Lp REGULARITY THEORY FOR EVEN ORDER ELLIPTIC SYSTEMS WITH ANTISYMMETRIC FIRST ORDER POTENTIALS

CHANG-YU GUO, CHANG-LIN XIANG* AND GAO-FENG ZHENG

Abstract. Motivated by a challenging expectation of Rivière [35], in the recent interesting work [5], de Longueville and Gastel proposed the following geometrical even order elliptic system

\[ \Delta^m u = \sum_{l=0}^{m-1} \Delta^l \langle V_l, du \rangle + \sum_{l=0}^{m-2} \Delta^l \delta(w_l du) \quad \text{in } B^{2m} \]

which includes polyharmonic mappings as special cases. Under minimal regularity assumptions on the coefficient functions and an additional algebraic antisymmetry assumption on the first order potential, they successfully established a conservation law for this system, from which everywhere continuity of weak solutions follows. This beautiful result amounts to a significant advance in the expectation of Rivière.

In this paper, we seek for the optimal interior regularity of the above system, aiming at a more complete solution to the aforementioned expectation of Rivière. Combining their conservation law and some new ideas together, we obtain optimal Hölder continuity and sharp Lp regularity theory, similar to that of Sharp and Topping [40], for weak solutions to a related inhomogeneous system. Our results can be applied to study heat flow and bubbling analysis for polyharmonic mappings.

Keywords: Even order elliptic system, Conservation law, Polyharmonic mappings, Lp theory, Riesz potential theory

2020 Mathematics Subject Classification: 35J48, 35G35, 35B65

Contents

1. Introduction and main results 2
2. Preliminaries and auxiliary results 8
3. Hölder continuity via decay estimates 12
4. Higher order regularity 20
5. Optimal local estimates 26
Appendix A. Higher order Lp-theory 33
Appendix B. Proof of equation (3.3) 34
References 35

*Corresponding author: Chang-Lin Xiang.
C.-Y. Guo is supported by the Qilu funding of Shandong University (No. 6255008963197). The corresponding author C.-L. Xiang was financially supported by the National Natural Science Foundation of China (No. 11701045). G.-F. Zheng is supported by the National Natural Science Foundation of China (No. 11571131).
1. Introduction and main results

1.1. Motivation. It is well known that geometric partial differential equations are usually nonlinear in nature, for instance, the equations of harmonic mappings and the prescribed mean curvature equations. In his recent remarkable work [34], to study conformally invariant variational problems in dimension two, Riviére introduced the following second order linear elliptic system

$$- \Delta u = \Omega \cdot \nabla u \quad \text{in } B^2,$$

where $u \in W^{1,2}(B^2, \mathbb{R}^n)$ and $\Omega = (\Omega_{ij}) \in L^2(B^2, so_n \otimes \Lambda^1 \mathbb{R}^2)$. As was verified in [34], (1.1) includes the Euler-Lagrange equations of critical points of all second order conformally invariant variational functionals which act on mappings $u \in W^{1,2}(B^2, N)$ from $B^2 \subset \mathbb{R}^2$ into a closed Riemannian manifold $N \subset \mathbb{R}^n$. In particular, (1.1) includes the equations of weakly harmonic mappings and the prescribed mean curvature equations.

Notice that the coefficient function $\Omega$ in (1.1) is independent of the solution $u$. The square integrability assumption on $\Omega$ makes the system critical in the sense that $\Omega \cdot \nabla u \in L^1(B^2)$, which may allow discontinuous weak solutions. Thus, from the point view of analysis, finding minimal additional assumptions on $\Omega$ so that every solution would have full regularity is a very interesting and important problem. During the exploration of conformally invariant problems, Riviére [34] found an additional assumption which is nowadays acknowledged as the optimal one. That is, assume in addition that $\Omega$ is antisymmetric as a matrix-value function. Under this additional assumption, Riviére found functions $A \in L^\infty \cap W^{1,2}(B^2, G\ell(n))$ and $B \in W^{1,2}(B^2, M_n)$ which satisfies $\nabla A - A\Omega = \nabla \perp B$, such that system (1.1) can be written equivalently as the conservation law

$$\text{div} (A \nabla u + B \nabla \perp u) = 0,$$

from which everywhere continuity of weak solutions of system (1.1) can be derived. As applications, this recovered the famous regularity result of Hélein [18], and confirmed affirmatively two long-standing conjectures by Hildebrandt and Heinz on conformally invariant geometrical problems and prescribed bounded mean curvature equations respectively; see [34] for details.

Conformally invariant variational problems in dimensions greater than two also attracted extensive research in recent years. For fourth order conformally invariant variational problems, see for instance the works of Chang, Wang and Yang [3] and Wang [48, 49] on biharmonic mappings from an Euclidean ball $B^n$ $(n \geq 4)$ into closed Riemannian manifolds. To extend the aforementioned powerful theory of Riviére [34], in the interesting work [24], Lamm and Riviére proposed the following fourth order elliptic system

$$\Delta^2 u = \Delta (V \cdot \nabla u) + \text{div}(w \nabla u) + W \cdot \nabla u \quad \text{in } B^4,$$

where $V \in W^{1,2}(B^4, M_n \otimes \Lambda^1 \mathbb{R}^4)$, $w \in L^2(B^4, M_n)$, and $W \in W^{-1,2}(B^4, M_n \otimes \Lambda^1 \mathbb{R}^4)$ is of the form

$$W = \nabla \omega + F,$$

with $\omega \in L^2(B^4, so_n)$ and $F \in L^{1,4}(B^4, M_n \otimes \Lambda^1 \mathbb{R}^4)$. System (1.3) includes both extrinsic and intrinsic biharmonic mappings from $B^4$ into closed Riemannian manifolds as special cases.
Note that the first order potential $\omega$ is antisymmetric as a matrix-valued function. Following the approach of [34], Lamm and Rivière [24] found $A \in W^{2,2} \cap L^\infty(B_{1/2}^4, M_n)$ and $B \in W^{1,4/3}(B_{1/2}^4, M_n \otimes \wedge^2 \mathbb{R}^3)$ such that $u$ is a solution of (1.3) in $B_{1/2}^4$ if and only if it satisfies the conservation law
\[
\text{div}[\nabla(A\Delta u) - 2\nabla A\Delta u + \Delta A\nabla u - Aw\nabla u + \nabla A(V \cdot \nabla u) - A\nabla(V \cdot \nabla u) - B \cdot \nabla u] = 0. 
\]
As in the second order case, everywhere continuity of weak solutions follows from (1.4).

In a recent conference proceeding [35], Rivière gave an inspiring survey on the role of integrability by compensation in conformal geometric analysis, where in particular he presented a proof (based on his work [34]) on Hölder regularity of weak solutions of (1.1). Then he put up the following challenging expectation.

Rivière’s expectation. "It is natural to believe that a general result exists for $m$-th order linear systems in $m$ dimension whose 1st order potential is antisymmetric."

A similar expectation was also given earlier by Lamm and Rivière [24, Remark 1.4].

It is natural to consider conformally invariant geometrical problems in general even dimensions, which in fact have already attracted great attention in the last decades. For instance, Gastel and Scheven [10] obtained, among other results, a regularity theory for both extrinsic and intrinsic polyharmonic mappings in critical dimensions. For more progress in this respect, see e.g. [14, 26] and the references therein. Thus, a positive solution of Rivière’s expectation would give, among many other applications, a new and unified approach for the Hölder regularity of (extrinsic and intrinsic) polyharmonic mappings.

However, there was no essential progress on the expectation until quite recently. In 2019 in the interesting work [5], de Longueville and Gastel proposed the following even order linear elliptic system
\[
\Delta^m u = \sum_{l=0}^{m-1} \Delta^l \langle V_l, du \rangle + \sum_{l=0}^{m-2} \Delta^l \delta(w_l du) 
\]
in the unit ball $B^{2m} \subset \mathbb{R}^{2m}$. The coefficient functions are assumed to satisfy
\[
w_k \in W^{2k+2-m,2}(B^{2m}, \mathbb{R}^{n \times n}) \quad \text{for } k \in \{0, \ldots, m-2\} \\
V_k \in W^{2k+1-m,2}(B^{2m}, \mathbb{R}^{n \times n} \otimes \wedge^1 \mathbb{R}^{2m}) \quad \text{for } k \in \{0, \ldots, m-1\}. 
\]

Moreover, the first order potential $V_0$ has the decomposition $V_0 = d\eta + F$ with
\[
\eta \in W^{2-m,2}(B^{2m}, \text{so}(n)) , \quad F \in W^{-m,2m-2m-1}(B^{2m}, \mathbb{R}^{n \times n} \otimes \wedge^1 \mathbb{R}^{2m}). 
\]
Note that $\eta$ is an antisymmetric matrix-valued function. System (1.5) includes both extrinsic and intrinsic $m$-polyharmonic mappings as well. It also includes equations of non-variational type, for example “the fake polyharmonic equations”:
\[
\Delta^m u + |\nabla u|^{2m} u = 0,
\]
provided that $u$ is bounded. For more examples, see e.g. Strzelecki and Zatorska-Goldstein [46] for $m = 2$. 
As that of Rivière [34] and Lamm-Rivière [24], de Longueville and Gastel [5] successfully established a conservation law for system (1.5). More precisely, set
\[
\theta_D := \sum_{k=0}^{m-2} \|w_k\|_{W^{2k+2-m,2}(D)} + \sum_{k=1}^{m-1} \|V_k\|_{W^{2k+1-m,2}(D)} + \|\eta\|_{W^{2-m,2}(D)} + \|F\|_{W^{2-m,2m+2r-1}(D)}
\]
for \(D \subset \mathbb{R}^{2m}\). Under a smallness assumption
\[
\theta_{B_{2m}^1} < \epsilon_m,
\]
they found \(A \in W^{m,2} \cap L^\infty(B_{1/2}^{2m}, Gl(n))\) and \(B \in W^{2-m,2}(B_{1/2}^{2m}, \mathbb{R}^{n \times n} \otimes \wedge^2 \mathbb{R}^{2m})\) which satisfies
\[
\Delta^{m-1}dA + \sum_{k=0}^{m-1} (\Delta^k A)V_k - \sum_{k=0}^{m-2} (\Delta^k dA)w_k = \delta B,
\]
such that \(u\) solves (1.5) in \(B_{1/2}^{2m}\) if and only if it is a distributional solution of the conservation law
\[
0 = \delta \left[ \sum_{l=0}^{m-1} \left( \Delta^l A \right) \Delta^{m-l-1} du - \sum_{l=0}^{m-2} \left( d\Delta^l A \right) \Delta^{m-l-1} u 
- \sum_{k=0}^{m-1} \sum_{l=0}^{k-1} \left( \Delta^l A \right) \Delta^{k-l-1} \langle V_k, du \rangle + \sum_{k=0}^{m-1} \sum_{l=0}^{k-1} \left( d\Delta^l A \right) \Delta^{k-l-1} \langle V_k, du \rangle 
- \sum_{k=0}^{m-2} \sum_{l=0}^{k-2} \left( \Delta^l A \right) d\Delta^{k-l-1} \delta \langle w_k, du \rangle + \sum_{k=0}^{m-2} \sum_{l=0}^{k-2} \left( d\Delta^l A \right) \Delta^{k-l-1} \delta \langle w_k, du \rangle 
- \langle B, du \rangle \right],
\]
where \(d\Delta^{-1}\delta\) denotes the identity map. As an application of (1.10), they obtained everywhere continuity of weak solutions of (1.5) on \(B_{1/2}^{2m}\). It is worth pointing out that in another interesting recent work of Hörter and Lamm [22], the authors constructed a slightly different conservation law using a small perturbation of Uhlenbeck’s gauge transform.

Before proceeding further, we would like to take a closer look at the system (1.5) and the conservation law (1.10). The first thing one may note is the difference of coefficient functions from that of the second order system (1.1) and the fourth order system (1.3): almost half of the coefficient functions in (1.5) are Sobolev functions with negative exponents (for definitions, see Section 2). This shall cause serious problems in establishing the conservation law (1.10). To be more accurate, to find \(A, B\) as above, the authors have to solve a huge system of partial differential equations. In fact, even in the fourth order case, the task of finding \(A, B\) is already quite difficult. Secondly, the regularity of all the coefficient functions lies on the borderline of elliptic regularity theory which prevent an application of the usual \(L^p\)-theory, and furthermore, Sobolev functions with negative exponents require a much more general \(L^p\) theory then the usual one. Due to these reasons, the authors wrote in [5, the last paragraph on page 19] that:
"……. But here, we consider a very general equation with rather irregular coefficients, so maybe we cannot expect much regularity in general."

However, there are still several works and results in the literature that encourage us to consider the possibility of better regularity theory than merely continuity of weak solutions to the system (1.5).

i) As was pointed out by Gastel and Scheven [10, Theorem 1.2], for polyharmonic mappings, Hölder continuity implies smoothness. Thus, Hölder continuity of solutions to system (1.5), if held, could be directly applied to give smoothness of weak solutions to polyharmonic mappings.

ii) The coefficients of the system (1.5) are rather irregular, however, one observes an additional structural feature from the system: apart from the highest order term $\Delta^m u$, all the left terms consist of multiplications of some derivatives of $u$, which may imply a potential iteration on the regularity of $u$. In particular, one notes that there exists no inhomogeneous term which is independent of $u$. Such an observation gives us one more support to study better regularity of weak solutions via a suitable iteration scheme.

Indeed, motivated by its geometric applications, Sharp and Topping [40] considered the inhomogeneous second order system (i.e., $m = 1$)

$$-\Delta u = \Omega \cdot \nabla u + f \quad \text{in } B^2,$$

and obtained optimal Hölder continuity and sharp interior $W^{2,p}$ estimates under the assumption $f \in L^p(B^2)$ for some $1 < p < 2$, which applies to Rivière’s system (1.1) directly. In the recent paper [17], motivated also by geometric applications, we extend the regularity result of Sharp and Topping [40] to the case $m = 2$, where the following inhomogeneous fourth order system was studied:

$$\Delta^2 u = \Delta(V \cdot \nabla u) + \text{div}(w \nabla u) + W \cdot \nabla u + f \quad \text{in } B^4$$

for some $f \in L^p(B^4)$. These two works naturally inspire us to study the regularity theory of the following inhomogeneous problem

$$\Delta^m u = \sum_{l=0}^{m-1} \Delta^l (V_i, du) + \sum_{l=0}^{m-2} \Delta^l \delta (w_l du) + f \quad \text{in } B^{2m} \quad (1.11)$$

for some $f \in L^p(B^{2m})$.

iii). $L^p$ regularity theory for (1.11) (when $m = 1, 2$) have rich geometric applications such as the energy identity (or bubbling analysis), which dates back to Sacks-Uhlenbeck [38] on harmonic mappings, and heat flow which dates back to e.g. Struwe [42, 43]. Indeed, the work of Sharp and Topping [40] has many interesting applications in these problems. For instance, it has been successfully applied in the study of angular energy quantization of weak solutions of system (1.1) by Laurain and Rivière [28], which largely extended the bubbling analysis of Sacks-Uhlenbeck [38]. In the fourth order case, $L^p$ estimates have also been applied to bubbling analysis in e.g. [27, 21, 51]. $L^p$ regularity estimates have also found applications in global estimates, which leads to the full energy quantization results; see for instance the interesting work and Lamm and Sharp [25]. For extension and applications to supercritical dimensional problems, see e.g. [39, 32].
Based on the above considerations and seeking for a more complete regularity theory for Rivièrè’s expectation, it is natural to ask

**Problem.** What is the optimal Hölder continuity and \( L^p \) regularity for weak solutions of (1.11)? Can we derive optimal interior estimates for (1.11) as that of [40] for \( m = 1 \) and [17] for \( m = 2 \)?

The aim of this paper is to give an affirmative answer to this problem, and thus, provide a complete solution to Rivièrè’s expectation in the even order case.

1.2. **Main results.** Our first theorem deals with the optimal Hölder continuity of weak solutions to (1.5).

**Theorem 1.1** (Hölder continuity). Suppose \( f \in L^p(B_1^{2m}) \) for some \( p \in (1, \frac{2m}{2m-1}) \). If \( u \in W^{m,2}(B_1^{2m}, \mathbb{R}^n) \) is a weak solution of (1.11), then

\[
u \in C_{\text{loc}}^{0,\alpha}(B_1^{2m})
\]

with \( \alpha = 2m(1 - 1/p) \). Moreover, there exist \( r_0, C > 0 \) depending only on \( m, n, p \) and the coefficient functions \( V_k, w_k \) such that for all \( 0 < r < r_0 \), there holds

\[
\sum_{j=1}^{m} \| \nabla^j u \|_{L^{2m/j,2}(B_1^{2m})} \leq C r^\alpha \left( \sum_{j=1}^{m} \| \nabla u \|_{L^{2m/j,2}(B_1^{2m})} + \| f \|_{L^p(B_1^{2m})} \right).
\]

(1.12)

The Hölder continuity is optimal, as one can see from the simplest case \( \Delta^m u = f \) by noticing the Sobolev embedding \( W^{2m,p}(B_1^{2m}) \subset C^{0,2m(1-1/p)}(B_1^{2m}) \). Moreover, applying Theorem 1.1 to the case \( f \equiv 0 \) yields that every weak solution \( u \in W^{m,2}(B_1^{2m}) \) of system (1.5) is locally \( \alpha \)-Hölder continuous for all \( \alpha \in (0,1)^1 \). On the other hand, the Hölder continuity is the best possible regularity that one can expect for weak solutions of (1.5). In the case \( m = 2 \), we constructed an example in [17] which fails to be locally Lipschitz continuous. Using a similar idea, it is not difficulty to construct such an example for the general case and we leave it to the interested readers.

