TIMELIKE RICCI BOUNDS FOR LOW REGULARITY
SPACETIMES BY OPTIMAL TRANSPORT

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Abstract. We prove that a globally hyperbolic smooth spacetime endowed with a $C^1$-Lorentzian metric whose Ricci tensor is bounded from below in all timelike directions, in a distributional sense, obeys the timelike measure-contraction property. This result includes a class of spacetimes with borderline regularity for which local existence results for the vacuum Einstein equation are known in the setting of spaces with timelike Ricci bounds in a synthetic sense. In particular, these spacetimes satisfy timelike Brunn–Minkowski, Bonnet–Myers, and Bishop–Gromov inequalities in sharp form, without any timelike nonbranching assumption.

If the metric is even $C^{1,1}$, in fact the stronger timelike curvature-dimension condition holds. In this regularity, we also obtain uniqueness of chronological optimal couplings and chronological geodesics.

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1. Introduction

Background. In the last two decades, optimal transport theory has been applied to a large variety of mathematical areas, including PDEs, Riemannian geometry, numerical analysis, etc. More recently, it has revealed promising links to general relativity, i.e. Einstein’s theory of gravity, as follows. Let $\mathcal{M}$ be a smooth spacetime of dimension $n \in \mathbb{N}_{\geq 2}$, endowed with a globally hyperbolic Lorentzian metric $g$ —
physically, one should always think of \((M, g)\) to solve the Einstein equation with given cosmological constant \(\Lambda \in \mathbb{R}\) and energy-momentum tensor \(T\). If \(g\) is smooth\(^1\), \(\text{[33, 36]}\) and later \(\text{[3, 4]}\) showed convexity properties of certain entropy functionals with respect to the volume measure \(\text{vol}_g\) along “chronological” geodesics in \(\mathcal{P}(M)\) to characterize the condition

\[\text{Ric}_g \geq K\] in all timelike directions. \hspace{1cm} (1.1)

Here, \(\mathcal{P}(M)\) is the space of Borel probability measures on \(M\). The relevant geometry thereon is described by a certain Lorentzian transport cost \(\ell_{g,p}\), where \(p \in (0,1)\), which plays the role of a time separation function.

The condition (1.1) has high relevance in general relativity. Indeed, for \(\Lambda = 0\), (1.1) for \(K = 0\) is equivalent to the strong energy condition of Hawking and Penrose \(\text{[19, 20]}\). Moreover \(\text{[7]}\), for arbitrary \(\Lambda \in \mathbb{R}\), if \(\inf_{M} \text{scal}_{g}(M) > -\infty\) then (1.1) for \(K = \inf_{M} \text{scal}_{g}(M)/2 - \Lambda\) is implied by the weak energy condition \(\text{“} T \geq 0 \text{“ in all timelike directions} \). The latter is believed to hold for most physically reasonable \(T\) \(\text{[51, p. 218]}\), and clearly holds in the vacuum case \(T = 0\). See \(\text{[5, 19, 33, 36, 49, 51]}\) for further discussions about (1.1).

On the other hand, said groundbreaking discoveries of \(\text{[33, 36]}\) have lead to the synthetic theory of TCD and TMCP spaces via the Boltzmann entropy \(\text{[7]}\) and later via the Rényi entropy \(\text{[3]}\) in the framework of measured Lorentzian spaces \(\text{[7, 27]}\), i.e. natural generalizations of spacetimes. The definitions in \(\text{[3, 7]}\) are partly equivalent \(\text{[3, Thm. 3.35, Thm. 4.20]}\), yet the approach in \(\text{[3]}\) yields quantitatively stronger geometric properties \(a \text{ priori}\), as made precise below. TCD and TMCP spaces are Lorentzian analogues of CD and MCP metric measure spaces \(\text{[12, 31, 38, 46, 47]}\). Roughly speaking, to make sense the respective formulations of TCD and TMCP only require a space with an abstract notion of time separation function, encoding how geodesics of points on \(M\) and probability measures look like, and a reference measure, encoding how to “average sectional curvatures”. Hence, these conditions are suitable to give a meaning to (1.1) even when no smooth structure is available to define the inherent Ricci tensor itself.

**Objective.** The aim of this work is to provide a first link of this novel synthetic point of view to the more customary analytic approach to (1.1) for Lorentzian metrics of low regularity by distribution theory; see e.g. \(\text{[15, 17, 30, 42, 45]}\) for previous works on distributional energy conditions.

We will start with a Lorentzian metric \(g\) on \(M\) of regularity at least \(C^1\) obeying (1.1) — and weighted versions thereof — in a distributional sense. Then we prove that the measured Lorentzian space canonically induced by \((M, g)\) according to Section 2.2 below has timelike (Bakry–Émery–)Ricci curvature bounded from below by \(K\) in the indicated synthetic senses. As applications, inter alia we derive timelike geometric inequalities. Notably, these are obtained in sharp form even though the regularity of \(g\) might be below \(C^{1,1}\), where \(g\) might admit timelike branching \(\text{[7, Def. 1.10]}\), and the localization technique \(\text{[7, Ch. 4, Sec. 5.3, Sec. 5.4]}\) used in \(\text{[7]}\) to prove these sharp inequalities in the synthetic setting does not apply.

This partly answers a question raised in \(\text{[25]}\). There \(\text{[25, Thm. 5.4]}\), smooth manifolds with \(C^1\)-Riemannian metrics and distributional Ricci bounds are shown to be CD spaces. A partial converse holds as well \(\text{[25, Thm. 6.3]}\), yet a Lorentzian analogue of this is beyond the scope of our work.

Our main results provide a set of concrete examples of TCD and TMCP spaces beyond “sufficiently regular” spacetimes. Moreover, as concretized further below, the proofs of our main results — Theorem 3.1 and Theorem 3.2 — are based on an approximation argument and are, as such, heavily inspired by the proof of stability.

\(^1\)In fact, \(C^2\)-regularity suffices for the arguments in \(\text{[3, 33, 36]}\).
of TCD and TMCP [7], see also [3], under the novel notion of weak convergence of measured Lorentzian spaces introduced in [7, Thm. 3.12]; see Remark 1.4 for a discussion of how this relates to (open) stability questions.

The mathematical relevance of our setting comes from the PDE point of view, where standard local existence results for the vacuum Einstein equation, together with Sobolev’s embedding theorem, in four dimensions just grant $C^1$-regularity of $g$ [40, p. 10], see also [9, 24]. In general, since the Einstein equation is hyperbolic, its solutions are typically not smooth, which makes the synthetic TCD and TMCP framework interesting to study its rough solutions. From a geometric perspective, $C^1$ [15] and $C^{1,1}$ [28] are the lowest regularities under which the classical Hawking singularity theorem [18, 19] has been proven under distributional timelike Ricci bounds. (See [15, 26] for $C^1$-versions of the Hawking–Penrose singularity theorem, and [43] for an overview over singularity theorems in general relativity.) Incidentally, our results build a first bridge between [15, 28] and the synthetic Hawking singularity theorem for timelike nonbranching low regularity spacetimes from [7, Thm. 5.6, Cor. 5.13]. Indeed, by Theorem 3.2 and Remark 2.3, the distributional $C^{1,1}$-versions from [28] are included in [7] (in the sense that the assumptions in [7] really extend those of [28]). In a similar kind, by Theorem 3.1, timelike nonbranching $C^1$-spacetimes with distributional timelike Ricci bounds as in [15] are covered by [7]. However, unlike the $C^{1,1}$-case, $C^1$-spacetimes are generally expected to admit timelike branching, hence [7] remains unknown to apply to some spaces from [15].

Results. Now we outline our main results. In order to keep the presentation light, we postpone technicalities and more precise definitions to Chapter 2.

Let $g$ be a globally hyperbolic Lorentzian metric on $M$ with regularity at least $C^1$. We write $\leq_g$ and $\ll_g$ for the future-directed $g$-causality and future-directed $g$-chronology on $M$, respectively. Let $\tau_g$ denote the usual time separation function induced by $g$, i.e. if $x, y \in M$ obey $x \leq_g y$ then $\tau_g(x, y)$ constitutes the maximal $g$-length of all future-directed $g$-causal curves from $x$ to $y$. Let $V \in C^1(M)$ and $N \in [n, \infty)$, and define the $N$-Bakry–Émery–Ricci tensor by

$$\text{Ric}_{g, V} := \text{Ric}_g + \text{Hess}_g V + \frac{1}{N - n} DV \otimes DV.$$  

It is understood in a distributional sense; lower bounds on it à la (1.1) are then formulated by requiring $(\text{Ric}_{g, V}^N(X, X), \mu) \geq K \langle g(X, X), \mu \rangle$ for every smooth $g$-timelike vector field $X$ on $M$ and every nonnegative, compactly supported volume density $\mu$. Of course, if $g$ is of class $C^2$, all expressions and inequalities make sense in the ordinary pointwise way.

To further ease the presentation, in the subsequent outline we restrict ourselves to the case $K = 0$, $V = 0$, and $N = n$. It is already interesting since it represents the strong energy condition of Hawking and Penrose in low regularity, because $\text{Ric}_{g, V}^N$ then simply becomes the usual Ricci tensor $\text{Ric}_g$. Our results — with appropriate modifications — will hold and be shown in all generality.

Under these simplifications, the formulation of our results still requires some further notation. For $p \in (0, 1)$, let $L_{g,p}: \mathcal{P}(M)^2 \to [0, \infty] \cup \{-\infty\}$ be the so-called $p$-Lorentz–Wasserstein distance [11] defined through

$$L_{g,p}^p(\mu, \nu) := \sup \left\| \tau_g \right\|_{L^p(M^2, \pi)}.$$  

Here, the supremum is taken over all $g$-causal couplings $\pi$ of $\mu$ and $\nu$. The latter notion means that $\pi$ is a Borel probability measure on $M^2$ with marginals $\mu$ and $\nu$ such that $x \leq_g y$ for $\pi$-a.e. $(x, y) \in M^2$. The role of $L_{g,p}$ is to “lift” the $g$-causal structure of $M$ to the space of mass distributions on $M$. And it enables us to define
timelike proper-time parametrized $\ell_{g,p}$-geodesics which, roughly speaking, are curves $(\mu_t)_{t \in [0,1]}$ in $\mathcal{P}(\mathcal{M})$ such that for every $s, t \in [0,1]$ with $s < t$,
\[
\ell_{g,p}(\mu_s, \mu_t) = (t - s) \ell_{g,p}(\mu_0, \mu_1) > 0.
\]
Lastly, the canonical volume measure $\text{vol}_g$ induced by $g$ is invoked through the $n$-Rényi entropy $S^n_g: \mathcal{P}(\mathcal{M}) \to [-\infty, 0]$; subject to the Lebesgue decomposition $\mu = \rho \text{vol}_g + \mu^\perp$, the latter is defined by
\[
S^n_g(\mu) := -\int_{\mathcal{M}} \rho^{1/N} \, d\mu = -\int_{\mathcal{M}} \rho^{1-1/N} \, d\text{vol}_g.
\]

**Theorem 1.1.** Assume $g$ to be $C^1$, and suppose it satisfies the distributional strong energy condition (1.1). Then for every $p \in (0,1)$ and every compactly supported, $m$-absolutely continuous mass distributions $\mu_0, \mu_1 \in \mathcal{P}(\mathcal{M})$ such that $x \ll_g y$ for every $x \in \text{spt} \, \mu_0$ and every $y \in \text{spt} \, \mu_1$, there exists a timelike proper-time parametrized $\ell_{g,p}$-geodesic $(\mu_t)_{t \in [0,1]}$ connecting $\mu_0$ to $\mu_1$ such that every $t \in [0,1]$ satisfies
\[
S^n_g(\mu_t) \leq (1 - t) S^n_g(\mu_0) + t S^n_g(\mu_1).
\]
In other words, the distributional strong energy condition for $g$ implies displacement convexity [32] of the $n$-Rényi entropy with respect to $\text{vol}_g$ along appropriate “timelike” geodesics in $\mathcal{P}(\mathcal{M})$.

Theorem 1.1 is our main technical ingredient. It immediately implies our main results. First, the measured Lorentzian space induced by $(\mathcal{M}, g, \text{vol}_g)$ satisfies the $p$-independent timelike measure-contraction property TMCP$(0, n)$ under the same assumptions on $g$ as in Theorem 1.1, cf. Theorem 3.1. Second, if $g$ is $C^{1,1}$, we even get the stronger timelike curvature-dimension condition TCD$_p(0, n)$ in Theorem 3.2. As indicated above, both assert the displacement convexity of $S^n_g(\mu)$ along “timelike” geodesics in $\mathcal{P}(\mathcal{M})$ which connect an $m$-absolutely continuous $\mu_0$ to a Dirac mass $\mu_1$ (TMCP), or two $m$-absolutely continuous $\mu_0$ and $\mu_1$ (TCD), respectively, obeying a chronology relation called $g$-timelike $p$-dualizability [7], cf. Definition 2.10. In general, the latter is strictly weaker than the chronology hypothesis on $\mu_0$ and $\mu_1$ in Theorem 1.1, which forces all geodesics starting in $\text{spt} \, \mu_0$ and ending in $\text{spt} \, \mu_1$ to stay away from the $g$-light cone in a “uniform” manner.

