Liouville-type theorem for Kirchhoff equations involving Grushin operators

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Abstract

The aim of this paper is to prove the Liouville-type theorem of the following weighted Kirchhoff equations:

\[
-M \left( \int_{\mathbb{R}^N} \omega(z) |\nabla G(u)|^2 \, dz \right) \text{div}_G (\omega(z) \nabla G(u)) = f(z) e^\alpha, \quad z = (x, y) \in \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}
\]

(0.1)

and

\[
M \left( \int_{\mathbb{R}^N} \omega(z) |\nabla G(u)|^2 \, dz \right) \text{div}_G (\omega(z) \nabla G(u)) = f(z) u^q,
\]

(0.2)

where \(M(t) = a + bt^k\), \(t \geq 0\), with \(a > 0\), \(b, k \geq 0\), \(k = 0\) if and only if \(b = 0\), \(q > 0\) and \(\omega(z), f(z) \in L^1_{\text{loc}}(\mathbb{R}^N)\) are nonnegative functions satisfying \(\omega(z) \leq C_1 \|z\|_G^d\) and \(f(z) \geq C_2 \|z\|_G^d\) as \(\|z\|_G \geq R_0\) with \(d > \theta - 2\), \(R_0\), \(C_i (i = 1, 2)\) are some positive constants, here \(\alpha \geq 0\) and \(\|z\|_G = (|x|^{\theta_1 + \alpha} + |y|^{\theta_2 + \alpha})^{1/(\theta_1 + \alpha)}\) is the norm corresponding to the Grushin distance. \(N_\alpha = N_1 + (1 + \alpha)N_2\) is the homogeneous dimension of \(\mathbb{R}^N\), \text{div}_G\) (resp., \(\nabla G\)) is Grushin divergence (resp., Grushin gradient). Under suitable assumptions on \(k, \theta, d,\) and \(N_\alpha\), the nonexistence of stable weak solutions to equations (0.1) and (0.2) is investigated.

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1 Introduction and main result

In this paper, we study the nonexistence of stable weak solutions for the weighted Kirchhoff equations

\[
-M \left( \int_{\mathbb{R}^N} \omega(z) |\nabla G(u)|^2 \, dz \right) \text{div}_G (\omega(z) \nabla G(u)) = f(z) e^\alpha, \quad z = (x, y) \in \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}
\]

(1.1)
and

\[
M \left( \int_{\mathbb{R}^N} \omega(z) |\nabla_G u|^2 \, dz \right) \text{div}_G (\omega(z) \nabla_G u) = f(z) u^q, \\
z = (x, y) \in \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2},
\]

where \( M(t) = a + bt^k, \ t \geq 0, \) with \( a > 0, b, k \geq 0, k = 0 \) if and only if \( b = 0. q > 0 \) and \( \omega(z), f(z) \in L^1_{\text{loc}}(\mathbb{R}^N) \) are nonnegative functions verifying \( \omega(z) \leq C_1 \| z \|_G^\theta \) and \( f(z) \geq C_2 \| z \|_G^\theta \) as \( \| z \|_G \geq R_0 \) with \( d > \theta - 2 \), where \( R_0, C_i (i = 1, 2) \) are some positive constants. Here \( \alpha \geq 0 \) and

\[
\| z \|_G = \left( |x|^{2(1+\alpha)} + |y|^2 \right)^{\frac{1}{1+\alpha}}, \ z = (x, y) \in \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}
\]

is the norm corresponding to the Grushin distance, where \( |x| \) and \( |y| \) are the usual Euclidean norms in \( \mathbb{R}^{N_1} \) and \( \mathbb{R}^{N_2} \), respectively.

Set \( \nabla_x \) and \( \nabla_y \) as Euclidean gradients with respect to the variables \( x \in \mathbb{R}^{N_1} \) and \( y \in \mathbb{R}^{N_2} \), respectively. The Grushin gradient is defined by

\[
\nabla_G = (\nabla_x, (1+\alpha)|x|^{\alpha} \nabla_y).
\]

Moreover, we define

\[
\text{div}_G (g, h) = \sum_{i=1}^{N_1} \frac{\partial g_i}{\partial x_i} + (1+\alpha)|x|^{\alpha} \sum_{j=1}^{N_2} \frac{\partial h_j}{\partial y_j}
= \text{div}_x g + (1+\alpha)|x|^{\alpha} \text{div}_y h, \quad (g, h) \in C^1(\mathbb{R}^N, \mathbb{R}^{N_1} \times \mathbb{R}^{N_2})
\]

as the Grushin divergence. Then the Grushin operator is given by

\[
\Delta_G u = \text{div}_G (\nabla_G u) = \Delta_x u + (1+\alpha)^2|x|^{2\alpha} \Delta_y u,
\]

where \( \Delta_x \) and \( \Delta_y \) represent the usual Laplacians on \( \mathbb{R}^{N_1} \) and \( \mathbb{R}^{N_2} \) respectively. This operator is uniformly elliptic for \( x \neq 0 \) and degenerate when \( x = (x_1, x_2, \ldots, x_{N_1}) \) goes to 0.

The anisotropic dilation attached to \( \Delta_G \) is defined by

\[
\delta_\lambda(z) = (\lambda x, \lambda^{1+\alpha} y), \quad \lambda > 0, z = (x, y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}.
\]

It is not hard to see that

\[
d\delta_\lambda(z) = \lambda^{N_0} \, dx \, dy = \lambda^{N_0} \, dz,
\]

where

\[
N_0 = N_1 + (1+\alpha)N_2
\]

is usually called the homogeneous dimension of \( \mathbb{R}^N \), \( dx \, dy \) denotes the Lebesgue measure in \( \mathbb{R}^N \). For more details about Grushin operators and their basic properties, we refer the reader for instance to [34].
In the case \( \alpha = 0 \) and \( \omega(z) \equiv 1 \), problems (1.1) and (1.2) are related to the stationary analogue of the following Kirchhoff model:

\[
\frac{\partial^2 u}{\partial t^2} - \left( a + b \int_{\mathbb{R}^N} |\nabla u|^2 \, dz \right) \Delta u = h(z, u),
\]

which was proposed by Kirchhoff in 1883 as a generalization of the well-known D’Alembert wave equation

\[
\rho \frac{\partial^2 u}{\partial t^2} - \left( \frac{p_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial z} \right|^2 \, dz \right) \frac{\partial^2 u}{\partial z^2} = h(z, u)
\]

for free vibrations of elastic strings, see [26], where \( \rho, p_0, h, E, L \) are constants which represent some physical meanings respectively. Indeed, Kirchhoff’s model considers the changes in length of the string produced by transverse vibrations. Up to now, a great attention has been paid to the study of the Kirchhoff-type problems involving nonlocal operators, because nonlinear equations with nonlocal operators have a broad application background and play an important role in physics, probability, biology, finance, etc. With the help of variational calculus, some important and interesting results for this direction, especially those concerning the existence and multiplicity of solutions, have been established, we refer the interested reader to [17–19, 24, 31, 37] and the references therein.

