FAMILIES OF DIRECTED GRAPHS AND TOPOLOGICAL CONJUGACY OF THE ASSOCIATED MARKOV-DYCK SHIFTS

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Abstract. We describe structural properties of strongly connected finite directed graphs, that are invariants of the topological conjugacy of their Markov-Dyck shifts. For strongly connected finite directed graphs with these properties topological conjugacy of their Markov-Dyck shifts implies isomorphism of the graphs.

1. Introduction

Let $\Sigma$ be a finite alphabet, and let $S$ be the left shift on $\Sigma^\mathbb{Z}$,

$$S((x_i)_{i \in \mathbb{Z}}) = (x_{i+1})_{i \in \mathbb{Z}}, \quad (x_i)_{i \in \mathbb{Z}} \in \Sigma^\mathbb{Z}. $$

The closed shift-invariant subsystems of the shifts $S$ are called subshifts. For an introduction to the theory of subshifts see [Ki] and [LM]. A finite word in the symbols of $\Sigma$ is called admissible for the subshift $X \subset \Sigma^\mathbb{Z}$ if it appears somewhere in a point of $X$. A subshift $X \subset \Sigma^\mathbb{Z}$ is uniquely determined by its language of admissible words.

In this paper we continue the study [KM] of the topological conjugacy of Markov-Dyck shifts. The Markov-Dyck shift of a strongly connected finite directed graph is constructed via the graph inverse semigroup. We denote a finite directed graph $G$ with vertex set $V$ and edge set $E$ by $G(V, E)$. The source vertex of an edge $e \in E$ we denote by $s$ and its target vertex by $t$. Given a finite directed graphs $G = G(V, E)$, let $E^- = \{ e^- : e \in E \}$ be a copy of $E$. Reverse the directions of the edges in $E^-$ to obtain the edge set $E^+ = \{ e^+ : e \in E \}$ of the reversed graph of $G(V, E^-)$. In this way one has defined a directed graph $G(V, E^- \cup E^+)$, that has the directed graphs $G(V, E^-)$ and $G(V, E^+)$ as subgraphs. With idempotents $1_U, V \in V$, the set $E^- \cup \{ 1_V : V \in V \} \cup E^+$ is the generating set of the graph inverse semigroup $S(G)$ of $G$ (see [L Section 10.7]), where, besides $1^2_V = 1_V, V \in V$, the relations are

$$1_U 1_W = 0, \quad U, W \in V, U \neq W,$$

$$f^- g^+ = \begin{cases} 1_{s(f)}, & \text{if } f = g, \\ 0, & \text{if } f \neq g, \quad f, g \in E, \end{cases}$$

$$1_{s(f)} f^- = f^- 1_{t(f)}, \quad 1_{t(f)} f^+ = f^+ 1_{s(f)}, \quad f \in E.$$

The directed graphs with a single vertex and $N > 1$ loops yield the Dyck inverse monoids (the "polycycliques" of [NP]), that we denote by $D_N$.

We consider strongly connected finite directed graphs $G = G(V, E)$, that we assume not to be a cycle. From the graph $G$ one obtains a Markov-Dyck shift $MD(G)$, that has as alphabet the set $E^- \cup E^+$, and a word $(e_k)_{1 \leq k \leq K}$ in the symbols of $E^- \cup E^+$ is admissible for $MD(G)$ precisely if

$$\prod_{1 \leq k \leq K} e_k \neq 0.$$

The directed graphs with a single vertex and $N > 1$ loops yield the Dyck shifts $D_N$ [Kr1].
For a directed graph \( G(V, E) \) we denote by \( R_G \) the set of vertices of \( G \), that have at least two incoming edges, and we denote by \( F_G \) the set of edges that are the only incoming edges of their target vertices. The graph \( G(V, F_G) \) is a directed subgraph of \( G \), that decomposes into directed trees, that we refer to as the subtrees of \( G \). The set \( R_G \) is the set of roots of the subtrees. A subtree is equal to the one-vertex tree with vertex \( V \in R_G \), if \( V \) is not the source vertex of an edge in \( F_G \). Contracting the subtrees of \( G \), that are not one-vertex, to their roots yields a directed graph, that we denote by \( \hat{G} \). In [Kr2] a Property (A) of subshifts, that is an invariant of topological conjugacy, was introduced and a semigroup, that is invariantly associated with a subshift with Property (A), was constructed. That the Markov-Dyck shifts have Property (A) was shown in [HK] Section 2. The semigroup \( S(Md(G)) \) that is associate to \( Md(G) \) is \( S(\hat{G}) \) [HK] Section 3], which implies, that the isomorphism class of the graph \( \hat{G} \) is an invariant of topological conjugacy of the Markov-Dyck shift of \( G \) [Kr1 Corollary 3.2, [Kr5] Theorem 2.1].

In [KM] three families, \( F_I, F_{II} \) and \( F_{III} \) of directed graphs \( G = G(V, E) \), such that \( G(V, F_G) \) is a tree, were considered. The Markov-Dyck shifts of the graphs in each of these families were characterized among the Markov-Dyck shifts by invariants of topological conjugacy. It was shown for the graphs in each of these families, that the topological conjugacy class of their Markov-Dyck shifts determines the isomorphism class of the graphs. The family \( F_I \) contains the graphs, such that \( G(V, F_G) \) is a tree, and such that all of its vertices, except the root of the subtree, have out-degree one. The family \( F_{II} \) contains the graphs, such that \( G(V, F_G) \) is a tree such that all leaves of the subtree are at level one. The family \( F_{III} \) contains the graphs \( G \), such that \( G(V, F_G) \) is a tree, that has the shape of a "V", and that are such that the two leaves of the subtree have the same out-degree in \( G \), and all interior vertices of the subtree have out-degree one in \( G \).

Continuing in this line we consider two families of graphs \( G(V, E) \), such that \( G(V, F_G) \) is a tree, that we name \( F_{IV} \) and \( F_V \). We characterize the Markov-Dyck shifts of the graphs in these families among the Markov-Dyck shifts by invariants of topological conjugacy, and we show, that the topological conjugacy class of their Markov-Dyck shifts in these families determines the isomorphism class of the graphs. The method of proof is the same as in [KM]. We choose canonical models for the graphs \( G = G(V, E) \) and obtain sufficient information from certain topological conjugacy invariants of \( Md(G) \) to reconstruct the canonical model of \( G \).

In a preliminary Section 2 we introduce notation and recall the relevant invariants of topological conjugacy, that we use. In Section 3 we consider the family \( F_{IV} \), that we define as the family of graphs \( G(V, E) \), such that \( G(V, F_G) \) is a tree, such that \( \text{card}(E \setminus F_G) = 4 \), and such that there is an \( H \in \mathbb{N} \), such that the leaves of the subtree are at level \( H \), and such that one finds two branch points, each of out-degree two, on the directed cycles of minimal length, that pass through the root of the subtree. Also a restriction on the cardinality of the set \( F_G \) is to be satisfied. In Section 4 we consider the family \( F_V \) of graphs \( G(V, E) \) such that \( G(V, F_G) \) is a tree, that satisfy a strong structural condition: The allowed subtrees are line graphs, or they can be obtained by replacing in a caterpillar tree the "legs" of the caterpillar by line graphs.

A rooted tree is called spherically homogeneous, if all of its leaves are at the same level, and if all vertices at the same level have the same out-degree. We say that a graph \( G(V, E) \) such that \( G(V, F_G) \) is a tree, is spherically homogeneous if its subtree is spherically homogeneous, if all source vertices of the edges \( e \in E \setminus F_G \) are leaves of the subtree, and if all of the leaves of the subtree have the same out-degree in \( G \) (see [Kr5]). In Section 5 we derive for graphs \( G(V, E) \) such that \( G(V, F_G) \) is a tree of height two, a criterion for spherical homogeneity in terms of invariants.
of topological conjugacy of the Markov-Dyck shifts of the graphs. For spherically homogeneous graphs with a subtree of height two we show, that the topological conjugacy class of their Markov-Dyck shifts determines the isomorphism class of the graphs.

In Section 6 we consider the graphs $G = G(V, E)$, such that the semigroup $S(Md(G))$ is the graph inverse semigroup of a two-vertex graph, and such that the graph $G(V, F_G)$ decomposes into a one-edge tree and a one-vertex tree. Also for these graphs we show, that the topological conjugacy class of their Markov-Dyck shifts determines the isomorphism class of the graphs, providing at the same time a characterization of these graphs among the Markov-Dyck shifts by invariants of topological conjugacy.

2. Preliminaries

Let there be given a graph $G = G(V, E)$. For a directed path $(e_i)_{1 \leq i \leq l}, I \in \mathbb{N}$, in $G$ we set
$$ c^{-} = ((c_i^-)_{1 \leq i \leq l}), \quad c^{+} = ((c_i^+)_{1 \leq i \leq l}). $$

We use as vertex set of $G$ the set $V$. Every edge $e \in E \setminus F_G$ maps into an edge $\hat{e}$ of $\hat{G}$, such that $t(\hat{e}) = t(e)$, and, if $s(e) \in R_G$, such that $s(\hat{e}) = s(e)$, and, if $s(e) \notin R_G$, such that $s(e)$ is the root of the subtree, to which $s(e)$ belongs. The edge set of $\hat{G}$ is $\hat{E} = \{ \hat{e} : e \in E \setminus F_G \}$.

