THE CONTINUITY EQUATION WITH CUSP SINGULARITIES

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Abstract. In this paper we study a special case of the completion of cusp Kähler-Einstein metric on the regular part of varieties by taking the continuity method proposed by La Nave and Tian. The differential geometric and algebro-geometric properties of the noncollapsing limit in the continuity method with cusp singularities will be investigated.

1. Introduction

The Yau-Tian-Donaldson conjecture for Fano manifolds has revealed deep connections among complex Monge-Ampère equation, metric geometry and complex algebraic geometry. Some specialists develop many techniques to deal with the celebrated conjecture, see [22] or [6], [7], [8]. These methods also play an important role in studying many other problems. For instance, in [21], J. Song proves that the metric completion of the regular set of Calabi-Yau varieties and canonical models of general type with crepant singularities is a compact length space which homeomorphic to the original variety. In [18], G. La Nave and G. Tian introduce a new continuity equation to consider the analytic minimal model program. Later in [19], G. La Nave, G. Tian and Z. L. Zhang study the differential geometric and algebro-geometric properties of the noncollapsing limit in the continuity equation. These fundamental results focus on the compact Kähler manifolds. For noncompact case, let us recall some facts. Suppose $M$ is a compact complex manifold, $D$ is an effective divisor with only normal crossings and $K_M + D$ is ample, where $K_M$ is the canonical line bundle over $M$. A well known result achieved by Kobayashi [16] and Tian-Yau [23] asserts that there exists a complete negative Kähler-Einstein metric on $M \setminus D$. Recently, in [2], two authors generalize this result. For the convenience, this consequence is stated as following (Theorem C [2])

Theorem 1.1. Let $\overline{M}$ be a compact Kähler manifold and $D$ is a simple normal crossings $\mathbb{R}$-divisor on $\overline{M}$ with coefficients in $[-1, +\infty)$ such that $K_{\overline{M}} + D$ is semi-positive and big. Then there exists a unique $\omega$ in $c_1(K_{\overline{M}} + D)$ which is smooth on a Zariski open set $U$ of $\overline{M}$ and such that

$$\text{Ric}(\omega) = -\omega + [D].$$

More precisely, $U$ can be taken to the $\overline{M} \setminus (D \cup S)$, where $S$ is the intersection of all effective $\mathbb{Q}$-divisors $E$ such that $K_{\overline{M}} + D - E$ is ample.

Motivated by [21], a natural problem is to ask what the completion of $(U, \omega)$ is. In this article, a special case is investigated. More precisely, suppose $\overline{M}$ is a projective manifold, $D$ is a smooth hypersurface and $K_{\overline{M}} + D$ is big and semi-ample. According to the Kawamata base point free theorem, there exists an integer $K \in \mathbb{Z}^+$ such that an orthonormal basis of $K(K_{\overline{M}} + D)$ gives a holomorphic map

$$\Phi : \overline{M} \to \mathbb{C}P^N.$$
where \( N = \dim H^0(\overline{M}, K(\overline{M} + D)) - 1 \). The mainly result in this article is that the completion of \((U, \omega)\) in the sense of Theorem 1 homeomorphic to \(\Phi(\overline{M} \setminus D)\). If the divisor \(D\) is simple normal crossing with coefficients 1, then there is a similar result. To dealing with this problem, the continuity method is taken proposed by G. La Nave and G. Tian in [18].

To begin with, let \(\overline{M}\) be a projective manifold with a Kähler metric \(\omega_0\) and \(D\) be a smooth hypersurface in \(\overline{M}\) such that \(K_{\overline{M}} + D\) is big and semi-ample. \(h_D\) is denoted by the hermitian metric on \(L_D\), the associated line bundle of \(D\) such that \(\omega_0 - \sqrt{-1} \partial \overline{\partial} \log \log_2 |s_D|_{h_D}^2 > 0\), where \(s_D\) is the defining section of \(D\). The following 1-parameter family equations are considered:

\[
(1 + t)\omega = \omega_0 - \sqrt{-1} \partial \overline{\partial} \log \log_2 |s_D|_{h_D}^2 - t(\operatorname{Ric}(\omega) - [D]),
\]

where \([D]\) is the current of integration along \(D\).

Recall that \(\omega\) is said to have cusp singularities along \(D\) if, whenever \(D\) is locally given by \((z_1 = 0)\), \(\omega\) is quasi-isometric to the cusp metric

\[
\omega_{\text{cusp}} = \frac{\sqrt{-1} d z_1 \wedge d \overline{z}_1}{|z_1|^2 \log^2 |z_1|^2} + \sum_{k=2}^n d z_k \wedge d \overline{z}_k.
\]

Since \(\omega_0 - \sqrt{-1} \partial \overline{\partial} \log \log_2 |s_D|_{h_D}^2\) is a Kähler metric on \(\overline{M} \setminus D\) with cusp singularities, the equation (1.2) essentially state the variation of cusp Kähler metric along \(t\). Therefore, the equation (1.2) is called the cusp continuity equation.

**Theorem 1.3.** The cusp continuity equation (1.2) is solvable for all \(t \in [0, +\infty)\).

\(\omega_t\) is denoted by the solution of (1.2), then we have the following convergence result.

**Theorem 1.4.** \(\omega_t\) converge to a unique weakly Kähler metric \(\omega_1\) such that \(\omega_1\) is smooth on \(\overline{M} \setminus (D \cup S)\) and satisfies

\[
\operatorname{Ric}(\omega_1) = -\omega_1, \text{ on } \overline{M} \setminus (\overline{S_{\overline{M}}} \cup D),
\]

where

\[
S_{\overline{M}} = \bigcap \{E|E\text{ is an effective divisor such that } K_{\overline{M}} + D - \rho E > 0 \text{ for some } \rho > 0\}.
\]

If \(G\) is a big divisor, we denoted \(B_+(G)\) by the intersection of all effective \(\mathbb{Q}\)-divisors \(E\) such that \(G - E\) is ample. Then \(S_{\overline{M}}\) appeared in Theorem 1.4 is \(B_+(K_{\overline{M}} + D)\). Observing that \(\Phi : (\overline{M}, D) \to (\Phi(\overline{M}), \Phi(D))\) can be viewed as a resolution of \((\Phi(\overline{M}), \Phi(D))\) and \(K_{\overline{M}} + D = \Phi^*(K_{\Phi(\overline{M})} + \Phi(D))\).

According to Theorem 0-3-12 [15], \(K_{\Phi(\overline{M})} + \Phi(D)\) is ample. Let \(\Phi(\overline{M})_{\text{reg}}\) be the regular part of \(\Phi(\overline{M})\) and \(\Phi(\overline{M})_{\text{sing}}\) be the singular part of \(\Phi(\overline{M})\). The following Proposition illustrate the connection between \(B_+(K_{\overline{M}} + D)\) and \(\Phi(\overline{M})_{\text{sing}}\), due to Proposition 2.3 [3].

**Proposition 1.5.** Let \(\pi : X \to Y\) be a birational morphism between normal projective varieties. For any big \(\mathbb{R}\)-divisor \(G\) on \(Y\) and any effective \(\pi\)-exceptional divisor \(\mathbb{R}\)-divisor \(F\) on \(X\), then we have

\[
B_+(\pi^* G + F) = \pi^{-1}(B_+(G)) \cup \text{Exc}(\pi),
\]

where \(\text{Exc}(\pi) \subset X\) is the set of points \(x \in X\) such that \(\pi\) is not birregular.

From Theorem 1.4 and Proposition 1.5 the following Corollary is derived immediately.

**Corollary 1.6.**

\[
S_{\overline{M}} = B_+(K_{\overline{M}} + D) = \Phi^{-1}(B_+(K_{\Phi(\overline{M})} + \Phi(D))) \cup \text{Exc}(\Phi) = \Phi^{-1}(\Phi(\overline{M})_{\text{sing}})
\]
and
\[ \overline{M \setminus (S_M \cup D)} = \overline{M_{\text{reg}} \setminus D}, \]

where \( \overline{M_{\text{reg}}} \) represents \( \Phi^{-1}(\Phi(M)_{\text{reg}}) \). Furthermore the metric \( \omega_1 \) is smooth on \( \overline{M_{\text{reg}} \setminus D} \).

Remark 1.7. If the codimension of \( \Phi(D) \) is not 1, then \( D \) is an exceptional divisor of the resolution \( \Phi \). Thus, \( \overline{M \setminus (S_M \cup D)} = \overline{M_{\text{reg}}} \).

\( M \) is denoted by \( \overline{M \setminus D} \). The next result states that the limit space \( (M, \omega_t, x) \) converge to in the Gromov-Hausdorff topology has more regular properties, such as metric structure, algebraic structure.

Theorem 1.8. The following results are hold.

1. \( (M, \omega_t, x) \) converges in the Gromov-Hausdorff topology to a length space \( (M_1, d_1, x_1) \) which is the metric completion \( (M_{\text{reg}} \setminus D, \omega_1) \).
2. \( M_1 = R \cup S \) and \( R = M_{\text{reg}} \setminus D \), where \( R \) is the regular part and \( S \) is the singular part.
3. \( R \) is geodesically convex and \( S \) is closed set which has codimension \( \geq 2 \).
4. \( M_1 \) homeomorphic to a normal quasi-subvariety \( \Phi(M \setminus D) \).

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2. Preliminaries

In this section, we list some fundamental definitions and results which will be used in the later.

Definition 2.1. Let \( V \) be an open set in \( \mathbb{C}^n \). A holomorphic map from \( V \) into a complex manifold \( M \) of complex dimension \( n \) is called a quasi-coordinate map if it is of maximal rank everywhere in \( V \).

This open set \( V \) is called a local quasi-coordinate of \( M \).

Definition 2.2. Let \( M \) be a complete Kähler manifold and \( \omega \) is the Kähler form. \( (M, \omega) \) is called bounded geometry if there is a quasi-coordinates \( \Gamma = \{(V; v^1, \ldots, v^n)\} \) which satisfies the following three conditions:

1. \( M \) is covered by the image of \( (V; v^1, \ldots, v^n) \).
2. The complement of some open neighborhood of \( D \) is covered by a finite of \( (V; v^1, \ldots, v^n) \) which are local coordinates in the usual sense.
3. There exist positive constants \( C \) and \( A_k \) \( (k = 0, 1, 2, \ldots) \) independent of \( \Gamma \) such that at each \( (V; v^1, \ldots, v^n) \), the inequalities

\[
\frac{1}{C} \delta_{ij} < (g_{ij}) < C \delta_{ij},
\]

\[
\left| \frac{\partial^{p+q}}{\partial v^p \partial \bar{v}^q} g_{ij} \right| < A_{|p|+|q|}, \quad \text{for any multiindices } p 
\]

hold, where \( g_{ij} \) denote the component of \( \omega \) with respect to \( V \).

Now we define the Hölder space of \( C^{k,\lambda} \)-functions on a complete Kähler manifold \( (M, \omega) \) which cover by the image of quasi-coordinates. For a nonnegative integer \( k \), \( \lambda \in (0,1) \) and \( u \in C^k(M) \), we
define
\[
||u||_{k,\lambda} = \sup_{V \in \mathcal{F}} \left\{ \sup_{z \in V} \left( \sum_{|p|+|q| \leq k} \left| \frac{\partial |p|+|q|}{\partial v^p \partial \bar{v}^q} u(z) \right| \right) \right\} + \sup_{z,z' \in V} \left( \sum_{|p|+|q| = k} \left| z - z' \right|^{-\lambda} \left| \frac{\partial |p|+|q|}{\partial v^p \partial \bar{v}^q} u(z) - \frac{\partial |p|+|q|}{\partial v^p \partial \bar{v}^q} u(z') \right| \right).
\]

The function space $C^{k,\lambda}(M)$ is, by definition,
\[
C^{k,\lambda}(M) = \{ u \in C^k(M); ||u||_{k,\lambda} < \infty \},
\]
which is a Banach space with respect to the norm $|| \cdot ||_{k,\lambda}$.

Next we state the generalized maximum principle, due to Yau (Proposition 1.6 [9]).

**Theorem 2.3.** Suppose $(M,\omega)$ is a complete Kähler manifold with bounded geometry and $f$ is a function on $M$ which is bounded from above. Then there exists a sequence $x_i$ in $M$ such that $\lim_{i \to \infty} f(x_i) = \sup f$, $\lim_{i \to \infty} |\nabla f(x_i)| = 0$ and $\lim_{i \to \infty} \text{Hess}(f)(x_i) \leq 0$, where the Hessian is taken with respect to $\omega$.

Now we introduce the Bochner formula on a general line bundle. Let $(M,\omega)$ be a Kähler manifold of dimension $n$ and $(L,h)$ be a Hermitian Line bundle over $M$. Let $\Theta_h$ be the Chern curvature form of $h$. Let $\nabla$ and $\nabla$ denote the $(1,0)$ and $(0,1)$ part of a connection respectively. The connection appeared in this paper is usually known as the Chern connection or Levi-Civita connection.

For a holomorphic section $\tau \in \mathcal{H}^0(M,L)$ we write for simplicity
\[
|\tau| = |\tau|_h, \quad |\nabla \tau|_{h \otimes \omega} = |\nabla \tau|,
\]
and
\[
|\nabla \nabla \tau|^2 = \sum_{i,j} |\nabla_i \nabla_j \tau|^2, \quad |\nabla \nabla \tau|^2 = \sum_{i,j} |\nabla_i \nabla_j \tau|^2.
\]

By direct computation we have

**Lemma 2.4.** (Bochner formulas). For any $\tau \in \mathcal{H}^0(M,L)$ one has
\[
\Delta_\omega |\tau|^2 = |\nabla \nabla \tau|^2 - |\tau|^2 \cdot tr_\omega \Theta_h \tag{2.5}
\]
and
\[
\Delta_\omega |\nabla \tau|^2 = |\nabla \nabla \tau|^2 + |\nabla \nabla \tau|^2 - \nabla_j(\Theta_h)_{ij} \langle \tau, \nabla_i \tau \rangle - \nabla_j(tr_\omega \Theta_h)(\nabla_i \tau, \tau) + R_{ij}(\nabla_j \tau, \nabla_i \tau) - 2(\Theta_h)_{ij}(\nabla_j \tau, \nabla_i \tau) - |\nabla \tau|^2 \cdot tr_\omega \Theta_h \tag{2.6}
\]
where $R_{ij}$ is the Ricci curvature of $\omega$, $\langle \cdot, \cdot \rangle$ is the inner product defined by $h$.

