ON UNCERTAINTY PRINCIPLE FOR THE MORSE–SARD THEOREM

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Abstract

The classical Morse–Sard theorem claims that for a mapping \( v : \mathbb{R}^n \to \mathbb{R}^m \) of class \( C^k \) the measure of critical values \( v(Z_{v,m}) \) is zero under condition \( k > \max(n - m, 0) \). Here the critical set, or \( m \)-critical set is defined as \( Z_{v,m} = \{ x \in \mathbb{R}^n : \text{rank} \nabla v(x) < m \} \). Further Dubovitskii in 1957 and independently Federer and Dubovitskiı in 1967 found some elegant extensions of this theorem to the case of other (e.g., lower) smoothness assumptions. They also established the sharpness of their results within the \( C^k \) category.

Here we formulate and prove a bridge theorem that includes all the above results as particular cases: namely, if a function \( v : \mathbb{R}^n \to \mathbb{R}^d \) belongs to the Holder class \( C^{k,\alpha}, 0 \leq \alpha \leq 1 \), then for every \( q > m - 1 \) the identity \( \mathcal{H}^q(Z_{v,m} \cap v^{-1}(y)) = 0 \) holds for \( \mathcal{H}^q \)-almost all \( y \in \mathbb{R}^d \), where \( \mu = n - m + 1 - (k + \alpha)(q - m + 1) \).

Intuitively, the sense of this bridge theorem is very close to Heisenberg’s uncertainty principle in theoretical physics: the more precise is the information we receive on measure of the image of the critical set, the less precisely the preimages are described, and vice versa.

The result is new even for the classical \( C^k \)-case (when \( \alpha = 0 \)); similar result is established for the Sobolev classes of mappings \( W^k_p(\mathbb{R}^n, \mathbb{R}^d) \) with minimal integrability assumptions \( p = \max(1, n/k) \), i.e., it guarantees in general only the continuity (not everywhere differentiability) of a mapping. However, using some \( N \)-properties for Sobolev mappings, established in our previous paper, we obtained that the sets of nondifferentiability points of Sobolev mappings are fortunately negligible in the above bridge theorem. We cover also the case of fractional Sobolev spaces.

The proofs of the most results are based on our previous joint papers with J. Bourgain and J. Kristensen (2013, 2015). We also crucially use very deep Y. Yomdin’s entropy estimates of near critical values for polynomials (based on algebraic geometry tools).

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1 Introduction

The Morse–Sard theorem in its classical form states that the image of the set of critical points of a $C^{n-m+1}$ smooth mapping $v: \mathbb{R}^n \to \mathbb{R}^m$ has zero Lebesgue measure in $\mathbb{R}^m$. More precisely, assuming that $n \geq m$, the set of critical points for $v$ is $Z_v = \{x \in \mathbb{R}^n : \text{rank } \nabla v(x) < m\}$ and the conclusion is that

$$\mathcal{L}^m(v(Z_v)) = 0. \quad (1.1)$$

The theorem was proved by Morse [39] in the case $m = 1$ and subsequently by Sard [42] in the general vector–valued case. The celebrated results of Whitney [47] show that the $C^{n-m+1}$ smoothness assumption on the mapping $v$ is sharp. However, the following result gives valuable information also for less smooth mappings.

**Theorem A (Dubovitskiǐ 1957 [14]).** Let $n, m, k \in \mathbb{N}$, and let $v: \mathbb{R}^n \to \mathbb{R}^m$ be a $C^k$–smooth mapping. Put $\nu = n - m - k + 1$. Then

$$\mathcal{H}^\nu(Z_v \cap v^{-1}(y)) = 0 \quad \text{for a.a. } y \in \mathbb{R}^m, \quad (1.2)$$

where $\mathcal{H}^\nu$ denotes the $\nu$–dimensional Hausdorff measure.

Here and in the following we interpret $\mathcal{H}^\beta$ as the counting measure when $\beta \leq 0$. Thus for $k \geq n - m + 1$ we have $\nu \leq 0$, and $\mathcal{H}^\nu$ in (1.2) becomes simply the counting measure, so the Dubovitskiǐ theorem contains the Morse–Sard theorem as particular case$^1$.

A few years later and almost simultaneously, Dubovitskiǐ [15] in 1967 and Federer [18, Theorem 3.4.3] in 1969$^2$ published another important generalization of the Morse–Sard theorem.

**Theorem B (Dubovitskiǐ–Federer).** For $n, k, d \in \mathbb{N}$ let $m \in \{1, \ldots, \min(n, d)\}$ and $v: \mathbb{R}^n \to \mathbb{R}^d$ be a $C^k$–smooth mapping. Put $q_0 = m + \frac{n}{k} = m - 1 + \frac{n-m+1}{k}$. Then

$$\mathcal{H}^{q_0}(v(Z_{v,m})) = 0,$$

where $Z_{v,m}$ denotes the set of $m$–critical points of $v$ defined as

$$Z_{v,m} = \{x \in \mathbb{R}^n : \text{rank } \nabla v(x) < m\}.$$ 

In view of the wide range of applicability of the above results it is a natural and compelling problem to extend the results to the classes of Sobolev mappings.

$^1$It is interesting to note that this first Dubovitskiǐ theorem remained almost unnoticed by West mathematicians for a long time; another proof was given in the recent paper [6] covering also some extensions to the case of Hölder spaces; see also [21] for the Sobolev case.

$^2$Federer announced [17] his result in 1966, this announcement (without any proofs) was sent on 08.02.1966. For the historical details, Dubovitskiǐ sent his paper [15] (with complete proofs) a month earlier, on 10.01.1966.
In the recent paper [22] by Hajląd Sz, Korobkov M.V., and Kristensen J. for \( k \leq n \) and for Sobolev classes \( W^k_p(\mathbb{R}^n, \mathbb{R}^d) \) it was proved a bridge theorem that includes all the above results as particular cases (see below Theorem 1.2). In the present paper we extend this result for the Holder classes \( C^{k,\alpha} \) and for Sobolev spaces \( W^k_p(\mathbb{R}^n, \mathbb{R}^d) \) with arbitrary integer \( k \geq 1 \), and also for fractional Sobolev spaces \( \mathcal{L}^{k+\alpha}_p \) and \( B^{k+\alpha}_{p,s} \) (e.g., for Bessel potential spaces and Besov spaces respectively).

The integrability assumptions here are very minimal and sharp, they are of kind \( p(k + \alpha) \geq n \), i.e., they guarantee in general only the continuity (not everywhere differentiability) of a mapping. However, we prove that the 'bad' set of nondifferentiability points of Sobolev mappings is fortunately negligible in the above bridge theorem because of some Luzin type \( N \)–properties with respect to lower dimensional Hausdorff measures established in our previous papers [9, 19, 28].

Let us note, in the conclusion, that the Morse–Sard theorem for the Sobolev spaces was very fruitful in mathematical fluid mechanics, in particular, it was used in the recent solution of the so-called Leray's problem for the steady Navier–Stokes system (see [29]).

1.1 Bridge F.-D.-theorems for the Holder classes of mappings

We say that a mapping \( v: \mathbb{R}^n \to \mathbb{R}^d \) belongs to the class \( C^{k,\alpha} \) for some positive integer \( k \) and \( 0 < \alpha \leq 1 \) if there exists a constant \( L \geq 0 \) such that

\[
|\nabla^k v(x) - \nabla^k v(y)| \leq L |x - y|^{\alpha}
\]

for all \( x, y \in \mathbb{R}^n \).

To simplify the notation, let us make the following agreement: for \( \alpha = 0 \) we identify \( C^{k,\alpha} \) with usual spaces of \( C^k \)-smooth mappings. The following theorem is one of the main results of the paper.

**Theorem 1.1.** Let \( m \in \{1, \ldots, n\} \), \( k \geq 1 \), \( d \geq m \), \( 0 \leq \alpha \leq 1 \), and \( v \in C^{k,\alpha}(\mathbb{R}^n, \mathbb{R}^d) \). Then for any \( q \in (m - 1, \infty) \) the equality

\[
\mathcal{H}^{\mu_q}(Z_{v,m} \cap v^{-1}(y)) = 0
\]

for \( \mathcal{H}^d \)-a.a. \( y \in \mathbb{R}^d \) holds, where

\[
\mu_q = n - m + 1 - (k + \alpha)(q - m + 1),
\]

and \( Z_{v,m} \) denotes the set of \( m \)-critical points of \( v \): \( Z_{v,m} = \{x \in \mathbb{R}^n : \text{rank } \nabla v(x) \leq m-1\} \).

Let us note, that for the classical \( C^k \)-case, i.e., when \( \alpha = 0 \), the behavior of the function \( \mu_q \) is very natural:

- \( \mu_q = 0 \) for \( q = q_0 = m-1 + \frac{n-m+1}{k} \) (Dubovitskiĭ–Federer Theorem B);
- \( \mu_q < 0 \) for \( q > q_0 \) [ibid.];
- \( \mu_q = \nu \) for \( q = m \) (Dubovitskiĭ Theorem A);
- \( \mu_q = n-m+1 \) for \( q = m-1 \).
The last value cannot be improved in view of the trivial example of a linear mapping $L: \mathbb{R}^n \to \mathbb{R}^d$ of rank $m - 1$ (in this limiting case the Coarea formula allows to estimate the nonzero $\mathcal{H}^\mu$-size of the corresponding preimages, see Theorem 1.4 below).

Thus, Theorem 1.1 contains all the previous theorems (Morse–Sard, A, B and even the Bates theorem for $C^{k,1}$-Lipschitz functions [4]) as particular cases.

Intuitively, the sense of the Bridge Theorem 1.1 is very close to the Heisenberg’s uncertainty principle in theoretical physics: the more precisely information we received on measure of the image of the critical set, the less precisely the preimages are described, and vice versa.

Remark 1.1. Note that for partial case $q = m = d$ (as in the Dubovitskiĭ theorem A) and under additional assumption that

$$|\nabla^k v(x) - \nabla^k v(y)| \leq \omega(|x - y|) \cdot |x - y|^\alpha \quad \text{with} \quad \omega(r) \to 0 \quad \text{as} \quad r \to 0,$$

the assertion of Theorem 1.1 was proved in the paper [6]. Further, under the same asymptotic assumption (1.3) and for the partial case $q = q_0 = m - 1 + \frac{n - m + 1}{k + \alpha}$, $\mu_q = 0$ (as in the Dubovitskiĭ-Federer Theorem B) the assertion of Theorem 1.1 was proved in the paper [48]. Finally, for the minimal rank value $m = 1$ (i.e., when the gradient totally vanishes on the critical set) and $q = q_0 = \frac{n}{k + \alpha}$, $\mu_q = 0$ (as in the Dubovitskiĭ-Federer Theorem B), the assertion of Theorem 1.1 was proved in the paper [30] (without additional assumption (1.3)).

1.2 Bridge F.-D.-theorems for the Sobolev spaces

In the recent paper [22] the following link between above classical Theorems A–B was established for $k \leq n$:

**Theorem 1.2** ([22]). Let $k, m \in \{1, \ldots, n\}$, $d \geq m$ and $v \in W^{k,1}_{p_0,1}(\mathbb{R}^n, \mathbb{R}^d)$ with $p_0 = \frac{n}{k}$. Then for any $q \in (m - 1, \infty)$ the equality

$$\mathcal{H}^{\mu_q}(Z_{v,m} \cap v^{-1}(y)) = 0 \quad \text{for} \quad \mathcal{H}^{q}\text{-a.a.} \quad y \in \mathbb{R}^d$$

holds, where again

$$\mu_q = n - m + 1 - k(q - m + 1),$$

and $Z_{v,m}$ denotes the set of $m$-critical points of $v$: $Z_{v,m} = \{x \in \mathbb{R}^n \setminus A_v : \text{rank} \nabla v(x) \leq m - 1\}$.

Here $W^{k,1}_{p_0,1}(\mathbb{R}^n, \mathbb{R}^d)$ means the space of Sobolev mappings with derivatives of $k$-th order belonging to the Lorentz space $L^{p_0,1}$ (this is the very sharp borderline condition which guarantees the continuity of mappings), and $A_v$ means the set of nondifferentiability points (it is known, that $\mathcal{H}^{p_0}(A_v) = 0$, see, e.g., [28]). We refer to section 2 for relevant definitions and notation for Lorentz spaces.
Note that in the pioneering paper by De Pascale [11] the assertion of the initial Morse–Sard theorem (1.1) (i.e., when \( k = n - m + 1, q = q_0 = m, \mu_q = 0 \)) was obtained for the Sobolev classes \( W_p^k(\mathbb{R}^n, \mathbb{R}^m) \) under additional assumption \( p > n \) (in this case the classical embedding \( W_p^k(\mathbb{R}^n, \mathbb{R}^m) \hookrightarrow C^{k-1} \) holds, so there are no problems with nondifferentiability points).

