CR-INVARIANTS AND THE SCATTERING OPERATOR FOR COMPLEX MANIFOLDS WITH BOUNDARY

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1. Introduction

The purpose of this paper is to describe certain CR-covariant differential operators on a strictly pseudoconvex CR manifold $M$ as residues of the scattering operator for the Laplacian on an ambient complex Kähler manifold $X$ having $M$ as a `CR-infinity.' We also characterize the CR $Q$-curvature in terms of the scattering operator. Our results parallel earlier results of Graham and Zworski [14], who showed that if $X$ is an asymptotically hyperbolic manifold carrying a Poincaré-Einstein metric, the $Q$-curvature and certain conformally covariant differential operators on the `conformal infinity' $M$ of $X$ can be recovered from the scattering operator on $X$. The results in this paper were announced in [18].

To describe our results, we first recall some basic notions of CR geometry and recent results [8, 9] concerning CR-covariant differential operators and CR-analogues of $Q$-curvature. If $M$ is a smooth, orientable manifold of real dimension $(2n + 1)$, a CR-structure on $M$ is a real hyperplane bundle $H$ on $TM$ together with a smooth
bundle map \( J : H \to H \) with \( J^2 = -1 \) that determines an almost complex structure on \( H \). We denote by \( T_{1,0} \) the eigenspace of \( J \) on \( H \otimes \mathbb{C} \) with eigenvalue \( +i \);
\[ \text{we will always assume that the CR-structure on } M \text{ is integrable in the sense that } [T_{1,0}, T_{1,0}] \subset T_{1,0}. \]
We will assume that \( M \) is orientable, so that the line bundle \( H^+ \subset T^* M \) admits a nonvanishing global section. A pseudo-Hermitian structure on \( M \) is smooth, nonvanishing section \( \theta \) of \( H^+ \). The Levi form of \( \theta \) is the Hermitian form \( L_\theta(v, w) = d\theta(v, Jw) \) on \( H \). The CR structure on \( M \) is called strictly pseudoconvex if the Levi form is positive definite. Note that this condition is actually independent of the choice of \( \theta \) compatible with a given orientation of \( M \).
We will always assume that \( M \) is strictly pseudoconvex in what follows. It follows from strict pseudoconvexity that \( \theta \) is a contact form, and the form \( \theta \wedge (d\theta)^n \) is a volume form that defines a natural inner product on \( C^\infty(M) \) by integration.
The pseudo-Hermitian structure on \( M \) also determines a connection on \( TM \), the Tanaka-Webster connection \( \nabla_\theta \); the basic data of pseudo-Hermitian geometry are the curvature and torsion of this connection (see [27], [29]).

Given a fixed CR-structure \((H, J)\) on \( M \), any nonvanishing section \( \overline{\theta} \) of \( H^+ \) compatible with a given orientation takes the form \( e^{2T} \theta \) for a fixed section \( \theta \) of \( H^+ \) and some function \( T \in C^\infty(M) \). The corresponding Levi form is given by \( L_{\overline{\theta}} = e^{2T}L_\theta \). In this sense the CR-structure determines a conformal class of pseudo-Hermitian structures on \( M \).

For strictly pseudoconvex CR-manifolds, Fefferman and Hirachi [8] proved the existence of CR-covariant differential operators \( P_k \) of order \( 2k \), \( k = 1, 2, \ldots, n + 1 \), whose principal parts are \( \Delta_k^\theta \), where \( \Delta_\theta \) is the sub-Laplacian on \( M \) with respect to the pseudo-Hermitian structure \( \theta \). They exploit Fefferman’s construction (formulated intrinsically by Lee in [21]) of a circle bundle \( C \) over \( M \) with a natural conformal structure and a mapping \( \theta \mapsto g_\theta \) from pseudo-Hermitian structures on \( M \) to Lorentz metrics on \( C \) that respects conformal classes. They then construct the conformally covariant differential operators found in [12] (referred to here as GJMS operators) on \( C \), and show that these operators pull back to CR-covariant differential operators on \( M \). The CR \( Q \)-curvature may be similarly defined as a pullback to \( M \) of Branson’s \( Q \)-curvature on the circle bundle \( C \). Here we will show that the operators \( P_k \) on \( M \) occur as residues for the scattering operator associated to a natural scattering problem with \( M \) as the boundary at infinity, and that the CR \( Q \)-curvature \( Q_\theta^\text{CR} \) can be computed from the scattering operator.

To describe the scattering problem, we first discuss its geometric setting. Recall that if \( M \) is an integrable, strictly pseudoconvex CR-manifold of dimension \( (2n+1) \) with \( n \geq 2 \), there is a complex manifold \( X \) of complex dimension \( m = n + 1 \) having \( M \) as its boundary so that the CR-structure on \( M \) is induced from the complex structure on \( X \) (this result is false, in general, when \( n = 1 \); see [17]). Let \( \varphi \) be a defining function for \( M \) and denote by \( \bar{X} \) the interior of \( X \) (we take \( \varphi < 0 \) in \( \bar{X} \)). The associated Kähler metric \( g \) on \( X \) is the Kähler metric with Kähler form

\[
\omega_\varphi = -\frac{i}{2} \partial \bar{\partial} \log(-\varphi)
\]

in a neighborhood of \( M \), extended smoothly to all of \( X \). The metric has the form

\[
g_\varphi = -\frac{\eta}{\varphi} + (1 - r \varphi) \left( \frac{d\varphi^2}{\varphi^2} + \frac{\Theta^2}{\varphi^2} \right).
\]

(1.1)
in a neighborhood of $M$, where $\eta$ and $\Theta$ have Taylor series to all orders in $\varphi$ at $\varphi = 0$. The boundary values $\Theta|_M = \theta$ and $\eta|_H = h$ induce respectively a contact form on $M$ and a Hermitian metric on $H$. The function $r$ is a smooth function, the transverse curvature, which depends on the choice of $\varphi$ (see [13]). Thus, the conformal class of a Hermitian metric $h$ on $H$, a subbundle of $TM$, is a kind of ‘Dirichlet datum at infinity’ for the metric $g_\varphi$, that is $(−\varphi)g_\varphi|_H = h$.

A motivating example for our work is the case of a strictly pseudoconvex domain $X \subset \mathbb{C}^m$ with Hermitian metric

$$g = \sum_{j,k=1}^{m} \frac{\partial^2}{\partial z_j \partial \overline{z}_k} \log \left( -\frac{1}{\varphi} \right) dz_j \otimes d\overline{z}_k,$$

where $\varphi$ is a defining function for the boundary of $X$ with $\varphi < 0$ in the interior of $X$. In this example, observe that if

$$\Theta = \frac{i}{2} (\overline{\partial} \varphi - \partial \varphi)$$

and $\iota : M \rightarrow X$ is the natural inclusion, then $\theta = \iota^* \Theta$ is a contact form on $M$ that defines the CR-structure $H = \ker \theta$. The form $d\theta$ induces the Levi form on $M$ and so defines a pseudo-Hermitian structure on $M$. Denote by $J$ the almost complex structure on $H$; the two-form $h = d\theta(\cdot, J \cdot)$ is a pseudo-Hermitian metric on $M$. It is not difficult to see that the conformal class of the pseudo-Hermitian structure on $M$, i.e., its CR-structure, is independent of the choice of defining function $\varphi$.

It is natural to consider scattering theory for the Laplacian, $\Delta_g$, on $(\tilde{X}, g)$, where $X$ is a complex manifold with boundary $M$. As discussed in what follows, the metric $g$ belongs to the class of $\Theta$-metrics considered by Epstein, Melrose, and Mendoza [4]; see also the recent paper of Guillarmou and Sá Barreto [15] where scattering theory for asymptotically complex hyperbolic manifolds (a class which includes those considered here) is analyzed in depth. Thus, the full power of the Epstein-Melrose-Mendoza analysis of the resolvent $R(s) = (\Delta_g - s(m - s))^{-1}$ of $\Delta_g$ is available to study scattering theory on $(\tilde{X}, g)$.

For $f \in C^\infty(M)$, $\Re(s) = m/2$, and $s \neq m/2$, there is a unique solution $u$ of the ‘Dirichlet problem’

$$(\Delta_g - s(m - s)) u = 0$$

$$u = (-\varphi)^{m-s} F + (-\varphi)^s G$$

$$F|_M = f.$$  

where $F, G \in C^\infty(X)$. The uniqueness follows from the absence of $L^2$ solutions of the eigenvalue problem for $\Re(s) = m/2$; this may be proved, for example, using [28] (see the comments in [13]). Here we will use the explicit formulas for the Kähler form and Laplacian obtained in [13] to obtain the asymptotic expansions of solutions to the generalized eigenvalue problem.

Unicity for the ‘Dirichlet problem’ (1.3) implies that the Poisson map

$$(1.4) \quad \mathcal{P}(s) : C^\infty(M) \rightarrow C^\infty(\tilde{X})$$

$$f \mapsto u$$
and the scattering operator

\[ S_X(s) : C^\infty(M) \to C^\infty(M) \]

\[ f \mapsto G|_M \]

are well-defined. The operator \( S_X(s) \) depends \textit{a priori} on the boundary defining function \( \varphi \) for \( M \). If \( \overline{\varphi} = e^\nu \varphi \) is another defining function for \( M \) and \( \nu|_M = \Upsilon \), the corresponding scattering operator \( \overline{S}_X(s) \) is given by

\[ \overline{S}_X(s) = e^{-s \Upsilon} S_X(s) e^{(s - m)\Upsilon}. \]

The operator \( S_X(s) \) admits a meromorphic continuation to the complex plane, possibly with singularities at \( s = 0, -1, -2, \ldots \); see \([25]\) where the scattering operator is described and the problem of studying its poles and residues is posed, and see \([15]\) for a detailed analysis of the scattering operator. The scattering operator is self-adjoint for \( s \) real. We will show that, with a geometrically natural choice of the boundary defining function \( \varphi \), the residues of certain poles of \( S_X(s) \) are CR-covariant differential operators.

To describe the setting for this result, recall that for strictly pseudoconvex domains \( \Omega \) in \( \mathbb{C}^m \), Fefferman \([6]\) proved the existence of a defining function \( \varphi \) for \( \partial \Omega \) which is an approximate solution of the complex Monge-Ampère equation.

The complex Monge-Ampère equation for a function \( \varphi \in C^\infty(\Omega) \) is the equation

\[ J[\varphi] = 1 \]

\[ \varphi|_{\partial \Omega} = 0 \]

where \( J \) is the complex Monge-Ampère operator

\[ J[\varphi] = \det \left[ \begin{array}{cc} \varphi & \varphi_j \\ \overline{\varphi} & \overline{\varphi}_j \end{array} \right] \]

We say that \( \varphi \in C^\infty(\Omega) \) is an approximate solution of the complex Monge-Ampère equation if

\[ J[\varphi] = 1 + \mathcal{O}(\varphi^{m+1}) \]

\[ \varphi|_{\partial \Omega} = 0 \]

The Kähler metric \( g \) associated to such an approximate solution \( \varphi \) is an approximate Kähler-Einstein metric on \( \Omega \), i.e., \( g \) obeys

\[ \text{(1.5)} \quad \text{Ric}(g) = -(m + 1)\omega + \mathcal{O}(\varphi^{m-1}). \]

where \( \omega \) is the Kähler form associated to \( \varphi \), and Ric is the Ricci form.

Under certain conditions, Fefferman’s result can be ‘globalized’ to the setting of complex manifolds \( X \) with strictly pseudoconvex boundary \( M \), as we discuss below. It follows that \( \overline{X} \) carries an approximate Kähler-Einstein metric \( g \) in the sense that \textbf{(1.5)} holds.

We will call a smooth function \( \varphi \) defined in a neighborhood of \( M \) a \textit{globally defined approximate solution} of the Monge-Ampère equation on \( X \) if for each \( p \in M \) there is a neighborhood \( U \) of \( p \) in \( X \) and a holomorphic coordinate system in \( U \) for which \( \varphi \) is an approximate solution of the Monge-Ampère equation. As we will show, such a solution exists if and only if \( M \) admits a pseudo-Hermitian structure \( \theta \) which is volume-normalized with respect to some locally defined, closed \((n + 1, 0)\)-form in a neighborhood of any point \( p \in M \) (see section \textbf{2.4.2} where we defined
“volume-normalized”, and see Burns-Epstein [2] where a similar condition is used to construct a global solution of the Monge-Ampère equation when dim \( M = 3 \). If dim \( M \geq 5 \), we can give a more geometric formulation of this condition. Recall that a CR-manifold is pseudo-Einstein if there is a pseudo-Hermitian structure \( \theta \) for which the Webster Ricci curvature is a multiple of the Levi form (see Lee [22] where this geometric notion is introduced and studied). In [22], Lee proved that if dim \( M \geq 5 \), then \( M \) admits a pseudo-Einstein, pseudo-Hermitian structure \( \theta \) if and only if \( \theta \) is volume-normalized with respect to a closed \((n+1,0)\)-form in a neighborhood of any point \( p \in M \). If dim \( M = 3 \), the pseudo-Einstein condition is vacuous and must be replaced by a more stringent condition; see section 2.4.2 in what follows. If \( X \) is a pseudoconvex domain in \( \mathbb{C}^m \), this condition is trivially satisfied since the pseudo-Hermitian structure induced by the Fefferman approximate solution is volume-normalized with respect to the restriction of \( \zeta = dz^1 \wedge \cdots \wedge dz^m \) to \( M \).

