Nonrigidity of piecewise-smooth circle maps

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Abstract

Let \( f_i, i = 1, 2 \) be piecewise-smooth \( C^1 \) circle homeomorphisms with two break points, \( \log Df_i, i = 1, 2 \) are absolutely continuous on each continuity intervals of \( Df_i \) and \( D \log Df_i \in L^p \) for some \( p > 1 \). Suppose, the jump ratios of \( f_1 \) and \( f_2 \) at their break points do not coincide but have the same total jumps (i.e. the product of jump ratios) and identical irrational rotation number of bounded type. Then the conjugation \( h \) between \( f_1 \) and \( f_2 \) is a singular function, i.e. it is continuous on \( S^1 \), but \( Dh(x) = 0 \) a.e. with respect to Lebesgue measure.

1 Introduction

This work continues and in some sense completes our study of conjugations between circle homeomorphisms with break type singularities. Let \( S^1 = \mathbb{R}/\mathbb{Z} \) with clearly defined orientation, metric, Lebesgue measure and the operation of addition be the unit circle. Let \( \pi : \mathbb{R} \to S^1 \) denote the corresponding projection mapping that "winds" a straight line on the circle. An arbitrary homeomorphism \( f \) that preserves the orientation of the unit circle \( S^1 \) can "be lifted" on the straight line \( \mathbb{R} \) in the form of the homeomorphism \( L_f : \mathbb{R} \to \mathbb{R} \) with property \( L_f(x + 1) = L_f(x) + 1 \) that is connected with \( f \) by relation \( \pi \circ L_f = f \circ \pi \). This homeomorphism \( L_f \) is called the lift of the homeomorphism \( f \) and is defined up to an integer term. The most important arithmetic characteristic of the homeomorphism \( f \) of the unit circle \( S^1 \) is the rotation number

\[
\rho(f) = \lim_{i \to \infty} \frac{L_f^i(x)}{i} \mod 1,
\]

where \( L_f \) is the lift of \( f \) with \( S^1 \) to \( \mathbb{R} \). Here and below, for a given map \( F \), \( F^i \) denotes its \( i \)-th iteration. The classical Denjoy's theorem states \([5]\), that if \( f \) is a circle diffeomorphism with irrational rotation number \( \rho = \rho(f) \) and \( \log Df \) is of bounded variation, then \( f \) is conjugate to the linear rotation \( f_\rho : x \to x + \rho \mod 1 \), that is, there exists a unique (up to additional constant) homeomorphism \( \varphi \) of the circle with \( f = \varphi^{-1} \circ f_\rho \circ \varphi \). Since the conjugating map \( \varphi \) and the unique \( f \)-invariant measure \( \mu_f \) are related by \( \varphi(x) = \mu_f([0, x]) \) (see \([3]\)), regularity properties of the conjugating map \( \varphi \) imply corresponding properties of the density of the absolutely continuous invariant measure \( \mu_f \). This problem of smoothness of the conjugacy of smooth diffeomorphisms is now very well understood (see for instance \([2, 16, 11, 12, 13, 14, 18]\)).

A natural extension of diffeomorphisms of the circle are piecewise-smooth homeomorphisms with break points, that is, maps that are smooth everywhere except for several singular points at which the first derivative has a jump. Notice that Denjoy’s result can

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be extended to circle homeomorphisms with break points. Below we present the exact statement of the corresponding theorem. The regularity properties of invariant measures of such maps are quite different from the case diffeomorphisms. Namely, invariant measure of piecewise-smooth circle homeomorphisms with break points and with irrational rotation number is singular w.r.t. Lebesgue measure (see [6, 7, 8, 10]). In this case, the conjugacy $\varphi$ between $f$ and linear rotation $f_\rho$ is singular function. Here naturally arises the question on regularity of conjugacy between two circle maps with break points. Consider two piecewise-smooth circle homeomorphisms $f_1, f_2$ which has break points with the same order on the circle and the same irrational rotation numbers. On what conditions is the conjugacy between two such homeomorphisms smooth? This is the rigidity problem for circle homeomorphisms with break points. Denote by $\sigma_f(b) := Df_-(b)/Df_+(b)$ the jump ratio or jump of $f$ at the break point $b$. The case of circle maps with one break point and the same jump ratios were studied in detail by K. Khanin and D. Khmelev [15], A. Teplinskii and K. Khanin [17]. Let $\rho = 1/(k_1 + 1/(k_2 + ... + 1/(k_n + ...))) := [k_1, k_2, \ldots, k_n, \ldots]$ be the continued fraction expansion of the irrational rotation number $\rho$. Define

$$M_o = \{\rho : \exists C > 0, \forall n \in \mathbb{N}, k_{2n-1} \leq C\}, \quad M_e = \{\rho : \exists C > 0, \forall n \in \mathbb{N}, k_{2n} \leq C\}.$$  

We formulate the main result of [17].

**Theorem 1.1.** Let $f_i \in C^{2+\alpha}(S^1\setminus\{b_i\}), \ i = 1, 2, \alpha > 0$ be two circle homeomorphisms with one break point that have the same jump ratio $\sigma$ and the same irrational rotation number $\rho \in (0, 1)$. In addition, let one of the following restrictions be true: either $\sigma > 1$ and $\rho \in M_e$ or $\sigma < 1$ and $\rho \in M_o$. Then the map $h$ conjugating the homeomorphisms $f_1$ and $f_2$ is a $C^1$-diffeomorphism.

In the case homeomorphisms with different jump ratios the following theorem was proved by A. Dzhalilov, H. Akin, S. Temir in [9].

**Theorem 1.2.** Let $f_i \in C^{2+\alpha}(S^1\setminus\{b_i\}), \ i = 1, 2, \alpha > 0$ be two circle homeomorphisms with one break point that have different jump ratio and the same irrational rotation number $\rho \in (0, 1)$. Then the map $h$ conjugating the homeomorphisms $f_1$ and $f_2$ is a singular function.

Now consider two piecewise-smooth circle homeomorphisms $f_1$ and $f_2$ with $m$ ($m \geq 2$) break points and the same irrational rotation number. Denote $BP(f_1)$ and $BP(f_2)$ the sets of break points of $f_1$ and $f_2$.

**Definition 1.3.** The homeomorphisms $f_1, f_2$ are said to be break equivalent if there exists a bijection $\psi_0$ such that

1. $\psi_0(BP(f_1)) = BP(f_2)$;
2. $\sigma_{f_2}(\psi_0(b)) = \sigma_{f_1}(b)$, for all $b \in BP(f_1)$.

The rigidity problem for the break equivalent $C^{2+\alpha}$-homeomorphisms and with trivial total jumps (i.e. it is equal to 1) was studied by K. Cunha and D. Smania in [1]. It was proved that any two such homeomorphisms with some combinatorial conditions are $C^1$-conjugated. The main idea of this work is to consider piecewise-smooth circle homeomorphisms as generalized interval exchange transformations. The case of non break equivalent homeomorphisms with two break points was studied by H. Akhadkulov, A. Dzhalilov and D. Mayer in [1]. The main result of [1] is the following theorem.
Theorem 1.4. Let \( f_i \in C^{2+\alpha}(S^1 \setminus \{a_i, b_i\}), i = 1, 2 \) be circle homeomorphisms with two break points \( a_i, b_i \). Assume that

1. their rotation numbers \( \rho(f_i), i = 1, 2 \) are irrational and coincide i.e. \( \rho(f_1) = \rho(f_2) = \rho, \rho \in \mathbb{R}^1 \setminus \mathbb{Q} \);
2. there exists a bijection \( \psi \) such that \( \psi(BP(f_1)) = BP(f_2) \);
3. \( \sigma_{f_1}(a_1)\sigma_{f_1}(b_1) \neq \sigma_{f_2}(a_2)\sigma_{f_2}(b_2) \).

Then the map \( h \) conjugating \( f_1 \) and \( f_2 \) is a singular function.

Now we consider a wider class of circle homeomorphisms with break points. We say that a circle homeomorphism \( f \) with finite number break points satisfies generalized conditions of Katznelson-Ornstein (K.O), if \( \log Df \) is absolutely continuous on each continuity intervals of \( Df \) and \( D \log Df \in L^p \) for some \( p > 1 \). In this work we study the conjugating map \( h \) between two circle homeomorphisms \( f_1 \) and \( f_2 \) with two break points and satisfying (K.O) conditions. Now we formulate the main result of present paper.

Theorem 1.5. Let \( f_i, i = 1, 2 \) be piecewise-smooth \( C^1 \) circle homeomorphisms with two break points \( a_i, b_i \). Assume that

1. the rotation numbers \( \rho(f_i) \) of \( f_i \), \( i = 1, 2 \) are irrational of bounded type and coincide;
2. \( \sigma_{f_1}(a_1)\sigma_{f_1}(b_1) = \sigma_{f_2}(a_2)\sigma_{f_2}(b_2) \);
3. \( \sigma_{f_1}(a_1) \neq \sigma_{f_2}(b) \) for all \( b \in BP(f_2) \);
4. the break points of \( f_i, i = 1, 2 \) do not lie on the same orbit;
5. \( f_i, i = 1, 2 \) satisfy (K.O) conditions for the same \( p > 1 \).

Then the map \( h \) conjugating \( f_1 \) and \( f_2 \) is a singular function.

The main approach for proving theorem 1.5 plays to study the behaviours of sequence \( \left\{ \log \frac{Df_{n+1}^{n}(b(x))}{Df_{n}^{n}(b(x))} \right\}_{n}^{\infty} \) where \( q_n, n = 1, 2, \ldots \) are first return times. This argument has been used by M. Herman in [11] for investigating conjugations between piecewise linear circle homeomorphisms with two break points. Recently it has been discussed by A. Dzhalilov and I. Liousse [7], to study invariant measures of circle homeomorphisms with two break points.

2 The Denjoy theory

We use the continued fraction \( \rho = [k_1, k_2, \ldots, k_n, \ldots) \) of the irrational number which is understood as the limit of the sequence of convergents \( p_n/q_n = [k_1, k_2, \ldots, k_n] \). The sequence of positive integer \( k_n \) with \( n \geq 1 \), which are called incomplete multiples, is uniquely determined for irrational \( \rho \). The coprimes \( p_n \) and \( q_n \) satisfy the recurrence relations \( p_n = k_n p_{n-1} + p_{n-2} \) and \( q_n = k_n q_{n-1} + q_{n-2} \) for \( n \geq 1 \), where we set for convenience, \( p_{-1} = 0, q_{-1} = 1, \) and \( p_0 = 1, q_0 = k_1 \).

The class of \textbf{P-homeomorphisms} consists of orientation preserving circle homeomorphisms \( f \) differentiable except in finite number break points at which left and right derivatives, denoted respectively by \( Df_- \) and \( Df_+ \), exist, and such that
- there exist constants $0 < c_1 < c_2 < \infty$ with $c_1 < Df(x) < c_2$ for all $x \in S^1 \setminus BP(f)$, $c_1 < Df_-(x_b) < c_2$ and $c_1 < Df_+(x_b) < c_2$ for all $x_b \in BP(f)$, with $BP(f)$ the set of break points of $f$ in $S^1$;
- log $Df$ has bounded variation in $S^1$.

If log $Df$ has bounded variation, in this situation log $Df^{-1}$ also have the same total variation and denote by $v = Var_{S^1} \log Df$. Let $\xi \in S^1$, we define the $n$-th generator interval $\Delta^{n}_0(\xi)$ as the circle arc $[\xi, f^{q_n}(\xi)]$ for even $n$ and as $[f^{q_n}(\xi), \xi]$ for odd $n$. The assertions listed below, which are valid for any $P$-homeomorphism $f$ with irrational rotation number $\rho = \rho(f)$. Their proofs can be found in \[11\], \[7\] and \[14\].