In the present paper we extend Theorem 1.2 to the case \( k > n \):

**Theorem 1.3.** Let \( m \in \{1, \ldots, n\}, \ k > n, \ d \geq m \) and \( v \in W_1^k(\mathbb{R}^n, \mathbb{R}^d) \). Then for any \( q \in (m - 1, \infty) \) the equality

\[
\mathcal{H}^q(Z_{v,m} \cap v^{-1}(y)) = 0 \quad \text{for } \mathcal{H}^q\text{-a.a. } y \in \mathbb{R}^d
\]

(1.4)

holds, where again

\[
\mu_q = n - m + 1 - k(q - m + 1).
\]

(1.5)

Note that for \( k > n \) the Sobolev \( W_1^k(\mathbb{R}^n) \)-functions are \( C^1 \)-smooth, so we can define the critical set \( Z_{v,m} = \{ x \in \mathbb{R}^n : \text{rank } \nabla v(x) \leq m - 1 \} \) in the usual way.

We emphasize the fact that in stating Theorems 1.1–1.3 we skipped the borderline case \( q = m - 1, \mu_q = n - m + 1 \). Of course, for this case we cannot assert that \( \mathcal{H}^{m-1} \)-almost all preimages in the \( m \)-critical set \( Z_{v,m} \) have zero \( \mathcal{H}^{n-m+1} \)-measure as the above mentioned counterexample with a linear mapping \( L: \mathbb{R}^n \to \mathbb{R}^d \) of rank \( m - 1 \) shows. But for this borderline case we obtain instead the following analog of the classical coarea formula:

**Theorem 1.4** (see [22]). Let \( n, d \in \mathbb{N}, \ m \in \{1, \ldots, \min(n, d)\} \), and \( v \in W_{n,1}^1(\mathbb{R}^n, \mathbb{R}^d) \). Then for any Lebesgue measurable subset \( E \) of \( Z_{v,m+1} = \{ x \in \mathbb{R}^n \setminus A_v : \text{rank } \nabla v(x) \leq m \} \) we have

\[
\int_E J_m v(x) \, dx = \int_{v(E)} \mathcal{H}^{n-m}(E \cap v^{-1}(y)) \, d\mathcal{H}^m(y),
\]

(1.6)

where \( J_m v(x) \) denotes the \( m \)-Jacobian of \( v \) defined as the product of the \( m \) largest singular values of the matrix \( \nabla v(x) \).

Thus, to study the level sets for the borderline case \( q = m - 1 \) in Theorems 1.2–1.3, one must take \( m' = m - 1 \) instead of \( m \) in Theorem 1.4. This Coarea formula plays a very important role below in the proof of central Theorem 1.1.

**Remark 1.2.** Note, that (1.6) is an extension of the classical Coarea formula: the last one is stated usually for \( m = d \) or under similar dimensional assumptions (see, e.g., [23], [35]–[36]). This extension is nontrivial because it could be that the Hausdorff dimension of \( v(E) \) in (1.6) is much larger than \( m \). Nevertheless, fortunately it turns out that the integrand function on the righthand side of (1.6) is nonzero only on \( \mathcal{H}^{m-\sigma} \)-finite subset of \( v(E) \), i.e., the integration on the righthand side of (1.6) makes sense.

Further, in view of the embedding \( W_{p_0,1}^k(\mathbb{R}^n) \hookrightarrow W_{n,1}^1(\mathbb{R}^n) \) for \( k \in \{1, \ldots, n\} \), \( p_0 = \frac{n}{k} \) (see, e.g., [34], §8), the assertion of Theorem 1.4 is in particular valid for the mappings \( v \in W_{p_0,1}^k(\mathbb{R}^n, \mathbb{R}^d), k \leq n \), and of course for \( C^1 \)-functions as well, i.e., under the conditions of Theorems 1.2–1.3.
1.3 Bridge F.-D.-theorems for mappings of fractional Sobolev spaces

Let $k \in \mathbb{N}$, $1 < p < \infty$ and $0 < \alpha < 1$. There exist two main types of fractional Sobolev spaces (which is a Sobolev analog of classical Holder classes $C^{k,\alpha}$), namely, (Bessel) potential spaces $\mathcal{L}^{k+\alpha}_p$ and Besov spaces $B^{k+\alpha}_{p,s}$.

Recall, that a function $v : \mathbb{R}^n \to \mathbb{R}^d$ belongs to the space $\mathcal{L}^\alpha_p(\mathbb{R}^n)$, if it is a convolution of a function $g \in L^p(\mathbb{R}^n)$ with the Bessel kernel $K_\alpha$, where $\hat{K}_\alpha(\xi) = (1 + 4\pi^2 \xi^2)^{-\alpha/2}$. It is well known that $\mathcal{L}^\alpha_p(\mathbb{R}^n) = W^\alpha_p(\mathbb{R}^n)$ if $\alpha \in \mathbb{N}$ and $1 < p < \infty$.

Similarly, $\mathcal{L}^{k+\alpha}_{p,1}(\mathbb{R}^n, \mathbb{R}^d)$ denotes the space of functions which could be represented as a convolution of the Bessel potential $K_\alpha$ with a function $g$ from the Lorentz space $L^p_{p,1}$ (see the definition of these spaces in the section 2).

There are many elegant relations between potential spaces and Besov spaces (see the definition 2.1). For our purposes it is sufficient to indicate that

$$\mathcal{L}^{k+\alpha}_p \subset B^{k+\alpha}_{p,\infty},$$

$$\forall s \in (1, \infty) \quad B^{k+\alpha}_{p,s} \subset B^{k+\alpha}_{p,\infty}. \quad (1.7)$$

So when $(k + \alpha)p > n$, we need to prove the Bridge F.-D.-theorem for the largest space $B^{k+\alpha}_{p,\infty}(\mathbb{R}^n, \mathbb{R}^d)$, then automatically the result will be true for any other $(k + \alpha, p)$-fractional Sobolev space of above kind. But if $(k + \alpha)p = n$, then functions from potential spaces $\mathcal{L}^{k+\alpha}_p(\mathbb{R}^n)$ are discontinuous in general (and the same is true, of course, for the larger space $B^{k+\alpha}_{p,\infty} \supset \mathcal{L}^{k+\alpha}_p$). Thus for this limiting case we need to consider the Bessel–Lorentz potential space $\mathcal{L}^{k+\alpha}_{p,1}(\mathbb{R}^n)$ to have the continuity.

The main result of this section is the following theorem, where for convenience we summarise the previous results for the Sobolev case $\alpha = 0$ (see the items (i)–(ii)) and the new results for the fractional Sobolev case (items (iii)–(iv)).

**Theorem 1.5.** Let $m \in \{1, \ldots, n\}$, $k \geq 1$, $d \geq m$, $0 \leq \alpha < 1$, $p \geq 1$ and let $v : \mathbb{R}^n \to \mathbb{R}^d$ be a mapping for which one of the following cases holds:

(i) $\alpha = 0$, $kp > n$, and $v \in W^k_p(\mathbb{R}^n, \mathbb{R}^d)$;

(ii) $\alpha = 0$, $kp = n$, and $v \in W^k_{p,1}(\mathbb{R}^n, \mathbb{R}^d)$;

(iii) $0 < \alpha < 1$, $p > 1$, $(k + \alpha)p > n$, and $v \in B^{k+\alpha}_{p,\infty}(\mathbb{R}^n, \mathbb{R}^d)$;

(iv) $0 < \alpha < 1$, $p > 1$, $(k + \alpha)p = n$, and $v \in \mathcal{L}^{k+\alpha}_{p,1}(\mathbb{R}^n, \mathbb{R}^d)$. 

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Then the mapping $v$ is continuous and for any $q \in (m-1, \infty)$ the equality
\[ \mathcal{H}^{\mu_q}(Z_{v,m} \cap v^{-1}(y)) = 0 \quad \text{for } \mathcal{H}^q\text{-a.a. } y \in \mathbb{R}^d \]
holds, where again
\[ \mu_q = n - m + 1 - (k + \alpha)(q - m + 1), \]
and $Z_{v,m}$ denotes the set of $m$-critical points of $v$: $Z_{v,m} = \{ x \in \mathbb{R}^n \setminus A_v : \text{rank } \nabla v(x) \leq m - 1 \}$.

Here $A_v$ means the set of nondifferentiability points for $v$. Recall, that by approximation results (see, e.g., [44] and [28]) under conditions of Theorem 1.5 the equalities
\[ \mathcal{H}^\tau(A_v) = 0 \quad \forall \tau > \tau_* := n - (k + \alpha - 1)p \quad \text{in cases (i), (iii)}; \]
\[ \mathcal{H}^{\tau_*}(A_v) = \mathcal{H}^p(A_v) = 0 \quad \tau_* := n - (k + \alpha - 1)p = p \quad \text{in cases (ii), (iv)} \]
are valid (in particular, $A_v = \emptyset$ if $(k + \alpha - 1)p > n$). However, it was proved in [19] that the impact of the "bad" set $A_v$ is negligible in the Bridge D.-F. Theorem 1.5, i.e., the following statement holds:

**Theorem 1.6** ([19]). Let the conditions of Theorem 1.5 be fulfilled for a function $v : \mathbb{R}^n \to \mathbb{R}^d$. Then
\[ \mathcal{H}^{\mu_q}(A_v \cap v^{-1}(y)) = 0 \quad \text{for } \mathcal{H}^q\text{-a.a. } y \in \mathbb{R}^d \]
for any $q \geq m - 1$.

**Remark 1.3.** Note, that since $\mu_q \leq 0$ for $q \geq q_0 = m - 1 + \frac{n-m+1}{k+\alpha}$, the assertions of Theorems 1.5–1.6 are equivalent to the equality $0 = \mathcal{H}^q[v(A_v \cup Z_{v,m})]$ for $q \geq q_0$, so it is sufficient to check the assertions of Theorems 1.5–1.6 for $q \in (m - 1, q_0]$ only.

Finally, let us comment briefly that the merge ideas for the proofs are from our previous papers [9], [27, 28] and [22]. In particular, the joint papers [8, 9] by one of the authors with J. Bourgain and J. Kristensen contain many of the key ideas that allow us to consider nondifferentiable Sobolev mappings. As in [9] (and subsequently in [27]) we also crucially use Y. Yomdin’s (see [48]) entropy estimates of near critical values for polynomials (recalled in Theorem 2.2 below). These Yomdin’s results seems to be very deep and fruitful in the topic, see, e.g., the very recent paper [3] where the Morse-Sard theorems were proved for min-type functions and for Lipschitz selections.

In addition to the above mentioned papers there is a growing number of papers on the topic, including [1, 2, 4, 6, 10, 20, 21, 38, 40, 41, 45, 46].

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2 Preliminaries

By an \( n \)-dimensional interval we mean a closed cube in \( \mathbb{R}^n \) with sides parallel to the coordinate axes. If \( Q \) is an \( n \)-dimensional cubic interval then we write \( \ell(Q) \) for its sidelength.

For a subset \( S \) of \( \mathbb{R}^n \) we write \( \mathcal{L}^n(S) \) for its outer Lebesgue measure (sometimes we use the symbol \( \text{meas} \, S \) for the same purpose). The \( m \)-dimensional Hausdorff measure is denoted by \( \mathcal{H}^m \) and the \( m \)-dimensional Hausdorff content by \( \mathcal{H}^m_\infty \). Recall that for any subset \( S \) of \( \mathbb{R}^n \) we have by definition

\[
\mathcal{H}^m(S) = \lim_{t \to 0} \mathcal{H}^m_t(S) = \sup_{t > 0} \mathcal{H}^m_t(S),
\]

where for each \( 0 < t \leq \infty \),

\[
\mathcal{H}^m_t(S) = \inf \left\{ \sum_{i=1}^{\infty} (\text{diam } S_i)^m : \text{diam } S_i \leq t, \ S \subset \bigcup_{i=1}^{\infty} S_i \right\}.
\]

It is well known that \( \mathcal{H}^n(S) = \mathcal{H}^n_\infty(S) \sim \mathcal{L}^n(S) \) for sets \( S \subset \mathbb{R}^n \).

To simplify the notation, we write \( \|f\|_{L^p} \) instead of \( \|f\|_{L^p(\mathbb{R}^n)} \), etc.

The Sobolev space \( W^k_p(\mathbb{R}^n, \mathbb{R}^d) \) is as usual defined as consisting of those \( \mathbb{R}^d \)-valued functions \( f \in L^p(\mathbb{R}^n) \) whose distributional partial derivatives of orders \( l \leq k \) belong to \( L^p(\mathbb{R}^n) \) (for detailed definitions and differentiability properties of such functions see, e.g., \([16],[37],[49],[12]\)). Denote by \( \nabla^k f \) the vector-valued function consisting of all \( k \)-th order partial derivatives of \( f \) arranged in some fixed order. However, for the case of first order derivatives \( k = 1 \) we shall often think of \( \nabla f(x) \) as the Jacobi matrix of \( f \) at \( x \), thus the \( d \times n \) matrix whose \( r \)-th row is the vector of partial derivatives of the \( r \)-th coordinate function.

