NON TRIVIAL EXAMPLES OF COUPLED EQUATIONS FOR KÄHLER METRICS AND YANG-MILLS CONNECTIONS

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Abstract. We provide non trivial examples of solutions to the system of coupled equations introduced by M. García-Fernández for the uniformization problem of a triple $(M,L,E)$ where $E$ is a holomorphic vector bundle over a polarized complex manifold $(M,L)$, generalizing the notions of both constant scalar curvature Kähler metric and Hermitian-Einstein metric.

1. Introduction

In his Ph.D thesis [7] (see also [1]), M. García-Fernández introduced a natural system of equations, called “coupled equations” that is related to a very simple data, composed by a Kähler manifold, a Kähler class and a vector bundle on the underlying manifold. Let us recall briefly how this system appears through a moment map construction.

Let $(M,\omega)$ be a compact symplectic manifold, $G$ a compact Lie group and $\pi: E \to M$ a smooth principal $G$-bundle on $M$. Let $J$ be the space of almost complex structures compatible with $\omega$, and $A(E)$ be the space of connections on $E$. One can consider the extended gauge group $\tilde{G}$ of $E$. It is the group of automorphisms of $E$ that cover the hamiltonian symplectomorphisms $\mathcal{H}$ of $M$. We mean that for such an automorphism $a$, $\pi\circ a = h\circ \pi$ where $h \in \mathcal{H}$ is a symplectomorphism of $M$, and in particular this group $\tilde{G}$ contains the gauge group of $E$. It is possible to check that there is a surjection map $p: \tilde{G} \to \mathcal{H}$ [7, Section 2.2]. Using this surjection, one can define an action of $a \in \tilde{G}$ on $J \times A(E)$, by

$$a \cdot (J, A) = (p(g)_* J, g \cdot A) = (dp(g) \circ J \circ (dp(g))^{-1}, g \circ A \circ g^{-1}).$$

Using the symplectic structures $\omega_J, \omega_{A(E)}$ on $J$ and $A(E)$, one can fix a symplectic form on $J \times A(E)$ by

$$\Omega = \alpha_0 \omega_J + \frac{\alpha_1}{(n-1)!} \omega_{A(E)},$$

for $\alpha_0, \alpha_1$ two real constants. Of course this choice is not canonical a priori, but this is probably the simplest one. In this setting, there is a moment map
\( \mu : \mathcal{J} \times \mathcal{A}(E) \to \text{Lie}(\tilde{G})^* \) associated to the action of \( \tilde{G} \) and the symplectic structure \( \Omega \) we have just described \cite[Proposition 2.3.1]{7}.

Let us assume from now that \( M \) has a Kähler structure, \( \omega \) is a Kähler form. We can restrict our attention to \( J_i \subset J \) the set of integrable almost-complex structures that are compatible with \( \omega \), and \( \mathcal{A}^{1,1}_J(E) \), the subset of compatible connections \( A \) on \( E \) such that \( F^{0,2}_A = F^{2,0}_A = 0 \) with respect to the structure \( J \). There is a \( \tilde{G} \)-invariant complex submanifold \( S \subset \mathcal{J} \times \mathcal{A}(E) \) consisting of pairs \( (J, A) \) such that \( J \in J_i \) and \( A \in \mathcal{A}^{1,1}_J(E) \). There is a Kähler form on the non singular part of this complex submanifold \( S \) obtained by restriction of \( \Omega \) and thus an induced moment map \( \mu \) from the \( \tilde{G} \)-action by holomorphic isometries. A pair \( (J, A) \in S \) is solution of the following system of two equations, namely the coupled equations

\[
\begin{align*}
\Lambda_\omega F_A &= z, \\
\alpha_0 \text{Scal}(g_J) + \alpha_1 \Lambda_\omega^2 (F_A \wedge F_A) &= \alpha_2.
\end{align*}
\]

if its orbit belongs to the Kähler reduction \( \mu^{-1}(0)/\tilde{G} \). Here \( g_J \) is the metric \( g_J = \omega(\cdot, J \cdot) \) induced by the Kähler form \( \omega \) and the structure \( J \in J_i \), \( \text{Scal}(g_J) \) the scalar curvature of \( g_J \), \( \Lambda_\omega \) the contraction operator with respect to the form \( \omega \), \( F_A \) the curvature of \( A \), and \( \alpha_2 \) a constant dependent on \( \alpha_0 \), \( \alpha_1 \), the Kähler class, and topological constants. Likewise \( z \) is determined by the Kähler class, and topological constants (see Remark 1.2 in \cite{1}).

Of course when one considers a trivial bundle \( E \), it turns out that the coupled system can be solved by finding a constant scalar curvature Kähler metric and a flat connection. From Fujiki and Donaldson’s work, it is well-known that the constant scalar curvature Kähler equation (cscK equation in short) appears as prescribing a zero of the moment map induced by the action of the hamiltonian symplectomorphisms \( \mathcal{H} \) on the integrable complex structures \( J_i \), see \cite{3} for details. On the other hand, the first equation of (2) appears naturally when one is considering the action of the gauge group of \( E \) \cite[Chapter 6]{6} in view of the Kobayashi-Hitchin correspondence. Certainly, the motivation to study the coupled equations (2) is coming from the natural question: what is a good moduli problem for a tuple \( (M, L, E) \) where \( (M, L) \) is a compact polarised manifold with a Kähler class \( 2\pi \sigma_1(L) \) and \( E \) is a holomorphic \( \mathcal{G}^\text{C} \)-bundle over \( M \)? It is also natural to wonder what are the natural geometric perturbations of the cscK equation, and to analyze its perturbations in terms of K-stability \cite[Chapter 4]{7}, in order to test the so-called Yau-Tian-Donaldson conjecture.

In order to do so, one is lead to construct non trivial examples \( (\alpha_0 \alpha_1 \neq 0) \) of solutions to the coupled equations (2) for the tuple \( (M, L, E) \) chosen as before. M. García-Fernández showed that it is possible to obtain examples by deformations (when \( \frac{\alpha_1}{\alpha_0} \) is small enough) of a manifold \( M \) that carries a constant scalar curvature Kähler metric and a Hermitian-Yang-Mills holomorphic vector bundle, if the automorphism group of \( M \) is finite \cite[Theorem]{7}. 

The proof reduces to an implicit function theorem and the assumption on the automorphism group allows to invert the linearization operator. Also, when $M$ has complex dimension 1, the term $F_A \wedge F_A$ vanishes and one can provide solutions to the coupled equations by considering the Kobayashi-Hitchin correspondence for holomorphic bundles, i.e. Mumford polystable bundles on the complex curve. On compact homogeneous Kähler-Einstein surfaces, examples can be provided by considering anti-self-dual connections. In higher dimensions, examples can be found by using projectively flat bundles over a manifold with constant scalar curvature metric and satisfying a natural topological condition. As one can remark, all these examples are very specific since they hold on manifolds that carry a constant scalar curvature Kähler metric. It is natural to wonder if one can find new examples of solutions to the coupled equations on complex manifolds such that there is no solution to the cscK equation in the class $2\pi c_1(L)$. The main goal of this paper is to construct such examples over a ruled surface and a ruled threefold.