(a) Generalized Finzi inequality; suppose $\xi \in S^1$, $\eta, \zeta \in \Delta^{q_n-1}_0(\xi)$ and $\eta$, $\zeta$ are continuity points of $Df^k$, $0 \leq k < q_n$. Then the following inequality holds: $|\log Df^k(\eta) - \log Df^k(\zeta)| \leq v$.

(b) Generalized Denjoy inequalities; let $\xi_0 \in S^1$ be a continuity point of $Df^{q_n}$, then the following inequality holds: $e^{-v} \leq Df^{q_n}(\xi_0) \leq e^v$.

From generalized Denjoy inequalities it follows that the trajectory of every point $\xi \in S^1$ is the dense set in $S^1$. This together with monotonicity of the homeomorphism $f$ implies the following theorem.

(c) Generalized Denjoy theorem; let $f$ be a $P$-homeomorphism with irrational rotation number $\rho$. Then $f$ is conjugate to the linear rotation $f_\rho$.

**Remark 2.1.** The same assertions as (a) – (c) holds for $f^{-1}$.

### 3 Absolute continuity of conjugating map

Consider two $P$-homeomorphisms on two copies of the circle $S^1$ with identical irrational rotation number $\rho$. Let $\varphi_1$ and $\varphi_2$ be maps conjugating $f_1$ and $f_2$ with the pure rotation $f_\rho$, i.e. $\varphi_1 \circ f_1 = f_\rho \circ \varphi_1$ and $\varphi_2 \circ f_2 = f_\rho \circ \varphi_2$. It is easy to check that the map $h = \varphi_2^{-1} \circ \varphi_1$ conjugates $f_1$ and $f_2$, i.e.

\[ h(f_1(x)) = f_2(h(x)) \]

for all $x \in S^1$.

**Lemma 3.1.** Let $f_1$ and $f_2$ are $P$-homeomorphisms with identical irrational rotation number. Then conjugating map $h$ between $f_1$ and $f_2$ is either absolutely continuous or singular function.

**Proof.** The conjugating homeomorphism $h$ is strictly increasing function on $S^1$. Then $Dh$ exists almost everywhere on $S^1$. Denote by $\mathcal{A} = \{ x : x \in S^1, \; Dh(x) > 0 \}$. It is clear that the set $\mathcal{A}$ is mod 0 invariant set with respect to $f_1$ i.e. $\mathcal{A} = f_1^{-1}(\mathcal{A})$ almost everywhere on $\mathcal{A}$. As $P$-homeomorphism the map $f_1$ is ergodic with respect to Lebesgue measure. Hence the Lebesgue measure of the set $\mathcal{A}$ is 0 or 1. The conjugation $h$ is singular function if $\ell(\mathcal{A}) = 0$ and it is absolutely continuous if $\ell(\mathcal{A}) = 1$. □

The following theorem gives the necessary condition of absolute continuity of conjugation.


Theorem 3.2. Let \( f_i, i = 1, 2 \) are \( P \)-homeomorphisms with identical irrational rotation number \( \rho \). If conjugation map \( h \) between \( f_1 \) and \( f_2 \) is absolutely continuous function, then for all \( \delta > 0 \)

\[
\lim_{n \to \infty} \ell(x : |\log Df_2^{q_n}(h(x)) - \log Df_1^{q_n}(x)| \geq \delta) = 0.
\]

Proof. First we prove that the sequence \( f_1^{q_n}(x) \) uniformly converges to \( x \). It is clear that \( |f_1^{q_n}(x) - x| = |\varphi_1^{-1} \circ f_1^{q_n} \circ \varphi_1(x) - x| \). By setting \( y = \varphi_1(x) \) we get \( |f_1^{q_n}(x) - x| = |\varphi_1^{-1}(f_1^{q_n}(y)) - \varphi_1^{-1}(y)| \). Furthermore \( |f_1^{q_n}(x) - x| \leq 1/q_n \) does not depend on \( x \) and tends to 0. This and the uniform continuity of \( \varphi_1^{-1} \) on \( S^1 \) implies that the sequence \( f_1^{q_n}(x) \) uniformly converges to \( x \). Denote by \( \| \cdot \|_1 \) the norm in \( L^1(S^1, d\ell) \). Now we show that

\[
(2) \quad \lim_{n \to \infty} \| \psi \circ f_1^{q_n} - \psi \|_1 = 0, \quad i = 1, 2 \text{ for all } \psi \in L^1(S^1, d\ell).
\]

Well known fact that the class \( C([a, b]) \) of continuous functions on \([a, b]\) is dense (in \( \| \cdot \|_1 \)) in \( L^1([a, b], d\ell) \). From this fact implies that if \( \psi \in L^1(S^1, d\ell) \), then for any sufficiently small \( \epsilon > 0 \) there exists a continuous function \( \psi_\epsilon \in C(S^1) \) and \( \phi_\epsilon \in L^1(S^1, d\ell) \) such that \( \psi = \psi_\epsilon + \phi_\epsilon \) and \( \| \phi_\epsilon \|_1 \leq \epsilon \). Using this and Denjoy inequalities we obtain

\[
\| \psi \circ f_1^{q_n} - \psi \|_1 \leq \| \psi_\epsilon \circ f_1^{q_n} - \psi_\epsilon \|_1 + (\sup |Df_1^{q_n}|^{-1} + 1)\| \phi_\epsilon \|_1 \leq \| \psi_\epsilon \circ f_1^{q_n} - \psi_\epsilon \|_{L_1} + (1 + \epsilon^2)\| \phi_\epsilon \|_1.
\]

As \( \psi_\epsilon \) is uniformly continuous on \( S^1 \) and by exponential refinement \( f_1^{q_n}(x) \) uniformly tends to \( x \), there exists a positive integer \( n_0 = n_0(\epsilon) \) such that for all \( n \geq n_0 \), the \( \| \psi_\epsilon \circ f_1^{q_n} - \psi_\epsilon \|_1 \leq \epsilon \). Therefore, \( \| \psi \circ f_1^{q_n} - \psi \|_1 \leq (2 + \epsilon^2)\epsilon \). Since \( \epsilon > 0 \) was arbitrary and sufficiently small.

Now we prove theorem 3.2. Assume that conjugation map \( h \) is absolutely continuous function then \( Dh \in L^1(S^1, d\ell) \) and \( Dh(x) > 0, x \in A \). Using equation (1) it is easy to see that for all natural number \( n \), the function \( Dh \) satisfies the following equation

\[
Dh(f_1^{q_n}(x))Df_1^{q_n}(x) = Dh(f_2^{q_n}(h(x)))Df_2^{q_n}(x) \quad \text{a.e.}
\]

Taking the logarithm, we obtain

\[
\log Dh(f_1^{q_n}(x)) - \log Dh(x) = \log Df_2^{q_n}(h(x)) - \log Df_1^{q_n}(x) \quad \text{a.e.}
\]

multiplying by \( 2i\pi/2(v_1 + v_2) \), where \( v_j = Var_{S^1 \log Df_j}, j = 1, 2 \) we obtain

\[
\frac{2i\pi \log Dh(f_1^{q_n}(x))}{2(v_1 + v_2)} - \frac{2i\pi \log Dh(x)}{2(v_1 + v_2)} = \frac{2i\pi \log Df_2^{q_n}(h(x))}{2(v_1 + v_2)} - \frac{2i\pi \log Df_1^{q_n}(x)}{2(v_1 + v_2)} \quad \text{a.e.}
\]

Consider the following function

\[
\psi(x) = \begin{cases} 
\exp\left(\frac{2i\pi \log Dh(x)}{2(v_1 + v_2)}\right) & \text{if } x \in A, \\
0 & \text{if } x \in S^1 \setminus A.
\end{cases}
\]

It is clear that \( \psi \) is measurable, \( \psi \in L^1(S^1, d\ell) \) and \( \| \psi \|_1 = \ell(A) \). Taking the exponential from last equation we obtain

\[
\psi(f_1^{q_n}(x)) - \psi(x) = \left[ \exp\left(\frac{2i\pi \left(\log Df_2^{q_n}(h(x)) - \log Df_1^{q_n}(x)\right)}{2(v_1 + v_2)}\right)\right] - 1 \psi(x) \quad \text{a.e.}
\]
Integrating the module of this equality, we have

\[
\int_{S^1} |\psi(f_1^{q_n}(x)) - \psi(x)| dx = \int_{S^1} \left| 2i\pi \left( \frac{\log Df_2^{q_n}(h(x)) - \log Df_1^{q_n}(x)}{2(v_1 + v_2)} \right) \right| - 1 | dx = \\
= \int_{S^1} 2 \sin \pi \left| \frac{\log Df_2^{q_n}(h(x)) - \log Df_1^{q_n}(x)}{2(v_1 + v_2)} \right| dx
\]

Suppose, by contradiction, that there exists \( \delta > 0 \) (we may suppose that \( \delta \in (0, v_1 + v_2) \)) such that \( \ell(S^n_\delta) \) does not converge to 0 when \( n \) goes to infinity, where \( S^n_\delta = \{ x : | \log Df_2^{q_n}(h(x)) - \log Df_1^{q_n}(x) | \geq \delta \} \). It follows from Denjoy’s inequality \( | \log Df_2^{q_n}(h(x)) - \log Df_1^{q_n}(x) | \leq v_1 + v_2 \), therefore \( \pi | \log Df_2^{q_n}(h(x)) - \log Df_1^{q_n}(x) | / (2(v_1 + v_2)) \) and \( \pi \delta / (2(v_1 + v_2)) \) belong to \([0, \pi/2]\), an interval where "\( \sin \)" is an increasing function. Hence, for all natural \( n \):

\[
\int_{S^1} |\psi(f_1^{q_n}(x)) - \psi(x)| dx \geq \int_{S^n_\delta} |\psi(f_1^{q_n}(x)) - \psi(x)| dx \geq \left[ 2 \sin \frac{\pi \delta}{2(v_1 + v_2)} \right] \ell(S^n_\delta).
\]

But \( \ell(S^n_\delta) \) does not tend to 0 when \( n \) goes to \( +\infty \). Hence \( \| \psi \circ f_1^{q_n} - \psi \|_1 \) does not tend to 0 when \( n \) goes to \( +\infty \), this is contradicts to (2) and ends the proof of theorem 3.2 \( \square \)

4 Dynamical partitions and universal estimates

In this section we will introduce two types of dynamical partitions and we will get some estimations for the ratios of length of elements of these partitions. Given a circle homeomorphism \( f \) with irrational rotation number \( \rho \), one may consider a positive marked trajectory (i.e. the positive trajectory of a marked point) \( \xi_i = f^i(\xi_0) \in S^1 \), where \( i \geq 0 \), and pick out of it the sequence of the dynamical convergents \( \xi_{q_n}, n \geq 0 \), indexed by the denominators of consecutive rational convergents to \( \rho \). We will also conventionally use \( \xi_{q_{-1}} = \xi_0 - 1 \). The well-understood arithmetical properties of rational convergents and the combinatorial equivalence between \( f \) and linear rotation \( f_\rho \) imply that the dynamical convergents approach the marked point, alternating their order in the following way:

\[
\xi_{q_{-1}} < \xi_{q_1} < \xi_{q_3} < \ldots < \xi_{q_{2m+1}} < \ldots < \xi_0 < \ldots < \xi_{q_{2m}} < \ldots < \xi_{q_2} < \xi_{q_0}.
\]

