DISCRETE $G$-SPECTRA AND EMBEDDINGS OF MODULE SPECTRA

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Abstract. In this paper we study the category of discrete $G$-spectra for a profinite group $G$. We consider an embedding of module objects in spectra into a category of module objects in discrete $G$-spectra, and study the relationship between the embedding and the homotopy fixed points functor. We also consider an embedding of module objects in terms of quasi-categories, and show that the two formulations of embeddings are essentially equivalent.

1. Introduction

Let $G$ be a profinite group. The theory of $G$-Galois extensions of structured ring spectra was introduced by Rognes in [17], and he gave an interpretation of the results in [4] in terms of $G$-Galois extensions. The category of discrete $G$-spectra was introduced by Davis in [5], and $G$-Galois extensions were also studied by Behrens-Davis [1] in terms of discrete $G$-spectra. The algebraic Galois theory is related to the descent theory, which has been important in algebraic geometry, number theory, category theory and homotopy theory. The framework of homotopical descent theory was developed by Hess [8] and Lurie [15]. In this paper we discuss some kind of homotopical descent theory related to $G$-Galois extensions of spectra.

Let $\Sigma Sp(G)$ be the model category of discrete symmetric $G$-spectra and let $\Sigma Sp(G)_k$ be its left Bousfield localization with respect to a spectrum $k$. Suppose we have a map $A \to B$ of monoids in discrete symmetric $G$-spectra, where $G$ acts on $A$ trivially. There is a functor

$$\text{Ex} : \text{Mod}_A(\Sigma Sp_k) \to \text{Mod}_B(\Sigma Sp(G)_k)$$

from the category $\text{Mod}_A(\Sigma Sp_k)$ of $A$-modules in $\Sigma Sp_k$ to the category $\text{Mod}_B(\Sigma Sp(G)_k)$ of $B$-modules in $\Sigma Sp(G)_k$ by the extension of scalars. This functor has a right adjoint, which we can regard as a homotopy fixed points functor $(\cdot)^{hG}$. Furthermore, the pair of functors is a $\Sigma Sp$-Quillen adjunction. Let $T$ be the full subcategory of the homotopy category $\text{Ho}(\text{Mod}_A(\Sigma Sp_k))$ consisting of $X$ such that the unit map $X \to L\text{Ex}(X)^{hG}$ is an equivalence. We show that the restriction of the total left derived functor $L\text{Ex}$ to $T$ is fully faithful as an $\text{Ho}(\Sigma Sp)$-enriched functor (Proposition 1.3). In particular, we have a $K(n)$-local $G_n$-Galois extension $L_{K(n)} F_n \to F_n$ by [1], where $K(n)$ is the $n$th Morava $K$-theory at a prime $p$ and $G_n$ is the $n$th extended Morava stabilizer group. Using the result in [6], we obtain the following theorem.

Theorem 1.1 (Theorem 1.6). The total left derived functor

$$L\text{Ex} : \text{Ho}(\Sigma Sp_{K(n)}) \to \text{Ho}(\text{Mod}_{F_n}(\Sigma Sp(G_n)_{K(n)}))$$

is fully faithful as an $\text{Ho}(\Sigma Sp)$-enriched functor.

Next we consider embeddings of modules in quasi-categories. Let $\text{Sp}_k$ be the underlying quasi-category of the model category $\Sigma Sp_k$. Suppose we have a map $A \to E$ of monoid objects in
Sp_k. We have a functor from the quasi-category Mod_A(Sp_k) of A-modules to the quasi-category Mod_E(Sp_k) of E-modules by the extension of scalars. This defines a comonad Θ on the E-modules, and we can consider the quasi-category Comod_{E,Θ}(Sp_k) of comodules over the comonad Θ. The extension of scalars functor factors through Comod_{E,Θ}(Sp_k), and we obtain a functor

\[ \text{Coex} : \text{Mod}_A(Sp_k) \to \text{Comod}_{(E,Θ)}(Sp_k). \]

This functor has a right adjoint P, which is an homotopical analogue of the functor taking primitive elements. Let T be the full subcategory of Mod_A(Sp_k) consisting of X such that the unit map X → PCoex(X) is an equivalence. We show that the restriction of the functor Coex to T is fully faithful (Proposition 5.2).

Finally, we compare the two formulations of embeddings. Let A → B be a map of monoids in discrete symmetric G-spectra, where G acts on A trivially. Let Mod_B(Sp(G)_k) be the underlying quasi-category of the model category Mod_B(ΣSp(G)_k). Under some mild conditions we show that there is an equivalence of quasi-categories

\[ \text{Mod}_B(Sp(G)_k) \simeq \text{Comod}_{(UB,Θ)}(Sp_k) \]

(Corollary 6.12), where UB is the underlying monoid object of B in Sp_k. As a corollary, we obtain that the functor Ex is equivalent to the functor Coex under the equivalence between Mod_B(Sp(G)_k) and Comod_{(UB,Θ)}(Sp_k) (Corollary 6.13), where we regard Ex as a functor of the underlying quasi-categories. This shows that the two formulations of embeddings are essentially equivalent. In particular, we show that the two formulations are essentially equivalent if A → B is a k-local G-Galois extension (Theorem 7.3).

The organization of this paper is as follows: In §3 we discuss the model structure on the category of discrete symmetric G-spectra. We show that the category of discrete symmetric G-spectra is a proper, combinatorial, symmetric monoidal ΣSp-model category satisfying the monoid axiom. We also discuss the Bousfield localization with respect to a spectrum with trivial G-action. In §4 we discuss embeddings of modules into the category of discrete symmetric G-spectra. We also discuss the relationship between the embeddings and the homotopy fixed points functors. In §5 we consider embeddings of quasi-categories of modules in spectra. For an adjunction of quasi-categories, we can consider the quasi-category of comodules over the comonad associated to the adjunction. We show that some full subcategory can be embedded into the quasi-category of comodules. In §6 we study the quasi-category of discrete G-spectra. We show that the quasi-category of discrete G-spectra can be described as a quasi-category of comodules. Finally we show that the two formulations of embeddings of module categories are essentially equivalent under some conditions. In §7 we discuss embeddings associated to profinite G-Galois extensions. We show that the two formulations are essentially equivalent for profinite G-Galois extensions.

2. Notation

For a model category M, we denote by Ho(M) the homotopy category of M. For X, Y ∈ M, we denote by [X, Y]_M the set of morphisms in Ho(M). If M is a simplicial model category, we denote by Map_M(X, Y) the mapping simplicial set for X, Y ∈ M. We denote by M° the full subcategory of M consisting of X such that X is both fibrant and cofibrant. If M is a simplicial model category, N(M°) is the underlying quasi-category of M, where N(−) is the simplicial nerve functor.

We denote by ΣSp the category of symmetric spectra. We give ΣSp the stable model structure. We denote by Sp the underlying quasi-category of ΣSp. For a spectrum k, we denote by ΣSp_k the left Bousfield localization of ΣSp with respect to k, and by Sp_k the underlying quasi-category of ΣSp_k. We denote by S the quasi-category of spaces. For a quasi-category C, we denote by Map_C(X, Y) the mapping space for X, Y ∈ C.
3. Model structure on the category of discrete symmetric $G$-spectra

Let $G$ be a profinite group. In this section we discuss model structure on the category of discrete symmetric $G$-spectra. We also study the Bousfield localization with respect to a spectrum with trivial $G$-action.

3.1. Discrete symmetric $G$-spectra. We denote by $\mathrm{Set}(G)$ the category of discrete $G$-sets. A simplicial discrete $G$-set is a simplicial object in $\mathrm{Set}(G)$. The model structure on the category of simplicial discrete $G$-sets was studied in [7]. We denote by $\mathrm{sSet}(G)$, the category of pointed simplicial discrete $G$-sets.

Let $\mathrm{sSet}(G)^{\Sigma}$ be the category of symmetric sequences in $\mathrm{sSet}(G)$. The category $\mathrm{sSet}(G)^{\Sigma}$ is a closed symmetric monoidal category. Let $S$ be a symmetric sequence given by $S = (S^0, S^1, S^2, \ldots)$, where $S^n$ is the $n$-sphere with trivial $G$-action. The symmetric sequence $S$ is a commutative monoid object in $\mathrm{sSet}(G)^{\Sigma}$. A discrete symmetric $G$-spectrum is a module object in $\mathrm{sSet}(G)^{\Sigma}$ over the commutative monoid $S$. A map of discrete symmetric $G$-spectra is a map of module objects. We denote by $\Sigma \mathrm{Sp}(G)$ the category of discrete symmetric $G$-spectra.

The category $\Sigma \mathrm{Sp}(G)$ is a complete, cocomplete, closed symmetric monoidal category with $S$ as the unit object. We denote the monoidal structure by $X \land Y = X \otimes_S Y$. We have an adjoint pair

$$\mathrm{triv} : \Sigma \mathrm{Sp} \rightleftarrows \Sigma \mathrm{Sp}(G) : (-)^G,$$

where $(-)^G$ is the $G$-fixed points functor and the functor $\mathrm{triv}(-)$ associates to a symmetric spectrum $X$ the discrete symmetric $G$-spectrum $X$ with trivial $G$-action. Notice that $\mathrm{triv}$ is a strong symmetric monoidal functor and that $(-)^G$ is a lax symmetric monoidal functor.

