FOURIER CHARACTERIZATIONS OF PILIPOVIĆ SPACES

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Abstract. Let \( f \) be a function or distribution on \( \mathbb{R}^d \). We show that \( f \) belongs to a certain Pilipović space, if and only if \( f \) and suitable partial fractional Fourier transforms of \( f \) satisfy certain types of estimates.

0. Introduction

In the paper we find characterizations of Pilipović spaces in terms of estimates of suitable fractional Fourier transforms of the involved functions. Pilipović spaces is a family of function spaces which contains any standard Fourier invariant Gelfand-shilov space. The Pilipović spaces \( H_s(\mathbb{R}^d) \) for \( s \geq 0 \) and \( H_{0,s}(\mathbb{R}^d) \) for \( s > 0 \), contain all finite linear combinations of Hermite functions and are dense in the Schwartz space \( \mathcal{S}(\mathbb{R}^d) \) (See [17] or Section 1 for notations.) One has

\[
H_{s_1}(\mathbb{R}^d) \subseteq H_{0,s_2}(\mathbb{R}^d) \subseteq H_{s_2}(\mathbb{R}^d), \quad 0 \leq s_1 < s_2.
\]

The Pilipović spaces increase with the parameter \( s \) above and are strongly related to the Gelfand-Shilov spaces \( S_s(\mathbb{R}^d) \) and \( \Sigma_s(\mathbb{R}^d) \) of Roumieu and Beurling types, respectively, which consist of all \( f \in \mathcal{S}(\mathbb{R}^d) \) such that

\[
\sup_{x \in \mathbb{R}^d} \sup_{\alpha, \beta \in \mathbb{N}^d} \left( \frac{|x^\alpha D^\beta f(x)|}{h^{\alpha + \beta}(|\alpha||\beta|^s)} \right) < \infty \tag{0.1}
\]

holds true for some \( h > 0 \) respectively for every \( h > 0 \). In fact, we have

\[
H_{s_1}(\mathbb{R}^d) = S_{s_1}(\mathbb{R}^d), \quad H_{0,s_2}(\mathbb{R}^d) = \Sigma_{s_2}(\mathbb{R}^d), \quad s_2 > s_1 \geq \frac{1}{2}
\]

but

\[
H_{s_1}(\mathbb{R}^d) \neq S_{s_1}(\mathbb{R}^d) = \{0\}, \quad H_{0,s_2}(\mathbb{R}^d) \neq \Sigma_{s_2}(\mathbb{R}^d) = \{0\}, \quad s_1 < s_2 \leq \frac{1}{2}
\]

In particular, it follows that the functions in Pilipović spaces obey strong ultra-differentiability conditions and strong exponential type decay bounds at infinity, because similar facts hold true for Gelfand-Shilov spaces.

There are several characterizations of Gelfand-Shilov spaces. One of the most important is described in the following proposition, which characterize such spaces in terms of estimates of the involved elements and their Fourier transforms, established by Eijndhoven in [10] and Chung, Chung and Kim in [8].

Proposition 0.1. Let \( s \geq 0 \) and \( f \in \mathcal{S}(\mathbb{R}^d) \). Then the following conditions are equivalent:

1. \( f \in S_s(\mathbb{R}^d) \) (\( f \in \Sigma_s(\mathbb{R}^d) \));
2. \(|f(x)| \lesssim e^{-r|x|^s} \) and \(|f(\xi)| \lesssim e^{-r|\xi|^s} \) for some \( r > 0 \) (for every \( r > 0 \)).

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Pilipović spaces can be defined in similar ways when \( s \geq 0 \) is real. In fact, for such \( s \), Pilipović spaces can be defined by replacing the operators \( x^\alpha \) and \( D_x^\beta \) in (0.1) with \( H^{\frac{s}{2}} \), where \( H = |x|^2 - \Delta_x \) is the harmonic oscillator. That is, (0.1) should be replaced by

\[
\sup_{x \in \mathbb{R}^d} \sup_{N \geq 0} \left( \frac{|H^N f(x)|}{h^N N!^{2s}} \right) < \infty. \tag{0.4}
\]

In the following proposition we characterize of Pilipović spaces in terms of estimates of the Hermite coefficients and the short-time Fourier transform \( V_\vartheta f \) of \( f \). Here we recall that any \( f \in \mathcal{S}'(\mathbb{R}^d) \) may in a unique way be expressed as a Hermite series expansion

\[
f(x) = \sum_{\alpha \in \mathbb{N}^d} c(f; \alpha) h_\alpha,
\]

where \( h_\alpha \) is the Hermite function of order \( \alpha \in \mathbb{N}^d \).

**Proposition 0.2.** Let \( s \geq 0, \phi(x) = \pi^{-\frac{d}{2}} e^{-\frac{1}{2}|x|^2} \),

\[
\vartheta_{r,s}(x, \xi) = \begin{cases} \ e^{-r|x|^2 + |\xi|^2} & , \ s > \frac{1}{2}, \ r > 0, \\ \ e^{-\left(\frac{1}{2} - r\right)|x|^2 + |\xi|^2} & , \ s = \frac{1}{2}, \ 0 < r < \frac{1}{2}, \\ \ e^{-\frac{1}{4}(|x|^2 + |\xi|^2) + r(\log(1+|x|+|\xi|)) \frac{1}{1 - 2r}} & , \ s < \frac{1}{2}, \ r > 0, \end{cases}
\]

\( f \in \mathcal{S}(\mathbb{R}^d) \) and let \( c(f; \alpha) \) be given by (0.5). Then the following conditions are equivalent:

1. \( f \in \mathcal{H}_s(\mathbb{R}^d) \) (\( f \in \mathcal{H}_{0,s}(\mathbb{R}^d) \));
2. (0.4) holds for some \( h > 0 \) (for every \( h > 0 \));
3. \( |c(f; \alpha)| \lesssim e^{-r|\alpha|^2} \) for some \( r > 0 \) (for every \( r > 0 \));
4. \( |V_\vartheta f(x, \xi)| \lesssim \vartheta_{r,s}(x, \xi) \) for some \( r > 0 \) (for every \( r > 0 \)).

In Proposition 0.2, the equivalence between (1) and (2) essentially follows from the definitions, the equivalence between (1) and (3) was established in [19] [20] for \( s \geq \frac{1}{2} \) and in [28] for general \( s \). The equivalence between (1) and (4) was established in [16] by Gröchenig and Zimmermann for \( \mathcal{H}_s(\mathbb{R}^d) \) when \( s \geq \frac{1}{2} \) and for \( \mathcal{H}_{0,s}(\mathbb{R}^d) \) when \( s > \frac{1}{2} \). For general \( s \), the equivalence between (1) and (4) is obtained in [28].

We observe that in view of (0.2), it follows that Proposition 0.2 give some further characterizations for Gelfand-Shilov spaces.

