Spectral properties of sample covariance matrices arising from random matrices with independent non identically distributed columns

Cosme Louart*  Romain Couillet†

Abstract

Given a random matrix \( X = (x_1, \ldots, x_n) \in \mathcal{M}_{p,n} \) with independent columns and satisfying concentration of measure hypotheses and a parameter \( z \) whose distance to the spectrum of \( \frac{1}{n} XX^T \) should not depend on \( p, n \), it was previously shown that the functionals \( \text{Tr}(AR(z)) \), for \( R(z) = (\frac{1}{n} XX^T - zI_p)^{-1} \) and \( A \in M_p \) deterministic, have a standard deviation of order \( O(\|A\|_*/\sqrt{n}) \). Here, we show that \( \|E[R(z)] - \tilde{R}(z)\|_F \leq O(1/\sqrt{n}) \), where \( \tilde{R}(z) \) is a deterministic matrix depending only on \( z \) and on the means and covariances of the column vectors \( x_1, \ldots, x_n \) (that do not have to be identically distributed). This estimation is key to providing accurate fluctuation rates of functionals of \( X \) of interest (mostly related to its spectral properties) and is proved thanks to the introduction of a semi-metric \( d_s \) defined on the set \( D_n(\mathbb{H}) \) of diagonal matrices with complex entries and positive imaginary part and satisfying, for all \( D, D' \in D_n(\mathbb{H}) \):

\[
d_s(D, D') = \max_{i \in \{1, \ldots, n\}} |D_i - D'_i|/((\Im(D_i) \Im(D'_i)))^{1/2}.
\]

Possibly most importantly, the underlying concentration of measure assumption on the columns of \( X \) finds an extremely natural ground for application in modern statistical machine learning algorithms where non-linear Lipschitz mappings and high number of classes form the base ingredients.

Keywords: random matrix theory ; concentration of measure ; covariance matrices ; contractivity and stability of quadratic equations.

MSC2020 subject classifications: Mathematics Subject Classification 2000: 15A52, 60B12, 62J10.

Notations

For any integer \( d \geq 1 \), we will denote for simplicity \( [d] \equiv \{1, \ldots, d\} \). Given a complex variable \( z \in \mathbb{C} \), we denote \( \bar{z} = \Re(z) - i \Im(z) \) its conjugate. We denote \( \mathbb{R}_+ = \{x \in \mathbb{R}, x \geq 0\} \) and \( \mathbb{H} = \{z \in \mathbb{C}, \Im(z) > 0\} \), the hyperbolic space. We denote \( \|\cdot\| \) the Euclidean norm on \( \mathbb{C}^p \) defined for any \( x = (x_1, \ldots, x_p) \in \mathbb{C}^p \) as \( \|x\| = (\sum_{i=1}^p |x_i|^2)^{1/2} \). We denote \( \mathcal{M}_{p,n} \) the set of real matrices of size \( p \times n \), \( \mathcal{M}_p \) the set of squared matrices of size \( p \) and \( \mathcal{O}_p \subset \mathcal{M}_p \) the set of orthogonal matrices. Given a set \( A \subset \mathbb{C} \), \( \mathcal{M}_{p,n}(A) \) is the set of matrices with entries in \( A \) and \( \mathcal{D}_n(A) \) the set of diagonal matrices with entries in \( A \). Given a diagonal matrix \( \Delta \in \mathcal{D}_n(A) \), we denote \( \Delta_1, \ldots, \Delta_n \) its diagonal elements. We

*GIPSA-lab. E-mail: cosmelouart@gmail.com
†LIG-lab, GIPSA-lab. E-mail: romain.couillet@gipsa-lab.grenoble-inp.fr
Sample covariance - independent columns

introduce on \( \mathcal{M}_{p,n}(\mathbb{C}) \) (and on \( \mathcal{D}_n(\mathbb{C}) \)) the spectral norm \( \| \cdot \| \) and the Frobenius norm \( \| \cdot \|_F \) defined for any \( M \in \mathcal{M}_{p,n}(A) \) as

\[
\|M\| = \sup_{\|x\| \leq 1} \|Mx\| \quad \text{and} \quad \|M\|_F = \sqrt{\text{Tr}(MM^T)}.
\]

Given two sets \( A, B \), we denote \( \mathcal{F}(A) \), the set of functions from \( A \) to \( \mathbb{R} \) and \( \mathcal{F}(A, B) \), the set of functions from \( A \) to \( B \). We consider several times the order on Hermitian matrices defined for any \( A, B \) hermitian as:

\[ A \leq B \quad \iff \quad B - A \text{ is non negative.} \]

**Introduction**

Considering a sample covariance matrix \( \frac{1}{n}XX^T \), where \( X = (x_1, \ldots, x_n) \in \mathcal{M}_{p,n} \) is the data matrix, we denote \( \text{Sp}(\frac{1}{n}XX^T) \) the spectrum of \( \frac{1}{n}XX^T \). The spectral distribution of \( \frac{1}{n}XX^T \), denoted \( \mu \equiv \frac{1}{n} \sum_{\lambda \in \text{Sp}(\frac{1}{n}XX^T)} \delta_\lambda \), is classically studied through its Stieltjes transform expressed as:

\[
g : \mathbb{C} \setminus \text{Sp}\left( \frac{1}{n}XX^T \right) \rightarrow \mathbb{C} \quad \text{with} \quad z \mapsto \int_{\mathbb{R}} \frac{d\mu(\lambda)}{\lambda - z}.
\]

The relevance of the Stieltjes transform has been extensively justified in some seminal works [MP67, Sil85] by the Cauchy integral that provides for any analytical mapping \( f \) defined on a neighborhood of a subset \( B \subset \text{Sp}(\frac{1}{n}XX^T) \) the identity:

\[
\int_B f(\lambda)d\mu(\lambda) = \frac{1}{2i\pi} \oint_{\gamma} f(z)g(z)dz,
\]

where \( \gamma : [0,1] \rightarrow \mathbb{C} \setminus \text{Sp}(\frac{1}{n}XX^T) \) is a closed path on which \( f \) is defined and whose interior \( I_\gamma \) satisfies \( I_\gamma \cap \text{Sp}(\frac{1}{n}XX^T) = B \cap \text{Sp}(\frac{1}{n}XX^T) \). But we can go further and approximate linear functionals of the eigenvectors thanks to the resolvent. If we denote \( E_B \) the random eigenspace associated to the eigenvalues of \( \frac{1}{n}XX^T \) belonging to \( B \), and \( \Pi_B \) the orthogonal projector on \( E_B \), then for any deterministic matrix \( A \in \mathcal{M}_p \):

\[
\text{Tr}(\Pi_B A) = \frac{1}{2i\pi} \int_{\gamma} \text{Tr}(AR(z))dz \quad \text{with} \quad R(z) \equiv \left( \frac{1}{n}XX^T - zI_p \right)^{-1}.
\]  

The matrix \( R(z) \) is commonly called the resolvent of \( \frac{1}{n}XX^T \). It satisfies in particular that, for all \( z \in \mathbb{C} \setminus \text{Sp}(\frac{1}{n}XX^T) \), \( g(z) = \frac{1}{n} \text{Tr}(R(z)) \). It thus naturally becomes the central element of the study of the spectral distribution. One of the first tasks in random matrix theory is to devise a so called “deterministic equivalent” for \( R(z) \) ([HLN07]), that we will denote here \( \tilde{R}(z) \). Specifically, we look for a deterministic matrix computable from the first statistics of our problem (the means and covariances of the \( x_i \)’s) and close to \( \text{E}[R] \). Two questions then arise:

1. Is \( R(z) \) close to \( \text{E}[R(z)] \)?
2. What does this notion of closeness really mean?

The first questions relate to concentrations properties on \( R(z) \) that arise from concentration properties on \( X \). The study of random matrices originally studied with i.i.d. entries ([MP67, Yin86]), mere Gaussian hypotheses ([BKV96]), or with weaker hypotheses concerning the first moments of the entries (supposed to be independent or at
least independent up to an affine transformation). Some more recent works showed the concentration of the spectral distribution of Wishart or Wigner matrices with bounding assumption on the entries of the random matrix under study -or at least a linear transformation of it- allows to employ Talagrand results in [GZ00] that also treat some log-concave hypotheses allowing to relax some independence assumptions as it is done in [AC15]. One can also find very light hypotheses on the quadratic functionals of the columns ([BZ08]), improved in [Yas16] or on the norms of the columns and the rows ([Ada11]). In the present work, we do not consider the case of what is called convexly concentrated random vectors in [VW14, MS11, Ada11] as it is done in [GZ00] because it requires a different approach that we conducted in [Lou22]. We adopt slightly more general hypotheses than the one adopted by [AC15] that allows to provide more precise result than [BZ08] and [Ada11].

The exact concentration hypothesis on $X$ is described in Assumption 0.4: for simplicity of exposition in the introduction, we will just assume here that the matrix $X$ is a $λ$-Lipschitz transformation of a Gaussian vector $Z \sim N(0, I_d)$ for any given $d \in \mathbb{N}$ but for a Lipschitz parameter $λ \ll n, p$ (we will write $λ \leq O(1)$). We also require the columns $x_1, \ldots, x_p$ of $X$ to be independent, but the entries of the columns may have intricate dependencies, as long as they remain $λ$-Lipschitz transformations of a Gaussian vector. This represents a wide range of random vectors and, among the most commonly studied random vectors, this mainly merely excludes the heavy tailed distributions and the discrete distributions. Some large description of the concentration of measure phenomenon can be found in [Led05] and [BLM13] and more specifically, some elementary applications to random matrices are provided in [Tao12, Ver18] and [LC18]. We further provide in subsection “Practical implications” a series of arguments demonstrating the relevance of this approach in classical problems of statistical machine learning.

The concentration inequality on $X$ in a sense “propagates” to the resolvent that then satisfies, for all deterministic matrices $A$ such that $\|A\|_F \equiv \sqrt{\text{Tr}(AA^T)} \leq 1$ and for all $z$ not too close to the spectrum of $\frac{1}{n}XX^T$:

$$P\left(|\text{Tr}(A (R(z) - E[R(z)]))| \geq t\right) \leq Ce^{-cnt^2} + Ce^{-cn},$$

for some numerical constants $C, c$ independent of $n, p$. We see from (0.2) the important benefit gained with our concentration hypothesis on $X$: it provides simple quasi-asymptotic results on the convergence of the resolvent, while most of the results on random matrices are classically expressed in the limiting regime where $n, p \rightarrow \infty$.

The condition $\|A\|_F \leq 1$ answers our second question: a specificity of our approach is to control the convergence of the resolvent with the Frobenius norm at a speed of order $O(1/\sqrt{n})$. The concentration inequality (0.2) means that all linear forms of $R(z)$, which are 1-Lipschitz for the Frobenius norm, have a standard deviation of order $O(1/\sqrt{n})$; this is crucial to be able to estimate quantities expressed in (0.1). Generally, the only studied linear forms of the resolvent are the Stieltjes transform $g(z) = -\frac{1}{p} \text{Tr}(R(z))$ (it is $1/\sqrt{n}$-Lipschitz so its standard deviation is of order $O(1/\sqrt{n})$ which is a classical result although not exactly under a concentration of measure assumption) or projections on deterministic vectors $u^T R(z)u$, for which only the concentration in spectral norm with a speed of order $O(1/\sqrt{n})$ is needed.

Those remarks gain a real importance when we are able to estimate the expectation of $R(z)$ with a deterministic equivalent (that we can compute). In this article, we look

---

1Some of these random vectors still satisfy what is referred to as “convex concentration” hypotheses and their sample covariance matrix may still be studied but to the expense of more advanced control (see [LC18] and a coming follow-up work).

2or $λ$-Lipschitz with $λ \leq O(1)$
for a closeness relation in Frobenius norm:
\[
\left\| E[R(z)] - \hat{R}(z) \right\|_F \leq O \left( \frac{1}{\sqrt{n}} \right).
\]

We may then replace in (0.2) the term “\(E[R(z)]\)” by \(\hat{R}(z)\) which we are able to compute from the expectations and covariances of the columns \(x_1, \ldots, x_n\).

Note that we do not assume that the columns are identically distributed; in particular, the means and covariances can be all different (although they have to satisfy some boundedness properties expressed in Assumptions 0.6 and 0.7). This remark may be related to the studies made of matrices \(X\) with a variance profile; but this is here even more general because the laws of the columns are not solely defined from their means and covariances (although the spectral distribution of \(\frac{1}{n}XX^T\) just depends on these quantities).

The extension of Marcenko Pastur result to non-identically distributed columns was well known; one can cite for instance [WCDS12, Yin20] treating a very similar problem but with different assumption of concentration (just some moments of the entries need to be bounded) the result is then given in the form of a limit (not a concentration lemma but with different assumption of concentration (just some moments of the entries need to be bounded) the result is then given in the form of a limit (not a concentration inequality), [KA16] showing that no eigenvalues lie outside of the support with Gaussian hypotheses and [DKL22] imposing some weak isotropic conditions on the different covariances.

The main issue raised by the multiple distribution hypothesis is that the deterministic equivalent is then defined from the \(n\) diagonal entries of a diagonal matrix\(^3\)

\[
\hat{\Lambda}^z = \text{Diag}(\hat{\Lambda}^z_i)_{i \in [n]} \in \mathcal{D}_n(C)
\]

solution to:

\[
\forall i \in [n] : \quad \hat{\Lambda}^z_i = z - \frac{1}{n} \text{Tr} \left( \Sigma_i \tilde{Q}^z \right) \quad \text{with} \quad \tilde{Q}^z = I_p - \frac{1}{n} \sum_{i=1}^n \frac{\Sigma_i}{\tilde{\Lambda}^z_i}^{-1}, \quad (0.3)
\]

in which \(\Sigma_i = E[x_i x_i^T]\). To our knowledge, the proof existence and uniqueness of these equations is an original contribution of the present paper. It seems that the resort in [WCDS12] to Vitali’s theorem to extend the definition of the deterministic equivalent of the resolvent to the whole set \(C_+\) gives a rigorous definition to the Stieltjes transform but does not set the existence and uniqueness of the solutions to (0.3) that is effectively used to estimate the Stieltjes transform in practice – besides, this approach also do not provide any converging bound).

The difficulties are (i) to prove the existence and uniqueness of \(\hat{\Lambda}^z\) and (ii) to ensure some stability properties\(^4\) on this equation eventually allowing us to assert that \(\|E[R(z)] - \hat{R}(z)\|_F \leq O(1/\sqrt{n})\), where \(\hat{R}(z) \equiv \frac{1}{z} \tilde{Q}^{\hat{\Lambda}^z}\). Those two difficulties disappear when most of the \(\Sigma_1, \ldots, \Sigma_n\) are equal; in other words, when we have a finite number of distinct distributions for the \(x_i\)’s (see [LC18]). When they are all different, our solution consists in introducing a convenient semi-metric\(^5\) \(d_s\) on which the fixed point equation satisfied by \(\hat{\Lambda}^z\) is contractive, leading (after still some work since a semi-metric is not as easy to treat as if \(d_s\) were a true metric) to existence, uniqueness and stability properties. This semi-metric, quite similar to the one already introduced in [LC20] to study robust estimators, is defined for any \(D, D' \in \mathcal{D}_n(H)\) as:

\[
d_s(D, D') = \frac{\| D - D' \|}{\sqrt{3(D)\cdot 3(D')}}.
\]

\(^3\)The interest to resort to a diagonal matrix of \(\mathcal{M}_n\) rather than to a vector of \(\mathbb{R}^n\) will be clearer later – mainly to employ Proposition 1.10 in a natural formalism.

\(^4\)Conceptually, it means that if we have a diagonal matrix \(L \in \mathcal{M}_n\) satisfying \(\forall i \in [n] : L_i \equiv z - \frac{1}{n} \text{Tr} \left( \Sigma_i \tilde{Q} \right)\) then \(L\) is “close” to \(\hat{\Lambda}^z\).

\(^5\)A semi metric is defined as a metric that does not satisfy the triangular inequality.
This semi-metric appears as a central object in random matrix theory, one can indeed point out the fact that any Stieltjes transform is 1-Lipschitz under this semi-metric (see Proposition A.1). It relates to Poincaré metric, the hyperbolic metric writes indeed $d_{\Pi}(z, z') = \cosh(d(z, z')^2 - 1)$. Apart from the book [EH70] that provided a groundwork for the introduction of such a metric, this approach already gained some visibility in [HFS07] [KLW13] or in a context closer to ours for the study of Wishart matrices and in a squared form in [AKE16]. We prefer an expression proportional to $\|D - D'\|$ because it is more adapted to comparison with classical distance on $D_n(\mathbb{H})$, as it is done in Proposition 6.1 that subsequently provides bounds to the convergence speed. Let us outline that once the appropriate semi-metric is identified, contractivity properties are not sufficient to prove the existence and uniqueness to (0.3): one also needs to introduce the correct space over which the mapping is contractive ($D_{I_*} \equiv \{ D \in D_n(\mathbb{H}), \frac{D}{I} \in D_n(\mathbb{H})\}$).

Let us now present more precisely our assumptions and main results.

Assumptions and Main Results

Let us start with the deterministic results concerning the diagonal matrices $\tilde{\lambda}^z$, which do not require any particularly constraining assumption.

**Theorem 0.1.** Given $n$ nonnegative symmetric matrices $\Sigma_1, \ldots, \Sigma_n \in \mathcal{M}_p$, for all $z \in \mathbb{H}$, the equation:

$$\forall i \in [n], L_i = z - \frac{1}{n} \text{Tr} \left( \Sigma_i \left( I_p - \frac{1}{n} \sum_{i=1}^{\frac{n}{p}} \Sigma_i \right)^{-1} \right)$$

(0.4)

admits a unique solution $L \in D_n(\mathbb{H})$ that we denote $\tilde{\lambda}^z$.

Letting $\tilde{R} : z \mapsto \left( \frac{1}{n} \sum_{i=1}^{\frac{n}{p}} \frac{\Sigma_i}{L_i} - zI_p \right)^{-1}$, we can construct with $\tilde{\lambda}^z$ a Stieltjes transform whose associated distribution converges towards the spectral distribution of $\frac{1}{n}XX^T$, where $X = (x_1, \ldots, x_n)$ and for all $i \in [n]$, $\Sigma_i = E[x_i x_i^T]$.

**Theorem 0.2.** The mapping $\tilde{g} : z \mapsto \frac{1}{\nu} \Pi(\tilde{R})$ is the Stieltjes transform of a measure $\tilde{\mu}$ of compact support $\tilde{S} \subset \mathbb{R}_+$.

Let us now set the stage for the results on $X$. We want to track the speed of convergence of our various quantities of interest as a function of $n$ to provide a quasi asymptotic result that can be used for sufficiently large values of $n$. We will then assume that all our quantities depend on $n$, $X = X_n$ but also $p = p_n$ that should not be too large compared to $n$.

**Assumption 0.3.** There exists a constant $K > 0$ such that $\forall n \in \mathbb{N}, p \leq Kn$.

The concentration hypothesis on $X = X_n$ is a bit involved: it is issued from the concentration of measure theory and concerns a wide range of random vectors, in particular any random matrix $X = \Phi(Z)$ with $Z \sim \mathcal{N}(0, I_d)$, $d \in \mathbb{N}$, and $\Phi : (\mathbb{R}^d, \| \cdot \|) \rightarrow (\mathcal{M}_{p,n}, \| \cdot \|_F)$, $\lambda$-Lipschitz. For more examples, see [Led05].

**Assumption 0.4.** There exist two constants $C, c > 0$ such that for all $n \in \mathbb{N}$ and for all 1-Lipschitz mapping $f : (\mathcal{M}_{p,n}, \| \cdot \|_F) \rightarrow (\mathbb{R}, | \cdot |)$:

$$\forall t > 0 : \mathbb{P}(\|f(X) - E[f(X)]\| \geq t) \leq Ce^{-(t/c)^2}.$$
A third natural and fundamental hypothesis is to assume that the $n$ columns of $X = (x_1, \ldots, x_n)$ are independent. Again, we do not assume that $x_1, \ldots, x_n$ are identically distributed: we can possibly have $n$ different distributions for the columns of $X$.