**Theorem 1.1.** Let \( X \) be a complex manifold of complex dimension \( m = n+1 \) with strictly pseudoconvex boundary \( M \). Let \( g \) be the Kähler metric on \( X \) associated to the Kähler form (1.1), and let \( S_X(s) \) be the scattering operator for \( \Delta_\varphi \). Finally, suppose that \( \Delta_\varphi \) has no \( L^2 \)-eigenvalues. Then \( S_X(s) \) has simple poles at the points \( s = m/2 + k/2, k \in \mathbb{N} \), and

\[
\text{Res}_{s=m/2+k/2} S_X(s) = c_k P_k,
\]

where the \( P_k \) are differential operators of order \( 2k \), and

\[
c_k = \frac{(-1)^k}{2^k k!(k-1)!}.
\]

If \( g \) is an approximate Kähler-Einstein metric given by a globally defined approximate solution of the Monge-Ampère equation, then for \( 1 \leq k \leq m \), the operators \( P_k \) are CR-covariant differential operators.

**Remark 1.2.** It is not difficult to show that, for generic compactly supported perturbations of the metric, \( L^2 \)-eigenvalues are absent. Our analysis applies if only the metric \( g \) has the form (1.2) in a neighborhood of \( M \).

**Remark 1.3.** We view the operators \( P_k \) as operators on \( C^\infty(M) \); if one instead views these operators as acting on appropriate density bundles over \( M \) they are actually invariant operators. Gover and Graham [9] showed that the CR-covariant differential operators \( P_k \) are logarithmic obstructions to the solution of the Dirichlet problem (1.3) when \( X \) is a pseudoconvex domain in \( \mathbb{C}^m \) with a metric of Bergman type, but did not identify them as residues of the scattering operator.

It follows from the self-adjointness (s real) and conformal covariance of \( S_X(s) \) that the operators \( P_k \) are self-adjoint and conformally covariant. As in [14], the analysis centers on the Poisson map \( \mathcal{P}(s) \) defined in (1.4). As shown in [4], the Poisson map is analytic in \( s \) for \( \text{Re}(s) > m/2 \). Moreover, at the points \( s = m/2 + k/2, k = 1, 2, \cdots \), the Poisson operator takes the form

\[
\mathcal{P}(s)f = (\varphi)^{m/2-k/2} F + [(\varphi)^{m/2+k/2} \log(-\varphi)] G
\]

for functions \( F, G \in C^\infty(X) \) with

\[
F|_M = f, \ G|_M = c_k P_k f.
\]
Here $P_k$ are differential operators determined by a formal power series expansion of the Laplacian (see Lemma 3.4), and are the same operators that appear as residues of the scattering operator at points $s = m/2 + k/2$. An important ingredient in the analysis is the asymptotic form of the Laplacian due to Lee and Melrose [23] and refined by Graham and Lee in [13].

If the defining function $\varphi$ is an approximate solution of the complex Monge-Ampère equation, the differential operators $P_k$, $1 \leq k \leq m$, can be identified with the GJMS operators owing to the characterization of $P(s)f$ described above (see Proposition 5.4 in [9]; the argument given there for pseudoconvex domains easily generalizes to the present setting).

Explicit computation shows that, for an approximate Kähler-Einstein metric $g$, the first operator has the form

$$P_1 = c_1(\Delta_b + n(2(n + 1))^{-1}R),$$

where $\Delta_b$ is the sub-Laplacian on $X$ and $R$ is the Webster scalar curvature, i.e., $P_1$ is the CR-Yamabe operator of Jerison and Lee [19].

The CR $Q$-curvature is a pseudo-Hermitian invariant realized as the pullback to $M$ of the $Q$-curvature of the circle bundle $\mathcal{C}$.

**Theorem 1.4.** Suppose that $X$ is a complex manifold with strictly pseudoconvex boundary $M$, and suppose that $g$ is an approximate Kähler-Einstein metric given by a globally defined approximate solution of the Monge-Ampère equation. Let $S_X(s)$ be the associated scattering operator. The formula

$$c_m Q^C_R = \lim_{s \to m} S_X(s)$$

holds, where $c_m$ is given by (1.7).

It follows from Theorem 1.1 and the conformal covariance of $S_X(s)$ that if $\overline{\theta} = e^{2\Upsilon} \theta$, then

$$e^{2m\Upsilon} Q^C_R = Q^C_R + P_m \Upsilon$$

as was already shown in Fefferman-Hirachi [8]. From this it follows that the integral $\int_M Q^C_R \psi$ is a CR-invariant (recall that $\psi$ is the natural volume form on $M$ defined by the contact form $\theta$). We remark that the integral of $Q^C_R$ vanishes for any three-dimensional CR-manifold because the integrand is a total divergence (see [8], Proposition 3.2 and comments below), while under the condition of our Theorem 1.4 there is a pseudo-Hermitian structure for which $Q^C_R = 0$ (see [8], Proposition 3.1). In our case, if $\varphi$ is a globally defined approximate solution of the Monge-Ampère equation, the induced contact form $\theta = (i/2)(\bar{\partial} \varphi - \partial \varphi)$ on $M$ is an ‘invariant contact form’ in the language of [8], and they show in Proposition 3.1 that $Q^C_R = 0$ for an invariant contact form. Thus it is not clear at present under what circumstances this invariant is nontrivial for a general, strictly pseudoconvex manifold.

Finally, we prove a CR-analogue of Graham and Zworski’s result ([14], Theorem 3) using scattering theory.

**Theorem 1.5.** Suppose that $X$ is a compact complex manifold with strictly pseudoconvex boundary $M$, and $g$ is an approximate-Kähler-Einstein metric given by a globally defined approximate solution of the Monge-Ampère equation. Then

$$\text{vol}_g \{ -\varphi > \varepsilon \} = c_0 \varepsilon^{-n-1} + c_1 \varepsilon^{-n} + \cdots + c_n \varepsilon^{-1} + L \log(-\varepsilon) + V + o(1).$$
where

\[ L = c \int_M Q^\mathcal{CR}_\theta \psi = 0 \]

We remark that Seshadri [26] already showed that \( L \) is, up to a constant, the integral of \( Q^\mathcal{CR}_\theta \). It is worth noting that our choice of defining function differs from Seshadri’s.

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2. **Geometric Preliminaries**

2.1. **CR Manifolds.** Suppose that \( M \) is a smooth orientable manifold of real dimension \( 2n + 1 \), and let \( \mathcal{C}TM = TM \otimes_{\mathbb{R}} \mathbb{C} \) be the complexified tangent bundle on \( M \). A **CR-structure** on \( M \) is a complex \( n \)-dimensional subbundle \( H \) of \( \mathcal{C}TM \) with the property that \( H \cap \overline{H} = \{0\} \). If, also, \([H, H] \subseteq H\), we say that the CR-structure is **integrable**. If we set \( H = \text{Re} \mathcal{H} \), then the bundle \( H \) has real codimension one in \( TM \). The map

\[ J : H \rightarrow H \]

\[ V + \overline{V} \mapsto iv(V - \overline{V}) \]

satisfies \( J^2 = -I \) and gives \( H \) a natural complex structure.

Since \( M \) is orientable, there is a nonvanishing one-form \( \theta \) on \( M \) with \( \ker \theta = H \). This form is unique up to multiplication by a positive, nonvanishing function \( f \in \mathcal{C}^\infty(M) \). A choice of such a one-form \( \theta \) is called a **pseudo-Hermitian structure** on \( M \). The **Levi form** is given by

\[ L_{\theta}(V, W) = -i d\theta(V, \overline{W}). \]

for \( V, W \in \mathcal{H} \) (here \( d\theta \) is extended to \( \mathcal{H} \) by complex linearity). Note that

\[ L_{f\theta} = f L_{\theta} \]

since \( \theta \) annihilates \( \mathcal{H} \). If \( d\theta \) is nondegenerate, then there is a unique real vector field \( T \) on \( M \), the **characteristic vector field**, with the properties that \( \theta(T) = 1 \) and \( T \cdot d\theta = 0 \). If \( \{W_\alpha\} \) is a local frame for \( \mathcal{H} \) (here \( \alpha \) ranges from 1 to \( n \)), then the vector fields \( \{W_\alpha, \overline{W_\alpha}, T\} \) form a local frame for \( \mathcal{C}TM \). If we choose \((1, 0)\)-forms \( \theta^\alpha \) dual to the \( W_\alpha \) then \( \{\theta^\alpha, \overline{\theta^\alpha}, \theta\} \) forms a dual coframe for \( \mathcal{C}TM \). We say that \( \{\theta^\alpha\} \) forms an admissible coframe dual to \( \{W^\alpha\} \) if \( \theta^\alpha(T) = 0 \) for all \( \alpha \). The integrability condition is equivalent to the condition that

\[ d\theta = d\theta^\alpha = 0 \mod \{\theta, \theta^\alpha\} \]

The Levi form is then given by

\[ L_{\theta} = h_{\alpha\overline{\beta}} \theta^\alpha \wedge \overline{\theta^\beta} \]

for a Hermitian matrix-valued function \( h_{\alpha\overline{\beta}} \). We will use \( h_{\alpha\overline{\beta}} \) to raise and lower indices in this article.
We will say that a given CR-structure is \textit{strictly pseudoconvex} if $L_{\theta}$ is positive definite. Note that (up to sign) this condition is independent of the choice of pseudo-Hermitian structure $\theta$.

In what follows, we will always suppose that $M$ is orientable and that $M$ carries a strictly pseudoconvex, integrable CR-structure. In this case, the pseudo-Hermitian geometry of $M$ can be understood in terms of the Tanaka-Webster connection on $M$ (see Tanaka \cite{27} and Webster \cite{29}). With respect to the frame discussed above, the Tanaka-Webster connection is given by

\begin{equation}
\nabla W_{\alpha} = \omega_{\alpha}^{\beta} \otimes W_{\beta}, \quad \nabla T = 0
\end{equation}

for connection one-forms $\omega_{\alpha}^{\beta}$ obeying the structure equations

\begin{align*}
d\theta^{\alpha} &= \theta^{\beta} \wedge \omega_{\alpha}^{\beta} + \theta \wedge \tau^{\alpha} \\
d\theta &= ih_{\alpha} \theta^{\beta} \wedge \theta^{\alpha}
\end{align*}

where the torsion one-forms are given by

\[ \tau^{\alpha} = A^{\alpha} \theta^{\beta}, \]

with $A_{\alpha \beta} = A_{\beta \alpha}$. The connection obeys the compatibility condition

\[ dh_{\alpha} = \omega_{\alpha}^{\beta} + \omega_{\beta}^{\alpha}. \]

with the Levi form described in (2.1) and (2.4).

\section{Complex Manifolds with CR Boundary.}

Now suppose that $X$ is a compact complex manifold of dimension $m = n + 1$ with boundary $\partial X = M$. We will denote by $\hat{X}$ the interior of $X$. The manifold $M$ inherits a natural CR-structure from the complex structure of the ambient manifold. We will suppose that that $M$ is strictly pseudoconvex; such a structure, induced by the complex structure of the ambient manifold, is always integrable.

We will suppose that $\varphi \in C^{\infty}(X)$ is a defining function for $M$, i.e., $\varphi < 0$ in $\hat{X}$, $\varphi = 0$ on $M$, and $d\varphi(p) \neq 0$ for all $p \in M$. We will further suppose that $\varphi$ has no critical points in a collar neighborhood of $M$ so that the level sets $M^{\varepsilon} = \varphi^{-1}(-\varepsilon)$ are smooth manifolds for all $\varepsilon$ sufficiently small.

Associated to the defining function $\varphi$ is the Kähler form

\begin{equation}
\omega_{\varphi} = -\frac{i}{2} \partial \bar{\partial} \log(-\varphi) = i \left( \frac{\partial \varphi}{-\varphi} + \frac{\partial \varphi \wedge \bar{\partial} \varphi}{\varphi^2} \right)
\end{equation}

We will study scattering on $X$ with the metric induced by the Kähler form (2.6). Since we can cover a neighborhood of $M$ in $X$ by coordinate charts, it suffices to consider the situation where $U$ is an open subset of $\mathbb{C}^{m}$ and $\varphi : U \to \mathbb{R}$ is a smooth function with no critical points in $U$, the set $\{ \varphi < 0 \}$ is biholomorphically equivalent to a boundary neighborhood in $X$, and $\{ \varphi = 0 \}$ is diffeomorphic to the corresponding boundary neighborhood in $M$. We will now describe the asymptotic geometry near $M$, recalling the ambient metric of \cite{13} and computing the asymptotic form of the metric and volume form.

