Orbital classification in an N-body bar

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ABSTRACT
The dynamics and evolution of any galactic structure are strongly influenced by the properties of the orbits that constitute it. In this paper, we compare two orbit classification schemes, one by Laskar (NAFF), and the other by Carpintero and Aguilar (CA), by applying both of them to orbits obtained by following individual particles in a numerical simulation of a barred galaxy. We find that, at least for our case and some provisos, the main frequencies calculated by the two methods are in good agreement: for 80% of the orbits the difference between the results of the two methods is less than 5% for all three main frequencies. However, it is difficult to evaluate the amount of regular or chaotic bar orbits in a given system. The fraction of regular orbits obtained by the NAFF method strongly depends on the critical frequency drift parameter, while in the CA method the number of fundamental frequencies strongly depends on the frequency difference parameter $L_r$ and the maximum integer used for searching the linear independence of the fundamental frequencies. We also find that, for a given particle, in general the projection of its motion along the bar minor axis is more regular than the other two projections, while the projection along the intermediate axis is the least regular.

Key words: Galaxies: structure - galaxies: kinematics and dynamics - Galaxy: bulge

1 INTRODUCTION
Nearly two-thirds of spiral galaxies in the Universe have a bar structure (e.g., Lee et al. 2012, Buta et al. 2010, 2015). Bars are one of the main drivers for the secular evolution of disc galaxies (see Athanassoula 2013 for a review), and can transport material from the bar region to the center and redistribute angular momentum within the galaxy. This is emitted by the resonant regions in the bar and its vicinity, and absorbed by the outer parts of the disc and, mainly, by the spheroidal components (halo and bulge). Moreover, there is a strong correlation between the strength of the bar and the amount of angular momentum thus redistributed (Athanassoula 2003). Therefore, understanding the structure and the dynamical properties of bars is one of the most important issues in the formation and evolution of disc galaxies.

Orbits are the fundamental building blocks of any galactic structure and therefore their properties greatly influence those of the structure. Moreover, it is difficult to describe the phase-space distribution for the chaotic orbits, which can not be adopted to construct torus models (McMillan & Binney 2008). The orbit families and, more generally, the orbital structure in a fixed bar potential have been considered by many studies (Contopoulos & Papayannopoulos 1980, Zhao 1996, Häfner et al. 2000, Manos & Athanassoula 2011, Wang et al. 2012, 2013). Different methods of orbit classification have been used:

The Lyapunov exponent method (see e.g. Benettin et al. 1976, 1978 for a description). The Lyapunov exponents describe the time-averaged exponential rate of divergence of two orbits with close initial conditions in the phase space. Orbits with significantly non-zero Lyapunov exponents are chaotic.

The Small ALignment Index method (SALI, Skokos 2001, Voglis et al. 2002, Skokos et al. 2004, Carpintero et al. 2014). This method can be considered as an extension of the Lyapunov one, as it relies on the properties of two arbitrary different initial deviation vectors of an orbit, in order to distinguish efficiently between chaotic and regular orbits. The Generalized ALignment Index (GALI, Skokos et al. 2007) is similar to SALI, but uses a set of at least three initially linearly independent deviation vectors.

The NAFF method, short for Numerical Analysis of Fundamental Frequencies, relies on the fact that the regular orbits move on a torus-like manifold and are thus quasi-periodic (Laskar 1990, 1993). We will describe it further in Sect. 3.1.

The spectral analysis method uses the Fourier transform of the time series of each coordinate of a given orbit (Carpintero & Aguilar 1998, hereafter CA98). We will hereafter refer to this method as CA, from the initials of its authors, and describe it further in Sect. 3.2.

While each method has its advantages, each also suffers from disadvantages. For example, the Lyapunov method necessitates...
very long integration times and the fraction of chaotic orbits also depends on the integration time \cite{Merritt:Fridman:1999}; the SALI method also needs relatively long integration times, albeit much shorter than the Lyapunov method. The CA method has some problems for rotating systems \cite{Carraro:etal:2005} and depends strongly on the orbit integration time \cite{Wang:etal:2012}. Finally in the NAFF method whether an orbit is regular or not depends on the drift of its frequencies, so that a critical value needs to be adopted (See Sect. 5 in the present paper). Compared to other methods, CA and NAFF have an important advantage, namely they give more information for the regular orbits, such as their fundamental frequencies, from the ratios of which it is possible to define orbital families. Both of them have been successfully applied to various potential systems \cite[e.g.][]{Papaphilippou:Laskar:1998,Valluri:etal:2010,Bryan:etal:2012,Valluri:etal:2016}.

Most studies so far have relied on simple analytic potentials, which, however, are not very realistic. In particular, real bars as well as N-body bars are composed of two parts: an inner part which is thick both horizontally and vertically, and an outer part which is thin in both these directions, while as yet no analytical potential with such a property has been developed \cite[for a review]{Athanassoula:2016}. N-body bar potentials, however, are much more complex to use and there are therefore relatively few studies relying on them, compared to the large number of studies relying on analytic potentials. \cite{Manos:Machado:2014} and \cite{Machado:Manos:2016} took an intermediate path, using analytical time-dependent potentials modelled after an N-body simulation of a strongly barred galaxy. The disadvantage of this approach is that both the disc and the bar potentials are rigid and have not responded to each other, which is not realistic.

An alternative route, much nearer to the N-bodies, is to freeze the simulation potential at a representative time and then follow in it orbits with initial conditions obtained from the positions and velocities of the simulation particles at that chosen time \cite{Athanassoula:2002,Athanassoula:2003,Athanassoula:2005,Martinez-Valpuesta:etal:2006,Voglis:etal:2007,Wozniak:Michel-Dansac:2009,Valluri:etal:2012,Valluri:etal:2016}. This approach has a number of advantages. The corresponding potentials are realistic, and allow for orbital structure studies in bars with a thick inner part and a thin outer part. It also provides a unique and correct definition of the orbital sample which will be used, whereas in rigid potentials this sample is arbitrary, thus rendering any estimate of the fraction of chaos in a given system also entirely arbitrary. Indeed, whether a given orbit is regular or chaotic depends on its location within the galaxy’s phase space, and different samples may populate this space differently. This severe drawback of analytical potentials is easily avoided by relying on the simulation to provide the initial conditions of the orbits. Concerning disadvantages, let us mention that a correct description of the potential from the simulation particles is not trivial and also that the potential has been frozen i.e. does not depend on time. It is nevertheless possible to obtain information on time evolution by considering a series of consecutive times and of corresponding frozen potentials. Thus full time information can be obtained, but in a very time consuming manner.

A third alternative is to use directly the orbits of a preselected number of particles during the simulation \cite{Ceverino:Klypin:2007,Gaida:etal:2013,Gaida:etal:2016}. This attractively straightforward way has a number of difficulties, not the least being the fact that most of the available techniques and information on orbital structure have been obtained for non-evolving potentials. As we will show here, however, this third alternative can still be very useful if one chooses carefully the time interval over which one follows the orbits so that it has as little evolution as possible.

In this paper, we will give a detailed comparison of the CA and NAFF orbit classification methods by studying orbits in a simulated bar. The outline of the paper is as follows. In §2 we describe briefly our numerically simulated bar. In §3 we outline different methods of orbit classifications. In §4 we present the main frequencies from two methods. In §5 we present the fraction of regular orbits from different classification schemes. In §6 we give a brief discussion. In §7 we present the summary and conclusions.