Despite Proposition 0.2 contains several characterizations for Pilipović spaces, there are no characterization of the form Proposition 0.1 in terms of estimates of the involved functions and their Fourier transforms. By modifying the weight function \( \vartheta_{r,s}(x, \xi) \) in (0.6) into

\[
\omega_{r,s}(x) = \begin{cases} \ e^{-r|x|^2} & , \ s > \frac{1}{2}, \ r > 0, \\ \ e^{-\left(\frac{1}{2} - r\right)|x|^2} & , \ s = \frac{1}{2}, \ 0 < r < \frac{1}{2}, \\ \ e^{-\frac{1}{4}|x|^2 + r(\log(1+|x|)) \frac{1}{1 - 2r}} & , \ s < \frac{1}{2}, \ r > 0, \end{cases}
\]

it follows by some manipulations that

\[
f \in \mathcal{H}_s(\mathbb{R}^d) \ (f \in \mathcal{H}_{0,s}(\mathbb{R}^d)) \ \Rightarrow \ |f(x)| \lesssim \omega_{r,s}(x), \ |\hat{f}(\xi)| \lesssim \omega_{r,s}(\xi), \tag{0.8}
\]

for some \( r > 0 \) (for every \( r > 0 \)). More generally, since \( \mathcal{H}_s(\mathbb{R}^d) \) and \( \mathcal{H}_{0,s}(\mathbb{R}^d) \) are invariant under any (partial) fractional Fourier transform (0.8) generalizes into

\[
f \in \mathcal{H}_s(\mathbb{R}^d) \ (f \in \mathcal{H}_{0,s}(\mathbb{R}^d)) \ \Rightarrow \ |(\mathcal{F} f)(x)| \lesssim \omega_{r,s}(x), \ t \in \mathbb{R}^d, \tag{0.8'}
\]
for some $r > 0$ (for every $r > 0$). (See Proposition 2.4 in Section 2.) The originally searched result was to show that equivalence occurs in (0.8), which should give characterizations of Pilipović spaces in terms of convenient estimates of the involved functions and their Fourier transforms. Unfortunately, those techniques which we are aware of, seem to be insufficient for such equivalence. On the other hand, by some straightforward estimates it turns out that equivalence occurs in (0.8)\textsuperscript{′}, i.e.
\[ |(\mathcal{F}_t f)(x)| \lesssim \omega_{r,s}(x), \quad t \in \mathbb{R}^d, \quad (0.8)\textsuperscript{′} \]
for some $r > 0$ (for every $r > 0$), giving a weaker form of searched characterization.

In fact, for suitable lattices $\Lambda \subseteq \mathbb{R}^d$ we improve (0.8)\textsuperscript{′} into
\[ f \in \mathcal{H}_s(\mathbb{R}^d) \quad (f \in \mathcal{H}_{0,s}(\mathbb{R}^d)) \iff |(\mathcal{F}_t f)(x)| \lesssim \omega_{r,s}(x), \quad t \in \Lambda, \quad (0.8)\textsuperscript{″} \]
for some $r > 0$ (for every $r > 0$). (See Theorems 2.2 and 2.3.)

In order to reach (0.8)\textsuperscript{″}, we observe that the right implication follows from (0.8)\textsuperscript{′}. The left implication is managed in a different way, based on a multi-dimensional version of Phragmén-Lindelöf’s theorem, which seems not so easy to be found in the literature. For this reason we have included an induction proof of this result in Appendix A, which might be of some independent interest.

The paper is organized as follows. In Section 1 we recall the definition and some basic properties of Gelfand-Shilov spaces and Pilipović spaces. In Section 2 we show the equivalences (0.8)\textsuperscript{″} and (0.8)\textsuperscript{″}. In Appendix A we deduce multi-dimensional estimates of Phragmén-Lindelöf type. In Appendix B we recall some important links between Bargmann transforms and short-time Fourier transforms, and discuss compositions of such transforms with fractional Fourier transforms.

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1. Preliminaries

1.1. Gelfand-Shilov spaces. Let $0 < s \in \mathbb{R}$ be fixed. Then the (Fourier-invariant) Gelfand-Shilov space $\mathcal{S}_s(\mathbb{R}^d)$ (of Roumieu type (Beurling type)) consists of all $f \in C^\infty(\mathbb{R}^d)$ such that
\[ \|f\|_{\mathcal{S}_s} \equiv \sup_{x, \alpha, \beta} \frac{|x^\alpha \partial^\beta f(x)|}{h^{\alpha+\beta}|\alpha!\beta!|}, \quad (1.1) \]
is finite for some $h > 0$ (for every $h > 0$). Here the supremum should be taken over all $\alpha, \beta \in \mathbb{N}^d$ and $x \in \mathbb{R}^d$. The semi-norms $\| \cdot \|_{\mathcal{S}_s}$ induce an inductive limit topology for the space $\mathcal{S}_s(\mathbb{R}^d)$ and projective limit topology for $\Sigma_s(\mathbb{R}^d)$, and the latter space becomes a Fréchet space under this topology.

The space $\mathcal{S}_s(\mathbb{R}^d) \neq \{0\}$ ($\Sigma_s(\mathbb{R}^d) \neq \{0\}$), if and only if $s \geq \frac{1}{2}$ ($s > \frac{1}{2}$).

The Gelfand-Shilov distribution spaces $\mathcal{S}'_s(\mathbb{R}^d)$ and $\Sigma'_s(\mathbb{R}^d)$ are the dual spaces of $\mathcal{S}_s(\mathbb{R}^d)$ and $\Sigma_s(\mathbb{R}^d)$, respectively.
We have
\[
S_{1/2}(\mathbb{R}^d) \hookrightarrow \Sigma_s(\mathbb{R}^d) \hookrightarrow S_s(\mathbb{R}^d) \hookrightarrow \Sigma_t(\mathbb{R}^d)
\]
\[
\hookrightarrow \mathcal{S}(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d) \hookrightarrow \Sigma_t'(\mathbb{R}^d)
\]
\[
\hookrightarrow S_{t}^\prime(\mathbb{R}^d) \hookrightarrow \Sigma_s^\prime(\mathbb{R}^d) \hookrightarrow S_{t/2}^\prime(\mathbb{R}^d), \quad \frac{1}{2} < s < t.
\]

Here and in what follows we use the notation \( A \hookrightarrow B \) when the topological spaces \( A \) and \( B \) satisfy \( A \subseteq B \) with continuous embeddings.

A convenient family of functions concerns the Hermite functions
\[
h_\alpha(x) = \pi^{-\frac{d}{4}} (-1)^{|\alpha|} (2^{|\alpha|} \alpha!)^{-\frac{1}{2}} e^{-\frac{|x|^2}{4}} (\partial^\alpha e^{-|x|^2}), \quad \alpha \in \mathbb{N}^d.
\]
The set of Hermite functions on \( \mathbb{R}^d \) is an orthonormal basis for \( L^2(\mathbb{R}^d) \). It is also a basis for the Schwartz space and its distribution space, and for any \( \Sigma_s \) when \( s > \frac{1}{2} \), \( \mathcal{S}_s \) when \( s \geq \frac{1}{2} \) and their distribution spaces. They are also eigenfunctions to the Harmonic oscillator \( H = H_\delta \equiv |x|^2 - \Delta \) and to the Fourier transform \( \mathcal{F} \), given by
\[
(\mathcal{F} f)(\xi) = \hat{f}(\xi) \equiv (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(x)e^{-i\langle x,\xi \rangle} \, dx, \quad \xi \in \mathbb{R}^d,
\]
when \( f \in L^1(\mathbb{R}^d) \). Here \( \langle \cdot, \cdot \rangle \) denotes the usual scalar product on \( \mathbb{R}^d \). In fact, we have
\[
H_\delta h_\alpha = (2|\alpha| + d) h_\alpha.
\]

The Fourier transform \( \mathcal{F} \) extends uniquely to homeomorphisms on \( \mathcal{S}'(\mathbb{R}^d) \), \( S_s^\prime(\mathbb{R}^d) \) and on \( \Sigma_t^\prime(\mathbb{R}^d) \). Furthermore, \( \mathcal{F} \) restricts to homeomorphisms on \( \mathcal{S}'(\mathbb{R}^d) \), \( S_s(\mathbb{R}^d) \) and on \( \Sigma_t(\mathbb{R}^d) \), and to a unitary operator on \( L^2(\mathbb{R}^d) \). Similar facts hold true when the Fourier transform is replaced by a partial Fourier transform.

