The metric transformations and modified Newtonian gravity

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Abstract:
The metric tensor in the four dimensional flat space-time is represented as the matrix form and then the transformation is performed for successive Lorentz boost. After extending or more generalizations the equivalent transformation of metric is derived for the curved space-time, manifested after the synergy of different sources of mass. The transformed metric in linear perturbation interestingly reveals a shift from Newtonian gravity for two or more than two body system.

Keywords: space-time metric, Lorentz boost, linearized metric, modified gravity

Mathematics classification codes: 83; 53; 53A; 53A4

1. Introduction:
In general relativity and in its application it is extensively studied only for the space-time curvature for single massive body and the dynamics of the test body is presented considering it like a small point mass. If we took the test mass enough massive the scenario would be quite different. As now the synergy between two massive bodies would generate different thing. Here in the following we will proceed to define the shift from usual Newtonian gravity to some extent of modification over linearized metric transformation in curved space-time.

In the four dimensional flat space-time or Minkowskian space-time the invariant line element \((ds^2)\) \([1, 2]\) in different inertial coordinates systems (here \(X, X'\) and \(X''\))

\[
ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu = -(dx^0)^2 + \sum_{k=1}^{3}(dx^k)^2 = -(dx^0)^2 + \sum_{k=1}^{3}(dx'^k)^2 = -(dx''^0)^2 + \sum_{k=1}^{3}(dx''^k)^2\]  

(1)

Where k stands for three spatial coordinates and the temporal elements are

\(dx^0 = cdt, \quad dx'^0 = cdt', \quad dx''^0 = cdt'', \quad c\) is the light velocity in free space

The transformation of coordinates system \(X\) to \(X'\) can be represented in matrix as

\[
\begin{bmatrix}
dx^0 \\
dx^1 \\
dx^2 \\
dx^3
\end{bmatrix}
= \begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
dx^0 \\
dx^1 \\
dx^2 \\
dx^3
\end{bmatrix}
= \begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
dx^0 \\
dx^1 \\
dx^2 \\
dx^3
\end{bmatrix}
\]  

(2)
The Lorentz transformation \([1, 2]\) by using Lorentz boost \([1, 2]\) also deals with the coordinates transformation as

\[
\begin{bmatrix}
dx^0 \\
dx^1 \\
dx^2 \\
dx^3
\end{bmatrix} = [L_1] \begin{bmatrix}
dx^0 \\
dx^1 \\
dx^2 \\
dx^3
\end{bmatrix} \tag{4}
\]

For the sake of simplicity let us denote, Column vector

\[
\begin{bmatrix}
dx^0 \\
dx^1 \\
dx^2 \\
dx^3
\end{bmatrix} = [dX]; 
\begin{bmatrix}
dx^0 \\
dx^1 \\
dx^2 \\
dx^3
\end{bmatrix} = [dX'], \text{ and row vector as transpose}
\]

\[
\begin{bmatrix}
dX^T \eta [dX] = [dX']^T \eta [dX']
\end{bmatrix} \tag{3}
\]

The Lorentz transformation matrix, \([L_1]\) = \(
\begin{bmatrix}
-\gamma \beta_1 & \gamma_1 & 0 & 0 \\
\gamma_1 & -\gamma \beta_1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\)

\[
\gamma_1 = \frac{1}{\sqrt{1-\frac{v_1^2}{c^2}}}; \quad \beta_1 = \frac{v_1}{c}; \quad v_1 \text{ the uniform velocity unidirectional axis of } X' \text{ relative to } X
\]

So that,

\[
[dX'] = [L_1][dX] \tag{5}
\]

2. The metric transformations under Lorentz boost:

Now plugging equation (5) to equation (3)

\[
[dX]^T [\eta][dX] = [dX']^T [\eta'][L_1][dX] \tag{6}
\]

We have the transformed basis

\[
[\eta]^T [dX] = ([\eta'][L_1])^T[dX'] \tag{7}
\]

[\eta], [L_1], and [\eta'] are symmetric.
\[ [\eta][dX] = [L_1][\eta][dX'] \quad (8) \]

Equation (8) can be used to check Lorentz invariance of metric tensor directly, but it tells that representing a vector equally in one spacetime to another \(X \rightarrow X'\), the metric get transform as,

\[ [\eta] \rightarrow [L_1][\eta'] \quad (9) \]

The above metric transformation is done for coordinate transformations in velocity space and is a linear transformation in Euclidean space. If we go for transformations in rapidity space [3] we can replace it in a Non-Euclidean style [3, 4], and the equation

\[
\begin{bmatrix}
\frac{dx^0}{d\tau^0} \\
\frac{dx^1}{d\tau^1} \\
\frac{dx^2}{d\tau^2} \\
\frac{dx^3}{d\tau^3}
\end{bmatrix} = \Lambda(w) \begin{bmatrix}
dx^0 \\
dx^1 \\
dx^2 \\
dx^3
\end{bmatrix} 
\]

(11)

