Spectral Proofs of Maximality of Some Seidel Matrices

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Lin and Yu verified maximality of some Seidel matrices in 2020 by calculating clique numbers with a computer. In this paper, we show that maximality of these matrices follows by investigating their spectra, without using a computer.

KEYWORDS: Seidel matrix, equiangular lines, maximality of Seidel matrices, Seidel spectrum

1. Introduction

A set of lines through the origin in a Euclidean space is equiangular if any pair from these lines forms the same angle. The problem to determine the maximum cardinality $N(d) (d \in \mathbb{Z}_{\geq 2})$ of a set of equiangular lines in $\mathbb{R}^d$ dates back to the result of Haantjes [8]. Also, the value $N(d)$ is known for every $d \leq 17$ (see [5, Table 1]). Some lower bounds of the values $N(d)$ are given by constructing sets of equiangular lines for larger values of $d$. We are interested in whether the bounds can be improved, and in particular we will check whether some sets of equiangular lines can be extended. Lin and Yu [9, 10] defined a set $X$ of equiangular lines of rank $r$ to be saturated if there is no line $l \notin X$ such that the union $X \cup \{l\}$ is a set of equiangular lines of rank $r$. Here, the rank of a set of equiangular lines is the smallest dimension of Euclidean spaces into which these lines are isometrically embedded. By using a computer implementing their algorithm [10, p. 274], they verified in [10, Theorem 1 and the end of Sect. 3.2] that seven sets of equiangular lines are saturated. Their algorithm requires the computation of the clique numbers of graphs, which is known to be an NP-complete problem. We will verify their results by investigating spectra, without a computer.

We introduce Seidel matrices in connection with equiangular lines. A Seidel matrix is a symmetric matrix with zero diagonal and all off-diagonal entries $\pm 1$. Note that if a Seidel matrix $S$ has largest eigenvalue $\lambda$, then there exist vectors whose Gram matrix equals $\lambda I - S$, which span a set of equiangular lines with common angle $\arccos(1/\lambda)$. Cao, Koolen, Munemasa and Yoshino [2] defined a Seidel matrix $S$ with largest eigenvalue $\lambda$ to be maximal if there is no Seidel matrix $S'$ containing $S$ as a proper principal submatrix with largest eigenvalue $\lambda$ such that $\text{rank}(\lambda I - S') = \text{rank}(\lambda I - S)$. In other words, the Seidel matrix obtained from a saturated set of equiangular lines is maximal.

In this paper, we prove Theorem 1.1, which shows maximality of Seidel matrices with spectra in Table 1, with only the aid of spectra instead of a computer. Specifically, we use Cauchy's interlacing theorem and the angles of matrices, which are used by Greaves and Yatsyna [6] in order to show some Seidel spectra do not exist. This method enables us to simultaneously verify maximality of some Seidel matrices having common spectra. For example, Szöllösi and Östergård in [11, Theorem 5.2] showed that there exist, up to switching equivalence, at least 1045 Seidel matrices of order 28 with spectrum $\{[5]^{14}, [-3]^7, [-7]^7\}$. Actually, Theorem 1.1 implies that these Seidel matrices are maximal.

Theorem 1.1. Seidel matrices with spectrum in Table 1 are maximal.

Remark 1.2. In this paper, the maximality of Seidel matrices having some given spectrum $\alpha$ is verified in such a way that we firstly enumerate the spectra of Seidel matrices properly containing a Seidel matrix having spectrum $\alpha$, and show that there are no Seidel matrices with these spectra. Lin and Yu [10] verified that seven Seidel matrices are maximal. Six of them have spectra in Table 1, and the other has spectrum $\beta = \{[5]^{16}, [-12 \pm \sqrt{37}]^6, [-7]^8, [-11]^8, [-13]^7\}$. We do not treat Seidel matrices with spectrum $\beta$ because there are many possible spectra of Seidel matrices which properly containing a Seidel matrix with spectrum $\beta$.