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2. Examples of solutions to coupled equations on Hirzebruch type ruled surfaces

Let us consider a ruled manifold of the form $M = \mathbb{P}(\mathcal{O} \oplus \mathcal{L}) \to \Sigma$, where $\Sigma$ is a compact Riemann surface, $\mathcal{L}$ is a holomorphic line bundle of degree $k \in \mathbb{Z}^*$ on $\Sigma$, and $\mathcal{O}$ is the trivial holomorphic line bundle. Let $g_\Sigma$ be the Kähler metric on $\Sigma$ of constant scalar curvature $2s_\Sigma$, with Kähler form $\omega_\Sigma$, such that $c_1(\mathcal{L}) = \frac{[\omega_\Sigma]}{2\pi}$. Let $K_\Sigma$ denote the canonical bundle of $\Sigma$. Since $c_1(K_\Sigma^{-1}) = [\rho_\Sigma/2\pi]$, where $\rho_\Sigma$ denotes the Ricci form, we have the relation $s_\Sigma = 2(1 - h)/k$, where $h$ denotes the genus of $\Sigma$.

The natural $\mathbb{C}^*$-action on $\mathcal{L}$ extends to a holomorphic $\mathbb{C}^*$-action on $M$. The open and dense set $M_0$ of stable points with respect to the latter action has the structure of a principal $\mathbb{C}^*$-bundle over the stable quotient. The hermitian norm on the fibers induces via a Legendre transform a function $\zeta : M_0 \to (-1, 1)$ whose extension to $M$ consists of the critical manifolds $E_0 := \zeta^{-1}(1) = P(\mathcal{O} \oplus 0)$ and $E_\infty := \zeta^{-1}(-1) = P(0 \oplus \mathcal{L})$.

These zero and infinity sections, $E_0$ and $E_\infty$, of $M \to \Sigma$ have the property that $E_0^2 = k$ and $E_\infty^2 = -k$, respectively. If $C$ denotes a fiber of the ruling $M \to \Sigma$, then $C^2 = 0$, while $C \cdot E_i = 1$ for both, $i = 0$ and $i = \infty$. Any real cohomology class in the two dimensional space $H^2(M, \mathbb{R})$ may be written as a linear combination of (the Poincaré duals of) $E_0$ and $C$,

$$m_1E_0 + m_2C.$$
Thus, we may think of $H^2(M, \mathbb{R})$ as $\mathbb{R}^2$, with coordinates $(m_1, m_2)$. The Kähler cone $K$ may be identified with $\mathbb{R}^2_+ = \{(m_1, m_2) \mid m_1 > 0, m_2 > 0\}$ (see [8] or Lemma 1 in [13]). To calculate $m_1$ and $m_2$ for a real cohomology class $\Gamma \in H^2(M, \mathbb{R})$ it is useful to notice that we have

$$
\begin{align*}
\int_{E_0} \Gamma &= \Gamma \cdot E_0 = (m_1 E_0 + m_2 C) \cdot E_0 = km_1 + m_2, \\
\int_C \Gamma &= \Gamma \cdot C = (m_1 E_0 + m_2 C) \cdot C = m_1.
\end{align*}
$$

Thus, we get $m_1 = \int_C \Gamma$ and $m_2 = \int_{E_0} \Gamma - k \int_C \Gamma$.

We shall use the techniques developed in [2, 3] to build a solution to the coupled equations (2) and check that the manifold $M$ does not carry a constant scalar curvature Kähler metric. To build the so-called admissible metrics [3] on $M$ we proceed as follows. Let $\theta$ be a connection one form for the Hermitian metric on $M_0$, with curvature $d\theta = \omega_\Sigma$. Let $\Theta$ be a smooth real function with domain containing $(-1, 1)$. Let $x$ be a real number such that $0 < x < 1$. Then an admissible Kähler metric is given on $M_0$ by

$$
g = \frac{1 + x^3}{x} g_\Sigma + \frac{d\zeta^2}{\Theta(\zeta)} + \Theta(\zeta) \theta^2
$$

with Kähler form

$$
\omega = \frac{1 + x^3}{x} \omega_\Sigma + d\zeta \wedge \theta.
$$

The complex structure yielding this Kähler structure is given by the pullback of the base complex structure along with the requirement

$$
J d\zeta = \Theta \theta
$$

The function $\zeta$ is hamiltonian with $K = J \text{grad} \zeta$ a Killing vector field. Observe that $K$ generates the circle action which induces the holomorphic $\mathbb{C}^*$-action on $M$ as introduced above. In fact, $\zeta$ is the moment map on $M$ for the circle action, decomposing $M$ into the free orbits $M_0 = \zeta^{-1}((-1, 1))$ and the special orbits $\zeta^{-1}(\pm 1)$. Finally, $\theta$ satisfies $\theta(K) = 1$. In order that $g$ (be a genuine metric and) extend to all of $M$, $\Theta$ must satisfy the positivity and boundary conditions

$$
\begin{align*}
(i) \quad & \Theta(\zeta) > 0, \quad -1 < \zeta < 1, & (ii) \quad & \Theta(\pm 1) = 0, & (iii) \quad & \Theta'(\pm 1) = \mp 2.
\end{align*}
$$

The last two of these are together necessary and sufficient for the compactification of $g$. Define a function $F(\zeta)$ by the formula

$$
\Theta(\zeta) = \frac{F(\zeta)}{1 + x^3}
$$

Since $1 + x^3$ is positive for $-1 < \zeta < 1$, conditions (7) imply the following equivalent conditions on $F(\zeta)$:

$$
\begin{align*}
(i) \quad & F(\zeta) > 0, \quad -1 < \zeta < 1, & (ii) \quad & F(\pm 1) = 0, & (iii) \quad & F'(\pm 1) = \mp 2(1 \pm x).
\end{align*}
$$
The volume form of $g$ in (4) is given by
\[ d\mu_g = \frac{\omega \wedge \omega}{2} = \frac{1 + x_3}{x} \omega_\Sigma \wedge d_3 \wedge \theta, \]
while the Ricci form is given by
\[ \rho_g = \left( s_\Sigma - \frac{F'(3)}{2(1 + x_3)} \right) \omega_\Sigma \wedge d \left( \frac{F'(3)}{2(1 + x_3)} \right) \wedge \theta, \]
and the scalar curvature is given by
\[ \text{Scal}(g) = \frac{2s_\Sigma x}{1 + x_3} - \frac{F''(3)}{1 + x_3}. \]

The calculations of these geometrical terms can be found in [2].

Now, since $E_0 = \tilde{z}^{-1}(1)$ and $k = c_1(\mathcal{L}) = \left[ \frac{2\pi}{\omega} \right]$ we have that
\[ \int_{E_0} [\omega] = \int_{-1}^{1} \omega_\Sigma = \frac{2\pi k(1 + x)}{x}. \]

It is also easy to see that
\[ \int_{C} [\omega] = \int_{0}^{2\pi} \int_{-1}^{1} d_3 \wedge dt = 4\pi, \]
where $t \in [0, 2\pi]$ is a fibre coordinate of $M_0 \to \Sigma \times (-1, 1)$ in a gauge chosen such that the connection form $\theta$ has no $d_3$ components. Therefore, from (3), we have that
\[ [\omega] = 4\pi E_0 + \frac{2\pi (1 - x)k}{x} C. \]

and we fix $x = \frac{k}{k + k'}$ with $k' \in \mathbb{Z}_+$. The $(1, 1)$ form
\[ \rho_g = \frac{\text{Scal}(g)}{4} \omega = \left( \frac{2s_\Sigma x(1 + x_3) + F''(3)(1 + x_3) - 2xF'(3)}{4(1 + x_3)^2} \right) \times \left( \frac{1 + x_3}{x} \omega_\Sigma - d_3 \wedge \theta \right) \]
is traceless and therefore anti-self-dual. Using this, we easily check that the form
\[ \alpha := \frac{x^2}{(1 + x_3)^2} \left( \frac{1 + x_3}{x} \omega_\Sigma - d_3 \wedge \theta \right), \]
which does not depend on $F(3)$, is both closed and anti-self-dual. Since the second Betti number $b_2(M)$ of our ruled surface $M$ is two, while the signature, $\sigma$, is zero, a basis for the vector space of harmonic real $(1, 1)$-forms on $(M, g)$ would be given by \{\omega, \alpha\}. Now,
\[ \int_{E_0} [\alpha] = \frac{x}{(1 + x)} \int_{-1}^{1} \omega_\Sigma = \frac{2\pi k x}{(1 + x)} \]
and
\[ \int_{C} [\alpha] = -x^2 \int_{0}^{2\pi} \int_{-1}^{1} (1 + x_3)^{-2} d_3 \wedge dt = \frac{-4\pi x^2}{1 - x^2}. \]
We therefore have that
\[
[\alpha] = \frac{2\pi x}{1 - x^2} (-2xE_0 + k(1 + x)C).
\]