For $f \in F_G$ we set $\hat{f}$ equal to $1_R$, where $R \in R_G$ is the root of the subtree, to which the edge $f$ belongs. We set
$$ e^{-} = \hat{e}^{-}, \quad e^{+} = \hat{e}^{+}, \quad e \in E \setminus F_G. $$

A periodic point $p$ of period $\pi$ of $Md(G)$ and its orbit are said to be neutral, if there exists an $R \in R_G$, which then is uniquely determined by $p$, such that for some $i \in \mathbb{Z}$
$$ \prod_{i \leq j < i + \pi(p)} \hat{p}_j = 1_R. $$

We denote by $I_k^0(Md(G))$ the cardinality of the set of neutral periodic orbits of length $k$ of $Md(G)$.

For a non-neutral periodic point $p$ of period $\pi$ of $Md(G)$ there exists a simple cycle $(\hat{a}_i)_{0 \leq i \leq L}$ in the graph $\hat{E} = G(R_G, \hat{E})$, such that for some $i \in \mathbb{Z}$ either

\begin{equation}
\prod_{i \leq j < i + \pi(p)} \hat{p}_j = (\prod_{0 \leq i \leq L} \hat{a}_i^-)^M,
\end{equation}

or

\begin{equation}
\prod_{i \leq j < i + \pi(p)} \hat{p}_j = (\prod_{L \geq j \geq 0} \hat{a}_i^+)^M.
\end{equation}

The simple cycle $\hat{a}$ is unique up to a cyclic permutation of its edges \cite[Section 2]{HI} \cite[Section 4]{HIK}. Following the terminology, that was introduced in \cite{HIK}, we refer to the equivalence class of simple cycles in the graph $\hat{E} = G(R_G, \hat{E})$, that is assigned in this way to the periodic point $p$, as the multiplier of $p$. In the case $(6.1)/(6.2)$ the periodic point $p$ is said to have a negative (positive) multiplier. The length of the multiplier is $L$. A topological conjugacy carries neutral periodic points into neutral periodic points. Also the map, that assigns to a non-neutral periodic point its multiplier, is an invariant of topological conjugacy \cite[Section 4]{HIK}. Given a directed graph $G = G(V, E)$ we denote the set of multipliers of $Md(G)$ by $\mathcal{M}(Md(G))$, and the set multipliers of $Md(G)$ of length $L$ of $Md(G)$ by $\mathcal{M}_L(Md(G))$. 

We denote for a multiplier $\mu \in \mathcal{M}(Md(G))$, by $\mathcal{O}_k^{(\mu)}(Md(G))$ the set of orbits with negative multiplier $\mu$ of length $k$, and we set

$$I_k^{(\mu)}(Md(G)) = \text{card}(\mathcal{O}_k^{(\mu)}(Md(G))).$$

In sections 3 - 5 we consider the case of a graph $G = (\mathcal{V}, \mathcal{E})$ such that $G(\mathcal{V}, \mathcal{F}_G)$ is a tree. In this case

$$\mathcal{M}_1(Md(G)) = \mathcal{E}.$$  

We set

$$\nu(Md(G)) = \text{card}(\mathcal{E}).$$

Contracting the tree $G(\mathcal{V}, \mathcal{F}_G)$ to its root yields the directed graph with a single vertex and $\nu(Md(G))$ loops. It follows from [HIK, Section 5], that the graphs $G = (\mathcal{V}, \mathcal{E})$, such that the graph $G(\mathcal{V}, \mathcal{F}_G)$ is a tree, are precisely the directed graphs that have a Dyck inverse monoid associated to them, and that

$$\mathcal{S}(Md(G)) = \mathcal{D}_{\nu(Md(G))}.$$  

We introduce notation. We set

$$\Lambda^{(\hat{\nu})}(Md(G)) = \min\{k \in \mathbb{N} : I_k^{(\hat{\nu})}(Md(G)) > 0\}, \quad \hat{\nu} \in \mathcal{E}.$$

We will use of the notation $\Lambda(Md(G))$ to indicate, that all $\Lambda^{(\hat{\nu})}(Md(G)), \hat{\nu} \in \mathcal{E}$, are equal, and $\Lambda(Md(G))$ denotes their common value. We will also use of the notation $I_k(Md(G))$ to indicate, that all $I_k^{(\hat{\nu})}(Md(G)), \hat{\nu} \in \mathcal{E}$, are equal and $I_k(Md(G))$, denotes their common value, $k \in \mathbb{N}$.

We denote the level of a vertex $V$ of the tree $G(\mathcal{V}, \mathcal{F}_G)$ by $\lambda(V)$. For $e \in \mathcal{E} \setminus \mathcal{F}_G$ we define the length of a word $\mathcal{W}$ in $G$ of length $\lambda(s(e)) + 1$, that starts at the root of the subtree and passes through $e$ as $((f_i)_{1 \leq i \leq \lambda(s(e))) \in \mathcal{E})$, that leaves the vertex $V$. We set

$$V_l^{(e)} = s(f_{e,l}), \quad 0 \leq l \leq \lambda(s(e))$$ 

and

$$V_{\lambda(s(e))}^{(e)} = s(e), \quad \mathcal{V}^{(e)} = \{V_l^{(e)} : 0 \leq l \leq \lambda(s(e))\}.$$ 

We also set

$$\Delta^{(\hat{\nu})} = I_{\Lambda^{(\hat{\nu})}+2}(Md(G)), \quad \hat{\nu} \in \mathcal{E}.$$ 

We note that

(\text{A}) \quad I_2^{\text{sh}}(Md(G)) = \text{card}(\mathcal{E}) = \nu(Md(G)) + \text{card}(\mathcal{F}_G).\]

3. $F_{IV}$.

Given a directed graph $G = (\mathcal{V}, \mathcal{E})$, such that $G(\mathcal{V}, \mathcal{F}_G)$, is a tree, we denote for $e \in \mathcal{E} \setminus \mathcal{F}_G$, $V \in \mathcal{V}^{(e)}$ and $m \in \mathbb{N}$, by $\mathcal{D}_m(V)[G]$ the set of directed paths of length $m$ in the graph $G(V, \mathcal{E})$, that leave the vertex $V$. We set

$$D_m^{(e)}[G] = \sum_{V \in \mathcal{V}^{(e)}} \text{card}(\mathcal{D}_m(V)[G]), \quad e \in \mathcal{E} \setminus \mathcal{F}_G, \quad m \in \mathbb{N}.$$ 

We also define codes

$$C_m(V)[G] = \bigcup_{1 \leq k \leq m} \{d^-d^+ : d \in \mathcal{D}_k(V)[G]\}, \quad e \in \mathcal{E} \setminus \mathcal{F}_G, \quad V \in \mathcal{V}^{(e)}, \quad m \in \mathbb{N}.$$ 

With the notation $\ell$ for the length of a word we set

$$\Gamma_m^{(\ell)}[G] = \{ cravings \} \in \prod_{V \in \mathcal{V}^{(e)}} C_m(V)[G] : \sum_{V \in \mathcal{V}^{(e)}} \ell(c_V) = 2(m + 1),$$

