Equilibrium Asset Pricing with Transaction Costs∗

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Abstract

We study a risk-sharing equilibrium where heterogenous agents trade subject to quadratic transaction costs. The corresponding equilibrium asset prices and trading strategies are characterised by a system of nonlinear, fully–coupled forward–backward stochastic differential equations. We show that a unique solution generally exists provided that the agents’ preferences are sufficiently similar. In a benchmark specification, the illiquidity discounts and liquidity premia observed empirically correspond to a positive relationship between transaction costs and volatility.

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1 Introduction

How does the introduction of a transaction tax affect the volatility of a financial market? Such questions about the interplay of liquidity and asset prices need to be tackled with equilibrium models, where prices are not exogenous inputs but determined endogenously by matching supply and demand. However, equilibrium analyses lead to notoriously intractable feedback loops. Indeed, if the optimal strategies for a given candidate price do not clear the market, then the price needs to adjust until this iteration converges. Trading costs compound these difficulties, because they severely complicate the corresponding optimisation problems.

Accordingly, the literature on equilibrium asset prices with transaction costs has focused either on numerical methods [25, 11, 10], or on models where the market volatility is either zero [46, 34, 47] or given exogenously [45, 42, 22, 7]. In the present study, we analyse a risk–sharing equilibrium where price levels, expected returns and volatilities are determined endogenously, by both balancing supply and demand and matching an exogenous terminal condition for the risky asset.

We consider two agents with mean–variance preferences who trade a safe and a risky asset to hedge the fluctuations of their random endowment streams. By developing new well–posedness results for fully-coupled systems of nonlinear forward-backward stochastic differential equations (FBSDEs), we show that a unique equilibrium with transaction costs generally exists provided the agents’ risk aversions are sufficiently similar.

In a concrete example with linear state dynamics, this characterisation reduces to a system of four coupled Riccati ODEs. These lead to explicit asymptotic formulas for similar risk aversions, which reveal close connections

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between the effects of transaction costs on asset prices, expected returns, and volatilities. To wit, the “liquidity discount” of asset prices compared to their frictionless counterparts, the “liquidity premia” that distinguish their expected returns, and the adjustment of the corresponding volatilities all have the same sign in our model, determined by the difference of the agents’ risk aversion parameters. In the empirically relevant case of positive illiquidity discounts and liquidity premia \([2, 8, 40]\), our model predicts a positive relation between transaction costs and volatility, corroborating empirical evidence of \([24]\) and numerical results of \([10]\). In addition to these systematic shifts, transaction costs also endogenously lead to mean-reverting expected returns as in the reduced-form models of \([29, 14, 36, 21]\): for illiquid assets, supply-demand imbalances do not offset immediately but only gradually, thereby leading to partially predictable returns.

Without transaction costs, the equilibrium dynamics of the risky asset are determined by a purely quadratic BSDE in our model, which leads to explicit formulas in concrete examples. With quadratic transaction costs on the agents’ trading rates, we show that the corresponding equilibria are characterised by fully-coupled systems of FBSDEs. To wit, the optimal risky positions evolve forward from the given initial allocations. In contrast, the corresponding trading rates controlling these positions need to be determined from their zero terminal values – near the terminal time, trading stops since additional trades can no longer earn back the costs that would need to be paid to implement them. If a constant volatility is given exogenously as in \([7]\), then these forward–backward dynamics suffice to pin down the equilibrium returns. In this case, the FBSDEs are linear, and therefore can be solved explicitly in terms of Riccati equations and conditional expectations of the endowment processes \([22, 5]\). In the present context, where the volatility is determined endogenously from the terminal condition for the risky asset, the corresponding FBSDEs are coupled to an additional backward equation arising from this extra constraint. Due to the quadratic preferences and trading costs, the resulting forward-backward system is still linear in the trading rates and positions. However, it also depends quadratically on the volatility of the risky asset, which is now no longer an exogenous constant but needs to be determined as part of the solution.

Accordingly, explicit solutions are no longer possible and existence and uniqueness are beyond the scope of the extant literature. Indeed, there is no general well–posedness theory for fully coupled systems of FBSDEs. In fact, even for linear equations, one can obtain either well–posedness, or infinitely many solutions, or no solutions at all, see the example in the introduction of \([35]\). Under a variety of additional monotonicity, non-degeneracy, or Lipschitz assumptions or for scalar forward and backward components, well-posedness results have been obtained by \([4, 15, 16, 17, 39, 51, 48, 49, 50, 26, 41, 35]\). However, none of these results are applicable to our fully–coupled system, which is not Lipschitz and has a bivariate backward component.

To overcome these difficulties, we focus on the case where both agents’ risk aversions are sufficiently similar. If these parameters coincide, then the BSDE for the equilibrium price decouples from the FBSDEs for the optimal position and trading rate, and in fact reduces to its frictionless counterpart. For distinct but similar risk aversions, we in turn establish the existence of a unique solution. Our proof is based on a Picard iteration under smallness conditions inspired by \([43]\). However, due to the coupling between forward and backward components, this standard argument only yields existence here if the time horizon is sufficiently short – a degenerate result in the present context since the cost on the trading rate then essentially imposes a no–trade equilibrium. Proving existence on arbitrary time horizons requires more subtle arguments tailored to the structure of the equations. Here, the key insight is that, for a given volatility process, the BSDE for the corresponding optimal positions and trading rates can be solved in terms of stochastic Riccati equations as in \([30, 3, 6]\). We develop a number of novel stability estimates for such equations. These in turn allow us to devise a one–dimensional Picard iteration for the equilibrium price process only – the corresponding positions and trading rates are constructed using the stochastic Riccati equations of \([30]\) in each step. If the agents’ risk aversions are sufficiently similar, we are in turn able to establish the existence of a solution, which is unique in a neighbourhood of its frictionless counterpart.

This well–posedness result applies in general settings without requiring a Markovian structure. However, it crucially exploits that all primitives of the model belong to suitable BMO spaces. This assumption ensures that the optimal positions remain uniformly bounded and our BSDEs are of quadratic growth, but rules out concrete specifications based on Brownian motions, for example. To show how our approach can be adapted to such settings, we therefore also study a concrete model with Brownian trading targets as in \([34]\). With such linear state dynamics and a linear terminal condition for the risky asset, the FBSDEs characterising the equilibrium can
be reduced to a system of four coupled scalar Riccati ODEs. For sufficiently similar risk aversions, existence for this system can in turn be established by adapting our Picard iteration. Again, the key idea is not to work with the full multidimensional system, but instead focus on only one component (the others are in turn constructed from this source term in each step of the iteration).

The remainder of this article is organised as follows. Section 2 introduces our model, both in the frictionless baseline version and with quadratic transaction costs on the trading rate. The agents’ individual optimisation problems for given price dynamics are then discussed in Section 3. Our main results on equilibrium asset prices without and with transaction costs are subsequently presented in Section 4. For better readability, all proofs are delegated to Sections 5–7 as well as Appendices A and B.

**Notations** Throughout, we fix a filtered probability space \((\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})\) with finite time horizon \(T > 0\); the filtration is generated by a standard Brownian motion \((W_t)_{t \in [0,T]}\). For \(0 \leq s \leq t \leq T\), the set of \([s,t]–\)valued stopping times is denoted by \(\mathcal{T}_{s,t}\); for \(\tau \in \mathcal{T}_{0,T}\), we write \(\mathbb{E}_\tau[\cdot]\) for the \(\mathcal{F}_\tau–\)conditional expectation. The \(\mathbb{R}–\)valued, progressively measurable processes \((X_t)_{t \in [0,T]}\) satisfying \(\|X\|_{\mathbb{H}_p}^p := \mathbb{E}\left[(\int_0^T X_t^2 dt)^{p/2}\right] < \infty\) for some \(p \in [1, \infty)\) are denoted by \(\mathbb{H}^p\). We also write \(\mathbb{H}^2_{\text{BMO}}\) for the \(\mathbb{R}–\)valued, progressively measurable processes \((X_t)_{t \in [0,T]}\) satisfying

\[
\|X\|^2_{\mathbb{H}^2_{\text{BMO}}} := \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}_\tau \left[ \int_\tau^T X_t^2 dt \right] < \infty.
\]

Finally, for any \(p \in [1, \infty]\), \(S^p\) denotes the \(\mathbb{R}–\)valued, \(\mathbb{F}–\)progressively measurable processes \(X\) with continuous paths for which \(\sup_{0 \leq t \leq T} |X_t| \in L^p\). The associated norm is denoted by \(\|\cdot\|_{S^p}\). For any other probability measure \(Q\) on \((\Omega, \mathcal{F})\), we define similarly \(L^p(Q), \mathbb{H}^p(Q), \mathbb{H}^2_{\text{BMO}}(Q),\) and \(S^p(Q)\).

## 2 Model

### 2.1 Financial Market

We consider a financial market with two assets. The first one is safe, with exogenous price normalised to one. The second one is risky, with price dynamics

\[
dS_t = \mu_t dt + \sigma_t dW_t. \tag{2.1}
\]

Here, the initial asset price \(S_0 \in \mathbb{R}\) as well as the (progressively measurable) expected returns process \((\mu_t)_{t \in [0,T]}\) and volatility process \((\sigma_t)_{t \in [0,T]}\) are to be determined in equilibrium by matching demand to the supply \(s \in \mathbb{R}\) of the risky asset. To pin down the equilibrium volatility – unlike in [45, 7],\(^1\) where this process is an exogenous constant – the terminal value of the risky asset is also required to match an exogenous \(\mathcal{F}_T\)–measurable random variable as in [23]:

\[
S_T = \mathcal{G}.
\]

This can be interpreted as a fundamental liquidation value [33], a terminal dividend [31], or the payoff of a derivative depending on an exogenous underlying [12].

### 2.2 Agents

The assets are traded by two agents \(n = 1, 2\) with mean–variance preferences over wealth changes as in [27, 37, 14, 21, 22]. The agents have risk aversions \(\gamma^n > 0, n = 1, 2\) and trade to hedge the fluctuations of their (cumulative) random endowments,

\[
dY^n_t = \beta^n_t dW_t, \quad \beta^n_t \in \mathbb{H}^2. \tag{2.2}
\]

Agent \(n\)’s initial position in the risky asset is fixed throughout and denoted by \(x^n\). To clear the market initially, we naturally assume that \(x^1 + x^2 = s\).

\(^1\)In [42], a particular value is singled out by focusing on linear equilibria.
2.3 Frictionless Trading

Suppose that \( \mu = \sigma \kappa \) where the market price of risk \( \kappa \) belongs to \( \mathbb{H}^2 \). Without transaction costs, agents’ trading strategies are described by the number \( \varphi_t \) of risky shares held at each time \( t \in [0, T] \). Taking into account each agent’s random endowment, their frictionless wealth dynamics are \( \varphi_t dS_t + dY^n_t \). For admissible strategies \( \varphi \) which satisfy \( \varphi_0 = x^n \) and \( \varphi \sigma \in \mathbb{H}^2 \), the corresponding mean–variance goal functional is

\[
J^n(\varphi) := \mathbb{E} \left[ \int_0^T \left( \varphi_t dS_t + dY^n_t - \frac{\gamma^n}{2} d\left( \int_0^t \varphi_s dS_s + Y^n_s \right) \right) \right]
= \mathbb{E} \left[ \int_0^T \left( \mu_t \varphi_t - \frac{\gamma^n}{2} \left( \sigma_t \varphi_t + \beta^n_t \right)^2 \right) dt \right] \to \text{max!} \quad (2.3)
\]

Accordingly, for \( \sigma > 0 \), the process \( -\beta^n/\sigma \) can also be interpreted as agent \( n \)'s target position in the risky asset. Related models where deviations from an exogenous target are directly penalised by an exogenous deterministic weight rather than the infinitesimal variance of the corresponding asset are studied by [13, 42].

2.4 Trading with Transaction Costs

Now suppose as in [1] that an exogenous quadratic transaction cost \( \lambda/2 > 0 \) is levied on the turnover rate \( \dot{\varphi}_t := d\varphi_t/dt \) of each agent’s portfolio. Then, the corresponding position \( \varphi \) becomes a state variable that can only be influenced gradually by adjusting the control \( \dot{\varphi} \). We focus on admissible trading rates \( \dot{\varphi} \in \mathbb{H}^2 \) for which the corresponding position \( \varphi = x^n + \int_0^t \dot{\varphi}_s ds \) satisfies \( \varphi \sigma \in \mathbb{H}^2 \), in analogy to the frictionless case. The frictional version of the mean–variance goal functional (2.3) is

\[
J^n_\lambda(\dot{\varphi}) := \mathbb{E} \left[ \int_0^T \left( \mu_t \varphi_t - \frac{\gamma^n}{2} \left( \sigma_t \varphi_t + \beta^n_t \right)^2 - \lambda \left( \varphi_t \right)^2 \right) dt \right] \to \text{max!} \quad (2.4)
\]

Note that each agent’s P&L is only affected by their own trading rate. Accordingly, the trading cost should be interpreted as a tax or the fees charged by an exchange rather than as a temporary price impact cost here. The assumption of quadratic rather than proportional costs is made for tractability. However, in light of the partial equilibrium literature [38], we expect the qualitative properties of our results to be robust across different small transaction costs.

3 Individual Optimisation

The first step towards solving for the equilibrium is to determine each agent’s individually optimal trading strategy for given asset prices. To this end, fix an initial risky asset price \( S_0 \in \mathbb{R} \), an expected return process \( \langle \mu_t \rangle_{t \in [0, T]} \), and a volatility process \( \langle \sigma_t \rangle_{t \in [0, T]} \) for which \( \mu = \sigma \kappa \) with a market price of risk \( \kappa \in \mathbb{H}^2 \). For better readability, all proofs are delegated to Section 5.

3.1 Frictionless Optimisation

Agent \( n \)'s optimiser for the frictionless model (2.3) can be computed directly by pointwise optimisation\(^3\),

\[
\varphi^n_t := \begin{cases} \frac{\mu_t}{\gamma^n \sigma_t^2} - \frac{\beta^n_t}{\sigma_t}, & \sigma_t \neq 0, \\ x^n, & \sigma_t = 0, \end{cases} \quad t \in (0, T]. \quad (3.1)
\]

\(^2\)By the Cauchy–Schwarz inequality, we then also have \( \varphi \mu \in \mathbb{H}^1 \) since \( \kappa \in \mathbb{H}^2 \).

\(^3\)Note that the optimal strategy is not determined uniquely on the set \( \{ \sigma = 0 \} \), since these values do not contribute to the P&L (2.3). We therefore choose arbitrary values that ensure market clearing. All subsequent result are independent of this choice.
3.2 Optimisation with Transaction Costs

Unlike its frictionless counterpart, the frictional optimisation problem (2.4) is no longer myopic and therefore cannot be solved directly using pointwise optimisation. However, (2.4) can be rewritten as

$$J^n(\phi) = -\mathbb{E} \left[ \int_0^T \left( \frac{\gamma^2 \sigma^2}{2} (\phi_t - \bar{\phi}^n_t)^2 + \frac{\lambda}{2} \dot{\phi}^2_t \right) dt \right] + \mathbb{E} \left[ \int_0^T \frac{\gamma^2}{2} (\sigma_t \dot{\phi}^n_t)^2 - (\beta^n_t)^2 dt \right].$$

Note that the second expectation on the right–hand side of this decomposition is finite for $\kappa, \beta \in \mathbb{R}^2$. Therefore, maximising the frictional mean–variance functional $J^n_\lambda$ is equivalent to solving a quadratic tracking problem, where the target is the frictionless optimiser (3.1):

$$\mathbb{E} \left[ \int_0^T \left( \frac{\gamma^2 \sigma^2}{2} (\phi_t - \bar{\phi}^n_t)^2 + \frac{\lambda}{2} \dot{\phi}^2_t \right) dt \right] \rightarrow \min! \quad (3.2)$$

Problems of this type have been studied by [30, 3, 6]. By strict convexity, each agent’s optimal trading rate is characterised by the first–order condition that its Gâteaux derivative vanishes in all directions [20, Proposition II.2.1]. A calculus–of–variations argument (compare [5, 7]) in turn shows that the optimal trading rate $\dot{\phi}^n_t$ of agent $n$, and the corresponding position $\phi^0_t$ are characterised by a forward–backward stochastic differential equation (FBSDE)\(^4\):

$$d\phi^0_t = \dot{\phi}^n_t dt, \quad \phi^0_0 = x^n, \quad (3.3)$$

$$d\dot{\phi}^n_t = \frac{\gamma^2 \sigma^2}{\lambda} (\phi^n_t - \phi^0_t) dt + \dot{Z}^n_t dW_t, \quad \dot{\phi}^n_T = 0. \quad (3.4)$$

Observe that the process $\dot{Z}^n_t$ needs to be determined as part of the solution here. Unlike for the constant volatilities $\sigma$ considered in [5, 7], this equation cannot be solved by reducing to standard Riccati equations. Instead, a backward stochastic Riccati equation (BSRDE) plays a crucial role in the analysis of [30, 3, 6]. It is shown in [30] that for bounded $\sigma$, this equation has a unique solution. A localisation argument shows that this remains true for $\sigma \in \mathbb{R}^{2}_{\text{BMO}}$, which will be the natural space for our equilibrium analysis in Section 4:

**Lemma 3.1.** For $\gamma, \lambda > 0$ and $\sigma \in \mathbb{R}^2_{\text{BMO}}$, the BSRDE

$$c_t = \int_t^T \left( \frac{\gamma^2 \sigma^2}{\lambda} - c_s^2 \right) ds - \int_t^T \dot{Z}_s^2 dW_s, \quad t \in [0, T], \quad (3.5)$$

has a unique solution $(c, Z) \in \mathcal{S}^\infty \times \mathbb{R}^{2}_{\text{BMO}}$. It satisfies

$$0 \leq c_t \leq \frac{\gamma}{\lambda} ||\sigma||^2_{\text{BMO}}, \quad t \in [0, T]. \quad (3.6)$$

With the auxiliary process $c$ at hand, the solution of the FBSDE (3.3–3.4) characterising the optimal trading rate for the tracking problem (3.2), or equivalently the original mean–variance optimisation (2.4), can in turn be constructed as follows:\(^5\)

**Lemma 3.2.** For $\gamma, \lambda > 0$ and $\sigma \in \mathbb{R}^2_{\text{BMO}}$, let $c$ be the solution of the corresponding BSRDE (3.5). For a progressively measurable process $\xi$ satisfying $\sigma \xi \in \mathbb{H}^2$, define

$$\tilde{\xi}_t := \frac{\gamma}{\lambda} \mathbb{E} \left[ \int_0^T e^{J_t} c_s d\sigma^2_s \xi_s ds \right], \quad t \in [0, T], \quad (3.7)$$

\(^4\)Here, the terminal condition for the trading rate is zero, because trades close to the terminal time $T$ can no longer earn back the trading costs that would need to be paid to implement them. More general terminal conditions are studied in [3, 6], for example.

