A micromagnetic theory of skyrmion lifetime in ultrathin ferromagnetic films:
SI Appendix

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1  Model

The starting point of our study is the continuum micromagnetic energy in the SI units [1–3]:

$$\mathcal{E}(M) = \frac{A}{M_s^2} \int_{\Omega \times (0,d)} |\nabla M|^2 \, d^3 r + \frac{K}{M_s^2} \int_{\Omega \times (0,d)} |M_\perp|^2 \, d^3 r + \mu_0 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\nabla \cdot M(r) \cdot \nabla \cdot M(r')}{8\pi |r - r'|} \, d^3 r \, d^3 r' + \frac{Dd}{M_s^2} \int_{\Omega} (M_\parallel \cdot \nabla \cdot M_\perp - M_\perp \cdot \nabla M_\parallel) \, d^2 r, \quad (1.1)$$

which consists of the sum of the exchange, magnetocrystalline anisotropy, magnetostatic and the interfacial Dzyaloshinskii-Moriya interaction (DMI) energies, respectively. Here $\Omega \subset \mathbb{R}^2$ is a two-dimensional domain specifying the shape of the ferromagnetic film of thickness $d$ in the plane, and the magnetization vector $M : \Omega \times (0,d) \to \mathbb{R}^3$ has constant length $|M| = M_s$, the saturation magnetization. The magnetization vector $M$ is extended by zero to the whole space in the definition of the stray field energy, and $\nabla \cdot M$ is understood distributionally. We write $M = (M_\perp, M_\parallel)$, where $M_\perp \in \mathbb{R}^2$ and $M_\parallel \in \mathbb{R}$ are, respectively, the in-plane and out-of-plane components of $M$, and use $\overline{M} = (\overline{M}_\perp, \overline{M}_\parallel)$ to denote the magnetization at one of the film surfaces. The constants $A, K, D$ and $\mu_0$ appearing in (1.1) are the exchange stiffness, the bulk uniaxial magnetocrystalline anisotropy constant with the easy axis normal to the film plane, the interfacial DMI constant normalized by the film thickness, and the vacuum permeability, respectively.

Our analysis is based on the stochastic Landau-Lifshitz-Gilbert (sLLG) equation describing the evolution of the magnetization vector $M = M(r,t)$ [4–6]:

$$\frac{\partial M}{\partial t} = \gamma \mu_0 \textbf{H}_{\text{eff}} \times M + \frac{\alpha}{M_s} M \times \frac{\partial M}{\partial t}, \quad (1.2)$$

where the effective field is given by $\textbf{H}_{\text{eff}} = -\frac{1}{\mu_0} \frac{\delta \mathcal{E}}{\delta M} + \sqrt{\sigma} \Xi$ and

$$\sigma = \frac{2\alpha k_B T}{\gamma \mu_0^2 M_s}, \quad (1.3)$$

and $\Xi$ is a suitably regularized three-dimensional spatiotemporal white noise in three space dimensions [7].

In the following, we consider a film of infinite extent, $\Omega = \mathbb{R}^2$, and use a standard simplification of the model appropriate for thin films, in which $M$ is assumed to be independent of thickness. We also invoke the local approximation to the stray field energy, in which the magnetostatic energy may be approximated by an effective shape anisotropy
term for \( d \lesssim \ell_{\text{ex}} \), where \( \ell_{\text{ex}} = \sqrt{2A/(\mu_0 M_s^2)} \) is the exchange length. Measuring lengths in the units of \( \ell_{\text{ex}} \) and the energy in the units of \( 2Ad \), and introducing \( m = M/M_s \) then yields the reduced energy \([3, 8–10]\)

\[
E(m) = \frac{1}{2} \int_{\mathbb{R}^2} \left\{ |\nabla m|^2 + (Q - 1)|m_\perp|^2 - 2\kappa m_\perp \cdot \nabla m_\parallel \right\} d^2r, \tag{1.4}
\]

where

\[
Q = \frac{2K}{\mu_0 M_s^2}, \quad \kappa = D \sqrt{\frac{2}{\mu_0 M_s^2 A}}. \tag{1.5}
\]

In turn, the sLLG equation, with the noise averaged over the film thickness reads

\[
\frac{\partial m}{\partial t} = -m \times h_{\text{eff}} + \alpha m \times \frac{\partial m}{\partial t}, \tag{1.6}
\]

where the time is measured in the units of \( \tau_0 = (\gamma \mu_0 M_s)^{-1} \) and

\[
h_{\text{eff}} = -\frac{\delta E}{\delta m} + \sqrt{2\alpha \varepsilon} \xi, \tag{1.7}
\]

where

\[
\varepsilon = \frac{k_B T}{2Ad} \tag{1.8}
\]

is an effective noise strength and \( \xi \) is a suitably regularized three-dimensional spatiotemporal white noise in two space dimensions.

Using the definition of \( E(m) \) we have

\[
\delta E(m) = \int_{\mathbb{R}^2} \left\{ \nabla m \cdot \nabla \delta m - (Q - 1)m_\parallel \delta m_\parallel - \kappa \left( \delta m_\perp \cdot \nabla m_\parallel + m_\perp \cdot \nabla \delta m_\parallel \right) \right\} d^2r
\]

\[
= -\int_{\mathbb{R}^2} \left\{ \Delta m_\parallel + (Q - 1)m_\parallel - \kappa \nabla \cdot m_\perp \right\} \delta m_\parallel d^2r
\]

\[
-\int_{\mathbb{R}^2} \left\{ \Delta m_\perp + \kappa \nabla m_\parallel \right\} \cdot \delta m_\perp d^2r, \tag{1.9}
\]

for any \( \delta m = (\delta m_\perp, \delta m_\parallel) \) such that \( m \cdot \delta m = 0 \). An explicit calculation yields

\[
h_{\text{eff}} = \Delta m + (Q - 1)m_\parallel \hat{z} - \kappa (\nabla \cdot m \hat{z} - \nabla m_\parallel) + \sqrt{2\alpha \varepsilon} \xi, \tag{1.10}
\]

where \( \nabla m_\parallel \) is extended by zero in the out-of-plane direction.

The obtained sLLG equation should be understood in the Stratonovich sense. However, since we are interested in the asymptotically small noise regime, \( \varepsilon \ll 1 \), when skyrmions are long-lived metastable states, we will neglect the Itô corrections to the sLLG equation and mostly not distinguish between the Itô and the Stratonovich formulations.
2 Equations in spherical coordinates

For each point in space and time, we define the following orthonormal basis:

\[ m = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta), \]  
\[ \hat{t} = (\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta), \]  
\[ \hat{p} = (-\sin \phi, \cos \phi, 0). \]  

We also define \( \mathbf{p} = (-\sin \phi, \cos \phi), \mathbf{p}^\perp = (\cos \phi, \sin \phi) \) and \( \mathbf{t} = (\cos \theta \cos \phi, \cos \theta \sin \phi). \) Observe that \( \mathbf{m} \times \hat{t} = \hat{p}, \mathbf{m} \times \hat{p} = -\hat{t}. \) It may be verified by a direct computation that

\[ \mathbf{m}_t = \theta_t \hat{t} + \sin \theta \phi_t \hat{p} \]  
and

\[ \Delta \mathbf{m} = \left[ \Delta \theta - \sin \theta \cos \theta |\nabla \phi|^2 \right] \hat{t} + \left[ 2 \cos \theta \nabla \theta : \nabla \phi + \sin \theta \Delta \phi \right] \hat{p} \]
\[ - \left[ |\nabla \theta|^2 + \sin^2 \theta |\nabla \phi|^2 \right] \mathbf{m}. \]  

Furthermore, we have

\[ \hat{z} = (\cos \theta) \mathbf{m} - (\sin \theta) \hat{t}, \]  
\[ \hat{y} = \sin \phi [(\sin \theta) \mathbf{m} + (\cos \theta) \hat{t}] + (\cos \phi) \hat{p}, \]  
\[ \hat{x} = \cos \phi [(\sin \theta) \mathbf{m} + (\cos \theta) \hat{t}] - (\sin \phi) \hat{p} \]  
and

\[ \nabla \cdot \mathbf{m}_\perp = \sin \theta \nabla \phi \cdot \mathbf{p} + \cos \theta \nabla \theta \cdot \mathbf{p}^\perp, \quad \nabla \mathbf{m}_\parallel = -\sin \theta \nabla \theta, \]

where the gradients have been extended by zero into the \( z \)-direction when necessary.

