Gaining or Losing Perspective for Convex Multivariate Functions on a Simplex

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Abstract MINLO (mixed-integer nonlinear optimization) formulations of the disjunction between the origin and a polytope via a binary indicator variable have broad applicability in nonlinear combinatorial optimization, for modeling a fixed cost $c$ associated with carrying out a set of $d$ activities and a convex variable cost function $f$ associated with the levels of the activities. The perspective relaxation is often used to solve such models to optimality in a branch-and-bound context, especially in the context in which $f$ is univariate (e.g., in Markowitz-style portfolio optimization). But such a relaxation typically requires conic solvers and are typically not compatible with general-purpose NLP software which can accommodate additional classes of constraints. This motivates the study of weaker relaxations to investigate when simpler relaxations may be adequate. Comparing the volume (i.e., Lebesgue measure) of the relaxations as means of comparing them, we lift some of the results related to univariate functions $f$ to the multivariate case. Along the way, we survey, connect and extend relevant results on integration over a simplex, some of which we concretely employ, and others of which can be used for further exploration on our main subject.

Keywords mixed-integer nonlinear optimization · global optimization · convex relaxation · perspective · simplex · polytope · volume · integration

1 Introduction

The “perspective reformulation” technique is used to obtain strong relaxations of the MINLO (mixed-integer nonlinear optimization) formulations modeling indicator variables: when an indicator variable is “off”, a vector of $d$ decision

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variables is forced to some specific point (often \(0 \in \mathbb{R}^d\)), and when it is “on”,
the vector of decision variables must belong to a specific convex set in \(\mathbb{R}^d\) (see
[GL10,LSS22] and the many references therein).

Perspective relaxations typically contain conic constraints, but not all NLP
solvers are equipped to handle conic constraints correctly. Conic solvers (like
MOSEK and SDPT3; see [MOS21b] and [TTT99], respectively) handle such con-
straints coming from well-known classes of cones (e.g., second-order cones,
power cones, exponential cones), by providing associated barrier functions.
But they do not have the capability to handle all such constraints. Even in
cases where a conic solver can handle the perspectivization of a given convex
function, there may be other (even convex) constraints that such a solver can-
not handle. In such a situation, we may hope to use a general NLP solver,
which we might also expect to be faster than a conic solver, but these are not
typically able to handle perspective functions correctly (see [LSS22, Sec. 1.2]).

For the univariate case of a continuous variable \(x\) being either 0 or in
a positive interval \([\ell, u]\), [LSS22,LSSX23] studied the trade-off between the
tightness and tractability of alternative relaxations, and proposed several nat-
ural and simpler non-conic-programming relaxations. For the specific case of
\(f(x) := x^p, p > 1\), they obtained concrete results, considering the relative
tightness of formulations as functions of \(\ell, u,\) and \(p\). These results apply to
the situation where indicator variables manage terms in a separable
objective function, with each continuous variable being either 0 or in an interval (not
containing 0).

In what follows, we consider the multivariate case in which the decision
variable (vector) \(x\) is either \(0 \in \mathbb{R}^d\) or in a simplex \(J \subset \mathbb{R}^d_{\geq 0}\) (not containing \(0\)).
Our goal is to lift results related to univariate functions from [LSS22,LSSX23]
to the multivariate case. The idea of comparing relaxations via their volumes
(i.e., Lebesgue measure) was introduced in [LM94] (also see [LSS18], and the
many references therein). [LSS22,LSSX23] first developed these ideas in the
context of perspective relaxation, for the univariate case. Following [LSS22,
LSSX23], we also use \((d + 2)\)-dimensional volume as a measure for comparing
relaxations. We have \(x \in \mathbb{R}^d_{\geq 0}\), a binary indicator variable \(z\) keeping track of
whether \(x = 0\) or \(x \in J\), and a further variable \(y \in \mathbb{R}\) which “captures” \(f(x)\); so \(d + 2\) variables in total.

**Organization and contributions.** In what follows, we formally define our
sets of interest: a disjunctive set, the perspective relaxation, and the naïve
relaxation. In Section 2, we derive general formulae for the volumes of the
perspective relaxation and naïve relaxation. These formulae both require in-
tegrating over a simplex, and so in Section 3, we survey, connect and extend
relevant results in the literature concerning integration over a simplex. This
survey is an important contribution of our work as it collects these results for
optimizers, in one place and in an accessible form; we rely on some of these
results in Section 4, and other results are natural tools that could be used
to push forward further on our motivating topic. In Section 4, we derive the
formula for the volume of the naïve relaxation for two natural families of func-
tions, generalizing what is known for the univariate case. We also demonstrate how to work numerically, when there is no closed-form integration formula. In Section 5, we make some brief conclusions.

Notation. In what follows, we use boldface lower-case for vectors and boldface upper-case for matrices. $e^d_n$ denotes the $n$-th unit vector in $\mathbb{R}^d$, and the superscript $d$ is often dropped if the dimension is clear from the context. $\Delta_d := \{ x \in \mathbb{R}_{\geq 0}^d : \sum_{j=1}^d x_j \leq 1 \}$ denotes the standard $d$-simplex in $\mathbb{R}^d$. $J := \{ v_0, v_1, \ldots, v_d \} \subset \mathbb{R}_{\geq 0}^d$ denotes an arbitrary $d$-simplex in $\mathbb{R}_{\geq 0}^d$, where $v_0, v_1, \ldots, v_d$ are the $d + 1$ (affinely independent) vertices of $J$. An affine transformation from $J$ to $\Delta_d$ can be used to extend integration results from $\Delta_d$ to a general $J$ (see for example, [Las21, RMTC19]).

We can define the (higher-dimensional) perspective relaxation $(f,J)$ at $(0_d,0)$ (Sec. IV, Remark 2.2.3, Page 160). Importantly, if we evaluate the closure of $\tilde{f}$ (whose epigraph is the closure of the epigraph of $f$, see [HUL93, Sec. IV, Definition 1.2.4, Page 149]) at $(0_d,0)$, we get 0 (see [HUL93, Sec. IV, Remark 2.2.3, Page 162]). So we can define the (higher-dimensional) perspective relaxation $P(f,J) := \text{cl} \{ (x,y,z) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} : \mu(x,z) \geq y \geq \tilde{f}(x,z), x \in z \cdot J, 1 \geq z > 0 \}$,
where the upper bound $\tilde{\mu}(x,z)$ is the perspective function of the linear function $\mu(x)$, and is thus linear itself.

Some comments on $P(f,J)$:

- $P(f,J)$ intersects the hyperplane defined by $z = 0$ at the single point $0_{d+2}$, and it intersects the hyperplane defined by $z = 1$ at $D_1(f,J)$. It is clear that $P(f,J)$ is the convex hull of $D(f,J)$.
- When $d = 1$, $P(f,J)$ is the perspective relaxation of [LSS22] (and others).
- If the simplex $J \subset \mathbb{R}_{\geq 0}^d$ is described by linear inequalities $Ax \leq b$, then we can write $x \in z \cdot J$ as the homogeneous system $Ax \leq bz$.
- The constraint $y \geq f(x)$ is equivalent to $(x,y,1) \in K_f$, where

$$K_f := \left\{ (x,y,z) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} : y \geq \bar{f}(x,z), \ z > 0 \right\}$$

is a convex cone, whose closure is $P(f,J)$ without the upper bound $\tilde{\mu}(x,z)$.

So relaxing the disjunction $D(f,J)$ to $P(f,J)$ enables us to use an interior-point conic solvers (like MOSEK and SDPT3) whenever appropriate barriers are available in the solver.

- Considering $f(x)$ at each of the $d + 1$ vertices of $J$, there is a unique hyperplane in the variables $(x,y) \in \mathbb{R}^{d+1}$ passing through these $d + 1$ points. Suppose that $J := \text{conv}\{v_0,v_1,\ldots,v_d\} \subset \mathbb{R}^d$. Then $\mu(x)$ can be defined as

$$\mu(x) := w^T B^{-1}(x - v_0) + f(v_0), \quad (2)$$

where $B$ is from (1), and $w^T := [f(v_1) - f(v_0), \ldots, f(v_d) - f(v_0)] \in \mathbb{R}^{1 \times d}$. Therefore,

$$\tilde{\mu}(x,z) = z\mu(x/z) = w^T B^{-1}(x - zv_0) + f(v_0)z.$$

Extending a key setting from [LSS22], we consider the following special case: the domain of the convex function $f$ is $\text{conv}(J \cup \{0\}) = \{z \cdot J : 0 \leq z \leq 1\}$, and $f(0) = 0$. We can then define the (higher-dimensional) na"ive relaxation:

$$P^0(f,J) := \left\{ (x,y,z) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} : \tilde{\mu}(x,z) \geq y \geq f(x), \ x \in z \cdot J, \ 1 \geq z \geq 0 \right\},$$

where the upper bound $\tilde{\mu}(x,z)$ is defined as in the perspective function of $\mu(x)$. It is clear that $P^0(f,J)$ is convex, due to the convexity of $f$ and the linearity of the other constraints.

Given any $x,z$ such that $x \in z \cdot J$ and $1 \geq z \geq 0$, we have $zf(x/z) + (1 - z)f(0) \geq f(x)$ because of the convexity of $f$, which implies that $zf(x/z) \geq f(x)$. From this, we can see that $P(f,J) \subseteq P^0(f,J)$, i.e., the naïve relaxation contains the perspective relaxation (as holds when $d = 1$), which implies that $P^0(f,J)$ is also a relaxation of $D(f,J)$. We can readily see that $P^0(f,J)$ is easier to handle than $P(f,J)$, because it involves $f$ rather than the perspective function of $f$. So it is natural to try and understand, depending on $f$ and $J$, how much stronger $P(f,J)$ is compared to $P^0(f,J)$.
Figure 1 shows the comparison of the naïve and perspective relaxation for an example in the univariate case. The left subfigure shows $D(f,J) = \{0_{d+2}\} \cup D_1(f,J)$ (in black) and the lower bounds $f(x)$ (in blue) and $\tilde{f}(x,z)$ (in yellow); the right subfigure is a cross section on the plane $z = \frac{1}{2}$ for the upper bound $\tilde{\mu}(x,z)$ (in black) and two lower bounds in the relaxations.

The univariate case can be used to handle separable multivariate convex functions. In what follows, we are more interested in nonseparable convex functions, like $(c^T x)^n$, $e^{c^T x}$, and the following example.

Example 1.1 A nice nonseparable convex function is the “log-sum-exp” function $f(x) := \log \sum_{j=1}^d e^{x_j}$, which is a smooth under-estimator of the function $\max\{x_1, \ldots, x_d\} \left( f(x) \leq \max\{x_1, \ldots, x_d\}, \lim_{u \to \infty} \frac{f(u x)}{u} = \max\{x_1, \ldots, x_d\} \right)$.

The “log-sum-exp” inequality $y \geq f(x)$ could be modeled with exponential-cone constraints (see [MOS21b, Sec. 5.2.6] and [MOS21a]):

$$\sum_{j=1}^d w_j \leq d,$$

$$(w_j, 1, x_j - y) \in K_{\exp} := \left\{ (u_1, u_2, u_3) : u_1 \geq u_2 e^{u_3/u_2}, u_2 \geq 0 \right\},$$

for $j = 1, \ldots, d$.

But in fact, even general nonlinear-programming solvers can comfortably work directly with $y \geq \log \sum_{j=1}^d e^{x_j}$. So, if we are satisfied with the associated naïve relaxation (notice that $f(0) = 0$), we can reliably use general nonlinear-programming solvers.

Going further, perspectivizing, we are led to the stronger inequality $y \geq z \log \frac{1}{z} \sum_{j=1}^d e^{x_j/z}$, which is not well handled by general nonlinear-programming solvers.

\footnote{We subtract a constant $\log d$ from the usual “log-sum-exp” function $\log \sum_{j=1}^d e^{x_j}$ to satisfy $f(0) = 0$. See https://docs.scipy.org/doc/scipy/reference/generated/scipy.special.logsumexp.html}
solving. But, similarly to how we conically modeled $y \geq \log \frac{1}{d} \sum_{j=1}^{d} e^{x_j}$, we can model
\[ y \geq z \log \frac{1}{d} \sum_{j=1}^{d} e^{x_j}/z \]
by
\[ \sum_{j=1}^{d} w_j \leq dz, \]
\[ (w_j, z, x_j - y) \in K_{\exp} := \{(u_1, u_2, u_3) : u_1 \geq u_2 e^{u_3/2u_2}, u_2 \geq 0\}, \]
for $j = 1, \ldots, d$,
which is nicely handled in MOSEK, but which now restricts what other types of (even nonconvex) constraints can be in a model and could generally lead to slower solves.

2 Volumes of relaxations

The well-known volume formula for the $d$-simplex $J := \text{conv}\{v_0, v_1, \ldots, v_d\} \subset \mathbb{R}^d$ is
\[ \text{vol}(J) = \int_J 1 \, dx = \frac{1}{d!} \det [v_1 - v_0, \ldots, v_d - v_0] = \frac{1}{d!} \det \left[ \begin{array}{cccc} v_0 & v_1 & \cdots & v_d \\ 1 & 1 & \cdots & 1 \end{array} \right]. \]

Lemma 2.1 Suppose that the $d$-simplex $J := \text{conv}\{v_0, v_1, \ldots, v_d\} \subset \mathbb{R}^d$, and $\mu(x)$ is defined by (2). Then
\[ \int_J \mu(x) \, dx = \frac{1}{d+1} \text{vol}(J) \sum_{j=0}^{d} f(v_j) = \frac{1}{(d+1)!} \det \left[ \begin{array}{cccc} v_0 & v_1 & \cdots & v_d \\ 1 & 1 & \cdots & 1 \end{array} \right] \sum_{j=0}^{d} f(v_j). \]

Proof
\[ \int_J \mu(x) \, dx = \int_J (w^T B^{-1}(x - v_0) + f(v_0)) \, dx \]
\[ = d! \text{vol}(J) \int_{\Delta_d} (w^T t + f(v_0)) \, dt, \]
where $\Delta_d = \text{conv}\{0, e_1, \ldots, e_d\}$. We use Lemma 3.2 (from the next section) to calculate the exact integral of a linear form over a simplex:
\[ \int_{\Delta_d} w^T t \, dt = \frac{1}{(d+1)!} \sum_{j=1}^{d} w_j. \]
Therefore,
\[ \int_J \mu(x) \, dx = \text{vol}(J) \left( \frac{1}{d+1} \sum_{j=0}^{d} (f(v_j) - f(v_0)) + f(v_0) \right), \]
and the result follows. \qed
Generalizing from the univariate case [LSS22, Thm. 1], we have the following simple formula for the volume of $P(f, J)$.

