E6Tensors: A Mathematica Package for E$_6$ Tensors

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Abstract

We present the Mathematica package E6Tensors, a tool for explicit tensor calculations in $E_6$ gauge theories. In addition to matrix expressions for the group generators of $E_6$, it provides structure constants, various higher rank tensors and expressions for the representations $27$, $78$, $351$ and $351'$. This paper comes along with a short manual including physically relevant examples. I further give a complete list of gauge invariant, renormalisable terms for superpotentials and Lagrangians.

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1 Introduction

The exceptional Lie group $E_6$ may be a suitable candidate for describing fundamental symmetries in particle physics [1]. In the discussion of $E_6$, most authors rely on abstract group theoretic methods. Besides, there is a paper by Kephart and Vaughn [2] that describes the generators of $E_6$ in terms of its maximal subgroup $SU(3) \times SU(3) \times SU(3)$. In practice, applying these methods may be cumbersome. For writing down Lagrangians and superpotentials of such models it is useful to have explicit expressions for irreducible representations and invariants, preferably in a way suited for computer use. To our knowledge, such a tool is still missing in the literature and the package E6Tensors tries to fill this gap.

Our package provides such explicit expressions for the representations $27, 78, 351, 351'$ as well as structure constants and higher rank tensors. E6Tensors enables the user to study Lagrangians and superpotentials including the aforementioned representations explicitly by components.

The outline of the paper is the following: In Section 2, we state the transformation law for the fundamental representation $27$. From that we construct the matrix expressions for the 78 group generators forming the adjoint representation $78$. Then it is possible to compute the (symmetric) structure constants and other properties of the matrix generators. Since higher-order tensors can be built from the tensor products of (anti-)fundamental representations, we derive expressions for the irreducible representations $351$ and $351'$ in Section 3. Section 4 then provides a small manual for the Mathematica package including some remarks and examples. A summary of possible gauge invariant terms in superpotentials and Lagrangians can be found in Appendix A.

2 Matrix Expressions for the Group Generators

In [2], the authors give a prescription how the fundamental representation can be expressed as three $3 \times 3$ matrices $L, M, N$ and how the group generators act on them. We briefly revise that
prescription and point out that we performed the calculations described there using symbolic
manipulation in Mathematica. Throughout the paper we use Greek indices for the fundamental
representation of $E_6$
\[ \mu, \nu, \rho, \cdots = 1, \ldots, 27. \]  
(1)
For the adjoint representation we use Latin indices $k, l, m, \ldots$
\[ k, l, m, \cdots = 1, \ldots, 78. \]  
(2)

For describing $E_6$ by its maximal subgroup $SU(3)_C \times SU(3)_L \times SU(3)_R$ we also use the $SU(3)$ indices
\[ \alpha, \beta, \gamma, \ldots = 1, 2, 3. \]  
(3)
\[ a, b, c, \ldots = 1, 2, 3. \]  
(4)
\[ p, q, r, \ldots = 1, 2, 3. \]  
(5)
\[ A, B, C, \ldots = 1, \ldots, 8. \]  
(6)

2.1 Transformation Law of the Fundamental Representation

With respect to its maximal subgroup $SU(3)_C \times SU(3)_L \times SU(3)_R$, the fundamental representation of $E_6$ can be decomposed as
\[ 27 \rightarrow (3, \bar{3}, 1) + (1, 3, \bar{3}) + (\bar{3}, 1, 3), \]  
(7)
and its adjoint representation can be decomposed as
\[ 78 \rightarrow (8, 1, 1) + (1, 8, 1) + (1, 1, 8) + (3, 3, 3) + (\bar{3}, \bar{3}, \bar{3}). \]  
(8)

First, the generators $T^A$ of the three $SU(3)$ subgroups corresponding to the first three representations in eq. (7) can be represented by the eight Gell-Mann matrices $\lambda^A/2$ and their action onto the three matrices $(L, M, N)$ according to eq. (7) can be expressed by
\[ T^A_C \left( L, \ M, \ N \right) = \left( \frac{1}{2} \lambda^A L, \ 0, \ -\frac{1}{2} N \lambda^A \right), \]
\[ T^A_L \left( L, \ M, \ N \right) = \left( -\frac{1}{2} L \lambda^A, \ \frac{1}{2} \lambda^A M, \ 0 \right), \]
\[ T^A_R \left( L, \ M, \ N \right) = \left( 0, \ -\frac{1}{2} M \lambda^A, \ \frac{1}{2} \lambda^A N \right). \]  
(9)
Second, there are the generators $T^{aap}$ and $\bar{T}^{aap}$ that mediate shifts between the matrices
\[ L, M, N \]

\[
T_{a\alpha p} L^b_{\beta} = \varepsilon_{\alpha\beta\gamma} \delta^b_a N^\gamma_p, \quad \bar{T}^{a\alpha p} L^b_{\beta} = -\varepsilon^{abc} \delta^a_b M_c^p, \\
T_{a\alpha p} M^q_{\beta} = \varepsilon_{abc} \delta^q_a L_c^\alpha, \quad \bar{T}^{a\alpha p} M^q_{\beta} = -\varepsilon^{pqr} \delta^q_a N_r^\alpha, \\
T_{a\alpha p} N^\beta_q = \varepsilon_{pqr} \delta^\beta_a M_r^\alpha, \quad \bar{T}^{a\alpha p} N^\beta_q = -\varepsilon^{\alpha\beta\gamma} \delta^q_p L^\gamma_a. \tag{10}
\]

