ASYMPTOTIC HECKE ALGEBRAS AND LUSZTIG-VOGAN BIJECTION VIA AFFINE MATRIX-BALL CONSTRUCTION

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Abstract. Affine matrix-ball construction (abbreviated AMBC) was developed by Chmutov, Lewis, Pylyavskyy, and Yudovina as an affine generalization of Robinson-Schensted correspondence. We show that AMBC gives a simple way to compute a distinguished (or Duflo) involution in each Kazhdan-Lusztig cell of affine symmetric groups. We then use AMBC to give the first known canonical presentation for the asymptotic Hecke algebras of extended affine symmetric groups. As an application, we show that AMBC gives a conceptual way to compute Lusztig-Vogan bijection. For the latter we build upon prior works of Achar and Rush.

1. Introduction

The Robinson-Schensted algorithm provides a bijection between the symmetric group and the set of pairs of standard Young tableaux of the same shape. It has become ubiquitous since its invention and now it appears in the theory of crystals, symmetric functions, representation of symmetric groups, Kazhdan-Lusztig cells, etc. Furthermore, it is generalized in many ways, i.e. Robinson-Schensted-Knuth correspondence [Knu70], domino insertion algorithm for hyperoctahedral groups [Gar90, Gar92, Gar93], exotic Robinson-Schensted algorithm [HT12], to name a few.

Recently, Chmutov, Lewis, Pylyavskyy, and Yudovina [CPY18, CLP17] introduced its affine generalization, called the affine matrix-ball construction (abbreviated AMBC), which extends the work of Shi [Shi91]. It provides a bijection $\Phi : w \mapsto (P(w), Q(w), \tilde{\rho}(w))$ from the extended affine symmetric group to the set of triples $(P, Q, \tilde{\rho})$ where $P$ and $Q$ are row-standard Young tableaux of the same shape and $\tilde{\rho}$ is an integer vector satisfying certain properties. This algorithm also enjoys a lot of properties which the usual Robinson-Schensted algorithm possesses.

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On the other hand, an (extended) affine symmetric group (or more generally an affine Weyl group) plays a central role in representation theory. It is strongly connected to the Springer theory (and its affine analogue), representation of Lie groups and Lie algebras, geometry of flag varieties, etc. (See e.g. [Hum91] [CG97] [LP07] [Yum16].) In order to understand this group and its representation theory, it is desirable to find not only its characters but the parametrization and realizations of its (irreducible) representations.

To this end, Lusztig [Lus87a] introduced the notion of asymptotic Hecke algebras, conventionally denoted \( J \) (see [Lus14] Chapter 18 for more details). It is a free abelian group with basis parametrized by elements of the corresponding affine Weyl group, say \( t_w \), and the multiplication is given by
\[
\sum_{w \in W} \gamma_{u,v,w}t_w 
\]
where the structure constants \( \gamma_{u,v,w} \) come from those of the affine Hecke algebra with respect to the canonical basis. Usually, it has a much simpler structure than the original affine Hecke algebra. For example, if we start with the extended affine symmetric group then the structure constants of such \( J \) can be obtained from Littlewood-Richardson coefficients of some (smaller) symmetric groups. Also, we will later observe that in this case \( J \) is a direct sum of certain matrix algebras. Nonetheless, it contains a lot of information of the representation theory of corresponding affine Hecke algebra. Thus, in order to study affine Hecke algebras it is often very useful to first understand the structure of the corresponding asymptotic version \( J \).

This paper develops such understanding of the structure of the asymptotic Hecke algebra \( J \) for \( GL_n \) in terms of AMBC. Here we briefly explain our results. For each integer partition \( \lambda \vdash n \), we define \( F_\lambda \) to be \( GL_{m_1} \times GL_{m_2} \times \cdots \) where \( m_i \) is the number of \( i \)'s appearing in the parts of \( \lambda \). Also, let \( \xi_\lambda \) be the two-sided Kazhdan-Lusztig cell in the extended affine symmetric group parametrized by \( \lambda \) in the sense of [Lus89]. Xi [Xi02] proved that for a certain left cell \( \Gamma^{an} \subset \xi_\lambda \), there exists a ring isomorphism between the representation ring \( R(F_\lambda) \) and \( J(\Gamma^{an}) \cap \Gamma^{an} \) where the latter is defined to be the restriction of \( J \) to the intersection of \( \Gamma^{an} \) and its inverse. Our first main result is the following theorem.

**Theorem (Theorem 4.2).** The ring isomorphism of Xi is equivalent to the map \( w \mapsto \overline{\rho}(w) \) where \( \overline{\rho}(w) \) is given by AMBC \( \Phi : w \mapsto (P(w), Q(w), \overline{\rho}(w)) \).

Originally, in his paper Xi used his bijection above to prove the conjecture of Lusztig [Lus89]. It states that \( J_{\xi_\lambda} \), or the asymptotic Hecke algebra restricted to \( \xi_\lambda \), is isomorphic to a matrix algebra with base ring \( R(F_\lambda) \). He gave a description of such an isomorphism, but his construction was not canonical; it depends on the choice of identifications of left cells using star operations and multiplication by a shift element. In this paper, we describe such an isomorphism that naturally comes from AMBC, which is therefore more canonical. More precisely, we have the following. Here, \( D \) is the set of elements in the extended affine symmetric group which correspond to primitive idempotents in \( J \), called the distinguished (or Duflo) involutions.

**Theorem (Theorem 6.1 and 7.2).** The AMBC map \( \Phi : w \mapsto (P(w), Q(w), \overline{\rho}(w)) \) gives an isomorphism between \( J_{\xi_\lambda} \) and a matrix algebra over \( R(F_\lambda) \). In particular, if \( w \) and \( w' \) are elements in \( \xi_\lambda \), then the multiplication of basis elements corresponding to \( w \) and \( w' \) in \( J_{\xi_\lambda} \) is nonzero if and only if \( Q(w) = P(w') \). Moreover, \( w \in D \) if and only if \( P(w) = Q(w) \) and \( \overline{\rho}(w) = 0 \).

Meanwhile, the conjecture of Lusztig on the structure of the asymptotic Hecke algebra \( J \) provides a certain bijection between the set of irreducible representations of \( GL_n \), denoted \( \text{Dom}(GL_n) \), and the union of \( \text{Dom}(F_\lambda) \) where \( \lambda \) runs over all the partitions of \( n \). This is now called the Lusztig-Vogan bijection, and is studied by Achar, Bezrukavnikov, Ostrik, and Rush. As AMBC also gives
a solution of Lusztig’s conjecture, it induces a certain kind of such bijections. Our next result is that these bijections are all equal; we have

**Theorem** (Theorem 8.1). The Lusztig-Vogan bijection induced from AMBC is equal to that of Achar, Bezrukavnikov, Ostrik, and Rush.

This paper is organized as follows; in Section 2 we recall basic notations and definitions used in this paper; in Section 3 we introduce the notion of canonical and anti-canonical left cells; in Section 4 we prove the theorem stating that the bijection of Xi is compatible with the result of AMBC; in Section 5 we study how multiplication by a shift element and a star operation behave under AMBC; in Section 6 we describe the image of distinguished involutions under AMBC; in Section 7 we provided an isomorphism between the asymptotic Hecke algebra attached to a two-sided cell and a certain matrix algebra using the AMBC map; in Section 8 we describe the Lusztig-Vogan bijection, explain some of its properties, and state that the bijection which is derived from AMBC is equal to the one previously studied by Achar, Bezrukavnikov, Ostrik, and Rush, whose proof is given in Section 9. In the appendix, we briefly discuss how our results can be applied to other affine Weyl groups of type $A$, especially those of $SL_n$ and $PGL_n$.

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2. Notations and definitions

Let us recall some notation and terminology from [CPY18, CLP17]. Denote the extended affine symmetric group

$$\tilde{S}_n := \{ w : \mathbb{Z} \to \mathbb{Z} \mid w \text{ is bijective, } w(i + n) = w(i) + n \}.$$ 

We shall use the window notation for elements of $\tilde{S}_n$: $w = [w(1), w(2), \ldots, w(n)]$. An important distinguished element $\omega \in \tilde{S}_n$ is the shift permutation $\omega = [2, 3, \ldots, n, n+1]$. We shall often identify $w \in \tilde{S}_n$ with the set $B_w \subset \mathbb{Z} \times \mathbb{Z}$ of all pairs $(i, w(i))$, $i \in \mathbb{Z}$. We refer to elements of $\mathbb{Z} \times \mathbb{Z}$ that are of the form $(i, w(i))$ as balls.

A partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a sequence of positive integers satisfying $\lambda_1 \geq \lambda_2 \geq \cdots$. We denote $\lambda'$ the transpose of $\lambda$, often also referred to as conjugate partition. For a partition $\lambda$ of size $n = \sum_i \lambda_i$, denoted also $\lambda \vdash n$, one can consider tabloids $T = (T_1, T_2, \ldots)$ of shape $\lambda$. Those are Young diagrams of shape $\lambda$ filled with residues from the set $[n] = \{1, 2, \ldots, n\}$, so that $T_i$ is the set of $\lambda_i$ residues in the $i$-th row of $T$, and each residue appears exactly once. The order of elements in each $T_i$, does not matter, it is often convenient however to consider elements of $T_i$ to be ordered in increasing order according to $1 < \cdots < n$. In this case we say that $T$ is row-standard.

For $n \geq 1$, we define $\text{Dom}(GL_n)$ to be the set of integer vectors

$$\text{Dom}(GL_n) := \{(a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n \mid a_1 \geq a_2 \geq \cdots \geq a_n\}.$$ 

We naturally identify this with the set of dominant weights of $GL_n$. Also let $\text{Rep}(GL_n)$ be the category of finite dimensional rational representations of $GL_n$. Then for any $\vec{\mu} \in \text{Dom}(GL_n)$, there is an irreducible representation $V(\vec{\mu}) \in \text{Rep}(GL_n)$ of highest weight $\vec{\mu}$, and

$$\{ V(\vec{\mu}) \in \text{Rep}(GL_n) \mid \vec{\mu} \in \text{Dom}(GL_n) \}$$
is a complete collection of irreducible objects in $\Rep(GL_n)$ up to isomorphism. If we further define $\mathcal{R}(GL_n)$ to be the Grothendieck ring of $\Rep(GL_n)$, then the classes of $V(\mu)$ form a $\mathbb{Z}$-basis of $\mathcal{R}(GL_n)$.

A partition $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash n$ can also be denoted $\lambda = (m_1, 2m_2, \ldots)$, where $m_i$ is the number of parts of size $i$ among the $\lambda_i$'s. We define

$$F_{\lambda} := GL_{m_1} \times GL_{m_2} \times \cdots \quad \text{and} \quad L_{\lambda} := GL_{\lambda_1} \times GL_{\lambda_2} \times \cdots.$$ 

Then $\Dom(F_{\lambda}) = \Dom(GL_{m_1}) \times \Dom(GL_{m_2}) \times \cdots$ is naturally a subset of $\mathbb{Z}^{(\lambda)}$.

For $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\tilde{\nu} = (\rho_1, \ldots, \rho_t)$, we define $\rev_{\lambda}(\tilde{\nu})$ (or $\rev(\tilde{\nu})$ if there is no confusion) to be the integer vector obtained from $\tilde{\nu}$ by reversing the order of elements corresponding to the parts of the same length in $\lambda$.

**Example 2.1.** For $\lambda = (2, 2, 1, 1)$ and $\tilde{\nu} = (3, 1, 5, 2, 4)$ we have $\rev_{\lambda}(\tilde{\nu}) = (1, 3, 4, 2, 5)$.

For a partition $\lambda \vdash n$, define $\RSYT(\lambda)$ to be the set of row-standard Young tabloid of shape $\lambda$. For any $P, Q \in \RSYT(\lambda)$, let $\overline{s}_{P,Q}$ be the symmetrized offset constraints with respect to $(P, Q)$ defined in [CPY18]. We shall not repeat the definition here, instead we refer the reader to either [CPY18] Definition 5.8 or [CLP17] Theorem 5.10 for two ways to compute it.

**Example 2.2.** Let $P = \begin{bmatrix} 2 & 1 & 7 \\ 3 & 7 & 8 \\ 1 & 3 & 9 \end{bmatrix}$ and $Q = \begin{bmatrix} 5 & 7 & 7 \\ 1 & 7 & 8 \\ 4 & 6 & 9 \end{bmatrix}$. Then $\overline{s}_{P,Q} = (0, -2, -1)$.

Then we say that $\tilde{\nu} \in \mathbb{Z}^{(\lambda)}$ is dominant with respect to $(P, Q)$ if $-(\tilde{\nu} - \overline{s}_{P,Q}) \in \Dom(F_{\lambda})$, or equivalently $\rev_{\lambda}(\tilde{\nu} - \overline{s}_{P,Q}) \in \Dom(F_{\lambda})$. In other words, $\tilde{\nu} \in \mathbb{Z}^{(\lambda)}$ is dominant with respect to $(P, Q)$ if and only if $\tilde{\nu} - \overline{s}_{P,Q}$ is “anti-dominant”.

**Example 2.3.** In the Example 2.2 vector $\tilde{\nu} = (2, 0, 2) \in \mathbb{Z}^3$ is dominant because $\tilde{\nu} - \overline{s}_{P,Q} = (2, 2, 3)$ and $2 \leq 2 \leq 3$.

We set

$$\Omega := \bigsqcup_{\lambda \vdash n} \RSYT(\lambda) \times \RSYT(\lambda) \times \mathbb{Z}^{(\lambda)},$$

$$\Omega_{\text{dom}} := \{(P, Q, \tilde{\nu}) \in \Omega \mid \tilde{\nu} \text{ is dominant with respect to } (P, Q)\}.$$ 

We define $\Phi: \tilde{S}_n \to \Omega_{\text{dom}}: w \mapsto (P(w), Q(w), \tilde{\nu}(w))$ to be the bijection defined in [CPY18] using the affine matrix-ball construction (abbreviated AMBC). Also, let $\Psi: \Omega \to \tilde{S}_n$ be the surjection defined by the backward AMBC. Then by [CPY18] Theorem 5.12 we have $\Psi|_{\Omega_{\text{dom}}} = \Phi^{-1}$. Both AMBC and backward AMBC were explained in great detail in both [CPY18] and [CLP17]. We shall not repeat the construction here for that reason. We do include the following example the reader can use to test her understanding.

**Example 2.4.** For $P$ and $Q$ as in Example 2.2 and for $\tilde{\nu} = (2, 0, 2)$ as in Example 2.3 we have $\Psi(P, Q, \rho) = [3, 7, 14, 2, 18, 4, 19, 8, 6]$.

Let $\mathcal{J}$ be the asymptotic Hecke algebra of $\tilde{S}_n$ defined in [Lus87a Section 2]. Then it is a free $\mathbb{Z}$-module with a basis $\{t_w \mid w \in \tilde{S}_n\}$, and $(\mathcal{J}, \{t_w \mid w \in \tilde{S}_n\})$ is a based ring in the sense of [Lus87b].
For a subset $X \subset \tilde{S}_n$, we define $J_X := \bigoplus_{x \in X} \mathbb{Z}t_x$ to be the free sub-$\mathbb{Z}$-module of $J$ generated by $\{t_x \mid x \in X\}$. Then for a two-sided cell $\mathcal{C} \subset \tilde{S}_n$, $J_{\mathcal{C}}$ is a two-sided ideal of $J$. Also, $(J_{\mathcal{C}}, \{t_w \mid w \in \mathcal{C}\})$ is a based ring and $\sum_{w \in D} t_w$ is its unit. In other words, we have a decomposition of based rings

$$J = \bigoplus_{\mathcal{C} \subset \tilde{S}_n} J_{\mathcal{C}},$$

where the sum is over all the two-sided cells $\mathcal{C}$ of $\tilde{S}_n$. Also, if $\Gamma$ is a left cell in $\mathcal{C}$, then $J_{\Gamma}$ (resp. $J_{\Gamma^{-1}}$, $J_{\Gamma^{-1} \cap \Gamma}$) is a left (resp. right, two-sided) ideal of $J_{\mathcal{C}}$. It is known [Lus87a] that $J_{\Gamma^{-1} \cap \Gamma}$ is a commutative ring with unit $t_w$ where $w$ is the unique element in $\Gamma \cap D$.

Recall some facts about nilpotent orbits, we refer the reader to [CM93] for an exposition. Nilpotent orbits of Lie $GL_n$ are parametrized by partitions of $n$, where to each $\lambda \vdash n$ one associates a nilpotent orbit $O_\lambda$ such that the Jordan type of any element in $O_\lambda$ is $\lambda$. Also, there exists a canonical bijection between nilpotent adjoint orbits of Lie $GL_n$ and two-sided cell in $\tilde{S}_n$ defined in [Lus99]. We denote the two-sided cell corresponding to $O_\lambda$ by $\mathcal{S}_\lambda$. Then it essentially follows from [Shi86] that $w \in \mathcal{S}_\lambda$ if and only if the shape of both $P(w)$ and $Q(w)$ is $\lambda$.

It follows from [Shi91] and [CPY18] that for each tabloid $P$, the union of fibers $\Psi(P, Q, \rho)$ for $(P, Q, \rho) \in \Omega$ where we vary $Q$ and $\rho$ are exactly a right cell of $\tilde{S}_n$. Similarly, if we fix a tabloid $Q$, the union of fibers is a left cell of $\tilde{S}_n$. If we want to restrict to a non-extended affine symmetric group $\tilde{S}_n$ (see Section 1.2 for the definition of $\tilde{S}_n$), then by [CPY18] Theorem 10.3] the same claims hold verbatim if we add condition $\sum_i \rho_i = 0$.

For $N \in O_\lambda$, we may identify $F_\lambda$ with the reductive part of $Z_{GL_n}(N)$, the stabilizer subgroup of $N$ in $G$. Then it follows from [X02] that there exists an isomorphism of rings from $J_{\Gamma^{-1} \cap \Gamma}$ to $\mathcal{R}(F_\lambda)$ under which each $t_w$ maps to an irreducible representation, say $V(\epsilon(w))$. Here, $\Gamma$ is a certain left cell contained in $\mathcal{S}_\lambda$. (Such $\Gamma$ will be called the anti-canonical left cell of $\mathcal{S}_\lambda$; see Section 3 and 4) Furthermore, he proved that there exists a (non-canonical) isomorphism $J_{\mathcal{S}_\lambda} \cong \text{Mat}_{\chi \chi}(\mathcal{R}(F_\lambda))$ compatible with the isomorphism above. Here, $\chi = \frac{n!}{\prod_{j=1}^{\lambda} j^{\lambda_j}}$ is the Euler characteristic of the Springer fiber of some/any $N \in O_\lambda$.

For $w \in \tilde{S}_n$, we define

$$R(w) = \{|i| \in [n] \mid w(i) > w(i + 1)\} \quad \text{and} \quad L(w) = \{|i| \in [n] \mid w^{-1}(i) > w^{-1}(i + 1)\},$$

called the right (resp. left) descent set of $w$. Also for a tabloid $T$, we define its $\tau$-invariant by

$$\tau(T) := \{|i| \in [n] \mid \text{\overline{T} lies in a strictly higher row of T than \overline{i + 1}\} \}.$$

Then by [CLP17] Proposition 3.6], we have $L(w) = \tau(P(w))$ and $R(w) = \tau(Q(w))$.

3. Canonical and anti-canonical left cells

Let $w_0 \in S_n$ be the longest element in $S_n$ and define

$$\tilde{S}_n := \{w \in \tilde{S}_n \mid l(w) + l(w_0) = l(ww_0)\} = \{w \in \tilde{S}_n \mid R(w) \subset \{1\}\},$$

i.e. the set of elements each of which has minimal length in its left $S_n$-coset. Then for each two-sided cell $\mathcal{C} \subset \tilde{S}_n$, the intersection $\Gamma_{\text{can}} := \mathcal{C} \cap \tilde{S}_n$ is a single left cell and is called the canonical left cell of $\mathcal{C}$ [LX88]. If $\mathcal{C} \not\subset \mathcal{S}_\lambda$, then we denote its canonical left cell by $\Gamma^\text{can}_\lambda$. 

Section 1]. Also if we set $D \subset \tilde{S}_n$ to be the set of distinguished (Duflo) involutions, then $D$ is finite and the unit of $J$ is given by $\sum_{w \in D} t_w$. 


Lemma 3.1. Let $T_{\lambda}^{\text{can}}$ be the reverse row superstandard tabloid of shape $\lambda$ with start at 1 in the sense of [CLP17, 2.1]. If $w \in \Gamma_{\lambda}^{\text{can}}$, then we have $Q(w) = T_{\lambda}^{\text{can}}$.

Proof. By assumption we have $\tau(Q(w)) = R(w) \subset \{\pi\}$. Since $T_{\lambda}^{\text{can}}$ is the unique tabloid of shape $\lambda$ which satisfies $\tau(T_{\lambda}^{\text{can}}) \subset \{\pi\}$, the result follows. □

Example 3.2. We have

$$T_{(4,3,1)}^{\text{can}} = \begin{array}{llll}
3 & 7 & 6 & 8 \\
2 & 5 & 4 & \\
1 & 3 & & \\
\end{array}.$$ 

This time, let $w_0^\lambda$ be the longest element in $\mathcal{S}_\lambda := \mathcal{S}_{\lambda_1} \times \mathcal{S}_{\lambda_2} \times \cdots$. Thus for example $w_0^{(1,1,\ldots,1)} = w_0$ and $w_0^{(n)} = \text{id}$. By direct calculation, we see that the shape of $P(w_0^\lambda)$ and $Q(w_0^\lambda)$ is $\lambda$, thus $w_0^\lambda \in \mathcal{L}_\lambda$.