**Theorem 2.2** Suppose that $f$ is a continuous and convex function on the $d$-simplex $J := \text{conv}\{v_0, v_1, \ldots, v_d\} \subset \mathbb{R}^d_{\geq 0} \setminus \{0\}$. Then

$$\text{vol}(P(f, J)) = \frac{1}{(d+2)!} \left| \det \begin{bmatrix} v_0 & v_1 & \cdots & v_d \\ 1 & 1 & \cdots & 1 \end{bmatrix} \right| \sum_{j=0}^{d} f(v_j) - \frac{1}{d+2} \int_J f(x) dx.$$  

**Proof** Notice that $P(f, J)$ is a hyperpyramid in $\mathbb{R}^{d+2}$ with apex $0^{d+2}$ and base a $(d+1)$-dimensional convex set in the $z = 1$ hyperplane defined by the system of inequalities,

$$\mu(x) \geq y \geq f(x) \quad \text{for } x \in J.$$

The volume of such a hyperpyramid is $\frac{1}{d+2} \mathcal{B} \mathcal{H}$, where $\mathcal{B}$ is the $(d+1)$-dimensional volume of the base, and $\mathcal{H}$ is the perpendicular height of the apex over the affine span of the base. In this hyperpyramid, $\mathcal{H} = 1$ because the apex is $0^{d+2}$ and the hyperplane containing the base is defined by the equation $z = 1$. We only need to compute the volume of the base via the integral

$$\mathcal{B} = \int_J (\mu(x) - f(x)) dx$$

$$= \frac{1}{(d+1)!} \left| \det \begin{bmatrix} v_0 & v_1 & \cdots & v_d \\ 1 & 1 & \cdots & 1 \end{bmatrix} \right| \sum_{j=0}^{d} f(v_j) - \int_J f(x) dx.$$  

The second equation follows from Lemma 2.1. Therefore,

$$\text{vol}(P(f, J)) = \frac{1}{(d+2)!} \left| \det \begin{bmatrix} v_0 & v_1 & \cdots & v_d \\ 1 & 1 & \cdots & 1 \end{bmatrix} \right| \sum_{j=0}^{d} f(v_j) - \frac{1}{d+2} \int_J f(x) dx.$$  

$\square$

Theorem 2.2 reduces calculation of $\text{vol}(P(f, J))$ to the calculation of the integral $\int_J f(x) dx$. We will make a detailed exploration of the fundamental problem of integration over a simplex in Section 3. We note that calculation of the volume of $P^0(f, J)$ is generally more complicated than the integral $\int_J f(x) dx$ (see Theorem 2.3). We address some relevant special cases in Section 4.

**Theorem 2.3** Suppose that $f(0) = 0$ and $f$ is continuous and convex on $\text{conv}(J \cup \{0\})$, where the $d$-simplex $J := \text{conv}\{v_0, v_1, \ldots, v_d\} \subset \mathbb{R}^d_{\geq 0} \setminus \{0\}$. Then

$$\text{vol}(P^0(f, J)) = \frac{1}{(d+2)!} \left| \det \begin{bmatrix} v_0 & v_1 & \cdots & v_d \\ 1 & 1 & \cdots & 1 \end{bmatrix} \right| \sum_{j=0}^{d} f(v_j) - \int_J z^d f(zx) dx dz.$$
Proof Considering the definition of the naïve relaxation, its volume is
\[
\text{vol}(\mathcal{P}^0(f, J)) = \int_{x \in z \cdot J, 0 \leq z \leq 1} (\mu(x, z) - f(x)) dx dz
\]
\[
= \int_{0 \leq z \leq 1} z^d \int_{x \in J} (z\mu(x) - f(x)) dx dz
\]
\[
= \int_{0 \leq z \leq 1} z^{d+1} dz \int_{x \in J} \mu(x) dx - \int_{0}^{1} z^d \int_{J} f(zx) dx dz.
\]
\[
= \frac{1}{(d+2)!} \left| \det \begin{bmatrix} v_0 & v_1 & \cdots & v_d \\ 1 & 1 & \cdots & 1 \end{bmatrix} \right| \sum_{j=0}^{d} f(v_j) - \int_{0}^{1} z^d \int_{J} f(zx) dx dz.
\]

The last equation follows from Lemma 2.1. \qed

Remark 2.4 Theorem 2.3 provides a different formula to compute the volume of the naïve relaxation from [LSS22, Thm. 2 and Cor. 4], by slicing along variable $z$ instead of variable $y$.

3 Integration over a simplex

Integration over a simplex is a well-researched topic. In this section, we survey the main results for integrating over a simplex. We add some missing details to the proofs in the literature and unify the notations to make the results more accessible to the optimization community. Our goal is to provide the most general results to serve as a toolbox for future research.

[BBD+10, Thm. 1 and Cor. 3] proved that integrating polynomials over a simplex is NP-hard, while there exists a polynomial-time algorithms for integrating polynomials of fixed total degree. We are mainly interested in closed-form formulae to be carried to later analyses for the integration of some convex function over a general simplex, e.g., power of linear forms ($c^T x$) and exponential of linear functions $e^{c^T x}$. This section is organized by method, including the monomial formula, series expansion, symmetric multilinear form, Fourier transformation, and cubature rules. Table 1 summarizes the functions considered as the integrands in this section, where generalized polynomials refers to sums of monomials whose exponents are nonnegative real numbers, and symmetric multilinear forms refers to functions $H : (\mathbb{R}^d)^q \to \mathbb{R}$ such that $H(x_1, x_2, \ldots, x_q) = H(x_{j_1}, x_{j_2}, \ldots, x_{j_q})$ for any permutation $(j_1, j_2, \ldots, j_q)$ of $(1, 2, \ldots, q)$ (symmetric) and $H(\lambda x_1 + \lambda_0 y, x_2, \ldots, x_q) = \lambda H(x_1, x_2, \ldots, x_q) + \lambda_0 H(y, x_2, \ldots, x_q)$ for any $\lambda, \lambda_0 \in \mathbb{R}$ (multilinear).
3.1 Monomial formula over a standard simplex

There is a well-known formula to integrate a particular generalized polynomial over the standard $d$-simplex $\Delta_d$ in $\mathbb{R}^d$ (see, for example [Las21]).

**Proposition 3.1**

\[
\int_{\Delta_d} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d} (1 - x_1 - \cdots - x_d)^{\alpha_{d+1}} \, dx = \frac{\prod_{j=1}^{d+1} \Gamma(\alpha_j + 1)}{\Gamma \left( \sum_{j=1}^{d+1} \alpha_j + d + 1 \right)},
\]

where $\alpha_j \in \mathbb{R}$, $\alpha_j > -1$, and the usual gamma function $\Gamma(z) := \int_0^\infty x^{z-1} e^{-x} \, dx$ for $z > 0$.

To integrate a polynomial over a standard simplex, we can represent it as a sum of monomials and then employ Proposition 3.1 for each monomial, noting that we can take $\alpha_{d+1} = 0$ (see for example [GM78, Eqn. 2.3]). This idea also appears in [Las21] with a different interpretation associated with “Bombieri-type polynomials”.

3.2 Series expansion

[BBD+10, Sec. 3.2] provides several polynomial-time algorithms to calculate the exact integral of a polynomial with fixed degree over a general simplex. These algorithms are based on integration formulae over a general simplex, for powers of linear functions with positive integer exponent and for the exponential of linear functions.
Lemma 3.2 ([BBD+10, Lem. 8 and Rem. 9]) Suppose that $J := \text{conv}\{v_0, v_1, \ldots, v_d\} \subset \mathbb{R}^d$. Then we have
\[
\int_J (c^T x)^n dx = d! \frac{1}{n!} \sum_{\sum_{k \in \mathbb{Z}^d_{\geq 0}} (c^T v_0)^{ka} \cdots (c^T v_d)^{kd}} \frac{\text{vol}(J)^n}{n! (n + d)!}.
\]
\[
\int_J e^{c^T x} dx = d! \frac{1}{\prod_{j=0}^d (1 - t(c^T v_j))} \sum_{n=0}^\infty \frac{\left(\prod_{k \neq j} (c^T (v_j - v_k))^d\right) t^n}{n! \prod_{j=0}^d (c^T v_j)^n}.
\]

By treating the sequence $\frac{(n+d)!}{n!} \int_J (c^T x)^n dx$ as the coefficient of a formal power series, [BBD+10] gives a useful generating function using Lemma 3.2:

Theorem 3.3 ([BBD+10, Thm. 10]) Suppose that $J := \text{conv}\{v_0, v_1, \ldots, v_d\} \subset \mathbb{R}^d$. Then we have
\[
\int_J (c^T x)^n dx = d! \frac{1}{\prod_{j=0}^d (1 - t(c^T v_j))} \sum_{n=0}^\infty \frac{\left(\prod_{k \neq j} (c^T (v_j - v_k))^d\right) t^n}{n! \prod_{j=0}^d (c^T v_j)^n}.
\]

Therefore, we can obtain the “short formulae” of Brion (in the case of a simplex) under a genericity assumption.

Theorem 3.4 ([Bri88]) Suppose that $J := \text{conv}\{v_0, v_1, \ldots, v_d\} \subset \mathbb{R}^d$, and $c^T v_j \neq c^T v_k$ for all $j \neq k$. Then we have
\[
\int_J (c^T x)^n dx = d! \frac{1}{\prod_{j=0}^d (1 - t(c^T v_j))} \sum_{n=0}^\infty \frac{\left(\prod_{k \neq j} (c^T (v_j - v_k))^d\right) t^n}{n! \prod_{j=0}^d (c^T v_j)^n}.
\]

Proof We can decompose the following function into partial fractions and apply the expansion of geometric series:
\[
\frac{1}{\prod_{j=0}^d (1 - t(c^T v_j))} = \sum_{j=0}^d \frac{1}{1 - t(c^T v_j)} \prod_{k \neq j} (c^T v_j - c^T v_k)
\]
\[
= \sum_{j=0}^d \left[ \sum_{n=0}^\infty t^n (c^T v_j)^n \prod_{k \neq j} (c^T v_j - c^T v_k) \right] t^n.
\]
Then (3) immediately follows from Theorem 3.3. For (4), we have

\[
\int_J e^{\gamma x} d\mathbf{x} = \sum_{n=0}^{+\infty} \int_J \frac{(\mathbf{e}^T \mathbf{x})^n}{n!} d\mathbf{x}
\]

\[
= d! \operatorname{vol}(J) \sum_{n=0}^{+\infty} \frac{1}{(n+d)!} \prod_{k: k \neq j} (\mathbf{e}^T \mathbf{v}_j - \mathbf{v}_k)^n + d \sum_{n=0}^{+\infty} \frac{(\mathbf{e}^T \mathbf{v}_j)^{n+d}}{(n+d)!} \prod_{k: k \neq j} (\mathbf{e}^T \mathbf{v}_j - \mathbf{v}_k)^n.
\]

where the last equation follows from the fact that

\[
\sum_{j=0}^{d} \frac{(\mathbf{e}^T \mathbf{v}_j)^n}{\prod_{k: k \neq j} (\mathbf{e}^T \mathbf{v}_j - \mathbf{v}_k)} = 0, \text{ for } n = 0, 1, \ldots, d - 1. \tag{5}
\]

This is due to the fact that the remainder of Lagrange interpolation polynomials is zero for polynomials of degree at most \(d\):

\[
t^n - \sum_{j=0}^{d} \frac{\prod_{k: k \neq j} (t - a_k)}{\prod_{k: k \neq j} (a_j - a_k)} a_j^n = 0, \text{ for } n = 1, \ldots, d. \tag{6}
\]

Letting \(a_j := 1/(\mathbf{e}^T \mathbf{v}_j)\) in (6) and evaluating at \(t = 0\), we get (5):

\[
t^n = \sum_{j=0}^{d} \frac{\prod_{k: k \neq j} (a_j - a_k)^n}{\prod_{k: k \neq j} (\mathbf{e}^T \mathbf{v}_j - \mathbf{e}^T \mathbf{v}_k)} \left(\frac{1}{\mathbf{e}^T \mathbf{v}_j}\right)^n
\]

\[
= \sum_{j=0}^{d} \prod_{k: k \neq j} (1 - (\mathbf{e}^T \mathbf{v}_j) t) \prod_{k: k \neq j} (\mathbf{e}^T \mathbf{v}_j - \mathbf{e}^T \mathbf{v}_k) (\mathbf{e}^T \mathbf{v}_j)^{d-n}
\]

\[
\Rightarrow 0 = \sum_{j=0}^{d} \prod_{k: k \neq j} (\mathbf{e}^T \mathbf{v}_j - \mathbf{e}^T \mathbf{v}_k)^{d-n}, \text{ for } n = 1, \ldots, d.
\]

In the general case, when the genericity assumption \((\mathbf{e}^T \mathbf{v}_j \neq \mathbf{e}^T \mathbf{v}_k\) for all \(j \neq k\) fails, we can take \(K \subset \{0, 1, \ldots, d\}\) to be an index set of different poles \(t = 1/(\mathbf{e}^T \mathbf{v}_k)\), and for \(k \in K\), we let \(m_k := |\{j \in \{0, 1, \ldots, d\} : \mathbf{e}^T \mathbf{v}_j = \mathbf{e}^T \mathbf{v}_k\}|\), which is the order of the pole.
Corollary 3.5 ([BBD+10, Cor. 13]) Suppose that $J := \text{conv}\{v_0, v_1, \ldots, v_d\} \subset \mathbb{R}^d$. Then we have

\[
\int_J (c^T x)^n dx = d! \text{vol}(J) \frac{n!}{(n+d)!} \sum_{k \in K} \text{Res} \left( \frac{1}{(n+d)!} \frac{\epsilon + (c^T v_k)^n + d}{\prod_{j \in K \setminus \{k\}} (\epsilon + (c^T (v_k - v_j))^n), \epsilon = 0} \right),
\]

where $\text{Res}$ denotes the residue (here at $\epsilon = 0$).

If there are poles with high order, the residue can be calculated by the Laurent series expansion.

Next, we establish an affine generalization of Theorem 3.3.