\( \delta^b_a \) is the Kronecker symbol and \( \varepsilon^{abc} \) the Levi-Civita symbol with \( \varepsilon_{123} = \varepsilon^{123} = 1 \). With these set of generators an infinitesimal, unitary \( E_6 \) transformation reads

\[
U(u, v, w, x, y) = 1 + i u_A T^A_C + i v_A T^A_L + i w_A T^A_R + i x_{a\alpha p} T^{a\alpha p} + i y_{a\alpha p} \bar{T}^{a\alpha p} + \ldots \tag{11}
\]

In total, \( u_A, v_A, w_A, x_{a\alpha p}, y_{a\alpha p} \) are 78 parameters.

2.2 Explicit Matrix Expressions for the Group Generators

We now aim at writing the transformation in eq. (11) with one set of 78 matrices \( kT \) and parameters \( \varepsilon^k \) that act on a 27-dimensional vector \( \psi \)

\[
U(\varepsilon) \psi = (1 + i \varepsilon^k kT + \ldots) \psi. \tag{12}
\]

For that purpose, we rearrange the transformation parameters and the matrices \( L, M, N \) into column vectors in the following way:

\[
\psi = (L_{11}, L_{12}, \ldots, M_{11}, \ldots, N_{11}, \ldots, N_{33})^T \\
\varepsilon = (u^1, u^2, \ldots, v^1, \ldots, w^1, \ldots, x^{111}, \ldots, y^{111}, \ldots, y^{333})^T. \tag{13}
\]

By comparing coefficients in (11) and (12), the \( 27 \times 27 \) matrices \( kT \) can be constructed. To adjust the normalisation and obtain Hermiticity, we perform the following change of basis

\[
k\tilde{T} = \frac{1}{2} (kT + k+27T), \quad 24 < k < 52 \\
k\tilde{T} = \frac{i}{2} (kT - k+27T), \quad 51 < k < 78. \tag{14}
\]

The group generators are included in the Mathematica package as an \( 78 \times 27 \times 27 \) dimensional array called \( E6\text{gen} \). They obey

\[
\text{tr}(kT_iT) = 3 \delta_{kl} \quad \text{and} \quad \sum_{k=1}^{78} kT_k = \frac{26}{3} 1_{27}. \tag{15}
\]

This sets the Dynkin index and the quadratic Casimir invariant to

\[
C(27) = 3 \quad \text{and} \quad C_2(27) = \frac{26}{3}, \tag{17}
\]
satisfying the well-known identity

\[ C_2(R) = \frac{\dim(G)}{\dim(R)} C(R), \tag{18} \]

for a representation \( R \) and the adjoint representation \( G \). In addition to this consistency check, \( C(27) \) and \( C_2(27) \) match the same values Kephart and Vaughn state in their paper \[2\].

By construction, the generators are ordered in the following way

\begin{align*}
_1T \ldots _8T : & \quad SU(3)_C, \\
_9T \ldots _{16}T : & \quad SU(3)_L, \\
_{17}T \ldots _{24}T : & \quad SU(3)_R. 
\end{align*}

Therefore, the diagonal generators representing the Cartan subalgebra are

\[ _3T, _8T, _{11}T, _{16}T, _{19}T, _{24}T. \tag{22} \]

The generators \( _{25}T \) to \( _{78}T \) are the shifting operators defined in eq. (10). The structure constants \( f_{klm} \) of a Lie Algebra are defined by

\[ [_kT, _lT] = i f_{klm} mT. \tag{23} \]

Applying the normalisation condition gives

\[ f_{klm} = -\frac{i}{C(27)} \text{tr}( [_kT, _lT] mT). \tag{24} \]

In the Mathematica package they are encoded in the array E6f. As a cross-check we calculated the normalisation to be

\[ f_{knn} f_{lmn} = 12 \delta_{kl} \tag{25} \]

which also matches the value in \[2\]. Since the structure constants are the generators of the adjoint representation, its quadratic Casimir and Dynkin index are

\[ C(G) = C_2(G) = 12. \tag{26} \]

The symmetric structure constants \( C_{klm} \) are defined by

\[ \{ _kT, _lT \} = _kT_iT + _lT_kT = i C_{klm} mT. \tag{27} \]

This can be rewritten as

\[ C_{klm} = -\frac{i}{C(27)} \text{tr}( \{ _kT, _lT \} mT). \tag{28} \]
Explicit computation then yields

\[ C_{klm} = 0 \quad \forall \ k, l, m = 1, \ldots, 78. \tag{29} \]

Hence, \( E_6 \) GUT models are in general free of chiral anomalies.