3. Existence and uniqueness of cusp continuity equation

This section is devoted to prove the Theorem [1.3]. When $t = 0$ the equation [1.2] has a solution $\omega(0) = \omega_0 - \sqrt{2dI(\partial \log \log^2 |S_{D_{h_0}}|)}$. For a fixed $t \neq 0$, we reduce [1.2] to a scalar equation. First, the background metric will be constructed. Since $K_M + D$ is semi-ample, the Kawamata base point free Theorem claims that there exists an integer $K_0$ such that $K_0(K_M + D)$ has no base point. Then a basis of $\mathcal{H}^0(M,K_0(K_M + D))$ gives a holomorphic map
\[
\Phi : \mathbb{M} \to \mathbb{C}P^N
\]
where \( N = \text{dim} H^0(M, K_0(K_M + D)) - 1 \). \( \omega_{FS} \) is denoted by the Fubini-Study metric on \( \mathbb{CP}^N \). Set \( \eta_l = \frac{1}{\log \omega_{FS}} \). Since \( \eta_l \in C^1(K_M + D) \), there exist a smooth volume form \( \Omega \) on \( M \) and hermitian metric \( h_D^l \) on \( L_D \) such that \( \eta_l = -\text{Ric}(\Omega) + \Theta h_D^l \) and \( \frac{1}{\log \omega_{FS}} \eta_l \in -10 \text{grad} \log^2 |s_D|_{h_D^l} > 0 \) for \( t' \in [0, t] \), where \( \Theta h_D^l \) is the curvature form of \( L_D \) with the metric \( h_D^l \). Set \( l = \frac{t}{1-t} \), then \( \tilde{\eta}_l := (1-t)\omega_0 + t\eta_l - \sqrt{-1} \partial \bar{\partial} |s_D|_{h_D^l}^2 \) is chosen as the background metric. Therefore the equation (1.2) is reduced to the following scalar equation

\[
(\tilde{\omega}_l + \sqrt{-1} \partial \bar{\partial} u_l)^n = e^{ul} \frac{\Omega_l}{|s_D|_{h_D^l}^2 \log^2 |s_D|^2},
\]

where \( |s_D|_{h_D^l}^2 \) is denoted by \( |s_D|_{h_D^l}^2 \) and \( \Omega_l = \Omega \left( \frac{\log^2 |s_D|_{h_D^l}^2}{\log^2 |s_D|^2} \right)^{\frac{1}{1-t}} \) is a smooth volume form on \( M \). For the convenience, we simplified the notation of the above equation as following

\[
(\tilde{\omega}_l + \sqrt{-1} \partial \bar{\partial} u_l)^n = e^{ul} e^{\varepsilon F} \cdot \tilde{\omega}_l^n. \tag{3.1}
\]

To get a complete metric, we define an open subset \( U \) in \( C^{k,\lambda}(M) \) by

\[
U = \{ v \in C^{k,\lambda}(M) | \frac{1}{\Omega} \tilde{\omega}_l < \tilde{\omega}_l + \sqrt{-1} \partial \bar{\partial} v < C \tilde{\omega}_l, \text{ for some positive constant } C \},
\]

where \( M = \overline{M} \setminus D \). If \( u_l \) belongs to \( U \) and satisfies (3.1), then \( \tilde{\omega}_l + \sqrt{-1} \partial \bar{\partial} u_l \) is a complete Kähler metric.

Now we take the continuity method to solve the equation (3.1). Consider the following equations

\[
(\tilde{\omega}_l + \sqrt{-1} \partial \bar{\partial} u_{l,s})^n = e^{ul_s} \cdot e^{\varepsilon F} \cdot \tilde{\omega}_l^n, \tag{3.2}
\]

where \( F = \frac{\Omega_l}{\tilde{\omega}_l^n |s_D|^2 \log |s_D|^2} \). We consider

\[
\text{We consider the } C^0 \text{ map } \Psi : C^{k,\lambda}(M) \to C^{k-2,\lambda}(M) \text{ defined by } \Psi(v) = e^{-ul_s} \cdot \frac{(\tilde{\omega}_l + \sqrt{-1} \partial \bar{\partial} v)^n}{\tilde{\omega}_l^n}. \text{ Define}
\]

\[
S = \{ s \in [0, 1] | \text{there is a solution } u_{l,s} \text{ satisfies } \Psi(u_{l,s}) = e^{sF} \}.
\]

Obviously, \( 0 \in S \). To prove \( 1 \in S \), it is sufficient to show that \( S \) is open and closed. The inverse mapping theorem implies the openness. The Fréchet derivative \( \Psi'(u_{l,s}) : C^{k,\lambda}(M) \to C^{k-2,\lambda}(M) \) at \( u_{l,s} \in U \) is given by

\[
h \to e^{\varepsilon F} (\triangle u_{l,s} h - \frac{h}{t}),
\]

where \( \omega_{l,s} = \tilde{\omega}_l + \sqrt{-1} \partial \bar{\partial} u_{l,s} \). Due to Kobayashi [16], \( F \in C^{k-2,\lambda}(M) \). Therefore, we have to show that, for any \( w \in C^{k-2,\lambda}(M) \),

\[
\triangle u_{l,s} h - \frac{h}{t} = w \tag{3.3}
\]

can be solved for \( h \in C^{k,\lambda}(M) \) and that \( |h|_{C^{k,\lambda}} \leq C |w|_{C^{k-2,\lambda}} \) for some constant \( C \) independent of \( w \).

We first to show that there is at most one function \( h \in C^{k,\lambda}(M) \) solving the equation (3.3). It suffices to verify that \( \triangle u_{l,s} h - \frac{h}{t} = 0 \) and \( h \in C^{k,\lambda}(M) \) imply \( h \equiv 0 \). Note that \( \omega_{l,s} \) is complete Kähler metric with bounded geometry due to Lemma 2 [16] and Proposition 1.4 [9]. For such a metric we can use the generalized maximum principle. Suppose \( h \in C^{k,\lambda}(M) \), \( h \) is in particular bounded. The generalized maximum principle implies that there exists a sequence of points \( \{x_i \} \) in \( M \) such that
\[ \lim_{i \to \infty} h(x_i) = \sup h \text{ and } \lim_{i \to \infty} \Delta_{\omega_{l,s}} h(x_i) \leq 0. \] We immediately see that \( \sup h \leq 0 \) according to the equation \( \Delta_{\omega_{l,s}} h - \frac{h}{l} = 0 \). Similarly, \( \inf h \geq 0 \) and \( h \equiv 0 \).

Now we prove the existence of \( h \). Let \( \{ \Omega_i \} \) be an exhaustion of \( M \) by compact subdomains. Suppose \( w \in C^{k-2,\lambda}(M) \) and let \( h_i \) be the unique solution to

\[ \Delta_{\omega_{l,s}} h_i - \frac{h_i}{l} = w \text{ in } \Omega_i, \]
\[ h_i = 0 \text{ on } \partial \Omega_i. \]

The maximum principle applied to \( \Omega_i \) shows that

\[ \sup_{\Omega_i} |h_i| \leq l \cdot \sup_{\Omega_i} |w|. \]

Interior Schauder estimates shows that a sequence of \( h_i \) converge to some \( h \in C^{k,\lambda}(M) \) which solves the equation (3.3) and that the estimate

\[ |u| \leq C \text{ on } \Omega_i \]

Next, it remains to show that \( S \) is closed. Assume that \( \{ s_i \} \subset E \) is a sequence with \( \lim_{i \to \infty} s_i = \bar{s} \) and \( u_{l,s_i} \) is the solution of (3.2) with \( s = s_i \). We want to prove \( \bar{s} \in E \). It amounts to getting a prior \( C^{k,\lambda}(M) \)-estimate for each \( u_{l,s_i} \). By applying the generalized maximum principle to (3.2), we have

\[ \sup_{M} |u_{l,s_i}| \leq l \cdot s_i \sup_{M} |F| \leq C \sup_{M} |F|. \]

So we have the \( C^{0} \)-estimate due to Lemma 1 [16]. For the \( C^{2} \)-estimate, since \( (M, \bar{\omega}_l) \) is a bounded geometry, by the standard calculation we have

\[ \text{Ric} \left( \omega_{l,s_i} \right) = -\frac{1}{l} \omega_{l,s_i} + \frac{1}{l} \bar{\omega}_l - s_i \sqrt{-1} \partial \bar{\partial} F + \text{Ric} \left( \bar{\omega}_l \right) \leq -\frac{1}{l} \omega_{l,s_i} + C \bar{\omega}_l \]

and

\[ \Delta_{\omega_{l,s_i}} \log \text{tr} \bar{\omega}_{l,s_i} \geq \frac{1}{\text{tr} \bar{\omega}_{l,s_i}} \left( -g^{il} \left( \bar{\omega}_l \right) R_{ij}(\omega_{l,s_i}) + g^{il} \left( \omega_{l,s_i} \right) g_{kl}(\omega_{l,s_i}) R_{ij}^{kl}(\bar{\omega}_l) \right). \]

Then

\[ \Delta_{\omega_{l,s_i}} \log \text{tr} \bar{\omega}_{l,s_i} \geq - a \text{tr} \omega_{l,s_i} \bar{\omega}_l - \frac{A}{\text{tr} \bar{\omega}_{l,s_i}} - C, \]

where \(-a\) is the lower bound of holomorphic bisectional curvature of metric \( \bar{\omega}_l \). Note that

\[ \Delta_{\omega_{l,s_i}} u_{l,s_i} = n - \text{tr} \omega_{l,s_i} \bar{\omega}_l. \]

Let \( H = \log \text{tr} \bar{\omega}_{l,s_i} - (a + 1) u_{l,s_i} \), then

\[ \Delta_{\omega_{l,s_i}} H \geq \text{tr} \omega_{l,s_i} \bar{\omega}_l - \frac{A}{\text{tr} \bar{\omega}_{l,s_i}} - C. \]

By the generalized maximum principle, there exists a sequence \( \{ x_i \} \) such that \( \lim_{i \to \infty} H(x_i) = \sup H \) and \( \lim_{i \to \infty} \sqrt{-1} \partial \bar{\partial} H(x_i) \geq 0 \). So we have a subsequence also denoted by \( \{ x_i \} \) such that

\[ \lim_{i \to \infty} \text{tr} \bar{\omega}_{l,s_i} \left( \text{tr} \omega_{l,s_i} \bar{\omega}_l - C \right)(x_i) \leq A. \]

Nota that \( (\text{tr} \bar{\omega}_{l,s_i})^{\frac{1}{n-1}} \leq C' \text{tr} \omega_{l,s_i} \bar{\omega}_l \). Then we get

\[ \lim_{i \to \infty} \text{tr} \bar{\omega}_{l,s_i} \left( \frac{1}{C'} (\text{tr} \bar{\omega}_{l,s_i})^{\frac{1}{n-1}} - C \right)(x_i) \leq A. \]  \hspace{1cm} (3.4)

If

\[ \lim_{i \to \infty} (\text{tr} \bar{\omega}_{l,s_i})^{\frac{1}{n-1}}(x_i) \leq 2C'C, \]
then we see
\[
\lim_{i \to \infty} tr_{\tilde{\omega}_i} \omega_{l,s_i}(x_i) \leq C.
\]

Otherwise
\[
\lim_{i \to \infty} (tr_{\tilde{\omega}_i} \omega_{l,s_i})^{1/2}(x_i) \geq 2C'C,
\]
then by (3.4) we have
\[
\lim_{i \to \infty} tr_{\tilde{\omega}_i} \omega_{l,s_i}(x_i) \leq C.
\]

Therefore, \( H \leq C \). This implies \( tr_{\tilde{\omega}_i} \omega_{l,s_i} \leq C \). Furthermore by a standard inequality, we get \( C^{-1} \tilde{\omega}_i \leq \omega_{l,s_i} \leq C \tilde{\omega}_i \).

For the 3-order estimate, let \( T = \sum_{i} \omega_{l,s_i} \omega_{l,s_i} \omega_{l,s_i} \), where \( \tilde{\omega}_i \) and \( \omega_{l,s_i} \) represent Riemannian metrics associated with Kähler forms \( \tilde{\omega}_i \) and \( \omega_{l,s_i} \). By a standard computations (c.f. Proposition 4.3 [9]), we have
\[
\Delta_{\omega_i}(T + C\omega_i \omega_i) \geq C_1 T - C_2.
\]

By the Laplace estimate of \( u_{l,s_i} \) and generalized maximum principle, we get \( T \leq C \). Thus, by taking a subsequence if necessary, \( u_{l,s_i} \) \( C^{2,\lambda} \)-converge to a solution with \( s = \tilde{s} \). This implies \( S \) is closed.

Next we prove the uniqueness of equation (3.1). Suppose that \( u_{l,1} \) and \( u_{l,2} \) are solutions to (3.1). Set \( \omega_2 = \tilde{\omega}_1 + \sqrt{-1} \partial \bar{\partial} u_{l,2} \), then we have
\[
\frac{(\omega_2 + \sqrt{-1} \partial \bar{\partial} (u_{l,1} - u_{l,2}))^n}{\omega_2^n} = e^{u_{l,1} - u_{l,2}}.
\]

Since \((M, \omega_2)\) is a complete Kähler manifold with bounded geometry (c.f. Proposition 1.4 [9]), applying the generalized maximum principle, there exists a sequence \( \{x_i\} \) such that \( \lim_{i \to \infty} (u_{l,1} - u_{l,2})(x_i) = \sup_M (u_{l,1} - u_{l,2}) \) and \( \lim_{i \to \infty} \text{Hess}(u_{l,1} - u_{l,2})(x_i) \leq 0 \). Furthermore, we obtain \( u_{l,1} \leq u_{l,2} \). By the same argument, we have \( u_{l,1} \geq u_{l,2} \). Therefore, the equation (3.1) has only one solution. Finally, the cusp continuity equation is solvable for all \( t \in [0, \infty) \) i.e., \( l \in [0, 1) \).

4. Convergence of Cusp Continuity Equation

In this section we investigate the regular properties of limit metric.

**Lemma 4.1.** Let \( F \) be a divisor on \( \overline{M} \). If \( F \) is nef and big, then there is an effective divisor \( E = \sum_i a_i E_i \) such that \( F - \epsilon E > 0 \) for all sufficiently small \( \epsilon > 0 \).