We denote by $G^\delta$ the group $G$ with discrete topology. Let $\Sigma \mathrm{Sp}(G^\delta)$ be the category of symmetric spectra with (continuous) $G^\delta$-action. The continuous homomorphism $G^\delta \to G$ induces a functor $(-)^\delta : \Sigma \mathrm{Sp}(G) \to \Sigma \mathrm{Sp}(G^\delta)$. For $X \in \Sigma \mathrm{Sp}(G^\delta)$, we denote by $dX$ the largest discrete $G$-subspectrume of $X$, that is,

$$dX = \operatorname{colim}_{H} X^H,$$

where $H$ ranges over all open subgroups $H$ of $G$. We can regard $d$ as a functor $d : \Sigma \mathrm{Sp}(G^\delta) \to \Sigma \mathrm{Sp}(G)$. Notice that we have an adjoint pair

$$(-)^\delta : \Sigma \mathrm{Sp}(G) \rightleftarrows \Sigma \mathrm{Sp}(G^\delta) : d.$$

We recall the model structure on $\Sigma \mathrm{Sp}(G)$. Let $U : \Sigma \mathrm{Sp}(G) \to \Sigma \mathrm{Sp}$ be the forgetful functor. A map $f : X \to Y$ in $\Sigma \mathrm{Sp}(G)$ is said to be

- a cofibration if $U(f)$ is a cofibration of symmetric spectra,
- a weak equivalence if $U(f)$ is a stable equivalence of symmetric spectra, and
- a fibration if it has the right lifting property with respect to all maps which are both cofibrations and weak equivalences.

With these definitions, $\Sigma \mathrm{Sp}(G)$ is a left proper cellular model category by [11] Theorem 2.3.2).

We shall show that $\Sigma \mathrm{Sp}(G)$ is a proper combinatorial model category. Since $\Sigma \mathrm{Sp}(G)$ is a cofibrantly generated model category, it suffices to show that $\Sigma \mathrm{Sp}(G)$ is locally presentable in order to show that $\Sigma \mathrm{Sp}(G)$ is combinatorial.

**Theorem 3.1.** The category $\Sigma \mathrm{Sp}(G)$ is locally presentable. Hence $\Sigma \mathrm{Sp}(G)$ is a combinatorial model category.

**Proof.** It is easy to see that the category $\mathrm{Set}(G)$ of discrete $G$-sets is locally presentable. Since $\Sigma \mathrm{Sp}(G)$ is the category of symmetric spectra based on the category of pointed discrete $G$-sets, we see that $\Sigma \mathrm{Sp}(G)$ is also locally presentable. \qed

**Proposition 3.2.** The model category $\Sigma \mathrm{Sp}(G)$ is proper.
Proof. It suffices to show that $\Sigma \text{Sp}(G)$ is right proper. Since $\Sigma \text{Sp}$ is right proper by [13, Theorem 5.5.2], this follows from the fact that the forgetful functor $U$ preserves fiber products and detects weak equivalences.

Next we consider the compatibility of the monoidal structure and the model structure on $\Sigma \text{Sp}(G)$. Note that the adjoint pair of functors

$$\text{triv} : \Sigma \text{Sp} \rightleftarrows \Sigma \text{Sp}(G) : (-)^G$$

is a Quillen adjunction by [5, Corollary 3.9].

**Theorem 3.3.** The category $\Sigma \text{Sp}(G)$ is a symmetric monoidal $\Sigma \text{Sp}$-model category. The adjoint pair $(\text{triv}, (-)^G)$ is a symmetric monoidal $\Sigma \text{Sp}$-Quillen adjunction.

**Proof.** The model category $\Sigma \text{Sp}(G)$ is a symmetric monoidal model category by [11, Theorem 8.11]. The theorem follows from the fact that $\text{triv} : \Sigma \text{Sp} \to \Sigma \text{Sp}(G)$ is a strong symmetric monoidal left Quillen functor.

**Corollary 3.4.** The symmetric monoidal model category $\Sigma \text{Sp}(G)$ satisfies the monoid axiom.

**Proof.** We have to show that every map in

$$(\{\text{trivial cofibrations}\} \land \Sigma \text{Sp}(G))\text{-cof}_{\text{reg}}$$

is a weak equivalence. By definition of the model structure on $\Sigma \text{Sp}(G)$, it is sufficient to show that the underlying map is a stable equivalence of symmetric spectra. Since the underlying map of a trivial cofibration is a trivial cofibration of symmetric spectra, the proposition follows from the fact that the category of symmetric spectra with stable model structure satisfies the monoid axiom by [13, Theorem 5.4.1].

The forgetful functor $U : \Sigma \text{Sp}(G) \to \Sigma \text{Sp}$ has a right adjoint

$$V : \Sigma \text{Sp} \to \Sigma \text{Sp}(G).$$

The discrete symmetric $G$-spectrum $V(X)$ for $X \in \Sigma \text{Sp}$ is given by

$$V(X) = d(\text{Map}(G, X)) = \text{Map}_c(G, X).$$

We can easily verify the following proposition.

**Proposition 3.5** (cf. [5, Corollary 3.8]). The adjoint pair of functors

$$U : \Sigma \text{Sp}(G) \rightleftarrows \Sigma \text{Sp} : V$$

is a symmetric monoidal $\Sigma \text{Sp}$-Quillen adjunction.

3.2. **Bousfield localization of** $\Sigma \text{Sp}(G)$. Let $k$ be a symmetric spectrum. We say that a morphism $f$ in $\Sigma \text{Sp}(G)$ is a $k$-local equivalence if $U(f)$ is a $k$-local equivalence in $\Sigma \text{Sp}$. Let $W_k$ be the class of $k$-local equivalences in $\Sigma \text{Sp}(G)$. As in [11 page 5015], there exists a left Bousfield localization $\Sigma \text{Sp}(G)_k$ with respect to $W_k$, and $\Sigma \text{Sp}(G)_k$ is left proper and cellular.

**Proposition 3.6.** The Bousfield localization $\Sigma \text{Sp}(G)_k$ is a combinatorial model category.

**Proof.** By Theorems 3.1 and 3.3, we see that $\Sigma \text{Sp}(G)$ is a left proper combinatorial simplicial model category. As in [11 page 5015], the class of $k$-local equivalences are that of $f$-local equivalences for some map $f$. Hence the Bousfield localization $\Sigma \text{Sp}(G)_k$ is combinatorial by [14 Proposition A.3.7.3].

In the following of this paper we assume that $k$ is cofibrant for simplicity.

**Theorem 3.7.** The model category $\Sigma \text{Sp}(G)_k$ is a symmetric monoidal $\Sigma \text{Sp}$-model category.
Proof. First, we show that \( \Sigma \text{Sp}(G)_k \) is a symmetric monoidal model category. Let \( A \to B \) be a cofibration and let \( X \to Y \) be a trivial cofibration in \( \Sigma \text{Sp}(G)_k \). Since \( k \) is cofibrant, we see that \( X \wedge k \to Y \wedge k \) is a trivial cofibration in \( \Sigma \text{Sp}(G) \). By Theorem 3.3, \( \Sigma \text{Sp}(G) \) is a monoidal model category, and hence the map

\[
(A \wedge Y \wedge k) \coprod_{(A \wedge X \wedge k)} (B \wedge X \wedge k) \to B \wedge Y \wedge k
\]

is a trivial cofibration. We have an isomorphism

\[
\begin{pmatrix}
(A \wedge Y) \coprod_{(A \wedge X)} (B \wedge X) \\
\end{pmatrix} \wedge k \cong \begin{pmatrix}
(A \wedge Y \wedge k) \coprod_{(A \wedge X \wedge k)} (B \wedge X \wedge k) \\
\end{pmatrix},
\]

and hence the map

\[
(A \wedge Y) \coprod_{(A \wedge X)} (B \wedge X) \to B \wedge Y
\]

is a trivial cofibration in \( \Sigma \text{Sp}(G)_k \). Therefore we see that \( \Sigma \text{Sp}(G)_k \) is a symmetric monoidal model category.

Next we show that \( \Sigma \text{Sp}(G)_k \) is a monoidal \( \Sigma \text{Sp} \)-model category. Recall that the functor \( \text{triv} : \Sigma \text{Sp} \to \Sigma \text{Sp}(G)_k \) is a strong symmetric monoidal left Quillen functor. Furthermore, the identity functor \( \text{id} : \Sigma \text{Sp}(G)_k \to \Sigma \text{Sp}(G)_k \) is a strong symmetric monoidal left Quillen functor. Hence the composition \( \text{id} \circ \text{triv} : \Sigma \text{Sp} \to \Sigma \text{Sp}(G)_k \) is also a strong symmetric monoidal left Quillen functor. This shows that \( \Sigma \text{Sp}(G)_k \) is a monoidal \( \Sigma \text{Sp} \)-model category. \( \Box \)

Proposition 3.8. The symmetric monoidal model category \( \Sigma \text{Sp}(G)_k \) satisfies the monoid axiom.

Proof. Recall that \( W_k \) is the class of \( k \)-local equivalences. We let \( C \) be the class of cofibrations in \( \Sigma \text{Sp}(G)_k \). Since the functor \( (\_ \wedge k) \) preserves all colimits, we see that

\[
(((C \cap W_k) \wedge \Sigma \text{Sp}(G)) \text{-cof}_{\text{reg}}) \wedge k \subset (((C \cap W) \wedge \Sigma \text{Sp}(G)) \text{-cof}_{\text{reg}}),
\]

where \( W \) is the class of weak equivalences in \( \Sigma \text{Sp}(G) \). By Proposition 3.4, \( \Sigma \text{Sp}(G) \) satisfies the monoid axiom. Hence \( \Sigma \text{Sp}(G) \text{-cof}_{\text{reg}} \subset W \). This shows that \( \Sigma \text{Sp}(G)_k \text{-cof}_{\text{reg}} \subset W_k \).

This completes the proof. \( \Box \)

Proposition 3.9. The adjoint pair of functors

\[
\text{triv} : \Sigma \text{Sp}_k \rightleftarrows \Sigma \text{Sp}(G)_k : (-)^G
\]

is a symmetric monoidal \( \Sigma \text{Sp} \)-Quillen adjunction.

Proof. It suffices to show that \( \text{triv} \) preserves \( k \)-local equivalences. Let \( f \) be a \( k \)-local equivalence in \( \Sigma \text{Sp} \). Since \( U(\text{triv}(f)) = f \), we see \( \text{triv}(f) \in W_k \). Hence \( \text{triv} \) preserves \( k \)-local equivalences. \( \Box \)

Proposition 3.10. The adjoint pair of functors

\[
U : \Sigma \text{Sp}(G)_k \rightleftarrows \Sigma \text{Sp}_k : V
\]

is a symmetric monoidal \( \Sigma \text{Sp} \)-Quillen adjunction.