2 THE SIMULATION AND BAR ORBITS

The initial conditions of this simulation comprise two components a disc and a halo. Both are live, i.e. described self-consistently, in order to allow exchange of angular momentum and thus a full bar growth \cite{Athanassoula:2002,Athanassoula:2003}. The initial density distribution of the disc is

$$\rho_d(R,z) = \frac{M_d}{4\pi h^2 z_0} \exp(-R/h) \sech^2 \left( \frac{z}{z_0} \right), \quad (1)$$

where $R$ is the cylindrical radius, $h$ is the disc radial scale length, $z_0$ is the disc vertical scale thickness and $M_d$ is the disc mass. The corresponding numerical values are $h = 3$ kpc, $z_0 = 0.6$ kpc and $M_d = 5 \times 10^{10} M_\odot$. For the halo we used an initial volume density of

$$\rho_h(r) = \frac{M_h}{2\pi r_c^2} \frac{\alpha}{r_c} \exp(-r^2/r_c^2) \sech^2 \left( \frac{r}{\gamma} \right),$$

where $r$ is the radius, $M_h$ is the halo mass, $\gamma$ and $r_c$ are the halo core and cut-off radii, respectively, and the constant $\alpha$ is given by

$$\alpha = \left[ 1 - \sqrt{q} \exp(q^2) \left[ 1 - \text{erf}(q) \right] \right]^{-1},$$

where $q = \gamma/r_c$ \cite{Hernquist:1993}. The numerical values used in this run are $r_c = 42.4$ kpc, $\gamma = 15$ kpc and $M_h = 19.54 \times 10^{10} M_\odot$. The halo is described by 1 million particles and the disc has 200 000 particles.

The initial conditions were built using the iterative method of \cite{Rodionov:etal:2009}, and to run the simulation we used a version of the GADGET3 code kindly made available to us by V. Springel. For a full description of GADGET see \cite{Springel:etal:2001,Springel:2005}. We adopted a softening length of 100 pc for the disc and of 200 pc for the halo and an opening angle of 0.5.

With these initial conditions, the disc dominates the potential in the inner parts, so that the bar forms very early on in the simulation.

The bar strength is defined as in \cite{Athanassoula:etal:2013}. More specifically, the Fourier components of the two-dimensional mass distribution can be written as

$$a_m(R) = \sum_{i=0}^{N_R} m_i \cos(m\theta_i), \quad m = 0, 1, 2, \ldots \quad (2)$$

$$b_m(R) = \sum_{i=0}^{N_R} m_i \sin(m\theta_i), \quad m = 1, 2, \ldots \quad (3)$$

where $N_R$ is the number of the particles inside a given annulus around the cylindrical radius $R$, $m_i$ is the $i$th particle mass and $\theta_i$
is its azimuthal angle. The \( a_m(R) \) and \( b_m(R) \) are a function of the cylindrical radius. The bar strength is measured by the maximum amplitude of the relative \( m = 2 \) component,

\[
A_2 = \max \left( \frac{\sqrt{a_2^2 + b_2^2}}{a_0} \right)
\]

where \( a_0 \) is given by equation (2) with \( m = 0 \). The evolution of the bar strength and the pattern speed with time are given in Fig. 1. We note that in the time interval 6 to 10 Gyr the bar strength and the pattern speed evolve little with time, so we analyse the orbits in this time interval. We selected a number of orbits visually, making sure that they were in the bar at the time of selection (6 Gyr). We then reran the simulation over the time range 6.0005-10.096 Gyr outputting only the positions, velocities and accelerations of the selected particles, but for a very large number of times (8192 outputs). We finally analysed 3094 orbits, whose initial positions at time 6.0005 Gyr are shown in Figure 2. The full disk at the nearby time (6.005 Gyr) is also presented in Figure 2. It is seen that the disk has a more extended range than that of the selected orbits. Here and elsewhere in this paper, the positions of these orbits are normalized by the corotation radius \( R_{CR} \).

### 3 ORBIT CLASSIFICATION BASED ON THE FREQUENCY MAPS

The Fourier spectral analysis technique was pioneered by Binney & Spergel (1982, 1984) to classify regular and chaotic orbits, and was then extended in different forms by Laskar (1993) and CA98. The key point of this method is that regular orbits are quasi-periodic, thus the Fourier spectra should consist of discrete lines and their frequencies can be expressed as integer linear combinations of \( N \) fundamental frequencies (where \( N \) is the dimension of the model). Chaotic orbits, however, are not quasi-periodic and the corresponding frequencies of the Fourier spectra cannot be reduced to integer combinations of up to only \( N \) basic frequencies.

Suppose that we have \( N_d \) consecutive sampled values \( z_{k'} \equiv z(t_k') \), where \( t_k' = k' \eta \), where \( \eta \) is sampling interval, and \( k' = 0, \ldots, N_d - 1 \). The discrete Fourier transform of \( z_{k'} \) can be written as

\[
Z_j = \frac{1}{N_d} \sum_{k' = 0}^{N_d-1} z_{k'} \exp \left( -i 2 \pi j k' \eta \right),
\]

where \( j = -N_d/2 + 1, \ldots, N_d/2 \). The Fourier spectrum consists of \( N_d \) waves with amplitudes \( |Z_j| \) and frequencies \( \Omega_j = 2\pi j / (N_d \eta) \). We also define three amplitudes \( |Z_{j,p}|, |Z_{j,v}|, \) and \( |Z_{j,pv}| \), where \( |Z_{j,p}| \) and \( |Z_{j,v}| \) correspond to the amplitudes from the position and velocity components, respectively, and \( |Z_{j,pv}| \) is given by \( \sqrt{|Z_{j,p}|^2 + |Z_{j,v}|^2} \). In this paper, we use \( |Z_j| \) to represent \( \sqrt{|Z_{j,p}|^2 + |Z_{j,v}|^2} \) unless stated otherwise. In order to facilitate the following discussions, we denote the time range 6.0005-8.048 Gyr as \( t_1 \), 8.0485-10.096 as \( t_2 \) and 6.0005-10.096 as \( t_{total} \) (\( t_1 = t_2 = \frac{t_{total}}{2} \)).

#### 3.1 NAFF

The numerical analysis of fundamental frequencies (NAFF) was pioneered by Laskar (1990, 1993), and developed further by Papaphilippou & Laskar (1996, 1998) for both two and three dimensional models. The key point of NAFF is that regular orbits move on a torus-like manifold and are thus quasi-periodic.

In an integrable system with \( N \) degrees of freedom, the Hamiltonian \( H(J, \theta) \) depends only on the actions \( J_j, H(J, \theta) = H(J_j) \),
and the equations of motion of the system are given by
\[ \dot{J}_j = 0, \quad \dot{\theta}_j(t) = \frac{\partial H}{\partial J_j} = \omega_j(J), \] (6)
where \( \theta_j \) are angle variables, and \( j = 1, 2, \ldots, N \). The orbit in the system can be written in terms of the complex variables
\[ z_j'(t) = J_j e^{i\omega_j t} = z_j'(0) \] (7)
where \( z_j(0) = z_j'(0) \). The motions in phase space take place on the surface of tori that are products of true circles with constant radii \( J_j = |z_j'(0)| \). The rate of the motions around a torus is determined by the frequency vector \( (\omega_1, \omega_2, \ldots, \omega_N) \). Generally, we do not know the precise action-angle variables \( (J_j, \theta_j) \), but we can find approximations \( (J'_j, \theta'_j) \). In the new coordinates, the motion can be written as
\[ f(t) = z_j'(t) + \sum_{k=1}^{\infty} A_k e^{i(k,\omega)t} \] (8)
where \( A_k \) are the complex amplitudes, and \( (k, \omega) = k_1 \omega_1 + k_2 \omega_2 + \ldots + k_N \omega_N \). In the limiting case, the coordinates \( (J'_j, \theta'_j) \) are action-angle variables, and the amplitudes \( A_k \) are close to zero.

In general, a system with more than one degree of freedom is not integrable. The Hamiltonian can be expressed as a perturbation of an integrable Hamiltonian \( H_0 \),
\[ H(J, \theta) = H_0(J) + \epsilon H_1(J, \theta), \] (9)
If the perturbation \( \epsilon \) is small, the Kolmogorov-Arnold-Moser (KAM) theorem suggests that a large fraction of the tori still exist and that the motion of most orbits is still quasi-periodic.

