On the well-posedness of Reynolds-Rayleigh-Plesset coupling

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Abstract

In the lubrication area, which is concerned with thin film flow, cavitation has been considered as a fundamental element to correctly describe the characteristics of lubricated mechanisms. Here, the well-posedness of a cavitation model that can explain the interaction between viscous effects and micro-bubbles of gas is studied. This cavitation model consists in a coupled problem between the compressible Reynolds PDE (that describes the flow) and the Rayleigh-Plesset ODE (that describes micro-bubbles evolution). Local times existence results are proved and stability theorems are obtained based on the continuity of the spectrum for bounded linear operators. Numerical results are presented to illustrate these theoretical results. Particularly, a lost of well-posedness depending upon a geometrical parameter is shown numerically.

Keywords: Cavitation modeling, Reynolds equation, Rayleigh-Plesset equation, stability.

1 Introduction

Cavitation is observed in various engineering devices, ranging from hydraulic systems to turbo pumps for space applications. It is a challenging issue linked with various phenomenon: acoustic, thermodynamic and fluid dynamic. In the lubrication area, which is concerned with thin film flow, cavitation has been considered as a fundamental element to correctly describe the characteristics of lubricated mechanisms [1, 2]. Cavitation has often been primary associated with a diminution of the pressure $p$ in the liquid falling below the vapor pressure. Numerous models have been introduced to couple this unilateral condition with the Reynolds equation, which is usually used to model the pressure evolution in thin film flow. Mathematical studies of these meso models can be found in [3, 4, 5] in which existence and uniqueness results are given for both the stationary and transient cases. However, it is physically recognized that the cavitation phenomenon is linked with the existence and evolution of micro-bubbles in a liquid. This aspect has been ignored in these models, until the works of Someya’s group [6, 7] who proposed to couple the full Rayleigh-Plesset equation (which describes the evolution of a bubble) with the Reynolds equation (which describes the fluid). In some sense, the resulting system of equations can be described as a micro model. Numerous works follow in the lubrication literature using simplified forms of the Rayleigh-Plesset equation for various kind of applications [8, 9, 10, 11]. The paper of Snyder et al. [12] can be considered as a review paper in this field.

The Reynolds and Rayleigh-Plesset equations

Physically, the fluid is contained in a domain in $\mathbb{R}^3$ that is limited by a flat plane in the $x_1-x_2$ plane, an upper surface defined by the gap function $h(x_1, x_2)$ and with vertical lateral boundary as shown in Fig. 1. The relative velocity between the surfaces in the $x_1-x_2$ plane, denoted $U$, is assumed to be known. Also, in this work we assume the relative velocity of the surfaces in the $x_3$-axis to be null. The thin film hypothesis allows to approximate the Navier-Stokes equations by means of the Reynolds equation [13]. In that context, the pressure is assumed to be a function $p = p(x_1, x_2, t)$. The compressible form of Reynolds equation reads

$$\nabla_x \cdot \left( \rho h^3 \frac{\nabla p}{12\mu} \right) = \nabla_x \cdot \left( \frac{U}{2} \rho h \right) + h \frac{\partial p}{\partial t} \quad \text{in } \Omega, \quad (1.1)$$
where $\rho$ is the density of the fluid (a mixture liquid/gas) and $\mu$ its viscosity. The Reynolds-Rayleigh-Plesset cavitation model assumes $\rho$ and $\mu$ to depend upon the fraction of gas present in the liquid, denoted $\alpha$ and computed as:

$$\alpha = \frac{\text{volume of gas}}{\text{volume of gas and liquid}}.$$ 

As example of this relations we have $[9, 12]$

$$\rho(\alpha) = (1-\alpha)\rho_l + \alpha \rho_g,$$

$$\mu(\alpha) = (1-\alpha)\mu_l + \alpha \mu_g,$$

where $\rho_l$ and $\rho_g$ ($\mu_l$ and $\mu_g$) are the densities (viscosities) of the liquid and the gas respectively.

The local bubbles’ radii is denoted by $R = R(x_1, x_2, t)$ and the relation $\alpha = \alpha(R)$ is asserted by means of geometrical arguments. Several choices of $\alpha(R)$ can be found in the literature (e.g., $[14, 12]$), all of them accomplishing that $\alpha$ is smooth and $\frac{d\alpha}{dR} > 0$. In Section 5 we will use the relation assumed by Someya et al. $[14]$, which reads:

$$\alpha(R) = \frac{\alpha_0}{\alpha_0 + (1-\alpha_0)(R_0/R)^3} \quad \forall R > 0,$$ 

where $\alpha_0$ is a data corresponding to the gas fraction for $R = R_0$, and $R_0$ is a reference radius also known.

The evolution of a small bubble of gas, spherical of radius $R$, immersed in a Newtonian fluid in adiabatic conditions is governed by the Rayleigh-Plesset (RP) equation $[15]$, which reads:

$$\rho_l \left[ \frac{3}{2} \left( \frac{\partial R}{\partial t} \right)^2 + R^2 \frac{\partial^2 R}{\partial t^2} \right] = P_0 \left( \frac{R_0}{R} \right)^{3k} - (p + p_0) - \frac{2\sigma}{R} - 4 \left( \frac{\mu_l + \kappa^*}{R} \right) \frac{\partial R}{\partial t},$$

where $P_0$ is the inner pressure of the bubble when its radius is equal to $R_0$, $k$ is the polytropic exponent, $p_0$ is the pressure at the boundary of $\Omega$, $\sigma$ is the surface tension and $\kappa^*$ is the surface dilatational viscosity $[12]$. The terms of the right hand side of Eq. (1.5) are called inertial terms.

It is noteworthy that there exist many works in Mechanics’ literature concerning the numerical resolution and modeling aspects of the coupling of the Rayleigh-Plesset equation with fluid flow equations (e.g., $[16, 17, 15, 19, 20, 21]$). The well known software FLUENT for Fluid Mechanics uses also this type of modeling $[22, 23, 24]$. On the other hand, in the mathematical field few works are concerned with this problem. The Rayleigh-Plesset equation alone without coupling (in which the pressure is a known data) has been subject of interest as differential equations with singularities $[25, 26]$. However, to the knowledge of the authors no mathematical analysis of the full coupling of the Rayleigh-Plesset equation with a flow equation (Euler, Stokes or Reynolds) so far appeared.

The structure of this document is as it follows: after the introduction section, the mathematical framework is described in Section 2 where notations and some previous required results are given. Section 3 is devoted to the study of the full system (1.1) to (1.5) including inertial terms; existence of a stationary solution is gained by way of the Implicit Function Theorem around some particular data for which a
stationary solution is easy to compute; a stability result is obtained with a small data assumption by studying the spectrum of a differential operator, and the continuity of that spectrum around the particular data; at last, an instability result is gained in the one dimensional case by means of the Routh-Hurwitz Theorem. In Section 3 a simplified Rayleigh-Plesset equation neglecting the inertial terms is considered; unlike the previous section, existence of the (local) solution of the system is not obvious and requires to use the Fredholm Alternative Theorem; stability results of the stationary solution for small data are obtained using also the spectrum’s continuity of a differential operator. The last section is devoted to some numerical examples. A comparison with a classical meso solution is performed and convergence properties are illustrated.

2 Mathematical framework

In this section we introduce some notations and previous results to be used along this document.