Consider the $(1,1)$ form $\gamma_{a,b} = a\omega + b\alpha$ for some constants $a, b$. Since
\[
\frac{[\gamma_{a,b}]}{2\pi} = \frac{2(a(1 - x^2) - bx^2)}{1 - x^2} E_0 + \frac{k(a(1 - x)^2 + bx^2)}{x(1 - x)} C
\]
it is easy to see that for appropriate choices of $a$ and $b$, $[\gamma_{a,b}]$ is an integral class and thus $\gamma_{a,b}$ may be viewed as the curvature form $F_A$ of some connection $A$ on some stable vector, $E$, bundle over $M$ (of course in that case $E$ is a line bundle). Actually any choice of $a \in \mathbb{Z}$ and $b$ integer multiple of $(2k + k')k'$ will imply that $[\gamma_{a,b}]$ belongs to $H^2(M, \mathbb{Z})$.

We easily calculate that
\[
\Lambda_\omega \gamma_{a,b} = 2a,
\]
which corresponds to the first equation of the system (2). The second equation of the coupled equations (2) in the variables $(g, \omega, J)$ on $M$ and $\gamma_{a,b}$ on $E$, corresponds to
\[
\alpha_0 \text{Scal}(g) + \alpha_1 \Lambda_\omega^2 (\gamma_{a,b} \wedge \gamma_{a,b}) = \alpha_2,
\]
for some constants $\alpha_0, \alpha_1, \alpha_2 \in \mathbb{R}$. It is straightforward to verify that
\[
\gamma_{a,b} \wedge \gamma_{a,b} = 2\left(a^2 - \frac{b^2x^4}{(1 + x^3)^4}\right) d\mu_g,
\]
and since $\Lambda_\omega^2 d\mu_g = 2$ (see e.g. 2.77 in [4]), we have that
\[
\Lambda_\omega^2 (\gamma_{a,b} \wedge \gamma_{a,b}) = 4\left(a^2 - \frac{b^2x^4}{(1 + x^3)^4}\right).
\]
Assuming that $(g, \omega, J)$ is admissible, and hence determined by $F(\tilde{z})$ satisfying (9), we get that (13) is equivalent to
\[
\alpha_0 \left(\frac{2\Sigma x}{1 + x^3} - \frac{F''(\tilde{z})}{1 + x^3}\right) + 4\alpha_1 \left(a^2 - \frac{b^2x^4}{(1 + x^3)^4}\right) = \alpha_2.
\]
Unless $b = 0$ we have that $\alpha_0$ must be non-zero. Otherwise, if $b = 0$, we only get a trivial solution since $M$ admits no constant scalar curvature Kähler metrics. We therefore arrive at the following ODE
\[
F''(\tilde{z}) = 2\Sigma x + \frac{4\alpha_1}{\alpha_0} \left(a^2 - \frac{b^2x^4}{(1 + x^3)^4}\right) (1 + x^3) - \frac{\alpha_2}{\alpha_0} (1 + x^3).
\]
Integrating twice we see that this has a solution, satisfying (9), if and only if
\[
\frac{\alpha_1}{\alpha_0} = -\frac{(1 - a^2)^2(2 - \Sigma x)}{8b^2x^4}
\]
and
\[
\frac{\alpha_2}{\alpha_0} = \frac{3b^2x^4(2 + \Sigma x) - a^2(2 - \Sigma x)(1 - x^2)^2}{2b^2x^4}.
\]
In that case we find a unique solution

\[(18) \quad F(\mathcal{J}) = \frac{(1 - \delta^2)(x^2(2 + s_x \Sigma x)^2 + 8x_3 + 4 + 2x^2 - s_x \Sigma x^3)}{4(1 + x_3)}\]

and then

\[(19) \quad \text{Scal}(g) = \frac{3(2 + s_x \Sigma x)}{2} - \frac{(2 - s_x \Sigma x)(1 - x^2)^2}{2(1 + x_3)^4}\]

Since \(s_x \Sigma x \lt 2\), \(\text{Scal}(g)\) is not an affine function of \(\mathcal{J}\) and hence not an extremal Kähler metric \([2]\) (remark that the bundle \(\mathcal{O} \oplus \mathcal{L} \to \Sigma\) is not Mumford polystable so \(M\) does not have a constant scalar curvature Kähler metric). For the same reason, we also notice that \(\frac{\alpha_1}{\alpha_0} < 0\) and thus the form \(\Omega\) defined by (1) is symplectic and not Kähler. Finally, with notations above, we set

\[(20) \quad x = \frac{k}{k + k'}, \quad a = k_1, \quad b = \frac{k_2(2k + k')k'}{k^2},\]

in the previous equations with \(k' \in \mathbb{Z}_+^*, k_1 \in \mathbb{Z}, k_2 \in \mathbb{Z}^*\). Furthermore, the condition \(F(\mathcal{J}) > 0\) is satisfied for all \(-1 \leq \mathcal{J} \leq 1\). Actually, if we set \(f(\mathcal{J}) = x^2(2 + s_x \Sigma x)^2 + 8x_3 + 4 + 2x^2 - s_x \Sigma x^3\), then \(f(-1) = 4(1 - x)^2 > 0\) and \(f(1) = 4(x + 1)^2 > 0\). If \((2 + s_x \Sigma x) \leq 0\) it is then straightforward to see that \(f(\mathcal{J}) > 0\) for \(-1 \leq \mathcal{J} \leq 1\). Since \(f'(-1) = 2x(4 - 2x - s_x \Sigma x^2) > 0\) the same can be concluded if \((2 + s_x \Sigma x) > 0\). Eventually, in both cases, \(F\) is strictly positive.

Thus, we have obtained that conditions (9), equations (13) and (12) are all satisfied and the system of coupled equation admits a solution in integral classes from (10) and (11). This leads to the following result.

**Proposition - Example 1.** Assume that \(M = \mathbb{P}(\mathcal{O} \oplus \mathcal{L}) \to \Sigma\) is a ruled manifold with \(\mathcal{L}\) of degree \(k \in \mathbb{Z}_+\). Fix \(k' \in \mathbb{Z}_+^*, k_1 \in \mathbb{Z}, k_2 \in \mathbb{Z}^*\). Consider the integral classes \(L := 2E_0^1 + k'C\) and \(E := 2(k_1 - k_2)E_0 + (k_1 k' + k_2(2k + k'))C\). Then, there exists an admissible Kähler metric \(\omega \in 2\pi c_1(L)\), a complex structure \(J\) defined by (6), (8), (18), and a connection \(A \in A_{1,1}^1(E)\) such that the triple \((\omega, J, A)\) is a solution to the coupled equations (2). The constants \((\alpha_0, \alpha_1, \alpha_2)\) satisfy \(\frac{\alpha_1}{\alpha_0} = -\frac{8k(2 - s_x \Sigma k + 2k')}{8k_2(k + k')} < 0\). Furthermore there is no constant scalar curvature Kähler metric in \(2\pi c_1(L)\).

