Coxeter group actions on Saalschützian $4F_3(1)$ series and very-well-poised $7F_6(1)$ series

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Abstract

In this paper we consider a function $L(\vec{x}) = L(a, b, c, d; e; f, g)$, which can be written as a linear combination of two Saalschützian $4F_3(1)$ hypergeometric series or as a very-well-poised $7F_6(1)$ hypergeometric series. We explore two-term and three-term relations satisfied by the $L$ function and put them in the framework of group theory.

We prove a fundamental two-term relation satisfied by the $L$ function and show that this relation implies that the Coxeter group $W(D_5)$, which has 1920 elements, is an invariance group for $L(\vec{x})$. The invariance relations for $L(\vec{x})$ are classified into six types based on a double coset decomposition of the invariance group. The fundamental two-term relation is shown to generalize classical results about hypergeometric series. We derive Thomae’s identity for $3F_2(1)$ series, Bailey’s identity for terminating Saalschützian $4F_3(1)$ series, and Barnes’ second lemma as consequences.

We further explore three-term relations satisfied by $L(a, b, c, d; e; f, g)$. The group that governs the three-term relations is shown to be isomorphic to the Coxeter group $W(D_6)$, which has 23040 elements. Based on the right cosets of $W(D_5)$ in $W(D_6)$, we demonstrate the existence of 220 three-term relations satisfied by the $L$ function that fall into two families according to the notion of $L$-coherence.

1 Introduction

Hypergeometric series were first introduced by Gauss [11], who studied $2F_1$ series. Generalized hypergeometric series of type $A_F B$, where $A$ and $B$ are positive integers, were studied in the late nineteenth and early twentieth century by Thomae [27], Barnes [3, 4], Ramanujan (see [14]), Whipple [30, 31, 32], Bailey [1, 2], and others.

There has been a renewed interest in hypergeometric series over the last twenty-five years. Relations among hypergeometric and basic hypergeometric series were put into a group-theoretic framework in papers by Beyer et al. [5], Srinivasa Rao et al. [21], Formichella et al. [10], Van der Jeugt and Srinivasa

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Rao [29], Livens and Van der Jeugt [17, 18]. Other papers include Groenevelt [12], van de Bult et al. [6], and Krattenthaler and Rivoal [16].

Hypergeometric series have also appeared in recent papers by Bump [7], Stade [22, 23, 24, 25], and Stade and Taggart [26] with applications in the theory of automorphic functions. Other recent works, with applications in physics, were written by Drake [8], Grozin [13], and Raynal [20].

The goal of this paper is to describe two-term and three-term relations among Saalschützian 4F3(1) hypergeometric series and put them in the framework of group theory. We examine a function $L(a, b, c, d; e; f, g)$ (see (2.2) for the definition) which is a linear combination of two Saalschützian 4F3(1) series. This particular linear combination of two Saalschützian 4F3(1) series appears in [23] in the evaluation of the Mellin transform of a spherical principal series $GL(4, \mathbb{R})$ Whittaker function.

In Section 3 we derive a fundamental two-term relation (see (3.3)) satisfied by $L(a, b, c, d; e; f, g)$. The fundamental two-term relation (3.3) is derived through a Barnes integral representation of $L(a, b, c, d; e; f, g)$ and generalizes both Thomae’s identity (see [2, p. 14]) and Bailey’s identity (see [31, Eq. (10.11)] or [2, p. 56]) in the sense that the latter two identities can be obtained as limiting cases of our fundamental two-term relation (see Section 5).

In Section 4 we show that the two-term relation (3.3) combined with the trivial invariances of $L(a, b, c, d; e; f, g)$ under permutations of $a, b, c, d$ and interchanging $f, g$ implies that the function $L(a, b, c, d; e; f, g)$ has an invariance group $G_L$ isomorphic to the Coxeter group $W(D_5)$, which is of order 1920. (See [15] for general information on Coxeter groups.) The invariance group $G_L$ is given as a matrix group of transformations of the affine hyperplane

$$V = \{(a, b, c, d, e, f, g)^T \in \mathbb{C}^7 : e + f + g - a - b - c - d = 1\}.$$  \hspace{1cm} (1.1)

The 1920 invariances of the $L$ function that follow from the invariance group $G_L$ are classified into six types based on a double coset decomposition of $G_L$ with respect to its subgroup $\Sigma$ consisting of all the permutation matrices in $G_L$.

To the best of the author’s knowledge, using such a double coset decomposition is a new way of describing all the relations induced by an invariance group and does not appear explicitly in the literature before.

Some consequences of the fundamental two-term relation (3.3) are shown in Section 5. In particular, as already mentioned, we show that Thomae’s and Bailey’s identities follow as limiting cases of (3.3). We also show that Barnes’ second lemma (see [1] or [2, p. 42]) follows as a special case of (3.3) when we take $d = g$.

In Section 6 we lay the group-theoretic foundation for the three-term relations satisfied by $L(a, b, c, d; e; f, g)$. The three-term relations are governed by the right coset space $G_L \backslash M_L$ described in that section. We also introduce the notion of $L$-coherence (see Definition 6.5), which will be used in the classification of the three-term relations in the following section.

Section 7 describes the three-term relations satisfied by $L(a, b, c, d; e; f, g)$. We show that for every $\sigma_1, \sigma_2, \sigma_3 \in M_L$ such that $\sigma_1, \sigma_2, \sigma_3$ are in different right
cosets of $G_L$ in $M_L$, there exists a three-term relation involving the functions $L(\sigma_1 \vec{x}), L(\sigma_2 \vec{x})$, and $L(\sigma_3 \vec{x})$. Thus we demonstrate the existence of 220 three-term relations. The 220 three-term relations are shown to fall into two families based on the notion of $L$-coherence. We explicitly find a three-term relation of each family in (7.1) and (7.7), and then show that every other three-term relation is obtained from one of those two through a change of variable of the form $\vec{x} \rightarrow \mu \vec{x}$ applied to all terms and coefficients.

Versions of the $L$ function (in terms of very-well-poised $7F_6(1)$ series, see (2.3)) were examined in the past by Bailey [1], Whipple [32], and Raynal [20]. The $L$ function appears as a Wilson function (a nonpolynomial extension of the Wilson polynomial) in [14]. Van de Bult et al. [6] examine generalizations to elliptic, hyperbolic, and trigonometric hypergeometric functions.

A basic hypergeometric series analog of the $L$ function (in terms of $8\varphi_7$ series) was studied by Van der Jeugt and Srinivasa Rao [29] and by Livens and Van der Jeugt [17]. The authors establish an invariance group isomorphic to $W(D_5)$ for the $8\varphi_7$ series, but do not classify all two-term relations, or consider how they could imply results about lower-order series. They also do not observe the idea of $L$-coherence and how it divides the three-term relations into two families.