On the other hand, the nonexistence and stability of solutions to nonlinear elliptic equations have drawn much attention in the last decades. For some physical motivation and recent development on the topic of stable solutions, we refer to [13]. Also, see [2, 33] for related problems.

The motivation of writing this article is to prove a Liouville-type theorem for stable solutions of equations (1.1) and (1.2). We recall that Liouville-type theorem focuses on the nonexistence of nontrivial solution in the entire space \( \mathbb{R}^N \). In 1981, in their pioneering article [22], Gidas and Spruck established the optimal Liouville-type result for positive solutions to the equation

\[
-\Delta u = |u|^{q-1}u \quad \text{in} \quad \mathbb{R}^N.
\]  

They proved that (1.3) has no positive solution if and only if \( 1 < q < q_s = \frac{N+2}{N-2} (= \infty) \) if \( N = 2 \). Farina [14, 16] also considered problem (1.3). He proved that there is no nontrivial stable solution if \( 1 < q < q_c(N) \), where \( q_c(N) \) is explicitly given and is always greater than the classical critical exponent \( \frac{N+2}{N-2} \). It is worth pointing out that his proof makes a delicate application of the classical Moser iterative method. Later, these results were extended to the quasilinear case \(-\Delta_p u = |u|^{q-1}u \) in [7] and the weighted quasilinear case \(-\Delta_p u = f(z)|u|^{q-1}u \) in [3].

Obviously, equation (1.1) becomes the following Laplace equation with exponential nonlinearity:

\[
-\Delta u = e^u \quad \text{in} \quad \mathbb{R}^N
\]

for the case \( M(t) \equiv 1, \alpha = 0, \) and \( \omega(z) \equiv 1 \equiv f(z) \). Problem (1.4) has been studied by several experts; for example, Farina [15] proved that all stable \( C^2 \) solutions of (1.4) must be
zero if $2 \leq N \leq 9$; Dancer and Farina [9] proved that equation (1.4) admits classical entire solutions which are stable outside a compact set of $\mathbb{R}^N$ if and only if $N \geq 10$. Later, Wang and Ye [38] proved the following theorem.

**Theorem 1.1** Let $\alpha = 0, M(t) \equiv 1 \equiv \omega(z)$, and $f(z) = |z|^d$ with $d > -2$. If $2 \leq N < 10 + 4d$, then equation (1.1) admits no stable weak solution.

For the case of negative exponent nonlinearity, the authors [32] obtained the following.

**Theorem 1.2** Let $q > 0, M(t) \equiv 1, \alpha = 0, and \omega(z) \equiv 1 \equiv f(z)$ in (1.2). If

$$2 \leq N < 2 + \frac{4}{1 + q} \left( q + \sqrt{q(q + 1)} \right),$$

then there are no positive stable solutions to (1.2) in $\mathbb{R}^N$.

**Remark 1.3** It is clear that if $2 < N < 10$, then from the above inequality we have

$$q > p_0 := \frac{N^2 - 8N + 4 + 8\sqrt{N - 1}}{(N - 2)(10 - N)}.$$ 

Recently, Du and Guo [10] studied equation (1.2) with $\alpha = 0, M(t) \equiv 1 \equiv \omega(z)$ and $f(z) = |z|^d$ with $d > -2$. It was proved that there is no positive stable solution provided $2 \leq N < 10 + 4d$ and $q > p_c(d)$ hold, where $p_c(d)$ is a critical exponent depending on $d$ and $N$.

Very recently, by Farina’s approach, Cowan and Fazly [6] established the nonexistence of nontrivial stable solution of the weighted elliptic equation

$$-\text{div} \left( \omega_i(z) \nabla u \right) = \omega_2(z)g(u) \quad \text{in} \ \mathbb{R}^N$$

with positive smooth weights $\omega_i(z), i = 1, 2$, where the nonlinearity $g(u) = e^u, |u|^{p-1}u$ with $p > 1$ and $-u^{-p}$ with $p > 0$. After that, these results were extended to the quasilinear case $-\Delta_p u = f(z)g(u)$ in [4, 27] and the weighted quasilinear case $-\text{div}(\omega(z)|\nabla u|^{p-2}\nabla u) = f(z)g(u)$ in [28, 29], where $g(u) = e^u$ or $g(u) = -u^{-q}, q > 0$. Similar works can be found in [5, 21, 23, 25, 41].

We now turn to the case where $\alpha > 0$, equations (1.1) and (1.2) become nonlinear elliptic equations involving Grushin operator. It is well known that the Grushin operator belongs to the wide class of subelliptic operators studied by Franchi et al. [20] (also see [1]). The Liouville-type theorem has been recently proved by Monticelli [35] for nonnegative classical solutions and by Yu [40] for nonnegative weak solutions of the problem

$$-\Delta_G u = u^r \quad \text{in} \ \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}.$$ 

The optimal exponent is $r < \frac{N_1 + 1}{N_2 - 1}$, where $N_a = N_1 + (1 + \alpha)N_2$ is the homogeneous dimension. The main tool they used [35, 40] is the Kelvin transform combined with the moving planes technique. On the other hand, Monti and Morbidelli [34] considered the classification results for equation

$$-\Delta_G u = u^{\frac{N_1 + 1}{N_2 - 1}} \quad \text{in} \ \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}.$$
Based on this reality, we define it natural to study weak solutions of (1.1) and (1.2) in a suitable weighted Sobolev space.

In the present paper, we try to establish the Liouville property for the class of stable weak solutions of (1.1) and (1.2).

Since solutions to elliptic equations with Hardy potentials may possess singularities, it is natural to study weak solutions of (1.1) and (1.2) in a suitable weighted Sobolev space. Based on this reality, we define

\[ \| \varphi \|_{\omega} = \left( \int_{\mathbb{R}^N} \omega(z) |\nabla_G \varphi|^2 \, dz \right)^{1/2} \]

for \( \varphi \in C_c^\infty(\mathbb{R}^N) \) and denote by \( H_1^1(\mathbb{R}^N, \omega) \) the closure of \( C_c^\infty(\mathbb{R}^N) \) with respect to the \( \| \cdot \|_{\omega} \)-norm. Note that, for \( \omega(z) \in L^1_{\text{loc}}(\mathbb{R}^N) \), we have \( C_c^1(\mathbb{R}^N) \subset H_1^1(\mathbb{R}^N, \omega) \). Denote also by \( H^1_{\text{loc}}(\mathbb{R}^N, \omega) \) the space of all functions \( u \) such that \( u\varphi \in H_1^1(\mathbb{R}^N, \omega) \) for all \( \varphi \in C_c^1(\mathbb{R}^N) \). Here and in the following \( C^k(\mathbb{R}^N) \) denotes the set of \( C^k \) functions with compact support in \( \mathbb{R}^N \).

