On A-numerical radius inequalities for $2 \times 2$ operator matrices-II

Satyajit Sahoo

$^a$P.G. Department of Mathematics, Utkal University, Vanisihar, Bhubaneswar-751004, India

Abstract

The main goal of this article is to establish several new upper and lower bounds for the A-numerical radius of $2 \times 2$ operator matrices, where A be the $2 \times 2$ diagonal operator matrix whose diagonal entries are positive bounded operator $A$.

Keywords: A-numerical radius; Positive operator; Semi-inner product; Inequality; Operator matrix

1. Introduction

Let $H$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and $\mathcal{L}(H)$ be the $C^*$-algebra of all bounded linear operators on $H$. The numerical range of $T \in \mathcal{B}(H)$ is defined as

$$W(T) = \{ \langle Tx, x \rangle : x \in H, \|x\| = 1 \}.$$ 

The numerical radius of $T$, denoted by $w(T)$, is defined as $w(T) = \text{sup}\{|z| : z \in W(T)\}$. It is well-known that $w(\cdot)$ defines a norm on $H$, and is equivalent to the usual operator norm $\|T\| = \text{sup}\{\|Tx\| : x \in H, \|x\| = 1\}$. In fact, for every $T \in \mathcal{B}(H)$,

$$\frac{1}{2}\|T\| \leq w(T) \leq \|T\|.$$ (1.1)

An interested reader is referred to the recent articles [4, 13, 20, 21, 22] for different generalizations, refinements and applications of numerical radius inequalities.

Let $\|\cdot\|$ be the norm induced from $\langle \cdot, \cdot \rangle$. An operator $A \in \mathcal{L}(H)$ is called selfadjoint if $A = A^*$, where $A^*$ denotes the adjoint of $A$. A selfadjoint operator $A \in \mathcal{L}(H)$ is called positive if $\langle Ax, x \rangle \geq 0$ for all $x \in H$, and is called strictly positive if $\langle Ax, x \rangle > 0$ for all non-zero $x \in H$. We denote a positive (strictly positive) operator $A$ by $A \geq 0$ ($A > 0$). We denote $\mathcal{R}(A)$ as...
the range space of $A$ and $\overline{\mathcal{R}(A)}$ as the norm closure of $\mathcal{R}(A)$ in $\mathcal{H}$. Let $A$ be a $2 \times 2$ diagonal operator matrix whose diagonal entries are positive operator $A$. Then $A \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ and $A \geq 0$. If $A \geq 0$, then it induces a positive semidefinite sesquilinear form, $\langle \cdot, \cdot \rangle_A : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$ defined by $\langle x, y \rangle_A = \langle Ax, y \rangle$, $x, y \in \mathcal{H}$. Let $\| \cdot \|_A$ denote the seminorm on $\mathcal{H}$ induced by $\langle \cdot, \cdot \rangle_A$, i.e., $\|x\|_A = \sqrt{\langle x, x \rangle_A}$ for all $x \in \mathcal{H}$. Then $\|x\|_A$ is a norm if and only if $A > 0$. Also, $(\mathcal{H}, \| \cdot \|_A)$ is complete if and only if $\mathcal{R}(A)$ is closed in $\mathcal{H}$. Here onward, we fix $A$ and $\mathbb{A}$ for positive operators on $\mathcal{H}$ and $\mathcal{H} \oplus \mathcal{H}$, respectively. We also reserve the notation $I$ and $O$ for the identity operator and the null operator on $\mathcal{H}$ in this paper.

$\|T\|_A$ denotes the $A$-operator seminorm of $T \in \mathcal{L}(\mathcal{H})$. This is defined as follows:

$$\|T\|_A = \sup_{x \in \mathcal{R}(A), \ x \neq 0} \frac{\|Tx\|_A}{\|x\|_A} = \inf \left\{ c > 0 : \|Tx\|_A \leq c \|x\|_A, 0 \neq x \in \overline{\mathcal{R}(A)} \right\} < \infty.$$  

Let

$$\mathcal{L}^A(\mathcal{H}) = \{ T \in \mathcal{B}(\mathcal{H}) : \|T\|_A < \infty \}.$$ 

Then $\mathcal{L}^A(\mathcal{H})$ is not a subalgebra of $\mathcal{B}(\mathcal{H})$, and $\|T\|_A = 0$ if and only if $AT A = O$. For $T \in \mathcal{L}^A(\mathcal{H})$, we have

$$\|T\|_A = \sup \{ \langle Tx, y \rangle_A : x, y \in \overline{\mathcal{R}(A)}, \|x\|_A = \|y\|_A = 1 \}.$$ 

If $AT \geq 0$, then the operator $T$ is called $A$-positive. Note that if $T$ is $A$-positive, then

$$\|T\|_A = \sup \{ \langle Tx, x \rangle_A : x \in \mathcal{H}, \|x\|_A = 1 \}.$$ 

An operator $X \in \mathcal{B}(\mathcal{H})$ is called an $A$-adjoint operator of $T \in \mathcal{B}(\mathcal{H})$ if $\langle Tx, y \rangle_A = \langle x, Xy \rangle_A$ for every $x, y \in \mathcal{H}$, i.e., $AX = T^* A$. By Douglas Theorem \[9\], the existence of an $A$-adjoint operator is not guaranteed. An operator $T \in \mathcal{B}(\mathcal{H})$ may admit none, or one or many $A$-adjoints. $A$-adjoint of an operator $T \in \mathcal{L}(\mathcal{H})$ exists if and only if $\mathcal{R}(T^* A) \subseteq \mathcal{R}(A)$. Let us now denote

$$\mathcal{L}_A(\mathcal{H}) = \{ T \in \mathcal{B}(\mathcal{H}) : \mathcal{R}(T^* A) \subseteq \mathcal{R}(A) \}.$$ 

Note that $\mathcal{L}_A(\mathcal{H})$ is a subalgebra of $\mathcal{B}(\mathcal{H})$ which is neither closed nor dense in $\mathcal{B}(\mathcal{H})$. Moreover, the following inclusions

$$\mathcal{L}_A(\mathcal{H}) \subseteq \mathcal{L}^A(\mathcal{H}) \subseteq \mathcal{L}(\mathcal{H})$$

hold with equality if $A$ is injective and has a closed range.