**Remark 1.2.** To be precise, three different timelike curvature-dimension conditions have been proposed in [3, 7] for general $K \in \mathbb{R}$ and $N \in [1, \infty)$. While [7] sets up the so-called entropic timelike curvature-dimension condition TCD$^p_e(K, N)$ by concavity properties of an exponentiated Boltzmann entropy similar to [12], [3] defines the timelike curvature-dimension condition TCD$_p(K, N)$ and its reduced version TCD$_p^*(K, N)$ after [1, 47]. We refer the reader to Remark 2.15 for general (and expected) relations between these notions.

We will concentrate on showing the second named version in our work. Our proofs can easily be adapted to derive the other conditions as well.

An analogous note applies to the three timelike measure-contraction properties TMCP$^e(K, N)$, TMCP$(K, N)$, and TMCP$^*(K, N)$ from [3, 7].

From this, we directly infer the subsequent timelike geometric inequalities under the assumption on $g$ from Theorem 1.1. They are stated in a precise and slightly more general form in Corollary 3.9, Corollary 3.10, and Corollary 3.11.

- **Brunn–Minkowski inequality.** Let $A_0, A_1 \subset \mathcal{M}$ be two relatively compact Borel sets obeying $x \ll_g y$ for every $x \in A_0$ and every $y \in A_1$. Given any $t \in [0,1]$, the set $A_t \subset \mathcal{M}$ of all $t$-intermediate points of future directed $g$-timelike geodesics starting in $A_0$ and ending in $A_1$ obeys
\[
\text{vol}_g[A_t]^{1/n} \geq (1 - t) \text{vol}_g[A_0]^{1/n} + t \text{vol}_g[A_1]^{1/n}.
\]
• Bishop–Gromov inequality. Let \( x \in \mathcal{M} \), and let \( E \subseteq \mathcal{M} \) be a compact set which is \( \tau_g \)-star shaped with respect to \( x \). For \( r > 0 \), let \( v_r \) denote the measure of the truncated hyperboloid \( \{ y \in \mathcal{M} : x \leq_y y, \tau_g(x, y) \leq r \} \cap E \) with respect to \( \text{vol}_g \). Then the quantity \( \frac{v_r}{R^n} \) is nonincreasing in \( r > 0 \). In other words, for every \( r, R > 0 \) with \( R > r \),
\[
\frac{v_r}{v_R} \geq \frac{r^n}{R^n}.
\]
Assuming the more restrictive distributional energy condition \( \text{Ric}_g \geq K \) in all timelike directions for \( K > 0 \), we also obtain the

• Bonnet–Myers inequality. We have
\[
\sup \tau_g(\mathcal{M}^2) \leq \pi \sqrt{\frac{n-1}{K}}.
\]

In our low regularity framework, starting from a distributional energy condition à la (1.1) the only comparable result we are aware of is a Bishop–Gromov inequality in the \( C^{1,1} \)-case \[14\]. On the other hand, in the synthetic approach to timelike Ricci curvature lower bounds all these estimates are standard consequences, as shown in [7] and later in [3]. By virtue of the approach from [3] we follow as outlined in Remark 1.2 above, all three named inequalities are obtained in sharp form; cf. Remark 3.12 for further details.

Lastly, if \( g \) is \( C^{1,1} \), in Corollary 3.13 we obtain uniqueness of chronological \( \ell_{g,\rho} \)-optimal couplings, uniqueness of “timelike” geodesics in \( \mathcal{P}(\mathcal{M}) \), and solvability of a Lorentzian Monge problem under distributional timelike (Bakry–Émery–)Ricci curvature lower bounds. Here, as already outlined above, the crucial feature of working with a \( C^{1,1} \)-metric is that it ensures timelike nonbranching, a natural condition to get said uniquenesses. In analogy to the Riemannian fact \[25\], Thm. 2.3, these uniqueness claims should be true without curvature assumptions, yet we do not address this generalization in our work. An advantage of the curvature hypothesis plus timelike nonbranching is that the TCD inequality automatically holds pathwise along “timelike” geodesics of probability measures \[3\], Thm. 3.41).

Outline of the proof of Theorem 1.1. The argument for our main results relies on a suitable approximation of \( g \) by smooth Lorentzian metrics which locally do not violate (1.1) too much. A very rough version of the relevant Lemma 2.8, yet conveying the key ideas for now, is the following.

**Lemma 1.3.** Assume \( g \) to satisfy (1.1). Then there exist smooth Lorentzian metrics \( \{ \tilde{g}_\varepsilon : \varepsilon > 0 \} \) such that \( \tilde{g}_\varepsilon \to g \) in \( C^{1,\text{loc}}(\mathcal{M}) \) as \( \varepsilon \to 0 \) and with the following property. For every compact \( C \subseteq \mathcal{M} \) and every \( \delta, \kappa > 0 \), there exists \( \varepsilon_0 > 0 \) such that for every \( \varepsilon \in (0, \varepsilon_0) \) and every \( v \in TX|_C \), we have
\[
|v|_{\tilde{g}_\varepsilon} \geq \sqrt{\kappa} \implies \text{Ric}_{\tilde{g}_\varepsilon}(v, v) \geq -\delta |v|_{\tilde{g}_\varepsilon}^2.
\]

This approximation result itself, at least in the unweighted case, is not new. In \[15, 28\], it has been employed to prove Hawking’s singularity theorem in \( C^1 \) and \( C^{1,1} \)-regularity, respectively. It is the technical reason for our imposed regularity on \( g \); a version of it e.g. for Lipschitz metrics remains unknown. The mentioned Riemannian result in \( C^{1,1} \)-regularity \[25\], Thm. 5.4] has been derived from a similar approximation procedure \[25\], Thm. 4.3].

Our argument for Theorem 1.1 follows the proof of \([3\), Thm. 3.29] for the weak stability of the TCD condition. In Subsection 3.2.3 below, given \( \{ \tilde{g}_\varepsilon : \varepsilon > 0 \} \) as in Lemma 1.3, for \( \mu_0 \) and \( \mu_1 \) as hypothesized with \( \kappa \propto \inf \tau_g(\text{spt } \mu_0 \times \text{spt } \mu_1) > 0 \),
we construct a recovery family \( \{ (\mu_0, \mu_1) : \varepsilon > 0 \} \) of \( g_{0,\varepsilon} \)-timelike \( p \)-dualizable pairs \( (\mu_0, \mu_1) \) for \((\mu_0, \mu_1)\), where \( \mu_0 \) and \( \mu_1 \) are absolutely continuous with respect to the volume measure \( \text{vol}_{g_{0,\varepsilon}} \). This is done in such a way that the unique \[ \ell_{g_{0,\varepsilon}} \]-optimal transport from \( \mu_0 \) to \( \mu_1 \) only matches points with \( \tau_{g_{0,\varepsilon}} \)-distance larger than \( \kappa \). This property, combined with Lemma 1.3, ensures displacement semiconvexity — with \( \hat{g}_{\varepsilon} \)-timelike Ricci lower bound \( -\delta \) — of the \( n \)-Rényi entropy with respect to \( \text{vol}_{g_{0,\varepsilon}} \) between \( \mu_0 \) and \( \mu_1 \) for sufficiently small \( \varepsilon > 0 \) (depending on the values of \( \delta \) and \( \kappa \)), which we establish by hand in Subsection 3.2.4 following the argument for [3, Prop. A.3]. Up to subsequences, it then remains to first let \( \varepsilon \to 0 \) and then \( \delta \to 0 \); the relevant inequalities are stable under these limits essentially because as a statement about convexity, they are of zeroth order nature. In Subsection 3.2.5 we then conclude the desired Theorem 3.1 and Theorem 3.2.

Remark 1.4. Despite the similarity of the outlined argument with [3, Thm. 3.29], we stress that the measured Lorentzian space induced by \((M, \hat{g}_{\varepsilon}, \text{vol}_{g_{\varepsilon}})\), for fixed \( \varepsilon > 0 \), is unclear to obey a TCD or TMCP condition, with lower bound \( -\delta \) or otherwise. Indeed, among others the possible range of \( \varepsilon \) in Lemma 1.3 depends on the parameter \( \kappa \), which describes how far away mass distributions have to lie from each other in order for the timelike Ricci bound (1.2) to be satisfied along their optimal transport. In particular, Theorem 1.1 does not follow from weak stability of the TCD condition (only from a similar proof). Hence, \( C^1 \)-spacetimes are still unclear to fall into the class of “timelike Ricci limit spaces” after the convergence of [7, Thm. 3.12], whose structure thus remains completely unstudied.

Organization. In Chapter 2, we review basic notions of \( C^1 \)-Lorentzian spacetimes and their Lorentzian geodesic structure, recall and slightly extend the approximation results from [15, 28, 25], and outline basics of Lorentzian optimal transport. Chapter 3 contains the proofs of Theorem 3.1, Theorem 3.2, and consequences of these main results.

2. Spacetimes of low regularity

2.1. Terminology. By convention, all Lorentzian metrics in this paper will have signature +,−,−,−. By \( M \) we denote a topological manifold (connected, Hausdorff, second countable) of class \( C^\infty \). The latter is no loss of generality, since for generic vector fields to be continuous, \( C^1 \)-regularity would be a natural assumption on \( M \), yet any \( C^1 \)-manifold possesses a unique \( C^\infty \)-structure that is \( C^1 \)-compatible with the given \( C^1 \)-structure [21, Thm. 2.9], and we would then simply work with that smooth atlas.

All over this chapter, let \( g \) be a Lorentzian metric on \( M \) of regularity at least \( C^1 \). Furthermore, let \( h \) be a complete Riemannian metric on \( M \) [37], with induced length distance \( d^h \), fixed throughout the paper. For \( v \in TM \), we write

\[
|v|_h := \sqrt{h(v,v)},
\]

and we define \( |v|_h \) analogously provided \( g(v,v) \geq 0 \).

We call \( v \in TM \) \( g \)-timelike if \( g(v,v) > 0 \), and \( g \)-causal if \( g(v,v) \geq 0 \). Henceforth, we fix a continuous timelike vector field \( Z \) on \( TM \), and we term \( v \in TM \setminus \{0\} \) future-directed if \( g(v,Z) > 0 \), and past-directed if \( g(v,Z) < 0 \).

A curve \( \gamma : [0,1] \to M \) is called future-directed \( g \)-timelike, respectively future-directed \( g \)-causal, if \( \gamma \) is \( d^h \)-Lipschitz continuous and \( \dot{\gamma}_t \) has the respective properties for \( \mathcal{L}^1 \)-a.e. \( t \in [0,1] \). Compared to absolute continuity, Lipschitz continuity is no restriction [35, p. 17]. We mostly consider the future orientation by \( Z \) and hence drop the prefix “future-directed” — see Remark 2.16 below, though — and, if clear from the context, the metric \( g \) for terminological convenience.
Let the $g$-length of a $g$-causal curve $\gamma : [0, 1] \to \mathcal{M}$ [39, Def. 5.11] be given by
\[
\text{Len}_g(\gamma) := \int_0^1 |\dot{\gamma}|_g \, dt,
\]
and define $l_g : \mathcal{M}^2 \to [0, \infty] \cup \{-\infty\}$ by
\[
l_g(x, y) := \sup\{\text{Len}_g(\gamma) : \gamma : [0, 1] \to \mathcal{M} \text{ $g$-causal curve, } \gamma_0 = x, \gamma_1 = y\}, \tag{2.1}
\]
setting $\sup \emptyset := -\infty$. Slightly deviating from other common definitions — cf. e.g. [15, Def. 2.1] and Remark 2.3 below — and rather following [27, Def. 3.27] we use the following notion of geodesics.

**Definition 2.1.** Given $(x, y) \in l_g^{-1}([0, \infty))$, a maximizer $\gamma : [0, 1] \to \mathcal{M}$ of $l_g(x, y) = l_g^+(x, y)$ in (2.1) is called $g$-geodesic.

For arbitrary sets $C_0, C_1 \subset \mathcal{M}$, we define
- the $g$-causal future of $C_0$ by
  \[
  J^+_g(C_0) := \{ y \in \mathcal{M} : l_g(x, y) \geq 0 \text{ for some } x \in C_0 \}, \tag{2.2}
  \]
- the $g$-causal past of $C_1$ by
  \[
  J^-_g(C_1) := \{ x \in \mathcal{M} : l_g(x, y) \geq 0 \text{ for some } y \in C_1 \}, \tag{2.3}
  \]
- the $g$-causal diamond of $C_0$ and $C_1$ by
  \[
  J_g(C_0, C_1) := J^+(C_0) \cap J^-(C_1).
  \]

Given $x, y \in \mathcal{M}$ and probability measures $\mu$ and $\nu$ on $\mathcal{M}$, we set $J^+_g(x) := J^+_g(\{x\})$ and $J^+_g(\mu) := J^+_g(\text{spt} \mu)$; accordingly, we define $J^-_g(x, y)$ and $J^-_g(\mu, \nu)$. By replacing “$\geq$” by “$>$”, we define the $g$-chronological future $I^+_g(C_0)$ of $C_0$, the $g$-chronological past $I^-_g(C_1)$ of $C_1$, etc.