Very recently Formichella et al. [10] explored a function $K(a; b, c, d; e, f, g)$ which is a different linear combination of two Saalschützian $4F_3(1)$ series from the function $L(a, b, c, d; e, f, g)$. The particular linear combination of $4F_3(1)$ series studied by Formichella et al. appears in the theory of archimedean zeta integrals for automorphic L functions (see [25][26]). The function $K(a; b, c, d; e, f, g)$ behaves very differently from $L(a, b, c, d; e, f, g)$. Formichella et al. obtain in [10] a two-term relation satisfied by $K(a; b, c, d; e, f, g)$ and show that their two-term relation implies that the symmetric group $S_6$ is an invariance group for $K(a; b, c, d; e, f, g)$. In addition, the existence of 4960 three-term relations satisfied by the $K$ function is demonstrated and the 4960 three-term relations are classified into five families based on the notion of Hamming type. In a future work by the author of the present paper and by Green and Stade, the connection between the $K$ and the $L$ functions will be studied.

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2 Hypergeometric series and Barnes integrals

The hypergeometric series of type $p+1F_p$ is the power series in the complex variable $z$ defined by

$$p+1F_p \left[ \begin{array}{c} a_1, a_2, \ldots, a_{p+1}; \\ b_1, b_2, \ldots, b_p; \end{array} \right] z = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n \cdots (a_{p+1})_n}{n!(b_1)_n(b_2)_n \cdots (b_p)_n} z^n, \quad (2.1)$$

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where \( p \) is a positive integer, the numerator parameters \( a_1, a_2, \ldots, a_{p+1} \) and
the denominator parameters \( b_1, b_2, \ldots, b_p \) are complex numbers, and the rising
factorial \((a)_n\) is given by

\[
(a)_n = \begin{cases} 
   a(a + 1) \cdots (a + n - 1) = \frac{\Gamma(a + n)}{\Gamma(a)}, & n > 0, \\
   1, & n = 0.
\end{cases}
\]

The series in (2.1) converges absolutely if \(|z| < 1\). When \(|z| = 1\), the series
converges absolutely if \(\text{Re}(\sum_{i=1}^{p} b_i - \sum_{i=1}^{p+1} a_i) > 0\) (see [2] p. 8). We assume
that no denominator parameter is a negative integer or zero. If a numerator
parameter is a negative integer or zero, the series has only finitely many nonzero
terms and is said to terminate.

When \( z = 1 \), the series is said to be of unit argument and of type \( p+1 \) \( F_6 \) (1).
If \( \sum_{i=1}^{p} b_i = \sum_{i=1}^{p+1} a_i + 1 \), the series is called Saalschützian. If \( 1 + a_1 = b_1 + a_2 =
\ldots = b_p + a_{p+1} \), the series is called well-poised. A well-poised series that satisfies
\( p_2 = 1 + \frac{1}{2} a_1 \) is called very-well-poised.

Our main object of study in this paper will be the function \( L(a, b, c, d; e; f, g) \) defined by

\[
L(a, b, c, d; e; f, g) = \frac{4F3 \left[ \begin{array} {c} a, b, c, d; \\ e, f, g \end{array} \right]}{\sin \pi e \Gamma(e) \Gamma(f) \Gamma(g) \Gamma(1 + a - e) \Gamma(1 + b - e) \Gamma(1 + c - e) \Gamma(1 + d - e)},
\]

where \( a, b, c, d, e, f, g \in \mathbb{C} \) satisfy \( e + f + g - a - b - c - d = 1 \).

The function \( L(a, b, c, d; e; f, g) \) is a linear combination of two Saalschützian
\( 4F3 \) (1) series. Other notations we will use for \( L(a, b, c, d; e; f, g) \) are \( L \left[ \begin{array} {c} a, b, c, d; \\ e; f, g \end{array} \right] \)
and \( L(\vec{x}) \), where we will always have \( \vec{x} = (a, b, c, d, e, f, g)^T \in V \) (see (1.1)).

It should be noted that by [2] Eq. (7.5.3)], the \( L \) function can be expressed as a very-well-poised \( \gamma F_6 \) (1) series:

\[
L(a, b, c, d; e; f, g) = \frac{\Gamma(1 + d + g - e)}{\pi \Gamma(g) \Gamma(1 + g - e) \Gamma(f - d) \Gamma(1 + a + d - e) \Gamma(1 + b + d - e) \Gamma(1 + c + d - e)} \cdot \gamma F_6 \left[ \begin{array} {c} d + g - e, 1 + \frac{1}{2}(d + g - e), g - a, g - b, g - c, d, 1 + d - e; \\ 1/2(d + g - e), 1 + a + d - e, 1 + b + d - e, 1 + c + d - e, 1 + g - e, g; \end{array} \right],
\]

provided that \( \text{Re}(f - d) > 0 \). Therefore our results on the \( L \) function can also be interpreted in terms of the very-well-poised \( \gamma F_6 \) (1) series given in (2.3).
Fundamental to the derivation of a nontrivial two-term relation for the $L$ function will be the notion of a Barnes integral, which is a contour integral of the form

$$\int_{\text{path}} \prod_{i=1}^{n} \Gamma(\epsilon_i(a_i + t)) \prod_{j=1}^{m} \Gamma(\epsilon_j(b_j - t)) \, dt,$$  \hspace{1cm} (2.4)

where $n, m \in \mathbb{Z}^+$; $\epsilon_i, \epsilon_j = \pm 1$; and $a_i, b_j, t \in \mathbb{C}$. The path of integration is the imaginary axis, indented if necessary, so that any poles of $\prod_{i=1}^{n} \Gamma(\epsilon_i(a_i + t))$ are to the left of the contour and any poles of $\prod_{j=1}^{m} \Gamma(\epsilon_j(b_j - t))$ are to the right of the contour. This path of integration always exists, provided that, for $1 \leq i \leq n$ and $1 \leq j \leq m$, we have $a_i + b_j \notin \{0, -1, -2, \ldots\}$ whenever $\epsilon_i = \epsilon_j = 1$.

From now on, when we write an integral of the form (2.4), we will always mean a Barnes integral with a path of integration as just described.

