Conjugacy classes and rational period functions for the Hecke groups

Wendell Ressler
February 12, 2021

Abstract

We establish a one-to-one correspondence between conjugacy classes of any Hecke group and irreducible systems of poles of rational period functions for automorphic integrals on the same group. We use this correspondence to construct irreducible systems of poles and to count poles. We characterize Hecke-conjugation and Hecke-symmetry for poles of rational period functions in terms of the transpose of matrices in conjugacy classes. We construct new rational period functions and families of rational period functions.

Key words: conjugacy classes, Hecke groups, Hecke-symmetry, rational period functions

2020 Mathematics Subject Classification: Primary 11F67; Secondary 11F12, 11E45.

1 Introduction

Marvin Knopp first defined and studied rational period functions (RPFs) for automorphic integrals [Kno74] and gave the first example of an RPF with nonzero poles [Kno78].

Knopp [Kno81], Hawkins [Haw], and Choie and Parson [CP90, CP91] took the main steps toward an explicit characterization of RPFs on the modular group \( \Gamma(1) \). An important tool for this work was Hawkins’ idea of studying irreducible systems of poles (ISPs) for RPFs. Ash [Ash89] gave an abstract characterization, and then Choie and Zagier [CZ93] and Parson [Par93] provided a more explicit characterization of the RPFs on \( \Gamma(1) \). The explicit characterizations use continued fractions to establish a connection between the poles of RPFs and binary quadratic forms.

Schmidt [Sch96] generalized Ash’s work and gave an abstract characterization of RPFs on any finitely generated Fuchsian group of the first kind with parabolic elements, a class of groups which includes the Hecke groups. Schmidt [Sch93] and Schmidt and Sheingorn [SS95] gave explicit descriptions of certain
RPFs on the Hecke groups using generalizations of the classical continued fractions and binary quadratic forms. This author continued that work in [CR01], [Res09], and [Res16] and has explicitly characterized RPFs on the Hecke groups for two cases. The question of a complete explicit characterization remains open.

In this paper we construct new RPFs on Hecke groups, including one in every case for the first four Hecke groups. We also construct families of RPFs across all (or for some families all but several) Hecke groups. Our primary tool is an explicit correspondence between poles of an RPF and conjugacy classes of its Hecke group. We show that by using this correspondence we may construct all ISPs of a given size from products of generators for corresponding conjugacy classes. We give a formula for the number of such products and thus for the number of corresponding poles. We also use the correspondence to conjugacy classes to characterize Hecke-conjugation and Hecke-symmetry for poles of rational period functions.

2 Background

We first summarize background ideas and definitions necessary to work with conjugacy classes and rational period functions. More details can be found in [CR01], [Res09], [HR13], and [Res16].

2.1 Hecke groups and Hecke symmetry

Let $S = S_\lambda = \begin{pmatrix} \lambda & 1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, where $\lambda$ is a positive real number. Put $G(\lambda) = \langle S, T \rangle / \{ \pm I \} \subseteq \text{PSL}(2, \mathbb{R})$. Erich Hecke [Hec36] showed that the values of $\lambda$ for which $G(\lambda)$ is discrete are $\lambda = \lambda_p = 2 \cos(\pi/p)$, for $p = 3, 4, 5, \ldots$, and for $\lambda \geq 2$. The groups with $\lambda = \lambda_p$, for $p \geq 3$ have come to be known as the Hecke groups; we denote them by $G_p = G(\lambda_p)$ for $p \geq 3$. The first several of these Hecke groups are $G_3 = G(1) = \Gamma(1)$ (the modular group), $G_4 = G(\sqrt{2})$, $G_5 = G(\frac{1 + \sqrt{5}}{2})$, and $G_6 = G(\sqrt{3})$.

A Hecke group $G_p$ is the free product of the cyclic group of order 2 generated by $T$ and the cyclic group of order $p$ generated by $U = U_\lambda = S_\lambda T = \begin{pmatrix} \lambda & -1 \\ 1 & 0 \end{pmatrix}$, so the group relations of $G_p$ are $T^2 = U^p = I$.

Elements of $G_p$ act on the Riemann sphere as Möbius transformations. An element $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_p$ is hyperbolic if $|a + d| > 2$, parabolic if $|a + d| = 2$, and elliptic if $|a + d| < 2$. We designate fixed points accordingly. Hyperbolic Möbius transformations each have two distinct real fixed points.

The action of $G_p$ induces an equivalence relation for points on the Riemann sphere. We say that $z_1$ and $z_2$ are $G_p$-equivalent if there exists $M \in G_p$ such that $z_2 = Mz_1$. Equivalence classes contain either all fixed points of the same kind, or no fixed points.

The stabilizer in $G_p$ of a complex number $z$, $\text{stab}(z) = \{ M \in G_p \mid Mz = z \}$, is a cyclic subgroup of $G_p$. We define the Hecke-conjugate of any hyperbolic fixed
point $\alpha$ to be the other fixed point of the elements in its stabilizer and denote it by $\alpha'$. If $R$ is a set of hyperbolic fixed points of $G_p$ we write $R' = \{x' \mid x \in R\}$. We say that a set $R$ has Hecke-symmetry if $R = R'$. We observe that $R \cup R'$ has Hecke-symmetry for any set of hyperbolic points $R$.

In the case of the modular group $G_3$, hyperbolic fixed points are elements of $\mathbb{Z}[\sqrt{D}]$ for $D = (a + d)^2 - 4$ and Hecke-conjugation reduces to algebraic conjugation. For other groups, a Hecke-conjugate $\alpha'$ is one of the algebraic conjugates of $\alpha$.

2.2 Conjugacy class generators

Schmidt and Sheingorn [SS95] observed that the non-elliptic conjugacy classes in $G_p$ have representatives that are products of the generators $V_j = U_j^{-1}S$ for $1 \leq j \leq p - 1$. In [HH13] we show that every non-elliptic element in $G_p$ is conjugate to a product of conjugacy class generators $V_j$ that is unique up to cyclic permutation. We define the block length of a product of generators $W = V_{j_1}V_{j_2} \cdots V_{j_t}$ to be $t$, the length of that product. Every conjugacy class in $G_p$ has an associated block length.

For a given $p \geq 3$ we have the generators

\[
V_1 = S = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix},
\]

\[
V_2 = US = \begin{pmatrix} \lambda & \lambda^2 - 1 \\ 1 & \lambda \end{pmatrix},
\]

\[
V_3 = U^2S = \begin{pmatrix} \lambda^2 - 1 & \lambda^3 - 2\lambda \\ \lambda & \lambda^2 - 1 \end{pmatrix},
\]

\[
V_{p-3} = U^{p-4}S = \begin{pmatrix} \lambda^2 - 1 & \lambda \\ 1 & \lambda^2 - 1 \end{pmatrix},
\]

\[
V_{p-2} = U^{p-3}S = \begin{pmatrix} \lambda & 1 \\ \lambda^2 - 1 & \lambda \end{pmatrix},
\]

\[
V_{p-1} = U^{p-2}S = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}.
\]

Generators $V_1$ and $V_{p-1}$ are parabolic, as are powers of $V_1$ and powers of $V_{p-1}$. All other generators and generator products are hyperbolic.

2.3 $\lambda$-continued fractions

We will use a modification of Rosen’s continued fractions [Ros54], which are closely associated with the Hecke groups.

For real $\alpha$ we put $\alpha_0 = \alpha$ and define $r_j = \lfloor \frac{\alpha_j}{\lambda} \rfloor + 1$ and $\alpha_{j+1} = \frac{1}{r_j \lambda - \alpha_j}$ for $j \geq 0$. Then $\alpha_j = r_j \lambda - \frac{1}{\alpha_{j+1}}$ for $j \geq 0$ and $\alpha = r_0 \lambda - \frac{1}{r_1 \lambda - \cdot \cdot \cdot} = [r_0; r_1, \ldots]$ is
the $\lambda_p$-continued fraction ($\lambda$-CF) for $\alpha$. An admissible $\lambda$-CF is one that arises from a finite real number by this algorithm.

An admissible $\lambda$-CF has at most $p - 3$ consecutive ones in any position but the beginning, and at most $p - 2$ consecutive ones at the beginning \cite{Res09}. Schmidt and Sheingorn \cite{SS95} show that a real number is a non-elliptic fixed point of $G_p$ if and only if it has a periodic $\lambda$-CF; the number is parabolic if its $\lambda$-CF has period $[2,1,\ldots,1]$, and hyperbolic if its $\lambda$-CF has any other period.

Thus every hyperbolic number $\alpha$ has a $\lambda$-CF expansion of the form

$$\alpha = [r_0; r_1, \ldots, r_n, \overline{r_{n+1}, \ldots, r_{n+m}}],$$

with a period that is not a cyclic permutation of $[2,1,\ldots,1]$. Two hyperbolic numbers are $G_p$-equivalent if and only if they have the same $\lambda$-CF period.

### 2.4 \(\mathbb{Z}[\lambda]\)-binary quadratic forms

We let $Q_{p,D}$ denote the set of binary quadratic forms

$$Q(x, y) = Ax^2 + Bxy + Cy^2,$$

with coefficients in $\mathbb{Z}[\lambda_p]$ and discriminant $D$. We also denote a form by $Q = [A, B, C]$ and refer to it as a $\lambda$-BQF. We restrict our attention to hyperbolic forms, which are indefinite forms associated with hyperbolic elements of $G_p$ as described below.

A Hecke group acts on $Q_{p,D}$ by $(Q \circ M)(x, y) = Q(ax + by, cx + dy)$ for $Q \in Q_{p,D}$ and $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_p$. This action preserves the discriminant and induces an equivalence relation on $Q_{p,D}$. We say that $Q_1$ and $Q_2$ are $G_p$-equivalent if there exists $M \in G_p$ such that $Q_2 = Q_1 \circ M$.

### 2.5 Matrices, forms, and fixed points

We will use a correspondence between primitive hyperbolic elements of $G_p$, certain indefinite $\lambda$-BQFs, and hyperbolic fixed points. Because the Hecke groups are projective we may assume that all matrices have positive trace. Because all positive powers of a matrix have the same attracting and repelling fixed points we restrict our attention to primitive matrices. We use subscripts to indicate connections between matrices, forms and numbers. The formal details of the isomorphisms appear in \cite{Res09}.