\(^5\)For uniformly bounded $\sigma$, this result is proved in [30]. For $\sigma \in \mathbb{R}^2_{\text{BMO}}$, we provide a short self–contained proof in Section 5. As a side product, we obtain that the solution coincides with its counterpart for the time–truncated “auxiliary problem” considered by [6].
and the linear (random) ODE
\[ \dot{\varphi}_t = \xi_t - c_t \varphi_t, \quad t \in [0, T], \quad \varphi_0 = x, \] (3.8)
which has the explicit solution
\[ \varphi_t = e^{-\int_0^t c_s \, ds} x + \int_0^t e^{-\int_s^t c_u \, du} \xi_s \, ds, \quad t \in [0, T]. \] (3.9)

Then, for \( \gamma = \gamma^n, \ x = x^n, \) and \( \xi = \varphi^n \) from (3.1), the corresponding solution \((\varphi^n, \varphi^\gamma)\) is optimal for (3.2) or equivalently (2.4). Moreover, if \( \sigma|\xi|^1/2 \in \mathbb{H}^{2 \text{BMO}} \), then \( \dot{\varphi} \) and \( \varphi \) are uniformly bounded.

Lemma 3.2 shows that for \( t \in [0, T) \), the optimal strategy with transaction costs trades towards the “signal process” \( \bar{\xi}_t/c_t \) at a (time–dependent and random) speed \( c_t \) determined by the BSRDE (3.5).\(^6\) For each agent’s individual optimisation problem (3.2), the signal is obtained from the corresponding frictionless optimiser (3.1), by appropriate discounting of its expected future values at a rate also derived from the BSRDE. For our equilibrium analysis in Section 4, the same construction will be applied to a different target strategy, see (4.7).

4 Equilibrium

With the characterisation of each agent’s individually optimal strategy at hand, we now turn to the determination of the equilibrium asset prices for which the agents’ aggregate demand for the risky asset equals its supply \( s \). For better readability, all proofs are deferred to Section 6.

4.1 Frictionless equilibrium

We first consider the frictionless case.

**Definition 4.1.** A price process \( S \) for the risky asset with initial value \( S_0 \in \mathbb{R} \), expected returns \((\mu_t)_{t \in [0, T]} \) and volatility \((\sigma_t)_{t \in [0, T]} \) is called a (Radner) equilibrium, if:

(i) \( \mu = \sigma \kappa \) for \( \kappa \in \mathbb{H}^2 \);

(ii) the terminal condition \( S_T = \mathcal{G} \) is satisfied;

(iii) the agents’ optimal trading strategies (3.1) for the given price process clear the market for the risky asset at all times,
\[ \varphi^1_t + \varphi^2_t = s, \quad t \in [0, T]. \]

For any equilibrium \((S_0, \mu, \sigma)\), market clearing and representation (3.1) for the agents’ individually optimal strategies implies
\[ \mu_t = \gamma \left( s \sigma_t^2 + \sigma_t (\beta^1_t + \beta^2_t) \right), \quad t \in [0, T], \quad \text{where} \quad \gamma := \frac{\gamma^1 \gamma^2}{\gamma^1 + \gamma^2}. \]

Accordingly, \((S, \sigma)\) solves the following quadratic BSDE:
\[ dS_t = \gamma (s \sigma_t^2 + \sigma_t (\beta^1_t + \beta^2_t)) \, dt + \sigma_t \, dW_t, \quad S_T = \mathcal{G}. \] (4.1)

Conversely, the individually optimal strategies (3.1) corresponding to the dynamics (4.1) are admissible if \( \sigma \in \mathbb{H}^2 \) and evidently clear the market. Whence, existence and uniqueness of Radner equilibria are generally equivalent to existence and uniqueness of solutions of the quadratic BSDE (4.1). Given that the measure
\[ \mathbb{P}^\beta \sim \mathbb{P}, \text{ with density process } Z^\beta := \mathcal{E} \left( - \int_0^T \gamma (\beta^1_t + \beta^2_t) \, dW_t \right), \] (4.2)

\(^6\)In particular, since \( \xi \) only depends on \( \sigma^2 \xi \), the optimiser for (2.4) in independent of the (arbitrary) values chosen for the frictionless optimiser on \{\( \sigma = 0 \)\}. 

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is well defined, the BSDE (4.1) can be rewritten in terms of the $\mathbb{P}^\beta$–Brownian motion $W^\beta = W - \int_0^\tau \gamma (\beta_1 + \beta_2)dt$ as a purely quadratic BSDE,

$$dS_t = \gamma \sigma_t^2 dt + \sigma_t dW_t^\beta, \quad S_T = \mathcal{G}. \quad (4.3)$$

Provided the terminal condition $\mathcal{G}$ is sufficiently integrable, it is well known that (4.3) has an explicit solution in terms of the Laplace transform of $\mathcal{G},$

$$S_t = -\frac{1}{2\gamma s} \log \mathbb{E}_t^\beta \left[ e^{-2\gamma s \mathcal{G}} \right], \quad t \in [0, T]. \quad (4.4)$$

To make sure the measure $\mathbb{P}^\beta$ is well defined and verify that (4.4) is indeed the unique solution of (4.3) in a suitable class, the aggregate trading target $\beta_1 + \beta_2$ and the terminal condition $\mathcal{G}$ need to be sufficiently integrable, e.g., uniformly bounded.

**Assumption 4.2.** $\beta_1 + \beta_2 \in \mathbb{H}_{BMO}^2$ and $|\mathcal{G}|$ has finite exponential moments of all orders.

**Proposition 4.3.** Suppose Assumption 4.2 is satisfied. Then, (4.4) is the unique solution of (4.3) among continuous, progressively measurable processes $S$ for which $(e^{-2\gamma s S_T})_{\tau \in T_0,T}$ is uniformly $\mathbb{P}^\beta$–integrable. In particular, the price process (4.4) is the unique Radner equilibrium in this class.

**Remark 4.4.** As already observed in, e.g., [18], the class of price processes for which $(e^{-2\gamma s S_T})_{\tau \in T_0,T}$ is uniformly $\mathbb{P}^\beta$–integrable is the largest possible class for uniqueness. Indeed, if this family is not uniformly $\mathbb{P}^\beta$–integrable, then $e^{-2\gamma s S}$ is a strict local $\mathbb{P}^\beta$–martingale by Itô’s formula and the dynamics (4.3), and hence a strict $\mathbb{P}^\beta$–supermartingale since it is also positive. As a result, the corresponding price process $S$ is strictly larger than (4.4).

The non–uniqueness described in Remark 4.4 can only arise for price processes that are unbounded from below. In fact, uniqueness always holds among price processes $S$ which admit an equivalent martingale measure with square–integrable density process $Z$ with respect to $\mathbb{P}^\beta$. Indeed, in view of the dynamics (4.3), we necessarily have $Z = \mathcal{E}(\gamma s \int_0^\tau \sigma_t dW_t^\beta)$ and in turn

$$0 \leq e^{-2\gamma s S_T} = e^{-2\gamma^2 s^2 \int_0^\tau \sigma_t^2 dt - 2\gamma s \int_0^\tau \sigma_t dW_t^\beta} \leq e^{-\gamma^2 s^2 \int_0^\tau \sigma_t^2 dt - 2\gamma s \int_0^\tau \sigma_t dW_t^\beta} = Z_\tau^2, \quad \text{for any } \tau \in T_0,T.$$

Whence uniform $\mathbb{P}^\beta$–integrability of $(e^{-2\gamma s S_T})_{\tau \in T_0,T}$ follows from Doob’s maximal inequality in this case. If the terminal condition is bounded, uniqueness even holds all among price processes $S$ admitting an equivalent martingale measure, since $S$ is then automatically bounded.

**Corollary 4.5.** Suppose Assumption 4.2 is satisfied and, moreover, $\mathcal{G} \in \mathbb{L}^\infty$. Then, (4.4) is the unique solution of (4.3) in $\mathcal{S}^\infty \times \mathbb{H}_{BMO}^2$, and therefore the unique Radner equilibrium among bounded price processes.

### 4.2 Equilibrium with transaction costs

We now turn to the main subject of the present study, equilibria with transaction costs.

**Definition 4.6.** A price process (2.1) for the risky asset with initial value $S_0 \in \mathbb{R}$, expected returns $(\mu_t)_{t \in [0,T]}$ and volatility $(\sigma_t)_{t \in [0,T]}$ is called a (Radner) equilibrium with transaction costs $\lambda$, if

1. $\mu = \sigma \kappa$ for $\kappa \in \mathbb{H}^2$ and $\sigma \in \mathbb{H}_{BMO}^2$;
2. the terminal condition $S_T = \mathcal{G}$ is satisfied;
3. the agents’ individually optimal trading strategies from Lemma 3.2 clear the market for the risky asset at all times, $\varphi^1_t + \varphi^2_t = s$, $t \in [0, T]$.

7Such a notion of uniqueness is used in [32], for example.
To clear the market, purchases must equal sales at all times, i.e., all individual trading rates must sum to zero. After summing the backward equations (3.4) for both agents’ optimal trading rates and using the market clearing condition \( \varphi^2_t = s - \varphi^1_t \), this leads to

\[
0 = \left( \frac{\sigma_t}{\lambda} (\gamma^1 \beta^1_t + \gamma^2 \beta^2_t) + \frac{\sigma_t^2}{\lambda} (\gamma^2 s + (\gamma^1 - \gamma^2) \varphi^1_t) - \frac{2 \mu_t}{\lambda} \right) dt + (\dot{Z}^1_t + \dot{Z}^2_t) dW_t.
\]

Since any local martingale of finite variation is constant, it follows that

\[
\mu_t = \sigma_t \left( \frac{\gamma^1 \beta^1_t + \gamma^2 \beta^2_t}{2} + \sigma_t \left( \frac{\gamma^2 s + (\gamma^1 - \gamma^2) \varphi^1_t}{2} \right) \right), \quad t \in [0, T]. \tag{4.5}
\]

Plugging this back into agent 1’s individual optimality condition (3.4) and recalling the terminal condition \( \varphi^1_T = 0 \) as well as the forward equation (3.3), we obtain the following FBSDE:

\[
\begin{align*}
    d\varphi^1_t &= \varphi^1_t, & &\varphi^1_0 = x^1, \tag{4.6} \\
    d\dot{\varphi}^1_t &= \left( \frac{\gamma^1 + \gamma^2}{2 \lambda} \left( \frac{\gamma^1 \beta^1_t - \gamma^2 \beta^2_t}{\gamma^1 + \gamma^2} - \sigma_t - \frac{\gamma^2 s + \gamma^1 \beta^1_t + \gamma^2 \beta^2_t}{\gamma^1 + \gamma^2} \right) \right) dt + \dot{Z}^1_t dW_t, & &\dot{\varphi}^1_T = 0. \tag{4.7}
\end{align*}
\]

The corresponding optimal strategy for agent 2 is determined by market clearing. As in the frictionless case discussed in Section 4.1, the corresponding equilibrium volatility is pinned down by the terminal condition \( S_T = \mathcal{S} \). More specifically, inserting (4.5) into (2.1), we obtain the following BSDE, which is coupled to the forward–backward system (4.6 – 4.7):

\[
    dS_t = \left( \frac{\gamma^1 - \gamma^2}{2} \varphi^1_t \sigma^2_t + \frac{\gamma^2 s}{2} \sigma^2_t + \frac{\gamma^1 \beta^1_t + \gamma^2 \beta^2_t}{2} \sigma_t \right) dt + \sigma_t dW_t, \quad S_T = \mathcal{S}. \tag{4.8}
\]

By reversing these arguments, it is straightforward to verify that sufficiently integrable solutions of the FBSDE (4.6 – 4.8) indeed identify Radner equilibria with transaction costs:

**Proposition 4.7.** Suppose that there exists a solution of the FBSDE (4.6 – 4.8) with \( (\varphi^1, \sigma) \in \mathbb{H}^2 \times \mathbb{H}^2_{\text{BMO}} \). Then, \( (S_0, \mu, \sigma) \) with \( \mu \) as in (4.5) is a Radner equilibrium with transaction costs.

Due to the coupling between the forward–backward equations (4.6 – 4.8) a direct existence proof by fixed-point iteration is elusive, unless the time horizon is sufficiently short so that very little trading is possible with costs on the trading rate. Establishing existence for sufficiently small transaction costs is also delicate, since the corresponding trading rates explode, which needs to be handled by a suitable renormalisation. Inspired by [42], we therefore focus on a different smallness condition, namely the case where both agents risk aversions are similar, \( \gamma^1 \approx \gamma^2 \).

For \( \gamma^1 = \gamma^2 \), the BSDE (4.8) for the frictional equilibrium price decouples from (4.6 – 4.7) and reduces to its frictionless counterpart (4.3). Accordingly, for \( \gamma^1 \approx \gamma^2 \), we expect the frictional equilibrium price \( S \) and its volatility \( \sigma \) to be close to their frictionless versions \( \bar{S} \) and \( \sigma_\ast \), respectively. To make this precise, the frictionless equilibrium volatility \( \sigma_\ast \) and the volatilities \( \beta^1, \beta^2 \) of the agents’ random endowments need to be sufficiently integrable:

**Assumption 4.8.** (i) the frictionless equilibrium volatility \( \sigma_\ast \) from Proposition 4.3 belongs to \( \mathbb{H}^2_{\text{BMO}} \); (ii) \( \beta^1, \beta^2 \in \mathbb{H}^2_{\text{BMO}} \), so that we can define the measure

\[
    Q^\beta \sim P \quad \text{with density process} \quad \frac{dQ^\beta}{dP} := \mathcal{E} \left( - \int_0^T \left( \gamma^2 s \sigma_t + \frac{\gamma^1 \beta^1_t + \gamma^2 \beta^2_t}{2} \right) dt \right)
\]

(iii) for some \( p > 2 \), we have \( \mathbb{E}^{Q^\beta} \left[ \exp \left( p \int_0^T \left( \gamma^2 s \sigma_t + \frac{\gamma^1 \beta^1_t + \gamma^2 \beta^2_t}{2} \right)^2 dt \right) \right] < \infty \).
We can now formulate our main result. It shows that an equilibrium with transaction costs exists, provided that the agents’ risk aversions $\gamma^1$, $\gamma^2$ are sufficiently similar. This equilibrium is also unique in a neighbourhood of the frictionless equilibrium price $\bar{S}$ and volatility $\bar{\sigma}$. To make these statements precise we define, for any $R > 0$, the following set of progressively measurable processes:

$$\mathcal{B}_\infty(R) := \{(S, \sigma) : \|S - \bar{S}\|_{\mathcal{S}_\infty}^2 + \|\sigma - \bar{\sigma}\|_{\mathcal{H}^{BMO}(Q^\beta)}^2 \leq R^2\}.$$ 

Our main result then can be formulated as follows:

**Theorem 4.9.** Suppose Assumptions 4.2 and 4.8 are satisfied. Then, there exists $R_{\max} > 0$ such that for any $R < R_{\max}$ the system of coupled FBSDEs (4.6–4.8) has a unique solution $(S, \sigma) \in \mathcal{B}_\infty(R)$ provided that $|\gamma^1 - \gamma^2|$ is small enough.\(^8\)

Theorem 4.9 is a special case of our more general well–posedness result Theorem 6.3 and applies, for example, if the endowment volatilities $\beta^1, \beta^2$ and the terminal condition $\mathcal{G}$ are uniformly bounded. More generally, the BMO assumptions from Assumption 4.8 guarantee that the equilibrium positions $\varphi^1$ and trading rates $\dot{\varphi}^1$ are uniformly bounded, which is crucial for the Picard iteration we use to prove Theorem 4.9. However, Assumption 4.8 does not cover specifications where the primitives $\beta^1, \beta^2$ follow certain unbounded diffusion processes such as Brownian motion. As a complement to Theorem 4.9, we now therefore discuss such a concrete example and show that the FBSDE system (4.6–4.8) can be reduced to a system of deterministic but coupled Riccati equations in this case. For sufficiently similar risk aversions $\gamma^1$ and $\gamma^2$, existence can in turn be established by adapting the Picard iteration used to prove Theorem 4.9 to these ODEs.