We call $(\mathcal{M}, g)$, or simply $g$, strongly causal [39, Def. 14.11] if for every $x \in \mathcal{M}$ and every open neighborhood $U \subset \mathcal{M}$ of $x$, there is another open neighborhood $V \subset U$ of $x$ such that every $g$-causal curve with endpoints in $V$ does not leave $U$.

We term the spacetime $(\mathcal{M}, g)$ causal [39, p. 407] if it has no closed nonconstant causal curves.

**Definition 2.2.** The spacetime $(\mathcal{M}, g)$, or simply $g$, is termed globally hyperbolic if it is causal, and $J_g(x, y)$ is compact for every $x, y \in \mathcal{M}$.

In our setting, this definition is equivalent to the traditional definition of global hyperbolicity in terms of strong causality plus compactness of causal diamonds [2, Thm. 3.2]. In fact, in spacetimes with dimension at least three and $g$ of class $C^{1,1}$, causality can be dropped completely in Definition 2.2 [22, Thm. 2.7].

If not explicitly stated otherwise, in the following we will always assume global hyperbolicity of any considered $g$.

**Remark 2.3.** Important facts used at several occasions inherited by the regularity imposed on $g$ are the following.
- Every $g$-geodesic $\gamma : [0, 1] \to \mathcal{M}$ has a causal character [29, Prop. 1.2]. More strongly, either $|\dot{\gamma}|_g > 0$ for every $t \in [0, 1]$, or $|\dot{\gamma}|_g = 0$ for every $t \in [0, 1]$. (A similar statement had been obtained before in [16, Thm. 1.1], see also [44, Thm. 2].)
- Every $g$-geodesic $\gamma : [0, 1] \to \mathcal{M}$ admits a proper-time reparametrization $\eta : [0, 1] \to \mathcal{M}$ with regularity $C^2$ [29, Thm. 1.1], see also [15, Prop. 2.13] and [39, Prop. 4.19]. This means that for every $s, t \in [0, 1]$ with $s < t$,
  \[
  \tau_g(\eta_s, \eta_t) = (t - s) \tau_g(\eta_0, \eta_1). \tag{2.4}
  \]
By the Cauchy–Lipschitz theorem, the latter yields that if the Christoffel symbols of \( g \) are locally Lipschitz continuous, i.e. provided \( g \) is \( C^{1,1} \), \( g \)-timelike \( g \)-geodesics parametrized by proper-time admit no forward or backward branching. That is, if two \( C^2 \)-curves \( \eta^1, \eta^2 : [0, 1] \to \mathcal{M} \) arising from the above procedure coincide on some nontrivial subinterval of \([0, 1]\), then \( \eta^1 = \eta^2 \). In particular, the Lorentzian geodesic space induced by \((M, g)\) according to Section 2.2 is \( g \)-timelike nonbranching [7, Def. 1.10].

2.2. \( C^1 \)-spacetimes as Lorentzian geodesic spaces. In this section, following [27, Sec. 5.1] we review the construction of a Lorentzian geodesic space [27, Def. 2.8, Def. 3.27] from the given spacetime \( \mathcal{M} \) with a globally hyperbolic \( C^1 \)-Lorentzian metric \( g \). As summarized in Proposition 2.4 below, this links our setting to the synthetic frameworks of [3, 7]. In fact, many of the results in this section hold for merely continuous, strongly causal, and causally plain [10, Def. 1.16] metrics. Since every Lipschitz metric is causally plain [10, Cor. 1.17], we only discuss the case of metric regularity at least \( C^1 \) to streamline the presentation.

Define two relations \( \ll_g \) and \( \leq_g \) on \( M \) by

\[
\begin{align*}
\text{• } x \ll_g y & \text{ if there is a } g \text{-timelike curve } \gamma : [0, 1] \to \mathcal{M} \text{ with } \gamma_0 = x \text{ and } \gamma_1 = y, \\
\text{or equivalently } l_g(x, y) & > 0, \text{ and} \\
\text{• } x \leq_g y & \text{ if there is a } g \text{-causal curve } \gamma : [0, 1] \to \mathcal{M} \text{ with } \gamma_0 = x \text{ and } \gamma_1 = y, \\
\text{or equivalently } l_g(x, y) & \geq 0.
\end{align*}
\]

Given any subset \( M \subset \mathcal{M} \), define

\[
M^2_{\ll_g} := M^2 \cap l_g^{-1}((0, \infty)) = \{(x, y) \in M^2 : x \ll_g y\},
\]

\[
M^2_{\leq_g} := M^2 \cap l_g^{-1}([0, \infty)) = \{(x, y) \in M^2 : x \leq_g y\}.
\]

Clearly, \( \ll_g \) is transitive and contained in \( \leq_g \), i.e. \( M^2_{\ll_g} \subset M^2_{\leq_g} \), and \( \leq_g \) is reflexive and transitive, which makes \((M, \ll_g, \leq_g)\) a causal structure after [27, Def. 2.1].

The positive part \( \tau_g := l_g^+ \) of the function \( l_g \) in (2.1) is a time separation function [27, Def. 2.8]: it is lower semicontinuous [27, Prop. 5.7], and for every \( x, y, z \in \mathcal{M} \),

\[
\begin{align*}
\text{a. } \tau_g(x, y) & = 0 \text{ provided } x \leq_g y, \\
\text{b. } \tau_g(x, y) & > 0 \text{ if and only if } x \ll_g y [27, \text{Lem. 5.6}], \text{ and} \\
\text{c. if } x \leq_g y \leq_g z, \text{ we have the reverse triangle inequality} \\
\tau_g(x, z) & \geq \tau_g(x, y) + \tau_g(y, z). \quad (2.5)
\end{align*}
\]

In particular, the quintuple \((M, d^h, \ll_g, \leq_g, \tau_g)\) forms a Lorentzian pre-length space [27, Prop. 5.8] in the sense of [27, Def. 2.8].

Global hyperbolicity of \( g \) entails further fine properties and non-ambiguities of \((M, d^h, \ll_g, \leq_g, \tau_g)\) as described now. The notion of \( g \)-causal curves in Section 2.1 coincide with the nonsmooth one from [27, Def. 2.18] (evidently defined solely in terms of \( \leq_g \), cf. [27, Prop. 5.9]. Moreover, their \( g \)-length agrees with their \( \tau_g \)-length \( \text{Len}_{\tau_g} \) [27, Def. 2.24], cf. [27, Rem. 5.1, Lem. 5.10]. In fact, \((M, d^h, \ll_g, \leq_g, \tau_g)\) is a strongly localizable Lorentzian length space after [27, Def. 3.16, Def. 3.22], cf. [27, Thm. 5.12]. By the causal ladder for Lorentzian length spaces [27, Thm. 3.26], global hyperbolicity of \( g \) after Definition 2.2 is then equivalent to global hyperbolicity of \((M, d^h, \ll_g, \leq_g, \tau_g)\) [27, Def. 2.35], i.e. we have

\[
\begin{align*}
\text{a. compactness of causal diamonds between any } x, y \in \mathcal{M}, \text{ and} \\
\text{b. non-total imprisonment, i.e. for every compact } C \subset \mathcal{M}, \\
\sup\{\text{Len}_{d^h}(\gamma) : \gamma : [0, 1] \to \mathcal{M} \text{ } g \text{-causal curve, } \gamma|_{[0,1]} \subset C\} & < \infty. \quad (2.6)
\end{align*}
\]

In particular, \( \tau_g \) is finite and continuous [27, Thm. 3.28].

Lastly, a combination of [41, Prop. 3.3, Cor. 3.4] with [27, Thm. 3.26, Thm. 3.28] and Remark 2.3 gives that \((M, d^h, \ll_g, \leq_g, \tau_g)\) satisfies all regularity properties required for the most important synthetic results in [3, 7] as follows.
Proposition 2.4. The space $(M, d^h, \ll g, \leq g, \tau_g)$ is a regular Lorentzian length space [27, Def. 3.16, Def. 3.22] with the following properties.

(i) **Causal closedness.** The set $M^2_{\leq g}$ is closed in $\mathcal{M}^2$.

(ii) **$k$-global hyperbolicity.** For every compact $C_0, C_1 \subset M$, the causal diamond $J_g(C_0, C_1)$ is compact in $\mathcal{M}$.

(iii) **Geodesy.** Every $x, y \in M$ with $x \leq g y$ are joined by a $g$-geodesic.

2.3. **Approximation.** Since $g$ is of class $C^1$ and since curvature quantities involve second derivatives of the metric components, these have to be defined distributionally in a sense that we briefly recall from [15, Sec. 3], see also [17, 25, 42].

2.3.1. **Distributional curvature bounds.** The space of distributions $\mathcal{D}'(M)$ is defined as the topological dual of the space of smooth, compactly supported sections of the volume bundle $\text{Vol}(M)$, i.e.

$$\mathcal{D}'(M) := \Gamma_c(\text{Vol}(M))^\prime.$$ 

An element $\mu \in \Gamma_c(\text{Vol}(M))$ is called volume density. The pairing of $u \in \mathcal{D}'(M)$ with $\mu$ will be denoted $(u, \mu)$.

We naturally regard $C^\infty(M)$ as subspace of $\mathcal{D}'(M)$ by identification of a given $f \in C^\infty(M)$ with the functional $\mu \mapsto \int_M f \mu$ on $\Gamma_c(\text{Vol}(M))$.

The above definition can be generalized to tensor distributions. More precisely, given $r, s \in \mathbb{N}_0$ the space of $T^r_s M$-valued distributions — with $r$ covariant and $s$ contravariant slots — is defined by

$$\mathcal{D}'(T^r_s M) := \Gamma_c(T^r_s M \otimes \text{Vol}(M))^\prime \cong \mathcal{D}'(M) \otimes C^\infty(M) T^r_s M.$$ 

In particular, every tensor distribution is locally defined by its proper coefficients in $\mathcal{D}'(M)$. That is, for a given atlas $(U_\alpha, \psi_\alpha)_{\alpha \in \mathcal{A}}$, the restriction $T\big|_{U_\alpha}$ of $T \in \mathcal{D}'T^r_s(M)$ to $U_\alpha$ can be written as

$$T\big|_{U_\alpha} = (\alpha \mathcal{T})_{i_1 \cdots i_r j_1 \cdots j_s} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s} \quad (2.7)$$

using Einstein’s summation convention, with local coefficients $(\alpha \mathcal{T})_{i_1 \cdots i_r j_1 \cdots j_s} \in \mathcal{D}'(U_\alpha)$.

Via the chart map $\psi_\alpha$, the latter can both be pushed forward to and recovered by pullback from a distribution on $\mathbb{R}^n$; cf. [15, Prop. 3.1] for details.

In view of the next definition [15, Def. 3.2], we call $\mu \in \Gamma_c(\text{Vol}(M))$ nonnegative provided $\int_U \mu \geq 0$ for every open $U \subset M$.

**Definition 2.5.** Let $u \in \mathcal{D}'(M)$. We write $u \geq 0$ if $(u, \mu) \geq 0$ for every nonnegative volume density $\mu \in \Gamma_c(\text{Vol}(M))$. Analogously, given any $v \in \mathcal{D}'(M)$ we write $u \geq v$ provided $u - v \geq 0$.

Given the $C^1$-metric $g$ with Christoffel symbols $\Gamma^k_{ij}$, a smooth vector field $X$ over $M$ with local components $X^1, \ldots, X^n$, some $V \in C^1(M)$, and $N \in [n, \infty)$, the following quantities are locally well-defined in $\mathcal{D}'(M)$ by the usual formulas:

\[
\begin{align*}
\text{Ric}_g(X, X) &:= \left[ \frac{\partial \Gamma^m_{ij}}{\partial x^m} - \frac{\partial \Gamma^m_{im}}{\partial x^j} + \Gamma^r_{ij} \Gamma^m_{rk} - \Gamma^m_{ir} \Gamma^r_{jk} \right] X^i X^j, \\
\text{Hess}_g(V, X, X) &:= \left[ \frac{\partial^2 V}{\partial x^j \partial x^j} - \Gamma^k_{ij} \frac{\partial V}{\partial x^k} \right] X^i X^j, \\
\text{Ric}^{N,V}_g(X, X) &:= \text{Ric}_g(X, X) + \text{Hess}_g(V, X, X) - \frac{1}{N-n} \text{DV}(X)^2.
\end{align*}
\]

If $N = n$, we assume $V$ to be constant by default, so that $\text{DV}(X)^2/(N-n) := 0$. Evidently, these definitions give rise to nonrelabeled tensor distributions

$$\text{Ric}_g, \text{Hess}_g V, \text{Ric}^{N,V}_g \in \mathcal{D}'(T^2_0(M)).$$
Definition 2.6. Given \( V \) and \( N \) as above and any \( K \in \mathbb{R} \), we say
\[
\text{Ric}^N_{g,V} \geq K \text{ in all timelike directions}
\]
if for every smooth \( g \)-timelike vector field \( X \) on \( \mathcal{M} \),
\[
\text{Ric}^N_{g,V}(X, X) \geq K |X|^2_g
\]
holds in the sense of Definition 2.5.

