VECTOR BUNDLES AND $SO(3)$-INVARIANTS
FOR ELLIPTIC SURFACES I

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1. Introduction.

Beginning with Donaldson’s seminal paper on the failure of the $h$-cobordism theorem in dimension 4 [4], the techniques of gauge theory have proved to be highly successful in analyzing the smooth structure of simply connected elliptic surfaces. Recall that a simply connected elliptic surface $S$ is specified up to deformation type by its geometric genus $p_g(S)$ and by two relatively prime integers $1 \leq m_1 \leq m_2$, the multiplicities of its multiple fibers. Here, if $p_g(S) = 0$, a surface $S$ such that $m_1 = 1$ is rational, and thus all surfaces $S$ with $p_g(S) = 0$ and $m_1 = 1$ are deformation equivalent and in particular diffeomorphic. Moreover, if $p_g(S) = 1$ and $m_1 = m_2 = 1$, then $S$ is a $K3$ surface. In all other cases, $S$ is a surface with Kodaira dimension one.

Our goal in this series of three papers is to prove the following result:

**Theorem.** Two possibly blown up simply connected elliptic surfaces are diffeomorphic if and only if they are deformation equivalent. More precisely, suppose that $S$ and $S'$ are relatively minimal simply connected elliptic surfaces. Suppose that $S$ has multiple fibers of multiplicities $m_1$ and $m_2$, with $1 \leq m_1 \leq m_2$, and that $S'$ has multiple fibers of multiplicities $m'_1$ and $m'_2$, with $1 \leq m'_1 \leq m'_2$. Let $\tilde{S}$ be a blowup of $S$ at $r$ points and $\tilde{S}'$ a blowup of $S'$ at $r'$ points. Suppose that $\tilde{S}$ and $\tilde{S}'$ are diffeomorphic. Then $r = r'$ and $p_g(S) = p_g(S')$, and moreover:

(i) If $p_g(S) > 0$, then $m_1 = m'_1$ and $m_2 = m'_2$.
(ii) If $p_g(S) = 0$, then $S$ is rational, i.e. $m_1 = 1$, if and only if $S'$ is rational if and only if $m'_1 = 1$. If $S$ and $S'$ are not rational, then $m_1 = m'_1$ and $m_2 = m'_2$.

There is also a routine generalization to the case of a finite cyclic fundamental group. The statements in the theorem that $r = r'$ and $p_g(S) = p_g(S')$ are easy consequences of the fact that $\tilde{S}$ and $\tilde{S}'$ are homotopy equivalent, and the main point is to determine the multiplicities. Before discussing the proof of the theorem in more detail, we shall review some of the history of the classification of simply connected elliptic surfaces:

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Theorem 1.1 [8]. There is a function \( f(m_1, m_2) \) defined on pairs of relatively prime positive integers \((m_1, m_2)\) such that \( f \) is symmetric and finite-to-one provided that neither \( m_1 \) nor \( m_2 \) is 1, with the following property: Let \( S \) and \( S' \) be two simply connected surfaces with \( p_g(S) = 0 \). Denote the multiplicities of the multiple fibers of \( S \) by \( m_1, m_2 \) and the multiplicities for \( S' \) by \( m_1', m_2' \). If \( S \) and \( S' \) are diffeomorphic, then \( f(m_1, m_2) = f(m_1', m_2') \). Moreover, let \( \tilde{S} \) and \( \tilde{S}' \) be blowups of \( S \) and \( S' \) at \( r \) points. Then:

(i) Every diffeomorphism \( \psi: \tilde{S} \rightarrow \tilde{S}' \) pulls back the cohomology class of an exceptional curve on \( \tilde{S}' \) to \( \pm \) the cohomology class of an exceptional curve on \( \tilde{S} \).

(ii) Every diffeomorphism \( \psi: \tilde{S} \rightarrow \tilde{S}' \) pulls back the cohomology class of a general fiber on \( \tilde{S}' \) to a rational multiple of the cohomology class of a general fiber on \( \tilde{S} \).

(iii) If \( \tilde{S} \) and \( \tilde{S}' \) are diffeomorphic, then \( f(m_1, m_2) = f(m_1', m_2') \). □

The function \( f(m_1, m_2) \) was then determined by S. Bauer [1] (the case \( m_1 = 2 \) is also in [8]):

**Theorem 1.2.** In the above notation,

\[
f(m_1, m_2) = \frac{(m_1^2 - 1)(m_2^2 - 1)}{3} - 1. \quad \square
\]

For the case \( p_g(S) > 0 \), there is the following result [9]:

**Theorem 1.3.** Let \( S \) and \( S' \) be two simply connected surfaces with \( p_g(S) > 0 \). Denote the multiplicities of the multiple fibers of \( S \) by \( m_1, m_2 \) and the multiplicities for \( S' \) by \( m_1', m_2' \). If \( S \) and \( S' \) are diffeomorphic, then \( m_1m_2 = m_1'm_2' \). Moreover, let \( \tilde{S} \) and \( \tilde{S}' \) be blowups of \( S \) and \( S' \) at \( r \) points. Then:

(i) Every diffeomorphism \( \psi: \tilde{S} \rightarrow \tilde{S}' \) pulls back the cohomology class of an exceptional curve on \( \tilde{S}' \) to \( \pm \) the cohomology class of an exceptional curve on \( \tilde{S} \).

(ii) Except possibly for \( p_g(S) = 1 \), every diffeomorphism \( \psi: \tilde{S} \rightarrow \tilde{S}' \) pulls back the cohomology class of a general fiber on \( \tilde{S}' \) to a rational multiple of the cohomology class of a general fiber on \( \tilde{S} \).

(iii) If \( \tilde{S} \) and \( \tilde{S}' \) are diffeomorphic, then \( m_1m_2 = m_1'm_2' \). □

The crux of the argument involves calculating a coefficient of a suitable Donaldson polynomial invariant \( \gamma_c(S) \). In fact, it is shown in [9] that \( \gamma_c(S) \) can be written as a polynomial in the intersection form \( q_S \) and the primitive class \( \kappa \) such that the class of a general fiber \([f]\) of \( S \) is equal to \( m_1m_2\kappa \), and that, for \( c \) sufficiently large, the first nonzero coefficient of this polynomial is given as follows: let \( n = 2c - 2p_g(S) - 1 \) and \( d = 4c - 3p_g(S) - 3 \). If \( \gamma_c(S) = \sum_{i=0}^{[d/2]} a_i q_S^i \kappa^{d-2i} \), then \( a_i = 0 \) for \( i > n \) and

\[
a_n = \frac{d!}{2^n n!} (m_1m_2)^{p_g(S)}. \]

The proof of this statement involves showing that the moduli space of stable vector bundles \( V \) with \( c_1(V) = 0 \) and \( c_2(V) = c \) fibers holomorphically over a Zariski open subset of a projective space, and that the fiber consists of \( m_1m_2 \) copies of a complex torus. It is natural to wonder if the techniques of [9] can be pushed to...
determine some of the remaining terms. However, it seems to be difficult to use the vector bundle methods used in [9] to make the necessary calculations, even in the case of no multiple fibers. Thus, it is natural to look for other techniques to complete the $C^\infty$ classification of elliptic surfaces.

Using a detailed analysis of certain moduli spaces of vector bundles, Morgan and O’Grady [15] together with Bauer [2] were able to calculate the coefficient $a_{n-1}$ in case $p_g(S) = 1$ and $c = 3$. The calculation is long and involved for the following reason: the moduli spaces are nonreduced, not necessarily of the correct dimension, and (in the case of trivial determinant) the integer $c$ is not in the “stable range.” The final answer is that, up to a universal combinatorial factor, $a_{n-1} = m_1m_2(2m_1^2m_2^2 - m_1^2 - m_2^2)$. From this and from the knowledge of $m_1m_2$, it is easy to determine the unordered pair $\{m_1, m_2\}$. In addition, the calculation shows that the class of a fiber of $S$ is preserved up to rational multiples in case $p_g(S) = 1$ as well (the possible exception in (ii) of Theorem 1.3 above), provided that not both of $m_1$ and $m_2$ are 1.

In the proof of the main theorem, we shall use the following results. Aside from standard techniques in the theory of vector bundles, and the gauge theory results that are described in the book [9], we use only the results of this series of papers and of [9] to handle the case $p_g(S) > 0$. In case $p_g(S) = 0$, we use the results in this series and in [8], as well as the calculation of Bauer described in Theorem 1.2 in case $m_1m_2 \equiv 0 \mod 2$. In case $m_1m_2 \equiv 1 \mod 2$, our proof does not depend on Bauer’s results.

Next we outline the strategy of the argument. Following a well-established principle [6], [10], we shall work with $SO(3)$-invariants instead of $SU(2)$-invariants since these are often much easier to calculate. Moreover, in case $b_2^+ = 1$ a good choice of an $SO(3)$ invariant can simplify the problem that the invariant depends on the choice of a certain chamber. Thus we must choose a class $w \in H^2(S; \mathbb{Z}/2\mathbb{Z})$ to be the second Stiefel-Whitney class of a principal $SO(3)$ bundle, although it will usually be more convenient to work with a lift of $w$ to $\Delta \in H^2(S; \mathbb{Z})$. One possible choice of a lift of $w$ would be the class $\kappa$, the primitive generator of $\mathbb{Z}^+ \cdot [f]$, or perhaps $c_1(S)$, or even $[f]$. All of these classes are rational multiples of $[f]$, and they do not simplify the problem.

