ON THE DISTRIBUTION OF POSITIVE AND NEGATIVE VALUES OF HARDY’S Z-FUNCTION

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Abstract. We investigate the distribution of positive and negative values of Hardy’s function

\[ Z(t) := \zeta\left(\frac{1}{2} + it\right)\chi\left(\frac{1}{2} + it\right)^{-1/2}; \quad \zeta(s) = \chi(s)\zeta(1 - s). \]

In particular we prove that

\[ \mu(I_+(T, H)) \gg T \quad \text{and} \quad \mu(I_-(T, H)) \gg T, \]

where \( \mu(\cdot) \) denotes the Lebesgue measure and

\[ I_+(T, H) = \{ T < t \leq T + H : Z(t) > 0 \}, \]
\[ I_-(T, H) = \{ T < t \leq T + H : Z(t) < 0 \}. \]

1. Introduction and statement of results

Hardy’s function \( Z(t) \) is defined as

\[ Z(t) := \zeta\left(\frac{1}{2} + it\right)\chi\left(\frac{1}{2} + it\right)^{-1/2}, \]

where \( \chi(s) \) \((s \in \mathbb{C})\) is the factor from the functional equation for \( \zeta(s) \), namely, \( \zeta(s) = \chi(s)\zeta(1 - s) \). Thus

\[ \chi(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1 - s), \quad \chi(s)\chi(1 - s) = 1. \]

(See the second author’s monograph [3] for an extensive account of the \( Z \)-function.) It follows that

\[ \chi\left(\frac{1}{2} + it\right) = \chi\left(\frac{1}{2} - it\right) = \chi^{-1}\left(\frac{1}{2} + it\right), \]

so that \( Z(t) \in \mathbb{R} \) when \( t \in \mathbb{R} \), \( Z(t) = Z(-t) \), and \( |Z(t)| = |\zeta(\frac{1}{2} + it)| \). Thus the zeros of \( \zeta(s) \) on the “critical line” \( \Re s = 1/2 \) correspond to the real zeros of \( Z(t) \), which makes \( Z(t) \) an invaluable tool in the study of the zeros of the zeta-function on the critical line.

Our main interest here is in the distribution of positive and negative values of \( Z(t) \), a topic previously discussed in [2], Chapter 11 of [3], and in [4]. If one looks at graphs of \( Z(t) \) in various \( t \) ranges, it is difficult to detect a bias toward positive or negative values. Let \( 2 \leq H \leq T \) and set

\[ I_+(T, H) = \{ T < t \leq T + H : Z(t) > 0 \} \]
and
\[ I_+(T, H) = \{ T < t \leq T + H : Z(t) < 0 \}. \]

Also let \( \mu(\cdot) \) denote Lebesgue measure. Mathematica calculations of \( I_+(T, H) \) and \( I_-(T, H) \) for diverse values of \( H \) and \( T \) suggest the conjecture that the measure of these sets is approximately \( H/2 \), even when \( H \) is quite small relative to \( T \) (see Tables 1 and 2 below). The purpose of this paper is to lend theoretical support to this conjecture by showing that \( Z(t) \) takes positive and negative values a positive proportion of the time on intervals that are not too short.

**Theorem 1.** We have
\[
\mu(I_+(T, T)) \gg T \quad \text{and} \quad \mu(I_-(T, T)) \gg T.
\]

Our method would also allow us to prove that \( \mu(I_\pm(T, H)) \gg H \) for \( H \) somewhat smaller than \( T \). Moreover, with more effort we could replace the \( \gg \) symbols by explicit inequalities. However, a heuristic argument suggests that the values we would obtain for the constants, even using the best currently available mean value estimates, would be rather small, so we have not bothered to calculate them. Note also that it follows from (1.2) that \( \mu(I_\pm(0, T)) \gg T \).

By a different argument we can prove a conditional result with reasonably good constants.

