Genuine Four Tangle for Four Qubit States

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Abstract. We report a four qubit polynomial invariant that quantifies genuine four-body correlations. The four qubit invariants are obtained from transformation properties of three qubit invariants under a local unitary on the fourth qubit.

Keywords: Four tangle, Polynomial invariants, multipartite entanglement

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Two multipartite pure states are equivalent under stochastic local operations and classical communication (SLOCC) [1] if one can be obtained from the other with some probability using SLOCC. Attempts [2, 3] to classify four-qubit pure states under SLOCC, have revealed that several entanglement classes contain a continuous range of strictly nonequivalent states, although with similar structure. In view of this, we proposed classification criteria [4] based on nature of multiqubit correlations in N-qubit pure states. In this article, we examine the three qubit invariants of four qubit states and derive higher degree invariants to quantify four and three-way correlations.

For a two qubit state, negative eigenvalue of partially transposed state operator is the invariant that distinguishes between a separable and an entangled state. In three qubit state space, two qubit subspace (for a selected pair of qubits) is characterized by a pair of two qubit invariants, while new two qubit invariants arise due to three body correlations in the composite space. The most important three qubit polynomial invariant is a degree four combination of two qubit invariants. The entanglement monotone constructed from this is Wootter’s three tangle [5]. Four qubit states sit in the space $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ with three qubit subspaces for each set of three qubits. If there were no four body correlations, then three tangles should determine the entanglement of a four qubit state. When four body correlations are present, additional three qubit invariants that depend on four way negativity fonts [6] exist. Three qubit invariants, for a given set of three qubits, constitute a five dimensional space. In this article, we obtain four qubit invariants using transformation properties of three qubit invariants under a local unitary applied to the fourth qubit. One can continue the process to a higher number of qubits.

FIVE THREE TANGLES

Two qubit unitary invariants for pair of qubits $A_1A_2$ in the most general four qubit state

$$|\Psi^{A_1A_2A_3A_4}\rangle = \sum_{i_1i_2i_3i_4} a_{i_1i_2i_3i_4} |i_1i_2i_3i_4\rangle; \quad (i_m = 0, 1),$$

(1)
are $D_{(A_3)_{i_3}}^{00} (A_4)_{i_4}$, $D_{(A_3)_{i_3}}^{00i_4} (A_4)_{i_4} - D_{(A_3)_{i_3}}^{01i_4} (A_4)_{i_4}$, $D_{(A_4)_{i_4}}^{00i_4} - D_{(A_4)_{i_4}}^{01i_3}$, $D_{(A_4)_{i_4}}^{0000} - D_{(A_4)_{i_4}}^{0010}$, $D_{(A_4)_{i_4}}^{0001} - D_{(A_4)_{i_4}}^{0101}$, where

$$D_{(A_3)_{i_3}}^{00} (A_4)_{i_4} = \text{det} \begin{bmatrix} a_{0i_3i_4} & a_{01i_3i_4} \\ a_{10i_3i_4} & a_{11i_3i_4} \end{bmatrix}, D_{(A_3)_{i_3}}^{0i_2i_4} = \text{det} \begin{bmatrix} a_{0i_2i_4} & a_{01i_2i_4} \\ a_{1i_2i_4} & a_{11i_2i_4} \end{bmatrix}, \tag{2}$$

$$D_{(A_3)_{i_3}}^{0i_2i_0} = \text{det} \begin{bmatrix} a_{0i_2i_0} & a_{0i_2+1i_0} \\ a_{1i_2i_0} & a_{1i_2+1i_0} \end{bmatrix}, D_{(A_4)_{i_4}}^{0i_2i_0} = \text{det} \begin{bmatrix} a_{0i_2i_4} & a_{0i_2+1i_4} \\ a_{1i_2i_4} & a_{1i_2+1i_4} \end{bmatrix}. \tag{3}$$

For qubits $A_1A_2A_3$ in $|\Psi^{A_1A_2A_3}\rangle$, three qubit invariants

$$\left( I_{3}^{A_1A_2A_3} \right)_{(A_4)_{i_4}} = \left( D_{(A_4)_{i_4}}^{00} (A_4)_{i_4} - D_{(A_4)_{i_4}}^{010} (A_4)_{i_4} \right)^2 - 4D_{(A_3)_{i_3}}^{00} (A_4)_{i_4} D_{(A_3)_{i_3}}^{00} (A_4)_{i_4} ; \quad i_4 = 0, 1. \tag{4}$$

quantify GHZ state like three-way correlations in three qubit state space. We examine the action of $U^{A_4} = \frac{1}{\sqrt{1+|y|^4}} \begin{bmatrix} 1 & -y^* \\ y & 1 \end{bmatrix}$ on invariant $\left( I_{3}^{A_1A_2A_3} \right)_{(A_4)_{i_4}}$. The transformed invariant is a combination of five three qubit invariants that is

$$\left( I_{3}^{A_1A_2A_3} \right)_{(A_4)_{i_4}}' = \frac{1}{(1+|y|^4)} \left[ (y^*)^4 \left( I_{3}^{A_1A_2A_3} \right)_{(A_4)_{i_4}} - 4(y^*)^3 P_{(A_4)_{i_4}}^{A_1A_2A_3} \\ + 6(y^*)^2 T_{A_4}^{A_1A_2A_3} - 4y^* P_{(A_4)_{i_4}}^{A_1A_2A_3} + \left( I_{3}^{A_1A_2A_3} \right)_{(A_4)_{i_4}} \right]. \tag{5}$$