The Moore-Penrose inverse of $A \in \mathcal{B}(\mathcal{H})$ \[10\] is the operator $X : R(A) \oplus R(A)^\perp \rightarrow \mathcal{H}$ which satisfies the following four equations:

1. $AX A = A$, 
2. $X A X = X$, 
3. $X A = P_{N(A)^\perp}$, 
4. $A X = P_{\mathcal{R}(A)^\perp | R(A) \oplus R(A)^\perp}$. 


Here $N(A)$ and $P_L$ denote the null space of $A$ and the orthogonal projection onto $L$, respectively. The Moore-Penrose inverse is unique, and is denoted by $A^\dagger$. In general, $A^\dagger \notin \mathcal{B}(H)$. It is bounded if and only if $R(A)$ is closed. If $A \in \mathcal{B}(H)$ is invertible, then $A^\dagger = A^{-1}$. If $T \in \mathcal{L}_A(H)$, the reduced solution of the equation $AX = T^*A$ is a distinguished $A$-adjoint operator of $T$, which is denoted by $T^{\#A}$ (see [2, 14]). Note that $T^{\#A} = A^\dagger T^*A$. If $T \in \mathcal{L}_A(H)$, then $AT^{\#A} = T^*A$, $R(T^{\#A}) \subseteq R(A)$ and $N(T^{\#A}) = N(T^*A)$ (see [9]). An operator $T \in \mathcal{B}(H)$ is said to be $A$-selfadjoint if $AT$ is selfadjoint, i.e., $AT = T^*A$. Observe that if $T$ is $A$-selfadjoint, then $T \in \mathcal{L}_A(H)$. However, in general, $T \neq T^{\#A}$. But, $T = T^{\#A}$ if and only if $T$ is $A$-selfadjoint and $R(T) \subseteq R(A)$. If $T \in \mathcal{L}_A(H)$, then $T^{\#A} \in \mathcal{L}_A(H)$, $(T^{\#A})^{\#A} = \overline{R(A)T P_{R(A)}}$, and $((T^{\#A})^{\#A})^{\#A} = T^{\#A}$. Also, $T^{\#A}T$ and $TT^{\#A}$ are $A$-positive operators, and

$$\|T^{\#A}T\|_A = \|TT^{\#A}\|_A = \|T\|_A^2 = \|T^{\#A}\|_A^2 = w_A(TT^{\#A}) = w_A(T^{\#A}T).$$

(1.2)

An operator $T$ is called $A$-bounded if there exists $\alpha > 0$ such that $\|Tx\|_A \leq \alpha \|x\|_A, \ \forall x \in H$. By applying Douglas theorem, one can easily see that the subspace of all operators admitting $A^{1/2}$-adjoints, denoted by $\mathcal{L}_{A^{1/2}}(H)$, is equal the collection of all $A$-bounded operators, i.e.,

$$\mathcal{L}_{A^{1/2}}(H) = \{T \in \mathcal{L}(H) ; \ \exists \alpha > 0; \ \|Tx\|_A \leq \alpha \|x\|_A, \ \forall x \in H\}.$$

Notice that $\mathcal{L}_A(H)$ and $\mathcal{L}_{A^{1/2}}(H)$ are two subalgebras of $\mathcal{L}(H)$ which are, in general, neither closed nor dense in $\mathcal{L}(H)$. Moreover, we have $\mathcal{L}_A(H) \subset \mathcal{L}_{A^{1/2}}(H)$ (see [2, 3]).

An operator $U \in \mathcal{L}_A(H)$ is said to be $A$-unitary if $\|Ux\|_A = \|U^{\#A}x\|_A = \|x\|_A$ for all $x \in H$. For $T, S \in \mathcal{L}_A(H)$, we have $(TS)^{\#A} = S^{\#A}T^{\#A}$, $(T + S)^{\#A} = T^{\#A} + S^{\#A}$, $\|TS\|_A \leq \|T\|_A \|S\|_A$ and $\|Tx\|_A \leq \|T\|_A \|x\|_A$ for all $x \in H$. In 2012, Saddi [19] introduced $A$-numerical radius of $T$ for $T \in \mathcal{B}(H)$, which is denoted as $w_A(T)$, and is defined as follows:

$$w_A(T) = \sup \{\|Tx\|_A : x \in H, \|x\|_A = 1\}.$$  

(1.3)

From (1.3), it follows that

$$w_A(T) = w_A(T^{\#A}) \text{ for any } T \in \mathcal{L}_A(H).$$

A fundamental inequality for the $A$-numerical radius is the power inequality (see [15]) which says that for $T \in \mathcal{B}(H)$,

$$w_A(T^n) \leq w_A^n(T), \quad n \in \mathbb{N}.$$  

(1.4)

Notice that the $A$-numerical radius of semi-Hilbertian space operators satisfies the weak $A$-unitary invariance property which asserts that

$$w_A(U^{\#A}TU) = w_A(T),$$  

(1.5)
for every $T \in \mathcal{L}_A(H)$ and every $A$-unitary operator $U \in \mathcal{L}_A(H)$ (see [4, Lemma 3.8]).

An interested reader may refer [1, 2] for further properties of operators on Semi-Hilbertian space.

Let
\[
\mathcal{R}_A(T) := \frac{T + T^\#_A}{2} \quad \text{and} \quad \mathcal{I}_A(T) := \frac{T - T^\#_A}{2i},
\]
for any arbitrary operator $T \in \mathcal{B}_A(H)$. Recently, in 2019 Zamani [24, Theorem 2.5] showed that if $T \in \mathcal{L}_A(H)$, then
\[
w_A(T) = \sup_{\theta \in \mathbb{R}} \left\| \mathcal{R}_A(e^{i\theta}T) \right\|_A = \sup_{\theta \in \mathbb{R}} \left\| \mathcal{I}_A(e^{i\theta}T) \right\|_A.
\]

In 2019, Zamani [24] showed that if $T \in \mathcal{L}_A(H)$, then
\[
w_A(T) = \sup_{\theta \in \mathbb{R}} \left\| \mathcal{R}_A(e^{i\theta}T + (e^{i\theta}T)^\#_A) \right\|_A.
\]

The author then extended the inequality (1.1) using $A$-numerical radius of $T$, and the same is produced below:
\[
\frac{1}{2} \|T\|_A \leq w_A(T) \leq \|T\|_A.
\]

Furthermore, if $T$ is $A$-selfadjoint, then $w_A(T) = \|T\|_A$. In 2019, Moslehian et al. [15] again continued the study of $A$-numerical radius and established some inequalities for $A$-numerical radius. Further generalizations and refinements of $A$-numerical radius are discussed in [5, 6, 17]. In 2020, Bhunia et al. [8] obtained several $A$-numerical radius inequalities. For more results on $A$-numerical radius inequalities we refer the reader to visit [10, 18, 23, 12].