$$C_m^{(\hat{\nu})}[G] = \text{card}(\Gamma_m^{(\ell)}[G]), \quad e \in \mathcal{E} \setminus \mathcal{F}_G, \quad m \in \mathbb{N}.$$
For $H, h, h_0, h_1 \in \mathbb{N}$, such that
\[ 0 \leq h < H, \quad h_0 \leq h_1 \leq H - h, \]
we denote by $G(H, h, h_0, h_1)$ the graph with vertex set
\[
\mathcal{V}(H, h, h_0, h_1) = \{V(i) : 0 \leq i \leq h\} \cup \bigcup_{\alpha \in \{0, 1\}} \{V_\alpha(i_\alpha) : 0 \leq i_\alpha \leq h_\alpha\} \cup \left( \bigcup_{\alpha, \beta \in \{0, 1\}} \{V_{\alpha,\beta}(i_{\alpha,\beta}) : 0 \leq i_{\alpha,\beta} \leq H_{\alpha,\beta} - h - h_\alpha\} \right),
\]
and edge set
\[
\mathcal{E}(H, h, h_0, h_1) = \{e_{\alpha,\beta} : \alpha, \beta \in \{0, 1\}\} \cup \{f(i) : 0 \leq i \leq h\} \cup \left( \bigcup_{\alpha \in \{0, 1\}} \{f_\alpha(i_\alpha) : 0 \leq i_\alpha \leq h_\alpha\} \cup \left( \bigcup_{\alpha, \beta \in \{0, 1\}} \{f_{\alpha,\beta}(i_{\alpha,\beta}) : 0 \leq i_{\alpha,\beta} \leq H_{\alpha,\beta} - h - h_\alpha\} \right) \right),
\]
and source and target mappings given by
\[
s(f(i)) = V(i - 1), t(f(i)) = V(i), \quad 0 < i \leq h,
\]
\[
s(f_\alpha(1)) = V(h),
\]
\[
s(f_\alpha(i_\alpha)) = V_\alpha(i_\alpha - 1), \quad 1 < i_\alpha \leq h_\alpha,
\]
\[
t(f_\alpha(i_\alpha)) = V_\alpha(i_\alpha), \quad 0 < i_\alpha \leq h_\alpha, \quad \alpha \in \{0, 1\},
\]
\[
s(f_{\alpha,\beta}(1)) = V_{\alpha,\beta}(h),
\]
\[
s(f_{\alpha,\beta}(i_{\alpha,\beta})) = V_{\alpha,\beta}(i_{\alpha,\beta} - 1), \quad 1 < i_{\alpha,\beta} \leq H_{\alpha,\beta} - h - h_\alpha,
\]
\[
t(f_{\alpha,\beta}(i_{\alpha,\beta})) = V_{\alpha,\beta}(i_{\alpha,\beta}), \quad 0 < i_{\alpha,\beta} \leq H_{\alpha,\beta} - h - h_\alpha, \quad \alpha, \beta \in \{0, 1\},
\]
\[
s(e_{\alpha,\beta}) = \begin{cases} V_\alpha(h_\alpha), & \text{if } H_{\alpha,\beta} - h - h_\alpha = 0, \\ V_{\alpha,\beta}(h_\alpha), & \text{if } H_{\alpha,\beta} - h - h_\alpha > 0, \end{cases}
\]
\[
t(e_{\alpha,\beta}) = V(0), \quad \alpha, \beta \in \{0, 1\}.
\]
We set
\[
\Pi_{IV} = \{(H, h, h_0, h_1) \in \mathbb{N}^4 : 0 \leq h < H, h_0 \leq h_1 < H - h - h_0, 3h + h_0 + h_1 > \frac{5}{2}H \},
\]
and define the family $F_{IV}$ by
\[
F_{IV} = \{G(H, h, h_0, h_1) : (H, h, h_0, h_1) \in \Pi_{IV}\}.
\]
For $(H, h, h_0, h_1) \in \Pi_{IV}$ one has $H > 5$ and
\[
(B) \quad h > \frac{H}{2}.
\]
We set
\[ C_m[H + 1] = C_m^{(c_e,1)}(G(H, \lceil \frac{H}{2} \rceil, \lfloor \frac{H}{2} \rfloor - 1, \lfloor \frac{H}{2} \rfloor - 1)), \quad H > 5, \quad 1 \leq m < \frac{H}{2} - 1. \]
Note, that
\[ C_m[H + 1] = H + 1 + 2(m + 1), \quad 1 \leq m < \frac{H}{2} - 1. \]

**Lemma 3.1.** For \((H, h, h_0, h_1) \in \Pi IV\), and \(G = G(H, h, h_0, h_1)\),

\[ f_{H+1+2(m+1)}^{(g_1,1)}(Md(G)) = C_m[H + 1] + H + 1 + 2(m + 1), \quad 0 \leq m < h_0, \]
\[ f_{H+1+2(h_0+1)}^{(g_1,1)}(Md(G)) = C_{h_0}[H + 1] + D_{h_0+1}^{(c_e,1)}[G(H, h, h_0, h_1)]. \]

**Proof.** Let \(\varphi\) be a permutation of the index set \([0, H]\) that maps the interval \((h - h_0, h_0]\) in an order preserving way onto the interval \((\lfloor \frac{H}{2} \rfloor, \lfloor \frac{H}{2} \rfloor - h_0, \lfloor \frac{H}{2} \rfloor, \lfloor \frac{H}{2} \rfloor]\), and that also maps the interval \((h, h + h_0]\) in an order preserving way onto the interval \((H - 1 - h_0, H]\). By (B) one has or \(l \in [0, H]\), that the generating functions of the code
\[ C(V_l^{(c_e,0)})(G(H, h, h_0, h_1)) \]
and of the code
\[ C(V_{\varphi(l)}^{(c_e,0)})(G(H, \lfloor \frac{H}{2} \rfloor, \lfloor \frac{H}{2} \rfloor - 1, \lfloor \frac{H}{2} \rfloor - 1)), \]
are the same, and that the generating functions of the code
\[ C(V_l^{(c_e,0)})(G(H, h, h_0, h_1)) \]
and of the code
\[ C(V_{\varphi(l)}^{(c_e,0)})(G(H, \lfloor \frac{H}{2} \rfloor, \lfloor \frac{H}{2} \rfloor - 1, \lfloor \frac{H}{2} \rfloor - 1)), \]
are also the same. Also
\[ \text{card}(\mathcal{D}_m(V_l^{(c_e,0)})(G(H, h, h_0, h_1))) = \text{card}(\mathcal{D}_m(V_{\varphi(l)}^{(c_e,0)})(G(H, \lfloor \frac{H}{2} \rfloor, \lfloor \frac{H}{2} \rfloor - 1, \lfloor \frac{H}{2} \rfloor - 1))), \quad 0 \leq l \leq H, m < h_0. \]
For \(e \in \mathcal{E} \setminus \mathcal{F}_G\) and \(0 \leq m \leq h_0\) one has the injective map of \(\mathcal{O}_{A+2(m+1)}^{(e)}\) into the set of directed paths in \(G\), that assigns to an orbit \(p \in \mathcal{O}_{A+2(m+1)}^{(e)}\) the path of length \(A + 2(m+1)\) that appears as a word in \(p\) after the symbol \(e\). We write the range of this map as
\[ \{(c(l))e^+(l)f(e^-)(l)d^+(h)d^-\langle H\rangle d^-\langle H\rangle e^-(c(l)) : (c(l))_0 \leq l \leq 1 G \in \mathcal{G}^{(g_1)} \} \cup \]
\[ \{ (f_{e,k})_0 \leq k < j, d^+d^+f_{e,l}(f_{e,k})_0 \leq k < j ) : d \in \mathcal{D}_{m+1}(V_l^{(e)}), 0 \leq l < H \} \cup \]
\[ \{ (f_{e,l}^+(l))_0 \leq j \leq f_{e,l}^-d^+d^-d^+e^- : d \in \mathcal{D}_{m+1}(c(s))). \}
\]
The lemma follows. \(\square\)

**Lemma 3.2.** For \((H, h, h_0, h_1) \in \Pi IV, h_0 < h_1,\) and \(G = G(H, h, h_0, h_1)\),

\[ f_{H+1+2(h_0+1)}^{(g_1,1)}(Md(G)) = C_{h_0}[H + 1] + 2(h_0 + 1) + H + 2, \]
\[ f_{H+1+2(h_0+1)}^{(g_0,0)}(Md(G)) = C_{h_0}[H + 1] + 2(h_0 + 1) + H + 1. \]

**Proof.** One finds from (B), that
\[ \text{card}(\mathcal{D}_{h_0+1}(V_{(i)}))(G) = 3, \quad h - h_0 \leq i < h, \]
\[ \text{card}(\mathcal{D}_{h_0+1}(V_{(0,i)})\mathcal{D}_G) = 2, \quad 1 \leq i_0 \leq h_0, \]
\[ \text{card}(\mathcal{D}_{h_0+1}(V_{(i,1)}))(G) = 2, \quad h_1 - h_0 \leq i_1 \leq h_1, \]
and
\[ \text{card}(\mathcal{D}_{h_0+1}(V_h))(G) = 3, \]
in fact,
\[ \mathcal{D}(V_0)[G] = \{(f_0(i_0))_{1 \leq i \leq h_0}, f_0.0(1), (f_0(i_0))_{1 \leq i \leq h_0}, f_0.0(1), (f_1(i_1))_{1 \leq i_1 \leq h_0 + 1}\} \]

For the remaining vertices there is only one directed path of length \( h_0 + 1 \) leaving them. Apply Lemma 1(b) to prove the lemma.

\[ \square \]

**Lemma 3.3.** For \((H, h, h_0, h_0) \in I_{IV}, \) and \( G = G(H, h, h_0, h_0), \)
\[ I_{H+1+2(h_0+1)}^{(h_0, h)}(M \delta(G)) = C_{h_0}[G(H, \lfloor \frac{H}{2} \rfloor, \lceil \frac{H}{2} \rceil - 1, \lfloor \frac{H}{2} \rfloor - 1)] + 2(h_0 + 1) + H + 2, \]

**Proof.** One finds from (B), that
\[
\text{card}(\mathcal{D}_{h_0+1}(V_i))[G] = 2, \quad h - h_0 \leq i < h,
\text{card}(\mathcal{D}_{h_0+1}(V_{0,i}))[G] = 2, \quad 1 < i_0 \leq h_0,
\text{card}(\mathcal{D}_{h_0+1}(V_{1,i}))[G] = 2, \quad 1 < i_1 \leq h_0,
\]
and
\[ \text{card}(\mathcal{D}_{h_0+1}(V_{h}))[G] = 4, \]

in fact,
\[ \mathcal{D}(V_0)[G] = \{(f_0(i_0))_{1 \leq i \leq h_0}, f_0.0(1), (f_0(i_0))_{1 \leq i \leq h_0}, f_0.0(1), \}
\{(f_1(i_1))_{1 \leq i_1 \leq h_0}, f_1.0(1), (f_1(i_1))_{1 \leq i_1 \leq h_0}, f_1.0(1)\}. \]

Apply Lemma 1(b) to prove the lemma.

\[ \square \]

**Theorem 3.4.** For a finite directed graph \( G = G(\mathcal{V}, \mathcal{E}) \) there exist data
\[(h, h_0, h_1, H) \in I_{IV},\]
such that there is a topological conjugacy
\[ (C) \quad \text{Md}(G) \simeq \text{Md}(G(h, h_0, h_1, H)), \]
if and only if
\[ (D1) \quad S(\text{Md}(G)) = \mathcal{D}, \]
and
\[ (D2) \quad \Lambda^{(e)} = H + 1, \quad \Delta^{(e)} = 2, \quad e \in \mathcal{M}(\text{Md}(G)). \]

**Proof.** The invariant conditions (D1 - 2) are the translation of a description of the family \( F_{IV} \) that is in terms of the subtree, of its height, its number of its leaves and its branch points.