For the marked trajectory, we use the notations \( \Delta^n_0 = \Delta_0^n(\xi_0) \) and \( \Delta^n_j = f^i(\Delta^n_0) \), where \( \Delta^n_0(\xi_0) \) is \( n \)-th generator interval. It is well known, that the set of intervals \( P_n = P_n(\xi_0, f) \) with mutually disjoint interiors defined as

\[
P_n = \{ \Delta^n_{i-1}, 0 \leq i < q_n; \Delta^n_j, 0 \leq j < q_{n-1} \},
\]

determines a partition of the circle for any \( n \). The partition \( P_n \) is called the \( n \)-th dynamical partition of the point \( \xi_0 \). Obviously the partition \( P_{n+1} \) is a refinement of the partition \( P_n \): indeed the intervals of order \( n \) are members of \( P_{n+1} \) and each interval \( \Delta^n_{i-1} \in P_n \ 0 \leq i < q_n \), is partitioned into \( k_{n,i} + 1 \) intervals belonging to \( P_{n+1} \) such that

\[
\Delta^n_{i-1} = \Delta^n_{i-1} \cup \bigcup_{s=0}^{k_{n,i}-1} \Delta^n_{i+q_{n-1} + s q_n},
\]
In (I), the consecutive atoms are $\Delta^i_0$ and $\Delta^i_{q_n+1}$, (II) $\Delta^i_n$ and $\Delta^i_{n+1}$, (III) $\Delta^i_n$ and $\Delta^i_{n+1-q_n-q_{n-1}}$. Two consecutive atoms of $P_n$ can be of the following three types:

$I)$ $\Delta^i_{n-1}$ and $\Delta^i_{n+q_n-1}$, (II) $\Delta^i_{n}$ and $\Delta^i_{n-1}$, (III) $\Delta^i_{n}$ and $\Delta^i_{n+1-q_n-q_{n-1}}$.

In (I), the consecutive atoms are $I$ and $f^{q_n-1}(I)$. Since, $f$ has finite number break points, by the mean value theorem, we have

$$\inf_{I}(Df^{q_n-1}) \leq \frac{|f^{q_n-1}(I)|}{|I|} \leq \sup_{I}(Df^{q_n-1}).$$

From Denjoy inequalities, it follows that $e^{-v} \leq |f^{q_n-1}(I)|/|I| \leq e^{v}$. Consider the case (II). Using equality (II) we get

$$\frac{|\Delta^i_{n-1}|}{|\Delta^i_{n}|} = \frac{|\Delta^i_{n+1}| + \sum_{s=0}^{k_{n+1}-1} |\Delta^i_{n+q_{n-1}+sq_n}|}{|\Delta^i_{n}|}.$$

It is clear that $\Delta^i_{n+1} \subset \Delta^i_{n-q_{n+1}}$. Using similar arguments (I) we have

$$\sum_{s=0}^{k_{n+1}-1} \inf_{\Delta^i_{n}}(Df^{q_n-1+sq_n}) \leq \frac{|\Delta^i_{n-1}|}{|\Delta^i_{n}|} \leq \sum_{s=0}^{k_{n+1}-1} \sup_{\Delta^i_{n}}(Df^{q_n-1+sq_n}) + \sup_{\Delta^i_{n}}(Df^{-q_n}).$$

Using Denjoy inequalities, we get $\inf_{\Delta^i_{n}}(Df^{q_n-1+sq_n}) \geq e^{-(s+1)v}$ and $\sup_{\Delta^i_{n}}(Df^{q_n-1+sq_n}) \leq e^{(s+1)v}$. Furthermore, the rotation number is bounded type i.e. $k_{n+1} \leq Q$, hence it follows $|\Delta^i_{n-1}|$ and $|\Delta^i_{n}|$ are $[(Q + 1)e^{(Q+1)v}]^{-1}$-comparable. Now, we consider the case (III). By Denjoy inequalities, $|\Delta^i_{n-1}|$ and $|\Delta^i_{n+1-q_n-q_{n-1}}|$ are $e^{-2v}$-comparable. Using above argument $|\Delta^i_{n-1}|$ and $|\Delta^i_{n}|$ are $[(Q + 1)e^{(Q+1)v}]^{-1}$-comparable. Finally, any two consecutive atoms $P_n$ are $C_2 = [(Q + 1)e^{(Q+3)v}]^{-1}$-comparable.

**Property 4.2.** There exists universal constant $C_2 = C_2(Q, f)$ such that two consecutive atoms of $P_n$ are $C_2$-comparable.

**Proof.** Two consecutive atoms of $P_n$ can be of the following three types:

$I)$ $\Delta^i_{n-1}$ and $\Delta^i_{n+q_n-1}$, (II) $\Delta^i_{n}$ and $\Delta^i_{n-1}$, (III) $\Delta^i_{n}$ and $\Delta^i_{n+1-q_n-q_{n-1}}$.

In (I), the consecutive atoms are $I$ and $f^{q_n-1}(I)$. Since, $f$ has finite number break points, by the mean value theorem, we have

$$\inf_{I}(Df^{q_n-1}) \leq \frac{|f^{q_n-1}(I)|}{|I|} \leq \sup_{I}(Df^{q_n-1}).$$

From Denjoy inequalities, it follows that $e^{-v} \leq |f^{q_n-1}(I)|/|I| \leq e^{v}$. Consider the case (II). Using equality (II) we get

$$\frac{|\Delta^i_{n-1}|}{|\Delta^i_{n}|} = \frac{|\Delta^i_{n+1}| + \sum_{s=0}^{k_{n+1}-1} |\Delta^i_{n+q_{n-1}+sq_n}|}{|\Delta^i_{n}|}.$$

It is clear that $\Delta^i_{n+1} \subset \Delta^i_{n-q_{n+1}}$. Using similar arguments (I) we have

$$\sum_{s=0}^{k_{n+1}-1} \inf_{\Delta^i_{n}}(Df^{q_n-1+sq_n}) \leq \frac{|\Delta^i_{n-1}|}{|\Delta^i_{n}|} \leq \sum_{s=0}^{k_{n+1}-1} \sup_{\Delta^i_{n}}(Df^{q_n-1+sq_n}) + \sup_{\Delta^i_{n}}(Df^{-q_n}).$$

Using Denjoy inequalities, we get $\inf_{\Delta^i_{n}}(Df^{q_n-1+sq_n}) \geq e^{-(s+1)v}$ and $\sup_{\Delta^i_{n}}(Df^{q_n-1+sq_n}) \leq e^{(s+1)v}$. Furthermore, the rotation number is bounded type i.e. $k_{n+1} \leq Q$, hence it follows $|\Delta^i_{n-1}|$ and $|\Delta^i_{n}|$ are $[(Q + 1)e^{(Q+1)v}]^{-1}$-comparable. Now, we consider the case (III). By Denjoy inequalities, $|\Delta^i_{n-1}|$ and $|\Delta^i_{n+1-q_n-q_{n-1}}|$ are $e^{-2v}$-comparable. Using above argument $|\Delta^i_{n-1}|$ and $|\Delta^i_{n}|$ are $[(Q + 1)e^{(Q+1)v}]^{-1}$-comparable. Finally, any two consecutive atoms $P_n$ are $C_2 = [(Q + 1)e^{(Q+3)v}]^{-1}$-comparable.

**Property 4.3.** There exists universal constant $C_3 = C_3(Q, f) \leq 1$ such that an atom $\Delta^i_{n+1}$ of $P_{n+1}$ is $C_3$-comparable to the atom $\Delta^i_{n}$ of $P_n$ that contains it.
Proof. The atom $\Delta^{n+1}_i$ of $P_{n+1}$ can be of following three types:

- either $\Delta^n_i$ such that $\Delta^n_i \in P_n$ or
- $\Delta^n_{i+q_{n-1}+s_q}$ such that $\Delta^n_{i+q_{n-1}+s_q} \subset \Delta^{n-1}_i$ and $\Delta^{n-1}_i \in P_n$ or
- $\Delta^{n+1}_i$ such that $\Delta^{n+1}_i \subset \Delta^{n-1}_i$ and $\Delta^{n-1}_i \in P_n$.

In the first case the atoms $\Delta^{n+1}_i$ and $\Delta^n_i$ are 1-comparable. In the second case, using similar argument to the above property we get the atoms $\Delta^{n+1}_i$ and $\Delta^n_i$ are $[(Q+1)e^{(Q+1)v}]^{-1}$, comparable and the third case we get the atoms $\Delta^{n+1}_i$ and $\Delta^n_i$ are $[(Q+1)e^{(Q+1)v}]^{-2}$, comparable. Finally, if we take $C_3 = [(Q+1)e^{(Q+1)v}]^{-2}$ then we are done.

Note that in the second and third cases of the above property the number $C_3$ is the greatest lower bound but $C_3^{-1}$ can not be the least upper bound. Using Denjoy inequalities and relation (14) can be found the number $C_4 = (1 + e^{-v})^{-1}$ is the least upper bound.

**Property 4.4.** There exist constants $0 < \kappa < \lambda < 1$ such that for all $n, m \in \mathbb{N}$ holds the following inequality $\kappa^m |\Delta^n_0| \leq |\Delta^{n+m}_0| \leq (1 + e^v)\lambda^m |\Delta^n_0|$.

Proof. By definition of dynamical partition, it is easy to see that $|\Delta^{n-1}_0| \geq |\Delta^{n+1}_0| + |\Delta^{n+1}_{q_{n-1}+m}|$ and $\Delta^{n+1}_0 \subset \Delta^{n+1}_{q_{n+1}}$. Using property 4.3 and proper last two relations together with Denjoy inequalities implies

$$C_3^{-1} \geq \frac{|\Delta^{n-1}_0|}{|\Delta^{n+1}_0|} \geq 1 + \frac{|\Delta^{n+1}_{q_{n+1}}|}{|\Delta^{n+1}_{q_{n-1}+m}|} \geq 1 + e^{-v}.$$ 

By induction way we can show that if $m$ is even then

$$C_3^\frac{m}{2} |\Delta^n_0| \leq |\Delta^{n+m}_0| \leq (1 + e^{-v})^{-\frac{m}{2}} |\Delta^n_0|.$$ 

If $m$ is odd then, using property 4.2 we have

$$C_3^\frac{m-1}{2} C_2 |\Delta^n_0| \leq |\Delta^{n+m}_0| \leq e^v (1 + e^{-v})^{-\frac{m-1}{2}} |\Delta^n_0|.$$ 

If we take $\kappa = \min\{\sqrt{C_3}, C_2\}$ and $\lambda = 1/\sqrt{1 + e^{-v}}$ then we are done.

Apply at most three times Finzi inequality to property 4.4 we get following remark.

**Remark 4.5.** For all $n, m \in \mathbb{N}$ holds the following inequality $e^{-3v} \kappa^m |\Delta^n| \leq |\Delta^{n+m}| \leq (1 + e^v) e^{3v} \lambda^m |\Delta^n|$, where $\Delta^n \in P_n$, $\Delta^{n+m} \in P_{n+m}$ and $\Delta^{n+m} \subset \Delta^n$.

Using this remark we show that the oscillation of log $Df^k$ tends to zero with exponential fast on every exponential small continuity intervals of $Df^k$.