We use the norm

\[
\|f\|_{W^k_p} = \|f\|_{L^p} + \|\nabla f\|_{L^p} + \cdots + \|\nabla^k f\|_{L^p},
\]

and unless otherwise specified all norms on the spaces \( \mathbb{R}^s \) \((s \in \mathbb{N})\) will be the usual euclidean norms.

Working with locally integrable functions, we always assume that the precise representatives are chosen. If \( w \in L_{1,\text{loc}}(\Omega) \), then the precise representative \( w^* \) is defined for all \( x \in \Omega \) by

\[
w^*(x) = \begin{cases} 
\lim_{r \to 0} \int_{B(x,r)} w(z) \, dz, & \text{if the limit exists and is finite,} \\
0 & \text{otherwise,}
\end{cases}
\]

where the dashed integral as usual denotes the integral mean,

\[
\int_{B(x,r)} w(z) \, dz = \frac{1}{\mathcal{L}^n(B(x,r))} \int_{B(x,r)} w(z) \, dz,
\]

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and $B(x, r) = \{ y : |y - x| < r \}$ is the open ball of radius $r$ centered at $x$. Henceforth we omit special notation for the precise representative writing simply $w^* = w$.

If $k < n$, then it is well-known that functions from Sobolev spaces $W^k_p(\mathbb{R}^n)$ are continuous for $p > \frac{n}{k}$ and could be discontinuous for $p \leq p_0 = \frac{n}{k}$ (see, e.g., [37, 49]). The Sobolev–Lorentz space $W^k_{p_0, 1}(\mathbb{R}^n) \subset W^k_{p_0}(\mathbb{R}^n)$ is a refinement of the corresponding Sobolev space. Among other things functions that are locally in $W^k_{p_0, 1}$ on $\mathbb{R}^n$ are in particular continuous (see, e.g., [27]).

Here we only mentioned the Lorentz space $L^{p, 1}$, and in this case one may rewrite the norm as (see for instance [34, Proposition 3.6])

$$\|f\|_{L^{p, 1}} = \int_0^{+\infty} \left[ \mathcal{L}^n(\{ x \in \mathbb{R}^n : |f(x)| > t \}) \right]^{\frac{1}{p}} \, dt.$$ 

Of course, we have the inequality

$$\|f\|_{L^p} \leq \|f\|_{L^{p, 1}}. \tag{2.1}$$

Denote by $W^k_{p, 1}(\mathbb{R}^n)$ the space of all functions $v \in W^k_p(\mathbb{R}^n)$ such that in addition the Lorentz norm $\|\nabla^k v\|_{L^{p, 1}}$ is finite.

### 2.1 On the largest Besov spaces $B^{k+\alpha}_{p, \infty}(\mathbb{R}^n, \mathbb{R}^d)$

Recall the following definition which was used in the Bridge–Morse–Sard Theorem 1.5.

**Definition 2.1.** Let $k \in \mathbb{N}$, $1 < p < \infty$, and $0 < \alpha < 1$. We will say that a mapping $v : \mathbb{R}^n \to \mathbb{R}^d$ belongs to the class $B^{k+\alpha}_{p, \infty}(\mathbb{R}^n, \mathbb{R}^d)$, if $v \in W^k_p(\mathbb{R}^n, \mathbb{R}^d)$ and there exists a constant $C$ such that for any $t > 0$ the estimate

$$\|\Omega^k_v(\cdot, t)\|_{L^p(\mathbb{R}^n)} \leq C \, t^\alpha$$

holds, where

$$\Omega^k_v(x, t) = \int_{Q(x, t)} |\nabla^k v(x) - \nabla^k v(Q)| \, dx,$$

and $\nabla^k v(Q)$ denotes the mean value of $\nabla^k v$ over the $n$-dimensional cube $Q = Q(x, t)$ centered at $x$ with $t = \ell(Q)$.

This is the largest space among other $(k + \alpha, p)$-fractional Sobolev space $B^{k+\alpha}_p$ and $L^{k+\alpha}_p$ (see (1.7)).
2.2 On potential spaces $L^\alpha_p$

In the paper we deal with (*Bessel*)-potential space $L^\alpha_p$, if it is a convolution of the Bessel kernel $K_\alpha$ with a function $g \in L_p(\mathbb{R}^n)$:

$$v = G_\alpha(g) := K_\alpha * g,$$

where $\hat{K}_\alpha(\xi) = (1 + 4\pi^2\xi^2)^{-\alpha/2}$. In particular,

$$\|v\|_{L^\alpha_p} := \|g\|_{L^p}.$$ 

It is well known that $L^\alpha_p(\mathbb{R}^n) = W^\alpha_p(\mathbb{R}^n)$ if $\alpha \in \mathbb{N}$ and $1 < p < \infty$.

2.3 On Lorentz potential spaces $L^\alpha_{p,1}$

To cover some other limiting cases, denote by $L^\alpha_{p,1}(\mathbb{R}^n, \mathbb{R}^d)$ the space of functions which could be represented as a convolution of the Bessel potential $K_\alpha$ with a function $g$ from the Lorentz space $L^p_{p,1}$; respectively,

$$\|v\|_{L^\alpha_{p,1}} := \|g\|_{L^p_{p,1}}.$$ 

Because of inequality (2.1), we have an evident inclusion

$$L^\alpha_{p,1}(\mathbb{R}^n) \subset L^\alpha_p(\mathbb{R}^n).$$

**Theorem 2.1** (see, e.g., Theorem 2.2 in [19], cf. with Lemma 3 on page 136 in [43]). Let $\alpha \geq 1$ and $1 < p < \infty$. Then $f \in L^\alpha_{p,1}(\mathbb{R}^n)$ iff $f \in L^{\alpha-1}_{p,1}(\mathbb{R}^n)$ and $\frac{\partial f}{\partial x_j} \in L^{\alpha-1}_{p,1}(\mathbb{R}^n)$ for every $j = 1, \ldots, n$.

(Here for convenience we use the agreement that $L^\alpha_p(\mathbb{R}^n) = L_p(\mathbb{R}^n)$ when $\alpha = 0$.)

**Corollary 2.1.** Let $k \in \mathbb{N}$ and $1 < p < \infty$. Then $L^\alpha_{p,1}(\mathbb{R}^n)$ is the space of functions such that all its distributional partial derivatives of order $\leq k$ belong to $L^\alpha_{p,1}(\mathbb{R}^n)$.

Note, that the space $W^k_{p,1}(\mathbb{R}^n)$ admits also a simpler (but equivalent) description: it consists of functions $f$ from the usual Sobolev space $W^k_p(\mathbb{R}^n)$ satisfying the additional condition $\nabla^k f \in L^k_{p,1}(\mathbb{R}^n)$ (i.e., this condition is on the highest derivatives only), see, e.g., [34].

We need some standard estimates (an analog of Sobolev type inequality for Lorentz spaces).
Lemma 2.1. Let $0 < \alpha < 1$, $\alpha p < n$, and $p > 1$. Suppose that $f \in L^\alpha_p(\mathbb{R}^n)$, i.e., $f = G_\alpha(g)$ for some $g \in L_{p,1}(\mathbb{R}^n)$. Then for every $n$-dimensional cubic interval $Q \subset \mathbb{R}^n$ with $\ell(Q) \leq 1$ the estimate
\[
\| \bar{f}_Q \|_{L^p_{\ast,q}(Q)} \leq C \| Mg \|_{L^p_{p,1}(Q)}
\]
holds, where $C$ depends on $n, p, \alpha$ only, $p_\ast = \frac{np}{n-\alpha p}$ and we denote
\[
\bar{f}_Q(x) = f(x) - \int_Q f(y) \, dy.
\]

Here by definition $\| g \|_{L^p_{p,1}(E)} := \| 1_E \cdot g \|_{L^p_{p,1}}$, where $1_E$ is the indicator function of $E$.
For reader’s convenience, we prove Lemma 2.1 in the Appendix 4.

Remark 2.1. Recall, that by properties of Lorentz spaces, the standard estimate
\[
\| Mg \|_{L^{p,q}} \leq C \| g \|_{L^{p,q}}
\]
holds for $1 < p < \infty$ (see, e.g., [34, Theorem 4.4]).

2.4 Approximation of Sobolev functions by polynomials

For a mapping $u \in L^1(Q, \mathbb{R}^d)$, $Q \subset \mathbb{R}^n$, $m \in \mathbb{N}$, define the polynomial $P_{Q,m}[u]$ of degree at most $m$ by the following rule:
\[
\int_Q y^\gamma (u(y) - P_{Q,m}[u](y)) \, dy = 0
\]
for any multi-index $\gamma = (\gamma_1, \ldots, \gamma_n)$ of length $|\gamma| = \gamma_1 + \cdots + \gamma_n \leq m$. Denote $P_{Q}[u] = P_{Q,k-1}[u]$.

The following well–known bounds will be used on several occasions.

Lemma 2.2 (see, e.g., [27]). Suppose $v \in W^k_1(\mathbb{R}^n, \mathbb{R}^d)$ with $k > n$. Then the gradient $\nabla v$ is a continuous mapping and for any $n$-dimensional cubic interval $Q \subset \mathbb{R}^n$ the estimate
\[
\| \nabla (v - P_Q[v]) \|_{L^\infty(Q)} \leq C \ell(Q)^{k-1-n} \| \nabla^k v \|_{L^1(Q)}
\]
holds, where $C$ is a constant depending on $n, d, k$ only. Moreover, the mapping $v_Q(y) = v(y) - P_Q[v](y)$, $y \in Q$, can be extended from $Q$ to the entire $\mathbb{R}^n$ such that the extension (denoted again) $v_Q \in W^k_1(\mathbb{R}^n, \mathbb{R}^d)$ and
\[
\| \nabla^k v_Q \|_{L^1(\mathbb{R}^n)} \leq C_0 \| \nabla^k v \|_{L^1(Q)},
\]
where $C_0$ also depends on $n, d, k$ only.
2.5 On Yomdin’s entropy estimates for the nearcritical values of polynomials

For a subset $A$ of $\mathbb{R}^m$ and $\varepsilon > 0$ the $\varepsilon$–entropy of $A$, denoted by $\text{Ent}(\varepsilon, A)$, is the minimal number of closed balls of radius $\varepsilon$ covering $A$. Further, for a linear map $L: \mathbb{R}^n \to \mathbb{R}^d$ we denote by $\lambda_j(L), j = 1, \ldots, d$, its singular values arranged in decreasing order: $\lambda_1(L) \geq \lambda_2(L) \geq \cdots \geq \lambda_d(L)$. Geometrically the singular values are the lengths of the semiaxes of the ellipsoid $L(\partial B(0, 1))$. We recall that the singular values of $L$ coincide with the eigenvalues repeated according to multiplicity of the symmetric nonnegative linear map $\sqrt{LL^*}: \mathbb{R}^d \to \mathbb{R}^d$. Also for a mapping $f: \mathbb{R}^n \to \mathbb{R}^d$ that is approximately differentiable at $x \in \mathbb{R}^n$ put $\lambda_j(f, x) = \lambda_j(d_x f)$, where by $d_x f$ we denote the approximate differential of $f$ at $x$. The next result is the basic ingredient of our proof.

**Theorem 2.2 ([48]).** For any polynomial $P: \mathbb{R}^n \to \mathbb{R}^d$ of degree at most $k$, for each $n$-dimensional cube $Q \subset \mathbb{R}^n$ of size $\ell(Q) = r > 0$, and for any number $\varepsilon > 0$ we have that

$$\text{Ent}(\varepsilon r, \{P(x) : x \in Q, \lambda_1 \leq 1 + \varepsilon, \ldots, \lambda_m - 1 \leq 1 + \varepsilon, \lambda_m \leq \varepsilon, \ldots, \lambda_d \leq \varepsilon\}) \leq C_Y (1 + \varepsilon^{1-m}),$$

where the constant $C_Y$ depends on $n, d, k, m$ only and for brevity we wrote $\lambda_j = \lambda_j(P, x)$.

2.6 On Fubini type theorems for graphs of continuous functions

Recall that by usual Fubini theorem, if a set $E \subset \mathbb{R}^2$ has a zero plane measure, then for $H^1$-almost all straight lines $L$ parallel to coordinate axes we have $H^1(L \cap E) = 0$. The next result could be considered as functional Fubini type theorem.