Let \( \Omega \subset \mathbb{R}^N \), \( N = 1, 2 \) be a regular domain, we consider the abstract problem of finding \( p(x,t), R(x,t) > 0 \), with \( x \in \Omega \) and \( t \geq 0 \), such that

\[
\frac{3}{2} \frac{1}{R} \left( \frac{\partial R}{\partial t} \right)^2 + \frac{\partial^2 R}{\partial t^2} = \frac{f_1(R) - p}{R} - \frac{\partial R}{\partial t} f_2(R) \tag{2.1}
\]

and

\[
\nabla_x \cdot (f_3(R) h^3 \nabla p) = \nabla_x \cdot (f_4(R) U h) + h f_5(R) \frac{\partial R}{\partial t},
\]

\[
p = 0 \quad \text{on} \ \partial \Omega. \tag{2.2}
\]

where \( U \in \mathbb{R}^N \). Along the initial conditions for every \( x \in \Omega \):

\[
R(x,0) = r_1(x),
\]

\[
\frac{\partial R}{\partial t}(x,0) = r_2(x) \tag{2.3}
\]

and \( r_1, r_2 \) are regular known functions. The terms in the left hand side of Eq. (2.1) are named \textit{inertial terms}. In the next sections we study the wellposedness of problem (2.1)–(2.3) when including or disregarding the inertial terms.

For \( \alpha, \beta \in \mathbb{R} \) with \( \alpha < \beta \) we define:

\[
B_{\alpha,\beta} = \{ w \in L^\infty(\Omega) : \alpha \leq w \leq \beta \ a.e. \ \text{on} \ \Omega \}.
\]

We make also the following hypotheses:

H1: \( f_1 \in C^2(\mathbb{R}_+^3; \mathbb{R}); \exists \delta_1 \in \mathbb{R}_+^3 \) such that \( f_1(R) = 0 \) and \( f_1'(R) < 0 \ \forall R \in \left[ R - \delta_1, R + \delta_1 \right] \). We denote \( m_3 = \min_{R \in [R - \delta_1, R + \delta_1]} |f_1'(R)| \) and \( M_3 = \max_{R \in [R - \delta_1, R + \delta_1]} |f_1'(R)| \);

H2: \( f_2 \in C^2(\mathbb{R}_+^3; \mathbb{R}_+^3) \);

H3: \( f_3 \in C^2(\mathbb{R}_+^3; \mathbb{R}) \) and \( \exists m_3, M_3 > 0 \) such that \( m_3 \leq f_3(r) \leq M_3 \ \forall r \in \mathbb{R}_+^3 \);

H4: \( f_4 \in C^2(\mathbb{R}_+^3; \mathbb{R}_+^3) \), \( f_4'(r) < 0 \ \forall r > 0 \);

H5: \( f_5 \in C^2(\mathbb{R}_+^3; \mathbb{R}_-^3) \);

H6: \( h \in B_{m_0, M_0} \) for \( 0 < m_0 < M_0 \) constants. We denote \( h_0 = \text{ess inf} \ h \).

\textbf{Remark 1.} \textit{The physical model given by Eqs. (1.1) to (1.5) is a particular case of problem (2.1) to (2.3) for which}

\[
f_1(R) = P_0 \left( \frac{R_0}{R} \right)^{3 \kappa} - p_\sigma - \frac{2\sigma}{R}, \tag{2.4}
\]

\[
f_2(R) = 4 \left( \frac{\mu_t + \kappa^\sigma / R^2}{R^2} \right), \quad f_3(R) = \frac{1}{12} \frac{(1 - \alpha(R)) \rho_t + \alpha(R) \rho_g}{\mu_t + \alpha(R) \mu_g},
\]

\[
f_4(R) = \frac{1}{2} [\rho_t + \alpha(R) (\rho_g - \rho_t)], \quad f_5(R) = f_4'(\alpha(R)) \alpha'(R).
\]
The next result is a particular case of Theorem 4.2 in [27].

**Lemma 1.** Let Ω be a smooth domain on \( \mathbb{R}^N \), \( f \in H^{-1}(\Omega) \) and \( u \in H^1_0(\Omega) \) be the unique solution of the elliptic problem\(^1\)

\[
\nabla \cdot (a \nabla u) = f \quad \text{in} \quad \Omega, \\
u = 0 \quad \text{on} \quad \partial \Omega,
\]

for \( a \in B_{\alpha,\beta}, \ 0 < \alpha < \beta \). Then there exists \( q > 2 \) (which depends on \( \alpha, \beta, \Omega \) and on the dimension \( N \)) such that, if \( f \in W^{-1,q}(\Omega) \), then \( u \) belongs to \( W^{1,q}_0(\Omega) \) and satisfies

\[
\| u \|_{W^{1,q}_0(\Omega)} \leq C \| f \|_{-1,q},
\]

where \( C = C(\alpha, \beta, \Omega, N) \).

Now, to fix henceforth a Sobolev space \( W^{1,q}(\Omega) \), we define the open subset \( Q \subset C(\bar{\Omega}) \) as

\[
Q = \{ R \in C(\bar{\Omega}) : R(x) > 0 \quad \forall x \in \Omega \},
\]

and set \( q > 2 \) given by Lemma 1 with \( \alpha = m_0^3 m_3 \min\{m_1,1\} \) and \( \beta = M_0^3 M_3 \max\{M_1,1\} \).

We define also the mapping

\[
A : \ Q \times C(\bar{\Omega}) \rightarrow C(\bar{\Omega}) \quad (R_1, R_2) \rightarrow A_1(R_1) + A_2(R_1, R_2),
\]

where \( A_1 : Q \rightarrow C(\bar{\Omega}) \) is such that \( A_1(R_1) \) is the unique solution of the elliptic problem

\[
\nabla \cdot (\alpha R_1 \nabla u) = \nabla \cdot (U h f_4(R_1)) \quad \text{in} \quad \Omega,
A_1(R_1) = 0 \quad \text{on} \quad \partial \Omega,
\]

and \( A_2 : Q \times C(\bar{\Omega}) \rightarrow C(\bar{\Omega}) \) is such that \( A_2(R_1, R_2) \) is the unique solution of the elliptic problem

\[
\nabla \cdot (\beta R_1 \nabla u) = h f_5(R_1) R_2 \quad \text{in} \quad \Omega,
A_2(R_1, R_2) = 0 \quad \text{on} \quad \partial \Omega.
\]

**Remark 2.** Both the solutions of (2.7) and (2.8) are in \( C(\bar{\Omega}) \) since \( W^{1,p}(\Omega) \subset C(\bar{\Omega}) \) continuously for any \( p > N \).

**Remark 3.** For any \( R_1 \in Q \), \( A_2(R_1, \cdot) \) is a bounded linear operator.

**Lemma 2.** The application \( A \) is of class \( C^2 \) from \( Q \times C(\bar{\Omega}) \) into \( C(\bar{\Omega}) \).

**Proof.** Let us define \( \phi : Q \times C(\bar{\Omega}) \times W^{1,q}_0(\Omega) \rightarrow W^{-1,q}(\Omega) \) by

\[
\phi(R_1, R_2, p) = \nabla \cdot (\alpha R_1 \nabla p) - \nabla \cdot (U h f_4(R_1)) - h f_5(R_1) R_2.
\]

We show first that \( \phi \) is of class \( C^2 \). Since \( f_3, f_4 \) and \( f_5 \) are of class \( C^2 \), it is enough to prove that the application \( \phi_1 : C(\bar{\Omega}) \times W^{1,q}_0(\Omega) \rightarrow W^{-1,q}(\Omega) \) defined by

\[
\phi_1(\xi_1, \xi_2, \xi_3, \xi_4, w) = \nabla \cdot (\alpha \nabla w) - \nabla \cdot (U h \xi_2) - h \xi_3 \xi_4
\]

is of class \( C^2 \), which follows from observing that its first and third terms are quadratic and the second one is linear.