**Lemma 4.6.** *(Universal estimates)* Let log $Df$ be an absolutely continuous each continuity interval of $Df$ and $D \log Df \in L_p$ for some $p > 1$, then for all $n, l$ and natural numbers, for all integer $k$ such that $0 \leq k \leq q_0$ and for all $\xi \in S^1$, $\eta \in \Delta^{n+1}_0(\xi)$ in the same continuity interval of $Df^k$, there exists a universal constant $C_5 = C_5(f) > 0$ such that

$$| \log Df^k(\xi) - \log Df^k(\eta) | \leq C_5 \lambda^{1/q},$$

where $q^{-1} = 1 - p^{-1}$. 
Proof. Fix \( n \in \mathbb{N} \) and \( k \in \{0, 1, ..., q_n\} \). Let \( \xi \in S^1 \) and \( \eta \in \Delta_n^{n+1}(\xi) \) be two circle points lying in the same continuity interval of \( D f^k \). Then, we have

\[
| \log D f^k(\xi) - \log D f^k(\eta) | \leq \sum_{j=0}^{k-1} | \log D f(f^k(\xi)) - \log D f(f^k(\eta)) |
\]

\[
\leq \sum_{j=0}^{k-1} \int_{f^k(\eta)}^{f^k(\xi)} | D \log D f(s) | ds = \int \{ 1_U D \log D f(s) | ds \leq \| 1_U \|_q \| D \log D f \|_p,
\]

where \( U = \bigcup_{j=0}^{k-1} f^j([\xi, \eta]) \). Using remark 4.5 we get

\[
\ell(U) = \sum_{j=0}^{k-1} | f^k(\eta) - f^k(\xi) | \leq \sum_{j=0}^{k-1} \frac{\Delta_j^{n+1}(\xi)}{| \Delta_j^n(\xi) |} \leq (1 + e^v)e^{3v} \chi^j.
\]

If we take \( C_5 = [(1 + e^v)e^{3v}]^{\frac{1}{2}} \| D \log D f \|_p \), then we are done. 

Now we introduce a new partition \( D_n = D_n(\xi_0, f) \) of circle. It will be known at section 6, that why we need introduce the new partition \( D_n \). So, we consider a \textit{full marked trajectory} (i.e. the full trajectory of a marked point) \( \xi_i = f^i(\xi_0) \in S^1, i \in \mathbb{Z} \) and pick out of it the sequence of the dynamical convergents \( \xi_{\pm q_n}, n \geq 0 \). It is well known

\[
\xi_{q-1} < ... < \xi_{q_{2m-1}} < \xi_{q_{2m}} < \xi_{q_{2m+1}} < ... < \xi_0 < ... < \xi_{q_{2m}} < \xi_{q_{2m-1}} < \xi_{q_{2m-2}} ... < \xi_{q_0}.
\]

Using the \( n \)-th fundamental interval of partition \( P_n = P_n(\xi_0, f) \), we define the \( n \)-th fundamental intervals of dynamical partition \( D_n \) as the following: \( I_0^n = \Delta_0^n(\xi_{-q_n}) \) and

\[
I_{n-1,n}^n = \Delta_{n-1}^0 \setminus \Delta_0^n(\xi_{-q_n}).
\]

It is well known, that the set of intervals

\[
D_n = \left\{ I_{i-1,n}^n, 0 \leq i < q_n; I_j^n, 0 \leq j < q_n + q_{n-1} \right\}
\]

with mutually disjoint interiors defined as determines a partition of the circle for any \( n \). It is clear that the partition \( D_n \) is a subpartition of \( P_n \) obtained by adding some negative iterates of \( \xi_0 \). The partition \( D_{n+1} \) is a refinement of the partition \( D_n \). As we go to \( D_{n+1} \) the following occurs. Each of intervals \( I_j^n, 0 \leq j < q_n + q_{n-1} \) is partitioned into two intervals belonging to \( D_{n+1} \), one of them \( I_{n+1}^n \) form and second is \( I_{n,n+1}^n \) form. Similarly, each of intervals \( I_{i-1,n}^n, 0 \leq i < q_n \) is partitioned into \( 2k_{n+1} - 1 \) intervals belonging to \( D_{n+1}, k_{n+1} - 1 \) of them \( I_{n,n+1}^n \) form and \( k_{n+1} \) of them \( I_{n+1}^n \) form. Moreover, any two consecutive subintervals of \( I_{i-1,n}^n, 0 \leq i < q_n \) are \( I_{n,n+1}^n \) and \( I_{n+1}^n \) form. Each \( I_{n+1} \) (or \( I_{n,n+1}^n \)) form interval goes to the next \( I_{n+1}^n \) (or \( I_{n,n+1}^n \)) form interval by \( f^{\pm q_n} \).

The transition scheme from dynamical partition \( D_n \) to \( D_{n+1} \) for the case \( i = 0 \).
The partition $D_n$ is a subpartition of $P_n$ in the sense that each interval of the partition $P_n$ consists of entire of intervals of the partition $D_n$. Now we prove some properties of the partition $D_n$.

**Property 4.7.** There exists universal constant $C_6 = C_6(Q, f) \leq 1$ such that any interval of $D_n$ is $C_6$-comparable to the interval of $P_n$ that contains it.

**Proof.** By definitions of dynamical partitions $P_n$ and $D_n$, it is easy to see that between intervals of dynamical partitions $P_n$ and $D_n$ has following relations:

$$
\Delta_j^{n-1} = I^n_j \cup I^{n-1,n}_j, \quad 0 \leq j < q_n \quad \text{and} \quad \Delta^n_i = I^n_{i+q_n}, \quad 0 \leq i < q_{n-1}.
$$

From this relations implies that the intervals $\Delta^n_i$ and $I^n_{i+q_n}$, $0 \leq i < q_{n-1}$ are $1$-comparable. Now we prove that both intervals $I^n_j$ and $I^{n-1,n}_j$ comparable to the $\Delta_j^{n-1}$, $0 \leq j < q_n$. It is clear that

$$
\bigcup_{s=0}^{k_n+1-1} f^{-sq_n}(I^n_j) \subset \Delta_j^{n-1} \subset \bigcup_{s=0}^{k_n+1} f^{-sq_n}(I^n_j).
$$

Considering this and using Denjoy inequalities together with relations (6) we get

$$
1 + e^{-v} \leq \frac{|\Delta_j^{n-1}|}{|I^n_j|} \leq 1 + Q e^{Q_v} \quad \text{and} \quad 1 + \frac{1}{Q e^{Q_v}} \leq \frac{|\Delta_j^{n-1}|}{|I^{n-1,n}_j|} \leq 1 + e^v.
$$

Finally, if we take $C_6 = [1 + Q e^{Q_v}]^{-1}$ then we are done. \qed

Since, the partition $D_n$ is a subpartition of $P_n$ using property 4.2 and property 4.7 we get the following remark.

**Remark 4.8.** Any two consecutive intervals of $D_n$ are $C_2 C_6^2$-comparable.

**Property 4.9.** There exists universal constants $C_7 = C_7(Q, f)$ and $C_8 = C_8(Q, f)$ which is $0 < C_7 < C_8 \leq 1$ such that $C_7 |I^n| \leq |I^{n+1}| \leq C_8 |I^n|$ where $I^{n+1} \in D_{n+1}$, $I^n \in D_n$ and $I^{n+1} \subset I^n$.

**Proof.** First we obtain all this $|I_i^{n+1}|/|I_i^{n,n+1}|$, $0 \leq i < q_{n+1}$ ratios, where $I_i^{n+1}$ and $I_i^{n,n+1}$ are two consecutive intervals of $D_{n+1}$. It is clear that

$$
\bigcup_{s=1}^{k_n+2-1} f^{sq_n+1}(I_i^{n+1}) \subset I_i^{n,n+1} \subset \bigcup_{s=1}^{k_n+2} f^{sq_n+1}(I_i^{n+1}).
$$

Using Denjoy inequalities we get

$$
(Q e^{Q_v})^{-1} |I_i^{n,n+1}| \leq |I_i^{n+1}| \leq e^v |I_i^{n,n+1}|.
$$

If $I^n = I^n_i \in D_n$, $0 \leq i < q_n + q_{n-1}$ then the interval $I^n_i$ is partitioned into two intervals belonging to $D_{n+1}$, one of them $I^{n+1}$ form and second is $I^{n,n+1}$ form. Using (9) we have

$$
\frac{1}{1 + Q e^{Q_v}} \leq \frac{|I^n|}{|I_i^n|} \leq e^v \frac{1}{1 + e^v} \quad \text{and} \quad \frac{1}{1 + e^v} \leq \frac{|I_i^{n+1}|}{|I^n_i|} \leq \frac{Q e^{Q_v}}{1 + Q e^{Q_v}}.
$$

If $I^n = I^{n-1,n}_i \in D_n$, $0 \leq i < q_n$ then the interval $I^{n-1,n}_i$ is partitioned into $2k_{n+1}$-1 intervals belonging to $D_{n+1}$, $k_{n+1} + 1$ of them $I^{n,n+1}$ form and $k_{n+1}$ of them $I^{n+1}$ form.
form. In particular if $k_{n+1} = 1$ then $I^{n+1} = I^{n-1,n}_i$, if $k_{n+1} \geq 2$ then for any two consecutive subintervals $I^{n+1}$, $I^{n,n+1}$ of $I^{n-1,n}_i$ using properties of partition $D_{n+1}$ and Denjoy inequalities we have

\[
(11) \quad |I^{n+1}|(\sum_{s=0}^{k_{n+1}-2} e^{-sv}) + |I^{n,n+1}|(\sum_{s=0}^{k_{n+1}-2} e^{-sv}) \leq |I^{n-1,n}_i| \leq |I^{n+1}|(\sum_{s=0}^{k_{n+1}-2} e^{sv}) + |I^{n,n+1}|(\sum_{s=0}^{k_{n+1}-2} e^{sv})
\]

Considering (9), (10) and (11) we get

\[
(12) \quad \frac{1}{Qe^{Qv}(1 + Qe^{Qv})} \leq \frac{|I^{n+1}|}{|I^{n-1,n}_i|} \leq \frac{e^v}{2 + e^v}
\]

and

\[
(13) \quad \frac{1}{(1 + e^v)Qe^{Qv}} \leq \frac{|I^{n,n+1}|}{|I^{n-1,n}_i|} \leq \frac{Qe^{(Q+1)v}}{1 + e^v + Qe^{(Q+1)v}}.
\]

Denote by $C_7 = \frac{[Qe^{Qv}(1 + Qe^{Qv})]^{-1}}{1}$ and $C_8 = Qe^{Qv}[1 + Qe^{Qv}]^{-1}$. Finally, if $k_{n+1} = 1$ then $C_7|I^n| \leq |I^{n+1}| \leq |I^n|$, if $k_{n+1} \geq 2$ then $C_7|I^n| \leq |I^{n+1}| \leq C_8|I^n|$. \hfill \Box

Therefore, every interval of $D_n$ contains at least two interval of $D_{n+2}$, using this we get following remark.

