Periodic Gibbs Measures for Models with Uncountable Set of Spin Values on a Cayley Tree

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Abstract We consider models with nearest-neighbor interactions and with the set [0, 1] of spin values, on a Cayley tree of order \( k \geq 1 \). We study periodic Gibbs measures of the model with period two. For \( k = 1 \) we show that there is no any periodic Gibbs measure. In case \( k \geq 2 \) we get a sufficient condition on Hamiltonian of the model with uncountable set of spin values under which the model have not any periodic Gibbs measure with period two. We construct several models which have at least two periodic Gibbs measures.

Keywords Cayley tree · Configuration · Gibbs measures · Non existence · Existence.

1 Introduction

The structure of the lattice (graph) plays an important role in investigations of spin systems. For example, in order to study the phase transition problem for a system on \( \mathbb{Z}^d \) and on Cayley tree there are two different methods: Pirogov-Sinai theory on \( \mathbb{Z}^d \), Markov random field theory and recurrent equations of this theory on Cayley tree. In [1-5,8,11-13, 15-17] for several models on Cayley tree, using the Markov random field theory Gibbs measures are described.

These papers are devoted to models with a finite set of spin values. Mainly were shown that these models have finitely many translation-invariant and uncountable numbers of the non-translation-invariant extreme Gibbs measures. Also for several models (see, for example, [6,8,12]) it were proved that there exist three periodic Gibbs measures (which are invariant with respect to normal subgroups of finite index of the group representation of the Cayley tree) and there are uncountable number of non-periodic Gibbs measures.

In [7] the Potts model with a countable set of spin values on a Cayley tree is considered and it was showed that the set of translation-invariant splitting Gibbs measures of the model contains at most one point, independently on parameters of the Potts model with countable set of spin values on the Cayley tree. This is a crucial difference from the models with a finite set of spin values, since the last ones may have more than one translation-invariant Gibbs measures.

In [3], [4], [13] models with an uncountable set of spin values are considered. Our paper is continuation of these papers.

The paper is organized as follows. Section 2 introduces the main definitions. In
Sect. 3 we prove non-existence Gibbs measures with period two on Cayley tree of order one. In Sect. 4, the Hammerstein’s nonlinear integral equation is presented. In Sect. 5, we give a sufficient condition on Hamiltonian of the model have not any periodic Gibbs measure. In Sect 6, 7 and 8 the existence of at least two periodic Gibbs measures for several models with uncountable set of spin values are proved respectively in cases $k = 2$, $k = 3$, $k \geq 4$. In Sect. 9, the existence of at least four periodic Gibbs measures for the models with uncountable set of spin values are proved in cases $k \geq k_0$.

2 Preliminaries

A Cayley tree $\Gamma^k = (V, L)$ of order $k \geq 1$ is an infinite homogeneous tree, i.e., a graph without cycles, with exactly $k + 1$ edges incident to each vertices. Here $V$ is the set of vertices and $L$ that of edges (arcs).

Consider models where the spin takes values in the set $[0, 1]$, and is assigned to the vertexes of the tree. For $A \subset V$ a configuration $\sigma_A$ on $A$ is an arbitrary function $\sigma_A : A \to [0, 1]$. Denote $\Omega_A = [0, 1]^A$ the set of all configurations on $A$. A configuration $\sigma$ on $V$ is then defined as a function $x \in V \mapsto \sigma(x) \in [0, 1]$; the set of all configurations is $[0, 1]^V$. The (formal) Hamiltonian of the model is:

$$H(\sigma) = -J \sum_{\langle x, y \rangle \in L} \xi_{\sigma(x), \sigma(y)},$$

where $J \in R \setminus \{0\}$ and $\xi : (u, v) \in [0, 1]^2 \to \xi_{u,v} \in R$ is a given bounded, measurable function. As usually, $\langle x, y \rangle$ stands for nearest neighbor vertices.

Let $\lambda$ be the Lebesgue measure on $[0, 1]$. On the set of all configurations on $A$ the a priori measure $\lambda_A$ is introduced as the $|A|$ fold product of the measure $\lambda$. Here and further on $|A|$ denotes the cardinality of $A$. We consider a standard sigma-algebra $\mathcal{B}$ of subsets of $\Omega = [0, 1]^V$ generated by the measurable cylinder subsets. A probability measure $\mu$ on $(\Omega, \mathcal{B})$ is called a Gibbs measure (with Hamiltonian $H$) if it satisfies the DLR equation, namely for any $n = 1, 2, \ldots$ and $\sigma_n \in \Omega_{V_n}$:

$$\mu\left(\left\{ \sigma \in \Omega : \sigma|_{V_n} = \sigma_n \right\}\right) = \int_{\Omega} \mu(d\omega) \nu_{\omega|_{W_{n+1}}^{V_n}} (\sigma_n),$$

where $\nu_{\omega|_{W_{n+1}}^{V_n}}$ is the conditional Gibbs density

$$\nu_{\omega|_{W_{n+1}}^{V_n}} (\sigma_n) = \frac{1}{Z_n (\omega|_{W_{n+1}}^{V_n})} \exp \left(-\beta H\left(\sigma_n || \omega|_{W_{n+1}}^{V_n}\right)\right),$$

and $\beta = \frac{1}{T}, T > 0$ is temperature. Here and below, $W_l$ stands for a ‘sphere’ and $V_l$ for a ‘ball’ on the tree, of radius $l = 1, 2, \ldots$, centered at a fixed vertex $x^0$ (an origin):

$$W_l = \{x \in V : d(x, x^0) = l\}, \quad V_l = \{x \in V : d(x, x^0) \leq l\};$$

and

$$L_n = \{\langle x, y \rangle \in L : x, y \in V_n\};$$

distance $d(x, y), x, y \in V$, is the length of (i.e. the number of edges in) the shortest path connecting $x$ with $y$. $\Omega_{V_n}$ is the set of configurations in $V_n$ (and $\Omega_{W_n}$ that in $W_n$; see
below). Furthermore, $\sigma \mid V_n$ and $\omega \mid W_{n+1}$ denote the restrictions of configurations $\sigma, \omega \in \Omega$ to $V_n$ and $W_{n+1}$, respectively. Next, $\sigma_n : x \in V_n \mapsto \sigma_n(x)$ is a configuration in $V_n$ and $H \left( \sigma_n \mid \omega \mid W_{n+1} \right)$ is defined as the sum $H(\sigma_n) + U \left( \sigma_n, \omega \mid W_{n+1} \right)$ where

$$H(\sigma_n) = -J \sum_{(x,y) \in E_n} \xi_{\sigma_n(x),\sigma_n(y)},$$

$$U \left( \sigma_n, \omega \mid W_{n+1} \right) = -J \sum_{(x,y) : x \in V_n, y \in W_{n+1}} \xi_{\sigma_n(x),\omega(y)}.$$

Finally, $Z_n \left( \omega \mid W_{n+1} \right)$ stands for the partition function in $V_n$, with the boundary condition $\omega \mid W_{n+1}$:

$$Z_n \left( \omega \mid W_{n+1} \right) = \int_{\Omega_{V_n}} \exp \left( -\beta H(\sigma_n) \right) \lambda_{V_n}(d\tilde{\sigma}_n).$$

Due to the nearest-neighbor character of the interaction, the Gibbs measure possesses a natural Markov property: for given a configuration $\omega_n$ on $W_n$, random configurations in $V_{n-1}$ (i.e., ‘inside’ $W_n$) and in $V \setminus V_{n+1}$ (i.e., ‘outside’ $W_n$) are conditionally independent.

We use a standard definition of a periodic measure (see, e.g. [9], [15]). The main object of study in this paper are periodic Gibbs measures for the model (2.1) on Cayley tree. In [13] the problem of description of such measures was reduced to the description of the solutions of a nonlinear integral equation. For finite and countable sets of spin values this argument is well known (see, e.g. [1-7, 9-10, 15-17]).