By the Lax-Milgram Theorem and Lemma 1 we have also that the partial derivative

\[
\frac{\partial \phi}{\partial p}(R_1, R_2, p) (z) : \ W^{1,q}_0(\Omega) \rightarrow W^{-1,q}(\Omega)
\]

is an isomorphism. Therefore, the result follows from noticing that \( \phi(R_1, R_2, A(R_1, R_2)) = 0 \quad \forall (R_1, R_2) \in Q \times C(\bar{\Omega}) \) and applying the Implicit Function Theorem\(^2\) to the application \( \phi \).

For a linear operator \( L \) we denote by \( V_p(L) \) its set eigenvalues and by \( \text{Sp}(L) \) its spectrum.

**Lemma 3.** Let \( X \) be a Banach space, let \( A \) be a bounded linear operator on \( X \) and \( \epsilon > 0 \). Then there exists \( \delta > 0 \) such that, if \( B \) is a bounded linear operator on \( X \) and \( \| A - B \| < \delta \), then for every \( \lambda \in \text{Sp}(B) \) there exists \( \xi \in \text{Sp}(A) \) such that \( | \lambda - \xi | < \epsilon \).

For a detailed proof of the previous Lemma the reader is referred to Lemma 3 in [29].

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\(^1\)Henceforth we denote \( \nabla \cdot \) by \( \nabla \cdot \).

\(^2\)e.g., [28]
3 Well-posedness with inertial terms

3.1 Existence of a local solution

Let us denote $R_1 = R$, $R_2 = \frac{\partial R}{\partial t}$ and $\tilde{R} = \left( \frac{R_1}{R_2} \right)$. Then, the problem \[ 2.1, 2.2, 2.3 \] can be rewritten as

\[
\begin{aligned}
\frac{d\tilde{R}}{dt} &= F(\tilde{R}), \\
\tilde{R}(0) &= \tilde{R}_0,
\end{aligned}
\]  

(3.1)

where $\tilde{R}_0 = \left( \frac{r_1}{r_2} \right) \in Q \times C(\bar{\Omega})$ and $F : Q \times C(\bar{\Omega}) \rightarrow (C(\bar{\Omega}))^2$ with

\[
F(R_1, R_2) = \left( -\frac{3}{2} \frac{R_2}{R_1} - R_2 f_2(R_1) + \frac{f_1(R_1) - A(R_1, R_2)}{R_1} \right).
\]  

(3.2)

By means of Lemma \[ \ref{lemma2} \] we have that $F$ is of class $C^2$. Thus, from the Cauchy-Lipschitz Theorem we obtain the next local existence and uniquenes result:

**Theorem 1.** There exists $T > 0$ such that the problem \[ \ref{3.1} \] has a unique solution in $C^3 \left( [0, T]; Q \times C(\bar{\Omega}) \right)$. 

3.2 Existence of stationary solutions

Observe that a stationary solution $(R_s, p_s)$ of problem \[ 2.1, 2.2 \] satisfies $p_s = f_1(R_s)$. For the next result we denote $h^+ = h - h_0$ (notice that $h^+ = 0$ if and only if $h$ is constant). Thus $(R_s, p_s)$ is solution of the system

\[
\nabla \cdot \left( (h^+ + h_0)^3 f_3(R_s) \nabla p_s \right) = \nabla \cdot ( U (h^+ + h_0) f_4(R_s) ) \quad \text{in} \ Ω,
\]

\[
p_s = f_1(R_s) \quad \text{in} \ Ω,
\]

\[
p_s = 0 \quad \text{on} \ ∂Ω.
\]  

(3.3)

Notice that in the particular case $h^+ = 0$ or $U = 0$, $(R_s, p_s) = (\tilde{R}, 0)$ is solution of \[ \ref{3.3} \], with $\tilde{R}$ given in (H1).

**Theorem 2.** Fix $U \in \mathbb{R}^2$ and $h_0 > 0$. Then the problem \[ \ref{3.3} \] has a unique solution $(R_s, p_s)$ with $R_s > 0$ whenever $\|h^+\|_\infty$ is small enough. Moreover, the solution $(R_s, p_s)$ depends continuously on $h^+$.

**Proof.** First we use the relation $p_s = f_1(R_s)$ to rewrite the stationary problem. Since $\nabla p_s = f_1'(R_s) \nabla R_s$, making the change of variable $R_s = \tilde{R} + \xi$ problem \[ \ref{3.3} \] can be written in function of $\xi$ as

\[
\nabla \cdot \left( (h^+ + h_0)^3 a_0(\xi) \nabla \xi \right) = \nabla \cdot ( U (h^+ + h_0) b_0(\xi) ) + \nabla \cdot ( U h_0 b_0(\xi) ) \quad \text{in} \ Ω,
\]

\[
\xi > -\tilde{R} \quad \text{in} \ Ω,
\]

\[
\xi = 0 \quad \text{on} \ ∂Ω,
\]  

(3.4)

where $a_0(\xi) = -f_3(\tilde{R} + \xi) f_1'(\tilde{R} + \xi)$, and $b_0(\xi) = f_4(\tilde{R} + \xi)$. We introduce the set

\[
W = \left\{ \xi \in W_0^{1, q}(Ω) : \xi > -\tilde{R} \text{ a.e. on } Ω \right\},
\]

which is open since the continuous embedding $W_0^{1, q} \subset C(\bar{Ω})$, and the application

\[
\phi_2 : W \times L^\infty(Ω) \longrightarrow W^{-1, q}(Ω) \quad (ξ, δ) \quad \mapsto \quad \nabla \cdot \left( (δ + h_0)^3 a_0(ξ) \nabla ξ \right) + \nabla \cdot ( U δ b_0(ξ) ) + \nabla \cdot ( U h_0 b_0(ξ) ).
\]

(3.5)

Using an argument analogous to the one used in Lemma \[ \ref{lemma2} \] to prove that $φ^1$ is of class $C^2$, it is possible to prove that $φ_2$ is of class $C^2$. Now noticing that $\phi_2(0, 0) = 0$, let us assume that $\frac{∂φ_2}{∂ξ}(0, 0)$ is invertible. Then, by means of the Implicit Function Theorem, we have that...
• \( \exists V_1 \subset W \) neighborhood of 0 on \( W_{01}^1 (\Omega) \); \( V_2 \) neighborhood of 0 on \( L^\infty (\Omega) \);

• \( \exists \psi : V_2 \mapsto V_1 \) function of class \( C^1 \) such that \( \forall \delta \in V_2, \psi (\delta) \) is solution of problem (3.4). Equivalently, \( (R_s, p_s) = (R + \psi (\delta), f_1 (R + \psi (\delta))) \) is solution of problem (3.3),

which is the result we want as the existence of \( V_2 \) can also be described as \( \| h^+ \|_\infty \) small enough.

