On the ensemble dependence in black hole geometrothermodynamics

Hernando Quevedo¹,², María N Quevedo³, Alberto Sánchez⁴ and Safia Taj⁵

¹ Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, AP 70543, México, DF 04510, Mexico
² Instituto de Cosmologia, Relatividade e Astrofísica and ICRANet—CBPF, Rua Dr. Xavier Sigaud, 150, CEP 22290-180, Rio de Janeiro, Brazil
³ Departamento de Matemáticas, Universidad Militar Nueva Granada, Cra. 11 No. 101-80, Bogotá D.E., Colombia
⁴ Departamento de Posgrado, CIIDET, AP 752, Querétaro, QRO 76000, Mexico
⁵ NUST College of Electrical and Mechanical Engineering, National University of Sciences and Technology, Rawalpindi, Pakistan

E-mail: quevedo@nucleares.unam.mx, maria.quevedo@unimilitar.edu.co, asanchez@nucleares.unam.mx and safiataaj@gmail.com

Received 30 July 2013, revised 19 December 2013
Accepted for publication 23 December 2013
Published 11 July 2014

Abstract

We investigate the dependence of the thermodynamic properties of black holes on the choice of statistical ensemble for a particular class of Einstein–Maxwell–Gauss–Bonnet black holes with the cosmological constant. We use partial Legendre transformations in the thermodynamic limit in order to compare the results in different ensembles, and show that the phase transition structure depends on the choice of thermodynamic potential. This result implies that thermodynamic metrics that are partially Legendre invariant cannot be employed to describe black hole thermodynamics, and partly explains why a particular thermodynamic metric has been used so far in the framework of black hole geometrothermodynamics.

Keywords: blackholes, geometrothermodynamics, phase transitions

1. Introduction

In the Euclidean path-integral approach to black hole thermodynamics, originally developed by Hawking et al [1–4], the thermodynamic partition function is computed from the path integral in the saddle-point approximation, obtaining as a result the laws of black hole thermodynamics. Originally, only the microcanonical ensemble was investigated [1, 4] because the canonical ensemble led to difficulties related to the stability of the black hole under consideration. Later on, the canonical ensemble was investigated by York et al [5–10] by using appropriate boundary conditions. It turns out that the Euclidean path-integral approach is not well defined until the boundary conditions of the system are properly defined. The various ensembles available to the system are defined by identifying the corresponding boundary conditions. York’s approach was used to analyze other ensembles [11] and to study particular systems like the charged black hole in the grand canonical ensemble [10], black holes in asymptotically anti-de Sitter spacetimes [12, 13], and black holes in two and three dimensions [12, 14].

Under certain circumstances, the results obtained by using the path-integral approach turned out to depend on the boundary conditions [12, 15, 16]. The stability of black holes also turned out to depend on the choice of boundary conditions and, consequently, on the ensemble [17]. In fact, it was found that in one ensemble the black hole can never be stable, independently of the values of the black hole parameters, and in another ensemble the black hole is almost always stable. This result shows that the stability properties of a black hole are drastically influenced by the boundary conditions that determine the ensemble. Moreover, it follows that this behavior must also be present in the thermodynamic limit in which different ensembles correspond to different thermodynamic potentials, related by Legendre transformations. On the other hand, an important characteristic of a black hole is its phase transition structure which is closely related to the stability properties of the system. It then follows that the phase
transition structure can, in principle, also be ensemble dependent. In the thermodynamic limit, this would imply a dependence on the thermodynamic potential.

The formalism of geometrothermodynamics (GTD) has been proposed in [18] as a Legendre invariant approach to describe thermodynamics in terms of geometric concepts. One of the conjectures of GTD is that curvature singularities of the equilibrium space are related to phase transitions of the system. One can then wonder whether GTD is able to handle the dependence of the phase transition structure on the statistical ensemble. This is the main goal of the present work. For the sake of concreteness, we will consider here a particular black hole configuration in the Einstein–Maxwell–Gauss–Bonnet (EMGB) theory. We will see that in fact the phase transition structure, as dictated by the behavior of the specific heats, drastically depends on the thermodynamic potentials which are related by partial Legendre transformations. Consequently, the metrics that in GTD are invariant under partial Legendre transformations cannot be used to describe in an invariant manner the properties of such black hole configurations.

This paper is organized as follows. In section 2, we present a brief introduction into the formalism of GTD. In section 3, we review the main aspects of a particular spherically symmetric black hole in the EMGB theory with cosmological constant. In section 4, we apply the formalism of GTD with a particular thermodynamic metric to reproduce the thermodynamic properties of the black hole configuration. Finally, in section 5, we discuss our results.

