Non-uniqueness for the Euler equations: 
the effect of the boundary

C. Bardos, L. Székelyhidi, Jr., and E. Wiedemann

Abstract. Rotational initial data is considered for the two-dimensional incompressible Euler equations on an annulus. With use of the convex integration framework it is shown that there exist infinitely many admissible weak solutions (that is, with non-increasing energy) for such initial data. As a consequence, on bounded domains there exist admissible weak solutions which are not dissipative in the sense of Lions, as opposed to the case without physical boundaries. Moreover, it is shown that admissible solutions are dissipative if they are Hölder continuous near the boundary of the domain.

Bibliography: 34 titles.

Keywords: Euler equations, non-uniqueness, wild solutions, dissipative solutions, boundary effects, convex integration, inviscid limit, rotational flows.

Contents

1. Introduction 190
2. Statement of the main results 191
   2.1. Formulation of the equations 191
   2.2. Rotationally symmetric data 192
3. Subsolutions and convex integration 193
4. Non-uniqueness for rotational initial data 195
5. Uniqueness of the zero viscosity limit 198
6. Dissipative solutions 200
7. A criterion for admissible solutions to be dissipative 201
Bibliography 205

The research of the second author was supported by ERC Grant Agreement No. 277993. Part of this work was done while the third author was a visitor to the project “Instabilities in Hydrodynamics” of the Fondation Sciences Mathématiques de Paris. He gratefully acknowledges the Fondation’s support.

AMS 2010 Mathematics Subject Classification. Primary 35D30, 35Q35, 76B03.
1. Introduction

The study of weak solutions of the incompressible Euler equations is motivated by (at least) two aspects of fluid flow: the presence of instabilities, most notably the Kelvin–Helmholtz instability, and fully developed 3-dimensional turbulence. Concerning the latter, an important problem arises in connection with the famous $5/3$ law of Obukhov–Kolmogorov and the conjecture of Onsager regarding energy conservation. We refer to [3], [15] and [6], [9], [19] for more information and recent progress regarding this problem.

Concerning the former, it has been the subject of intensive research to define a physically meaningful notion of weak solution that can capture the basic features of such instabilities and be analytically well behaved at the same time. Due to the lack of a theorem analogous to the existence of Leray–Hopf weak solutions of the Navier–Stokes equations, several weaker notions have been considered.

Dissipative solutions of the incompressible Euler equations were introduced by Lions [22] as a concept of solution with two desirable properties: (i) existence for arbitrary initial data, and (ii) weak-strong uniqueness, meaning that a dissipative weak solution agrees with the strong solution as long as the latter exists. Dissipative solutions have been shown to arise, among others, as viscosity [22] or hydrodynamic [25] limits of the incompressible Euler equations. The major drawback of dissipative solutions is that, in general, the velocity field does not solve the Euler equations in the sense of distributions.

Weak solutions (that is, distributional solutions with some additional properties), on the other hand, have been constructed by various techniques (see [10], [11], [26], [28], [29], [31], [32], [34]). Many of these results come with a high level of non-uniqueness, even violating the weak-strong uniqueness property (we refer to the survey [12]). In particular, in [34] the existence of global (in time) weak solutions was shown for arbitrary initial data.

Due to the high level of non-uniqueness, a natural question is whether there are any selection criteria among weak solutions. In this regard, it has been noted in [11] and [13] that in the absence of boundaries a weak solution is dissipative in the sense of Lions, provided that the weak energy inequality

\[
\int |v(x, t)|^2 \, dx \leq \int |v(x, 0)|^2 \, dx \quad \text{for almost every } t > 0 \quad (1)
\]

holds. In [11] this condition is referred to as an admissibility condition, in analogy with the entropy condition used in hyperbolic conservation laws [8]. Admissibility turned out to be a useful selection criterion among weak solutions, since already in the weak form in (1) it implies the weak-strong uniqueness property of dissipative solutions (stronger versions of the energy inequality are discussed in [11]). This is even the case not just for distributional solutions but also for measure-valued solutions (see [5]).

Despite the weak-strong uniqueness property, there exists a large, in fact $L^2$-dense, set of initial data on the whole space or with periodic boundary conditions [32] (see also [31]), for which the initial and boundary value problem admits infinitely many admissible weak solutions. Such initial data, called ‘wild initial data’, necessarily has to be irregular.
The non-uniqueness of admissible weak solutions is intimately related to the presence of instabilities. For instance, in [30] the non-uniqueness of admissible weak solutions was shown for the flat vortex sheet initial data

\[ v_0(x) = \begin{cases} e_1 & \text{if } x_d \in \left(0, \frac{1}{2}\right), \\ -e_1 & \text{if } x_d \in \left(-\frac{1}{2}, 0\right), \end{cases} \tag{2} \]

extended periodically to the torus \( T^d \). Note that the stationary vector field is an obvious solution in this case, but the statement in [30] is that there exist infinitely many non-stationary solutions. A common feature in these solutions is that for time \( t > 0 \) they exhibit an expanding ‘turbulent’ region around the initial vortex sheet, much akin to the propagation of a singularity in the classical Kelvin–Helmholtz problem. Further examples of this nature appeared in [4] and recently in [7] for the compressible Euler system.

Motivated by the idea that it is the underlying Kelvin–Helmholtz instability that is responsible for the non-uniqueness of admissible weak solutions, we study in this note the case of domains with boundary. We show that the presence of a (smooth) boundary can lead to the same effect of an expanding turbulent region as in [30]. As a corollary, we observe that admissibility does not imply the weak-strong uniqueness property in domains with boundary.

2. Statement of the main results

2.1. Formulation of the equations. We study weak solutions of the initial and boundary value problem for the incompressible Euler equations

\[ \begin{aligned} \partial_t v + v \cdot \nabla v + \nabla p &= 0, \\
\text{div } v &= 0, \\
v \big|_{t=0} &= v_0, \end{aligned} \tag{3} \]

complemented by the usual kinematic boundary condition

\[ v \big|_{\partial \Omega} \cdot \nu = 0. \]

Here \( \Omega \subset \mathbb{R}^d, d \geq 2, \) is a domain with sufficiently smooth boundary, \( T > 0 \) is a finite time, \( v: \Omega \times [0, T) \to \mathbb{R}^d \) is the velocity field, \( p: \Omega \times (0, T) \to \mathbb{R} \) is the (scalar) pressure, \( v_0 \) is the initial velocity, and \( \nu \) is the inner unit normal to the boundary of \( \Omega \).

In order to give the precise definition of weak solutions, consider the space of solenoidal vector fields on \( \Omega \) (cf. [16], Chap. III):

\[ H(\Omega) = \left\{ v \in L^2(\Omega; \mathbb{R}^d) : \int_{\Omega} v \cdot \nabla p \, dx = 0 \right\} \]

for every \( p \in W^{1,2}_{\text{loc}}(\Omega) \) such that \( \nabla p \in L^2(\Omega) \).
Let $v_0 \in H(\Omega)$. An admissible weak solution of (3) with initial data $v_0$ is defined to be a vector field $v \in L^\infty(0,T;H(\Omega))$ such that for every test function $\phi \in C^\infty_c(\Omega \times [0,T];\mathbb{R}^d)$ with $\text{div} \phi = 0$ we have

$$\int_0^T \int_\Omega (\partial_t \phi \cdot v + \nabla \phi : v \otimes v) \, dx \, dt + \int_\Omega v_0(x) \cdot \phi(x,0) \, dx = 0,$$

and the energy inequality (1) holds.

