GUP parameter and black-hole temperature

Elias C. Vagenas\textsuperscript{1(a)}, Salwa M. Alsaleh\textsuperscript{2(b)} and Ahmed Farag Ali\textsuperscript{3,4(c)}

\textsuperscript{1} Theoretical Physics Group, Department of Physics, Kuwait University - P.O. Box 5969, Safat 13060, Kuwait
\textsuperscript{2} Department of Physics and Astronomy, King Saud University - Riyadh 11451, Saudi Arabia
\textsuperscript{3} Netherlands Institute for Advanced Study - Korte Spinhuissteeg 3, 1012 CG Amsterdam, The Netherlands
\textsuperscript{4} Department of Physics, Faculty of Science, Benha University - Benha, 13518, Egypt

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Abstract – Motivated by a recent work of Scardigli, Lambiase and Vagenas (SLV), we derive the GUP parameter, i.e., $\alpha_0$, when the GUP has a linear and quadratic term in momentum. The value of the GUP parameter is obtained by conjecturing that the GUP-deformed black-hole temperature of a Schwarzschild black hole and the modified Hawking temperature of a quantum-corrected Schwarzschild black hole are the same. The leading term in both cases is the standard Hawking temperature and since the corrections are considered as thermal, the modified and deformed expressions of temperature display a slight shift in the Hawking temperature. Finally, by equating the first correction terms, we obtain a value for the GUP parameter. In our analysis, the GUP parameter is not a pure number but depends on the ratio $m_p/M$ with $m_p$ the Planck mass and $M$ the black-hole mass.

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Introduction. – Gravitation, described by the relation between space-time curvature and the presence of stress-energy in General Relativity (GR), is incompatible with the Heisenberg uncertainty principle (HUP), as the latter predicts that the measurement of the position to high accuracy would introduce a huge amount of energy to the system, thus, causing the space-time structure to break down due to gravity. Therefore, the HUP should be generalized to incorporate quantum-gravitational effects. Such generalizations were studied extensively in the past decade \cite{1}. Moreover, they were suggested by several candidate theories of quantum gravity, such as string theory, loop quantum gravity, deformed special relativity, and studies of black-hole physics \cite{2-7}.

There are many possible deformations of the HUP to a generalized uncertainty principle (GUP) by a dimensionless deformation parameter $\alpha_0$. This parameter can be calculated on a theoretical basis (as in some models of string theory, see ref.\cite{2}), or from a phenomenological approach that would set a bound on this deformation parameter, for example see refs.\cite{8-10}. In most of the studies above, a specific non-linear representation of the operators in the deformed canonical commutation relations (CCR) is used,

\begin{equation}
[\hat{X}, \hat{P}] = i\hbar \left(1 + \alpha_0^2 \frac{\hat{p}^2}{m_p^2} \right),
\end{equation}

where $\alpha_0$ is the dimensionless GUP parameter and $m_p$ is the Planck mass\cite{1}.

However, a more general deformation was explored in \cite{11}, that has a linear and a quadratic term in momentum $P$

\begin{equation}
[\hat{X}, \hat{P}] = i\hbar (1 - 2\alpha \hat{P} + 4\alpha^2 \hat{P}^2)
\end{equation}

with $\alpha = \alpha_0/m_p = 2\alpha_0\ell_p/\hbar$.

The deformed representations of the Heisenberg algebra of the CCR lead to deformed quantum mechanics, which leads to deducing discreteness of space-time as in ref.\cite{11}. Moreover, the deformed CCR were used in calculations of the corrected energy shift of the hydrogen atom spectrum, the Lamb shift, the Landau levels, and the Scanning Tunneling Microscope (STM), all leading to estimates on $\alpha_0$ ranging between $10^{-11}$ and $10^{-25}$\cite{9,12}.

\footnote{\textsuperscript{1}Here we set $c = k_B = 1$, thus the Planck length is given as $\ell_p = G\hbar$ while the Planck energy will be $\varepsilon_p = m_p$. Moreover, the Planck energy satisfies the HUP, i.e., $\varepsilon_p\ell_p = \hbar/2$, and thus, the Planck mass and Planck length are connected through $\hbar = 2\ell_p m_p$.}
A more refined approach for evaluating bounds on $\alpha_0$ from gravitational interaction was made in ref. [13]. In this work, the Poisson brackets and classical Newtonian mechanics were not deformed, unlike the previous approaches. Moreover, the authors were able to recover standard GR and quantum mechanics when $\alpha_0 \to 0$. Hence, general covariance is preserved, and there is no effect on the geodesic equation for a test particle in the gravitational field. The bounds on $\alpha_0$ obtained from the gravitational interaction range between $\alpha_0 < 10^{10}$ and $\alpha_0 < 10^{35}$.