By the assumption that \( K_{\overline{M}} + D \) is big and semi-ample, there exists an effective divisor \( E = \sum_i a_i E_i \) such that \( K_{\overline{M}} + D - \epsilon E > 0 \) for all sufficiently small \( \epsilon > 0 \) according to Lemma 4.1. Thus we choose a volume form \( \Omega \), a hermitian metric \( h_D' \) on \( L_D \) and hermitian metrics \( h_{E_i} \), such that
\[
-\text{Ric}(\Omega) + \Theta' - \sum_i \epsilon a_i \Theta_{E_i} > 0,
\]

where \( \Theta' \) and \( \Theta_{E_i} \) represent curvature forms of line bundles \( L_D \) and \( L_{E_i} \) associated with metrics \( h_D' \) and \( h_{E_i} \) respectively. \( s_D \) and \( s_{E_i} \) are denoted by the defining sections of \( L_D \) and \( L_{E_i} \). For simplicity, we write \( log |s_E|^2 = \sum_i a_i log |s_{E_i}|^2 \). By taking appropriate \( \Omega \), \( h_D' \) and \( h_{E_i} \), we can assume that
\[
-\text{Ric}(\Omega) + \Theta' - \sum_i \epsilon a_i \Theta_{E_i} > 0,
\]

and
\[
\omega_{\ell, E} := (1 - l) \omega_0 + l(-\text{Ric}(\Omega) - \sqrt{-1} \partial \bar{\partial} \log |s_D|^2_{h_D'} + \epsilon \sqrt{-1} \partial \bar{\partial} \log |s_E|^2) - \sqrt{-1} \partial \bar{\partial} \log |s_D|^2_{h_D'} > 0
\]
for $l \in [1/2, 1]$. Let $\tilde{\omega}_l := (1 - l)\omega_0 + l(-\text{Ric}(\Omega) + \Theta_D) - \sqrt{-1} \partial \bar{\partial} \log^2 |s_D|_{h_D}^2$ (may not be a metric), where the hermitian metric $h_D$ is defined as $\omega(0) = \omega_0 - \sqrt{-1} \partial \bar{\partial} \log^2 |s_D|_{h_D}^2 > 0$ and $\Theta_D$ is the curvature form of $h_D$. Then the equation (4.2) is written as

$$\left(\tilde{\omega}_l + \sqrt{-1} \partial \bar{\partial} u_l\right)^n = e^{\frac{\Omega}{s_D|_{h_D}^2 \log^2 |s_D|_{h_D}^2}}.$$  

This equation is also equivalent to

$$(\tilde{\omega}_l + \sqrt{-1} \partial \bar{\partial} u_l)^n = e^{\frac{\Omega}{s_D|_{h_D}^2 \log^2 |s_D|_{h_D}^2}} + \frac{\Omega}{s_D|_{h_D}^2 \log^2 |s_D|_{h_D}^2}.$$  

where $w_l = u_l - \epsilon l \log |s_E|^2 + l \log \frac{|s_D|_{h_D}}{|s_D|_{h_D}^2} + \log \frac{\log^2 |s_D|_{h_D}^2}{\log^2 |s_D|_{h_D}^2}$ and $\Omega_l = \Omega + \Omega \left(\frac{\log^2 |s_D|_{h_D}^2}{\log^2 |s_D|_{h_D}^2}\right)^\frac{\epsilon}{l}$.  

**Lemma 4.4.** There exists a constant $C$ independent of $l$ such that $-C \leq w_l \leq C - \epsilon \log |s_E|^2$.

**Proof.** For the lower bound, we note that $\tilde{\omega}_l + \sqrt{-1} \partial \bar{\partial} u_l = (1 - l)\omega_0 + l(-\text{Ric}(\Omega) + \Theta_D - \sum_i \epsilon a_i \Theta E_i) - \sqrt{-1} \partial \bar{\partial} \log^2 |s_D|_{h_D}^2$ is a complete Kähler metric with bounded geometry on $M$. Applying the generalized maximum principle to (4.3), we get $w_l \geq -C - \epsilon \log |s_E|^2 \geq -C$.

For the upper bound, we differentiate $l$ at both side of equation (4.2), then

$$\nabla \omega \tilde{u}_l + \frac{n+1}{l} \nabla \omega u_l \geq \frac{1}{l} (u_l - \frac{u_l}{l}).$$

where $\omega_l = \tilde{\omega}_l + \sqrt{-1} \partial \bar{\partial} u_l$.

By the simple calculation, we get

$$\nabla \omega_l \left(\frac{u_l}{l} - n \log l\right) \geq \frac{1}{l} (\frac{u_l}{l} - n \log l)'.$$

According to the generalized maximum principle, $(\frac{u_l}{l} - n \log l)$ decrease when $l$ tends to 1. Therefore, there exists a constant $C$ such that $u_l \leq C$. By the definition of $w_l$, we see $w_l \leq C - \epsilon \log |s_E|^2$.  

**Lemma 4.5.** There exist two constants $C$ and $a$ independent of $l$ such that $C^{-1}|s_E|^{2l(a+1)}\tilde{\omega}_l \geq \omega_l := \tilde{\omega}_l + \sqrt{-1} \partial \bar{\partial} u_l \leq C|s_E|^{-2l(a+1)(n-1)}\tilde{\omega}_l$.

**Proof.** Since $\text{Ric}(\omega) \geq -\frac{1}{l} \omega_l$, by Yau’s Schwarz Lemma [25], we have

$$\nabla \omega \log \text{tr} \omega \tilde{\omega}_l \geq -a \cdot \text{tr} \omega \tilde{\omega}_l - \frac{1}{l},$$

where $a$ is a positive upper bound of the holomorphic bisectional curvature of $\tilde{\omega}_l$ for $l \in [0, 1]$. Put $H = \log \text{tr} \omega \tilde{\omega}_l - (a+1)\tilde{u}_l$, then we get

$$\nabla \omega H \geq \text{tr} \omega \tilde{\omega}_l - C.$$  

By the generalized maximum principle, there exists a sequence $\{x_i\}$ such that $\lim_{i \to \infty} H(x_i) = \sup_M H$ and $\lim_{i \to \infty} \Delta \omega H(x_i) \leq 0$. Thus by the Lemma [16, 3] we have $H \leq C$. This implies

$$\text{tr} \omega \tilde{\omega}_l \leq \frac{C}{|s_E|^{2l(a+1)}}.$$  

Note that

$$\text{tr} \omega \tilde{\omega}_l \leq \frac{1}{(n-1)!} \left(\text{tr} \omega \tilde{\omega}_l\right)^{n-1} \cdot \omega_{\tilde{\omega}_l}^n \leq \frac{C}{|s_E|^{2l(a+1)(n-1)}}.$$  

Hence this Lemma is proved.  

$\square$
According to Lemma 4.3 we know that for any compact subset $K \subset M \setminus (D \cup \text{Supp}E)$, there exists a constant $C_K > 0$ independent of $l$ such that $C_K^{-1}\omega_l \leq \omega_l \leq C_K\omega_l$, i.e., $|\Delta\omega_l u_l| \leq C_K$. By Theorem 17.14 of [14], we have $|u_l|_{C^{2,\lambda}} \leq C_K'$ on $K \times [\frac{1}{2}, 1]$. Furthermore, by the standard bootstrapping argument, for any $m > 0$, $|u_l|_{C^{m,\lambda}} \leq C_{K,m}$ on $K \times [\frac{1}{2}, 1]$. By the standard diagonal argument and passing to a subsequence $\{l_i\}$ such that $u_{l_i} \stackrel{C^\infty}{\longrightarrow} u$ on each compact $K$ when $l_i$ tends to 1. The monotonicity of $\left(\frac{d}{dr} - n \log l\right)$ implies that $u_l \stackrel{C^\infty}{\longrightarrow} u$ on each compact $K$ when $l$ tends to 1. Therefore, the Theorem 1.3 is proved.

5. Algebraic structure of the limit space

5.1. Gromov-Hausdorff convergence: global convergence. In this subsection we consider a family of manifolds $(M,\omega_l)$ on which the lower bound of Ricci curvature can be controlled, i.e., $\text{Ric}(\omega_l) \geq -\frac{1}{l}\omega_l$ for $l \in [\frac{1}{2}, 1)$. By Gromov precompactness theorem [4], passing to a subsequence $l_i \to 1$ and fix $x_0 \in M_{\text{reg}} \setminus D$, we may assume that $(M,\omega_{l_i},x_0) \stackrel{d_{GH}}{\longrightarrow} (M_{1},d_{1},x_1)$.

The limit $(M_{1},d_{1})$ is a complete length metric space. It has a regular/singular decomposition $M_{1} = \mathcal{R} \cup \mathcal{S}$, a point $x \in \mathcal{R}$ iff the tangent cone at $x$ is the Euclidean space $\mathbb{R}^{2n}$. The following lemma is the same as Lemma 3.3 in [19].

**Lemma 5.1.** There is a sufficiently small constant $\delta > 0$ such that for any $l \in [\frac{1}{2}, 1)$, if a metric ball $B_{\omega_l}(x,r)$ satisfies $\text{Vol}(B_{\omega_l}(x,r)) \geq (1-\delta)\text{Vol}(B^0_r)$, where $\text{Vol}(B^0_r)$ is the volume of a metric ball of radius $r$ in $2n$-Euclidean space, then $\text{Ric}(\omega_l) \leq (2n-1)r^{-2}\omega_l$ in $B_{\omega_l}(x,\delta r)$.

**Lemma 5.2.** The regular set $\mathcal{R}$ is open in the limit space $(M_{1},d_{1},x_{1})$.

**Proof.** If $x \in \mathcal{R}$, then by Colding’s volume convergence theorem [10], there exists $r = r(x) > 0$ such that $\mathcal{H}^{2n}(B_{d_{1}}(x,r)) \geq (1-\frac{\delta}{2})\text{Vol}(B^0_r)$, where $\mathcal{H}^{2n}$ denotes the Hausdorff measure. Suppose $x_i \in M$ satisfying $x_i \stackrel{d_{GH}}{\longrightarrow} x$, then by the volume convergence theorem again, $\text{Vol}_{\omega_{l_i}}(B_{\omega_{l_i}}(x_i,r)) \geq (1-\delta)\text{Vol}(B^0_r)$ for sufficiently large $i$. According to Lemma 5.1 and Anderson’s harmonic radius estimate [4], there is a constant $\delta' = \delta'(\alpha)$ for any $0 < \alpha < 1$ such that the $C^{1,\alpha}$ harmonic radius at $x_i$ is bigger than $\delta'\delta r$. Passing to the limit, it gives a harmonic coordinate on $B_{d_{1}}(x,\delta'\delta r)$. This implies $B_{d_{1}}(x,\delta'\delta r) \subset \mathcal{R}$. So $\mathcal{R}$ is open with a $C^{1,\alpha}$ Kähler metric $\omega_{\mathcal{R}}$, moreover $\omega_{l_i}$ converges to $\omega_{\mathcal{R}}$ in $C^{1,\alpha}$ topology on $\mathcal{R}$. \hfill $\Box$

Since $\mathcal{R}$ is dense in $M_{1}$, so we have the following Lemma.

**Lemma 5.3.** $(M_{1},d_{1}) = (\mathcal{R},\omega_{\mathcal{R}})$, the metric completion of $(\mathcal{R},\omega_{\mathcal{R}})$.

**Lemma 5.4.** $\mathcal{R}$ is geodesically convex in $M_{1}$ in the sense that any minimal geodesic with endpoints in $\mathcal{R}$ lies in $\mathcal{R}$.

**Proof.** It is simply a consequence of Colding-Naber’s Hölder continuity of tangent cones along a geodesic in $M_{1}$ [11]. Actually, if $x,y \in \mathcal{R}$, then for any minimal geodesic connecting $x$ and $y$, a neighborhood of endpoints lies in $\mathcal{R}$, so the geodesic will never touch the singular set. \hfill $\Box$
Let \( \overline{D} \) be any divisor in \( \overline{M} \) such that \( D \cup S_M \subset \overline{D} \). Define the Gromov-Hausdorff limit of \( \overline{D} \)

\[
\overline{D}_1 := \{ x \in M | \text{there exists } x_i \in \overline{D} \text{ such that } x_i \xrightarrow{d_{GH}} x \}.
\]

**Proposition 5.5.** \((M_1, d_1)\) is isometric to \( (\overline{M} \setminus \overline{D}, \omega_1) \), where \( \omega_1 \) is defined as Theorem 1.4.

**Proof.** First, we prove the following Claim.

**Claim 5.6.** \( \overline{D}_1 \setminus S \) is a subvariety of dimension \((n - 1)\) if it is not empty.

**Proof.** Let \( x \in \overline{D}_1 \setminus S \) and \( x_i \in \overline{D} \) such that \( x_i \xrightarrow{d_{GH}} x \). By the \( C^{1,\alpha} \) convergence of \( \omega_i \) around \( x \), there are \( C, r > 0 \) independent of \( i \) and a sequence of harmonic coordinates in \( B_{\omega_i}(x_i, r) \) such that \( C^{-1} \omega_E \leq \omega_i \leq C \omega_E \) where \( \omega_E \) is the Euclidean metric in the coordinates. Furthermore, according to Lemma 3.11 [24], any \( x_i \in M \) converging to \( x \) has a holomorphic coordinate \((z^1, z^2, \ldots, z^n)\) on \( B_{\omega_i}(x_i, r) \) such that \( C^{-1} \omega_E(\partial/\partial z^1, \partial/\partial z^2) \leq \omega_i(\partial/\partial z^1, \partial/\partial z^2) \leq C \omega_E(\partial/\partial z^1, \partial/\partial z^2) \). Since the total volume of \( \overline{D} \) is uniformly bounded for any \( \omega_i \), the local analytic \( \overline{D} \cap B_{\omega_i}(x_i, r) \) have a uniform bound of degree and so converge to an analytic set \( \overline{D}_1 \cap B_{d_1}(x, r) \).

From the above Claim we know that \( \dim_M(\overline{D}_1) = \dim_M(S \cup (\overline{D}_1 \setminus S)) \leq 2n - 2 \). By the argument of [20], \((M_1 \setminus \overline{D}_1, \omega_1)\) homeomorphic and locally isometric to \((\overline{M} \setminus \overline{D}, \omega_1)\). Since \( M_1 \) is a length space and \( \dim_M(\overline{D}_1) \leq 2n - 2 \), \((M_1 \setminus \overline{D}_1, \omega_1)\) isometric to \((\overline{M} \setminus \overline{D}, \omega_1)\). Furthermore, we have

\[
(M_1, d_1) = (\overline{M}_1 \setminus \overline{D}_1, \omega_1) = (\overline{M} \setminus \overline{D}, \omega_1).
\]

A direct corollary is

**Corollary 5.7.** \((M, \omega_1, x_0)\) converges globally to \((M_1, d_1, x_1)\) in the Gromov-Hausdorff topology as \( l \to l \).

**Corollary 5.8.** Let \( M_{\text{reg}} = M_{\overline{\text{reg}}} \setminus D \), then \( \omega_1 \) is smooth on \( M_{\text{reg}} \). \((M_1, d_1)\) is isometric to \((M_{\overline{\text{reg}}}, \omega_1)\).

