A Grauert Type Theorem and Extension of Matrices with Entries in $H^\infty$

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Abstract

In the paper we prove an extension theorem for matrices with entries in $H^\infty(U)$ for $U$ being a Riemann surface of a special type. One of the main components of the proof is a Grauert type theorem for "holomorphic" vector bundles defined over maximal ideal spaces of certain Banach algebras.

1. Introduction.

1.1. Let $N \subset M$ be a relatively compact domain in an open Riemann surface $M$ such that

$$\pi_1(N) \cong \pi_1(M). \quad (1.1)$$

Let $R$ be an unbranched covering of $N$ and $i: U \hookrightarrow R$ be a domain in $R$. Assume that

the induced homomorphism of the fundamental groups

$$i_*: \pi_1(U) \longrightarrow \pi_1(R) \text{ is injective.} \quad (1.2)$$

In this paper we continue to study the space $H^\infty(U)$, of bounded holomorphic functions on $U$ satisfying (1.1) and (1.2), started in [Br]. One of the main results proved in [Br, Th.1.1] was a Forelli type theorem on projections for $H^\infty(U)$. In the present paper we prove an extension theorem for matrices with entries in $H^\infty(U)$.

To formulate the result let us recall the following definition.

We say that a collection $f_1, \ldots, f_n$ of functions from $H^\infty(U)$ satisfies the corona condition if

$$|f_1(z)| + |f_2(z)| + \ldots + |f_n(z)| \geq \delta > 0 \quad \text{for all } z \in U. \quad (1.3)$$

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In [Br,Corol.1.5] we proved that the corona problem is solvable in $H^\infty(U)$ meaning that for any $f_1, \ldots, f_n$ satisfying (1.3) there are $g_1, \ldots, g_n \in H^\infty(U)$ such that

$$f_1g_1 + f_2g_2 + \ldots + f_ng_n \equiv 1.$$  \hfill (1.4)

For instance, for $U$ being the open unit disc $\mathbb{D} \subset \mathbb{C}$ the solvability of the corona problem follows from the celebrated Carleson’s Corona Theorem [C]. In this paper we consider a matrix version of the corona problem.

**Theorem 1.1** Let $A = (a_{ij})$ be a $n \times k$ matrix, $k < n$, with entries in $H^\infty(U)$. Assume that the family of determinants of submatrices of $A$ of order $k$ satisfies the corona condition. Then there exists an $n \times n$ matrix $\tilde{A} = (\tilde{a}_{ij}), \tilde{a}_{ij} \in H^\infty(U)$, so that $\tilde{a}_{ij} = a_{ij}$ for $1 \leq j \leq k$, and $\det(\tilde{A}) = 1$.

In fact, we can estimate the norm of $\tilde{A}$ in terms of the norm of $A$, $\delta$ (from (1.3)), $n$ and $N$. (This estimate does not depend of the choice of $U$.)

**Remark 1.2** For $k = 1$ we have a column of functions from $H^\infty(U)$ satisfying the corona condition. The conclusion of the theorem in this case is essentially stronger than just the solvability of equation (1.4).

Note that a similar to Theorem 1.1 result for $H^\infty(U)$ with $U$ being the interior of a bordered Riemann surface was proved first by Tolokonnikov [T, Th.3] (see also this paper for further results and references concerning the extension problem for matrices with entries in different function algebras).

Let us give an example of a Riemann surface $U$ satisfying (1.1) and (1.2).

**Example 1.3** Consider the standard action of the group $\mathbb{Z} + i\mathbb{Z}$ on $\mathbb{C}$ by shifts. The fundamental domain of the action is the square $R := \{z = x + iy \in \mathbb{C} : \max\{|x|, |y|\} \leq 1\}$. By $R_t$ we denote the square similar to $R$ with the length of the side $t$. Let $O$ be the orbit of $0 \in \mathbb{C}$ with respect to the action of $\mathbb{Z} + i\mathbb{Z}$. For any $x \in O$ we will choose some $t(x) \in [1/2, 3/4]$ and consider the square $R(x) := x + R_t(x)$ centered at $x$. Let $V \subset \mathbb{C}$ be a simply connected domain satisfying the property:

there is a subset $\{x_i\}_{i \in I} \subset O$ such that $V \cap (\bigcup_{x \in O} R(x)) = \bigcup_{i \in I} R(x_i)$.

We set $U := V \setminus (\bigcup_{i \in I} R(x_i))$. Then $U$ satisfies the required conditions. In fact, the quotient space $\mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ is a torus $\mathbb{T}$. Let $S$ be the image of $R_{1/3}$ in $\mathbb{T}$. Then $U$ belongs to the covering $C$ of $\mathbb{T} \setminus S$ with the covering group $\mathbb{Z} + i\mathbb{Z}$. The condition that embedding $U \hookrightarrow C$ induces an injective homomorphism of fundamental groups follows from the construction of $U$.

1.2. Two essential components of our proof of Theorem 1.1 are [Br, Th.1.1] (a Forelli type theorem), and a new Grauert type theorem formulated in this section. Assume that

$$\tilde{N} \text{ is holomorphically convex in } M, \text{ and } \pi_1(N) \cong \pi_1(M).$$ \hfill (1.5)

Let $G$ be a family of subgroups of $\pi_1(M)$. For any $G \in G$ by $p_G : M_G \rightarrow M$ we denote the unbranched covering of $M$ corresponding to $G$, that is, $\pi_1(M_G) = G$.\vspace{1cm}
Let $U$ be a domain in $M$ satisfying $\pi_1(U) \cong \pi_1(M)$. According to the covering homotopy theorem (see e.g. [Hu]), we have $U_G := p_G^{-1}(U) \subset M_G$ is the covering of $U$ corresponding to $G$. In particular, it is valid for $N$. Further, disjoint union $U_G := \sqcup_{G \in G} U_G$ is an open subset of the complex space $M_G := \sqcup_{G \in G} M_G$. By $H^\infty(U_G)$ we denote the Banach algebra of bounded holomorphic functions on $U_G$ equipped with the supremum norm. Assume now that $U$ is such that $\overline{N} \subset U$. Let $r_U : H^\infty(U_G) \to H^\infty(N_G)$ be the restriction homomorphism. By $H^\infty(N_G)$ we denote the closure in $H^\infty(U_G)$ of the algebra generated by all $r_U(H^\infty(U_G))$ with $\overline{N} \subset U$.

Let $M_G(N)$ be the maximal ideal space of $H^\infty(N_G)$, that is, the set of all non-trivial homomorphisms $\phi : H^\infty(N_G) \to \mathbb{C}$ equipped with the weak $\ast$ topology (which is called the Gelfand topology). It is a compact Hausdorff space. Evaluation at an $x \in N_G$ determines an element of $M_G(N)$. Hence there is a continuous embedding $i : N_G \hookrightarrow M_G(N)$. In what follows we regard $N_G$ as a subset of $M_G(N)$. Then we prove the corona theorem for $H^\infty(N_G)$.

**Theorem 1.4** $N_G$ is an open everywhere dense subset of $M_G(N)$.

Let $U \subset \subset M$ be a relatively compact domain such that $\overline{N} \subset U$, and $\overline{U}$ satisfies conditions (\ref{item3}). Clearly $M_G(N) \subset M_G(U)$. Let $E$ be a continuous vector bundle over $M_G(U)$ of complex rank $n$. We say that $E$ is holomorphic if $E|_{U_G}$ is holomorphic in the usual sense. A homomorphism $h : E_1 \to E_2$ of holomorphic vector bundles over $M_G(U)$ is said to be holomorphic if $h|_{U_G} : E_1|_{U_G} \to E_2|_{U_G}$ is a holomorphic map. If, in addition, $h$ is a homeomorphism we say that $E_1$ and $E_2$ are holomorphically isomorphic.

**Theorem 1.5** (A Grauert Type Theorem) Assume that holomorphic vector bundles $E_1$ and $E_2$ over some $M_G(U)$ as above are isomorphic as continuous bundles. Then their restrictions to $M_G(N)$ are holomorphically isomorphic.

1.3. In this section we formulate some corollaries of Theorem \ref{item3} that will be used in the proof of Theorem \ref{item4}.

Let $U_n$ denote the group of unitary $n \times n$ matrices. Assume that $U$ is an open Riemann surface satisfying the conditions of Theorem \ref{item4}. Clearly, the universal covering of $U$ is the open unit disk $\mathbb{D} \subset \mathbb{C}$, and $\pi_1(U)$ acts holomorphically on $\mathbb{D}$ by Möbius transformations. Further, for a Riemann surface $X$ let us denote by $||f||_\infty$ the norm of $f \in H^\infty(X)$. We say that a matrix $a = (a_{ij})$ with entries in $H^\infty(X)$ is (left/right) invertible if $a^{-1}(z)$ exists for each $z \in X$ and $a^{-1}$ has entries in $H^\infty(X)$. By $||a|| := \max_{i,j} ||a_{ij}||_\infty$ we denote the norm of $a$.

**Theorem 1.6** Let $\rho : \pi_1(U) \to U_n$ be a homomorphism. There are a constant $C = C(n, N) > 0$ depending only on $n$ and $N$ and an invertible $n \times n$ matrix $a = (a_{ij})$, $a_{ij} \in H^\infty(\mathbb{D})$, such that

1. $a(g(z)) = a(z) \cdot \rho(g)$ for any $g \in \pi_1(U)$, $z \in \mathbb{D}$;

2. $\max\{||a||, ||a^{-1}||\} \leq C$
Now, by $H^\infty_n(U)$ we denote the $H^\infty(U)$-module consisting of the columns $(f_1, \ldots, f_n)$, $f_i \in H^\infty(U)$, $i = 1, \ldots, n$. Any $H^\infty(U)$-invariant subspace of $H^\infty_n(U)$ will be called a submodule. We say that a submodule $M \subset H^\infty_n(U)$ is closed in the topology of the pointwise convergence on $U$ if for any net $\{f_\alpha\} \subset M$ that pointwise converges on $U$ to an $f \in H^\infty_n(U)$ we have $f \in M$. As an application of Theorem 1.6 we obtain.