In 2020, the concept of the $A$-spectral radius of $A$-bounded operators was introduced by Feki in [11] as follows:
\[
r_A(T) := \inf_{n \geq 1} \|T^n\|_A^{\frac{1}{n}} = \lim_{n \to \infty} \|T^n\|_A^{\frac{1}{n}}.
\]

Here we want to mention that the proof of the second equality in (1.9) can also be found in [11, Theorem 1]. Like the classical spectral radius of Hilbert space operators, it was shown in [11] that $r_A(\cdot)$ satisfies the commutativity property, i.e.
\[
r_A(TS) = r_A(ST),
\]
for all $T, S \in \mathcal{L}_{A^{1/2}}(H)$. For the sequel, if $A = I$, then $\|T\|$, $r(T)$ and $\omega(T)$ denote respectively the classical operator norm, the spectral radius and the numerical radius of an operator $T$.

The objective of this paper is to present a few new $A$-numerical radius inequalities for $2 \times 2$ operator matrices. In this aspect, the rest of the paper is broken down as follows. In
section 2, we collect a few results about $A$-numerical radius inequalities which are required to state and prove the results in the subsequent section. Section 3 contains our main results, and is of two parts. Motivated by the work of Hirzallah et al. [13], the first part presents several $A$-numerical radius inequalities of $2 \times 2$ operator matrices while the next part focuses on some $A$-numerical radius inequalities.

2. Preliminaries

We need the following lemmas to prove our results.

**Lemma 2.1.** [Theorem 7 and corollary 2, 11] If $T \in \mathcal{L}_{A^{1/2}}(\mathcal{H})$. Then

$$w_A(T) \leq \frac{1}{2}(\|T\|_A + \|T^2\|_A^{1/2}).$$

**(2.1)**

Further, if $AT^2 = 0$, then

$$w_A(T) = \frac{\|T\|_A}{2}.$$

**(2.2)**

**Lemma 2.2.** [Corollary 3, 11] Let $T \in \mathcal{L}(\mathcal{H})$ is an $A$-self-adjoint operator. Then,

$$\|T\|_A = w_A(T) = r_A(T).$$

**Lemma 2.3.** [Lemma 6, 7] Let $T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$ be such that $T_1, T_2, T_3, T_4 \in \mathcal{L}_{A^{1/2}}(\mathcal{H})$. Then, $T \in \mathcal{L}_{A^{1/2}}(\mathcal{H} \oplus \mathcal{H})$ and

$$r_A(T) \leq r \left( \begin{pmatrix} \|T_1\|_A & \|T_2\|_A \\ \|T_3\|_A & \|T_4\|_A \end{pmatrix} \right).$$

The following lemma is already proved by Bhunia et al. [8] for the case strictly positive operator $A$. Very recently the same result proved by Rout et al. [18] without the condition $A > 0$ is stated next for our purpose.

**Lemma 2.4.** [Lemma 2.4, 18] Let $T_1, T_2 \in \mathcal{L}_A(\mathcal{H})$. Then

(i) $w_A\left( \begin{pmatrix} T_1 & O \\ O & T_2 \end{pmatrix} \right) = \max\{w_A(T_1), w_A(T_2)\}$.

(ii) $w_A\left( \begin{pmatrix} O & T_1 \\ T_2 & O \end{pmatrix} \right) = w_A\left( \begin{pmatrix} O & T_2 \\ T_1 & O \end{pmatrix} \right)$.

(iii) $w_A\left( \begin{pmatrix} O & e^{i\theta}T_2 \\ e^{-i\theta}T_2 & O \end{pmatrix} \right) = w_A\left( \begin{pmatrix} O & T_1 \\ T_2 & O \end{pmatrix} \right)$ for any $\theta \in \mathbb{R}$.
(iv) \( w_A \left( \begin{bmatrix} T_1 & T_2 \\ T_2 & T_1 \end{bmatrix} \right) \) = \( \max \{ w_A(T_1+T_2), w_A(T_1-T_2) \} \). In particular, \( w_A \left( \begin{bmatrix} O & T_2 \\ T_2 & O \end{bmatrix} \right) = w_A(T_2) \).

The following Lemma is proved by Rout et al. \[18\].

**Lemma 2.5.** [Lemma 2.2, \[18\]] Let \( T_1, T_2, T_3, T_4 \in \mathcal{L}_A(\mathcal{H}) \). Then

(i) \( w_A \left( \begin{bmatrix} T_1 & 0 \\ 0 & T_4 \end{bmatrix} \right) \leq w_A \left( \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \right) \).

(ii) \( w_A \left( \begin{bmatrix} O & T_2 \\ T_3 & O \end{bmatrix} \right) \leq w_A \left( \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \right) \).

**Lemma 2.6.** [Lemma 2.4 and Lemma 3.1, \[10, 7\]] Let \( T_1, T_4 \in \mathcal{L}_{A^{\#A}}(\mathcal{H}) \). Then, the following assertions hold

(i) \( \left\| \begin{bmatrix} T_1 & 0 \\ 0 & T_4 \end{bmatrix} \right\|_A = \left\| \begin{bmatrix} 0 & T_1 \\ T_4 & 0 \end{bmatrix} \right\|_A = \max \{ \|T_1\|_A, \|T_4\|_A \} \).

(ii) If \( T_1, T_2, T_3, T_4 \in \mathcal{L}_A(\mathcal{H}) \), then \( \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \) \( \#_A \) \( = \begin{bmatrix} T_1^{\#_A} & T_3^{\#_A} \\ T_2^{\#_A} & T_4^{\#_A} \end{bmatrix} \).

In order to prove our main result the following identity is essential for our purpose. If \( T \in \mathcal{L}_{A^{\#_A}}(\mathcal{H}) \) and \( \begin{bmatrix} T & T \\ -T & -T \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \), so by \[22\]

\[
\begin{align*}
\quad w_A \left( \begin{bmatrix} T & T \\ -T & -T \end{bmatrix} \right) &= \frac{1}{2} \left\| \begin{bmatrix} T & T \\ -T & -T \end{bmatrix} \right\|_A = \|T\|_A. \\
\text{ (2.3)}
\end{align*}
\]

**3. Results**

We will split our results into two subsections. The first part deals with \( A \)-numerical radius of \( 2 \times 2 \) operator matrices. The second part concerns some upper bound for \( A \) numerical radius inequalities.

