Integrable open-boundary conditions for the supersymmetric t-J model.
The quantum group invariant case.

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Abstract

We consider integrable open–boundary conditions for the supersymmetric t–J model commuting with the number operator $n$ and $S^z$. Four families, each one depending on two arbitrary parameters, are found. We find the relation between Sklyanin’s method of constructing open boundary conditions and the one for the quantum group invariant case based on Markov traces. The eigenvalue problem is solved for the new cases by generalizing the Nested Algebraic Bethe ansatz of the quantum group invariant case (which is obtained as a special limit). For the quantum group invariant case the Bethe ansatz states are shown to be highest weights of $spl_q(2,1)$.

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1 Introduction

During the last years there has been an increasing activity on the study of systems in the finite interval. For the case of one dimensional models integrable by the quantum inverse scattering method the pioneering works of Cherednik and Sklyanin [1, 2] were the starting point. There, the so called reflection equations, appeared as the new ingredient to the Yang-Baxter equation when dealing with independent boundary conditions. A lot of integrable models arised using this method, see for example [3, 4]. Some of them have been shown to have physical applications as for example the Hofstadter problem [5] or the reaction diffusion equations [6]. Some of these models have been solved and other would need non trivial generalizations of the methods developed for periodic and twisted boundary conditions (for a review see [7]). Certain limits of this open boundary conditions where shown to give quantum group invariant transfer matrices and hamiltonians [8, 3, 8, 1, 10].

On the other hand the t-J model has attracted much interest in connection with high-$T_c$ superconductivity. This model is obtained from the Hubbard model as an effective hamiltonian for the low-energy states in the strong correlation limit. In this limit double occupancy of fermions is forbidden, leading to only three possible states at each lattice site. For a given ratio of the coupling constants the model becomes supersymmetric. In one dimension the model is exactly solvable by means of the nested Bethe ansatz method, see for example [11]. Recently the quantum inverse scattering method was used to solve this model [12] and the completeness of the Bethe states was shown [13]. Also a $spl_q(2, 1)$ invariant t-J model was proposed and solved by a generalization of the nested Bethe ansatz method to quantum group invariant open boundary conditions [14]. This method has been also generalized for the case of the $SU_q(n)$ invariant chains [13]. Up to this moment for the susy t-J model only the periodic and the quantum group invariant cases have been solved by means of the nested Bethe ansatz.

Some fundamental questions remain open, among them: the solution of the generic open chain with independent boundary conditions, the highest weight property for the quantum group invariant chain, the problem of completeness of Bethe states and the relation of the Markov trace method for finding quantum group invariant chains with the method of reflection equations.

It is interesting to find the widest possible class of open boundary conditions compatible with the integrability of the model. This terms could give count of impurities or magnetic
fields located at the boundaries. In the present work we find new integrable boundary
conditions for the supersymmetric t-J hamiltonian. We look only for solutions commuting
with the operators \( n \) and \( S^z \) because these are the ones keeping the structure of the
low-energy states. Other boundary conditions would give rise to low-energy states of the
domain wall type \([18, 4]\). These new conditions survive in the limit of zero anisotropy,
which is not the case for those of the quantum group invariant case. We find the necessary
generalization of the nested Bethe ansatz construction to the case of independent
boundary conditions in the edges of the chain.

For the quantum group invariant chain we prove the highest weight property for the Bethe
eigenvectors. This is the first proof of this property in an invariant chain with respect
to a quantum group of rank greater than one and is the first step in order to treat the
completeness problem (recently this property has been also shown to hold in the \( SU_q(n) \)
invariant case \([16]\)). We will also see that there is a degeneration in the eigenvalues.

A general method based on Markov traces to find quantum group invariant chains has
been proposed recently \([14, 24]\). We find the connection with the method of reflection
equations modified for non symmetric \( S \)-matrices in \([3]\). The proof of the commutativity
of the transfer matrices and of the quantum group invariance is immediate from this point
of view.

The paper is organized as follows. In section 2 all the diagonal solutions to the reflection
equations are found, giving the relation between the transfer matrix and the new integrable families of hamiltonians. We also discuss the relation of Markov traces with reflection equations in the quantum group invariant case. In section 3 we diagonalize the transfer matrices using the quantum inverse scattering method generalizing Sklyanin’s approach to open boundary conditions for the Nested Bethe Ansatz construction. Section 4 is devoted to probe the highest weight property for the Bethe vectors in the quantum group invariant case. Section 5 contains a summary of the main results and some further possible investigations.

## 2 Integrable open–boundary conditions

As is by now well known the supersymmetric t–J model is related to the graded 15 vertex
model hamiltonian, see for example \([14, 13]\). This is a vertex model with three states
per link which can be bosonic or fermionic. The matrix of vertex weights has the form:
where \( a, b, c \) and \( d \) are indexes running from 1 to 3 and
\( a = \sin(v + \gamma), \ b = \sin v, \ c_+ = e^{iv} \sin \gamma, \ c_- = e^{-iv} \sin \gamma, \ w = \sin(-v + \gamma). \)

We have adopted the convention of making fermionic the third state.

This matrix is a trigonometric solution to the quantum Yang-Baxter equation:

\[
S_{12}(u)S_{13}(u + v)S_{23}(v) = S_{23}(v)S_{13}(u + v)S_{12}(u),
\]

defined in the space \( V_1 \otimes V_2 \otimes V_3 \) with the standard notation \( S_{ij} \in \text{End}(V_i \otimes V_j). \)

This \( S \)-matrix does not enjoy \( P \) and \( T \) symmetry but just \( PT \) invariance:

\[ PS_{12}P := S_{21} = S_{12}^{t_1t_2}. \]

It is not crossing invariant either but it obeys the weaker property \([9]\):  

\[
\left\{ \left[ S_{12}(v)^{t_2} \right]^{-1} \right\}^{t_2} = L(v, \gamma)M_2S_{12}(v + 2\eta)M_2^{-1}, \]

where \( L(v, \gamma) \) is a \( c \)-number function, \( \eta \) a constant and \( M \) a symmetry of the \( S \)-matrix:

\[
[M_1 \otimes M_2, S_{12}(v)] = 0.
\]

We find by direct calculation from \([1]\) and \([2]\):
\[ \eta = \frac{\gamma}{2} \]

\[ M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & q^2 & 0 \\ 0 & 0 & -q^2 \end{pmatrix}; \quad q = e^{i\gamma}, \quad (3) \]

\[ L(v, \gamma) = -\frac{b(v)}{w(v)}. \]