A Barnes integral can often be evaluated in terms of hypergeometric series using the Residue Theorem, provided that we can establish the necessary convergence arguments. This is the approach we take in the next section. We will make use of the extension of Stirling’s formula to the complex numbers (see [28, Section 4.42] or [33, Section 13.6]):

$$\Gamma(a + z) = \frac{\sqrt{2\pi}z^{a+z-1/2}e^{-z}(1 + O(1/|z|))}{n!} \text{ uniformly as } |z| \to \infty,$$  \hspace{1cm} (2.5)

provided that $-\pi + \delta \leq \arg(z) \leq \pi - \delta$, $\delta \in (0, \pi)$.

When applying the Residue Theorem, we will use the fact that the gamma function has simple poles at $t = -n, n = 0, 1, 2, \ldots$, with

$$\text{Res}_{t=-n} \Gamma(t) = \frac{(-1)^n}{n!}. \hspace{1cm} (2.6)$$

When simplifying expressions involving gamma functions, the reflection formula for the gamma function will often be used:

$$\Gamma(t)\Gamma(1-t) = \frac{\pi}{\sin \pi t}. \hspace{1cm} (2.7)$$

Finally, we will use a result about Barnes integrals known as Barnes’ lemma (see [3] or [2, p. 6]):

**Lemma 2.1** (Barnes’ lemma). If $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, we have

$$\frac{1}{2\pi i} \int_{\text{path}} \Gamma(\alpha + t)\Gamma(\beta + t)\Gamma(\gamma - t)\Gamma(\delta - t) \, dt = \frac{\Gamma(\alpha + \gamma)\Gamma(\beta + \gamma)\Gamma(\beta + \delta)}{\Gamma(\alpha + \beta + \gamma + \delta)},$$  \hspace{1cm} (2.8)

provided that none of $\alpha + \gamma, \alpha + \delta, \beta + \gamma, \beta + \delta$ belongs to $\{0, -1, -2, \ldots\}$. 

5
3 Fundamental two-term relation

In this section we show that the function $L(a, b, c, d; e; f, g)$ defined in (2.2) can be represented as a Barnes integral. The Barnes integral representation will then be used to derive a fundamental two-term relation satisfied by the $L$ function.

Proposition 3.1.

\[
L(a, b, c, d; e; f, g) = \frac{1}{\pi \Gamma(a) \Gamma(b) \Gamma(c) \Gamma(d) \Gamma(1 + a - e) \Gamma(1 + b - e) \Gamma(1 + c - e) \Gamma(1 + d - e)} \frac{1}{2\pi i} \int_{\Gamma} \frac{\Gamma(a + t) \Gamma(b + t) \Gamma(c + t) \Gamma(d + t) \Gamma(1 - e - t) \Gamma(-t)}{\Gamma(f + t) \Gamma(g + t)} \, dt. \tag{3.1}
\]

Proof. Let

\[
I[a, b, c, d; e; f, g] = \frac{1}{2\pi i} \int_{\Gamma} \frac{\Gamma(a + t) \Gamma(b + t) \Gamma(c + t) \Gamma(d + t) \Gamma(1 - e - t) \Gamma(-t)}{\Gamma(f + t) \Gamma(g + t)} \, dt. \tag{3.2}
\]

For $N \geq 1$, let $C_N$ be the semicircle of radius $\rho_N$ on the right side of the imaginary axis and center at the origin, chosen in such a way that $\rho_N \to \infty$ as $N \to \infty$ and $\varepsilon := \inf N \dist(C_N, \mathbb{Z} \cup (\mathbb{Z} - e)) > 0$.

The formula (2.7) gives

\[
G(t) := \frac{\Gamma(a + t) \Gamma(b + t) \Gamma(c + t) \Gamma(d + t) \Gamma(1 - e - t) \Gamma(-t)}{\Gamma(f + t) \Gamma(g + t)} = \frac{-\pi^2 \Gamma(a + t) \Gamma(b + t) \Gamma(c + t) \Gamma(d + t)}{\Gamma(f + t) \Gamma(g + t) \Gamma(e + t) \Gamma(1 + t) \sin \pi t \sin \pi(e + t)}.
\]

By Stirling’s formula (2.5),

\[
\frac{\Gamma(a + t) \Gamma(b + t) \Gamma(c + t) \Gamma(d + t)}{\Gamma(f + t) \Gamma(g + t) \Gamma(e + t) \Gamma(1 + t)} \sim t^{a+b+c+d-e-f-g-1} = t^{-2}.
\]

It follows that we can find a constant $K > 0$ such that

\[
|G(t)| \leq K/|t|^2 \quad \text{if} \quad t \in C_N, \quad N = 1, 2, \ldots,
\]

which implies

\[
\int_{C_N} G(t) \, dt \to 0 \quad \text{as} \quad N \to \infty.
\]

Therefore the integral given by $I[a, b, c, d; e; f, g]$ is equal to the sum of the residues of the integrand at the poles of $\Gamma(1 - e - t)$ and $\Gamma(-t)$, which yields the result.

\[\blacksquare\]
The fundamental two-term relation satisfied by $L(a, b, c; d; e; f, g)$ is given in the next proposition.

**Proposition 3.2.**

\[
L\left[\begin{array}{c}
a, b, c, d; \\
e; f, g
\end{array}\right] = L\left[\begin{array}{c}
a, b, g - c, g - d; \\
1 + a + b - f; 1 + a + b - e, g
\end{array}\right].
\]

**Proof.** Let $I\left[\begin{array}{c}
a, b, c, d; \\
e; f, g
\end{array}\right]$ be as given in (3.2). As a first step, we will prove that

\[
I\left[\begin{array}{c}
a, b, c, d; \\
e; f, g
\end{array}\right] = \frac{\Gamma(c)\Gamma(d)\Gamma(1 + a - c)\Gamma(1 + b - e)}{\Gamma(f - a)\Gamma(f - b)\Gamma(g - c)\Gamma(g - d)}.
\]

By Barnes’ lemma,

\[
\frac{\Gamma(a + t)\Gamma(b + t)}{\Gamma(f + t)} = \frac{1}{2\pi i} \frac{\Gamma(c + t)\Gamma(d + t)}{\Gamma(g + t)} \int_u \frac{\Gamma(t + u)\Gamma(f - a - b + u)\Gamma(a - u)\Gamma(b - u)}{\Gamma(1 - e + u)} du
\]

and

\[
\frac{\Gamma(a + t)\Gamma(b + t)}{\Gamma(f + t)} = \frac{1}{2\pi i} \frac{\Gamma(c + t)\Gamma(d + t)}{\Gamma(g + t)} \int_v \frac{\Gamma(t + v)\Gamma(g - c - d + v)\Gamma(c - v)\Gamma(d - v)}{\Gamma(1 - e + v)} dv.
\]