However, it is believed that $\alpha_0$ should be of order unity, as predicted by string theory [2]. Closer bounds to this were obtained from considering the proton decay by GUP-deformed virtual black holes [14] which obtained a bound of $\alpha_0 > 10^{-3}$, and another closer to unity $\alpha_0 \sim 1$ when higher dimensions are considered.

In a previous work [15], a computation of the value of $\alpha_0$ was performed by comparing two different low-energy (first order in $\hbar$) corrections for the expression of the Hawking temperature. The first is due to a quadratic GUP, and therefore involves $\beta_0$ which is equivalent to $\alpha_0^5$. The second correction is obtained by including the deformation of the metric due to quantum corrections to the Newtonian potential. The value obtained was also of order of unity $\alpha_0^5 = \frac{\beta_0}{\hbar}$, in agreement with the predictions of string theory.

In this paper, we extend these computations to a linear and quadratic GUP, obtaining a GUP parameter $\alpha_0$ which depends on the ratio $m_p/M$ with $m_p$ the Planck mass and $M$ the black-hole mass.

**GUP-deformed black-hole temperature.** In this section, we use a very general deformation of the HUP [11], with linear and quadratic terms in momentum $p$

$$\Delta x \Delta p \geq \frac{\hbar}{2} \left[ 1 + \left( \frac{\alpha}{\sqrt{\langle p \rangle}} + 4\alpha^2 \right) \langle p \rangle^2 \right] + 4\alpha^2 \langle p \rangle^2 - 2\alpha \sqrt{\langle p \rangle^2} \] . (3)$$

Since we are interested in mirror-symmetric states, such that $\langle p \rangle^2 = 0$, hence $\Delta p = \sqrt{\langle p^2 \rangle}$, and $\Delta x \Delta p \geq (1/2)\langle [x, p] \rangle$, the above-mentioned GUP in terms of commutators reads

$$[x, p] = i\hbar (1 - 2\alpha p + 4\alpha^2 p^2) .$$

(4)

Based on the Heisenberg microscope argument [16], if one wants to locate a particle of size $\delta x$, then a photon of energy $E$ has to be shot towards this particle. Employing the GUP version given by eq. (3) and following the arguments in refs. [17–23], the size of the particle will roughly be

$$\delta x \sim \hbar \frac{2E}{2} - \hbar \alpha \frac{2}{2} + 2\hbar \alpha^2 E.$$  

(5)

Of course, given the particle’s (average) wavelength $\lambda \approx \delta x$, one can use eq. (5) to compute the energy $E$ of the particle. Following this syllogism, one can compute the GUP-deformed black-hole temperature. In particular, one considers a Schwarzschild black hole of mass $M$ and event horizon at position $r_H = 2MG$. An ensemble of unpolarized photons which are the Hawking radiation particles, are coming out of this event horizon. The position uncertainty $\delta z$ of these photons is proportional to the size of the event horizon, namely $\delta z = 2\mu r_H$, with $\mu$ a dimensionless constant which will be determined later. Based on the equipartition principle, the energy $E$ of the photons of the Hawking radiation is actually the temperature $T$ of the Schwarzschild black hole, and, thus, in this case eq. (5) now reads

$$4\mu GM \approx \frac{\hbar}{2T} - \frac{\hbar \alpha}{2} + 2\hbar \alpha^2 T$$

or, equivalently,

$$4\mu GM \approx \frac{\hbar}{2T} - \frac{\hbar \alpha}{2m_p} + 2\hbar \alpha^2 T.$$  

(7)

At this point, we can fix the parameter $\mu$ by taking the deformation parameter $\alpha_0 \to 0$ and requiring to recover the standard Hawking temperature, i.e.,

$$T_{BH} = \frac{\hbar}{8\pi GM} .$$

(8)

Therefore, the dimensionless constant has to be $\mu = \pi$.