**Theorem 2.3 (see Theorem 5.3 in [22]).** Let $\mu \geq 0$, $q > 0$, and $v : \mathbb{R}^n \to \mathbb{R}^d$ be a continuous function. For a set $E \subset \mathbb{R}^n$ define the set function

$$\Phi(E) = \inf_{E \subset \bigcup_j D_j} \sum_j (\text{diam } D_j)^\mu [\text{diam } v(D_j)]^q,$$

where the infimum is taken over all countable families of compact sets $\{D_j\}_{j \in \mathbb{N}}$ such that $E \subset \bigcup_j D_j$. Then $\Phi(\cdot)$ is a countably subadditive and the implication

$$\Phi(E) = 0 \Rightarrow \left[ H^\mu(E \cap v^{-1}(y)) = 0 \text{ for } H^q\text{-almost all } y \in \mathbb{R}^d \right]$$

holds.
2.7 On local properties of considered potential spaces

Let \( \mathcal{B} \) be some space of functions defined on \( \mathbb{R}^n \). For a set \( \Omega \subset \mathbb{R}^n \) define the space \( \mathcal{B}_{\text{loc}}(\Omega) \) in the following standard way:

\[
\mathcal{B}_{\text{loc}}(\Omega) := \{ f : \Omega \to \mathbb{R} : \text{for any compact set } E \subset \Omega \exists g \in \mathcal{B} \text{ such that } f(x) = g(x) \forall x \in E \}.
\]

Put for simplicity \( \mathcal{B}_{\text{loc}} = \mathcal{B}_{\text{loc}}(\mathbb{R}^n) \).

It is easy to see that for \( \alpha > 0 \) and \( q > p > 1 \) the following inclusions hold:

\[
\mathcal{L}^{\alpha}_{q,\text{loc}} \subset \mathcal{L}^{\alpha}_{p,1,\text{loc}} \subset \mathcal{L}^{\alpha}_{p,\text{loc}} \subset B_{p,\infty,\text{loc}}^{\alpha}.
\]

Since the Morse–Sard type theorems have a local nature, if we prove some such of these theorems for \( \mathcal{L}^{\alpha}_{p} \), then the same result will be valid for the spaces \( \mathcal{L}^{\alpha}_{p,1} \) and \( \mathcal{L}^{\alpha}_{q} \) for all \( q > p \). Similarly, if we prove some Morse–Sard type theorems for \( \mathcal{L}^{\alpha}_{p,1} \), then the same result will be valid for the spaces \( \mathcal{L}^{\alpha}_{q} \) with \( q > p \), etc.

2.8 Approximation by Holder–smooth functions

We need also the following approximation result.

**Theorem 2.4** (see, e.g., Chapter 3 in [49] or [6]). Let \( p > 1, k \in \mathbb{N}, \alpha \in (0, 1) \). Then for any \( f \in B_{p,\infty,\text{loc}}^{k+\alpha}(\mathbb{R}^n) \) and for each \( \varepsilon > 0 \) there exist an open set \( U \subset \mathbb{R}^n \) and a function \( g \in C^{k,\alpha}(\mathbb{R}^n) \) such that

(i) \( \mathcal{L}^n(U) < \varepsilon \);

(ii) each point \( x \in \mathbb{R}^n \setminus U \) is a Lebesgue point for \( f \);

(iii) \( f \equiv g \) on \( \mathbb{R}^n \setminus U \).

Note, that in the cited references the approximation property is discussed for the case of Sobolev spaces \( W^{k}_p \), but the proof for the \( B_{p,\infty}^{k+\alpha} \) space easily follows from the just mentioned Sobolev case and some standard arguments on real analysis concerning approximation limits and Whitney-type extension theorems for Holder classes (see, e.g., Theorem 4 in [43, §2.3, Chapter 6]).

3 Proofs of the main results

3.1 Bridge Federer–Dubovitskiǐ theorem for Sobolev mappings

The purpose here is to prove the assertion of the *bridge Dubovitskiǐ–Federer Theorem 1.3*. Fix integers \( m \in \{1, \ldots, n\}, d \geq m, k > n \), and a mapping \( v \in W^{k}_1(\mathbb{R}^n, \mathbb{R}^d) \). Then, by Lemma 2.2 the function \( v \) is \( C^1 \)-smooth.
Denote $Z_{v,m} = \{ x \in \mathbb{R}^n : \text{rank} \nabla v(x) < m \}$. Fix a number $q > m - 1$. Denote in this subsection

$$\mu = \mu_q = n - m + 1 - k(q - m + 1).$$

**CASE I.** Suppose now that

$$q \in (m - 1, q_o],$$

where $q_o = m - 1 + \frac{n - m + 1}{k}$. Then by direct calculation we have $\mu \geq 0$.

The required assertion of the *bridge Dubovitskiĭ–Federer Theorem 1.3* is equivalent (by virtue of Theorem 2.3) to the identity

$$\Phi(Z_{v,m}) = 0,$$

where by definition

$$\Phi(E) := \inf_{E \subset \bigcup_j D_j} \sum_j (\text{diam } D_j)^\mu \left[ \text{diam } v(D_j) \right]^q.$$

As indicated the infimum is taken over all countable families of compact sets $\{D_j\}_{j \in \mathbb{N}}$ such that $E \subset \bigcup_j D_j$.

Before embarking on the detailed proof we make some preliminary observations that allow us to make a few simplifying assumptions. We could assume without loss of generality that

$$|\nabla v(x)| \leq 1 \quad \forall x \in \mathbb{R}^n.$$

Denote $Z_v = Z_{v,m}$. The following lemma contains the main step in the proof of Theorems 1.3.

**Lemma 3.1.** Let $q \in (m - 1, q_o]$. Then for any sufficiently small $n$-dimensional dyadic interval $I \subset \mathbb{R}^n$ the estimate

$$\Phi(Z_v \cap I) \leq C \ell(I)^n (m - q) \cdot \| \nabla^k v \|^q_{L^1(I)}$$

holds, where the constant $C$ depends on $n, m, k, d$ only.

Note, that by our assumptions $k > n$, therefore

$$q \leq q_o = m - 1 + \frac{n - m + 1}{k} < m.$$

**Proof.** By virtue of (2.3) it suffices to prove that

$$\Phi(Z_v \cap I) \leq C \ell(I)^n (m - q) \cdot \| \nabla^k v \|^q_{L^1(\mathbb{R}^n)}$$

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for the mapping $v_I$ defined in Lemma 2.2, where $C = C(n, m, k, d)$ is a constant.

Fix an $n$-dimensional dyadic interval $I \subset \mathbb{R}^n$ and recall that $v_I(x) = v(x) - P_I(x)$ for all $x \in I$. Denote $Z'_v = Z_v \cap I$ and

$$\sigma = \|\nabla^k v_I\|_{L_1}, \quad r = \ell(I), \quad \varepsilon = \sup_{x \in I} |\nabla v_I(x)|.$$

Then by Lemma 2.2 we have

$$\varepsilon \leq c \sigma r^{k-n-1}. \quad (3.2)$$

Of course, for sufficiently small cubes $I$ the trivial inequality

$$\varepsilon < 1 \quad (3.3)$$

holds.

Since $\nabla P_I(x) = \nabla v(x) - \nabla v_I(x)$, $|\nabla v_I(x)| \leq \varepsilon$, $|\nabla v(x)| \leq 1$, and $\lambda_m(v, x) = 0$ for $x \in Z'_v$, we have

$$Z'_v \subset \{x \in I : \lambda_1(P_I, x) \leq 1 + \varepsilon, \ldots, \lambda_{m-1}(P_I, x) \leq 1 + \varepsilon, \lambda_m(P_I, x) \leq \varepsilon\}.$$

Applying Theorem 2.2 to polynomial $P_I$ with $Q = I$ and using the Lipschitz condition $\text{diam} v_I(I) \leq \sqrt{n} \varepsilon r$, we find a finite family of balls $T_j \subset \mathbb{R}^d$, $j = 1, \ldots, N$ with $N \leq C_Y(1 + \varepsilon^{1-m})$, each of radius $(1 + \sqrt{n})\varepsilon r$, such that

$$\bigcup_{j=1}^N T_j \supset v(Z'_v).$$

Therefore, we have

$$\Phi(Z'_v) \leq C N \varepsilon^{q \mu} \leq C_1(1 + \varepsilon^{1-m}) \varepsilon^{q \mu} \leq C \varepsilon^{1-m+q \mu}. \quad (3.3)$$

It implies, by (3.2), that

$$\Phi(Z'_v) \leq C \sigma^{1-m+q \mu} r^{(k-1-n)(1-m+q)+q+\mu}. \quad (3.4)$$

By direct calculation, since by definition $\mu = \mu_q = n - m + 1 - k(q - m + 1)$, we have

$$(k - 1 - n)(1 - m + q) + q + \mu = n(m - q),$$

thus, the estimate (3.4) turns out to be

$$\Phi(Z'_v) \leq C \sigma^{1-m+q \mu} r^{n(m-q)},$$

where the constant $C$ depends on $n, m, k, d$ only. The Lemma is proved. \qed

---

3Here we use the following elementary fact: for any linear maps $L_1 : \mathbb{R}^n \to \mathbb{R}^d$ and $L_2 : \mathbb{R}^n \to \mathbb{R}^d$ the estimates $\lambda_l(L_1 + L_2) \leq \lambda_l(L_1) + \|L_2\|$ hold for all $l = 1, \ldots, m$, see, e.g., [48, Proposition 2.5 (ii)].
Corollary 3.1. Let \( q \in (m - 1, q_0] \). Then for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for any subset \( E \) of \( \mathbb{R}^n \) we have \( \Phi(Z_\varepsilon \cap E) \leq \varepsilon \) provided \( \mathcal{L}^n(E) \leq \delta \). In particular, \( \Phi(Z_\varepsilon \cap E) = 0 \) whenever \( \mathcal{L}^n(E) = 0 \).

Proof. Let \( \mathcal{L}^n(E) < \delta \). Then we can find a family of nonoverlapping \( n \)-dimensional dyadic intervals \( I_j \) such that \( E \subset \bigcup_j I_j \) and \( \sum_j \ell^n(I_j) < C\delta \). Of course, for sufficiently small \( \delta \) the estimates

\[
\|\nabla^k v\|_{L^1(I_j)} < 1, \quad \ell(I_j) \leq \delta^{\frac{1}{n}}
\]

are fulfilled for every \( j \). Denote

\[
r_j = \ell(I_j), \quad \sigma_j = \|\nabla^k v\|_{L^1(I_j)}, \quad \sigma = \|\nabla^k v\|_{L^1}.
\]

In view of Lemma 3.1 we have

\[
\Phi(E) \leq C \sum_j r_j^{n(m-q)} \sigma_j^{q-m+1}.
\]

Since by our assumptions

\[
0 < m - q_\circ \leq m - q < 1,
\]

we have

\[
\sum_j r_j^{n(m-q)} \sigma_j^{q-m+1} \leq C \left( \sum_j r_j^n \right)^{m-q} \left( \sum_j \sigma_j \right)^{q-m+1} \leq C' \sigma^{q-m+1} \cdot \delta^{m-q}.
\]

The lemma is proved. \( \square \)

By the classical approximation results (see, e.g., Chapter 3 in [49] or [6]), our mapping \( v \) coincides with a mapping \( g \in C^k(\mathbb{R}^n, \mathbb{R}^d) \) off an exceptional set of small \( n \)-dimensional Lebesgue measure. This fact, together with Corollary 3.1 and Dubovitskiĭ Theorem A, finishes the proof of the classical Dubovitskiĭ Theorem A for the Sobolev case when \( d = m \). But since Theorem 1.3 was not proved for \( C^k \)-smooth mappings, we have to do this step now.

Lemma 3.2. Let \( q \in (m - 1, q_0] \) and \( g \in C^k(\mathbb{R}^n, \mathbb{R}^d) \), \( k > n \). Then

\[
\Phi_g(Z_{g,m}) = 0,
\]

where \( \Phi_g \) is calculated by the same formula (3.1) with \( g \) instead of \( v \) and \( Z_{g,m} = \{ x \in \mathbb{R}^n : \text{rank} \nabla g(x) < m \} \).
Proof. We can assume without loss of generality that $g$ has compact support and that $|\nabla g(x)| \leq 1$ for all $x \in \mathbb{R}^n$. We then clearly have that $g \in W^k_1(\mathbb{R}^n, \mathbb{R}^d)$, hence we can in particular apply the above results to $g$. The following assertion plays the key role: 

\[(\ast) \quad \text{For any } n\text{-dimensional dyadic interval } I \subset \mathbb{R}^n \text{ the estimate}
\]

\[\Phi(Z_{g,m} \cap I) \leq C \ell(I)^{n(m-q)} \|\nabla^k \tilde{g}_I\|_{L_1(I)}^{q-m+1}
\]

holds, where the constant $C$ depends on $n, m, k, d$ only, and we denoted

\[\nabla^k \tilde{g}_I(x) = \nabla^k g(x) - \int_I \nabla^k g(y) \, dy.
\]

The proof of $(\ast)$ is almost the same as that of Lemma 3.1, with evident modifications (we need to take the approximation polynomial $P_I(x)$ of degree $k$ instead of $k - 1$, etc.).