### 3 Higher Rank Tensors

The transformation laws for fundamental \( 27 \), anti-fundamental \( \overline{27} \) and adjoint representation \( 78 \) are as follows

\[
\begin{align*}
\psi_\mu &\rightarrow \psi_\mu + i \epsilon^k (kT)_\mu^\nu \psi_\nu, \\
\bar{\psi}^\mu &\rightarrow \bar{\psi}^\mu - i \epsilon^k (kT)_\nu^\mu \bar{\psi}^\nu, \\
\phi_l &\rightarrow \phi_l + \epsilon^k f_{klm} \phi_m,
\end{align*}
\]

with Greek indices running from 1 to 27 and Latin indices running from 1 to 78, cf. eq. (1) and eq. (2).

#### 3.1 Higher Dimensional Representations

The representations \( 351, 351' \) and \( 650 \) are included in the tensor products

\[
\begin{align*}
27 \otimes 27 &= 27 \oplus 351 \oplus 351', \\
27 \otimes 27 &= 1 \oplus 78 \oplus 650.
\end{align*}
\]

Therefore, they can be represented as rank-two tensors. Their transformation properties are implicitly given by (30) and (31). For \( 351 \) and \( 351' \), we labelled the entries for that tensors \( \chi_1, \ldots, \chi_{351} \) and choose them in a way that

\[
\bar{X}^{\mu\nu} X_{\mu\nu} = \bar{\chi}^1 \chi_1 + \cdots + \bar{\chi}^{351} \chi_{351}, \quad \mu, \nu = 1, \ldots, 27, \tag{35}
\]

ensuring a canonical normalisation of the kinetic term. A comment on this normalisation is given in Appendix B.

\( 351 \) can be represented by a second rank tensor \( A_{\mu\nu} \) antisymmetric in its indices. In the Mathematica package it is included as an \( 27 \times 27 \) dimensional array called E6A.

\( 351' \) is also a rank two tensor \( S_{\mu\nu} \) but symmetric in its indices and additionally satisfying

\[
d^{\mu\nu\lambda} S_{\mu\nu} = 0, \quad \forall \ \lambda = 1, \ldots, 27, \tag{36}
\]

with \( d^{\mu\nu\lambda} \) defined below in eq. (39). \( 351' \) is named E6S in the package.

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\(^2\)The notation for \( 351 \) and \( 351' \) differs in the literature. In our notation, \( 351 \) is symmetric, whereas \( 351' \) is anti-symmetric.
has a fundamental and an anti-fundamental index with vanishing trace \( \psi_\mu^\nu = 0 \) and

\[
k T_\mu^\mu \psi_\mu^\nu = 0. \tag{37}
\]

It is not (yet) included in the package due to its memory usage.

### 3.2 Invariants for the Fundamental Representation

The Kronecker symbol \( \delta_\mu^\nu \) is the most simple way to define a quadratic invariant of a fundamental and an anti-fundamental representation

\[
\delta_\mu^\nu \bar{\psi}_\nu \psi_\mu. \tag{38}
\]

A cubic invariant can be defined in the following way [2]

\[
d_{\mu\nu\lambda} \psi_\mu \psi_\nu \psi_\lambda = \det (L + M + N) - \operatorname{tr}(LMN). \tag{39}
\]

The entries of the tensor \( d_{\mu\nu\lambda} \) can be obtained by comparing the coefficients of the field components on each side of the equation. It is provided as \texttt{E6d} in the \texttt{Mathematica} package. Together with \( d_{\mu\nu\lambda} \), defined by

\[
d_{\mu\nu\lambda} \bar{\psi}_\nu \bar{\psi}_\mu \psi_\lambda = \det (L^\dagger + M^\dagger + N^\dagger) - \operatorname{tr}(N^\dagger M^\dagger L^\dagger), \tag{40}
\]

it is normalised to

\[
d_{\mu\nu\lambda} d_{\mu\nu\sigma} = 10 \delta_\sigma^\lambda. \tag{41}
\]

Additionally there are some compound tensors that are used to construct the invariants in Appendix A. A tensor with four indices is

\[
D^\sigma_\mu = d_{\mu\nu\lambda} d^{\lambda\sigma}. \tag{42}
\]

In the package it is called \texttt{E6D}.