**Remark 4.10.** Let $I^{n+2} \in D_{n+2}$, $I^n \in D_n$ and $I^{n+2} \subset I^n$. Then the following inequalities holds

\[
(14) \quad C_7^2|I^n| \leq |I^{n+2}| \leq C_8|I^n|.
\]

## 5 Universal bounds to the barycentric coefficients

Let $f$ be a P-homeomorphism with two break points $a$, $b$ which is does not lie on the same orbit and with irrational rotation number of bounded type. Consider partition $D_n = D_n(a,f)$. Denote by $I^n(b)$ the interval of $D_n$ that contains the point $b$. In the following discussion we have to compare different intervals. Let $[\alpha, \beta]$ be an interval in $S^1$ and $\gamma \in [\alpha, \beta]$. The barycentric coefficient of $\gamma$ in $[\alpha, \beta]$ is the ratio

\[
\mathfrak{B}(\gamma; [\alpha, \beta]) = \frac{|\alpha, \gamma|}{|\alpha, \beta|}.
\]

A universal bound for $f$ is a constant that does not depend on $n$ and does not depend on point. Now we show that there exists a subsequence $(n_s)$ of $\mathbb{N}$ such that the barycentric coefficient of the point $b$ in $I^{n_s} = I^{n_s}(b)$ is universal bounded in $(0,1)$.

**Proposition 5.1.** Let $f$ be a P-homeomorphism with two break points $a$, $b$ which is does not lie on the same orbit and with irrational rotation number of bounded type. Then there exists a subsequence $n_s \in \mathbb{N}$ such that for all $n_s$ holds the following inequalities $C_7^2 \leq \mathfrak{B}(b; I^{n_s}) \leq 1 - C_7^2$. 

11
Proof. We argue by contradiction and suppose that there exists a natural number \( n_0 \geq 1 \) such that for all \( n \geq n_0 \) hold

\[
\mathcal{B}(b; I^n) < C_7^2 \quad \text{or} \quad \mathcal{B}(b; I^n) > 1 - C_7^2.
\]

This inequalities ensure that, if the interval \( I^n \) partitioned into at least two intervals belonging to \( D_{n+1} \), then the point \( b \) is always in one of the extremal intervals \( (I_{j}^{n+1}, \text{first} \) or \( I_{k}^{n+1}, \text{last} \) among the \( D_{n+1} \) intervals that is contained \( I^n \). Actually, if \( b \) belongs to none of the two extremal intervals of \( D_{n+1} \) which are contained in \( I^n \), then the point \( b \) is separated from left edge of \( I^n \) by at least \( |I_{j}^{n+1}| \) distance and from right edge of \( I^n \) by at least \( |I_{k}^{n+1}| \) distance. But, using property 4.9, we have gotten be the following inequalities

\[
C_7^2 < C_7 \leq \frac{|I_{j}^{n+1}|}{I^n} \leq \mathcal{B}(b; I^n) \leq 1 - \frac{|I_{k}^{n+1}|}{I^n} < 1 - C_7 \leq 1 - C_7^2.
\]

If the interval \( I^n \) is member of partition \( D_{n+1} \), then it is surely partitioned into at least two intervals belonging to \( D_{n+2} \). In this case the point \( b \) also lies in one of the extremal \( I_{j}^{n+2} \) or \( I_{k}^{n+2} \) intervals among the \( D_{n+2} \) intervals that is contained \( I^n \). Otherwise using remark 4.10 we have gotten be the following inequalities

\[
C_7^2 \leq \frac{|I_{j}^{n+2}|}{I^n} \leq \mathcal{B}(b; I^n) \leq 1 - \frac{|I_{k}^{n+2}|}{I^n} \leq 1 - C_7^2.
\]

So, both inequalities (16) and (17) are contradiction to (15). Continue, the point \( b \) cannot be indefinitely in the first interval of the partition \( D_{n+1} \) which is contained in \( I^n \). That is, for all positive integers \( n' \), there exists \( n \geq n' \) such that \( I^{n+1} \) is not the first intervals of the partition \( D_{n+1} \) that are contained \( I^n \). If not, since the length of intervals of partition \( D_{n} \) tends to zero when \( n \) goes to infinity, the point \( b \) would be arbitrary near the common left extremity of all \( I^n \), for all \( n \geq n' \). Therefore, \( b \) would equal this point which is the left end point of \( I^{n'} \) and hence an iterate of the point \( a \), which contradicts the hypothesis that the break points \( a \) and \( b \) of \( f \) have disjoint orbits. In a like manner, we can show that the point \( b \) cannot be indefinitely in the last interval of the partition \( D_{n+1} \) which is contained in \( I^n \).

So, there exists a subsequence on \( n_s \in \mathbb{N} \), such that the point \( b \) is in the first of the \( D_{n+1} \) intervals that are contained in \( I_{n_s} \) and in the last of the \( D_{n+2} \) intervals that are contained \( I^{n+1}_{n_s} \). In this situation, the point \( b \) is separated from left edge of \( I_{n_s} \) by at least \( |I_{j}^{n+2}| \) distance and from right edge of \( I_{n_s} \) by at least \( |I_{k}^{n+2}| \) distance. By remark 4.10 we get

\[
C_7^2 \leq \frac{|I_{j}^{n+2}|}{I^{n_s}} \leq \mathcal{B}(b; I^{n_s}) \leq 1 - \frac{|I_{k}^{n+2}|}{I^{n_s}} \leq 1 - C_7^2.
\]

\[\square\]

6 Universal bounds to the consecutive break points of \( f^{q_m} \)

In this section we consider a \( P \)-homeomorphism \( f \) with two break points \( a, b \) which is does not lie on the same orbit. It is clear that the map \( f^{q_m} \) has for break points \( a_j = f^{-j}(a) \) and \( b_j = f^{-j}(b) \) with \( j = 0, 1, ..., q_m - 1 \). If we consider \( m \)-th dynamical partition
\[ D_m = D_m(a, f) \] of the break point \( a \), then the break points \( a_j, 0 \leq j < q_m \) of \( f^{q_m} \) lies on the endpoints of intervals \( I_{j-m}^{m-1} \) of \( D_m \) for every natural number \( m \). Eventually, we must study behavior of only the break points \( b_j, 0 \leq j < q_m \) of \( f^{q_m} \). Thats why we have introduced the partition \( D_m \). Now we consider \( m \)-th dynamical partition \( P_m = P_m(a, f) \) of the break point \( a \). The break points \( a_j, 0 \leq j < q_m \) of \( f^{q_m} \) can not be the endpoints of intervals of \( P_m \) and the atoms of \( P_m \) containing the points \( a_j \) in its interior denoted dy \( \Delta^m(a_j) \). Next theorem show that if for some \( m = n_k \geq n_0 \) the barycentric coefficient of the point \( b \) in \( D^m \) is universally bounded then for such \( m \) there exists the universal bounds between any two consecutive break points of \( f^{q_m} \) i.e. any two consecutive break points of \( f^{q_m} \) can not be very close to each other. More precisely:

**Theorem 6.1.** Let \( f \) be a \( P \)-homeomorphism with break points \( a \) and \( b \). Suppose for some \( m \geq n_0 \) the barycentric coefficient of the point \( b \) in \( D^m \) is universally bounded with constants \( 0 < C^2 \leq 1 - C^2_7 < 1 \), i.e. \( C^2_7 \leq B(b; D^m) \leq 1 - C^2_7 \). Then there exists a natural number \( l = l(f) \) such that:

(i) for all \( 0 \leq j < q_m \) the interior of the atom \( \Delta^{m+1}(a_j) \) contains only one break point of \( f^{q_m} \); the point \( a_j \), hence \( D f^{q_m} \) is continuous on each component of \( \Delta^{m+1}(a_j) \setminus \{a_j\} \);

(ii) for all \( 0 \leq j < q_m \) the barycentric coefficients of the points \( a_j \) in \( \Delta^{m+1}(a_j) \) is universally bounded with constants \( 0 < C_6 \leq 1 - C_6 < 1 \), i.e.

\[ C_6 \leq B(a_j; \Delta^{m+1}(a_j)) \leq 1 - C_6. \]

**Proof.** To prove first the statement of this theorem first we find consecutive break points of \( f^{q_m} \) and then we estimate distance between consecutive break points of \( f^{q_m} \). W.l.o.g. choose \( m \) to be odd. Then \( m \)-th and \( m - 1 \)-th generator intervals are the following form:

\[ \Delta^m_0(a) = [f^{q_m}(a), a] \] and \( \Delta^{m-1}_0(a) = [a, f^{q_m-1}(a)] \). By definition of dynamical partition \( D_m = D_m(a, f) \) the interval \( I^m(b) \) is either an interval \( I^m_{k0} = f^{k0}[a, f^{-q_m}(a)] \) for some \( 0 \leq k0 < q_m \) or an interval \( I^{m-1,1}_{j0} = f^{j0}[f^{-q_m}(a), f^{q_m-1}(a)] \) for some \( 0 \leq j0 < q_m \). First we suppose \( I^m(b) = f^{k0}[a, f^{-q_m}(a)] \). It is easy to see that if \( 0 \leq k0 < q_m \) then the consecutive break points of \( f^{q_m} \) are following:

\[ \{ f^{-k_0+i}(b), f^{-q_m+i}(a) \in \Delta^m_{-1}, \quad 1 \leq i \leq k0; \}

\[ f^{-q_m+i}(a), f^{-q_m+i-k_0}(b) \in \Delta^m_{-1}, \quad k_0 < i < q_m. \]

From relations (19) implies that if \( 1 \leq i \leq k0 \) then between break point \( f^{-q_m+i}(a) \) of \( f^{q_m} \) and right endpoint \( f^{q_m-1+i}(a) \) of \( \Delta^{m-1} \) can not lies another break points of \( f^{q_m} \). Similarly, if \( k0 < i \leq q_m \) then between break point \( f^{-q_m+i}(a) \) of \( f^{q_m} \) and left endpoint \( f^i(a) \) of \( \Delta^{m-1} \) can not lies another break points of \( f^{q_m} \). By definition of dynamical partition \( D_m \) we have \( I^{m-1,1}_{i} = f^i[f^{-q_m}(a), f^{q_m-1}(a)] \) and \( I^m_i = f^i[a, f^{-q_m}(a)] \). Using inequalities (17) we get \( |I^{m-1,1}_{i}| \geq C_6|\Delta^{m-1}_i|, \quad 1 \leq i \leq k0 \) and \( |I^m_i| \geq (1 + e^v)^{-1}|\Delta^{m-1}_i|, \quad k0 < i \leq q_m \). Now we estimate the distance between this consecutive break points of \( f^{q_m} \). By assumption of theorem 6.1

\[ C_7^2 \leq B(b; D^m) = \frac{|[f^{k0}(a), b]|}{|[f^{k0}(a), f^{-q_m+k0}(a)]|} \leq 1 - C_7^2, \]

from this imply

\[ C_7^2 \leq 1 - B(b; D^m) = \frac{|[b, f^{-q_m+k0}(a)]|}{|[f^{k0}(a), f^{-q_m+k0}(a)]|} \leq 1 - C_7^2. \]

13
So, by the Denjoy estimate, we get

\[
e^{-2v} C^2_7 \leq \frac{|f-q_m[f^{k_0}(a), b]|}{|f-q_m[f^{k_0}(a), f^{q_m+k_0}(a)]|} \leq e^{2v}(1 - C^2_7).
\]

Using Finzi inequality (generalized Finzi inequality) to the inequalities (21) and (22) we can show that the distance between consecutive break points \( f^{-k_0+i}(b) \) and \( f^{-q_m+i}(a) \), \( 1 \leq i \leq k_0 \) of \( f^{q_m} \) greater then \( C^2_7 e^{-v}[f^{i}[a, f^{-q_m}(a)]] = C^2_7 e^{-v} |I_i^{m}| \). Similarly, the distance between consecutive break points \( f^{q_m+i}(a) \) and \( f^{-q_m+i-k_0}(b) \), \( k_0 < i \leq q_m \) of \( f^{q_m} \) greater then \( C^2_7 e^{-2v}[f^{-i}[f^{q_m}(a), f^{-2q_m}(a)] = C^2_7 e^{-2v} |f^{-q_m}(I_i^{m})| \). Using inequalities (7) together with Denjoy inequalities it is easy to see that the distance between consecutive break points \( f^{-k_0+i}(b) \) and \( f^{-q_m+i}(a) \), \( 1 \leq i \leq k_0 \) of \( f^{q_m} \) greater then \( C^2_7 C_9 e^{-v} |\Delta_i^{m-1}| \) and the distance between consecutive break points \( f^{-q_m+i}(a) \) and \( f^{-q_m+i-k_0}(b) \), \( k_0 < i \leq q_m \) of \( f^{q_m} \) greater then \( C^2_7 C_9 e^{-3v} |\Delta_i^{m-1}| \).