Write $x < y$ if the path from $x^0$ to $y$ goes through $x$. Call vertex $y$ a direct successor of $x$ if $y > x$ and $x, y$ are nearest neighbors. Denote by $S(x)$ the set of direct successors of $x$. Observe that any vertex $x \neq x^0$ has $k$ direct successors and $x^0$ has $k + 1$.

Let $h : x \in V \mapsto h_x = (h_{t,x}, t \in [0,1]) \in R^{[0,1]}$ be mapping of $x \in V \setminus \{ x^0 \}$. Given $n = 1, 2, \ldots$, consider the probability distribution $\mu^{(n)}$ on $\Omega_{V_n}$ defined by

$$\mu^{(n)}(\sigma_n) = Z_n^{-1} \exp \left( -\beta H(\sigma_n) + \sum_{x \in W_n} h_{\sigma_n(x),x} \right).$$  \hspace{1cm} (2.2)

Here, as before, $\sigma_n : x \in V_n \mapsto \sigma(x)$ and $Z_n$ is the corresponding partition function:

$$Z_n = \int_{\Omega_{V_n}} \exp \left( -\beta H(\tilde{\sigma}_n) + \sum_{x \in W_n} h_{\tilde{\sigma}_n(x),x} \right) \lambda_{V_n}(\tilde{\sigma}_n).$$  \hspace{1cm} (2.3)

The probability distributions $\mu^{(n)}$ are compatible if for any $n \geq 1$ and $\sigma_{n-1} \in \Omega_{V_{n-1}}$:

$$\int_{\Omega_{W_n}} \mu^{(n)}(\sigma_{n-1} \vee \omega_n) \lambda_{W_n}(d(\omega_n)) = \mu^{(n-1)}(\sigma_{n-1}).$$  \hspace{1cm} (2.4)

Here $\sigma_{n-1} \vee \omega_n \in \Omega_{V_n}$ is the concatenation of $\sigma_{n-1}$ and $\omega_n$. In this case there exists a unique measure $\mu$ on $\Omega_V$ such that, for any $n$ and $\sigma_n \in \Omega_{V_n}$, $\mu \left( \left\{ \sigma \mid V_n = \sigma_n \right\} \right) = \mu^{(n)}(\sigma_n)$.

**Definition 2.1** The measure $\mu$ is called splitting Gibbs measure corresponding to Hamiltonian (2.1) and function $x \mapsto h_x$, $x \neq x^0$.  

The following statement describes conditions on $h_x$ guaranteeing compatibility of the corresponding distributions $\mu^{(n)}(\sigma_n)$.

**Proposition 2.2**[13] The probability distributions $\mu^{(n)}(\sigma_n)$, $n = 1, 2, \ldots$, in (2.2) are compatible iff for any $x \in V \setminus \{x^0\}$ the following equation holds:

$$f(t, x) = \prod_{y \in S(x)} \frac{\int_0^1 \exp(J\beta_0 x u) f(u, y) du}{\int_0^1 \exp(J\beta_0 u) f(u, y) du}.$$  

(2.5)

Here, and below $f(t, x) = \exp(h_{t,x} - h_{0,x})$, $t \in [0, 1]$ and $du = \lambda(du)$ is the Lebesgue measure.

From Proposition 2.2 it follows that for any $h = \{h_x \in R^{[0,1]} : x \in V\}$ satisfying (2.5) there exists a unique Gibbs measure $\mu$ and vice versa. However, the analysis of solutions to (2.5) is not easy. This difficulty depends on the given function $\xi$.

$x \in V$ is called even (odd) if $d(x, x_0)$ - even (odd). In this paper we shall study special periodic solutions to (2.5), which are in the form $f(t, x) = f(t)$ if $x$ - even and $f(t, x) = g(t)$ if $x$ - odd. For such functions equation (2.5) can be written as

$$f(t) = \left(\frac{\int_0^1 K(t, u) g(u) du}{\int_0^1 K(0, u) g(u) du}\right)^k, \quad g(t) = \left(\frac{\int_0^1 K(t, u) f(u) du}{\int_0^1 K(0, u) f(u) du}\right)^k,$$

(2.6)

where $K(t, u) = \exp(J\beta_\xi_{tu})$, $f(t), g(t) > 0$, $t, u \in [0, 1]$.

We put

$$C^+[0, 1] = \{\varphi \in C[0, 1] : \varphi(x) \geq 0\}.$$  

We are interested to positive continuous solutions to (2.6), i.e. such that

$$f, g \in C^+_0[0, 1] = \{\varphi \in C[0, 1] : \varphi(x) \geq 0\} \setminus \{\theta \equiv 0\}.$$  

Define the operator $A_k : C^+_0[0, 1] \rightarrow C^+_0[0, 1]$ by

$$(A_k f)(t) = \left[\frac{(W f)(t)}{(W f)(0)}\right]^k, \quad k \in N,$$

where $W : C[0, 1] \rightarrow C[0, 1]$ is linear operator, which is defined by :

$$(W f)(t) = \int_0^1 K(t, u) f(u) du,$$  

(2.7)

and defined the linear functional $\omega : C[0, 1] \rightarrow R$ by

$$\omega(f) = (W f)(0) = \int_0^1 K(0, u) f(u) du.$$  

Then Eq.(2.6) can be written as

$$A_k f = g, \quad A_k g = f, \quad f, g \in C^+_0[0, 1]$$  

(2.8)
3 Non Existence of periodic Gibbs Measures for the Model (2.1): Case $k = 1$.

At first we are going to consider (2.8) for $k = 1$. The system of equations (2.8) is equivalent to linear equations

$$(Wf)(t) = w(f)g(t), \quad (Wg)(t) = w(g)f(t), \quad f, g \in C^+[0, 1] \quad (3.1)$$

**Lemma 3.1** Let $(f, g)$ satisfies (3.1) with $f \neq g$, $\delta_0 = sup\{\delta \in (0, \infty) : f - \delta g > 0\}$. Then $W(f - \delta_0 g) > 0$.

**Proof** We have $f - \delta_0 g \geq 0 \Rightarrow W(f - \delta_0 g) \geq 0$. Suppose $W(f - \delta_0 g) = 0$ then

$$f - \delta_0 g \equiv 0 \Rightarrow \frac{f(t)}{g(t)} = \delta_0, \quad t \in [0, 1].$$

For $t = 0$

$$g(0) = \frac{(Wf)(0)}{w(f)} = 1 = \frac{(Wg)(0)}{w(g)} = f(0).$$

Then $\delta_0 = 1$. This contradicts to $f \neq g$. Thus we have proved $W(f - \delta_0 g) > 0$.

**Theorem 3.2** The system of equations $(A_1 f)(t) = g(t), (A_1 g)(t) = f(t)$ has not any solution $(f, g) \in (C^+[0, 1])^2$ with $f \neq g$.

**Proof** Let $(f_1(x), g_1(x))$ be a solution of the system of equations:

$$(A_1 f)(t) = g(t), \quad (A_1 g)(t) = f(t).$$

Then

$$(Wf_1)(t) = w(f_1)g_1(t), \quad (Wg_1)(t) = w(g_1)f_1(t).$$

Put

$$\lambda_1 = w(f_1), \quad \lambda_2 = w(g_1) \Rightarrow \lambda_i > 0, \quad i \in \{1, 2\}.$$

Denote

$$\delta_1 = sup\{\delta \in (0, \infty) : f - \delta g > 0\}, \quad \delta_2 = sup\{\delta \in (0, \infty) : g - \delta f > 0\}.$$

By Lemma 3.1

$$\lambda_1 g(t) - \lambda_2 \delta_1 f(t) = W(f - \delta_1 g) > 0,$$

$$\lambda_2 f(t) - \lambda_1 \delta_2 g(t) = W(g - \delta_2 f) > 0.$$

Hence

$$\frac{\lambda_2}{\lambda_1} \delta_1 < \frac{g(t)}{f(t)}, \quad \frac{\lambda_1}{\lambda_2} \delta_2 < \frac{f(t)}{g(t)}, \quad t \in [0, 1].$$

There exists $t_0, t_1 \in [0, 1]$ such that $\delta_2 \geq \frac{g(t_0)}{f(t_0)}$ and $\delta_1 \geq \frac{f(t_1)}{g(t_1)}$. Then

$$\frac{g(t)}{f(t)} \geq \frac{g(t_0)}{f(t_0)} = \delta_2 \geq \frac{\lambda_2}{\lambda_1} \delta_1, \quad \frac{f(t)}{g(t)} \geq \frac{f(t_1)}{g(t_1)} = \delta_1 \geq \frac{\lambda_1}{\lambda_2} \delta_2.$$

Thus we have $\frac{\lambda_1}{\lambda_2} \delta_1 < \delta_2$ and $\frac{\lambda_2}{\lambda_1} \delta_2 < \delta_1$ this is a contradiction.
4 The Hammerstain’s nonlinear equation.