Lemma 3.3. Let $T_{\lambda}^{\text{an}}$ be the standard Young tableau of shape $\lambda$ with $\{1, 2, \ldots, \lambda_1\}$ on the first column, $\{\lambda_1 + 1, \ldots, \lambda_1 + \lambda_2\}$ on the second column, etc. Then we have $\Phi(w_0^\lambda) = (T_{\lambda}^{\text{an}}, T_{\lambda}^{\text{an}}, 0)$.

Proof. It is clear that $R(w) = [n] - \{\lambda_1, \lambda_1 + \lambda_2, \ldots\}$. Since $T_{\lambda}^{\text{an}}$ is the unique tabloid of shape $\lambda$ which satisfies $\tau(T_{\lambda}^{\text{an}}) = [n] - \{\lambda_1, \lambda_1 + \lambda_2, \ldots\}$, we have $Q(w_0^\lambda) = T_{\lambda}^{\text{an}}$. Since $w_0^\lambda$ is an involution, we also have $P(w_0^\lambda) = T_{\lambda}^{\text{an}}$ by [CLP17, Proposition 3.1]. Moreover, $w_0^\lambda \in \mathcal{S}_n$ thus $\overline{\rho}(w) = 0$ by [CPY18, Theorem 10.2]. □

We define $\Gamma_{\lambda}^{\text{an}} \subset \mathcal{L}_\lambda$ to be the left cell containing $w_0^\lambda$, called the anti-canonical left cell of $\mathcal{L}_\lambda$. Then the lemma above implies that $w \in \Gamma_{\lambda}^{\text{an}}$ if and only if $Q(w) = T_{\lambda}^{\text{an}}$.

Example 3.4. We have

$$T_{(4,3,1)}^{\text{an}} = \begin{array}{llll}
1 & 4 & 6 & 8 \\
2 & 5 & 7 & \\
3 & & & \\
\end{array}.$$ 

4. Bijection of Xi and AMBC

Our goal in this section is to show that the bijection $\mathcal{Z} : (\Gamma_{\lambda}^{\text{an}})^{-1} \cap \Gamma_{\lambda}^{\text{an}} \rightarrow \text{Dom}(F_{\lambda})$ defined by Xi [Xi102] can be interpreted in terms of AMBC. First we describe his bijection $\mathcal{Z} : (\Gamma_{\lambda}^{\text{an}})^{-1} \cap \Gamma_{\lambda}^{\text{an}} \rightarrow \text{Dom}(F_{\lambda})$ following [Xi102, 5.2.1].

Recall the following notions from [CPY18, Definition 3.20]. The $<_{SE}$ ordering on $\mathbb{Z} \times \mathbb{Z}$ is given by $(i, j) <_{SE} (i', j')$ if and only if $i < i'$ and $j < j'$. A stream is a subset of $\mathcal{B}_w$ invariant under translation $(i, j) \mapsto (i + n, j + n)$ and forming a chain in $<_{SE}$-ordering. The density of a stream is the number of distinct translation classes in it. Now we can state the following definition.

Definition 4.1. For $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash n$, we say that $\{S_1, S_2, \ldots, S_l(\lambda)\}$ is a complete stream family of $w \in \mathcal{L}_\lambda$ (or complete antichain family in the sense of [Xi102]) if $S_i$ are pairwise disjoint streams, the density of each $S_i$ is equal to $\lambda_i$, and $\bigcup_{i=1}^{l(\lambda)} S_i = \mathcal{B}_w$. 


Let \( \{S_1, S_2, \ldots, S_{l(\lambda)}\} \) be a complete stream family of \( w \in (\Gamma^{an}_\lambda)^{-1} \cap \Gamma^{an}_\lambda \), which always exists by [Xi02 Theorem 5.1.12]. Set \( D(S_i) \) to be the altitude of \( S_i \) as in [CLPT17 Definition 2.11], i.e.

\[
D(S_i) := \sum_{(x,y) \in S_i, 1 \leq x \leq n} \left\lfloor \frac{y}{n} \right\rfloor - 1.
\]

After rearranging \( S_i \)'s if necessary, we may assume that \( D(S_i) \geq D(S_{i+1}) \) whenever \( \lambda_i = \lambda_{i+1} \). Now we define \( \overline{\tau}(w) = (D(S_1), D(S_2), \ldots, D(S_{l(\lambda)})) \in \text{Dom}(F_\lambda) \), which gives the bijection \( \overline{\tau} : (\Gamma^{an}_\lambda)^{-1} \cap \Gamma^{an}_\lambda \to \text{Dom}(F_\lambda) \) of \( \Xi_i \). By [Xi02 Lemma 5.2.4], this value does not depend on the choice of the complete stream family \( \{S_1, S_2, \ldots, S_{l(\lambda)}\} \) of \( w \).

We state the main theorem of this section.

**Theorem 4.2.** For \( w \in (\Gamma^{an}_\lambda)^{-1} \cap \Gamma^{an}_\lambda \), we have \( \overline{\tau}(w) = \text{rev}_\lambda(\overline{\rho}(w)) \).

The rest of this section is devoted to its proof. To this end, first we define

\[
\mathcal{P}_\lambda = \left\{ \left. an + \sum_{j=1}^{i-1} \lambda_j' + 1, an + \sum_{j=1}^i \lambda_j' \right| a \in \mathbb{Z}, 0 \leq i \leq \lambda_1 \right\}.
\]

For example, \( \mathcal{P}_{(3,2,1,1)} \) consists of \( \{1, 2, 3, 4\}, \{5, 6\}, \{7\} \) and their shifts by multiples of 7. It is clear that \( \mathcal{P}_\lambda \) is a partition of \( \mathbb{Z} \).

**Lemma 4.3.** Let \( w \in (\Gamma^{an}_\lambda)^{-1} \cap \Gamma^{an}_\lambda \).

   (a) \( \mathcal{B}_w \cap (A \times \mathbb{Z}) \) for any \( A \in \mathcal{P}_\lambda \) is totally ordered with respect to the northeast ordering.

   (b) For any stream \( S \subset \mathcal{B}_w \), \( \#(S \cap (A \times \mathbb{Z})) \leq 1 \).

**Proof.** Since \( w \in (\Gamma^{an}_\lambda)^{-1} \cap \Gamma^{an}_\lambda \), we have \( Q(w) = T^{an}_\lambda \) which means that \( R(w) = \tau(Q(w)) = [n] \setminus \{\lambda_1', \lambda_1 + \lambda_2', \ldots\} \). It implies that \( w(\sum_{j=1}^{i-1} \lambda_j' + 1) > w(\sum_{j=1}^i \lambda_j') \) for any \( 0 \leq i \leq \lambda_1 \).

Now the statement easily follows from this.

For a stream \( S \subset \mathcal{B}_w \), we define

\[
S^{\square} := \{ (A, B) \in \mathcal{P}_\lambda^2 \mid S \cap (A \times B) \neq \emptyset \}.
\]

Suppose that \( w \in (\Gamma^{an}_\lambda)^{-1} \cap \Gamma^{an}_\lambda \) and assume that \( \{S_2', S_3', \ldots, S_{l(\lambda)}'\} \) is a complete stream family of the partial permutation \( \text{fw}(w) \). Then there exists a complete stream family \( \{S_1, S_2, \ldots, S_{l(\lambda)}\} \) of \( w \) such that

1. \( S_1 \) is a channel of \( \mathcal{B}_w \),
(2) $D(S_1) = \rho_1$ where $\overline{\varphi}(w) = (\rho_1, \rho_2, \ldots)$, and

(3) $S_i^{\square} = S_i^{\square}$.

Proof. Recall the construction of the backward AMBC algorithm. In particular, $B_w$ is obtained by following the steps below.

(a) Let $S := \mathfrak{st}_{\mu_1}(T_1^{an}, T_2^{an})$ and give $S$ some proper numbering; here $T_\lambda^{an} = (T_1^{an}, T_2^{an}, \ldots)$. (See [CPY18] Section 3.4 for the definition of $\mathfrak{st}$.)

(b) Calculate the backward numbering $d_{w, S}^{\text{bk}}$ on $B_{\text{fw}(w)}$ (see [CPY18] Definition 4.4 for the definition of $d_{w, S}^{\text{bk}}$).

(c) For each $i \in \mathbb{Z}$, consider the zig-zag $Z_i$ whose outer corner-posts are balls in $B_{\text{fw}(w)}$ labeled by $i$.

(d) The union of all inner corner-posts of $Z_i$ for all $i \in \mathbb{Z}$ is $B_w$.

Now suppose that some proper numbering on $S = \mathfrak{st}_{\mu_1}(T_1^{an}, T_2^{an})$ is given, say $S = \{ S^{(i)} \mid i \in \mathbb{Z} \}$ where $S^{(i)} = (a_i, b_i)$ is $S$. We claim that if $(x, y) \in B_{\text{fw}(w)}$ satisfies $(x, y) \leq_{\text{NW}} S^{(i)}$ but $(x, y) \not\leq_{\text{NW}} S^{(i+1)}$, then $d_{w, S}^{\text{bk}}(x, y) = i$. In other words, we have $d_0 = d_{w, S}^{\text{bk}}$ where the notations are from [CPY18] Definition 4.1. Indeed, suppose $(a, b), (c, d) \in B_{\text{fw}(w)}$ satisfies $d_0(a, b) = d_0(c, d)$. In other words,

$$(a, b), (c, d) \in \{(x, y) \in \mathbb{Z}^2 \mid x \geq a_i, y \geq b_i\} - \{(x, y) \in \mathbb{Z}^2 \mid x \geq a_i + 1, y \geq b_i + 1\}.$$ 

However, the balls in

$$B_{\text{fw}(w)} \cap \{(x, y) \in \mathbb{Z}^2 \mid x \geq a_i, y \geq b_i\} - \{(x, y) \in \mathbb{Z}^2 \mid x \geq a_i + 1, y \geq b_i + 1\}$$

are totally ordered with respect to southwest ordering by Lemma 4.3. Note that this lemma applies to $\text{fw}(w)$ since $\text{fw}(T_\lambda^{an}) = (T_2^{an}, \ldots)$ is also anticanonical when restricted to indices not in $T_1^{an}$. Therefore, if $(a, b) \leq_{\text{NW}} (c, d)$ then $(a, b) = (c, d)$. Thus the claim follows by looking at the construction of $d_{w, S}^{\text{bk}}$ [CPY18] Definition 4.1.

Now let $I(Z_i)$ be the set of inner corner-posts of $Z_i$ and $O(Z_i)$ be the set of outer corner-posts of $Z_i$. Then the backward AMBC algorithm replaces balls in $O(Z_i) \cup \{ S^{(i)} \}$ with those in $I(Z_i)$. By Lemma 4.3, it is easy to check that for $(x, y) \in I(Z_i)$ and $(z, w) \in O(Z_i)$ we have

- if $y = w$ and $a_i \leq x < a_i + 1$, then $z = x + 1$ and
- if $x = z$ and $b_i \leq y, w < b_i + 1$, then $w = y + 1$.

From this, it follows that for any $A, B \in \mathcal{P}_\lambda$ we have

$$\#((A \times B) \cap I(Z_i)) = \#((A \times B) \cap O(Z_i) \cup \{ S^{(i)} \}).$$

In other words, we have

- $\#((A \times B) \cap B_w) = \#((A \times B) \cap B_{\text{fw}(w)}) + 1$ if $(A, B) \in S^{\square},$ and
- $\#((A \times B) \cap B_w) = \#((A \times B) \cap B_{\text{fw}(w)})$ otherwise.

Now we can choose $S_1, S_2, \ldots$ so that $S_1^{\square} = S^{\square}$ and $S_i^{\square} = S_i^{\square}$ for $2 \leq i \leq l(\lambda)$. Indeed, for any pair $A, B \in \mathcal{P}_\lambda$, and for any stream $S_i$ we know that at most one element of this stream lies in $A \times B$. We also know the same for the stream $S$, as can be seen by examining the rows of $T_\lambda^{an}$. Let
us arbitrarily biject elements of $O(Z_i) \cup \{S^{(i)}\}$ lying inside a particular $A \times B$ with those of $I(Z_i)$ lying inside the same $A \times B$. This maps each of the streams $S, S'_i$ into a new stream $S_1, S_i$. It is clear that $\{S_1, S_2, \ldots \}$ satisfy the desired properties.

To illustrate this proof, we give the following example.

**Example 4.5.** Let $n = 16$ and $w = [11, 5, 4, 3, 2, -9, 13, 10, 9, 8, 1, 15, 12, 22, 14, 16]$. Then $fw(w) = [0, 11, 5, 4, 3, -8, 13, 10, 9, 2, 0, 15, 0, 22, 0]$. It is easy to see that $w \in (\Gamma^\alpha)^{-1} \cap \Gamma^\alpha$ for $\lambda = (5, 4, 2, 2, 1)$. Let $A = B = \{7, 8, 9, 10, 11\}$. Then

$$\left(\mathbb{Z} \times B\right) \cap B_w = \{(8, 10), (9, 9), (10, 8)\} \text{ and } \left(\mathbb{Z} \times B\right) \cap (B_{fw(w)} \cup S) = \{(7, 7), (9, 10), (10, 9)\}.$$ 

We choose an arbitrary bijection between the two sets, and repeat it for other pairs $(A, B)$. Then streams of any complete stream family of $B_{fw(w)}$ together with $S$ biject to streams of a complete stream family of $B_w$.

We are ready to prove the main theorem in this section.

**Proof of Theorem 4.2.** We proceed by induction on $l = l(\lambda) = \lambda'$. When $l = 1$, then $w = \omega^d$ for some $d \in \mathbb{Z}$ and in this case it is easy to see that $\varepsilon(w) = \text{rev}(\bar{\rho})(w) = (d)$. (Recall that $\omega$ is the shift permutation $\{2, 3, \ldots, n, n + 1\}$.) In general, we choose a complete stream family $\{S_1, S_2, \ldots, S_l\}$ of $fw(w)$ and $\{S_1, S_2, \ldots, S_l\}$ of $w$ as Lemma 4.3. Then $\rho_1 = D(S_1)$, and for any $2 \leq i \leq l$ we have

$$D(S_i) = \sum_{(x, y) \in S_{i-1}, 1 \leq x \leq n} \left[ \frac{y}{n} \right] - 1 = \sum_{(x, y) \in S_i, 1 \leq x \leq n} \left[ \frac{y}{n} \right] - 1 = D(S'_i)$$

since $S_i \cong S'_i$. In other words, as a multiset we have

$$\{\rho_1, \rho_2, \ldots, \rho_l\} = \{D(S_1)\} \cup \{\rho_2, \ldots, \rho_l\} = \{D(S_1)\} \cup \{D(S'_2), \ldots, D(S'_l)\} = \{D(S_1)\} \cup \{D(S_2), \ldots, D(S_l)\}$$

by induction assumption. After reordering $D(S_i)$ if necessary, we have $\varepsilon(w) = \text{rev}_\lambda(\bar{\rho}(w))$ as desired. \qed

5. Shift, Knuth moves, and AMBC

In this section we discuss how multiplication by $\omega = [2, 3, \ldots, n, n + 1]$ and Knuth moves behave under the AMBC. We start with some definitions.

**Definition 5.1.** For $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\bar{\rho} = (\rho_1, \ldots, \rho_r)$, we say that $\bar{\rho}$ is determinantal (with respect to $\lambda$) if $\rho_i = \rho_j$ whenever $\lambda_i = \lambda_j$.

Note that if $\bar{\rho}$ is determinantal with respect to $\lambda$ then both $\bar{\rho} \in \text{Dom}(F_\lambda)$ and $\text{rev}_\lambda(\bar{\rho}) \in \text{Dom}(F_\lambda)$. Also it is a highest weight of a tensor product of powers of determinant representations of factors of $F_\lambda$.

**Definition 5.2.** Let $T = (T_1, T_2, \ldots)$ be a tabloid of shape $\lambda$ and suppose that $\bar{t} \in T_i$ for some $1 \leq i \leq n$ and $1 \leq t \leq l(\lambda)$. We define $\delta(T, i) = (\delta_1, \delta_2, \ldots, \delta_{l(\lambda)})$ and $\bar{t}(T, i) = (t_1, t_2, \ldots, t_{l(\lambda)})$ as follows. Here $[-]$ is the Iverson bracket.
Multiplication by \( \omega \). Here we discuss how to relate \( \Phi(\omega w), \Phi(w^\omega - 1) \) and \( \Phi(w) \). Let us start with the following lemma.

**Lemma 5.3.** Let \( T = (T_1, T_2, \ldots) \) be a tabloid of shape \( \lambda \) and suppose that \( \pi \) is contained in \( T_t \).

1. \( \text{lch}_j(\omega(T)) = \text{lch}_j(T) \) if \( j \neq t, t - 1 \) and \( \lambda_j = \lambda_{j+1} \).
2. If \( \lambda_t = \lambda_{t+1} \), then \( \text{lch}_t(\omega(T)) = \text{lch}_t(T) - 1 \).
3. If \( \lambda_{t-1} = \lambda_t \), then \( \text{lch}_{t-1}(\omega(T)) = \text{lch}_t(T) + 1 \).

**Proof.** The first part is straightforward. For (2), we choose the ordering on \( \omega(T_t) \) to be the standard activation ordering as in [CLP17, Definition 5.1] and that on \( T_t \) to be the same one except that \( \pi \) is set to be the smallest entry. Then \( \omega : T_t \rightarrow \omega(T_t) \) is order-preserving and \( \omega \) also preserves the charge matching of \( (T_t, T_{t+1}) \), i.e. if \( \bar{s} \in T_t \) is matched to \( \bar{b} \in T_{t+1} \), then \( \omega(\bar{s}) = \bar{a} + 1 \in \omega(T_t) \) is matched to \( \omega(\bar{b}) = \bar{b} + 1 \in \omega(T_{t+1}) \). (See [CLP17, Definition 5.2] for the definition of the charge matching.) However, \( \pi \in T_t \) contributes to the charge whereas \( \omega(\pi) = \bar{T} \in \omega(T_t) \) does not, from which (2) follows. (3) is also similarly proved. 

**Lemma 5.4.** For tabloids \( P,Q \) of the same shape \( \lambda \), we have

\[
\mathfrak{s}_{P,\omega(Q)} = \mathfrak{s}_{P,Q} - \vec{\delta}(Q,n) + \vec{\gamma}(Q,n) \quad \text{and} \quad \mathfrak{s}_{\omega(P),Q} = \mathfrak{s}_{P,Q} + \vec{\gamma}(P,n) - \vec{\delta}(P,n).
\]

**Proof.** It follows from Lemma 5.3 and [CLP17, Theorem 5.10]. 

**Proposition 5.5.** Suppose that \( \Phi(w) = (P,Q, \mathfrak{s}_{P,Q} + \vec{\gamma}) \). Then we have

\[
\Phi(w^{-1}) = (P,\omega(Q), \mathfrak{s}_{P,\omega(Q)} + \vec{\gamma} - \vec{\delta}(Q,n))
\]

\[
\Phi(w) = (\omega(P),Q, \mathfrak{s}_{\omega(P),Q} + \vec{\gamma} + \vec{\delta}(P,n))
\]

In particular, if \( \Phi(w) = (P,P, \vec{\gamma}) \) then \( \Phi(\omega w^{-1}) = (\omega(P), \omega(P), \vec{\gamma}) \).
Proof. By Lemma 5.4 it is enough to argue the following:
\[
\Phi(w^{-1}) = (P, \omega(Q), \overrightarrow{s}_{P, Q} + \overrightarrow{\beta} - \overleftarrow{\tau}(Q, n)),
\]
\[
\Phi(\omega w) = (\omega(P), Q, \overrightarrow{s}_{P, Q} + \overrightarrow{\beta} + \overleftarrow{\tau}(P, n)).
\]
This follows from the definition of the altitude of a stream given in [CLP17 Definition 2.11]. Indeed, if we compare the process of applying the AMBC to \( w \) and \( \omega w \), they are identical up to the \( \omega \) of residues modulo \( n \), except there is a unique forward step of the AMBC when one of the \( D(x) \) is larger by 1 in the shifted version. It is exactly the step \( t \) such that \( t \in P_i \). Similar consideration works for \( w\omega^{-1} \).

We finish with an example.

Example 5.6. Let \( n = 9 \) and \( w = [-1, 3, 10, -5, 14, -3, 18, 7, 2] \). Then \( \Phi(w) = (P, Q, \overrightarrow{s}_{P, Q} + \overrightarrow{\beta}) \), where \( P, Q, \overrightarrow{s}_{P, Q} \) are as in the Example 5.2 and \( \overrightarrow{\beta} = (0, 1, 2) \). Then \( \omega w = [0, 4, 11, -4, 15, -2, 19, 8, 3] \) and \( \Phi(\omega w) = (\omega(P), Q, \overrightarrow{s}_{P, Q} + \overrightarrow{\beta'}) \), where \( \overrightarrow{\beta'} = (0, 1, 3) = \overrightarrow{\beta} + (0, 0, 1) = \overrightarrow{\beta} + i(P, 9) \), and
\[
\omega(P) = \begin{pmatrix} 1 & 2 & 6 \\ 4 & 8 & 9 \\ 3 & 7 & 5 \end{pmatrix}. \]
One checks that \( \overrightarrow{s}_{\omega(P), Q} = (0, -2, 0) \), in agreement with Proposition 5.5.

5.2. Knuth moves and star operations. Recall the definition of Knuth moves in [CLP17 3.3] which corresponds to the star operations in [KL79]. Until the end of this section we assume \( n \geq 3 \), since if \( n = 2 \) then the order of \( s_0 s_1 \) is infinity and the star operation is not well-defined.

