TIME CRYSTALLINE ORDER IN A 1D PCA – THE π-GÁCS CONSTRUCTION

As with conventional phases of matter, the possibility of a stable, discrete time crystal depends on spatial dimension. Thus, it is natural to ask if a PCA time-crystal can also exist in 1D. Note that in the quantum case, while the 1D MBL discrete time crystal is fairly well established \cite{1-5}, the stability of MBL in 2D, and hence the existence of a 2D MBL DTC phase, remains controversial \cite{6, 7}.

Unfortunately, Toom’s route to stability, discussed in the main text, cannot be generalized to 1D. Since each island of errors is only separated by two domain walls in one dimension, locally one cannot efficiently tell which side corresponds to the error and which side to the correct region. This intuition is not specific to the Toom model, and in fact for many decades it was conjectured that all 1D, finite-range and finite-state PCAs were generically ergodic, i.e. the so-called “positive-rates conjecture” \cite{8}.

Surprisingly, in 1998, this longstanding conjecture was proven incorrect by Gács \cite{9}. Gács constructed a 1D translation invariant PCA, with nearest neighbor interactions, with the following remarkable property: on a chain of length $L$, the dynamics exhibit $2^L$ stable stationary measures (intuitively, one can think of these as fixed points) in the limit $L \to \infty$; said another way, Gács’ PCA can “remember” one bit per unit length! Each cell/site of the PCA has a large state space, likely somewhere between $2^{24}$ and $2^{400}$ \cite{8, 9}. Roughly speaking, each cell contains one bit that it is trying to remember, and the remaining 399 bits (taking e.g. the $2^{400}$ state space) are involved in a highly collective error correction protocol. As in the Toom model, the stochastic errors can be biased so long as they remain below some finite threshold, above which a dynamical phase transition will restore ergodicity.

Even more remarkably, not only is the Gács model an error-corrected memory, it can execute Turing-complete operations on the protected state space. In other words, his construction demonstrates that a stochastic 1D PCA can be used to simulate a deterministic CA, and hence error-corrected classical computing is possible in 1D. This immediately implies the existence of the “$\pi$-Gács time crystal”: In particular, one can simply use the Gács construction to emulate a CA with the rule: $1 \leftrightarrow -1$. His mathematical results then imply that this is an absolutely stable discrete time crystal, with infinitely long-lived temporal order as $L \to \infty$.

Gács’ result is mathematically rigorous, and as such, there are assumptions about the error model \cite{9, 11}. Crucially, as discussed in the main text, this construction is stable to very generic noise models: the noise need not be Markov, and needs only to satisfy:

$$P_{\bigwedge_{\ell=1}^k E_{u_\ell}} = \prod_{\ell=1}^k P_{E_{u_\ell} | E_{u_{\ell-1}} \cdots E_1} \leq \epsilon^k,$$

where $E_{u_\ell}$ denotes the indicator function of whether an error has occurred at the space-time point $u_\ell$. Thus, a sufficient condition is simply to require that each $P_{E_{u_\ell} | E_{u_{\ell-1}} \cdots E_1} \leq \epsilon$; if we choose $\ell$ to be ordered in time, this will be satisfied if the probability for an error to occur at $\ell$, conditioned on all of the past errors, is below some constant $\epsilon$. This will always be the case for a Markov model of the form, $M = T + \epsilon \Delta M$, for arbitrary (local) entries $|\Delta M_{\eta \rightarrow \eta'}| \leq 1$. Thus, the “stability to errors” of the Gács model can be understood, more generally, as the stability to local perturbations.

As a result, as long as the error can be made small enough and satisfy Eq. (1), one has a path towards building an absolutely stable one dimensional time crystal.
FIG. S1. Schematic of two π-Toom Floquet cycles using our protocol. In steps 1 and 3, a pinning potential is applied to allow the bath to extract energy from the system. In step 2, the CA rule is applied on the oscillators B based upon the state in oscillators A; in step 4, the role of the two types of oscillators is reversed.