For every \( k \in \mathbb{N} \) we consider an integral operator \( H_k \) acting in \( C^+[0,1] \) as

\[
(H_k f)(t) = \int_0^1 K(t,u) f^k(u) du \tag{4.1}
\]

If \( k \geq 2 \) then the operator \( H_k \) is a nonlinear operator which is called Hammerstain’s operator of order \( k \). For a nonlinear homogenous operator \( A \) it is known that if there are positive solutions of the operator \( A \) then the number of the positive solutions are continuum. (see[10], p.186).

Denote \( \mathcal{M}_0 = \{ \varphi \in C^+[0,1] : \varphi(0) = 1 \} \).

Lemma 4.1 The system of equations:

\[
(A_k f)(t) = g(t), \quad (A_k g)(t) = f(t) \quad k \geq 2. \tag{4.2}
\]

has a positive solution iff the system of equations:

\[
(H_k f)(t) = \lambda_1 g(t), \quad (H_k g)(t) = \lambda_2 f(t), \quad k \geq 2 \tag{4.3}
\]

has a positive solution in \( (\mathcal{M}_0)^2 \).

Proof Necessity. Let \( (f_0, g_0) \in (C^+_0[0,1])^2 \) be a solution of the system of equations (4.2). We have

\[
(Wf_0)(t) = w(f_0) \sqrt[k]{g_0(t)}, \quad (Wg_0)(t) = w(g_0) \sqrt[k]{f_0(t)}.
\]

From this equality we get

\[
(H_k f_1)(t) = \lambda_1 g_1(t), \quad (H_k g_1)(t) = \lambda_2 f_1(t).
\]

where \( f_1(t) = \sqrt[k]{f_0(t)}, \; g_1(t) = \sqrt[k]{g_0(t)} \) and \( \lambda_1 = w(f_0) > 0, \; \lambda_2 = w(g_0) > 0 \). It is easy to see that \( (f_1, g_1) \in (\mathcal{M}_0)^2 \).

Sufficiency. Let \( k \geq 2 \) and \( (f_1, g_1) \in (\mathcal{M}_0)^2 \) be a solution of the system (4.3). From \( f_1(0) = 1, \; g_1(0) = 1 \) we get

\[
1 = g_1(0) = (H_k f_1)(0) = w(f_1^k), \quad 1 = f_1(0) = (H_k g_1)(0) = w(g_1^k).
\]

Then

\[
f_1 = \frac{H_k g_1}{w(g_1^k)}, \quad g_1 = \frac{H_k f_1}{w(f_1^k)}.
\]

From this equalities we get \( A_k f_0 = g_0, \; A_k g_0 = f_0 \) with \( f_0 = f_1^k \in C^+_0[0,1], \; g_0 = g_1^k \in C^+_0[0,1] \). This completes the proof.

Lemma 4.2 The system of equations (4.3) has a positive solution iff the system of equations:

\[
(H_k f)(t) = g(t), \quad (H_k g)(t) = f(t), \quad k \geq 2 \tag{4.4}
\]
has a positive solution.

**Proof** Necessity. Let \((f_0(t), g_0(t))\) be a positive solution of the system (4.3). Define functions:

\[
\begin{align*}
  f_1(t) &= \frac{1}{C_1} f_0(t), \\
  g_1(t) &= \frac{1}{C_2} g_0(t).
\end{align*}
\]

Then

\[
(H_k f_1)(t) = \frac{1}{C_1^k} (H_k f_0)(t) = \frac{\lambda_1}{C_1^k} g_0(t) = \frac{\lambda_1 C_2}{C_1^k} g_1(t).
\]

Put

\[
C_1 = (\lambda_1)^{\frac{1}{k+1}} (\lambda_1 \lambda_2)^{\frac{1}{k^2-1}}, \quad C_2 = (\lambda_2)^{\frac{1}{k+1}} (\lambda_1 \lambda_2)^{\frac{1}{k^2-1}}.
\]

We have

\[
(H_k f_1)(t) = \frac{\lambda_1 (\lambda_2)^{\frac{1}{k+1}} (\lambda_1 \lambda_2)^{\frac{1}{k^2-1}}}{(\lambda_1 \lambda_2)^{\frac{k}{k^2-1}}} g_1(t) = \frac{(\lambda_1 \lambda_2)^{\frac{1}{k+1}}}{(\lambda_1 \lambda_2)^{\frac{k}{k^2-1}}} g_1(t) = g_1(t).
\]

Similarly we get

\[
H_k g_1(t) = f_1(t).
\]

**Sufficiency.** Let \((f_1(t), g_1(t))\) be a positive solution of the system (4.4). We get

\[
\begin{align*}
  f_0(t) &= C_1 f_1(t), \\
  g_0(t) &= C_2 g_1(t),
\end{align*}
\]

where \(C_1, C_2\) are given in (4.5) It is easy to verify

\[
(H_k f_0)(t) = \lambda_1 g_0(t), \quad (H_k g_0)(t) = \lambda_2 f_0(t), \quad k \geq 2.
\]

This completes the proof.

Denote

\[
\mathcal{K} = \left\{ f \in C^+[0,1] : M \cdot \min_{t \in [0,1]} f(t) \geq m \cdot \max_{t \in [0,1]} f(t) \right\},
\]

\[
\mathcal{P}_k = \left\{ \varphi \in C[0,1] : \frac{m}{M} \cdot \left( \frac{1}{M} \right)^{\frac{1}{k+1}} \leq \varphi(t) \leq \frac{M}{m} \cdot \left( \frac{1}{m} \right)^{\frac{1}{k+1}} \right\}, \quad k \geq 2,
\]

where

\[
M = \max_{t,u \in [0,1]^2} K(t,u), \quad m = \min_{t,u \in [0,1]^2} K(t,u).
\]

**Proposition 4.3** Let \(k \geq 2\). Then

a) \(H_k(C^+[0,1]) \subset \mathcal{K}\).

b) If \((f_0, g_0) \in (C^+_0[0,1])^2\) is a solution of the system (4.4) then \((f_0, g_0) \in (\mathcal{P}_k)^2\).