**Assumption 0.5.** $X$ has independent columns $x_1, \ldots, x_n \in \mathbb{R}^p$.

Let us note for simplicity, for any $i \in [n]$:

$$\mu_i \equiv E[x_i] \quad \Sigma_i \equiv E[x_i x_i^T] \quad \text{and} \quad C_i \equiv \Sigma_i - \mu_i \mu_i^T.$$  

It is easy to deduce from Assumption 0.4 (see [LC18]) that there exists a constant $K > 0$ such that for all $n \in \mathbb{N}$, $\|C_i\| \leq O(1)$. But to have the best convergence bounds, we also need to impose\(^6\)

**Assumption 0.6.** $\exists K > 0$ such that $\forall n \in \mathbb{N}, \forall i \in [n] : \|\mu_i\| \leq K$.

We conclude with a last assumption which is likely central to precisely approximate the support of the spectral distribution.\(^7\) Although we are unsure of its importance, our line of arguments could not avoid it; it is nonetheless quite a weak constraint in view of the practical use of our result.

**Assumption 0.7.** $\exists K > 0$ such that $\forall n \in \mathbb{N}, \forall i \in [n] : \Sigma_i \geq K I_p$.

Recalling that our goal is to estimate the spectral distribution of $\frac{1}{n}XX^T$, we denote $\lambda_1 \geq \cdots \geq \lambda_p \geq 0$ the $p$ eigenvalues of $\frac{1}{n}XX^T$. We wish to estimate:

$$\mu = \frac{1}{p} \sum_{i=1}^{p} \delta_{\lambda_i}.$$  

Now, let us remark that the spectrum of $\frac{1}{n}XX^T$ is closely related to the spectrum of $\frac{1}{n}X^TX$ via the equivalence:

$$\lambda \in \operatorname{Sp} \left( \frac{1}{n}XX^T \right) \setminus \{0\} \iff \lambda \in \operatorname{Sp} \left( \frac{1}{n}X^TX \right) \setminus \{0\}.$$  

As a consequence, one of the two matrices has $|n - p|$ supplementary zeros in its spectrum. Since those zeros do not carry any specific information about the distribution of $\frac{1}{n}XX^T$, we will remove them from our study to avoid unnecessary complications. The $\min(p,n)$ first entries of $\sigma(\frac{1}{n}XX^T)$ and $\sigma(\frac{1}{n}X^TX)$ are the same (and some of them can cancel), we thus naturally introduce the sets:

$$S \equiv \{ E[\lambda_i], i \in [n] \} \quad \text{and} \quad S_{-0} \equiv \{ E[\lambda_i], i \in [\min(p,n)] \},$$

(we have the inclusion $S \subset S_{-0} \cup \{0\}$, but possibly, $0 \in S_{-0}$). To avoid the issue on zero, instead of studying, as it is usually done, the resolvent $R(z) = (\frac{1}{n}XX^T - zI_p)^{-1}$, we rather look at:

$$Q^z \equiv (I_p - \frac{1}{zn}XX^T)^{-1},$$

that has the advantage of satisfying $\|Q^z\| \leq O(1)$, for all $z \in S_{-0}$, which will allow us to show the concentration of $Q^z$ even for $z$ close to zero when $n < p$ (and $0 \notin S_{-0}$).

Given a set $T \subset C$, we note for any $\varepsilon > 0$, $T^\varepsilon = \{ z \in C, \exists t \in T, |z - t| \leq \varepsilon \}$. We introduce the semi-norm $\|f\|_{S^a}$, defined for any $f \in F(C)$ as:

$$\|f\|_{S^a} = \sup_{z \in C, \exists t \in S^a} |f(z)|.$$  

\(^6\)In [LC18], we only supposed that $\|\mu_i\| \leq O(\sqrt{n})$, however, then we only had an approximation of the resolvent with the spectral norm, here we will provide a similar convergence result with the Frobenius norm.

\(^7\)The values of the Stieltjes distribution $g(z)$ can be approximated for $z \in C$ sufficiently far from the real axis or for $z \in \mathbb{R}$ sufficiently far from the support without this assumption.
Theorem 0.8. Under Assumptions $\text{0.3 0.7}$ given $\epsilon > 0$, there exist two constants $C, c > 0$, such that for all linear mapping $u : F(C) \to C \ L$-Lipschitz for the norm $\| \cdot \|_{S^c, u}$:

$$\forall t > 0: \quad P \left( \| u(g - \tilde{g}) \| \geq t \right) \leq Ce^{-cnpt^2} + Ce^{-cn}.$$  

In particular, following the preliminary inferences of the introduction, for any analytical mapping $f : C \to C$, since the integration on bounded paths of $C \ \setminus S_{\epsilon, o}$ is Lipschitz for the norm $\| \cdot \|_{S^c, u}$, we can approximate:

$$P \left( \left\| \int f(\lambda) d\mu(\lambda) - \frac{1}{2i\pi} \int f(z) g(z) dz \right\| \geq t \right) \leq Ce^{-cnpt^2} + Ce^{-cn},$$

for any closed path $S_{\epsilon, o} \subset \gamma \subset C \ \setminus S_{\epsilon, o}$ with length $l_\gamma$ satisfying $l_\gamma \leq O(1)$. There exists a correspondence between a distribution and its Stieltjes transform: denoting $\mu$ the spectral distribution of $\frac{1}{n} XX^T$, we have indeed for any real $a < b$:

$$\mu([a, b]) = \lim_{y \to 0} \frac{1}{\pi} \int_a^b \Im(g(x + iy)) dx.$$  

As seen on Figure 1, this measure is naturally close to $\tilde{\mu}$ defined for any real $a < b$ as $\tilde{\mu}(a, b) = \lim_{y \to 0} \frac{1}{\pi} \int_a^b \Im(\tilde{g}(x + iy)) dx$. For computation issues, the quantities $p$ and $n$ are chosen relatively small; the convergence of the iteration of the fixed point equation (0.4) defining $\Lambda$ is in particular very slow when the covariances $\Sigma_1, \ldots, \Sigma_n$ are all different from one another. Although the contractivity of (0.4) for the semi metric $d_\Lambda$ ensures the convergence of the iterations, the Lipschitz parameter is very close to 1 when $z = x + iy$ is close to the spectrum (that happens naturally when $y$ is chosen close to 0 to estimate $d\tilde{\mu}$).

We even have a stronger result that provides inferences on eigenvectors thanks to the approximation of the mapping $R : z \mapsto \left( \frac{1}{n} XX^T - z I_p \right)^{-1}$. Here $z$ must be far from zero when $0 \in S$ (even if $0 \notin S_{\epsilon, o}$). For this result, we introduce the semi-norm $\| \cdot \|_{F, S^c}$, defined for every $f \in F(C, M_p)$ as:

$$\| f \|_{F, S^c} = \sup_{z \in C \ \setminus S^c} \| f(z) \|_F.$$  

Theorem 0.9. Under Assumptions $\text{0.3 0.7}$ given $\epsilon > 0$, there exist two constants $C, c > 0$, such that for all linear form $u : F(C, M_p) \to \mathbb{R}$, $1$-Lipschitz for the norm $\| \cdot \|_{F, S^c}$:

$$P \left( \| u \left( R - \tilde{R} \right) \| \geq t \right) \leq Ce^{-cnpt^2} + Ce^{-cn}.$$  

The projections on deterministic vectors provide us with good estimates on isolated eigenvectors, but a concentration in spectral norm would have been sufficient for this kind of result. A key consequence of Theorem 0.9 lies in its also providing accurate estimates of projections on high dimensional subspaces $F \subset \mathbb{R}^p$; this is shown in Figure 2 that depicts some of these projections with increasing numbers of classes. Given $k \in \mathbb{N}$, we consider $B \equiv \{ \theta_1, \ldots, \theta_k \} \subset \mathbb{R}$, a (random) subset of $k$ eigenvalues of $\frac{1}{n} XX^T$, $E_B$ the eigenspace associated to those eigenvalues and $\Pi_B$ and $\Pi_F$, respectively the orthogonal projection on $E_B$ and $F$. If one can construct a deterministic path $\gamma$ such that

---

*It must be noted that the setting of Figure 2 does not exactly fall under the hypotheses of the paper (since here $\| E[|x|] \| \geq O(\sqrt{p})$, as the amplitude of the signals must be sufficiently large for the resulting eigenvalues to isolate from the bulk of the distribution when the number of classes is high ($\sqrt{p} \approx 14$ is not so large). However, even in this extreme setting the prediction are good.

*The number of classes is the number of different distributions that can follow the column vectors of $X$.

---

Page 7/39
Sample covariance - independent columns

Figure 1: Spectral distribution of $\frac{1}{n}XX^T$ and its deterministic estimate obtained from $\tilde{\Lambda}$ for $n = 160$ and $p = 80$. Introducing $P$ an orthogonal matrix chosen randomly and $\Sigma \in D_p$ such that for $j \in \{1, \ldots, 20\}, \Sigma_j = 1$ and for $j \in \{21, \ldots, 80\}, \Sigma_j = 8$, we chose (left) $\forall i \in [n], x_i \sim \mathcal{N}(0, \Sigma)$ and (right) $\forall i \in [n], x_i \sim \mathcal{N}(0, \Sigma_i)$, where $\Sigma_i = \Sigma$ and $\Sigma_{i+1} = P^T \Sigma_i P$ for all $i \in [n]$. The histograms would have been similar for any other concentrated vectors $x_1, \ldots, x_n$ having the same covariances and comparable observation diameter (see Definition 1.1)

$$\{E[\theta_i], i \in [k]\}^c \subset \gamma \subset C \setminus S_{\varepsilon_0}$$

then we can bound (since $\|\Pi_F\|_F = \sqrt{\dim(F)} \leq O(\sqrt{p})$):

$$P\left(\left| \frac{1}{p} \text{Tr}(\Pi_F \Pi_A) - \frac{1}{2ip\pi} \oint_{\gamma} \text{Tr}(\Pi_F \tilde{R}(z))dz \right| \geq t \right) \leq Ce^{-cnp\varepsilon^2} + Ce^{-cn},$$

Practical implications

Results on the large dimensional behavior of sample covariance matrix models are far from a novelty and the present article may at first sight be seen as yet another variation stacked on the pile of previous generalizations to the seminal work of Marčenko and Pastur [MP67].

The core motivation for the present development of a concentration of measure approach to random matrices does not arise as a mere additional mathematical exercise but more fundamentally as a much needed new toolbox to address the modern problem of efficient big data processing. Specifically, the question of mastering artificial intelligence and automated data processing in the era of the data deluge comes along with two structural difficulties: (i) efficient machine learning algorithms (starting with neural networks) inherently rely on a succession of non-linear operators which, as it appears, guarantee algorithm stability when these operators are Lipschitz; (ii) while hitting several instances of the curse of dimensionality (which make modern neural networks extremely resource consuming), algorithm stability also appears to be guaranteed when large and numerous data are considered; besides, and possibly most importantly, as evidenced in recent works [SLTC20], assuming the large dimensional data to be instances of concentrated random vectors is a satisfying model to theoretically track the behavior of real algorithms applied on real data (precisely, it can be shown, as with the example of artificial images produced with generative adversarial networks, that real data of practical interest – such as images, sounds, or modern natural language embeddings – are akin to concentrated random vectors).
Figure 2: Prediction of the alignment of the signals in the data towards the eigen space of the biggest eigen values of $\frac{1}{n}XX^T$ for $p = 200$, $n = kn_k$ where $n_k = 20$ and $k$ is the number of classes taking the values $k = 10, 15, 20, 30, 35, 42, 50, 62, 75$. The signals $u_1, \ldots, u_k$ are drawn randomly and independently following a law $\mathcal{N}(0, I_p)$. Then, for all $j \in [k]$ and $l \in [20]$, $x_{j+ln_k} \sim \mathcal{N}(u_j, I_p)$. (top and bottom left) representation of the spectral distribution of $\frac{1}{n}XX^T$ and its prediction with $\hat{\Lambda}$. (bottom right) representation (with marks) of the quantities $\text{Tr}(\Pi_{E_A}UU^T)$ and $\Pi_{E_A} \parallel$ and their prediction (smooth line) with the integration of, respectively, $\Im(\frac{1}{2\pi} \text{Tr}(\hat{Q}^{\hat{\Lambda}}))$ and $\Im(\frac{1}{2\pi} \Pi_{E_A} \parallel)$ on the path drawn in red on the other graphs. The line $y = k$ represents instances of $\text{Tr}(\Pi_{A}) = k$ that can be approximated integrating $\frac{1}{2\pi} \text{Tr}(\hat{Q}^{\hat{\Lambda}})$. The projection $\text{Tr}(\Pi_{E_A}UU^T)$ is a little lower than $k$ because the eigen vectors associated to the highest eigen values of $\frac{1}{n}XX^T$ are not perfectly aligned with the signal due to the randomness of $X$. 
Sample covariance - independent columns

These two features of modern data processing forcefully call for a revision of large dimensional statistics, starting with random matrix theory, towards a generic framework of concentration of measure theory for large dimensional vectors and matrices. The present work, buttressed on our detailed contribution [LC20], provides the mathematical base ground for applying this framework to elementary (yet already quite rich) models of large dimensional “realistic sample” covariance matrices. Note in particular that one important contribution of this paper is to generalize the results to unlimited number of classes (in classification tasks for instance) which is a classical question raised in machine learning problems.

Among other applications, our results are applicable to:

- track the behavior of the successive non-linear layers of neural networks applied on (concentrated random vector-modelled) real data; the main interest of our framework in this setting is that concentration of measure propagates seamlessly from layer to layer (which would not be valid if assumptions of independence between the data vector entries were made);
- evaluate the performance of most standard machine learning algorithms for classification and regression, all based on Lipschitz operators: from hyperplane-based (support vector machines) to graph-based (semi-supervised learning) classifications.

We consequently believe that the present work may serve, in the long run, as a springboard to simplify and systematize the analysis, improvement, and cost efficiency (thus reduce the carbon footprint) of modern large data processing algorithms.

The remainder of the article is structured as follows. We first provide some basic notations and results to handle concentrated random vectors (Section 1). Second, we explain why and how we have to place ourselves on an event of overwhelming probability where $Q^z$ is bounded to show our convergence results (Section 2). Third, we prove the concentration of the resolvent (Section 3) and provide a first, intermediary, deterministic equivalent which at this stage will not be numerically satisfying (Section 4). Then we introduce the semi-metric $d_s$ and prove Theorem 0.1 (the existence and uniqueness of $\tilde{\Lambda}$) (Section 5) and devise our second deterministic equivalent $\tilde{Q}^{\tilde{\Lambda}}$ that we can compute from $\tilde{\Lambda}$ and that satisfies Theorem 0.9 (Section 6). Eventually, we show that $\tilde{g}$ is a Stieltjes transform (Theorem 0.2) and that it approaches the Stieltjes transform of our spectral distribution (Theorem 0.8) (Section 7).
1 Concentration of Measure notations and general inferences

Concentration of measure is a phenomenon that appears for random vectors in high dimensions, in our case, for \( X \in \mathcal{M}_{p,n} \) or even for the column vectors \( x_i \in \mathbb{R}^p \). The idea is to track the concentration of real “observations” (or “functionals”) of those random vectors through dimensionality (here when \( n \) gets big since \( p \) is seen as an integer sequence depending on \( n \)). More precisely, we say that a random vector \( Z \in \mathbb{R}^n \) is concentrated if all the \( f(Z) \) for any \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) 1-Lipschitz satisfies interesting concentration inequality detailed below.

In what follows, we simplify greatly the approach, for a larger coverage of these notions, we advice the reader to consult our previous works in \cite{E18, E21}. To stay in the setting that interest us, we keep the index notation \( n \), the next section either \( M_{p,n} \), \( \mathbb{R}^p, \mathcal{M}_{p,n}, \mathcal{M}_n, \mathcal{D}_n \) and vector spaces of functions taking value in those vector spaces. In all those cases, the associated norm (or semi-norm) must be specified, but it will generally be the euclidean norm (i.e. the Frobenius norm or matricial spaces).

To control the convergence speeds, we will extensively use the notation \( O(a_n) \) or \( o(a_n) \) in inequalities, they are defined followingly, for two given sequences \( (a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{R}} \in \mathbb{R}_+^n \):

\[
\begin{align*}
    b_n &\leq O(a_n) \iff \exists K > 0, \forall n \in \mathbb{N}, b_n \leq K a_n \\
    b_n &\geq O(a_n) \iff a_n \leq O(b_n) \\
    b_n &\leq o(a_n) \iff \forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}, \forall n \geq n_\varepsilon, b_n \leq \varepsilon a_n
\end{align*}
\]

**Definition 1.1.** Given a sequence of normed (or semi-normed) vector spaces \( (E_n, \| \cdot \|_n)_{n \geq 0} \), a sequence of random vectors \( (Z_n)_{n \geq 0} \in \prod_{n \geq 0} E_n \), a sequence of positive reals \( (\sigma_n)_{n \geq 0} \in \mathbb{R}_+^n \) and a parameter \( q > 0 \), we say that \( Z_n \) is concentrated with an observable diameter of order \( O(\sigma_n) \) iff there exist \( C, c > 0 \) such that for all \( n \in \mathbb{N} \), for all 1-Lipschitz mapping \( f : E_n \rightarrow \mathbb{R} \):

\[
\forall t > 0 : \quad \mathbb{P} \left( \| f(Z_n) - E[f(Z_n)] \| \geq t \right) \leq C e^{-\left(\frac{t^q}{c \sigma_n}\right)^q} \tag{1.1}
\]

We note in that case \( Z \sim \mathcal{E}_q(\sigma) \). When \( \sigma_n \leq O(1) \), we have \( Z \sim \mathcal{E}_q(1) \), that we denote more simply \( Z \sim \mathcal{E}_q \).

With this new notation, we can rewrite Assumption \ref{0.4} followingly:

\[ X \sim \mathcal{E}_2. \]

We deduce immediately from this definition, that once he have a concentrated random vector (for instance \( Z \sim \mathcal{N}(0, I_n) \)) then we can construct an infinite number of them thanks to Lipschitz transformations as explained in next proposition.

**Proposition 1.2.** In the setting of Definition \ref{1.1} if we are given a second sequence of normed vector spaces \( (E'_n, \| \cdot \|'_n)_{n \geq 0} \), a sequence of positive parameters \( (\lambda_n)_{n \geq 0} \in \mathbb{R}_+^n \) and a sequence of \( \lambda_n \)-Lipschitz transformation \( \phi_n : E_n \rightarrow E'_n \), then we have the implication:

\[ Z \sim \mathcal{E}_q(\sigma) \implies \phi(Z) \sim \mathcal{E}_q(\lambda \sigma). \]

\[ ^{11} \text{A random vector } Z \text{ of } E \text{ is a measurable function from a probability space } (\Omega, \mathcal{F}, \mathbb{P}) \text{ to the normed vector space } (E, \| \cdot \|) \text{ (endowed with the Borel } \sigma\text{-algebra); one should indeed write } \tilde{Z} : \Omega \rightarrow E, \text{ but we abusively simply denote } Z \in E. \]
We pursue this first definition with an important notion called the deterministic equivalent, which is basically a relevant approximation of the expectation of a concentrated vector. It is in particular important when we look at linear forms on the random vector (our main objective is to find a deterministic equivalent for $R(z)$).

**Definition 1.3.** In the setting of Definition 1.1 and given a sequence of deterministic vectors $\tilde{Z} \in E_n$, we say that $Z$ is linearly concentrated around the deterministic equivalent $\tilde{Z}$ with an observable diameter of order $\sigma$ iff there exist two constants $C, c > 0$ such that for all $n \in \mathbb{N}$ and all $u : E_n \to \mathbb{R}$ 1-Lipschitz and linear:
\[
\forall t > 0 : \quad P \left( \left| u(Z_n - \tilde{Z}_n) \right| \geq t \right) \leq Ce^{-(t/c\sigma n)^g}.
\]

We note then $Z \in \tilde{Z} \pm E_q(\sigma)$.

This notation will be extensively employed on random variables (depending on $X$ and thus on $n$) for which the notion of Lipschitz concentration and of linear concentration is equivalent. Of course, $\tilde{Z} \propto E_q(\sigma) \implies Z \in E[Z] \pm E_q(\sigma)$. We have in particular the simple but important result:

**Lemma 1.4** ([LC18], Lemma 2.6). In the setting of Definition 1.1, given two deterministic vectors $\tilde{Z}, \tilde{Z}' \in E_n$, if $\|Z_n - Z_n'\|_n \leq O(\sigma_n)$, then we have the equivalence:
\[
Z \in \tilde{Z} \pm E_q(\sigma) \iff Z \in \tilde{Z}' \pm E_q(\sigma).
\]

This Lemma gives the global scheme of our paper, we first show that $R(z) \propto E_2(1/\sqrt{m})$, then we look for a deterministic matrix $R(z)$ such that $\|E[R(z)] - \tilde{R}(z)\|_F \leq O(1/\sqrt{m})$. We provide then a useful control on the norm of linearly concentrated matrices (the same bound is true for vectors of $\mathbb{R}^n$ since they can be seen as matrices and the spectral norm then coincides with the euclidean norm). While being a stronger property, the lipschitz concentration described in Definition 1.1 does not allow us to get a better bound.