**Theorem 2.** Assume the Riemann hypothesis and Montgomery’s pair correlation conjecture are true. Then for all \( T \) sufficiently large we have
\[
\mu(I_+(0, T)) \geq .32909T \quad \text{and} \quad \mu(I_-(0, T)) \geq .32909T.
\]

The well-known Riemann hypothesis is the statement that all complex zeros of \( \zeta(s) \) have real parts equal to 1/2, and for a formulation of Montgomery’s pair correlation conjecture, see [5] and (4.2).

It is worth noting that the answer to the corresponding question for \( \log |Z(t)| = \log |\zeta(\frac{1}{2} + it)| \), that is, how often \( \log |Z(t)| \) is positive and how often it is negative, is known. For Selberg [7] (also see Tsang [9]) has shown that \( \log |\zeta(\frac{1}{2} + it)/((\frac{1}{2} \log \log t)^{1/2} \text{ is normally distributed with mean zero. Thus, the measure of the set of } t \in [T, 2T] \text{ for which } \log |Z(t)| \text{ is either positive or negative is } \sim T/2 \text{ as } T \to \infty. \)

2. **Lemmas for the Proof of Theorem 1**

In this section we set down the lemmas necessary for the proof of Theorem 1.

Define the arithmetic function \( \alpha_\nu \) by
\[
\frac{1}{\sqrt{\zeta(s)}} = \sum_{\nu=1}^{\infty} \alpha_\nu \nu^{-s} \quad (\Re s > 1).
\]

\[1^{\text{From Tsang's version of the result one can deduce that the measure of the set of } t \in [T, T+H] \text{ for which } \log |Z(t)| \text{ is either positive or negative is } \sim H/2, \text{ where } T^{1/2+\varepsilon} \leq H \leq T \text{ and } 0 < \varepsilon \leq 1/2.}\]
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| \( T \) | \( \mu(I_+(T, T))/\left(\frac{T}{2}\right) \) |
|---|---|
| 100 | 0.943850 |
| 200 | 0.987534 |
| 500 | 0.963277 |
| 1,000 | 0.981253 |
| 5,000 | 0.986981 |
| 10,000 | 0.990367 |
| 50,000 | 0.968667 |

Table 1. Ratios of the measures of sets in dyadic intervals where \( Z(t) > 0 \) to the conjectured values.

| \( T \) | \( \mu(I_+(T, 100))/50 \) |
|---|---|
| 100 | 0.943850 |
| 200 | 0.989211 |
| 500 | 0.967649 |
| 1,000 | 0.876483 |
| 5,000 | 1.04117 |
| 10,000 | 0.967802 |
| 100,000 | 1.05694 |
| 1,000,000 | 0.959324 |
| 10,000,000 | 1.00084 |
| 100,000,000 | 1.00168 |

Table 2. Ratios of the measures of sets in intervals of length 100 where \( Z(t) > 0 \) to the conjectured value.

For \( 1 \leq \nu \leq X \) let

\[
\beta_\nu = \alpha_\nu \left(1 - \frac{\log \nu}{\log X}\right)
\]

and set

\[
B_X(s) = \sum_{\nu \leq X} \beta_\nu \nu^{-s}.
\]

In his famous proof that a positive proportion of the zeros of the zeta-function are on the critical line, Selberg [6] used \( |B_X(\frac{1}{2} + it)|^2 \) to mollify (smooth) \( Z(t) \). This function serves the same purpose for us here.