Next, we rearrange eq. (7) in order to make it a quadratic equation of the black-hole temperature $T$

$$\left( \frac{4\hbar \alpha^2}{m_p^2} \right) T^2 - \left( \frac{8\pi GM}{m_p} + \frac{\hbar \alpha}{m_p} \right) T + \frac{\hbar}{4} = 0$$

(9)

with roots

$$T = \frac{(8\pi GM + \hbar \alpha)}{m_p} \pm \sqrt{\left( \frac{8\pi GM + \hbar \alpha}{m_p} \right)^2 - 4 \left( \frac{4\hbar \alpha^2}{m_p^2} \right) \frac{\hbar}{2}}$$

(10)

Now, we can expand eq. (10) near $\alpha_0 \to 0$ and up to second order in $\alpha_0$, and, thus, we obtain

$$T = \frac{\hbar}{8\pi GM} \left[ 1 + \frac{\alpha_0}{2\pi} \left( \frac{m_p}{M} \right) + 5 \frac{\alpha_0}{2\pi} \left( \frac{m_p}{M} \right)^2 \right] .$$

(11)

At this point it should be stressed that we assumed that the GUP corrections have a thermal character. Thus, it was expected that the GUP corrections will produce a slight shift in the Hawking spectrum, and, therefore, the GUP-deformed black-hole temperature is a shifted Hawking temperature.
Quantum-corrected Schwarzschild metric. – General relativity can be considered as an effective field theory, when considering two heavy objects at rest. This leads to tree diagrams for graviton exchange between these objects as first studied by Duff [24]. The quantum correction to Newton’s potential from these diagrams was obtained (up to a leading term) by Donoghue [25]. Furthermore, it was found that the gravitational interaction between the two above-mentioned objects can be described by a potential energy which is produced by the potential generated from the mass $M$ [26],

$$V(r) = -\frac{GM}{r} \left(1 + \frac{3GM}{r} \left(1 + \frac{m}{M} + \frac{41}{10\pi} \frac{m^2}{r^2}\right)\right). \quad (12)$$

Since the quantum corrections were made to the Schwarzschild solution, we expect, as mentioned in [24], that the classical Schwarzschild metric would be deformed. This can be found easily from the weak-field limit approximation, considering a large mass black hole, e.g., a solar-mass black hole. We have the relation

$$V(r) \simeq \frac{1}{2} (g_{tt}(r) - 1) \quad (14)$$

or, equivalently,

$$g_{tt}(r) \simeq 1 + 2V(r). \quad (15)$$

Thus, we can find the $tt$ component of the deformed metric

$$g_{tt}(r) = 1 - \frac{2GM}{r} + \epsilon(r). \quad (16)$$

Here we have introduced $\epsilon(r)$,

$$\epsilon(r) = -\frac{6G^2M^2}{r^2} \left(1 + \frac{m}{M} - \frac{41}{5\pi} \frac{G^3M^3}{r^3} \right) \left(\frac{\ell P}{GM}\right)^2. \quad (17)$$

It can be easily seen that the general form of the Schwarzschild metric can be written as [27]

$$ds^2 = F(r)dt^2 - F(r)^{-1}dr^2 - C(r)d\Omega^2, \quad (18)$$

where $F(r) = g_{tt}$. The effective Newtonian potential can be derived from a deformed metric for a point particle that moves at non-relativistic velocities, in a stationary gravitational field, i.e., asymptotically quasi-Minkowskian $r \to \infty$. Hence, we obtain the quantum-corrected metric

$$ds^2 = \left(1 - \frac{2GM}{r} + \epsilon(r)\right) dt^2 - \left(1 - \frac{2GM}{r} + \epsilon(r)\right)^{-1} dr^2 - r^2d\Omega^2. \quad (19)$$

Computing the GUP parameter. – In this section, we follow the methodology developed in ref. [15] in order to compute the quantum-corrected Hawking temperature from the metric given by eq. (19). We employ the general formula for the black-hole temperature [28]

$$T = \frac{\hbar}{4\pi} \lim_{r \to r_H} [g_{tt}(r)]' \quad (20)$$

with the prime “’” to denote the partial derivative with respect to the radial coordinate, i.e., $r$, and $r_H$ the solution of the horizon equation

$$r - 2GM + \epsilon(r) r = 0. \quad (21)$$

It should be noted that this deformation is valid in the limit $|\epsilon(r)| \ll GM/r$. Next, we “regularize” $\epsilon(r)$ by writing $\epsilon(r) \equiv \delta \phi(r)$ with $\phi(r)$ being a smooth function of $r$, and $\delta$ a regularization parameter. Hence, the solution to eq. (21) reads [15]

$$r_H = a - \frac{\delta \phi(a)}{1 + \delta \phi(a) + a' \phi(a)} \quad (22)$$

where $a = 2GM$. Moreover, the value of $[g_{tt}(r_H)]'$ is equal to

$$[g_{tt}(r_H)]' = \frac{1}{a(1 - \lambda)^2} + \delta \phi'[a(1 - \lambda)] \quad (23)$$

with the parameter $\lambda$ being

$$\lambda = \frac{\delta \phi(a)}{1 + \delta \phi(a) + a' \phi(a)} \quad (24)$$

Therefore, the quantum-corrected Hawking temperature of the Schwarzschild black hole is given by

$$T = \frac{\hbar}{4\pi} g_{tt}(r_H) = \frac{\hbar}{4\pi a} \left\{1 + \delta \left[2\phi(a) + a\phi'(a)\right] \right.$$  

$$+ \delta^2 \phi(a)[\phi(a) - 2a\phi'(a) - a^2\phi''(a) + \cdots \right\}. \quad (25)$$