We remark in passing that in fact one may assume that admissible weak solutions are in the space $C([0,T);H^w(\Omega))$, where $H^w(\Omega)$ is the space $H(\Omega)$ equipped with the weak $L^2$-topology. Indeed, the dissipative solutions of Lions are also defined in this space. Nevertheless, for simplicity we will just treat the velocity fields as elements in the larger space $L^\infty(0,T;H(\Omega))$.

2.2. Rotationally symmetric data. In the present paper, we consider rotationally symmetric initial data in two dimensions. It should be noted that the restriction to two dimensions is purely for simplicity of presentation—the constructions and the methods can be easily extended to higher dimensions. Similarly, we will consider an annulus as the domain purely for simplicity of presentation—the non-trivial topology of the domain does not play a role in our results.

By ‘rotational’ we mean initial data of the form

$$v_0(x) = \alpha_0(r)(\sin \theta, - \cos \theta)$$

on an annulus

$$\Omega = \{x \in \mathbb{R}^2: \rho < |x| < R\},$$

where $0 < \rho < R < \infty$. Vector fields as in (4) are known to define stationary solutions of the Euler equations regardless of the choice of $\alpha_0$, and are frequently used as explicit examples in the study of incompressible flows [1], [24], [27].

Fix a radius $r_0$ with $\rho < r_0 < R$ and consider the initial data on the annulus given by (4) with

$$\alpha_0(r) = \begin{cases} 
-\frac{1}{r^2} & \text{if } \rho < r < r_0, \\
\frac{1}{r^2} & \text{if } r_0 < r < R,
\end{cases}$$

which corresponds to a rotational flow with a jump discontinuity on the circle $\{r = r_0\}$.

**Theorem 1.** Let $\Omega$ be an annulus as in (5), let $T > 0$ be a finite time, and let $v_0$ be rotational as in (4) and (6). Then apart from the stationary solution $v(\cdot,t) = v_0$ there exist infinitely many non-stationary admissible weak solutions of the Euler equations on $\Omega \times (0,T)$ with initial data $v_0$. Among these, infinitely many have strictly decreasing energy, and infinitely many conserve the energy.

Our proof, given in § 4 below, relies on the techniques from [11] and is similar to the construction in [30].

Regarding the quest for suitable selection principles, a much-discussed criterion is the viscosity solution, defined to be a solution obtained as a weak limit of
Leray–Hopf solutions as the viscosity converges to zero. In the case of the initial data in (2) it is an easy exercise (see for instance [4]) to show that the viscosity solution agrees with the stationary solution. In the rotational case (6) the same is true, as has been observed, for instance, in [23] (where in fact a much deeper result was obtained, related to confined eddies as initial data). For the reader’s convenience, we provide an elementary proof of this fact in §5 below.

**Proposition 2.** Let $\Omega \subset \mathbb{R}^2$ be an annulus and let the initial data be given by (4). Then every sequence of Leray–Hopf solutions of the Navier–Stokes equations with viscosities tending to zero which correspond to this initial data will converge strongly to the stationary solution $v(\cdot, t) = v_0$ of the Euler equations.

Finally, we discuss the relation between admissible weak solutions and the dissipative solutions of Lions in bounded domains. For the convenience of the reader we recall in §6 the precise definition of dissipative solutions. As a corollary to Theorem 1 we show in §6 that, contrary to the case without boundaries, admissible weak solutions need not be dissipative.

**Corollary 3.** On $\Omega$ there exist admissible weak solutions which are not dissipative solutions.

This corollary says that in the presence of a boundary the weak-strong uniqueness might fail for admissible weak solutions. On the technical level the explanation for this lies in the observation that the notion of strong solution in a bounded domain does not allow any control of the boundary behaviour. Therefore, in §7 we study what happens when additional boundary control is available.

**Theorem 4.** Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with $C^2$ boundary. Suppose that $v$ is an admissible weak solution of (3) on $\Omega$ for which there exist some $\delta > 0$ and $\alpha > 0$ such that $v$ is Hölder continuous with exponent $\alpha$ on the set

$$\Gamma_\delta = \{ x \in \overline{\Omega} : \text{dist}(x, \partial \Omega) < \delta \}$$

uniformly with respect to $t$. Then $v$ is a dissipative solution.

### 3. Subsolutions and convex integration

In order to prove Theorem 1 we recall the basic framework developed in [10], [11], with slight modifications to accommodate for domains with boundary. For further details we refer to the survey [12] and the recent lecture notes [31].

To start with, we recall the definition of subsolution. To this end let us fix a non-negative function

$$\bar{e} \in L^\infty(0, T; L^1(\Omega)),$$

which will play the role of the (kinetic) energy density. We will work in the space-time domain

$$\Omega_T := \Omega \times (0, T),$$

where $\Omega \subset \mathbb{R}^d$ is either an open domain with Lipschitz boundary or $\Omega = \mathbb{T}^d$. 

Definition 5 (subsolution). A subsolution of the incompressible Euler equations with respect to the kinetic energy density $\varepsilon$ is a triple 

$$(\overline{v}, \overline{u}, \overline{q}): \Omega_T \to \mathbb{R}^d \times \mathcal{S}_0^{d \times d} \times \mathbb{R},$$

with $\overline{v} \in L^\infty(0,T;H(\Omega))$, $\overline{u} \in L^1_{\text{loc}}(\Omega_T)$, and $\overline{q} \in \mathcal{D}'(\Omega_T)$ (that is, $\overline{q}$ is a distribution), such that

$$\begin{cases} 
\partial_t \overline{v} + \text{div} \overline{u} + \nabla \overline{q} = 0 \\
\text{div} \overline{v} = 0 
\end{cases} \quad \text{in the sense of distributions} \quad (7)$$

and moreover,

$$\overline{v} \otimes \overline{v} - \overline{u} \leq \frac{2}{d} \overline{\varepsilon} I \quad \text{a.e.} \ (x,t). \quad (8)$$

Here $\mathcal{S}_0^{d \times d}$ denotes the set of symmetric traceless $d \times d$ matrices and $I$ is the identity matrix. Observe that subsolutions automatically satisfy $|\overline{v}|^2/2 \leq \varepsilon$ a.e. If in addition (8) is an equality a.e., then $\overline{v}$ is a weak solution of the Euler equations.

A convenient way to express the inequality (8) is obtained by introducing the generalized energy density

$$e(\overline{v}, \overline{u}) = \frac{d}{2} |\overline{v} \otimes \overline{v} - \overline{u}|_\infty,$$

where $| \cdot |_\infty$ is the operator norm of the matrix (= the largest eigenvalue for symmetric matrices). The inequality (8) can then be equivalently written as

$$e(\overline{v}, \overline{u}) \leq \varepsilon \quad \text{a.e.} \quad (9)$$

The key point of convex integration is that a strict inequality in (8) gives enough room so that high-frequency oscillations can be ‘added’ on top of the subsolution—of course in a highly non-unique way—so that one obtains weak solutions. It is important also to note that, since in the process of convex integration only compactly supported (in space-time) perturbations are added to the subsolution, the boundary and initial conditions of the weak solutions so obtained agree with the corresponding data of the subsolution. This is the content of the following theorem, which is essentially Proposition 2 from [11].

