CLASSIFICATION OF SUPERSOLUTIONS AND LIOUVILLE THEOREMS FOR SOME NONLINEAR ELLIPTIC PROBLEMS

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Abstract. In this paper we consider positive supersolutions of the elliptic equation 

\[-\Delta u = f(u)|\nabla u|^q\]

posed in exterior domains of \(\mathbb{R}^N\) \((N \geq 2)\), where \(f\) is continuous in \([0, +\infty)\) and positive in \((0, +\infty)\) and \(q > 0\). We classify supersolutions \(u\) into four types depending on the function \(m(R) = \inf_{|x| = R} u(x)\) for large \(R\), and give necessary and sufficient conditions in order to have supersolutions of each of these types. As a consequence, we also obtain Liouville theorems for supersolutions depending on the values of \(N, q\) and on some integrability properties on \(f\) at zero or infinity. We also describe these questions when the equation is posed in the whole \(\mathbb{R}^N\).

1. Introduction and results. The purpose of the present paper is to analyze the existence and nonexistence of supersolutions of the elliptic problem

\[-\Delta u = f(u)|\nabla u|^q\]

in \(\mathbb{R}^N \setminus B_{R_0}\), \((1)\)

where \(N \geq 2, q > 0\) and \(B_{R_0}\) stands for the ball with radius \(R_0\) centered at zero. The nonlinearity \(f\) is a continuous function defined in \([0, +\infty)\) and positive in \((0, +\infty)\).

In particular, we are interested in obtaining Liouville type theorems for (1).

Nonlinear Liouville theorems go back to the pioneering reference [25], where the model problem

\[-\Delta u = f(u)\]

\(\text{in } \mathbb{R}^N\)

\((2)\)

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with \( f(t) = t^p, \ p > 1 \) was considered. Later, some other works have dealt with the same problem, either considering alternative proofs to that in [25] (see [16]) or obtaining similar results for more general nonlinearities (cf. [32]) and operators (see [35] for the context of the \( p \)-Laplacian).

On the other hand, the obtention of Liouville theorems has also been extended to deal with supersolutions of (2) instead of solutions, or with the same problem when posed in exterior domains of \( \mathbb{R}^N \), rather than in the whole \( \mathbb{R}^N \) (cf. [24], [10], [4]). Numerous works have dealt with the question of nonexistence of supersolutions with some more general nonlinearities and operators. Without being exhaustive with the references, we mention [5], [6], [7], [8], [9], [11], [12], [15], [18], [19], [20], [28], [30] and [33] (and references therein). We also refer to the survey [29] for a more complete list.

A natural question related to (2) is to know how the nonexistence results are modified when the equation is perturbed in some way, for instance with the introduction of a gradient term. Of course, this can be done in multiple ways, but a problem that has been somewhat studied is

\[-\Delta u + |\nabla u|^q = f(u) \quad \text{in} \ \mathbb{R}^N, \]

where \( q > 0 \). This problem has been considered in the reference case \( f(u) = u^p, \ p > 0 \), in [17] and later in [34] and [37] (cf. also the extension to the \( p \)-Laplacian setting considered in [22], [23]). In most of these works the study was restricted to radial solutions, but this restriction was later dropped in the works [2], [3] and [1], where the case \( q \geq 1 \) was analyzed, obtaining nonexistence results for general positive supersolutions (see also [36] for a survey on the problem).

Our intention in this paper is to analyze (2) when it is perturbed with a gradient term in a different way. Thus we will restrict to the study of problem (1) in the rest of the paper. We will perform a complete analysis of the positive supersolutions of (1), inspired by the results in [4]. Let us mention that this precise problem has been studied in [13] for some more general operators when \( 0 < q < 1 \), but only the case \( f(t) = t^p \) is dealt with there. Also, in [21], the more general problem

\[-\text{div}(h(x)g(u)A(|\nabla u|)\nabla u) \geq f(x, u, \nabla u) \quad \text{in} \ \mathbb{R}^N \]

was analyzed, but the nonlinearity \( f(x, u, \nabla u) \) is essentially like \( u^p|\nabla u|^q \), so that again only the power case is known.

Before stating precise results, let us clarify that we will always be dealing with continuous weak supersolutions, that is, functions \( u \in H^1_{\text{loc}}(\mathbb{R}^N \setminus B_{R_0}) \cap C(\mathbb{R}^N \setminus B_{R_0}) \) verifying

\[
\int_{\mathbb{R}^N \setminus B_{R_0}} \nabla u \nabla \phi \geq \int_{\mathbb{R}^N \setminus B_{R_0}} f(u)|\nabla u|^q \phi
\]

for every nonnegative \( \phi \in C_0^\infty(\mathbb{R}^N \setminus B_{R_0}) \).

Our analysis of positive supersolutions \( u \) of problem (1) is based upon the fact that the function

\[ m(R) = \inf_{|x|=R} u(x), \]

which is positive for \( R > R_0 \), is monotone for large \( R \) (see Lemma 2.1 in Section 2). This allows to classify all possible supersolutions into four types:

Type 1: \( m(R) \) is decreasing and \( \lim_{R \to +\infty} m(R) = 0; \)

Type 2: \( m(R) \) is increasing and \( \lim_{R \to +\infty} m(R) = \ell \) for some \( \ell > 0; \)
Type 3: \( m(R) \) is decreasing and \( \lim_{R \to +\infty} m(R) = \ell \) for some \( \ell > 0 \);
Type 4: \( m(R) \) is increasing and \( \lim_{R \to +\infty} m(R) = +\infty \).

In order to simplify the exposition, we will assume throughout the paper that whenever a positive supersolution \( u \) of (1) exist, the radius \( R_0 \) has been chosen so that \( m(R) \) is monotone for \( R > R_0 \).

We are intentionally excluding from our classification all constant solutions of (1). Also, it is important to observe that the singular nature of the problem allows the existence of supersolutions which are not constant, but are eventually constant in the sense that they are constant for \( |x| > R_1 \) for some \( R_1 > R_0 \), at least when \( 0 < q < 1 \) (see Remark 1 in Section 2). Thus these supersolutions are also excluded from our classification. Let us mention in passing that this phenomenon does not seem to be possible for \( q \geq 1 \).

It is perhaps interesting to note that the presence of supersolutions of types 2 and 3 is in contrast with problems (2) and (3), where only supersolutions of types 1 and 4 are possible. This is related to the fact that all constants are indeed solutions of (1).

The existence of each type of supersolution depends first of all on the dimension \( N \). The cases \( N \geq 3 \) and \( N = 2 \) –as in multiple well-known situations– are genuinely different, as can be seen for instance from the fact that supersolutions of type 4 are never possible when \( N \geq 3 \), while for \( N = 2 \), the only types which can arise are 2 and 4.

Let us begin with the case of higher dimensions \( N \geq 3 \). Since \( f \) is assumed to be positive in \((0, +\infty)\), the existence of supersolutions of types 2 and 3 does not really depend on \( f \), and it is only related to the relative values of \( q \) and \( N \). However, when considering supersolutions of type 1, the relevant condition is

\[
\int_0^\delta \frac{f(t)}{t^\theta} \, dt < +\infty,
\]

for some \( \delta > 0 \), where

\[
\theta = \frac{(2 - q)(N - 1)}{N - 2}.
\]

This condition resembles the one found in [4] for the case \( q = 0 \) (and actually reduces to that one in this particular case).

Our results for problem (1) in the case \( N \geq 3 \) can be summarized as follows.

**Theorem 1.1.** Assume \( N \geq 3 \) and \( f \in C([0, +\infty)) \) is positive in \((0, +\infty)\). Then:

(a) If \( q > \frac{N}{N - 1} \), there exist positive supersolutions of (1) of types 1, 2 and 3.