In our second theorem, we derive optimal higher order regularity of weak solutions.

**Theorem 1.2** (local \( L^p \) estimates). Let \( u \in W^{m,2}(B_1^{2m}, \mathbb{R}^n) \) be a weak solution of (1.11) with \( f \in L^p(B_1^{2m}) \) for \( p \in (1, \frac{2m}{2m-1}) \). Then

\[
u \in W^{m+1, 2m/(m-1)p}_{\text{loc}}(B_1^{2m}).
\]

Moreover, there exist \( \epsilon > 0 \) and \( C > 0 \) depending only on \( m, n, p \) and the coefficient functions \( V_k, w_k \) such that, if the smallness condition (1.9) is satisfied on \( B_1^{2m} \) with \( \epsilon_m = \epsilon \), then

\[
\| u \|_{W^{m+1, 2m/(m-1)p}(B_1^{2m})} \leq C \left( \| f \|_{L^p(B_1^{2m})} + \| u \|_{L^1(B_1^{2m})} \right).
\]

(1.13)

\(^1\)In [16] we proved \( \alpha \)-Hölder continuity for solutions to (1.5) for some \( \alpha \in (0,1) \) without using conservation law.
Theorem 1.2 can be regarded as a counterpart of Sharp-Topping [40, Theorem 1.1] (corresponding to the case \( m = 1 \)) and our work [17, Theorem 1.2] (corresponding to the case \( m = 2 \)) to the general even order system (1.5). The order \( m + 1 \) is also the best possible; see Example 4.4 below (and the case \( m = 2 \) was explained in [17]).

As a direct application of Theorem 1.2, we have the following improvement of the regularity result of de Longueville and Gastel [5].

**Corollary 1.3.** If \( u \in W^{m,2}(B_1, \mathbb{R}^n) \) is a weak solution of (1.5), then
\[
u \in \bigcap_{0<\epsilon<1} W^{m+1,2-\epsilon}_\text{loc}(B_1^{2m}) \subset \bigcap_{0<\alpha<1} C^{0,\alpha}_\text{loc}(B_1^{2m}).
\]

One more application of Theorem 1.2 yields the following energy gap, which was known to be useful in deducing the energy quantization results; see for instance [29, 28, 21].

**Corollary 1.4** (Energy gap). Let \( u \in W^{m,2}(\mathbb{R}^{2m}, \mathbb{R}^n) \) be a weak solution of (1.5) in \( \mathbb{R}^{2m} \). There exists some \( \epsilon = \epsilon(m, n) > 0 \) such that if
\[
\theta_{\mathbb{R}^{2m}} < \epsilon,
\]
then \( u \equiv 0 \) in \( \mathbb{R}^{2m} \), where \( \theta_{\mathbb{R}^{2m}} \) is defined as in (1.8).

To control the size of the paper, we do not consider in this paper the compactness problem of weak solutions of system (1.11) under the weak \( L\log L \)-integrability of \( f \), as that of Sharp-Topping [40] (when \( m = 1 \)) and Guo-Xiang-Zheng [17] (when \( m = 2 \)).

### 1.3. Strategy of the proof.
In spirit, we follow the scheme of Sharp and Topping [40] and [17] to divide the proof into three main steps.

1. In the first step, we derive the Morrey type decay estimate in Theorem 1.1 via the conservation law (1.10), from which Hölder continuity follows. The decay estimates show that \( \nabla^j u \) (\( 1 \leq j \leq m \)) belong to some Morrey spaces.
2. In the second step, we combine the above Morrey type estimate, together with the Riesz potential theory of Adams [1] (see Lemma 2.8) on Morrey spaces, to deduce an almost optimal higher order Sobolev regularity. That is, we prove that \( u \in W^{m+1,q}_\text{loc} \) for all \( q \) smaller than the optimal exponent \( \bar{q} = \frac{2mp}{2m-(m-1)p} \).
3. In the final step, we show that the concerned local \( W^{m+1,q} \) estimates are uniform with respect to \( q < \bar{q} \) so that by passing to the limit we can obtain the asserted optimal higher order Sobolev regularity.

As we are dealing with a very general even order system with highly irregular coefficients, it is expected that the situation will become much more complicated than that of the second order system considered in Sharp and Topping [40], where several parts require more delicate treatments.

The first step is fundamental for the following steps and it is based on an iteration procedure. In Sharp and Topping [40], their smart iteration argument requires a very precise control on some coefficients of related estimates so that the key coefficients are sufficiently small. Such a precise control is based on the monotonicity of the function \( r \mapsto |B_r|^{-1} f|_{B_r} \) for subharmonic functions \( f \). Thus it fails even in the fourth order case \( m = 2 \). In our case, it is far more subtle and a more general treatment is required: we
have to use an extremely complicated conservation law \( (1.10) \) to figure out the system for \( A \Delta u \) and \( A \nabla u \). As there are many terms with low regularity appearing, we have to do a very careful case study in order to properly apply the Riesz potential theory to gain the desired decay estimates.

Another significant difference occurs in the last step. In the case of Sharp-Toppings, they run a similar scaling and iteration argument as in the first step due to the well control of coefficient of decay of energy of \( \nabla u \). Moreover, in the second order case every solution belongs to \( W_{2,p}^{2,p} \) locally, which allows them to view the system \(-\Delta u = \Omega \cdot \nabla u\) as a pointwise identity from that they can deduce estimate for \( \nabla^2 u \) directly. In our case, it is far more difficult to obtain a uniform estimate for \( \|\nabla^{m+1} u\|_q \). To overcome this difficulty, we combine a duality argument, together with some elliptic estimates on polyharmonic maps with Navier boundary condition, to obtain an estimate such that Simon’s iteration method can be applied to deduce a quantitative and uniform control on the norm \( \|\nabla^{m+1} u\|_{q, B_{1/2}} \) for all \( q \) smaller than the optimal exponent. A disadvantage of this argument is that we only obtain an upper bound of the form \( \|u\|_{W^{m,2}} + \|f\|_{L^p} \). To improve this bound, we will have to run an additional interpolation argument to obtain the better bound in \( (1.13) \).

This paper is organized as follows. Section 2 contains some preliminaries and auxiliary results for later proofs. Our main theorems are proved in Sections 3, 4 and 5. We also add an appendix to include certain auxiliary results that was used in the proofs of our main theorems.

Our notations are standard. By \( A \lesssim B \) we mean there exists a universal constant \( C > 0 \) such that \( A \leq CB \).

Acknowledgements. We are grateful to professor Xiao Zhong for many useful suggestions and discussions during the preparation of this manuscript.

2. Preliminaries and auxiliary results

2.1. Function spaces and related. Let \( \Omega \subset \mathbb{R}^n \) be a bounded smooth domain, \( 1 \leq p < \infty \) and \( 0 \leq s < n \). The Morrey space \( M^{p,s}(\Omega) \) consists of functions \( f \in L^p(\Omega) \) such that

\[
\|f\|_{M^{p,s}(\Omega)} \equiv \left( \sup_{x \in \Omega, r > 0} r^{-s} \int_{B_r(x) \cap \Omega} |f|^p \right)^{1/p} < \infty.
\]

Denote by \( L^p \) the weak \( L^p \) space and define the weak Morrey space \( M^{p,s}_w(\Omega) \) as the space of functions \( f \in L^p_w(\Omega) \) such that

\[
\|f\|_{M^{p,s}_w(\Omega)} \equiv \left( \sup_{x \in \Omega, r > 0} r^{-s} \|f\|_{L^p_w(B_r(x) \cap \Omega)}^p \right)^{1/p} < \infty,
\]

where

\[
\|f\|_{L^p_w(B_r(x) \cap \Omega)}^p \equiv \sup_{t > 0} t^p \left| \{ x \in B_r(x) \cap \Omega : |f(x)| > t \} \right|.
\]

For a measurable function \( f : \Omega \to \mathbb{R} \), denote by \( \delta_f(t) = |\{ x \in \Omega : |f(x)| > t \}| \) its distributional function and by \( f^*(t) = \inf \{ s > 0 : \delta_f(s) \leq t \} \), \( t \geq 0 \), the nonincreasing
The Lorentz space $L^{p,q}(\Omega)$ ($1 < p < \infty, 1 \leq q \leq \infty$) is the space of measurable functions $f : \Omega \to \mathbb{R}$ such that

$$\|f\|_{L^{p,q}(\Omega)} \equiv \left( \int_{0}^{\infty} (t^{1/p} f^{**}(t))^{q} \frac{dt}{t} \right)^{1/q}, \quad \text{if } 1 \leq q < \infty,$$

$$\sup_{t > 0} t^{1/p} f^{**}(t) \quad \text{if } q = \infty$$

is finite.

It is well-known that $L^{p,p} = L^p$ and $L^{p,\infty} = L^\infty_p$. We will need the following Hölder’s inequality in Lorentz spaces.

**Proposition 2.1** ([33]). Let $1 < p_1, p_2 < \infty$ and $1 \leq q_1, q_2 \leq \infty$ be such that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \leq 1 \quad \text{and} \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} \leq 1.$$

Then, $f \in L^{p_1,q_1}(\Omega)$ and $g \in L^{p_2,q_2}(\Omega)$ implies $fg \in L^{p,q}(\Omega)$. Moreover,

$$\|fg\|_{L^{p,q}(\Omega)} \leq \|f\|_{L^{p_1,q_1}(\Omega)} \|g\|_{L^{p_2,q_2}(\Omega)}.$$

**Proposition 2.2** ([52]). For $1 < p < \infty$ and $1 \leq q_1 \leq q_2 \leq \infty$, there holds $L^p(\Omega) = L^{p,q_1}(\Omega)$ and $L^{p,q_1}(\Omega) \subset L^{p,q_2}(\Omega)$ with

$$\|f\|_{L^{p,q_2}(\Omega)} \leq C(p, q_1, q_2) \|f\|_{L^{p,q_1}(\Omega)}.$$

Moreover, if $|\Omega| < \infty$, then $L^{p,q}(\Omega) \supset L^{r,s}(\Omega)$ for $1 < p < r < \infty$, $1 \leq q, s \leq \infty$, and

$$\|f\|_{L^{p,r}(\Omega)} \leq C_{r,p} |\Omega|^{\frac{1}{r} - \frac{1}{p}} \|f\|_{L^{r,\infty}(\Omega)}.$$

Lorentz spaces can be used to define the following Lorentz-Sobolev spaces. For $k \in \mathbb{N}, 1 < p < \infty, 1 \leq q \leq \infty$, the Lorentz-Sobolev space $W^{k,p,q}(\Omega)$ consists of measurable functions $f : \Omega \to \mathbb{R}$ which are weakly differentiable up to order $k$ with $\nabla^\alpha f \in L^{p,q}(\Omega)$ for all $|\alpha| \leq k$, and its norm is defined by

$$\|f\|_{W^{k,p,q}(\Omega)} = \left( \sum_{|\alpha| \leq k} \|\nabla^\alpha f\|_{L^{p,q}(\Omega)}^p \right)^{1/p}.$$

**Proposition 2.3** ([47]). Let $k \in \mathbb{N}, 1 < p < \infty$ and $1 \leq q \leq \infty$. Then

1. $W^{k,p}(\Omega) = W^{k,p,p}(\Omega)$;
2. If $\Omega$ is bounded and smooth and $kp < n$, then $W^{k,p,q}(\Omega)$ embeds continuously into $L^{p,q}(\Omega)$, where $1/p^* = 1/p - k/n$.
3. $W^{k,n/k,1}(\Omega) \subset C(\Omega)$.

**Proposition 2.4.** ([7, Theorem 2], [4, Page 19]) Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain and $V \subset \mathbb{R}^n$ an open set such that $\Omega \subset \subset V$, $k \in \mathbb{N}, 1 < p < \infty, 1 \leq q \leq \infty$. There exists a bounded linear operator $E : W^{k,p,q}(\Omega) \to W^{k,p,q}(\mathbb{R}^n)$, such that for every $f \in W^{k,p,q}(\Omega)$,

(i) $Ef = f$ a.e. in $\Omega$,
(ii) If $f$ is compactly supported in $V$. Moreover, there exists a constant $C > 0$ depending only on $n, k, p$ and $\Omega$, such that for all $f \in W^{k,p,q}(\Omega)$, there holds

$$
\|Ef\|_{W^{k,p,q}(\mathbb{R}^n)} \leq C\|f\|_{W^{k,p,q}(\Omega)}.
$$

For later use, we furthermore need Lorentz-Sobolev spaces with negative exponents. For $k \in \mathbb{N}$, $1 \leq p, q \leq \infty$, the Lorentz-Sobolev space $W^{-k,p,q}(\Omega)$ is defined to be the space of all distributions $f$ on $\Omega$ of the form $f = \sum_{|\alpha| \leq k} \nabla^\alpha f_\alpha$ with $f_\alpha \in L^{p,q}(\Omega)$. The corresponding norm is defined as

$$
\|f\|_{W^{-k,p,q}(\Omega)} := \inf \sum_{|\alpha| \leq k} \|f_\alpha\|_{L^{p,q}(\Omega)},
$$

where the infimum is taken over all decompositions of $f$ as given in the definition. The following facts on Lorentz-Sobolev spaces with negative exponents can be found in [4, 5].

**Proposition 2.5.** Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain.

1. (Generalized Hölder’s inequality) Suppose $f \in W^{-k_0,p_0,q_0}(\Omega)$ and $g \in W^{k_1,p_1,q_1}(\Omega)$ with $k_0, k_1 \in \mathbb{N}$, $1 \leq p_0, p_1 < \infty$ and $1 \leq q_0, q_1 < \infty$. If $k_0 \leq k_1$, $\frac{1}{p_0} + \frac{1}{p_1} \leq 1$ and $k_1 p_1 < n$, then $fg \in W^{-k_0,s,l}(\Omega)$ with $s = \frac{np_0}{n(p_0 + p_1) - k_1 p_1}$ and $\frac{1}{l} = \min \{1, \frac{1}{q_0} + \frac{1}{q_1}\}$. Moreover,

$$
\|fg\|_{W^{-k_0,s,l}(\Omega)} \leq C\|f\|_{W^{-k_0,p_0,q_0}(\Omega)}\|g\|_{W^{k_1,p_1,q_1}(\Omega)}.
$$

The assertion continues to hold in the case $k_1 p_1 = n$ if we additionally assume $g \in L^\infty$.

2. (Sobolev embedding) If $k \in \mathbb{Z}$, $l \in \mathbb{N}$, $1 < p < \frac{n}{l}$ and $1 \leq q \leq \infty$, then $W^{k,p,q}(\Omega)$ embeds continuously into $W^{k-l,\frac{np}{n-lp},q}(\Omega)$ with

$$
\|f\|_{W^{k-l,\frac{np}{n-lp},q}(\Omega)} \leq C\|f\|_{W^{k,p,q}(\Omega)}.
$$

2.2. Fractional Riesz operators. Let $0 < \alpha < n$ and $I_\alpha = c_{n,\alpha}|x|^{n-\alpha}$, $x \in \mathbb{R}^n$, be the usual fractional Riesz operators, where $c_{n,\alpha}$ is a positive normalization constant. The following well-known estimates on fractional Riesz operators in Morrey spaces were proved by Adams [1].

**Proposition 2.6.** Let $0 < \alpha < n$, $0 \leq \lambda < n$ and $1 \leq p < \frac{n-\lambda}{\alpha}$. There exists a constant $C > 0$ depending only $n, \alpha, \lambda$ and $p$ such that, for all $f \in M^{p,\lambda}(\mathbb{R}^n)$, there holds

(i) If $p > 1$, then

$$
\|I_\alpha(f)\|_{M^{\frac{(n-\lambda)p}{n-\lambda-p\lambda}(\mathbb{R}^n)}} \leq C\|f\|_{M^{p,\lambda}(\mathbb{R}^n)}.
$$

(ii) If $p = 1$, then

$$
\|I_\alpha(f)\|_{M^\frac{n-\lambda}{n-\lambda-\lambda}(\mathbb{R}^n)} \leq C\|f\|_{M^{1,\lambda}(\mathbb{R}^n)}.
$$

Note that when $\lambda = 0$ it reduces to the usual Riesz potential theories between $L^p$ spaces.
Proposition 2.7. For $0 < \alpha < n$, $1 < p < n/\alpha$, $1 \le q \le q' \le \infty$, the fractional Riesz operators

$$I_\alpha : L^{p,q}(\mathbb{R}^n) \to L^{\frac{np}{n-\alpha q'}}(\mathbb{R}^n)$$

and

$$I_\alpha : L^1(\mathbb{R}^n) \to L^{\frac{n}{n-\alpha}}(\mathbb{R}^n)$$

are bounded.