2. Geometrothermodynamics

In equilibrium thermodynamics, to describe a system with \( n \) thermodynamic degrees of freedom, one needs a thermodynamic potential \( \Phi \), a set of \( n \) extensive variables \( \{ E^a \} \) \((a = 1, \ldots, n)\), and the corresponding dual intensive variables \( \{ I^a \} \). Classical thermodynamics is invariant with respect to a change of thermodynamic potential \( \Phi \rightarrow \tilde{\Phi} \) which is defined by means of the Legendre transformations [19]

\[
\{ Z^a \} \rightarrow \{ \tilde{Z}^a \} = \{ \tilde{\Phi}, E^a, I^a \},
\]

\( \Phi = \tilde{\Phi} - \delta_{a b} E^a E^b \), \( E^a = -I^a \), \( E^a = \tilde{E}^a \), \( I^a = \tilde{I}^a \).

(1)

where \( i \cup j \) is any disjoint decomposition of the set of indices \( \{ 1, \ldots, n \} \), and \( k, l = 1, \ldots, i \). In particular, for \( i = \varnothing \) we obtain the identity transformation. Moreover, for \( i = \{ 1, \ldots, n \} \), equation (2) defines a total Legendre transformation, i.e.,

\[
\Phi = \tilde{\Phi} - \delta_{a b} E^a E^b \), \( E^a = -I^a \), \( I^a = \tilde{E}^a \).

(3)

Notice that in order to apply a Legendre transformation to a tensorial object in a particular coordinate system, it is necessary to use the corresponding matrix representation which can be computed by considering all the coordinates as independent, that is, as coordinates of a \( (2n + 1) \)-dimensional space [18]. It then turns out that to investigate the mathematical structure of thermodynamics in general it is necessary to use contact geometry which is constructed as follows. Consider a \( (2n + 1) \)-dimensional differential manifold \( T \) together with its tangent manifold \( T' \). Let \( V \subset T' \) be an arbitrary field of hyperplanes on \( T \). It can be shown that there exists a non-vanishing differential 1-form \( \Theta \) on the cotangent manifold \( T' \) such that \( \mathcal{V} \equiv \ker \Theta \). If the Frobenius integrability condition \( \Theta \wedge d\Theta = 0 \) is satisfied, the hyperplane field \( \mathcal{V} \) is said to be completely integrable. On the contrary, if \( \Theta \wedge d\Theta \neq 0 \), then \( \mathcal{V} \) is non-integrable. In the limiting case \( \Theta \wedge (d\Theta)^n \neq 0 \), the hyperplane field \( \mathcal{V} \) becomes maximally non-integrable and is said to define a contact structure on \( T \). The pair \( (T, \Theta) \) determines a contact manifold [20]. Consider \( G \) as a non-degenerate metric on \( T \). The set \( (T, \Theta, G) \) defines a Riemannian contact manifold.

Let us choose the coordinates of \( T \) as \( Z^a = \{ \Phi, E^a, I^a \} \) with \( A = 0, 1, \ldots, 2n \). According to Darboux theorem, the contact 1-form can be written as

\[
\Theta = d\Phi - \delta_{a b} E^a E^b \), \( \delta_{a b} = \text{diag} \{1, 1, \ldots, 1\},
\]

(4)

where we assume the convention of summation over repeated indices. Under a Legendre transformation, the contact 1-form transforms as \( \Theta \rightarrow \tilde{\Theta} = d\Phi - \delta_a dE^b E^c \). Consequently, the contact manifold \( (T, \Theta) \) is a Legendre invariant structure. If we now impose Legendre invariance on the metric \( G \), the Riemannian contact manifold \( (T, \Theta, G) \) is Legendre invariant. Any Riemannian contact manifold \( (T, \Theta, G) \) which satisfies the condition of Legendre invariance is called a thermodynamic phase manifold and constitutes the starting point for a description of thermodynamic systems in terms of geometric concepts.

