Test of the Universal Rise of Hadronic Total Cross Sections at Super-high Energies

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Received: date / Revised version: date

Abstract. Saturation of the Froissart-Martin unitarity bound that the total cross sections increase like \( \log^2(s/s_0) \) appears to be confirmed. Due to this statement, the \( \log^2(s/s_0) \) was assumed to extend the universal rise of all the total hadronic cross sections to reduce the number of adjustable parameters by the COMPETE Collaboration in the Particle Data Group (2006). Based on this assumption of parametrization, we test if the assumption on the universality of the coefficient \( B \) is justified through investigations of the value of \( B \) for \( \pi^\pm p(K^\pm p) \) and \( \bar{p}p, pp \) scatterings. We search for the simultaneous best fit to \( \sigma_{\text{tot}} \) and \( \rho \) ratios, using a constraint from the FESR of the \( P' \) type for \( \pi^\mp p \) scatterings and constraints which are free from unphysical regions for \( \bar{p}p, pp \) and \( K^\pm p \) scatterings. By including rich informations of the low-energy scattering data, the errors of \( B \) parameters decreases especially for \( \pi p \). The resulting value of \( B_{\pi p} \) is consistent with \( B_{\pi p} \) within two standard deviation, which appears to support the universality hypothesis.

PACS. 11.55.Hx Sum rules – 13.85.Lg Total cross sections

Purpose of this Paper

It is well-known that the Froissart-Martin unitarity bound \([1]\) that the increase of total cross sections is at most \( \log^2 \nu \). It had not been possible, however, to discriminate between asymptotic \( \log \nu \) and \( \log^2 \nu \) fits if one uses \( \pi N \) high-energy data alone above 70 GeV. Therefore, we have proposed \([2]\) to use rich informations of \( \pi p \) total cross sections at low- and intermediate-energy regions through the finite-energy sum rules (FESR) of the \( P' \) type \([3]\) as well as \([4,5]\) in addition to total cross sections, and have arrived at the conclusion that \( \log^2 \nu \) behavior is preferred, i.e., the Froissart-Martin bound \([1]\) is saturated. Cudell et al., (COMPETE Collab.) \([6]\) have considered several classes of analytic parametrizations of hadronic scattering amplitudes, and compared their predictions to all available forward data \((pp, \bar{p}p, \pi p, Kp, \gamma p, \gamma\gamma, \Sigma^- p)\). Although these parametrizations were very close for \( \sqrt{s} \geq 9 \) GeV, it turned out that they differ markedly at low energy, where \( \log^2 \) enables one to extend the fit down to \( \sqrt{s} = 4 \) GeV \([4]\).

The statement that the \( \log^2 \nu \) behaviour is preferred have been confirmed in \([7]\) and \([8]\). In Ref. \([6]\), the \( \log^2(s/s_0) \) was assumed to extend the universal rise of all the total hadronic cross sections. This resulted in reducing the number of adjustable parameters. Recently, however, it was pointed out in Ref. \([9]\) that \([7,8]\) gave different predictions for the value of \( B \) for \( \pi N \) and \( NN \), i.e., different predictions at superhigh energies: \( \sigma_{\pi N}^a > \sigma_{NN}^a [7] \) and \( \sigma_{\pi N}^a \sim 2/3 \sigma_{NN}^a [8] \).

The purpose of this article is to investigate the value of \( B \) for \( \pi^\pm p(K^\pm p) \) and \( \bar{p}p, pp \) cases in order to check if the assumption on the universality of the coefficient \( B \) is justified. We search for the simultaneous best fit to \( \sigma_{\text{tot}} \) and \( \rho \) for \( \pi^\pm p \) scatterings and constraints which are free from unphysical regions for \( \bar{p}p, pp \) and \( K^\pm p \) scatterings \([10]\).