We can rewrite the stochastic LLG equation by cross-multiplying both sides by \( \mathbf{m} \) as follows:

\[ \mathbf{m} \times \frac{\partial \mathbf{m}}{\partial t} + \alpha \frac{\partial \mathbf{m}}{\partial t} = -\mathbf{m} \times (\mathbf{m} \times \mathbf{h}_{\text{eff}}) = \mathbf{h}_{\text{eff}}(\mathbf{m}) - \mathbf{m}(\mathbf{h}_{\text{eff}}(\mathbf{m}) \cdot \mathbf{m}). \]  

We note that \( \mathbf{m} \times \mathbf{m}_t = \theta_t \hat{p} - (\sin \theta) \phi_t \hat{t}. \) In order to obtain the equations for \( \theta \) and \( \phi \), we project onto the two basis vectors \( \hat{t} \) and \( \hat{p} \) perpendicular to \( \mathbf{m} \). For the first one, we obtain

\[ (\mathbf{m} \times \mathbf{m}_t) \cdot \hat{t} + \alpha \mathbf{m}_t \cdot \hat{t} = \mathbf{h}_{\text{eff}} \cdot \hat{t}, \]

which is equivalent to

\[ -\sin \theta \phi_t + \alpha \theta_t = \Delta \theta - \sin \theta \cos \theta |\nabla \phi|^2 - (Q - 1) \cos \theta \sin \theta \]
\[ + \kappa \sin \theta (\sin \theta \nabla \phi \cdot \mathbf{p} + \cos \theta \nabla \theta \cdot \mathbf{p}^\perp - \nabla \theta \cdot \mathbf{t}) + \sqrt{2 \alpha \varepsilon} \eta, \]  

4
where \( \eta \) is a suitably regularized spatiotemporal white noise in two space dimensions. We note that since \( \cos \theta \nabla \theta \cdot p^\perp = \nabla \theta \cdot t \), we obtain

\[
- \sin \theta \phi_t + \alpha \theta_t = \Delta \theta - \sin \theta \cos \theta |\nabla \phi|^2 - (Q - 1) \cos \theta \sin \theta + \kappa \sin^2 \theta \nabla \phi \cdot p + \sqrt{2\alpha \varepsilon} \eta.
\]  

(2.13)

For the second one, we have

\[
(m \times m_t) \cdot \hat{p} + \alpha m_t \cdot \hat{p} = h_{\text{eff}} \cdot \hat{p},
\]

(2.14)

which leads to

\[
\alpha \sin \theta \phi_t + \theta_t = \sin \theta \Delta \phi + 2 \cos \theta \nabla \theta \cdot \nabla \phi - \kappa \sin \theta \nabla \theta \cdot p + \sqrt{2\alpha \varepsilon} \zeta,
\]

(2.15)

where \( \zeta \) is another suitably regularized spatiotemporal white noise in two space dimensions independent of \( \eta \).

3 Integral identities

Below we manipulate the equations for \( \theta \) and \( \phi \) to obtain a series of integral identities in which the contribution of the exchange energy vanishes. This amounts to testing the weak form of the equations with the infinitesimal generators of the continuous symmetry groups of the exchange energy acting on the solution.

3.1 Rotations

We multiply (2.15) by \( \sin \theta \) and integrate over space:

\[
\int \sin \theta \theta_t d^2r + \alpha \int \sin^2 \theta \phi_t d^2r
= -\kappa \int \sin^2 \theta \nabla \theta \cdot p d^2r + \sqrt{2\alpha \varepsilon} \int \zeta(\mathbf{r}, t) \sin \theta d^2r.
\]

(3.1)

We define \( A^2(t) = \int \sin^2 \theta d^2r \) and obtain

\[
\int \sin \theta \theta_t d^2r + \alpha \int \sin^2 \theta \phi_t d^2r
= -\kappa \int \sin^2 \theta \nabla \theta \cdot p d^2r + \sqrt{2\alpha \varepsilon} A(t) \dot{W}_1(t),
\]

(3.3)

where \( W_1(t) \) is a Wiener process. This equation is explicitly

\[
\int \sin \theta \theta_t d^2r + \alpha \int \sin^2 \theta \phi_t d^2r + \kappa \int \sin^2 \theta \nabla \theta \cdot p d^2r
= \sqrt{2\alpha \varepsilon} \left( \int \sin^2 \theta d^2r \right)^{1/2} \dot{W}_1(t).
\]

(3.5)
3.2 Dilations

We next multiply (2.13) by \((r - r_0(t)) \cdot \nabla \theta\) and integrate to obtain

\[
- \int_{\mathbb{R}^2} (r - r_0(t)) \cdot \nabla \theta \sin \phi t \, d^2r + \alpha \int_{\mathbb{R}^2} (r - r_0(t)) \cdot \nabla \theta \, d^2r
= \int_{\mathbb{R}^2} (r - r_0(t)) \cdot \nabla \theta (\Delta \theta - \sin \theta \cos \theta |\nabla \phi|^2 \right) (3.6)
\]

\[
- (Q - 1) \cos \theta \sin \theta + \kappa \sin^2 \theta \nabla \phi \cdot \mathbf{p} \right) \, d^2r
+ \sqrt{2\alpha \varepsilon} \int_{\mathbb{R}^2} (r - r_0(t)) \cdot \nabla \eta(t) \, d^2r.
\] (3.7)

Similarly, multiplying (2.15) by \((r - r_0(t)) \cdot \nabla \phi \sin \theta\) yields

\[
\alpha \int_{\mathbb{R}^2} (r - r_0(t)) \cdot \nabla \phi \sin^2 \theta \phi t \, d^2r + \int_{\mathbb{R}^2} (r - r_0(t)) \cdot \nabla \phi \sin \theta \, d^2r
= \int_{\mathbb{R}^2} (r - r_0(t)) \cdot \nabla \phi \sin(\sin \theta \Delta \phi + 2 \cos \theta \nabla \theta \cdot \nabla \phi - \kappa \sin \theta \nabla \theta \cdot \mathbf{p}) \, d^2r
+ \sqrt{2\alpha \varepsilon} \int_{\mathbb{R}^2} (r - r_0(t)) \cdot \nabla \phi \sin \theta \zeta(r, t) \, d^2r.
\] (3.8)

We now work to eliminate the contributions of the exchange energy from the above identities. It is easy to see that an integration by parts yields

\[
\int_{\mathbb{R}^2} (r - r_0(t)) \cdot \nabla \theta \Delta \theta \, d^2r = 0.
\] (3.9)