**Lemma 1.** Let \( X = T^\theta \) with \( 0 < \theta < 1/4 \). Then

\[
(2.1) \quad \int_T^{2T} Z(t) |B_X(\frac{1}{2} + it)|^2 \, dt = o(T) \quad (T \to \infty).
\]
Proof. By (1.1) we can write the integral in question as
\[
\frac{1}{i} \int_{\frac{1}{2} + iT}^{\frac{1}{2} + 2iT} \chi(s)^{-1/2} \zeta(s) B_X(s) B_X(1-s) \, ds.
\]
By Cauchy’s theorem we may replace the segment of integration \([\frac{1}{2} + iT, \frac{1}{2} + 2iT]\) by the other three sides of the rectangle with vertices \(\frac{1}{2} + iT, c + iT, c + 2iT, \) and \(\frac{1}{2} + 2iT\), where \(c = 1 + 1/\log T\), traversed in that order. It is not difficult to see that the coefficients of \(B_X(s)\) satisfy \(|\beta_\nu| \leq 1\) (since \(\alpha_\nu\) is multiplicative and \(0 \leq 1 - \log \nu / \log X \leq 1\)). Thus for \(\sigma \geq -1\) we have
\[
B_X(s) \ll \max(X^{1-\sigma}, \log X).
\]
Moreover, for \(\sigma \geq \frac{1}{2}, t \geq 2,\)
\[
\zeta(\sigma + it) \ll (t^{(1-\sigma)/3} + 1) \log t,
\]
which is the standard convexity bound for \(\zeta(s)\) and follows from \(\zeta(\frac{1}{2} + it) \ll t^{1/6} \log t\) and \(\zeta(1+it) \ll \log t\). Also for \(-1 \leq \sigma \leq 2, t \geq 2,\) by Stirling’s formula for the gamma-function, we have
\[
\chi(s) = \left(\frac{2\pi}{t}\right)^{s+1/2} e^{it(t+\pi/4)} \left(1 + O\left(\frac{1}{t}\right)\right).
\]
The contribution of the horizontal sides of the rectangle, \([\frac{1}{2} + iT, c + iT]\) and \([\frac{1}{2} + 2iT, c + 2iT]\), is therefore
\[
\ll \int_{1/2}^{c} \max(X^{1-\sigma}, \log X) X^{\sigma T^{(\sigma-1)/2}} (T^{(1-\sigma)/3} + 1) \log T \, d\sigma \ll XT^{1/4} \log T.
\]
On the right-hand side of the rectangle the series for \(\zeta(s)\) is absolutely convergent. Therefore, by using (2.5), we see that the integral over this side equals
\[
\int_T^{2T} \left(\sum_{n=1}^{\infty} n^{-c-it}\right) \left(\sum_{\nu \leq X} \beta_\nu \nu^{-c-it}\right) \left(\sum_{\mu \leq X} \beta_\mu \mu^{c-1+it}\right) \left(\frac{t}{2\pi}\right)^{(c-1/2+it)/2} e^{-i(t+\pi/4)/2} \left(1 + O\left(\frac{1}{t}\right)\right) \, dt.
\]
Using
\[
\sum_{n=1}^{\infty} n^{-c-it} \leq \sum_{n=1}^{\infty} n^{-c} = \zeta \left(1 + \frac{1}{\log T}\right) \ll \log T
\]
and with (2.3) and (2.4), it is seen that the \(O\)-term contributes \(O(T^{1/4} X \log^2 T)\). The remaining expression is
\[
e^{-\pi i/8} \sum_{n=1}^{\infty} \sum_{\nu \leq X} \sum_{\mu \leq X} \frac{\beta_\nu \beta_\mu \mu^{c-1}}{n^c \nu^c} \int_T^{2T} \left(\frac{t}{2\pi}\right)^{(c-1/2)/2} e^{i(t/2) \log(\mu^2/2\pi \nu \mu^2)} \, dt.
\]
By the second derivative bound for exponential integrals (see Lemma 2.2 of [1] or Lemma 4.5 of [8]) the integral is \(\ll T^{3/4}\). Therefore the entire expression is
\[
\ll T^{3/4} \sum_{n=1}^{\infty} n^{-c} \sum_{\nu \leq X} \nu^{-c} \sum_{\mu \leq X} \mu^{c-1} \ll T^{3/4} X \log^2 T.
\]
Combining our estimates, we find that the integral in (2.2) is
\[
\ll XT^{1/4} \log T + T^{1/4} X \log^2 T + T^{3/4} X \log^2 T \ll T^{3/4} X \log^2 T.
\]
Thus, if we take \( X = T^\theta \) with \( \theta < 1/4 \), (2.1) follows.

\[\square\]

**Lemma 2.** Let \( X = T^\theta \) with \( 0 < \theta < 1/2 \). Then

\[
(2.6) \quad \int_T^{2T} |Z(t)||B_X(\frac{1}{2} + it)|^2 \, dt \geq T + o(T).
\]