Total cross sections, \( \rho \) ratios and constraints

Let us consider the forward \( \bar{p}p, pp, \pi^\pm p \) and \( K^\pm p \) scatterings. We take both the crossing-even and crossing-odd forward scattering amplitudes, \( F^{(+)} \) and \( F^{(-)} \), defined by

\[
F^{(\pm)}(\nu) = \frac{f^{\pm p}(\nu) \pm f^{\bar{\pm}p}(\nu)}{2};
\]

\[
f^{\bar{\pm}p}(\nu) = F^{(+)}(\nu) + F^{(-)}(\nu),
\]

\[
f^{\pm p}(\nu) = F^{(+)}(\nu) - F^{(-)}(\nu),
\]

where \((a, a) = (\bar{p}, p), (\pi^-, \pi^+), (K^-, K^+), \) respectively. We assume

\[
\text{Im} F^{(+)}(\nu) = \text{Im} R(\nu) + \text{Im} F_{\nu'}(\nu) = \frac{\nu}{m^2} (c_0 + c_1 \log \frac{\nu}{m} + c_2 \log^2 \frac{\nu}{m}) + \frac{\beta_{\nu'}}{m} \left( \frac{\nu}{m} \right)^{\alpha_{\nu'}}
\]

\[
\text{Im} F^{(-)}(\nu) = \text{Im} F_{\nu'}(\nu) = \frac{\beta_{\nu'}}{m} \left( \frac{\nu}{m} \right)^{\alpha_{\nu'}}
\]
at high energies for $\nu > N$. Here $m = M$ (proton mass), $m = \mu$ (pion mass), and $m = m_K$ (kaon mass) for $p(p)p$, $p\pi$ and $K\pi$ scatterings, respectively. The $\nu, k$ are the incident $p(p), \pi$ and $K$ energies, momenta in the laboratory system, respectively. Using the crossing-even/odd property, $F^{(\pm)}(-\nu) = \pm F^{(\pm)}(\nu)^*$, the real parts are given by

\[
\text{Re} F^{(\pm)}(\nu) = \frac{\pi \nu}{2m^2} \left( c_1 + 2c_2 \ln \frac{\nu}{m} \right) - \frac{\beta \nu}{m} \left( \nu \right)^{\alpha \nu} \cot \frac{\pi \alpha \nu}{2} + F^{(\pm)}(0) ,
\]

(5)

\[
\text{Re} F^{(-)}(\nu) = \frac{\beta \nu}{m} \left( \frac{\nu}{m} \right)^{\alpha \nu} \tan \frac{\pi \alpha \nu}{2} .
\]

(6)

The total cross sections $\sigma_{ap}^{\pi p}$ and $\rho_{ap}^{\pi p}$ are given by

\[
\text{Im} f^{\pi p, \rho p}(\nu) = \frac{\nu}{4\pi} \text{Im} f^{\pi p} - \text{Re} f^{\pi p} = \frac{\nu}{4\pi} \text{Im} f^{\rho p} - \text{Re} f^{\rho p} .
\]

respectively.

The total cross sections $\sigma_{tot}^{\alpha}$, $\sigma_{tot}^{ap}$, and the $\rho$ ratios $\rho_{ap}^{\pi}$, $\rho_{ap}^{\rho}$ are given by

\[
\text{Im} f^{\alpha} = \frac{k}{4\pi} \sigma_{tot}^{\alpha} , \quad \rho_{ap}^{\pi} = \frac{\text{Re} f^{\pi p}}{\text{Im} f^{\pi p}} , \quad \rho_{ap}^{\rho} = \frac{\text{Re} f^{\rho p}}{\text{Im} f^{\rho p}} .
\]

The Eq. (8) gives directly a constraint for $p\pi$ scattering,

\[
\frac{2}{\pi} \int_{0}^{N} \frac{\nu}{k^2} \{ \text{Im} R(\nu) + \text{Im} F_{\nu}^{\pi p}(\nu) \} \, d\nu - 2 \frac{P}{\pi} \int_{0}^{N} \frac{\nu}{k^2} \{ \text{Im} R(\nu) + \text{Im} F_{\nu}^{\rho p}(\nu) \} \, d\nu + \frac{1}{\pi} \int_{0}^{N} \sigma_{tot}^{(+)}(k) \, dk .
\]

(9)

For $p\pi$ scattering, RHS can be estimated with sufficient accuracy, regarding Eq. (9) as an exact constraint. [2] Re$F^{(\pm)}(\nu)$ is represented by scattering lengths and pole term comes only from nucleon. The last term is estimated from the rich data of experimental $\sigma_{tot}^{ap}$.