To proceed further, without loss of generality we set \(r_0(t) = 0\). With the help of the identity in (3.9), we add the terms coming from the exchange in (3.6) and (3.8) to obtain

\[
- \int_{\mathbb{R}^2} r \cdot \nabla \theta \sin \theta \cos \theta |\nabla \phi|^2 \, d^2r + \int_{\mathbb{R}^2} r \cdot \nabla \phi \sin(\sin \theta \Delta \phi + 2 \cos \theta \nabla \theta \cdot \nabla \phi) \, d^2r
= -\frac{1}{2} \int_{\mathbb{R}^2} r \cdot \nabla (\sin^2 \theta) |\nabla \phi|^2 \, d^2r + \int_{\mathbb{R}^2} r \cdot \nabla \phi \nabla \cdot (\sin^2 \theta \nabla \phi) \, d^2r
= -\frac{1}{2} \int_{\mathbb{R}^2} r \cdot \nabla (\sin^2 \theta) |\nabla \phi|^2 \, d^2r - \int_{\mathbb{R}^2} \nabla (r \cdot \nabla \phi) \cdot (\sin^2 \theta \nabla \phi) \, d^2r,
\] (3.10)

after an integration by parts. We note that

\[
\nabla (r \cdot \nabla \phi) = \nabla \phi + H(\phi) r,
\] (3.11)

where \(H(\phi)\) is the Hessian of \(\phi\). Therefore, we have

\[
\int_{\mathbb{R}^2} \nabla (r \cdot \nabla \phi) \cdot (\sin^2 \theta \nabla \phi) \, d^2r = \int_{\mathbb{R}^2} \sin^2 \theta |\nabla \phi|^2 \, d^2r + \int_{\mathbb{R}^2} \sin^2 \theta \nabla \phi \cdot H(\phi) r \, d^2r.
\] (3.12)
On the other hand, there holds
\[ r \cdot \nabla (\sin^2 \theta) = \nabla \cdot (r \sin^2 \theta) - 2 \sin^2 \theta, \]  
(3.13)
and, hence, after integrating by parts we get
\[
\int_{\mathbb{R}^2} r \cdot \nabla (\sin^2 \theta) |\nabla \phi|^2 d^2r = \int_{\mathbb{R}^2} \nabla \cdot (r \sin^2 \theta) |\nabla \phi|^2 d^2r - 2 \int_{\mathbb{R}^2} \sin^2 \theta |\nabla \phi|^2 d^2r \\
= - \int_{\mathbb{R}^2} (r \sin^2 \theta) \cdot \nabla (|\nabla \phi|^2) d^2r - 2 \int_{\mathbb{R}^2} \sin^2 \theta |\nabla \phi|^2 d^2r \\
= -2 \left( \int_{\mathbb{R}^2} \sin^2 \theta \nabla \phi \cdot \phi d^2r + \int_{\mathbb{R}^2} 2 \sin^2 \theta |\nabla \phi|^2 d^2r \right). 
(3.14)

Combining everything together, as expected we arrive at
\[
- \int_{\mathbb{R}^2} r \cdot \nabla \sin \theta \cos \theta |\nabla \phi|^2 d^2r + \int_{\mathbb{R}^2} r \cdot \nabla \phi \sin \theta (2 \cos \theta \nabla \theta \cdot \nabla \phi + \sin \theta \Delta \phi) d^2r = 0. 
(3.15)
We now use the above identity to add up (3.6) and (3.8) to obtain
\[
- \int_{\mathbb{R}^2} (r - r_0(t)) \cdot \nabla \phi \sin \theta \cos \theta |\nabla \phi|^2 d^2r + \int_{\mathbb{R}^2} (r - r_0(t)) \cdot \nabla \phi \sin \theta \cos \theta \phi d^2r \\
+ \alpha \int_{\mathbb{R}^2} \sin^2 \theta \nabla \phi \cdot (r - r_0(t)) \phi d^2r + \int_{\mathbb{R}^2} (r - r_0(t)) \cdot \nabla \phi \sin \theta \cos \theta \phi d^2r \\
= \int_{\mathbb{R}^2} (r - r_0(t)) \cdot \nabla \theta \cdot (-Q - 1) \cos \theta \sin \theta + \kappa \sin^2 \theta \nabla \phi \cdot \cdot \phi d^2r \\
+ \int_{\mathbb{R}^2} \sin \theta \nabla \phi \cdot (r - r_0(t)) (-\kappa - \sin \theta \nabla \theta \cdot \cdot \phi d^2r \\
+ \sqrt{2} \alpha \epsilon \int_{\mathbb{R}^2} (r - r_0(t)) \cdot \nabla \phi \eta(r, t) d^2r \\
+ \sqrt{2} \alpha \epsilon \int_{\mathbb{R}^2} \sin \theta \nabla \phi \cdot (r - r_0(t)) \cdot \zeta(r, t) d^2r. 
(3.16)
We define \( B^2(t) = \int_{\mathbb{R}^2} |(r - r_0(t)) \cdot \nabla \theta|^2 d^2r + \int_{\mathbb{R}^2} \sin^2 \theta |(r - r_0(t)) \cdot \nabla \phi|^2 d^2r \). Then the
above equation reads

\[-\int_{\mathbb{R}^2} (r - r_0(t)) \cdot \nabla \sin \theta \phi_t \, d^2r + \alpha \int_{\mathbb{R}^2} (r - r_0(t)) \cdot \nabla \theta \, d^2r \]
\[+ \alpha \int_{\mathbb{R}^2} \sin^2 \theta \nabla \cdot (r - r_0(t)) \phi_t \, d^2r + \int_{\mathbb{R}^2} (r - r_0(t)) \cdot \nabla \phi_t \sin \theta \, d^2r \]
\[= \int_{\mathbb{R}^2} (r - r_0(t)) \cdot \nabla \theta \left( -(Q - 1) \cos \theta \sin \theta + \kappa \sin^2 \theta \nabla \phi \cdot p \right) \, d^2r \]
\[+ \int_{\mathbb{R}^2} \sin \theta \nabla \phi \cdot (r - r_0(t)) (-\kappa \sin \theta \nabla \phi \cdot p) \, d^2r \]
\[+ \sqrt{2\alpha \varepsilon B(t)} \dot{W}_2(t), \quad (3.17)\]

where \(W_2(t)\) is another Wiener process. This equation is explicitly

\[-\int_{\mathbb{R}^2} (r - r_0(t)) \cdot \nabla \sin \theta \phi_t \, d^2r + \alpha \int_{\mathbb{R}^2} (r - r_0(t)) \cdot \nabla \theta \, d^2r \]
\[+ \alpha \int_{\mathbb{R}^2} \sin^2 \theta \nabla \cdot (r - r_0(t)) \phi_t \, d^2r + \int_{\mathbb{R}^2} (r - r_0(t)) \cdot \nabla \phi_t \sin \theta \, d^2r \]
\[= \int_{\mathbb{R}^2} (r - r_0(t)) \cdot \nabla \theta \left( -(Q - 1) \cos \theta \sin \theta + \kappa \sin^2 \theta \nabla \phi \cdot p \right) \, d^2r \]
\[+ \int_{\mathbb{R}^2} \sin \theta \nabla \phi \cdot (r - r_0(t)) (-\kappa \sin \theta \nabla \phi \cdot p) \, d^2r \]
\[+ \sqrt{2\alpha \varepsilon} \int_{\mathbb{R}^2} \theta \eta(r, t) \, d^2r \quad (3.18)\]