Remark 2.7. If \( g \) and \( V \) are of class \( C^{1,1} \), \( \text{Ric}^N_{g,V}(X, X) \) is well-defined as an element of \( L^\infty_{\text{loc}}(\mathcal{M}, \text{vol}_g) \), cf. Subsection 2.4.1. In this case, the condition
\[
\text{Ric}^N_{g,V} \geq K \text{ in all timelike directions}
\]
holds if and only if for every \( X \) as in Definition 2.6, (2.8) holds \( \text{vol}_g \)-a.e.; if \( g \) and \( V \) are of class \( C^2 \), this characterization improves to a pointwise statement of (2.8).

2.3.2. Regularization of the metric. Now we show how to approximate \( g \) in a “nice” way. That is, if \( g \) obeys distributional curvature bounds, recall Definition 2.6, it will even be possible to almost preserve these bounds, at least locally, cf. Lemma 2.8.

In order to approximate \( g \), we need to clarify how to regularize a distribution over \( \mathcal{M} \). Fix a standard mollifier \( \{ \rho_\varepsilon : \varepsilon > 0 \} \) in \( \mathbb{R}^n \), a countable atlas \( \{(U_\alpha, \psi_\alpha) : \alpha \in \mathbb{N} \} \) with relatively compact \( U_\alpha \), a subordinate partition of unity \( \{ \chi_\alpha \}_{\alpha \in \mathbb{N}} \), as well as functions \( \chi_\alpha \in C^\infty_0(U_\alpha) \) with \( |\chi_\alpha|_{(M)} = [0, 1] \) and \( \chi_\alpha = 1 \) on an open neighborhood of \( \text{spt} \xi_\alpha \) in \( U_\alpha \).

As usual, the convolution of a Euclidean distribution \( u \) with compact support \cite[p. 1434]{15} in an open set \( \Omega \subset \mathbb{R}^n \) with \( \rho_\varepsilon, \varepsilon \in (0, d_E(\text{spt} u, \partial \Omega)) \), is the smooth function \( u * \rho_\varepsilon \) on \( \Omega \) given by
\[
(u * \rho_\varepsilon)(x) := (u, \rho_\varepsilon(x - \cdot)).
\]
Then for \( T \in \mathcal{D}'(\mathcal{M}) \) we define a smooth \((r, s)\)-tensor field \( T *_M \rho_\varepsilon \) by
\[
T *_M \rho_\varepsilon := \sum_{\alpha \in \mathbb{N}} \chi_\alpha (\psi^{-1}_\alpha)_* [(\psi_\alpha)_*(\xi_\alpha)_* T) * \rho_\varepsilon],
\]
where the convolution on the right-hand side is understood componentwisely in terms of the push-forwards of the local coefficients from (2.7) to \( \mathbb{R}^n \) via (2.9).

Clearly, for every \( u \in \mathcal{D}'(\mathcal{M}) \) and every \( \varepsilon > 0 \), we have \( u *_M \rho_\varepsilon \geq 0 \) if \( u \geq 0 \).

In the most relevant case for our purposes, namely \( T := g \in \mathcal{D}'T^2_2(\mathcal{M}) \), it follows from basic properties of mollification in \( \mathbb{R}^n \) \cite[Prop. 3.5]{15} that \( g *_M \rho_\varepsilon \to g \) in \( C^1_{\text{loc}}(\mathcal{M}) \) as \( \varepsilon \to 0 \). However, this convergence is too weak to ensure mollification of \( g \) to (almost) preserve distributional curvature bounds. Neither are light cones with respect to \( g *_M \rho_\varepsilon \) “narrower” than those of \( g \), a property which will be used multiple times below. Both issues are resolved in the following crucial Lemma 2.8 which summarizes \cite[Lem. 4.1, Lem. 4.2, Lem. 4.5]{15}. (The arguments therein can easily be adapted to cover the case of arbitrary curvature lower bounds \( K \in \mathbb{R} \) and arbitrary \( C^1 \)-weights \( V \), cf. \cite[Thm. 4.3, Rem. 4.4]{25}.)

Recall that for a Lorentzian metric \( \tilde{g} \), by \( \tilde{g} < g \) we mean that every \( \tilde{g} \)-causal tangent vector \( v \) is \( g \)-timelike (more visually, that \( \tilde{g} \)-light cones are strictly “narrower” than \( g \)-light cones). Also, for \( T \in \mathcal{D}'T^2_2(\mathcal{M}) \) and a compact \( C \subset \mathcal{M} \) we set
\[
\|T\|_{\infty, C} := \sup \{|T(x)(v, w)| : x \in C, v, w \in T_x \mathcal{M} \text{ with } |v|_h = |w|_h = 1\}.
\]
Lastly, let \( h \nabla \) denote the Levi-Civita connection with respect to \( h \).

Lemma 2.8. There exist smooth Lorentzian metrics \( \{ \tilde{g}_\varepsilon : \varepsilon > 0 \} \) on \( \mathcal{M} \), time-orientable by the same timelike vector field \( Z \) as \( g \), with the following properties.

(i) We have \( \tilde{g}_\varepsilon < g \) for every \( \varepsilon > 0 \).
(ii) We have \( \tilde{g}_\varepsilon - g \ast_M \rho_\varepsilon \to 0 \) in \( C^\infty_{loc}(M) \) as \( \varepsilon \to 0 \). That is, for every compact \( C \subset M \) and every \( \varepsilon \in \mathbb{N} \), we have

\[
\lim_{\varepsilon \to 0} \| (h^\nabla)^i \tilde{g}_\varepsilon - (h^\nabla)^i (g \ast_M \rho_\varepsilon) \|_{\infty, C} = 0.
\]

In particular, \( \tilde{g}_\varepsilon \to g \) in \( C^1_{loc}(M) \) as \( \varepsilon \to 0 \), i.e. for every compact \( C \subset M \) and every \( \varepsilon \in \{0, 1\} \),

\[
\lim_{\varepsilon \to 0} \| (h^\nabla)^i \tilde{g}_\varepsilon - (h^\nabla)^i g \|_{\infty, C} = 0.
\]

Moreover, let \( V \in C^1(M) \) and \( N \in [n, \infty) \), and assume

\[
\text{Ric}^N_g \geq K \text{ in all timelike directions}.
\]

Then \( \{ \tilde{g}_\varepsilon : \varepsilon > 0 \} \) can be constructed to have the following further property. For every compact \( C \subset M \) and every \( c, \delta, \kappa > 0 \), there exists \( \varepsilon_0 > 0 \) such that for every \( \varepsilon \in (0, \varepsilon_0) \) and every \( v \in T_M|_C \),

\[
|v|_{\tilde{g}_\varepsilon} \geq \sqrt{\kappa}, \quad |v|_h \leq \sqrt{\varepsilon} \implies \text{Ric}^N_{\tilde{g}_\varepsilon}(v, v) \geq (K - \delta) |v|_{\tilde{g}_\varepsilon}^2.
\]

Knowledge of the following consequence of (i) above and [15, Rem. 1.1] will be relevant e.g. in the proofs of Lemma 3.4, and Proposition 3.8.

Lemma 2.9. For every \( \varepsilon > 0 \), \( \tilde{g}_\varepsilon \) is globally hyperbolic.

2.4. Lorentzian optimal transport. Lastly, we recall basic elements of Lorentzian optimal transport theory, referring to [7, 11, 23, 33, 36, 48] for details.

Evidently, all subsequent notions with background space \( M \) will make sense on any closed subset \( M \subset M \).

2.4.1. Measure-theoretic notation. Let \( \mathcal{P}(M) \) be the class of Borel probability measures on \( M \), and let \( \mathcal{P}_c(M) \) consist of all \( \mu \in \mathcal{P}(M) \) with compact support \( \text{spt} \mu \subset M \).

Let \( vol_g \) be the Lorentzian volume measure on \( M \) associated to \( g \). It arises from the volume form \( dvol_g \) induced by \( g \) by the formula

\[
dvol_g|_U := \sqrt{|\det g|} \, dx^1 \wedge \cdots \wedge dx^n \quad (2.10)
\]
on a coordinate chart \( (U, \psi) \), where \( \{dx^1(x), \ldots, dx^n(x)\} \) is a positively oriented basis of \( T^*_x M \) for every \( x \in U \). Let \( \mathcal{P}_{ac}^\mathcal{P}(M, vol_g) \) be the set of all \( vol_g \)-absolutely continuous elements of \( \mathcal{P}(M) \), and set \( \mathcal{P}^\mathcal{P}_c(M, vol_g) := \mathcal{P}_c(M) \cap \mathcal{P}_{ac}^\mathcal{P}(M, vol_g) \).

Given \( \mu, \nu \in \mathcal{P}(M) \), let \( \Pi(\mu, \nu) \) be the set of all their couplings, i.e. all \( \pi \in \mathcal{P}(M^2) \) such that \( \pi[\cdot \times M] = \mu \) and \( \pi[M \times \cdot] = \nu \). This concept of couplings conveniently makes sense of chronology and causality relations between \( \mu \) and \( \nu \) in terms of their supports, namely in terms of the sets \( \Pi_{\leq y}^>(\mu, \nu) \) and \( \Pi_{\leq y}^\leq(\mu, \nu) \), respectively, which consist of all \( \pi \in \Pi(\mu, \nu) \) with \( \pi[M^2_{\leq y}] = 1 \) and \( \pi[M^2_{\geq y}] = 1 \), respectively.

2.4.2. The \( \ell_{g,p}-\)optimal transport problem. Given \( p \in (0, 1) \), define \( \ell_{g,p}: \mathcal{P}(M)^2 \to [0, \infty] \cup \{-\infty\} \) through

\[
\ell_{g,p}(\mu, \nu) := \sup\{\|\pi\|_{L^p(M^2, \pi)} : \pi \in \Pi_{\leq y}^>(\mu, \nu)\}, \quad (2.11)
\]

subject to the usual convention \( \ell_{g,p}(\mu, \nu) := -\infty \) if \( \Pi_{\leq y}^>(\mu, \nu) = \emptyset \). This quantity is morally interpreted as a time separation function on \( \mathcal{P}(M) \), compare with (2.5): indeed [7, Prop. 2.5], for every \( \mu, \nu, \sigma \in \mathcal{P}(M) \),

\[
\ell_{g,p}(\mu, \sigma) \geq \ell_{g,p}(\mu, \nu) + \ell_{g,p}(\nu, \sigma).
\]

Given \( \mu, \nu \in \mathcal{P}(M) \), we call \( \pi \in \Pi(\mu, \nu) \) \( \ell_{g,p}-\)optimal if \( \pi \in \Pi_{\leq y}^>(\mu, \nu) \) and \( \pi \) realizes the supremum in (2.11). Concerning existence of such \( \pi \), for our purposes it will suffice to know that if \( \mu, \nu \in \mathcal{P}_c(M) \) with \( \Pi_{\leq y}^>(\mu, \nu) \neq \emptyset \) — a condition which holds
for \( \pi := \mu \otimes \nu \) if \( \text{spt} \mu \times \text{spt} \nu \subset M^2_{\geq 0} \) — admit an \( \ell_{g,p} \)-optimal coupling; also, by compactness of \( \text{spt} \mu \times \text{spt} \nu \) we clearly have
\[
\ell_{g,p}(\mu, \nu) \leq \sup \tau_g(\text{spt} \mu \times \text{spt} \nu) < \infty.
\]

In view of our intended synthetic treatment of \( g \)-timelike Ricci curvature bounds we recall the following definition by [7, Def. 2.18], see also [33, Def. 4.1].

**Definition 2.10.** We call a pair \((\mu, \nu) \in \mathcal{P}_c(\mathcal{M})^2\) \( g \)-timelike \( p \)-dualizable if
\[
\{ \pi \in \Pi(\mu, \nu) : \pi \text{ is } \ell_{g,p}\text{-optimal} \} \cap \Pi_{\leq_p}(\mu, \nu) \neq \emptyset.
\]

Any element of the set on the left-hand side is called \( g \)-timelike \( p \)-dualizing.

