Fokker–Planck equation and path integral representation of the fractional Ornstein–Uhlenbeck process with two indices

Chai Hok Eab and S C Lim

1 Department of Chemistry Faculty of Science, Chulalongkorn University Bangkok 10330, Thailand
2 Faculty of Engineering, Multimedia University 63100 Cyberjaya, Selangor Darul Ehsan, Malaysia

E-mail: Chaihok.E@chula.ac.th and sclim47@gmail.com

Received 12 May 2014, revised 17 September 2014
Accepted for publication 30 September 2014
Published 17 November 2014

Abstract
This paper considers the Fokker–Planck equation and path integral formulation of the fractional Ornstein–Uhlenbeck process parametrized by two indices. The effective Fokker–Planck equation of this process is derived from the associated fractional Langevin equation. The path integral representation of the process is constructed, and the basic quantities are evaluated.

Keywords: Fokker–Planck equation, fractional Ornstein–Uhlenbeck process, path integral
PACS numbers: 02.50.Ey, 05.10.Gg, 05.40.-a

1. Introduction
The two Gaussian Markov processes, Brownian motion and the Ornstein–Uhlenbeck process, have been used extensively in various applications from natural sciences to financial mathematics. However, many systems in the real world are more complex, and they are non-Markovian in character with memory. Therefore it is necessary to go beyond these simple Markovian models based on Brownian motion and the Ornstein–Uhlenbeck process. For example, Brownian motion as the model for the normal diffusion process can be generalized to fractional Brownian motion [1–3] in order to describe anomalous diffusion [4]. Similarly, models in financial time series and meteorology based on the ordinary Ornstein–Uhlenbeck process has to use its Gaussian fractional generalization, the fractional Ornstein–Uhlenbeck process [5–8].

3 Author to whom any correspondence should be addressed.
In many applications of Brownian motion and the Ornstein–Uhlenbeck process, the path integral method has played an important role [9–11]. One would expect path the integral technique to have a similar role for fractional Brownian motion and the fractional Ornstein–Uhlenbeck process. Lately, there has been considerable interest in applications of the path integral method in quantum mechanics and quantum field theory in fractal and multifractal spacetime. Applications of the path integral in fractional quantum mechanics was studied by Laskin [12] and several authors [13–15]. Path integral formulation has also been used in fractional quantum field theory [16, 17]. The connection between fractional stochastic calculus and constructive field theory has also been considered [18, 19]. Another motivation for studying fractional and multifractional path integrals is that several candidate theories of quantum gravity share the idea that spacetime is multifractal, with integer dimension 4 at large scales, while it is two-dimensional in the ultraviolet limit or small scales [20–22]. This leads to the necessity to consider quantum theory in multifractal spacetime [23–25].

Due to the presence of fractional integro-differential operators, one would expect the evaluation of the path integrals of the fractional stochastic processes to be more complicated. Sebastian [26] was first to study the path integral representation of fractional Brownian motion. Subsequently, the path integral of fractional Brownian motion has been applied to model polymers [27, 28]. Lately, there has been renewed interest in path integral formulation of fractional Brownian motion and the fractional Levy process by several authors [29–32]. The path integral for the fractional Ornstein–Uhlenbeck process or fractional oscillator process with single index has been considered in [33]. Another path integral approach has been considered by Friedrich and Eule, who used a discrete time path integral representation for the continuous time random walk [34, 35]. The Eule–Friedrich approach is quite different from the one used in our work, which follows that of Sebastian [26]. A brief summary of the path integral representations of various fractional processes is given in the book [11].

In the application to systems which have variable memory, it is necessary to use a fractional process parametrized by an index which varies in time or space, for example, the multifractional Brownian motion [36, 37] and the multifractional Ornstein–Uhlenbeck process [38]. However, such multifractional processes are more complex and less adapted to practical purposes. It may be good to have a process that can provide a more flexible model and yet is mathematically more tractable. In the case of the Ornstein–Uhlenbeck process, instead of the multifractional Ornstein–Uhlenbeck process, one can use a fractional Ornstein–Uhlenbeck process indexed by two parameters. The additional index for the fractional Ornstein–Uhlenbeck process allows more flexibility in applications. For the fractional Ornstein–Uhlenbeck process with a single index, both its long-time and short-time properties are characterized by a single parameter. The main advantage of the fractional Ornstein–Uhlenbeck process with two indices over that of a single index is that both its long-time and short-time behaviour can have separate characterization by two different parameters. Possible applications of the fractional Ornstein–Uhlenbeck process with two indices include the modeling of the Von Karman wind speed spectrum [39] and the study of Casimir energy for a fractional quantum field [40, 41].

In this paper we first consider briefly the properties of the fractional Ornstein–Uhlenbeck process with two indices, which will be followed by the derivation of the Fokker–Planck equation for the process. A subsequent section contains the path integral formulation for the fractional Ornstein–Uhlenbeck process with two indices.
2. Fractional Ornstein–Uhlenbeck process with two indices

First we recall that the ordinary Ornstein–Uhlenbeck process can be obtained as the solution of the usual Langevin equation

\[ D_x x(t) + \lambda x(t) = \xi(t), \]

where \( \xi(t) \) is standard white noise with its mean zero and covariance given by \( \langle \xi(t) \xi(s) \rangle = \delta(t - s) \). The solution of (1) is

\[ x(t) = x_0 e^{-\lambda t} + \int_0^t G(t - u) \xi(u) du, \]

where \( x_0 = x(0) \) and \( G(t) = e^{-\lambda t} \). One gets

\[ \langle x(t) \rangle = x_0 e^{-\lambda t}, \]

\[ \langle x(t) x(s) \rangle = \frac{1}{2\lambda} \left( e^{-\lambda |t - s|} - e^{-\lambda |t + s|} \right). \]

In the long-time limit, the Ornstein–Uhlenbeck process becomes a stationary process with covariance

\[ \langle x(t) x(s) \rangle = \frac{e^{-\lambda |t - s|}}{2\lambda}. \]

In other words, the Ornstein–Uhlenbeck process is a stationary process if the starting time is \( t = -\infty \), and a non-stationary process if it begins at a finite time. Such a dependence on the initial time of the time integro-differential operators is reflected in the distinction between the Riemann–Liouville (and also Caputo) fractional derivative and the Weyl fractional derivative [42].