In this correspondence a primitive hyperbolic matrix with positive trace

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

corresponds to the indefinite $\lambda$-BQF

$$Q_M = [A, B, C] = [c, d - a, -b],$$
with discriminant \( D = (a + d)^2 - 4 \). Because of its connection to \( M \), we say that such a \( \lambda \)-BQF \( Q_M \) is hyperbolic. The form \( Q_M \) in turn corresponds to

\[
\alpha_{Q_M} = \alpha_M = \frac{-B + \sqrt{D}}{2A} = \frac{a - d + \sqrt{D}}{2c},
\]

which is the attracting fixed point of \( M \). We complete the correspondence by using the fact that a hyperbolic fixed point \( \alpha \) has a \( \lambda \)-CF expansion of the form

\[
\alpha = [r_0; r_1, \ldots, r_n, r_{n+1}, \ldots, r_{n+m}].
\]

If we put \( V = S^{r_0}T S^{r_1}T \cdots S^{r_n}T \) and \( W = S^{r_{n+1}}T \cdots S^{r_{n+m}}T \) the element of \( G_p \) that corresponds to \( \alpha \) is

\[
M_\alpha = VWV^{-1}.
\]

The matrix \( M_\alpha \) is primitive, hyperbolic, and has \( \alpha \) as an attracting fixed point. The \( \lambda \)-BQF that corresponds to \( \alpha \) is \( Q_{M_\alpha} = Q_\alpha \).

### 2.6 Hecke-conjugation

Given a hyperbolic matrix \( M \), its inverse \( M^{-1} \) is also hyperbolic and has the same fixed points, but with attracting and repelling points reversed. As a result the Hecke-conjugate of \( \alpha_M \) is

\[
\alpha_M' = \alpha_{M^{-1}}.
\]

A calculation shows that for any hyperbolic \( \alpha \) and \( V \in G_p \) we have \( (V \alpha)' = V \alpha' \).

Given a hyperbolic \( \lambda \)-BQF \( Q = [A, B, C] \) its negative \( -Q = [-A, -B, -C] \) is also hyperbolic. A simple calculation shows that if \( Q = Q_M \) then \( -Q = Q_{M^{-1}} \). Thus the Hecke-conjugate of \( \alpha_Q \) is

\[
\alpha_Q' = \alpha_{-Q}.
\]

### 2.7 Reduction and simplicity

The positive poles of every rational period function are “simple” numbers associated with simple \( \lambda \)-BQFs, which are closely related to reduced \( \lambda \)-BQFs.

Zagier’s theory of reduction for classical binary quadratic forms in [Zag81] uses negative classical continued fractions. We will use a generalization to \( \lambda \)-BQFs developed in [Res09].

We say that a real number \( \beta \) is \( G_p \)-reduced if its \( \lambda_p \)-CF expansion is purely periodic with a period that is not \( [2, 1, \ldots, 1] \). If \( \beta \) is \( G_p \)-reduced it is hyperbolic, and we also say that the associated hyperbolic \( \lambda \)-BQF \( Q_\beta \) is \( G_p \)-reduced.

Any hyperbolic \( \lambda \)-BQF \( Q \) can be transformed into a reduced form by the action of finitely many elements of \( G_p \). This process maps reduced forms to reduced forms and so produces a cycle of reduced forms in the same equivalence class.
class as $Q$. For the associated hyperbolic number $\beta_Q$, this reduction removes the pre-period then cyclically permutes the period to produce a cycle of reduced numbers in the same equivalence class as $\beta_Q$. Every equivalence class of hyperbolic numbers has a unique cycle of reduced numbers.

We may characterize reduced numbers in terms of their size alone. We show in [Res09] that a hyperbolic fixed point $\beta$ is $G_p$-reduced with $j$ leading ones in its $\lambda$-CF if and only if

$$0 < \beta' < U^j + 2(0) < \beta < U^{j+1}(0),$$

for $0 \leq j \leq p - 3$. If the $\lambda$-CF for a reduced number $\beta$ does not have a leading 1, then from (1) we have $\beta > \lambda = U^2(0)$. If the $\lambda$-CF for $\beta$ has a leading 1 we have $\beta < \lambda$.

We say that a hyperbolic $\lambda$-BQF $Q = [A, B, C]$ is $G_p$-simple if $A > 0 > C$; we also say that the associated hyperbolic number $\alpha_Q$ is $G_p$-simple. A hyperbolic number $\alpha$ is $G_p$-simple if and only if $\alpha' < 0 < \alpha$. Every reduced number $\beta$ that is greater than $\lambda$ (so its $\lambda$-CF does not have a leading 1) is associated with at least one simple number because $0 < \beta' < \lambda < \beta$ implies $(S^{-1}\beta)' < 0 < S^{-1}\beta$.

If $A$ is an equivalence class of hyperbolic $\lambda$-BQFs, we define the corresponding set of simple numbers $Z_A = \{ \alpha : Q_\alpha \in A, \alpha G_p$-simple $\}$. These sets are nonempty because every hyperbolic equivalence class has a cycle of reduced numbers, at least one of which has a $\lambda$-CF with a leading entry greater than 1. In [Res09] we show that the set of simple numbers for a hyperbolic $\lambda$-BQF equivalence class $A$ is

$$Z_A = \left\{ S^{-i}\beta : Q_\beta \in A \text{ is } G_p \text{-reduced, } 1 \leq i \leq \left\lfloor \frac{\beta}{\lambda} \right\rfloor \right\}. \quad (2)$$

### 2.8 Rational period functions

For the matrix $M = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$ and the function $f(z)$, we define the weight $2k$ slash operator $f \mid_{2k} M = f \mid M$ by

$$(f \mid M)(z) = (cz + d)^{-2k} f(Mz).$$

For a fixed $p \geq 3$ and positive integer $k$ we define a rational period function (RPF) of weight $2k$ for $G_p$ to be a rational function that satisfies the relations

$$q + q \mid T = 0,$$

and

$$q + q \mid U + \cdots + q \mid U^{p-1} = 0. \quad (4)$$

This definition is equivalent to Marvin Knopp’s original definition of rational period functions for automorphic integrals [Kno74]. The set of rational functions that satisfy (3) and (4) forms a vector space.

Following Hawkins’ insight for RPFs on the modular group [Haw] we define an irreducible system of poles (ISP) to be the minimal set of nonzero poles.
forced to occur together by the relations (3) and (4). For some weights an RPF with poles in a given ISP must also have a pole at 0.

The poles of an RPF on $G_p$ are all real, and the nonzero poles are all hyperbolic fixed points of $G_p$. The set of positive poles in any given ISP is $Z_A$ for some equivalence class $\mathcal{A}$ of hyperbolic forms.

If $q$ is an RPF of weight $2k$ on $G_p$ with a pole \emph{only} at zero, then $q$ must have the form \cite{Kno81}

$$q_{k,0}(z) = \begin{cases} a_0(1 - z^{-2k}), & \text{if } 2k \neq 2, \\ a_0(1 - z^{-2}) + b_1 z^{-1}, & \text{if } 2k = 2. \end{cases} \tag{5}$$

For a nonzero pole $\alpha$ we define

$$q_{k,\alpha}(z) = PP_\alpha \left[ \frac{D^{k/2}}{Q_\alpha(z,1)^k} \right] = PP_\alpha \left[ \frac{(\alpha - \alpha')^k}{(z - \alpha)^k(z - \alpha')^k} \right], \tag{6}$$

where $D$ is the discriminant of the corresponding $\lambda$-BQF $Q_\alpha$. Using this notation we have the following expression for any RPF on $G_p$ \cite{CR01}.

\textbf{Theorem 1.} Fix $p \geq 3$ and let $\lambda = \lambda_p$. An RPF of weight $2k \in 2\mathbb{Z}^+$ on $G_p$ is of the form

$$q(z) = \sum_{\ell=1}^L \left( \sum_{\alpha \in Z_{A_\ell}} q_{k,\alpha}(z) - \sum_{\alpha \in Z_{-A_\ell}} q_{k,\alpha}(z) \right) + c q_{k,0}(z) + \sum_{n=1}^{2k-1} c_n z^n, \tag{7}$$

where each $A_\ell$ is a $G_p$-equivalence class of $\lambda$-BQFs, $Z_{A_\ell}$ is the set of positive poles associated with $A_\ell$, $q_{k,\alpha}$ is given by (6), $q_{k,0}$ is given by (5), and the $C_\ell$ and $c_n$ are all constants.

The last sum in (7) is required by the partial fraction decompositions in (6).

If $A = -A$ then the associated ISP $P_A$ has Hecke-symmetry. If an RPF of weight $2k$ with poles in $P_A$ exists for this case it has the form

$$q(z) = \sum_{\alpha \in Z_A} \left( q_{k,\alpha}(z) - q_{k,\alpha'}(z) \right) + \sum_{n=1}^{2k-1} c_n z^n, \tag{8}$$

If $k$ is odd then

$$q(z) = \sum_{\alpha \in Z_A} Q_\alpha(z,1)^{-k}, \tag{9}$$

is an RPF of weight $2k$ with poles in $P_A$ \cite{CR01}.

If $A \neq -A$ then the associated ISPs $P_A$ and $P_{-A}$ do not have Hecke-symmetry. If an RPF of weight $2k$ with poles in $P_A$ exists for this case it has the form

$$q(z) = \sum_{\alpha \in Z_A} q_{k,\alpha}(z) - \sum_{\alpha \in Z_{-A}} q_{k,\alpha'}(z) + \sum_{n=1}^{2k-1} c_n z^n. \tag{10}$$
The union $P_A \cup P_{-A}$ does have Hecke-symmetry. Then

$$q(z) = \sum_{\alpha \in \mathbb{Z}_A} Q_\alpha(z,1)^{-k} - (-1)^k \sum_{\alpha \in \mathbb{Z}_{-A}} Q_\alpha(z,1)^{-k},$$

is an RPF of weight $2k$ for any $k$ with poles in $P_A \cup P_{-A}$ [Res16].

For a fixed $k$ and ISP $P_A$, the results quoted above characterize nontrivial RPFs for certain values of $k$ and certain kinds of ISPs. In particular, (9) and (11) imply the existence of a nontrivial RPF for

(i) $k$ odd and Hecke-symmetric pole set $P_A$,

(ii) any $k$ and pole set $P_A \cup P_{-A}$.

Existence is not guaranteed and few RPFs have been constructed for the other cases

(iii) $k$ even and Hecke-symmetric pole set $P_A$,

(iv) any $k$ and non-Hecke-symmetric pole set $P_A$.

We will write several new examples of RPFs in case (iii) and (iv) in Section 5.