To facilitate the writing, we unify equations (1.1) and (1.2) into the following equation:

\[
-zM\left( \int_{\mathbb{R}^N} \omega(z) |\nabla_G u|^2 \, dz \right) \text{div}_G(\omega(z) \nabla_G u) = f(z)g(u),
\]

\[
z = (x, y) \in \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2},
\]

where \( g(u) = e^u \) or \( g(u) = -u^{-q} \).

**Definition 1.4** Let \( X = H^1_{\text{loc}}(\mathbb{R}^N, \omega) \cap H^1_{\text{loc}}(\mathbb{R}^N, \omega) \), we say that \( u \in X \) is a weak solution of (1.7) if \( f(z)g(u) \in L^1_{\text{loc}}(\mathbb{R}^N) \) and

\[
(a + b\| u \|_{\omega}^{2k}) \int_{\mathbb{R}^N} \omega(z) \nabla_G u \cdot \nabla_G \varphi \, dz = \int_{\mathbb{R}^N} f(z)g(u)\varphi \, dz, \quad \forall \varphi \in C_c^1(\mathbb{R}^N),
\]

where \( g(u) = e^u \) or \( g(u) = -u^{-q} \).

The energy functional \( \mathcal{J} : X \to \mathbb{R} \) corresponding to (1.7) is

\[
\mathcal{J}(u) = \frac{a}{2} \| u \|_{\omega}^2 + \frac{b}{2(k + 1)} \| u \|_{\omega}^{2(k+1)} - \int_{\mathbb{R}^N} f(z)G(u) \, dz,
\]

where \( G(u) = \int_0^u g(s) \, ds \).

Obviously, if \( u \in X \) is a weak solution of (1.7), then for any \( \varphi \in C_c^1(\mathbb{R}^N) \), the function \( E(t) := \mathcal{J}(u + t\varphi) \) satisfies \( E'(0) = 0 \). As in [7], we say that the solution \( u \) of (1.7) is stable if \( E''(0) \geq 0 \). Now, we compute \( E''(0) \). Observe that

\[
\frac{E'(t) - E'(0)}{t} = \frac{\mathcal{J}'(u + t\varphi)\varphi - \mathcal{J}'(u)\varphi}{t}
\]
\[ E''(0) = \lim_{t \to 0} \frac{E'(t) - E'(0)}{t} = 2bk\|u\|_{w_2}^{2(k-1)} \left( \int_{\mathbb{R}^N} \omega(z)\nabla G u \cdot \nabla G \varphi \, dz \right)^2 + (a + b\|u\|_{w_2}) \int_{\mathbb{R}^N} \omega(z)|\nabla G \varphi|^2 \, dz \]

we obtain

\[ E''(0) = \lim_{t \to 0} \frac{E'(t) - E'(0)}{t} = 2bk\|u\|_{w_2}^{2(k-1)} \left( \int_{\mathbb{R}^N} \omega(z)\nabla G u \cdot \nabla G \varphi \, dz \right)^2 + (a + b\|u\|_{w_2}) \int_{\mathbb{R}^N} \omega(z)|\nabla G \varphi|^2 \, dz \]

We are ready to state the stability as follows.

**Definition 1.5** We say that a weak solution \( u \) of (1.7) is stable if \( f(z)g'(u) \in L^1_{\text{loc}}(\mathbb{R}^N) \) and

\[ 2bk\|u\|_{w_2}^{2(k-1)} \left( \int_{\mathbb{R}^N} \omega(z)\nabla G u \cdot \nabla G \varphi \, dz \right)^2 + (a + b\|u\|_{w_2}) \int_{\mathbb{R}^N} \omega(z)|\nabla G \varphi|^2 \, dz \]

\[ \geq \int_{\mathbb{R}^N} f(z)g'(u)\varphi^2 \, dz \]  

(1.9)

for every \( \varphi \in C^1_c(\mathbb{R}^N) \), where \( g(u) = e^u \) or \( g(u) = -u^{-q} \).

**Remark 1.6** If \( u \) is a stable weak solution of (1.1), in view of (1.9) with \( g(u) = e^u \), it can be deduced that

\[ \int_{\mathbb{R}^N} f(z)e^u \varphi^2 \, dz \leq A \int_{\mathbb{R}^N} \omega(z)|\nabla G \varphi|^2 \, dz, \quad \forall \varphi \in C^1_c(\mathbb{R}^N) \]  

(1.10)

with

\[ A = a + b(1 + 2k)\|u\|_{w_2}^{2k}. \]  

(1.11)

Similarly, if \( u \) is a positive stable weak solution of (1.2), by virtue of (1.9) with \( g(u) = -u^{-q} \), it follows that

\[ q \int_{\mathbb{R}^N} f(z)u^{-(1+q)} \varphi^2 \, dz \leq A \int_{\mathbb{R}^N} \omega(z)|\nabla G \varphi|^2 \, dz, \quad \forall \varphi \in C^1_c(\mathbb{R}^N), \]  

(1.12)

where \( A \) is given in (1.11). Note that (1.8)–(1.10) and (1.12) hold for all \( \varphi \in H^1_0(\mathbb{R}^N, \omega) \) by density arguments.

Throughout this paper, the functions \( \omega(z), f(z) \) satisfy the following assumptions:

(H1) \( \omega(z), f(z) \in L^1_{\text{loc}}(\mathbb{R}^N) \) are nonnegative functions. In addition, there exist \( d > \theta - 2 \), \( C_i > 0 \) \((i = 1, 2)\) and \( R_0 > 0 \) such that

\[ \omega(z) \leq C_1\|z\|_{G}^\theta, \quad f(z) \geq C_2\|z\|_{G}^\theta, \quad \forall \|z\|_{G} \geq R_0. \]

To simplify the notations, we denote

\[ \mu_0(k, \theta, d) = 2 - \theta + \frac{4(2 - \theta + d)}{1 + 2k}, \quad \mu_1(k, \theta, d) = 2 - \theta + \frac{1 + \sqrt{1 + 2k}}{2k}(2 - \theta + d). \]
Our results can be stated as follows.

**Theorem 1.7** Let \( M(t) = a + bt^k, t \geq 0, a > 0, b, k \geq 0, k = 0 \) if and only if \( b = 0 \). Suppose that (H1) and \( N_a < \mu_k(k, \theta, d) \) hold. Then there is no stable weak solution \( u \in X \) to (1.1).