We set
\[ C_{m}^{(h_0)} = C_{m}^{(h_0, 0)}[G(H, \lfloor \frac{H}{2} \rfloor, h_0, \lfloor \frac{H}{2} \rfloor - 1)], \quad 0 \leq m \leq \lfloor \frac{H}{2} \rfloor - 1. \]

**Theorem 3.5.** For a directed graph \( G = G(\mathcal{V}, \mathcal{E}) \), such that (D1 - 2) hold, there exists a \( h_0 \in \mathbb{N}, \)
\[ 1 \leq h_0 \leq \frac{1}{2}(\Lambda(\text{Md}(G)) - 3), \]
such that
\[ (3.1) \quad I^{(\delta)}_{\Lambda(\text{Md}(G)) + 2(m+1)} = C_{m}[\Lambda(\text{Md}(G))] + \Lambda(\text{Md}(G)) + 2(m + 1), \quad 0 \leq m < h_0, \]
and such that the set
\[(3.2) \quad \mathcal{M}_1^{(\bar{h}_0)}(MD(G)) = \{ \tilde{e} \in \mathcal{M}_1(MD(G)) : I_{\bar{h}_0+1}^{(\bar{h}_0)}(MD(G)) + \Lambda(MD(G)) + 2(\tilde{h}_0 + 1) \}
\]
is not empty, and such that either
\[\mathcal{M}_1^{(\bar{h}_0)}(MD(G)) = \mathcal{M}_1(MD(G)),\]
in which case (C) holds for
\[(3.3) \quad H + 1 = \Lambda(MD(G)), \quad h = \frac{1}{3}(4\Lambda(MD(G)) - I_2^{(0)}(MD(G)) - 2\tilde{h}_0),\]
or else
\[I_{\bar{h}_0+1}^{(\bar{h}_0)}(MD(G)) = C_{\bar{h}_0}^{(\bar{h}_0)} + I_{\bar{h}_0+1}^{(\bar{h}_0)}(MD(G)) + 1, \quad \tilde{e} \in \mathcal{M}_1(MD(G)),\]
and there exists a $\tilde{h}_1$,
\[\tilde{h}_0 < \tilde{h}_1 \leq \frac{1}{3}(\Lambda(MD(G)) - 3),\]
such that
\[(3.4) \quad I_{\bar{h}_0+1}^{(\bar{h}_0)}(MD(G)) = C_{\bar{h}_0}^{(\bar{h}_0)} + \Lambda(MD(G)) + 4(\tilde{h}_0 + m), \quad \tilde{h}_0 < m < \tilde{h}_1, \quad \tilde{e} \in \mathcal{M}_1^{(\bar{h}_0)}(MD(G)),\]
and
\[(3.5) \quad I_{\bar{h}_0+1}^{(\bar{h}_0)}(MD(G)) = C_{\bar{h}_0}^{(\bar{h}_0)} + \Lambda(MD(G)) + 4(\tilde{h}_0 + \tilde{h}_1), \quad \tilde{e} \in \mathcal{M}_1^{(\bar{h}_0)}(MD(G)),\]
in which case (C) holds for
\[(3.6) \quad H + 1 = \Lambda(MD(G)), \quad h = \frac{1}{3}(4\Lambda(MD(G)) - I_2^{(0)}(MD(G)) - \tilde{h}_0 - \tilde{h}_1).\]

Proof. The existence of a $\tilde{h}_0$ as in (3.1) and (3.2) follows from Lemma 1. The dichotomy follows from Lemma 2 and Lemma 3. To obtain the $\tilde{h}_1$ as in (3.4) and (3.5), use a permutation $\psi$ of the index set $[0, H]$ that maps the interval $[h - \lceil \frac{H}{2} \rceil, H]$ in an order preserving way onto the interval $[0, \lceil \frac{H}{2} \rceil + h]$. As is seen from (B), for $l \in [0, H]$ the generating functions of the code
\[C(V^{(c_n, o)}_{\psi(l)}[G(H, h, h_0, h_1)])\]
and of the code
\[C(V^{(c_n, o)}_{\psi(l)}[G(H, \lceil \frac{H}{2} \rceil, \lceil \frac{H}{2} \rceil - 1, \lceil \frac{H}{2} \rceil - 1)])\]
are the same. The proof also involves counting the directed paths of length $h_1 + 1$ in $G(h, h_0, h_0)$ and in $G(h, \lceil \frac{H}{2} \rceil, h_0, \lceil \frac{H}{2} \rceil - 1)$, that leave the vertices in $\mathcal{V}^{(c_n, o)}$.

(3.3) and (3.6) follow from (A).

\[\square\]

**Corollary 3.6.** For directed graphs $G = G(\mathcal{V}, \mathcal{E})$ such that the Markov-Dyck shifts $MD(G)$ satisfies (D 1 - 2), the topological conjugacy of the Markov-Dyck shifts $MD(G)$ implies the isomorphism of the graphs $G$.

Proof. A graph $G = G(\mathcal{V}, \mathcal{E})$ belongs to the family $\mathcal{F}_{IV}$ precisely if $MD(G)$ satisfies (D 1 - 2), and in Theorem 3.5 the data $(h, h_0, h_1, H)$ of the canonical model of $G$ are expressed in terms of invariants of topological conjugacy.

\[\square\]
4. \( \mathbf{F}_V \)

Let \( G = G(\mathcal{V}, \mathcal{E}) \), be a directed graph, such that \( G(\mathcal{V}, \mathcal{F}_G) \), is a tree. A vertex \( V \in \mathcal{V} \) determines a path

\[
b(V) = (f_l(V))_{1 \leq l \leq \lambda(V)},
\]

from the root \( V(0) \) of the subtree to \( V \). We denote the out-degree of a vertex by \( D \), and we set

\[
\beta(V) = D(V(0)) + \sum_{1 \leq l \leq \lambda(V)} D(t(f_l(V))), \quad V \in \mathcal{V}.
\]

We define a family \( \mathbf{F}_V \) of directed graphs, as the family, that contains the graphs \( G = G(\mathcal{V}, \mathcal{E}) \), such that \( G(\mathcal{V}, \mathcal{F}_G) \), is a tree and that have an edge \( \epsilon \in \mathcal{E} \setminus \mathcal{F}_G \), such that

\[
(4.1) \quad \beta(s(\epsilon)) = \lambda(s(\epsilon)) + \text{card}(\mathcal{E} \setminus \mathcal{F}_G).
\]

We describe the canonical models that we use for the graphs of this type. Let there be given \( \ell \in \mathbb{Z}_+ \) and \( K \in \mathbb{N} \), and let there be given

\[\eta_k \in [0, \ell], \quad 1 \leq k \leq K,\]

such that

\[\eta_k < \eta_{k+1}, \quad 0 \leq k < K.\]

Also let there be given

\[M_k \in \mathbb{N}, \quad 1 \leq k \leq K.\]

and

\[(\mu_k(L))_{L \in \mathbb{Z}_+} \in \mathbb{Z}_+^{\mathbb{Z}_+}, \quad 1 \leq k \leq K,\]

such that

\[
\sum_{L \in \mathbb{Z}_+} \mu_k(L) = M_k, \quad 1 \leq k \leq K.
\]

and such that

\[\mu_K(L) = 0, \quad L > \ell - \eta_K.\]

Set

\[\Pi = \{(\ell, K, \Omega) : \ell \in \mathbb{Z}_+, K \in \mathbb{N}, \Omega = (\eta_k, (\mu_k(L))_{L \in \mathbb{Z}_+})_{1 \leq k \leq K}\} \]