3 Conjugacy classes and irreducible systems of poles

We have outlined a correspondence between primitive hyperbolic matrices, hyperbolic $\lambda$-BQFs, and hyperbolic fixed points. The next lemma shows that the action of $G_p$ on numbers and $\lambda$-BQFs corresponds to conjugation of matrices.

**Lemma 1.** Fix $p \geq 3$ and let $\lambda = \lambda_p$. Suppose that $\alpha$ and $\beta$ are hyperbolic numbers associated with primitive hyperbolic matrices $M_\alpha, M_\beta \in G_p$ and with hyperbolic $\lambda$-BQFs $Q_\alpha$ and $Q_\beta$. Then for any $V \in G_p$ the following statements are equivalent.

(a) $\beta = V \alpha$, and

(b) $Q_\beta = Q_\alpha \circ V^{-1}$.

(c) $M_\beta = VM_\alpha V^{-1}$

**Proof.** Lemma 7 in [Res09] showed that (a) and (b) are equivalent. In order to prove that (a) implies (c) we use the fact that if (a) holds then $\alpha$ and $\beta$ must have the same $\lambda$-CF period, and calculate the matrices $M_\beta$ and $VM_\alpha V^{-1}$. The proof of the converse is another direct calculation.

We will use $\sim$ to denote equivalence in all three settings: equivalence of numbers with respect to $G_p$, equivalence of $\lambda$-BQFs with respect to $G_p$, and conjugacy of elements of $G_p$. So Lemma 1 means that $\alpha \sim \beta$ (hyperbolic
numbers) if and only if \( Q_\alpha \sim Q_\beta \) (quadratic forms) if and only if \( M_\alpha \sim M_\beta \) (conjugation).

The correspondence in Section 2.5 holds only for primitive hyperbolic elements of \( G_p \). Matrices in a conjugacy class are either all primitive or all non-primitive, so we will describe conjugacy classes themselves as primitive or not. Lemma 1 has a corollary for equivalence classes.

**Corollary 1.** Fix \( p \geq 3 \) and let \( \lambda = \lambda_p \). The following sets are in one-to-one correspondence for the Hecke group \( G_p \):

- equivalence classes of hyperbolic numbers,
- equivalence classes of hyperbolic \( \lambda \)-BQFs, and
- primitive hyperbolic conjugacy classes.

### 3.1 Connecting ISPs and conjugacy classes

A primitive hyperbolic conjugacy class in \( G_p \) corresponds to a unique \( G_p \)-equivalence class of \( \lambda \)-BQFs \( \mathcal{A} \), which in turn is associated with a unique rational period function ISP \( P_A \). The elements of \( P_A \) all have the same \( \lambda \)-CF period and thus lie in the same \( G_p \)-equivalence class of numbers. We reverse this to find the unique conjugacy classes for a given ISP.

We can make this explicit. Given a primitive hyperbolic conjugacy class in \( G_p \) we write its product of conjugacy class generators. We pay particular attention to \( V_1 = S \) because it plays a special role in the correspondence. We cyclically permute the generators (by conjugation) if necessary to write a generator product in the form

\[
W = V_1^{m_1} V_{j_1}^{m_2} V_{j_2}^{m_3} \cdots V_1^{m_\ell} V_{j_\ell},
\]

where \( m_t \geq 0 \) and \( 2 \leq j_t \leq p - 1 \) for \( 1 \leq t \leq \ell \). This \( W \) has \( \ell \) generators that are not \( V_1 \) and \( m_t \) copies of \( V_1 \) preceding each \( V_{j_t} \). Then \( W \) is conjugate to

\[
M = S W S^{-1} = S^{m_1 + 2} T(ST)^{j_1 - 2} S^{m_2 + 2} T(ST)^{j_2 - 2} \cdots S^{m_\ell + 2} T(ST)^{j_\ell - 2},
\]

which corresponds to the reduced fixed point

\[
\beta_1 = \overline{[m_1 + 2; 1, 1, \ldots, 1, m_2 + 2, 1, 1, \ldots, 1, \ldots, m_\ell + 2, 1, 1, \ldots, 1]}.
\]

We observe that the \( \lambda \)-CF is admissible because \( j_t - 2 \leq p - 3 \) for each \( t \) and that its period is minimal because the original matrix \( W \) is primitive. We let \( \mathcal{A} \) denote the \( G_p \)-equivalence class of \( \lambda \)-BQFs that corresponds to the \( G_p \)-equivalence class of numbers \( \beta_1 \).

By (2), the simple numbers in \( \beta_1 \) are images under powers of \( S^{-1} \) of reduced numbers that have \( \lambda \)-CF leading entry greater than 1. These are the \( \ell \) reduced numbers

\[
\beta_\ell = \overline{[m_\ell + 2; 1, 1, \ldots, 1]}.
\]
for \(1 \leq t \leq \ell\). The simple number(s) for each \(\beta_t\) are the \(m_t + 1\) number(s)

\[
\alpha_i^{(t)} = S^{-i} \beta_t = [m_t + 2 - i; 1, 1, \ldots, 1, \ldots],
\]

for \(1 \leq i \leq \lfloor \frac{\beta_t}{\lambda} \rfloor = m_t + 1\). Thus the set of simple numbers in \([\beta_1]\) is

\[
Z_A = \{\alpha_1^{(1)}, \alpha_{m_1 + 1}^{(1)}, \alpha_1^{(2)}, \ldots, \alpha_{m_2 + 1}^{(2)}, \ldots, \alpha_1^{(\ell)}, \ldots, \alpha_{m_\ell + 1}^{(\ell)}\}. \quad (14)
\]

These numbers are the positive poles for the ISP \(P_A = Z_A \cup T Z_A\).

Conversely, the positive poles of a given ISP \(P_A\) are the simple numbers \(Z_A\). The equivalence class for \(Z_A\) contains at least one reduced number \(\beta_1\) of the form \((13)\). As a result, matrices corresponding to poles in \(P_A\) are all in the same conjugacy class as \(M_{\beta_1} \sim W\), where \(W\) is a conjugacy class generator block. The product \(W\) (and so the conjugacy class) is primitive and hyperbolic because \(M_{\beta}\) is primitive and hyperbolic.

We summarize our result the following Theorem.

**Theorem 2.** Fix \(p \geq 3\) and let \(\lambda = \lambda_p\). There is a one-to-one correspondence between primitive hyperbolic conjugacy classes of \(G_p\) and ISPs for RPFs on \(G_p\). In particular, a conjugacy class generator product of the form \((12)\) corresponds to a set of simple numbers \(Z_A\) of the form \((13)\).

We simplify subsequent calculations by observing that when we translate between generator products and \(\lambda\)-CFs every generator (except \(V_1\)) corresponds to the sequence of \(\lambda\)-CF entries listed in Table 1. The occurrence of \(V_m\) corresponds to the addition of \(m\) to the following \(\lambda\)-CF entry, making that entry \(2 + m\).

**Table 1: Translation between generators and \(\lambda\)-CF periods**

| Generator | \(\lambda\)-CF entry |
|-----------|----------------------|
| \(V_1^m\) | add \(m\) |
| \(V_2\)   | \([2]\) |
| \(V_3\)   | \([2, 1]\) |
| \(\vdots\) | \(\vdots\) |
| \(V_{p-1}\) | \([2, 1, 1, \ldots, 1]\) 
|           | \(_{p-3}\) |

### 3.2 Counting poles and ISPs

One consequence of our construction is that the number of positive poles in an ISP is the same as the block length for the corresponding conjugacy class.
Corollary 2. Fix $p \geq 3$ and let $\lambda = \lambda_p$. A primitive hyperbolic conjugacy class in $G_p$ has block length $n$ if and only if the corresponding RPF ISP has $n$ positive poles.

Proof. We observe that $\sum_{i=1}^{\ell} (m_i + 1)$ is the block length of the product (12) and the cardinality of $Z_d$ in (11).

Given a Hecke group $G_p$ we let $B_p(n)$ denote the number of ISPs for $G_p$ that have $n$ positive poles. By Corollary 2, $B_p(n)$ is also the number of primitive hyperbolic conjugacy classes in $G_p$ with block length $n$. Calculating $B_p(n)$ is (essentially) the problem of counting the number of primitive cyclic words of length $n$ on $q$ letters.

Cyclic words are sometimes called necklaces because we identify each word with its cyclic permutations, as we have done with conjugacy class generators by conjugation. A cyclic word of length $n$ has period $n$, and possibly sub-period $d$ for some $d | n$. A word is primitive if it has no nontrivial sub-period.

Let $P_q(n)$ denote the number of primitive words of length $n$ on $q$ letters, and $C_q(n)$ denote the number of primitive cyclic words of length $n$ on $q$ letters. Then $C_q(n)$ is the number of primitive necklaces with $n$ beads chosen from $q$ colors. The following result is well-known.

Lemma 2. The number of primitive cyclic words of length $n$ with $q$ letters is

$$C_q(n) = \frac{1}{n} \sum_{d | n} \mu(d) q^{n/d}.$$ 

We give a short proof inspired by a proof of the formula for $P_q(n)$ in [GGW58].

Proof. There are $q^n$ words of length $n$ with $q$ letters. Every word has a sub-period that divides $n$, so

$$q^n = \sum_{d | n} P_q(d).$$

By Möbius inversion we have

$$P_q(n) = \sum_{d | n} \mu \left( \frac{n}{d} \right) q^{d} = \sum_{d | n} \mu(d) q^{n/d},$$

where $\mu(n)$ is the Möbius function. If we identify cyclic permutations every word has $n$ equivalent words, so

$$C_q(n) = \frac{1}{n} P_q(n) = \frac{1}{n} \sum_{d | n} \mu(d) q^{n/d}. $$
For $n > 1$ we have $B_p(n) = C_{p-1}(n)$. But there are two parabolic conjugacy class generator products of block length 1 ($V_1$ and $V_{p-1}$), so
$$B_p(1) = C_{p-1}(1) - 2 = p - 3.$$ 

**Corollary 3.** Fix $p \geq 3$ and let $\lambda = \lambda_p$. The number of ISPs with $n$ positive poles for the Hecke group $G_p$ is
$$B_p(n) = \begin{cases} 
p - 3, & n = 1 \\
C_{p-1}(n), & n > 1. \end{cases}$$

In Table 2 we list the number of ISPs with small numbers of positive poles for several Hecke groups.