We re-write the integral for $I\left[\begin{array}{c}
a, b, c, d; \\
e; f, g
\end{array}\right]$ by substituting for the above expressions, changing the order of integration, so that we integrate with respect to $t$ first (the change in the order of integration is readily justified using Stirling’s formula and Fubini’s theorem), and then applying Barnes’ Lemma again to the integral with respect to $t$. We obtain

\[
I\left[\begin{array}{c}
a, b, c, d; \\
e; f, g
\end{array}\right] = -\frac{1}{4\pi^2 \Gamma(c)\Gamma(d)\Gamma(1 + a - c)\Gamma(1 + b - e)} \cdot \int_u \frac{\Gamma(f - a - b + u)\Gamma(a - u)\Gamma(b - u)\Gamma(1 - e + u)}{\Gamma(1 - e + u + v)} du.
\]

(3.5)
After the substitution $v \mapsto c + d - f + v$ in the inside integral, it is easily checked (using the Saalschützian condition $e + f + g - a - b - c - d = 1$) that the right-hand side of (3.3) is invariant under the transformation

$$(a, b, c, d; e; f, g) \mapsto (a, b, g - c, g - d; 1 + a + b - f; 1 + a + b - e, g),$$

which proves (3.4). The result in the proposition now follows immediately from (3.4) upon writing the two $L$ functions in (3.3) in terms of their Barnes integral representations (3.1).

4 Invariance group

In the previous section we showed that the function $L(a, b, c, d; e; f, g)$ satisfies the two-term relation (3.3). If we define

$$A = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 & 1 \\
0 & 0 & -1 & -1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix} \in GL(7, \mathbb{C}), \quad (4.1)$$

then (3.3) can be expressed as $L(\bar{x}) = L(A\bar{x})$.

If $\sigma \in S_7$, we will identify $\sigma$ with the matrix in $GL(7, \mathbb{C})$ that permutes the standard basis $\{e_1, e_2, \ldots, e_7\}$ of the complex vector space $\mathbb{C}^7$ according to the permutation $\sigma$. For example,

$$(123) = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.$$

Let

$$G_L = \langle (12), (23), (34), (67), A \rangle \leq GL(7, \mathbb{C}). \quad (4.2)$$

The two-term relation (3.3) along with the trivial invariances of the function $L(a, b, c, d; e; f, g)$ under permutations of $a, b, c, d$ and interchanging $f, g$ implies that $G_L$ is an invariance group for $L(a, b, c, d; e; f, g)$, i.e. $L(\bar{x}) = L(\alpha \bar{x})$ for every $\alpha \in G_L$.

The goal of this section is to find the isomorphism type of the group $G_L$ and further to describe the two-term relations for the $L$ function in terms of
a double coset decomposition of $G_L$ with respect to its subgroup $\Sigma$ defined as follows:

$$\Sigma = \langle (12), (23), (34), (67) \rangle.$$  \hfill (4.3)

The group $\Sigma$ is a subgroup of $G_L$ consisting of permutation matrices. It is clear that $\Sigma \cong S_4 \times S_2$ and so $|\Sigma| = 48$. We note that if $\sigma \in \Sigma, \alpha \in G_L$, the multiplication $\sigma \alpha$ permutes the rows of $\alpha$, and the multiplication $\alpha \sigma$ permutes the columns of $\alpha$. A double coset of $\Sigma$ in $G_L$ is a set of the form

$$\Sigma \alpha \Sigma = \{ \sigma \alpha \tau : \sigma, \tau \in \Sigma \},$$

for some $\alpha \in G_L$. \hfill (4.4)

The distinct double cosets of the form (4.4) partition the group $G_L$ and give us a double coset decomposition of $G_L$ with respect to $\Sigma$. (See [9, p. 119] for more on double cosets.)

In Theorem 4.1 below we show that the group $G_L$ is isomorphic to the Coxeter group $W(D_5)$, which is of order 1920. In Theorem 4.2 we show that the subgroup $\Sigma$ is the largest permutation subgroup of $G_L$ and obtain a double coset decomposition of $G_L$ with respect to $\Sigma$. We list a representative for each of the six double cosets obtained and give the six invariance relations induced by those representatives (see (4.6)–(4.11)). The six invariance relations listed are all the “different” types of invariance relations in the sense that every other invariance relation can be obtained by permuting the first four entries and permuting the last two entries on the right-hand side of a listed invariance relation (which corresponds to permuting the rows of the accompanying matrix), and by permuting $a, b, c, d$ and permuting $f, g$ on the right-hand side of a listed invariance relation (which corresponds to permuting the columns of the accompanying matrix).

**Theorem 4.1.** The group $G_L$ is isomorphic to the Coxeter group $W(D_5)$, which is of order 1920.

**Proof.** The Dynkin diagram of the Coxeter group $W(D_n)$ is given by the graph with vertices labeled $1', 1, 2, \ldots, n$, where $i, j \in \{1, 2, \ldots, n\}$ are connected by an edge if and only if $|i - j| = 1$, and $1'$ is connected to 2 only. The presentation of $W(D_n)$ is given by

$$W(D_n) = \langle s_1, s_1', s_2, \ldots, s_n : (s_is_j)^{m_{ij}} = 1 \rangle,$$

where $m_{ii} = 1$ for all $i$; and for $i$ and $j$ distinct, $m_{ij} = 3$ if $i$ and $j$ are connected by an edge, and $m_{ij} = 2$ otherwise. It is well-known that the order of $W(D_n)$ is $2^{n-1}n!$ (see [15, Section 2.11]).

Consider the generators of $G_L$ given by

$$a_1 = (34), a_2 = (23), a_3 = (34)A, a_4 = (67), a_{1'} = (12).$$ \hfill (4.5)

A direct computation shows that

$$(a_ia_j)^{m_{ij}} = 1, \quad \text{for all } i, j \in \{1, 2, 3, 4, 1'\}.$$
Therefore if we define \( \varphi(s_i) = a_i \) for every \( i \in \{1, 2, 3, 4, 1'\} \), \( \varphi \) extends (uniquely) to a surjective homomorphism from \( W(D_5) \) onto \( G_L \). Since \( W(D_5) \) is a finite group, we need to show that \( G_L \) and \( W(D_5) \) have the same order to complete the proof. To that end, it is enough to just show that \( |G_L| > 960 = \frac{|W(D_5)|}{2} \). An estimate on the order of \( G_L \) is obtained by computing the sizes of the distinct double cosets \( \Sigma A \Sigma \) and \( \Sigma [(123)(67)A]^3 \Sigma \) of \( \Sigma \) in \( G_L \), where \( \Sigma \) is as given in (4.3). Indeed, by permuting columns that are different as multisets and then permuting the rows of the resulting matrices in every possible way, we see that the number of matrices that belong to each of those double cosets is 12 \cdot 48\cdot 3, which shows that \( |G_L| > 960 \) and completes the proof.