### 3.3 Translations

We now multiply (2.13) and (2.15) by \(\theta_x\) and \(\sin \theta \phi_x\), respectively, to obtain

\[-\int_{\mathbb{R}^2} \theta_x \sin \theta \phi_{\xi} \, d^2r + \alpha \int_{\mathbb{R}^2} \theta_x \theta_{\xi} \, d^2r \]
\[= \int_{\mathbb{R}^2} \theta_x (\Delta \theta - \sin \theta \cos \theta |\nabla \phi|^2 \]
\[-(Q - 1) \cos \theta \sin \theta + \kappa \sin^2 \theta \nabla \phi \cdot p \, d^2r \]
\[+ \sqrt{2\alpha \varepsilon} \int_{\mathbb{R}^2} \theta_x \eta(r, t) \, d^2r \quad (3.19)\]
\[
\alpha \int_{\mathbb{R}^2} \sin^2 \theta \phi_x \phi_t \, d^2 r + \int_{\mathbb{R}^2} \sin \theta \phi_x \theta_t \, d^2 r \\
= \int_{\mathbb{R}^2} \phi_x \sin \theta (\sin \theta \Delta \phi + 2 \cos \theta \nabla \theta \cdot \nabla \phi - \kappa \sin \theta \nabla \theta \cdot p) \, d^2 r \\
+ \sqrt{2\alpha \varepsilon} \int_{\mathbb{R}^2} \sin \theta \phi_x \zeta(r, t) \, d^2 r.
\]

(3.20)

We once again use the identity
\[
\int_{\mathbb{R}^2} \theta_x \Delta \theta \, d^2 r = 0
\]
(3.21)
to simplify a combination of the exchange contributions from (2.13) and (2.15):
\[
- \int_{\mathbb{R}^2} \theta_x \sin \theta \cos \theta |\nabla \phi|^2 \, d^2 r + \int_{\mathbb{R}^2} \phi_x \sin \theta (\sin \theta \Delta \phi + 2 \cos \theta \nabla \theta \cdot \nabla \phi) \, d^2 r \\
= - \frac{1}{2} \int_{\mathbb{R}^2} \partial_x (\sin^2 \theta) |\nabla \phi|^2 \, d^2 r + \int_{\mathbb{R}^2} \phi_x \nabla \cdot (\sin^2 \theta \nabla \phi) \, d^2 r \\
= \int_{\mathbb{R}^2} \sin^2 \theta \nabla \phi \cdot \nabla \phi_x \, d^2 r - \int_{\mathbb{R}^2} \sin^2 \theta \nabla \phi \cdot \nabla \phi \, d^2 r = 0.
\]

(3.22)

Therefore, we obtain
\[
- \int_{\mathbb{R}^2} \theta_x \sin \theta \phi_t \, d^2 r + \alpha \int_{\mathbb{R}^2} \theta_x \theta_t \, d^2 r + \alpha \int_{\mathbb{R}^2} \sin^2 \theta \phi_x \phi_t \, d^2 r + \int_{\mathbb{R}^2} \sin \theta \phi_x \theta_t \, d^2 r \\
= \int_{\mathbb{R}^2} \theta_x \left( -(Q - 1) \cos \theta \sin \theta + \kappa \sin^2 \theta \nabla \phi \cdot p \right) \, d^2 r \\
+ \int_{\mathbb{R}^2} \sin \theta \phi_x (-\kappa \sin \theta \nabla \theta \cdot p) \, d^2 r \\
+ \sqrt{2\alpha \varepsilon} \int_{\mathbb{R}^2} \theta_x \eta(r, t) \, d^2 r + \sqrt{2\alpha \varepsilon} \int_{\mathbb{R}^2} \sin \theta \phi_x \zeta(r, t) \, d^2 r.
\]

(3.23)

Defining \( C^2(t) = \int_{\mathbb{R}^2} |\theta_x|^2 \, d^2 r + \int_{\mathbb{R}^2} \sin^2 \theta |\phi_x|^2 \, d^2 r \), we then write
\[
- \int_{\mathbb{R}^2} \theta_x \sin \theta \phi_t \, d^2 r + \alpha \int_{\mathbb{R}^2} \theta_x \theta_t \, d^2 r + \alpha \int_{\mathbb{R}^2} \sin^2 \theta \phi_x \phi_t \, d^2 r + \int_{\mathbb{R}^2} \sin \theta \phi_x \theta_t \, d^2 r \\
= \int_{\mathbb{R}^2} \theta_x \left( -(Q - 1) \cos \theta \sin \theta + \kappa \sin^2 \theta \nabla \phi \cdot p \right) \, d^2 r \\
+ \int_{\mathbb{R}^2} \sin \theta \phi_x (-\kappa \sin \theta \nabla \theta \cdot p) \, d^2 r \\
+ \sqrt{2\alpha \varepsilon} C(t) \dot{W}_3(t).
\]

(3.24)
where \( W_3(t) \) is another Wiener process. Finally, following the same argument and defining
\[
D^2(t) = \int_{\mathbb{R}^2} |\theta_g|^2 d^2r + \int_{\mathbb{R}^2} \sin^2 \theta |\phi_y|^2 d^2r,
\]
we then write
\[
-\int_{\mathbb{R}^2} \theta_y \sin \theta \phi_t d^2r + \alpha \int_{\mathbb{R}^2} \theta_y \theta_t d^2r + \alpha \int_{\mathbb{R}^2} \sin^2 \theta \phi_y \phi_t d^2r + \int_{\mathbb{R}^2} \sin \theta \phi_y \theta_t d^2r
= \int_{\mathbb{R}^2} \theta_y (- (Q - 1) \cos \theta \sin \theta + \kappa \sin^2 \theta \nabla \phi \cdot p) d^2r
+ \int_{\mathbb{R}^2} \sin \theta \phi_y (- \kappa \sin \theta \nabla \theta \cdot p) d^2r
+ \sqrt{2\alpha \varepsilon} D(t) W_4(t),
\]
where \( W_4(t) \) is yet another Wiener process. The last two equations may be explicitly written as
\[
-\int_{\mathbb{R}^2} \theta_x \sin \theta \phi_t d^2r + \alpha \int_{\mathbb{R}^2} \theta_x \theta_t d^2r + \alpha \int_{\mathbb{R}^2} \sin^2 \theta \phi_x \phi_t d^2r + \int_{\mathbb{R}^2} \sin \theta \phi_x \theta_t d^2r
= \int_{\mathbb{R}^2} \theta_x (- (Q - 1) \cos \theta \sin \theta + \kappa \sin^2 \theta \nabla \phi \cdot p) d^2r
+ \int_{\mathbb{R}^2} \sin \theta \phi_x (- \kappa \sin \theta \nabla \theta \cdot p) d^2r
+ \sqrt{2\alpha \varepsilon} \sqrt{\int_{\mathbb{R}^2} |\theta_x|^2 d^2r + \int_{\mathbb{R}^2} \sin^2 \theta |\phi_x|^2 d^2r} W_3(t),
\]
and
\[
-\int_{\mathbb{R}^2} \theta_y \sin \theta \phi_t d^2r + \alpha \int_{\mathbb{R}^2} \theta_y \theta_t d^2r + \alpha \int_{\mathbb{R}^2} \sin^2 \theta \phi_y \phi_t d^2r + \int_{\mathbb{R}^2} \sin \theta \phi_y \theta_t d^2r
= \int_{\mathbb{R}^2} \theta_y (- (Q - 1) \cos \theta \sin \theta + \kappa \sin^2 \theta \nabla \phi \cdot p) d^2r
+ \int_{\mathbb{R}^2} \sin \theta \phi_y (- \kappa \sin \theta \nabla \theta \cdot p) d^2r
+ \sqrt{2\alpha \varepsilon} \sqrt{\int_{\mathbb{R}^2} |\theta_y|^2 d^2r + \int_{\mathbb{R}^2} \sin^2 \theta |\phi_y|^2 d^2r} W_4(t).
\]