Proof. By definition, \( U \) preserves weak equivalences and cofibrations. \( \Box \)

Proposition 3.11. If \( K \) is a cofibrant object in \( \Sigma \text{Sp}(G)_k \), then the functor \( K \wedge (\_ : \Sigma \text{Sp}(G)_k \to \Sigma \text{Sp}(G)_k \) preserves weak equivalences.
Proof. The forgetful functor \( U : \Sigma Sp(G)_k \to \Sigma Sp_k \) is monoidal. Furthermore, \( U \) preserves and detects weak equivalences. Hence it suffices to show that the functor \( U(K) \wedge (-) : \Sigma Sp_k \to \Sigma Sp_k \) preserves weak equivalences. Since \( U \) is a left Quillen functor, \( U(K) \) is cofibrant in \( \Sigma Sp_k \), and hence \( U(K) \wedge f \) is a weak equivalence in \( \Sigma Sp \). By [12, Lemma 5.4.4], \( U(K) \wedge f \wedge k \) is also a weak equivalence in \( \Sigma Sp \). This implies that \( U(K) \wedge f \) is a \( k \)-local equivalence. This completes the proof. \( \square \)

3.3. Filtered colimits in \( \Sigma Sp(G)_k \). In this subsection we shall show that filtered colimit preserves weak equivalences in \( \Sigma Sp(G)_k \).

Let \( \Lambda \) be a filtered category, and let \( F : X \to Y \) be a natural transformation of functors from \( \Lambda \) to \( \Sigma Sp(G)_k \).

Lemma 3.12. If \( F(\lambda) : X(\lambda) \to Y(\lambda) \) is a trivial fibration in \( \Sigma Sp(G)_k \) for all \( \lambda \in \Lambda \), then the induced map on colimits

\[
\colim_{\lambda \in \Lambda} X \longrightarrow \colim_{\lambda \in \Lambda} Y
\]

is also a trivial fibration.

Proof. Let \( I \) be a set of maps of the form \( \langle \partial \Delta^k \times G/N \rangle_{+} \to \langle \Delta^k \times G/N \rangle_{+} \) in \( \Sigma Sp(G)_k \), where \( k \geq 0 \) and \( N \) is an open subgroup of \( G \). Let \( F_n : \text{sSet}(G)_* \to \Sigma Sp(G) \) be the left adjoint to the evaluation functor \( Ev_n \). We can take \( I^\Sigma = \bigcup_{n \geq 0} F_n(I) \) as a set of generating cofibrations of \( \Sigma Sp(G)_k \). Hence it suffices to show that the map colimit \( \colim X(\lambda) \to \colim Y(\lambda) \) has the right lifting property with respect to \( I^\Sigma \). By adjointness, we see that it suffices to show that the map \( \colim Ev_n X(\lambda)^N \to \colim Ev_n Y(\lambda)^N \) in \( \text{sSet}_* \) has the right lifting property with respect to all the maps \( \partial \Delta^k_+ \to \Delta^k_+ \) for \( k \geq 0 \). Since \( X(\lambda) \to Y(\lambda) \) is a trivial fibration, the induced map \( Ev_n X(\lambda)^N \to Ev_n Y(\lambda)^N \) is a trivial fibration in \( \text{sSet}_* \). Since \( \partial \Delta^k_+ \) and \( \Delta^k_+ \) are compact objects in \( \text{sSet}_* \), we see that \( \colim Ev_n X(\lambda)^N \to \colim Ev_n Y(\lambda)^N \) has the right lifting property with respect to \( \partial \Delta^k_+ \to \Delta^k_+ \). This completes the proof. \( \square \)

By Proposition 3.10, \( \Sigma Sp(G)_k \) is a combinatorial model category. By [14, Proposition A.2.8.2], the diagram category \( \text{Fun}(\Lambda, \Sigma Sp(G)_k) \) supports the projective model structure. Since the colimit functor

\[ \colim : \text{Fun}(\Lambda, \Sigma Sp(G)_k) \to \Sigma Sp(G)_k \]

is a left Quillen functor, it preserves trivial cofibrations.

Proposition 3.13. If \( F(\lambda) : X(\lambda) \to Y(\lambda) \) is a \( k \)-local equivalence for all \( \lambda \in \Lambda \), then the induced map on colimits

\[
\colim_{\lambda \in \Lambda} X \longrightarrow \colim_{\lambda \in \Lambda} Y
\]

is also a \( k \)-local equivalence.

Proof. Let \( X \to Z \to Y \) be a factorization of \( F \) in \( \text{Fun}(\Lambda, \Sigma Sp(G)_k) \) such that \( X \to Z \) is a trivial cofibration and \( Z \to Y \) is a trivial fibration. Since the colimit functor preserves trivial cofibrations, \( \colim X \to \colim Z \) is a trivial cofibration. The proposition follows from the fact that the colimit functor preserves trivial fibrations by Lemma 3.12. \( \square \)

Corollary 3.14. For any diagram \( X \in \text{Fun}(\Lambda, \Sigma Sp(G)_k) \), the canonical map

\[
\hocolim_{\lambda \in \Lambda} X \longrightarrow \colim_{\lambda \in \Lambda} X
\]

in \( \text{Ho}(\Sigma Sp(G)_k) \) is an isomorphism.
Proof. Let $X' \to X$ be a cofibrant replacement in $\text{Fun}(\Lambda, \Sigma \text{Sp}(G)_k)$. The homotopy colimit $\text{hocolim} X$ is represented by $\text{colim} X'$. The map $\text{colim} X' \to \text{colim} X$ induced on the colimits is a $k$-local equivalence by Proposition 3.13. \qed

4. Embeddings of modules in $\Sigma \text{Sp}(G)_k$

In this section we discuss embeddings of modules in $\Sigma \text{Sp}(G)_k$. Let $A$ be a monoid in $\Sigma \text{Sp}_k$. We regard $A$ as a monoid in $\Sigma \text{Sp}(G)_k$ with trivial $G$-action. For a map $\varphi : A \to B$ of monoids in $\Sigma \text{Sp}(G)_k$, we show that a certain full subcategory of $\text{Ho}(\text{Mod}_A(\Sigma \text{Sp}_k))$ can be embedded into $\text{Ho}(\text{Mod}_B(\Sigma \text{Sp}(G)_k))$ as an $\text{Ho}(\Sigma \text{Sp})$-enriched category. We also discuss the relationship between the embedding and the homotopy fixed points functor.

4.1. Model structure on module categories. Let $R$ be a monoid object in a combinatorial symmetric monoidal $\Sigma \text{Sp}$-model category $\mathcal{M}$ satisfying the monoid axiom. We denote by $\text{Mod}_R(\mathcal{M})$ the category of left $R$-module objects in $\mathcal{M}$. The category $\text{Mod}_R(\mathcal{M})$ is a $\Sigma \text{Sp}$-module in the sense of [10] §4.1. If $R$ is a commutative monoid object, then $\text{Mod}_R(\mathcal{M})$ is a closed symmetric monoidal category and a symmetric $\Sigma \text{Sp}$-algebra in the sense of [10] §4.1.

A map $f : M \to N$ in $\text{Mod}_R(\mathcal{M})$ is said to be

- a weak equivalence if it is a weak equivalence in $\mathcal{M}$,
- a fibration if it is a fibration in $\mathcal{M}$, and
- a cofibration if it has the left lifting property with respect to all maps which are both fibrations and weak equivalences.

With these definitions, $\text{Mod}_R(\mathcal{M})$ is a $\Sigma \text{Sp}$-module category by [13] Theorem 4.1. Note that the unit object $R$ is cofibrant in $\text{Mod}_R(\mathcal{M})$. If $R$ is commutative monoid object, then $\text{Mod}_R(\mathcal{M})$ is a symmetric monoidal $\Sigma \text{Sp}$-model category.

Let $\varphi : A \to B$ be a map of monoids in $\mathcal{M}$. We have an adjoint pair of functors

$$B \otimes_A (-) : \text{Mod}_A(\mathcal{M}) \rightleftarrows \text{Mod}_B(\mathcal{M}) : \varphi^*,$$

where $\varphi^*$ is the restriction of scalars functor. We can verify that $\varphi^*$ preserves fibrations and weak equivalences. Hence we obtain the following lemma.

Lemma 4.1. The adjoint pair $(B \otimes_A (-), \varphi^*)$ is a $\Sigma \text{Sp}$-Quillen adjunction.

4.2. Adjoint between module categories. Let $\mathcal{M}$ and $\mathcal{N}$ be combinatorial symmetric monoidal $\Sigma \text{Sp}$-model categories satisfying the monoid axiom. We suppose that we have a strong symmetric monoidal left Quillen functor $i : \mathcal{M} \to \mathcal{N}$. We denote by $j : \mathcal{N} \to \mathcal{M}$ its right adjoint. We take monoid objects $A$ in $\mathcal{M}$ and $B$ in $\mathcal{N}$. We suppose that there is a morphism of monoid objects

$$\varphi : i(A) \to B.$$

By Lemma 4.1 the morphism $\varphi$ induces a $\Sigma \text{Sp}$-Quillen adjunction $(B \otimes_{i(A)} (-), \varphi^*)$. Since $i$ is strong symmetric monoidal, it induces a functor $i : \text{Mod}_A(\mathcal{M}) \to \text{Mod}_{i(A)}(\mathcal{N})$. We see that $j$ induces a right adjoint $j : \text{Mod}_{i(A)}(\mathcal{N}) \to \text{Mod}_A(\mathcal{M})$ to $i$. Composing these two adjunctions, we obtain an adjoint pair of functors

$$\text{Ex} : \text{Mod}_A(\mathcal{M}) \rightleftarrows \text{Mod}_B(\mathcal{N}) : \text{Re}.$$

Lemma 4.2. The pair $(\text{Ex}, \text{Re})$ is a $\Sigma \text{Sp}$-Quillen adjunction.

