Fermion determinant with dynamical chiral symmetry breaking

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Abstract

One-loop fermion determinant is discussed for the case in which dynamical chiral symmetry breaking caused by momentum dependent fermion self energy $\Sigma(p^2)$ take place. The obtained series generalizes the heat kernel expansion for hard fermion mass.

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In quantum field theory, 1-loop quantum correction of fermions is from fermion determinant. It is known how to derive an exact non-perturbative representation for the chiral fermion determinant with hard fermion mass \cite{1} and the technique has been widely used in the literature such as to derive the effective action of quarks aiming at modeling QCD at low-energy \cite{2,3}. The central object in the calculation of the effective action is $\text{ln} \det(D+m)$, where in the Euclidean space, the differential Dirac operator $D$ depends on hermitian external flavor sources (or collective meson fields) which have well defined transformation laws with respect to the action of the chiral group

$$D \equiv \nabla - s + ip\gamma_5 \quad \quad \nabla_\mu \equiv \partial_\mu - iv_\mu - ia_\mu \gamma_5 = -\nabla^\dagger_\mu , \quad (1)$$

$m$ is a hard fermion mass. Ignoring anomaly, the real part of $\text{ln} \det(D+m)$ in terms of a proper time integral is

\textsuperscript{1}Mailing address
\[ \text{Re ln Det}(D + m) = \frac{1}{2} \text{Tr ln}[(D^\dagger + m)(D + m)] \]
\[ = -\frac{1}{2} \lim_{\Lambda \to \infty} \int d^4x \int_0^\infty \frac{d\tau}{\tau} \text{Tr} e^{-m^2\tau} \langle x | e^{-\tau(E - \nabla^2)} | x \rangle \] (2)

with
\[ E - \nabla^2 = D^\dagger D + D^\dagger m + mD \] (3)

and
\[ (\nabla)^\dagger \equiv -\partial + i\gamma \cdot p - i\gamma_5 \gamma \]
\[ D^\dagger = \nabla^\dagger - s - ip\gamma_5 \]
\[ E = -2ms - 2im\gamma \cdot p + \frac{i}{4}[\gamma^\mu, \gamma^\nu] \mathcal{R}_{\mu\nu} + \gamma^\mu d^\mu(s + ip\gamma_5) + i\gamma^\mu[a_\mu\gamma_5(s + ip\gamma_5) + (s + ip\gamma_5)a_\mu\gamma_5] \]
\[ + s^2 + p^2 + [s, p]i\gamma_5 \] (4)
\[ \mathcal{R}_{\mu\nu} \equiv i[\nabla_\mu, \nabla_\nu] = [d_\mu a_\nu - d_\nu a_\mu]\gamma_5 + V_{\mu\nu} - i[a_\mu, a_\nu] \]
\[ d_\mu f = \partial_\mu f -i[v_\mu, f] \quad d_\mu(fg) = (d_\mu f)g + f(d_\mu g) . \]

With help of standard Seely-DeWitt expansion,
\[ \langle x | e^{-\tau(E - \nabla^2)} | x \rangle = \frac{1}{16\pi^2} \left[ \frac{1}{\tau^2} - \frac{E}{\tau} + \frac{1}{2} E^2 - \frac{1}{6} [\nabla_\mu, [\nabla_\mu, E]] - \frac{1}{12} \mathcal{R}_{\mu\nu} \mathcal{R}^{\mu\nu} - \frac{\tau}{6} E^3 \right. \]
\[ + \frac{\tau^2}{12} \left. \{ E[\nabla_\mu, [\nabla_\mu, E]] + [\nabla_\mu, [\nabla_\mu, E]]E + [\nabla_\mu, E][\nabla_\mu, E] \right] + \frac{\tau^2}{24} E^4 + \cdots \right] , \] (5)

the integration of \( \tau \) in (2) can be finished, we then get the expansion of Re ln det\( (D + m) \)
with hard fermion mass \( m \),