**Proposition 1.5** ([LC18], Corollary 2.13). In the setting of Definition 1.3 if $E_n = \mathcal{M}_{p,n}$:
\[
Z \in \tilde{Z} \pm E_2(\sigma) \implies E[\|Z - \tilde{Z}\|] \leq O(\sqrt{n}\sigma)
\]

The concentration $Q^x \propto E_2(1/\sqrt{m})$ can not be rigorously stated with Proposition 1.2 because when $1/nXX^T$ is too close to zero, then $Q^x$ diverges and is far from a Lipschitz transformation of $X$ (see the beginning of subsection 2 for more details on this issue). We need here to remove the highly improbable event that some eigen values of $1/nXX^T$ get close to 1 (this is the term $Ce^{-cn}$ appearing in Theorem 0.8 and 0.9). This little correction is made possible thanks to next lemma.

**Lemma 1.6.** In the setting of Definition 1.7 if we are given a sequence of events $A_n$ such that $P(A_n) \geq O(1)$, then:
\[
Z \propto E_q(\sigma) \implies (Z | A) \propto E_q(\sigma).
\]

In particular, that implies that if a functional $f : E_n \to \mathbb{R}$ is only $\lambda$-Lipschitz on a subset $F_n \subset E_n$, and $P(Z_n \in F_n) \geq O(1)$, then the concentration $Z \propto E_q(\sigma)$ implies the existence of two constants $C, c > 0$, such that for all $n \in \mathbb{N}$:
\[
\forall t > 0 : \quad P \left( |f(Z_n) - E[f(Z_n) | Z_n \in F_n]| \geq t | Z_n \in F_n \right) \leq Ce^{-(t/c\lambda\sigma_n)^g}.
\]

Lemma 1.6 is a simple consequence of:

**Lemma 1.7** (Concentration of locally Lipschitz observations). Given a sequence of random vectors $Z_p : \Omega \to E_p$, satisfying $Z_p \propto E_q(\sigma_p)$, for any sequence of mappings $f_p : E_p \to \mathbb{R}$, which are $1$-Lipschitz on $Z_p(\Omega)$, we have the concentration $f_p(Z_p) \propto E_q(\sigma_p)$.
Sample covariance - independent columns

Proof. considering a sequence of median \( m_{f_p} \) of \( f_p(Z_p) \) and the (sequence of) sets \( S_p = \{ f_p \leq m_{f_p} \} \subset E_p \), if we note for any \( z \in E_p \) and \( U \subset E_p \), \( U \neq \emptyset \), \( d(z, U) = \inf \{ \| z - y \|, y \in U \} \), then we have for any \( z \in A \) and \( t > 0 \):

\[
\begin{align*}
  f_p(z) &\geq m_{f_p} + t \\
  f_p(z) &\leq m_{f_p} - t
\end{align*}
\]

since \( f_p \) is 1-Lipschitz on \( A \). Therefore, since \( z \mapsto d(z, S_p) \) and \( z \mapsto d(z, S_p^c) \) are both 1-Lipschitz on \( E \) and both admit 0 as a median \( \mathbb{P}(d(Z_p, S_p) \geq 0) = 1 \geq \frac{1}{2} \) and \( \mathbb{P}(d(Z_p, S_p) \leq 0) \geq \mathbb{P}(f_p(Z_p) \leq m_{f_p}) \geq \frac{1}{2} \),

\[
\mathbb{P}(\{|f_p(Z_p) - m_{f_p}| \geq t\}) \leq \mathbb{P}(d(Z_p, S_p) \geq t) + \mathbb{P}(d(Z_p, S_p) \leq t) \leq 2Ce^{-(t/c_{p})}.
\]

Proof. (of Lemma [1.6]) One just has to bound \( \forall t > 0 \):

\[
\begin{align*}
  \mathbb{P}(\{|f(Z_p) - f(Z_p')| \geq t | A\}) &\leq \frac{1}{\mathbb{P}(A)}\mathbb{P}(\{|f(Z_p) - f(Z_p')| \geq t\}) \\
  &\leq \frac{1}{\mathbb{P}(A)}\mathbb{P}(\{|f(Z_p) - f(Z_p')| \geq t\}) \leq Ce^{-(t/c_{p})^r},
\end{align*}
\]

We end this preliminary section with three important results of concentration of measure theory that will give us the basis for the approximation of \( \mathbb{E}[Q^2] \).

The first one is a combination of Hölder inequality with the characterization of the exponential concentration with the moments.

**Lemma 1.8.** Given an integer \( m \in \mathbb{N}, 2m \) random variables \( Z_1, \ldots, Z_m \) and \( m \) parameters \( a_1, \ldots, a_m, \sigma > 0, \ldots, \sigma > 0 \) such that for all \( i \in [m] \), \( Z_i \in a_i \pm \mathcal{E}_q(\text{sigma}_i) \):

\[
\mathbb{E}[Z_1 \cdots Z_m - a_1 \cdots a_m | A] \leq C2^{m}\left(\frac{\mathbb{E}[Z_1] \cdots \mathbb{E}[Z_m]}{q}\right)^{\frac{1}{m}} (\sigma_1 + |a_1|) \cdots (\sigma_m + |a_m|),
\]

for some constant \( C, c > 0 \).

**Proposition 1.9** (Hanson-Wright, [Ada15], [LC21], Proposition 8). Given a random matrix \( X \in \mathcal{M}_{p,n} \), if we assume that \( X \propto \mathcal{E}_2 \) and \( \|\mathbb{E}[X]\|_F \leq O(1) \), then for all deterministic matrix \( A \in \mathcal{M}_{p,n} \), we have the concentration\(^\text{12}\):

\[
X^TAX \in \mathbb{E}[X^TAX] \pm \mathcal{E}_2(\|A\|_F) + \mathcal{E}_1(\|A\|) \quad \text{in} \quad (\mathcal{M}_{n,}, \| \cdot \|_F)
\]

In particular, given any \( r \leq O(1) \) and any 1-Lipschitz and linear mapping \( u : \mathcal{M}_n \to \mathbb{R} \):

\[
\mathbb{E}\left[\left\|u(X^TAX - \mathbb{E}[X^TAX])\right\|_F^r\right] \leq O(\|A\|_F^r)
\]

**Proposition 1.10** ([LC21], Proposition 11). Given three random matrices \( D \in \mathcal{D}_n, X = (x_1, \ldots, x_n), Y = (y_1, \ldots, y_n) \in \mathcal{M}_{p,n} \) and a deterministic matrix \( \hat{D} \in \mathcal{D}_n \), such that:

- \( D \in \mathbb{E}[D] \pm \mathcal{E}_2 \) in \( (\mathcal{D}_n, \| \cdot \|) \),
- \( \|D - \hat{D}\|_F \leq O(1) \),
- \( \forall i \in [n], x_i \propto \mathcal{E}_2 \) and \( y_i \propto \mathcal{E}_2 \) in \( (\mathbb{R}^p, \| \cdot \|) \),
- \( \sup_{i \in [n]} \|\mathbb{E}[x_i]\|, \sup_{i \in [n]} \|\mathbb{E}[y_i]\| \leq O(1) \),

we can bound:

\[
\frac{1}{n}\|\mathbb{E}[X(D - \hat{D})Y^T]\|_F \leq O(1)
\]

\(^{12}\text{The notation } \mathcal{E}_2(\|A\|_F) + \mathcal{E}_1(\|A\|) \text{ means that we have the same concentration inequality as in [1.1] but with } Ce^{-(t/c_{p})^2} + Ce^{-(t/c_{p})}, \text{ replacing the right-hand term}\)
2 Resorting to a “concentration zone” for $Q^2$

Before studying the matricial case, let us first place ourselves in $\mathbb{R}$. We consider $X \in \mathbb{R}$, a Gaussian random variable with zero mean and variance equal to $\sigma^2 (X \sim \mathcal{N}(0, \sigma^2))$. In particular, although we work with unidimensional variables, there can still be a possible dependence on $n$, and we can write $X \sim \mathcal{E}_2$. The random variable $Q \equiv 1/(1 - X)$ is only defined if $X \neq 1$ and its law $f_Q$ can be computed on $\mathbb{R} \setminus \{1\}$ and satisfies:

$$f_Q(q) = \frac{e^{-(1 - q)^2/\sigma^2}}{\sqrt{2\pi}\sigma q^2}.$$  

Thus $Q$ is clearly not exponentially concentrated (when $q \to \infty$, $f_Q(q) \sim e^{-1/q^2}$ therefore the expectation of $Q$ is not even defined). However, if $\sigma$ is small enough (at least $\sigma \leq o(1)$), it can be interesting to consider the event $A_Q \equiv \{X \leq \frac{1}{2}\}$ satisfying $P(A_Q^c) \leq C e^{-1/2\sigma^2}$. The mapping $f : z \mapsto \frac{1}{\sqrt{z}}$ being 4-Lipschitz on $(-\infty, \frac{1}{2}]$, one sees that $(Q \mid A_Q) \in \mathcal{E}_2$. Following this setting, in the matricial cases, we also need to place ourselves in a concentration zone $A_Q$ where the fixed point $Q$ is defined; sufficiently small to retrieve an exponential concentration with $Q \mid A_Q$ but large enough to be highly probable.

The same resort to a concentration zone will take place for the resolvent matrix $Q^2 = (I_p - \frac{1}{n}XX^T)^{-1}$, for that purpose, we introduce in this section an event of high probability, $A_Q$, under which the eigen values of $\frac{1}{n}XX^T$ are far from 1, for all $z \in S^2$.

Let us start with a bound on $\|X\|$, Assumption 0.6 leads us to:

$$\|E[X]\| \leq \sqrt{n} \sup_{1 \leq i \leq n} \|E[x_i]\| \leq O(\sqrt{n})$$

then, we deduce from Proposition 1.5 applied to Assumption 0.4 that $E[\|X\|] \leq \|E[X]\| + O(\sqrt{n}) \leq O(\sqrt{n})$. Now, introducing a constant $\varepsilon > 0$ such that $O(1) \leq \varepsilon \leq O(1)$ and $\nu > 0$ defined with:

$$\sqrt{\nu} \equiv E\left[\frac{1}{\sqrt{n}}\|X\|\right] \leq O(1),$$

we denote:

$$A_\nu \equiv \left\{\frac{1}{n}\|XX^T\| \leq \nu + \varepsilon\right\}. \tag{2.1}$$

Since $\|X\|/\sqrt{n} \in \nu \pm \mathcal{E}_2(1/\sqrt{n})$, we know that there exist two constants $C, c > 0$ such that $P(A_\nu^c) \leq C e^{-cn}$. The mapping $M \rightarrow \frac{1}{n}MM^T$ is $O(1/\sqrt{n})$ Lipschitz on $X(A_\nu) \subset M_{p,n}$ and therefore, thanks to Lemma 1.6

$$\frac{1}{n}XX^T \mid A_\nu \sim \mathcal{E}_2\left(\frac{1}{\sqrt{n}}\right)$$

Let us note $\sigma : M_p \rightarrow \mathbb{R}^p$, the mapping that associates to any matrix the sequence of its eigen values in decreasing order. It is well known (see [CL96, Theorem 8.1.15]) that $\sigma$ is 1-Lipschitz (from $(M_p, \|\cdot\|_F)$ to $(\mathbb{R}^p, \|\cdot\|)$) and therefore if we denote $\lambda_1, \ldots, \lambda_p$ the eigen values of $\frac{1}{n}XX^T$ such that $\lambda_1 \geq \cdots \geq \lambda_n$, we have the concentration:

$$(\lambda_1, \ldots, \lambda_p) \mid A_\nu \in (E_{A_\nu}[\lambda_1], \ldots, E_{A_\nu}[\lambda_p]) \pm \mathcal{E}_2\left(\frac{1}{\sqrt{n}}\right).$$
From now on, to avoid unnecessary complications, we denote $S \equiv \{E_{A_{i}}[\lambda_{1}], \ldots, E_{A_{i}}[\lambda_{p}]\}$ and $S_{0} \equiv \{E_{A_{i}}[\lambda_{1}], \ldots, E_{A_{i}}[\lambda_{\min(n,p)}]\}$ (with the expectation taken on $A_{i}$). We show in the next lemma that the event

$$A_{Q} \equiv A_{\nu} \cap \{\forall i \in [\min(p, n)] : \lambda_{i} \in S^{c/2}\}$$

has an overwhelming probability.

**Lemma 2.1.** There exist $C, c > 0$ such that $\forall n \in \mathbb{N} \ P(A_{Q}^{c}) \leq Ce^{-cn}$.

**Proof.** Starting from the identity $S_{0}^{c} \cup i \in [p] [E_{A_{i}}[\lambda_{i}] - \frac{\varepsilon}{2}, E_{A_{i}}[\lambda_{i}] + \frac{\varepsilon}{2}]$, we see that $S_{0}^{c}$ is a union of, say, $d$ intervals of $\mathbb{R}$. There exists $2d$ indexes $i_{1} \leq \cdots \leq i_{d}$ and $j_{1} \leq \cdots \leq j_{d}$ in $[\min(p, n)]$ such that:

$$S^{c} = \bigcup_{k \in [d]} \left[ E_{A_{i_{k}}}[^{\lambda_{i_{k}}}] - \frac{\varepsilon}{2}, E_{A_{i_{k}}}[^{\lambda_{j_{k}}}] + \frac{\varepsilon}{2} \right].$$

Since $E_{A_{i}}[^{\lambda_{i}}] \geq 0$ and $E_{A_{i}}[^{\lambda_{j_{k}}}] \leq \sqrt{\nu} + O(e^{-cn}) \leq O(1)$, we can bound:

$$\varepsilon d \leq E_{A_{i}}[^{\lambda_{j_{k}}}] + \varepsilon \leq O(1),$$

That implies in particular that $d \leq O(1)$ because $\varepsilon \geq O(1)$. We can then bound thanks to the concentration of the $2d$ random variables $\lambda_{i_{1}}, \ldots, \lambda_{i_{d}}$ and $\lambda_{j_{1}}, \ldots, \lambda_{j_{d}}$:

$$P(A_{Q}^{c}) = P(\exists k \in [d] | \lambda_{i_{k}} - E_{A_{i}}[^{\lambda_{i_{k}}}]| > \varepsilon/2 \text{ or } |\lambda_{j_{k}} - E_{A_{i}}[^{\lambda_{j_{k}}}]| > \varepsilon/2)$$

$$\leq \sum_{k = 1}^{d} \left( P\left(|\lambda_{i_{k}} - E_{A_{i}}[^{\lambda_{i_{k}}}]| > \varepsilon/2\right) + P\left(|\lambda_{j_{k}} - E_{A_{i}}[^{\lambda_{j_{k}}}]| > \varepsilon/2\right) \right) \leq 2dC e^{-cn\varepsilon^{2}/4},$$

where $C$ and $c$ are the two constants appearing in the concentration inequality of $(\lambda_{1}, \ldots, \lambda_{p}) = \sigma(\frac{1}{n}XX^{T})$.

At that point of the paper we know that the eigenvalues of $\frac{1}{n}XX^{T}$ are most likely lying in the union of compact intervals $S^{c} \subset [-\varepsilon, \nu + \varepsilon]$.

### 3 Concentration of the resolvent $Q^{z} = (I_{n} - \frac{1}{n}XX^{T})^{-1}$.

Given a matrix $A \in \mathcal{M}_{p,n}(\mathbb{C})$, we denote $|A| = \sqrt{AA^{T}}$ ($AA^{T}$ is a nonnegative Hermitian matrix). With a simple diagonalization procedure, one can show for any Hermitian matrix $A$ the simple characterization of the spectrum of $|A|$:

$$\text{Sp}(|A|) = \{ |\lambda|, \lambda \in \text{Sp}(A) \} \quad (3.1)$$

It is possible to go further than the mere bound $|Q^{z}| \leq O(1)$ when $z$ is close to 0 and $p \leq n$, we are then able to show that $|Q^{z}| \leq O(|z|/(|z| + 1))$. When $p \geq n$ no such bound is true and it is then convenient to rather look at the coresolvent $\hat{Q}^{z} \equiv (I_{p} - \frac{1}{n\varepsilon}XX^{T})^{-1}$ that satisfies in that regime $|\hat{Q}^{z}| \leq O(|z|/(|z| + 1))$.

To formalize this approach, we introduce two quantities that will appear in our convergence speeds:

$$\kappa_{z} \equiv \begin{cases} |z| & \text{if } p \leq n \\ 1 + |z| & \text{if } n \leq p \end{cases} \quad \bar{\kappa}_{z} \equiv \begin{cases} 1 & \text{if } p \leq n \\ |z| & \text{if } n \leq p \end{cases}$$

Since $P(A_{Q}^{c}) \leq Ce^{-cn}$, for all $i \in [p], |E_{A_{i}}[^{\lambda_{i}}] - E_{A_{i}}[^{\lambda_{i}}]| \leq 2\sqrt{\sigma}Ce^{-cn}$, and we see that the definition is almost the same.
Sample covariance - independent columns

Note that both of them are bounded by 1 but they can tend to zero with $|z|$ depending on the sign of $p-n$, besides:

$$\kappa z \bar{\kappa} = \frac{|z|}{1 + |z|},$$

(3.2)

Note than in our formalism, the parameter $z$, as most quantities of the paper, is varying with $n$ (we do not assume that $O(1) \leq |z| \leq O(1)$ like $\varepsilon$. It is not a "constant".

Lemma 3.1. Under $A_Q$, $Q^z$ and $\bar{Q}^z$ can be defined on 0 and for any $z \in \mathbb{C} \setminus S^z_{-0}$:

$$O(\kappa z) I_p \leq |Q^z| \leq O(\kappa z) I_p \quad \text{and} \quad O(\bar{\kappa} z) I_p \leq |\bar{Q}^z| \leq O(\bar{\kappa} z) I_p$$

(for the classical order relation on hermitian matrices).

Proof. We can diagonalize the nonnegative symmetric matrix: $\frac{1}{n} XX^T = PD\Phi T$, with $D = \text{Diag}(\lambda_1, \ldots, \lambda_p)$ and $\Phi \in O_p$, an orthogonal matrix. There exists $q \in [p]$ such that $\lambda_{q+1} = \cdots = \lambda_p = 0$, and for all $1 \leq q, \lambda_i \neq 0$ (possibly $q = p$ or $q < \min(p, n)$). Then, if we denote $P_0 \in M_{p-p-q}$, the matrix composed of the $p-q$ last columns of $P$, $P_+$ the matrix composed of the rest of the $q$ columns, and $D_+ = \text{Diag}(\lambda_1, \ldots, \lambda_q)$, we can decompose:

$$Q^z = P \left( \frac{zI_p}{zI_p - D} \right) P^T = P_0 P_0^T + P_+ \left( \frac{zI_q}{zI_q - D_+} \right) P_+^T.$$

Since $P_0 P_0^T = 0$ and $P_+ P_+^T = 0$, we have: $|Q^z|^2 = P_0 P_0^T + P_+ \left( \frac{|z| I_q}{|zI_q - D_+|^2} \right) P_+^T$. We can bound:

$$O\left( \frac{|z|}{1 + |z|} \right) \leq \frac{|z|}{|z| + \nu + \varepsilon} \leq P_+ \left( \frac{|z| I_q}{|zI_q - D_+|^2} \right) P_+^T \leq \frac{|z|}{d(z, S_{-0}) - \frac{\varepsilon}{2}} \leq O\left( \frac{|z|}{1 + |z|} \right)$$

Therefore, in all cases ($p \leq n$ or $p \geq n$) $O(1) \leq |Q^z| \leq O(1)$. However, when $p \leq n$, we can precise those bounds:

- if $E_{A_Q}[\lambda_{\min(p, n)}] > \frac{\varepsilon}{2}$, then $P_0$ is empty, $P_+ = P$ and we see that $O\left( \frac{|z|}{1 + |z|} \right) \leq |Q^z| \leq O\left( \frac{|z|}{1 + |z|} \right)$,

- if $E_{A_Q}[\lambda_{\min(p, n)}] \leq \frac{\varepsilon}{2}$, the bound $d(z, S_{-0}) \geq \varepsilon$ implies that $|z| \geq \frac{\varepsilon}{2}$ and $O(\kappa z) = O\left( \frac{|z|}{1 + |z|} \right) \leq O(1) \leq O(\kappa z)$, therefore:

$$O(\kappa z) \leq O\left( \min(1, \kappa z) \right) I_p \leq |Q^z| \leq O(1 + \kappa z) I_p \leq O(\kappa z).$$

The inequalities on $\bar{Q}^z$ are proven the same way.

\[ \Box \]

Proposition 3.2. Given $z \in \mathbb{C} \setminus S^z_{-0}$, we have the concentrations $Q^z$ | $A_Q \propto E_2(\kappa z / \sqrt{n})$ in $(M_p, \| \cdot \|_F)$ and $\bar{Q}^z$ | $A_Q \propto E_2(\kappa z / \sqrt{n})$ in $(M_n, \| \cdot \|_F)$.