The Fokker–Planck equation corresponds to the Ornstein–Uhlenbeck process is given by

\[ \frac{\partial}{\partial t} P(x, t | x_0, t_0) = \frac{\partial}{\partial x} \left[ \lambda x P(x, t | x_0, t_0) \right] + \frac{D}{2} \frac{\partial^2}{\partial x^2} P(x, t | x_0, t_0), \]

which corresponds to the Lagrangian

\[ L(x, \dot{x}) = \frac{1}{2} (\dot{x} + \lambda x)^2, \]

where \( \dot{x} \) denotes an ordinary time derivative of \( x \). The Euler–Langrange equation is

\[ \ddot{x} - \lambda^2 x = 0. \]

The path integral representation of the Ornstein–Uhlenbeck process, which is a Markov process, can be obtained quite easily just like the case for Brownian motion [43]. By using the Kolmogorov equation, the infinitesimal propagator can be chained into a path integration. One gets

\[ G_0(x_N, t_N | x_0, t_0) = \int \cdots \int \left\{ \prod_{k=1}^{N-1} G_0(x_{k+1}, t_{k+1} | x_k, t_k) dx_k \right\} G_0(x_1, t_1 | x_0, t_0), \]

where

\[ G_0(x, t | x_0, t_0) = \frac{1}{\sqrt{2\pi D (1 - e^{2\lambda (t-t_0)})}} \exp \left\{ - \frac{(x - x_0 e^{-\lambda (t-t_0)})^2}{2\lambda D (1 - e^{2\lambda (t-t_0)})} \right\}. \]

By taking limit \( N \to \infty \) and \( |t_{k+1} - t_k| \to 0 \) while keeping \( \sum_{k=0}^{N-1} |t_{k+1} - t_k| < \infty \) in (9), one gets the path integration of the Ornstein–Uhlenbeck process [43].
There are several ways to generalize the Ornstein–Uhlenbeck process to its fractional counterpart. One way is to replace the white noise by a fractional Gaussian noise \([5, 8]\), or one can apply the Lamperti transformation to fractional Brownian motion \([6, 7]\). Note that if the white noise in the Langevin equation is replaced by the fractional Levy motion, its solution is the fractional Ornstein–Uhlenbeck process of \(\alpha\)-stable-type \([44, 45]\). In this paper we shall not consider all these variants of fractional Ornstein–Uhlenbeck processes. Instead, we shall restrict ourselves to the fractional Ornstein–Uhlenbeck process obtained from the fractional Langevin equation by keeping the white noise and replace the time-differential operator \(D_t\) by a fractional one \(D_t^\alpha\) in the Langevin equation (1). There exist the following three possible fractional generalizations of the Langevin equation:

\[
\begin{align*}
D_t^\alpha x(t) + \lambda^\alpha x(t) &= \xi(t), \quad (11a) \\
(D_t + \lambda)^\alpha x(t) &= \xi(t), \quad (11b) \\
(D_t^\beta + \lambda^\beta)^\gamma x(t) &= \xi(t), \quad (11c)
\end{align*}
\]

where \(0 < \alpha \leq 1\) and \(0 < \beta \leq 1\). Note that one can formally define the ‘shifted’ fractional derivative \((D_t^\alpha + \lambda^\alpha)^\gamma\) in terms of the unshifted derivative \(D_t^\alpha\). By using binomial expansion, it is possible to express the shifted fractional derivative in terms of unshifted ones:

\[
(D_t^\alpha + \lambda^\alpha)^\gamma f(t) = \sum_{j=0}^{\infty} \binom{\gamma}{j} \lambda^j D_t^j f(t),
\]

with the fractional derivative of the Caputo and Riemann–Liouville type defined as follows. The fractional derivative of order \(\alpha\) denoted by \(D_t^\alpha\) can be defined in terms of its inverse operator or the fractional integral \([46]\):

\[
\begin{align*}
\alpha D_t^{-\alpha} f(t) &= D_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} f(u)\,du. \quad (13)
\end{align*}
\]

When \(a = -\infty\), the fractional derivative is known as a Weyl fractional derivative; when \(a = 0\) (for simplicity we write \(\alpha D_t^\alpha \equiv D_t^\alpha\)), one gets the Riemann–Liouville and Caputo fractional derivative, according to the following definitions. A fractional derivative of arbitrary order \(\alpha\), with \(n - 1 < \alpha < n\), can be defined through fractional integration of order \(n - \alpha\) and successive ordinary derivative of order \(n\):

\[
\begin{align*}
D_t^\alpha f(t) &= \left(\frac{d}{dt}\right)^n D_t^{\alpha-n} f(t), \quad \text{Riemann – Liouville} \quad (14a) \\
C D_t^\alpha f(t) &= D_t^{\alpha-n} \left(\frac{d}{dt}\right)^n f(t), \quad \text{Caputo.} \quad (14b)
\end{align*}
\]

The Lagrangian associated with the Ornstein–Uhlenbeck process is given by

\[
L = x(t)\Lambda_{\text{af}}(D_t)x(t), \quad (15a)
\]

with

\[
\Lambda_{\text{af}}(D_t) = \left((D_t^\alpha)^\gamma + \lambda^\alpha\right)^\gamma (D_t^\alpha + \lambda^\alpha)^\gamma. \quad (15b)
\]

Here we remark that a more rigorous treatment of fractional operators such as \((D_t^\alpha + \lambda^\alpha)^\gamma\) and \(\Lambda_{\text{af}}(D_t)\) can be obtained by using hypersingular integrals \([47]\). Path integral
representation of the fractional Ornstein–Uhlenbeck process with a single index given by (11b) has been considered in [33]. In this paper we intend to study the general case (11c) with \(\alpha \neq 1\) and \(\gamma \neq 1\). We shall restrict our work to the case of the Riemann–Liouville type of fractional oscillator, as it is usually done for path integral formulation of fractional processes.

### 2.1. Basic properties of fractional Ornstein–Uhlenbeck process

Since \(x_{\alpha\gamma}(t)\) is a Gaussian process, it can be characterized by its mean and covariance. For convenience, the process is assumed to be centred with zero mean. The covariance of the fractional Ornstein–Uhlenbeck process with two indices based on Riemann–Liouville fractional derivatives is given by [39, 40]:

\[
C_{\alpha\gamma}(t, s) = \left\langle x_{\alpha\gamma}(t)x_{\alpha\gamma}(s) \right\rangle = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \binom{\gamma + m - 1}{m} \binom{\gamma + n - 1}{n} (\lambda^\alpha)^{m+n} \\
\times \frac{\Gamma(1 + \alpha(m + \gamma + n))}{\Gamma(1 + \alpha(\gamma + n))} \\
\times {}_2F_1 \left( 1 - \alpha(\gamma + n), 1, 1 + \alpha(\gamma + m); \frac{s - t}{\tau} \right).
\] (16)

The variance of \(x_{\alpha\gamma}\) is

\[
\sigma_{\alpha\gamma}^2(t) = \left\langle \left( x_{\alpha\gamma}(t) \right)^2 \right\rangle = \sum_{q=0}^{\infty} \left( -\lambda^\alpha \right)^q \frac{\Gamma(2\gamma + q - 1)}{\Gamma(2\gamma + q)} A_q,
\] (17)

where

\[
A_q = \sum_{m+n=q} \binom{\gamma + m - 1}{m} \binom{\gamma + n - 1}{n} \frac{1}{\Gamma(\alpha(m + \gamma + n))}.
\] (18)

Before we proceed to derive the path integral representation of \(x_{\alpha\gamma}(t)\), we consider briefly some basic properties of this process [39, 40]. First we note that just like all the Riemann–Liouville processes (such as the case of fractional Brownian motion [48] and the fractional Ornstein–Uhlenbeck process with single index [42]), \(x_{\alpha\gamma}(t)\) is a non-stationary process, and it does not have stationary increments.

