Abstract
We consider here the Feynman amplitudes of renormalizable non-commutative quantum field theory models. Different representations (the parametric and the Mellin one) are presented. The latter further allows the proof of meromorphy of an amplitude in the space-time dimension.

1 Introduction
This paper presents different representations of Feynman amplitudes of non-commutative quantum field theory (NCQFT) models. The models considered here are the renormalizable models, namely the Grosse-Wulkenhaar model [1, 2] or the "covariant models" (which include the non-commutative Gross-Neveu model or the Langmann-Szabo-Zarembo - LSZ - model).

2 Renormalizable NCQFT models
We consider the 4−dimensional Moyal space [xμ, xν] = iΘμν, where the the matrix Θ is

\[
Θ = \begin{pmatrix}
0 & \theta & 0 & 0 \\
-\theta & 0 & 0 & 0 \\
0 & 0 & 0 & \theta \\
0 & 0 & -\theta & 0
\end{pmatrix} .
\]

We also consider an Euclidean metric. Let us now introduce the two types of renormalizable non-commutative models.

2.1 The Grosse-Wulkenhaar model
The results established in the sequel hold for models with interactions of type \( \tilde{\phi} \ast \phi \ast \tilde{\phi} \ast \phi \). One has the Grosse-Wulkenhaar model of a complex scalar field

\[
S_{GW} = \int d^4 x \left( \partial_{\mu} \tilde{\phi} \ast \partial^{\mu} \phi + \Omega^2 (\tilde{x}_{\mu} \tilde{\phi}) \ast (\tilde{x}^{\mu} \phi) + \tilde{\phi} \ast \phi \ast \tilde{\phi} \ast \phi \right)
\]

where \( \tilde{x}_{\mu} = 2(\Theta^{-1})_{\mu\nu} x^\nu \). This action leads to the following propagator from a point \( x \) to a point \( y \):

\[
C(x, y) = \int_0^{\infty} \frac{d\alpha}{2\pi} \coth(\alpha) \left( e^{-\frac{\alpha}{4} \coth(\alpha)(x-y)^2} - e^{-\frac{\alpha}{4} \tanh(\alpha)(x+y)^2} \right).
\]
Let us now introduce the short and long variables: \( u = \frac{1}{\sqrt{2}} (x - y) \) and \( v = \frac{1}{\sqrt{2}} (x + y) \). Let \( t_\ell = \tanh \frac{\alpha}{2} \). The propagator (2.3) becomes

\[
C(x, y) = \int_0^\infty \frac{\hat{\Omega} d\alpha}{(2\pi \sinh(\alpha))^2} e^{-\frac{\hat{\Omega}}{\sqrt{2}} u^2 - \frac{\hat{\Omega}}{\sqrt{2}} t_\ell v^2}.
\]

(2.4)

### 2.2 The covariant models

As already stated in the introduction, amongst this type of models one has the non-commutative Gross-Neveu model and the LSZ model. The results we present in the sequel hold for the latter but they can be however extended for the Gross-Neveu model also. The LSZ action writes:

\[
S_{LSZ} = \int d^4x \left( (\partial^\mu \bar{\phi} - i \Omega \tilde{\epsilon}_\mu \phi) \star (\partial^\mu \phi - i \Omega \tilde{\epsilon}_\mu \bar{\phi}) + \bar{\phi} \star \phi \star \bar{\phi} \star \phi \right).
\]

(2.5)

This action leads to the propagator

\[
C(x, y) = 2 \int_0^1 dt_\ell \frac{\hat{\Omega}(1 - t_\ell^2)}{(4\pi t_\ell)^2} e^{-\frac{\hat{\Omega}}{\sqrt{2}} t_\ell u^2 + \frac{i}{4} \hat{\Omega} \frac{u \wedge v}{u^2 + v^2}},
\]

where \( u \wedge v = u_1 v_2 - u_2 v_1 + u_3 v_4 - u_4 v_3 \).