DETAILS OF THE FLOQUET LANGEVIN EVOLUTION AND NUMERICAL SIMULATION

More detailed description of the mapping

In this section we discuss in some additional detail our construction for mapping the CA dynamics within a noisy Floquet-Langevin classical system. We begin with the oscillators at site $x$ in the state $(q^A_x, p^A_x) = \left( Q(s), 0 \right)$, where $Q(s)$ is the function that maps a particular CA state to a position of the oscillator. Given the discrete nature of the $S$, one can map each state $s \in S$ to a different integer in the real line of positions. From there, the dynamics evolve via a 4-step process, Fig. S1.

**Step 1: Relaxation.** The goal of the first step is to leverage dissipation in order to reduce fluctuations in the system. In particular, we envision turning on a one-body potential, $V_{\text{pin}}(q)$, which has a local minimum at $Q(s)$ for all $s \in S$. At sufficiently low temperatures, the dissipative dynamics will relax the oscillator’s positions, $q_x$, toward valid values of $Q(s)$ with low momenta (with fluctuations of order the equipartition scale $\sim k_B T$). The precise form of $V_{\text{pin}}$ is not important; however, for concreteness we will utilize

$$V_{\text{pin}}(q) = v_{\text{pin}} \prod_{s \in S} (Q(s) - q)^2$$  \hspace{1cm} (2)

where the overall magnitude of the pinning potential is set by $v_{\text{pin}}$.

**Step 2: Fix A move B.** As illustrated in Fig. S2, the second step of the Floquet dynamics implements the cellular automata transition $B = T(A)$. We will keep $q^B$ fixed using the pinning potential, $V_{\text{pin}}$. For the $B$ oscillators, however, we turn off $V_{\text{pin}}$, and turn on an interaction $V_I$ between $q^A$ and $q^B$. This interaction is engineered such that each $q^B_x$ sees only a single potential minimum corresponding to the desired CA update rule; in general, this will depend on the state of the $A$ oscillators in the associated neighborhood, $\{q^A_{x+N_x} \}$. 

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The above explanation outlines the details of the mapping process, particularly the relaxation step and the interaction step, which are crucial for understanding the Floquet-Langevin evolution in the context of cellular automata dynamics.
FIG. S2. Schematic depicting the translation between a cellular automata step and the corresponding Hamiltonian simulation step. (a) Shows a single step of a one dimensional version of the “π-Toom” rule, which consists of a majority vote and a bit flip. (b) In the Hamiltonian setting, we consider two sets of oscillators, A (blue) and B (red). The corresponding Hamiltonian simulation proceeds in two steps. First, there is a “relaxation” step followed by local interactions which implement the “majority vote”. In this second step, the B oscillators are fixed, while the state of the A oscillators is updated. (c) Trajectory $q_{A/B}(t)$ during an error correction step of the Toom model for a $32 \times 32$ lattice with $v = 50$ and $T = 2$. The vertical dashed lines divide time into four steps as described in the text; an error of the form $q \sim -1$ gets corrected to be $q \sim 1$.

Defining the location of this minimum to be $\mathcal{T}(\{q_{A+N}^A\})$, we can then specify an interaction of the form:

$$V_I(\{q_{A+N}^A\}, q_N^B) = \frac{v_I}{2} \left( \mathcal{T}(\{q_{A+N}^A\}) - q_N^B \right)^2,$$

where the interaction strength is characterized by $v_I$. This is a highly non-linear but local interaction between each $q_N^B$ and a finite set of A oscillators, $\{q_{A+N}^A\}$, within the neighborhood, $N$. In particular, as shown in Fig. S2(b), for the example of $\mathcal{T}$ being an “anti-majority vote”, the interaction would correspond to an $|N| + 1$ body coupling [12].

**Step 3: Relaxation.** In the third step, we turn off the interaction, $V_I$, while ramping up the pinning potential, $V_{\text{pin}}$. As in the first step, dissipation relaxes and pins the positions of the oscillators.