Proof. Noting $\Phi : M_{p,n} \rightarrow M_p(\mathbb{C})$ and $\bar{\Phi} : M_{p,n} \rightarrow M_n(\mathbb{C})$ defined as:

$$\Phi(M) = \left( I_p - \frac{MM^T}{zn} \right)^{-1} \quad \text{and} \quad \bar{\Phi}(M) = \left( I_n - \frac{M^TM}{zn} \right)^{-1},$$

\[14\]In theory, both the parameter $z$ and the set $S^z_{-0}$ depends on our asymptotic parameter $n$, so one should rigorously write $z_n \in S^z_{-0}(n)$
it is sufficient to show that $\Phi$ (resp. $\hat{\Phi}$) is $O(\kappa_z / \sqrt{n})$-Lipschitz (resp. $O(\tilde{\kappa}_z / \sqrt{n})$-Lipschitz) on $M_{n,p}^A \equiv X(A_Q)$. For any $M \in M_{n,p}^A$ and any $H \in M_{p,n}$, we can bound $\|M\| \leq (\nu + \varepsilon)\sqrt{n} \leq O(\sqrt{n})$ and:

$$
\|d\Phi_M \cdot H\|_F = \left\| \Phi(M) \right\|_F \left( \frac{1}{n} MH^T + H M^T \right) \Phi(M) \left\|_F
$$

We can now distinguish two different cases:

- if $p \leq n$ then:

  $$
  \|d\Phi_M \cdot H\|_F \leq O\left( \frac{\|H\|_F |z|}{(1 + |z|)^2 \sqrt{n}} \right) \leq O\left( \frac{\|H\|_F |z|}{(1 + |z|) \sqrt{n}} \right).
  $$

- if $p \geq n$, we know that $\|\Phi(M)\| \leq O(\kappa_z) \leq O(1)$ and $\|\hat{\Phi}(M)\| \leq O(\tilde{\kappa}_z) \leq O(\frac{1}{1 + |z|})$.

  We employ then the classical identity:

  $$
  \Phi(M)M = M^T \Phi(M)
  $$

  to be able to bound:

  $$
  \|d\Phi_M \cdot H\|_F \leq O\left( \frac{\|H\|_F |z|}{(1 + |z|)^2 \sqrt{n}} \right) \leq O\left( \frac{\|H\|_F |z|}{(1 + |z|) \sqrt{n}} \right).
  $$

Thus, in all cases, under $X(A_Q)$, $Q^z$ is an $O(\kappa_z / \sqrt{n})$-Lipschitz transformation of $X \propto \mathcal{E}_2(1)$ and as such, it satisfies the concentration inequality of the proposition thanks to Lemma 1.6. The same holds for $\hat{Q}^z$.

4 A first deterministic equivalent

One is often merely working with linear functionals of $Q^z$, and since Proposition 3.2 implies that $Q^z | A_Q \in E_{A_Q} Q^z \pm \mathcal{E}_2$, one naturally wants to estimate the expectation $E_{A_Q}[Q^z]$. In [LC18] is provided a deterministic equivalent $\tilde{Q}^z \in M_{p}(C)$ satisfying $\|E[Q^z] - \tilde{Q}^z\| \leq O(1 / \sqrt{n})$ for any $z \in \mathbb{R}^-$, we are going to show below a stronger result,

- with a Frobenius norm replacing the spectral norm,
- for any complex $z \in C \setminus S_{\varepsilon_0}$,
- for random vectors $x_1, \ldots, x_n$ having possibly different distributions (it was assumed in [LC18] that there was a finite number of classes)

An efficient approach, developed in particular in [Sil86], is to look for a deterministic equivalent of $Q^z$ depending on a deterministic diagonal matrix $\Delta \in \mathbb{R}^n$ and having the form:

$$
\tilde{Q}^z = (I_p - \Sigma^z)^{-1}
$$

where

$$
\Sigma^z \equiv \frac{1}{n} \sum_{i=1}^{n} \Sigma_i^z = \frac{1}{n} E[X \Delta^{-1} X^T].
$$

$^{15} M_{n,p}^A \subset \{ M \in M_{n,p}, \frac{1}{n} \|MM^T\| \leq \nu + \varepsilon \}$
Sample covariance - independent columns

One can then express the difference with the expectation of $Q_z$ under $A_Q$, $E_{A_Q}[Q_z]$ followingly:

$$E_{A_Q}[Q_z] - \hat{Q}^\Delta = E_{A_Q} \left[ Q_z \left( \frac{1}{zn} XX^T - \Sigma^\Delta \right) \hat{Q}^\Delta \right]$$

$$= \frac{1}{n} \sum_{i=1}^{n} E_{A_Q} \left[ Q_z \left( \frac{x_i x_i^T}{z} - \frac{\Sigma_i}{\Delta_i} \right) \hat{Q}^\Delta \right].$$

To pursue the estimation of the expectation, one needs to control the dependence between $Q_z$ and $x_i$. For that purpose, one uses classically the Schur identities:

$$Q_z = Q_{z-1} + \frac{1}{zn} Q_z^z x_i x_i^T Q_{z-1}$$

and

$$Q_z x_i \equiv \frac{Q_z^z x_i^T}{1 - \frac{1}{zn} x_i^T Q_z^z x_i}, \quad (4.2)$$

for $Q_{z-1} = (I_n - \frac{1}{zn} X_{-i} X_{-i}^T)^{-1}$ (recall that $X_{-i} = (x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) \in \mathcal{M}_{p,n}$).

The Schur identities can be seen as simple consequences to the the so called "resolvent identity" that can be generalized to any, possibly non commuting, square matrices $A, B \in \mathcal{M}_p$ with the identity:

$$A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1} \quad \text{or} \quad A^{-1} + B^{-1} = A^{-1}(A + B)B^{-1} \quad (4.3)$$

(it suffices to note that $A(A^{-1} + A^{-1}(B - A)B^{-1}) = I_p$).

Introducing the notation

$$A^z \equiv \text{Diag}_{1 \leq i \leq n}(z - \frac{1}{n} x_i^T Q_z^z x_i),$$

one can express thanks to the independence between $Q_z^z$ and $x_i$:

$$E_{A_Q}[Q_z] - \hat{Q}^\Delta = \frac{1}{n} \sum_{i=1}^{n} E_{A_Q} \left[ Q_{z-1} \left( \frac{x_i x_i^T}{A_i^z} - \frac{\Sigma_i}{\Delta_i} \right) \hat{Q}^\Delta \right]$$

$$+ \frac{1}{zn^2} \sum_{i=1}^{n} \frac{1}{\Delta_i} E_{A_Q} \left[ Q_z x_i x_i^T Q_{z-1} \Sigma_i \hat{Q}^\Delta \right]$$

$$= \epsilon_1 + \delta_1 + \delta_2 + \epsilon_2 \quad (4.4)$$

with:

$$\epsilon_1 = \frac{1}{n} E_{A_Q} \left[ Q_z X \left( \frac{\Delta - A^z}{z \Delta} \right) X^T \hat{Q}^\Delta \right]$$

$$\epsilon_2 = \frac{1}{zn^2} \sum_{i=1}^{n} \frac{1}{\Delta_i} E_{A_Q} \left[ Q_z x_i x_i^T Q_{z-1} \Sigma_i \hat{Q}^\Delta \right]$$

$$\delta_1 = \frac{1}{n} \sum_{i=1}^{n} E_{A_Q} \left[ Q_{z-1} \left( \frac{x_i x_i^T - E_{A_Q}[x_i x_i^T]}{\Delta_i} \right) \hat{Q}^\Delta \right]$$

$$\delta_2 = \frac{1}{n} \sum_{i=1}^{n} E_{A_Q} \left[ Q_{z-1} \left( \frac{E_{A_Q}[x_i x_i^T] - \Sigma_i}{\Delta_i} \right) \hat{Q}^\Delta \right].$$

From this decomposition, one is enticed into choosing, in a first step $\Delta \approx E_{A_Q}[A^z] \in \mathcal{D}_n(\mathbb{C})$ so that $\epsilon_1$ would be small. We will indeed take for $\Delta$, the deterministic diagonal matrix:

$$\hat{A}^z \equiv \text{Diag} \left( z - \frac{1}{n} \text{Tr}(\Sigma_i E_{A_Q}[Q_z]) \right)_{1 \leq i \leq n} \in \mathcal{D}_n(\mathbb{C}).$$

**Lemma 4.1.** Given $z \in \mathbb{C} \setminus S^{\epsilon}_{-\theta}$, $(A^z | A_Q) \propto E_z(\kappa_{z} / \sqrt{n})$ in $(\mathcal{D}_n(\mathbb{C}), \| \cdot \|)$. 

Page 18/39
Sample covariance - independent columns

Proof. Inspiring from the proof of Proposition 3.2, one can show easily that for all \( z \in \mathbb{C} \setminus S^2\), the mapping \( X \to \Lambda^z = \text{Diag}_i \in [n](z - \frac{1}{n} x_i^T Q z x_i) \) is a \( O(\kappa z / \sqrt{n}) \)-Lipschitz transformation from \( (\mathcal{M}_{p,n}, \| \cdot \| F) \) to \( (\mathcal{D}_n, \| \cdot \|) \) under \( \mathcal{A}_Q \) (since then \( \| x_i \| \leq O(\sqrt{n}) \)).

Putting the Schur identities (4.2), the relation (3.3) and the formula (4.6),

together one obtains:

\[
\frac{1}{z n} \hat{Q}^z X X^T = \hat{Q}^z - I_n,
\]

(4.5)

Remark 4.2. Although it is not needed, one also have a small variation of Lemma 4.1:

\[(\Lambda^z \mid A_Q) \in \hat{\Lambda}^z \pm \mathcal{E}_z(|z|/\sqrt{n}) \quad \text{in} \quad (\mathcal{D}_n, \| \cdot \|_F),
\]

since we know from (4.6) that \( \hat{\Lambda}^z = \frac{z}{\text{Diag}(\hat{Q}^z)} \) and, as seen in the proof of Proposition 3.2, one can show that the mapping \( X \to \frac{z}{\text{Diag}(\hat{Q}^z)} X \) is a \( O(|z|/\sqrt{n}) \)-Lipschitz transformation of \( X \). This new concentration is better for the class of Lipschitz observations it concerns (the observation Lipschitz for the Frobenius norm \( \| \cdot \|_F \)) but worse for the observable diameter when \( p \leq n \) and \( |z| \geq 1 > \kappa z = \frac{1+|z|}{|z|} \).

To be able to use Proposition 3.10 with \( \hat{\Lambda}^z \), one further needs:

Lemma 4.3. \( \| E_{A_Q}[\Lambda^z] - \hat{\Lambda}^z \| \leq O(\kappa z / n) \).

This lemma that seems quite simple actually requires three preliminary results whose main aim is to show that \( Q z \) is close to \( z \). Let us first try and bound \( \Lambda z \) thanks to (4.6).

Lemma 4.4. Given \( z \in \mathbb{C} \setminus S^2\):

\[ O(\|z\|) I_n \leq O \left( \frac{|z|}{\kappa z} \right) I_n \leq |\Lambda^z| \leq O \left( \frac{|z|}{\kappa z} \right) I_n \leq O(1 + |z|) I_n \]

Proof. The characterization (3.1) and Lemma 3.1 imply:

\[ O(\kappa z) \leq \inf_{z \in [n]} \text{Sp}(\hat{Q}^z) \leq |\text{Diag}(\hat{Q}^z)| = |\text{Diag}_i \in [n](e_i^T \hat{Q}^z e_i)| \leq \sup_{z \in [n]} \text{Sp}(\hat{Q}^z) \leq O(\kappa z) \]

where \( e_1, \ldots, e_n \) are the \( n \) vectors of the canonical basis of \( \mathbb{R}^n \) (\( e_j \in \mathbb{R}^n \) is full of 0 except in the \( j^{th} \) entry where there is a 1). One can then deduce the result of the lemma thanks to (4.6).

Then to be able to neglect the dependence relation between \( x_i \) and \( X \) under \( A_Q \) we introduce the following lemma.

Lemma 4.5 (Independence under \( A_Q \)). Given two mappings \( f : \mathbb{R}^p \to \mathcal{M}_p \) and \( g : \mathcal{M}_{p,n} \to \mathcal{M}_p \) such that under \( A_Q \), \( \| f(x_i) \|_F \leq O(\kappa f) \) and \( \| g(X_{-i}) \| \leq O(\kappa g) \), we can approximate:

\[ \| E_{A_Q}[f(x_i)g(X_{-i})] - E_{A_Q}[f(x_i)]E_{A_Q}[g(X_{-i})] \|_F \leq O \left( \kappa f \kappa g e^{-cn} \right), \]

for some constant \( c \geq O(1) \).
Sample covariance - independent columns

Proof. Let us continue $f|_{x(AQ)}$ and $g|_{X_i(AQ)}$ respectively on $\mathbb{R}^p$ and on $M_{p,n}$ defining for any $x \in \mathbb{R}^p$ and $M \in M_{p,n}$:

$$
\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in x_i(AQ) \\
0 & \text{otherwise}
\end{cases}
\quad \text{and} \quad
\tilde{g}(M) = \begin{cases} g(M) & \text{if } M \in X_i(AQ) \\
0 & \text{otherwise}
\end{cases}
$$

Let us estimate:

$$
E_{AQ} [f(x_i)g(X_{-i})] = E \left[ (\mathbb{1}_{x_i \in AQ}) f(x_i) g(X_{-i}) \right] / P(AQ)
$$

We know that:

$$
\mathbb{1}_{x_i \in AQ} (\omega) \leq \mathbb{1}_{x_i \in x_i(AQ) \setminus x_i(AQ)} (\omega),
$$

thus the inequality $\mathbb{1}_{x_i \in AQ} (\omega) \neq \mathbb{1}_{x_i \in x_i(AQ)} (\omega)$ only happens for $\omega \in A \setminus AQ$ which has a probability lower than $Ce^{-cn}$ for some constants $C, c > 0$. We can then bound:

$$
|E [\mathbb{1}_{x_i \in x_i(AQ) \setminus x_i(AQ)} f(x_i) g(X_{-i})] | \leq O(\kappa_f \kappa_g e^{-cn}),
$$

which allows us to see thanks to the independence between $X_i$ and $x_i$:

$$
E_{AQ} [f(x_i)g(X_{-i})] = E \left[ f(x_i) \tilde{g}(X_{-i}) \right] + \frac{P(\mathbb{1}_{x_i \in AQ})}{P(AQ)} E \left[ f(x_i) \tilde{g}(X_{-i}) \right] + O(\kappa_f \kappa_g e^{-cn})
$$

$$
= E \left[ f(x_i) \right] E \left[ \tilde{g}(X_{-i}) \right] + O(\kappa_f \kappa_g e^{-cn})
$$

$$
= E_{AQ} [f(x_i)] E_{AQ} [g(X_{-i})] + O(\kappa_f \kappa_g e^{-cn})
$$

As it was done in Proposition 3.2 one can show that $u^TQ^z\cdot x_i$ and $u^TQ_{-i}^z x_i$ are both $\kappa_z$-Lipschitz transformations of $X \sim \mathcal{E}_2$ and deduce:

Lemma 4.6. Given a deterministic vector $u \in \mathbb{R}^p$, we have the concentration:

$$
u^TQ_{-i}^z x_i | A_Q \in O(\kappa_z) \pm \mathcal{E}_2(\kappa_z)
$$

Proof. As it was done in Proposition 3.2 one can show that $u^TQ_{-i}^z x_i$ is a $\kappa_z$-Lipschitz transformations of $X \sim \mathcal{E}_2$ and deduce the concentration. Besides, one can bound thanks to Lemma 4.5 and our assumptions:

$$
|E_{AQ} [u^TQ_{-i}^z x_i] | \leq |u^T E_{AQ} [Q_{-i}] x_i| + O(\kappa_z) \leq O(\kappa_z)
$$

Lemma 4.7. Given $z \in \mathbb{C} \setminus S_{e_0}^c$, for all $i \in [n]$, $\|E_{AQ} [Q^z - Q^z_{-i}] \| \leq O(\sqrt{\frac{n}{i}})$, the same way, one has $\|E_{AQ} [Q^z - Q^z_{-i}]^2 \| \leq O(\frac{\sqrt{n}}{i})$.

Proof. As we saw with Lemma 4.5 we can consider that under $AQ$, $X_{-i}$ and $x_i$ are almost independent. We now omit the exponent “$z$” on $Q$ and $Q_{-i}$ for simplicity. Given two deterministic vectors $u, v \in \mathbb{R}^p$, we can deduce from Lemma 4.6 and the estimation of the product of concentrated random variables given in Lemma 1.8:

$$
|u^T E_{AQ} [Q - Q_{-i}] v| \leq \left| E_{AQ} \left[ \frac{u^T Q_{-i} x_i a_i^T Q_{-i} v}{n \Lambda_i} \right] \right|
$$

$$
\leq \bar{\kappa}_z \frac{E_{AQ} [u^T Q_{-i} x_i | x_i^T Q_{-i} v]}{n \Lambda_i} \leq O \left( \frac{\bar{\kappa}_z}{\bar{\kappa}_z} \right) \leq O \left( \frac{\bar{\kappa}_z}{\bar{\kappa}_z} \right).
$$

\footnote{for $x_p, y_p \in M_p$ and $(a_p)_{p \in R^n}$ the notation $x_p = y_p + O(\|x_p - y_p\|)$ signifies that $\|x_p - y_p\| \leq O(a_p)$}
thanks to the formula $\kappa_{z}k_{z} = \frac{|z|}{|z|^{2}}$.

For the difference of the squares, let us bound with the same justifications:

$$\left| u^{T}E_{A_{Q}}[Q]^{2} - |Q_{-i}|v \right| = \left| u^{T}E_{A_{Q}}[QQ - Q_{-i}Q_{-i}]v \right|$$

$$= \left| E_{A_{Q}} \left[ \frac{u^{T}Q_{-i}x_{i}x_{i}^{T}Q_{-i}v}{n\Lambda_{i}} \right] \right| + \left| E_{A_{Q}} \left[ \frac{u^{T}Q_{-i}Q_{-i}x_{i}x_{i}^{T}Q_{-i}v}{n\Lambda_{i}} \right] \right|$$

$$\leq O \left( \frac{\kappa_{z}k_{z}^{2}}{|z|^{2}} \right) \leq O \left( \frac{\kappa_{z}^{2}}{n} \right),$$

$$\blacksquare$$

**Proof of Lemma 4.8.** The proof of the concentration of $\Lambda^z$ was already presented before the lemma. We are just left to bound:

$$\left\| \hat{\Lambda}^z - E_{A_{Q}}[\Lambda^z] \right\|$$

$$\leq \frac{1}{n} \sup_{i \in [n]} \left( \left| \text{Tr} \left( \Sigma_i E_{A_{Q}}[Q - Q_{-i}] \right) \right| + \left| \text{Tr} \left( (\Sigma_i - E_{A_{Q}}[x_i x_i^{T}]) E_{A_{Q}}[Q_{-i}] \right) \right| + \left| \text{Tr} \left( E_{A_{Q}}[x_i x_i^{T}] E_{A_{Q}}[Q_{-i}] - E_{A_{Q}}[x_i x_i^{T} Q_{-i}] \right) \right| \right) \leq O \left( \frac{\kappa_{z}}{n} \right) \quad (4.7)$$

with the same arguments as in the proof of Lemma 4.7.

$$\blacksquare$$

One can directly deduce from the concentration of $\Lambda^z$ and Lemma 4.1 a bound on its expectation $\hat{\Lambda}^z$.

**Lemma 4.8.** Given $z \in C \setminus S_{0}^{c}$:

$$O \left( |z|I_{n} \right) I_{n} \leq O \left( \frac{|z|}{\kappa_{z}} \right) I_{n} \leq \left| \hat{\Lambda}^z \right| \leq O \left( \frac{|z|}{\kappa_{z}} \right) I_{n} \leq O \left( 1 + |z| \right) I_{n}$$

**Proof.** One already know from Lemma 4.4 that $O \left( \frac{|z|}{\kappa_{z}} \right) \leq |\hat{\Lambda}^z| \leq O \left( \frac{|z|}{\kappa_{z}} \right)$ then it suffices to bound thanks to (4.7): $\left\| \hat{\Lambda}^z - E_{A_{Q}}[\Lambda^z] \right\| \leq O \left( \frac{\kappa_{z}}{n} \right) \leq O \left( \frac{1}{n} \right)$.  

$$\blacksquare$$

Let us add a simple corollary that will be of multiple use since $\Lambda^z_i$ is generally appearing in the denominator.