There is also a compound tensor carrying five indices:

\[
D^{\mu\nu,\sigma\tau}_\lambda = (d^{\mu\sigma\xi} d^{\nu\tau\eta} - d^{\mu\tau\xi} d^{\nu\sigma\eta}) d_{\xi\eta\lambda}; \tag{43}
\]

\[
D_{\mu\nu\sigma\tau}^\lambda = (d_{\mu\sigma\xi} d_{\nu\tau\eta} - d_{\mu\tau\xi} d_{\nu\sigma\eta}) d_{\xi\eta\lambda}. \tag{44}
\]

They are not included in the package to avoid excessive memory usage.

### 3.3 Invariants for the Adjoint Representation

The normalisation condition for the generators can be used to define a quadratic invariant for the adjoint representation \( \phi \).

\[
\delta_{kl} \phi_k \phi_l = \frac{1}{3} \operatorname{tr}(kT_l T) \phi_k \phi_l \tag{45}
\]
With the structure constants one can form an invariant of three different adjoint representations \( \phi, \phi', \phi'' \):

\[
f_{klm} \phi_k \phi'_l \phi''_m.
\]

(46)

The completely symmetric tensor

\[
\chi^5_{klmn} = \delta_{kl} \delta_{mn} + \delta_{kl} \delta_{mn} + \delta_{kl} \delta_{mn}
\]

(47)

in the package is called \( \text{E6chi} \) and can be used to form an quartic invariant \( \chi^5_{klmn} \phi_k \phi'_l \phi''_m \phi'''_n \).

### 3.4 Mixed Invariants

The generators \( (kT)_\nu^\mu \) form a tensor \( kT^\nu_\mu \) with an adjoint, a fundamental and an anti-fundamental index.

Further, there is a tensor that can be constructed from the anti-commutator and Kronecker symbols

\[
k_l H^\nu_\mu = \{T_k, T_l\}^\nu_\mu - \frac{2}{9} \delta_{kl} \delta^\nu_\mu,
\]

(48)
called \( \text{E6H} \) in the package.

Contracting \( kT^\nu_\mu \) with \( d_\nu\rho\lambda \) gives the tensor

\[
k_A^\mu\nu,\lambda = kT^\sigma_\mu d_\sigma\nu\lambda - kT^\sigma_\nu d_\mu\sigma\lambda
\]

(49)

which is antisymmetric w.r.t. \( \mu \leftrightarrow \nu \) and the tensor

\[
kS^\mu\nu,\lambda = -kT^\sigma_\lambda d_\mu\nu\sigma
\]

(50)

which is symmetric w.r.t. \( \mu \leftrightarrow \nu \). They are called \( \text{E6kA} \) and \( \text{E6kS} \), respectively.

### 4 E6Tensors.m - A short Manual

#### 4.1 Download and Installation

\texttt{E6Tensors} can be downloaded from \url{http://e6tensors.hepforge.org}. On that page, there are also some instructions on how to install it. Currently, there are two versions at the download section: \texttt{e6tensors_full-1.0.0.tar.gz} and \texttt{e6tensors_small-1.0.0.tar.gz}.

\texttt{e6tensors_small-1.0.0.tar.gz} contains the following files: \texttt{install.sh} calls the command line version of \texttt{Mathematica}® and runs \texttt{create_E6Tensors.m}. This script uses \texttt{E6gen.m} an \texttt{E6d.m} to create the higher dimensional tensors and saves them as arrays to \texttt{E6Tensors.m}. \texttt{examples.nb} shows some well-documented examples how \texttt{E6Tensors.m} can be used. Note that \texttt{E6Tensors.m} has a size of roughly 150 MB. Therefore make sure to provide enough RAM for

\footnote{We assume it to be called \texttt{math}. Change that if it has another name on your system.}
loading it. Running `create_E6Tensors.m` also may need some time. On a quad-core i5 ma-
chine this took about half an hour working on four subkernels. To use parallelisation, change 
`LaunchKernels[1]` in `create_E6Tensors.m` to the appropriate value.

For most users, we recommend to download `e6tensors_full-1.0.0.tar.gz`. After ex-
tracting the tarball, it is ready to use and contains all files of `e6tensors_small-1.0.0.tar.gz` 
including `E6Tensors.m`.

You probably will not need all tensors in a single project. In that case you can comment 
out the unnecessary tensors in `create_E6Tensors.m` and run it to get your own customised file 
`E6Tensors.m`.

### 4.2 Structure of the Package

`E6Tensors.m` is a simple text file. It contains the definitions of all tensors as nested lists. In 
this way, it is very flexible to use: You can write your own functions and procedures that fit 
the problem you want to solve. As an example, the Pauli matrices would look like

\[
\{\{0,1\},\{1,0\}\},\{\{0,-I\},\{I,0\}\},\{\{1,0\},\{0,-1\}\}\}.
\]

The generators `E6gen` have exactly the same structure. Hence,

\[
E6gen[[k,mu,nu]]
\]

is the element in the \(\mu\)th row and the \(\nu\)th column of the \(k\)th generator. All tensors are listed 
in Table 1. There, you can also find their symbolic name, the indices they carry and a short 
explanation.

