EXISTENCE AND GENERAL DECAY OF BALAKRISHNAN-TAYLOR VISCOELASTIC EQUATION WITH NONLINEAR FRICITIONAL DAMPING AND LOGARITHMIC SOURCE TERM

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Abstract. In this paper, we consider a Balakrishnan-Taylor viscoelastic wave equation with nonlinear frictional damping and logarithmic source term. By assuming a more general type of relaxation functions, we establish explicit and general decay rate results, using the multiplier method and some properties of the convex functions. This result is new and generalizes earlier results in the literature.

1. Introduction. In this paper we are concerned with the following Balakrishnan-Taylor viscoelastic wave equation with nonlinear frictional damping and logarithmic source term in \( \Omega \times \mathbb{R}^+ \),

\[
|u_t|^p u_{tt} - \left( \xi_1 + \xi_2 \| \nabla u \|^2 + \sigma (\nabla u, \nabla u_t) \right) \Delta u - \Delta u_{tt} + \int_0^t g(t-s) \Delta u(s) ds + h(u_t) = ku \ln |u|.
\]

(1)

To Eq. (1), we consider the boundary condition

\[
u = 0, \quad \text{on} \quad \partial \Omega \times \mathbb{R}^+,
\]

(2)

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and initial conditions
\[ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega. \] (3)
Here \( \Omega \subseteq \mathbb{R}^n \) \( (n \geq 1) \) is a bounded domain with a smooth boundary \( \partial \Omega \) and \( \rho \) is a positive real number such that
\[ 0 < \rho \leq \frac{2}{n-2} \text{ if } n \geq 3, \quad \rho > 0 \text{ if } n = 1, 2. \]
The coefficients \( \xi_1, \xi_2 \) and \( \sigma \) are positive constants. The integral term is the memory responsible for the viscoelastic damping. The function \( g \), usually called relaxation function, is a real function, \( h(u_t) \) represents the nonlinear frictional damping, where \( h \) is a specific function.

In [4], Balakrishnan and Taylor first introduced Balakrishnan-Taylor damping \( \sigma(\nabla u(t), \nabla u_t(x, t)) \), see also Bass and Zes [6]. The system is the well-known wave equation with \( \xi_2 = \sigma = 0 \) and Kirchhoff type wave equation with \( \sigma = 0 \), which have been extensively studied. The general form of viscoelastic wave equation with Balakrishnan-Taylor damping reads as follows
\[ u_{tt} - (a + b \|\nabla u\|^2 + \sigma(\nabla u, \nabla u_t)) \Delta u + \int_0^t h(t-s)\Delta u(s)ds + g(u_t) = f(u). \] (4)

In absence of viscoelastic term, Zarai and Tatar [44] proved the global existence and a polynomial decay for the problem with \( f(u) = |u|^p u \). Tatar and Zarai [42] established an exponential decay and proved blow up in finite time. Park [40] established a general decay result of the problem with \( f(u) = 0 \). For viscoelastic Balakrishnan-Taylor wave equation (4), Ha [21] proved a general decay result of energy without imposing any restrictive growth assumption on the damping term. Recently, Feng and Kang [15] studied a viscoelastic wave equation with Balakrishnan-Taylor and frictional dampings and established a general decay rate of solution. For more results concerning wave equation with Balakrishnan-Taylor damping, one can refer to Clark [14], Feng and Soufyane [16], Ha [21, 19, 20], Tatar and Zarai [43], Wu [45], You [47] and Zarai and Tatar [49] and others. To motivate our work, we also recall some known results related to (1), in absence of Balakrishnan-Taylor damping. Precisely for the viscoelastic wave equation
\[ u_{tt} - \rho \Delta u_{tt} - \Delta u + \int_0^t h(t-s)\Delta u(s)ds + g(u_t) = f(u), \] (5)
many stability results have been established. For instance, if \( h = 0 \), we know that the energy of the solution decays exponentially (polynomially) if \( g \) decays exponentially (polynomially). Messaoudi [32, 33] introduced the assumption \( g'(t) \leq -\xi(t)g(t) \) to study two related model of (5) and established general decay rates of solution for a wider class of relaxation function. In [1], Alabau-Boussouira and Cannarsa considered a semilinear abstract second-order equation with a memory and claimed various general decay rates of solution by assuming
\[ g'(t) \leq -H(g(t)), \] (6)
where \( H \) is a positive, strictly increasing and strictly convex function satisfying \( H(0) = H'(0) = 0 \). Mustafa and Messaoudi [39] considered a variant of (5) with (6) and established a general and explicit decay result under much weaker conditions than those of [1]. Since then, many interesting results on decay of solutions to viscoelastic equation have been established by using the condition (6). For instance, Cavalcanti et al. [9, 10], Lasiecka et al. [26, 27], Mustafa [34], Mustafa
and Messaoudi [39] and Xiao and Liang [46]. The energy decay rate first established by Lasiecka and Wang [27] is not only general but also optimal in which the decay rates are characterized by an ODE of the same type as the one generated by the inequality (6) satisfied by $g$. Mustafa [35, 36] introduced a more general assumption on $g$ than (6); namely $g'(t) \leq -\xi(t)H(g(t))$, to study two classes of viscoelastic wave equation and proved the optimal decay of energy. In the presence of the time-dependent coefficient $\eta(t)$, Mustafa and Messaoudi [38] and Mustafa and Abusharkh [37] considered a viscoelastic wave equation and a viscoelastic plate equation, respectively, and established two general energy decay results of the two systems, which depend on both $h$ and $\eta$. Cavalcanti and Oquendo [12] considered a viscoelastic wave equation with a complementary frictional damping. They obtained an exponential decay of energy for $g$ decaying exponentially and $h$ linear. In addition, they also obtained polynomial stability of energy for $g$ decaying polynomially and $h$ having a polynomial growth near zero. If $g = 0$, we refer the reader to Lasiecka and Tataru [28], Liu and Zuazua [29], and Martinez [30, 31] to find some decay results for damped wave equations.

The logarithmic nonlinearity usually appears in inflation cosmology and supersymmetric filed theories, quantum mechanics and nuclear physics. This type of problems has applications in many branches of physics such as nuclear physics, optics and geophysics. Birula and Mycielski [7, 8] considered a relativistic version of logarithmic quantum mechanics in a bounded interval $[a, b], \quad u_{tt} - u_{xx} + u = \varepsilon u \ln |u|^2$.

Cazenave and Haraux [13] studied the Cauchy problem of a 3-D wave equation with logarithmic nonlinearity and proved the global existence and uniqueness of solutions. Gorka [17] considered an initial-boundary value problem to a 1-D case of the model in [13], and proved the global existence of weak solutions. Bartkowski and Gorka [5] proved the existence of global classical solutions and also studied the Cauchy problem of the 1-D case of the model. When studying the dynamics of Q-ball in theoretical physics, Hiramatsu et al. [23] introduced the following equation

$$u_{tt} - \Delta u + u + u_t + |u|^2 u = u \ln |u|,$$

and obtained some numerical results. However, the global existence of weak solutions was proved by Han [22]. Hu et al. [24] considered

$$u_{tt} - \Delta u + u_t = u \ln |u|^k,$$

and established some energy decay rates. Peyravi [41] improved Hu et al’s result and established a general decay of energy.

In this paper, we intend to consider (1)-(3) with both weak frictional damping and viscoelastic damping acting simultaneously and complementarily in the domain. We establish a general decay rate for the energy without imposing any growth assumption near the origin on $h$ and strongly weakening the usual assumptions on $g$. In other words, the result here holds for a larger class of functions $g$ and $h$, from which the energy decay rates are not necessarily of exponential or polynomial types, and hence improve some previous related results.

The paper is organized as follows. In Section 2, we introduce some assumptions needed in this paper to establish our main results and give a local existence result of solutions. In Section 3, we state the global existence result. In Section 4, we give some technical lemmas to prove our stability result. Section 5 is devoted to the proof of our stability result.
2. Preliminaries. First of all we give some assumptions:

(A1) \( g : \mathbb{R}^+ \to (0, \infty) \) is a \( C^1 \) nonincreasing function satisfying

\[
g(0) > 0, \quad \xi_1 - \int_0^{+\infty} g(s) ds = \ell > 0,
\]

and there exists a \( C^1 \) function \( G : (0, \infty) \to (0, \infty) \) which is linear or it is strictly increasing and strictly convex \( C^2 \) function on \( (0, r_1], \ r_1 \leq g(0) \), with \( G(0) = G'(0) = 0 \), such that

\[
g'(t) \leq -\xi(t) G(g(t)), \quad \forall t \geq 0,
\]

where \( \xi(t) \) is a positive nonincreasing differentiable function.

(A2) \( h : \mathbb{R} \to \mathbb{R} \) is a nondecreasing \( C^0 \) function such that there exists a strictly increasing function \( h_0 \in C^1(\mathbb{R}^+) \), with \( h_0(0) = 0 \), and positive constants \( c_1, c_2, \varepsilon \) such that

\[
\begin{align*}
    h_0(|s|) &\leq |h(s)| \leq h_0^{-1}(|s|) \quad \text{for all } |s| \leq \varepsilon, \\
    c_1 |s| &\leq |h(s)| \leq c_2 |s| \quad \text{for all } |s| \geq \varepsilon.
\end{align*}
\]

In addition, we assume that the function \( H \), defined by \( H(s) = \sqrt{s} h_0(\sqrt{s}) \), is a strictly convex \( C^2 \) function on \((0, r_2)\), for some \( r_2 > 0 \), when \( h_0 \) is nonlinear.

**Remark 1.** Hypothesis (A2) implies that \( sh(s) > 0 \), for all \( s \neq 0 \).

**Remark 2.** It is worth noting that condition (9), with \( \varepsilon = 1 \), was first introduced by Lasiecka and Tataru [28].

**Remark 3.** If \( G \) is a strictly increasing and strictly convex \( C^2 \) function on \((0, r_1]\), with \( G(0) = G'(0) = 0 \), then it has an extension \( \overline{G} \), which is strictly increasing and strictly convex \( C^2 \) function on \((0, \infty)\). For instance, if \( G(r_1) = a, G'(r_1) = b, G''(r_1) = c \), we can define \( \overline{G} \), for \( t > r_1 \), by

\[
\overline{G}(t) = \frac{c}{2} t^2 + (b - cr_1) t + \left( a + \frac{c}{2} r_1^2 - br_1 \right).
\]

The same remark goes for \( \overline{H} \). See [36].