In the later proof, we will mainly use the special cases

$$I_\alpha : L^{\frac{2m}{j},2}(\mathbb{R}^{2m}) \to L^{\frac{2m}{j},2}(\mathbb{R}^{2m})$$

for $0 < \alpha < 2m$ and $m \ge j \ge 1$ such that $\alpha + j < 2m$.

The result of Proposition 2.7 continues to hold for $\alpha = 0$ (see Theorems V.3.15 and VI.3.1 of Stein and Weiss [41]), provided we admit to use the notation $I_0$ to denote singular integral operators of the type $\nabla^{2m-k}f_k$ for integers $1 \le k \le 2m - 1$. In these cases, it will be convenient to introduce the notation $I_0$.

The following lemma plays a crucial role in the proofs of Sharp-Topping [40], and [17], and also in our approach here. It can be viewed as an improved version of the classical Riesz potential theory.

Lemma 2.8 ([1], Proposition 3.1). Let $0 < \alpha < \beta \le n$ and $f \in M^{1,n-\beta}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ for some $1 < p < \infty$. Then, $I_\alpha f \in L^{\frac{\beta}{\beta-\alpha}}(\mathbb{R}^n)$ with

$$\|I_\alpha f\|_{L^{\frac{\beta}{\beta-\alpha}}(\mathbb{R}^n)} \le C_{\alpha,\beta,n,p}\|f\|_{M^{1,n-\beta}(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)}.$$ 

In particular, in the case $1 < p < n/\beta$, we have

$$\frac{\beta p}{\beta - \alpha} > \frac{np}{n - \alpha p},$$

which implies that $I_\alpha f$ has better integrability than the usual one from $L^p$ boundedness. This is due to the fact that $f$ has additional fine property, that is, $f$ also belongs to some Morrey space. For a local version of Lemma 2.8, see [40, Lemma A.3].

2.3. Scaling invariance of (1.11). We shall use a scaling argument in our later proofs. Let $u$ be a weak solution of (1.11). For any $B_{2R}(x_0) \subset B_1 \subset \mathbb{R}^{2m}$ and $x \in B_{1/2} = B_{1/2}(0)$, set

$$u_R(x) = u(x_0 + Rx), \quad V_{l,R}(x) = R^{2m-2l-1}V_l(x_0 + Rx),$$

$$w_{l,R}(x) = R^{2m-2l-2}w_l(x_0 + Rx), \quad f_R(x) = R^{2m}f(x_0 + Rx).$$

It is straightforward to verify that $u_R$ satisfies

$$\Delta^m u_R = \sum_{l=0}^{m-1} \Delta^l \langle V_{l,R}, du_R \rangle + \sum_{l=0}^{m-2} \Delta^l \delta (w_{l,R} du_R) + f_R \quad \text{in } B_{1/2}. \quad (2.4)$$

Moreover, for any $0 < r < 1$ and $1 \le q \le \infty$, there hold

$$\|\nabla^i u_R\|_{L^{\frac{2m}{i}q}(B_r(0))} = \|\nabla^i u\|_{L^{\frac{2m}{i}q}(B_{rR}(x_0))}, \quad \|f_R\|_{L^p(B_r(0))} = R^{2m(1-1/p)}\|f\|_{L^p(B_{rR}(x_0))}.$$
3. Hölder continuity via decay estimates

In this section, we prove Theorem 1.1. Throughout this section, we use $B_r(x)$ to denote the ball with centre $x$ and radius $r$ in $\mathbb{R}^{2m}$, and write $B_r$ instead of $B_r(0)$ when $x = 0$. Set

$$1 < p < 2m/(2m - 1) \quad \text{and} \quad \alpha_0 = 2m(1 - 1/p).$$

To prove Theorem 1.1, it suffices to establish the following decay estimate.

**Lemma 3.1** (Decay estimate). Suppose $f \in L^p(B_1)$ for some $1 < p < 2m/(2m - 1)$ and $u \in W^{m,2}(B_1, \mathbb{R}^n)$ is a solution to system (1.11). There exist $r_0, \tau \in (0,1)$ and $C > 0$ depending only on $m, n, p$ and the coefficient functions $V_k, w_k$ such that for $x \in B_{1/2}$ and $0 < r < r_0$,

$$\sum_{j=1}^m \|\nabla^j u\|_{L^{2m/j,2}(B_r(x))} \leq C \left( \sum_{j=1}^m \|\nabla^j u\|_{L^{2m/j,2}(B_1)} + \|f\|_{L^p(B_1)} \right)^{r \alpha_0}.$$

Once the above lemma is proved, Theorem 1.1 follows easily.

**Proof of Theorem 1.1.** By Lemma 3.1 and Proposition 2.2, we have

$$\int_{B_r(x)} |\nabla u|^{2m} \leq C \|\nabla u\|_{L^{2m,j,2}(B_r(x))}^{2m} \leq C \left( \|u\|_{W^{m,2}(B_1)} + \|f\|_{L^p(B_1)} \right)^{2m} \tau^{2m \alpha_0}$$

for all $x \in B_{1/2}$ and $0 < r < r_0$. Then Morrey’s Dirichlet growth theorem [11] implies that $u \in C^{0,\alpha_0}(B_{1/2})$. The proof is complete. \hfill \Box

Due to the scaling invariance exhibited in section 2.3, Lemma 3.1 is a consequence of the following estimate.

**Lemma 3.2.** Suppose $f \in L^p(B_1)$ for some $1 < p < 2m/(2m - 1)$ and $u \in W^{m,2}(B_1, \mathbb{R}^n)$ is a solution to system (1.11). There exist $\epsilon, C > 0$ depending only on $m, n, p$ and the coefficient functions $V_k, w_k$ such that, if

$$\theta_{B_1} < \epsilon,$$

then

$$\sum_{j=1}^m \|\nabla^j u\|_{L^{2k/j,2}(B_r)} \leq \tau^\beta \sum_{j=1}^m \|\nabla^j u\|_{L^{2k/j,2}(B_1)} + C r^{\alpha_0} \|f\|_{L^p(B_1)}, \quad (3.1)$$

where $\theta_{B_1}$ is defined in (1.8) and $\beta = (\alpha_0 + 1)/2$.

The proof of Lemma 3.2 is lengthy. We divide it into four steps. First choose a sufficiently small $\epsilon$ in Lemma 3.2 such that the conservation law (1.10) holds for some $A, B$ in $B_1^2$.

**Step 1.** Deduce the equation of $Adu$. 

2Recall that under the smallness assumption $\theta_{B_1} < \epsilon$ the conservation law (1.10) only holds on $B_{1/2}$, but here for simplicity we directly assume it hold on the whole ball $B_1$ since otherwise we can do everything below on $B_{1/2}$.
Proposition 3.3. Adu satisfies the system

\[ \delta\Delta^{m-1}(Adu) = \sum_{i=1}^{m-1} \delta^i \left( \sum_{j=m-i}^{m} \nabla^j A \nabla^{2m-i-j}u \right) + \delta K + Af, \]  

(3.2)

where \( K \) is the last five terms of the conservation law (1.10).

In below \( \delta \) denotes the divergence operator and \( \delta^i \) means taking divergence for \( i \) times. By \( \nabla^j A \nabla^k u \) we mean it is a linear combination of products of the type \( \nabla^\alpha A \nabla^\beta u \) for which \( |\alpha| = j \) and \( |\beta| = k \).

Proof. Repeatedly using Leibniz rule gives

\[ \delta\Delta^{m-1}(Adu) = \delta(A\Delta^{m-1}du) + \sum_{i=1}^{m-1} \delta^i \left( \sum_{j=m-i}^{m} \nabla^j A \nabla^{2m-i-j}u \right). \]  

(3.3)

Then the conservation law (1.10) yields

\[ \delta(A\Delta^{m-1}du) = \delta \left( \sum_{l=1}^{m-1} \Delta^l A\Delta^{m-1-l}du + \sum_{l=0}^{m-2} d\Delta^l A\Delta^{m-1-l}u \right) + \delta K + Af, \]

where \( K \) is the last five terms of the conservation law of (1.10). Hereafter, we use \( \sum_i a_i \) to denote a linear combination of \( a_i \)’s, i.e., \( \sum_i a_i = \sum_i c_i a_i \) for some harmless absolute constants \( c_i \). We will write down the coefficient explicitly when necessary.

Furthermore, note that the first term in the right hand side of the above equality can be written as

\[ \delta \left( \sum_{l=1}^{m-1} \Delta^l A\Delta^{m-1-l}du + \sum_{l=0}^{m-2} d\Delta^l A\Delta^{m-1-l}u \right) = \sum_{i=1}^{m-1} \delta^i \left( \sum_{j=m-i}^{m} \nabla^j A \nabla^{2m-i-j}u \right). \]

For instance, repeatedly using Leibniz rule gives (with \( l = 1 \))

\[ \Delta A\Delta^{m-2}du = \sum_{i=0}^{m-2} \delta^i(\nabla^{m-i} A \nabla^{m-1}u), \]

and (with \( l = m - 1 \))

\[ \Delta^{m-1}Adu = \sum_{i=0}^{m-2} \delta^i(\nabla^{m} A \nabla^{m-i}u), \]

and (with \( l = 0 \))

\[ dA\Delta^{m-1}u = dA\Delta^{m-2}du = \sum_{i=0}^{m-2} \delta^i(\nabla^{m-1-i} A \nabla^{m}u) \]

and (with \( l = m - 2 \))

\[ d\Delta^{m-2}Adu = \Delta^{m-2}dA\Delta u = \sum_{i=0}^{m-2} \delta^i(\nabla^{m-1} A \nabla^{m-i}u). \]

This leads to (3.2). The proof is complete. \( \square \)

---

3A proof of this equality can be found in the Appendix.
By noting that
\[ \Delta^{m-1}(A\Delta u) = \delta \Delta^{m-1}(Adu) - \Delta^{m-1}(dAdu), \]
we infer from Proposition 3.3 that

**Corollary 3.4.** \( A\Delta u \) satisfies system
\[ \Delta^{m-1}(A\Delta u) = \sum_{i=1}^{m-1} \delta^i \left( \sum_{j=m-i}^{m} \nabla^j A \nabla^{2m-i-j} u \right) + \delta K + Af, \tag{3.4} \]
where \( K \) is the last five terms of the conservation law (1.10).

**Step 2.** Decomposition of \( Adu \).
We use Hodge decomposition to obtain
\[ Adu = d\tilde{u}_1 + d^*\tilde{u}_2 + \tilde{h} \quad \text{in } B_1, \]
such that
\[ \Delta^m \tilde{u}_1 = \sum_{i=1}^{m-1} \delta^i \left( \sum_{j=m-i}^{m} \nabla^j A \nabla^{2m-i-j} u \right) + \delta K + Af, \]
\[ \Delta^m \tilde{u}_2 = \Delta^{m-1}(dA \wedge du) \]
in \( B_1 \) and \( \tilde{h} \) is a harmonic one form in \( B_1 \). Here to be more precise, we use \( d^* \) to mean the codifferential operator.

Now we extend all the functions from \( B_1 \) into the whole space \( \mathbb{R}^{2m} \) in a bounded way. Still we use the same notations to denote the extended functions. We define
\[ u_1 = c \log * \left( \sum_{i=1}^{m-1} \delta^i \left( \sum_{j=m-i}^{m} \nabla^j A \nabla^{2m-i-j} u \right) + \delta K + Af \right) \tag{3.5} \]
and
\[ u_2 = c \log * \Delta^{m-1}(dA \wedge du) \tag{3.6} \]
such that \( \Delta^m (\tilde{u}_1 - u_1) = \Delta^m (\tilde{u}_2 - u_2) = 0 \) in \( B_1 \), where \( c \log \) is the fundamental solution of \( \Delta^m \) in \( \mathbb{R}^{2m} \), and then define \( h = \tilde{u}_1 - u_1 + \tilde{u}_2 - u_2 + \tilde{h} \). Consequently, we obtain
\[ Adu = du_1 + d^*u_2 + h \quad \text{in } B_1, \tag{3.7} \]
with \( u_1 \) given by (3.5), \( u_2 \) given by (3.6), and \( h \) is an \( m \)-polyharmonic 1-form in \( B_1 \), i.e.,
\[ \Delta^m h = 0 \quad \text{in } B_1. \]

**Step 3.** Estimates of \( u_1, u_2 \) and \( h \).
To estimate \( u_1 \), let
\[ u_{11} \equiv \log * \left( \sum_{i=1}^{m-1} \delta^i \left( \sum_{j=m-i}^{m} \nabla^j A \nabla^{2m-i-j} u \right) + \delta K \right) \]
and
\[ u_{12} = \log * (Af) \]
such that
\[ u_1 = u_{11} + u_{12}. \]
The estimate of $u_{12}$ is simple and standard: since $f \in L^p(\mathbb{R}^{2m})$ and $A \in L^\infty$, we have $u_{12} \in W^{2m,p}(\mathbb{R}^{2m})$ with $\nabla^ju_{12} = I_{2m-j}(Af) \in L^{p_j}(\mathbb{R}^{2m})$ for all $1 \leq j \leq 2m$, where $p^{-1} = (2m-j)/2m$ and
\[
\|\nabla^ju_{12}\|_{L^{p_j}(B_1)} \leq \|\nabla^ju_{12}\|_{L^{p_j}(\mathbb{R}^{2m})} \leq C_{m,j,p} \|f\|_{L^p(\mathbb{R}^{2m})} \leq C_{m,j,p} \|f\|_{L^p(B_1)} \tag{3.8}
\]
Here, when $j = 2m$, we use $I_0$ to denote the singular integral operators $\nabla^m \log$ on $L^p(\mathbb{R}^{2m})$.

We estimate $u_{11}$ as follows. Note that for each $1 \leq t \leq m$, we have
\[
\nabla^t u_{11} = I_{2m-t} \left( \sum_{i=1}^{m-1} \delta^i \left( \sum_{j=m-i}^m \nabla^j A \nabla^{2m-i-j} u \right) + \delta K \right).
\]

Hence
\[
|\nabla^t u_{11}| \lesssim \sum_{i=1}^{m-1} \sum_{j=m-i}^m I_{2m-t-i} \left( |\nabla^j A \nabla^{2m-i-j} u| \right) + I_{2m-t-1}(|K|).
\]

Since $A, u \in W^{m,2}(\mathbb{R}^{2m})$, we have $\nabla^j A \nabla^{2m-i-j} u \in L^{2m-2,j} L^{2m-2,2} \subset L^{2m-2,1} \subset L^{2m-2,2}$. With
\[
|\nabla^j A \nabla^{2m-i-j} u|_{L^{2m-2,j}} \leq |\nabla^j A|_{L^{2m,j}}^2 |\nabla^{2m-i-j} u|_{L^{2m-2,2}} \lesssim \|\nabla^{2m-i-j} u\|_{L^{2m-2,2}}.
\]
Thus, it holds
\[
\|I_{2m-t-i} \left( |\nabla^j A \nabla^{2m-i-j} u| \right)\|_{L^{2m_t,2}(\mathbb{R}^{2m})} \lesssim \|\nabla^{2m-i-j} u\|_{L^{2m-2,2}}.
\]

Taking summation over $i$ and $j$, and then over $t$, we obtain
\[
\sum_{t=1}^{m} \sum_{i=1}^{m-1} \sum_{j=m-i}^m I_{2m-t-i} \left( |\nabla^j A \nabla^{2m-i-j} u| \right)_{L^{2m_t,2}(\mathbb{R}^{2m})} \lesssim \epsilon \sum_{j=1}^{m} \|\nabla^j u\|_{L^{2m,j}(\mathbb{R}^{2m})} \lesssim \epsilon \sum_{j=1}^{m} \|\nabla^j u\|_{L^{2m,j}(B_1)},
\]
where we used the boundedness of the extensions of $u$ from $B_1$ into $\mathbb{R}^{2m}$.