(b) When \( \frac{N}{N - 1} \geq q \geq 1 \), no supersolutions of (1) of type 3 exist, while there always exist supersolutions of type 2. Supersolutions of type 1 exist if and only if (4) holds.

(c) For \( 0 < q < 1 \), there never exist supersolutions of (1) of types 2 and 3, while supersolutions of type 1 exist if and only if (4) holds.

Moreover, positive supersolutions of type 4 never exist in this case.

As a consequence of the statements above, we have a Liouville theorem for (1).

**Corollary 1** (Liouville theorem). Assume \( N \geq 3 \) and \( f \in C([0, +\infty)) \) is positive in \((0, +\infty)\). If \( q < 1 \) and (4) does not hold, then every positive supersolution of problem (1) is eventually constant.
As for the case \( N = 2 \), our results are:

**Theorem 1.2.** Assume \( N = 2 \) and \( f \in C([0, +\infty)) \) is positive in \((0, +\infty)\). Then:

(a) If \( q \geq 2 \), there exist positive supersolutions of (1) of types 2 and 4.

(b) When \( 2 > q \geq 1 \), there always exist supersolutions of type 2. Supersolutions of type 4 exist if and only if there exist \( a, M > 0 \) such that

\[
\int_M^\infty e^{at} f(t) \, dt < +\infty.
\]

(c) For \( 0 < q < 1 \), there never exist supersolutions of (1) of type 2, while supersolutions of type 4 exist if and only if (6) holds.

Moreover, positive supersolutions of types 1 and 3 never exist in this case.

**Corollary 2** (Another Liouville theorem). Assume \( N = 2 \) and let \( f \in C([0, +\infty)) \) is positive in \((0, +\infty)\). If \( q < 1 \) and (6) does not hold, then every positive supersolution of problem (1) is eventually constant.

Of particular interest in (1) is the special case where \( f \) is a power, \( f(t) = t^p \), \( p > 0 \), that is,

\[
-\Delta u = u^p |\nabla u|^q \quad \text{in} \quad \mathbb{R}^N \setminus B_{R_0},
\]

(7)

The above theorems directly apply to this case to obtain for instance: when \( N \geq 3 \), if \( 0 < q < 1 \) and \( 0 < p \leq \frac{N-q(N-1)}{N-2} \), then every positive supersolution of (7) is eventually constant; for \( N = 2 \), the nonexistence of not eventually constant positive supersolutions holds if \( 0 < q < 2 \). Both results are sharp, and coincide with those in [21] when the equation is considered in \( \mathbb{R}^N \) (for \( N \geq 3 \)).

Observe that all the nonexistence results stated above apply equally to the equation in (1) when it is posed in the whole \( \mathbb{R}^N \), namely

\[
-\Delta u = f(u) |\nabla u|^q \quad \text{in} \quad \mathbb{R}^N,
\]

(8)

where \( N \geq 3 \) (it is well known that the only positive, superharmonic functions in \( \mathbb{R}^2 \) are constants, so that the case \( N = 2 \) is uninteresting). However, it is to be stressed that the maximum principle implies \( m(R) = \inf_{|x| \leq R} u(x) \) for all positive supersolutions, so that the function \( m(R) \) is always strictly decreasing, unless \( u \) is constant. Therefore, only supersolutions of types 1 and 3 are possible, and it also follows that eventually constant supersolutions which are not constant do not exist. On the other hand, if \( u \) is a positive supersolution of (1) of one of these types, then it is easily checked that the function \( \tilde{u} = \min\{u, m(R_1)\} \) is a weak \( H^1_{\text{loc}} \) supersolution of (8) when \( R_1 \) is large enough. Therefore,

**Theorem 1.3.** Assume \( N \geq 3 \) and \( f \in C([0, +\infty)) \) is positive in \((0, +\infty)\). Then:

(a) If \( q > \frac{N}{N-1} \), there exist positive supersolutions of (8) of types 1 and 3.

(b) When \( 0 < q \leq \frac{N}{N-1} \), no supersolutions of (8) of type 3 exist, while supersolutions of type 1 exist if and only if (4) holds.

The corresponding Liouville theorem is:

**Corollary 3** (Liouville Theorem in \( \mathbb{R}^N \)). Assume that \( N \geq 3 \) and \( f \in C([0, +\infty)) \) is positive in \((0, +\infty)\). If \( 0 < q \leq \frac{N}{N-1} \) and (4) does not hold, then the only positive supersolutions of problem (8) are constants.

To conclude the discussion of our results, let us mention that the very interesting question of existence and nonexistence of positive solutions of (8) seems to be very
delicate for general functions $f$. However, as a consequence of the uniqueness theo-
rem for ode’s, the only radially symmetric, positive solutions of (8) are constants
if $f$ is locally Lipschitz in $(0, +\infty)$ and $q \geq 1$. The case $0 < q < 1$ seems more
difficult to deal with, even with radial symmetry. We refer the reader to [27] for an
Emden-Fowler analysis in the particular case $f(t) = t^p$, $p > 0$.

Finally, let us say a word about our methods of proof. As far as the nonexistence
goes, the idea in higher dimensions ($N \geq 3$) is similar as the one in [4]. It is possible
to show that if there exists a positive supersolution $u$ of types 1, 2 or 3, then there
exists a positive, radially symmetric solution $v$ which is of the same type as $u$. Thus
the problem is reduced to a radial one, and a further change of variables transforms
it into a one-dimensional, singular, sort-of initial value problem. The analysis of
the latter is then performed along similar lines as in [4]. As for the case $N = 2$, the
previous reduction is not possible. Therefore, we only perform it in annulus whose
outer radius goes to infinity, which involves some extra estimates for the obtained
radially symmetric solutions.

With regard to the existence results, they are obtained either by explicit con-
bstruction in some cases, or by employing Schauder’s fixed point theorem for an
integral operator in a suitable space of functions.

The rest of the paper is organized as follows: in Section 2 we gather some pre-
liminaries, which have to do with the classification of positive supersolutions of (1)
and with the reduction to the radial setting. Sections 3 and 4 are dedicated to the
cases $N \geq 3$ and $N = 2$, respectively.

2. Preliminaries. In this section we consider several questions related to the clas-
sification of positive supersolutions of (1) and the reduction to the radial setting.
We begin with an extension of a result in [4] (see also [2]), which gives sense to
the classification introduced in the previous section. Remember that, for a positive
supersolution $u$ of (1) we always denote

$$ m(R) = \inf_{|x|=R} u(x). $$

**Lemma 2.1.** Let $u \in H^1_{\text{loc}}(\mathbb{R}^N \setminus B_{R_0}) \cap C(\mathbb{R}^N \setminus B_{R_0})$ be a nonnegative function verifying $-\Delta u \geq 0$ in $\mathbb{R}^N \setminus B_{R_0}$ in the weak sense. Then, there exists $R_1 > R_0$
such that the function $m(R)$ is either strictly increasing, or strictly decreasing or
constant in $(R_1, +\infty)$.

**Proof.** According to Lemma 3 in [4], there exists $R_0' \geq R_0$ such that $m(R)$ is
monotone for $R > R_0'$. As we have observed in the introduction, we can always
assume $R_0' = R_0$.

Suppose first that $m$ is nondecreasing. Let us show that if there exists an interval
$[R_1, R_2]$ where $m$ is constant, then $m$ is constant for $R > R_1$. Indeed, assume on
the contrary there exists $R_3 > R_2$ such that $m(R_3) > m(R_1)$. For any $R \in [R_1, R_2]$
the function $m(R)$ attains an interior minimum in the annulus $A(R, R_3) = \{ x \in
\mathbb{R}^N : R < |x| < R_3 \}$. Thus by the maximum principle $u$ has to be constant
in $A(R, R_3)$, contradicting that $m(R_3) > m(R_1)$. We deduce that either $m$ is constant
for $R > R_0$ or $m$ is strictly increasing in $(R_0, +\infty)$.