**Lemma 2.1.** (Logarithmic Sobolev inequality, [17, 18]) Let \( u \) be any function in \( H^1_0(\Omega), \ \Omega \subseteq \mathbb{R}^n \) be a bounded smooth domain and \( a > 0 \) be any number. Then

\[
\int_\Omega |u|^2 \log \frac{|u|}{\|u\|_2} dx \leq \left( \frac{3}{4} \log \frac{4a}{\varepsilon} \right) \|u\|_2^2 + \frac{a}{4} \|\nabla u\|_2^2.
\]

To establish our decay result, we need the following lemma whose proof can be found in [2].

**Lemma 2.2.** Let \( \varepsilon_0 \in (0, 1) \). Then there exists \( d_\varepsilon > 0 \) such that

\[
s |\ln s| \leq s^2 + d_\varepsilon s^{1-\varepsilon_0}, \quad \forall s > 0.
\]

By combining the arguments of [11], [2] and [48], we can establish the local existence of solutions of problem (1)-(3).

**Theorem 2.3.** Suppose that (7) and (9) hold. Then given \( (u_0, u_1) \in H^1_0(\Omega) \times H^1_0(\Omega) \), there exists \( T > 0 \) and a unique solution \( u \) of problem (1)-(3) such that

\[
u \in C([0, T); H^1_0(\Omega)) \cap C^1([0, T); H^1_0(\Omega)).
\]
The total energy of problem (1)-(3) is given by
\[
E(t) = \frac{1}{2} \left( \xi_1 - \int_0^t g(s) ds \right) \| \nabla u \|_2^2 + \frac{1}{2} \| \nabla u_t \|_2^2 + \frac{1}{2} (g \circ \nabla u)(t) + \frac{k}{2} \int \Omega u^2 \ln |u| dx \\
+ \frac{k}{4} \| u \|_2^2 + \frac{\xi_2}{4} \| \nabla u \|_4^4 + \frac{1}{\rho + 2} \| u_t \|_{\rho + 2}^{\rho + 2},
\]
where
\[
(g \circ v)(t) = \int_0^t g(t - s) \int_\Omega |v(t, x) - v(s, x)|^2 dx dt.
\]

For the energy functional \(E(t)\) we have the following lemma.

**Lemma 2.4.** Suppose that assumptions (A1) and (A2) hold. Then \(E(t)\) is a non-increasing function and satisfies for any \(t < T\),
\[
E'(t) = \frac{1}{2} (g' \circ \nabla u)(t) - \frac{1}{2} g(t) \int_\Omega |\nabla u|^2 dx - \sigma \left( \frac{1}{2} \frac{d}{dt} \| \nabla u \|^2 \right)^2 - \int_\Omega h(u_t) u_t dx \leq 0
\]
(12)

**Proof.** Multiplying (1) by \(u_t\), integrating by parts over \(\Omega\), we find (12). \(\square\)

3. **Global existence.** In this section, we establish the global existence of the weak solution of (1)-(3). We define the following functionals on \(H_0^1(\Omega)\):
\[
I(u(t)) = (\xi_1 - \int_0^t g(s) ds) \| \nabla u \|_2^2 + \| \nabla u_t \|_2^2 + (g \circ \nabla u)(t) - k \int_\Omega u^2 \ln |u| dx,
\]
\[
J(u(t)) = \frac{k}{2} \| u \|_2^2 + \frac{\xi_2}{4} \| \nabla u \|_4^4 + \frac{1}{2} I(u(t)),
\]
then
\[
E(t) = \frac{1}{\rho + 2} \| u_t \|_{\rho + 2}^{\rho + 2} + J(u(t)).
\]

We also, introduce the following:
\[
E_1 = \frac{k}{4} \gamma_*, \quad \gamma_* = e^{\frac{C_0}{k}}, \quad C_0 = \frac{k}{2} - \frac{3k^3}{4} \log \left( \frac{4\hat{\lambda}}{e} \right), \quad 0 < \hat{\lambda} < \frac{2l}{k}.
\]
(13)

**Lemma 3.1.** Assume that (A1) holds. Let \(u_0, u_1 \in H_0^1(\Omega)\), such that \(\| u_0 \|_2 < \gamma_*\) and \(0 < E(0) < E_1\). Then, \(I(u(t)) \geq 0\) for all \(t \in [0, T)\).

**Proof.** We follow the computations of the proof of Lemma 3.3 of [41]. First, we prove, under the above assumptions, that \(\| u(\cdot, t) \|_2 < \gamma_*\) \(\forall t \in [0, T)\).

By the definition of \(E(t)\) and using Lemma 2.1 and assumption (A1), we have
\[
E(t) \geq J(u(t)) \geq \frac{1}{2} \left( \xi_1 - \int_0^t g(s) ds \right) \| \nabla u \|_2^2 + \frac{1}{2} (g \circ \nabla u)(t) - \frac{k}{2} \int_\Omega u^2 \ln |u| dx + \frac{k}{4} \| u \|_2^2 \\
\geq \frac{1}{2} \left( l - \frac{ka}{4} \right) \| \nabla u \|_2^2 + \frac{1}{2} (g \circ \nabla u)(t) + \frac{k}{2} \left( \frac{3k^3}{4} + \log \left( \frac{4\hat{\lambda}}{e} \right) \right) \| u \|_2^2.
\]

Then letting \(a = \hat{\lambda}\) and using the fact that \(0 < \hat{\lambda} < \frac{2l}{k}\), we obtain, from the last inequality,
\[
E(t) \geq \frac{1}{2} C_0 \gamma^2 - \frac{k}{2} (\log \gamma) \gamma^2 = Q(\gamma),
\]
(14)
where \(\gamma = \| u(\cdot, t) \|_2\) and \(C_0\) is defined in (13). It is not difficult to check that \(Q\) is increasing on \((0, \gamma_*)\), decreasing on \((\gamma_*, +\infty)\) and \(Q(\gamma) \to -\infty\) as \(\gamma \to +\infty\).
Moreover,
\[
\max_{0 \leq \gamma < +\infty} Q(\gamma) = \frac{1}{2} C_0 \gamma_*^2 - \frac{k}{2} (\log \gamma_*) \gamma_*^2 = Q(\gamma_*) = E_1,
\]
where \( \gamma_* \) is defined in (13). Assume \( \|u(., t)\|_2 < \gamma_* \) is not true in \([0, T)\). Therefore, using the continuity of \( u(t) \), it follows that there exists \( 0 < t_0 < T \) such that \( \|u(x, t_0)\|_2 = \gamma_* \). From (14) we can see that \( E(t_0) \geq Q(\|u(., t_0)\|_2) = Q(\gamma_*) = E_1 \). But this is impossible because \( E(t) \leq E(0) < E_1 \) for all \( t \geq 0 \).

Now by the definition of \( I(u(t)) \), and using Lemma 2.1 with \( a = \hat{a} \), for all \( t \in (0, T) \) we have
\[
I(u(t)) \geq l\|\nabla u\|_2^2 - k \int_\Omega u^2 \log |u|dx \\
\geq \left( t - \frac{k\hat{a}}{4} \right) \|\nabla u\|_2^2 + \left( k - \frac{3k}{4} \log \left( \frac{4\hat{a}}{e} \right) - k \log \|u\|_2 \right) \|u\|_2^2 \\
\geq \left( t - \frac{k\hat{a}}{4} \right) \|\nabla u\|_2^2 + \frac{k}{2} \|u\|_2^2 \geq 0.
\]
This completes the proof. □

Remark 4. We can see that if \( \|u_0\|_2 < \gamma_* \) and \( E(0) < E_1 \), then \( J(u(t)) \geq 0 \) and consequently \( E(t) \geq 0 \) for all \( t \in [0, T) \). Therefore, from (15), for \( t \in [0, T) \) we have
\[
\|u_t(t)\|_{\rho+2}^2 \leq (\rho + 2)E(t) \leq (\rho + 2)E(0), \\
\|u_t(t)\|_2 \leq C\|u_t(t)\|_{\rho+2} \leq C((\rho + 2)E(0))^{\frac{\rho+2}{2}}, \\
\|\nabla u(t)\|_2^2 \leq \frac{4}{4l - \hat{a}k} I(t) \leq \frac{8}{4l - \hat{a}k} E(t) \leq \frac{8}{4l - \hat{a}k} E(0),
\]
which show that the solution is global and bounded in time.

4. Technical lemmas. In this section, we establish several lemmas needed for the proof of our main result.

Lemma 4.1. [32] Assume that \( g \) satisfies (A1). Then, for \( u \in H_0^1(\Omega) \), we have
\[
\int_\Omega \left( \int_0^t g(t-s)(u(t)-u(s))ds \right)^2 dx \leq c(g \circ \nabla u)(t)
\]
and
\[
\int_\Omega \left( \int_0^t g'(t-s)(u(t)-u(s))ds \right)^2 dx \leq -c(g' \circ \nabla u)(t),
\]
where, from now on, \( c \) denotes a generic positive constant.