Theorem 6 (Subsolution criterion). Let $\varepsilon \in L^\infty(\Omega_T)$ and let $(\overline{v}, \overline{u}, \overline{q})$ be a subsolution. Furthermore, let $\mathcal{U} \subset \Omega_T$ be a subdomain such that $(\overline{v}, \overline{u}, \overline{q})$ and $\varepsilon$ are continuous on $\mathcal{U}$ and

$$e(\overline{v}, \overline{u}) < \varepsilon \quad \text{on} \ \mathcal{U},$$

$$e(\overline{v}, \overline{u}) = \varepsilon \quad \text{a.e. in} \ \Omega_T \setminus \mathcal{U}. \quad (10)$$

Then there exist infinitely many weak solutions $v \in L^\infty(0,T;H(\Omega))$ of the Euler equations such that

$$v = \overline{v} \quad \text{a.e. in} \ \Omega_T \setminus \mathcal{U},$$

$$\frac{1}{2} |v|^2 = \varepsilon \quad \text{a.e. in} \ \Omega_T,$$

$$p = \overline{q} - \frac{2}{d} \varepsilon \quad \text{a.e. in} \ \Omega_T.$$
If in addition
\[ \bar{v}(\cdot, t) \to v_0(\cdot) \quad \text{in } L^2(\Omega) \text{ as } t \to 0, \]
then \( v \) solves the Cauchy problem (3).

We also refer to [31], where a detailed discussion of the convex integration technique can be found—in particular, the above theorem is Theorem 7 of [31].

4. Non-uniqueness for rotational initial data

In this section we wish to apply the framework of §3 to prove Theorem 1. Thus, we set
\[ \Omega := \{x \in \mathbb{R}^2 : \rho < |x| < R\} \]
to be an annulus, fix \( r_0 \in (\rho, R) \) and let
\[ v_0(x) = \begin{cases} 
- \frac{1}{|x|^3} x^\perp, & |x| < r_0, \\
\frac{1}{|x|^3} x^\perp, & |x| > r_0,
\end{cases} \]
where \( x^\perp = \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} \). We will construct subsolutions by a method similar to that in [30].

Owing to Theorem 6 of the previous section, it suffices to show the existence of certain subsolutions. We fix two small constants \( \lambda > 0 \) (‘turbulent propagation speed’) and \( \epsilon > 0 \) (‘energy dissipation rate’), to be determined later.

We look for subsolutions \((v, u, q)\) (cf. Definition 5—the energy density function \( \bar{e} \) is still to be fixed) of the form
\[ v(x, t) = \alpha(r, t) \left( \begin{array}{c} \sin \theta \\ -\cos \theta \end{array} \right), \]
where \( \alpha(r, 0) = \alpha_0(r) \) and \((r, \theta)\) denotes polar coordinates on \( \mathbb{R}^2 \),
\[ u(x, t) = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} \left( \begin{array}{c} \beta(r, t) \\ \gamma(r, t) \end{array} \right) \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} \]
\[ = \begin{pmatrix} \beta \cos(2\theta) + \gamma \sin(2\theta) & \beta \sin(2\theta) - \gamma \cos(2\theta) \\ \beta \sin(2\theta) - \gamma \cos(2\theta) & -\beta \cos(2\theta) - \gamma \sin(2\theta) \end{pmatrix}, \]
and
\[ q = \bar{q}(r). \]

As a side remark, note that the choice \( \alpha(r, t) = \alpha_0(r) \) for all \( t \geq 0 \), \( \beta = -\alpha^2/2 \), \( \gamma = 0 \), and
\[ \bar{q}(r) = \frac{1}{2} \alpha^2 + \int_\rho^r \frac{\alpha(s)^2}{s} \, ds, \]
yields the well-known stationary solution (the integral in the formula for \( \bar{q} \) represents the physical pressure).
We insert this ansatz into (7) to arrive at two equations. More precisely, using the formulae

\[ \nabla_x r = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \quad \text{and} \quad \nabla_x \theta = \frac{1}{r} \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}, \]

we obtain

\[
\begin{align*}
\partial_t \alpha \sin \theta + \partial_r \beta &\left[ \cos \theta \cos(2\theta) + \sin \theta \sin(2\theta) \right] + \partial_r \gamma \left[ \cos \theta \sin(2\theta) - \sin \theta \cos(2\theta) \right] \\
&+ \frac{2}{r} \beta \left[ \sin \theta \sin(2\theta) + \cos \theta \cos(2\theta) \right] + \frac{2}{r} \gamma \left[ -\sin \theta \cos(2\theta) + \cos \theta \sin(2\theta) \right] \\
&+ \partial_r \bar{q} \cos \theta = 0
\end{align*}
\]

and

\[
\begin{align*}
-\partial_t \alpha \cos \theta + \partial_r \beta &\left[ \cos \theta \sin(2\theta) - \sin \theta \cos(2\theta) \right] + \partial_r \gamma \left[ -\cos \theta \cos(2\theta) - \sin \theta \sin(2\theta) \right] \\
&+ \frac{2}{r} \beta \left[ -\sin \theta \cos(2\theta) + \cos \theta \sin(2\theta) \right] + \frac{2}{r} \gamma \left[ -\sin \theta \sin(2\theta) - \cos \theta \cos(2\theta) \right] \\
&+ \partial_r \bar{q} \sin \theta = 0.
\end{align*}
\]

If we multiply the first equation by \( \sin \theta \) and add it to the second one multiplied by \( \cos \theta \), use the identities \( \cos^2 \theta - \sin^2 \theta = \cos(2\theta) \) and \( 2 \sin \theta \cos \theta = \sin(2\theta) \), and then separate by terms involving \( \sin(2\theta) \) and \( \cos(2\theta) \), respectively, we will eventually get the two equations

\[
\begin{align*}
\partial_r \beta + \frac{2}{r} \beta + \partial_r \bar{q} &= 0, \\
\partial_t \alpha + \partial_r \gamma + \frac{2}{r} \gamma &= 0.
\end{align*}
\]

It can be easily verified that these equations are equivalent to the original system (7) for our ansatz.

If we set \( \bar{q}(r) \) as in (14) and \( \beta = -\alpha^2/2 \), then the first equation will be satisfied, in nice analogy with [30] (up to a sign). Also, the second equation is similar to the one in [30], but it involves the additional ‘centrifugal’ term \( (2/r)\gamma \). Therefore, we cannot simply set \( \gamma = \alpha^2/2 \) as in [30] to obtain Burgers’ equation. However, observing that \( \partial_r (r^2 \gamma) = 2r \gamma + r^2 \partial_r \gamma \), we set

\[
\alpha(r, t) = \frac{1}{r^2} f(r, t)
\]

and

\[
\gamma = -\frac{\lambda}{2r^2} (1 - f^2) = -\frac{\lambda}{2} \left( \frac{1}{r^2} - r^2 \alpha^2 \right),
\]

so that the second equation in (15), after multiplication by \( r^2 \), turns into Burgers’ equation

\[
\partial_t f + \frac{\lambda}{2} \partial_r (f^2) = 0.
\]

The initial data (6) for \( \alpha \) then corresponds to

\[
f(r, 0) = \begin{cases} 
-1 & \text{if } \rho < r < r_0, \\
1 & \text{if } r_0 < r < R.
\end{cases}
\]
Then, for this initial data, Burgers’ equation (17) has a rarefaction wave solution for \( t \in [0, T] \), provided that \( \lambda > 0 \) is sufficiently small (depending on \( T \) and \( \rho < r_0 < R \)), and it can be explicitly written as

\[
f(r, t) = \begin{cases} 
-1 & \text{if } \rho < r < r_0 - \lambda t, \\
\frac{r - r_0}{\lambda t} & \text{if } r_0 - \lambda t < r < r_0 + \lambda t, \\
1 & \text{if } r_0 + \lambda t < r < R.
\end{cases}
\] (18)

Therefore, by setting \( \alpha(r, t) = f(r, t)/r^2 \) for \( f \) as in (18), \( \beta = -\alpha^2/2 \), \( \gamma \) as in (16), and \( \bar{v} \) as in (14), we obtain a solution of the equations (7) with initial data corresponding to (12).