The manifolds $M^{\varepsilon}$ inherit a natural CR-structure from the ambient manifold $X$ with

\[ \mathcal{H}^{\varepsilon} = \mathcal{C}TM^{\varepsilon} \cap T^{1,0}U. \]
Given a defining function $\varphi$, we define a one-form
\[
\Theta = \frac{i}{2} (\bar{\partial} - \partial) \varphi
\]
and let
\[
\theta_{\varepsilon} = \iota_{\varepsilon}^{*} \Theta
\]
where $\iota_{\varepsilon} : M^\varepsilon \to U$ is the natural embedding. The contact form $\theta_{\varepsilon}$ gives $M^\varepsilon$ a pseudo-Hermitian structure. We will denote by $\mathcal{H}$ the subbundle of $T^{1,0} U$ whose fibre over $M^\varepsilon$ is $\mathcal{H}$. Note that
\[
d\Theta = i \partial \varphi.
\]
and the Levi form on $M^\varepsilon$ is given by
\[
L_{\theta_{\varepsilon}} = -i d\theta_{\varepsilon}
\]
We will assume that each $M^\varepsilon$ is strictly pseudoconvex, i.e., $L_{\theta_{\varepsilon}}$ is positive definite for all sufficiently small $\varepsilon > 0$. To simplify notation, we will write $\theta$ for $\theta_{\varepsilon}$, suppressing the $\varepsilon$, as the meaning will be clear from the context.

2.2.1. Ambient Connection. In order to describe the asymptotic geometry of $X$, we recall the ambient connection defined by Graham and Lee \cite{13} that extends the Tanaka-Webster connection on each $M^\varepsilon$ to $\mathcal{C} \mathcal{T} U$. First we recall the following lemma from \cite{23}.

Lemma 2.1. There exists a unique $(1,0)$-vector field $\xi$ on $U$ so that:
\[(2.7)\] $\partial \varphi(\xi) = 1$
\[
\text{and}
\[(2.8)\] $\xi \lrcorner \partial \bar{\partial} \varphi = r \bar{\partial} \varphi$
for some $r \in \mathcal{C}^\infty(U)$.

The smooth function $r$ in \[(2.8)\] is called the transverse curvature.
We decompose $\xi$ into real and imaginary parts,
\[(2.9)\] $\xi = \frac{1}{2} (N - iT)$,
where $N$ and $T$ are real vector fields on $U$. It easily follows from \[(2.9)\] that
\[d\varphi(N) = 2, \ \theta(N) = 0\]
and
\[\theta(T) = 1, \ T \lrcorner d\theta = 0.\]
Thus $T$ is the characteristic vector field for each $M^\varepsilon$, and $N$ is normal to each $M^\varepsilon$.
Let $\{W_\alpha\}$ be a frame for $\mathcal{H}$. It follows from Lemma \[(2.1)\] that $\{W_\alpha, W_\bar{\alpha}, T\}$ forms a local frame for $\mathcal{C} \mathcal{T} M^\varepsilon$, while $\{W_\alpha, W_\bar{\alpha}, \xi, \bar{\xi}\}$ forms a local frame for $\mathcal{C} \mathcal{T} U$. If $\{\theta^\alpha\}$ is a dual coframe for $\{W_\alpha\}$, then $\{\theta^\alpha, \theta^\bar{\alpha}, \theta\}$ is a dual coframe for $\mathcal{C} \mathcal{T} M^\varepsilon$, while $\{\theta^\alpha, \theta^\bar{\alpha}, \theta, \theta^\bar{\alpha}\}$ is a dual coframe for $\mathcal{C} \mathcal{T} U$. The Levi form on each $\mathcal{H}^\varepsilon$ is given by
\[L_{\theta} = h_{\alpha \bar{\beta}} \theta^\alpha \wedge \theta^\bar{\beta}\]
for a Hermitian matrix-valued function $h_{\alpha \bar{\beta}}$. We will use $h_{\alpha \bar{\beta}}$ to raise and lower indices. We will set
\[
W_m = \xi, \ W_{\bar{m}} = \bar{\xi}, \ \theta^m = \partial \varphi, \ \theta^\bar{m} = \bar{\partial} \varphi.
\]
In what follows, repeated Greek indices are summed from 1 to \( n \) and repeated Latin indices are summed from 1 to \( m = n + 1 \).

The following important lemma decomposes the form \( d\Theta \) into ‘tangential’ and ‘transverse’ components.

**Lemma 2.2.** The formula

\[
\partial \varphi = h_{\alpha\beta} \theta^\alpha \wedge \theta^\beta + r \partial \varphi \wedge \bar{\partial} \varphi
\]

holds.

Graham and Lee \[13\] proved:

**Proposition 2.3.** There exists a unique linear connection \( \nabla \) on \( U \) so that

(a): For any vector fields \( X \) and \( Y \) on \( U \) tangent to some \( M^\varepsilon \), \( \nabla_X Y = \nabla^\varepsilon_X Y \) where \( \nabla^\varepsilon \) is the pseudo-Hermitian connection on \( M^\varepsilon \).

(b): \( \nabla \) preserves \( \mathcal{H} \), \( N \), \( T \), and \( L_\theta \); that is, \( \nabla_X \mathcal{H} \subset \mathcal{H} \) for any \( X \in C\mathcal{T}U \), and \( \nabla T = \nabla N = \nabla L_\theta = 0 \).

(c): If \( \{W_\alpha\} \) is a frame for \( \mathcal{H} \), and \( \{\theta^\alpha, \partial \varphi\} \) is the dual \((1,0)\)-coframe on \( U \), then

\[
d\theta^\alpha = \theta^\beta \wedge \partial \varphi^\beta - i\partial \varphi \wedge \tau^\alpha + i(W^\alpha r)d\varphi \wedge \theta + \frac{1}{2}r d\varphi \wedge \theta^\alpha.
\]

The connection \( \nabla \) is called the ambient connection.

2.2.2. **Kähler Metric.** Using Lemma 2.2, we can also compute the Kähler form

\[
\omega = \frac{i}{2} \left( \frac{1}{\varphi} h_{\alpha\beta} \theta^\alpha \wedge \theta^\beta + \frac{1-r\varphi}{\varphi^2} \partial \varphi \wedge \bar{\partial} \varphi \right).
\]

The induced Hermitian metric is

\[
g_{\varphi} = \frac{1}{\varphi} h_{\alpha\beta} \theta^\alpha \otimes \theta^\beta + \frac{1-r\varphi}{\varphi^2} \partial \varphi \otimes \bar{\partial} \varphi.
\]

It is easily computed that

\[
g_{\varphi}(N, N) = 4 \frac{1-r\varphi}{\varphi^2}
\]

so that the outward unit normal field associated to the surfaces \( M^\varepsilon \) is

\[
\nu = \frac{-\varphi}{2\sqrt{1-r\varphi}} N
\]

We note for later use that the induced volume form \( \omega_{\varphi}^m \) is given by

\[
\omega_{\varphi}^m = \left( \frac{i}{2} \right)^m \left( \frac{1-r\varphi}{(-\varphi)^m+1} \det (h_{\alpha\beta}) \right) \theta^1 \wedge \theta^2 \wedge \cdots \wedge \theta^m \wedge \theta^m
\]

while

\[
\nu \wedge \omega_{\varphi}^m |_{M^\varepsilon} = \frac{m}{2n-1} \frac{1-r\varepsilon}{\varepsilon^n} (d\theta^\varepsilon) \wedge \theta^\varepsilon.
\]

We will set

\[
\psi = \frac{m}{2n-1} (d\theta)^n \wedge \theta.
\]

We also note for later use that if \( u \in \mathcal{C}\infty(X) \) and

\[
\text{du} = u_\alpha \theta^\alpha + u_{\alpha\beta} \theta^\alpha \wedge \theta^\beta + u_m \partial \varphi + u_m \bar{\partial} \varphi
\]
then
\[(2.18) \quad |du\rangle_{\varphi}^2 = -\varphi h^{\alpha\overline{\beta}} u_\alpha u_{\overline{\beta}} + \frac{\varphi^2}{1-r\varphi} u_m u_{m}\]

### 2.3. The Laplacian on \(X\)

The Laplacian on the Kähler manifold \((X, \omega, \varphi)\) is the operator
\[(2.19) \quad \Delta_{\varphi} u = \text{Tr} (i\partial \overline{\partial} u) = g^{\overline{\alpha}\overline{\beta}} u_{\overline{\alpha}}\]
for \(u \in C^\infty(X)\), where we now write \(\Delta_{\varphi}\) rather than \(\Delta_g\) to emphasize the dependence of \(\Delta\) on the boundary defining function \(\varphi\).

Graham and Lee [13] computed the Laplacian in a collar neighborhood of \(M\), separating ‘normal’ and ‘tangential’ parts. To state their results, recall that the sub-Laplacian is defined on each \(M^\varepsilon\) by
\[(2.20) \quad \Delta_b u = \left( u^{\alpha}_\alpha + u^{\overline{\beta}}_{\overline{\beta}} \right)\]
where covariant derivatives are taken with respect to the Tanaka-Webster connection on \(M^\varepsilon\).

Graham and Lee [13] proved:

**Theorem 2.4.** The formula
\[(2.21) \quad \Delta_{\varphi} = \frac{\varphi}{4} \left[ \frac{-\varphi}{1-r\varphi} \left( N^2 + T^2 + 2rN + 2X_r \right) - 2\Delta_b + 2nN \right]\]
holds, where
\[X_r = r^\alpha W_\alpha + r^\overline{\beta} W_{\overline{\beta}}.\]

It will be useful to recast the above formula for \(\Delta_{\varphi} u\) in terms of \(x = -\varphi\). Note that
\[(2.22) \quad N = 2 \frac{\partial}{\partial \varphi} = -2 \frac{\partial}{\partial x}\]
so that
\[\Delta_{\varphi} u = \left( \frac{1}{1+rx} \right) \left( x \frac{\partial}{\partial x} \right)^2 u - (n+1) x \frac{\partial}{\partial x} u + \frac{1}{4} \left( \frac{x^2}{1+rx} \right) \left( T^2 u - 2ru_x + 2X_ru \right) + \frac{1}{4} x \left( -2\Delta_b u \right)\]
We think of \(\Delta_{\varphi}\) as a variable-coefficient differential operator with respect to vector fields \((x \partial_x)\) and vector fields tangent to the boundary \(M\). In a neighborhood of \(M\) we have
\[(2.23) \quad \Delta_{\varphi} \sim \sum_{k \geq 0} x^k L_k\]
for differential operators \(L_k\), where the indicial operator \(L_0\) is
\[(2.24) \quad L_0 = - \left( \left( x \frac{\partial}{\partial x} \right)^2 - mx^2 \frac{\partial}{\partial x} \right)\]

\[\text{Note that our definition differs from that of Graham and Lee by an overall factor of } -1/4.\]
and the operator $L_1$ is

\begin{equation}
L_1 = \frac{1}{4} \left( -2\Delta_b u - 4r_0 x \frac{\partial}{\partial x} - 4r_0 \left[ \left( x \frac{\partial}{\partial x} \right)^2 - x \frac{\partial}{\partial x} \right] \right)
\end{equation}

where

$$r = r_0 + O(x).$$

### 2.4. The Complex Monge-Ampère Equation.

#### 2.4.1. Local Theory.

Let $\Omega$ be a domain in $\mathbb{C}^m$ with smooth boundary. The complex Monge-Ampère equation is the nonlinear equation

$$J[u] = 1$$

$$u|_{\partial \Omega} = 0$$

for a function $u \in C^\infty(\Omega)$, $u > 0$ on $\Omega$, where $J[u]$ is the Monge-Ampère operator:

\begin{equation}
J[u] = (-1)^m \det \left( \begin{array}{cc}
u & \overline{\nu} \\
u & \overline{\nu}
\end{array} \right).
\end{equation}

If $u$ solves the complex Monge-Ampère equation then

$$- \left( \log \left( \frac{1}{u} \right) \right)_{\overline{\nu}} dz^j \otimes d\overline{z}^k$$

is a Kähler-Einstein metric.

Fefferman [6] showed that there is a smooth function $\psi \in C^\infty(\Omega)$ that satisfies

$$J[\varphi] = 1 + O(\varphi^{m+1})$$

and that $\psi$ is uniquely determined up to order $m + 1$. Cheng and Yau [3] showed the existence of an exact solution belonging to $C^\infty(\Omega) \cap C^{m+3/2-\epsilon}(\overline{\Omega})$, while Lee and Melrose [23] showed that the exact solution has an asymptotic expansion with logarithmic terms beginning at order $m + 2$.

We will show that Fefferman’s local approximate solution of the Monge-Ampère equation [6] can be globalized to an approximate solution of the Monge-Ampère equation near the boundary of a complex manifold $X$. We will see later that, to globalize Fefferman’s construction, we need to impose a geometric condition on the CR-structure of $M$ inherited from the complex structure of $X$. For the convenience of the reader, we review the properties of the operator $J$ under a holomorphic coordinate change and the connection between solutions of the Monge-Ampère equation and Kähler-Einstein metrics.

If $f : \Omega \subset \mathbb{C}^m \to \mathbb{C}^m$ is holomorphic, then $f'$ denotes the matrix

$$\left( f' \right)_{jk} = \frac{\partial f_j}{\partial z_k}.$$

**Lemma 2.5.** Let $f$ be a local biholomorphism. Then, for any local, smooth function $u$ on $\Omega$,

$$J \left[ (\det(f'))^{-2/(m+1)} (u \circ f) \right] = J [u] \circ f.$$

A proof was given by Fefferman in [6]. Here we give an alternative proof using the following identity.
Lemma 2.6. The formula

\[ J[u] = u^{m+1} \det \left[ \log \left( \frac{1}{u} \right) \right]_{\overline{\mathbf{k}}} \]

holds.