\[ \text{Re ln Det}(\partial + m) \]
\[ = \ln \text{Det}(\partial + m) - \frac{N_c}{32\pi^2} \lim_{\Lambda \to \infty} \int d^4x \text{tr}_f \left[ 8m[\Lambda^2 + m^2(\ln \frac{m^2}{\Lambda^2} + \gamma - 1)]s - 8m^2(\ln \frac{m^2}{\Lambda^2} + \gamma)a^2 \right. \]
\[ - \frac{4}{3}[d_\mu a^\mu]^2 - \frac{2}{3}(\ln \frac{m^2}{\Lambda^2} + \gamma + 1)(d_\mu a_\nu - d_\nu a_\mu)(d^\mu a^\nu - d^\nu a^\mu) - \left[ \frac{4}{3}(\ln \frac{m^2}{\Lambda^2} + \gamma) + \frac{16}{3} \right] a^4 \]
\[ + \frac{4}{3}(\ln \frac{m^2}{\Lambda^2} + \gamma) + \frac{8}{3}[a_\mu a_\nu a^\mu a^\nu - 4[\Lambda^2 + m^2(3(\ln \frac{m^2}{\Lambda^2} + 3\gamma - 1)]s^2 \right. \]
\[ - 4[\Lambda^2 + m^2(\ln \frac{m^2}{\Lambda^2} + \gamma - 1)]p^2 + (16 \ln \frac{m^2}{\Lambda^2} + 16\gamma + 16)m\sigma a^2 - \frac{2}{3}(\ln \frac{m^2}{\Lambda^2} + \gamma)V_{\mu\nu}V^{\mu\nu} \right. \]
\[ + \left. i[\frac{8}{3}(\ln \frac{m^2}{\Lambda^2} + \gamma) + \frac{16}{3}a^\mu a^\nu V_{\mu\nu} + 8m(\ln \frac{m^2}{\Lambda^2} + \gamma)pd^\mu a_\mu + O(p^6) \right] , \] (6)

where \( N_c \) is number of color (which in present discussion is a global degree of freedom) and \( \text{tr}_f \) is trace for flavor indices. The expansion terms are arranged according to its momentum power, \( \partial_\mu, v_\mu, a_\mu \) are treated as order \( p \) and \( s, p \) as order \( p^2 \).
The hard fermion mass $m$ play the role of breaking chiral symmetry and infrared cutoff. It can be generalized to a matrix [4] to respect further breaking of the chiral symmetry. But the formulation cannot deal with the case of dynamical symmetry breaking in which the most general situation is not the appearance of a hard fermion mass, but a momentum dependent fermion self energy $\Sigma(p^2)$. Even in NJL [5] or ENJL [2] models, the hard mass will be replaced by momentum dependent self energy in high order loop calculations. Taking hard fermion mass often cause extra ultraviolet divergence. For example, in the case of QCD, the fermion self energy damping at least as $1/p^2$ at ultraviolet momentum region, if we take a hard fermion mass to substitute self energy, we will over estimate its contribution to physics. To respect this momentum dependence of fermion self energy in the theory, in this paper, we develop a generalized proper time formulation to calculate fermion determinant with momentum dependent fermion self energy.

Since momentum dependent fermion self energy $\Sigma(p^2)$ in coordinate space is represented by $\Sigma(-\partial^2)\delta(x-y)$, the naive generalization of Indet$(D+m)$ should be calculating Indet$[D+\Sigma(-\partial^2)]$. We argue this is not suitable since original Indet$(D+m)$ is invariant under following local symmetry transformations,

$$
\begin{align*}
s(x) &\rightarrow s'(x) = V(x)s(x)V^\dagger(x) \\
p(x) &\rightarrow p'(x) = V(x)p(x)V^\dagger(x) \\
v_\mu(x) &\rightarrow v'_\mu(x) = V(x)v_\mu(x)V^\dagger(x) + V(x)[i\partial_\mu V^\dagger(x)] \\
a_\mu(x) &\rightarrow a'_\mu(x) = V(x)a_\mu(x)V^\dagger(x)
\end{align*}
$$

which leads to

$$
D_x \rightarrow D'_x \equiv \partial_x - i\gamma^5 \gamma_5 - s'(x) + ip'(x)\gamma_5
$$

$$
= V(x)[\partial_x - i\gamma^5 \gamma_5 - s(x) + ip(x)\gamma_5]V^\dagger(x) = V(x)D_x V^\dagger(x)
$$

and

$$
\text{Indet}(D+m) \rightarrow \text{Indet}(D'+m) = \text{Indet}(VDV^\dagger + m) = \text{Indet}[V(D+m)V^\dagger] = \text{Indet}(D+m)
$$

(9)

Here the invariance is due to the property of constant $m$ which leads

$$
V(x)mV^\dagger(x) = m
$$

(10)

If we change $m$ to $\Sigma(-\partial^2)$, above relation no longer valid due to differential operator dependence of $\Sigma$. 