Lemma 4.2. The functional
\[
\phi(t) = \frac{1}{\rho + 1} \int_\Omega |u_t|^\rho u_t dx + \frac{\sigma}{4} \|\nabla u\|^4 + \int_\Omega \nabla u_t \nabla u dx
\]
satisfies, along the solution of (1) and for any \( t \geq 0 \),
\[
\phi'(t) \leq -\frac{l}{4} \|\nabla u\|_2^2 + \frac{1}{\rho + 1} \int_\Omega |u_t|^{\rho+2} dx - \xi_2 \|\nabla u\|_2^2 + c(g \circ \nabla u)(t) \\
+ c \int_\Omega h^2(u_t) dx + k \int_\Omega u^2 \log |u| dx + \|\nabla u_t\|_2^2.
\]
Proof. Direct differentiation of \( \phi \), using (1), yields
\[
\phi'(t) = \frac{1}{\rho + 1} \int_{\Omega} |u_t|^{\rho + 2} dx + \int_{\Omega} |u_t|^\rho u_t u dx + \sigma \|\nabla u\|^2 (\nabla u, \nabla u_t) + \|\nabla u_t\|^2
+ \int_{\Omega} \nabla u \nabla u_t dx
= \frac{1}{\rho + 1} \int_{\Omega} |u_t|^{\rho + 2} dx - (\xi_1 + \xi_2 \|\nabla u\|^2) \|\nabla u\|^2
+ \int_{\Omega} \nabla u(t) \int_0^t g(t-s) \nabla u(s) ds dx - \int_{\Omega} u h(u_t) dx
+ k \int_{\Omega} u^2 \log |u| dx + \|\nabla u_t\|^2
\]
By exploiting Young’s inequality and Hölder’s inequality, we obtain
\[
\int_{\Omega} \nabla u(t) \int_0^t g(t-s) \nabla u(s) ds dx
= \int_{\Omega} \nabla u(t) \int_0^t g(t-s)(\nabla u(s) - \nabla u(t)) ds dx
+ \int_{\Omega} \nabla u(t) \int_0^t g(t-s) \nabla u(t) ds dx
\leq \frac{l}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{1}{2l} \int_{\Omega} \left( \int_0^t g(t-s)(\nabla u(s) - \nabla u(t)) ds \right)^2 dx
+ (\xi_1 - l) \int_{\Omega} |\nabla u|^2 dx
\leq \frac{l}{2} ||\nabla u||^2_2 + \frac{c}{2l} (g \circ \nabla u)(t) + (\xi_1 - l) \|\nabla u\|_2^2
\]
and
\[
- \int_{\Omega} u h(u_t) dx \leq \frac{l}{4} ||\nabla u||^2_2 + \frac{c}{l} \int_{\Omega} h^2(u_t) dx.
\]
Inserting (19)-(20) into (18), we get the desired inequality (17). \( \square \)

**Lemma 4.3.** The functional
\[
\chi(t) = \int_{\Omega} \left( \Delta u_t - \frac{1}{\rho + 1} |u_t|^\rho u_t \right) \int_0^t g(t-s)(u(t) - u(s)) ds dx
\]
satisfies, along the solution of (1) and for any \( t \geq 0 \),
\[
\chi'(t) \leq \delta (1 + 2(\xi_1 - l)^2) \|\nabla u\|^2 + \left( \delta - \frac{1}{\rho + 1} \int_0^t g(s) ds \right) \int_{\Omega} |u_t|^{\rho + 2} dx
+ \left( \frac{3c}{\delta} + (2\delta + \frac{1}{4\delta})(\xi_1 - l) \right) (g \circ \nabla u)(t) + \sigma^2 \frac{2\delta}{l} E(0) \left( \frac{1}{d dt} \|\nabla u\|^2 \right)^2
+ \left( \delta - \int_0^t g(s) ds \right) \|\nabla u_t\|_2^2 + c_{c_0, \delta}(g \circ \nabla u)^{-\delta} (t) + \delta \int_{\Omega} h^2(u_t) dx
+ \left( \frac{g(0)}{4\delta} + c(\delta) (g(0))^{\rho+1} \left( \frac{8}{l} E(0) \right)^{\frac{2}{2}} \right) (-g' \circ \nabla u)(t).
\]
Proof. By using (1), we have
\[
\chi'(t) = \int_{\Omega} (\Delta u_t - |u_t|^\rho u_t) \int_0^t g(t-s) (u(t) - u(s)) ds dx \\
+ \int_0^t g(s) ds \int_{\Omega} (\Delta u_t - |u_t|^\rho u_t) u_t dx \\
+ \int_0^t (\Delta u_t - |u_t|^\rho u_t) \int_0^t g'(t-s) (u(t) - u(s)) ds dx
\]
\[
= \int_\Omega (\xi_1 + \xi_2 \|\nabla u\|^2 + \sigma (\nabla u, \nabla u_t)) \nabla u(t) \int_0^t g(t-s)(\nabla u(t) - \nabla u(s)) ds dx \\
- \int_\Omega \int_0^t g(t-s) \nabla u(s) ds \int_0^t g(t-s)(\nabla u(t) - \nabla u(s)) ds dx \\
+ \int_\Omega h(u_t) \int_0^t g(t-s)(u(t) - u(s)) ds dx \\
- k \int_\Omega u \ln |u| \int_0^t (g(t-s)(u(t) - u(s))) ds dx \\
- \frac{1}{\rho + 1} \int_\Omega |u_t|^\rho u_t \int_0^t g'(t-s)(u(t) - u(s)) ds dx \\
- \frac{1}{\rho + 1} \int_\Omega |u_t|^\rho+2 dx \left( \int_0^t g(s) ds \right) - \int_0^t g(s) ds \|\nabla u_t\|^2_2 \\
- \int_\Omega \nabla u_t \int_0^t g'(t-s)(\nabla u(t) - \nabla u(s)) ds dx.
\] (22)

Nothing that \( E(t) \) is non-increasing, then
\[
\left( \xi_1 - \int_0^{+\infty} g(s) ds \right) \|\nabla u\|^2 \leq \int_\Omega \left( \xi_1 - \int_0^t g(s) ds \right) |\nabla u|^2 dx \\
\leq 2E(t),
\] (23)

which gives us
\[
\|\nabla u\|^2 \leq \frac{2}{l} E(0).
\] (24)

By using Young’s inequality, Hölder’s inequality and (24), we obtain, for any \( \delta > 0 \),
\[
\int_\Omega (\xi_1 + \xi_2 \|\nabla u\|^2) \nabla u \int_0^t g(t-s) (\nabla u(t) - \nabla u(s)) ds dx \\
\leq \frac{\delta}{2} \|\nabla u\|^2_2 + \frac{C}{\delta} (g \circ \nabla u)(t),
\] (25)
\[
\int_\Omega \sigma (\nabla u, \nabla u_t) \nabla u(t) \int_0^t g(t-s)(\nabla u(t) - \nabla u(s)) ds dx \\
\leq \sigma^2 \|\nabla u, \nabla u_t\|^2 \frac{\delta}{2} \|\nabla u\|^2_2 + \frac{C}{\delta} (g \circ \nabla u)(t) \\
\leq \frac{2\delta \sigma^2}{l} E(0) \left( \frac{1}{2} \frac{d}{dt} \|\nabla u\|^2_2 \right)^2 + \frac{C}{\delta} (g \circ \nabla u)(t),
\] (26)
\[
\int_{\Omega} \int_{0}^{t} g(t-s) \nabla u(s) ds \int_{0}^{t} g(t-s) (\nabla u(t) - \nabla u(s)) ds dx \\
\leq \delta \int_{\Omega} \left( \int_{0}^{t} g(t-s) (|\nabla u(t) - \nabla u(s)| + |\nabla u(t)|) ds \right)^2 dx \\
+ \frac{1}{4\delta} \int_{\Omega} \left( \int_{0}^{t} g(t-s) (|\nabla u(t) - \nabla u(s)|) ds \right)^2 dx \\
\leq (2\delta + \frac{1}{4\delta}) \int_{\Omega} \left( \int_{0}^{t} g(t-s) (|\nabla u(t) - \nabla u(s)|) ds \right)^2 dx \\
+ 2\delta (\xi_1 - l)^2 \|\nabla u\|^2 \\
\leq (2\delta + \frac{1}{4\delta})(\xi_1 - l)(g \circ \nabla u)(t) + 2\delta (\xi_1 - l)^2 \|\nabla u\|^2. \quad (27)
\]

The third term of the right-hand side of (22) can be estimated as follows:

\[
\int_{\Omega} h(u_t) \int_{0}^{t} g(t-s) (u(t) - u(s)) ds dx \\
\leq \delta \int_{\Omega} h^2(u_t) dx + \frac{c}{\delta}(g \circ \nabla u)(t). \quad (28)
\]

By using Lemma 2.2, we get for any \( \delta_1 > 0 \) and \( \epsilon_0 \in (0, 1) \)

\[
-k \int_{\Omega} u \ln |u| \int_{0}^{t} g(t-s) (u(t) - u(s)) ds dx \\
\leq k \int_{\Omega} \left( u^2 + d_{\epsilon_0} |u|^{1-\epsilon_0} \right) \left| \int_{0}^{t} g(t-s) (u(t) - u(s)) ds \right| dx \\
\leq c \int_{\Omega} |u|^2 \left| \int_{0}^{t} g(t-s) (u(t) - u(s)) ds \right| dx + \delta_1 \int_{\Omega} |u|^2 dx \\
+ c_{\epsilon_0, \delta_1} \int_{\Omega} \left| \int_{0}^{t} g(t-s) (u(t) - u(s)) ds \right|^{\frac{2}{1+\epsilon_0}} dx \\
\leq c\delta_1 \|\nabla u\|_2 + \frac{c}{\delta_1} \int_{\Omega} \left| \int_{0}^{t} g(t-s) (u(t) - u(s)) ds \right|^2 dx \\
+ c_{\epsilon_0, \delta_1} \int_{\Omega} \left| \int_{0}^{t} g(t-s) (u(t) - u(s)) ds \right|^{\frac{2}{1+\epsilon_0}} dx, \quad (29)
\]

then, putting \( \frac{\delta}{4} = c\delta_1 \) and using Hölder’s inequality and Young’s inequality, we find

\[
-k \int_{\Omega} u \ln |u| \int_{0}^{t} g(t-s) (u(t) - u(s)) ds dx \\
\leq \frac{\delta}{4} \|\nabla u\|_2^2 + \frac{c}{\delta_1}(g \circ \nabla u)(t) + c_{\epsilon_0, \delta_1}(g \circ \nabla u)^{\frac{1}{1+\epsilon_0}}. \quad (30)
\]
For the fifth term of the right-hand side of (22), one has
\[
\frac{1}{\rho+1} \int_\Omega |u|^\rho u_t \int_0^t g'(t-s)(u(t) - u(s))dsdx
\]
\[
\leq c(\delta) \int_\Omega \left(\int_0^t g'(t-s)u(t-s)ds\right)^{\rho+2} dx + \delta \int_\Omega |u|^{\rho+2} dx
\]
\[
\leq c(\delta) (g(0))^{\rho+1} \int_0^t -g'(t-s)u(t-s) dsdx + \delta \int_\Omega |u|^{\rho+2} dx
\]
\[
\leq c(\delta) (g(0))^{\rho+1} c \int_0^t -g'(t-s) \left( \int_\Omega |\nabla u(t) - \nabla u(s)|^2 \right)^{\rho+2} dx ds + \delta \int_\Omega |u|^{\rho+2} dx,
\]
where we have used the fact that
\[
\int_\Omega |\nabla u(t) - \nabla u(s)|^2 dx \leq 2 \int_\Omega |\nabla u(t)|^2 dx + 2 \int_\Omega |\nabla u(s)|^2 dx
\]
\[
\leq 4 \sup_{s > 0} \|\nabla u(s)\|_2^2 \leq \frac{8}{7} E(0).
\]