From these data \( (\ell, K, \Omega) \in \Pi \), we build a directed graph \( G[\ell, K, \Omega] \), with vertex set

\[\mathcal{V}[\ell, K, \Omega] = \{V(h) : 0 \leq h \leq \ell\} \cup \{V_{k,L}(m, l) : 1 \leq l \leq L, 1 \leq m \leq \mu_k(L), L \in \mathbb{N}, 1 \leq k \leq K\},\]

and edge set

\[\{f(h) : 1 \leq h \leq \ell\} \cup \{f_{k,L}(m, l) : 1 \leq l \leq L, 1 \leq m \leq \mu_k(L), L \in \mathbb{N}, 1 \leq k \leq K\} \cup \{e_k(m) : 1 \leq k \leq K, 1 \leq m \leq \mu_k(0)\} \cup \{\epsilon\}.\]

The source and target maps are given by

\[
s(f(h)) = V(h - 1), \quad 1 \leq h \leq \ell,
\]

\[
t(f(h)) = V(h), \quad 0 \leq h < \ell,
\]

\[
s(f_{k,L}(m, l)) = V(\eta_k), \quad 1 \leq m \leq M_k, 1 \leq k \leq K,
\]

\[
s(f_{k,L}(m, l)) = V_{k,L}(m, l - 1), \quad 1 < l \leq L, 1 \leq m \leq \mu_k(L), 1 \leq k \leq K,
\]

\[
t(f_{k,L}(m, l)) = V_{k,L}(m, l), \quad 1 \leq l \leq L, 1 \leq m \leq \mu_k(L), 1 \leq k \leq K,
\]

and

\[
s(e_{k,L,m}) = V_{k,L}(m, L), \quad 1 \leq m \leq \mu_k(L), 1 \leq k \leq K,
\]
Let $G_{10}$ TOSHIHIRO HAMACHI AND WOLFGANG KRIEGER

Proof.

Lemma 4.1. Let $e$ and (4.1) holds for $G$ of (4.1) holds for $V$ such that the path $b$ are touched by the path $b$ and for the edges $e_{K, \ell - \eta_{K}}(m), 1 \leq m \leq \mu_{K}(\ell - \eta_{K})$.

For $\ell \in \mathbb{N}, K = 1, \eta_{K} = \ell$ one obtains the single vertex graphs with the common point of the circles as the root of the subtree.

The graphs $G[0, K, \Omega]$ and $G[1, K, \Omega]$ are also contained in $F_{I}$.

Lemma 4.2. Let $G = G(V, E)$ be a directed graph such that $G(V, F_{G})$ is a tree. Let $G$ have an edge $e' \in E \setminus F_{G}$ for which (4.1) holds. Then $G(V, F_{G})$ has a leave for which (4.1) holds.

Proof. In the case, that $s(e')$ is not an leave of $G(V, F_{G})$, there is a unique leave $V$ of $G(V, F_{G})$, that can be reached from the vertex $s(e')$ by a directed path, and $D(V) = 1$, and one sees that (4.1) holds for the edge that leaves $V$.

Lemma 4.3. For a graph $G(V, E)$ such that $G(V, F_{G})$ is a tree and an edge $e \in E \setminus F$ such that

\[ \beta(s(e)) = \lambda(s(e)) + \text{card}(E \setminus F_{G}) \]

there exist uniquely data $[\ell, K, \Omega]$ such that $G(V, E)$ is isomorphic to $G[\ell, K, \Omega]$.

Proof. By Lemma 4.3 we can choose a edge $e_{0} \in E \setminus F_{G}$, such that $s(e_{0})$ is a leave of $G(V, F_{G})$, such that (4.1) holds for $e_{0}$ and such that $\lambda(s(e_{0}))$ is maximal. The parameter $K$ is given as the cardinality of the set $V_{0}$ that contains the vertices that are touched by the path $b(s(e_{0}))$ (including the root of $G(V, F_{G})$ and the source vertex of $e_{0}$), and that have more than one outgoing edge. One also has

\[ \{\eta_{0} : 1 \leq k \leq K\} = \{\lambda(V) : V \in V_{0}\}. \]

Note, that the path $b(s(e_{0}))$ shares the vertices $V(h), 0 \leq h \leq \eta_{K}$, (and only these) with any other path of maximal length, that connects the root to the source vertex of an edge $e$ that satisfies (4.1). Denoting by $E^{(k)}$ the set of edges in $E \setminus F_{G}$, that can be reached from $V(\eta_{k})$, one has $M_{k}(L), 1 \leq k \leq K, L \in \mathbb{Z}_{+}$, given by

\[ M_{k}(L) = \text{card}(\{e \in E^{(k)} : \lambda(s(e)) = \eta_{k} + L\}). \]

For given data $[\ell, K, \Omega], \Omega = ((\eta_{k}, (\mu_{k}(L))_{L \in \mathbb{Z}_{+}})_{1 \leq k \leq K}$, we introduce a sequence $[K^{(\kappa)}, \Omega^{(\kappa)}], 1 \leq \kappa \leq K$, of auxiliary data. We set

\[ K^{(\kappa)} = \kappa, \quad 1 \leq \kappa \leq K, \]

and

\[ \Omega^{(\kappa)} = (\ell, (\eta_{k}^{(\kappa)}, (\mu_{k}^{(\kappa)}(L))_{L \in \mathbb{Z}_{+}})_{1 \leq k \leq K}), \]

with

\[ \eta_{1}^{(\kappa)} = 0, \quad 1 \leq \kappa \leq K, \]

and with

\[ \eta_{k}^{(\kappa)} = \eta_{k} - \eta_{1}, \quad M_{k}^{(\kappa)} = M_{k}, \quad 1 \leq k \leq \kappa, \quad 1 \leq \kappa \leq K. \]
and

\[ \mu_k^{(e)}(L) = \begin{cases} 
0, & \text{if } L < \eta_1, \\
\mu_k(L - \eta_1), & \text{if } L \geq \eta_1,
\end{cases} \quad 1 \leq k \leq \kappa, \quad 1 \leq \kappa \leq K. \]

Inspection of the graph \( G[\ell, K, \Omega] \), \( \Omega = ((\eta_k, (\mu_k(L)))_{L \in \mathbb{Z}_+})_{1 \leq k \leq K} \), shows that

\[ \ell = \max_{e \in E \setminus F_G} \Lambda^{(e)}, \quad K = \text{card}\{\Delta^{(e)} : e \in E \setminus F_G\}, \]

and that, with the enumeration

\[ \{\Delta^{(e)} : e \in E \setminus F_G\} = \{\tau_k : 1 \leq k \leq K\}, \]

\( \tau_k < \tau_{k+1}, \quad 1 \leq k \leq K \),

that

\[ M_k = \text{card}\{e \in E \setminus F_G : \Delta^{(e)} = \tau_k\}, \quad 1 \leq k \leq K. \]

Set

\[ \delta_k = \eta_{k+1} - \eta_k, \quad 1 \leq k < K. \]

One finds that

\[ \delta_1 = \min\{\delta \in \mathbb{N} : \Sigma_{\Lambda^{(e)} \beta, e} (MD(G)) > I_{\Lambda^{(e)} + 2\delta}^{(e)}(MD(G[\ell, 1, \Omega^{(1)}])), e \in E^{(1)}\}, \]

and then inductively that

\[ \delta_k = \min\{\delta \in \mathbb{N} : I_{\Lambda^{(e)} + 2\delta}^{(e)}(MD(G)) > I_{\Lambda^{(e)} + 2\delta}^{(e)}(MD(G[\ell, \kappa, \Omega^{(e)}])), e \in E^{(k)}\}, \quad 1 < \kappa \leq K. \]

One finds also that

\[ \eta_1 = \frac{1}{K}(1 - \frac{1}{2}I^{(0)}_2 + \sum_{e \in E \setminus F_G} \Lambda^{(e)} - \sum_{1 \leq k \leq K} \sum_{k < k' < K} M_{k'} \delta_k), \]

and

\[ \mu_k(L) = \text{card}\{e \in E_k : \Lambda^{(e)} = \eta_k = L\}, \quad M \in \mathbb{Z}_+, \quad 1 \leq k \leq K. \]

**Theorem 4.3.** For a finite directed graph \( G = G(V, E) \) there exist data \([\ell, K, \Omega]\) such that there is a topological conjugacy

\[ MD(G) \simeq MD(G[\ell, K, \Omega]), \]

if and only if \( S(MD(G)) \) is a Dyck inverse monoid, and if there exists \( e \in E \setminus F_G \) such that

\[ I_{\Lambda^{(e)} + 2\delta}^{(e)}(MD(G)) = \Lambda^{(e)} + \nu(MD(G)) - 1, \]

and in this case (4.8) holds for \( K, \Omega \) given by (3 - 7).

**Proof.** One has

\[ I_{\Lambda^{(e)} + 2\delta}(MD(G)) = \beta(s(e)), \quad \Lambda(e) = \lambda(s(e)) + 1, \quad e \in E \setminus F_G. \]

It follows by (A) that (4.1) hold for any \( e \in E \setminus F_G \) that satisfies (4.8). As a consequence of Lemma 4.4 there exists a set of data \([\ell, K, \Omega]\) such that \( MD(G(V, E)) \) is topologically conjugate to \( MD(G(K, \Omega)) \). These data are given by (4.3 -7). \( \Box \)
Corollary 4.4. For directed graphs $G = G(V, E)$ such that to $MD(G)$ there is associated a Dyck inverse monoid, and such that there is an $e \in E \setminus F_G$, such that 

$$I_e^{(c)} + 2(MD(G)) = \Lambda^{(c)}(MD(G)) + v(MD(G)) - 1,$$

the topological conjugacy of the Markov-Dyck shifts $MD(G)$ implies the isomorphism of the graphs $G$.