**Theorem 1.7** Let $M \subset H^\infty_n(U)$ be a submodule closed in the topology of the pointwise convergence on $U$. Then for some $k$ the module $M$ can be represented as $M = H \cdot H^\infty_k(U)$, where $H$ is a left invertible $n \times k$ matrix with entries in $H^\infty(U)$. Moreover, there is a constant $c = c(n, N) > 0$ depending on $n$ and $N$ only such that

$$\max\{||H||, ||H^{-1}||\} \leq c.$$  

Another possible application of Theorem 1.6 is the definition of analogs of Blaschke products on $U$.

Let $r : \mathbb{D} \to U$ be the universal covering map. By $\mathcal{Z}(g)$ we denote the divisor of zeros of a non-zero $g$.

**Corollary 1.8** Let $\{z_i\} \subset U$ be a sequence of not necessarily distinct points. Assume that $r^{-1}(\{z_j\}) = \mathcal{Z}(f)$ for some $f \in H^\infty(\mathbb{D})$. Then there are a positive constant $A = A(N)$ depending on $N$ only and a function $h \in H^\infty(U)$ such that

$$\mathcal{Z}(h) = \{z_i\} \quad \text{and} \quad \sup_{z \in U} |h(z)| \leq A.$$  

2. Proof of the Corona Theorem for $H^\infty(\mathbb{N}_G)$.

In this section we will prove Theorem 1.4.

2.1. First, we describe $M_G$ as a fibre bundle over $M$ with a discrete fibre. We start with the description of the covering $p_G : M_G \to M$ corresponding to a group $G \in \mathcal{G}$.

Let $\mathcal{U} = (U_i)_{i \in I}$ be an open acyclic cover of $M$ by sets biholomorphic to open Euclidean balls. For a complex Lie group $S$ by $Z^1(\mathcal{U}, S)$ we denote the set of holomorphic $S$-valued $\mathcal{U}$-cocycles. By definition, $s = \{s_{ij}\}$, $s_{ij} \in \mathcal{O}(U_i \cap U_j, S)$, is an element of $Z^1(\mathcal{U}, S)$ if

$$s_{ij}s_{jk} = s_{ik} \quad \text{on} \quad U_i \cap U_j \cap U_k.$$  

Now let $X_G := \pi_1(M)/G$ be the set of cosets of $\pi_1(M)$ with respect to the (left) action of $G$ on $\pi_1(M)$ defined by left multiplications. By $[Gq] \in X_G$ we denote the coset containing $q \in \pi_1(M)$. Consider the complex Lie group $H(X_G)$ of all homeomorphisms of $X_G$ (equipped with discrete topology). We define the homomorphism $\tau_G : \pi_1(M) \to H(X_G)$ by the formula:

$$\tau_G([Gq]) := [Gqq^{-1}], \quad q \in \pi_1(M).$$  

Set $Q(G) := \pi_1(M)/\text{Ker}(\tau_G)$ and let $[g]_G$ be the image of $g \in \pi_1(M)$ in $Q(G)$. Finally by $\tau_G : Q(G) \to H(X_G)$ denote the unique homomorphism whose pullback
to \( \pi_1(M) \) coincides with \( \tau_G \). Then from the basic facts of the theory of fibre bundles (see e.g. [Hi]) it follows that

There is a cocycle \( c = \{ c_{ij} \} \in Z^1_b(U, \pi_1(M)) \) such that \( M_G \) is biholomorphic to the quotient space of \( \sqcup_{i \in I} U_i \times X_G \) by the equivalence relation:

\[
U_i \times X_G \ni x \times \tau'_G([c_{ij}]_G)(h) \sim x \times h \in U_j \times X_G.
\]

Projection \( p_G : M_G \to M \) is defined by the coordinate projections \( U_i \times X_G \to U_i \).

Consider now \( M_G := \sqcup_{G \in G} M_G \). Let \( X_G := \prod_{G \in G} X_G \) and \( H_G := \prod_{G \in G} H_G \). For any \( a \in H_G \), \( a = \{ a_G \}_{G \in G}, a_G \in H_G \), we define \( a : X_G \to X_G \) by the formula

\[
a(x) := \{ a(x_G) \}_{G \in G}, \quad x \in X_G, \quad x = \{ x_G \}_{G \in G}, \quad x_G \in X_G.
\]

Further, set \( \tau'_G := \{ \tau'_{G'} \}_{G' \in G} \). Then \( \tau'_G \) is a homomorphism from \( Q(G) := \prod_{G \in G} Q(G) \) into \( H_G \). We also set \( [c_{ij}]_G := \{ [c_{ij}]_G \}_{G \in G} \in \mathcal{O}(U_i \cap U_j, Q(G)) \). Now using the above construction of \( M_G \) we have that \( M_G \) is biholomorphic to the quotient space of \( \sqcup_{i \in I} U_i \times X_G \) by the equivalence relation:

\[
U_i \times X_G \ni x \times \tau'_G([c_{ij}]_G)(h) \sim x \times h \in U_j \times X_G.
\]

In particular, \( M_G \) is a bundle over \( M \) with fibre \( X_G \). By \( p_G : M_G \to M \) we denote the corresponding projection.

Similarly, for a domain \( U \subset M \) satisfying \( \pi_1(U) \cong \pi_1(M) \), the complex space \( U_G := \sqcup_{G \in G} U_G \subset M_G \) is a bundle over \( U \) with discrete fibre \( X_G \).

2.2. As the next step we define a compact Hausdorff space \( E(\overline{N}, \beta X_G) \). Then in the proof of Theorem [4], we will show that this space is homeomorphic to \( M_G(N) \).

Let \( l^\infty(X_G) \) be the algebra of bounded complex-valued functions \( f \) on the discrete space \( X_G \) with pointwise multiplication and norm \( ||f|| = \sup_{x \in X_G} |f(x)| \). Let \( \beta X_G \) be the Stone-\( \v{C} \)ech compactification of \( X_G \), i.e., the maximal ideal space of \( l^\infty(X_G) \) equipped with the Gelfand topology. Then \( X_G \) is naturally embedded into \( \beta X_G \) as an open everywhere dense subset, and the topology on \( X_G \) induced by this embedding coincides with the original one, i.e., is discrete. Every function \( f \in l^\infty(X_G) \) has a unique extension \( \hat{f} \in C(\beta X_G) \). Further, any homeomorphism \( h \in H_G \) of \( X_G \) determines an isometric isomorphism of Banach algebras \( h^* : l^\infty(X_G) \to l^\infty(X_G) \). Therefore \( h \) can be extended to a homeomorphism \( \hat{h} : \beta X_G \to \beta X_G \). This extension shows that now we can think of \( H_G \) as a subgroup of the group of homeomorphisms of \( \beta X_G \).

We retain the notation of Section 2.1. Let us define the bundle \( E(M, \beta X_G) \) over \( M \) with the fibre \( \beta X_G \) as the quotient space of \( \sqcup_{i \in I} U_i \times \beta X_G \) by the equivalence relation:

\[
U_i \times \beta X_G \ni x \times \tau'_G([c_{ij}]_G)(\xi) \sim x \times \xi \in U_j \times \beta X_G.
\]

Let \( \hat{p}_G : E(M, \beta X_G) \to M \) be the corresponding projection. Since \( U_i \times X_G \) is an open everywhere dense subset of \( U_i \times \beta X_G \), the definitions of \( M_G \) and \( E(M, \beta X_G) \) show that \( M_G \) is an open everywhere dense subbundle of \( E(M, \beta X_G) \) and \( \hat{p}_G|_{M_G} = p_G \).

Similarly we can define the bundle \( E(U, \beta X_G) \to U \) for any domain \( U \subset M \) satisfying \( \pi_1(U) \cong \pi_1(M) \), such that \( U_G \) is an open everywhere dense subset of \( E(U, \beta X_G) \).

The proof of the next result repeats the arguments of the proof of [Br1, Th.2.2].
Proposition 2.1 Let $U \subset M$ be a domain satisfying $\pi_1(U) \cong \pi_1(M)$. For every $h \in H^\infty(U_G)$ there is a unique $\hat{h} \in C(E(U, \beta X_G))$ such that $\hat{h}|_{U_G} = h$.

We just recall how to construct $\hat{h}$. Consider the restriction of $h$ to the set $U_i \times X_G$. Then for each $z \in U_i$ we extend the function $h(z, \cdot) \in l^\infty(X_G)$ to $\hat{h}(z, \cdot) \in C(\beta X_G)$ by continuity. The collection of all such extended functions for any $z \in U_i$ and $i \in I$ forms the required function $\hat{h}$.

Let $E(\mathcal{N}, \beta X_G) := \hat{\mathcal{O}}_G^{-1}(\mathcal{N})$. Then it is easy to see that $N_G \subset E(\mathcal{N}, \beta X_G)$ is an open everywhere dense subset, and $E(\mathcal{N}, \beta X_G)$ is a Hausdorff compact. We will prove that $E(\mathcal{N}, \beta X_G)$ is homeomorphic to $\mathcal{M}_G(N)$ which gives us the proof of Theorem 1.4.