As stated before Theorem 4.1, we are interested in the complete double coset decomposition of \( G_L \) with respect to \( \Sigma \) since this will classify all the invariance relations for the function \( L(a, b, c, d; e; f, g) \) in a convenient way. We use the same technique as in the proof of Theorem 4.1 given by permuting columns that are different as multisets and then permuting the rows of the resulting matrices in every possible way. We find that there are six double cosets of \( \Sigma \) in \( G_L \). Representative matrices for the double cosets are \( I_7, A, [(123)(67)A]^2, [(123)(67)A]^3, [(123)A]^3, [(123)(67)A]^4 \). The corresponding double coset sizes are \( 1 \cdot 48, 12 \cdot 48, 12 \cdot 48, 12 \cdot 48, 2 \cdot 48, 1 \cdot 48 \). Furthermore, the representative matrices are all seen to have different entries, so that \( \Sigma \) must indeed be the largest permutation subgroup of \( G_L \). Each representative matrix gives rise to an invariance relation. Theorem 4.2 summarizes the result.

**Theorem 4.2.** Let \( \Sigma \) be as defined in (4.3). Then \( \Sigma \) consists of all the permutation matrices in \( G_L \). There are six double cosets in the double coset decomposition of \( G_L \) with respect to \( \Sigma \). Representative matrices for the double cosets are \( I_7, A, [(123)(67)A]^2, [(123)(67)A]^3, [(123)A]^3, [(123)(67)A]^4 \) and the corresponding double coset sizes are \( 1 \cdot 48, 12 \cdot 48, 12 \cdot 48, 12 \cdot 48, 2 \cdot 48, 1 \cdot 48 \). The corresponding invariances of the \( L \) function are given by

\[
L \begin{bmatrix}
  a, b, c, d; \\
  e; f, g
\end{bmatrix} = L \begin{bmatrix}
  a, b, c, d; \\
  e; f, g
\end{bmatrix},
\]

\[
L \begin{bmatrix}
  a, b, c, d; \\
  e; f, g
\end{bmatrix} = L \begin{bmatrix}
  a, b, g - c, g - d; \\
  1 + a + b - f; 1 + a + b - e, g
\end{bmatrix},
\]

\[
L \begin{bmatrix}
  a, b, c, d; \\
  e; f, g
\end{bmatrix} = L \begin{bmatrix}
  1 + a - e, g - c, a, f - c; \\
  1 + a - c; 1 + a + b - e, 1 + a + d - e
\end{bmatrix},
\]

\[
L \begin{bmatrix}
  a, b, c, d; \\
  e; f, g
\end{bmatrix} = L \begin{bmatrix}
  1 + d - e, 1 + a - e, g - c, g - b; \\
  1 + g - b - c, 1 + a + d - e, 1 + g - e
\end{bmatrix},
\]

\[
L \begin{bmatrix}
  a, b, c, d; \\
  e; f, g
\end{bmatrix} = L \begin{bmatrix}
  g - a, g - b, g - c, g - d; \\
  1 + g - f; 1 + g - e, g
\end{bmatrix},
\]

\[
L \begin{bmatrix}
  a, b, c, d; \\
  e; f, g
\end{bmatrix} = L \begin{bmatrix}
  1 + c - e, 1 + d - e, 1 + a - e, 1 + b - e; \\
  2 - e, 1 + g - e, 1 + f - e
\end{bmatrix}.
\]
5 Applications of the fundamental two-term relation

In this section we prove some consequences of the fundamental two-term relation given in Proposition 3.2. As a first step, we write the two $L$ functions in (3.3) in terms of their definitions as linear combinations of two $4F3(1)$ series. We obtain

\[
4F3\left[ \frac{a, b, c, d;}{e, f, g; 1} \right] - 4F3\left[ \frac{1 + a - e, 1 + b - e, 1 + c - e, 1 + d - e;}{1 + f - e, 1 + g - e, 2 - e;} \right] \sin \pi e \Gamma(e) \Gamma(f) \Gamma(g) \Gamma(1 + b - e) \Gamma(1 + a - e) \Gamma(1 + c - e) \Gamma(1 + d - e) \Gamma(1 + f - e) \Gamma(1 + g - e) \Gamma(2 - e)
\]

\[
= 4F3\left[ \frac{a, b, g - c, g - d;}{1 + a + b - f, 1 + a + b - e, g;} \right] - \sin \pi(1 + a + b - f) \Gamma(1 + a + b - f) \Gamma(1 + a + b - e) \Gamma(1 + c - e) \Gamma(f - b) \Gamma(f - a) \Gamma(1 + d - e) \Gamma(1 + c - e) \Gamma(f - a)
\]

\[
= 3F2\left[ \frac{b, g - c, g - d;}{f + g - c - d, g; 1} \right] - \sin \pi(1 + a + b - f) \Gamma(1 + a + b - f) \Gamma(1 + a + b - e) \Gamma(1 + c - e) \Gamma(g - d) \Gamma(1 + f - e) \Gamma(f + g - a - b) \Gamma(1 + f - a)
\]

\[
\cdot \Gamma(1 + f - e) \Gamma(f + g - a - b) \Gamma(1 + f - a - b).
\]

(5.1)

We fix $b, c, d, f, g \in \mathbb{C}$ in such a way that

\[
\Re(f + g - b - c - d) > 0, \quad \Re(f - b) > 0.
\]

(5.2)

Let $a \in \mathbb{C}$ and let $e = 1 + a + b + c + d - f - g$ depend on $a$. In equation (5.1) we let $|a| \to \infty$. Using Stirling’s formula (2.5) and the conditions (5.2), we obtain

\[
3F2\left[ \frac{b, c, d;}{f, g; 1} \right] \Gamma(f) \Gamma(g) \Gamma(f + g - b - c - d) \Gamma(f + g - b - c)
\]

\[
= 3F2\left[ \frac{b, g - c, g - d;}{f + g - c - d, g; 1} \right] \Gamma(f + g - c - d) \Gamma(g) \Gamma(f - b) \Gamma(f + g - c - d)
\]

\[
= 3F2\left[ \frac{f - b, g - b, f + g - b - c - d;}{f + g - b - d, f + g - b - c; 1} \right] \Gamma(f + g - b - d) \Gamma(f + g - b - c) \Gamma(f + g - b - c)
\]

(5.3)

