Magnetised black hole as a gravitational lens

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We use the Ernst-Schwarzschild solution for a black hole immersed in a uniform magnetic field to estimate corrections to the bending angle and time delay due to presence of weak magnetic fields in galaxies and between galaxies, and also due to influence of strong magnetic field near supermassive black holes. The magnetic field creates a kind of confinement in space, that leads to increasing of the bending angle and time delay for a ray of light propagating in the equatorial plane.

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I. 1. INTRODUCTION.

One of the most intriguing observable test of general relativity is deflection of light exerting gravitational attraction. This effect is feasible for observation nowadays in many scales: deflection induced by compact object, such as stars or black holes, by galaxies, cluster of galaxies, and even by the large scale structure of the universe [1]. At the same time, an important factor which influences the dynamic of the interacting matter in the Universe is existing of magnetic fields: from weak fields ($\sim 5 \mu G$) in galaxies, cluster of galaxies ($\sim 6 \mu G$) and between clusters of galaxies ($\sim 50 \mu G$), until very strong magnetic fields near supermassive black holes. This fields should influence not only propagation of charged matter, but also of rays of light at large distances or in the regime of strong gravitational field, when the magnetic field is one of the parameters of the space-time metric.

In this letter, we shall study the corrections to the deflection angle and time delay for a ray of light in vicinity of black holes immersed in a magnetic field. The only known exact solution of the Einstein-Maxwell equations describing the non-rotating black hole immersed in a magnetic field is the Ernst solution [2]. Thermodynamical and geometrical properties of this solution were investigated in [3]. Some interesting generalisations of the solution were found in [4], and different effects near magnetised black hole were studied in [5].

We shall use Ernst solution as a model for our estimations of the corrections. The problem is solvable in the equatorial plane where the equations of motion allow separation of variables. The main feature we observed is that the magnetic field, even being small as a space-time metric parameter, i.e. small enough not to deform the geometry of the compact objects, leads to considerable increasing of deflection angle and time delay when distance from an observer to a source is large enough. The strong magnetic field localised near black holes also induces increasing of deflection angle and time delay.

II. NULL GEODESICS AROUND ERNST-SCHWARZSCHILD BLACK HOLES

The exact solution of the Einstein-Maxwell equations, represented by the Ernst generalisation for a Schwarzschild metric [2], describes a non-rotating black hole immersed in an asymptotically homogeneous external magnetic field, and the corresponding metric has the form: [2]:

$$ds^2 = \Lambda^2 \left( 1 - \frac{2M}{r} \right) dt^2 - \left( 1 - \frac{2M}{r} \right)^{-1} dr^2 - r^2 d\Omega^2$$ (1)

where the external magnetic field is determined by the real parameter $B$, and

$$\Lambda = 1 + B^2 r^2 \sin^2 \theta.$$ (2)

The vector potential for the magnetic field is given by the formula:

$$A_\mu dx^\mu = -\frac{Br^2 \sin^2 \theta}{2\Lambda} d\phi.$$ (3)

As a magnetic field is assumed to exist everywhere in space, the above metric is not asymptotically flat. The event horizon is again $r_h = 2M$, and the surface gravity at the event horizon is the same as that for a Schwarzschild metric, namely

$$\chi = 2\pi T_H = \frac{1}{4M}.$$ (4)

This leads to the same thermodynamic properties [1] as for the case of Schwarzschild black hole.

To analyse the lens parameters of the above black hole, we need to consider the null geodesic equations for Ernst-Schwarzschild space-time [8]. For this, let us introduce the momenta:

$$p_\mu = g_{\mu\nu} \frac{dx^\nu}{ds}.$$ (5)
where $s$ is an invariant affine parameter. The Hamiltonian is

$$H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu. \quad (6)$$

The action can be represented in the form:

$$S = Et - L\phi - S_r(\mathbf{r}) - S_\theta(\theta), \quad (7)$$

where $E$ and $L$ are the particle's energy and angular momentum respectively.