Notice that to construct a concrete phase manifold we only need to specify the metric \( G \). It turns out that Legendre invariance does not fix completely the metric \( G \). All the metrics we have found so far can be classified as invariant under total Legendre transformations

\[
G^{\text{II}} = \left( d\Phi - I_a dE^a \right)^2 + L \left( Z^a \right) \left( \xi_{a b} E^a E^b \right) \left( \chi_{a b} dE^a dI^b \right),
\]

(5)

or invariant also under partial Legendre transformations

\[
G^{\text{III}} = \left( d\Phi - I_a dE^a \right)^2 + L \left( Z^a \right) \left( E_f \right)^{2+1} dE^a dI^a
\]

(6)

Here \( L \) is an arbitrary Legendre invariant real function of \( Z^a \) or a constant, and \( \xi_{a b} \) and \( \chi_{a b} \) are diagonal constant matrices that can be expressed in terms of the Euclidean and pseudo-Euclidean metrics \( \delta_{a b} = \text{diag} \{1, 1, \ldots, 1\} \) and \( \eta_{a b} = \text{diag} \{-1, 1, \ldots, 1\} \), respectively. The choice \( \xi_{a b} = \delta_{a b} \) and \( \chi_{a b} = \delta_{a b} \) leads to the metric \( G' \) that has been used to describe systems with first-order phase transitions [18, 21]. The
alternative choice $\xi_{ab} = \delta_{ab} \chi_{ab} = \eta_{ab}$, which is denoted by $G^\mu$, turned out to describe correctly second-order phase transitions especially in black hole thermodynamics [21–23]. Moreover, in the metric $G^\mu$ the additional choice $\varepsilon_{ab} = \frac{1}{2} (\delta_{ab} - \eta_{ab})$ can be used to interpret the thermodynamic limiting case $T \to 0$ as a curvature singularity.

In classical thermodynamics, a particular thermodynamic system is completely determined by its fundamental equation $\Phi = \Phi (E^\nu)$ which determines an n-dimensional surface in $T$. In GTD, we use the same idea to construct the manifold which should describe a particular thermodynamic system. However, some technical details must be considered in order to preserve the Legendre invariance of $T$. To this end, consider a (smooth) embedding map $\varphi : E \to T$, where $E$ is an n-dimensional submanifold of the phase manifold $(T, \Theta, G)$. If we consider the set $\{ E^\nu \}$ as the coordinates of $E$, then the embedding map reads $\varphi : \{ E^\nu \} \mapsto \{ Z^\nu (E^\nu) \} = \{ \Phi (E^\nu), E^\nu, L^\nu (E^\nu) \}$. In this manner, the fundamental equation $\Phi = \Phi (E^\nu)$ appears in a natural way as the result of introducing a well-defined embedding map $\varphi$. Moreover, the metric $G$ induces in $E$ the canonical metric $g$ by means of $g = \varphi^* (G)$, where $\varphi^*$ is the pullback of $\varphi$. The pair $(E, g)$ is called equilibrium manifold if the map $\varphi : E \to T$ satisfies the condition $\varphi^* (\Theta) = \varphi^* (d\Phi - \delta_{ab} L^a dE^b) = 0$, (7) which implies that $d\Phi = I_d dE^\nu$, $\frac{\partial \Phi}{\partial E^\nu} = I_\nu$. (8)

The first of these equations corresponds to the first law of thermodynamics whereas the second one is usually known as the condition for thermodynamic equilibrium [24].

In the case of the metric $G^\mu$ discussed above with $\xi_{ab} = \frac{1}{2} (\delta_{ab} - \eta_{ab})$ and $\chi_{ab} = \eta_{ab}$, the corresponding induced metric of $E$ can be written as

$$g^\mu = \frac{1}{2} \left( E^\nu \frac{\partial \Phi}{\partial E^\nu} - \eta_{\nu\lambda} \frac{\partial \Phi}{\partial E^\lambda} \right) \left( \eta^\nu_{~\lambda} \frac{\partial \Phi}{\partial E^\lambda} dE^\nu \right).$$

where $\eta^\nu_{~\lambda} = \text{diag} (-1, 1, \ldots, 1)$. The geometric properties of the equilibrium manifold $E$ described by the metric $g^\mu$ should be related to the thermodynamic properties of the system described by the fundamental equation $\Phi (E^\nu)$. In GTD, it is conjectured that $E$ is curved for systems with thermodynamic interaction and that curvature singularities in $E$ correspond to phase transitions of the corresponding thermodynamic system.

Notice that the arbitrary function $L$ represents a Legendre invariant conformal factor. In practice, it does not affect the main behavior of the curvature of $g^\mu$ which is the main geometric quantity in GTD. Nevertheless, this function plays an important role if we demand that our results be invariant not only under Legendre transformations, but also with respect to changes of representation. In fact, one can show [25] that a particular choice of $L$ makes the metric $G^\mu$ invariant under changes of representation in general. In the case of the metric $G^\mu$, an additional conformal factor is needed [26].