### Table 2: Number of ISPs in $G_p$ with $n$ positive poles

| $n$ | $p=3$ | $p=4$ | $p=5$ | $p=6$ | $p=7$ |
|-----|-------|-------|-------|-------|-------|
| 1   | 0     | 1     | 2     | 3     | 4     |
| 2   | 1     | 3     | 6     | 10    | 15    |
| 3   | 2     | 8     | 20    | 40    | 70    |
| 4   | 3     | 18    | 60    | 150   | 315   |
| 5   | 6     | 48    | 204   | 624   | 1554  |
| 6   | 9     | 116   | 670   | 2580  | 7735  |
| 7   | 18    | 312   | 2340  | 11160 | 39990 |
| 8   | 30    | 810   | 8160  | 48750 | 209790 |

### 3.3 ISPs with few poles

We can write all ISPs that have $n$ positive poles by finding all conjugacy class generator products with block length $n$. We illustrate this by writing the ISPs with fewest number of poles for several groups. In Section 5 we write RPFs for each of these ISPs.

**Example 1.** Each of the following ISPs has the smallest possible cardinality for the given group.

1. The modular group $G_3$ has no hyperbolic conjugacy class generators, so it has no ISPs with 2 poles. The single hyperbolic product of length 2 in $G_3$ is $V_1V_2$. The corresponding reduced number is $[3] = \frac{3 + \sqrt{5}}{2}$ and the two simple numbers in that class are $\alpha_1 = [2; \overline{3}] = \frac{1 + \sqrt{5}}{2}$ and $\alpha_2 = [1; \overline{3}] = \frac{-1 + \sqrt{5}}{2}$. The ISP is $P_A = Z_A \cup T Z_A = \left\{ \frac{1 + \sqrt{5}}{2}, \frac{-1 + \sqrt{5}}{2} \right\}$. 

2. The group $G_4$ has a single hyperbolic generator $V_2$. The corresponding reduced number is $[\overline{2}] = 1 + \sqrt{2}$ and the simple number in that class is $\alpha = \lfloor 1; \overline{2} \rfloor = 1$. The ISP is $P_A = Z_A \cup TZ_A = \{1, -1\}$.

3. The group $G_5$ has two hyperbolic generators $V_2$ and $V_3$.

   (a) The generator $V_2$ corresponds to the reduced number $[\overline{2}] = \lambda + \sqrt{\lambda}$. The simple number in the class is $\alpha = \lfloor 1; \overline{2} \rfloor = \sqrt{\lambda}$, and the ISP is $P_{A_2} = Z_{A_2} \cup TZ_{A_2} = \{\sqrt{\lambda}, -\sqrt{\lambda}\}$.

   (b) The generator $V_3$ corresponds to the reduced number $[\overline{2}, 1] = \lambda + \frac{1}{\sqrt{\lambda}}$. The simple number in the class is $\beta = \lfloor 1; \overline{2}, 1 \rfloor = \frac{1}{\sqrt{\lambda}}$, and the ISP is $P_{A_3} = Z_{A_3} \cup TZ_{A_3} = \{\frac{1}{\sqrt{\lambda}}, -\frac{1}{\sqrt{\lambda}}\}$.

4. The group $G_6$ has three hyperbolic generators $V_2, V_3,$ and $V_4$.

   (a) The generator $V_2$ corresponds to the reduced number $[\overline{2}] = \sqrt{3} + \sqrt{2}$. The simple number in the class is $\alpha = \lfloor 1; \overline{2} \rfloor = \sqrt{2}$, and the ISP is $P_{A_4} = Z_{A_4} \cup TZ_{A_4} = \{\sqrt{2}, -\sqrt{2}\}$.

   (b) The generator $V_3$ corresponds to the reduced number $[\overline{2}, 1] = \sqrt{3} + 1$. The simple number in the class is $\alpha = \lfloor 1; \overline{2}, 2 \rfloor = 1$, and the ISP is $P_{A_5} = Z_{A_5} \cup TZ_{A_5} = \{1, -1\}$.

   (c) The generator $V_4$ corresponds to the reduced number $[\overline{2}, 1, 1] = \sqrt{3} + \frac{1}{\sqrt{2}}$. The simple number in the class is $\alpha = \lfloor 1; \overline{2}, 1, 2 \rfloor = \frac{1}{\sqrt{2}}$, and the ISP is $P_{A_6} = Z_{A_6} \cup TZ_{A_6} = \{\frac{1}{\sqrt{2}}, -\sqrt{2}\}$.

4 Hecke-conjugation

In this section we study Hecke-conjugation and Hecke-symmetry, which play important roles in the structure of ISPs and RPFs. We use the results to give several characterizations of Hecke-symmetry. Of particular importance is a connection between Hecke-conjugate ISPs and the transpose of conjugacy class generators.

4.1 Background

We first state several facts from [CR01] and [Res09].

Suppose $A$ is an equivalence class of $\lambda_\nu$-BQFs. Then $-A = \{-Q|Q \in A\}$ is another equivalence class of forms, not necessarily distinct from $A$. The numbers associated with the forms in $-A$ are the Hecke-conjugates of the numbers associated with the forms in $A$. The ISP associated with $A$ is

$$P_A = Z_A \cup TZ_A = Z_A \cup Z'_{-A}.$$
An ISP $P_A$ has an even number of poles; the positive half is in $Z_A$ and the negative half is in $Z_A'\setminus Z_A = Z_{-A} \cup Z'_{(-A)} = P_{-A}$.

If we take Hecke-conjugates of an ISP $P_A$ for $G_p$ we get

$$P'_A = Z'_A \cup Z_{-A} = Z_{-A} \cup Z'_{(-A)} = P_{-A},$$

which is another ISP not necessarily distinct from $P_A$. If $P_A = P_{-A}$ then $P_A$ has Hecke-symmetry. If $P_A \neq P_{-A}$ then $P_A$ and $P_{-A}$ are Hecke-conjugate ISPs and $P_A \cup P_{-A}$ has Hecke-symmetry.

### 4.2 Characterizing Hecke-conjugation

It will be useful to have several characterizations of Hecke-conjugate ISPs.

**Lemma 3.** Fix $p \geq 3$ and let $\lambda = \lambda_p$. Suppose that $A_1$ and $A_2$ are two hyperbolic equivalence classes of $\lambda$-BQFs with associated ISPs (respectively) $P_{A_1}$ and $P_{A_2}$.

The following statements are equivalent.

1. $P'_{A_1} = P_{A_2}$.
2. $-A_1 = A_2$.
3. $TZ'_{A_1} = Z_{A_2}$.
4. $\alpha'_1 \sim \alpha_2$ for every $\alpha_1 \in Z_{A_1}$ and $\alpha_2 \in Z_{A_2}$.

**Proof.** The equivalence of (a), (b) and (c) is contained in Section 4.1. We show that (b) and (d) are also equivalent.

(b) $\Rightarrow$ (d) We suppose that $-A_1 = A_2$ and let $\alpha_1 \in Z_{A_1}$ and $\alpha_2 \in Z_{A_2}$. Then (c) also holds, so $T\alpha'_1 \in TZ'_{A_1} = Z_{A_2}$, so $\alpha'_1 \sim \alpha_2$.

(d) $\Rightarrow$ (b) Suppose that $\alpha'_1 \sim \alpha_2$ for every $\alpha_1 \in Z_{A_1}$ and $\alpha_2 \in Z_{A_2}$. Fix two such numbers $\alpha_1$ and $\alpha_2$. Then by Lemma 1 the corresponding $\lambda$-BQFs satisfy $Q_{\alpha'_1} \sim Q_{\alpha_2}$. But $Q_{\alpha'_1} \in -A_1$ and $Q_{\alpha_2} \in A_2$, and because non-disjoint equivalence classes are identical we have $-A_1 = A_2$.

The next lemma shows that Hecke-conjugation is related to taking the transpose of corresponding matrices.
Lemma 4. Fix $p \geq 3$ and let $\lambda = \lambda_p$. Suppose that $M \in G_p$ is hyperbolic, that $Q_M$ is the associated $\lambda$-BQF, and that $\alpha_M$ is the associated fixed point. Then

$$Q_M^\top = -Q_M \circ T,$$

and

$$\alpha_M^\top = T \alpha'_M.$$

Proof. If we write $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $M^\top = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$. For the first part we use $Q_M = \begin{pmatrix} c & -a \\ d & -b \end{pmatrix}$ and $Q_M^\top = \begin{pmatrix} b & d \\ a & c \end{pmatrix}$ to calculate that

$$-Q_M \circ T = \begin{pmatrix} -c & a \\ -a & d \end{pmatrix} \circ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & d \\ a & c \end{pmatrix} = Q_M^\top.$$

For the second result we use the facts that $\alpha_M = a - d + \sqrt{D}$ and $\alpha_M^\top = a - d + \sqrt{D}$, where $D = (a + d)^2 - 4$, to calculate that

$$T \alpha'_M = T \left( \frac{a - d - \sqrt{D}}{2c} \right) = \frac{a - d + \sqrt{D}}{2b} = \alpha_M^\top.$$

We let $\mathcal{M}_A$ denote the set of matrices that corresponds to $Z_A$, that is,

$$\mathcal{M}_A = \{ M_\alpha : \alpha \in Z_A \}.$$

By Corollary 1 the elements of a given $\mathcal{M}_A$ are all in the same conjugacy class. We also use the transpose symbol to denote the set of all transpose matrices, as in

$$\mathcal{M}_A^\top = \{ M^\top : M \in \mathcal{M}_A \}.$$

The next lemma shows that the transposes of the matrices for the positive poles in an ISP are associated with the positive poles in the conjugate ISP.

Lemma 5. Fix $p \geq 3$ and let $\lambda = \lambda_p$. Suppose that $A_1$ and $A_2$ are two hyperbolic equivalence classes of $\lambda$-BQFs with associated ISPs (respectively) $P_{A_1}$ and $P_{A_2}$. Then $P_{A_1}$ and $P_{A_2}$ are Hecke-conjugate ISPs if and only if $M_{A_1}^\top = M_{A_2}$.

Proof. We first suppose that $P_{A_1}$ and $P_{A_2}$ are Hecke-conjugate ISPs and let $M \in M_{A_1}^\top$. Then $M^\top \in \mathcal{M}_{A_1}$, so $\alpha_M^\top \in Z_{A_1}$. Now $\alpha_M^\top = T \alpha'_M$ by Lemma 4 so $T \alpha'_M \in Z_{A_1}$. Thus $\alpha_M \in TZ_{A_1}$ by Lemma 3 so $M \in \mathcal{M}_{A_2}$. This shows that $\mathcal{M}_{A_1}^\top \subseteq \mathcal{M}_{A_2}$. For containment in the other direction we can reverse the steps above. Containment in both directions means that $\mathcal{M}_{A_1}^\top = \mathcal{M}_{A_2}$.