**Remark 2.11.** By the preceding discussion, it is evident that if \( \mu, \nu \in \mathcal{P}_c(\mathcal{M}) \) satisfy \( \text{spt} \mu \times \text{spt} \nu \subset M^2_{\geq 0} \), then the pair \((\mu, \nu)\) is \( g \)-timelike \( p \)-dualizable (even in a stronger sense [7, Def. 2.27], cf. [7, Cor. 2.29]).

2.4.3. **Timelike proper-time parametrized \( \ell_{g,p} \)-geodesics.** Next, we review the technical definition of geodesics with respect to \( \ell_{g,p} \), referring to [3, Subsec. 2.3.6, App. B] for details. The idea is to construct the latter as “proper-time reparametrizations” of plans concentrated on \( g \)-geodesics, i.e. \( \text{Len}_{g,p} \)-maximizing \( g \)-causal curves. Compared to the weaker notion of timelike \( \ell_{g,p} \)-geodesics from [33, Def. 1.1], in a more general synthetic setting this procedure allows for good compactness properties more evidently [3, Prop. B.11], as implicitly used many times in Chapter 3. (Although with some more effort, it should be possible to prove the notion from [33] has similar properties.) If \( g \) is smooth, no ambiguity occurs in all relevant cases [3, Rem. B.10].

Let \( \text{Geo}_g(\mathcal{M}) \) be the set of \( g \)-geodesics \( \gamma : [0, 1] \to \mathcal{M} \); it is Polish by Proposition 2.4 and non-total imprisonment, cf. (2.6). Furthermore, let \( e_t : \text{Geo}_g(\mathcal{M}) \to \mathcal{M} \) be the evaluation map \( e_t(\gamma) := \gamma_t, t \in [0, 1] \). Set
\[
\text{TGeo}_g(\mathcal{M}) := \{ \gamma \in \text{Geo}_g(\mathcal{M}) : \tau_g(\gamma_0, \gamma_1) > 0 \},
\]
which precisely consists of \( g \)-timelike \( g \)-geodesics by Remark 2.3. By the proof of [29, Prop. 9.1], see also [3, Lem. B.6] and [27, Cor. 3.35], there exists a continuous reparametrization map \( r : \text{TGeo}_g(\mathcal{M}) \to \mathcal{C}([0, 1]; \mathcal{M}) \) such that \( \eta := r(\gamma) \) obeys (2.4) for every \( \gamma \in \text{TGeo}_g(\mathcal{M}) \). With this said, given \( \mu_0, \mu_1 \in \mathcal{P}(\mathcal{M}) \) we set
\[
\text{OptTGeo}^{\tau_{r,\delta}}_{\mu_0, \mu_1}(\mu_0, \mu_1) := \tau_{r_2} \{ \pi \in \mathcal{P}(\text{Geo}_g(\mathcal{M})) : (e_0, e_1)_2 \pi \text{ is } \ell_{g,p}\text{-optimal} \},
\]
with \( (e_0, e_1)_2 \pi \in \text{OptTGeo}^{\tau_{r,\delta}}_{\mu_0, \mu_1}(\mu_0, \mu_1) \).

**Definition 2.12.** A curve \((\mu_t)_{t \in [0, 1]} \) in \( \mathcal{P}(\mathcal{M}) \) is called **timelike proper-time parametrized \( \ell_{g,p} \)-geodesic** if it is represented by some \( \pi \in \text{OptTGeo}^{\tau_{r,\delta}}_{\mu_0, \mu_1}(\mu_0, \mu_1), \) i.e.
\[
\mu_t = (e_t)_2 \pi
\]
for every \( t \in [0, 1] \); such a \( \pi \) is called **timelike \( \ell_{g,p} \)-optimal geodesic plan.**

By construction, every timelike \( \ell_{g,p} \)-optimal geodesic plan \( \pi \) is concentrated on \( g \)-causal curves which satisfy (2.4). As a corollary of (2.5), every timelike proper-time parametrized \( \ell_{g,p} \)-geodesic \((\mu_t)_{t \in [0, 1]} \) is a timelike \( \ell_{g,p} \)-geodesic in the sense of [33, Def. 1.1] if \( \ell_{g,p}(\mu_0, \mu_1) < \infty \); indeed, for every \( s, t \in [0, 1] \) with \( s < t \),
\[
\ell_{g,p}(\mu_s, \mu_t) = (t - s) \ell_{g,p}(\mu_0, \mu_1) \in (0, \infty).
\]
2.4.4. Synthetic timelike lower Ricci curvature bounds. The subsequent synthetic definitions of timelike Ricci curvature lower bounds — foreshadowed by the works [7, 33, 36] which studied a different entropy functional — have been set up for general measured Lorentzian spaces [7, Def. 1.11] in [3, Def 3.3, Def. 4.1]. These constitute Lorentzian counterparts of analogous notions for metric measure spaces, cf. [38, Def. 2.1] and [47, Def. 1.3, Def. 5.1].

This is where a reference measure comes into play: given \( V \in C^1(\mathcal{M}) \), set

\[
n^V_g := e^{-V \text{ vol}_g}.
\]

The associated measured Lorentzian structure, recall Section 2.2, is written

\[
\mathcal{X}^V_g := (\mathcal{M}, d^h, n^V_g, \ll_g, \leq_g, \tau_g).
\] (2.12)

For \( N \in [1, \infty) \), subject to the Lebesgue decomposition \( \mu = \rho n^V_g + \mu_{\perp} \) of \( \mu \in \mathcal{P}(\mathcal{M}) \), the \( N \)-Rényi entropy \( S^N_{\rho^V} : \mathcal{P}(\mathcal{M}) \to [0, 0] \) with respect to \( n^V_g \) is

\[
S^N_{\rho^V}(\mu) := -\int_{\mathcal{M}} \rho^{-1/N} \, d\mu = -\int_{\mathcal{M}} \rho^{1-1/N} \, d\rho^V_g.
\] (2.13)

If \( V = 0 \) and \( N = n \), it reduces to the \( n \)-Rényi entropy \( S^n(\rho) \) defined in Chapter 1.

Moreover, for \( t \in [0, 1] \) and \( K \in \mathbb{R} \), we define the \textit{distortion coefficients} \( \tau_{K,N}^{(t)} \) [47, p. 137] as follows. Given any \( \vartheta \geq 0 \), set

\[
\varrho_{K,N}(\vartheta) := \begin{cases} 
\sin(\sqrt{KN^{-1}} \vartheta) / \sqrt{KN^{-1}} & \text{if } K > 0, \\
\vartheta & \text{if } K = 0, \\
\sinh(\sqrt{KN^{-1}} \vartheta) / \sqrt{KN^{-1}} & \text{otherwise},
\end{cases}
\] (2.14)

\[
\sigma_{K,N}^{(t)}(\vartheta) := \begin{cases} 
t & \text{if } K \vartheta^2 = 0, \\
\varrho_{K,N}(t \vartheta) / \varrho_{K,N}(\vartheta) & \text{otherwise,}
\end{cases}
\]

\[
\tau_{K,N}^{(t)}(\vartheta) := t^{1/N} \sigma_{K,N}^{(t)}(\vartheta)^{1-1/N}.
\]

We always have the inequality

\[
\sigma_{K,N}^{(t)}(\vartheta) \leq \tau_{K,N}^{(t)}(\vartheta).
\]

In the nondegeneracy cases \( K \vartheta^2 = 0 \) or \( K \vartheta^2 < N \pi^2 \), the function \( u : [0, 1] \to \mathbb{R} \) given by \( u(t) := \sigma_{K,N}^{(t)}(\vartheta) \) is the unique solution to the ODE

\[
u''(t) + \frac{K}{N} \vartheta^2 u(t) = 0
\]

with boundary data \( u(0) = 0 \) and \( u(1) = 1 \). The quantity \( \tau_{K,N}^{(t)}(\vartheta) \) is a geometric average of two distortion coefficients. Roughly speaking, it encodes the behavior of the optimal transport Jacobian in timelike directions: it is affected by curvature in “\( N - 1 \)” directions orthogonal to the transport, represented by \( \sigma_{K,N}^{(t)}(\vartheta) \), while the contribution tangential to the transport does not see any curvature, represented by \( \tau_{K,N}^{(t)}(\vartheta) \).

Compare with the proof of [4, Thm. 5.9].

**Definition 2.13.** Let \( p \in (0, 1) \), \( K \in \mathbb{R} \), and \( N \in [1, \infty) \). We say \( \mathcal{X}^V_g \) satisfies the timelike curvature-dimension condition \( \text{TCR}_p(K,N) \) if for every \( g \)-timelike p-dualizable pair \((\mu_0, \mu_1) = (\rho_0 n^V_g, \rho_1 n^V_g) \in \mathcal{P}^p(\mathcal{M}, \text{vol}_g) \), there exist

- a timelike proper-time parametrized \( \ell_{\rho,p} \)-geodesic \( (\mu_t)_{t \in [0,1]} \) connecting \( \mu_0 \) to \( \mu_1 \), and
- a \( g \)-timelike p-dualizing coupling \( \pi \in \Pi^{\leq p}(\mu_0, \mu_1) \).
such that for every \( t \in [0, 1] \) and every \( N' \geq N \),

\[
S_{\gamma}^{N' \vee} (\mu_t) \leq -\int_{\mathcal{M}^2} \tau_{K,N'}^{(1-t)}(x_0, x_1) \rho_0(x_0)^{-1/N'} \, d\pi(x_0, x_1)
- \int_{\mathcal{M}^2} \tau_{K,N'}^{(t)}(x_0, x_1) \rho_1(x_1)^{-1/N'} \, d\pi(x_0, x_1).
\]

**Definition 2.14.** Let \( K \in \mathbb{R} \), and \( N \in [1, \infty) \). We say \( \mathcal{X}_g^V \) satisfies the timelike measure-contraction property \( \text{TMCP}(K,N) \) if for every \( \mu_0 = \rho_0 n_g^V \in \mathcal{P}_{c}^\infty (\mathcal{M}, \text{vol}_g) \) and every \( x_1 \in \mathcal{M} \) with \( \mu_0[I_g^{-} (x_1)] = 1 \), there exist \( p \in (0, 1) \) and a timelike proper-time parametrized \( \ell_{g,p} \)-geodesic \( (\mu_t)_{t \in [0,1]} \) from \( \mu_0 \) to \( \mu_1 := \delta_{x_1} \) such that for every \( t \in [0, 1] \) and every \( N' \geq N \),

\[
S_{\gamma}^{N' \vee} (\mu_t) \leq -\int_{\mathcal{M}^2} \tau_{K,N'}^{(1-t)}(x_0, x_1) \rho_0(x_0)^{-1/N'} \, d\mu_0(x_0).
\]

These conditions are compatible with the smooth case, in the sense that if \( g \) is smooth, roughly speaking, \( \text{TCD}_p(K,N) \) and \( \text{TMCP}(K,N) \) characterize \( g \)-timelike Ricci curvature lower bounds by \( K \in \mathbb{R} \) and upper dimension bounds by \( N \in [1, \infty) \) for \((M,g)\) [4, Thm. 5.9, Thm. 6.1]; Theorem 3.1 and Theorem 3.2 will extend these results to lower regularity Lorentzian metrics. (The results from [4] are even proven for more general smooth Finsler spacetimes.)

We note that \( \text{TMCP}(K,N) \) does not depend on the given transport exponent \( p \) in Definition 2.14. Indeed, the collection \((\mu_t)_{t \in [0,1]}\) therein is a timelike proper-time parametrized \( \ell_{g,p} \)-geodesic if and only if it constitutes a timelike proper-time parametrized \( \ell_{g,p'} \)-geodesic for every \( p' \in (0,1) \). This simply follows because \( \Pi_{\leq g} (\mu_0, \mu_1) = \Pi (\mu_0, \mu_1) \) is a singleton; cf. [3, Rem. 4.3] and [8, Rem. 2.4].

Moreover, the following basic properties hold.

- Both notions are consistent in the “curvature parameter” \( K \) and the “dimensional parameter” \( N \) [3, Prop. 3.7, Prop. 4.5].
- Moreover, \( \text{TCD}_p(K,N) \) implies \( \text{TMCP}(K,N) \) [3, Prop. 4.8], yet the latter condition is strictly weaker in general [3, Rem. A.5].

**Remark 2.15.** The condition \( \text{TCD}_p(K,N) \) implies the reduced timelike curvature-dimension condition \( \text{TCD}_p^\lambda(K,N) \) from [3, Def. 3.2], cf. [3, Prop. 3.6]. Under \( g \)-timelike nonbranching according to Remark 2.3, the latter condition is equivalent to the \( \text{TCD}_p^\lambda(K,N) \) condition introduced in [7, Def. 3.2] after [33, 36], which is formulated in terms of the Boltzmann entropy, by [3, Thm. 3.35]. Analogous chains of implications are satisfied by \( \text{TMCP}(K,N) \) [3, Prop. 4.5, Thm. 4.20].