Next consider the first term $K_1$ of $K$, i.e.
\[
K_1 = \sum_{k=1}^{m-1} \sum_{l=0}^{k-1} \Delta^l A \Delta^{k-l-1} d(V_k du).
\]

Repeatedly using Leibniz rule yields
\[
\Delta^l A \Delta^{k-l-1} d(V_k du) = \sum_{i=1}^{2(k-l)} \Delta^l A \nabla^{2(k-l)-i} V_k \nabla^i u.
\]

The estimate of $\|I_{2m-t-i} \left( \Delta^l A \nabla^{2(k-l)-i} V_k \nabla^i u \right)\|_{L^{2m_t,2}(\mathbb{R}^{2m})}$ is far more complicated than the previous term. We divide this term into two cases according to the order of the derivatives of $A$:

**Case 1**: $2l > m$. Then $2(k-l) < 2(m - 1 - \frac{m}{2}) = m - 2 < m$ and $2(k-l) - 1 < 2k + 1 - m$. Note that $\Delta^l A \in W^{2l-m,2}$ is of negative exponent. Since $i \leq 2(k-l)$,
\[ \nabla^{2(k-l)-i}V_k \in W^{2l-m+i+1,2} \text{ and } \nabla^i u \in W^{m-i,2} \] are of positive power. Moreover, it is direct to verify that
\[
\min_{1 \leq i \leq 2(k-l)} (2l - m + i + 1, m - i) \geq 2l - m,
\]
which means, by Proposition 2.5, that the product \[ \Delta^l A \nabla^{2(k-l)-i}V_k \nabla^i u \in W^{2l-m,2} \]. Using Leibniz rule, we obtain
\[
\Delta^l A \nabla^{2(k-l)-i}V_k \nabla^i u = \sum_{|\alpha| \leq 2l-m} \partial^\alpha A_\alpha \left( \nabla^{2(k-l)-i}V_k \nabla^i u \right)
\]
\[
= \sum_{|\alpha| \leq 2l-m} \sum_{0 \leq \beta \leq \alpha} \partial^\beta \left( A_\alpha \partial^{\alpha-\beta} \left( \nabla^{2(k-l)-i}V_k \nabla^i u \right) \right)
\]
\[
= \sum_{|\alpha| \leq 2l-m} \sum_{0 \leq \beta \leq \alpha} \sum_{0 \leq \gamma \leq \alpha - \beta} \partial^\beta \left( A_\alpha \nabla^{2(k-l)-i+|\gamma|}V_k \nabla^{\alpha-|\beta|-|\gamma|+i} u \right),
\]
where \( A_\alpha \in L^2(\mathbb{R}^m) \). So \( I_{2m-1-t} \) acts on this summation gives a summation of the form
\[
I_{2m-1-t-|\beta|} \left( A_\alpha \nabla^{2(k-l)-i-|\gamma|}V_k \nabla^{\alpha-|\beta|-|\gamma|+i} u \right).
\]
The integrability of the function inside the potential is as follows: \( A_\alpha \in L^2 \), and
\[
\nabla^{2(k-l)-i-|\gamma|}V_k \in W^{2l-m+i+1-\gamma,2} \subset L^{2m-(2l+i+1+\gamma),2}, \nabla^{\alpha-|\beta|-|\gamma|+i} u \in L^{2m-|\alpha|-|\beta|-|\gamma|+i,2}
\]
and so
\[
A_\alpha \nabla^{2(k-l)-i-|\gamma|}V_k \nabla^{\alpha-|\beta|-|\gamma|+i} u \in L^{p,2}
\]
with
\[
\frac{1}{p} = \frac{1}{2} + \frac{2m - (2l + i + 1) + |\gamma|}{2m} + \frac{|\alpha| - |\beta| - |\gamma| + i}{2m} = 3m + |\alpha| - |\beta| - 2l - 1.
\]
Recall that all the functions are extended in such a way that they are supported in the ball \( B_2 \). Hence we only need \( p \geq \frac{2m}{2m-1-t-|\beta|+i} \) such that
\[
I_{2m-1-t-|\beta|} \left( A_\alpha \nabla^{2(k-l)-i-|\gamma|}V_k \nabla^{\alpha-|\beta|-|\gamma|+i} u \right) \in L^{2m/t,2}.
\]
This does hold by noticing that
\[
(3m + \alpha - \beta - 2l - 1) - (2m - 1 - \beta) = m - 2l + \alpha \leq 0.
\]
Thus, there holds
\[
\left\| I_{2m-1-t-|\beta|} \left( A_\alpha \nabla^{2(k-l)-i-|\gamma|}V_k \nabla^{\alpha-|\beta|-|\gamma|+i} u \right) \right\|_{L^{2m,2}} \leq \epsilon \sum_{i=1}^{m} \left\| \nabla^i u \right\|_{L^{2m/i,2}(B_1)}.
\]
In conclusion, in Case 1 we get
\[
\left\| I_{2m-t-1} \left( \Delta^l A \nabla^{2(k-l)-i}V_k \nabla^i u \right) \right\|_{L^{2m/t,2}(\mathbb{R}^{2m})} \leq \epsilon \sum_{i=1}^{m} \left\| \nabla^i u \right\|_{L^{2m/i,2}(B_1)}, \tag{3.11}
\]
Case 2: \( 2l \leq m \). In this case, we have two sub-cases:
\textbf{Subcase 2.1:} \(2l \leq m\) and \(2(k-l) - i > 2k+1-m\). That is, \(\nabla^{2(k-l)-i} V_k \in W^{2l+i-1-m, 2}\) is of negative exponent. Then, \(i < m - 2l - 1 < m\). In this case
\[
\min_{1 \leq i \leq 2(k-l)} (m - 2l, m - i) \geq |2l + i + 1 - m| = m - (2l + i + 1).
\]
So \(\Delta^i A \nabla^{2(k-l)-i} V_k \nabla^i u\) makes sense, and
\[
\Delta^i A \nabla^{2(k-l)-i} V_k \nabla^i u = \sum_{|\alpha| \leq m-(2l+i+1)} \Delta^i A \partial^\alpha V_{k, \alpha} \nabla^i u
\]
\[
= \sum_{\alpha} \sum_{\beta} \partial^\beta \left( V_{k, \alpha} \partial^{\alpha-\beta} (\Delta^i A \nabla^i u) \right)
\]
\[
= \sum_{\alpha} \sum_{\beta} \sum_{\gamma} \partial^\beta \left( V_{k, \alpha} \nabla^{2l+|\gamma|} A \nabla^{\alpha-\beta-|\gamma|} u \right). \quad (3.12)
\]
Then
\[
V_{k, \alpha} \nabla^{2l+|\gamma|} A \nabla^{\alpha-\beta-|\gamma|+i} u \in L^{p, 2}
\]
with
\[
\frac{1}{p} = \frac{1}{2} + \frac{2l + |\gamma|}{2m} + \frac{|\alpha| - |\beta| - |\gamma| + i}{2m} = \frac{m + 2l + |\alpha| - |\beta| + i}{2m}.
\]
Thus \(p \geq \frac{2m}{2m-1-t-\beta+i}\) is equivalent to
\[
m + 2l + |\alpha| - |\beta| + i \leq 2m - 1 - |\beta|,
\]
which is equivalent to
\[
\alpha \leq m - 2l - i - 1.
\]
This holds by our assumption. So in \textbf{Subcase 2.1} we also have the estimate \((3.11)\).

\textbf{Subcase 2.2:} \(2l \leq m\) and \(2(k-l) - i \leq 2k+1-m\). That is, \(\Delta^i A \) and \(\nabla^{2(k-l)-i} V_k \in W^{2l+i-1-m, 2}\) are usual Sobolev functions with positive exponent. In this case we have two more cases:

1. \(i \leq m\). Then
\[
\Delta^i A \nabla^{2(k-l)-i} V_k \nabla^i u \in L^{p, 2}
\]
with
\[
\frac{1}{p} = \frac{2l}{2m} + \frac{2m - (2l + i + 1)}{2m} + \frac{i}{2m} = \frac{2m - 1 - t + i}{2m}.
\]
Then \(I_{2m-1-t(L^{2m-1-t}, l)} \subset L^{2m, 2}\) holds.

2. \(i > m\). Then \(\nabla^i u \in W^{m-i, 2}\) is of negative exponent. In this case
\[
\min(m - 2l, 2l + i + 1 - m) \geq i - m.
\]
So
\[
\Delta^i A \nabla^{2(k-l)-i} V_k \nabla^i u = \sum_{\alpha \leq m} \Delta^i A \nabla^{2(k-l)-i} V_k \partial^\alpha u_{\alpha}
\]
\[
= \sum_{\alpha \leq m} \sum_{\beta} \partial^\beta \left( u_{\alpha} \partial^{\alpha-\beta} \left( \Delta^i A \nabla^{2(k-l)-i} V_k \right) \right)
\]
\[
= \sum_{\alpha \leq m} \sum_{\beta} \sum_{\gamma} \partial^\beta \left( u_{\alpha} \nabla^{2l+|\gamma|} A \nabla^{2(k-l)-i+|\alpha|-|\beta|-|\gamma|} V_k \right). \quad (3.13)
\]
We have
\[ u_\alpha \nabla^{2l+|\gamma|} A \nabla^{2(k-l)-i+|\alpha|-|\beta|-|\gamma|} V_k \in L^{p,2} \]
with
\[ \frac{1}{p} = \frac{1}{2} + \frac{2l + |\gamma|}{2m} + \frac{2m - 1 - 2l - i + (|\alpha| - |\beta| - |\gamma|)}{2m} = \frac{3m - 1 + i + |\alpha| - |\beta|}{2m}. \]
Thus \( p \geq \frac{2m}{2m-2-|\beta|+1} \) is equivalent to
\[ 3m - 1 + i - |\beta| \leq 2m - 1 - |\beta|, \]
is equivalent to
\[ \alpha \leq i - m, \]
which holds. Therefore, in **Subcase 2.2** the estimate (3.11) holds as well.

In conclusion, we obtain the estimate
\[ \sum_{l=1}^{m} \left\| I_{2m-1-t} \left( \sum_{k=1}^{m-1} \sum_{l=0}^{k-1} \Delta^l A \nabla^{2(k-l)-i+|\alpha|-|\beta|-|\gamma|} V_k \nabla^i u \right) \right\|_{L^{2m/i,2}(\mathbb{R}^{2m})} \lesssim \epsilon \sum_{i=1}^{m} \| \nabla^i u \|_{L^{2m/i,2}(B_1)}. \]
Arguing exactly as in the above, we can obtain the same estimate for the remaining terms of \( K \). Therefore, combining (3.9) and (3.14) together, we deduce
\[ \sum_{j=1}^{m} \| \nabla^j u_1 \|_{L^{2m/j,2}(\mathbb{R}^{2m})} \lesssim \epsilon \sum_{i=1}^{m} \| \nabla^i u \|_{L^{2m/i,2}(B_1)}. \]
(3.15)

Using (3.15) and (3.8), we obtain
\[ \sum_{j=1}^{m} \| \nabla^j u_1 \|_{L^{2m/j,2}(\mathbb{R}^{2m})} \lesssim \epsilon \sum_{i=1}^{m} \| \nabla^i u \|_{L^{2m/i,2}(B_1)} + \| f \|_{L^p(B_1)}. \]
(3.16)

Next we estimate the function \( u_2 \) and \( h \). By the definition (3.6) of \( u_2 \), we use the same method as that of (3.9) to obtain
\[ \sum_{j=1}^{m} \| \nabla^j u_2 \|_{L^{2m/j,2}(\mathbb{R}^{2m})} \lesssim \epsilon \sum_{i=1}^{m} \| \nabla^i u \|_{L^{2m/i,2}(B_1)}. \]
(3.17)

As to the polyharmonic function \( h \), we can apply [10, Lemma 6.2] to find that, for any \( 0 < r < 1 \),
\[ \sum_{j=1}^{m} \| \nabla^j h \|_{L^{2m/j,2}(B_r)} \lesssim r \sum_{i=1}^{m} \| \nabla^i h \|_{L^{2m/i,2}(B_1)}. \]
(3.18)

**Step 4.** Conclusion.

Now we prove the decay estimate (3.1) as follows. Let \( 0 < \tau < 1 \) be determined later. We have, for every \( 1 \leq j \leq m \)
\[ \| \nabla^{j-1} (A^{-1} du_{11}) \|_{L^{2m/j,2}(B_r)} \lesssim \sum_{i=1}^{j} \| \nabla^i u_{11} \|_{L^{2m/i,2}(B_r)} \]
\[ \lesssim \sum_{i=1}^{j} \| \nabla^i u_{11} \|_{L^{2m/i,2}(B_1)} \lesssim \epsilon \sum_{i=1}^{m} \| \nabla^i u \|_{L^{2m/i,2}(B_1)}; \]
Therefore, from the above four estimates we derive, for each $1 \leq j \leq m$,

\[
\|\nabla^{j-1}(A^{-1}du_2)\|_{L^{2m/j,2}(B_r)} \lesssim \sum_{i=1}^{j} \|\nabla^i u_2\|_{L^\frac{2m}{j,2}(B_r)} \lesssim \tau \|f\|_{L^p(B_1)};
\]

\[
\|\nabla^{j-1}(A^{-1}h)\|_{L^{2m/j,2}(B_r)} \lesssim \sum_{i=1}^{j} \|\nabla^i h\|_{L^\frac{2m}{j,2}(B_r)} \lesssim \tau \|f\|_{L^p(B_1)};
\]

\[
\|\nabla^{j-1}(A^{-1}h)\|_{L^{2m/j,2}(B_r)} \lesssim \sum_{i=1}^{j} \|\nabla^i h\|_{L^\frac{2m}{j,2}(B_r)} \lesssim \tau \|f\|_{L^p(B_1)}.
\]

Therefore, from the above four estimates we derive, for each $1 \leq j \leq m$,

\[
\|\nabla^j u\|_{L^\frac{2m}{j,2}(B_r)} \leq \|\nabla^{j-1}(A^{-1}h)\|_{L^\frac{2m}{j,2}(B_r)} + \|\nabla^{j-1}(A^{-1}du_1)\|_{L^\frac{2m}{j,2}(B_r)} + \|\nabla^{j-1}(A^{-1}du_2)\|_{L^\frac{2m}{j,2}(B_r)}
\]

\[
\lesssim \tau \sum_{i=1}^{m} \|\nabla^i h\|_{L^{2m/i,2}(B_1)} + \epsilon \sum_{i=1}^{m} \|\nabla^i u\|_{L^{2m/i,2}(B_1)} + \tau \|f\|_{L^p(B_1)}.
\]

Then scaling invariance implies

\[
\sum_{i=1}^{m} \|\nabla^i u\|_{L^{2m/i,2}(B_r)} \leq \tau^\beta \sum_{i=1}^{m} \|\nabla^i u\|_{L^{2m/i,2}(B_1)} + C\tau^{\alpha_0} \|f\|_{L^p(B_1)}
\]

for some $C > 0$ independent of $\tau$ and $\epsilon$.

Now set $\alpha_0 = 2m(1 - 1/p)$ and $\beta = (\alpha_0 + 1)/2$. First choose $\tau$ such that $2C\tau \leq \tau^\beta$, and then choose $\epsilon < \tau$. Taking summation over $j$, we obtain

\[
\sum_{i=1}^{m} \|\nabla^i u\|_{L^{2m/i,2}(B_r)} \leq \tau^\beta \sum_{i=1}^{m} \|\nabla^i u\|_{L^{2m/i,2}(B_1)} + C\tau^{\alpha_0} \|f\|_{L^p(B_1)}
\]

for some $C > 0$ independent of $\tau$. This proves (3.1).

Finally, apply an iteration argument as follows. Write $\Phi(u, B_r) = \sum_{i=1}^{m} \|\nabla^i u\|_{L^\frac{2m}{j,2}(B_r)}$ and $F = \|f\|_{L^p(B_1)}$. Then scaling invariance implies

\[
\Phi(u_\tau, B_1) = \Phi(u, B_\tau) \quad \text{and} \quad \|f_\tau\|_{L^p(B_1)} = \tau^{\alpha_0} \|f\|_{L^p(B_1)}.
\]

Thus, for any $k \in \mathbb{N}$, there holds

\[
\Phi(u, B_{r^k}) = \Phi(u_{r^{k-1}}, B_{r^k}) \leq \tau^\beta \Phi(u_{r^{k-1}}, B_{r^k}) + C\tau^{\alpha_0} \|f_{r^{k-1}}\|_{L^p(B_1)}
\]

\[
= \tau^\beta \Phi(u, B_{r^{k-1}}) + C\tau^{\alpha_0} \|f\|_{L^p(B_{r^{k-1}})}
\]

\[
\leq \tau^\beta \Phi(u, B_{r^{k-1}}) + C\|f\|_{L^p(B_1)} \tau^{\alpha_0}.
\]
Therefore, using an iteration arguments we obtain

\[ \Phi(u, B_{r^k}) \leq \tau^k \Phi(u, B_1) + C \| f \|_{L^p(B_1)} \tau^{k\alpha_0} \sum_{i=0}^{k-1} \tau^i (\beta - \alpha_0). \]

Since \( \beta > \alpha_0 \), we obtain, for any \( k \geq 1 \),

\[ \Phi(u, B_{r^k}) \leq C \tau^k \alpha_0 \left( \Phi(u, B_1) + \| f \|_{L^p(B_1)} \right). \]

The proof of Lemma 3.2 is complete taking into account of the monotonicity of \( r \mapsto \Phi(u, B_r) \).

