Nineteen vortex equations and integrability

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Abstract

The class of five integrable vortex equations discussed recently by Manton is extended so it includes the relativistic BPS Chern-Simons vortices, yielding a total of nineteen vortex equations. Not all the nineteen vortex equations are integrable, but four new integrable equations are discovered and we generalize them to infinitely many sets of four integrable vortex equations, with each set denoted by its integer order $n$. Their integrability is similar to the known cases, but give rise to different (generalized) Baptista geometries, where the Baptista metric is a conformal rescaling of the background metric by the Higgs field. In particular, the Baptista manifolds have conical singularities. Where the Jackiw-Pi, Taubes, Popov and Ambjørn-Olesen vortices have conical deficits of $2\pi$ at each vortex zero in their Baptista manifolds, the higher-order generalizations of these equations are also integrable with larger constant curvatures and a $2\pi n$ conical deficit at each vortex zero. We then generalize a superposition law, known for Taubes vortices of how to add vortices to a known solution, to all the integrable vortex equations. We find that although the Taubes and the Popov equations relate to themselves, the Ambjørn-Olesen and Jackiw-Pi vortices are added by using the Baptista metric and the Popov equation. Finally, we find many further relations between vortex equations, e.g. we find that the Chern-Simons vortices can be interpreted as Taubes vortices on the Baptista manifold of their own solution.

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1 Introduction

Abelian vortices play an important role as perhaps the simplest nontrivial solitons in gauge theories and have applications in a variety of fields, e.g. in superconductors or as cosmic strings. The relativistic generalization of Ginzburg-Landau theory describing the vortices in type II superconductors is called the Abelian Higgs model. It has a single parameter describing the physics of the model, namely the ratio of the Higgs mass to the photon mass. When the ratio is unity, the vortices are said to be critically coupled and their governing equations reduce from two coupled second-order partial differential equations (PDEs) to two first-order PDEs; the latter two can then be combined, yielding a single vortex equation of second order, which is known as the Taubes equation [1, 2, 3, 4].

The Taubes vortex equation, although simpler than the generic coupling case, is not integrable and hence no closed-form analytic expression for its solution is known. Modification of the base geometry, however, changes this fact and as first found by Witten, the Taubes equation becomes integrable when the base manifold is tuned to a particular constant negative curvature, turning the manifold into a hyperbolic plane [5]. Integrability requires that the curvature of the base manifold matches the constant term in the Taubes equation. Further integrable vortex equations are known, such as that describing the Jackiw-Pi vortices in nonrelativistic Chern-Simons theory [6] and that describing Popov vortices [7]. These three vortex equations and further two integrable vortex equations for the Ambjørn-Olesen [8] and Bradlow vortices, were then put on equal footing by Manton [9].

The five vortex equations studied by Manton can nicely be understood geometrically in terms of a quantity called the Baptista metric, which is the background metric after a conformal rescaling by the Higgs field squared, yielding $g_2 = g_0 |\phi|^2$. The integrability of the vortex equations are then understood as the condition that the background Gauss curvature is constant and equal to the constant term in the vortex equation and the curvature corresponding to the Baptista metric is constant and equal to the coefficient of the Higgs field squared. Up to rescaling of lengths and field redefinitions, the types of integrable vortex equations are thus simply given by the signs of these two coefficients – in this paper we shall call them vortex coefficients (VCs).

Although it turns out that all the possible vortex equations that one can write down satisfying the positive (magnetic) flux condition and being linear in the Higgs field squared (i.e. $|\phi|^2$) are integrable, one may wonder why stop at the linear order in $|\phi|^2$? In fact, a well known vortex equation (at critical coupling) that is not linear but quadratic in $|\phi|^2$ is the relativistic BPS Chern-Simons vortex equation [10]. It is, however, not integrable. Nevertheless, in this paper we will extend the vortex equations from the five vortex equations of Manton to the next order in $|\phi|^2$, yielding nineteen vortex equations and the title of the paper. If we also include the vanishing vortex polynomial, which we shall call the Laplace vortex equation, there will be twenty vortex equations.

In this paper, we first perform a classification of the vortex equations and count how many types of vortex equations are possible under the condition that the magnetic flux remains positive definite as a function of the highest power of the Higgs field squared, $L$. The nineteen or rather twenty vortex equations studied in this paper correspond to $L = 2$ and includes the Chern-Simons equation. One may expect that the equations become increasingly more difficult and less likely to be integrable, but we find that for each increment of $L$ by one, there are 4 new integrable vortex equations; in particular they can be thought of as the higher-order generalizations of the Jackiw-Pi, the Taubes, the Popov and the Ambjørn-Olesen
vortex equations. They can be understood geometrically by generalizing the Baptista metric to a higher-order Baptista metric as \( g_{2n} = g_0 |\phi|^{2n} \). The new integrable vortex equations correspond to cases where the higher-order Baptista manifolds have constant curvature. The higher order of the metric also impacts the geometry locally as the higher-order Baptista manifolds for integrable vortices are constantly curved spaces with conical singularities, where each single vortex gives rise to a conical excess of \( 2\pi n \), with \( n \) corresponding to the order of the Baptista manifold. We also integrate the newly found integrable vortex equations, but the corresponding Bradlow-type bounds turn out to be independent of the order of the Baptista manifold. We then consider superposition rules and the geometric interpretation of adding vortices to known vortex solutions for all of the integrable types of vortices, generalizing the known result of Baptista for Taubes vortices. Finally, we contemplate further relations among the vortex equations and find for example that the Chern-Simons vortex can be interpreted geometrically as a Taubes vortex on the Baptista background of itself. Using this relation we contemplate the geometry that the integrable solutions of the Taubes equation gives rise to for the Chern-Simons vortices, which turns out to be singular.

This paper is organized as follows. In Sec. 2 we set up the geometry, the vortex equations and the Baptista metric. In Sec. 3 we classify the vortex equations for \( L = 2 \), i.e. the simplest twenty vortex equations on Taubes form. In Sec. 4 we review the five integrable vortex equations of Ref. [9] and present four new ones. In Sec. 5 we integrate the vortex equations and find Bradlow-type bounds that for some vortex equations limit the number of vortices for a given volume, but put a lower bound on others. In Sec. 6 we prove that the vortex equations imply an analytic form of the solutions in a small open disc around a vortex zero and use this result to calculate the conical deficits of the Baptista manifolds at those vortex zeros. In Sec. 7 we generalize the vortex superposition law of Baptista for Taubes vortices to all the integrable vortex equations, including the other four equations of Ref. [9] as well as the newly found integrable vortex equations. A summary of our relations and results can be seen in Fig. 1. In Sec. 8, we find the singular geometry for which the integrable Taubes vortices on the hyperbolic plane give rise to Chern-Simons vortices. Finally, we conclude the paper with a discussion and outlook in Sec. 9.

2 Vortex equations on Riemann surfaces

2.1 The geometry

Consider a Riemann surface \((M_0, g_0)\), equipped with a metric \( g_0 \), having a constant Gauss curvature \( K_0 \) and we will use a local complex coordinate \( z \). The Riemann surface has the metric \((ds_0^2)\):

\[
g_0 = \Omega_0 dz d\bar{z},
\]

with \( \Omega_0 \) being the conformal factor

\[
\Omega_0 = \frac{4}{(1 + K_0 |z|^2)^2},
\]

admits a local complexified frame

\[
e = \frac{2}{1 + K_0 |z|^2} dz \in \Omega^{1,0}(M_0),
\]
which obeys the structure equation
\[ de - i\Gamma \wedge e = 0, \]  
(4)
and \( \Gamma \) is the spin connection 1-form
\[ \Gamma = iK_0 \frac{zd\bar{z} - \bar{z}dz}{1 + K_0|z|^2}d\bar{z} \in \Omega^1(M_0), \]  
(5)
where we denote generic \( r \)-forms on \( M_0 \) by \( \Omega^r(M_0) \) and holomorphic \( r \)-forms on \( M_0 \) by \( \Omega^r(M_0) \). The curvature 2-form is related to the spin connection via
\[ R = d\Gamma = i\frac{1}{2}K_0 e \wedge \bar{e} \in \Omega^{1,1}(M_0), \]  
(6)
with \( K_0 \) being the Gauss curvature, which can also be obtained directly from the conformal factor of the metric \( g_0 \) on \( M_0 \) as
\[ K_0 = \frac{1}{2} \Delta_{g_0} \log \Omega_0 = \frac{1}{2} (d\delta + \delta d) \log \Omega_0 = -\frac{2}{\Omega_0} \partial_z \bar{\partial}_z \log \Omega_0, \]  
(7)
where \( \delta = -* d* \) is the coderivative and \( * \) is the Hodge “star” operation, \( * : \Omega^{s,p}(M_0) \rightarrow \Omega^{1-s,1-p}(M_0) \). \( \Delta_{g_0} = d\delta + \delta d \) is the Hodge Laplacian with respect to the metric \( g_0 \). \( \partial_z = \frac{\partial}{\partial z} \) and \( \partial_{\bar{z}} = \frac{\partial}{\partial \bar{z}} \) are partial derivatives of the local coordinates.