2.3. Let $U \subset M$ be a domain satisfying $\pi_1(U) \cong \pi_1(M)$. We study some analytic properties of $E(U, \beta X_G)$. First, it follows from the definition that the base of the topology on $E(U, \beta X_G)$ consists of the sets $S_{O,H}$ homeomorphic to $O \times H$, where $O$ is an open subset of $U$ biholomorphic to an open Euclidean ball and $H$ is a clopen subset of $\beta X_G$. Also, $S_{O,H} \cap U_G$ is an open everywhere dense subset of $S_{O,H}$.

Definition 2.2 A function $f \in C(S_{O,H})$ is said to be holomorphic if its restriction to $S_{O,H} \cap U_G$ is holomorphic in the usual sense. For any open set $W \subset U_G$ the function $f \in C(W)$ is holomorphic if its restriction to each $S_{O,H} \subset W$ is holomorphic. The sheaf of germs of holomorphic on $E(U, \beta X_G)$ functions will be denoted by $\hat{\mathcal{O}}_U$.

A sheaf $\mathcal{F}$ of $\hat{\mathcal{O}}_U$-modules on $E(U, \beta X_G)$ is called syzygetic if for any fibre $F$ of $\hat{\mathcal{O}}_G$ there is an open neighbourhood of $F$ over which $\mathcal{F}$ admits a finite free resolution

$$0 \rightarrow (\hat{\mathcal{O}}_U)^{n_k} \rightarrow \ldots \rightarrow (\hat{\mathcal{O}}_U)^{n_1} \rightarrow \mathcal{F} \rightarrow 0$$

(2.1)

of sheaves of $\hat{\mathcal{O}}_U$-modules.

In the next result by $H^*(E(U, \beta X_G), \mathcal{F})$ we denote the Čech cohomology groups with values in the sheaf $\mathcal{F}$.

Theorem 2.3 Let $U \subset M$ be a Stein domain satisfying $\pi_1(U) \cong \pi_1(M)$. Let $\mathcal{F}$ be a syzygetic sheaf on $E(U, \beta X_G)$. Then for any $i \geq 1$

$$H^i(E(U, \beta X_G), \mathcal{F}) = 0.$$ 

Proof. First, note that for any open cover $\mathcal{V}$ of $E(U, \beta X_G)$ one can find a refinement $\mathcal{V}'$ of $\mathcal{V}$ satisfying the properties:

(a) there is an open countable cover of a finite multiplicity $\mathcal{V}' = (\tilde{V}_i)_{i \in I}$ of $U$ by sets biholomorphic to complex Euclidean balls such that each element of $\mathcal{V}'$ is a subset of one of $\hat{\mathcal{O}}_G^{-1}(\tilde{V}_i)$ of the form $S_{\tilde{V},H}$, where $H$ is a clopen subset of $\beta X_G$, and $\hat{\mathcal{O}}_G(S_{\tilde{V},H}) = \tilde{V}_i$;

(b) for a fixed $i$ the sets $S_{\tilde{V},H} \in \mathcal{V}'$ form a finite open cover of $\hat{\mathcal{O}}_G^{-1}(\tilde{V}_i)$ such that any two distinct sets of the cover are non-intersecting;

(c) $\mathcal{F}$ admits a finite free resolution (2.1) over each $\hat{\mathcal{O}}_G^{-1}(\tilde{V}_i)$.

Existence of the above cover follows from the fact that the fibre $\beta X_G$ is a compact totally disconnected set and that $U$ has a countable exhaustion by compact subsets.
Thus it suffices to calculate the cohomology groups with respect to the family of open covers $\mathcal{V}'$.

For a fixed cover $\mathcal{V}'$ let us consider the presheaf of sections of $\mathcal{F}$ defined on all possible intersections of sets from $\mathcal{V}'$ and its direct image $\mathcal{F}' := (\hat{p}_G)_*(\mathcal{F})$ with respect to $\hat{p}_G$. Then $\mathcal{F}'$ is a presheaf generated by groups of sections $\Gamma(\hat{p}_G^{-1}(O), \mathcal{F})$ where $O$ is intersection of some sets $\hat{V}_i$ from (a). Now property (b) of $\mathcal{V}'$ shows that the groups $H^i(\mathcal{V}', \mathcal{F})$ and $H^i(\hat{\mathcal{V}}, \mathcal{F}')$ are isomorphic (see also the arguments in [Br,Prop.2.4]). Moreover, according to the construction of $E(U, \beta X_G)$, the sheaf generated by $\mathcal{F}'$ can be identified with the sheaf of germs of holomorphic sections of a holomorphic bundle $B$ over $U$ with the fibre $C(\beta X_G)$. Here $B$ is the bundle associated with the right isometric action of $Q(\mathcal{G})$ on $C(\beta X_G)$ (the shift of the argument) obtained from the right action of $Q(\mathcal{G})$ on $\beta X_G$. Thus by (c) $\mathcal{F}'$ admits a finite free resolution

$$0 \rightarrow (\mathcal{O}_U^{C(\beta X_G)})^{n_k} \rightarrow \ldots \rightarrow (\mathcal{O}_U^{C(\beta X_G)})^{n_1} \rightarrow \mathcal{F}' \rightarrow 0$$

of sheaves of $\mathcal{O}_U^{C(\beta X_G)}$-modules over each $\hat{V}_i$ (it is the direct image of the corresponding resolution for $\mathcal{F}$ over $\hat{p}_G^{-1}(\hat{V}_i)$). Here $\mathcal{O}_U^{C(\beta X_G)}$ is sheaf of germs of holomorphic $C(\beta X_G)$-valued functions. Now since $U$ is Stein and $\hat{\mathcal{V}}$ consists of Stein manifolds, by Bungart [B,Sect.3] and the classical Leré theorem (about calculating cohomology groups by acyclic covering) we have (for any $i \geq 1$)

$$H^i(U, \mathcal{F}') = H^i(\hat{\mathcal{V}}, \mathcal{F}') = 0.$$

This completes the proof of the theorem. \(\square\)

We prove now

**Lemma 2.4** The algebra $H^\infty(\overline{\mathcal{N}_G})$ separates points of $E(\overline{\mathcal{N}}, \beta X_G)$.

**Proof.** Let $F_1, F_2$ be fibres of $\hat{p}_G$ over distinct points $z_1, z_2 \in M$. We will show that for any $f_i \in C(F_i)$, $i = 1, 2$, there is a holomorphic function $H \in \hat{\mathcal{O}}_M(E(M, \beta X_G))$ such that $H|_{F_i} = f_i$, $i = 1, 2$. Since the restriction of a holomorphic function defined on $E(M, \beta X_G)$ to $E(\overline{\mathcal{N}}, \beta X_G)$ is a function from $H^\infty(\overline{\mathcal{N}_G})$, this will give the required statement.

Consider the restriction homomorphism of the sheaves $r : \hat{\mathcal{O}}_M \rightarrow C(F_1) \sqcup C(F_2)$. Kernel $\text{Ker}(r)$ is a syzygetic sheaf (see e.g. [B,Lm.3.3]). Thus by Theorem 2.3, $H^1(E(M, \beta X_G), \text{Ker}(r)) = 0$. Then the homomorphism of the global sections

$$r : \hat{\mathcal{O}}_M(E(M, \beta X_G)) \rightarrow C(F_1) \sqcup C(F_2)$$

is surjective. \(\square\)

2.4. We are ready to prove Theorem 4. It is just a routine to check that the statement of the theorem is equivalent to the following one:

Let $U \subset M$ be a Stein domain such that $\pi_1(U) \cong \pi_1(M)$ and $\overline{\mathcal{N}} \subset U$. Let $f_1, \ldots, f_n$ be a collection of bounded holomorphic functions defined on $E(U, \beta X_G)$ satisfying the corona condition (1.3) at each point of $E(U, \beta X_G)$. Then there are an open domain $V \supset \overline{\mathcal{N}}$ in $U$ with $\pi_1(V) \cong \pi_1(U)$ and bounded holomorphic functions $h_1, \ldots, h_n$ defined on $E(V, \beta X_G)$ such that $\sum_{i=1}^n h_i f_i \equiv 1$. 

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Consider the homomorphism $t : (\hat{\mathcal{O}}_U)^n \rightarrow \hat{\mathcal{O}}_U$ defined as
\[ t(s_{1x}, \ldots, s_{nx}) := f_{1x}s_{1x} + \ldots + f_{nx}s_{nx} \quad x \in E(U, \beta X_{\mathcal{G}}). \]
Here $s_x$ denotes the germ of section $s$ at the point $x$. Let us check that $\text{Ker}(t)$ is a syzygetic sheaf on $E(U, \beta X_{\mathcal{G}})$.

Let $z \in U$ be a point and $F_z := \hat{\mathcal{p}}_{\mathcal{G}}^{-1}(z) \subset E(U, \beta X_{\mathcal{G}})$ be the fibre over $z$. Then there is an open neighbourhood $O \subset U$ of $z$ biholomorphic to an open Euclidean ball such that $\hat{\mathcal{p}}_{\mathcal{G}}^{-1}(O)$ is disjoint union of sets $S_{O,H_i}$, $i = 1, \ldots, n$, so that $|f_i(v)| \geq \delta/n$ for any $v \in S_{O,H_i}$. Here $\delta > 0$ is the constant from the corona condition (1.3) for $f_1, \ldots, f_n$. (We also admit that for some $i$ the sets $S_{O,H_i}$ can be empty.) Existence of such $O$ and $S_{O,H_i}$ follows from the continuity of $f_1, \ldots, f_n$ and the fact that $F_z$ is compact and totally disconnected. Now let us define the homomorphism $k : (\hat{\mathcal{O}}_U)^{n-1} \rightarrow \text{Ker}(t)$ over $\hat{\mathcal{p}}_{\mathcal{G}}^{-1}(O)$ by the formula: if $x \in S_{O,H_i}$ then
\[ k(s_{1x}, \ldots, s_{n-1x}) := (s_{1x}, \ldots, s_{i-1x}, \sum_{l=1}^{i-1}(-f_{lx}/f_{ix})s_{lx} + \sum_{l=i+1}^{n-1}(-f_{l+1x}/f_{ix})s_{lx}, s_{ix}, \ldots, s_{n-1x}). \]

Since $H_i$ are clopen non-intersecting subsets of $\beta X_{\mathcal{G}}$, we can glue together the holomorphic matrices that define $k$ on each $S_{O,H_i}$ to obtain a global holomorphic homomorphism over $\hat{\mathcal{p}}_{\mathcal{G}}^{-1}(O)$. Clearly, this homomorphism determines an isomorphism between $(\hat{\mathcal{O}}_U)^{n-1}$ and $\text{Ker}(t)$ over $\hat{\mathcal{p}}_{\mathcal{G}}^{-1}(O)$.