**Remark 1.8** If \( \alpha = \theta = 0 \) and \( k = 1 \), then the result in Theorem 1.7 coincides with that in [39]. If \( \alpha = k = 0 \), we get a result similar to that in [6, 23]. If \( \alpha = k = \theta = 0 \) we derive the result in [38]. Finally, if \( \alpha = k = \theta = d = 0 \), we have the Liouville theorem in the pioneering article [15].

**Theorem 1.9** Let \( M(t) = a + bt^k, t \geq 0, a > 0, b, k \geq 0, k = 0 \) if and only if \( b = 0 \). Assume that (H1) holds. Then (1.2) has no positive stable weak solution \( u \in X \) provided that one of the following conditions is satisfied:

\[
(H_2) \ k \geq 0 \text{ and } N_a \leq 2 - \theta, q > 0;
\]

\[
(H_3) \ 0 \leq q < \frac{4}{3} \text{ and } 2 - \theta < N_a < \mu_k(k, \theta, d), q > q_c;
\]

\[
(H_4) \ k > \frac{2}{3} \text{ and } 2 - \theta < N_a < \mu_k(k, \theta, d), q > \tilde{q}_c;
\]

\[
(H_5) \ k > \frac{2}{3} \text{ and } N_a = \mu_k(k, \theta, d), q > \frac{4}{N - 3};
\]

\[
(H_6) \ k > \frac{2}{3} \text{ and } \mu_k(k, \theta, d) < N_a < \mu_1(k, \theta, d), q_1 < q < q_2,
\]

where

\[
q_c = -1 - \frac{2(2 - \theta + d)[N_a - 4 + 2\theta - d + \sqrt{(N_a + d)^2 - (N_a - 2 + \theta)^2(1 + 2k)}]}{(N_a - 2 + \theta)(1 + 2k)(N_a - \mu_k(k, \theta, d))};
\]

\[
\tilde{q}_c = -1 - \frac{2(2 - \theta + d)[N_a - 4 + 2\theta - d + \sqrt{(N_a + d)^2 - (N_a - 2 + \theta)^2(1 + 2k)}]}{(N_a - 2 + \theta)(1 + 2k)(N_a - \mu_k(k, \theta, d))};
\]

\[
q_{1,2} = -1 - \frac{2(2 - \theta + d)[N_a - 4 + 2\theta - d \pm \sqrt{(N_a + d)^2 - (N_a - 2 + \theta)^2(1 + 2k)}]}{(N_a - 2 + \theta)(1 + 2k)(N_a - \mu_k(k, \theta, d))}.
\]

**Remark 1.10** If \( \alpha = k = \theta = 0 \), we obtain

\[
q_c = \frac{2(2 + d)^2 + 2(N - 2)(2 + d) - (N - 2)^2 - 2(2 + d)\sqrt{(2 + d)(d + 2N - 2)}}{(N - 2)(N - 10 - 4d)},
\]

which equals the critical exponent \( p_c \) in [10]. If \( \alpha = k = \theta = d = 0 \), we find

\[
q_c = \frac{N^2 - 8N + 4 + 8\sqrt{N - 1}}{N - 2)(10 - N)},
\]

which is the critical exponent \( p_0 \) in (1.5) and equals the exponent in [32]. Obviously, equations (1.1) and (1.2) are an extension of problems in [5, 6, 10, 15, 23, 24, 28, 29, 32, 38, 39], respectively. Therefore, our conclusions in Theorem 1.7 and Theorem 1.9 extend some results in the above references.

This paper is organized as follows. In Sect. 2, we give the proof of Theorem 1.7. The proof of Theorem 1.9 is finally finished in Sect. 3. In the sequel, we denote by \( C \) some constant, which may vary from line to line. If this constant depends on an arbitrary small number \( \varepsilon \), then we denote it by \( C_\varepsilon \).
2 Proof of Theorem 1.7

We first give the following proposition which plays a crucial role in the proof of Theorem 1.7.

**Proposition 2.1** Assume that \( u \) is a stable weak solution of (1.1). Then, for any \( s \in (0, \frac{1}{1+2k}) \), there exists a constant \( C = C(k, s) > 0 \) such that

\[
\int_{\mathbb{R}^N} f(z)e^{(1+s)u} \psi^{2(1+s)} \, dz \leq CA^{1+s} \int_{\mathbb{R}^N} \omega(z)^{1+s} |\nabla_G \psi|^{2(1+s)} \, dz
\]

holds for all functions \( \psi \in C^1_c(\mathbb{R}^N) \) satisfying \( 0 \leq \psi \leq 1 \) and \( \nabla_G \psi = 0 \) in a neighborhood of \( \{z \in \mathbb{R}^N \mid f(z) = 0 \} \).

**Proof** We will use some of the ideas in [23, 28] to complete the proof. For each \( i \in \mathbb{N} \), we define

\[
\beta_i(t) = \begin{cases} \frac{t^2}{e^t}, & t < i, \\ (\frac{i}{2}t - i + 1)e^{\frac{t^2}{2}}, & t \geq i, \end{cases}
\]

and

\[
\gamma_i(t) = \begin{cases} e^{at}, & t < i, \\ s(t - i + 1)e^{at}, & t \geq i. \end{cases}
\]

It is not difficult to verify that \( \beta_i, \gamma_i \) are increasing positive \( C^1(\mathbb{R}) \) functions and

\[
\beta_i^2(t) \geq \gamma_i(t), \quad \beta_i'(t)^2 = \frac{s^2}{4} \gamma_i(t) \quad \text{and} \quad \beta_i^2(t) + \gamma_i^2(t) \gamma_i'(t)^{-1} \leq Ca^t \tag{2.2}
\]

for all \( t \in \mathbb{R} \), where \( C > 0 \) depends only on \( s \). Owing to \( u \in H^1_\text{loc}(\mathbb{R}^N, \omega) \), we conclude \( \beta_i(u), \gamma_i(u) \in H^1_\text{loc}(\mathbb{R}^N, \omega) \) for any \( i \in \mathbb{N} \).

Let \( \varepsilon \in (0, 1) \) and \( \psi \in C^1_c(\mathbb{R}^N) \) be a nonnegative function.