4. Higher order regularity

In this section, we will prove an almost optimal higher order regularity result. The main tool is Lemma 2.8. Throughout this section, we set

\[ 1 < p < \frac{2m}{2m - 1}, \quad \alpha_0 = 2m(1 - 1/p) \quad \text{and} \quad M \equiv \| u \|_{W^{m,2}(B_1)} + \| f \|_{L^p(B_1)}. \]

4.1. \( W^{m,q} \)-estimate with any \( q < \frac{2p}{2 - p} \).

**Proposition 4.1.** Suppose \( f \in L^p(B_1) \) for some \( p \in (1, 2m/(2m - 1)) \) and \( u \in W^{m,2}(B_1, \mathbb{R}^n) \) is a solution of the inhomogeneous system (1.11). Then \( u \in W^{m,q}_{\text{loc}}(B_1) \) for any \( q < \frac{2p}{2 - p} \).

**Proof.** As in the previous section, it suffices to assume that \( \theta_{B_1} < \epsilon \) for some sufficiently small \( \epsilon \), such that the conservation law (1.10) holds for some functions \( A, B \). And then extend all the related functions from \( B_{1/2} \) to the whole space with compact support in \( B_2 \) in a bounded way. Then apply the Hodge decomposition for \( Adu \) as follows.

Set

\[ v = \log * \left( \sum_{i=1}^{m-1} \delta^i \left( \sum_{j=m-i}^{m} \nabla^j A \nabla^{2m-i-j} u \right) + \delta K + Af \right), \]

\[ g = \log * \Delta^{m-1}(dA \wedge du) \]

and

\[ h = Adu - dv - *dg \]

such that \( h \) is \( m \)-polyharmonic in \( B_1 \).

To begin with, recall that we have proved in Theorem 1.1 that

\[ \nabla^i u \in M^{\frac{2m}{i}, \frac{2m}{i} - \alpha_0} (\mathbb{R}^{2m}) \cap L^{\frac{2m}{i}}(\mathbb{R}^{2m}), \quad i = 1, \ldots, m. \]

**Estimate of \( g \).** By Hölder’s inequality, for every \( 1 \leq i \leq m \),

\[ \nabla^i A \nabla^{m+1-i} u \in L^{\frac{2m}{m+1}} \cap M^{1, m+1+\alpha_0} (\mathbb{R}^{2m}) = L^{\frac{2m}{m+1}} \cap M^{1, 2m-(m+1-\alpha_0)} (\mathbb{R}^{2m}). \]

Thus

\[ \nabla^m g \approx I_1 \left( \sum_{i=1}^{m} \nabla^i A \nabla^{m+1-i} u \right) \in L^{q_0}(\mathbb{R}^{2m}) \]

with

\[ q_0 = \frac{2m}{m+1} \frac{m+1-\alpha_0}{m-\alpha_0} > 2 \quad (4.1) \]

since \( 0 < \alpha_0 < m \).
To estimate $v$, we introduce the following notations to decompose $\nabla^mv$:

- $v_1 := I_m \ast \sum_{i=1}^{m-1} \delta^i \left( \sum_{j=m-i}^m \nabla^j A \nabla^{2m-i-j} u \right)$;
- $v_2 := I_m \ast \delta(K) = I_{m-1}(K)$;
- $v_3 := I_m \ast (Af)$.

**Estimate of $v_1$.** Note that

$$v_1 = I_m \sum_{i=1}^{m-1} \delta^i \left( \sum_{j=m-i}^m \nabla^j A \nabla^{2m-i-j} u \right) \approx \sum_{i=1}^{m-1} \sum_{j=m-i}^m I_{m-i} \left( \nabla^j A \nabla^{2m-i-j} u \right)$$

and that

$$\nabla^j A \nabla^{2m-i-j} u \in L^{\frac{2m}{2m-i}} \cap M^{1,i+\alpha_0} = L^{\frac{2m-i}{2m}} \cap M^{1,2m-(2m-i-\alpha_0)}(\mathbb{R}^m).$$

Hence $I_{m-i} \left( \nabla^j A \nabla^{2m-i-j} u \right) \in L^{q_i}$ with

$$\frac{1}{q_i} = \frac{2m-i}{2m} \left( 1 - \frac{m-i}{2m-i-\alpha_0} \right) = \frac{(2m-i)(m-\alpha_0)}{2m(2m-i-\alpha_0)} < \frac{1}{2}$$

for all $1 \leq i \leq m-1$. Moreover, we have $q_1 > q_2 > \cdots > q_{m-1} = q_0 > 2$. Thus we obtain

$$v_1 \in L^{q_{m-1}} = L^{q_0}(\mathbb{R}^m),$$

where $q_0$ is the number defined as in (4.1).

**Estimate of $v_2$.** Next we shall establish similar estimates for the terms involving $K$. Let $K_1$ be the first term of $K$, i.e.

$$K_1 = \sum_{k=0}^{m-1} \sum_{\delta=0}^{m-1-k-1} \sum_{l=0}^{m-1-k-1} \sum_{i=1}^{2(k-l)} (\Delta^l A) ^{k-l-1} d(V_k du) = \sum_{k=0}^{m-1} \sum_{l=0}^{m-1-k-1} \sum_{i=1}^{2(k-l)} \Delta^l A \nabla^{2(k-l)-i} V_k \nabla^i u.$$ 

by repeatedly using Leibniz rules. As in the proof of Lemma 3.2, we divide this term into two cases.

**Case 1:** $2l > m$. Then by (3.10) there holds

$$\Delta^l A \nabla^{2(k-l)-i} V_k \nabla^i u = \sum_{|\alpha| \leq 2l-m} \sum_{0 \leq \beta \leq \alpha} \sum_{0 \leq \gamma \leq \alpha-\beta} \partial^\beta \left( A_\alpha \nabla^{2(k-l)-i-|\gamma|} V_k \nabla^{\alpha-|\beta|-|\gamma|+i} u \right),$$

for some $A_\alpha \in L^2(\mathbb{R}^{2m})$. So $I_{m-1-|\beta|}$ acts on this term giving a summation of the type

$$I_{m-1-|\beta|} \left( A_\alpha \nabla^{2(k-l)-i-|\gamma|} V_k \nabla^{\alpha-|\beta|-|\gamma|+i} u \right).$$

The integrability of the function inside the potential is as follows:

$$A_\alpha \in L^2, \nabla^{2(k-l)-i-|\gamma|} V_k \in W^{2l-m+i+1-\gamma} \subset L^{\frac{2m}{2l-m+i+1-\gamma}}, \nabla^{\alpha-|\beta|-|\gamma|+i} u \in L^{\frac{2m}{m+|\alpha|-|\beta|-2l-1}}$$

and so

$$T := A_\alpha \nabla^{2(k-l)-i-|\gamma|} V_k \nabla^{\alpha-|\beta|-|\gamma|+i} u \in L^p$$

with

$$\frac{1}{p} = \frac{1}{2} + \frac{2m-(2l+i+1)+|\gamma|}{2m} + \frac{|\alpha|-|\beta|-|\gamma|+i}{2m} = \frac{3m+|\alpha|-|\beta|-2l-1}{2m}.$$

Note that since $|\alpha| - 2l \leq -m$,

$$3m + |\alpha| - |\beta| - 2l - 1 \leq 2m - 1 - |\beta|.$$
and so $T \subset L^p \subset L^{2m-1-|\beta|}$.

By Hölder’s inequality and the decay estimate in Theorem 1.1, we can verify that
\[
\sup_{x \in B_{1/2}, \text{dist}(x, \partial B) < \frac{1}{2}} r^{-q\alpha_0} \int_{B_r(x)} |T|^q \leq CM,
\]
where $q = q_\beta = \frac{2m}{2m-1-|\beta|}$. By Hölder’s inequality, this implies that $K_1 \in M^{1,\alpha_0+1}(B_{1/2})$, that is,
\[
\sup_{x \in B_{1/2}, \text{dist}(x, \partial B) < \frac{1}{2}} r^{-(1+\alpha_0)} \int_{B_r(x)} |T| \leq CM,
\]
where $1 + \alpha_0 = \alpha_0 + 2m(1 - 1/q_\beta) = \alpha_0 + 1 - |\beta|$. Hence
\[
I_{m-1-|\beta|} (T) \in L^{q_\beta} \frac{2m-1-|\beta|-\alpha_0}{m-\alpha_0} = L^{\frac{2m}{2m-1-|\beta|-\alpha_0}}.
\]
Note that as a function of $\beta$, $\frac{2m}{2m-1-|\beta|-\alpha_0}$ is decreasing and
\[
|\beta| \leq |\alpha| \leq 2l - m \leq 2(m-2) - m = m-4 \leq m-2.
\]
This implies
\[
\frac{2m}{2m-1-|\beta|} \frac{2m-1-|\beta|-\alpha_0}{m-\alpha_0} \geq q_0
\]
for all $|\beta| \leq |\alpha| \leq 2l - m$. So
\[
I_{m-1}(K_1) \approx I_{m-1-|\beta|} (T) \in L^{q_0}.
\]

Case 2: $2l \leq m$. In this case, we have two subcases:

Subcase 2.1: $2l \leq m$ and $2(k-l) - i > 2k + 1 - m$. Then, by (3.12) there holds
\[
\Delta^l A \nabla^{2(k-l)-i} V_k \nabla^i u = \sum_{\alpha} \sum_{\beta} \sum_{\gamma} \partial^{\beta} \left(V_{k,\alpha} \nabla^{2l+|\gamma|} A \nabla^{|\alpha|-|\beta|-|\gamma|+i} u\right).
\]
Thus,
\[
T := V_{k,\alpha} \nabla^{2l+|\gamma|} A \nabla^{|\alpha|-|\beta|-|\gamma|+i} u \in L^p
\]
with
\[
\frac{1}{p} = \frac{1}{2} + \frac{2l + |\gamma|}{2m} + \frac{|\alpha| - |\beta| - |\gamma| + i}{2m} = \frac{m + 2l + |\alpha| - |\beta| + i}{2m}.
\]
Since $|\alpha| \leq m - (2l + i + 1)$,
\[
m + 2l + |\alpha| - |\beta| - 1 \leq 2m - 1 - |\beta|
\]
and we are in the same situation as in Case 1, that is,
\[
T \in L^{\frac{2m}{2m-1-|\beta|} \cap M^{1,\alpha_0+1+|\beta|}}.
\]
So
\[
I_{m-1-|\beta|} (T) \in L^{\frac{2m}{2m-1-|\beta|} \frac{2m-1-|\beta|-\alpha_0}{m-\alpha_0}}.
\]
Note that
\[
|\beta| \leq m - (2l + i + 1) \leq m - i - 1 \leq m - 2.
\]
Thus we have
\[
I_{m-1}(K_1) \approx I_{m-1-|\beta|} (T) \in L^{q_0}.
\]
Subcase 2.2: \( 2l \leq m \) and \( 2(k - l) - i \leq 2k + 1 - m \). That is, \( \Delta^i A \) and \( \nabla^{2(k-l)-i} V_k \in W^{2l+i+1-m,2} \) are the usual Sobolev functions of positive exponents. In this case we have two more cases: (1) \( i \leq m \) and (2) \( i > m \).

(1). When \( i \leq m \), we have

\[
\Delta^i A \nabla^{2(k-l)-i} V_k \nabla^i u \in L^{p,2}
\]

with

\[
\frac{1}{p} = \frac{2l}{2m} + \frac{2m - (2l + i + 1)}{2m} + \frac{i}{2m} = \frac{2m - 1}{2m}.
\]

Then \( I_{m-1}(\Delta^i A \nabla^{2(k-l)-i} V_k \nabla^i u) \in L^{\frac{2m}{2m-1}, \frac{2m-1-\alpha_0}{m-\alpha_0}} \subset L^{q_0} \) holds.

(2). When \( i > m \), by (3.13) we have

\[
\Delta^i A \nabla^{2(k-l)-i} V_k \nabla^i u = \sum_{\alpha \leq i-m} \sum_{\beta} \sum_{\gamma} \partial^\beta \left( u_\alpha \nabla^{2l+|\gamma|} A \nabla^{2(k-l)-i+|\alpha|-|\beta|-|\gamma|} V_k \right).
\]

So

\[
T := u_\alpha \nabla^{2l+|\gamma|} A \nabla^{2(k-l)-i+|\alpha|-|\beta|-|\gamma|} V_k \in L^{p,2}
\]

with

\[
\frac{1}{p} = \frac{1}{2} + \frac{2l + |\gamma|}{2m} + \frac{2m - 2l - i + (|\alpha| - |\beta| - |\gamma|)}{2m} = \frac{3m - 1 - i + |\alpha| - |\beta|}{2m}.
\]

Since \( |\alpha| \leq i - m \),

\[
3m - 1 - i + |\alpha| - |\beta| \leq 2m - 1 - |\beta|
\]

and so

\[
T \in L^{\frac{2m}{2m-1-|\beta|}, M^{1, \alpha_0+1+|\beta|}}
\]

and

\[
I_{m-1-|\beta|}(T) \in L^{\frac{2m}{2m+1-|\beta|}, M^{1, \alpha_0-|\beta|}}
\]

Note that

\[
\beta \leq i - m \leq 2k - m \leq 2(m - 1) - m = m - 2,
\]

which again is similar to the Case 1. Therefore, in Subcase 2.2 we conclude that

\[
I_{m-1}(K_1) \approx I_{m-1-|\beta|}(T) \in L^{q_0}.
\]

Combining all the above estimate together, we conclude that \( I_{m-1}(K_1) \in L^{q_0} \). One can estimate similarly for the other terms in \( K \) to arrive finally at

\[
v_2 \in L^{q_0}.
\]

**Estimate of \( v_3 \).** It follows from standard elliptic regularity theory that

\[
v_3 \in W^{m,p}(\mathbb{R}^{2m}) \subset L^{2q}_p(\mathbb{R}^{2m}).
\]

In conclusion, we obtain

\[
v \in L^{q_0}(\mathbb{R}^{2m})
\]

for some \( q_0 > 2 \).

Since the polyharmonic function \( h \) is smooth in \( B_1 \), we deduce

\[
u \in W^{m,q_0}_{\text{loc}}(B_1).
\]
Note that \( p > 1 \) and \( \alpha_0 > 0 \) implies \( 2 < q_0 < \frac{2p}{2-p} \). Thus we have improved the regularity of \( u \) from \( W^{m,2} \) to \( W^{m,q_0} \).

**Iteration.** Next we use a bootstrapping argument to continue improving the regularity of \( u \). We claim that

\[
 u \in W^{m,q}_{\text{loc}} \quad \text{with} \quad q < \frac{2p}{2-p} \quad \implies \quad u \in W^{m,\frac{2mq_0}{q_0+2m} + 1 - \alpha}_{\text{loc}}.
\]  

(4.2)

This is true because if \( u \in W^{m,q}_{\text{loc}} \) with \( q < \frac{2p}{2-p} \), then

\[
v_0, v_1, v_2 \in L^{\frac{2mq_0}{q_0+2m} + 1 - \alpha}_{m-\alpha}.
\]  

(4.3)

Indeed, consider first \( v_0 \) and we know

\[
\nabla^{m+1-i} u \in W^{i-1,q_0} \subset L^{\frac{2mq_0}{q_0+2m}}
\]

and so \( T = \sum_i \nabla^i A \nabla^{m+1-i} u \in L^{2mq_0} \cap M^{1,m+1-\alpha} \). Lemma 2.8 implies

\[
v_0 \approx I_1(T) \in L^{\frac{2mq_0}{q_0+2m} + 1 - \alpha}_{m-\alpha}.
\]

For the term \( v_1 \), we have

\[
\nabla^{2m-i-j} u \in W^{m-(2m-i-j),q_0} \subset L^{\frac{2mq_0}{q_0+2m} + 1 - \alpha}_{m-\alpha}
\]

and so \( T = \nabla^j A \nabla^{2m-i-j} u \in L^{q_1} \), with \( q_1 = \frac{2mq_0}{(m-i)q_0 + 2m} \). Since \( T \in M^{1,1+\alpha}(B_1) \), Lemma 2.8 implies that

\[
v_1 \approx I_{m-i}(T) \in L^{\frac{2mq_0}{q_0+2m} + 1 - \alpha}_{m-\alpha}.
\]

Note that as a function of \( i \), \( \frac{2mq_0}{q_0+2m} + 1 - \alpha \) is decreasing, and so it attains the minimum \( \frac{2mq_0}{q_0+2m} + 1 - \alpha \) when \( i = m-1 \), and thus

\[
v_1 \subset L^{\frac{2mq_0}{q_0+2m} + 1 - \alpha}_{m-\alpha}.
\]

Finally, we shall consider \( I_{m-1}(K_1) \) case by case as in the previous study. Corresponding to Case 1 above, we have

\[
T = A_\alpha \nabla^{2(k-l)-i-|\gamma|} V_k \nabla^{|\alpha|-|\beta|-|\gamma|+i} u \in L^p
\]

with \( p = \frac{2mq_0}{2m+(2m-2l+|\alpha|-|\beta|-1)q_0} \). Since in this case, \( 2m-2l+|\alpha|-1 \leq 2m-m-1 = m-1 \), we further conclude that

\[
T \in L^p \subset L^{\frac{2mq_0}{q_0+2m} + 1 - \alpha}_{m-\alpha}.
\]

Since \( T \in M^{1,\alpha_0+1+|\beta|} \) as well, Lemma 2.8 gives

\[
I_{m-1-|\beta|}(T) \in L^{\frac{2mq_0}{q_0+2m} + 1 - |\beta|}_{m-\alpha}.
\]

Since as a function of \( \beta \), \( \frac{2mq_0}{q_0+2m} + 1 - |\beta| \) is decreasing, it attains the minimum \( \frac{2mq_0}{q_0+2m} + 1 - |\beta| \) when \( |\beta| = m-2 \). This gives

\[
I_{m-1}(K_1) \subset L^{\frac{2mq_0}{q_0+2m} + 1 - \alpha}_{m-\alpha}.
\]

Similarly, one can prove the other cases as well. For instance, in Subcase 2.1, we know

\[
\nabla^{|\alpha|-|\beta|-|\gamma|+i} u \in L^{\frac{2mq_0}{q_0+2m} + 1 - \alpha}_{m-\alpha}.
\]
and thus
\[ T = V_{k,\alpha} \nabla^{2l+|\gamma|} A \nabla^{\alpha-|\beta|-|\gamma|+i} u \in L^p \]
with \( p = \frac{2mq_0}{2m+(2|\alpha|-|\beta|+1)q_0} \). Since
\[ 2l+|\alpha|-|\beta|+i \leq 2l+m-(2l+i+1)-|\beta|+i = m-1-|\beta|, \]
\[ T \in L^{\frac{2mq_0}{2m+(m-1-|\beta|)q_0}} \cap M^{1+\alpha_0+|\beta|} \]
and so similar to the reason for Case 1, we conclude
\[ I_{m-1}(K) \in \frac{2mq_0}{2m+(m-1-|\beta|)q_0} \cap M^{1+\alpha_0+|\beta|} \]
All together leads to (4.3).