**Step 4: Fix B move A.** In the final step, we implement “$A = \mathcal{T}(B)$” by repeating step two with the role of A and B reversed.

After these four steps, our Floquet dynamics have implemented two steps of the cellular automata update rule, $\mathcal{T}$. This block naturally forms a single period of the Floquet drive, which can then be repeated. As mentioned in the main text, one can also replace the transition $\mathcal{T}$ in the last step with the “do-nothing” CA rule if one wants to implement only a single CA update, $\mathcal{T}$, per Floquet cycle.

**Estimating temperature dependence of error $P_E$**

To predict the rate of such errors, suppose that the interaction potential is driving an oscillator to the state $q = -1$, so that $V_I(q) = \frac{v_I}{2}(q + 1)^2$. When the dynamics switch to the pinning potential $V_{\text{pin}}(q)$, an error will occur if the oscillator has a position $q \in [0, \infty)$. Assuming the system reaches local equilibrium with respect to $V_I$, the probability of this error can be estimated from the Boltzmann distribution as:

$$P_E \left( \frac{v_I}{T} \right) = \frac{1}{2} \text{Erfc} \left( \sqrt{\frac{v_I}{2T}} \right),$$

which asymptotically gives exponential decay $P_E \sim e^{-v_I/2T}$. This is the functional form used in Fig. 2(a) of the main text, in excellent agreement with the numerical data.
From left to right, the temperature is varied such that $T = 0.5, 2, 10$. From top to bottom, we change the damping ratio, $\kappa_f = 0.5, 1, 1.5$

**Effect of the choice of damping parameter $\gamma$**

In the “fix $A$, move $B$” step, if we fix the $A$ oscillators, then the $B$ oscillators are critically damped harmonic oscillators with additional noise. If there is no noise, then replacing $q_k^B = \exp(\lambda t)$ gives two eigenvalues $\lambda = -(1/2) \left( \gamma \pm \sqrt{\gamma^2 - 4\nu_I} \right)$. Thus, the decay of oscillation amplitude is fastest when the damping has the critical value $\gamma^*=2\nu_I$. When $\gamma > \gamma^*$, it is over-damped, and the relaxation is slow because the velocity is low. When $\gamma < \gamma^*$, oscillation remain and the damping is slow due to small friction coefficient. Similarly, in the “relaxation” steps, both oscillators correspond to damped harmonic oscillators for the potential $V_{\text{pin}}(q) = v_{\text{pin}}(q^2 - 1)^2 \simeq 4v_{\text{pin}}(q - 1)^2$ near $q = +1$. The critical damping during the relaxation step is then given by $\gamma^*_r = 4\sqrt{2v_{\text{pin}}}$. We tune the damping ratio $\kappa_f = \gamma^*/\gamma^*_r = \gamma^-/\gamma^*_r$ such all steps within the Floquet evolution are tuned to critical damping. For simplicity, we let $v_{\text{pin}} = 4\nu_I = \nu$, and then the critical relaxation time when $\kappa_f = 1$ is $t^*_r \sim 1/\sqrt{\nu}$. The critical damped solution when the initial oscillator position is $q_0$ at rest is given by $q(t) = q_0 e^{-\sqrt{\nu}t} \left( \sqrt{\nu}t + 1 \right)$.

We now study whether this construction can effectively correct implement CA rules. To this end, we study the correction of a single error under $\pi$-Toom dynamics. We set the initial position of the $A, B$ oscillators to be uniformly $q = +1$ in all cells, except at cell $(1,1)$, where we choose $q = -1$ to create an error. All initial momenta are zero. In Fig. S3 we monitor the position of the $A$ and $B$ oscillator at cell $(1,1)$. In this plot, between $[0,1],[2,3],[4,5],\cdots$ we have the relaxation steps; between $[1,2],[5,6],[9,10],\cdots$ we have Toom steps; while between $[3,4],[7,8],[11,12],\cdots$ we have $\pi$-Toom steps. We see that in all cases the error gets corrected within a duration of $t = 4$, after which the black and blue curves have similar shapes at low temperature; however, the critical damping case [Fig. S3 b1-b3] relaxes the fastest and exhibits the smallest fluctuations, compared with the under-damped [Fig. S3 a1-a3] and over-damped [Fig. S3 c1-c3] cases. We have verified that the same behavior holds for various pinning potentials $\nu$ and temperatures $T$.