**Proof** a) Let \(\varphi \in H_k(C^+[0,1])\) be an arbitrary function. There exists a function \(\psi \in C^+[0,1]\) such that \(\varphi = H_k \psi\). Since \(\varphi\) is continuous on \([0,1]\), there are \(t_1, t_2 \in [0,1]\) such that

\[
\varphi_{\min} = \min_{t \in [0,1]} \varphi(t) = \varphi(t_1) = (H_k \psi)(t_1),
\]
\[
\varphi_{\text{max}} = \max_{t \in [0,1]} \varphi(t) = \varphi(t_2) = (H_k \psi)(t_2).
\]

Hence
\[
\varphi_{\text{min}} \geq m \int_0^1 \psi^k(u) du \geq m \int_0^1 \frac{K(t_2, u)}{M} \psi^k(u) du = \frac{m}{M} \varphi_{\text{max}},
\]
i.e. \( \varphi \in \mathcal{K} \).

b) Let \((f, g) \in (C^+_0[0,1])^2\) be a solution of the system (4.4). Then we have
\[
\|f\| \leq M\|g\|^k, \quad \|g\| \leq M\|f\|^k,
\]
where
\[
\|f\| = \max_{t \in [0,1]} |f(t)|.
\]
Hence
\[
\|f\| \leq M^{k+1}\|f\|^{k^2}.
\]
Consequently
\[
\|f\| \geq \left(\frac{1}{M}\right)^{\frac{1}{k^2}}.
\]
By the property a)
\[
f(t) \geq f_{\text{min}} = \min_{t \in [0,1]} f(t) \geq \frac{m}{M} \|f\|.
\]
Consequently
\[
f(t) \geq \frac{m}{M} \left(\frac{1}{M}\right)^{\frac{1}{k^2}}.
\]
Also we have
\[
f(t) = (H_k g)(t) \geq m \int_0^1 g^k(u) du \geq m g_{\text{min}}^k
\]
and
\[
g(t) = (H_k f)(t) \geq m \int_0^1 f^k(u) du \geq m f_{\text{min}}^k.
\]
Then
\[
f_{\text{min}} \geq m g_{\text{min}}^k, \quad g_{\text{min}} \geq m f_{\text{min}}^k \Rightarrow f_{\text{min}} \geq m^{k+1} f_{\text{min}}^{k^2} \quad \text{i.e.}
\]
\[
f_{\text{min}} \leq \left(\frac{1}{m}\right)^{\frac{1}{k^2}}.
\]
By the property a)
\[
f(t) \leq f_{\text{max}} \leq \frac{M}{m} f_{\text{min}} \leq \frac{M}{m} \left(\frac{1}{m}\right)^{\frac{1}{k^2}}.
\]
Thus we have \(f \in \mathcal{P}_k\). Similarly one can prove that \(g \in \mathcal{P}_k\).

**Lemma 4.4** Let \(f \neq g\). Put
\[
\delta_1 = \sup\{\delta \in [0, \infty) : f(t) - \delta g(t) \in C^+[0,1]\}
\]
and
\[
\delta_2 = \sup\{\delta \in [0, \infty) : g(t) - \delta f(t) \in C^+[0,1]\}.
\]
If \( \max\{\delta_1, \delta_2\} \geq 1 \) then \((f, g) \in (C^+_0[0,1])^2 \) can not be solution to the system (4.4).

**Proof** Let \((f_1, g_1) \in (C^+_0[0,1])^2 \) be a solution of system (4.4) and assume \( \max\{\delta_1, \delta_2\} = \delta_1 \geq 1 \) (the case \( \max\{\delta_1, \delta_2\} = \delta_2 \) is similar). Then

\[
g(t) - \delta_k^t f(t) = \int_0^1 K(x,t)(f^k(x) - \delta_k^t g^k(x))dx \geq 0.
\]

There exists \( t \exists_0 \in [0,1] \) such that \( \delta_1 = \frac{g(t_0)}{f(t_0)} \). Moreover we have

\[
\frac{g(t)}{f(t)} \geq \delta_1^k, \quad t \in [0,1].
\]

Then

\[
\delta_1 = \frac{g(t_0)}{f(t_0)} \geq \delta_1^k \Rightarrow \delta_1 = 1.
\]

It is clear that if \( \delta_1 = 1 \) then \( f(x) = g(x) \). But this contradicts to \( f(x) \neq g(x) \).

**Theorem 4.5** Let \((f_1(t), g_1(t)) \) be a solution of system (4.4) with \( f_1 \neq g_1 \). Put \( \phi(t) = f_1(t) - g_1(t) \). The function \( \phi(t) \) changes its sign in \([0,1] \).

**Proof** Assume that \( f_1(t) - g_1(t) \geq 0 \) (the case \( g_1(t) - f_1(t) \geq 0 \) is similar). Consider

\[
\phi^{(1)}_{\delta}(t) = f_1(t) - \delta g_1(t), \quad \phi^{(2)}_{\delta}(t) = g_1(t) - \delta f_1(t), \quad \delta \in [0, \infty).
\]

Put

\[
\delta_1 = \sup\{\delta \in [0, \infty) : \phi^{(1)}_{\delta}(t) \in C^+[0,1]\}
\]

and

\[
\delta_2 = \sup\{\delta \in [0, \infty) : \phi^{(2)}_{\delta}(t) \in C^+[0,1]\}.
\]

One can easily check that \( \delta_1 \geq 1 \). By Lemma 4.4 \((f_1(t), g_1(t)) \) can not be solution of system (4.4). This contradicts our assumption \( f_1(t) - g_1(t) \geq 0 \).

5 Non Existence of periodic Gibbs Measures for Model(2.1): Case \( k \geq 2 \).

**Lemma 5.1** Assume function \( \varphi \in C[0,1] \) changes its sign on \([0,1] \). Then for every \( a \in R \) the following inequality holds

\[
\|\varphi_a\| \geq \frac{1}{n+1}\|\varphi\|, \quad n \in N,
\]

where \( \varphi_a = \varphi_a(t) = \varphi(t) - a, \quad t \in [0,1] \). (see [3], p.9)

**Proposition 5.2** Let \( k \geq 2 \). If the kernel \( K(t,u) \) satisfies the condition

\[
\left(\frac{M}{m}\right)^k - \left(\frac{m}{M}\right)^k < \frac{1}{k}, \quad (5.1)
\]

then the system (4.4) has not any solution \((f, g) \) in \((C^+_0[0,1])^2 \) with \( f \neq g \).
Proof Assume that there is a solution \((f_1, g_1) \in (C_0^+[0,1])^2\). Denote \(h(t) = f_1(t) - g_1(t)\). Then by Theorem 4.5 the function \(h(t)\) changes its sign on \([0,1]\). By Lemma 5.1 we get

\[
\max_{t \in [0,1]} |h(t) + \frac{k}{2} (\gamma_1 + \gamma_2) \int_0^1 f(s)ds| \geq \frac{1}{2} \|h\|,
\]

where

\[
\gamma_1 = \left(\frac{m}{M}\right)^k, \quad \gamma_2 = \left(\frac{M}{m}\right)^k.
\]

By a mean value Theorem we have

\[
-h(t) = \int_0^1 K(t,u)k\xi^{k-1}(u)h(u)du,
\]

here \(\xi \in C^+[0,1]\) and

\[
\min\{f_1(t), g_1(t)\} \leq \xi(t) \leq \max\{f_1(t), g_1(t)\}, \quad t \in [0,1].
\]

By Proposition 4.3 we have \(\xi \in \mathcal{P}_k\), i.e.

\[
\frac{m}{M} \left(\frac{1}{M}\right)^{\frac{1}{k+1}} \leq \xi(t) \leq \frac{M}{m} \left(\frac{1}{m}\right)^{\frac{1}{k+1}}, \quad t \in [0,1].
\]

Hence

\[
\gamma_1 \leq K(t,u)\xi^{k-1}(u) \leq \gamma_2, \quad t, u \in [0,1].
\]

Therefore

\[
\left| k \cdot K(t,u)\xi^{k-1}(u) - k \frac{\gamma_1 + \gamma_2}{2} \right| \leq k \frac{\gamma_2 - \gamma_1}{2}.
\]

Then

\[
\left| h(t) - \frac{k}{2} (\gamma_1 + \gamma_2) \int_0^1 h(u)du \right| \leq \frac{k}{2} (\gamma_2 - \gamma_1) \|h\|. \tag{5.2}
\]

Assume the kernel \(K(t,u)\) satisfies the condition (5.1). Then \(k(\gamma_2 - \gamma_1) < 1\) and the inequality (5.2) contradicts to Lemma 5.1. This completes the proof.

**Proposition 5.3** Let \(k \geq 2\). Let the kernel \(K(t,u)\) satisfies the condition (5.1). For every \(\lambda_1 > 0, \lambda_2 > 0\) the Hammerstein’s system of equations

\[
H_k f = \lambda_1 g, \quad H_k g = \lambda_2 f \tag{5.3}
\]

has not solution \((f,g) \in (C_0^+[0,1])^2, f \neq g\).