Where, \(\Lambda(w) = e^{Z w}\)

rapidity \(w_1 = -\ln[\gamma_1(1 - \beta_1)]\)

\[
Z = \begin{bmatrix}
0 & -1 \\
-1 & 0
\end{bmatrix}
\]

We have,

\[ [\eta] \rightarrow e^{Z w}[\eta'] \quad (12) \]

Turning equation (12) into nonlinear fashion,

\[ [\eta] \rightarrow [1 + Z w + \frac{(Z w)^2}{2!} + \cdots ][\eta'] \quad (13) \]

Likewise for reference frame \(X''\) using equation (5)

\[
[dX''][^T][\eta''][L_2][L_1][dX'] = [dX][^T][\eta][dX] \quad (14)
\]

\[
[dX''][^T][\eta''][L_2][L_1] = [dX][^T][\eta] \quad (15)
\]

Taking transpose both side of equation (15)

\[
[\eta][dX] = ([\eta'][L_2][L_1])[^T][dX'] \quad (16)
\]

\([\eta], [L_1], [L_2]\) and \([\eta'']\) are symmetric. Finally leads to

\[ [\eta][dX] = [L_1][L_2][\eta'][dX'] \quad (17) \]

Thus metric get transform for successive two Lorentz transformations [2] in rapidity space.

\[ [\eta] \rightarrow [e^{Z w_1}][e^{Z w_2}][\eta'] \quad (18) \]
3. Generalizing flat space-time metric to curved space-time metric:

From Einstein’s principle of equivalence the effects of gravity is equivalent to the experience in a curved space-time [5, 6, and 7]. Introduction of massive body in a flat space-time or Minkowskian space-time [1] turns out it to curved space-time, and line elements is given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu$$  \hspace{1cm} (19)

Where the metric, $[g_{\mu\nu}] = [\eta_{\mu\nu} + h_{\mu\nu} + \text{higher order terms in } h_{\mu\nu}]$  \hspace{1cm} (20)

$$[g_{\mu\nu}] = [1 + [h_{\mu\nu}][\eta_{\mu\nu}]^{-1} + \text{higher order terms in } h_{\mu\nu}][\eta_{\mu\nu}]^{-1}][\eta_{\mu\nu}]$$  \hspace{1cm} (21)

Now for metric transformation from flat spacetime to curved spacetime, equations (13) and (21) are equivalent in representation.

The perturbation $h_{\mu\nu}$ are assumed to be so small that all expressions can be linearised with respect to the $h_{\mu\nu}$ and their derivatives of $h_{\mu\nu}$. Choosing orthogonal coordinate system, the flat space-time Minkowski metric [8] in matrix form, the linear perturbation in matrix form,

$$[h_{\mu\nu}] = \begin{pmatrix}
h_{00} & 0 & 0 & 0 \\
0 & h_{11} & 0 & 0 \\
0 & 0 & h_{22} & 0 \\
0 & 0 & 0 & h_{33}
\end{pmatrix}$$  \hspace{1cm} (22)

From principle of equivalence we have the freedom to choose “locally inertial coordinates system” [1, 7] at every space-time point in an arbitrary gravitational field. Owing to this in weak gravitational field we can think for point to point an equivalent Lorentz transformation.

Generalizing this concept and implying the fact of metric transformations as in equation (9) we can build suitable form of the linearized metric in a curved space-time. This can be done in a little tricky way, where initial metric that is the Minkowski metric is placed as the multiplicative factor of metric deformation. Such that the local coordinates system faced an equivalent Lorentz transformation with a factor of initial metric

Where the space-time metric in Non-Euclidean curved spacetime for weak gravity theory is nearly Cartesian that of perturbed over flat space-time [5, 7, 8, 9], it implies neglecting higher order term in matrix form,

$$[g_{\mu\nu}] = [1 + [h_{\mu\nu}][\eta_{\mu\nu}]^{-1}][\eta_{\mu\nu}]$$  \hspace{1cm} (23)

Square bracket containing the perturbative term tells about the deformation of unit scale in curved space-time corresponds to the flat space-time or Minkowski space-time, not using Einstein summation convention.
Where identity matrix, \( \mathbf{1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \) and \( [\eta_{\mu\nu}]^{-1} \) is the inverse of matrix, \( [\eta_{\mu\nu}] \)

\[
[h_{\mu\nu}] [\eta_{\mu\nu}]^{-1} = \begin{pmatrix}
  h_{00} & 0 & 0 & 0 \\
  0 & h_{11} & 0 & 0 \\
  0 & 0 & h_{22} & 0 \\
  0 & 0 & 0 & h_{33}
\end{pmatrix}
\begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
-h_{00} & 0 & 0 & 0 \\
0 & h_{11} & 0 & 0 \\
0 & 0 & h_{22} & 0 \\
0 & 0 & 0 & h_{33}
\end{pmatrix} \tag{24}
\]

Implying equation (21) for two body system where the fabric of background spacetime for the mass \( M^\prime \) (source 1) is perturbed by the weak field self-gravity of the mass \( m^\prime \) (source 2) and considering the initial metric get changes as the same fashion equivalently as the successive two Lorentz transformations described in equation (18) in local coordinates system. Since building of equations (13) and (23) are equivalent so far introduction of the multiplicative factors of initial metric.