Proof. By definition of the model structures, $\varphi^*$ and $j$ are right Quillen functors. Hence the composition $\text{Re} = j \varphi^*$ is also a right Quillen functor. \qed
Since \( \text{Ho}(\text{Mod}_A(M)) \) and \( \text{Ho}(\text{Mod}_B(N)) \) are \( \text{Ho}(\Sigma \text{Sp}) \)-modules, they are triangulated categories. By Lemma 4.2 we obtain an adjunction of \( \text{Ho}(\Sigma \text{Sp}) \)-modules

\[
\text{L} \text{Ex} : \text{Ho}(\text{Mod}_A(M)) \rightleftarrows \text{Ho}(\text{Mod}_B(N)) : \text{Re}.
\]

Note that \( \text{L} \text{Ex} \) and \( \text{Re} \) are exact functors. Let \( T \) be the full subcategory of \( \text{Ho}(\text{Mod}_A(M)) \) consisting of \( X \) such that the unit map \( X \to \text{Re} \text{L} \text{Ex}(X) \) is an isomorphism

\[
T = \{ X \in \text{Ho}(\text{Mod}_A(M)) | X \xrightarrow{\sim} \text{Re} \text{L} \text{Ex}(X) \}.
\]

It is easy to see that \( T \) is a thick subcategory of \( \text{Ho}(\text{Mod}_A(M)) \).

**Proposition 4.3.** The restriction of \( \text{L} \text{Ex} \) to \( T \) is fully faithful as an \( \text{Ho}(\Sigma \text{Sp}) \)-enriched functor.

**Proof.** This follows from the natural isomorphism

\[
\text{RMap}_{\text{Mod}_B}(\text{L} \text{Ex}(X), \text{L} \text{Ex}(Y)) \cong \text{RMap}_{\text{Mod}_A}(X, \text{R} \text{Re} \text{L} \text{Ex}(Y))
\]

in \( \text{Ho}(\Sigma \text{Sp}) \). \( \square \)

### 4.3. Homotopy fixed points functor

By Proposition 4.3, we have a \( \Sigma \text{Sp} \)-Quillen adjunction

\[
\text{triv} : \Sigma \text{Sp}_k \rightleftarrows \Sigma \text{Sp}(G)_k : (-)^G,
\]

where \( \text{triv} \) is strong symmetric monoidal. Let \( A \) be a monoid object in \( \Sigma \text{Sp}_k \) and we regard \( A \) as a monoid object in \( \Sigma \text{Sp}(G)_k \) with trivial \( G \)-action. Let \( \varphi : A \to B \) be a map of monoid objects in \( \Sigma \text{Sp}(G)_k \). By Lemma 4.2, we have a \( \Sigma \text{Sp} \)-Quillen adjunction

\[
\text{Ex} : \text{Mod}_A(\Sigma \text{Sp}_k) \rightleftarrows \text{Mod}_B(\Sigma \text{Sp}(G)_k) : \text{Re}.
\]

This induces an adjunction

\[
\text{L} \text{Ex} : \text{Ho}(\text{Mod}_A(\Sigma \text{Sp}_k)) \rightleftarrows \text{Ho}(\text{Mod}_B(\Sigma \text{Sp}(G)_k)) : \text{R} \text{Re}
\]

of \( \text{Ho}(\Sigma \text{Sp}) \)-modules. We define a homotopy fixed points functor \( (-)^{hG} \) to be the total right derived functor of \( \text{Re} \):

\[
(-)^{hG} = \text{R} \text{Re} : \text{Ho}(\text{Mod}_B(\Sigma \text{Sp}(G)_k)) \to \text{Ho}(\text{Mod}_A(\Sigma \text{Sp}_k)).
\]

Although the definition of the homotopy fixed points spectrum \( X^{hG} \) depends on the map \( \varphi \), we shall show that the homotopy type of the underlying spectrum of \( X^{hG} \) is independent from \( \varphi \). There is a commutative diagram

\[
\begin{array}{ccc}
\text{Mod}_B(\Sigma \text{Sp}(G)_k) & \xrightarrow{(-)^G} & \text{Mod}_A(\Sigma \text{Sp}_k) \\
F_B \downarrow & & \downarrow F_A \\
\Sigma \text{Sp}(G)_k & \xrightarrow{(-)^G} & \Sigma \text{Sp}_k,
\end{array}
\]

where \( F_A : \text{Mod}_A(\Sigma \text{Sp}_k) \to \Sigma \text{Sp}_k \) and \( F_B : \text{Mod}_B(\Sigma \text{Sp}(G)_k) \to \Sigma \text{Sp}(G)_k \) are forgetful functors. Since \( F_A \) and \( F_B \) preserve weak equivalences, they induce functors \( hF_A : \text{Ho}(\text{Mod}_A(\Sigma \text{Sp}_k)) \to \text{Ho}(\Sigma \text{Sp}_k) \) and \( hF_B : \text{Ho}(\text{Mod}_B(\Sigma \text{Sp}(G)_k)) \to \text{Ho}(\Sigma \text{Sp}(G)_k) \), respectively. Since \( F_A \) and \( F_B \) preserve fibrations, we obtain a natural isomorphism

\[
hF_A(X^{hG}) \cong (hF_BX)^{hG}
\]

in \( \text{Ho}(\Sigma \text{Sp}_k) \). This means the homotopy type of the underlying spectrum of \( X^{hG} \) is independent from \( \varphi \).

Since \( A \) is cofibrant in \( \text{Mod}_A(\Sigma \text{Sp}_k) \), we have an isomorphism \( \text{L} \text{Ex}(A) \cong B \). Hence the unit of the adjunction \( (\text{L} \text{Ex}, \text{R} \text{Re}) \) gives a map \( A \to B^{hG} \) in \( \text{Ho}(\text{Mod}_A(\Sigma \text{Sp}_k)) \). Recall that \( T \) is the full
subcategory of $\text{Ho}(\text{Mod}_A(\Sigma \text{Sp}_k))$ consisting of $X$ such that the unit map $X \to \text{RRe} \text{LEx}(X)$ is an isomorphism. Hence $A \in T$ if the map $A \to B^{hG}$ is an isomorphism.

**Proposition 4.4.** If $A \to B^{hG}$ is an isomorphism, then $T$ contains all dualizable objects in $\text{Ho}(\text{Mod}_A(\Sigma \text{Sp}_k))$.

**Proof.** We put $M = \text{Mod}_A(\Sigma \text{Sp}_k)$ and $N = \text{Mod}_B(\Sigma \text{Sp}(G)_k)$. Let $X$ be a dualizable object in $\text{Ho}(M)$. We denote by $DX$ its dual. Since $\text{LEx}$ is strong symmetric monoidal, $\text{LEx}(X)$ is a dualizable object in $\text{Ho}(N)$ and its dual is $\text{LEx}(DX)$. For any $W \in \text{Ho}(M)$, we have a natural isomorphism

$$[W, \text{RRe} \text{LEx}(X)]_M \cong [\text{LEx}(W) \otimes^A_B \text{LEx}(DX), \text{LEx}(A)]_N.$$  

We have $\text{LEx}(W) \otimes^B_A \text{LEx}(DX) \cong \text{LEx}(W \otimes^A_A DX)$. This implies the following isomorphism

$$[\text{LEx}(W) \otimes^B_A \text{LEx}(DX), \text{LEx}(A)]_N \cong [W, \text{RRe} \text{LEx}(A) \otimes^A_A X]_M.$$  

By the Yoneda lemma, we obtain $\text{RRe} \text{LEx}(X) \cong \text{RRe} \text{LEx}(A) \otimes^A_A X$. By the assumption, we have $A \cong \text{RRe} \text{LEx}(A)$, and hence $\text{RRe} \text{LEx}(A) \otimes^A_A X \cong X$. This shows $X \cong \text{RRe} \text{LEx}(X)$. \qed

**Remark 4.5.** Suppose the localization functor $L_k$ satisfies Assumption 6.2 further. Furthermore, we suppose $G$ has a finite virtual cohomological dimension. If $B$ is a consistent $k$-local $G$-Galois extension of $L_k A$ in the sense of [1, Definition 6.2.1], then $\varphi$ induces an isomorphism $A \cong B^{hG}$ by [1 Proposition 6.1.7(3) and Corollary 6.3.2].

### 4.4. The $K(n)$-local category

Let $E_n$ be the $n$th Morava $E$-theory spectrum and let $K(n)$ be the $n$th Morava $K$-theory spectrum at a prime $p$. We denote by $G_n$ the extended Morava stabilizer group. Note that $G_n$ has a finite virtual cohomological dimension.

In [5 Definition 2.3] Davis constructed a discrete $G_n$-spectrum $F_n$ by using results in [1]. The spectrum $E_n$ is recovered by the equivalence

$$E_n \simeq L_{K(n)} F_n$$

in $\text{Ho}(\Sigma \text{Sp})$, where $L_{K(n)}$ is the Bousfield localization with respect to $K(n)$. In [1 §8.1] Behrens-Davis upgraded $F_n$ to be a commutative monoid object in $\Sigma \text{Sp}(G_n)$ and they showed that $F_n$ is a consistent $K(n)$-local $G_n$-Galois extension of $L_{K(n)} S$. This implies that the unit map $\varphi : S \to F_n$ induces an isomorphism $S \cong (F_n)^{hG_n} \in \text{Ho}(\Sigma \text{Sp}_{K(n)})$.

We consider the adjunction

$$\text{LEx} : \text{Ho}(\Sigma \text{Sp}_{K(n)}) \rightleftarrows \text{Ho}(\text{Mod}_{F_n}(\Sigma \text{Sp}(G_n)_{K(n)})) : \text{RRe}.$$  

By [5 Theorem 9.7] and [6 Theorem 1.1], we see that the map $X \to \text{RRe} \text{LEx}(X)$ is an isomorphism for any $X \in \text{Ho}(\Sigma \text{Sp}_{K(n)})$. Hence we obtain the following theorem by Proposition 4.3.