Theorem 3.6 Suppose that $J := \text{conv}\{v_0, v_1, \ldots, v_d\} \subset \mathbb{R}^d$, we have

\[
d! \text{vol}(J) \frac{1}{\prod_{j=0}^d (1 - t(c^T v_j + b))} = \sum_{n=0}^{\infty} \left( \frac{n+d}{n!} \frac{1}{n!} \int_J (c^T x + b)^n dx \right) t^n.
\]

Proof We first prove an affine generalization of Lemma 3.2,

\[
\frac{(n+d)!}{n!} \int_J (c^T x + b)^n dx = d! \text{vol}(J) \sum_{k \in \mathbb{Z}_{\geq 0}^{d+1}, \|k\|_1 = n} \prod_{j=0}^d (c^T v_j + b)^{k_j}.
\]  

(7)

By the binomial theorem and Lemma 3.2, we have

\[
\frac{(n+d)!}{n!} \int_J (c^T x + b)^n dx = \frac{(n+d)!}{n!} \sum_{m=0}^n \binom{n}{m} b^{n-m} \int_J (c^T x)^m dx
\]

\[
= d! \text{vol}(J) \sum_{m=0}^n \frac{(n+d)!}{(n-m)!(m+d)!} b^{n-m} \sum_{k \in \mathbb{Z}_{\geq 0}^{d+1}, \|k\|_1 = m} \prod_{j=0}^d (c^T v_j + b)^{k_j}.
\]

Now we apply the binomial theorem to the right-hand-side of (7) and collect terms with the same degree of $b$:

\[
d! \text{vol}(J) \sum_{k \in \mathbb{Z}_{\geq 0}^{d+1}, \|k\|_1 = n} \sum_{m=0}^n b^{n-m} \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{d+1}, \|\alpha\|_1 = n-m} \prod_{j=0}^d \binom{k_j}{\alpha_j} (c^T v_j)^{k_j-\alpha_j}
\]

\[
= d! \text{vol}(J) \sum_{m=0}^n b^{n-m} \sum_{k \in \mathbb{Z}_{\geq 0}^{d+1}, \|k\|_1 = m} \prod_{\alpha \in \mathbb{Z}_{\geq 0}^{d+1}, \|\alpha\|_1 = n-m} \prod_{j=0}^d \binom{k_j + \alpha_j}{\alpha_j} (c^T v_j)^{k_j}.
\]
It can be shown to be the same as the right-hand side by the well-known generalized Vandermonde identity:

$$\sum_{\alpha \in \mathbb{Z}_{d+1}^d, \|\alpha\|_1 = n-m}^{d} \prod_{j=0}^{d} \left( k_j + \alpha_j \right) = \left( \|k\|_1 + n - m + d \right).$$

A short proof of this identity is by double counting the coefficient of $t^{n-m}$ on both sides of $\prod_{j=0}^{d} (1 + t)^{-(k_j+1)} = (1 + t)^{-(\|k\|_1 + d+1)}$.

Therefore, we can verify that (7) holds.

Then by multiplying (7) by $t^n$ and summing from 0 to $\infty$, we have

$$\sum_{n=0}^{\infty} \frac{(n+d)!}{n!} \int_J (c^T x + b)^n dx \cdot t^n$$

$$= d! \cdot \text{vol}(J) \sum_{n=0}^{\infty} t^n \sum_{k \in \mathbb{Z}_{d+1}^d, \|k\|_1 = n} (c^T v_0 + b)^k \cdots (c^T v_d + b)^k$$

$$= d! \cdot \text{vol}(J) \prod_{j=0}^{d} \sum_{k_j=0}^{\infty} (c^T v_j + b)^{k_j} t^{k_j}$$

$$= d! \cdot \text{vol}(J) \frac{1}{\prod_{j=0}^{d} (1 - t(c^T v_j + b))}.$$

Therefore, the result holds.

Using the same method as (3), we obtain the following corollary.

**Corollary 3.7** Suppose that $J := \text{conv}\{v_0, v_1, \ldots, v_d\} \subset \mathbb{R}^d$. Suppose further that $c^T v_j + b \neq c^T v_k + b$. Then we have

$$\int_J (c^T x + b)^n dx = d! \cdot \text{vol}(J) \frac{n!}{(n+d)!} \sum_{j=0}^{d} \prod_{k: k \neq j} (c^T v_j - v_k).$$

We can extend Theorem 3.6 to calculate the integration of a product of powers of $D$ affine forms. By replacing $tc$ by $\sum_{j=1}^{D} t_j c_j$ and $tb$ by $\sum_{j=1}^{D} t_j b_j$ in Theorem 3.6 and taking the expansion in powers $t_1^{\alpha_1} \cdots t_D^{\alpha_D}$, we obtain:

**Corollary 3.8** Suppose that $J := \text{conv}\{v_0, v_1, \ldots, v_d\} \subset \mathbb{R}^d$, we have

$$d! \cdot \text{vol}(J) \prod_{j=0}^{d} \frac{1}{(1 - t_1(c^T v_j + b_1) - \cdots - t_D(c^T v_j + b_D))}$$

$$= \sum_{\alpha \in \mathbb{Z}_{0}^{D}} \frac{((\|\alpha\|_1 + d)!)}{\alpha_1! \cdots \alpha_D!} \int_J (c^T v_0 + b_0)^{\alpha_1} \cdots (c^T v_d + b_d)^{\alpha_D} dx \cdot t_1^{\alpha_1} \cdots t_D^{\alpha_D}.$$
Next, we review the three algorithms implemented in [BBD+10].

The first algorithm is called the Taylor-expansion method. By Corollary 3.8, letting \( D := d \) and \( c_j := e_j, b_j := 0, \) for \( j = 1, \ldots, d, \) we can compute the integral \( \int_J x_1^{\alpha_1} \cdots x_d^{\alpha_d} \, dx \) of a monomial by multiplying \( \frac{d^! \text{vol}(J)}{(\|\alpha\|_1 + d)^!} \) by the coefficient of \( t_1^{\alpha_1} \cdots t_d^{\alpha_d} \) in the Taylor expansion of

\[
\frac{1}{\prod_{j=0}^d (1 - t^j v_j)}.
\]

Using this method, we can integrate a polynomial with fixed degree \( q \) in polynomial time (see [BBD+10, Proof of Cor. 3]).

The second algorithm is called the linear-form decomposition method. We first decompose arbitrary monomial as sums of powers of linear forms

\[
x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d} = \frac{1}{(\|\alpha\|_1)!} \sum_{0 \leq \beta \leq \alpha} (-1)^{\|\alpha\|_1 - \|\beta\|_1} \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_d}{\beta_d} (\beta_1 x_1 + \cdots + \beta_d x_d)^{\|\alpha\|_1},
\]

and then we use Corollary 3.5 to compute the integral of each linear form. This is also a polynomial-time algorithm for fixed degree \( q \) (See [BBD+10, Alternative proof of Cor. 3]).

The third algorithm is called the iterated-Laurent method. The equation (4) can also be viewed as an equation with respect to the variables \( c \). We know that \( \int_J x_1^{\alpha_1} \cdots x_d^{\alpha_d} \, dx \) is the coefficient of \( \frac{c_1^{\alpha_1} \cdots c_d^{\alpha_d}}{\alpha_1! \cdots \alpha_d!} \) in the Taylor expansion of the left-hand side of (4) \( \int_J e^{c^T x} \, dx \). By expanding the right-hand side of (4) into an iterated Laurent series with respect to the variables \( c_1, \ldots, c_d \), we can compute the integral by comparing the coefficient of \( \frac{c_1^{\alpha_1} \cdots c_d^{\alpha_d}}{\alpha_1! \cdots \alpha_d!} \) (see [BBD+10, Rem. 15]).

From the numerical experiment in [BBD+10], we know that for low dimensions (\( d \leq 5 \)), the iterated-Laurent method is faster than the two other methods; for high dimensions, the linear-form decomposition method is faster than the other two methods.

### 3.3 Symmetric multilinear form

A (multivariate) polynomial \( f(x) \) is \( q \)-homogeneous if \( f(\lambda x) = \lambda^q f(x) \). Besides representing a polynomial as a sum of monomials, we may also write a polynomial as a sum of homogeneous polynomials. [LA01] provides a nice formula for the integration of a \( q \)-homogeneous (\( q \) is a positive integer) polynomial on a simplex by associating with the symmetric multilinear form.

**Lemma 3.9** For a \( q \)-homogeneous polynomial \( f(x) : \mathbb{R}^d \to \mathbb{R} \), there exists a symmetric multilinear form \( H_f : ([\mathbb{R}^d]^q) \to \mathbb{R} \) by the polarization formula

\[
H_f(x_1, x_2, \ldots, x_q) = \frac{1}{2^{q!}} \sum_{\epsilon \in \{-1, 1\}^q} \epsilon_1 \epsilon_2 \cdots \epsilon_q f(\sum_{j=1}^q \epsilon_j x_j),
\]
such that $H_f(x, x, \ldots, x) = f(x)$.

Proof The symmetry follows from the definition of $H_f$ because any permutation between the $x_j$'s would result in the same $H_f$. Given $q$-homogeneous $f(x)$, we can easily check that $H_f(x, x, \ldots, x) = f(x)$:

$$H_f(x, x, \ldots, x) = \frac{1}{2^q q!} \sum_{\epsilon \in \{\pm 1\}^q} \epsilon_1 \epsilon_2 \ldots \epsilon_q f(\sum_{j=1}^q \epsilon_j x_j)$$

$$= f(x) \frac{1}{2^q q!} \sum_{\epsilon \in \{\pm 1\}^q} \epsilon_1 \epsilon_2 \ldots \epsilon_q (\sum_{j=1}^q \epsilon_j)^q$$

$$= f(x) \frac{1}{2^q q!} \sum_{k=0}^q (-1)^k \binom{q}{k} (q - 2k)^q$$

$$= f(x) \frac{1}{2^q q!} \sum_{j=0}^q \binom{q}{j} q^{q-j} (-2)^j \sum_{k=0}^q (-1)^k \binom{q}{k} k^j$$

$$= f(x).$$

The last equation follows from a well-known identity (see [Rui96], for example):

$$\sum_{k=0}^q (-1)^k \binom{q}{k} k^j = \begin{cases} 0, & 0 \leq j \leq q - 1; \\ q!(q-1)^q, & j = q. \end{cases}$$

$\square$

**Theorem 3.10 ([LA01, Thm. 2.1])** Suppose that $J := \text{conv}\{v_0, v_1, \ldots, v_d\} \subset \mathbb{R}^d$. Suppose that $H : (\mathbb{R}^d)^q \to \mathbb{R}$ is a symmetric multilinear form. Then we have

$$\int_J H(x, x, \ldots, x) dx = \frac{\text{vol}(J)}{(q+d)/q} \sum_{0 \leq i_1 \leq \cdots \leq i_q \leq d} H(v_{i_1}, v_{i_2}, \ldots, v_{i_q}). \tag{8}$$

Suppose that $f : \mathbb{R}^d \to \mathbb{R}$ is a $q$-homogeneous polynomial ($q \in \mathbb{Z}_{\geq 1}$), then

$$\int_J f(x) dx = \frac{\text{vol}(J)}{2q! q^{d+q}} \sum_{0 \leq i_1 \leq \cdots \leq i_q \leq d} \sum_{\epsilon \in \{\pm 1\}^q} \epsilon_1 \cdots \epsilon_q f(\sum_{j=1}^q \epsilon_j v_{i_j}). \tag{9}$$

Proof For (8), we make an affine bijection between $J$ and the standard $d$-simplex $\Delta_d$ via (1),

$$x \in J \iff t = B^{-1}(x - v_0) \in \Delta_d.$$
Then the integration becomes
\[
\int_J H(x, \ldots, x) \, dx = \left| \det(B) \right| \int_{\Delta_d} H(Bt + v_0, \ldots, Bt + v_0) \, dt
\]
\[
= d! \vol(J) \int_{\Delta_d} H \left( \sum_{j=1}^d t_j v_j + \left(1 - \sum_{j=1}^d t_j \right) v_0, \ldots, \sum_{j=1}^d t_j v_j + \left(1 - \sum_{j=1}^d t_j \right) v_0 \right) \, dt
\]
\[
= d! \vol(J) \sum_{\alpha \in \mathbb{Z}_{\geq 0}^d, ||\alpha|| = q} q! \prod_{j=1}^d \theta_j \int_{\Delta_d} \left( \frac{1}{\theta_1 \cdots \theta_d} \prod_{j=1}^d \prod_{j=1}^d (1 - t_1 - \cdots - t_d)^{\theta_j} \right) \, dt
\]
where the last equality follows from the multilinearity and symmetry of \( H \), and
\[
H(v_0^{\alpha_0}, \ldots, v_d^{\alpha_d}) := H(v_0, \ldots, v_0, \ldots, v_d, \ldots, v_d).
\]
Therefore, by Proposition 3.1, we get (8). For (9), we can use Lemma 3.9 to construct the associated \( H_f \). Then it follows from (8) directly.

By Theorem 3.10, we can integrate a polynomial with fixed degree \( q \) in polynomial time.

3.4 Fourier transformation

There is a Fourier-transformation method in [Bar91,Bar94] for integration of a class of exponential functions over a polytope in standard form. We can recover (4) via this method.

**Theorem 3.11 ([Bar91,Bar94])** Let \( P := \{ x \in \mathbb{R}_0^n : Ax = b \} \), where \( A \in \mathbb{R}^{m \times n} \) has rank \( A = m < n \), \( b \in \mathbb{R}^m \), and suppose that \( \dim P = n - m \). That is, \( P \) is full dimensional in the \((n - m)\)-dimensional hyperplane \( \{ x \in \mathbb{R}^n : Ax = b \} \). Then, for all \( c \in \mathbb{R}_0^n \), we have
\[
\int_P e^{-c^T x} \, dx = \sqrt{\det(AA^T)} \int_{\mathbb{R}^n} e^{2\pi i a^T y} \prod_{j=1}^n \frac{1}{2\pi i a_j + c_j} \, dy,
\]
where \( a_1, \ldots, a_n \) are the columns of \( A \).

**Proof** For \( \varphi : \mathbb{R}^n \to \mathbb{R} \), we denote the Fourier transform by \( \hat{\varphi}(\xi) := \int e^{-2\pi i \xi^T x} \varphi(x) \, dx \).