On the other hand, Eq. (8) for $\bar{p}(p)p$ scattering suffers from the unphysical regions coming from boson poles below the $\bar{p}p$ threshold. Reliable estimates, however, are difficult. Similarly in $K\pi$ scattering, poles of $\Lambda, \Sigma$ resonant states contribute below $K^-\pi$ threshold. In ref. [10], we have presented a new constraint, called FESR(1)(N1-N2), free from unphysical regions. We consider Eq. (9) with $N = N_1$ and $N = N_2$ ($N_2 > N_1$). Taking the difference between these two relations, we obtain the relation

\[
\frac{2}{\pi} \int_{0}^{N_2} \frac{\nu}{k^2} \{ \text{Im} R(\nu) + \text{Im} F_{\nu}^{\pi p}(\nu) \} \, d\nu = \frac{1}{2\pi} \int_{0}^{N_2} \sigma_{tot}^{(+)}(k) \, dk .
\]

(10)

The RHS can be estimated from the experimental $\sigma_{tot}^{\bar{p}p,pp}$ and $\sigma_{tot}^{K^-p}$ data regarding Eq. (10) as an exact constraint.

The general approach

The formula, Eqs. (1)-(7), and the constraints, Eqs. (9) and (10), are our starting points. The $\sigma_{tot}^{ap}$ and $\rho_{ap}^{\pi,\rho}$ are fitted simultaneously for respective processes of $p(p)p$, $p\pi$, $K\pi$ scatterings. The high-energy parameters $c_2$, $c_3$, $\alpha$, $\beta$, $\beta_{\nu}$ are treated as process-dependent, while $\alpha_{\nu}$ and $\alpha_{V}$ are fixed with common values for every process. The FESR(1) (N1-N2) (Eq. (10)) and FESR(1)(0-N) (Eq. (9)) give constraints between $c_2$, $c_3$ and $\beta_{\nu}$ for $\bar{p}(p)p$, $K\pi$ and $p\pi$ scatterings, respectively. $F^{(+)}(0)$ is treated as an additional parameter, and the number of fitting parameters is 5 for each process. The resulting $c_2$ are related to the $B$ parameters, defined by $\log \chi^2(s/s_0) + \cdots$, through the equation

\[
B_{ap} = \frac{4\pi}{m_{\pi}} c_2 ,
\]

where $m = M, \mu, m_K$ for $a = p, \pi, K$, (11) and we can test the universality of $B$ parameters for the relevant processes.

Result of the analyses

The $\sigma_{tot}^{ap}$ for $k \geq 20$ GeV and $\rho_{ap}^{\pi,\rho}$ for $k \geq 5$ GeV are fitted simultaneously. We take two cases, ($\alpha_{\nu}, \alpha_{V}$) = (0.500, 0.497) and (0.542, 0.496). These values are selected by considering the $\chi^2$ behaviours of the fit to $\bar{p}p, pp$ data: The total $\chi^2$ is almost independent of the input value of $\alpha_{\nu}$, while it is sensitive to the value of $\alpha_{V}$. We select two values of $\alpha_{\nu}$ as typical examples while $\alpha_{V}$ is selected from the minima of $\chi^2$. The $\chi^2$ takes its minimum at $\alpha_{V} \sim 0.50$ independently of $\alpha_{\nu}$-value.

The FESR(1)(10-20GeV)(Eq. (10)) for $\bar{p}(p)p$, $K\pi$ and FESR(1)(0-20GeV)(Eq. (11)) for $p\pi$ are given respectively by

1.837(2.061)$/3_{\nu} + 7.247c_4 + 19.96c_1 + 55.27c_2 = 58.54 (12)

4.810(5.542)$/3_{\nu} + 26.14c_4 + 88.74c_1 + 302.3c_2 = 25.41 (13)

109.2(124.1)$/3_{\nu} + 653.6c_3 + 2591c_1 + 10928c_2 = 71.12 (14)

Practically it is estimated from the fit to $\sigma_{tot}$ in 2.5GeV$< k < 100$GeV through phenomenological formula. See ref. [10] for detail.

3. In the actual analysis we fit data of $Re f$ instead of $f$. $Re f$ data are made from the original $f$ data multiplied by $\sigma_{tot}$, which is given by the fit in ref. [9].