We note that the conditions (5.2) are needed for the absolute convergence of the two $3F2(1)$ series in (5.3). Applying (5.3) twice yields Thomae’s identity

\[
3F2\left[ \frac{b, c, d;}{f, g; 1} \right] \Gamma(f) \Gamma(g) \Gamma(f + g - b - c - d) \Gamma(f + g - b - c)
\]

\[
= 3F2\left[ \frac{f - b, g - b, f + g - b - c - d;}{f + g - b - d, f + g - b - c; 1} \right] \Gamma(f + g - b - d) \Gamma(f + g - b - c) \Gamma(f + g - b - c)
\]

(5.4)

In fact, applying (5.4) twice gives (5.3), so that (5.3) and (5.4) are equivalent.
Next in equation (5.1) we let $a \to -n$, where $n$ is a nonnegative integer. Using the fact that $\lim_{a \to -n} \frac{1}{\Gamma(n)} = 0$ and then formula (2.7) to simplify the result, we obtain Bailey’s identity

$$
\frac{4F_3}{(e)_n(f)_n} \left[ \frac{-n, b, c, d; e, f, g}{e, f, g; 1} \right] = \frac{(e-b)_n(f-b)_n}{(e)_n(f)_n} \frac{4F_3}{1-n+b-f, 1-n+b-e, g; 1},
$$

(5.5)

which holds provided that $e + f + g - b - c - d + n = 1$.

Thomae’s and Bailey’s identities have been shown in [10] in a similar way to be limiting cases of a fundamental two-term relation satisfied by the function $K(a; b, c, d; e, f, g)$.

As a final application, in the fundamental two-term relation (3.3) we let $d = g$. We express the left-hand side as a Barnes integral according to Proposition 3.1, and we write the right-hand side in terms of two $4F_3(1)$ series according to the definition of the $L$ function. The condition $d = g$ causes one of the terms on the right-hand side to go to zero and the $4F_3(1)$ series in the other term to be trivially equal to one. If we simplify the result further using (2.7), we obtain

$$
\frac{1}{2\pi i} \int \Gamma(a + t) \Gamma(b + t) \Gamma(c + t) \Gamma(1 - e - t) \Gamma(-t) dt
$$

$$
\frac{\Gamma(a) \Gamma(b) \Gamma(1 + a - e) \Gamma(1 + b - e) \Gamma(1 + c - e)}{\Gamma(f - a) \Gamma(f - b) \Gamma(f - c)},
$$

(5.6)

which holds provided that $e + f - a - b - c = 1$. The equation (5.6) is precisely the statement of Barnes’ second lemma.

### 6 Group-theoretic structure of the three-term relations

Let

$$
M_L = \langle (12), (23), (34), (56), (67), A \rangle \leq GL(7, \mathbb{C}).
$$

(6.1)

The group $M_L$ is generated by the invariance group $G_L$ and the transposition matrix (56). We will show that the right cosets of $G_L$ in $M_L$ govern the three-term relations for the function $L(a, b, c, d; e; f, g)$. In particular, in the next section we will show that for every $\sigma_1, \sigma_2, \sigma_3 \in M_L$ such that $\sigma_1, \sigma_2, \sigma_3$ are in different right cosets of $G_L$ in $M_L$, there exists a three-term relation involving the functions $L(\sigma_1 \vec{x}), L(\sigma_2 \vec{x})$, and $L(\sigma_3 \vec{x})$.

In this section we lay the foundation of the group-theoretic structure of the three-term relations satisfied by the $L$ function by studying the right coset space $G_L \backslash M_L$ and then introducing the notion of $L$-coherence (see Definition 6.5), which will lead us to the classification of the three-term relations through Proposition 6.6.
Theorem 6.1. The group $M_L$ is isomorphic to the Coxeter group $W(D_6)$, which has 23040 elements.

**Proof.** The proof follows the same lines as the proof of Theorem 4.1. We let $a_1, a_2, a_3, a_4, a_1'$ be as in (4.5), and we also let $a_5 = (56)$. We obtain a surjective homomorphism from $W(D_6)$ onto $M_L$, which homomorphism is then shown to be an isomorphism by estimating the order of $M_L$ using the counting technique from the proof of Theorem 4.1.

There are $23040/1920 = 12$ right cosets of $G_L$ in $M_L$. The group $M_L$ acts by right multiplication on those right cosets through

$$(G_L \mu) \cdot \nu = G_L (\mu \nu), \quad \text{for } \mu, \nu \in M_L.$$  

It is well-known that the Coxeter group $W(D_6)$ has a center consisting of two elements (see [15, pp. 82 and 132]). We let $w_0$ denote the unique nonidentity element in the center of $M_L$. The element $w_0$ is called the central involution. The following proposition follows from standard facts about the Coxeter group $W(D_n)$ (see [15, Section 2.10]):

**Proposition 6.2.** We can label the 12 right cosets of $G_L$ in $M_L$ by

$$1, 2, \ldots, 6, \bar{1}, \bar{2}, \ldots, \bar{6}$$

in such a way that $i$ and $\bar{i}$ are interchanged under the action of $w_0$, for every $i \in \{1, 2, \ldots, 6\}$.

Indeed, the central involution $w_0$ is computed to be

$$w_0 = (12)(34)[[(1234)(567)]^2 A]^4,$$

and from here, representatives of the twelve right cosets $6, \ldots, 1, \bar{6}, \ldots, \bar{1}$ are computed to be

$$\mu_6 = I_7, \quad \mu_5 = (56), \quad \mu_4 = (57),$$

$$\mu_3 = w_2, \quad \mu_2 = (56)w_2, \quad \mu_1 = (57)w_2,$$

$$\mu_0 = w_0, \quad \mu_5 = (56)w_0, \quad \mu_4 = (57)w_0,$$

$$\mu_3 = w_1, \quad \mu_2 = (56)w_1, \quad \mu_1 = (57)w_1,$$

respectively, where

$$w_1 = (1234)[(1234)(567)A]^3(1432),$$

$$w_2 = w_0 w_1.$$  

Every matrix $\mu \in M_L$ induces a permutation of the twelve element set $\{1, \ldots, 6, \bar{1}, \ldots, \bar{6}\}$. Let $\Phi : M_L \to S_{12}$ be the induced permutation representation. We note that if $\mu \in M_L$, then the permutation $\Phi(\mu)$ can be uniquely described by specifying its effect on the set $\{1, \ldots, 6\}$, since the elements in $\{1, \ldots, 6\}$ will be permuted based on where $\{1, \ldots, 6\}$ are permuted with the addition or omission of a bar. We give the images under $\Phi$ of the generators of $M_L$ in the following proposition:
Proposition 6.3. The images of the generators $a_1, a_2, a_3, a_4, a_5, \text{ and } a_1'$ of $M_L$ under the permutation representation $\Phi : M_L \to S_{12}$ are given by

$\Phi(a_1) = \Phi((34)) = (1\ 2\ 3\ 4\ 5\ 6)$,

$\Phi(a_2) = \Phi((23)) = (1\ 3\ 2\ 4\ 5\ 6)$,

$\Phi(a_3) = \Phi((34)A) = (1\ 2\ 4\ 3\ 5\ 6)$,

$\Phi(a_4) = \Phi((67)) = (1\ 2\ 3\ 5\ 4\ 6)$,

$\Phi(a_5) = \Phi((56)) = (1\ 2\ 3\ 4\ 6\ 5)$,

$\Phi(a_1') = \Phi((12)) = (2\ 1\ 3\ 4\ 5\ 6)$.