The frequency map analysis consists of obtaining a quasi-periodic approximation of the numerical solutions of the Hamiltonian system in Eq. (9) in the form of a finite number of terms without searching for an explicit transformation of coordinates in action-angle variables
\[ f(t) = z_j'(t) + \sum_{k=1}^{k_{\text{max}}} A_k e^{i(k,\omega)t} \] (10)
where \( k_{\text{max}} \) is the number of terms, and \( A_k \) are of decreasing amplitude.

A regular orbit is quasi-periodic, and the complex function combining its positions and velocities \( f(t) = X(t) + iV(t) \) can be expanded in a Fourier series (Binney & Tremaine 2008)
\[ f(t) = \sum_{k=1}^{k_{\text{max}}} A_k \exp(i\omega_k t) \] (11)
where \( \omega_k \) are the linear combinations of the fundamental frequencies, \( \omega_k = \omega_1 * \omega_1 + m_2 * \omega_2 + m_N * \omega_N \). \( A_k \) are the complex amplitudes and \( k_{\text{max}} \) is the number of terms. The NAFF algorithm is designed to obtain an approximate form of \( f(t) \)
\[ f'(t) = \sum_{k=1}^{k_{\text{max}}} A'_k \exp(i\omega'_k t) \] (12)
where the frequencies \( \omega'_k \) and complex amplitudes \( A'_k \) can be obtained by an iterative scheme. The first frequency \( \omega'_1 \) is searched by computing the maximum amplitude of \( \phi(\sigma) = \langle f(t), \exp(i\sigma t) \rangle \) where the scalar product \( \langle f(t), g(t) \rangle \) is given by
\[ \langle f(t), g(t) \rangle = \frac{1}{T} \int_{-T/2}^{T/2} f(t)g(t) \chi(t) dt, \] (13)
where \( T \) is the time interval, \( \chi(t) \) is the conjugate of \( g(t) \), and \( \chi(t) = 1 + \cos(2\pi t/T) \) is the Hannings window function. In the NAFF routine, the location of the primary frequency corresponds to the largest amplitude among the position spectrum \( |Z_{j,p}| \) and the velocity spectrum \( |Z_{j,v}| \). The location of the first frequency is around the primary frequency. Once the first frequency has been found, its complex amplitude \( A'_1 \) is obtained by the orthogonal projection \( A'_1 = \langle f(t), \exp(i\omega_1 t) \rangle \). The first frequency component is subtracted and the process is restarted on the remaining part of the \( f_1(t) = f(t) - A'_1 \exp(i\omega_1 t) \) to find the second frequency \( \omega_2 \). The process is repeated to find the third \( \omega_3 \), fourth \( \omega_4 \) and more frequency components until the residual function does not significantly decrease when subtracting the following term. The fundamental frequencies are from these selected frequencies.

For the regular orbits, the fundamental frequencies do not change with time. Therefore, the frequency drift of the fundamental frequencies in two intervals provides us the regular behavior of the orbits. The frequency drift is defined as (Valluri et al. 2010, 2012, 2016)
\[ \log(\Delta f_1) = \log \left| \frac{\omega_1(t_2) - \omega_1(t_1)}{\omega_1(t_1)} \right|, \] (14)
\[ \log(\Delta f_2) = \log \left| \frac{\omega_2(t_2) - \omega_2(t_1)}{\omega_2(t_1)} \right|, \] (15)
\[ \log(\Delta f_3) = \log \left| \frac{\omega_3(t_2) - \omega_3(t_1)}{\omega_3(t_1)} \right|. \] (16)
and the frequency drift parameter \( \log(\Delta f) \) is the largest value of \( \log(\Delta f_1) \), \( \log(\Delta f_2) \) and \( \log(\Delta f_3) \). The orbit will be chaotic if the frequency drift parameter is large. Usually, a critical value \( \log(\Delta f_0) \) is used to distinguish chaotic from regular orbits. If the frequency drift is smaller than the critical value \( \log(\Delta f_0) \), the orbit is classified as regular, otherwise, the orbit is chaotic. It is seen that the frequency drift in this definition is a relative drift; a shortcoming of this definition occurs when the fundamental frequency is large. In particular, the accuracy of the determination of the main frequencies of the ordinary FFT is of the order of \( 1/T \), and the NAFF method uses a Hannings window to search for the maximum peak in the spectrum, which increases the accuracy of the main frequencies to the order of \( 1/T^4 \) (Papamphilippou & Laskar 1996). Thus, the frequencies of the orbits can be recovered with high accuracy even for the chaotic orbits (Valluri & Merritt 1998). If the absolute values of the fundamental frequency in the first and second intervals are large, the relative value of \( |(\omega_1(t_1) - \omega_1(t_2))/\omega_1(t_1)| \) will still be small. Therefore, we also use a different definition of the frequency drift, which is given by
\[ \Delta F_1 = \left| \frac{\omega_1(t_1) - \omega_1(t_2)}{\delta \omega} \right|, \] (17)
\[ \Delta F_2 = \left| \frac{\omega_2(t_1) - \omega_2(t_2)}{\delta \omega} \right|, \] (18)
\[ \Delta F_3 = \left| \frac{\omega_3(t_1) - \omega_3(t_2)}{\delta \omega} \right|. \] (19)
where \( \delta \omega = 2\pi/(N\delta \eta) \), where \( N\delta \) is again the number of sampled points and \( \eta \) is the sampling interval. The frequency drift parameter \( \Delta F \) is taken as the largest value of \( \Delta F_1 \), \( \Delta F_2 \) and \( \Delta F_3 \). We take this frequency drift as the absolute frequency drift. A critical value \( \Delta F_0 \) is used to distinguish regular orbits from chaotic ones.
Figure 4. Distribution of the absolute frequency drift parameter $\Delta F$ (top) and the relative frequency drift parameter $\log(\Delta f)$ (bottom) for our selected 3094 orbits. The solid, dotted and dashed lines correspond to the comparison of different time ranges as labelled at the top right of the top panel.

Figure 4 shows the distribution of the absolute frequency drift parameter (top) and the relative frequency drift parameter (bottom) from the NAFF method between $t_1$ and $t_2$ (solid line). It is seen that most orbits have an absolute frequency drift smaller than $2\delta \omega$. The peak of the distribution of the relative frequency drift $\log \Delta f$ is around -1, which indicates a 10% frequency drift.

Generally, the frequency drift can be considered between any two different intervals, therefore, we also study the cases from the $t_1$ time range to $t_{\text{total}}$ and $t_2$ to $t_{\text{total}}$. In Figure 4 we show the frequency drift parameter from time $t_1$ to $t_{\text{total}}$ (dotted lines), and one from time $t_2$ to $t_{\text{total}}$ (dashed lines), respectively. It is seen that most orbits have smaller absolute frequency drift parameters in $t_1 - t_{\text{total}}$ and $t_2 - t_{\text{total}}$ than those in $t_1 - t_2$, which can be explained in the following way: The frequency resolution is twice higher for time $t_{\text{total}}$ time range than that for the $t_1$ and $t_2$ ones since we use the same time step to output the orbits. The absolute frequency drift parameter is calculated using $\delta \omega$ rather than $0.5\delta \omega$ in cases from time $t_1$ to $t_{\text{total}}$ and $t_1$ to $t_2$. In this paper, the orbit types are given by using the drift parameter from time $t_1$ to $t_2$ unless stated otherwise.

3.2 The CA method

The key point of CA is to find the number of the fundamental frequencies. In its initial form this method used only the position to do the Fourier transform (CA98). An updated version of the code uses the Frequency Modified Fourier Transform (Sidlichovsky & Nesvorný, 1998) (FMFT) to extract lines, and the spectral analysis is performed on both the position and velocity component $X(t) + iV(t)$, which is similar to what is done in the NAFF scheme.