Instead we shall consider the case where $\Delta$ is transverse to $f$, more specifically where $\Delta \cdot \kappa = 1$. Of course, we shall need to choose $\Delta$ to be the class of a holomorphic divisor as well in order to be able to apply algebraic geometry. As we shall see in Section 2, we can always make the necessary choices and the final calculation will show that the answer does not depend on the choices made. Note that $\Delta$ is well-defined up to a multiple of $\kappa$, and that the choices $\Delta$ and $\Delta - \kappa$ correspond to different choices for $w^2 \equiv p \mod 4$. Finally, as we shall show in Section 3, in case $b_2^+(S) = 1$ or equivalently $p_g(S) = 0$, there is a special chamber $\mathcal{C}(w,p)$ which is natural in an appropriate sense under diffeomorphisms.

With this choice of $\Delta$, the study of the relevant vector bundles divides into two very different cases, depending on whether $m_1m_2 \equiv 0 \mod 2$ or $m_1m_2 \equiv 1 \mod 2$. In this paper we shall collect results which are needed for both cases and show how the main theorems follow from the calculations in Parts II and III. In Part II of this series, we shall consider the case where $\Delta \cdot \kappa = 1$ and $m_1m_2 \equiv 0 \mod 2$. In this case, $m_1$, say, is even. Since $\Delta \cdot f = m_1m_2\Delta \cdot \kappa$, a vector bundle $V$ with $c_1(V) = \Delta$ has even degree on a general fiber $f$. At first glance, then, it seems as if we are again in the situation of [7] and [9] and that there is no new information to be
gained from the Donaldson polynomial. However, it turns out that the asymmetry between \( m_1 \) and \( m_2 \) appears in the moduli space as well. In this case, the moduli space again fibers holomorphically over a Zariski open subset of a projective space. But the fibers now consist of just \( m_2 \) copies of a complex torus. We then have, by an analysis that closely parallels [9], the following result:

**Theorem 1.4.** Let \( w \) and \( p \) be as above, and set \( d = -p - 3(p_g(S) + 1) \) and \( n = (d-p_g(S))/2 \). Suppose that \( \gamma_{w,p}(S) \) is the Donaldson polynomial for the \( SO(3) \)-bundle \( P \) over \( S \) with \( w_2(P) = w \) and \( p_1(P) = p \) where if \( p_g(S) = 0 \) this polynomial is associated to the chamber \( \mathcal{C}(w,p) \) defined in (3.6) below. Then, assuming that \( m_1 \) is even and writing \( \gamma_{w,p}(S) = \sum_{i=0}^{[d/2]} a_i q_i \kappa_S^{d-2i} \), we have, for all \( p \) such that \(-p \geq 2(4p_g(S) + 2)\), \( a_i = 0 \) for \( i > n \) and

\[
    a_n = \frac{d!}{2^m q!} (m_1 m_2)^{p_g(S)} m_2.
\]

In particular, the leading coefficient contains an “extra” factor of \( m_2 \). Using this and either [9] in case \( p_g(S) > 0 \) or [1] in case \( p_g(S) = 0 \), we may then determine \( \{m_1, m_2\} \). Note that in case \( p_g(S) = 0 \) and one of \( m_1, m_2 \) is 1, then it cannot be \( m_1 \) since \( m_1 \) is even. Thus \( m_2 = 1 \) and the leading coefficient does not determine \( m_1 \) (as well it cannot).

Finally, in Part III we shall discuss the case where \( \Delta \cdot \kappa = 1 \) and \( m_1 m_2 \equiv 1 \mod 2 \). If \( m_1 m_2 \equiv 1 \mod 2 \), then vector bundles \( V \) with \( c_1(V) = \Delta \) have odd degree when restricted to a general fiber, and the general methods for studying vector bundles on elliptic surfaces described in [7] and [9] Chapter 7 do not apply. Thus we must develop new techniques for studying such bundles, and this is the subject of Part III of this series. Fortunately, it turns out that this moduli problem is in many ways much simpler to study than the case of even degree on the general fiber. For example, as long as the expected dimension is nonnegative, for a suitable choice of ample line bundle the moduli space is always nonempty, irreducible, and smooth of the expected dimension. Moreover a Zariski open subset of the moduli space is independent of the multiplicities, and from this one can show easily that the leading coefficient of the Donaldson polynomial for the corresponding \( SO(3) \)-bundle is (up to the usual combinatorial factors) equal to 1. At first glance, this rather disappointing result suggests that no new information can easily be gleaned from the Donaldson polynomial. However, this suggestion is misleading: in some sense, the structure of the moduli space allows the contribution of the multiple fibers to be localized around the multiple fibers, enabling us to calculate the next two coefficients in the Donaldson polynomial. By contrast, in the case of trivial determinant, the moduli space for a surface with two multiple fibers of multiplicities \( m_1 \) and \( m_2 \) looks roughly like a branched cover of the corresponding moduli space for a surface without multiple fibers. A further simplification is that we can work with moduli spaces of small dimension, for example dimension two or four. Using the vector bundle results, we shall show:

**Theorem 1.5.** Let \( S \) be a simply connected elliptic surface with two multiple fibers \( m_1 \) and \( m_2 \), with \( m_1 m_2 \equiv 1 \mod 2 \). Let \( w \in H^2(S; \mathbb{Z}/2\mathbb{Z}) \) satisfy \( w \cdot \kappa = 1 \). Suppose that \( \gamma_{w,p}(S) \) is the Donaldson polynomial for the \( SO(3) \)-bundle \( P \) over \( S \) with \( w_2(P) = w \) and \( p_1(P) = p \) where if \( p_g(S) = 0 \) this polynomial is associated to
the chamber $C(w, p)$ defined in (3.6) below.

(i) Suppose $w$ and $p$ are chosen so that the expected complex dimension of the moduli space $-p - 3(p_g(S) + 1)$ is 2. Then for all $\Sigma \in H_2(S; \mathbb{Z})$,
\[
\gamma_{w,p}(S)(\Sigma, \Sigma) = \Sigma^2 + ((m_1^2m_2^2)(p_g(S) + 1) - m_1^2 - m_2^2)(\Sigma \cdot \kappa)^2.
\]

(ii) Suppose $w$ and $p$ are chosen so that the expected complex dimension of the moduli space $-p - 3(p_g(S) + 1)$ is 4. Then for all $\Sigma \in H_2(S; \mathbb{Z})$,
\[
\gamma_{w,p}(S)(\Sigma, \Sigma, \Sigma, \Sigma) = 3(\Sigma^2)^2 + 6C_1(\Sigma^2)(\Sigma \cdot \kappa)^2 + (3C_1^2 - 2C_2)(\Sigma \cdot \kappa)^4,
\]
where
\[
C_1 = (m_1^2m_2^2)(p_g(S) + 1) - m_1^2 - m_2^2;
\]
\[
C_2 = (m_1^2m_2^2)(p_g(S) + 1) - m_1^4 - m_2^4.
\]

Here $C_1$ is the second coefficient of the degree two polynomial.

Note that the final answer has the following self-checking features. First, it is a polynomial in $g_S$ and $\kappa$. If $p_g(S) = 1$ and $m_1 = m_2 = 1$, so that $S$ is a $K3$ surface, then the term $(\Sigma \cdot \kappa)$ does not appear. This is in agreement with the general result that $\gamma_{w,p}(S)$ is a multiple of a power of $g_S$ alone. If $p_g(S) = 0$ and $m_1 = 1$, then the answer is independent of $m_2$, since in this case all of the surfaces $S$ for various choices of $m_2$ are diffeomorphic. In fact, we shall turn this remark around and use the knowledge of $\gamma_{w,p}(S)$ for $p_g = 0$, $m_2 = 1$ and $m_1$ arbitrary, to determine $\gamma_{w,p}(S)$ in general.

The techniques used to prove Theorem 1.5 should be capable of further generalization. For example, these methods should give in principle (that is, up to the knowledge of the multiplication table for divisors in $\text{Hilb}^n S$ ), the full polynomial invariant in case $m_1$ and $m_2$ are odd. One might make a conjectural formula for $\gamma_{w,p}(S)$ in general along the lines suggested by Kronheimer and Mrowka in [13]. In our case the formula should conjecturally read as follows: let $\gamma_t(\Sigma)$ be the Donaldson polynomial $\gamma_{w,p}(S)(\Sigma, \ldots, \Sigma)$ for $w = \Delta \mod 2$ or $w = \Delta - \kappa \mod 2$ and $p$ chosen so that $w^2 \equiv p \mod 4$ and $-p - 3\chi(\mathcal{O}_S) = 2t$, so that the complex dimension of the moduli space is $2t$. It follows from Proposition 2.1 below that $\gamma_t$ depends only on $t$. Then the natural analogue of the conjectures in [13] is the conjecture that
\[
\sum_{t \geq 0} \frac{\gamma_t(\Sigma)}{(2t)!} = \exp \left( \frac{g_S}{2} \cdot \frac{\cosh(m_1m_2(\kappa \cdot \Sigma))}{\cosh(m_1(\kappa \cdot \Sigma)) \cosh(m_2(\kappa \cdot \Sigma))} \right)^{p_g + 1}.
\]

It essentially follows from Theorem 1.5 that this formula is correct through the first three terms, including the case $p_g = 0$ where the quotient is not given by a finite sum of exponentials, and it is likely that a further extension of the methods in Part III and the knowledge of the multiplication in $\text{Hilb}^n S$ can establish the general formula. Finally we should add that many of these techniques have applications to the $SU(2)$ case.