**3.1. Certain \( A \)-numerical radius inequalities of operator matrices**

Here, we establish our main results dealing with different upper and lower bounds for \( A \)-numerical radius of \( 2 \times 2 \) block operator matrices. The very first result is stated next.
Theorem 3.1. Let \( T_2, T_3 \in \mathcal{L}_A(H) \). Then

\[
    w_{\Lambda} \left( \begin{bmatrix} 0 & T_2 \\ T_3 & 0 \end{bmatrix} \right) \leq \min \{ w_A(T_2), w_A(T_3) \} + \min \left\{ \frac{\|T_2 + T_3\|_A}{2}, \frac{\|T_2 - T_3\|_A}{2} \right\}.
\]

Proof. Let \( U = \frac{1}{\sqrt{2}} \begin{bmatrix} I & -I \\ I & I \end{bmatrix} \). To show that \( U \) is \( \Lambda \)-unitary, we need to prove that \( \|x\|_\Lambda = \|U^*x\|_\Lambda \). So,

\[
    U^* = \Lambda^* U^* \Lambda
    = \frac{1}{\sqrt{2}} \begin{bmatrix} A^\dagger & O \\ O & A^\dagger \end{bmatrix} \begin{bmatrix} I & I \\ -I & I \end{bmatrix} \begin{bmatrix} A & O \\ O & A \end{bmatrix}
    = \frac{1}{\sqrt{2}} \begin{bmatrix} A^\dagger A & A^\dagger A \\ -A^\dagger A & A^\dagger A \end{bmatrix}
    = \frac{1}{\sqrt{2}} \begin{bmatrix} P_{\mathcal{R}(A)} & P_{\mathcal{R}(A)} \\ -P_{\mathcal{R}(A)} & P_{\mathcal{R}(A)} \end{bmatrix}
    \quad \therefore \quad (A^*)^\dagger = \mathcal{R}(A^*) \quad \& \quad \mathcal{R}(A^*) = \mathcal{R}(A).
\]

This in turn implies \( UU^* = \begin{bmatrix} P_{\mathcal{R}(A)} & O \\ O & P_{\mathcal{R}(A)} \end{bmatrix} = U^*U \). Now, for \( x = (x_1, x_2) \in H \oplus H \), we have

\[
    \|Ux\|_\Lambda^2 = \langle Ux, Ux \rangle_\Lambda = \langle U^*Ux, x \rangle_\Lambda
    = \left( \begin{bmatrix} P_{\mathcal{R}(A)} & O \\ O & P_{\mathcal{R}(A)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right)_\Lambda
    = \left( \begin{bmatrix} A P_{\mathcal{R}(A)} & O \\ O & A P_{\mathcal{R}(A)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right)_\Lambda
    = \left( \begin{bmatrix} AA^\dagger A & O \\ O & AA^\dagger A \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right)_\Lambda
    = \|x\|_\Lambda^2.
\]

So, \( \|Ux\|_\Lambda = \|x\|_\Lambda \). Similarly, it can be proved that \( \|U^*x\|_\Lambda = \|x\|_\Lambda \). Thus, \( U \) is an \( \Lambda \)-unitary operator.
Using the identity $w_A(T) = w_A(U^\# A^{\#} T U)$, we have

$$w_A\left(\begin{bmatrix} 0 & T_2 \\ T_3 & 0 \end{bmatrix}\right) = w_A\left(\begin{bmatrix} 0 & T_2 \\ T_3 & 0 \end{bmatrix}^\# \right) = w_A\left(\begin{bmatrix} 0 & T_2 \\ T_3 & 0 \end{bmatrix}^{\#} U\right)$$

$$= \frac{1}{2} w_A\left(\begin{bmatrix} I & -I \\ I & I \end{bmatrix} \begin{bmatrix} 0 & T_2^{\# A} \\ T_3^{\# A} & 0 \end{bmatrix} \begin{bmatrix} I & -I \\ I & I \end{bmatrix}\right)$$

$$= \frac{1}{2} w_A\left(\begin{bmatrix} \frac{P_A}{\|P_A\|} & \frac{P_A}{\|P_A\|} \\ -\frac{P_A}{\|P_A\|} & \frac{P_A}{\|P_A\|} \end{bmatrix} \begin{bmatrix} 0 & T_2^{\# A} \\ T_3^{\# A} & 0 \end{bmatrix} \begin{bmatrix} I & -I \\ I & I \end{bmatrix}\right)$$

$$= \frac{1}{2} w_A\left(\begin{bmatrix} T_2^{\# A} + T_2 & T_2^{\# A} - T_2 \\ -T_3^{\# A} + T_3 & -T_3^{\# A} - T_3 \end{bmatrix}\right)$$

$$= \frac{1}{2} w_A\left(\begin{bmatrix} T_2 + T_3 & T_2 - T_3 \\ -(T_2 - T_3) & -(T_2 + T_3) \end{bmatrix}\right)$$

$$= \frac{1}{2} w_A\left(\begin{bmatrix} T_2 + T_3 & T_2 - T_3 \\ -(T_2 - T_3) & -(T_2 + T_3) \end{bmatrix}\right) + \frac{1}{2} w_A\left(\begin{bmatrix} 0 & -2T_3 \\ 2T_3 & 0 \end{bmatrix}\right)$$

$$\leq \frac{1}{2} \left\{ w_A\left(\begin{bmatrix} T_2 + T_3 & T_2 + T_3 \\ -(T_2 + T_3) & -(T_2 + T_3) \end{bmatrix}\right) + w_A\left(\begin{bmatrix} 0 & -2T_3 \\ 2T_3 & 0 \end{bmatrix}\right)\right\}$$

Now, using identity (2.3) and Lemma 2.4, we have

$$w_A\left(\begin{bmatrix} 0 & T_2 \\ T_3 & 0 \end{bmatrix}\right) = \frac{\|T_2 + T_3\|_A}{2} + w_A(T_3). \quad (3.1)$$

Replacing $T_3$ by $-T_3$ in the inequality (3.1) and using Lemma 2.4, we get

$$w_A\left(\begin{bmatrix} 0 & T_2 \\ T_3 & 0 \end{bmatrix}\right) = \frac{\|T_2 - T_3\|_A}{2} + w_A(T_3). \quad (3.2)$$