Since \( v_3 \in L^{2p} \), the claim (4.3) implies \( \nabla^m u \in L^{\frac{2mq_0}{2m+(m-1-|\beta|)q_0}} \cap M^{1+\alpha_0+|\beta|} \) (B1). That is,
\[ u \in W^{m,\frac{2mq_0}{2m+(m-1-|\beta|)q_0}} \cap M^{1+\alpha_0+|\beta|} \]
Finally, noticing that
\[ \frac{2mq_0}{q_0+2m} \frac{m+1-\alpha_0}{m-\alpha_0} < \frac{2p}{2-p} \iff q_0 < \frac{2p}{2-p} \]
and that
\[ q_0 \not\geq \frac{2p}{2-p} \implies p \frac{2mq_0}{q_0+2m} \frac{m+1-\alpha_0}{m-\alpha_0} \not\geq \frac{2p}{2-p} \]
Thus, by iterating the bootstrapping claim (4.2), we eventually find that
\[ u \in W^{m,q}_{loc} \quad \text{for all } q < \frac{2p}{2-p}. \]
The proof of Proposition 4.1 is complete. \( \square \)

4.2. \( W^{m+1,q} \)-estimate with any \( q < \frac{2mp}{2m-(m-1)p} \).

**Proposition 4.2.** Let \( u \in W^{m,2}(B_1, \mathbb{R}^n) \) be a weak solution of the inhomogeneous system (1.5) with \( f \in L^p(B_1) \) for \( p \in (1, \frac{2m}{2m-1}) \). Then \( u \in W^{m+1,q}_{loc}(B_1) \) for any \( q < \frac{2mp}{2m-(m-1)p} \).

**Proof.** Note that \( u \in W^{m,q} \) implies \( \nabla^m u \in L^{\frac{2mq}{2m-(m-1)q}} \) and so the proof of Proposition 4.1 implies
\[ \nabla^m g, v_1, v_2, v_3 \in W^{1,\frac{2mq}{2m+q}} \]
or equivalently \( u \in W^{m+1,\frac{2mq}{2m+q}} \). Note that \( \frac{2mq}{2m+q} \not\geq \frac{2mp}{2m-(m-1)p} \) when \( q \not\geq \frac{2p}{2-p} \). The proof is complete. \( \square \)

**Remark 4.3.** Once we prove that \( u \) belongs to \( W^{m,\frac{2p}{2m+(m-1)p}}_{loc} \), then the above proof implies that \( u \in W^{m,\frac{2p}{2m+(m-1)p}}_{loc} \).

The following example shows that the \((m+1)\)-th order (Sobolev) regularity is the best one can expect for the general system (1.5).

**Example 4.4** (Solutions without \( W^{m+2,p} \)-regularity). Let \( g: \mathbb{R} \to \mathbb{R} \) be a continuous function with the following properties:
- \( g \in W^{m+1,2}((-2,2)) \) but \( g \notin W^{m+2,1}((-1,1)) \);
- \( g \geq 1 \) on \((-1,1)\).
Consider the map $u : B_1 \to \mathbb{R}$, $B_1 \subset \mathbb{R}^{2m}$, defined by
$$u(x) = x_1 g(x_2).$$

Set
$$V_1(x) = x_1 \frac{g''(x_2)}{g(x_2)} \quad \text{and} \quad V(x) = (V_1(x), 0, \ldots, 0).$$

It is straightforward to verify that $V \in W^{m-1,2}(B_1)$ and
$$\Delta^m u = \Delta^{m-1}(V \cdot \nabla u) \quad \text{in} \ B_1.$$

However, the regularity of $g$ implies that $u \not\in W^{m+2,1}(B_1)$.

5. Optimal local estimates

In this section, we complete the proof of Theorem 1.2.

5.1. Some estimates on polyharmonic maps with Navier boundary condition.

We begin with a simple lemma concerning the a priori estimate of polyharmonic maps with Navier boundary condition.

**Lemma 5.1.** Let $B_1 \subset \mathbb{R}^n$ be the unit ball and $g \in W^{m-1,q}(B_1)$ for some $1 < q < 2$. Then, there exists a unique $h \in W^{m-1,q}(B_1)$ such that
$$\begin{cases}
\Delta^{m-1} h = 0 & \text{in} \ B_1, \\
\Delta^i h = \Delta^i g & \text{on} \ \partial B_1, \quad 0 \leq i \leq m-2.
\end{cases} \quad (5.1)$$

Moreover, there exists a constant $C = C(n, m, q) > 0$ such that
$$\|h\|_{W^{m-1,q}(B_1)} \leq C\|g\|_{W^{m-1,q}(B_1)}.$$ 

**Proof.** By a standard approximation argument, we may assume that $g \in C^\infty(\bar{B}_1)$. Then the existence and uniqueness of $h$ for equation $(5.1)$ can be deduced easily. The point is to deduce the a priori estimate.

We prove it by induction. When $m = 2$, this has been proved in [17]. Assume the lemma holds with $m$ replaced by $m - 1$.

Suppose now $h$ solves equation $(5.1)$. Put $h_1 = \Delta h$. Then,
$$\begin{cases}
\Delta^{m-2} h_1 = 0 & \text{in} \ B_1, \\
\Delta^i h_1 = \Delta^i \Delta g & \text{on} \ \partial B_1, \quad 0 \leq i \leq m-3.
\end{cases}$$

Since $\Delta g \in W^{m-3,q}(B_1)$, by induction, we have $h_1 \in W^{m-3,q}(B_1)$ and
$$\|h_1\|_{W^{m-3,q}(B_1)} \leq C\|\Delta g\|_{W^{m-3,q}(B_1)}.$$ 

Returning to the equation of $h$, we have
$$\begin{cases}
\Delta h = h_1 & \text{in} \ B_1, \\
h = g & \text{on} \ \partial B_1.
\end{cases}$$

Thus by the $L^p$ regularity theory of elliptic equations, we obtain $h \in W^{m-1,q}(B_1)$ together with the estimate
$$\|h\|_{W^{m-1,q}(B_1)} \leq C (\|h_1\|_{W^{m-3,q}(B_1)} + \|g\|_{W^{m-1,q}(B_1)}).$$
Proof. Suppose \( \alpha \) where

Then by Lemma 5.2, completes the proof.

Next we use the above lemma to deduce the following uniform estimate, which will be used in the proof of Theorem 1.2 later. Let \( u, A \) be the functions given in Theorem 1.2. We have proved that \( u \in W^{m+1, \frac{2m}{m+1}}_{\text{loc}} \) in the previous section. Hence \( A \Delta u \in W^{m-1, \frac{2m}{m+1}}_{\text{loc}} \).

**Lemma 5.2.** Let \( u, A \) be the functions given in Theorem 1.2. There exists a constant \( C = C(m, p) > 0 \) satisfying the following property. For any \( B_R(z) \subset B_1(0) \), let \( h \in W^{m-1,q}(B_R(z)) \), \( q = \frac{2m}{m+1} \), be the unique solution of the equation

\[
\begin{aligned}
\Delta^{m-1} h &= 0 & \text{in } B_R(z), \\
\Delta^i h &= \Delta^i (A \Delta u) & \text{on } \partial B_R(z), \quad 0 \leq i \leq m - 2.
\end{aligned}
\]

Then, we have

\[
\| \nabla^{m-2} h \|_{L^{2p/(2-p)}(B_{R/2}(z))} \leq C \left( \| u \|_{W^{m,2}(B_1)} + \| f \|_{L^p(B_1)} \right).
\]

**Proof.** Suppose \( h \) solves equation (5.2). Put \( h_R(x) = R^2 h(z + R x) \), \( A_R = A(z + R x) \), \( u_R = u(z + R x) \) for \( x \in B_1 = B_1(0) \). Then,

\[
\begin{aligned}
\Delta^{m-1} h_R &= 0 & \text{in } B_1, \\
\Delta^i h_R &= \Delta^i (A_R \Delta u_R) & \text{on } \partial B_1, \quad 0 \leq i \leq m - 2.
\end{aligned}
\]

Then by Lemma 5.1, \( h_R \in W^{m-1,q}(B_1) \) with

\[
\| h_R \|_{W^{m-1,q}(B_1)} \leq C_m \| A_R \Delta u_R \|_{W^{m-1,q}(B_1)}.
\]

Then, using the Sobolev embedding \( W^{m-1,q}(B_1) \subset W^{m-2,2}(B_1) \), we obtain a constant \( C = C(m, p) > 0 \) such that

\[
\| \nabla^{m-2} h_R \|_{L^2(B_1)} \leq C \| \nabla u_R \|_{W^{m,2}(B_1)} \leq CR^\alpha \left( \| u \|_{W^{m,2}(B_1)} + \| f \|_{L^p(B_1)} \right),
\]

where \( \alpha = 2m(1 - 1/p) \). Thus, the previous estimate together with [10, Lemma 6.2] implies that there exists \( C = C(m, p) > 0 \) such that

\[
\| \nabla^{m-2} h_R \|_{L^p(B_{R/2})} \leq C \| \nabla^{m-2} h_R \|_{L^2(B_1)} \leq CR^\alpha \left( \| u \|_{W^{m,2}(B_1)} + \| f \|_{L^p(B_1)} \right),
\]

which is equivalent to the estimate in the lemma. The proof is complete. \( \Box \)

Now, we are ready to prove Theorem 1.2.

5.2. **Proof of Theorem 1.2.** Set

\[
\bar{q} = \frac{2mp}{2m - (m - 1)p}, \quad \bar{p} = \frac{2p}{2 - p}, \quad 1 < p < \frac{2m}{2m - 1}.
\]

The idea is to establish uniform estimate for \( \| \nabla^{m+1} u \|_{L^q(B_{R/2})} \) with respect to all \( q < \bar{q} \) in terms of \( \left( \| f \|_{L^p(B_1)} + \| u \|_{L^1(B_1)} \right) \). The approach of Sharp and Topping [40] does not work. We follow the approach of [17]. The proof consists of two steps. In the first step, we shall prove \( u \in W^{m+1, q}_{\text{loc}} \). By Remark 4.3, it suffices to show \( u \in W^{m, \bar{p}}_{\text{loc}} \). In the second step, we shall derive the optimal interior estimate (1.13) via the conservation law.

For any

\[
\max \{ 2, \bar{p}/2 \} < \gamma < \bar{p},
\]
we have, for any $1 \leq i \leq m$,
\[
\nabla^i u \in W^{m-i,\gamma}(B^{2m}) \subset L^{\gamma_i}(B^{2m}), \quad \text{with} \quad \frac{1}{\gamma_i} = \frac{1}{\gamma} - \frac{m-i}{2m}.
\]

Let $\gamma' = \gamma/(\gamma - 1)$. Note that $p' < \gamma' < 2$. There holds
\[
W_0^{m,p'}(B_{1/2}) \subset L^{p'}(B_{1/2}).
\]
Thus $W_0^{m,\gamma'}(B_{1/2}) \subset L^{\pi/\gamma'}(B_{1/2})$ with
\[
\frac{2\gamma'}{\pi} = \left(\frac{1}{\gamma'} - \frac{1}{\gamma} \right)^{-1} = \left(\frac{1}{\gamma} - \frac{1}{\gamma'} \right)^{-1} > p'.
\]
There are two cases.

5.2.1. \textit{m is an even integer.} We want to estimate the norm of $\|\nabla^m u\|_{\gamma, B_{1/2}}$. By [17, Propositions A.1 and B.1], and a direct iteration, we infer that there exists $C_p > 0$ independent of $\gamma$, such that
\[
\|\nabla^m u\|_{\gamma, B_{1/2}} \leq C_p \left(\|\Delta^{m/2} u\|_{\gamma, B_1} + \|u\|_{1, B_1}\right). \tag{5.4}
\]

To estimate $\|\Delta^{m/2} u\|_{\gamma, B_1}$, we use the system for $A\Delta u$. More precisely, by Corollary 3.4, we have
\[
\Delta^{m-1}(A\Delta u) = \sum_{i=1}^{m-1} \delta^i \left(\sum_{j=m-i}^m \nabla^j A \nabla^{2m-i-j} u\right) + \delta K + Af \quad \text{in } B_1. \tag{5.5}
\]

Split $A\Delta u$ as $A\Delta u = v + h$ in $B_1$ such that
\[
\begin{cases}
\Delta^{m-1} h = 0 & \text{in } B_1, \\
\Delta^i h = \Delta^i (A\Delta u) & \text{on } \partial B_1, \quad 0 \leq i \leq m-2
\end{cases}
\]
and
\[
\begin{cases}
\Delta^{m-1} v = \Delta^{m-1} (A\Delta u) & \text{in } B_1, \\
v = \Delta^{m-2} v = \cdots = \Delta^2 v = 0 & \text{on } \partial B_1.
\end{cases}
\]

Then $v$ satisfies the system
\[
\Delta^{m-1} v = \sum_{i=1}^{m-1} \delta^i \left(\sum_{j=m-i}^m \nabla^j A \nabla^{2m-i-j} u\right) + \delta K + Af.
\]

\textbf{Step 1: a duality argument.} We first estimate $\|\Delta^{m-2} v\|_{\gamma, B_1}$. By duality, we have
\[
\|\Delta^{m-2} v\|_{\gamma, B_1} = \sup_{\varphi \in \mathcal{A}_1} \int_{B_1} (\Delta^{m-2} v) \varphi,
\]
where
\[
\mathcal{A}_1 = \{ \varphi \in C_0^\infty(B_1, \mathbb{R}^m) : \|\varphi\|_{\gamma'} \leq 1 \}.
\]

For any $\varphi \in \mathcal{A}_1$, let $\Phi \in W^{\pi/\gamma',\gamma'}(B_1)$ satisfy
\[
\begin{cases}
\Delta^{\pi/\gamma'} \Phi = \varphi & \text{in } B_1, \\
\Phi = \Delta \Phi = \cdots = \Delta^{m-1} \Phi = 0 & \text{on } \partial B_1.
\end{cases}
\]

By a similar argument as that of (5.4), there exists a constant $C_p > 0$ (independent of $\gamma$) such that
\[
\|\Phi\|_{W^{m,\gamma'}(B_1)} \leq C_p \|\varphi\|_{\gamma'} \leq C_p.
\]
Note that integration by parts gives
\[
\int_{B_1} \Delta^{m-2} v \varphi = \int_{B_1} \Delta^{m-2} v \Delta^{\frac{m}{2}} \Phi \\
= \int_{B_1} \Delta^{\frac{m}{2}} v \Delta^{m-2} \frac{\partial}{\partial \nu} \Phi + \int_{\partial B_1} \left( \Delta^{\frac{m}{2}} v \frac{\partial \Delta^{m-2} \Phi}{\partial \nu} - \frac{\partial \Delta^{m-2} v}{\partial \nu} \Delta^{m-2} \Phi \right) \\
= \int_{B_1} \Delta^{\frac{m}{2}} v \Delta^{m-2} \Phi \quad \text{since } \Delta^{\frac{m}{2}} v = \Delta^{m-2} \Phi = 0 \text{ on } \partial B_1 \\
= \int_{B_1} \Delta^{\frac{m}{2}+1} v \Delta^{m-2} \Phi \quad \text{since } \Delta^{\frac{m}{2}} v = \Delta^{m-2} \Phi = 0 \text{ on } \partial B_1 \\
= \cdots \\
= \int_{B_1} \Delta^{m-1} v \Phi \quad \text{since } \Delta^{m-2} v = \Phi = 0 \text{ on } \partial B_1.
\]
Thus, we have
\[
\sup_{\varphi \in \mathcal{A}_1} \int_{B_1} \Delta^{m-2} v \cdot \varphi \leq C \sup_{\Phi \in \mathcal{A}_2} \int_{B_1} \Delta^{m-2} v \Delta^{\frac{m}{2}} \Phi = \sup_{\Phi \in \mathcal{A}_2} \int_{B_1} \Delta^{m-1} v \Phi,
\]
where
\[
\mathcal{A}_2 = \left\{ \Phi \in W^{m,\gamma'}(B_1) : \|\Phi\|_{W^{m,\gamma'}(B_1)} \leq 1, \Phi = \Delta \Phi = \cdots = \Delta^{\frac{m}{2}-1} \Phi = 0 \text{ on } \partial B_1 \right\}.
\]