Also, since $N(14) = 28$, $N(16) = 40$ and $N(17) = 48$ are proved in [4, 5], we immediately obtain maximality of Seidel matrices having one of the three spectra at the beginning of Table 1. However, to determine the values $N(d)$ (for some $d$) is harder than proving maximality of these Seidel matrices.
Lin and Yu describe a strategy to determine $N(d)$ by using maximality in [10, Section 4]. In their strategy, the first step is to classify the sets of $N(d)$ equiangular lines in $\mathbb{R}^d$, and the second is to show their maximality. The technique of proving maximality will become meaningful after such a classification is achieved.

2. Preliminaries

In this section, we give some definitions and preliminary results. Every Seidel matrix $S$ of order $n$ satisfies $\text{tr} S = 0$ and $\text{tr} S^2 = n(n-1)$. This together with Cauchy’s interlacing theorem (see [1, Corollary 2.5.2]) immediately implies the following lemma.

Lemma 2.1. Let $m$ be a positive integer at least 2. Let $S$ be a Seidel matrix of order $n$ having eigenvalues $\lambda_1 > \lambda_2 > \ldots > \lambda_n$ with respective multiplicities $m(\lambda_1), \ldots , m(\lambda_m)$. If $S$ is not maximal, then there exists a Seidel matrix $S$ of order $n + 1$ which contains $S$ as a principal submatrix, with spectrum

$$\{[\lambda_i]^{m(\lambda_i)+1} \cup \{\lambda_i\mid 2 \leq i \leq m\} \cup \{\mu_i \mid 2 \leq i \leq m\}$$

(2.1)

for some real numbers $\mu_2, \ldots , \mu_m$ satisfying that

$$\lambda_2 \geq \mu_2 \geq \lambda_3 \geq \mu_3 \geq \ldots \geq \lambda_m \geq \mu_m .$$

(2.2)

Furthermore,

$$\sum_{i=2}^{m} \mu_i = -\lambda_1 + \sum_{i=2}^{m} \lambda_i, \quad \text{and} \quad \sum_{i=2}^{m} \mu_i^2 = 2n - \lambda_1^2 + \sum_{i=2}^{m} \lambda_i^2 .$$

(2.3)

Following [6], we define the angles of a matrix and describe their properties. Let $M$ be a real symmetric matrix of order $n$. Let $\Lambda(M)$ be the set of eigenvalues of $M$, and $\chi(M)$ the characteristic polynomial of $M$. For each $i \in \{1, \ldots , n\}$, let $M[i]$ denote the matrix obtained from $M$ by deleting the $i$-th row and $i$-th column. For $\lambda \in \Lambda(M)$ and $j \in \{1, \ldots , n\}$, the angle $\alpha_{\lambda,j}$ is defined to be the norm of the orthogonal projection of $e_j$ onto the eigenspace of $\lambda$. Here $e_j$ denotes the vector of which the $j$-th entry is 1 and the others are 0.

Proposition 2.2 ([6, Proposition 5.1]). Let $M$ be a real symmetric matrix of order $n$. Then $\sum_{\lambda \in \Lambda(M)} \alpha_{\lambda,j}^2 = 1$ for each $j \in \{1, \ldots , n\}$. For each $\lambda \in \Lambda(M)$, the multiplicity of $\lambda$ equals $\sum_{j=1}^{n} \alpha_{\lambda,j}^2$.

Proposition 2.3 ([6, Proposition 5.2]). Let $M$ be a real symmetric matrix of order $n$. Then for each $j \in \{1, \ldots , n\}$,

$$\chi_{M,j}(x) = \chi_M(x) \sum_{\lambda \in \Lambda(M)} \frac{\alpha_{\lambda,j}^2}{x - \lambda} .$$

Let $M_n(\mathbb{Z})$ denote the set of integer matrices of order $n$, and $J \in M_n(\mathbb{Z})$ be the all-one matrix. Since every Seidel matrix is congruent to $J - I$ modulo $2M_n(\mathbb{Z})$, the following lemma can be applied to Seidel matrices.