Motivated by an analogous result for (essentially) nonbranching metric measure spaces [6], all three timelike curvature-dimension conditions are in fact expected to coincide under appropriate timelike nonbranching hypotheses.

**Remark 2.16.** Starting from \((M,g)\), one can define a Lorentzian geodesic space \((\mathcal{M}, d^\lambda, \ll_{\gamma}^\lambda, \leq_{\gamma}^\lambda, \tau_{\gamma}^\lambda)\) in complete analogy to Section 2.2 relative to past-directed — in the evident sense — in place of future-directed \( g \)-timelike and \( g \)-causal curves. This is called the \( g \)-causally reversed structure of \( (M, d^\lambda, \ll_{\gamma}, \leq_{\gamma}, \tau_{\gamma}) \) [7, Sec. 1.1]. The regularity properties from Proposition 2.4 transfer to it.

Replacing \( t \) by \( 1 - t \) in Definition 2.13 and employing that this definition is “symmetric” in the regularity properties asked for \( \mu_0 \) and \( \mu_1 \), it is clear that \( \mathcal{X}_g^V \) satisfies \( \text{TCD}_p(K,N) \) if and only if \((\mathcal{X}_g^V)^{\tau_{\gamma}^\lambda}\) does. A similar property for \( \text{TMCP}(K,N) \) is unclear, for \( \text{TMCP}(K,N) \) for \((\mathcal{X}_g^V)^{\tau_{\gamma}^\lambda}\) encodes semiconvexity of the Rényi entropy along timelike \( \ell_{g,p} \)-optimal transport [sic] from a Dirac measure to an \( n_g^V \)-absolutely continuous mass distribution.
3. Main results and consequences

3.1. Statements. Now we are in a position to state our main results in a more complete form than outlined in Chapter 1.

**Theorem 3.1.** Assume $g$ to be $C^1$. Let $K \in \mathbb{R}$ and $N \in [n, \infty)$, and suppose
\[ \text{Ric}^{N,V}_g \geq K \text{ in all timelike directions}. \] (3.1)
Then the measured Lorentzian space $\mathcal{X}_g^V$ from (2.12) induced by $(M, g, n_g^V)$ satisfies TMCP$(K, N)$ according to Definition 2.14.

That is, for every $\mu_0 = \rho_0 n_g^V \in \mathcal{P}^{ac}_c(M, \text{vol}_g)$ and every $x_1 \in M$ with $\int_{g_0}^{I_g^{-1}}(x_1) = 1$, there are $p \in (0, 1)$ and a timelike proper-time parametrized $t_{g,p}$-geodesic $(\mu_t)_{t \in [0,1]}$ from $\mu_0$ to $\mu_1 := \delta_{x_1}$ such that for every $t \in [0,1)$ and every $N' \geq N$,
\[ S^{N',V}_g(\mu_t) \leq -\int_{\mathcal{M}^2} \tau^{(1-t)}_{K,N'}(t_g(x_0, x')) \rho_0(x_0)^{-1/N'} \text{d}\mu_t(x_0, x'). \] (3.2)

**Theorem 3.2.** Assume $g$ to be $C^{1,1}$. Let $K \in \mathbb{R}$ and $N \in [n, \infty)$, and suppose (3.1). Then for every $p \in (0,1)$, the measured Lorentzian space $\mathcal{X}_g^V$ from (2.12) induced by $(M, g, n_g^V)$ satisfies TCD$_p(K, N)$ according to Definition 2.13.

That is, for all $g$-timelike $p$-dualizable $(\mu_0, \mu_1) = (\rho_0 n_g^V, \rho_1 n_g^V) \in \mathcal{P}^{ac}_c(M, \text{vol}_g)^2$, there exist
- a timelike proper-time parametrized $t_{g,p}$-geodesic $(\mu_t)_{t \in [0,1]}$ connecting $\mu_0$ to $\mu_1$, and
- a $g$-timelike $p$-dualizing coupling $\pi \in \Pi_{\leq \pi}(\mu_0, \mu_1)$

such that for every $t \in [0,1]$ and every $N' \geq N$,
\[ S^{N',V}_g(\mu_t) \leq -\int_{\mathcal{M}^2} \tau^{(1-t)}_{K,N'}(t_g(x_0, x')) \rho_0(x_0)^{-1/N'} \text{d}\pi(x_0, x') \] (3.3)

3.2. Proofs of Theorem 3.1 and Theorem 3.2. In order to prove the two preceding claims, the main work has to be performed for the more general version of Theorem 1.1 stated now in Proposition 3.3, adapted to general $K$, $V$, and $N$.

For convenience, we will write
\[ g_\infty := g. \]

To relax notation a bit, we will agree that whenever a Lorentzian metric, say $g_k$, has a subscript $k \in \mathbb{N}_\infty$, we endow corresponding quantities defined by that metric with the same subscript, e.g. we write $| \cdot |_k$ instead of $| \cdot |_{g_k}$, etc.

**Proposition 3.3.** Given any $K \in \mathbb{R}$ and $N \in [n, \infty)$, suppose
\[ \text{Ric}^{N,V}_\infty \geq K \text{ in all timelike directions}. \] (3.4)
Assume $(\mu_{\infty,0}, \mu_{\infty,1}) = (\rho_{\infty,0} n_{\infty}^V, \rho_{\infty,1} n_{\infty}^V) \in \mathcal{P}^{ac}_c(M, \text{vol}_\infty)^2$ to satisfy
\[ \text{spt} \mu_{\infty,0} \times \text{spt} \mu_{\infty,1} \subset M^2_{\leq \infty} \]
and $\rho_{\infty,0}, \rho_{\infty,1} \in L^\infty(M, \text{vol}_\infty)$. Then for every $p \in (0,1)$ there exist
- a timelike proper-time parametrized $t_{\infty,p}$-geodesic $(\mu_t)_{t \in [0,1]}$ from $\mu_{\infty,0}$ to $\mu_{\infty,1}$, and
- an $L^\infty_p$-optimal coupling $\pi \in \Pi_{\leq \pi}(\mu_{\infty,0}, \mu_{\infty,1})$

such that for every $t \in [0,1]$ and every $N' \geq N$,
\[ S^{N',V}_\infty(\mu_{\infty,t}) \leq -\int_{\mathcal{M}^2} \tau^{(1-t)}_{K,N'}(t_{\infty}(x_0, x')) \rho_{\infty}(x_0)^{-1/N'} \text{d}\pi(x_0, x'). \]
The proof of this proposition, in turn, is sub divided into various subsections incorporated in the body of the text below.

To streamline the exposition, in this chapter we adopt the subsequent notational convention. If a quantity is not introduced explicitly in a specific result or proof, it automatically refers to the respective object defined in one of the results or proofs listed in this chapter. Also, until Subsection 3.2.5 various subsequences will be extracted, which is not notationally reflected either for readability.

3.2.1. Setup and notation. Given the estimate (3.4), let \((\varepsilon_k)_{k\in\mathbb{N}}\) be a fixed sequence in \((0,\infty)\) decreasing to 0, let \(\{\delta_k : \varepsilon > 0\}\) be a family of smooth Lorentzian metrics satisfying all properties of Lemma 3.2.1, and set

\[\varphi_k := \delta_{\varepsilon_k}.\]

For \(k \in \mathbb{N}\), according to (2.12) we write

\[X_k := (M, d^h, n_k, \ll_k, \kappa_k).\]

In the sequel, we set

\[\kappa := \inf \tau_\infty(spt \mu_{\infty,0} \times spt \mu_{\infty,1}) > 0.\] (3.5)

3.2.2. Uniform convergence. In this technical section, we recapitulate the uniform convergence of \((\kappa_k)_{k\in\mathbb{N}}\) to \(\kappa\) on compact subsets of \(\mathcal{M}_{\infty}\), cf. Corollary 3.5. This will be needed in the proof of Lemma 3.7, cf. (3.6). For a similar result coming from approximation of the reference metric by smooth metrics with wider light cones, see [34, Prop. A.2].

The proofs of the corresponding results are standard, hence omitted.

Lemma 3.4. For every \(\varepsilon > 0\) and every compact \(C \subset \mathcal{M}^2_{\infty}\), there exists \(k_0 \in \mathbb{N}\) such that for every \(k \geq k_0\) and every \((x, y) \in C\),

\[\tau_\infty(x, y) \leq \tau_k(x, y) + \varepsilon.\]

Corollary 3.5. For every set \(C\) as in Lemma 3.4, the sequence \((\kappa_k)_{k\in\mathbb{N}}\) converges to \(\kappa\) uniformly on \(C\).

3.2.3. Construction of a recovery sequence. Before getting to Lemma 3.7, some further notational preparations are in order.

Let \(M\) be a \(d^h\)-closed ball in \(M\) which compactly contains \(J_\infty(\mu_{\infty,0}, \mu_{\infty,1})\). Since \(n_{\infty}[\partial M] = 0\), by Portmanteau’s theorem the sequence \((\kappa_k)_{k\in\mathbb{N}}\) converges weakly to \(\kappa_{\infty}\), where we set, for \(k \in \mathbb{N}\),

\[\kappa_k := n_k^h[M]^{-1} n_k^\mathcal{L} M.\]

Since \(M\) is compact, \(W_2(\kappa_k, \kappa_{\infty}) \to 0\) as \(k \to \infty\), where \(W_2\) is the 2-Wasserstein metric on \(\mathcal{P}(M)\) with respect to the restriction of \(d^h\) to \(M\). Given any \(k \in \mathbb{N}\), let \(\varphi_k \in \mathcal{P}(M^2)\) be a fixed \(W_2\)-optimal coupling of \(\kappa_k\) and \(\kappa_{\infty}\) [50, Thm. 4.1]. Let \(p^k : M \to \mathcal{P}(M)\) denote the disintegration of \(\varphi_k\) with respect to \(\text{spt} \mu_1\), given by the formula \(d\varphi_k(x, y) = dp^k(y) d\mu_k(x)\). Let \(p^k : \mathcal{P}(M, \kappa_{\infty}) \to \mathcal{P}(M, \kappa_k)\) denote the canonically induced (and nonrelabeled) map.

The proof of Lemma 3.7 below follows Step 1 to Step 3 for [3, Thm. 3.29]. It involves the subsequent Lemma 3.6 [7, Lem. 3.15]. Various items therein do not explicitly appear in our arguments below, but are used in the outsourced parts of the proof of Proposition 3.3 in Subsection 3.2.5.

Lemma 3.6. Let \(\pi_\infty \in \mathcal{P}(\mu_{\infty,0}, \mu_{\infty,1})\) be \(g_{\kappa_{\infty}}\)-timelike \(p\)-dualizing, \(p \in (0, 1]\). Then there exist sequences \((\pi_n)_{n\in\mathbb{N}}\) in \(\mathcal{P}(M^2)\) and \((a_n)_{n\in\mathbb{N}}\) in \([1, \infty)\) such that
(i) the sequence \((a_n)_{n \in \mathbb{N}}\) converges to 1,
(ii) \(\pi^n_{\infty}[M^2_{\infty}] = 1\) for every \(n \in \mathbb{N}\),
(iii) \(\pi_\infty^n = \rho_\infty^n m_\infty^2 \in \mathcal{P}_a(M^2, m_\infty^2)\) and \(\rho_\infty^n \in L^\infty(M^2, m_\infty^2)\) for every \(n \in \mathbb{N}\),
(iv) the sequence \((\pi^n)_{n \in \mathbb{N}}\) converges weakly to \(\pi_\infty\),
(v) writing \(\rho^n_{\infty,0}\) and \(\rho^n_{\infty,1}\) for the density of the first and second marginal of \(\pi^n_\infty\) with respect to \(m_\infty\), we have
\[
\rho^n_{\infty,0} \leq a_n \rho_\infty,0 \quad \text{m}_\infty\text{-a.e.},
\]
\[
\rho^n_{\infty,1} \leq a_n \rho_\infty,1 \quad \text{m}_\infty\text{-a.e.},
\]
(vi) \(\rho^n_{\infty,0} \to \rho_\infty,0\) and \(\rho^n_{\infty,1} \to \rho_\infty,1\) in \(L^1(M, m_\infty)\) as \(n \to \infty\).