4 \ Dynamic skyrmion profile

We introduce the polar coordinates \((r, \psi)\) relative to \(r_0\):
\[
r = r_0 + (r \cos \psi, r \sin \psi).
\]
With the skyrmion radius $\rho > 0$ and angle $\varphi \in \mathbb{R}$ fixed, the skyrmion profile $(\theta(r, \psi), \phi(r, \psi))$ in the core reads

$$\theta = 2 \arctan(r/\rho), \quad \phi = \psi + \varphi - \pi, \quad (4.2)$$

while the profile in the skyrmion tail is

$$\theta = \pi - 2\rho_0 \sqrt{Q - 1} K_1(r \sqrt{Q - 1}), \quad \phi = \psi + \varphi - \pi. \quad (4.3)$$

It is useful to note that in the polar coordinates we have for the core:

$$\sin \theta = \frac{2r}{\rho^2 + r^2}, \quad \cos \theta = \frac{\rho^2 - r^2}{\rho^2 + r^2}, \quad \theta_r = \frac{2r}{\rho^2 + r^2}. \quad (4.4)$$

We also note that

$$\theta_t = -\theta r \hat{r}_0 \cdot (\cos \psi, \sin \psi) - \frac{2r \rho}{\rho^2 + r^2} \quad (4.5)$$

and

$$\phi_t = \frac{1}{r} \hat{r}_0 \cdot (\sin \psi, -\cos \psi) + \dot{\varphi}. \quad (4.6)$$

In addition, we have

$$\mathbf{p} \cdot \nabla \theta = \theta_r \sin \varphi, \quad \mathbf{p} \cdot \nabla \phi = -\frac{1}{r} \cos \varphi, \quad (4.7)$$

and

$$\theta_x = \theta_r \cos \psi, \quad \theta_y = \theta_r \sin \psi, \quad \phi_x = -\frac{1}{r} \sin \psi, \quad \phi_y = \frac{1}{r} \cos \psi. \quad (4.8)$$

We next calculate a number of integrals of these expressions that we will use in the sequel, introducing an upper cutoff length $L = (Q - 1)^{-1/2}$ whenever the obtained integrals exhibit logarithmic divergences. We get

$$\int_{\mathbb{R}^2} \sin^2 \theta \, d^2r \simeq 8\pi \rho^2 \int_0^{L/\rho} \frac{z^3}{(1 + z^2)^2} \, dz \simeq 8\pi \rho^2 \ln(L/\rho), \quad (4.9)$$

$$\int_{\mathbb{R}^2} r \theta_r \cos \theta \sin \theta \, d^2r \simeq 8\pi \rho^2 \int_0^{L/\rho} \frac{z^3}{(1 + z^2)^2} \, dz \simeq -8\pi \rho^2 \ln(L/\rho), \quad (4.10)$$

$$\int_{\mathbb{R}^2} \theta_r \sin^2 \theta \, d^2r \simeq 16\pi \rho \int_0^{L/\rho} \frac{z^3}{(1 + z^2)^3} \, dz \simeq 4\pi \rho, \quad (4.11)$$

$$\int_{\mathbb{R}^2} r \theta_r \theta_r \, d^2r \simeq -8\pi \rho \int_0^{L/\rho} \frac{z^3}{(z^2 + 1)^2} \, dz \simeq -8\pi \rho \dot{\varphi} \ln(L/\rho), \quad (4.12)$$

$$\int_{\mathbb{R}^2} r^2 \theta_r^2 \, d^2r \simeq 8\pi \rho^2 \int_0^{L/\rho} \frac{z^3}{(1 + z^2)^2} \, dz \simeq 8\pi \rho^2 \ln(L/\rho), \quad (4.13)$$
where we always set \( z = r/\rho \). Notice that due to the weak logarithmic dependence of some of the above integrals on the cutoff length \( L \), these formulas remain valid on the time scale \( \tau_{\text{relax}} \), even if the outer solution does not have time to fully relax to its steady state profile in (4.3).

The above considerations may be compared with the result of a direct numerical solution for the radial skyrmion profile, in which \( \theta = \theta(r) \) and \( \phi = \psi - \pi \), that satisfies the Euler-Lagrange equation associated with the micromagnetic energy

\[
\frac{d^2 \theta}{dr^2} + \frac{1}{r} \frac{d \theta}{dr} - \frac{1}{r^2} \sin \theta \cos \theta - \frac{\kappa}{r} \sin^2 \theta - (Q - 1) \sin \theta \cos \theta = 0,
\]

and the conditions at infinity

\[
\theta(0) = 0, \quad \theta(\infty) = \pi.
\]

This nonlinear boundary value problem may be solved with the help of the shooting method [1], yielding a very accurate numerical approximation for the profile. The energy of this profile is given by

\[
E(\theta) = \pi \int_0^\infty \left[ \left( \frac{d \theta}{dr} \right)^2 + \frac{1}{r^2} \sin^2 \theta + (Q - 1) \sin^2 \theta - \frac{\kappa}{r} (\sin \theta \cos \theta - \theta + \pi) \right] r \, dr.
\]

A comparison of the resulting values of the skyrmion energy with an approximation [8,9]

\[
E(\theta) \approx 4\pi - \frac{\pi \kappa^2}{(Q - 1) \ln (a\kappa^{-1}\sqrt{Q - 1})}
\]

shows that the latter reproduces the values obtained numerically in the interval of \( \kappa/\sqrt{Q - 1} < 0.8 \) within 17%, provided that \( a = 2.8 \). A comparison of the two is presented in Fig. S1.

5 Reduction to a finite-dimensional system

We now derive an equation for \( \varphi(t) \) by evaluating (3.5) on the above solution. With the help of the obtained formulas for the respective integrals, we obtain

\[
-\frac{\dot{\rho}}{\rho} + \alpha \dot{\varphi} = -\frac{\kappa \sin \varphi}{2 \rho \ln(L/\rho)} + \sqrt{\frac{\alpha \varepsilon}{4\pi \rho^2 \ln(L/\rho)}} W_1(t).
\]

Next we evaluate (3.18):

\[
\dot{\varphi} + \frac{\alpha \dot{\rho}}{\rho} = -(Q - 1) + \frac{\kappa \cos \varphi}{2 \rho \ln(L/\rho)} + \sqrt{\frac{\alpha \varepsilon}{4\pi \rho^2 \ln(L/\rho)}} W_2.
\]
Fig. S1: A comparison of the skyrmion energy from (4.16) with the profile solving (4.14) and (4.15) (solid line through the numerical values shown as dots) and the approximation in (4.17) with $a = 2.8$ (dashed line).