For
\[
g(x) := \begin{cases} e^{-c^T x}, & x \in \mathbb{R}_0^n ; \\ 0, & \text{otherwise}, \end{cases}
\]
we have
\[
\hat{g}(y) = \int_{\mathbb{R}^n} e^{-2\pi i y^T x} g(x) \, dx = \int_{\mathbb{R}_0^n} e^{-2\pi i y^T x - c^T x} \, dx = \prod_{j=1}^n \frac{1}{2\pi i y_j + c_j}.
\]
Choose \( x_0 \in \mathbb{R}^n \) such that \( Ax_0 = b \). Let \( f(x) := g(x + x_0) \). Then

\[
\hat{f}(y) = e^{2\pi i x_0^T y} \hat{g}(y) = e^{2\pi i x_0^T y} \prod_{j=1}^n \frac{1}{2\pi i y_j + c_j}.
\]

We can see that

\[
e^{2\pi i x_0^T y} \hat{g}(y) = \int_{L+x_0} g(x) dx = \int_L g(x+ x_0) dx = \int_L f(x) dx,
\]

where \( L := \{ x \in \mathbb{R}^n : Ax = 0 \} \) denotes the null space of \( A \).

By the formula

\[
\int_L f(x) dx = \int_{L^+} \hat{f}(y) dy,
\]

where \( L^+ := \{ A^T x : x \in \mathbb{R}^m \} \) denotes the row space of \( A \) (see [BS07, H03]), we obtain

\[
\int_{\Delta_d} e^{-c^T x} dx = \int_{L^+} \hat{f}(y) dy = \int_{L^+} \hat{f}(y) dy = \sqrt{\det(AA^T)} \int_{\mathbb{R}^m} e^{2\pi i b^T A^T z} \prod_{j=1}^n \frac{1}{2\pi i a_j^T z + c_j} dz = \sqrt{\det(AA^T)} \int_{\mathbb{R}^m} e^{2\pi i b^T z} \prod_{j=1}^n \frac{1}{2\pi i a_j^T z + c_j} dz,
\]

where the penultimate equation follows from a change of variables \( y := A^T z \).

We can connect the integration over \( \Delta_d \) in \( \mathbb{R}^d \) with the integration over \( \Delta_d' := \{ x = (x_0, x_1, x_2, \ldots, x_d) \in \mathbb{R}^{d+1} : \sum_{j=0}^d x_j = 1 \} \) in \( \mathbb{R}^{d+1} \), via

\[
\sqrt{d+1} \int_{\Delta_d} f(1 - \sum_{j=1}^d x_j, x_1, x_2, \ldots, x_d) dx = \int_{\Delta_d'} f(x_0, x_1, x_2, \ldots, x_d) dx.
\]

To obtain the above equation, we observe that the affine transformation \( \varphi : \Delta_d \to \Delta_d' \) given by

\[
\varphi(x) := \left[ \begin{array}{c} -1_d^T \\ 1_d \\ 0_d \end{array} \right] x + \left[ \begin{array}{c} 1 \\ 0_d \end{array} \right] =: Qx + \gamma,
\]

satisfies \( x \in \Delta_d \iff \varphi(x) \in \Delta_d' \). Therefore, by performing the affine transformation \( \varphi \), the ratio of the volumes is \( \sqrt{\det(Q^T Q)} \) (see [GK10], for example), which here becomes

\[
\sqrt{\det(Q^T Q)} = \sqrt{\det(1_d 1_d^T + I_d)} = \sqrt{d+1}.
\]
Corollary 3.12 ([Bar91, Bar94]) We can recover (4) from Theorem 3.11.

Proof We first show that for $\Delta'_d = \{ x \in \mathbb{R}^{d+1} : \sum_{j=0}^d x_j = 1 \}$ in $\mathbb{R}^{d+1}$, $c_j \neq c_k$ for all $j \neq k$,

$$\int_{\Delta'_d} e^{-\lambda x} dx = \sqrt{d+1} \sum_{j=0}^d \frac{e^{c_j}}{\prod_{k: k \neq j}(c_j - c_k)}.$$ 

Letting $P := \Delta'_d$ in Theorem 3.11, for $c > 0$, we obtain

$$\int_{\Delta'_d} e^{-c\lambda x} dx = \sqrt{d+1} \int_R \prod_{j=0}^d e^{2\pi i y} dy .$$

We can use Cauchy’s residue theorem to calculate the integral. Take the contour $C$ consisting of the segment $[-R, R]$ ($\pi R > \max c_j$) and the upper semicircle $R(\cos \theta + i \sin \theta)$ ($\theta \in [0, \pi]$). Assuming that $c_j \neq c_k$ (for $j \neq k$), we have that the function has the isolated singularities $\frac{ic_j}{2\pi}$. By the residue theorem, we have

$$\int_C e^{2\pi i z} dz = 2\pi i \sum_{j=0}^d \frac{e^{2\pi i z}}{\prod_{k=0}^d (2\pi i z + c_k)} \frac{ic_j}{2\pi}$$

$$= \sum_{j=0}^d \lim_{z \to \frac{i c_j}{2\pi}} \prod_{k=0}^d e^{2\pi i z} (2\pi i z + c_k)$$

$$= \sum_{j=0}^d e^{-c_j} \prod_{k: k \neq j}(c_k - c_j)$$

In general, assume that $K \subset \{0, 1, \ldots, d\}$ is the index set of different singularity $\frac{ic_j}{2\pi}$, and for $k \in K$, let $m_k := |\{ j \in \{0, 1, \ldots, d\} : c_j = c_k \}|$. Then

$$\int_C \frac{e^{2\pi i z}}{\prod_{k=0}^d (2\pi i z + c_k)} dz = 2\pi i \sum_{k \in K} e^{2\pi i z} \frac{ic_k}{2\pi}$$

$$= \sum_{k \in K} \lim_{z \to \frac{i c_k}{2\pi}} \prod_{j=0}^d e^{2\pi i z} (z + c_j)$$

$$= \sum_{k \in K} e^{z} \frac{e^{i c_k}}{\prod_{j=0}^d (z + c_j)}.$$ (11)

Also we have

$$\int_C \frac{e^{2\pi i z}}{\prod_{k=0}^d (2\pi i z + c_k)} dz = \int_{\pi R}^R \frac{e^{2\pi iy}}{\prod_{k=0}^d (2\pi iy + c_k)} dy + \int_{|z|=R} \frac{e^{2\pi iz}}{\prod_{k=0}^d (2\pi iz + c_k)} dz.$$ (12)

For $|z| = R$, we have $|e^{2\pi iz}| = |e^{2\pi i R \cos \theta - 2\pi i R \sin \theta}| = e^{-2\pi R \sin \theta} \leq 1$. Then we have

$$\left| \int_{|z|=R} \frac{e^{2\pi iz}}{\prod_{k=0}^d (2\pi iz + c_k)} dz \right| \leq \frac{\pi R}{(\pi R)^{d+1}} = \frac{1}{(\pi R)^d}.$$
Therefore, by taking $R \to +\infty$ in (12), we obtain
\[
\int_{\mathbb{R}} \frac{e^{2\pi i y}}{\prod_{k=0}^{d}(2\pi i y + c_k)} dy = \sum_{j=0}^{d} \prod_{k: \ k \neq j}(c_k - c_j).
\]
Thus, for $c < 0$, we have
\[
\int_{\Delta_d} e^{\mathbf{c}^T \mathbf{x}} d\mathbf{x} = \sqrt{d + 1} \sum_{j=0}^{d} \prod_{k: \ k \neq j}(c_j - c_k).
\]
For general $c$, there exists $M > \max c_j$ such that $c - M \mathbf{1} < 0$. Thus
\[
\int_{\Delta_d} e^{\mathbf{c}^T \mathbf{x}} d\mathbf{x} = e^M \int_{\Delta_d} e^{(e^{-M \mathbf{1}})\mathbf{x}} d\mathbf{x} = \sqrt{d + 1} \sum_{j=0}^{d} \prod_{k: \ k \neq j}(c_j - c_k).
\]

Then by affine transformation, we have
\[
\mathbf{x} \in J \quad \iff \quad \mathbf{B}^{-1}(\mathbf{x} - \mathbf{v}_0) \in \Delta_d \quad \iff \quad \mathbf{Q}\mathbf{B}^{-1}(\mathbf{x} - \mathbf{v}_0) + \gamma \in \Delta_d',
\]
where $\mathbf{B} = [\mathbf{v}_1 - \mathbf{v}_0, \ldots, \mathbf{v}_d - \mathbf{v}_0]$ from (1), $\mathbf{Q} = \begin{bmatrix} I_d & -1_d \end{bmatrix}$, and $\gamma = \begin{bmatrix} 0_d & 1 \end{bmatrix}$ from (10).

So, for $\tilde{\mathbf{c}}^T \mathbf{v}_j \neq \tilde{\mathbf{c}}^T \mathbf{v}_k$ $(j \neq k)$, we have
\[
\int_{\mathbb{R}^d} e^{\mathbf{c}^T \mathbf{x}} d\mathbf{x} = |\det(\mathbf{B})| \int_{\Delta_d} e^{\mathbf{c}^T (\mathbf{B}^{-1}\mathbf{y} + \mathbf{v}_0)} d\mathbf{y} = |\det(\mathbf{B})| \int_{\Delta_d} e^{\mathbf{c}^T (\mathbf{B}^{-1}\mathbf{y} + \mathbf{v}_0)} d\mathbf{y}
\]
\[
= \frac{d! \vol(J)}{\sqrt{d+1}} \int_{\Delta_d} e^{\mathbf{c}^T (\mathbf{B}^{-1}\mathbf{y} + \mathbf{v}_0) \sum_{j=0}^{d} y_j} d\mathbf{y}
\]
\[
= \frac{d! \vol(J)}{\sqrt{d+1}} \int_{\Delta_d} e^{\mathbf{c}^T (\mathbf{B}^{-1}\mathbf{y}) \mathbf{v}_j} d\mathbf{y}
\]
\[
= d! \vol(J) \sum_{j=0}^{d} \prod_{k: \ k \neq j}(\mathbf{e}^{\mathbf{c}^T \mathbf{v}_j} - \mathbf{e}^{\mathbf{c}^T \mathbf{v}_0}).
\]

Next, we provide a corollary that is useful for Theorem 4.12. It follows from the same proof idea as Corollary 3.12 to connect the integration over $\Delta_d$ with the integration over $\Delta_d'$, but the residues in (11) are computed differently.

We consider the integration of the exponential function $e^{\mathbf{c}^T \mathbf{x}}$ that has exactly two distinct values when evaluated at the vertices of the simplex: one for the vertex $\mathbf{v}_0$, and another for all the other vertices.

**Corollary 3.13** Suppose that $J := \text{conv}\{\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_d\} \subset \mathbb{R}^d$. Suppose further that $u := \tilde{\mathbf{c}}^T \mathbf{v}_0 - \tilde{\mathbf{c}}^T \mathbf{v}_j$ is a nonzero constant for $j = 1, \ldots, d$, then
\[
\int_{\mathbb{R}^d} e^{\mathbf{c}^T \mathbf{x}} d\mathbf{x} = d! \vol(J) \frac{e^{\mathbf{c}^T \mathbf{v}_0}}{u^d} \left( e^u - \sum_{j=0}^{d-1} \frac{u^j}{j!} \right).
\]
Proof We claim that for \( \Delta'_d \) and \( c \) satisfies \( c_j = c_0 + u \) for some \( u \neq 0 \) and \( j = 1, \ldots, d \),
\[
\int_{\Delta'_d} e^{-c^T x} dx = \sqrt{d + 1} \cdot \frac{e^{-(c_0 + u)}}{u^d} \left( e^u - \sum_{j=0}^{d-1} \frac{u^j}{j!} \right).
\]

Using the same proof as for Corollary 3.12 to obtain equation (11), we obtain the above claim by computing the residues in equation (11)
\[
\frac{1}{\sqrt{d + 1}} \int_{\Delta'_d} e^{-c^T x} dx = \text{Res} \left( \prod_{k=0}^{d-1} \frac{e^x}{x + c_k}, -c_0 \right) + \text{Res} \left( \prod_{k=0}^{d-1} \frac{e^x}{x + (c_0 + u)}, -c_0 + u \right)
= \lim_{z \to -c_0} \frac{e^z}{z + c_0 + u} + \lim_{z \to -c_0 + u} \frac{1}{(d-1)!} \frac{1}{z - (c_0 + u)} \frac{d^{d-1}}{dz^{d-1}} \left( \frac{e^z}{z + c_0} \right)
= \frac{e^{-c_0}}{u^d} - \frac{e^{-(c_0 + u)}}{u^d} \sum_{j=0}^{d-1} \frac{u^j}{j!} = \frac{e^{-(c_0 + u)}}{u^d} \left( e^u - \sum_{j=0}^{d-1} \frac{u^j}{j!} \right).
\]

So, for \( c^T v_0 - c^T v_j = u \) (i.e., \( c^T B = -u I \)), applying the affine transformations (1) and (10), we have
\[
\int_J e^{c^T x} dx = |\text{det}(B)| \int_{\Delta'_d} e^{c^T (By + v_0)} dy = |\text{det}(B)| \int_{\Delta'_d} e^{-(u - (1^T y) + c^T v_0)} dy
= \frac{d! \text{vol}(J)}{\sqrt{d + 1}} \int_{\Delta'_d} e^{-c^T y} dy \quad (\text{let } c \text{ satisfy } c_j - c_0 = u, c_0 = -c^T v_0)
= d! \text{vol}(J) \frac{e^{-c^T v_0 - u}}{u^d} \left( e^u - \sum_{j=0}^{d-1} \frac{u^j}{j!} \right).
\]
The last equation follows from the above claim. \( \square \)

3.5 Cubature-rule formulae

In this subsection, we survey the calculation of multidimensional integrals over a simplex via approximate integration formulae, a.k.a., cubature rules (see [Str71, CR93], [GM78]). These formulae are of the form:
\[
\int_{\mathcal{R}_d} f(x) dx = \sum_{j=1}^{M} \lambda_j f(w_j) + Rf, \quad (13)
\]
where \( \mathcal{R}_d \) is a given region in \( \mathbb{R}^d \), the points \( w_j \in \mathbb{R}^d \), the coefficients \( \lambda_j \in \mathbb{R} \) are given, and \( Rf \) is the approximation error. We call \( \sum_{j=1}^{M} \lambda_j f(w_j) \) in (13) an integration formula of degree \( q \) if the approximation error \( Rf = 0 \) for all polynomials \( f : \mathbb{R}^d \to \mathbb{R} \) of degree at most \( q \). For the univariate case, i.e., \( d = 1 \), the theory of approximate integration is well-established (these formulae in one-dimension are also referred to as quadrature formulae, see [DR07]).

For integration of polynomials of degree at most \( q \), we can leverage an approximate integration formula of degree \( q \) over a simplex. Because affine...
transformation does not change the degree of the polynomial, we focus on integration over the standard simplex \( \Delta_d \) in the following and present two formulae for general \( d \).