Next, we consider the link between the fractional Ornstein–Uhlenbeck process and fractional Brownian motion. Let the increment process \(x_{\alpha\gamma}(t + \tau) - x_{\alpha\gamma}(t)\) be denoted by \(\Delta x_{\alpha\gamma}(t, \tau)\). By direct computation one shows that as \(\tau \to 0\), the covariance of \(\Delta x_{\alpha\gamma}(t, \tau)\) approaches that of fractional Brownian motion of Hurst index \(H = \alpha\gamma - 1/2\). That is, as \(\tau_1 \to 0\) and \(\tau_2 \to 0\),

\[
\left\langle \Delta x_{\alpha\gamma}(t, \tau_1) \Delta x_{\alpha\gamma}(t, \tau_2) \right\rangle \sim |\tau_1|^{2\alpha\gamma - 1} + |\tau_2|^{2\alpha\gamma - 1} - |\tau_1 - \tau_2|^{2\alpha\gamma - 1}.
\] (19)

Recall that a stochastic process \(w(t)\) is locally asymptotically self-similar, with index \(\kappa\) at the point \(t_0\) if there exists a non-degenerate process \(T_{t_0}^\kappa(u)\) such that
where \( \varepsilon \) denotes equality of finite dimensional distributions. Again, by direct computation of the covariance of the increment process on the l.h.s. of (20), one can verify that \( x_{\alpha \gamma}(t) \) is locally asymptotically self-similar and its tangent process \( T_{\kappa}(u) \) is a fractional Brownian motion with Hurst index \( H = \kappa = \alpha \gamma - 1/2 \). It can also be verified that the fractal dimension of \( x_{\alpha \gamma}(t) \) is \( 5/2 - \alpha \gamma \), since it locally behaves like a fractional Brownian motion which has a fractal dimension of \( 2 - H \).

Both Brownian motion and the Ornstein–Uhlenbeck process are Markov processes. When Brownian motion is generalized to a fractional Brownian motion, it loses the Markovian property and becomes a long memory (or long-range dependent) process. On the other hand, the particular type of fractional extension of the Ornstein–Uhlenbeck process considered here is a non-Markov process with short memory (short-range dependence) [42]. There exists another type of fractional Ornstein–Uhlenbeck process which has a long–range dependent property [5, 7, 8]. See reference [7] for other definitions of the fractional Ornstein–Uhlenbeck processes and their properties. Here we would like to add that if the Langevin equation is driven by fractional levy motion, its solution will give a fractional Ornstein–Uhlenbeck process of \( \alpha \)-stable-type, which is non-Gaussian, and its memory structure requires somewhat different characterization [49, 50].

Another interesting property is the ergodic property, which has recently attracted considerable attention. For examples, the validity or violation of ergodic property has been considered for various stochastic processes such as continuous time random walk [44], fractional Brownian motion [45], and the fractional Ornstein–Uhlenbeck process of \( \alpha \)-stable-type [50]. Ergodicity is a fundamental property of a dynamical system, and its underlying idea is that for a system, the ensemble average of its properties equals the time average.

Here, we shall verify the ergodicity of the fractional Ornstein–Uhlenbeck process with two indices using the Khinchin theorem on the condition of its auto-correlation function [51, 52]. The autocorrelation function of a centred Gaussian process \( y(t) \) is given by

\[
r(s, t) = \frac{\langle y(s)y(t) \rangle}{\left\langle (y(s)^2) \right\rangle^{1/2}}. \tag{21}
\]

According to the Khinchin theorem, if the autocorrelation function of a stationary Gaussian \( y(t) \) satisfies \( \lim_{t \to \infty} r(t) = 0 \), then \( y(t) \) is ergodic. The fractional Ornstein–Uhlenbeck process of Riemann–Liouville type \( x_{\alpha \gamma}(t) \) is non-stationary. However, it has been shown that if \( x_{\alpha \gamma}(t) \) is allowed to evolve for a sufficiently long time, it will achieve stationarity and become a fractional Ornstein–Uhlenbeck process of the Weyl type, which is stationary [39]. Since the condition for ergodicity on autocorrelation is in the large-time limit, we can verify the ergodic property for the Weyl-type fractional Ornstein–Uhlenbeck process to infer that same property also holds for \( x_{\alpha \gamma}(t) \). It has been shown in [39] that the large time behaviour of the covariance \( C(t, t + \tau) \) of \( x_{\alpha \gamma}(t) \) varies as \( \tau^{-\alpha-1} \), which tends to 0 as \( \tau \to \infty \). With this property, together with the finite variance of the process, one verifies that the autocorrelation of \( x_{\alpha \gamma}(t) \) satisfies the condition \( \lim_{t \to \infty} r(t) = 0 \), and the process is ergodic [51, 52]. Thus we see, just as in the case of fractional Brownian motion, that the fractional Ornstein–Uhlenbeck process with two indices also satisfies the ergodic property.
2.2. Fokker–Planck equation of the fractional Ornstein–Uhlenbeck process

In this section we would like to derive the Fokker–Planck equation for the fractional Ornstein–Uhlenbeck process \( x_\alpha(t) \) from the fractional Langevin equation (11c) or any equations derived from (11c). By adapting an argument similar to that given in reference [53] (see page 268, chapter 5, S5.C.3.), which assumes a strong frictional limit with a large frictional coefficient such that the particle relaxes to its stationary state very rapidly. Consequently, one can assume velocity \( v \) does not vary with time such that \( \dot{v} \approx 0 \) and one can then write the corresponding Langevin equation as

\[
K_{\alpha} x = F(x) + \xi_t,
\]

where \( K_{\alpha} \) is a differential-integral operator. For a Brownian particle, the fractional Brownian motion and the fractional Ornstein–Uhlenbeck process with two indices \( K_t \) is respectively given by (23a)–(23c) if \( \xi \) is white noise:

\[
K_t = \begin{cases} D_t, & (23a) \\ D_t^H + 1/2, & (23b) \\ (D_t^H + \lambda_\alpha^H)^{\gamma_t}, & (23c) \end{cases}
\]

Equation (22) can also be written as

\[
D_t x = D_t L F(x) + D_t L \xi_t,
\]

where \( L \) is the left inverse operator of \( K_t \) defined by

\[
L K_t = I + \pi_0,
\]

and \( \pi_0 \) is the projection operator given by \( \pi_0 f(t) = f(0) \). For the three cases in (23), one has

\[
L = \begin{cases} I, & (26a) \\ I^H + 1/2, & (26b) \\ (1 + \lambda_\alpha^H)^{\gamma_t} I_{\alpha_0}, & (26c) \end{cases}
\]

Note that \( D_t \pi_0 = 0 \), since the projection operator to the initial point of any function of time \( t \) is a constant.

Recall that the probability of finding the particle in the interval \( x \) and \( x + dx \) is given by \( P(x, t)dx \), where the probability density is defined by the average \( P(x, t) = \langle \rho(x, t) \rangle_\xi \), and the corresponding ‘equation of motion’ is

\[
\frac{\partial}{\partial t} \rho(x, t) = -\partial_x \left[ D_t L F(x) + D_t L \xi_t \right] \rho(x, t) = -\left[ L_0 + L' \right] \rho(x, t),
\]

where

\[
L_0 = D_t L \xi_t \partial_x, \quad L' = \partial_x D_t L F(x),
\]

Note that the operator \( L' \) contains random a variable at time \( \tau < t \), which means it has to be averaged over random variable \( x(\tau) \).