**Step 2: estimate** \( \int_{B_1} \Delta^{m-1} v \Phi \) **for** \( \Phi \in \mathcal{A}_2 \). Recall that
\[
\int_{B_1} \Delta^{m-1} v \Phi = \int_{B_1} \left\{ \sum_{i=1}^{m-1} \delta^i \left( \sum_{j=m-i}^{m} \nabla^j A \nabla^{2m-i-j} u \right) + \delta K + A f \right\} \Phi.
\]

**Part 1.** For any \( 1 \leq i \leq m-1, m-i \leq j \leq m \), we have
\[
\int_{B_1} \nabla^j A \nabla^{2m-i-j} u \nabla^i \Phi \leq \|\nabla^j A\|_{2m} \|\nabla^{2m-i-j} u\|_{\gamma_{2m-i-j}^*} \|\nabla^i \Phi\|_{\gamma_{i}^*}
\]
\[
\lesssim \epsilon \|\nabla^{2m-i-j} u\|_{\gamma_{2m-i-j}^*, B_1}.
\]
So
\[
\int_{B_1} \sum_{i=1}^{m-1} \delta^i \left( \sum_{j=m-i}^{m} \nabla^j A \nabla^{2m-i-j} u \right) \Phi \lesssim \epsilon \sum_{i=1}^{m} \|\nabla^i u\|_{\gamma_{i}^*, B_1}.
\]

**Part 2.** By Hölder’s inequality,
\[
\int_{B_1} (Af) \Phi \leq \|Af\|_p \|\Phi\|_{p', B_1}.
\]
Since \( \gamma < \hat{p} \), we have \( \gamma' > \hat{p}' \). Note that \( W_0^{m,\gamma'}(B_1) \subset L^{\hat{p}'}(B_1) \). Thus \( W_0^{m,\gamma'}(B_1) \subset L^{2\gamma'/2-\gamma}(B_1) \) with \( 2\gamma'/2-\gamma = \left( \frac{1}{\gamma'} - \frac{1}{2} \right)^{-1} = \left( \frac{1}{\gamma} - \frac{1}{2} \right)^{-1} > p' \). Here we used the assumption \( \gamma > 2 \). Thus
\[
\int_{B_1} (Af) \Phi \leq C_p \|f\|_{p, B_1}
\]
for some constant \( C_p \) independent of \( \gamma \).
Part 3. Estimates concerning $K$. We write the first term of $K$ as

$$K_1 = \sum_{k=0}^{m-1} \sum_{l=0}^{k-1} (\Delta^t A) \Delta^{k-l-1} d(V_k, du)$$

and estimate

$$\int_{B_1} \delta K_1 \Phi = - \int_{B_1} K_1 \cdot \nabla \Phi$$

as follows. Repeatedly using the Leibniz rule yields

$$\Delta^t A \Delta^{k-l-1} d(V_k, du) = \sum_{i=1}^{2(k-l)} \Delta^t A \nabla^{2(k-l)-i} V_k \nabla^i u.$$ 

Case 1: $2l > m$. Then by (3.10) there holds

$$\Delta^t A \nabla^{2(k-l)-i} V_k \nabla^i u = \sum_{|\alpha| \leq 2l-m} \sum_{0 \leq \beta \leq \alpha} \sum_{0 \leq \eta \leq \alpha - \beta} \partial^\beta \left( A_\alpha \nabla^{2(k-l)-i+|\eta|} V_k \nabla^{\alpha-|\beta|-|\eta|+i} u \right),$$

where $A_\alpha \in L^2(\mathbb{R}^{2m})$. Hence in this case, $K_1$ is a linear combination of terms like

$$\sum_{|\alpha| \leq 2l-m} \sum_{0 \leq \beta \leq \alpha} \sum_{0 \leq \eta \leq \alpha - \beta} \int_{B_{1/2}} A_\alpha \nabla^{2(k-l)-i+|\eta|} V_k \nabla^{\alpha-|\beta|-|\eta|+i} u \nabla^{\beta+1} \Phi.$$

We know that

$$\nabla^{2(k-l)-i+|\eta|} V_k \in W^{2l-m+i+1-\eta} \subset L^{2m-(\beta+1+i)\gamma}, \quad \nabla^{\alpha-|\beta|-|\eta|+i} u \in L^{\gamma_{|\alpha-|\beta|-|\eta|+i}}$$

and so

$$A_\alpha \nabla^{2(k-l)-i+|\eta|} V_k \nabla^{\alpha-|\beta|-|\eta|+i} u \in L^{q,2}$$

with

$$\frac{1}{q} = 1 + \frac{2m - (2l + i + 1) + |\eta|}{2m} + \frac{1}{\gamma} - \frac{m - (|\alpha| - |\beta| - |\eta| + i)}{2m} = 1 + \frac{1}{\gamma} - \frac{2l - |\alpha| + |\beta| + 1}{2m}.$$ 

Note that $\frac{2l - |\alpha| + |\beta| + 1}{2m} \geq \frac{m+1}{2m}$ for all $\beta, \alpha$ in the above choice and recall that $\gamma > 2$. Hence

$$0 < \frac{1}{q} < 1 + \frac{1}{\gamma} - \frac{m+1}{2m} < 1.$$

Also we have

$$\nabla^{\beta+1} \Phi \in W^{m-|\beta|-1, \gamma'}(B_1) \subset L^{2m-(m-|\beta|-1)\gamma'}$$

such that

$$\frac{1}{q} + \frac{2m - (m - |\beta| - 1)\gamma'}{2m\gamma'} = 2 - \frac{2l + m - |\alpha|}{2m} \leq 1$$

since $|\alpha| \leq 2l - m$. Furthermore, the equality attained only if $|\alpha| = 2l - m$. 

Thus

\[
\int_{B_{1/2}} A_\alpha \nabla^{2(k-l)-i+|\eta|} V_k \nabla^{\alpha-|\beta|-|\eta|+i} u \nabla^{|\beta|+1} \Phi \\
\leq \|A_\alpha\|_2 \|V_k\| \left\| \nabla^{\alpha-|\beta|-|\eta|+i} u \right\|_{L^\gamma_{\alpha-|\beta|-|\eta|+i}} \left\| \nabla^{|\beta|+1} \Phi \right\|_{L^{2m-(m-|\beta|-1)\gamma}} \\
\leq C_p \|A_\alpha\|_2 \left\| \nabla^{\alpha-|\beta|-|\eta|+i} u \right\|_{L^\gamma_{\alpha-|\beta|-|\eta|+i}} \\
\leq C_p \epsilon \left\| \nabla^{\alpha-|\beta|-|\eta|+i} u \right\|_{L^\gamma_{\alpha-|\beta|-|\eta|+i}}.
\]

Summing over all indexes we achieve

\[
\sum_{|\alpha| \leq 2l-m} \sum_{0 \leq \beta \leq \alpha} \sum_{0 \leq \eta \leq \alpha-\beta} \int_{B_{1/2}} A_\alpha \nabla^{2(k-l)-i+|\eta|} V_k \nabla^{\alpha-|\beta|-|\eta|+i} u \nabla^{|\beta|+1} \Phi \lesssim \epsilon \sum_{i=1}^m \|\nabla^i u\|_{\gamma_i, B_{1/2}}.
\]

We can estimate all the remaining terms exactly in the same way as above. Then, summarizing all the estimates gives

\[
\| \Delta_{m-2}^\gamma v \|_{\gamma, B_1} \lesssim \epsilon_m \sum_{i=1}^m \|\nabla^i u\|_{\gamma_i, B_1} + C_p \|f\|_{p, B_1}.
\tag{5.6}
\]

**Step 3: Iteration.** Now, we have

\[
\|\nabla^{m-2}(A\Delta u)\|_{L^\gamma(B_{1/2})} \lesssim \|\nabla^{m-2}v\|_{L^\gamma(B_{1/2})} + \|\nabla^{m-2} h\|_{L^\gamma(B_{1/2})} \\
\leq C \left( \|\Delta_{m-2}^\gamma v\|_{L^\gamma(B_1)} + \|u\|_{L^1(B_1)} \right) + \|\nabla^{m-2} h\|_{L^\gamma(B_{1/2})} \\
\leq C_p \epsilon_m \sum_{i=1}^m \|\nabla^i u\|_{\gamma_i, B_1} + C_p \left( \|f\|_{L^p(B_1)} + \|u\|_{L^1(B_1)} + \|\nabla^{m-2} h\|_{L^p(B_{1/2})} \right).
\tag{5.7}
\]

Notice the following interpolation inequality from [2, Chapter 5] (see also [40]):

\[
\sum_{i=1}^{m-1} \|\nabla^i u\|_{\gamma_i, B_1} \lesssim C \frac{1}{\gamma} \left( \|\nabla^m u\|_{\gamma, B_1} + \|u\|_{1, B_1} \right)
\]

for all \( \gamma > 1 \), with \( C > 0 \) independent of \( \gamma \). We obtain from the above interpolation inequality and (5.7) that

\[
\|\nabla^m u\|_{\gamma, B_{1/2}} \lesssim \epsilon_m \|\nabla^m u\|_{\gamma, B_1} + \|u\|_{1, B_1} + \|f\|_{p, B_1} + \|\nabla^{m-2} h\|_{p, B_{1/2}}.
\tag{5.8}
\]

With (5.8) at hand, the remaining step is to use a standard scaling technique as that of [40, Proof of Lemma 7.2]. Namely, we first use scaling to deduce, for any \( B_{R}(z) \subset B_1 \),

\[
\|\nabla^m u\|_{L^\gamma(B_{R}(z))} \leq C \|\nabla^m u\|_{L^\gamma(B_{R}(z))} + CR^{-m} \left( \|f\|_{L^p(B_R(z))} + \|u\|_{L^1(B_R(z))} + \|\nabla^{m-2} h\|_{L^p(B_{R/2})} \right),
\]

for \( \beta = 2mp > 0 \) (independent of \( \gamma \)). Then, we apply (5.3) to find that

\[
\|\nabla^{m-2} h\|_{L^p(B_{R/2})} \leq C \left( \|u\|_{W^{m,2}(B_1)} + \|f\|_{L^p(B_1)} \right)
\]

for all \( B_R(z) \subset B_1 \). Hence

\[
\|\nabla^m u\|_{L^\gamma(B_{R/2}(z))} \leq C \|\nabla^m u\|_{L^\gamma(B_R(z))} + CR^{-m} \left( \|u\|_{W^{m,2}(B_1)} + \|f\|_{L^p(B_1)} \right).
\]
Then, we use an iteration lemma of Simon (see e.g. [40, Lemma A.7]) to derive the following uniform estimate with respect to $\gamma$:

$$\|\nabla^m u\|_{L^\gamma(B_{1/4})} \leq C \left( \|u\|_{W^{m,2}(B_1)} + \|f\|_{L^p(B_1)} \right)$$

with a constant $C = C(p,m)$ independent of $\gamma$. Sending $\gamma \to \bar{p}$ yields $\nabla^m u \in L^\bar{p}(B_{1/4})$. Consequently, $u \in W^{m+1,\bar{p}}(B_{1/4})$. Thus we obtain the optimal regularity for $u$.

In the next step, we shall refine estimate (5.9) to obtain the following quantitative estimate:

$$\|u\|_{W^{m+1,\bar{p}}(B_{1/2})} \leq C \left( \|f\|_{L^p(B_1)} + \|u\|_{L^1(B_1)} \right).$$

For this, we use an interpolation argument.

First of all, note that by the conservation law (5.5),

$$\Delta^{m-1}(A \Delta u) = \sum_{i=1}^{m-1} \delta^i \left( \sum_{j=m-i}^m \nabla^j A \nabla^{2m-i-j} u \right) + \delta K + Af.$$

We may write the right-hand side of the above equation as the form $\sum_{i=0}^{m-1} \delta^i f_i$ so as to apply the regularity Lemma A.2, where $f_i \in L^{p_i}(B_1)$ are certain vector valued functions with $p_i = \frac{2mp}{2m-2mp}$ for all $0 \leq i \leq m-1$. Let us consider for instance, the term $\sum_{j=m-i}^m \nabla^j A \nabla^{2m-i-j} u$. Note that $\nabla^j A \in L^\frac{2m}{s}$ and $\nabla^{2m-i-j} u \in W^{i+j-m,\bar{p}} \subseteq L^{\frac{2m}{2m-(i+j-m)p}}$. Thus $\nabla^j A \nabla^{2m-i-j} u \in L^s$ with

$$\frac{1}{s} = \frac{j}{2m} + \frac{2m - (i + j - m)\bar{p}}{2m \bar{p}} = \frac{2m - ip}{2mp} = \frac{1}{p_i}.$$

Similarly, we can verify the terms inside $\delta K$ as in the previous calculation. Furthermore, direct computation shows

$$\|f_i\|_{L^{p_i}} \lesssim \|u\|_{W^{m,\bar{p}}}.$$

Combining the above estimate with Lemma A.2 from the appendix, we conclude

$$\|\nabla^{m+1} u\|_{L^{\bar{p}}(B_{1/2})} \lesssim \|u\|_{W^{m,\bar{p}}(B_1)} + \|u\|_{L^1(B_1)},$$

By standard interpolation and the Sobolev embedding, we have

$$\|u\|_{W^{m,\bar{p}}(B_1)} \leq \epsilon \|\nabla^{m+1} u\|_{L^{\bar{p}}(B_1)} + C \|u\|_{L^1(B_1)}.$$ 

Combining all these estimates, we thus conclude that

$$\|\nabla^{m+1} u\|_{L^{\bar{p}}(B_{1/2})} \lesssim \epsilon \|\nabla^{m+1} u\|_{L^{\bar{p}}(B_1)} + C \left( \|f\|_{L^p(B_1)} + \|u\|_{L^1(B_1)} \right).$$

From (5.11), a scaling argument as that of (5.9) gives (5.10).

5.2.2. $m$ is an odd integer. In the case, $u \in W^{m-1,\eta}_{loc}$ for any $\eta < \bar{r} = \frac{2mp}{2m-(m+1)p}$ and $m - 1$ is an even integer. Furthermore, we have

$$\|\nabla^{m-1} u\|_{q,B_{1/2}} \approx \|\Delta^{m-3\gamma}(A \Delta u)\|_{q,B_{1/2}},$$

In this case, we may repeat exactly what we have done in Case I. The only difference is that instead of first showing that $u \in W^{m,\bar{p}}_{loc}$, we first show that $u \in W^{m-1,\bar{r}}_{loc}$ and

$$\|u\|_{W^{m-1,\bar{r}}(B_{1/2})} \leq C \left( \|f\|_{L^p(B_1)} + \|u\|_{W^{m-1,2}(B_1)} \right).$$
Finally, as in the even case dealt above, combining \((5.12)\), a standard scaling argument, together with an iteration lemma of Simon, gives the desired estimate \((1.13)\). The proof is thus complete.

**Proof of Corollary 1.4.** By the previous proof, we know there exists a constant \(C = C(p, m) > 0\) such that
\[
\sum_{i=1}^{m} \| \nabla^i u \|_{L^{\bar{q}_i}(B_{R})} \leq C(p, m) \| u \|_{L^{2}(B_1)},
\]
where \(\bar{q}_i = \frac{2mp}{2m - (2m - 1)p} \), \(i = 1, \ldots, m\). Using a simple scaling, we then deduce
\[
\sum_{i=1}^{m} \| \nabla^i u \|_{L^{\bar{q}_i}(B_{R^n})} \leq C(p, m) R^{m \left(\frac{2m - 2}{p} - 2\right)} \| u \|_{W^{m,2}(\mathbb{R}^{2m})}.
\]
(5.13)

Note that \(\frac{2m}{2p} - 2 < 0\). Sending \(R \to \infty\) gives \(\nabla u = 0\) and so \(u\) is a constant. Since \(u \in L^2(\mathbb{R}^{2m})\), \(u \equiv 0\) in \(\mathbb{R}^{2m}\).