For concreteness, suppose similarly as in [34] that the aggregate endowment is zero and both agents’ endowment volatilities follow Brownian motions:

$$\beta^1_t = -\beta^2_t = \beta W_t, \quad \beta > 0.$$ 

The terminal condition for the risky asset also is a linear function of the underlying Brownian motion:

$$\mathcal{G} = bT + a W_T, \quad a > 0, \quad b \in \mathbb{R}.$$ 

Then, the frictionless equilibrium price from Proposition 4.3 is a Bachelier model,

$$\bar{S}_t = (b - \bar{\gamma} s a^2)T + \bar{\gamma} s a^2 t + a W_t, \quad t \in [0, T].$$ 

In this Markovian setting, the FBSDE system (4.6–4.8) can be reformulated as a PDE by the standard Markovian ansatz that the backward components are smooth functions of time $t$ and the forward components $W_t$ and $\varphi^1_t$. Itô’s formula and comparison of the diffusion and drift terms in turn leads to a semilinear PDE. For the linear state dynamics and terminal conditions considered here, a linear ansatz finally allows to reduce this PDE to a system of coupled Riccati equations. If these have a solution, it identifies an equilibrium with transaction costs:

**Proposition 4.10.** Suppose the following system of coupled Riccati equations has a solution on $[0, T]$:

$$B'(t) = \frac{\gamma^1 - \gamma^2}{2} \beta (a + B(t)) - C(t) E(t), \quad B(T) = 0,$$

$$C'(t) = \frac{\gamma^1 - \gamma^2}{2} (a + B(t))^2 - C(t) F(t), \quad C(T) = 0,$$

$$E'(t) = \frac{\gamma^1 + \gamma^2}{2\lambda} \beta (a + B(t)) - E(t) F(t), \quad E(T) = 0,$$

$$F'(t) = \frac{\gamma^1 + \gamma^2}{2\lambda} (a + B(t))^2 - F(t)^2, \quad F(T) = 0,$$

\(^8\)An exact upper bound for $\gamma^1 - \gamma^2$ depending on $R$ and an explicit expression for $R_{\max}$ are provided in Theorem 6.3 below.
and define, for \( t \in [0, T] \),
\[
A(t) = \int_t^T \left( C(u)D(u) + \gamma^2 a^2 - \frac{\gamma^2}{2}(a + B(u))^2 \right) du,
\]
\[
D(t) = \int_t^T \left( e^{\int_u^T F(r dr)} \frac{\gamma^2}{2\lambda}(a + B(u))^2 \right) du.
\]

Then, an equilibrium price with transaction costs and the corresponding optimal trading rates are given by
\[
S_t = \bar{S}_t + A(t) + B(t)W_t + C(t)\varphi_t^1, \quad \dot{\varphi}_t^1 = -\dot{\varphi}_t^2 = D(t) + E(t)W_t + F(t)\varphi_t^1, \quad t \in [0, T],
\]
where
\[
\varphi_t^1 = e^{\int_0^t F(r) dr} x^1 + \int_0^t e^{\int_u^t F(r) dr} (D(u) + E(u)W_u) du, \quad t \in [0, T].
\]

Similarly as in Theorem 4.9, a solution of the ODE system is guaranteed to exist provided the agents’ risk aversions are sufficiently similar:

**Theorem 4.11.** Suppose that \( |\gamma^1 - \gamma^2| \) is sufficiently small. Then, the system of Riccati equations from Proposition 4.10 has a solution on \([0, T]\), which in turn identifies an equilibrium with transaction costs.

The Riccati equations from Proposition 4.10 can readily be solved numerically with standard ODE solvers. To shed some light on their comparative statics, it is nevertheless instructive to consider the asymptotics as the difference
\[
\varepsilon = \gamma^1 - \gamma^2,
\]
of the agents’ risk aversions tends to zero. For \( \varepsilon = 0 \), we evidently have \( B(t; 0) = C(t; 0) = 0 \) and in turn \( A(t; 0) = 0 \). Moreover, the explicit formulas for scalar Riccati equations and linear ODEs, as well as an elementary integration show that
\[
F(t; 0) = -\sqrt{\frac{(\gamma^1 + \gamma^2)a^2}{2\lambda}} \tanh (\delta(T - t)), \quad E(t; 0) = -\beta \sqrt{\frac{\gamma^1 + \gamma^2}{2\lambda}} \tanh (\delta(T - t))
\]
\[
D(t; 0) = -\frac{\gamma^2 sa}{\sqrt{2\lambda(\gamma^1 + \gamma^2)}} \tanh (\delta(T - t)), \quad \delta := \sqrt{\frac{(\gamma^1 + \gamma^2)a^2}{2\lambda}}.
\]

With these limiting functions at hand, one then readily verifies that, for \( \varepsilon \to 0 \), the first-order asymptotics of \( C(t; \varepsilon), B(t; \varepsilon), \) and \( A(t; \varepsilon) \) are
\[
C(t; \varepsilon) = -\frac{\varepsilon a^2}{2} \int_t^T e^{\int_u^T F(r; 0) dr} ds + o(\varepsilon) = -\varepsilon a \sqrt{\frac{\lambda}{2(\gamma^1 + \gamma^2)}} \tanh (\delta(T - t)) + o(\varepsilon),
\]
and in turn
\[
B(t; \varepsilon) = \int_t^T \left( -\frac{\varepsilon a}{2} + C(s; \varepsilon)E(s; 0) \right) ds + o(\varepsilon) = -\frac{\varepsilon \beta a \sqrt{\lambda}}{\sqrt{2(\gamma^1 + \gamma^2)}} \tanh (\delta(T - t)) + o(\varepsilon), \quad (4.9)
\]
as well as
\[
A(t; \varepsilon) = \int_t^T \left( C(s; \varepsilon)D(s; 0) + \frac{\varepsilon \gamma^2 sa^2}{2(\gamma^1 + \gamma^2)} - \gamma^2 saB(s, \varepsilon) \right) ds + o(\varepsilon)
\]
\[
= \frac{\varepsilon \gamma^2 s \beta a}{\gamma^1 + \gamma^2} \log \cosh (\delta(T - t)) + \frac{\varepsilon \gamma^2 s \sqrt{\lambda a^2}}{\sqrt{2(\gamma^1 + \gamma^2)^{3/2}}} \tanh (\delta(T - t)) + o(\varepsilon).
\]
We see that, as \( \varepsilon \to 0 \), the equilibrium trading rate \( \dot{\varphi}_t \) from Proposition 4.10 converges to

\[
\dot{\varphi}_t = F(t; 0) \times (\varphi_t - \varphi_t^1), \quad \dot{\varphi}_t^1 := \frac{r^2 - \gamma^2 s}{\gamma^1 + \gamma^2} - \frac{\beta}{a} W_t.
\]

Whence, at the leading order for small \( \varepsilon \), the equilibrium position of agent 1 tracks its frictionless counterpart \( \varphi_t^1 \) with the relative trading speed \( -F(t; 0) \). Accordingly, for small \( \varepsilon \), the corresponding deviation \( \varphi_t - \varphi_t^1 \) approximately has Ornstein–Uhlenbeck dynamics,

\[
d(\varphi_t - \varphi_t^1) \approx (F(t; 0)(\varphi_t - \varphi_t^1))dt + \frac{\beta}{a}dW_t. \tag{4.10}
\]

Let us now discuss what this implies for the corresponding equilibrium price of the risky asset. It’s initial level \( S_0 \) is adjusted by \( A(0, \varepsilon) + C(0, \varepsilon)\varphi_0^1 \) compared to the frictionless case. Here, the second term quickly converges to a stationary value as the time horizon \( T \) grows. In contrast, the first term approximately grows linearly and therefore dominates for long time horizons,

\[
A(0, \varepsilon) + C(0, \varepsilon)\varphi_0^1 = \frac{(\gamma^1 - \gamma^2)\gamma^2 s}{\sqrt{2(\gamma^1 + \gamma^2)}} T \beta a \sqrt{X} + O(1), \quad \text{as} \ T \to \infty. \tag{4.11}
\]

Therefore, as in the overlapping-generations model of [45], the stock price can be either increased or decreased due to transaction costs here. In the present context, the sign of this correction term is determined by the difference \( \gamma^1 - \gamma^2 \) of the agents’ risk aversions. If we choose \( \gamma^2 > \gamma^1 \) to match the illiquidity discounts observed empirically [2], then the discount (4.11) is concave in the transaction cost consistent with the empirical findings of [2].

Next, let us turn to the drift rate of the risky asset. The difference to its frictionless counterpart is\(^9\)

\[
A'(t) + B'(t)W_t + C'(t)\varphi_t^1 + C(t)\dot{\varphi}_t^1
= (A'(t) + C(t)D(t)) + (B'(t) + C(t)E(t))W_t + (C'(t) + C(t)F(t))\dot{\varphi}_t^1
= \left(-\gamma s a^2 + \frac{\gamma^2}{2} (a + B(t))^2\right)W_t + \left(\frac{\gamma^1 - \gamma^2}{2} (a + B(t))^2\right)\dot{\varphi}_t^1
= \frac{\gamma^1 - \gamma^2}{2} a^2 (\varphi_t^1 - \varphi_t^1) + \gamma^2 s a B(t) + o(\gamma^1 - \gamma^2).
\]

We see that the “liquidity premium” compared to the frictionless case consists of two parts. The first is a rescaling of the Ornstein–Uhlenbeck process (4.10): like in [42, 7], transaction costs endogenously lead to a mean-reverting “momentum factor” as in the reduced form models of [29, 14, 36, 21].

However, unlike in [42, 7] where the difference between frictional and frictionless expected returns fluctuates around zero, an additional deterministic component appears here. Up to rescaling with the factor \( \gamma^2 s a_0 \), it coincides with the volatility correction \( B(t) \) for small \( \gamma^1 - \gamma^2 \). As a consequence, the illiquidity discount of the initial price \( S_0 \), the average liquidity premium in the expected returns, and the volatility correction all have the same sign in our model, which is determined by the difference \( \gamma^2 - \gamma^1 \) of the agents’ risk aversion coefficients. The empirical literature consistently finds positive illiquidity discounts [2] and liquidity premia [2, 8, 40]. If we choose \( \gamma^2 > \gamma^1 \) to reproduce this in our model, then it follows that the corresponding volatility correction due to transaction costs is also positive. This theoretical result that illiquidity should lead to higher volatilities corroborates the empirical results of [24] and numerical findings of [10].

To understand the intuition behind this result, recall that \( \beta > 0 \), so that price shocks are positively correlated with shocks to agent 1’s endowment exposure. For a positive price shock, agent 1 then has to sell risky assets to hedge against the increased exposure to future price shocks. Conversely, agent 2 has to buy shares of the risky

\(^9\)Here, we have used integration by parts for the first step, the ODEs satisfied by \( A, B, \) and \( C \) for the second, and the asymptotics (4.9) of the function \( B(t) \) in the third step.
asset. Accordingly, agent 2 can be interpreted as a “trend follower”, whereas agent 1 follows a “contrarian” strategy. If \( \gamma^2 > \gamma^1 \), the trend follower’s buying motive after a positive price shock is stronger than the contrarian’s motive to sell. To clear the market, the expected return of the risky asset therefore has to decrease compared to the frictionless benchmark to make selling more attractive. Accordingly, positive price shocks are dampened and an analogous argument shows that the same effect persists for negative price shocks. Since price shocks are dampened, the equilibrium volatility therefore has to increase in order to match the fixed terminal condition.

5 Proofs for Section 3

This section contains the proofs of the results on Riccati BSRDEs and FBSDEs from Section 3. First, we prove Lemma 3.1, which ensures existence and uniqueness of suitably integrable solutions of the BSRDE (3.5) for volatility processes \( \sigma \in \mathbb{H}^2_{\text{BMO}} \).

**Proof of Lemma 3.1.** For each \( n \in \mathbb{N} \), consider the truncated process \( \sigma^n := \sigma \wedge n \). Since this process is uniformly bounded, the truncated BSRDE

\[
c^n_t = \int_t^T \left( \frac{\gamma}{\lambda} (\sigma^n_s)^2 - (c^n_s)^2 \right) ds - \int_t^T Z^n_s dW_s, \quad t \in [0, T],
\]

has a unique solution \( (c^n, Z^n) \in \mathcal{S}^\infty \times \mathbb{H}^2 \) for each \( n \) with \( c^n \geq 0 \) by [30, Theorem 2.1]. Indeed, in their notation, our case corresponds to

\[
A = C = D = 0, \quad N = B = 1, \quad Q(t) = \frac{\gamma}{\lambda} (\sigma^n)^2, \quad M = 0.
\]

Since \( N \) is positive and uniformly bounded away from 0, \( M \) is bounded and non–negative, and \( Q \) is bounded and non–negative, [30, Theorem 2.1] indeed does apply.

Then, by taking conditional expectations, we see that all of these solutions are uniformly bounded from above, since

\[
c^n_t = \mathbb{E}_t \left[ \int_t^T \left( \frac{\gamma}{\lambda} (\sigma^n_s)^2 - (c^n_s)^2 \right) ds \right] \leq \frac{\gamma}{\lambda} \| \sigma \|_{\mathbb{H}^2}^2.
\]

By the comparison theorem for Lipschitz BSDEs [44, Theorem 9.4], \( c^n_t \geq 0 \) for any \( t \in [0, T] \), since \( (0, 0) \) is the unique solution of the BSDE with terminal condition 0 and generator \(-y^2\). Whence, the solutions of the truncated equations satisfy

\[
0 \leq c^n_t \leq \frac{\gamma}{\lambda} \| \sigma \|_{\mathbb{H}^2}^2, \quad t \in [0, T].
\]

Also record for future reference that the corresponding martingale parts are given by

\[
M^n_t = \int_0^t Z^n_s dW_s = -c^n_0 + c^n_t + \int_0^t \left( \frac{\gamma}{\lambda} (\sigma^n_s)^2 - (c^n_s)^2 \right) ds.
\]

The family \( (\sup_{t \in [0,T]} |M^n_t|)_{n \in \mathbb{N}} \) is bounded in \( L^2 \). Indeed, using that each \( c^n \) satisfies (3.6), we obtain

\[
\sup_{t \in [0,T]} |M^n_t| \leq 2 \frac{\gamma}{\lambda} \| \sigma \|_{\mathbb{E}^4}^2 + \frac{\gamma}{\lambda} \int_0^T \sigma^n_s^2 ds + T \frac{\gamma^2}{\lambda^2} \| \sigma \|_{\mathbb{E}^4}^4, \quad n \in \mathbb{N}.
\]

Now use the elementary inequality \((a+b+c)^2 \leq 3(a^2+b^2+c^2)\) and the energy inequality for BMO martingales [28, p.26] to obtain the desired bound:

\[
\mathbb{E} \left[ \left( \sup_{t \in [0,T]} |M^n_t| \right)^2 \right] \leq 3 \frac{\gamma^2}{\lambda^2} \| \sigma \|_{\mathbb{E}^4}^2 + 6 \frac{\gamma^2}{\lambda^2} \| \sigma \|_{\mathbb{E}^4}^4 + 3T^2 \frac{\gamma^4}{\lambda^4} \| \sigma \|_{\mathbb{E}^8}^8, \quad n \in \mathbb{N}.
\]
Next, note that since the solutions of (5.1) are bounded uniformly for all $n$, the pair $(c^n, Z^n)$ also solves the BRSDE
\[
c^n_t = \int_t^T \left( \frac{\gamma}{\lambda} (\sigma^n_s)^2 - \left( c^n_s \right)^+ \left( \frac{\gamma}{\lambda} \|\sigma\|_{\text{BMO}}^2 \right) \right) ds - \int_t^T Z^n_s dW_s.
\]
Since the generator of this BRSDE is uniformly Lipschitz continuous, and its value at 0 is bounded, the standard comparison theorem for Lipschitz BSDEs (see, e.g., [44, Theorem 9.4]) shows that the solutions $c^n$ are nondecreasing in $n$.

Therefore, the monotone limit $c = \lim_{n \to \infty} c^n$ is well defined, and satisfies $c_T = 0$ and (3.6) by construction. Now set
\[
M_t := -c_0 + c_t + \int_0^t \left( \frac{\gamma}{\lambda} \sigma^2_s - c^2_s \right) ds, \quad t \in [0, T].
\]
Recalling that both $\sigma^n$ and $c^n$ are nonnegative and nondecreasing in $n$, the monotone convergence theorem gives
\[
\lim_{n \to \infty} \int_0^t \frac{\gamma}{\lambda} (\sigma^n_s)^2 ds = \int_0^t \frac{\gamma}{\lambda} \sigma^2_s ds, \quad \lim_{n \to \infty} \int_0^t (c^n_s)^2 ds = \int_0^t c^2_s ds.
\]
Therefore, $M$ is the pointwise limit of $M^n$. Since the family $(\sup_{t \in [0, T]} |M_t^n|)_{n \in \mathbb{N}}$ is bounded in $\mathbb{L}^2$, $M_t^n$ therefore converges to $M_t$ in $\mathbb{L}^1$ for each $t \in [0, T]$. Hence, it follows that $M$ is a square-integrable martingale. Hence, the martingale representation theorem shows that $M = \int_0^T Z_t dW_t$ for a process $Z \in \mathbb{H}^2$.

In summary, recalling that $c_T = 0$, we have
\[
\int_t^T Z_s dW_s = M_T - M_t = -c_t + \int_t^T \left( \frac{\gamma}{\lambda} \sigma^2_s - c^2_s \right) ds,
\]
that is, $(c, Z) \in \mathcal{S}^{\infty} \times \mathbb{H}^2$ solves the original BSDE. Moreover, by Itô isometry and the conditional version of the argument used in (5.4), it follows that for any $\tau \in \mathcal{T}_{0, T}$,
\[
\mathbb{E} \left[ \int_\tau^T Z_s^2 ds \right] = \mathbb{E} \left[ \left( \int_\tau^T Z_s dW_s \right)^2 \right] \leq \mathbb{E} \left[ \sup_{t \in [\tau, T]} \left| \int_\tau^t Z_s dW_s \right|^2 \right]
\leq 3 \frac{\gamma^2}{\lambda^4} \|\sigma\|_{\text{BMO}}^2 + 6 \frac{\gamma^2}{\lambda^2} \|\sigma\|_{\text{BMO}}^4 + 3T^2 \frac{\gamma^4}{\lambda^6} \|\sigma\|_{\text{BMO}}^8
\]
Thus, $Z$ is also in $\mathbb{H}^2_{\text{BMO}}$. Uniqueness among bounded solutions with $c \geq 0$ follows from the standard comparison theorem for Lipschitz BSDEs [44, Theorem 9.4] by considering the equivalent BRSDE
\[
c_t = \int_t^T \left( \frac{\gamma}{\lambda} \sigma^2_s - \left( c^+_s \right) \left( \frac{\gamma}{\lambda} \|\sigma\|_{\text{BMO}}^2 \right) \right) ds - \int_t^T Z_s dW_s.
\]

Next, we prove Lemma 3.2, which solves the FBSDE (3.3–3.4) describing the optimiser of the quadratic tracking problem (3.2).