Proof. Using row-column operations, one proves that

\[ \det \left( \begin{array}{cc} u & u_{\overline{\mathbf{k}}} \\ u_j & u_{j{\overline{\mathbf{k}}}} \end{array} \right) = u \det \left( u_{\overline{\mathbf{k}}} - \frac{u_j u_{\overline{\mathbf{k}}}}{u} \right). \]

On the other hand, the identity

\[ \left( \log \left( \frac{1}{u} \right) \right)_{j{\overline{\mathbf{k}}} \left( \log \left( \frac{1}{u} \right) \right)_{\overline{\mathbf{k}}} = -\frac{u_{\overline{\mathbf{k}}}}{u} + \frac{u_j u_{\overline{\mathbf{k}}}}{u^2} \]

shows that

\[ J[u] = (-1)^m u \det \left( u_{\overline{\mathbf{k}}} - \frac{u_j u_{\overline{\mathbf{k}}}}{u} \right) = u^{m+1} \det \left( \left( \log \left( \frac{1}{u} \right) \right)_{\overline{\mathbf{k}}} \right). \]

Combining (2.28) and (2.29) shows that (2.27) holds.

We can use the formula (2.27) to show that if \( u \) solves the Monge-Ampère equation, then \( u \) is the Kähler potential of a Kähler-Einstein metric. Recall that if

\[ g = v_{\overline{\mathbf{k}}} dz^j \otimes d\overline{z}^\mathbf{k} \]

then the Ricci curvature is

\[ R_{a\overline{b}} = -\left[ \log \det(v_{\overline{\mathbf{k}}}) \right]_{a\overline{b}} \]

Now let

\[ v = \log \left( \frac{1}{u} \right) \]

where \( J[u] = 1 \). Then

\[ R_{a\overline{b}} = -\left[ \log \det(v_{\overline{\mathbf{k}}}) \right]_{a\overline{b}} \]

\[ = -\left[ \log \left( u^{-1/(m+1)} \right) \right]_{a\overline{b}} \]

\[ = -(m + 1) \left( \log \left( \frac{1}{u} \right) \right)_{a\overline{b}} \]

\[ = -(m + 1)g_{a\overline{b}} \]

which is the Einstein equation.

Now we prove Lemma 2.5. First, we compute

\[ \left[ \log \left( \det f' \right)^{-2/(m+1)} u \circ f \right]_{\overline{\mathbf{k}}} = \frac{-1}{m + 1} \left[ \log \left( \left| \det f' \right|^2 \right) \right]_{\overline{\mathbf{k}}} \]

\[ + \left[ \log (u \circ f) \right]_{\overline{\mathbf{k}}} \]

\[ = \left[ \log (u \circ f) \right]_{\overline{\mathbf{k}}} \]

where the first right-hand term vanishes because \( \left| \det f' \right|^2 = (\det f') (\det \overline{f'}) \) and \( \det f' \) is holomorphic. We note that the vanishing of the first term also shows that the Kähler metric with Kähler potential \( u \) (when \( u \) solves the Monge-Ampère equation) is invariant whether \( u \) is considered as a scalar function or a density.
To compute the nonzero right-hand term in (2.30) we first note that if \( f \) is a holomorphic map then we have the identity
\[
(v \circ f)_\partial = \left[ (f')^t \left( v_{\bar{\partial}} \circ f \right) (\bar{f}) \right]_\partial.
\]
Thus, using (2.27), we compute
\[
\begin{align*}
J \left[ |\det(f')|^{-2/(m+1)} u \circ f \right] &= |\det(f')|^{-2} (u \circ f)^{m+1} \\
& \quad \times \det \left( (f')^t \right) \det \left( \log \left( \frac{1}{u} \right)_{\bar{\partial}} \circ f \right) \det(\bar{f}) \\
& = (u \circ f)^{m+1} \det \left( \log \left( \frac{1}{u} \right)_{\bar{\partial}} \circ f \right).
\end{align*}
\]
as was to be proved.

It is essential for our globalization argument that an approximate solution to the Monge-Ampère equation be determined uniquely up to a certain order. This proof was given by Fefferman \cite{6} and we repeat it for the reader’s convenience.

**Lemma 2.7.** Any smooth, local, approximate solution \( \psi \in \mathcal{C}^\infty(\Omega) \) to the Monge-Ampère equation is uniquely determined up to order \( m+1 \).

**Proof.** Suppose that \( \rho \) is a smooth function on \( \Omega \) defined in a neighborhood of \( \partial \Omega \) with \( \rho = 0 \) on \( \partial \Omega \) and \( \rho'(p) \neq 0 \) for all \( p \in \partial \Omega \). We recall Fefferman’s iterative construction of an approximate solution \( u \) to the Monge-Ampère equation, i.e., a function \( u \in \mathcal{C}^\infty \) with \( u|\partial \Omega = 0 \) and \( J[u] = 1 + \mathcal{O}(u^{m+2}) \). To obtain a first approximation, note that for \( \rho \) as above, and for any smooth function \( \eta \), we have
\[
J[\eta \rho] = \eta^{m+1} J[\rho],
\]
when \( \rho = 0 \), so that the function
\[
\psi^{(1)} = \rho \cdot J[\rho]^{-1/(m+1)}
\]
satisfies \( J[\psi^{(1)}] = 1 \) on \( \partial \Omega \), and \( J[\psi^{(1)}] = 1 + \mathcal{O}(\psi^{(1)}) \). The fact that \( J[\rho] \) is nonzero on \( \partial \Omega \) follows from pseudoconvexity that implies that \( \rho_{\partial \Omega} \) is positive definite on \( \ker \partial \rho \) on \( \partial \Omega \), and that \( \rho' \neq 0 \) on \( \partial \Omega \). Note that if \( \varphi \) and \( \psi \) are two functions vanishing on \( \partial \Omega \), it follows that \( \varphi = \eta \psi \) for some smooth function \( \eta \). Thus, by (2.31), \( J[\varphi] = \eta^{m+1} J[\psi] \). From this computation it follows that any approximate solution \( u \) is uniquely determined up to first order.

We now iterate this construction. Suppose that for an integer \( s \geq 2 \), we have an approximate solution to the Monge-Ampère equation to order \( s-1 \). That is, we have a smooth function \( \psi \) with \( \psi = 0 \) on \( \partial \Omega \), \( \psi'(p) \neq 0 \), for all \( p \in \partial \Omega \), and \( J[\psi] = 1 + \mathcal{O}(\psi^{s-1}) \). We seek a function of the form \( v = \psi + \eta \psi^s \), where \( \eta \in \mathcal{C}^\infty \) is chosen so that \( J[v] = 1 + \mathcal{O}(\psi^s) \). The iteration is based on formula
\[
J[\psi + \eta \psi^s] = J[\psi] + [s(m + 2 - s)] \eta \psi^{s-1} + \mathcal{O}(\psi^s),
\]
for smooth functions \( \psi \) and \( \eta \), again with the property that \( \psi \) vanishes on \( \partial \Omega \). This formula is a straightforward computation using the formula (2.26). From this formula it follows that the desired function \( v \) is given by
\[
v = \psi + \left[ \frac{1 - J(\psi)}{s(m + 2 - s)} \right] \psi^s.
\]
The iteration clearly works up to \( s = m + 1 \) and produces an approximate solution with the desired properties. It also follows that any function \( \tilde{u} \) with \( u - \tilde{u} = O(\psi^{m+2}) \) satisfies \( J[\tilde{u}] = J[u] + O(\psi^{m+2}) \). Thus, in particular, any smooth function having the same \((m+1)\)-jet on \( \partial \Omega \) as an approximate solution is also an approximate solution.

On the other hand, it is clear that any two approximate solutions must have the same \((m+1)\)-jet on \( \partial \Omega \). If \( \psi \) and \( \tilde{\psi} \) satisfy \( \psi - \tilde{\psi} = \eta \psi^s \) then \( J[\psi] - J[\tilde{\psi}] = s(m + 2 - s)\eta \psi^{s-1} + O(\psi^s) \). In particular, if \( s < m+2 \) and \( J[\psi] - J[\tilde{\psi}] = O(\psi^{m+2}) \) then \( \psi \) and \( \tilde{\psi} \) are approximate solutions uniquely determined up to order \( m + 2 \).

\[ \square \]

2.4.2. Global Theory. Now suppose \( X \) is a compact complex manifold of dimension \( m = n + 1 \) with boundary \( M = \partial X \). Note that \( M \) has real dimension \( 2n + 1 \) and inherits an integrable CR-structure from \( X \). As always, we assume that \( M \) with this CR-structure is strictly pseudoconvex. We first say what it means for a single smooth function \( \varphi \) defined in a neighborhood of \( M \) to be an approximate solution of the complex Monge-Ampère equation. We denote by \( C^\infty(X) \) the smooth functions on \( X \).

**Definition 2.8.** We will say that a function \( \varphi \in C^\infty(X) \) is a globally defined approximate solution of the complex Monge-Ampère equation near \( M = \partial X \) if for any \( p \in M \), there is a neighborhood \( V \) of \( p \) in \( X \) and holomorphic coordinates \( z \) on \( V \) so that \( \varphi \) is an approximate solution of the complex Monge-Ampère equation in the chosen coordinates.

As we will see later, we will need such a globally defined approximate solution in order to identify the residues of the scattering operator on \( X \) with CR-covariant differential operators.

If \( \varphi \) is a defining function for \( M \) with \( \varphi < 0 \) in the interior of \( X \), we associate to \( \varphi \) a Kähler form

\[
\omega_\varphi = \frac{i}{2} \partial \overline{\partial} \log(-1/\varphi)
\]

and a pseudo-Hermitian structure

\[
\theta = \frac{i}{2} (\overline{\partial} - \partial) \varphi \bigg|_M.
\]

Observe that two defining functions \( \varphi \) and \( \rho \) generate the same Kähler metric if and only if \( \rho = e^{F} \varphi \) for a pluriharmonic function \( F \), i.e. \( \partial \overline{\partial} F = 0 \). It is known that a pluriharmonic function \( F \) is uniquely determined by its boundary values (see, for example, Bedford [I]). If \( \theta_\rho \) and \( \theta_\varphi \) are the corresponding pseudo-Hermitian structures on \( M \) then \( \theta_\rho = e^F \theta_\varphi \), where \( f = F|_M \).

We give a necessary and sufficient condition on \( M \) for a globally defined approximate solution of the Monge-Ampère equation to exist. Recall that the canonical bundle of \( M \) is the bundle generated by forms \( f \theta^1 \wedge \cdots \wedge \theta^m \wedge \theta \) where \( f \) is smooth, \( \theta \) is a contact form, and \( \{\theta^\alpha\}_{\alpha=1}^n \) is an admissible co-frame. If \( M \) is the boundary of a strictly pseudoconvex domain in \( \mathbb{C}^m \), the canonical bundle is generated by restrictions of forms \( f dz^1 \wedge \cdots \wedge dz^m \) to \( M \). The sections of the canonical bundle are \((n + 1, 0)\)-forms \( \zeta \) on \( M \).
If \( \theta \) is a contact form, \( T \) is the characteristic vector field, and \( \zeta \) is any nonvanishing section of the canonical bundle, it is not difficult to see that
\[
\theta \wedge (T \wedge \zeta) \wedge (T \wedge \zeta) = \lambda \theta \wedge (d\theta)^n
\]
for a smooth positive function \( \lambda \). We say that the contact form \( \theta \) is volume-normalized with respect to a nonvanishing section \( \zeta \) of the canonical bundle if
\[
\theta \wedge (d\theta)^n = (i)^n n! \theta \wedge (T \wedge \zeta) \wedge (T \wedge \zeta)
\]
where \( T \) is the characteristic vector field. The following criterion will be useful.

**Lemma 2.9.** The contact form \( \theta \) given by (2.33) is volume-normalized with respect to the form \( \zeta = dz^1 \wedge \cdots \wedge dz^m \mid_M \) if and only if
\[
J[\varphi] = 1 + O(\varphi)
\]
in the coordinates \( (z_1, \ldots, z_m) \).

For the proof see Farris [5], Proposition 5.2. Using the lemma, we can prove:

**Proposition 2.10.** Suppose that \( X \) is a compact complex manifold with boundary \( M = \partial X \). There is a globally defined approximate solution \( \varphi \) of the Monge-Ampère equation in a neighborhood of \( M \) if and only if \( M \) admits a pseudo-Hermitian structure \( \theta \) with the following property: In a neighborhood of any point \( p \in M \), there is a local, closed \((n + 1, 0)\) form \( \zeta \) such that \( \theta \) is volume-normalized with respect to \( \zeta \).

**Proof.** (i) First, suppose that \( X \) admits a globally defined approximate solution \( \varphi \) of the Monge-Ampère equation. Let \( \theta \) be the associated contact form on \( X \), i.e., \( \theta \) is given by (2.33). Pick \( p \in M \) and let \( z \equiv (z_1, \ldots, z_m) \) be holomorphic coordinates near \( p \) so chosen that \( \varphi \) is an approximate solution of the Monge-Ampère equation near \( p \) in these coordinates. Let
\[
\zeta = dz^1 \wedge \cdots \wedge dz^m \mid_M .
\]
Then \( \theta \) is volume-normalized with respect to \( \zeta \) by Lemma 2.9.