The last term of the right-hand side of (22) is handled as follows
\[
\int_\Omega \nabla u_t \int_0^t g'(t-s)(\nabla u(t) - \nabla u(s))dsdx
\]
\[
\leq \delta \|\nabla u_t\|_2^2 + \frac{g(0)}{4\delta} (-g' \circ \nabla u)(t).
\]
Combining all the previous estimates, we obtain the desired inequality (21). \qed

Now, we define the functional
\[
L(t) = NE(t) + \phi(t) + N_1 \chi(t).
\]
We have
\[
|L(t) - NE(t)| \leq |\phi(t)| + N_1 |\chi(t)|
\]
\[
\leq \frac{1}{\rho+2} \int_\Omega |u|^{\rho+2} dx + \frac{1}{(\rho + 1)(\rho + 2)} \int_\Omega |u|^{\rho+2} dx
\]
\[
+ \frac{\sigma}{4} \|\nabla u\|_2^4 + \frac{1}{2} \|\nabla u_t\|_2^2 + \frac{1}{2} \|\nabla u\|_2^2
\]
\[
+ N_1 \left| \int_\Omega \Delta u_t \int_0^t g(t-s)(u(t) - u(s))dsdx \right|
\]
\[
+ N_1 \frac{1}{\rho+1} \int_\Omega |u|^{\rho+1} \int_0^t g(t-s)|u(t) - u(s)|dsdx.
\]
Similarly to (31), we infer that, for any \(\delta > 0\),
\[
\int_\Omega |u|^{\rho+1} \int_0^t g(t-s)|u(t) - u(s)|dsdx \leq \delta (\epsilon_1 - l)^{\rho+1} c \left( \frac{8}{7} E(0) \right)^{\frac{\sigma}{4}} (g \circ \nabla u)(t)
\]
\[
+ c(\delta) \int_\Omega |u|^{\rho+2} dx.
\]
Besides, we have
\[
\int_{\Omega} |u|^{\rho+2} \, dx \leq e^{\rho+2} \left( \int_{\Omega} |\nabla u|^2 \right)^{\frac{\rho+2}{2}} \\
\leq e^{\rho+2} \left( \frac{2}{l} E(0) \right)^{\frac{\rho+2}{2}} \frac{2}{l} E(t). \tag{36}
\]

Then, we get
\[
|L(t) - N E(t)| \leq \frac{1}{\rho + 2} \int_{\Omega} |u_t|^{\rho+2} \, dx + e^{\rho+2} \left( \frac{2}{l} E(0) \right)^{\frac{\rho+2}{2}} \frac{2}{l} E(t) \\
+ \frac{\sigma}{4} ||\nabla u||^4 + \left( \frac{N_1 + 1}{2} \right) ||\nabla u_t||^2 + \left( \frac{1}{2} \right) ||\nabla u||^2 \\
+ \left( \frac{N_1}{2} (\xi_1 - l) \right) + \frac{N_1 \delta}{\rho + 1} (\xi_1 - l)^{\rho+1} c \left( \frac{8}{l} E(0) \right)^{\frac{\rho+1}{2}} (g \circ \nabla u)(t) \\
+ \frac{N_1}{\rho + 1} c(\delta) \int_{\Omega} |u_t|^{\rho+2} \, dx \\
\leq E(t) + e^{\rho+2} \left( \frac{2}{l} E(0) \right)^{\frac{\rho+2}{2}} \frac{2}{l} E(t) \\
+ \frac{\sigma}{\xi_2} E(t) + (N_1 + 1) E(t) + \frac{1}{l} E(t) \\
+ \left( N_1 (\xi_1 - l) \right) + \frac{2N_1 \delta}{\rho + 1} (\xi_1 - l)^{\rho+1} c \left( \frac{8}{l} E(0) \right)^{\frac{\rho+1}{2}} E(t) \\
+ N_1 \frac{\rho + 2}{\rho + 1} c(\delta) E(t) \\
\leq c(1 + N_1) E(t), \tag{37}
\]

which yields
\[
(N - c(1 + N_1)) E(t) \leq L(t) \leq (N + c(1 + N_1)) E(t).
\]

By taking $N$ large enough so that $N > c(1 + N_1)$, we obtain $E \sim L$.

**Lemma 4.4.** For any $t_0 > 0$, there exists a constant $\lambda > 0$ such that, for any $t \geq t_0$,
\[
L'(t) \leq -\lambda E(t) + c(g \circ \nabla u)(t) + c_a(g \circ \nabla u)^{\frac{1}{2-a}} + c \int_{\Omega} h^2(u_t) \, dx. \tag{38}
\]

**Proof.** Since $g$ is a positive and $g(0) > 0$ then, for any $t_0 > 0$, we have
\[
\int_{0}^{t} g(s) \, ds \geq \int_{0}^{t_0} g(s) \, ds = g_0 > 0, \quad \forall t \geq t_0.
\]
By using (12), (17) and (21), we obtain for $t \geq t_0$,
\[ L'(t) \leq \left( \frac{N}{2} - \frac{N_1g(0)}{4\delta} - N_1c(\delta)(g(0))^{\rho+1}c \left( \frac{8}{7}E(0) \right)^{\frac{2}{2}} \right) (g' \circ \nabla u)(t) \]

\[ - \frac{N_1(g_0 - \delta)}{\rho + 1} - \lambda \int_{\Omega} |u_t|^{\rho+2} dx \]

\[ - \left( \frac{1}{4} - N_1\delta (1 + 2(\xi_1 - l)^2) - \frac{\lambda(\xi_1 - g_0)}{2} \right) \| \nabla u \|_2^2 \]

\[ + \left( c + N_1 \left( \frac{3c}{\delta} + (2\delta + \frac{1}{4\delta})(\xi_1 - l) \right) + \frac{\lambda}{2} \right) (g \circ \nabla u)(t) \]

\[ - \left( N\sigma - 2N_1\sigma^2\delta E(0) \right) \left( \frac{1}{2} \frac{d}{dt} \| \nabla u \|_2^2 \right)^2 \]

\[ + (c + N_1\sigma) \int_{\Omega} h^2(u_t) dx - \lambda \| \nabla u \|_2^2 + \lambda \| u \|_2^2 \]

\[ + \frac{\lambda k}{4} \| u \|_2^2 + N_1c_{0,\delta}(g \circ \nabla u)^\frac{1}{1-\alpha} (t) - (N_1(g_0 - \delta) - 1 - \lambda) \| \nabla u_t \|_2^2. \]

Using the definition of \( E(t) \), we obtain for any \( \lambda > 0 \),

\[ L'(t) \leq -\lambda E(t) + \left( \frac{N}{2} - \frac{N_1g(0)}{4\delta} - N_1c(\delta)(g(0))^{\rho+1}c \left( \frac{8}{7}E(0) \right)^{\frac{2}{2}} \right) (g' \circ \nabla u)(t) \]

\[ - \left( \frac{N_1(g_0 - \delta)}{\rho + 1} - \lambda \right) \int_{\Omega} |u_t|^{\rho+2} dx \]

\[ - \left( \frac{1}{4} - N_1\delta (1 + 2(\xi_1 - l)^2) - \frac{\lambda(\xi_1 - g_0)}{2} \right) \| \nabla u \|_2^2 \]

\[ + \left( c + N_1 \left( \frac{3c}{\delta} + (2\delta + \frac{1}{4\delta})(\xi_1 - l) \right) + \frac{\lambda}{2} \right) (g \circ \nabla u)(t) \]

\[ - \left( N\sigma - 2N_1\sigma^2\delta E(0) \right) \left( \frac{1}{2} \frac{d}{dt} \| \nabla u \|_2^2 \right)^2 \]

\[ + (c + N_1\sigma) \int_{\Omega} h^2(u_t) dx \]

\[ + (1 - \frac{\lambda}{2}) k \left( \ln \| u \|_2 + \frac{3}{4} \ln \left( \frac{4a}{\xi} \right) \right) \| u \|_2^2 \]

\[ + \frac{\lambda k}{4} \| u \|_2^2 + N_1c_{0,\delta}(g \circ \nabla u)^\frac{1}{1-\alpha} (t) - (N_1(g_0 - \delta) - 1 - \lambda) \| \nabla u_t \|_2^2. \]
At this point, we choose our constant carefully. First, we choose \( N_1 \) large enough so that
\[
N_1 \frac{g_0}{2} > 1,
\]
then \( 0 < \delta < \frac{g_0}{2} \) so small that
\[
\frac{l}{4} - N_1 \delta \left( 1 + 2(\xi_1 - l)^2 \right) > 0.
\]
Consequently, we get
\[
N_1(g_0 - \delta) - 1 > 0.
\]
After, we pick \( N \) sufficiently large so that \( E \sim L \),
\[
N > c(1 + N_1),
\]
\[
\frac{N}{\tau} - N_1 c(\delta) (g(0))^{\rho+1} c \left( \frac{8}{7} E(0) \right)^2 > 0
\]
and
\[
N\sigma - 2N_1 \sigma^2 \delta E(0) > 0
\]
Finally, we choose \( \lambda \) and \( k \) small enough so that \( \lambda \leq 1 \),
\[
\frac{N_1(g_0 - \delta) - 1 - \frac{\lambda}{\rho + 2}}{\rho + 1} > 0,
\]
\[
N_1(g_0 - \delta) - 1 - \frac{\lambda}{2} > 0,
\]
and
\[
\frac{l}{4} - N_1 \delta \left( 1 + 2(\xi_1 - l)^2 \right) - \frac{\lambda(\xi_1 - g_0)}{2} - \left( 1 - \frac{\lambda}{2}, \frac{ka}{4} \right) > 0.
\]
Hence, we arrive at
\[
L'(t) \leq -\lambda E(t) + c(g \circ \nabla u)(t) + c_v(g \circ \nabla u) \frac{h^2}{\Omega}(t)
+ c \int_\Omega h^2(u_t) dx + \left( 1 - \frac{\lambda}{2} \right) k \left( 1 + \ln \|u\|_2 + \frac{3}{4} \ln \left( \frac{4a}{e} \right) \right) \|u\|_2.
\]
(40)
Using Lemma 3.1, we have
\[
\ln \|u\|_2 < \ln(\gamma^*),
\]
so by taking \( a \) satisfying
\[
0 < a < e^{-\frac{4}{3}} \hat{a},
\]
we guarantee
\[
1 + \ln \|u\|_2 + \frac{3}{4} \ln \left( \frac{4a}{e} \right) < 0
\]
which completes the proof. \( \Box \)
5. Stability. In this section we state and prove the main result of our work. For this purpose, we have the following lemmas and remarks.