**Lemma 3.7.** Let \(p \in (0, 1]\). Then there exists a sequence \((\mu_{k,0}, \mu_{k,1})_{k \in \mathbb{N}}\) of pairs \((\mu_{k,0}, \mu_{k,1}) = (\rho_{k,0} m_k, \rho_{k,1} m_k) \in \mathcal{P}_a(M, m_k)\) such that
(i) \((\mu_{k,0}, \mu_{k,1})_{k \in \mathbb{N}}\) converges weakly to \((\mu_\infty,0, \mu_\infty,1)\), and
(ii) for every \(k \in \mathbb{N}\), the pair \((\mu_{k,0}, \mu_{k,1})\) is \(g_k\)-timelike \(p\)-dualizable by a coupling \(\tilde{\pi}_k \in \Pi_{\leq k}(\mu_{k,0}, \mu_{k,1})\) satisfying
\[
\tilde{\pi}_k[\{\tau_k > \kappa\}] = 1.
\]

**Proof.** Given a \(g\)-timelike \(p\)-dualizing coupling \(\pi_\infty \in \Pi_{\leq \kappa}(\mu_\infty,0, \mu_\infty,1)\), let \((\pi^n)_{n \in \mathbb{N}}\) be as in Lemma 3.6. Define \(\mu^n_{k,0}, \mu^n_{k,1} \in \mathcal{P}_a(M, m_k), k \in \mathbb{N}\), by
\[
\mu^n_{k,0} := \phi^k(\pi^n_{\infty,0}) = \rho^n_{k,0} m_k,
\]
\[
\mu^n_{k,1} := \phi^k(\pi^n_{\infty,1}) = \rho^n_{k,1} m_k.
\]
Moreover, define \(\pi^n_k \in \Pi(\mu^n_{k,0}, \mu^n_{k,1}) \cap \mathcal{P}_a(M^2, m_\infty^2)\) by
\[
\pi^n_k := \left(\text{proj}_{k} \circ \text{proj}_{\infty}\right)[\left(\rho^n_{\infty,0} \circ \left(\text{proj}_{k} \circ \text{proj}_{\infty}\right)\right) q_k \otimes q_k].
\]
Using tightness of \((q_k)_{k \in \mathbb{N}}\) [50, Lem. 4.3, Lem. 4.4], we obtain the weak convergence of \((\pi^n_k)_{k \in \mathbb{N}}\) to \(\pi^n_\infty, n \in \mathbb{N}\), up to a nonrelabeled subsequence. Then Lemma 3.6, a compactness argument, and a diagonal procedure yield a sequence \((\tilde{\pi}_k)_{k \in \mathbb{N}}\) of probability measures \(\tilde{\pi}_k \in \mathcal{P}_a(M^2, m_\infty^2)\) converging weakly to \(\pi_\infty\), with
\[
\tilde{\pi}_k := \pi^n_k.
\]

Let \(U_0, U_1 \subset M\) be relatively compact open sets with \(\text{spt} \mu_\infty,0 \subset U_0\), \(\text{spt} \mu_\infty,1 \subset U_1\), and \(\inf \tau_\infty(\Omega) > 2\kappa\), where
\[
\Omega := U_0 \times U_1.
\]
By Lemma 3.4 applied to \(\varepsilon := \kappa\) and \(C := \Omega\), we have
\[
\Omega \subset \{\tau_k > \kappa\}
\]
for large enough \(k \in \mathbb{N}\). By Portmanteau’s theorem, since \(\Omega\) is open,
\[
1 = \pi_\infty[\Omega] \leq \liminf_{k \to \infty} \pi_k[\Omega].
\]
Up to passage to a subsequence, we may and will thus assume \(\pi_k[\Omega] > 0\) for every \(k \in \mathbb{N}\). Let the marginals \(\bar{\mu}_{k,0}, \bar{\mu}_{k,1} \in \mathcal{P}_a(M, m_k)\) of \(\tilde{\pi}_k\) be given by
\[
\bar{\mu}_{k,0} := \bar{\rho}_{k,0} m_k = \rho^n_{k,0} m_k,
\]
\[
\bar{\mu}_{k,1} := \bar{\rho}_{k,1} m_k = \rho^n_{k,1} m_k.
\]
Define \(\hat{\pi}_k \in \mathcal{P}_a(M, m_k)\) through
\[
\hat{\pi}_k := \pi_k[\Omega]^{-1} \tilde{\pi}_k \, d\Omega
\]
with marginals \(\hat{\mu}_{k,0}, \hat{\mu}_{k,1} \in \mathcal{P}_a(M, m_k)\) given by
\[
\hat{\mu}_{k,0} = \hat{\rho}_{k,0} m_k,
\]
\[
\hat{\mu}_{k,1} = \hat{\rho}_{k,1} m_k.
\]
Albeit these measures admit a $g_k$-chronological coupling by construction, it is not clear whether these are $g_k$-timelike $p$-dualizable, i.e. their $\ell_{k,p}$-cost is maximized by a coupling concentrated on the set $M^{k}_{\leq \infty}$. To modify $\mu_{k,0}$ and $\mu_{k,1}$ accordingly, let $\pi_k \in \mathbb{P}(\mu_{k,0}, \mu_{k,1})$ be an $\ell_{k,p}$-optimal coupling; by choosing the previous coupling $\pi_k$ as a competitor, and using compactness of $M^2$, its cost is strictly positive and finite. Since $(\pi_k)_{k\in\mathbb{N}}$ is weakly convergent, its marginal sequences $(\mu_{k,0})_{k\in\mathbb{N}}$ and $(\mu_{k,1})_{k\in\mathbb{N}}$ are tight; so is $(\pi_k)_{k\in\mathbb{N}}$ by [50, Lem. 4.4]. Thus, a nonrelabeled subsequence of the latter converges weakly to some $\pi_\infty \in \mathbb{P}(\mu_{\infty,0}, \mu_{\infty,1})$. By (3.5),

$$1 = \pi_\infty[\Omega] \leq \liminf_{k \to \infty} \pi_k[\Omega].$$

Up to passing to a subsequence, we may and will thus assume that $\pi_k[\Omega] > 0$ for every $k \in \mathbb{N}$. Then we define $\bar{\pi}_k \in \mathbb{P}(M^2)$ through

$$\bar{\pi}_k := \pi_k[\Omega]^{-1} \pi_k \mathbf{1}_{\Omega}.$$ 

By the restriction property of $\ell_{k,p}$-optimal couplings [7, Lem. 2.10], $\pi_k$ constitutes a chronological $\ell_{k,p}$-optimal coupling of its marginals $\mu_{k,0}, \mu_{k,1} \in \mathbb{P}_{ac}(M, m_k)$; in fact, $\bar{\pi}_k$ will even be uniquely determined by that property, see e.g. the proof of Proposition 3.8. Moreover, we have $\pi_k[\{\tau_k > \kappa\}] = 1$ for large enough $k \in \mathbb{N}$ thanks to (3.6). Hence, the pair $(\mu_{k,0}, \mu_{k,1})$ and $\bar{\pi}_k$ obey the desired requirements. ☐

3.2.4. Displacement semiconvexity. Now we prove displacement semiconvexity of Rényi’s entropy with respect to $m_k$ between $\mu_{k,0}$ and $\mu_{k,1}$.

In view of Lemma 2.8, this is the point where the additional property $\bar{\pi}_k[\{\tau_k > \kappa\}] = 1$ for every $k \in \mathbb{N}$, independently of the value $\kappa$ from (3.5), from Lemma 3.7 comes into play.

In the sequel, let $S^N_k$ denote the $N$-Rényi entropy with respect to $m_k$, $k \in \mathbb{N}_\infty$, defined analogously to (2.13).

**Proposition 3.8.** Let $\delta > 0$. Then there exists $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$, there exists a timelike proper-time parametrized $\ell_{k,p}$-geodesic $(\mu_{k,t})_{t \in [0,1]}$ from $\mu_{k,0}$ to $\mu_{k,1}$ such that for every $t \in [0,1]$ and every $N' \geq N$,

$$S^N_k(\mu_{k,t}) \leq -\int_{M^2} \tau^{-(1-\delta)}_{K-\delta, N'}(\tau_k(x^0, x^1)) \rho_{k,0}(x^0)^{-1/N'} d\bar{\pi}_k(x^0, x^1) - \int_{M^2} \tau^{(t)}_{K-\delta, N'}(\tau_k(x^0, x^1)) \rho_{k,1}(x^1)^{-1/N'} d\bar{\pi}_k(x^0, x^1).$$

(3.7)

**Proof.** The claim follows from essentially the same computations as [33, Ch. 6] and [4, Thm. 5.9]. We only describe the setting and the necessary modifications.

Let $c > 0$ be a given constant with respect to which all $g_\infty$-causal curves passing through the compact set $M$ have $d^0$-length no larger than $c$. (Thus, all $g_k$-causal curves with endpoints in $M$ that set by Lemma 2.8, $k \in \mathbb{N}$, which will be used several times without explicit notice below.) For such $c$, $\delta$ as hypothesized, $M$ as given, and $\kappa$ as in (3.5), let $k_0 \in \mathbb{N}$ be as provided by Lemma 2.8. Let $k \in \mathbb{N}$ with $k \geq k_0$, and recall from Lemma 2.9 that $g_k$ is globally hyperbolic. Hence, the theory developed in [33] applies as follows. As $\bar{\pi}_k$ is chronological and $\ell_{k,p}$-optimal, standard Kantorovich duality, cf. [50, Thm. 5.10] and [7, Rem. 2.2, Prop. 2.8, Prop. 2.19], entails the $p$-separation of $(\mu_{k,0}, \mu_{k,1})$ according to [33, Def. 4.1]. Since $\mu_{k,0} \ll m_k \ll vol_k$, $\bar{\pi}_k$ is the unique chronological $\ell_{k,p}$-optimal coupling of $\mu_{k,0}$ and $\mu_{k,1}$ relative to the Lorentzian spacetime $(M, g_k)$ [33, Thm. 5.8]. In particular, there is a sufficiently regular vector field $X_k$ on $M$ such that

$$\bar{\pi}_k = (\text{Id}, T_{k,1})_\sharp \mu_{k,0},$$

where $T_{k,1} : [0,1] \times M \to M$ is given by

$$T_{k,1}(x) := \exp_x t X_k(x).$$
Moreover, by [3, Rem. B.10] and [33, Cor. 5.9], there exists a unique timelike proper-time parametrized $\ell_{k,p}$-geodesic $(\mu_{k,t})_{t \in [0,1]}$ from $\mu_{k,0}$ to $\mu_{k,1}$. It is given by
$$\mu_{k,t} = (T_{k,t})_{[0,1]}.$$ (3.8)

Lastly, let $A_{k,t} := \tilde{D} T_{k,t} : TM|_{h} \to (T_{k,t}), TM$ be the approximate derivative [33, Def. 3.8] of $T_{k,t}$ as given by [33, Prop. 6.1]. It is invertible and depends smoothly on $t \in [0,1]$ at vol$_k$-a.e. $x \in M$. For such $x$ and a given $t \in [0,1]$, set
$$J_{k,t}(x) := |\det A_{k,t}(x)| e^{-V(T_{k,t}(x))},
\varphi_{k,t}(x) := \log J_{k,t}(x) = \log |\det A_{k,t}(x)| - V(T_{k,t}(x)).$$

Assume $N' \geq N > n$; the case $N = n$ can be treated similarly. Evaluated at any fixed point in $M$, the curve $(T_{k,t})_{t \in [0,1]}$ is a $g_k$-timelike geodesic passing through $M$. In particular, its $d^h$-length is no larger than $c$, whence $|T_{k,t}|_h \leq c$ for every given $t \in [0,1]$. Moreover, geodesy [33, Thm. 6.4], (3.8), and $\tilde{\pi}_k[\{\tau_k \geq \kappa\}] = 1$ imply
$$\vartheta_k := |\tilde{T}_{k,t}|_{h} = \tau_k(\cdot, T_{k,1}) > \kappa \quad \text{vol}_k$-a.e. (3.9)

Computing as in Step 2 for [3, Prop. A.2] and using Lemma 2.8 with (3.9),
$$\tilde{\varphi}_{k,t} + \frac{1}{N'} \tilde{\varphi}_{k,t}^2 \leq \tilde{\varphi}_{k,t} + \frac{1}{N'} \tilde{\varphi}_{k,t}^2
\leq -\text{Ric}_{k,t}^{N\prime}(\tilde{T}_{k,t}, \tilde{T}_{k,t}) \leq -(K - \delta) \vartheta_k \text{ vol}_k-a.e.$$(3.9)

This is a version of (A.4) in [3]. From here, we follow the proof of [4, Thm. 5.9] verbatim to conclude the statement. 