Proof. In (4.3–7) the data $[l, K, \Omega]$ are expressed in terms of invariants of topological conjugacy. □

5. Spherically homogeneous case directed graphs of height two

We consider a directed graph $G(V, E)$ such that $G(V, F_G)$ is a tree, that has uniform height two. For spherically homogeneous graphs with a subtree of arbitrary height see [K&N].

5.1. A criterion for spherical homogeneity for height two. We denote the root of the tree $G(V, F_G)$ by $V_0$. For an edge $e \in E \setminus F_G$ we denote by $U_k^{(c)}$ the set of cycles $a$ in $G(V, E^+ \cup E^-)$ at $V_0$ of length $k \in \mathbb{N}$, such that the bi-infinite concatenation of $a$ yields a periodic point with multiplier $\tilde{e}$. The out-degree of $V_0$ we denote by $K$, and for $e \in E \setminus F_G$ we denote the out-degree of $V_{e, 1}$ by $L_e$, and the out-degree of $s(e)$ by $M_e$.

Lemma 5.1.

$$I_k^{(c)} = K + L_e + M_e, \quad e \in E \setminus F_G. \tag{5.1}$$

Proof. For all $e \in E \setminus F_G(G)$ every cycle in $U_k^{(c)}$ is obtained by inserting into the cycle $f_{\infty, 0} f_{\infty, 1} e$ either a loop $\tilde{f}_0 \tilde{f}_0^{-1}$, $s(\tilde{f}_0) = V_0$, at $V_0$, or a loop $\tilde{f}_1^{-1} \tilde{f}_1$, $s(\tilde{f}_1^{-1}) = V_{e, 1}$, at $V_{e, 1}$, or else a loop $\tilde{e} \leftarrow \tilde{e}^{-1}$, $s(\tilde{e}^{-1}) = s(e^{-})$ at $s(e^{-})$. □

Lemma 5.2. Let $I_k^{(c)}$ have the same value for all $e \in E \setminus F_G$. Then

$$\text{card} \{\{\tilde{f} \in F_G, s(\tilde{f}) = V_{e, 1}, \tilde{e} \in E \setminus F_G, s(\tilde{e}) = t(\tilde{f})\}\} = L_e M_e, \quad e \in E \setminus F_G.$$

Proof. By Lemma 5.1

$$\text{card} \{\{\tilde{f} \in F_G, s(\tilde{f}) = V_{e, 1}, \tilde{e} \in E \setminus F_G, s(\tilde{e}) = t(\tilde{f})\}\} =$$

$$\sum_{\{\tilde{f} \in F_G : s(\tilde{f}) = V_{e, 1}\}} \text{card} \{\{\tilde{e} \in E \setminus F_G, s(\tilde{e}) = t(\tilde{f})\}\} =$$

$$\sum_{\{\tilde{f} \in F_G : s(\tilde{f}) = V_{e, 1}\}} M_{\tilde{e}} = \sum_{\{\tilde{f} \in F_G : s(\tilde{f}) = V_{e, 1}\}} (I_0 - L_{\tilde{e}} - K) =$$

$$\sum_{\{\tilde{f} \in F_G : s(\tilde{f}) = V_{e, 1}\}} (I_0 - L_{\tilde{e}} - K) = \sum_{\{\tilde{f} \in F_G : s(\tilde{f}) = V_{e, 1}\}} M_e = L_e M_e. \quad \square$$

Lemma 5.3.

$$I_9^{(c)} = K(K^2 + 3KL_e + 2KM_e + 7L_e M_e + 2L_e^2 + 2M_e^2) + L_e(L_e^2 + 3L_e M_e + 2M_e^2) + M_e^3, \quad e \in E \setminus F_G. \tag{5.2}$$

Proof. Count the cycles in $U_9^{(c)}$ by applying Lemma 5.2. □

Lemma 5.4.

$$I_{10}^{(c)} = K^2 + L_e^2 + M_e^2 + 3KL_e + 3L_e M_e + 3K M_e, \quad e \in E \setminus F_G. \tag{5.3}$$
Proof. For all $e \in E \setminus F_G$ every cycle in $U^{(c)}_{10}$ transverses the edge $e$ twice. Count these cycles by applying Lemma 5.2. □

**Lemma 5.5.** Let $I^{(c)}_5$, $I^{(c)}_9$, and $I^{(c)}_{10}$ have the same value for all $e \in E \setminus F_G$. Then

\[ L_e = 6I_5 + 4K - 4 - \frac{I_5}{K}(I_5 + 1) + \frac{1}{I_5}(1 + 4I_{10} - 3K^2) + \frac{1}{KI_5}(I_9 + I_{10}). \]  

(5.4)

Proof. Insert (5.1), (5.2) and (5.3) into (5.4). □

**Theorem 5.6.** Let $I^{(c)}_5$, $I^{(c)}_9$, and $I^{(c)}_{10}$ have the same value for all $e \in E \setminus F_G$. Then $G$ is rotationally homogeneous.

Proof. The theorem follows from Lemma 5.1 and Lemma 5.5. □

5.2. Spherically homogeneous directed graphs with a subtree of height two. We denote the out-degree of the root of $F_G$ by $K$, the out-degree of the vertices of $F_G$ at level one by $L$ and the out-degree of of the vertices of $F_G$ at level two by $M$. We suppress the Markov-Dyck shift $M\sigma(G)$ of $G$ in the notation, and set

\[ \tau = \text{card}(F_G). \]

We note that

\[ \nu = KLM, \]  

(5.5)

\[ \tau = K(1 + L), \]  

(5.6)

\[ L = \frac{\tau - K}{K}, \]  

(5.7)

\[ M = \frac{\nu}{\tau - K}, \]  

(6.8)

\[ L = \frac{\nu}{M\tau - \nu}, \]  

(5.9)

\[ K = \tau - \frac{\nu}{M}. \]  

(5.10)

**Lemma 5.7.** $M = 1$ if and only if

\[ (\tau - \nu)(3 - 1) = (\tau - \nu)^2 + \nu. \]  

(5.11)

Proof. By lemma 5.1 and by (5.5) and (5.7) $M = 1$ implies (5.11). Conversely, let (5.11) hold for $K, L, M \in \mathbb{N}$. By Lemma 5.1 and by (5.5) and (5.7), (5.11) yields the equation

\[ L(1 + KL)M^2 - (1 + KL + 2KL^2 + L - L^2)M + KL^2 + KL + 1 - L^2 = 0. \]

with its root $M = 1$. For its other root $M'$ one finds that it cannot be a positive integer:

\[ \frac{1 + KL + KL^2 - L^2}{L(1 + KL)} < \frac{1}{L} + \frac{L(K - 1)}{1 + KL} < \frac{1}{L} + 1. \]  

(5.12)

Lemma 5.8.

\[ K^3 - (1 + \tau + I_5)K^2 + (2\tau - \nu + \tau I_5)K - \tau^2 = 0. \]
Proof. By Lemma 5.1 and by (5.7) and (5.8)
\[ I_5 = K + \frac{\tau - K}{K} + \frac{\nu}{\tau - K}, \]
or
\[ K(\tau - K)I_5 = K^2(\tau - K) + (\tau - K)^2 + \nu K, \]
which is (5.12). \hfill \Box

Lemma 5.9.
\[ I_0^4 = K^2 - K + KL^2 + KLM^2 + K^2LM. \] (5.13)
Proof. The Markov-Dyck shift of $G$ has $K(K - 1) + KL + KL(L - 1)$ neutral periodic orbits of length four, that transverse only edges in $F_G$, and it has $KLM(M - 1)$ neutral periodic orbits of length four, that transverse only edges in $E \setminus F_G$, and it has $KLM + KLM(M - 1)$ other neutral periodic orbits of length four. \hfill \Box

Lemma 5.10.
\[ K^3 + (\nu I_5 + \nu - 2\tau - I_0^4)K + \tau(\tau - \nu) = 0. \] (5.14)
Proof. To obtain (5.14) from (5.13), apply Lemma 5.1 and use (5.7). \hfill \Box

We will use the notation
\[ a = 1 + \tau + I_5, \quad b = (\nu - \tau)(I_5 + 2) - 2\tau - I_0^4, \quad c = \tau(2\tau - \nu). \]

Lemma 5.11. In the case that $M \geq 2$, one has that
\[ K = \frac{1}{2a}(b + \sqrt{b^2 - 4ac}), \] (5.15)
\[ L = \frac{\tau - K}{K}, \] (5.16)
\[ M = \frac{\nu}{\tau - K}. \] (5.17)
Proof. From Lemma 5.8 and Lemma 5.10 one has the equation
\[ aK^2 + bK + c = 0. \] (5.18)
If $c = 0$, then $K = -\frac{b}{2a}$. If $c < 0$ then $K$ is equal to the positive root of (5.18). If $c > 0$, then $M < \frac{b}{2a} + 2$, which leaves the possibilities that $M = 2$, or that $M = 3$ and $L = 1$, and in both cases one confirms that $\frac{b}{2a} > -2K$. We have shown that (5.15) holds in all cases, and (5.16) and (5.17) hold by (5.7) and (5.8). \hfill \Box

Theorem 5.12. For spherically homogeneous directed graphs $G = G(V,E)$, if
\[ (\tau - \nu)(I_5 - 1) = (\tau - \nu)^2 + \nu, \]
then
\[ K = \tau - \nu, \quad L = \frac{\nu}{K}, \quad M = 1, \]
and if
\[ (\tau - \nu)(I_5 - 1) \neq (\tau - \nu)^2 + \nu, \]
then
\[ K = \frac{1}{2a}(b + \sqrt{b^2 - 4ac}), \quad L = \frac{\tau - K}{K}, \quad M = \frac{\nu}{\tau - K}. \]
Proof. See Lemma 5.7 and Theorem 5.11 and also (5.9) and (5.10). \hfill \Box
Corollary 5.13. (a) Spherical homogeneity is an invariant of topological conjugacy for the Markov-Dyck shifts of directed graphs with a subtree of uniform depth two.