**Theorem 4.6.** The functor

$$\text{LEx} : \text{Ho}(\Sigma \text{Sp}_{K(n)}) \longrightarrow \text{Ho}(\text{Mod}_{F_n}(\Sigma \text{Sp}(G_n)_{K(n)}))$$

is fully faithful as an $\text{Ho}(\Sigma \text{Sp})$-enriched functor.

Let $W$ be the class of morphisms $f$ in $\text{Ho}(\text{Mod}_{F_n}(\Sigma \text{Sp}(G_n)_{K(n)}))$ such that $\text{RRe}(f)$ is an isomorphism.

**Corollary 4.7.** The $K(n)$-local category $\text{Ho}(\Sigma \text{Sp}_{K(n)})$ is equivalent to the localization of the homotopy category $\text{Ho}(\text{Mod}_{F_n}(\Sigma \text{Sp}(G_n)_{K(n)}))$ with respect to $W$ as an $\text{Ho}(\Sigma \text{Sp})$-enriched category

$$\text{Ho}(\Sigma \text{Sp}_{K(n)}) \simeq \text{Ho}(\text{Mod}_{F_n}(\Sigma \text{Sp}(G_n)_{K(n)}))[W^{-1}].$$
5. Embeddings of quasi-categories of modules over $\text{Sp}_k$

Let $\text{Sp}_k$ be the underlying quasi-category of the model category $\Sigma \text{Sp}_k$. Let $\psi : A \to E$ be a map of monoid objects in $\text{Sp}_k$. We have an adjunction of underlying quasi-categories

$$E \land_A (-) : \text{Mod}_A(\text{Sp}_k) \rightleftarrows \text{Mod}_E(\text{Sp}_k) : \psi^*.$$ 

In this section we discuss embedding of certain full subcategory of $\text{Mod}_A(\text{Sp}_k)$ into the quasi-category of comodules associated to the adjunction.

5.1. Quasi-category of comodules. Suppose we have an adjunction of quasi-categories $L : C \rightleftarrows D : R$.

This induces an adjunction of opposite quasi-categories $R^{\text{op}} : D^{\text{op}} \rightleftarrows C^{\text{op}} : L^{\text{op}}$.

By [15, Proposition 4.7.4.3], there is an endomorphism monad $\Theta \in \text{Alg}(\text{End}(D^{\text{op}}))$. We regard $\Theta$ as a comonad on $D$ and we define a quasi-category of left $\Theta$-comodules $\text{Comod}_\Theta(D)$ to be $L\text{Mod}_\Theta(D^{\text{op}})$.

Note that the functor $L : C \to D$ factors through a functor $L' : C \to \text{Comod}_\Theta(D)$ so that $UL' \simeq L$, where $U : \text{Comod}_\Theta(D) \to D$ is the forgetful functor. We say that $C$ is comonadic over $D$ if $L'$ is an equivalence of quasi-categories.

For a monoid object $A$ in $\text{Sp}_k$, we denote by $\text{Mod}_A(\text{Sp}_k)$ the category of left $A$-modules in $\text{Sp}_k$. For a map $\psi : A \to E$ of monoid objects in $\text{Sp}_k$, we have an adjunction of quasi-categories

$$L : \text{Mod}_A(\text{Sp}_k) \rightleftarrows \text{Mod}_E(\text{Sp}_k) : R,$$

where $L = E \land_A (-)$ and $R = \psi^*$. Hence we obtain a comonad $\Theta$ on $\text{Mod}_E(\text{Sp}_k)$ and a quasi-category of left $\Theta$-comodules $\text{Comod}_{\{E, \Theta\}}(\text{Sp}_k) = \text{Comod}_\Theta(\text{Mod}_E(\text{Sp}_k))$.

The functor $L$ factors through

$$\text{Coex} : \text{Mod}_A(\text{Sp}_k) \to \text{Comod}_{\{E, \Theta\}}(\text{Sp}_k),$$

so that $U\text{Coex} \simeq L$. We set

$$\Theta' = \text{Coex} R.$$

The functor $\Theta'$ is informally given by $\Theta'(X) = \Theta X$ with $\Theta$-comodule structure for $X \in \text{Mod}_E(\text{Sp}_k)$. Note that $\Theta'$ is a right adjoint to the forgetful functor $U$.

For $X \in \text{Comod}_{\{E, \Theta\}}(\text{Sp}_k)$, we have a cosimplicial object

$$C^\bullet(R, \Theta, UX)$$

in $\text{Mod}_A(\text{Sp}_k)$ by the cobar construction. We define a functor

$$P : \text{Comod}_{\{E, \Theta\}}(\text{Sp}_k) \to \text{Mod}_A(\text{Sp}_k)$$

by $PX = \lim C^\bullet(R, \Theta, UX)$. For $Y \in \text{Mod}_A(\text{Sp}_k)$, we have a coaugmented cosimplicial object

$$Y \to E^\bullet + 1 Y$$

in $\text{Mod}_A(\text{Sp}_k)$ given by

$$E^{k+1}Y = \bigwedge_A E \land_A \cdots \land_A \bigwedge_A Y.$$
with the usual cosimplicial structure. There is an equivalence of cosimplicial objects
\[ C^*(R, \Theta, U\text{Coex}(Y)) \simeq E^{*+1}Y. \]
Note that the map \( Y \to \lim E^{*+1}Y \) is an analogue of the derived completion of \( Y \) along \( \psi \) in the sense of [2]. In particular, if \( k = A = S \), then \( Y \to \lim E^{*+1}Y \) is an analogue of the \( E \)-nilpotent completion in the sense of [2].

**Proposition 5.1.** The functor \( \text{Coex} : \text{Mod}_A(\text{Sp}_k) \rightleftharpoons \text{Comod}_{(E, \Theta)}(\text{Sp}_k) : P \).

**Proof.** Let \( \mathcal{X} \) be the full subcategory of \( \text{Comod}_{(E, \Theta)}(\text{Sp}_k) \) consisting of \( X \) such that the functor
\[ \text{Map}_{\text{Comod}_{(E, \Theta)}(\text{Sp}_k)}(\text{Coex}(-), X) : \text{Mod}_A(\text{Sp}_k) \to S \]
is representable, where \( S \) is the quasi-category of spaces. We denote by \( \widetilde{P}(X) \) the representing object in \( \text{Mod}_A(\text{Sp}_k) \) for \( X \in \mathcal{X} \). In this case, \( \widetilde{P}(X) \) is well-defined up to canonical equivalence and we obtain a functor
\[ \widetilde{P} : \mathcal{X} \to \text{Mod}_A(\text{Sp}_k). \]

First, we shall show that \( \Theta'Z \in \mathcal{X} \) for any \( Z \in \text{Mod}_E(\text{Sp}_k) \). Since \( \Theta' \) is a right adjoint to \( U \) and \( U\text{Coex} \simeq L \) is a left adjoint to \( R \), we see that there is a natural equivalence
\[ \text{Map}_{\text{Comod}_{(E, \Theta)}(\text{Sp}_k)}(\text{Coex}(Y), \Theta'Z) \simeq \text{Map}_{\text{Mod}_A(\text{Sp}_k)}(Y, RZ) \]
for any \( Y \in \text{Mod}_E(\text{Sp}_k) \), and hence \( \Theta'Z \in \mathcal{X} \). Note that \( \widetilde{P}\Theta'Z \simeq RZ \).

Next, we shall show that \( X \in \mathcal{X} \) for any \( X \in \text{Comod}_{(E, \Theta)}(\text{Sp}_k) \). Let \( C^*(\Theta', \Theta, UX) \) be a cosimplicial object in \( \text{Comod}_{(E, \Theta)}(\text{Sp}_k) \) given by the cobar construction. By the dual of [15, Lemma 4.7.4.12], we see that \( \lim C^*(\Theta', \Theta, UX) \simeq X \).

Therefore, we obtain \( \mathcal{X} = \text{Comod}_{(E, \Theta)}(\text{Sp}_k) \). Since there is an equivalence \( \widetilde{P}C^*(\Theta', \Theta, UX) \simeq C^*(R, \Theta, UX) \) of cosimplicial objects, we see that \( \widetilde{P}X \simeq PX \) for any \( X \). This completes the proof.

Let \( \mathcal{T} \) be a full subcategory of \( \text{Mod}_A(\text{Sp}_k) \) consisting of \( X \) such that the unit map \( X \to P\text{Coex}(X) \) is an equivalence
\[ \mathcal{T} = \{ X \in \text{Mod}_A(\text{Sp}_k) | X \xrightarrow{\simeq} P\text{Coex}(X) \}. \]
In the same way as in Proposition \[4.3\] we obtain the following proposition.

**Proposition 5.2.** The restriction of \( \text{Coex} \) to \( \mathcal{T} \) is a fully faithful functor of quasi-categories.

5.2. Examples.

(1) Let \( k = S \) and let \( \psi : S \to MU \) be the unit map, where \( MU \) is the complex cobordism spectrum. We have an adjunction \( MU \wedge (-) : \text{Sp} \rightleftharpoons \text{Mod}_{MU}(\text{Sp}) : \psi^\ast \). We denote by \( MU \wedge MU \) the comonad on \( \text{Mod}_{MU}(\text{Sp}) \) associated to the adjoint pair \( (MU \wedge (-), \psi^\ast) \). If \( X \in \text{Sp} \) is connective, then the map \( X \to \lim MU^{*+1}X \) is an equivalence by [2, Theorem 6.5]. Hence \( X \in \mathcal{T} \). Let \( \text{Sp}^{\geq 0} \) be the full subcategory of \( \text{Sp} \) consisting of connective spectra. By Proposition \[5.2\] the functor
\[ MU \wedge (-) : \text{Sp}^{\geq 0} \to \text{Comod}_{(MU,MU \wedge MU)}(\text{Sp}) \]
is fully faithful.
Lemma 6.1. If \( \Sigma \text{Sp}((G) \rightarrow \Sigma \text{Sp}(G)) \) is comonadic over \( \text{Sp}_k \), then \( \text{Sp}(G)_k \) is equivalent to \( \text{Comod}_\Gamma(\text{Sp}_k) \), under some mild conditions.