[GM78] gives an invariant integration formula under all affine transformations of \( \Delta_d \) onto itself, i.e., under the mappings \( \varphi_\sigma: (x_1, \ldots, x_d) \to (x_{\sigma_1}, \ldots, x_{\sigma_d}) \), where \( \sigma = (\sigma_0, \sigma_1, \ldots, \sigma_d) \) is a permutation of \((0,1,\ldots,n)\) and \( x_0 = 1 - \sum_{j=1}^d x_j \). They use a combinatorial identity and consider the basis

\[
\left\{ (1 - \sum_{j=1}^d x_j)^{\alpha_0} x_1^{\alpha_1} \cdots x_d^{\alpha_d} : \|\alpha\|_1 = q, \alpha \in \mathbb{Z}_{\geq 0}^d \right\},
\]

instead of the standard monomial basis

\[
\left\{ x_1^{\alpha_1} \cdots x_d^{\alpha_d} : \|\alpha\|_1 \leq q, \alpha \in \mathbb{Z}_{\geq 0}^d \right\}
\]

for polynomials of degree at most \( q \).

**Theorem 3.14 ([GM78, Thm. 4])** Let \( q = 2s + 1, s \in \mathbb{Z}_{\geq 0} \). Then

\[
\sum_{j=0}^s (-1)^j 2^{-2s} (q + d - 2j)^q \sum_{\|k\|_1 = s-j, \ k \in \mathbb{Z}_{\geq 0}^d} f \left( \frac{2k_1 + 1}{q + d - 2j}, \ldots, \frac{2k_d + 1}{q + d - 2j} \right)
\]

\[
= \int_{\Delta_d} f(x)dx - Rf \quad (14)
\]

is an integration formula of degree \( q \).

This formula is invariant under \( \varphi_\sigma \) because

\[
\sum_{\|k\|_1 = s-j, \ k \in \mathbb{Z}_{\geq 0}^d} f \left( \frac{2k_1 + 1}{q + d - 2j}, \ldots, \frac{2k_d + 1}{q + d - 2j} \right) = \sum_{\|k\|_1 = s-j, \ k \in \mathbb{Z}_{\geq 0}^d} f \left( \left\{ \frac{2k + 1}{q + d - 2j} \right\} \right),
\]

where for any point \( y = (y_1, \ldots, y_d) \in \mathbb{R}^d \), we define \( \{(1 - \sum_{j=1}^d y_j, y)\} \) as the image of all points which are images of \( y \) under the mappings \( \varphi_\sigma \), and we denote \( f((y)) := \sum_{w \in \{y\}} f(w) \).

For example, in the case \( s = 0 \), (14) reduces to the formula “\( T_d : 1-1 \)” of [Str71, p. 307]:

\[
\frac{1}{d} f \left( \frac{1}{\Delta^1}, \ldots, \frac{1}{\Delta^1} \right) = \int_{\Delta_d} f(x)dx - Rf.
\]

In the case \( s = 1 \), (14) reduces to the formula “\( T_d : 3-1 \)” of [Str71, p. 308]:

\[
\frac{(d+3)^2}{d(d+1)} f \left( \left\{ \frac{3}{\Delta^1}, \frac{1}{\Delta^1}, \ldots, \frac{1}{\Delta^1} \right\} \right) - \frac{(d+1)^2}{d(d+2)} f \left( \frac{1}{\Delta^1}, \ldots, \frac{1}{\Delta^1} \right) = \int_{\Delta_d} f(x)dx - Rf.
\]

In general, (14) in Theorem 3.14 requires the evaluation of \( f \) at \( \sum_{j=0}^s (s-j+d)^4 = (s+d+1) \) points.

By a composition of approximate integration formulae in one dimension, [Str71] gives another formula, called the “conical product formula”. 
Theorem 3.15 ([Str71, pp. 28–31]) There exist \( d \) many approximate integration formulae in one dimension of degree \( 2s + 1 \):

\[
\int_0^1 (1 - y_k)^{d-k} f(y_k) dy_k = \sum_{j=1}^{s+1} \lambda_{k,j} f(w_{k,j}) + Rf, \quad \text{for } k = 1, \ldots, d. \tag{15}
\]

Therefore, we can obtain the conical product formula of degree \( 2s + 1 \) for \( \int_{\Delta_d} f(x) dx \) with the evaluation of \( f \) at \((s+1)^d\) points \( w_{j_1,j_2,\ldots,j_d} = (\nu_{j_1}, \nu_{j_2}, \ldots, \nu_{j_2,\ldots,j_d}) \) and the corresponding coefficients \( \lambda_{j_1,j_2,\ldots,j_d} = \lambda_{1,j_1} \lambda_{2,j_2} \cdots \lambda_{d,j_d} \), where

\[
\nu_{j_1,j_2,\ldots,j_d} = (1 - w_{1,j_1})(1 - w_{2,j_2}) \cdots (1 - w_{k-1,j_{k-1}}) w_{k,j_k},
\]

for \( k = 1, \ldots, d, 1 \leq j_k \leq s + 1 \).

Proof Recall the Gauss-Jacobi quadrature formula of degree \( 2s + 1 \) (see [HT13, GST19])

\[
\int_{-1}^1 (1 - x)^\alpha (1 + x)^\beta f(x) dx = \sum_{j=1}^{s+1} \lambda_j f(w_j) + Rf,
\]

where \( w_1, \ldots, w_s, w_{s+1} \) are the roots of the Jacobi polynomial

\[
P_{s+1}^{(\alpha,\beta)}(x) := \frac{(-1)^{s+1}}{2^{s+1}(s+1)!} \frac{d^{s+1}}{dx^{s+1}} \left( (1 - x)^\alpha (1 + x)^\beta (1 - x^2)^{s+1} \right),
\]

and

\[
\lambda_j := \int_{-1}^1 (1 - x)^\alpha (1 + x)^\beta \prod_{k \neq j} \frac{x - w_k}{w_j - w_k} dx
\]

\[
= \frac{\Gamma(s+1+\alpha+1)\Gamma(s+1+\beta+1)}{\Gamma(s+1+\alpha+\beta+1)(s+1)!} \frac{2^{\alpha+\beta+1}}{(1-w_j^2)^{p_{s+1}^{(\alpha,\beta)}(w_j)}}.
\]

where \( p_{s+1}^{(\alpha,\beta)}(w_j) \) is the derivative of \( P_{s+1}^{(\alpha,\beta)}(x) \) (with respect to \( x \)) evaluated at \( w_j \).

For an arbitrary interval \([a, b]\),

\[
\int_a^b (b - x)^\alpha (x - a)^\beta f(x) dx
\]

\[
= \int_{-1}^1 (1 - t)^\alpha (1 + t)^\beta f \left( \frac{(b-a) t + (a+b)}{2} \right) \left( \frac{b-a}{2} \right)^{\alpha+\beta+1} dt
\]

\[
= \sum_{j=1}^{s+1} \left( \frac{b-a}{2} \right)^{\alpha+\beta+1} \lambda_j f \left( \frac{(b-a) w_j + (a+b)}{2} \right) + Rf.
\]

Thus, letting \( a = 0, b = 1, \alpha = d - k \) for \( k = 1, \ldots, d, \beta = 0 \), we obtain (15).

Now we rewrite \( \int_{\Delta_d} f(x) dx \) via iterated univariate integration:

\[
\int_{\Delta_d} f(x) dx = \int_0^1 \int_0^{1-x_1} \cdots \int_0^{1-x_1-\cdots-x_{d-1}} f(x) dx_d \cdots dx_2 dx_1.
\]
Then we apply the transformation \( x_1 = y_1, x_2 = (1-x_1)y_2, \ldots, x_d = (1-x_1-\cdots-x_{d-1})y_d \). Notice that \( 1 - \sum_{k=1}^d x_k = \prod_{k=1}^d (1-y_k) \) and \( x_j = y_j \prod_{k=1}^{j-1} (1-y_k) \). The Jacobian determinant of the transformation is
\[
(1-y_1)^{d-1}(1-y_2)^{d-2}\ldots(1-y_{d-1}).
\]
So the integration becomes
\[
\int_{\Delta_d} f(x) \, dx = \int_0^1 \int_0^1 \cdots \int_0^1 (1-y_1)^{d-1}(1-y_2)^{d-2}\ldots(1-y_{d-1}) f(x) \, dy_d \ldots dy_2 dy_1.
\]
Notice that \( x_1^{\alpha_1} \cdots x_d^{\alpha_d} = \prod_{j=1}^d y_j^{\alpha_j} (1-y_j)^{\alpha_j+1+\cdots+\alpha_d} \). Thus, the degree of the integrand with respect to \( y_j \) is \( \alpha_j+\cdots+\alpha_d \), which is at most the degree of \( f \). Applying (15) for \( k = d, d-1, \ldots, 1 \) sequentially, we can obtain the conical product formula of degree \( 2s \).

For example, in the case \( d = 2 \), the conical product formula of degree \( 2s + 1 \) requires \((s + 1)^2\) points, which is no more than \( \binom{s+3}{2} = \frac{(s+3)(s+2)(s+1)}{6} \)
employed by (14). In the case \( s = 0 \), the conical product formula reduces to the formula “\( T_d : 1\)-1” of [Str71, p. 307]. In the case \( s = 1 \), following the proof of Theorem 3.15, we obtain a formula different from (14):
\[
\int_{\Delta_2} f(x) dx = \sum_{j_1=1}^2 \sum_{j_2=1}^2 \lambda_{1,j_1} \lambda_{2,j_2} f(w_{1,j_1}, (1-w_{1,j_1})w_{2,j_2}), \quad \text{with}
\]
\[\lambda_{1,1} := \frac{2+\sqrt{3}}{36}, \quad w_{1,1} := \frac{4-\sqrt{3}}{10}, \quad \lambda_{1,2} := \frac{2-\sqrt{3}}{36}, \quad w_{1,2} := \frac{2+\sqrt{3}}{10}, \quad \lambda_{2,1} := \frac{1}{2}, \quad w_{2,1} := \frac{1}{2} \left( 1 + \sqrt{\frac{3}{4}} \right), \quad \lambda_{2,2} := \frac{1}{2}, \quad w_{2,2} := \frac{1}{2} \left( 1 - \sqrt{\frac{3}{4}} \right).\]

For larger \( s \), the exact expressions for the points and weights are tedious, but we can work them out numerically. For example, [Str71, p. 314] has “\( T_2 : 7\)-1” for \( d = 2, s = 3 \) (16 points) and [Str71, p. 315] has “\( T_3 : 7\)-1” for \( d = 3, s = 3 \) (64 points). When \( d \) is large, the conical product formula of degree \( 2s + 1 \) requires \((s + 1)^d\) points, which is more than \( \binom{s+d+1}{d} \) points in (14).

The minimum number of points required in an integration formula of degree \( q \) is still open for general \( q \). There is a lower bound \( \binom{s+d}{d} \) [Str71, p. 118–120] for a formula of degree \( 2s \). There is a small improvement for the odd degree case \( 2s + 1 \). [GM78] conjectures that Theorem 3.14 has the minimum number of points for a formula of degree \( 2s + 1 \) when \( d \geq 2s \). The coefficients in the conical product formula are always positive and sum to \( 1/d! \), which is a desirable property for numerical stability. As mentioned in [GM78], a significant fraction of the weights in (14) are negative, which might amplify the roundoff errors in the approximation.

Numerically, there are also adaptive algorithms based on a cubature rule and a subdivision strategy to automatically achieve the desired precision of an integral over a general simplex; see [GC03].

The cubature formulae problem is also closely related to the symmetric tensor decomposition and Waring’s problem; see [Col15, CGLM08]. We can
view the construction of a cubature formula of degree \( q \) as the construction of \( \lambda_j, w_j \) satisfying

\[
\sum_{j=1}^{r} \lambda_j p_k(w_j) = \int_{\Delta_d} p_k(x) \, dx := A(p_k), \quad k = 1, \ldots, \binom{d+q}{q},
\]

where the \( p_k \) form a basis of the polynomials with degree at most \( q \).

A tensor \( T \in \mathbb{R}^{(d+1)\times \cdots \times (d+1)} \) is called symmetric if \( t_{j_\sigma(1) \ldots j_\sigma(q)} = t_{j_1 \ldots j_q} \) for all permutations \( \sigma \) on \( \{1, \ldots, q\} \). We construct the following tensor \( T = [t_{j_1 \ldots j_q}] \in \mathbb{R}^{(d+1)\times \cdots \times (d+1)} \), \( j_k \in \{0, 1, \ldots, d\} \) such that

\[
t_{j_1 \ldots j_q} = \int_{\Delta_d} x_{j_1} \cdots x_{j_q} \, dx, \quad \text{where, } x_0 = 1 - \sum_{j=1}^{d} x_j = 1 - \|x\|_1.
\]

It is easy to see that \( T \) is symmetric because

\[
t_{j_1 \ldots j_q} = \int_{\Delta_d} x_0^{\alpha_0} \cdots x_d^{\alpha_d} \, dx,
\]

where \( \alpha_k, k \in \{0, 1, \ldots, d\} \) is the number of index \( k \) appearing in \( j_1, \ldots, j_q \), which does not change under the permutations \( \sigma \). And because \( \sum_{k=0}^{d} \alpha_k = q \), a cubature formula of degree \( q \) shows that

\[
t_{j_1 \ldots j_q} = \sum_{j=1}^{r} \lambda_j (w_j)_{j_1} \cdots (w_j)_{j_q},
\]

which yields a rank-\( r \) symmetric tensor decomposition of \( T \):

\[
T = \sum_{j=1}^{r} \lambda_j \left(1 - \|w_j\|_1, w_j \right) \otimes \left(1 - \|w_j\|_1, w_j \right) \cdots \otimes \left(1 - \|w_j\|_1, w_j \right). \tag{q times}
\]

On the other hand, if there exists a rank-\( r \) symmetric tensor decomposition of \( T \) with \( y_j \neq 0 \):

\[
T = \sum_{j=1}^{r} \lambda_j \underbrace{y_j \otimes y_j \cdots \otimes y_j}_{q \text{ times}}.
\]

We can scale \( y_j \) to satisfy \( \|y_j\|_1 = 1 \), i.e.,

\[
T = \sum_{j=1}^{r} \lambda_j \frac{y_j}{\|y_j\|_1} \otimes \frac{y_j}{\|y_j\|_1} \cdots \otimes \frac{y_j}{\|y_j\|_1} \tag{q times}
\]

Therefore, the construction of a cubature formula with minimum points is equivalent to the calculation of the symmetric rank of the corresponding tensor \( T \).
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[CGLM08] points out the equivalence between symmetric tensor and homogeneous polynomials in $\mathbb{R}[x_0, x_1, \ldots, x_d]^q$. The associated homogeneous polynomial to the symmetric tensor $T$ is

$$p_T(x) = \sum_{j_1, \ldots, j_q} t_{j_1, \ldots, j_q} x_0^{\alpha_0(j)} x_1^{\alpha_1(j)} \ldots x_d^{\alpha_d(j)}$$

$$= \sum_{\|\alpha\|_1 = q} q! \alpha_0! \alpha_1! \ldots \alpha_d! t_{\alpha_0 \alpha_1 \ldots \alpha_d} x_0^{\alpha_0} x_1^{\alpha_1} \ldots x_d^{\alpha_d},$$

where $\alpha_k(j), k \in \{0, 1, \ldots, d\}$ is the number of index $k$ appearing in $j_1, \ldots, j_q$, and $t_{\alpha_0 \alpha_1 \ldots \alpha_d} := \int_{\Delta_d} x_0^{\alpha_0} \ldots x_d^{\alpha_d} dx$.