(b) For Markov-Dyck shifts of directed graphs with a single subtree of uniform depth two, that are spherically homogeneous, topological conjugacy of the Markov-Dyck shifts implies isomorphism of the graphs.

Proof. See Theorem 5.6 and Theorem 5.11. □

Corollary 5.14. Let \( \tilde{G} = G(\tilde{V}, \tilde{E}) \) and \( G = G(V, E) \) be finite strongly connected directed graphs and let
\[
\varphi : \text{Md}(G) \to \text{Md}(G),
\]
be a topological conjugacy. Let \( S(\text{Md}(G)) \) be a Dyck inverse monoid. Assume that

(a) \( \Lambda(e) = 3, \quad e \in \mathcal{M} \),

(b) \( I_5^{(e)}, I_9^{(e)}, \) and \( I_{10}^{(e)} \) have the same value for all \( e \in \mathcal{M} \),

(c) \( (\tau - \nu)(I_5 - 1) = (\tau - \nu)^2 + \nu. \)

Then there exists an automorphism \( \beta \) of \( \text{Md}(G) \) and an isomorphism
\[
\pi : \tilde{G} \to G,
\]
such that one has for the topological conjugacy \( \varphi \pi \) of \( \text{Md}(G) \) onto \( \text{Md}(G) \), that is induced by \( \pi \), that
\[
\varphi = \beta \varphi \pi.
\]

Proof. The hypothesis on the associated semigroup implies that \( G \) and therefore also \( \tilde{G} \) has a single subtree. Hypothesis (a) implies that \( G(V, \mathcal{F}^G) \), and therefore also \( G(\tilde{V}, \mathcal{F}^{\tilde{G}}) \), has uniformly depth two, and hypothesis (b) implies by Theorem 5.6 that \( G \), and therefore also \( \tilde{G} \), are rotationally homogeneous. Hypothesis (c) implies by Lemma 5.7 that \( \varphi \) induces a bijection
\[
\pi_{\circ} : \tilde{E} \setminus \mathcal{F}^{\tilde{G}} \to E \setminus \mathcal{F}_G,
\]
which can be extended to an isomorphism \( \pi : \tilde{G} \to G \), by setting
\[
\pi(\tilde{V}_0) = V_0, \quad \pi(\tilde{V}_1^{(\varepsilon)}) = V_1^{(\pi_{\circ}(\varepsilon))}, \quad \pi(s(\varepsilon)) = s(\pi_{\circ}(\varepsilon)),
\]
\[
\pi(\tilde{f}_0^{(\varepsilon)}) = f_0^{(\pi_{\circ}(\varepsilon))}, \quad \pi(\tilde{f}_1^{(\varepsilon)}) = f_1^{(\pi_{\circ}(\varepsilon))}.
\]
By construction
\[
\hat{\varphi}_{\pi} = \hat{\varphi}.
\]
Set
\[
\beta = \varphi \varphi_{\pi}^{-1}. \quad \square
\]

6. A FAMILY OF THREE-VERTEX GRAPHS

We consider directed graphs \( G = G(V, E) \), such that the semigroup \( S(\text{Md}(G)) \) is the graph inverse semigroup of a two-vertex graph \( \tilde{G} \), and such that the graph \( G(V, \mathcal{F}^G) \) decomposes into a one-edge tree and a one-vertex tree. We denote the source vertex of the edge of the one-edge tree by \( \alpha_0 \), and its target vertex by \( \alpha_1 \),
and we denote the vertex of the one-vertex tree by $\beta$. For the adjacency matrix $A_G$ of the graph $G$ we choose the notation

\[
\begin{pmatrix}
A_G(\alpha_0, \alpha_0) & A_G(\alpha_0, \alpha_1) & A_G(\alpha_0, \beta) \\
A_G(\alpha_1, \alpha_0) & A_G(\alpha_1, \alpha_1) & A_G(\alpha_1, \beta) \\
A_G(\beta, \alpha_0) & A_G(\beta, \alpha_1) & A_G(\beta, \beta)
\end{pmatrix} =
\begin{pmatrix}
T_{\alpha\alpha} - \Delta(\alpha) & 1 & \Delta_{\alpha} \\
\Delta(\alpha) & 0 & T_{\alpha\beta} - \Delta_{\alpha} \\
T_{\beta\alpha} & 0 & T_{\beta\beta}
\end{pmatrix}.
\]

Also, if $\Delta_{\alpha} = 0$, then it is required, that $\Delta_{\alpha} < T_{\alpha\beta}$, and if $T_{\alpha\alpha} = T_{\beta\beta} = \Delta_{\alpha} = 0$, then it is required that $T_{\alpha\beta} \geq 2, T_{\beta\alpha} \geq 2$.

Given a graph $G$ with adjacency matrix (6.1) set

\begin{align*}
(6.2) & \quad s = T_{\alpha\alpha} + T_{\beta\beta}, \\
(6.3) & \quad a = T_{\alpha\beta} + T_{\alpha\beta}, \\
(6.4) & \quad b = T_{\alpha\beta} T_{\beta\alpha}, \\
(6.5) & \quad c = \Delta_{\alpha} + T_{\alpha\beta}, \\
(6.6) & \quad d = \Delta_{\alpha} T_{\alpha\beta}.
\end{align*}

One has that

\[
T_{\beta\alpha} = \frac{b - d}{a - c}.
\]

As isomorphism invariants of the graph $\hat{G}$, the numbers $a$ and $b$, as well as the number $s$ are invariants of topological conjugacy. We note, that once also $c$ and $d$ are shown to be invariants of topological conjugacy the graph $G$ can be reconstructed from its Markov-Dyck shift by (6.7) and (6.2) or (6.3), and by (6.4) or (6.5). We also note that the Markov-Dyck shifts of graphs $G = G(\mathcal{V}, \mathcal{E})$ such that $\hat{G}$ is a two-vertex graph, and such that $G(\mathcal{V}, \mathcal{F}_G)$ decomposes into a one-edge tree and a one-vertex tree, are characterized by

\[
I_2^0(Md(G)) = 1 + s(Md(G)) + a(Md(G)).
\]

Given a graph $G$ with adjacency matrix (6.1) we say that multipliers $\hat{e}, \hat{\mathcal{E}}$, in $\mathcal{M}_1(Md(G))$ are compatible, and write $\hat{e} \sim \hat{\mathcal{E}}$, if $\hat{e} - \hat{\mathcal{E}} \in S^{-}(Md(G))$. In terms of the graph $\hat{G}$ the compatibility of $\hat{e}, \hat{\mathcal{E}} \in \mathcal{M}_1(Md(G))$ means that $\hat{e}$ and $\hat{\mathcal{E}}$ are loops at the same vertex of $\hat{G}$. We denote by $\mathcal{M}_{1,1}(Md(G))$ the set of multipliers of fixed points of $Md(G)$, and we denote by $\mathcal{M}_{2,1}(Md(G))(\mathcal{M}_{2,2}(Md(G)))$ the set of multipliers of the orbits of length two of $Md(G)$ of length one (two).

Consider the set of graphs $G$ with adjacency matrix (6.1) such that

\[
\mathcal{M}_{2,1}(Md(G)) \neq \emptyset,
\]

which is equivalent to the condition, that

\[
\Delta^{(\alpha)} > 0.
\]
In this case the graph $G$ is reconstructed from its Markov-Dyck shift by
\[
\Delta^{(\alpha)} = \text{card}(\mathcal{M}_{2,1}(Md(G))),
\]
\[
T_{\alpha\alpha} - \Delta^{(\alpha)} = \text{card}(\{\hat{e} \in M_{1,1}(Md(G)) : \hat{e} \sim \hat{e}\}), \quad \hat{e} \in \mathcal{M}_{2,1}(Md(G)),
\]
\[
T_{\alpha\beta} = I_1'(Md(G)) - T_{\alpha\alpha} - 1, \quad \hat{e} \in \mathcal{M}_{2,1}(Md(G)),
\]
\[
\Delta_{\alpha} T_{\beta\alpha} = I_2'(Md(G)) - T_{\alpha\alpha}(T_{\alpha\alpha} - 1) - T_{\beta\beta}(T_{\beta\beta} - 1).
\]

We partition the set of graphs with adjacency matrix (6.1) such that $\Delta^{(\alpha)} = 0$ into three subsets.