Now according to Theorem 2.3, $H^1(E(U, \beta X_{\mathcal{G}}), \text{Ker}(t)) = 0$. Since also the homomorphism $t$ is locally surjective (because it is surjective at each point), the standard argument (that associates to the short exact sequence of sheaves the long exact sequence of cohomology groups with values in these sheaves) shows that the homomorphism of global holomorphic sections $(\hat{\mathcal{O}}_U(E(U, \beta X_{\mathcal{G}})))^n \rightarrow \hat{\mathcal{O}}_U(E(U, \beta X_{\mathcal{G}}))$ induced by $t$ is surjective. In particular, there are $g_1, \ldots, g_n \in \hat{\mathcal{O}}_U(E(U, \beta X_{\mathcal{G}}))$ such that $\sum g_if_i \equiv 1$. It remains then to restrict $g_1, \ldots, g_n$ to any $E(V, \beta X_{\mathcal{G}})$ with $V \supset \overline{V}$, $\overline{V} \subset U$ and $\pi_1(V) \cong \pi_1(U)$. This restriction produces the required bounded holomorphic functions $h_1, \ldots, h_n$.

The above arguments together with Lemma 2.4 show that $\mathcal{M}_{\mathcal{G}}(N)$ is homeomorphic to $E(\overline{V}, \beta X_{\mathcal{G}})$.

The proof of the theorem is complete. \( \square \)

3. Proof of the Grauert Type Theorem.

In this section we will prove Theorem 1.5.

Let $E$ be a holomorphic vector bundle defined in an open neighbourhood $O$ of $\mathcal{M}_{\mathcal{G}}(N)$. Without loss of generality we can consider $O$ as $E(U, \beta X_{\mathcal{G}})$, where $U$ is a Stein domain in $M$ such that $\overline{N} \subset U$ and $\pi_1(M) \cong \pi_1(U)$. Since the fibre of $E(U, \beta X_{\mathcal{G}})$ is totally disconnected, the arguments similar to those used in the proof of Theorem 1.3 (see Section 2.4) show that the sheaf $\hat{\mathcal{O}}(E)$ of germs of holomorphic sections of $E$ is coherent. (In fact, in an open neighbourhood of a fibre of $E(U, \beta X_{\mathcal{G}})$ it is holomorphically isomorphic to $(\hat{\mathcal{O}}_U)^k$, where $k = \text{rank}(E)$.) By the same reason
the sheaf of germs \( S(E; F) \) of holomorphic sections of \( E \) vanishing on a fibre \( F \) of \( E(U, \beta X_G) \) is also coherent. In particular, \( H^1(E(U, \beta X_G), S(E; F)) = 0 \). Now from the short exact sequence of sheaves

\[
0 \rightarrow S(E; F) \rightarrow \hat{O}(E) \rightarrow \hat{O}(E)|_F \rightarrow 0
\]

by the standard argument involving the long exact sequence of cohomology groups we obtain

**Proposition 3.1** There are global holomorphic sections \( s_1, ..., s_k \) of \( E \), \( k = \text{rank}(E) \), whose restrictions to \( F \) give a trivialization of \( E|_F \). \( \square \)

Recall that \( \text{det}(E) \) is a complex rank 1 vector bundle which is determined by taking the determinant of a cocycle defining \( E \).

**Proposition 3.2** Assume that \( \text{det}(E) \) is holomorphically trivial. Then there are an open neighbourhood \( V \subset O \) of \( M_G(N) \), a positive integer \( t \), and a holomorphic map \( f_E : V \rightarrow \mathbb{C}^{t+1} \) such that

1. \( f_E(V) \) belongs to a closed analytical subset \( X_E \) of \( \mathbb{C}^{t+1} \) defined as the set of zeros of a finite family of holomorphic homogeneous polynomials;
2. \( X_E \setminus \{0\} \) is a smooth manifold;
3. there is a holomorphic vector bundle \( E' \) over \( X_E \setminus \{0\} \) such that \( f_E^* E = E' \).

**Proof.** From Proposition 3.1 and compactness of \( M_G(N) \) it follows that there is an open neighbourhood \( V \subset O \) of \( M_G(N) \) and a finite number of linearly independent holomorphic sections \( h_1, ..., h_n \) of \( E|_V \) such that their restrictions to each point \( x \in V \) generate \( E_x \). As before we can choose \( V \) in the form \( E(U, \beta X_G) \), where \( U \) is a Stein domain containing \( N \) such that \( \pi_1(U) \cong \pi_1(M) \). We now recall the following construction (see, e.g., [GH, Ch.1, Sect. 5]).

Let \( W \) be the \( n \)-dimensional complex vector space of holomorphic sections of \( E|_V \) generated by \( h_1, ..., h_n \). Since for any \( x \in V \) sections from \( W \) generate the fibre \( E_x \), the subspace \( \Lambda_x \subset W \) of sections vanishing at \( x \) is \( n-k \)-dimensional. Below we also use the following definitions.

Let \( X \) be a \( p \)-dimensional complex vector space. By \( G(s, X) \) we denote the corresponding complex Grassmanian, i.e., the set of \( s \)-dimensional complex linear subspaces in \( X \). We also define the universal bundle \( S \rightarrow G(s, X) \) of complex rank \( s \), whose fibre over each \( \Lambda \in G(s, X) \) is the subspace \( \Lambda \). Clearly, \( S \) is a holomorphic subbundle of the trivial bundle \( G(s, X) \times \mathbb{C}^p \). By \( S^* \) we denote the dual bundle which under identification \( * : G(s, X) \rightarrow G(p-s, X^*) \) is isomorphic to the universal bundle over \( G(p-s, X^*) \).

Now there is a natural map

\[
\tau_W : V \rightarrow G(n-k, W) = G(k, W^*)
\]

such that

\[
E = \tau_W^*(S^*) \quad \text{and} \quad W = \tau_W^*(H^0(G(k,n), \mathcal{O}(S^*)))
\]
One can express the map $\tau_W$ explicitly. Indeed, let $e_1, ..., e_k$ be a local reper of holomorphic sections of $E$ (defined in some open subset of $V$). Then for $i = 1, ..., n$,

$$h_i = \sum_{a=1}^k a_{ia} e_a$$

for some holomorphic functions $a_{ia}$. Under the identification $G(n-k, W) \cong G(k, W^*)$, the map $\tau_W$ is locally given by

$$\tau_W(x) := \begin{pmatrix}
a_{11}(x) & \cdots & a_{1k}(x) \\
\vdots & \ddots & \vdots \\
a_{n1}(x) & \cdots & a_{nk}(x)
\end{pmatrix}.$$ 

From this formula it is clear that $\tau_W$ is holomorphic. Let $P_k : G(k, W^*) \rightarrow \mathbb{P}^t$, $t = \binom{n}{k} - 1$, be the Plücker embedding into the projective space. Let $\{A_i\}_{i \in I}$ be the set of all $k \times k$-minors of the above matrix $(a_{ij})$. Then locally the map $P_k \circ \tau_W$ is given as

$$(P_k \circ \tau_W)(x) := (A_1(x) : ... : A_{t+1}(x)).$$

This formula shows that $P_k \circ \tau_W$ is also holomorphic. Moreover, there is a holomorphic vector bundle $\tilde{E}$ over the complex projective manifold $X := P_k(G(k, W^*))$ such that $(P_k \circ \tau_W)^*(\tilde{E}) = E|_V$. Since $det(E)$ is holomorphically trivial, the above functions $A_s$ are just the local representation of global holomorphic functions $h_i, \wedge ... \wedge h_{ik}$ (which are the sections of $det(E)$). In particular, one can define the holomorphic map $f_E : V \rightarrow \mathbb{C}^{t+1}$, $f_E(x) = (A_1(x), ..., A_{t+1}(x))$, such that $P_k \circ \tau_W = \pi \circ f_E$, where $\pi : \mathbb{C}^{t+1} \rightarrow \mathbb{P}^t$ is the natural projection. Also the image of $f_E$ belongs to the complex manifold $X_k := \pi^{-1}(X) \in \mathbb{C}^{t+1} \setminus \{0\}$. Since $X \subset \mathbb{P}^t$ is a smooth projective manifold, by Chow’s theorem $X_k$ is defined as the set of zeros of a finite family of holomorphic homogeneous polynomials. Thus $X_E := X_k \cup \{0\} \subset \mathbb{C}^{t+1}$ is the set of zeros of the same family of polynomials. It remains to set $E' := \pi^*(\tilde{E})$. Then according to our construction, $f_E^*(E') = E|_V$.

The proof of the proposition is complete.