**Corollary 4.9.** Given $z \in C \setminus S_{0}^{c}$:

$$\frac{1}{\Lambda_{i}^{z}} \left| A_{Q} \right| = \frac{1}{\Lambda_{i}^{z}} \pm F_{2} \left( \frac{k_{z}}{\sqrt{n}|z|} \right) \quad \text{in} \quad (D_{n}, ||.||_{F}).$$

**Proof.** The concentration is just a consequence of identity 4.6 and the concentration of $\hat{Q}^z$ given in Proposition 3.2. We can further bound thanks to Lemma 4.1, 4.4, 4.3 and 4.8:

$$\left\| E_{A_{Q}} \left[ \frac{1}{\Lambda^z} \right] \right\|_{F} \leq \left\| E_{A_{Q}} \left[ \Lambda^z - E_{A_{Q}}[\Lambda^z] \right] \right\|_{F} + \left\| E_{A_{Q}}[\Lambda^z - \hat{\Lambda}^z] \right\|_{F}$$

$$\leq O \left( \frac{\kappa_{z}k_{z}^{2}}{|z|^{2}\sqrt{n}} \right) + O \left( \frac{\kappa_{z}k_{z}^{2}}{|z|^{2}\sqrt{n}} \right) \leq O \left( \frac{\kappa_{z}}{|z|\sqrt{n}} \right)$$

$$\blacksquare$$

We can now prove the main result of this subsections that allows us to set that $\hat{Q}^\hat{\Lambda}^z$ is a deterministic equivalent of $\hat{Q}^z$ (thanks to Lemma 1.3).
Sample covariance - independent columns

**Proposition 4.10.** Given $z \in \mathbb{C} \setminus S_{<0}$:

$$\| \tilde{Q}^{\Lambda} \| \leq O(\kappa_z) \quad \text{and} \quad \| E_{A_0}[Q^z] - \tilde{Q}^{\Lambda} \|_F \leq O \left( \frac{\kappa_z}{\sqrt{n}} \right).$$

To prove this proposition, we will bound the different elements of the decomposition (4.4). To bound $\varepsilon_1$, we will need the following lemma. The concentration is quiet sharp since we have:

$$\kappa_z \tilde{\kappa}_z = \frac{|z|}{1 + |z|}$$

**Lemma 4.11.** Given $z \in \mathbb{C} \setminus S_{<0}$, under $A_Q$,

$$Q^z X = X^T \tilde{Q}^z | A_Q \propto \mathcal{E}_2(\kappa_z \tilde{\kappa}_z)$$

and $\forall i \in [n]$, $\| E_{A_Q}[Q^z x_i] \| \leq O(\kappa_z \tilde{\kappa}_z)$. 

**Proof.** We follow the steps of the proof of Proposition 3.2. Depending on the sign of $p - n$, it is more convenient to work with the expression $Q^z X$ (when $p \leq n$) or with $X^T Q^z$ (when $p \geq n$). We just treat here the case $p \leq n$ and therefore look at the variations of the mapping $\Psi : \mathcal{M}_{p,n} \rightarrow \mathcal{M}_{p,n}(\mathbb{C})$ defined as:

$$\Psi(M) = \frac{1}{z} \left( I_p - \frac{M M^T}{z n} \right)^{-1} M.$$ 

to show the concentration of $Q^z X = \Psi(X)$. For all $H, M \in \mathcal{M}_{n,p} \equiv X(A_Q)$ and with the notation $\Phi(M) = \left( I_p - \frac{M M^T}{z n} \right)^{-1}$ given in the proof of Proposition 3.2:

$$\| d\Phi |_M \cdot H \| \leq \left\| \Psi(M) \frac{1}{nz}(M H^T + H M^T) \Psi(M) M \right\| + \| \Psi(M) H \| \leq O \left( \frac{|z| \| H \|}{(1 + |z|)^2} \right) + O \left( \frac{|z| \| H \|}{1 + |z|} \right) \leq O(\kappa_z \tilde{\kappa}_z \| H \|).$$

Thus, under $A_Q$, $\Psi$ is $O(\kappa_z \tilde{\kappa}_z)$-Lipschitz (for the Frobenius norm) and therefore $Q^z X \propto \mathcal{E}_2(\kappa_z \tilde{\kappa}_z)$.

To control the norm of $E_{A_Q}[Q^z x_i]$, let bound for any deterministic $u \in \mathbb{R}^p$ such that $\| u \| \leq 1$:

$$\| E_{A_Q}[u^T Q^z x_i] \| = |z| \| E_{A_Q} \left[ \frac{u^T Q^z x_i}{\Lambda_i^z} \right] \| $$

(thanks to Schur identities (4.2)). Then, we apply Lemma 1.8 to the concentrations given by Lemma 4.6 and Corollary 4.9 to obtain:

$$\| E_{A_Q}[u^T Q^z x_i] \| = |z| \| \frac{E_{A_Q}[u^T Q^z x_i]}{\Lambda_i^z} \| + O \left( \frac{|z| \kappa_z \tilde{\kappa}_z}{|z| \sqrt{n}} \right) \leq O \left( \frac{\kappa_z \tilde{\kappa}_z}{\sqrt{n}} \right)$$

**Proof of Proposition 4.11.** Let us note for simplicity $\tilde{Q} \equiv \| \tilde{Q}^{\Lambda} \|$. Looking at decomposition (4.4) we can start with the bound:

$$\| \varepsilon_1 \|_F = \left\| \frac{1}{zn} E_{A_Q} \left[ Q^z X (\tilde{\Lambda} - \Lambda) (\tilde{\Lambda})^{-1} X^T \right] \tilde{Q}^{\Lambda} \right\|_F \leq O \left( \frac{\kappa_{\tilde{Q}} \kappa_z \tilde{\kappa}_z}{|z| \sqrt{n}} \right) \leq O \left( \frac{\kappa_{\tilde{Q}}}{\sqrt{n}} \right)$$

obtained with the bound $\frac{1}{\Lambda_i^z} \leq O(\frac{1}{|z|}) \leq O(1)$ given by Lemma 4.8 and applying Proposition 4.10 with the hypotheses:
Sample covariance - independent columns

- $X \mid A_Q \propto \mathcal{E}_2$ and $\|E_{A_Q} [x_i]\| \leq O(1)$ Assumption 0.4
- $Q^2 X \mid A_Q \propto \mathcal{E}_2(\kappa_i \kappa_2)$ and $\|E_{A_Q} [Q^2 x_i]\| \leq O(\kappa_i \kappa_2)$ given by Lemma 4.1
- $\Lambda^2 \mid A_Q \in E_{A_Q}[\Lambda^2] + \mathcal{E}_2 \left( \frac{\kappa^2}{\sqrt{n}} \right)$ in $(D_n, \|\cdot\|)$ given by Lemma 4.1
- $\|E_{A_Q} [\Lambda^2] - \bar{\Lambda}^2\|_F \leq O(\kappa^2 / \sqrt{n})$ thanks to Lemma 4.3

Second, for any matrix $A \in \mathcal{M}_p(C)$ satisfying $\|A\|_F \leq 1$, let us bound thanks to Cauchy-Schwarz inequality:

\[
|\text{Tr}(A\varepsilon_2)| \leq \sqrt{\frac{1}{n^2}} E_{A_Q} \left[ \text{Tr} \left( A Q^2 X | \bar{\Lambda}^2 \right)^2 \right] \cdot \sqrt{\frac{1}{n^2}} \left[ \text{Tr} \left( \bar{Q}^\Lambda x_i x_i^T \Xi_i \Sigma_i \bar{Q} \bar{\Lambda}^\Xi \right) \right] \leq O \left( \frac{\kappa_{\hat{Q}}}{\sqrt{n}} \right)
\]

thanks to the bounds provided by our assumptions, and Lemmas 2.1, 4.3 and 4.11.

Third, we bound easily with Lemma 4.5 the quantity:

\[
||\text{Tr}(A\delta_1)|| = \frac{1}{n} \sum_{i=1}^n \|\text{Tr} \left( \bar{Q}^\Lambda A E_{A_Q} [Q^2 x_i x_i^T] \right) - \text{Tr} \left( \bar{Q}^\Lambda A E_{A_Q} [Q^2 x_i x_i^T] E_{A_Q} [x_i x_i^T] \right) \| \leq O \left( \frac{\kappa_{\hat{Q}}}{\sqrt{n}} \right)
\]

And we can bound $||\text{Tr}(A\delta_2)|| \leq O \left( \frac{\kappa_{\hat{Q}}}{\sqrt{n}} \right)$ since $||\Sigma_i - E_{A_Q} [x_i x_i^T]|| \leq O(\frac{1}{n})$ (as explained in the proof of Lemma 4.7).

Taking the supremum on $A \in \mathcal{M}_{p,n}(C)$ and putting the bounds on $\|\varepsilon_1\|_F, \|\varepsilon_2\|_F, \|\delta_1\|_F$ and $\|\delta_2\|_F$ together, we obtain:

\[
\left\| E_{A_Q} [Q^2] - \bar{Q}^\Lambda \right\|_F \leq O \left( \frac{\kappa_{\hat{Q}}}{\sqrt{n}} \right)
\]

So, in particular $\kappa_{\hat{Q}} \equiv \left\| \bar{Q}^\Lambda \right\| \leq \left\| E_{A_Q} [Q^2] \right\| + O \left( \frac{\kappa_{\hat{Q}}}{\sqrt{n}} \right)$, which implies that $\kappa_{\hat{Q}} \leq O(\kappa_2)$ as $\|E_{A_Q} [Q^2]\|$ since $\frac{1}{\sqrt{n}} = O(1)$. We obtain then directly the second bound of the proposition.

5 Introduction of the semi-metric $d_q$ and proof of Theorem 0.1

Proposition 4.10 slightly simplified the problem because while we initially had to estimate the expectation of the whole matrix $Q^2$, now, we just need to approach the diagonal matrix $\bar{\Lambda}^2 \equiv \text{Diag}(z - \frac{1}{n} \text{Tr}(\Sigma_i Q^2 x_i))_{i \in [n]}$. One is tempted to introduce from the pseudo identity (where $\hat{Q}$ was defined in (4.1)):

\[
\hat{\Lambda}^2 \approx z - \frac{1}{n} \text{Tr}(\Sigma_i \hat{Q}^2 x_i) \approx z - \frac{1}{n} \text{Tr}(\Sigma_i \hat{Q} \hat{\Lambda}^\Xi)
\]

a fixed point equation whose solution would be a natural estimate for $\bar{\Lambda}^2$. This equation is given in Theorem 5.1 we chose $\bar{\Lambda}^2$ satisfying

\[
\bar{\Lambda}^2 = z - \frac{1}{n} \text{Tr}(\Sigma_i \hat{Q} \hat{\Lambda}^\Xi).
\]
We are now going to prove that $\tilde{\Lambda}$ is well defined for any $z \in \mathbb{H}$ (where we recall that $\mathbb{H} \equiv \{z \in \mathbb{C}, \Re(z) > 0\}$) to prove Theorem 0.1.

Introducing the mapping:

$$\forall L \in \mathcal{D}_n(\mathbb{H}) : \ T(L) = z I_n - \text{Diag} \left( \frac{1}{n} \text{Tr} \left( \Sigma_i \hat{Q}^{L_i} \right) \right)_{1 \leq i \leq n},$$

we want to show that $T$ admits a unique fixed point. For that purpose, let us adapt tools already introduced in \cite{LC20} that rely on the introduction of a semi-metric $d_s$ defined\footnote{In \cite{LC20}, $d_s$ is only defined on $\mathcal{D}_n(\mathbb{R}_+)$ and the denominator appearing in the definition is then $\sqrt{D_i D_i'}$ instead of $\sqrt{\text{Tr}(D_i D_i')}$: the present adaptation does not change the fundamental properties of the semi-metric; the only objective with this new choice is to be able to set that $T$ is contractive for this semi-metric.} for any $D, D' \in \mathcal{D}_n(\mathbb{H})$ as:

$$d_s(D, D') = \sup_{1 \leq i \leq n} \frac{|D_i - D_i'|}{\sqrt{\text{Tr}(D_i D_i')}}$$

(it lacks the triangular inequality to be a true metric). This semi-metric is introduced to set Banach-like fixed point theorems.

**Definition 5.1.** Given $\lambda > 0$, we denote $\mathcal{C}^\lambda(\mathcal{D}_n(\mathbb{H}))$ (or more simply $\mathcal{C}^\lambda$ when there is no ambiguity), the class of functions $f : \mathcal{D}_n(\mathbb{H}) \rightarrow \mathcal{D}_n(\mathbb{H})$, $\lambda$-Lipschitz for the semi-metric $d_s$; i.e. satisfying for all $D, D' \in \mathcal{D}_n(\mathbb{H})$:

$$d_s(f(D), f(D')) \leq \lambda d_s(D, D').$$

when $\lambda < 1$, we say that $f$ is contracting for the semi-metric $d_s$.

This class presents an important number of stability properties that we list in the next proposition (the stability of the class through the sum is proven in \cite{A}) it was already done with a slightly different setting in \cite{LC20}, this new results allows to extend some results of \cite{AKE16} and provide more direct proof as it is done in Proposition A.5):

**Proposition 5.2.** Given three parameters $\alpha, \lambda, \theta > 0$ and two mappings $f \in \mathcal{C}^\lambda_\alpha$ and $g \in \mathcal{C}^\lambda_\theta$,

$$f^{-1} \in \mathcal{C}^\lambda_\alpha, \quad \alpha f \in \mathcal{C}^\lambda_\alpha, \quad f \circ g \in \mathcal{C}^{\lambda \theta}_\alpha, \quad \text{and} \quad f + g \in \mathcal{C}^{\max(\lambda, \theta)}_\alpha.$$

We can now present our fixed point theorem that has been demonstrated once again in \cite{LC20}:

**Theorem 5.3 (\cite{LC20} Theorem 3.15).** Given a subset $\mathcal{D}_f$ of $\mathcal{D}_n(\mathbb{H})$ and a mapping $f : \mathcal{D}_f \rightarrow \mathcal{D}_f$ with the imaginary part bounded from above and below (in $\mathcal{D}_f$), if it is furthermore contracting for the stable semi-metric $d_s$ on $\mathcal{D}_f$, then there exists a unique fixed point $\Delta^* \in \mathcal{D}_f$ satisfying $\Delta^* = f(\Delta^*)$.

We are going to employ this theorem on the mapping $T$ for $z \in \mathbb{H}$. We first restrict our study on a subset of $\mathcal{D}_n(\mathbb{H})$:

$$\mathcal{D}_T \equiv \{ D \in \mathcal{D}_n(\mathbb{H}) ; D/z \in \mathcal{D}_n(\mathbb{H}) \}$$

**Lemma 5.4.** For any $z \in \mathbb{H}$, $T(D_T) \subset D_T$.

**Proof.** Considering $z \in \mathbb{H}$, and $L \in D_T$ and $i \in [n]$, the decomposition $\hat{Q}^{L_i} = \left( I_p - \frac{1}{n} \sum_{i=1}^{\max(n, n)} \frac{\text{Tr}(L_i) \Sigma_i}{|L_i|^2} - \frac{1}{n} \sum_{i=1}^{\max(n, n)} \frac{\text{Tr}(L_i) \Sigma_i}{|L_i|^2} \right)^{-1}$ allows us to set thanks to the resolvent iden-
Let us bound for any $D$ on the quantity (4.3):

$$\exists (I^z(L)) = \exists(z) + \frac{1}{2mn} \text{Tr} \left( \Sigma_i \left( \bar{Q}^L + \tilde{Q}^L \right) \right)$$

$$= \exists(z) + \frac{1}{n} \text{Tr} \left( \Sigma_i \tilde{Q}^L \sum_{i=1}^{n} \frac{\exists(L_i) \Sigma_i}{|L_i|^2} \tilde{Q}^L \right) > 0$$

(since $\tilde{Q}^L \Sigma_i \tilde{Q}^L$ is a non-negative Hermitian matrix). The same way, one can show:

$$\exists (I^z(L), I^z(L')) = \exists(z) + \frac{1}{n} \text{Tr} \left( \Sigma_i \tilde{Q}^L \sum_{i=1}^{n} \frac{\exists(L_i) \Sigma_i}{|L_i|^2} \tilde{Q}^L \right) > 0$$

□

Let us now express the Lipschitz parameter of $I^z$ for the semi-metric $d_s$.

**Proposition 5.5.** For any $z \in \mathbb{H}$, the mapping $I^z$ is 1-Lipschitz for the semi-metric $d_s$ on $\mathcal{D}_{I^z}$ and satisfies for any $L, L' \in \mathcal{D}_{I^z}$:

$$d_s(I^z(L), I^z(L')) \leq \sqrt{(1 - \phi(z, L))(1 - \phi(z, L'))} d_s(L, L'),$$

where for any $w \in \mathbb{H}$ and $L \in \mathcal{D}_{I^z}$:

$$\phi(w, L) = \frac{\exists(w)}{\sup_{1 \leq i \leq n} \exists(I^w(L))} \in (0, 1).$$

**Proof.** Let us bound for any $L, L' \in \mathcal{D}_{I^z}$:

$$|I^z(L), I^z(L')| = \frac{1}{n} \text{Tr} \left( \Sigma_i \tilde{Q}^L \left( \frac{1}{n} \sum_{j=1}^{n} \frac{L_j - L'_j}{L_j L'_j} \Sigma_j \right) \tilde{Q}^L \right)$$

$$= \frac{1}{n} \text{Tr} \left( \Sigma_i \tilde{Q}^L \left( \frac{1}{n} \sum_{j=1}^{n} \frac{\exists(L_j) \Sigma_j}{\sqrt{\exists(L_j) \exists(L'_j)}} \sqrt{\frac{\exists(L_j) \exists(L'_j)}{L_j L'_j}} \Sigma_j \right) \tilde{Q}^L \right)$$

$$\leq d_s(L, L') \sqrt{\frac{1}{n} \text{Tr} \left( \Sigma_i \tilde{Q}^L \left( \frac{1}{n} \sum_{i=1}^{n} \frac{\exists(L'_i) \Sigma_i}{|L'_i|^2} \right) \tilde{Q}^L \right)}$$

$$\cdot \frac{1}{n} \text{Tr} \left( \Sigma_i \tilde{Q}^L' \left( \frac{1}{n} \sum_{i=1}^{n} \frac{\exists(L'_i) \Sigma_i}{|L'_i|^2} \right) \tilde{Q}^L' \right)$$

$$\leq d_s(L, L') \sqrt{(\exists(I^z(L), I^z(L')) - \exists(z)) (\exists(I^z(L'), I^z(L)) - \exists(z))},$$

thanks to Cauchy-Schwarz inequality and the identity

$$0 \leq \frac{1}{n} \text{Tr} \left( \Sigma_i \tilde{Q}^L \sum_{i=1}^{n} \frac{\exists(L'_i) \Sigma_i}{|L'_i|^2} \tilde{Q}^L (L) \right) = \exists(I^z(L), I^z(L')) - \exists(z)$$

issued from the proof of Lemma 5.4.

Dividing both sides of (5.2) by $\sqrt{\exists(I^z(L), I^z(L'))}$, we retrieve the wanted Lipschitz parameter.