**Proof** By Lemma 4.2 the system of equations (5.3) is equivalent to the following system of equations

\[
\int_0^1 K(t,u)f_k(u)du = g_1(t), \quad \int_0^1 K(t,u)g_k(u)du = f_1(t) \tag{5.4}
\]

By Theorem 5.1 the equation (5.4) has not solution in \((C_0^+[0,1])^2\). Hence the equation (5.3) has not solution in \((C_0^+[0,1])^2\).
Theorem 5.4 Let \( k \geq 2 \). If the kernel \( K(t, u) \) satisfies the condition (5.1), then the system of equations (2.6) has not solution in \((C_0^+[0,1])^2, f \neq g\).

Proof Assume there is solution \((f_1, g_1) \in (C^+[0,1])^2\), i.e.
\[
A_k f_1 = g_1, \quad A_k g_1 = f_1.
\]
By Lemma 4.1 the functions \( f_2(t) = \sqrt[j]{f_1(t)} \) and \( g_2(t) = \sqrt[j]{g_1(t)} \), \( t \in [0, 1] \) satisfy the Hammerstein’s system of equations, i.e.
\[
H_k f_2 = \lambda_1 g_2, \quad H_k g_2 = \lambda_2 f_2
\]  
(5.5)
where \( \lambda_1 = \omega(f_1) > 0 \), \( \lambda_2 = \omega(f_2) > 0 \) and \((f_2, g_2) \in (M_0)^2\).
On the other hand by Lemma 4.2 there exists \((f_3, g_3)\) a solution of the Hammerstein’s system of equations:
\[
H_k f_3 = g_3, \quad H_k g_3 = f_3.
\]
But this is contradicts to Proposition 5.2. This completes the proof.

6 Existence of periodic Gibbs Measures for Model (2.1): Case \( k = 2 \).
In this section we construct a function \( K(t, u) \) such that corresponding equation (2.6) has a solution \((f, g)\) with \( f \neq g\). Put
\[
K_n(t, u) = \frac{1 - b_n c_n^3}{n} \sqrt[n]{u - \frac{1}{2}} \left( \sqrt[n]{(u - \frac{1}{2})^2 - 4} \sqrt[n]{t - \frac{1}{2}} \right)^2, \quad t, u \in [0, 1]
\]  
(6.1)
where
\[
b_n = \left( \frac{1}{\sqrt[n]{4}} \right)^{(n-1)} \left( 1 + \frac{2}{n} \right), \quad c_n^3 = \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{(2 + \sqrt[n]{4})^2} du.
\]

Lemma 6.1 For all \( t, u \in [0, 1] \), the following holds:
\[
\lim_{n \to \infty} K_n(t, u) > 0.
\]
Proof It is easy to see
\[
\lim_{n \to \infty} K_n(t, u) > 0 \iff \lim_{n \to \infty} \left( 1 - b_n c_n^3 \sqrt[n]{u - \frac{1}{2}} \left( \sqrt[n]{(u - \frac{1}{2})^2 - 4} \sqrt[n]{t - \frac{1}{2}} \right)^2 \right) > 0.
\]
We have
\[
\lim_{n \to \infty} b_n = \lim_{n \to \infty} \left( \frac{1}{\sqrt[n]{4}} \right)^{(n-1)} \left( 1 + \frac{2}{n} \right) = \frac{1}{4},
\]
\[
\lim_{n \to \infty} c_n = \lim_{n \to \infty} \sqrt[n]{\frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{(2 + \sqrt[n]{4})^2} du} \geq \sqrt[n]{\frac{1}{8}}.
\]
Then
\[
\lim_{n \to \infty} K_n(t, u) > 0 \iff \lim_{n \to \infty} \left( 1 - b_n c_n^3 \sqrt[n]{u - \frac{1}{2}} \left( \sqrt[n]{(u - \frac{1}{2})^2 - 4} \sqrt[n]{t - \frac{1}{2}} \right)^2 \right) \geq
\]
Corollary 6.2  There exists $n_0$ such that for every $n \geq n_0$ the function $K_{n_0}(t, u)$ is a positive function.

Proof  Straightforward.

Theorem 6.3  The system of Hammerstain’s equation:

\[
\int_0^1 K_{n_0}(t, u) f^2(u) du = g(t), \quad \int_0^1 K_{n_0}(t, u) g^2(u) du = f(t)
\]  \hspace{1cm} (6.2)

in the space $(C[0, 1])^2$ has at least two positive solutions with $f \neq g$.

Proof  Let

\[
f_1^{(n_0)}(t) = c_{n_0} \left( \sqrt[3]{u} - \frac{1}{2} + 2 \right), \quad g_1^{(n_0)}(t) = 1, \quad t \in [0, 1],
\]

Then $(f_1^{(n_0)}, g_1^{(n_0)}) \in (C[0, 1])^2$ and positive.

(a) Consider the first equation:

\[
\int_0^1 K_{n_0}(t, u) f^2(u) du = g(t).
\]

\[
\int_0^1 K_{n_0}(t, u) \left( f_1^{(n_0)}(u) \right)^2 du = 1 - \int_0^1 b_{n_0} c_{n_0}^3 \sqrt[3]{u - \frac{1}{2}} \left( \sqrt[3]{u - \frac{1}{2}}^2 - 4 \right) \sqrt[3]{t - \frac{1}{2}} du =
\]

\[
= 1 - b_{n_0} c_{n_0}^3 \sqrt[3]{u_1} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\sqrt[3]{u_1} - 4} \left( \sqrt[3]{u_1^2} - 4 \right)^2 du_1 = 1 = g_1^{(n_0)}(t).
\]

where $u_1 = u - \frac{1}{2}, t_1 = t - \frac{1}{2}$.

(b) Now we consider the second equation:

\[
\int_0^1 K_{n_0}(t, u) \left( g_1^{(n_0)}(u) \right)^2 du = f_1^{n_0}(t).
\]

\[
\int_0^1 K_{n_0}(t, u) \left( g_1^{(n_0)}(u) \right)^2 du = \int_0^1 \frac{1 - b_{n_0} c_{n_0}^3 \sqrt[3]{u - \frac{1}{2}} \left( \sqrt[3]{u - \frac{1}{2}}^2 - 4 \right) \sqrt[3]{t - \frac{1}{2}}}{c_{n_0}^2 \left( \sqrt[3]{u - \frac{1}{2}} + 2 \right)^2} du =
\]

Let $u_1 = u - \frac{1}{2}, t_1 = t - \frac{1}{2}$. Then

\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{c^2 \left( \sqrt[3]{u_1} + 2 \right)^2} du_1 - b_{n_0} c_{n_0} \sqrt[3]{u_1} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\sqrt[3]{u_1} \left( \sqrt[3]{u_1} - 4 \right)^2}{\left( \sqrt[3]{u_1} + 2 \right)^2} du_1 =
\]
\[
\frac{1}{c_{n_0}^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\sqrt{|u_1+2|}} du_1 - b_{n_0} c_{n_0} \sqrt{t_1} \int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{u_1} (\sqrt{u_1} - 2)^2 du_1 = \\
= 2c_{n_0} + 4b_{n_0} c_{n_0} \sqrt{t_1} \int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{u_1^2} du_1 = c_{n_0} \left( \sqrt{t - \frac{1}{2} + 2} \right) = f_1^{n_0}(t).
\]

By symmetry of \((f, g)\) we have \((g_1^{n_0}(t), f_1^{n_0}(t))\) is also solution of (6.2).
This completes the proof.
From this we get

**Theorem 6.4** The model:

\[
H(\sigma) = -\frac{1}{\beta} \sum_{<x,y>} \ln \left( \frac{1 - b_{n_0} c_{n_0}^3 \sqrt{\sigma(x) - \frac{1}{2}} \left( \frac{\sqrt{\sigma(x) - \frac{1}{2}} - 4}{\sqrt{\sigma(y) - \frac{1}{2}}} \right)^2}{c_{n_0}^2 \left( \sqrt{\sigma(x) - \frac{1}{2} + 2} \right)^2} \right)
\]

on the Cayley tree \(\Gamma^2\) has at least two periodic Gibbs measures.