The second law of thermodynamics can be expressed as

$$\pm \frac{\partial^2 \Phi}{\partial E^\nu \partial E^\nu} \geq 0,$$ (10)

where the sign depends on the chosen thermodynamic potential. For instance, if $\Phi$ coincides with the entropy $S$ the sign is positive (concavity condition), but if $\Phi$ is associated with the internal energy, the sign is negative (convexity condition). It is interesting that the second law can be associated with the orientability of the equilibrium space $E$. Indeed, since $E$ is a Riemannian manifold, the volume form

$$\omega = \sqrt{\text{det} (g^\nu)} \ dE^1 \wedge dE^2 \wedge \cdots \wedge dE^n$$ (11)

is well defined only if $E$ is orientable. On the other hand, the determinant of $g^\nu$ is proportional to $\frac{\partial \Phi}{\partial E^\nu} \frac{\partial \Phi}{\partial E^\nu} \cdots$. If the second derivative $\frac{\partial^2 \Phi}{\partial E^\nu \partial E^\nu}$ happens to change its sign, vanishing at a given point, then the volume form becomes ill-defined and the manifold is not orientable. Moreover, at the same point, the concavity/convexity condition (10) breaks down, indicating a violation of the second law. Thus, we see that a breakdown of the orientability of $E$ with metric $g^\nu$ can be interpreted as a violation of the second law of thermodynamics.

### 3. A spherically symmetric charged black hole

In this section, we study a particular black hole solution which is appropriate to illustrate the ensemble dependence of black hole thermodynamics. The Einstein–Gauss–Bonnet (EGB) theory is the most general theory in five dimensions that leads to second-order differential equations, although the corresponding Lagrangian density contains quadratic powers of the curvature. The most general action of the EGB theory is obtained by adding the Gauss–Bonnet (GB) invariant and the matter Lagrangian $L_{\text{matter}}$ to the Einstein–Hilbert action

$$I = \kappa \int d^5 \sqrt{\hat{g}} \left[ R + \alpha (R^2 - 4R^{ab} R_{ab}) + R^{apq} \left( \frac{\partial \Phi}{\partial E^a} \left( \frac{\partial \Phi}{\partial E^p} \right) \right) + L_{\text{matter}} \right].$$ (12)

where $\kappa$ is related to the Newton constant, and $\alpha$ is the GB coupling constant. In the case of the EMGB theory with cosmological constant, the matter component of the action (12) is given by

$$L_{\text{matter}} = F^{\mu \nu} F_{\mu \nu} - 2 \Lambda , \quad F_{\mu \nu} = A_{\mu, \nu} - A_{\nu, \mu} ,$$ (13)

where $\Lambda$ is the cosmological constant, and $F_{\mu \nu}$ represents the electromagnetic Faraday tensor.
The low energy limit of certain string theories leads to the EMGB theory with cosmological constant; therefore, it is important to study the physical properties of exact solutions like black hole solutions. A particular solution was obtained in [27] (see also [28–30]) by using the following 5D static spherically symmetric line element

\[ ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 \left[ d\theta_1^2 + \sin^2 \theta_1 \left( d\theta_2^2 + \sin^2 \theta_2 d\psi_2^2 \right) \right], \]  

(14)

The coordinate \( r \) has the dimension of length while the angular coordinates \((\theta_1, \theta_2) \in [0, \pi] \) and \( \psi_2 \in [0, 2\pi] \). A 5D spherically symmetric solution in EMGB gravity with \( \Lambda \) was obtained by Wiltshire [31], using the metric ansatz (14) and the metric function

\[ f(r) = 1 + \frac{r^2}{4\alpha} \left[ 1 + \frac{8\alpha M}{r^4} - \frac{8\alpha Q^2}{3r^6} + \frac{4\alpha \Lambda}{3} \right]. \]  

(15)

The two parameters \( M > 0 \) and \( Q \) are identified as the mass and electric charge of the system. It is easy to see that the conditions for the solution (15) to describe a black hole spacetime are \( f(r_h) = 0 \) and

\[ 1 + \frac{8\alpha M}{r_h^4} - \frac{8\alpha Q^2}{3r_h^6} + \frac{4\alpha \Lambda}{3} > 0, \]  

(16)

where \( r_h \) is the radius of the outermost horizon. In this work, we will limit ourselves to the investigation of positive \( \alpha \) and negative definite \( \Lambda \) in order for the mass of the black hole to be always positive.