When the weak condition (2) holds, if looking for integrable boundary conditions, it is necessary to solve for \( K^\pm(v) \) the following reflection equations [3]:

\[ S_{12}(u - v)K_1^-(u)S_{21}(u + v)K_2^-(v) = \\
K_2^-(v)S_{12}(u + v)K_1^-(u)S_{21}(u - v), \quad (4) \]

\[ S_{12}(-u + v)K_1^+(u)^{l_1}M_1^{-1}S_{21}(-u - v - 2\eta)M_1K_2^+(v)^{l_2} = \\
K_2^+(v)^{l_2}M_1S_{12}(-u - v - 2\eta)M_1^{-1}K_1^+(u)^{l_1}S_{21}(-u + v). \quad (5) \]

There is an automorphism between \( K^- \) and \( K^+ \):

\[ K^+(v) = K^-(-v - \eta)^tM. \quad (6) \]

As explained in sect. 1, it is interesting to find open boundary conditions commuting with the operators \( n \) and \( S^z \) which are diagonal. For that reason we will only look for diagonal solutions to the equation (4) :

\[ K^-(v)_{ab} = K_a^-(v)\delta_{ab} \quad (7) \]

Inserting this in equation (4) the only nontrivial equations are:

\[ \sin(u + v)[K_b^-(u)K_a^-(v)e^{\text{sign}(a-b)(u-v)} - K_a^-(u)K_b^-(v)e^{-\text{sign}(a-b)(u-v)}] \\
+ \sin(u - v)[K_a^-(u)K_a^-(v)e^{-\text{sign}(a-b)(u+v)} - K_b^-(u)K_b^-(v)e^{\text{sign}(a-b)(u+v)}] = 0, \]

\[ a, b = 1, 2, 3. \]
Deriving this equation with respect to \( u \) and making \( u = 0 \) the solutions to these equations are found to be:

\[
K_a(v) = e^{iv} \sin(\xi_+ - v)
\]
\[
K_b(v) = e^{-iv} \sin(\xi_- + v); \quad a > b
\]

with \( \xi_- \) an arbitrary parameter.

This gives two families of solutions for the equation (2) and using the automorphism (3) the same number for the equation (4), these are:

\[
K_a^{-}(v) = \frac{1}{\sin \xi_-} \begin{pmatrix}
    e^{-iv} \sin(\xi_+ + v) & 0 & 0 \\
    0 & e^{-iv} \sin(\xi_- + v) & 0 \\
    0 & 0 & e^{iv} \sin(\xi_- - v)
\end{pmatrix},
\]

\[
K_b^{-}(v) = \frac{1}{\sin \xi_-} \begin{pmatrix}
    e^{-iv} \sin(\xi_+ + v) & 0 & 0 \\
    0 & e^{iv} \sin(\xi_- - v) & 0 \\
    0 & 0 & e^{iv} \sin(\xi_- - v)
\end{pmatrix},
\]

\[
K_a^{+}(v) = \frac{1}{\sin \xi_+} \begin{pmatrix}
    e^{iv} \sin(\xi_- - v) & 0 & 0 \\
    0 & q^2 e^{iv} \sin(\xi_+ - v) & 0 \\
    0 & 0 & -q e^{-iv} \sin(\xi_+ + v + \gamma)
\end{pmatrix},
\]

\[
K_b^{+}(v) = \frac{1}{\sin \xi_+} \begin{pmatrix}
    e^{iv} \sin(\xi_- - v) & 0 & 0 \\
    0 & q e^{-iv} \sin(\xi_+ + v + \gamma) & 0 \\
    0 & 0 & -q e^{-iv} \sin(\xi_+ + v + \gamma)
\end{pmatrix},
\]

where \( \xi_+ \) and \( \xi_- \) are arbitrary independent parameters. As it is obvious from equations (3,5) these solutions can be multiplied by arbitrary factors, these have been chosen in (8),(9) in order to have \( K^{-}(0) = 1 \). It is important to note at this point that the fact of having several families of diagonal solutions is a caracteristic of \( S \)-matrices corresponding to algebras with rank bigger than one [4]. The number of independent solutions equals the rank of the algebra as in the \( A_{n-1} \) [4].

For fixed boundary conditions described by the matrices \( K^{\pm}(v) \), one uses the monodromy matrix:

\[
U_{ab}(v) = \sum_{cd} T_{ac}(v)K_{cd}^{-}(v)T_{db}^{-1}(-v).
\]
Where $T_{ac}(v)$ is the standard monodromy matrix for a $L \times L$ square lattice defined as the matrix product over the $S$’s:

$$T_{ab(c)}^{(d)}(v) = S_{b_{2}c_{1}}^{a_{d_{1}}}(v)S_{b_{3}c_{2}}^{a_{d_{2}}}(v)S_{b_{4}c_{3}}^{a_{d_{3}}}(v)...S_{b_{L}c_{L}}^{a_{d_{L}}}(v),$$

indexes in parenthesis act in the quantum space $\mathbb{C}^3 \times \mathbb{C}^3 \times ... \mathbb{C}^3$ and $a$ and $b$ in the horizontal auxiliary space $\mathbb{C}^3$ in the usual convention.

The operator $T^{-1}(v)$ is the inverse of $T(v)$ in both, horizontal and the quantum space and is given by:

$$T_{ab(c)}^{-1(d)}(v) = S_{b_{2}c_{1}}^{a_{d_{1}}}(v)S_{b_{3}c_{2}}^{a_{d_{2}}}(v)S_{b_{4}c_{3}}^{a_{d_{3}}}(v)...S_{b_{L}c_{L}}^{a_{d_{L}}}(v),$$

where:

$$\tilde{S}_{cd}^{ab}(v) = \frac{S_{dc}^{ba}(-v)}{\sin(\gamma + v)\sin(\gamma - v)}.$$

The elements of $\tilde{S}$ will be denoted with a tilde "\(\tilde{\}\)".

The operator $U$ can be represented as a $3 \times 3$ matrix of operators acting in the quantum space:

$$U = \begin{pmatrix} A & B_2 & B_3 \\ C_2 & D_{22} & D_{23} \\ C_3 & D_{32} & D_{33} \end{pmatrix}.$$

(11)

This operator matrix satisfies the reflection equation [4]:

$$S_{12}(u - v)U_{1}^{-}(u)S_{21}(u + v)U_{2}^{-}(v) = U_{2}^{-}(v)S_{12}(u + v)U_{1}^{-}(u)S_{21}(u - v).$$

(12)

The fixed boundary condition transfer matrix is then defined as:

$$t(v) = \sum_{ab} K_{ab}^{+}(v)U_{ba}(v).$$

(13)
Thanks to equations (2, 3, 12, 13) the transfer matrix $t(v)$ defines a one parameter family of commuting operators [3, 4]:

$$[t(v), t(u)] = 0,$$  \hspace{1cm} (14)

showing the integrability of the model.