3
\[ V(x)\Sigma(-\partial_x^2)V^\dagger(x) = \Sigma[-V(x)\partial_x^2V(x)] = \Sigma\left[ -\left( \partial_\mu + V(x)[\partial_\mu V^\dagger(x)] \right) \right]^2 \neq \Sigma(-\partial_x^2) . \]  \hspace{1cm} (11)

To implement local symmetry (7) in our generalization, instead of considering \( \Sigma(-\partial^2) \), we need to consider \( \Sigma(-\nabla^2) \) where
\[
\nabla^\mu \equiv \partial^\mu - iv^\mu(x) ,
\]  \hspace{1cm} (12)

where the bar over \( \nabla_\mu \) is to specify the difference of present derivative with that introduced in (1). Use (7), we find
\[
\nabla_\mu \overset{\text{trans}}{\rightarrow} \nabla'_\mu \equiv \partial'_\mu - iv^\mu'(x) = V(x)\nabla'_\mu V(x) = \gamma^\mu \left( \partial_\mu - iv_\mu \right) \gamma^\mu \nabla^\mu V^\dagger(x) .
\]  \hspace{1cm} (13)

Then
\[
\Sigma(-\nabla_x^2) \rightarrow \Sigma(-\nabla'_x^2) = \Sigma[-V(x)\nabla'_x^2V(x)] = V(x)\Sigma(-\nabla'_x^2)V^\dagger(x) \]  \hspace{1cm} (14)

and
\[
\ln\det[D + \Sigma(\nabla^2)] \rightarrow \ln\det[D' + \Sigma(\nabla'^2)] = \ln\det\left[ V[D + \Sigma(\nabla^2)]V^\dagger \right] = \ln\det[D + \Sigma(\nabla^2)] . \]  \hspace{1cm} (15)

So, \( \ln\det[D + \Sigma(\nabla^2)] \) is invariant under transformation (7) and can be thought as correct generalization of \( \ln\det(D + m) \). The generalized fermion determinant now is
\[
\Re \ln\Det[D + \Sigma(-\nabla^2)] = \frac{1}{2} \Tr \ln \left[ [D^\dagger + \Sigma(-\nabla^2)]D + \Sigma(-\nabla^2) \right] = -\frac{1}{2} \lim_{\Lambda \rightarrow \infty} \int_{\frac{1}{4\Lambda^2}}^\infty \frac{d\tau}{\tau} \Tr e^{-\tau(\overline{E} - \nabla^2 + \Sigma(\nabla^2)) + Jg(\nabla^2) + g'(\nabla^2)K - \Phi \Sigma(-\nabla^2)} \]  \hspace{1cm} (16)

where
\[
\overline{E} = \nabla^2 + \Sigma^2(-\nabla^2) + Jg(\nabla^2) + g'(\nabla^2)K - \Phi \Sigma(-\nabla^2) \]  \hspace{1cm} (17)

and
\[
[\Phi \Sigma(-\nabla^2)] \equiv \gamma^\mu [d_\mu, \Sigma(-\nabla^2)] = \gamma^\mu \left( \partial_\mu \Sigma(-\nabla^2) - iv^\mu \gamma^\mu \right) \]
\[
\overline{E} \equiv \frac{i}{4} [\gamma^\mu, \gamma^\nu] R_{\mu\nu} + \gamma^\mu d_\mu(s + ip\gamma_5) + i\gamma^\mu [a_\mu \gamma_5(s + ip\gamma_5) + (s + ip\gamma_5)a_\mu \gamma_5] + s^2 + p^2 + [s, p]i\gamma_5 \]
\[
g(x) = g'(x) \equiv \Sigma(-x) \hspace{1cm} J = -i\Phi \gamma_5 - s - ip\gamma_5 \hspace{1cm} K = -i\Phi \gamma_5 - s + ip\gamma_5 .
\]