A similar argument shows that, if $m$ is nonincreasing and it is constant in an
interval $[R_1, R_2]$, then $m = m(R_1)$ if $R \in (R_0, R_1)$. Assume such an interval exists
(if not $m$ is strictly decreasing and we are done) and $m$ is not constant for $R > R_2$
Choose $R_2$ with the property that the interval $[R_1, R_2]$ is the maximal interval
where $m$ is constant. Then $m$ is decreasing for $R > R_2$. If not, there would exist
an interval $[R_3, R_4]$ with $R_3 > R_2$ and $m(R) = m(R_3) < m(R_2)$ if $R \in [R_3, R_4]$. But this would imply $m$ is constant for $R < R_4$, a contradiction. Therefore either $m$ is strictly decreasing for $R > R_2$ or $m$ is constant for $R > R_2$. This concludes the proof. 

**Remark 1.** As mentioned in the introduction, if $q < 1$ there always exist positive nonconstant supersolutions of (1) which are constant for large $|x|$. Indeed, fix $\lambda > 0$ and denote $M = \sup_{[0,\lambda]} f(t)$. Consider the function $u = \lambda - A(R_0 + \delta - |x|)^{\alpha}$, where $\alpha = (2-q)/(1-q) > 1$. It will be a supersolution in the interval $(R_0, R_0 + \delta)$ provided that

$$\alpha - 1 - (N-1)\frac{R_0 + \delta - r}{r} \geq M\alpha^{q-1}A^{q-1}$$

where $r = |x|$, as usual. Since $(R_0 + \delta - r)/r \leq \delta/R_0$, the previous inequality can always be achieved if $\delta$ is chosen small enough and then $A$ is large enough. Finally, the function

$$v(x) = \begin{cases} u(x) & R_0 < |x| < R_0 + \delta \\ \lambda & |x| \geq R_0 + \delta \end{cases}$$

will be a positive $C^1$ supersolution of (1).

Our next step is to make a little more precise the classification of supersolutions. Depending on the dimension $N$, only some types of supersolutions are possible. This is the content of the following lemma.

**Lemma 2.2.** Assume $q > 0$ and $f \in C([0, +\infty))$ is nonnegative. When $N = 2$, there do not exist positive supersolutions of (1) of types 1 and 3, while for $N \geq 3$, no supersolutions of type 4 exist.

**Proof.** Assume $u$ is a positive, not eventually constant supersolution of (1) and consider initially $N \geq 3$. Choose $R_2 > R_1 > R_0$ and define the function

$$\Phi(x) = \frac{m(R_1) - m(R_2)}{R_1^{2-N} - R_2^{2-N}}(|x|^{2-N} - R_2^{2-N}) + m(R_2)$$

for $x \in A(R_1, R_2) := \{x \in \mathbb{R}^N : R_1 < |x| < R_2\}$. Observe that $\Phi$ is harmonic in $A(R_1, R_2)$, with $u \geq \Phi$ on $\partial A(R_1, R_2)$. Therefore, by the maximum principle it follows that $u \geq \Phi$ in $A(R_1, R_2)$. In particular, for every $R \in (R_1, R_2)$,

$$m(R) \geq \frac{m(R_1) - m(R_2)}{R_1^{2-N} - R_2^{2-N}}(R_2^{2-N} - R_1^{2-N}) + m(R_2)$$

$$= m(R_2) \left(\frac{R_2^{2-N} - R_1^{2-N}}{R_1^{2-N} - R_2^{2-N}}\right) + m(R_1) \left(\frac{R_1^{2-N} - R_2^{2-N}}{R_1^{2-N} - R_2^{2-N}}\right).$$

Now assume that $u$ is of type 4, that is, $\lim_{R \to +\infty} m(R) = +\infty$. Letting $R_2 \to +\infty$ in (10) with fixed $R$ and $R_1$ we arrive at a contradiction.

Next we analyze the case $N = 2$. We consider the function which is obtained by replacing in the definition of $\Phi$ the power $2-N$ by a logarithm, that is

$$\Psi(x) = \frac{m(R_1) - m(R_2)}{\log R_1 - \log R_2}(|x| - \log R_2) + m(R_2)$$

(11)
for \( x \in A(R_1, R_2) \). As before, \( u \geq \Psi \) in \( A(R_1, R_2) \), so that
\[
m(R) \geq \frac{m(R_1) - m(R_2)}{\log R_1 - \log R_2} (\log R - \log R_2) + m(R_2)
\]

\[
= m(R_2) \left( \frac{\log R_1 - \log R}{\log R_1 - \log R_2} \right) + m(R_1) \left( \frac{\log R - \log R_2}{\log R_1 - \log R_2} \right)
\]

(12)

for \( R \in (R_1, R_2) \). Assume \( u \) is not of type 4, so that \( \lim_{R \to +\infty} m(R) = \ell \in [0, +\infty) \). We can let \( R_2 \to +\infty \) in (12) to obtain \( m(R) \geq m(R_1) \). Since \( R \) and \( R_1 \) are arbitrary, it follows that \( m \) is nondecreasing. Therefore \( u \) has to be of type 2. The proof is concluded.

Finally, we show that when \( N \geq 3 \), and for a suitable range of \( q \), it is always possible to reduce problem (1) to a radial setting.

**Lemma 2.3.** Assume \( N \geq 3 \), \( 0 < q \leq 2 \) and \( f \in C([0, +\infty)) \) is positive in \( (0, +\infty) \). If there exists a positive supersolution \( u \) of (1), then there exists \( R_1 > R_0 \) and a positive, radially symmetric solution \( v \) of (1) in \( \mathbb{R}^N \setminus B_{R_1} \), which is of the same type as \( u \).

**Proof.** For \( R_2 > R_1 > R_0 \), consider the function \( \Phi \) given by (9) in \( A(R_1, R_2) \). To stress the dependence of \( \Phi \) on \( R_2 \) we will temporarily denote it by \( \Phi_{R_2} \). As before, \( u \geq \Phi_{R_2} \) in \( A(R_1, R_2) \). Now we analyze the problem

\[
\begin{cases}
-\Delta v = f(v)|\nabla v|^q & \text{in } A(R_1, R_2) \\
v = \Phi_{R_2} & \text{on } \partial A(R_1, R_2).
\end{cases}
\]

(13)

It is clear that \( u \) is a supersolution of (13) while \( \Phi_{R_2} \) is a subsolution. Therefore, by the method of sub and supersolutions there exists a minimal solution \( v_{R_2} \) of (13), which is radially symmetric (see for instance the Appendix in [4]; only the case \( q = 0 \) is dealt with there, but the proofs are easily adapted to cover the full range \( q \in (0, 2] \) with the aid of Theorem 2.1 in [14] and the standard estimates in Section 4.3 of [31]).