It only remains to show that \( \frac{\partial \phi_2}{\partial \xi} (0, 0) \) is invertible. Indeed, we have \( \forall z \in W_{01}^1 (\Omega) \):

\[
\frac{\partial \phi_2}{\partial \xi} (0, 0) (z) = \nabla \cdot \left((h_0 + \delta)^3 (a_0) (\xi) \nabla z + a_0' (\xi) \nabla \xi) + (h_0 + \delta) b_0' (\xi) U z \right) \big|_{(\xi, \delta) = (0, 0)}
\]

Fixing an arbitrary \( g \in W^{-1, q} (\Omega) \) and denoting \( \ell = h_0 b_0' (0) U \in \mathbb{R}^2 \) we will prove that there exists a unique \( z \in W_{01}^1 (\Omega) \) such that

\[
\nabla \cdot (h_0^3 a_0 (0) \nabla z + \ell z) = g \text{ in } \Omega.
\]

Since \( g \in H^{-1} (\Omega) \), \( a_0 (0) > 0 \) and \( h_0 > 0 \) (see (H1) and (H3)), by means of the Lax-Milgram Theorem the variational problem

\[
- \int_\Omega (h_0^3 a_0 (0) \nabla z + \ell z) \cdot \nabla \phi d\Omega = \int_\Omega g \phi d\Omega \quad \forall \phi \in H_0^1 (\Omega),
\]

has a unique solution \( z \in H_0^1 (\Omega) \). Moreover, from the continuous inclusion \( H^1 (\Omega) \subset L^q (\Omega) \), we have \( \nabla \cdot (\ell z) \in W^{-1, q} (\Omega) \) and thus by Lemma 1 we obtain \( z \in W_{01}^1 (\Omega) \).

A proof analogous to the one of Theorem 2 may be written for the next result:

**Theorem 3.** Fix \( h \in B_{\mu_0, \mu_0} \), \( 0 < \mu_0 < \mu_0 \). Then there exists \( \epsilon (h) > 0 \) such that the problem (3.3) has a unique solution \((R_s, p_s)\) with \( R_s > 0 \) whenever \( ||U|| < \epsilon (h) \). Moreover, the solution \((R_s, p_s)\) depends continuously on \( U \).

### 3.3 Stability Analysis

Recalling the application \( F \) given by (3.2) and the stationary solution \((R_s, p_s)\) introduced in the previous section, we denote by \( \mathcal{L}_F \) the differential of \( F \) at \((R_s, 0)\), i.e.,

\[
\mathcal{L}_F : \begin{cases} (C (\Omega))^2 \rightarrow (C (\Omega))^2 \\
(S_1, S_2) \rightarrow DF (R_s, 0) (S_1, S_2)
\end{cases}
\]

We will show the stability of the stationary solution in some particular cases. For this, we will show that the spectrum of \( \mathcal{L}_F \) is such that \( \Re (\lambda) < 0 \ \forall \lambda \in \text{Sp} (\mathcal{L}_F) \setminus \{0\} \). Previously, we perform some computations.

Recalling that \( f_1 (R_s) = p_s = A (R_s, 0) \) we obtain:

\[
(\mathcal{L}_F (S_1, S_2))_1 = S_2,
(\mathcal{L}_F (S_1, S_2))_2 = \frac{f'_1 (R_s) S_1}{R_s} - \frac{1}{R_s} \left(D_1 A (R_s, 0) (S_1) + D_2 A (R_s, 0) (S_2)\right) + \frac{1}{R_s^2} A (R_s, 0) S_1 - f_2 (R_s) S_2.
\]

Now, since \( A_2 (R, 0) = 0 \) for any \( R \) in \( Q \), we have that \( D_1 A (R_s, 0) = DA_1 (R_s) \). With this, deriving (2.7) with respect to \( R_1 \) and denoting \( \pi_1 (S_1) = D_1 A (R_s, 0) (S_1) \) we obtain that \( \pi_1 (S_1) \) satisfies

\[
- \nabla \cdot \left(h^3 f_3 (R_s) \nabla \pi_1 (S_1)\right) = \nabla \cdot \left(h^3 f'_3 (R_s) S_1 \nabla A_1 (R_s) - Uh f'_3 (R_s) S_1\right),
\]

\[\pi_1 (S_1) = 0 \text{ on } \partial \Omega.\]

Similarly, we have \( D_2 A (R_s, 0) (S_2) = DA_2 (R_s, 0) (S_2) = A_2 (R_s, S_2) \). Thus, denoting \( \pi_2 (S_2) = D_2 A (R_s, 0) (S_2) \) we have that \( \pi_2 (S_2) \) accomplishes

\[
- \nabla \cdot \left(h^3 f_3 (R_s) \nabla \pi_2 (S_2)\right) = - h f_3 (R_s) S_2 \text{ in } \Omega,
\]

\[\pi_2 (S_2) = 0 \text{ on } \partial \Omega.\]

For the next results we denote \( b_1 = - f'_1 (R) R^{-1} > 0, b_2 = f_2 (R) > 0, b_r = 1/R, b_3 = f_3 (R), b_4 = - f'_4 (R) \) and \( b_5 = - f_5 (R) \), all positive constants as follows from (H1)-(H5).
Remark 4. If \( h^+ = 0 \) or \( U = 0 \) we have \( A_1 (R_\ast) = 0 \). Thus \( A(R_\ast, 0) = p_\ast = 0 \), \( R_\ast = \bar{R} \) and \( \mathcal{L}_F (S_1, S_2) \) can be written
\[
\mathcal{L}_F (S_1, S_2) = B \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} - b_r \begin{pmatrix} 0 \\ \pi_1 (S_1) + \pi_2 (S_2) \end{pmatrix},
\]
where \( B = \begin{pmatrix} 0 & 1 \\ -b_1 & -b_2 \end{pmatrix} \), and Eq. (3.9) reads
\[
-b_3 \nabla \cdot (h^3 \nabla \pi_1 (S_1)) = b_4 \nabla \cdot (U h S_1) \quad \text{in } \Omega, \\
\pi_1 (S_1) = 0 \quad \text{on } \partial \Omega.
\]
We denote by \( \{ \lambda^B_1, \lambda^B_2 \} \) the set of eigenvalues of \( B \) and notice that \( \text{Re} (\lambda^B_1) < 0 \) and \( \text{Re} (\lambda^B_2) < 0 \).

Lemma 4. Let \( h^+ = 0 \) or \( U = 0 \). Then
\[
\text{Sp} (\mathcal{L}_F) \subset \text{Vp} (\mathcal{L}_F) \cup \{ \lambda^B_1, \lambda^B_2 \}.
\]
Moreover if \( \lambda \in \text{Vp} (\mathcal{L}_F) \setminus \{ \lambda^B_1, \lambda^B_2 \} \) with associated eigenfunction \( (S_1, S_2) \in C (\Omega)^2 \) then \( (S_1, S_2) \in H^1_0 (\Omega)^2 \), \( S_2 = \lambda S_1 \) and \( S_1 \) is solution of the problem
\[
\frac{b_3}{b_r} \xi (\lambda) \nabla \cdot (h^3 \nabla S_1) = b_4 U \cdot \nabla (h S_1) + \lambda b_3 h S_1 \quad \text{in } \Omega, \\
S_1 = 0 \quad \text{on } \partial \Omega,
\]
where \( \xi (\lambda) = \lambda^2 + b_2 \lambda + b_1 \) with roots \( \{ \lambda^B_1, \lambda^B_2 \} \).