Gelfand-Shilov spaces and their distribution spaces can also be characterized by estimates of short-time Fourier transform, (see e.g. [16,23,28]). More precisely, let \( \phi \in \mathcal{S}(\mathbb{R}^d) \) be fixed. Then the short-time Fourier transform \( V_\phi f \) of \( f \in \mathcal{S}'(\mathbb{R}^d) \) with respect to the window function \( \phi \) is the Gelfand-Shilov distribution on \( \mathbb{R}^{2d} \), defined by
\[
V_\phi f(x,\xi) = \mathcal{F} \left( f \phi(\cdot - x) \right)(\xi), \quad x, \xi \in \mathbb{R}^d.
\]
If \( f, \phi \in \mathcal{S}(\mathbb{R}^d) \), then it follows that
\[
V_\phi f(x,\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(y)\phi(y-x)e^{-i\langle y,\xi \rangle} \, dy, \quad x, \xi \in \mathbb{R}^d.
\]

By [27, Theorem 2.3] it follows that the definition of the map \( (f, \phi) \mapsto V_\phi f \) from \( \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \) to \( \mathcal{S}(\mathbb{R}^{2d}) \) is uniquely extendable to a continuous map from \( S_s^\prime(\mathbb{R}^d) \times S_s^\prime(\mathbb{R}^d) \) to \( S_s^\prime(\mathbb{R}^{2d}) \), and restricts to a continuous map from \( S_s(\mathbb{R}^d) \times S_s(\mathbb{R}^d) \) to \( S_s(\mathbb{R}^{2d}) \). The same conclusion holds with \( \Sigma_s \) in place of \( \mathcal{S}_s \), at each place.

In the following propositions we give characterizations of Gelfand-Shilov spaces and their distribution spaces in terms of estimates of the short-time Fourier transform. We omit the proof since the first part follows from [16, Theorem 2.7] and the second part from [28, Proposition 2.2]. See also [9] for related results.

**Proposition 1.1.** Let \( s \geq \frac{1}{2} \) (\( s > \frac{1}{2} \)), \( \phi \in S_s(\mathbb{R}^d) \setminus 0 \) \((\phi \in \Sigma_s(\mathbb{R}^d) \setminus 0)\) and let \( f \) be a Gelfand-Shilov distribution on \( \mathbb{R}^d \). Then the following is true:

1. \( f \in S_s(\mathbb{R}^d) \) (\( f \in \Sigma_s(\mathbb{R}^d) \)), if and only if
\[
|V_\phi f(x,\xi)| \lesssim e^{-r(|x|^{\frac{1}{2}} + |\xi|^{\frac{1}{2}})}, \quad x, \xi \in \mathbb{R}^d,
\]
for some \( r > 0 \) (for every \( r > 0 \)).
(2) \( f \in \mathcal{S}_h'(\mathbb{R}^d) \) \((f \in \Sigma_s'(\mathbb{R}^d))\), if and only if
\[
|V_\phi f(x, \xi)| \leq e^{r(|\xi|^2 + |x|^2)}, \quad x, \xi \in \mathbb{R}^d,
\]
for every \( r > 0 \) (for some \( r > 0 \)).

1.2. Spaces of Hermite series expansions. Next we recall the definitions of topological vector spaces of Hermite series expansions, given in [28]. As in [28], it is convenient to use suitable extensions of \( \mathbb{R}_+ \) when indexing our spaces.

**Definition 1.2.** The sets \( \mathbb{R}_s \) and \( \overline{\mathbb{R}}_s \) are given by
\[
\mathbb{R}_s = \mathbb{R}_+ \bigcup \{b_r\} \quad \text{and} \quad \overline{\mathbb{R}}_s = \mathbb{R}_s \bigcup \{0\}.
\]
Moreover, beside the usual ordering in \( \mathbb{R} \), the elements \( b_r \) in \( \mathbb{R}_s \) and \( \overline{\mathbb{R}}_s \) are ordered by the relations \( x_1 < b_{\sigma_1} < b_{\sigma_2} < x_2 \), when \( \sigma_1, \sigma_2, x_1 \) and \( x_2 \) are positive real numbers such that \( x_1 < \frac{1}{2} \) and \( x_2 \geq \frac{1}{2} \).

**Definition 1.3.** Let \( p \in [1, \infty] \), \( s \in \mathbb{R}_s \), \( r \in \mathbb{R} \), \( \sigma \) be a weight on \( \mathbb{N}^d \), and let
\[
\vartheta_{r,s}(\alpha) = \begin{cases} 
 e^{r|\alpha|} , & \text{when } s \in \mathbb{R}_+, \\
 e^{r|\alpha|} , & \text{when } s = b_r, \quad \alpha \in \mathbb{N}^d.
\end{cases}
\]

Then,
1. \( \ell^p_0(\mathbb{N}^d) \) is the set of all sequences \( \{c(\alpha)\}_{\alpha \in \mathbb{N}^d} \subseteq \mathbb{C} \) on \( \mathbb{N}^d \);
2. \( \ell^p_0(\mathbb{N}^d) \equiv \{0\} \), and \( \ell^p_0(\mathbb{N}^d) \) is the set of all sequences \( \{c(\alpha)\}_{\alpha \in \mathbb{N}^d} \subseteq \mathbb{C} \) such that \( c(\alpha) \neq 0 \) for at most finite numbers of \( \alpha \);
3. \( \ell^p_0(\mathbb{N}^d) \) is the Banach space which consists of all sequences \( \{c(\alpha)\}_{\alpha \in \mathbb{N}^d} \subseteq \mathbb{C} \) such that
\[
\|\{c(\alpha)\}_{\alpha \in \mathbb{N}^d}\|_{\ell^p_0} \equiv \|\{c(\alpha)\}_{\alpha \in \mathbb{N}^d}\|_{\ell^p} < \infty;
\]
4. \( \ell_{0,s}(\mathbb{N}^d) \equiv \bigcap_{r>0} \ell^p_{\vartheta_{r,s}}(\mathbb{N}^d) \) and \( \ell_{s}(\mathbb{N}^d) \equiv \bigcup_{r>0} \ell^p_{\vartheta_{r,s}}(\mathbb{N}^d) \), with projective respective inductive limit topologies of \( \ell^p_{\vartheta_{r,s}}(\mathbb{N}^d) \) with respect to \( r > 0 \);
5. \( \ell_{0,s}(\mathbb{N}^d) \equiv \bigcap_{r>0} \ell^p_{\vartheta_{r,s}}(\mathbb{N}^d) \) and \( \ell_{s}'(\mathbb{N}^d) \equiv \bigcup_{r>0} \ell^p_{\vartheta_{r,s}}(\mathbb{N}^d) \), with inductive respective projective limit topologies of \( \ell^p_{\vartheta_{r,s}}(\mathbb{N}^d) \) with respect to \( r > 0 \).

Let \( p \in [1, \infty] \), and let \( \Omega_N \) be the set of all \( \alpha \in \mathbb{N}^d \) such that \( |\alpha| \leq N \). Then the topology of \( \ell^p_0(\mathbb{N}^d) \) is defined by the inductive limit topology of the sets
\[
\{ \{c(\alpha)\}_{\alpha \in \mathbb{N}^d} \in \ell^p_0(\mathbb{N}^d) \mid c(\alpha) = 0 \text{ when } \alpha \not\in \Omega_N \}
\]
with respect to \( N \geq 0 \), and whose topology is given through the semi-norms
\[
\|\{c(\alpha)\}_{\alpha \in \mathbb{N}^d}\|_{\ell^p_0(\Omega_N)} \equiv \|\{c(\alpha)\}_{\alpha \in \mathbb{N}^d}\|_{\ell^p} < \infty.
\]
(1.5)

It is clear that these topologies are independent of \( p \). Furthermore, the topology of \( \ell^p_0(\mathbb{N}^d) \) is defined by the semi-norms (1.5). It follows that \( \ell^p_0(\mathbb{N}^d) \) is a Fréchet space, and that the topology is independent of \( p \).