**Proof.** Note that \( M_{\overline{\text{reg}}} \setminus (\overline{M} \setminus \overline{D}) = M_{\text{reg}} \cap \overline{D} \) has real codimension larger than 2 in \((M_{\overline{\text{reg}}}, \omega_1)\). So \( \overline{M} \setminus \overline{D} \) is dense in \( M_{\text{reg}} \). We conclude

\[
(M_1, d_1) = (\overline{M} \setminus \overline{D}, \omega_1) = (M_{\overline{\text{reg}}}, \omega_1).
\]

**Proposition 5.9.** \( M_{\text{reg}} = \mathcal{R} \), the regular set of \( M_1 \).

**Proof.** Since \( M_{\text{reg}} \) has smooth structure in \( M_1 \), we have \( M_{\text{reg}} \subset \mathcal{R} \). Next we show the converse. We argue by contradiction. Suppose \( p \in \mathcal{R} \setminus M_{\text{reg}} \), then there exists a family of points \( p_i \in M_{\text{sing}} \) such that \( p_i \xrightarrow{d_{GH}} p \), where \( M_{\text{sing}} = (\Phi^{-1}(\overline{M}(\overline{\mathcal{S}}(\Phi)))) \setminus D \). By \( C^{1,\alpha} \) convergence on \( \mathcal{R} \), there exist \( C, r > 0 \) independent of \( l \) and a sequence of harmonic coordinates on \( B_{\omega}(p_i, r) \) such that \( C^{-1} \omega_E \leq \omega_i \leq C \omega_E \) where \( \omega_E \) is the Euclidean metric in this coordinate. Furthermore, the sequence of harmonic coordinate can be perturbed to a holomorphic coordinate on \( B_{\omega_i}(p_i, r) \) [24]. Denote \( m = \dim_{\mathcal{C}}(M_{\text{sing}}) \). Then

\[
\text{Vol}_{\omega_i}(M_{\text{sing}} \cap B_{\omega_i}(p_i, r)) = \int_{M_{\text{sing}} \cap B_{\omega_i}(p_i, r)} \omega_i^m 
\geq \int_{M_{\text{sing}} \cap B_{\omega_E}(C^{-\epsilon}r)} (C^{-1} \omega_E)^m
\]

Let \( \mathcal{D} \) be any divisor in \( \overline{M} \) such that \( D \cup \overline{S_M} \subset \mathcal{D} \). Define the Gromov-Hausdorff limit of \( \mathcal{D} \)

\[
\mathcal{D}_1 := \{ x \in M | \text{there exists } x_i \in \mathcal{D} \text{ such that } x_i \xrightarrow{d_{GH}} x \}.
\]
which has a uniform lower bound. However, this contradicts with the degeneration of the limit metric \( \eta_1 \) along \( M_{\text{sing}} \):

\[
\text{Vol}_{\omega}(M_{\text{sing}} \cap B_{\omega}(p_l, r)) \leq \text{Vol}_{\omega}(M_{\text{sing}})
= \int_{M_{\text{sing}}} \omega^m = \int_{M_{\text{sing}}} ((1 - l)\omega_0 + l\eta)^m
\]

which tends to 0 as \( l \to 1 \), where the last equality bases on a Lemma \([16] P410\). So we have \( M_{\text{reg}} \supset R \).

5.2. \( L^\infty \) estimate and gradient estimate to holomorphic sections. In this subsection we obtain the \( L^\infty \) estimate and gradient estimate to holomorphic section \( s \in H^0(R, k(K + D)) \). \( h = \omega_1^{-nk} \) is chosen as the Hermitian metric of line bundle \( k(K + D) \), where \( k \in \mathbb{Z} \). The curvature form \( \Theta_h \) of Hermitian metric \( h = \omega_1^{-nk} \) is \( k\omega_1 \). By Lemma \([2,4] \) we have the following formulas.

Lemma 5.10. For \( s \in H^0(R, k(K + D)) \), there exists a constant \( C \) such that

\[
\Delta_{\omega_1} |s|^2 = |\nabla s|^2 - kn|s|^2
\]

and

\[
\Delta_{\omega_1} |\nabla s|^2 \geq |\nabla \nabla s|^2 + |\nabla \nabla s| - Ck|\nabla s|^2 - k\nabla_j(\omega_1)_{ij}(s, \nabla_i s).
\]

Proof. Since on \( R \), \( \text{Ric}(\omega_1) = -\omega_1 \). So these formulas are directly derived from Lemma \([2,4] \). □

In order to applying Moser iteration, the Sobolev inequality on \( R \) is needed. The following two Lemmas are due to Song (Lemma 3.7 and 4.6 \([21] \)).

Lemma 5.11. There is a family of cut-off functions \( \rho_\epsilon \in C_0^\infty(R) \) with \( 0 < \rho_\epsilon < 1 \) such that \( \rho_\epsilon^{-1}(1) \) forms an exhaustion of \( R \) and

\[
\int_R |\nabla \rho_\epsilon|^2 \omega_1^n \to 0.
\]

Lemma 5.12. Fix any \( 0 < r < R \), the Sobolev constant on \( B_{\omega_1}(x, r) \) is uniformly bounded by a constant \( C_S \) depending on upper bound \( R \), \( R^{-1} \) and \( (R - r)^{-1} \). More precisely, for any \( l \in [\frac{1}{2}, 1) \) and \( f \in C_0^1(B_{\omega_1}(x, r)) \),

\[
C_S \left( \int_{B_{\omega_1}(x, r)} |f|^{\frac{2n}{n-1}} \omega_1^n \right)^{\frac{n-1}{n}} \leq \int_{B_{\omega_1}(x, r)} (|f|^2 + |\nabla f|^2) \omega_1^n.
\]

Fix \( 0 < r < R \) such that \( B_{\omega_1}(x, r) \subset B_{\omega_1}(x, 2r) \subset B_{\omega_1}(x, R) \).

Lemma 5.13. If \( f \in C_0^1(B_{\omega_1}(x, r) \cap R) \), then there exists a constant \( C \) depending on \( R \), \( R^{-1} \) and \( (R - r)^{-1} \) such that

\[
C \left( \int_{B_{\omega_1}(x, r) \cap R} |f|^{\frac{2n}{n-1}} \omega_1^n \right)^{\frac{n-1}{n}} \leq \int_{B_{\omega_1}(x, r) \cap R} (|f|^2 + |\nabla f|^2) \omega_1^n.
\]

Proof. Let \( f_\epsilon = \rho_\epsilon f \), where \( \rho_\epsilon \) is constructed as Lemma \([5,11] \) and \( \Omega_\epsilon = \text{Supp} f_\epsilon \). Then \( \omega_1 \) uniformly converge to \( \omega_1 \) on \( \Omega_\epsilon \) as \( l \) tends to 1 for a fixed \( \epsilon \). Therefore \( \Omega_\epsilon \subset B_{\omega_1}(x, r) \) for \( l \) sufficiently close to 1. By Lemma \([5,12] \) we have

\[
C_S \left( \int_{B_{\omega_1}(x, r)} |f_\epsilon|^{\frac{2n}{n-1}} \omega_1^n \right)^{\frac{n-1}{n}} \leq \int_{B_{\omega_1}(x, r)} (|f_\epsilon|^2 + |\nabla f_\epsilon|^2) \omega_1^n.
\]
Let \( l \to 1 \), the above inequality gives

\[
CS \left( \int_{B_{r^l}(x,r)} |f_\epsilon|^\frac{2n}{n-1} \omega_1^n \right)^{\frac{n-1}{n}} \leq \int_{B_{r^l}(x,r)} (|f_\epsilon|^2 + |\nabla f_\epsilon|^2) \omega_1^n.
\]

Note that by letting \( \epsilon \to 0 \), we get

\[
\int_{B_{r^l}(x,r)} |f_\epsilon|^{\frac{2n}{n-1}} \omega_1^n \to \int_{B_{r^l}(x,r)} |f|^{\frac{2n}{n-1}} \omega_1^n
\]

and

\[
\int_{B_{r^l}(x,r)} |f_\epsilon|^2 \omega_1^n \to \int_{B_{r^l}(x,r)} |f|^2 \omega_1^n.
\]

By some calculations we have

\[
\left| \int_{B_{r^l}(x,r)} |\nabla f_\epsilon|^2 \omega_1^n - \int_{B_{r^l}(x,r)} |\nabla f|^2 \omega_1^n \right| = \left| \int_{B_{r^l}(x,r)} (|\nabla \rho_1|^2 |f|^2 + (|\rho_1|^2 |\nabla f|^2 - |\nabla f|^2)) \omega_1^n \right|
\]

which tends to 0. So this Lemma is proved. \( \square \)

**Lemma 5.14.** There exists a constant \( C \) independent of \( k \) such that if \( s \in H^0(\mathcal{R}, k(K_{\mathcal{M}} + D)) \), then

\[
\int_{B_{r^l}(x,\frac{7}{4}r) \cap \mathcal{R}} |\nabla s|^2 \omega_1^n \leq Ckr^{-2} \int_{B_{r^l}(x,2r) \cap \mathcal{R}} |s|^2 \omega_1^n
\]

and

\[
\int_{B_{r^l}(x,\frac{7}{4}r) \cap \mathcal{R}} (|\nabla \nabla s|^2 + |\nabla \nabla s|^2) \omega_1^n \leq Ck^2r^{-4} \int_{B_{r^l}(x,2r) \cap \mathcal{R}} |s|^2 \omega_1^n.
\]

**Proof.** Let \( \vartheta \in C_0^\infty (B_{r^l}(x,\frac{15}{8}r) \cap \mathcal{R}) \) be any cut-off function such that \( 0 \leq \vartheta \leq 1 \), \( |\nabla \vartheta| \leq 10r^{-2} \) and \( \vartheta = 1 \) on \( B_{r^l}(x,\frac{7}{4}r) \cap \mathcal{R} \), then by Bochner formula we have

\[
\int_{\mathcal{R}} \vartheta^2 |\nabla s|^2 \omega_1^n = nk \int_{\mathcal{R}} \vartheta^2 |s|^2 \omega_1^n + \int_{\mathcal{R}} \vartheta^2 \Delta |s|^2 \omega_1^n.
\]

Note that

\[
\int_{\mathcal{R}} \vartheta^2 \Delta |s|^2 \omega_1^n = -2 \int_{\mathcal{R}} \vartheta \nabla_i \vartheta \langle \nabla_i s, \bar{s} \rangle \omega_1^n \leq \frac{1}{2} \int_{\mathcal{R}} \vartheta^2 |\nabla s|^2 \omega_1^n + 2 \int_{\mathcal{R}} |\nabla \vartheta|^2 |s|^2 \omega_1^n.
\]

Therefore,

\[
\int_{B_{r^l}(x,\frac{7}{4}r) \cap \mathcal{R}} |\nabla s|^2 \omega_1^n \leq Ckr^{-2} \int_{B_{r^l}(x,2r) \cap \mathcal{R}} |s|^2 \omega_1^n.
\]

For the second inequality, also by the Bochner formula

\[
\int_{\mathcal{R}} \vartheta^2 (|\nabla \nabla s|^2 + |\nabla \nabla s|^2) \omega_1^n \leq \int_{\mathcal{R}} \vartheta^2 (|\nabla \nabla s|^2 + Ck|\nabla s|^2 + k \nabla_j (\omega_1)_{ij} \langle s, \nabla_i \bar{s} \rangle |s|^2 \omega_1^n.
\]

Note that

\[
\int_{\mathcal{R}} \vartheta^2 \Delta |\nabla s|^2 \omega_1^n = -2 \int_{\mathcal{R}} \vartheta \nabla_i \vartheta \nabla_i |s|^2 \omega_1^n \leq \frac{1}{4} \int_{\mathcal{R}} \vartheta^2 (|\nabla \nabla s|^2 + |\nabla \nabla s|^2) \omega_1^n + C \int_{\mathcal{R}} |\nabla \vartheta|^2 + |s|^2 \omega_1^n
\]
and
\[
\int_R k \partial^2 \nabla_j (\omega)_{ij}(s, \nabla_i s) \omega^n_1 = -k \int_R \partial^2 (\omega)_{ij}(s, \nabla_i s) + \langle s, \nabla_j \nabla_i s \rangle \omega^n_1 - 2k \int_R \partial \nabla_j \partial (\omega)_{ij}(s, \nabla_i s) \omega^n_1
\]
\[
\leq \frac{1}{4} \int_R \partial^2 (|\nabla s|^2 + |\nabla s|^2) \omega^n_1 + Ck \int_R \partial^2 |s|^2 \omega^n_1 + Ck \int_R \partial^2 |s|^2 \omega^n_1
\]
\[
+ Ck \int_R |\nabla \partial^2 |s|^2 \omega^n_1
\]

Summing up these estimates we have
\[
\int_{B_{\omega_1}(x, 2r) \cap R} (|\nabla s|^2 + |\nabla s|^2) \omega^n_1 \leq Ckr^{-2} \int_{B_{\omega_1}(x, 2r) \cap R} |s|^2 \omega^n_1 + Ck \int_{B_{\omega_1}(x, 2r) \cap R} |s|^2 \omega^n_1.
\]

Applying the first inequality we obtain the second estimate.

Proposition 5.15. There exists a constant \( C(R, r) \) independent of \( k \) such that if \( s \in H^0(R, k(K_{\overline{M}} + D)) \), then
\[
\sup_{B_{\omega_1}(x, 2r) \cap R} |s|^2 \leq C(R, r)k^n p^{-2n} \int_{B_{\omega_1}(x, 2r) \cap R} |s|^2 \omega^n_1
\]
and
\[
\sup_{B_{\omega_1}(x, r) \cap R} |\nabla s|^2 \leq C(R, r)k^{n+1} r^{-2n-2} \int_{B_{\omega_1}(x, 2r) \cap R} |s|^2 \omega^n_1.
\]

Proof. Choose a cut-off function \( \vartheta \in C^\infty_0(B_{\omega_1}(x, 2r) \cap R) \). Then for any \( p \geq \frac{n}{n-1} \), by Lemma 5.10 we have
\[
\int_R \vartheta^2 |s|^2 |\nabla |s|^2 \omega^n_1 = \frac{p^2}{4(p-1)} \int_R \vartheta^2 |\nabla |s|^2| |s|^2 |\nabla s|^2 \omega^n_1
\]
\[
= \frac{p^2}{4(p-1)} \int_R (\vartheta^2 |s|^2 |\nabla |s|^2| |s|^2 |\nabla s|^2 \omega^n_1 - 2 \vartheta \cdot \nabla |s|^2 |\nabla s|^2 \omega^n_1
\]
\[
\leq \frac{p^2}{4(p-1)} \int_R (\vartheta^2 |s|^2 |\nabla s|^2 \omega^n_1 + nk \frac{p^2}{4(p-1)} \int_R \vartheta^2 |s|^2 |\nabla s|^2 |\nabla s|^2 \omega^n_1
\]
\[
+ \frac{p^2}{4(p-1)} \int_R \vartheta \cdot \nabla \vartheta \cdot |s|^2 |\nabla s|^2 \omega^n_1
\]

By Cauchy-Schwarz inequality,
\[
\int_R \vartheta \cdot |\nabla \vartheta| \cdot |s|^2 |\nabla s|^2 \omega^n_1 \leq \int_R \vartheta^2 |s|^2 |\nabla s|^2 \omega^n_1 + \frac{1}{4} \int_R |\nabla \vartheta|^2 |s|^2 |\nabla s|^2 \omega^n_1
\]

Therefore
\[
\int_R \vartheta^2 |s|^2 |\nabla |s|^2 \omega^n_1 \leq Cpk \int_R (\vartheta^2 + |\nabla \vartheta|^2) |s|^2 |\nabla s|^2 \omega^n_1.
\]

By Lemma 5.13
\[
\left( \int_R (\vartheta |s|^p)^{\frac{2n}{n-1}} \omega^n_1 \right)^{\frac{n-1}{n}} \leq Cpk \int_R (\vartheta^2 + |\nabla \vartheta|^2) |s|^2 |\nabla s|^2 \omega^n_1.
\]
Put $p_j = \nu^{j+1}$ for $j \geq 0$, where $\nu = \frac{n}{n-1}$. Define a family of radius inductively by $r_0 = \frac{3}{2}r$ and $r_j = r_{j-1} - 2^{-j-1}r$. $B_j$ is denoted by $B_{\omega_j}(x, r_j) \cap \mathcal{R}$. We choose a family of cut-off functions $\vartheta_j \in C_0^\infty(B_j)$ such that

$$0 \leq \vartheta_j \leq 1, \quad |\nabla \vartheta_j| \leq 2^{j+2}r^{-1} \text{ and } \vartheta_j = 1 \text{ on } B_{j+1}.$$ 

Thus (6.2) gives, by setting $\vartheta = \vartheta_j$

$$\left( \int_{B_{j+1}} |s|^{2p_j+1}\omega_1^n \right)^{\frac{1}{p_j+1}} \leq (C p_j k)^{\frac{1}{p_j+1}} 2^{j+1} r^{-1} \left( \int_{B_j} |s|^{2p_j} \omega_1^n \right)^{\frac{1}{p_j}}.$$ 

By the iteration argument, we see

$$\sup_{B_{\omega_1}(x, r) \cap \mathcal{R}} |s|^2 \leq C k^{n-1} r^{-2(n-1)} \left( \int_{B_0} |s|^{\frac{2n}{n+1}} \omega_1^n \right)^{\frac{n+1}{2n}} \leq C k^{n-1} r^{-2(n-1)} \int_{B_0} (|s|^2 + |\nabla s|^2) \omega_1^n.$$ 

According to Lemma [5.14] we get the first estimate.