**Proof of Lemma 3.2.** First note that since $\sigma \in H^2_{\text{BMO}}$, Lemma 3.1 shows that there is a unique solution $c$ of the BSRDE (3.5) which is nonnegative and bounded. Next, as $c$ is nonnegative, the (conditional version of the) Cauchy–Schwarz inequality, $\sigma \in H^2_{\text{BMO}}$ and Fubini’s theorem give
\[
\mathbb{E} \left[ \int_0^T \xi^2_t dt \right] \leq \frac{\gamma^2}{\lambda^2} \mathbb{E} \left[ \int_0^T \mathbb{E} \left[ \int_t^T \sigma_s dW_s \|\sigma\|_{\text{BMO}}^2 \right] ds \right] \leq \frac{\gamma^2}{\lambda^2} \mathbb{E} \left[ \int_0^T \|\sigma\|_{\text{BMO}}^2 \mathbb{E} \left[ \int_0^T \sigma_s^2 ds \right] ds \right] \leq \frac{\gamma^2}{\lambda^2} \|\sigma\|_{\text{BMO}}^2 \int_0^T \mathbb{E} \left[ \int_t^T \sigma_s^2 ds \right] dt \leq \frac{\gamma^2}{\lambda^2} \|\sigma\|_{\text{BMO}}^2 \int_0^T \sigma_s^2 ds dt \leq \frac{\gamma^2}{\lambda^2} \|\sigma\|_{\text{BMO}}^2 T \mathbb{E} \left[ \int_0^T \sigma_s^2 ds \right].
\]
Together with \( \sigma \xi \in \mathbb{H}^2 \), this shows that \( \xi \) also belongs to \( \mathbb{H}^2 \). Notice now that \( \xi \) can also be directly characterised as the unique solution of the linear BSDE

\[
\tilde{\xi}_t = \int_t^T \left( \frac{\gamma}{\lambda} \sigma_s^2 \xi_s - c_s \tilde{\xi}_s \right) ds - \int_t^T Z_s^\xi dW_s, \quad t \in [0, T].
\]  

(5.5)

Similarly, for \( \gamma = \gamma^n, x = x^n \) and \( \xi = \phi^n, (\varphi, \hat{\varphi}) \) solves the FBSDE \( (3.3) - (3.4) \) characterising the optimisers for \( (3.2) \). Indeed, the forward equation \( (3.3) \) is evidently satisfied by definition. The terminal condition \( \hat{\varphi}_T = 0 \) follows from \( cT = 0 \), the fact that \( \xi \in \mathbb{H}^2 \), and \( \sigma^2 \xi \in \mathbb{H}^1 \). It therefore remains to show that \( \hat{\varphi} \) also has the backward dynamics \( (3.4) \). The ODE \( (3.8) \) for \( \hat{\varphi} \), integration by parts, and the dynamics \( (5.5) \) and \( (3.5) \) of \( \xi \) and \( c \) show that

\[
d\hat{\varphi}_t = d\xi_t - c_t \hat{\varphi}_t dt - \varphi_t dc_t
\]

\[
= \left( -\frac{\gamma}{\lambda} \sigma_t^2 \xi_t + c_t \hat{\xi}_t \right) dt + Z_t^\xi dW_t - c_t \hat{\varphi}_t dt - \varphi_t \left( c_t^2 - \frac{\gamma}{\lambda} \sigma_t^2 \right) dt - \varphi_t Z_t^\xi dW_t.
\]

Again using the ODE \( (3.8) \) to replace \( \tilde{\xi}_t \) with \( \hat{\varphi}_t + c_t \hat{\varphi}_t \), it follows that the trading rate \( (3.8) \) indeed has the required dynamics:

\[
d\hat{\varphi}_t = \frac{\gamma \sigma_t^2}{\lambda} (\varphi_t - \xi_t) dt + (Z_t^\xi - \varphi_t Z_t^\xi) dW_t.
\]

Since \( c \) is nonnegative and \( \tilde{\xi} \in \mathbb{H}^2 \), we have \( \varphi \in \mathcal{S}_2 \). As \( \sigma \in \mathbb{H}^2_{\text{BMO}} \), Lemma A.3 (with \( A_t = \sup_{s \in [0, t]} \varphi_s^2 \) and \( \beta_t = (Z_t^\beta)^2 \)) in turn shows that the local martingale in this decomposition is in fact a square–integrable martingale. The same argument shows that \( \sigma \varphi \in \mathbb{H}^2 \), and \( \hat{\varphi} \) also belongs to \( \mathbb{H}^2 \) by \( (3.8) \) because \( (\xi, \varphi) \in \mathbb{H}^2 \times \mathbb{H}^2 \) and \( c \) is bounded. As a consequence, the admissible trading rate \( \hat{\varphi} \) and the corresponding position \( \varphi \) are optimal for \( (3.2) \). In particular, the solution is unique. Finally, if \( \sigma \xi \in \mathbb{H}^2_{\text{BMO}}, \xi \) is bounded since \( c \) is nonnegative. In view of \( (3.9) \), \( \varphi \) therefore is uniformly bounded as well as \( \xi \) is bounded and \( c \) is nonnegative. The boundedness of \( \hat{\varphi} \) in turn follows from \( (3.8) \) since \( \xi, c \), and \( \varphi \) are bounded.

\[\square\]

### 6 Proofs for Section 4

We first prove Proposition 4.3 on the existence and uniqueness of frictionless Radner equilibria under the following weaker (but more involved) version of Assumption 4.2.

**Assumption 6.1.** (i) \( \beta^1 + \beta^2 \in \mathbb{H}^2 \) and the local martingale \( Z^\beta \) from \( (4.2) \) is a martingale;

(ii) \( \mathbb{E}^\beta [e^{-\gamma s \xi}] < \infty \);

(iii) \( \mathbb{E}^\beta \left[ (Z_t^\beta)^{\frac{\beta^1}{1+\varepsilon}} \right] + \mathbb{E}^\beta \left[ e^{-4s(1+\varepsilon)\gamma \xi} \right] + \mathbb{E}^\beta \left[ e^{4s\gamma(1+\varepsilon)\xi} \right] \leq \infty \) for some \( \varepsilon > 0 \) and \( p > 1 \).

**Remark 6.2.** Notice that if Assumption 4.2 holds, then it is immediate that Assumption 6.1(i) is satisfied, since \( \mathbb{H}^2_{\text{BMO}} \subset \mathbb{H}^2 \) and stochastic exponentials of stochastic integrals (with respect to a Brownian motion) of processes in \( \mathbb{H}^2_{\text{BMO}} \) are uniformly integrable martingales. Moreover, 6.1(ii) and (iii) also hold as \( \xi \) has exponential moments of any order, and since \( Z^\beta \) satisfies the so–called “Muckenhoupt condition” by [28, Theorem 2.4] because \( \beta^1, \beta^2 \in \mathbb{H}^2_{\text{BMO}} \).

**Proof of Proposition 4.3.** The existence of a solution to \( (4.3) \) with the appropriate properties is immediate from direct calculations or [18, Theorem 2.1].\(^{10} \) For uniqueness, notice that for any such solution, the martingale

\[
M_t := \mathbb{E}^\beta_t \left[ e^{-2\gamma s \xi} \right], \quad t \in [0, T],
\]

\(^{10}\)Note that the assumption \( \xi \in L^1 \) is not needed here.
is uniformly integrable. Itô’s formula gives
\[ e^{-2\gamma s S_t} = e^{-2\gamma s \mathbb{E}} - \int_t^T e^{-2\gamma s S_u} \sigma_u \, dW_u^\beta, \quad t \in [0, T]. \]

The stochastic integral on the right-hand side must be a martingale, since the left-hand side is. We can thus take conditional expectations to deduce that
\[ S_t = -\frac{1}{2\gamma s} \log \left( \mathbb{E}_t^\beta \left[ e^{-2\gamma s \mathbb{E}} \right] \right), \quad t \in [0, T]. \]

Uniqueness of \( \sigma \) in turn follows from the martingale representation theorem.

Let us now verify that this price process \( S \) indeed defines a Radner equilibrium. Its drift under \( \mathbb{P} \) is immediately given by Girsanov’s theorem,
\[ \mu_t = \gamma s \sigma_t^2 + \gamma (\beta_1^1 + \beta_1^2) \sigma_t, \quad t \in [0, T]. \]

Since \( \beta_1^1 + \beta_2^2 \in \mathbb{H}_2 \), we just need to verify that \( \sigma \in \mathbb{H}_2 \). To this end, notice that since the martingale \( M \) satisfies, by Doob’s inequality
\[ \mathbb{E}^\beta \left[ \sup_{0 \leq t \leq T} M_t^{2(1+\varepsilon)} \right] \leq \left( \frac{2(1+\varepsilon)}{1+2\varepsilon} \right)^{2(1+\varepsilon)} \mathbb{E}^\beta \left[ e^{-4(1+\varepsilon)\gamma s \mathbb{E}} \right] < \infty, \]
then the martingale representation property implies the existence of a process \( Z \in \mathbb{H}^{2+\varepsilon}(\mathbb{P}^\beta) \), such that
\[ dM_t = Z_t dW_t^\beta, \]
from which we deduce that
\[ \sigma_t = -\frac{1}{2\gamma s M_t} Z_t. \]

We then estimate that
\[
\mathbb{E} \left[ \int_0^T \sigma_t^2 \, dt \right] = \frac{1}{4\gamma^2} \mathbb{E}^\beta \left[ (Z_T^T)^{-1} \int_0^T Z_t^2 \, dt \right] \\
\leq \frac{1}{4\gamma^2} \mathbb{E}^\beta \left[ (Z_T^T)^{-1} \sup_{t \in [0, T]} M_t^{2\varepsilon/(1+\varepsilon)} \right]^{\varepsilon/(1+\varepsilon)} \mathbb{E}^\beta \left[ \left( \int_0^T Z_t^2 \, dt \right)^{1+\varepsilon} \right]^{1/(1+\varepsilon)} \\
\leq \frac{1}{4\gamma^2} \left( \frac{2(1+\varepsilon)}{1+2\varepsilon} \right)^{\varepsilon/(1+\varepsilon)} \mathbb{E}^\beta \left[ \left( Z_T^T \right)^{-\varepsilon/(1+\varepsilon)} \right]^{\varepsilon/(1+\varepsilon)} \mathbb{E}^\beta \left[ e^{-4\varepsilon/(1+\varepsilon)\gamma s \mathbb{E}} \right]^{\varepsilon/(1+\varepsilon)} \|Z\|_{\mathbb{H}^{2+\varepsilon}(\mathbb{P}^\beta)} < \infty.
\]

Since the market also obviously clears, this completes the proof.

\( \square \)

**Proof of Corollary 4.5.** The uniqueness is clear by Proposition 4.3, and the existence of a solution in \( S^\infty \times \mathbb{H}^2_{\text{BMO}} \) is classical, see for instance [9, Corollary 2.1].

\( \square \)

Next, we show that sufficiently integrable solutions of the FBSDE system (4.6–4.8) indeed identify equilibria with transaction costs:

**Proof of Proposition 4.7.** First, Property (ii) and market clearing in Property (iii) from Definition 4.6 hold by assumption. Next, \( \varphi_1 \in \mathbb{H}_2 \) gives \( \varphi_1 \in \mathcal{S}^2 \). Thus, using that \( \sigma \in \mathbb{H}^2_{\text{BMO}} \) it follows from Lemma A.3 that \( \sigma \varphi_1 \in \mathbb{H}_2 \) and in turn also \( \sigma \varphi_2 \in \mathbb{H}_2 \). Now, using that \( \beta_1^1, \beta_2^2, \sigma \varphi_1 \in \mathbb{H}_2 \) and \( \sigma \in \mathbb{H}^2_{\text{BMO}} \subset \mathbb{H}_2 \) gives Property (i).

It remains to show that \( \hat{\varphi}_1, \hat{\varphi}_2 \) are indeed optimal for agents 1 and 2. By Lemma 3.2, we need to check that \((\hat{\varphi}_1^n, \hat{\varphi}_2^n)\) solves the FBSDEs characterisation of agent \( n \)’s individually optimal trading in (3.3–3.4). This follows immediately from the forward–backward dynamics (4.6–4.7) by inserting the definition (4.5) of \( \mu \).

\( \square \)
Finally, we provide a well-posedness result for the FBSDE system characterising the frictional equilibrium price, positions, and trading rates. In order to work with small processes for $\gamma^1 \approx \gamma^2$, we pass from from the frictional equilibrium price $S$ to its deviation $Y = S - \bar{S}$ from its frictional counterpart $\bar{S}$. Subtracting (4.1) from (4.8) and denoting the frictionless equilibrium volatility by $\bar{\sigma}$, we obtain the following BSDE for $Y$ which is coupled to (4.6 - 4.7):

$$
dY_t = \left( \frac{\gamma^1 - \gamma^2}{2}(\bar{\sigma}_t + Z_t^2)\varphi_t + \frac{\gamma^2 s}{2}(Z_t^2)^2 + Z_t^2 \left( \gamma^2 s \bar{\sigma}_t + \frac{\gamma^1 \beta^1_t + \gamma^2 \beta^2_t}{2} \right) - \frac{\gamma^1 - \gamma^2}{2} \bar{\sigma}_t^2 \varphi_t \right) dt + Z_t^2 dW_t, \quad Y_T = 0,
$$

(6.1)

where

$$
\varphi^1 := \frac{\gamma^2 s}{\gamma^1 + \gamma^2} + \frac{\gamma^2 \beta^2 - \gamma^1 \beta^1}{\gamma^1(\gamma^1 + \gamma^2)},
$$

denotes the frictionless equilibrium position of agent 1. Well-posedness of the system (4.6 - 4.7, 6.1) will be a special case of Theorem 6.3 below. The crux of its proof lies in the fact that a naive Picard iteration for all three components of the FBSDE (6.2 - 6.4) does not work. Indeed, because of the quadratic nature of the problem, we want to be able to use BMO-like arguments which require first to ensure that each step of the iteration remains in a ball (for the appropriate norms) whose radius is small enough. This is feasible for (6.4), since we assume that $\gamma^1 - \gamma^2$ is small. However, there is no reason why successive Picard iterations of (6.2) and (6.3) would remain small unless the time horizon is also sufficiently short. The key idea to overcome this issue is to use the specific structure of our problem and to realise that one should only perform the iteration on (6.4), and use our well-posedness result for (4.6 - 4.7), when $Z$ is given, in each step. This is crucial as we then have very precise estimates and stability results given in Section 7, that allow us to obtain the desired contraction in the end.

**Theorem 6.3.** Let $(\gamma^1, \gamma^2, \bar{\gamma}, \bar{\alpha}, \sigma, \nu, \nu') \in (0, \infty)^4 \times (H^2_BMO)^3 \times S^\infty$. Define the measure $P^\alpha \sim P$ by $\frac{dP^\alpha}{dP} := \mathcal{E} \left( \int_0^T \alpha_s dW_s \right)_T$, and assume that for some $p \in (1, 2)$, $\mathbb{E}^{P^\alpha} \left[ \frac{1}{2} \sigma^2 \int_0^T \alpha_s^2 du \right] < \infty$. Let

$$
R \leq \min \left\{ \| \bar{\sigma} \|_{\text{BMO}^{(P^\alpha)}}^{-1}, \frac{1}{4\sqrt{2}\kappa} \right\} =: R_{\text{max}},
$$

and assume that

$$
|\gamma^1 - \gamma^2| < \| \bar{\sigma} \|_{\text{BMO}^{(P^\alpha)}}^{-1} \min \left\{ \frac{R(1/\sqrt{2} - 2\kappa R)}{4\| \bar{\sigma} \|_{\text{BMO}^{(P^\alpha)}}^2 h(\varphi, \bar{\gamma}, \bar{\alpha}, \sigma, \nu, \nu') + 8\nu' \| S^\infty \| \bar{\sigma} \|_{\text{BMO}^{(P^\alpha)}}^2}, \frac{1 - 8\kappa R}{\sigma^2 g(\varphi, \bar{\gamma}, \bar{\alpha}, \nu)} \right\} =: \varepsilon_{\text{max}},
$$

where

$$
h(\varphi, \bar{\gamma}, \bar{\alpha}, \sigma, \nu, \nu') := |x| + 32\bar{\gamma} \left( \left\| \nu \right\|_{\text{BMO}^{(P^\alpha)}} \| \bar{\sigma} \|_{\text{BMO}^{(P^\alpha)}}^2 + \| \nu' \|_{S^\infty} \| \bar{\sigma} \|_{\text{BMO}^{(P^\alpha)}}^2 \right) (\| \alpha \|_{\text{BMO}^{(P)}}^2 + 1)^2,
$$

and $g(\varphi)$ is defined in Theorem 7.5. Then, the system of coupled forward–backward SDEs

$$
d\varphi_t = \varphi_t dt, \quad \varphi_0 = x, \quad (6.2)
dd\varphi_t = \bar{\gamma}(\bar{\sigma}_t + Z_t) \left( \varphi_t - \frac{\nu_t}{\bar{\sigma}_t + Z_t} - \nu'_t \right) dt + Z_t dW_t, \quad \varphi^1_T = 0, \quad (6.3)
ddY_t = \left( \frac{\gamma^1 - \gamma^2}{2}(\bar{\sigma}_t + Z_t)^2 \varphi_t + \frac{\gamma^2 s}{\gamma^1 + \gamma^2} \varphi_t + \kappa Z_t^2 - \alpha_t Z_t - \frac{\gamma^1 - \gamma^2}{2} \bar{\sigma}_t^2 \left( \frac{\nu_t}{\bar{\sigma}_t} + \nu'_t \right) \right) + Z_t dW_t, \quad Y_T = 0, \quad (6.4)
$$

has a unique solution for $(Y, Z)$ lying inside a ball of radius $R$ for the norm on $S^\infty \times H^2_BMO(P^\alpha)$. Moreover $\varphi^1$ and $\varphi^2$ are both uniformly bounded. For

$$
\bar{\gamma} := \frac{\gamma^1 + \gamma^2}{2\lambda}, \quad \nu := \frac{\gamma^2 \beta^2 - \gamma^1 \beta^1}{\gamma^1 + \gamma^2}, \quad \nu' := \frac{\gamma^2 s}{\gamma^1 + \gamma^2}, \quad \kappa := \frac{\gamma^2 s}{\gamma^1 + \gamma^2}, \quad \alpha := -\frac{\gamma^2 s}{\gamma^1 + \gamma^2}, \quad \gamma^1 := \frac{\gamma^1 + \gamma^2}{2},
$$

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the solution to (6.2 – 6.4) provides the unique solution of the FBSDEs (4.6 – 4.8) for which \((S – S, \sigma – \bar{\sigma})\) lies inside a ball of radius \(R\) on \(\mathcal{S}^{\infty} \times \mathbb{H}^2_{\text{BMO}}(\mathbb{P}^\alpha)\) by defining \(S := \bar{S} + Y_t, \sigma := \bar{\sigma} + Z_t\).