Next we introduce spaces of formal Hermite series expansions
\[
f = \sum_{\alpha \in \mathbb{N}^d} c(f, \alpha) h_\alpha, \quad \{c(f, \alpha)\}_{\alpha \in \mathbb{N}^d} \in \ell^p_0(\mathbb{N}^d),
\]
which correspond to
\[
\ell_{0,s}(\mathbb{N}^d), \ell_{s}(\mathbb{N}^d), \ell_{0,s}'(\mathbb{N}^d) \quad \text{and} \quad \ell_{s}'(\mathbb{N}^d).
\]
We consider the mappings
\[ T_H : \{ c(\alpha) \}_{\alpha \in \mathbb{N}^d} \mapsto \sum_{\alpha \in \mathbb{N}^d} c(\alpha) h_\alpha \]  \hspace{1cm} (1.8)

between sequences and formal Hermite series expansions.

**Definition 1.4.** If \( s \in \mathbb{R}_0^{+} \), then
\[ \mathcal{H}_{0,s}(\mathbb{R}^d), \mathcal{H}_s(\mathbb{R}^d), \mathcal{H}_s'(\mathbb{R}^d) \text{ and } \mathcal{H}_{0,s}'(\mathbb{R}^d), \]  \hspace{1cm} (1.9)
are the images of \( T_H \) respectively in \( \mathcal{L}_{\mathcal{S}}(\mathbb{R}^d) \) of corresponding spaces in \( \mathcal{L}_{\mathcal{S}}(\mathbb{R}^d) \). The topologies of the spaces in \( \text{(1.9)} \) are inherited from the corresponding spaces in \( \text{(1.7)} \).

We recall that \( f \in \mathcal{S}(\mathbb{R}^d) \) if and only if it can be written as \( (1.6) \) such that
\[ |c(f, \alpha)| \lesssim \langle \alpha \rangle^{-N}, \]
for every \( N \geq 0 \) (cf. e.g. \[21\]). In particular it follows from the definitions that the inclusions
\[ \mathcal{H}_0(\mathbb{R}^d) \hookrightarrow \mathcal{H}_{0,s}(\mathbb{R}^d) \hookrightarrow \mathcal{H}_s(\mathbb{R}^d) \hookrightarrow \mathcal{H}_{0,t}(\mathbb{R}^d) \]
\[ \hookrightarrow \mathcal{S}(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d) \hookrightarrow \mathcal{H}_{0,t}'(\mathbb{R}^d) \hookrightarrow \mathcal{H}_s'(\mathbb{R}^d) \]
\[ \hookrightarrow \mathcal{H}_{0,s}'(\mathbb{R}^d) \hookrightarrow \mathcal{H}_0'(\mathbb{R}^d), \]  \hspace{1cm} (1.10)
are dense.

**Remark 1.5.** By the definition it follows that \( T_H \) in \( \text{(1.8)} \) is a homeomorphism between any of the spaces in \( \text{(1.7)} \) and corresponding space in \( \text{(1.9)} \).

The next results give some characterizations of \( \mathcal{H}_s(\mathbb{R}^d) \) and \( \mathcal{H}_{0,s}(\mathbb{R}^d) \) when \( s \) is a non-negative real number.

**Proposition 1.6.** Let \( 0 \leq s \in \mathbb{R} \) and let \( f \in \mathcal{H}_0'(\mathbb{R}^d) \). Then \( f \in \mathcal{H}_s(\mathbb{R}^d) \) \((f \in \mathcal{H}_{0,s}(\mathbb{R}^d))\), if and only if \( f \in \mathcal{C}^{\infty}(\mathbb{R}^d) \) and satisfies
\[ \| H_d^N \mathcal{F}_d \mathcal{F} f \|_{L^\infty} \lesssim h^N N! 2^s, \]  \hspace{1cm} (1.11)
for some \( h > 0 \) (every \( h > 0 \)). Moreover, it holds
\[ \begin{align*}
\mathcal{H}_s(\mathbb{R}^d) &= \mathcal{S}_s(\mathbb{R}^d) \neq \{0\}, & \mathcal{H}_{0,s}(\mathbb{R}^d) &= \mathcal{S}_s(\mathbb{R}^d) \neq \{0\} & \text{when } s \in (\frac{1}{2}, \infty), \\
\mathcal{H}_s(\mathbb{R}^d) &= \mathcal{S}_s(\mathbb{R}^d) \neq \{0\}, & \mathcal{H}_{0,s}(\mathbb{R}^d) &= \mathcal{S}_s(\mathbb{R}^d) = \{0\} & \text{when } s = \frac{1}{2}, \\
\mathcal{H}_s(\mathbb{R}^d) &= \mathcal{S}_s(\mathbb{R}^d) = \{0\}, & \mathcal{H}_{0,s}(\mathbb{R}^d) &= \mathcal{S}_s(\mathbb{R}^d) = \{0\} & \text{when } s \in (0, \frac{1}{2}), \\
\mathcal{H}_s(\mathbb{R}^d) &\neq \mathcal{S}_s(\mathbb{R}^d) = \{0\}, & \mathcal{H}_{0,s}(\mathbb{R}^d) &\neq \mathcal{S}_s(\mathbb{R}^d) \neq \{0\} & \text{when } s = 0.
\end{align*} \]

We refer to \[28\] for the proof of Proposition \[1.6\].

Due to the pioneering investigations related to Proposition \[1.6\] by Pilipović in \[19\] \[20\], we call the spaces \( \mathcal{H}_s(\mathbb{R}^d) \) and \( \mathcal{H}_{0,s}(\mathbb{R}^d) \) Pilipović spaces of Roumieu and Beurling types, respectively. In fact, in the restricted case \( s \geq \frac{1}{2} \), Proposition \[1.6\] was proved already in \[19\] \[20\].

2. Characterizations of Pilipović spaces by estimates of the Fourier transform

In this section we deduce characterizations of Pilipović spaces in terms of estimates of the involved functions and some of their fractional Fourier transforms. These main issues are given in Theorems \[2.2\] and \[2.3\].
Theorem 2.3. Let \( \omega \in H \) and let \( \omega_d, \omega_r, \omega_s \) be given by (2.1) with \( s \leq \frac{1}{2} \). Then the following is true:
\[
C^{-1} \omega_{d,r,s}(x) \leq \omega_{d,r,s}(x + y) \leq C \omega_{d,r,s}(x) \quad \text{when} \quad R \leq |x| \leq c/|y|,
\]
for some \( R \geq 2 \) and positive constants \( c \) and \( C \). This is needed for applying Theorem 3.2 in [27].

Proposition 2.1. Let \( p \in (0, \infty] \), \( s \in \mathbb{R} \) be such that \( s \leq \frac{1}{2} \), \( \phi = \pi^{-\frac{d}{2}} e^{-\frac{1}{2}|x|^2} \), \( f \in H^0_0(\mathbb{R}^d) \), and let \( \omega_{d,r,s} \) be given by (2.1). Then the following is true:

1. if in addition \( s < \frac{1}{2} \), then \( f \in H_s(\mathbb{R}^d) \), if and only if
   \[
   \|V_0 f \cdot e^{\frac{d}{2}|\cdot|^2} \omega_{2d,r,s}\|_{L^p} < \infty
   \]
   is finite for some \( r > 0 \);

2. if in addition \( s > 0 \), then \( f \in H_{0,s}(\mathbb{R}^d) \), if and only if (2.2) is finite for every \( r > 0 \).