4. Geomterothermodynamic analysis

To find the fundamental equation of the black hole under consideration, we first note that the surface area of the event horizon is

\[ A = r_h^3 \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \sin^2 \theta \sin \phi d\theta d\phi d\psi = 2\pi^2 r_h^3. \]  

(17)

Moreover, the Bekenstein–Hawking entropy is

\[ S = \frac{\hbar^2}{2\pi \alpha} = \frac{8\alpha r_h^2}{3\pi}, \]  

which becomes \( S = r_h^2 \) by choosing the appropriate constants [32]. Since the black hole condition \( f(r_h) = 0 \) implies that

\[ \frac{\Lambda}{3} r_h^6 - 2r_h^2 + 2(M - 2\alpha) r_h^2 - \frac{2}{3} Q^2 = 0, \]  

(18)

the corresponding thermodynamic fundamental equation in the mass representation becomes

\[ M = 2\alpha + S^{2/3} + \frac{Q^2}{3S^{2/3}} - \frac{\Lambda}{6} S^{4/3}. \]  

(19)

Notice that to guarantee the positiveness of the mass, we must demand that \( \alpha > 0 \) and \( \Lambda < 0 \).

From the energy conservation law for black holes, \( dM = TdS + \phi dQ \), we can derive the expressions for the temperature and electric potential of the black hole on the event horizon as

\[ T = \frac{2}{9} \frac{3S^{4/3} - \Lambda S^{2/3} - Q^2}{S^{5/3}}, \quad \phi = \frac{2}{3} \frac{Q}{S^{2/3}}. \]  

(20)

According to Davies [33], a black hole undergoes a second-order phase transition at those points where the specific heat \( C \) diverges. Strictly speaking, this means that we must introduce the concept of ‘heat’, say \( \Omega_{\text{heat}} \), for a black hole. Of course, this is a problem that can be handled correctly only within the context of a physically meaningful statistical model which is probably the most important unsolved problem in black hole thermodynamics. The simplest available solution of this problem is to use the analogy with classical thermodynamics. Indeed, the first law of thermodynamics \( dM = TdS + \phi dQ \) allows us to define the ‘heat’ through the relationship \( d\Omega_{\text{heat}} \equiv TdS \) so that \( dM = d\Omega_{\text{heat}} + \phi dQ \). Then, following the standard approach of ordinary thermodynamics, we introduce the specific heat as

\[ C_\theta \equiv \left( \frac{d\Omega_{\text{heat}}}{d\theta} \right)_Q \equiv \left( \frac{dM}{d\theta} \right)_Q. \]  

(21)

which in this case is given by

\[ C_\theta = 3S \left( \frac{3S^{2/3} - \Lambda S^{2/3} - Q^2}{5Q^2 - 3S^{2/3} - \Lambda S^{2/3}} \right). \]  

(22)

The temperature (20) is positive only in the range \( 3S^{4/3} - \Lambda S^{2/3} > Q^2 \) and, therefore, the specific heat can take either positive \( \left( 5Q^2 - 3S^{2/3} - \Lambda S^{2/3} > 0 \right) \) or negative \( \left( 5Q^2 - 3S^{2/3} - \Lambda S^{2/3} < 0 \right) \) values, indicating the possibility of stable and unstable states. In the limiting case \( 5Q^2 - 3S^{2/3} - \Lambda S^{2/3} = 0 \), the black hole undergoes a second-order phase transition during which it changes its state of thermodynamic stability.