4. Our $\alpha_{\nu}$ corresponds to $1 - \eta_1$ in parametrization of COM-PETE collab.[1]. $\alpha_{\nu} = 0.542$ corresponds to their best fit value, $\eta_1 = 0.468$.
Table 1. Values of parameters and $\chi^2$ in the best fits in $(\alpha_{p'}, \alpha_V) = (0.500, 0.497)$. Both total $\chi^2$ and respective $\chi^2$ for each data with the number of data points are given. The result of $p(p)p$ scattering is in the 1st row, $p\pi$ scattering in the 2nd row and $Kp$ scattering in the 3rd row. The errors are given only for $c_2$. The values of $\beta_{p'}$ are obtained from FESR.

| process | $B$ | $\alpha_{p'} = 0.500$ | $\alpha_{p'} = 0.542$ |
|---------|-----|---------------------|---------------------|
| $pp, pp$ | $B_{pp}$ | $0.289 \pm 0.023$ | $0.368 \pm 0.024$ |
| $\pi^+ p$ | $B_{p\pi}$ | $0.351 \pm 0.036$ | $0.333 \pm 0.039$ |
| $K^+ p$ | $B_{Kp}$ | $0.37 \pm 0.21$ | $0.37 \pm 0.22$ |

Table 2. Values of $B$ parameters in unit of mb. The results are given in two cases $\alpha_{p'} = 0.500, 0.542$.

| process | $B$ | $\alpha_{p'} = 0.500$ | $\alpha_{p'} = 0.542$ |
|---------|-----|---------------------|---------------------|
| $pp, pp$ | $B_{pp}$ | $0.0520 \pm 0.0041$ | $-0.259$ |
| $\pi^+ p$ | $B_{p\pi}$ | $0.0185 \pm 0.0013$ | $-0.142$ |
| $K^+ p$ | $B_{Kp}$ | $0.0185 \pm 0.0013$ | $-0.142$ |

By using Eq. (11), we can derive the $B$ parameters from $c_2$ in Table 1. The result is given in Table 2 in two cases $\alpha_{p'} = 0.500, 0.542$. As seen in Table 2, $B_{pp}$ is somewhat smaller than the $B_{p\pi}$, but is consistent within two standard deviation, although its central value changes slightly depending upon the choice of $\alpha_{p'}$. Central value of $B_{Kp}$ is consistent with $B_{p\pi}$, although its error is very large, due to the present situation of $Kp$ data. Based on these results, present experimental data are consistent with the hypothesis of the universal rise of the total cross section in super-high energies. On the other hand, $\sigma_{\pi N}^{\text{tot}} \sim 2/3 \sigma_{\pi N}^{\text{tot}}$ appears not to be favoured in our analysis. This is our main result.

Remarks on the analysis of $p\pi$

In order to obtain the above conclusion, it is essential to determine $c_2$ in $p\pi$ (or $B_{p\pi}$) with enough accuracy. However, it is very difficult task, since the experimental $\sigma_{\text{tot}}^{\pi p}$ are reported only in very limited regions with momenta $k < 400$ GeV, in contrast with the $\alpha_{\text{tot}}^{p\pi}$ data obtained up to $k = 1.7266 \cdot 10^5$ GeV. Actually, if we fit the same data in the fit of Table 1 using 6 (not 5) parameters with no use of the FESR, Eq. (14), we obtain

$c_2 = (120 \pm 46) \cdot 10^{-5} \rightarrow B_{p\pi} = 0.301 \pm 0.116 \text{ mb}, (15)$

where $(\alpha_{p'}, \alpha_V) = (0.500, 0.497)$. The above value is consistent with the one given in Table 2. $B_{p\pi} = 0.351 \pm 0.036 \text{ mb}$, within its large error. However, this error is very large, and the $B_{p\pi}$ in Eq. (15) is consistent with both $B_{pp} = 0.289 \text{ mb}$ and $2/3 B_{pp} = 0.193 \text{ mb}$. So by using this value we cannot obtain any definite conclusion. In other words, by including the rich informations of the low-energy $p\pi$ scattering data through FESR, the error of $B_{p\pi}$ is reduced to be less than one third($0.116 \text{ mb} \rightarrow 0.036 \text{ mb}$), and as a result, the universality of $B$ ($B_{pp} = B_{p\pi}$) appears to be preferred.