**Proof.** We begin by noting that

\[
\int_T^{2T} |Z(t)||B_X(\frac{1}{2} + it)|^2 \, dt = \int_T^{2T} |\zeta(\frac{1}{2} + it)||B_X(\frac{1}{2} + it)|^2 \, dt \geq \left| \int_T^{2T} \zeta(\frac{1}{2} + it)B_X(\frac{1}{2} + it)^2 \, dt \right|.
\]

By the well known approximate formula (see Chapter 1 of [1])

\[
\zeta(\frac{1}{2} + it) = \sum_{n \leq T} n^{-1/2-it} - \frac{T^{1/2-it}}{1/2-it} + O(T^{-1/2}) \quad (|t| \leq 2T),
\]

and for the range \( T \leq t \leq 2T \) this becomes

\[
\zeta(\frac{1}{2} + it) = \sum_{n \leq T} n^{-1/2-it} + O(T^{-1/2}).
\]

Hence,

\[
\int_T^{2T} |Z(t)||B_X(\frac{1}{2} + it)|^2 \, dt \geq \left| \int_T^{2T} \left( \sum_{n \leq T} n^{-1/2-it} + O(T^{-1/2}) \right)B_X(\frac{1}{2} + it)^2 \, dt \right|.
\]

By the mean value theorem for Dirichlet polynomials and since \( |\beta_\nu| \leq 1 \), the \( O \)-term contributes

\[
\ll T^{-1/2} \sum_{\nu \in \mathbb{X}} \frac{\beta_\nu^2}{\nu} (T + \nu) \ll T^{1/2} \log X + T^{-1/2} X \ll T^{1/2} \log T
\]

for \( X = T^\theta \) with \( \theta < 1/2 \). To treat the other term let

\[
B_X(s)^2 = \sum_{m \leq X^2} b(m)m^{-s},
\]

where \( b(m) = \sum_{d|m} \beta_d \beta_{m/d} \). Note that \( b(1) = 1 \) and \( |b(m)| \leq d(m) \), the divisor function of \( m \).

Thus, we find that

\[
\int_T^{2T} \left( \sum_{n \leq T} n^{-1/2-it} \right)B_X(\frac{1}{2} + it)^2 \, dt = \int_T^{2T} \left( \sum_{n \leq T} n^{-1/2-it} \right) \left( \sum_{m \leq X^2} b(m)m^{-1/2-it} \right) \, dt
\]

\[
= T + \sum_{n \leq T} \frac{b(m)}{(mn)^{1/2}} \int_T^{2T} (mn)^{-it} \, dt
\]

\[
= T + O \left( \sum_{n \leq T} \frac{|b(m)|}{(mn)^{1/2} \log mn} \right).
\]
The $O$-term is
\[
\ll \sum_{n \leq T} \frac{1}{n^{1/2}} \sum_{m \leq X} \frac{d(m)}{m^{1/2}} \ll T^{1/2} X \log X.
\]

Combining our estimates, we find that
\[
\int_T^{2T} |Z(t)||B_X(\frac{1}{2} + it)|^2 \, dt \geq T + O(T^{1/2} X \log X).
\]

The result now follows provided that $0 < \theta < 1/2$.

\[\Box\]

Lemma 3. Let $X = T^\theta$ with $0 < \theta < 1/100$. Then
\[
(2.7) \quad \int_T^{2T} Z(t)^2 |B_X(\frac{1}{2} + it)|^4 \, dt \ll T.
\]

Proof. This estimate is implicit in the proof of Lemma 15 of Selberg [6]. His notation differs from ours, so we shall briefly indicate the differences and describe how to obtain (2.7) from his argument.