2.2 Vortex equations

We will denote vortices by the pair \((\phi, A)\) of a connection 1-form \( A \) and a smooth section \( \phi \) on a line bundle over the Riemann surface \( M_0 \). The vortex equation will be taken to be
\[ dA \wedge e = (d\phi - iA\phi) \wedge e = 0, \]  
(8)
\[ F = dA = P(|\phi|^2)\omega_0, \]  
(9)
where \( \omega_0 = \frac{i}{2} \Omega_0 dz \wedge d\bar{z} \in \Omega^{1,1}(M_0) \) is the Kähler form on \( M_0 \) and \( P(|\phi|^2) \) is the vortex polynomial. Using the Maurer-Cartan solution of Eq. (8),
\[ A_{\bar{z}} = -i\partial_{\bar{z}} \log \phi, \]  
(10)
and the decomposition of the Higgs field \( \phi = e^{u+ix} \), we arrive at the vortex equation on “Taubes” form
\[ \Delta_{g_0} u = -\frac{4}{\Omega_0} \partial_z \partial_{\bar{z}} u = P(e^{2u}) - \frac{2\pi}{\Omega_0} \sum_{i=1}^{N} \delta^{(2)}(z - z_i), \]  
(11)
where \( z_i, i = 1, \ldots, N \) are zeros of the Higgs field \( \phi \). We will denote the set of not-necessarily distinct zeros by \( \{z_i\} \) and the distinct zeros \( z_k \in \{z_i\} \) with multiplicities \( N_k \). The sum over Dirac-delta functions will be abbreviated by
\[ \delta_D := 2\pi \sum_{i=1}^{N} \delta^{(2)}(z - z_i), \]  
(12)
where $D = \phi^{-1}(0) \equiv \{z_i\}, i = 1, \ldots, N$ is the effective divisor. For reference, we note that the field strength 2-form is given by
\[ F = -2i \partial \bar{z} u \, dz \wedge d\bar{z} \in \Omega^{1,1}(M_0), \] (13)
whereas the magnetic flux is
\[ B = *F = -\frac{4}{\Omega_0} \partial \bar{z} u = \Delta_g u. \] (14)

The integral of the magnetic flux is the first Chern class of the line bundle
\[ \frac{1}{2\pi} \int_{M_0} F = N. \] (15)
Restricting to $N > 0$ positive, without loss of generality, we also have that
\[ 0 < N = -\frac{2}{\pi} \int \partial \bar{z} u \, dx^1 \wedge dx^2, \] (16)
which implies that
\[ \int_{M_0} P(e^{2u}) \omega_0 > 0, \] (17)
which we henceforth shall call the positive flux condition.

### 2.3 The Baptista metric

A geometric interpretation of the vortex moduli space is given by considering the Baptista metric, which we will denote by $g_2$ and may define as [11, 12, 9]
\[ g_2 = \Omega_2 \, dz d\bar{z}, \quad \Omega_2 := \Omega_0 |\phi|^2 = \Omega_0 e^{2u}, \] (18)
which is, of course, degenerate at the zeros of the Higgs field $\{z_i\}$. Its corresponding curvature is given by
\[ K_2 = \frac{1}{2} \Delta_{g_2} \log \Omega_2 = -\frac{2}{\Omega_2} \partial \bar{z} \partial z \log \Omega_2 \]
\[ = \frac{\Omega_0}{\Omega_2} \left( K_0 + \Delta_g u \right) = \frac{\Omega_0}{\Omega_2} \left( K_0 - \frac{4}{\Omega_0} \partial \bar{z} u \right), \] (19)
which can be expressed in terms of the background curvature $K_0$ and the magnetic flux 2-form $F$; it is only defined on $M = M_0 \setminus \{z_i\}$. In particular, the metric possesses conical singularities at the points $\{z_i\}$ where the Higgs field vanishes [12, 9].

We will now define a generalized Baptista metric, which is given by
\[ \Omega_{2n} \equiv \Omega_0 e^{2nu}, \] (20)
where $n \in \mathbb{Z}_{\geq 0}$ is a positive semi-definite integer. For $n > 0$, it is degenerate at the points $\{z_i\}$. The trivial case, $n = 0$, is just the background metric of course and is regular. For
\( n = 1 \) it is the normal Baptista metric and \( n = 2 \) is the highest-order Baptista metric that we will consider in this paper. The corresponding Gauss curvatures then become

\[
K_{2n} = \frac{1}{2} \Delta_{g_{2n}} \log \Omega_{g_{2n}} = -\frac{2}{\Omega_{2n}} \partial_z \partial_{\bar{z}} \log \Omega_{2n}
\]

\[
= \frac{\Omega_0}{\Omega_{2n}} (K_0 + n \Delta_{g_0} u) = \frac{\Omega_0}{\Omega_{2n}} \left( K_0 - \frac{4n}{\Omega_0} \partial_z \partial_{\bar{z}} u \right),
\]

(21)

and are only defined on \( M = M_0 \{ z_i \} \), except for \( n = 0 \). Notice that there are nontrivial relations between the higher-order Baptista metrics, e.g.,

\[
\Omega_4 = \frac{\Omega_2^2}{\Omega_0}.
\]

(22)

An interesting relation can be found by combining the curvatures \( K_2 \) of Eq. (19) and \( K_4 \) of Eq. (21) with \( n = 2 \) in such a way as to eliminate the field strength 2-form, yielding

\[
\Omega_0 K_0 = 2\Omega_2 K_2 - \Omega_4 K_4.
\]

(23)

It illustrates what is clear from the construction, namely that the curvatures of the Baptista manifolds are related to each other via the background curvature.

### 2.4 The vortex polynomial

The vortex polynomial, \( P \), for the standard Taubes vortices is proportional to the square root of the Higgs potential. In this paper, we will not concern us with the underlying theory giving rise to the equations or whether they pose a well-defined variational problem and if they do, in what is the well-defined underlying theory that does.

The vortex polynomial may be written as

\[
P (e^{2u}) = -C_0 + \sum_{n=1}^L C_{2n} e^{2nu},
\]

(24)

with \( C_{2n} \in \mathbb{R} \) and \( e^{2u} \) being real coefficients and the square of the norm of the Higgs field.

\( L = 1 \) corresponds to the case covered in Ref. [9] giving rise to \( Q = 5 \) vortex equations. If the case of \( C_{2n} = 0 \ \forall n \) is included, there are \( Q = 6 \) vortex equations at \( L = 1 \). The vortex equation with a vanishing vortex polynomial, \( P = 0 \), was considered in Ref. [13] and the corresponding vortices may be called Laplace vortices.

In this paper, we will take \( L = 2 \) motivated by the Chern-Simons vortex equation at critical coupling, which corresponds to \( C_0 = 0, C_2 = 1 \) and \( C_4 = -1 \). We will see shortly that this case gives rise to \( Q = 19 \) vortex equations as well as the title of the paper or rather \( Q = 20 \) vortex equations, if we include the Laplace vortices. We can calculate the number of vortex equations \( Q \) as a function of the highest power, \( L \) of \( e^{2u} \), in the series (24) as follows. The total number of possibilities is classified by each coefficient being positive, negative or zero\(^1\). Counting the number of possibilities is rather easy. The relevant space is the sign function

\[
\text{sign} : \mathbb{R} \to \{-1, 0, +1\} \equiv \oplus,
\]

(25)

\(^1\)When three or more coefficients are nonvanishing, there will be families of equations; however, we will still classify them according to the signs of the coefficients that cannot be fixed in magnitude by rescaling.
where we have defined the latter symbol to be the set of $-1$, 0 and $+1$. For each term in
the polynomial, there are thus 3 possible values of the coefficient (in our classification of the
equations) and hence $3^{L+1}$ different equations (recall that the sum is zero based). Requiring
now that the first Chern class, $N$ of Eq. (15), is positive semi-definite, eliminates the equations
with the coefficients, $-C_0$, $C_2$, ..., $C_{2L}$, in the subset $\{-1, 0\}$, of which there are
\[
\sum_{n=0}^{L+1} \binom{L+1}{n} = 2^{L+1}.
\]  
We will however allow for all the coefficients to vanish, of which there is exactly one equation
and it is the Laplace vortex equation. Summing up, the final result is thus
\[
HF(L) = 3^{L+1} - 2^{L+1} + 1,
\]  
which is the Hilbert function in the counting of equations. We can now write down the Hilbert
series, which reads
\[
H(t) = \sum_{L=-1}^{\infty} HF(L) t^L = \sum_{L=-1}^{\infty} (3^{L+1} - 2^{L+1} + 1) t^L
= \frac{1}{t} \left( \frac{1}{1-3t} - \frac{1}{1-2t} + \frac{1}{1-t} \right).
\]  
Expanding the first few terms of the Hilbert series yields
\[
H(t) = \frac{1}{t} + 2 + 6t + 20t^2 + 66t^3 + 212t^4 + 666t^5 + O(t^6),
\]
from which we can read off how many vortex equations there are at each level, $L$. Having no
right hand side $L = -1$ (in this notation), gives only one equation $HF(-1) = 1$, namely the
Laplace-vortex equation; allowing for one coefficient ($L = 0$), there is additionally the possi-
bility of the Bradlow vortex equation [9] and hence $HF(0) = 2$; allowing for two coefficients
($L = 1$), there are now four more vortex equations: the Taubes or hyperbolic, the Popov, the
Jackiw-Pi and the Ambjørn-Olesen vortex equations [9] and hence $HF(1) = 6$. The $L = 2$
case is the next step, which contains the relativistic Chern-Simons vortex equation and is the
class of vortex equations that we will study in this paper. In this case there are $HF(2) = 20$
vortex equations as promised.
Only for the case $L = 1$, yielding six vortex equations, it is possible to normalize all
coefficients to $\pm 1$ if they are nonvanishing. In the general case, there are $L$ real coefficients
but only two ways of absorbing constants (see below for details). Therefore, we will in general
have at most
\[
W(L) \leq L - 1,
\]  
families of equations, meaning that there are $W(L)$ real parameters in the equation whose
nonvanishing magnitude can be varied (but not set to zero).
3 Classification of the vortex equations

In this paper, we will study the vortex equation in the class of equations that can be put in the following form

$$\Delta_{g_0} u + \frac{\delta D}{\Omega_0} = -C_0 + C_2 e^{2u} + C_4 e^{4u}. \quad (31)$$

The left-hand side is the magnetic flux and the right hand side is a polynomial in the Higgs field squared ($e^{2u}$). Since the magnetic flux is a real quantity and so is the vortex field $u : \mathbb{C} \to \mathbb{R}$, all three coefficients must be real valued $C_{0,2,4} \in \mathbb{R}$. This class of vortex equations is the next order, compared to the five equations considered in Ref. [9].