### 2.3 Non-local interaction

Using the explicit form of the Moyal product, the interaction term of both (2.2) and (2.5) lead to the following contribution in position space

\[
\delta(x_1^V - x_2^V + x_3^V - x_4^V) e^{2i \sum_{1 \leq i < j \leq 4} (-1)^i+j+1 x_i^V \Theta^{-1} x_j^V}
\]

(2.7)

where \( x_1^V, \ldots, x_4^V \) are the 4–vectors of the positions of the 4 fields incident to the respective vertex \( V \).

To end this section, let us remark that the Moyal space, as a linear space of infinite dimension, admits a particular base, the matrix base, which in two dimensions can be indexed by two natural numbers. All the NCQFT entities expressed in this section can be rewritten in this base (see for example [1, 2]). However, we do not introduce it here, since this is not requested to present our results.

### 3 Parametric representation

In the case of commutative QFT, one has translation invariance in position space. As a consequence of this invariance, the first polynomial vanishes when integrating over all internal positions. Therefore, one has to integrate over all internal positions (which correspond to vertices) save one, which is thus singularized. However, the polynomial is a still a canonical object, i. e. it does not depend of the choice of this particular vertex.

In the non-commutative case the translation invariance is lost (see previous section). Therefore, one can integrate over all internal positions and hypermomenta, without vanishing of the first polynomial. However, in order to be able to recover the commutative limit, we also singularize a particular vertex. We call this particular vertex the root. Because there is no translation invariance, the polynomial does depend on the choice of the root; however the leading UV terms do not.

From the propagator (2.4) and the vertices contributions (2.7) one is able to write the Feynman amplitude \( A \) as function of the non-commutative polynomials \( H U \) and \( H V \) as

\[
A = K \int_0^1 \prod_{\ell=1}^L [dt_\ell(1 - t_\ell^2)] |H U(t)|^{-2} e^{-\frac{H V(t)}{D}},
\]

(3.1)

where \( K \) is some constant, unessential for this calculus and by \( x_e \) and \( D \) we mean the external positions of the graph and resp. the space-time dimension. In [3] it was furthermore proved that \( H U \) and \( H V \) are polynomials in the set of variables \( t \).
Let us state that, even the formulas above hold also for non-orientable graphs (that is graphs corresponding to interactions $\hat{\phi} \bullet \hat{\phi} \bullet \hat{\phi} \bullet \hat{\phi}$), for simplicity reasons we restrict ourselves to the study of polynomials for orientable graphs (that is graphs corresponding to interactions $\hat{\phi} \bullet \hat{\phi} \bullet \hat{\phi} \bullet \hat{\phi}$, as already mentioned in the previous section).

### 3.1 The parametric representation for the Grosse-Wulkenhaar model

In [2], non-zero leading terms (i.e., terms which have the smallest global degree in the $t$ variables) of $HU$ were identified. These terms are dominant in the UV regime.

In order to characterize some of them, we need the following definition:

**Definition 3.1** Let a subset $J$ of the set $\{1, \ldots, L\}$ of internal lines of a Feynman graph. Then $J$ is a hyper-tree if it contains a tree in the dual graph and its complement contains a tree in the direct Feynman graph.

Let $|J|$ be the cardinal of the set $J$. Considering now a Feynman graph of genus $g$ and $F$ faces. In [2] it was proven the theorem:

**Theorem 3.1** One has the following lower limit on the polynomial $HU$

$$HU(t) \geq \sum_{J_{\text{hyper-tree}}} (2s)^{2g-k_J} \prod_{\ell \in J} t_\ell,$$

where $s = \frac{1}{\Omega}$ and $k_J = |J| - F + 1$.

### 3.2 Parametric representation for the Langmann-Szabo-Zarembo Model

It was proven in [4] that one can compute some leading terms for this type of model also.