**Proof.** We first establish two a priori estimates that will be used throughout the proof. Let \(Z \in \mathbb{H}^2_{\text{BMO}}(\mathbb{P}^\alpha)\) with \(\|Z\|_{\mathbb{H}^2_{\text{BMO}}(\mathbb{P}^\alpha)} \leq \|\bar{\sigma}\|_{\mathbb{H}^2_{\text{BMO}}(\mathbb{P}^\alpha)}\). Then by the elementary inequality \((a + b)^2 \leq 2(a^2 + b^2)\) for \((a, b) \in \mathbb{R}^2\), we have

\[
\|\bar{\sigma} + Z\|^2_{\mathbb{H}^2_{\text{BMO}}(\mathbb{P}^\alpha)} \leq 2 \left( \|\bar{\sigma}\|^2_{\mathbb{H}^2_{\text{BMO}}(\mathbb{P})} + \|Z\|^2_{\mathbb{H}^2_{\text{BMO}}(\mathbb{P})} \right) \leq 4\|\bar{\sigma}\|^2_{\mathbb{H}^2_{\text{BMO}}(\mathbb{P})},
\]

(6.5)

Moreover, Corollary 7.4, Lemma A.1, and (6.5) show that the FBSDE (6.2)–(6.3) (with this fixed \(Z\)) has a bounded solution such that \(\varphi\) satisfies the estimate

\[
\|\varphi\|_{\mathcal{S}^{\infty}} \leq |x| + \bar{\gamma}T \left( \|\nu\|^2_{\mathbb{H}^2_{\text{BMO}}(\mathbb{P})} \|\bar{\sigma} + Z\|^2_{\mathbb{H}^2_{\text{BMO}}(\mathbb{P})} + \|\nu'\|_{\mathcal{S}^{\infty}} \|\bar{\sigma} + Z\|^2_{\mathbb{H}^2_{\text{BMO}}(\mathbb{P})} \right) \left( \|\alpha\|^2_{\mathbb{H}^2_{\text{BMO}}(\mathbb{P})} + 1 \right)^2 
\leq |x| + 32\gamma T \left( \|\nu\|^2_{\mathbb{H}^2_{\text{BMO}}(\mathbb{P})} \|\bar{\sigma} + Z\|^2_{\mathbb{H}^2_{\text{BMO}}(\mathbb{P})} + \|\nu'\|_{\mathcal{S}^{\infty}} \|\bar{\sigma} + Z\|^2_{\mathbb{H}^2_{\text{BMO}}(\mathbb{P})} \right) \left( \|\alpha\|^2_{\mathbb{H}^2_{\text{BMO}}(\mathbb{P})} + 1 \right)^2 
=: h_{\varphi}(x, \bar{\gamma}, \alpha, \bar{\sigma}, \nu, \nu').
\]

Next, let \(Z^n := 0\), and define \((\varphi^n, \psi^n)\) as the solution of the FBSDEs (6.2)–(6.3), corresponding to the volatility \(\bar{\sigma} + Z^n\), and \((Y^n, Z^n)\) as the solution of

\[
dY^n_t = \left( (\bar{\sigma}_t + Z^n_{t})^2 \gamma_1^2 - \gamma_2^2 \frac{\varphi^n_t}{2} + \kappa(Z^n_t)^2 - \gamma_1^2 - \gamma_2^2 \frac{\sigma^n_t}{2} \right) dt + Z^n_t dW^n_t, \quad Y^n_0 = 0.
\]

By the a priori estimate (6.7), we know that \(\varphi^n\) is bounded. This implies that \((Y^n, Z^n)\) is well defined and belongs to \(\mathcal{S}^{\infty} \times \mathbb{H}^2_{\text{BMO}}(\mathbb{P}^\alpha)\).

For \(n \geq 2\), we continue by induction. Given \((Y^{n-1}, Z^{n-1}) \in \mathcal{S}^{\infty} \times \mathbb{H}^2_{\text{BMO}}(\mathbb{P}^\alpha)\), let \(\varphi^n, \psi^n\) be defined as the solution of the FBSDEs (6.2)–(6.3) corresponding to the volatility \(\bar{\sigma} + Z^{n-1}\), and \((Y^n, Z^n) \in \mathcal{S}^{\infty} \times \mathbb{H}^2_{\text{BMO}}(\mathbb{P}^\alpha)\) as the solution of

\[
dY^n_t = \left( (\bar{\sigma}_t + Z^{n-1}_{t})^2 \gamma_1^2 - \gamma_2^2 \frac{\varphi^n_t}{2} + \kappa(Z^{n-1}_t)^2 - \gamma_1^2 - \gamma_2^2 \frac{\bar{\sigma}_t}{2} \right) dt + Z^n_t dW^n_t, \quad Y^n_0 = 0.
\]

We proceed to show that for sufficiently small \(|\gamma_1 - \gamma_2|\), this iteration is a contraction on \(\mathcal{S}^{\infty} \times \mathbb{H}^2_{\text{BMO}}(\mathbb{P}^\alpha)\). By the Banach fixed point theorem, it therefore has a unique fixed point \((Y, Z)\). Together with the pair \((\varphi, \psi)\) that solves the tracking problem corresponding to the volatility \(\bar{\sigma} + Z\), we have in turn constructed the desired solution of (6.2 – 6.4).

To establish that our mapping is indeed a contraction, we first show as in [43] that it maps sufficiently small balls in \(\mathcal{S}^{\infty} \times \mathbb{H}^2_{\text{BMO}}(\mathbb{P}^\alpha)\) into themselves. To this end, suppose that

\[
\|Y^{n-1}\|^2_{\mathcal{S}^{\infty}} + \|Z^{n-1}\|^2_{\mathbb{H}^2_{\text{BMO}}(\mathbb{P}^\alpha)} \leq R^2,
\]

where we recall that \(R < \min\{\|\bar{\sigma}\|_{\mathbb{H}^2_{\text{BMO}}(\mathbb{P}^\alpha)}, \frac{1}{4\sqrt{2\kappa}}\}\). Apply Itô’s formula to \((Y^n)^2\) and use that \(Y^n_T = 0\). Then take conditional \(\mathcal{Q}\)-expectation and use that \(Y^n_t\) is bounded and \(Z^n_t \in \mathbb{H}^2_{\text{BMO}}(\mathbb{P}^\alpha)\). For any stopping time \(\tau\) with values in \([0, T]\), this gives

\[
0 = (Y^n_T)^2 + \mathbb{E}^{\mathbb{P}_T}_{\tau} \left[ \int_{\tau}^{T} (Z^n_s)^2 ds \right] + \mathbb{E}^{\mathbb{P}_T}_{\tau} \left[ \int_{\tau}^{T} 2Y^n_s(\bar{\sigma}_s + Z^{n-1}_s)^2 \left( \gamma_1^2 - \gamma_2^2 \right) \varphi^n_s + 2Y^n_s \kappa(Z^{n-1}_s)^2 ds \right] 
- \mathbb{E}^{\mathbb{P}_T}_{\tau} \left[ \int_{\tau}^{T} 2Y^n_s \frac{\gamma_1^2 - \gamma_2^2}{2} (\bar{\sigma}_s)^2 \left( \frac{\nu}{\bar{\sigma}_s} + \nu' \right) ds \right].
\]

(6.8)
Now use that \( Y^n \in S^\infty \) and \( \|Z^{n-1}\|_{BMO(\mathbb{P})} \leq R \). Together with the a priori estimates (6.5) and (6.7), this yields
\[
(Y_\tau^n)^2 + \mathbb{E}_{\tau}^{\mathbb{P}} \left[ \int_\tau^T (Z_s^n)^2 \, ds \right] \leq |\gamma_1 - \gamma_2|^2 \|Y^n\|_{S^\infty} \|\sigma + Z^{n-1}\|^2_{BMO(\mathbb{P})} \|\varphi^n\|_{S^\infty} + 2\kappa \|Y^n\|_{S^\infty} \|Z^{n-1}\|^2_{BMO(\mathbb{P})} \\
+ |\gamma_1 - \gamma_2| |Y^n|_{S^\infty} (\|\nu\|_{BMO(Q)} \|\sigma\|_{BMO(\mathbb{P})} + \|\nu\|_{S^\infty} \|\sigma\|_{BMO(\mathbb{P})}^2) \leq \|Y^n\|_{S^\infty} (|\gamma_1 - \gamma_2|^2 \left( 4\|\sigma\|^2_{BMO(\mathbb{P})} \|\varphi^n\|_{S^\infty} + \|\nu\|_{BMO(\mathbb{P})} \|\sigma\|_{BMO(\mathbb{P})}^2 + \|\nu\|_{S^\infty} \|\sigma\|_{BMO(\mathbb{P})}^2 + 2\kappa R^2 \right) \\
=: \|Y^n\|_{S^\infty} (|\gamma_1 - \gamma_2|^2 |h_R(x, \tilde{\gamma}, \alpha, \tilde{\sigma}, \nu, \nu') + 2\kappa R^2) ,
\]
where
\[
h_R(x, \tilde{\gamma}, \alpha, \tilde{\sigma}, \nu, \nu') := 4\|\overline{\sigma}\|^2_{BMO(\mathbb{P})} h_\varphi(x, \tilde{\gamma}, \alpha, \tilde{\sigma}, \nu, \nu') + \|\nu\|_{BMO(\mathbb{P})} \|\overline{\sigma}\|^2_{BMO(\mathbb{P})} + \|\nu\|_{S^\infty} \|\overline{\sigma}\|^2_{BMO(\mathbb{P})} .
\]
Taking the supremum over all \( \tau \) (for \( Y^n \)) and rearranging yields
\[
\|Y^n\|_{S^\infty} \leq |\gamma_1 - \gamma_2| \|h_R(x, \tilde{\gamma}, \alpha, \tilde{\sigma}, \nu, \nu') + 2\kappa R^2|.
\]
Now taking the supremum over all \( \tau \) in (6.9) (for \( Z^n \)) and using (6.10), we obtain
\[
\|Z^n\|^2_{BMO(\mathbb{P})} \leq (|\gamma_1 - \gamma_2|^2 \|h_R(x, \tilde{\gamma}, \alpha, \tilde{\sigma}, \nu, \nu') + 2\kappa R^2) \leq 2 R^2 .
\]
Using our bounds on \( |\gamma_1 - \gamma_2| \), and the fact that \( R \leq \frac{1}{4\sqrt{2} \kappa} \), we deduce that
\[
\|Y^n\|^2_{S^\infty} + \|Z^n\|^2_{BMO(\mathbb{P})} \leq 2 \left( |\gamma_1 - \gamma_2|^2 \|h_R(x, \tilde{\gamma}, \alpha, \tilde{\sigma}, \nu, \nu') + 2\kappa R^2 \right)^2 \leq R^2 .
\]
We now show that our iteration is a contraction on the ball \( B_R \times \mathbb{H}^2_{BMO}(\mathbb{P}) \). To this end, consider \((y, z), ((y', z')) \in B_R^2 \times \mathbb{H}^2_{BMO}(\mathbb{P}) \), and write \((Y, Z), (Y', Z')\) for their images produced by our iteration. Also denote by \((\varphi, \tilde{\varphi}), (\varphi', \tilde{\varphi}')\) the corresponding optimal tracking strategies (corresponding to volatilities \( \sigma + z \) and \( \tilde{\sigma} + z' \), respectively). To verify that our iteration is indeed a contraction, we have to show that for some \( \eta \in (0,1) \),
\[
\|Y - Y'\|^2_{S^\infty} + \|Z - Z'\|^2_{BMO(\mathbb{P})} \leq \eta \left( \|y - y'\|^2_{S^\infty} + \|z - z'\|^2_{BMO(\mathbb{P})} \right) .
\]
To ease notation, set
\[
\delta y := y - y', \quad \delta z := z - z', \quad \delta Y := Y - Y', \quad \delta Z := Z - Z'.
\]
Applying Itô’s formula on \([\tau, T]\) for any \([0, T]\)-valued stopping time \( \tau \), inserting the dynamics of \( Y \) and \( Y' \), taking \( \mathbb{P} \)-conditional expectations, and using the identity \( ab - cd = a(b - d) + (a - c)d \) for \((a, b, c, d) \in \mathbb{R}^4 \), we obtain
\[
\delta Y^2 + \mathbb{E}_{\tau}^{\mathbb{P}} \left[ \int_\tau^T \delta Z_t^2 \, dt \right] = \mathbb{E}_{\tau}^{\mathbb{P}} \left[ (\gamma_2 - \gamma_1) \int_\tau^T \delta Y_t \left( (\tilde{\sigma}_t + z_t)^2 \varphi_t - (\tilde{\sigma}_t + z'_t)^2 \varphi'_t \right) dt - 2\kappa \int_\tau^T \delta Y_t ((z_t)^2 - (z'_t)^2) dt \right] \leq \|\delta Y\|_{S^\infty} |\gamma_1 - \gamma_2| \left( \mathbb{E}_{\tau}^{\mathbb{P}} \left[ \int_\tau^T (\tilde{\sigma}_t + z_t)^2 |\varphi_t - \varphi'_t| dt + \int_\tau^T |2\tilde{\sigma}_t + z_t + z'_t| |\delta z_t| |\varphi'_t| dt \right] \right) + 2\kappa \|\delta Y\|_{S^\infty} \mathbb{E}_{\tau}^{\mathbb{P}} \left[ \int_\tau^T |z_t + z'_t| |\delta z_t| dt \right] .
\]
To estimate the conditional expectation in the first term on the right–hand side of (6.12), define the process
\[
A_t := \sup_{u \leq t} |\varphi_u - \varphi'_u|, \quad t \in [0, T].
\]
Lemma A.3, (6.5), Jensen’s inequality, and Theorem 7.5 in turn yield

\[ \mathbb{E}_{\tau}^{\mathbb{P}^\alpha} \left[ \int_{\tau}^{T} (\bar{\sigma}_t + z_t)^2 |\varphi_t - \varphi'|^2 \, dt \right] \leq \mathbb{E}_{\tau}^{\mathbb{P}^\alpha} \left[ \int_{\tau}^{T} (\bar{\sigma}_t + z_t)^2 A_t \, dt \right] \leq 4 \| \bar{\sigma} \|_{BMO}^2 \mathbb{E}_{\tau}^{\mathbb{P}^\alpha} \left[ \sup_{u \in [0,T]} |\varphi_u - \varphi_u'| \right] \]

\[ \leq 4 \| \bar{\sigma} \|_{BMO}^2 \mathbb{P}^\alpha g_{\varphi}(x, \bar{\gamma}, \alpha, \sqrt{2}\bar{\sigma}, \sqrt{2}\bar{\sigma}, \nu, \nu') \frac{1}{2} \| \delta z \|_{BMO}^2. \]  

To estimate the conditional expectation in the second term on the right-hand side of (6.12), we use that \( \varphi' \in \mathcal{S}^\infty \), the conditional version of the Cauchy–Schwarz inequality and the elementary inequality \((a + b + c)^2 \leq 2a^2 + 4b^2 + 4c^2\). Together with the fact that both \( \| z \|_{BMO}^2 \) and \( \| z' \|_{BMO}^2 \) are smaller than \( \| \bar{\sigma} \|_{BMO}^2 \) and the a priori estimate (6.7), this yields

\[ \mathbb{E}_{\tau}^{\mathbb{P}^\alpha} \left[ \int_{\tau}^{T} |z_t - z'_t||\delta z_t||\varphi'_t| \, dt \right] \leq \| \varphi' \|_{\mathcal{S}^\infty} \mathbb{E}_{\tau}^{\mathbb{P}^\alpha} \left[ \int_{\tau}^{T} (2\bar{\sigma}_t + z_t + z'_t)^2 \, dt \right] \]

\[ \leq 4\mathbb{E}_{\tau}^{\mathbb{P}^\alpha} \left[ \int_{\tau}^{T} \delta z_t^2 \, dt \right] ^{\frac{1}{2}} \leq 4h_{\varphi}(x, \bar{\gamma}, \alpha, \bar{\sigma}, \nu, \nu') \| \bar{\sigma} \|_{BMO}^2 \| \delta z \|_{BMO}^2. \]  

To estimate the conditional expectation in the third term on the right-hand side of (6.12), we argue in a similar fashion and obtain

\[ \mathbb{E}_{\tau}^{\mathbb{P}^\alpha} \left[ \int_{\tau}^{T} |z_t + z'_t||\delta z_t| \, dt \right] \leq 2R \| \delta z \|_{BMO}^2. \]  