**Proof of Theorem 1.5.** Let $E_1, E_2$ be holomorphic vector bundles of complex rank $k$ defined in an open neighbourhood $O$ of $\mathcal{M}_G(N) = E(N, \beta X_G)$. Assume also that $E_1 \cong E_2$ as continuous bundles. We will prove that there is an open neighbourhood $V \subset O$ of $\mathcal{M}_G(N)$ such that $E_1|_V$ and $E_2|_V$ are holomorphically isomorphic.

First, we will prove the theorem under the additional assumption that the bundles $det(E_i)$, $i = 1, 2$, are holomorphically trivial. Then as in Proposition 3.2 we can construct holomorphic maps $\tau_i : U \rightarrow G(k, n) \rightarrow \mathbb{P}^t$ (with the same $n$) for some open neighbourhood $U \subset O$ of $\mathcal{M}_G(N)$ such that $E_i|_U = \tau_i^*(S)$, $i = 1, 2$. Since $U$ is a compact and $E_1, E_2$ are topologically isomorphic, we can choose $n$ so big that $\tau_1$ and $\tau_2$ are homotopic (see, e.g., Husemoller [H]). Denote by $H_t : \overline{U} \rightarrow G(k, n)$, $t \in [0, 1]$, this homotopy. Then $B := \{B_t := H_t^*(S), t \in [0, 1]\}$, is a continuous bundle on $\overline{U} \times [0, 1]$ such that $B_0 = E_1$ and $B_1 = E_2$. Let $s_1, ..., s_n$ be a basis in $H^0(G(k, n), O(S))$. Then $s_i(t) := H_t^*(s_i)$, $i = 1, ..., n$, is a family of linearly independent continuous sections of $B$, such that for any $t$, $s_1(t), ..., s_n(t)$ generate each
fibre of $B_t$. Moreover, each $H_t$ is defined by $s_1(t), ..., s_n(t)$ as in the construction of Proposition 3.2. Since also $\det(B)$ is topologically trivial (because it determines a continuous homotopy between $\det(E_1)$ and $\det(E_2)$), we obtain a continuous homotopy between holomorphic maps $f_{E_i}: \mathcal{U} \to \mathbb{C}^t$, $i = 1, 2$, that covers $H_t$. Indeed, the homotopy map is defined by the family of global complex-valued continuous functions \{s_1(t) \wedge ... \wedge s_n(t)\} on $\mathcal{U} \times [0, 1]$.  

Now the remaining part of the proof can be extracted from Novodvorskii’s theorem [No], however, for the sake of completeness we will present a more detailed exposition.

Consider $\mathcal{M}_G(N)$ as the inverse limit of compacts. Namely, let $\Gamma$ be the set of all finite collections of holomorphic functions defined in some open neighbourhoods of $\overline{N}$ such that each collection from $\Gamma$ contains also the functions from $f_{E_1}$ and $f_{E_2}$. Let us fix some order in each $\gamma \in \Gamma$ such that $\gamma$ is started with the functions from $f_{E_1}$ and then from $f_{E_2}$. For $\gamma_i \in \Gamma$, $i = 1, 2$, we will say that $\gamma_1 \leq \gamma_2$ if the ordered set $\gamma_2$ contains $\gamma_1$ as an ordered subset. Clearly, each $\gamma \in \Gamma$ can be considered as a holomorphic map into some $\mathbb{C}^l(\gamma)$, where $l(\gamma)$ is the cardinality of $\gamma$. Then we set $N^\prime_{\gamma} := \gamma(\mathcal{M}_G(N)) \subset \mathbb{C}^l(\gamma)$. Also by $N^\prime_{\gamma} \in \mathbb{C}^{2t+2}$ we denote the image of $\mathcal{M}_G(N)$ by the map $e = (f_{E_1}, f_{E_2})$. Further, by $N_\gamma$ we will denote the polynomial hull of $N^\prime_{\gamma}$. Clearly, if $\gamma_1 \leq \gamma_2$, the projection $\pi^\gamma_{\gamma_1} : \mathbb{C}^{l(\gamma_1)} \to \mathbb{C}^{l(\gamma_1)}$ to the first $l(\gamma_1)$ coordinates maps $N_{\gamma_2}$ into $N_{\gamma_1}$. According to Proposition 3.2 (1), the set $X_1 := X_{E_1} = X_{E_2} \subset \mathbb{C}^{t+1}$ is polynomially convex. Thus $N_\gamma$ belongs to the polynomially convex set $Y_\gamma := (\pi^\gamma_{\gamma})^{-1}(X_1 \times X_1)$. Let $B^k(r) \subset \mathbb{C}^k$ be the open Euclidean ball centered at 0. We set

$$\tilde{N}_\gamma := Y_\gamma \cap (N_\gamma + B^{l(\gamma)}(1/l(\gamma))).$$  

Since for $\gamma_1 \leq \gamma_2$ we have $l(\gamma_1) \leq l(\gamma_2)$, the projection $\pi^\gamma_{\gamma_1}$ maps $\tilde{N}_{\gamma_2}$ into $\tilde{N}_{\gamma_1}$. Moreover, each $\tilde{N}_\gamma$ is an open neighbourhood of $N_\gamma$ in $Y_\gamma$. Thus we have the inverse limiting system generated by $\tilde{N}_\gamma$ and $\pi^\gamma_{\gamma_1}$. Since $\mathcal{M}_G(N)$ is the maximal ideal space of $H^\infty(N_G)$, the inverse limit of this system coincides with $\mathcal{M}_G(N)$. By $\pi_{\gamma} : \mathcal{M}_G(N) \to \tilde{N}_\gamma$ we denote the corresponding map. Let $h_1 : \mathbb{C}^{2t+2} \to \mathbb{C}^{t+1}$, $h_2 : \mathbb{C}^{2t+2} \to \mathbb{C}^{t+1}$ be the projections to the first and to the last $t + 1$ coordinates, respectively. Then as we have already proved, $f_{E_i} := h_1 \circ \pi_{\gamma}$ is homotopic to $f_{E_2} := h_2 \circ \pi_{\gamma}$ inside of the smooth manifold $X_1 \setminus \{0\}$. Now it is easy to prove (see, e.g., [L, Lm.1]) that there is a $\gamma \in \Gamma$ such that $h_1 \circ \pi_{\gamma}$ maps $\tilde{N}_\gamma$ into $X_1 \setminus \{0\}$, $i = 1, 2$, and $h_1 \circ \pi_{\gamma}$ is homotopic to $h_2 \circ \pi_{\gamma}$ inside of $X_1 \setminus \{0\}$. In particular, we obtain that $\tilde{N}_\gamma$ belongs to the smooth part of $Y_\gamma$. Let $F_i := (h_1 \circ \pi_{\gamma})^\ast (E')$, $i = 1, 2$, with $E'$ as in Proposition 3.2 (3). Then $F_1$ and $F_2$ are isomorphic as continuous vector bundles and $E_i = (h_1 \circ \pi_{\gamma})^\ast (E')$, $i = 1, 2$. Since $\tilde{N}_\gamma \subset Y_\gamma$ is an open smooth neighbourhood of the Stein compact $N_\gamma$, there is an open smooth Stein neighbourhood $X \subset \tilde{N}_\gamma$ of $N_\gamma$. Now $F_1|_X \cong F_2|_X$ as continuous bundles and therefore by Grauert’s theorem [Gr], they are holomorphically isomorphic too. Then $E_1$ and $E_2$ are holomorphically isomorphic because they are pullbacks of $F_1|_X$ and $F_2|_X$ by a holomorphic map.

This completes the proof of the theorem in the case when $\det(E_i)$, $i = 1, 2$, are holomorphically trivial.
Consider now the case when $det(E_1)$ and $det(E_2)$ are not necessarily holomorphically trivial. From the conditions of the theorem it follows that $det(E_1)$ and $det(E_2)$ are topologically isomorphic. Thus the holomorphic complex rank 1 vector bundle $det(E_1) \otimes det(E_2)^{-1}$ is topologically trivial. Now according to the first part of the proof of the theorem, $det(E_1) \otimes det(E_2)^{-1}$ is holomorphically trivial too. Consider now the holomorphic vector bundles $H_i := E_i \otimes det(E_i)^{-1}$, $i = 1, 2$. Then $H_1$ and $H_2$ are topologically isomorphic and $det(H_i)$, $i = 1, 2$, are topologically trivial. Again from the first part of the proof it follows that $H_1$ and $H_2$ are holomorphically isomorphic. Since as it was shown above $det(E_1)$ and $det(E_2)$ are holomorphically isomorphic, the bundles $E_1 = H_1 \otimes det(E_1)$ and $E_2 = H_2 \otimes det(E_2)$ are holomorphically isomorphic too.

The proof of the theorem is complete. \( \square \)

4. Some Additional Properties of $E(M, \beta X_G)$.

In this section we describe some topological properties of the space $E(M, \beta X_G)$ which will be used in the proof of Theorem \([\mathbb{4}]\).

Let $H$ be a connected Hausdorff space. Assume also that $H$ is locally arcwise connected, locally simply connected and admits an exhaustion by at most countable number of compact subsets. Then the fundamental group $\pi_1(H)$ is well defined. Now the bundle $E(H, \beta X_G)$ over $H$ can be defined using the coverings corresponding to some family of subgroups $G \in \mathcal{G}$ exactly as in Section 2.2 (with $H$ instead of $M$). Assume, in addition, that the topological (covering) dimension $dim(H)$ of $H$ is finite.