**Theorem 3.2** One has the following lower limit on the polynomial $HU$

$$HU(t) \geq \sum_{J_0_{\text{hyper-tree}}} s^{2g+(F-1)} \prod_{\ell \in K} \left( \frac{1 + t_\ell^2}{2t_\ell} \prod_{\ell' \in \{1, \ldots, L\}} t_{\ell'}, \right)^2,$$

where $K = \{1, \ldots, L\} - J_0$, with $J_0$ some admissible set.

Note that the product of the factors $(\Omega \pm 1)$ depends on the topology of the graph (see [4] for details).

Theorem 3.2 or Theorem 3.3 allow to obtain the following power counting for both these models

$$\omega = 4g + \frac{1}{2}(N - 4),$$

where $\omega$ is the superficial degree of convergence and $N$ is the number of external legs of the respective graph.

Let us now make some comments on the results of this section. First of all, one notices an improvement in the power counting (3.4), improvement given by the presence of a new term in the graph genus. Moreover, let us recall that, in commutative QFT the parametric representation leads naturally to the topological notion of trees and to some “democracy” between them (one sums over all trees, with the same weight for each of them). The non-commutative equivalent of these properties is the natural appearance of the more involved topological notion of hyper-trees and a corresponding “democracy” between them. Another important issue to stress on is (as in the commutative case) the explicit positivity of the formulas. Finally, let us state that in all the formulas of the non-commutative parametric representation the space-time dimension $D$ is just a parameter. It is again the exact same situation as for the parametric representation for commutative QFT.

Note that for both type of models, when considering second polynomial $HV$, similar leading $UV$ terms, similar results of positivity, boundness, “democracy” between adapted topological entities and finally power counting have been obtained.
4 Mellin representation; meromorphy in $D$

Following [5] we present here the Mellin representation for the Feynman amplitudes of a graph corresponding to the Grosse-Wulkenhaar or the LSZ model. The polynomial $DU$ can be written as

$$HU = \sum_{K_U} a_{K_U} \prod_{\ell=1}^{L} t^{u_{K_U}} = \sum_{K_U} HU_{K_U},$$

(4.1)

where $K_U$ is a reunion of subsets of internal lines, $a_{K_U}$ is some constant (depending on the topology) and $u_{K_U}$ is an exponent which can take the values 0, 1 or 2 (see [4] for details). The difference with the commutative case comes from the presence of the constants $a_{K_U}$ as well as from the fact that the exponents $u_{K_U}$ are allowed to take the value 2.

The second polynomial $HV$ has both a real $HV^R$ and an imaginary part $HV^I$. This also is a major difference with respect to the commutative case. One now writes down formulas analogous to (4.1), formulas involving the reunion of subsets of internal lines $K_V$, the monomials $HV^R_{K_V}$ and $HV^I_{K_V}$, the constants $s^R_{K_V}$ and $s^I_{K_V}$ and the exponents $u_{K_V}$.

In order to introduce the Mellin representation, one writes for the real part

$$e^{-HV^R_{K_V}/U} = \int_{\tau^R_{K_V}} \Gamma(-y_{K_V}) \left(\frac{HV^R_{K_V}}{U}\right)^{y_{K_V}^R},$$

(4.2)

where $\int_{\tau^R_{K_V}}$ is a short notation for $\int_{-\infty}^{+\infty} \frac{d(x_{K_V})}{2\pi}$, with $\Re x_{K_V}$ fixed at $\tau^R_{K_V} < 0$. This formula introduces the set of Mellin parameters $y^R_{K_V}$.

A similar formula is written for the imaginary part $HV^I$ of $HV$, which introduces the set of Mellin parameters $y^I_{K_V}$. Note that in this case this will hold in the sense of distributions. This comes from the fact that the non-commutative vertex contribution (see [24]) has a distributional form. This is the major difference with respect to the commutative case.