Next we suppose that $\mathcal{M}_{A_1}^\top = \mathcal{M}_{A_2}$. Let $\alpha_1 \in Z_{A_1}$ and $\alpha_2 \in Z_{A_2}$. Put $M = M_{\alpha_1}$, so $\alpha_1 = \alpha_M$. Then $M^\top \in \mathcal{M}_{A_1}^\top = \mathcal{M}_{A_2}$. But $\alpha_M^\top = T \alpha'_M = T \alpha'_1$ by Lemma 4 so $\alpha'_1 \in Z_{A_2}$ and $\alpha'_1 \sim \alpha_2$. Then Lemma 3 gives us that $P_{A_1}$ and $P_{A_2}$ are Hecke-conjugate ISPs. □
This allows us to characterize conjugate ISPs using the generators of their corresponding conjugacy classes.

**Lemma 6.** Fix \( p \geq 3 \) and let \( \lambda = \lambda_p \). Suppose that \( W_1 \) and \( W_2 \) are products of conjugacy class generators in \( G_p \) with associated ISPs (respectively) \( P_{A_1} \) and \( P_{A_2} \). Then \( P_{A_1} \) and \( P_{A_2} \) are Hecke-conjugate ISPs if and only if \( W_1^T \sim W_2 \).

**Proof.** By Theorem 2 we have that elements of \( M_{A_i} \) are in the same conjugacy class as \( W_1 \) and elements of \( M_{A_2} \) are in the same conjugacy class as \( W_2 \). Then the result follows from Lemma 5.

The transpose of a product of matrices is the product (in reverse order) of the transpose matrices, so in order to use Lemma 6 we need to calculate the transpose of each generator. But the transpose of each conjugacy class generator is another conjugacy class generator.

**Lemma 7.** Fix \( p \geq 3 \) and let \( \lambda = \lambda_p \). The conjugacy class generators satisfy \( V_j^T = V_{p-j} \) for \( 1 \leq j \leq p-1 \).

**Proof.** In [HR13] we show that \( V_j = \begin{pmatrix} a_j & a_{j+1} \\ a_{j-1} & a_j \end{pmatrix} \), where \( a_j = \frac{\sin(j\pi/p)}{\sin(\pi/p)} \).

Because \( \sin((p-j)\pi/p) = \sin(\pi - j\pi/p) = \sin(j\pi/p) \) we have \( a_{p-j} = a_j \), \( a_{p-j+1} = a_{j-1} \), and \( a_{p-j-1} = a_{j+1} \). Thus

\[
V_{p-j} = \begin{pmatrix} a_{p-j} & a_{p-j+1} \\ a_{p-j-1} & a_{p-j} \end{pmatrix} = \begin{pmatrix} a_j & a_{j-1} \\ a_{j+1} & a_j \end{pmatrix} = V_j^T.
\]

If a conjugacy class generator product is \( W_1 = V_{j_1}V_{j_2} \cdots V_{j_{n-1}}V_{j_n} \), the transpose is

\[
W_1^T = V_{j_n}^T V_{j_{n-1}}^T \cdots V_{j_2}^T V_{j_1}^T = V_{p-j_n} V_{p-j_{n-1}} \cdots V_{p-j_2} V_{p-j_1}.
\]

another product of conjugacy class generators. Two generator products in the same conjugacy class must be cyclic permutations of each other. Thus \( W_1^T \sim W_2 \) if and only if \( W_1^T \) is a cyclic permutation of \( W_2 \). This gives us another way to characterize conjugate ISPs using conjugacy class generators.

**Theorem 3.** Fix \( p \geq 3 \) and let \( \lambda = \lambda_p \). Suppose that \( W_1 \) and \( W_2 \) are products of conjugacy class generators in \( G_p \) with associated ISPs (respectively) \( P_{A_1} \) and \( P_{A_2} \). Then \( P_{A_1} \) and \( P_{A_2} \) are Hecke-conjugate ISPs if and only if \( W_1^T \) is a cyclic permutation of \( W_2 \).
4.3 Hecke-symmetry

An ISP $P_A$ is Hecke-symmetric if it is its own Hecke-conjugate, so our characterizations of Hecke-conjugation give us characterizations of Hecke-symmetry.

**Corollary 4** (to Lemma 3). Fix $p \geq 3$ and let $\lambda = \lambda_p$. Suppose that $A$ is a hyperbolic equivalence class of $\lambda$-BQF\textrsfs with associated ISP $P_A$. The following statements are equivalent.

(a) $P'_A = P_A$.
(b) $-A = A$.
(c) $TZ'_A = Z_A$.
(d) $\alpha' \sim \alpha$ for every $\alpha \in Z_A$.

There is a connection between Hecke-symmetry of ISPs and transpose properties of conjugacy class generators. The next theorem is a corollary to Lemma 5, Lemma 6, and Theorem 3.

**Theorem 4.** Fix $p \geq 3$ and let $\lambda = \lambda_p$. Suppose that $A$ is a hyperbolic equivalence class of $\lambda$-BQF\textrsfs with associated ISP $P_A$. The following statements are equivalent.

(a) $P_A$ has Hecke-symmetry.
(b) $M_A^\top = M_A$.
(c) $W_A^\top \sim W$.
(d) $W^\top$ is a cyclic permutation of $W$.

We can easily determine the symmetry properties of an ISP $P_A$ from the associated conjugacy class generator product $W$. If $W^\top$ is a cyclic permutation of $W$ then $P_A$ has Hecke-symmetry. If $W^\top$ is not a cyclic permutation of $W$ then $P_A$ does not have Hecke-symmetry, $P_{-A}$ is the ISP that is Hecke-conjugate to $P_A$, and the union $P_A \cup P_{-A} = P_A \cup P'_A$ does have Hecke-symmetry.

**Example 2.** We use conjugacy class generators to determine Hecke-symmetry for the ISPs in Example 1.

1. In $G_3$ we found that $W = V_1V_2$ gave us the ISP $P_A = \left\{ \frac{1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2} \right\}$. We calculate that $W^\top = V_2^\top V_1^\top = V_1V_2 = W$, so $P_A$ has Hecke-symmetry.
2. In $G_4$ we found that $W = V_2$ gave us the ISP $P_A = \{1, -1\}$. We calculate that $W^\top = V_2^\top = V_2 = W$, so $P_A$ has Hecke-symmetry.
3. In $G_5$ the hyperbolic generators are $V_2$ and $V_3$.  

17
(a) The generator $W = V_2$ gave us the ISP $\mathcal{P}_{A_1} = \{\sqrt{2}, \frac{1}{\sqrt{2}}\}$. We calculate that $W^\top = V_2^\top = V_3 \not\sim W$, so $\mathcal{P}_{A_1}$ does not have Hecke-symmetry.

(b) The generator $W = V_3$ gave us the ISP $\mathcal{P}_{A_2} = \{\frac{1}{\sqrt{2}}, -\sqrt{2}\}$. We calculate that $W^\top = V_3^\top = V_2 \not\sim W$, so $\mathcal{P}_{A_2}$ does not have Hecke-symmetry.

The fact that $V_2^\top = V_3$ means that ISPs $\mathcal{P}_{A_1}$ and $\mathcal{P}_{A_2}$ are Hecke-conjugate and their union has Hecke-symmetry.

4. In $G_6$ the hyperbolic generators are $V_2, V_3$, and $V_4$.

(a) The generator $W = V_2$ gave us the ISP $\mathcal{P}_{A_1} = \{\sqrt{2}, \frac{1}{\sqrt{2}}\}$. We calculate that $W^\top = V_2^\top = V_4 \not\sim W$, so the corresponding ISP $\mathcal{P}_{A_1} = \{\sqrt{2}, \frac{1}{\sqrt{2}}\}$ does not have Hecke-symmetry.

(b) The generator $W = V_3$ gave us the ISP $\mathcal{P}_{A_2} = \{1, -1\}$. We calculate that $W^\top = V_3^\top = V_3 = W$, so $\mathcal{P}_{A_2}$ has Hecke-symmetry.

(c) The generator $W = V_4$ gave us the ISP $\mathcal{P}_{A_3} = \{\frac{1}{\sqrt{2}}, -\sqrt{2}\}$. We calculate that $W^\top = V_4^\top = V_2 \not\sim W$, so $\mathcal{P}_{A_3}$ does not have Hecke-symmetry.

The fact that $V_2^\top = V_4$ means that ISPs $\mathcal{P}_{A_1}$ and $\mathcal{P}_{A_3}$ are Hecke-conjugate and their union has Hecke-symmetry.

5 Examples of rational period functions

Our procedure for finding ISPs takes us a long way toward constructing RPFs for Hecke groups. In this section we write several rational period functions for the ISPs we've already found, as well as for two new ISPs. Most of the RPFs are new.

5.1 Simple constructions

It is straightforward to write rational period functions for two classes of poles and weights.

(i) If $\mathcal{P}_A$ has Hecke-symmetry, an RPF with poles $\mathcal{P}_A$ is given by (9) for any odd $k$.

(ii) If $\mathcal{P}_{A_1}$ and $\mathcal{P}_{A_2}$ non-Hecke-symmetric but conjugate to each other then an RPF with poles $\mathcal{P}_{A_1} \cup \mathcal{P}_{A_2}$ is given by (11) for any $k$.

Example 3. Consider the ISPs from Examples 1 and 2. The Hecke-symmetric ISPs give us RPFs with the smallest number of poles in $G_3$, $G_4$, and $G_6$, but
only for \( k \) odd. The non-Hecke-symmetric ISPs give us RPFs with more poles for any \( k \) when combined with Hecke-conjugate ISPs for \( G_5 \) and \( G_6 \). The RPF for \( G_3 \) is well-known; the others are new.

1. In \( G_3 \) we found the Hecke-symmetric ISP \( P_A = \left\{ \frac{1 + \sqrt{5}}{2}, \frac{1 - \sqrt{5}}{2} \right\} \). If \( k \) is odd an RPF of weight \( 2k \) and poles \( P_A \) is

\[
q(z) = \frac{1}{(z^2 - z - 1)^k} + \frac{1}{(z^2 + z - 1)^k}.
\]

This was the first known RPF with nonzero poles for the modular group, constructed by Marvin Knopp \[\text{[Kno78]}\].