For instance, `E6gen` has indices \(78, 27, 27\) refering to the gauge index \(k = 1, \ldots, 78\), the row 
index \(\mu = 1, \ldots, 27\) and the column index \(\nu = 1, \ldots, 27\). The order follows the convention 
of the `Part[]` function in Mathematica. Keep in mind that Mathematica does not make any 
difference between row and column vectors.

### 4.3 Known Issues

It is not recommended to open `E6Tensors.m` via the graphical frontend of Mathematica. To 
load it use

\[
\text{Get["\#\#/E6Tensors.m"]}
\]

instead, where `\#\#` refers to the correct path.

### 4.4 Examples

In the download version, there is a notebook `examples.nb` that contains some possible appli-
cations of the package. It starts with loading `E6gen.m` and `E6d.m` which are sufficient for the 
first examples.
| Name in [2] | Name in E6Tensors | Indices | Comment |
|-------------|-------------------|---------|---------|
| $kT_{\mu}^{\nu}$ | E6gen | 78, 27, 27 | adjoint representation 78 |
| | | | $\text{tr} (kT_i T) = 3 \delta_{kl}$ |
| $A_{\mu\nu}$ | E6A | 27, 27 | antisymmetric 351 |
| | | | $27 \times 27$ matrix with entries labelled $\chi_1 \ldots \chi_{351}$ |
| $S_{\mu\nu}$ | E6S | 27, 27 | symmetric 351' |
| | | | $d^{\mu\sigma\nu} S_{\nu\sigma} = 0$ |
| | | | $27 \times 27$ matrix with entries labelled $\chi_1 \ldots \chi_{351}$ |
| $d_{\mu\nu \lambda}$ | E6d | 27, 27, 27 | fully symmetric invariant |
| | | | $d_{\mu\nu \lambda} d^{\mu\nu \sigma} = 10 \delta^\sigma_\lambda$ |
| $f_{k\ell m}$ | E6f | 78, 78, 78 | structure constants: $[kT, iT] = i f_{k\ell m} m T$ |
| | | | $f_{k\ell m} f_{l\ell n} = 12 \delta_{kl}$ |
| $D^{\sigma\tau}_{\mu\nu}$ | E6D | 27, 27, 27, 27 | $D^{\sigma\tau}_{\mu\nu} = \delta^\sigma_\mu \delta^\tau_\nu + \delta^\tau_\mu \delta^\sigma_\nu$ |
| $\chi_{k\ell m n}$ | E6chi | 78, 78, 78, 78 | $\chi_{k\ell m n} = \delta_{k\ell} \delta_{m n} + \delta_{k m} \delta_{\ell n} + \delta_{k n} \delta_{\ell m}$ |
| $k H_{\mu}^{\nu}$ | E6H | 27, 27, 78, 78 | $k H_{\mu}^{\nu} = \{ T_k, T_l \} \nu_{\mu} - \frac{2}{9} \delta_{k\ell} \delta_{\mu}^{\nu}$ |
| $k A_{\mu\nu \lambda}$ | E6kA | 78, 27, 27, 27 | antisymmetric in $\mu, \nu$ |
| | | | $k A_{\mu\nu \lambda} = k T_{\mu \sigma}^{\nu} d_{\sigma \nu \lambda} - k T_{\nu \sigma}^{\mu} d_{\mu \sigma \lambda}$ |
| $k S_{\mu\nu \lambda}$ | E6kS | 78, 27, 27, 27 | symmetric in $\mu, \nu$ |
| | | | $k S_{\mu\nu \lambda} = -k T_{\lambda \sigma}^{\mu} d_{\mu \sigma \nu}$ |

Table 1: Overview of Tensors in E6Tensors.m.
We identify the Standard Model generators among the $E_6$ generators: By construction, we can choose the gluons to be the first eight generators. The generators of $SU(2)_L$ can then be defined as

$$T^9, T^{10}, T^{11},$$

and hypercharge as

$$Y = \sqrt{\frac{3}{5}} \left(-\sqrt{\frac{1}{3}}T^{16} - T^{19} - \sqrt{\frac{1}{3}}T^{24}\right).$$

There are two additional U(1) charges, which we can define by

$$Y' = \sqrt{\frac{1}{40}} \left(-2\sqrt{3}T^{16} - T^{19} + 3\sqrt{3}T^{24}\right),$$
$$Y'' = \sqrt{\frac{1}{40}} \left(-2\sqrt{3}T^{16} + 4T^{19} - 2\sqrt{3}T^{24}\right).$$

In this basis, there is a singlet in the fundamental representation for $Y'$ and $Y''$ each.