The transfer matrix $t(v)$ is related to a quantum one dimensional hamiltonian through its first derivative in the usual way [4]:

$$\dot{t}(0) = -\frac{1}{4} \sin \gamma H \ tr K^+(0) + tr \dot{K}^+(0),$$  \hspace{1cm} (15)

where $H$ is the hamiltonian given by:

$$H = \sum_{j=1}^{L-1} h_{j,j+1} + \frac{1}{2} \dot{K}_j^{-1}(0) + \frac{tr[K_0^+(0)h_{L0}]}{tr[K^+(0)]}. $$  \hspace{1cm} (16)

The subscript 0 means the horizontal or auxiliary space, and the two sites hamiltonian $h$ is given by:

$$h_{j,j+1} = P \dot{S}_{j,j+1}(0).$$  \hspace{1cm} (17)

We see from (16) that the effect of the previous construction in the hamiltonian is to add terms depending on the matrices $K^\pm$ on the edges of the chain. As we have two families of solutions for the reflection matrices at each boundary, and they are independent, we will have four kinds of boundary terms.

Using (16) and omiting a term proportional to the identity operator we obtain the following hamiltonians:

$$H = -P \left\{ \sum_{j=1}^{L-1} \sum_{\sigma} \left( c_{j,\sigma}^+ c_{j+1,\sigma}^0 + c_{j,\sigma} c_{j+1,\sigma}^+ \right) \right\} P - 2 \sum_{j=1}^{L-1} \left( S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \cos \gamma \left( S_j^z S_{j+1}^z - \frac{n_j n_{j+1}}{4} \right) \right) - \cos \gamma \sum_{j=1}^{L} n_j + i \sin \gamma \left( n_1 - n_L \right) - i \sin \gamma \sum_{j=1}^{L-1} \left( n_j S_j^z - S_{j+1}^z n_{j+1} \right) + i \sin \gamma H_{a\beta}^{b}, \ \alpha, \beta = a, b$$

$$H_{aa}^b = (\cot \xi_+ - 1) n_1 - (\cot(\xi_+ - \gamma) - 1) n_L$$  \hspace{1cm} (18)
This is the supersymmetric t-J hamiltonian with four different kinds of integrable boundary conditions depending each one on two arbitrary parameters $\xi_{\pm}$. The operators $c_{j}^{(i)}$ are spin up or down annihilation (creation) operators. The $S_{j}$ are spin matrices and $n_{j} (n_{j}^{h})$ the occupation number of electrons (holes) at lattice site $j$. The operator $P = \prod_{j=1}^{L} (1 - n_{j}^{\uparrow} n_{j}^{\downarrow})$ forbids the double occupancy of electrons at one lattice site. There are only three possibilities $(1, 2, 3) = (\uparrow, \downarrow, 0)$, an electron with spin up, down or a hole.

The boundary hamiltonians $H_{ab}^{b}$ correspond to choosing the families $\alpha$ for the $K^{+}$ matrix and $\beta$ for the $K^{-}$.

The previous boundary conditions can be interpreted as the effect of impurities or magnetic fields located at the edges of the chain. There appear in this hamiltonian imaginary boundary terms as in the case of the open XXZ hamiltonian \cite{2, 4}. But in this case imaginary terms appear also in the bulk part, this was first noticed for the $SU_{q}(n)$ invariant models in \cite{4} and in the case of the quantum group invariant t-J model in \cite{14}. This seems to be a characteristic of models with underlying quantum group of rank larger than one. For this case, the imaginary bulk part is relevant for configurations $\uparrow \downarrow$ or $\downarrow \uparrow$ which are separated by holes. This term also breaks the spin parity of the model. At one electron per site (half filling) the imaginary bulk term in (18) reduce to that of the XXZ model.

Although the hamiltonians obtained are non hermitean they have real eigenvalues in some cases. In the quantum group invariant cases the eigenvalues are real as shown in \cite{14, 15}. For the general boundary conditions discussed in this article this is still true for the cases $(\alpha, \beta) = (a, a), (b, b)$ as we will see in next section. In this cases the hamiltonians become hermitean under an auxiliary scalar product as shown in \cite{14} for the quantum group invariant case. In the hiperbolic regime the eigenvalues are real in all the cases. In the rational limit (the $spl(2, 1)$ t-J model with open boundary conditions) all the cases give hermitian hamiltonians.

We will now make some comments concerning the quantum group invariant case. The $spl_{q}(2, 1)$ invariant hamiltonian of reference \cite{14} is obtained after taking in (18) the limit $\Xi_{\pm} = e^{i \xi_{\pm}} \rightarrow \infty$. When looking in this limit to the reflection matrices we have for the wo families of solutions $K^{-} = 1$ and:

$$
H_{ab}^{b} = (\cot(\xi_{-}) - 1)(S_{1}^{z} - n_{1}^{h}/2) - (\cot(\xi_{+} - \gamma) - 1) n_{L}
$$

$$
H_{ba}^{b} = (\cot(\xi_{-}) - 1)n_{1} - (\cot(\xi_{+} - 1))(S_{L}^{z} - n_{L}^{h}/2)
$$

$$
H_{bb}^{b} = (\cot(\xi_{-}) - 1)(S_{1}^{z} - n_{1}^{h}/2) - (\cot(\xi_{+} - 1))(S_{L}^{z} - n_{L}^{h}/2).
$$
\[
K^+ = M = \begin{pmatrix}
1 & 0 & 0 \\
0 & q^2 & 0 \\
0 & 0 & -q^2
\end{pmatrix}.
\]

Then we recover the construction of Foerster and Karowski defining the transfer matrix as the Markov trace associated with the superalgebra \(spl_q(2,1)\) of the monodromy matrix (10) (with \(K^- = 1\)) in the auxiliary space:

\[
t_q(v) = \sum_a M_{aa} U_{aa}(v) = \sum_{abc} M_{ab} T_{bc}(v) T_{ca}^{-1}(-v).
\]

This interpretation has been further developed also in [24]. In fact it is known [23] that for all solution to the Yang-Baxter equation corresponding to a non exceptional affine Lie algebra \(g^{(k)}\) in the fundamental representation, PT symmetry, unitarity and equation (4) are obeyed. Also, for all cases except \(D_{n}^{(2)}\), \(K^-(v) = 1\) is a solution to eqn. (4) and using the automorfism (6) \(K^+(v) = M\) is a solution to eqn. (5). Then for all this Lie algebras, except \(D_{n}^{(2)}\) the transfer matrix (19) forms a commuting family, see [3]. This transfer matrix commutes with all the generators of the quantum algebra \(U_q(g_0)\) where \(g_0\) is the maximal finite-dimensional subalgebra of \(g^{(k)}\) [22, 3].