□

The contractivity of $I^z$ is not fully stated in the previous proposition because, the Lipschitz parameter depends on the values of $L, L'$ and we want a bound uniform on $\mathcal{D}_{I^z}$. This will be done thanks to the two lemmas.
**Lemma 5.6** (Commutation between inversion and modulus of matrices). *Given an invertible matrix* \(M \in \mathcal{M}_p\), \(|M^{-1}| = |M|^{-1}\) *and for any* \(K > 0:\)

\[
\Im M^{-1} \geq K I_p \text{ or } \Re M^{-1} \geq K I_p \quad \Rightarrow \quad |M| \leq \frac{1}{K} I_p.
\]

*Proof.* We have the identity:

\[
|M^{-1}|^2 = M^{-1} M^{-1} = (M M)^{-1} = (|M|^2)^{-1}.
\]

Then we take the square root on both sides to obtain the first identity (recall that the modulus of a matrix is a nonnegative hermitian matrix). Now let us assume that \(\Im(M^{-1}) \geq K\), we know that:

\[
|M^{-1}|^2 = \Im(M^{-1})\Im(M^{-1})^T + \Re(M^{-1})\Re(M^{-1})^T
\]

\[
- i\Re(M^{-1})\Im(M^{-1})^T + i\Im(M^{-1})\Re(M^{-1})^T,
\]

is a nonnegative hermitian matrix satisfying for all \(x \in \mathbb{C}^p\) (the cross terms cancel):

\[
\bar{x}^T |M^{-1}|^2 x = \bar{x}^T \Im(M^{-1})\Im(M^{-1})^T x + \bar{x}^T \Re(M^{-1})\Re(M^{-1})^T x
\]

\[
\geq \bar{x}^T \Im(M^{-1})\Im(M^{-1})^T x \geq K^2 \|x\|^2.
\]

Thus the lower eigen value of \(|M^{-1}| = |M|^{-1}\) is bigger than \(K\) and therefore \(|M| \leq \frac{1}{K} I_p.\)

**Lemma 5.7.** *Given* \(L \in \mathcal{D}_I^*, \text{ we can bound:}*

\[
\Im(z) I_n \leq |I^*(L)| \leq O \left( |z| + \frac{|z|}{\Im(z)} \right) I_n
\]

*and:*

\[
O \left( \frac{\Im(z)}{1 + \Im(z)} \right) I_p \leq \left| Q^{I^*(L)} \right| \leq \frac{|z| I_p}{\Im(z)}.
\]

*Proof.* The lower bound of \(I^*(L)\) is immediate (see the proof of Lemma [5.4]). If \(L \in \mathcal{D}_I^*\), then we know that \(L/z \in \mathcal{D}_n(\mathbb{H})\), and therefore, noting that:

\[
\Im \left((\hat{Q}^L/z)^{-1}\right) = \Im \left(z I_p - \frac{1}{n} \sum_{i=1}^n \Sigma_i L_i/z \right) = \Im(z) I_p + \frac{1}{n} \sum_{i=1}^n \Im(L_i/z) \Sigma_i \geq \Im(z) I_p,
\]

we can deduce from Lemma [5.6] that \(|\hat{Q}^L/z| \leq \frac{1}{\Im(z)}\) and thus \(\|\hat{Q}^L\| \leq \frac{|z|}{\Im(z)}\) which gives us directly the upper bound on \(I^*(L)\) since \(\text{Tr}(\Sigma_i) \leq O \left( |z| + \frac{|z|}{\Im(z)} \right).\)

Finally, we can bound:

\[
\left\| I_n - \frac{1}{n} \sum_{i=1}^n \frac{\Sigma_i}{I^*(L)_i} \right\| \leq 1 + \frac{1}{n} \sum_{i=1}^n \left( \frac{\|\Sigma_i\|}{\Im(I^*(L)_i)} \right) \leq 1 + O \left( \frac{1}{\Im(z)} \right),
\]

which provides the lower bound on \(|\hat{Q}^L|\) since \(O \left( \frac{\Im(z)}{1 + \Im(z)} \right) \leq \frac{1}{1 + \Im(z)}.\)

We now have all the elements to prove Theorem [0.1].
Proof of Theorem 0.1 On the domain $I^2(D_1)$, the mapping $I^2$ is bounded and contract- ing for the semimetric $d_s$ thanks to Proposition 5.5 and Lemma 5.7. The hypotheses of Theorem 5.3 are thus satisfied, and we know that there exists a unique diagonal matrix $\tilde{\Lambda}^2 \in I^2(D_1)$ such that $I^2(\tilde{\Lambda}^2) = \tilde{\Lambda}^2$. There can not exist a second diagonal matrix $\Lambda' \in D_n(\mathbb{H})$ such that $I^2(\Lambda') = I^2(\tilde{\Lambda}^2)$ because then Proposition 5.5 (true on the whole domain $D_n(\mathbb{H})$) would imply that $d_s(\Lambda', \tilde{\Lambda}^2) < d_s(\Lambda', \tilde{\Lambda}^2)$.

\[ \square \]

We end this section with an interesting result on $\tilde{\Lambda}^2$ that is proven in $\mathbb{E}$ since it will not be of any use for our main results.

**Lemma 5.8.** $\sup_{i \in [n]} |\tilde{\Lambda}^2_i| \leq O(1 + |z|)$.

### 6 Convergence of $\tilde{\Lambda}^2$ towards $\tilde{\Lambda}^2$, proof of Theorem 0.9

To show the convergence of $\tilde{\Lambda}^2$ towards $\tilde{\Lambda}^2 = I^2(\tilde{\Lambda}^2)$, we need an important result bounding the distance to a fixed point of a contracting mapping for the semimetric $d_s$. This sets what we call the stability of the equation. First allowing us to bound $\|\tilde{\Lambda}^2 - \tilde{\Lambda}^2\|$, it will then be employed to show the continuity of $z \mapsto \tilde{\Lambda}^2$. In the former application, the convergence parameter is $n$, while in the latter application it is a parameter $t \in C$ in the neighbourhood of $0$. For the sake of generality, we thus introduce here a new notation $s \in \mathbb{N}$ to track the convergence (we will then consider $s = n$ or sequences $(t_s)_{s \in \mathbb{N}} \in C^\mathbb{N}$ depending on the applications).

**Proposition 6.1.** Let us consider a family of mappings of $D_n(\mathbb{H})$, $(f^s)_{m \in \mathbb{N}}$, each $f^s$ being $\lambda$-Lipschitz\(^1\) for the semi-metric $d_s$ with $\lambda < 1$ and admitting the fixed point $\tilde{\Lambda}^s = f^s(\tilde{\Gamma}^s)$ and a family of diagonal matrices $\Gamma^s$. If one assumes that $\tilde{\Lambda}^2 d_s(3(\Gamma^s), 3(f^s(\Gamma^s))) \leq o_{s \to \infty}(1)$, then:

\[ d_s(\Gamma^s, \tilde{\Lambda}^2) \leq O_{s \to \infty} \left( \left| f^s(\Gamma^s) - \Gamma^s \right| \frac{\sqrt{3}(\tilde{\Gamma}^s)}{\sqrt{3}(\Gamma^2)} \right). \]

This proposition is just a mere application of the following elementary result.

**Lemma 6.2.** Given three diagonal matrices $\Gamma^1, \Gamma^2, \Gamma^3 \in D_n(\mathbb{H})$:

\[ \frac{\Gamma^3}{\sqrt{3}(\Gamma^1)} \leq \frac{\Gamma^3}{\sqrt{3}(\Gamma^2)} \left( 1 + d_s(3(\Gamma^1), 3(\Gamma^2)) \right). \]

**Proof.** Let us simply bound:

\[ \frac{\Gamma^3}{\sqrt{3}(\Gamma^1)} \leq \frac{\Gamma^3}{\sqrt{3}(\Gamma^2)} + \frac{\Gamma^3 \left( \sqrt{3}(\Gamma^2) - \sqrt{3}(\Gamma^1) \right)}{\sqrt{3}(\Gamma^1) \sqrt{3}(\Gamma^3)} \]

\[ \leq \frac{\Gamma^3}{\sqrt{3}(\Gamma^2)} + \frac{\Gamma^3}{\sqrt{3}(\Gamma^1)} \sqrt{3}(\Gamma^2) \left( \sqrt{3}(\Gamma^1) - \sqrt{3}(\Gamma^1) \right), \]

\(^1\)Actually, $f^s$ does not need to be $\lambda$-Lipschitz on the whole set $D_n(\mathbb{H})$, but we need to be able to bound for all $s \in \mathbb{N}$:

\[ d_s(f^s(\tilde{\Lambda}^s), f^s(\tilde{\Gamma}^s)) \leq \lambda d_s(\tilde{\Lambda}^s, \tilde{\Gamma}^s) \]

\(^1\)Usually the notations $O(1)$ and $o(1)$ are used for quasi-asymptotic studies when $n$ tends to infinity but in this proposition, the relevant parameter is $s$, thus $d_s(f^s(\tilde{\Lambda}^s), f^s(\Gamma^s)) \leq o_{s \to \infty}(1)$ means that for all $K > 0$, there exists $S \in \mathbb{N}$ such that for all $s \geq S$, $d_s(f^s(\tilde{\Lambda}^s), f^s(\Gamma^s)) \leq K$. 

Page 27/39
we can then conclude with the bound $\sqrt{3(G^1)} \left( \sqrt{3(G^2)} + \sqrt{3(G^1)} \right) \geq \sqrt{3(G^1)3(G^2)}$. □

**Proof of Proposition [6.1]** Let us simply bound thanks to Lemma [6.2]:

$$d_s(\hat{\Gamma}^*, \Gamma^*) \leq \frac{\left\| \hat{\Gamma}^* - f^*(\Gamma^*) \right\|}{\sqrt{3(\hat{\Gamma}^*)3(\Gamma^*)}} + \frac{\left\| f^*(\Gamma^*) - \Gamma^* \right\|}{\sqrt{3(\hat{\Gamma}^*)3(\Gamma^*)}} \leq d_s(\hat{\Gamma}^*, f^*(\Gamma^*)) \left( 1 + d_s(3(\Gamma^*), 3(f^*(\Gamma^*))) \right) + \frac{\left\| f^*(\Gamma^*) - \Gamma^* \right\|}{\sqrt{3(\hat{\Gamma}^*)3(\Gamma^*)}} \leq \lambda d_s(\hat{\Gamma}^*, \Gamma^*) \left( 1 + d_s(3(\Gamma^*), 3(f^*(\Gamma^*))) \right) + \frac{\left\| f^*(\Gamma^*) - \Gamma^* \right\|}{\sqrt{3(\hat{\Gamma}^*)3(\Gamma^*)}} \leq \frac{\left\| f^*(\Gamma^*) - \Gamma^* \right\|}{1 - \lambda - \lambda d_s(3(\Gamma^*), 3(f^*(\Gamma^*)))} \leq O \left( \frac{f^*(\Gamma^*) - \Gamma^*}{\sqrt{3(\hat{\Gamma}^*)3(\Gamma^*)}} \right)$$

since $d_s(3(\Gamma^*), 3(f^*(\Gamma^*))) \leq o(1) \leq \frac{1}{\lambda^2}$ for $s$ sufficiently big. □

To be employ Proposition [6.1] on the matrices $\hat{\Gamma}^n = \hat{A}^z$ and $\Gamma^r = \hat{A}^z$ and on the mapping $f^n = I^2$, we first need to set:

**Proposition 6.3.** Given $z \in \mathbb{C}$ such that $d(z, S^c) \geq O(1)$:

$$d_s(3(I^2(\hat{A}^z)), 3(\hat{A}^z)) \leq O \left( \frac{K}{n} \right)$$

**Proof.** We can bound thanks to Lemmas [4.7] and [6.4]:

$$d_s(3(I^2(\hat{A}^z)), 3(\hat{A}^z)) = \sup_{1 \leq i \leq n} \left| \frac{1}{n} \text{Tr} \left( \Sigma_i 3 \left( \hat{Q}^z - E/Aq \left[ Q^z \right] \right) \right) \right| + O \left( \frac{K}{n} \right) \leq O \left( \frac{1}{\sqrt{n}} \left| 3(\hat{Q}^z) - 3 \left( E/Aq \left[ Q^z \right] \right) \right|_{F} \right) + O \left( \frac{K}{n} \right),$$

since $\|\Sigma_i\|_F \leq O(\sqrt{n})$. The identity $\frac{1}{\sqrt{n}}Q^zXX^T = Q - I_n$ allows us to write $\frac{3(E/Aq(Q^z))}{3(z)} = -\frac{1}{\sqrt{n}E/Aq}[Q^z]^2 \hat{Q}^z = -\frac{1}{\sqrt{n}E/Aq}[Q^z]^2 \hat{Q}^z$. We already know how to estimate $E/Aq[Q^z]$ thanks to Proposition [4.10] we are thus left to estimate $E/Aq[Q^z]$. We do not give the full justifications that are closely similar to those presented in the proof of Proposition [4.10] – mainly application of Schur identities [4.2], Proposition [1.10] and Lemmas [4.4.1] [4.7]. To complete this estimation, we consider a deterministic matrix $A \in M_p,$
Sample covariance - independent columns

and we estimate:

\[
\text{Tr} \left( AE_{A_Q} \left[ \hat{Q}^z \left( Q^z - \hat{Q}^\ Lambda \right) \right] \right) \\
= \frac{1}{n} \sum_{i=1}^{n} \text{Tr} \left( AE_{A_Q} \left[ \hat{Q}^z Q^z_i x_i x_i^T \hat{Q}^\ Lambda \right] \right) \\
= \frac{1}{n} \sum_{i=1}^{n} \text{Tr} \left( AE_{A_Q} \left[ \frac{\hat{Q}^z Q^z_i x_i x_i^T}{\Lambda_i^2} \right] \right) \\
= \frac{1}{n} \sum_{i=1}^{n} \text{Tr} \left( AE_{A_Q} \left[ \frac{\hat{Q}^z Q^z_i x_i x_i^T}{\Lambda_i^2} \right] \right) - \frac{1}{n^2} \sum_{i=1}^{n} \text{Tr} \left( AE_{A_Q} \left[ \frac{Q^z_i x_i x_i^T Q^z_i x_i x_i^T}{\Lambda_i^2} \right] \right) \\
= -\frac{1}{n^2} \sum_{i=1}^{n} \text{Tr} \left( AE_{A_Q} \left[ \frac{\hat{Q}^z Q^z_i x_i x_i^T}{\Lambda_i^2} \right] \right) + O \left( \frac{\kappa_z}{\sqrt{n}} \right) \\
= -E_{A_Q} \left[ \frac{1}{n} \text{Tr} \left( A\hat{Q}^z \left( \hat{Q}^\ Lambda - \hat{Q}^\ Lambda \right) \right) \right] + O \left( \frac{\kappa_z}{\sqrt{n}} \right)
\]

with the introduction of the notation:

\[
\Delta^z \equiv \text{Diag}_{i \in [n]} \left[ \frac{1}{n} x_i^T \right] |Q^z_i|^2 x_i).
\]

The random diagonal matrix \( \Delta^z \) being a \( O(\kappa_z^2 / \sqrt{n}) \) Lipschitz transformation of \( X \) for the spectral norm \( \| \cdot \| \) on \( D_n \), we know that \( \Delta^z \mid A_Q \in \hat{\Delta}^z \pm \varepsilon_{\Delta}(\kappa_z / \sqrt{n}) \), where we noted \( \hat{\Delta}^z \equiv E_{A_Q}[\Delta^z] \). One can then pursue the estimation, thanks again to Proposition 6.10.

\[
\text{Tr} \left( AE_{A_Q} \left[ \hat{Q}^z \left( Q^z - \hat{Q}^\ Lambda \right) \right] \right) \\
= \frac{1}{n} \sum_{i=1}^{n} \frac{\hat{\Delta}^z_i}{\Lambda_i^2} \text{Tr} \left( AE_{A_Q} \left[ \frac{\hat{Q}^z Q^z_i x_i x_i^T}{\Lambda_i^2} \right] \right) + O \left( \frac{\kappa_z}{\sqrt{n}} \right)
\]

Now, taking the expectation under \( A_Q \) on the identity valid for any \( i \in [n] \):

\[
\exists(\Lambda_i^2) = \exists(z) - \frac{1}{n} \exists \left( x_i^T Q^z_i x_i \right) = \exists(z) \left( 1 + \frac{1}{n z} (x_i^T Q^z_i x_i - x_i^T Q^z_i x_i) \right)
\]

we deduce that \( \hat{\Delta}^z = \frac{1}{\exists(z)} \exists(\hat{\Lambda}^z) - \hat{\Lambda}^z + O(\kappa_z/n) \). Therefore, with the identity \( E_{A_Q}[\hat{Q}^z \hat{Q}^\ Lambda] = |\hat{Q}^\ Lambda|^2 + O_z \| \hat{Q}^\ Lambda \| (\kappa_z / \sqrt{n}) \), we can estimate (see the proof of Lemma 5.4 for the identification of \( \exists(\text{Tr}(\hat{Q}^\ Lambda)) \)):

\[
\text{Tr} \left( AE_{A_Q} \left[ |Q^z|^2 \right] \right) = \frac{1}{n} \sum_{i=1}^{n} \left( \frac{\exists(\hat{\Lambda}_i^2)}{\exists(z) \left( \exists(\hat{\Lambda}_i^2) \right)} - \frac{1}{\Lambda_i^2} \right) \text{Tr} \left( AE_{A_Q} \left[ \hat{Q}^z_i \Sigma_i \hat{Q}^\ Lambda \right] \right) \\
+ \text{Tr} \left( A |\hat{Q}^\ Lambda|^2 \right) + O \left( \frac{\kappa_z}{\sqrt{n}} \right)
\]

\[
= \frac{z}{\exists(z)} \exists \left( \text{Tr} \left( A\hat{Q}^\ Lambda \right) \right) + \frac{1}{n} \text{Tr} \left( A\hat{Q}^\ Lambda \right) + O \left( \frac{\kappa_z}{\sqrt{n}} \right)
\]
Sample covariance - independent columns

(note that \(|z|/\Im(z) \leq 1\)). In conclusion:

\[
\left\| \frac{\Im(E_{A_Q} \{Q^i\}) - \Im(\hat{\Lambda}^i)}{\Im(z)} \right\|_F \leq O \left( \frac{\Im(z)\kappa_z}{|z|\sqrt{n}} \right) \leq O \left( \frac{\kappa_z}{\sqrt{n}} \right),
\]

and we retrieve the result of the proposition.

And to show that the mapping \(I^z\) is contracting, we will need:

**Lemma 6.4.** Given \(z \in \mathcal{C} \setminus \mathcal{S}_c^z\):

\[
\Im(z) \leq \inf_{i \in [n]} \Im(\hat{\Lambda}^z_i) \leq \sup_{i \in [n]} \Im(\hat{\Lambda}^z_i) \leq O(\Im(z)).
\]

**Proof.** The lower bound is obvious since for all \(i \in [n]\):

\[
\Im(\hat{\Lambda}^z_i) = \Im(z) + \frac{\Im(z)}{|z|^2 n^2} E_{A_Q}[\text{Tr}(Q^z_i X_{-i} X_i^T X_{-i}^T Q^z_i \Sigma_i)] \geq \Im(z).
\]

The upper bound is obtained thanks to the bound, valid under \(A_Q, Q^z_i X_{-i} / |z|\sqrt{n} \leq O(1)\) provided by Lemma 4.11.

We now have all the elements to show:

**Proposition 6.5.** For any \(z \in H\) such that \(d(z, S_{-0}) \geq O(1)\):

\[
\|\hat{\Lambda}^z - \hat{\Lambda}^z\| \leq O \left( \frac{\kappa_z}{n} \right) \quad \text{and} \quad O \left( \frac{|z|}{\kappa_z} \right) \leq |\hat{\Lambda}^z| \leq O \left( \frac{|z|}{\kappa_z} \right).
\]

**Proof.** We already know from Proposition 6.3 that \(d_z(\Im(I^z(\hat{\Lambda}^z)), \Im(\hat{\Lambda}^z)) \leq o(1)\). Besides, the Lipschitz parameter \(\lambda\) of \(I^z\) on the set \(\{\hat{\Lambda}^z, \Lambda^z, n \in \mathbb{N}\}\) is such that \(1 - \lambda \geq O(1)\). Recall indeed from Proposition 6.5 that:

\[
\lambda \leq \sqrt{1 - \frac{\Im(z)}{\text{sup}_{i \in [n]} \Im(\Lambda^z_i)}} \leq \sqrt{1 - O(1)} \leq 1 - O(1),
\]

thanks to Lemma 6.4. Therefore, we can employ Proposition 6.1 to set that:

\[
\left\| \frac{\hat{\Lambda}^z - \hat{\Lambda}^z}{\sqrt{\Im(\hat{\Lambda}^z)\Im(\hat{\Lambda}^z)}} \right\| = d_z(\hat{\Lambda}^z, \hat{\Lambda}^z) \leq O \left( \frac{\hat{\Lambda}^z - I^{\hat{\Lambda}^z}}{\sqrt{\Im(\hat{\Lambda}^z)\Im(\hat{\Lambda}^z)}} \right),
\]

which implies thanks to Lemma 6.4 that:

\[
\|\hat{\Lambda}^z - \hat{\Lambda}^z\| \leq O \left( \sqrt{\frac{\text{sup}_{i \in [n]} \Im(\Lambda^z_i)}{\text{inf}_{i \in [n]} \Im(\Lambda^z_i)}} \|\hat{\Lambda}^z - I^{\hat{\Lambda}^z}\| \right).
\]

We reach here the only point of the whole paper where we will employ Assumption 6.7.