Now we would like to pass to the limit as \( R_2 \to +\infty \). We have the inequalities \( 0 < v_{R_2} \leq u \), so this gives local bounds for the set \( \{ v_{R_2} \}_{R_2 > R_1} \). Since \( q \leq 2 \), we can also obtain bounds for the gradient of the solutions using the results in Section 4.3 of [31]. Then it is standard to get local \( C^{1,\alpha} \) bounds (cf. [26]), so that by means of a diagonal procedure we obtain a sequence \( R_{2,n} \to +\infty \) such that \( v_{R_{2,n}} \to v \) in \( C^1_{\text{loc}}(\mathbb{R}^N \setminus B_{R_1}) \). In particular, we obtain that \( v \) is a radially symmetric weak solution of the equation \(-\Delta v = f(v)|\nabla v|^q \) in \( \mathbb{R}^N \setminus B_{R_1} \), verifying also \( v = m(R_1) \) on \( \partial B_{R_1} \). Since \( \Phi_{R_2} \leq v_{R_2} \leq u \), letting \( R_2 \to +\infty \) and employing the radial symmetry of \( v \) we have
\[
\frac{m(R_1) - \ell}{R_1^{2-N}} + \ell \leq v(r) \leq m(r), \quad r > R_1,
\]

(14)

where \( \ell = \lim_{R \to +\infty} m(R) \), which is finite by Lemma 2.2. Now reaching the desired conclusion is easy: if \( u \) is of type 1, then \( \ell = 0 \), so that \( v(r) \to 0 \) as well, therefore \( v \) is decreasing for large \( r \) and it is of type 1. If \( u \) is of type 2 then \( \ell > 0 \) and \( v(r) < m(r) < \ell \) for \( r > R_1 \), and we obtain \( \lim_{r \to +\infty} v(r) = \ell \) by (14). Hence \( v \) is of type 2. Finally, when \( u \) is of type 3 we have \( m(R_1) > \ell \), so that again by (14) \( v > \ell \) and \( \lim_{r \to +\infty} v(r) = \ell \); thus \( v \) is of type 3.

**Remark 2.** When \( N = 2 \) and there exists a positive supersolution of (1), it is equally possible to obtain a positive, radially symmetric solution \( v \) as in Lemma
2.3. Unfortunately, it is not possible to show with the same methods that $v$ is of the same type as $u$. Actually, it can be shown in some cases that $v$ is not!

3. The case $N \geq 3$. The purpose of the present section is to prove Theorem 1.1. For the sake of clarity, we split the proof into a series of lemmas, dealing in turn with each type of supersolutions. Of course we will assume throughout the section that $N \geq 3$ and $f \in C([0, +\infty))$ is positive in $(0, +\infty)$.

We begin by considering the case $q \geq 2$.

**Lemma 3.1.** Assume $q \geq 2$. Then there exist positive, radially symmetric solutions of (1) of types 1, 2 and 3.

**Proof.** Since we are looking for radially symmetric solutions, we assume $u(x) = v(r)$, $r = |x|$, so that we need to solve the equation

$$-v'' - \frac{N-1}{r}v' = f(v)|v'|^q \quad \text{for large } r. \quad (15)$$

With the change of variables $s = r^{2-N}/(N-2)$, $v(r) = w(s)$, the previous equation gets transformed into

$$-w'' = cs^\nu f(w)|w'|^q \quad \text{for small } s,$$

for some $c = c(q,N) > 0$, where $\nu = -\theta \geq 0$. For $\lambda > 0, \mu \in \mathbb{R}$ or $\lambda = 0, \mu > 0$, we consider the Cauchy problem:

$$\begin{cases} -w'' = cs^\nu f(w)|w'|^q \quad s > 0, \\
w(0) = \lambda \\
w'(0) = \mu. \end{cases} \quad (17)$$

Since $f$ is continuous and $\nu \geq 0$, it follows by Cauchy-Peano’s theorem that there exists at least a local solution $w$ of this problem, defined in an interval $[0, s_0]$ for some small positive $s_0$. This solution is in addition positive if $s_0$ is small enough. Thus there exists a positive, radially symmetric solution $u$ of (1) in the complement of a ball. Finally, observe that $u$ is of type 1 when $\lambda = 0, \mu > 0$, of type 2 when $\lambda > 0, \mu < 0$ and of type 3 if $\lambda > 0, \mu > 0$. The proof is concluded.

We next turn to the subquadratic case $0 < q < 2$, and consider supersolutions of type 1. The proof of our next lemma is an adaptation of that of Theorem 6 in [4].

**Lemma 3.2.** Assume $0 < q < 2$. Then there exist positive supersolutions of (1) of type 1 if and only if

$$\int_0^{s_0} \frac{f(t)}{t^\theta} \, dt < +\infty \quad (16)$$

for some $\delta > 0$, where

$$\theta = \frac{(2-q)(N-1)}{N-2}.$$

**Proof.** Assume first that (16) does not hold, and there exists a positive supersolution of type 1 of (1). By Lemma 2.3 there exists $R_1 > R_0$ and a positive, radially symmetric solution $v$ of (1) in $\mathbb{R}^N \setminus B_{R_1}$ of type 1. We make the change of variables $s = r^{2-N}/(N-2)$, $w(s) = v(r)$ in the ordinary differential equation satisfied by $v$.

Then the function $w$ is nondecreasing and verifies, for some $c > 0, s_0 > 0$,

$$\begin{cases} -w'' = cs^{-\theta} f(w)(w')^q \quad s \in (0, s_0) \\
w(0) = 0. \end{cases} \quad (17)$$
We claim that \( w' > 0 \) in \((0, s_0)\). Indeed, if \( w'(s_1) = 0 \) for some \( s_1 \in (0, s_0) \), then since \( w'' \leq 0 \), we would obtain \( w'(s) = 0 \) for \( s \in (s_1, s_0) \), so that \( w \) would be constant in \((s_1, s_0)\), against our assumptions. Thus \( w' > 0 \).

Also, the mean value theorem gives \( w(s) = w'(\xi)s \geq w'(s)s \), where \( \xi \) is some point in the interval \((0, s)\). Hence

\[
0 < w'(s) \leq \frac{w(s)}{s} \quad \text{in} \quad (0, s_0).
\]

(18)

The monotonicity of \( w' \) implies that \( w(s) \geq C_0s \) for some \( C_0 > 0 \) and every \( s \in (0, s_0) \). We divide the equation in (17) by \((w')^{q-1}\) and integrate in \((s, s_0)\) to arrive at

\[
w'(s)^{2-q} \geq (2-q)c \int_s^{s_0} \frac{f(w(t))}{w(t)^\theta} w'(t)dt
\]

for every \( s \in (0, s_0) \). Hence, using (18):

\[
\left( \frac{w(s)}{s} \right)^{2-q} \geq (2-q)c \int_s^{s_0} \frac{f(w(t))}{w(t)^\theta} \left( \frac{w(t)}{t} \right)^\theta w'(t)dt
\]

(19)

for \( s \in (0, s_0) \). Taking into account that \( w(s) \geq C_0s \), (19) implies

\[
\left( \frac{w(s)}{s} \right)^{2-q} \geq (2-q)cC_0^{q-\theta} \int_s^{s_0} \frac{f(w(t))}{w(t)^\theta} w'(t)dt = (2-q)cC_0^{q-\theta} \int_{w(s)}^{w(s_0)} \frac{f(\tau)}{\tau^\theta} d\tau
\]

We deduce, since \( w(0) = 0 \) and (16) does not hold, that \( \lim_{s \to 0} w(s)/s = +\infty \). Thus, we can diminish \( s_0 \) to ensure that \( w(s) \geq s \) in \((0, s_0)\).