(ii) Suppose that \( \theta \) is a given contact form on \( M \) with the property that, for each point \( p \in M \), there is a neighborhood of \( p \) and a closed, locally defined section \( \zeta \) of the canonical bundle with respect to which \( \theta \) is volume-normalized. Write
\[
\zeta = f \quad dz^1 \wedge \cdots \wedge dz^m \mid_M
\]
for holomorphic coordinates \( \{z_1, \ldots, z_m\} \) defined in a neighborhood of \( p \) and a smooth function \( f \). The condition \( d\zeta = 0 \) is equivalent to the condition
\[
\frac{\partial}{\partial z_p} f = 0
\]
i.e., \( f \) is a CR-holomorphic function. By the strict pseudoconvexity of \( M \), there is a holomorphic extension \( F \) to a neighborhood \( V \) of \( p \) in \( X \), i.e., there is an \( F \) defined near \( p \) with \( \overline{\partial} F = 0 \) and \( F\mid_{M \cap V} = f \) (see [20]). We claim that we can find new holomorphic coordinates \( w \equiv (w_1, \ldots, w_m) \) near \( p \) with the property that
\[
\frac{\partial (w_1, \ldots, w_m)}{\partial (z_1, \ldots, z_m)} = F(z)
\]
(2.34)
If so then
\[
\zeta = dw^1 \wedge \cdots \wedge dw^m \mid_M
\]
Constructing in $V$ an approximate solution $\psi_V$ of the Monge-Ampère equation in the $w$-coordinates (as in Lemma 2.7, following Fefferman [6]), we conclude from Lemma 2.9 that the induced contact form

$$\theta_V = \frac{i}{2} (\overline{\partial} - \partial) \psi_V \bigg|_{M \cap V}$$

on $M \cap V$ is volume-normalized with respect to $\zeta$, and thus coincides with $\theta$.

We now claim that the local approximate solutions $\psi_V$ can be glued together to form a globally defined approximate solution to the Monge-Ampère equation in the sense of Definition 2.8. We first note an important property of the transition map for two local coordinates. Let $V_1$ and $V_2$ be neighborhoods of $M$ in $X$ with nonempty intersection, let $z$ and $w$ be holomorphic coordinates on $V_1$ and $V_2$, and suppose that $\psi_1$ and $\psi_2$ are approximate solutions of the complex Monge-Ampère equation in these respective coordinates. More precisely, $u_1 = \psi_1 \circ z$ and $u_2 = \psi_2 \circ w$ are approximate solutions to the Monge-Ampère equation on coordinate patches $U_1$ and $U_2$ in $\mathbb{C}^m$, and there is a biholomorphic map $g : U_2 \cap w^{-1}(V_1 \cap V_2) \to U_1 \cap z^{-1}(V_1 \cap V_2)$. The function $u_2 = \frac{1}{2} |g'|^{2/(m+1)} u_1 \circ g$ is also an approximate solution of the complex Monge-Ampère equation in $U_2 \cap w^{-1}(V_1 \cap V_2)$ by Lemma 2.5 so by uniqueness we have $u_2 = e^{F} u_1 \circ g$, up to order $m+1$, where $F = (2/(m+1)) \log |g'|$ is pluriharmonic. Moreover, since $u_1$ and $u_2$ both induce the contact form $\theta$ it follows that

$$\overline{\partial} - \partial) u_2 \big|_{U_2 \cap w^{-1}(M \cap V_1 \cap V_2)} = \overline{\partial} - \partial) u_1 \big|_{U_2 \cap w^{-1}(M \cap V_1 \cap V_2)}$$

from which we deduce that $F \big|_{U_2 \cap w^{-1}(M \cap V_1 \cap V_2)} = 0$, and hence $F = 0$ by the uniqueness of pluriharmonic extensions. In particular, the map $g$ is unimodular, $|g'| = 1$. Thus $u_2 = u_1 \circ g$ on $U_2 \cap w^{-1}(V_1 \cap V_2)$ up to order $m+1$.

We now fix a boundary defining function $\rho$. Suppose that $\{U_i\}$ is a finite cover of a neighborhood of the boundary by holomorphic charts. Denote by $F_i$ the map from $\mathbb{C}^m$ into $U_i$ and set $F_{ij} = F_i^{-1} \circ F_j$. As proved above, the cover and holomorphic coordinates $(U_i, F_i)$ may be chosen so that the transition maps are unimodular, i.e., $|F_{ij}| = 1$. Using Fefferman’s construction, we can produce in each $U_i$ an approximate solution $u_i$ in the sense that

$$J [u_i] = 1 + O(\rho^{m+1})$$

Now suppose that $\{\chi_i\}$ is a $C^\infty$ partition of unity subordinate to the cover $\{U_i\}$. We claim that the smooth function $u = \sum_i \chi_i u_i$ is an approximate solution of the Monge-Ampère equation in the sense of Definition 2.8. Choose $U_i$ so that $p \in U$. We may write $u = \sum_i (\chi_i \circ F_i)(z) \frac{1}{2} |g_i'| u_j \circ F_j$. Since $u_j \circ F_i = (u_j \circ F_j) \circ F_{ij}$ we see that $(u_j \circ F_i)$ is also an approximate solution to the Monge-Ampère equation in the $F_i$-coordinates. Thus, there is a smooth function $\eta_{ji}$ so that

$$(u_j \circ F_i)(z) - (u_i \circ F_i)(z) = \eta_{ji}(z) (\rho \circ F_i)^{m+2} (z)$$

where $\eta_{ji}$ is smooth. We conclude that

$$u(z) - u_i(z) = O((\rho \circ F_i)^{m+2}).$$

This shows that $u$ is also an approximate solution of the Monge-Ampère equation in the $F_i$-coordinates as claimed.

To finish the proof it suffices to establish that such a holomorphic coordinate change $z \mapsto w$, as in 2.31, exists. We consider a coordinate transformation given
by

\[(2.35) \quad w(z) = (h(z), z_2, \ldots, z_m), \]

where \(h(z)\) is the unknown holomorphic function. Condition (2.34) is equivalent to

\[(2.36) \quad \frac{\partial h}{\partial z^1}(z_1, \ldots, z_m) = F(z_1, z_2, \ldots, z_m). \]

Here, \(F\) is the holomorphic extension of the CR-function \(f\). We solve this equation for \(h\) as follows. We set the convention that a boundary chart in \(\mathbb{C}^m\) is the intersection of an open ball about 0 with the (real) half-space \(\text{Im } z_m \geq 0\). We assume that the boundary point \(p\) corresponds to \(0 \in \partial \mathbb{C}^m\). The unknown function \(h\) is a complex-valued function defined in a neighborhood \(V\) of \(0 \in \mathbb{C}^m\), is holomorphic in \(V \cap \{ \text{Im } z_m > 0 \}\), has CR boundary values, and satisfies \(h(0) = 0\). Thus, the map \(w(z)\), defined in (2.35), preserves the boundary \(\text{Im}(z_n) = 0\).

Consequently, the desired change of coordinates is obtained by solving the initial value problem

\[(2.37) \quad \frac{\partial h}{\partial z^1}(z_1, \ldots, z_m) = F(z_1, z_2, \ldots, z_m) \quad h(0, z_2, \ldots, z_m) = 0, \]

by simple integration. \(\Box\)

We can also express the basic criterion in Proposition 2.10 in geometric terms. Recall that the contact form \(\theta\) defines a pseudo-Hermitian, pseudo-Einstein structure on \(M\) if the Webster Ricci tensor is a multiple of the Levi form. Lee\cite{22} proved:

**Theorem 2.11.** Suppose that \(M\) is a CR-manifold of dimension \(\geq 5\). A contact form \(\theta\) on \(M\) is pseudo-Einstein if and only if for each \(p \in M\) there is a neighborhood of \(p\) in \(M\) and a locally defined closed section \(\zeta\) of the canonical bundle with respect to which \(\theta\) is volume-normalized.

As an immediate consequence of Theorem 2.11, we have:

**Theorem 2.12.** Suppose that \(M\) is a CR-manifold of dimension \(\geq 5\). There is a globally defined approximate solution \(\varphi\) of the complex Monge-Ampère equation in a neighborhood of \(M\) if and only if \(M\) carries a contact form \(\theta\) for which the corresponding pseudo-Hermitian structure is pseudo-Einstein. In this case, the contact form \(\theta\) is induced by the globally defined approximate solution to the Monge-Ampère equation \(\varphi\).

**Remark 2.13.** If \(\varphi\) is a global approximate solution to the Monge-Ampère equation, then so is \(e^F \varphi\) where \(F\) is any pluriharmonic function. The effect of the factor \(F\) is simply to change the choice of local coordinates needed to obtain a local approximate solution of the Monge-Ampère equation in any chart, as the argument in the proof of Proposition 2.10 easily shows. As observed above, the Kähler form \(\omega_{\varphi}\) is invariant under the change \(\varphi \mapsto e^F \varphi\).

### 3. Poisson Operator and Scattering Operator

In this section we study the Dirichlet problem (1.3) following a standard technique in geometric scattering theory (see, for example, Melrose\cite{24}; we follow closely the analysis of the Poisson operator and scattering operator on conformally
compact manifolds by Graham and Zworski in [14]). Note that Epstein, Melrose, and Mendoza [4] had previously studied the Poisson operator for a class of manifolds that includes compact complex manifolds with strictly pseudoconvex boundaries. More recently, Guillarmou and Sá Barreto [15] studied scattering theory and radiation fields for asymptotically complex hyperbolic manifolds, a class which also includes that studied here.

We will set \( x = -\varphi \) and we will denote by \( \mathcal{C}^\infty(X) \) the set of smooth functions on \( X \) having Taylor series to all orders at \( x = 0 \), and by \( \dot{\mathcal{C}}^\infty(X) \) the space of functions vanishing to all orders at \( x = 0 \). The space \( \mathcal{C}^\infty(\dot{X}) \) consists of smooth functions on \( \dot{X} \) with no restriction on boundary behavior. We will denote by \( x^s \mathcal{C}^\infty(X) \) the set of functions in \( \mathcal{C}^\infty(\dot{X}) \) having the form \( x^s F \) for \( F \in \mathcal{C}^\infty(X) \).

Since \( N = -2 \frac{\partial}{\partial x} \)

it follows that

\[
\nu = -\frac{x}{\sqrt{1+rx}} \frac{\partial}{\partial x}
\]

is the outward normal to the hypersurface \( x = \varepsilon \). Green’s theorem implies that

\[
\int_{x>\varepsilon} (u_1 \Delta \varphi u_2 - u_2 \Delta \varphi u_1) \ w^m = \int_{x=\varepsilon} (u_1 \nu u_2 - u_2 \nu u_1) \ \nu \cdot w^m
\]

We first note the ‘boundary pairing formula’ (recall the definition (2.16)).

**Proposition 3.1.** Suppose \( \text{Re}(s) = m/2 \), that \( u_1 \) and \( u_2 \) belong to \( \mathcal{C}^\infty(\dot{X}) \) and there are functions \( F_i, G_i \in \mathcal{C}^\infty(X) \) so that \( u_i = x^{m-s} F_i + x^s G_i, \ i = 1, 2 \). Finally, suppose that \( (\Delta \varphi - s(m-s)) u_i = r_i \in \mathcal{C}^\infty(X), \ i = 1, 2 \). Then, the formula

\[
\int_X (u_1 r_2 - u_2 r_1) \ w^m = (2s - m) \int_M (F_1 G_2 - F_2 G_1) \ \psi
\]

holds.

**Proof.** A standard computation using (3.2) and (3.1) together with (2.16) and (2.21).

**Remark 3.2.** For \( \text{Re}(s) = m/2 \) complex conjugation reverses the roles of \( s \) and \( m-s \). Thus we obtain the formula

\[
\int_X (u_1 \overline{r_2} - \overline{u_2} r_1) \ w^m = (2s - m) \int_M (F_1 \overline{G_2} - \overline{F_2} G_1) \ \psi
\]

For later use, we note an extension of the boundary pairing formula analogous to Proposition 3.3 of [14].