By elementary facts of the Lebesgue integration theory, for an arbitrary family of nonoverlapping $n$-dimensional dyadic intervals $I_j$ one has

\[
\sum_j \|\nabla^k \tilde{g}_{I_j}\|_{L_1(I_j)} \to 0 \quad \text{as} \quad \sup_j \ell(I_j) \to 0 \quad (3.7)
\]

The proof of this estimate is really elementary since now $\nabla^k g$ is a continuous and compactly supported function, and, consequently, is uniformly continuous and bounded.

From $(\ast)$ and $(3.7)$, repeating the arguments of Corollary 3.1, using the assumptions on $g$ and taking

\[\sigma_j = \|\nabla^k \tilde{g}_{I_j}\|_{L_1(I_j)}, \quad \sigma = \sum_j \sigma_j \]

in definitions $(3.5)$, we obtain that $\Phi_g(Z_{g,m}) < \varepsilon$ for any $\varepsilon > 0$, hence the sought conclusion $(3.6)$ follows.

By the above–mentioned approximation results, the investigated mapping $v$ equals a mapping $g \in C^k(\mathbb{R}^n, \mathbb{R}^d)$ off an exceptional set of small $n$–dimensional Lebesgue measure. This fact together with Lemma 3.2 readily implies

Corollary 3.2 (cp. with [11]). Let $q \in (m - 1, q_\circ]$. Then there exists a set $\tilde{Z}_v$ of $n$-dimensional Lebesgue measure zero such that $\Phi(Z_v \setminus \tilde{Z}_v) = 0$. In particular, $\Phi(Z_v) = \Phi(\tilde{Z}_v)$.

From Corollaries 3.1 and 3.2 we conclude that $\Phi(Z_v) = 0$, and this finishes the proof of Theorem 1.3 for the case $q \in (m - 1, q_\circ]$. In particular, since $\mu_q = 0$ for $q = q_\circ$, we have proved that

\[\mathcal{H}^{q_\circ}(v(Z_{v,m})) = 0.
\]

Case II. Suppose now that

\[q > q_\circ, \quad (3.8)
\]
where again \( q_\circ = m - 1 + \frac{n-m+1}{k} \). Then by direct calculation \( \mu_q = n-m+1-(q-m+1)k < 0 \). Therefore, the assertion of Theorem 1.3 for this case is equivalent to the equality
\[
\mathcal{H}^q(v(Z_{v,m})) = 0.
\] (3.9)

But we proved on the previous step that \( \mathcal{H}^{q_\circ}(v(Z_{v,m})) = 0 \). Evidently, it implies the required assertion (3.9) under above assumption \( q > q_\circ \). This finishes the proof of Theorem 1.3 for the general case.

**Remark 3.1.** As we could see from the above proofs, the assertion of Theorem 1.3 is valid also under assumption \( v \in BV_k(\mathbb{R}^n, \mathbb{R}^d) \) (instead of \( W_1^k \)) with the same \( k \). Here \( BV_k \) means the space of functions \( v \in W_1^{k-1} \) such that its \( k \)-th (distributional) derivatives are Radon measures.

### 3.2 Bridge F.-D. Theorem for Holder classes of mappings

This subsection is devoted to the proof of Theorem 1.1. First of all, let us introduce the main idea of our approach which is extremely simple. In a sense, it is 'lazy way': on the first stage, we used the estimates of critical values through the Sobolev norms of highest \( k \)-derivatives, which were obtained before. On the second stage, we improve this estimates "by \( \alpha \)" using Holder continuity condition of these \( k \)-derivatives. Finally, the Coarea formula is very helpful to analyse the structure of level sets.

Fix \( m \in \{1, \ldots, n\} \), \( k \geq 1 \), \( d \geq m \), \( 0 < \alpha < 1 \), and \( v \in C^{k,\alpha}(\mathbb{R}^n, \mathbb{R}^d) \). Take also parameter \( q > m - 1 \). Of course, as in the previous subsection, it is sufficient to consider the case
\[
q \in (m - 1, q_\circ],
\]
where \( q_\circ = m - 1 + \frac{n-m+1}{k+\alpha} \). By definition of the space \( C^{k,\alpha} \) and since the result has the local nature, we may assume without loss of generality that
\[
|\nabla^k v(x) - \nabla^k v(y)| \leq |x - y|^\alpha \quad \text{for all } x, y \in \mathbb{R}^n.
\]
(3.10)
\[
|\nabla v(x)| \leq 1 \quad \text{for all } x \in \mathbb{R}^n,
\]
(3.11)
\[
v(x) \equiv 0 \quad \text{if } |x| > 1.
\]
(3.12)

Denote again \( Z_v = Z_{v,m} = \{ x \in \mathbb{R}^n : \text{rank} \nabla v(x) < m \} \). Now the parameter \( \mu \) is different from the previous subsection:
\[
\mu = n - m + 1 - (k + \alpha)(q - m + 1).
\]
As before, the assertion of the Bridge Dubovitskii–Federer Theorem 1.1 is equivalent (by virtue of Theorem 2.3) to
\[
\Phi(Z_v) = 0,
\]
where we denoted

\[ \Phi(E) = \inf_{E \subset \bigcup_j D_j} \sum_j \left( \text{diam } D_j \right)^\mu \left[ \text{diam } v(D_j) \right]^q. \]

As indicated the infimum is taken over all countable families of compact sets \( \{D_j\}_{j \in \mathbb{N}} \) such that \( E \subset \bigcup_j D_j \).

The proof of Theorem 1.1 consists of several steps.

**Step I.** The following assertion is the main tool.

\((**)\) Under above assumptions on \( v \), for an arbitrary \( n \)-dimensional cube \( I \) of size \( r = \ell(I) \leq 1 \) the estimate

\[ \Phi(Z_v \cap I) \leq C r^n \]  \hspace{1cm} (3.13)

holds, where the constant \( C \) depends on \( n, m, k, d \) only.

The proof of (**) also splits into the several cases:

*Case I.* Suppose now that \( k > n \).

Then the following intermediate estimates hold.

**Lemma 3.3.** Under above assumptions on \( v \) for any \( n \)-dimensional dyadic interval \( I \subset \mathbb{R}^n \) the estimate

\[ \Phi(Z_v \cap I) \leq C \left\| \nabla^k \tilde{v}_I \right\|_{L^1(I)}^{q-m+1} \cdot \ell(I)^{n(m-q)-\alpha(q-m+1)} \]  \hspace{1cm} (3.14)

holds, where the constant \( C \) depends on \( n, m, k, d \) only, and we denoted

\[ \nabla^k \tilde{v}_I(x) = \nabla^k v(x) - \int_I \nabla^k v(y) \, dy. \]

The proof of this lemma is almost the same as that of Lemma 3.1, with evident modifications (we need to take the approximation polynomial \( P_I(x) \) of degree \( k \) instead of \( k-1 \), etc.).

Using our assumptions on Holder conditions (3.10), we obtain from (3.14) for an arbitrary cube \( I \) centred at \( x \) of size \( r = \ell(I) \leq 1 \) the estimate

\[ \Phi(Z_v \cap I) \leq C r^{(n+\alpha)(q-m+1)} \cdot r^{n(m-q)-\alpha(q-m+1)} = C r^n, \]

as required in (3.13). So the case \( k > n \) is finished.

*Case II.* Suppose now that \( 1 \leq k \leq n \).

The following intermediate estimate from our previous paper plays the important role here:
Lemma 3.4 (see Lemma 5.4 in [22]). Under above assumptions on \( v \) for any \( n \)-dimensional cube \( I \subset \mathbb{R}^n \) the estimate

\[
\Phi(Z_v \cap I) \leq C \left( \ell(I)^{\mu} \| \nabla^k \tilde{v}_I \|_{L^{p_0,1}(I)}^q + \ell(I)^{\mu + m - 1} \| \nabla^k \tilde{v}_I \|_{L^{p_0,1}(I)}^{q - m + 1} \right)
\]

holds, where \( p_0 = \frac{n}{k} \), the constant \( C \) depends on \( n, m, k, d \) only, and we denoted

\[
\nabla^k \tilde{v}_I(x) = \nabla^k v(x) - \int_I \nabla^k v(y) \, dy,
\]

\[
\| f \|_{L^{p_0,1}(E)} := \| 1_E \cdot f \|_{L^{p_0,1}}.
\]

Using our assumptions on Holder conditions (3.10), we obtain from the above formulas for an arbitrary cube \( I \) centred at \( x \) of size \( r = \ell(I) \leq 1 \) the estimate

\[
\Phi(Z_v \cap I) \leq C \left( r^{(\alpha + \frac{n}{p_0})q} \cdot r^{n - m + 1 - (k + \alpha)(q - m + 1)} + r^{(\alpha + \frac{n}{p_0})(q - m + 1)} \cdot r^{n - (k + \alpha)(q - m + 1)} \right)
\]

\[
= C \left( r^{(\alpha + k)q} \cdot r^{n - m + 1 - (k + \alpha)(q - m + 1)} + r^{(\alpha + k)(q - m + 1)} \cdot r^{n - (k + \alpha)(q - m + 1)} \right)
\]

\[
= C \left( r^{n + (m - 1)(k + \alpha - 1)} + r^n \right) \leq 2C r^n.
\]

as required in (3.13).

So we have proved the key property (**), i.e., the estimate

\[
\Phi(Z_v \cap I) \leq C \ell(I)^n
\]

with \( C = C(n, m, k, d) \) for arbitrary \( k \).

**Step II: the case \( m = 1 \).**

Suppose now that

\[
m = 1.
\]

In other words, now \( Z_v = \{ x \in \mathbb{R}^n : \nabla v(x) = 0 \} \). Thus \( \nabla^k v(x) \equiv 0 \) for almost all \( x \in Z_v \). Then we have the decomposition:

\[
Z_{v,m} = E_0 \cup E_1,
\]

where \( \mathcal{L}^n(E_0) = 0 \), and every \( x \in E_1 \) is a density point for the set \( \{ x \in \mathbb{R}^n : \nabla^k v(x) = 0 \} \). It implies, by elementary arguments, that

\[
\lim_{y \to x} \frac{\| \nabla^k v(y) - \nabla^k v(x) \|}{|x - y|^\alpha} \to 0 \quad \forall x \in E_1.
\]
Then, checking the proof of the basic estimate (3.15), we see that for any point $x \in E_1$ the identity
\[
\lim_{r \to 0} \frac{\Phi(Z_v \cap I(x, r))}{r^n} = 0
\] (3.16)
holds, where $I(x, r)$ denotes the cube centered at $x$ with $\ell(I) = r$. By usual elementary facts of real analysis and by subadditivity of $\Phi(\cdot)$, the convergence (3.16) implies that $\Phi(E_1) = 0$. The equality $\Phi(E_0) = 0$ follows from the condition $\mathcal{L}^n(E_0) = 0$ and (3.15). So $\Phi(Z_v) = 0$ as required. The case $m = 1$ is finished completely.

**Step III.**
From this point, for all the steps below we assume that $m \geq 2$. On this stage we have to establish the following assertion:

**Lemma 3.5.** There exists a Borel set $T \subset \mathbb{R}^d$ with $\mathcal{H}^q(T) = 0$ such that
\[
\mathcal{H}^\mu(Z_{v,m} \cap v^{-1}(y)) < \infty \quad \text{for every } y \in \mathbb{R}^d \setminus T.
\] (3.17)

**Proof.** Consider the set function $\Psi$, which is a slightly modification of $\Phi$ and is defined as follows:
\[
\Psi(E) = \lim_{\delta \to 0} \inf_{E \subset \bigcup_j D_j, \text{diam } D_j \leq \delta} \sum_j (\text{diam } D_j)^\mu \left[\text{diam } v(D_j)\right]^q.
\] (3.18)

Here the infimum is taken over all countable families of compact sets $\{D_j\}_{j \in \mathbb{N}}$ such that $E \subset \bigcup_j D_j$ and $\text{diam } D_j \leq \delta$ for all $j$.

By [22, Theorem 5.4] the above defined $\Psi(\cdot)$ is a countably subadditive set–function and for any $\lambda > 0$ the estimate
\[
\mathcal{H}^q\left(\{y \in \mathbb{R}^d : \mathcal{H}^\mu(E \cap v^{-1}(y)) \geq \lambda\}\right) \leq \frac{5\Psi(E)}{\lambda}
\] (3.19)
holds$^4$.

But the previous inequality (3.15) implies
\[
\Psi(E \cap Z_{v,m}) \leq C \cdot \mathcal{L}^n(E)
\] (3.20)
for any measurable set $E \subset \mathbb{R}^n$.