**Smoothing discontinuities**

The interaction $V_I$ in Eq. (3) involves two set of coordinates: $q_{x+N}^B$ and $\{q_{x+N}^A\}$. In the numerical simulation, one needs to implement the motion by applying a force on the $A$ oscillators and $B$ oscillators. Naively, the $T$ of Eq. (3) would be taken as $T(S(q)))$. However, due to the discontinuities in $S$, this choice would cause $V_I$ to depend discontinuously on $\{q_{x+N}^A\}$, leading to $\delta$-function forces. This would make the numerical simulation of Hamilton’s equations challenging. To avoid this, we first evaluate the values of $V_I$ at a discrete set of points of $q_{x+N}^A = \pm 1$, while keeping $q_{x+N}^B$ arbitrary. We then create a linear interpolation between these discrete points to create a smoothed version of $V_I$ for oscillators $A$, which allows the evaluation of the force in the numerical simulation.
FIG. S4. Floquet Langevin simulation of the $\pi$-Toom model for a two dimensional lattice of size $32 \times 32$, with $v = 50$. For each panel, the inset depicts the initial configuration (for both $A$ and $B$ oscillators), where red indicates $q = +1$ and blue indicates $q = -1$. (a) For a uniform initial state, at low temperatures, the time crystalline order parameter remains finite at late times. At high temperatures, the time crystal quickly melts into a disordered phase. (b) For an initial state with a central island of errors (i.e. oscillators in $q = -1$), at low temperatures, the Floquet Langevin dynamics “error correct” and the time crystalline order grows toward a plateau at late times. At high temperatures, the time crystal again melts into a disordered phase. (c) For an initial state with stripes of errors, one sees the same qualitative behavior as in panel (b). As expected, near the transition (data set with $v/T = 15.75$), it becomes difficult to tell whether the time crystalline order will eventually decay or plateau to a finite value.

EFFECT OF INITIAL STATE ON THE FLOQUET-LANGEVIN $\pi$-TOOM DYNAMICS

To investigate the emergence of DTC order, we compute the Floquet-Langevin dynamics starting from three distinct initial states: (i) a uniform input state with all oscillators in $q = +1$ [Fig. S4(a)], (ii) a state which contains an island of $q = -1$ oscillators in the center [Fig. S4(b)], and (iii) a state which consists of diagonal stripes of $q = -1$ oscillators [Fig. S4(c)] [13]. In the language of the $\pi$-Toom PCA, for each of these initial states, one can think of the oscillators with $q = -1$ as “errors”, which will either be “corrected” by our Floquet Langevin dynamics (for sufficiently low bath temperatures) or not.

For the uniform initial state [Fig. S4(a)], the DTC order parameter, $(-1)^{[t/\tau]} \langle M_A \rangle$, begins at unity for all temperatures. At high temperatures, the order parameter quickly decays to zero, indicating that the Floquet Langevin dynamics drive the system toward the disordered phase. On the other hand, for sufficiently low temperatures, the time crystalline order evolves toward a finite plateau value at late-times, indicative of a DTC. In Fig. S4(b), we show the analogous dynamics starting from an initial state with a central island of errors. For low temperatures, the Floquet Langevin dynamics correct these errors and the DTC order parameter grows, with the system approaching a time crystalline state. Again, above a critical temperature, time crystalline order “melts” and the stroboscopic magnetization decays to zero. Finally, Fig. S4(c) depicts the dynamics starting from a striped error configuration; the qualitative features are identical to Fig. S4(b), although the competition between the DTC phase and the disordered phase is more apparent at intermediate temperatures.