Let \( Y^\bullet \) be a cosimplicial object in \( \Sigma \text{Sp}(G)^\circ \). We denote by \( Y \) the homotopy limit of \( Y^\bullet \) in \( \Sigma \text{Sp}(G) \), and by \( Z \) the homotopy limit of \( UY^\bullet \) in \( \Sigma \text{Sp} \). We use a model of homotopy limits in [2] Chapter 18. There is a canonical map \( UY \rightarrow Z \) in \( \Sigma \text{Sp} \).

**Lemma 6.1.** If \( G \) has finite cohomological dimension, then the map \( UY \rightarrow Z \) is a weak equivalence.
Proof. By functoriality of homotopy limits, we can regard $Z$ as an object in $\Sigma Sp(G)$. By the construction of homotopy limits, we see that $Y = dZ$, where $dZ$ is the largest discrete $G$-subspecrum of $Z$. Hence we obtain

$$Y = \text{colim}_N \text{holim}_\Delta (Y^\bullet)^N,$$

where $N$ ranges over all open normal subgroups of $G$.

The discrete $G$-spectrum $Y^r$ is fibrant in $\Sigma Sp(G)$ for any $r \geq 0$. By [1] Proposition 3.3.1(2)], we see that the fixed points spectrum $(Y^r)^N$ is equivalent to the homotopy fixed points spectrum $(Y^r)^{hN}$. By [5] Theorem 4.2.4.1, we see that $(Y^r)^N \simeq \text{holim}_\Delta \text{Map}_c(G^{\bullet+1},Y^r)^N$. Hence we obtain

$$\text{colim}_N \text{holim}_\Delta (Y^\bullet)^N \simeq \text{colim}_N \text{holim}_\Delta \text{holim}_\Delta \text{Map}_c(G^{\bullet+1},Y^\bullet)^N \simeq \text{colim}_N \text{holim}_\Delta \text{Map}_c(G^{\bullet+1},Z)^N.$$

By the proof of [5] Theorem 7.4], we have a spectral sequence

$$E_2^{p,q} \cong \text{colim}_N H^p_c(N;\pi_q(Z)) \implies \pi_{q-p}(Y).$$

Since $E_2^{0,q} \cong \pi_q(Z)$ and $E_2^{0,q} = 0$ for $p > 0$, we see that the inclusion map $Y \to Z$ is a weak equivalence in $\Sigma Sp$.

We consider the following assumption on the localization functor $L_k$.

**Assumption 6.2** (cf. [1] Assumption 1.0.3)). The localization functor $L_k$ is given as a composite of two localization functors $L_M L_T$, where $L_T$ is a smashing localization and $L_M$ is a localization with respect to a finite spectrum $M$.

**Theorem 6.3.** If $G$ has finite cohomological dimension and the localization functor $L_k$ satisfies Assumption 6.2, then $Sp(G)_k$ is comonadic over $Sp_k$, that is, we have an equivalence of quasi-categories

$$Sp(G)_k \cong \text{Comod}_L(Sp_k).$$

**Proof.** By definition of $k$-local equivalences in $\Sigma Sp(G)_k$, the forgetful functor $U : Sp(G)_k \to Sp_k$ is conservative. Let $X^\bullet$ be a cosimplicial object in $Sp(G)_k$ that is $U$-split. By [14] Corollary 4.2.4.8], the limit of $X^\bullet$ exists. Hence it is sufficient to show that $U$ preserves the limit of $X^\bullet$ by [15] Theorem 4.7.4.5]. By [14] Proposition 4.2.4.4], there is a cosimplicial object $Y^\bullet$ in $\Sigma Sp(G)_k$ such that $N(Y^\bullet) \simeq X^\bullet$. Applying the forgetful functor $U$, we obtain a cosimplicial object $UY^\bullet$ in $\Sigma Sp$. Let $UY^\bullet \to L_k UY^\bullet$ be a functorial fibrant replacement in $\Sigma Sp_k$. Note that $UY^r$ is not $k$-local in general but $T$-local by [1] Proposition 6.1.7(2)]. By [1] Corollary 6.1.3], we obtain a $k$-local equivalence

$$\text{holim}_\Delta UY^\bullet \cong \text{holim}_\Delta L_k UY^\bullet.$$

Let $Y = \text{holim}_\Delta Y^\bullet$ be the homotopy limit of $Y^\bullet$ in $\Sigma Sp(G)_k$. Using Lemma 6.1, we see that $UY$ is equivalent to $\text{holim}_\Delta UY^\bullet$ in $\Sigma Sp$. Hence $UY$ is $k$-locally equivalent to $\text{holim}_\Delta L_k UY^\bullet$. By [14] Theorem 4.2.4.1], we see that $U$ preserves the limit of $X^\bullet$.

### 6.2. The quasi-category of algebra objects in $Sp(G)_k$.

Let $\text{Alg}(\Sigma Sp(G)_k)$ be the category of monoid objects in $\Sigma Sp(G)_k$, and let $F : \text{Alg}(\Sigma Sp(G)_k) \to \Sigma Sp(G)_k$ be the forgetful functor. By [15] Theorem 4.1(3)], $\text{Alg}(\Sigma Sp(G)_k)$ supports a model structure as follows. A map $f : X \to Y$ in $\text{Alg}(\Sigma Sp(G)_k)$ is said to be

- a weak equivalence if $F(f)$ is a weak equivalence in $\Sigma Sp(G)_k$,
- a fibration if $F(f)$ is a fibration in $\Sigma Sp(G)_k$, and
- a cofibration if it has the right lifting property with respect to all maps which are both fibrations and weak equivalences.
By Proposition 3.6, Theorem 3.7 and Proposition 3.8, $\Sigma Sp(G)_k$ is a combinatorial symmetric monoidal model category which satisfies the monoid axiom. By [15, Proposition 4.1.4.3], we see that $\text{Alg}(\Sigma Sp(G)_k)$ is a simplicial model category and that the forgetful functor $F : \text{Alg}(\Sigma Sp(G)_k) \to \Sigma Sp(G)_k$ is a simplicial right Quillen functor.

Let $\text{Alg}(Sp(G)_k)$ be the quasi-category of algebra objects in $Sp(G)_k$. By [15, Theorem 4.1.4.4] and [15, Theorem 1.3.4.20], there is an equivalence of quasi-categories

$$N(\text{Alg}(\Sigma Sp(G)_k)^{op}) \simeq \text{Alg}(Sp(G)_k).$$

For $Y \in \text{Alg}(\Sigma Sp_k)$, we have an object $\text{Map}_c(G,Y)$ in $\Sigma Sp(G)_k$. We consider a map

$$\text{Map}_c(G,Y) \land \text{Map}_c(G,Y) \longrightarrow \text{Map}_c(G,Y)$$

in $\Sigma Sp(G)_k$, which is an adjoint of the map

$$U(\text{Map}_c(G,Y) \land \text{Map}_c(G,Y)) \cong U(\text{Map}_c(G,Y)) \land U(\text{Map}_c(G,Y)) \xrightarrow{ev(e) \land ev(e)} Y \land Y \xrightarrow{m} Y,$$

where $ev(e)$ is the evaluation map at the identity element $e \in G$ and $m$ is the multiplication map on $Y$. By this map, we can regard $\text{Map}_c(G,Y)$ as an object in $\text{Alg}(\Sigma Sp(G)_k)$. Hence we obtain a functor

$$V : \text{Alg}(\Sigma Sp_k) \longrightarrow \text{Alg}(\Sigma Sp(G)_k)$$

given by $V(Y) = \text{Map}_c(G,Y)$. Note that $V$ is a right adjoint of the forgetful functor $U$.

**Proposition 6.4.** The adjoint pair of functors

$$U : \text{Alg}(\Sigma Sp(G)_k) \rightleftarrows \text{Alg}(\Sigma Sp_k) : V$$

is a simplicial Quillen adjunction.

**Proof.** Let $F : \text{Alg}(\Sigma Sp(G)_k) \to \Sigma Sp(G)_k$ be the forgetful functor. We consider the following commutative diagram

$$\begin{array}{ccc}
\text{Alg}(\Sigma Sp_k) & \xrightarrow{V} & \text{Alg}(\Sigma Sp(G)_k) \\
\downarrow F & & \downarrow F \\
\Sigma Sp_k & \xrightarrow{V} & \Sigma Sp(G)_k.
\end{array}$$

By Proposition 3.10, $V : \Sigma Sp_k \to \Sigma Sp(G)_k$ is a right Quillen functor. This implies that $V : \text{Alg}(\Sigma Sp_k) \to \text{Alg}(\Sigma Sp(G)_k)$ is also a right Quillen functor. 

By Proposition 6.4, we have an adjunction of quasi-categories

$$U : \text{Alg}(Sp(G)_k) \rightleftarrows \text{Alg}(Sp_k) : V.$$ 

This induces a map of quasi-categories

$$\text{Alg}(Sp(G)_k) \longrightarrow \text{Comod}_\Gamma(\text{Alg}(Sp_k)),$$

where $\Gamma$ is a comonad on $\text{Alg}(Sp_k)$ associated to the adjunction $(U,V)$.