Definition 5.7. Let \( n \geq 3 \). For a tabloid \( T \) and \( 1 \leq i \leq n \), suppose that the rows of \( T \) containing \( i \) and \( i+1 \) are different. Let \( S \) be the tabloid obtained from \( T \) by exchanging \( i \) and \( i+1 \). If they satisfy either
\[
\{\tau(T) \cap \{i, i+1\}, \tau(S) \cap \{i, i+1\}\} = \{(i), (i+1)\} \text{ or } \{(i-1), (i)\}, \tau(S) \cap \{i-1, i\}\} = \{(i-1), (i)\},
\]
then we say that \( T^* \) is well-defined for \( \sim i \), and define \( T^* := S \).

Note that \( T^* \) is well-defined then \( (T^*)^* \) is also well-defined and \( (T^*)^* = T \).

Definition 5.8. For an element \( w \in \mathcal{S}_n \) such that \( Q(w)^* \) is well-defined for \( \sim \overrightarrow{\pi} \), we define \( \overrightarrow{w} \in \mathcal{S}_n \) to be the element obtained from \( w \) by a right star operation such that \( Q(\overrightarrow{w}) = Q(w)^* \). Similarly, if \( P(w)^* \) is well-defined for \( \sim \overrightarrow{\pi} \), then we define \( \overleftarrow{w} \in \mathcal{S}_n \) to be the element obtained from \( w \) by a left star operation such that \( P(\overleftarrow{w}) = P(w)^* \).

It follows from [CLP17 Theorem 3.11] that \( \overrightarrow{w} \) and \( \overleftarrow{w} \) are well-defined for such \( i \). The same [CLP17 Theorem 3.11] also provides the following description of how star operations changes the result of AMBC. Suppose that \( \Phi(w) = (P, Q, \overrightarrow{\pi}) \) for some \( w \in \mathcal{S}_n \). If \( \sim \overrightarrow{\pi} \), then we have
\[
\Phi(\overrightarrow{w}) = (P, Q^*, \overrightarrow{\pi}) \text{ and } \Phi(\overleftarrow{w}) = (P^*, Q, \overrightarrow{\pi})
\]
whenever \( w^*, \overrightarrow{w} \) are well-defined. If \( \sim \overrightarrow{\pi} \), then we have
\[
\Phi(\overrightarrow{w}) = (P, Q^*, \overrightarrow{\pi} + \overrightarrow{\tau}(Q, 1) - \overrightarrow{\tau}(Q, n)) \text{ and } \Phi(\overleftarrow{w}) = (P^*, Q, \overrightarrow{\pi} - \overrightarrow{\tau}(P, 1) + \overrightarrow{\tau}(P, n))
\]
whenever \( w^*, \overrightarrow{w} \) are well-defined.

Now we discuss how the star operation affects symmetrized offset constraints.
Lemma 5.9. For \( n \geq 3 \), let \( T \) be a tabloid of shape \( \lambda \vdash n \) such that \( T^* \) is well-defined for \( \ast \sim \triangledown \). If \( \triangledown \neq \Pi, \) then \( \text{lch}_j(T) = \text{lch}_j(T^*) \) for any \( j \). Now suppose that \( \triangledown = \Pi, \) \( \triangledown \in T_s, \) and \( \Pi \in T_t. \)

(a) If \( |t - s| \geq 2 \), then \( \text{lch}_j(T^*) = \text{lch}_j(T) \) if \( j \notin \{ s - 1, s, t - 1, t \} \) and

- if \( \lambda_s = \lambda_s + 1 \), then \( \text{lch}_s(T^*) = \text{lch}_s(T) + 1 \).
- if \( \lambda_{s-1} = \lambda_s \), then \( \text{lch}_{s-1}(T^*) = \text{lch}_{s-1}(T) - 1 \).
- if \( \lambda_t = \lambda_{t+1} \), then \( \text{lch}_t(T^*) = \text{lch}_t(T) - 1 \).
- if \( \lambda_{t-1} = \lambda_t \), then \( \text{lch}_{t-1}(T^*) = \text{lch}_{t-1}(T) + 1 \).

(b) If \( t = s + 1 \), then either \( \triangledown \in T_t \) or \( n - 1 \in T_s \). In this case \( \text{lch}_j(T^*) = \text{lch}_j(T) \) if \( j \notin \{ s - 1, s = t - 1, t \} \) and

- if \( \lambda_{s-1} = \lambda_s \), then \( \text{lch}_{s-1}(T^*) = \text{lch}_{s-1}(T) - 1 \).
- if \( \lambda_t = \lambda_s + 1 \), then \( \text{lch}_t(T^*) = \text{lch}_t(T) + 2 \).
- if \( \lambda_t = \lambda_{t+1} \), then \( \text{lch}_t(T^*) = \text{lch}_t(T) - 1 \).
- if \( \lambda_{t-1} = \lambda_t \), then \( \text{lch}_{t-1}(T^*) = \text{lch}_{t-1}(T) + 1 \).

(c) If \( s = t + 1 \), then either \( \triangledown \in T_s \) or \( n - 1 \in T_t \). In this case \( \text{lch}_j(T^*) = \text{lch}_j(T) \) if \( j \notin \{ t - 1, t = s - 1, s \} \) and

- If \( \lambda_{t-1} = \lambda_t \), then \( \text{lch}_{t-1}(T^*) = \text{lch}_{t-1}(T) + 1 \).
- If \( \lambda_t = \lambda_{s+1} \), then \( \text{lch}_{s+1}(T^*) = \text{lch}_{s+1}(T) - 2 \).
- If \( \lambda_s = \lambda_{s+1} \), then \( \text{lch}_s(T^*) = \text{lch}_s(T) + 1 \).

Proof. Let us first assume that \( \ast \sim \triangledown \neq \Pi \) and suppose that \( \triangledown \in T_t \) and \( \triangledown + 1 \in T_s \). Then \( s \neq t \), and if \( |s - t| \geq 2 \) then it is clear that \( \text{lch}_j(T^*) = \text{lch}_j(T) \) for any \( j \). Thus it remains to check the case when \( |s - t| = 1 \), i.e. \( t = s + 1 \) or \( s = t + 1 \). But since \( T^{**} = T \), by symmetry it suffices to show the case when \( t = s + 1 \) which we assume from now on.

It is still clear that \( \text{lch}_j(T^*) = \text{lch}_j(T) \) if \( j \neq s \). Thus we only need to show that \( \text{lch}_s(T^*) = \text{lch}_s(T) \) when \( \lambda_s = \lambda_{s+1} \). Since in this case we have \( \triangledown \in T^*(T^*) - \tau(T) \), for the star operation to be well-defined we should have either \( \triangledown + 2 \in T_t = T_{s+1} \) so that \( \triangledown + 2 \in T^*(T^*) - \tau(T) \) or \( \triangledown + 1 \in T_s = T_{t-1} \) so that \( \triangledown + 1 \in T^*(T^*) - \tau(T) \).

Here we only deal with the case when \( \triangledown + 2 \in T_t = T_{s+1} \), but the other case can also be similarly proved. We take activation orderings on \( T_s \) and \( T_{s+1}^* \) which are the same as the standard activation ordering except that \( \triangledown + 1 \in T_s \) and \( \triangledown \in T_s^* \) are the smallest element in each row. Then \( \triangledown + 1 \in T_s \) is matched to \( \triangledown + 2 \in T_{s+1} \) and \( \triangledown \in T_s^* \) is matched to \( \triangledown + 1 \in T_{s+1}^* \). Furthermore, the matchings between \( T_s \) and \( T_{s+1} \) are the same as those between \( T_s^* \) and \( T_{s+1}^* \) except one; there exists \( \jmath \) such that \( \jmath \in T_s \) is matched to \( \jmath \in T_{s+1} \) and \( \jmath \in T_s^* \) is matched to \( \jmath + 2 \in T_{s+1}^* \). However, this does not affect the local charge since \( \jmath \neq \jmath + 1 \), i.e. either \( j < i \) or \( i + 2 < j \). Thus we have \( \text{lch}_s(T^*) = \text{lch}_s(T) \).

It remains to consider the case when \( \ast \sim \Pi \). We assume that \( \Pi \in T_s \) and \( \Pi \in T_t \), and first suppose that \( |t - s| \geq 2 \). It is still clear that \( \text{lch}_j(T^*) = \text{lch}_j(T) \) if \( j \notin \{ s - 1, s, t - 1, t \} \). Now if \( \lambda_s = \lambda_{s+1} \), then we choose activation orderings on \( T_s \) and \( T_{s+1}^* \) which are the same as the standard activation ordering except that \( \Pi \in T_s^* \) becomes the smallest element. Then it is clear that the star
operation preserves matchings between \((T_s, T_{s+1})\) and \((T_s^*, T_{s+1}^*)\), and \(\bar{T} \in T_s\) does not contribute to the charge but \(\bar{\pi} \in T_s^*\) always does. Thus it follows that \(\text{lch}_s(T^*) = \text{lch}_s(T) + 1.\)

On the other hand, if \(\lambda_{s-1} = \lambda_s\) then we take the standard activation ordering on both \(T_{s-1}\) and \(T_s^*\). Then it is also clear that the star operation is matching-preserving, but the element \(\bar{k} \in T_{s-1}\) which is matched to \(\bar{T}\) contributes to the charge whereas \(\bar{k} \in T_{s-1}^*\) which is necessarily matched to \(\bar{\pi}\) does not. In other words, we have \(\text{lch}_{s-1}(T^*) = \text{lch}_{s-1}(T) - 1.\) Now the statements about the local charges on \(T\)-th and \((t-1)\)-th row follow from applying this argument to \(T^*.\)

If \(|t-s| = 1\), then it suffices to consider the case when \(t = s + 1\) because the other case is proved by switching \(T\) and \(T^*.\) Now if \(t = s + 1\), i.e. if \(\bar{T} \in T_s\) and \(\bar{\pi} \in T_{s+1}\), then clearly \(\text{lch}_j(T^*) = \text{lch}_j(T)\) when \(j \notin \{s-1, s, s+1\}.\) First suppose that \(\lambda_{s-1} = \lambda_s.\) If we pose the standard activation ordering on both \(T_{s-1}\) and \(T_{s+1}\), then the star operation is matching-preserving. Also, there exists \(2 \leq k \leq n - 1\) such that \(\bar{k} \in T_{s-1}\) is matched to \(\bar{T} \in T_s\) but \(\bar{k} \in T_{s+1}^*\) is matched to \(\bar{\pi} \in T_s^*\). Thus it is clear that \(\text{lch}_{s-1}(T^*) = \text{lch}_{s-1}(T) - 1.\) Now when \(\lambda_{s+1} = \lambda_{s+2}\), then similar argument applies and one can show that \(\text{lch}_{s+1}(T^*) = \text{lch}_{s+1}(T) - 1.\)

This time we assume that \(\lambda_s = \lambda_{s+1}\) and prove that \(\text{lch}_s(T^*) = \text{lch}_s(T) + 2.\) Note that either \(\bar{T} \in T_{s+1}\) or \(\bar{T} \notin T_s\) for the star operation to be well-defined for \(* \sim \bar{\pi}.\) Here we only discuss the case when \(\bar{T} \in T_{s+1}\), but the other case is proved similarly. We choose the activation ordering of \(T_s\) and \(T_s^*\) to be the standard one except that \(\bar{\pi} \in T_s^*\) becomes the smallest element. Then \(\bar{T} \in T_s\) is matched to \(\bar{T} \in T_{s+1}\), whereas \(\bar{\pi} \in T_s^*\) is matched to \(\bar{T} \in T_{s+1}^*.\) Besides these matchings, there is no difference between \(T_s\) and \(T_s^*\), whereas \(T_{s+1}^*\) is obtained from \(T_{s+1}\) by replacing \(\bar{\pi}\) with \(\bar{T}.\) This causes the local charge of \(T^*\) at row \(s\) to be bigger by 1 than that of \(T,\) together with considering the matching \((\bar{T}, \bar{T})\) in \(T\) and \((\bar{\pi}, \bar{T})\) in \(T^*,\) we see that \(\text{lch}_s(T^*) = \text{lch}_s(T) + 2\) as desired. \(\square\)

**Lemma 5.10.** For \(n \geq 3,\) let \(T, P, Q\) be tabloids of shape \(\lambda \vdash n\) such that \(T^*\) is well-defined for \(* \sim \bar{i}.\) If \(\bar{i} \notin \bar{\pi},\) then we have \(\bar{\mathbf{S}}_{P,T} = \bar{\mathbf{S}}_{P,T^*}\) and \(\bar{\mathbf{S}}_{T,Q} = \bar{\mathbf{S}}_{T^*,Q}.\) If \(\bar{i} = \bar{\pi},\) then we have

\[
\begin{align*}
\bar{\mathbf{S}}_{P,T^*} &= \bar{\mathbf{S}}_{P,T} + \bar{\mathbf{S}}(T, 1) - \delta(T, n) - \delta(T, 1) + \delta(T, n) \\
\bar{\mathbf{S}}_{T^*,Q} &= \bar{\mathbf{S}}_{T,Q} - \bar{\mathbf{S}}(T, 1) + \delta(T, n) + \delta(T, 1) - \delta(T, n).
\end{align*}
\]

**Proof.** It follows from Lemma 5.9 and [CLP17, Theorem 5.10]. \(\square\)

**Proposition 5.11.** For \(n \geq 3,\) suppose that \(w \in \bar{\bar{S}}_n\) satisfies \(\Phi(w) = (P, Q, \bar{\mathbf{S}}_{P,Q} + \bar{\rho})\) for some tabloids \(P, Q\) of shape \(\lambda \vdash n\) and \(\text{rev}_\lambda(\bar{\rho}) \in \text{Dom}(F_\lambda).\)

1) If \(Q^*\) is well-defined for \(* \sim \bar{i} \notin \bar{\pi}\) then \(\Phi(w) = (P, Q^*, \bar{\mathbf{S}}_{P,Q^*} + \bar{\rho}).\) Similarly if \(P^*\) is well-defined for \(* \sim \bar{i} \notin \bar{\pi}\) then \(\Phi(w) = (P^*, Q, \bar{\mathbf{S}}_{P^*,Q} + \bar{\rho}).\)

2) If \(Q^*\) is well-defined for \(* \sim \bar{\pi}\) then \(\Phi(w) = (P, Q^*, \bar{\mathbf{S}}_{P,Q^*} + \bar{\rho} - \delta(Q, 1) - \delta(Q, n)).\) Similarly if \(P^*\) is well-defined for \(* \sim \bar{\pi}\) then \(\Phi(w) = (P^*, Q, \bar{\mathbf{S}}_{P^*,Q} + \bar{\rho} - \delta(P, 1) + \delta(P, n)).\)

In particular, if \(\Phi(w) = (P, P, \bar{\rho})\) for some \(P\) such that \(P^*\) is well-defined then \(\Phi(w^*) = (P^*, P^*, \bar{\rho}).\)

**Proof.** By symmetry, it suffices only to prove the statements for \(w^*\) when \(Q^*\) is well-defined. If \(* \sim \bar{i} \notin \bar{\pi},\) then by [CLP17, Theorem 3.11] we have \(\Phi(w^*) = (P, Q^*, \bar{\mathbf{S}}_{P,Q} + \bar{\rho}) = (P, Q^*, \bar{\mathbf{S}}_{P,Q^*} + \bar{\rho}).\)
Now if $* \sim \pi$, then again by [CLP17, Theorem 3.11] we have
\[
\Phi(w^*) = (P, Q^*, \mathfrak{F}_{P, Q} + \mathfrak{F}'(Q, 1) - \mathfrak{F}(Q, n)) = (P, Q^*, \mathfrak{F}_{P, Q} + \mathfrak{F}'(Q, 1) - \mathfrak{F}(Q, n))
\]
by Lemma 5.10.

**Example 5.12.** Let $n = 9$ and $w = [−3, 10, −5, 14, −3, 18, 7, 2]$, as in the Example 5.6. Recall that $\Phi(w) = (P, Q, \mathfrak{F}_{P,Q})$, where $P, Q, \mathfrak{F}_{P,Q}$ are as in the Example 2.2, and $\mathfrak{F}(Q, 1) = (0, 1, 2)$. For $* \sim \bar{9}$ we have $w^* = [−7, 3, 10, −5, 14, −3, 18, 7, 8]$. Then one checks that $\Phi(w^*) = (P, Q^*, (0, 0, 0))$. Since for $P = \begin{pmatrix} 2 & 4 & 6 \\ 3 & 7 & 8 \\ 1 & 5 & 9 \end{pmatrix}$ and $Q^* = \begin{pmatrix} 3 & 5 & 7 \\ 2 & 8 & 9 \\ 1 & 4 & 6 \end{pmatrix}$ we have $\mathfrak{F}_{P, Q} = (0, −1, −2)$, the claim of Proposition 5.11 reduces to $(0, 0, 0) = (0, −1, −2) + (0, 1, 2) + \mathfrak{F}(Q, 1) - \mathfrak{F}(Q, 9)$. The latter is true since $\mathfrak{F}(Q, 1) = \mathfrak{F}(Q, 9) = (0, 0, 0)$.

### 6. Distinguished involutions

Here we study the set of distinguished (or Duflo) involutions defined in [Lus87a], denoted $D \subset \widetilde{S}_n$. The main result in this section is as follows.

**Theorem 6.1.** $w \in D$ if and only if $\Phi(w) = (T, T, 0)$ for some $T$.

Its proof consists of the following three lemmas.

**Lemma 6.2.** Let $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ be left cells of $\widetilde{S}_n$ contained in the same two-sided cell. Then,

(a) $\Gamma_1^{-1} \cap \Gamma_3$ can be obtained from $\Gamma_2^{-1} \cap \Gamma_4$ by applying (left and right) star operations and (left and right) multiplication by $\omega$.

(b) $\Gamma_1$ can be obtained from $\Gamma_2$ by applying right star operations and right multiplication by $\omega$.

(c) $\Gamma_1^{-1} \cap \Gamma_1$ can be obtained from $\Gamma_2^{-1} \cap \Gamma_2$ by applying the map $w \mapsto *w^*$ (where the left and right star operation corresponds to the same $i$ for some $1 \leq i \leq n$) and conjugation by $\omega$.

**Proof.** It is a reformulation of [Xi02, Lemma 2.2.1, Corollary 2.2.2, Proposition 2.2.3] based on the result of [Shi86]. Also note that (b) follows from (a) and (c) follows from (b).

**Lemma 6.3.** If $w \in D$, then $w^* \in D$ (where the left and right star operation corresponds to the same $i$ for some $1 \leq i \leq n$) and $\omega w \omega^{-1} \in D$.

**Proof.** See e.g. [Xi02, Proposition 1.4.6].

**Lemma 6.4.** $D \cap S_n$ consists of all the involutions in $S_n$. In particular, $w_0^\lambda \in D$.

**Proof.** This follows from [Lus87a, Corollary 1.9(d)] and the fact that every involution in $S_n$ is distinguished.
Proof of Theorem 6.1. By Lemma 6.2(c) and Lemma 6.3 any distinguished involution can be obtained from another one in the same two-sided cell by applying \( w \mapsto \omega w \) and \( w \mapsto \omega w \omega^{-1} \) several times. Therefore, by Proposition 6.3 and 6.11 it suffices to show that \( \Phi(w) = (T, T, 0) \) for at least one distinguished involution in each two-sided cell. Now the result follows from that \( w_0^\lambda \in \mathcal{Z}_\lambda \cap D \) by Lemma 6.1 and \( \Phi(w_0^\lambda) = (T_\lambda^a, T_\lambda^a, 0) \) by Lemma 6.3.

Example 6.5. Let \( n = 9 \) and consider tabloid \( T = \begin{array}{ccc} 3 & 7 & 8 \\ 1 & 5 & 9 \end{array} \). Applying the reverse AMBC construction to \((T, T, 0)\) we get the Duflo involution \([-3, 5, 3, 7, 2, 10, 4, 8, 9] \in \widetilde{S}_9\).

7. Structure of asymptotic Hecke algebras attached to two-sided cells

Let \( \mathfrak{c} = \mathfrak{c}_\lambda \) for some \( \lambda \vdash n \) and recall the definition of \( \mathfrak{c} : (T_\lambda^a)^{-1} \cap T_\lambda^a \rightarrow \text{Dom}(F_\lambda) \) in Section 4. A conjecture of Lusztig \cite{Lus89}, proved by Xi \cite{Xi02} for \( G = GL_n \), states that there exists a non-canonical isomorphism between \( J_\mathfrak{c} \) and \( \text{Mat}_{\lambda \times \lambda}(\mathcal{R}(F_\lambda)) \) where \( \lambda = \lambda_\mathfrak{c} \). Furthermore, it restricts to an isomorphism \( J_1(\Gamma_\lambda) \rightarrow \mathcal{R}(F_\lambda) \) which maps \( t_w \) to \( V(\mathfrak{c}(w)) \).

One problem of Xi’s construction is that the isomorphism \( J_\mathfrak{c} \simeq \text{Mat}_{\lambda \times \lambda}(\mathcal{R}(F_\lambda)) \) is not canonical; it depends on the choice of the identification of left cells using star operation and multiplication by \( \omega \). Here we propose a more canonical construction of such an isomorphism using AMBC.

Definition 7.1. For an element \( w \in \widetilde{S}_n \) such that \( \Phi(w) = (P, Q, \overline{p}) \), we denote \( t_w \) by \( t(P, Q, \overline{p}) \).

Our claim is that there exists an isomorphism such that \( t_{P,Q} \) corresponds to an “elementary matrix”. The main result of this section is the following theorem.