Therefore, taking the unit ball $B$ in $\mathbb{R}^n$, putting $E = B \cap Z_{v,m}$, using assumption (3.12) (that $v$ is compactly supported in $B$), from (3.19)–(3.20) we obtain
\[
\mathcal{H}^q\left(\{y \in \mathbb{R}^d : \mathcal{H}^\mu(Z_{v,m} \cap v^{-1}(y)) \geq \lambda\}\right) \leq \frac{C}{\lambda}.
\]

Thus, the set $T_0 := \{y \in \mathbb{R}^d : \mathcal{H}^\mu(Z_{v,m} \cap v^{-1}(y)) = \infty\}$ has zero $\mathcal{H}^q$-measure. Of course, there exists a Borel set $T \supset T_0$ with $\mathcal{H}^q(T) = 0$. This finishes the proof of the Lemma.

$^4$Really in [22, Theorem 6.2] this estimate was proved for arbitrary continuous function $v : \mathbb{R}^n \to \mathbb{R}^d$ and only for positive $\mu > 0$; but the proof for the limiting case $\mu = 0$ could be done the same way as in [22] with evident simplifications.
**Step IV.**

Now put $E_* = Z_{v,m} \cap v^{-1}(\mathbb{R}^d \setminus T)$, where the $H^q$-negligible Borel set $T$ was defined on the previous step. Of course, the set $E_*$ is Borel (Lebesgue) measurable, and, from the previous estimate (3.17) we have

\[ H^\mu(E_* \cap v^{-1}(y)) < \infty \quad \text{for any } y \in v(E_*). \] (3.21)

We claim the following assertion.

**Lemma 3.6.** There exists a decomposition $E_* = E_0 \cup E_1$ such that

- $\mathcal{L}^n(E_0) = 0$,
- $\text{rank } \nabla v(x) \leq m - 2$ for all $x \in E_1$.

**Proof.** Apply to the measurable set $E_*$ the Coarea formula (1.6) with $m' = m - 1$ instead of $m$:

\[ \int_{E_*} J_{m-1}v(x) \, dx = \int_{v(E_*)} H^{n-m+1}(E_* \cap v^{-1}(y)) \, dH^{m-1}(y), \] (3.22)

where $J_{m-1}v(x)$ denotes the $(m-1)$-Jacobian of $v$ defined as the product of the $(m-1)$ largest singular values of the matrix $\nabla v(x)$. By construction (see (3.21)) and trivial inequality $\mu = n-m+1-(k+\alpha)(q-m+1) < n-m+1$ we have that $H^{n-m+1}(E_* \cap v^{-1}(y)) = 0$ for every $y \in v(E_*)$. Therefore, both integrals in (3.22) vanish, i.e.,

\[ \int_{E_*} J_{m-1}v(x) \, dx = 0. \]

This implies evidently the required assertion of the Lemma. \(\square\)

**Step V.**

Now it is easy to finish the proof of Theorem 1.1 by elementary calculations:

**Lemma 3.7.** The identity $\Phi(E_*) = 0$ holds, where the set $E_*$ was defined on the previous step.

**Proof.** The required identity is equivalent to the following equalities:

- $\Phi(E_0) = 0$,
- $\Phi(E_1) = 0$,

where $E_0$ and $E_1$ were defined on the previous step. First equality $\Phi(E_0) = 0$ follows from (3.15) and from the assumption $\mathcal{L}^n(E_0) = 0$. Now consider the set $E_1$. By construction,
\( E_1 \subset Z_{v,m-1} \), so we could apply the previous estimate (3.20) for \( m' = m - 1 \) instead of \( m \) to obtain

\[
\tilde{\Psi}(E_1) < \infty,
\]

where \( \tilde{\Psi} \) is defined as \( \Psi \) (see (3.18)) with \( \mu \) replaced by

\[
\tilde{\mu} = n - m' + 1 - (k + \alpha)(q - m' + 1) = \mu - (k + \alpha - 1) < \mu.
\]

In other words,

\[
\tilde{\Psi}(E) = \lim_{\delta \to 0} \inf_{E \subset \bigcup_j D_j, \text{diam} D_j \leq \delta} \sum_j (\text{diam} D_j)^{\tilde{\mu}} [\text{diam} v(D_j)]^q.
\]

Of course, (3.23) implies \( \tilde{\Psi}(E_1) \geq \Psi(E_1) = 0 \), and, consequently, \( \Psi(E_1) \geq \Phi(E_1) = 0 \). \( \square \)

From the last Lemma, by construction and equality \( \mathcal{H}^q(T) = 0 \) we have

\[
\mathcal{H}^n(Z_{v,m} \cap v^{-1}(y)) = \mathcal{H}^n(E_\ast \cap v^{-1}(y)) = 0 \quad \text{for} \mathcal{H}^q\text{-almost all } y \in \mathbb{R}^d.
\]

The Theorem 1.1 is proved completely.

### 3.3 Bridge F.-D. Theorem for mappings of Besov spaces \( B^{k+\alpha}_{p,\infty}(\mathbb{R}^n, \mathbb{R}^d) \)

This subsection is devoted to the proof of Theorem 1.5, case (iii). Similar to the previous subsection, we apply here ”the lazy way”: on the first stage, we use the estimates of critical values through the Sobolev norms of highest \( k \)-derivatives, which were obtained before. On the second stage, we improve this estimates ”by \( \alpha \)” using Holder continuity condition ”in average” of these \( k \)-derivatives. Finally, we use the fact that functions from the Besov spaces \( B^{k+\alpha}_{p,\infty}(\mathbb{R}^n) \) coincides with the classical Holder \( C^{k,\alpha} \)-functions with exceptional set of small measure, so we could apply the results of the previous subsection.

Fix \( m \in \{1, \ldots, n\} \), \( k \geq 1 \), \( d \geq m \), \( 0 < \alpha < 1 \), and \( v \in B^{k+\alpha}_{p,\infty}(\mathbb{R}^n, \mathbb{R}^d) \). Take also a parameter \( q > m - 1 \). Of course, as in the previous subsection, it is sufficient to consider the case

\[
q \in (m - 1, q_o],
\]

where \( q_o = m - 1 + \frac{n-m+1}{k+\alpha} \).

By definition 2.1 of the space \( B^{k+\alpha}_{p,\infty} \) we may assume without loss of generality that for any \( t > 0 \) the estimate

\[
\| \Omega^k_v(\cdot, t) \|_{L^p(\mathbb{R}^n)} \leq t^\alpha
\]

(3.24)
holds, where
\[
\Omega_{v}^{k}(x, t) = \sup_{Q} \left\{ \int_{Q} |\nabla^{k}v(x) - \nabla^{k}v(Q)| \, dx : \ x \in Q, \ \ell(Q) = t \right\},
\]
(3.25)
and \(\nabla^{k}v(Q)\) denotes the mean value of \(\nabla^{k}v\) over the \(n\)-dimensional cube \(Q\).

Also, since the result has local nature, we may assume also that
\[
v(x) \equiv 0 \quad \text{if} \quad |x| > 1.
\]

Denote again \(Z_{v,m} = \{x \in \mathbb{R}^{n} \setminus A_{v} : \text{rank} \nabla v(x) < m\}\). Here \(A_{v}\) means the set of nondifferentiability points of \(v\); recall that this set has a small size, namely,
\[
\mathcal{H}^{r}(A_{v}) = 0 \quad \forall \tau > \tau_{\ast} := n - (k + \alpha - 1)p,
\]
in particular, \(A_{v} = \emptyset\) if \((k + \alpha - 1)p > n\).

Now the parameter \(\mu\) is the same as in the previous subsection:
\[
\mu = n - m + 1 - (k + \alpha)(q - m + 1).
\]
As before, the assertion of the Bridge Dubovitski˘ı–Federer Theorem 1.5 (iii) is equivalent (by virtue of Theorem 2.3) to
\[
\Phi(Z_{v,m}) = 0,
\]
where we denoted
\[
\Phi(E) = \inf_{E \subset \bigcup_{j} D_{j}} \sum_{j} (\text{diam } D_{j})^{\mu} [\text{diam } v(D_{j})]^{q}.
\]
As indicated the infimum is taken over all countable families of compact sets \(\{D_{j}\}_{j \in \mathbb{N}}\) such that \(E \subset \bigcup_{j} D_{j}\).

Moreover, since \(\Phi\) is a subadditive set-function and \(Z_{v,m}\) is given by the countable union
\[
Z_{v,m} := \bigcup_{j=1}^{\infty} \{x \in Z_{v,m} : |\nabla v(x)| \leq j\},
\]
it is sufficient to check the equality
\[
\Phi(Z_{v}) = 0,
\]
where now we use the notation
\[
Z_{v} := \{x \in Z_{v,m} : |\nabla v(x)| \leq 1\}.
\]

The proof of Theorem 1.5 (iii) splits into the two main cases \(k > n\) and \(k \leq n\).
CASE I.
Suppose now that
\[ k > n. \]

STEP I.1 (ESTIMATES ON SINGLE CUBES).
By Lemma 3.3 we obtain

**Lemma 3.8.** Under above assumptions on \( v \) for any \( n \)-dimensional interval \( Q \subset \mathbb{R}^n \) the estimate
\[
\Phi(Z_v \cap Q) \leq C \| \nabla^k \bar{v}_Q \|_{L^1(Q)}^{q-m+1} \cdot \ell(Q)^{n(m-q) - \alpha(q-m+1)}
\]
holds, where the constant \( C \) depends on \( n, m, k, d \) only, and we denoted
\[
\nabla^k \bar{v}_Q(x) = \nabla^k v(x) - \int_Q \nabla^k v(y) \, dy.
\]

Put \( r = \ell(Q) \). Using definition of \( \Omega^k_v(x, t) \) (see (3.25)) and some very elementary integral estimates, we could rewrite the formula (3.26) in the following equivalent form:
\[
\Phi(Z_v \cap Q) \leq C \| \Omega^k_v(\cdot, 2r) \|_{L^1(Q)}^{q-m+1} \cdot r^{n(m-q) - \alpha(q-m+1)} \leq C \| \Omega^k_v(\cdot, 2r) \|_{L^p(Q)}^{q-m+1} \cdot r^{n(1 - \frac{1}{p}(q-m+1))} \cdot r^{\alpha(1 - \frac{1}{p}(q-m+1))}.
\]

STEP I.2 (ESTIMATES ON UNION OF SIMILAR CUBES).

**Lemma 3.9.** Under above assumptions on \( v \) for any family of disjoint \( n \)-dimensional intervals \( Q_j \subset \mathbb{R}^n \) of the same size \( \ell(Q_j) \equiv r \) the estimate
\[
\sum_j \Phi(Z_v \cap Q_j) \leq C \| \frac{1}{r^\alpha} \Omega^k_v(\cdot, 2r) \|_{L^p(\cup_j Q_j)}^{q-m+1} \cdot \text{meas}(\cup_j Q_j)^{1 - \frac{1}{p}(q-m+1)}.
\]
holds.

The above Lemma can be deduced easily from (3.27) applying the Holder inequality, and we use here also the fact that
\[
q - m + 1 \leq q_\circ - m + 1 = \frac{n - m + 1}{k + \alpha} < p.
\]

STEP I.3 (ESTIMATES ON FINITE UNION OF CUBES).

Since every finite union of disjoint dyadic intervals could be represented as a union of a finite family of disjoint intervals having the same (sufficiently small) \( r \)-size, Lemma 3.9 easily implies
**Lemma 3.10.** Under above assumptions on $v$ for any finite family of disjoint dyadic $n$-dimensional interval $Q_j \subset \mathbb{R}^n$, $j = 1, \ldots, N$, the estimate

$$
\sum_{j=1}^{N} \Phi(Z_v \cap Q_j) \leq C \left\| \frac{1}{r^\alpha} \Omega_v^k(\cdot, 2r) \right\|_{L_p(\bigcup_j Q_j)}^{q-m+1} \cdot \text{meas}(\bigcup_j Q_j)^{1-\frac{1}{p}(q-m+1)} \tag{3.29}
$$

holds, where $r$ is sufficiently small.

Of course, (3.29) implies

$$
\sum_{j=1}^{N} \Phi(Z_v \cap Q_j) \leq C \sup_{r > 0} \left\| \frac{1}{r^\alpha} \Omega_v^k(\cdot, 2r) \right\|_{L_p(\mathbb{R}^n)}^{q-m+1} \cdot \text{meas}(\bigcup_j Q_j)^{1-\frac{1}{p}(q-m+1)} \tag{3.30}
$$

where $C$ depends on $n, m, k, d$ only.

**Step I.4 (estimates on general sets of finite measure).**

Taking $N \to \infty$ in (3.30), we have

$$
\Phi(Z_v \cap (\bigcup_{j=1}^{\infty} Q_j)) \leq \sum_{j=1}^{\infty} \Phi(Z_v \cap Q_j) \leq C \cdot \text{meas}(\bigcup_j Q_j)^{1-\frac{1}{p}(q-m+1)}
$$

for any family of disjoint $n$-dimensional dyadic intervals $Q_j$.