For safety of further calculation, we limit the ultraviolet behavior of \( \Sigma(k^2) \) as
\[
\frac{\Sigma^2(k^2)}{k^2} \xrightarrow{k^2 \rightarrow \infty} 0 \]  \hspace{1cm} (18)
The key now is to calculate
\[
\text{Tr} e^{-\tau[E - \nabla^2 + \Sigma^2(-\nabla^2) + Jg(\nabla^2) + g'(\nabla^2)\Sigma(-\nabla^2)]} = \int d^4x \text{tr}\langle x| e^{-\tau[E - \nabla^2 + \Sigma^2(-\nabla^2) + Jg(\nabla^2) + g'(\nabla^2)\Sigma(-\nabla^2)]}|x\rangle
\] (19)
in which term \(\langle x| e^{-\tau[E - \nabla^2 + \Sigma^2(-\nabla^2) + Jg(\nabla^2) + g'(\nabla^2)\Sigma(-\nabla^2)]}|x\rangle\) is much more complex than (18), since except the constraint (18), the differential operator dependent function \(\Sigma\) is still unknown.

\[
\langle x| e^{-\tau[E - \nabla^2 + \Sigma^2(-\nabla^2) + Jg(\nabla^2) + g'(\nabla^2)\Sigma(-\nabla^2)]}|x\rangle
= \int \frac{d^4p}{(2\pi)^4} \exp \left\{ -\tau \left[ E(x) - \nabla^2 - 2ip \cdot \nabla_x + p^2 + \Sigma^2(-\nabla^2 - 2ip \cdot \nabla_x + p^2) + Jg(\nabla^2 + 2ip \cdot \nabla_x - p^2) + g'(\nabla^2 - 2ip \cdot \nabla_x + p^2)\right] \right\}. \tag{20}
\]

Assign \(E, J, K\) are order of \(p\), we can take low energy expansion for (20). Substitute the result of momentum expansion into (16), we finally get

\[
\text{Re} \ln \text{Det}[D + \Sigma(-\nabla^2)]
= \int d^4x \text{tr}_f \left[ C_0 s + C_1 a^2 + C_2 [d_\mu a^\mu]^2 + C_3 (d^\mu a^\nu - d^\nu a^\mu)(d_\mu a_\nu - d_\nu a_\mu) + C_4 a^4 + C_5 a^\nu a_\nu a_\nu + C_6 s^2 + C_7 p^2 + C_8 s^2 + C_9 V^\mu_\nu V_{\mu\nu} + C_{10} V^\mu_\nu a_\mu a_\nu + C_{11} p d_\mu a^\mu \right] + O(p^6)
\] (21)

with coefficients related to \(\Sigma\) by

\[
C_0 = -4 \int d\tilde{k} \Sigma_k X_k
C_1 = 2 \int d\tilde{k} \left[ (-2\Sigma_k^2 + k^2 \Sigma_k \Sigma'_k) X_k^2 + (-2\Sigma_k^2 + k^2 \Sigma_k \Sigma'_k) \frac{X_k}{\Lambda^2} \right]
C_2 = -2 \int d\tilde{k} \left[ 2A_k X_k^2 + 2A_k \frac{X_k^2}{\Lambda^2} + A_k \frac{X_k}{\Lambda^4} + \frac{k^2}{2} \Sigma_k^2 X_k + \frac{k^2}{2} \Sigma'_k X_k^2 \right]
C_3 = - \int d\tilde{k} \left[ 2B_k X_k^3 + 2B_k \frac{X_k^3}{\Lambda^2} + B_k \frac{X_k^3}{\Lambda^4} + \frac{k^2}{2} \Sigma_k^2 X_k + \frac{k^2}{2} \Sigma'_k X_k^2 \right]
C_4 = 2 \int d\tilde{k} \left[ \left( \frac{4\Sigma_k^4}{3} + \frac{2k^2 \Sigma_k^2}{3} + \frac{k^4}{18} \right) (6X_k^4 + \frac{6X_k^3}{\Lambda^2} + \frac{3X_k^2}{\Lambda^4} + X_k) \right] - \left( 4\Sigma_k^2 + \frac{k^2}{2} \right) (2X_k^3 + \frac{2X_k^2}{\Lambda^2} + \frac{X_k}{\Lambda^4}) \tag{22}
C_5 = \int d\tilde{k} \left[ (-4\Sigma_k^4 + \frac{2k^2 \Sigma_k^2}{3} + \frac{k^4}{18}) (6X_k^4 + \frac{6X_k^3}{\Lambda^2} + \frac{3X_k^2}{\Lambda^4} + X_k) \right] + 4\Sigma_k^2 (2X_k^3 + \frac{2X_k^2}{\Lambda^2} + \frac{X_k}{\Lambda^4}) \tag{23}
\]