Theorem 7.2. Let us label the left cells in \( \mathfrak{c} = \mathfrak{c}_\lambda \) by \( \Gamma_1, \Gamma_2, \ldots, \Gamma_\lambda \). Then there exists an algebra isomorphism \( \Upsilon' = \Upsilon_\lambda : J_\mathfrak{c} \rightarrow \text{Mat}_{\lambda \times \lambda}(\mathcal{R}(F_\lambda)) \) such that if \( w \in \Gamma_j \cap \Gamma_\lambda \) and \( \Phi(w) = (P, Q, \overline{s}_{P,Q} + \overline{p}) \), then \( \Upsilon'(t_w) \) is the matrix whose \((i, j)\)-entry is \( V(\text{rev}_\lambda(\overline{p})) \) and other entries are zero.

It is clear that \( \Upsilon \) gives a well-defined isomorphism of abelian groups. Therefore, the theorem is true if for any \( \overline{p}, \overline{p}' \in \text{rev}_\lambda(\text{Dom}(F_\lambda)) \) we have

\[
t(P, Q, \overline{s}_{P,Q} + \overline{p}) \cdot t(Q', R, \overline{s}_{Q', R} + \overline{p}') = \delta_{Q, Q'} \sum_{\overline{p}''} m_{\overline{p}, \overline{p}', \overline{p}''} t(P, R, \overline{s}_{P, R} + \overline{p}'')
\]

where \( V(\text{rev}_\lambda(\overline{p})) \otimes V(\text{rev}_\lambda(\overline{p}'')) \simeq \bigoplus_{\overline{p}''} V(\text{rev}_\lambda(\overline{p}''))^{m_{\overline{p}, \overline{p}', \overline{p}''}}. \) This equation holds when \( Q \neq Q' \) by \cite{Lus87} Corollary 1.9. Thus we may assume that \( Q = Q' \), i.e. we only need to show that

\[
t(P, Q, \overline{s}_{P,Q} + \overline{p}) \cdot t(Q, R, \overline{s}_{Q,R} + \overline{p}') = \sum_{\overline{p}''} m_{\overline{p}, \overline{p}', \overline{p}''} t(P, R, \overline{s}_{P, R} + \overline{p}'')
\]

Example 7.3. Let \( n = 9 \) and \( w = [-1, 3, 10, -5, 14, -3, 18, 7, 2] \). Then \( \Phi(w) = (P, Q, \overline{s}_{P,Q} + \overline{p}) \), where \( P = \begin{array}{ccc} 2 & 4 & 6 \\ 7 & 8 & 9 \end{array} \), \( Q = \begin{array}{ccc} 3 & 5 & 7 \\ 1 & 2 & 8 \end{array} \), \( \overline{s}_{P,Q} = (0, -2, -1) \), and \( \overline{p} = (0, 1, 2) \). Let \( w' = \begin{array}{ccc} 2 & 4 & 6 \\ 7 & 8 & 9 \end{array} \).
Then $\Phi(w') = (Q,R, \vec{s}_{Q,R} + \rho')$, where $R = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 7 & 9 \\ 1 & 3 & 5 \end{bmatrix}$ and $\rho' = (0,0,2)$. Tensoring $GL_3$ representations with highest weights $(2,1,0)$ and $(2,0,0)$ we get

$$V(2,1,0) \otimes V(2,0,0) = V(4,1,0) \oplus V(3,2,0) \oplus V(3,1,1) \oplus V(2,2,1).$$

Taking into account that $\vec{s}_{P,R} = (0,1,-1)$, and $\rho' = (0,0,2)$. Thus by Lemma 6.2, it suffices to show that

$$\begin{bmatrix} 7 & 8 & 10 \\ -5 & -3 & 19 \\ -10 & 14 & 23 \end{bmatrix}.$$

For example, $(0,0,2) + (0,1,4) = (0,0,2)$ and $\Phi^{-1}((P,R,(0,0,2))) = [-7,3,-5,18,19,-3,7,23,8]$. On the other hand $t_{w'} \cdot t_w = 0$ since $R \neq P$.

The rest of this section is devoted to the proof of Equation [7.1]. We start with the following lemma.

**Lemma 7.4.** For some $u,v \in \mathcal{S}_n$, suppose that $t_u t_v = \sum_{w \in \mathcal{S}_n} \gamma_{u,v,w}^{-1} t_w$.

1. If $v^*$ is well-defined for some $* \sim \tilde{i}$, then $v^*$ is also well-defined when $\gamma_{u,v,w}^{-1} \neq 0$, and we have $t_u t_{v^*} = \sum_{w \in \mathcal{S}_n} \gamma_{u,v,w}^{-1} t_{v^*}$.

2. If $u^*$ is well-defined for some $* \sim \tilde{i}$, then $u^*$ is also well-defined when $\gamma_{u,v,w}^{-1} \neq 0$, and we have $t_{u^*} t_v = \sum_{w \in \mathcal{S}_n} \gamma_{u,v,w}^{-1} t_{u^*} t_v$.

3. Suppose that $t_u t_v \neq 0$. Then for some $* \sim \tilde{i}$, $u^*$ is well-defined if and only if $v^*$ is well-defined. In this case we have $t_{u^*} t_v = \sum_{w \in \mathcal{S}_n} \gamma_{u,v,w}^{-1} t_w$.

4. For any $i,j,k \in \mathbb{Z}$, we have $t_{w^i u^j w^{-k}} = \sum_{w \in \mathcal{S}_n} \gamma_{u,v,w}^{-1} t_{w^i u^j w^{-k}}$.

**Proof.** The first three statements follows from [X02 Theorem 1.6.2]. The last one follows from that $t_{w^i} = t_{w^j} t_{w^j}$, $t_{w^j} = t_{w^j} t_{w^j}$, and $t_{w^i} = t_{w^i} t_{w^i}$. \( \square \)

Now suppose that one of $\rho', \rho'' \in \text{rev}_\lambda(\text{Dom}(F_\lambda))$ is determinantal. Then we know that

$$V(\text{rev}(\rho')) \otimes V(\text{rev}(\rho'')) \simeq V(\text{rev}(\rho') + \text{rev}(\rho'')).$$

Here we prove [7.1] in an analogous situation.

**Lemma 7.5.** Suppose that one of $\rho', \rho'' \in \text{rev}_\lambda(\text{Dom}(F_\lambda))$ is determinantal. Then

$$t(P,Q, \vec{s}_{P,Q} + \rho') \cdot t(Q,R, \vec{s}_{Q,R} + \rho'') = t(P,R, \vec{s}_{P,R} + \rho + \rho').$$

In particular, we have $t_{P,Q} t_{Q,R} = t_{P,R}$.

**Proof.** For tabloids $P,Q,R$ of the same shape we denote by $\Psi(P,Q,R)$ the statement that the equation above is true for the triple $(P,Q,R)$ and any $\rho, \rho''$ such that one of $\rho, \rho''$ is determinantal. The proof of this lemma consists of three steps.

1. $\Psi(T,T,T)$ holds for any tabloid $T$. We already know that $\Psi(T_{\lambda^a}, T_{\lambda^a}, T_{\lambda^a})$ holds by Theorem 4.2 and [X02 Theorem 8.2.1]. Thus by Lemma 6.2, it suffices to show that

$$\Psi(T,T,T) \Rightarrow \Psi(T^*, T^*, T^*), \Psi(\omega(T), \omega(T), \omega(T))$$

where $T^*$ is well-defined for some $* \sim \tilde{i}$. This follows from Proposition 5.9 and Lemma 7.4.
(2) $\mathcal{P}(T, T, T')$ holds for any tabloid $T$ and $T'$ of the same shape. We know that it holds when $T = T'$ by the first step. Thus again by Lemma 6.2 it suffices to show that

$$\mathcal{P}(T, T, T') \Rightarrow \mathcal{P}(T, T, T'), \mathcal{P}(T, T', \omega(T')).$$

where $T''$ is well-defined for some $* \sim \overline{7}$. We again use Proposition 5.5, 5.11 and Lemma 7.4. First of all if $* \not\sim \overline{7}$, then from

$$t(T, T, \overline{7}) \cdot t(T, T', \overline{7}, T'') = t(T, T, \overline{7} + \overline{7}, T' + \overline{7})$$

we have

$$t(T, T, \overline{7}) \cdot t(T, T', \overline{7}, T'') = t(T, T, \overline{7} + \overline{7}, T' + \overline{7})$$

thus $\mathcal{P}(T, T, T'')$ holds. Now if $* \sim \overline{7}$ then we have

$$t(T, T, \overline{7}) \cdot t(T, T', \overline{7}, T'') = t(T, T, \overline{7} + \overline{7}, T' + \overline{7})$$

Thus if we replace $\overline{7}$ by $\overline{7} + \overline{7}$, then we see that $\mathcal{P}(T, T, T')$ also holds in this case. Note that $\overline{7} + \overline{7}$ is determinantal, thus one of $\overline{7}$ is determinantal if and only if one of $\overline{7}, \overline{7}$ is determinantal. Finally, we also have

$$t(T, T, \overline{7}) \cdot t(T, \omega(T'), \overline{7} + \overline{7} + \overline{7} - \overline{7} + \overline{7} - \overline{7}) = t(T, \omega(T'), \overline{7} + \overline{7} + \overline{7} - \overline{7} + \overline{7} - \overline{7}).$$

Thus if we replace $\overline{7}$ by $\overline{7} + \overline{7}$, then it follows that $\mathcal{P}(T, T, \omega(T'))$ holds.

(3) $\mathcal{P}(P, Q, R)$ holds for any tabloids $P, Q, R$ of the same shape. Similarly to above, it suffices to show that

$$\mathcal{P}(P, Q, R) \Rightarrow \mathcal{P}(P, Q^*, R), \mathcal{P}(P, \omega(Q), R)$$

where $Q^*$ is well-defined for some $* \sim \overline{7}$. We again use Proposition 5.5, 5.11 and Lemma 7.4. If $* \not\sim \overline{7}$, then from

$$t(P, Q, P, Q + \overline{7}) \cdot t(Q, R, Q + \overline{7}) = t(P, R, Q + \overline{7} + \overline{7})$$

we have

$$t(P, Q, P, Q + \overline{7}) \cdot t(Q, R, Q + \overline{7}) = t(P, R, Q + \overline{7} + \overline{7})$$

which implies $\mathcal{P}(P, Q^*, R)$. On the other hand, if $* \sim \overline{7}$ then we have

$$t(P, Q, P, Q + \overline{7}) \cdot t(Q, R, Q + \overline{7}) = t(P, R, Q + \overline{7} + \overline{7})$$

Thus by replacing $\overline{7}$ and $\overline{7}$ with $\overline{7} + \overline{7}$, we see that $\mathcal{P}(P, Q^*, R)$ holds. Finally, we also have

$$t(P, \omega(Q), \overline{7} + \overline{7}) \cdot t(Q, R, \overline{7} + \overline{7}) = t(P, \omega(Q), R + \overline{7} + \overline{7})$$

Thus by replacing $\overline{7}$ and $\overline{7}$ with $\overline{7} + \overline{7}$, respectively, we see that $\mathcal{P}(P, \omega(Q), R)$ holds.

The lemma is proved. \hfill $\square$

Now we prove (7.4) in the case when $P = Q = R$ without any restriction on $\overline{7}$ or $\overline{7}$.
Lemma 7.6. For any tabloid $T$ we have
\[ t(T, T, \bar{\tau}) \cdot t(T, T, \bar{\tau}') = \sum_{\bar{\tau}''} m_{\bar{\tau}, \bar{\tau}', \bar{\tau}''} t(T, T, \bar{\tau}'') \]
where $V(\text{rev}_\lambda(\bar{\tau})) \otimes V(\text{rev}_\lambda(\bar{\tau}')) \simeq \bigoplus_{\bar{\tau}''} V(\text{rev}_\lambda(\bar{\tau}''))^{\oplus m_{\bar{\tau}, \bar{\tau}', \bar{\tau}''}}$.

Proof. By Theorems 4.2 and [X02, Theorem 8.2.1], at least we know that the statement holds when $T = T^n$. Now it follows from Propositions 5.5, 5.11 andLemma 7.4.

Proof of Theorem 7.2. Here we prove that equation (7.1) holds. By Lemma 7.5 we have
\[
\sum_{\bar{\tau}''} m_{\bar{\tau}, \bar{\tau}', \bar{\tau}''} t(P, P, \bar{\tau}'') \cdot t(P, R, \bar{s}_{P, R})
\]
which is the same as
\[
\sum_{\bar{\tau}''} m_{\bar{\tau}, \bar{\tau}', \bar{\tau}''} t(P, P, \bar{\tau}'') \cdot t(P, P, \bar{P} + \bar{\tau}'')
\]
by Lemma 7.6. Again by Lemma 7.5 it is equal to
\[
\sum_{\bar{\tau}''} m_{\bar{\tau}, \bar{\tau}', \bar{\tau}''} t(P, P, \bar{s}_{P, R} + \bar{\tau}''),
\]
which is what we want to prove.

\[ \square \]

8. Equality of two Lusztig-Vogan bijections

Let $G$ be a reductive group over $\mathbb{C}$. In [Lus89], Lusztig defined a conjectural bijection between $\text{Dom}(G)$ and
\[ \mathcal{O} := \{(N, \rho) \mid N \in \text{Lie } G, \rho \in \text{Dom}(F_N)\}/G \]
where $F_N$ is the reductive part of the stabilizer $\text{Z}_G(N)$ of $N \in \text{Lie } G$ and the quotient by $G$ is with respect to adjoint action on $\text{Lie } G$. Later, a similar bijection is also independently conjectured by Vogan [Vog91]. We call such a correspondence the Lusztig-Vogan bijection.

In this section we focus on the case when $G = GL_n(\mathbb{C})$. Then we may and shall identify $\mathcal{O}$ with $\bigsqcup_{\lambda \in \mathbb{C}} \text{Dom}(F_\lambda)$. Let us describe some properties of this bijection as described in [Lus87a, 10.8]. Suppose that $\Theta : \text{Dom}(GL_n) \rightarrow \mathcal{O} = \bigsqcup_{\lambda \in \mathbb{C}} \text{Dom}(F_\lambda)$ is such a bijection. We regard $\text{Dom}(GL_n)$ as a subset of $\widetilde{S_n}$ by identifying $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ with $w_\mu := [n\mu_1 + 1, n\mu_2 + 2, \ldots, n\mu_n + n] \in \widetilde{S_n}$. Then we have a decomposition of $\widetilde{S_n}$ into double $\mathcal{S_n}$ cosets
\[ \widetilde{S_n} = \bigsqcup_{\mu \in \text{Dom}(GL_n)} \mathcal{S_n}w_\mu \mathcal{S_n}. \]
In particular, there exists an one-to-one correspondence between $\text{Dom}(GL_n)$ and double $\mathcal{S_n}$-cosets of $\widetilde{S_n}$, i.e. $\text{Dom}(GL_n) \simeq \mathcal{S_n} \setminus \widetilde{S_n}/\mathcal{S_n}$. 
Recall that $\tilde{S}_{nf}$ is defined to be the set of minimal length elements in each left $S_n$-coset. We define $j_Snf:=(S_{nf})^{-1} \cap S_{nf}$, i.e. the set of minimal length elements in each double $S_n$-coset. Then clearly we have a bijection between $j_Snf$ and double $S_n$-cosets in $S_n$, i.e. $j_Snf \simeq S_n \backslash \tilde{S}_n / S_n$.

On the other hand, from the result of Section 3 we have $j_Snf = \bigsqcup_{\lambda \vdash n} (\Gamma_\lambda^{can})^{-1} \cap \Gamma_\lambda^{can}$. Therefore, we have a chain of bijections

$$\tilde{\Theta}: \bigsqcup_{\lambda \vdash n} (\Gamma_\lambda^{can})^{-1} \cap \Gamma_\lambda^{can} = j_Snf \rightarrow S_n \backslash \tilde{S}_n / S_n \rightarrow \text{Dom}(GL_n) \overset{\Theta}{\rightarrow} \bigsqcup_{\lambda \vdash n} \text{Dom}(F_\lambda).$$

Now [Lus87a, 10.8] asserts that $\tilde{\Theta}$ should satisfy that $\tilde{\Theta}(\Gamma_\lambda^{can})^{-1} \cap \Gamma_\lambda^{can} = \text{Dom}(F_\lambda)$, and furthermore it induces an algebra isomorphism

$$\mathcal{J}(\Gamma_\lambda^{can})^{-1} \cap \Gamma_\lambda^{can} \simeq \mathcal{R}(F_\lambda) : t_w \mapsto V(\tilde{\Theta}(w)).$$

We already have one candidate deduced from the result of Section 4. Indeed, recall that there is an algebra isomorphism $\Upsilon_\lambda: \mathcal{J}_\lambda \rightarrow \text{Mat}_\chi \chi(\mathcal{R}(F_\lambda))$ which restricts to an isomorphism $\Upsilon_\lambda: \mathcal{J}(\Gamma_\lambda^{can})^{-1} \cap \Gamma_\lambda^{can} \rightarrow \mathcal{R}(F_\lambda)$. It sends $t_w$ for $w \in (\Gamma_\lambda^{can})^{-1} \cap \Gamma_\lambda^{can}$ to $\text{rev}(\tilde{\rho}(w)) \in \text{Dom}(F_\lambda)$. Let us define

$$\tilde{\Theta}_1: \bigsqcup_{\lambda \vdash n} (\Gamma_\lambda^{can})^{-1} \cap \Gamma_\lambda^{can} \rightarrow \bigsqcup_{\lambda \vdash n} \text{Dom}(F_\lambda)$$

be the disjoint union of such bijections. Then it is clear that $\tilde{\Theta}_1$ satisfies the properties of $\tilde{\Theta}$ above.

In other words, the composition

$$\Theta_1: \text{Dom}(GL_n) \simeq S_n \backslash \tilde{S}_n / S_n \simeq \bigsqcup_{\lambda \vdash n} (\Gamma_\lambda^{can})^{-1} \cap \Gamma_\lambda^{can} \overset{\tilde{\Theta}_1}{\rightarrow} \bigsqcup_{\lambda \vdash n} \text{Dom}(F_\lambda) = \mathcal{O}$$

is a Lusztig-Vogan bijection.

There is another realization of such a bijection studied by Achar, Bezrukavnikov, Ostrik, and Rush. First of all, for a reductive group $G$ over $\mathbb{C}$, Ostrik established such a bijection using a $G \times \mathbb{C}^*$-equivariant $K$-theory of the nilpotent cone of $\text{Lie} G$ [Ost00]. Also, Bezrukavnikov defined two bijections in terms of a bounded derived category of $G$-equivariant coherent sheaves on the nilpotent cone of $\text{Lie} G$ [Bez03] and a tensor category attached to each two-sided cell in the affine Weyl group of $G$ [Bez04]. Later, it is proved that they are indeed the same as one another [Bez09].

For $G = GL_n(\mathbb{C})$, Achar also described such a bijection in a combinatorial way [Ach01] and proved that this bijection is the same as the one defined by Bezrukavnikov and Ostrik [Ach04]. Later, his method is simplified by Rush [Rus17a, Rus17b]. From now on we denote this bijection by $\Theta_2 : \text{Dom}(GL_n(\mathbb{C})) \rightarrow \mathcal{O}$.

Note that the bijection $\tilde{\Theta}_2: \bigsqcup_{\lambda \vdash n} (\Gamma_\lambda^{can})^{-1} \cap \Gamma_\lambda^{can} \rightarrow \bigsqcup_{\lambda \vdash n} \text{Dom}(F_\lambda)$ induced from $\Theta_2$ satisfies the properties that $\tilde{\Theta}$ above possesses. Indeed, $\tilde{\Theta}_2$ is derived from the equivalence of two tensor categories $\text{Rep}(Z_\infty) \rightarrow A_\infty$ as in [Bez09, 1.7]. (Here $Z_\infty$ is the stabilizer of a nilpotent element $N \in \text{Lie} GL_n(\mathbb{C})$ whose orbit corresponds to $\infty$.) If we take the $K$-theory of this equivalence, then we have an algebra isomorphism $\text{R}(F_\infty) \simeq \text{R}(Z_\infty) \rightarrow \mathcal{J}(\Gamma_\infty^{can})^{-1} \cap \Gamma_\infty^{can}$ which sends $\tilde{\rho} = \tilde{\Theta}_2(w)$ to $t_w$ for $w \in (\Gamma_\infty^{can})^{-1} \cap \Gamma_\infty^{can}$, from which the claim follows. (Since $F_\infty$ is the reductive part of $Z_\infty$, we may identify $\text{R}(F_\infty)$ and $\text{R}(Z_\infty)$ canonically.)

We claim that these two realizations of the Lusztig-Vogan bijection coincide, i.e.
Theorem 8.1. We have $\Theta_1 = \Theta_2$, or equivalently $\tilde{\Theta}_1 = \tilde{\Theta}_2$.

Let us show how this can be used to compute Lusztig-Vogan bijection in practice.

Example 8.2. Let $n = 7$ and take a dominant weight $\tilde{\mu} = (5, 1, 1, 1, -2, -2, -2) \in \text{Dom}(GL_7)$. Then $w_{\tilde{\mu}} = [36, 9, 10, 11, -9, -8, -7]$. Rearranging this window in increasing order we find the element $[-9, -8, -7, 9, 10, 11, 36] \in j\tilde{S}_{7f}$ in the same double $S_f$-coset $S_f \tilde{S}_7/S_7$ as $w_{\tilde{\mu}}$. Applying to it the AMBC construction we get $(T, T, (-2, 1, 3))$, where $T = \begin{bmatrix} 5 & 6 & 7 \\ 2 & 3 & 4 \\ 1 & 0 & 0 \end{bmatrix} = T_{(3,3,1)}^{\text{can}}$ is a canonical tableau, as expected. We conclude that the Lusztig-Vogan bijection maps $\tilde{\mu}$ to $\tilde{\rho} = (5, 1, 1, 1, -2, -2, -2)$ to the pair $(\lambda, \tilde{\rho})$, where $\lambda = (3,3,1)$ and $\tilde{\rho} = (1, -2, 3) \in \text{Dom}(F_\lambda)$.