The symmetric tensor decomposition is closely related to secant varieties of the Veronese variety if the field is $\mathbb{C}$. Consider the map from a vector to a $k$-th power of a linear form:

$$\nu_{n,k} : \mathbb{C}^n \rightarrow \mathbb{C}[x_1, \ldots, x_n]^k$$

$$w \rightarrow (w_1 x_1 + \ldots + w_n x_n)^k.$$  

The image $\nu_{n,k}(\mathbb{C}^n)$ is called the Veronese variety and is denoted by $V_{n,k}$. Recall the equivalence between symmetric tensor and homogeneous polynomials. We see that a symmetric tensor of rank 1 corresponds to a point on the Veronese variety. A symmetric tensor of rank at most $r$ lies in the linear space spanned by $r$ points of the Veronese variety. The closure of the union of all linear spaces spanned by $r$ points of the Veronese variety $V_{n,k}$ is called the $(r-1)$-th secant variety of $V_{n,k}$.

Therefore, the construction of a cubature formula with minimum points is equivalent to decompose the corresponding homogeneous polynomial to a sum of powers of linear form with minimum number of summands, which is known as the polynomial Waring problem.

4 Comparing naïve and perspective relaxations

In this section, we present some concrete results comparing volumes of naïve and perspective relaxations. Quantities of interest are the cut-off amount $\text{vol}(P_0(f,J)) - \text{vol}(P(f,J))$ and the cut-off ratio $\frac{\text{vol}(P_0(f,J)) - \text{vol}(P(f,J))}{\text{vol}(P(f,J))}$. For a family of examples, understanding when the cut-off ratio is bounded below by a positive quantity or when it tends to zero and at what rate, gives us information on when the naïve is an adequate approximation of the perspective relaxation.

4.1 $q$-homogeneous functions

Suppose that $f(x)$ is $q$-homogeneous, i.e., $f(\lambda x) = \lambda^q f(x)$ for $\lambda \geq 0$ ($\lambda = 0$ implies that $f(0) = 0$). Then, for a general simplex, we can compute the
volume of the naïve relaxation and compare it to the volume of the perspective relaxation.

**Lemma 4.1** Suppose that $f$ is continuous, $q$-homogeneous ($q \geq 1$) and convex on $\text{conv}(J \cup \{0\})$ where the $d$-simplex $J := \text{conv}\{v_0, v_1, \ldots, v_d\} \subset \mathbb{R}^d_{\geq 0} \setminus \{0\}$. Then

$$\text{vol}(P^0(f, J)) = \frac{1}{(d + 2)!} \left| \det \begin{bmatrix} v_0 & v_1 & \ldots & v_d \\ 1 & 1 & \ldots & 1 \end{bmatrix} \right| \sum_{j=0}^d f(v_j) - \frac{1}{q + d + 1} \int_J f(x) dx.$$ 

**Proof** By Theorem 2.3, we have

$$\text{vol}(P^0(f, J)) = \frac{1}{(d + 2)!} \left| \det \begin{bmatrix} v_0 & v_1 & \ldots & v_d \\ 1 & 1 & \ldots & 1 \end{bmatrix} \right| \sum_{j=0}^d f(v_j) - \int_0^1 z^d \int_J f(zx) dx dz.$$ 

Because $f(x)$ is $q$-homogeneous, we have $f(zx) = z^q f(x)$, and we obtain

$$\int_0^1 z^d \int_J f(zx) dx dz = \int_{0 \leq z \leq 1} z^{q+d} dz \int_{x \in J} f(x) dx = \frac{1}{q + d + 1} \int_J f(x) dx.$$ 

The result follows. \qed

**Theorem 4.2** Suppose that $f$ is continuous, $q$-homogeneous ($q \geq 1$) and convex on $\text{conv}(J \cup \{0\})$ where the $d$-simplex $J := \text{conv}\{v_0, v_1, \ldots, v_d\} \subset \mathbb{R}^d_{\geq 0} \setminus \{0\}$. Then

$$\text{vol}(P^0(f, J)) - \text{vol}(P(f, J)) = \frac{q - 1}{(q + d + 1)(d + 2)} \int_J f(x) dx.$$ 

**Proof** Recall from Theorem 2.2 that the volume of the perspective relaxation $P(f, J)$ is

$$\text{vol}(P(f, J)) = \frac{1}{(d + 2)!} \left| \det \begin{bmatrix} v_0 & v_1 & \ldots & v_d \\ 1 & 1 & \ldots & 1 \end{bmatrix} \right| \sum_{j=0}^d f(v_j) - \frac{1}{d + 2} \int_J f(x) dx.$$ 

Combining this with Lemma 4.1, the result follows. \qed

**Remark 4.3** Concerning Theorem 4.2, as a reality check: (i) when $q = 1$, we obtain $\text{vol}(P^0(f, J)) = \text{vol}(P(f, J))$, which agrees with the fact that both of these volumes are zero; (ii) when $q > 1$, because $zf(x) \geq f(zx) = z^q f(x)$ for any $z \in [0, 1]$, we have that $f$ is nonnegative, which implies from the theorem that $\text{vol}(P^0(f, J)) \geq \text{vol}(P(f, J))$ (which we know anyway because $P(f, J) \subset P^0(f, J)$).

Theorem 4.2 is a broad generalization of the following key result of [LSS22] giving an expression for the difference of volumes for convex power functions.
Corollary 4.4 ([LSS22, Cor. 14]) For $d := 1$, $J := [\ell, u] \ (u > \ell > 0)$, and $f(x) := x^q$, with $q > 1$, we have

\[
\text{vol}(P_0(f, J)) - \text{vol}(P(f, J)) = \frac{(q-1)(u^{q+1} - \ell^{q+1})}{3(q+2)(q+1)}.
\]

For the remainder of Section 4.1, as compared to the hypotheses of Theorem 4.2, we restrict our attention to $f(x) := (c^T x)^q \ (c \neq 0)$ satisfying either

(i) $q > 1$ ($q \in \mathbb{R}$) and $c^T v_j \geq 0$, or
(ii) $q$ is an even integer (without the assumption $c^T v_j \geq 0$).

Note that (i) and (ii) each ensure that $f(x)$ is convex on $J$.

We establish in these cases that the cut-off ratio has a positive lower bound. This demonstrates that in these cases, the excess volume of the naïve relaxation, as compared to the perspective relaxation, is substantial.

4.1.1 Case (i)

$f(x) = (c^T x)^q \ (q > 1$ with further conditions on $c$)

Suppose that $q > 1$, and $c^T v_j \geq 0$ for $j = 0, 1, \ldots, d$.

Lemma 4.5 For $J := \text{conv}\{v_0, v_1, \ldots, v_d\}$, if $c \neq 0$, $q \geq 1$ and $c^T v_j \geq 0$, then

\[
\int_J (c^T x)^q dx \geq d! \text{vol}(J) \frac{\Gamma(q+1)}{\Gamma(q+d+1)} \sum_{j=0}^{d} (c^T v_j)^q.
\]

The inequality becomes tight when $\frac{c^T v_j}{c^T v_k} \to 0$ for all $j \neq k$, where $c^T v_k = \max_j c^T v_j$.

Proof $x \in J \Leftrightarrow y = B^{-1}(x - v_0) \in \Delta_d$, where $B := [v_1 - v_0, \ldots, v_d - v_0]$.

\[
\int_J (c^T x)^q dx = d! \text{vol}(J) \int_{\Delta_d} (c^T By + c^T v_0)^q dy
\]

\[
= d! \text{vol}(J) \int_{\Delta_d} [(c^T v_1)y_1 + \cdots + (c^T v_d)y_d + (c^T v_0)(1 - y_1 - \cdots - y_d)]^q dy
\]

\[
\geq d! \text{vol}(J) \sum_{j=0}^{d} \int_{\Delta_d} (c^T v_j)^q(y_j)^q dy, \quad \text{with } y_0 := 1 - y_1 - \cdots - y_d
\]

\[
= d! \text{vol}(J) \frac{\Gamma(q+1)}{\Gamma(q+d+1)} \sum_{j=0}^{d} (c^T v_j)^q.
\]

The inequality follows from $(\sum_{j=1}^n x_j)^q \geq \sum_{j=1}^n x_j^q$ when $x_j \geq 0$ and $q \geq 1$. The last equality follows from Proposition 3.1. Because $c \neq 0$, $c^T v_k = \max_j c^T v_j > 0$. The inequality holds tight only when $\frac{c^T v_j}{c^T v_k} \to 0$ for all $j \neq k$, where $c^T v_k = \max_j c^T v_j$. □
**Theorem 4.6** Suppose that \( f(x) = (c^T x)^q \) (\( c \neq 0 \)) with \( q > 1 \) and \( c^T v_j \geq 0 \), \( j = 0, 1, \ldots, d \), where the \( d \)-simplex \( J := \text{conv}\{v_0, v_1, \ldots, v_d\} \subset \mathbb{R}^d_+ \setminus \{0\} \). Then

\[
\frac{\text{vol}(P_0(f, J)) - \text{vol}(P(f, J))}{\text{vol}(P_0(f, J))} \geq \frac{(q - 1)}{\frac{f(q+d+2)}{(d+1)f(q+1)} - (d + 2)} > 0.
\]

The lower bound becomes tight when \( \frac{c^T v_j}{c^T v_k} \to 0 \) for all \( j \neq k \), where \( c^T v_k = \max_j c^T v_j \).

**Proof** By Theorem 2.3, Theorem 4.2 and Lemma 4.5,

\[
\frac{\text{vol}(P_0(f, J)) - \text{vol}(P(f, J))}{\text{vol}(P_0(f, J))} = \frac{q - 1}{d + 2} \cdot \frac{\int_J f(x) dx}{\frac{f(q+d+2)}{(d+1)f(q+1)} \sum_{j=0}^d f(v_j) - \int_J f(x) dx} \geq \frac{(q - 1)}{\frac{f(q+d+2)}{(d+1)f(q+1)} - (d + 2)} > 0.
\]

For example, when the simplex is parametrized by \( u > 0 \), with \( v_0 \) fixed and \( v_j = v_0 + u e_j \), for \( j = 1, \ldots, d \), and \( c = \lambda^{1/q} e_k \) for any \( \lambda > 0 \) and \( k = 1, \ldots, d \), i.e., \( f(x) = \lambda x_k^q \), we have \( c^T v_k = \max_j c^T v_j \) and \( \lim_{u \to +\infty} \frac{c^T v_j}{c^T v_k} = \lim_{u \to +\infty} \frac{c^T v_0}{c^T v_0 + u} = 0 \) for all \( j \neq k \). In this example, the lower bound is asymptotically tight.

For fixed \( d \), the lower bound in Theorem 4.6 has the order of \( O\left(\frac{1}{q^d}\right)\). Theorem 4.6 recovers the lower bound on the cut-off ratio from [LSS22].

**Corollary 4.7** ([LSS22, Cor. 17]) For \( d = 1 \), \( J = [\ell, u] \) (\( u > \ell > 0 \)), and \( f(x) = x^q \), with \( q > 1 \), we have

\[
\frac{\text{vol}(P_0(f, J)) - \text{vol}(P(f, J))}{\text{vol}(P_0(f, J))} \geq \frac{2}{q + 4}.
\]

The lower bound becomes tight only as \( \ell/u \to 0 \).

4.1.2 Case (ii) \( f(x) = (c^T x)^q \) (\( q \) is an even integer)

Suppose that \( q \) is an even integer. Similarly, we would prove a lower bound on the ratio between \( \int_J (c^T x)^q dx \) and \( \sum_{j=0}^d (c^T v_j)^q \), aiming at providing a lower bound on the cut-off ratio.

By Lemma 3.2,

\[
\int_J (c^T x)^q dx = d! \frac{q!}{(q + d)!} h_q(c^T v_0, c^T v_1, \ldots, c^T v_d),
\]
where \( h_q(x_1, \ldots, x_d) := \sum_{|k|_1 = q} x_1^{k_1} \ldots x_d^{k_d} \), which is called a complete homogeneous symmetric polynomial \([Hun77]\). Note that when \( J = \Delta_d \), we have

\[
\int_{\Delta_d} (c^T x)^q \, dx = \frac{q!}{(q + d)!} h_q(0, c_1, \ldots, c_d) = \frac{q!}{(q + d)!} h_q(c_1, \ldots, c_d).
\]

There are interests in proving even-degree complete homogeneous symmetric polynomials are positive definite (i.e., \( h_q(x) \geq 0 \) for all \( x \), and \( h_q(x) = 0 \) if and only if \( x = 0 \)) via different techniques like generating functions, Schur convexity and divided differences \([Hun77, Tao, RT19]\). The integration formula

\[
h_q(c_1, \ldots, c_d) = \frac{q!}{q!} \int_{\Delta_d} (c^T x)^q \, dx
\]

would give a simple proof of positive definiteness of \( h_q(c_1, \ldots, c_d) \) when \( q \) is even. This is related to the probability interpretation using i.i.d exponentially distributed random variables mentioned in the comments of the blog \([Tao]\):

\[
h_q(c_1, \ldots, c_d) = \frac{1}{q!} \int_{\mathbb{R}^d_{\geq 0}} (c^T x)^q e^{-1^T x} \, dx.
\]

The two formulas are connected via a simplification of the multidimensional Laplace form of \( f \) (see \([Las21, Thm. 2.1]\)).