Given the decomposition of an element $\mu \in M_L$ in terms of the generators $a_1, a_2, a_3, a_4, a_5, a_1'$, Proposition 6.3 can be used to find the permutation corresponding to $\Phi(\mu)$.

The next proposition is a restatement of standard facts about the construction of the Coxeter group $W(D_n)$ as a semidirect product (see [15, Section 2.10] for more details).

Proposition 6.4. The permutation representation $\Phi : M_L \to S_{12}$ is faithful, i.e. $\ker(\Phi) = \{I_7\}$. The embedding of $M_L$ into $S_{12}$ consists of all permutations of the form $(j_1\ j_2\ j_3\ j_4\ j_5\ j_6)$, where each $j_i$ belongs to the set $\{1, \ldots, 6, \bar{1}, \ldots, \bar{6}\}$, the $j_i$'s are all distinct if we remove the bars, and an even number, i.e. 0, 2, 4, or 6, of the $j_i$'s contain a bar.

Central to the classification of the three-term relations in the next section is the notion of $L$-coherence defined as follows:

Definition 6.5. If $S$ is a subset of $G_L \setminus M_L$, we say that $S$ is $L$-coherent if no two elements of $S$ are interchanged by the action of the central involution $w_0$. If $S$ is not $L$-coherent, then $S$ is called $L$-incoherent.

It is clear that a set $S$ is $L$-coherent if and only if $S$ does not contain both elements of the form $i$ and $\bar{i}$, for any $i \in \{1, \ldots, 6\}$.

The group $M_L$ acts by right multiplication on triples of cosets of $G_L \setminus M_L$ according to

$\{i, j, k\} \cdot \mu = \{i \cdot \mu, j \cdot \mu, k \cdot \mu\}$,

for $i, j, k \in G_L \setminus M_L$ and $\mu \in M_L$.

There are $\binom{12}{3} = 220$ triples of right cosets of $G_L \setminus M_L$. The final proposition of this section describes the orbits of the action of $M_L$ on those triples. It will be used in the next section in the classification of the three-term relations satisfied by the $L$ function.
Proof. From Proposition 6.4, it is easily seen that there are two orbits given by Proposition 6.6, it is easily seen that there are two orbits given by the $L$-coherent and $L$-incoherent triples. A simple counting argument establishes the lengths of the two orbits.

\[ \square \]

7 Three-term relations

For every $i \in G_L \backslash M_L$, we define the function

\[ L_i(\hat{x}) = L(\mu \hat{x}), \]

where $\hat{x} = (a, b, c, d, e, f, g)^T \in V$ and $\mu$ is any representative of the right coset $i$. A three-term relation involving the functions $L_i(\hat{x}), L_j(\hat{x}),$ and $L_k(\hat{x})$ is said to be of type $\{i, j, k\}$.

**Proposition 7.1.** A three-term relation of the $L$-coherent type $\{6, 5, 4\}$ is given by

\[
\begin{align*}
\frac{\sin \pi (f - g)}{\Gamma(e - a)\Gamma(e - b)\Gamma(e - c)\Gamma(e - d)} & L \left[ \begin{array}{c}
a, b, c, d; e, f, g \\
\end{array} \right] \\
+ \frac{\sin \pi (g - e)}{\Gamma(f - a)\Gamma(f - b)\Gamma(f - c)\Gamma(f - d)} & L \left[ \begin{array}{c}
a, b, c, d; f, e, g \\
\end{array} \right] \\
+ \frac{\sin \pi (e - f)}{\Gamma(g - a)\Gamma(g - b)\Gamma(g - c)\Gamma(g - d)} & L \left[ \begin{array}{c}
a, b, c, d; g, f, e \\
\end{array} \right] = 0. 
\end{align*}
\]

**Proof.** We have

\[
\frac{1}{2\pi i} \int \frac{\Gamma(a + t)\Gamma(b + t)\Gamma(c + t)\Gamma(d + t)\Gamma(1 - e - t)\Gamma(1 - f - t)\Gamma(-t)}{\Gamma(g + t)} dt \\
\cdot [\sin \pi e \sin \pi (1 - f - t) - \sin \pi f \sin \pi (1 - e - t) - \sin \pi (f - e) \sin \pi (t)] dt = 0,
\]

since the quantity in square brackets in the above integral is equal to zero, which can be seen by applying elementary trigonometric identities. We break up the integral into three parts and use Formula (2.7) to simplify the result to obtain

\[
\begin{align*}
\frac{1}{2\pi i} & \int \frac{\sin \pi e \Gamma(a + t)\Gamma(b + t)\Gamma(c + t)\Gamma(d + t)\Gamma(1 - e - t)\Gamma(-t)}{\Gamma(f + t)\Gamma(g + t)} dt \\
- \frac{1}{2\pi i} & \int \frac{\sin \pi f \Gamma(a + t)\Gamma(b + t)\Gamma(c + t)\Gamma(d + t)\Gamma(1 - f - t)\Gamma(-t)}{\Gamma(e + t)\Gamma(g + t)} dt \\
- \frac{1}{2\pi i} & \int \frac{\sin \pi (f - e) \Gamma(a + t)\Gamma(b + t)\Gamma(c + t)}{\Gamma(d + t)\Gamma(1 - e - t)\Gamma(1 - f - t)} dt = 0.
\end{align*}
\]
After the substitution $t \mapsto t + 1 - e$ in the third integral, we express each of the three integrals as an $L$ function according to (3.1). The end result is a three-term relation of type $\{6, 5, 4\}$ which can be written as

$$\frac{\sin \pi e}{\Gamma(1 + a - f)\Gamma(1 + b - f)\Gamma(1 + c - f)\Gamma(1 + d - f)} L^\left[a, b, c, d; \right. e, f, g \left. \right] + \frac{\sin \pi(-f)}{\Gamma(1 + a - e)\Gamma(1 + b - e)\Gamma(1 + c - e)\Gamma(1 + d - e)} L^\left[a, b, c, d; \right. f, e, g \left. \right] + \frac{\sin \pi(e - f)}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(d)} L^\left[1 - a, 1 - b, 1 - c, 1 - d; \right. 2 - g, 2 - f, 2 - e \left. \right] = 0. \quad (7.2)$$

If we let

$$\mu = (14)(23)(123)^3 \in M_L,$$

we have

$$\{6, 5, 4\} \cdot \mu = \{6, 5, 4\}.$$  

Applying the change of variable $\vec{x} \mapsto \mu \vec{x}$ to all terms and coefficients in the relation (7.2) yields the result.