7 **Existence of periodic Gibbs Measures for Model (2.1): Case \(k = 3\).**

**Lemma 7.1** Let \(a \in R\). Then for every odd (even) function \(\varphi(x) \in C[0, 1]\) the following equation holds:

\[
\int_{-a}^{a} \frac{\varphi(x)}{(1 + \sin x)^3} dx = -2 \int_{0}^{a} \frac{\varphi(x) \sin x (3 + \sin^2 x)}{\cos^6 x} dx.
\]

\[
\left( \int_{-a}^{a} \frac{\varphi(x)}{(1 + \sin x)^3} dx = 2 \int_{0}^{a} \frac{\varphi(x)(1 + 3 \sin^2 x)}{\cos^6 x} dx \right).
\]

**Proof** Let \(\varphi(x)\) be odd (the case even is similar) function

\[
\int_{-a}^{a} \frac{\varphi(x)}{(1 + \sin x)^3} dx = \int_{0}^{a} \frac{\varphi(x)}{(1 + \sin x)^3} dx + \int_{-a}^{0} \frac{\varphi(x)}{(1 + \sin x)^3} dx = \\
\int_{0}^{a} \frac{\varphi(x)}{(1 + \sin x)^3} dx - \int_{0}^{a} \frac{\varphi(x)}{(1 - \sin x)^3} dx = -2 \int_{0}^{a} \frac{\varphi(x) \sin x (3 + \sin^2 x)}{\cos^6 x} dx.
\]

Put

\[
K(t, u) = \frac{1 - \frac{22}{17} \sin \frac{\pi (2u-1)}{3} \sin \frac{\pi (2u-1)}{3}}{a^3 (1 + \sin \frac{\pi (2u-1)}{3})^3}, \quad t, u \in [0, 1], \quad (7.1)
\]

where \(a = \sqrt[5]{\frac{198\sqrt{3}}{5\pi}}\). It is easy to see that \(K(t, u)\) is a positive and continuous function.

**Theorem 7.2** The system of Hammerstain's equations

\[
\int_{0}^{1} K(t, u) f^3(u) du = g(t), \quad \int_{0}^{1} K(t, u) g^3(u) du = f(t), \quad (7.2)
\]
in the space \((C[0,1])^2\) has at least two positive solutions with \(f \neq g\).

**Proof** (a) Denote

\[ f_1(t) = a \left( 1 + \sin \frac{\pi(2t-1)}{3} \right), \quad g_1(t) = 1, \quad t \in [0,1], \]

where \(a = \frac{4}{5\sqrt{3}}\). Then \((f_1, g_1) \in (C[0,1])^2\) and the functions \(f_1\) and \(g_1\) are positive. Consider the first equation of (7.2)

\[ \int_0^1 K(t,u)f_1^3(u)du = 1 - \frac{22}{17} \sin \frac{\pi(2t-1)}{3} \int_0^1 \sin \frac{\pi(2u-1)}{3}du = 1. \]

(b) Now we check the second equation.

\[ \int_0^1 K(t,u)g_1^3(u)du = \int_0^1 \frac{1 - \frac{22}{17} \sin \frac{\pi(2t-1)}{3} \sin \frac{\pi(2u-1)}{3}}{a^3(1 + \sin \frac{\pi(2u-1)}{3})^3}du. \]

Let \(t_1 = \frac{\pi}{3}(2t-1), u_1 = \frac{\pi}{3}(2u-1).\) Then

\[ \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} K(t_1,u_1)g_1^3(u_1)du_1 = \frac{3}{2a^3\pi} \left( \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 1 - \frac{22}{17} \sin t_1 \sin u_1 \right)du_1 = \]

\[ = \frac{3}{2a^3\pi} \left( \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{(1 + \sin u_1)^3}du_1 - \frac{22}{17} \sin t_1 \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{\sin u_1}{(1 + \sin u_1)^3}du_1 \right). \]

By Lemma 7.1 LHS of this equality is

\[ \frac{3}{2a^3\pi} \left[ \int_0^{\frac{\pi}{3}} \left( 1 + \sin^2 u_1 \right)du_1 - \frac{22}{17} \sin t_1 \int_0^{\frac{\pi}{3}} \sin^2 u_1(3 + \sin^2 u_1)du_1 \right] = \]

\[ = \frac{3}{a^3\pi} \left[ \int_0^{\sqrt{3}} (1 + 4y^2)(1 + y^2)dy + \frac{22}{17} \sin t_1 \int_0^{\sqrt{3}} y^2(3 + 4y^2)dy \right] = \]

\[ = \frac{198\sqrt{3}}{a^3\pi} (1 + \sin t_1) = \frac{198\sqrt{3}}{a^3\pi} \left( 1 + \sin \frac{\pi(2t-1)}{3} \right) = f(t). \]

By symmetry of \((f_1, g_1)\) we have \((g_1(t), f_1(t))\) is also solution to (6.2).

This completes the proof.

**Theorem 7.3** The model:

\[ H(\sigma) = -\frac{1}{\beta} \sum_{<x,y>} \ln \left( 1 - \frac{22}{17} \sin \frac{\pi(2\sigma(x)-1)}{3} \sin \frac{\pi(2\sigma(y)-1)}{3} \right) \frac{a^3(1 + \sin \frac{\pi(2\sigma(x)-1)}{3})^3}{1 + \sin \frac{\pi(2\sigma(x)-1)}{3}} \]

on the Cayley tree \(\Gamma^3\) has at least two periodic Gibbs measures.
8 Existence of periodic Gibbs Measures for Model (2.1): Case $k \geq 4$.

Denote

$$c_k = \frac{2 \left( 1 - \left( \frac{1}{3} \right)^{k-1} \right)}{k-2} \left( 1 - \left( \frac{1}{3} \right)^{k-2} \right) - 2 \left( 1 - \left( \frac{1}{3} \right)^{k-1} \right).$$

(8.1)

Lemma 8.1 For every $k \in \mathbb{N}$, $k \geq 4$ the following inequality holds: $|c_k| < 4$.

**Proof** For $k \geq 4$ we have

$$|c_k| = \frac{2 \left( 1 - \left( \frac{1}{3} \right)^{k-1} \right)}{k-2} + \frac{2 \left( \frac{1}{3} \right)^{k-1} (k+1) k - 2 \left( \frac{1}{3} \right)^{k-1} (k+1) (k-2)}{k-3} < \frac{2(k-2)}{k-3}.$$  

Thus

$$|c_k| < \frac{2(k-2)}{k-3} = 2 + \frac{2}{k-3} < 4. \quad (8.2)$$

Hence $|c_k| < 4$ for $k \geq 4$.

For each $k \geq 4$, $a > 0$ we define the continuous function

$$K(t, u, k) = \frac{1 + c_k (t - \frac{1}{2}) (u - \frac{1}{2})}{a^k (u + \frac{1}{2})^k}, \quad t, u \in [0, 1].$$

By the inequality (8.2) it follows that the function $K(t, u, k)$ is positive.

**Theorem 8.2** For each $k \geq 4$ the Hammerstein’s system of equations:

$$\int_0^1 K(t, u, k) f_k(u) du = g(t), \quad \int_0^1 K(t, u, k) g_k(u) du = f(t) \quad (8.3)$$

in $(\mathbb{C}[0, 1])^2$ have at least two positive solutions with $f \neq g$.

**Proof** Let $k \geq 4$. Define the positive continuous functions $f_1(t)$, $g_1(t)$ on $[0, 1]$ by the equality

$$f_1(t) = a \left( t + \frac{1}{2} \right), \quad g_1(t) = 1$$

where

$$a = a(k) = \sqrt[k]{2 k - 1 \left( 1 - \left( \frac{1}{3} \right)^{k-1} \right)}, \quad k \geq 4.$$  

It is easy to see that $a > 0$. We shall show that $(f_1, g_1)$ is a solution to the Hammerstein’s system of equations (8.2).