FINITE-SIZE SCALING BEHAVIOR OF THE $\pi$-TOOM TIME CRYSTALLINE DYNAMICS

In this section, we complement our analysis of the robustness of the $\pi$-Toom time crystal by demonstrating that the lifetime of the time crystalline dynamics scales exponentially in system size, and does not appear to host any saturation—this further strengthens our claims of the stability of the $\pi$-Toom time crystal in the $L \rightarrow \infty$ thermodynamic limit.

We begin by considering the PCA dynamics of the $\pi$-Toom model, Fig. S6. In this case, the stability of the $\pi$-Toom model immediately inherits from the proven stability of the underlying Toom model [10]. Nevertheless, its numerical analysis provides a controlled setting where we can better understand the signatures of the stability of the $\pi$-Toom time crystal when considering the full Langevin dynamics.

For different error rates $\epsilon$, we consider the PCA dynamics for different system sizes $N = L \times L$. Crucially, we observe that by increasing the system size of the system, the lifetime of the time crystalline behavior grows as well. This growth is also enhanced upon decreasing the error rate $\epsilon$ of the dynamics since $\epsilon$ reduces the probability of an
FIG. S5. Cumulants, \( \langle N^0 \rangle_c \), of the error distribution from a Floquet Langevin simulation of the \( \pi \)-Toom model for \( T = 11.94, v = 100 \). We find that accurately estimating higher cumulants, e.g. \( c_3 \) and \( c_4 \), is more challenging at temperatures above the critical temperature, \( T_c = 9.6 \).

FIG. S6. Left Panels: Dynamics of the two-point correlation function \( \langle M_A(t)M_A(0) \rangle \) of the magnetization \( M_A \) under the PCA dynamics for even (full lines) and odd periods (light dashed lines) and varying system size. Regardless of the choice of error rate \( \epsilon \), the system exhibits a robust period doubling behavior whose lifetime grows with the system size of the system. Right Panel: Dependence of the time crystalline lifetime \( \tau_{TC} \) on the system size for different error rates \( \epsilon \). For all \( \epsilon < 0.1 \) within the \( \pi \)-Toom time crystal phase, \( \tau_{TC} \) grows exponentially with system size, and does not exhibit any saturation behavior. This supports the stability of the \( \pi \)-Toom time crystal in the thermodynamic limit.

extensive error occurring and thus destroying the synchronized behavior that is stabilized by Toom’s rule.

For a more quantitative analysis of this decay, we extract the lifetime of the time crystalline behavior \( \tau_{TC} \) as the time at which \( \langle M_A(t)M_A(0) \rangle \) first becomes smaller than 0.75\( \langle M_A(0) \rangle^2 \). Plotting this lifetime as a function of system size, we observe that the enhancement of the lifetime is exponential in the system size and does not display any signs of saturating—this demonstrates the robustness of the \( \pi \)-Toom time crystal within the PCA model.

We then perform the same analysis in the full Langevin dynamics. Owing to computational cost, we are limited to a smaller number of parameters, however we observe the same behavior, regardless of the depth of the potentials \( v \), Figs. S7 and S8. For different temperatures (at fixed potential depth), we observe that increasing the system size substantially increases the lifetime of the time-crystalline behavior and preserved the period doubling dynamics. Crucially, upon extracting the lifetime of the time crystalline order, we observe the same exponential scaling with system size, demonstrating the robustness of the \( \pi \)-Toom time crystal. Since there are no signatures of saturation, this behavior is expected to extend to the thermodynamic limit, justifying our description of the \( \pi \)-Toom time crystal as an out-of-equilibrium phase of matter.
FIG. S7. Analogous data as in Fig. S6 but considering the full Langevin dynamics with \( v = 50 \) and varying the temperature \( T \) of the dynamics.

FIG. S8. Analogous data as in Fig. S6 but considering the full Langevin dynamics with \( v = 100 \) and varying the temperature \( T \) of the dynamics.