Example 8.3. Let $n = 7$, take nilpotent orbit corresponding to $\lambda = (2, 2, 1, 1, 1)$, and take $\tilde{\rho} = (0, 0, 1, 0, -1) \in \text{Dom}(F_\lambda)$. We wish to apply the inverse Lusztig-Vogan bijection to the pair $(\lambda, \tilde{\rho})$. We have $T_{(2,2,1,1,1)}^{\text{can}} = \begin{bmatrix} 6 & 7 \\ 3 & 5 \\ 1 & 1 \end{bmatrix}$, and applying the inverse AMBC construction to the triple $(T_{(2,2,1,1,1)}^{\text{can}}, T_{(2,2,1,1,1)}^{\text{can}}, (0, 0, -1, 0, 1))$ we get $[-28, -8, -2, 4, 10, 16, 36] \in j\tilde{S}_{7f}$. Rearranging this window in the order increasing modulo 7 we get $w = [36, 16, 10, 4, -2, -8, -28]$. This happens to be $w_{\tilde{\mu}}$ for $\tilde{\mu} = (5, 2, 1, 0, -1, -2, -5) \in \text{Dom}(GL_7)$, which is the desired answer.

9. Proof of Theorem 8.1

This section is devoted to the proof of Theorem 8.1. We start with the following lemma.

Lemma 9.1. For $\lambda = (1^{m_1} 2^{m_2} \ldots) \vdash n$, we identify $\text{Dom}(F_\lambda) = \prod_{i \geq 1} \text{Dom}(GL_{m_i})$ and define $\tilde{\rho}_\lambda(r, s) = (\rho_i)_{i \geq 1} \in \text{Dom}(F_\lambda)$ to be $\rho_i = 0$ if $i \neq r$ and $\rho_r = (s, 0, \ldots, 0)$. Set $Y_\lambda := \{0\} \sqcup \{\tilde{\rho}_\lambda(r, s) \in \text{Dom}(F_\lambda) \mid r \in \lambda, s \geq 1\}$.

If $\Theta_1^{-1}$ and $\Theta_2^{-1}$ coincide on $\bigsqcup_{\lambda \vdash n} Y_\lambda \subset \mathcal{O}$, then $\Theta_1 = \Theta_2$.

Proof. Let $\Lambda_m$ be the ring of symmetric Laurent polynomials of $m$ variables. Then $\Lambda_m = \mathbb{Z}[e_1, e_2, \ldots, e_{m-1}, e_m]$ where $e_i$ are elementary symmetric functions. Thus an algebra automorphism of $\Lambda_m$ stabilizing $e_1, e_2, \ldots, e_{m-1}, e_m$ is the identity on $\Lambda_m$. Since $\mathbb{Z}[e_1, e_2, \ldots, e_{m-1}, e_m] = \mathbb{Z}[h_1, h_2, \ldots, h_{m-1}, h_m]$ where $h_i$ are complete homogeneous symmetric functions, an algebra automorphism of $\Lambda_m$ stabilizing $h_1, h_2, \ldots, h_{m-1}, h_m$ is also the identity on $\Lambda_m$.

Since there exists a ring isomorphism $\mathcal{R}(GL_m) \simeq \Lambda_m$ which sends $V(s, 0, 0, \ldots)$ to $h_s$ (see e.g. [Sta86], Chapter 7: Appendix 2]), if we identify $\mathcal{R}(F_\lambda)$ with $\prod_{i \geq 1} \mathcal{R}(GL_{m_i}) \simeq \prod_{i \geq 1} \Lambda_{m_i}$, then an algebra automorphism of $\mathcal{R}(F_\lambda)$ stabilizing each element in $Y_\lambda$ is the identity. Now the lemma follows from the fact that $\mathcal{R}(F_\lambda) \to \mathcal{R}(F_\lambda) : V(\tilde{\rho}) \mapsto V((\Theta_2 \circ \Theta_1^{-1}))(\tilde{\rho})$ is an algebra automorphism by the argument in the previous section.

Remark. Indeed, the proof is still valid if we replace $Y_\lambda$ by $Y_\lambda := \{\tilde{\rho}_\lambda(r, s) \in \text{Dom}(F_\lambda) \mid r \in \lambda, 1 \leq s \leq m_r\}$.
For a partition $\lambda$, we define $\mathfrak{M}_\lambda(0)$ to be the Young tableau of shape $\lambda$ whose $i$-th column is filled with $\lambda'_i - 1, \lambda'_i - 3, \ldots, 3 - \lambda'_i, 1 - \lambda'_i$ from top to bottom. For example, we have

$$\mathfrak{M}_{(3,3,2,1)}(0) = \begin{pmatrix} 4 & 3 & 1 \\ 2 & 1 & -1 \\ 0 & -1 \\ -2 & -3 \\ -4 \end{pmatrix}.$$ 

Also, when $i \in \lambda$ we define the Young tableau $\mathfrak{M}_\lambda(r, s)$ of shape $\lambda$ as follows. The entries of $\mathfrak{M}_\lambda(r, s)$ are the same as those of $\mathfrak{M}_\lambda(0)$ except $(1, i)$-entries for $1 \leq i \leq r$. Let us write $a_i, b_i$ to be the $(1, i)$-th entry of $\mathfrak{M}_\lambda(0)$ and $\mathfrak{M}_\lambda(r, s)$, respectively, for $1 \leq i \leq r$. Then $b_i$ are uniquely determined by the following conditions.

- $a_i \leq b_i$ for $1 \leq i \leq r$.
- $b_1 \geq b_2 \geq \cdots \geq b_r$.
- $\sum_{i=1}^r b_i = s + \sum_{i=1}^r a_i$.
- $\sum_{i=1}^r b_i^2$ is the minimum among the choice of $b_i$ satisfying the properties above.

For example, we have

$$\mathfrak{M}_{(3,3,2,1)}(2, 5) = \begin{pmatrix} 6 & 6 & 1 \\ 2 & 1 & -1 \\ 0 & -1 \\ -2 & -3 \\ -4 \end{pmatrix}.$$ 

We define $\mu_\lambda(0)$ (resp. $\mu_\lambda(r, s)$) to be a dominant weight of $GL_n$ consisting of entries of $\mathfrak{M}_\lambda(0)$ (resp. $\mathfrak{M}_\lambda(r, s)$). For example we have $\mu_{(3,3,2,1)}(0) = (4, 3, 2, 1, 1, 0, -1, -1, -2, -3, -4)$ and $\mu_{(3,3,2,1)}(2, 5) = (6, 6, 2, 1, 1, 0, -1, -1, -2, -3, -4)$.

**Lemma 9.2.** $\Theta_2(\mu_\lambda(0)) = 0$ and $\Theta_2(\mu_\lambda(r, s)) = \rho_\lambda(r, s)$.

**Proof.** We briefly review some properties of $\Theta_2$ following [Rus17b]. For $\lambda \vdash n$ and $1 \leq j \leq \lambda_1$, it is clear that $\sum_{i \geq j} m_i = \lambda'_j$. Thus we may consider a standard block diagonal embedding $\zeta: \prod_{i \geq j} GL_{m_i} \to GL_{\lambda'_j}$. Let $\zeta = \prod_j \zeta_j := \prod_{1 \leq j \leq \lambda_1} (\prod_{i \geq j} GL_{m_i} \to GL_{\lambda'_j})$ be a product of such morphisms. On the other hand, we also define $\xi_j : GL_{m_i} \to (GL_{m_i})^i$ to be the diagonal embedding, and let $\xi = \prod_{i \in \lambda} \xi_i : \prod_{i \in \lambda} (GL_{m_i} \to (GL_{m_i})^i)$ be their product. Then since $\prod_{i \in \lambda} (GL_{m_i})^i = \prod_{i \geq j} GL_{m_i} \to \prod_{1 \leq j \leq \lambda_1} (\prod_{i \geq j} GL_{m_i})$, the composition $\zeta \circ \xi : \prod_{i \in \lambda} GL_{m_i} \to \prod_{1 \leq j \leq \lambda_1} GL_{\lambda'_j}$ is well-defined. If we let $L_\lambda$ to be $\prod_{1 \leq j \leq \lambda_1} GL_{\lambda'_j}$, then $\zeta \circ \xi$ defines a morphism $F_\lambda \simeq \prod_{i \in \lambda} GL_{m_i} \to L_\lambda$. For example, if $\lambda = (4, 2, 1, 1)$ then $\zeta \circ \xi : GL_2 \times GL_1 \times GL_2 \to GL_5 \times GL_3 \times GL_2 \times GL_2$ is defined by

$$(A, B, C) \mapsto \left( \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}, \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, A, A \right).$$
Let us investigate \((\zeta \circ \xi)^*: \mathcal{R}(L_\lambda) \to \mathcal{R}(F_\lambda)\) in terms of Laurent symmetric functions. Recall that 
\(\mathcal{R}(GL_m) \simeq \Lambda_m\) where \(\Lambda_m\) is the ring of symmetric Laurent polynomials of \(m\) variables. Therefore we may identify \(\mathcal{R}(F_\lambda)\) with \(\bigotimes_i \Lambda(\vec{x}_i)\) where \(\Lambda(\vec{x}_i)\) is the ring of Laurent symmetric functions with variables \(\vec{x}_i = (x_{i1}, x_{i2}, \ldots, x_{im_i})\). Similarly, we identify \(\mathcal{R}(L_\lambda)\) with \(\bigotimes_j \Lambda(\vec{y}_j)\) where \(\Lambda(\vec{y}_j)\) is the ring of Laurent symmetric functions with variables \(\vec{y}_j = (y_{j1}, y_{j2}, \ldots, y_{j\lambda_j})\). Then direct calculation shows that \((\zeta \circ \xi)^* f(\vec{y}_j) \in \bigotimes_j \Lambda(\vec{y}_j)\) to \(f(\vec{x}_i)\) for \(i \geq j\). Furthermore, if \(P \in \bigotimes_j \Lambda(\vec{y}_j)\) is homogeneous in \(\bigcup_j \vec{y}_j\) then the image \((\zeta \circ \xi)^*(P)\) is also homogeneous in \(\bigcup_i \vec{x}_i\) and \((\zeta \circ \xi)^*\) is degree-preserving. For example, if \(\lambda = (4, 4, 2, 1, 1)\), then
\[
(\zeta \circ \xi)^*(f(y_{21}, y_{22}, y_{23})) = f(x_{11}, x_{12}, x_{21})
\]

Now suppose that we are given \(\vec{\mu} \in \text{Dom}(F_\lambda)\). For \(\vec{\nu} \in \text{Dom}(GL_m)\), defined \(\vec{\mu}' \in \text{Dom}(L_\lambda)\) which satisfies the following conditions.

(a) As multisets, \(\vec{\nu}\) is the same as the union of parts in \(\vec{\mu}'\).

(b) Let \(\Delta_\lambda \in \text{Dom}(L_\lambda)\) be the half sum of positive roots of \(L_\lambda\). Then \(\vec{\mu}' - 2\Delta_\lambda\) is dominant.

(c) Let \(V(\vec{\mu}' - 2\Delta_\lambda)\) be the irreducible representation of \(L_\lambda\) of highest weight \(\vec{\mu}' - 2\Delta_\lambda\). Then \(V(\vec{\mu}')\) appears as an irreducible constituent of \((\zeta \circ \xi)^*(V(\vec{\mu}' - 2\Delta_\lambda))\), i.e. the restriction of \(V(\vec{\mu}' - 2\Delta_\lambda)\) to \(F_\lambda\) under \(\zeta \circ \xi\).

Let \(X_{\vec{\mu}} = \{\vec{v} \in \text{Dom}(GL_m) \mid \text{such } \vec{\mu}' \text{ exists}\}\). Then according to \cite{Ach04} and \cite{Rus77}, we have that \(\Theta_2(\vec{\mu}') = \vec{\mu}\) if and only if \(\vec{\mu} \in X_{\vec{\mu}}\) and \(\|\vec{\mu}\| = \min\{\|\vec{\mu}'\| \mid \vec{\nu} \in X_{\vec{\mu}}\}\). Also in this case \(\vec{\mu}'\) is uniquely determined.

We set \(\vec{\mu}' = \vec{\alpha} + 2\Delta_\lambda\) where \(\vec{\alpha} = (\alpha_j)_{1 \leq j \leq \lambda_1}\) is such that \(\alpha_j = (\alpha_{j1}, \alpha_{j2}, \ldots, \alpha_{j\lambda_j}) \in \text{Dom}(GL_{\lambda_j})\). Thus we have \(\alpha_{j1} \geq \alpha_{j2} \geq \cdots \geq \alpha_{j\lambda_j}\) by (b). Under the identification \(\mathcal{R}(F_\lambda) \simeq \bigotimes_i \Lambda(\vec{x}_i)\) and \(\mathcal{R}(L_\lambda) \simeq \bigotimes_j \Lambda(\vec{y}_j)\), the irreducible representation \(V(\vec{\alpha}) \in \mathcal{R}(L_\lambda)\) corresponds to \(\prod_j s_{\alpha_j}(\vec{y}_j) \in \bigotimes_j \Lambda(\vec{y}_j)\) where \(s_{\alpha_j}\) is the Schur function corresponding to \(\alpha_j\). Therefore, \((\zeta \circ \xi)^* V(\vec{\alpha}) \in \mathcal{R}(F_\lambda)\) is identified with \(\prod_j s_{\alpha_j}(\vec{x}_i) \in \bigotimes_i \Lambda(\vec{x}_i)\).

We first suppose that \(\Theta_2(\vec{\mu}') = 0\). Then (c) implies that \(\prod_j s_{\alpha_j}(\vec{x}_i) \in \bigotimes_i \Lambda(\vec{x}_i)\) has a nonzero constant term. Let \(b\) be the smallest part in \(\lambda\) then we have a factorization \(\prod_{j \leq b} s_{\alpha_j}(\vec{x}_i) = \prod_{j \geq b} s_{\alpha_j}(\vec{x}_i) \prod_{j \geq 1} s_{\alpha_j}(\vec{x}_i)\). Note that the variables \(\vec{x}_b\) does not appear in \(\prod_{j \geq b} s_{\alpha_j}(\vec{x}_i)\).

Thus for \(\prod_j s_{\alpha_j}(\vec{x}_i)\) to have a constant term, we also have that \(\vec{x}_b\) does not appear in \(\prod_{j \leq b} s_{\alpha_j}(\vec{x}_i)\).

Now direct calculation shows that it is only possible when there exists \(\beta_j \in \mathbb{Z}\) for \(1 \leq j \leq b\) such that \(\alpha_j = (\beta_j, \beta_j, \ldots, \beta_j)\) and \(\sum_{j=1}^b \beta_j = 0\). Then \(\prod_{j \leq b} s_{\alpha_j}(\vec{x}_i) = 1\) and \(\prod_j s_{\alpha_j}(\vec{x}_i) = 1\).

We iterate the above argument by enlarging \(b\) to cover all the parts in \(\lambda\) and conclude that there exists \(\beta_j \in \mathbb{Z}\) for \(1 \leq j \leq \lambda_1\) such that \(\alpha_j = (\beta_j, \beta_j, \ldots, \beta_j)\) for all \(j\) and \(\sum_{j=1}^k \beta_j = 0\) whenever \(k \in \lambda\). In this case we have \(\prod_j s_{\alpha_j}(\vec{x}_i) = 1\), thus the assumption is satisfied. On the other hand, if we let \((\Delta_\lambda)_j\) be the component of \(\Delta_\lambda\) corresponding to \(GL_{\lambda_j}\), then from the definition we have
\[
2(\Delta_{\lambda_j})_j = (\lambda'_j - 1, \lambda'_j - 3, \ldots, 3 - \lambda'_j, 1 - \lambda'_j).
\]

Therefore in order to find \(\vec{\mu}'\), we need to minimize
\[
\sum_j ((\beta_j + \lambda'_j - 1)^2 + (\beta_j + \lambda'_j - 3)^2 + \cdots + (\beta_j + 1 - \lambda'_j)^2)\]
subject to the condition $\sum_{j=1}^k \beta_j = 0$ whenever $k \in \lambda$. Now it is easy to show that the minimum is achieved when $\alpha = 0$, i.e., $\hat{\mu} = 2\Delta_\lambda$. Thus it follows that $\hat{\mu} = \hat{\mu}_\lambda(0)$ by the definition of $\mathcal{M}_\lambda(0)$.

We proceed to the case when $\Theta_2(\hat{\mu}) = \hat{\nu}_\lambda(r, s)$ and argue similarly to above. The condition (c) implies that $\langle \prod_i s_{a_i}(\mathcal{F}_{\geq j}), s_{a}(\mathcal{F}_r) \rangle \neq 0$ where $(\ , \ )$ is the pairing on $\bigotimes_k \Lambda(\mathcal{F}_k)$ induced from the standard bilinear form on the ring of (Laurent) symmetric functions. In this case it is not hard to show that there exists $\beta_j \in \mathbb{Z}$ for $1 \leq j \leq \lambda_1$ and $\gamma_j \in \mathbb{N}$ for $1 \leq j \leq r$ such that

$$\alpha_j = \begin{cases} (\beta_j + \gamma_j, \beta_j, \ldots, \beta_j) & \text{if } 1 \leq j \leq r, \\ (\beta_j, \beta_j, \ldots, \beta_j) & \text{if } r < j \leq \lambda_1, \end{cases}$$

$\sum_{j=1}^k \beta_j = 0$ whenever $k \in \lambda$, and $\sum_{j=1}^r \gamma_j = s$. Now in order to find $\hat{\mu}$, we need to minimize

$$\sum_{1 \leq j \leq r} ((\beta_j + \gamma_j + \lambda_j' - 1)^2 + (\beta_j + \lambda_j' - 3)^2 + \cdots + (\beta_j + 1 - \lambda_j'))^2$$

$$+ \sum_{r < j \leq \lambda_1} ((\beta_j + \lambda_j' - 1)^2 + (\beta_j + \lambda_j' - 3)^2 + \cdots + (\beta_j + 1 - \lambda_j'))^2$$

subject to the condition that $\sum_{j=1}^k \beta_j = 0$ whenever $k \in \lambda$ and $\sum_{j=1}^r \gamma_j = s$.

It is again easy to show that $\beta_j = 0$ for each $j$, thus it suffices to minimize $\sum_{1 \leq j \leq r} (\gamma_j + \lambda_j' - 1)^2$. Also it follows that $\alpha + 2\Delta_\lambda$ is obtained from $\hat{\mu}_\lambda(0)$ by replacing $\lambda_1' - 1, \lambda_2' - 1, \ldots, \lambda_r' - 1$ with $\gamma_1 + \lambda_1' - 1, \gamma_2 + \lambda_2' - 1, \ldots, \gamma_r + \lambda_r' - 1$. Now it is clear that $\{b_1, b_2, \ldots, b_r\} = \{\gamma_1 + \lambda_1' - 1, \gamma_2 + \lambda_2' - 1, \ldots, \gamma_r + \lambda_r' - 1\}$ by considering the defining conditions of $b_i$. (Recall that $b_i$ is the $(1, i)$-entry of $\mathcal{M}_\lambda(r, s)$.) Thus it follows that $\hat{\mu} = \hat{\mu}_\lambda(r, s)$ as desired. \hfill $\Box$

By Lemma 9.1 and 9.2, in order to prove Theorem 8.1 it suffices to show that $\Theta_1(\hat{\mu}_\lambda(0)) = 0$ and $\Theta_1(\hat{\mu}_\lambda(r, s)) = \hat{\nu}_\lambda(r, s)$. We analyze this condition more directly in terms of AMBC.

**Lemma 9.3.** Suppose that we are given $\lambda \vdash n$, $\hat{\mu} = (\mu_1, \mu_2, \ldots, \mu_n) \in \text{Dom}(GL_n)$, and $\hat{\nu} \in \text{Dom}(F_\lambda)$. Then the following are equivalent:

1. $\Theta_1(\hat{\mu}) = \hat{\nu}$.
2. Let $w = \Psi(T_{\lambda}^\lambda, T_{\lambda}^\lambda, \text{rev}_{\lambda}(\hat{\mu}))$. Then $w$ and $w_\mathcal{F}$ are in the same double $S_n$-coset.
3. Let $w = \Psi(T_{\lambda}^\lambda, T_{\lambda}^\lambda, \text{rev}_{\lambda}(\hat{\nu}))$. Then $\{w(1), w(2), \ldots, w(n)\} = \{n\mu_1+1, n\mu_2+2, \ldots, n\mu_n+n\}$.
4. Let $w = \Psi(T_{\lambda}^\lambda, T_{\lambda}^\lambda, \text{rev}_{\lambda}(\hat{\nu}))$ and $R \subset B_w$ be a set of representatives in each translation class by $(n, n)$. Then $\{D(x) \mid x \in R\} = \hat{\mu}$ as multisets, where $D(x)$ is the block diagonal defined in [CLP17, Definition 2.10].