For the polynomial $HU$ one has

$$\Gamma\left(\sum_{K_V} y^R_{K_V} + y^I_{K_V} + \frac{D}{2}\right) (HU)^{-\sum_{K_V}(y^R_{K_V} + y^I_{K_V}) - \frac{D}{2}} = \int_{\sigma} \prod_{K_U} \Gamma(-x_{K_U}) U^{x_{K_U}},$$

(4.3)

where $\int_{\sigma}$ is a short notation for $\int_{-\infty}^{+\infty} \frac{d(x_{K_U})}{2\pi}$ with $\sum_{K_U} x_{K_U} + \sum_{K_V} (y^R_{K_V} + y^I_{K_V}) = -\frac{D}{2}$. Furthermore, let

$$\phi_{\ell} = \sum_{K_U} u_{\ell K_U} x_{K_U} + \sum_{K_V} (v^R_{\ell K_V} y^R_{K_V} + v^I_{\ell K_V} y^I_{K_V}) + 1.$$  

(4.4)

and the convex domain

$$\Delta = \left\{ \sigma, \tau^{R, I}, \phi_{\ell} \left| \begin{array}{l}
\sigma_{K_U} < 0; \tau^{R, I}_{K_V} < 0; -1 < \tau^I_{K_V} < 0; \\
\sum_{K_U} x_{K_U} + \sum_{K_V} (y^R_{K_V} + y^I_{K_V}) = -\frac{D}{2} \\
\forall \ell, \Re \phi_{\ell} \equiv \sum_{K_U} u_{\ell K_U} \sigma_{K_U} \\
+ \sum_{K_V} (v^R_{\ell K_V} \tau^{R, I}_{K_V} + v^I_{\ell K_V} \tau^{R, I}_{K_V}) + 1 > 0
\end{array} \right. \right\}$$

(4.5)

where $\sigma, \tau^{R, I}$ and $\phi_{\ell}$ stand for $\Re x_{K_U}, \Re y^R_{K_V}$ and $\Re y^I_{K_V}$.

Putting all these together, one is able to prove (see again [3]) the following theorem:

**Theorem 4.1** A Feynman amplitude of a Grosse-Wulkenhaar or LSZ graph is analytic in the strip $0 < \Re D < 2$ where it writes

$$A_{\beta} = K' \int_{\Delta} \prod_{K_U} \frac{x^{x_{K_U}}}{\Gamma(-x_{K_U})} \left(\prod_{K_V} (s^R_{K_V})^{y^R_{K_V}} \Gamma(-y^R_{K_V})\right) \left(\prod_{K_V} (s^I_{K_V})^{y^I_{K_V}} \Gamma(-y^I_{K_V})\right) \left(\prod_{\ell=1}^{L} \Gamma(K(\frac{\phi^R_{\ell}}{2})\Gamma(K(\frac{\phi^I_{\ell}}{2})\right).$$

(4.6)

where $\int_{\Delta}$ is a short notation for integration over the variables $\frac{x_{K_U}}{2\pi i}, \frac{y^R_{K_V}}{2\pi i}$ and $\frac{y^I_{K_V}}{2\pi i}$ in the domain $\Delta$. 


This theorem holds as tempered distribution of the external invariants. It is this the main difference with the commutative case: this integral representation (previously true in the sense of functions of the external invariants) now holds only in the sense of distributions. Indeed, the distributional character of commutative amplitudes reduces to a single overall \(\delta\)-function of momentum conservation. This is no longer true for these non-commutative amplitudes, which must be seen as distributions smeared against test functions of the external variables.

Furthermore, let us note here that the representation given by Theorem 4.1 allows the study of asymptotic behavior under rescaling of arbitrary subsets of external invariants of a Feynman amplitude.

Finally, let us end this paper by a theorem regarding the meromorphy of a Feynman amplitude in \(D\):

**Theorem 4.2.** Any Feynman amplitude is a tempered meromorphic distribution in \(D\), i.e. the amplitude smeared against any fixed Schwarz-class test function of the external invariants yields a meromorphic function in \(D\) in the entire complex plane, with singularities located among a discrete rational set which depends only on the graph and not on the test function.

The fact that all the formulas in this paper present \(D\) as a simple parameter, as well as Theorem 4.2 above, pave the road for dimensional regularization and renormalization of these theories.

**References**

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