Next we prove the second inequality. Let $\vartheta$ and $p$ as above. By Lemma [5.11] we have

$$\int_{\mathcal{R}} \vartheta^2 |\nabla |s| |s|^2 |\nabla s|^2 \omega_1^n = \frac{p^2}{4(p-1)} \int_{\mathcal{R}} \vartheta^2 \cdot |\nabla |s| |s|^2 \cdot |\nabla_s| |s|^2 \omega_1^n$$

$$= \frac{p^2}{4(p-1)} \int_{\mathcal{R}} (-\vartheta^2 |\nabla s|^2(\omega_1^n) \Delta |s|^2 - 2\vartheta \cdot \nabla \vartheta \cdot |\nabla |s| |s|^2) \omega_1^n$$

$$\leq \frac{p^2}{4(p-1)} \int_{\mathcal{R}} \left( -\vartheta^2 |\nabla s|^2(\omega_1^n) \Delta |s|^2 + |\nabla s|^2 \right) + k \nabla_j(\omega_1)_{ij} \langle s, \nabla_i \bar{s} \rangle \cdot \vartheta^2 |\nabla s|^2(\omega_1^n)$$

$$+ C k \vartheta^2 |\nabla s|^2(\omega_1^n) \Delta |s|^2 - 2\vartheta \cdot \nabla \vartheta \cdot |\nabla |s| |s|^2 \cdot |\nabla_s| |s|^2 \omega_1^n.$$ 

The term $\int_{\mathcal{R}} k \nabla_j(\omega_1)_{ij} \langle s, \nabla_i \bar{s} \rangle \cdot \vartheta^2 |\nabla s|^2(\omega_1^n)$ can be estimate by integration by parts as follows

$$\int_{\mathcal{R}} k \nabla_j(\omega_1)_{ij} \langle s, \nabla_i \bar{s} \rangle \cdot \vartheta^2 |\nabla s|^2(\omega_1^n)$$

$$= -k \int_{\mathcal{R}} (\omega_1)_{ij} \left( \vartheta^2 |\nabla s|^2(\omega_1^n) (\langle \nabla_j s, \nabla_i \bar{s} \rangle + \langle s, \nabla_j \nabla_i \bar{s} \rangle) + (p - 1) \vartheta^2 |\nabla s|^2 |\nabla_j s|^2 \langle s, \nabla_i \bar{s} \rangleight.$$ 

$$+ 2\vartheta \nabla_j \vartheta |\nabla s|^2(\omega_1^n) \langle s, \nabla_i \bar{s} \rangle\right)$$

$$\leq -\frac{1}{2} \int_{\mathcal{R}} \vartheta^2 |\nabla s|^2(\omega_1^n) (|\nabla \nabla s|^2 + |\nabla \nabla s|^2) \omega_1^n + C(p - 1)^2 k^2 \int_{\mathcal{R}} \vartheta^2 |\nabla s|^2(\omega_1^n)$$

$$+ C k \int_{\mathcal{R}} (|\nabla \vartheta|^2 |\nabla s|^2(\omega_1^n) + \vartheta^2 |\nabla s|^2(\omega_1^n).$$

Note that

$$\int_{\mathcal{R}} \vartheta \cdot \nabla \vartheta \cdot |\nabla s|^2(\omega_1^n) \cdot |\nabla s|^2(\omega_1^n) \leq \frac{1}{2} \int_{\mathcal{R}} \vartheta^2 |\nabla s|^2(\omega_1^n) (|\nabla \nabla s|^2 + |\nabla \nabla s|^2) \omega_1^n + C \int_{\mathcal{R}} |\nabla \vartheta|^2 |\nabla s|^2(\omega_1^n).$$

Summing up these estimates we conclude

$$\int_{\mathcal{R}} |\nabla |s| |s|^2 |\nabla s|^2(\omega_1^n \leq C k^3 \int_{\mathcal{R}} \left( k \vartheta^2 |\nabla s|^2(\omega_1^n) \cdot |\nabla s|^2 + |\nabla \vartheta|^2 |\nabla s|^2(\omega_1^n + \vartheta^2 |\nabla s|^2 + |\nabla s|^2(\omega_1^n).$$

Applying the Lemma [5.13]
By setting

Case 1: If

Then (5.17) gives

Put \( p_j = \nu^{j+1} \) for \( j \geq 0 \), where \( \nu = \frac{n}{n-1} \). Define a family of radius inductively by \( r_0 = \frac{3}{2}r \) and \( r_j = r_{j-1} - 2^{-j-1}r \). \( B_j \) is denoted by \( B_{\omega_1}(x, r_j) \cap \mathcal{R} \). We choose a family of cut-off functions \( \vartheta_j \in C_{0}^{\infty}(B_j) \) such that

By setting \( \vartheta = \vartheta_j \), the above inequality gives

Case 1: If \( \left( \int_{B_j} |\nabla s|^{2p_j} \omega_1^n \right)^{\frac{1}{p_j}} \geq k \left( \int_{B_j} |s|^{2p_j} \omega_1^n \right)^{\frac{1}{p_j}} \) for all \( j \geq 0 \). Then

Then (5.17) gives

By iteration argument we get

By Lemma 5.13 and a cut-off argument, we have

According to Lemma 6.14 we get

Case 2: There exists \( j_0 \) such that \( \left( \int_{B_j} |\nabla s|^{2p_j} \omega_1^n \right)^{\frac{1}{p_j}} \geq k \left( \int_{B_j} |s|^{2p_j} \omega_1^n \right)^{\frac{1}{p_j}} \) for all \( j > j_0 \), but

Then

\[ \int_{B_{j_0}} k|\nabla s|^{2(p_{j_0} - 1)}|s|^{2} \omega_1^n \leq k \left( \int_{B_{j_0}} |\nabla s|^{2p_{j_0}} \omega_1^n \right)^{\frac{p_{j_0}-1}{p_{j_0}}} \left( \int_{B_{j_0}} |s|^{2p_{j_0} \omega_1^n} \right)^{\frac{1}{p_{j_0}}} \leq k^{p_{j_0}} \int_{B_{j_0}} |s|^{2p_{j_0} \omega_1^n} \]
By the iteration argument and (5.17), we have

\[ \sup_{B_{r_0}(x,r) \cap R} |\nabla s|^2 \leq C(kr^{-2})^{m/n}(\int_{B_{r_0}} |s|^{2p_j \omega_1^n})^{1/p_j}. \]

The supremum of $|\nabla s|$ follows from

\[ \left( \int_{B_{r_0}} |s|^{2p_j \omega_1^n} \right)^{1/p_j} \leq \left( \sup_{B_{r_0}} |s| \right)^{2p_j \omega_1^n} \left( \int_{B_{r_0}} |s|^{2 \omega_1^n} \right)^{1/p_j} \leq C(kr^{-2})^{n-\omega} \int_{B_{r_0}} |s|^{2 \omega_1^n}. \]

Case 3: If \( \left( \int_{B_j} |\nabla s|^{2p_j \omega_1^n} \right)^{1/p_j} \leq k \left( \int_{B_j} |s|^{2p_j \omega_1^n} \right)^{1/p_j} \) for infinite \( i \), then

\[ \sup_{B_{r_1}(x,r) \cap R} |\nabla s|^2 \leq k \sup_{B_{r_1}(x,r) \cap R} |s| \leq C(k^{n+1}\nu^{-2n}) \int_{B_{r_1}(x,2r) \cap R} |s|^2 \omega_1^n. \]

\[ \square \]

5.3. \( L^2 \) estimate. In order to construct global holomorphic section on line bundle \( k(K_{\overline{M}} + D) \), we need the following version of \( L^2 \)-estimate due to Demailly (Theorem 5.1 [12]).

**Theorem 5.18.** Let \((M, \omega)\) be a \( n \)-dimensional complete Kähler manifold and \( L \) be a holomorphic line bundle over \( M \) equipped with a smooth hermitian metric such that \( \Theta_h \geq \delta \omega \). Then for every \( L \)-value \((n, 1)\) form \( \tau \) satisfying

\[ \bar{\partial} \tau = 0, \quad \int_M |\tau|^2_{h, \omega} \omega^n < \infty, \]

there exists a \( L \)-valued \((n, 0)\) form \( u \) such that \( \bar{\partial} u = \tau \) and

\[ \int_M |u|^2_{h, \omega} \omega^n \leq \frac{1}{\delta} \int_M |\tau|^2_{h, \omega} \omega^n. \]

For the singular hermitian metric \( h \) on \( L \), by the approximation argument, we have

**Corollary 5.19.** Let \((M, \omega)\) be a \( n \)-dimensional complete Kähler manifold and \( L \) be a holomorphic line bundle over \( M \) equipped with a singular hermitian metric such that \( \Theta_h \geq \delta \omega \) in the current sense. Then for every \( L \)-value \((n, 1)\) form \( \tau \) satisfying

\[ \bar{\partial} \tau = 0, \quad \int_M |\tau|^2_{\tilde{h}, \omega} \omega^n < \infty, \]

there exists a \( L \)-valued \((n, 0)\) form \( u \) such that \( \bar{\partial} u = \tau \) and

\[ \int_M |u|^2_{\tilde{h}, \omega} \omega^n \leq \frac{1}{\delta} \int_M |\tau|^2_{\tilde{h}, \omega} \omega^n. \]

**Proposition 5.20.** \((\mathcal{R} = M_{reg}, k\omega_1)\) is a Kähler manifold (not complete). \( k(K_{\overline{M}} + D) \) is a holomorphic line bundle over \( \mathcal{R} \). Choosing a hermitian metric \( h = \omega_1^{-nk} \), then the curvature form \( \Theta_h = k\omega_1 \). For any smooth \( k(K_{\overline{M}} + D) \)-valued \((0, 1)\) form \( \tau \) satisfying

\[ \bar{\partial} \tau = 0, \quad \text{Supp} \tau \subset \mathcal{R} \]

there exists a \( k(K_{\overline{M}} + D) \)-valued section \( \zeta \) such that \( \bar{\partial} \zeta = \tau \) and

\[ \int_\mathcal{R} |\zeta|^2_{h, k\omega_1} \omega_1^n \leq \int_\mathcal{R} |\tau|^2_{h, k\omega_1} (\omega_1^n). \]
Proof. Since $K_M + D$ is big and semi-ample over $M$, by Lemma 4.1, there exists an effective divisor $E$ on $M$ such that $K_M + D - \epsilon E$ is ample for all sufficiently small $\epsilon > 0$. Let $s_E$ be the defining section of $E$ and $h_E$ be a smooth hermitian metric satisfying $\eta_1 - \epsilon \Theta_E > 0$, where $\eta_1$ is constructed as section 3 and $\Theta_E$ is the curvature form. We consider the following Monge-Ampère equation

$$(\eta_1 - \epsilon \Theta_E - \sqrt{-1}\partial\overline{\partial} \log \log^2 |s_D|^2 + \sqrt{-1}\partial\overline{\partial} u_{1,\epsilon})^n = e^{u_{1,\epsilon}} \cdot \frac{\Omega}{|s_D|^2 \log^2 |s_D|^2}.$$  

Fixed a small $\alpha > 0$, this equation is rewritten as

$$((1 - \alpha)\eta_1 + \alpha(\eta_1 - \frac{\epsilon}{\alpha} \Theta_E) - \sqrt{-1}\partial\overline{\partial} \log \log^2 |s_D|^2 + \sqrt{-1}\partial\overline{\partial} u_{1,\epsilon})^n = e^{u_{1,\epsilon}} \cdot \frac{\Omega}{|s_D|^2 \log^2 |s_D|^2}.$$  

By the same argument of subsection 5.5, we know that $\omega_{1,\epsilon} = \eta_1 - \epsilon \Theta_E - \sqrt{-1}\partial\overline{\partial} \log \log^2 |s_D|^2 + \sqrt{-1}\partial\overline{\partial} u_{1,\epsilon} C^{c\omega}(M_{\text{reg}})$-converge to $\omega_1$ as $\epsilon$ tends to 0. Now we define a family of hermitian metric

$$h_\epsilon = e^{-ku_{1,\epsilon}} \left( \frac{\Omega}{|s_D|^2 \log^2 |s_D|^2} \right)^{-k} e^{-\epsilon k \log |s_E|^2}.$$  

By a direct calculation, $\text{Ric}(h_\epsilon) \geq k\omega_{1,\epsilon}$ in the current sense. $\tau$ has compact support and

$$\lim_{\epsilon \to 0} \int_{M = M \setminus \hat{D}} |\tau|_{h, k\omega_{1,\epsilon}}^2 (k\omega_{1,\epsilon})^n = \int_M |\tau|_{h, k\omega_{1}}^2 (k\omega_{1})^n < \infty.$$  

So by the above corollary, there exists $\zeta_\epsilon$ on $M$ such that

$$\partial_\epsilon \zeta_\epsilon = \tau, \quad \int_M |\zeta_\epsilon|^2_{h_{1, k\omega_{1,\epsilon}}} (k\omega_{1,\epsilon})^n \leq \int_M |\tau|_{h, k\omega_{1,\epsilon}}^2 (k\omega_{1,\epsilon})^n$$  

for each $\epsilon$. This also implies

$$\int_M |\zeta_\epsilon|^2_{h_{1, k\omega_{1}}} (k\omega_{1})^n < \infty.$$  

Hence we can take a subsequence of $\zeta_\epsilon$ converging weakly in $L^2(M, (k\omega_{1})^n)$ to $\zeta$ and

$$\partial_\epsilon \zeta = \tau, \quad \int_M |\zeta|^2_{h_{1, k\omega_{1}}} (k\omega_{1})^n \leq \int_M |\tau|_{h, k\omega_{1}}^2 (k\omega_{1})^n$$  

on $M$. The proof is complete after pushing $\zeta$ to $M_{\text{reg}}$.  