On the other hand it can be shown in general [19] that:

\[
[S, M \otimes M] = 0,
\]

and,

\[
\text{tr}_2(M_2 P S_{12}) \propto 1,
\]

\[
\text{tr}_2(M_2 (PS)^{-1}_{12}) \propto 1,
\]

giving to the transfer matrix (13) the interpretation of a Markov trace. The relation between reflection equations an Markov traces has also been pointed out in [17].

3 Algebraic Bethe Ansatz

We want to diagonalize the hamiltonians (18). Equation (13) shows that the eigenvalues and eigenvectors of the hamiltonians with open boundary conditions can be obtained as
derivatives of those of the transfer matrix (13). Then we have to solve the eigenvalue problem:

$$ t^{\alpha\beta} \Psi = \lambda \Psi, \quad (20) $$

where $t^{\alpha\beta}(v)$ will denote the transfer matrix constructed using family $\alpha$ of solutions for the $K^+$ and family $\beta$ for $K^-$, $\alpha, \beta = a$ or $b$.

We will find for the transfer matrices in the preceding section a generalization of the nested Bethe ansatz, only found previously for quantum group invariant conditions [14, 15]. As Bethe ansatz for the transfer matrix eigenvectors, a linear combination of $B_a$'s acting on a ferromagnetic ground state state and summed over the indices $a$ is proposed. Then one should find the coefficients in such linear combination from the eigenvalue condition. Surprisingly enough, these coefficients turn to obey an eigenvector problem analogous to the original one but with a new transfer matrix. This new transfer matrix is built from statistical weights obtained from the original ones deleting the first row and column. The reflection matrices give also new reflection matrices for the reduced problem. The problem is solved in the sense that it reduces to a set of algebraic equations: the nested Bethe Ansatz equations (NBAE).

To follow these steps it is necessary to make use of the commutation relations for the operators $U_{ab}$ given in equation (12) which are the same whatever the matrix $K^-$ is if it obeys (4) and $T, T^{-1}$ are Yang-Baxter operators as defined previously. In order to simplify this commutation relations instead of using the operators $D_{ab}$ it is more convenient to work with new operators $\hat{D}_{ab}$ such that [14]:

$$ D_{ab}(v) = \chi(v) \frac{S_{ac}(2v + \gamma)}{b(2v + \gamma)} \hat{D}_{dc}(v) + \delta_{ab} \frac{c_{+}(2v)}{a(2v)} A(v), \quad (21) $$

$$ \chi(v) = \frac{b(2v)a(2v + \gamma)}{a(2v)b(2v + \gamma)}. $$

Using this change and eqns. (11, 12) the commutation relations are obtained as [14]:

$$ A(v) B_a(v') = \frac{a(v' - v)b(v' + v)}{b(v' - v)a(v' + v)} B_a(v') A(v) $$

$$ - \frac{c_{+}(v' - v)b(2v')}{b(v' - v)a(2v')} B_a(v) A(v') $$

10
\[
\hat{D}_{bd}(v)B_a(v') = \frac{c_-(v' + v)}{a(v' + v)} \chi(v') S_{da}^{bc}(2v' + \gamma) B_b(v) \hat{D}_{dc}(v'),
\]
(22)

\[
\frac{S_{fd}^{ce}(v - v')}{b(v - v')} \frac{S^{-1}_{ab}(v - v' - \gamma)}{b(-v - v' - \gamma)} B_c(v') \hat{D}_{ge}(v) + \frac{1}{\chi(v)} \frac{c_-(v + v') b(2v')}{a(v + v') a(2v')} \delta_{ab} B_b(v') A(v')
\]
(23)

\[
\frac{\chi(v') c_+(v - v') S_{ea}^{bd}(2v' + \gamma)}{\chi(v) b(v - v') b(2v' + \gamma)} B_d(v) \hat{D}_{ef}(v')
\]

where all indexes in eqns. (21, 22, 23) assume only the values 2 and 3.

Using the change (21) and the definition of the transfer matrix (13) it can be written as:

\[
t^{a\beta} = \frac{q}{\sin \xi_+} \left( \sin(\xi_+ - v - \gamma) e^{iv} A(v) + \chi(v) K_{a(1)cd}(v) \hat{D}_{dc}(v) \right),
\]

\[
t^{b\beta} = \frac{1}{\sin \xi_+} \left( \sin(\xi_+ - v) e^{iv} A(v) + \chi(v) \sin(\xi_+ + v + \gamma) e^{-iv} K_{b(1)cd}(v) \hat{D}_{dc}(v) \right),
\]

with \(\beta = a, b\) and:

\[
K_{a(1)}^{-}(v) = \begin{pmatrix} q e^{iv} \sin(\xi_+ - v - \gamma) & 0 \\ 0 & e^{-iv} \sin(\xi_+ + v) \end{pmatrix},
\]

\[
K_{b(1)}^{-}(v) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

The notation will become clear below, where we will see that these are \(K^-\) matrices for a reduced problem corresponding to a \(spl_q(2, 1)\) algebra.

It is easy to find an eigenstate of these transfer matrices, the first level pseudovacuum, given by:

\[
\Phi = \otimes_{i=1}^{L} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]

This ferromagnetic state is an eigenvector of \(A(v)\) and \(\hat{D}_{dd}(v)\). Let us look at this point.