The first classification that we will make is to count the number of nonvanishing vortex coefficients (VC) and we will denote the type as type $m$, with $m$ being the number of nonvanishing VCs.

3.1 Type 0

No nonvanishing VCs reduces the vortex equation to the simplest possible equation:

$$-4 \partial_z \partial_{\bar{z}} u + \delta_D = 0, \quad (32)$$

which may be called the Laplace vortex equation. This equation is of course independent of $L$. The solutions depend on the choice of base geometry and the boundary conditions. Suitability of the boundary conditions depend on whether $M_0$ is compact or not.

A simple class of solutions can be constructed out of holomorphic functions with singularities at $\{z_i\}$:

$$u = \sum_{i=1}^{N} c_i \log |z - z_i|, \quad (33)$$

with $c_i > 0$ positive coefficients.

3.2 Type I

One nonvanishing VC means that there are $L + 1$ vortex equations in this class and for $L = 2$ this implies that there are 3 equations: $C_0 < 0$, $C_2 > 0$ and $C_4 > 0$, which correspond to the Bradlow equation [9]

$$\Delta_{g_0} u + \frac{\delta D}{\Omega_0} = 1, \quad (34)$$

the Jackiw-Pi equation [6]

$$\Delta_{g_0} u + \frac{\delta D}{\Omega_0} = e^{2u}, \quad (35)$$

and a new equation which we may call type $\Gamma_4^*$:

$$\Delta_{g_0} u + \frac{\delta D}{\Omega_0} = e^{4u}. \quad (36)$$
The reason for the signs of the coefficients is that the magnetic flux is positive \((N > 0)\) and \(u : \mathbb{C} \to \mathbb{R}\) is a real field and hence \(e^{2u}\) is non-negative. For the latter two equations, the coefficients \(C_2\) and \(C_4\) can be set to unity, respectively, by a rescaling in \(u \to u - \frac{1}{4} \log |C_2|\) and \(u \to u - \frac{1}{4} \log |C_4|\). For the Bradlow equation, the coefficient can be rescaled to unity by a coordinate transformation \(z \to |C_0|^{-\frac{1}{2}} z\) (the 2-dimensional delta function picks up the same factor as \(\partial_z \partial_{\bar{z}}\) under the local coordinate rescaling). Notice that the coordinate rescaling changes the conformal factor as follows
\[
\Omega_0 = \frac{4}{(1 + K_0|z|^2)^2} \to \frac{4}{(1 + K_0|C_0|^{-1}|z|^2)^2} = \frac{4}{(1 + K'_0|z|^2)^2},
\]
with \(K'_0\) the constant curvature of \(M_0\) in the new coordinates. In order not to clutter the notation, we will simply drop the primes of \(K_0\) upon rescaling the coordinates by \(|C_0|^{-\frac{1}{2}}\).

### 3.3 Type II

The type of vortex equation with two nonvanishing VCs can for \(L > 1\) be further classified into which of the coefficients are nonvanishing: type \(\Pi_{02}\), type \(\Pi_{04}\) and type \(\Pi_{24}\), which we will discuss in turn below. Using a scaling argument, the two nonvanishing coefficients can take the values \(\{-1, 1\}\) which gives a total of \(2^2\) vortex equations, and the positive flux condition eliminates the choice of both of them being \(-1\).\(^2\) Since there are \(\binom{L+1}{2}\) ways to set \(L - 1\) of the \(L + 1\) coefficients to zero, the total number of type II vortex equations is thus:
\[
\left(\binom{L + 1}{2}\right)(2^2 - 1) = \frac{3}{2} L(L + 1).
\]

#### 3.3.1 Type \(\Pi_{02}\)

This type of vortex equation was already covered in Ref. [9]
\[
\Delta_{g_0} u + \frac{\delta D}{\Omega_0} = -C_0 + C_2 e^{2u}.
\]

The scaling argument combining the shift in \(u\) and the scaling of the local coordinate \(z\), yields \(u \to u - \frac{1}{2} \log |C_2/C_0|\) and \(z \to |C_0|^{-\frac{1}{2}} z\), after which the Gauss curvature becomes \(K'_0 = K_0|C_0|^{-1}\) (but we will drop the prime as usual) and the only possible values for the coefficients \(C_0\) and \(C_2\) are: \((C_0, C_2) = (-1, -1)\) for Taubes vortices, \((C_0, C_2) = (1, 1)\) for Popov vortices and \((C_0, C_2) = (-1, 1)\) for Ambjørn-Olesen vortices.

#### 3.3.2 Type \(\Pi_{04}\)

The next type of vortex equation is new, but is conceptually similar to the ones in the previous subsubsection
\[
\Delta_{g_0} u + \frac{\delta D}{\Omega_0} = -C_0 + C_4 e^{4u}.
\]

We will see in the next section that it is integrable, in the same way as the Taubes, Popov and Ambjørn-Olesen vortices are, but with different geometric interpretation. The scaling

\(^2\)To put the positiveness of the VCs on the same footing, we will talk about the signs of \(-C_0, C_2,\) and \(C_4\).
argument is similar to the one above and is given by $u \rightarrow u - \frac{1}{4} \log |C_4/C_0|$ and $z \rightarrow |C_0|^{-\frac{1}{2}} z$. The possible values of the coefficients are thus $(C_0, C_4) = (-1, -1)$ for type $\Pi_{04}^-$ vortices, $(C_0, C_4) = (1, 1)$ for type $\Pi_{04}^+$ vortices and $(C_0, C_4) = (-1, 1)$ for type $\Pi_{04}^*$ vortices.

These vortex equations can be thought of as generalizations of the Taubes, the Popov and the Ambjørn-Olesen vortex equations with the Higgs field squared replaced by a quartic power of the Higgs field.

### 3.3.3 Type $\Pi_{24}$

This type of vortex equation is known from the relativistic BPS Chern-Simons theory [10] and takes the form

$$\Delta_{g_0} u + \frac{\delta_D}{\Omega_0} = C_2 e^{2u} + C_4 e^{4u}. \tag{41}$$

This time $C_0 = 0$ so the rescaling is done as $u \rightarrow u - \frac{1}{2} \log |C_2/C_4|$ and $z \rightarrow |C_4|^\frac{1}{2} |C_2|^{-1} z$. The possible coefficients, keeping in mind that the right-hand side of the vortex equation must be able to have a positive integral, are: $(C_2, C_4) = (1, 1)$ for the type $\Pi_{24}^+$ vortices, $(C_2, C_4) = (1, -1)$ for the relativistic BPS Chern-Simons vortices and $(C_2, C_4) = (-1, 1)$ for the type $\Pi_{24}^*$ vortices.

The type $\Pi_{24}^*$ and type $\Pi_{24}^+$ vortex equations can be thought of as being to the Ambjørn-Olesen and Popov vortex equations what the Chern-Simons vortex equations are to the Taubes equation.

### 3.4 Type III

Finally, we have the most complicated type of vortex equations for which none of the 3 VCs vanish

$$\Delta_{g_0} u + \frac{\delta_D}{\Omega_0} = -C_0 + C_2 e^{2u} + C_4 e^{4u}. \tag{42}$$

It is a choice which VCs to scale to unity; we choose $C_0$ and $C_2$ and hence we perform a rescaling by shifting $u \rightarrow u - \frac{1}{2} \log |C_2/C_0|$ and $z \rightarrow |C_0|^{-\frac{1}{2}} z$ obtaining

$$\Delta_{g_0} u + \frac{\delta_D}{\Omega_0} = -\text{sign}(C_0) + \text{sign}(C_2)e^{2u} + C_4' e^{4u}, \tag{43}$$

where $C_4' = \frac{C_4|C_0|}{|C_2|^2} \neq 0$ and after the rescaling we still have $\text{sign}(C_4') = \text{sign}(C_4)$. We can now drop the prime on $C_4$ and recall that $C_0$ and $C_2$ can be only $\pm 1$. There are now two possibilities: either $C_4 > 0$ or $C_4 < 0$, which we will consider in turn below.

The number of type III vortex equations can be calculated as follows. There are now 3 nonvanishing coefficients, which we classify according to their sign, giving $2^3$ vortex equations. The positive flux condition eliminates again one of these, since all coefficients cannot be negative. For general $L$, there are now $\binom{L+1}{3}$ ways of setting $L - 2$ of $L + 1$ coefficients to zero, yielding a total of

$$\binom{L+1}{3} (2^3 - 1) = \frac{7}{6} (L - 1)L(L + 1), \tag{44}$$

vortex equations of type III.
3.4.1 Type III \(_{024}^{+++}\)

There are 4 vortex equations that obey the positive flux condition for \(C_4 > 0\): the type III \(_{024}^{+++}\) vortex equation with \((C_0, C_2) = (-1, -1)\), the type III \(_{024}^{++-}\) vortex equation with \((C_0, C_2) = (-1, 1)\), the type III \(_{024}^{-+-}\) vortex equation with \((C_0, C_2) = (1, -1)\) and the type III \(_{024}^{+++}\) vortex equation with \((C_0, C_2) = (1, 1)\).