Lemma 2.4 ([7, Lemma 2.2]). Let $M$ be a matrix congruent to $J - I$ modulo $2M_n(\mathbb{Z})$. Then modulo $2\mathbb{Z}[x]$, we have

$$\chi_M(x) = \begin{cases} (x+1)^n & \text{if } n \in 2\mathbb{Z}, \\ x(x+1)^{n-1} & \text{if } n \in 2\mathbb{Z} + 1. \end{cases}$$

3. Maximal Seidel Matrices with Exactly Three Eigenvalues

In Lemma 2.1, suppose $m = 3$. Rewriting the inequality $(\mu_2 - \mu_3)^2 \geq 0$ by (2.3), we have the following lemma.

Lemma 3.1. A Seidel matrix of order $n$ with three eigenvalues $\lambda_1 > \lambda_2 > \lambda_3$ is maximal if

$$2(\lambda_2^2 + \lambda_3^2 - \lambda_1^2) - (\lambda_2 + \lambda_3 - \lambda_1)^2 + 4n < 0.$$  

(3.1)
The adjacency matrix of the line graph $L$ holds for $n$.

**Table 2. Potential spectra of hypothetical Seidel matrices of order $n$ up to 24 with exactly three integral eigenvalues $\lambda_1 > 0 \geq \lambda_2 > \lambda_3$.**

| $n$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\#$ | (3.1) | $n$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\#$ | (3.1) |
|-----|-------------|-------------|-------------|-----|------|-----|-------------|-------------|-------------|-----|------|
| 6   | 3           | -1          | -3          | 1   | true| 20  | 9           | -1          | -11         | 1   | true|
| 8   | 3           | -1          | -5          | 1   | true| 20  | 7           | -1          | -5          | 4   | true|
| 9   | 5           | -1          | -4          | 1   | true| 20  | 7           | -1          | -13         | 1   | true|
| 9   | 3           | 0           | -3          | 1   | false| 20  | 5           | -1          | -5          | 8   | true|
| 10  | 3           | -1          | -7          | 1   | false| 20  | 3           | -1          | -17         | 1   | false|
| 12  | 7           | -1          | -5          | 1   | true| 21  | 13          | -1          | -8          | 1   | true|
| 12  | 5           | -1          | -7          | 1   | true| 21  | 5           | -1          | -16         | 1   | false|
| 12  | 3           | -1          | -5          | 1   | true| 21  | 3           | 0           | -7          | 1   | false|
| 12  | 3           | -1          | -9          | 1   | false| 22  | 3           | -1          | -19         | 1   | false|
| 14  | 3           | -1          | -11         | 1   | false| 24  | 15          | -1          | -9          | 1   | true|
| 15  | 9           | -1          | -6          | 1   | true| 24  | 11          | -1          | -5          | 1   | true|
| 15  | 5           | -1          | -10         | 1   | true| 24  | 7           | -1          | -5          | $\geq 1^*$ | true|
| 15  | 3           | -2          | -5          | 1   | false| 24  | 7           | -1          | -9          | 3   | true|
| 16  | 7           | -1          | -9          | 1   | true| 24  | 7           | -1          | -17         | 1   | true|
| 16  | 3           | -1          | -13         | 1   | false| 24  | 5           | -1          | -7          | ?   | true|
| 18  | 11          | -1          | -7          | 1   | true| 24  | 5           | -1          | -19         | 1   | false|
| 18  | 5           | -1          | -7          | 1   | true| 24  | 5           | -3          | -7          | $\geq 1^*$ | true|
| 18  | 5           | -1          | -13         | 1   | false| 24  | 3           | -1          | -21         | 1   | false|
| 18  | 3           | -1          | -15         | 1   | false| 24  | 3           | -5          | -9          | 1   | false|
| 18  | 3           | -3          | -9          | 1   | false|     |              |              |              |     |      |