It is easy to see from the definition of \(A(v)\) that:
\[ A(v)\Phi = a^L(v)\tilde{a}^L(-v)K_1^-(v). \]

For the case of \( D_{aa}(v) \) it is necessary to make use of the fact that 
\[ T_{a1}(v)\Phi = T_{ab}^{-1}(-v)\Phi = 0 \]
when \( a \neq b \) and commute \( T_{b1}(v)T_{1b}^{-1}(-v) \) using the Yang-Baxter equation for the monodromy matrices \( T \) and \( T^{-1} \). The result is:

\[ D_{aa}(v)\Phi = \left( K_a^-(v) - \frac{c_+(2v)}{a(2v)}K_1^-(v) \right) b^L(v)\tilde{b}^L(-v)\Phi + \frac{c_+(2v)}{a(2v)}A(v)\Phi. \]

After using the formula (21) we obtain:

\[ \hat{D}_{cd}(v)\Phi = \frac{\tau(v)}{q}K_{a(1)cd}^+(v), \]
\[ \hat{D}_{cd}(v)\Phi = \tau(v)\Phi \sin(\xi_--v-\gamma)e^{iv}K_{b(1)cd}^+(v), \]

for the families \( a \) and \( b \) of \( K^- \) matrices respectively, where:

\[
K_{a(1)}^+(v) = \begin{pmatrix}
qe^{-iv}\sin(\xi_-+v-\gamma) & 0 \\
0 & -q^2e^{iv}\sin(\xi_-+v-2\gamma)
\end{pmatrix},
\]
\[
K_{b(1)}^+(v) = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix},
\]

and:

\[
\tau(v) = \frac{b^2(2v+\gamma)}{qa(2v+\gamma)w(2v+\gamma)\sin\xi_-}\left( \frac{c_+(2v)}{a(2v)} - 1 \right) b^L(v)\tilde{b}^L(-v).
\]

The notation will become clear below, where we will see that these are \( K^+ \) matrices for the reduced problem.

Also it is easily seen that \( C_b(v)\Phi = 0 \) and that \( \mathcal{B}_b(v)\Phi \) \((b=2,3)\) is not proportional to \( \Phi \) and different from zero. Then we can use linear combinations of these last operators to create excitations over the first level pseudovacuum in order to look for the eigenvectors of the transfer matrices. Then we will use for the first level Bethe ansatz the vectors:
\[ \Psi = \mathcal{B}_{i_1}(v_1) \mathcal{B}_{i_2}(v_2) \ldots \mathcal{B}_{i_N}(v_N) \Phi \Psi^{(i)}_{(1)} , \]

with subindexes running from 2 to 3. The coefficients \( \Psi^{(i)}_{(1)} \) will be determined by the second-level Bethe ansatz.

Commuting this vector with the operators of the transfer matrices and using the rules given by formulas (22,23) the result obtained is:

\[ t^{\alpha \beta}(v) \Psi = \Gamma^{\alpha \beta}(v) \prod_{i=1}^{N} \frac{a(v_i - v)b(v_i + v)}{b(v_i - v)a(v_i + v)} a_L(v) \tilde{a}_L(-v) \Psi \]

\[ + \Delta^{\alpha \beta} \frac{b(2v)b(2v + \gamma)}{a(2v)w(2v + \gamma)} \left( \frac{c_+(2v)}{a(2v)} - 1 \right) \prod_{i=1}^{N} \left( \frac{1}{b(v - v_i)b(-v - v_i - \gamma)} \right) \times \]

\[ b_L(v) \tilde{b}_L(-v) \mathcal{B}_{j_1}(v_1) \mathcal{B}_{j_2}(v_2) \ldots \mathcal{B}_{j_N}(v_N) \Phi \Psi^{(i)}_{(1)}(v + \gamma/2, \{v_i + \gamma/2\})_{(j)} \Psi^{(i)}_{(1)} \]

\[ + \text{u.t.} , \tag{24} \]

with:

\[ \Gamma^{aa}(v) = \Gamma^{ab}(v) = \frac{1}{\sin \xi_+ \sin \xi_-} \sin(\xi_+ - v - \gamma) \sin(\xi_- + v) , \]

\[ \Gamma^{ba}(v) = \Gamma^{bb}(v) = \frac{1}{\sin \xi_+ \sin \xi_-} \sin(\xi_+ - v) \sin(\xi_- + v) , \]

\[ \Delta^{aa}(v) = \frac{q^{-1}}{\sin \xi_+ \sin \xi_-} , \]

\[ \Delta^{ab}(v) = \frac{1}{\sin \xi_+ \sin \xi_-} \sin(\xi_- - v - \gamma)e^{iv} , \]

\[ \Delta^{ba}(v) = \frac{q^{-1}}{\sin \xi_+ \sin \xi_-} \sin(\xi_+ + v + \gamma)e^{-iv} , \]

\[ \Delta^{bb}(v) = \frac{1}{\sin \xi_+ \sin \xi_-} \sin(\xi_+ + v + \gamma) \sin(\xi_- - v - \gamma) . \]

The wanted terms are obtained after using the first term of the commutation rules (22,23). The unwanted terms \( \text{u.t.} \) come from the second and third terms of the commutation rules, for a detailed study of how to deal these terms see [15]. There is no summation over the indexes \( \alpha, \beta \). The operator \( t^{\alpha \beta}_{(1)}(v + \gamma/2, \{v_i + \gamma/2\}) \) is the second level transfer matrix coming from the family \( \alpha \) of \( K^+ \) matrices and \( \beta \) of \( K^- \). The matrices \( K^+_{(1)} \) and
\[ K_{(1)}^- \] turn out to be the corresponding reflection matrices for this reduced second level problem, this explains the notation. The second level transfer matrix is then given by:

\[ t_{(1)}^{\alpha\beta}(v + \gamma/2, \{v_i + \gamma/2\}) = \sum_{c=2}^{3} K_{(1)cc}^+(v)U_{\alpha(1)cc}(v + \gamma/2, \{v_i + \gamma/2\}) , \]

where the second level \( U_{\alpha(1)}(v, \{v_i\}) \) operator is formed from the weights given in the first term in (23) and the \( K_{(1)}^- \) matrix.

We are led from equation (24) to a new eigenvalue problem for a reduced transfer matrix with only two states per link corresponding to the superalgebra \( sl_{q(1,1)} \):

\[ t_{(1)}^{\alpha\beta}(v + \gamma/2, \{v_i + \gamma/2\}) \Psi_{(1)} = \lambda_{(1)}^{\alpha\beta}(v) \Psi_{(1)}. \]  

(25)

It can be seen that the unwanted terms in (24) cancel if:

\[ \frac{\Gamma^{\alpha\beta}}{\Delta^{\alpha\beta}} \prod_{i \neq k}^N \frac{a(v_i - v_k)b(v_i + v_k)}{b(v_i - v_k)a(v_i + v_k)} a^L(v_k) b^L(-v_k) + \frac{\lambda_{(1)}^{\alpha\beta}(v_k)}{\sin \gamma a(2v_k + \gamma) a(2v_k) - 1} \prod_{i \neq k}^N \frac{1}{b(v_k - v_i)b(-v_k - v_i - \gamma)} b^L(v_k) b^L(-v_k) = 0, \quad k = 1, \ldots, N \]  

(26)

notice that the factor depending on \( \alpha, \beta \) is equal to 1 in the quantum group invariant case.