From the inequalities (3.1) and (3.2), we have

$$w_A\left(\begin{bmatrix} 0 & T_2 \\ T_3 & 0 \end{bmatrix}\right) \leq w_A(T_3) + \min\left\{ \frac{\|T_2 + T_3\|_A}{2}, \frac{\|T_2 - T_3\|_A}{2} \right\}. \quad (3.3)$$
Again, in the inequality (3.3), interchanging $T_2$ and $T_3$ and using Lemma 2.4(ii), we get
\[
 w_A\left(\begin{bmatrix} 0 & T_2 \\ T_3 & 0 \end{bmatrix}\right) \leq w_A(T_2) + \min\left\{ \frac{\|T_2 + T_3\|_A}{2}, \frac{\|T_2 - T_3\|_A}{2} \right\}.
\] (3.4)

From the inequalities (3.3) and (3.4), we get
\[
 w_A\left(\begin{bmatrix} 0 & T_2 \\ T_3 & 0 \end{bmatrix}\right) \leq \min\{w_A(T_2), w_A(T_3)\} + \min\left\{ \frac{\|T_2 + T_3\|_A}{2}, \frac{\|T_2 - T_3\|_A}{2} \right\}.
\]

This completes the proof.

**Theorem 3.2.** Let $T_2, T_3 \in \mathcal{L}_A(\mathcal{H})$. Then
\[
 w_A\left(\begin{bmatrix} 0 & T_2 \\ T_3 & 0 \end{bmatrix}\right) \geq \max\{w_A(T_2), w_A(T_3)\} - \min\left\{ \frac{\|T_2 + T_3\|_A}{2}, \frac{\|T_2 - T_3\|_A}{2} \right\}.
\]

and
\[
 w_A\left(\begin{bmatrix} 0 & T_2 \\ T_3 & 0 \end{bmatrix}\right) \geq \max\left\{ \frac{\|T_2 + T_3\|_A}{2}, \frac{\|T_2 - T_3\|_A}{2} \right\} - \min\{w_A(T_2), w_A(T_3)\}.
\]

**Proof.** Let $U = \frac{1}{\sqrt{2}}\begin{bmatrix} I & -I \\ I & I \end{bmatrix}$. It can be shown that $U$ is $\mathcal{A}$-unitary. Then
\[
 \frac{1}{2}\begin{bmatrix} T_2 + T_3 & T_2 + T_3 \\ -(T_2 + T_3) & -(T_2 + T_3) \end{bmatrix}^{\#_A} = U^{\#_A}\begin{bmatrix} 0 & T_2 \\ T_3 & 0 \end{bmatrix}^{\#_A} U - \frac{1}{2}\begin{bmatrix} T_2 + T_3 & T_2 + T_3 \\ -(T_2 + T_3) & -(T_2 + T_3) \end{bmatrix}^{\#_A}.
\] (3.5)

So,
\[
\begin{bmatrix} 0 & -T_3 \\ T_3 & 0 \end{bmatrix}^{\#_A} = U^{\#_A}\begin{bmatrix} 0 & T_2 \\ T_3 & 0 \end{bmatrix}^{\#_A} U - \frac{1}{2}\begin{bmatrix} T_2 + T_3 & T_2 + T_3 \\ -(T_2 + T_3) & -(T_2 + T_3) \end{bmatrix}^{\#_A}.
\] (3.6)

This implies
\[
 w_A\left(\begin{bmatrix} 0 & -T_3 \\ T_3 & 0 \end{bmatrix}^{\#_A}\right) \leq w_A\left(U^{\#_A}\begin{bmatrix} 0 & T_2 \\ T_3 & 0 \end{bmatrix}^{\#_A} U\right) + \frac{1}{2} w_A\left( -\begin{bmatrix} T_2 + T_3 & T_2 + T_3 \\ -(T_2 + T_3) & -(T_2 + T_3) \end{bmatrix}^{\#_A}\right).
\]

Which in turn implies that
\[
 w_A\left(\begin{bmatrix} 0 & -T_3 \\ T_3 & 0 \end{bmatrix}\right) \leq w_A\left(\begin{bmatrix} 0 & T_2 \\ T_3 & 0 \end{bmatrix}^{\#_A}\right) + \frac{1}{2} w_A\left( -\begin{bmatrix} T_2 + T_3 & T_2 + T_3 \\ -(T_2 + T_3) & -(T_2 + T_3) \end{bmatrix}\right)
\]
\[
 = w_A\left(\begin{bmatrix} 0 & T_2 \\ T_3 & 0 \end{bmatrix}\right) + \frac{1}{2} w_A\left( -\begin{bmatrix} T_2 + T_3 & T_2 + T_3 \\ -(T_2 + T_3) & -(T_2 + T_3) \end{bmatrix}\right).
\]
Thus, using inequality (3.7) and Lemma 2.4

$$w_A(T_3) \leq w_A\left(\begin{bmatrix} 0 & T_2 \\ T_3 & 0 \end{bmatrix}\right) + \frac{\|T_2 + T_3\|_A}{2}. \quad (3.7)$$

Replacing $T_3$ by $-T_3$ in the inequality (3.7), we have

$$w_A(T_3) \leq w_A\left(\begin{bmatrix} 0 & T_2 \\ T_3 & 0 \end{bmatrix}\right) + \frac{\|T_2 - T_3\|_A}{2}. \quad (3.8)$$

Now from inequality (3.7) and (3.8) that

$$w_A(T_3) \leq w_A\left(\begin{bmatrix} 0 & T_2 \\ T_3 & 0 \end{bmatrix}\right) + \min\left\{\frac{\|T_2 + T_3\|_A}{2}, \frac{\|T_2 - T_3\|_A}{2}\right\}. \quad (3.9)$$

Interchanging $T_2$ and $T_3$ in the inequality (3.9), we get

$$w_A(T_2) \leq w_A\left(\begin{bmatrix} 0 & T_2 \\ T_3 & 0 \end{bmatrix}\right) + \min\left\{\frac{\|T_2 + T_3\|_A}{2}, \frac{\|T_2 - T_3\|_A}{2}\right\}. \quad (3.10)$$

From inequalities (3.9) and (3.10), we have

$$\max\{w_A(T_2), w_A(T_3)\} \leq w_A\left(\begin{bmatrix} 0 & T_2 \\ T_3 & 0 \end{bmatrix}\right) + \min\left\{\frac{\|T_2 + T_3\|_A}{2}, \frac{\|T_2 - T_3\|_A}{2}\right\}. \quad (3.11)$$

Which proves the first inequality.