The generators for $SU(2)_R$ can be defined as

$$T^{17}, T^{18}, T^{19},$$

and $B - L$ as

$$B - L = -\sqrt{\frac{1}{2}} (T^{16} + T^{24}) = \sqrt{\frac{5}{2}} Y + \sqrt{\frac{3}{2}} T^{19}.$$  

As a first check, we can write the fundamental representation as a list of field names and show their quantum numbers in a table. We also check for the GUT normalisation of the $U(1)$ charges and the correct commutation relation for $SU(2)_L$, $SU(2)_R$ and $Y$.

In a next step, we use $d^{\mu\nu\lambda}$ to write down the trilinear coupling in the superpotential

$$W = \frac{\lambda}{6} d^{\mu\nu\rho}\psi_\mu \psi_\nu \psi_\rho.$$  

For instruction, we once write out the explicit sum over all indices and once use the Dot[] operator. In many cases, the latter one will be the faster way to do it. The same holds for functions like TensorContract[].

For the next examples E6Tenors.m must be located in the same directory. We first test the tensors for the higher dimensional representations 351 and 351', i.e. their defining properties

$$d^{\mu\nu\lambda}S_{\mu\nu} = 0, \quad S_{\mu\nu} = S_{\nu\mu},$$

and

$$A_{\mu\nu} = -A_{\nu\mu}.$$  

The normalisation of the kinetic terms gives the wanted result.

The couplings to matter fields can be described by $W = \bar{S}^{\mu\nu}\psi_\mu \psi_\nu$ and $W = \bar{A}^{\mu\nu}\psi_\mu \psi_\nu$. Now, we can read off the fields that couple to down quarks. Since $A^{\mu\nu}$ is anti-symmetric, it does not
couple fields of the same representation (e.g. the same flavour) to each other.

For a more advanced example, we discuss possible vacuum expectation values (VEVs) for $351' (S^{\mu \nu})$ and $351 (A^{\mu \nu})$. Since they are contained in the tensor product $27 \otimes 27$, we can write an infinitesimal $E_6$ transformation as

$$S^{\mu \nu} \rightarrow (\delta^\mu_\rho + i \alpha_k k T^\rho_\mu) (\delta^\nu_\sigma + i \alpha_k k T^\sigma_\nu) S^{\sigma \rho} = S^{\mu \nu} + i \alpha_k (k T^\sigma_\nu S_{\rho \sigma} + k T^\rho_\mu S_{\rho \nu}). \quad (60)$$

$\alpha_k$ is a set of parameters. For a VEV $s^{\mu \nu} = \langle S^{\mu \nu} \rangle$ that transforms trivially under a set of generators $\{k T\}$, the last term in (60) must vanish. Using the permutation symmetry of $S^{\mu \nu}$, this can be written as a matrix equation

$$(k T \cdot s)^T + k T \cdot s = 0. \quad (61)$$

$A^{\mu \nu}$ is antisymmetric, therefore the condition reads

$$(k T \cdot a)^T - k T \cdot a = 0. \quad (62)$$

The conditions are implemented in `example.nb`. As set of generators we use the gluons and

$$Q = I_L^3 + \sqrt{\frac{5}{3}} Y. \quad (63)$$

This ensures that the vacuum does not carry electric charge or colour. We then calculate the resulting mass terms that are generated by this VEV.

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A Renormalisable Potentials

For completeness, we write down all possible renormalisable and gauge invariant terms that may occur in superpotentials and Lagrangians. They can also be found in [2].
Mass Terms

Possible mass terms for the various representations are:

\[
\begin{align*}
27 \cdot 27 & : \bar{\psi}^\mu \psi_\mu \\
78^2 & : \text{tr}(k_T T) \phi_k \phi'_l = 3 \phi_k \phi'_k \\
351 \cdot 351 & : A^{\mu\nu} A_{\mu\nu} \\
351' \cdot 351' & : \bar{S}^{\mu\nu} S_{\mu\nu}
\end{align*}
\]

Cubic Terms

\[
\begin{align*}
27^3 & : d^{\mu\nu\lambda} \psi_\mu \psi_\nu \psi_\lambda \\
351^{3/2} & : d^{\mu\nu\lambda} d^{\sigma\tau\rho} S_{\mu\nu} S_{\sigma\tau} S_{\lambda\rho} \\
27 \cdot 351' & : \psi_\mu \psi_\nu \bar{S}^{\mu\nu} \\
351^2 \cdot 27 & : D^{\mu,\nu;\sigma} A_{\mu\nu} A_{\sigma\tau} \bar{\psi}^\lambda \\
27 \cdot 351 \cdot 78 & : k T_\mu ^{\nu} d^{\mu\sigma\lambda} A_{\sigma\tau} \phi_k \\
78^3 & : f_{klm} \phi_k \phi'_l \phi''_m \\
27 \cdot 27 \cdot 78 & : k T_\mu ^{\nu} \bar{\psi}^\mu \psi_\nu \phi_k \\
351 \cdot 351 \cdot 78 & : k T_\mu ^{\nu} \bar{A}^{\mu\lambda} A_{\nu\lambda} \phi_k \\
351' \cdot 351' \cdot 78 & : k T_\mu ^{\nu} \bar{S}^{\mu\lambda} S_{\nu\lambda} \phi_k
\end{align*}
\]