It remains to study the generalized energy. Since \( \bar{v} \) is given by (13) and moreover

\[
\bar{v} \otimes \bar{v} = \alpha(r, t)^2 \begin{pmatrix} \cos^2 \theta & -\sin \theta \cos \theta \\ -\sin \theta \cos \theta & \cos^2 \theta \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & -\cos \theta \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \alpha(r, t)^2 \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix},
\]

and since the eigenvalues of a matrix are invariant under conjugation by an orthogonal transformation, in order to determine \( e(\bar{v}, \bar{u}) = |\bar{v} \otimes \bar{v} - \bar{u}|_\infty \), it suffices to find the largest eigenvalue of the matrix

\[
\begin{pmatrix} -\beta & -\gamma \\ -\gamma & \alpha^2 + \beta \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \alpha^2 & \frac{\lambda}{2} \left( \frac{1}{r^2} - r^2 \alpha^2 \right) \\ \frac{\lambda}{2} \left( \frac{1}{r^2} - r^2 \alpha^2 \right) & \frac{1}{2} \alpha^2 \end{pmatrix}.
\]

Taking \( |\alpha| \leq 1/r^2 \) and \( \lambda \geq 0 \) into account, we easily calculate that

\[
e(\bar{v}, \bar{u}) = \frac{1}{2} \alpha^2 + \frac{\lambda}{2} \left( \frac{1}{r^2} - r^2 \alpha^2 \right) = \frac{1}{2r^4} \left[ 1 - (1 - r^2 \lambda)(1 - f(r, t)^2) \right]. \tag{19}
\]

Finally, we set

\[
\bar{e}(r, t) = \frac{1}{2r^4} \left[ 1 - \varepsilon(1 - r^2 \lambda)(1 - f(r, t)^2) \right],
\]

where \( \varepsilon \) is small enough that \( \bar{e} > 0 \). Observe that

\[
e(\bar{v}, \bar{u}) \leq \bar{e} \leq \frac{1}{2} |v_0|^2 \quad \text{in } \Omega_T.
\]

More precisely, summarizing the calculations in this section, we have the following result.

**Proposition 7.** For any choice of constants \( \varepsilon, \lambda \) satisfying

\[
0 < \lambda < \min \left\{ \frac{1}{R^2}, \frac{r_0 - \rho}{T}, \frac{R - r_0}{T} \right\},
\]

\[
0 \leq \varepsilon \leq \frac{1}{1 - \rho^2 \lambda},
\]

the solution \( \bar{v} \) of the equations (7) is well-defined and unique in the sense of generalized energy.
there exists a subsolution \((\bar{v}, \bar{u}, \bar{q})\) in \(\Omega_T\) with respect to the kinetic energy density
\[
\bar{e}(r, t) = \frac{1}{2r^4} [1 - \varepsilon(1 - r^2 \lambda)(1 - f(r, t)^2)]
\]
and with initial data \(v(x, 0) = v_0(x)\) from (12) such that
\[
e(\bar{v}, \bar{u}) < \bar{e} \quad \text{in} \quad U,
\]
e(\bar{v}, \bar{u}) = \bar{e} \quad \text{in} \quad \Omega_T \setminus U
\]
with
\[
U := \{x \in \mathbb{R}^2 : r_0 - \lambda t < |x| < r_0 + \lambda t\}.
\]

We can now conclude with the proof of Theorem 1.

Proof of Theorem 1. We apply Proposition 7 above with \(\varepsilon > 0\) to obtain a subsolution \((\bar{v}, \bar{u}, \bar{q})\). According to Theorem 6 with this subsolution, there exist infinitely many weak solutions \(v \in L^\infty(0, T; H(\Omega))\) such that
\[
|v|^2 = 2\bar{e}\quad \text{almost everywhere in} \quad \Omega_T
\]
and with initial data \(v_0\). To check that these are admissible, observe that
\[
\int_{\Omega} |v(x, t)|^2 \, dx = \int_{\Omega} 2\bar{e}(x, t) \, dx \leq \int_{\Omega} \frac{1}{|x|^4} \, dx = \int_{\Omega} |v_0(x)|^2 \, dx.
\]
Finally, observe that we obtain strictly energy-decreasing solutions by choosing \(\varepsilon > 0\) and energy-conserving solutions for \(\varepsilon = 0\).

\[\square\]

5. Uniqueness of the zero viscosity limit

Proof of Proposition 2. Consider the Navier-Stokes equations with viscosity \(\varepsilon > 0\):
\[
\partial_t v_\varepsilon + v_\varepsilon \cdot \nabla v_\varepsilon + \nabla p_\varepsilon = \varepsilon \Delta v_\varepsilon,
\]
div \(v_\varepsilon = 0,
\]
\(v_\varepsilon(x, 0) = v_0\),
\(v_\varepsilon|_{\partial\Omega} = 0\).

It is known that the Navier–Stokes equations in two space dimensions admit a unique weak solution (the Leray–Hopf solution) which satisfies the energy equality
\[
\frac{1}{2} \int_\Omega |v_\varepsilon(x, t)|^2 \, dx + \varepsilon \int_0^t \int_\Omega |\nabla v_\varepsilon(x, s)|^2 \, dx \, ds = \frac{1}{2} \int_\Omega |v_0(x)|^2 \, dx
\]
for every \(t \in [0, T]\) (see, for instance, [17] for details). It turns out that if the initial data \(v_0\) has the rotational symmetry in (4), then the (unique) Leray–Hopf solution will have the same symmetry.