Now we consider the case \( q_m \leq k_0 < q_m + q_m - 1 \). In this case the consecutive break points of \( f^{q_m} \) are following:

\[
\begin{align*}
& f^{-(k_0-q_m+i)}(b), f^{-q_m+i}(a) \in \Delta_i^{m} \cup \Delta_i^{m-1}, \quad 1 \leq i \leq k_0 - q_m; \\
& f^{k_0+i}(b), f^{-q_m+i}(a) \in \Delta_i^{m-1}, \quad k_0 - q_m < i \leq q_m.
\end{align*}
\]

The same manner as above we can show that the distance between of the break points \( f^{-(k_0-q_m+i)}(b) \) and \( f^{-q_m+i}(a) \), \( 1 \leq i \leq k_0 \) of \( f^{q_m} \) greater then \( C^2_7 + e^{-v} C_9 e^{-v} |\Delta_i^{m-1}| \) and the distance between of the break points \( f^{-k_0+i}(b) \) and \( f^{-q_m+i}(a) \), \( k_0 - q_m < i \leq q_m \) of \( f^{q_m} \) greater then \( C^2_7 C_9 e^{-3v} |\Delta_i^{m-1}| \). Moreover, for all \( 1 \leq i \leq q_m \) between break point \( f^{-q_m+i}(a) \) of \( f^{q_m} \) and right endpoint \( f^{q_m+1}(a) \) of \( \Delta_i^{m-1} \) can not lies another break points of \( f^{q_m} \) and \( |I_i^{m-1,m}| \geq C_6 |\Delta_i^{m-1}| \) for all \( 1 \leq i \leq q_m \).

Now we suppose \( I_i^{m}(b) = f^{j_0}[f^{-q_m}(a), f^{q_m-1}(a)] \) for some \( 0 \leq j_0 < q_m \). In this case to find the consecutive break points of \( f^{q_m} \) we consider two cases: \( b \in f^{j_0}[f^{-q_m}(a), f^{q_m+1}(a)] \subset \bigcup u \in f^{j_0}[f^{-q_m+1}(a), f^{q_m-1}(a)] \subset \bigcup m \). If \( b \in f^{j_0}[f^{-q_m}(a), f^{q_m+1}(a)] \) then consecutive break points of \( f^{q_m} \) are following:

\[
\begin{align*}
& f^{-q_m+i}(a), f^{-j_0+i}(b) \in \Delta_i^{m-1}, \quad 1 \leq i \leq j_0; \\
& f^{-q_m+i}(a), f^{-q_m+i-j_0}(b) \in \Delta_i^{m-1}, \quad j_0 < i \leq q_m.
\end{align*}
\]

It is clear that, for all \( 1 \leq i \leq q_m \) between break point \( f^{-q_m+i}(a) \) of \( f^{q_m} \) and left endpoint \( f^{j_0}(a) \) of \( \Delta_i^{m-1} \) can not lies another break points of \( f^{q_m} \) and \( |I_i^{m}| \geq (1 + e^{v})^{-1} |\Delta_i^{m-1}| \).

Now we estimate the distance between consecutive break points of \( f^{q_m} \). By assumption of theorem (6.1) the barycentric coefficient of the point \( b \) in \( I_i^{m} \) is universally bounded i.e.

\[
C^2_7 \leq \mathfrak{B}(b; I_i^{m}) = \frac{|f^{-q_m+j_0}(a), b|}{|f^{-q_m+j_0-2q_m}(a), f^{q_m+j_0}(a)|} \leq 1 - C^2_7.
\]

By above notation \( I_0^{m-1,m} = [f^{-q_m+j_0}(a), f^{q_m-1+j_0}(a)] \). Using inequalities (7) and (25) together with Finzi inequality we get \( C^2_7 e^{-v} (1 + e^{v})^{-1} \geq |f^{-q_m+i}(a), f^{-j_0+i}(b)| \) for all \( 1 \leq i \leq j_0 \). The same manner as above we can show that \( C^2_7 (1 + e^{v})^{-1} e^{-v} |\Delta_i^{m-1}| \leq |f^{-q_m+i}(a), f^{-j_0+i}(b)| \leq |f^{-q_m+i}(a), f^{-j_0+i}(b)| \) for all \( 1 \leq i \leq q_m \).

Now we consider the case \( b \in f^{j_0}[f^{q_m+1}(a), f^{q_m-1}(a)] \subset \bigcup m \). In this case if \( 0 \leq j_0 < q_m - q_m - 1 \) then consecutive break points of \( f^{q_m} \) are following:

\[
\begin{align*}
& f^{-j_0-q_m+i}(b), f^{-q_m+i}(a) \in \Delta_i^{m-1} \cup \Delta_i^{m}, \quad 1 \leq i \leq j_0 + q_m - 1; \\
& f^{-j_0-q_m+i}(b), f^{-q_m+i}(a) \in \Delta_i^{m-1}, \quad j_0 + q_m - 1 < i \leq q_m.
\end{align*}
\]

14
Similarly we can show that $C^2(1 + e^v)^{-1}e^{-3v}|\Delta_i^m| \leq |f-i-q_m+i(b), f-q_m+i(a)|$ for all $1 \leq i \leq j_0 + q_{m-1}$ and $C^2(1 + e^v)^{-1}e^{-4v}|\Delta_i^m| \leq |f-i-q_m-1+i(b), f-q_m-1+i(a)|$ for all $j_0 + q_{m-1} < i \leq q_m$. If $q_m - q_{m-1} < j_0 < q_m$ then consecutive break points of $f^m$ are following:

\begin{align}
\begin{cases}
f-q_m+i(a), f-j_0+i(b) \in \Delta_i^m, \\
f-q_m+i(a), f-q_m-j_0+i(b) \in \Delta_i^m \cup \Delta_{q_{m-1}-q_m+i},
\end{cases}
\end{align}

1 \leq i \leq j_0; \quad j_0 < i \leq q_m.

In this case $C^2(1 + e^v)^{-1}e^{-v}|\Delta_i^m| \leq |f-q_m+i(a), f-j_0+i(b)|$ for all $1 \leq i \leq j_0$ and $C^2(1 + e^v)^{-1}e^{-4v}|\Delta_i^m| \leq |f-q_m+i(a), f-q_m-1+i(b)|$ for all $j_0 < i \leq q_m$. Using above concepts it is easy to see that distance between the consecutive break points of $f^m$ greater then $\min\{C^2(1 + e^v)^{-1}e^{-4v}, C^2C_2e^{-3v}\} |\Delta_i^m|$ for all $0 \leq i \leq q_m$. By remark 4.5 there exist such $l \in \mathbb{N}$ that hold this $|\Delta^{m+l}(a_j)| \leq (1 + e^v)e^{3v}\lambda^l|\Delta^m(a_j)|$ inequality. If we take a natural number $l$ such that $2(1 + e^v)e^{3v}\lambda^l < \min\{C^2(1 + e^v)^{-1}e^{-4v}, C^2C_2e^{-3v}\}$ then the interior of the atom $\Delta^{m+l}(a_j)$ contains only one break point of $f^m$, because distance between the nearest break points of $f^m$ greater then $|\Delta^{m+l}(a_j)|$.

Now we prove the second assertion of theorem 6.1. Using property of dynamical partition $P_m$ for every $0 < j \leq q_m$ we can written explicit form of the intervals $\Delta^{m+l}(a_j)$ as the following form:

- $\Delta^{m+l}(a_j) = [f^m+i-j(a), f^m+i+q_m+i-1-j(a)]$, if $l$ is even,
- $\Delta^{m+l}(a_j) = [f^m+i+q_m+i-1-j(a), f^m+i-j(a)]$, if $l$ is odd.

If $l$ is even then the barycentric coefficient of point $a_j$ in $\Delta^{m+l}(a_j)$ is equal to the following ratio:

$$
\mathcal{B}(a_j; \Delta^{m+l}(a_j)) = \frac{||f^m+i-j(a), a_j||}{||f^m+i-j(a), f^m+i+q_m+i-1-j(a)||} = \frac{||f^m+i-j(a), f^{-j}(a)||}{||f^m+i-j(a), f^m+i+q_m+i-1-j(a)||}.
$$

In the case $l$ is odd then the barycentric coefficient of point $a_j$ is equal to the following ratio:

$$
\mathcal{B}(a_j; \Delta^{m+l}(a_j)) = \frac{||f^m+i+q_m+i-1-j(a), a_j||}{||f^m+i+q_m+i-1-j(a), f^m+i-j(a)||} = \frac{||f^m+i+q_m+i-1-j(a), f^{-j}(a)||}{||f^m+i+q_m+i-1-j(a), f^m+i-j(a)||}.
$$

Let us take change of variable $z = f^m+i-j(a)$, then

$$
\mathcal{B}(f^{-q_m+l}(z); \Delta^{m+l}(f^{-q_m+l}(z))) = \frac{||z, f^{-q_m+l}(z)||}{||z, f^m+i-1-j(z)||} = \frac{|r^l(z)|}{|\Delta^{n+l-1}(z)|},
$$

if $l$ is even and if $l$ is odd, then

$$
\mathcal{B}(f^{-q_m+l}(z); \Delta^{m+l}(f^{-q_m+l}(z))) = \frac{|r^{n+l-1,n+l}(z)|}{|\Delta^{n+l-1}(z)|}.
$$

Using inequalities 7 it is easy to see that the following inequalities hold for both cases of $l$

$$
C_6 \leq \mathcal{B}(f^{-q_m+l}(z); \Delta^{m+l}(f^{-q_m+l}(z))) \leq 1 - C_6.
$$

\[\square\]
7 Estimates for differences of \( \log D f^{q_n} \)

Let \( f_i, i = 1, 2 \) be circle homeomorphisms with two break points \( a_i, b_i \) satisfying the conditions (1) – (5) of theorem 6.1.5. We introduce the following function on the circle

\[
F_n(x) = \frac{D f_2^{q_n}(h(x))}{D f_1^{q_n}(x)} = \frac{D f_2(h(x)) \cdot D f_2(f_2(h(x))) \cdots D f_2(f_2^{q_n-1}(h(x)))}{D f_1(x) \cdot D f_1(f_1(x)) \cdots D f_1(f_1^{q_n-1}(x))}.
\]