Selberg writes $\eta(t)$ for our $B_X(\frac{1}{2} + it)$ and $\eta_h(t)$ for $B_X(\frac{1}{2} + it + h)$. Moreover, he takes $\xi = T^{(2a-1)/20}$ with $1/2 < a < 3/5$ for the length of $\eta(t)$, whereas we write $X = T^\theta$ for the length of $B_X(s)$. Instead of $Z(t)$, Selberg works with
\[
X(t) = -\left(\frac{\pi}{2}\right)^{1/4} Z(t) \left(1 + O\left(\frac{1}{t}\right)\right)
\]
for $t$ positive (see equations (2.1)–(2.3) on p. 92 of [6]). In the course of the proof of Lemma 15, Selberg estimates the integral
\[
\int_T^{T+U} X(t+h)X(t+k)|\eta_h(t)\eta_h(t)|^2 \, dt
\]
for $0 \leq h, k \leq H$, where $H \leq 1/\sqrt{\log \xi}$ and $T^a \leq U \leq T^{3/5}$ (see [6], p. 100, just below equation (4.3)). If we take this with $h = k = 0$, we see that
\[
(2.8) \quad \int_T^{T+U} X(t)^2|\eta(t)|^4 \, dt = \left(\frac{\pi}{2}\right)^{1/2} \left(1 + O\left(\frac{1}{T}\right)\right) \int_T^{T+U} Z(t)^2 |B_X(\frac{1}{2} + it)|^4 \, dt.
\]

Now, Selberg [6] (see the bottom of p. 108) shows that
\[
(2.9) \quad \int_T^{T+U} X(t+h)X(t+k)|\eta_h(t)\eta_h(t)|^2 \, dt = \sqrt{2\pi} UK(h-k) + O(T^{1/2} \xi^7),
\]
where
\[
K(u) = \Re \left(\tau^{iu} \sum_{\nu_1,\nu_2,\nu_3,\nu_4 < \xi} \frac{\beta_{\nu_1} \beta_{\nu_2} \beta_{\nu_3} \beta_{\nu_4}}{\nu_1 \nu_2 \nu_3 \nu_4} \frac{\kappa^{1+iu}}{(\nu_2 \nu_3)^{iu}} \sum_{n < \tau n / \nu_2 \nu_4} n^{-1-iu}\right)
\]
with $\tau = \sqrt{1/2\pi}$ and $\kappa = (\nu_1 \nu_3, \nu_2 \nu_4)$. Over the course of the next five pages Selberg proves that $K(u) = O(1)$ for $0 < u \leq 1/\log \xi$ (see near the bottom of p. 113). We need this with $u = 0$, but that also follows because, as is apparent from its definition, $K(u)$ is continuous at $u = 0$. (Selberg excludes $u = 0$ because there are poles in an expression he uses to approximate a truncation of the zeta function; see the displayed equation just after (4.23).)
Taking $1/2 < a < 3/5$ corresponds to taking $X = \xi = T^\theta$ with $0 < \theta = (2a - 1)/20 < 1/100$. Then, if $U \gg T^{1/2-7\theta}$, we find that $U$ satisfies $T^a \leq U \leq T^{3/5}$, as required, and from (2.8) and (2.9) we have

$$\int_T^{T+U} Z(t)|B_X(\frac{1}{2} + it)|^4 \, dt \ll U.$$  

Splitting the interval $[T, 2T]$ into subintervals of length $U$ and adding the results, we finally obtain (2.7).

3. Proof of Theorem 1

We prove only the first estimate in (1.2) as the proof of the second is similar.

Clearly we have, setting $I_\pm(T) = I_\pm(T, T)$ for shortness,

$$\int_T^{2T} Z(t)|B_X(\frac{1}{2} + it)|^2 \, dt = \int_{I_+(T)} Z(t)|B_X(\frac{1}{2} + it)|^2 \, dt + \int_{I_-(T)} Z(t)|B_X(\frac{1}{2} + it)|^2 \, dt,$$

and

$$\int_T^{2T} |Z(t)||B_X(\frac{1}{2} + it)|^2 \, dt = \int_{I_+(T)} Z(t)|B_X(\frac{1}{2} + it)|^2 \, dt - \int_{I_-(T)} Z(t)|B_X(\frac{1}{2} + it)|^2 \, dt.$$  