To show this, we take the ansatz
\[
v_\varepsilon(x, t) = \alpha_\varepsilon(r, t) \begin{pmatrix} \sin \theta \\ -\cos \theta \end{pmatrix}
\]
and \(p_\varepsilon = p_\varepsilon(r)\), again using polar coordinates. Insertion of this ansatz into the first equation of (20) yields
\[
\partial_t \alpha_\varepsilon \sin \theta - \frac{\alpha_\varepsilon^2}{r} \cos \theta + \partial_r p_\varepsilon \cos \theta = \varepsilon \left( \frac{\partial_r \alpha_\varepsilon}{r} + \frac{\partial_r^2 \alpha_\varepsilon}{r^2} - \frac{\alpha_\varepsilon}{r^2} \right) \sin \theta.
\]
If we choose
\[ p_\varepsilon(r) = \int_\rho^r \frac{\alpha_\varepsilon(s)^2}{s} \, ds \]
and divide by \( \sin \theta \), then we end up with the parabolic equation
\[
\partial_t \alpha_\varepsilon = \varepsilon \left( \frac{\partial_r \alpha_\varepsilon}{r} + \partial_r^2 \alpha_\varepsilon - \frac{\alpha_\varepsilon}{r^2} \right). \tag{22}
\]
Insertion of our ansatz into the second equation of (20) also gives (22), as one can easily check by a similar computation. Moreover, the divergence-free condition is automatically satisfied, the initial condition becomes
\[
\alpha_\varepsilon(\cdot, 0) = \alpha_0, \tag{23}
\]
with \( \alpha_0 \) defined by (6), and the boundary condition translates into
\[
\alpha_\varepsilon(\rho) = \alpha_\varepsilon(R) = 0. \tag{24}
\]
Thus, we obtain the well-posed parabolic initial and boundary value problem (22)–(24). By well-known results (cf., for instance, [14], §7.1), this parabolic problem admits a unique weak solution for each \( \varepsilon > 0 \). But our calculations so far show that if \( \alpha_\varepsilon \) is a solution of the parabolic problem, then the corresponding \( v_\varepsilon \) defined by (21) is the (unique) Leray–Hopf solution of the Navier–Stokes problem (20), and at the same time it satisfies the initial and boundary value problem for the heat equation:
\[
\begin{align*}
\partial_t v_\varepsilon &= \varepsilon \Delta v_\varepsilon, \\
\text{div } v_\varepsilon &= 0, \\
v_\varepsilon(\cdot, 0) &= v_0, \\
v_\varepsilon|_{\partial\Omega} &= 0.
\end{align*}
\]
Since the solutions of the heat equation converge strongly to the stationary solution, and since we have shown that for our particular initial data the heat equation coincides with the Navier–Stokes equations, the proposition is thus proved. □

**Remark 8.** The previous discussion can be extended to initial data on a cylinder of the form \( Z = \Omega \times \mathbb{T} \subset \mathbb{R}^2 \times \mathbb{T} \), where \( \Omega \subset \mathbb{R}^2 \) is still the annulus. Indeed, for so-called 2\( \frac{1}{2} \)-dimensional initial data \( V_0(x_1, x_2) = (v_0(x_1, x_2), w(x_1, x_2)) \) on \( Z \), where \( v_0 \) is as in (4), there may exist infinitely many admissible weak solutions, but only the solution given by
\[
V(x_1, x_2, t) = (v_0(x_1, x_2), w(x_1 - (v_0)_1 t, x_2 - (v_0)_2 t))
\]
arises as a zero viscosity limit. We omit details, but remark that this can be shown along the lines of [4], where a similar analysis was carried out for the case of shear flows.
6. Dissipative solutions

Let \( S(w) = (\nabla w + \nabla w^t)/2 \) denote the symmetric gradient of a vector field \( w \), and let
\[
E(w) = -\partial_t w - P(w \cdot \nabla w),
\]
with \( P \) denoting the Leray–Helmholtz projection on \( H(\Omega) \).

The following definition is from [22], given here in the version of [2] for bounded domains. The reader may consult these references also for a motivation of the definition.

**Definition 9.** Let \( \Omega \) be a bounded domain with \( C^1 \) boundary. A vector field \( v \in C([0, T]; H_w(\Omega)) \) is said to be a dissipative solution of the Euler equations (3) if for every divergence-free test vector field \( w \in C^1(\bar{\Omega} \times [0, T]) \) with \( w \cdot \nu \mid_{\partial \Omega} = 0 \) one has
\[
\int_{\Omega} |v - w|^2 \, dx \leq \exp \left( 2 \int_0^t \|S(w)\|_{\infty} \, ds \right) \int_{\Omega} |v(x, 0) - w(x, 0)|^2 \, dx
+ 2 \int_0^t \int_{\Omega} \exp \left( 2 \int_s^t \|S(w)\|_{\infty} \, d\tau \right) E(w) \cdot (v - w) \, dx \, ds \tag{25}
\]
for all \( t \in [0, T] \).

An immediate consequence of this definition is the weak-strong uniqueness (Proposition 4.1 in [22]):

**Proposition 10.** Suppose there exists a solution \( v \in C^1(\bar{\Omega} \times [0, T]) \) of the Euler equations (3). Then \( v \) is unique in the class of dissipative solutions with the same initial data.

This follows simply by choosing \( w = v \) as a test function in the definition of dissipative solutions.

Next, we prove Corollary 3, showing that admissible solutions may fail to be unique in bounded domains even for smooth initial data.

**Proof of Corollary 3.** Recall the construction from §4 and define
\[
\tilde{\Omega} = \{ x \in \mathbb{R}^2 : \rho < |x| < r_0 \} \subset \Omega.
\]
It follows immediately from the definition that the restriction of a subsolution to a subdomain is itself a subsolution. Therefore, we may consider the subsolution \( (\bar{v}, \bar{u}, \bar{q}) \) constructed in §4 as a subsolution on \( \tilde{\Omega} \) with energy density \( \bar{e} \) as in Proposition 7, and with initial data given by
\[
\bar{v}(x, 0) = -\frac{x^\perp}{|x|^3} \quad \text{for} \quad x \in \tilde{\Omega}
\]
(cf. (12)). Application of Theorem 6 now in \( \tilde{\Omega} \) with this subsolution yields infinitely many admissible weak solutions as in the proof of Theorem 1.

Since the initial data \( \bar{v}(x, 0) \) is smooth in \( \tilde{\Omega} \), there exists a unique strong solution (indeed, this is the stationary solution). Thus, weak-strong uniqueness fails, a fortiori implying that the non-stationary admissible weak solutions are not dissipative in the sense of Lions. \( \square \)
7. A criterion for admissible solutions to be dissipative

We have seen that, on bounded domains, an admissible weak solution may fail to be dissipative. However, this will not happen provided that such a solution is H"older continuous near the boundary of the domain, as claimed in Theorem 4 above. The aim of this last section is to prove this theorem. We follow Appendix B of [11], but have to take into account that we need to deal with test functions which are not necessarily compactly supported in $\Omega$ in the definition of dissipative solutions.

So let $\Omega$ be a bounded domain in $\mathbb{R}^2$ with $C^2$ boundary and let $v$ be an admissible weak solution of the Euler equations (3) as in the statement of Theorem 4. Assume for the moment that for every divergence-free $w \in C^1(\Omega \times [0, T])$ satisfying the boundary condition we have

$$
\frac{d}{dt} \int_\Omega v \cdot w \, dx = \int_\Omega (S(w)(v-w) \cdot (v-w) - E(w) \cdot v) \, dx \tag{26}
$$

in the sense of distributions, where $E(w)$ is the quantity defined at the beginning of §6. We claim that (26) already implies that $v$ is a dissipative solution. Indeed, this can be shown exactly as in [11]: On the one hand, since $v$ is admissible,

$$
\frac{d}{dt} \int_\Omega |v|^2 \, dx \leq 0 \tag{27}
$$

in the sense of distributions. On the other hand, using the definition of $E(w)$ and the identity $\int_\Omega (w \cdot \nabla w) \cdot w \, dx = 0$ (which follows from $w \cdot \nu |_{\partial \Omega} = 0$), we have

$$
\frac{d}{dt} \int_\Omega |w|^2 \, dx = -2 \int_\Omega E(w) \cdot w \, dx. \tag{28}
$$

Since

$$
\int_\Omega |v-w|^2 \, dx = \int_\Omega |v|^2 \, dx + \int_\Omega |w|^2 \, dx - 2 \int_\Omega v \cdot w \, dx,
$$

we infer from this together with (26), (27), and (28) that

$$
\frac{d}{dt} \int_\Omega |v-w|^2 \, dx \leq 2 \int_\Omega \left( E(w) \cdot (v-w) - S(w)(v-w) \cdot (v-w) \right) \, dx
$$

$$
\leq 2 \int_\Omega E(w) \cdot (v-w) \, dx + 2 \|S(w)\|_{\infty} \int_\Omega |v-w|^2 \, dx
$$

in the sense of distributions. We can then apply Gronwall’s inequality as in [11] to obtain (25) for every $t \in [0, T]$. Therefore, it remains to prove (26) for every test function $w$.