Then, the Hamilton-Jacoby equations for null geodesics read

$$\frac{1}{2} g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} = - \frac{\partial S}{\partial s} = 0. \quad (8)$$

It is evident that the equations of motions allow separation of variables in the equatorial plane $\theta = \pi/2$ [3]. After making use of the equation $p^1 = -p_0/g_{00}(dr/dt)$ and equation (8), we easily find the propagation equation

$$\left(\frac{dr}{dt}\right)^2 = -\frac{g_{00}}{g_{11}} \left(1 + \frac{g_{33}(p_3/p_0)^2}{g_{00}}\right), \quad (9)$$

where $p_0 = E$, and $p_3 = -L$ are constants of motion, and $g_{ik}$ are metric coefficients given by formulas (1) and (2) at $\theta = \pi/2$. In a similar fashion we find the geodesic trajectory equation

$$\left(\frac{dr}{d\phi}\right)^2 = -\frac{g_{33}}{g_{11}} \left(1 + \frac{g_{33}(p_3/p_0)^2}{g_{00}}\right). \quad (10)$$

We see that propagation and trajectory equations contain only ratio $b = L/E$, which is called the impact parameter. The qualitative description of the motion can be made by considering the effective potential of the motion:

$$U_{eff} = \pm \frac{g_{00}L^2}{g_{33}}. \quad (11)$$

The solution of the equations $U_{eff} = \partial U_{eff}/\partial r = 0$ gives the radius the closed circular geodesic orbits. The essential feature of this potential for Ernst-Schwarzschild case is that for non-zero $b$, the escape of the massless particles in the equatorial plane is impossible. This is stipulated by existence of an additional turning point (where $dr/dt = 0$). For small magnetic fields, we will be interested here, this additional turning point occurs at a sufficiently large $r$.

III. BENDING ANGLE AND TIME DELAY FOR ERNST-SCHWARZSCHILD BLACK HOLES

Passing near black hole ray of light approaches the black hole at some minimal distance from it, called distance of closest approach $r_{min}$. In Schwarzschild case, $r_{min}$ is determined as the largest real root of the equation $dr/dt = 0$. Yet, in presence of magnetic fields, $r_{min}$ is not the largest root anymore: In the regime of small $B$, the largest root corresponds to another turning point very far from black hole. Thus, $dr/dt = 0$ gives

$$r^3 + (2M - r)(1 + B^2 r^2) + b^2 = 0 \quad (12)$$

If we know $r_{min}$ with great accuracy, we can perform integrations for finding bending angle:

$$\alpha = \phi_s - \phi_o = - \int_{r_s}^{r_{min}} \frac{d\phi}{dr} dr + \int_{r_{min}}^{r_o} \frac{d\phi}{dr} dr - \pi. \quad (13)$$

Here $r_o$ is radial coordinate of an observer and $r_s$ is radial coordinate of the source.

In a similar fashion one can find the time delay, which is the difference between the light travel time for the actual ray, and the travel time for the ray the light would have taken in the Minkowskian space-time:

$$t_s - t_o = - \int_{r_s}^{r_{min}} \frac{dt}{dr} dr + \int_{r_{min}}^{r_o} \frac{dt}{dr} dr - \frac{d\phi}{cos B}. \quad (14)$$

Here the term $\frac{d\phi}{cos B}$ represents the propagation time for a ray of light, if the black hole is absent (see for instance [10]).

We shall also use the following designation: the difference in time delay between Schwarzschild and Ernst-Schwarzschild space-times

$$\delta = (t_s - t_o)E - S - (t_s - t_o)S. \quad (15)$$

Equations (9) and (10) can be re-written in the following form:

$$\left(\frac{dt}{dr}\right) = \frac{(1 - \frac{2M}{r})^{-1}}{\sqrt{1 + (\frac{2M - r}{1 + B^2 r^2})^{\frac{1}{2}}}} \quad (16)$$

$$\left(\frac{d\phi}{dr}\right) = \frac{(1 + B^2 r^2)^{\frac{1}{2}}}{\sqrt{(2M - r)(1 + B^2 r^2)^{\frac{1}{2}} + \frac{r^2}{4}}} \quad (17)$$

Now, using (16), (17) in integrals (13) and (14), we are in position to compute the bending angles and time delays for different values of the impact parameter $b$ and magnetic field $B$. First, the parameter $B$ of the external magnetic field in (1-2), and magnetic field in units of Gauss are connected by the following relation:

$$B = 4.25 \cdot 10^{-21} \frac{B_0}{M_\odot}, \quad (18)$$

where $B_0$ is the the magnetic field in units of Gauss. As $B$ has units of the inverse length, sometimes natural units are used $B = 1/r_h$, were $r_h = 2M$. A super-massive black hole in the centre of our galaxy has the mass $M = (3.6 \pm 0.2) \times 10^9 M_\odot$ (what corresponds to $M \approx 3.4 \cdot 10^{-7}$ pc, in units $c = G = 1$), and is situated at a distance $r_o =$
(7.9±0.4)×10^3 pc from the Earth. The "unit" magnetic field \( B = 1/(2M) \) for the above super-massive black hole is \( B_0 = 3.26810^{13}G \), according to the formula (18).