We have the following, where we show that elements in \( H^0_0(\mathbb{R}^d) \) belong to \( H_s(\mathbb{R}^d) \) or \( H_{0,s}(\mathbb{R}^d) \), if and only if they satisfy estimates of the form
\[
\sup_{t \in \Lambda_{t_0,u}} \left( \|(\mathcal{F}_t f) \cdot e^{\frac{d}{2}|\cdot|^2} \omega_{d,r,s}\|_{L^\infty} \right) < \infty,
\]
\[
\Lambda_{t_0,u} = \{ t_0 + (k_1u_1, \ldots, k_du_d); 0 \leq k_ju_j < 2, \ k_j \in \mathbb{Z} \},
\]
or
\[
\sup_{t \in \mathbb{R}^d} \left( \|(\mathcal{F}_t f) \cdot e^{\frac{d}{2}|\cdot|^2} \omega_{d,r,s}\|_{L^\infty} \right) < \infty.
\]

Theorem 2.2. Let \( s \in \mathbb{R} \) be such that \( s < \frac{1}{2} \), \( t_0 \in \mathbb{R}^d \), \( u \in (0, 1)^d \), \( f \in H^0_0(\mathbb{R}^d) \), and let \( \omega_{d,r,s} \) be given by (2.1). Then the following conditions are equivalent:

1. \( f \in H_s(\mathbb{R}^d) \);
2. (2.3) holds for some \( r > 0 \);
3. (2.4) holds for some \( r > 0 \).

Theorem 2.3. Let \( s \in \mathbb{R} \) be such that \( s \leq \frac{1}{2} \), \( t_0 \in \mathbb{R}^d \), \( u \in (0, 1)^d \) for \( s = \frac{1}{2} \) and \( u \in (0, 1)^d \) for \( s < \frac{1}{2} \), \( f \in H^0_0(\mathbb{R}^d) \), and let \( \omega_{d,r,s} \) be given by (2.1). Then the following conditions are equivalent:

1. \( f \in H_{0,s}(\mathbb{R}^d) \);
2. (2.3) holds for every \( r > 0 \);
3. (2.4) holds for every \( r > 0 \).

We need some preparations for the proofs. First we have the following proposition.
Proposition 2.4. Let $s \in \mathbb{R}_0$ be such that $s < \frac{1}{2}$, $f \in \mathcal{H}_s(\mathbb{R}^d)$, and $\omega_{d,r,s}$ be given by (2.1). Then
\[ \sup_{t \in \mathbb{R}^d} \| \mathcal{F}_t f \cdot e^{\frac{i}{2} |x|^2} \omega_{d,r,s} \|_{L^\infty} < \infty \tag{2.5} \]
holds for some $r > 0$ (for every $r > 0$).

Proof. We only prove the result for $r > 0$. The case $s = 0$ follows by similar arguments and is left for the reader.

Let \( \phi(x) = \pi^{-\frac{d}{4}} e^{-\frac{1}{2} |x|^2} \). By Fourier inversion formula we get
\[ \left| \int_{\mathbb{R}^d} V_\phi f(x, \xi) e^{\frac{i}{2} \langle x, \xi \rangle} \, d\xi \right| \leq |f(\frac{1}{2} x)\phi(\frac{1}{2} x - x)| = |f(\frac{1}{2} x)| e^{-\frac{1}{4} |x|^2}. \]
This gives
\[ |f(\frac{1}{2} x)| e^{-\frac{1}{4} |x|^2} \leq \int_{\mathbb{R}^d} V_\phi f(x, \xi) e^{\frac{i}{2} \langle x, \xi \rangle} \, d\xi \leq \int_{\mathbb{R}^d} |V_\phi f(x, \xi)| \, d\xi \leq \int_{\mathbb{R}^d} e^{-\frac{1}{4} (|x|^2 + |\xi|^2)} \omega_{2d, r, s}(x, \xi) \, d\xi \leq \int_{\mathbb{R}^d} e^{-\frac{1}{4} (|x|^2 + |\xi|^2)} \omega_{2d, r, s}(x) \omega_{d, c, r, s}(\xi) \, d\xi \leq e^{-\frac{1}{4} |x|^2} \omega_{d, c, r, s}(x) \]
for some $r > 0$ (every $r > 0$). Hence,
\[ |f(\frac{1}{2} x)| \leq e^{-\frac{1}{4} |x|^2} \omega_{d, c, r, s}(x) e^{\frac{i}{2} |x|^2} = e^{-\frac{1}{4} |x|^2} \omega_{d, c, r, s}(x) \]
for some $r > 0$ (every $r > 0$). Here the positive constant $c_1$ can be chosen independently on $r$. This gives
\[ |f(x)| \leq e^{-\frac{1}{4} |x|^2} \omega_{d, c, r, s}(x) \]
for some $r > 0$ (every $r > 0$), and the assertion holds for $t = 0$. Since \( \{ \mathcal{F}_t \}_{t \in \mathbb{R}^d} \) is an equicontinuous set of homeomorphisms on $\mathcal{H}_s(\mathbb{R}^d)$ and $\mathcal{H}_0(\mathbb{R}^d)$ in view of [PS, Proposition 7.1] and its proof, it follows by replacing $f$ by $\mathcal{F}_t f$ in the previous estimates that (2.5) holds for some $r > 0$ (every $r > 0$).

Proposition 2.4 shows one of the directions in Theorems 2.2 and 2.3. The converse needs more steps and we begin with the following. Here we invoke the Bargmann transform because later investigations are based on analyticity arguments. In what follows we let
\[ A_t(x, \xi) = \left( \cos(t, \frac{\xi}{2}) x + \sin(t, \frac{\xi}{2}) \xi, -\sin(t, \frac{\xi}{2}) x + \cos(t, \frac{\xi}{2}) \xi \right), \quad x, t, \xi \in \mathbb{R} \]
\[ A_{d,t}(x, \xi) = (A_t(x, \xi_1), \ldots, A_t(x, \xi_d)), \quad x, t, \xi \in \mathbb{R}^d. \tag{2.6} \]

Proposition 2.5. Let $s \in \mathbb{R}_0$ be such that $s < \frac{1}{2}$, $t \in \mathbb{R}^d$, $\phi(x) = \pi^{-\frac{d}{4}} e^{-\frac{1}{2} |x|^2}$ and let $\omega_{d,r,s}$ and $\phi_{d,t,s}(x, \xi)$ be given by (2.1) and (2.6). Then there is a constant $c > 0$, and for every $r > 0$, a constant $C = C_r > 0$ such that
\[ \sup_{x, \xi \in \mathbb{R}^d} |V_\phi f(A_{d,t}(x, \xi)) e^{\frac{i}{2} |x|^2} \omega_{d,r,s}(x)| \leq C_r \left\| \mathcal{F}_t f \cdot e^{\frac{i}{2} |x|^2} \omega_{d,c,r,s} \right\|_{L^\infty} \tag{2.7} \]
and
\[ \sup_{z \in \mathbb{C}^d} |A_{d,t}(e^{-it} \frac{\xi}{2} z_1, \ldots, e^{-it} \frac{\xi}{2} z_d) \cdot \omega_{d,r,s}(\text{Re}(z)) e^{-\frac{i}{4} |\text{Im}(z)|^2} | \]
\[ \leq C_r \left\| \mathcal{F}_t f \cdot e^{\frac{i}{2} |x|^2} \omega_{d,c,r,s} \right\|_{L^\infty} \tag{2.8} \]
for every $f \in \mathcal{H}_0(\mathbb{R}^d)$.
Proof. The estimates (2.7) and (2.8) are equivalent in view of Appendix [13] (see also [28]). Since 
\( (\mathcal{M}_d(\mathcal{P}_t f))(z) = (\mathcal{M}_d f)(e^{-it/2}z_1, \ldots, e^{-it/2}z_d), \quad z = (z_1, \ldots, z_d), \quad t = (t_1, \ldots, t_d), \) 
the proposition follows if we prove (2.7) in the case \( t = 0. \)