Now, plugging (6.13) – (6.15) into (6.12), taking the supremum over all \( \tau \) (both for \( Y \) and \( Z \)), then taking conditional \( \mathbb{P}^\alpha \)-expectations, applying Lemma B.1 and Theorem 7.5 (together with (6.5)), and using the elementary inequality \( 2ab \leq \frac{1}{\varepsilon}a^2 + \varepsilon b^2 \) yields

\[ \| \delta Y \|_{\mathcal{S}^\infty}^2 + \| \delta Z \|_{BMO}^2 \leq 8|\gamma^1 - \gamma^2|\| \delta Y \|_{\mathcal{S}^\infty} \| \bar{\sigma} \|_{BMO}^2 g_{\varphi}(x, \bar{\gamma}, \alpha, \sqrt{2}\bar{\sigma}, \sqrt{2}\bar{\sigma}, \nu, \nu') \frac{1}{2} \| \delta z \|_{BMO}^2 + 8|\gamma^1 - \gamma^2|\| \delta Y \|_{\mathcal{S}^\infty} \| \bar{\sigma} \|_{BMO}^2 k_{\varphi}(x, \bar{\gamma}, \alpha, \bar{\sigma}, \nu, \nu') \| \delta z \|_{BMO}^2 + 8\kappa \| \delta Y \|_{\mathcal{S}^\infty} R \| \delta z \|_{BMO}^2 \]

\[ \leq \frac{1}{\varepsilon} \| \delta Y \|_{\mathcal{S}^\infty}^2 + \varepsilon \eta^2 \| \delta z \|_{BMO}^2, \]

where

\[ \eta := 4|\gamma^1 - \gamma^2|\| \bar{\sigma} \|_{BMO}^2 \left( g_{\varphi}(x, \bar{\gamma}, \alpha, \sqrt{2}\bar{\sigma}, \sqrt{2}\bar{\sigma}, \nu, \nu') \frac{1}{2} + h_{\varphi}(x, \bar{\gamma}, \alpha, \bar{\sigma}, \nu, \nu') \right) + 4\kappa R. \]

We deduce that for any \( \varepsilon > 1 \),

\[ \| \delta Y \|_{\mathcal{S}^\infty}^2 + \| \delta Z \|_{BMO}^2 \leq \| \delta Y \|_{\mathcal{S}^\infty}^2 + \frac{\varepsilon}{\varepsilon - 1} \| \delta Z \|_{BMO}^2 \leq \frac{\varepsilon^2}{\varepsilon - 1} \eta^2 \| \delta z \|_{BMO}^2, \]

We choose \( \varepsilon = 2 \) and deduce the desired result, since by our assumptions

\[ \frac{\varepsilon^2}{\varepsilon - 1} \eta^2 = 4\eta^2 < 1. \]

For the last part of the result, observe that these specific parameter choices satisfy all the requirements in Theorem 6.3 in view of Assumptions 4.2 and 4.8. This gives us a unique solution to the associated FBSDE system (4.6 – 4.7, 6.1). Any solution of (4.6 – 4.7, 6.1) in turn provides a solution to (4.6 – 4.8) by defining \( S := S_t + Y_t \) and \( \sigma := \bar{\sigma} + Z \). The converse is obviously true for solutions as in Theorem 4.9.

We now prove Proposition 4.10, which characterises equilibria with transaction costs via coupled systems of Riccati ODEs in a particular model with linear state dynamics and terminal condition.
Proof of Proposition 4.10. First notice that the functions $A(t)$, $D(t)$ satisfy the following Riccati equations:

\[ A'(t) = -\gamma s a^2 + \frac{\gamma^2}{2} (a + B(t))^2 - C(t) D(t), \quad A(T) = 0, \]

\[ D'(t) = -\frac{\gamma^2 s}{2\lambda} (a + B(t))^2 - F(t) D(t), \quad D(T) = 0. \]

Together with the Riccati ODEs for the functions $B(t), C(t), E(t), F(t)$, it follows that the functions

\[ f(t, x, y) = A(t) + B(t)x + C(t)y, \quad g(t, x, y) = D(t) + E(t)x + F(t)y, \]

solve the following semilinear PDEs (here, the arguments $(t, x, y)$ are omitted to ease notation):

\[ f_t + \frac{1}{2} f_{xx} + f_y g = -\gamma s a^2 + \frac{\gamma^2}{2} (a + B)^2 + \frac{\gamma^1 - \gamma^2}{2} \beta (a + B)x + \frac{\gamma^1 - \gamma^2}{2} (a + B)^2 y \]

\[ = \frac{\gamma^1 - \gamma^2}{2} (a + f_x)^2 y + \frac{\gamma^2}{2} f_x^2 + f_x \left( \gamma^2 a + \frac{\gamma^1 - \gamma^2}{2} \beta x \right) - \frac{\gamma^1 - \gamma^2}{2} a^2 \left( \frac{\gamma^2 s}{\gamma^1 + \gamma^2} - \frac{\beta}{a} x \right), \]

\[ g_t + \frac{1}{2} g_{xx} + g_y g = \frac{\gamma^1 + \gamma^2}{2\lambda} (a + f_x) \beta x - \frac{\gamma^2 s}{2\lambda} (a + f_x)^2 + \frac{\gamma^1 + \gamma^2}{2\lambda} (a + f_x)^2 y, \]

on $[0, T) \times \mathbb{R}^2$, with terminal conditions $f(T, x, y) = g(T, x, y) = 0$. By definition of $\varphi^1$,

\[ \varphi^1_t = g(t, W_t, \varphi^1_t). \]

Now set

\[ Y_t = f(t, W_t, \varphi^1_t), \quad Z_t = f_x(t, W_t, \varphi^1_t) = B(t). \]

Then, Itô’s formula, the PDEs for $f(t, x, y), g(t, x, y)$, and the definition of $Z$ show that $\varphi^1, Y, Z$ satisfy the BSDEs

\[ d\varphi^1_t = \frac{\gamma^1 + \gamma^2}{2\lambda} \left( \beta W_t (a + Z_t) - \frac{\gamma^2 s}{\gamma^1 + \gamma^2} (a + Z_t)^2 + (a + Z)^2 \varphi^1_t \right) dt + E(t)dW_t, \]

\[ dY_t = \left( \frac{\gamma^1 - \gamma^2}{2} (a + Z)^2 \varphi^1_t + \frac{\gamma^2 s}{2} Z_t^2 + Z_t \left( \gamma^2 sa + \frac{\gamma^1 - \gamma^2}{2} \beta W_t \right) - \frac{\gamma^1 - \gamma^2}{2} a^2 \left( \frac{\gamma^2 s}{\gamma^1 + \gamma^2} - \frac{\beta}{a} W_t \right) \right) dt + Z_t dW_t, \]

with terminal conditions $\varphi^1_T = Y_T = 0$. Together with the forward equation $d\varphi^1_t = \dot{\varphi}^1_t dt$, as well as the BSDE for the frictionless equilibrium price $\bar{S}$ from Proposition 4.3, it follows that $S = \bar{S} + Y, \sigma = a + Z = \bar{\sigma} + Z, \dot{\varphi}^1, E$, and $\varphi^1$ indeed solve the forward-backward equations (4.6 – 4.8). Since the frictionless equilibrium volatility is constant here, $\bar{\sigma} = a$ and $Z_t = B(t)$ is deterministic, we evidently have $\sigma \in \mathbb{H}^2_{\text{BMO}}$. Since the Brownian motion $W$ has finite moments and zero autocorrelation function, one also readily verifies that $\varphi^1 \in \mathbb{H}^2$. The assertion in turn follows from Proposition 4.7.

We now turn to the proof of Theorem 4.11, which guarantees existence of the Riccati system from Proposition 4.9 for sufficiently similar risk aversion parameters. The argument is very close in spirit to that of Theorem 6.3. Indeed, we also obtain well-posedness of the system by a Picard iteration scheme which is devised so that the successive iterations remain in a sufficiently small ball. And in order to achieve this, a naive direct iteration of the four equations does not work unless the time horizon is sufficiently short. Instead, we have to start by studying separately the system satisfied by $C, E, F$ for fixed $B$, exactly as we did for (6.2 – 6.3), when $Z$ is fixed, in the proof of Theorem 6.3. After developing the necessary stability estimates, we can then proceed to the iteration for $B$ and obtain the desired result. This shows that the approach underlying Theorem 6.3 is not crucially tied to the stringent integrability assumptions imposed there to deal with a general setting, but can also be adapted to other specific settings on a case–by–case basis.
Proof of Theorem 4.11. To ease notation, set
\[ \dot{\gamma} := \frac{\gamma^1 + \gamma^2}{2}, \quad \varepsilon := \gamma^1 - \gamma^2, \]
as well as
\[ \tilde{B}(t) := B(t) + a, \quad t \in [0, T]. \]

Step 1: Dealing with \((C, E, F)\). We start by giving ourselves some bounded map \(\tilde{B} : [0, T] \rightarrow \mathbb{R}\) and analyse the following coupled system of ODEs on \([0, T]\):
\[
\begin{align*}
C\tilde{B}(t) &= -\int_t^T \left( \frac{\varepsilon}{2} \tilde{B}(s)^2 - F\tilde{B}(s)C\tilde{B}(s) \right) ds, \\
E\tilde{B}(t) &= -\int_t^T \left( \frac{\beta}{2} \tilde{B}(s) - F\tilde{B}(s)E\tilde{B}(s) \right) ds, \\
F\tilde{B}(t) &= -\int_t^T \left( \frac{\gamma}{2} \tilde{B}(s)^2 - (F\tilde{B})^2(s) \right) ds.
\end{align*}
\]  
(6.17)

As \(\tilde{B}\) is bounded, the equation for \(F\tilde{B}\) has a unique solution. Using that \(0 \leq \frac{\gamma}{2} |\tilde{B}(s)|^2 \leq \frac{\gamma}{2} (|\tilde{B}|_\infty)^2\), the comparison theorem for ODEs gives the estimate
\[
- \sqrt{\frac{\gamma}{\lambda}} (|\tilde{B}|_\infty) \leq - \sqrt{\frac{\gamma}{\lambda}} (|\tilde{B}|_\infty) \tanh\left( \sqrt{\frac{\gamma}{\lambda}} (|\tilde{B}|_\infty)(T - t) \right) \leq F\tilde{B}(t) \leq 0, \quad t \in [0, T].
\]  
(6.18)

The ODEs for \(E\tilde{B}\) and \(C\tilde{B}\) are linear and have the unique solutions
\[
E\tilde{B}(t) = -\frac{\gamma}{\lambda} \int_t^T \tilde{B}(s) e^{\int_s^t F\tilde{B}(r) dr} ds, \quad C\tilde{B}(t) = -\frac{\varepsilon}{2} \int_t^T \tilde{B}(s)^2 e^{\int_s^t F\tilde{B}(r) dr} ds, \quad t \in [0, T].
\]

In particular, non–positivity of \(F\) implies
\[
|E\tilde{B}(t)| \leq \frac{\gamma}{\lambda} |\tilde{B}|_\infty (T - t), \quad |C\tilde{B}(t)| \leq \frac{\varepsilon}{2} |\tilde{B}|^2_\infty (T - t), \quad \text{for all } t \in [0, T].
\]  
(6.19)

We will also need some stability results for these solutions with respect to variations of \(\tilde{B}\). Fix thus two bounded functions \(\tilde{B}\) and \(\tilde{B}'\). Using that \(F\tilde{B} - F\tilde{B}'\) satisfies the ODE
\[
\left( F\tilde{B} - F\tilde{B}' \right)(t) = -\int_t^T \left( \frac{\gamma}{\lambda} (\tilde{B}(s) + \tilde{B}'(s))(\tilde{B}(s) - \tilde{B}'(s)) - (F\tilde{B}(s) + F\tilde{B}'(s))(F\tilde{B} - F\tilde{B}')(s) \right) ds,
\]
we obtain
\[
F\tilde{B}(t) - F\tilde{B}'(t) = -\frac{\gamma}{\lambda} \int_t^T e^{\int_s^t (F\tilde{B}(r) + F\tilde{B}'(r)) dr} (\tilde{B}(s) + \tilde{B}'(s))(\tilde{B}(s) - \tilde{B}'(s)) ds.
\]

Non–positivity of \(F\tilde{B}\) and \(F\tilde{B}'\) gives
\[
|F\tilde{B}(t) - F\tilde{B}'(t)| \leq \frac{\gamma}{\lambda} (|\tilde{B}|_\infty + |\tilde{B}'|_\infty) \||\tilde{B} - \tilde{B}'|\|_\infty (T - t), \quad \text{for all } t \in [0, T].
\]

Using that the \(x \mapsto e^x\) is \(1\)-Lipschitz continuous on \((-\infty, 0]\), this implies
\[
\left| e^{\int_t^s F\tilde{B}(r) dr} - e^{\int_t^s F\tilde{B}'(r) dr} \right| \leq \frac{\gamma}{\lambda} (|\tilde{B}|_\infty + |\tilde{B}'|_\infty) \||\tilde{B} - \tilde{B}'|\|_\infty (T - t)^2, \quad \text{for } 0 \leq t \leq s \leq T.
\]  
(6.20)
Taking into account the explicit expressions for $E^\tilde{B}$ and $C^\tilde{B}$, we also deduce that
\[
E^\tilde{B}(t) - E^{\tilde{B}'}(t) = -\frac{\tilde{\gamma} \beta}{\lambda} \int_t^T \left( e^{\int_r^s F^\tilde{B}(r) dr} (\tilde{B}(s) - \tilde{B}'(s)) + \tilde{B}'(s) (e^{\int_r^s F^\tilde{B}(r) dr} - e^{\int_r^s F^{\tilde{B}'}(r) dr}) \right) ds,
\]
\[
C^\tilde{B}(t) - C^{\tilde{B}'}(t) = -\frac{\varepsilon}{2} \int_t^T \left( e^{\int_r^s F^\tilde{B}(r) dr} (\tilde{B}(s) + \tilde{B}'(s)) (\tilde{B}(s) - \tilde{B}'(s)) + \tilde{B}'(s)^2 (e^{\int_r^s F^\tilde{B}(r) dr} - e^{\int_r^s F^{\tilde{B}'}(r) dr}) \right) ds.
\]
Together with (6.20), this yields
\[
\|E^\tilde{B} - E^{\tilde{B}'}\|_\infty \leq \frac{\tilde{\gamma} \beta T}{\lambda} \left( 1 + \frac{\tilde{\gamma} T^2}{\lambda} \|\tilde{B}'\|_\infty (\|\tilde{B}\|_\infty + \|\tilde{B}'\|_\infty) \right) \|\tilde{B} - \tilde{B}'\|_\infty, \tag{6.21}
\]
as well as
\[
\|C^\tilde{B} - C^{\tilde{B}'}\|_\infty \leq \frac{\varepsilon T}{2} \left( 1 + \frac{\tilde{\gamma} T^2}{\lambda} \|\tilde{B}'\|_\infty^2 (\|\tilde{B}\|_\infty + \|\tilde{B}'\|_\infty) \right) \|\tilde{B} - \tilde{B}'\|_\infty. \tag{6.22}
\]

**Step 2: A priori estimate for $\|\tilde{B}\|_\infty$.** Now, fix some $R > a$, define $\tilde{B}^0 = a$ and, for a fixed integer $n \geq 1$, consider a continuous function $\tilde{B}^{n-1}$ with $\|\tilde{B}^{n-1}\|_\infty \leq R$. Let $(C^n, E^n, F^n)$ be the unique solution of the system (6.17) with $\tilde{B} := \tilde{B}^{n-1}$. We then define $\tilde{B}^n$ as the unique solution of the following (linear) ODE (well–posedness is clear since $\tilde{B}^{n-1}$, $C^n$, $E^n$ and $F^n$ are all uniformly bounded):
\[
\tilde{B}^n(t) = a - \int_t^T (\varepsilon \beta \tilde{B}^{n-1}(s) - E^n(s)C^n(s)) ds, \quad t \in [0,T]. \tag{6.23}
\]
Using the estimates on $E^n$ and $C^n$ from (6.19), we obtain
\[
\|\tilde{B}^n\|_\infty \leq a + \varepsilon \beta T \|\tilde{B}^{n-1}\|_\infty + \frac{\varepsilon \tilde{\gamma} \beta}{2\lambda} T^3 \|\tilde{B}^{n-1}\|_\infty^3.
\]
Now, choose $\varepsilon$ small enough so that
\[
a + \varepsilon T R + \frac{\varepsilon \tilde{\gamma} \beta}{2\lambda} T^3 R^3 \leq R.
\]
Then we have
\[
\|\tilde{B}^n\|_\infty \leq R.
\]

**Step 3: Picard iteration for $\tilde{B}$.** Finally, using the fact that
\[
\tilde{B}^n(t) - \tilde{B}^m(t) = -\int_t^T \varepsilon \beta (\tilde{B}^{n-1}(s) - \tilde{B}^{m-1}(s)) ds
\]
\[
- \int_t^T C^n(s)(E^n(s) - E^m(s)) + E^m(s)(C^n(s) - C^m(s)) ds,
\]
it follows from (6.19), (6.21), and (6.22) that we indeed have a contraction provided that $\varepsilon$ is small enough. \[\square\]

### 7 Stability results

We now derive a number of stability results. These are the key ingredients for the convergence of the Picard iteration that allows us to prove existence for the FBSDE (6.2 – 6.4) in Theorem 6.3.

We first consider the process $c$ from Lemma 3.1. Since it is positive, it also solves the counterpart of the BSDE (3.5) where the quadratic generator $f_t(y) = \frac{\sigma^2_t}{2} - y^2$ is replaced by the monotone generator $g_t(y) = \frac{\sigma^2_t}{2} - (y^+)^2$. The same argument can be applied to the $y$–derivative of the generator. Stability of the solution
in turn follows from results for monotone BSDEs. To apply these estimates in the body of the paper, we develop them under an equivalent probability measure \( \mathbb{P}^\alpha \sim \mathbb{P} \) with density process

\[
Z^\alpha := \mathcal{E} \left( \int_0^T \alpha_t dW_t \right), \quad \text{for } \alpha \in \mathbb{H}^2_{\text{BMO}}. \tag{7.1}
\]

Under \( \mathbb{P}^\alpha \), the BSDE for \( c \) can be rewritten as

\[
c_t = \int_t^T \left( \frac{\gamma}{\chi} \sigma_s^2 - c_s^2 - \alpha_s Z_s \right) ds - \int_t^T Z_s dW_s^\alpha,
\]

for a \( \mathbb{P}^\alpha \)–Brownian motion \( W^\alpha \). Writing \( \mathbb{E}^\alpha \left[ \cdot \right] \) for the expectation under \( \mathbb{P}^\alpha \) to ease notation, we in turn have the following stability estimate.