In classical thermodynamics, once the internal energy of the system is well defined, the analysis at the level of thermodynamic variables can be associated with a particular statistical ensemble. In the case of black holes, however, there is no unique definition of internal energy. For instance, the potential \( M(S, Q) \) has been associated with the micro-canonical ensemble [34] and with the canonical ensemble [35]. In the path-integral method, in which the ensemble is fixed through the boundary conditions, one could also associate this potential with the grand canonical ensemble [36]. Using this last option, we can say that the phase transition structure we have found above for the EMGB black hole can be associated with the grand canonical ensemble. Now we can consider the ensemble dependence already found in black hole thermodynamics. Indeed, since the stability properties of a black hole can drastically change from one ensemble to another, in the thermodynamic limit, in which the change of ensemble corresponds to a change of thermodynamic potential, the same effect can occur.
In the case of the EMGB black hole we are considering here, performing Legendre transformations on $M(S, Q)$ one can derive the potentials
\[ H \equiv M - \phi Q, \quad F \equiv M - TS, \quad G \equiv M - TS - \phi Q, \] (23)
that are known as the enthalpy, the Helmholtz free energy, and the Gibbs free energy, respectively. The enthalpy
\[ H = M - \phi Q = 2\alpha + S^{1/3} - \frac{3}{4}\phi^2 S^{2/3} - \frac{\Lambda}{6} S^{4/3} \] (24)
satisfies the first law of thermodynamics, $dH = TdS - Qd\phi$, and can be associated with the canonical ensemble. Then, if we assume again that the ‘heat’ of the black hole is defined by $dQ_{\text{heat}} = TdS$, the specific heat for fixed $\phi$ is given by
\[ C_\phi \equiv \left( \frac{\partial Q_{\text{heat}}}{\partial T} \right)_\phi = -3S^{1/3} 2\phi S^{2/3} - 9\phi^2 S^{1/3} + 4AS \]
(25)
Taking into account the condition $12S^{1/3} - 9\phi^2 S^{1/3} + 4AS > 0$, which follows from the condition $T > 0$, we find that stable states ($C_\phi > 0$) are allowed for entropies in the range $S^{1/3} > \frac{1}{4\phi}(4 - 3\phi^2)$. Furthermore, for $\phi^2 > 4/3$ and $S > 0$ the last condition is always satisfied, indicating that stable states always exist in this case. Moreover, the specific heat (25) shows that the roots of the equation $12S^{1/3} - 9\phi^2 S^{1/3} + 4AS = 0$ determine the locations where second-order phase transitions occur. Notice that the singularities of $C_\phi$ are different from those of $C_G$; consequently, the corresponding phase transition structures do not coincide. In the limiting case $A \to 0$, the difference is more obvious since the specific heats
\[ C_\phi = 3S \left( \frac{3S^{2/3} - Q^2}{5Q^2 - 3S^{2/3}} \right), \quad C_G = -3S, \] (26)
indicate the existence of phase transitions in the potential $M(S, Q)$ with no transitions at all in the potential $H(S, \phi)$. This shows that the thermodynamic properties of this black hole are not invariant with respect to the partial Legendre transformation that relates the potentials $M(S, Q)$ and $H(S, \phi)$.

Now we turn to the description of the above results within the framework of GTD. As mentioned in section 2, we must choose a metric to derive the geometric properties of the equilibrium manifold. In general, we have three different options, namely, $G^I$, $G^II$ and $G^III$. Since $G^III$ is invariant with respect to partial Legendre transformations, the above result shows that $G^III$ cannot be used to describe the thermodynamics of the EMGB black hole. It follows that the only invariance that can be imposed in the case of black holes is with respect to total Legendre transformations for which we can use the metrics $G^I$ and $G^II$. In our experience, we have seen that $G^II$ correctly describes systems with second-order phase transitions and now we want to see whether it can also correctly handle the dependence on the ensemble.

In the grand canonical ensemble, the fundamental equation $M = M(S, Q)$ is given in equation (19). Then, the coordinates of the equilibrium manifold $E$ are $E^a = (S, Q)$ and the thermodynamic potential is $\Phi = M$. The thermodynamic metric (9) can then be written as (we choose the conformal factor $L = 1$ for simplicity)
\[ g = \frac{4}{27S^{4/3}} \left( 3S^{4/3} + 3S^{2/3} - 2Q^2 - 2\Lambda S^2 \right) + 4Q^2 \]
(27)
A straightforward computation results in the following scalar curvature
\[ R = \frac{27}{2} \left( 3S^{4/3} - Q^2 - 2\Lambda S^2 \right)^2 \]
(28)
with
\[ N_a(S, Q, \Lambda) = 42Q^2S^{2/3}A^2 - 34SQ^2A - 5SQ^2S^2A^2 - 18Q^4S^{2/3} - 7S^2A^3 + 36S^{4/3}A \]
\[ + 15S^{1/3}A^2 - 162S^3 + 108Q^2S^2A^2 \]
(29)
From the expression for the scalar curvature it is obvious that the singularities are located at the points satisfying the equation $3 S^{4/3} - Q^2 - \Lambda S^2 \to 0$, which coincide with the points where $T \to 0$, and at the points satisfying the equation $3 S^{4/3} - 5 Q^2 + \Lambda S^2 \to 0$, which are the points where $C_G \to \infty$. This proves that the curvature scalar correctly reproduces the thermodynamic behavior in the grand canonical ensemble.

It would be interesting to prove explicitly the invariance of the above results with respect to a total Legendre transformation, i.e., when the thermodynamic potential is the Gibbs potential $G(T, \phi) = M - TS - \phi Q$, which is also associated with the grand canonical ensemble [37, 38]. Unfortunately, it is not possible to express $S$ and $Q$ in terms of $T$ and $\phi$ so that the Gibbs potential cannot be written explicitly. However, in the limiting case of a vanishing cosmological constant, it is possible to find explicitly the Gibbs potential and the corresponding thermodynamic metric for which the invariance with respect to the total Legendre transformation can be shown at the level of the scalar curvature.