Note that once the bundle \(E\) and the class \(L\) are fixed as in our proposition, the solution \((\omega, J)\) is unique in the set of admissible Kähler metrics, up to automorphisms.

### 2.1. About the Calabi-Yang-Mills functional

We are now going to consider the coupled equations from a variational point of view but in a slightly different setup than in \([7]\) and \([1]\) where the constants \(\alpha_i\) are all positive. First of all, using the fact that for any \(A \in A_{1,1}^1(E)\)

\[(21) \quad |F_A|^2 = |A_\omega F_A|^2 - \frac{1}{2} A_\omega^2 (F_A \wedge F_A)\]
Remark 2.1. Our functional \( \Lambda_\omega F_\alpha \) which appears when one is integrating over the manifold \( M \) and its sum of the Calabi functional and the Yang-Mills functional.

\[
\{ \begin{align*}
\Lambda_\omega F_\alpha & = z, \\
\alpha_0 \text{Scal}(g) - 2\alpha_1 |F_\alpha|^2 & = \alpha_2 - 2\alpha_1 |z|^2,
\end{align*} \]

and we shall consider the case \( \alpha_0 \neq 0 \). It is natural to introduce the Calabi-Yang-Mills type functional

\[
CYM(g, A) = \int_M \left( \text{Scal}(g) - 2\frac{\alpha_1}{\alpha_0} |F_\alpha|^2 \right)^2 d\mu_g + \|F_\alpha\|^2,
\]

and the constant

\[
\alpha_2 = \frac{4\pi\alpha_0}{(n-1)!} \frac{\langle c_1(M) \cup [\omega]^{n-1}, [M] \rangle}{Vol_M([\omega])} + \frac{2\alpha_1}{(n-2)!} \frac{\langle c(E) \cup [\omega]^{n-2}, [M] \rangle}{Vol_M([\omega])}
\]

which appears when one is integrating over the manifold \( M \) the second equation of the system (2). Here we assume that the complex structure is fixed and \( g \) varies among Kähler metrics with a fixed Kähler class \( 2\pi c_1(L) \).

**Remark 2.1.** Our functional \( CYM \) differs from [7]. Indeed we choose our functional such that, up to a renormalization, it is globally invariant if we do the change of metric \( \omega \to tw \) and change accordingly the constants \( \alpha_0, \alpha_1 \) by \( t\alpha_0, t^2\alpha_1 \) in (2). Furthermore, when \( \alpha_1 \to 0 \), it reduces to precisely the sum of the Calabi functional and the Yang-Mills functional.

Now, with (21), we get that

\[
CYM(g, A) = \int_M \left( \text{Scal}(g) - 2\frac{\alpha_1}{\alpha_0} |F_\alpha|^2 - \frac{\alpha_2}{\alpha_0} + 2\frac{\alpha_1}{\alpha_0} |z|^2 \right)^2 d\mu_g
\]

\[
+ \|F_\alpha\|^2
\]

\[
+ 2 \left( \frac{\alpha_2}{\alpha_0} - 2\frac{\alpha_1}{\alpha_0} |z|^2 \right) \int_M (\text{Scal}(g) - 2\frac{\alpha_1}{\alpha_0} |F_\alpha|^2) d\mu_g
\]

\[
- \left( \frac{\alpha_2}{\alpha_0} - 2\frac{\alpha_1}{\alpha_0} |z|^2 \right)^2 Vol_M([\omega]),
\]

\[
\| \text{Scal}(g) - 2\frac{\alpha_1}{\alpha_0} |F_\alpha|^2 - \frac{\alpha_2}{\alpha_0} + 2\frac{\alpha_1}{\alpha_0} |z|^2 \|^2_{L^2}
\]

\[
+ \left( 1 - 4\frac{\alpha_2\alpha_1}{\alpha_0^2} + 8\frac{\alpha_1^2}{\alpha_0^2} |z|^2 \right) \|\Lambda_\omega F_\alpha\|^2_{L^2}
\]

\[
+ \delta(E, [\omega], M, \alpha),
\]

where \( \delta(E, [\omega], M, \alpha) \) is a constant dependant only on topological constants of \( (E, [\omega], M) \) and the triple \( \alpha = (\alpha_0, \alpha_1, \alpha_2) \). Remark that we can write

\[
\|\Lambda_\omega F_\alpha\|^2_{L^2} = \|\Lambda_\omega F_\alpha - z\|^2_{L^2} + 2z \int_M F_\alpha \wedge \omega^{n-1} - |z|^2 Vol_M([\omega]),
\]

\[
\|\Lambda_\omega F_\alpha - z\|^2_{L^2} + \delta'(E, M, [\omega]),
\]
where $\delta'(E, M, [\omega])$ depends only on $[\omega]$ and the topology of $(E, M)$. Therefore, we obtain

$$
CYM(g, A) = \left\| Scal(g) - \frac{2\alpha_1}{\alpha_0}|F_A|^2 - \frac{\alpha_2}{\alpha_0} + 2\alpha_1|z|^2\right\|_{L^2}
+ \left(1 - 4\frac{\alpha_2\alpha_1}{\alpha_0^2} + 8\frac{\alpha_1^2}{\alpha_0^2}|z|^2\right)\||\omega F_A - z\|_{L^2}^2
+ \delta''(E, [\omega], M, \alpha),
$$

with $\delta''(E, [\omega], M, \alpha)$ a topological constant. We claim that for several choices of $\omega, L, E$ in Theorem 1, there exists a solution to the coupled equations (2) that minimize the CYM functional.

Actually, we remark that if one has the inequality

$$(23) \quad \left(1 - 4\frac{\alpha_2\alpha_1}{\alpha_0^2} + 8\frac{\alpha_1^2}{\alpha_0^2}|z|^2\right) > 0,$$

then we get

$$CYM(g, A) \geq \delta''(E, [\omega], M, \alpha),$$

and the equality is achieved precisely for a solution to (2). From (17) and (20), we remark by a direct computation that the limit when $k' \to +\infty$ of the LHS of (23) is

$$1 + \frac{3}{k_2^2} + \frac{a^2}{k_4^2}.$$

Thus, if we choose $k'$ large enough (hence $x > 0$ small enough), we get the required inequality (23) and independently of the choice of the other parameters.

Another possible choice is to take the limit $k_2 \to \pm\infty$ and in that case the LHS of (23) tends to 1. We can also do the following choice : $\alpha_2 = 0$ with $k_1 = k_2 = a, k' = k, b = 3a, x = \frac{1}{2}$ and $s_{\Sigma} = -2$ which implies again that the solutions of the coupled equations minimize the CYM functional.

From our discussion we obtain the following corollary.

**Corollary 1.** In Theorem 1, there exists for $k'$ large enough or $|k_2|$ large enough (or for the choice $s_{\Sigma} = -2, b = 3a, k' = k$) a solution to the coupled equations (2) that is the absolute minimum of the CYM functional.