Morgan and Mrowka [14] have independently determined the second coefficient $a_{n-1}$ for all $S$ such that $p_g(S) \geq 1$, for the case of the $SU(2)$-invariant $\gamma_c(S)$. The answer is that, up to combinatorial factors,
\[
a_{n-1} = (m_1m_2)^{p_g(S)}((m_1^2m_2^2)(p_g(S) + 1) - m_1^2 - m_2^2).
\]
2. Preliminaries on elliptic surfaces.

Let \( S \) be a simply connected elliptic surface with at most two multiple fibers of multiplicities \( m_1 \leq m_2 \). We shall allow \( m_1 \) or both \( m_1 \) and \( m_2 \) to be one. Let \([f]\) denote the class in homology of a smooth nonmultiple fiber of \( S \). There is a unique homology class \( \kappa_S = \kappa \) such that \([f] = m_1 m_2 \kappa \), and \( \kappa \) is primitive \([9]\). Let \( P \) be a principal \( SO(3) \)-bundle over \( S \) with \( w_2(P) = w \) and \( p_1(P) = p \). Note that \( w^2 \equiv p \mod 4 \). We shall be concerned with bundles \( P \) such that \( w \cdot \kappa \mod 2 = 1 \).

In this section, we shall show that, modulo diffeomorphism, the choice of \( w \) is not essential. Indeed, we shall prove that, given a class \( w \), there is a diffeomorphism \( \psi : S \to S' \), where \( S' \) is again a simply connected elliptic surface with two multiple fibers of multiplicities \( m_1 \) and \( m_2 \), such that \( \psi^* \kappa_{S'} = \kappa_S \) and such that there exists a holomorphic divisor \( \Delta \) with \( w = \psi^*[\Delta] \mod 2 \). Thus we may always assume that \( w \) is the reduction of a \((1,1)\) class. We begin with an arithmetic result, which is not in fact needed in what follows but which helps to clarify the role of the choice of \( w \) modulo diffeomorphisms. In the arguments below, we shall sometimes blur the distinction between \( H_2(S) \) and \( H^2(S) \) using the canonical identification between these two groups.

**Proposition 2.1.** Let \( S \) be a simply connected elliptic surface.

(i) Suppose that \( m_1 m_2 \equiv 1 \mod 2 \), and let \( a \in \mathbb{Z}/4\mathbb{Z} \). Then the group of orientation-preserving diffeomorphisms \( \psi : S \to S \) such that \( \psi_*([f]) = [f] \) acts transitively on the set of \( w \in H^2(S; \mathbb{Z}/2\mathbb{Z}) \) such that \( w \cdot \kappa = 1 \) and \( w^2 \equiv a \mod 4 \).

(ii) If \( m_1 m_2 \equiv 0 \mod 2 \) and \( a \in \mathbb{Z}/4\mathbb{Z} \), then there are at most three orbits of the set

\[
\{ w \in H^2(S; \mathbb{Z}/2\mathbb{Z}) : w \cdot \kappa = 1 \text{ and } w^2 \equiv a \mod 4 \}
\]

under the group of diffeomorphisms of \( S \) which fix \( \kappa \).

**Proof.** Let \( L \) be the image of \( H_2(S - \pi^{-1}(D)) \) in \( H_2(S) \), where \( D \) is a small disk in \( \mathbb{P}^1 \) which we may assume contains the multiple fibers and no other singular fiber. Thus \( L \subseteq (\kappa^+) \), and in fact \( L \) has index \( m_1 m_2 \) in \((\kappa^+)\). Let \( \varphi \) be an automorphism...
of the lattice $H_2(S;\mathbb{Z})$ fixing $\kappa$. Thus by restriction $\varphi$ induces an automorphism of $(\kappa^+)$. The method of proof of Theorem 6.5 of Chapter 2 of [9] shows that there is a diffeomorphism $\psi$, automatically orientation-preserving, inducing $\varphi$ provided that $\varphi(L) \subseteq L$ and that $\varphi$ has real spinor norm one.

Clearly we may write $L = \mathbb{Z}[\kappa] \oplus W$, where $W$ is an even unimodular lattice. Moreover $(\kappa^+) = \mathbb{Z} \cdot \kappa \oplus W$, with the inclusion $L \subseteq (\kappa^+)$ the natural inclusion given by $[f] = m_1 m_2 \kappa$. If $W^\perp$ denotes the orthogonal complement of $W$ in $H_2(S;\mathbb{Z})$, then $W^\perp = \text{span}\{\kappa, x\}$ for some class $x$ with $x \cdot \kappa = 1$. Given $a \mod 4$, we can always assume after replacing $x$ by $x + \kappa$ that $x^2 \equiv a \mod 4$. Now it is easy to describe all automorphisms of $H_2(S;\mathbb{Z})$ fixing $\kappa$: choosing an isometry $\tau$ of $W$, $\varphi$ is given by

$$
\varphi(\kappa) = \kappa;
\varphi(\alpha) = \tau(\alpha) + \ell(\alpha)\kappa, \quad \alpha \in W;
\varphi(x) = x + cx + \beta.
$$

Here $\ell$ is an arbitrary homomorphism $W \to \mathbb{Z}$ and $\beta$ is the unique element of the unimodular lattice $W$ such that $-\beta \cdot \alpha = \ell(\tau(\alpha))$ for all $\alpha \in W$. Furthermore $c = -\beta^2/2$. It is clear that every choice of $\tau$ and $\ell$ (or equivalently $\beta$) produces an automorphism $\varphi$, and that $\varphi(L) = L$ if and only if $m_1 m_2$ divides $\ell$ or equivalently $\beta$. If $x'$ is another class such that $x' \cdot \kappa \equiv 1 \mod 2$ and $(x')^2 \equiv x^2 \mod 4$, we can write $x' = nx + bk + \beta$, where $b \in W$. Since we only care about $x' \mod 2$, we may assume that $n = 1$. Note that $2b + \beta^2 \equiv 0 \mod 4$ and thus $b \equiv \beta^2/2 \mod 2$.

First assume that $m_1 m_2$ is odd. Then since $\beta \equiv m_1 m_2 \beta \mod 2$, we may assume that $\beta$ is divisible by $m_1 m_2$. Choosing $\tau = \text{Id}$ and $\ell$, $c$ in the definition of $\varphi$ as specified by $\beta$ gives $\varphi$ such that $\varphi(x) \equiv x' \mod 2$. As $\varphi$ is unipotent, it is easy to see that $\varphi$ has spinor norm one, i.e. that $\varphi$ is in the same connected component of the group of automorphisms of the quadratic form of $H_2(S;\mathbb{R})$ as the identity. Thus there is a diffeomorphism $\psi$ realizing $\varphi$.

Next suppose that $2|m_1 m_2$. In fact in this case the class $x$ defined above is fixed mod 2 by every isometry $\varphi$ as above which satisfies $\varphi(L) = L$. Since $m_1 m_2 | \beta$ and $c \equiv \beta^2/2 \mod 2$, it follows that $\varphi(x) \equiv x \mod 2$. Now let $x'$ be a class with $x' \cdot \kappa \equiv 1 \mod 2$ and $(x')^2 \equiv x^2 \mod 4$. We may assume that $x' \not\equiv x$. First consider the case where $x' = x + bk + \alpha$ and $b \equiv 0 \mod 2$. Thus we may replace $x'$ by $x + \alpha$. By assumption $\alpha^2 \equiv 0 \mod 4$. We may assume that $\alpha$ is primitive (otherwise $\alpha \equiv 0 \mod 2$ or $\alpha$ is congruent to a primitive nonzero element mod 2). Replacing $\alpha$ by $\alpha + 2\beta$, where $\beta \in W^\perp$, replaces $\alpha^2$ by $\alpha^2 + 4(\alpha \cdot \beta) + 4\beta^2$. Since $\alpha$ is primitive, it is easy to see that there is a choice of $\beta$ so that $(\alpha + 2\beta)^2 = 0$. Thus we may assume that $\alpha$ is primitive and that $\alpha^2 = 0$. The group $SO(W)$ includes into the automorphism group of $L$, and every element of $SO(W)^*$, the set all elements of $SO(W)$ with spinor norm one, is realized by a diffeomorphism. Moreover an easy exercise shows that $SO(W)^*$ acts transitively on the set of primitive $\alpha \in W$ with $\alpha^2 = 0$. Thus the set of all possible $x + \alpha$, with $\alpha \not\equiv 0$, is contained in a single orbit under the diffeomorphism group.

In case $x' = x + \kappa + \alpha$ with $\alpha^2 \equiv 2 \mod 4$, an argument similar to that given above shows that we may assume that $\alpha^2 = 2$ and that every two classes $x_1 = x + \kappa + \alpha_1$ and $x_2 = x + \kappa + \alpha_2$ with $\alpha_1^2 = 2$ are conjugate under the group of diffeomorphisms of $S$ which fix $\kappa$. Thus there are at most three orbits in this case. \(\square\)
The following result is really only needed in the case where $m_1m_2$ is even, since in case $m_1m_2$ is odd we can appeal to (i) of (2.1) above.

**Proposition 2.2.** Let $S$ be a simply connected elliptic surface and $w$ be a class in $H^2(S;\mathbb{Z}/2\mathbb{Z})$ with $w \cdot \kappa = 1$. Then after replacing $S$ with a deformation equivalent elliptic surface, we may assume that there is a divisor $\Delta$ on $S$ with $\Delta \cdot \kappa = 1$, and such that all singular fibers of $S$ are irreducible rational curves with a singular ordinary double point, i.e. $S$ is nodal.