3.4.2 Type III \(_{024}^{++-}\)

There are only 3 vortex equations that obey the positive flux condition for \(C_4 < 0\): the type III \(_{024}^{---}\) vortex equation with \((C_0, C_2) = (-1, -1)\), the type III \(_{024}^{-+ -}\) vortex equation with \((C_0, C_2) = (-1, 1)\) and the type III \(_{024}^{++-}\) vortex equation with \((C_0, C_2) = (1, 1)\).

3.5 Summary of types

We will now summarize the types of vortex equations in the following table.

| type | name                  | \(C_0\) | \(C_2\) | \(C_4\) |
|------|-----------------------|---------|---------|---------|
| type 0 | “Laplace”            | 0       | 0       | 0       |
| type I\(_0^\) | Bradlow            | -1      | 0       | 0       |
| type I\(_2^+\) | Jackiw-Pi        | 0       | +1      | 0       |
| type I\(_4^\) | –                  | 0       | 0       | +1      |
| type II\(_{02}^--\) | Taubes         | -1      | -1      | 0       |
| type II\(_{02}^{+-}\) | Popov           | +1      | +1      | 0       |
| type II\(_{02}^{++}\) | Ambjørn-Olesen | -1      | +1      | 0       |
| type II\(_{04}^--\) | –                  | -1      | 0       | -1      |
| type II\(_{04}^{+-}\) | –                  | +1      | 0       | +1      |
| type II\(_{04}^{++}\) | –                  | -1      | 0       | +1      |
| type II\(_{24}^{--}\) | Chern-Simons     | 0       | +1      | -1      |
| type II\(_{24}^{+-}\) | –                  | 0       | -1      | +1      |
| type II\(_{24}^{++}\) | –                  | 0       | +1      | +1      |
| type III\(_{024}^{--}\) | –                  | -1      | -1      | +1      |
| type III\(_{024}^{+-}\) | –                  | -1      | +1      | +1      |
| type III\(_{024}^{++}\) | –                  | +1      | -1      | +1      |
| type III\(_{024}^{+++}\) | –                  | +1      | +1      | +1      |
| type III\(_{024}^{---}\) | –                  | -1      | -1      | -1      |
| type III\(_{024}^{-+-}\) | –                  | -1      | +1      | -1      |
| type III\(_{024}^{++-}\) | –                  | +1      | +1      | -1      |

Table 1: The classification of vortex equations for \(L = 2\), i.e. the class of twenty vortex equations described by Eq. (31).

For \(L = 2\), we only have four major types of vortex equations in the classification, namely type 0, type I, type II and type III; summing up the number of different vortex equation in each type, we get

\[
1 + (L + 1) + \frac{3}{2}L(L + 1) + \frac{7}{6}(L - 1)L(L + 1)\bigg|_{L=2} = 20,
\] (45)
as expected. As a consistency check, we can contemplate the situation with $L + 1$ being general and not equal to two, for which there would now be $L + 2$ major types (the first being type 0). The counting can easily be seen to be done by

$$1 + \sum_{i=1}^{L+1} \binom{L+1}{i} (2^i - 1) = 3^{L+1} - 2^{L-1} + 1,$$

as it should, see Eq. (27).

## 4 Integrability

We will now turn to the question of integrability of a subset of the vortex equations, reviewing the known ones and presenting the new integrable vortex equations.

### 4.1 Type II

We will review the integrability in this type of vortex equations following Ref. [9]. Using the scaling arguments described in the previous section, the coefficients $C_0 = \pm 1$ and $C_2 = \pm'1$ and the vortex equation is given in Eq. (39). The strategy of the integrability utilized in this type of equation is to remove the constant term $C_0$ by finding a suitable geometry for $M_0$ and then integrate the remaining vortex field à la Liouville. In particular, we note that

$$\Delta_{g_0} u = -\frac{4}{\Omega_0} \partial_z \partial_{\bar{z}} u = -\frac{4}{\Omega_0} \partial_z \partial_{\bar{z}} u' + \frac{2}{\Omega_0} \partial_z \partial_{\bar{z}} \log \Omega_0 = \Delta_{g_0} u' - K_0,$$

with

$$u = u' - \frac{1}{2} \log \Omega_0.$$

It is thus clear that a constant curvature manifold $M_0$ with Gauss curvature $K_0 = C_0$ effectively eliminates the constant term as well as the conformal factor from the vortex equation:

$$-4\partial_z \partial_{\bar{z}} u' + \delta_D = C_2 e^{2u'}.$$

Exponentiating Eq. (48) yields

$$e^{2u'} = \Omega_0 e^{2u} = \Omega_2,$$

so indeed the vortex equation in the form of Eq. (49) can be interpreted as a constant Baptista curvature (19):

$$-\frac{2}{\Omega_2} \partial_z \partial_{\bar{z}} \log \Omega_2 + \frac{\delta_D}{\Omega_2} = K_2 + \frac{\delta_D}{\Omega_2} = C_2,$$

with the exception of certain singularities at $\{z_i\}$ where the Baptista metric is degenerate.

An elegant way of writing the vortex equation of this type, is to notice the form of the Baptista curvature (away from vortex singularities $\{z_i\}$) of Eq. (19) can be rewritten as

$$\frac{\Omega_2}{\Omega_0} K_2 - K_0 = -K_0 + K_2 e^{2u} = -\frac{4}{\Omega_0} \partial_z \partial_{\bar{z}} u,$$
which is exactly the vortex equation (39) on \( M_0 \setminus \{ z_i \} \). Equating the above equation with the vortex equation, yields the Baptista equation

\[-K_0 + K_2 e^{2u} = -C_0 + C_2 e^{2u}, \tag{53}\]
or on Baptista form

\[\Omega_0 (K_0 - C_0) = \Omega_2 (K_2 - C_2), \tag{54}\]

which is only defined in \( M_0 \setminus \{ z_i \} \). Integrability corresponds to the “trivial” case where each side of the equation vanishes separately.

The constant curvature equation for \( K_0 = C_0 = \pm 1 \) is simply solved by

\[\Omega_0 = \frac{4}{(1 + C_0 |z|^2)^2}, \tag{55}\]

and likewise, so is the constant Baptista curvature condition or vortex equation (49):

\[\Omega_2 = \frac{4}{(1 + C_2 |z|^2)^2}, \tag{56}\]

but it bears no vortices. A simple remedy is to use the chain rule, giving

\[\Omega_2 = e^{2u'} = \frac{4}{(1 + C_2 |z|^2)^2} \left| \frac{df}{dz} \right|^2, \tag{57}\]

where the ramification points of \( f(z) \) (i.e. the points where \( f'(z) = 0 \)) correspond to vortex positions or vortex centers. The original Higgs field is related to the above solution via the solution to the constant curvature solution on \( M_0 \) as

\[|\phi|^2 = e^{2u} = \frac{\Omega_2}{\Omega_0} = \frac{(1 + C_0 |z|^2)^2}{(1 + C_2 |f(z)|^2)^2} \left| \frac{df}{dz} \right|^2. \tag{58}\]

This expression lends an easy way to calculate \( \phi \) in a certain gauge (holomorphic gauge), but the gauge can be chosen arbitrarily, of course.

### 4.2 Type II\(_{04}\)

The vortex equation is now given by Eq. (40) with \( C_0 = \pm 1 \) and \( C_4 = \pm 1' \) which are found by using the scaling arguments of Sec. 3; the integrability in this type of vortex equation is to the best of our knowledge new. They can be thought of as the higher-order generalizations of the Taubes, Ambjørn-Olesen and Popov vortex equations with \( e^{2u} \) replaced by \( e^{4u} \). The strategy for uncovering integrability is the same as in the type II\(_{02}\) vortices, but we have to modify the change of variables as

\[\Delta_{g_0} u = \Delta_{g_0} u' - \frac{K_0}{2}, \tag{59}\]

with

\[u = u' - \frac{1}{4} \log \Omega_0. \tag{60}\]
The change of the fraction in the above change of variables from \( u \) to \( u' \) is dictated by the exponentiation

\[
e^{4u'} = \Omega_0 e^{4u} = \Omega_4, \tag{61}
\]

where we want a linear dependence on \( \Omega_0 \) for not mocking up integrability. Interestingly, this changes the coefficient in Eq. (59) from unity to a half. Setting now \( K_0 = 2C_0 \) eliminates the constant term in the vortex equation and leaves us with

\[
-4\partial_z\partial\bar{z}u' + \delta_D = C_4 e^{4u'}. \tag{62}
\]

This equations can now be interpreted as a constant generalized Baptista curvature (21):

\[
-\frac{1}{\Omega_4} \partial_z\partial\bar{z} \log \Omega_4 + \frac{\delta_D}{\Omega_4} = \frac{K_4}{2} + \frac{\delta_D}{\Omega_4} = C_4, \tag{63}
\]

with the exception of the singularities at \( \{z_i\} \) where the generalized Baptista metric is degenerate. As in section 4.1, we recognize that the generalized Baptista curvature (21) (away from vortex singularities and for \( n = 2 \)) can be written as

\[
\frac{\Omega_4}{2\Omega_0} - \frac{K_0}{2} = -\frac{K_0}{2} + \frac{K_4}{2} e^{4u} = -\frac{4}{\Omega_0} \partial_z\partial\bar{z}u, \tag{64}
\]

which has the same form as the vortex equation (40) on \( M_0\{z_i\} \). Equating the above equation with the vortex equation yields the higher-order Baptista equation

\[
\frac{K_0}{2} + \frac{K_4}{2} e^{4u} = -C_0 + C_4 e^{4u}, \tag{65}
\]

which can be written on Baptista form as

\[
\Omega_0(K_0 - 2C_0) = \Omega_4(K_4 - 2C_4), \tag{66}
\]

which is again only defined on \( M_0\{z_i\} \). Integrability corresponds to the “trivial” solution of setting each side of the above equation to zero.