In NAFF, the frequencies are calculated sequentially and any later frequency and amplitude depend on the previous ones. Once the previous ones are found, they will not change in the subsequent steps. After $k - 1$ cycles, the $k_{\text{th}}$ frequency is shifted from $\omega_k$ mostly due to the existence of close frequencies which have significant amplitudes. After a number of cycles this can lead to differences of the order of several $\delta \omega$. The FMFT method consists of the NAFF process but gives a correction of frequencies via Eq. (36) in Sidlichovsky & Nesvorný (1998). It is important to note that the frequencies and amplitudes in FMFT can change with the number of extracted lines because every frequency and amplitude are corrected by the primary selected peaks in the FFT spectrum. This is a major difference between the FMFT and the corresponding method used in NAFF (See Table 4 for an example).

The rightmost panels of Figure 5 show 10 lines extracted by the CA method with FMFT and 10 extracted by the NAFF method in the spectra of the three ($x, y, z$) components for orbit 2745. It is seen that most lines from the two methods agree, but some lines are significantly different. It is also noted that the primary frequency in the CA code is found by the largest amplitude $|Z_{j,p,v}|$ (defined below eq. 2) in the FFT spectrum, which is slightly different from that done in the NAFF method. For most orbits, the frequency with the largest amplitude $|Z_{j,p,v}|$ is consistent with the frequency with the largest amplitude among $|Z_{j,p}|$ and $|Z_{j,v}|$. However, for some orbits this is not true. In Table 2 we show the frequencies and amplitudes of the first 20 strongest lines in the FFT spectra of orbit 1315. It is seen that the frequency with the largest $|Z_{j,p,v}|$ is 78.242571, while the frequency with the largest amplitude among $|Z_{j,p}|$ and $|Z_{j,v}|$ is 119.665109 in the $x$ component.

We refer the interested readers to CA98 for a full description of their technique. Here we only give a brief overview and some modifications on the new version of their code. There is a clean distinction between the main and fundamental frequencies in this new version. The main frequencies are the frequencies whose amplitudes are the maximum (or second maximum) on each coordinate. These frequencies are used to determine whether or not the orbit is resonant. The fundamental frequencies are the independent frequencies. We will take an example to illustrate this difference. If there is no integer non-zero vector $(l, m, n)$ to satisfy $l\omega_1 + m\omega_2 + n\omega_3 = 0$, these main frequencies are independent. If the rest of the spectral lines can be expressed as the linear combinations of them, then the fundamental frequencies are the same as the main frequencies. If there are more than three independent frequencies, the number of fundamental frequencies will be 4, and thus the orbit is classified as an irregular type in CA. If there is one resonance, then the three main frequencies are not independent, the main frequencies are not the fundamental frequencies.

4 MAIN FREQUENCIES IN NAFF AND CA

There are two different conceptual frequencies in the literature, one is the fundamental frequency, the other is the main frequency. Unfortunately, these two are sometimes confusingly used.

In NAFF, the fundamental frequencies are frequencies of the angle variables in the case of a regular orbit for which the action/angle variables exist. In that case any coordinate time series will have a spectrum made of discrete lines at frequencies that can be written as linear combinations with integer coefficients of three independent “fundamentals frequencies”. However, unless the coordinates used are close to angle variables, there is no reason why the dominant line in one spectrum should be one of those fundamental frequencies. For box orbits, the fundamental frequencies are identified by the highest amplitude terms in the Cartesian coordinates. On the other hand, for tube orbits, the terms with the second or subsequent highest amplitudes are taken as the fundamental frequencies (Valluri & Merritt, 1998).

In the CA method, the main frequencies are frequencies with the maximum or subsequent highest amplitudes of each coordinate, which is the same as the “fundamental” frequencies in NAFF. The main frequencies in CA are used further to determine whether or
not there are resonances. If there is no resonance among the main frequencies, they may be taken as fundamental frequencies too. If, however, there are resonances, then the main frequencies are used to determine one to three linearly independent fundamental frequencies for regular orbits, or more than three for irregular orbits. Therefore, the main frequencies in CA coincide with the “fundamental” frequencies in NAFF. In the remaining of the paper, we will use the “main” frequencies and “fundamental” frequencies as defined in CA.

The first step to get the main frequencies is to extract the lines from the Fourier spectra. We use both the position and velocity components \( X(t) + iV(t) \) to get the spectrum for each component. In order for positions and velocities to contribute in a comparable manner, we use a normalized, dimensionless position and velocity to do the Fourier transform. The original position and velocity are divided by \( R_{\text{max}} \) and \( V_{\text{max}} \), respectively, where \( R_{\text{max}} = \sqrt{x^2 + y^2 + z^2} \) and \( V_{\text{max}} = \sqrt{v_x^2 + v_y^2 + v_z^2} \), \( x, y \) and \( z \) are the three positions, \( v_x, v_y \) and \( v_z \) are the three velocities and \( \{ \cdot \cdot \cdot \} \) denotes the average over time along the orbit.

The detailed method to extract the spectrum is given in Laskar (2003), we refer the reader to his paper for further details. Here we just point out that the strongest spectral lines in each component are obtained using an accurate numerical technique. In NAFF, all extracted lines are sorted by amplitude in descending order. The first main frequency \( \omega_1 \) corresponds to the line with the largest amplitude, the second main frequency \( \omega_2 \) is the next highest peak coming from a different component and a value different from the first main frequency. The third main frequency is one of the remaining frequencies, should come from the remaining component, and should not be any linear combination of \( \omega_1 \) and \( \omega_2 \).

In CA, the \( x, y \) and \( z \) axes should be aligned with the major, intermediate and minor axes of the system. Then the main frequency from each component should yield \( \omega_x < \omega_y < \omega_z \), where \( \omega_x, \omega_y \) and \( \omega_z \) are the highest peaks from the spectrum of the \( x \), \( y \) and \( z \) components, respectively. Therefore, if the frequency from the largest peak in each component does not satisfy \( \omega_x < \omega_y < \omega_z \), the CA method switches the corresponding coordinates, unless the two corresponding amplitudes are very close to each other. The first main frequency is the smallest frequency among \( \omega_x, \omega_y \) and \( \omega_z \). The second and third main frequencies are from the frequency components with intermediate and largest values among \( \omega_x, \omega_y \) and \( \omega_z \), respectively. In principle, when \( \omega_x < \omega_y < \omega_z \), then the second main frequency is from the spectrum of the \( y \) component, and the third main frequency is from the spectrum of the \( z \) component. However, in practice, when \( \omega_y \) is quite close to \( \omega_z \), then \( \omega_x \) is searched in descending order of amplitude in the \( y \) spectrum until \( \omega_x \) is significantly larger than \( \omega_x \). A similar treatment is adopted for the third main frequency.