**Proof.** Fix a nodal simply connected elliptic surface with a section $B$ such that $p_g(B) = p_g(S)$. Using [9], $S$ is deformation equivalent through elliptic surfaces to a logarithmic transform of $B$ at two smooth fibers, where the multiplicities of the logarithmic transforms are $m_1$ and $m_2$. Fix one such logarithmic transform $S_0$, and let $\psi: S \to S_0$ be a diffeomorphism preserving the class of the fiber. Using this diffeomorphism, we shall identify $S$ and $S_0$. Let $\Delta$ be an element in $H^2(S;\mathbb{Z})$ whose mod 2 reduction is $w$ and such that $\Delta \cdot \kappa = 1$. We shall show that, by further modifying the complex structure on $S$, we may assume that $\Delta$ is of type $(1,1)$.

Given $\Delta$, we have the image $i_*(\lbrack \Delta \rbrack) \in H^2(S;\mathcal{O}_S)$, where $i_*$ is the map induced on sheaf cohomology by the inclusion $\mathbb{Z} \subset \mathcal{O}_S$. The set of all complex structures of an elliptic surface on $S$ for which the associated Jacobian surface is $B$ and which are locally isomorphic to $S$ is a principal homogeneous space over $H^1(\mathbb{P}^1;B)$, where $B$ is the sheaf of local holomorphic cross sections of $B$ ([9] Chapter 1 Theorem 6.7). Moreover there is a surjective map $H^2(S;\mathcal{O}_S) \cong H^2(B;\mathcal{O}_B) \to H^1(\mathbb{P}^1;B)$ ([9], Chapter 1, Lemma 5.11). Thus given a cohomology class $\eta \in H^2(S;\mathcal{O}_S)$, we can form the associated surface $S^\eta$ and consider the element $i^{\eta}_*(\lbrack \Delta \rbrack) \in H^2(S^\eta;\mathcal{O}_{S^\eta}) \cong H^2(S;\mathcal{O}_S)$.

**Lemma 2.3.** In the above notation, $i^{\eta}_*(\lbrack \Delta \rbrack) = i_*(\lbrack \Delta \rbrack) + m_1m_2\eta$.

**Proof.** This presumably could be proved by a rather involved direct calculation. For another argument, note that the map $H^2(S;\mathcal{O}_S) \to H^2(S;\mathcal{O}_S)$ defined by

$$\eta \mapsto i^\eta_*(\lbrack \Delta \rbrack) - i_*(\lbrack \Delta \rbrack) - m_1m_2\eta$$

is holomorphic, since it arises from a variation of Hodge structure. The argument of Lemma 6.13 in Chapter 1 of [9] shows that $i^\eta_*(\lbrack \Delta \rbrack) - i_*(\lbrack \Delta \rbrack) - m_1m_2\eta$ lies in the countable (not necessarily discrete) subgroup $H^1(\mathbb{P}^1;R^1\pi_*\mathcal{O}_S)$ of $H^2(S;\mathcal{O}_S) = H^1(\mathbb{P}^1;R^1\pi_*\mathcal{O}_S)$. This is only possible if the image of the map is contained in a single point, and since the image contains the origin the map is identically zero. □

Returning to the proof of (2.2), since $H^2(S;\mathcal{O}_S)$ is divisible, there is a choice of $\eta$ so that $i^\eta_*(\lbrack \Delta \rbrack) = 0$. For the corresponding complex structure, $\lbrack \Delta \rbrack$ is then a $(1,1)$ class. □

Finally we shall describe a way to associate new elliptic surfaces to $S$ which generalizes the construction of the Jacobian surface. Suppose that $S$ is an elliptic surface over $\mathbb{P}^1$. Let $\eta = \text{Spec } k$ be the generic point of $\mathbb{P}^1$, where $k = k(\mathbb{P}^1)$ is the function field of the base curve, let $\bar{\eta} = \text{Spec } \bar{k}$, where $\bar{k} = \overline{k(\mathbb{P}^1)}$ is the algebraic closure of $k$, and let $S_\eta$ and $S_{\bar{\eta}}$ be the restrictions of $S$ to $\eta$ and $\bar{\eta}$. Thus $S_\eta$ is a curve of genus one over $k$.

Given an algebraic elliptic surface with a section $\pi: B \to \mathbb{P}^1$, it has an associated Weil-Chatelet group $WC(B)$ [3], which classifies all algebraic elliptic surfaces $S$.
whose Jacobian surface is $B$. As above we let $B_\eta$ be the elliptic curve over $k$ defined by the generic fiber of $B$. By definition $\text{WC}(B)$ is the Galois cohomology group $H^1(G, B_\eta(k))$, where $G = \text{Gal}(k/k)$ and $B_\eta(k)$ is the group of points of the elliptic curve $B_\eta$ defined over $k$. There is an exact sequence

$$0 \to \text{III}(B) \to \text{WC}(B) \to \bigoplus_{t \in C} H_1(\pi^{-1}(t); \mathbb{Q}/\mathbb{Z}) \to 0.$$ 

The subgroup $\text{III}(B)$ corresponds to elliptic surfaces without multiple fibers whose Jacobian surface is isomorphic to $B$, and the quotient describes the possible local forms for the multiple fibers. Thus if $\xi \in \text{WC}(B)$ corresponds to the surface $S$, then $S$ has a multiple fiber of multiplicity $m$ at $t \in \mathbb{P}^1$ if and only if the projection of $\xi$ to $H_1(\pi^{-1}(t); \mathbb{Q}/\mathbb{Z})$ has order $m$.

The surface $S$ is specified by an element $\xi$ of $\text{WC}(J(S))$, where $J(S)$ is the Jacobian surface associated to $S$. Let us recall the recipe for $\xi$ [18]: we have the curve $S_\eta$ and its Jacobian $J(S_\eta)$ defined over $k$. The curve $S_\eta$ is a principal homogeneous space over $J(S_\eta)$, and thus defines a class $\xi \in \text{WC}(J(S))$, by the following rule: let $\sigma$ be a point of $S_\eta$. Given $g \in \text{Gal}(k/k)$, the divisor $g(\sigma) - \sigma$ has degree zero on $S_\eta$ and so defines an element of $J(S_\eta)$, which is easily checked to be a 1-cocycle. The induced cohomology class is $\xi$.

For every integer $d$ there is an algebraic elliptic surface $J^d(S)$, whose restriction to the generic fiber $\eta$ is the Picard scheme of divisors of degree $d$ on the curve $S_\eta$. Thus $J^0(S) = J(S)$ and $J^1(S) = S$. We claim that, if $S$ corresponds to the class $\xi \in \text{WC}(J(S))$, then $J^d(S)$ corresponds to the class $d\xi$. Indeed, using the above notation, if $\sigma$ defines a point of $S_\eta$, then $d\sigma$ is a point of $J^d(S_\eta)$. Thus, the corresponding cohomology class is represented by $d(\eta(\sigma) - \sigma)$ and so is equal to $d\xi$. In particular, if $S$ has a multiple fiber of multiplicity $m$ at $t$, then $J^d(S)$ has a multiple fiber of multiplicity $m/\gcd(m,d)$. Of course if $\text{gcd}(m,d)$ then the multiplicity is one. Finally note that $J(S)$ is the Jacobian surface of $J^d(S)$ for every $d$ and that $p_\eta(J^d(S)) = p_\eta(S)$.

Ideally we would like there to be a Poincaré line bundle $\mathcal{P}_d$ over $S \times_{\pi^1} J^d(S)$ such that the restriction of $\mathcal{P}_d$ to the slice $S \times_{\pi^1} \{\lambda\}$ is the line bundle of degree $d$ on the fiber of $S$ over $\pi(\lambda)$ corresponding to $\lambda$. In general this is too much to ask. However such a bundle exists locally around every smooth nonmultiple fiber: if $X$ is the inverse image in $S$ of a small disk $D$ in $\mathbb{P}^1$ such that all fibers on $X$ are smooth and nonmultiple, and $X_d$ is the corresponding preimage in $J^d(S)$, then there is a Poincaré line bundle over $X \times_D X_d$. There is also an analogous statement where we replace a small classical open set in $\mathbb{P}^1$ with an étale open set. The proof for this result is essentially contained in the proof of Theorem 1.3 of Chapter 7 in [9]. Another construction is given in Section 7 of Part III of this series.

### 3. Suitable line bundles

Suppose that we are given a class $w \in H^2(S; \mathbb{Z}/2\mathbb{Z})$ with $w \cdot \kappa \equiv 2 \mod 2$ and an integer $p$ with $w^2 \equiv p \mod 4$. Choose once and for all a complex structure on $S$ for which there is a divisor $\Delta$ with $w = \Delta \mod 2$. Let $c$ be the integer $(\Delta^2 - p)/4$. The principal $\text{SO}(3)$-bundle $P$ over $S$ with $w_2(P) = w$ and $p_1(P) = p$ lifts uniquely to a principal $U(2)$-bundle $P'$ over $S$ with $c_1(P') = \Delta$ and $c_2(P') = c$. Moreover, by Donaldson’s theorem, if $g$ is a Hodge metric on $S$ corresponding to the ample
line bundle $L$, we can identify the moduli space of gauge equivalence classes of $g$-anti-self-dual connections on $P$ with the moduli space of $L$-stable rank two vector bundles $V$ over $S$ with $c_1(V) = \Delta$ and $c_2(V) = c$.