We shall check the first equation.

$$\int_0^1 K(t, u, k) f_1^k(u) du = \int_0^1 \frac{1 + c_k (t - \frac{1}{2}) (u - \frac{1}{2})}{a^k (u + \frac{1}{2})^k} \left( a \left( u + \frac{1}{2} \right) \right)^k du =$$
1 \int_0^1 \left( 1 + c_k \left( t - \frac{1}{2} \right) \left( u - \frac{1}{2} \right) \right) du = 1 + c_k t_1 \int_{-\frac{1}{2}}^{\frac{1}{2}} u_1 du_1 = 1.

Where \( t_1 = t - \frac{1}{2} \) and \( u_1 = u - \frac{1}{2} \). Hence

\[
1 \int_0^1 K(t, u, k) f_1^k(u) du = g_1(t).
\]

Now we shall check the second equation.

\[
1 \int_0^1 K(t, u, k) g_1^k(u) du = 1 \int_0^1 \left( \frac{1 + c_k (t - \frac{1}{2} \left( u - \frac{1}{2} \right) \right)}{a^k (u + \frac{1}{2})^k} \right) du =
\]

\[
\frac{1}{a^k} \int_0^1 c_k (t - \frac{1}{2} \left( u - \frac{1}{2} \right) du + \int_0^1 \frac{c_k (t - \frac{1}{2} \left( u - \frac{1}{2} \right) du = \frac{1}{a^k} \left( \int_{-\frac{1}{2}}^{\frac{1}{2}} (u + 1)^k du + c_k t_1 \int_{-\frac{1}{2}}^{\frac{1}{2}} u_1 (u + 1)^k du_1 \right) =
\]

(\text{where } t_1 = t - \frac{1}{2}, \ u_1 = u - \frac{1}{2})

\[
= \frac{a + at_1}{k - 2} \left( 1 - \left( \frac{1}{3} \right)^{k-1} \right) - 2 \left( 1 - \left( \frac{1}{3} \right)^{k-1} \right) \left( \frac{k - 1 - \left( \frac{1}{3} \right)^{k-2}}{2(k - 2)} \left( \frac{1 - \left( \frac{1}{3} \right)^{k-1}}{2(k - 2)} \right) \right) = a + at_1 = a \left( t + \frac{1}{2} \right) = f_1(t).
\]

Moreover, \((g_1(t), f_1(t))\) is also solution to (8.3).

**Theorem 8.3** Let \( k \geq 4 \). The model

\[
H(\sigma) = -\frac{1}{\beta} \sum_{<x,y>} \ln \left( \frac{1 + c_k (\sigma(x) - \frac{1}{2}) (\sigma(y) - \frac{1}{2})}{a^k (\sigma(x) + \frac{1}{2})^k} \right)
\]

on the Cayley tree \( \Gamma^k \) has at least two periodic Gibbs measures.

9 Existence of four periodic Gibbs Measures for Model (2.1).

Denote

\[
c_{ij}(m) = \frac{1}{m + 2(i - 1) + 2(j - 1)}, \quad (n, m, p) \in N \times N \times N, \quad 1 \leq i, j \leq n,
\]

\[
A^{(m,p)}_n = \left( \frac{c_{ij}(m)}{4^{(p+1)}} \right)_n \text{ be } n \times n \text{ square matrix.}
\]

If \( n \in \{2, 3\} \) then it’s easy to check \( \det(A^{(m,p)}_n) \neq 0 \).

Put

\[
\begin{pmatrix}
a_{11} \\
a_{12} \\
a_{13}
\end{pmatrix} = (A^{(1,0)}_3)^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix}
a_{21} \\
a_{22} \\
a_{23}
\end{pmatrix} = (A^{(3,1)}_3)^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\]

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and
\[
\left( \begin{array}{c}
  b_{11} \\
  b_{12}
\end{array} \right) = \left( A_{2(5,2)} \right)^{-1} \left( \begin{array}{c}
  1 \\
  0
\end{array} \right), \quad \left( \begin{array}{c}
  b_{21} \\
  b_{22}
\end{array} \right) = \left( A_{2(7,3)} \right)^{-1} \left( \begin{array}{c}
  0 \\
  1
\end{array} \right),
\]
where \( A_n^{(m,p)} \)^{-1} is inverse of \( A_n^{(m,p)} \).
So we define following functions:
\[
\psi_1(u) = a_{11} + a_{12}u^2 + a_{13}u^4, \quad \psi_2(u) = a_{21}u^2 + a_{22}u^4 + a_{23}u^6,
\]
\[
\psi_3(u) = b_{11}u + b_{12}u^3, \quad \psi_4(u) = b_{21}u^3 + b_{22}u^5.
\]
Finally
\[
K_1(t, u; k) = \psi_1(u) \left( \sqrt[k]{20t^4 + \frac{3}{4}} - 1 \right) + \psi_2(u) \left( \sqrt[k]{6t^2 + \frac{1}{2}} - 1 \right),
\]
\[
K_2(t, u; k) = \psi_3(u) \left( \sqrt[k]{t^3 + 1} - 1 \right) + \psi_4(u) \left( \sqrt[k]{t^5 + 1} - 1 \right),
\]
\[
\tilde{K}(t, u; k) = 1 + K_1(t, u; k) + K_2(t, u; k).
\]

**Remark 9.1** There exist \( k_0 \in N \) such that for all \( k \geq k_0 \) the following inequality holds
\[
\tilde{K} \left( t - \frac{1}{2}, u - \frac{1}{2}; k \right) > 0, \quad (t, u) \in [0, 1]^2.
\]

**Proof** It is sufficient to show:
\[
\lim_{k \to \infty} \tilde{K} \left( t - \frac{1}{2}, u - \frac{1}{2}; k \right) > 0, \quad (t, u) \in [0, 1]^2.
\]
Let \( \gamma : [0, 1] \to [m, M] \) be a function, \( m > 0 \).
We have
\[
0 = \lim_{k \to \infty} (\sqrt[k]{m} - 1) \leq \lim_{k \to \infty} (\sqrt[k]{\gamma(t)} - 1) \leq \lim_{k \to \infty} (\sqrt[k]{M} - 1) = 0.
\]
Hence
\[
\lim_{k \to \infty} (\sqrt[k]{\gamma(t)} - 1) = 0 \Rightarrow \lim_{k \to \infty} K_i \left( t - \frac{1}{2}, u - \frac{1}{2}; k \right) = 0, \quad i \in \{1, 2\}
\]
and
\[
\lim_{k \to \infty} \tilde{K} \left( t - \frac{1}{2}, u - \frac{1}{2}; k \right) = 1 > 0.
\]
This completes the proof.

**Lemma 9.2** If \( k \in \{1, 2\} \), then
\[
(i) \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_1(u)u^{2k}du = \frac{1}{12} \left( 1 + (-1)^{k+1} \right),
\]
\[
(ii) \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_2(u)u^{2k}du = \frac{1}{12} \left( 1 + (-1)^k \right),
\]
\[
(iii) \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_3(u)u^{2k+1}du = \frac{1}{12} \left( 1 + (-1)^k \right),
\]

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(iv) $\int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_4(u)u^{2k+1}du = \frac{1}{12} (1 + (-1)^{k+1})$.

Proof For $k \in \{1, 2\}$

(i) $\int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_1(u)u^{2k}du = a_{11} \int_{-\frac{1}{2}}^{\frac{1}{2}} u^{2k}du + a_{12} \int_{-\frac{1}{2}}^{\frac{1}{2}} u^{2(k+1)}du + a_{13} \int_{-\frac{1}{2}}^{\frac{1}{2}} u^{2(k+2)}du = \frac{a_{11}}{4^k(2k + 1)} + \frac{a_{12}}{4^{k+1}(2k + 3)} + \frac{a_{13}}{4^{k+2}(2k + 5)} = a_{11} \times c_{k+1,1}^{(0)}(1) + a_{12} \times c_{k+1,2}^{(0)}(1) + a_{13} \times c_{k+1,3}^{(0)}(1) = \frac{1}{12} (1 + (-1)^{k+1})$.