The method used by Reichl [53] is not applicable to the fractional Langevin equation (22). Instead, we consider the following alternative Langevin equation

\[
J. Phys. A: Math. Theor. 47 (2014) 495203 C H Eab and S C Lim
\]
\[ K_{x,t} = -K_d F(x) + \xi, \]  

or

\[ D_{x,t} = -F(x) + \zeta. \]

Equation (28b) is a Langevin equation with Gaussian fractional noise \( \zeta = D_\tau L \xi \). Note that the force \( F(x) \) is local, and this is crucial for the use of Reichl’s procedure [53] in deriving the Fokker–Planck equation. Equation (27) still holds if \( L' = \partial_x F(x) \) is replaced by \( L' = \partial_x F(x) \), which has no explicit time dependence. Now we define

\[ U(x, t) = e^{-dL}. \]  

and introduce a new probability density \( \theta(x, t) \) by

\[ \rho(x, t) = U(x, t) \theta(x, t). \]

From (27) and (30), it can be shown that

\[ \partial_x \theta(x, t) = -\Theta_x(t) \theta(x, t), \]

where

\[ \Theta_x(t) = e^{i\partial_x F(x)} [D_L \xi(t) \partial_x] e^{-i\partial_x F(x)}. \]

The solution of (31) is given by

\[ \theta(x, t) = e^{-\int_0^t d\theta_x(c)} \theta(x, 0). \]

By taking the average of \( \theta(x, t) \) over the white noise \( \xi \) gives

\[ \langle \theta(x, t) \rangle = \left\{ e^{-\int_0^t d\theta_x(c)} \right\} \theta(x, 0) \]

\[ = e^{i t} \left\{ \int_0^t du \int_0^t dv \Theta_x(u) \Theta_y(v) \right\} \theta(x, 0). \]

The second cumulant of \( \Theta_x(t) \) is given by

\[ \int_0^t du \int_0^t dv \Theta_x(u) \Theta_y(v) \]

\[ = \int_0^t du \int_0^t dv U^{-1}(u) \partial_x U(u)(\zeta(u) \zeta(v))_x U^{-1}(v) \partial_x U(v) \]

\[ = \int_0^t du \int_0^t dv U^{-1}(u) \partial_x U(u) C_\zeta(u, v) U^{-1}(v) \partial_x U(v), \]

where \( C_\zeta(u, v) \) is the covariance of the noise \( \zeta \).

Now the average of \( \theta(x, t) \) in (34) can be written as

\[ \langle \theta(x, t) \rangle = e^{it} \int_0^t du \int_0^t dv \Theta_x(u) \Theta_y(v) \theta(x, 0). \]

Differentiating (36) with respect to time gives

\[ \partial_t \langle \theta(x, t) \rangle = \left\{ \int_0^t dt U^{-1}(t) \partial_x U(t) C_\zeta(t, \tau) U^{-1}(\tau) \partial_x U(\tau) \right\} \]

\[ e^{it} \int_0^t du \int_0^t dv U^{-1}(u) \partial_x U(u) C_\zeta(u, v) U^{-1}(v) \partial_x U(v) \theta(x, 0) \]

\[ = \left\{ \int_0^t dt U^{-1}(t) \partial_x U(t) C_\zeta(t, \tau) U^{-1}(\tau) \partial_x U(\tau) \right\} \langle \theta(x, t) \rangle. \]
By taking the average of (30) with respect to white noise \( \xi \), one obtains
\[
P(x, t) = \langle \rho(x, t) \rangle = U(t) \langle \delta(x, t) \rangle. \tag{38}
\]
Differentiating (38) with respect to time to get
\[
\partial_t P(x, t) = -\partial_x F(x) P(x, t) + U(t) \left\{ \int_0^t \mathrm{d}r U^{-1}(t) \partial_x U(t) C_{\xi}(t, \tau) U^{-1}(\tau) \partial_x U(\tau) \right\} U^{-1}(t) P(x, t). \tag{39}
\]
Let us define operators \( S_+(t) \) and \( Y^x_+(t) \), which are operators in \( x \) and functions of \( t \):
\[
Y_+^x(t) = U^{-1}(t) \partial_x U(t), \tag{40a}
\]
\[
Y_-^x(t) = U(t) \partial_x U^{-1}(t), \tag{40b}
\]
\[
S_+(t) = U(t) \left[ \int_0^t \mathrm{d}r Y_+^x(t) C_{\xi}(t, \tau) Y_+^x(\tau) \right] U^{-1}(t). \tag{40c}
\]
Using these operators, one can write the Fokker–Planck equation corresponding to the Langevin equation (28) as
\[
\partial_t P(x, t) = -\partial_x F(x) P(x, t) + S_+(t) P(x, t). \tag{41}
\]
For the special case with \( F(x) = 0 \), the Langevin equation becomes
\[
Kx(t) = \xi(t), \tag{42}
\]
or
\[
D_x x(t) = D_x L_x \xi(t) = \xi(t). \tag{43}
\]
Since \( L' = \partial_t F(x) = 0 \) we have \( U(t) = 1 \); thus in this case one gets \( Y_+^x(t) = \partial_x \) from (40a), and
\[
S'_+(t) = \int_0^t \mathrm{d}r \partial_x C_{\xi}(t, \tau) \partial_x = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \mathrm{d}u \int_0^u \mathrm{d}v \partial_x C_{\xi}(u, v) \partial_x
\]
\[
= \frac{1}{2} \frac{d}{dt} \{ C(t, t) - C(t, 0) - C(0, t) + C(0, 0) \} \partial_x^2
\]
\[
= \frac{1}{2} \frac{d}{dt} \sigma^2(t). \tag{44}
\]
Thus the Fokker–Planck equation for the force-free case is given by
\[
\partial_t P(x, t) = \frac{1}{2} \frac{d}{dt} \sigma^2(t) \partial_x^2 P(x, t). \tag{45}
\]
For Brownian motion with variance \( \sigma^2(t) = t \), (45) reduces to the ordinary diffusion equation. In the case of a fractional Brownian motion with variance \( t^{2H} \left[ 2\Gamma(H - 1/2) \right]^{-1} \), the corresponding Fokker–Planck equation is
\[
\partial_t P(x, t) = \frac{1}{2} \frac{t^{2H-1}}{\Gamma(H - 1/2)^2} \partial_x^2 P(x, t), \tag{46}
\]
which is in agreement with the result obtained by other authors [54–56].
It needs to be pointed out that the effective Fokker–Planck equation does not fully characterize non-Markovian processes such as fractional Brownian motion and the fractional Ornstein–Uhlenbeck process. For example, both the standard and Riemann–Liouville type fractional Brownian motion and the scaled Brownian motion with appropriate scaling factor
all have the same variance (up to a multiplicative constant), so they all have the same effective Fokker–Planck equation. Reference [57] provides a more detailed discussion on this point. Now consider the case of the fractional Ornstein–Uhlenbeck process with two indices. From its variance given by (17), one obtains the effective Fokker–Planck equation as

\[
\partial_t P(x, t) = \frac{1}{2} \left[ \sum_{q=0}^{\infty} (-\lambda^q)^q A_q t^{(2\gamma + q) - 2} \right] \partial_x^2 P(x, t), \tag{47}
\]

where

\[
\sum_{q=0}^{\infty} (-\lambda^q)^q A_q t^{(2\gamma + q) - 2} = \frac{t^{2\gamma - 2}}{\Gamma^2(\alpha\gamma)} - \lambda^\alpha \frac{2t^{2\gamma(\alpha + 1) - 2}}{\Gamma(\alpha\gamma) \Gamma(\alpha(\gamma + 1))} + \lambda^{2\alpha} \left[ \frac{\gamma(\gamma + 1)}{\Gamma(\alpha(\gamma + 2))} + \frac{\gamma^2}{\Gamma(\alpha(\gamma + 1)) \Gamma(\alpha(\gamma + 1))} \right] t^{(2\gamma + 2) - 2} \ldots. \tag{48}
\]