The map \( F_n \) has for jump points (i.e. the map \( F_n \) has jump) \( a_k^1 = f_1^{-k}(a_1) \), \( b_k^1 = f_1^{-k}(b_1) \) and \( a_k^2 = h^{-1}(f_2^{-k}(a_2)) \), \( b_k^2 = h^{-1}(f_2^{-k}(b_2)) \) with \( 0 \leq k < q_n \). To prove the theorem 1.5 we will consider the following two cases:

either \( \mu_1[a_1, b_1] = \mu_2[a_2, b_2] \) and or \( \mu_1[a_1, b_1] \neq \mu_2[a_2, b_2] \),

where \( \mu_i \) is an invariant probability measure of \( f_i, i = 1, 2 \). Consider first the case \( \mu_1[a_1, b_1] = \mu_2[a_2, b_2] \). Since conjugation map \( h \) is unique up to additive constant we choose \( h \) such that \( h(a_1) = a_2 \), then by assumption \( \mu_1[a_1, b_1] = \mu_2[a_2, b_2] \) implies that \( h(b_1) = b_2 \). Using this we get \( f_2(j_1^{-k}(a_1)) = f_2^{-k}(a_2) \) and \( h(j_1^{-k}(b_1)) = f_2^{-k}(b_2) \) for all \( 0 \leq k < q_n \). It is easy to see the jump points of \( F_n \) are \( a_k^1 = f_1^{-k}(a_1) \), \( b_k^1 = f_1^{-k}(b_1) \) \( 0 \leq k < q_n \) i.e. the jump points of \( F_n \) composed only of the break points of \( f_1^{q_n} \). The jumps of \( F_n \) at these points are following:

\[
\sigma_{F_n}(a_k^1) = \frac{\sigma_{f_2^{q_n}}(h(a_k^1))}{\sigma_{f_1^{q_n}}(a_k^1)} = \frac{\sigma_{f_2}(h(a_1))}{\sigma_{f_1}(a_1)} = \frac{\sigma_{f_2}(a_2)}{\sigma_{f_1}(a_1)},
\]

\[
\sigma_{F_n}(b_k^1) = \frac{\sigma_{f_2^{q_n}}(h(b_k^1))}{\sigma_{f_1^{q_n}}(b_k^1)} = \frac{\sigma_{f_2}(h(b_1))}{\sigma_{f_1}(b_1)} = \frac{\sigma_{f_2}(b_2)}{\sigma_{f_1}(b_1)},
\]

and by assumption theorem 1.5 implies that \( \sigma_{F_n}(a_k^1) \neq \sigma_{F_n}(b_k^1) \), and \( \sigma_{F_n}(a_k^1) \times \sigma_{F_n}(b_k^1) = 1 \) for all \( 0 \leq k < q_n \). Denote by \( \delta_0 = |\log \sigma_{f_2}(a_2) - \log \sigma_{f_1}(a_1)| > 0 \). Apply theorem 6.1 to the function \( f_1 \) we can find subsequence \( n_s = n_s(f_1) \in \mathbb{N} \) such that break points of \( f_1^{n_s} \) far from each other. Let \( l = l(f_1) \) be the natural number which is defined in theorem 6.1. The following proposition is formulated for a suitable subsequence \( n_s \in \mathbb{N} \) and natural number \( l \).

**Proposition 7.1.** Assume the homeomorphisms \( f_i, i = 1, 2 \) satisfy the conditions of Theorem 6.1. Then there exists a natural number \( l_0 = l_0(f_1, f_2) \) such that \( l_0 \geq l \) and for all \( 0 \leq k < q_n \), on one of the two connected components of \( \Delta^{n_s+l_0}_{f_1} \{a_k^1\} \), the following inequality holds:

\[
|\log D f_2^{q_n}(h(x)) - \log D f_1^{q_n}(x)| \geq \delta_0.
\]

**Proof.** Let us take a positive integer \( T \) such that \( C_5(f_1) \lambda_1^{T/q} + C_5(f_2) \lambda_2^{T/q} \leq \frac{\delta_0}{2} \), where \( C_5(f_i) \) and \( \lambda_i, i = 1, 2 \) appropriate constants of \( f_i, i = 1, 2 \) which are satisfies lemma 4.6. Denote by \( l_0 = \max\{T, l\} \). According to theorem 6.1 (i), the interior of the atom \( \Delta^{n_s+l_0}_{f_1} \{a_k^1\} \) contains only one jump point of \( F_n \) the point \( a_k^1 \) hence, \( F_n \) is continuous on each component \( \Delta^{n_s+l_0}_{f_1} \{a_k^1\} \) \( \{a_k^1\} \).

If \( |\log F_{n_s}(x)| \geq \delta_0 \) on the left component of \( \Delta^{n_s+l_0}_{f_1} \{a_k^1\} \) \( \{a_k^1\} \), then we are done. If not, there exists at least one point \( x \) on the left component of \( \Delta^{n_s+l_0}_{f_1} \{a_k^1\} \) \( \{a_k^1\} \) such that

\[
|\log F_{n_s}(x)| < \delta_0.
\]
Now, for any $y$ in the left component of $\Delta_{f_1}^{n_s+l_0}(a^1_k) \setminus \{a^1_k\}$ we have

$$|\log F_{n_s}(y)| \leq |\log F_{n_s}(x)| + |\log F_{n_s}(x) - \log F_{n_s}(y)| < \delta_0 + \frac{\delta_0}{2},$$

by lemma 4.6 with $k = q_{n_s}$ and $l = l_0$. Then in particular $|\log F_{n_s}(a^1_k - 0)| < 3\delta_0/2$ and

$$|\log F_{n_s}(a^1_k + 0)| = |\log(\sigma_{F_{n_s}}(a^1_k)F_{n_s}(a^1_k - 0))| \geq \frac{3\delta_0}{2}.$$

Finally, for $y$ in the right component of $\Delta_{f_1}^{n_s+l_0}(a^1_k) \setminus \{a^1_k\}$, we have

$$|\log F_{n_s}(y)| \geq |\log(F_{n_s}(a^1_k + 0))| - |\log(F_{n_s}(a^1_k + 0)) - \log F_{n_s}(y)| \geq \delta_0.$$  

Hence on the right component of $\Delta_{f_1}^{n_s+l_0}(a^1_k) \setminus \{a^1_k\}$, we have $|\log F_{n_s}(y)| \geq \delta_0$. \hfill $\square$

Now we consider second the case $\mu_1[a_1, b_1] \neq \mu_2[a_2, b_2]$. Without loss of generality, we can suppose that $\mu_1[a_1, b_1] < \mu_2[a_2, b_2]$ the opposite case can be handled similarly. We choose $h$ such that $h(a_1) = a_2$. Using this together with above inequality we get $h(b_1) < b_2$. Since $h$ is continuous and strictly increasing function, then there exist a unique point $c_1$ such that $b_1 < c_1$ and $h(c_1) = b_2$. Using this it is easy to see the map $F_n$ has for jump points $a^1_k = f^{-1}_1(a_1)$, $b^1_k = f^{-1}_1(b_1)$, $c^1_k = f^{-1}_1(c_1)$. In this case the jump points of $F_n$ obtained by adding some negative iterates of $c_1$ to the break points of $f_1^{n_s}$. Moreover, the break points $c^1_k$ goes to the points $f^{-1}_2(b_2)$ by $h$. i.e. $h(f^{-1}_1(c_1)) = f^{-1}_2(b_2)$. The jumps of $F_n$ at these points are:

$$\sigma_{F_n}(a^1_k) = \frac{\sigma_{f_1^{n_s}}(h(a^1_k))}{\sigma_{f_1^{n_s}}(a^1_k)} = \frac{\sigma_{f_2}(h(a_1))}{\sigma_{f_1}(a_1)} = \frac{\sigma_{f_2}(a_2)}{\sigma_{f_1}(a_1)},$$

$$\sigma_{F_n}(b^1_k) = (\sigma_{f_1^{n_s}}(b^1_k))^{-1} = (\sigma_{f_1}(b_1))^{-1}, \quad \sigma_{F_n}(c^1_k) = \sigma_{f_2^{n_s}}(b^2_k) = \sigma_{f_2}(b_2)$$

and by assumption theorem 15 implies that $\sigma_{F_n}(a^1_k) \times \sigma_{F_n}(b^1_k) \neq 1$, $\sigma_{F_n}(a^1_k) \times \sigma_{F_n}(c^1_k) \neq 1$, $\sigma_{F_n}(a^1_k) \times \sigma_{F_n}(b^1_k) \times \sigma_{F_n}(c^1_k) = 1$, for all $0 \leq k, s, t < q_{n_s}$. Let $n_s = n_s(f_1) \in \mathbb{N}$ the subsequence such that break points of $f_1^{n_s}$ far from each other. The main changes in this case is $F_{n_s}$ may not be continuous on one of the two connected components of $\Delta_{f_1}^{n_s+l_0}(a^1_k) \setminus \{a^1_k\}$ and one passes from every continuity interval of $F_{n_s}$ to the next one by multiplying $F_{n_s}$ by the jump at the common extremity of these two consecutive intervals. Denote by $3\delta_1 = \min\{|\log \sigma_{f_2}(a_2) - \log \sigma_{f_1}(a_1) + \log \sigma_{f_2}(b_2)|, |\log \sigma_{f_2}(a_2) - \log \sigma_{f_1}(a_1)|\}$. It is clear that $\delta_1$ is positive. Next, we will show that for any $0 \leq k < q_{n_s}$ there exists a subinterval in $\Delta_{f_1}^{n_s+l_0}(a^1_k)$ such that on this subinterval $|\log F_{n_s}|$ is $\delta_1$- far from 0.

**Proposition 7.2.** Assume the homeomorphisms $f_i, i = 1, 2$ satisfy the conditions of Theorem 15. Let $l_0$ be the constant which is defined in proposition 7.1.

(i) Then for any $0 \leq k < q_{n_s}$ there exists a subinterval $I^s_k \subset \Delta_{f_1}^{n_s+l_0}(a^1_k)$ such that, on the interval $I^s_k$ the following inequality holds:

$$|\log D_{f_2^{n_s}}(h(x)) - \log D_{f_1^{n_s}}(x)| \geq \delta_1.$$ 

(ii) There exists a universal constant $C_9 = C_9(f_1, f_2) > 0$ such that the intervals $I^s_k$ and $\Delta_{f_1}^{n_s+l_0}(a^1_k)$ are $C_9$- comparable.
Proof. Let us take a natural number $\tilde{t}$ such that $C_5(f_1)\lambda_1^{\tilde{t}/q} + C_5(f_2)\lambda_2^{\tilde{t}/q} \leq \frac{2}{9}$, where $C_5(f_i)$ and $\lambda_i$, $i = 1, 2$ appropriate constants of $f_i$, $i = 1, 2$ which are satisfies universal estimates. Denote by $l_1 = \max\{\tilde{t}, l_0\}$. Let $\Delta_{f_1}^{n \rightarrow l_1}(a_k^1)$ be the atoms of $P_{n \rightarrow l_1}(a, f_1)$ containing the points $a_k^1$ in its interior. It is clear that $\Delta_{f_1}^{n \rightarrow l_1}(a_k^1) \subset \Delta_{f_1}^{n \rightarrow l_0}(a_k^1)$ and by remark 14.5 there exists a constant $\kappa_1 = \kappa(f_1) > 0$ such that $|\Delta_{f_1}^{n \rightarrow l_1}(a_k^1)| \geq e^{-3\sigma_1} \kappa_1^{1-l_0} |\Delta_{f_1}^{n \rightarrow l_0}(a_k^1)|$ for all $0 \leq k < q_n$, where $\sigma_1 = \arg\max_{\ell} \log D_f \ell$. Now we will construct an intervals $I_{k}^{\alpha}$ which is comparable with $\Delta_{f_1}^{n \rightarrow l_1}(a_k^1)$. For this we define following sets $B_{n \alpha} = \{c_1^\alpha, c_2^\alpha = f_1^{-k}(c_1), 0 \leq k < q_n\}$, $G_{n \alpha} = \{k : 0 \leq k < q_n, \Delta_{f_1}^{n \rightarrow l_1}(a_k^1) \cap B_{n \alpha} \neq \emptyset\}$ and $\overline{G}_{n \alpha}$ is complement of $G_{n \alpha}$. Let for definiteness $G_{n \alpha}$ is non empty. If $k \in \overline{G}_{n \alpha}$ then the atom $\Delta_{f_1}^{n \rightarrow l_1}(a_k^1)$ contains only one jump point of $F_{n \alpha}$ the point $a_k^1$ hence $F_{n \alpha}$ is continuous on each component $\Delta_{f_1}^{n \rightarrow l_1}(a_k^1) \setminus \{a_k^1\}$. Since $\delta_1 \leq \delta_0$ and $l_0 \leq l_1$, according to proposition 6.1 (ii) there exists $C_6 = C_6(f_1)$ such that

$$\left|I_{k}^{\alpha}\right| \geq C_6|\Delta_{f_1}^{n \rightarrow l_1}(a_k^1)| \geq C_6\kappa_1^{1-l_0}e^{-3\sigma_1}|\Delta_{f_1}^{n \rightarrow l_0}(a_k^1)|$$