3.2.5. Conclusions. For notational convenience, given any $\pi \in \Pi(\mu_{\infty,0}, \mu_{\infty,1})$, $t \in [0,1]$, $K \in \mathbb{R}$, and $N \in [1, \infty)$, we define
$$\tau_{K,N}(\pi) := -\int_{M^2} \tau_{K,N}(\pi_{\infty}(x^0, x^1)) \rho_{\infty,0}(x^0)^{-1/N'} \, d\pi(x^0, x^1)
- \int_{M^2} \tau_{K,N}(\pi_{\infty}(x^0, x^1)) \rho_{\infty,1}(x^1)^{-1/N'} \, d\pi(x^0, x^1).$$

Proof of Proposition 3.3. The estimate obtained in Proposition 3.8 is a version of (3.9) in [3], with $\pi_k := \tilde{\pi}_k$, $k \in \mathbb{N}$ with $k \geq k_0$. Given any $\eta > 0$, by Corollary 3.5 we can modify $k_0$ in such a way that for every $k \geq k_0$, $\tau_k$ can be replaced by $\tau_{\infty}$ on the right-hand side of (3.7), and the two respective expressions differ at most by $\eta$. From there, letting $k \to \infty$ for fixed $\delta, \eta > 0$ we follow verbatim the proof of [3, Thm. 3.29] — with $\tau_{\infty}$ in place of $\tau$ therein — and get the following property. Given $\delta$ and $\eta$ as above, there exist a timelike proper-time parametrized $\ell_{\infty,p}$-geodesic $(\mu_{\infty,t})_{t \in [0,1]}$ from $\mu_{\infty,0}$ to $\mu_{\infty,1}$ and a $g_{\infty}$-timelike $p$-dualizing coupling $\pi_{\infty}^\delta \in \Pi_{\infty}(\mu_{\infty,0}, \mu_{\infty,1})$ such that for every $t \in [0,1]$ and every $N' \geq N$, we have
$$S_{\infty}^{N'}(\mu_{\infty,t}) \leq \tau_{K,N}(\pi_{\infty}) + \eta.$$ (3.10)

Note that the inherent objects do not depend on $\eta$.

Fix a sequence $(\delta_n)_{n \in \mathbb{N}}$ in $(0, \infty)$ decreasing to 0, and let $(\mu_{\infty,t}^{\delta_n})_{t \in [0,1]}$ and $\pi_{\infty}^{\delta_n}$ be the above objects with respect to $\delta_n$, $n \in \mathbb{N}$. Let $\pi^n \in \text{OptTGeo}_{\infty}^{2\infty}(\mu_{\infty,0}, \mu_{\infty,1})$ represent $(\mu_{\infty,t}^{\delta_n})_{t \in [0,1]}$. By our assumption
$$\text{spt} \mu_{\infty,0} \times \text{spt} \mu_{\infty,1} \subset M_{\infty,\infty}^2$$
and by compactness of timelike $\ell_{\infty,p}$-optimal geodesic plans relative to $X^\prime$, constructed in Section 2.2 [3, Prop. B.11], cf. Proposition 2.4, a nonrelabeled subsequence of $(\pi^n)_{n \in \mathbb{N}}$ converges weakly to some $\pi \in \text{OptTGeo}_{\infty}^{2\infty}(\mu_{\infty,0}, \mu_{\infty,1})$. The latter represents a timelike proper-time parametrized $\ell_{\infty,p}$-geodesic $(\mu_{\infty,t})_{t \in [0,1]}$ from $\mu_{\infty,0}$ to $\mu_{\infty,1}$. Moreover, by a tightness argument and stability of $\ell_{\infty,p}$-optimal couplings [7, Lem. 2.11], a nonrelabeled subsequence of $(\pi_{\infty}^{\delta_n})_{n \in \mathbb{N}}$ converges weakly
to some $\ell_{\infty,p}$-optimal coupling $\pi_{\infty} \in \Pi_{\infty}(\mu_{\infty,0}, \mu_{\infty,1})$. Thus, given $\varepsilon > 0$, $t \in [0,1]$, and $N' \geq N$ we obtain

$$
S_{\infty}^t(\mu_{\infty,t}) \leq \limsup_{n \to \infty} S_{\infty}^t(\mu_{\infty,t}) \leq \limsup_{n \to \infty} T_{K-\varepsilon,N'}^t(\pi_{\infty}) + \eta \\
\leq \limsup_{n \to \infty} T_{K-\varepsilon,N'}^t(\pi_{\infty}) + \eta \leq T_{K-\varepsilon,N'}^t(\pi_{\infty}) + \eta.
$$

Here we have successively used weak lower semicontinuity of the Rényi entropy on $\mathcal{P}(M)$ [31, Thm. B.33], the estimate (3.10), nondecreasingness of the distortion coefficient $\tau_{K,N'}^t(\vartheta)$ in $K \in \mathbb{R}$ for fixed $r \in [0,1]$, $N' \geq N$, and $\vartheta \geq 0$, as well as upper semicontinuity of $T_{K-\varepsilon,N'}^t$ after [3, Lem. 3.27]. Finally, sending $\varepsilon \to 0$ and $\eta \to 0$ in the previous inequality via Fatou’s lemma gives the result.

**Proof of Theorem 3.1.** Combining Proposition 3.3 with [3, Prop. 4.9], we directly obtain the TMCP($K,N$) condition for $\mathcal{X}^\varphi_{\infty}$. Indeed, albeit [3, Prop. 4.9] assumes the *weak* timelike curvature-dimension condition from [3, Def. 3.3], its proof needs displacement semiconvexity of the Rényi entropy only between mass distributions satisfying the assumptions of Proposition 3.3.

**Proof of Theorem 3.2.** Recall from Remark 2.3 that if $g_{\infty}$ is of class $C^{1,1}$, then $\mathcal{X}^\varphi_{\infty}$ is $g_{\infty}$-timelike nonbranching. Up to a change of the involved distortion coefficients, the identical argument as for [3, Prop. 3.38] — note that the reductions in Step 1 therein are precisely the assumptions on the marginals in Proposition 3.3 — entails a pathwise version of TCD$_p(K,N)$. This verifies TCD$_p(K,N)$ by integration.

### 3.3. Consequences of Theorem 3.1 and Theorem 3.2.

#### 3.3.1. Sharp timelike geometric inequalities

Having established displacement semiconvexity of $S_{\infty}^\varphi$ along appropriate timelike proper-time parametrized $f_{g,\varphi}$-geodesics, the following three geometric inequalities are derived in a standard way, cf. [7, Prop. 3.4, Prop. 3.5, Prop. 3.6] or [3, Prop. 3.11, Cor. 3.14, Thm. 3.16]. For instance, Corollary 3.9 is a simple consequence of Jensen’s inequality.

**Corollary 3.9** (Sharp Brunn–Minkowski). *Let the assumptions of Theorem 3.1 hold. Let $p \in (0,1)$, let $A_0 \subset M$ be a relatively compact Borel set with $n^p_y[A_0] > 0$, and let $\mu_0 \in \mathcal{P}_c^\infty(M, \text{vol}_g)$ be the uniform distribution on $A_0$. For a specified Borel set $A_1 \subset M$ and $t \in [0,1]$, we set

$$
A_t := \{\gamma_1 : \gamma \in T\text{Geo}^{\varphi}(M), \gamma_0 \in A_0, \gamma_1 \in A_1\}
$$

as well as

$$
\Theta := \begin{cases}
\sup_{A_t} \tau_g(A_0 \times A_1) & \text{if } K < 0, \\
\inf_{A_t} \tau_g(A_0 \times A_1) & \text{otherwise}.
\end{cases}
$$

(i) Let $x_1 \in M$ such that $\mu_0[I_g^- (x_1)] = 1$, and set $A_1 := \{x_1\}$. Then for every $t \in [0,1)$ and every $N' \geq N$,

$$
n^p_y[A_t]^{1/N'} \geq \tau_{K,N'}^{1-t}(\Theta) n^p_y[A_0]^{1/N'}.
$$

(ii) Let the assumptions of Theorem 3.2 hold. Let $A_1 \subset M$ be a relatively compact Borel set with $n^p_y[A_1] > 0$. Let $\mu_1 \in \mathcal{P}_c^\infty(M, \text{vol}_g)$ be the uniform distribution on $A_1$, and assume $g$-timelike $p$-dualizability of $(\mu_0, \mu_1)$. Then for every $t \in [0,1)$ and every $N' \geq N$,

$$
n^p_y[A_t]^{1/N'} \geq \tau_{K,N'}^{1-t}(\Theta) n^p_y[A_0]^{1/N'} + \tau_{K,N'}^t(\Theta) n^p_y[A_1]^{1/N'}.
$$

For a general $\mu_0$ as above, there might be no $x_1 \in M$ with $\mu_0[I_g^- (x_1)] = 1$, thus its existence in the first item of Corollary 3.9 is an additional assumption.*
**Corollary 3.10** (Sharp Bonnet–Myers). Let the assumptions from Theorem 3.1 hold, and further suppose \( K > 0 \). Then

\[
\sup \tau_g(M^2) \leq \pi \sqrt{\frac{N-1}{K}}.
\]

For the third corollary, we refer to (2.14) for the definition of the function \( s_{K,N} \). Moreover, we call a set \( E \subset M \) \( \tau_g \)-star-shaped with respect to \( x \in M \) if for every \( \gamma \in \text{TGeo}^\tau_x(M) \) with \( \gamma_0 = x \) and \( \gamma_1 \in E \) we have \( \gamma_t \in E \) for every \( t \in (0,1) \). Given such \( E \) and \( x \) as well as \( r > 0 \), set

\[
\mathcal{B}^\tau(x,r) := \{ y \in M : \tau_g(x,y) \in (0, r) \} \cup \{ x \},
\]

and define

\[
v_r := n_{g_\gamma}^V [\mathcal{B}^\tau(x,r) \cap E], \\
s_r := \limsup_{\delta \to 0} -\frac{1}{\delta} n_{g_\gamma}^V [ (\mathcal{B}^\tau(x,r+\delta) \setminus \mathcal{B}^\tau(x,r)) \cap E].
\]

**Corollary 3.11** (Sharp Bishop–Gromov). Let the assumptions of Theorem 3.1 hold. Let \( E \subset M \) be a compact set which is \( \tau_g \)-star-shaped with respect to \( x \in M \). Then for every \( r,R > 0 \) with \( r < R \leq \pi \sqrt{(N-1)/\max\{K,0\}} \)

\[
\frac{s_r}{s_R} \geq \frac{s_{K,N-1}(r)}{s_{K,N-1}(R)} N^{-1}
\]

as well as

\[
\frac{v_r}{v_R} \geq \frac{\int_0^r s_{K,N-1}(s)^{N-1} \, ds}{\int_0^R s_{K,N-1}(s)^{N-1} \, ds}.
\]

**Remark 3.12.** These three corollaries explain why we chose to derive \( \text{TMCP}(K,N) \) and \( \text{TCD}_p(K,N) \) in Theorem 3.1 and Theorem 3.2 instead of their reduced or entropic versions, cf. Remark 1.2. Indeed, Corollary 3.11 is sharp in the sense that model spaces attain equality therein [7, Rem. 5.11]. More generally, Corollary 3.9, Corollary 3.10, and Corollary 3.11 are sharp in the sense of dimensional improvements: recall that if a globally hyperbolic \( C^1 \)-spacetime of dimension \( n \) obeys \( \text{TCD}_p(K,N) \) for some \( N \in [1, \infty) \), then

\[ n = \dim^\tau \ M \leq N. \]

Here \( \dim^\tau \ M \) is the Lorentzian Hausdorff dimension of \((M,g)\) from [34, Def. 3.1]. Under \( \text{TCD}_p^\tau(K,N) \) or \( \text{TCD}_p^\tau(K,N) \), the above statements do \textit{a priori} only hold for \( N \) replaced by \( N + 1 \), cf. [3, Rem. 3.19] and [34, Thm. 5.2].

Under the stronger assumptions of Theorem 3.2, Corollary 3.10 and Corollary 3.11 follow alternatively from Remark 1.2 and [7, Prop. 5.9, Prop. 5.10]. The latter have been derived by using the localization technique from [7, Ch. 4], itself reliant on \( g \)-timelike nonbranching, which may fail below \( C^{1,1} \)-regularity under synthetic timelike Ricci bounds [13].

**3.3.2. Uniqueness of chronological optimal couplings and chronological geodesics.** A further direct implication of Theorem 3.2, together with the implicit \( g \)-timelike nonbranching property, are the following uniqueness results about the \( \ell_{g,p} \)-optimal transport problem, cf. [7, Thm. 3.19, Thm. 3.20] or [3, Thm. 4.16, Thm. 4.17].

**Corollary 3.13.** Let the assumptions of Theorem 3.2 hold. Given \( p \in (0,1) \), suppose \( g \)-timelike \( p \)-dualizability of the pair \((\mu_0,\mu_1) \in \mathcal{P}_c^g(M,\text{vol}_g) \times \mathcal{P}_c(M)\).
(i) **Uniqueness of chronological optimal couplings.** The set of \( \ell_{g,p} \)-optimal couplings of \( \mu_0 \) and \( \mu_1 \) which also lie in \( \Pi_{g,p}(\mu_0,\mu_1) \) is a singleton \( \{\pi\} \). Moreover, there exists a \( \mu_0 \)-measurable map \( T : \text{spt} \mu_0 \to M \) such that
\[
\pi = (\text{Id}, T)_* \mu_0.
\]

(ii) **Uniqueness of chronological geodesics.** The set \( \text{OptGeo}^{r_a}_{g,p}(\mu_0,\mu_1) \) is a singleton \( \{\pi\} \). Furthermore, there exists a \( \mu_0 \)-measurable map \( \Sigma : \text{spt} \mu_0 \to T\text{Geo}^{r_a}(M) \) such that
\[
\pi = \Sigma_* \mu_0.
\]

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