**Lemma 4.8** For \( J := \text{conv}\{v_0, v_1, \ldots, v_d\} \), if \( c \neq 0 \) and \( q \) is an even integer, then

\[
\int_J (c^T x)^q \, dx \geq d! \text{vol}(J) \frac{q!}{(q + d)!} \frac{1}{2^d \left(\frac{d}{2}\right)!} \sum_{j=0}^d (c^T v_j)^q.
\]

**Proof** Because \( q \) is even, \([Hun77, Thm. 1]\) gives the bound \( h_q(c_1, \ldots, c_d) \geq \frac{1}{2^d \left(\frac{d}{2}\right)!} (\sum_{j=1}^d c_j^2)^{\frac{d}{2}} \). Then, we have

\[
h_q(c_1, \ldots, c_d) \geq \frac{1}{2^d \left(\frac{d}{2}\right)!} \left( \sum_{j=1}^d c_j^2 \right)^{\frac{d}{2}} \geq \frac{1}{2^d \left(\frac{d}{2}\right)!} \sum_{j=1}^d c_j^2
\]

because \( c_j^2 \geq 0 \). Therefore, we obtain the following lower bound

\[
\int_J (c^T x)^q \, dx = d! \text{vol}(J) \frac{q!}{(q + d)!} h_q(c^T v_0, c^T v_1, \ldots, c^T v_d)
\]

\[
\geq d! \text{vol}(J) \frac{q!}{(q + d)!} \frac{1}{2^d \left(\frac{d}{2}\right)!} \sum_{j=0}^d (c^T v_j)^q.
\]

\( \square \)

It is still an open question whether this bound is tight or not. The bound of \([Hun77]\) is conjectured to be tight and the second inequality is tight. However, the equality conditions are different.
Theorem 4.9 Suppose that \( f(x) = (c^T x)^q \) \((c \neq 0)\) with \( q \) an even integer, where the \( d \)-simplex \( J := \text{conv}\{v_0, v_1, \ldots, v_d\} \subset \mathbb{R}^d_0 \setminus \{0\} \). Then

\[
\frac{\text{vol}(P^0(f, J)) - \text{vol}(P(f, J))}{\text{vol}(P^0(f, J))} \geq \frac{q - 1}{\frac{q+1}{q(d+1)!} 2^q \left( \frac{q}{2} \right)! - (d + 2)}.
\]

Proof By Theorem 2.3, Theorem 4.2 and Lemma 4.8,

\[
\frac{\text{vol}(P^0(f, J)) - \text{vol}(P(f, J))}{\text{vol}(P^0(f, J))} = \frac{q - 1}{d + 2} \frac{\int_J f(x)dx}{\frac{q+1}{q(d+1)!} \sum_{j=0}^{d} f(v_j) - \int_J f(x)dx} \geq \frac{q - 1}{\frac{q+1}{q(d+1)!} 2^q \left( \frac{q}{2} \right)! - (d + 2)}.
\]

\( \square \)

For fixed \( d \), the lower bound in Theorem 4.9 has the order of \( O\left( \frac{1}{2^q \left( \frac{q}{2} \right)!!} \right) \).

We can improve the coefficient \( \gamma \) from \( \frac{1}{2^q \left( \frac{q}{2} \right)!} \) in the inequality \( h_q(x_1, \ldots, x_d) \geq \gamma \sum_{j=1}^{d} x_j^q \) for some special cases, and thus improve the lower bound in Theorem 4.9.

Proposition 4.10 Suppose \( h_q(x_1, \ldots, x_d) := \sum_{|k|_k = q} x_1^{k_1} \cdots x_d^{k_d} \) is the complete homogeneous symmetric polynomial of even degree \( q \) with \( d \) variables. In the following three cases (a) \( q = 2 \), or (b) \( d = 2 \), or (c) \( d = 3 \) and \( q = 4 \), we have \( h_q(x_1, \ldots, x_d) \geq \frac{1}{2} \sum_{j=1}^{d} x_j^q \).

Proof (a) If \( q = 2 \), then the bound is \( h_2(x_1, x_2, \ldots, x_d) \geq \frac{1}{2} \sum_{j=1}^{d} x_j^2 \) and the bound is tight when \( \sum_{j=1}^{d} x_j = 0 \). The result follows from \( h_2(x_1, x_2, \ldots, x_d) = \frac{1}{2} \sum_{j=1}^{d} x_j^2 + \frac{1}{2} (\sum_{j=1}^{d} x_j)^2 \).

(b) If \( d = 2 \), then the bound is \( h_2(x_1, x_2) \geq \frac{1}{2} (x_1^2 + x_2^2) \) and the bound is tight when \( x_1 + x_2 = 0 \). Because \( c \geq 0 \) implies \( h_2(x_1, x_2) \geq (x_1^2 + x_2^2) \), we consider the case \( x_1 > 0 > x_2 \).

\[
\frac{h_2(x_1, x_2)}{x_1^q + x_2^q} = \frac{x_1^{q+1} - x_2^{q+1}}{(x_1 - x_2)(x_1^q + x_2^q)} := R(t) = \frac{1 + t^{q+1}}{(1+t)(1+t^q)}.
\]

where \( t := -\frac{x_2}{x_1} > 0 \). We have

\[
(1 + t)^2 (1 + t^q)^2 R'(t) = (q + 1)t^q(1 + t)(1 + t^q) - (1 + t^{q+1})(1 + qt^{q-1} + (q + 1)t^q)
\]

\[
= t^{2q} + qt^{q+1} - qt^{q-1} - 1 = t^q \left( t^q - \frac{1}{t^q} + q(t - \frac{1}{t}) \right).
\]
Because \( t^2 - \frac{1}{t^2} \), \( t - \frac{1}{t} \) is increasing on \((0, \infty)\) and obtain 0 if and only if \( t = 1 \). Therefore, we know that \( R(t) \) is decreasing on \((0, 1]\) and increasing on \([1, \infty)\), which implies \( \min R(t) = R(1) = \frac{1}{2} \).

(c) If \( d = 3 \) and \( q = 4 \), then we have
\[
h_4(x_1, x_2, x_3) = \frac{1}{2}(x_1^4 + x_2^4 + x_3^4)
\]

Because this function evaluated at the vertices of the simplex: one for the vertex \( c \) where
\[
J = \text{conv}\{v_0, v_1, \ldots, v_d\},
\]
satisfies \( J = 0 \) for \( j = 1, \ldots, d \). In other word, the exponential function \( f(x) = e^{c^T x} - 1 \) has exactly two distinct values when evaluated at the vertices of the simplex: one for the vertex \( v_0 \), and another for all the other vertices. Because this function \( f(x) \) is not \( q \)-homogeneous, we need a new theorem (Theorem 4.12) to compute the volume of the perspective and naive relaxations.

**Theorem 4.12** Suppose that \( J := \text{conv}\{v_0, v_1, \ldots, v_d\} \subset \mathbb{R}^d \), \( f(x) := e^{c^T x} - 1 \) and \( c \) satisfies \( c^T v_0 - c^T v_j = u \neq 0 \), for \( j = 1, \ldots, d \). Then,
\[
\text{vol}(P(f, J)) = \frac{d! \text{vol}(J)}{(d + 2)!} (e^{c^T v_0} + de^{c^T v_0 - u}) - \frac{d! \text{vol}(J)}{d + 2} \frac{e^{c^T v_0 - u}}{u^d} \left( e^u - \sum_{j=0}^{d-1} \frac{u^j}{j!} \right),
\]
and
\[
\text{vol}(P^0(f, J)) =
\]
\[
\frac{d! \text{vol}(J)}{(d + 2)!} (e^{c^T v_0} + de^{c^T v_0 - u} + 1) - \frac{d! \text{vol}(J)}{(c^T v_0)u^d} \left( e^u - \sum_{j=0}^{d-1} \frac{u^j}{j!} \right) + \frac{d! \text{vol}(J)}{(c^T v_0)[-c^T v_0]u]^d} \left( e^{-(c^T v_0 - u)} - \sum_{j=0}^{d-1} \frac{[-(c^T v_0 - u)]^j}{j!} \right).
\]
Proof

\[
\left| \begin{vmatrix} v_0 & v_1 & \ldots & v_d \\ 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 1 \end{vmatrix} \right| \sum_{j=0}^{d} f(v_j) = d! \, \text{vol}(J) \sum_{j=0}^{d} f(v_j) = d! \, \text{vol}(J) (e^{c^T v_0} + d e^{c^T v_0 - u} - (d + 1)).
\]

By Corollary 3.13, we obtain

\[
\int_J f(x)dx = d! \, \text{vol}(J) \frac{e^{c^T v_0 - u}}{u^d} \left( e^u - \sum_{j=0}^{d-1} \frac{u^j}{j!} \right) - \text{vol}(J).
\]

By Theorem 2.2, we obtain the volume of the perspective relaxation. By Theorem 2.3, we only need to compute \( \int_0^1 \int_J f(zx)dxdz \) to calculate the volume of the naïve relaxation. We use Corollary 3.13 and obtain

\[
\int_0^1 \int_J f(zx)dxdz = d! \, \text{vol}(J) \int_0^1 \frac{e^{c^T v_0 - u}}{u^d} \left( e^{zu} - \sum_{j=0}^{d-1} \frac{(zu)^j}{j!} \right) dz - \text{vol}(J)
\]

Let \( I(j) := \int_0^1 z^j e^{c^T v_0 - u}dz \), using integration by parts, we have

\[
I(j + 1) = \frac{1}{c^T v_0 - u} \int_0^1 z^{j+1} e^{(c^T v_0 - u)z} dz = \frac{1}{c^T v_0 - u} e^{c^T v_0 - u} \int_0^1 z^{j+1} e^{(c^T v_0 - u)z} \bigg|_0^1 - \frac{j + 1}{c^T v_0 - u} I(j)
\]

\[
= \frac{e^{c^T v_0 - u}}{c^T v_0 - u} - \frac{j + 1}{c^T v_0 - u} I(j).
\]

Solving this recursive equation with \( I(0) = \frac{1 - e^{c^T v_0 - u}}{c^T v_0 - u} \), we obtain

\[
\frac{[-(c^T v_0 - u)]^{j+1}}{j!} I(j) = 1 - \sum_{k=0}^{j} \frac{[-(c^T v_0 - u)]^{k} e^{c^T v_0 - u}}{k!}.
\]
Thus,
\[
\frac{e^{c^Tv_0} - 1}{c^Tv_0} - \frac{1}{c^Tv_0} \sum_{j=0}^{d-1} \frac{u_j}{j!} \int_0^1 z^j e^z (e^{c^Tv_0 - u}) dz = e^{c^Tv_0} - 1 - \sum_{j=0}^{d-1} \frac{u_j}{j!} J(j)
\]

\[
= e^{c^Tv_0} - 1 - \sum_{j=0}^{d-1} \frac{u_j}{j!} \frac{1}{[-(c^Tv_0 - u)]^{j+1}} \left( 1 - \sum_{\ell=0}^{d-1} \frac{[-(c^Tv_0 - u)]^{\ell}}{\ell!} \sum_{j=0}^{d-1} \frac{u^j}{[-(c^Tv_0 - u)]^{j+1}} \right)
\]

\[
= e^{c^Tv_0} - 1 + \frac{1 - u^d[-(c^Tv_0 - u)]^{-d}}{c^Tv_0} - e^{c^Tv_0 - u} \sum_{\ell=0}^{d-1} \frac{u^\ell - u^d[-(c^Tv_0 - u)]^{\ell-d}}{\ell!(c^Tv_0)}
\]

\[
= e^{c^Tv_0 - u} \left( e^{u} - \sum_{j=0}^{d-1} \frac{u_j}{j!} \right) - \frac{u^d e^{c^Tv_0 - u}}{(c^Tv_0)[- (c^Tv_0 - u)]^d} \left( e^{-(c^Tv_0 - u)} - \sum_{j=0}^{d-1} \frac{[-(c^Tv_0 - u)]^{j}}{j!} \right),
\]

where the second last equality follows from the geometric series. Therefore, we obtain the formula for $\text{vol}(P^0(f, J))$. \qed

Next, we present two families of simplices for the exponential function $f(x) = e^{x^T} - 1$ such that the cut-off ratio asymptotically goes to 0, and we establish the rate of convergence for each by Theorem 4.12.

**Theorem 4.13** (a) Suppose that $J := \{x : x \leq ku, \|x - ku\| \leq u\} = \text{conv}\{v_0, v_0 - u\epsilon_1, \ldots, v_0 - u\epsilon_d\}$, where $v_0 := ku \in \mathbb{R}^d_{\geq 0}$, and $f(x) := e^{x^T} - 1$. Then,

\[
\lim_{u \to \infty} e^d \cdot \frac{\text{vol}(P^0(f, J)) - \text{vol}(P(f, J))}{\text{vol}(P^0(f, J))} = (d + 1)!
\]

(b) Suppose $J := \text{conv}\{v_0, v_0 + u\epsilon_1, \ldots, v_0 + u\epsilon_d\}$, where $0 \neq v_0 \in \mathbb{R}^d_{\geq 0}$, and $f(x) = e^{x^T} - 1$. Suppose that $v_0$ is fixed and $u$ tends to infinity. Then,

\[
\lim_{u \to \infty} u \cdot \frac{\text{vol}(P^0(f, J)) - \text{vol}(P(f, J))}{\text{vol}(P^0(f, J))} = d + 1.
\]

**Proof** (a) Because $1^Tv_0 = kdu$, $1^T(v_0 - u\epsilon_j) = (kd - 1)u$, by Theorem 4.12, we collect the highest-order term as $u$ tends to infinity and obtain,

\[
\text{vol}(P^0(f, J)) - \text{vol}(P(f, J)) \sim \frac{d! \text{vol}(J)}{d + 2} \frac{e^{kd}u}{u^d}
\]

\[
P^0(f, J) \sim \frac{d! \text{vol}(J)}{(d + 2)!} u^d e^{kd}
\]

\[
\lim_{u \to \infty} u^d \cdot \frac{\text{vol}(P^0(f, J)) - \text{vol}(P(f, J))}{\text{vol}(P^0(f, J))} = (d + 1)!
\]
(b) Because $1^T(v_0 + u e_j) = 1^T v_0 + u$, by Theorem 4.12, we collect the highest-order term as $u$ tends to infinity and obtain,

$$
\text{vol}(P^0(f, J)) - \text{vol}(P(f, J)) \sim \frac{d! \text{vol}(J)}{d + 2} \frac{e^{1^T v_0} e^u}{(d + 1)! u}
$$

$$
\text{vol}(P^0(f, J)) \sim \frac{d! \text{vol}(J)}{(d + 2)!} d^{1^T v_0} e^u
$$

$$
\lim_{u \to \infty} u \cdot \frac{\text{vol}(P^0(f, J)) - \text{vol}(P(f, J))}{\text{vol}(P^0(f, J))} = d + 1.
$$

\[\square\]

Theorem 4.13(a) recovers the following key result of \[LSS22\].