Let us discuss now briefly the uniqueness of the solutions we found. We know that at the level of Chern classes,

$$c_1(M) = 2H - p^*c_1(\mathcal{O} \oplus L) + p^*c_1(\Sigma),$$

where $H \in |\mathcal{O}_M(1)|$ and $p : M \to \Sigma$ is the canonical projection of the ruled manifold $M$ to the surface $\Sigma$. Thus for our last choice above ($\alpha_2 = 0$ and $\Sigma$ has genus $h = 1 + k, k \in \mathbb{Z}_+$), we are under the conditions of [7, Proposition 3.5.3 (2)] since $c_1(M) \leq 0$ for $k$ large enough. Thus we have constructed a family $((M, J), E, 2\pi c_1(L))$ with solutions $(g, A)$ to the coupled equations (2) such that the associated Kähler form $\omega$ is unique in the Kähler class $2\pi c_1(L)$. 
3. Examples of Solutions to Coupled Equations on the Total Space of a Projective Bundle over a Product of Two Riemann Surfaces

In order to obtain a form \( \Omega \) defined by (1) which is Kähler we would like to construct an example of solutions to (2) where \( \alpha_1/\alpha_0 > 0 \). To that end, we change the setting a little bit to gain more flexibility.

Let us consider a ruled manifold of the form \( M = \mathbb{P}(O \oplus \mathcal{L}) \to \Sigma_1 \times \Sigma_2 \), where \( \Sigma_i \), \( i = 1, 2 \) is a compact Riemann surface, \( \mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2 \), where \( \mathcal{L}_i \) is a holomorphic line bundle of degree \( k_i \in \mathbb{Z}^* \) on \( \Sigma_i \), and \( O \) is the trivial holomorphic line bundle. Let \( \pm g_i \) be the Kähler metric on \( \Sigma_i \) of constant scalar curvature \( \pm 2s_i \), with Kähler form \( \pm \omega_i \), such that \( c_1(\mathcal{L}_i) = \frac{[\omega_i]}{2\pi} \). If we denote the genus of \( \Sigma_i \) by \( h_i \), we have the relation \( s_i = 2(1 - h_i)/k_i \). Similarly to the previous section, the zero and infinity sections of \( M \to \Sigma_1 \times \Sigma_2 \) are denoted by \( E_0 \) and \( E_\infty \). Further, \( M_0 \) and \( 3 : M_0 \to (-1,1) \) are defined as before, while \( \theta \) now satisfies \( d\theta = \omega_1 + \omega_2 \). Let \( x_1 \neq x_2 \) be real numbers such that \( 0 < |x_i| < 1 \) and \( \frac{\omega_i}{x_i} \) is positive. If, again, \( \Theta \) is a smooth real function with domain containing \((-1,1)\) and satisfying (7) we now have an admissible metric on \( M \) which on \( M_0 \) is given by

\[
g = \frac{1 + x_1\bar{\delta}}{x_1}g_1 + \frac{1 + x_2\bar{\delta}}{x_2}g_2 + \frac{d\bar{\delta}^2}{\Theta(\delta)} + \Theta(\delta)\theta^2
\]

with Kähler form

\[
\omega = \frac{1 + x_1\bar{\delta}}{x_1}\omega_1 + \frac{1 + x_2\bar{\delta}}{x_2}\omega_2 + d\bar{\delta} \wedge \theta,
\]

and the complex structure given as in the previous section. If we set

\[
\Theta(\delta) = \frac{F(\delta)}{(1 + x_1\bar{\delta})(1 + x_2\bar{\delta})},
\]

then the boundary conditions (7) now become equivalent to

\[
(i) \quad F(\delta) > 0, \quad -1 < \delta < 1,
(ii) \quad F(\pm 1) = 0,
(iii) \quad F'(\pm 1) = \mp 2(1 \pm x_1)(1 \pm x_2).
\]

and the scalar curvature of \( g \) equals

\[
\text{Scal}_g = \frac{2s_1x_1}{1 + x_1\bar{\delta}} + \frac{2s_2x_2}{1 + x_2\bar{\delta}} - \frac{F''(\delta)}{(1 + x_1\bar{\delta})(1 + x_2\bar{\delta})}.
\]

Consider the 2-form \( \eta = d(\delta \theta) \) on \( M_0 \). By the discussion of Section 1.3 in [3] this form is well-defined and closed on \( M \). In fact, \( [\eta] \) is the Poincaré dual of \( 2\pi[E_0 + E_\infty] \) and \( H^2(M,\mathbb{R}) \) is generated by \([\eta]\) and pullbacks from \( \Sigma_1 \times \Sigma_2 \). Further \([\eta/(4\pi)]\) generates \( H^2(p^{-1}(x),\mathbb{Z}) \) where \( p^{-1}(x) \) denotes a fibre of \( p : M \to \Sigma_1 \times \Sigma_2 \) while \([\omega_i/(2\pi k_i)]\) (appropriately lifted) is a primitive
integer class as well. We may write the Kähler class of our admissible metrics as

\[
\omega = \frac{2\pi}{x_1} [\omega_1/(2\pi)] + \frac{2\pi}{x_2} [\omega_2/(2\pi)] + 4\pi [\eta/(4\pi)].
\]

Notice that as long as \(x_1\) and \(x_2\) are rational numbers, \([\omega/(2\pi)]\) is a rational class and hence by rescaling the Kähler metric as necessary, we obtain an integer class and a corresponding line bundle \(L\). Such a rescaling would rescale \(\alpha_1/\alpha_0\) from the coupled equations by the same factor, but would not change its sign and would not change the qualitative properties of e.g. the CYM functional. In what follows we shall therefore ignore this rescaling factor.

Let

\[
\alpha = (\omega_1 + \omega_2) f + df \wedge \theta,
\]

where \(f = \frac{1}{(1+x_1)(1+x_2)}\). It is easy to see that \(\alpha\) is closed. If \(\langle \cdot, \cdot \rangle\) denotes the inner product on 2-forms induced by the metric \(g\), we observe that \(\langle \alpha, \omega \rangle = 0\) using the following facts

\[
\langle \frac{1 + x_i\delta}{x_i} \omega_i, \frac{1 + x_j\delta}{x_j} \omega_j \rangle = \delta_{ij},
\]

\[
\langle \frac{1 + x_i\delta}{x_i} \omega_i, d_3 \wedge \theta \rangle = 0,
\]

\[
\langle d_3 \wedge \theta, d_3 \wedge \theta \rangle = 1.
\]

Now \([\alpha] = m_1[\omega_1/(2\pi)] + m_2[\omega_2/(2\pi)] + n[\eta/(4\pi)]\) for some \(n, m_1, m_2 \in \mathbb{R}\). By integrating along the fibre \(C\) of the ruling and along \(\Sigma_i\) as embedded in \(E_0 = z^{-1}(1)\), we can determine the value of \(n\) and \(m_i, i = 1, 2\). Indeed, if \([\alpha] = m_1[\omega_1/(2\pi)] + m_2[\omega_2/(2\pi)] + n[\eta/(4\pi)]\) then (allowing for a slight abuse of notation)

\[
\int_C [\alpha] = n
\]

while

\[
\int_{\Sigma_i \subset E_0} [\alpha] = (m_i + \frac{n}{2}) \int_{\Sigma_i} \frac{\omega_i}{2\pi} = (m_i + \frac{n}{2}) k_i
\]