The constant curvature solution for \( K_0 = 2C_0 = \pm 2 \) thus reads

\[
\Omega_0 = \frac{4}{(1 + 2C_0|z|^2)^2}, \tag{67}
\]

whereas the constant generalized Baptista curvature is solved by

\[
\Omega_4 = e^{4u'} = \frac{4}{(1 + 2C_4|f(z)|^2)^2} \left| \frac{df}{dz} \right|^2, \tag{68}
\]

where ramification points of \( f(z) \) correspond to vortex positions. The original Higgs field is now related to the solution by

\[
|\phi|^2 = e^{2u} = \sqrt{\frac{\Omega_4}{\Omega_0}} = \frac{|1 + 2C_0|z|^2|}{1 + 2C_4|f(z)|^2} \left| \frac{df}{dz} \right|. \tag{69}
\]

This solution does not lend an easy choice of gauge that fixes the phase in terms of \( f(z) \).
4.3 Type I

4.3.1 Bradlow

Integrability of the Bradlow or type $I_0^-$ equation was discussed in Ref. [9] on hyperbolic space and further in Ref. [14] on nontrivial geometries with nonconstant curvature. As noticed in Ref. [9], it is possible to simply set $C_2 := 0$ in the solution (58) which yields a class of solutions on the hyperbolic plane.

4.3.2 Jackiw-Pi

Integrability of the Jackiw-Pi or type $I_2^+$ equation was discussed in Ref. [9]. On flat space ($K_0 = 0$) it is possible to simply set $C_0 := 0$ in the solution (58) which yields a class of solutions on $\mathbb{R}^2$ or $T^2$.

4.3.3 Type $I_4^+$

Integrability of the type $I_4^+$ equation is, to the best of our knowledge, new. The solution can be found by simply setting $C_0 := 0$ in Eq. (69) which yields a class of solutions on $\mathbb{R}^2$ or $T^2$. This equation can be thought of as the higher-order generalization of the Jackiw-Pi vortex equation with $e^{2u}$ replaced by $e^{4u}$.

4.4 Summary of integrable vortex equations

We now summarize the integrable vortex equations in Tab. 2.

| type    | name               | $C_0$ | $C_2$ | $C_4$ | $M_0$   | $M_2$   | $M_4$   |
|---------|--------------------|-------|-------|-------|---------|---------|---------|
| type 0  | “Laplace”          | 0     | 0     | 0     | $\mathbb{R}^2$ | $\mathbb{R}^2$ | $\mathbb{R}^2$ |
| type $I_0^-$ | Bradlow         | -1    | 0     | 0     | $\mathbb{H}^2$ | $\mathbb{R}^2$ | $\mathbb{R}^2$ |
| type $I_2^+$ | Jackiw-Pi        | 0     | +1    | 0     | $\mathbb{R}^2$ | $S^2$ | -       |
| type $I_4^+$ | –                 | 0     | 0     | +1    | $\mathbb{R}^2$ | -       | $S^2$ |
| type $\Pi_{02}^-$ | Taubes         | -1    | -1    | 0     | $\mathbb{H}^2$ | $\mathbb{H}^2$ | -       |
| type $\Pi_{02}^{++}$ | Popov       | +1    | +1    | 0     | $S^2$ | $S^2$ | -       |
| type $\Pi_{02}^{-+}$ | Ambjørn-Olesen | -1    | +1    | 0     | $\mathbb{H}^2$ | $S^2$ | -       |
| type $\Pi_{04}^-$ | –                 | -1    | 0     | -1    | $\mathbb{H}^2$ | -       | $\mathbb{H}^2$ |
| type $\Pi_{04}^{++}$ | –             | +1    | 0     | +1    | $S^2$ | -       | $S^2$ |
| type $\Pi_{04}^{-+}$ | –             | -1    | 0     | +1    | $\mathbb{H}^2$ | -       | $S^2$ |

Table 2: The integrable subset of vortex equations for $L = 2$, i.e. among the class of twenty vortex equations described by Eq. (31). The last three columns display the geometries of the (constant curvature) manifolds $M_0$, Baptista manifolds $M_2$ and higher-order Baptista manifolds $M_4$. The vanishing constant curvature manifolds are denoted by $\mathbb{R}^2$, but they can equally well be tori: $T^2$ etc. The – denotes manifolds that do not have a constant curvature and hence a more complicated geometry. The Baptista manifolds have conical singularities at the vortex zeros $\{z_i\}$. 

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5 Volumes

For Taubes vortices, the integral of the vortex equation leading to a bound between the number of vortices and surface area of $M_0$ was considered by Bradlow in Ref. [15]. Here we will make similar considerations for the class of vortex equations at hand. In particular, by integrating Eq. (9) on $M_0$, we get

$$\int_{M_0} F = 2\pi N = \int_{M_0} P(|\phi|^2) \omega_0 = \int_{M_0} P(|\phi|^2) \Omega_0 \, dx^1 \wedge dx^2, \tag{70}$$

where we have used the first Chern class of the line bundle (15) being the winding number $N$ or simply the vortex number. Inserting the vortex polynomial (31), we have

$$2\pi N = -C_0 \int_{M_0} \omega_0 + C_2 \int_{M_0} e^{2u} \omega_0 + C_4 \int_{M_0} e^{4u} \omega_0$$

$$= -C_0 \text{Vol}(M_0) + C_2 \text{Vol}(M_2) + C_4 \text{Vol}(M_4), \tag{71}$$

that is the winding number $N$ is related to the volumes of the manifolds $M_0$ (the Riemann surface), $M_2$ and $M_4$, where the latter two manifolds are $M_0$ equipped with the conformally transformed metrics $g_2$ and $g_4$, respectively.

Let us consider the case where $M_0$ is smooth and compact with genus $g_0$. Then by the Gauss-Bonnet theorem, we have

$$\int_{M_0} K_0 \omega_0 = \int_{M_0} K_0 \Omega_0 \, dx^1 \wedge dx^2 = 2\pi \chi_E(M_0) = 4\pi(1 - g_0), \tag{72}$$

where $\chi_E(M_0)$ is the Euler characteristic of $M_0$. Similarly, for the Baptista metric we have

$$\int_{M_0} K_2 e^{2u} \omega_0 = \int_{M_0} K_2 \Omega_2 \, dx^1 \wedge dx^2 = 4\pi(1 - g_0). \tag{73}$$

Now relating it to the Baptista area, $\text{Vol}(M_2)$, using Eq. (51) we have

$$C_2 \int_{M_0} e^{2u} \omega_0 = C_2 \text{Vol}(M_2) = \int_{M_0} K_2 e^{2u} \omega_0 + 2\pi N = 2\pi(2 - 2g_0 + N). \tag{74}$$

The additional term $2\pi N$ comes from the $N$ points where the Baptista metric is degenerate.

5.1 Integrable vortices of type $\Pi_{02}$, type $I_0$ and type $I_2$

With the above integrals in hand, we can now integrate the Baptista equation (54), valid for the integrable vortex equations of type $\Pi_{02}$, type $I_0$ and type $I_2$, obtaining

$$4\pi(1 - g_0) - C_0 \text{Vol}(M_0) = 2\pi(2 - 2g_0 + N) - C_2 \text{Vol}(M_2). \tag{75}$$

The Euler characteristic times $2\pi$ cancels out, leaving us with

$$C_2 \text{Vol}(M_2) = C_0 \text{Vol}(M_0) + 2\pi N, \tag{76}$$

and the values of $C_0$ and $C_2$ for the mentioned types of vortex equations are given in Tab. 1. This latter equation is in fact true also on noncompact $M_0$, which can be seen from the
integral of the vortex equation (71) with $C_4 := 0$. In that case the volumes would diverge and should be considered as areas of a disc $B \subset M_0$ in the limit of the radius being sent to infinity.