\qed}

5.4. local separation of points. Recall that $\Phi : \overline{M} \to \Phi(\overline{M})$ is defined as in section 3. Naturally, $\Phi$ induce a map $\Phi : R \to \Phi(\overline{M})$. If $s \in H^0(R, k(K_M + D))$, then by Proposition 5.14 we know that $s$ is local bounded and local Lipschitz. So $s$ can be continuous extended to the limit space $M_1$. Furthermore, the map $\Phi_1 : (R, \omega_1) \to (\Phi(\overline{M}), \omega_{FS})$ defined by $\Phi$ can be continuously extend to $\Phi_1 : (M_1, d_1) \to (\Phi(\overline{M}), \omega_{FS})$. This subsection is devoted to demonstrate that this map is injective. First we recall some notations and results which originate from [13].

Definition 5.21. We consider the following data $(p_*, O, U, J, g, L, h, A)$ satisfying

1. $(p_*, O, U, J, g)$ is an open Kähler manifold with a complex structure $J$, a Riemannian metric $g$ and a base point $p_* \in O \subset U$ for an open set $O$.

2. $L \to U$ is a hermitian line bundle equipped with a hermitian metric $h$ and $A$ is the connection induced by the hermitian metric $h$ on $L$, with its curvature $\Theta(A) = \omega$ which is a Kähler form of $g$.

The data $(p_*, O, U, J, g, L, h, A)$ is said to satisfy the $H$-condition if there exist a constant $C$ and a compactly supported smooth section $\sigma : U \to L$ such that
For any integer $Y \leq \frac{n}{2}$, we have

\[
\|\mathbf{L}_{2}(\mathbf{U})\| \leq C(||\mathbf{L}_{2}^{2+1}(\mathbf{O}) + ||\mathbf{L}_{2}^{2}(\mathbf{O})||).
\]

(4) $H_{4}: ||\mathbf{L}_{2}(\mathbf{U})|| < \min\left(\frac{1}{2^{|\psi|}}, 10^{-20}\right)$,

(5) $H_{5}: ||\mathbf{L}_{2}^{2+1}(\mathbf{O}) < \frac{1}{2}G$.

Fix any point $p$ in $M_{1}$, $(M_{1}, p, k_{d_{1}})$ converges in pointed Gromov-Hausdorff topology to a tangent cone $C(Y)$ over the cross section when $k \to \infty$. We still use $p$ for the vertex of $C(Y)$. Let $Y_{reg}$ and $Y_{sing}$ be the regular part and singular part of $Y$ respectively. By [4], $Y_{sing}$ has Hausdorff dimension equal or less than $2n-2$. $C(Y_{reg})\{p\}$ has a natural complex structure induced from the Gromov-Hausdorff limit and the cone metric $g_{C}$ on $C(Y)$ is given by

\[
\omega_{C} = \frac{1}{2}\sqrt{-1}\partial \bar{\partial}r^{2},
\]

where $r$ is the distance function for any point $z \in C(Y)\setminus p$. According to Proposition 5.23, the singular set $S$ of $M_{1}$ must be a locally analytic set by taking the limit of a divisor on $M$. So we also have the following cut-off function on $Y$.

**Proposition 5.22.** For any $\epsilon > 0$, there exists a cut-off function $\gamma$ on $Y$ such that

1. $\gamma \in C^{\infty}(Y_{reg})$ and $0 \leq \gamma \leq 1$,
2. $\gamma$ is supported in the $\epsilon$ neighborhood of $Y_{sing}$,
3. $\gamma = 1$ on a neighborhood of $Y_{sing}$,
4. $\||\mathbf{L}_{2}(Y_{reg})\| < \epsilon$.

We consider the trivial line bundle $L_{C}$ on $C(Y)$ equipped the hermitian metric $h_{C} = e^{-|z|^{2}}$, where $|z|^{2} = r^{2}$. Then the curvature coincides with $\omega_{C}$. $A_{C}$ is denoted by the connection of $L_{C}$ with metric $h_{C}$.

**Lemma 5.23.** Let $p_{*} \in C(Y_{reg})\{p\}$ such that $\frac{1}{2} < e^{-|p_{*}|^{2}}$. Then there exists $U \subset C(Y_{reg})\{p\}$ and an open neighborhood $O \subset U$ of $p_{*}$ such that $(p_{*}, O, J_{C}, g_{C}, L_{C}, h_{C}, A_{C})$ satisfies the $H$-condition.

From the construction in [13], $U$ is a product in $C(Y_{reg})\{p\}$ i.e., there exists $U_{Y} \subset Y_{reg}$ such that $U = \left\{ z = (y, r) \in C(Y) | y \in U_{Y}, r \in (r_{U}, R_{U}) \right\}$. For $m \in \mathbb{Z}^{+}$ defined as [13] (P79), we define

\[
U(m) = \left\{ z = (y, r) \in C(Y) | y \in U_{Y}, r \in (m^{-\frac{1}{2}}r_{U}, R_{U}) \right\}.
\]

For any integer $t$ and $1 \leq t \leq m$, $\mu_{t} : U \to U(m)$ is defined by $\mu_{t}(z) = t^{\frac{1}{2}}z$. The following proposition is due to [13].

**Proposition 5.24.** Suppose $(p_{*}, O, U(m), J_{C}, g_{C}, L_{C}, h_{C}, A_{C})$ constructed as in Lemma 5.23 satisfies the $H$-condition. If $(p_{*}, O, U(m), J, g, L, h, A)$ satisfies $H$-condition and there exists a small constant $\epsilon > 0$ such that

\[
\|g - g_{C}\|_{\mathbf{C}_{0}(U(m))} + ||J - J_{C}\|_{\mathbf{C}_{0}(U(m))} < \epsilon
\]

Then we can find some $1 \leq t \leq m$ such that $(p_{*}, O, U, \mu_{t}^{*}J, \mu_{t}^{*}(tg), \mu_{t}^{*}(L^{t}), \mu_{t}^{*}(h^{t}), \mu_{t}^{*}(A^{t}))$ satisfies $H$-condition.
Fix any point $p$, we can assume that $(M_1, k_r^2 d_1, p)$ converge to a tangent cone $C(Y_\rho)$ for some sequence $k_r$ in pointed Gromov-Hausdorff topology. For an open set $U \subset \subset C(Y_{reg}) \setminus \{p\}$, there is an embedding $\chi_{k_r}: U \to \mathcal{R} = M_{reg}$. Note that $d_1|\mathcal{R} = \omega_1$. The following Lemma follows from the convergence of $(M_1, k_r^2 d_1, p)$.

**Lemma 5.25.** There exist $v$ and $\epsilon > 0$ such that we can find an embedding $\chi_{k_r}$ which satisfies

1. $\frac{1}{2}|z| \leq k_r^2 d_1(p, \chi_{k_r}(z)) \leq 2|z|$,
2. $||\chi_{k_r}^*(k_r \omega_1) - \omega_C||_{C^0(U)} + ||\chi_{k_r}^*(J_{R}) - J_{C}||_{C^0(U)} < \epsilon$.

**Proposition 5.26.** For any two distinct points $p$ and $q$ in $M_1$, we have

$\Phi_1(p) \neq \Phi_1(q)$.

**Proof.** Step 1: For any two distinct points $p$ and $q$, there exist $r$ and $R$ such that $p, q \in B_{d_1}(x_1, r) \subset B_{d_1}(x_1, 2r) \subset B_{d_1}(x_1, R)$. Suppose $C(Y_p)$ and $C(Y_q)$ are two tangent cones of $p$ and $q$ after rescaling $(M_1, d_1)$ at $p$ by $k_{v_p} \to \infty$ and at $q$ by $k_{v_q} \to \infty$. Then according to Lemma 5.23 we can construct two collection of data $(p_*, O_p, U_p(m_p), J_p, g_1, L_p, h_p, A_p)$ and $(q_*, O_q, U_q(m_q), J_q, g_q, L_q, h_q, A_q)$ which satisfy the $H$-condition, where $U_p(m_p) \subset C(Y_p)$ and $U_q(m_q) \subset C(Y_q)$. In addition, we can always assume that

1. the constant $C$ appeared in the $H$-condition for $U_p(m_p)$ and $U_q(m_q)$ are the same,
2. $k_{v_p} = k_{v_q} = k_{v_{p,q}}$,
3. $r_{p_*} := d_{C(Y_p)}(p, p_*)$ and $r_{q_*} := d_{C(Y_q)}(q, q_*)$ are small enough, which define below.

Step 2: From Lemma 5.26 there exist $k_{v_{p,q}}$ such that $\chi_{p,k_{v_{p,q}}} : U_p(m_p) \to \mathcal{R} \text{ and } \chi_{q,k_{v_{p,q}}} : U_q(m_q) \to \mathcal{R}$ satisfy the following:

1. $\frac{1}{2}|z| \leq k_{v_{p,q}}^2 d_1(p, \chi_{p}(z)) \leq 2|z|$,
2. $\frac{1}{2}|z| \leq k_{v_{p,q}}^2 d_1(q, \chi_{q}(z)) \leq 2|z|$,
3. $\chi_{p}(U_p(m_p)) \cap \chi_{q}(U_q(m_q)) = \emptyset$,
4. $||\chi_{p}^*(k_{v_{p,q}} \omega_1) - \omega_p||_{C^0(U_p(m_p))} + ||\chi_{p}^*(J_{R}) - J_p||_{C^0(U_p(m_p))} < \epsilon$,
5. $||\chi_{q}^*(k_{v_{p,q}} \omega_1) - \omega_q||_{C^0(U_q(m_q))} + ||\chi_{q}^*(J_{R}) - J_q||_{C^0(U_q(m_q))} < \epsilon$.

where for the convenience, $\chi_{p,k_{v_{p,q}}}$ and $\chi_{q,k_{v_{p,q}}}$ are denoted by $\chi_p$ and $\chi_q$, respectively.

Step 3: By the Proposition 5.24 and sufficiently small $\epsilon$ in Step 2, there exists $1 \leq t_p \leq m_p$ such that $(p_*, O_p, U_p, \mu_{t_p} \chi_{p}(J_{R}), \mu_{t_p} \chi_{p}^*(k_{v_{p,q}} \omega_1), \mu_{t_p} \chi_{p}(L_{p}^{t_p}), \mu_{t_p} \chi_{p}^*(h_{p}^{t_p}), \mu_{t_p} \chi_{p}^*(A_{p}^{t_p}))$ satisfies the $H$-condition. Thus there is a compactly smooth section $\sigma_p$ such that $\sigma_p$ has properties $H_1$, $H_2$, $H_4$ and $H_5$. By the same argument, there exists $1 \leq t_q \leq m_q$ such that $(q_*, O_q, U_q, \mu_{t_q} \chi_{q}(J_{R}), \mu_{t_q} \chi_{q}^*(k_{v_{p,q}} \omega_1), \mu_{t_q} \chi_{q}^*(L_{q}^{t_q}), \mu_{t_q} \chi_{q}^*(h_{q}^{t_q}), \mu_{t_q} \chi_{q}^*(A_{q}^{t_q}))$ satisfies the $H$-condition. Thus there is a compactly smooth section $\sigma_q$ such that $\sigma_q$ has properties $H_1$, $H_2$, $H_4$ and $H_5$.

Step 4: There is an embedding from $(\mu_{t_p} \chi_{p}^*(L_{p}^{t_p}), U_p)$ to $(k_p(K_M + D), \mathcal{R})$, where $k_p = t_p k_{v_{p,q}}$. So $\sigma_p$ can be viewed as a compactly smooth section of $k_p(K_M + D)$. We now apply Proposition 5.24 to $\tau_p = \tilde{\sigma}_p$. Then there exists a $(k_p(K_M + D))$ valued section $\varsigma_p$ solving the $\tilde{\sigma}$ equation $\tilde{\sigma}_p = \tau_p$ with $\int_{\mathcal{R}} |\varsigma_p|^2(k_p \omega_1)^n \leq \int_{\mathcal{R}} |\tau_p|^2(k_p \omega_1)^n \leq \min \left( \frac{1}{8\sqrt{2}C}, 10^{-20} \right)$.

Let $z_{p_*} = \chi_{p}(p_*)$, then from $H_3$ and $H_5$, we have:

$$\varsigma_p(z_{p_*}) \leq C(||\tilde{\sigma}_p||_{L^{2n+1}(O_p)} + ||\varsigma_p||_{L^2(O_p)}) \leq \frac{1}{8\sqrt{2}C} + \frac{1}{8C} \leq \frac{1}{4}.$$
Set $\sigma'_p = \sigma_p - \varsigma_p$. Then $\sigma'_p$ is a holomorphic section of $k_p(K_M + D)$ over $\mathcal{R}$ and from Proposition 5.15, $\sigma'_p$ can be continuously extended to $M_1$. By the $H$-condition, we have the following relations:

1. $|\sigma'_p(z_p)| > \frac{1}{2}$,
2. $||\sigma'_p||_{L^2(\mathcal{R}, k_pomega_1, h^{kp}_p)} \leq 2(2\pi)^{\frac{n}{2}},$
3. $||\sigma'_p||_{L^2(\mathcal{R}\setminus U_p, k_pomega_1, h^{kp}_p)} = ||\sigma_p||_{L^2(\mathcal{R}\setminus U_p, k_pomega_1, h^{kp}_p)} \leq \min\left(\frac{1}{8\sqrt{2C}}, 10^{-20}\right)$.

Then by Proposition 5.15,

$$|\sigma'_p(p)| \geq |\sigma'_p(z_p)| - \sup_{B_{d_1}(x_1, r)} |\nabla \sigma'_p| h^{kp}_p d_1(p, z_p) \geq \frac{2}{3}$$

when $r_\delta$ is sufficiently small.

Now we restrict $\sigma'_p$ on $U_q$. By $H_3$,

$$|\sigma'_p(z_q)| \leq C||\sigma'_p||_{L^2(\mathcal{R}|U_p, k_pomega_1, h^{kp}_p)} \leq C \min\left(\frac{1}{8\sqrt{2C}}, 10^{-20}\right).$$

Similarly,

$$|\sigma'_p(q)| \leq |\sigma'_p(z_q)| + \sup_{B_{d_1}(x_1, r)} |\nabla \sigma'_p| h^{kp}_p d_1(q, z_q) \leq 2C \min\left(\frac{1}{8\sqrt{2C}}, 10^{-20}\right),$$

when $r_\delta$ is sufficiently small.