In [11] the identity (26) is proved for the case that $w$ is compactly supported in $\Omega$ at almost every time (see the considerations after the equality (96) in [11]). Now let $w \in C^1(\overline{\Omega} \times [0, T])$ be a divergence-free vector field with $w \cdot \nu |_{\partial \Omega} = 0$ which does not necessarily have compact support in space. We will suitably approximate $w$ by vector fields that do have compact support, much in the spirit of Kato [21] (in particular, see §4 therein).
Assume for the moment that $\Omega$ is simply connected, so that $\partial \Omega$ has only one connected component. Since $w$ is divergence-free, there exists a function $\psi \in C([0, T]; C^2(\overline{\Omega})) \cap C^1(\overline{\Omega} \times [0, T])$ such that

$$w(x, t) = \nabla^\perp \psi(x, t)$$

and $\psi|_{\partial \Omega} = 0$. Now let $\chi: [0, \infty) \to \mathbb{R}$ be a non-negative smooth function such that

$$\chi(s) = \begin{cases} 0 & \text{if } s < 1, \\ 1 & \text{if } s > 2, \end{cases}$$

and let

$$w_\varepsilon(x, t) = \nabla^\perp \left( \chi \left( \frac{\text{dist}(x, \partial \Omega)}{\varepsilon} \right) \psi(x, t) \right).$$

Then by Lemma 14.16 in [18] there exists an $\eta > 0$ depending on $\Omega$ such that the function $x \mapsto \text{dist}(x, \partial \Omega)$ is $C^2$ on

$$\Gamma_\eta = \{ x \in \overline{\Omega}: \text{dist}(x, \partial \Omega) < \eta \},$$

and hence $w_\varepsilon \in C^1_c(\Omega \times [0, T])$ for sufficiently small $\varepsilon > 0$. Therefore, (26) is true for $w_\varepsilon$:

$$\frac{d}{dt} \int_\Omega v \cdot w_\varepsilon \, dx = \int_\Omega (S(w_\varepsilon)(v - w_\varepsilon) \cdot (v - w_\varepsilon) - E(w_\varepsilon) \cdot v) \, dx. \quad (30)$$

We will now let $\varepsilon$ tend to zero to recover (26).

Writing $d(x) = \text{dist}(x, \partial \Omega)$, we get from the definition of $w_\varepsilon$ that

$$w_\varepsilon = \chi \left( \frac{d}{\varepsilon} \right) \nabla^\perp \psi + \frac{1}{\varepsilon} \chi' \left( \frac{d}{\varepsilon} \right) \psi \nabla^\perp d,$$

and since $\psi \in C([0, T]; C^2(\overline{\Omega}))$ and $\psi|_{\partial \Omega} = 0$, there is a constant $C$ independent of $t$ and $\varepsilon$ such that

$$|\psi(x, t)| \leq C d(x)$$

for all $x \in \overline{\Omega}$. Moreover, since the support of $\chi'(\cdot/\varepsilon)$ is contained in $(\varepsilon, 2\varepsilon)$ and since $|\nabla d| \leq 1$, it follows from (31) that

$$w_\varepsilon \to w \text{ strongly in } L^\infty([0, T]; L^2(\Omega))$$

as $\varepsilon \to 0$. For the left-hand side of (30) this immediately implies that

$$\frac{d}{dt} \int_\Omega v \cdot w_\varepsilon \, dx \to \frac{d}{dt} \int_\Omega v \cdot w \, dx$$
in the sense of distributions. Moreover, recalling the definition of $E(w_\varepsilon)$, we can write the right-hand side of (30) as

$$
\int_\Omega \left( S(w_\varepsilon)(v - w_\varepsilon) \cdot (v - w_\varepsilon) - E(w_\varepsilon) \cdot v \right) dx
= \int_\Omega \left[ \partial_t w_\varepsilon \cdot v + (v \cdot \nabla w_\varepsilon) \cdot v - ((v - w_\varepsilon) \cdot \nabla w_\varepsilon) \cdot w_\varepsilon \right] dx,
$$

and the right-hand side of (26) is given by a similar expression.

Next observe that, again by (32),

$$
\int_\Omega \partial_t w_\varepsilon \cdot v dx \to \int_\Omega \partial_t w \cdot v dx
$$
in the sense of distributions, and also that

$$
\int_\Omega ((v - w_\varepsilon) \cdot \nabla w_\varepsilon) \cdot w_\varepsilon dx = \int_\Omega ((v - w) \cdot \nabla w) \cdot w dx = 0,
$$
thanks to the formula $((v - w) \cdot \nabla w) \cdot w = (v - w) \cdot (1/2) \nabla |w|^2$ and the fact that $v - w \in H(\Omega)$ (and similarly for $((v - w_\varepsilon) \cdot \nabla w_\varepsilon) \cdot w_\varepsilon$).

To complete the proof of (26) and therefore of Theorem 4, it remains to show that

$$
\int_\Omega (v \cdot \nabla w_\varepsilon) \cdot v dx \to \int_\Omega (v \cdot \nabla w) \cdot v dx \tag{33}
$$
in the sense of distributions as $\varepsilon \to 0$.

To this end, note that for every $x \in \Omega$ sufficiently close to $\partial \Omega$ there exists a unique closest point $\hat{x} \in \partial \Omega$, and then

$$
x = \hat{x} + d(x)\nu(\hat{x}).
$$

We denote by $\tau(\hat{x}) = (-\nu_2(\hat{x}), \nu_1(\hat{x}))$ the unit vector at $\hat{x}$ tangent to $\partial \Omega$ and use the notation $v_\tau(x) = v(x) \cdot \tau(x)$, $\partial_\tau w_\nu(x) = \nabla w_\nu(x) \cdot \tau(x)$, and so on (recall that $\hat{x}$ is uniquely determined by $x$). If $\varepsilon$ is sufficiently small, then we can write (recall (29))

$$
\int_\Omega (v \cdot \nabla (w_\varepsilon - w)) \cdot v dx = \int_{\Gamma_{2\varepsilon}} v_\nu \partial_\nu (w_\varepsilon - w)_\nu v_\nu dx + \int_{\Gamma_{2\varepsilon}} v_\tau \partial_\tau (w_\varepsilon - w)_\tau v_\tau dx
+ \int_{\Gamma_{2\varepsilon}} v_\tau \partial_\tau (w_\varepsilon - w)_\nu v_\nu dx + \int_{\Gamma_{2\varepsilon}} v_\tau \partial_\tau (w_\varepsilon - w)_\tau v_\tau dx
=: I_1 + I_2 + I_3 + I_4.
$$
Recalling (31) as well as $\nabla \psi = w$ and observing that $\nabla d = \nu$, we compute