The solution of the Fokker–Planck equation (45) subjected to the initial condition \(P(x, 0) = \delta(x - x_0)\) is given by

\[
P(x, t) = \frac{1}{\sqrt{2\pi\sigma^2(t)}} e^{\frac{-|x - x_0|^2}{2\sigma^2(t)}}. \tag{49}
\]

Note that (48) contains an infinite number of what appear to be scaled Brownian motion. The Fokker–Planck equation with appropriate boundary conditions is widely used in solving the first passage time problem. In theory it is possible to obtain the first passage time distribution of the fractional Ornstein–Uhlenbeck process by using the effective Fokker–Planck equation (48), in particular its asymptotically large time limit. Such a problem is currently under study.

Finally we remark that in the small time limit, (47) reduces to (45) if the Hurst index is taken to be \(\alpha\gamma - 1/2\). This does not come as a surprise, since it has been stated earlier that both fractional Brownian motion and the Ornstein–Uhlenbeck process satisfy the same local property.

3. Path integral representation of fractional Ornstein–Uhlenbeck process

In this section we will obtain the path integral formulation of the fractional Ornstein–Uhlenbeck process with two indices. For convenience, let us denote

\[
K_{\alpha, \gamma}(\lambda) = \left( D_t^\alpha + \lambda^\alpha \right)^\gamma,
\]

or simply \(K\), since \(\alpha, \gamma,\) and \(\lambda\) are fixed throughout this paper.

3.1. Solution of classical path

Let us introduce the action for the fractional Ornstein–Uhlenbeck process as follows:

\[
S[x] = \frac{1}{2} \int_0^\beta \left[ Kx(t) \right]^2 dt, \tag{51}
\]
where
\[ \ddot{x}(t) = x(t) - x(0). \] (52)

The classical solution can be obtained by variational principle,
\[ 0 = \delta S[x] = \int_0^\beta \left\{ \left[ D^{\alpha \tau} + \dot{\lambda}^a \right] \left[ D^a + \dot{\lambda}^a \right] \ddot{x}(t) \right\} \delta \ddot{x}(t) \, dt, \] (53)
where the variation at two end points is zero; that is, \( \delta \ddot{x}(0) = \delta \ddot{x}(\beta) = 0. \)

Denote by \( D^{\alpha \tau} = (D^a)^\dagger \) the adjoint of \( D^a \) defined by
\[ \int_0^\beta f(t) D^a g(t) \, dt = \int_0^\beta g(t)(D^a)^\dagger f(t) \, dt. \] (54)

Let \([\alpha]\) be the lowest integer that is greater than or equal to \( \alpha \). We have for the Riemann–Liouville fractional derivative that
\[ D^a_t = D^{[\alpha]}_t I^{\alpha-a}_t \Rightarrow (D^a)^\dagger = D^{[\alpha]-a}_t I^{1-\alpha}_t \dot{\lambda}^a = \zeta D^a, \] (55)
which gives a left-fractional derivative of Caputo type \( D^a_t \). On the other hand, the adjoint of the right-fractional derivative of Caputo type \( (D^a_t)^\dagger \) is equal to the left-fractional derivative of Riemann–Liouville type \( D^a_t \). Since we consider only the case with zero initial value, \( \ddot{x}(0) = 0 \), both the right-derivatives of the Caputo and Riemann–Liouville type are equivalent. This is, however, not true for the left-derivative.

Using the notation introduced in (50), let us consider the adjoint operator \( K^\dagger \). Now the equation of motion can be written as
\[ K^\dagger K \ddot{x}(t) = 0. \] (56)
Consider the derivative of Riemann–Liouville type \( D^{\alpha}_t \), for \( 0 < \alpha \gamma \leq 1 \), its adjoint \( D^{\alpha \tau} \) is of Caputo-type. The solution of (56) can be obtained by first noting
\[ K^\dagger y(t) = \left[ D^{\alpha \tau} + \dot{\lambda}^a \right] y(t) = \left[ 1 + \dot{\lambda}^a \right] I^\alpha y(t) = 0, \] (57)
which then gives
\[ y(t) = A \sum_{n=0}^{\infty} \frac{(\gamma + n - 1)_{(\beta - 1)}}{\Gamma(\alpha(\gamma + n))} (-\lambda^a)^{\gamma+n} \left( \frac{\beta - 1}{\Gamma(\alpha(\gamma + n))} \right)^{n}, \]
(58)
We can now solve for \( \ddot{x} \) from the equation
\[ K \ddot{x}(t) = y(t), \] (59a)
where \( K \) can be reexpressed as
\[ K = \left[ D^a_t + \dot{\lambda}^a \right] = D^{[\alpha]}_t \left[ 1 + \dot{\lambda}^a \right] = D^{[\alpha]-a}_t \left[ 1 + \dot{\lambda}^a \right]. \] (59b)

It is then straightforward to get the solution by applying (58) to the right-hand side of (59). Since the term in summation involved powers of \( \frac{(\beta - 1)^{n-1}}{\Gamma(\gamma)}, \) it is convenient to consider the following fractional differential equation
\[ D_t^{\alpha} \left[ 1 + \lambda^\alpha T_t^\alpha \right] \bar{x}_\mu(t) = \frac{(\beta - t)^{\mu-1}}{\Gamma(\mu)}, \quad (60) \]

which gives

\[ \bar{x}_\mu(t) = B_\mu + \sum_{m=0}^{\infty} \left( \gamma + m - 1 \right) \left( \gamma + n - 1 \right) (-\lambda^\alpha)^m \frac{t^{\mu(m+n)}}{\Gamma(\mu)} \frac{t^m}{\Gamma(\alpha(\gamma + m) + 1)\Gamma(\gamma + n)} \binom{m+n}{m} \binom{m+n}{n} \frac{1}{\Gamma(\alpha(\gamma + m))\Gamma(\alpha(\gamma + n))}. \quad (61) \]

By using the formula given in [58], page 317, #3.197.3, one gets

\[ I_\mu^{\alpha(\gamma + m)} \frac{(\beta - t)^{\mu-1}}{\Gamma(\mu)} = \frac{t^{\mu(\gamma + m)}\beta^{\mu-1}}{\Gamma(\alpha(\gamma + m) + 1)\Gamma(\gamma + n)} \binom{n}{m} _2F_1 \left( 1 - \alpha(\gamma + n), 1, 1 + \alpha(\gamma + m); \frac{t}{\beta} \right). \quad (62) \]