This implies, that for any measurable set $U$ the inequality

$$
\Phi(Z_v \cap U) \leq C \cdot \text{meas}(U)^{1-\frac{1}{p}(q-m+1)}, \tag{3.31}
$$

holds, where $C$ does not depend on $U$.

**Step I.5 (finishing of the proof of Theorem 1.5 (iii) for the case $k > n$).**

From Theorem 2.4 it follows that for any $\varepsilon > 0$ there exists a decomposition $\mathbb{R}^n = U \cup E$, $E = \mathbb{R}^n \setminus U$, where $\text{meas}(U) < \varepsilon$ and the identities $v = g$ and $\nabla v = \nabla g$ hold on the set $E$, where the mapping $g$ belongs to the class $C^{k,\alpha}(\mathbb{R}^n, \mathbb{R}^d)$. By Theorem 1.1, proved in the previous subsection, we have $\Phi(Z_v \cap E) = \Phi_g(Z_v \cap E) = 0$. On the other hand, by (3.31) the value $\Phi(U) \leq C \cdot \varepsilon^{1-\frac{1}{p}(q-m+1)}$ could be made arbitrary small. Therefore, $\Phi(Z_v) = 0$ as required.

**Case II.**

Suppose now that

$$
k \leq n.
$$

This case splits into two possibilities: $\alpha p \leq n$ and $\alpha p > n$. 

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Case II.A.

\[ k \leq n, \quad \alpha p \leq n. \]

Since \((k + \alpha)p > n\), we have, in particular, that

\[
\frac{pn}{n - p\alpha} > \frac{n}{k}. \tag{3.32}
\]

**Step II.A.1 (Estimates on Single Cubes).**

By Lemma 3.4 we obtain

**Lemma 3.11.** Under above assumptions on \(v\) for any \(n\)-dimensional cube \(Q \subset \mathbb{R}^n\) the estimate

\[
\Phi(Z_v \cap Q) \leq C(\ell(Q)^\mu \|\nabla^k \bar{u}_Q\|_{L_{p_0,1}(Q)}^q + \ell(Q)^{\mu + m - 1} \|\nabla^k \bar{v}_Q\|_{L_{p_0,1}(Q)}^{q-m+1})
\]

holds, where \(p_0 = \frac{n}{k}\), the constant \(C\) depends on \(n, m, k, d\) only, and we denoted

\[
\nabla^k \bar{v}_Q(x) = \nabla^k v(x) - \int_Q \nabla^k v(y) \, dy.
\]

Denote \(r = \ell(Q)\). Take \(s < \frac{pn}{n - p\alpha}\) such that \(s > p_0\) (this is possible by (3.32)). Then by virtue of elementary inequality \(\|f\|_{L_{p_0,1}(E)} \leq C \|f\|_{L_s(E)} \cdot \left(\text{meas}(E)\right)^{\frac{1}{s} - \frac{1}{p}}\) the estimate (3.33) implies

\[
\Phi(Z_v \cap Q) \leq C(r^{\mu + (k - \frac{n}{s})q} \|\nabla^k \bar{v}_Q\|_{L_s(Q)}^q + r^{\mu + m - 1 + (k - \frac{n}{s})(q - m + 1)} \|\nabla^k \bar{v}_Q\|_{L_s(Q)}^{q-m+1}).
\]

Then, using the definition \(\mu = n - m + 1 - (k + \alpha)(q - m + 1)\), we obtain

\[
\Phi(Z_v \cap Q) \leq C(r^{n + (m-1)(k + \alpha - 1) - (\alpha + \frac{n}{s})q} \|\nabla^k \bar{v}_Q\|_{L_s(Q)}^q + r^{n - (\alpha + \frac{n}{s})(q - m + 1)} \|\nabla^k \bar{v}_Q\|_{L_s(Q)}^{q-m+1}). \tag{3.34}
\]

Take \(1 < \tau < p\) and \(0 < \beta < \alpha\) such that

\[
\frac{1}{s} = \frac{1}{\tau} - \frac{\beta}{n}. \tag{3.35}
\]

By [13] (see first display formula on p. 30) we have

\[
\|\nabla^k \bar{v}_Q\|_{L_s(Q)} \leq C \int_0^r t^{-\beta} \left(\int_Q \Omega^k_v(z, t)^\tau \, dz\right)^{\frac{1}{\tau}} t^{-1} \, dt \leq C r^{\frac{n}{s} - \frac{n}{\tau}} \int_0^r t^{\alpha - \beta} \left(\frac{1}{t^{ap}} \int_Q \Omega^k_v(z, t)^p \, dz\right)^{\frac{1}{p}} t^{-1} \, dt
\]

Holder ineq.

\[
\leq C r^{\frac{n}{s} - \frac{n}{\tau} + (\alpha - \beta)(1 - \frac{1}{p})} \left(\int_0^r t^{\alpha - \beta - 1} \cdot t^{-ap} \int_Q \Omega^k_v(z, t)^p \, dz \, dt\right)^{\frac{1}{p}} \tag{3.35}
\]

\[
= C r^{\alpha + \frac{n}{s} - \frac{n}{\tau}} \left(\int_0^r t^{\alpha - \beta - 1} \cdot t^{-ap} \int_Q \Omega^k_v(z, t)^p \, dz \, dt\right)^{\frac{1}{p}}.
\]

By technical reasons, now our parameters \(s\) and \(\tau\) play the role of parameters \(r\) and \(q\) from [13, p.30].
Denote by $\Lambda_r(Q)$ the value in parenthesis in the last formula. Using this information in previous estimate (3.34), we get
\[
\Phi(Z_v \cap Q) \leq C \left( r^{(m-1)(k+\alpha-1)+n(1-\frac{q}{p})} \Lambda_r(Q)^{\frac{q}{p}} + r^{n(1-\frac{2m+1}{p})} \Lambda_r(Q)^{\frac{q-m+1}{p}} \right). \tag{3.37}
\]

**Step II.A.2 (estimates on union of similar cubes).**

**Lemma 3.12.** Under above assumptions on $v$ for any family of disjoint $n$-dimensional intervals $Q_j \subset \mathbb{R}^n$ of the same size $\ell(Q_j) \equiv r$ the estimate
\[
\sum_j \Phi(Z_v \cap Q_j) \leq C \left( \text{meas}(\bigcup_j Q_j)^{\frac{1}{p}} \Lambda_r(\bigcup_j Q_j)^{\frac{q}{p}} + \text{meas}(\bigcup_j Q_j)^{1-\frac{2m+1}{p}} \Lambda_r(\bigcup_j Q_j)^{\frac{q-m+1}{p}} \right)
\]
holds.

Here we denoted with
\[
\Lambda_r(E) = r^{\beta-\alpha} \cdot \int_0^r t^{n-\beta-1} \cdot t^{-\alpha p} \int_E \Omega_k^p(z,t)^p \, dz \, dt
\]
for any measurable set $E \subset \mathbb{R}^n$. Note that, by definition, $\Lambda_r$ is an additive function, i.e.,
\[
\sum_j \Lambda_r(E_j) = \Lambda_r(\bigcup_j E_j) \tag{3.38}
\]
for any family of disjoint measurable sets $E_j$. Moreover, by definition (3.24), the uniform estimate
\[
\Lambda_r(E) \leq C \tag{3.39}
\]
is valid, where $C$ does not depend on $v, E$ and $r$.

**Proof.** By (3.37),
\[
\sum_j \Phi(Z_v \cap Q_j) \leq C(I_1 + I_2),
\]
where
\[
I_1 := \sum_j r^{n(1-\frac{q-m+1}{p})} \Lambda_r(Q_j)^{\frac{q-m+1}{p}}, \tag{3.40}
\]
\[
I_2 := \sum_j r^{(m-1)(k+\alpha-1)+n(1-\frac{q}{p})} \Lambda_r(Q_j)^{\frac{2}{p}}. \tag{3.41}
\]
Let us estimate $I_1$ first. Since $q \leq q_0 = m - 1 + \frac{n-m+1}{k+\alpha} < p + (m - 1)(1 - \frac{1}{k+\alpha})$, we have
\[ q - m + 1 \leq q_0 - m + 1 = \frac{n-m+1}{k+\alpha} < p. \tag{3.42} \]

Therefore, we could apply Holder inequality to (3.40) to obtain
\[ I_1 \leq \left( \sum_{j} r^n \right)^{\frac{1-q_0}{p}} \left( \sum_{j} \Lambda_r(Q_j) \right)^{\frac{q_0}{p}} \leq \text{meas}(\bigcup_j Q_j)^{\frac{1-q_0}{p}} \Lambda_r(\bigcup_j Q_j)^{\frac{q_0}{p}}. \tag{3.38} \]

Now let us estimate $I_2$. Since $q \leq q_0 = m - 1 + \frac{n-m+1}{k+\alpha} < p + (m - 1)(1 - \frac{1}{k+\alpha})$, we have
\[ n(1 - \frac{q}{p}) \geq - (m - 1) \frac{n}{p} \cdot \frac{k + \alpha - 1}{k + \alpha} > -(m - 1)(k + \alpha - 1), \]
and so,
\[ (m - 1)(k + \alpha - 1) + n(1 - \frac{q}{p}) > 0. \tag{3.43} \]

Now consider two possible cases.

Case $I_{21}$. $q < p$. We could apply Holder inequality to (3.41) to obtain
\[ I_2 \leq r^{(m-1)(k+\alpha-1)} \left( \sum_{j} r^n \right)^{\frac{1-q}{p}} \left( \sum_{j} \Lambda_r(Q_j) \right)^{\frac{q}{p}} \leq r^{(m-1)(k+\alpha-1)} \text{meas}(\bigcup_j Q_j)^{\frac{1-q}{p}} \Lambda_r(\bigcup_j Q_j)^{\frac{q}{p}}. \tag{3.38} \]

Case $I_{22}$. $q \geq p$. Here we have
\[ I_2 \leq r^{(m-1)(k+\alpha-1)+n(1-\frac{q}{p})} \left( \sum_{j} \Lambda_r(Q_j) \right)^{\frac{q}{p}} \leq \text{meas}(\bigcup_j Q_j)^{\frac{1}{p}(m-1)(k+\alpha-1)+(1-\frac{q}{p})} \Lambda_r(\bigcup_j Q_j)^{\frac{q}{p}}. \tag{3.43}, (3.38) \]

Hence, the Lemma is proved. \qed

**Step II.A.3.** (estimates on finite union of dyadic cubes).

**Lemma 3.13.** Under above assumptions on $v$ for any finite union of $n$-dimensional dyadic disjoint intervals $Q_j \subset \mathbb{R}^n$, $j = 1, \ldots, N$ the estimate
\[ \sum_{j=1}^{N} \Phi(Z_v \cap Q_j) \leq C \left( \text{meas}(\bigcup_j Q_j)^{\frac{1}{p}(m-1)(k+\alpha-1)+(1-\frac{q}{p})} \Lambda_r(\bigcup_j Q_j)^{\frac{q}{p}} + \text{meas}(\bigcup_j Q_j)^{\frac{1-q_0}{p}} \Lambda_r(\bigcup_j Q_j)^{\frac{q_0}{p}} \right). \tag{3.44} \]
holds, for sufficiently small $r > 0$. 29
This assertion follows immediately from the previous Lemma 3.12 and from the fact that any finite union of disjoint dyadic cubes could be represented as finite union of some finite family of disjoint dyadic cubes of the same size $r$.

**Step II.A.4 (estimates on general sets of finite measure).**

Taking $N \to \infty$ in (3.44) and using (3.39), we have

$$
\Phi(Z_v \cap (\cup_{j=1}^{\infty} Q_j)) \leq \sum_{j=1}^{\infty} \Phi(Z_v \cap Q_j) \leq C \left( \text{meas}(\cup_j Q_j)^{\frac{1}{n}(m-1)(k+\alpha-1)+(1-\frac{2}{p})} + \text{meas}(\cup_j Q_j)^{1-\frac{q-m+1}{p}} \right),
$$

for any family of disjoint $n$-dimensional intervals $Q_j$.

This implies that, for any measurable set $U$, the inequality

$$
\Phi(Z_v \cap U) \leq C \left( \text{meas}(U)^{\frac{1}{n}(m-1)(k+\alpha-1)+(1-\frac{2}{p})} + \text{meas}(U)^{1-\frac{q-m+1}{p}} \right),
$$

(3.45)

holds, where $C$ does not depend on $U$.

**Step II.A.5 (finishing of the proof of Theorem 1.5 (iii) for the case $k \leq n$, $\alpha p \leq n$).**

From Theorem 2.4 it follows that for any $\varepsilon > 0$ there exists a decomposition $\mathbb{R}^n = U \cup E$, $E = \mathbb{R}^n \setminus U$, where $\text{meas}(U) < \varepsilon$ and the identities $v = g$ and $\nabla v = \nabla g$ hold on the set $E$, where the mapping $g$ belongs to the class $C^{k,\alpha}(\mathbb{R}^n, \mathbb{R}^d)$. By Theorem 1.1, proved in the previous subsection, we have $\Phi(Z_v \cap E) = \Phi(g(Z_g \cap E) = 0$. On the other hand, by (3.45) the value $\Phi(U)$ could be made arbitrary small (as $\varepsilon$ is small). Therefore, $\Phi(Z_v) = 0$ as required.