We shall also have to make a choice of the ample line bundle $L$. If $p_g(S) > 0$, then the resulting Donaldson polynomial invariant does not depend on the choice of $L$, whereas if $p_g(S) = 0$, then the invariant depends on the chamber containing $c_1(L)$ [11], [12]. We then make the following definition [17]:

**Definition 3.1.** A wall of type $(\Delta, c)$ is a class $\zeta \in H^2(S; \mathbb{Z})$ such that $\zeta \equiv \Delta \pmod{2}$ and $\Delta^2 - 4c \leq \zeta^2 < 0$.

In particular there are no such walls unless $\Delta^2 - 4c < 0$. Clearly this definition depends only on $\Delta \pmod{2}$ and $p = \Delta^2 - 4c$, and we shall also refer to walls of type $(w, p)$.

Now suppose that $p_g(S) = 0$, i.e. that $b_2^+(S) = 1$. Let

$$\Omega_S = \{x \in H^2(S; \mathbb{R}) : x^2 > 0\}.$$ Let $W^\zeta = \Omega_S \cap (\zeta)^\perp$. A chamber of type $(\Delta, c)$ (or of type $(w, p)$) for $S$ is a connected component of the set

$$\Omega_S - \bigcup \{W^\zeta : \zeta \text{ is a wall of type } (\Delta, c'), c' \leq c\}.$$ For the purposes of algebraic geometry, walls of type $(\Delta, c)$ arise as follows: let $L$ be an ample line bundle and let $V$ be a rank two bundle over $S$ with $c_1(V) = \Delta$ and $c_2(V) = c$ which is strictly $L$-semistable. Let $O_S(F)$ be a destabilizing sub-line bundle. Thus there is an exact sequence

$$0 \to O_S(F) \to V \to O_S(-F + \Delta) \otimes I_Z \to 0,$$

where $I_Z$ is the ideal sheaf of a codimension two local complete intersection sub-scheme. Thus

$$c_2(V) = c = -F^2 + F \cdot \Delta + \ell(Z).$$ Since $\ell(Z)$ is nonnegative, we can rewrite this as

$$-F^2 + F \cdot \Delta \leq c.$$ Moreover

$$(2F - \Delta)^2 = -4(-F^2 + F \cdot \Delta) + \Delta^2 \leq \Delta^2 - 4c,$$

so that we can rewrite the last condition by

$$\Delta^2 - 4c \leq (2F - \Delta)^2.$$ Using the fact that $L \cdot F = (L \cdot \Delta)/2$, we have

$$L \cdot (2F - \Delta) = 0,$$

and so by the Hodge index theorem $(2F - \Delta)^2 \leq 0$, with equality holding if and only if $2F - \Delta = 0$ (recall that $S$ is simply connected). This case cannot arise for us since $\Delta \cdot \kappa = 1$ and thus $\Delta$ is primitive. In particular $\zeta = 2F - \Delta$ is a wall of type $(\Delta, c)$. Of course, it is also the cohomology class of a divisor, and thus has type $(1, 1)$. It then follows easily that, if $L_1$ and $L_2$ are two ample line bundles such that $c_1(L_1)$ and $c_1(L_2)$ lie in the interior of the same chamber of type $(\Delta, c)$, then a rank two vector bundle $V$ with $c_1(V) = \Delta$ and $c_2(V) = c$ is $L_1$-stable if and only if it is $L_2$-stable.

With this said, we can make the following definition:
Lemma 3.2. Let $c$ be an integer, and set $w = \Delta \mod 2$ and $p = \Delta^2 - 4c$. An ample line bundle $L$ is $(\Delta, c)$-suitable or $(w, p)$-suitable if, for all walls $\zeta$ of type $(\Delta, c)$ which are the classes of divisors on $S$, we have $\text{sign } f \cdot \zeta = \text{sign } L \cdot \zeta$.

Remark. 1) Suppose that $\zeta$ is a $(1, 1)$ class satisfying $\zeta^2 \geq 0$. It follows from the Hodge index theorem that if $\zeta \cdot f > 0$, then $\zeta \cdot L > 0$ as well. Thus we can drop the requirement that $\zeta^2 < 0$.

2) In our case $\zeta \equiv \Delta \mod 2$ and thus $\zeta \cdot \kappa \equiv 1 \mod 2$. It follows that $\zeta \cdot \kappa \neq 0$ and thus that $\zeta \cdot f \neq 0$. Thus the condition $\zeta \cdot f \neq 0$ (which was included as part of the definition in [9]) is always satisfied in our case.

3) In case $b^+_2(S) = 1$, $L$ is $(\Delta, c)$-suitable if and only if the class $\kappa$ lies in the closure of the chamber containing $c_1(L)$.

Lemma 3.3. For every $c$, $(\Delta, c)$-suitable ample line bundles exist.

Proof. Let $L_0$ an ample line bundle. For $n \geq 0$, let $L_n = L_0 \otimes O_S(nf)$. It follows from the Nakai-Moishezon criterion that $L_n$ is ample as well. We claim that if $n > -p(L_0 \cdot f)/2$, then $L_n$ is $(\Delta, c)$-suitable.

To see this, let $\zeta = 2F - \Delta$ be a wall of type $(\Delta, c)$ with

$$\Delta^2 - 4c \leq \zeta^2 < 0.$$ 

We may assume that $a = \zeta \cdot f > 0$, and must show that $\zeta \cdot L_n > 0$ as well. The class $ac_1(L_0) - (L_0 \cdot f)\zeta$ is perpendicular to $f$. Since $f^2 = 0$, we may apply the Hodge index theorem to conclude:

$$0 \geq ac_1(L_0) - (L_0 \cdot f)\zeta = a^2L_0^2 - 2a(L_0 \cdot f)(L_0 \cdot \zeta) + (L_0 \cdot f)^2\zeta^2.$$ 

Using the fact that $\zeta^2 \geq \Delta^2 - 4c = p$, we find that

$$L_n \cdot \zeta \geq \frac{a(L_0^2)}{2(L_0 \cdot f)} + \frac{\zeta^2}{2a}(L_0 \cdot f) > \frac{p}{2a}(L_0 \cdot f).$$ 

Thus

$$L_n \cdot \zeta = (L_0 \cdot \zeta) + n(\zeta \cdot f) > \frac{p}{2a}(L_0 \cdot f) - \frac{pa}{2}(L_0 \cdot f) = \frac{p}{2}(L_0 \cdot f)(a - \frac{1}{a}) \geq 0.$$ 

Thus $L_n$ is $(\Delta, c)$-suitable. \(\square\)

In the case where $b^+_2(S) = 1$, we have the following interpretation of $(\Delta, c)$-suitability.

Lemma 3.4. Suppose that $p_3(S) = 0$. If $L_1$ and $L_2$ are both $(\Delta, c)$-suitable, then $c_1(L_1)$ and $c_1(L_2)$ lie in the same chamber of type $(\Delta, c)$. Thus there is a unique chamber $C(w, p)$ of type $(\Delta, c)$ which contains the first Chern classes of $(\Delta, c)$-suitable ample line bundles. Conversely, if $L$ is ample and $c_1(L) \in C(w, p)$, then $L$ is $(\Delta, c)$-suitable.

Proof. Let $L_1$ and $L_2$ be $(\Delta, c)$-suitable. Since $p_3(S) = 0$, every cohomology class is of type $(1, 1)$. Thus if $\zeta$ is a wall of type $(\Delta, c)$, then

$$\text{sign } L_1 \cdot \zeta = \text{sign } f \cdot \zeta = \text{sign } L_2 \cdot \zeta.$$
This exactly implies that $c_1(L_1)$ and $c_1(L_2)$ are not separated by any wall $(\zeta)$. Conversely suppose that $c_1(L) \in C(w, p)$, where $C(w, p)$ is the unique chamber containing the first Chern classes of $(\Delta, c)$-suitable ample line bundles. This means in particular that $L \cdot \zeta \neq 0$ for every $\zeta$ of type $(\Delta, c)$. The proof of (3.4) shows that $c_1(L) + N[f] \in C(w, p)$ for all sufficiently large $N$. Thus, for all $\zeta$ of type $(\Delta, c)$,

$$\text{sign } L \cdot \zeta = \text{sign } L \cdot \zeta + N \text{ sign } f \cdot \zeta.$$ 

Since $f \cdot \zeta \neq 0$, sign $L \cdot \zeta + N \text{ sign } f \cdot \zeta = \text{sign } f \cdot \zeta$ for all $N \gg 0$. Thus sign $f \cdot \zeta = \text{sign } L \cdot \zeta$, and $L$ is $(\Delta, c)$-suitable. □

**Lemma 3.5.** Suppose that $p_g(S) = 0$. The chamber $C(w, p)$ is the unique chamber of type $(w, p)$ which contains $\kappa$ in its closure. Thus every diffeomorphism $\psi$ of $S$ satisfies $\psi^*C(w, p) = \pm C(\psi^* w, p)$. More generally, if $S$ and $S'$ are two elliptic surfaces with $p_g = 0$ and $\psi : S \to S'$ is a diffeomorphism, then $\psi^*C(w, p) = \pm C(\psi^* w, p)$.

**Proof.** Let $C_1$ and $C_2$ be two distinct chambers which contain $\kappa$ in their closures. Let $\zeta$ be a wall separating $C_1$ and $C_2$. We may assume that $\zeta \cdot x > 0$ for all $x \in C_1$ and $\zeta \cdot x < 0$ for all $x \in C_2$. Thus $0 \leq \zeta \cdot \kappa \leq 0$, so that $\zeta \cdot \kappa = 0$. However this contradicts the fact that $\zeta \cdot \kappa \neq 0$. Thus there is at most one chamber containing $\kappa$ in its closure. We have seen in the proof of Lemma 3.3 that, for all ample line bundles $L$ and integers $N \gg 0$, $c_1(L) + N \kappa \in C(w, p)$. Thus $\kappa + (1/N)c_1(L) \in C(w, p)$. It follows that $\kappa$ indeed lies in the closure of $C(w, p)$, so that $C(w, p)$ is the unique chamber with this property.