**Proposition 3.3.** Suppose that \( \text{Re}(s) > m/2 \) and \( 2s - m \notin \mathbb{N} \). Suppose that \( u_i \in \mathcal{C}^\infty(\dot{X}) \) takes the form

\[
uu_i = x^{m-s} F_i + x^s G_i \]

and \( (\Delta \varphi - s(m-s)) u_i = 0, \ i = 1, 2 \). Then

\[
\text{FP}_{\varepsilon^0} \left( \int_{x>\varepsilon} [\nabla u_1, \nabla u_2] - s(m-s) u_1 u_2 \right) \ w^m = -m \int_M G_1 F_2 \ \psi = -m \int_M F_1 G_2 \ \psi
\]

where FP denotes the Hadamard finite part of the integral as \( \varepsilon \downarrow 0 \).
are differential operators of order at most two depending holomorphically on
for ordinates $y$
the operators $\Omega$. Observe that
Similarly, if $u \sim \sum_{k=0}^{\infty} x^{m+s+k} f_k$ for $f_k \in \mathcal{C}^\infty(M)$, we have
\begin{equation}
(\Delta - s(m-s)) u \in \mathcal{C}^\infty(X)
\end{equation}
f for $s \in \mathbb{C}$ with $2s - m \notin \mathbb{Z}$. Then the Taylor expansions of $F$ and $G$ at $x = 0$
are formally determined respectively by $F|_M$ and $G|_M$. In particular, we have
\begin{equation}
f_k = \frac{1}{k!} \Gamma(2s-m-k) \Gamma(2s-m) P_{k,s}, f_0
\end{equation}
and
\begin{equation}
g_k = \frac{1}{k!} \Gamma(m-2s-k) \Gamma(m-2s) P_{k,m-s}, g_0
\end{equation}
where $P_{k,s}$ are differential operators of order $2k$ holomorphic in $s$ with leading symbol
\begin{equation}
\sigma(P_{k,s}) = \frac{1}{2k} \sigma(-\Delta^k)
\end{equation}
Proof. Recall the asymptotic development (2.23) for the Laplacian which we use to
derive a recurrence for the Taylor coefficients $f_k$ and $g_k$ of $F$ and $G$. For $2s-m \notin \mathbb{Z}$,
we may consider the terms involving $F$ and $G$ separately. We first consider $F$.
Observe that
\begin{equation}
(L_0 - s(m-s)) \left(x^{m-s+k} f\right) = k(2s-m-k)x^{s+k} f
\end{equation}
for $f \in \mathcal{C}^\infty(M)$. Since $L_k = P(x\partial_x, \partial_y)$ for a defining function $x$ and boundary co-
ordinates $y$ where $P$ is a polynomial of degree at most two with smooth coefficients,
the operators
\begin{equation}
Q_{k,\ell}(s) = x^{-m+s-\ell} L_{k-\ell} x^{m-s+\ell}
\end{equation}
are differential operators of order at most two depending holomorphically on $s$. If
$u \sim \sum_{k=0}^{\infty} x^{m+s+k} f_k$, it follows from (3.6) and (2.23) that for any $k \geq 1$,
\begin{equation}
f_k = -\frac{1}{k(2s-m-k)} \sum_{\ell=0}^{k-1} Q_{k,\ell}(s) f_\ell
\end{equation}
Similarly, if $u \sim \sum_{k=0}^{\infty} x^{s+k} g_k$ for $g_k \in \mathcal{C}^\infty(M)$, we have
\begin{equation}
g_k = -\frac{1}{k(m-2s-k)} \sum_{\ell=0}^{k-1} Q_{k,\ell}(m-s) g_\ell
\end{equation}
\text{Lemma 3.4. Suppose that } u \in \mathcal{C}^\infty(\hat{X}) \text{ satisfies } u = x^{m-s} F + x^s G \text{ for functions } F \text{ and } G \text{ belonging to } \mathcal{C}^\infty(X), \text{ and that}
\begin{equation}
(\Delta - s(m-s)) u \in \mathcal{C}^\infty(X)
\end{equation}
for $s \in \mathbb{C}$ with $2s - m \notin \mathbb{Z}$. Then the Taylor expansions of $F$ and $G$ at $x = 0$
are formally determined respectively by $F|_M$ and $G|_M$. In particular, we have
\begin{equation}
f_k = \frac{1}{k!} \Gamma(2s-m-k) \Gamma(2s-m) P_{k,s}, f_0
\end{equation}
and
\begin{equation}
g_k = \frac{1}{k!} \Gamma(m-2s-k) \Gamma(m-2s) P_{k,m-s}, g_0
\end{equation}
where $P_{k,s}$ are differential operators of order $2k$ holomorphic in $s$ with leading symbol
\begin{equation}
\sigma(P_{k,s}) = \frac{1}{2k} \sigma(-\Delta^k)
\end{equation}
Proof. Recall the asymptotic development (2.23) for the Laplacian which we use to
derive a recurrence for the Taylor coefficients $f_k$ and $g_k$ of $F$ and $G$. For $2s-m \notin \mathbb{Z}$,
we may consider the terms involving $F$ and $G$ separately. We first consider $F$.
Observe that
\begin{equation}
(L_0 - s(m-s)) \left(x^{m-s+k} f\right) = k(2s-m-k)x^{s+k} f
\end{equation}
for $f \in \mathcal{C}^\infty(M)$. Since $L_k = P(x\partial_x, \partial_y)$ for a defining function $x$ and boundary co-
ordinates $y$ where $P$ is a polynomial of degree at most two with smooth coefficients,
the operators
\begin{equation}
Q_{k,\ell}(s) = x^{-m+s-\ell} L_{k-\ell} x^{m-s+\ell}
\end{equation}
are differential operators of order at most two depending holomorphically on $s$. If
$u \sim \sum_{k=0}^{\infty} x^{m+s+k} f_k$, it follows from (3.6) and (2.23) that for any $k \geq 1$,
The formulas for \( f_k, g_k, \) and \( P_{k,s} \) follow easily from these formulas and the fact that

\[
Q_{k,k-1}(s) = \frac{1}{4} \left( -2\Delta s u - 4r_0 (m - s + 1) - 4r_0 \left((m-s+1)^2 - (m-s+1)\right) \right)
\]

\[\square\]

**Remark 3.5.** We will write \( p_{k,s} \) for the operator with \( f_k = p_{k,s} f_0 \), so that \( p_{k,s} \) is meromorphic with poles at \( s = m/2 + k/2, \ldots, m/2 + 1/2 \). We will denote

\[
p_{\ell} = \operatorname{Res}_{s=m/2+\ell/2} p_{\ell,s}
\]

The operator \( p_{\ell} \) is a differential operator of order at most \( 2\ell \) with principal symbol

\[
\sigma(p_{\ell}) = \frac{1}{2^\ell \ell!(\ell-1)!} \sigma(-\Delta_0^s)
\]

For \( \text{Re}(s) > m/2 \), let

\[
R(s) = (\Delta_\varphi - s(m-s))^{-1}
\]

be the \( L^2(X) \) resolvent, let \( \sigma_p(\Delta_\varphi) \) denote the set of \( L^2 \)-eigenvalues of \( \Delta_\varphi \), and let

\[
\Sigma = \{ s : \text{Re}(s) > m/2, s(m-s) \in \sigma_p(\Delta_\varphi) \}.
\]

We will now solve the Dirichlet problem (1.3) for \( \text{Re}(s) \geq m/2 \) and \( s \notin \Sigma \).

The following result is an easy consequence of the work of Epstein, Melrose, and Mendoza [4], noting that in our case the Kähler metric is an even metric, i.e., depends smoothly on the defining function \( \varphi \) (and not simply on its square root).

**Proposition 3.6.** The set \( \Sigma \) contains at most finitely many points, and the resolvent operator \( R(s) \) is a meromorphic operator-valued function for \( \text{Re}(s) > m/2 - 1/2 \) having at most finitely many, finite-rank poles at \( s \in \Sigma \). Moreover, for \( s \notin \Sigma \), and \( \text{Re}(s) > m/2 - 1/2 \),

\[
R(s) : \mathcal{C}^\infty(X) \rightarrow x^s \mathcal{C}^\infty(X).
\]

First, we prove uniqueness of solutions to the Dirichlet problem (1.3) for \( s \) with \( \text{Re}(s) \geq m/2, s \notin \Sigma \), and \( 2s - m \notin \mathbb{Z} \).

**Proposition 3.7.** Suppose that \( \text{Re}(s) \geq m/2, s \notin \Sigma \), and \( 2s - m \notin \mathbb{Z} \). Suppose that \( u \in \mathcal{C}^\infty(X) \) with \( (\Delta_\varphi - s(m-s))u = 0 \), and that \( u = x^{m-s}F + x^sG \) with \( F|_M = 0 \). Then \( u = 0 \).

**Proof.** First, suppose that \( \text{Re}(s) > m/2 \) and \( s \notin \Sigma \). It follows from Lemma 3.4 that \( u = x^sG \) for \( G \in \mathcal{C}^\infty(X) \). Since \( \text{Re}(s) > m/2 \) it is clear that \( \int_X |u|^2 \omega^m < \infty \), hence \( u \in L^2(X) \), hence \( u = 0 \).

If \( \text{Re}(s) = m/2 \) but \( s \neq m/2 \), we may again assume that \( u = x^sG \) for \( G \in \mathcal{C}^\infty(X) \). Next, we set \( u_1 = u_2 = u \) in (3.4) to conclude that \( \int_M |G|^2 \psi = 0 \) so that \( G|_M = 0 \). Using Lemma 3.4 again we conclude that \( G \in \mathcal{C}^\infty(X) \), hence \( u \in \mathcal{C}^\infty(X) \). As in [15], we can now deduce from [28] that \( u = 0 \).

To prove existence of a solution of the Dirichlet problem (1.3), we follow the method of Graham and Zworski [14]. Given \( f \in \mathcal{C}^\infty(M) \) we can construct a formal power series solution \( u = x^{m-s}F \) modulo \( \mathcal{C}^\infty(X) \), and then use the resolvent to correct this approximate solution to an exact solution. Using Borel’s lemma we can sum the asymptotic series \( \sum_{j \geq 0} f_j x^j \) (where \( f_j \) is computed via (3.8) with \( f_0 = f \)) to a function \( F \in \mathcal{C}^\infty(X) \). As in [14], we obtain:
Lemma 3.8. There is an operator $\Phi(s) : \mathcal{C}^\infty(M) \to x^{-s}\mathcal{C}^\infty(X)$ with
\[
(\Delta_\varphi - s(m - s)) \circ \Phi : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(X)
\]
so that $\Gamma(m - 2s)^{-1}\Phi(s)$ is holomorphic in $s$.

Note that $\Phi(s)$ need not be linear as the construction of $F$ depends on the choice of cutoff functions in the application of Borel’s lemma. Now define an operator
\[
\mathcal{P}(s) : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(\hat{X})
\]
for $s$ with $\text{Re}(s) \geq m/2$, $s \neq m/2$ and $s \notin \Sigma$ by
\[
\mathcal{P}(s) = [I - R(s)(\Delta_\varphi - s(m - s))] \circ \Phi(s)
\]

Lemma 3.9. For any $f \in \mathcal{C}^\infty(M)$, the function $u = \mathcal{P}(s)f$ solves the Dirichlet problem (1.3), and $f \mapsto \mathcal{P}(s)f$ is a linear operator.

Proof. The linearity of $\mathcal{P}(s)$ will follow from the unicity of the solution to (1.3). It is immediate from the definition that $(\Delta_\varphi - s(m - s))u = 0$, and from the mapping property in Proposition 3.8, $u = x^{m-s}F + x^sG$ with $F = x^{s-m}\Phi(s)f$ and $G = -x^{-s}R(s)[(\Delta_\varphi - s(m - s))\Phi(s)f]$. \hfill $\square$

We now have:

Theorem 3.10. For $\text{Re}(s) \geq m/2$, $2s - m \notin \mathbb{Z}$, and $s \notin \Sigma$, there exists a unique solution of the Dirichlet problem (1.3).

3.2. The Scattering Operator. The scattering operator for $\Delta_\varphi$ is the linear mapping
\[
S_X(s) : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)
\]
\[
f \mapsto G|_M
\]
where $u = x^{m-s}F + x^sG$ solves (1.3). It is well-defined by Theorem 3.10.

The scattering operator has infinite-rank poles when $\text{Re}(s) > m/2$ and $2s - m \in \mathbb{Z}$ owing to the crossing of indicial roots for the normal operator $L_0$. At the exceptional points $s = m/2 + k$ one expects solutions of the eigenvalue equation $(\Delta_\varphi - s(m - s))u = 0$ having the form
\[
u = x^{m/2-k}F + \left(x^{m/2+k}\log x\right)G
\]
In order to study the singularities of the scattering operator at these points we modify the construction of the Poisson operator following the lines of [14], section 4.

Let $f_1$ and $f_2$ belong to $\mathcal{C}^\infty(M)$ and let $u_1$ and $u_2$ solve the corresponding Dirichlet problems for some $s$ with $\text{Re}(s) > m/2$ and $2s - m \notin \mathbb{N}$. Applying the generalized boundary pairing formula (see Proposition 3.6) to $u_1$ and $u_2$ for $s$ real, we conclude that
\[
\int_M f_1S_X(s)f_2 = \int_M [S_X(s)f_1][S_X(s)f_2]
\]
so $S_X(s)$ is self-adjoint in the natural inner product on $\mathcal{C}^\infty(M)$.

Now we study the scattering operator near the exceptional points. The arguments used here are exactly those of section 3 in [14] but we summarize them here for the reader’s convenience.