**Lemma 7.1.** Fix \((\gamma, \lambda, p, \alpha) \in (0, \infty)^2 \times (1, 2) \times \mathbb{H}^2_{\text{BMO}}(\mathbb{P})\), with corresponding measure \( \mathbb{P}^\alpha \) given by (7.1), and suppose that \( \mathbb{E}^\alpha \left[ e^{\frac{2p}{p-1} \int_0^T \alpha_s^2 du} \right] < \infty \). For \((\sigma, \tilde{\sigma}) \in \mathbb{H}^2_{\text{BMO}}(\mathbb{P}) \times \mathbb{H}^2_{\text{BMO}}(\mathbb{P})\), denote by \( c^\alpha \) and \( \tilde{c}^\alpha \) the solutions of the BRSDE (3.5). Then

\[
\mathbb{E}^\alpha \left[ \sup_{t \in [0, T]} |c_t^\alpha - \tilde{c}_t^\alpha|^2 \right] \leq g_c(\gamma/\lambda, \alpha, \sigma, \tilde{\sigma}) \|\sigma - \tilde{\sigma}\|^2_{\mathbb{H}^2_{\text{BMO}}(\mathbb{P}^\alpha)},
\]

where

\[
g_c(\gamma/\lambda, \alpha, \sigma, \tilde{\sigma}) := \left( \frac{p}{p-1} \right)^2 g(\gamma/\lambda, \alpha, \sigma, \tilde{\sigma})^2 \mathbb{E}^\alpha \left[ e^{\frac{2p}{p-1} \int_0^T \alpha_s^2 du} \right]^{\frac{2p}{p-1}},
\]

with

\[
g(\gamma/\lambda, \alpha, \sigma, \tilde{\sigma}) := \frac{\gamma}{\lambda} \sigma^4/2 (\|\sigma\|^2_{\mathbb{H}^2_{\text{BMO}}(\mathbb{P}^\alpha)} + \|\tilde{\sigma}\|^2_{\mathbb{H}^2_{\text{BMO}}(\mathbb{P}^\alpha)})^{\frac{1}{2}}.
\]

**Proof.** For \( t \in [0, T] \), apply Itô’s formula to \( e^{\int_0^t \alpha_s^2 du} (c_s^\alpha - \tilde{c}_s^\alpha)^2 \) and use that \( e^{\int_0^T \alpha_s^2 du} (c_T^\alpha - \tilde{c}_T^\alpha)^2 = 0 \). Together with the BRSE dynamics (3.5), this gives

\[
e^{\int_0^t \alpha_s^2 du} (c_s^\alpha - \tilde{c}_s^\alpha)^2 = \int_t^T e^{\int_u^t \alpha_s^2 du} \left( 2 (c_s^\sigma - \tilde{c}_s^\tilde{\sigma}) \left( \frac{\gamma}{\chi} (\sigma_s^2 - \tilde{\sigma}_s^2) - (c_s^\sigma)^2 + (\tilde{c}_s^\tilde{\sigma})^2 - \alpha_s (Z_s^\sigma - Z_s^\tilde{\sigma}) \right) - \alpha_s^2 (c_s^\sigma - \tilde{c}_s^\tilde{\sigma})^2 \right) ds
\]

\[\quad - 2 \int_t^T e^{\int_u^t \alpha_s^2 du} (c_s^\sigma - \tilde{c}_s^\tilde{\sigma}) (Z_s^\sigma - Z_s^\tilde{\sigma}) dW_s^\alpha - \int_t^T e^{\int_u^t \alpha_s^2 du} (Z_s^\sigma - Z_s^\tilde{\sigma})^2 ds. \tag{7.2}\]

Note that \( \int_0^T e^{\int_u^t \alpha_s^2 du} (c_s^\sigma - \tilde{c}_s^\tilde{\sigma}) (Z_s^\sigma - Z_s^\tilde{\sigma}) dW_s^\alpha \) is an \( \mathbb{H}^1(\mathbb{P}^\alpha) \)–martingale. Indeed, using that \( c^\sigma, \tilde{c}^\tilde{\sigma} \in S^\infty \) and \( Z^\sigma, Z^\tilde{\sigma} \in \mathbb{H}^2(\mathbb{P}^\alpha) \) together with the elementary inequality \( ab \leq a^2/2 + b^2/2 \) and the fact that \( \mathbb{E}^\alpha \left[ e^{\int_0^T \alpha_s^2 du} \right] < \infty \) (since \( p > 1 \) implies that \( 2p/(2-p) > 2 \)), we obtain

\[
\mathbb{E}^\alpha \left[ \left( \int_0^T e^{\int_u^t \alpha_s^2 du} (c_s^\sigma - \tilde{c}_s^\tilde{\sigma}) (Z_s^\sigma - Z_s^\tilde{\sigma})^2 ds \right)^{\frac{1}{2}} \right] \leq \left\| c^\sigma - \tilde{c}^\tilde{\sigma} \right\|_{S^\infty} \mathbb{E}^\alpha \left[ e^{\int_0^T \alpha_s^2 du} \left( \int_0^T (Z_s^\sigma - Z_s^\tilde{\sigma})^2 ds \right)^{\frac{1}{2}} \right]
\]

\[\leq \frac{1}{2} \left\| c^\sigma - \tilde{c}^\tilde{\sigma} \right\|_{S^\infty} \mathbb{E}^\alpha \left[ e^{\int_0^T \alpha_s^2 du} + \int_0^T (Z_s^\sigma - Z_s^\tilde{\sigma})^2 ds \right] < \infty.
\]

Now, take conditional \( \mathbb{P}^\alpha \)-expectations on both sides of (7.2), use that \( c^\sigma \), and \( \tilde{c}^\tilde{\sigma} \) are nonnegative to apply the elementary inequality \((x-y)(-x^2+y^2) = -(x-y)^2(x+y) \leq 0 \) for \( x, y \geq 0 \), and take into account the elementary inequality \(-2ab \leq a^2 + b^2 \). This yields the estimate

\[
e^{\int_0^t \alpha_s^2 du} (c_s^\sigma - \tilde{c}_s^\tilde{\sigma})^2 \leq \mathbb{E}^\alpha \left[ \int_t^T e^{\int_u^t \alpha_s^2 du} \frac{\gamma}{\chi} (c_s^\sigma - \tilde{c}_s^\tilde{\sigma}) (c_s^\sigma - \tilde{c}_s^\tilde{\sigma}) ds \right]. \tag{7.3}\]
Define the non-decreasing process

\[ A_t = e^{f_t^j \alpha_u^2 du} \sup_{u \in [0,t]} |c_u^\sigma - c_u^{\tilde{\sigma}}|, \quad 0 \leq t \leq T. \]

Then by Lemma A.3, we obtain

\[
\mathbb{E}_t^\alpha \left[ \int_t^T e^{f_t^j \alpha_u^2 du} \frac{2}{\lambda} (c_u^\sigma - c_u^{\tilde{\sigma}}) (\sigma_u^2 - \tilde{\sigma}_u^2) ds \right] \leq 2 \frac{\gamma}{\lambda} \mathbb{E}_t^\alpha \left[ \int_t^T A_s |\sigma_u^2 - \tilde{\sigma}_u^2| ds \right] \leq 2 \frac{\gamma}{\lambda} \left( \mathbb{E}_t^\alpha \left[ A_t \int_t^T |\sigma_u^2 - \tilde{\sigma}_u^2| ds + \int_t^T \mathbb{E}_s^\alpha \left[ \int_t^T |\sigma_u^2 - \tilde{\sigma}_u^2| du \right] dA_s \right) \tag{7.4}\]

Next, for any \( \tau \in \mathcal{T}_{0,T} \), the conditional version of the Cauchy–Schwarz inequality and the elementary inequality \((a+b)^2 \leq 2(a^2 + b^2)\) show that

\[
\mathbb{E}_\tau^\alpha \left[ \int_\tau^T |\sigma_s^2 - \tilde{\sigma}_s^2| ds \right] \leq \left( \mathbb{E}_\tau^\alpha \left[ \int_\tau^T 2(|\sigma_s|^2 + |\tilde{\sigma}_s|^2) ds \right] \right)^{\frac{1}{2}} \left( \mathbb{E}_\tau^\alpha \left[ \int_\tau^T |\sigma_s - \tilde{\sigma}_s|^2 ds \right] \right)^{\frac{1}{2}} \leq \sqrt{2} \left( \|\sigma\|_{H^{2}_{BMO}(P^\alpha)}^2 + \|	ilde{\sigma}\|_{H^{2}_{BMO}(P^\alpha)}^2 \right)^{\frac{1}{2}} \|\sigma - \tilde{\sigma}\|_{H^{2}_{BMO}(P^\alpha)}.
\]

Plugging this into (7.4), we obtain

\[
\mathbb{E}_t^\alpha \left[ \int_t^T e^{f_t^j \alpha_u^2 du} \frac{2}{\lambda} (c_u^\sigma - c_u^{\tilde{\sigma}}) (\sigma_u^2 - \tilde{\sigma}_u^2) ds \right] \leq g_1(\gamma/\lambda, \alpha, \sigma, \tilde{\sigma}) \|\sigma - \tilde{\sigma}\|_{H^{2}_{BMO}(P^\alpha)} \mathbb{E}_t^\alpha [A_T]. \tag{7.5}
\]

Inserting (7.5) back into (7.3) gives

\[
|c_t^\sigma - c_t^{\tilde{\sigma}}|^2 \leq e^{-f_t^j \alpha_u^2 du} g_1(\gamma/\lambda, \alpha, \sigma, \tilde{\sigma}) \|\sigma - \tilde{\sigma}\|_{H^{2}_{BMO}(P^\alpha)} \mathbb{E}_t^\alpha [A_T] \leq g_1(\gamma/\lambda, \alpha, \sigma, \tilde{\sigma}) \|\sigma - \tilde{\sigma}\|_{H^{2}_{BMO}(P^\alpha)} \mathbb{E}_t^\alpha [A_T]
\]

Now, take the supremum over \( t \in [0,T] \) on both sides, then \( \mathbb{P}^\alpha \)-expectations, and finally use Lemma B.2 for a fixed \( p \in (1,2) \). It follows that, for any \( \varepsilon > 0 \),

\[
\mathbb{E}^\alpha \left[ \sup_{t \in [0,T]} |c_t^\sigma - c_t^{\tilde{\sigma}}|^2 \right] \leq g_1(\gamma/\lambda, \alpha, \sigma, \tilde{\sigma}) \|\sigma - \tilde{\sigma}\|_{H^{2}_{BMO}(P^\alpha)} \cdot \left( \frac{1}{\varepsilon^2} \mathbb{E}^\alpha \left[ \sup_{0 \leq s \leq T} |c_u^\sigma - c_u^{\tilde{\sigma}}|^2 \right] + \frac{\varepsilon}{4} \left( \frac{p}{p-1} \right)^2 \mathbb{E}^\alpha \left[ e^{\frac{2p}{p-1} \int_0^T \alpha_u^2 du} \right] ^{\frac{2-p}{p}} \right).
\]

This implies that, for any \( \varepsilon > g_1(\gamma/\lambda, \alpha, \sigma, \tilde{\sigma}) \|\sigma - \tilde{\sigma}\|_{H^{2}_{BMO}(P^\alpha)} \),

\[
\mathbb{E}^\alpha \left[ \sup_{t \in [0,T]} |c_t^\sigma - c_t^{\tilde{\sigma}}|^2 \right] \leq \frac{\varepsilon^2 g_1(\gamma/\lambda, \alpha, \sigma, \tilde{\sigma}) \|\sigma - \tilde{\sigma}\|_{H^{2}_{BMO}(P^\alpha)}^2}{4(\varepsilon - g_1(\gamma/\lambda, \alpha, \sigma, \tilde{\sigma}) \|\sigma - \tilde{\sigma}\|_{H^{2}_{BMO}(P^\alpha)})} \left( \frac{p}{p-1} \right)^2 \mathbb{E}^\alpha \left[ e^{\frac{2p}{p-1} \int_0^T \alpha_u^2 du} \right] ^{\frac{2-p}{p}}.
\]

The asserted estimate in turn corresponds to the optimal choice \( \varepsilon = 2g_1(\gamma/\lambda, \alpha, \sigma, \tilde{\sigma}) \|\sigma - \tilde{\sigma}\|_{H^{2}_{BMO}(P^\alpha)} \). \( \square \)

We now move on to the process

\[
\tilde{\xi}_t^\sigma = \frac{\gamma}{\lambda} \mathbb{E}_t^\sigma \left[ \int_t^T e^{-f_s^j c_u^\sigma \sigma_s^2 ds} ds \right], \tag{7.6}
\]

from Lemma 3.2. The linear (and, in particular, monotone since \( c \) is nonnegative) BSDE (5.5) for this process rewrites as follows under the measure \( \mathbb{P}^\alpha \):

\[
\tilde{\xi}_t^\sigma = \int_t^T \left( \frac{\gamma}{\lambda} \sigma_s^2 \tilde{\xi}_s - c_s \tilde{\xi}_s - \alpha_s Z_s^\sigma \right) ds - \int_t^T Z_s^\sigma dW_s^\alpha, \quad t \in [0,T].
\]

We first record some uniform estimates, which are a direct consequence of the non-negativity of \( c \) established in Lemma 3.1.
Corollary 7.2. Suppose that the process \( \xi = (\xi_t)_{t \in [0,T]} \) satisfies \( \sigma|\xi|^\frac{1}{2} \in \mathbb{H}^2_{\text{BMO}}(\mathbb{P}) \). Then for \( (\gamma, \lambda) \in (0, \infty)^2 \), the process \( \xi \) from (7.6) satisfies
\[
\|\xi^\sigma\|_{\mathcal{S}^2} \leq \frac{\gamma}{\lambda} \|\sigma\xi^\frac{1}{2}\|_{\mathbb{H}^2_{\text{BMO}}(\mathbb{P})}.
\]

Next, we show that the stability result for \( c \) established in Lemma 7.1 and another application of the stability theorem for monotone BSDEs yield the following stability result for \( \xi \).

Corollary 7.3. Fix \( (\gamma, \lambda, p, \alpha, \sigma) \in (0, \infty)^2 \times (1, 2) \times \mathbb{H}^2_{\text{BMO}}(\mathbb{P}) \), with corresponding measure \( \mathbb{P}^\alpha \) given by (7.1), and suppose that \( \mathbb{E}^\alpha \left[ \int_0^T \alpha_0^2 du \right] < \infty \). For \( (\nu, \nu', \sigma, \sigma') \in \mathbb{H}^2_{\text{BMO}}(\mathbb{P}) \times \mathbb{S}^\infty \times \mathbb{H}^2_{\text{BMO}}(\mathbb{P}) \times \mathbb{H}^2_{\text{BMO}}(\mathbb{P}) \), set \( \xi^\sigma := \frac{\gamma}{\lambda} + \nu' \) and \( \xi^\sigma := \frac{\gamma}{\lambda} + \nu' \), and denote by \( \xi^\sigma \) and \( \xi^\sigma \) the corresponding processes from (7.6). Then
\[
\mathbb{E}^\alpha \left[ \sup_{0 \leq t \leq T} |\xi^\sigma - \xi^\sigma|^2 \right] \leq g_\xi(\gamma/\lambda, \alpha, \sigma, \bar{\sigma}, \nu, \nu') \|\sigma - \bar{\sigma}\|^2_{\mathbb{H}^2_{\text{BMO}}(\mathbb{P}^\alpha)},
\]
where
\[
g_\xi(\gamma/\lambda, \alpha, \sigma, \bar{\sigma}, \nu, \nu') := \left( \frac{p}{p - 1} \right) ^2 (g_2(\gamma/\lambda, \alpha, \sigma, \bar{\sigma}))^2 \mathbb{E}^\alpha \left[ e^{2p \int_0^T \alpha_0^2 du} \right] ^{\frac{2p}{p - 1}}
\]
with
\[
g_2(\gamma/\lambda, \alpha, \sigma, \bar{\sigma}, \nu, \nu') := 2 \frac{\gamma}{\lambda} \|\nu\|_{\mathbb{H}^2_{\text{BMO}}(\mathbb{P}^\alpha)} + \|\nu'\|_{\mathbb{S}^\infty} g_1(\gamma/\lambda, \alpha, \sigma, \bar{\sigma}) + 2 \frac{\gamma}{\lambda} \|\xi\|_{\mathbb{S}^\infty} \|\bar{\sigma}\|^2_{\mathbb{H}^2_{\text{BMO}}(\mathbb{P}^\alpha)} T g_\xi(\gamma/\lambda, \alpha, \sigma, \bar{\sigma}).
\]