If $\vec{x} = (a, b, c, d, e, f, g)^T \in V$, we define $\gamma_1(\vec{x}), \gamma_2(\vec{x}),$ and $\gamma_3(\vec{x})$ to be the respective coefficients in front of the three $L$ functions in (7.1). This way the three-term relation (7.1) can be written as

$$\gamma_1(\vec{x})L_6(\vec{x}) + \gamma_2(\vec{x})L_5(\vec{x}) + \gamma_3(\vec{x})L_4(\vec{x}) = 0, \quad (7.3)$$

or, equivalently, as

$$\sum_{i=0}^{2} \gamma_1((576)^i\vec{x})L((576)^i\vec{x}) = 0. \quad (7.4)$$

Let $\{i, j, k\}$ be any $L$-coherent triple. Since, by Proposition 6.6, $\{i, j, k\}$ is in the same orbit as $\{6, 5, 4\}$, there exists $\mu \in M_L$ such that

$$\{6, 5, 4\} \cdot \mu = \{i, j, k\}.$$  

Then a three-term relation of type $\{i, j, k\}$ is given by

$$\gamma_1(\mu \vec{x})L_i(\vec{x}) + \gamma_2(\mu \vec{x})L_j(\vec{x}) + \gamma_3(\mu \vec{x})L_k(\vec{x}) = 0, \quad (7.5)$$

or, equivalently, by

$$\sum_{i=0}^{2} \gamma_1((576)^i\mu \vec{x})L((576)^i\mu \vec{x}) = 0. \quad (7.6)$$
Proposition 7.2. A three-term relation of the $L$-incoherent type $\{6,5,\bar{6}\}$ is given by

$$\sin \pi g \sin \pi (f - g) \left[ \frac{\Gamma(1 + a - f)\Gamma(1 + b - f)\Gamma(1 + c - f)}{\Gamma(1 + d - f)\Gamma(e - a)\Gamma(e - b)\Gamma(e - c)\Gamma(e - d)} \cdot L \left[ a, b, c, d; \right] e; f, g \right] + \frac{1}{\pi^2} \left[ \sin \pi g \sin \pi (g - e) \sin \pi (f - a) \cdot \sin \pi (f - b) \sin \pi (f - c) \sin \pi (f - d) \cdot \sin \pi f \sin \pi (e - f) \sin \pi (g - a) \cdot \sin \pi (g - b) \sin \pi (g - c) \sin \pi (g - d) \right] \cdot L \left[ a, b, c, d; \right] f; e, g \right] + \frac{\sin \pi (e - f) \sin \pi (f - g)}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(d)\Gamma(g - a)\Gamma(g - b)\Gamma(g - c)\Gamma(g - d)} \cdot L \left[ 1 - a, 1 - b, 1 - c, 1 - d; \right] 2 - e; 2 - f, 2 - g = 0. \quad (7.7)$$

Proof. Let

$$\mu = (14)(23)[(123)4]^3(576) \in M_L.$$ 

We have

$$\{6,5,4\} : \mu = \{5,4,\bar{6}\}.$$ 

This and (7.3) lead to a three-term relation of type $\{5,4,\bar{6}\}$ given by

$$\gamma_1(\mu \bar{x})L_5(\bar{x}) + \gamma_2(\mu \bar{x})L_4(\bar{x}) + \gamma_3(\mu \bar{x})L_6(\bar{x}) = 0.$$ 

We combine the above three-term relation of type $\{5,4,\bar{6}\}$ with the three-term relation of type $\{6,5,4\}$ given in (7.3) and cancel the terms involving the function $L_4(\bar{x})$ to obtain (7.7).

If $\bar{x} = (a,b,c,d,e,f,g)^T \in V$, we define $\beta_1(\bar{x}),\beta_2(\bar{x}),$ and $\beta_3(\bar{x})$ to be the respective coefficients in front of the three $L$ functions in (7.7). This way the three-term relation (7.7) can be written as

$$\beta_1(\bar{x})L_6(\bar{x}) + \beta_2(\bar{x})L_5(\bar{x}) + \beta_3(\bar{x})L_6(\bar{x}) = 0. \quad (7.8)$$

Let $\{i,j,\bar{i}\}$ be any triple that is $L$-incoherent. By Proposition $6.6$ $\{i,j,\bar{i}\}$ is in the same orbit as $\{6,5,\bar{6}\}$ and so there exists $\mu \in M_L$ such that

$$\{6,5,\bar{6}\} : \mu = \{i,j,\bar{i}\}.$$ 

Then a three-term relation of type $\{i,j,\bar{i}\}$ is given by

$$\beta_1(\mu \bar{x})L_i(\bar{x}) + \beta_2(\mu \bar{x})L_j(\bar{x}) + \beta_3(\mu \bar{x})L_i(\bar{x}) = 0. \quad (7.9)$$
To summarize our results, we have shown that for every \( \sigma_1, \sigma_2, \sigma_3 \in M_L \) such that \( \sigma_1, \sigma_2, \sigma_3 \) are in different right cosets of \( G_L \) in \( M_L \), there exists a three-term relation involving the functions \( L(\sigma_1 \vec{x}), L(\sigma_2 \vec{x}), \) and \( L(\sigma_3 \vec{x}) \). We have a total of 220 three-term relations. The three-term relations fall into two families based on \( L \)-coherence. The two fundamental three-term relations for \( L \)-coherent and for \( L \)-incoherent triples are given in (7.3) and (7.8) respectively. Any other three-term relation can be obtained from one of those two through a change of variable of the form \( \vec{x} \mapsto \mu \vec{x} \) applied to all terms and coefficients. The result is a three-term relation of the form (7.5) or (7.9). The appropriate matrix \( \mu \in M_L \) can be found through Proposition 6.3, which describes the actions of the generators \( a_1, a_2, a_3, a_4, a_5, a_1' \) of the group \( M_L \).

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