is

\[
\Delta c_\psi (p) = \tilde{p} c_\psi + [\tilde{p} - \text{terms}]
\]

\[
c_\psi = iy_j \left[ \frac{1}{36} - \frac{1}{3\epsilon} P_iP_i y_j + (\frac{1}{8} - \frac{1}{2\epsilon}) i\gamma_5 (P_j y_j - y_j P_j) y_i + \left( \frac{43}{54} - \frac{10}{9\epsilon} \right) Tr (P_i P_j) y_i \right]
\]

The two-loop fermion-fermion-scalar counterterm is

\[
\Delta c_i = \sum_{n=1}^{7} c_i(n)
\]

\[
c_i(1) = iy_k \left[ \left( -\frac{7}{6} + \frac{2}{\epsilon} \right) Tr (P_j P_k) S_i + i\gamma_5 \left( \frac{1}{2} - \frac{2}{\epsilon} \right) Tr (S_j S_k) P_i + i\gamma_5 \left( \frac{22}{27} - \frac{32}{9\epsilon} \right) Tr (P_j P_k) P_i y_j \right]
\]

\[
c_i(2) = iy_k [i\gamma_5 (\frac{1}{4} - \frac{1}{\epsilon}) (S_j S_k P_j + P_j S_i S_j) + i\gamma_5 \left( \frac{2}{3} - \frac{2}{\epsilon} \right) P_j P_k P_i y_k]
\]

\[
c_i(3) = iy_k [i\gamma_5 (\frac{1}{8} - \frac{1}{2\epsilon}) (S_j Y_j P_j + P_j Y_j S_j) + i\gamma_5 (\frac{1}{9} - \frac{4}{3\epsilon}) (P_i P_j P_j + P_j P_j P_i)] y_k
\]
\[ c_i(4) = iy_k \gamma_5 \left( \frac{1}{4} - \frac{1}{\epsilon} \right) (S_i Y_j P_k + P_j Y_k S_i) - i \gamma_5 \left( \frac{1}{4} + \frac{1}{\epsilon} \right) (P_i Y_j S_k + S_j Y_k P_i) - \frac{2}{\epsilon} (P_i S_j P_k + P_j S_k P_i) - i \gamma_5 \left( \frac{2}{3} + \frac{2}{\epsilon} \right) (P_i P_j P_k + P_j P_k P_i) \]

\[ c_i(5) = iy_k \gamma_5 (2S_j P_s S_k - \frac{1}{2} S_j y_i P_k - \frac{2}{3} P_j P_k P_i - \frac{1}{2} P_j y_i S_k) \]

\[ c_i(6) = 0 \]

\[ c_i(7) = -\frac{1}{2} y_j P_k y_l h_{ijkl} \]

The two-loop quartic scalar counterterm is

\[ \Delta c_{ijkl} = \sum_{n=1}^{9} c_{ijkl}^{(2)}(n) \]

\[ c(1) = 16 \text{Tr} Y_m \left[ i \left( \frac{8}{\epsilon} - \frac{2}{3} \right) S Y_m P^3 + i \left( \frac{3}{\epsilon} - \frac{3}{4} \right) S Y_m (SY P + PYS) + \right. \]

\[ \left. \left( \frac{8}{\epsilon} + \frac{13}{3} \right) P Y_m P^3 + P Y_m (SY P + P P S P Y S) \right] + \]

\[ \left( \frac{6}{\epsilon} - 1 \right) P Y_m (SY P + P Y S) + i \left( \frac{3}{\epsilon} - \frac{9}{4} \right) P Y_m (S Y S - P Y P) + p.c. \]

\[ c(2) = 4 \text{Tr} Y_m \left[ \frac{3}{2} (Y Y_m Y_m Y Y \dagger - Y^2 Y_m Y^2) - \right. \]

\[ \left. 8 P Y_m (Y^2 + \frac{11}{8} P^2) + (8 Y Y - P Y Y)^\dagger P^2 \right] + p.c. \]

\[ c(3) = \text{Tr} Y_m \left[ \left( \frac{12}{\epsilon} - 3 \right) (Y^4 Y_m - (Y Y \dagger)^3 Y_m) + 12 P^4 Y_m - \left( \frac{64}{\epsilon} + \frac{20}{3} \right) P^4 Y_m + \right. \]

\[ \left. \left( \frac{32}{\epsilon} - \frac{8}{3} \right) 2i S Y P P_m + 2i P S Y P P_m - P^2 Y Y_m^\dagger + P Y Y S Y_m \right] + p.c. \]

\[ c(4) = \text{Tr} \left[ - \left( \frac{48}{\epsilon} - 12 \right) (S_m S P P_n + S_n P^2 S_n + \right. \]

\[ \left. P_m S^2 P_n + P_m P S S_n) - \left( \frac{128}{\epsilon} - \frac{160}{3} \right) P_m P^2 P_n \right] \]

\[ c(5) = \text{Tr} \left[ - \left( \frac{64}{\epsilon} - \frac{128}{3} \right) P_m P P_n P - 48 S_m P S_n P - \right. \]

\[ \left. \left( \frac{24}{\epsilon} - 18 \right) (S_m S P P_n + S_n P P_n P + S_m S S_n P + P_m P S_n S) \right] \]

\[ c(6) = \text{Tr} \left[ \left( \frac{6}{\epsilon} - \frac{7}{2} \right) P_m P_n \right] h_{mp} h_{np} \]

\[ c(7) = c(8) = c(9) = 0 \]

where the indices \( i, j, k, l \), which are totally symmetrized, are understood.