Insert (62) into (61) and then (59) to obtain the solution, which can be written in the following form:

\[ \bar{x}(t) = B + AU(t), \quad (63) \]

where \( A \) and \( B \) are constants to be determined and

\[ U(t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( \gamma + m - 1 \right) \left( \gamma + n - 1 \right) (-\lambda^\alpha)^m \frac{t^{\mu(m+n)}}{\Gamma(\alpha(\gamma + m) + 1)\Gamma(\gamma + n)} \binom{m+n}{m} \binom{m+n}{n} \frac{1}{\Gamma(\alpha(\gamma + m))\Gamma(\alpha(\gamma + n))}. \quad (64) \]

The values for \( A \) and \( B \) can be obtained by considering (63) for \( t = 0 \) and \( t = \beta \). It is obvious that \( U(0) = 0 \), which gives \( B = 0 \). Considering the value of the hypergeometric function at unity, from ([58], #9.122.1, page 1008), one gets

\[ _2F_1(\alpha, \nu; \gamma; 1) = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \nu)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \nu)}, \quad [\text{Re} \gamma > \text{Re}(\alpha + \nu)]. \quad (65) \]

Thus,

\[ U(\beta) = \sum_{q=0}^{\infty} (-\lambda^\alpha)^q \Omega_q \frac{\beta^{(2q+1)-1}}{\alpha(2q + q) - 1}. \quad (66) \]

where

\[ \Omega_q = \sum_{m+n=q} \left( \gamma + m - 1 \right) \left( \gamma + n - 1 \right) \frac{1}{\Gamma(\alpha(\gamma + m))\Gamma(\alpha(\gamma + n))}. \quad (67) \]

Finally, from (63), \( t = \beta \) gives \( A = \frac{\pi(\beta)}{U(\beta)} \), so the classical path is given by

\[ x_c(t) = x_0 + \frac{x_\beta - x_0}{U(\beta)} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( \gamma + m - 1 \right) \left( \gamma + n - 1 \right) (-\lambda^\alpha)^m \frac{t^{\mu(m+n)}\beta^{\mu(\gamma + n)-1}}{\Gamma(\alpha(\gamma + m) + 1)\Gamma(\alpha(\gamma + n))} \binom{n}{m} _2F_1 \left( 1 - \alpha(\gamma + n), 1, 1 + \alpha(\gamma + m); \frac{t}{\beta} \right). \quad (68) \]
3.2. Propagator

Next, we want to evaluate the propagator, which is given by

$$G(x_\beta, \beta; x_\alpha, 0) = \frac{1}{x_\beta} \int H[x] \delta(x(\beta) - x_\beta) \delta(x(0) - x_\alpha) e^{-S[x]}, \quad (69)$$

where the action is given by (51).

We first consider the transformation

$$x(t) = x_c(t) + q(t), \quad (70)$$

where $x_c(\cdot)$ is the classical solution as obtained in (68), and the variable $q(\cdot)$ is the fluctuation around the classical path, satisfying the conditions $q(0) = q(\beta) = 0$. Substituting (69) into (51) gives the action as

$$S[x] = \frac{1}{2} \int_0^\beta \left[ K\dot{x}_c(t) \right]^2 dt + \int_0^\beta \left[ K\dot{x}_c(t) \right]|Kq(t)| dt + \frac{1}{2} \int_0^\beta |Kq(t)|^2 dt \quad (71)$$

The cross terms disappear, since $K^T K x_c = 0$ (see (56)). Now substitute (72) into (69) to get

$$G(x_\beta, \beta; x_\alpha, 0) = e^{-S[x_c]} G(x_\alpha, \beta; x_\alpha, 0), \quad (73)$$

where $G(x_\alpha, \beta; x_\alpha, 0)$ is the loop part, i.e., the propagator that begins and ends at the same point $x(\beta) = x(0) = x_\alpha$.

Recall that $\chi_c(t) = K\dot{x}_c(t)$, and it is given by (58) with the constant $A$ previously determined. To be more precise, one can write

$$\chi_c(t) = (x_\beta - x_\alpha) W(\beta - t). \quad (74a)$$

$$W(\beta - t) = \frac{1}{U(\beta)} \sum_{n=0}^\infty \left( \gamma + n - 1 \right) (-\lambda)^n \frac{\beta - t)^{\alpha(\gamma + n) - 1}}{\Gamma(\alpha(\gamma + n))}. \quad (74b)$$

Thus, the action becomes

$$S[x_c] = \frac{1}{2} \int_0^\beta W^2(\beta - t). \quad (75)$$

The integral term in the above equation can be computed:

$$\int_0^\beta W^2(\beta - t) = \frac{1}{[U(\beta)]^2} \sum_{m=0}^\infty \sum_{n=0}^\infty \left( \gamma + m - 1 \right) \left( \gamma + n - 1 \right) (-\lambda)^{m+n} \times \frac{1}{\Gamma(\alpha(\gamma + m))\Gamma(\alpha(\gamma + n))} \int_0^\beta dt (\beta - t)^{\alpha(2\gamma + m + n) - 2}. \quad (76)$$

For $\alpha > 1/2$, the summation term exactly equals $U(\beta)$, which gives

$$\int_0^\beta W^2(\beta - t) = \frac{1}{U(\beta)}. \quad (77)$$

Thus, one obtains

$$G(x_\beta, \beta; x_\alpha, 0) = e^{-\frac{1}{2U(\beta)} \int_0^\beta W^2(\beta - t)} G(x_\alpha, \beta; x_\alpha, 0). \quad (78)$$
and
\[ 1 = \int_{-\infty}^{\infty} dx_0 e^{-\frac{1}{2U(\beta)} |x_0|^2} G(x_0, \beta; x_0, 0) = \sqrt{2\pi U(\beta)} G(x_0, \beta; x_0, 0). \] (79)

Finally, the propagator is obtained as
\[ G(x_\beta, \beta; x_0, 0) = \frac{1}{\sqrt{2\pi U(\beta)}} e^{-\frac{1}{2U(\beta)} |x_\beta|^2}. \] (80)

Here we note that the variance
\[ \mathbb{E}[(x(\beta) - x(0))^2] = \int_{-\infty}^{\infty} dx_\beta |x_\beta|^2 G(x_\beta, \beta; x_0, 0) = U(\beta). \] (81)

### 3.3. Partition function

The partition function, which is defined as the trace of the propagator, is given by
\[ Z(\beta) = \int_V dx G(x, \beta; x, 0) = \frac{V}{\sqrt{2\pi U(\beta)}}, \] (82)

where \( V \) is the volume enclosing the particle.

### 3.4. Generating function

The generating function is defined as a logarithm of the partition function with source term \( \langle h, \bar{h} \rangle \):
\[ Z(\beta, h) = Z(\beta) e^{\frac{1}{2} \langle h, \bar{R} R h \rangle}, \] (83)

where \( R \) is the right inverse of \( K \); i.e., \( KR = I \), an identity.