Step 5: By the same argument of Step 4, let $q_k = t_q k_{p,q}$, we construct a holomorphic section $\sigma'_q$ such that

$$|\sigma'_q(q)| \geq \frac{2}{5}, \quad |\sigma'_q(p)| \leq 2C \min\left(\frac{1}{8\sqrt{2C}}, 10^{-20}\right).$$

Step 6: Set $K = t_q k_p = t_p k_q$. Then $(\sigma'_p)^{t_q}$ and $(\sigma'_q)^{t_p}$ are holomorphic section of $K(K_M + D)$ which can be continuously extended to $M_1$. Modifying the constant $10^{-20}$ as small as enough, we have $|(\sigma'_p)^{t_q}(p)| \gg |(\sigma'_q)^{t_p}(q)|$ and $|(\sigma'_q)^{t_q}(q)| \gg |(\sigma'_p)^{t_p}(q)|$. Therefore, we conclude that $\Phi_1$ is injective. \qed

5.5. Surjectivity of $\Phi_1$. In this subsection we will complete the proof of Theorem 1.8. Let $u_1$ be the solution to the following equation in the current sense

$$(\eta_1 - \sqrt{-1} \partial \bar{\partial} \log \log^2 |s_D|^2 + \sqrt{-1} \partial \bar{\partial} u_1)^n = e^{u_1} \frac{\Omega}{|s_D|^2 \log^2 |s_D|^2}.$$  

Since $K_M + D$ is big and semi-ample, there exists an effective divisor $E = \sum a_i E_i$ such that $K_M + D - \epsilon E > 0$ for all sufficiently small $\epsilon > 0$.

Let $p \in \text{Supp} E \setminus D$ and $\pi : \overline{M} \rightarrow M$ be the blow up at $p$ with exceptional divisor $\pi^{-1}(p) = F$. Set $\tilde{D} = \pi^{-1}(D)$ and $\tilde{E} = \sum a_i E_i$, where $\tilde{E}_i = \pi^{-1}(E_i) - F$. $s_{\tilde{E}_i}$, $s_F$ and $s_D$ are denoted by the defining sections of line bundles $L_{\tilde{E}_i}$, $L_F$ and $L_D$ respectively. Let $\chi$ be fixed Kähler metric on $\overline{M}$. We choose appropriate hermitian metrics $h_{\tilde{E}_i}$ and $h_F$ such that

$$\pi^* \eta_1 + \delta \sqrt{-1} \partial \bar{\partial} \log |s_F|^2 + \delta \sum a_i \sqrt{-1} \partial \bar{\partial} \log |s_{\tilde{E}_i}|^2 \geq \mu \chi$$
for some small constants \( \delta \) and \( \mu \). Note that \( \widetilde{\Omega} = \frac{\pi^*\Omega}{(s_F)_B} \) defines a smooth volume form on \( \widetilde{M} \). We consider the following Monge-Ampère equation on \( \widetilde{M} \):
\[
(\tilde{\eta}_1 + \epsilon \chi + \sqrt{-1} \partial \bar{\partial} \varphi_{\epsilon})^n = e^{\varphi_{\epsilon}} (\epsilon^2 + |s_F|^2)^{n-1} \frac{\widetilde{\Omega}}{|s_D|^2 h_B \log^2 |s_D|^2 h_B},
\]
where \( \tilde{\eta}_1 = \pi^* \eta_1 - \sqrt{-1} \partial \bar{\partial} \log |s_D| h_B^2 \) and \( h_B = \pi^* h_D \). By Theorem 1 of [10], the equation has a unique smooth solution \( \varphi_{\epsilon} \) for each \( \epsilon \); moreover
\[
\tilde{\omega}_{\epsilon} := \tilde{\eta}_1 + \epsilon \chi + \sqrt{-1} \partial \bar{\partial} \varphi_{\epsilon}
\]
is a smooth complete Kähler metric on \( \widetilde{M} \setminus \bar{D} \).

**Lemma 5.28.** For any \( \delta \) and \( \epsilon \), there exist two constants \( C(\delta) \) and \( C \) independent of \( \epsilon \) such that
\[
-C(\delta) + \delta \log |s_F|^2 + \delta \sum_i a_i \log |s_{E_i}|^2 \leq \varphi_{\epsilon} \leq C + \log |s_D|^2 h_B^2.
\]

**Proof.** For the upper bound, let
\[
V_\epsilon = \int \frac{(\epsilon^2 + |s_F|^2)^{n-1} \frac{\widetilde{\Omega}}{|s_D|^2 h_B \log^2 |s_D|^2 h_B}}{\Omega},
\]
so we have \( V_1 \geq V_\epsilon \geq V_0 \). Hence \( V_\epsilon \) is uniformly bounded. Denote \( (\epsilon^2 + |s_F|^2)^{n-1} \frac{\widetilde{\Omega}}{|s_D|^2 h_B \log^2 |s_D|^2 h_B} \) by \( \widetilde{\Omega}_\epsilon \), then we have the following calculation
\[
\frac{1}{V_\epsilon} \int \varphi_{\epsilon} \widetilde{\Omega}_\epsilon = \frac{1}{V_\epsilon} \int \log \left( \frac{\tilde{\omega}_{\epsilon}^n}{\Omega} \right) \widetilde{\Omega}_\epsilon \leq \log \int \tilde{\omega}_{\epsilon}^n - \log V_\epsilon
\]
\[
= \log \int (\pi^* \eta + \epsilon \chi)^n - \log V_\epsilon \leq C,
\]
where the third equality bases on a Lemma ([10] P410). Since \( \varphi_{\epsilon} - \log |s_D|^2 h_B^2 \in PSH(\widetilde{M}, \pi^* \eta + \epsilon \chi) \), the mean inequality implies
\[
\sup \varphi_{\epsilon} \leq C + \log |s_D|^2 h_B^2.
\]

For the lower bound, we set \( \varphi_{\epsilon, \delta} = \varphi_{\epsilon} - \delta \log |s_F|^2 - \delta \sum_i a_i \log |s_{E_i}|^2 \) and denote \( |s_D|^2 h_B^2 = |s_D|^2 \delta \), then the equation (5.27) is equivalent to
\[
(\eta_1^{\delta} + \delta \sqrt{-1} \partial \bar{\partial} \log |s_F|^2 + \delta \sum_i a_i \sqrt{-1} \partial \bar{\partial} \log |s_{E_i}|^2 + \epsilon \chi + \sqrt{-1} \partial \bar{\partial} \varphi_{\epsilon, \delta} + \sqrt{-1} \partial \bar{\partial} \log \frac{|s_D|^2 h_B^2}{|s_D|^2})^n
\]
\[
= e^{\varphi_{\epsilon, \delta} + \frac{\log |s_D|^2 h_B^2}{|s_D|^2}} \prod_i |s_{E_i}|^{2a_i \delta} \cdot |s_F|^2 \delta \cdot (\epsilon^2 + |s_F|^2)^{n-1} \cdot \frac{\widetilde{\Omega}'}{|s_D|^2 h_B^2 \log^2 |s_D|^2 h_B^2}.
\]
where $\eta^\delta = \pi^* \eta_1 - \sqrt{-1} \partial \bar{\partial} \log |s_D|^2$ satisfying $
abla^\delta - \delta \sqrt{-1} \partial \bar{\partial} \log |s_F|^2 + \delta \sum_i a_i \sqrt{-1} \partial \bar{\partial} \log |s_{E_i}|^2 > 0$ and $\widetilde{\Omega}' = \frac{|s_D|^2}{|s_D|^2} \widetilde{\Omega}$. We introduce the following equation

$$(\eta^\delta - \delta \sum_i \Theta_{E_i} - \delta \Theta_F + \epsilon \chi + \sqrt{-1} \partial \bar{\partial} \psi_{\epsilon, \delta})^n = \epsilon^{\psi_{\epsilon, \delta}} \cdot (\epsilon^2 + |s_F|^2)^{n-1} \cdot \frac{\widetilde{\Omega}'}{|s_D|^2 |s_D|^2}.$$ 

By the generalized maximum principle, there exists a sequence $\{x_i\}$ such that $\lim_{i \to \infty} \psi_{\epsilon, \delta}(x_i) = \inf \psi_{\epsilon, \delta}$ and $\lim_{i \to \infty} \sqrt{-1} \partial \bar{\partial} \psi_{\epsilon, \delta}(x_i) > 0$. Then we have

$$\inf \psi_{\epsilon, \delta} \geq (n-1) \log \frac{1}{\epsilon^2 + |s_F|^2} + \log \frac{|s_D|^2 |s_D|^2 (\eta^\delta - \delta \sum_i \Theta_{E_i} - \delta \Theta_F + \epsilon \chi)^n}{\Omega'} \geq -C(\delta).$$

Set $H_{\epsilon, \delta} = \varphi_{\epsilon, \delta} - \psi_{\epsilon, \delta}$ and $v^\delta = \eta^\delta - \delta \sum_i \Theta_{E_i} - \delta \Theta_F + \epsilon \chi$, then

$$\log \left( \frac{v^\delta + \sqrt{-1} \partial \bar{\partial} \psi_{\epsilon, \delta} + \sqrt{-1} \partial \bar{\partial} H_{\epsilon, \delta} + \sqrt{-1} \partial \bar{\partial} \log \frac{|s_D|^2 |s_D|^2}{|s_F|^2}}{v^\delta + \sqrt{-1} \partial \bar{\partial} \psi_{\epsilon, \delta}} \right)^n = H_{\epsilon, \delta} + \log \frac{|s_D|^2 |s_D|^2}{|s_F|^2} + \delta \log |s_F|^2 + \delta \sum_i a_i \log |s_{E_i}|^2.$$ 

By the generalized maximum principle again

$$\inf \left( H_{\epsilon, \delta} + \log \frac{|s_D|^2 |s_D|^2}{|s_F|^2} \right) \geq -C(\delta).$$

Note that $\log \frac{|s_D|^2 |s_D|^2}{|s_F|^2}$ is a smooth function on $\widetilde{M}$, so it can be bounded by $C(\delta)$. Moreover we get the lower bound of $\varphi_{\epsilon}$. 

**Lemma 5.29.** There exists a constant $C$ independent of $\epsilon$ such that on $\widetilde{M} \setminus \bar{D}$, we have

$$\text{Ric}(\tilde{\omega}_e) \leq -\tilde{\omega}_e + C \chi.$$ 

**Proof.** We observe some following consequences:

1. $\pi^* \eta_1 \leq C \chi$,
2. Since $\Omega$ is a smooth volume form, $\text{Ric}(\Omega) \leq C \chi$,
3. $\Theta_{\bar{D}} \geq -C \chi$,
4. $\sqrt{-1} \partial \bar{\partial} \log (\epsilon^2 + |s_F|^2) \geq -C \chi.$

Thus by a simple calculation we get the Lemma. 

Set $\chi' = \chi - \sqrt{-1} \partial \bar{\partial} \log \frac{|s_D|^2}{|s_D|^2}$, then by a calculation we have

$$\chi' = \chi - 2 \sqrt{-1} \partial \bar{\partial} \log \frac{|s_D|^2}{|s_D|^2} + 2 \frac{\sqrt{-1} \partial \bar{\partial} \log |s_D|^2 \wedge \bar{\partial} \log |s_D|^2}{\log^2 |s_D|^2}.$$ 

Take an appropriate hermitian metric $|\cdot|$, we can assume that

$$\frac{1}{2} \chi \leq \frac{1}{2} \chi + 2 \frac{\sqrt{-1} \partial \bar{\partial} \log |s_D|^2 \wedge \bar{\partial} \log |s_D|^2}{\log^2 |s_D|^2} \leq \chi' \leq 2 \chi + 2 \frac{\sqrt{-1} \partial \bar{\partial} \log |s_D|^2 \wedge \bar{\partial} \log |s_D|^2}{\log^2 |s_D|^2}.$$
So, by Lemma 5.29 we have
\[ \text{Ric}(\tilde{\omega}_\epsilon) \leq -\tilde{\omega}_\epsilon + C\chi'. \]

On the other hand, we can choose a sufficiently large \( A' \), a sufficiently small \( \alpha \) and a hermitian metric \( |.| \) such that
\[
\begin{align*}
A'\pi^*\eta_1 - \Theta_F - \sum a_i\Theta_{\tilde{E}_i} - A'\sqrt{-1}\tilde{\omega}_\epsilon \log |s_{\tilde{D}}|^2 &
\geq 3\chi - 2A'\sqrt{-1}\tilde{\omega}_\epsilon \log |s_{\tilde{D}}|^2 + 2A'\sqrt{-1}\tilde{\omega}_\epsilon \log |s_{\tilde{D}}|^2 \\
&\geq 2\chi + 2A'\sqrt{-1}\tilde{\omega}_\epsilon \log |s_{\tilde{D}}|^2 \geq \alpha'.
\end{align*}
\]

From now on we always assume that the hermitian metric \( |.| \) on \( L_{\tilde{D}} \) satisfy \( A'\pi^*\eta_1 - \Theta_F - \sum a_i\Theta_{\tilde{E}_i} - A'\sqrt{-1}\tilde{\omega}_\epsilon \log |s_{\tilde{D}}|^2 \geq \alpha' \).

**Lemma 5.30.** There exist \( C \) and \( \lambda \) independent of \( \epsilon \) such that
\[
\tilde{\omega}_\epsilon \leq \frac{C(\log |s_{\tilde{D}}|^2)^C}{|s_{\tilde{D}}|^{2\lambda} \cdot |s_{\tilde{F}}|^{2\lambda^2} \cdot \prod |s_{\tilde{E}_i}|^{2\lambda^2} \chi'}.
\]

**Proof.** By Yau’s Schwarz Lemma [25] and Lemma 5.29 we have
\[
\Delta_{\tilde{\omega}_\epsilon} \log tr_{\chi'}\tilde{\omega}_\epsilon \geq -Ctr_{\chi'}\chi' - \frac{C}{tr_{\chi'}\tilde{\omega}_\epsilon}.
\]

There is a fact that is
\[
\Delta_{\tilde{\omega}_\epsilon} \varphi \leq n - tr_{\tilde{\omega}_\epsilon}(\pi^*\eta_1 - \sqrt{-1}\tilde{\omega}_\epsilon \log |s_{\tilde{D}}|^2).
\]

Let \( H = \log(|s_{\tilde{D}}|^{2A} \cdot |s_{\tilde{F}}|^{2A^2} \cdot \prod |s_{\tilde{E}_i}|^{2A^2} \cdot tr_{\chi'}\tilde{\omega}_\epsilon) - A^2 A' \varphi \), where \( A' \) is chosen as above and \( A \) is defined below. Then on \( \overline{M \setminus (\tilde{D} \cup F \cup \text{Supp}E)} \), we have
\[
\Delta_{\tilde{\omega}_\epsilon} H \geq -Ctr_{\tilde{\omega}_\epsilon}A' - A^2 A'n + Atr_{\tilde{\omega}_\epsilon}(AA'\pi^*\eta_1 - A\Theta_F - A \sum a_i\Theta_{\tilde{E}_i} - \Theta_{\tilde{D}} - AA'\sqrt{-1}\tilde{\omega}_\epsilon \log |s_{\tilde{D}}|^2).
\]

When \( A \) is sufficiently large we observed that
\[
Atr_{\tilde{\omega}_\epsilon}(A(A'\pi^*\eta_1 - \Theta_F - \sum a_i\Theta_{\tilde{E}_i} - A'\sqrt{-1}\tilde{\omega}_\epsilon \log |s_{\tilde{D}}|^2) - \Theta_{\tilde{D}}) \geq (C + 1)\chi'.
\]