Proof. Remind that \( p_\ast = A(R_\ast, 0) = 0 \) and \( R_\ast = \bar{R} \). For any \( \lambda \in \mathbb{C} \setminus \{ \lambda^B_1, \lambda^B_2 \} \), from Eq. (3.11) we have
\[
(L_F - \lambda I) \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} = (B - \lambda I) \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} - b_r (B - \lambda I)^{-1} \begin{pmatrix} 0 \\ \pi_1 (S_1) + \pi_2 (S_2) \end{pmatrix}.
\]
Since the map \( (S_1, S_2) \mapsto \pi_1 (S_1) + \pi_2 (S_2) \) is compact, by means of the Fredholm's Alternative Theorem the mapping at the right hand side of this equation (from \( C (\Omega)^2 \) into itself) is injective if and only if it is surjective, from where we have the \( \text{Sp} (\mathcal{L}_F) \subset \text{Vp} (\mathcal{L}_F) \cup \{ \lambda^B_1, \lambda^B_2 \} \).

Fix now \( \lambda \in \text{Vp} (\mathcal{L}_F) \setminus \{ \lambda^B_1, \lambda^B_2 \} \) with associated eigenvector \( (S_1, S_2) \neq (0, 0) \) so we can write
\[
S_2 = \lambda S_1, \\
-b_1 S_1 - b_2 S_2 - b_r [\pi_1 (S_1) + \pi_2 (S_2)] = \lambda S_2.
\]
Then we obtain
\[
\pi_2 (\lambda S_1) + \pi_1 (S_1) = \frac{- \xi (\lambda)}{b_r} S_1.
\]
Since \( \xi (\lambda) \neq 0 \) and from the definitions of \( \pi_1 \) and \( \pi_2 \) we deduce that \( (S_1, S_2) \in H^1_0 (\Omega)^2 \). Thus, using this last equation, Eq. (3.10) and Eq. (3.12) we obtain the Eqs. (3.13)-(3.14).

Theorem 4. Let \( h \) be as in Theorem 3. Then there exists \( \epsilon = \epsilon (h) > 0 \) such that \( \| U \|_\infty < \epsilon \) is asymptotically stable for the evolution problem (3.1).

Proof. Assume first \( U = 0 \) and denote \( \mathcal{L}_F^0 = \mathcal{L}_F |_{U=0} \). Then due to Lemma 4 it is enough to study the eigenvalues of \( \mathcal{L}_F \). Thus, take \( \lambda \in \text{Vp} (\mathcal{L}_F) \setminus \{ \lambda^B_1, \lambda^B_2 \} \) with associated eigenfunction \( (S_1, S_2) \), from Lemma 4 we have \( S_2 = \lambda S_1 \) and \( S_1 \in H^1_0 (\Omega) \) accomplishing Eqs. (3.13) and (3.14), which read
\[
\frac{b_3}{b_r} \xi (\lambda) \nabla \cdot (h^3 \nabla S_1) = \lambda b_3 h S_1 \quad \text{in } \Omega, \\
S_1 = 0 \quad \text{on } \partial \Omega.
\]
Since \( \xi (\lambda) \) is not null we deduce that \( \lambda \neq 0 \), otherwise \( (S_1, S_2) \) would be null. Then we obtain that \( S_1 \) accomplishes the next variational formulation
\[
- \frac{b_3}{b_r} \frac{\xi (\lambda)}{\lambda} \int_{\Omega} h^3 \nabla S_1 \nabla \phi \, d\Omega = b_5 \int_{\Omega} h S_1 \phi \, d\Omega \quad \forall \phi \in H^1_0 (\Omega).
\]
Taking \( \phi = S_1 \) we obtain that \( \gamma = -\xi (\lambda) / \lambda \in \mathbb{R}^+ \) and since \( \lambda \) accomplishes the equation \( \lambda^2 + (\gamma + b_2) \lambda + b_1 = 0 \) we conclude that \( \text{Re}(\lambda) < 0 \). We have shown the result for the case \( U = 0 \).

For the general case, we observe from Theorem 3 that the mapping \( U \rightarrow \mathbb{R}_e(U) \) is continuous in a neighborhood \( V_1 \ni 0 \in \mathbb{R}^s \), thus if \( U \rightarrow 0 \) in \( \mathbb{R}^s \) then \( \| DF(\mathbb{R}_e(U), 0) - DF(\mathbb{R}, 0) \|^2 \rightarrow 0 \) in the space of linear continuous operators from \( C (\bar{\Omega}) \) into itself. Then the result follows from Lemma 3.

We give now a result of instability for \( \|U\| \) big enough.

**Theorem 5.** Set \( h^+ = 0 \) and consider the one-dimensional case \( (N = 1) \). Then there exists \( M > 0 \) such that if \( \|U\|_\infty > M \) the solution \((R_e, p_u)\) of problem (3.3) is asymptotically unstable for the evolution problem (3.1).

**Proof.** Let us assume \( \Omega = [0, 1] \). Due to Lemma 4 it is enough to study the eigenvalues of \( \mathcal{L}_F \). Fix now \( \lambda \in \mathcal{V}(\mathcal{L}_F) \setminus \{\lambda_b, \lambda_b^2\} \) with associated eigenvector \((S_1, S_2) \neq (0, 0)\). Now defining \( \gamma_1, \gamma_2 \in \mathbb{C} \) by

\[
\gamma_1 = -\frac{b_4 b_5}{h_0^2 b_5 \xi (\lambda)}, \quad \gamma_2 = -\frac{b_4 b_5 r_1 \lambda}{h_0^2 b_5 \xi (\lambda)},
\]

from Eqs. (3.13), (3.14) we deduce that \( S_1 \) satisfies

\[
S_1'' + \gamma_1 U S_1' + \gamma_2 S_1 = 0, \quad S_1 (0) = S_1 (1) = 0.
\]

We deduce from this that \( \lambda \neq 0 \). In fact, if \( \lambda = 0 \) then \( \gamma_2 = 0 \) and we deduce that \( S_1 = 0, S_2 = \lambda S_1 = 0 \), which is not possible. Denoting by \( r_1, r_2 \) the roots of the characteristic polynomial \( P(\lambda) = r^2 + \gamma_1 U r + \gamma_2 \) of the last equation. We notice that \( r_1 \neq r_2 \), otherwise \( S_1 \) would be null, then \( S_1 \) can be written

\[
S_1 (x) = C_1 \exp (r_1 x) + C_2 \exp (r_2 x).
\]

Thus the boundary conditions imply

\[
C_1 + C_2 = 0, \quad C_1 \exp r_1 + C_2 \exp r_2 = 0.
\]

Thus, since \((C_1, C_2) \neq (0, 0)\) we have

\[
\det \left( \begin{array}{cc} 1 & 1 \\ \exp r_1 & \exp r_2 \end{array} \right) = 0,
\]

hence \( r_1 \) and \( r_2 \) satisfy the equation \( r_2 - r_1 = 2k \pi i \forall k \in \mathbb{N}^* \), from which we deduce that

\[
(r_1 + r_2)^2 - 4r_1 r_2 = -4k^2 \pi^2, \quad \forall k \in \mathbb{N}^*.
\]