Now consider the free energy
\[ F(\beta, h) = -\frac{1}{\beta} \log Z(\beta, h) = F(\beta, 0) - \frac{1}{2\beta} \langle h, \bar{R} R h \rangle. \] (84)

We then have
\[ \frac{\delta F(\beta, h)}{\delta h(t)} = -\frac{1}{\beta Z(\beta, h)} \frac{\delta Z(\beta, h)}{\delta h(t)} = \frac{1}{\beta} \mathcal{E}_h[\bar{x}(t)]. \] (85)

The mean of \( \bar{x} \) is given by
\[ M(t) = \mathcal{E}_h[\bar{x}(t)] = \beta \lim_{h \to 0} \frac{\delta F(\beta, h)}{\delta h(t)} = \lim_{h \to 0} \bar{R} R h(t) = 0. \] (86)

Here, \( \mathcal{E}_h(\cdot) \) and \( \mathcal{E}_0(\cdot) \) are the expectations, with source \( h \) and zero, respectively. Similarly, the covariance can be evaluated as
\[ C(s, t) = \beta \lim_{h \to 0} \frac{\delta^2}{\delta h(s) \delta h(t)} F(\beta, h) = R \bar{R} \delta(t - s). \] (87)

This is the same as the result obtained by using the Langevin equation (with the probability distribution of the random variable restricted to the finite region \([0, \beta]\)), (see appendix B.2).
4. Concluding remarks

We have obtained the Fokker–Planck equation for the fractional Ornstein–Uhlenbeck process with two indices. This effective Fokker–Planck equation is more complicated than that for fractional Brownian motion due to the complex structure of the variance of the fractional Ornstein–Uhlenbeck process. The path integral representation of the fractional Ornstein–Uhlenbeck process with two indices of Caputo and Riemann–Liouville type and various relevant physical quantities are derived.

Physical applications of the fractional Ornstein–Uhlenbeck process with two indices to the Von Karman wind speed spectrum [39] and Casimir energy for fractional quantum field have been considered [40, 41]. Note that in the latter application, one makes use of the fact that the fractional Ornstein–Uhlenbeck process with two indices can be regarded as a one-dimensional, fractional Klein-Gordon scalar massive Euclidean field. In view of the recent interest in quantum field theories with multifractal spacetime structure—in the sense that spacetime is of integer dimension 4 at large scales, and it is two-dimensional in the small scales [20–25]—it is hoped that the path integral formulation given here may have relevance in some of these theories.

Appendix A. Fokker–Planck equation with some simple potentials

Here we derive the values of $\Upsilon$, hence $S_x(t)$, which allows one to obtain the associated Fokker–Planck equation as given by (41). From (29), with $L' = \partial_x F(x)$, and (40a), we write explicitly

$$Y_\pm(t) = e^{\pm \partial_x F(x)} \partial_x e^{\pm \partial_x F(x)}. \tag{A.1}$$

Differentiating (A.1) with respect to time gives

$$\partial_t Y_\pm(t) = \mp e^{\pm \partial_x F(x)} \partial_x \left[ \partial_x, \partial_x F(x) \right] e^{\pm \partial_x F(x)}$$

$$= \mp e^{\pm \partial_x F(x)} \partial_x \left[ \partial_x, F(x) \right] e^{\pm \partial_x F(x)}. \tag{A.2}$$

The cases to be considered are given in the table 1. For the free and linear cases, (A.1) becomes

$$\partial_t Y_\pm(t) = 0,$$  \tag{A.3}

$$Y_\pm(t) = Y_\pm(0) = \partial_x.$$  \tag{A.4}

Thus the corresponding operator $S_x(t)$ is

$$S_x(t) = e^{\partial_x} \frac{1}{2} \frac{d \sigma^2(t)}{dt} \partial_x^2 e^{-t \partial_x} = \frac{1}{2} \frac{d \sigma^2(t)}{dt} \partial_x^2,$$  \tag{A.5}
and so it is the same as free case (44). The Fokker–Planck equation for the linear case is given by

$$\partial_t P(x, t) = -g \partial_x P(x, t) + \frac{1}{2} \frac{d\sigma^2(t)}{dt} \partial^2_x P(x, t). \quad (A.6)$$

Finally, for the case of harmonic potential, (A.1) can be written as

$$\partial_t Y^\pm(t) = \pm \omega Y^\pm(t), \quad (A.7)$$

$$Y^\pm(t) = Y^\pm(0)e^{\pm \omega t} = \partial_x e^{\pm \omega t}. \quad (A.8)$$

Therefore one gets

$$S_x(t) = \left[ \int_0^t \int_0^\tau \right] U(t) \partial^2_x U^{-1}(t). \quad (A.9)$$

Note that

$$U(t) \partial^2_x U^{-1}(t) = \left[ Y^\pm(t) \right]^2 = \partial^2_x e^{-2 \omega t}. \quad (A.10)$$

Thus we have

$$S_x(t) = \left[ \int_0^t \int_0^\tau \right] \partial^2_x . \quad (A.11)$$

Thus the Fokker–Planck equation for this case is

$$\partial_t P(x, t) = \omega \partial_x P(x, t) + \left[ \int_0^t \int_0^\tau \right] \partial^2_x P(x, t). \quad (A.12)$$

### Appendix B. Evaluations of generating function and covariance

#### B.1. Generating function

The following argument is used to obtain (83). Add the source $\langle h, x \rangle$ to the action, and use the same procedure as in subsection 3.1. Instead of (56), we get

$$x(t) = x^\circ(t) - RR^\dagger h(t). \quad (B.1)$$

Thus,

$$x(t) = x^\circ(t) - RR^\dagger h(t), \quad (B.2)$$

where $x^\circ$ is the homogeneous solution; i.e., when $h = 0$. The similar procedure is repeated till we get the propagator with source $h$:

$$G\left[ x^\beta, \beta; x_0, 0; h \right] = G\left[ x^\beta, \beta; x_0, 0; 0 \right] e^{\pm h RR^\dagger h}, \quad (B.3)$$

where the propagator $G\left[ x^\beta, \beta; x_0, 0; 0 \right]$ is the same as in (80). By taking the trace of $\int_v d\tau G\left[ x, \beta; x, 0; h \right]$ the same as in (82), we get (83).
B.2. Covariance

Let us express the last term of (87) in details:

$$\mathbb{E} \delta(t - s) = \int_0^t du R(t, u) \int_u^\beta dv R(v, u) \delta(v - s)$$

$$= \int_0^\beta du \theta(t - u) R(t, u) \int_0^\beta dv \theta(v - u) R(v, u) \delta(v - s)$$

$$= \int_0^\beta du \theta(t - u) R(t, u) \theta(s - u) R(s, u)$$

$$= \int_0^{\min(t, s)} du R(t, u) R(s, u), \quad (B.4)$$

which is the same as that obtained from the Langevin equation:

$$\mathbb{E} \delta(t - s) = \int_0^t du R(t, u) \int_0^\beta dv R(s, v) \delta(u - v). \quad (B.5)$$

Appendix C. Caputo type equation

In this appendix, we discuss the solution of an equation of the motion (57) of the Caputo type. It is straightforward to show that (58) is the solution of the Riemann–Liouville type

$${\mathcal{D}}^{\alpha} \left[ 1 + \lambda \alpha^\alpha I^\alpha \right] y(t) = 0. \quad (C.1)$$

Now, we insert identity $(-D)_t I = I$ into the l.h.s. of (57) to give

$$(-D)_t \left[ 1 + \lambda \alpha^\alpha I^\alpha \right] I^{1 - \alpha}(-D)_t y_C(t) = (-D)_t \left[ 1 + \lambda \alpha^\alpha I^\alpha \right] I^{1 - \alpha} y_C(t)$$

$$= (-D)_t \left[ 1 + \lambda \alpha^\alpha I^\alpha \right] I^{1 - \alpha} \left[ y_C(t) - y_C^\beta \right]$$

$$= (-D)_t I^{1 - \alpha} \left[ 1 + \lambda \alpha^\alpha I^\alpha \right] \frac{1}{\Gamma(1 - \alpha)} \left[ y_C(t) - y_C^\beta \right]$$

$$= \mathcal{D}^{\alpha} \left[ 1 + \lambda \alpha^\alpha I^\alpha \right] \frac{1}{\Gamma(1 - \alpha)} \left[ y_C(t) - y_C^\beta \right], \quad (C.2)$$

which is clear that

$$y_C(t) = y_C^\beta + y(t). \quad (C.3)$$

The undetermined constant $y_C^\beta$ can be chosen to be zero, thus minimizes the action (51). This verifies that the solution of the Riemann-Liouville type is also a solution of the Caputo type.