At this point, we restore $\epsilon(r)$ and take only the first-order terms in $\delta$, thus,

$$T = \frac{\hbar}{8\pi GM} (1 + [2\epsilon(a) + a\epsilon'(a)]). \quad (26)$$

Now, we may conjecture that the GUP-deformed black-hole temperature given by eq. (11) is equal to the quantum-corrected Hawking temperature given by eq. (26). Thus, the GUP deformation corresponds to the tree-diagram quantum correction to gravity. Using this conjecture, we can compare eq. (11) with eq. (26) to obtain an estimate for the GUP parameter $\alpha_0$. Utilizing eq. (17), for the RHS of eq. (26), we obtain

$$2\epsilon(a) + a\epsilon'(a) = \frac{B}{8G^2M^3} \quad (27)$$

and $B = \frac{4\alpha G^2M\hbar}{5\pi}$. 40001-p3
We can now obtain a value for $\alpha_0$ by comparing the first- and second-order terms in $\alpha_0$ of eq. (11) with the first-order corrections in $\epsilon$ of eq. (26), i.e.,

$$-\frac{\alpha_0}{2\pi} \left( \frac{m_p}{M} \right) + 5 \left( \frac{\alpha_0}{2\pi} \right)^2 \left( \frac{m_p}{M} \right)^2 = \frac{41}{10\pi} \left( \frac{m_p}{M} \right)^2. \quad (28)$$

The solutions of this quadratic equation read

$$\alpha_0 = \frac{\sqrt{2 \left( \frac{m_p}{M} \right)} \pm \sqrt{\left( \frac{1}{\sqrt{2}} \left( \frac{m_p}{M} \right) \right)^2 + 4 \left( \frac{41}{10\pi} \right) \left( \frac{m_p}{M} \right)^2}}{2} \quad (29)$$

It is noteworthy that eq. (29) depends on $M$. Since $M \gg m_p$, we may expand the negative solution in expression (29) and the leading terms give the value for $\alpha_0$,

$$\alpha_0 = \frac{82}{10} \left( \frac{m_p}{M} \right) + \frac{1681}{10\pi} \left( \frac{m_p}{M} \right)^3. \quad (30)$$

Conclusions. – In this paper, we have calculated the value of the deformation parameter $\alpha_0$ of a linear and quadratic GUP. We obtained this value by conjecturing that the deformed Hawking temperature due to GUP is equivalent to the Hawking temperature obtained from a quantum-corrected Schwarzschild solution when considering a tree diagram between two massive objects gravitationally interacting.

The GUP-deformed Hawking temperature was obtained by assuming a particle localized near the horizon, $\Delta x \sim r_H$, and using a GUP with linear and quadratic terms in $p$, instead of the standard HUP. In this way, we obtained an expression for the black-hole temperature with terms depending on the GUP deformation parameter $\alpha_0$ (see eq. (11)).

The quantum corrections of the Schwarzschild black-hole metric stem from the quantum correction to the Newtonian potential derived by Donoghue and collaborators. Specifically, the corrections to the Newtonian potential imply naturally a quantum correction to the Schwarzschild metric, in the weak-field limit. Using this corrected metric, we were able to obtain an expression for the quantum-corrected temperature (see eq. (26)).

Finally, we compared both expressions for the Hawking temperature and equated the subleading terms, since the leading terms are the same and equal to the standard Hawking temperature. As we conjectured that the GUP-deformed black-hole temperature is equivalent to the quantum-corrected temperature, we have obtained an expression for $\alpha_0$.

This work is an extension of the work done in ref. [15] in the sense that in ref. [15] a GUP with a quadratic term in momentum is used while in this work we have employed a GUP with linear and quadratic terms in momentum. However, the GUP parameter $\alpha_0$ obtained in this work is proportional to powers of the dimensionless ratio $(m_p/M)$ and this means that the GUP parameter $\alpha_0$ depends on the specific system under study. Therefore, this result can be interpreted as a loss of or “less” universality of GUP compared to the one described in ref. [15].

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