Therefore
\[
\Delta_{\tilde{\omega}_\epsilon} H \geq tr_{\tilde{\omega}_\epsilon}A' - \frac{C}{tr_{\chi'}\tilde{\omega}_\epsilon} - A^2 A'n.
\]

By the generalized maximum principle, there exists a sequence \( \{x_i\} \) such that \( \lim_{i \to \infty} H(x_i) = \sup H \) and \( \lim_{i \to \infty} \sqrt{-1}\partial\bar{\partial} H(x_i) \leq 0 \). Thus,
\[
\lim_{i \to \infty} tr_{\chi'}\tilde{\omega}_\epsilon \cdot (tr_{\tilde{\omega}_\epsilon}A' - A^2 A'n)(x_i) \leq C.
\]

Since
\[
\tilde{\omega}_\epsilon^n = e^{\varepsilon_s} \frac{\Omega}{|s_{\tilde{D}}|^{2\log^2 |s_{\tilde{D}}|^2}} \leq C \log^2 |s_{\tilde{D}}|^2 (\chi')^n,
\]
then we have
\[
\frac{1}{\log^2|s_D|^2} \left( \frac{1}{C} \log^2 |s_D|^2 (tr_{\tilde{\omega}_t})^{\frac{1}{n-1}} \right) \leq Ctr_{\tilde{\omega}} \chi'.
\]
Furthermore,
\[
\lim_{i \to \infty} tr_{\tilde{\omega}_t} \left( \frac{1}{C} \log^2 |s_D|^2 (tr_{\tilde{\omega}_t})^{\frac{1}{n-1}} - A^2 A'n \right)(x_i) \leq C.
\] (5.31)

If
\[
\lim_{i \to \infty} (tr_{\tilde{\omega}_t})^{\frac{1}{n-1}}(x_i) \leq \lim_{i \to \infty} 2A^2 A'nC \log^2 |s_D|^2(x_i),
\]
then
\[
\lim_{i \to \infty} (tr_{\tilde{\omega}_t})(x_i) \leq \lim_{i \to \infty} (2A^2 A'nC)^{n-1}(\log^2 |s_D|^2)^{n-1}(x_i).
\]
Otherwise
\[
\lim_{i \to \infty} (tr_{\tilde{\omega}_t})^{\frac{1}{n-1}}(x_i) \geq \lim_{i \to \infty} 2A^2 A'nC \log^2 |s_D|^2(x_i).
\]

From (5.31) we know
\[
\lim_{i \to \infty} A^2 A'ntr_{\tilde{\omega}_t}(x_i) \leq C.
\]

In general we have
\[
\lim_{i \to \infty} (tr_{\tilde{\omega}_t})(x_i) \leq \lim_{i \to \infty} C(\log^2 |s_D|^2)^C(x_i).
\]

By the definition of \( H \) and Lemma 5.28, we have
\[
H(x) \leq \lim_{i \to \infty} \left( \log (|s_D|^{2A} \cdot |s_E|^{2A} \cdot \prod_i |s_{E_i}|^{2A^2} \cdot C(\log^2 |s_D|^2)^C) + A^2 A'C(\delta) - A^2 A'\delta \log |s_E|^2 - A^2 A' \delta \sum_i a_i \log |s_{E_i}|^2 \right)(x_i) \leq C
\]
when choosing \( A >> A' \) and sufficiently small \( \delta \). So we get this Lemma from the upper bound of \( \varphi_t \).

Let \( B \) be a disk in \( M \setminus D \) centered at \( p \). Denote \( f_1, f_2, \ldots, f_N \) by the defining functions of divisors \( E_1, E_2, \ldots, E_N \) on \( B = \pi^{-1}(B) \). From Lemma 5.30, we obtain the following corollary.

**Corollary 5.32.** There exist \( C \) and \( \lambda \) independent of \( \epsilon \) such that
\[
(tr_{\tilde{\omega}_t})_{|\partial B} \leq \frac{C}{\prod_i |f_i|^{2\lambda^2}} .
\]

Let \( \tilde{\chi} \) be the pull back of the Euclidean metric \( \sqrt{-1} \sum_j dz_j \wedge d\bar{z}_j \) on \( B \). Then \( \tilde{\chi} \) is a smooth closed nonnegative \((1, 1)\) form and is a Kähler metric on \( \tilde{\mathcal{B}} \setminus F \). The following Lemma is due to Song [21].

**Lemma 5.33.** There exist a constant \( C > 0 \), a sufficiently small \( \epsilon_0 > 0 \) and a smooth hermitian metric \( h_F' \) on \( L_F \) such that on \( \tilde{B} \)
\[
C^{-1} \tilde{\chi} \leq \chi' \leq C \frac{\tilde{\chi}}{|s_F|_{h_F'}^2}
\]
and
\[
\pi^* \eta_1 - \sqrt{-1} \partial \bar{\partial} \log |s_D|^2 - \epsilon_0 \Theta_{h_F'} > C^{-1} \chi'.
\]
Lemma 5.34. There exist $0 < \beta < 1$, $C > 0$ and $\Lambda > 0$ independent of $\epsilon$ such that

$$\tilde{\omega}_\epsilon \leq \frac{C}{|s_F|_{h_p}^{2(1-\beta)} \cdot \prod_i |f_i|^{2\Lambda} \chi'}, \text{ in } \tilde{B}.$$  

Moreover, we have

$$\pi^* \omega_1 \leq \frac{C}{|s_F|_{h_p}^{2(1-\beta)} \cdot \prod_i |f_i|^{2\Lambda} \chi'}, \text{ in } \tilde{B},$$

where $\omega_1 = \eta_1 - \sqrt{-1} \partial \bar{\partial} \log log^2 |s_D|^2 + \sqrt{-1} \partial \bar{\partial} u_1$.

Proof. Let $H = \log \left( |s_F|_{h_p}^{2(1+r)} \cdot \prod_i |f_i|^{2\lambda^2} \cdot tr_\chi \tilde{\omega}_\epsilon \right) - A \varphi_\epsilon$ for some sufficiently large $A$ and sufficiently small $r$. There are some facts on $\tilde{B}\setminus(F \cup \text{Supp}\ E)$:

1. $\Delta_{\tilde{\omega}_\epsilon} \log |s_F|_{h_p}^2 = -tr_\tilde{\omega}_\epsilon \Theta_{h_p}$,
2. $\Delta_{\tilde{\omega}_\epsilon} \log \prod_i |f_i|^{2\lambda^2} = 0$,
3. $\Delta_{\tilde{\omega}_\epsilon} \varphi_\epsilon = n - tr_{\tilde{\omega}_\epsilon} (\pi^* \eta_1 - \sqrt{-1} \partial \bar{\partial} \log log^2 |s_D|^2) - c tr_{\tilde{\omega}_\epsilon} \chi$,
4. $\Delta_{\tilde{\omega}_\epsilon} tr_\chi \tilde{\omega}_\epsilon \geq \frac{c tr_\chi (\text{Ric}(\tilde{\omega}_\epsilon))}{tr_\chi \tilde{\omega}_\epsilon} \geq 1 - \frac{C}{|s_F|_{h_p}^2 tr_\chi \tilde{\omega}_\epsilon}$.

Thus, on $\tilde{B}\setminus(F \cup \text{Supp}\ E)$, we have

$$\Delta_{\tilde{\omega}_\epsilon} H \geq 1 - \frac{C}{|s_F|_{h_p}^2 tr_\chi \tilde{\omega}_\epsilon} - An - (r + 1) tr_{\tilde{\omega}_\epsilon} \Theta_{h_p} + A tr_{\tilde{\omega}_\epsilon} (\pi^* \eta_1 - \sqrt{-1} \partial \bar{\partial} \log log^2 |s_D|^2)$$

$$\geq c tr_{\tilde{\omega}_\epsilon} \chi' - \frac{C}{|s_F|_{h_p}^2 tr_\chi \tilde{\omega}_\epsilon} - C,$$

where the last inequality bases on Lemma 5.33 and choosing sufficiently large $A$ and small $r$. By Yau’s Schwarz Lemma [25],

$$\Delta_{\tilde{\omega}_\epsilon} \log tr_\chi \tilde{\omega}_\epsilon \geq -C_1 tr_{\tilde{\omega}_\epsilon} \chi' - \frac{C_1}{tr_\chi \tilde{\omega}_\epsilon}.$$  

Let $G = H + \frac{c}{2tr_\chi \tilde{\omega}_\epsilon} \log \left( \prod_i |f_i|^{2\lambda^2 + 2} \cdot tr_\chi \tilde{\omega}_\epsilon \right)$. Then

$$\Delta_{\tilde{\omega}_\epsilon} G \geq c tr_{\tilde{\omega}_\epsilon} \chi' - \frac{C}{|s_F|_{h_p}^2 tr_\chi \tilde{\omega}_\epsilon} - C - \frac{c}{2} tr_{\tilde{\omega}_\epsilon} \chi' - \frac{c}{2tr_\chi \tilde{\omega}_\epsilon}$$

$$\geq \frac{c}{2} tr_{\tilde{\omega}_\epsilon} \chi' - \tilde{C} - \frac{C}{|s_F|_{h_p}^2 tr_\chi \tilde{\omega}_\epsilon}.$$  

For a fixed sufficiently large $\lambda > 0$, there exists a constant $C > 0$ such that

$$\sup_{\partial \tilde{B}} G \leq C$$

from the estimate in Corollary 5.32 and Lemma 5.28.

So we can assume that

$$\sup_{\tilde{B}} G = G(p_{\text{max}})$$

for some $p_{\text{max}} \in \tilde{B}\setminus(F \cup \text{Supp}\ E)$. Then at $p_{\text{max}}$, we have

$$|s_F|_{h_p}^2 tr_\chi \tilde{\omega}_\epsilon (c tr_{\tilde{\omega}_\epsilon} \chi' - 2\tilde{C})(p_{\text{max}}) \leq C.$$
Note that
\[
\frac{1}{\log^2|s_D|^2} (tr_{\chi'} \tilde{\omega}_e)^{\frac{1}{n-1}} \leq C tr_{\omega} \chi'.
\]
Then according to the boundedness of \( \frac{1}{\log^2|s_D|^2} \) in \( \tilde{B} \), we get
\[
|s_F|_{h_F'}^2 tr_{\chi'} \tilde{\omega}_e(\chi'(tr_{\chi'} \tilde{\omega}_e)^{\frac{1}{n-1}} - 2\tilde{C})(p_{\max}) \leq C. \tag{5.35}
\]
If \((tr_{\chi'} \tilde{\omega}_e)^{\frac{1}{n-1}}(p_{\max}) \leq \frac{2\tilde{C}}{3} \), then \( G \) is bounded from above by a uniform constant.
Otherwise \((tr_{\chi'} \tilde{\omega}_e)^{\frac{1}{n-1}}(p_{\max}) \geq \frac{2\tilde{C}}{3} \), i.e., \( C \leq \frac{2\tilde{C}}{3} (tr_{\chi'} \tilde{\omega}_e)^{\frac{1}{n-1}}(p_{\max}) \). Then by equation (5.35) we get
\[
|s_F|_{h_F'}^2 \cdot tr_{\chi'} \tilde{\omega}_e \cdot \frac{C'}{3} (tr_{\chi'} \tilde{\omega}_e)^{\frac{1}{n-1}}(p_{\max}) \leq C,
\]
i.e.
\[
\log |s_F|_{h_F'}^2 + \log tr_{\chi'} \tilde{\omega}_e + \frac{1}{n-1} \log tr_{\chi'} \tilde{\omega}_e(p_{\max}) \leq C.
\]
According to the definition of \( G \), Lemma \[3.28\] and Lemma \[3.30\] we have \( G \leq C \) when we choose large \( C_1 \).

In sum, in all cases, we have \( G \leq C \). Then
\[
|s_F|_{h_F'}^2(2^{1+r}) \cdot \prod_i |f_i|^2 \lambda^2 + \frac{2\tilde{C}}{3} (2\lambda^2 + 2) \cdot tr_{\chi'} \tilde{\omega}_e \cdot (tr_{\chi'} \tilde{\omega}_e)^{\frac{1}{n-1}} \leq C.
\]
Note that \( tr_{\chi'} \tilde{\omega}_e \geq C^{-1} tr_{\chi'} \tilde{\omega}_e \), then we observe
\[
(tr_{\chi'} \tilde{\omega}_e)^{1+\frac{2\tilde{C}}{3}} \leq \frac{C}{|s_F|_{h_F'}^2(2^{1+r}) \cdot \prod_i |f_i|^2 \lambda^2 + \frac{2\tilde{C}}{3} (2\lambda^2 + 2)}.
\]
If we choose \( r = \frac{\tilde{C}}{12\lambda^2} \), then \( 1 - \beta = \frac{1 + \tilde{C}}{12\lambda^2} \) for some \( \beta \in (0, 1) \). Furthermore, there exists a constant \( \Lambda > 0 \) such that
\[
\tilde{\omega}_e \leq \frac{C}{|s_F|_{h_F'}^2(2^{1-\beta}) \cdot \prod_i |f_i|^2 \chi'}. \tag{5.36}
\]

From now on we turn to the Gromov-Hausdorff convergence. Recall
\[
\overline{D_1} := \{ x \in M_1 | \text{there exists } x_i \in \overline{D} \text{ such that } x_i \xrightarrow{d_{GH}} x \},
\]
where \( \overline{D} \) is a divisor such that \( D \cup S_M \subset \overline{D} \). By the Proposition \[5.34\] \((M \setminus \overline{D}, \omega_1)\) is isometric to \((M \setminus \overline{D}, \omega_1)\).

**Lemma 5.36.** \( \Phi_1 : M_1 \setminus \overline{D_1} \rightarrow \Phi(M \setminus \overline{D}) \) is bijective.

**Proof.** Note that \((M_1 \setminus \overline{D_1}) \subset R = M_{\text{reg}} \) and \( \Phi|_{M_{\text{reg}}} \) is biholomorphic, so \( \Phi_1 \) is bijective. \( \square \)

**Lemma 5.37.** \( \Phi_1 : \overline{D_1} \rightarrow \Phi(\overline{D}) \) is surjective.

**Proof.** For any \( x' \in \overline{D_1} \), there exists a curve \( \gamma : [0, 1] \rightarrow M \setminus \overline{D} \) with \( \gamma(0) = x' \) and \( \gamma([0, 1]) \subset M \setminus \overline{D} \) such that \( \int_0^1 |\gamma'|_{\omega_1} dt < \infty \) by Lemma \[5.34\]. The curve \( \gamma(t) \) gives a curve \( \tilde{\gamma}(t) \) for \( 0 < t < 1 \) through an isometry from \((M \setminus \overline{D}, \omega_1)\) to \((M_1 \setminus \overline{D_1}, \omega_1)\). Hence there is a limit \( x'' = \lim_{t \to 0} \tilde{\gamma}(t) \) in \( M_1 \). Then
\[
\Phi_1(x'') = \lim_{t \to 0} \Phi_1(\tilde{\gamma}(t)) = \lim_{t \to 0} \Phi(\gamma(t)) = \Phi(x').
\]
Therefore, $\Phi_1$ is surjective.

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