**Proposition 4.1** $E(H, \beta X_G)$ is a paracompact space satisfying

$$dim(E(H, \beta X_G)) = dim(H).$$

**Proof.** The above conditions imply that $H$ can be covered by at most countable number of simply connected relatively compact subsets $U_i$. By the definition over each $U_i$ the bundle $E(H, \beta X_G)$ is homeomorphic to $U_i \times \beta X_G$. Therefore $E(H, \beta X_G)$ can be covered by at most countable number of open relatively compact subsets $E(H, \beta X_G)|_{U_i}$ which implies that $E(H, \beta X_G)$ is a paracompact space. Since $dim(U_i) \leq dim(H)$ and $dim(\beta X_G) = 0$, we obtain that $dim(U_i \times \beta X_G) \leq dim(H)$. But for any $x \in \beta X_G$ we have $U_i \times \{x\} \subset U_i \times \beta X_G$. Thus $dim(U_i) \leq dim(U_i \times \beta X_G)$. Now the required result follows from existence of the open cover of $E(H, \beta X_G)$ by sets homeomorphic to $U_i \times \beta X_G$ and the fact that there is some $i$ for which $dim(U_i) = dim(H)$. \( \square \)

Let $K$ be a connected finite-dimensional locally compact Hausdorff space satisfying the same properties as $H$. Assume also that $K$ is homotopically equivalent to $H$. Then $\pi_1(K) \cong \pi_1(H)$ and so $E(K, \beta X_G)$ is well defined too.

**Proposition 4.2** Under the above assumptions, $E(H, \beta X_G)$ and $E(K, \beta X_G)$ are homotopically equivalent.
Proof. Let $F : H \to K$ and $G : K \to H$ be such that $F \circ G : K \to K$ and $G \circ F : H \to H$ are homotopic to the identity maps. Since $F_\ast$ and $G_\ast$ induce isomorphisms of fundamental groups, the covering homotopy theorem (see e.g. [Hu]) implies that there are continuous maps $\tilde{F} : H_G \to K_G$ and $\tilde{G} : K_G \to H_G$ that cover $F$ and $G$, respectively, such that $\tilde{F} \circ \tilde{G}$, and $\tilde{G} \circ \tilde{F}$ are homotopic to the identity maps. (Here $H_G$ and $K_G$ are defined similarly to $M_G$ from Section 2.1.) Locally the maps $\tilde{F}$ and $\tilde{G}$ can be described as follows. Let $U \subset H$ and $V \subset K$ be open simply connected subsets such that $F(U) \subset V$. Then by the definition $H_G$ is homeomorphic to $U \times X_G$ over $U$ and $K_G$ is homeomorphic to $V \times X_G$ over $V$. Moreover, in appropriate local coordinates we have $\tilde{F}(u \times x) := F(u) \times x \in K_G|_V$ for $u \times x \in U \times X_G \cong H_G|_U$. A similar description is valid for $\tilde{G}$. Since the local description is clearly equivariant with respect to the equivalence relations defining $E(H, \beta X_G)$ and $E(K, \beta X_G)$, we can extend $\tilde{F}$ and $\tilde{G}$ to continuous maps $\tilde{F}' : E(H, \beta X_G) \to E(K, \beta X_G)$ and $\tilde{G}' : E(K, \beta X_G) \to E(H, \beta X_G)$. For instance, for $\tilde{F}'$ the required extension over $U$ has the form

$$\tilde{F}'(u \times \xi) := F(u) \times \xi \in E(K, \beta X_G)|_V, \quad u \times \xi \in U \times \beta X_G \cong E(H, \beta X_G)|_U.$$  

It remains to check that $\tilde{F}'$ and $\tilde{G}'$ determine a homotopic equivalence of $E(H, \beta X_G)$ and $E(K, \beta X_G)$. Consider $G \circ F$. Let $R : H \times [0, 1] \to H$ be a continuous map such that $R(., 0) = G \circ F$ and $R(., 1) = id$. Let us prove that we can lift $R$ to a homotopy of $E(H, \beta X_G)$. As before, by the covering homotopy theorem, there is a homotopy $\tilde{R} : H_G \times [0, 1] \to H_G$ that covers $R$. Let $U_i \subset H, i = 1, 2$, be open simply connected sets such that $R(U_1 \times I) \subset U_2$ for an open subinterval $I \subset [0, 1]$. Then in appropriate local coordinates $\tilde{R} : H_G|_{U_1} \times I \to H_G|_{U_2}$ is defined by the formula

$$\tilde{R}(u_1 \times x \times t) := R(u_1 \times t) \times x \in H_G|_{U_2}, \quad u_1 \times x \times t \in H_G|_{U_1} \times I.$$  

Clearly, $\tilde{R}(., 0) = \tilde{G} \circ \tilde{F}$ and $\tilde{R}(., 1) = id$. Since $\tilde{R}$ is equivariant with respect to the equivalence relation that defines $E(H, \beta X_G)$, we can determine the homotopy $\tilde{R}' : E(H, \beta X_G) \to E(H, \beta X_G)$. Locally it is given by the formula

$$\tilde{R}'(u_1 \times \xi \times t) := R(u_1 \times t) \times \xi \in E(H, \beta X_G)|_{U_2}, \quad u_1 \times \xi \times t \in E(H, \beta X_G)|_{U_1} \times I.$$  

Clearly $\tilde{R}'$ is a continuous extension of $\tilde{R}$, and $\tilde{R}'(., 0) = \tilde{G}' \circ \tilde{F}'$, $\tilde{R}'(., 1) = id$. Similar arguments can be applied to $\tilde{F}' \circ \tilde{G}'$. This shows that $E(H, \beta X_G)$ and $E(K, \beta X_G)$ are homotopically equivalent. □

In the next result we will assume that $M$ is a complex manifold. Recall that $M_G$ is a fibre bundle over $M$ with the fibre $X_G$ associated with a representation $\tau_G^\ast$ of $Q(G)$ into the group of bijective homomorphisms of $X_G$. Let $T := \prod_{G \in G} G$ and $\rho : T \to U_n$ be a unitary representation. By $E_\rho$ we denote the flat vector bundle over $M_G$ associated with $\rho$.

**Proposition 4.3** There is a unique holomorphic vector bundle $\tilde{E}_\rho$ over $E(M, \beta X_G)$ whose restriction to $M_G$ coincides with $E_\rho$. 

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Proof. According to the construction of Section 2.1, we can describe $E_{\rho}$ as follows (cf. [Br,Prop.2.3]). There is an open acyclic cover $\mathcal{U} = \{U_i\}_{i \in I}$ of $M$ such that $E_{\rho}$ is defined on the cover $\mathcal{V} = \{V_i\}_{i \in I}$ of $M$. Define $V_i := p^{-1}_G(U_i)$, by a cocycle $d = \{d_{ij}\} \in \mathbb{Z}^1_\mathbb{B}(\mathcal{V}, T)$, i.e., $E_{\rho}$ is defined by the equivalence relation

$$V_i \times \mathbb{C}^n \ni x \times \rho(d_{ij})(v) \sim x \times v \in V_j \times \mathbb{C}^n.$$ 

Observe that each $V_i$ is biholomorphic to $U_i \times X_G$ and therefore $\rho(d_{ij})$ can be thought of as a function defined on $U_i \cap U_j$ with values in the space of maps $X_G \rightarrow U_n$. In fact, $\rho(d_{ij})$ is a constant multivalued function. Further, because $U_n$ is a compact subset of some $\mathbb{C}^N$, each map $r : X_G \rightarrow U_n$ admits the natural continuous extension $\tilde{r} : \beta X_G \rightarrow U_n$ (obtained by the extension of coordinates of $r$). Therefore we obtain an extended function $\tilde{\rho}(d_{ij}) : U_i \cap U_j \rightarrow C(\beta X_G, U_n)$. Now denote by $\mathcal{V}_k$ the set obtained from $V_k$ by taking the closure of each fibre of $V_k \subset E(M, \beta X_G)$. By definition, $\mathcal{V}_k = \tilde{\rho}^{-1}_G(U_k) \cong U_n \times \beta X_G$ is an open subset of $E(M, \beta X_G)$. Rewriting according to this identification $\tilde{\rho}(d_{ij})$ in coordinates on $E(M, \beta X_G)$, we can think of $\tilde{\rho}(d_{ij})$ as a continuous function $\mathcal{V}_i \cap \mathcal{V}_j \rightarrow U_n$ such that $\tilde{\rho}(d_{ij})|_{\mathcal{V}_i \cap \mathcal{V}_j} = \rho(d_{ij})$. Moreover, since $\rho(d_{ij}) \cdot \rho(d_{jk}) = \rho(d_{ik})$ on $V_i \cap V_j \cap V_k$, we have $\tilde{\rho}(d_{ij}) \cdot \tilde{\rho}(d_{jk}) = \tilde{\rho}(d_{ik})$ on $\mathcal{V}_i \cap \mathcal{V}_j \cap \mathcal{V}_k$ by continuity. This shows that $\{\tilde{\rho}(d_{ij})\}$ determines a cocycle defined on the cover $(\mathcal{V}_i)_{i \in I}$ of $E(M, \beta X_G)$ with values in $U_n$. Thus we can define $\tilde{E}_{\rho}$ on $E(M, \beta X_G)$ by the equivalence relation

$$\mathcal{V}_i \times \mathbb{C}^n \ni x \times \tilde{\rho}(d_{ij})(v) \sim x \times v \in \mathcal{V}_j \times \mathbb{C}^n.$$ 

Clearly $\tilde{E}_{\rho}|_{\mathcal{V}_G} = E_{\rho}$ and so $\tilde{E}_{\rho}$ is holomorphic by the definition. \hfill\Box

Assume now that $M$ is a non-compact complex Riemann surface and $E_{\rho}$, $\tilde{E}_{\rho}$ are the bundles from Proposition 4.3.