Typical distance from a source to a black hole is of order \( r_o = 1 \) pc. Now let us take \( M = 1 \), then we can re-scale the corresponding values for \( r_o \) and \( r_s \)

\[
r_o = 2.3235 \cdot 10^{10}, \quad r_s = 2.9412 \cdot 10^6, \quad M = 1. \quad (19)
\]

For Ernst model to be valid we need a small enough value of \( B \), so that the distant turning point, responsible for "reflecting" of a massless particle from infinity, would be at least a few orders larger than the largest of \( r_o \) and \( r_s \). Thereby we imply that region of asymptotic behaviour of the magnetic field is far enough, not to create confinement of light rays. Indeed, for instance for \( B \sim 10^{-10}, \) that turning point occur at \( r = r_{\text{turn}} \sim 10^{14} \), i.e. four orders greater than \( r_o \). In fact the applicability of the Ernst solution to real situations depends on two factors: the value of the magnetic field \( B \), and the distance at which the Ernst space-time is considered: if \( B \) is large or even of order \( 1/M \) this strong magnetic field everywhere in space creates large potential barrier around black hole. This induces strong confinement even at a distance not far from black hole. This situation should be remedied by matching the homogeneous magnetic field with decreasing magnetic field at some distance from black hole. At the same time, near the event horizon Ernst solution should be adequate. When \( B \) is much less than \( 1/M \), we can apply Ernst model even for sufficiently large distances, when confining properties of the effective potential show themselves at even larger scale of distances. When \( B \rightarrow 0 \), we are approaching the Schwarzschild limit and can consider larger and larger distances within Ernst solution as a model for magnetic field in our Universe.

First, let us consider weak magnetic fields for supermassive black holes. The results of numerical computation of the integrals (13) and (14) are given in tables I and II. We integrated expressions (13) and (14) directly without changing variable using built-in functions in Mathematica. The integration up to very large values of limits of integrals is stable, provided one controls the precision of all incoming data and intermediate procedures. To check this we can integrate, for instance, (13) for \( r_o = r_s = 10^{15} \) (\( M = 1 \)), and some large \( b \): For Schwarzschild limit the results show excellent agreement with formula (24) of [10] in PPN-expansion approach. Thus, for \( b = 20 \) from (13) we get, \( \alpha + \pi = 3.777285 \), while (24) in [10] gives \( \alpha + \pi = 3.777144 \). This also shows that weak field limit expanded in higher order in \( M/b \) is indeed very good approximation.

Now let us try to estimate the effect of the strong magnetic field in the central part of the galactic black hole. For this one cannot integrate equations (13), (14) up to some large distances, because as is well-known, the magnetic field near black hole decreases quickly as the distance is increased. Nevertheless we are able to consider the bending of light ray and delay in propagation time near black hole, implying that far from black hole the magnetic field influences the situation according to the estimation scheme described before. For this we put an "observer" and a "source" not far from black hole: on illustrative example in Table III \( r_o = r_s = 20 \) (\( M = 1 \)).

Let us come back in our estimation from geometrical to ordinary units. We have used Ernst solution for estimation of the influence of the magnetic fields in two regimes: weak galactic magnetic field existing everywhere in the galaxy and strong magnetic field in the region close to the supermassive black hole.

Weak magnetic field is supposed to influence the propagation of light during the whole way from a source to an observer, and therefore is expected to make considerable correction to bending angle and time delay. From table I we see that the increasing in bending angle due to magnetic field for a case of supermassive black hole can be considered as observable at the present time only if the magnetic field would be at least \( B \sim 10^{-10} \) or greater, i.e. about \( 10^2G \). This is much larger than the real galactic magnetic field, which is \( \sim 10\mu G \). The time delay is affected by weak magnetic field non-negligibly, also only for \( B_0 \sim 10^2G \) or larger. Yet, even seeming corrections to time delay and bending angles for \( B_0 \sim 10^2G \) is quite surprising, because such a field is too weak to deform the geometry of the Schwarzschild black hole: the space-time near Ernst black hole with \( B_0 \sim 10^2G \) is, in fact, a Schwarzschild space-time with very high accuracy. The magnetic field should be much larger, about \( \sim 1/M \) in geometric units, to create considerable effect near black hole. Therefore we can conclude that the correction effect at \( B_0 \sim 10^2G \) is due-to the existence of the non-vanishing magnetic field everywhere in the way of a ray of light, and not because of peculiarities of black hole geometry near the event horizon.