Set \( \phi = \gamma_i(u) \psi^2 \) as a test function in (1.8) with \( g(u) = e^u \), we have

\[
B \int_{\mathbb{R}^N} \omega(z)|\nabla_G u|^2 \gamma_i'(u) \psi^2 \, dz + 2B \int_{\mathbb{R}^N} \omega(z)\gamma_i(u) \psi \nabla_G u \cdot \nabla_G \psi \, dz \\
= \int_{\mathbb{R}^N} f(z)e^u \gamma_i(u) \psi^2 \, dz,
\]

where \( B = a + b\|\mu\|_2^{2k} \). By Young’s inequality, it yields

\[
B \int_{\mathbb{R}^N} \omega(z)|\nabla_G u|^2 \gamma_i'(u) \psi^2 \, dz \\
\leq 2B \int_{\mathbb{R}^N} \omega(z)\gamma_i(u) \psi |\nabla_G u||\nabla_G \psi| \, dz + \int_{\mathbb{R}^N} f(z)e^u \gamma_i(u) \psi^2 \, dz \\
\leq \varepsilon B \int_{\mathbb{R}^N} (\omega(z)^{1/2}|\nabla_G u|\gamma_i'(u)^{1/2})^2 \, dz \\
+ C \int_{\mathbb{R}^N} (\omega(z)^{1/2} \gamma_i'(u) \gamma_i(u)^{-1/2}|\nabla_G \psi|)^2 \, dz + \int_{\mathbb{R}^N} f(z)e^u \gamma_i(u) \psi^2 \, dz
\]
\[
= \varepsilon B \int_{\mathbb{R}^N} \omega(z) |\nabla_G u|^2 \gamma_i'(u) \psi^2 \, dz \\
+ C_\varepsilon B \int_{\mathbb{R}^N} \omega(z) \gamma_i^2(u) \gamma_i'(u) \psi^{-1} |\nabla_G \psi|^2 \, dz + \int_{\mathbb{R}^N} f(z) e^u \gamma_i(u) \psi^2 \, dz,
\]

which implies that
\[
(1 - \varepsilon) B \int_{\mathbb{R}^N} \omega(z) |\nabla_G u|^2 \gamma_i'(u) \psi^2 \, dz
\leq C_\varepsilon B \int_{\mathbb{R}^N} \omega(z) \gamma_i^2(u) \gamma_i'(u) \psi^{-1} |\nabla_G \psi|^2 \, dz + \int_{\mathbb{R}^N} f(z) e^u \gamma_i(u) \psi^2 \, dz. \tag{2.3}
\]

On the other hand, according to the stability assumption, we take \( \varphi = \beta_i(u) \psi \) in (1.10) and get
\[
\int_{\mathbb{R}^N} f(z) e^u \beta_i^2(u) \psi^2 \, dz
\]
\[
\leq A \int_{\mathbb{R}^N} \omega(z) |\nabla_G u|^2 \beta_i^2(u) \psi^2 \, dz + 2A \int_{\mathbb{R}^N} \omega(z) \beta_i(u) \beta_i'(u) \psi |\nabla_G u||\nabla_G \psi| \, dz
\]
\[
+ A \int_{\mathbb{R}^N} \omega(z) \beta_i^2(u) |\nabla_G \psi|^2 \, dz. \tag{2.4}
\]

We use Young’s inequality to estimate the middle term of the right-hand side of the above inequality:
\[
2A \int_{\mathbb{R}^N} \omega(z) \beta_i(u) \beta_i'(u) \psi |\nabla_G u||\nabla_G \psi| \, dz
\]
\[
\leq \varepsilon A \int_{\mathbb{R}^N} (\omega(z))^{1/2} |\nabla_G u| \beta_i'(u) \psi^2 \, dz + C_\varepsilon A \int_{\mathbb{R}^N} (\omega(z))^{1/2} \beta_i(u) |\nabla_G \psi| \, dz
\]
\[
= \varepsilon A \int_{\mathbb{R}^N} \omega(z) |\nabla_G u|^2 \beta_i'(u) \psi^2 \, dz + C_\varepsilon A \int_{\mathbb{R}^N} \omega(z) \beta_i^2(u) |\nabla_G \psi|^2 \, dz.
\]

Substituting this estimate into (2.4), there holds
\[
\int_{\mathbb{R}^N} f(z) e^u \beta_i^2(u) \psi^2 \, dz
\]
\[
\leq (1 + \varepsilon) A \int_{\mathbb{R}^N} \omega(z) |\nabla_G u|^2 \beta_i'(u) \psi^2 \, dz + C_\varepsilon A \int_{\mathbb{R}^N} \omega(z) \beta_i^2(u) |\nabla_G \psi|^2 \, dz. \tag{2.5}
\]

Together with (2.2), (2.3), (2.5), we obtain
\[
\int_{\mathbb{R}^N} f(z) e^u \beta_i^2(u) \psi^2 \, dz
\]
\[
\leq \frac{(1 + \varepsilon) A}{4} \int_{\mathbb{R}^N} \omega(z) |\nabla_G u|^2 \gamma_i'(u) \psi^2 \, dz + C_\varepsilon A \int_{\mathbb{R}^N} \omega(z) \beta_i^2(u) |\nabla_G \psi|^2 \, dz
\]
\[
\leq \frac{(1 + \varepsilon) A}{4(1 - \varepsilon)B} \int_{\mathbb{R}^N} f(z) e^u \gamma_i(u) \psi^2 \, dz + C_\varepsilon A \int_{\mathbb{R}^N} \omega(z)[\beta_i^2(u) + \gamma_i^2(u) \gamma_i'(u)^{-1}] |\nabla_G \psi|^2 \, dz
\]
\[
\leq \frac{(1 + \varepsilon) A}{4(1 - \varepsilon)B} \int_{\mathbb{R}^N} f(z) e^u \beta_i^2(u) \psi^2 \, dz + C_\varepsilon A \int_{\mathbb{R}^N} \omega(z)e^u |\nabla_G \psi|^2 \, dz.
\]
This combined with the expressions of $A$ and $B$ gives
\[
\int_{\mathbb{R}^N} f(z)e^{\beta^2(z)}(u)^2\,dz \leq \frac{(1+\varepsilon)(1+2k)}{4(1-\varepsilon)} \int_{\mathbb{R}^N} f(z)e^{\beta^2(z)}(u)^2\,dz + C(A) \int_{\mathbb{R}^N} \omega(z)e^{\mu}\vert\nabla G\psi\vert^2\,dz,
\]
that is,
\[
\lambda \varepsilon \int_{\mathbb{R}^N} f(z)e^{\beta^2(z)}(u)^2\,dz \leq C(A) \int_{\mathbb{R}^N} \omega(z)e^{\mu}\vert\nabla G\psi\vert^2\,dz,
\]
(2.6)
where $\lambda \varepsilon = 1 - (1+\varepsilon)(1+2k)s^4(1-\varepsilon)$. Since $\lim_{\varepsilon \to 0^+} \lambda \varepsilon = 1 - (1+2k)s^4$. Thanks to $s \in (0, 4\frac{1+2k}{\varepsilon})$, we let $\varepsilon > 0$ be so small that $\lambda \varepsilon > 0$.

Letting $i \to \infty$ in (2.6), by Fatou’s lemma, we have
\[
\int_{\mathbb{R}^N} f(z)e^{(1+s)\mu}\,dz \leq CA \int_{\mathbb{R}^N} \omega(z)e^{\mu}\vert\nabla G\psi\vert^2\,dz
\]
(2.7)
for some constant $C > 0$ depending only on $k$ and $s$.