**Proof.** (1) and (2) are equivalent by the definition of $\Theta_1$. Now if (3) holds, then since $w_\mathcal{F} = [n\mu_1+1, n\mu_2+2, \ldots, n\mu_n+n]$, there exists $\sigma \in S_n$ such that $w_\mathcal{F}\sigma = w$. Thus (2) holds. Now let us assume (2). Then $w \in (T_{\lambda}^\lambda)^{-1} \cap T_{\lambda}^\lambda$, thus it is of the minimal length in its double $S_n$-coset. On the other hand, it is easy to show that $w_\mathcal{F}$ is of the minimal length in its right $S_n$-coset. Thus there exists $\sigma \in S_n$ such that $w_\mathcal{F}\sigma = w$ which implies that $\{w(1), w(2), \ldots, w(n)\} = \{w_\mathcal{F}(1), w_\mathcal{F}(2), \ldots, w_\mathcal{F}(n)\}$. As $w_\mathcal{F} = [n\mu_1+1, n\mu_2+2, \ldots, n\mu_n+n]$ (3) holds.
If we assume (3), then (4) clearly holds. Now we assume (4). Let \( \vec{\mu}' = (\mu'_1, \mu'_2, \ldots, \mu'_n) \in \text{Dom}(GL_n) \) be such that \( w \) and \( w_{\vec{\mu}'} \) are in the same double \( S_n \)-coset, which always exists by the bijection \( \text{Dom}(GL_n) \simeq S_n \backslash S_n / S_n \simeq \mathcal{L}_{2,1}(\Gamma^\text{can})^{-1} \cap \Gamma^\text{can} \). As (2) implies (3), we have \( \{w(1), w(2), \ldots, w(n)\} = \{n\mu'_1+1, n\mu'_2+2, \ldots, n\mu'_n+n\} \). But then \( \{D(w(1)), D(w(2)), \ldots, D(w(n))\} = \vec{\mu}' \) as multisets, thus \( \vec{\mu} = \vec{\mu}' \) by assumption and thus (3) holds.

Therefore, in order to prove Theorem 8.1, it suffices to find a set of representative \( R \subset B_w \) in each translation class by \( (n, n) \) such that

\[
\{D(x) \mid x \in R\} = \begin{cases} 
\vec{\mu}_\lambda(r, s) & \text{when } w = \Psi(T^\text{can}_\lambda, T^\text{can}_\lambda, \text{rev}_\lambda(\vec{\rho}_\lambda(r, s))), \\
\vec{\mu}_\lambda(0) & \text{when } w = \Psi(T^\text{can}_\lambda, T^\text{can}_\lambda, 0).
\end{cases}
\]

From now on we proceed by induction on the number of rows in \( \lambda \). We first consider one-row case, i.e. \( \lambda = (n) \). For \( s \geq 0 \), there exist \( a, b \in \mathbb{N} \) such that \( 0 \leq b < n \) and \( s = an+b \). Then \( \vec{\mu}_\lambda(n, s) \) (when \( s > 0 \)) consists of \( b \) numbers of \((a+1)\)'s and \((n-b)\) numbers of \( a \)'s. Thus by Lemma 9.3, it suffices to show that

\[
\{w(1), w(2), \ldots, w(n)\} = \{(a+1)i + 1 \mid 1 \leq i \leq b\} \cup \{an + i \mid b < i \leq n\} = \{s+1, s+2, \ldots, s+n\}
\]

where \( w = \Psi(T^\text{can}_\lambda, T^\text{can}_\lambda, (s)) \). But this is obvious because \( w = \omega^s \) in this case.

Now suppose that we are given \( \lambda \vdash n \) such that \( l(\lambda) \geq 2 \) and assume that Theorem 8.1 is true for partitions of \( \lambda \vdash l(\lambda) \) parts. We set \( w = \Psi(T^\text{can}_\lambda, T^\text{can}_\lambda, \text{rev}_\lambda(\vec{\rho})) \) for \( \vec{\rho} = \vec{\rho}_\lambda(r, s) \) or \( \vec{\rho} = 0 \) and let \( \vec{\mu} \in \text{Dom}(GL_n) \) such that \( \Theta_1(\vec{\mu}) = \vec{\rho} \). Also we let \( n' := n - \lambda_1, \lambda := (\lambda_2, \lambda_3, \ldots) \vdash n' \) and \( w' := \text{fw}(w) \). Then there exists \( \vec{\rho}' \in \text{Dom}(\check{F}_\lambda) \) such that \( w' = \Psi(T^\text{can}_\lambda, T^\text{can}_\lambda, \vec{\rho}') \) as a partial permutation. Indeed, \( \vec{\rho}' \) is defined such that \( \text{rev}_\lambda(\vec{\rho}') \) is obtained by removing the first entry from \( \text{rev}_\lambda(\vec{\rho}) \). Thus in particular we have

\[
\vec{\rho}' = \begin{cases} 
\vec{\rho}_\lambda(r, s) & \text{if } \vec{\rho} = \vec{\rho}_\lambda(r, s) \text{ where } r \neq \lambda_1 \text{ or } r = \lambda_1 = \lambda_2, \\
0 & \text{otherwise}.
\end{cases}
\]

Let \( \vec{\mu}' \in \text{Dom}(GL_n) \) such that \( \Theta_1(\vec{\mu}') = \vec{\rho}' \). Then by induction assumption, \( \vec{\mu}' = \vec{\mu}_\lambda(0) \) if \( \vec{\rho}' = 0 \) and \( \vec{\mu}' = \vec{\mu}_\lambda(r, s) \) if \( \vec{\rho}' = \vec{\rho}_\lambda(r, s) \). Also, for some \( r > 0 \) we have

\[
\mathfrak{s}(w) = \begin{cases} 
\mathfrak{s}_t(a \{n'+1, n'+2, \ldots, n\}, \{n'+1, n'+2, \ldots, n\}) & \text{if } \vec{\rho} = \vec{\rho}_\lambda(\lambda_1, s) \text{ and } \lambda_1 > \lambda_2, \\
\mathfrak{s}_0(a \{n'+1, n'+2, \ldots, n\}, \{n'+1, n'+2, \ldots, n\}) & \text{otherwise}.
\end{cases}
\]

(See [CPY13] Definition 3.26) for the definition of \( \mathfrak{s}(w) \).

Let us first consider the case when \( \mathfrak{s}(w) = \mathfrak{s}_t(a \{n'+1, n'+2, \ldots, n\}, \{n'+1, n'+2, \ldots, n\}) \) for some \( s > 0 \) (and thus \( \vec{\rho}' = 0 \)). We know that Theorem 8.1 holds when \( \rho = 0 \) since \( \Theta_1^{-1}(0) = \Theta_2^{-1}(0) \); see Lemma 9.1 and the following remark. We are going to prove the statement by induction on \( S \), increasing it by 1.

Consider tableau \( \mathfrak{M}_\lambda(r, s) \) and distinguish a \textbf{W-channel} in it as follows: it is a subset of boxes such that (1) each column has exactly one box in the \textbf{W-channel}; (2) an integer \( k \) exists such that the content of each box in the \textbf{W-channel} is either \( k \) or \( k+1 \); (3) each next box of the \textbf{W-channel} in the left to right order is either in the same row or in a higher row.
Example 9.4. The two possible \( \mathcal{W} \)-channels of the tableau above are shown here in red:

\[
\begin{array}{c|c}
3 & 0 \\
2 & 1 \\
0 & 1 \\
\hline
2 & 3 \\
4 & 4 \\
\end{array}
\quad \text{and} \quad
\begin{array}{c|c}
3 & 0 \\
2 & 1 \\
0 & 1 \\
\hline
2 & 3 \\
4 & 4 \\
\end{array}
\]

Lemma 9.5. \( \mathcal{W}_{\lambda}(r, s) \) has either one or two \( \mathcal{W} \)-channels.

Proof. Let \( k \) be the filling of the single box in the leftmost column among those of size 1. By definition of \( \mathcal{W}_{\lambda}(r, s) \) we know \( k \) is non-negative, since \( b_i \geq a_i = 0 \) if \( i \) is the index of the column of size 1. It is clear that any \( \mathcal{W} \)-channel must contain this \( k \), and therefore it can only be filled with either \( k \)-s and \( k+1 \)-s, or with \( k \) and \( k-1 \). Furthermore, for each of those two choices there is at most one \( \mathcal{W} \)-channel filled with them, this is because no two entries in the same column of \( \mathcal{W}_{\lambda}(r, s) \) are consecutive integers. What remains to argue is that at least one of those two \( \mathcal{W} \)-channels does exist.

Assume that \( i \) is the smallest column index such that \( b_i = k \). First we suppose that we can find one of the entries \( k-1, k, k+1 \) in the column \( i-1 \) below the first row. If it is \( k-1 \), then we can find either \( k \) or \( k-1 \) in each column to the left of it. If it is \( k \) or \( k+1 \), we can find either \( k \) or \( k+1 \) in each column to the left of it. Assume now that all entries below the first row in column \( i-1 \) are at most \( k-2 \). This means that \( a_{i-1} \leq k \). If we had \( b_{i-1} \geq k+2 \), we could reduce \( b_{i-1} \) by 1 and increase \( b_i \) by 1 while decreasing the sum of squares of \( b \)-s. Thus \( b_{i-1} = k+1 \). Let \( j \) be the smallest column index such that \( b_j = k+1 \). Then if we can find an entry equal to \( k \) or \( k+1 \) below the first row in column \( j-1 \), we can proceed to find such entry in each next column to the left. If not, we know that \( a_{j-1} \leq k+1 \). On the other hand, by assumption \( b_{j-1} \geq k+2 \). Reducing \( b_{j-1} \) by 1 and increasing \( b_i \) by 1 decreases the sum of squares of \( b \)-s — contradiction. This completes the proof.

Let \( w = \Psi(T^\text{can}_\lambda, T^\text{can}_\lambda, \text{rev}_\lambda(\mathcal{W}_{\lambda}(r, s))) \) and assume that the entries of \( \mathcal{W}_{\lambda}(r, s) \) coincide with the multiset of block coordinates \( D(x) \) as we vary \( x \) over the equivalence classes of balls in \( w \). (This holds when \( s = 0 \), i.e. when \( w = \Psi(T^\text{can}_\lambda, T^\text{can}_\lambda, 0) \) as observed above.)

Lemma 9.6. Given a channel of \( w \), there is an integer \( k \) such that block coordinates of all balls in it are \( k \) or \( k+1 \); there is either one or two channels in \( w \), they form a river and the block coordinates of (equivalence classes of) balls in them correspond exactly to (either one or two) \( \mathcal{W} \)-channels of \( \mathcal{W}_{\lambda}(r, s) \). (See [CPY18 Definition 3.18] for the definition of a river.)

Proof. Throughout the proof we are using the fact that \( w \) lies in both left and right canonical cells, i.e. \( w(1) < w(2) < \cdots < w(n) \) and \( w^{-1}(1) < w^{-1}(2) < \cdots < w^{-1}(n) \). It is not hard to see that only balls with equal or consecutive block coordinates can belong to a channel, this proves the first statement. Once we have chosen such a \( k \), the balls with block coordinates \( k \) and \( k+1 \) do form a saturated chain in \( SE \) order. It is a channel if and only if its length is equal to \( \lambda_1 \). This however is only possible if each column contains at least one entry \( k \) or \( k+1 \), which by definition makes it into a \( \mathcal{W} \)-channel. In the opposite direction, any \( \mathcal{W} \)-channel gives a saturated chain in the \( SE \) order, and it is of the correct length. Finally, since \( \lambda_1 > \lambda_2 \), all channels must belong to the same river. \( \Box \)
We proceed with the step of induction. Let us increase $s$ by 1. By Theorem [CPY18, Theorem 16.9] this results in a shift of size 1 of NE-most channel of the indexing river.

**Remark.** Theorem [CPY18, Theorem 16.9] is stated in the generality when the streams have the same flow as the width of the Shi poset. This is equivalent to $\lambda_1 = \lambda_2$. We are interested in the case when $\lambda_1 > \lambda_2$. The proof of all statement in [CPY18, Section 16.1] however extends verbatim to this generality as well.

It is easy to see that the multiset of block coordinates $D(x)$ as we vary $x$ over equivalence classes in this channel changes as follows: we get one less of $D(x) = k$ and one more of $D(x) = k + 1$, where $k$ is such that the set of block coordinates of the balls in this channel is either $\{k\}$ or $\{k, k+1\}$.

**Lemma 9.7.** This $k$ is the smallest entry in the first row of $\mathcal{W}_\lambda(r, s)$.

**Proof.** We know by construction that the smallest entry in the first row belongs to all (i.e. one or two) $\mathcal{W}$-channels. It is easy to see that if the smallest entry of the first row is the larger of the two indexes of the $\mathcal{W}$-channel, then the latter corresponds to the SE-most channel in the river of $w$, different from its NE-most channel. The statement follows. $\square$

We now argue that the same change happens to the total content of $\mathcal{W}_\lambda(r, s)$ as we increase $s$ by 1.

**Lemma 9.8.** $\mathcal{W}_\lambda(r, s + 1)$ is obtained from $\mathcal{W}_\lambda(r, s)$ by increasing the smallest entry of the first row by 1.

**Proof.** Keeping in mind that we are trying to minimize $\sum b_i^2$ conditioned on knowing $\sum b_i$ and the fact that $b_i \geq a_i \geq 0$, it is not hard to check that first, one is indeed related to the other by a single increase of an entry by 1, and second, it has to be the smallest entry. $\square$

This completes the step of induction.

**Example 9.9.** The following illustrates how the $\mathcal{W}$-channels change as we start with $s = 0$ and increase it till $s = 5$. On each step we show in red the union of all boxes belonging to some $\mathcal{W}$-channel.

\[
\begin{array}{cccccc}
4 & 3 & 3 & 0 & \rightarrow & 4 & 3 & 3 & 1 \\
2 & 1 & 1 & \rightarrow & 2 & 1 & 1 & \rightarrow & 4 & 3 & 3 & 2 \\
0 & -1 & -1 & \rightarrow & 0 & -1 & -1 & \rightarrow & 0 & -1 & -1 & \rightarrow & 4 & 3 & 3 & 3 \\
-3 & -3 & -3 & \rightarrow & -3 & -3 & -3 & \rightarrow & -3 & -3 & -3 & \rightarrow & 4 & 4 & 4 & 3 \\
4 & -4 & -4 & -4 & \rightarrow & 4 & -4 & -4 & -4 & \rightarrow & 4 & -4 & -4 & -4 & .
\end{array}
\]

It remains to consider the case when $\mu(w) = \mu_0(\{n' + 1, n' + 2, \ldots, n\})$ and $\vec{\mu}' = \vec{\mu}'(r, s)$ for some $r$ and $s$. Here we define some ad-hoc notations; for an integer vector $\vec{\nu}$ and $i \in \mathbb{Z}$, we define $\vec{\nu}^{(i)}$ to be the number of $i$ in $\vec{\nu}$. Also let $\vec{\nu}^{(>i)} = \sum_{j > i} \nu(j)$ and define $\vec{\nu}^{(>i)}$, $\vec{\nu}^{(\leq i)}$, and $\vec{\nu}^{(<i)}$ similarly. We start with the following lemma.

**Lemma 9.10.** Let $\vec{\mu}'_\lambda(0), \vec{\mu}'_\lambda(r, s)$ be as before. Here $\vec{\nu}'$ can be any of $\vec{\mu}'_\lambda(0)$ or $\vec{\mu}'_\lambda(r, s)$.

1. For $k \in \mathbb{Z}$, we have $\vec{\nu}'^{(k)} + \vec{\nu}'^{(k+1)} \leq \lambda_2$. 

(2) For \( k \in \mathbb{Z} \), we have \( \tilde{\mu}_\lambda(0)^{(k)} = \tilde{\mu}_\lambda(0)^{(-k)} \).

(3) For \( k < 0 \), we have \( \tilde{\mu}_\lambda(r, s)^{(k)} = \tilde{\mu}_\lambda(0)^{(k)} \).

(4) \( \tilde{\mu}_\lambda(r, s)^{(\geq 0)} = \tilde{\mu}_\lambda(0)^{(\geq 0)} \).

(5) \( \tilde{\mu}_\lambda(r, s)^{(0)} \leq \tilde{\mu}_\lambda(0)^{(0)} \).

(6) \( \tilde{\mu}^{(\geq 0)} \geq \tilde{\mu}^{(\leq 0)} \) or equivalently \( \tilde{\mu}^{(\geq 0)} \geq \tilde{\mu}^{(\leq 0)} \).

(7) For \( k > 0 \), we have \( \tilde{\mu}^{(\geq 0)} - \tilde{\mu}^{(\leq -k)} \leq k \lambda_2 \).

Proof. (1) holds since each column of \( \mathfrak{W}_\lambda(0) \) and \( \mathfrak{W}_\lambda(r, s) \) does not contain any two consecutive integers. (2) is obvious. (3) is also straightforward since the negative entries of \( \mathfrak{W}_\lambda(0) \) and \( \mathfrak{W}_\lambda(r, s) \) are the same. (4) follows immediately from (3). (5) follows from the description of \( \mathfrak{W}_\lambda(r, s) \). (6) holds because of (2), (4), and (5). For (7), by (3) and (4) it suffices to prove the inequality when \( \tilde{\mu}' = \tilde{\mu}_\lambda(0) \). But \( \tilde{\mu}_\lambda(0)^{(\geq 0)} = \tilde{\mu}_\lambda(0)^{(\leq 0)} \) by (2), thus the result follows from (1).

Suppose that the window notation of \( w' \) is given by

\[
[c_1, c_2, \ldots, c_{\tilde{\mu}'^{(\leq 0)}}, b_{\tilde{\mu}'^{(0)}}, \ldots, b_2, b_1, a_{\tilde{\mu}'^{(0)}}, \ldots, a_2, a_1, \emptyset, \emptyset, \ldots, \emptyset].
\]

In particular, we have

\[
c_1 < c_2 < \cdots < c_{\tilde{\mu}'^{(\leq 0)}} < b_{\tilde{\mu}'^{(0)}} < \cdots < b_2 < b_1 < a_{\tilde{\mu}'^{(0)}} < \cdots < a_2 < a_1.
\]

Furthermore, it is clear that

\[
\tilde{\mu}' = \left( \left[ \frac{a_1}{n} \right] - 1, \left[ \frac{a_2}{n} \right] - 1, \ldots, \left[ \frac{a_{\tilde{\mu}'^{(0)}}}{n} \right] - 1, \left[ \frac{b_1}{n} \right] - 1, \left[ \frac{b_2}{n} \right] - 1, \ldots, \left[ \frac{b_{\tilde{\mu}'^{(0)}}}{n} \right] - 1, \right.
\]

\[
\left. \left[ \frac{c_{\tilde{\mu}'^{(\leq 0)}}}{n} \right] - 1, \ldots, \left[ \frac{c_2}{n} \right] - 1, \left[ \frac{c_1}{n} \right] - 1 \right).
\]

It follows that \( \left[ \frac{a_1}{n} \right] \geq 2, \left[ \frac{b_1}{n} \right] = 1, \) and \( \left[ \frac{c_1}{n} \right] \leq 0 \) for any \( a_i, b_i, c_i \).

Example 9.11. Let \( n = 11, \lambda = (4, 3, 3, 1) \), and \( \tilde{\mu}' = (0, 2, 0) = \tilde{\mu}_{(3,3,1)}(3, 2) \). Then

\[
w' = [-15, -6, -5, 4, 23, 24, 25, \emptyset, \emptyset, \emptyset, \emptyset].
\]

Thus we have

\[
a_1 = 25, \ a_2 = 24, \ a_3 = 23, \ b_1 = 4, \ c_4 = -5, \ c_2 = -6, \ c_1 = -15,
\]

\[
\tilde{\mu}' = (2, 2, 2, 0, -1, -1, -2) = \tilde{\mu}_{(3,3,1)}(3, 2).
\]

Indeed, \( \mathfrak{W}_{(3,3,1)}(3, 2) \) is given by

\[
\begin{array}{ccc}
2 & 2 & 2 \\
0 & -1 & -1 \\
-2 & & \\
\end{array}
\]

The following lemma will turn out to be useful later on.

Lemma 9.12. Let \( k \in \mathbb{Z}_{\geq 0} \).

(1) For \( 1 \leq i \leq \tilde{\mu}^{(\geq 0)} \) and \( k \in \mathbb{Z}_{\geq 0} \), \( \left[ \frac{a_i}{n} \right] - 1 \leq k \) if and only if \( i > \tilde{\mu}^{(\geq k)} \).

(2) For \( 1 \leq i \leq \tilde{\mu}^{(\leq 0)} \) and \( k \in \mathbb{Z}_{\geq 0} \), \( 1 - \left[ \frac{a_i}{n} \right] \leq k \) if and only if \( i > \tilde{\mu}^{(\leq -k)} \).
(3) For any $1 \leq i \leq j \leq \tilde{r}^\prime(^{(<0)}, \tilde{r}^\prime(^{>0}))$, we have $2 \left\lfloor \frac{j-1}{\lambda_2} \right\rfloor \leq \left\lceil \frac{c_i}{n} \right\rceil - \left\lfloor \frac{c_i}{n} \right\rfloor$.