Recall the operators $p_{k,s}$ and $p_{\ell}$ defined in Remark 3.3. First, we prove:
Lemma 3.11. At the points $s = m/2 + \ell/2$, $\ell = 1, 2, \cdots$, $s \notin \Sigma$, the Poisson map takes the form

$$P(m/2 + \ell/2)f = x^{m/2-\ell/2}F + \left(x^{m/2+\ell/2}\log x\right)G$$

where

$$F|_M = f$$

and

$$G|_M = -2p_\ell f$$

where

$$(3.10) \quad p_\ell = \text{Res}_{s=m/2+\ell/2} p_{\ell,s}$$

is a differential operator of order $2\ell$ with

$$\sigma(p_\ell) = \frac{1}{2^{\ell!} (\ell - 1)!} \sigma(\Delta^\ell)$$

Proof. We first show that the Poisson map $P(s)$ is also regular at $s = m/2 + \ell/2$, $\ell = 1, 2, \cdots$ so long as these points do not belong to $\Sigma$. As in [14] we introduce the operator

$$(3.11) \quad \Phi_\ell(s) = \Phi(s) - \Phi(m - s) \circ p_{\ell,s}$$

where $p_{\ell,s}$ is a differential operator of order $2\ell$ defined in Remark 3.5. Each of the right-hand terms in (3.11) has at most a first-order pole at $s = m/2 + \ell/2$; the operators $p_{j,s}$ occurring in the definition of $\Phi(s)$ have at most first-order poles, while $\Phi(m - s)$ is analytic in $s$ for $\text{Re}(s) > m/2$. For given $f \in C^\infty(M)$, we compute the residue of $\Phi_\ell(s)f$ at $s = m/2 + \ell/2$. First

$$\lim_{s \to m/2 + \ell/2} \left( s - \frac{m}{2} - \frac{\ell}{2} \right) \Phi(s)f = x^{m/2+\ell/2} \text{Res}_{s=m/2+\ell/2} (p_{\ell,s}f) + O\left(x^{m/2+\ell/2+1}\right)$$

since the remaining terms in the asymptotic expansion for $\Phi(s)f$ are holomorphic near $s = m/2 + \ell/2$. Second,

$$\lim_{s \to m/2 + \ell/2} \left( s - \frac{m}{2} - \frac{\ell}{2} \right) \Phi(m - s)(p_{\ell,s}f) = x^{m/2+\ell/2} \text{Res}_{s=m/2+\ell/2} (p_{\ell,s}f) + O\left(x^{m/2+\ell/2+1}\right).$$

It follows that

$$(3.12) \quad \text{Res}_{s=m/2+\ell/2} \Phi_\ell(s)f = O\left(x^{m/2+\ell/2+1}\right)$$

so that, by Lemma [3.4], $\text{Res}_{s=m/2+\ell/2} \Phi_\ell(s)f \in \mathcal{L}^\infty(X)$.

Now let us define

$$\mathcal{P}_\ell(s) = [I - R(s)(\Delta - s(m - s))] \circ \Phi_\ell(s)$$

Clearly, $\mathcal{P}_\ell(s)$ is holomorphic in a deleted neighborhood of $s = m/2 + \ell/2$ (with at most a first-order pole at $s = m/2 + \ell/2$) and maps $C^\infty(M)$ into $C^\infty(X)$. If $s \notin \Sigma$, it follows from the definition of $\mathcal{P}_\ell(s)$, equation (3.12), and Proposition 3.6 that

$$\text{Res}_{s=m/2+\ell/2} \mathcal{P}_\ell(s)f \in x^2C^\infty(X),$$

hence the residue is an $L^2(X)$ function, and hence is zero. Thus $\mathcal{P}_\ell(s)$ is holomorphic at $s = m/2 + \ell/2$. It follows from the uniqueness of solutions to the Dirichlet
problem that \( \mathcal{P}_\ell(s) = \mathcal{P}(s) \) wherever the former is defined. Exactly as in [14] we can compute \( \mathcal{P}(m/2 + \ell/2)f \) by using \( \mathcal{P}_\ell(s) \), the formula
\[
\lim_{t \to 0} \frac{e^{-t} - e^{t}}{t} = -2 \log x
\]
and the fact that the \( p_{k,s} \) have at most simple poles at \( s = m/2 + \ell/2 \). This computation shows that \( \mathcal{P}(m/2 + \ell/2) \) has the stated form. \( \square \)

Next, we prove:

**Proposition 3.12.** Suppose that \( \Delta_X \) has no eigenvalues of the form \( s(m-s) \) with \( s = m/2 + \ell/2, \ell = 1, 2, \cdots \). Then, the scattering operator \( S_X(s) \) has a first-order pole at \( s = m/2 + \ell/2, \ell = 1, 2, \cdots \) with
\[
\text{Res}_{s=m/2+\ell/2} S_X(s) = p_\ell.
\]
where \( p_\ell \) is the differential operator given by (3.10).

**Proof.** From the formula for the \( \mathcal{P}_\ell(s) \), it is clear that for \( 2s - m \notin \mathbb{N} \), we can compute the scattering operator from
\[
S_X(s)f = \left. \left[ -x^{-s} R(s) (\Delta_X - s(m-s)) \Phi(s)f \right] \right|_{x=0}.
\]
Since \( \mathcal{P}(s) \) is holomorphic at \( s = n/2 + \ell/2 \) (unless \( s \in \Sigma \)), it follows that
\[
\text{Res}_{s=m/2+\ell/2} [S_X(s)f] = \text{Res}_{s=m/2+\ell/2} \left[ x^{-s} \Phi(s)f \right]_{x=0}.
\]
But
\[
\text{Res}_{s=m/2+\ell/2} \left[ x^{-s} \Phi(s)f \right]_{x=0} = \text{Res}_{s=m/2+\ell/2} \left[ x^{-s} \Phi(m-s)p_\ell f \right]_{x=0}
\]
and the claimed formula holds. \( \square \)

To connect the scattering operator and the CR \( Q \)-curvature, we will also need the following result about the pole of the scattering operator at \( s = m \); this result is a direct analogue of Proposition 3.7 in [14] but we give the short proof for the reader’s convenience.

**Proposition 3.13.** Let \( 1 \) denote the constant function on \( M \). Then, the formula
\[
S_X(m)1 = -\lim_{s \to m} p_{m,s}(1)
\]
holds.

**Proof.** As \( s \to m \) we have \( \mathcal{P}(s)1 \to 1 \). On the other hand, for \( s \) with \( |s - m| < 1/2 \),
\[
\mathcal{P}(s)1 = \sum_{k=0}^{m} x^{m-s+k} p_{k,s}(1) + x^s S_X(s)1 + O(x^{m+1/2}).
\]
This implies that
\[
\lim_{s \to m} \left[ x^{2m-s} p_{m,s}(1) + x^m S_X(s)1 \right] = 0
\]
from which the claimed formula follows. \( \square \)
Remark 3.14. Note that, although $p_{m,s}$ has a pole at $s = m$, the limit $\lim_{s \to m} p_{m,s}(1)$ exists. This implies that $P_{m,s,1}$ (see (3.7)) has a first-order zero at $s = m$, i.e., $P_{m,s,1} = (m - s)Q_{m,s}$ for a scalar function $Q_{m,s}$. The CR $Q$-curvature is then given by $Q_{m,m}$ [8].

4. CR-Covariant Operators

In this section we show that if $\varphi$ is an approximate solution of the complex Monge-Ampère equation in the sense discussed above, then the residues of the scattering operator at $s = m/2 + \ell/2, \ell = 1, \cdots, m$ are the CR-covariant differential operators $P_k$ defined in [8]. In order to do this we first recall Fefferman and Graham’s [7] set-up for studying conformal invariants of compact manifolds and the construction of the GJMS [12] operators. We then recall its application to CR-manifolds taking care that the arguments carry over from pseudoconvex domains in $\mathbb{C}^m$ to the manifold setting studied here.

4.1. The GJMS Construction. We begin by recalling Fefferman and Graham’s construction of the ambient metric and ambient space for a conformal manifold and the GJMS conformally covariant operators on $\mathcal{C}$ obtained from this construction. Suppose that $(\mathcal{C}, [g])$ is a conformal manifold of signature $(p,q)$, i.e., a smooth manifold of dimension $N = p + q$ together with a conformal class of pseudo-Riemannian metrics of signature $(p,q)$ on $\mathcal{C}$. Fix a conformal representative $g_0$. The metric bundle $G \subset S^2T^*\mathcal{C}$ is a bundle on $\mathcal{C}$ with fibres

$$G_p = \{t^2 g_0(p) : t > 0\}$$

We denote by $\pi : G \to M$ the natural projection. The tautological metric $G$ on $G$ is given by

$$G(X,Y) = g(\pi_*X, \pi_*Y)$$

for tangent vectors $X$ and $Y$ to $(p,g) \in G$. There is a natural $\mathbb{R}^+$-action $\delta_s$ on $G$ given by $\delta_s(p,g) = (p, s^2 g)$.

The ambient space over $\mathcal{C}$ is the space $\tilde{G} = G \times (-1, 1)$. Note that the map $g \mapsto (g,0)$ imbeds $G$ in $\tilde{G}$.

Fefferman and Graham proved the existence of a unique metric $\tilde{g}$ of signature $(p+1,q+1)$ on $\tilde{G}$, the ambient metric on $\tilde{G}$ having the following three properties:

(a) $\delta^*_s \tilde{g} = \tilde{g}$
(b) $\delta^*_s \tilde{g} = s^2 \tilde{g}$
(c) $\text{Ric}(\tilde{g}) = 0$ along $\tilde{G}$ to infinite order if $N$ is odd, and up to order $N/2$ if $N$ is even.

Here the uniqueness is meant in the sense of formal power series.

To define the GJMS operators, we first define spaces of homogeneous functions on $\tilde{G}$. For $w \in \mathbb{R}$ let $\mathcal{E}(w)$ denote the functions $f$ on $\tilde{G}$ homogeneous of degree $w$ with respect to $\delta_s$ and smooth away from 0. The GJMS operators $\mathcal{P}_k$ may be defined in two ways.

1. Given $f \in \mathcal{E}(-N/2 + k)$, extend $f$ to a function $\tilde{f}$ homogeneous of the same degree on $\tilde{G}$, and set

$$\mathcal{P}_k f = \tilde{\Delta}^k \tilde{f}$$

where $\tilde{\Delta}$ is the Laplacian for the ambient metric $\tilde{g}$ on $\tilde{G}$.
(2) Given \( f \in \mathcal{E}(-N/2 + k) \), \( P_k \) is the normalized obstruction to extending \( f \) to a smooth function \( \tilde{f} \) on \( \mathcal{G} \) having the same homogeneity and satisfying \( \tilde{\Delta}^k \tilde{f} = 0 \).

The existence of GJMS operators was proven in \([12]\) for \( k = 1, 2, \cdots \) if \( N \) is odd, and for \( k = 1, 2, \cdots, N/2 \) if \( N \) is even.

4.2. Application to CR-Manifolds. Following \([9]\) we describe how the GJMS construction \([12]\) can be used to prove the existence of CR-covariant differential operators. We begin with a CR-manifold \( M \) of dimension \( 2n + 1 \) and show how to construct a conformal manifold \( C \) of dimension \( 2n + 2 \) and a conformal class of metrics with signature \((2n + 1, 1)\) to which the GJMS construction may be applied. One then “pulls back” the GJMS operators to \( M \).

Recall that the canonical bundle \( K \) over \( M \) is the bundle of holomorphic \((n+1)\)-forms generated by holomorphic forms of the type \( \theta \wedge \theta_1 \wedge \cdots \wedge \theta_n \) where \( \theta \) is a contact form and \( \{\theta^\alpha\} \) is a basis for \( \mathcal{H} \) of admissible \((1,0)\)-forms. We denote by \( K^* \) the canonical bundle of \( M \) with the zero section removed. The circle bundle \( C \) over \( M \) is the bundle \( C = (K^*)^{1/(n+2)}/\mathbb{R}^+ \).

The circle bundle is an \( S^1 \)-bundle over \( M \), having real dimension \( 2m \) if \( m = n + 1 \). If we fix a contact form \( \theta \) on \( M \) (and hence a pseudo-Hermitian structure on \( M \)), there is a corresponding section \( \zeta \) of \( K^* \) chosen so that \( \theta \) is volume-normalized with respect to \( \zeta \). We denote by \( \psi \) the angle determined by \( \zeta(p) \) in each fibre of \( C \) and define a fibre variable

\[
\gamma = \frac{\psi}{n + 2}
\]

Note that \( \gamma \) is canonically determined by \( \theta \). Following Lee \([21]\), let us define a canonical one-form \( \sigma \) on \( C \) by

\[
(n + 2)\sigma = (n + 2)d\gamma + i\omega_\alpha^\beta - \frac{1}{2(n + 1)}R\theta
\]

where \( \omega_\alpha^\beta \) is the connection one-form and \( R \) is the Webster scalar curvature of the pseudo-Hermitian structure \( \theta \). The mapping \( \theta \mapsto g_\theta \) given by

\[
g_\theta = h_{\alpha\beta} \theta^\alpha \cdot \theta^\beta + 2\theta \cdot \sigma
\]

(where \( \cdot \) denotes the symmetric product) defines a mapping of pseudo-Hermitian structures to Lorenz metrics which respects conformal classes. One can now obtain GJMS operators on \( C \) using the Fefferman-Graham construction.

**Remark 4.1.** It is immediate from formulas \((4.2)\) and \((4.3)\) that

\[
g_\theta(T, T) = -\frac{1}{(n + 1)(n + 2)}R.
\]

On the other hand, Farris \([5]\) computed that, if \( \theta \) is the contact form induced by an approximate solution of the complex Monge-Ampère equation, then

\[
g_\theta(T, T) = 2r
\]

where \( r \) is the transverse curvature. It follows that the transverse curvature is, in this case, an intrinsic pseudo-Hermitian invariant.
To compute their pullbacks to $M$, we first note that the metric bundle $\mathcal{G}$ of $(\mathcal{C}, [g])$ is diffeomorphic to $(K^*)^{1/(n+2)}$ and $\mathcal{G} \simeq (K^*)^{1/(n+1)} \times (-1, 1)$. We define spaces of functions

$$\mathcal{E}(w, w') = \left\{ f \in \mathcal{C}^\infty(K^*)^{1/(n+2)} : f(\lambda \xi) = \lambda^w \overline{\lambda}^{w'} f(\xi) \text{ for } \lambda \in \mathbb{C}^* \right\}$$

$$= \left\{ f \in \mathcal{E}(w + w') : (e^{i\phi})^* f(\xi) = e^{i\phi(w - w')} f(\xi) \right\}$$

We will primarily be concerned with functions in

$$\mathcal{E}(w, w) = \left\{ f \in \mathcal{E}(w + w) : (e^{i\phi})^* f(\xi) = f(\xi) \right\}$$

which descend to smooth functions on $M$.