On the other hand, replacing $\psi$ by $\psi^{1+s}$ in (2.7) and using Hölder’s inequality, we derive
\[
\int_{\mathbb{R}^N} f(z)e^{(1+s)\mu}\,dz \leq C(A) \int_{\mathbb{R}^N} \omega(z)e^{\mu}\vert\nabla G\psi\vert^2\,dz
\]
(2.8)
\[
\int_{\mathbb{R}^N} f(z)e^{(1+s)\mu}\,dz \leq C(A) \int_{\mathbb{R}^N} \omega(z)e^{\mu}\vert\nabla G\psi\vert^2\,dz
\]
(2.9)
Hence, (2.1) is obtained immediately. \hfill \Box

Set $R > 0$, $\Omega_{2R} = B_1(0, 2R) \times B_2(0, 2R^{1+s})$, where $B_i \subset \mathbb{R}^N$, $i = 1, 2$, are open balls centered at 0, the radii are $2R$ and $2R^{1+s}$, respectively. Let $\chi(t) \in C_{\infty}(\mathbb{R}; [0, 1])$ be cut-off functions such that
\[
\chi(t) = \begin{cases} 
1, & 0 \leq t \leq 1, \\
0, & t \geq 2.
\end{cases}
\]
Define
\[
\psi_{1,R}(x) = \chi\left(\frac{|x|}{R}\right), \quad x \in \mathbb{R}^N, \quad \psi_{2,R}(y) = \chi\left(\frac{|y|}{R^{1+s}}\right), \quad y \in \mathbb{R}^N
\]
and
\[
\psi_R(x, y) = \psi_{1,R}(x)\psi_{2,R}(y), \quad (x, y) \in \mathbb{R}^N = \mathbb{R}^N \times \mathbb{R}^N.
\]
(2.8)
By a series of calculations, one can verify that there exists a constant $C > 0$ independent of $R$ such that

\[
\begin{align*}
|\nabla_x \psi_{1,R}| & \leq CR^{-1}, & |\nabla_y \psi_{2,R}| & \leq CR^{-(1+\alpha)}, \\
|\Delta_x \psi_{1,R}| & \leq CR^{-2}, & |\Delta_y \psi_{2,R}| & \leq CR^{-2(1+\alpha)}, \\
|\nabla G \psi_R|^2 + |\Delta G \psi_R| & \leq CR^{-2}, \\
R & \leq \|z\|_G \leq CR, & \forall z = (x, y) & \in \Omega_2 \setminus \Omega_R.
\end{align*}
\]  

(2.9)

**Proof of Theorem 1.7** Arguing by contradiction, we assume that $u$ is a stable weak solution of (1.1). Let $\psi = \psi_R(x, y) = \psi_{1,R}(x)\psi_{2,R}(y)$ with $R \geq R_0$ in (2.1), then there exists a positive constant $C$ independent of $R$ such that

\[
\int_{\Omega_2 \setminus \Omega_R} f(z) e^{(1+s)u} \, dz \leq CA^{1+\alpha}R^{-2(1+s)} \int_{\Omega_2 \setminus \Omega_R} |z|^{\Theta(1+s)-ds} \, dz
\]

(2.10)

where assumption (H$_1$) and (2.9) have been used.

Since $N_\alpha < \mu_0(k, \theta, d)$ and

\[
\lim_{s \to \frac{4}{1+2k}} \left[ N_\alpha + \theta (1 + s) - ds - 2(1 + s) \right] = N_\alpha - \mu_0(k, \theta, d) < 0,
\]

we may choose some $s \in (0, \frac{4}{1+2k})$ suitably near $\frac{4}{1+2k}$ such that $N_\alpha + \theta (1 + s) - ds - 2(1 + s) < 0$. Letting $R \to +\infty$ in (2.10), we have

\[
\int_{\mathbb{R}^N} f(z) e^{(1+s)u} \, dz = 0.
\]

A contradiction! The proof is completed. \(\square\)

### 3 Proof of Theorem 1.9

In this section, we give the proof of Theorem 1.9, which mainly relies on the following a priori estimate.

**Proposition 3.1** Suppose that $u$ is a positive stable weak solution of (1.2) with $q > 0$. Then, for every $s \in (h(q), -1)$, where

\[
h(t) = -\frac{1 + 2k + 2t + 2\sqrt{t(t + 1 + 2k)}}{1 + 2k}, \quad t > 0,
\]

(3.1)

and for any constant $\tau \geq \frac{q+4}{q+1}$, there exists a constant $C > 0$ depending on $q, s, \tau, \alpha, k$ such that

\[
\int_{\mathbb{R}^N} \left( f(z) u^{\tau-q} + \omega(z) |\nabla G u|^{\frac{2}{q}} \right) \psi^{2\tau} \, dz
\]

\[
\leq CA^{\frac{q}{q+1}} \int_{\mathbb{R}^N} \omega(z) \frac{q+4}{q+1} f(z) \frac{2\alpha+1}{\tau+1} |\nabla G \psi|^{\frac{2\alpha+1}{\tau+1}} \, dz
\]

(3.2)
holds for all functions $\psi \in C^1_c(\mathbb{R}^N)$ verifying $0 \leq \psi \leq 1$ and $\nabla_G \psi = 0$ in a neighborhood of $\{z \in \mathbb{R}^N \mid f(z) = 0\}$.

**Proof** Some ideas in this proof are inspired by [29]. For each $i \in \mathbb{N}\setminus\{0\}$, we define

$$
\delta_i(t) = \begin{cases} 
\frac{1 - \varepsilon}{2} (t + \frac{1 - \varepsilon}{\delta_i}) & 0 \leq t < \frac{1}{i}, \\
\frac{1 - \varepsilon}{\delta_i} & t \geq \frac{1}{i},
\end{cases}
$$

and

$$
\eta_i(t) = \begin{cases} 
\frac{\varepsilon}{i} (t + \frac{1 - \varepsilon}{\delta_i}), & 0 \leq t < \frac{1}{i}, \\
\frac{1}{i} & t \geq \frac{1}{i}.
\end{cases}
$$

A direct calculation yields that $\delta_i, \eta_i$ are decreasing positive $C^4([0, +\infty))$ functions and

$$
\delta_i^2(t) \geq t \delta_i(t), \quad \delta_i'(t)^2 = \left(\frac{1 + s}{4|s|}\right)^2 |\eta_i'(t)|^2, \quad \delta_i^2(t) + \eta_i^2(t)|\eta_i'(t)|^{-1} \leq Ct^{1+s} \quad (3.3)
$$

for all $t \geq 0$, where $C > 0$ is a constant depending only on $s$. Owning to $u \in H^1_{\text{loc}}(\mathbb{R}^N, \omega)$, we have $\delta_i(u), \eta_i(u) \in H^1_{\text{loc}}(\mathbb{R}^N, \omega)$ for any $i \in \mathbb{N}\setminus\{0\}$.