\[
[g_{\mu\nu}] = [\mathbf{1} + [h_{\mu\nu}^{(1)}] [\eta_{\mu\nu}]^{-1}] [\mathbf{1} + [h_{\mu\nu}^{(2)}] [\eta_{\mu\nu}]^{-1}] [\eta_{\mu\nu}] \tag{25}
\]

The new perturbed metric for two different massive sources is thus multiplication of one isolated background metric with the others metric deformation. The equation (25) is obvious in regard that offing any one of the sources from the system reproduces the corresponding metric for the existing single source in the linearized perturbation theory.

For \( n \) sources using the law of induction transformed metric is

\[
[g_{\mu\nu}] = \prod_{i=1}^{n} [\mathbf{1} + [h_{\mu\nu}^{(i)}] [\eta_{\mu\nu}]^{-1}] [\eta_{\mu\nu}] \tag{26}
\]

4. The perturbation of metric and modified gravity:

The geodesic motion from least action principle [8, 10] in a curved space-time is

\[
\frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau} = 0 \tag{27}
\]

\( d\tau \rightarrow \) proper time, \( (x^{\mu}; \ x^0 = ct, x^1, x^2, x^3) \)

The affine connection,

\[
\Gamma^\mu_{\rho\sigma} = \frac{1}{2} g^{\mu\nu} \left( \frac{\partial g_{\nu\sigma}}{\partial x^\rho} + \frac{\partial g_{\nu\rho}}{\partial x^\sigma} - \frac{\partial g_{\rho\sigma}}{\partial x^\nu} \right) \tag{28}
\]

For weak field approximation in nearly Cartesian coordinates system [11]

\[
\frac{d^2x_1}{dt^2} = \frac{1}{2} c^2 \nabla g_{00} \tag{29}
\]
\[
\frac{d^2x^i}{dt^2} = \frac{1}{2}c^2 \nabla h_{00} \quad (30)
\]
Equation of motion using Newtonian mechanics for freely falling body,

\[
\frac{d^2x^i}{dt^2} = -\nabla \phi \quad (31)
\]

Gravitational potential, \( \phi = -\frac{GM}{r} \)

Gravitational constant, \( G = 6.673 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ sec}^{-2} \)

\[
\nabla \phi = -\frac{1}{2}c^2 \nabla h_{00} \quad (32)
\]

In context of the Einstein field equations [5, 7, 11]

\[
R_{\mu\nu} - g_{\mu\nu} R = -8\pi G T_{\mu\nu} \quad (33)
\]

The static isotropic metric solution of the equations is given by Schwarzschild [5, 7, 9, 11]

\[
d s^2 = -\left[1 + \frac{2\phi}{c^2}\right] c^2 dt^2 + \left[1 + \frac{2\phi}{c^2}\right]^{-1} dr^2 + r^2 d\theta + r^2 \sin^2 \theta d\phi^2 \quad (34)
\]

The perturbed component from equations (32) and (34)

\[
h_{00} = -\frac{2\phi}{c^2} \quad (35)
\]

For a suitable example let us imagine two different masses \( M \) and \( m \), at positions \( r_1 \) and \( r_2 \), from the observation point then from equation (25) the perturbed metric has the component

\[
g_{00} = -1 + \left(h_{00}^{(1)} + h_{00}^{(2)}\right) - h_{00}^{(1)} h_{00}^{(2)} \quad (36)
\]

\[
\Phi = \left(\phi_1 + \phi_2 + \frac{2\phi_1 \phi_2}{c^2}\right) \quad (37)
\]

Where potential \( \phi_1 = -\frac{GM}{r_1} \) and \( \phi_2 = -\frac{Gm}{r_2} \)

\[
\Phi = -\left(\frac{GM}{r_1} + \frac{Gm}{r_2}\right) + \frac{2G^2Mm}{c^2r_1r_2} \quad (38)
\]

The equation express an interesting matter that in the light of linearized perturbation of metric tensor in general relativity the synergy between two massive sources, the binary system shows anomaly in regard of Newtonian gravity enhanced by an additional potential term. In single massive source equation (38) results no extra potential term and the potential is in agreement of Newtonian gravity.
5. Conclusion:

In this paper we studied how the metric in Minkowskian space-time gets transformed from one coordinates system to another after successive Lorentz transformation. And likewise this idea is generalized to achieve metric transformation from one curved space-time to another. The result shows significant shift from Newtonian gravity for two or more massive bodies. As an example for highly massive terrestrial bodies in a binary system the dynamics will be totally different. The Newtonian gravity now is not hold good and is replaced by its modified form as in equation (38), where a clear additional term for two body gravitational potential has been showed. This modification of gravity can be alternative to MOND [15] and may be applied to study the anomalous behaviour of galactic dynamics [16, 17 and 18] regarding massive terrestrial bodies.

Data Availability Statements:

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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