\[
+ \frac{X_k}{\Lambda^2} - X_k^2
\]
\[C_6 = 2 \int \! dk \left( (3\Sigma_k^2 - 2k^2\Sigma_k \Sigma'_k)X_k^2 + [2\Sigma_k^2 - k^2(1 + 2\Sigma_k \Sigma'_k)]\frac{X_k}{\Lambda^2} \right)\]

\[C_7 = 2 \int \! dk \left( (\Sigma_k^2 - 2k^2\Sigma_k \Sigma'_k)X_k^2 - k^2(1 + 2\Sigma_k \Sigma'_k)\frac{X_k}{\Lambda^2} \right)\]

\[C_8 = -4 \int \! dk \left( (4\Sigma_k^3 + k^2\Sigma_k)X_k^2 \right)\]

\[C_9 = \int \! dk \left( \frac{1}{2}k^2\Sigma'_k \Sigma''_k + \frac{1}{6}k^2\Sigma_k \Sigma''_k \right)X_k + (-C_k + D_k)\frac{X_k}{\Lambda^2} - (C_k - D_k)X_k^2 + 2E_kX_k^3\]

\[+ 2E_k \frac{X_k^2}{\Lambda^2} - E_k \frac{X_k^2}{\Lambda^4}\]

\[C_{10} = -4i \int \! dk \left[ 2F_kX_k^3 + 2F_k \frac{X_k}{\Lambda^2} + F_k \frac{X_k}{\Lambda^4} + \frac{k^2}{2}\Sigma'_k \frac{X_k}{\Lambda^2} + \frac{k^2}{2}\Sigma_k X_k^2 \right]\]

\[C_{11} = 4 \int \! dk \left[ (\Sigma_k - \frac{1}{2}k^2\Sigma'_k)\frac{X_k}{\Lambda^2} + (\Sigma_k - \frac{1}{2}k^2\Sigma'_k)X_k^2 \right]\] (22)

where

\[\int \! dk \equiv N_c \int \! \frac{d^4k}{(2\pi)^4} e^{-\frac{k^2+\Sigma^2(k^2)}{\Lambda^2}}\] (23)

\[\Sigma_k \equiv \Sigma(k^2) \quad X_k \equiv \frac{1}{k^2 + \Sigma^2(k^2)}\] (24)

and \(A_k, B_k, C_k, D_k, E_k, F_k\) are given in appendix A. We see that asymptotic behavior of \(\Sigma(k^2)\) (18) insure factor \(e^{-\frac{k^2+\Sigma^2(k^2)}{\Lambda^2}}\) appeared in integration measure (23) is an ultraviolet suppression factor which will keep our momentum integration be convergent.

(21) and (22) are our final result for the real part of fermion determinant with presence of dynamical quark self energy. The result in this paper only given up to order of \(p^4\), one can easily generalize the calculation to higher orders of the momentum expansion. As a self check of theory, take \(\Sigma(k^2)\) be constant \(m\), in the limit of \(\Lambda^2 \to \infty\), the momentum integration in (22) can be finished, the result gives

\[C_0 \xrightarrow{\Sigma=m} -\frac{N_c}{4\pi^2} m[\Lambda^2 + m^2(ln\frac{m^2}{\Lambda^2} + \gamma - 1)]\]

\[C_1 \xrightarrow{\Sigma=m} \frac{N_c}{4\pi^2} m^2(ln\frac{m^2}{\Lambda^2} + \gamma)\]

\[C_2 \xrightarrow{\Sigma=m} \frac{N_c}{24\pi^2}\]

\[C_3 \xrightarrow{\Sigma=m} \frac{N_c}{48\pi^2} (ln\frac{m^2}{\Lambda^2} + \gamma + 1)\]

\[C_4 \xrightarrow{\Sigma=m} \frac{N_c}{24\pi^2} (ln\frac{m^2}{\Lambda^2} + \gamma + 4)\]