Quartic Terms

For the fundamental and adjoint representations, there are

\[
\begin{align*}
27^2 \cdot 27^2 & : \bar{\psi}^\mu \psi_\mu \bar{\psi}^\nu \psi_\nu \\
D^{\mu\nu} \bar{\psi}^\sigma \bar{\psi}^\tau \psi_\mu \psi_\nu & \\
78^4 & : (\phi_1 \phi_l)^2 (\phi_k \phi'_k)^2 \\
& (\phi_1 \phi'_l)^2 (\phi_k \phi'_k)^2 \\
& \phi_k \phi_l \phi_m \phi_n \text{tr}\{k T_1 T\{m T_n T\}} \\
& \chi_{klmn} \phi_k \phi_l \phi_m \phi_n \\
27 \cdot 27 \cdot 78^2 & : \bar{\psi}^\mu \psi_\mu \phi_k \phi_k \\
& (\bar{\psi}^\mu \psi_\nu \phi_k \phi_l)
\end{align*}
\]
Including $351$ and $351'$ gives

$$351^2 - 351^2 : \quad (A_{\mu \nu} \bar{A}^{\mu \nu})^2$$

$$A_{\mu \nu} \bar{A}^{\mu \nu} A_{\rho \tau} \bar{A}^{\rho \tau}$$

$$d^{\mu \nu \lambda} d_{\xi \eta \lambda} A_{\mu \sigma} A_{\nu \tau} \bar{A}^{\xi \sigma} \bar{A}^{\eta \tau}$$

$$d^{\mu \nu \alpha} d^{\rho \sigma \beta} d_{\xi \eta \alpha} d_{\lambda \rho \beta} A_{\mu \sigma} A_{\nu \tau} \bar{A}^{\xi \sigma} \bar{A}^{\eta \tau}$$

$$d^{\mu \nu \alpha} d^{\rho \sigma \beta} d_{\alpha \beta \gamma} d^{\xi \eta \alpha} d_{\xi \eta \sigma} d_{\lambda \rho \beta} A_{\mu \sigma} A_{\nu \tau} \bar{A}^{\xi \sigma} \bar{A}^{\eta \tau}$$

$$351^2 - 351^2 : \quad (A_{\mu \nu} \bar{A}^{\mu \nu})^2$$

$$S_{\mu \nu} \bar{S}^{\mu \nu} S_{\sigma \tau} \bar{S}^{\sigma \tau}$$

$$d^{\mu \nu \lambda} d_{\xi \eta \lambda} S_{\mu \sigma} S_{\nu \tau} \bar{S}^{\xi \sigma} \bar{S}^{\eta \tau}$$

$$d^{\mu \nu \alpha} d^{\rho \sigma \beta} d_{\xi \eta \alpha} d_{\lambda \rho \beta} S_{\mu \sigma} S_{\nu \tau} \bar{S}^{\xi \sigma} \bar{S}^{\eta \tau}$$

$$351 \cdot 351 \cdot 78^2 : \quad \bar{A}^{\mu \nu} A_{\mu \nu} \phi_k \phi_k$$

$$(k_l H_{\mu}^\nu) \bar{A}^{\mu \lambda} A_{\nu \lambda} \phi_k \phi_l$$

$$(k_l H_{\mu}^\nu) d^{\mu \sigma \alpha} d_{\nu \tau \alpha} \bar{A}^{\tau \sigma} A_{\alpha \lambda} \phi_k \phi_l$$

$$(k_l H_{\mu}^\nu) d^{\mu \sigma \alpha} (T_{\nu}^\tau) d_{\nu \rho \alpha} \bar{A}^{\rho \sigma} A_{\alpha \lambda} \phi_k \phi_l$$

$$351' \cdot 351' \cdot 78^2 : \quad \bar{S}^{\mu \nu} S_{\mu \nu} \phi_k \phi_k$$

$$(k_l H_{\mu}^\nu) \bar{S}^{\mu \lambda} S_{\nu \lambda} \phi_k \phi_l$$

$$(k_l H_{\mu}^\nu) d^{\mu \sigma \alpha} d_{\nu \tau \alpha} \bar{S}^{\tau \lambda} S_{\sigma \lambda} \phi_k \phi_l$$

$$27 \cdot 27 \cdot 351 \cdot 351 : \quad \bar{\psi}^{\mu \nu} \bar{\psi}^{\mu} A_{\sigma \tau}$$