**Lemma 5.1.** [38] Under the assumptions (A1) and (A2), the solution of (1)-(3) satisfies the estimates

\[
\int_{\Omega} h^2(u_t) dx \leq c \int_{\Omega} u_t h(u_t) dx, \quad \text{if } h_0 \text{ is linear} \tag{41}
\]

\[
\int_{\Omega} h^2(u_t) d\Gamma \leq cH^{-1}(\chi_0(t)) - cE'(t), \quad \text{if } h_0 \text{ is nonlinear} \tag{42}
\]

where

\[
\chi_0(t) := \frac{1}{|\Omega_2|} \int_{\Omega_2} u_t(t) h(u_t(t)) dx \leq -cE'(t) \tag{43}
\]

and

\[
\Omega_2 = \{ x \in \Omega : |u_t(t)| \leq \varepsilon_1 \}, \quad \text{where } \varepsilon_1 = \min\{r_2, h_0(r_2)\}.
\]

Let’s define

\[
I(t) := (-g' o \nabla u)(t) \leq -cE'(t). \tag{44}
\]

**Lemma 5.2.** Under the assumptions (A1) and (A2), we have the following estimate

\[
(g o \nabla u)(t) \leq \frac{t}{q} \overline{G}^{-1} \left( \frac{qI(t)}{t\xi(t)} \right), \tag{45}
\]

where \( q \in (0,1) \) and \( \overline{G} \) is an extension of \( G \) such that \( \overline{G} \) is strictly increasing and strictly convex \( C^2 \) function on \((0, \infty)\); see remark 3.

**Proof.** For the proof of (45), we define the following quantity

\[
\lambda(t) := \frac{q}{t} \int_{0}^{t} \int_{\Omega} |\nabla u(t) - \nabla u(t-s)|^2 dx ds, \quad t > 0.
\]

By using (11) and (12), we easily see that

\[
\lambda(t) \leq \frac{8qE(0)}{t},
\]

then choosing \( q \in (0,1) \) small enough so that, for all \( t > 0 \),

\[
\lambda(t) < 1. \tag{46}
\]

Since \( G \) is strictly convex on \((0, r_1)\) and \( G(0) = 0 \), then

\[
G(\theta z) \leq \theta G(z), \quad 0 \leq \theta \leq 1 \quad \text{and} \quad z \in (0, r_1]. \tag{47}
\]

The use of (8), (46), (47) and Jensen’s inequality leads to

\[
I(t) = \frac{1}{q \lambda(t)} \int_{0}^{t} \lambda(t) (-g'(s)) \int_{\Omega} q|\nabla u(t) - \nabla u(t-s)|^2 dx ds
\]

\[
\geq \frac{1}{q \lambda(t)} \int_{0}^{s} \lambda(t) \xi(t) G(\xi(t)) \int_{\Omega} q|\nabla u(t) - \nabla u(t-s)|^2 dx ds
\]

\[
\geq \frac{\xi(t)}{q \lambda(t)} \int_{0}^{s} G(\lambda(t) g(s)) \int_{\Omega} q|\nabla u(t) - \nabla u(t-s)|^2 dx ds
\]

\[
\geq \frac{t \xi(t)}{q} G \left( \frac{q}{t} \int_{0}^{s} g(s) \int_{\Omega} |\nabla u(t) - \nabla u(t-s)|^2 dx ds \right)
\]

\[
= \frac{t \xi(t)}{q} G \left( \frac{q}{t} \int_{0}^{s} g(s) \int_{\Omega} |\nabla u(t) - \nabla u(t-s)|^2 dx ds \right).
\]
This gives (45).

**Remark 5.** Using the fact that \((g \circ \nabla u)(t) \leq E(t) \leq E(0)\), we obtain
\[
(g \circ \nabla u)(t) = (g \circ \nabla u) \frac{\xi}{1+\epsilon_0} (t)(g \circ \nabla u) \frac{1}{1+\epsilon_0} (t)
\leq c(g \circ \nabla u) \frac{1}{1+\epsilon_0} (t)
\] (49)

**Remark 6.** In the case of \(G\) is linear and since \(\xi\) is nonincreasing, we have
\[
\xi(t)(g \circ \nabla u) \frac{1}{1+\epsilon_0} (t) = (\xi^\epsilon_0(t)\xi(t)(g \circ \nabla u)(t)) \frac{1}{1+\epsilon_0}
\leq (\xi^\epsilon_0(0)\xi(t)(g \circ \nabla u)(t)) \frac{1}{1+\epsilon_0}
\leq c(\xi(t)(g \circ \nabla u)(t)) \frac{1}{1+\epsilon_0}
\leq c(-E'(t)) \frac{1}{1+\epsilon_0}.\]

**Theorem 5.3.** Let \((u_0, u_1) \in H^1_0(\Omega) \times H^1(\Omega)\) be given. Assume that (A1) and (A2) are satisfied and \(b_0\) is linear. Then there exist strictly positive constants \(c, k_2\) and \(\epsilon_1\) such that the solution of (1)-(3) satisfies, for all \(t > t_1 = \max\{t_0, 1\},
\[
E(t) \leq c\left(1 + \int_{t_0}^{t} \xi^{1+\epsilon_0}(s)ds\right) \frac{1}{1+\epsilon_0}, \quad \text{if } G\text{ is linear.}
\]
\[
E(t) \leq ct \frac{1}{1+\epsilon_0} K_2^{-1} \left( \frac{k_2}{t \int_{t_1}^{t} \xi(s)ds} \right), \quad \text{if } G\text{ is nonlinear}
\]
where \(K_2(s) = sK'(\epsilon_1 s)\) and \(K = \left(\left[\overline{G}\right]^{-1} \frac{1}{1+\epsilon_0}\right)^{-1}.

**Proof. Case 1:** \(G\) is linear

We multiply (38) by \(\xi(t)\) and use (8), (12), (41), (5), (45) and (49) to get
\[
\xi(t)L'(t) \leq -m\xi(t)E(t) + c(-E'(t)) \frac{1}{1+\epsilon_0} - cE'(t), \quad \forall t \geq t_0.
\] (53)

Multiply (53) by \(\xi^\epsilon_0(t)E^\epsilon_0(t)\), and recall that \(\xi' \leq 0\), to obtain
\[
\xi^{\epsilon_0+1}(t)E^{\epsilon_0}(t)L'(t) \leq -m\xi^{\epsilon_0+1}(t)E^{\epsilon_0+1}(t) + c(\xi E)^{\epsilon_0}(t)(-E'(t)) \frac{1}{1+\epsilon_0} - cE'(t), \quad \forall t \geq t_0.
\]

Use of Young’s inequality, with \(q = \epsilon_0 + 1\) and \(q^* = \frac{\epsilon_0+1}{\epsilon_0}\), gives, for any \(\epsilon' > 0\),
\[
\xi^{\epsilon_0+1}(t)E^{\epsilon_0}(t)L'(t) \leq -m\xi^{\epsilon_0+1}(t)E^{\epsilon_0+1}(t) + c(\epsilon'\xi E^{\epsilon_0+1}(t)E^{\epsilon_0+1} - c\epsilon' E'(t))
= -(m - \epsilon'c)\xi^{\epsilon_0+1}(t)E^{\epsilon_0+1} - cE'(t), \quad \forall t \geq t_0.
\]

We then choose \(0 < \epsilon' < \frac{m}{c}\) and use that \(\xi' \leq 0\) and \(E' \leq 0\), to get, for \(c_1 = m - \epsilon'c\),
\[
(\xi^{\epsilon_0+1} E^{\epsilon_0} L)'(t) \leq \xi^{\epsilon_0+1}(t)E^{\epsilon_0}(t)L_1'(t) \leq -c_1\xi^{\epsilon_0+1}(t)E^{\epsilon_0+1}(t) - cE'(t), \quad \forall t \geq t_0,
\]
which implies
\[
(\xi^{\epsilon_0+1} E^{\epsilon_0} L + cE)'(t) \leq -c_1\xi^{\epsilon_0+1}(t)E^{\epsilon_0+1}(t), \quad \forall t \geq t_0.
\]

Let \(L_1 = \xi^{\gamma+1} E^\gamma L + cE\). Then \(L_1 \sim E\) (thanks to (37)) and
\[
L_1'(t) \leq -c\xi^{\epsilon_0+1}(t)L_1^{\epsilon_0+1}(t), \quad \forall t \geq t_0.
\]
Integrating over \((t_0, t)\) and using the fact that \(L_1 \sim E\), we obtain (51).