Let us now consider the canonical ensemble with fundamental equation (24). The enthalpy $H$ is in this case the thermodynamic potential and the coordinates of the equilibrium manifold are $E^a = (S, \phi)$. The corresponding thermodynamic metric is obtained from (9) as (we choose the
conformal factor $L = 1$ for simplicity)

$$g = \frac{S^{4/3}}{12} \left[ (12 - 9\phi^2 - 4AS^{2/3}) \left\{ \frac{1}{81S^3}(12 - 9\phi^2 + 4AS^{2/3}) dS \right\} \right],$$

which leads to the curvature scalar

$$R = \frac{5184}{S^{13/3}} \times \left[ \frac{N_{II}(S, \phi, \Lambda)}{S(12 - 9\phi^2 - 4AS^{2/3})(12 - 9\phi^2 + 4AS^{2/3})^2} \right].$$

with

$$N_{II}(S, \phi, \Lambda) = \left( 12 - 9\phi^2 - 4AS^{2/3} \right)^2 + AS^{2/3} \left\{ 27\phi^2 (4 - 3\phi^2) + 22AS^{2/3} (9\phi^2 - 4AS^{2/3} - 4) \right\}.$$  \hspace{1cm} (32)

The curvature singularities which follow from the limit $12 - 9\phi^2 + 4AS^{2/3} \rightarrow 0$ determine the phase transition structure of the black hole, because they coincide with the divergences of the specific heat $C_\phi$. The second set of singularities for which $12 - 9\phi^2 - 4AS^{2/3} \rightarrow 0$ corresponds to the limit $T \rightarrow 0$ and indicates the breakdown of the thermodynamic description of the black hole. This shows that the thermodynamic behavior of this black hole which follows from the canonical ensemble is correctly reproduced in GTD.

5. Conclusions

In this work, we analyzed the problem of ensemble dependence in the context of GTD. In the Euclidean path-integral approach, it is known that the stability properties of a black hole can depend on the statistical ensemble. In the thermodynamic limit, a change of ensemble can be simply performed as a Legendre transformation that acts on the thermodynamic potential. It then follows that the thermodynamic properties of a black hole can depend on the choice of thermodynamic potential.

On the other hand, the invariance with respect to Legendre transformations plays an essential role in GTD. So far, the metrics used in GTD can be either invariant under total Legendre transformations ($G^{I}$ and $G^{II}$) or invariant also with respect to partial transformations ($G^{III}$). In all the previous applications of GTD in black hole thermodynamics, we found that it is necessary to choose $G^{II}$ to correctly reproduce the phase transition structure of black holes. However, it was not clear at all why such a particular choice was necessary. This was also a source of some criticism against the GTD formalism. In this work, we show for the first time that a particular choice is mandatory because the physical nature of black holes implies certain mathematical conditions that affect Legendre invariance. In fact, the important question is whether the ensemble dependence affects the formalism of GTD. We have shown in this work that indeed GTD can handle correctly this dependence and, moreover, it explains partially why in GTD a particular thermodynamic metric must be chosen in order to correctly reproduce the thermodynamic behavior of black holes. In fact, by analyzing explicitly a particular black hole in the EMGB gravity theory with cosmological constant, we showed that the phase transition structure is not invariant with respect to partial Legendre transformations.

Notice that in this work we use only a particular black hole solution, which was analyzed before in [38], to explicitly show the ensemble dependence. Nevertheless, one could also use any one of the solutions previously investigated in GTD, for instance, black holes in two and three dimensions, the Kerr–Newman black hole and its generalizations in Horava–Lifshitz gravity, higher dimensional black holes and black rings, other Einstein–Gauss–Bonnet black hole configurations, etc. The main result of this work does not depend on the choice of any particular example. In fact, the ensemble dependence is a consequence of the non-additivity of the entropy, a property that characterizes all black hole configurations.

The above results imply that the metric $G^{III}$, which is invariant with respect to partial Legendre transformations, cannot be used to describe black hole thermodynamics. However, the family of metrics invariant under total transformations contains two metrics, $G^{I}$ and $G^{II}$, but only $G^{II}$ can be used for black holes with second-order phase transitions. In practice, the difference between $G^{I}$ and $G^{II}$ is only the signature: the first one is Euclidean and the second one is pseudo-Euclidean. The question is how such a particular distinctness can differentiate between systems with first-order and second-order phase transitions. Preliminary results seem to indicate that this can be used to formulate an invariant definition of phase transitions [39].