References

[1] Biagini F, Hu Y, Øksendal B and Zhang T 2008 Stochastic Calculus for Fractional Brownian Motion and Applications (New York: Springer)

[2] Mishura Y 2008 Stochastic Calculus for Fractional Brownian Motion and Related Processes (New York: Springer)

[3] Nourdin I 2012 Selected Aspects of Fractional Brownian Motion (New York: Springer)
[4] Klages R, Radons G and Sokolov I (ed) 2008 Anomalous Transport: Foundations and Applications (Weinheim, Germany: Wiley-VCH)
[5] Cheridito P, Kawaguchi H and Maejima M 2003 Electronic J. Probability 8 1–14
[6] Lim S C and Muniandy S V 2003 J. Phys. A: Math. Gen. 36 3961–82
[7] Magdziarz M 2008 Physica A 387 123–33
[8] Yan L, Lu Y and Xu Z 2008 J. Phys. A: Math. Theor. 41 145007
[9] Chaichian M and Demichev A 2001 Path Integrals in Physics Volume I: Stochastic Processes and Quantum Mechanics (Bristol: IOP Publishing)
[10] Kleinert H 2009 Path Integral, in Quantum Mechanics, Statistics Polymer Physics and Financial Markets 5th edn (Singapore: World Scientific)
[11] Wio H S 2013 Path Integrals for Stochastic Processes: An Introduction (Singapore: World Scientific)
[12] Laskin N 2000 Phys. Lett. A 268 298–305
[13] Tarasov V E and Zaslavsky G M 2008 Communications in Nonlinear Science and Numerical Simulation 13 248–58
[14] Calzoner M, Scudellari G and Scalisi M 2012 J. Math. Phys 53 102110
[15] Kleinert H and Zatloukal V 2011 Eur. Phys. Lett. 100 10001
[16] Ferrante D D, Guralnik G S, Guralnik Z and Pehlevan C 2013 arXiv:1301.4233
[17] Maghrebi J and Unterberger J 2011 Ann. H. Poincaré 12 1199–226
[18] Maghazi J and Unterberger J 2012 Ann. H. Poincaré 13 209–70
[19] Modeco L 2009 Class. Quant. Grav. 26 242002
[20] Ambjørn J, Görlach A, Jurkiewicz J and Loll R 2010 Phys. Lett. B 690 420–6
[21] Calzoner G 2010 Phys. Rev. Lett. 104 251301
[22] Calzoner G 2013 Phys. Rev. D 88 065005
[23] Reuter M and Saueressig F 2012 New J. Phys. 14 055022
[24] Calzoner G 2012 J. High Energy Phys. JHEP01(2012)065
[25] Sebastian K 1995 J. Phys. A: Math. Gen. 28 4305–11
[26] Chakravarti N and Sebastian K L 1997 Physica A 264 6309–4
[27] Calvino I and Sánchez R 2009 J. Phys. A: Math. Theor. 42 055003
[28] Calvino I, Sánchez R and Carreras B 2009 J. Phys. A: Math. Theor. 42 055003
[29] Janakiraman D and Sebastian K 2012 Phys. Rev. E 86 061105
[30] Wio H 2013 J. Phys. A: Math. Theor. 46 115005
[31] Eab C H and Lim S C 2009 Physica A 371 303–16
[32] Friedrich R and Eule S 2011 (arXiv:1110.5771)
[33] Friedrich R and Eule S 2007 Towards a path-integral formulation of continuous time random walks Path Integrals—New Trends and Perspectives (Singapore: World Scientific) pp 581–4
[34] Pelletier R and Vehel J L 1995 Multifractional brownian motion : definition and preliminary results. Tech. Rep. Rapport de Recherche de l’INRIA 2645
[35] Benassi A, Jaffard S and Roux D 1997 Rev. Mat. Iberoam 13 19–81
[36] Lim S and Teo L 2007 J. Phys. A: Math. Theor. 40 6035–60
[37] Lim S C, Li M and Teo L P 2008 Phys. Lett. A 372 6309–20
[38] Lim S C and Teo L P 2009 J. Phys. A: Math. Theor. 42 065208
[39] Lim S and Teo L 2011 Casimir effect associated with fractional klein–gordon field fractional dynamics, recent advances ed Y Klafter, S C Lim and R Metzler (Singapore: World Scientific) pp 483–506
[40] Lim S C and Eab C H 2006 Phys. Lett. A 355 87–96
[41] Ezawa H 2000 Acta Appl. Math. 63 119–35
[42] Bel G and Barkai E 2005 Phys. Rev. Lett. 94 240602
[43] Deng W and Barkai E 2009 Phys. Rev. E 79 011112
[44] Samko S G, Kilbas A A and Marichev O I 1993 Fractional Integral and Derivatives: Theory and Applications (Amsterdam: Gordon and Breach)
[45] Samko S G 2002 Hypersingular integrals and their applications Series: Analytical Methods and Special Functions vol 5 (London: Taylor and Francis)
[46] Lim S C 2001 J. Phys. A: Math. Gen. 34 1301–10
[47] Magdziarz M and Weron A 2007 Studia Mathematica 181 47–60
[48] Magdziarz M 2009 Stochastic Processes and their Applications 119 3416–34
[51] Magdziarz M and Weron A 2011 Ann. Phys. NY 326 2431–43
[52] Khinchin A 1948 Mathematical Foundations of Statistical Mechanics (New York: Dover)
[53] Reichl L E 1998 A Modern Course in Statistical Physics 2nd edn (New York: John Wiley and Sons)
[54] Wang K and Lung C 1990 Phys. Lett. A 151 119–21
[55] Ünal G 2007 Fokker–Planck–Kolmogorov equation for fBm: derivation and analytical solutions Mathematical Physics, Proc. 12th Regional Conference, Islamabad, Pakistan, 27 March–1 April 2006 (Singapore: World Scientific) pp 53–60
[56] Hahn M G, Kobayashi K and Umarov S 2011 Proc. Amer. Math. Soc. 139 691–705
[57] Lim S C and Muniandy S V 2002 Phys. Rev. E 66 021114
[58] Gradshteyn I S and Ryzhik I M 1980 Table of Integrals, Series, and Products 4th edn (New York: Academic)