It is to set that:

\[
\inf_{i \in [n]} \Im(\hat{\Lambda}^z_i) = \Im(z) + \inf_{i \in [n]} \sum_{j=1}^n \frac{\Im(\hat{\Lambda}^z_j)}{n^2|A^z_j|^2} \text{Tr}(\Sigma_i \tilde{Q}^{\Lambda^z} \Sigma_j \tilde{Q}^{\Lambda^z})
\]

\[
\geq \Im(z) + \sum_{j=1}^n \frac{\Im(\hat{\Lambda}^z_j)}{n^2|A^z_j|^2} O \left( \text{Tr}(\tilde{Q}^{\Lambda^z} \Sigma_j \tilde{Q}^{\Lambda^z}) \right) \geq O \left( \text{sup}_{i \in [n]} \Im(\Lambda^z_i) \right).
\]
Sample covariance - independent columns

As a conclusion:

\[ \left\| \lambda^z - \tilde{\lambda}^z \right\| \leq O \left( \left\| \lambda^z - T(\tilde{\lambda}^z) \right\| \right) \leq O \left( \frac{1}{\sqrt{n}} \left\| \tilde{Q} - \tilde{Q}^\lambda \right\| \right) \leq O \left( \frac{\kappa_z}{\sqrt{n}} \right). \]

Besides, when can deduce that \( \lambda^z \) and \( \tilde{\lambda}^z \) have the same upper and lower bound of order \( O(|z|/\kappa_z) \) since \( O(\kappa_z) = O(|z|/\kappa_z) \).

We can then deduce a result on the resolvent:

**Corollary 6.6.** For any \( z \in \mathbb{C} \) such that \( d(z, S_z) \geq O(1) \), \( \| \tilde{Q}^\lambda \| \leq O(\kappa_z) \) and:

\[ \left\| E_{A_Q} [Q^z] - \tilde{Q}^\lambda \right\|_F \leq O \left( \frac{\kappa_z}{\sqrt{n}} \right) \]

**Proof.** We already know from Proposition 4.10 that \( \left\| E_{A_Q} [Q^z] - \tilde{Q}^\lambda \right\|_F \leq O(\kappa_z/\sqrt{n}) \), thus we are left to bound:

\[ \left\| \tilde{Q}^\lambda - \tilde{Q}^\lambda \right\|_F \leq \left\| \tilde{Q}^\lambda \left( \frac{1}{n} \sum_{i=1}^n \frac{\lambda^z - \tilde{\lambda}^z}{\lambda_i^z \lambda_i^z} \Sigma_i \right) \tilde{Q}^\lambda \right\|_F \]

\[ \leq \sup_{z \in \mathbb{C}} \left\| \frac{\lambda^z - \tilde{\lambda}^z}{\lambda_i^z \lambda_i^z} \Sigma_i \right\|_F \left\| \tilde{Q}^\lambda \right\|_F \left\| \tilde{Q}^\lambda \right\|_F \]

\[ \leq O \left( \frac{\kappa_z^2 \sqrt{\kappa_z}}{|z| n} \right) \leq O \left( \frac{\kappa_z}{\sqrt{n}} \right) \]

(thanks to Lemma 4.4 3.1 5.8 and Proposition 6.5) We can then deduce that:

\[ \left\| \tilde{Q}^\lambda \right\|_F \leq \left\| \tilde{Q}^\lambda - \tilde{Q}^\lambda \right\|_F + \left\| \tilde{Q}^\lambda \right\|_F \leq O(\kappa_z). \]

We can now prove our second main Theorem (on the linear concentration of \( \frac{1}{z} Q^z \) around \( \tilde{\frac{1}{z} Q^z} \)):

**Proof of Theorem 7.9.** We saw in the proof of Proposition 3.2 that the mapping \( R \in (\mathcal{F}(\mathbb{C} \setminus S^e), A_Q) \), defined, under \( A_Q \), for any \( z \in \mathbb{C} \setminus S^e \) as \( R(z) = -\frac{1}{z} Q^z \) has a Lipschitz parameter bounded by:

\[ \sup_{z \in \mathbb{C} \setminus S^e} O \left( \frac{\kappa_z}{|z| \sqrt{n}} \right) = O \left( \frac{1}{\sqrt{n}} \right) \]

(since when \( n \geq p, \kappa_z = \frac{|z|}{1 + |z|} \), and when \( n < p, \kappa_z = 1 \) but \( E[\lambda_p] = 0 \) and therefore \( \inf_{z \in \mathbb{C} \setminus S^e} 1/|z| \geq O(1) \). As a \( O(1/\sqrt{n}) \)-Lipschitz transformation of \( X \propto E_2, (R \mid A_Q) \propto E_2(1/\sqrt{n}) \), we can then conclude thanks to Corollary 6.6 with the bound:

\[ \left\| E_{A_Q} [R] - \tilde{R} \right\|_{F,S^e} \leq \sup_{z \in \mathbb{C} \setminus S^e} \frac{1}{|z|} \left\| E_{A_Q} [Q^z] - \tilde{Q}^\lambda \right\| \leq \sup_{z \in \mathbb{C} \setminus S^e} O \left( \frac{\kappa_z}{\sqrt{n}|z|} \right) \leq O \left( \frac{1}{\sqrt{n}} \right) \]

Although it is not particularly needed for practical use, we are now going to show that the mapping \( \tilde{g} : z \to \frac{1}{p} \text{Tr}(\tilde{R}(z)) \) is a Stieltjes transform. One of the main result of next section is also to show the convergence of the Stieltjes transform \( g \) towards \( \tilde{g} \) on \( \mathbb{C} \setminus S^e \) (which is a bigger set than \( \mathbb{C} \setminus S^e \) on which we approximated \( R(z) \)).
7 Approach with the Stieltjes formalism, proofs of Theorems 6.2 and 0.8

We start with an interesting identity that gives a direct link between the Stieltjes transforms \( g \) and \( \tilde{g} \) with the diagonal matrix \( \Lambda^z \) and \( \tilde{\Lambda}^z \). From the equality \( Q^z - \frac{1}{np} XX^T Q^z = I_p \), and the Schur identities (1.2) we can deduce that:

\[
g(z) = -\frac{1}{pz} \text{Tr}(Q^z) = -\frac{1}{z} - \frac{1}{npz^2} \sum_{i=1}^n x_i^T Q^z x_i \\
= -\frac{1}{z} - \frac{1}{npz} \sum_{i=1}^n x_i^T Q^z x_i \Lambda_i \] 

with the notation \( g^{D^z} \), defined for any mapping \( D : \mathbb{H} \ni z \mapsto D^z \in D_n(\mathbb{H}) \) as\(^{20}\).

Interestingly enough, if we denote \( \tilde{g} \equiv g^\Lambda^z \), then one can readily check that we have the equality \( \tilde{g} = -\frac{1}{pz} \text{Tr}(\tilde{Q}^z) \). To show that \( \tilde{g} \) is a Stieltjes transform, we will employ the following well known theorem that can be found for instance in [Bol97]:

**Theorem 7.1.** Given an analytic mapping \( f : \mathbb{H} \to \mathbb{H} \), if \( \lim_{y \to +\infty} -iyf(iy) = 1 \) then \( f \) is the Stieltjes transform of a probability measure \( \mu \) that satisfies the two reciprocal formulas:

- For any continuous point\(^{21}\) \( a < b \): \( \mu([a,b]) = \lim_{y \to 0^+} \frac{1}{\pi} \int_a^b \Im(f(x + iy))dx \).
- If, in particular, \( \forall z \in \mathbb{H} \), \( zf(z) \in \mathbb{H} \), then \( \mu(\mathbb{R}^-) = 0 \) and \( f \) admits an analytic continuation on \( C \setminus \{ \mathbb{R} \cup \{0\} \} \).

The first hypothesis to verify in order to employ Theorem 7.1 is the analyticity of \( \tilde{g} \), that originates from the analyticity of \( z \to \tilde{\Lambda}^z \). We can show with limiting arguments that \( \tilde{\Lambda}^z \) is analytical as a limit of a sequence of analytical mappings. However, although it is slightly more laborious, we prefer here to prove the analyticity from the original definition. First let us show the continuity with Proposition 6.1.

**Proposition 7.2.** The mapping \( z \mapsto \tilde{\Lambda}^z \) is continuous on \( \mathbb{H} \).

**Proof.** Given \( z \in \mathbb{H} \), we consider a sequence \( (t_s)_{s \in \mathbb{N}} \in \{ w \in \mathbb{C} \mid w + z \in \mathbb{H} \} \) such that \( \lim_{s \to \infty} t_s = 0 \). Let us verify the assumption of Proposition 6.1 where for all \( s \in \mathbb{N} \), \( f^s = f^{z + t_s} \), \( \Gamma^s = \tilde{\Lambda}^{z + t_s} \), and \( I^s = \tilde{\Lambda}^z \) (it does not depend on \( s \)). We already know from Proposition 5.5 that \( f^s \) are all contracting for the stable semi-metric with a Lipschitz parameter \( \lambda < 1 \) that can be chosen independent from \( s \) for \( s \) big enough. Let us express for any \( s \in \mathbb{N} \) and any \( i \in [n] \):

\[
f^s(\Gamma^s)_i - \Gamma^s_i = f^{z + t_s}(\tilde{\Lambda}^z)_i - \tilde{\Lambda}^z_i = t_s 
\tag{7.1}
\]

Noting that for \( s \) sufficiently big, \( \Im(f^{z + t_s}(\tilde{\Lambda}^z)) = \Im(t_s) + \Im(\tilde{\Lambda}^z) \geq \frac{\Im(\tilde{\Lambda}^z)}{4} \geq \frac{\Im(\lambda)}{4} \), we see that \( d_s(\Im(f^s(\Gamma^s)_i), \Im(\Gamma^s)_i) \leq \frac{\Im(\lambda)}{4} \to 0 \). Therefore, the assumptions of Proposition 6.1 are satisfied and we can conclude that there exists \( K > 0 \) such that for all

---

\(^{20}\)The Stieltjes transform of the spectral distribution of \( \frac{1}{n} XX^T X \) is \( \tilde{g} \equiv g^\Lambda^z \), where for all \( D : \mathbb{H} \ni z \mapsto D^z \in D_n(\mathbb{H}) \), \( \tilde{g}^{D^z} : z \mapsto -\frac{1}{2} \sum_{i=1}^n \frac{1}{\pi} T_i 
\)

\(^{21}\)We can add the property \( \forall \epsilon \in \mathbb{R}, \mu([\epsilon]) = \lim_{y \to 0^+} y \Im(f(x + iy)) \), here for \( \mu \) to be continuous in \( a, b \), we need \( \mu([a]) = \mu([b]) = 0 \).
Sample covariance - independent columns

$s \in \mathbb{N}$:

$$\left\| \frac{\tilde{A}^{z+t} - \tilde{A}^z}{\sqrt{\Im(\tilde{A}^{z+t})\Im(\tilde{A}^z)}} \right\| \leq \frac{K|t_s|}{\inf_{i \in [n]} \sqrt{\Im(\tilde{A}^{z+t})\Im(\tilde{A}^z)}} \leq \frac{2K|t_s|}{\Im(z)}. $$

Besides, we can also bound:

$$\sqrt{\Im(\tilde{A}^{z+t})} \leq \frac{2\sqrt{\Im(\tilde{A}^z)}}{\Im(z)}(\Im(\tilde{A}^z) + Kt_s) \leq O(1),$$

That directly implies that $\|\tilde{A}^{z+t} - \tilde{A}^z\| \leq O(t_s) \xrightarrow{s \to \infty} 0$, and consequently, $z \mapsto \tilde{A}^z$ is continuous on $\mathbb{H}$. 

Let us now show that $z \mapsto \tilde{A}^z$ it is differentiable. Employing again the notation $f^t = T^{z+t}$, we can decompose (noting for $D \in \mathcal{D}_n$, $R(D) \equiv (zI_n - \frac{1}{n} \sum_{i=1}^{n} \Lambda^z_n)$:

$$\left(\tilde{A}^{z+t} - \tilde{A}^z\right) = \left(f^t(\tilde{A}^{z+t}) - f^t(\tilde{A}^z) + f^t(\tilde{A}^z) - f^0(\tilde{A}^z)\right)$$

$$= \text{Diag}_{i \in [n]} \left(\frac{1}{n} \text{Tr} \left(\sum_j Q_j^{\tilde{A}^{z+t}} \frac{1}{n} \sum_{j=1}^{n} \Lambda_j^{z+t} - \tilde{A}_j^z \sum_j Q_j^{\tilde{A}^z}\right)\right) + tI_n$$

Now, if we introduce the vector $a(t) = \left(\Lambda_i^z - \tilde{A}_i^z\right)_{1 \leq i \leq n} \in \mathbb{R}^n$, and for any $D, D' \in \mathcal{D}_n(\mathbb{H})$, the matrix:

$$\Psi(D, D') = \left(\frac{1}{n} \text{Tr}(\sum_j R(D)\sum_j R(D'))_{D_jD_j'}\right)_{1 \leq i,j \leq n} \in \mathcal{M}_n$$

We have the equation:

$$a(t) = \Psi(\tilde{A}^z, \tilde{A}^{z+t})a(t) + tI.$$

To be able to solve this equation we need:

**Lemma 7.3.** *Given any $z, z' \in \mathbb{H}$, $I_n - \Psi(\tilde{A}^z, \tilde{A}^{z'})$ is invertible.*

**Proof.** We are going to show the injectivity of $I_n - \Psi(\tilde{A}^z, \tilde{A}^{z'})$. Let us introduce a vector $x \in \mathbb{R}^n$ such that $x = \Psi(\tilde{A}^z, \tilde{A}^{z'})x$. We can bound thanks to Cauchy-Schwartz inequality, with similar calculus as in the proof of Proposition 5.3:

$$|x_i| = \left| \frac{1}{n} \text{Tr} \left(\sum_j R(\tilde{A}^z) \sum_{j=1}^{n} x_j \Sigma_j \sqrt{\Im(\tilde{A}^z)\Im(\tilde{A}^{z'})} R(\tilde{A}^{z'})\right) \right|$$

$$\leq \sup_{j \in [n]} \left| \frac{x_j}{\sqrt{\Im(\tilde{A}^z)\Im(\tilde{A}^{z'})}} \right| \sqrt{\Im(\tilde{A}_i^z) - \Im(z)\sqrt{\Im(\tilde{A}^z) - \Im(z)}}$$

therefore, if we denote $\|x\|_{\tilde{A}^z, \tilde{A}^{z'}} \equiv \sup_{j \in [n]} \left| \frac{x_j}{\sqrt{\Im(\tilde{A}^z)\Im(\tilde{A}^{z'})}} \right|$, we have then the bound:

$$\|x\|_{\tilde{A}^z, \tilde{A}^{z'}} \leq \|x\|_{\tilde{A}^z, \tilde{A}^{z'}} \sqrt{(1 - \phi(z, \tilde{A}^z))(1 - \phi(z, \tilde{A}^{z'}))}$$

which directly implies that $x = 0$ since we know that $\phi(z, \tilde{A}^{z'}) = \frac{\Im(w)}{\sup_{1 \leq i \leq n} \Im(\tilde{A}^{z'})_{ii}} \in (0, 1).$
Sample covariance - independent columns

From the continuity of \((z, z') \mapsto \Psi(\tilde{\Lambda}^z, \tilde{\Lambda}^{z'})\), and the limit \(\lim_{z \to \infty} \|\Psi(\tilde{\Lambda}^z, \tilde{\Lambda}^{z'})\| = 0\) (see the proof of Proposition 7.8 in [B]), we can deduce as a side result from Lemma 7.3.

**Lemma 7.4.** Given any \(z, z' \in \mathbb{H}\), \(\|\Psi(\tilde{\Lambda}^z, \tilde{\Lambda}^{z'})\| < 1\)

The continuity of \(z \mapsto \tilde{\Lambda}^z\) given by Proposition 7.2 and the continuity of the inverse operation on matrices (around \(I_n - \Psi(\tilde{\Lambda}^z, \tilde{\Lambda}^z)\) which is invertible), allows us to let \(t\) tend to zero in the equation

\[
\frac{1}{t} \bar{a}(t) = (I_n - \Psi(\tilde{\Lambda}^z, \tilde{\Lambda}^{z+t}))^{-1} \mathbb{1},
\]

to obtain:

**Proposition 7.5.** The mapping \(z \mapsto \tilde{\Lambda}^z\) is analytic on \(\mathbb{H}\), and satisfies:

\[
\frac{\partial \tilde{\Lambda}^z}{\partial z} = \text{Diag} \left( (I_n - \Psi(\tilde{\Lambda}^z, \tilde{\Lambda}^z))^{-1} \mathbb{1} \right)
\]

We can then conclude first that for all \(i \in [n]\), the mappings \(z \mapsto \frac{1}{\tilde{\Lambda}^z_{ii}}\) are Stieltjes transforms.

**Proposition 7.6.** For all \(i \in [n]\), there exists a distribution \(\tilde{\mu}_i\) with support on \(\mathbb{R}_+\) whose Stieltjes transform is \(z \mapsto -\frac{1}{\tilde{\Lambda}^z_{ii}}\).

**Proof.** We just check the hypotheses of Theorem 7.1. We already know that \(z \mapsto -\frac{1}{\tilde{\Lambda}^z_{ii}}\) is analytical thanks to Proposition 7.5 and the lower bound \(\tilde{\Lambda}^z_{ii} \geq \Im(z) > 0\). Besides, \(\forall z \in \mathbb{H}:\)

\[
\Im \left( -\frac{1}{\tilde{\Lambda}^z_{ii}} \right) = \frac{\Im(\tilde{\Lambda}^z)}{|\tilde{\Lambda}^z_{ii}|} > 0 \quad \text{and} \quad \Im \left( -\frac{z}{\tilde{\Lambda}^z_{ii}} \right) = \frac{\Im(\tilde{\Lambda}^z/z)}{|\tilde{\Lambda}^z_{ii}/z|} > 0,
\]

since \(\tilde{\Lambda}^z \in \mathcal{D}_{I_n}\). Finally recalling from Lemma 5.7 that for all \(y \in \mathbb{R}_+,\) \(\|\tilde{\Lambda}^{iy}\| \leq \frac{\|y\|}{\Im(y)} = 1\), we directly see that for all \(j \in [n]:\)

\[
\frac{\tilde{\Lambda}^{iy}}{iy} = 1 + \frac{1}{iy^n} \text{Tr}(\Sigma_j \tilde{\Lambda}^{iy}) \xrightarrow{y \to +\infty} 1.
\]

We can then conclude with Theorem 7.1. \(\square\)

We can then deduce easily that \(\tilde{\gamma}\) is also a Stieltjes transform (it is the Theorem 0.2) with an interesting characterization of its measure with the \(\tilde{\mu}_i\) (defined in Proposition 7.6).

**Proposition 7.7.** The mapping \(\tilde{\gamma}\) is the Stieltjes transform of the measure:

\[
\tilde{\mu} = \left( \frac{n}{p} - 1 \right) \delta_0 + \frac{1}{p} \sum_{i=1}^{n} \tilde{\mu}_i,
\]

where \(\delta_0\) is the Dirac measure on 0 (if \(p > n\), then the measures \(\tilde{\mu}_1, \ldots, \tilde{\mu}_n\) contains Dirac weights on zero that cancel the term \(-\frac{\delta_0}{p}\delta_0\)).

Recall that \(\tilde{\mu}\), satisfies \(\tilde{\gamma}(z) = \int_0^{+\infty} \frac{1}{\lambda - z} d\tilde{\mu}(\lambda)\), and let us denote \(\tilde{S}\), its support. This formula implies that \(\tilde{\gamma}\) is analytic on \(\mathbb{C}\setminus\tilde{S}\) and that for all \(z \in \mathbb{C}\setminus\tilde{S}\), \(\tilde{\gamma}(z) = \frac{\tilde{\gamma}(z)}{\tilde{\gamma}(z)}\). To precise the picture let us provide a result of compactness of \(\tilde{S}\) proven in the Appendix.

**Proposition 7.8.** The measure \(\tilde{\mu}\) has a compact support \(\tilde{S} \subset \mathbb{R}_+\) and \(\text{sup} \tilde{S} \leq O(1)\).