2. In \( G_4 \) we found the Hecke-symmetric ISP \( P_A = \{1, -1\} \). If \( k \) is odd an RPF of weight \( 2k \) and poles \( P_A \) is

\[
q(z) = \frac{1}{(z^2 - 1)^k}.
\]

3. In \( G_5 \) we found the ISPs \( P_{A_1} = \left\{ \sqrt{\lambda}, -\sqrt{\lambda} \right\} \) and \( P_{A_2} = \left\{ \frac{1}{\sqrt{\lambda}}, -\sqrt{\lambda} \right\} \). Neither ISP has Hecke-symmetry but they are conjugate to each other. A rational period function of weight \( 2k \) (for any \( k \)) with poles in \( P_{A_1} \cup P_{A_2} \) is

\[
q(z) = \frac{1}{(z^2 - \lambda)^k} - \frac{(-1)^k}{(\lambda z^2 - 1)^k}.
\]

4. In \( G_6 \) we found the ISPs \( P_{A_1} = \left\{ \sqrt{2}, -\sqrt{2} \right\} \), \( P_{A_2} = \{1, -1\} \), and \( P_{A_3} = \left\{ \frac{1}{\sqrt{2}}, -\sqrt{2} \right\} \). The ISP \( P_{A_2} \) has Hecke-symmetry, so if \( k \) is odd an RPF of weight \( 2k \) and poles \( P_{A_2} \) is

\[
q(z) = \frac{1}{(z^2 - 1)^k}.
\]

The ISPs \( P_{A_1} \) and \( P_{A_3} \) do not have Hecke-symmetry but they are conjugate to each other. A rational period function of weight \( 2k \) (for any \( k \)) with poles in \( P_{A_1} \cup P_{A_3} \) is

\[
q(z) = \frac{1}{(z^2 - 2)^k} - \frac{(-1)^k}{(2z^2 - 1)^k}.
\]

We could start with any conjugacy class generator block, write the corresponding ISP, then use (9) or (11) to write an RPF. We illustrate this in the next example.

**Example 4.** We start with two similar generator products of block length 3 in \( G_6 \) and write their corresponding ISPs corresponding RPFs.
(a) The generator block \( W = V_1 V_3 V_5 \) satisfies \( W^\top = (V_1 V_3 V_5)^\top = V_1 V_3 V_5 = W \), so the corresponding ISP has Hecke-symmetry and the corresponding RPF has 3 positive poles. The reduced number for \( W = V_1 V_3 V_5 \) is \([3,1,2,1,1,1]\), so the simple numbers in the class are \([2;1,2,1,1,1,3]\), \([1;1,2,1,1,1,3]\), and \([1;1,1,1,3,1,2]\). The ISP is

\[
P_A = \mathcal{Z}_A \cup T \mathcal{Z}_A
\]

\[
= \left\{ \frac{2 + \sqrt{7}}{\sqrt{3}}, \frac{-1 + \sqrt{7}}{\sqrt{3}}, \frac{-1 + \sqrt{7}}{2\sqrt{3}} \right\} \cup \left\{ \frac{2 - \sqrt{7}}{\sqrt{3}}, \frac{-1 - \sqrt{7}}{\sqrt{3}}, \frac{-1 - \sqrt{7}}{2\sqrt{3}} \right\}.
\]

If \( k \) is odd an RPF on \( G_6 \) of weight \( 2k \) and poles \( P_{A_1} \cup P_{A_2} \) is

\[
q_1(z) = \frac{1}{(3\sqrt{3}z^2 - 12z - 3\sqrt{3})^k} + \frac{1}{(3\sqrt{3}z^2 + 6z - 6\sqrt{3})^k} + \frac{1}{(6\sqrt{3}z^2 + 6z - 3\sqrt{3})^k},
\]

which we simplify to the RPF

\[
q(z) = 3^k q_1(z)
\]

\[
= \frac{1}{(\sqrt{3}z^2 - 4z - \sqrt{3})^k} + \frac{1}{(\sqrt{3}z^2 + 2z - 2\sqrt{3})^k} + \frac{1}{(2\sqrt{3}z^2 + 2z - \sqrt{3})^k}.
\]

(b) The generator block \( W_1 = V_1 V_2 V_5 \) satisfies \( W_1^\top = (V_1 V_2 V_5)^\top = V_1 V_2 V_5 = W_1 \). Now \( W_1 \not\sim W_2 \) so the conjugate ISPs \( P_{A_1} \) and \( P_{A_2} \) do not have Hecke-symmetry. Each ISP has 3 positive poles so the union has 6 positive poles.

The reduced number for \( W_1 = V_1 V_2 V_5 \) is \([3,2,1,1,1]\), so the simple numbers in \( P_{A_1} \) are \([2;2,1,1,1,3]\), \([1;2,1,1,1,3]\), and \([1;1,1,1,3,1,2]\). The ISP for \( W_1 \) is

\[
P_{A_1} = \mathcal{Z}_A \cup T \mathcal{Z}_A
\]

\[
= \left\{ \frac{3\sqrt{3} + \sqrt{47}}{4}, \frac{-\sqrt{3} + \sqrt{47}}{4}, \frac{-2\sqrt{3} + \sqrt{47}}{7} \right\}
\]

\[
\cup \left\{ \frac{3\sqrt{3} - \sqrt{47}}{5}, \frac{-\sqrt{3} - \sqrt{47}}{11}, \frac{-2\sqrt{3} - \sqrt{47}}{5} \right\}.
\]

The reduced number for \( W_2 = V_1 V_2 V_5 \) is \([3,1,1,2,1,1]\), so the simple numbers in \( P_{A_2} \) are \([2;1,1,2,1,1,3]\), \([1;1,1,2,1,1,3]\), and \([1;1,1,1,3,1,1,2]\). The ISP for \( W_2 \) is

\[
P_{A_2} = \left\{ \frac{3\sqrt{3} + \sqrt{47}}{5}, \frac{-2\sqrt{3} + \sqrt{47}}{11}, \frac{-\sqrt{3} + \sqrt{47}}{5} \right\}
\]

\[
\cup \left\{ \frac{3\sqrt{3} - \sqrt{47}}{4}, \frac{-2\sqrt{3} - \sqrt{47}}{7}, \frac{-\sqrt{3} - \sqrt{47}}{4} \right\}.
\]
An RPF on $G_6$ of weight $2k$ (for any $k$) with poles $P_A \cup P_B$ is

$$q(z) = \frac{1}{(4z^2 - 6\sqrt{3}z - 5)^k} + \frac{1}{(4z^2 + 2\sqrt{3}z - 11)^k} + \frac{1}{(7z^2 + 4\sqrt{3}z - 7)^k} - \frac{1}{(-1)^k(5z^2 - 6\sqrt{3}z - 4)^k} - \frac{1}{(5z^2 + 4\sqrt{3}z - 7)^k} - \frac{1}{(-1)^k(11z^2 + 2\sqrt{3}z - 4)^k}.$$

5.2 More complicated constructions

It is more challenging to write rational period functions with poles in a single ISP $P_A$ that satisfies

(iii) $P_A$ has Hecke-symmetry and $k$ is even, or
(iv) $P_A$ does not have Hecke-symmetry.

The only previously published examples known to this author are

- several RPFs of weight 2 or 4 on $G_3$ with non-Hecke-symmetric ISPs [Haw],
- an RPF of weight 4 on $G_3$ with a Hecke-symmetric ISP [Haw, Kno89], and
- an RPF of weight 2 on $G_4$ with a non-Hecke-symmetric ISP [Res16].

We offer new examples for these cases for the ISPs in Examples 1 and 2. We use the fact that if there is an RPF of weight $2k$ with $k$ even and poles in a Hecke-symmetric ISP $P_A$ it must have the given by (8). If there is an RPF of weight $2k$ and poles in a non-Hecke-symmetric ISP $P_A$ it must have the given by (10).

**Example 5.** We again consider the ISPs from Examples 1 and 2.

1. In $G_3$ we found one Hecke-symmetric ISP with two positive poles, $P_A = \{\alpha_1, \alpha_2\} \cup \{\alpha'_1, \alpha'_2\} = \{\frac{1+i\sqrt{3}}{2}, -\frac{1+i\sqrt{3}}{2}\}$. In Example 3 we wrote an RPF of weight $2k$ for any odd $k$ and this ISP.

Now we write an RPF of weight 4 ($k = 2$) whose ISP is $P_A$. By (8) such an RPF must have the form

$$q(z) = q_{2,\alpha_1}(z) - q_{2,\alpha'_1}(z) + q_{2,\alpha_2}(z) - q_{2,\alpha'_2}(z) + \sum_{n=1}^{3} \frac{c_n}{z^n}.$$ 

We use use (8) and partial fraction decomposition to write

$$q(z) = \frac{-2/\sqrt{3}}{z - \alpha_1} + \frac{1}{(z - \alpha_1)^2} - \frac{2/\sqrt{3}}{z - \alpha'_1} - \frac{1}{(z - \alpha'_1)^2} + \frac{-2/\sqrt{3}}{z - \alpha_2} + \frac{1}{(z - \alpha_2)^2} - \frac{2/\sqrt{3}}{z - \alpha'_2} - \frac{1}{(z - \alpha'_2)^2} + \sum_{n=1}^{3} \frac{c_n}{z^n}.$$
Substitution into the two relations (3) and (4) shows both are satisfied if $c_1 = 8/\sqrt{5}$ and $c_2 = c_3 = 0$. Thus

$$q(z) = \frac{-2/\sqrt{5}}{z - \alpha_1} + \frac{1}{(z - \alpha_1)^2} - \frac{2/\sqrt{5}}{z - \alpha_1'} - \frac{1}{(z - \alpha_1')^2}$$

$$+ \frac{-2/\sqrt{5}}{z - \alpha_2} + \frac{1}{(z - \alpha_2)^2} - \frac{2/\sqrt{5}}{z - \alpha_2'} - \frac{1}{(z - \alpha_2')^2} + \frac{8/\sqrt{5}}{z},$$

is a rational period function of weight 4 on the modular group $G_3$ with the Hecke-symmetric ISP $P_A = \left\{ \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2} \right\}$. Hawkins first wrote this RPF in [Haw] and Knopp published it in [Kno89].