**Proposition 4.4** $\tilde{E}_{\rho} \rightarrow E(M, \beta X_G)$ is a topologically trivial holomorphic vector bundle.

**Proof.** Since any non-compact complex Riemann surface is a Stein manifold, $M$ is homotopically equivalent to a one-dimensional CW-complex $K$ (which satisfies conditions of Proposition 4.2). Then according to Proposition 4.2, the paracompact spaces $E(M, \beta X_G)$ and $E(K, \beta X_G)$ are homotopically equivalent. Let $s : E(K, \beta X_G) \rightarrow E(M, \beta X_G)$ be one of the maps determining the equivalence. Then according to the general theory of vector bundles (see [H]) the statement of the proposition will follow from triviality of $s^*(\tilde{E}_{\rho})$ over $E(K, \beta X_G)$. Now Proposition 4.1 implies that $\dim(E(K, \beta X_G)) = 1$. Therefore the only obstacle to triviality of $s^*(\tilde{E}_{\rho})$ is the first Chern class $c_1(s^*(\tilde{E}_{\rho})) \in H^2(E(K, \beta X_G), \mathbb{Z})$. Since $E(K, \beta X_G)$ is one dimensional, the latter cohomology group is trivial which implies that $c_1(s^*(\tilde{E}_{\rho})) = 0$, and so $\tilde{E}_{\rho}$ is topologically trivial. \hfill\Box

5. **Proof of Theorem 1.4 and Corollary 1.6.**

**Proof of Theorem 1.6.** Let $N \subset M$ be a relatively compact domain in an open Riemann surface $M$ such that $\pi_1(N) \cong \pi_1(M)$. Let $R$ be an unbranched covering
of $N$ and $i : U \hookrightarrow R$ be a domain in $R$. Assume that the induced homomorphism of the fundamental groups $i_* : \pi_1(U) \to \pi_1(R)$ is injective. Without loss of generality we may also assume that $i_*$ is surjective. Indeed, if $i_*(\pi_1(U))$ is a proper subgroup of $\pi_1(R)$, then consider the covering $\tilde{R}$ of $R$ corresponding to $i_*(\pi_1(U))$. Clearly $\tilde{R}$ is a covering of $N$. Now, by the covering homotopy theorem, there is a holomorphic embedding $i : U \hookrightarrow \tilde{R}$ that covers $i$ and such that $i_* : \pi_1(U) \to \pi_1(\tilde{R})$ is a bijection. So we can work with the triple $(U, \tilde{i}, \tilde{R})$ satisfying conditions of the theorem.

Now, according to the above assumptions, any homomorphism $\rho : \pi_1(R) \to U_n$ coincides with $\rho \circ i_* : \pi_1(U) \to U_n$.

Let $G \subset \pi_1(N)$ be a subgroup and $\rho : G \to U_n$ be a homomorphism. Denoting $G$ by $G_\rho$ we emphasize that $G$ is the domain of the definition of $\rho$. Then we set

$$G := \{ \rho : G \subset \pi_1(N), \rho \in \text{Hom}(G, U_n) \}.$$

Since $\pi_1(M) \cong \pi_1(N)$, we can construct the associated with a $\rho \in \text{Hom}(G, U_n)$ complex vector bundle $E_\rho$ over $M_{G_\rho}$. Here $M_{G_\rho}$ is the covering of $M$ with the fundamental group $G_\rho$. Let $\mathcal{U} = (U_i)_{i \in I}$ be the acyclic cover of $M$ from the construction of $M_{G_\rho}$ and $M_G$ (see Section 2.1). Consider the cover $\mathcal{V}_{G_\rho} := p^{-1}_{G_\rho} (\mathcal{U})$ of $M_{G_\rho}$. Since $\mathcal{V}_{G_\rho}$ is acyclic too, there is a cocycle $\{ c_{ij,G_\rho} \} \in Z^1(\mathcal{V}_{G_\rho}, U_n)$ such that $E_\rho$ is equivalent to the quotient space of $\bigsqcup_{i \in I} p^{-1}_{G_\rho}(U_i) \times \mathbb{C}^n$ by the equivalence relation

$$p^{-1}_{G_\rho}(U_i) \times \mathbb{C}^n \ni x \times c_{ij,G_\rho}(v) \sim x \times v \in p^{-1}_{G_\rho}(U_j) \times \mathbb{C}^n.$$

The same is valid for any $\rho$ and $G_\rho$. Thus we can construct a holomorphic bundle $E_\rho$ over $M_G$ defined on the cover $p^{-1}(\mathcal{U})$ such that $E_\rho|_{M_{G_\rho}} = E_\rho$. Clearly, the bundle $E_\rho$ has the unitary structure group. In fact, $E_\rho$ is a vector bundle associated with a representation $\tilde{R} : \prod_{G_\rho \in G} G_\rho \to U_n$, $R|_{G_\rho} = \rho$. Now according to Propositions 4.3 and 4.4, the extension $\tilde{E}_\rho$ of $E_\rho$ to $E(M, \beta X_G)$ is topologically trivial. Since $N \subset M$ is a Stein compact, by Theorem 1.3 $\tilde{E}_\rho$ is holomorphically trivial in an open neighbourhood of $E(N, \beta X_G) \subset E(M, \beta X_G)$. Going back to $E_\rho$ we obtain that cocycle $\{ c_{ij,G_\rho} \}$ is holomorphically trivial on $\mathcal{N}_{G_\rho} \subset M_{G_\rho}$. We can say even more. Let $\mathcal{U}' = (U'_i)$ be a finite acyclic cover of $N$ by compact Euclidean balls such that each $U'_i$ is a compact subset of one of $U_j \in \mathcal{U}$. Let us restrict $\{ c_{ij,G_\rho} \}$ to $p^{-1}_{G_\rho}(\mathcal{U}')$. From the triviality of $\tilde{E}_\rho$ it follows that there are holomorphic functions $c_{i,G_\rho} \in \mathcal{O}(p^{-1}_{G_\rho}(U'_i), GL_n(\mathbb{C}))$ such that

$$c_{i,G_\rho}^{-1}(z) \cdot c_{j,G_\rho}(z) = c_{ij,G_\rho}(z), \quad \text{for any } z \in p^{-1}_{G_\rho}(U'_i) \cap p^{-1}_{G_\rho}(U'_j),$$

and there is a constant $C > 0$ depending only on $n$ and $N$ such that

$$\sup_{i,G_\rho} \max_{i,G_\rho} \{ ||c_{i,G_\rho}||, ||c_{i,G_\rho}^{-1}|| \} \leq C.$$

(Here $||.||$ is defined as in Section 1.3.)

Let us take now an unbranched covering $R$ of $N$ and $U \subset R$ as in the beginning of this section. Consider the universal covering $r : \mathbb{D} \to R$. (Recall that $R \subset M_{G_\rho}$ and $\pi_1(R) = \pi_1(M_{G_\rho}) = G_\rho$.) Then $r^{-1}(p^{-1}_{G_\rho}(U'_i)) = \bigsqcup_{g \in G_\rho} S_{ig}$, where $S_{ig}$ is biholomorphic
to $p_{G_\rho}^{-1}(U_i')$. Consider the pullback $r^*\{\{c_{ij,G_{\rho}}\}\}$ to the cover $(S_{ij})$ of $\mathbb{D}$. Since the latter is an acyclic cover and the bundle $E_p$ is obtained from the representation $\rho : G_\rho \to U_n$, there are locally constant functions $\rho_{ig,G_{\rho}} : S_{ij} \to U_n$ such that $\rho_{ig,G_{\rho}}(z) \cdot \rho_{ijh,G_\rho}^{-1}(z) = r^*\{c_{ij,G_{\rho}}\}(z)$ for any $z \in S_{ij} \cap S_{ijh} \neq \emptyset$, and $\rho_{il,g,G_{\rho}} = \rho_{ig,G_{\rho}} \cdot \rho(l)$. Define $F_{ig,G_\rho} = r^*(c_{ig,G_{\rho}}) \cdot \rho_{il,g,G_{\rho}}$ on $S_{ij}$. Then from the above equations it follows that $F_{ig,G_\rho}(z) = F_{ijh,G_\rho}(z)$ for any $z \in S_{ij} \cap S_{ijh}$. That is the family $\{F_{ig,G_\rho}\}$ determines a global function $F_{G_\rho} \in \mathcal{O}(\mathbb{D},GL_n(\mathbb{C}))$. Now the group $G_\rho$ acts holomorphically on $\mathbb{D}$ by Möbius transformations such that each $l \in G_\rho$ maps any $S_{ij}$ biholomorphically onto $S_{il,g}$. In particular, for $z \in S_{ij}$ we have

$$F_{G_\rho}(l(z)) = r^*(c_{ij,G_{\rho}})(l(z)) \cdot \rho_{il,g,G_{\rho}} = F_{G_\rho}(z) \cdot (\rho_{ijh,G_{\rho}}^{-1} \cdot \rho_{il,g,G_{\rho}}) = F_{G_\rho}(z) \cdot \rho(l).$$

Clearly $F_{G_\rho}$ satisfies the required estimates of Theorem 1.4. Since the universal covering $\bar{U}$ of $U$ admits a holomorphic embedding into $\mathbb{D}$ equivariant with respect to the actions of $G_\rho$ on $\bar{U}$ and $\mathbb{D}$, respectively, the restriction $F_{G_\rho}|\bar{U}$ determines the required matrix function $a$. \qed

**Proof of Corollary 1.8.** Let $\{z_i\} \subset U$ be a sequence such that $r^{-1}(\{z_i\}) \subset \mathbb{D}$ is the set of zeros of a non-zero bounded holomorphic function. In particular, there is a Blaschke product $B$ whose set of zeros (counted with their multiplicities) is exactly $r^{-1}(\{z_i\})$. Since the set $\{z_i\}$ is invariant with respect to the action of $G := \pi_1(U)$ and $B$ is an interior function, we have $B(g(z)) = B(z) \cdot \rho(g)$, $g \in G$, for some representation $\rho : G \to U_1$. Now according to Theorem 1.4, there is a holomorphic function $a \in \mathcal{O}(\mathbb{D},C^*)$ such that $a(g(z)) = a(z) \cdot \rho(g)$, $g \in G$, and $\max\{||a||_\infty, ||a^{-1}||_\infty\} \leq C$ for some constant $C$ depending on $N$ only. Set $\tilde{h}(z) := B(z)/a(z)$. Then $\tilde{h}$ is invariant with respect to the action of $G$, the set of zeros (counted with their multiplicities) of $\tilde{h}$ is $r^{-1}(\{z_i\})$, and $||\tilde{h}||_\infty \leq C$. Clearly, $\tilde{h} = r^*(h)$ for some $h \in H^\infty(U)$ satisfying the required properties. \qed

### 6. Proof of Theorems 1.1 and 1.5.

**Proof of Theorem 1.7.** In the proof we use the Lax-Halmos theorem (see e.g. [T]). By $S^1$ we denote the boundary of $\mathbb{D}$.