Acknowledgments

We would like to thank the members of the GTD-group at UNAM for fruitful comments and discussions. This work was supported by CONACyT-Mexico, grant no. 166391, DGAPA-UNAM and by CNPq-Brazil.

References

[1] Hawking S W 1976 Phys. Rev. D 13 191
[2] Hartle J B and Hawking S W 1976 Phys. Rev. D 13 2188
[3] Gibbons G W and Hawking S W 1977 Phys. Rev. D 15 2752
[4] Hawking S W 1979 General Relativity: An Einstein Centenary Survey ed S W Hawking and W Israel (Cambridge: Cambridge University Press)
[5] York J W 1986 Phys. Rev. D 33 2092
[6] York J W 1989 Physica A 158 425
[7] Whiting B F and York J W 1988 Phys. Rev. Lett. 61 1336
[8] Whiting B F 1990 Class. Quantum Grav 7 15
[9] Brown J D and York J W 2014 The Black Hole: 25 Years After ed C Teitelboim and J Zanelli (New York: Plenum) at press (arXiv:gr-qc/9405024)
[10] Braden H W, Brown J D, Whiting B F and York J W 1990 Phys. Rev. D 42 3376
[11] Brown J D, Comer G L, Martinez E A, Melmed J, Whiting B F and York J W Jr 1990 Class. Quantum Grav. 7 1433
[12] Brown J D, Creighton J and Mann R B 1994 Phys. Rev. D 50 6394
[13] Louko J and Winters-Hilt S N 1996 Phys. Rev. D 54 2647
[14] Lemos J P S 1996 Phys. Rev. D 54 6206
[15] Hawking S W and Page D N 1983 Commun. Math. Phys. 87 577
[16] Page D N 1992 Black Hole Physics ed V D Sabbata and Z Zhang (Dordrecht: Kluwer Academic)
[17] Comer G L 1992 Class. Quantum Grav. 9 947
[18] Quevedo H 2007 J. Math. Phys. 48 013506
[19] Arnold V I 1980 Mathematical Methods of Classical Mechanics (New York: Springer)
[20] Dillen F and Verstraelen L 2006 Handbook of Differential Geometry (Amsterdam: Elsevier)
[21] Quevedo H, Sánchez A, Taj S and Vázquez A 2011 Gen. Rel. Grav. 43 1153
[22] Álvarez J L, Quevedo H and Sánchez A 2008 Phys. Rev. D 77 084004
[23] Vázquez A, Quevedo H and Sánchez A 2010 J. Geom. Phys. 60 1942
[24] Callen H B 1985 Thermodynamics and an Introduction to Thermostatics (New York: Wiley)
[25] Bravetti A, Nettel F, Lopez-Monsalvo F and Quevedo H 2013 The conformal metric structure of geometrothermodynamics J. Math. Phys. 54 033513
[26] Bravetti A, Momeni D, Myrzakulov R and Quevedo H 2013 Gen. Rel. Grav 45 1603
[27] Chakraborty S and Bandyopadhyay T 2008 Class. Quant. Grav. 25 245015
[28] Myers R C and Simon J Z 1988 Phys. Rev. D 38 2434
[29] Cai R G and Guo Q 2004 Phys. Rev. D 69 104025
[30] Nojiri S and Odintsov S 2001 Phys. Lett. B 521 87
[31] Cai R G 2002 Phys. Rev. D 65 084014
[32] Wiltshire D L 1986 Phys. Lett. B 169 36
[33] Wiltshire D L 1988 Phys. Rev. D 38 2445
[34] Falcke H and Hehl F W 2003 The Galactic Black Hole: Lectures on General Relativity and Astrophysics (Bristol: Institute of Physics)
[35] Davies P C W 1978 Rep. Prog. Phys 41 1313
[36] Wei Y H 2009 Phys. Rev. D 80 024029
[37] Myung Y S, Kim Y W and Park Y J 2008 Phys. Lett. B 663 342
[38] Peca C S and Lemos S P S 1999 Phys. Rev. D 59 124007
[39] Caldarelli M, Cognola G and Klem D 2000 Class. Quantum Grav. 17 399
[40] Taj S, Quevedo H and Sánchez A 2012 Gen. Rel. Grav. 44 1489
[41] Quevedo H 2014 An invariant classification of phase transitions, in preparation