**Case 2:** \(G\) is non-linear.

Using (38), (42), (45) and (49), we obtain, \(\forall t \geq t_0,\)

\[
L'(t) \leq -mE(t) + ct^\frac{1}{1+\epsilon_0} \left[ \left( \frac{qI(t)}{t}\right)^{-1} \left( \frac{qI(t)}{t^{1+\epsilon_0} t_0} \right) \right]^\frac{1}{1+\epsilon_0} - cE'(t). \tag{54}
\]

Combining the strictly increasing property of \(G\) and the fact that \(\frac{t}{t_0} < 1\) whenever \(t > 1\), we obtain

\[
\left(\frac{qI(t)}{t}\right)^{-1} \left( \frac{qI(t)}{t^{1+\epsilon_0} t_0} \right) \leq - \frac{qI(t)}{t^{1+\epsilon_0} t_0} - cE'(t), \quad \forall t > t_1, \tag{55}
\]

where \(t_1 = \max\{t_0, 1\}\).

Let \(F_1(t) = L(t) + cE(t) \sim E\), then (56) takes the form

\[
F'_1(t) \leq -mE(t) + ct^\frac{1}{1+\epsilon_0} \left[ \left( \frac{qI(t)}{t}\right)^{-1} \left( \frac{qI(t)}{t^{1+\epsilon_0} t_0} \right) \right]^\frac{1}{1+\epsilon_0}, \quad \forall t > t_1. \tag{57}
\]

Set

\[
K = \left( \left( \frac{qI(t)}{t}\right)^{-1} \right)^{-1} \bigg[ \left( \frac{qI(t)}{t^{1+\epsilon_0} t_0} \right) \bigg], \quad \chi(t) = \frac{qI(t)}{t^{1+\epsilon_0} t_0} \tag{58}
\]

then,

\[
K' = (1 + \epsilon_0)G' \left( \frac{qI(t)}{t^{1+\epsilon_0} t_0} \right) > 0 \quad \text{on} \ (0, r_1],
\]

\[
K'' = \frac{\epsilon_0}{(G^{-1})^{1+\epsilon_0}} + (1 + \epsilon_0) \left( G^{-1} \right)^{\frac{1+\epsilon_0}{1+\epsilon_0}} G'' > 0 \quad \text{on} \ (0, r_1].
\]

So, (57) reduces to

\[
F'_1(t) \leq -mE(t) + ct^\frac{1}{1+\epsilon_0} K^{-1}(\chi(t)), \quad \forall t > t_1. \tag{59}
\]

Now, for \(\epsilon_1 < r_1\) and using (59) and the fact that \(E' \leq 0, K' > 0, K'' > 0 \) on \(0, r_1\],

we find that the functional \(F_2\), defined by

\[
F_2(t) := K' \left( \frac{\epsilon_1}{t^{1+\epsilon_0}} \cdot \frac{E(t)}{E(0)} \right) F_1(t) + E(t), \quad \forall t > t_1,
\]

satisfies, for some \(\alpha_1, \alpha_2 > 0\),

\[
\alpha_1 F_2(t) \leq E(t) \leq \alpha_2 F_2(t) \tag{60}
\]

and, for all \(t > t_1\),

\[
F'_2(t) \leq -mE(t) K' \left( \frac{\epsilon_1}{t^{1+\epsilon_0}} \cdot \frac{E(t)}{E(0)} \right) + ct^\frac{1}{1+\epsilon_0} K' \left( \frac{\epsilon_1}{t^{1+\epsilon_0}} \cdot \frac{E(t)}{E(0)} \right) K^{-1}(\chi(t)). \tag{61}
\]

Let \(K^*\) be the convex conjugate of \(K\) in the sense of Young, then

\[
K^*(s) = s(K')^{-1}(s) - K \left[ (K')^{-1}(s) \right], \quad \text{if} \ s \in (0, K'(r_1]), \tag{62}
\]

and \(K^*\) satisfies the following generalized Young inequality

\[
AB \leq K^*(A) + K(B), \quad \text{if} \ A \in (0, K'(r_1)), \ B \in (0, r_1]. \tag{63}
\]
Therefore, by setting \( A = K' \left( \frac{\varepsilon_1}{t^{\frac{1}{m+\alpha}}} \cdot \frac{E(t)}{E(0)} \right) \) and \( B = K^{-1}(\chi(t)) \), we arrive at

\[
F_2'(t) \leq -mE(t)K' \left( \frac{\varepsilon_1}{t^{\frac{1}{m+\alpha}}} \cdot \frac{E(t)}{E(0)} \right) + c\varepsilon E \left( \frac{\varepsilon_1}{t^{\frac{1}{m+\alpha}}} \cdot \frac{E(t)}{E(0)} \right) \chi(t)
\]

or

\[
F_2'(t) \leq -mE(t)K' \left( \frac{\varepsilon_1}{t^{\frac{1}{m+\alpha}}} \cdot \frac{E(t)}{E(0)} \right) + c\varepsilon E \left( \frac{\varepsilon_1}{t^{\frac{1}{m+\alpha}}} \cdot \frac{E(t)}{E(0)} \right) \chi(t)
\]

Then, multiplying (64) by \( \xi(t) \) and using (44), (58), we get

\[
\xi(t)F_2'(t) \leq -m\xi(t)E(t)K' \left( \frac{\varepsilon_1}{t^{\frac{1}{m+\alpha}}} \cdot \frac{E(t)}{E(0)} \right) + c\varepsilon \xi(t) \cdot \frac{E(t)}{E(0)} K' \left( \frac{\varepsilon_1}{t^{\frac{1}{m+\alpha}}} \cdot \frac{E(t)}{E(0)} \right)
\]

Using the non-increasing property of \( \xi \), we obtain, for all \( t > t_1 \),

\[
(\xi F_2 + cE)'(t) \leq -m\xi(t)E(t)K' \left( \frac{\varepsilon_1}{t^{\frac{1}{m+\alpha}}} \cdot \frac{E(t)}{E(0)} \right) + c\varepsilon \xi(t) \cdot \frac{E(t)}{E(0)} K' \left( \frac{\varepsilon_1}{t^{\frac{1}{m+\alpha}}} \cdot \frac{E(t)}{E(0)} \right).
\]

Therefore, by setting \( F_3 := \xi F_2 + cE \sim E \), we conclude that

\[
F_3'(t) \leq -m\xi(t)E(t)K' \left( \frac{\varepsilon_1}{t^{\frac{1}{m+\alpha}}} \cdot \frac{E(t)}{E(0)} \right) + c\varepsilon \xi(t) \cdot \frac{E(t)}{E(0)} K' \left( \frac{\varepsilon_1}{t^{\frac{1}{m+\alpha}}} \cdot \frac{E(t)}{E(0)} \right)
\]

This gives, for a suitable choice of \( \varepsilon_1 \),

\[
F_3'(t) \leq -k\xi(t) \left( \frac{E(t)}{E(0)} \right) K' \left( \frac{\varepsilon_1}{t^{\frac{1}{m+\alpha}}} \cdot \frac{E(t)}{E(0)} \right) , \quad \forall t > t_1,
\]

or

\[
k \left( \frac{E(t)}{E(0)} \right) K' \left( \frac{\varepsilon_1}{t^{\frac{1}{m+\alpha}}} \cdot \frac{E(t)}{E(0)} \right) \xi(t) \leq -F_3'(t) , \quad \forall t > t_1. \tag{65}
\]

An integration of (65) yields

\[
\int_{t_1}^{t} k \left( \frac{E(s)}{E(0)} \right) K' \left( \frac{\varepsilon_1}{s^{\frac{1}{m+\alpha}}} \cdot \frac{E(s)}{E(0)} \right) \xi(s) ds \leq - \int_{t_1}^{t} F_3'(s) ds \leq F_3(t_1). \tag{66}
\]

Using the facts that \( K', K'' > 0 \) and the non-increasing property of \( E \), we deduce that the map \( t \mapsto E(t)K' \left( \frac{\varepsilon_1}{t^{\frac{1}{m+\alpha}}} \cdot \frac{E(t)}{E(0)} \right) \) is non-increasing and consequently, we have

\[
k \left( \frac{E(t)}{E(0)} \right) K' \left( \frac{\varepsilon_1}{t^{\frac{1}{m+\alpha}}} \cdot \frac{E(t)}{E(0)} \right) \int_{t_1}^{t} \xi(s) ds \leq - \int_{t_1}^{t} F_3'(s) ds \leq F_3(t_1), \quad \forall t \geq t_1. \tag{67}
\]
Multiplying each side of (67) by \( \frac{1}{t^{\frac{1}{\Gamma(\alpha)}}} \), we have
\[
k \left( \frac{1}{t^{\frac{1}{\Gamma(\alpha)}}} \frac{E(t)}{E(0)} \right) K' \left( \frac{\varepsilon_1}{t^{\frac{1}{\Gamma(\alpha)}}} \frac{E(t)}{E(0)} \right) \int_{t_1}^{t} \xi(s)ds \leq \frac{k_2}{t^{\frac{1}{\Gamma(\alpha)}}}, \quad \forall t \geq t_1. \tag{68}
\]

Next, we set \( K_2(s) = sK'(\varepsilon_1 s) \) which is strictly increasing, and consequently, we obtain,
\[
kK_2 \left( \frac{1}{t^{\frac{1}{\Gamma(\alpha)}}} \cdot \frac{E(t)}{E(0)} \right) \int_{t_1}^{t} \xi(s)ds \leq \frac{k_2}{t^{\frac{1}{\Gamma(\alpha)}}}, \quad \forall t > t_1. \tag{69}
\]

Finally, for two positive constants \( k_2 \) and \( k_3 \), we infer
\[
E(t) \leq k_3 t^{\frac{1}{\Gamma(\alpha)}} K_2^{-1} \left( \frac{k_2}{t^{\frac{1}{\Gamma(\alpha)}}} \int_{t_1}^{t} \xi(s)ds \right), \quad \forall t > t_1. \tag{70}
\]

This finishes the proof. \( \square \)