Using the fact that the \( r_1 + r_2 = -\gamma_1 U \) and \( r_1 r_2 = \gamma_2 \) we obtain that \( \lambda \) is root of the fourth degree polynomial given by

\[
P_k (\lambda) = 4k^2 \pi^2 \lambda^4 + (4 \sigma_2 + 8k^2 \sigma_2) \lambda^3 + (4 \sigma_2 \sigma_2 + 4 \pi^2 k^2 (b_2^2 + b_1 h)) \lambda^2 +
+ (4 \sigma_2 \sigma_2 + 8 \pi^2 k^2 b_1 b_2) \lambda + 4 \pi^2 k^2 b_2^2 + \sigma_1 U^2,
\]

rewritten as

\[
P_k (\lambda) = a_0 \lambda^4 + b_0 \lambda^3 + a_1 \lambda^2 + b_1 \lambda + a_2,
\]

where \( \sigma_1 = \frac{b_2^2 \pi^2 k^2}{b_5 h_0^2} \) and \( \sigma_2 = \frac{b_0 b_4 b_5}{b_3 h_0^2} \) are both positive constants. Let us now denote the Hurwitz determinants associated to \( P_k \):

\[
\Delta_1 = \det (b_0), \quad \Delta_2 = \det \begin{vmatrix} b_0 & b_1 \\ a_0 & a_1 \end{vmatrix}, \quad \Delta_3 = \det \begin{vmatrix} b_0 & b_1 & 0 \\ a_0 & a_1 & a_2 \\ 0 & b_0 & b_1 \end{vmatrix}, \quad \Delta_4 = \det \begin{vmatrix} b_0 & b_1 & 0 & 0 \\ a_0 & a_1 & a_2 & 0 \\ 0 & b_0 & b_1 & 0 \\ 0 & a_0 & a_1 & a_2 \end{vmatrix}.
\]

We have \( \Delta_1 = 4 \sigma_2 + 8 \pi^2 k^2 b_2 > 0 \), \( \Delta_2 = (4 \sigma_2 + 8 \pi^2 k^2 b_2) (4 \sigma_2 \sigma_2 + 4 \pi^2 k^2 (b_2 + b_1)) > 0 \) and

\[
\Delta_3 = (320 b_1 b_2^2 \pi^2 k^2 - 16 \sigma_1 U^2) \sigma_2^2 + (512 b_1 b_2^2 \pi^4 k^4 - 64 b_2 \pi^2 \sigma_1 U^2) \sigma_2 - 64 b_2^4 \pi^4 k^4 \sigma_1 U^2^2 + 256 b_1 b_2^2 \pi^6 k^6 + 64 b_1 b_2 \sigma_3^2.
\]

Notice that \( \Delta_4 = a_2 \Delta_3 \). Thus, \( \text{sign} (\Delta_4) = \text{sign} (\Delta_3) \). According to the Routh-Hurwitz Theorem [60], the number of roots of the polynomial \( P_k \) with positive real part is equal to the number of changes of sign in the sequence \( \{\Delta_1, \Delta_2, \Delta_3, \Delta_4\} \). Therefore, the proof ends by noticing that \( \Delta_3 \) is negative for \( \|U\| \) big enough.
4 Well-posedness without inertial terms

Disregarding the inertial terms in Eq. (2.1) (as done in [7, 12, 31]) we obtain the following simplified version of the Rayleigh-Plesset equation

$$\frac{\partial R}{\partial t} = \frac{f_1(R) - p}{R f_2(R)},$$

(4.1)

along the initial condition

$$R(x, 0) = r_1(x) \quad \forall x \in \Omega,$$

(4.2)

where $r_1 \in C(\bar{\Omega})$ known and $p \in W^{1,q}_0(\Omega)$ is the solution of (2.2).

4.1 Existence of a local solution

Next we prove that from (4.1) we can express $\partial R/\partial t$ as a function of $R$. Denoting $R_1 = R$, $R_2 = \frac{\partial R}{\partial t}$, we recall the decomposition

$$p = A(R_1, R_2) = A_1(R_1) + A_2(R_1, R_2),$$

with $A_1$ and $A_2$ as in (2.7) and (2.8) respectively. Now, defining $\Pi : Q \times C(\bar{\Omega}) \mapsto C(\bar{\Omega})$ by

$$\Pi(R_1, R_2) = \frac{f_1(R_1) - A_1(R_1) - A_2(R_1, R_2)}{R_1 f_2(R_1)},$$

(4.3)

we have the next result:

**Lemma 5.** Given $R \in Q$, there exists a unique $G(R) \in C(\bar{\Omega})$ such that

$$G(R) = \Pi(R, G(R)),$$

and the mapping $R \mapsto G(R)$ is of class $C^2$.

**Proof.** Let us fix $R \in Q$, we will show that there exists a unique $S \in C(\bar{\Omega})$ such that $S = \Pi(R, S)$. Using (4.3), we first notice that the equation $S = \Pi(R, S)$ is equivalent to

$$S + \frac{A_2(R, S)}{R f_2(R)} = \frac{f_1(R) - A_1(R)}{R f_2(R)}.$$

We denote by $J : C(\bar{\Omega}) \mapsto C(\bar{\Omega})$ the linear mapping $S \mapsto S + A_2(R, S) / (R f_2(R))$. To prove the existence of a unique solution for the last equation we will show that $J$ is bijective, which will give us the existence of $G$ by taking $G(R) = S$. Now, since the mapping $S \mapsto A_2(R, S)$ is compact, by means of the Fredholm Alternative Theorem it is enough to prove that $J$ is injective. Indeed, let us take $w \in C(\bar{\Omega})$ such that $J(w) = 0$, then we have

$$R f_2(R) w + A_2(R, w) = 0.$$

Multiplying this equation by $-f_5(R) h w$ and integrating by parts we obtain

$$\int_{\Omega} R f_2(R) (-f_5(R)) h w^2 \, d\Omega + \int_{\Omega} (-f_5(R)) h A_2(R, w) \, w \, d\Omega = 0.$$  

(4.4)

Now, multiplying (2.8) by $A_2(R_1, R_2)$, integrating and using (H5) we have for any $(R_1, R_2) \in Q \times C(\bar{\Omega})$

$$\int_{\Omega} (-f_5(R_1)) h A_2(R_1, R_2) R_2 \, d\Omega \geq 0.$$  

(4.5)

Taking $R_1 = R$ and $R_2 = w$ in the last equation and carrying that into Eq. (4.4) we obtain $w = 0$, so we conclude $J$ is injective.

Next, we prove that $G$ is of class $C^2$. Let us define the mapping $\Phi : Q \times C(\bar{\Omega}) \mapsto C(\bar{\Omega})$ such that

$$\Phi(R, S) = S - \Pi(R, S),$$

(4.6)
which is of class $C^2$ since all the involved functions are regular enough. Now, fixing some arbitrary $(R_0, S_0) \in Q \times C(\bar{\Omega})$ such that $\Phi (R_0, S_0) = 0$ we have for any $w \in C(\bar{\Omega})$

$$\frac{\partial \Phi}{\partial S} (R_0, S_0) (w) = w + \frac{A_2 (R_0, w)}{R_0 f_2 (R_0)} = J (w).$$

From where we obtain that $\frac{\partial \Phi}{\partial S} (R_0, S_0)$ is an automorphism on $C(\bar{\Omega})$. Thus, we conclude that $G$ is of class $C^2$ by means of the Implicit Function theorem.

**Theorem 6.** There exists $T > 0$ such that problem (2.2)-(4.1)-(4.2) has a unique solution in $C^3 ([0, T]; Q)$.

**Proof.** The result follows directly from applying the Cauchy-Lipschitz Theorem to the equivalent evolution problem

$$\frac{\partial R}{\partial t} = G (R),$$

along the initial condition (4.2).

### 4.2 Stability analysis

Let us notice the stationary solution of (4.1) is also the couple $(R_s, p_s)$ obtained in Section 3.2. Here we study the stability of that solution for the evolution problem (4.6).