(using that \(\eta = \frac{3}{2} d\theta + d_3 \wedge \theta = \frac{3}{2}(\omega_1 + \omega_2) + d_3 \wedge \theta\)). On the other hand, since \(\alpha = (\omega_1 + \omega_2) f + df \wedge \theta\), we also have that

\[
\int_C [\alpha] = \int_C df \wedge \theta = 2\pi (f(1) - f(-1))
\]

and

\[
\int_{\Sigma_i \subset E_0} [\alpha] = f(1) \int_{\Sigma_i} \omega_i = 2\pi f(1) k_i,
\]

and so, using that \(f = \frac{1}{(1+x_1)(1+x_2)}\), we get

\[
n = \frac{-4\pi (x_1 + x_2)}{(1 - x_1^2)(1 - x_2^2)}
\]
while
\[ m_1 = m_2 = \frac{2\pi (1 + x_1 x_2)}{(1 - x_1^2)(1 - x_2^2)}. \]

Similarly to the previous section we now define \( \gamma_{a,b} = a\omega + b\alpha \). From the discussion above we see that
\[
[\gamma_{a,b}] = 2\pi \left( \frac{a}{x_1} + \frac{b(1 + x_1 x_2)}{(1 - x_1^2)(1 - x_2^2)} \right) \left[ \omega_1/(2\pi) \right] 
+ 2\pi \left( \frac{a}{x_2} + \frac{b(1 + x_1 x_2)}{(1 - x_1^2)(1 - x_2^2)} \right) \left[ \omega_2/(2\pi) \right] 
+ 4\pi \left( a - \frac{b(x_1 + x_2)}{(1 - x_1^2)(1 - x_2^2)} \right) \left[ \eta/(4\pi) \right].
\]

Given values of \( x_1 \) and \( x_2 \), it is now clear that we can choose \( a, b \in \mathbb{R} \) such that \([\gamma_{a,b}]\) is an integer class.

Using that for any \((1, 1)\) forms \( \beta, \delta \), we have
\[
\Lambda_2^2(\beta \wedge \delta) = 2(\langle \beta, \omega \rangle \langle \delta, \omega \rangle - \langle \beta, \delta \rangle),
\]
we calculate that
\[
\Lambda_2^2(\gamma_{a,b} \wedge \gamma_{a,b}) = 12a^2 - 2b^2 x_1^2 (1 + x_2 \bar{\delta})^2 + x_1^2 (1 + x_2 \bar{\omega})^2 \frac{(1 + x_2 \bar{\omega})^2 + (x_1(1 + x_2) + x_2(1 + x_3))^2}{(1 + x_1)^2 (1 + x_2 \bar{\delta})^2}.
\]

Since \( \Lambda_2 \gamma_{a,b} = 3a \), and the scalar curvature is given by (27) the coupled equations (2) are satisfied, in the case \( \alpha_0 \neq 0 \), if and only if
\[
F''(\mathfrak{z}) = 2s_1 x_1 (1 + x_2 \bar{z}) + 2s_2 x_2 (1 + x_1 \bar{z})
+ (12a^2 \frac{\alpha_1}{\alpha_0} - \frac{\alpha_2}{\alpha_0}) (1 + x_1 \bar{z})(1 + x_2 \bar{z})
- 2b^2 \frac{\alpha_1}{\alpha_0} x_1^2 (1 + x_2 \bar{\delta})^2 + x_1^2 (1 + x_1 \bar{\omega})^2 \frac{(1 + x_2 \bar{\omega})^2 + (x_1(1 + x_2) + x_2(1 + x_3))^2}{(1 + x_1)^2 (1 + x_2 \bar{\delta})^2}.
\]

Set \( \kappa_1 = (12a^2 \frac{\alpha_1}{\alpha_0} - \frac{\alpha_2}{\alpha_0}) \) and \( \kappa_2 = 4b^2 \frac{\alpha_1}{\alpha_0} \). The the above equation can be written as
\[
F''(\mathfrak{z}) = 2s_1 x_1 (1 + x_2 \bar{z}) + 2s_2 x_2 (1 + x_1 \bar{z})
+ \kappa_1 (1 + x_1 \bar{z})(1 + x_2 \bar{z})
- \kappa_2 (1 + x_1 \bar{z})^{-3} - 2s_2(1 + x_2 \bar{z})^{-3}/(x_1 - x_2).
\]

Let us fix
\[
P(t) = \int_{-1}^{t} (2s_1 x_1 (1 + x_2 \bar{z}) + 2s_2 x_2 (1 + x_1 \bar{z})) \, d\mathfrak{z}
+ \kappa_1 \int_{-1}^{t} (1 + x_1 \bar{z})(1 + x_2 \bar{z}) \, d\mathfrak{z}
- \kappa_2 \int_{-1}^{t} (1 + x_1 \bar{z})^{-3} - 2s_2(1 + x_2 \bar{z})^{-3} \frac{1}{(x_1 - x_2)} \, d\mathfrak{z}
+ 2(1 - x_1)(1 - x_2).
\]

Then
\[
F(\mathfrak{z}) = \int_{-1}^{\mathfrak{z}} P(t) \, dt
\]
gives a bona fide solution if and only if $\kappa_1$ and $\kappa_2$ are such that
\begin{equation}
(30) \quad P(1) = -2(1 + x_1)(1 + x_2), \quad \int_{-1}^{1} P(t) \, dt = 0,
\end{equation}
and $F(\beta) > 0$ for $-1 < \beta < 1$. We calculate that
\begin{align*}
P(t) &= (2s_1x_1 + 2s_2x_2)(t + 1) + (s_1 + s_2)x_1x_2(t^2 - 1) + 2(1 - x_1)(1 - x_2) + \kappa_1 \left( t + 1 + (x_1 + x_2)(\frac{t^2 - 1}{2} + x_1x_2(\frac{t^3 + 1}{3}) \right) + \frac{\kappa_2}{2(x_1-x_2)} \left( x_1^2 (\frac{1}{(1+x_1)^2} - \frac{1}{(1-x_1)^2}) - x_2^2 (\frac{1}{(1+x_2)^2} - \frac{1}{(1-x_2)^2}) \right).
\end{align*}
Now the first equation of \((30)\) becomes
\begin{align*}
(1 + \frac{x_1x_2}{3})\kappa_1 + \frac{x_1^3x_2 + 2x_1^2x_2^2 - x_1x_2 + x_2^3}{(1-x_1)^2(1-x_2)^2} \kappa_2 &= -2(1 + x_1x_2) - 2(s_1x_1 + s_2x_2),
\end{align*}
while the second is
\begin{align*}
(1 + \frac{x_1x_2}{3} - \frac{x_1+x_2}{3})\kappa_1 + \frac{x_1^3x_2 + 2x_1^2x_2^2 - x_1x_2 + x_2^3}{(1-x_1)^2(1-x_2)^2} \kappa_2 &= -2(1 + x_1x_2 - (x_1 + x_2)) - 2(s_1x_1 + s_2x_2) + \frac{2}{3}(s_1 + s_2)x_1x_2.
\end{align*}
Thus we have a linear system in the variables $(\kappa_1, \kappa_2)$ with coefficients determined only by $s_1, s_2, x_1,$ and $x_2$. In particular, they do not depend on $a$ and $b$. We spot right away that if $x_1 = -x_2$, but $s_1 \neq -s_2$ this system is inconsistent. If $x_1 = -x_2$ with $s_1 = s_2$, the system has an infinite number of solutions with say $\kappa_2$ as the free parameter. In particular the Kähler class determined by $x_1 = -x_2$ has a constant scalar curvature Kähler metric ($\kappa_2 = 0$) in this case. On the other hand, if $x_1 \neq -x_2$, it is elementary to check that the linear system has a unique solution $(\kappa_1, \kappa_2)$. In particular, in that case
\begin{align*}
\kappa_2 &= \frac{2(1 - x_1)^2(1 - x_2)^2(6(x_1 + x_2) - 3(s_1x_1^2 + s_2x_2^2) + (s_1 + s_2)x_1^2x_2^2)}{3(x_1 + x_2)(-4x_1^2 - x_1x_2 - x_1^3x_2 - 4x_2^2 + 8x_1^2x_2^2 - x_1x_2^3 + 3x_1^3x_2^3)},
\end{align*}
where $(-4x_1^2 - x_1x_2 - x_1^3x_2 - 4x_2^2 + 8x_1^2x_2^2 - x_1x_2^3 + 3x_1^3x_2^3)$ in the above expression is always less than zero for $0 < |x_i| < 1$. Now, using that $s_ix_i < 2$ it is not hard to check that for $0 < x_1, x_2 < 1$ ($\mathcal{L}$ from $M = \mathbb{P}(\mathcal{O} \oplus \mathcal{L}) \to \Sigma_1 \times \Sigma_2$ being positive definite), $\kappa_2$, and hence $\frac{\alpha}{\alpha_0}$, is never positive. Likewise, $\kappa_2$ is never positive for $-1 < x_1, x_2 < 0$. Moving forwards we shall, without loss of generality, assume that $0 < x_1 < 1$ and $-1 < x_2 < 0$. Unfortunately, (i) of (26) is hard to check in general and by experimenting with some examples we discovered that in some cases it is simply not satisfied. This is a situation not unlike the extremal Kähler metric situation on e.g. ruled surfaces of higher genus. For now, we shall focus on a few token examples, taking us through the various genera that may occur for $\Sigma_1$ and $\Sigma_2$, where the positivity of $F(\beta)$ may be verified directly.