\[C_5 \xrightarrow{\Sigma=m} -\frac{N_c}{24\pi^2} (ln\frac{m^2}{\Lambda^2} + \gamma + 2)\]
\[
\begin{align*}
C_6 & \rightarrow N_c \frac{\Lambda^2}{8\pi^2} \left[ \Lambda^2 + m^2 \left( 3 \ln \frac{m^2}{\Lambda^2} + 3\gamma - 1 \right) \right] \\
C_7 & \rightarrow N_c \frac{\Lambda^2}{8\pi^2} \left[ \Lambda^2 + m^2 \left( \ln \frac{m^2}{\Lambda^2} + \gamma - 1 \right) \right] \\
C_8 & \rightarrow -\frac{N_c}{2\pi^2} m \left( \ln \frac{m^2}{\Lambda^2} + \gamma + 1 \right) \\
C_9 & \rightarrow \frac{N_c}{48\pi^2} \left( \ln \frac{m^2}{\Lambda^2} + \gamma \right) \\
C_{10} & \rightarrow -\frac{iN_c}{12\pi^2} \left( \ln \frac{m^2}{\Lambda^2} + \gamma + 2 \right) \\
C_{11} & \rightarrow -\frac{N_c}{4\pi^2} m \left( \ln \frac{m^2}{\Lambda^2} + \gamma \right).
\end{align*}
\]

(25)

Substitute them into (21), we reproduce original result (8).

In summary, we have generalized the traditional proper time method for calculation of fermion determinant to include dynamical chiral symmetry breaking caused by momentum dependent fermion self energy. The physical application of this formulation will be discussed elsewhere.

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APPENDIX A: COEFFICIENTS DEFINITIONS

\[ A_k = \frac{2}{3} k^2 \Sigma_k \Sigma_{k}' (-1 - 2 \Sigma_k \Sigma_{k}') - \frac{1}{3} \Sigma_k^2 (-1 - 2 \Sigma_k \Sigma_{k}') + \frac{1}{3} k^2 \Sigma_k^2 (\Sigma_k^2 + \Sigma_k \Sigma_{k}'') \]
\[ + \frac{1}{6} k^4 (\Sigma_k^2 + \Sigma_k \Sigma_{k}'') \]

\[ B_k = \frac{2}{3} k^2 \Sigma_k \Sigma_{k}' (-1 - 2 \Sigma_k \Sigma_{k}') - \frac{1}{3} \Sigma_k^2 (-1 - 2 \Sigma_k \Sigma_{k}') + \frac{1}{3} k^2 \Sigma_k^2 (\Sigma_k^2 + \Sigma_k \Sigma_{k}'') \]
\[ + \frac{1}{18} k^4 (\Sigma_k^2 + \Sigma_k \Sigma_{k}'') + \frac{1}{6} k^2 (-1 - 2 \Sigma_k \Sigma_{k}') \]

\[ C_k = \frac{1}{3} - \frac{1}{3} \Sigma_k \Sigma_{k}' + \frac{1}{2} k^2 \Sigma_k^2 \]

\[ D_k = -\frac{1}{2} k^2 \Sigma_k^2 + \frac{1}{3} k^2 \Sigma_k \Sigma_k'' (-1 - 2 \Sigma_k \Sigma_k') + \frac{2}{9} k^4 \Sigma_k^2 (1 + 2 \Sigma_k \Sigma_k') \]
\[ - \frac{2}{9} k^4 \Sigma_k^2 (-\Sigma_k^2 - \Sigma_k \Sigma_{k}'') \]
\[ + \frac{1}{3} k^2 \Sigma_k \Sigma_k' \]

\[ E_k = \frac{1}{3} k^2 \Sigma_k \Sigma_k' (-1 - 2 \Sigma_k \Sigma_k')^2 + \frac{1}{9} k^4 \Sigma_k^2 (1 + 2 \Sigma_k \Sigma_k')^2 \]

\[ F_k = \frac{4}{3} k^2 \Sigma_k \Sigma_k' - \frac{4}{3} k^2 (\Sigma_k \Sigma_k')^2 - \frac{2}{3} \Sigma_k^2 + \frac{2}{3} \Sigma_k \Sigma_k' + \frac{1}{3} k^2 \Sigma_k^2 (\Sigma_k^2 + \Sigma_k \Sigma_k'') + \frac{1}{9} k^4 (\Sigma_k^2 + \Sigma_k \Sigma_k'') \]
\[ + \frac{1}{3} k^2 (-1 - 2 \Sigma_k \Sigma_k') + \frac{1}{2} k^2 \]