We will reach a contradiction by iterating the use of (19). The inequality \( w(s) \geq s \) in \((0, s_0)\) gives:

\[
\left( \frac{w(s)}{s} \right)^{2-q} \geq (2-q)c \int_{w(s)}^{w(s_0)} \frac{f(\tau)}{\tau^\theta} d\tau =: H(w(s)) \quad \text{in} \quad (0, s_0).
\]

Taking this inequality again in (19) we have the improved inequality

\[
\left( \frac{w(s)}{s} \right)^{2-q} \geq (2-q)c \int_{w(s)}^{w(s_0)} \frac{f(w(t))}{w(t)^\theta} H(w(t))^{\frac{\theta}{2-q}} w'(t)dt
\]

\[
= (2-q)c \int_{w(s)}^{w(s_0)} \frac{f(w(t))}{w(t)^\theta} H(w(t))^{\frac{\theta}{2-q}} d\tau
\]

\[
= - \int_{w(s)}^{w(s_0)} H'(\tau) H(\tau)^{\frac{2-q}{\theta}} d\tau = \frac{H(w(s))^{\frac{2-q}{\theta}}}{\frac{2-q}{\theta} + 1},
\]

where we have used \( H(w(s_0)) = 0 \). It is possible to iterate this procedure to obtain two sequences \( \{a_k\}_{k=1}^\infty \) and \( \{b_k\}_{k=1}^\infty \) given by \( a_1 = 1 \), \( a_k = \left( \frac{\theta}{2-q} \right)^{2-q} a_{k-1} + 1 \), \( b_1 = 1 \), \( b_k = b_{k-1}^{\frac{\theta}{2-q}} a_k \) such that

\[
\left( \frac{w(s)}{s} \right)^{2-q} \geq \frac{H(w(s))^{a_k}}{b_k} \quad \text{in} \quad (0, s_0) \quad \text{for} \quad k.
\]

(20)

It is not hard to obtain an explicit expression for \( a_k \):

\[
a_k = \sum_{j=0}^{k-1} \left( \frac{\theta}{2-q} \right)^j \left( \frac{\theta}{2-q} \right)^k - 1.
\]
From this expression, it easily follows that

$$\left(\frac{\theta}{2-q}\right)^{k-1} \leq a_k \leq C_1 \left(\frac{\theta}{2-q}\right)^{k-1}$$

(21)

for some positive constant $C_1$, which in turn gives for $b_k$ the inequality $b_k \leq C_1 b_{k-1}^\theta \left(\frac{\theta}{2-q}\right)^{k-1}$ for $k \geq 1$. Iterating this inequality from $k = 1$ we see that

$$b_k \leq C_1 \sum_{j=0}^{k-1} \left(\frac{\theta}{2-q}\right)^j \sum_{j=0}^{k-1} (k-j) \left(\frac{\theta}{2-q}\right)^j$$

(22)

for $k \geq 1$. The sum in the last exponent is an arithmetic-geometric sum, hence we can explicitly evaluate it:

$$\sum_{j=0}^{k-1} (k-j) \left(\frac{\theta}{2-q}\right)^j = \left(\frac{\theta}{2-q}\right)^k - 1 \left(1 + \frac{1}{\frac{\theta}{2-q} - 1}\right) - \frac{k}{\frac{\theta}{2-q} - 1},$$

(23)

and it follows finally from (22) and (23) that

$$b_k \leq C_2 \left(\frac{\theta}{2-q}\right)^{k-1}$$

(24)

for some $C_2 > 1$.

To conclude the proof of this part take $\delta \in (0, s_0)$ such that $H(w(s)) > 2C_2$ in $(0, \delta)$. Plugging (21) and (24) into (20) we see that

$$\left(\frac{w(s)}{s}\right)^{2-q} \geq 2 \left(\frac{\theta}{2-q}\right)^{k-1}$$

in $(0, \delta)$ for every $k$.

Letting $k \to +\infty$, and observing that $\theta > 2 - q$, we reach a contradiction, so that there are no positive supersolutions of (1) of type 1 when condition (16) does not hold.

Next, let us prove the converse implication and assume that (16) holds. For $\lambda > 0$, consider the Cauchy problem

$$\begin{cases}
-w'' = cs^{-\theta} f(w)(w')^q & s \in (0, s_0) \\
w(0) = 0 \\
w'(0) = \lambda.
\end{cases}$$

(25)

Observe that any positive solution of (25) gives rise, with the change of variables $s = \tau^{2-N}/(N-2)$, $w(s) = v(\tau)$, to a positive, radially symmetric solution of (1) of type 1. Therefore our proof is reduced to show that there actually exists a positive solution of (25) when $\lambda$ is small enough.

Denote $\zeta_\lambda(s) = \lambda s$. In the Banach space $X = \{z \in C^1[0, s_0] : z(0) = 0\}$ endowed with the standard $C^1$ norm $\|z\|_{C^1} = \max\{|z|\infty, |z'|\infty\}$, consider the set $B = \{z \in X : |z - \zeta_\lambda|_{C^1} \leq \frac{\lambda}{2}\}$, which is closed and convex, and for $z \in B$ define the integral operator

$$Tz(s) = \lambda s - c \int_0^s \int_0^t \tau^{-\theta} f(z(\tau))|z'(\tau)|^q \, d\tau \, dt, \quad s \in [0, s_0].$$

We claim that $T$ is well-defined, maps $B$ into $B$ and is compact. To show the first two assertions, notice that for every $z \in B$, we have the inequalities $\frac{\lambda}{2}s \leq z \leq \frac{3\lambda}{2}s$. 


\[ \frac{1}{2} \leq z' \leq \frac{3 \lambda}{2} \text{ in } [0, s_0], \text{ so that} \]
\[
|\langle Tz \rangle'(s) - \lambda| = c \int_0^s \tau^{-\theta} f(z(\tau))|z'(\tau)|^q d\tau
\]
\[
\leq c \left( \frac{3\lambda}{2} \right)^{q+\theta-1} \int_0^s \frac{f(z(\tau))}{\tau^\theta} z'(\tau) d\tau
\]
\[
= c \left( \frac{3\lambda}{2} \right)^{q+\theta-1} \int_0^z f(t) \frac{t}{t^\theta} dt
\]
\[
\leq c \left( \frac{3\lambda}{2} \right)^{q+\theta-1} \int_0^{\frac{3\lambda}{2}} f(t) \frac{t}{t^\theta} dt \leq \frac{\lambda}{2}
\]
in \([0, s_0]\), taking \(\lambda\) small enough (observe that \(\theta + q - 1 > 1\) since \(N \geq 3\) and \(0 < q < 2\)). It follows in a similar way that \(|\langle Tz(s) \rangle - z\lambda| \leq \frac{1}{2}\). Thus \(T\) is well defined and maps \(B\) into \(B\).

To show that \(T\) is compact, let \(\{z_n\}_{n=1}^\infty\) be an arbitrary sequence and denote \(w_n = Tz_n\). It follows by (26) that \(\{w_n\}_{n=1}^\infty\) is uniformly bounded in \([0, s_0]\), so that \(\{w_n\}\) is equicontinuous and uniformly bounded, and we may assume that \(w_n \rightarrow w\) uniformly in \([0, s_0]\), for some \(w \in C[0, s_0]\). We claim that \(w \in C^1[0, s_0]\) and \(w_n' \rightarrow w'\) uniformly in \([0, s_0]\).

Observe that \(|w_n'(s)| = s^{-\theta} f(z_n(s))\) is uniformly bounded in compacts of \((0, s_0]\). Hence by means of Arzelá-Ascoli’s theorem and a diagonal argument we may assume that also \(w_n' \rightarrow \tilde{w}\) uniformly in compacts of \((0, s_0]\) for some \(\tilde{w} \in C[0, s_0]\). We readily get that \(w \in C^1[0, s_0]\) and \(\tilde{w} = w'\). But the convergence is indeed uniform in \([0, s_0]\) (defining \(w'(0) = \lambda\)). To prove it, take \(\varepsilon > 0\). By (26):
\[
|w_n'(s) - \lambda| \leq c \left( \frac{3\lambda}{2} \right)^{q+\theta-1} \int_0^{\frac{3\lambda}{2}} f(t) \frac{t}{t^\theta} dt,
\]
and the same is true for \(w(s)\) by passing to the limit. Hence
\[
|w_n'(s) - w'(s)| \leq 2c \left( \frac{3\lambda}{2} \right)^{q+\theta-1} \int_0^{\frac{3\lambda}{2}} f(t) \frac{t}{t^\theta} dt \leq \varepsilon
\]
provided \(s \in [0, \delta]\) for \(\delta < s_0\) small enough. Since \(w_n' \rightarrow w'\) uniformly in \([\delta, s_0]\), we also have \(|w_n'(s) - w'(s)| \leq \varepsilon\) for \(s \in [\delta, s_0]\) if \(n\) is large enough. Thus \(w_n' \rightarrow w'\) uniformly in \([0, s_0]\), that is, \(w_n \rightarrow w\) in \(X\) and \(T\) is compact.