Here we denote the derivative

$$\mathcal{L}_G : C (\bar{\Omega}) \ni w \mapsto C (\bar{\Omega}) = D G (R_s) (w).$$

Using the definition of $\Pi (R, S)$ we compute the derivative with respect to $R$ in the equation $S = \Pi (R, S)$ and make the evaluation at $R = R_s$, $S = 0$, so we obtain that $\mathcal{L}_G (w)$ satisfies:

$$R_s f_2 (R_s) \mathcal{L}_G (w) - f'_1 (R_s) w + \pi_1 (w) + \pi_2 (\mathcal{L}_G (w)) = 0,$$

with $\pi_1$ and $\pi_2$ as in Eqs. (3.9) and (3.10) respectively.

For the next results we denote $d_1 = -f'_1 (\bar{R}) / (\bar{R} f_2 (\bar{R}))$, $d_2 = (\bar{R} f_2 (\bar{R}))^{-1}$, $d_3 = \bar{R} f_3 (\bar{R}) f_2 (\bar{R})$, $d_4 = -f'_4 (\bar{R})$ and $d_5 = -f_5 (\bar{R})$. All these constants are positive as follows from (H1)-(H5).

**Lemma 6.** Assume $h^+ = 0$ or $U = 0$. Then

$$\text{Sp} (\mathcal{L}_G) \subset \text{Vp} (\mathcal{L}_G) \cup \{-d_1 \}.$$

Moreover, if $w \in C (\bar{\Omega})$ is an eigenvector of $\mathcal{L}_G$ with associated eigenvalue $\lambda$, then $w \in H^1_0 (\Omega)$ and it satisfies

$$d_3 (d_1 + \lambda) \nabla \cdot (h^3 \nabla w) = d_4 U \cdot \nabla (hw) + \lambda d_5 hw \quad \text{in } \partial \Omega,$nabla \cdot (h^3 w) = 0 \quad \text{on } \partial \Omega. \quad (4.9)$$

**Proof.** From Remark 4 we have $(R_s, p_s) = (\bar{R}, 0)$. Putting this into Eq. (4.8) we obtain that for any $\lambda \in \mathbb{C}$:

$$\mathcal{L}_G (w) - \lambda w = (\lambda + d_1) \left[ -w - \frac{d_2}{\lambda + d_1} [\pi_1 (w) + \pi_2 (\mathcal{L}_G (w))] \right], \quad (4.10)$$

with $\pi_1 (w)$ given by (3.12). Since the map $w \mapsto \pi_1 (w) + \pi_2 (\mathcal{L}_G (w))$ is compact, by means of the Fredholm’s Alternative Theorem we obtain that $\text{Sp} (\mathcal{L}_G) \subset \text{Vp} (\mathcal{L}_G) \cup \{-d_1 \}$.

Take now $w \in C (\bar{\Omega})$ eigenvector of $\mathcal{L}_G$ with associated eigenvalue $\lambda$, carrying this into equation (4.10) we obtain

$$\lambda + d_1 - \frac{d_2}{\lambda + d_1} w = \pi_1 (w) + \lambda \pi_2 (w).$$

then $w \in H^1_0 (\Omega)$ and Equation (4.9) follows from this last relation and Eqs. (3.12) and (3.10).

**Theorem 7.** For every $U \in \mathbb{R}^2$ there exists $\epsilon = \epsilon (U) > 0$ such that if $\| h^+ \|_\infty < \epsilon$, then the solution $(R_s, p_s)$ of problem (3.3) is asymptotically stable for the evolution problem (2.2)-(4.1)-(4.2).
Proof. Let us assume first that $h^+ = 0$. By Lemma 6 it is enough to study the eigenvalues of $\mathcal{L}_G$. Hence, take $\lambda \in \mathbb{C} \setminus \{-d_1\}$ such that $\mathcal{L}_G(w) = \lambda w$ for some $w \neq 0$. Then (4.9) reads
\[
\begin{align*}
    h^2 d_3 (d_1 + \lambda) \Delta w &= d_4 U \cdot \nabla w + \lambda d_5 w & \text{in } \Omega, \\
    w &= 0 & \text{on } \partial \Omega.
\end{align*}
\] (4.11)

We notice that $\lambda \neq 0$. In fact, if $\lambda = 0$ then multiplying the Eq. (4.11) by $w$ and integrating by parts we obtain $w = 0$, which is not possible. Decomposing $\lambda = \lambda_1 + i \lambda_2$ and $w = w_1 + i w_2$, writing the differential equation for the real and imaginary parts we obtain the equations
\[
\begin{align*}
    -h^2 d_3 (d_1 + \lambda_1) \Delta w_1 + h^2 d_3 \lambda_2 \Delta w_2 &= d_4 U \cdot \nabla w_1 + d_5 (\lambda_1 w_1 - \lambda_2 w_2) = 0, \\
    -h^2 d_3 (d_1 + \lambda_1) \Delta w_2 - h^2 d_3 \lambda_2 \Delta w_1 &= d_4 U \cdot \nabla w_2 + d_5 (\lambda_1 w_2 + \lambda_2 w_1) = 0.
\end{align*}
\]

Multiplying the first equation by $w_1$, the second equation by $w_2$ and integrating by parts we may obtain
\[
\begin{align*}
    h^2 d_3 (d_1 + \lambda_1) \int_\Omega |\nabla w_1|^2 d\Omega &= -h^2 d_3 \lambda_2 \int_\Omega \nabla w_2 \nabla w_1 d\Omega + d_5 \lambda_1 \int_\Omega w_1^2 d\Omega - d_5 \lambda_2 \int_\Omega w_2 d\Omega = 0, \\
    h^2 d_3 (d_1 + \lambda_1) \int_\Omega |\nabla w_2|^2 d\Omega &= h^2 d_3 \lambda_2 \int_\Omega \nabla w_2 \nabla w_1 d\Omega + d_5 \lambda_1 \int_\Omega w_2^2 d\Omega + d_5 \lambda_2 \int_\Omega w_1 d\Omega = 0.
\end{align*}
\]

Adding up both equations we have
\[
\begin{align*}
    h^2 d_3 (d_1 + \lambda_1) \int_\Omega (|\nabla w_1|^2 + |\nabla w_2|^2) d\Omega &= d_5 \lambda_1 \int_\Omega (|w_1|^2 + |w_2|^2) d\Omega = 0.
\end{align*}
\]

Observing that $\lambda_1 \geq 0$ implies $w = 0$, which is not possible, we conclude that $\text{Re} (\lambda) = \lambda_1 < 0$.

We have shown the stability for $h^+ = 0$. Now from Theorem 2 we have that the mapping $h^+ \mapsto R_s (h^+)$ is continuous in a neighborhood $V_1 \ni 0$ in $L^\infty (\Omega)$. Thus, if $h^+ \to 0$ in $L^\infty (\Omega)$ then
\[
\|DG (R_s (h^+), 0) - DG (R_s, 0)\| \to 0
\]
in the space of linear continuous operators from $C (\bar{\Omega})$ into itself. Then the result follows from Lemma 3.

Theorem 8. Fix $h \in B_{M_0} = \{0 < m_0 < M_0\}$. Then there exists $\epsilon > 0$ such that if $\|U\| < \epsilon$ then the solution $(R_s, p_s)$ of problem (3.3) is asymptotically stable for the evolution problem (2.2)-(4.1)-(4.2).