Since the areas are non-negative, the signs of $C_0$ and $C_2$ determine whether there is an upper or lower bound on the number of vortices $N$. In particular, for the Taubes equation ($C_0 = C_2 = -1$) we have

$$\text{Vol}(M_2) = \text{Vol}(M_0) - 2\pi N,$$

indicating that the Baptista volume is smaller than the volume (area) of $M_0$ [9]. Since $\text{Vol}(M_2) \geq 0$ Bradlow’s bound follows

$$N \leq \frac{\text{Vol}(M_0)}{2\pi}.$$  \hspace{1cm} (77)

For Popov vortices $C_0 = C_2 = 1$ means that the Baptista volume is bigger than the volume (area) of $M_0$:

$$\text{Vol}(M_2) = \text{Vol}(M_0) + 2\pi N.$$  \hspace{1cm} (79)

For the Ambjørn-Olesen vortices, $C_0 = -1$ and $C_2 = 1$, so

$$\text{Vol}(M_2) = -\text{Vol}(M_0) + 2\pi N,$$

yielding a lower bound on the vortex number

$$N \geq \frac{\text{Vol}(M_0)}{2\pi}.$$  \hspace{1cm} (81)

For the Jackiw-Pi vortices, $C_2 = 1$ and $C_0 = 0$, and hence the Baptista volume is the vortex number (times $2\pi$):

$$\text{Vol}(M_2) = 2\pi N.$$  \hspace{1cm} (82)

Similarly, for the Bradlow vortices, $C_0 = -1$ and $C_2 = 0$, so the vortex number is proportional to the volume (area) of $M_0$:

$$\text{Vol}(M_0) = 2\pi N.$$  \hspace{1cm} (83)

### 5.2 Integrable vortices of type $\Pi_{04}$ and type $I_4$

We will now turn to the higher-order integrable vortex equations, i.e. those that involve the higher-order Baptista metric $g_4$. Integrating now the higher-order Baptista equation (66), valid for the integrable vortex equations of type $\Pi_{04}$, and type $I_4$, we obtain

$$4\pi(1 - g_0) - 2C_0 \text{Vol}M_0 = 2\pi(2 - 2g_0 + 2N) - 2C_4 \text{Vol}(M_4),$$  \hspace{1cm} (84)

where the $4\pi N$ coming from the Gauss-Bonnet integral is due to the conical excess of $4\pi$ per vortex zero (see the next section). The Euler characteristic times $2\pi$ cancels out of the above equation, yielding

$$C_4 \text{Vol}(M_4) = C_0 \text{Vol}(M_0) + 2\pi N,$$  \hspace{1cm} (85)
so although the curvatures $K_0 = 2C_0$ and $K_4 = 2C_4$ have a factor of two in their relations to the VCs, this equation is formally identical to Eq. (76) and the values of $C_0$ and $C_4$ are given in Tab. 1. This equation is again true on noncompact $M_0$, which can be seen from the integral of the vortex equation (71) with $C_2 := 0$.

We now have analogous relations for the higher-order vortex equations, i.e. since the volumes (areas) are non-negative, we have for the type $\Pi_{04}^-$ vortices

$$\text{Vol}(M_4) = \text{Vol}(M_0) - 2\pi N,$$

yielding a Bradlow bound

$$N \leq \frac{\text{Vol}(M_0)}{2\pi}.$$  

For the type $\Pi_{04}^+$ vortices, the higher-order Baptista volume is instead bigger than the volume (area) of $M_0$:

$$\text{Vol}(M_4) = \text{Vol}(M_0) + 2\pi N.$$  

For the type $\Pi_{04}^\ast$ vortices, we have

$$\text{Vol}(M_4) = -\text{Vol}(M_0) + 2\pi N,$$

which like in the case of the Ambjørn-Olesen vortices yields a lower bound on the vortex number

$$N \geq \frac{\text{Vol}(M_0)}{2\pi}.$$  

Finally, for the type $I_4^\ast$ vortices, the higher-order Baptista volume is related directly to the winding number

$$\text{Vol}(M_4) = 2\pi N.$$  

6 Vortex cones and deficits

One may naively expect that the behavior of the field $u$ near a zero $z_k \in \{z_i\}$ for some $k$ depends on the vortex polynomial $P(e^{2u})$, but this is not so. In fact, it suffices to know that the selfdual equation (8) holds and that the vortex field satisfies any boundary condition

$$\lim_{z \to z_0} e^{2u} > 0, \quad \forall z_0 \in \partial M_0,$$  

then the following proposition holds [2]. Let $(A, \phi)$ be a smooth solution to the first order equation (8). Then the set of zeros $\{z_i\}$, $i = 1, \ldots, N$ is discrete and in some neighborhood of each $z_k \in \{z_i\}$,

$$\phi(z, \bar{z}) = (z - z_k)^{N_k} h_k(z, \bar{z}),$$

where $N_k \in \mathbb{Z}_+$ is the order of the zero and $h_k(z, \bar{z})$ is $C^\infty$ and nonvanishing on the neighborhood. The proof utilizes the $\bar{\partial}$-Poincaré lemma (for the proof see Ref. [16, page 5]), which states that $iA(z, \bar{z})$ is a $C^\infty$ function on a closed disc $B \subset \mathbb{R}^2$. Then the differential equation

$$\partial_z \beta(z, \bar{z}) = iA(z, \bar{z}),$$
has a $C^\infty$ solution $\beta$ in the interior of $B$:

$$\beta = \frac{1}{2\pi} \int_B \frac{A(\zeta, \bar{\zeta})}{\zeta - z} \, d\zeta \wedge d\bar{\zeta}. \quad (95)$$

Now using that $\eta(z) = e^{-\beta(z, \bar{z})} \phi(z, \bar{z})$ is a holomorphic function

$$\partial_{\bar{z}}(e^{-\beta} \phi) = e^{-\beta} (\partial_{\bar{z}} \phi - (\partial_{\bar{z}} \beta) \phi) = e^{-\beta} (\partial_{\bar{z}} - iA(z, \bar{z})) \phi = 0, \quad (96)$$

where $\phi = e^\beta \eta$ is a product of a $C^\infty$ nonvanishing function $e^\beta$ and a holomorphic function $\eta(z)$. Since the zeros of a complex holomorphic function are discrete, a finite number of zeros occur in any bounded set $B \subset \mathbb{R}^2$. We can thus write

$$\eta(z) = (z - z_k)^{N_k} \eta_k(z), \quad (97)$$

and in turn

$$h_k(z, \bar{z}) = \eta_k(z)e^{\beta(z, \bar{z})}. \quad (98)$$

Finally Eq. (93) follows.

Now that we have established that Eq. (93) holds and that $h_k$ is nonvanishing near the zero $z_k \in \{z_i\}$ for some $k$ and the multiplicity of the zero is $N_k$, we have [9]

$$e^{2u} = |\phi|^2 = |h_k|^2 |z - z_k|^{2N_k}, \quad (99)$$

and thus

$$\Omega_2 = \Omega_0 e^{2u} \simeq \Omega_0 (|z_k| |h_k(z_k, \bar{z}_k)|^2 |z - z_k|^{2N_k}), \quad (100)$$

in the vicinity of $z_k$. The Baptista metric can then in the near vicinity of $z_k$ be written in local coordinates $(z, \bar{z})$ as

$$g_2 = \Omega_2 dzd\bar{z} = \Omega_2 d(z - z_k)d(\bar{z} - \bar{z}_k)$$

$$\simeq \Omega_2 (|z_k| |h_k(z_k, \bar{z}_k)|^2)^{2N_k^2} \left(d\rho^2 + r^2d\theta^2\right)$$

$$\simeq \Omega_2 (|z_k| |h_k(z_k, \bar{z}_k)|^2)^2 \left(d\rho^2 + \rho^2d\chi^2\right), \quad (101)$$

where in the second equality we have shifted the coordinate system to place the origin at $z_k$, in the third equality we have changed to polar coordinates $z - z_k = re^{i\theta}$ and in the fourth equality we have changed variables to

$$\rho = \frac{1}{N_k + 1} r^{N_k + 1}, \quad \chi = (N_k + 1) \theta. \quad (102)$$

Since $\theta \in [0, 2\pi]$ the range of $\chi$ is given by $\chi \in [0, 2\pi(N_k + 1)]$. The geometry of the Baptista manifold is thus that of cones at the vortex zeros $\{z_i\}$ with a deficit angle of $2\pi N_k$ at each zero $z_k \in \{z_i\}$ with multiplicity $N_k$. The cones are not flat cones, except in the case of Bradlow vortices, as $C_2 = 0$ implies vanishing Baptista curvature $K_2 = 0$. On the other hand, they are cones with constant curvature $K_2 = C_2$, except at the vortex zeros $\{z_i\}$.

For higher-order Baptista manifolds, in particular for the integrable cases of type $\Pi_{04}$ and type $I_4$, the generalized Baptista curvature $K_4 = 2C_4$ is also constant, but the local geometry is different.
In particular, we have
\[
\Omega_{2n} = \Omega_0 e^{2nu} \simeq \Omega_0 (|z_k|) h_k (z_k, \bar{z}_k) |z - z_k|^{2nN_k},
\] (103)
in the vicinity of \( z_k \) and generalizing now to the \( n \)-th order Baptista metric, which near the vicinity of \( z_k \) can be written in local coordinates \((z, \bar{z})\) as
\[
g_{2n} = \Omega_0 e^{2nu} = \Omega_0 (|z_k|) h_k (z_k, \bar{z}_k) |z - z_k|^{2nN_k},
\] (104)
where now we have defined
\[
\rho = \frac{1}{nN_k + 1} r^{nN_k + 1}, \quad \chi = (nN_k + 1) \theta.
\] (105)

We can now readily see that the cone deficits of the higher-order Baptista metrics becomes higher by a factor of \( n \) and in particular for \( n = 2 \) (i.e. the case of constant \( K_4 \) Baptista curvature), the deficits are \( 4\pi N_k \) at each zero \( z_k \in \{z_i\} \) with multiplicity \( N_k \). For example for a single type II \( \Pi_{02} \) vortex, the geometry is described by a constant higher-order Baptista curvature manifold with cone deficits \( 4\pi \) at a single localized vortex zero.

7 Vortex superposition and relations among vortex equations

We will now consider some geometrically interesting consequences of the Baptista equation, in particular, how vortex solutions can be found iteratively, supposing that one knows how to find the solution to a given vortex equation. The main observation is that the Baptista equation (54) is symmetric in the metrics \( g_0 \) and \( g_2 \); this fact can be understood as the quantity on each side of the Baptista equation being invariant under a degenerate conformal transformation of the metric \( \Omega_0 \rightarrow e^{2u} \Omega_2 \) [12]. Due to this transitivity property, the invariance can be used to find new vortex solutions by “adding” vortices to known solutions.