$$\bar{\psi}^{\mu \nu} \bar{A}^{\nu \tau} A_{\mu \tau}$$

$$d_{\mu \lambda \gamma} \bar{\psi}^{\mu \nu} \bar{\psi}^{\nu \tau} A_{\sigma \tau}$$

$$27 \cdot 27 \cdot 351' \cdot 351' : \quad \bar{\psi}^{\mu \nu} \bar{\psi}^{\mu} \bar{S}^{\sigma \tau} S_{\sigma \tau}$$

$$\bar{\psi}^{\mu \nu} \bar{\psi}^{\mu} \bar{S}^{\sigma \tau} S_{\mu \tau}$$

$$27^2 351^2 : \quad d^{\mu \sigma \xi} d^{\rho \sigma \tau} \bar{\psi}^{\mu \xi} \bar{\psi}^{\mu} A_{\sigma \tau} A_{\xi \eta}$$

$$d^{\mu \sigma \alpha} d^{\rho \sigma \beta} d_{\alpha \beta \gamma} d^{\xi \eta \alpha} d_{\xi \eta \beta} \bar{\psi}^{\mu \xi} \bar{\psi}^{\mu} A_{\sigma \tau} A_{\xi \eta}$$

$$27^2 351^2 : \quad d^{\mu \sigma \alpha} d^{\rho \sigma \beta} d_{\alpha \beta \gamma} d^{\xi \eta \alpha} d_{\xi \eta \beta} \bar{\psi}^{\mu \xi} \bar{\psi}^{\mu} S_{\sigma \tau} S_{\xi \eta}$$

$$351^3 78 : \quad (k_l T_{\mu}^\nu) d^{\mu \sigma \alpha} d^{\rho \sigma \beta} d_{\alpha \beta \gamma} d^{\xi \eta \alpha} d_{\xi \eta \beta} \bar{\psi}^{\mu \xi} \bar{\psi}^{\mu} A_{\sigma \tau} A_{\xi \eta} \phi_k$$

**B On the Normalisation of 351'**

The symmetric tensor $351'$ ($S_{\mu \nu}$) is defined by

$$S_{\nu \mu} = S_{\mu \nu} \quad \text{and} \quad d^{\mu \nu \lambda} S_{\mu \nu} = 0 \quad \forall \lambda = 1, \ldots, 27. \quad (64)$$

The first condition is easy to construct: We label the off-diagonal entries $\phi_1, \ldots, \phi_{351}$ and the diagonal ones $\phi_{352}, \ldots, \phi_{378}$. The second condition then eliminates 27 entries. For $\lambda = 1$, it
reads
\[ \phi_{122} + \phi_{207} + \phi_{226} + \phi_{244} - \phi_{102} = 0. \]  
(65)

It is now tempting to solve e.g. for \( \phi_{102} \) and substitute that in \( S_{\mu\nu} \). But then the kinetic term \( \partial^\alpha S^{\mu\nu} \partial_\alpha S_{\mu\nu} \) is not canonically normalised.\(^4\)

Another solution is to introduce new field names \( \psi_1, \psi_2, \psi_3, \psi_4 \) with

\[
\begin{align*}
\phi_{102} &= a(\psi_1 + \psi_2 + \psi_3 + \psi_4) \\
\phi_{122} &= \psi_1 - b(\psi_1 + \psi_2 + \psi_3 + \psi_4) \\
\phi_{207} &= \psi_2 - b(\psi_1 + \psi_2 + \psi_3 + \psi_4) \\
\phi_{226} &= \psi_3 - b(\psi_1 + \psi_2 + \psi_3 + \psi_4) \\
\phi_{244} &= \psi_4 - b(\psi_1 + \psi_2 + \psi_3 + \psi_4)
\end{align*}
\]

For \( a = 1 - 4b \) and \( b = (5 + \sqrt{5})/20 \), the defining condition is fulfilled and the kinetic term for \( S^{\mu\nu} \) takes the form

\[ \partial_\alpha \tilde{S}^{\mu\nu} \partial^\alpha S_{\mu\nu} = \cdots + \partial_\alpha \tilde{\psi}_1 \partial^\alpha \psi_1 + \partial_\alpha \tilde{\psi}_2 \partial^\alpha \psi_2 + \partial_\alpha \tilde{\psi}_3 \partial^\alpha \psi_3 + \partial_\alpha \tilde{\psi}_4 \partial^\alpha \psi_4 + \cdots \]  
(71)

The same procedure also works for all other values of \( \lambda \). It is important, that the component with the relative minus sign (\( \phi_{102} \) in eq. 65) is replaced by the expression with \( a \) in it. This procedure is implemented in \texttt{createE6Tensors.m} and used to construct \texttt{E6S}.

**References**

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\(^4\)\( \alpha \) is a space-time index in this case.