2. In $G_4$ we found one Hecke-symmetric ISP with one positive pole, $P_A = \{1, -1\}$. We wrote an RPF of weight $2k$ for any odd $k$ and this ISP in Example 3. An RPF of weight $2k$ for even $k$ must have the form

$$q(z) = q_{k,1}(z) - q_{k,-1}(z) + \sum_{n=1}^{2k-1} c_n z^n.$$

We let $k = 2$, use (6) and then partial fractions to find that $q$ has the form

$$q(z) = \frac{-1}{z - 1} + \frac{1}{(z - 1)^2} - \frac{1}{z - 1} - \frac{1}{(z + 1)^2} + \frac{3}{z} = \sum_{n=1}^{3} c_n z^n.$$

We substitute and find that the two relations (3) and (4) are satisfied if $c_1 = 2$ and $c_2 = c_3 = 0$. Thus

$$q(z) = \frac{-1}{z - 1} + \frac{1}{(z - 1)^2} - \frac{1}{z - 1} - \frac{1}{(z + 1)^2} + \frac{2}{z},$$

is an RPF of weight 4 on $G_4$ with Hecke-symmetric ISP $P_A = \{1, -1\}$.

3. In $G_5$ we found non-Hecke-symmetric ISPs with one positive pole each, $P_{A_1} = \{\sqrt{\lambda}, \frac{1}{\sqrt{\lambda}}\}$ and $P_{A_2} = \{\sqrt{\lambda}, -\sqrt{\lambda}\}$. In Example 3 we wrote an RPF of weight $2k$ (for any $k$) with poles in $P_{A_1} \cup P_{A_2}$.

(a) By (10) an RPF of weight $2k$ with poles only in $P_{A_1}$ has the form

$$q(z) = q_{k,\sqrt{\lambda}}(z) - q_{k,-\sqrt{\lambda}}(z) + \sum_{n=1}^{2k-1} c_n z^n.$$

We let $k = 1$ and substitute to find that the two relations (3) and (4) are satisfied if $c_1 = 0$. This gives us that

$$q(z) = \frac{1}{z - \sqrt{\lambda}} - \frac{\sqrt{\lambda}}{\sqrt{\lambda}z + 1}$$

is a rational period function of weight 2 with the non-Hecke-symmetric ISP $P_{A_1}$. 

22
(b) By (10) an RPF of weight $2k$ with poles only in $P_{A_2}$ has the form

$$q(z) = q_{k,1/\sqrt{\lambda}}(z) - q_{k,-\sqrt{\lambda}}(z) + \sum_{n=1}^{2k-1} \frac{c_n}{z^n}.$$  

We let $k = 1$ and substitute to find that the two relations (3) and (4) are satisfied if $c_1 = 0$. Thus

$$q(z) = \frac{\sqrt{\lambda}}{\sqrt{\lambda}z - 1} - \frac{1}{z + \sqrt{\lambda}}$$  

(21)

is a rational period function of weight 2 with the non-Hecke-symmetric ISP $P_{A_2}$.

4. In $G_6$ we found three ISPs with one positive pole each. The ISP $P_{A_2} = \{1, -1\}$ is Hecke-symmetric, and ISP $P_{A_1} = \{\sqrt{2}, -\sqrt{2}\}$ and $P_{A_3} = \{\frac{1}{\sqrt{2}}, -\sqrt{2}\}$ do not have Hecke-symmetry but are conjugate to each other. In Example 3 we wrote an RPF of weight $2k$ for $k$ odd with poles in $P_{A_2}$ and an RPF of weight $2k$ (for any $k$) with poles $P_{A_1} \cup P_{A_3}$.

(a) An RPF of weight $2k$ with poles only in $P_{A_1}$ has the form

$$q(z) = q_{k,1/\sqrt{2}}(z) - q_{k,-1/\sqrt{2}}(z) + \sum_{n=1}^{2k-1} \frac{c_n}{z^n}.$$  

For $k = 1$ we have that

$$q(z) = \frac{1}{z - \sqrt{2}} - \frac{\sqrt{2}}{\sqrt{2}z + 1},$$  

(22)

is a rational period function of weight 2 with the non-Hecke-symmetric ISP $P_{A_1}$.

(b) By (6) an RPF of weight $2k$ for even $k$ with poles in the Hecke-symmetric ISP $P_{A_2}$ must have the form

$$q(z) = q_{k,1}(z) - q_{k,-1}(z) + \sum_{n=1}^{2k-1} \frac{c_n}{z^n}.$$  

We let $k = 2$, use (6) and then partial fraction decomposition to find that $q$ has the form

$$q(z) = \frac{-1}{z - 1} + \frac{1}{(z - 1)^2} - \frac{1}{z - 1} - \frac{1}{(z + 1)^2} + \sum_{n=1}^{3} \frac{c_n}{z^n}.$$  

We substitute and find that the two relations (3) and (4) are satisfied if $c_1 = 2$ and $c_2 = c_3 = 0$. Thus

$$q(z) = \frac{-1}{z - 1} + \frac{1}{(z - 1)^2} - \frac{1}{z - 1} - \frac{1}{(z + 1)^2} + \frac{2}{z},$$  

23
is an RPF of weight 4 on $G_6$ with Hecke-symmetric ISP $P_{A_2} = \{1, -1\}$.  
(c) An RPF of weight $2k$ with poles only in $P_{A_k}$ has the form 

$$q(z) = q_{k,1/\sqrt{2}}(z) - q_{k,-\sqrt{2}}(z) + \sum_{n=1}^{2k-1} \frac{c_n}{z^n}.$$ 

For $k = 1$ we have that 

$$q(z) = \frac{\sqrt{2}}{\sqrt{2}z - 1} - \frac{1}{z + \sqrt{2}}, \quad (23)$$ 

is a rational period function of weight 2 with the non-Hecke-symmetric ISP $P_{A_3}$.

6 Families of rational period functions

Schmidt and Sheingorn point out in [SS95] that conjugacy class generators are $\lambda_p$-invariant. By this they meant that as $p$ changes, the values of the entries change but as functions of $\lambda$ the generators do not change. Rather, increasing $p$ changes the set of generators by adding matrices. They called this phenomenon the “$q$-principle.” (Schmidt and Sheingorn used $q$ to index the Hecke groups, so in our context this would be the “$p$-principle.”)

The $p$-principle also holds for products of conjugacy class generators. If a particular Hecke group $G_{p_0}$ has a conjugacy class with a product of generators then that product (with each matrix a function of $\lambda$) is a conjugacy class block for every Hecke group $G_p$ with $p \geq p_0$. This gives us a natural method for constructing families of RPFs across Hecke groups. We illustrate this with an example.

Example 6. Fix $p \geq 3$. The product $V_1V_{p-1}$ is a hyperbolic conjugacy class generator block in $G_p$. Moreover, $(V_1V_{p-1})^T = V_1V_{p-1}$ so the corresponding ISP is Hecke-symmetric and has 2 positive poles. Now 

$$V_1V_{p-1} = S(ST)^{p-2}S \sim S^3T(ST)^{p-3},$$ 

so the corresponding reduced number is $[3, 1, \ldots, 1]_{p-3}$ and the set of positive poles in the ISP is 

$$Z_A = \left\{ [2,1,\ldots,1,3]_{p-3}, [1,1,\ldots,1,3]_{p-3} \right\} = \left\{ \frac{\lambda_p + \sqrt{\lambda_p^2 + 4}}{2}, \frac{-\lambda_p + \sqrt{\lambda_p^2 + 4}}{2} \right\}.$$ 

This defines a family of ISPs $P_A = Z_A \cup T \cdot Z_A$ for $G_p$. Then by (9) we have that 

$$q(z) = \frac{1}{(z^2 - \lambda_p z - 1)^k} + \frac{1}{(z^2 + \lambda_p z - 1)^k}, \quad (24)$$ 

24
is an RPF of weight $2k$ ($k$ odd) for any $G_p$ with poles $P_A = \mathbb{Z}_A \cup T \mathbb{Z}_A$.

If we allow $p$ to take on any value we have a family of ISPs $P_A = \mathbb{Z}_A \cup T \mathbb{Z}_A$ and a corresponding family of RPFs for the Hecke groups. This is the family constructed by Parson and Rosen [PR84] and Schmidt [Sch93]. Marvin Knopp’s RPF (15) for $G_3$ in Example 3 is a member of this family.

6.1 Families of Hecke-symmetric ISPs

The family of ISPs in Example 6 is one of a larger class of Hecke-symmetric ISP families for the conjugacy generator products $V_j V_{p-j}$. For every fixed $j \geq 1$ the product $V_j V_{p-j}$ produces a family of ISPs for the Hecke groups with $p \geq 2j+1$. The restriction on $p$ ensures that $V_j \neq V_{p-j}$ and that the various families produced are distinct.

We can produce other Hecke-symmetric classes of ISP families by writing generator products that are conjugate to themselves. We list blocks that produce other classes of families of ISPs.

1. Products of the form $V_j^s V_{p-j}^s$ for $j, s \geq 1$ produce distinct families of ISPs for the groups $G_p$ with $p \geq 2j+1$.

2. Embedded products of the form in part 1, such as $V_j^s V_{p-j}^t V_{p-j}^{t-1}$ for $j, \ell, s, t \geq 1$ produce distinct families of ISPs for the groups $G_p$ with $p \geq \max\{2j+1, 2\ell+1\}$.

3. The generator $V_{p/2}$ produces a family of ISPs for the groups $G_p$ with $p$ even.

4. The generator $V_{p/2}$ embedded in products of the form in parts 1 or 2, such as $V_j^s V_{p/2} V_{p-j}^s$ for $j, s \geq 1$ produces distinct families of ISPs for the groups $G_p$ with $p$ even, $p \geq 2j+2$.

The next example gives two new families of RPFs obtained from two of these self-conjugate generator products.

Example 7.

1. Fix an even number $p$ with $p \geq 4$. Then $V_{p/2}$ is a hyperbolic conjugacy class generator in $G_p$ that corresponds to a Hecke-symmetric ISP with one positive pole. Now

$$V_{p/2} = (ST)^{\frac{p}{2}-1} S \sim S^2 T (ST)^{\frac{p}{2}-2},$$

so the corresponding reduced number is

$$\beta = \left[\frac{2, 1, \ldots, 1}{p/2-2}\right].$$
and the positive pole is in

\[ Z_A = \left\{ \left[ 1, \frac{1}{p/2-2}, \ldots, 1 \right] \right\} = \{ 1 \} . \]

Then (9) gives us that

\[ q(z) = \frac{1}{(z^2 - 1)^k} \]

is a family of RPFs of weight \( 2k \) (\( k \) odd) for \( G_p \) (\( p \geq 4 \)) with poles \( P_A = Z_A \cup T Z_A \). The RPFs (10) for \( G_4 \) and (18) for \( G_6 \) in Example 3 are members of this family.