Again, by identity (3.5) and inequality (2.3) that

$$\frac{1}{2}\|T_2 + T_3\|_A = \frac{1}{2}w_A\left(\begin{bmatrix} T_2 + T_3 & T_2 + T_3 \\ -(T_2 + T_3) & -(T_2 + T_3) \end{bmatrix}\right)$$

$$= \frac{1}{2}w_A\left(\begin{bmatrix} T_2 + T_3 & T_2 + T_3 \\ -(T_2 + T_3) & -(T_2 + T_3) \end{bmatrix}\right)^\#^A$$

$$\leq w_A\left(U^\#^A\begin{bmatrix} 0 & T_2 \\ T_3 & 0 \end{bmatrix}^\#^A\right) + w_A\left(\begin{bmatrix} 0 & -T_3 \\ T_3 & 0 \end{bmatrix}\right)$$

$$= w_A\left(\begin{bmatrix} 0 & T_2 \\ T_3 & 0 \end{bmatrix}\right) + w_A(T_3) \text{ by Lemma 2.4}$$
Thus,
\[
\frac{1}{2} \|T_2 + T_3\|_A \leq w_A \left( \begin{bmatrix} 0 & T_2 \\ T_3 & 0 \end{bmatrix} \right) + w_A(T_3). \tag{3.12}
\]
Replacing \(T_3\) by \(-T_3\) in the inequality (3.12) and using Lemma 2.4, we get
\[
\frac{1}{2} \|T_2 - T_3\|_A \leq w_A \left( \begin{bmatrix} 0 & T_2 \\ T_3 & 0 \end{bmatrix} \right) + w_A(T_3). \tag{3.13}
\]
It follows from inequalities (3.12) and (3.13) that
\[
\max \left\{ \frac{\|T_2 + T_3\|_A}{2}, \frac{\|T_2 - T_3\|_A}{2} \right\} \leq w_A \left( \begin{bmatrix} 0 & T_2 \\ T_3 & 0 \end{bmatrix} \right) + w_A(T_3). \tag{3.14}
\]
Interchanging \(T_2\) and \(T_3\) in the inequality (3.14) and using Lemma 2.4, we get
\[
\max \left\{ \frac{\|T_2 + T_3\|_A}{2}, \frac{\|T_2 - T_3\|_A}{2} \right\} \leq w_A \left( \begin{bmatrix} 0 & T_2 \\ T_3 & 0 \end{bmatrix} \right) + w_A(T_2). \tag{3.15}
\]
Now combining (3.14) and (3.15), we have
\[
\max \left\{ \frac{\|T_2 + T_3\|_A}{2}, \frac{\|T_2 - T_3\|_A}{2} \right\} - \min \{w_A(T_2), w_A(T_3)\} \leq w_A \left( \begin{bmatrix} 0 & T_2 \\ T_3 & 0 \end{bmatrix} \right). \tag{3.16}
\]
This completes the proof.

\begin{proof}

Let us consider \(A\)-unitary operator \(U = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \); \(U^{\#A} = \begin{bmatrix} P_{\mathcal{K}(A)} & 0 \\ 0 & P_{\mathcal{K}(A)} \end{bmatrix} \); \(T = \begin{bmatrix} 0 & T_2 \\ T_3 & 0 \end{bmatrix} \).
\end{proof}

**Theorem 3.3.** Let \(T_2, T_3 \in \mathcal{L}_A(\mathcal{H})\). Then
\[
w_A^2 \left( \begin{bmatrix} 0 & T_2 \\ T_3 & 0 \end{bmatrix} \right) \geq \frac{1}{2} \left\{ w_A(T_2T_3 + T_3T_2), w_A(T_2T_3 - T_3T_2) \right\}.
\]

\begin{proof}

Let us consider \(A\)-unitary operator \(U = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \); \(U^{\#A} = \begin{bmatrix} P_{\mathcal{K}(A)} & 0 \\ 0 & P_{\mathcal{K}(A)} \end{bmatrix} \); \(T = \begin{bmatrix} 0 & T_2 \\ T_3 & 0 \end{bmatrix} \).
\end{proof}
Now,

\[
(T^\#)^2 + (U^\# T^\# U)^2 = \begin{bmatrix} 0 & T_3^\# \\ T_2^\# & 0 \end{bmatrix}^2 + \left( \begin{bmatrix} 0 & \frac{P_{R(A)}}{P_{R(A)}} \\ T_2^\# & 0 \end{bmatrix} \right)^2 \begin{bmatrix} 0 & T_3^\# \\ T_2^\# & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}^2
\]

\[
= \begin{bmatrix} T_3^\# T_2^\# + T_2^\# T_3^\# & 0 \\ T_3^\# T_2^\# & 0 \end{bmatrix}^2 + \begin{bmatrix} T_3^\# T_2^\# + T_2^\# T_3^\# & 0 \\ T_3^\# T_2^\# & 0 \end{bmatrix}^2
\]

\[
= \begin{bmatrix} T_2 T_3 + T_3 T_2 & 0 \\ 0 & T_3 T_2 + T_2 T_3 \end{bmatrix}^\#
\]

So,

\[
w_A \left( \begin{bmatrix} T_2 T_3 + T_3 T_2 & 0 \\ 0 & T_3 T_2 + T_2 T_3 \end{bmatrix} \right) = w_A \left( \begin{bmatrix} T_2 T_3 + T_3 T_2 & 0 \\ 0 & T_3 T_2 + T_2 T_3 \end{bmatrix}^\# \right)
\]

\[
= w_A ((T^\#)^2 + (U^\# T^\# U)^2)
\]

\[
\leq w_A ((T^\#)^2) + w_A ((U^\# T^\# U)^2)
\]

\[
= w_A^2 (T^\#) + w_A^2 (U^\# T^\# U)
\]

\[
= w_A^2 (T) + w_A^2 (T)
\]

\[
= 2w_A^2 (T) \quad \text{(as } w_A(T) = w_A(T^\#)\text{)}.
\]

Hence by using Lemma 2.4 we obtain

\[
w_A(T_2 T_3 + T_3 T_2) \leq 2 w_A^2 (T). \quad (3.17)
\]