Proof. Let us assume first that $U = 0$. By Lemma 6 it is enough to study the eigenvalues of $\mathcal{L}_G$. Hence, take $\lambda \in \mathbb{C} \setminus \{-d_1\}$ such that $\mathcal{L}_G(w) = \lambda w$ for some $w \neq 0$. If $\lambda = 0$ then from Eq. (4.9) we obtain $w = 0$, which is a contradiction. Thus, we have $\lambda \neq 0$ and this time Eq. (4.9) in its variational version reads
\[
\begin{align*}
    -d_3 \frac{(\lambda + d_1)}{\lambda} \int_\Omega h^3 \nabla w \nabla \phi d\Omega &= d_5 \int_\Omega h w \phi d\Omega & \forall \phi \in H^1_0 (\Omega).
\end{align*}
\]

Along the same arguments used in Theorem 4 this implies $\lambda \in \mathbb{R}^-$. The result follows analogously to the end of Theorem 4 proof, this time using the continuity of the mapping $U \mapsto R_s (U)$ asserted in Theorem 3.

5 Numerical examples

In this section we show some numerical examples for the evolution problem (2.2)-(4.1)-(4.2). The numerical method employed consists in a Finite Volume Method to discretize Eq. (2.2) and a backward Euler scheme to discretize Eq. (4.1). For more details on the numerical method the reader is referred to [11]. The domain $\Omega = [0, 2\pi J_R] \times [0, B]$ is divided into cells of size $\Delta x_1 = 2\pi J_r / 512$ and $\Delta x_2 = 2B / 64$ in the $x_1$ and $x_2$ axis respectively, where $J_R = B = 25.4 \times 10^{-3}$ m and. The time step was taken as $\Delta t = 4 \times 10^{-4}$ s. The number of time steps, denoted by $N^*$, is taken big enough in order to observe temporal convergence for each case. Thus, $t^{N^*} = \Delta t N^*$ corresponds to the final time simulated. The gap function $h$ was set as $h(x_1, x_2) = h_0 (1 - \epsilon \cos(x_1))$ with $h_0 > 0$ and $\epsilon \in [0, 1]$ is the eccentricity. Dirichlet boundary conditions are imposed, reading
\[
p (x_1, 0) = p(x_1, B) = 0 & \forall x_1 \in [0, 2\pi J_R],
\]
and the next periodic conditions

$$p(0,x_2) = p(2\pi J_R,x_2), \quad \frac{\partial p}{\partial x_1}(0,x_2) = \frac{\partial p}{\partial x_1}(2\pi J_R,x_2) \quad \forall x_2 \in [0,B].$$

The initial conditions are $\dot{R}(x_1,x_2,t = 0) = R(x_1,x_2,t = 0)/R_0 = 1$ and $\frac{\partial R}{\partial t} = 0$ in $\Omega$.

The geometrical setting corresponds to a journal bearing device, which scheme is shown in Fig. 2. The physical parameters setting is given in Table 1. To present the results we will use the non-dimensional fields $\hat{p} = p/P_0$, $\hat{R} = R/R_0$, and the variables $\hat{x}_1 = x_1/J_R$ and $\hat{x}_2 = x_2/B$.

| Symbol | Value | Units | Description |
|--------|-------|-------|-------------|
| $\rho_\ell$ | 854 | kg/m$^3$ | Liquid density |
| $\mu_\ell$ | $7.1 \times 10^{-3}$ | Pa·s | Liquid viscosity |
| $\rho_g$ | 1 | kg/m$^3$ | Gas density |
| $\mu_g$ | $1.81 \times 10^{-5}$ | Pa·s | Gas viscosity |
| $k$ | 1.4 | | Gas polytropic exponent |
| $\sigma$ | $3.5 \times 10^{-2}$ | N/m | Liquid surface tension |
| $P_0$ | 1 | atm | Reference pressure |
| $p_\partial$ | 1 | atm | Pressure at the boundary |
| $R_0$ | $3.85 \times 10^{-7}$ | m | Bubbles’ equilibrium radius at 1 atm |
| $\alpha_0$ | 0.1 | | Initial gas fraction |
| $J_R$ | $25.4 \times 10^{-3}$ | m | Journal radius |
| $B$ | $25.4 \times 10^{-3}$ | m | Journal width |
| $h_0$ | $0.001 \times J_R$ | m | Journal clearance |
| $\epsilon$ | 0 − 1 | | Journal eccentricity |

Table 1: Parameter values for the Journal Bearing.
Time-convergence towards a stationary solution

For this cases we fix the eccentricity \( \epsilon = 0.3 \) and rotational speed \( \omega = 1000 \) rpm. In Fig. 3 we show an example of time-evolution for the fields \( \hat{p} \) and \( \hat{R} \) along the middle of the bearing (\( \hat{x}_2 = 0.5 \)). Three times steps are shown corresponding to the begging of the simulation. A time-convergence of the fields is observed towards a stationary solution, which is obtained for \( N^* = 1000 \).

For \( \alpha_0 = 0 \) the density and viscosity fields are constants and the Reynolds equation is independent from \( R \). Thus, this case corresponds to the usual incompressible stationary Reynolds equation disregarding cavitation effects (known as the Sommerfeld model). Comparing \( \hat{p}_s \) for \( \alpha_0 = 0 \) and \( \alpha_0 = 0.1 \), it can be noticed how cavitation, modeled by the expansion/compression of the bubbles, modifies the stationary pressure field.

We want to compare the present model with the classical meso model of Reynolds variational inequality [3]. This model is essentially based on the fact that the pressure does not fall below the so-called cavitation pressure, denoted \( p_{\text{cav}} \) [32]. In the present model, the stationary solution \( (\hat{R}_s, p_s) \) satisfies \( p_s = f_1(\hat{R}_s) \), so that \( \hat{p}_{\text{cav}} \) is chosen as

\[
\hat{p}_{\text{cav}} = \min_{r > 0} \hat{f}_1(r).
\]

In Fig. 3 we show also the solution for the Reynolds model. This solution, denoted \( \hat{p}_R \), is obtained taken \( \rho = \rho_\ell, \mu = \mu_\ell \) in the stationary Reynolds equation along the condition \( \hat{p}_R \geq \hat{p}_{\text{cav}} \). Notice that a small area where \( \hat{p}_R = \hat{p}_{\text{cav}} \) is observed.

Varying the eccentricity \( \epsilon \): loosing well-posedness

We perform a set of computations for increasing values of the eccentricity \( \epsilon \) using the data of Table 1. Until a value of \( \epsilon \) around 0.41 the convergence of the transient solution towards the stationary one is similar of those of Fig. 3. However, for \( \epsilon > 0.41 \) the time-convergence is no longer obtained. To explain this phenomena, we remark that as \( \epsilon \) increases the solution \( \hat{p}_s = f_1(\hat{R}_s) \) becomes closer to the minimum

\[
\hat{p}_{\text{cav}} = \min_{r > 0} \hat{f}_1(r) = \hat{f}_1(\hat{R}_{\text{cav}}),
\]

as it is shown in Fig. 4 for \( \hat{x}^* = (\hat{x}_1^*, \hat{x}_2^*) = (4.85, 0.5) \). Also, we observe that around \( (\hat{p}_{\text{cav}}, \hat{R}_{\text{cav}}) \) the assumption (H1) is not satisfied and the coefficient \( a_0 \) changes of sign in problem (3.4). Then, the solution of the stationary problem can not be guaranteed.

Remark 5. As shown above, increasing the eccentricity is related to the possible ill-posedness of the stationary problem. However, it is also associated to an increase of \( h^+ \), so that the relation of this
numerical results to the instability of the solution can be posed for the cases where no time-convergence is observed (here for $\epsilon > 0.41$).

Figure 4: Equilibrium curve $\hat{f}_1(\hat{R})$ and evolution of the state $(R(\hat{x}^*, t), p(\hat{x}^*, t))$ for several values of $\epsilon$.

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