By Hausdorff-Young’s inequality we get for some constants \( c_1, c_2 > 0, \)

\[
sup_{x, \xi \in \mathbb{R}^d} |V_\alpha f(x, \xi) e^{\frac{1}{2}||x||^2} \omega_{d, r, s}(x)|
\]

\[
= \sup_{x, \xi \in \mathbb{R}^d} |\hat{f}(\xi - x)|(\xi) e^{\frac{1}{2}||x||^2} \omega_{d, r, s}(x)|
\]

\[
\leq \sup_{y \in \mathbb{R}^d} \left( \int |f(y)\phi(y - x)e^{\frac{1}{2}||x||^2} \omega_{d, r, s}(x)| \right) dy
\]

\[
= \sup_{y \in \mathbb{R}^d} \left( \int |(f(y)e^{\frac{1}{2}||y||^2})e^{-\frac{1}{2}||x - 2y||^2} \omega_{d, r, s}(y)| \right) dy
\]

\[
\lesssim \sup_{y \in \mathbb{R}^d} \left( \int |f(y)e^{\frac{1}{2}||y||^2} \omega_{d, r, s}(y)) | e^{-\frac{1}{2}||x - 2y||^2/\omega_{d, r, s}(y)}| \right) dy
\]

\[
\leq \sup_{y \in \mathbb{R}^d} \left( \int e^{-\frac{1}{2}||x - 2y||^2/\omega_{d, r, s}(y)} \right) \sup_{y \in \mathbb{R}^d} |f(y)e^{\frac{1}{2}||y||^2} \omega_{d, r, s}(y)|
\]

\[
\approx \sup_{y \in \mathbb{R}^d} |f(y)e^{\frac{1}{2}||y||^2} \omega_{d, r, s}(y)|.
\]

That is,

\[
\sup_{x, \xi \in \mathbb{R}^d} |V_\alpha f(x, \xi) e^{\frac{1}{2}||x||^2} \omega_{d, r, s}(x)| \lesssim \sup_{x \in \mathbb{R}} |f(x) e^{\frac{1}{2}||x||^2} \omega_{d, r, s}(x)|
\]

for some \( c > 0 \) which is independent of \( r > 0 \). This gives (2.7). \( \square \)

We also have the following.

**Proposition 2.6.** Let \( s \in \overline{\mathbb{R}}, \omega_{d, r, s} \) be given by (2.1), and let

\[
\Omega = \Gamma_1 \times \cdots \times \Gamma_d \quad \text{and} \quad \Omega_0 = \partial \Gamma_1 \times \cdots \times \partial \Gamma_d,
\]

where

\[
\Gamma_j = \{ z \in \mathbb{C} : \alpha_j \leq \text{Arg}(z) \leq \beta_j \}, \quad j = 1, \ldots, d,
\]

and \( \alpha_j, \beta_j \in \mathbb{R} \) satisfy \( 0 < \beta_j - \alpha_j < \frac{\pi}{2} \). If \( F \in A(\Omega) \) satisfies

\[
|F(z)| \lesssim e^{r_0|z|^2}, \quad z \in \Omega, \quad \text{and} \quad |F(z)| \lesssim \omega_{d, r, s}(z)^{-1}, \quad z \in \Omega_0,
\]

for some constants \( r_0, r > 0 \), then for some constant \( c \) which is independent of \( r \) and \( r_0 \), it holds

\[
|F(z)| \lesssim \omega_{d, r, s}(z)^{-1}, \quad z \in \Omega.
\]

For the proof we need the following lemma, which is a special case of Proposition A.2 in Appendix A. The proof is therefore omitted.

**Lemma 2.7.** Let \( \Omega \) and \( \Omega_0 \) be the same as in Proposition 2.6 and suppose that

\[
|F(z)| \lesssim e^{r_0|z|^2}, \quad z \in \Omega, \quad \text{and} \quad |F(z)| \leq M, \quad z \in \Omega_0,
\]

for some constants \( r_0, M > 0 \). Then

\[
|F(z)| \leq M, \quad z \in \Omega.
\]
Proof of Proposition 2.4. By choosing \( \theta_1, \ldots, \theta_d \in \mathbb{R} \) in suitable ways and considering

\[
G(z_1, \ldots, z_d) = F(e^{i\theta_1} z_1, \ldots, e^{i\theta_d} z_d),
\]

we reduce ourself to the case when \( \alpha_j = -\beta_j \) and \( 0 < \beta_j < \frac{\pi}{2} \).

Let

\[
H(z) = F(z) G_c(z),
\]

where

\[
G_c(z) = \prod_{j=1}^d e^{-c r_j \frac{z_j}{2^j}}, \quad s = 2^j \quad \text{and} \quad G_c(z) = \prod_{j=1}^d e^{-c r_j (\log(1+z_j)) \frac{1}{2^j}}, \quad s \in \mathbb{R}, \quad s < \frac{1}{2}.
\]

By choosing \( c > 0 \) large enough and independently of \( r \) we have that \( H \) is bounded on \( \Omega_0 \) and

\[
|H(z)| \lesssim e^{c r |z|^2}, \quad z \in \Omega.
\]

By Lemma 2.7 it follows that \( H \) is bounded on \( \Omega \). This implies

\[
|F(z)| \lesssim |G_c(z)|^{-1},
\]

which gives the desired estimate and thereby the result. \( \Box \)

Proof of Theorem 2.2. Evidently, (3) implies (2), and by Proposition 2.4 it follows that (1) implies (3). We need to prove that (2) implies (1).

By considering \( \mathcal{F}_{0} f \) and using the fact that \( \mathcal{H}_s(\mathbb{R}^d) \) and \( \mathcal{H}_{0,u}(\mathbb{R}^d) \), we reduce ourself to the case that \( t_0 = 0 \).

Let \( t = (k_1 u_1, \ldots, k_d u_d) \in \Lambda_{0,u} \) be fixed,

\[
\alpha_j = \pi k_j / 2, \quad \beta_j = \min(\pi(k_j+1)u_j, \pi),
\]

and let \( \Gamma_j, \Omega \) and \( \Omega_0 \) be as in Proposition 2.6 \( j = 1, \ldots, d \). Also let \( F = (\mathcal{F}_d f) \).

We claim that (2.9) holds.

In fact, if \( ((k_1+1) u_1, \ldots, (k_d+1) u_d) \in \Lambda_{0,u} \), then (2.9) follows from (2.3) and letting \( z_j = x_j \) be real in (2.8), \( j = 1, \ldots, d \).

We only prove the assertion in the Roumieu case. That is, (2.3) holds for some \( r > 0 \), then we shall prove that \( f \in \mathcal{H}_d(\mathbb{R}^d) \), or equivalently, that

\[
\|\mathcal{F}_d f \cdot \omega_{d,r,s} \|_{L^\infty} < \infty
\]

holds for some \( r > 0 \). The other cases follow by similar arguments and are left for the reader.