**Theorem 6.5.** Let $G$ be a profinite group that has finite cohomological dimension. We assume that the localization functor $L_k$ satisfies Assumption 6.2. The quasi-category $\text{Alg}(Sp(G)_k)$ is comonadic over $\text{Alg}(Sp_k)$, that is, we have an equivalence of quasi-categories

$$\text{Alg}(Sp(G)_k) \xrightarrow{\simeq} \text{Comod}_\Gamma(\text{Alg}(Sp_k)).$$
Proof. The forgetful functor $U : \text{Alg}(\Sigma \text{Sp}(G)_k) \to \text{Alg}(\text{Sp}_k)$ is conservative. Let $X^\bullet$ be a cosimplicial object in $\text{Alg}(\Sigma \text{Sp}(G)_k)$ such that $UX^\bullet$ is split. Since $\text{Alg}(\Sigma \text{Sp}(G)_k)$ is the underlying quasi-category of $\text{Alg}(\Sigma \text{Sp}(G)_k)$, $X^\bullet$ has a limit by [14 Corollary 4.2.4.8]. Hence it is sufficient to show that $U$ preserves the limit of $X^\bullet$ by [15 Theorem 4.7.4.5]. By [14 Proposition 4.2.4.4], there is a cosimplicial object $Y^\bullet$ in $\text{Alg}(\Sigma \text{Sp}(G)_k)^{op}$ such that $N(Y^\bullet) \simeq X^\bullet$. We have to show that the map $U(\text{holim}_\Delta Y^\bullet) \to \text{holim}_\Delta UY^\bullet$ is a weak equivalence in $\text{Alg}(\Sigma \text{Sp}_k)$.

Let $F : \text{Alg}(\Sigma \text{Sp}(G)_k) \to \Sigma \text{Sp}(G)_k$ be the forgetful functor. We consider the following commutative diagram

$$
\begin{array}{ccc}
\text{Alg}(\Sigma \text{Sp}(G)_k) & \xrightarrow{U} & \text{Alg}(\Sigma \text{Sp}_k) \\
F \downarrow & & \downarrow F \\
\Sigma \text{Sp}(G)_k & \xrightarrow{U} & \Sigma \text{Sp}_k.
\end{array}
$$

Since $F$ is conservative, it is sufficient to show that $FU(\text{holim}_\Delta Y^\bullet) \to F(\text{holim}_\Delta UY^\bullet)$ is a weak equivalence in $\Sigma \text{Sp}_k$. Since $F$ is a simplicial right Quillen functor, $F$ preserves homotopy limits. Hence we have $FU(\text{holim}_\Delta Y^\bullet) \simeq U(\text{holim}_\Delta FY^\bullet)$ and $F(\text{holim}_\Delta UY^\bullet) \simeq \text{holim}_\Delta UFY^\bullet$. By the proof of Theorem 6.3, the map $U(\text{holim}_\Delta FY^\bullet) \to \text{holim}_\Delta UFY^\bullet$ is a weak equivalence in $\Sigma \text{Sp}_k$.

This completes the proof.

6.3. The quasi-category of module objects in $\Sigma \text{Sp}(G)_k$. Let $B$ be an object in $\Sigma \text{Sp}(G)_k$. We assume that $B$ is cofibrant in $\Sigma \text{Sp}(G)_k$. We denote by $UB$ the underlying monoid object in $\Sigma \text{Sp}_k$. Note that $UB$ is cofibrant in $\Sigma \text{Sp}_k$ by Proposition [3.10]. By [15 Theorem 1.3.4.20 and Theorem 4.3.3.17], the underlying quasi-categories of $\text{Mod}_B(\Sigma \text{Sp}(G)_k)$ and $\text{Mod}_B(\Sigma \text{Sp}_k)$ are equivalent to $\text{Mod}_B(\text{Sp}(G)_k)$ and $\text{Mod}_B(\text{Sp}_k)$, respectively.

For $M \in \text{Mod}_B(\Sigma \text{Sp}_k)$, we regard $\text{Map}_c(G, M)$ as an object in $\Sigma \text{Sp}(G)_k$. We consider a map $B \wedge \text{Map}_c(G, M) \to \text{Map}_c(G, M)$ in $\Sigma \text{Sp}(G)_k$, which is an adjoint of the map

$$
U(B \wedge \text{Map}_c(G, M)) \cong UB \wedge U\text{Map}_c(G, M) \xrightarrow{\text{id} \wedge \text{ev}(e)} UB \wedge M \xrightarrow{a} M,
$$

where $\text{ev}(e)$ is the evaluation map at the identity element $e \in G$ and $a$ is the action map on $M$. This defines a $B$-module structure on $\text{Map}_c(G, M)$, and we see that $\text{Map}_c(G, M)$ is an object in $\text{Mod}_B(\Sigma \text{Sp}(G)_k)$. Hence we obtain a functor $V : \text{Mod}_B(\Sigma \text{Sp}_k) \to \text{Mod}_B(\Sigma \text{Sp}(G)_k)$ given by $V(M) = \text{Map}_c(G, M)$. Note that $V$ is the right adjoint of the forgetful functor $U$, and hence we have an adjunction

$$
U : \text{Mod}_B(\Sigma \text{Sp}(G)_k) \rightleftarrows \text{Mod}_B(\Sigma \text{Sp}_k) : V.
$$

Lemma 6.6. The adjoint pair $(U, V)$ is a $\Sigma \text{Sp}$-Quillen adjunction.

Proof. This follows from Proposition [3.10].

The $\Sigma \text{Sp}$-Quillen adjunction $(U, V)$ induces an adjunction of quasi-categories

$$
U : \text{Mod}_B(\Sigma \text{Sp}(G)_k) \rightleftarrows \text{Mod}_B(\text{Sp}_k) : V.
$$

Let $\Gamma$ be a comonad on $\text{Mod}_B(\text{Sp}_k)$ associate to the adjoint pair $(U, V)$, and let

$$
\text{Comod}_{(U,B,\Gamma)}(\text{Sp}_k) = \text{Comod}_\Gamma(\text{Mod}_B(\text{Sp}_k))
$$

be the quasi-category of comodules over $\Gamma$.
Theorem 6.7. Let \( G \) be a profinite group that has finite cohomological dimension. We assume that the localization functor \( L_k \) satisfies Assumption \( \ref{assumption} \). The quasi-category \( \text{Mod}_B(\Sigma Sp(G)_k) \) is comonadic over \( \text{Mod}_{UB}(\Sigma Sp_k) \), that is, we have an equivalence of quasi-categories

\[
\text{Mod}_B(\Sigma Sp(G)_k) \xrightarrow{\simeq} \text{Comod}_L(\text{Mod}_{UB}(\Sigma Sp_k)).
\]

Proof. This can be proved in the same way as in Theorem \( \ref{comonadic} \). The forgetful functor \( U : \text{Mod}_B(\Sigma Sp(G)_k) \to \text{Mod}_{UB}(\Sigma Sp_k) \) is conservative. Let \( X^\bullet \) be a cosimplicial object in \( \text{Mod}_B(\Sigma Sp(G)_k) \) such that \( UX^\bullet \) is split. By \( \cite{14} \) Corollary 4.2.4.8, \( X^\bullet \) has a limit. By \( \cite{14} \) Theorem 4.7.4.5, it is sufficient to show that \( U \) preserves the limit of \( X^\bullet \). By \( \cite{14} \) Proposition 4.2.4.4, there is a cosimplicial object \( Y^\bullet \) in \( \text{Mod}_B(\Sigma Sp(G)_k)^o \) such that \( N(Y^\bullet) \simeq X^\bullet \). We have to show that the map \( U(\text{holim}_\Delta Y^\bullet) \to \text{holim}_\Delta UY^\bullet \) is a weak equivalence in \( \text{Mod}_{UB}(\Sigma Sp_k) \).

Let \( F : \text{Mod}_B(\Sigma Sp(G)_k) \to \Sigma Sp(G)_k \) be the forgetful functor. We consider the following commutative diagram

\[
\begin{array}{ccc}
\text{Mod}_B(\Sigma Sp(G)_k) & \xrightarrow{U} & \text{Mod}_{UB}(\Sigma Sp_k) \\
\downarrow{F} & & \downarrow{F} \\
\Sigma Sp(G)_k & \xrightarrow{U} & \Sigma Sp_k.
\end{array}
\]

Since \( F \) is conservative, it is sufficient to show that \( FU(\text{holim}_\Delta Y^\bullet) \to F(\text{holim}_\Delta UY^\bullet) \) is a weak equivalence in \( \Sigma Sp_k \). Since \( F \) is a right \( \Sigma Sp \)-Quillen functor by Lemma \( \ref{right_quillen} \) \( F \) preserves homotopy limits. Hence we have \( FU(\text{holim}_\Delta Y^\bullet) \simeq U(\text{holim}_\Delta FY^\bullet) \) and \( F(\text{holim}_\Delta UY^\bullet) \simeq \text{holim}_\Delta UFY^\bullet \). By the proof of Theorem \( \ref{equiv} \) the map \( U(\text{holim}_\Delta FY^\bullet) \to \text{holim}_\Delta UFY^\bullet \) is a weak equivalence in \( \Sigma Sp_k \). This completes the proof. \( \square \)

6.4. Equivalence of the two formulations. In this subsection we shall show that Proposition \( \ref{equiv} \) and Proposition \( \ref{equiv1} \) are essentially equivalent under some conditions.

Let \( A \) be a monoid object in \( \Sigma Sp \). We regard \( A \) as a monoid object in \( \Sigma Sp(G) \) with trivial \( G \) action. Let \( \varphi : A \to B \) be a map of monoid objects in \( \Sigma Sp(G) \). We assume that \( A \) is cofibrant in \( \Sigma Sp \) and that \( UB \) is fibrant and cofibrant in \( \Sigma Sp \).