To solve the reduced problem we can follow parallel steps from those of the previous level. First define the operators \( A_{(1)} = U_{(1)22}, B_{(1)} = U_{(1)23}, C_{(1)} = U_{(1)32} \) and \( D_{(1)} = U_{(1)33} \). The second level Bethe ansatz for \( \Psi_1 \) is given by:

\[ \Psi_{(1)} = B_{(1)}(v_1 + \gamma/2, \{v_i + \gamma/2\}) B_{(1)}(v_2 + \gamma/2, \{v_i + \gamma/2\}) \ldots B_{(1)}(v_M + \gamma/2, \{v_i + \gamma/2\}) \Phi_{(1)} , \]

where:

\[ \Phi_{(1)} = \otimes_{i=1}^N \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \]
is the second level pseudo vacuum. This is annihilated by the \( C_{(1)} \) operator. As in the first level it is convenient to make the change:

\[
\hat{D}_{(1)}(v, \{v_i\}) = D_{(1)}(v, \{v_i\}) - \frac{c_{-}(2v)}{a(2v)} A_{(1)}(v, \{v_i\}). \tag{27}
\]

The action of \( A_{(1)} \) and \( \hat{D}_{(1)} \) on \( \Phi_{(1)} \) is given by:

\[
A_{(1)}(v + \gamma/2, \{v_{i\gamma/2}\})\Phi_{(1)} = K_{(1)11}^{-1}(v) a(v - v_i)\tilde{a}(-v - v_i - \gamma)\Phi_{(1)} \tag{28}
\]

\[
\hat{D}_{(1)}(v + \gamma/2, \{v_{i\gamma/2}\})\Phi_{(1)} = \mu_{a}(v)q b(v + \gamma) a(2v + \gamma) a_{(1)}(v) \prod_{i=1}^{N} b(v - v_i)\tilde{b}(-v - v_i - \gamma)\Phi_{(1)} \tag{29}
\]

\[
\mu_{a}(v) = q^{-1}e^{-iv}\sin(\xi_{+} + v + \gamma) \quad \mu_{b}(v) = 1.
\]

The reduced transfer matrices can be written after the change (27):

\[
t_{(1)} = \theta_{a}(v)q b(2v + \gamma) a(2v + \gamma) a_{(1)}(v) - \rho_{a}(v)\hat{D}_{(1)} \tag{30}
\]

\[
\theta_{a}(v) = e^{-iv}\sin(\xi_{-} + v), \quad \theta_{b}(v) = 1
\]

\[
\rho_{a}(v) = q^{2}e^{iv}\sin(\xi_{-} - v - 2\gamma), \quad \rho_{b}(v) = 1.
\]

The commutation relations for the operators \( A_{(1)}, \hat{D}_{(1)} \) and \( B_{(1)} \) follow from (12), see [14]. Using these and (30,28,29) a reasoning similar to the first level Bethe ansatz gives the final result for the eigenvalue problem (20):

\[
\lambda(v) = \lambda_{A}(v) + \lambda_{D_{I}}(v) + \lambda_{D_{II}}(v), \tag{31}
\]

with:

\[
\lambda_{A}(v) = \lambda_{(ab)A}(v) \prod_{i=1}^{N} a(v_{i} - v) b(v_{i} + v) a_{(1)}(v) \tilde{a}_{L}(-v), \tag{32}
\]

\[
\lambda_{(aa)A}(v) = \lambda_{(ab)A}(v) = \frac{q\sin(\xi_{+} - v - \gamma)\sin(\xi_{+} + v)}{\sin \xi_{+} \sin \xi_{-}}.
\]
The nested Bethe ansatz equations obtained from the cancelation of unwanted terms are:

\[
\lambda_{(ba),A}(v) = \lambda_{(bb),A}(v) = \frac{q \sin(\xi_+ - v) \sin(\xi_- + v)}{\sin \xi_+ \sin \xi_-},
\]

\[
\lambda_{D_1}(v) = \lambda_{(a\beta),D_1}(v) \frac{b(2v)b(2v + \gamma)}{a(2v)w(2v + \gamma)} \left( \frac{c_+(2v)}{a(2v)} - 1 \right) \left( 1 - \frac{c_-(2v + \gamma)}{a(2v + \gamma)} \right) b^L(v)\bar{b}^L(-v)
\]
\[
\prod_{i=1}^{N} a(v - v_i)\tilde{a}(-v - v_i - \gamma) \prod_{j=1}^{M} a(\nu_j - v)\tilde{b}(\nu_j + v + \gamma)
\]

(33)

\[
\lambda_{(aa),D_1}(v) = \frac{q \sin(\xi_+ - v - \gamma) \sin(\xi_- + v)}{\sin \xi_+ \sin \xi_-}
\]

\[
\lambda_{(ab),D_1}(v) = \frac{q^2 \sin(\xi_+ - v - \gamma) \sin(\xi_- + v - \gamma)e^{2iv}}{\sin \xi_+ \sin \xi_-}
\]

\[
\lambda_{(ba),D_1}(v) = \frac{q^{-1} \sin(\xi_+ + v + \gamma) \sin(\xi_- + v)e^{-2iv}}{\sin \xi_+ \sin \xi_-}
\]

\[
\lambda_{(bb),D_1}(v) = \frac{\sin(\xi_+ + v + \gamma) \sin(\xi_- + v - \gamma)}{\sin \xi_+ \sin \xi_-}
\]

\[
\lambda_{D_{II}}(v) = -\lambda_{(a\beta),D_{II}}(v) \frac{b(2v)b(2v + \gamma)}{a(2v)w(2v + \gamma)} \left( \frac{c_+(2v)}{a(2v)} - 1 \right) \left( 1 - \frac{c_-(2v + \gamma)}{a(2v + \gamma)} \right)
\]
\[
b^L(v)\bar{b}^L(-v) \prod_{j=1}^{M} a(\nu_j - v)\tilde{b}(\nu_j + v + \gamma)\]

(34)

\[
\lambda_{(aa),D_{II}}(v) = \lambda_{(ba),D_{II}}(v) = \frac{q \sin(\xi_+ + v + \gamma) \sin(\xi_- + v - 2\gamma)}{\sin \xi_+ \sin \xi_-}
\]

\[
\lambda_{(bb),D_{II}}(v) = \frac{\sin(\xi_- - v - \gamma) \sin(\xi_+ + v + \gamma)}{\sin \xi_+ \sin \xi_-}
\]

In the previous equations it is easily seen that for the case \((\alpha, \beta) = (b, b)\) the eigenvalue is real. For the case \((\alpha, \beta) = (a, a)\) there is an overall factor \(q\) which can be eliminated by the redefinition \(K^+_a \rightarrow q^{-1}K^+_a\), making the eigenvalue real, (we have maintained the present definition in order to make more clear the quantum group invariant limit). For the “mixed” cases \((a, b), (b, a)\), we have imaginary terms which are not possible to eliminate by redefinitions of the reflection matrices, or by gauge transformations of the \(S\) matrix. As explained in sect. 2 the eigenvalues are real for all the cases in the hyperbolic and rational regimes.