**Lax-Halmos Theorem.** Let $M$ be a weak * closed submodule of the $H^\infty(\mathbb{D})$-module $H^\infty_0(\mathbb{D})$. Then for some $k$ we have $M = \Psi \cdot H^\infty_k(\mathbb{D})$, where $\Psi$ is a left unimodular $n \times k$ matrix with entries in $H^\infty(\mathbb{D})$, that is, $\Psi^*\xi \cdot \Psi(\xi) = I_k$ for a.e. $\xi$, $\xi \in S^1$. If two such modules $\Psi \cdot H^\infty_k(\mathbb{D})$ and $\Theta \cdot H^m_\infty(\mathbb{D})$ are equal, then $k = m$ and $\Psi = \Theta \cdot V$, where $V \in U_k$.

The proof of a similar result for submodules of $H^2_n(\mathbb{D})$ (with the same conclusion but with $H^2_n(\mathbb{D})$ instead of $H^\infty_0(\mathbb{D})$) can be found, e.g., in [Ni,Lect.I, Corol.6]. The required result now can be obtained from the case of $H^2_n(\mathbb{D})$ similarly to the proof in [Ga,Ch.II, Th.7.5].

We proceed to the proof of Theorem 1.7. In our proof we use a scheme suggested in [T]. Here, however, instead of the Forelli theorem [F] we use Theorem 1.1 of [Br], and instead of the classical Grauert theorem [Gr] our Theorem 1.4.


Let $r : \mathbb{D} \rightarrow U$ be the universal covering map and $G = \pi_1(U)$. We can identify $H^\infty(U)$ with $H^\infty_G := r^*(H^\infty(\mathbb{D}))$, the subalgebra of $G$-invariant functions in the algebra $H^\infty(\mathbb{D})$. The module $M$ can be identified with an $H^\infty_G$-submodule in $H^\infty_{G,n} := r^*(H^\infty_n(U))$. Let $N$ be the weak $*$ closed $H^\infty(\mathbb{D})$-module generated by $M$. Then by the Lax-Halmos theorem, $N = \Psi \cdot H^\infty(\mathbb{D})$, where $\Psi$ is a left unimodular matrix. Therefore we have $\max H \mapsto H_{\Psi}$ satisfies the corona condition. According to Lemma 1 of [T], one can find a matrix $H \in \mathcal{O}(\mathbb{D}, GL_k(\mathbb{C}))$ such that $\Omega(A(z)) = \Omega(z) \cdot \alpha(A)$. Since $\alpha(A)^\ast = \alpha(A)^{-1}$, we have $\Omega'(A(z)) = \alpha(A)^\ast \cdot \Omega'(z)$ where by Theorem [L6] $\Omega' := \Omega^{-1}$ is a bounded matrix from $\mathcal{O}(\mathbb{D}, GL_k(\mathbb{C}))$. Let

$$H^\infty(\mathbb{D}, \alpha) := \{ f \in H^\infty_k(\mathbb{D}) : f(A(z)) = \alpha(A)^\ast \cdot f(z) \text{ for all } z \in \mathbb{D} \text{ and all } A \in G \}.$$ 

Then $N \cap H^\infty_{G,n} := \Psi \cdot H^\infty(\mathbb{D}, \alpha)$. Further, $\Psi \cdot \Omega'$ is a bounded $n \times k$ matrix with entries in $H^\infty_G$, $N \cap H^\infty_{G,n} = \Psi \cdot \Omega' \cdot H^\infty_{G,k}$, and $\Psi \cdot \Omega'$ is the pullback of a left invertible $n \times k$ matrix $H$ with entries in $H^\infty(U)$. It remains to check the equality $N \cap H^\infty_{G,n} = M$. By [Br, Th. 1.1] there exists a continuous $H^\infty_G$-linear projector $P : H^\infty(\mathbb{D}) \rightarrow H^\infty_G$ satisfying the property:

Let $o(x) := \{ g_x \}_{g \in G}$ be an orbit, and let $\{ f_a \} \subset H^\infty(\mathbb{D})$ be a net. Assume that the restriction $\{ f_a \}_{o(x)}$ converges in the weak * topology of $l^\infty(o(x))$ to $f|_{o(x)}$ for some $f \in H^\infty(\mathbb{D})$. Then $\lim f_a(x) = P(f)(x)$.

Now if $f \in N \cap H^\infty_{G,n}$, then $f$ is the limit in the weak * topology of a net $\{ f_a \}$ of the form $f_a = \sum_{i=1}^n g_{i_o} \cdot h_{i_o}$ where $g_{i_o} \in H^\infty(\mathbb{D}), h_{i_o} \in M$. Since any orbit $o(x)$ is an interpolating sequence for $H^\infty(\mathbb{D})$ (see, e.g. [Br1, Lm. 7.1]), the restriction of $\{ f_a \}$ to $o(x)$ converges in the weak * topology of $l^\infty(o(x))$ to $f|_{o(x)}$. Further, $P(g_{i_o} \cdot h_{i_o}) = P(g_{i_o}) \cdot h_{i_o}$, and so we have $P(f_a) \in M$. Finally, $P(f) = f \in M$ because, according to our assumptions, $M$ is closed in the topology of the pointwise convergence on $\mathbb{D}$.

To estimate the norms of $H$ and $H^{-1}$ it suffices to estimate the norms of the matrix $r^*(H) = \Psi \cdot \Omega'$ and its inverse. But then the required estimates follow from the fact that $\Psi$ is a left unimodular matrix and $\max \{ ||\Omega'||, ||(\Omega')^{-1}|| \} \leq C(n,N)$. (Here $|| \cdot ||$ is defined as in Section 1.3.) Therefore we have $\max \{ ||r^*(H)||, ||r^*(H^{-1})|| \} \leq n^{3/2}C(n,N)$. □

**Proof of Theorem [L7].** Let $A = (a_{ij})$ be a $n \times k$ matrix, $k < n$, with entries in $H^\infty(U)$. Assume that the family of determinants of submatrices $A$ of order $k$ satisfies the corona condition. According to Lemma 1 of [T], one can find a $k \times n$ matrix $G$ with entries in $H^\infty(U)$ such that

$$G \cdot A = I_k \quad (6.1)$$

The operator $G$ maps $v \in H^\infty_n(U)$ into $G \cdot v \in H^\infty_k(U)$. Let $\text{Ker}(G) \subset H^\infty_n(U)$ be its kernel. Clearly, $\text{Ker}(G)$ is a submodule of $H^\infty_n(U)$ closed in the topology of the pointwise convergence on $U$ (see the definition in Section 1.3). Then according to Theorem [L7] and (6.1), $\text{Ker}(G) = H \cdot H^\infty_{n-k}(U)$ for some left invertible $n \times (n-k)$ matrix $H$ with entries in $H^\infty(U)$. For the matrix $J := I_n - A \cdot G$, we have $G \cdot J = 0$. 

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Let us define $J_i$ as the $i$th column of the matrix $J$. Then $J_i \in H_n^\infty(U)$, $G \cdot J_i = 0$, $J_i \in \text{Ker}(G) = H \cdot H_{n-k}^\infty(U)$, and for some column $S_i \in H_{n-k}^\infty(U)$ we have $J_i = H \cdot S_i$. Consider the $(n-k) \times n$ matrix $S = (S_1, \ldots, S_n)$. Then $J = H \cdot S$ and $A \cdot G + H \cdot S = I_n$. So the $n \times n$ matrix $F := (A, H)$ is invertible and $F^{-1} = \left( \begin{smallmatrix} G \\ S \end{smallmatrix} \right)$. Since $k < n$, we can divide the last column of $F$ by $\det(F)$ to obtain the invertible matrix $\tilde{A}$ with $\det(\tilde{A}) = 1$ that extends $A$.

Now, let $\delta > 0$ be the number in the corona condition (1.3) for the minors of $A$ of order $k$ and $\|A\|$ be the norm of $A$. Then the effective estimate for solutions of the corona problem in $H^\infty(U)$ see [Br,Corol.1.5] and Lemma 1 of [T] imply that $\|G\| \leq c_1(n, N, \delta, \|A\|)$. From here and the estimate for $\|H\|, \|H^{-1}\|$ in Theorem 1.7, we obtain that $\|S\| \leq c_2(n, N, \delta, \|A\|)$. These estimates together imply that $\max\{\|\tilde{A}\|, \|(\tilde{A})^{-1}\|\} \leq c_3(n, N, \delta, \|A\|)$ (which does not depend of the choice of $U$).

The proof of the theorem is complete. \(\square\)

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