Proof. The argument is similar to the proof of Lemma 7.1. For \( t \in [0, T] \), apply Itô’s formula to \( e^{\int_0^T \alpha_0^2 ds} (\xi^\sigma_t - \bar{\xi}^\sigma_t)^2 \) and use that \( e^{\int_0^T \alpha_0^2 ds} (\xi^\sigma_t - \bar{\xi}^\sigma_t)^2 = 0 \). This gives
\[
e^{\int_0^T \alpha_0^2 du} (\xi^\sigma_t - \bar{\xi}^\sigma_t)^2 = \int_0^T e^{\int_0^u \alpha_0^2 ds} \left( -\alpha_0^2 (\xi^\sigma_t - \bar{\xi}^\sigma_t)^2 + 2(\xi^\sigma_t - \bar{\xi}^\sigma_t) \left( \frac{\gamma}{\lambda} \nu_\alpha(s - \bar{\sigma}_t) + \frac{\gamma}{\lambda} \nu_\alpha'(\sigma_t^\alpha - \bar{\sigma}_t^\alpha) \right) \right) ds
\]
\[
+ \int_0^T e^{\int_0^u \alpha_0^2 ds} 2(\xi^\sigma_t - \bar{\xi}^\sigma_t) \left( -c^\sigma_t \xi_t^\sigma - \bar{c}^\sigma_t \bar{\xi}_t^\sigma \right) - \alpha_0(s^\sigma_t - Z^\sigma_t) - \alpha_0(s^\sigma_t - Z^\sigma_t) \right) ds
\]
\[
+ \int_0^T e^{\int_0^u \alpha_0^2 ds} (Z^\sigma_t - Z^\sigma_t) dW^\alpha_s - \int_0^T e^{\int_0^u \alpha_0^2 ds} (Z^\sigma_t - Z^\sigma_t)^2 ds.
\]
It follows as in the proof of Lemma 7.1 that \( \int_0^T e^{\int_0^u \alpha_0^2 ds} (Z^\sigma_t - Z^\sigma_t) dW^\alpha_s \) is an \( \mathbb{H}^1(\mathbb{P}^\alpha) \)-martingale. Now also use the elementary inequality \(-2ab \leq a^2 + b^2\), the identity \( ab - cd = a(b - d) + d(a - c) \), and that \( c^\sigma \) is non-negative. As a consequence,
\[
e^{\int_0^T \alpha_0^2 ds} (\xi^\sigma_t - \bar{\xi}^\sigma_t)^2 \leq \mathbb{E}^\alpha \left[ \int_0^T e^{\int_0^u \alpha_0^2 ds} 2(\xi^\sigma_t - \bar{\xi}^\sigma_t) \left( \frac{\gamma}{\lambda} \nu_\alpha(s - \bar{\sigma}_t) + \frac{\gamma}{\lambda} \nu_\alpha'(\sigma_t^\alpha - \bar{\sigma}_t^\alpha) - \bar{c}^\sigma_t (c^\sigma_t - \bar{c}^\sigma_t) \right) ds \right].
\]
(7.7)

Next, the conditional versions of the inequalities of Cauchy–Schwarz and Jensen’s, the elementary inequality \((a + b)^2 \leq 2(a^2 + b^2)\), Lemma 7.1, and Corollary 7.2 yield that, for any stopping time \( \tau \),
\[
2\mathbb{E}^\alpha \left[ \int_\tau^T \left| \frac{\gamma}{\lambda} \nu_\alpha(s - \bar{\sigma}_t) + \frac{\gamma}{\lambda} \nu_\alpha'(\sigma_t^\alpha - \bar{\sigma}_t^\alpha) - \bar{c}^\sigma_t (c^\sigma_t - \bar{c}^\sigma_t) \right| ds \right]
\]
\[
\leq 2 \frac{\gamma}{\lambda} \mathbb{E}^\alpha \left[ \int_\tau^T (\sigma_t - \bar{\sigma}_t)^2 ds \right] ^\frac{1}{2} \left( \mathbb{E}^\alpha \left[ \int_\tau^T \nu_\alpha^2 ds \right] ^\frac{1}{2} + \mathbb{E}^\alpha \left[ \int_\tau^T 2(\sigma_t^2 + \bar{\sigma}_t^2) ds \right] ^\frac{1}{2} \right)
\]
\[
+ 2 \|\xi\|_{\mathbb{S}^\infty} \mathbb{E}^\alpha \left[ \sup_{t \in [0,T]} (e^\tau_t - c^\tau_t)^2 \right] ^\frac{1}{2}
\]
\[
\leq \|\sigma - \bar{\sigma}\|^2_{\mathbb{H}^2_{\text{BMO}}(\mathbb{P}^\alpha)} \left( \frac{2\gamma}{\lambda} \|\nu\|^2_{\mathbb{H}^2_{\text{BMO}}(\mathbb{P}^\alpha)} + \|\nu'\|_{\mathbb{S}^\infty} g_1(\gamma/\lambda, \alpha, \sigma, \bar{\sigma}) + 2 \frac{\gamma}{\lambda} \|\xi\|_{\mathbb{S}^\infty} \|\bar{\sigma}\|^2_{\mathbb{H}^2_{\text{BMO}}(\mathbb{P}^\alpha)} T g_\xi(\gamma/\lambda, \alpha, \sigma, \bar{\sigma}) \right).
\]

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As in the proof of Lemma 7.1, we deduce that

\[(\xi_t^\sigma - \xi_t^\tilde{\sigma})^2 \leq \frac{\gamma}{\lambda} \|\sigma - \tilde{\sigma}\|_{H^2_{\text{BMO}}(P)} \mathbb{E}^P_0[C_T] \times \left( \frac{2}{\nu} \|\nu\|_{H^2_{\text{BMO}}(P)} + \|\nu\|_S \|g_1(\gamma/\lambda, \alpha, \sigma, \tilde{\sigma}) + 2 \|\xi\|_S \|\tilde{\sigma}\|_{H^2_{\text{BMO}}(P)} T g_2(\gamma/\lambda, \alpha, \sigma, \tilde{\sigma}) \right),\]

where

\[C_t = e^{f_0^t a^2_\nu du} \sup_{u \in [0,t]} |\xi_u^\sigma - \xi_u^\tilde{\sigma}|, \quad t \in [0,T].\]

Then we can argue exactly as in the proof of Lemma 7.1 to conclude. \(\square\)

We finally turn to the optimal tracking strategies \(\varphi\) from Lemma 3.2. Recall that these solve the (random) linear ODE

\[\dot{\varphi}_t = \xi_t - c_t \varphi_t, \quad \varphi_0 = x,\]

which has the explicit solution

\[\varphi_t = e^{-\int_0^t c_u du} x + \int_0^t e^{-\int_s^t c_u du} \xi_s ds. \quad (7.8)\]

Together with Corollary 7.2, we obtain the following estimate:

**Corollary 7.4.** Let \((\gamma, \lambda) \in (0, \infty)^2\) and define \(\xi := \frac{\nu}{\sigma} + \nu'\) for processes \((\nu, \sigma, \nu') \in H^2_{\text{BMO}} \times H^2_{\text{BMO}} \times S^\infty\). Then, the process \(\varphi\) from (7.8) satisfies

\[\|\varphi\|_{S^\infty} \leq |\varphi_0| + T \|\xi\|_{S^\infty} \leq |x| + \frac{\gamma}{\lambda} T \left( \|\nu\|_{H^2_{\text{BMO}}(P)} \|\sigma\|_{H^2_{\text{BMO}}(P)} + \|\nu\|_{S^\infty} \|\sigma\|_{H^2_{\text{BMO}}(P)}^2 \right).\]

This uniform bound together with the stability results for \(c\) and \(\xi\) now allow to establish a stability result for the optimal tracking strategies in terms of the BMO–norm of the underlying volatility processes.

**Theorem 7.5.** Fix \((\gamma, \lambda, p, \alpha) \in (0, \infty)^2 \times (1, 2) \times H^2_{\text{BMO}}(P),\) with corresponding measure \(P^\alpha\) given by (7.1), and suppose that \(\mathbb{E}^\alpha \left[ e^{\frac{2p}{\alpha} \int_0^T a^2_\nu du} \right] < \infty.\) For \((\nu, \nu', \sigma, \tilde{\sigma}) \in H^2_{\text{BMO}} \times S^\infty \times \mathbb{R} \times H^2_{\text{BMO}} \times H^2_{\text{BMO}}(P),\) set \(\xi^\sigma := \frac{\nu}{\sigma} + \nu'\) and \(\xi^\tilde{\sigma} := \frac{\nu}{\tilde{\sigma}} + \nu',\) and denote by \(\varphi^\sigma\) and \(\varphi^\tilde{\sigma}\) the corresponding strategies from (7.6). Then

\[\mathbb{E}^\alpha \left[ \sup_{t \in [0,T]} |\varphi^\sigma_t - \varphi^\tilde{\sigma}_t|^2 \right] \leq g_\varphi(x, \gamma/\lambda, \alpha, \sigma, \tilde{\sigma}, \nu, \nu') \|\sigma - \tilde{\sigma}\|_{H^2_{\text{BMO}}(P)}^2;\]

where

\[g_\varphi(x, \gamma/\lambda, \alpha, \sigma, \tilde{\sigma}, \nu, \nu') := 3T^2 \left( x^2 + T^2 \frac{\gamma^2}{\lambda^2} \|\xi^\sigma\|_{S^\infty}^2 \|\sigma\|_{H^2_{\text{BMO}}(P)}^4 \right) g_c(\gamma/\lambda, \alpha, \sigma, \tilde{\sigma})^2 + g_{\tilde{\xi}}(\gamma/\lambda, \alpha, \sigma, \tilde{\sigma}, \nu, \nu')^2.\]

**Proof.** Observe that the map \(x \mapsto e^{-x}\) is Lipschitz continuous on \(\mathbb{R}_+\) with Lipschitz constant 1. Whence, for \(t \in [0,T],\)

\[|\varphi^\sigma_t - \varphi^\tilde{\sigma}_t| \leq |x| \int_0^t |c^\sigma_u - c^\tilde{\sigma}_u| du + \int_0^t \left( \int_s^t |c^\sigma_u - c^\tilde{\sigma}_u| du \right) \xi^\sigma_s ds + \int_0^t e^{-\int_s^t c^\sigma_u du} |\xi^\sigma_s - \xi^\tilde{\sigma}_s| ds \leq |x| T \sup_{u \in [0,T]} |c^\sigma_u - c^\tilde{\sigma}_u| + T^2 \|\xi^\sigma\|_{S^\infty} \sup_{u \in [0,T]} |c^\sigma_u - c^\tilde{\sigma}_u| + T \sup_{u \in [0,T]} |\xi^\sigma_u - \xi^\tilde{\sigma}_u|.

Now, take the supremum over \(t \in [0,T]\) and square the result. In view of the elementary inequality \((a + b + c)^2 \leq 3(a^2 + b^2 + c^2)\) for \(a, b, c \in \mathbb{R},\) the assertion then follows by taking \(P^\alpha\text{–}\text{expectations.}\) \(\square\)
A  BMO Results

This appendix collects some auxiliary results on BMO martingales that are used in the proofs of Theorem 6.3, Lemma 3.2, and Proposition 4.7.

Lemma A.1. Let \((\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})\) be a filtered probability space supporting a Brownian motion \((W_t)_{t \in [0,T]}\) and such that all \(\mathbb{F}\)-martingales are continuous. Let \((\alpha_t)_{t \in [0,T]} \in \mathbb{H}^2_{\text{BMO}}(\mathbb{P})\) and define \(\mathbb{P}^\alpha \sim \mathbb{P}\) on \(\mathcal{F}_T\) by

\[
\frac{d\mathbb{P}^\alpha}{d\mathbb{P}} = \mathcal{E}\left(\int_0^T \alpha_t dW_t\right)_T.
\]

Then \(\alpha \in \mathbb{H}^2_{\text{BMO}}(\mathbb{P}^\alpha)\). Moreover, if \((\sigma_t)_{t \in [0,T]} \in \mathbb{H}^2_{\text{BMO}}(\mathbb{P})\), then \((\sigma_t)_{t \in [0,T]} \in \mathbb{H}^2_{\text{BMO}}(\mathbb{P}^\alpha)\) and

\[
||\sigma||^2_{\mathbb{H}^2_{\text{BMO}}(\mathbb{P}^\alpha)} \leq 8(||\alpha||_{\mathbb{H}^2_{\text{BMO}}(\mathbb{P})} + 1)^2||\sigma||^2_{\mathbb{H}^2_{\text{BMO}}(\mathbb{P})}, \quad ||\sigma||^2_{\mathbb{H}^2_{\text{BMO}}(\mathbb{P})} \leq 8(||\alpha||_{\mathbb{H}^2_{\text{BMO}}(\mathbb{P})} + 1)^2||\sigma||^2_{\mathbb{H}^2_{\text{BMO}}(\mathbb{P}^\alpha)}.
\]

Proof. This follows immediately from the proof of [28, Theorem 3.6] and Lemma A.2 applied under \(\mathbb{P}^\alpha\) and \(\mathbb{P}\).

Lemma A.2. Let \((\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})\) be a probability space supporting a Brownian motion \((W_t)_{t \in [0,T]}\). Let \((\alpha_t)_{t \in [0,T]} \in \mathbb{H}^2_{\text{BMO}}(\mathbb{P})\) and define the \(\mathbb{P}\)-martingale \((M_t)_{t \in [0,T]}\) by \(M_t := \int_0^t \alpha_s dW_s\). Then for any \(p > 1\) with \(p \geq (||\alpha||_{\mathbb{H}^2_{\text{BMO}}(\mathbb{P})} + 1)^2\) and any stopping time \(\tau\), we have

\[
||E_{\tau} \left(\frac{(E(M)_{\tau})^{1/p}_{1/p}}{E(M)_{T}^{1/p}_{1/p}}\right)\|^\infty \leq 2.
\]

Proof. The condition on \(p\) implies that \(||\alpha/(\sqrt{2}(\sqrt{p} - 1))||_{\mathbb{H}^2_{\text{BMO}}(\mathbb{P})} \leq \frac{1}{2}\). Thus it follows from the John–Nirenberg inequality [28, Theorem 2.2] that, for any stopping \(\tau\),

\[
E_{\tau} \left[\frac{1}{2(\sqrt{p} - 1)^2} (\langle M\rangle_T - \langle M\rangle_\tau)\right] \leq 2.
\]

The claim now follows from the proof of (a) \(\Rightarrow\) (b) in [28, Theorem 2.4] with \(C_p = 2\).

Lemma A.3. Let \((\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})\) be a probability space, \((\beta_t)_{t \in [0,T]}\) a nonnegative process and \((A_t)_{t \in [0,T]}\) a nondecreasing process. Then, for any \([0,T]\)-valued stopping time \(\tau\),

\[
E_{\tau} \left[\int_\tau^T A_s \beta_s ds\right] \leq A_{\tau} E_{\tau} \left[\int_\tau^T \beta_s ds\right] + E_{\tau} \left[\int_\tau^T E_u \left[\int_u^T \beta_s ds\right] dA_u\right]. \tag{A.1}
\]

Moreover, if \(\sqrt{\beta} \in \mathbb{H}^2_{\text{BMO}}\), then

\[
E_{\tau} \left[\int_\tau^T A_s \beta_s ds\right] \leq \|\sqrt{\beta}\|_{\mathbb{H}^2_{\text{BMO}}}^2 E_{\tau} [A_T]. \tag{A.2}
\]

Proof. Write \(A_s = A_{\tau} + \int_\tau^s dA_u\) for \(s \geq \tau\). Fubini’s theorem in turn gives

\[
\int_\tau^T A_s \beta_s ds = A_{\tau} \int_\tau^T \beta_s ds + \int_\tau^T \int_\tau^s \beta_s dA_u ds = A_{\tau} \int_\tau^T \beta_s ds + \int_\tau^T \int_{u}^T \beta_s ds dA_u.
\]

Now, (A.1) follows from taking conditional expectations, using the conditional result corresponding to (the optional version of) [19, Theorem VI.57], and that the optional projection of the process \((\int_u^T \beta_s ds)_{u \in [0,T]}\) is \(E_u[\int_u^T \beta_s ds])_{u \in [0,T]}\). Moreover, (A.2) follows from (A.1) by the definition of the BMO norm.
B Variations on Doob’s inequality

The following versions of Doob’s inequality are used in the proofs of Theorem 6.3 and Lemma 7.1, respectively.

**Lemma B.1.** Let \((\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})\) be a filtered probability space, and \(X\) an \(\mathcal{F}_T\)-measurable nonnegative random variable with \(\mathbb{E}[X^2] < \infty\). Then

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \mathbb{E}_t[X] \right] \leq 2 \mathbb{E} [X^2]^{\frac{1}{2}}.
\]

**Proof.** The inequalities of Cauchy–Schwarz and Doob show that

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \mathbb{E}_t[X] \right] \leq \left( \mathbb{E} \left[ \sup_{t \in [0,T]} \mathbb{E}_t[X]^2 \right] \right)^{1/2} \leq 2 \mathbb{E} [X^2]^{\frac{1}{2}}.
\]

**Lemma B.2.** Let \((\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})\) be a filtered probability space, \(p \in (1,2)\), and let \(X\) and \(Y\) be \(\mathcal{F}_T\)-measurable nonnegative random variables, with \(\mathbb{E}[X^2] < \infty\) and \(\mathbb{E}[Y^{2p/(2-p)}] < \infty\). Then, for \(\varepsilon > 0\),

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \mathbb{E}_t[XY] \right] \leq \frac{1}{\varepsilon} \mathbb{E} [X^2] + \frac{\varepsilon}{4} \left( \frac{p}{p - 1} \right)^2 \mathbb{E} [Y^{2p/(2-p)}]^{2-p/p}.
\]

**Proof.** The inequalities of Hölder and Doob yield

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \mathbb{E}_t[XY] \right] \leq \left( \mathbb{E} \left[ \sup_{t \in [0,T]} \mathbb{E}_t[XY]^p \right] \right)^{1/p} \leq \frac{p}{p - 1} \mathbb{E} \left[ (XY)^p \right]^{1/p} \leq \frac{p}{p - 1} \mathbb{E} [X^2]^{\frac{p}{2}} \mathbb{E} [Y^{2p/(2-p)}]^{2-p/p}.
\]

Using the elementary inequality \(2ab \leq \varepsilon^{-1}a^2 + \varepsilon b^2\), valid for any \(\varepsilon > 0\) and \(a, b \in \mathbb{R}\), we in turn deduce

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \mathbb{E}_t[XY] \right] \leq \frac{1}{\varepsilon} \mathbb{E} [X^2] + \frac{\varepsilon}{4} \left( \frac{p}{p - 1} \right)^2 \mathbb{E} [Y^{2p/(2-p)}]^{2-p/p}.
\]

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