Let $\varepsilon \in (0, 1)$ and $\psi \in C^1_c(\mathbb{R}^N)$ be a nonnegative function.

Choosing $\varphi = \eta_i(u)\psi^2$ as a test function in (1.8) with $g(u) = -u^{-q}, q > 0$, we obtain

$$
B \int_{\mathbb{R}^N} \omega(z)|\nabla_G u|^2 \eta_i^2(u) \psi^2 \, dz + 2B \int_{\mathbb{R}^N} \omega(z)\eta_i(u) \psi \nabla_G u \cdot \nabla_G \psi \, dz \\
= -\int_{\mathbb{R}^N} f(z)u^{-q} \eta_i(u) \psi^2 \, dz.
$$

By use of Young’s inequality, we deduce that

$$
B \int_{\mathbb{R}^N} \omega(z)|\nabla_G u|^2 |\eta_i'(u)| \psi^2 \, dz \\
\leq 2B \int_{\mathbb{R}^N} \omega(z)\eta_i(u) \psi |\nabla_G u| |\nabla_G \psi| \, dz + \int_{\mathbb{R}^N} f(z)u^{-q} \eta_i(u) \psi^2 \, dz \\
\leq \varepsilon B \int_{\mathbb{R}^N} \omega(z)^{1/2} |\nabla_G u| |\eta_i'(u)|^{1/2} \psi^{1/2} \, dz \\
+ C_\varepsilon B \int_{\mathbb{R}^N} \omega(z)^{1/2} \eta_i(u) |\eta_i'(u)|^{-1/2} |\nabla_G \psi| \, dz + \int_{\mathbb{R}^N} f(z)u^{-q} \eta_i(u) \psi^2 \, dz \\
= \varepsilon B \int_{\mathbb{R}^N} \omega(z)|\nabla_G u|^2 |\eta_i'(u)| \psi^2 \, dz + C_\varepsilon B \int_{\mathbb{R}^N} \omega(z)\eta_i^2(u) |\eta_i'(u)|^{-1} |\nabla_G \psi| \psi^2 \, dz \\
+ \int_{\mathbb{R}^N} f(z)u^{-q} \eta_i(u) \psi^2 \, dz,
$$

that is,

$$
(1 - \varepsilon) B \int_{\mathbb{R}^N} \omega(z)|\nabla_G u|^2 |\eta_i'(u)| \psi^2 \, dz \\
\leq C_\varepsilon B \int_{\mathbb{R}^N} \omega(z)\eta_i^2(u) |\eta_i'(u)|^{-1} |\nabla_G \psi| \psi^2 \, dz + \int_{\mathbb{R}^N} f(z)u^{-q} \eta_i(u) \psi^2 \, dz. \quad (3.4)
$$
On the other hand, by virtue of the stability assumption, we take $\varphi = \delta_t(u)\psi$ in (1.12) and get

$$
q \int_{\mathbb{R}^N} f(z) u^{-(1+q)} \delta_t^2(u) \psi^2 \, dz
\leq A \int_{\mathbb{R}^N} \omega(z) |\nabla_G u| \delta_t(u) \psi^2 \, dz + 2A \int_{\mathbb{R}^N} \omega(z) \delta_t(u) \psi |\nabla_G u| |\delta_t(u)| |\nabla_G \psi| \, dz + A \int_{\mathbb{R}^N} \omega(z) \delta_t^2(u) |\nabla_G \psi|^2 \, dz.
$$

(3.5)

Moreover, by Young’s inequality, we conclude

$$
2 \int_{\mathbb{R}^N} \omega(z) \delta_t(u) \psi |\nabla_G u| |\delta_t(u)| |\nabla_G \psi| \, dz
\leq \varepsilon \int_{\mathbb{R}^N} (\omega(z)^{1/2} |\nabla_G u| |\delta_t(u)| \psi)^2 \, dz + C_\varepsilon \int_{\mathbb{R}^N} (\omega(z)^{1/2} \delta_t(u) |\nabla_G \psi|)^2 \, dz
= \varepsilon \int_{\mathbb{R}^N} \omega(z) |\nabla_G u|^2 \delta_t(u) \psi^2 \, dz + C_\varepsilon \int_{\mathbb{R}^N} \omega(z) \delta_t^2(u) |\nabla_G \psi|^2 \, dz.
$$

Substituting this inequality into (3.5), it holds that

$$
q \int_{\mathbb{R}^N} f(z) u^{-(1+q)} \delta_t^2(u) \psi^2 \, dz \leq (1+\varepsilon)A \int_{\mathbb{R}^N} \omega(z) |\nabla_G u|^2 \delta_t(u) \psi^2 \, dz + C_\varepsilon A \int_{\mathbb{R}^N} \omega(z) \delta_t^2(u) |\nabla_G \psi|^2 \, dz.
$$

(3.6)

Combining (3.6) with (3.3) and (3.4), we can find

$$
q \int_{\mathbb{R}^N} f(z) u^{-(1+q)} \delta_t^2(u) \psi^2 \, dz \leq \frac{(1+\varepsilon)(1+s)^2}{4|s|} A \int_{\mathbb{R}^N} \omega(z) |\nabla_G u|^2 |\eta_t(u)| \psi^2 \, dz + C_\varepsilon A \int_{\mathbb{R}^N} \omega(z) \delta_t^2(u) |\nabla_G \psi|^2 \, dz.
$$

that is,

$$
\mu \int_{\mathbb{R}^N} f(z) u^{-(1+q)} \delta_t^2(u) \psi^2 \, dz \leq C_\varepsilon A \int_{\mathbb{R}^N} \omega(z) u^{1+s} |\nabla_G \psi|^2 \, dz,
$$

(3.7)

where

$$
\mu = q - \frac{(1+\varepsilon)(1+s)^2(1+2k)}{4|s|(1-\varepsilon)}.
$$
Clearly,
\[
\lim_{\varepsilon \to 0^+} \mu_\varepsilon = \mu_0 := q - \frac{(1 + s)^2(1 + 2k)}{4|s|}.
\]

Thanks to \(s \in (\mu(q), -1)\), we get \(\mu_0 > 0\). Thus, we can fix some sufficiently small \(\varepsilon > 0\) such that \(\mu_\varepsilon > 0\). Furthermore, the monotone convergence theorem implies
\[
\int_{\mathbb{R}^N} f(z) u^{-q} \psi^2 \, dz \leq CA \int_{\mathbb{R}^N} \omega(z) u^{1+\tau} |\nabla_G \psi|^2 \, dz \tag{3.8}
\]
as \(i \to +\infty\) in (3.7), where \(C > 0\) depending on \(q, k,\) and \(s\).