The limit \(\Xi \rightarrow \infty\) leads to the formulas obtained in [14] for the quantum group invariant case.

The nested Bethe ansatz equations obtained from the cancelation of unwanted terms are:
\[ \eta_{\alpha\beta}(v_k) \left( \frac{a(v_k)\bar{a}(-v_k)}{b(v_k)\bar{b}(-v_k)} \right)^L \prod_{i \neq k}^{N} \frac{a(v_i - v_k)b(v_i + v_k)\bar{b}(v_i + v_k + \gamma)}{a(v_i - v_i)a(v_i + v_k)\bar{a}(-v_i - v_k - \gamma)} \]

\[ \prod_{j=1}^{M} \frac{a(v_j + v_k + \gamma)b(v_j - v_k)}{b(v_j + v_k + \gamma)a(v_j - v_k)} = 1, \quad k = 1, \ldots, N, \quad (35) \]

\[ \zeta_{\alpha\beta}(v_k) \prod_{i=1}^{N} \frac{a(v_i - v_i)\bar{a}(-v_i - v_i - \gamma)}{b(v_i - v_i)b(-v_i - v_i - \gamma)} = 1, \quad l = 1, \ldots M \quad (36) \]

\[ \eta_{aa}(v_k) = 1 \quad \zeta_{aa}(v_k) = \frac{\sin(v_i + \nu_i)\sin(v_i - \nu_i - \gamma)}{\sin(v_i - \nu_i - 2\gamma)\sin(v_i + \nu_i + \gamma)} \]
\[ \eta_{ab}(v_k) = \frac{\sin(v_i + v_k)\sin(v_i - v_k)}{\sin(v_i - v_k + \gamma)\sin(v_i + v_k + \gamma)} \quad \zeta_{ab}(v_k) = \frac{\sin(v_i + \nu_i + \gamma)}{\sin(v_i - \nu_i - 2\gamma)} \]
\[ \eta_{ba}(v_k) = \frac{\sin(v_i + v_k)\sin(v_i - v_k)}{\sin(v_i - v_k + \gamma)\sin(v_i + v_k + \gamma)} \quad \zeta_{ba}(v_k) = 1 \]

Then the solution to the eigenvalue problem (21) is given by equation (31), with \( v_i, \nu_i \) given by equations (33,36).

This equations are real for \((\alpha, \beta) = (a, a), (b, b)\) and imaginary for the ”mixed” cases.

This new fact is present only in the case of independent open boundary conditions for \(S\)-matrices of algebras with rank larger than one. This is the first model solved for these conditions and this property is also expected in other systems [15, 4]. This terms may deserve further study in the hyperbolic regime, but for the trigonometric regime do not seem to have physical applications. In the quantum group invariant limit all these terms dissapear. Imaginary eigenvalues appear also in the periodic case for this \(S\) matrix as noticed in [14].

4 Highest weight property for the \(spl_q(2, 1)\) invariant t-J model

In this section the highest weight property for the Bethe states of the quantum group invariant case is proved. This property has been also shown to hold for the Bethe states of the \(SU_q(2)\) invariant XXZ chain [9, 21], but it is shown for the first time for a quantum group invariant chain with more than two states per link.

In reference [14] a representation in the lattice of the generators of \(spl_q(2, 1)\) was obtained at certain limits of the spectral parameter \(v\). The important ones for what follows are:
\[ A(x \to \infty) \sim q^{-L} q^{2W_1}, \]
\[ D_{33}(x \to 0) \sim q^L q^{2W_3}, \]
\[ C_2(x \to \infty) \sim \alpha_- q^{-L/2} q^{-W_3/2} F_1 q^{W_1}, \]
\[ D_{32}(x \to 0) \sim -\alpha_+ q^{L/2} \tilde{\sigma} q^{W_1/2+2W_3} F_2, \]
\[ \alpha_{\pm} = q^{\pm 1/2} (q - q^{-1}), \]
\[ \tilde{\sigma} = \sigma \otimes \sigma \otimes \ldots \otimes \sigma, \quad \sigma = diag(1, 1, -1). \]

In the previous formulas, \( F_1, F_2 \) are generators of the quantum algebra, the rest of the generators are \( E_1, E_2 \), \( H_1 = W_1 - W_2 \) and \( H_2 = W_2 + W_3 \). These obey:

\[ q^{H_1} q^{H_2} = q^{H_2} q^{H_1}, \]
\[ q^{H_i} F_j q^{-H_i} = q^{\alpha_{ij}} F_j, \]
\[ q^{H_i} E_j q^{-H_i} = q^{\alpha_{ij}} E_j, \]
\[ [F_1, E_1] = \frac{q^{H_1} - q^{-H_1}}{q - q^{-1}}, \quad [F_1, E_2] = 0, \]
\[ [F_2, E_2] = \frac{q^{H_2} - q^{-H_2}}{q - q^{-1}}, \quad [F_2, E_1] = 0, \]
\[ E_2^2 = F_2^2 = 0, \]

plus \( q \)-Serre relations. Here \( a_{ij} \) are the elements of the graded Cartan matrix given by \( a_{11} = 2, a_{12} = a_{21} = -1, a_{22} = 0 \). These are the commutation relations defining the quantum group \( spl_q(2, 1) \). These generators can be shown to commute with the quantum group invariant transfer matrix \( t(x) \) using the method in references [22, 3] or by direct calculation [14], i.e:

\[ [t(x), q^{H_i}] = 0, \]
\[ [t(x), F_i] = 0, \]
\[ [t(x), E_i] = 0, \quad i = 1, 2. \]  (38)

To prove the highest weight property for the Bethe states \( \Psi \) it is necessary to show that \( F_1 \Psi = F_1 \Psi = 0 \). For that we need to know the commutation relations between...
the operators $B_d$ and the generators $F_i$. These are obtained using equations (37,12) and making the necessary limits. The result for $F_1$ is:

$$F_1 B_d(v) = q^{(2-d)/2}B_d(v)F_1 + q^{(1-i)/2}q^{-W_3/2+W_1} \times \left( \delta_{d2} \left( 1 - \frac{c_+(2v)}{a(2v)} \right) A(v) - \chi(v) \frac{S_{bc}^2(2v + \gamma)}{b(2v + \gamma)} \hat{D}_{bc}(v) \right),$$

where we have also used (21) and the commutation relations:

$$[q^{W_1}, A(x)] = 0, \quad [W_3, B_d(v)] = \delta_{d3}B_d(v), \quad [W_1, B_d(v)] = -B_d(v).$$