Using similar argument to \((T^\#)^2 - (U^\# T^\# U)^2\), we have

\[
w_A(T_2 T_3 - T_3 T_2) \leq 2 w_A^2 (T). \quad (3.18)
\]

Combining (3.17) and (3.18) we get

\[
w_A^2 \left( \begin{bmatrix} 0 & T_2 \\ T_3 & 0 \end{bmatrix} \right) \geq \frac{1}{2} \left\{ w_A(T_2 T_3 + T_3 T_2), w_A(T_2 T_3 - T_3 T_2) \right\}.
\]
Corollary 3.1. Let \( T_1, T_2, T_3, T_4 \in \mathcal{L}_A(\mathcal{H}) \). Then

\[
\begin{align*}
  w_A\left( \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \right) &\geq \max \left\{ w_A(T_1), w_A(T_4), \frac{1}{\sqrt{2}} (w_A(T_2T_3 + T_3T_2))^\frac{1}{2}, \frac{1}{\sqrt{2}} (w_A(T_2T_3 - T_3T_2))^\frac{1}{2} \right\}.
\end{align*}
\]

Proof. Based on Lemma 2.3, Lemma 2.4 and Theorem 3.3, we have

\[
\begin{align*}
  w_A\left( \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \right) &\geq \max \left\{ w_A\left( \begin{bmatrix} T_1 & 0 \\ T_3 & 0 \end{bmatrix} \right), w_A\left( \begin{bmatrix} 0 & T_2 \\ 0 & T_4 \end{bmatrix} \right) \right\} \\
  &\geq \max \left\{ w_A(T_1), w_A(T_4), \frac{1}{\sqrt{2}} (w_A(T_2T_3 + T_3T_2))^\frac{1}{2}, \frac{1}{\sqrt{2}} (w_A(T_2T_3 - T_3T_2))^\frac{1}{2} \right\}.
\end{align*}
\]

\( \blacksquare \)

Theorem 3.4. Let \( T_2, T_3 \in \mathcal{L}_A(\mathcal{H}) \). Then for \( n \in \mathbb{N} \)

\[
\begin{align*}
  w_A\left( \begin{bmatrix} 0 & T_2 \\ T_3 & 0 \end{bmatrix} \right) &\geq \left[ \max \{ w_A((T_2T_3)^n), w_A((T_3T_2)^n) \} \right]^\frac{1}{2n}.
\end{align*}
\] (3.19)

Proof. Let \( T = \begin{bmatrix} 0 & T_2 \\ T_3 & 0 \end{bmatrix} \). Then for \( n \in \mathbb{N} \), \( T^{2n} = \begin{bmatrix} (T_2T_3)^n & 0 \\ 0 & (T_3T_2)^n \end{bmatrix} \) and using Lemma 2.4, we obtain

\[
\begin{align*}
  \max \{ w_A((T_2T_3)^n), w_A((T_3T_2)^n) \} &= w_A\left( \begin{bmatrix} (T_2T_3)^n & 0 \\ 0 & (T_3T_2)^n \end{bmatrix} \right) \\
  &= w_A(T^{2n}) \\
  &\leq w_A^n(T) \quad \text{by inequality 1.4} \\
  &= w_A^n\left( \begin{bmatrix} 0 & T_2 \\ T_3 & 0 \end{bmatrix} \right).
\end{align*}
\]

\( \blacksquare \)

The following lemma is already proved by Hirzallah et al. [13] for the case of Hilbert space operators. Using similar technique we can prove this lemma for the case of semi-Hilbert space. Now we state here the result without proof for our purpose.

Lemma 3.5. Let \( T = \begin{bmatrix} T_1 & T_2 \\ T_2 & T_1 \end{bmatrix} \in \mathcal{L}_A(\mathcal{H} \oplus \mathcal{H}) \) and \( n \in \mathbb{N} \). Then \( T^n = \begin{bmatrix} P & Q \\ Q & P \end{bmatrix} \) for some \( P, Q \in \mathcal{L}_A(\mathcal{H}) \) such that \( P + Q = (T_1 + T_2)^n \) and \( P - Q = (T_1 - T_2)^n \).

The forthcoming result is analogous to Theorem 3.4.
Theorem 3.6. Let $T_1, T_2 \in \mathcal{L}_A(\mathcal{H})$. Then

$$w_A\left(\begin{bmatrix} T_1 & T_2 \\ -T_2 & -T_1 \end{bmatrix}\right) \geq \max\{w_A(((T_1 - T_2)(T_1 + T_2))^n), w_A(((T_1 + T_2)(T_1 - T_2))^n)\}^{\frac{1}{2n}}$$

(3.20)

for $n \in \mathbb{N}$ and

$$w_A\left(\begin{bmatrix} T_1 & T_2 \\ -T_2 & -T_1 \end{bmatrix}\right) \leq \frac{\max\{\|T_1 + T_2\|_A, \|T_1 - T_2\|_A\}}{2}$$

$$+ \frac{\left[\max\{\|(T_1 + T_2)(T_1 - T_2)\|_A, \|(T_1 - T_2)(T_1 + T_2)\|_A\}\right]^\frac{1}{2}}{2}.$$ 

(3.21)

Proof. Let $T = \begin{bmatrix} T_1 & T_2 \\ -T_2 & -T_1 \end{bmatrix}$ and $R = T^2 = \begin{bmatrix} T_1^2 - T_2^2 & T_1T_2 - T_2T_1 \\ T_1T_2 - T_2T_1 & T_1^2 - T_2^2 \end{bmatrix}$. Using Lemma 3.5 we have there exist $P, Q \in \mathcal{L}_A(\mathcal{H})$ such that $R^n = \begin{bmatrix} P & Q \\ Q & P \end{bmatrix}$ with $P+Q = ((T_1^2 - T_2^2) + (T_1T_2 - T_2T_1))^n$ and $P - Q = ((T_1^2 - T_2^2) - (T_1T_2 - T_2T_1))^n$. So, $T^{2n} = \begin{bmatrix} P & Q \\ Q & P \end{bmatrix}$ with $P+Q = ((T_1 - T_2)(T_1 + T_2))^n$ and $P - Q = ((T_1 + T_2)(T_1 - T_2))^n$. By using inequality (1.4), we have

$$w_A^{2n}(T) \geq w_A(T^{2n})$$

$$= w_A\left(\begin{bmatrix} P & Q \\ Q & P \end{bmatrix}\right)$$

$$= \max\{w_A(P + Q), w_A(P - Q)\} \text{ (by Lemma 2.4)}$$

$$= \max\{w_A(((T_1 - T_2)(T_1 + T_2))^n), w_A(((T_1 + T_2)(T_1 - T_2))^n)\}.$$ 

(3.22)

This proves the inequality (3.20). In order to prove the inequality (3.21), let $T = \begin{bmatrix} T_1 & T_2 \\ -T_2 & -T_1 \end{bmatrix}$.