In our analysis of Table 1 $\sigma_{\pi p}^{\text{tot}}$ in $k \geq 2N$ and $\rho_{\pi p}$ in $k \geq 4.95$ GeV were fitted simultaneously, using FESR(1)($0 - 2N$) with $N_2 = 20$ GeV. When we analyze the data by taking $N_2 = 25(30)$ GeV, the $B_{p\pi}$ are determined as $0.315 \pm 0.052(0.303 \pm 0.060) \text{ mb}$. The results are not so sensitive to the choice of $N_2$, although their errors become slightly larger. If we use the FESR(1)($10-20$ GeV) not ($0-20$ GeV) also for $p\pi$, similarly to $p\pi pp$ and $Kp$ and fit the same data, we obtain $B_{p\pi} = 0.314 \pm 0.075 \text{ mb}$, which is consistent with our result given in Table 2, but its error becomes about twice the larger of the value in Table 2. In order
Fig. 1. Results of the fits to (a) $\sigma^{\bar{p}p}_{\text{tot}}$, (b) $\rho^{\bar{p}p}$, (c) $\sigma^{\pi^-p}_{\text{tot}}$, (d) $\rho^{\pi^-p}$, (e) $\sigma^{K^-p}_{\text{tot}}$, (f) $\rho^{K^-p}$, In (b), red(orange) points Fajardo 80[11](Bellettini 65[12]) of $\rho^{pp}$. In (d), red(green) points are Apokin[13](Burq 78[14]) and blue points are the others in $\rho^{\pi^-p}$.
to obtain sufficiently small error of $B_{\pi}$ it appears to be important to include the informations of low-energy scattering data with $0 \leq k \leq 10\text{GeV}$ through FESR.

Finally we would like to add several remarks:

(i) Our $B_{pp}$, $B_{pp} = 0.289 \pm 0.023 \text{mb}$ (in case $\alpha_{PP} = 0.500$), is consistent with the value of $B$ by COMPETE collab.\cite{2817}, $0.308 \pm 0.010 \text{mb}$, which is obtained by assuming the universality of $B$ for various processes.

(ii) Our $B_{pp}$ is also consistent with the value by Block and Halzen\cite{2792}, $0.2817 \pm 0.0064 \text{mb}$ or $0.2792 \pm 0.0059 \text{mb}$ (from the $c_0$ parameter in Table III of ref.\cite{2792}). A present value of our $B_{pp}$ is located between the above two results.

(iii) The universality of $B$ parameter has some theoretical basis from QCD\cite{2817}.

(iv) Our predictions of $\sigma_{tot}^{pp}$ and $\rho^{pp}$ at LHC energy($\sqrt{s}=14\text{TeV}$) are

$$
\sigma_{tot}^{pp} = 109.5 \pm 2.8 \text{mb}, \quad \rho^{pp} = 0.133 \pm 0.004 \quad (16)
$$

This value is consistent with our previous one, $\sigma_{tot} = 107.1 \pm 2.6 \text{ mb}$, $\rho = 0.127 \pm 0.004\text{[10]}$, which was obtained through the analysis based on only the crossing-even amplitude, using restricted data sets. The values of Eq. (16) is also located between predictions of the relevant two groups, $\sigma_{tot}^{pp} = 111.5 \pm 1.2_{\text{syst}}^{+4.1}_{-2.4_{\text{stat}}} \text{mb}$, $\rho^{pp} = 0.1361 \pm 0.0015_{\text{syst}}^{+0.0058}_{-0.0025_{\text{stat}}}\text{[16]}$ and $\sigma_{tot}^{pp} = 107.3 \pm 1.2 \text{mb}$, $\rho^{pp} = 0.132 \pm 0.001\text[8]{}$.

(v) The fit to the $\pi p$ data given in Table 1 gives the prediction at $k = 610\text{GeV}$, $\sigma_{tot}^{\pi p} = 25.91 \pm 0.03 \text{mb}$ (in case of $\alpha_{\pi^+} = 0.500$)\footnote{At this energy, $\sigma_{tot}^{\pi^+p}$ is predicted with $25.62 \pm 0.03 \text{mb}$. The difference $\sigma_{tot}^{\pi^+p} - \sigma_{tot}^{\pi^-p} \simeq 0.3 \text{mb}$.} which is consistent with the recent observation by SELEX collaboration, $\sigma_{tot}^{\pi^-N} = 26.6 \pm 0.9 \text{mb.}\footnote{\text{[17]}}$

(vi) Finally we would like to emphasize the importance of precise measurements of $p$ ratios in $p\bar{p}$, $pp$, $\pi^+p$, $K^-p$ scatterings at intermediate energies above $k \geq 5\text{GeV}$ for further investigations of $B$ parameters.

Acknowledgements This work was (in part) supported by JSPS and French Ministry of Foreign Affairs under the Japan-France Integrated Action Program (SAKURA).

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