**EXPERIMENTAL REALIZATIONS**

In this section, we describe two experimental routes for implementing the PCA simulation protocol described in the main text. Before detailing these two avenues, let us recall the presented protocol, using that as a stepping stone to formalize the requirements of a suitable experimental platform Fig S1. The Floquet Langevin dynamics are divided into 4 steps: in steps (1) and (3), the potential is in a “pinning” configuration, whereby a strong double-well potential (in the case of a two-state cellular automata) is used to relax the oscillators and extract energy; while during steps (2)[(4)] a multi-particle interaction is used to implement the cellular automata rule on the B[A] oscillators, based upon the state of the A[B] oscillators. To this end, we require an experimental platform to:

(i) apply a non-harmonic potential to the oscillator;

(ii) apply a multi-particle potential to each particle and its neighborhood;

(iii) couple the system to a thermal bath.

We propose two different experimental settings where we envision our protocol being realized: ions trapped in an array of surface-electrode traps [Fig. S9(a)]; and an array of coupled superconducting circuits [Fig. S9(b)].

**Ions trapped in Pauli traps**

In the first setting, recent developments in surface traps have enabled the exquisite control over the motional potential of ions in space [14–20], offering the ability to prepare and dynamically control complicated anharmonic potentials for each individual ion [21]—this meets requirement (i). While electrostatic forces can be leveraged to induce pairwise interactions between the ions, such interactions would be highly dependent on the geometry of the array and, owing to their long-range Coulomb nature, not constrained to a particular finite neighborhood. As a result, ensuring
that they are compatible with the cellular automata of interest requires additional considerations with respect to the geometry of the ion traps. We propose an alternative approach, using measurements and feed-forward to dynamically generate the interaction potential \( V_i \). We envision the position of the ions being constantly monitored, and based upon the position of the “memory” oscillators, the potential of the “active” oscillators is controlled, effectively generating a multi-particle interaction potential—this meets requirement (ii). We note that this approach is possible owing to the ability to dynamically control the motional potential in real-time. Crucially, this approach is agnostic to the PCA being simulated, and thus simulating different PCAs (even embedded in different geometries and dimensions) does not require modifying the underlying hardware. Intriguingly, by adding a time delay between the observation and the update of the potential one can also alter the nature of the system’s dynamics—in this case, the system’s dynamics can be made to depend explicitly on their history and not just the current position of the particles leading to non-Markovian dynamics.

Interestingly, we note that, recent progress in the engineering of multi-body interactions between the internal degrees of freedom of trapped ions hints at the possibility of implementing the required potentials through the system’s native physical interactions \[22\] [23]. However, the extension of these ideas to interactions between motional modes, as well as their incorporation with the double-well potential requires additional exploration.

Finally, we discuss requirement (iii). The coupling of the ions to the outside world comes through different mechanisms (i.e. black-body radiation, electric current fluctuations in the electrodes, background gas collisions), all of which can be tuned (via temperature, pressure of the chamber and further engineering of the surface electrodes) \[24\]. At the same time, additional noise and dissipation can be explicitly added to the system, either by adding noise to the contact potentials and thus adding fluctuations to the ion’s potentials, or by using laser side-band cooling to remove kinetic energy from the ion’s motion. This level of control allows one to go beyond thermal baths to explore the effect of different spectral features of the noise on the system’s ability to simulate a PCA \[25\].

Putting all of these elements together, we can briefly describe the operation of the experiment through our four step simulation process \[Fig. S1\]. During the first step, the double-well potential is added to all the ions. The coupling to the bath relaxes the ions. During the second step, the double-well potential is removed from the active ions (B), while kept at the memory ones (A). The location of the memory ions is tracked through fluorescence imaging and the minimum of the harmonic potential is controlled so as to capture the PCA transition rule. During the third step, the double-well potential is again added to all the ions. During the fourth step, the second step is repeated exchanging the role of the A and B oscillators to be the active and the memory ions, respectively.