The continuity of \(T\) is shown by using a similar argument. Hence we can apply Schauder’s fixed point theorem to obtain a fixed point \(w \in B\) of \(T\), which is a solution of (25) with \(w > 0\) in \((0, s_0]\). As already observed, this concludes the proof of the lemma. \(\square\)

**Remark 3.** It can be easily deduced from the above proof that condition (16) is necessary and sufficient for existence of solutions even if \(f\) is not defined at zero.

We conclude our preliminary lemmas with supersolutions of type 2.

**Lemma 3.3.** There exist positive supersolutions of (1) of type 2 if and only if \(q \geq 1\).

**Proof.** Assume first \(q < 1\). Let \(u\) be a positive supersolution of (1) of type 2. By Lemma 2.3, there exists a positive, radial solution \(v\) of (1) of the same type defined in \(\mathbb{R}^N \setminus B_{R_1}\), for some \(R_1 > R_0\). Thus \(v\) verifies (15) in Lemma 3.1, while \(v' \geq 0\)
for large \( r \) and \( \lim_{r \to +\infty} v(r) = \ell > 0 \). We perform the same change of variables 
\[ s = r^{2-N} / (N-2) \] and \( w(s) = v(r) \). Then, for some small positive \( s_0 \) and some \( c > 0 \):
\[
\begin{cases}
-w'' = cs^{-\theta} f(w)|w'|^q & s \in (0, s_0), \\
w(0) = \ell,
\end{cases}
\]
where now \( w' \leq 0 \). Arguing as in Lemma 3.2 we can show that \( w' < 0 \) in \( (0, s_0) \).

Letting \( z = \ell - w \), we have that
\[
\begin{cases}
z'' = cs^{-\theta} f(\ell - z)(z')^q & s \in (0, s_0), \\
z(0) = 0.
\end{cases}
\]

Observe also that \( z' > 0 \). Then, since \( f \) is strictly positive in a neighborhood of \( \ell \), we obtain for some positive constants \( C \) and \( D \), after dividing the previous equation by \( (z')^q \) and integrating between \( s \) and \( s_0 \) for \( s \in (0, s_0) \):
\[
0 \leq \frac{(z')^{1-q}}{1-q} \leq -C s^{1-\theta} (1-q) + D.
\]

We arrive at a contradiction by letting \( s \) go to zero, since \( \theta > 1 \) in this case.

When \( q \geq 1 \), it is easy to construct a positive supersolution of (1) with radial symmetry. Indeed, we can look for \( u \) in the form \( u(x) = \ell - e^{-\alpha |x|} \), where \( \alpha > 0 \) is chosen suitably and \( \ell > 0 \) is arbitrary. It is not hard to check that \( u \) will be a supersolution of (1) provided that
\[
\alpha - \frac{N-1}{|x|} \geq f(\ell - e^{-\alpha |x|}) \alpha^{q-1} e^{(1-q)\alpha |x|}
\]
for large \( |x| \). Denoting \( M = \sup_{t \in [0,\ell]} f(t) \), the previous inequality is a consequence of
\[
\alpha - \frac{N-1}{|x|} \geq M \alpha^{q-1} e^{(1-q)\alpha |x|}.
\]

This inequality can be easily achieved for \( |x| \geq R_0 \) by choosing \( \alpha \) large enough. The proof is concluded.

We finally turn to the proof of our main result in the case \( N \geq 3 \).

**Proof of Theorem 1.1.** First of all observe that, by Lemma 2.2, there do not exist positive supersolutions of type 4 when \( N \geq 3 \). Also, part (a) is a direct consequence of Lemma 3.1 when \( q \geq 2 \).

On the other hand, all the assertions dealing with supersolutions of types 1 and 2 are already proved in Lemmas 3.2 and 3.3, respectively. Thus, only the statements regarding supersolutions of type 3 need to be shown. But observe that if \( u \) is a supersolution of type 3 and \( \ell = \lim_{R \to +\infty} m(R) \in (0, +\infty) \), then \( v = u - \ell \) is a positive supersolution of type 1 for the problem
\[
-\Delta v = g(v)|\nabla v|^q \quad \text{in} \quad \mathbb{R}^N \setminus B_{R_1},
\]
for some \( R_1 > R_0 \), where \( g(t) = f(t - \ell) \). By Lemma 3.2, such supersolutions exist if and only if
\[
\int_0^\delta \frac{g(t)}{t^\theta} \, dt < +\infty
\]
for some small positive \( \delta \). Since \( g(0) = f(\ell) > 0 \), the integral is convergent when \( q > \frac{N}{N-1} \), independent of \( f \). This shows the assertions on positive supersolutions of type 3 in parts (a), (b) and (c). The proof of Theorem 1.1 is concluded.
4. **Positive supersolutions in two dimensions.** In this final section we deal with positive supersolutions of (1) in the planar case $N = 2$. It is worthy of mention that the reduction to a radial setting as in Lemma 2.3 is not possible, so a slightly different approach is needed. In most proofs we will actually reduce the problem to a radial one, but in a finite interval $(R_1, R_2)$. This implies that some further estimates are needed before we can let $R_2 \to +\infty$.

By Lemma 2.2 we know that only supersolutions of types 2 and 4 are possible. Let us begin with the former ones.

**Lemma 4.1.** Problem (1) admits a positive supersolution of type 2 if and only if $q \geq 1$.

**Proof.** The proof is similar to that of Lemma 3.3 in Section 3. Indeed, the proof that there are positive supersolutions of (1) of type 2 when $q \geq 1$ is exactly the same as in that lemma, just setting $N = 2$.

Thus we assume in what follows that $q < 1$ and show that no positive supersolutions of (1) of type 2 exist. Assume for a contradiction that there is one such $u$. Choose $R_1 > R_0$ and for $R_2 > R_1$, consider the problem

$$
\begin{aligned}
-\Delta v &= f(v)|\nabla v|^q \quad \text{in } A(R_1, R_2) \\
v &= \Psi \\
&\quad \text{on } \partial A(R_1, R_2),
\end{aligned}
$$

where $\Psi$ is given by (11) in Lemma 2.2 (recall that $\Psi$ is harmonic in $A(R_1, R_2)$ and $\Psi(R_i) = m(R_i)$ for $i = 1, 2$). By the same arguments as in the proof of Lemma 2.3, there exists a radially symmetric solution $v \in C^1 [R_1, R_2]$ of this problem verifying $\Psi \leq v \leq u$ in $A(R_1, R_2)$ (the solution $v$ depends of course on $R_2$, but we are not making this dependence explicit for brevity).

By the maximum principle, and since $m$ is an increasing function, we have $v \geq m(R_1)$. Moreover, since $v$ is radially symmetric, $v(r) \leq m(r) \leq m(R_2) \leq \lim_{R \to +\infty} m(R) =: \ell$ if $R_1 \leq r \leq R_2$, where $\ell \in (0, +\infty)$. It also follows that $v$ attains its maximum at $R_2$.