The converse of this lemma does not always hold. For example, the Seidel matrix $S := J - I - 2A$ where $A$ is the adjacency matrix of the line graph $L(K_{2,n})$ of the complete bipartite graph $K_{2,n}$ has three eigenvalues $3 > -1 > 3 - 2n$ for $n \geq 3$. This was shown to be maximal in [2, Theorem 3.1]. However, $2((-1)^2 + (3 - 2n)^2 - 3^2) = (1 + (3 - 2n)^2 - 3^2) + 4 \cdot 2n - 20n + 1 \geq 0$
holds for $n \geq 5$.

Szöllösi and Östergård enumerated potential spectra of hypothetical Seidel matrices of order $n$ up to 24 in [11, Table 5]. In their table, the existence of some Seidel matrices remained open. However, since two of them exist due to the existence of regular graphs with spectrum $\{[8]^1, [2]^6, [0]^9, [-4]^6\}$ or $\{[10]^1, [2]^3, [-2]^6, [-4]^6\}$ in [14, Appendix A], we display their table as Table 2 together with slight changes (with a star as a superscript). Note that we do not give the multiplicities of eigenvalues in Table 2 since they are can be determined from the eigenvalues. We remark that if a Seidel matrix $S$ satisfies (3.1) then so does $-S$. Using Lemma 3.1, we obtain 2 \cdot 20 spectra of which Seidel matrices are maximal.

### 4. Maximal Seidel Matrices with Spectra in Table 1

In this section, we divide Theorem 1.1 into the following claims, and prove them. First, maximality of Seidel matrices with exactly three eigenvalues in Table 1 follows from Lemma 3.1. Below we treat the Seidel matrices with four eigenvalues in Table 1.

**Proposition 4.1.** A Seidel matrix with spectrum $\{[5]^3, [-7]^5, [-11]^10, [-15]^3\}$ is maximal.

**Proof.** By way of contradiction, we assume that such a Seidel matrix is not maximal. Applying Lemma 2.1, we find a Seidel matrix $S'$ and real numbers $\mu_2, \mu_3$ and $\mu_4$ satisfying (2.2) and (2.3), where $\lambda_1 := 5, \lambda_2 := -7, \lambda_3 := -11$ and $\lambda_4 := -15$. By (2.3), we see that $\mu_2, \mu_3$ and $\mu_4$ are the roots of $f(x) = x^3 + 38x^2 + 481x + q$ for some integer $q$. Then the discriminant $D(f)$ of $f$ is $-(q - 2028)(27q - 54760)$. Noting that $54760/27 \approx 2028.1$, we see that $q = 2028$ to fulfill $D(f) \geq 0$. Namely we have $f(x) = x^3 + 38x^2 + 481x + 2028 = (x + 12)(x + 13)^2$. This means that $\mu_2 = -12$ and $\mu_3 = \mu_4 = -13$. This contradicts $\lambda_4 \geq \mu_4$ in (2.2).

In the same way, we may show the following proposition.

**Proposition 4.2.** A Seidel matrix with spectrum $\{[5]^7, [-15]^5, [-19]^10, [-25]^3\}$ is maximal.

**Proof.** By way of contradiction, we assume that such a Seidel matrix is not maximal. Applying Lemma 2.1, we find a Seidel matrix $S'$ and real numbers $\mu_2, \mu_3$ and $\mu_4$ satisfying (2.2) and (2.3), where $\lambda_1 := 5, \lambda_2 := -15, \lambda_3 := -19$ and $\lambda_4 := -25$. By (2.3), we see that $\mu_2, \mu_3$ and $\mu_4$ are the roots of $f(x) = x^3 + 64x^2 + 1365x + q$ for some integer $q$. Then the discriminant $D(f)$ of $f$ is $-(q - 9702)(27q - 261950)$. Noting that $261950/27 \approx 9701.9$, we see that $q = 9702$ to...
fulfill $D(f) \geq 0$. Namely we have $f(x) = x^3 + 64x^2 + 1365x + 9702 = (x + 21)^2(x + 22)$. This means that $\mu_2 = \mu_3 = -21$ and $\mu_4 = -22$. This contradicts $\lambda_4 \geq \mu_4$ in (2.2).