To see the final statement, we use [8] to see that every diffeomorphism $\psi$ of $S$ satisfies $\psi^* \kappa = \pm \kappa$. Thus $\pm \kappa$ lies in the closure of $\psi^* C(w, p)$. Clearly, if $C$ is a chamber of type $(w, p)$, then $\psi^* C$ is a chamber of type $(\psi^* w, p)$. It follows that $\psi^* C(w, p) = \pm C(\psi^* w, p)$. The statement about two different surfaces is proved similarly. □

**Definition 3.6.** The chamber described in Lemma 3.4 will be called the suitable chamber of type $(\Delta, c)$ or of type $(w, p)$ or the $(\Delta, c)$-suitable or $(w, p)$-suitable chamber.

4. The geometric meaning of suitability.

The goal of this section is to describe the meaning of $(\Delta, c)$-suitability. Given the bundle $V$ on $S$, it defines by restriction a bundle $V|f$ on each fiber $f$. Our main result says essentially that $V$ is stable for one, or equivalently all, $(\Delta, c)$-suitable line bundles $L$ if and only if $V|f$ is semistable for almost all $f$.

It will be more convenient to use the language of schemes to state this result. As in Section 2, let $k(\mathbb{P}^1)$ denote the function field of $\mathbb{P}^1$ and let $\overline{k}(\mathbb{P}^1)$ be the algebraic closure of $k(\mathbb{P}^1)$. Set $\eta = \text{Spec } k(\mathbb{P}^1)$ and $\overline{\eta} = \text{Spec } \overline{k}(\mathbb{P}^1)$. Thus $\eta$ is the generic point of $\mathbb{P}^1$. Let $S_\eta = S \times_{\mathbb{P}^1} \eta$ be the generic fiber of $\pi$ and let $S_\overline{\eta} = S \times_{\mathbb{P}^1} \overline{\eta}$. Here $S_\eta$ is a curve of genus one over the field $k(\mathbb{P}^1)$ and $S_\overline{\eta}$ is the curve over $\overline{k}(\mathbb{P}^1)$ defined by extending scalars. Let $V_\eta$ and $V_\overline{\eta}$ be the vector bundles over $S_\eta$ and $S_\overline{\eta}$ respectively obtained by restricting $V$. We can then define stability and semistability for $V_\eta$ and $V_\overline{\eta}$; for $V_\eta$, a destabilizing subbundle must also be defined over $k(\mathbb{P}^1)$. Trivially, if $V_\eta$ is unstable (resp. not stable) then $V_\overline{\eta}$ is unstable (resp. not stable). Thus if $V_\eta$ is stable, then $V_\overline{\eta}$ is stable as well.
Lemma 4.1. $V_\eta$ is semistable if and only if $V_\bar{\eta}$ is semistable.

Proof. We have seen that, if $V_\eta$ is not semistable, then $V_\bar{\eta}$ is not semistable. Conversely suppose that $V_\bar{\eta}$ is not semistable. Then there is a canonically defined maximal destabilizing line subbundle of $V_\bar{\eta}$, which thus is fixed under by every element of $\text{Gal}(k(P^1)/k(P^1))$. By standard descent theory this line subbundle must then be defined over $k(P^1)$. Thus $V_\eta$ is not semistable. □

Remark. If $V_\eta$ is strictly semistable, it is typically the case that $V_\eta$ is actually stable.

Lemma 4.2. In case $\Delta \cdot \kappa = 1$, the bundle $V_\eta$ is semistable if and only if it is stable.

Proof. First assume that $m_1m_1 \equiv 1 \mod 2$. In this case $V_\eta$ has odd fiber degree, and so there are no strictly semistable bundles over $k(P^1)$. Hence, if $V_\eta$ is semistable, then by (4.1) $V_\bar{\eta}$ is semistable and therefore stable. Thus $V_\eta$ is stable by the remarks preceding (4.1).

In case $m_1m_1 \equiv 0 \mod 2$, suppose that $V_\eta$ is strictly semistable. Thus there is a line bundle on $S_\eta$ of degree $m_1m_2/2$. There would thus exist a divisor $D$ on $S$ with $D \cdot f = m_1m_2/2$. Since $f = m_1m_2\kappa$, this possibility cannot occur. Thus $V_\eta$ is stable. □

Here then is the theorem of this section:

Theorem 4.3. Let $V$ be a rank two vector bundle on $S$ with $c_1(V) = \Delta$ and $c_2(V) = c$ and let $L$ be a $(\Delta, c)$-suitable ample line bundle. Then $V$ is $L$-stable if and only if the restriction $V_\eta$ of $V$ to the generic fiber $S_\eta$ is stable.

Proof. First suppose that $V$ is $L$-stable. Let $F_\eta$ be a subbundle of $V_\eta$ of rank one. Then there is a divisor $F$ on $S$ such that $O_S(F)$ restricts to $F_\eta$ and an inclusion $O_S(F) \to V$. Hence there is an effective divisor $D$ and an inclusion $O_S(F+D) \to V$ and the cokernel is torsion free. Since $F_\eta$ is a subbundle of $V_\eta$, the divisor $D$ cannot have positive intersection number with $f$. As $D$ is effective it is supported in the fibers of $\pi$ and so $F$ and $F+D$ have the same restriction to the generic fiber. We may thus replace $F$ by $F+D$. Then $V/O_S(F)$ is torsion free. Hence there is an exact sequence

$$0 \to O_S(F) \to V \to O_S(\Delta - F) \otimes I_Z \to 0,$$

where $Z$ is a codimension two subscheme of $S$. Thus

$$\Delta^2 - 4c \leq (2F - \Delta)^2.$$

Since $V$ is $L$-stable, $L \cdot (2F - \Delta) < 0$. It follows from Definition 3.2 and 2) of the remark following it that $f \cdot (2F - \Delta) < 0$ as well. Thus $\deg F_\eta < \deg V_\eta/2$, which says that $V_\eta$ is stable.

Conversely suppose that $V_\eta$ is stable. Let $O_S(F)$ be a sub-line bundle of $V$, where we may assume that $V/O_S(F)$ is torsion free. Reversing the argument above shows that $f \cdot (2F - \Delta) < 0$ and therefore that $L \cdot (2F - \Delta) < 0$ as well. Thus $V$ is $L$-stable. □
corollary 4.4. Let $V$ be a rank two vector bundle on $S$ with $c_1(V) = \Delta$ and $c_2(V) = c$. Then the following are equivalent:

(i) There exists a $(\Delta, c)$-suitable ample line bundle $L$ such that $V$ is $L$-stable.
(ii) $V$ is $L$-stable for every $(\Delta, c)$-suitable ample line bundle $L$.
(iii) $V$ is stable.
(iv) $V$ is semistable.
(v) The restriction $V|_{\pi^{-1}(t)}$ is semistable for almost all $t \in \mathbb{P}^1$.
(vi) There exists a $t \in \mathbb{P}^1$ such that $\pi^{-1}(t)$ is smooth and the restriction $V|_{\pi^{-1}(t)}$ is semistable.

Proof. By (4.1) and (4.2), (iii) and (iv) are equivalent, and by (4.3) (i) $\implies$ (iii) $\implies$ (ii). The implication (ii) $\implies$ (i) is trivial. The implication (iv) $\implies$ (v) follows from the openness of semistability in the Zariski topology in the sense of schemes, and the implication (v) $\implies$ (vi) is trivial. To see that (vi) $\implies$ (iv), suppose that $V_t$ is not semistable. Then a destabilizing sub-line bundle extends to give a sub-line bundle over the pullback of $S$ to some finite base change of $\mathbb{P}^1$. Thus $V|_{\pi^{-1}(t)}$ is unstable for every $t \in \mathbb{P}^1$ such that $\pi^{-1}(t)$ is smooth, and so (vi) $\implies$ (iv).

Remark. In case $\Delta \cdot \kappa \equiv 0 \mod 2$, there can exist strictly semistable bundles on $S_\eta$ of degree $\Delta \cdot f$. There are examples of rank two bundles $V$ on $S$ with $c_1(V) = \Delta$ and $V$ strictly semistable such that $V$ is either stable, strictly semistable, or unstable (cf. [8]).

5. Donaldson polynomials and the main theorems.

As above we let $S$ denote a simply connected elliptic surface with $p_g(S) \geq 0$. Fix $w = \Delta \mod 2$ and let $p$ be an integer satisfying $w^2 \equiv p \mod 4$. For $p_g(S) > 0$, there is the Donaldson polynomial $\gamma_{w,p}(S)$ corresponding to the $SO(3)$-bundle $P$ with invariants $w$ and $p$. Here for simplicity we shall always choose the orientation on the moduli space which agrees with the natural complex orientation. The polynomial $\gamma_{w,p}(S)$ is invariant up to sign under self-diffeomorphisms $\psi$ of $S$ such that $\psi^*w = w$. If $p_g(S) = 0$, then we have the distinguished chamber $C(w, p)$ which contains $\kappa$ in its closure. We shall then use $\gamma_{w,p}(S)$ to denote the Donaldson polynomial for $S$ with respect to the chamber $C(w, p)$, again with the orientation chosen to be the complex orientation. Since $\psi^*C(w, p) = \pm C(\psi^*w, p)$, the invariant $\gamma_{w,p}(S)$ is again natural up to sign under orientation-preserving self-diffeomorphisms which fix $w$. Of course, there are only finitely many choices for $w$, so that there is a subgroup of finite index in the full group of diffeomorphisms fixing $[f]$ which will also fix $w$.