Using the previous results, the commutation relations (22,23) and the fact that $F_1 \Phi = 0$ we find:

$$F_1 \Psi = \sum_{i=1}^{N} q^p \delta_{i2} \frac{b(2v_k)}{a(2v_k)} \times \{ \prod_{i \neq k} \frac{a(v_i - v_k)b(v_i + v_k)}{b(v_i - v_k)a(v_i + v_k)} a^L(v_k)\hat{a}^L(-v_k) + \frac{\lambda_{(1)}(v_k)}{\sin \gamma a(2v_k + \gamma)} \left( \frac{c_+(2v_k)}{a(2v_k)} - 1 \right) \prod_{i \neq k} \frac{1}{b(v_k - v_i)b(-v_k - v_i - \gamma)} \times b^L(v_k)\hat{b}^L(-v_k) \} B_{i_k+1}(v_k) \ldots B_{i_k-1}(v_{k-1}) \Phi M_{(j)}^{(i)} M_{(j)}^{(i)} \Psi,$$

where $p$ is an operator irrelevant for what follows, and we have made use of equation (25). The matrix $M_{(j)}^{(i)}$ takes count of the reordering of the $B$ operators see [7, 15]. We see looking to the equation (26) that this last expression shows that $F_1 \Psi = 0$.

For the operator $F_2$ we use the relation (21) to find that:

$$\hat{D}_{32}(x \to 0) = -q^{k-1/2}(q - q^{-1})\tilde{\sigma} q^{W_1/2+2W_3} F_2.$$

Using the commutation relations (23) in the limit $x \to 0$ we see that the first summand is of order $x^{-2}$ with respect to the third and fourth and is the only one which survives in this limit. Using the obtained commutation relation N times:

$$F_2 \Psi = cB_{j_1}(v_1) \ldots B_{j_N}(v_N) \Phi F_{(i)2(i)}^{(j)} M_{(j)}^{(i)} \Psi_{(i)}^{(j)}.$$
where $F_{(1)2}$ is the $x \to 0$ limit of $U_{(1)23}$ and $c$ is an unimportant factor different from zero. We use again the relation (12) for the reduced problem and take the corresponding limits to find:

$$F_{(1)2}B_{(1)}(v) = q^2 B_{(1)}(v)F_{(1)2} + (q - q^{-1})D_{(1)}(x = 0) \left(D_{(1)}(v) - A_{(1)}(v)\right),$$

where we have also used $[D_{(1)}(x = 0), A_{(1)}(v)] = 0$. Using the previous commutation relation, the change (27) and the fact that $F_{(1)2}\Phi_{(1)} = 0$ we find:

$$F_{(1)2}\Psi_{(1)} = \tilde{c}M \sum_{l=1}^{M} \prod_{j \neq l} a(\nu_j - \nu_l) b(\nu_j + \nu_l + \gamma) \left(1 - \frac{c}{a(2\nu_l + \gamma)}\right) \times \left\{ \prod_{i=1}^{N} b(\nu_i - \nu_l) b(-\nu_i - \nu_l - \gamma) - \prod_{i=1}^{N} a(\nu_i - \nu_l) a(-\nu_i - \nu_l - \gamma) \right\} \times B_{(1)}(v_{l+1}) \cdots B_{(1)}(v_{l-1})\Phi_{(1)},$$

where $\tilde{c}$ is an operator that does not affect the following argument. It is clear, after looking to formula (36) in the quantum group invariant limit, that $F_2\Psi = 0$. This finish the proof of the highest weight property for the Bethe eigenvectors of the $spl_q(2,1)$ invariant t-J chain. As the kernel of $F_1, F_2$ is stable under variations of $\gamma$, the highest weight property holds even when the $\gamma/\pi$ is rational; this can be important for the study of the associated RSOS models [20].

From equations (38) we can see that the eigenvectors are classified by multiplets corresponding to irreducible representations of the quantum superalgebra $spl_q(2,1)$. The eigenvalues are degenerate, since the vectors $E_i\Psi, E_i^2\Psi, \ldots, E_i^J\Psi, \quad i = 1, 2$, with $J \neq 0$ in general, are all eigenvectors of $t(v)$ with the same eigenvalue. All these vectors are included in the same multiplet whose highest weight vector is given by a Bethe ansatz state. These vectors are not coming from the Bethe ansatz but all are directly obtained from it by applying the, lowering, $E_i$ operators. To deal with the problem of completeness it would be necessary to to find a general expression for the degeneracy of the eigenvalues and study the Bethe ansatz equations, and then to make an analysis paralell to that of the $spl(2,1)$ invariant t-J model [13].
5 Conclusions

We have presented open boundary conditions compatible with integrability for the one dimensional supersymmetric t-J model. It turns out that there are four families of boundary conditions commuting with $n$ and $S^z$ depending each one on two arbitrary parameters. A connection between the Markov trace method for finding open boundary conditions and reflection equations have been found, showing in this way its generality. We have also found the relation between the one dimensional t-J model with open boundary conditions and the open transfer matrix of the graded 15 vertex model. The models proposed have been diagonalized using a generalization of the quantum group invariant nested Bethe ansatz for this kind of boundary conditions. The $spl(2,1)$ invariant chain is obtained as a special limit. It turns out that for some ”mixed” cases the eigenvalues in the trigonometric regime are imaginary. This phenomenon is new and characteristic of open boundary conditions for systems with more than two states per link.

The highest weight property for the Bethe states of the quantum group invariant t-J model have been shown. Using the quantum group invariance of the chain we have found that the eigenvectors are classified by multiplets. These multiplets of highest weight given by the Bethe ansatz vectors are generated by the lowering $E_i$ operators applied to the corresponding Bethe state. We hope that this combination of Bethe ansatz and the quantum group properties of the model will give a complete set of eigenvectors as is the case for the $spl(2,1)$ periodic chain [13].

It would be interesting to make a parallel study for models with open boundary conditions associated to the algebras $A_{n-1}$. This question has been recently addressed in [16, 23]. The problem of completeness remains open, and further investigation on the quantum group properties of the model and the Bethe ansatz equations could bring some advance.

The proof of the highest weight property opens the study of the associated RSOS models.

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