Since the main frequencies are selected among the extracted lines in the spectrum in both methods, they may depend on the candidate number of the extracted lines \( L_{\text{max}} \). Figure 6 shows a comparison of the main frequencies obtained with \( L_{\text{max}} = 10 \) and with \( L_{\text{max}} = 12 \) for both methods. It is seen that only a small number (\(< 0.1\%\)) of the main frequencies in NAFF have been changed when using different values of \( L_{\text{max}} \), while about 6\% of the main frequencies have been changed in CA. Here the changes in the main frequencies from \( L_{\text{max}} = 10 \) to \( L_{\text{max}} = 12 \) mean that the largest frequency difference of \( |\omega_i(L_{\text{max}} = 12) - \omega_i(L_{\text{max}} = 10) - 1| \) (\( i = 1, 2, 3 \)) is larger than 0.01. It is easy to understand these changes of the main frequencies with the increasing number of \( L_{\text{max}} \) in both the NAFF and CA methods. In the CA method, the frequencies and amplitudes of the extracted lines are corrected by the next extracted lines, therefore, the frequency and amplitude from the extracted lines are changed when \( L_{\text{max}} \) is different. In the NAFF method, the increasing number of \( L_{\text{max}} \) may give new frequencies and amplitudes. To illustrate this, we show the frequencies and amplitudes of the extracted lines in orbit 1315 for both the NAFF and the CA methods in Table 1. From the definition of the main frequencies in the two methods, we know that \( \omega_1 = 79.181 \) (\( k = 3 \) in the \( x \) component) with \( L_{\text{max}} = 10 \), and \( \omega_1 = 125.802 \)
with $L_{\text{max}} = 12$ ($k = 11$) in NAFF. The shift of $\omega_1$ in NAFF is because a new line with a large amplitude is found in the eleventh step. In the CA code, the increasing number of $L_{\text{max}}$ changes the frequencies and amplitudes, therefore, $\omega_1$ can be changed. In Figure 7 we show the dependence of three main frequencies on the value of $L_{\text{max}}$ for orbit 1315 (Left) and orbit 1220 (Right). It is seen that the main frequencies from NAFF will not be changed if $L_{\text{max}} \geq 12$, while there is a small fluctuation along the $L_{\text{max}}$ value for orbit 1315. For orbit 1220, only $\omega_1$ has been changed at $L_{\text{max}} = 36$, and will be kept as a constant with $L_{\text{max}} > 36$. In order to avoid the missing lines and save the compute time, we adopt $L_{\text{max}} = 12$ in the remainder of the paper unless stated otherwise.

In NAFF, the absolute difference between the first and second main frequencies $|\omega_i - \omega_j|(i = 1, 2, 3, j = 1, 2, 3, \text{ and } i \neq j)$ must be larger than a critical value $L_{r,a}$, which we define as the critical absolute frequency difference. In CA, the parameter to distinguish two frequencies is similar to NAFF, but with the value of the relative frequency difference $|\omega_i - \omega_j|/\sqrt{\omega_i^2 + \omega_j^2}$ larger than a critical value $L_r$. In order to compare the main frequencies in these two methods, we introduced a definition consistent with that of CA, i.e. $|\omega_i - \omega_j|/\sqrt{\omega_i^2 + \omega_j^2} > L_r$. In Figure 8 we show the comparison of main frequencies from $L_r = 2 \times 10^{-4}$ and $L_r = 2 \times 10^{-3}$, where the first one is suggested by the CA method. We found that around 6% and 1.5% of the orbits have a different main frequency in the NAFF and the CA methods, respectively. Here we define two main frequencies as different if $|\omega_i(L_r = 2 \times 10^{-4})/\omega_i(L_r = 2 \times 10^{-3}) - 1| (i = 1, 2, 3)$ is larger than 0.01.

Since our orbits are extracted from a simulation, they are necessarily much noisier than those obtained from an analytic potential. In order to estimate this effect on the main frequency detection, we will vary the absolute critical value $L_{r,a}$ in the NAFF method, to check whether any lines with very small amplitude are taken as the main frequencies. In Figure 9 we compare the main frequencies with different $L_{r,a}$, and find that even when the value of $L_{r,a}$ is increased from $10^{-6}$ to 1, only 22.3% of the orbits change their main frequencies: the largest frequency change of $|\omega_i(L_{r,a} = 1)/\omega_i(L_{r,a} = 2 \times 10^{-6}) - 1|$ ($i = 1, 2, 3$) is larger than 0.01. We define nine parameters to describe the corresponding amplitude variation:

$$R_{i,a} = A_i(L_{r,a} = 10^{-4})/A_i(L_{r,a} = 10^{-6}),$$

$$R_{i,b} = A_i(L_{r,a} = 10^{-2})/A_i(L_{r,a} = 10^{-6}),$$

$$R_{i,c} = A_i(L_{r,a} = 1)/A_i(L_{r,a} = 10^{-6}),$$

where $i = 1, 2, 3$ and $A_i$ are the amplitudes of the main frequencies $\omega_i$. In Figure 10 we show the distribution of these parameters $R_{i,a}, R_{i,b}$ and $R_{i,c}$ in the NAFF method. We can see that some lines with low amplitude appear as new main frequencies as $L_{r,a}$ increases. However, the number of orbits for which the ratio of the amplitudes is considerably different from unity is quite small. This is true even when we change this parameter by 6 orders of magnitude, from $10^{-6}$ to 1 (rightmost panels). In other words, the simulation noise does not affect the main frequencies significantly.

We also check the distribution of the amplitude ratios in the CA method, and find results similar to those in NAFF. Therefore, the effect of the critical parameter to distinguish two frequencies is small in both methods. However, in the CA method, if we increase the value of $L_r$, the number of fundamental frequencies may be changed significantly, which increases the fraction of regular orbits significantly. Indeed in CA, the parameter $L_r$ has two meanings: One is the frequency difference, which is the same as shown in our paper, while the other is the critical value determining whether an orbit is resonant, or not. In the CA code, if $|\omega_1 + m\omega_2 + n\omega_3|/\sqrt{(m\omega_1)^2 + (m\omega_2)^2 + (m\omega_3)^2}$ is smaller than $L_r$, then a resonance has been found. Since the number of the fundamental frequencies depends on the resonance number of the orbits, $L_r$ can affect the number of fundamental frequencies. In order to give a more detailed comparison of the main frequencies between the NAFF and CA methods, we adopt a relative critical value $L_r = 2 \times 10^{-4}$ in the remainder of the paper, unless otherwise indicated.

Figure 11 shows the histogram of the ratios of the three main frequencies from both the NAFF (bottom) and CA (top) methods. It is seen that there are typical peaks in these distributions, which indicate the intrinsic orbit types in our N-body bar. Note that the ordinate is in a logarithmic scale, which means that the peaks are very high, i.e. that many orbits are in families with well defined frequency ratios. For both methods, the face-on view, $(x, y)$, has two

![Figure 6](image-url)

**Figure 6.** Comparison of main frequencies from different values of $L_{\text{max}}$. The solid line represents equality of two frequencies for two different parameters $L_{\text{max}}$ (12 and 10). The top and bottom panels represent the results from the CA and NAFF methods, respectively. The sample interval is $t_{\text{total}}$.

![Figure 7](image-url)

**Figure 7.** Dependence of three main frequencies on $L_{\text{max}}$ for both NAFF (solid line) and CA (dotted line) for orbit 1315 (left panel) and orbit 1220 (right panel). The top, middle and bottom panels represents the results for $\omega_3$, $\omega_2$ and $\omega_1$, respectively. For orbit 1220, $\omega_1$ and $\omega_2$ from two methods are same.
The parameters the scale for the ordinate is logarithmic.

Figure 8. Comparison of main frequencies from different values of $L_r$. The red solid line represents equality of two frequencies for two different parameters $L_r$. The top and bottom panels represent the results from the CA and NAFF methods, respectively. The time interval used here is $t_{\text{total}}$.

Figure 9. Comparison of main frequencies from different values of $L_{r,a}$ in NAFF. The solid line represents equality of two frequencies for two parameters $L_{r,a}$. The sample interval is $t_{\text{total}}$.

Figure 10. The distribution of the amplitude ratios. From top to bottom, the results are for $\omega_3$, $\omega_2$, and $\omega_1$, respectively. The sample interval is $t_{\text{total}}$. The parameters $R_{\text{to}}, R_{\text{tb}}$, and $R_{\text{tc}}$ are defined in eqs. (20)-(22). Note that the scale for the ordinate is logarithmic.

clear peaks. The highest peak is for 1:1, and the second highest for 2:3, the two having amplitudes of 1120 and 436 in NAFF, 1913 and 434 in CA98. The two edge-on views, (y, z) and (x, z), also have two clear peaks one at 4:5 and the other at 4:7. The implications of this result will be discussed elsewhere.