Lemma 5.1. For every choice of $w$ and $p$, $\gamma_{w,p}(S)$ lies in $\mathbb{Q}[q_S, \kappa_S]$. Moreover, if for some choice of $w$ and $p$, $\gamma_{w,p}(S)$ does not lie in $\mathbb{Q}[q_S]$, then every diffeomorphism $\psi$ from $S$ to another simply connected elliptic surface $S'$ satisfies $\psi^*\kappa_{S'} = \pm \kappa_S$.

Proof. The set of automorphisms of $H_2(S; \mathbb{Z})$ of the form $\psi_*$, where $\psi$ is a diffeomorphism satisfying $\psi_*(\lfloor f \rfloor) = \lfloor f \rfloor$, $\psi^*w = w$, and $\psi^*\gamma_{w,p}(S) = \gamma_{w,p}(S)$, is a subgroup of finite index in the group of all isometries of $H_2(S; \mathbb{Z})$ preserving $\lfloor f \rfloor$, by [8] Part I Theorem 6 and [9] Chapter 2 Theorem 6.5. Thus by [9] Chapter 6 Theorem 2.12, $\gamma_{w,p}(S) \in \mathbb{Q}[q_S, \kappa_S]$. Moreover $\kappa$ is the unique such class. The last statement of the lemma is then clear.
Next let us discuss the effect of blowing up. Suppose that $\rho: \tilde{S} \to S$ is the $r$-fold blowup of $S$, and let the exceptional classes in $H_2(\tilde{S})$ be denoted by $e_1, \ldots, e_r$. Likewise let $S'$ be another simply connected elliptic surface and let $\rho': \tilde{S}' \to S'$ be the $r$-fold blowup of $S'$, with exceptional classes $e'_1, \ldots, e'_{r'}$. If $\psi: S \to \tilde{S}'$ is a diffeomorphism, then $\psi^* e'_i = \pm e_j$ for a uniquely determined $j$, by [8] Part I Theorem 7 and [9] Chapter 6 Corollary 3.8. It follows that, if $w' \in H^2(\tilde{S}'; \mathbb{Z}/2\mathbb{Z})$ is of the form $(\rho')^* w'_0$ for some $w'_0 \in H^2(S'; \mathbb{Z}/2\mathbb{Z})$, then there is a $w_0 \in H^2(S; \mathbb{Z}/2\mathbb{Z})$, such that $\psi^* w' = \rho^* w_0$. Finally, we shall need the following extension of [9] Chapter 6 Theorem 3.1:

**Proposition 5.2.** Let $w_0 \in H^2(S; \mathbb{Z}/2\mathbb{Z})$ and let $\rho: \tilde{S} \to S$ be the $r$-fold blowup of $S$. If $b_2^+ (S) = 1$, assume moreover that $\gamma_{\rho^* w_0, p}$ is defined with respect to some chamber $D$. Let $C$ be a chamber of type $(w_0, p)$ on $H^2(S; \mathbb{R})$ such that $D$ contains $\rho^* C$ in its closure. Then

$$\gamma_{\rho^* w_0, p}|\rho^* H_2(S; \mathbb{Z}) = \gamma_{w_0, p},$$

where if $b_2^+ (S) = 1$, the polynomial $\gamma_{w_0, p}$ is defined with respect to the chamber $C$.

Here, in case $p_g(S) = 0$, the chamber $C$ does not in general determine a unique chamber $D$ on $\tilde{S}$. However the conclusion of the proposition implies in particular that the value of $\gamma_{\rho^* w_0, p}$ on classes in $\rho^* H_2(S; \mathbb{Z})$ is independent of the chamber for $\tilde{S}$ of type $(\rho^* w_0, p)$ which contains $C$ in its closure.

This result follows from standard gauge theory techniques [5]. It can also be proved in our case via algebraic geometry, using the blowup formulas for instance in [8]. Since it does not appear with an explicit proof in the literature, we shall outline a proof in the only case that concerns us, where the chamber $C$ contains the first Chern class of an ample line bundle. We shall just write down the argument in the most interesting case, where $p_g(S) = 0$. We shall also assume that the moduli space has the expected dimension, although the arguments given here can easily be extended to handle the general case.

By induction we may assume that $\rho: \tilde{S} \to S$ is the blowup of $S$ at a single point $p$. Let $E$ be the exceptional curve and $\epsilon$ be its cohomology class. We shall usually identify $H^2(S)$ with its image in $H^2(\tilde{S})$ under $\rho^*$. Let $D$ be a chamber for $\tilde{S}$ of type $(\rho^* w_0, p)$ containing $C$ in its closure and let $\zeta$ be a wall for $D$. Then $\zeta = \zeta' + a\epsilon$, where $\zeta' \in H^2(S; \mathbb{Z})$ and $a \in \mathbb{Z}$ (in fact $2a$ since $\zeta \equiv \Delta \mod 2$). After possibly reflecting in $\epsilon$, which is realized by an orientation-preserving diffeomorphism $r_\epsilon$ of $\tilde{S}$, we may assume that $a \geq 0$: Indeed, $r_\epsilon$ switches the two possible chambers corresponding to $\pm a\epsilon$, and so, if $\gamma_1$ and $\gamma_2$ are the two invariants corresponding to the two choices of chambers, then $r_\epsilon^* \gamma_1 = \gamma_2$. Since $r_\epsilon^* H_2(S; \mathbb{Z})$ is the identity, it suffices to prove the result for either chamber. So we can assume that $a \geq 0$.

Since $C$ is in the closure of $D$, if $x \in C$, then $x \cdot \zeta' = x \cdot \zeta \geq 0$. Conversely, if we start with an ample line bundle $L$ on $S$ such that $c_1(L) \in C$, then for all $N \gg 0$, $Nc_1(L) - \epsilon$ is the first Chern class of an ample line bundle $L_N$ on $\tilde{S}$. Moreover $(Nc_1(L) - \epsilon) \cdot \zeta' \geq Nc_1(L) \cdot \zeta' \geq 0$. It follows from this that $c_1(L_N)$ lies in $D$, and if $c_1(L)$ is in the interior of $C$ then $c_1(L_N)$ is in the interior of $D$.

Consider rank two vector bundles $V$ on $\tilde{S}$ with $c_1(V) = \rho^* \Delta$ and $c_2(V) = \epsilon$. Set $V = (\rho, \tilde{V})^\vee$. Then $V$ is a rank two vector bundle on $S$ with $c_1(V) = \Delta$ and $c_2(V) \leq c_2(\tilde{V})$, where equality holds if and only if $\tilde{V} = \rho^* V$. The arguments of
the proof of Theorem 5.5 in Part II of [8], which essentially just depend on the
determinant of $V$ being a pullback, show the following. There is a constant $N_0$, 
depending only on $L$ and $c$, such that, for all $N \geq N_0$, if $\tilde{V}$ is $L_N$-stable then $V$
is $L$-semistable, and conversely if $V$ is $L$-stable then $\tilde{V}$ is $L_N$-stable. Moreover 
the map $V \mapsto \varrho^* V$ defines an open immersion of schemes from the moduli space of 
$L$-stable rank two vector bundles on $S$ with $c_1 = \Delta$ and $c_2 = c$ to the corresponding 
moduli space for $\tilde{S}$ and $c_1 = \rho^* \Delta$.

To evaluate the Donaldson polynomial on $\tilde{S}$ on a collection of classes of the form 
$\rho^* \alpha$, represent $\alpha$ by a smoothly embedded Riemann surface $C$ on $S$ which does 
not pass through $p$, the center of the blowup, and choose a theta characteristic on $C$. 
This choice leads to a divisor $D_C$ on the moduli space. By definition $\tilde{V}$ lies in 
$D_C$ if and only if the Dirac operator coupled to the ASD connection induced on $C$ has a kernel. 
From this it is clear that $\tilde{V}$ lies in $D_C$ if and only if $V$ lies in the corresponding 
divisor on the moduli space for $S$ of bundles with $c_1 = \Delta$ and $c_2 = c_2(V) \leq c$. An easy counting argument then shows that, if $d$ is the dimension 
of the moduli space for $\tilde{S}$ and we choose $C_1, \ldots, C_d$ in general position and general 
theta characteristics on $C_i$, then $\tilde{V}$ lies in the intersection $D_{C_1} \cap \cdots \cap D_{C_d}$ 
if and only if $V = \rho^* V$ and $V$ lies in the corresponding intersection for the moduli 
space of $S$. As $#(D_{C_1} \cap \cdots \cap D_{C_d})$ calculates the value $\gamma_{\rho^* w_0, p}(\{C_1, \ldots, [C_d]\})$, it is then 
clear that

$$
\gamma_{\rho^* w_0, p}(\rho^* H_2(S; \mathbb{Z}) = \gamma_{w_0, p}.
$$

Assuming Theorems 1.4 and 1.5, we can now state the main results of this series 
of papers.

**Theorem 5.3.** Suppose that $S$ and $S'$ are two simply connected elliptic surfaces 
with $p_g(S) = p_g(S') = 1$. Suppose that neither $S$ nor $S'$ is a $K3$ surface. Let 
$\tilde{S}$ and $\tilde{S}'$ be two blowups of $S$ and $S'$, and let $\psi: \tilde{S} \rightarrow \tilde{S}'$ be a diffeomorphism. 
Identify $H^2(S'; \mathbb{Z})$ with its image in $H^2(\tilde{S}; \mathbb{Z})$ under the natural map, and similarly 
for $H^2(S'; \mathbb{Z})$. Then $\psi^* \kappa_{S'} = \pm \kappa_S$.