Proof. (1) and (2) follow from the description of $\tilde{r}^\prime$ above. (3) is equivalent to that if $j - i \geq k\lambda_2$ for some $k \in \mathbb{N}$ then $\left\lceil \frac{c_i}{n} \right\rceil - \left\lfloor \frac{c_i}{n} \right\rfloor \geq 2k$. This is easily obtained from Lemma 9.10. □

We define (recall that $n' = n - \lambda_1$)

$$A_i = (n' + 1 - i, a_i) \quad \text{for } 1 \leq i \leq \tilde{r}^\prime(^{<0})$$

$$B_i = (\tilde{r}^\prime(^{<0}) + 1 - i, b_i) \quad \text{for } 1 \leq i \leq \tilde{r}^\prime(^{>0})$$

$$C_i = \left(1 - \left\lfloor \frac{c_i}{n} \right\rfloor\right)n + i, \left(1 - \left\lfloor \frac{c_i}{n} \right\rfloor\right)n + c_i) \quad \text{for } 1 \leq i \leq \tilde{r}^\prime(^{<0}).$$

Then it is immediate that

$$A_i, B_i, C_i \in \mathcal{B}_{w'} \cap \{(a, b) \in \mathbb{Z}^2 \mid a \geq 1, b \geq 1\} - \{(a, b) \in \mathbb{Z}^2 \mid a \geq n' + 1, b \geq n' + 1\}$$

and that $\{A_i \mid 1 \leq i \leq \tilde{r}^\prime(^{>0})\} \cup \{B_i \mid 1 \leq i \leq \tilde{r}^\prime(^{<0})\} \cup \{C_i \mid 1 \leq i \leq \tilde{r}^\prime(^{<0})\}$ contains exactly one representative in each translation class by $(n, n)$ in $\mathcal{B}_{w'}$. Also, we have

$$A_1 >_{SE} A_2 >_{SE} A_3 >_{SE} A_{\tilde{r}^\prime(^{>0})} >_{SE} B_1,$$

$$C_1 >_{SE} C_2 >_{SE} C_3 >_{SE} C_{\tilde{r}^\prime(^{>0})} >_{SE} B_1,$$

$$B_1 >_{SE} B_2 >_{SE} \cdots >_{SE} B_{\tilde{r}^\prime(^{>0})}.$$}

However, $A_1$ and $C_3$ are not comparable with respect to southeast ordering.

**Example 9.13.** When $w' = [-15, -6, -5, 4, 23, 24, 25, 0, 0, 0]$ as above, we have

$$A_1 = (7, 25), A_2 = (6, 24), A_3 = (5, 23), B_1 = (4, 4), C_3 = (13, 5), C_2 = (14, 6), C_1 = (23, 7).$$

Thus $A_1 >_{SE} A_2 >_{SE} A_3 >_{SE} B_1$ and $C_1 >_{SE} C_2 >_{SE} C_3 >_{SE} B_1$. However, any of $A_i$ and $C_j$ are not comparable.

In order to proceed the backward AMBC, it is necessary to calculate the backward numbering $d_{w'}^{b, \text{st}(w)}$ as in [CPY18 Section 4.2]. For this, first we fix a proper numbering of $\text{st}(w)$ such that $\text{st}(w)^{(-1)} = (0, 0)$, i.e. $(0, 0)$ is labeled -1. Also we define the numbering $d$ on $A_i, B_i$, and $C_i$ by

$$d(A_i) = -i, \quad d(B_i) = -\left(\tilde{r}^\prime(^{>0}) + i\right), \quad d(C_i) = -i.$$}

and extend to the whole of $\mathcal{B}_{w'}$ by periodicity, i.e. if $b \in \mathcal{B}_{w'}$ then $d(b + k(n, n)) = d(b) + k\lambda_1$. Note that this is the largest monotone numbering on $\{A_i\} \cup \{B_i\} \cup \{C_i\}$, which respects $\leq_{SE}$ subject to the condition that $d(A_1) = d(C_1) = -1$. (Recall that $\tilde{r}^\prime(^{>0}) \geq \tilde{r}^\prime(^{<0})$ by Lemma 9.10)

**Example 9.14.** When $w' = [-15, -6, -5, 4, 23, 24, 25, 0, 0, 0]$ as above, $d$ is defined by

$$d(A_1) = d(C_1) = -1, \quad d(A_2) = d(C_2) = -2, \quad d(A_3) = d(C_3) = -3, \quad d(B_1) = -4.$$}

**Lemma 9.15.** Let $d$ be the numbering of $\mathcal{B}_{w'}$ defined above. Then $d = d_{w'}^{b, \text{st}(w)}$.

**Proof.** Suppose otherwise. Then from [CPY18 Remark 13.1], there exist $b, b' \in \mathcal{B}_{w'}$ such that $b <_{SE} b'$ and $d(b) \geq d(b')$. Without loss of generality we may assume that $b$ is one of $A_i, B_i$, or $C_i$. We consider each case in the following.

(a) Suppose that $b = A_i$ for some $1 \leq i \leq \tilde{r}^\prime(^{>0})$. In particular, $d(b) = -i$.
• If \( b' = A_j + k(n, n) \) for some \( 1 \leq j \leq \overline{p}^{(0)} \) and \( k \in \mathbb{Z} \), then we have \( d(b') = -j + k\lambda_1 \leq -i \), i.e. \( k \leq \frac{i - a_i}{\lambda_1} \). Comparing the y-coordinates, we also have \( a_j + kn > a_i \), i.e. \( k > \frac{a_i - a_j}{n} \). It follows that \( \frac{a_i - a_j}{j - 1} \leq \frac{n}{\lambda_1} < 1 \). But this is impossible since \( (a_1, a_2, \ldots) \) is a strictly decreasing sequence.

• \( b' = B_j + k(n, n) \) for some \( 1 \leq j \leq \overline{p}^{(0)} \) and \( k \in \mathbb{Z} \). This case we have \( d(b') = -(\overline{p}^{(0)} + j) + k\lambda_1 \leq -i \), i.e. \( k \leq \frac{i - \lambda_1}{\lambda_1} + \frac{j - 1}{\lambda_1} \). Comparing the y-coordinates, we also have \( b_j + kn > a_i \), i.e. \( k > \frac{a_i - b_j}{n} \). It follows that \( \frac{a_i - b_j}{\overline{p}^{(0)} + j - 1} \leq \frac{n}{\lambda_1} < 1 \). This is again impossible since \( (a_1, a_2, \ldots, d_{\overline{p}^{(0)}}, b_1, b_2, \ldots) \) is strictly decreasing.

• \( b' = C_j + k(n, n) \) for some \( 1 \leq j \leq \overline{p}^{(0)} \) and \( k \in \mathbb{Z} \). This case we have \( d(b') = -(\overline{p}^{(0)} + j) + k\lambda_1 \leq -i \), i.e. \( k \leq \frac{i - \lambda_1}{\lambda_1} + \frac{j - 1}{\lambda_1} \). Comparing the y-coordinates, we also have \( b_j + kn > a_i \), i.e. \( k > \frac{a_i - b_j}{n} \). It follows that \( \frac{a_i - b_j}{\overline{p}^{(0)} + j - 1} \leq \frac{n}{\lambda_1} < 1 \). This is again impossible since \( (a_1, a_2, \ldots, d_{\overline{p}^{(0)}}, b_1, b_2, \ldots) \) is strictly decreasing.

(b) Suppose that \( b = B_i \) for some \( 1 \leq i \leq \overline{p}^{(0)} \). In particular, \( d(b) = -(\overline{p}^{(0)} + i) \).

• If \( b' = A_j + k(n, n) \) for some \( 1 \leq j \leq \overline{p}^{(0)} \) and \( k \in \mathbb{Z} \), then we have \( d(b') = -(\overline{p}^{(0)} + j) + k\lambda_1 \leq -i \), i.e. \( k \leq \frac{i - \lambda_1}{\lambda_1} + \frac{j - 1}{\lambda_1} \). Comparing the y-coordinates, we also have \( b_j + kn > a_i \), i.e. \( k > \frac{a_i - b_j}{n} \). Thus \( \frac{a_i - b_j}{\overline{p}^{(0)} + j - 1} \leq \frac{n}{\lambda_1} < 1 \). This is impossible since it implies \( k' = b_j \).

• \( b' = B_j + k(n, n) \) for some \( 1 \leq j \leq \overline{p}^{(0)} \) and \( k \in \mathbb{Z} \). This case we have \( d(b') = -(\overline{p}^{(0)} + j) + k\lambda_1 \leq -i \), i.e. \( k \leq \frac{i - \lambda_1}{\lambda_1} + \frac{j - 1}{\lambda_1} \). Comparing the y-coordinates, we also have \( b_j + kn > a_i \), i.e. \( k > \frac{a_i - b_j}{n} \). Thus \( \frac{a_i - b_j}{\overline{p}^{(0)} + j - 1} \leq \frac{n}{\lambda_1} < 1 \). This is impossible since it implies \( k' = b_j \).

• \( b' = C_j + k(n, n) \) for some \( 1 \leq j \leq \overline{p}^{(0)} \) and \( k \in \mathbb{Z} \). This case we have \( d(b') = -(\overline{p}^{(0)} + j) + k\lambda_1 \leq -i \), i.e. \( k \leq \frac{i - \lambda_1}{\lambda_1} + \frac{j - 1}{\lambda_1} \). Comparing the y-coordinates, we also have \( b_j + kn > a_i \), i.e. \( k > \frac{a_i - b_j}{n} \). Thus \( \frac{a_i - b_j}{\overline{p}^{(0)} + j - 1} \leq \frac{n}{\lambda_1} < 1 \). This is again impossible since it implies \( k' = b_j \).

(c) Suppose that \( b = C_i \) for some \( 1 \leq i \leq \overline{p}^{(0)} \). In particular, \( d(b) = -i \).

• If \( b' = A_j + k(n, n) \) for some \( 1 \leq j \leq \overline{p}^{(0)} \) and \( k \in \mathbb{Z} \), then we have \( d(b') = -(\overline{p}^{(0)} + j) + k\lambda_1 \leq -i \), i.e. \( k \leq \frac{i - \lambda_1}{\lambda_1} + \frac{j - 1}{\lambda_1} \). Comparing the y-coordinates, we also have \( n' \leq n + j + kn > (1 - \left[ \frac{a_i}{n} \right])n + i + \lambda_1 > 1 \). Thus by Lemma \[9.10\] it follows that \( i > \overline{p}^{(0)} \). But then we have \( \overline{p}^{(0)} \leq \overline{p}^{(0)} \) which is not possible by Lemma \[9.10\].

• \( b' = B_j + k(n, n) \) for some \( 1 \leq j \leq \overline{p}^{(0)} \) and \( k \in \mathbb{Z} \). This case we have \( d(b') = -(\overline{p}^{(0)} + j) + k\lambda_1 \leq -i \), i.e. \( k \leq \frac{i - \lambda_1}{\lambda_1} + \frac{j - 1}{\lambda_1} \). Comparing the y-coordinates, we also have \( n' \leq n + j + kn > (1 - \left[ \frac{a_i}{n} \right])n + i + \lambda_1 > 1 \). Thus \( i > \overline{p}^{(0)} \) by Lemma \[9.12\] Now we have \( \overline{p}^{(0)} \leq \overline{p}^{(0)} \) and \( k \leq \frac{i - \lambda_1}{\lambda_1} + \frac{j - 1}{\lambda_1} \leq -i \). This is not possible by Lemma \[9.10\].

• \( b' = C_j + k(n, n) \) for some \( 1 \leq j \leq \overline{p}^{(0)} \) and \( k \in \mathbb{Z} \). This case we have \( d(b') = -(\overline{p}^{(0)} + j) + k\lambda_1 \leq -i \), i.e. \( k \leq \frac{i - \lambda_1}{\lambda_1} + \frac{j - 1}{\lambda_1} \). Comparing the y-coordinates, we also have \( n' \leq n + j + kn > (1 - \left[ \frac{a_i}{n} \right])n + i + \lambda_1 > 1 \). Thus \( i > \overline{p}^{(0)} \) and \( k \leq \frac{i - \lambda_1}{\lambda_1} + \frac{j - 1}{\lambda_1} \leq -i \). This is not possible by Lemma \[9.10\].
not possible we have \( k \geq 1 \). Also \( j \geq i \). This time we compare the \( x \)-coordinates and obtain 
\[
(1 - \left\lfloor \frac{j}{\lambda} \right\rfloor)n + j > (1 - \left\lfloor \frac{i}{\lambda} \right\rfloor)n + i, \text{ i.e. } k > \frac{j - i}{\lambda} + \left\lfloor \frac{j}{\lambda} \right\rfloor - \left\lfloor \frac{i}{\lambda} \right\rfloor.
\]
Thus \( k \geq \frac{j}{\lambda} - \frac{i}{\lambda} \) as \( |i - j| < n \). By Lemma 9.12 it implies that 
\[
2 \left\lfloor \frac{i - j}{\lambda} \right\rfloor \leq k \leq \frac{i - j}{\lambda},
\]
which is true only when \( \frac{i - j}{\lambda} < 1 \). This contradicts that \( 1 \leq k \leq \frac{i - j}{\lambda} \).

We considered all the possible cases. The lemma is proved. \( \square \)

We set \( R' := \bigsqcup_{i=1}^{\lambda_1} R'_i \) where \( R'_i = \{ b \in B_{w'} \mid d(b) = -i \} \). Then it is clear that \( R' \) is the set of representatives in each translation class by \((n, n) \) in \( B_{w'} \). Also for \( 1 \leq i \leq \lambda_1 \) let \( Z_i \) be the zig-zag with the back corner-post at \( a(w)^{(-i)} = (1 - i, 1 - i) \) and outer corner-posts at the balls in \( R'_i \). We set \( R := \bigsqcup_{i=1}^{\lambda_1} R_i \) where \( R_i = \{ \text{inner corner-posts of } Z_i \} \). Then it is also clear that \( R \) is the set of representatives in each translation class by \((n, n) \) in \( B_w \). Our goal is to show that \{ \( D(b) \mid b \in R \) \} is equal to \( \tilde{\mu}_\lambda(0) \) or \( \tilde{\mu}_\lambda(r, s) \), and by Lemma 9.13 it implies Theorem 8.1.

For \( k \in \mathbb{Z} \) and \( 1 \leq i \leq \lambda_1 \), let us set
\[
\begin{align*}
\hat{A}_{k,i} &:= A_{k\lambda_1+i} + (k, n, n) & \text{if } 1 \leq k\lambda_1+i \leq \tilde{\mu}^{(>0)}, \\
\hat{B}_{k,i} &:= B_{k\lambda_1+i} - \tilde{\mu}^{(>0)} + (k, n, n) & \text{if } 1 \leq k\lambda_1+i - \tilde{\mu}^{(>0)} \leq \tilde{\mu}^{(0)}, \\
\hat{C}_{k,i} &:= C_{k\lambda_1+i} + (k, n, n) & \text{if } 1 \leq k\lambda_1+i \leq \tilde{\mu}^{(<0)}.
\end{align*}
\]

Note that \( d(\hat{A}_{k,i}) = d(\hat{B}_{k,i}) = d(\hat{C}_{k,i}) = -i \). Also let \( N_i = \left\lfloor \frac{\tilde{\mu}^{(<0)} - i}{\lambda_1} \right\rfloor \) so that \( N_i\lambda_1+i \leq \tilde{\mu}^{(<0)} < (N_i+1)\lambda_1+i \). Since \( \tilde{\mu}^{(>0)} - \tilde{\mu}^{(<0)} < \lambda_1 \) by Lemma 9.10 \((N_i+2)\lambda_1+i > \tilde{\mu}^{(>0)}\). Now we consider the three possible cases below.

(a) \((N_i+1)\lambda_1+i \leq \tilde{\mu}^{(>0)}\). Then
\[
R'_i = \{ \hat{A}_{k,i} \mid 1 \leq k \leq N_i+1 \} \cup \{ \hat{C}_{k,i} \mid 1 \leq k \leq N_i \}
\]

(b) \( \tilde{\mu}^{(>0)} < (N_i+1)\lambda_1+i \leq \tilde{\mu}^{(>0)}\). Then
\[
R'_i = \{ \hat{A}_{k,i} \mid 1 \leq k \leq N_i \} \cup \{ \hat{B}_{N_i+1,i} \} \cup \{ \hat{C}_{k,i} \mid 1 \leq k \leq N_i \}
\]

(c) \((N_i+1)\lambda_1+i > \tilde{\mu}^{(>0)}\). Then
\[
R'_i = \{ \hat{A}_{k,i} \mid 1 \leq k \leq N_i \} \cup \{ \hat{C}_{k,i} \mid 1 \leq k \leq N_i \}
\]

**Example 9.16.** Suppose that \( n = 15, \lambda = (5, 4, 4, 2) \), and \( \mathcal{w}' = (0, 4, 0) \). Thus
\[
w' = \Psi(T^{can}_{(4,4,2)}(T^{can}_{(4,4,2)}(0, 3, 0))) = [-21, -20, -8, -7, 5, 6, 32, 33, 34, 46, 0, 0, 0, 0, 0].
\]

Then
\[
\begin{align*}
A_1 &= (10, 46), & A_2 &= (9, 34), & A_3 &= (8, 33), & A_4 &= (7, 32), & B_1 &= (6, 6), & B_2 &= (5, 5), \\
C_1 &= (18, 7), & C_2 &= (19, 8), & C_3 &= (31, 9), & & C_4 &= (32, 10)
\end{align*}
\]
and \( d(A_i) = d(C_i) = -i, d(B_i) = -5 - i \). Also,
\[
\begin{align*}
\hat{A}_{0,i} &= A_i \text{ for } 1 \leq i \leq 4, & \hat{C}_{0,i} &= C_i \text{ for } 1 \leq i \leq 4, \\
\hat{B}_{1,1} &= B_2 + (15, 15) = (20, 20), & \hat{B}_{0,5} &= B_1 = (6, 6).
\end{align*}
\]
Thus in particular \( d(\hat{A}_{0,i}) = d(\hat{C}_{0,i}) = -i \) for \( 1 \leq i \leq 4 \) and \( d(\hat{B}_{1,1}) = -1, d(\hat{B}_{0,5}) = -5 \).
Recall the process to calculate inner corner-posts of a zigzag from the back and outer corner-posts. If a zigzag $Z$ consists of the back corner post $(a_0, b_0)$ and outer corner-posts $(a_1, b_1), (a_2, b_2), \ldots, (a_r, b_r)$ such that $a_1 < a_2 < \cdots < a_r$, then the outer corner-posts of $Z$ is given by
\[
\{(a_0, b_1), (a_1, b_2), \ldots, (a_{r-1}, b_r), (a_r, b_0)\}.
\]

We wish to apply this rule to each $Z_i$ for $1 \leq i \leq \lambda_1$. However, in fact it is sufficient only to calculate the block coordinates $\{D(x) \mid x \in Z_i\}$ for each $Z_i$ for our purpose. To this end, we simplify the argument by exploiting the notion of block coordinates.

**Definition 9.17.** For $(x, y) \in \mathbb{Z}^2$, we define the $n$-block coordinate (or block coordinate if there is no confusion) of $(x, y)$ to be $([\frac{x}{n}] - 1, [\frac{y}{n}] - 1)$ and write $[(x, y)]_n = ([\frac{x}{n}] - 1, [\frac{y}{n}] - 1)$. For a subset $A \subset \mathbb{Z}^2$, we define $[A]_n := \{[b]_n \mid b \in A\}$. Note that if $[b]_n = (x, y)$ then $D(b) = y - x$.

For $1 \leq i \leq \lambda_1$, direct calculation shows that $[\text{st}(w)^{(−1)}]_n = (−1, −1)$ and also
\[
\left[\tilde{A}_{ki}\right]_n = (k, \left[\frac{\alpha k_{\lambda i + 1}}{n}\right] + k - 1), \quad \left[\tilde{B}_{ki}\right]_n = (k, k), \quad \left[\tilde{C}_{ki}\right]_n = (k + 1 - \left[\frac{c k_{\lambda i + 1}}{n}\right], k).
\]

Therefore, $[R'_i]_n$ and $\{D(x) \mid x \in R'_i\}$ for $1 \leq i \leq \lambda_1$ is given as follows.