We define 9 parameters to check whether a main frequency agrees in the two methods:

$$\delta f_i^{ij} = \left| \frac{\omega_i(\text{NAFF})}{\omega_j(\text{CA})} - 1 \right|,$$

with $i = 1, 2, 3$ and $j = 1, 2, 3$. We take the minimum value of $\delta f_i^{ij}$ as $\delta f_i$. If $\delta f_i$ is small, then at least one main frequency from CA is consistent with one from NAFF. The corresponding frequency to $\delta f_i$ is defined as $\omega_i'$. We find that 97% of the orbits have the minimum $\delta f_i$ smaller than 0.01, in other words, these orbits have at least one similar main frequency from the two methods, which can also be seen from the comparison of one similar main frequency in Figure 12. Then we define 18 parameters to check two main frequencies agree from two methods, these parameters are given as

$$\delta f_i^{ij} = \left| \frac{\omega_i(\text{NAFF})}{\omega_j(\text{CA})} - 1 \right| + \frac{\omega_i'(\text{NAFF})}{\omega_j'(\text{CA})} - 1.$$

with $i = 1, 2, 3$, $j = 1, 2, 3$, $i' = 1, 2, 3$, and $j' = 1, 2, 3$, and $i' \neq j$. The minimum value of $\delta f_i^{ij}$ is defined as $\delta f_i$.

Two main frequencies agree in the two methods if $\delta f_i$ is smaller than a critical value $\delta f_{2,0}$. If the remaining main frequencies from two methods are close (the frequency difference is smaller than $\delta f_{2,0}/2$), then all three main frequencies are in agreement. The first two identical main frequencies are defined as $\omega_i$ and $\omega_i'$, and the remaining main frequency as $\omega_i''$. We find 88% of the orbits have two frequencies in agreement and 39% of the orbits have three frequencies in agreement from two methods if $\delta f_{2,0} = 0.02$. If we increase $\delta f_{2,0}$ to 0.1, then 99% of the orbits have two frequencies in agreement and 80% of the orbits have three frequencies in agreement between the two methods. In other words, most orbits have an average difference in the main frequency smaller than 5%.

In Figure 13 we show the comparison of the three main frequencies from the two methods. The two first panels, referring to the two first frequencies, show a close equality, with all points distributed very close to the diagonal. The third panel, referring to the third frequency, has a different structure. About 80% of the points (2458 orbits) are around the diagonal, but with a considerably larger spread than for the first and second frequency. This may argue that this third frequency is less accurately defined than the other two. Note also that a considerable number of orbits (144 orbits, 5%) are located at the wings along the green and blue solid lines which follow $Y = AX$ with $A = 3/2$ (green) or $A = 2/3$ (blue), respectively. This could be due to a badly recognised third frequency.

As shown in Valluri et al. (2010), the accuracy of the frequency analysis decreases significantly when orbits were integrated for less than 20 oscillation periods, therefore, it is interesting to compare the main frequencies from the NAFF method with those from the CA method for orbits with more than 20 oscillation periods. In the top panel of Figure 14 we show the fraction of our orbits with fixed oscillation periods. We find that 70% orbits have more than 20 oscillation periods. In the bottom panel of Figure 14 we present the histogram of the oscillation periods. It is noted that the distribution peaks around 20. Therefore, the output interval for most orbits in our sample is reasonable for the frequency analysis. Figure 15 shows that the fraction of orbits having three frequencies...
Table 1. Frequencies and amplitudes extracted from the spectrum of orbit 1315 in the x (top), y (middle) and z (bottom) components using the NAFF and CA methods with different $L_{\text{max}}$.

| k | $\omega_k$ (NAFF) | $A_k$ (NAFF) | $L_{\text{max}} = 10$ | $\omega_k$ (NAFF) | $A_k$ (NAFF) | $L_{\text{max}} = 12$ | $\omega_k$ (CA) | $A_k$ (CA) | $L_{\text{max}} = 10$ | $\omega_k$ (CA) | $A_k$ (CA) | $L_{\text{max}} = 12$ |
|---|-----------------|-------------|---------------|-----------------|-------------|---------------|-----------------|-------------|---------------|-----------------|-------------|---------------|
| 1 | 118.649975     | 0.087493    | 118.649795    | 0.087493       | 123.906921  | 0.205454    | 123.903713    | 0.204517    |
| 2 | 118.649975     | 0.129730    | 118.649795    | 0.129730       | 123.906921  | 0.205454    | 123.903713    | 0.204517    |
| 3 | 79.180569      | 0.187178    | 79.180569     | 0.187178       | 78.497611   | 0.215303    | 78.493748     | 0.210555    |
| 4 | 83.774549      | 0.120298    | 83.774549     | 0.120298       | 121.152373  | 0.180770    | 121.152083    | 0.180511    |
| 5 | 76.471528      | 0.057949    | 76.471528     | 0.057949       | 132.023087  | 0.144923    | 83.037662     | 0.134594    |
| 6 | 122.210680     | 0.178263    | 122.210680    | 0.178263       | 80.38356   | 0.197389    | 80.38356      | 0.197389    |
| 7 | 79.180569      | 0.189038    | 79.180569     | 0.189038       | 130.690324  | 0.222754    | 130.690324    | 0.222754    |
| 8 | 83.774549      | 0.120298    | 83.774549     | 0.120298       | 121.152373  | 0.180770    | 121.152083    | 0.180511    |
| 9 | 76.471528      | 0.057949    | 76.471528     | 0.057949       | 132.023087  | 0.144923    | 83.037662     | 0.134594    |
| 10| 118.649975     | 0.129730    | 118.649795    | 0.129730       | 123.906921  | 0.205454    | 123.903713    | 0.204517    |
| 11| 125.801781     | 0.195953    | 133.928603    | 0.104530       | 124.332263  | 0.447271    | 124.359786    | 0.443748    |
| 12| 120.187879     | 0.289342    | 120.187879    | 0.289342       | 124.332263  | 0.447271    | 124.359786    | 0.443748    |

Figure 11. Histogram of the ratios of the main frequencies from the CA (top) and NAFF (bottom) methods. From left to right, the results are for the ratios with $\omega_1/\omega_2$, $\omega_2/\omega_3$, and $\omega_1/\omega_3$, respectively. From left to right, the vertical dashed lines represent ratios with 4/7, 2/3, 4/5 and 1, respectively. The sample interval is $t_{\text{total}}$. The ratio and the corresponding orbit numbers are indicated in the top-right corner of each panel.

Orbit classification in a N-body bar

Figure 11 shows the dependence of the fraction of regular orbits on the absolute critical frequency drift parameter (Left panel) and the relative critical frequency drift parameter (Right panel) from the NAFF method. It is seen that the fraction of regular orbits strongly depends on the critical frequency drift value, but it is difficult to give a reasonable choice.