Finally, we evaluate (3.26) and (3.27):

$$
\dot{y}_0 - \alpha \dot{x}_0 = \sqrt{\frac{\alpha \varepsilon}{2\pi}} \dot{W}_3, \quad (5.3)
$$

$$
- \dot{x}_0 - \alpha \dot{y}_0 = \sqrt{\frac{\alpha \varepsilon}{2\pi}} \dot{W}_4. \quad (5.4)
$$

Solving explicitly for the rates of change of the unknowns, we obtain

$$
\frac{d}{dt} \left( \ln \rho \varphi \right) = \frac{-1}{1 + \alpha^2} \left( \begin{array}{cc} \alpha & -1 \\ 1 & \alpha \end{array} \right) \left( \begin{array}{c} Q - 1 - \frac{\kappa \cos \varphi}{2\rho \ln(L/\rho)} \\ \frac{\kappa \sin \varphi}{2\rho \ln(L/\rho)} \end{array} \right) + \sqrt{\frac{\alpha \varepsilon}{4\pi \rho^2 \ln(L/\rho)}} \left( \begin{array}{c} \dot{W}_1(t) \\ \dot{W}_2(t) \end{array} \right) \quad (5.5)
$$

and

$$
\dot{x}_0(t) = -\sqrt{\frac{\alpha \varepsilon}{2\pi}} \times \frac{\alpha \dot{W}_3 + \dot{W}_4}{1 + \alpha^2}, \quad (5.6)
$$

$$
\dot{y}_0(t) = \sqrt{\frac{\alpha \varepsilon}{2\pi}} \times \frac{\dot{W}_3 - \alpha \dot{W}_4}{1 + \alpha^2}. \quad (5.7)
$$

We now show that the noises $W_1$ through $W_4$ are independent and simplify the expressions involving the noise terms. The independence of the noises follows from mutual
orthogonality, to the leading order, of the scalar multiples associated with the symmetry
groups used to derive our integral identities. For example, we have
\[
\langle \dot{W}_1(t)\dot{W}_2(t') \rangle = \frac{1}{AB} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \langle \sin \theta(r, t) \zeta(r, t)(r' \cdot \nabla \theta(r', t') \eta(r', t') + \sin \theta(r', t') r' \cdot \nabla \phi(r', t') \zeta(r', t')) \rangle d^2r d^2r' = \frac{\delta(t - t')}{AB} \int_{\mathbb{R}^2} \sin^2 \theta r \cdot \nabla \phi d^2r \simeq 0. \quad (5.8)
\]
The other correlators can be computed analogously. Then, noting that
\[
\tilde{W}_1 = -(1 + \alpha^2)^{-1/2}(\alpha W_1 - W_2) \quad \text{and} \quad \tilde{W}_2 = -(1 + \alpha^2)^{-1/2}(W_1 + \alpha W_2)
\]
are also two independent Wiener processes, we obtain
\[
\dot{x}_0 = \sqrt{\frac{\alpha \varepsilon}{2\pi(1 + \alpha^2)}} \tilde{W}_3, \quad \dot{y}_0 = \sqrt{\frac{\alpha \varepsilon}{2\pi(1 + \alpha^2)}} \tilde{W}_4, \quad (5.10)
\]
where we again dropped the tildes in the noises for simplicity of notation. Our analysis thus shows that
the skyrmion translational degree of freedom \( r_0(t) \) decouples from the radius \( \rho(t) \) and the
rotation angle \( \varphi(t) \) and yields the skyrmion diffusivity:
\[
D_{\text{eff}} = \frac{\alpha \varepsilon}{4\pi(1 + \alpha^2)}. \quad (5.11)
\]
Equation (5.5) should be supplemented by a reflecting boundary condition at \( \rho = L_0 \) with some \( 0 < L_0 < L \) in order to make the noise term meaningful, consistently with the
assumption of \( \rho \ll L \) made throughout the derivation. Furthermore, replacing \( \ln(L/\rho) \) by \( \Lambda \) allows to rewrite (5.5) in the following simplified form:
\[
\left( \begin{array}{c} \dot{\rho} \\ \dot{\varphi} \end{array} \right) = -\frac{1}{8\pi \Lambda(1 + \alpha^2)} \left( \begin{array}{cc} \alpha & -\rho^{-1} \\ \rho^{-1} & \alpha \rho^{-2} \end{array} \right) \left( \begin{array}{c} \frac{\partial H}{\partial \rho} \\ \frac{\partial H}{\partial \varphi} \end{array} \right) + \sqrt{\frac{\alpha \varepsilon}{4\pi \Lambda(1 + \alpha^2)}} \left( \begin{array}{c} \tilde{W}_1(t) \\ \rho^{-1} \tilde{W}_2(t) \end{array} \right), \quad (5.12)
\]
where

\[ H(\rho, \varphi) = 4\pi\Lambda(Q - 1)\rho^2 - 4\pi\kappa\rho\cos\varphi \]  

(5.13)

is the Hamiltonian associated with the dynamics in (5.12) for \( \alpha = 0 \). Finally, changing the variables to \( \bar{z} = \bar{x} + i\bar{y} \) with \( \bar{x} = \rho\cos\varphi \) and \( \bar{y} = \rho\sin\varphi \) yields

\[ d\bar{z}(t) = -\frac{\alpha + i}{1 + \alpha^2} \left[ (Q - 1)\bar{z}(t) - \frac{\kappa}{2\Lambda} \right] dt + \sqrt{\frac{\alpha\varepsilon}{4\pi\Lambda(1 + \alpha^2)}} d\bar{W}(t), \]  

(5.14)

where \( \bar{W}(t) = W_1(t) + iW_2(t) \) is a complex-valued Wiener process. Explicitly, this equation may be written in terms of \( \bar{x} \) and \( \bar{y} \) as

\[ d\bar{x} = \frac{1}{1 + \alpha^2} \left[ \frac{\alpha\kappa}{2\Lambda} - \alpha(Q - 1)\bar{x} + (Q - 1)\bar{y} \right] dt + \sqrt{\frac{\alpha\varepsilon}{4\pi\Lambda(1 + \alpha^2)}} dW_1(t), \]  

(5.15)

\[ d\bar{y} = \frac{1}{1 + \alpha^2} \left[ \frac{\kappa}{2\Lambda} - (Q - 1)\bar{x} - \alpha(Q - 1)\bar{y} \right] dt + \sqrt{\frac{\alpha\varepsilon}{4\pi\Lambda(1 + \alpha^2)}} dW_2(t), \]  

(5.16)

where \( W_1 \) and \( W_2 \) were suitably rotated. The associated Fokker-Planck equation reads

\[ (1 + \alpha^2)p_t = \left[ \left( \alpha(Q - 1)\bar{x} - \frac{\alpha\kappa}{2\Lambda} - (Q - 1)\bar{y} \right) p \right] \bar{x} + \left[ \left( (Q - 1)\bar{x} - \frac{\kappa}{2\Lambda} + \alpha(Q - 1)\bar{y} \right) p \right] \bar{y} + \frac{\alpha\varepsilon}{8\pi\Lambda} (p_{\bar{x}\bar{x}} + p_{\bar{y}\bar{y}}). \]  

(5.17)

By an explicit verification, the equilibrium solution of this equation is given by \( p = p_{\text{eq}}(\bar{x}, \bar{y}) \), where

\[ p_{\text{eq}}(\bar{x}, \bar{y}) = \frac{4\Lambda(Q - 1)}{\varepsilon} e^{-\frac{H(\bar{x} + i\bar{y})}{\varepsilon}}, \]  

(5.18)

and

\[ H(\bar{z}) = 4\pi\Lambda(Q - 1)|\bar{z} - \bar{z}_0|^2, \quad \bar{z}_0 = \frac{\kappa}{2\Lambda(Q - 1)}. \]  

(5.19)

### 6 Collapse rate in the small noise limit

There are several approaches to calculating the mean skyrmion lifetime from (5.17) for \( \varepsilon \to 0 \) and all other parameters fixed \[^{[12]}\]. Here we achieve this by considering a stationary stochastic process, in which a particle evolving according to (5.14) and hitting the absorber is reinjected into the neighborhood of \( \bar{z} = \bar{z}_0 \) with probability \( g(\bar{x}, \bar{y}) d\bar{x} d\bar{y}, \) for some positive function \( g \) with support in a small neighborhood of \( \bar{z}_0 \). This process is described by a
stationary version of (5.17) with \( C_\varepsilon (1 + \alpha^2) \bar{g} \) added to its right-hand side, provided that \( C_\varepsilon \ll 1 \) is chosen in such a way that the solution \( p \) integrates to unity over all \( |\bar{z}| > \delta \). The precise choice of \( \bar{g} \) is unimportant in the limit of \( \varepsilon \to 0 \), as the solution will approach \( p = p_{eq} \) everywhere, except for a small neighborhood around the closest point \((\bar{x}, \bar{y}) = (\delta, 0)\) on the absorber to \( \bar{z}_0 \).