**Case II.B.** Suppose now that

$$
k \leq n, \quad \alpha p > n
$$

(this is the very last case we need to check in Theorem 1.5 (iii))

**Step II.B.1 (estimates on single cubes).**

Again by Lemma 3.4 we obtain

**Lemma 3.14.** Under above assumptions on $v$, for any $n$-dimensional cube $Q \subset \mathbb{R}^n$ the estimate

$$
\Phi(Z_v \cap Q) \leq C \left( \ell(Q)^{\mu} \| \nabla^k \bar{v}_Q \|_{L^p_{\text{loc}}(Q)}^{\mu} + \ell(Q)^{\mu+m-1} \| \nabla^k \bar{v}_Q \|_{L^p_{\text{loc}}(Q)}^{q-m+1} \right),
$$

holds, where $p_0 = \frac{n}{k}$, the constant $C$ depends on $n, m, k, d$ only, and we denoted

$$
\nabla^k \bar{v}_Q(x) = \nabla^k v(x) - \int_Q \nabla^k v(y) \, dy.
$$
Denote \( r = \ell(Q) \). Take parameters \( \beta \) and \( \tau \) such that
\[
\alpha > \beta > \frac{n}{\tau} > \frac{n}{p}.
\]

By [13] (see 6 formula (12) on p. 30) we have
\[
\| \nabla^k \bar{v}_Q \|_{L^\infty(Q)} \leq C r^{\beta - \frac{n}{\tau}} \cdot \int_0^r t^{-\beta} \left( \frac{1}{t^p} \int_Q \Omega^k_v(z, t) \, dz \right)^{\frac{1}{p}} t^{-1} \, dt
\]

Holder ineq.
\[
\leq C r^{\beta - \frac{n}{\tau} + \frac{\alpha - \beta}{1 - \frac{1}{p}}} \left( \int_0^r t^{\alpha - \beta - 1} \cdot t^{-\alpha p} \int_Q \Omega^k_v(z, t) \, dz \, dt \right)^{\frac{1}{p}}
\]
\[
= C r^{\alpha - \frac{n}{\tau}} \left( r^{\beta - \alpha} \cdot \int_0^r t^{\alpha - \beta - 1} \cdot t^{-\alpha p} \int_Q \Omega^k_v(z, t) \, dz \, dt \right)^{\frac{1}{p}}.
\]

Consequently, for \( s \in (p, \infty) \) we have
\[
\| \nabla^k \bar{v}_Q \|_{L^s(Q)} \leq C r^{\alpha + \frac{n}{s}} \left( r^{\beta - \alpha} \cdot \int_0^r t^{\alpha - \beta - 1} \cdot t^{-\alpha p} \int_Q \Omega^k_v(z, t) \, dz \, dt \right)^{\frac{1}{p}}.
\]

Hence we received exactly the same estimate as previous (3.36) for the case II.A (when \( k \leq n, \alpha p \leq n \)). So the remaining part of the proof for the present case II.B could be made identically the same as Steps II.A.2–II.A.5 from above. Thus Theorem 1.5 (iii) is completely proved.

### 3.4 Bridge F.-D. Theorem for Lorentz potential spaces \( \mathcal{L}^{\alpha}_{p,1}(\mathbb{R}^n, \mathbb{R}^d) \).

Here we consider the last case (iv) of Theorem 1.5. This case is much simpler than the previous one (for Besov spaces).

Fix \( m \in \{1, \ldots, n\}, k \geq 1, d \geq m, 0 < \alpha < 1, \) and \( v \in \mathcal{L}^{k+\alpha}_{p,1}(\mathbb{R}^n, \mathbb{R}^d) \). Now, by assumptions of Theorem 1.5 (iv), we have
\[
(k + \alpha)p = n,
\]
and in particular,
\[
k < n \quad \text{and} \quad \alpha p < n.
\]

The continuity of \( v \) follows easily from the generalised Holder inequality for the Lorentz spaces (see, e.g., [27]).

\footnote{By technical reasons, now our parameter \( \tau \) plays the role of parameter \( q \) from [13, p.30].}
Take a parameter $q > m - 1$. Of course, as in previous subsection, it is sufficient to consider the case

$$q \in (m - 1, q_0],$$

where $q_0 = m - 1 + \frac{n-m+1}{k+\alpha}$.

Denote again $Z_{v,m} = \{x \in \mathbb{R}^n \setminus A_v : \text{rank } \nabla v(x) < m\}$. Here $A_v$ means the set of nondifferentiability points of $v$; recall that this set has a small size, namely,

$$\mathcal{H}^p(A_v) = 0.$$

Now the parameter $\mu$ is the same as in the previous subsection:

$$\mu = n - m + 1 - (k + \alpha)(q - m + 1).$$

As before, the assertion of the Bridge Dubovitskiĭ–Federer Theorem 1.5 is equivalent (by virtue of Theorem 2.3) to

$$\Phi(Z_{v,m}) = 0,$$

where

$$\Phi(E) := \inf_{E \subset \bigcup_j D_j} \sum_j \left(\text{diam } D_j\right)^\mu \left[\text{diam } v(D_j)\right]^q.$$

As before, since $\Phi$ is a subadditive set-function, it is sufficient to check only the equality

$$\Phi(Z_v) = 0,$$

where again we denote

$$Z_v := \{x \in Z_{v,m} : |\nabla v(x)| \leq 1\}.$$

Put $p_o = \frac{n}{k}$. Since $(k + \alpha)p = n$, we have, in particular, that

$$\frac{pn}{n - p\alpha} = p_o.$$

By Lemma 3.4 we obtain, that under above assumptions on $v$ for any $n$-dimensional cube $Q \subset \mathbb{R}^n$ the estimate

$$\Phi(Z_v \cap Q) \leq C(\ell(Q)^\mu \|\nabla^k \bar{v}_Q\|_{L_p,1}^q + \ell(Q)^{\mu + m - 1} \|\nabla^k \bar{v}_Q\|_{L_p,1}^{q - m + 1})$$

holds, where $p_o = \frac{n}{k}$, the constant $C$ depends on $n, m, k, d$ only, and we denoted

$$\nabla^k \bar{v}_Q(x) = \nabla^k v(x) - \int_Q \nabla^k v(y) \, dy.$$
From the inclusion \( v \in \mathcal{L}_{p,1}^{k+\alpha}(\mathbb{R}^n, \mathbb{R}^d) \) and from Theorem 2.1 it follows that \( \nabla^k v = \mathcal{G}_\alpha(g) := K_\alpha * g \) for some \( g \in L_{p,1}(\mathbb{R}^n) \). Then by Lemma 2.1 we have

\[
\| \nabla^k \bar{V}_Q \|_{L_{p,1}(Q)} \leq \| M g \|_{L_{p,1}(Q)}.
\]

Denote \( h = Mg \). The formula (2.2) implies

\[
h \in L_{p,1}(\mathbb{R}^n).
\]

Now we could rewrite our basic estimate as

\[
\Phi(Z_v \cap Q) \leq C \left( \tau^\mu \| h \|_{L_{p,1}(Q)}^q + \tau^{m+1-\alpha} \left( \| h \|_{L_{p,1}(Q)}^{q-\alpha} + \| h \|_{L_{p,1}(Q)}^{q-1} \right) \alpha \right).
\]

Then, using the definition \( \mu = n - m + 1 - (k+\alpha)(q-m+1) \), we obtain

\[
\Phi(Z_v \cap Q) \leq C \left( \tau^{(m-1)(k+\alpha-1)-n(1-\frac{q}{p})} \| h \|_{L_{p,1}(Q)}^q + \tau^{n-(k+\alpha)(q-m+1)} \| h \|_{L_{p,1}(Q)}^{q-\alpha} \right).
\]

Now we received the estimate of type (3.37) with \( \| h \|_{L_{p,1}(Q)}^p \) instead of \( \Lambda_r(Q) \). Since

\[
\sum_i \| h \|_{L_{p,1}(Q_i)}^p \leq \| h \|_{L_{p,1}(\cup_i Q_i)}^p,
\]

for any family of disjoint cubes \( Q_i \) (see, e.g., [34, Lemma 3.10]), the rest of the proof could be made almost ”word by word” repeating the same arguments as before (after the formula (3.37) ) with evident simplifications. Namely, now we do not need to consider separately the case of equal cubes of size \( r \), since (3.47) is applicable for any family of cubes. Another difference is that now we have the equality \( (k+\alpha)p = n \) instead of inequality \( (k+\alpha)p < n \), but it does not produce any obstacles for the arguments.

Thus Theorem 1.5 is proved for the case (iv). Since it was the last one, Theorem 1.5 is completely proved.

4 Appendix

Fix \( 0 < \alpha < 1 \) and \( p > 1 \) with \( \alpha p < n \). Suppose that \( f \in \mathcal{L}_{p,1}^\alpha(\mathbb{R}^n) \), i.e., there exists a function \( g \in L_{p,1}(\mathbb{R}^n) \) such that

\[
f(x) = \mathcal{G}_\alpha(x) := \int_{\mathbb{R}^n} g(y) K_\alpha(x-y) \, dy,
\]

where \( K_\alpha \) is the corresponding Bessel potential function. The purpose of this section is to prove the estimate of Lemma 2.1, i.e., that for every \( n \)-dimensional cubic interval \( Q \subset \mathbb{R}^n \) with \( \ell(Q) \leq 1 \) the inequality

\[
\| \bar{f}_Q \|_{L_{p,1}(Q)} \leq C \| Mg \|_{L_{p,1}(Q)}
\]

(4.1)
holds, where \( p_* = \frac{np}{n-\alpha p} \), the constant \( C \) depends on \( n, p, \alpha \) only, and we denote
\[
\bar{f}_{Q}(x) = f(x) - \int_{Q} f(y) \, dy.
\]
Here
\[
Mg(x) = \sup_{r > 0} \int_{B(x,r)} |g(y)| \, dy
\]
is the usual Hardy–Littlewood maximal function of \( g \).

Recall that by our notation \( \|g\|_{L_{p,1}(E)} := \|1_{E} \cdot g\|_{L_{p,1}} \), where \( 1_{E} \) is the indicator function of \( E \).

First of all, fix a cube \( Q \) of size \( r = \ell(Q) \leq 1 \), and split our function \( f \) into the sum
\[
f = f_1 + f_2,
\]
where
\[
f_1 := \int_{\mathbb{R}^n} g_1(y) K_{\alpha}(x - y) \, dy = \int_{2Q} g(y) K_{\alpha}(x - y) \, dy,
\]
\[
f_2 := \int_{\mathbb{R}^n} g_2(y) K_{\alpha}(x - y) \, dy = \int_{\mathbb{R}^n \setminus 2Q} g(y) K_{\alpha}(x - y) \, dy,
\]
\[
g_1 := g \cdot 1_{2Q}, \quad g_2 := g \cdot 1_{\mathbb{R}^n \setminus 2Q},
\]
and where \( 2Q \) means the cube with the same centre of double size \( \ell(2Q) = 2\ell(Q) \).

The proof of the estimate (4.1) splits into two steps.

**STEP 1.** It was proved in [19, Lemma 5.1] that for any \( x \in Q \) the inequality
\[
|f_1(x)| \leq C \int_{Q} \frac{Mg_1(y)}{|x - y|^{n-\alpha}} \, dy
\]
holds. This implies, by virtue of Hardy-Littlewood-Sobolev theorem (see [5, Theorem IV.4.18]), that
\[
\|f_1\|_{L_{p,1}(Q)} \leq C \|Mg_1\|_{L_{p,1}(Q)} \leq C \|Mg\|_{L_{p,1}(Q)} \quad (4.2)
\]
(here and henceforth we denote by \( C \) the general constants depending on the parameters \( n, p, \alpha \) only).

**STEP 2.** In [19, Lemma 5.2] it was proved the following estimate:
\[
\text{diam}[f_2(Q)] \leq C r^{\alpha-n} \int_{Q} Mg_2(y) \, dy.
\]
This implies, by virtue of the usual Hölder inequality, that
\[ \| \tilde{f}_2 \|_{L^{p*,1}(Q)} \leq C \| M g_2 \|_{L^p(Q)} \leq C \| M g \|_{L^p,1(Q)} \] (4.3)
where we denote
\[ \tilde{f}_2(x) := f_2(x) - \int_Q f_2(y) \, dy, \]

Evidently, the inequalities (4.2) and (4.3) imply the required estimate (4.1). The proof is complete.

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