Let \( k_0 \in \{1, \ldots, N\} \) be fixed and consider the sector \( \Omega_0 \), given by

\[
\Omega_0 = \left\{ (z_1, \ldots, z_d) \in \mathbb{C}^d ; (k_0 - 1) t_0 \frac{\pi}{2} \leq \argin(z_1) \leq k_0 t_0 \frac{\pi}{2} \right\}
\]

or

\[
\Omega_0 = \left\{ (z_1, \ldots, z_d) \in \mathbb{C}^d ; N t_0 \frac{\pi}{2} \leq \argin(z_1) \leq \pi \right\}.
\]

By (2.8) and the fact that \( (\mathcal{F}_d f)(x) = f(-x) \), it follows that

\[
\sup_{z \in \partial \Omega_0} |\mathcal{F}_d f(z) \cdot \omega_{d,r,s}(z)| \leq C_r, \quad j = 1, 2.
\]

Suppose that \( k = k_0 \) and \( z \) belongs to \( \Omega_0 \) in (2.11), or that \( k = N + 1 \) and \( z \) belongs to \( \Omega_0 \) in (2.12). Then let

\[
h_{k,c,w_1}(z_1) = \begin{cases} \mathcal{F}_d f(z) \cdot e^{-c r_j (\log(1 + e^{-i(k-1/2)u_j} z_1)) \frac{1}{2^j}}, & s \in \mathbb{R}, \quad 0 < s < \frac{1}{2}, \\ \mathcal{F}_d f(z) \cdot e^{-c r_j (\log(1 + e^{-i(k-1/2)u_j} z_1)) \frac{2}{2^j}}, & s = 2^j \end{cases},
\]

where \( w_1 = (z_2, \ldots, z_d) \), and \( c > 0 \) is independent of \( r \). Here we let the branch cut of \( \log(\cdot) \) be the negative real axis.
If we choose \( c \) large enough and use (2.13), then it follows that for some constant \( c_1 \) which only depends on \( d \) and \( s \), we have
\[
|h_{k,c,w_1}(z_1)| \leq C_r \omega_{d-1,c,r,s}(w_1)^{-1} e^{\frac{t}{2}|w_1|^2}, \quad z_1 \in \partial \Omega
\]
and
\[
|h_{k,c,w_1}(z_1)| \leq C_r \omega_{d-1,c,r,s}(w_1)^{-1} e^{\frac{t}{2}|z_1|^2} e^{\frac{t}{2}|w_1|^2}, \quad z_1 \in \Omega.
\]
By Phragmén-Lindelöf’s theorem (see Proposition A.1 in Appendix A) we have
\[
|h_{k,c,w_1}(z_1)| \leq C_r \omega_{d-1,c,r,s}(w_1)^{-1} e^{\frac{t}{2}|w_1|^2}, \quad z_1 \in \Omega,
\]
which implies that
\[
|\mathfrak{I}_d f(z) \omega_{1,c,r,s}(z_1) \omega_{d-1,c,r,s}(w_1) e^{-\frac{t}{2}|w_1|^2}| \leq C_r,
\]
when \( z_1 \in \Omega_0 \) and \( w_1 \in \mathbb{C}^{d-1} \), for some constant \( c > 0 \) which is independent of \( r \). Since the union of possible \( \Omega_0 \) equals
\[
\Omega = \{ z_1 \in \mathbb{C}; \im(z_1) \geq 0 \},
\]
the estimate (2.14) holds for all such \( z_1 \). By using \( \mathcal{F}_{2+t} f(x) = \mathcal{F}_t f(-x) \), we obtain similar estimates with \( 2 + t \) in place of \( t \) above, which in turn give (2.14) for \( z_1 \in \mathbb{C} \setminus \Omega \). Consequently, (2.14) holds for all \( z \in \mathbb{C}^d \).

By a combination of (2.8) and (2.14), similar arguments with \( z_2 \) in place of \( z_1 \) give
\[
|\mathfrak{I}_d f(z) \omega_{1,c,r,s}(z_1) \omega_{1,c,r,s}(z_2) \omega_{d-2,c,r,s}(w_2) e^{-\frac{t}{2}|w_2|^2}| \leq C_r,
\]
when \( z_1 \in \Omega_0 \) and \( w_2 = (z_3, \ldots, z_d) \in \mathbb{C}^{d-2} \), for some constant \( c > 0 \) which is independent of \( r \). By continuing in this inductively way we obtain after \( d \) steps that
\[
|\mathfrak{I}_d f(z) \omega_{1,c,r,s}(z_1) \cdots \omega_{1,c,r,s}(z_d)| \leq C_r, \quad z \in \mathbb{C}^d,
\]
for some constant \( c > 0 \) which is independent of \( r \). This gives
\[
|\mathfrak{I}_d f(z) \omega_{d,c,r,s}(z)| \leq C_r, \quad z \in \mathbb{C}^d,
\]
for some constant \( c > 0 \) which is independent of \( r \), and the result follows. \( \square \)

Appendix A. Estimates of Phragmén-Lindelöf types

First we recall the following result of Phragmén-Lindelöf. For completeness we give a proof.

Proposition A.1. Let \( \alpha, \beta, \rho \in \mathbb{R} \) be such that
\[
\beta > \alpha, \quad \rho > 0 \quad \text{and} \quad \rho(\beta - \alpha) < \pi.
\]
Also let \( F \) be analytic in
\[
\Omega = \{ z \in \mathbb{C}; \alpha < \arg(z) < \beta \}
\]
and continuous on \( \overline{\Omega} \) such that
\[
|F(z)| \leq M, \quad \arg(z) \in \{\alpha, \beta\}
\]
and
\[
|F(z)| \leq C e^{r|z|^\rho}, \quad z \in \Omega,
\]
for some constants \( C, M, r \geq 0 \). Then
\[
|F(z)| \leq M, \quad z \in \overline{\Omega}. \quad (A.1)
\]
Proof. By considering $F_0(z) = F(e^{\theta z})$ for suitable $t \in \mathbb{R}$, we reduce ourself to the case when $\alpha = -\beta < 0$. Let $\rho_0 > \rho$ be chosen such that
\[ \rho_0 \beta = \frac{\rho_0(\beta - \alpha)}{2} < \frac{\pi}{2}, \]
and let $\theta = \cos(\rho_0 \beta) > 0$ and
\[ H_\varepsilon(z) = F(z) e^{-\varepsilon z^\rho_0}, \quad \varepsilon > 0. \]
Then $|\text{Arg}(z^\rho_0)| < \frac{\pi}{2}$ which gives
\[ |H_\varepsilon(z)| = |F(z)| |e^{-\varepsilon z^\rho_0}| \leq |F(z)| |e^{-\varepsilon |z|^\rho_0}|. \]
This gives
\[ |H_\varepsilon(z)| \leq C e^{r |z|^\rho_0} \to 0 \quad \text{as} \quad z \to \infty, \]
because $\rho_0 > \rho$. Hence for some $R_0 > 0$ we have
\[ |H_\varepsilon(z)| < M \quad \text{as} \quad z \in \overline{\Omega} \quad \text{and} \quad |z| \geq R_0. \quad (A.2) \]
If $\Omega_0$ is the set of all $z \in \Omega$ such that $|z| \leq R_0$, then $H_\varepsilon$ is analytic in $\Omega_0$, continuous on $\overline{\Omega_0}$, and $|H_\varepsilon(z)| < M$ on the boundary $\partial \Omega_0$ of $\Omega_0$. By the maximum modulus principle we obtain
\[ |H_\varepsilon(z)| < M \quad \text{as} \quad z \in \overline{\Omega} \quad \text{and} \quad |z| \leq R_0, \quad (A.3) \]
and a combination of (A.2) and (A.3) shows that
\[ |H_\varepsilon(z)| < M \quad \text{as} \quad z \in \overline{\Omega}. \]
By letting $\varepsilon$ tending to 0 we finally get (A.1). \qed

We have now the following multi-dimensional version of the previous result.