7.1 Vortices of type \( \Pi_{02} \), type \( \Pi_0 \) and type \( \Pi_2 \)

We will make the derivation in a slightly different way with respect to the way it is presented in Ref. [12] (where only the Taubes vortices were discussed). Suppose we know a vortex solution \( u_1 \) which solves
\[
\Delta_{g_0} u_1 + \frac{\delta D_1}{\Omega_0} = -C_0 + C_2 e^{2u_1}.
\] (106)
Then consider the composite solution to the same equation on the same background \((M_0, g_0)\):
\[
\Delta_{g_0} (u_1 + u_2) + \frac{\delta D_1 + \delta D_2}{\Omega_0} = -C_0 + C_2 e^{2u_1 + 2u_2},
\] (107)
which we can rewrite using the fact that \( u_1 \) solves Eq. (106) as
\[
\Delta_{g_0} u_2 + \frac{\delta D_2}{\Omega_0} = -C_2 e^{2u_1} + C_2 e^{2u_1 + 2u_2},
\] (108)
where it is understood that $D_1$ is the effective divisor containing all the zeros of $u_1$ and $D_2$ is the same but for $u_2$. Denoting by $\Omega_2^{(1)} := e^{2u_1}\Omega_0$ and $g_2^{(1)} := \Omega_2^{(1)} d\bar{z}d\bar{z}$ the Baptista metric with the specific solution $u_1$, we can write the above equation as

$$\Delta_{g_2^{(1)}} u_2 + \frac{\delta D_2}{\Omega_2^{(1)}} = -C_2 + C_2 e^{2u_2}.$$  \hfill (109)

The observation made for the Taubes vortices in Ref. [12], is that the above equation is of the same form as that of $u_1$, namely Eq. (106), but the geometric interpretation of the vortex superposition is as follows: The solution for $u_1$ is found on the background metric $g_0$, from which the Baptista metric can be constructed, viz. $g_2^{(1)}$, and then the solution for $u_2$ can be found by solving exactly the same equation, however with the background metric $g_0$ replaced by the Baptista metric $g_2^{(1)}$ and $C_0$ replaced by $C_2$. In the case of Taubes vortices studied in Ref. [12], the signs of $C_0$ and $C_2$ are the same so the equation is exactly the same.

A twist to the story is exactly that $C_0$ is replaced by $C_2$ in Eq. (109): This has no consequence for the Taubes or the Popov vortices, but for the Ambjørn-Olesen vortices, this implies that the first solution is found for the Ambjørn-Olesen vortex equations, but an additional vortex added to the solution is found in the background of the previous Baptista metric, but using now the Popov vortex equation. The same holds for the Jackiw-Pi vortices, a vortex can be added by using the Baptista metric as the background metric and the Popov vortex equation. Bradlow vortices are added using the Laplace vortex equation. Since the latter equation is homogeneous, it is not important whether the background metric or the Baptista metric is used.

### 7.2 Vortices of type II$_{04}$ and type I$_4$

We will now consider the analogous calculation as in the previous subsection, but for the vortex equations involving a higher-order Baptista metric. Suppose we know a vortex solution $u_1$ which solves

$$\Delta_{g_0} u_1 + \frac{\delta D_1}{\Omega_0} = -C_0 + C_4 e^{4u_1}.$$  \hfill (110)

Then consider the composite solution to the same equation on the same background $(M_0, g_0)$:

$$\Delta_{g_0} (u_1 + u_2) + \frac{\delta D_1 + \delta D_2}{\Omega_0} = -C_0 + C_4 e^{4u_1+4u_2},$$  \hfill (111)

which we can rewrite using Eq. (110) as

$$\Delta_{g_0} u_2 + \frac{\delta D_2}{\Omega_0} = -C_4 e^{4u_1} + C_4 e^{2u_1+2u_2}.$$  \hfill (112)

where it is understood that $D_1$ is the effective divisor containing all the zeros of $u_1$ and $D_2$ is the same but for $u_2$. Denoting by $\Omega_4^{(1)} := e^{4u_1}\Omega_0$ and $g_4^{(1)} := \Omega_4^{(1)} d\bar{z}d\bar{z}$ the higher-order Baptista metric with the specific solution $u_1$, we can write the above equation as

$$\Delta_{g_4^{(1)}} u_2 + \frac{\delta D_2}{\Omega_4^{(1)}} = -C_4 + C_4 e^{4u_2}.$$  \hfill (113)

---

3In Ref. [12], a rescaling of the Baptista metric was performed so that the two vortex equations would have the same constants in magnitude. We do not perform this rescaling here, since we have already normalized the VCs and only the signs of $C_0$ and $C_2$ remain. Furthermore, we do not allow for the conformal transformation making the metric negative semi-definite.
The solution for \( u_1 \) is found on the background metric \( g_0 \), from which the higher-order Baptista metric can be constructed, viz. \( g_0^{(1)} \), and then the solution for \( u_2 \) can be found by solving the same type of vortex equation, however with the background metric \( g_0 \) replaced by the Baptista metric \( g_0^{(1)} \) and \( C_0 \) replaced by \( C_4 \).

For the type \( \Pi_{54}^- \) and type \( \Pi_{54}^+ \) vortices \( C_0 \) and \( C_4 \) have the same sign and are rescaled to unity, so the first vortex \( u_1 \) and the addition \( u_2 \) are solved by the same vortex equation. However, for the type \( \Pi_{54}^* \) vortex equation, the \( u_1 \) field is found by their corresponding equation whereas the addition of the \( u_2 \) vortices is governed instead by the type \( \Pi_{54}^* \) vortex equation and in the background of the higher-order Baptista metric \( g_0^{(1)} \).

### 7.3 Vortices of type \( \Pi_{024} \)

We will now consider the class of vortex equations of type \( \Pi_{024} \)

\[
\Delta_{g_0} u + \frac{\delta_D}{\Omega_0} = -C_0 + C_2 e^{2u} + C_4 e^{4u},
\]

with \( C_0 \neq 0, C_2 \neq 0 \) and \( C_4 \neq 0 \), which are not integrable. Let us first consider a superposition law for adding vortices to a known solution, say \( u_1 \). Then the composite solution \( u_1 + u_2 \) on the same type of vortex equation, however with the background metric \( g_0 \) replaced by \( g_0^{(1)} \) and \( C_0 \) replaced by \( C_4 \).

\[
\Delta_{g_0}(u_1 + u_2) + \frac{\delta_D}{\Omega_0} = -C_0 + C_2 e^{2u_1 + 2u_2} + C_4 e^{4u_1 + 4u_2},
\]

which using the equation for \( u_1 \) we can write as

\[
-4\partial_x \partial_{\bar{x}} u_2 + \delta_D = C_2 \Omega_2^{(1)} (e^{2u_2} - 1) + C_4 \Omega_4^{(1)} (e^{4u_2} - 1),
\]

where \( \Omega_n^{(1)} := e^{2nu_n} \Omega_0, \ n = 1, 2 \) are Baptista and higher-order Baptista metrics, respectively. We can see that for both \( C_2 \) and \( C_4 \) nonvanishing, the superposition law loses its nice geometric interpretation.

It is still possible to eliminate the constant factor, \(-C_0\), from the vortex equation by using a change of variables. As in the integrable cases, we can write

\[
u = u' - \frac{x}{2} \log \Omega_0, \quad x \in \mathbb{R},
\]

for which the vortex equation reduces to

\[
\Delta_{g_0} u' + \frac{\delta_D}{\Omega_0} = C_2 \frac{e^{2u'}}{\Omega_0^2 u',} + C_4 \frac{e^{4u'}}{\Omega_0^2},\]

if \( xK_0 = C_0 \). Two natural choices would be

\[
\begin{align*}
-4\partial_x \partial_{\bar{x}} u' + \delta_D &= \sqrt{\Omega_0} C_2 e^{2u'} + C_4 e^{4u'},
-4\partial_x \partial_{\bar{x}} u' + \delta_D &= C_2 e^{2u'} + C_4 \frac{e^{4u'}}{\Omega_0},
\end{align*}
\]

for \( x = \frac{1}{2} \) and \( x = 1 \), respectively. Hence, for the general case with \( C_0 \neq 0, C_2 \neq 0 \) and \( C_4 \neq 0 \), it is not possible to remove the constant term, \(-C_0\), by a change of variable without inducing a metric dependence into either of the coefficients \( C_2 \) or \( C_4 \) (or both of them). The vortex equation with \( K_0 = 2C_0 \) and \( K_0 = C_0 \) are thus similar to those with \( C_0 := 0 \), but not equivalent due to the metric or space dependence induced in the VCs, \( C_2 \) and \( C_4 \).
7.4 Relations between type \( \Pi_{24} \) and type \( \Pi_{02} \) vortices

Last but finally, we will consider a geometric reinterpretation of the vortex equations of type \( \Pi_{24} \) as being vortex equations of type \( \Pi_{02} \) on the metric being the Baptista metric of the latter equation. More precisely, for the vortex equations of type \( \Pi_{24} \):

\[
\Delta_{g_0} u + \frac{\delta_D}{\Omega_0} = C_2 e^{2u} + C_4 e^{4u},
\]

the observation is simply that using the Baptista metric \( \Omega_2 = e^{2u}\Omega_0 \), we can write the equation as

\[
\Delta_{g_2} u + \frac{\delta_D}{\Omega_2} = C_2 + C_4 e^{2u},
\]

which is exactly the vortex equation of type \( \Pi_{02} \) on the metric containing the vortex solution itself, but with \( C_2 \rightarrow C_4 \) and \( C_0 \rightarrow C_2 \). The relation is geometrically interesting.