Now, let $k \in G_{n \alpha}$. Then the atom $\Delta_{f_1}^{n \rightarrow l_1}(a_k^1)$ contains two jump points of $F_{n \alpha}$ the point $a_k^1$ and an element $c_1^\alpha$ of $B_{n \alpha}$. Hence $F_{n \alpha}$ is continuous on each component $\Delta_{f_1}^{n \rightarrow l_1}(a_k^1) \setminus \{a_k^1, c_1^\alpha\} = L_k^{\alpha} \cup M_k^{\alpha} \cup R_k^{\alpha}$. Let for definiteness the point $c_1^\alpha$ lie on the left hand side of the point $a_k^1$. Then $R_k^{\alpha}$ is right component of $\Delta_{f_1}^{n \rightarrow l_1}(a_k^1) \setminus \{a_k^1\}$ using above arguments the intervals $R_k^{\alpha}$ and $\Delta_{f_1}^{n \rightarrow l_0}(a_k^1)$ are $C_6\kappa_1^{1-l_0}e^{-3\sigma_1}$ comparable. If on the interval $R_k^{\alpha}$ holds this $|\log F_{n \alpha}(x)| < \delta_1$ inequality, then we take $I_{k}^{\alpha} = R_k^{\alpha}$ and desired result follows obviously. If not, there exists at least one point $z_0$ in $R_k^{\alpha}$ such that $|\log F_{n \alpha}(z_0)| \leq \delta_1$. It is easy to see for any $x \in L_k^{\alpha}$ and $y \in M_k^{\alpha}$ holds these inequalities:

$$|\log F_{n \alpha}(x) - \log F_{n \alpha}(z_0)| \geq$$

$$\geq |\log \sigma F_{n \alpha}(c_1^\alpha)\sigma F_{n \alpha}(a_k) - 3(C_5(f_1)\lambda_1^{l_1/q} + C_5(f_2)\lambda_2^{l_1/q})|,$n

(32)

$$|\log F_{n \alpha}(y) - \log F_{n \alpha}(z_0)| \geq |\log \sigma F_{n \alpha}(a_k) - 2(C_5(f_1)\lambda_1^{l_1/q} + C_5(f_2)\lambda_2^{l_1/q})|.$$

(33)

It is clear that the intervals $L_k^{\alpha} \cup M_k^{\alpha}$ and $R_k^{\alpha}$ are two consecutive intervals of $D_{n \alpha + l_1}$. By remark 14.8 at least one of the intervals $L_k^{\alpha}$ and $M_k^{\alpha}$ are $C_2C_6^2/2$ - comparable with $R_k^{\alpha}$. First, let the interval $L_k^{\alpha}$ be $C_2C_6^2/2$ - comparable with $R_k^{\alpha}$. Using (32) for any point $x \in L_k^{\alpha}$ we get

$$|\log F_{n \alpha}(x)| \geq |\log \sigma F_{n \alpha}(c_1^\alpha)\sigma F_{n \alpha}(a_k) - 3(C_5(f_1)\lambda_1^{l_1/q} + C_5(f_2)\lambda_2^{l_1/q}) - \delta_1.$$n

(34)

Of the determine integer $l_1$ and positive $\delta_1$, implies $|\log \sigma F_{n \alpha}(c_1^\alpha)\sigma F_{n \alpha}(a_k)| = |\log f_2(a_2) - \log f_1(a_1) + \log f_2(b_2)| \geq 3\delta_1$ and $3(C_5(f_1)\lambda_1^{l_1/q} + C_5(f_2)\lambda_2^{l_1/q}) \leq \delta_1$. The last two equations together with (34) imply $|\log F_{n \alpha}(x)| \geq \delta_1$ for any point $x \in L_k^{\alpha}$. If we take $I_{k}^{\alpha} = L_k^{\alpha}$ then the intervals $I_{k}^{\alpha}$ and $\Delta_{f_1}^{n \rightarrow l_1}(a_k^1)$ are $C_9 = 2^{-1}C_2C_6^2\kappa_1^{1-l_0}e^{-3\sigma_1}$ comparable. Secondly, let the interval $M_k^{\alpha}$ be $C_2C_6^2/2$ - comparable with $R_k^{\alpha}$. Similarly we can show that $|\log F_{n \alpha}(y)| \geq \frac{4\delta_1}{3}$ for any point $y \in M_k^{\alpha}$. In this case, if we take $I_{k}^{\alpha} = M_k^{\alpha}$ again the intervals $I_{k}^{\alpha}$ and $\Delta_{f_1}^{n \rightarrow l_1}(a_k^1)$ are $C_9$ - comparable. \qed
8 Proof of main theorem

Proof. Assume that the homeomorphisms $f_i, i = 1, 2$ satisfy the conditions of theorem 1.5. By lemma 3.1 the conjugation map $h$ between $f_1$ and $f_2$ is either absolutely continuous or singular function. Suppose $h$ is absolutely continuous function. Then by theorem 3.2 for all $\epsilon > 0$ there exists such natural number $n_0$ such that for $n > n_0$ holds this inequality for any $\delta > 0$. First, we assume $\mu_1[a_1, b_1] = \mu_2[a_2, b_2]$ where $\mu_i, i = 1, 2$ are invariant measures of $f_i$. In this case jump points of $F_n$ appear break points of $f_i^{q_{n_0}}$. Apply theorem 6.1 to the function $f_1$ we can find sufficiently large $n_s = n_s(f_1) > n_0$ such that the break points of $f_i^{q_{n_0}}$ far from each other. Let $\delta_0 = \frac{\epsilon}{3!}\log|\sigma_f(a_2) - \sigma_f(a_1)|$ and $l_0 = l_0(f_1, f_2)$ be the natural number which is defined in proposition 7.1. By proposition 7.1 on one of the two connected components of $\Delta_{f_1}^{q_{n_0}}(a_k) \setminus \{a_k\}$, we have $|\log Df_2^{q_{n_0}}(h(x)) - \log Df_1^{q_{n_0}}(x)| \geq \delta_0$. By theorem 6.1(ii) there exists $C_6 = C_6(f_1)$ such that the length of this component greater then $C_6|\Delta_{f_1}^{q_{n_0}+l_0}(a_k)|$. Hence, for a suitable subsequence $n_s$:

$$\ell(x : |\log Df_2^{q_{n_0}}(h(x)) - \log Df_1^{q_{n_0}}(x)| \geq \delta_0) \geq$$

$$\geq \ell(x \in \bigcup_{k=0}^{q_{n_0}-1} \Delta_{f_1}^{q_{n_0}+l_0}(a_k) : |\log Df_2^{q_{n_0}}(h(x)) - \log Df_1^{q_{n_0}}(x)| \geq \delta_0) \geq$$

$$\geq C_6\ell\left( \bigcup_{k=0}^{q_{n_0}-1} \Delta_{f_1}^{q_{n_0}+l_0}(a_k) \right) = C_6 \sum_{k=0}^{q_{n_0}-1} |\Delta_{f_1}^{q_{n_0}+l_0}(a_k)|$$

Using remark 4.5 we get

$$\sum_{k=0}^{q_{n_0}-1} |\Delta_{f_1}^{q_{n_0}+l_0}(a_k)| = \sum_{k=0}^{q_{n_0}-1} \frac{|\Delta_{f_1}^{q_{n_0}+l_0}(a_k)|}{|\Delta_{f_1}^{q_{n_0}-1}(a_k)|} |\Delta_{f_1}^{q_{n_0}-1}(a_k)| \geq$$

$$\geq e^{-3w_1}1_{l_0+1}^{q_{n_0}-1} \sum_{k=0}^{q_{n_0}-1} |\Delta_{f_1}^{q_{n_0}-1}(a_k)|$$

By property 4.2 two consecutive atoms of $P_{n_s}(a_1, f_1)$ are $C_2 = C_2(f_1)$-comparable and using this fact we get

$$\sum_{k=0}^{q_{n_0}-1} |\Delta_{f_1}^{q_{n_0}-1}(a_k)| = \sum_{k=0}^{q_{n_0}-1} \frac{1}{|\Delta_{f_1}^{q_{n_0}-1}(a_k)|} \sum_{k=0}^{q_{n_0}-1} |\Delta_{f_1}^{q_{n_0}-1}(a_k)| \geq$$

$$\geq \frac{C_2}{1 + C_2}.$$ 

Using (36), (37) and (38) we get:

$$\ell(x : |\log Df_2^{q_{n_0}}(h(x)) - \log Df_1^{q_{n_0}}(x)| \geq \delta_0) \geq \frac{k_{l_0+1}^{q_{n_0}}C_6C_2}{e^{3w_1}(1 + C_2)}.$$

Now we consider the second case $\mu_1[a_1, b_1] \neq \mu_2[a_2, b_2]$. Suppose $\mu_1[a_1, b_1] < \mu_2[a_2, b_2]$ the opposite case can be handled similarly. Let $\delta_1$ be the positive number which is defined
in proposition 7.2. Using by proposition 7.2 for any $0 \leq k < q_n$, there exists a subinterval $I^n_k \subset \Delta^{n_1+h_0}(a^1_k)$ such that, on the interval $I^n_k$ hold the following inequality

$$|\log Df_2^{q_n}(h(x)) - \log Df_1^{q_n}(x)| \geq \delta_1$$

and the intervals $I^n_k$ and $\Delta^{n_1+h_0}(a^1_k)$ are $C_9$-comparable. Using similar arguments as above we get:

$$\ell(x : |\log Df_2^{q_n}(h(x)) - \log Df_1^{q_n}(x)|) \geq \ell(x \in \bigcup_{k=0}^{q_n-1} I^n_k : |\log Df_2^{q_n}(h(x)) - \log Df_1^{q_n}(x)| \geq \delta_1) \geq \frac{\kappa_1^{l_0+1}C_0C_2}{e^{3v_1}(1+C_2)}.$$

If we take $\delta = \delta_1$ and

$$\epsilon = \min\left\{\frac{\kappa_1^{l_0+1}C_0C_2}{2e^{3v_1}(1+C_2)}, \frac{\kappa_1^{l_0+1}C_0C_2}{2e^{3v_1}(1+C_2)}\right\},$$

then it is a contradiction to (35).

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