3.1. Examples.
3.1.1. An example with $s_1 = -s_2 = 2$, $x_1 = 1/2$, $-1 < x_2 = x < 0$. Since $0 < x_1 < 1$ and $s_1 > 0$ while $-1 < x < 0$ and $s_2 < 0$ this corresponds to the case where $\Sigma_1$ and $\Sigma_2$ both have zero genus and $M = \mathbb{P}(O \oplus O(1, -1)) \to \mathbb{C}P^1 \times \mathbb{C}P^1$.

It is easy to check that the linear system has a unique solution unless $x = -1/2$.

If $x = -1/2$, the system simplifies to a unique equation
$$33\kappa_1 - 16\kappa_2 = -198$$
which has many solutions, one of them $\kappa_2 = 0$, i.e. a cscK metric (in fact Kähler-Einstein [12]) as it is confirmed by Theorem 9 in [3]. This theorem together with [3, Theorem 8] also tells us that no other values of $x$ correspond to Kähler classes admitting a cscK metric. If $x \neq -1/2$, we calculate that
$$\kappa_1 = \frac{6(x - 2)(3 + 2x + 7x^2)}{8 + 5x + 16x^2 + x^3}$$
and
$$\kappa_2 = \frac{-9(1 - x)^2(1 + x)^2(1 + 2x)}{8 + 5x + 16x^2 + x^3}.$$

We then observe that for $x < -1/2$, $\kappa_2 > 0$ and hence $\frac{\partial}{\partial t} > 0$. For instance, for $x = -3/4$,
$$P(t) = \frac{1}{3284(2 + t)^2(4 - 3t)^2} (-57636 - 396428t + 431692t^2 + 369508t^3 - 304629t^4 - 150804t^5 + 60291t^6 + 25839t^7)$$

We observe that $P(t)$ is positive at $t = -1$, negative at $t = 1$, and changes sign only once in the interval $-1 < t < 1$. Since $\int_{-1}^{1} P(t) \, dt = 0$, it is therefore clear that $F(\bar{z}) = \int_{-1}^{\bar{z}} P(t) \, dt > 0$ for $-1 < \bar{z} < 1$.

**Proposition - Example 2.** On $M = \mathbb{P}(O \oplus O(1, -1)) \to \mathbb{C}P^1 \times \mathbb{C}P^1$, with complex structure $J$, there exists integral classes $L$ and $E$, a Kähler metric $\omega \in 2\pi c_1(L)$, and a connection $A \in \mathcal{A}^{1,1}_J(E)$ such that the triple $(\omega, J, A)$ is a solution to the coupled equations (2) and for which constants $(\alpha_0, \alpha_1, \alpha_2)$ satisfy $\frac{\partial}{\partial \alpha} > 0$. Further, the Kähler class $2\pi c_1(L)$ admits no constant scalar curvature Kähler metric.

**Remark 3.1.** Note that $M$ is a toric bundle on compact homogeneous manifolds, so two torus invariant Kähler metrics can be joined by a smooth geodesic, see [10, Theorem 4]. We also remark that $M$ is a standard compact almost homogeneous space with two ends [11, Theorem 12.1]. One can apply [9, Theorem 2 & 3] to deduce the existence of an extremal metric in each Kähler class. Note that from [11, Section 12], we know that the geodesics on $M$ satisfy all stability principles and in particular are smooth. Unfortunately we cannot apply the results of [7, Section 3.5] that hold only in dimension 2 to deduce the uniqueness of the solution to the coupled equations. Nevertheless, we conjecture that the constructed solution $(g, A)$ is unique.
3.1.1. As before, we observe that \( \Psi_0 \) changes sign only once in the interval \( (-1,1) \) and for which constants \( (\alpha_0, \alpha_1, \alpha_2) \) satisfy \( \frac{\alpha_1}{\alpha_0} > 0 \). Furthermore

\[
P(t) = \frac{1}{1962(3-t)^2(2+t)^7} \left( 12636 - 120588t - 85289t^2 + 33646t^3 + 24982t^4 - 5012t^5 - 2033t^6 + 394t^7 \right)
\]

As before, we observe that \( P(t) \) is positive at \( t = -1 \), negative at \( t = 1 \), and changes sign only once in the interval \(-1 < t < 1\). Since \( f_{-1}^1 P(t) \, dt = 0 \), it is therefore clear that \( F(\delta) = \int_{-1}^{1} P(t) \, dt > 0 \) for \(-1 < \delta < 1\).

**Proposition - Example 3.** Let \( M = \mathbb{P}(O \oplus L_1 \otimes L_2) \rightarrow \mathbb{C}P^1 \times T^2 \), where \( L_1 = O(1) \rightarrow \mathbb{C}P^1 \) and \( L_2 \rightarrow T^2 \) is a negative line bundle on \( T^2 \), and let \( J \) denote the complex structure. Then there exists integral classes \( L \) and \( E \), a Kähler metric \( \omega \in 2\pi c_1(L) \), and a connection \( A \in A^{1,1}_J(E) \) such that the triple \( (\omega, J, A) \) is a solution to the coupled equations \( (2) \) and for which constants \( (\alpha_0, \alpha_1, \alpha_2) \) satisfy \( \frac{\alpha_1}{\alpha_0} > 0 \). Further, the Kähler class \( 2\pi c_1(L) \) admits no constant scalar curvature Kähler metric.