Let $\gamma = \min_{e \in [m(R_1), m]} f(t) > 0$. We deduce that $v$ verifies

$$
-v'' - \frac{1}{r} v' \geq \gamma |v'|^q \quad \text{if } R_1 < r < R_2.
$$

With the change of variables $s = \log r$, $w(s) = v(r)$, we arrive at the problem

$$
\begin{aligned}
-w'' &\geq \gamma e^{(2-q)s} |w'|^q \\
w(s_1) &= m(R_1) \\
w(s_2) &= m(R_2),
\end{aligned}
$$

where $s_i = \log R_i$, $i = 1, 2$.

Let us see that $w' > 0$ in $(s_1, s_2)$. Indeed, if we had $w'(\tilde{s}) = 0$ for some $\tilde{s} \in (s_1, s_2)$, using the monotonicity of $w'$ we would arrive at $w'(s) \leq 0$ for every $s \in (\tilde{s}, s_2)$. Since $w$ attains its maximum at $s_2$, this would imply that $w(s) = m(R_2)$ if $s \in (\tilde{s}, s_2)$. This in turn yields $v(r) = m(R_2)$ for $r \in (e^{\tilde{s}}, R_2)$, which leads to $m(r) = m(R_2)$ if $r \in (e^{\tilde{s}}, R_2)$, which is not possible.

Thus $w' > 0$. Dividing by $(w')^q$ and integrating between $s_1$ and $s_2$, we have

$$
w'(s_1)^{1-q} \geq w'(s_1)^{1-q} - w'(s_2)^{1-q} \\
\geq \gamma \frac{1-q}{2-q} (e^{(2-q)s_2} - e^{(2-q)s_1}) \\
= \gamma \frac{1-q}{2-q} (R_2^{2-q} - R_1^{2-q}).
$$

This completes the proof.
Next, we claim that
\[ w'(s_1) \leq C \] (30)
for a positive constant \( C \) independent of \( R_2 \). Indeed, choose a small \( \delta > 0 \), and let \( M = \sup_{t \in [m(R_1), t]} f(t) \). Arguing in a similar way as before, we have
\[ -w'' \leq M e^{(2 - q)(s_1 + \delta)} |w'|^q = K |w'|^q, \quad s \in (s_1, s_1 + \delta). \]
Dividing by \( (w')^{q-1} \) and integrating between \( s_1 \) and \( s \) we obtain:
\[ w'(s)^{2-q} \geq w'(s_1)^{2-q} - (2 - q)K(w(s) - w(s_1)) \]
\[ \geq w'(s_1)^{2-q} - (2 - q)K\ell. \]
If we assume the right-hand side is positive (otherwise there is nothing to prove), we can raise to the power \( \frac{1}{2-q} \) and integrate in \((s_1, s_1 + \delta)\) to have
\[ \ell \geq (w'(s_1)^{2-q} - (2 - q)K\ell)^{\frac{1}{2-q}}. \]
This shows (30).

Coming back to (29), we have:
\[ \gamma = 1 - \frac{q}{2-q} (R_2^{2-q} - R_1^{2-q}) \leq C^{1-q}. \]
Letting \( R_2 \to +\infty \), we obtain a contradiction, which shows that no positive supersolutions of (1) of type 2 exist. The proof is concluded.

The analysis of supersolutions of type 4 is slightly different since there are necessary and sufficient conditions for their existence only when \( q \in (0, 2) \). Let us deal with this case next.

**Lemma 4.2.** Assume \( 0 < q < 2 \). Then there exist positive supersolutions of (1) of type 4 if and only if
\[ \int \int e^{at} f(t) \, dt < +\infty \] (31)
for some \( M, a > 0 \).

**Proof.** Assume (31) does not hold and there exists a positive supersolution of (1) of type 4. We argue similarly as in the proof of Lemma 4.1. Let \( v \) be the solution of problem (27) obtained there. Performing in the radial version of (27) the same change of variables \( s = \log r \), we arrive at
\[ \begin{cases} 
-w'' = e^{(2 - q)s} f(w) |w'|^q & s \in (s_1, s_2) \\
w(s_1) = m(R_1) \\
w(s_2) = m(R_2),
\end{cases} \] (32)
where \( s_i = \log R_i, \, i = 1, 2 \). It is equally proved that \( w' > 0 \) in \((s_1, s_2)\). Hence we may divide the equation by \((w')^{q-1}\) and integrate between \( s_1 \) and \( s_2 \) to get
\[ \frac{w'(s_1)^{2-q}}{2-q} \geq \int_{s_1}^{s_2} e^{(2-q)t} f(w(t)) w'(t) \, dt. \]
With the aid of (30), this inequality gives
\[ \int_{s_1}^{s_2} e^{(2-q)t} f(w(t)) w'(t) \, dt \leq \frac{C^{2-q}}{2-q}. \] (33)
Next, we claim that $w(s) \leq Ks$ for large $s \in (s_1, s_2)$, where $K$ does not depend on $R_2$. Indeed, since $w'' \leq 0$, we deduce using (30):

\[ w(s) \leq m(R_1) + w'(s_1)(s - s_1) \leq m(R_1) + Cs \]

\[ \leq \left( \frac{m(R_1)}{\log R_1} + C \right) s = Ks, \]

where $K$ does not depend on $R_2$. This shows the claim.

Coming back to (33), since $s \geq \frac{1}{K}w(s)$, we see that

\[ \int_{m(R_1)}^{m(R_2)} e^{\frac{2-q}{2} \frac{s}{R} f(\tau)} d\tau = \int_{s_1}^{s_2} e^{\frac{2-q}{2} w(t)} f(w(t)) w'(t) dt \leq \frac{C^{2-q}}{2} s. \]

Letting $R_2 \to +\infty$ we obtain that (31) holds, against the assumption. Thus no positive supersolutions of (1) of type 4 can exist.

To conclude the proof, let us assume that (31) holds. Our intention is to show that the problem

\[ \begin{cases} -w'' = e^{(2-q)s} f(w)|w'|^q & \text{in } (s_0, +\infty) \\ \lim_{s \to +\infty} w(s) = +\infty \end{cases} \quad (34) \]

admits a positive solution. The change of variables $s = \log r$, $w(s) = v(r)$ will then provide with a positive, radially symmetric solution of (1) which is of type 4.

For this sake, we use again Schauder’s fixed point theorem, but with some important differences with respect to the proof of Lemma 3.2. Consider the Banach space $\tilde{X} = \{ z \in C^1([s_0, +\infty) : \|z\| < +\infty \}$, where

\[ \|z\| = \max \left\{ \sup_{[s_0, +\infty)} \frac{|z(s)|}{s}, \sup_{[s_0, +\infty)} |z'(s)| \right\}, \]

and the set $\tilde{B} = \{ z \in \tilde{X} : \|z - z_\lambda\| \leq \frac{1}{2} \}$, where $z_\lambda(s) = \lambda s$ and $\lambda > 0$ is fixed. Define the operator

\[ Tz(s) = \lambda s - \int_{s_0}^{s} \int_{s_0}^{s} e^{(2-q)\tau} f(z(\tau))|z'(\tau)|^q d\tau dt, \quad s \in [s_0, +\infty). \]

Let us prove that $T$ maps $\tilde{B}$ into $\tilde{B}$ if $\lambda$ is chosen large enough. To begin with, assume $\lambda \geq 2(2-q)/a$. Taking into account that $z(s) \geq \frac{\lambda}{2}s$, $\frac{\lambda}{2} \leq z'(s) \leq \frac{3\lambda}{2}$ in $[s_0, +\infty)$ for every $z \in \tilde{B}$:

\[ \|(Tz)'(s) - \lambda\| = \int_{s_0}^{s} e^{(2-q)\tau} f(z(\tau))z'(\tau)^q d\tau \]

\[ \leq \left( \frac{3\lambda}{2} \right)^{q-1} \int_{s_0}^{s} e^{a\tau} f(z(\tau))z'(\tau) d\tau \]

\[ = \left( \frac{3\lambda}{2} \right)^{q-1} \int_{z(s_0)}^{z(s)} e^{at} f(t) dt \]

\[ \leq \left( \frac{3\lambda}{2} \right)^{q-1} \int_{\frac{\lambda}{2}s_0}^{\frac{3\lambda}{2}s_0} e^{at} f(t) dt \leq \frac{\lambda}{2}, \quad (35) \]
if $\lambda$ is chosen large enough, since $0 < q < 2$. Integrating this inequality we also obtain:

$$\left| \frac{Tz(s)}{s} - \lambda \right| \leq \frac{\lambda s - s_0}{2} \leq \frac{\lambda}{2},$$

for every $s > s_0$, hence $T$ is well-defined and $T(\overline{B}) \subset \overline{B}$.