Adding (3.1) and (3.2), we deduce that

$$\int_{I_+(T)} Z(t)|B_X(\frac{1}{2} + it)|^2 \, dt = \frac{1}{2} \left( \int_T^{2T} Z(t)|B_X(\frac{1}{2} + it)|^2 \, dt + \int_T^{2T} |Z(t)||B_X(\frac{1}{2} + it)|^2 \, dt \right).$$

By Lemma 1

$$\int_T^{2T} Z(t)|B_X(\frac{1}{2} + it)|^2 \, dt = o(T) \quad (T \to \infty),$$

and by Lemma 2

$$\int_T^{2T} |Z(t)||B_X(\frac{1}{2} + it)|^2 \, dt \geq T + o(T) \quad (T \to \infty).$$

Thus, by (3.1)–(3.5) we obtain

$$\int_{I_+(T)} Z(t)|B_X(\frac{1}{2} + it)|^2 \, dt \geq \frac{1}{2} T + o(T) \quad (T \to \infty).$$

By the Cauchy-Schwarz inequality we then deduce that

$$\frac{1}{2} T + o(T) \leq \mu(I_+(T, T))^{1/2} \left( \int_T^{2T} |Z(t)|^2 |B_X(\frac{1}{2} + it)|^4 \, dt \right)^{1/2}.$$  

The first bound in (1.2) now follows from (3.7) and the estimate

$$\int_T^{2T} |Z(t)|^2 |B_X(\frac{1}{2} + it)|^4 \, dt \ll T$$

in Lemma 3.
4. Proof of Theorem 2

In this section we assume both the Riemann hypothesis and Montgomery’s pair correlation conjecture. To state the latter, let \( \gamma, \gamma' \) denote arbitrary ordinates of zeros of the zeta-function and let

\[
N(T) = \sum_{0 < \gamma \leq T} 1.
\]

As is well known (see e.g., [1] or [8]),

\[
(4.1) \quad N(T) \sim \frac{T}{2\pi} \log \frac{T}{2\pi} \quad (T \to \infty).
\]

Montgomery’s pair correlation conjecture [5] asserts that, if \( \alpha, \beta \) are fixed real numbers with \( \alpha < \beta \), then

\[
(4.2) \quad \sum_{0 < \gamma, \gamma' \leq T} 1 \sim \left( \int_{\alpha}^{\beta} \left[ 1 - \left( \frac{\sin \pi u}{\pi u} \right)^2 \right] du + \delta(\alpha, \beta) \right) N(T)
\]

as \( T \to \infty \). Here \( \delta(\alpha, \beta) = 1 \) if \( 0 \in [\alpha, \beta] \) and \( \delta(\alpha, \beta) = 0 \) otherwise. We define

\[
S_+(T) = \left\{ 0 < \gamma \leq T : Z'(\gamma) > 0 \right\},
\]

\[
S_-(T) = \left\{ 0 < \gamma \leq T : Z'(\gamma) < 0 \right\},
\]

and define, with \( |A| \) denoting the cardinality of the set \( A \),

\[
N_+(T) = |S_+(T)|, \quad N_-(T) = |S_-(T)|.
\]

It follows from (4.2) that almost all zeros \( \rho = \frac{1}{2} + i\gamma \) of the zeta-function are simple, that is, the number of them with ordinates in \( (0, T] \) is \( \sim N(T) \). Thus, consecutive ordinates almost always alternate between the two sets \( S_+(T) \) and \( S_-(T) \) and we have

\[
(4.3) \quad N_+(T) \sim N_-(T) \sim \frac{1}{2} N(T) \quad (T \to \infty).
\]

Suppose now that \( \gamma \) is the ordinate of a simple zero and that \( \gamma^* \) is the next ordinate greater than \( \gamma \). Setting

\[
f(\alpha) := \int_{0}^{\alpha} \left[ 1 - \left( \frac{\sin \pi u}{\pi u} \right)^2 \right] du
\]

with \( \alpha > 0 \), we see from (4.2) that

\[
\sum_{0 < \gamma, \gamma^* \leq T} 1 \leq \sum_{0 < \gamma - \gamma^* \leq \frac{2\pi \alpha}{\log T}} 1 \sim f(\alpha) N(T) \quad (T \to \infty).
\]