**Corollary 4.14 ([LSS22, Cor. 6])** For $d = 1$, $J := [\ell, u]$ ($\ell > 0$), and $f(x) := e^x - 1$. Let $\ell := ku$ for some fixed $k \in (0, 1)$, then we have

$$
\lim_{u \to \infty} u \cdot \frac{\text{vol}(P^0(f, J)) - \text{vol}(P(f, J))}{\text{vol}(P^0(f, J))} = \frac{2}{1 - k}.
$$

### 4.3 The log-sum-exp function and more

For some convex functions, there are no closed-form formulae for integration over a simplex. In such a case, we can numerically approximate the integration using the cubature formulae presented in Section 3.5 and compute the asymptotic ratio. Suppose that we have a cubature formula of degree $q$:

$$
\int_{\Delta_d} f(x) dx = \sum_{j=1}^{M} \lambda_j f(w_j) + Rf,
$$

where $Rf$ is the approximation error, and $Rf = 0$ for all polynomials $f$ of degree at most $q$. And we can use $\sum_{j=1}^{M} \lambda_j f(w_j)$ with $M$ summands to approximate the integration $\int_{\Delta_d} f(x) dx$. Then, after affine transformation, we can approximate $\int_{J} f(x) dx$ as follows:

$$
\int_{J} f(x) dx \approx \sum_{j=1}^{M} |\det B| \lambda_j f(B w_j + v_0),
$$

where $x \in J \iff B^{-1}(x - v_0) \in \Delta_d$ as in (1). Therefore, we can calculate $\text{vol}(P(f, J))$ (see Theorem 2.2)

$$
\text{vol}(P(f, J)) \approx \frac{|\det B|}{(d + 2)!} \sum_{j=0}^{d} f(v_j) - \frac{|\det B|}{d + 2} \sum_{j=1}^{M} \lambda_j f(B w_j + v_0).
$$

To compute $\text{vol}(P^0(f, J))$, we need a cubature formula for the region $\{(x, z) : x \in z \cdot J, 0 \leq z \leq 1\}$. 

Theorem 4.15 (Theorem 2.8-1 in [Str71]) Suppose that we have a cubature formula of degree \( q \) for a region \( J \)

\[
\int_J f(x) \, dx = \sum_{j=1}^M \lambda_j f(w_j) + R_1 f,
\]

and a cubature formula of degree \( q \)

\[
\int_0^1 z^d f(z) \, dz = \sum_{k=1}^N \nu_k f(r_k) + R_2 f.
\]

Then we have a cubature formula of degree \( q \)

\[
\iint_{x \in J, 0 \leq z \leq 1} f(x) \, dx \, dz = \sum_{j=1}^M \sum_{k=1}^N \lambda_j \nu_k f(r_k w_j) + R_3 f.
\]

Therefore, we can calculate \( \text{vol}(P^0(f, J)) \) (see Theorem 2.3)

\[
\text{vol}(P^0(f, J)) \approx \frac{|\det B|}{(d+2)!} \sum_{j=0}^d f(v_j) - |\det B| \sum_{j=1}^M \sum_{k=1}^N \lambda_j \nu_k f(r_k (B w_j + v_0)).
\]

Next, we test on an example with \( d = 3 \), the log-sum-exp function \( f(x) := \log e^{x_1} + e^{x_2} + e^{x_3} \) (see Example 1.1), and the scaled standard simplex \( J := u \cdot \Delta_d \).

We use the cubature formula of degree 5 in Theorem 3.15 for \( \Delta_d \) (https://www.mathworks.com/matlabcentral/fileexchange/9435-n-dimensional-simplex-quadrature) and the Gauss-Jacobi quadrature formula of degree 5 for \( \int_0^1 z^d f(z) \, dz \) (https://www.mathworks.com/matlabcentral/fileexchange/12854-gauss-jacobi-quadrature-rule-n-a-b) to numerically approximate the cut-off ratio. We have

\[
P(f, J) \approx \frac{u^d}{(d+2)!} \sum_{j=1}^d f(u e_j) - \frac{u^d}{d+2} \sum_{j=1}^M \lambda_j f(u w_j),
\]

\[
P^0(f, J) \approx \frac{u^d}{(d+2)!} \sum_{j=1}^d f(u e_j) - \frac{u^d}{d+2} \sum_{j=1}^M \sum_{k=1}^N \lambda_j \nu_k f(u r_k w_j),
\]

where \( M \) and \( N \) are the number of summands in the cubature formula of degree 5 in Theorem 3.15 for \( \Delta_d \) and the Gauss-Jacobi quadrature formula of degree 5 for \( \int_0^1 z^d f(z) \, dz \), respectively. Figure 2 shows that the approximated cut-off ratio is always small and quickly tends to decrease, thus demonstrating that for this family of examples, the naïve relaxation is quite good.
Fig. 2 The approximated cut-off ratio for a log-sum-exp function $f(x) = \log \frac{e^{x_1 + x_2 + x_3}}{3}$ with respect to an expanding simplex $J = u \cdot \Delta_d$

In fact, in what follows, we prove this apparent limiting behavior (for arbitrary $d$ and even when the base of the simplex is shifted), and at the same time providing some validation of the approximation that we made above using cubature.

**Lemma 4.16** Suppose that $v_j \in \mathbb{R}^d$, for $j = 1, \ldots, d$. Then

$$\lim_{u \to \infty} \frac{1}{u} \int_{\Delta_d} \log \left( \frac{1}{d} \sum_{j=1}^{d} e^{u x_j + v_j} \right) dx = \int_{\Delta_d} \max(x) dx = \frac{1}{(d+1)!} \sum_{j=1}^{d} \frac{1}{j}.$$

**Proof** Notice that

$$\log \left( \frac{1}{d} e^{u \max(x) + \min(v)} \right) \leq \log \left( \frac{1}{d} \sum_{j=1}^{d} e^{u x_j + v_j} \right) \leq \log \left( \frac{1}{d} e^{u \max(x)} \sum_{j=1}^{d} e^{v_j} \right).$$

We have

$$\max(x) + \frac{\min(v) - \log d}{u} \leq \log \left( \frac{1}{d} \sum_{j=1}^{d} e^{u x_j + v_j} \right) \leq \max(x) + \frac{\log \left( \sum_{j=1}^{d} e^{v_j} \right) - \log d}{u}.$$

Therefore, $\lim_{u \to \infty} \frac{1}{u} \int_{\Delta_d} \log \left( \frac{1}{d} \sum_{j=1}^{d} e^{u x_j + v_j} \right) dx = \int_{\Delta_d} \max(x) dx$. By [Las21, Thm. 2.1], we have

$$\int_{\Delta_d} \max(x) dx = \frac{1}{(d+1)!} \int_{\mathbb{R}_0^{d}} \max(x) e^{-\frac{1}{d} x} dx = \frac{1}{(d+1)!} \mathbb{E}(\max(X_1, \ldots, X_d)).$$

where $X_1, \ldots, X_d$ are i.i.d. exponential random variables with mean 1, and $\mathbb{E}(\cdot)$ denotes the expectation. By [Rén53, Eq. 1.9], the lemma follows. \hfill \square

**Theorem 4.17** Let $J := v_0 + u \Delta_d$, where $0 \neq v_0 \in \mathbb{R}_0^d$, and $f(x) := \log \left( \frac{1}{d} \sum_{j=1}^{d} e^{v_j} \right)$. Then,

$$\lim_{u \to \infty} \frac{\text{vol}(P_0(f,J)) - \text{vol}(P(f,J))}{\text{vol}(P_0(f,J))} = 0.$$
Proof By Lemma 4.16, \( \lim_{u \to \infty} \frac{1}{u} \int_{\Delta_d} f(u \mathbf{x} + \mathbf{v}_0) d\mathbf{x} = C_d \), where \( C_d = \frac{\sum_{j=1}^{d} \frac{1}{j}}{(d+1)!} \).

By Theorem 2.2 and 2.3, we have

\[
\frac{\text{vol}(P_0(f,J)) - \text{vol}(P(f,J))}{d! \text{vol}(J)} = \frac{1}{d+2} \int_{\Delta_d} f(u \mathbf{x} + \mathbf{v}_0) d\mathbf{x} - \int_0^1 z^d \int_{\Delta_d} f(z(u \mathbf{x} + \mathbf{v}_0)) d\mathbf{x} dz.
\]

Thus,

\[
\lim_{u \to \infty} \frac{1}{u} \frac{\text{vol}(P_0(f,J)) - \text{vol}(P(f,J))}{d! \text{vol}(J)} = \frac{1}{d+2} C_d - \frac{1}{d+2} \lim_{u \to \infty} \frac{1}{u} \int_{\Delta_d} f(z(u \mathbf{x} + \mathbf{v}_0)) d\mathbf{x} dz
\]

\[
= \frac{1}{d+2} C_d - \int_0^1 z^d C_d dz = 0.
\]

We can further compute

\[
\frac{\text{vol}(P_0(f,J))}{d! \text{vol}(J)} = \frac{1}{(d+2)!} \left( f(\mathbf{v}_0) + \sum_{j=1}^{d} f(u \mathbf{e}_j + \mathbf{v}_0) \right) - \int_0^1 \frac{1}{d+2} \int_{\Delta_d} f(z(u \mathbf{x} + \mathbf{v}_0)) d\mathbf{x} dz.
\]

Thus,

\[
\lim_{u \to \infty} \frac{1}{u} \frac{\text{vol}(P_0(f,J))}{d! \text{vol}(J)} = \frac{d}{(d+2)!} - \frac{1}{d+2} C_d = \frac{1}{(d+2)!} \left( d - \sum_{j=1}^{d} \frac{1}{j} \right) > 0.
\]

Therefore, the result follows. \( \square \)

5 Conclusions

We investigated the idea of using volume as a measure to compare the perspective relaxation and naive relaxation in the multivariate case, for a natural disjunctive model that received a lot of attention in the univariate case. Focusing on the natural and fundamental building-block case where the domain is a simplex, we extended some results for the univariate case.

- We provided a theorem to compute the volumes of the perspective relaxation and naive relaxation for general functions and connect the calculation to the integration over a simplex. We made an extensive survey of the relevant results on integration over a simplex, working out the connections and some extensions (which might additionally be of independent interest for the optimization community).
- We analyzed the cut-off ratio for several important classes of functions, generalizing results from the univariate case. Specifically, the cut-off ratio has a positive lower bound for powers of linear functions under some conditions, which implies that the difference between the two relaxations is substantial. On the other side, the cut-off ratio is small and tends to 0 for a class of exponential functions and the log-sum-exp function over a scaled standard simplex, which implies that the perspective and naive relaxations are close.
When the closed formula is not available, we provided an idea on how to use cubature formulae to numerically compute and compare the volumes.

For future directions, we believe that some technical improvements can be achieved, for example, the lower bound in Theorem 4.9, as well as understanding the asymptotic behavior of the cut-off ratio in terms of more general classes of functions and domains. A further interesting direction is to generalize and compare other relaxations for the multivariate setting, such as extending the original function, and perspective relaxation of the piecewise-linear under-estimators (see [LSSX23]). However, we probably need stronger assumptions on the functions and simplex to handle other relaxation in the multivariate setting.

Finally, we briefly discuss a general setting when the decision variable (vector) \( x \) is either \( 0 \in \mathbb{R}^d \) or in a polytope \( P \subset \mathbb{R}^d_{\geq 0} \) (not containing \( 0 \)), and we have a triangulation of the convex polytope \( P \). We are considering convex relaxations of the "disjunctive set"

\[
D(f, \mathcal{J}) := \{0_{d+1+|\mathcal{N}|}\} \cup \bigcup_{n \in \mathcal{N}} \left\{ (x, y, z) \in \mathbb{R}^d \times \mathbb{R} \times \{0, 1\}^{|\mathcal{N}|} : y = f(x), \ x \in J_n, \ z = e_{n}^{\mathcal{N}} \right\},
\]

where \( \mathcal{J} = \{ J_n : n \in \mathcal{N} \} \) is a triangulation of the convex polytope domain in \( \mathbb{R}^d \), and \( f \) is convex on \( J_n \), for \( n \in \mathcal{N} \). We assume that the polytope domain is a subset of \( \mathbb{R}^d_{\geq 0} \). The binary \( |\mathcal{N}| \)-vector \( z \) is either \( 0 \), if \( (x, y) = (0, 0) \) or \( z \) is the \( n \)-th standard unit vector, if \( x \in J_n \) for some \( n \in \mathcal{N} \). The special case with \( |\mathcal{N}| = 1 \) is what we analyzed in this work. In applications of the general case, \( D(f, \mathcal{J}) \) would be a substructure of a larger model, where the cost of \( x \in J_n \) is \( f(x) + c_n \), and is modeled by \( y + \sum_{n \in \mathcal{N}} c_n z_n \).

Let \( \mu_n(x) \) be a linear function that bounds \( y \) from above on \( J_n, \ n \in \mathcal{N} \). By introducing \( x_n \) and \( y_n \) for each simplex \( J_n, \ n \in \mathcal{N} \), we obtain the extended perspective relaxation

\[
P(f, \mathcal{J}) := \text{cl} \left\{ (\{x_n : n \in \mathcal{N}\}, y, z) \in \mathbb{R}^{d|\mathcal{N}|} \times \mathbb{R}^{|\mathcal{N}|} \times \{0, 1\}^{|\mathcal{N}|} : \tilde{\mu}_n(x_n, z_n) \geq y_n \geq \tilde{f}(x_n, z_n), \ 1^T z \leq 1, x_j \in z_n \cdot J_n, \ n \in \mathcal{N} \right\},
\]

where \( x = \sum_{n \in \mathcal{N}} x_n \), and \( y = \sum_{n \in \mathcal{N}} y_n \). It is only the constraint \( 1^T z \leq 1 \) that prevents the extended perspective relaxation from factoring across the set of simplices \( \mathcal{J} \), and hopefully we can still use the analysis for each subproblem on a single simplex. We see going deeper into analyzing \( P(f, \mathcal{J}) \) as a starting point for some important further work on our topic.

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Conflict of interest

The authors declare that they have no conflict of interest.
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