Introducing \( q = p/p_{eq} \), we get

\[
\left[ (Q - 1)x - \frac{\kappa}{2\Lambda} \right] (\alpha q_{\bar{x}} - q_{\bar{y}}) + y(Q - 1)(\alpha q_{\bar{y}} + q_{\bar{x}})
= \frac{\alpha \varepsilon}{8\pi \Lambda} (q_{\bar{x}x} + q_{\bar{y}y}) + gp_{eq}^{-1},
\]

where \( g = C_\varepsilon \bar{g} \). Then to the leading order in \( \varepsilon \ll 1 \) the function \( q(\bar{x}, \bar{y}) \) is independent of \( \bar{y} \), approaches unity for \( \bar{x} \gg \varepsilon \), and solves

\[
\left[ (Q - 1)\delta - \frac{\kappa}{2\Lambda} \right] q_{\bar{x}} = \frac{\varepsilon}{8\pi \Lambda} q_{\bar{x}x}
\]

in the \( O(\varepsilon) \) neighborhood of \( \bar{x} = \delta \). The solution \( q = q(\bar{x}) \) of (6.2) for \( \delta < \bar{x} < \infty \) with \( q(\delta) = 0 \) and \( q(+\infty) = 1 \) is

\[
q(\bar{x}) = 1 - \exp \left\{ -\frac{8\pi \Lambda}{\varepsilon} \left[ (Q - 1)\delta - \frac{\kappa}{2\Lambda} \right] (\bar{x} - \delta) \right\}.
\]

The normal probability flux into the absorbing boundary is then to the leading order

\[
j(\bar{y}) = \frac{\alpha \varepsilon}{8\pi \Lambda(1 + \alpha^2)} p_{eq}(\sqrt{\delta^2 - \bar{y}^2}, \bar{y})q_{\bar{x}}(\delta).
\]

The approximation above is valid when the scale \( |\bar{y}| \sim \sqrt{\varepsilon \delta / \kappa} \) on which \( j(\bar{y}) \) is concentrated is significantly less than the curvature radius \( \delta \) of the absorber, which translates into a consistency condition \( \varepsilon \ll \kappa \delta \).

Integrating the expression in (6.4) over \( y \) yields the leading order approximation for the collapse rate:

\[
J_\delta \simeq \frac{\alpha \Lambda(Q - 1)}{1 + \alpha^2} \left[ \frac{\kappa}{2\Lambda} - (Q - 1)\delta \right] \left( \frac{8\delta}{\varepsilon \kappa} \right)^{1/2} e^{-\frac{H(\delta)}{\varepsilon}}.
\]

Since the prefactor in (6.5) depends weakly on \( \delta \), we may write to the leading order in \( \delta \ll 1 \) as \( \varepsilon \to 0 \):

\[
J_\delta \simeq \frac{\alpha(Q - 1)A_\delta}{1 + \alpha^2} \left( \frac{2\kappa \delta}{\varepsilon} \right)^{1/2} \exp \left\{ -\frac{\pi \kappa^2}{\varepsilon \Lambda(Q - 1)} \right\},
\]

where

\[
A_\delta = \exp \left\{ \frac{4\pi \kappa \delta}{\varepsilon} \left[ 1 - \frac{\Lambda(Q - 1)\delta}{\kappa} \right] \right\}
\]

is the anomalous factor from the correction to the barrier height due to the finite size of the absorber.
7 Collapse rate in the small absorber limit

We now consider the limit $\delta \to 0$ with all other parameters fixed. In this limit the collapse rate $J_\delta$ vanishes due to the vanishing capture rate of a Brownian particle as $\delta \to 0$, independently of the noise strength $\varepsilon$ or other parameters. The solution of (6.1) will then be close to 1 everywhere, except in a diffusive boundary layer around the absorber. For $\delta \ll 1$, this solution is dominated by pure diffusion on the $O(\varepsilon/\kappa)$ length scale of the advection-diffusion boundary layer (for $\alpha \sim 1$). Therefore, to the leading order in $\delta \ll 1$ we have

$$q(\bar{x}, \bar{y}) \simeq \frac{\ln \left(\frac{\sqrt{\bar{x}^2 + \bar{y}^2}}{\delta}\right)}{\ln \left(\frac{b \varepsilon}{\kappa \delta \sqrt{1 + \alpha^2}}\right)},$$

(7.1)

where $b \approx 0.179$, which, up to constants in the logarithm, is the solution of the two-dimensional Laplace’s equation in an annulus that vanishes at $|z| = \delta$ and equals 1 at $|z| = O(\varepsilon/\kappa)$.

From (7.1) the probability flux into the boundary of the absorber is asymptotically

$$j \simeq \frac{\alpha \varepsilon p_{eq}(\delta, 0)}{8\pi \Lambda(1 + \alpha^2) \ln \left(\frac{b \varepsilon}{\kappa \delta \sqrt{1 + \alpha^2}}\right)}.$$

(7.2)

Therefore, integrating this expression over the boundary and using the definition of $p_{eq}$, we obtain that the collapse rate is

$$J_\delta \simeq \frac{\alpha (Q - 1) A_\delta}{(1 + \alpha^2) \ln \left(\frac{b \varepsilon}{\kappa \delta \sqrt{1 + \alpha^2}}\right)} \exp \left\{ -\frac{\pi \kappa^2}{\varepsilon \Lambda(1 + \alpha^2)} \right\}.$$

(7.3)

This formula is valid when $p_{eq}$ does not vary significantly over the absorber, which is equivalent to $\delta \ll \varepsilon/\kappa$. Notice that in this regime $A_\delta \simeq 1$.

We now revisit (7.1) and give more details concerning its derivation. Observe that near a small absorber the advection field is approximately constant. Then, up to a rotation, (6.1) in the vicinity of the absorber is, to the leading order in $\delta \ll 1$,

$$q_{\bar{x}\bar{x}} + q_{\bar{y}\bar{y}} + cq_x = 0, \quad c = \frac{4\pi \kappa \sqrt{1 + \alpha^2}}{\alpha \varepsilon}.$$

(7.4)

Introduce a new variable $u$ such that

$$q(\bar{x}, \bar{y}) = 1 - u(\bar{x}, \bar{y}) e^{-c\bar{x}/2}.$$

(7.5)

This new variable must vanish far from the absorber and is equal to 1 on the boundary of the absorber to the leading order in $\delta \ll 1$. After a change of variables $u$ solves

$$u_{\bar{x}\bar{x}} + u_{\bar{y}\bar{y}} - \frac{1}{4} c^2 u = 0.$$

(7.6)
With the above boundary conditions its solution is given by

\[ u(\bar{x}, \bar{y}) = \frac{K_0 \left( \frac{1}{2} c \sqrt{\bar{x}^2 + \bar{y}^2} \right)}{K_0 \left( \frac{1}{2} c \delta \right)}, \]  

(7.7)

where \( K_0(z) \) is the modified Bessel function of the second kind. Thus, to the leading order in \( \delta \ll 1 \) we have

\[ q(\bar{x}, \bar{y}) \simeq 1 - u(\bar{x}, \bar{y}) \simeq \frac{\ln \left( \frac{\sqrt{\bar{x}^2 + \bar{y}^2}}{\delta} \right)}{\ln \left( \frac{\alpha \epsilon}{\pi \gamma_0 \kappa \delta \sqrt{1 + \alpha^2}} \right)}, \]  

(7.8)

where \( \gamma_0 \approx 0.5772 \) is the Euler-Mascheroni constant. Finally, to the leading order in \( \kappa \delta / \epsilon \ll 1 \) this is equivalent to (7.1).