**Proof.** Arguing as in Corollary 3.6 of Chapter 6 of [9], we see that it suffices to show 
that some $\gamma_{w_0, p}(S)$ actually involves $\kappa_S$. If $m_1 m_2 \equiv 1 \pmod{2}$, then the coefficient 
of $\kappa_S^2$ in $\gamma_{w_0, p}(S)$, for the choice of $p$ given in Theorem 1.5 (i) corresponding to 
the two-dimensional moduli space, is $2m_1^2 m_2^2 - m_1^2 - m_2^2$. This number is zero if 
m_1 = m_2 = 1, in which case $S$ is a $K3$ surface. Otherwise $2m_1^2 m_2^2 - m_1^2 - m_2^2 > 0$. 
Thus the coefficient of $\kappa_S^2$ is nonzero.

If $m_1 m_2 \equiv 0 \pmod{2}$, then the expected dimension of the moduli space is $4c - 
\Delta^2 - 6 \equiv \Delta^2 \mod{2}$. Moreover $\Delta \cdot \kappa_S = 1$ and $K_S = (2m_1 m_2 - m_1 - m_2) \kappa$. Since 
exactly one of $m_1, m_2$ is even, $\Delta \cdot K_S \equiv 1 \mod{2}$. Thus by the Wu formula $\Delta^2 \equiv 1 
\mod{2}$, and the dimension of the moduli space is odd. It follows that every nonzero 
invvariant must involve $\kappa_S$. Since nonzero invariants exist by Theorem 1.4 or more 
generally by Donaldson’s theorem on the nonvanishing of the invariants, there are 
choices of $p$ for which $\gamma_{w_0, p}(S)$ actually involves $\kappa_S$. □

**Theorem 5.4.** Suppose that $S$ and $S'$ are two elliptic surfaces with $p_g(S) = 
p_g(S') \geq 1$ with finite cyclic fundamental group and multiple fibers of multiplici-
ties $\{m_1, m_2\}$ and $\{m'_1, m'_2\}$, respectively, that $\tilde{S}$ and $\tilde{S}'$ are two blowups of $S$ and 
$S'$, and that $\tilde{S}$ and $\tilde{S}'$ are diffeomorphic. Then $\{m_1, m_2\} = \{m'_1, m'_2\}$. Hence $S$ 
and $S'$ are deformation equivalent.
Proof. As in [9] we can reduce to the simply connected case. Using Theorem 1.3, we know that $m_1m_2 = m_1'm_2'$ and that, if $\psi: \tilde{S} \to \tilde{S}'$ is a diffeomorphism, then $\psi_*(H_2(S;\mathbb{Z})) = H_2(S';\mathbb{Z})$ under the identification of $H_2(S;\mathbb{Z})$ with a subspace of $H_2(\tilde{S};\mathbb{Z})$ and likewise for $S'$. Fix a class $w \in H^2(S';\mathbb{Z})$ with $w \cdot \kappa_{S'} = 1$. Thus

$$\gamma_{\psi^*w,p}(\tilde{S})|H_2(S;\mathbb{Z}) = \gamma_{w,p}(\tilde{S}')|H_2(S';\mathbb{Z})$$

under the natural identifications. Moreover $\psi^*w \cdot \kappa_{S} = w \cdot \kappa_{S'} = 1$. Thus the Donaldson polynomial invariants for the minimal surfaces $S$ and $S'$ for the values $\psi^*w$ and $w$ respectively are equal. If $m_1m_2 = m_1'm_2' \equiv 1 \mod 2$, then

$$(p_g(S) + 1)m_1^2m_2^2 - m_1^2 - m_2^2 = (p_g(S) + 1)(m_1')^2(m_2')^2 - (m_1')^2 - (m_2')^2.$$ 

Thus $m_1m_2 = m_1'm_2'$ and $m_1^2 + m_2^2 = (m_1')^2 + (m_2')^2$. It follows that $(m_1 + m_2)^2 = (m_1' + m_2')^2$ and so $m_1 + m_2 = m_1' + m_2'$. We can thus determine the elementary symmetric functions of $m_1$ and $m_2$ from the diffeomorphism type, and hence the unordered pair.

If $m_1m_2 = m_1'm_2' \equiv 0 \mod 2$, then, assuming that $2|m_1$, it follows from Theorem 1.4 that we can determine $m_1m_2$ and $m_2$. Thus, we can determine $m_1$ as well. □

Theorem 5.5. Suppose that $S$ and $S'$ are two nonrational elliptic surfaces with finite cyclic fundamental group and with $p_g(S) = p_g(S') = 0$ with multiple fibers of multiplicities $\{m_1, m_2\}$ and $\{m_1', m_2'\}$, respectively, that $\tilde{S}$ and $\tilde{S}'$ are two blowups of $S$ and $S'$, and that $\tilde{S}$ and $\tilde{S}'$ are diffeomorphic. Suppose further that $m_1m_2 \equiv 0 \mod 2$. Then $m_1m_2' \equiv 0 \mod 2$, and $\{m_1, m_2\} = \{m_1', m_2'\}$.

Proof. We may again reduce to the simply connected case. Note that every diffeomorphism $\psi$ from $\tilde{S}$ to $\tilde{S}'$ sends the subspace $H^2(S';\mathbb{Z})$ to $H^2(S;\mathbb{Z})$. Choose a class $w \in H^2(S';\mathbb{Z}/2\mathbb{Z})$ with $w \cdot \kappa_{S'} = 1$. We must have $\psi^*\mathcal{C}(w,p) = \pm \mathcal{C}(\psi^*w,p)$, by Lemma 3.5. As in the preceding argument, we are immediately reduced to comparing the Donaldson invariants for the surfaces $S$ and $S'$. Let us first show that $m_1m_2' \equiv 0 \mod 2$. First note that the two-dimensional invariant corresponds to $-p = 5 > 4 = 2(4p_g(S) + 2)$. Thus we are in the stable range and can apply Theorem 1.4 to conclude that the leading coefficient of $\gamma_{w,-5}(S) = m_2$. Since $S$ is not rational $m_2 > 1$. But if $m_1m_2' \equiv 1 \mod 2$, then by (i) of Theorem 1.5 the leading coefficient of $\gamma_{w,-5}(S')$ is 1, a contradiction. Hence $m_1m_2' \equiv 0 \mod 2$ and the leading coefficient of $\gamma_{w,-5}(S')$ is just $m_2'$. It follows that $m_2 = m_2'$ and, by Bauer’s result (Theorem 1.2), that $(m_1^2 - 1)(m_2^2 - 1) = ((m_1')^2 - 1)((m_2')^2 - 1)$. Thus $m_1 = m_1'$ as well. □

Theorem 5.6. Suppose that $S$ and $S'$ are two nonrational elliptic surfaces with finite cyclic fundamental group and with $p_g(S) = p_g(S') = 0$ with multiple fibers of multiplicities $\{m_1, m_2\}$ and $\{m_1', m_2'\}$, respectively, that $\tilde{S}$ and $\tilde{S}'$ are two blowups of $S$ and $S'$, and that $\tilde{S}$ and $\tilde{S}'$ are diffeomorphic. Suppose further that $m_1m_2 \equiv 1 \mod 2$. Then $m_1'm_2' \equiv 1 \mod 2$, and $\{m_1, m_2\} = \{m_1', m_2'\}$.

Proof. As before we pass to the simply connected case. If $m_1'm_2' \equiv 0 \mod 2$, then by (5.5) $m_1m_2 \equiv 0 \mod 2$ as well, a contradiction. Thus $m_1'm_2' \equiv 1 \mod 2$. Using
(i) and (ii) of Theorem 1.5, we see that the Donaldson polynomials determine the quantities

\[ A = m_1^2 m_2^2 - m_1^2 - m_2^2 + 1 = (m_1^2 - 1)(m_2^2 - 1); \]
\[ B = m_1^4 m_2^4 - m_1^4 - m_2^4 + 1 = (m_1^4 - 1)(m_2^4 - 1). \]

We must show that \( A \) and \( B \) determine \( \{m_1, m_2\} \) provided that both \( m_1 \) and \( m_2 \) are greater than one. This is just a matter of elementary algebra: let \( \sigma_1 = m_1^2 + m_2^2 \) and \( \sigma_2 = m_1^3 m_2^3 \). Then \( \sigma_1 \) and \( \sigma_2 \) are the elementary symmetric functions in \( m_1^2 \) and \( m_2^2 \) and thus determine \( \{m_1^2, m_2^2\} \). As \( m_1 \) and \( m_2 \) are positive the knowledge of \( \{m_1^2, m_2^2\} \) determines \( \{m_1, m_2\} \).

To read off \( \sigma_1 \) and \( \sigma_2 \) from \( A \) and \( B \), note that if \( A \neq 0 \) then

\[ B = (m_1^2 + 1)(m_2^2 + 1) = m_1^2 m_2^2 + m_1^2 + m_2^2 + 1. \]

Thus \( 2\sigma_2 = B/A + A - 2 \) and \( 2\sigma_1 = B/A - A \), provided that \( A \neq 0 \). Now \( A = 0 \) if \( m_1 \) or \( m_2 \) is one, and otherwise \( A \geq 1 \). Thus, provided neither of \( m_1 \) or \( m_2 \) is one, \( A \) and \( B \) determine \( \sigma_2 \) and \( \sigma_1 \). □

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