(a) $(N_i + 1)\lambda_1 + i \leq \tilde{\mu}^{(>)}$. Then
\[
[R'_i]_n = \left\{(k, \left[\frac{\alpha k_{\lambda i + 1}}{n}\right] + k - 1) \mid 1 \leq k \leq N_i + 1\right\} \cup \left\{(k + 1 - \left[\frac{c k_{\lambda i + 1}}{n}\right], k) \mid 1 \leq k \leq N_i\right\}
\]
\[
\{D(x) \mid x \in R'_i\} = \left\{\left[\frac{\alpha k_{\lambda i + 1}}{n}\right] - 1 \mid 1 \leq k \leq N_i + 1\right\} \cup \left\{\left[\frac{c k_{\lambda i + 1}}{n}\right] - 1 \mid 1 \leq k \leq N_i\right\}
\]

(b) $\tilde{\mu}^{(>)} < (N_i + 1)\lambda_1 + i \leq \tilde{\mu}^{(=)}$. Then
\[
[R'_i]_n = \left\{(k, \left[\frac{\alpha k_{\lambda i + 1}}{n}\right] + k - 1) \mid 1 \leq k \leq N_i\right\} \cup \{(N_i + 1, N_i + 1)\}
\]
\[
\cup \left\{(k + 1 - \left[\frac{c k_{\lambda i + 1}}{n}\right], k) \mid 1 \leq k \leq N_i\right\}
\]
\[
\{D(x) \mid x \in R'_i\} = \left\{\left[\frac{\alpha k_{\lambda i + 1}}{n}\right] - 1 \mid 1 \leq k \leq N_i\right\} \cup \{0\} \cup \left\{\left[\frac{c k_{\lambda i + 1}}{n}\right] - 1 \mid 1 \leq k \leq N_i\right\}
\]

(c) $(N_i + 1)\lambda_1 + i > \tilde{\mu}^{(>)}$. Then
\[
[R'_i]_n = \left\{(k, \left[\frac{\alpha k_{\lambda i + 1}}{n}\right] + k - 1) \mid 1 \leq k \leq N_i\right\} \cup \left\{(k + 1 - \left[\frac{c k_{\lambda i + 1}}{n}\right], k) \mid 1 \leq k \leq N_i\right\}
\]
\[
\{D(x) \mid x \in R'_i\} = \left\{\left[\frac{\alpha k_{\lambda i + 1}}{n}\right] - 1 \mid 1 \leq k \leq N_i\right\} \cup \left\{\left[\frac{c k_{\lambda i + 1}}{n}\right] - 1 \mid 1 \leq k \leq N_i\right\}
\]

Now we calculate $[R_i]_n$ for each $1 \leq i \leq \lambda_1$. Note that there are no two balls $b, b' \in R'_i$ such that $[b]_n$ and $[b']_n$ have the same $x$ or $y$ coordinate. Thus, to this end we may apply the usual backward AMBC process on $\{(1 - i, 1 - i)\} \cup [R_i]_n$ directly; it will give the same result as applying the backward AMBC process on $\{(1 - i, 1 - i)\} \cup R_i$ and calculating the block coordinates of the result. Now direct calculation shows the following.
Therefore

\[ (a) \quad (N_i + 1)\lambda_1 + i \leq \bar{\mu}^{(>0)}. \]

Then

\[ [R_i]_n = \left\{(k - 1, \left\lfloor \frac{\lambda_i + k}{n} \right\rfloor + k - 1) \mid 1 \leq k \leq N_i + 1\right\} \cup \{(N_i, N_i)\}
\]

\[ \cup \left\{(k + 1 - \left\lceil \frac{\lambda_i + 1}{n} \right\rceil, k - 1) \mid 1 \leq k \leq N_i\right\} \]

\[ \{D(x) \mid x \in R_i\} = \left\{\left\lfloor \frac{\lambda_i + 1}{n} \right\rfloor \mid 1 \leq k \leq N_i\right\} \cup \{-1\} \cup \left\{\left\lceil \frac{\lambda_i + 1}{n} \right\rceil - 2 \mid 1 \leq k \leq N_i\right\} \]

(b) \quad \bar{\mu}^{(>0)} < (N_i + 1)\lambda_1 + i \leq \bar{\mu}^{(\geq 0)}. \]

Then

\[ [R_i]_n = \left\{(k - 1, \left\lfloor \frac{\lambda_i + 1}{n} \right\rfloor + k - 1) \mid 1 \leq k \leq N_i\right\} \cup \{(N_i, N_i + 1), (N_i + 1, N_i)\}
\]

\[ \cup \left\{(k + 1 - \left\lceil \frac{\lambda_i + 1}{n} \right\rceil, k - 1) \mid 1 \leq k \leq N_i\right\} \]

\[ \{D(x) \mid x \in R_i\} = \left\{\left\lfloor \frac{\lambda_i + 1}{n} \right\rfloor \mid 1 \leq k \leq N_i\right\} \cup \{1, -1\} \cup \left\{\left\lceil \frac{\lambda_i + 1}{n} \right\rceil - 2 \mid 1 \leq k \leq N_i\right\} \]

(c) \quad (N_i + 1)\lambda_1 + i > \bar{\mu}^{(\geq 0)}. \]

Then

\[ [R_i]_n = \left\{(k, \left\lfloor \frac{\lambda_i + 1}{n} \right\rfloor + k - 1) \mid 1 \leq k \leq N_i\right\} \cup \{(N_i, N_i)\}
\]

\[ \cup \left\{(k + 1 - \left\lceil \frac{\lambda_i + 1}{n} \right\rceil, k) \mid 1 \leq k \leq N_i\right\} \]

\[ \{D(x) \mid x \in R_i\} = \left\{\left\lfloor \frac{\lambda_i + 1}{n} \right\rfloor \mid 1 \leq k \leq N_i\right\} \cup \{0\} \cup \left\{\left\lceil \frac{\lambda_i + 1}{n} \right\rceil - 2 \mid 1 \leq k \leq N_i\right\} \]

**Example 9.18.** Recall the example above when \( n = 15, \lambda = (5, 4, 4, 2), \bar{\rho}' = (0, 4, 0), \) and \( w' = \Psi(T_{(4, 4, 2)} T_{(4, 4, 2)} (0, 3, 0)) = [-21, -20, -8, -7, 5, 6, 32, 33, 34, 46, 0, 0, 0, 0]. \)

We have

\[ \{\text{st}(w)^{(-1)}\} \cup R'_1 = \{(0, 0), \tilde{A}_{0.1}, \tilde{B}_{1.1}, \tilde{C}_{0.1}\} = \{(0, 0), (10, 46), (20, 20), (32, 10)\}, \]

\[ \{\text{st}(w)^{(-2)}\} \cup R'_2 = \{(-1, -1), \tilde{A}_{0.2}, \tilde{C}_{0.2}\} = \{(-1, -1), (9, 34), (31, 9)\}, \]

\[ \{\text{st}(w)^{(-3)}\} \cup R'_3 = \{(-1, -1), \tilde{A}_{0.3}, \tilde{C}_{0.3}\} = \{(-2, -2), (8, 33), (19, 8)\}, \]

\[ \{\text{st}(w)^{(-4)}\} \cup R'_4 = \{(-1, -1), \tilde{A}_{0.4}, \tilde{C}_{0.4}\} = \{(-3, -3), (7, 32), (18, 7)\}, \]

\[ \{\text{st}(w)^{(-5)}\} \cup R'_5 = \{(-1, -1), \tilde{B}_{0.5}\} = \{(-4, -4), (6, 6)\}. \]

Therefore

\[ \begin{align*}
\{\text{st}(w)^{(-1)}\} \cup R'_1 & = \{(-1, -1), (0, 3), (1, 1), (2, 0)\}, \\
\{\text{st}(w)^{(-2)}\} \cup R'_2 & = \{(-1, -1), (0, 2), (2, 0)\}, \\
\{\text{st}(w)^{(-3)}\} \cup R'_3 & = \{(-1, -1), (0, 2), (1, 0)\}, \\
\{\text{st}(w)^{(-4)}\} \cup R'_4 & = \{(-1, -1), (0, 2), (1, 0)\}, \\
\{\text{st}(w)^{(-5)}\} \cup R'_5 & = \{(-1, -1), (0, 0)\}.
\end{align*} \]
By applying the backward AMBC algorithm to each $[R'_n]_n$, we get Therefore
\[
[R_1]_n = \{(-1, 3), (0, 1), (1, 0), (2, -1)\},
\]
\[
[R_2]_n = \{(-1, 2), (0, 0), (2, -1)\},
\]
\[
[R_3]_n = \{(-1, 2), (0, 0), (1, -1)\},
\]
\[
[R_4]_n = \{(-1, 2), (0, 0), (1, -1)\},
\]
\[
[R_5]_n = \{(-1, 0), (0, -1)\}.
\]

Note that this is consistent with
\[
R_1 = \{(0, 46), (10, 20), (20, 10), (32, 0)\},
\]
\[
R_2 = \{(-1, 34), (9, 9), (31, -1)\},
\]
\[
R_3 = \{(-2, 33), (8, 8), (19, -2)\},
\]
\[
R_4 = \{(-3, 32), (7, 7), (18, -3)\},
\]
\[
R_5 = \{(-4, 6), (6, -4)\}.
\]

Furthermore, we have
\[
\{D(x) \mid x \in R'_1\} = \{3, 0, -2\},
\]
\[
\{D(x) \mid x \in R'_2\} = \{2, -2\},
\]
\[
\{D(x) \mid x \in R'_3\} = \{D(x) \mid x \in R'_4\} = \{2, -1\},
\]
\[
\{D(x) \mid x \in R'_5\} = \{0\},
\]
and
\[
\{D(x) \mid x \in R'_1\} = \{4, 1, -1, -3\},
\]
\[
\{D(x) \mid x \in R'_2\} = \{3, 0, -3\},
\]
\[
\{D(x) \mid x \in R'_3\} = \{3, -0, -2\},
\]
\[
\{D(x) \mid x \in R'_4\} = \{1, -1\}.
\]

From this calculation, we derive the relations between $\tilde{\mu}$ and $\tilde{\mu}'$. (Recall that $N_i = \left\lfloor \frac{\tilde{\mu}(\leq 0) - i}{\lambda_i} \right\rfloor$.)

- For $j > 0$, $\tilde{\mu}^{(j+1)} = \tilde{\mu}^{(j)}$ and $\tilde{\mu}^{(-j-1)} = \tilde{\mu}^{(-j)}$.
- $\tilde{\mu}^{(0)} = \#\{i \mid 1 \leq i \leq \lambda_1, (N_i + 1)\lambda_1 + i > \tilde{\mu}(\geq 0)\}$
- $\tilde{\mu}^{(1)} = \#\{i \mid 1 \leq i \leq \lambda_1, \tilde{\mu}(\geq 0) < (N_i + 1)\lambda_1 + i \leq \tilde{\mu}(\geq 0)\}$
- $\tilde{\mu}^{(-1)} = \#\{i \mid 1 \leq i \leq \lambda_1, (N_i + 1)\lambda_1 + i \leq \tilde{\mu}(\geq 0)\}$

Now it is clear from the description that $\tilde{\mu}^{(1)} = \tilde{\mu}(\geq 0) - \tilde{\mu}(\leq 0) = \tilde{\mu}(0)$. Also by Lemma 9.11
\[
\tilde{\mu}^{(-1)} = \tilde{\mu}(\geq 0) - \tilde{\mu}(\leq 0) = \tilde{\mu}(0) - \tilde{\mu}(0)(\leq 0) = \tilde{\mu}(0)^{(0)} = \sum_{k \geq 1} (-1)^{k-1} \lambda_k.
\]

Since $\sum_{j \in \mathbb{Z}} \tilde{\mu}^{(j)} = \lambda_1 + \sum_{j \in \mathbb{Z}} \tilde{\mu}^{(j)}$, it follows that $\tilde{\mu}^{(0)} = \sum_{k \geq 2} (-1)^{k-1} \lambda_k$.

Now suppose that $\tilde{\mu}' = \tilde{\mu}(0)$. Then it is easy to see that $\tilde{\mu}(0)^{(j+1)} = \tilde{\mu}(0)^{(j)}$, $\tilde{\mu}(0)^{(-j-1)} = \tilde{\mu}(0)^{(-j)}$ for $j \geq 0$ and $\tilde{\mu}(0)^{(0)} = \#\{i \mid 1 \leq i \leq \lambda_1, \lambda'_i \in 2\mathbb{Z}\} = \sum_{k \geq 2} (-1)^{k-1} \lambda_k$. By comparing this with the relations between $\tilde{\mu}$ and $\tilde{\mu}'$, we conclude that $\tilde{\mu} = \tilde{\mu}(0)$. 

This time suppose that \( \overline{\mu}' = \overline{\mu}(r, s) \) where \( r \in \tilde{\lambda} \). Then by Lemma A.10 and the argument above it is still true that \( \overline{\mu}(s, r)^{(j-1)} = \overline{\mu}(s, r)^{(j-1)} \) for \( j > 0 \) and also

\[
\overline{\mu}(s, r)^{(1)} = \overline{\mu}(0)^{(1)} = \# \{ i \mid 1 \leq i \leq \lambda_1, \lambda'_1 - 1 \in 2\mathbb{Z} \} = \sum_{k \geq 1} (-1)^{k-1} \lambda_k
\]

\[
\overline{\mu}(s, r)^{(0)} = \overline{\mu}(0)^{(0)} = \# \{ i \mid 1 \leq i \leq \lambda_1, \lambda'_1 \in 2\mathbb{Z} \} = \sum_{k \geq 2} (-1)^k \lambda_k.
\]

(The second equation holds since \( r \leq \lambda_2 \).) Also from the construction of \( \mathcal{M}_\lambda(r, s) \) and \( \mathcal{M}_\lambda(r, s) \), it is also easy to show that \( \overline{\mu}(s, r)^{(j+1)} = \overline{\mu}(s, r)^{(j)} \) for \( j > 0 \). Now it follows from \( \sum_{j \in \mathbb{Z}} \overline{\mu}(s, r)^{(j)} = \lambda_1 + \sum_{j \in \mathbb{Z}} \overline{\mu}(s, r)^{(j)} \) that \( \overline{\mu}(s, r)^{(1)} = \overline{\mu}(s, r)^{(0)} \). But this also shows that \( \overline{\mu} = \overline{\mu}(s, r) \) by considering the relations between \( \overline{\mu} \) and \( \overline{\mu}' \).

This completes the proof of Theorem 8.1.

### Appendix A. Asymptotic Hecke algebras for SL\(_n\) and PGL\(_n\)

So far, we discussed the extended affine symmetric group and its asymptotic Hecke algebra. In terms of representation theory, it means that we only considered the affine Weyl group of \( GL_n \). In this section, we observe how our result can be extended to simple groups of type \( A \), especially \( SL_n \) and \( PGL_n \). For simplicity, from now on we assume that \( GL_n \), \( SL_n \), and \( PGL_n \) are all defined over \( \mathbb{C} \).

#### A.1. \( G = SL_n \) case

Note that \( \omega^n \in S_n \) is in the center of \( S_n \). The affine Weyl group of \( SL_n \) can be identified with \( S_n / \langle \omega^n \rangle \), and its two-sided cell (resp. left cell, resp. right cell) is given by the image of each cell of \( S_n \) under the quotient map \( \pi : S_n \rightarrow S_n / \langle \omega^n \rangle \). In particular, its two-sided cells are also parametrized by the partitions of \( n \).

Likewise, for a two-sided cell \( \underline{\mathcal{C}} \) of \( S_n \) we define \( \mathcal{J}_\underline{\mathcal{C}} := \mathcal{J}_{\underline{\mathcal{C}}}(\omega^n - 1) \). Then it is clear that this is an asymptotic Hecke algebra corresponding to \( SL_n \) attached to the two-sided cell \( \pi(\underline{\mathcal{C}}) \). Now the following theorem gives a structural description of \( \mathcal{J}_{\underline{\mathcal{C}}}^{SL} \) in terms of Theorem 7.2.

**Theorem A.1.** For \( \underline{\mathcal{C}} = \underline{\mathcal{C}}_\lambda \), the isomorphism in Theorem 7.2 factors through

\[
\mathcal{J}_{\underline{\mathcal{C}}}^{SL} \simeq \text{Mat}_{\chi \times \chi}(\mathcal{R}(F_\lambda))/(\omega^n - 1) \simeq \text{Mat}_{\chi \times \chi}(\mathcal{R}(F_\lambda))/(V(\lambda))
\]

where both \( \mathcal{J}_{\underline{\mathcal{C}}} \) and \( \text{Mat}_{\chi \times \chi}(\mathcal{R}(F_\lambda)) \) are regarded as \( \mathbb{Z}[\omega^n] \)-algebras and \( V(\lambda) \) is the irreducible representation of \( F_\lambda \) of highest weight \( \lambda = (\lambda_1, \lambda_2, \ldots) \in \text{Dom}(F_\lambda) \). In particular, \( \mathcal{J}_{\underline{\mathcal{C}}}^{SL} \) is a matrix algebra.

**Proof.** The first isomorphism is obvious. The second isomorphism follows from the fact that if \( \Phi(w) = (P, Q, \bar{\rho}) \) then \( \Phi(\omega^n) = (P, Q, \bar{\rho} + \lambda) \) which follows from Proposition 5.6. \( \square \)

**Remark.** A similar statement can also be found in [Xi02 Section 8.4].
A.2. $G = PGL_n$ case. Consider the case when $G = PGL_n$. Then its affine Weyl group is

$$\mathfrak{S}_n := \{w \in \mathfrak{S}_n | \sum_{i=1}^n w(i) = n(n + 1)/2\},$$

i.e. the (non-extended) affine symmetric group. For an integer sequence $(a_1, a_2, \ldots)$, let us define $|a_1, a_2, \ldots| := \sum_i a_i$. Then by [CPY18, Theorem 10.3], we have

$$\Phi(\mathfrak{S}_n) = \{(P, Q, \vec{\rho}) \in \Omega_{dom} | |\vec{\rho}| = 0\}$$

Also its two-sided cell (resp. left cell, resp. right cell) is obtained from the intersection of such a cell in $\mathfrak{S}_n$ with $\mathfrak{S}_n$. In particular, the two-sided cells of $\mathfrak{S}_n$ are also parametrized by the partitions of $n$.

Let $\mathcal{J}_P^{\text{PGL}} \subset \mathcal{J}_n$ be the asymptotic Hecke algebra corresponding to $PGL_n$ attached to $\mathcal{S} \cap \mathfrak{S}_n$. In general, $\mathcal{J}_P^{\text{PGL}}$ is no longer a matrix algebra; a counterexample is given in [Xi02, Section 8.3]. Here, we discuss a sufficient condition when $\mathcal{J}_P^{\text{PGL}}$ is indeed isomorphic to a matrix algebra.

**Theorem A.2.** Suppose that $\mathcal{S} = \mathcal{S}_\lambda$. If $\gcd(\lambda'_1, \lambda'_2, \ldots) = 1$, then there exists a ring isomorphism $\mathcal{J}_P^{\text{PGL}} \simeq \text{Mat}_{\lambda \times \lambda}(\mathcal{R}(\bar{F}_\lambda))$ where $\bar{F}_\lambda = F_\lambda/\{cI \in F \}$. 

**Proof.** As $\gcd(\lambda'_1, \lambda'_2, \ldots) = 1$, by [CLP17, Theorem 8.6] any tabloid $T$ of shape $\lambda$ can be obtained from $T'_\lambda$ by successive star operations (without applying $\omega$). Now for each $T \in \text{RSYT}(\lambda)$, we fix a series of such star operations. Then it induces a bijection $\phi_T : \Gamma^\lambda \to \Gamma_T$ defined by the composition of right star operations. Here, $\Gamma_T$ is the left cell parametrized by $T$, i.e. $w \in \Gamma_T$ if and only if $Q(w) = T$. Also note that star operations stabilizes $\mathfrak{S}_n \subset \mathfrak{S}_n$, which implies that it restricts to $\phi_T : \Gamma^\lambda \cap \mathfrak{S}_n \to \Gamma_T \cap \mathfrak{S}_n$.

Now for any tabloid $T$ of shape $\lambda$, we define $w_T := \phi_T(w_0^\lambda)$ where $w_0^\lambda$ is as in Section 3. Then it follows that $\Phi(w_T) = (T^\lambda, T, \vec{\rho}_T + \vec{s}_T)_{T^\lambda, T}$ for some $\vec{\rho}_T = (\rho_1, \rho_2, \ldots)$ such that $|\vec{\rho}_T| + |\vec{s}_T| = 0$. By similar argument to Lemma 7.2, we see that $\vec{\rho}_T$ is determinantal and $\tau_{w_T} = t(T^\lambda, T, \vec{\rho}_T)T_{w_T}$. Note that $t(T^\lambda, T, \vec{\rho})$ and $T_{w_T}$ are not in general contained in $\mathcal{J}_P^{\text{PGL}}$, but $\tau_{w_T}$ is an element of $\mathcal{J}_P^{\text{PGL}}$.

For $P, Q \in \text{RSYT}(\lambda)$, let us define $t_{P, Q} := (t_{w_T})^{-1}t_{w_T} \in \mathcal{J}_P^{\text{PGL}}$. Then it is clear that $t_{P, Q} = t(P, P, \vec{\rho}_Q - \vec{\rho}_P)_{T_{P, Q}}$. As $\vec{\rho}_Q - \vec{\rho}_P$ is determinantal, $\{t_{P, Q} | P, Q \in \text{RSYT}(\lambda)\}$ also gives a matrix basis of $\mathcal{J}_P^{\text{PGL}}$. i.e. we have an algebra isomorphism $\mathcal{Y} : \mathcal{J}_P^{\text{PGL}} \to \text{Mat}_{\lambda \times \lambda}(\mathcal{R}(\bar{F}_\lambda))$ where $\mathcal{Y}(t_{P, Q})$ is an elementary matrix for any $P, Q \in \text{RSYT}(\lambda)$. (In general $\mathcal{Y}'$ is different from $\mathcal{Y}$ defined in Theorem 7.2 since $\vec{\rho}_Q - \vec{\rho}_P$ needs not be zero.)

It remains to show that $\mathcal{Y}(\mathcal{J}_P^{\text{PGL}}) = \text{Mat}_{\lambda \times \lambda}(\mathcal{R}(\bar{F}_\lambda))$; here we identify $\mathcal{R}(\bar{F}_\lambda)$ with the subring of $\mathcal{R}(\bar{F}_\lambda)$ generated by $V(\bar{\mu})$ such that $|\bar{\mu}| = 0$. Recall that for any $w \in \mathfrak{S}_n$ we have $\Phi(w) = (P, Q, \vec{\rho})$ such that $|\vec{\rho}| = 0$. In particular,

$$t_w = t(P, P, \vec{\rho}_T) = t(P, P, \vec{\rho} - \vec{s}_{P, Q})t_{P, Q}$$

$$= t(P, P, \vec{\rho} - \vec{s}_{P, Q} + \vec{\rho}_P - \vec{\rho}_Q)t_{P, Q}$$

where

$$|\vec{\rho} - \vec{s}_{P, Q} + \vec{\rho}_P - \vec{\rho}_Q| = |\vec{\rho}_P| - |\vec{\rho}_Q| - |\vec{s}_{P, Q}| = -|\vec{s}_T| + |\vec{s}_{T^\lambda}| - |\vec{s}_{P, Q}| = 0$$
since $\mathbb{T}_{\lambda}^\text{n} - \mathbb{T}_{\lambda}^\text{m} = \mathbb{T}_{\lambda}^\text{Q}$. As $\Upsilon'(t_w)$ is a matrix whose only nonzero entry corresponds to $V(\mathbb{T}_{\lambda}^\text{n} - \mathbb{T}_{\lambda}^\text{m} + \mathbb{T}_{\lambda}^\text{P} - \mathbb{T}_{\lambda}^\text{Q}) \in R(\tilde{F}_{\lambda})$, we conclude that $\Upsilon'(\mathcal{J}_{\text{PGL}}^\text{L}) \subset \text{Mat}_{\chi \times \chi}(R(\tilde{F}_{\lambda}))$. The other inclusion also follows in almost the same manner. □

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