6.1. Consider the set of graphs $G$ with adjacency matrix (6.1), such that
\[
\mathcal{M}_{2,1}(Md(G)) = \emptyset,
\]
and such that there are $\hat{e}, \hat{e} \in \mathcal{M}_{1,1}(Md(G))$ that are incompatible, which is equivalent to the condition that
\[
\Delta^{(\alpha)} = 0, \quad T_{\alpha\alpha} > 0, \quad T_{\beta\beta} > 0.
\]

With a choice of $\hat{e}, \hat{e} \in \mathcal{M}_{1,1}(Md(G)), \hat{e} \neq \hat{e}$, set
\[
T_{\hat{e}} = \text{card}(\{\hat{e}' \in M_{1,1}(Md(G)) : \hat{e}' \sim \hat{e}\}), \quad T_{\hat{e}} = \text{card}(\{\hat{e}' \in M_{1,1}(Md(G)) : \hat{e}' \sim \hat{e}\}),
\]
The reconstruction of the graph $G$ from its Markov-Dyck shift is by
\[
I_{4}'(Md(G)) + I_4'(Md(G)) - T_{\hat{e}} - T_{\hat{e}} - 1 = \Delta_{\alpha} + T_{\beta\alpha} = c(Md(G)),
\]
\[
I_2'(Md(G)) - T_{\hat{e}}(T_{\hat{e}} - 1) - T_{\hat{e}}(T_{\hat{e}} - 1) = \Delta_{\alpha} T_{\beta\alpha} = d(Md(G)).
\]

6.2. Consider the set of graphs $G$ with adjacency matrix (6.1), such that
\[
\mathcal{M}_{2,1}(Md(G)) = \emptyset, \quad I_1'(Md(G)) > 0,
\]
and such that all $\hat{e} \in \mathcal{M}_{1,1}(Md(G))$ are compatible, which is equivalent to the condition, that
\[
\Delta^{(\alpha)} = 0, \quad T_{\alpha\alpha} + T_{\beta\beta} > 0, \quad T_{\alpha\alpha} T_{\beta\beta} = 0.
\]

Under these assumptions one has that
\[
-1 - I_1'(Md(G)) + \frac{1}{2}(\min I_4'(\mu) : \mu \in \mathcal{M}_2(Md(G)) + \max I_4'(\mu) : \mu \in \mathcal{M}_2(Md(G))) = \Delta_{\alpha} + T_{\beta\alpha} = c(Md(G)),
\]
\[
I_2'(Md(G)) - I_1'(Md(G))(I_1'(Md(G)) - 1) = \Delta_{\alpha} T_{\beta\alpha} = d(Md(G)).
\]

It follows, that
\[
A_G = \begin{pmatrix}
T_{\alpha\alpha} & 1 & \Delta_{\alpha} \\
0 & 0 & T_{\alpha\beta} - \Delta_{\alpha} \\
T_{\beta\alpha} & 0 & 0 \\
\end{pmatrix},
\]
or that
\[
A_G = \begin{pmatrix}
0 & 1 & \Delta_{\alpha} \\
0 & 0 & T_{\alpha\beta} - \Delta_{\alpha} \\
T_{\beta\alpha} & 0 & 0 \\
\end{pmatrix}.
\]

One distinguishes three cases.

6.2.a. Assume moreover, that
\[
1 + \Delta_{\alpha} \neq T_{\beta\alpha}.
\]

Under this additional assumption, if
\[
I_4'(\hat{e}) = I_2'(Md(G)) + 1 + \Delta_{\alpha}, \quad \hat{e} \in \mathcal{M}_{1,1}(Md(G)),
\]
then \( A_G \) is given by (6.7), and if
\[
I_4^{(e)} = I_7(Md(G)) + T_{\beta,\alpha}, \quad \tilde{e} \in \mathcal{M}_{1,1}(Md(G)),
\]
then \( A_G \) is given by (6.8).

6.2.b. Assume moreover that
\[
1 + \Delta_\alpha = T_{\beta,\alpha}, \quad T_{\alpha,\beta} - \Delta_\alpha \neq T_{\beta,\alpha}.
\]
Under this additional assumption, if
\[
I_4^{(e)}(Md(G)) = T_{\alpha,\beta} - \Delta_\alpha, \quad \tilde{e} \in \mathcal{M}_{1,1}(Md(G)),
\]
then \( A_G \) is given by (6.7), and if
\[
I_4^{(e)}(Md(G)) = T_{\beta,\alpha}, \quad \tilde{e} \in \mathcal{M}_{1,1}(Md(G)),
\]
then \( A_G \) is given by (6.8).

6.2.c. Assume moreover, that
\[
1 + \Delta_\alpha = T_{\beta,\alpha} = T_{\alpha,\beta} - \Delta_\alpha.
\]
One distinguishes two cases.

6.2.c.I. Assume further, that
\[
\Delta_\alpha = 0.
\]
Under this further assumption one has that
\[
A_G = \begin{pmatrix}
I_7(Md(G)) & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix},
\]
or that
\[
A_G = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & I_7(Md(G))
\end{pmatrix}.
\]
These two adjacency matrices yield isomorphic graphs.

6.2.c.II. Assume further, that
\[
\Delta_\alpha > 0.
\]
Under this further assumption one has from (6.9) that
\[
T_{\alpha,\beta} = 1 + \Delta_\alpha, \quad T_{\beta,\alpha} = 1 + 2\Delta_\alpha.
\]
If \( \tilde{G} \) is isomorphic to the graph with adjacency matrix
\[
\begin{pmatrix}
I_7(Md(G)) & 1 + \Delta_\alpha \\
1 + 2\Delta_\alpha & 0
\end{pmatrix},
\]
then
\[
A_G = \begin{pmatrix}
I_7(Md(G)) & 1 & \Delta_\alpha \\
0 & 0 & 1 \\
1 + 2\Delta_\alpha & 0 & 0
\end{pmatrix},
\]
and if \( \tilde{G} \) is isomorphic to the graph with adjacency matrix
\[
\begin{pmatrix}
I_7(Md(G)) & 1 + 2\Delta_\alpha \\
1 + \Delta_\alpha & 0
\end{pmatrix},
\]
then
\[
A_G = \begin{pmatrix}
0 & 1 & \Delta_\alpha \\
0 & 0 & 1 \\
1 + 2\Delta_\alpha & 0 & I_7(Md(G))
\end{pmatrix}.
\]
6.3. Consider the set of graphs $G$ with adjacency matrix (6.1), such that one has that

$$I_1^-(Md(G)) = 0,$$

which is equivalent to the condition, that

$$T_{\alpha\alpha} = T_{\beta\beta} = 0.$$

One distinguishes two cases.

6.3.a. Assume, that also

$$I_2^-(Md(G)) > 0,$$

which is equivalent to the condition, that

$$\Delta_\alpha > 0.$$

Under this further assumption the reconstruction if the graph $G$ from $Md(G)$ is by

$$-1 + I_4^{(\mu)}(Md(G)) = \Delta_\alpha + T_{\alpha\beta} = c(Md(G)), \quad \mu \in \mathcal{M}_{2,2}(Md(G)),$$

$$I_2^-(Md(G)) = \Delta_\alpha T_{\beta\alpha} = d(Md(G)).$$

6.3.b. Assume, that also

$$I_2^-(Md(G)) = 0,$$

which is equivalent to the condition, that

$$\Delta_\alpha = 0.$$

Denote by $I_2^-(\alpha)$ the cardinality of the set of points $(p_i)_{i \in \mathbb{Z}}$ of period two of $Md(G)$, such that $s(p_0) \in \{\alpha_0, \alpha_1\}$, and by $I_2^-(\beta)$ the cardinality of the set of points $(p_i)_{i \in \mathbb{Z}}$ of period two of $Md(G)$, such that $s(p_0) = \beta$. By [?, Corollary 2.3] the set $\{I_0^+(\alpha), I_0^+(\beta)\}$ is an invariant of topological conjugacy. One has, that

$$I_0^+(\alpha) = 1 + T_{\alpha\beta}, \quad I_0^+(\beta) = T_{\beta\alpha}. $$

The graph $G$ is reconstructed by

$$\{T_{\beta\alpha}\} = \{T_{\alpha\beta}, T_{\beta\alpha}\} \cap \{I_0^+(\alpha), I_0^+(\beta)\}.$$ 

In summary we state a theorem.

**Theorem 6.1.** For graphs $G = G(V, E)$ such that the semigroup $S(Md(G))$ is the graph inverse semigroup of a two-vertex graph, and such that the graph $G(V, F_G)$ decomposes into a one-edge tree and a one-vertex tree, the topological conjugacy of their Markov-Dyck shift implies the isomorphism of the graphs.

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