### 8 Upper bound for the collapse rate

Observe that, according to (5.14), in order for a skyrmion to collapse the Hamiltonian \( H(\bar{z}(t)) \) defined on the trajectories must reach the value of \( H(\delta) \). Therefore, conditioning the process on \( H(\bar{z}(t)) \) reaching \( H(\delta) \) at a specified time, one can rigorously estimate the collapse rate from above.

Starting with the exact equation for \( dH/dt \) obtained from the Itô formula [12]:

\[ dH = \frac{2\alpha(Q - 1)(\varepsilon - H)}{1 + \alpha^2} \, dt + \sqrt{\frac{4\alpha \varepsilon (Q - 1)H}{1 + \alpha^2}} \, dW_0(t), \]  

(8.1)

where \( W_0(t) \) is a Wiener process, the corresponding Fokker-Planck equation for the probability density \( p(H, t) \) is

\[ (1 + \alpha^2)p_t = 2\alpha(Q - 1)(Hp + \varepsilon Hp_H)_H. \]  

(8.2)

To calculate the rate \( \bar{J}_\delta \geq J_\delta \) with which \( H(\bar{z}(t)) \) reaches \( H(\delta) \), we again solve the stationary variant of (8.2) with an absorbing boundary condition at \( H = H(\delta) \). The solution may be written in the form \( p(H) = \varepsilon^{-1} e^{-H/\varepsilon} q(H) \), where to the leading order \( q(H) \simeq 1 - e^{(H-H(\delta))/\varepsilon} \) in the \( O(\varepsilon) \) boundary layer around \( H = H(\delta) \). This yields

\[ \bar{J}_\delta \simeq \frac{2\alpha(Q - 1)H(\delta)}{(1 + \alpha^2)\varepsilon} e^{-\frac{H(\delta)}{\varepsilon}}. \]  

(8.3)

Since the barrier height \( H(\delta) < H(0) \), to the leading order in \( \varepsilon \ll 1 \) the rate is bounded above by

\[ \bar{J}_\delta^0 \simeq \frac{2\pi \alpha \kappa^2 A_\delta}{\varepsilon \Lambda (1 + \alpha^2)} \exp \left\{ -\frac{\pi \kappa^2}{\varepsilon \Lambda (Q - 1)} \right\}. \]  

(8.4)
This expression gives an upper bound for $J_\delta$, provided the exponential factor in (8.4) is small. It does not involve any approximations in solving (6.1), apart from the usual Arrhenius asymptotics for the solution of a one-dimensional Fokker-Planck equation.

9 Collapse rates as functions of dimensionless parameters

The obtained Arrhenius rates $J_\delta$ in both regimes may be conveniently expressed in terms of the thermal stability factor

$$\Delta = \frac{\pi \kappa^2}{\varepsilon \Lambda (Q - 1)} \gg 1,$$

which measures the asymptotic barrier height to collapse in the units of $k_B T$:

$$J_\delta = \frac{\alpha (Q - 1)}{1 + \alpha^2} A_\delta \exp(-\Delta) \times \left\{ \begin{array}{ll} \left( \frac{2 \kappa \delta}{\varepsilon} \right)^{1/2}, & \varepsilon \ll \kappa \delta, \\ \ln^2 \left( \frac{\varepsilon \Delta}{2 \kappa \delta} \right), & \varepsilon \gg \kappa \delta. \end{array} \right.$$  

This formula reveals several properties of the skyrmion collapse rate. First, in both cases the dependence of the collapse rate on the Gilbert damping parameter $\alpha$ is proportional to $\alpha/(1 + \alpha^2)$. In particular, for small values of $\alpha$ the rate is linear in $\alpha$. Second, for a fixed value of $\Delta$ the collapse rate is proportional to $(Q - 1)A_\delta$ and depends on all the other dimensionless parameters of the problem via a combination $\kappa \delta / \varepsilon$. In particular, the formula in (9.2) gives the precise dependence of the effective Arrhenius prefactor $\Gamma_\delta = J_\delta e^\Delta / A_\delta$ on all the model parameters away from the crossover when $\varepsilon \sim \kappa \delta$.

When plotting the collapse rate as a function of the parameters, we restrict the values of $Q$ and $\kappa$ to those that satisfy the assumptions of validity of the respective formulas:

$$\frac{\kappa}{\sqrt{Q - 1}} < 1, \quad \frac{\kappa}{2 \Lambda (Q - 1)} > \delta,$$

corresponding to $\Lambda > 1$ and $\delta < \bar{z}_0$.

Finally, the assumption of applicability of (7.3) is taken to be

$$\delta < \frac{\alpha \varepsilon}{15 \kappa \sqrt{1 + \alpha^2}},$$

corresponding to the logarithm in (7.3) greater than unity, with (6.6) applied in the opposite case.

10 Large deviations

The optimal escape trajectory for (5.14) is obtained by minimizing the large deviation action

$$S = \frac{2 \pi \Lambda (1 + \alpha^2)}{\alpha} \int_0^T \left| \frac{\dot{z}}{1 + \alpha^2} \right|^2 + \frac{\alpha + i}{1 + \alpha^2} \left[ (Q - 1) \dot{z} - \frac{\kappa}{2 \Lambda} \right]^2 \, dt,$$  

(10.1)
over all trajectories $\bar{z}(t)$ starting at $\bar{z}(0) = \bar{z}_0$ and terminating at $\bar{z}(T) = 0$, and then sending $T \to \infty$ [13]. Expanding the square and recombining the terms above gives

$$S = \frac{\pi \kappa^2}{\Lambda(Q-1)} + \frac{2\pi \Lambda(1+\alpha^2)}{\alpha} \int_0^T \left| \dot{\bar{z}} - \frac{\alpha - i}{1+\alpha^2} \left[ (Q-1)\bar{z} - \frac{\kappa}{2\Lambda} \right] \right|^2 dt \geq \frac{\pi \kappa^2}{\Lambda(Q-1)},$$

(10.2)

where the lower bound in the right-hand side is the energy barrier. As $T \to \infty$, this lower bound is asymptotically achieved by the trajectory $\bar{z}(t)$ that solves

$$\ddot{\bar{z}} = \frac{\alpha - i}{1+\alpha^2} \left[ (Q-1)\bar{z} - \frac{\kappa}{2\Lambda} \right], \quad t < T,$$

(10.3)

with $\bar{z}(T) = 0$ and $\bar{z}(0) \simeq \bar{z}_0$. The solution is explicitly

$$\bar{z}_{\text{opt}}(t) = \bar{z}_0 \left( 1 - e^{\frac{\alpha - i}{1+\alpha^2} (Q-1)(t-T)} \right).$$

(10.4)

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