**Theorem 5.4.** Assume that (A1) and (A2) are satisfied and \( h_0 \) is nonlinear. Then there exist strictly positive constants \( c_3, c_4, k_2, k_3 \) and \( \varepsilon_2 \) such that the solution of \((1)-(3)\) satisfies, for all \( t > t_1 \),
\[
E(t) \leq J_1^{-1} \left( c_3 \int_{t_0}^{t} \xi(s)ds + c_4 \right), \quad \text{if } G \text{ is linear,} \tag{71}
\]

where
\[
J_1(t) = \int_{t}^{1} \frac{1}{J_2(s)} ds, \quad J_2(t) = tJ'(\varepsilon_1 t),
\]
\[
J = (F^{-1} + H^{-1})^{-1}, \quad F(t) = t^{1+k_0},
\]
and
\[
E(t) \leq k_3 t^{\frac{1}{\Gamma(\alpha)}} W_2^{-1} \left( \frac{k_2}{t^{\frac{1}{\Gamma(\alpha)}}} \int_{t_1}^{t} \xi(s)ds \right), \quad \text{if } G \text{ is non-linear,} \tag{72}
\]

where
\[
W_2(t) = tW'(\varepsilon_2 t) \text{ and } W = \left( \left[ (G)^{-1} \right]^{\frac{1}{\Gamma(\alpha)}} + H^{-1} \right)^{-1}.
\]

**Proof.** **Case 1:** \( G \) is linear

Multiplying (38) by \( \xi(t) \) and using (42), (49) and (50), we get, \( \forall t \geq t_0, \)
\[
\xi(t)L'(t) \leq -m\xi(t)E(t) + c(-E'(t))^{\frac{1}{\Gamma(\alpha)}}
\]
\[
+ c\xi(t) \int_{\Gamma_1} h^2(u_t(t))d\Gamma
\]
\[
\leq -m\xi(t)E(t) + c(-E'(t))^{\frac{1}{\Gamma(\alpha)}} + c\xi(t)H^{-1}(\chi_0(t)).
\]

Let \( F(t) = t^{1+k_0} \). Then, the last inequality can be written as
\[
\xi(t)L'(t) \leq -m\xi(t)E(t) + cF^{-1}(-E'(t)) + cH^{-1}(\chi_0(t)), \forall t \geq t_0. \tag{73}
\]
Therefore, (73) becomes
\[ \mathcal{L}'(t) \leq -m\xi(t)E(t) + c\xi(t)J^{-1}(\chi_1(t)), \forall t \geq t_0, \] (74)
where \( \mathcal{L} := \xi L + E \sim E, J = (F^{-1} + H^{-1})^{-1}, \) and \( \chi_1(t) = \max \{-E'(t), \chi_0(t)\}. \)
In fact,
\[ J' = \frac{F'H'}{F'' + H'} > 0 \text{ on } (0, r_2], \]
\[ J'' = \frac{(F')^2 H''}{(H' + F')^2} > 0 \text{ on } (0, r_2]. \]
Now, for \( \varepsilon_2 < r_2 \) and \( c_0 > 0 \), using (74) and the fact that \( E' \leq 0, J' > 0, J'' > 0 \) on \( (0, r_2] \), we find that the functional \( \mathcal{L}_1 \), defined by
\[ \mathcal{L}_1(t) := J' \left( \frac{\varepsilon_2 E(t)}{E(0)} \right) \mathcal{L}(t) + E(t) \]
satisfies, for some \( \alpha_3, \alpha_4 > 0 \)
\[ \alpha_3 \mathcal{L}_1(t) \leq E(t) \leq \alpha_4 \mathcal{L}_1(t) \] (75)
and, for all \( t \geq t_0 \)
\[ \mathcal{L}'_1(t) = \varepsilon_2 \frac{E'(t)}{E(0)} J'' \left( \frac{\varepsilon_2 E(t)}{E(0)} \right) \mathcal{L}(t) + J' \left( \frac{\varepsilon_2 E(t)}{E(0)} \right) \mathcal{L}'(t) + E'(t) \]
\[ \leq -mE(t)J' \left( \frac{\varepsilon_2 E(t)}{E(0)} \right) + c\xi(t)J' \left( \frac{\varepsilon_2 E(t)}{E(0)} \right) J^{-1}(\chi_1(t)) \] (76)
Let \( J^* \) be the convex conjugate of \( J \) in the sense of Young (see [3]), then, as in (62) and (63), with \( A = J' \left( \frac{\varepsilon_2 E(t)}{E(0)} \right) \) and \( B = J^{-1}(\chi_1(t)) \), using (43), we arrive at
\[ \mathcal{L}'_1(t) \leq -mE(t)J' \left( \frac{\varepsilon_2 E(t)}{E(0)} \right) + c\xi(t)J' \left( \frac{\varepsilon_2 E(t)}{E(0)} \right) + c\xi(t)\chi_0(t) \]
\[ \leq -mE(t)J' \left( \frac{\varepsilon_2 E(t)}{E(0)} \right) + c\xi(t)J' \left( \frac{\varepsilon_2 E(t)}{E(0)} \right) \]
Consequently, with a suitable choice of \( \varepsilon_2 \) and \( c_0 \), we obtain, for all \( t \geq t_0 \),
\[ \mathcal{L}'_1(t) \leq -c\xi(t)J' \left( \frac{\varepsilon_2 E(t)}{E(0)} \right) = -c\xi(t)J_2 \left( \frac{\varepsilon_2 E(t)}{E(0)} \right), \] (77)
where \( J_2(t) = tJ'(\varepsilon_2 t) \). Since \( J_2'(t) = J'(\varepsilon_2 t) + \varepsilon_2 tJ''(\varepsilon_2 t) \), then, using the strict convexity of \( J \) on \( (0, r_2] \), we find that \( J_2'(t), J_2(t) > 0 \) on \( (0, 1) \]. Thus, with
\[ R_1(t) = \varepsilon J_2(\varepsilon_2 E(t)), \quad 0 < \varepsilon < 1, \]
and taking in account (75) and (77), we easily see that
\[ R_1(t) \sim E(t) \] (78)
and, for some \( c_3 > 0 \),
\[ R_1'(t) \leq -c_3 \xi(t)J_2(R_1(t)), \quad \forall t \geq t_0. \]
Then, a simple integration gives, for some \( c_4 > 0 \),
\[ R_1(t) \leq J_1^{-1} \left( c_3 \int_{t_0}^{t} \xi(s)ds + c_4 \right), \quad \forall t \geq t_0, \] (79)
where \( J_1(t) = \int_t^1 \frac{1}{x^2(x)} \, ds \).

**Case 2.** \( G \) is non-linear.

Using (38), (42), (45) and (49), we obtain, \( \forall \ t \geq t_0 \),

\[
L'(t) \leq -mE(t) + ct^{\frac{1}{\tau + \sigma_0}} \left[ (\overline{G})^{-1} \left( \frac{qI(t)}{t^{\tau + \sigma_0} \xi(t)} \right) \right]^{\frac{1}{\tau + \sigma_0}} + cH^{-1}(\chi_0(t)) - cE'(t). \tag{80}
\]

Using the strictly increasing and strictly convex properties of \( \overline{H} \) and \( \overline{G} \), setting \( \theta = \left( \frac{1}{2} \right)^{\tau + \sigma_0} < 1 \), \( \forall t > 1 \) and using

\[
\overline{H}(\theta z) \leq \theta \overline{H}(z), \ 0 \leq \theta \leq 1 \quad \text{and} \quad z \in (0, r_2],
\]

we obtain

\[
\overline{H}^{-1}(\chi_0(t)) \leq t^{\frac{1}{\tau + \sigma_0}} \overline{H}^{-1} \left( \frac{\chi_0(t)}{t^{\tau + \sigma_0}} \right), \ \forall t > 1
\]

and

\[
(\overline{G})^{-1} \left( \frac{qI(t)}{t^{\tau + \sigma_0} \xi(t)} \right) \leq (\overline{G})^{-1} \left( \frac{qI(t)}{t^{\tau + \sigma_0} \xi(t)} \right), \ \forall t > 1,
\]

hence (80) becomes

\[
L'(t) \leq -mE(t) + ct^{\frac{1}{\tau + \sigma_0}} \left[ (\overline{G})^{-1} \left( \frac{qI(t)}{t^{\tau + \sigma_0} \xi(t)} \right) \right]^{\frac{1}{\tau + \sigma_0}} + cH^{-1}(\chi_0(t)) - cE'(t), \quad \forall t > 1. \tag{82}
\]

Let \( F(t) = L(t) + cE(t) \sim E \), then (82) takes the form

\[
F'(t) \leq -mE(t) + ct^{\frac{1}{\tau + \sigma_0}} \left[ (\overline{G})^{-1} \left( \frac{qI(t)}{t^{\tau + \sigma_0} \xi(t)} \right) \right]^{\frac{1}{\tau + \sigma_0}} + ct^{\frac{1}{\tau + \sigma_0}} \overline{H}^{-1} \left( \frac{\chi_0(t)}{t^{\tau + \sigma_0}} \right). \tag{83}
\]

Let \( r_0 = \min \{ r_1, r_2 \} \), \( \chi(t) = \max \left\{ \frac{qI(t)}{t^{\tau + \sigma_0} \xi(t)}, \frac{\chi_0(t)}{t^{\tau + \sigma_0}} \right\} \), \( \tag{84} \)

and

\[
W = \left( \left[ (\overline{G})^{-1} \right]^{\frac{1}{\tau + \sigma_0}} + \overline{H}^{-1} \right)^{-1}.
\]

The strictly increasing and strictly convex properties of \( \overline{H} \) and \( \overline{G} \) imply that \( (\overline{H}^{-1})'' < 0 \) and \( (\overline{G}^{-1})'' < 0 \) on \( [0, r_0) \). Hence,

\[
W'' = -\left[ \frac{1}{\tau + \sigma_0} \left[ (\overline{G})^{-1} \right]^{\frac{1}{\tau + \sigma_0}} \left[ (\overline{G})^{-1} \right]'' - \epsilon_0 \left[ (\overline{G})^{-1} \right]^2 \left[ (\overline{G})^{-1} \right]^{\frac{1}{\tau + \sigma_0}} + \left( \overline{H}^{-1} \right)'' \right] > 0.
\]