Here prime denotes the transformed invariant and additional invariants are

$$T_{A_4}^{A_1A_2A_3} = \frac{1}{6} \left( D_{(A_4)_{i_4}}^{0000} + D_{(A_4)_{i_4}}^{0001} + D_{(A_4)_{i_4}}^{0010} + D_{(A_4)_{i_4}}^{0011} \right)^2 - \frac{2}{3} \left( D_{(A_3)_{i_3}}^{0000} (A_4)_{i_4} + D_{(A_3)_{i_3}}^{0010} (A_4)_{i_4} \right) \left( D_{(A_3)_{i_3}}^{0000} (A_4)_{i_4} + D_{(A_3)_{i_3}}^{0010} (A_4)_{i_4} \right)$$

$$+ \frac{1}{3} \left( D_{(A_4)_{i_4}}^{0000} (A_4)_{i_4} + D_{(A_4)_{i_4}}^{0010} (A_4)_{i_4} \right) \left( D_{(A_4)_{i_4}}^{0000} (A_4)_{i_4} + D_{(A_4)_{i_4}}^{0010} (A_4)_{i_4} \right)$$

$$- \frac{2}{3} \left( D_{(A_3)_{i_3}}^{0000} (A_4)_{i_4} D_{(A_3)_{i_3}}^{0010} (A_4)_{i_4} + D_{(A_3)_{i_3}}^{0000} (A_4)_{i_4} D_{(A_3)_{i_3}}^{0010} (A_4)_{i_4} \right), \tag{6}$$

$$P_{(A_4)_{i_4}}^{A_1A_2A_3} = \frac{1}{2} \left( D_{(A_4)_{i_4}}^{0000} (A_4)_{i_4} + D_{(A_4)_{i_4}}^{0010} (A_4)_{i_4} \right) \left( D_{(A_4)_{i_4}}^{0000} + D_{(A_4)_{i_4}}^{0010} + D_{(A_4)_{i_4}}^{0011} \right)$$

$$- \left( D_{(A_3)_{i_3}}^{0000} (A_4)_{i_4} \left( D_{(A_3)_{i_3}}^{0000} + D_{(A_3)_{i_3}}^{0010} \right) + D_{(A_3)_{i_3}}^{0000} (A_4)_{i_4} \left( D_{(A_3)_{i_3}}^{0000} + D_{(A_3)_{i_3}}^{0011} \right) \right). \tag{7}$$

Five three tangles, constructed from invariants $\left( I_{3}^{A_1A_2A_3} \right)_{(A_4)_{i_4}}$, $\left( I_{3}^{A_1A_2A_3} \right)_{(A_4)_{i_4}}$, $P_{(A_4)_{i_4}}^{A_1A_2A_3}$, and $T_{A_4}^{A_1A_2A_3}$, capture the entanglement of $A_1A_2A_3$ due to three and four-way correlations.
GENUINE FOUR TANGLE

Continuing the search for a four qubit invariant that detects genuine four-way correlations, we notice that when a selected $U^{A_4}$ results in $\left(I_3^{A_1A_2A_3}\right)'(A_4)_0 = 0$, we have at hand a quartic equation. A quartic equation, $y^4a - 4by^3 + 6y^2c - 4dy + f = 0$, in variable $y$ has associated polynomial invariants $S = af - 4bd + 3c^2$, cubic invariant $T = acf - ad^2 - b^2f + 2bcd - c^3$, and discriminant $\Delta = S^3 - 27T^2$. Therefore, the degree eight polynomial invariant associated with $I_3^{A_1A_2A_3}(A_4)_0 = 0$ is

$$I_{(4,8)}^{A_1A_2A_3A_4} = 3 \left(T_{A_4}^{A_1A_2A_3}\right)^2 + \left(I_3^{A_1A_2A_3}(A_4)_0 \right)^2 - 4P^{A_1A_2A_3}(A_4)_0 P^{A_1A_2A_3}(A_4)_1. \tag{8}$$

The discriminant is given by $\Delta = \left(I_4^{A_1A_2A_3A_4}\right)^3 - 27 \left(J^{A_1A_2A_3A_4}\right)^2$, where

$$J^{A_1A_2A_3A_4} = \det \begin{bmatrix} I_3^{A_1A_2A_3}(A_4)_0 & P^{A_1A_2A_3}(A_4)_1 & T_{A_4}^{A_1A_2A_3} \\ P^{A_1A_2A_3}(A_4)_1 & T_{A_4}^{A_1A_2A_3} & P^{A_1A_2A_3}(A_4)_0 \\ T_{A_4}^{A_1A_2A_3} & P^{A_1A_2A_3}(A_4)_0 & I_3^{A_1A_2A_3}(A_4)_0 \end{bmatrix}. \tag{9}$$

We may mention here that since there are four ways in which a given set of three qubits may be selected, $\Delta$ can be expressed in terms of different sets of three qubit invariants.

The four tangle $\tau_{(4,8)} = 4 \left| 2I_{(4,8)}^{A_1A_2A_3A_4}\right|^2$ quantifies 4-way correlations [7]. If four tangle is zero then transformation equations acquire a simpler form and yield four qubit invariants that quantify 3-way correlations. Invariant to quantify entanglement of a four qubit state having purely two qubit correlations can also be easily obtained. What is the utility of these polynomial invariants? Quantum entanglement distributed between distinct parties is a physical resource for practical quantum information processing. Polynomial invariants are used to construct entanglement monotones to quantify entanglement.

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