Then $T^{\#A} = \begin{bmatrix} T_1^{\#A} & -T_2^{\#A} \\ T_2^{\#A} & -T_1^{\#A} \end{bmatrix}$, so $TT^{\#A} = \begin{bmatrix} T_1T_1^{\#A} + T_2T_2^{\#A} & -T_1T_2^{\#A} - T_2T_1^{\#A} \\ -T_2T_1^{\#A} - T_1T_2^{\#A} & T_2T_2^{\#A} + T_1T_1^{\#A} \end{bmatrix}$. Now it fol-
lows from \(1.2\) that
\[
\|T\|^2_A = \|TT^\#_A\|_A \\
= w_A(TT^\#_A) \\
= \max\{w_A(T_1T_1^\#_A + T_2T_2^\#_A - T_1T_2^\#_A - T_2T_1^\#_A), w_A(T_1T_1^\#_A + T_2T_2^\#_A + T_1T_2^\#_A + T_2T_1^\#_A)\} \\
\text{(by Lemma 2.4)} \\
= \max\{w_A((T_1 - T_2)(T_1 - T_2)^\#_A), w_A((T_1 + T_2)(T_1 + T_2)^\#_A)\} \\
= \max(\|(T_1 - T_2)(T_1 - T_2)^\#_A\|_A, \|(T_1 + T_2)(T_1 + T_2)^\#_A\|_A) \\
= \max(\|T_1 - T_2\|^2_A, \|T_1 + T_2\|^2_A).
\]
Thus
\[
\|T\|_A = \max(\|T_1 - T_2\|_A, \|T_1 + T_2\|_A).
\]
(3.23)

Similarly we can show that
\[
\|T^2\|_A = \max(\|(T_1 - T_2)(T_1 + T_2)\|_A, \|(T_1 + T_2)(T_1 - T_2)\|_A).
\]
(3.24)

From inequality \((2.1)\), combining inequality \((3.23)\) and \((3.24)\), we obtain
\[
w_A(T) \leq \frac{1}{2}(\|T\|_A + \|T^2\|_A^2) \\
= \max(\|T_1 + T_2\|_A, \|T_1 - T_2\|_A) \\
+ \frac{\left[\max(\|(T_1 + T_2)(T_1 - T_2)\|_A, \|(T_1 - T_2)(T_1 + T_2)\|_A)\right]^2}{2}.
\]
\(\square\)

### 3.2. Some \(A\)-numerical radius inequalities for operators

In this subsection we establish some upper bounds for \(A\)-numerical radius of operators. In the next result, we derive an upper bound for \(A\)-numerical radius of product of operators on semi-Hilbertian space.

**Theorem 3.7.** Let \(T_1, T_2 \in \mathcal{L}_A(H)\). Then
\[
w_A(T_1T_2) \leq \frac{1}{2}\left(\|T_2T_1\|_A + \|T_1\|_A\|T_2\|_A\right).
\]
Proof. It is not difficult to see that \( \mathcal{R}_A(e^{i\theta}T_1T_2) \) is an \( A \)-selfadjoint operator. So, by Lemma 2.2 we have

\[
\| \mathcal{R}_A(e^{i\theta}T_1T_2) \|_A = w_A(\mathcal{R}_A(e^{i\theta}T_1T_2)).
\]

So,

\[
\| \mathcal{R}_A(e^{i\theta}T_1T_2) \|_A = \frac{1}{2} w_A \left( e^{i\theta}T_1T_2 + e^{-i\theta}T_2^\# A T_1^\# A \right)
\]

\[
= \frac{1}{2} w_A \left( \begin{bmatrix} e^{i\theta}T_1T_2 + e^{-i\theta}T_2^\# A T_1^\# A & 0 \\ 0 & 0 \end{bmatrix} \right)
\]

It can observed that

\[
\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} e^{i\theta}T_1T_2 + e^{-i\theta}T_2^\# A T_1^\# A & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} e^{i\theta}AT_1T_2 + e^{-i\theta}AT_2^\# A T_1^\# A & 0 \\ 0 & 0 \end{bmatrix}
\]

\[
= \begin{bmatrix} e^{i\theta}(T_2^\# A T_1^\# A) + e^{-i\theta}(T_1T_2) & 0 \\ 0 & 0 \end{bmatrix}
\]

\[
= \begin{bmatrix} e^{-i\theta}T_2^\# A T_1^\# A + e^{i\theta}T_1T_2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}
\]

Hence 

\[
\begin{bmatrix} e^{i\theta}T_1T_2 + e^{-i\theta}T_2^\# A T_1^\# A & 0 \\ 0 & 0 \end{bmatrix}
\]

is \( A \)-selfadjoint operator.

So by applying Lemma 2.2 we see that

\[
\| \mathcal{R}_A(e^{i\theta}T_1T_2) \|_A = \frac{1}{2} r_A \left( \begin{bmatrix} e^{i\theta}T_1T_2 + e^{-i\theta}T_2^\# A T_1^\# A & 0 \\ 0 & 0 \end{bmatrix} \right)
\]

\[
= \frac{1}{2} r_A \left( \begin{bmatrix} e^{i\theta}T_1 & T_2^\# A \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_2 & 0 \\ T_1^\# A T_1 & T_2^\# A T_2^\# A \end{bmatrix} \right)
\]

\[
\leq \frac{1}{2} r \left( \begin{bmatrix} \| T_2T_1 \|_A & \| T_2T_2^\# A \|_A \\ \| T_1^\# A T_1 \|_A & \| T_1^\# A T_2^\# A \|_A \end{bmatrix} \right)
\]

(by Lemma 2.3)

\[
= \frac{1}{2} \left( \| T_2T_1 \|_A + \| T_1 \|_A \| T_2 \|_A \right).
\]

So by taking supremum over \( \theta \in \mathbb{R} \), then using (1.6) we get our desired result. \( \square \)
Acknowledgments.
We thank the Government of India for introducing the work from home initiative during the COVID-19 pandemic.

4. References

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