Taking advantage of (3.4) with \(\varepsilon = 1/2\), (3.3), and (3.7), it follows that
\[
\begin{align*}
B \int_{\mathbb{R}^N} & \omega(z) |\nabla_G u|^2 |\eta_i(u)| \psi^2 \, dz \\
& \leq CB \int_{\mathbb{R}^N} \omega(z) \eta_i^2(u) |\eta_i(u)|^{-1} |\nabla_G \psi|^2 \, dz + 2 \int_{\mathbb{R}^N} f(z) u^{-q} \eta_i(u) \psi^2 \, dz \\
& \leq CB \int_{\mathbb{R}^N} \omega(z) u^{1+\tau} |\nabla_G \psi|^2 \, dz + 2 \int_{\mathbb{R}^N} f(z) u^{-q} \eta_i(u) \psi^2 \, dz \\
& \leq C(A + B) \int_{\mathbb{R}^N} \omega(z) u^{1+\tau} |\nabla_G \psi|^2 \, dz. \tag{3.9}
\end{align*}
\]

Setting \(i \to +\infty\) in (3.9), the monotone convergence theorem leads to
\[
\begin{align*}
B \int_{\mathbb{R}^N} & \omega(z) |\nabla_G u|^2 |\eta_i(u)| \psi^2 \, dz \\
& \leq C(A + B) \int_{\mathbb{R}^N} \omega(z) u^{1+\tau} |\nabla_G \psi|^2 \, dz. \tag{3.10}
\end{align*}
\]

Consequently,
\[
\begin{align*}
\int_{\mathbb{R}^N} & \omega(z) |\nabla_G u|^2 u^{-1} \psi^2 \, dz \\
& \leq C \int_{\mathbb{R}^N} \omega(z) u^{1+\tau} |\nabla_G \psi|^2 \, dz. \tag{3.11}
\end{align*}
\]

Now we claim that (3.2) holds true. Indeed, we can choose some positive constant \(\tau \gg 1\) such that
\[
\frac{(\tau - 1)(s - q)}{s + 1} \geq \tau, \quad \text{or} \quad \tau \geq \frac{q - s}{q + 1}.
\]

By virtue of \(0 \leq \psi(z) \leq 1\) in \(\mathbb{R}^N\), it follows
\[
\left(\psi(z)^{\frac{2(\tau - 1)(s - q)}{s + 1}} \right) \leq \left(\psi(z)\right)^{2\tau}, \quad \forall z \in \mathbb{R}^N.
\]

Replacing \(\psi\) by \(\psi^\tau\) in (3.8) and applying Hölder’s inequality, we see that
\[
\begin{align*}
\int_{\mathbb{R}^N} f(z) u^{-q} \psi^{2\tau} \, dz \\
& \leq CA \int_{\mathbb{R}^N} \omega(z) u^{1+\tau} \psi^{2(\tau - 1)} |\nabla_G \psi|^2 \, dz
\end{align*}
\]
Definethefunction \( m \). Evidently, if
\[
\lim_{t \to 0^+} h(t) = -1, \quad \lim_{t \to +\infty} h(t) = -\infty, \quad h'(t) < 0, \forall t > 0
\]
and
\[
\lim_{t \to 0^+} l(t) = 2 - \theta, \quad \lim_{t \to +\infty} l(t) = \mu_0(k, \theta, d),
\]
\[
\frac{2 - \theta + d)(2\sqrt{t(t + 1 + 2k)} + 1 + 2k + t(1 - 2k))}{(1 + 2k)(t + 1)^2\sqrt{t + 1 + 2k}}, \quad t > 0.
\]
A straightforward calculation yields that if 0 ≤ k ≤ 3 2, then \( l(t) \) is increasing on \((0, \frac{1+2k+2\sqrt{2k}}{2k-3})\) and decreasing on \((\frac{1+2k+2\sqrt{2k}}{2k-3}, \infty)\). Moreover, \( l(\frac{1+2k+2\sqrt{2k}}{2k-3}) = \mu_1(k, \theta, d), l(\frac{4}{2k-3}) = \mu_0(k, \theta, d)\).

Thus, if \( N_\alpha \leq 2 - \theta \) and \( k \geq 0 \), then \( N_\alpha < l(t), \forall t > 0 \). Hence if we fix \( s \in (h(q), -1) \) sufficiently near to \( h(q) \), we have

\[
N_\alpha < \frac{(2 - \theta)(q - s) - d(s + 1)}{q + 1}, \quad q > 0,
\]

which means that \( m < 0 \). Then we reach a contradiction by letting \( R \to +\infty \) in (3.14).

If \( 2 - \theta < N_\alpha < \mu_0(k, \theta, d) \) and \( 0 \leq k \leq \frac{3}{2} \). Make use of the monotonicity of \( l(t) \), there exists unique \( q_c > 0 \) such that \( N_\alpha < l(t) \) for \( t > q_c \). So if we fix \( s \in (h(q), -1) \) sufficiently close to \( h(q) \), we obtain

\[
N_\alpha < \frac{(2 - \theta)(q - s) - d(s + 1)}{q + 1}, \quad q > q_c,
\]

which means that \( m < 0 \). Letting \( R \to +\infty \) in (3.14), we get a contradiction and the desired result is obtained. Obviously, \( q_c \) can be derived from the equation \( N_\alpha = l(q) \), which is given in (1.13).

If \( 2 - \theta < N_\alpha < \mu_0(k, \theta, d) \) and \( k > \frac{3}{2} \). According to the monotonicity of \( l(t) \), there exists unique \( \tilde{q}_c > 0 \) such that \( N_\alpha < l(t) \) for \( t > \tilde{q}_c \). Taking \( R \to +\infty \) in (3.14), we reach a contradiction, and \( q_c \) may be deduced from the equation \( N_\alpha = l(q) \), which is given in (1.14).

If \( N_\alpha = \mu_0(k, \theta, d) \) and \( k > \frac{3}{2} \). Combining \( l(\frac{4}{2k-3}) = \mu_0(k, \theta, d) \) and the monotonicity of \( h(t) \), we have \( l(t) > N_\alpha \) for \( t > \frac{4}{2k-3} \). We get a contradiction by tending \( R \to +\infty \) in (3.14).

Assume now \( \mu_0(k, \theta, d) < N_\alpha < \mu_1(k, \theta, d) \) and \( k > \frac{3}{2} \) by the monotonicity of \( l(t) \), there exists \( q_{1,2} > \frac{4}{2k-3} \) such that \( l(t) > N_\alpha \) for \( q_1 < t < q_2 \). We get a contradiction by tending \( R \to +\infty \) in (3.14). Evidently, \( q_{1,2} \) can be derived from the equation \( N_\alpha = l(q) \), which is given in (1.15). So we are done. \( \Box \)

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