**Superconducting circuits**

We now consider a superconducting circuit array as a system of tunable non-linear coupled oscillators. We consider a flux-qubit architecture, whereby the oscillator “position” corresponds to the phase \( \Phi \) generated by a persistent current within the superconducting loop \[28\]; physically, this persistent current enforces the requirement that exactly a multiple of the flux quantum \( \Phi_0 \) penetrates the superconducting loop. In this configuration, the potential landscape can be tuned by changing the threaded flux through different elements of the superconducting qubit \[27\]: this ability to straightforwardly engineer double-well potentials, immediately satisfies requirement (i).

The most direct way of coupling two superconducting oscillators is by proximatizing them and using their mutual conductance \((C_{mut})\) or inductance \((M_{mut})\) \[28\]. In our case, we are interested in leveraging inductance coupling which directly couples the phases between the oscillators \( H_I = M_{mut} \sin \Phi_i \sin \Phi_j \) in analogy to the position dependent coupling discussed in the main text. While proximitization enables the coupling between two oscillators, this coupling can also be mediated by an intermediate superconducting loop, allowing for additional tuning control of the interactions \[28\].

In general, implementing an arbitrary PCA will requires \( N \)-body interactions to capture the correct potential necessary in steps (2) and (4), where \( N \) is the number of cells associated with the transition rule (4 in the case of the \( \pi \)-Toom model). Fortunately, there has been extensive effort creating such interactions using superconducting qubits, which we can leverage when considering the continuum (classical) limit of these objects \[29\]. Such interactions arise when multiple superconducting oscillators are inductively coupled to a tunable, common oscillator that is far detuned—this satisfies requirement (ii). Let us note that, the additional connectivity, as well as the tuning of a larger number of interactions implies that the complexity of the object increases rapidly with the complexity of the PCA rule.

Curiously, the particular case of the \( \pi \)-Toom model can actually be implemented using only tunable two-oscillator
Classical feedback

FIG. S9. a) An array of trapped ions serves as oscillators for the simulation of the PCA dynamics (top). Local electrodes can control the motional potential and dynamically change it between a pinning potential or a single-well potential centered around the positions associated with each state of the cellular automata. The potential necessary for implementing the PCA transition rule can be obtained by measuring the memory ions, and feed-forwarding the transition rule onto the potential experienced by the active ions (bottom). b) An array of superconducting circuits provides a set of tunable, anharmonic oscillators, that serves as a platform for the studying the Langevin dynamics explored in the main text. For each cell of the PCA, one requires two circuits (blue and red), which can be coupled to each other and their neighbors (green). These additional coupling circuits enable the dynamically, tunable interactions necessary for implementing the PCA transition rules.

interactions. Given the anti-majority vote nature of the π-Toom transition, the potential:

\[
V_{I, \text{approx}} = \left( \frac{\Phi_A^{x,y} + \Phi_A^{x+1,y} + \Phi_A^{x,y+1}}{3} + \Phi_B^{x,y} \right)^2
\]

has a minimum where the sign of \( \Phi_B^{x,y} \) is opposite to the mean of the nearby memory cells. Fixing the A particles as the memories using the pinning potential, the B oscillator can equilibrate to the potential minimum, implementing the necessary anti-majority transition rule. Note that different values of the mean of the \( \Phi_A^{x,y} \) phases will place the minimum of the potential of \( \Phi_B^{x,y} \) at different locations; however, since the sign of the phase is correctly captured, we expect that the following pinning step causes the system to relax to the same phase value regardless of the differences in \( \Phi_A^{x,y} \).

Having discussed how interactions can be generated to implement the PCA transition rules, we turn to discussing the role of the bath in the platform. Much like the trapped ion case, one can directly control the temperature of the environment (e.g. by adjusting the cryostat temperature) in order to modify the bath coupled to the superconducting circuit array [27]. One can also artificially inject noise into the system and study how the ability to simulate the PCA dynamics is affected by the spectral properties of the injected noise [30]. These considerations satisfy requirement (iii).

Zooming out a bit, we note that this setup provides a flexible platform that can investigate broad questions related to the classical to quantum crossover in the dynamics of open systems. Indeed, by lowering the temperature and preparing smaller coherent states, one eventually approaches the regime where the quantum nature of the dynamics becomes important.

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