For the intergalactic magnetic field \( \sim 6\mu G \) (what is \( \sim 10^{20} \), when \( M = 1 \) and \( G = c = 1 \)), the distant turning point is situated at \( r_{\text{turn}} = 3.18 \cdot 10^{18} \)pc. Therefore we can expect considerable correction to the bending angle and time delay only if \( r_o \) is approaching by order to the radius of the distant turning point, i.e. at least about \( 10^{16} - 10^{17} \)pc, what is much greater than the size of our cluster of galaxies (\( \sim 1.5 \cdot 10^7 \)pc).

On the contrary, strong magnetic field in the centre of galaxy decay quickly with distance and does not make influence far from black hole, yet it increases seemingly the lens parameters already at \( B \sim 10^{-5} \) (\( \sim 10^7 - 10^8G \)).

The Ernst model we used here is certainly a rough approximation to a real situation, mainly because magnetic field is not uniform, but has rather complicated distribution in large scale space-time. Also, it would be difficult to get a source and an observer exactly in the equatorial plane. Therefore our estimations are more
Table I: Bending angle \( \alpha + \pi \) and value of the "far" turning point \( r_{\text{tun}} \), evaluated for different values of the impact parameter \( b \) and small magnetic field \( B \) for the super-massive black hole in the centre of our galaxy, \( r_o = 2.3235 \cdot 10^{10}, r_s = 2.9412 \cdot 10^6 \). (Geometrical units are used, \( M = 1 \)).

| \( B \) | \( \alpha + \pi (b = 6) \) | \( r_{\text{tun}} \cdot 10^{12} \) | \( \alpha + \pi (b = 10) \) | \( r_{\text{tun}} \cdot 10^{12} \) | \( \alpha + \pi (b = 20) \) | \( r_{\text{tun}} \cdot 10^{12} \) |
|-------|-----------------|---------------|-----------------|---------------|-----------------|---------------|
| \( 0 \cdot 10^{-10} \) | 4.8609788632 | \(-\) | 3.7319849720 | \(-\) | 3.37772180250 | \(-\) |
| \( 1 \cdot 10^{-10} \) | 4.8609789477 | 11.8563 | 3.7319851128 | 9.99999 | 3.37772208405 | 7.93700 |
| \( 2 \cdot 10^{-10} \) | 4.8609892297 | 4.70518 | 3.7320022496 | 3.96850 | 3.37775635750 | 3.14980 |
| \( 3 \cdot 10^{-10} \) | 4.8612094483 | 2.74023 | 3.733692805 | 2.31120 | 3.37849042386 | 1.83440 |
| \( 4 \cdot 10^{-10} \) | 4.8631698141 | 1.86725 | 3.735635568 | 1.57490 | 3.38502497645 | 1.25000 |
| \( 4.5 \cdot 10^{-10} \) | 4.8665248390 | 1.59588 | 3.741228265 | 1.34601 | 3.39620839297 | 0.92832 |
| \( 5 \cdot 10^{-10} \) | 4.8737382850 | 1.38672 | 3.753250674 | 1.16961 | 3.42025321276 | 0.92832 |
| \( 5.5 \cdot 10^{-10} \) | 4.8881343060 | 1.22123 | 3.777244084 | 1.03003 | 3.46824003166 | 0.81753 |
| \( 6 \cdot 10^{-10} \) | 4.9151533653 | 1.08746 | 3.822275808 | 0.91720 | 3.55830348461 | 0.72798 |

Figure 1: Radius of minimal approach \( r_{\text{min}} \) as a function of the impact magnetic field strength \( B \) for the impact parameters \( b = 6 \), \( b = 7 \), \( b = 20 \), and \( b = 40 \).