So, (83) reduces to

\[
F'(t) \leq -mE(t) + ct^{\frac{1}{\tau + \sigma_0}} W^{-1}(\chi(t)), \quad \forall t > t_1. \tag{85}
\]
Now, for \( \epsilon_2 < r_0 \) and using (80) and the fact that \( E' \leq 0, W' > 0, W'' > 0 \) on \((0, r_0)\), we find that the functional \( F_1 \), defined by

\[
F_1(t) := W' \left( \frac{\epsilon_2}{t - t_1} \cdot \frac{E(t)}{E(0)} \right) F(t), \quad \forall t > t_1,
\]

satisfies, for all \( t > t_1 \),

\[
F_1'(t) \leq -mE(t)W' \left( \frac{\epsilon_2}{t - t_1} \cdot \frac{E(t)}{E(0)} \right) + c\frac{1}{t^{\frac{1}{1+c}}} W'' \left( \frac{\epsilon_2}{t - t_1} \cdot \frac{E(t)}{E(0)} \right) W^{-1}(\chi(t)).
\]

Let \( W^* \) be the convex conjugate of \( W \) in the sense of Young [3], then

\[
W^*(s) = s(W^{-1}(s)) - W \left[ (W')^{-1}(s) \right], \quad \text{if } s \in (0, W'(r_0)]
\]

and \( W^* \) satisfies the following generalized Young inequality

\[
AB \leq W^*(A) + W(B), \quad \text{if } A \in (0, W'(r_0)], \ B \in (0, r_0].
\]

Then with \( A = W' \left( \frac{\epsilon_2}{t - t_1} \cdot \frac{E(t)}{E(0)} \right) \) and \( B = W^{-1}(\chi(t)) \), we arrive at

\[
F_1'(t) \leq -mE(t)W' \left( \frac{\epsilon_2}{t - t_1} \cdot \frac{E(t)}{E(0)} \right) + c\frac{1}{t^{\frac{1}{1+c}}} W^* \left( \frac{\epsilon_2}{t - t_1} \cdot \frac{E(t)}{E(0)} \right)
+ c\frac{1}{t^{\frac{1}{1+c}}} \chi(t).
\]

So, multiplying (89) by \( \xi(t) \) and using (43), (44), (84) give

\[
\xi(t)F_1'(t) \leq -m\xi(t)E(t)W' \left( \frac{\epsilon_2}{t - t_1} \cdot \frac{E(t)}{E(0)} \right) + c\epsilon_2 \xi(t) \cdot \frac{E(t)}{E(0)} W' \left( \frac{\epsilon_2}{t - t_1} \cdot \frac{E(t)}{E(0)} \right)
- cE'(t), \forall t > t_1.
\]

Using the non-increasing property of \( \xi \), we obtain, for all \( t > t_1 \),

\[
(\xiF_1 + cE)'(t) \leq -m\xi(t)E(t)W' \left( \frac{\epsilon_2}{t - t_1} \cdot \frac{E(t)}{E(0)} \right)
+ c\epsilon_2 \xi(t) \cdot \frac{E(t)}{E(0)} W' \left( \frac{\epsilon_2}{t - t_1} \cdot \frac{E(t)}{E(0)} \right).
\]

Therefore, by setting \( F_2 := \xiF_1 + cE \), we get, \( \forall t > t_1 \),

\[
F_2'(t) \leq -m\xi(t)E(t)W' \left( \frac{\epsilon_2}{t - t_1} \cdot \frac{E(t)}{E(0)} \right) + c\epsilon_2 \xi(t) \cdot \frac{E(t)}{E(0)} W' \left( \frac{\epsilon_2}{t - t_1} \cdot \frac{E(t)}{E(0)} \right).
\]

This gives, for a suitable choice of \( \epsilon_2 \),

\[
F_2'(t) \leq -k\xi(t) \left( \frac{E(t)}{E(0)} \right) W' \left( \frac{\epsilon_2}{t - t_1} \cdot \frac{E(t)}{E(0)} \right), \quad \forall t > t_1,
\]
or
\[ k \left( \frac{E(t)}{E(0)} \right) W' \left( \frac{\varepsilon_2}{t^{\frac{1}{1+\sigma}}} \cdot \frac{E(t)}{E(0)} \right) \xi(t) \leq -F_2'(t), \quad \forall t > t_1. \] (90)

An integration of (90) yields
\[ \int_{t_1}^{t} k \left( \frac{E(s)}{E(0)} \right) W' \left( \frac{\varepsilon_2}{s^{\frac{1}{1+\sigma}}} \cdot \frac{E(s)}{E(0)} \right) \xi(s) ds \leq - \int_{t_1}^{t} F_2''(s) ds \leq F_2(t_1), \quad \forall t > t_1. \] (91)

Using the facts that $W', W'' > 0$ and the non-increasing property of $E$, we deduce that the map $t \mapsto E(t)W' \left( \frac{\varepsilon_2}{t^{\frac{1}{1+\sigma}}} \cdot \frac{E(t)}{E(0)} \right)$ is non-increasing and consequently, we have
\[ k \left( \frac{E(t)}{E(0)} \right) W' \left( \frac{\varepsilon_2}{t^{\frac{1}{1+\sigma}}} \cdot \frac{E(t)}{E(0)} \right) \int_{t_1}^{t} \xi(s) ds \leq \int_{t_1}^{t} k \left( \frac{E(s)}{E(0)} \right) W' \left( \frac{\varepsilon_2}{s^{\frac{1}{1+\sigma}}} \cdot \frac{E(s)}{E(0)} \right) \xi(s) ds \leq F_2(t_1), \quad \forall t > t_1. \] (92)

Multiplying each side of (92) by $\frac{1}{t^{\frac{1}{1+\sigma}}}$, we have
\[ k \left( \frac{1}{t^{\frac{1}{1+\sigma}}} \cdot \frac{E(t)}{E(0)} \right) W' \left( \frac{\varepsilon_2}{t^{\frac{1}{1+\sigma}}} \cdot \frac{E(t)}{E(0)} \right) \int_{t_1}^{t} \xi(s) ds \leq \frac{k_2}{t^{\frac{1}{1+\sigma}}}, \quad \forall t > t_1. \] (93)

Next, we set $W_2(s) = sW'(\varepsilon_2 s)$ which is strictly increasing, then we obtain,
\[ kW_2 \left( \frac{1}{t^{\frac{1}{1+\sigma}}} \cdot \frac{E(t)}{E(0)} \right) \int_{t_1}^{t} \xi(s) ds \leq \frac{k_2}{t^{\frac{1}{1+\sigma}}}, \quad \forall t > t_1. \] (94)

Finally, for two positive constants $k_2$ and $k_3$, we obtain
\[ E(t) \leq k_3 t^{\frac{1}{1+\sigma}} W_2^{-1} \left( \frac{k_2}{t^{\frac{1}{1+\sigma}} \int_{t_1}^{t} \xi(s) ds} \right). \] (95)

This finishes the proof. \(\square\)

**Examples 5.5.** The following examples illustrate our results:

1. $h_0$ and $G$ are linear
   Let $g(t) = ae^{-b(1+t)}$, where $b > 0$ and $a > 0$ is small enough so that (7) is satisfied, then $g'(t) = -\xi(t)G(g(t))$ where $G(t) = t$ and $\xi(t) = b$. For the frictional nonlinearity, assume that $h_0(t) = ct$. So $H(t) = \sqrt{t}h_0(\sqrt{t}) = ct$. Therefore, we can use (51) to obtain
\[ E(t) \leq \frac{c_1}{(1 + t)^{\frac{1}{\alpha}}} \] (96)

2. $h_0$ is linear and $G$ is non-linear
   Let $g(t) = \frac{a}{(1 + t)^{\frac{1}{\alpha}}}$, where $q > 1 + \epsilon_0$ and $a$ is chosen so that hypothesis (7) remains valid. Then
\[ g'(t) = -bG(g(t)), \quad \text{with} \quad G(s) = s^{\frac{q+1}{q}}, \]
where $b$ is a fixed constant. For the frictional nonlinearity, we assume that $h_0(t) = ct$, and $H(t) = \sqrt{t}h_0(\sqrt{t}) = ct$. 

Since \( K(s) = s^{(\alpha+1)(q+1)} \). Then, (52) gives, \( \forall t \geq t_1 \)

\[
E(t) \leq \frac{c}{t^{(q-1+\epsilon_0)}}
\]

(97)

3. \( h_0 \) is non-linear and \( G \) is linear

Let \( g(t) = ae^{-(b(1+t))} \), where \( b > 0 \) and \( a > 0 \) is small enough so that (7) is satisfied, then \( g'(t) = -\xi(t)G(g(t)) \) where \( G(t) = t \) and \( \xi(t) = b \). Also, assume that \( h_0(t) = ct^2 \), where \( H(t) = \sqrt{t}h_0(\sqrt{t}) = ct^2 \). Then, after taking \( \epsilon_0 = \frac{1}{2} \), we have

\[
F(t) = t^{\frac{3}{2}}
\]

and

\[
J(t) = ct^{\frac{3}{2}}.
\]

Therefore, applying (71), we obtain

\[
E(t) \leq \frac{c}{(1+t)^2}
\]

(98)

4. \( h_0 \) is non-linear and \( G \) is non-linear

Let \( g(t) = \frac{a}{(1+t)^2} \), where \( a \) is chosen so that hypothesis (7) remains valid. Then

\[
g'(t) = -bG(g(t)), \quad \text{with} \quad G(s) = s^{\frac{5}{4}},
\]

where \( b \) is a fixed constant. For the frictional nonlinearity, let \( h_0(t) = ct^2 \) and \( H(t) = ct^2 \). Then, with \( \epsilon_0 = \frac{1}{5} \), we obtain

\[
W = \left( \left( \frac{1}{(G)^{-1}} \right)^{\frac{1}{5+\epsilon_0}} + H^{-1} \right)^{-1} = cs^{\frac{3}{2}}
\]

and

\[
W_2(s) = cs^{\frac{3}{2}}
\]

Therefore, applying (72), we obtain, \( \forall t \geq t_1 \)

\[
E(t) \leq \frac{c}{t^{\frac{3}{2}}}
\]

Remark 7. It is worth mentioning that, the decay rates obtained here are due to the viscoelastic and frictional dampings. However, if \( g = h = 0 \), then the Balakrishnan-Taylor damping stabilizes the system and we obtain \( E(t) \leq E(0) \), for all \( t > 0 \). An open question is whether this latter damping is strong enough to obtain some type of uniform decay.

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