Hence, the number of simple zeros \( \rho = \frac{1}{2} + i\gamma \) with \( 0 < \gamma \leq T \) and \( \gamma^* - \gamma > 2\pi \alpha/\log T \) is greater than or equal to

\[
(1 - f(\alpha) + o(1)) N(T) \quad (T \to \infty).
\]
By (4.3) the number of these γ that are in $S_+(T)$ (similarly, $S_-(T)$) is therefore
\[ \geq (1 - f(\alpha) + o(1))N(T) - \frac{1}{2}N(T) = \left(\frac{1}{2} - f(\alpha) + o(1)\right)N(T) \quad (T \to \infty). \]

Thus, if we define
\[ N_+(\alpha, T) := \sum_{\gamma \in S_+(T) \cap \gamma^* - \gamma > 2\pi \alpha / \log T} 1, \]
and $N_-(\alpha, T)$ similarly, then
\[ (4.4) \quad N_\pm(\alpha, T) \geq \left(\frac{1}{2} - f(\alpha) + o(1)\right)N(T) \quad (T \to \infty). \]

For $T$ large, let $B = B(T) > 1$ be such that every gap $\gamma^* - \gamma$ between consecutive ordinates of zeros with $\gamma \in (0, T]$ is less than $2\pi B / \log T$. Then we have
\[
\int_0^B N_+(\alpha, T) \, d\alpha = \int_0^B \left( \sum_{\gamma \in S_+(T) \cap \gamma^* - \gamma > 2\pi \alpha / \log T} 1 \right) \, d\alpha
= \sum_{\gamma \in S_+(T)} \int_0^{((\gamma^* - \gamma) \log T) / 2\pi} 1 \, d\alpha
= \frac{\log T}{2\pi} \sum_{\gamma \in S_+(T)} (\gamma^* - \gamma)
\leq \frac{\log T}{2\pi} \mu(I_+(0, T)).
\]

Now $N_+(\alpha, T)$ is nonnegative, so for any $A \in [0, B],
\[ \mu(I_+(0, T)) \geq \frac{2\pi}{\log T} \int_0^A N_+(\alpha, T) \, d\alpha. \]

By (4.1) and (4.4) we therefore see that
\[ \mu(I_+(0, T)) \geq T \int_0^A \left(\frac{1}{2} - f(\alpha)\right) \, d\alpha + o(T) \quad (T \to \infty). \]

Using Mathematica, we find that the right-hand side attains a maximum value slightly greater than .32909 $T$ when $A \approx .952$. The same argument works mutatis mutandis for $\mu(I_-(0, T))$, so the proof of Theorem 2 is complete.

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References

[1] A. Ivić, The Riemann zeta-function, John Wiley & Sons, New York 1985 (reissue, Dover, Mineola, New York, 2003).
[2] A. Ivić, On some problems involving Hardy’s function, Central European Journal of Mathematics 8(6)(2010), 1029-1040.
[3] A. Ivić, The theory of Hardy’s Z-function, Cambridge University Press, Cambridge, 2012, 245pp.
[4] A. Ivić, Hardy’s function $Z(t)$ - results and problems, to appear in the Steklov Math. Inst. Proc. in honour of the 125th anniversary of I.M. Vinogradov (2017).
[5] H.L. Montgomery, The pair correlation of zeros of the zeta-function, Proc. Symp. Pure Math. 24, AMS, Providence 1973, 181-193.
[6] A. Selberg, On the zeros of Riemann’s zeta-function, Collected works, vol. 1, Springer Verlag, Berlin etc., pp. 85-141.
[7] A. Selberg, Old and new conjectures and results about a class of Dirichlet series, Collected works, vol. 2, Springer Verlag, Berlin etc., pp. 47-63.
[8] E.C. Titchmarsh, The theory of the Riemann zeta-function (2nd edition), Oxford University Press, Oxford, 1986.
[9] K. M. Tsang, The distribution of the values of the Riemann zeta-function, P.D. dissertation, Princeton Univ., Princeton, NJ, 1984.

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