For $k \in \mathbb{Z}$, we define $P_{w, w'} : \mathcal{E}(w, w') \to \mathcal{E}(w - k, w' - k)$

$$f \mapsto 2^{-k} P_k f,$$

where $\mathcal{P}_k$ is defined in (4.1). Then choosing $w = w' = (k - (n + 1))/2$, we get operators $P_k$ defined on $\mathcal{E}(-N/2 + k)$, which are invariant under the circle action $(e^{i\phi})^*$ and hence may be viewed as smooth sections of a density bundle over $M$. These operators $P_k$ are the CR-covariant differential operators which we will connect to poles of the scattering operator.

If $X$ admits a globally defined approximate solution $\varphi$ of the Monge-Ampère equation, then for each $p \in M = \partial X$ there is a neighborhood $U$ of $p$ and holomorphic coordinates $(z_1, \cdots, z_m)$ near $p$ so that $\varphi$ is an approximate solution of the Monge-Ampère equation in $U$. Let

$$\theta = \frac{i}{2} \left( \overline{\partial} - \partial \right) \varphi \bigg|_M$$

be the induced pseudo-Hermitian structure on $M$, and let $\zeta = dz^1 \wedge \cdots \wedge dz^m \big|_M$. Then $\theta$ is volume-normalized with respect to $\zeta$.

Let us denote by $z_0$ the induced fibre coordinate of $(K^*)^{1/(n+2)}$ and let

$$Q = |z_0|^2 \varphi$$

Then $Q$ is a globally defined smooth function on $\tilde{G}$ (which is diffeomorphic to $\mathbb{C} \times N$ for a collar neighborhood $N$ of $M$ in $X$) and the ambient metric on $\tilde{G}$ is the Kähler metric associated to the Kähler form

$$\omega = i \partial \overline{\partial} Q$$

where the corresponding metric $g_\theta$ on $\mathcal{C}$ is given by (1.3). The key computation linking the GJMS operators to the Laplacian is given in Proposition 5.4 of [11] and clearly generalizes to our situation. Thus we have:

**Proposition 4.2.** If $u$ is a smooth function on $X$ then

$$\widetilde{\Delta} \left( |z_0|^{2w} \varphi^w u \right) = \left( |z_0|^{2w} \varphi^w \right) (\Delta_\varphi + w(n + 1 + w)) u$$

where $g$ is the metric associated to the Kähler form

$$\omega_\varphi = \frac{i}{2} \partial \overline{\partial} \log (-1/\varphi)$$
5. Proofs of the Main Theorems

Finally, we prove Theorems 1.1, 1.4, and 1.5.

Proof of Theorem 1.1. The statement about the poles of $S_X(s)$ and $s = m/2 + k/2$ is proved in Proposition 3.12. If $g$ is a metric on $X$ associated to the Kähler form $\omega = i\partial \bar{\partial} \log(-1/\varphi)$ for a globally defined approximate solution of the Monge-Ampère equation, then the identification of the residues of $S_X(s)$ with the CR-covariant differential operators of Fefferman and Hirachi is a consequence of Proposition 4.2 and the second characterization of the GJMS operators given in section 4.1. □

Proof of Theorem 1.4. Owing to Proposition 3.13, it suffices to identify $\lim_{s \to m} p_{m,s,1}$ with the CR $Q$-curvature. This is a consequence of Remark 3.14. □

Proof of Theorem 1.5. To prove Theorem 1.5, let $u_s = \mathcal{P}(s)1$ for $s$ real. Observe that $u_s$ is real and that $u_s \to 1$ as $s \to m$ uniformly on compact subsets of $X$. On the other hand, for $s \neq m$ but $s$ close to $m$, $u_s$ takes the form

\begin{equation}
 u_s \sim x^{m-s} F(s) + x^s G(s)
\end{equation}

where $F(s)$ and $G(s)$ belong to $C^\infty(X)$, $G(s) = S_X(s)1 + \mathcal{O}(x)$ uniformly in $s$ near $m$, and

\begin{equation}
 F(s) - 1 \sim \sum_{k \geq 1} x^k F_k(s)
\end{equation}

where $F_k \in C^\infty(M)$ and $F_k(s) \to 0$ as $s \to m$, save for the $F_m(s)$ term which obeys

\begin{equation}
 F_m(s) + S_X(s)1 \to 0 \text{ as } s \to m
\end{equation}

(see the proof of Proposition 3.13). Note that

\[ \int_M F_m(s) \psi = \int p_{m,s,1} \psi. \]

As in [14] we will prove Theorem 1.5 by computing

\[ \text{FP} \epsilon \int_{x > \epsilon} [du_s]^2 - s(m-s)u_s u_s \omega^m \]

(where FP denotes the Hadamard finite part) in two different ways. First, we use Proposition 3.3 with $u_1 = u_2 = u_s$ to conclude that

\begin{equation}
 \text{FP} \epsilon \int_{x > \epsilon} [du_s]^2 - s(m-s)u_s u_s \omega^m = -m \int_M S_X(s)1 \psi
\end{equation}

Secondly (and somewhat more painfully), we use the asymptotic expansion of $u_s$ and the asymptotic form of the volume form $\omega^m$ directly to conclude that

\begin{equation}
 \text{FP} \epsilon \int_{x > \epsilon} [du_s]^2 - s(m-s)u_s u_s \omega^m = m \frac{2}{L}
\end{equation}

from which the desired equality will follow. The computations follow along the lines of [14] with some trivial differences in the computation for (A.4) owing to the different form of the Laplacian on a complex manifold. We give a summary in Appendix A. □
Appendix A: CR Q-curvature and Asymptotic Volume

The purpose of this appendix is to summarize the calculations leading to the identity (5.5) used in the proof of Theorem 1.5. We will show that

\[(A.1) \lim_{s \to m} s(m-s) \text{FP} \left( \int_{x>\varepsilon} u_s^2 \omega^m \right) = -mc_m \int_M Q^{CR}_\theta \psi + mL/2 \]

and

\[(A.2) \lim_{s \to m} \text{FP} \left( \int_{\varepsilon<x<x_0} |du_s|^2 \omega^m \right) = -mc_m \int_M Q^{CR}_\theta \psi \]

As in [14], since \(u_s \to 1\) uniformly on compacts of \(X\), it suffices to compute the respective limits

\[(A.3) \lim_{s \to m} \left[ s(m-s) \text{FP} \left( \int_{\varepsilon<x<x_0} u_s^2 \omega^m \right) \right] \]

and

\[(A.4) \lim_{s \to m} \left[ \text{FP} \left( \int_{\varepsilon<x<x_0} |du_s|^2 \omega^m \right) \right] \]

for any \(x_0 > 0\). This reduction allows us to use boundary coordinates and introduce asymptotic expansions for \(u_s\) and \(\omega^m\). In the computations we make use of the simple formulas

\[(A.5) \text{FP} \int_{\varepsilon}^{x_0} \frac{x^{m-2s+j}}{x} \, dx = \frac{x_0^{m-2s+j}}{m-2s+j} \]

(note that the finite part is independent of \(x_0\) if \(j = m\)) and

\[(A.6) \text{FP} \int_{\varepsilon}^{x_0} \frac{dx}{x} = \log x_0 \]

Setting \(x = -\varphi\), we also have from (2.14) that

\[\omega^m_\varphi = \frac{\eta}{x^m} \frac{dx}{x} \wedge (d\theta)^n \wedge \theta \]

for \(\eta \in C^\infty(X)\) with \(\psi = (\eta|_M) (d\theta)^n \wedge \theta\) (here \(\eta|_M = m/2^{n-1}\) in accordance with (2.16)). We will write

\[\eta \sim \sum_{k \geq 0} x^k \eta_k \]

for \(\eta_k \in C^\infty(M)\), so that

\[L = \int_M \eta_m (d\theta)^n \wedge \theta. \]

First, we consider (A.3). In expanding the density

\[u_s^2 \omega^m = f_1 \frac{dx}{x} \wedge (d\theta)^n \wedge \theta \]

asymptotically in \(x\), we may neglect terms which are integrable, or terms which give rise to finite parts which are holomorphic at \(s = m\). It suffices then to compute the coefficient of \(x^{2m-2s}\) in the expansion for \(f_1\), since only the \(x^{2m-2s}\) term will
give rise to a finite part with pole at \( s = m \). Note that the resulting residue is independent of \( x_0 \) (see (A.5)). Since
\[
  u_s^2 = x^{2m-2s} \left[ 1 + 2(F(s) - 1) + (F(s) - 1)^2 \right]
  + 2x^{2m} F(s) G(s) + x^2 G(s)^2
\]
it suffices to examine the terms \( x^{2m-2s} F_m \) and \( 2x^{2m-2s} F_m(s) \) in \( f_1 \). The first of these contributes \((m/2) L\) to (A.3) and the second contributes
\[
- \int_M \mathcal{S}_X(m) \psi = -mc_m \int_M Q^C_R \psi.
\]
This leads to (A.1) as claimed.

Next, we consider (A.4). From (2.18), (2.22), and the fact that
\[
|u_m|^2 = \frac{1}{4} |(N - iT) u|^2
\]
it follows that the density
\[
(A.7) \quad |du_s|^2 \omega^m = \frac{1}{1 + r_x} |x \partial_x u_s|^2 \omega^m + x H^\alpha \overline{\theta} (u_s)_\alpha (u_s)_{\overline{\beta}} \omega^m
\]
for a tensor \( H^\alpha \) which is smooth in \( x \) down to \( x = 0 \). We will show that the first right-hand term in (A.7) leads to the right-hand side of (A.2) and the second right-hand term in (A.7) makes no contribution.

From the asymptotic form of \( u_s \) (see (5.1), (5.2), (5.3)) we have
\[
(A.8) \quad x \partial_x u_s = x^{m-s} K_1(s) + x^s K_2(s)
\]
where
\[
K_1(s) = (m-s) F(s) + x F_x(s)
\]
and \( K_1(s) \) and \( K_2(s) \) both approach zero as \( s \to m \). For this reason, writing
\[
|x \partial_x u_s|^2 \omega^m = f_2 \frac{dx}{x} \wedge (d\theta)^n \wedge \theta
\]
we need only consider the coefficient of \( x^{2m-2s} \) in the expansion of \( f_2 \) since all other terms give rise to terms whose finite parts vanish as \( s \to m \). On squaring (A.8) we have
\[
|x \partial_x u_s|^2 = x^{2m-2s} K_1(s)^2 + 2x^m K_1(s) K_2(s) + x^{2s} K_2(s)^2.
\]
We can drop terms containing \((m-s)^2\) times a holomorphic function since these will vanish as \( s \to m \), even if the power \( x^{2s-2m} \) occurs in \( f_2 \). Thus we need to compute the coefficient of \( x^m \) in \( K_1(s)^2 \) to order \((m-s)\). This is \( 2(2m-s)(m-s) F_m(s) \) which contributes \(-2(m-s) \int_M F_m(s) \psi\) to the finite part of \( \int_{x > \varepsilon} |x \partial_x u_s|^2 \omega^m \) and approaches \(-mc_m \int_M Q^C_R \psi\) as \( s \to m \).

It remains to show that
\[
(A.9) \quad \lim_{s \to m} \frac{\zeta}{\varepsilon} \left( \int_{\varepsilon}^{X_0} \frac{r_x}{1 + r_x} |x \partial_x u_s|^2 \omega^m \right) = 0
\]
and
\[
(A.10) \quad \lim_{s \to m} \frac{\zeta}{\varepsilon} \left( \int_{\varepsilon}^{X_0} x H^\alpha \overline{\theta} (u_s)_\alpha (u_s)_{\overline{\beta}} \omega^m \right) = 0
\]
In the first case, we can use the analysis above to show that the coefficient of \( x^{2m-2s} \) in the expansion for \( [(r \cdot x)/(1 + r_x)] |x \partial_x u_s|^2 \omega^m \) vanishes as \((s-m)^2\) when \( s \to m \),
implying (A.9). Introducing boundary local coordinates \( y_j \), to prove that (A.10) holds it suffices to show that

\[
\lim_{s \to m} \int_{\partial M} \phi \frac{\partial u}{\partial y_j} \frac{\partial u}{\partial y_k} \omega^m = 0
\]

where \( \phi \) is a smooth function supported in a local coordinate patch near the boundary \( M \). This follows from the fact that, in local coordinates \((x, y)\) on \( X \) in a neighborhood of \( M \),

\[
\frac{\partial u}{\partial y_j} = x^{m-s} L_1(s) + x^s L_2(s)
\]

where both \( L_1(s) \) and \( L_2(s) \) vanish to order \((m-s)\) as \( s \to m \).

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