$$
(w_\varepsilon - w)_\nu = \left( \chi \left( \frac{d}{\varepsilon} \right) - 1 \right) \partial_\nu \psi = \left( \chi \left( \frac{d}{\varepsilon} \right) - 1 \right) w_\nu, \\
(w_\varepsilon - w)_\tau = -\left( \chi \left( \frac{d}{\varepsilon} \right) - 1 \right) \partial_\nu \psi + \frac{1}{\varepsilon} \chi \left( \frac{d}{\varepsilon} \right) \psi
$$

$$
= \left( \chi \left( \frac{d}{\varepsilon} \right) - 1 \right) w_\tau + \frac{1}{\varepsilon} \chi \left( \frac{d}{\varepsilon} \right) \psi,
$$

$$
\partial_\nu (w_\varepsilon - w)_\nu = \frac{1}{\varepsilon} \chi \left( \frac{d}{\varepsilon} \right) w_\nu + \left( \chi \left( \frac{d}{\varepsilon} \right) - 1 \right) \partial_\nu w_\nu, 	ag{34}
$$

$$
\partial_\nu (w_\varepsilon - w)_\tau = \left( \chi \left( \frac{d}{\varepsilon} \right) - 1 \right) \partial_\nu w_\tau + \frac{1}{\varepsilon} \chi^\prime \left( \frac{d}{\varepsilon} \right) \psi, 	ag{35}
$$

$$
\partial_\tau (w_\varepsilon - w)_\nu = \left( \chi \left( \frac{d}{\varepsilon} \right) - 1 \right) \partial_\tau w_\nu, 	ag{36}
$$

$$
\partial_\tau (w_\varepsilon - w)_\tau = \left( \chi \left( \frac{d}{\varepsilon} \right) - 1 \right) \partial_\tau w_\tau + \frac{1}{\varepsilon} \chi^\prime \left( \frac{1}{\varepsilon} \right) w_\nu. 	ag{37}
$$

Before we estimate $I_1 - I_4$ using (34)–(37), let us collect some more information: As mentioned above, there is a constant $C$ independent of $t$ such that $|\psi(x)| \leq Cd(x)$. Moreover, since $w \in C^1(\overline{\Omega} \times [0, T])$ and $w_\nu = 0$ on $\partial \Omega$, we find similarly a constant independent of $t$ such that $|w_\nu(x)| \leq Cd(x)$. By assumption, if $\varepsilon$ is small enough, then $\nu$ is Hölder continuous with exponent $\alpha$ on $\Gamma_{2\varepsilon}$, uniformly with respect to $t$, and since $\nu \in H(\Omega)$ implies that $\nu = 0$ on $\partial \Omega$ (cf. [16], Chap. III), we obtain another time-independent constant such that $|v_\nu(x)| \leq C d(x)\alpha$ on $\Gamma_{2\varepsilon}$. Finally, note that $\nu, w, \text{ and } v$ are uniformly bounded on $\Gamma_{2\varepsilon}$ if $\varepsilon$ is small, and that there is a constant independent of $\varepsilon$ such that $|\Gamma_{2\varepsilon}| \leq C\varepsilon$.

In the light of these considerations we can use (34)–(37) to get the estimates

$$
|I_1| \leq \frac{1}{\varepsilon} \int_{\Gamma_{2\varepsilon}} v_\nu^2 |\chi^\prime| \|w_\tau\| \|\nabla \nu\| \, dx + \int_{\Gamma_{2\varepsilon}} v_\nu^2 |\chi - 1| \|\partial_\nu w_\nu\| \, dx
$$

$$
\leq C\varepsilon^{2\alpha + 1} + C\varepsilon^{2\alpha + 1} \rightarrow 0,
$$

$$
|I_2| \leq \int_{\Gamma_{2\varepsilon}} |v_\nu| |v_\tau| \|\chi - 1\| \|\partial_\nu w_\tau\| \, dx + \frac{1}{\varepsilon^2} \int_{\Gamma_{2\varepsilon}} |v_\nu| |v_\tau| \|\chi^\prime\| \|\psi\| \, dx
$$

$$
\leq C\varepsilon^{\alpha + 1} + C\varepsilon^\alpha \rightarrow 0,
$$

$$
|I_3| \leq \int_{\Gamma_{2\varepsilon}} |v_\nu| |v_\tau| \|\chi - 1\| \|\partial_\tau w_\nu\| \, dx \leq C\varepsilon^{\alpha + 1} \rightarrow 0,
$$

$$
|I_4| \leq \int_{\Gamma_{2\varepsilon}} v_\tau^2 \|\chi - 1\| \|\partial_\tau w_\tau\| \, dx + \frac{1}{\varepsilon} \int_{\Gamma_{2\varepsilon}} v_\tau^2 \|\chi^\prime\| |w_\nu| \, dx
$$

$$
\leq C\varepsilon + C\varepsilon \rightarrow 0,
$$

all of them uniform with respect to time. This proves Theorem 4 if $\Omega$ is simply connected.

As a final step, we convince ourselves that the proof can easily be modified in the spirit of §1.4 in [20] to the general case when $\partial \Omega$ has $N$ connected components.
\( \Gamma^1, \ldots, \Gamma^N \). There still exists a \( \psi \in C([0, T]; C^2(\overline{\Omega})) \cap C^1(\overline{\Omega} \times [0, T]) \) with \( \nabla^\perp \psi = w \), but we can no longer require that \( \psi |_{\partial \Omega} = 0 \). Instead, \( \psi \) will take a constant value \( \psi^i \) on \( \Gamma^i \), but the numbers \( \psi^i \) may be different. Now, if \( \varepsilon > 0 \) is small enough, then the sets

\[
\Gamma^i_{2\varepsilon} = \{ x \in \overline{\Omega}: \text{dist}(x, \Gamma^i) < 2\varepsilon \}, \quad i = 1, \ldots, N,
\]

will be mutually disjoint, so that \( w_\varepsilon \) is well-defined by setting

\[
w_\varepsilon(x) = \begin{cases} 
\nabla^\perp \left( \chi \left( \frac{\text{dist}(x, \partial \Omega)}{\varepsilon} \right) (\psi(x) - \psi^i) \right) & \text{if } x \in \Gamma^i_{2\varepsilon}, \\
\psi(x) & \text{if } x \in \overline{\Omega} \setminus \bigcup_i \Gamma^i_{2\varepsilon},
\end{cases}
\]

with \( \chi \) as in the simply connected case. With this choice of \( w_\varepsilon \) we can then employ the very same arguments as above.

Remark 11. Theorem 4 implies that there cannot be wild solutions on an annulus with smooth rotational initial data that are Hölder continuous. Indeed, any admissible Hölder continuous solution must be dissipative by our theorem, and the weak-strong uniqueness then gives us that this solution must coincide with the stationary one. This observation is particularly interesting in the light of recent results (for instance, [6], [9], [19]) where examples of Hölder continuous wild solutions are constructed.

One of the first papers of Professor Mark Vishik (“On general boundary problems for elliptic differential equations” [33]) was essential, in particular in France, for the training of mathematicians in the generation of the first author of the present contribution. Then when he turned to Navier–Stokes and turbulence he played an important role in progress over the last 60 years towards the mathematical understanding of turbulence in fluid mechanics. Hence we hope that this essay will contribute to the preservation of his memory and to the recognition of his influence on our community.

The authors would like to thank Professor Edriss Titi for interesting and valuable discussions.

Bibliography

[1] C. J. Amick, “Existence of solutions to the nonhomogeneous steady Navier–Stokes equations”, Indiana Univ. Math. J. 33:6 (1984), 817–830.