**Appendix A. Higher order \(L^p\)-theory**

We write a simple result for higher order \(L^p\) regularity theory that’s used in the proof of Theorem 1.2. It is a generalization of the second order equation \(-\Delta u = \text{div} f_1 + f_2\).

**Lemma A.1.** Suppose \(n \geq 2m\), \(1 < p < n/m\) and \(u \in W^{m,p}(\mathbb{R}^n)\) solves equation
\[
\Delta^m u = \sum_{i=0}^{m-1} \delta^i f_i \quad \text{in } \mathbb{R}^n.
\]
where \(f_i \in L^{p_i}(\mathbb{R}^n)\) with \(p_i = \frac{np}{n - mp} \) for all \(m - 1 \geq i \geq 0\). Then \(u \in W^{m+1, \frac{np}{n - (m-1)p}}(\mathbb{R}^n)\) and
\[
\| u \|_{W^{m+1, \frac{np}{n - (m-1)p}}(\mathbb{R}^n)} \leq C_{m,n,p} \sum_{i=0}^{m-1} \| f_i \|_{L^{p_i}(\mathbb{R}^n)}.
\]

**Proof.** Let \(I_{2m}\) be the fundamental solution of \(\Delta^m\) in \(\mathbb{R}^n\). We have
\[
u = I_{2m} \left( \sum_{i=0}^{m-1} \delta^i f_i \right) \approx \sum_{i=0}^{m-1} I_{2m-i}(f_i).
\]
Then
\[
I_{2m-i}(f_i) \in W^{2m-i,p_i}(\mathbb{R}^n) \subset W^{m+1,p_{m-1}}(\mathbb{R}^n)
\]
for each \(0 \leq i \leq m - 1\). Thus \(u \in W^{m+1,p_{m-1}}(\mathbb{R}^n)\). Moreover, for each \(j \leq m + 1\), there holds
\[
\nabla^j u \approx \sum_{i=0}^{m-1} I_{2m-j}(f_i).
\]
The Riesz potential theory implies that for each \(0 \leq i \leq m-1\), \(I_{2m-i-j}(f_i) \in L^q(\mathbb{R}^n)\) with

\[
\frac{1}{q_j} = \frac{1}{p_i} - \frac{2m-i-j}{n} = \frac{1}{p} - \frac{2m-j}{np},
\]

and

\[
\|\nabla^j u\|_{L^{\frac{np}{n-(2m-j)p}}(\mathbb{R}^n)} \leq C(n, m, p) \sum_{i=0}^{m-1} \|f_i\|_{L^{p_i}(\mathbb{R}^n)}.
\]

Therefore,

\[
\sum_{j=1}^{m+1} \|\nabla^j u\|_{L^{\frac{np}{n-(2m-j)p}}(\mathbb{R}^n)} \leq C(n, m, p) \sum_{i=0}^{m-1} \|f_i\|_{L^{p_i}(\mathbb{R}^n)}.
\]

This completes the proof. \(\Box\)

Consequently, we can deduce the following local estimate via a cut-off argument.

**Lemma A.2.** Suppose \(n \geq 2m\), \(1 < p < n/m\) and \(u \in W^{m,p}(\mathbb{R}^n)\) solves equation

\[
\Delta^m u = \sum_{i=0}^{m-1} \delta^i f_i \quad \text{in } B_1.
\]

where \(f_i \in L^{p_i}(B_1)\) with \(p_i = \frac{np}{n-2m} \text{ for all } m-1 \geq i \geq 0\). Then

\[
u \in W^{m+1, \frac{np}{n-(m-1)p}}_{\text{loc}}(B_1)
\]

and

\[
\|u\|_{W^{m+1, \frac{np}{n-(m-1)p}}_{\text{loc}}(B_{1/2})} \leq C_{m,n,p} \left( \sum_{i=0}^{m-1} \|f_i\|_{L^{p_i}(B_1)} + \|u\|_{L^p(B_1)} \right).
\]

**Appendix B. Proof of equation (3.3)**

**Proof.** Using induction method. For \(m = 2\), the claim holds by a direct computation:

\[
\delta \Delta (Adu) = \delta (A \Delta du + 2 \nabla A \cdot \nabla du + \Delta Adu) = \delta (A \Delta du) + \delta (A \nabla^2 u + \nabla^2 A \nabla u).
\]

Suppose the claim for \(m\) holds. Then, for \(m+1\) and \(u, A \in W^{m+1,2}\), we have by induction

\[
\delta \Delta^m (Adu) = \Delta \left( \delta (A \Delta^{m-1} du) + \sum_{i=1}^{m-1} \sum_{j=m-i}^m \delta^i \left( \nabla^j A \nabla^{2m-i-j} u \right) \right)
\]

\[
= \delta \Delta (A \Delta^{m-1} du) + \sum_{i=1}^{m-1} \sum_{j=m-i}^m \delta^i \Delta \left( \nabla^j A \nabla^{2m-i-j} u \right).
\]
For the last term, we have
\[
\begin{align*}
\delta^i \Delta (\nabla^j A \nabla^{2m-i-j} u) &= \delta^i (\nabla^j A \nabla^{2m-i-j} u + \nabla^j A \nabla^{2m-i-j} u + \nabla^j A \Delta \nabla^{2m-i-j} u) \\
&= \delta^i (\nabla^j A \nabla^{2m-i-j} u) + \delta^i (\nabla^j A \nabla^{2m-i-j} u) + \delta^i (\nabla^j A \nabla^{2m-i-j} u) \\
&= \delta^i+1 (\nabla^j A \nabla^{2m-i-j} u) - \delta^i (\nabla^j A \nabla^{2m-i-j} u) \\
&= \delta^i (\nabla^j A \nabla^{2m-i-j} u) + \delta^i (\nabla^j A \nabla^{2m-i-j} u) \\
&= \delta^i+1 (\nabla^j A \nabla^{2(m+1)-(i+1)-(j+1)} u + \nabla^j A \nabla^{2(m+1)-(i+1)-(j+1)} u) \\
&= \delta^i+1 (\nabla^j A \nabla^{2(m+1)-(i+1)+(j+1)} u).
\end{align*}
\]

This gives
\[
\sum_{i=1}^{m-1} \sum_{j=m-i}^{m} \delta^i \Delta (\nabla^j A \nabla^{2m-i-j} u) = \sum_{i=1}^{m+1} \sum_{j=m+1-i}^{m} \delta^i (\nabla^j A \nabla^{2(m+1)-(i+1)-(j+1)} u).
\]

For the term \(\delta \Delta (A \Delta^{m-1} du)\), it is much easier:
\[
\delta \Delta (A \Delta^{m-1} du) = \delta (A \Delta^{m} du) + \delta (\nabla A A \Delta^{m-1} \nabla^2 u) + \delta (\Delta A \Delta^{m-1} du).
\]

The last two terms can be easily disposed by noting that \(\Delta^{m-1} = \delta^{m-1} \nabla^{m-1}\). Transfer \(\delta^{m-1}\) from \(u\) to \(A\) gives
\[
\delta (\nabla A \Delta^{m-1} \nabla^2 u) + \delta (\Delta A \Delta^{m-1} du) = \sum_{i=1}^{m} \delta^i (\nabla^{m+1-i} A \nabla^{m+1} u + \nabla^{m+2-i} u \nabla^m u).
\]

Hence, in summary, we have
\[
\delta \Delta^m (A du) = \delta (A \Delta^m du) + \sum_{i=1}^{m+1} \sum_{j=m+1-i}^{m+1} \delta^i (\nabla^j A \nabla^{2(m+1)-(i+1)-(j+1)} u).
\]

The proof is complete.

\[\square\]

References

[1] D.R. Adams, A note on Riesz potentials. Duke Math. J. 42 (1975), no. 4, 765-778.
[2] R.C. Adams and J.F. Fournier, Sobolev spaces. Second edition. Pure and Applied Mathematics (Amsterdam), 140. Elsevier/Academic Press, Amsterdam, 2003.
[3] S.-Y.A. Chang, L. Wang and P.C. Yang, A regularity theory of biharmonic maps. Commun. Pure Appl. Math. 52(9) (1999), 1113-1137.
[4] F.L. de Longueville, Regularität der Lösungen von Systemen (2m)-ter Ordnung vom polyharmonischen Typ in kritischer Dimension. Dissertation Universität Duisburg-Essen 2018 (see https://d-nb.info/1191692124/34).
[5] F.L. de Longueville and A. Gastel, Conservation laws for even order systems of polyharmonic map type. Preprint 2019.
[6] A. Deruelle and T. Lamm, Existence of expanders of the harmonic map flow, to appear in Ann. Sci. Ec. Norm. Sup. (4), 2020.
[7] R. DeVore and K. Scherer, Interpolation of linear operators on Sobolev spaces. Ann. of Math. (2) 109 (1979), no. 3, 583-599.
[8] J. Eells and J.H. Sampson, Harmonic mappings of Riemannian manifolds. Amer. J. Math. 86 (1964), 109-160.

[9] A. Gastel, The extrinsic polyharmonic map heat flow in the critical dimension. Adv. Geom. 6 (2006), no. 4, 501-521.

[10] A. Gastel and C. Scheven, Regularity of polyharmonic maps in the critical dimension. Comm. Anal. Geom. 17 (2009), no. 2, 185-226.

[11] M. Giaquinta, Multiple integrals in the calculus of variations and nonlinear elliptic systems. Annals of Mathematics Studies, 105. Princeton University Press, Princeton, NJ, 1983.

[12] D. Gilbarg and N. Trudinger, Elliptic partial differential equations of second order. Reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001.

[13] L. Grafakos, Classical Fourier analysis. Second edition. Graduate Texts in Mathematics, 249. Springer, New York, 2008.

[14] P. Goldstein, P. Strzelecki and A. Zatorska-Goldstein, On polyharmonic maps into spheres in the critical dimension. Ann. Inst. H. Poincaré Anal. Non Linéaire 26 (2009), 1387-1405.

[15] C.-Y. Guo and C.-L. Xiang, Regularity of solutions for a fourth order linear system via conservation law. J. Lond. Math. Soc. (2) 101 (2020), no. 3, 907-922.

[16] C.-Y. Guo and C.-L. Xiang, Regularity of weak solutions to higher order elliptic systems in critical dimensions. Trans. Amer. Math. Soc. 374 (2021), no. 5, 3579-3602.

[17] C.-Y. Guo, C.-L. Xiang and G.-F. Zheng, The Lamm-Riviere system I: $L^p$ regularity theory. To appear in Cal. Var. PDEs, 2021.

[18] F. Hélein, Harmonic maps, conservation laws and moving frames. Cambridge Tracts in Mathematics, 150. Cambridge University Press, Cambridge, 2002.

[19] S. Hildebrandt, Nonlinear elliptic systems and harmonic mappings. Proceedings of the 1980 Beijing Symposium on Differential Geometry and Differential Equations, vol 1,2,3 (Beijing, 1980), 481-615, Science Press, Beijing, 1982.

[20] J. Hineman, T. Huang and C.-Y. Wang, Regularity and uniqueness of a class of biharmonic map heat flows. Calc. Var. Partial Differential Equations 50 (2014), no. 3-4, 491-524.

[21] P. Hornung and R. Moser, Energy identity for intrinsically biharmonic maps in four dimensions. Anal. PDE 5 (2012), no. 1, 61-80.

[22] J. Hörter and T. Lamm, Conservation laws for even order elliptic systems in the critical dimensions - a new approach. To appear in Calc. Var. Partial Differential Equations, 2021.

[23] T. Lamm, Biharmonic map heat flow into manifolds of nonpositive curvature. Calc. Var. Partial Differential Equations 22 (2005), 421-445.

[24] T. Lamm and T. Rivière, Conservation laws for fourth order systems in four dimensions. Comm. Partial Differential Equations 33 (2008), 245-262.

[25] T. Lamm and B. Sharp, Global estimates and energy identities for elliptic systems with antisymmetric potentials. Comm. Partial Differential Equations 41 (2016), no. 4, 579-608.

[26] T. Lamm and C. Wang, Boundary regularity for polyharmonic maps in the critical dimension. Adv. Calc. Var. 2 (2009), 1-16.

[27] P. Laurain and T. Rivière, Energy quantization for biharmonic maps. Adv. Calc. Var. 6 (2013), no. 2, 191-216.

[28] P. Laurain and T. Rivière, Angular energy quantization for linear elliptic systems with antisymmetric potentials and applications. Anal. PDE 7 (2014), no. 1, 1-41.

[29] F.-H Lin and T. Rivière, Energy quantization for harmonic maps. Duke Math. J. 111 (2002), no. 1, 177-193.

[30] X.-G. Liu, Partial regularity for weak heat flows into a general compact Riemannian manifold. Arch. Ration. Mech. Anal. 168 (2003), no. 2, 131-163.

[31] C.B. Morrey, The problem of plateau on a Riemannian manifold. Ann. Math. (2) 49 (1948), 807-851.

[32] R. Moser, An $L^p$ regularity theory for harmonic maps. Trans. Amer. Math. Soc. 367 (2015), no. 1, 1-30.

[33] R. O’Neil, Convolution operators and $L(p, q)$ spaces. Duke Math. J. 30 (1963) 129-142.
[34] T. Rivièrè, Conservation laws for conformally invariant variational problems. Invent. Math. 168 (2007), 1-22.
[35] T. Rivièrè, The role of integrability by compensation in conformal geometric analysis. Analytic aspects of problems in Riemannian geometry: elliptic PDEs, solitons and computer imaging, 93-127, Sémin. Congr., 22, Soc. Math. France, Paris, 2011.
[36] T. Rivièrè, Conformally invariant variational problems. Lecture notes at ETH Zurich, available at https://people.math.ethz.ch/ riviere/lecture-notes, 2012.
[37] M. Rupflin, An improved uniqueness result for the harmonic map flow in two dimensions. Calc. Var. Partial Differential Equations 33 (2008), no. 3, 329-341.
[38] J. Sacks and K. Uhlenbeck, The existence of minimal immersions of 2-spheres. Ann. of Math. (2) 113 (1981), no. 1, 1-24.
[39] B. Sharp, Higher integrability for solutions to a system of critical elliptic PDE. Methods Appl. Anal. 21 (2014), no. 2, 221-240.
[40] B. Sharp and P. Topping, Decay estimates for Rivièrè’s equation, with applications to regularity and compactness. Trans. Amer. Math. Soc. 365 (2013), no. 5, 2317-2339.
[41] E. Stein and G. Weiss, Introduction to Fourier analysis on Euclidean spaces. Princeton Math. Ser., 32, Princeton University Press, 1971.
[42] M. Struwe, On the evolution of harmonic maps of Riemannian surfaces. Comm. Math. Helv. 60 (1985), 558-581.
[43] M. Struwe, On the evolution of Harmonic maps in high dimensions. J. Differential Geom. 28 (1988), 485-502.
[44] M. Struwe, Partial regularity for biharmonic maps, revisited. Calc. Var. Partial Differential Equations 33 (2008), 249-262.
[45] P. Strzelecki and A. Zatorska-Goldstein, On a nonlinear fourth order elliptic system with critical growth in first order derivatives. Adv. Calc. Var. 1 (2008), no. 2, 205-222.
[46] L. Tartar, Imbedding theorems of Sobolev spaces into Lorentz spaces. Boll. Unione Mat. Ital., Sez. B, Artic. Ric. Mat. (8) 1 (1998), 479-500.
[47] C.Y. Wang, Biharmonic maps from $\mathbb{R}^4$ into a Riemannian manifold. Math. Z. 247 (2004), 65-87.
[48] C.Y. Wang, Stationary biharmonic maps from $\mathbb{R}^m$ into a Riemannian manifold. Comm. Pure Appl. Math. 57 (2004), 419-444.
[49] C.Y. Wang, Well-posedness for the heat flow of biharmonic maps with rough initial data. J. Geom. Anal. 22 (2012), no. 1, 223-243.
[50] C.Y. Wang and S.Z. Zhang, Energy identity of approximate biharmonic maps to Riemannian manifolds and its application. J. Funct. Anal. 263 (2012), no. 4, 960-987.
[51] W.P. Ziemer, Weakly differentiable functions. Graduate Texts in Mathematics, 120. Springer-Verlag, New York, 1989.

(Chang-Yu Guo) Research Center for Mathematics and Interdisciplinary Sciences, Shandong University 266237, Qingdao, P. R. China and Institute of Mathematics, École Polytechnique Fédérale de Lausanne (EPFL), Station 8, CH-1015 Lausanne, Switzerland
Email address: changyu.guo@sdu.edu.cn

(Chang-Lin Xiang) Three Gorges Mathematical Research Center, China Three Gorges University, 443002, Yichang, P. R. China, and School of Information and Mathematics, Yangtze University, 434023, Jingzhou, P. R. China
Email address: changlin.xiang@ctgu.edu.cn

(Gao-Feng Zheng) School of Mathematics and Statistics, Central China Normal University, Wuhan 430079, P. R. China
Email address: gzfeng@math.ccnu.edu.cn