For \( M \in \text{Mod}_{UB}(\Sigma Sp) \), we denote by

\[
B(UB, A, M) = |B_*(UB, A, M)|
\]

the geometric realization of the bar construction \( B_*(UB, A, M) \). We define a map

\[
\Psi_M : B(UB, A, M) \to U\text{Map}_e(G, M)
\]

by applying \( U \) to the map \( B(B, A, M) \to \text{Map}_e(G, M) \) in \( \Sigma Sp(G) \) that is adjoint to the map \( UB(B, A, M) \cong B(UB, A, M) \to M \) induced by the action of \( UB \) on \( M \). In particular, we have a map

\[
\Psi_{UB} : B(UB, A, UB) \to U\text{Map}_e(G, UB).
\]

We set \( \psi = U\varphi : A \to UB \). Recall that there is a Quillen adjunction

\[
UB \land_A (-) : \text{Mod}_A(\Sigma Sp_k) \rightleftarrows \text{Mod}_{UB}(\Sigma Sp_k) : \psi^*.
\]

Lemma 6.8. If \( M \) is a fibrant and cofibrant object in \( \text{Mod}_{UB}(\Sigma Sp_k) \), then \( B(UB, A, M) \) represents \( UB \land_A (R\psi^* M) \) in \( \text{Ho}(\text{Mod}_{UB}(\Sigma Sp_k)) \).

Proof. Since \( M \) is fibrant in \( \text{Mod}_{UB}(\Sigma Sp_k) \), \( M \) represents \( R\psi^* M \) in \( \text{Ho}(\text{Mod}_A(\Sigma Sp_k)) \). If \( QAM \to M \) is a cofibrant replacement in \( \text{Mod}_A(\Sigma Sp_k) \), then \( UB \land_A (R\psi^* M) \) is represented by \( UB \land_A QAM \). By \( \cite{19} \) Lemma 4.1.9, we have an equivalence \( B(UB, A, QAM) \cong UB \land_A QAM \).
We shall show that there is an equivalence \( B(UB, A, QAM) \xrightarrow{\sim} B(UB, A, M) \). For any \( r \geq 0 \), \( UB \wedge A^r \) is cofibrant in \( \Sigma Sp \). This implies an equivalence \( UB \wedge A^r \wedge QAM \xrightarrow{\sim} UB \wedge A^r \wedge M \) by [13 Lemma 5.4.4]. Hence we obtain an equivalence \( |B_\bullet(UB, A, QAM)| \xrightarrow{\sim} |B_\bullet(UB, A, M)| \) by [19 Corollary 4.1.6].

**Lemma 6.9.** Let \( M \) be a fibrant and cofibrant object in \( \text{Mod}_{UB}(\Sigma Sp_k) \). If \( \Psi_{UB} \) is a \( k \)-local equivalence, then \( \Psi_M \) is also a \( k \)-local equivalence.

**Proof.** We have an isomorphism between \( B(UB, A, M) \) and \( B(UB, A, UB) \wedge UB M \), and an equivalence between \( UMap_c(G, M) \) and \( UMap_c(G, UB) \wedge UB M \). Since \( M \) is cofibrant in \( \text{Mod}_{UB}(\Sigma Sp) \), \( \Psi_{UB} \) induces a \( k \)-local equivalence \( B(UB, A, UB) \wedge UB M \xrightarrow{\sim} UMap_c(G, UB) \wedge UB M \) by [13 Lemma 5.4.4]. This completes the proof. \( \square \)

There is an adjunction of quasi-categories \( UB \wedge A(-) : \text{Mod}_A(\text{Sp}_k) \rightleftarrows \text{Mod}_{UB}(\text{Sp}_k) : \psi^* \), and hence we obtain a comonad \( \Theta \) on \( \text{Mod}_{UB}(\text{Sp}_k) \) and a quasi-category of comodules \( \text{Comod}_{UB, \Theta}(\text{Sp}_k) = \text{Comod}_\Theta(\text{Mod}_{UB}(\text{Sp}_k)) \).

We set \( C = \text{Mod}_{UB}(\text{Sp}_k)^{\text{op}}, \)
\( C(G) = \text{Mod}_B(\text{Sp}(G)_k)^{\text{op}}, \)
\( D = \text{Mod}_A(\text{Sp}_k)^{\text{op}}. \)

We have an adjunction of quasi-categories \( V : C \rightleftarrows C(G) : U \). By [15 Proposition 4.7.4.3], we obtain an endomorphism monad \( \Gamma \in \text{Alg}(\text{End}(C)) \) and a left \( \Gamma \)-module \( U \in \text{LMod}_\Gamma(\text{Fun}(C(G), C)) \).

We set
\[
H = UB \wedge A(-) : D \to C,
\]
\[
H' = B \wedge A(-) : D \to C(G),
\]
\[
F = \psi^* : C \to D.
\]

Note that \( H = UH' \) is the right adjoint to \( F \). The functor \( H = UH' \) lifts to a left \( \Gamma \)-module \( \overline{UH'} \in \text{LMod}_\Gamma(\text{Fun}(D, C)) \). We consider the composite map
\[
\Gamma \xrightarrow{\text{id}_\Gamma \times u} \Gamma HF \xrightarrow{a \times \text{id}_F} HF,
\]
where \( u \) is the unit of the adjoint pair \( (F, H) \) and \( a \) is the action of \( \Gamma \) on \( H \). For any \( M \in \text{Mod}_{UB}(\text{Sp}_k) \), this map induces a natural map
\[
UMap_c(G, M) \leftarrow UMap_c(G, UB \wedge A M) \leftarrow UB \wedge A M
\]
in \( \text{Mod}_{UB}(\text{Sp}_k) \).

**Lemma 6.10.** If \( \Psi_{UB} \) is a \( k \)-local equivalence, then the composite map \( \Gamma \to \Gamma HF \to HF \) is an equivalence of functors.

**Proof.** It is sufficient to show that the induced map \( UB \wedge A M \to UMap_c(G, M) \) is an equivalence in \( \text{Mod}_{UB}(\text{Sp}_k) \) for any \( M \). This follows from Lemmas 6.8 and 6.9. \( \square \)

**Theorem 6.11.** If \( \Psi_{UB} \) is a \( k \)-local equivalence, then there is an equivalence of quasi-categories
\[
\text{Comod}_{(UB,F)}(\text{Sp}_k) \simeq \text{Comod}_{(UB,\Theta)}(\text{Sp}_k).
\]
Proof. By [15, Proposition 4.7.3.2] and Lemma 6.10 we see that $\Gamma$ is an endomorphism monad for $H$. Hence we obtain an equivalence between $\text{LMod}_\Theta(C)$ and $\text{LMod}_\Gamma(C)$. □

By Theorems 6.7 and 6.11 we obtain the following corollaries.

Corollary 6.12. Let $G$ be a profinite group that has finite cohomological dimension. We assume that the localization functor $L_k$ satisfies Assumption 6.2. If $\Psi_{UB}$ is a $k$-local equivalence, then there is an equivalence of quasi-categories

$$\text{Mod}_B(\text{Sp}(G)_k) \simeq \text{Comod}_{(UB,\Theta)}(\text{Sp}_k).$$

The functor $B \wedge A (\cdot) : \text{Mod}_A(\Sigma\text{Sp}_k) \to \text{Mod}_B(\Sigma\text{Sp}(G)_k)$ of $\Sigma\text{Sp}$-model categories induces a functor

$$\text{Ex} : \text{Mod}_A(\text{Sp}_k) \to \text{Mod}_B(\text{Sp}(G)_k)$$

of quasi-categories. Recall that we have a functor

$$\text{Coex} : \text{Mod}_A(\text{Sp}_k) \to \text{Comod}_{(UB,\Theta)}(\text{Sp}_k).$$

Corollary 6.13. Under the equivalence $\text{Mod}_B(\text{Sp}(G)_k) \simeq \text{Comod}_{(UB,\Theta)}(\text{Sp}_k)$, there is an equivalence of functors

$$\text{Ex} \simeq \text{Coex}.$$
Furthermore, we have an isomorphism $B(B_\alpha, A, B_\alpha) \cong B(B_\alpha, A, A) \land_A B_\alpha$. Since $A \to B_\alpha$ is a cofibration in the category of commutative symmetric ring spectra, we obtain an equivalence $B(B_\alpha, A, B_\alpha) \cong B_\alpha \land_A B_\alpha$ by [10] Proposition 15.12. Hence we obtain a $k$-local equivalence

$$B(B_\alpha, A, B_\alpha) \xrightarrow{\cong_k} \text{Map}(G_\alpha, B_\alpha).$$

Let $r_\alpha : Q_\alpha M \to M$ be a cofibrant replacement in $\text{Mod}_{B_\alpha}(\Sigma \text{Sp}_k)$ such that $r_\alpha$ is a trivial fibration. We obtain a $k$-local equivalence

$$B(B_\alpha, A, Q_\alpha M) \xrightarrow{\cong_k} \text{Map}(G_\alpha, Q_\alpha M)$$

as in Lemma 6.8. Since $A$ and $B_\alpha$ are cofibrant commutative symmetric ring spectra, we see that $r_\alpha$ induces a $k$-local equivalence between $B(B_\alpha, A, Q_\alpha M)$ and $B(B_\alpha, A, M)$ by using [10] Proposition 15.12. Since $r_\alpha$ is a trivial fibration, $\text{Map}(G_\alpha, Q_\alpha M) \to \text{Map}(G_\alpha, M)$ is also a trivial fibration. Hence

$$B(B_\alpha, A, M) \to \text{Map}(G_\alpha, M)$$

is a $k$-local equivalence. Since $\Psi_M$ is the colimit of the above maps over the directed system, the lemma follows from Proposition 3.13.

**Theorem 7.3.** There is an equivalence of quasi-categories

$$\text{Mod}_B(\text{Sp}(G)_k) \simeq \text{Comod}_{(UB, \Theta)}(\text{Sp}_k).$$

Under this equivalence, there is an equivalence of functors

$$\text{Ex} \simeq \text{Coex}.$$

*Proof.* By Theorem 6.7 we have an equivalence between $\text{Mod}_B(\text{Sp}(G)_k)$ and $\text{Comod}_{(UB, \Gamma)}(\text{Sp}_k)$. We can show that $\text{Comod}_{(UB, \Gamma)}(\text{Sp}_k)$ is equivalent to $\text{Comod}_{(UB, \Theta)}(\text{Sp}_k)$ as in Theorem 6.11 using Lemma 7.2. This completes the proof.

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