We end this section with the proof of the convergence of \(g\) towards \(\tilde{g}\) that will close our paper.
Sample covariance - independent columns

**Proof of Theorem 7.8.** Be careful that we want here a concentration for the infinite norm on $C \setminus S_{x_0}$. Two cases are under study:

- When $p \leq n$, $\kappa_z = \frac{|z|}{1 + |z|}$, and therefore we can show as in the proof of Proposition 3.2 that the mapping $g$ defined for any $z \in C \setminus (S_{x_0} \cup \{0\})$ with the identity 
  
  \[ g(z) = -\frac{1}{p} \text{Tr}(Q^2) \]

  is, under $A_Q$ a $O(1/\sqrt{np})$-Lipschitz transformation of $X$, thus 
  
  \[ (g \mid A_Q) \propto \mathcal{E}_2(1/\sqrt{np}) \text{ in } (F(C), \| \cdot \|_{S_{x_0}}). \]

  We can then conclude thanks to the bound:

  \[
  \sup_{z \in C \setminus S_{x_0}} |E_{A_Q}[g(z)] - \bar{g}(z)| \leq \sup_{z \in C \setminus S_{x_0}} \frac{1}{zp} \left| \text{Tr} \left( E_{A_Q} [Q^2] - \bar{Q} \tilde{\Lambda} \right) \right| 
  \]

  \[
  \leq \sup_{z \in C \setminus S_{x_0}} O \left( \frac{\kappa_z}{|z|} \frac{1}{\sqrt{np}} \right) \leq O \left( \frac{1}{\sqrt{np}} \right).
  \]

- When $p \geq n$, $\kappa_z = \frac{|z|}{1 + |z|}$, and we are going to employ the fact (see Lemma 4.4) that then $|\Lambda| \geq O(1)$. Indeed, thanks to this lower bound and the identity:

  \[ g(z) = \frac{1}{z} \left( \frac{n}{p} - 1 \right) - \frac{1}{p} \sum_{i=1}^{n} \frac{1}{\Lambda_i^2}, \]

  we can show as in the proof of Lemma 4.1 that $g$ is, under $A_Q$, a $O(1/\sqrt{np})$-Lipschitz transformation of $X$, and as such $(g \mid A_Q) \propto \mathcal{E}_2(1/\sqrt{np})$ in $(F(C), \| \cdot \|_{S_{x_0}})$. And this time we conclude with the bound:

  \[
  \sup_{z \in C \setminus S_{x_0}} |E_{A_Q}[g(z)] - \bar{g}(z)| \leq \sup_{z \in C \setminus S_{x_0}} \frac{1}{p} \sum_{i=1}^{n} E \left[ \left| \Lambda_i^2 - \bar{\Lambda}_i \right| \right] 
  \]

  \[
  \leq \sup_{z \in C \setminus S_{x_0}} O \left( \frac{1}{p} \sum_{i=1}^{n} E_{A_Q} \left[ \left| \Lambda_i^2 - \bar{\Lambda}_i \right| \right] \right) 
  \]

  \[
  \leq O \left( \frac{1}{p} \right) \leq O \left( \frac{1}{\sqrt{np}} \right)
  \]

We retrieve then the concentration inequality of the theorem thanks to the bound on $P(A_Q)$ given in Lemma 2.4.

---

**A Properties of the semi metric $d_s$ and consequences in random matrix theory**

In this appendix, we will employ the semi-metric $d_s$ indifferently on diagonal matrices $D_n(H)$ or vectors of $H^n$ or more simply with variables of $H$ as in next proposition.

**Proposition A.1.** All the Stieltjes transforms are 1-Lipschitz for the semi-metric $d_s$ on $H$.

**Proof.** We consider a Stieltjes transform $g : z \to \int \frac{d\mu(t)}{z - t}$ for a given measure $\mu$ on $\mathbb{R}$. Given $z, z' \in H$, we can bound thanks to Cauchy-Schwarz inequality:

\[
|g(z) - g(z')| \leq \left| \int \frac{z' - z}{(t - z)(t - z')} d\mu(t) \right| \leq \left| \frac{z' - z}{\sqrt{3(z)3(z')}} \right| \left| \int \frac{3(z)3(z')}{(t - z)(t - z')} d\mu(t) \right| 
\]

\[
\leq \left| \frac{z' - z}{\sqrt{3(z)3(z')}} \right| \sqrt{\int \frac{\overline{3(z)}3(z')}{|t - z|^2} d\mu(t)} \int \frac{3(z')}{|t - z'|^2} d\mu(t) 
\]

\[
= \sqrt{3(g(z))3(g(z'))} d_s(z, z')
\]

\[ \square \]
We did not find any particular use of this proposition (the idea could be to solve $g(z) = z$ but the stability it introduces looks interesting.

Let us start with some preliminary lemmas before proving the fourth point of Proposition 5.2, i.e. the stability towards the sum of the class of mappings, 1-Lipschitz for the semi-metric $d_s$. They are mere adaptation of results of [LC20].

**Lemma A.2.** Given four positive numbers $a, b, \alpha, \beta \in \mathbb{R}_+$:

$$\sqrt{ab} + \sqrt{\alpha \beta} \leq \sqrt{(a + \alpha)(b + \beta)} \quad \text{and} \quad \frac{a + \alpha}{b + \beta} \leq \max \left( \frac{a}{b}, \frac{\alpha}{\beta} \right)$$

**Lemma A.3.** Given four diagonal matrices $\Delta, \Delta', D, D' \in \mathbb{D}_n(\mathbb{H})$:

$$d_s(\Delta + D, \Delta' + D') \leq \max(d_s(\Delta, \Delta'), d_s(D, D')).$$

**Proof of Lemma A.3.** For any $\Delta, \Delta', D, D' \in \mathbb{D}_n(\mathbb{H})$, there exists $i_0 \in [n]$ such that:

$$d_s(\Delta + D, \Delta' + D') = \frac{|\Delta_{i_0} - \Delta'_{i_0} + D_{i_0} - D'_{i_0}|}{\sqrt{3(\Delta_{i_0} + D_{i_0}) \Delta'(\Delta'_{i_0} + D'_{i_0})}} \leq \frac{|\Delta_{i_0} - \Delta'_{i_0}| + |D_{i_0} - D'_{i_0}|}{\sqrt{3(\Delta_{i_0}) \Delta'_{i_0} + \sqrt{3(D_{i_0}) \Delta'(D'_{i_0})}}} \leq \max \left( \frac{|\Delta_{i_0} - \Delta'_{i_0}|}{\sqrt{3(\Delta_{i_0}) \Delta'_{i_0}}}, \frac{|D_{i_0} - D'_{i_0}|}{\sqrt{3(D_{i_0}) \Delta'(D'_{i_0})}} \right)$$

thanks to Lemma A.2.

We deduce directly from this lemma that if $f, g : \mathbb{D}_n(\mathbb{H}) \to \mathbb{D}_n(\mathbb{H})$ are $\lambda$-Lipschitz for $d_s$ then $f + g$ is also $\lambda$-Lipschitz.

This property gives a very fast proof to show the existence and uniqueness of solutions to the equations studied in [AKE16] (however their proof is not much longer). We start with a preliminary lemma:

**Proposition A.4.** Given a matrix $S \in \mathcal{M}_{n,p}(\mathbb{R}_+)$, $z \mapsto Sz$ goes from $\mathbb{H}^p$ to $\mathbb{H}^n$ and it is 1-Lipschitz for the semimetric $d_s$.

**Proof.** For any $z \in \mathbb{H}^p$, $\Im(Sz) = S\Im(z) \in \mathbb{R}_+^n$ since all the entries of $S$ are positive. If we denote $s_1, \ldots, s_n$, the columns of $S$, we can decompose $Sz = \sum_{i=1}^n s_i \pi_i(z)$, where $\pi_i(z) = z_i$. Each mapping $s_i \pi_i$ is 1-Lipschitz for $d_s$, since we have for any $z, z' \in \mathbb{H}^p$:

$$d_s(s_i \pi_i(z), s_i \pi_i(z')) = \sup_{j \in [p]} \left| \frac{s_i[j] z_i - s_i[j] z'_i}{\sqrt{3(s_i[j] z_i^2 s_i[j] z'_i^2)}} \right| = \frac{|z_i - z'_i|}{\sqrt{3(z_i) \Im(z'_i)}}$$

therefore, as a sum of 1-Lipschitz operators, we know that $z \mapsto Sz$ is also 1-Lipschitz for $d_s$.

**Proposition A.5.** Given any $a \in \mathbb{R}^n$ and any matrix $S \in \mathcal{M}_n(\mathbb{R}_+)$, and $z \in \mathbb{H}$, the equation:

$$-\frac{1}{m} = z \mathbb{1} + a + Sz$$

admits a unique solution $m \in \mathbb{C}_n^\alpha$.

**Note** that unlike in [AKE16], we do not suppose that $S$ is symmetric.
We conclude then with Theorem 5.3 that there exists a unique $x$.

Proof. Let us introduce $I : x \to zI + a - S_\frac{1}{4}$. To employ Theorem 5.3 let us first show that the imaginary part of $I(x)$ is bounded from below and above for all $x \in \mathbb{H}^n$. Given $x \in \mathbb{H}^n$ we see straightforwardly that $\Im(I(x)) \geq \Im(z)$, we can furthermore bound:

$$\Im(I(x)) \leq \Im(z) + S \frac{\Im(x)}{|x|^2} \leq \Im(z) + S \frac{1}{\Im(x)} \leq \left( \Im(z)I_p + \frac{1}{3}\frac{S}{\Im(z)} \right) \| \equiv \kappa I \|.$$

Besides, we already know from Proposition 5.2 that $I$ is 1-Lipschitz for $d_s$ but we need a Lipschitz parameter lower than 1. Given $x, y \in \mathbb{H}^n$ we can bound thanks to Proposition 5.2 and A.4

$$|I(x) - I(y)| \leq S \left( \frac{1}{x} - \frac{1}{y} \right) \leq \sqrt{\Im(S \left( \frac{1}{x} \right) \Im(S \left( \frac{1}{y} \right))) d_s(x, y)},$$

that implies that the Lipschitz parameter of $I$ is lower than:

$$\left( 1 - \frac{\Im(z)}{\Im(I(x))} \right) \left( 1 - \frac{\Im(z)}{\Im(I(y))} \right) \leq 1 - \frac{\Im(z)}{\kappa I} < 1$$

We conclude then with Theorem 5.3 that there exists a unique $x \in \mathbb{H}^n$ such that $x = I(x)$, from which we can deduce the existence and uniqueness of $m = \frac{1}{\kappa}$.

**B Proofs of the non necessary results**

Proof of Lemma 5.8. If we assume that $\forall i \in [n], |\bar{A}^i| \geq 2 \nu$, then we deduce that $\frac{1}{n} \sum_{i=1}^n \left| \frac{A^i}{|A^i|} \right| \leq \frac{1}{\nu}$ and $|\frac{1}{n} \sum_{i=1}^n \frac{A^i}{|A^i|} | \leq \frac{1}{2}$, and therefore, $|Q^L| \leq 2$. As a consequence, $\forall i \in [n]$:

$$|\bar{A}^i| \leq |z| + \frac{1}{n} \text{Tr} \left( \Sigma_i |Q^L| \right) \leq |z| + \frac{2 \nu}{n}.$$ 

We can conclude that:

$$\sup_{i \in [n]} |\bar{A}^i| \leq \max \left( \frac{\nu}{2}, |z| + \frac{2 \nu}{n} \right) \leq O(1 + |z|).$$

Proof of Proposition 7.8. We are going to show that for $x$ sufficiently big, $\lim_{y \to 0^+} \Im(y(x + iy)) = 0$, which will allow us to conclude thanks to the relation between $\bar{\mu}$ and $\bar{\nu}$ given in Theorem 7.1. Considering $z = x + iy \in \mathbb{H}$, for $x, y \in \mathbb{R}$ and such that $x \geq x_0 \equiv \max \left( \frac{8}{n} \sup_{i \in [n]} \text{Tr} \left( \Sigma_i \right), 4 \nu \right)$

let us show first that $\forall i \in [n], \Re(\bar{A}^i) \geq \frac{\nu}{4}$. This is a consequence of the fact that $\bar{I}^z$ is stable on $A \equiv D_n(\{ t \geq \frac{\nu}{4} \} + iR_\nu) \cap D_t$. Indeed, given $L \in A$

$$\Re \left( (\bar{Q}^L)^{-1} \right) = I_p - \frac{1}{n} \sum_{i=1}^n \frac{\Re(\bar{A}^i) \Sigma_i}{|L_i|^2} \geq I_p - \frac{1}{n} \sum_{i=1}^n \frac{\Sigma_i}{\Re(L_i)} \geq \frac{1}{2}$$

and as we already know, since $D_t$, $\Im \left( (\bar{Q}^L)^{-1} \right) \geq 0$, therefore, $\| \bar{Q}^L \| \leq 2$. We can then bound:

$$\Re(I^z(L)_i) = x - \frac{1}{n} \text{Tr} \left( \Sigma_i \bar{Q}^L \left( 1 - \frac{1}{n} \sum_{j=1}^n \frac{\Re(L_j) \Sigma_j}{|L_j|^2} \right) \bar{Q}^L \right) \geq x - \frac{1}{n} \text{Tr} \left( \Sigma_i \right) \geq \frac{x}{2}.$$
Thus as a limit of elements of \( A \), \( \Lambda^z \in A \), and \( \forall i \in [n] \), \( \Re(\Lambda^z_i) \geq \frac{x}{4} \).

Besides, let us bound:

\[
\Im(\Lambda^z_i) = y + \frac{1}{n} \text{Tr} \left( \Sigma_i R(\Lambda^z) \frac{1}{n} \sum_{j=1}^{n} \frac{\Im(\Lambda^z_j) \Sigma_j}{|\Lambda^z_j|^2} R(\Lambda^z) \right) \\
\leq y + \frac{4}{n} \text{Tr}(\Sigma_i) \left\| \frac{1}{n} \sum_{j=1}^{n} \Sigma_j \right\| \sup_{j \in [n]} \frac{\Im(\Lambda^z_j)}{\Re(\Lambda^z_j)^2} \\
\]

and with the bounds \( \left\| \frac{1}{n} \sum_{j=1}^{n} \Sigma_j \right\| \leq \nu \), \( \Re(\Lambda^z_j)^2 \geq \frac{xn}{4} \), we can eventually bound \( \sup_{j \in [n]} \Im(\Lambda^z_j) \leq y + \frac{4\nu}{x} \sup_{j \in [n]} \Im(\Lambda^z_j) \), which implies, for \( x \) sufficiently big:

\[
\sup_{j \in [n]} \Im(\Lambda^z_j) \leq \frac{y}{1 - \frac{4\nu}{x}} \xrightarrow{y \to 0^+} 0.
\]

We can then conclude letting \( y \) tend to 0 in the formulation \( \tilde{g} = g^z \):

\[
\Im(\tilde{g}(x + iy)) = \frac{y}{x^2 + y^2} \left( \frac{n}{p} - 1 \right) + \frac{1}{p} \sum_{i=1}^{n} \frac{\Im(\Lambda^z_i)}{\Re(\Lambda^z_i)^2 + \Im(\Lambda^z_i)^2} \xrightarrow{y \to 0^+} 0
\]

\( \square \)

References

[AC15] Radosław Adamczak and Djalil Chafaï. Circular law for random matrices with unconditional log-concave distribution. *Communications in Contemporary Mathematics*, 17(4), 2015.

[Ada11] Radoslaw Adamczak. On the marchenko-pastur and circular laws for some classes of random matrices with dependent entries. *Electronic Journal of Probability*, 16:1065–1095, 2011.

[Ada15] Radosław Adamczak. A note on the hanson-wright inequality for random vectors with dependencies. *Electronic Communications in Probability*, 20(72):1–13, 2015.

[AKE16] Oskari Ajanki, Torben Krüger, and Laszlo Erdös. Singularities of solutions to quadratic vector equations on the complex upper half-plane. *Communications on Pure and Applied Mathematics*, 70(9):1672–1705, 2016.

[BKV96] A. Boutet de Monvel, A. Khorunzhy, and V. Vasilchuk. Limiting eigenvalue distribution of random matrices with correlated entries. *Markov Processes and Related Fields*, 2(4):607–636, 1996.

[BLM13] Stéphane Boucheron, Gábor Lugosi, and Pascal Massart. *Concentration Inequalities: A Nonasymptotic Theory of Independence*. Oxford University Press, 2013.

[Bol97] Vladimir Bolotnikov. On a general moment problem on the half axis. *Linear algebra and its application*, 255:57–112, 1997.

[BZ08] Zhidong Bai and Wang Zhou. Large sample covariance matrices without independence structures in columns. *Statistica Sinica*, 18:425–442, 2008.

[DKL22] Alicja Dembczak-Kołodziejczyk and Anna Lytova. On the empirical spectral distribution for certain models related to sample covariance matrices with different correlations. *Random Matrices: Theory and Applications*, 11(03):2250030, 2022.

[EH70] Clifford J Earle and Richard S Hamilton. A fixed point theorem for holomorphic mappings. In *Proc. Sympos. Pure Math*, volume 16, pages 61–65, 1970.

[GL96] Gene H. Golub and Charles F. Van Loan. *Matrix computations*. Johns Hopkins University press, 1996.

[GZ00] Alice Guionnet and Ofer Zeitouni. Concentration of the spectral measure for large matrices. *Electronic Communications in Probability*, 5:119–136, 2000.
Sample covariance - independent columns

[HFS07] J William Helton, Reza Rashidi Far, and Roland Speicher. Operator-valued semicircular elements: solving a quadratic matrix equation with positivity constraints. *International Mathematics Research Notices*, 2007(9):rnm086–rnm086, 2007.

[HLN07] W. Hachem, P. Loubaton, and J. Najim. Deterministic equivalents for certain functionals of large random matrices. *Annals of Applied Probability*, 17(3):875–930, 2007.

[KA16] Abla Kammoun and Mohamed-Slim Alouini. No eigenvalues outside the limiting support of generally correlated gaussian matrices. *IEEE Transactions on Information Theory*, 62(7):4312–4326, 2016.

[KLW13] Matthias Keller, Daniel Lenz, and Simone Warzel. On the spectral theory of trees with finite cone type. *Israel Journal of Mathematics*, 194(1):107–135, 2013.

[LC18] Cosme Louart and Romain Couillet. Concentration of measure and large random matrices with an application to sample covariance matrices. *arXiv preprint arXiv:1805.08295*, 2018.

[LC20] Cosme Louart and Romain Couillet. A concentration of measure and random matrix approach to large dimensional robust statistics. *submitted*, 2020.

[LC21] Cosme Louart and Romain Couillet. Concentration of measure and generalized product of random vectors with an application to hanson-wright-like inequalities. *arXiv preprint arXiv:2102.08020*, 2021.

[Led05] Michel Ledoux. *The concentration of measure phenomenon*. Number 89. American Mathematical Soc., 2005.

[Lou22] Cosme Louart. Sharp bounds for the concentration of the resolvent in convex concentration settings. *arXiv preprint arXiv:2201.00284*, 2022.

[MP67] V. A. Marčenko and L. A. Pastur. Distribution of eigenvalues for some sets of random matrices. *Math USSR-Sbornik*, 1(4):457–483, 1967.

[MS11] Mark W. Meckes and Stanislaw J. Szqrek. Concentration for noncommutative polynomials in random matrices. *Proceedings of the American Mathematical Society*, 2011.

[Sil86] J. W. Silverstein. Eigenvalues and eigenvectors of large dimensional sample covariance matrices. *Random Matrices and their Applications*, pages 153–159, 1986.

[SLTC20] Mohamed El Amine Seddik, Cosme Louart, Mohamed Tamaazousti, and Romain Couillet. Random matrix theory proves that deep learning representations of gandate behave as gaussian mixtures. *ICML (submitted)*, 2020.

[Tao12] Terence Tao. *Topics in random matrix theory*, volume 132. American Mathematical Soc., 2012.

[Ver18] Roman Vershynin. *High-Dimensional Probability*. Cambridge University Press, 2018.

[VW14] Van Vu and Ke Wang. Random weighted projections, random quadratic forms and random eigenvectors. *Random Structures and Algorithms*, 2014.

[WCDS12] Sebastian Wagner, Romain Couillet, Mérouan Debbah, and Dirk TM Slock. Large system analysis of linear precoding in correlated miso broadcast channels under limited feedback. *IEEE transactions on information theory*, 58(7):4509–4537, 2012.

[Yas16] Pavel Yaskov. A short proof of the marchenko–pastur theorem. *Comptes Rendus Mathematique*, 354, 3:319–322, 2016.

[Yin86] Y. Yin. Limiting spectral distribution for a class of random matrices. *Journal of Multivariate Analysis*, 20:50–68, 1986.

[Yin20] Yanqing Yin. On the singular value distribution of large-dimensional data matrices whose columns have different correlations. *Statistics*, 54(2):353–374, 2020.