5 FRACTION OF REGULAR ORBITS AS OBTAINED FROM NAFF AND CA

Once we have the main frequencies, NAFF classifies orbits as regular or chaotic using the frequency drift. CA classifies the orbits by finding the number of the fundamental frequencies. Figure 16 shows the dependence of the fraction of regular orbits on the absolute critical frequency drift parameter (Left panel) and the relative critical frequency drift parameter (Right panel) from the NAFF method. It is seen that the fraction of regular orbits strongly depends on the critical frequency drift value, but it is difficult to give a reasonable choice.
respectively. The amplitude is normalized by the largest values of the frequencies and amplitudes of the first 20 strongest lines in the FFT spectra of orbit 1315. From top to bottom, the results for the x, y and z components are:

| j  | \( |Z_{j,pu}| \) | \( \Omega_j \) | \( |Z_{j,pv}| \) | \( \Omega_j \) | \( |Z_{j,v}| \) |
|----|----------------|--------------|----------------|--------------|----------------|
| 1  | 78.242571      | 0.376261     | 78.242571      | 0.397491     | 119.665109    | 0.441276     |
| 2  | 79.77639       | 0.341673     | 131.938454     | 0.388476     | 79.77639      | 0.372881     |
| 3  | 131.938454     | 0.341376     | 76.708403      | 0.341961     | 78.242571     | 0.353760     |
| 4  | 119.665109     | 0.322304     | 133.472622     | 0.330353     | 124.267613    | 0.323890     |
| 5  | 118.130941     | 0.306547     | 121.199277     | 0.310250     | 127.339494    | 0.322616     |
| 6  | 130.404285     | 0.290761     | 79.77639       | 0.307313     | 118.130941    | 0.321816     |
| 7  | 133.472622     | 0.277273     | 118.130941     | 0.290476     | 130.404285    | 0.309150     |
| 8  | 76.708403      | 0.261478     | 116.596773     | 0.283636     | 82.845075     | 0.305882     |
| 9  | 124.267613     | 0.260451     | 130.404285     | 0.271128     | 131.938454    | 0.286639     |
| 10 | 127.339494     | 0.230389     | 122.733445     | 0.241251     | 81.310907     | 0.239860     |
| 11 | 121.199277     | 0.222195     | 115.062605     | 0.190504     | 125.801781    | 0.220963     |
| 12 | 82.845075      | 0.216798     | 124.267613     | 0.175397     | 133.472622    | 0.211252     |
| 13 | 122.733445     | 0.206655     | 87.447580      | 0.144521     | 130.404285    | 0.203641     |
| 14 | 116.596773     | 0.203209     | 119.665109     | 0.114172     | 75.174235     | 0.171446     |
| 15 | 81.310907      | 0.171751     | 75.174235      | 0.096095     | 122.733445    | 0.164955     |
| 16 | 125.801781     | 0.157773     | 88.981748      | 0.093776     | 128.870117    | 0.159648     |
| 17 | 135.006790     | 0.144317     | 96.652588      | 0.090859     | 76.708403     | 0.140725     |
| 18 | 115.062605     | 0.143714     | 84.379244      | 0.090526     | 73.640607     | 0.120937     |
| 19 | 75.174235      | 0.138975     | 111.994269     | 0.066164     | 85.913412     | 0.116579     |
| 20 | 128.870117     | 0.113232     | 95.118420      | 0.062765     | 95.118420     | 0.105852     |

Table 2. Frequencies and amplitudes of the first 20 strongest lines in the FFT spectra of orbit 1315. From top to bottom, the results for the x, y and z components, respectively. The amplitude is normalized by the largest values of \( |Z_{j,pu}|, |Z_{j,pv}| \) and \( |Z_{j,v}| \) in the x, y and z components.
In order to compare the ranking of the various orbits in regularity by two kinds of the critical value in the NAFF method, we rank all orbits as a function of their $\Delta F$ values. The ranking is defined as $r_{\Delta F}$. The most regular orbit will have $r_{\Delta F} = 1$ and the most chaotic one $r_{\Delta F} = 3094$. We then rank 3094 orbits as function of their log($\Delta f$) values, which is called $r_{\Delta f}$. Again the most regular orbit will have $r_{\Delta f} = 1$ and the most chaotic one $r_{\Delta f} = 3094$.

In CA, an orbit is classified by the number of the fundamental frequencies. An orbit is irregular if it has more than three independent fundamental frequencies, otherwise it is regular. In this method, there are two important parameters $L_r$ and $I_n$. The former is used to determine whether the two main frequencies are the same, and whether the main frequencies are at resonance, while
For than 50 oscillations for the two methods with different parameters. The upper panel of Figure 19 shows the number of orbits with more than 50 oscillation periods. The solid, dotted and dashed lines represent the results from the absolute critical value in NAFF, relative critical value in NAFF and CA, respectively. The different lines represent the results with the different parameters in two methods. The region with \((R/R_{CR})\) beyond 0.2 is ignored because there are no orbits with more than 50 oscillation periods.

CA, we find in general very good agreement between all methods, with a fraction of regular orbits around 53%

It seems that the regular fraction along the radius from NAFF with log(\(\Delta f_0\)) = -1.09 is consistent with that from CA with \(I_n = 18\) if \((R/R_{CR})\) is smaller than 0.18. If, however, we compare the orbit types from the two methods one by one, we find a small decrease, so that only 47% of the orbits have the same type in both NAFF and CA methods. When we compare the two NAFF methods, we find that a very large fraction, 95%, the same types when we use \(\Delta F_0 = 2.9\) and log(\(\Delta f_0\)) = -1.09 in NAFF.

If we take log(\(\Delta f_0\)) = -1.09 and consider the frequency drift parameter only in the x, y, and z components, respectively, then the regular orbit fractions are 56.2%, 52.2% and 72.8%. If we consider the frequency drift parameter in two components, x and y, x and z, y and z, then the regular orbit fractions with log(\(\Delta f_0\)) = -1.09 are 43.2%, 50.7% and 46.7%, respectively. Therefore, the z component is most regular, while the y component is most chaotic in the bar system. This is similar to the fact that the intermediate tube orbits are unstable in the triaxial system (e.g. Merritt & Fridman 1996, Binney & Tremaine 2008).

6 DISCUSSION

It seems difficult to give definite values of \(\Delta F_0\) and log(\(\Delta f_0\)) in NAFF, and \(L_t\) and \(I_n\) in CA to classify orbits, but we can attempt to do this by selecting some likely regular and chaotic orbits. We use 196 orbits which have the same main frequencies from the NAFF and CA methods, and have a small frequency drift \(\Delta F < 0.5\). We find that when we choose \(I_n \geq 30\) and \(L_t = 2 \times 10^{-4}\), most of these orbits are regular in the CA method. Even so, a few of these orbits, are still irregular when we take \(I_n = 30\). For example, as shown in Figure 20, orbit 160 is a regular orbit in NAFF, but we find there are some chaotic property in the y-z plane and this could explain why CA classifies it irregular.

Next we select 40 orbits which have the same main frequencies from the NAFF and CA methods, but with large frequency drift \(\Delta F > 9.3\). Most of them are irregular when we take \(I_n = 16\).
but orbits such as orbit 865, are still regular in the CA method. From Figure 21 usually this orbit is regular in each interval, but the shape changes with time. Since the three main frequencies are independent and no extra fundamental frequencies are found, the CA method classifies it as a regular box orbit. On the other hand, if we use the frequency drift method to classify it, it will be classified as irregular. This frequency drift, however, could perhaps be due to the slight potential changes with time and may not necessarily be due to the fact that the orbit is irregular.

There is a further point related to the evolution of the potential. Namely, we find that the smallest $\Delta F$ among the full 3094 orbits is 0.016, while the resolution of the main frequencies is usually $10^{-4} - 10^{-3}$. Thus, even the smallest $\Delta F$ is still larger than the frequency resolution. If we take $\Delta F_0 = 0.01$, then every orbit is chaotic in NAFF; a small $I_0$ in CA with $L_e = 2 \times 10^{-4}$ can give similar results, so we can say both methods are in good agreement, but this is only an extreme case. Compared with the CA method, the results of NAFF only weakly depend on $L_e$ and $L_{\text{max}}$. For the parameter $L_{\text{max}}$, if we do not take into account CPU time limits, we can make it as large as possible. Also the $I_0$ value may have to be chosen differently for different potentials in the CA method. The advantage of the CA method is that it can give independent fundamental frequencies of orbits, which can yield more detailed information about regular orbits.