\( p.c. \) represents the terms obtained with the substitution rule: \( P \rightarrow -P, Y \rightarrow Y^\dagger, Y^\dagger \rightarrow Y \).
References

[1] G. ‘t Hooft and M. Veltman, Nucl. Phys. B44 (1972) 189.
[2] G. ‘t Hooft, Nucl. Phys. B61 (1973) 455.
[3] P. Breitenlohner and D. Maison, Comm. Math. Phys. 52 (1977) 11.
[4] M. Chanowitz, M. Furman and I. Hinchliffe, Nucl. Phys. B159 (1979) 225.
[5] G. Bonneau, Int. Journ. Mod. Phys. A5 (1990) 3831.
[6] M. Caffo, H. Czyż, S. Laporta and E. Remiddi, Nuovo Cim. 111A (1998) 365.
[7] C. Schubert, Nucl. Phys. B323 (1989) 478.
[8] T.L. Trueman, Phys. Lett. 88B (1979) 331; Z. Physik C69 (1996) 525.
[9] O. Piguet and S.P. Sorella, Algebraic Renormalization Lectures Notes in Physics Monographs, Springer-Verlag Berlin Heidelberg, 1995.
[10] P.A. Grassi, T. Hurth and M. Steinhauser, ‘Practical algebraic renormalization’, preprint hep-th/9907426.
[11] C.P. Martin and D. Sanchez-Ruiz, ‘Action principles, restoration of BRS symmetry and the renormalization group equation for chiral non-Abelian gauge theories in dimensional renormalization with a non-anticommuting γ5’, preprint hep-th/9905076.
[12] G. Bonneau, Nucl. Phys. B177 (1981) 523.
[13] J.G. Korner, N. Nasrallah and K. Schilcher, Phys. Rev. D41 (1990) 888; R. Ferrari, A. Le Yaouanc, L. Oliver and J.C. Raynal, Phys. Rev. D52 (1995) 3036.
[14] R. van Damme, Nucl. Phys. B244 (1984) 105.
[15] K. Chetyrkin, M. Misiak and M. Munz, Nucl. Phys. B518 (1998) 473.
[16] T. van Ritbergen, J.A.M. Vermaseren, S.A. Larin, Phys. Lett. B400 (1997) 379.
[17] J.C. Collins, Nucl. Phys. B92 (1975) 477; J.C. Collins, Renormalization Cambridge Monographs on Mathematical Physics, Cambridge University Press (1984).
[18] K.G. Wilson and J.G. Kogut, Phys. Rep. C12 (1974) 75.
[19] J. Polchinski, Nucl. Phys. B231 (1984) 269.
[20] C.M. Becchi, ‘On the construction of renormalized quantum field theory using renormalization group techniques’, in Elementary particles, Field theory and Statistical mechanics, Eds. M. Bonini, G. Marchesini and E. Onofri, Parma University 1993.
[21] A partial list of references is: G. Keller, C. Kopper and M. Salmhofer, Helv. Phys. Acta 65 (1992) 32; M. Bonini, M. D’Attanasio and G. Marchesini, Nucl. Phys. B409 (1993) 441; G. Keller, C. Kopper, Comm. Math. Phys. 161 (1994) 515 and 176 (1996) 193; R.D. Ball, R.S. Thorne, Ann. Phys. 236 (1994) 117 and 241 (1995) 337.

[22] M. Bonini, M. D’Attanasio and G. Marchesini, Nucl. Phys. B418 (1994) 81; B421 (1994) 429; B437 (1995) 163; Phys. Lett. B346 (1995) 87; M. D’Attanasio and T.R. Morris, Phys. Lett. B378 (1996) 213; M. Pernici, M. Raciti and F. Riva, Nucl. Phys. B520 (1998) 469.

[23] M. Pernici and M. Raciti, Nucl. Phys. B531 (1998) 560.

[24] M. E. Machacek and M.T. Vaughn, Nucl. Phys. B222 (1983) 83; B236 (1984) 221; B249 (1985) 70.

[25] S. Adler and W. Bardeen, Phys. Rev. 182 (1969) 1517.

[26] S. Wolfram, Mathematica, (Addison-Wesley, New York 1988).

[27] E. Braaten and J.P. Leveille, Phys. Rev. D24 (1981) 1369.

[28] M. Fischler and J. Oliensis, Phys. Rev. D28 (1983) 2027.

[29] A.I. Davydychev, ‘Explicit results for all orders of the $\epsilon$ expansion of certain massive and massless diagrams’, preprint hep-ph/9910224.

[30] M. Bonini and F. Vian, Nucl. Phys. B532 (1998) 473.

[31] W. A. Bardeen, Phys. Rev. 184 (1969) 1848.

[32] A.P. Balachandran, G. Marmo, V.P. Nair and C.G. Trahern, Phys. Rev. D25 (1982) 2713.

[33] S. Okubo, Phys. Rev. D16 (1977) 3528.