[2] К. Бардос, Э. С. Тити, “Уравнения Эйлера идеальной несжимаемой жидкости”, VMH 62:3(375) (2007), 5–46; English transl., C. Bardos and E. S. Titi, “Euler equations for incompressible ideal fluids”, Russian Math. Surveys 62:3 (2007), 409–451.

[3] C. Bardos and E. S. Titi, Mathematics and turbulence: where do we stand?, 2013, 40 pp., arXiv:1301.0273.

[4] C. Bardos, E. S. Titi, and E. Wiedemann, “The vanishing viscosity as a selection principle for the Euler equations: the case of 3D shear flow”, C. R. Math. Acad. Sci. Paris 350:15-16 (2012), 757–760.

[5] Y. Brenier, C. De Lellis, and L. Székelyhidi, Jr., “Weak-strong uniqueness for measure-valued solutions”, Comm. Math. Phys. 305:2 (2011), 351–361.
[6] T. Buckmaster, C. De Lellis, and L. Székelyhidi, *Transporting microstructure and dissipative Euler flows*, 2013, 35 pp., arXiv:1302.2815.

[7] E. Chiodaroli, C. De Lellis, and O. Kreml, *Global ill-posedness of the isentropic system of gas dynamics*, 2013, 30 pp., arXiv:1304.0123.

[8] C. M. Dafermos, *Hyperbolic conservation laws in continuum physics*, 3rd ed., Grundlehren Math. Wiss., vol. 325, Springer-Verlag, Berlin 2010, xxxvi+708 pp.

[9] S. Daneri, *Cauchy problem for dissipative Hölder solutions to the incompressible Euler equations*, 2013, 33 pp., arXiv:1302.0988.

[10] C. De Lellis and L. Székelyhidi, Jr., “The Euler equations as a differential inclusion”, *Ann. of Math. (2) 170*:3 (2009), 1417–1436.

[11] C. De Lellis and L. Székelyhidi, Jr., “On admissibility criteria for weak solutions of the Euler equations”, *Arch. Ration. Mech. Anal. 195*:1 (2010), 225–260.

[12] C. De Lellis and L. Székelyhidi, Jr., “The $h$-principle and the equations of fluid dynamics”, *Bull. Amer. Math. Soc. (N. S.) 49*:3 (2012), 347–375.

[13] J. Duchon and R. Robert, “Inertial energy dissipation for weak solutions of incompressible Euler and Navier–Stokes equations”, *Nonlinearity 13*:1 (2000), 249–255.

[14] L. C. Evans, *Partial differential equations*, 2nd ed., Grad. Stud. Math., vol. 19, Amer. Math. Soc., Providence, RI 2010, xxii+749 pp.

[15] G. L. Eyink and K. R. Sreenivasan, “Onsager and the theory of hydrodynamic turbulence”, *Rev. Modern Phys. 78*:1 (2006), 87–135.

[16] G. P. Galdi, *An introduction to the mathematical theory of the Navier–Stokes equations*, vol. I: Linearized steady problems, Springer Tracts Natur. Philos., vol. 38, Springer-Verlag, New York 1994, xii+450 pp.

[17] G. P. Galdi, “An introduction to the Navier–Stokes initial-boundary value problem”, *Fundamental directions in mathematical fluid mechanics*, Adv. Math. Fluid Mech., Birkhäuser, Basel 2000, pp. 1–70.

[18] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, reprint of the 1998 edition, Classics Math., Springer-Verlag, Berlin 2001, xiv+517 pp.

[19] P. Isett, *Hölder continuous Euler flows in three dimensions with compact support in time*, 2012 (v4 – 2014), 176 pp., arXiv:1211.4065.

[20] T. Kato, “On classical solutions of the two-dimensional non-stationary Euler equation”, *Arch. Ration. Mech. Anal. 25*:3 (1967), 188–200.

[21] T. Kato, “Remarks on zero viscosity limit for nonstationary Navier–Stokes flows with boundary”, *Seminar on nonlinear partial differential equations* (Berkeley, CA 1983), Math. Sci. Res. Inst. Publ., vol. 2, Springer, New York 1984, pp. 85–98.

[22] P.-L. Lions, *Mathematical topics in fluid mechanics*, vol. 1: *Incompressible models*, Oxford Lecture Ser. Math. Appl., vol. 3, The Clarendon Press, Oxford Univ. Press, New York 1996, xiv+237 pp.

[23] M. C. Lopes Filho, H. J. Nussenzveig Lopes, and Yuxi Zheng, “Convergence of the vanishing viscosity approximation for superpositions of confined eddies”, *Comm. Math. Phys. 201*:2 (1999), 291–304.

[24] A. J. Majda and A. L. Bertozzi, *Vorticity and incompressible flow*, Cambridge Texts Appl. Math., vol. 27, Cambridge Univ. Press, Cambridge 2002, xii+545 pp.

[25] L. Saint-Raymond, “Convergence of solutions to the Boltzmann equation in the incompressible Euler limit”, *Arch. Ration. Mech. Anal. 166*:1 (2003), 47–80.

[26] V. Scheffer, “An inviscid flow with compact support in space-time”, *J. Geom. Anal. 3*:4 (1993), 343–401.
Non-uniqueness for the Euler equations

[27] A. I. Shnirelman, “Lattice theory and flows of ideal incompressible fluid”, *Russian J. Math. Phys.* 1:1 (1993), 105–114.

[28] A. Shnirelman, “On the nonuniqueness of weak solution of the Euler equation”, *Comm. Pure Appl. Math.* 50:12 (1997), 1261–1286.

[29] A. Shnirelman, “Weak solutions with decreasing energy of incompressible Euler equations”, *Comm. Math. Phys.* 210:3 (2000), 541–603.

[30] L. Székelyhidi, Jr., “Weak solutions to the incompressible Euler equations with vortex sheet initial data”, *C. R. Math. Acad. Sci. Paris* 349:19-20 (2011), 1063–1066.

[31] L. Székelyhidi, From isometric embeddings to turbulence, Lecture note no. 41, Max-Planck-Institut für Mathematik in den Naturwissenschaften, Leipzig 2012, 54 pp., http://www.mis.mpg.de/preprints/in/lecturenote-4112.pdf.

[32] L. Székelyhidi Jr. and E. Wiedemann, “Young measures generated by ideal incompressible fluid flows”, *Arch. Ration. Mech. Anal.* 206:1 (2012), 333–366.

[33] М.И. Вишик, “Об общих краевых задачах для эллиптических дифференциальных уравнений”, Тр. ММО, 1, ГИТЛ, М.–Л. 1952, с. 187–246; English transl., M. I. Vishik, “On general boundary problems for elliptic differential equations”, *Eight papers on differential equations*, Amer. Math. Soc. Transl. Ser. 2, vol. 24, Amer. Math. Soc., Providence, RI 1963, pp. 107–172.

[34] E. Wiedemann, “Existence of weak solutions for the incompressible Euler equations”, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 28:5 (2011), 727–730.

Claude Bardos

Université Paris VII – Denis Diderot,
Paris, France

*E-mail: claudesbardos@gmail.com*

László Székelyhidi, Jr.

Universität Leipzig, Mathematisches Institut,
Leipzig, Germany

*E-mail: szekelyhidi@math.uni-leipzig.de*

Emil Wiedemann

University of British Columbia,
Vancouver, Canada;
Pacific Institute for the Mathematical Sciences,
Vancouver, Canada

*E-mail: emil@math.ubc.ca*