Let us finally show that $T$ is compact (as before, the continuity of $T$ is shown by arguing in a similar way). Take an arbitrary sequence $\{z_n\}_{n=1}^{\infty} \subset \overline{B}$ and let $w_n = Tz_n$. By Arzelà-Ascoli’s theorem and a diagonal argument, since the sequences $\{Tz_n\}$, $\{(Tz_n)'\}$ and $\{(Tz_n)^n\}$ are locally uniformly bounded, we may assume $w_n \to w$, $w_n' \to w'$ uniformly on compact sets for some function $w \in C^1[s_0, +\infty)$.

Let us show that the convergence $w_n' \to w'$ is actually uniform in $[s_0, +\infty)$. Indeed, observe that, if we fix $s_1 > s_0$, take $s > s_1$, and argue as in (35) we arrive at

$$|w_n'(s) - w'(s)| = \int_{s_1}^{s} e^{(2-q)\tau} f(z_n(\tau))|z_n'(\tau)|^q d\tau \leq \left( \frac{3\lambda}{2} \right)^{-1} \int_{\frac{s}{2}}^{s} e^{\alpha t} f(t) dt.$$

A similar equality holds for $w$, by passing to the limit. Hence

$$|w_n'(s) - w'(s)| \leq |w_n'(s_1) - w'(s_1)| + 2 \left( \frac{3\lambda}{2} \right)^{-1} \int_{\frac{s}{2}}^{s} e^{\alpha t} f(t) dt$$

for every $s > s_1$. Next take $\varepsilon > 0$. Choosing $s_1$ large enough we have the last term less than $\frac{\varepsilon}{2}$. Taking $n$ large enough we also have $|w_n'(s_1) - \bar{w}(s_1)| \leq \frac{\varepsilon}{2}$, hence $|w_n'(s) - \bar{w}(s)| \leq \varepsilon$ if $s > s_1$. Since this inequality also holds in $[s_0, s_1]$ for large enough $n$, we obtain that $w_n' \to w'$ uniformly in $[s_0, +\infty)$. Hence

$$\frac{|w_n(s) - w(s)|}{s} \leq \frac{1}{s} \int_{s_0}^{s} |w_n'(t) - w'(t)| dt \to 0$$

uniformly in $[s_0, +\infty)$, as $n \to +\infty$. This shows that $T$ is compact in $\overline{B}$.

Therefore we can apply Schauder’s fixed point theorem to obtain that $T$ has a fixed point $w$ in $\overline{B}$, which is a solution of (34) verifying $\frac{\lambda}{2}s \leq w(s) \leq \frac{3\lambda}{2}s$. Observe that this implies that $w$ is positive and $\lim_{s \to +\infty} w(s) = +\infty$. As remarked above, this function provides with a positive, radially symmetric solution of (1) of type 4. The proof is concluded.

We finally consider the only left case where $q \geq 2$. It is worth mentioning that, although positive supersolutions can always be constructed, they can be of different types, depending on the behavior at infinity of their derivatives. However, we are not exploring this distinction further.

**Lemma 4.3.** Assume $q \geq 2$. Then there always exist positive supersolutions of (1) of type 4.

**Proof.** As in the proof of Lemma 4.2, It suffices to obtain a positive supersolution of the problem

$$\begin{cases}
-w'' = e^{-(q-2)s}\frac{f(w)w'|^q}{s} & \text{in } (s_0, +\infty) \\
\lim_{s \to +\infty} w(s) = +\infty.
\end{cases}$$

(36)

We consider first the case $q = 2$. Observe that since $f(t) \geq 0$, we always have

$$\int_{\lambda}^{\infty} e^{F(t)} \, dt = +\infty$$

(37)
for every $\lambda > 0$, where $F(t) = \int_0^t f(\tau) d\tau$. In this case, equation (36) can be easily integrated. Indeed, a positive, increasing solution of problem (36) is given implicitly by

$$\int_{\lambda}^{w(s)} e^{F(t)} dt = s - s_0,$$

for $s > s_0$, where $s_0$ and $\lambda$ are positive and arbitrary. This equation can be solved because of condition (37), which also gives $\lim_{s \to +\infty} w(s) = +\infty$. Observe in passing that $w'(s) = e^{-F(w(s))}$, so that when $\lim_{t \to +\infty} F(t) = +\infty$ we also obtain $\lim_{s \to +\infty} w'(s) = 0$.

Now we turn to the case $q > 2$. Assume first that $f$ verifies a condition similar to (31): there exist $a, M > 0$ such that

$$\int_M^{\infty} e^{-at} f(t) dt < +\infty. \tag{38}$$

In the space $\tilde{X}$ introduced in the proof of Lemma 4.2, consider the set $\tilde{B} = \{ z \in \tilde{X} : \| z - z_\lambda \| \leq \frac{\lambda}{2} \}$, where $z_\lambda(s) = \lambda s$ and $\lambda > 0$ is fixed. On $\tilde{B}$ define the operator

$$Tz(s) = \lambda s - \int_{s_0}^s \int_{\tau}^t e^{-(q-2)\tau} f(z(\tau))|z'(\tau)|^q d\tau d\tau, \quad s \in [s_0, +\infty).$$

We can argue in a completely similar way as in (35), except that now the exponent in the exponential is negative, to obtain that, for $\lambda \leq 2/(3a)$:

$$|(Tz)'(s) - \lambda| = \int_{s_0}^s e^{-(q-2)\tau} f(z(\tau))z'(\tau)^q d\tau \leq \left( \frac{3\lambda}{2} \right)^{q-1} \int_{s_0}^s e^{-a \tau} f(z(\tau))z'(\tau) d\tau = \left( \frac{3\lambda}{2} \right)^{q-1} \int_{z(s_0)}^{z(s)} e^{-at} f(t) dt \leq \left( \frac{3\lambda}{2} \right)^{q-1} \int_{s_0}^s e^{-at} f(t) dt.$$

This difference can be made less than or equal to $\frac{\lambda}{2}$ provided $\lambda$ is chosen small enough, since $q > 2$. The rest of the proof in this case is essentially the same as that of Lemma 4.2 and therefore will be omitted.

To conclude the proof, consider the case where $f$ does not satisfy (38). In particular, $f$ verifies $\lim_{t \to +\infty} F(t) = +\infty$. Now let $w$ be the solution of (36) with $q = 2$, obtained at the beginning of the proof. Since $\lim_{s \to +\infty} w'(s) = 0$, it follows that

$$-w'' = f(w)(w')^2 \geq e^{-(q-2)s} f(w)(w')^q, \quad s \geq s_0,$$

if $s_0$ is large enough, so that $w$ is a positive supersolution of (36). The proof is concluded.

**Proof of Theorem 1.2.** It is immediate with the use of Lemmas 4.1, 4.2 and 4.3. □

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