Figure 2: (Left) Radius of minimal approach \( r_{\text{min}} \) as a function of the impact parameter \( b \) for \( B = 10^{-6} \). (Right). Radius of minimal approach \( r_{\text{min}} \) as a function of the impact parameter \( b \) for \( B = 10^{-6} \) (bottom), \( B = 5 \cdot 10^{-3} \), and \( B = 10^{-2} \) (top). We see that for relatively small \( B \), \( r_{\text{min}} \) does not change seemingly when changing \( B \), unless the impact parameter \( b \) is large enough.
representing the illustrative idea that the magnetic fields create some confinement which increase the bending angle and time delay, and are not expected to give some high precision results. We hope, nevertheless, that the influence of magnetic field upon black hole lensing might be observable in the future.

IV. CONCLUSION

We study lensing in the Ernst-Schwarzschild spacetime, to estimate lens effects of a black hole immersed in a magnetic field. We showed that already small magnetic field $B \sim 10^{-10}$ (in geometric units), which does not distort the geometry of the black hole near the event horizon, nevertheless increases seemingly the bending angle and time delay for a ray of light propagating in vicinity of a supermassive black hole. This happens because the magnetic field is supposed to exist even far from black hole, thereby exerting considerable metric influence on a ray of light during the whole way from a source to an observer. Yet, this magnetic field is many order less than the real galactic magnetic field $B_g \sim 10^{-18}$. On the contrary, estimations in the region close to a black hole, within the same Ernst-Schwarzschild model, show that strong magnetic field in the central region near a black hole should give rise to non-negligible increase in the bending angle and time delay.

As is known, large-scale magnetic field in Universe has both poloidal and toroidal components. Yet, it is poloidal component, which is dominant. Therefore Ernst solution, where the magnetic field stipulates a single direction in space is reasonable approximation. Another point is, that very strong magnetic field in the centre of a black hole near the event horizon, thereby exerting considerable metric influence on a ray of light during the whole way from a source to an observer. Yet, this magnetic field is many order less than the real galactic magnetic field $B_g \sim 10^{-18}$. On the contrary, estimations in the region close to a black hole, within the same Ernst-Schwarzschild model, show that strong magnetic field in the central region near a black hole should give rise to non-negligible increase in the bending angle and time delay.

The magnetic field in the central region of the galactic black hole: $r_o = r_s = 20$, $b = 6$.

Table II: Difference in time delay $\delta$ between Schwarzschild and Ernst-Schwarzschild space-times (in geometrical units, $M = 1$) for small magnetic field $B$ for the supermassive black hole in the centre of our galaxy, $r_o = 2.3235 \times 10^{10}$, $r_s = 2.9412 \times 10^8$, $b = 6$.

| $B$          | $\delta$ ($b = 6$) |
|--------------|-------------------|
| $1 \times 10^{-10}$ | 0                 |
| $2 \times 10^{-10}$ | 0.000031          |
| $3 \times 10^{-10}$ | 0.006990          |
| $4 \times 10^{-10}$ | 0.005673          |
| $5 \times 10^{-10}$ | 0.038277          |
| $6 \times 10^{-10}$ | 0.162521          |
| $7 \times 10^{-10}$ | 0.553574          |

Table III: Bending angle $\alpha$ and propagation time $\tau$ for Ernst-Schwarzschild space-times (in geometrical units, $M = 1$) for $b = 6$. “Observer” and “source” are supposed to be situated not far from the black hole in order to estimate influence of magnetic field in the central region of the galactic black hole: $r_o = r_s = 20$, $b = 6$.

| $B$          | $\alpha + \pi$ ($b = 6$) | $t_s - t_o$ + $d_{s-o}$/$\cos B$ ($b = 6$) |
|--------------|--------------------------|------------------------------------------|
| $0$          | 4.25233372               | 56.8455337725                           |
| $10^{-6}$    | 4.25233383               | 56.845537425                           |
| $10^{-5}$    | 4.25234010               | 56.845538048                           |
| $10^{-4}$    | 4.25234254               | 56.845559125                           |
| $10^{-3}$    | 4.25320549               | 56.847693341                           |
| $2 \times 10^{-3}$ | 4.25582382               | 56.854196524                           |
| $3 \times 10^{-3}$ | 4.26019947               | 56.865066714                           |
| $4 \times 10^{-3}$ | 4.26635039               | 56.880373524                           |
| $5 \times 10^{-3}$ | 4.27430179               | 56.900206814                           |

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