Lemma 4.3. There is no Seidel matrix with spectrum $\{5^2, -5^2, -7^2, -8^2, -11^2\}$.

Proof. By way of contradiction, we assume that such a Seidel matrix $S$ exists. Fix an arbitrary index $j \in \{1, \ldots, 41\}$ of $S$. By Cauchy's interlacing theorem (see [1, Corollary 2.5.2]), the spectrum of $S[j]$ is $\{5^2, -5^2, -7^2, -8^2, -11^2\}$ for some real numbers $\mu_1, \mu_2$ and $\mu_3$. Since $\text{tr} S[j] = 0$ and $\text{tr} S[j]^2 = 40 \cdot 39$, we have $\mu_1 + \mu_2 + \mu_3 = -21$ and $\mu_1^2 + \mu_2^2 + \mu_3^2 = 179$. Hence $\mu_1, \mu_2$ and $\mu_3$ are the roots of $f(x) := x^3 + 21x^2 + 131x + qj$ for some integer $qj$. Next we consider the angles $\alpha_{\lambda, j}$ of the Seidel matrix $S$. Proposition 2.3 implies that

$$f_j(x) = (x - \mu_1)(x - \mu_2)(x - \mu_3) = \prod_{\lambda \in \Lambda(S)} (x - \lambda)^{a_{\lambda, j}}.$$ 

Hence we have for each $\lambda \in \Lambda(S)$,

$$a_{\lambda, j}^2 = f_j(\lambda) \prod_{\lambda \in \Lambda(S) \setminus \{\lambda\}} (\lambda - v)^{-1}.$$ 

We obtain that $a_{-11, j}^2 = -(q_1 - 231)/192$ and $a_{-8, j}^2 = (q_1 - 216)/39$. Since $a_{-8, j}^2 \geq 0$, we have $q_1 \geq 216$. In addition, Lemma 2.4 implies that $q_1$ is odd, and hence $q_1 \geq 217$ follows. Then we see that $3a_{-8, j}^2 > a_{-11, j}^2$ holds. This together with Proposition 2.2 implies that

$$3 = \sum_{j=1}^{41} a_{-8, j}^2 > \sum_{j=1}^{41} a_{-11, j}^2 = 3.$$ 

This is a contradiction.

Proposition 4.4. A Seidel matrix with spectrum $\{5^2, -5^2, -7^2, -8^2, -11^2\}$ is maximal.

Proof. By way of contradiction, we assume that such a Seidel matrix is not maximal. Applying Lemma 2.1, we find a Seidel matrix $S'$ and real numbers $\mu_2, \mu_3$ and $\mu_4$ satisfying (2.2) and (2.3), where $\lambda_1 := 5, \lambda_2 := -5, \lambda_3 := -7$ and $\lambda_4 := -11$. By (2.3), we see that $\mu_2, \mu_3$ and $\mu_4$ are the roots of $f(x) = x^3 + 26x^2 + 221x + qj$ for some integer $qj$. Since $f(-7) = f(-11) = qj - 616$ holds, if $qj \neq 616$, then the number of roots of $f$ in the open interval $(-11, -7)$ is either 0 or 2. This contradicts (2.2). Thus we obtain that $qj = 616$. Hence we have $f(x) = x^3 + 26x^2 + 221x + 616 = (x + 7)(x + 8)(x + 11)$. This means that $\mu_2 = -7, \mu_3 = -8$ and $\mu_4 = -11$. Namely, the spectrum of $S'$ is $\{5^2, -5^2, -7^2, -8^2, -11^2\}$. However, this contradicts Lemma 4.3.

The above propositions complete the proof of Theorem 1.1.

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