7 SUMMARY AND CONCLUSIONS

Individual particle orbits are the backbone of any structure. It is thus important for understanding the formation and evolution of this structure to know whether the orbits that constitute it are chaotic or regular and, in the latter case, what family they are associated with. Bars, in particular, are a favourite field for such tests and thus many studies have addressed the orbital structure in bars. Most of them, however, use an analytic potential and are thus not very realistic (see e.g. Athanassoula 2016, for a review). A further disadvantage of such studies is that it is not trivial to choose the initial conditions for the orbits and the result can depend critically on this choice. Instead, we used here orbits taken directly from the simulation. This means that they have very realistic potentials, but at the expense of some noise and, particularly, some evolution of the potential.

As a first step towards understanding the orbital structure in bars, we compare in this paper two methods, the NAFF method of Laskar (1990) and the method of Carpintero & Aguilar (1998). We show how the main frequencies depend on the maximum extracted line number $L_{\text{max}}$ and on the parameter to distinguish two main frequencies $L_e$. We find that only a small number ($<0.1\%$) of the main frequencies in NAFF have been changed when using different values of $L_{\text{max}}$, while about 6% of the main frequencies have been changed in CA. If we change $L_e$ from $2 \times 10^{-4}$ to $2 \times 10^{-3}$, then around 6% and 1.5% of the orbits have a different main frequency in the NAFF and CA methods, respectively.

We find that, at least for our case, the main frequencies calculated by the two methods are in good agreement provided we use the same definitions and values for $L_{\text{max}}$ and $L_e$; for 80% of the orbits the differences between the results of the two methods are less than 5% for all three main frequencies. We also find that there are two clear peaks in the histogram of the ratios of the three main frequencies in both methods. The highest peak is 1:1, and the second highest is 2:3 for the face-on view $(x, y)$. The two edge-on views, $(y, z)$ and $(x, z)$ also have two clear peaks, one at 4:5 and the other at 4:7.

We find that the fraction of the regular orbits strongly depends on two parameters $L_e$ and $I_0$ in the CA method. The former is used to determine whether the two frequencies are the same and whether there are resonances among the main frequencies. The fraction of the regular orbits increases with increasing $L_e$ or $I_0$. In the NAFF methods, the fraction of the regular orbits strongly depends on the critical frequency drift parameter. The regular fraction is increased with increasing this parameter. However, it is difficult to give certain values of these parameters in both methods. The fact that there is no abrupt change from chaotic to regular reflects the fact that there is stickiness and confined chaos. We also find that, for a given particle, in general the projection of its motion along the bar minor axis is more regular than the other two projections, while the projection along the intermediate axis is the least regular.

Increasing the number of particles in the simulation will decrease the noise. In a future paper we plan to use a simulation with a considerably larger number of particles, to determine how noise may influence the results.
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REFERENCES

Athanassoula E., 2002, ApJ, 569, L83
Athanassoula E., 2003, MNRAS, 341, 1179
Athanassoula E., 2005, Annals of the New York Academy of Sciences, 1045, 168
Athanassoula E., 2013, Secular Evolution of Galaxies, ed. J. Falcón-Barroso & J. H. Knapen, Cambridge, UK: Cambridge University Press, p. 305
Athanassoula E., 2016, Galactic Bulges, 418, 391
Athanassoula E., Machado R. E. G., Rodionov S. A., 2013, MNRAS, 429, 1949
Benettin G., Galgani L., Giorgilli A., Strelcyn J.-M., 1978, Academie des Sciences Paris Comptes Rendus Serie B Sciences Physiques, 286, 431
Benettin G., Galgani L., Strelcyn J.-M., 1976, PhRvA, 14, 2338
Binney J., Spergel D., 1982, ApJ, 252, 308
Binney J., Spergel D., 1984, MNRAS, 206, 159
Binney J., Tremaine S., 2008, Galactic Dynamics: Second Edition. Princeton University Press
Bryan S. E., Mao S., Kay S. T., Schaye J., Dalla Vecchia C., Booth C. M., 2012, MNRAS, 422, 1863
Buta R. J., Sheth K., Athanassoula E., Bosma A., Knapen J. H., Laurikainen E., Salo H., Elmegreen D., Ho L. C., Zaritsky D., Courtois H., Hinz J. L., Muñoz-Mateos J.-C., Kim T., Regan M. W., Gadotti D. A., 2015, ApJS, 217, 32
Buta R. J., Sheth K., Regan M., Hinz J. L., Gil de Paz A., Menéndez-Delmestre K., Muñoz-Mateos J.-C., Seibert M., Laurikainen E., Salo H., Gadotti D. A., Athanassoula E., Bosma A., Knapen J. H., Ho L. C., 2010, ApJS, 190, 147
Carpintero D. D., Aguilar L. A., 1998, MNRAS, 298, 1
Carpintero D. D., Muzzio J. C., Navone H. D., 2014, MNRAS, 438, 2871
Carpintero D. D., Muzzio J. C., Vergne M. M., Wachlin F. C., 2003, Celestial Mechanics and Dynamical Astronomy, 85, 247
Ceverino D., Klypin A., 2007, MNRAS, 379, 1155
Contopoulos G., Papayanopoulos T., 1980, A&A, 92, 33
Gajda G., Loks E. L., Athanassoula E., 2015, ArXiv e-prints
Gajda G., Loks E. L., Athanassoula E., 2016, ArXiv e-prints
Häfner R., Evans N. W., Dehnen W., Binney J., 2000, MNRAS, 314, 433
Hernquist L., 1993, ApJS, 86, 389
Laskar J., 1990, Icar, 88, 266
Laskar J., 1993, Physica D Nonlinear Phenomena, 67, 257
Laskar J., 2003, ArXiv Mathematics e-prints
Lee G.-H., Park C., Lee M. G., Choi Y.-Y., 2012, ApJ, 745, 125
Machado R. E. G., Manos T., 2016, MNRAS, 458, 3578
Manos T., Athanassoula E., 2011, MNRAS, 415, 629
Manos T., Machado R. E. G., 2014, MNRAS, 438, 2201
Martinez-Valpuesta I., Shlosman I., Heller C., 2006, ApJ, 637, 214
McMillan P. J., Binney J. J., 2008, MNRAS, 390, 429
Merritt D., Fridman T., 1996, ApJ, 460, 136
Panapphilippou Y., Laskar J., 1996, A&A, 307, 427
Panapphilippou Y., Laskar J., 1998, A&A, 329, 451
Rodionov S. A., Athanassoula E., Sotnikova N. Y., 2009, MNRAS, 392, 904
Skokos C., 2001, Journal of Physics A Mathematical General, 34, 10029
Skokos C., Antonopoulos C., Bountis T. C., Vrahatis M. N., 2004, Journal of Physics A Mathematical General, 37, 6269
Skokos C., Bountis T. C., Antonopoulos C., 2007, Physica D Nonlinear Phenomena, 231, 30
Springel V., 2005, MNRAS, 364, 1105
Springel V., Yoshida N., White S. D. M., 2001, NewA, 6, 79
Šidlíkovský M., Nesvorný D., 1996, Celestial Mechanics and Dynamical Astronomy, 65, 137
Valluri M., Debattista V. P., Quinn T., Moore B., 2010, MNRAS, 403, 525
Valluri M., Debattista V. P., Quinn T. R., Roškar R., Wadsley J., 2012, MNRAS, 419, 1951
Valluri M., Merritt D., 1998, ApJ, 506, 686
Valluri M., Shen J., Abbott C., Debattista V. P., 2016, ApJ, 818, 141
Voglis N., Harsoula M., Contopoulos G., 2007, MNRAS, 381, 757
Voglis N., Kalapotharakos C., Stavropoulos I., 2002, MNRAS, 337, 619
Wang Y., Mao S., Long R. J., Shen J., 2013, MNRAS, 435, 1437
Wang Y., Zhao H., Mao S., Rich R. M., 2012, MNRAS, 427, 1429
Wozniak H., Michel-Dansac L., 2009, A&A, 494, 11
Zhao H. S., 1996, MNRAS, 283, 149