Proposition A.2. Let $\alpha_j, \beta_j, \rho_j \in \mathbb{R}$ be such that
\[ \beta_j > \alpha_j, \quad \rho_j > 0 \quad \text{and} \quad \rho_j(\beta_j - \alpha_j) < \pi, \quad j = 1, \ldots, d. \]
Also let $F$ be analytic in
\[ \Omega = \{ z \in \mathbb{C}^d ; \alpha_j < \text{Arg}(z_j) < \beta_j, \quad j = 1, \ldots, d \} \]
and continuous on $\overline{\Omega}$ such that
\[ |F(z)| \leq M, \quad \text{Arg}(z_j) \in \{\alpha_j, \beta_j\}, \]
for $j = 1, \ldots, d$, and
\[ |F(z)| \leq C \prod_{j=1}^{d} e^{r_j |z_j|^\rho_j}, \quad z \in \Omega, \quad (A.4) \]
for some constants $C, M, r_j \geq 0, \quad j = 1, \ldots, d$. Then
\[ |F(z)| \leq M, \quad z \in \overline{\Omega}. \]

Proof. We shall argue with induction and apply Phragmén-Lindelöf’s theorem of one dimension in the induction steps.

If we choose $\theta_1, \ldots, \theta_d \in \mathbb{R}$ in suitable ways and consider
\[ F(e^{i\theta_1} z_1, \ldots, e^{i\theta_d} z_d), \]
we reduce ourself to the case when $\alpha_j = -\beta_j$ and $\beta_j > 0$. By considering
\[ F(z_1^{2/r_1}, \ldots, z_d^{2/r_d}), \]
and letting $r = \max_{1 \leq j \leq d} (r_j)$, we then reduce ourself to the case when
\[ \rho_1 = \cdots = \rho_d = 2, \quad r_1 = \cdots = r_d = r > 0 \quad \text{and} \quad \beta_j < \frac{\pi}{4}. \]
We have
\[ \Omega = \Gamma_1 \times \cdots \times \Gamma_d, \quad \text{and let} \quad \Omega_0 = \partial \Gamma_1 \times \cdots \times \partial \Gamma_d, \]
where
\[ \Gamma_j = \{ z \in \mathbb{C} : \alpha_j \leq \text{Arg}(z) \leq \beta_j \}, \quad j = 1, \ldots, d. \]

Then it follows from the assumptions that
\[ |F(z)| \lesssim e^{r|z|^2}, \quad z \in \Omega, \quad \text{and} \quad |F(z)| \leq M, \quad z \in \Omega_0, \]
For any \( k \in \{1, \ldots, d\} \), let
\[ \Omega_k = \Gamma_1 \times \cdots \times \Gamma_k \times \partial \Gamma_{k+1} \times \cdots \times \partial \Gamma_d, \]
\[ w_k = (z_1, \ldots, z_{k-1}, z_{k+1}, \ldots, z_d) \in \mathbb{C}^{d-1}. \]
and let
\[ G_k(z_k) = F(z) = F(z_1, \ldots, z_d) \]
when \( w_k \) is fixed. We claim that
\[ |F(z)| \leq M, \quad z \in \Omega_k, \]
for every \( k \in \{1, \ldots, d\} \). This in turn gives the assertion, by choosing \( k = d \).

First suppose that \( k = 1 \) and let
\[ w_1 \in \partial \Gamma_2 \times \cdots \times \partial \Gamma_d \subseteq \mathbb{C}^{d-1} \]
be fixed. Then it follows from the assumptions that \( G_1 \) is analytic on \( \Gamma_1 \) and satisfies
\[ |G_1(z_1)| \leq C_{w_1} e^{r_0|z_1|^2}, \quad z_1 \in \Gamma_1 \quad \text{and} \quad |G_1(z_1)| \leq M, \quad z_1 \in \partial \Gamma_1, \]
for some constant \( C_{w_1} \) which only depends on \( w_1 \). By Phragmén-Lindelöf’s theorem (Proposition A.1) we get \( |G_1(z_1)| \leq M \) when \( z_1 \in \Gamma_1 \), which is the same as
\[ |F(z)| \leq M, \quad z \in \Omega_1, \]
since \( w_1 \) is arbitrarily chosen. This gives the claim for \( k = 1 \).

Suppose that the claim holds true for \( k \in \{1, \ldots, d-1\} \) and prove the claim for \( n = k + 1 \). Let
\[ w_n \in \Gamma_1 \times \cdots \times \Gamma_k \times \partial \Gamma_{k+2} \times \cdots \times \partial \Gamma_d \subseteq \mathbb{C}^{d-1}. \]
be fixed. Then
\[ |G_n(z_n)| \leq C_{w_n} e^{r_0|z_n|^2}, \quad z_n \in \Gamma_n \quad \text{and} \quad |G_n(z_n)| \leq M, \quad z_n \in \partial \Gamma_n, \]
Here the latter inequality follows from the induction hypothesis. By Phragmén-Lindelöf’s theorem we get \( |G_n(z_n)| \leq M \) when \( z_n \in \Gamma_n \), which is the same as
\[ |F(z)| \leq M, \quad z \in \Omega_n, \]
since \( w_n \) is arbitrarily chosen. This gives the claim and thereby the assertion. \( \Box \)
Appendix B. Links between the Bargmann transform, fractional Fourier transforms and compositions with short-time Fourier transform

We recall that if $\mathcal{S}_t$ denotes the fractional Fourier transform of order $t \in \mathbb{R}^d$ and $f \in H_0^1(\mathbb{R}^d)$, then

$$
(\mathcal{W}_d(\mathcal{S}_t f))(z) = (\mathcal{W}_d f)(e^{-it_1 \frac{d}{2} z_1}, \ldots, e^{-it_d \frac{d}{2} z_d}), \quad (B.1)
$$

and the link between the Bargmann transform and short-time Fourier transform is

$$
V_\phi f(x, \xi) = (2\pi)^{-\frac{d}{2}} e^{-\frac{1}{4}|\xi|^2} e^{-\frac{i}{2}(x, \xi)} (\mathcal{W}_d f)(2^{-\frac{1}{2}} \xi),
$$

$$
z = x + i\xi, \ x, \xi \in \mathbb{R}^d
$$

(see (1.28) in [27] and ) A combination of these relations give

$$
(V_\phi(\mathcal{S}_t f))(x, \xi) = (2\pi)^{-\frac{d}{2}} e^{-\frac{1}{4}|\xi|^2} e^{-\frac{i}{2}(x, \xi)} (\mathcal{W}_d f)(2^{-\frac{1}{2}} \xi),
$$

$$
U_{d,t}(z) = (U_{1,t}(z_1), \ldots, U_{d,t}(z_d))
$$

$$
U_{1,t}(z_j) = (\cos(t_j \frac{\pi}{2}) x_j + \sin(t_j \frac{\pi}{2}) \xi_j) + i(-\sin(t_j \frac{\pi}{2}) x_j + \cos(t_j \frac{\pi}{2}) \xi_j),
$$

$$
z = x + i\xi, \ x, \xi \in \mathbb{R}^d,
$$

(see (27) in [27] and ) A combination of these relations give

$$
(V_\phi(\mathcal{S}_t f))(x, \xi) = e^{\frac{i}{4}(\Phi_{1,t}(x_1, \xi_1) + \cdots + \Phi_{d,t}(x_d, \xi_d))} V_\phi f(A_{t_1}(x_1, \xi_1), \ldots, A_{t_d}(x_d, \xi_d)),
$$

$$
\Phi_{t_j}(x_j, \xi_j) = \sin(2 t_j \frac{\pi}{2}) (\xi_j^2 - x_j^2) + 2(\cos(2 t_j \frac{\pi}{2}) - 1)x_j \xi_j,
$$

$$
A_{t_j}(x_j, \xi_j) = (\cos(t_j \frac{\pi}{2}) x_j + \sin(t_j \frac{\pi}{2}) \xi_j) - \sin(t_j \frac{\pi}{2}) x_j + \cos(t_j \frac{\pi}{2}) \xi_j),
$$

$$
x, \xi \in \mathbb{R}^d.
$$

(B.4)

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