7.5 Summary of relations between vortex equations

We will now summarize the relations amongst the vortex equations, including the ones used for the integrable cases, the diagram of relations is shown in Fig. 1. The figure shows a map of relations between vortex equations for \( L = 2 \) (i.e. 20 vortex equations). The rightmost column of equations are type I integrable vortex equations and are hence integrable as they are all the Liouville equation, with either sign, except for the Laplace vortex equation, which is type 0 and is a harmonic problem. The square brackets \([\,]\) denote the geometry of the Baptista manifold and the bar denotes the switch of the overall sign of the right-hand side of the given equation. The third column are type II integrable vortex equations, except the Bradlow equation which is type I integrable. The first kind of relation, denoted by the arrow \( \Longrightarrow \) is the equivalence between the vortex equations by choosing a specific constant curvature for the background metric on the left-hand equation, yielding the vortex equation on the right-hand side of the arrow on flat space, see the third and fourth columns. The round brackets \( () \) denote the geometry on which the vortex equations are integrable. The second kind of relation, denoted by the arrow \( \odot \mapsto \) are superposition laws of adding vortices to a known solution, see the third and fourth columns. The third kind of relation, denoted by the arrow \( \oslash \mapsto \) is the map or geometric interpretation that the vortex equation on the right-hand side has the background metric replaced by the Baptista metric of the solution to the equation, which is exactly the vortex equation on the left-hand side of the arrow, see the second and third columns. Finally, the last relation, denoted by the arrow \( \hookrightarrow \) is the relation that maps the vortex equation on the left-hand side to the vortex equation on the right-hand side of the arrow, albeit with at least one metric-dependent coefficient, by choosing a constant background curvature, see the first and second columns. The curly brackets \( \{} \) denote the background geometry that reduces the vortex equations via the latter relation.

8 Further integrability and singular geometries

We may contemplate using the relations of Sec. 7.4 between type \( \Pi_{24} \) and type \( \Pi_{02} \) vortices to find the geometries that could bear an integrable vortex solution. This is somewhat related
Figure 1: Map of relations between vortex equations for $L = 2$ (i.e. 20 vortex equations). From right, the columns are type I integrable, type II integrable and integrable on a singular space, with the exception of type $II^{++}_{24}$ which is not integrable and the Bradlow equation which is type I (and not type II) integrable. The different relations are equivalence by tuning the curvature of the background metric (denoted by $\equiv$), superposition law (denoted by $\oplus \mapsto \oplus$), the geometric interpretation of the vortex living on a manifold described by its own Baptista metric (denoted by $\iff$) and finally, a change of metric that relates vortex equations – but with metric dependent coefficients (denoted by $\mapsto$). For more details, see the text.
to the Kazdan-Warner problem [17]. The vortex equation (122) is integrable if we tune the Baptista metric to have a constant curvature \(-C_2\), that is:

\[\Omega^2 = \frac{4}{(1 - C_2|z|^2)^2},\]  

(123)

for which the vortex equation reduces to

\[-4\partial_z\partial_{\bar{z}} u' + \delta_D = C_4 e^{2u'}, \quad u = u' - \frac{1}{2} \log \Omega^2.\]  

(124)

The solution in terms of \(u'\) is then simply

\[e^{2u'} = \frac{4}{(1 + C_4|f(z)|^2)^2} \left| \frac{df}{dz} \right|^2,\]  

(125)

and the original field \(u\) is thus

\[e^{2u} = \frac{(1 - C_2|z|^2)^2}{(1 + C_4|f(z)|^2)^2} \left| \frac{df}{dz} \right|^2.\]  

(126)

Since \(\Omega^2 = \Omega_0 e^{2u}\), the background geometry \(\Omega_0\) does not have a degenerate, but a singular metric

\[\Omega_0 = 4 \frac{(1 + C_4|f(z)|^2)^2}{(1 - C_2|z|^2)^4} \left| \frac{df}{dz} \right|^{-2},\]  

(127)

with singularities at the ramification points, \(f'(z) = 0\), i.e. the vortex positions. This class of singular geometries “solve” the Chern-Simons equation for \(C_2 = 1\), \(C_4 = -1\) as well as two further vortex equations, see Tab. 1. It might be possible to find some extensions of these metrics that are better behaved.

9 Discussion and outlook

In this paper, we have considered vortex equations of the type that naturally is derived from the Abelian Higgs model at critical coupling, but generalized to having an arbitrary “potential” or rather in our language an arbitrary vortex polynomial. Using the positive flux condition, we have classified the possible choices of vortex coefficients and found that for a vortex polynomial that is at most quadratic in the Higgs field squared (\(|\phi|^2\)), corresponding to \(L = 2\), there are nineteen vortex equations other than the (empty) Laplace vortex equation. Although the classification of vortex equations becomes more complicated with a drastically increasing number of vortex equations as \(L\) is increased, the number new of integrable vortex equations is always four as \(L\) is incremented by one: one new integrable equation of type \(I^2\), and three new of type \(I^2_0(2L)\). We then establish the lower or upper bounds on the vortex number as a function of the volume (area) of base manifold \((M_0, g_0)\), depending on the type of vortex equation. There are also vortex equations for which there are no such bound, for instance the Popov vortices. Then we consider the geometry of the Baptista and higher-order Baptista metrics and find that the Baptista manifolds always have conical singularities at the vortex zeros, but with the conical deficit increasing with the order of the Baptista metric. We then consider the superposition rules and geometric interpretation of finding new
vortex solutions in the background of a known solution for all the integrable types of vortices, generalizing the known result of Baptista for Taubes vortices. Finally, we contemplate further relations among the new vortex equations, including an interpretation of type $\Pi_{24}$ vortices as type $\Pi_{02}$ vortices in the Baptista background of their own solutions. For example, a Chern-Simons vortex can be interpreted geometrically as a Taubes vortex on the Baptista background of itself. We have used this relation to find the singular geometry that would be the solution to the Chern-Simons vortex.

This work leads to many questions for future work. The first of which is what are the theories that these new vortex equations could be derived from. For the $L = 1$ case, i.e. the five vortex equations of Ref. [9], the theories behind the equations were found in Ref. [13]. A possible future direction of research would be to extend such consideration to the new vortex equations (i.e. $L = 2$) studied in this paper. The theory behind the Chern-Simons equation is known [10] and in fact it involves an extra field, that is, the electric potential, in order to effectively get (at critical coupling of that theory) the two coupled vortex equations included in our class of equations. Hence, the theories giving rise to the vortex equations contemplated here may have to include further fields than simply a magnetic (gauge) potential and a complex Higgs field for some of the equations.

It would be interesting to study the singular geometries that are “solutions” to the Chern-Simons vortices and the other two equations of type $\Pi_{24}$, in particular whether the metric can be regularized or made better behaved in some way.

It is clear that the four new integrable vortices of type $\Pi_+^{2n}$, type $\Pi_0^{2n}$, type $\Pi_0^{*2n}$, and type $\Pi_0^{*+2n}$, for which the vortex equation becomes

$$\Delta g_{\nu \mu} + \frac{\delta D}{\Theta_0} = -C_0 + C_{2n} e^{2n \nu},$$

the Baptista equation reads

$$\Theta_0(K_0 - nC_0) = \Theta_2n(K_{2n} - nC_{2n}),$$

the Higgs field squared is

$$|\phi|^2 = \left| \frac{\Theta_2n}{\Theta_0} \right|^\frac{1}{n} = \frac{|1 + nC_0|^2 \frac{2}{n} |df|^2 \frac{2}{n} |dz|^2}{|1 + nC_{2n} |f(z)|^2 |z|^2} \right| (130)$$

the conical deficits become $2\pi nN_k$ (for multiplicity $N_k$ of a zero $z_k$) and the Bradlow bounds would be unchanged. This establishes an infinite family of integrable vortex equations for $n \in \mathbb{Z} > 0$ with coefficients $C_0 = 0$, $C_{2n} = 1$ for type $\Pi_+^{2n}$; $C_0 = -1$, $C_{2n} = -1$ for type $\Pi_0^{-2n}$; $C_0 = 1$, $C_{2n} = 1$ for type $\Pi_0^{**2n}$ and $C_0 = -1$, $C_{2n} = 1$ for type $\Pi_0^{*+2n}$. For $n = 1$ these equations are four of the five vortex equations of Ref. [9] and for $n = 2$, they are the new integrable vortex equations studied in this paper. The question remains whether there are further integrable vortex equations left in the Taubes-like class of vortex equations.

Abelian vortex equations on constant curvature Riemann surfaces, in particular the ones studied in Ref. [9], have been reinterpreted as flat non-Abelian Cartan connections [18] by lifting the vortex equations to three dimensional group manifolds. It would be interesting to see whether the new integrable vortex equations, found in this paper, have such interpretation and how their differences manifest themselves in the Cartan connections. For instance, how
the difference in the conical deficits are realized in the Cartan connections after lifting the equations.

Impurities and in particular magnetic impurities have been studied in vortex systems and a nice interpretation of the impurity is given by a set of coupled vortex equations, in the limit where one of the vortex fields becomes much heavier than the other; the “frozen” vortex may thus be interpreted in the remaining vortex equation as a magnetic impurity [19]. The magnetic type of impurity has been generalized to all Manton’s five integrable vortex systems in Ref. [20]. It would be interesting to study magnetic impurities in the newly found vortex equations.

By allowing the metric to depend on the Higgs field, it is known that other equations than the Liouville equation can be related to the Taubes equation for Abelian Higgs vortices [21]. It would be interesting to see what relations would be possible from the vortex equations considered in this paper.

Non-Abelian generalizations on the sphere for the known integrable vortices have been considered in Ref. [22]. It would be interesting to consider such generalizations for the new vortex equations found in this paper.

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