2. Fix \( p \geq 5 \). Then the product \( V_2 V_{p-2} \) is a primitive hyperbolic conjugacy class generator block in \( G_p \). The corresponding ISP is Hecke-symmetric and has 2 positive poles. Now

\[ V_2 V_{p-2} = STS(ST)^{p-3}S \sim S^2T^2T(ST)^{p-4}, \]

so the corresponding reduced number is

\[ \beta = \left[ 2, \frac{2}{p-4}, 1, \ldots, 1 \right], \]

and the set of positive poles in the ISP is

\[ Z_A = \left\{ \left[ 1, \frac{1}{p-4}, \ldots, 1, 2 \right], \left[ 1, \frac{1}{p-4}, \ldots, 1, 2, 2 \right] \right\} = \left\{ \lambda_p^2 - 2 + \frac{\lambda_p^4 + 4}{2\lambda_p}, \lambda_p^1 - 2 + \frac{\lambda_p^4 + 4}{2\lambda_p} \right\} . \]

If \( k \) is odd (9) gives us

\[ q_1(z) = \frac{1}{(\lambda_p z^2 + (-\lambda_p^2 + 2)z - \lambda_p)^k} + \frac{1}{(\lambda_p z^2 + (\lambda_p^2 - 2)z - \lambda_p)^k} . \]

We multiply \( q_1(z) \) by \( \lambda_p^k \) to conclude that

\[ q(z) = \frac{1}{(z^2 + (-\lambda_p + 2/\lambda_p)z - 1)^k} + \frac{1}{(z^2 + (\lambda_p - 2/\lambda_p)z - 1)^k} , \quad (25) \]

is a family of RPFs of weight \( 2k \) (\( k \) odd) for \( G_p \) (\( p \geq 5 \)) with poles \( P_A = Z_A \cup T Z_A \).

If we let \( p = 3 \), so \( \lambda_p = 1 \) in (25) we have Marvin Knopp’s RPF for the modular group (13), even though the calculations above do not hold for \( p = 3 \). Thus we could think of (15) as a member of both families (24) and (25).
6.2 Families of non-Hecke-symmetric ISPs

We could construct families of non-Hecke-symmetric ISPs $P_{A_1}$ by starting with any conjugacy class generator product that is not conjugate to its transpose. We can use this to write a family of RPFs with non-Hecke-symmetric poles, at least for small weights. We can also write a family of RPFs with poles in the Hecke-symmetric $P_{A_1} \cup P_{A_2}$, where $P_{A_2}$ is the ISP conjugate to $P_{A_1}$.

**Example 8.** Fix $p \geq 5$. Then $V_2$ is a hyperbolic conjugacy class generator in $G_p$ that is not conjugate to its transpose. ($V_2$ is parabolic in $G_3$ and equal to its transpose in $G_4$.) The corresponding ISP is non-Hecke-symmetric and has one positive pole. Now $V_2 = STS \sim S^2T$ so the corresponding reduced number is $[2]$, and $Z_{A_1} = \{[1, 2]\} = \{\sqrt{\lambda_p^2 - 1}\}$. The ISP is

$$P_{A_1} = \left\{ \sqrt{\lambda_p^2 - 1}, -1/\sqrt{\lambda_p^2 - 1} \right\}.$$

The conjugacy class generator $V_2^T = V_{p-2}$ gives us the ISP conjugate to $P_{A_1}$. Now $V_{p-2} = (ST)^{p-3}S \sim S^2T(ST)^{p-4}$, so the corresponding reduced number is $[2, 1, \ldots, 1]_{p-4}$, and $Z_{A_2} = \left\{ \overline{[1, 1, \ldots, 1]}_{p-4} \right\} = \{1/\sqrt{\lambda_p^2 - 1}\}$. The ISP is

$$P_{A_2} = \left\{ 1/\sqrt{\lambda_p^2 - 1}, -\sqrt{\lambda_p^2 - 1} \right\},$$

By (10) an RPF of weight $2k$ with poles only in $P_{A_1}$ has the form

$$q(z) = q_k \sqrt{\lambda_p^2 - 1}(z) - q_{k-1}/\sqrt{\lambda_p^2 - 1}(z) + \sum_{n=1}^{2k-1} c_n z^n.$$  

We let $k = 1$ and substitute to find that the two relations (3) and (4) are satisfied if $c_1 = 0$. This gives us that

$$q(z) = \frac{1}{z - \sqrt{\lambda_p^2 - 1}} - \frac{\sqrt{\lambda_p^2 - 1}}{\sqrt{\lambda_p^2 - 1}z + 1},$$

is a family of RPFs of weight 2 for $G_p$ ($p \geq 5$) with poles in the non-Hecke-symmetric ISP $P_{A_1}$. The RPFs (20) for $G_5$ and (22) for $G_6$ in Example 5 are members of this family.

In a similar fashion we can use (10) to find that

$$q(z) = \frac{\sqrt{\lambda_p^2 - 1}}{\sqrt{\lambda_p^2 - 1}z - 1} - \frac{1}{z + \sqrt{\lambda_p^2 - 1}},$$

is a family of RPFs of weight 2 for $G_p$ ($p \geq 5$) with poles in the non-Hecke-symmetric ISP $P_{A_1}$. The RPFs (20) for $G_5$ and (22) for $G_6$ in Example 5 are members of this family.
is a family of RPFs of weight 2 for $G_p$ ($p \geq 5$) with poles in the non-Hecke-symmetric ISP $P_{A_2}$. The RPFs (21) for $G_5$ and (23) for $G_6$ in Example 5 are members of this family.

Finally, we can write a family of RPFs with poles in the Hecke-symmetric $P_{A_1} \cup P_{A_2}$. We use (11) to get that

$$q(z) = \frac{1}{(z^2 - (\lambda_p^2 - 1))^k} - \frac{(-1)^k}{((\lambda_p^2 - 1))z^2 - 1)^k},$$

is a family of RPFs of weight $2k$ (for any $k$) in $G_p$ ($p \geq 5$) with poles in $P_{A_1} \cup P_{A_2}$. The RPFs (17) for $G_5$ and (19) for $G_6$ in Example 5 are members of this family.

References

[Ash89] Avner Ash. Parabolic cohomology of arithmetic subgroups of $SL(2, \mathbb{Z})$ with coefficients in the field of rational functions on the Riemann sphere. *Amer. J. Math.*, 111(1):35–51, 1989.

[Cho89] YoungJu Choie. Rational period functions for the modular group and real quadratic fields. *Illinois J. Math.*, 33(3):495–530, 1989.

[CP90] YoungJu Choie and L. Alayne Parson. Rational period functions and indefinite binary quadratic forms. I. *Math. Ann.*, 286(4):697–707, 1990.

[CP91] YoungJu Choie and L. Alayne Parson. Rational period functions and indefinite binary quadratic forms. II. *Illinois J. Math.*, 35(3):374–400, 1991.

[CR01] Wendell Culp-Ressler. Rational period functions on the Hecke groups. *Ramanujan J.*, 5(3):281–294, 2001.

[CZ93] Yj. Choie and D. Zagier. Rational period functions for $PSL(2, \mathbb{Z})$. In *A tribute to Emil Grosswald: number theory and related analysis*, volume 143 of *Contemp. Math.*, pages 89–108. Amer. Math. Soc., Providence, RI, 1993.

[GGW58] S. W. Golomb, Basil Gordon, and L. R. Welch. Comma-free codes. *Canadian J. Math.*, 10:202–209, 1958.

[GR61] E. N. Gilbert and John Riordan. Symmetry types of periodic sequences. *Illinois J. Math.*, 5:657–665, 1961.

[Haw] John Hawkins. On rational period functions for the modular group. unpublished manuscript.

[Hec36] E. Hecke. Über die Bestimmung Dirichletscher Reihen durch ihre Funktionalgleichung. *Math. Ann.*, 112(1):664–699, 1936.
[HR13] Giabao Hoang and Wendell Ressler. Conjugacy classes and binary quadratic forms for the Hecke groups. *Canad. Math. Bull.*, 56(3):570–583, 2013.

[Kno74] Marvin I. Knopp. Some new results on the Eichler cohomology of automorphic forms. *Bull. Amer. Math. Soc.*, 80:607–632, 1974.

[Kno78] Marvin I. Knopp. Rational period functions of the modular group. *Duke Math. J.*, 45(1):47–62, 1978. With an appendix by Georges Grinstein.

[Kno81] Marvin I. Knopp. Rational period functions of the modular group. II. *Glasgow Math. J.*, 22(2):185–197, 1981.

[Kno89] M. I. Knopp. Recent developments in the theory of rational period functions. In *Number theory (New York, 1985/1988)*, volume 1383 of *Lecture Notes in Math.*, pages 111–122. Springer, Berlin, 1989.

[Par93] L. Alayne Parson. Rational period functions and indefinite binary quadratic forms. III. In *A tribute to Emil Grosswald: number theory and related analysis*, volume 143 of *Contemp. Math.*, pages 109–116. Amer. Math. Soc., Providence, RI, 1993.

[PR84] L. Alayne Parson and Kenneth H. Rosen. Automorphic integrals and rational period functions for the Hecke groups. *Illinois J. Math.*, 28(3):383–396, 1984.

[Res09] Wendell Ressler. On binary quadratic forms and the Hecke groups. *Int. J. Number Theory*, 5(8):1401–1418, 2009.

[Res16] Wendell Ressler. Hecke-symmetry and rational period functions. *Ramanujan J.*, 41(1-3):323–334, 2016.

[Ros54] David Rosen. A class of continued fractions associated with certain properly discontinuous groups. *Duke Math. J.*, 21:549–563, 1954.

[Sch93] Thomas A. Schmidt. Remarks on the Rosen λ-continued fractions. In *Number theory with an emphasis on the Markoff spectrum (Provo, UT, 1991)*, volume 147 of *Lecture Notes in Pure and Appl. Math.*, pages 227–238. Dekker, New York, 1993.

[Sch96] Thomas A. Schmidt. Rational period functions and parabolic cohomology. *J. Number Theory*, 57(1):50–65, 1996.

[SS95] Thomas A. Schmidt and Mark Sheingorn. Length spectra of the Hecke triangle groups. *Math. Z.*, 220(3):369–397, 1995.

[Zag81] D. B. Zagier. *Zetafunktionen und quadratische Körper*. Springer-Verlag, Berlin, 1981. Eine Einführung in die höhere Zahlentheorie. [An introduction to higher number theory], Hochschultext. [University Text].