3.1.3. An example with \( x_1 = 1/2, x_2 = -1/3, s_1 = 2, s_2 = 2 \). We apply a similar method to Subsection 3.1.1, fixing \( \Sigma_1 \) with genus 0 (and \( k_1 = 1 \)) and \( \Sigma_2 \) with genus 1 (and \( k_2 < 0 \)). By applying condition (19) from [3], it is easy to check that the Kähler class corresponding to these parameters does not admit a constant scalar curvature Kähler metric. In that case \( \kappa_2 = \frac{608}{327} > 0 \) so \( \frac{\alpha_1}{\alpha_0} > 0 \). Furthermore

\[
P(t) = \frac{2}{981(3-t)^2(2+t)^7} \left( 3456 - 25860t - 21568t^2 + 3239t^3 + 6188t^4 - 319t^5 - 502t^6 + 50t^7 \right)
\]

As before, we observe that \( P(t) \) is positive at \( t = -1 \), negative at \( t = 1 \), and changes sign only once in the interval \(-1 < t < 1\). Since \( f_{-1}^1 P(t) \, dt = 0 \), it is therefore clear that \( F(\delta) = \int_{-1}^{1} P(t) \, dt > 0 \) for \(-1 < \delta < 1\).

**Proposition - Example 4.** Let \( M = \mathbb{P}(O \oplus L_1 \otimes L_2) \rightarrow \mathbb{C}P^1 \times \Sigma \), where \( \Sigma \) is a Riemann surface of genus at least 2, \( L_1 = O(1) \rightarrow \mathbb{C}P^1 \), and \( L_2 \rightarrow \Sigma \) is \( K_{\Sigma}^{\mathbb{C}} \) tensored by a flat line bundle on \( \Sigma \). If \( J \) denotes the complex structure, then there exists integral classes \( L \) and \( E \), a Kähler metric \( \omega \in 2\pi c_1(L) \), and a connection \( A \in A^{1,1}_J(E) \) such that the triple \( (\omega, J, A) \) is a solution to the coupled equations \( (2) \) and for which constants \( (\alpha_0, \alpha_1, \alpha_2) \) satisfy \( \frac{\alpha_1}{\alpha_0} > 0 \). Further, the Kähler class \( 2\pi c_1(L) \) admits no constant scalar curvature Kähler metric.

3.1.4. An example with \( x_1 = 1/2, x_2 = -2/5, s_1 = 0, s_2 = 2 \). We apply a similar method to Subsection 3.1.1, fixing \( \Sigma_1 \) with genus 1 (and \( k_1 > 0 \))
and $\Sigma_2$ with genus $h_2 > 1$ (and $k_2 = 1 - h_2$). By applying condition (19) from [3], it is easy to check that the Kähler class corresponding to these parameters does not admit a constant scalar curvature Kähler metric. In that case $\kappa_2 = \frac{1029}{1175} > 0$ so $\frac{\alpha_1}{\alpha_0} > 0$. Furthermore

$$P(t) = \frac{1}{5310(2+t)^2(5-2t)^2} \left(57622 - 777868t - 363069t^2 + 225660t^3 + 108333t^4 - 16656t^5 - 7852t^6 - 368t^7\right)$$

As before, we observe that $P(t)$ is positive at $t = -1$, negative at $t = 1$, and changes sign only once in the interval $-1 < t < 1$. Since $\int_{-1}^{1} P(t) \, dt = 0$, it is therefore clear that $F(3) = \int_{-1}^{3} P(t) \, dt > 0$ for $-1 < 3 < 1$.

**Proposition - Example 5.** Let $M = \mathbb{P}(\mathcal{O} \oplus \mathcal{L}_1 \otimes \mathcal{L}_2) \to T^2 \times \Sigma$, where $\Sigma$ is a Riemann surface of genus at least 2, $\mathcal{L}_1 \to T^2$ is a positive holomorphic line bundle, and $\mathcal{L}_2 \to \Sigma$ is $K_{\Sigma}^{-1}$ tensored by a flat line bundle on $\Sigma$. If $J$ denotes the complex structure, then there exists integral classes $L$ and $E$, a Kähler metric $\omega \in 2\pi c_1(L)$, and a connection $A \in \mathcal{A}^{1,1}(E)$ such that the triple $(\omega, J, A)$ is a solution to the coupled equations (2) and for which constants $(\alpha_0, \alpha_1, \alpha_2)$ satisfy $\frac{\alpha_1}{\alpha_0} > 0$. Further, the Kähler class $2\pi c_1(L)$ admits no constant scalar curvature Kähler metric.

3.1.5. An example with $x_1 = 1/2$, $x_2 = -4/9$, $s_1 = -1$, $s_2 = 2$. We apply a similar method to Subsection 3.1.1, fixing $\Sigma_1$ with genus $h_1 > 1$ (and $k_1 = 2(1 - h_1)$) and $\Sigma_2$ with genus $h_2 > 1$ (and $k_2 = 1 - h_2$). By applying condition (19) from [3], it is easy to check that the Kähler class corresponding to these parameters does not admit a constant scalar curvature Kähler metric. In that case $\kappa_2 = \frac{71825}{73680} > 0$ so $\frac{\alpha_1}{\alpha_0} > 0$. Furthermore

$$P(t) = \frac{1}{343408(2+t)^2(5-2t)^2} \left(6466113 - 159543216t - 40474082t^2 + 54232672t^3 + 11937913t^4 - 1961120t^5 - 731312t^6 - 579968t^7\right)$$

As before, we observe that $P(t)$ is positive at $t = -1$, negative at $t = 1$, and changes sign only once in the interval $-1 < t < 1$. Since $\int_{-1}^{1} P(t) \, dt = 0$, it is therefore clear that $F(3) = \int_{-1}^{3} P(t) \, dt > 0$ for $-1 < 3 < 1$.

**Proposition - Example 6.** Let $M = \mathbb{P}(\mathcal{O} \oplus \mathcal{L}_1 \otimes \mathcal{L}_2) \to \Sigma_1 \times \Sigma_2$, where $\Sigma_i$ is a Riemann surface of genus at least 2, $\mathcal{L}_1 \to \Sigma_1$ is $K_{\Sigma_1}$ tensored by a flat line bundle on $\Sigma_1$, and $\mathcal{L}_2 \to \Sigma_2$ is $K_{\Sigma_2}^{-1}$ tensored by a flat line bundle on $\Sigma_2$. If $J$ denotes the complex structure, then there exists integral classes $L$ and $E$, a Kähler metric $\omega \in 2\pi c_1(L)$, and a connection $A \in \mathcal{A}^{1,1}(E)$ such that the triple $(\omega, J, A)$ is a solution to the coupled equations (2) and for which constants $(\alpha_0, \alpha_1, \alpha_2)$ satisfy $\frac{\alpha_1}{\alpha_0} > 0$. Further, the Kähler class $2\pi c_1(L)$ admits no constant scalar curvature Kähler metric.

The computations in the beginning of Section 2.1 are still valid in the present setting and using that $\frac{\alpha_1}{\alpha_0} = \frac{\alpha_2}{b^2}$ and $\frac{\alpha_2}{\alpha_0} = \frac{3\alpha}{b^2}\kappa_2 - \kappa_1$, and $z = 3\alpha$, 

we get that

\[
CYM(g, A) = \left\| \text{Scal}(g) - 2\frac{\alpha_1}{\alpha_0}|F_A|^2 - \frac{\alpha_2}{\alpha_0} + 2\frac{\alpha_1}{\alpha_0}|z|^2 \right\|_{L^2}^2
\]

\[
+ \left( 1 + \frac{\kappa_1\kappa_2 + \frac{\alpha_1^2}{\alpha_0}\kappa_2^2}{b^2} \right) \left\| \Lambda \omega F_A - z \right\|_{L^2}^2
\]

\[+ \delta''(E, [\omega], M, \alpha). \]

Since, as we mentioned earlier, \( \kappa_1 \) and \( \kappa_2 \) depend only on \( x_1, x_2, s_1, \) and \( s_2, \) we may conclude that for \( |b| \) sufficiently large the solutions we constructed above to the coupled system are actually minima of the CYM functional.

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