(ii) $\int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_2(u)u^{2k}du = a_{21} \int_{-\frac{1}{2}}^{\frac{1}{2}} u^{2(k+1)}du + a_{22} \int_{-\frac{1}{2}}^{\frac{1}{2}} u^{2(k+2)}du + a_{23} \int_{-\frac{1}{2}}^{\frac{1}{2}} u^{2(k+3)}du = \frac{a_{21}}{4^k(2k + 1)} + \frac{a_{22}}{4^{k+1}(2k + 3)} + \frac{a_{23}}{4^{k+2}(2k + 5)} = a_{21} \times c_{k+1,1}^{(1)}(3) + a_{22} \times c_{k+1,2}^{(1)}(3) + a_{23} \times c_{k+1,3}^{(1)}(3) = \frac{1}{12} (1 + (-1)^{k})$.

(iii) $\int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_3(u)u^{2k+1}du = b_{11} \int_{-\frac{1}{2}}^{\frac{1}{2}} u^{2(k+1)}du + b_{12} \int_{-\frac{1}{2}}^{\frac{1}{2}} u^{2(k+2)}du = b_{11} \times c_{k,1}^{(2)}(5) + b_{12} \times c_{k,2}^{(2)}(5) = \frac{1}{12} (1 + (-1)^{k}) = \frac{1}{12} (1 + (-1)^{k})$.

(iv) $\int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_4(u)u^{2k+1}du = b_{21} \int_{-\frac{1}{2}}^{\frac{1}{2}} u^{2(k+2)}du + b_{22} \int_{-\frac{1}{2}}^{\frac{1}{2}} u^{2(k+3)}du = b_{21} \times c_{k,1}^{(3)}(7) + b_{22} \times c_{k,2}^{(3)}(7) = \frac{1}{12} (1 + (-1)^{k+1})$.

Lemma 9.3 The function $\varphi_0(u) = 1$ is a fixed point of the operator $H_k$:

$$(H_kf)(t) = \int_0^1 \tilde{K} \left( t - \frac{1}{2}, u - \frac{1}{2}; k \right) f^k(u)du, \quad k \geq 2 \quad (9.1)$$

Proof Let $u_1 = u - \frac{1}{2}, \quad v_1 = v - \frac{1}{2}$ then

$$\left( H_k\varphi_0 \right) \left( t - \frac{1}{2} \right) = \int_0^1 \tilde{K} \left( t - \frac{1}{2}, u - \frac{1}{2}; k \right) du = \int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{K}(t_1, u_1; k)du_1 =$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[ 1 + K_1(t_1, u_1; k) + K_2(t_1, u_1; k) \right] du_1 = 1 + \int_{-\frac{1}{2}}^{\frac{1}{2}} K_1(t_1, u_1; k)du_1 + \int_{-\frac{1}{2}}^{\frac{1}{2}} K_2(t_1, u_1; k)du_1.$$ 

Now we’ll prove the following

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} K_i(t_1, u_1; k)du_1 = 0, \quad i \in \{1, 2\}. \quad (9.2)$$
Case: $i = 1$

$$
\int_{-\frac{1}{2}}^{\frac{1}{2}} K_1(t, u_1; k) du_1 = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[ \psi_1(u_1) \left( \sqrt[k]{20t^4} + \frac{3}{4} - 1 \right) du_1 + \psi_2(u_1) \left( \sqrt[k]{6t^2} + \frac{1}{2} - 1 \right) \right] du_1 =
$$

$$
= \left( \sqrt[k]{20t^4} + \frac{3}{4} - 1 \right) \left( a_{11} + a_{12} \int_{-\frac{1}{2}}^{\frac{1}{2}} u^2 du + a_{13} \int_{-\frac{1}{2}}^{\frac{1}{2}} u^4 du \right) +
$$

$$
+ \left( \sqrt[k]{6t^2} + \frac{1}{2} - 1 \right) \left( a_{21} \int_{-\frac{1}{2}}^{\frac{1}{2}} u^2 du + a_{22} \int_{-\frac{1}{2}}^{\frac{1}{2}} u^4 du + a_{23} \int_{-\frac{1}{2}}^{\frac{1}{2}} u^6 du \right) =
$$

$$
= \left( \sqrt[k]{20t^4} + \frac{3}{4} - 1 \right) \left( a_{11} + a_{12} \left( \frac{3}{4} + \frac{a_{13}}{5 \cdot 4^2} \right) \right) + \left( \sqrt[k]{6t^2} + \frac{1}{2} - 1 \right) \left( \frac{a_{21}}{3 \cdot 4} + \frac{a_{22}}{5 \cdot 4^2} + \frac{a_{23}}{7 \cdot 4^{k+2}} \right) =
$$

$$
= \left( \sqrt[k]{20t^4} + \frac{3}{4} - 1 \right) \left( a_{11} \times c_1^{(0)}_{1,1} + a_{12} \times c_1^{(0)}_{1,2} + a_{13} \times c_1^{(0)}_{1,3} \right) +
$$

$$
+ \left( \sqrt[k]{6t^2} + \frac{1}{2} - 1 \right) \left( a_{21} \times c_2^{(1)}_{2,1} + a_{22} \times c_2^{(1)}_{2,2} + a_{23} \times c_2^{(1)}_{2,3} \right) = 0.
$$

Case: $i = 2$

$$
\int_{-\frac{1}{2}}^{\frac{1}{2}} K_2(t, u_1; k) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_3(u_1) \left( \sqrt[k]{t^2} + 1 \right) du_1 + \int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_4(u_1) \left( \sqrt[k]{t^5} + 1 \right) du_1
$$

It’s easy to check for $j \in \{3, 4\}$ the functions $\psi_j(u_1)$ is odd, i.e:

$$
\int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_j(u_1) du_1 = 0 \Rightarrow \int_{-\frac{1}{2}}^{\frac{1}{2}} K_2(t, u_1; k) du_1 = 0.
$$

Thus we have proved

$$(H_k \varphi_0) (t) = 1 + \int_{-\frac{1}{2}}^{\frac{1}{2}} K_1(t, u_1; k) du_1 + \int_{-\frac{1}{2}}^{\frac{1}{2}} K_2(t, u_1; k) du_1 = 1.$$ 

This completes the proof.

Denote

$$
f_1(u) = \sqrt[k]{6u^2} + \frac{1}{2}, \quad f_2(u) = \sqrt[k]{20u^4} + \frac{3}{4}, \quad g_1(u) = \sqrt[k]{u^3} + 1, \quad g_2(u) = \sqrt[k]{u^5} + 1.
$$

**Theorem 9.4** For all $k \geq k_0$ the Hammerstein’s system of equations:

$$
\int_{0}^{1} \tilde{K} \left( t - \frac{1}{2}, u - \frac{1}{2}; k \right) f^k(u) du = g(t), \quad \int_{0}^{1} \tilde{K} \left( t - \frac{1}{2}, u - \frac{1}{2}; k \right) g^k(u) du = f(t) \quad (9.3)
$$

in $(C[0; 1])^2$ have at least four positive solutions with $f \neq g.$
Proof We’ll show
\[
\begin{pmatrix}
    f_1 \left( u - \frac{1}{2} \right), & f_2 \left( u - \frac{1}{2} \right), & f_2 \left( u - \frac{1}{2} \right), & f_1 \left( u - \frac{1}{2} \right)
\end{pmatrix}
\]
and
\[
\begin{pmatrix}
    g_1 \left( u - \frac{1}{2} \right), & g_2 \left( u - \frac{1}{2} \right), & g_2 \left( u - \frac{1}{2} \right), & g_1 \left( u - \frac{1}{2} \right)
\end{pmatrix}
\]
are solutions to the system of equations (9.3).

At first we’ll prove \((f_1(u - \frac{1}{2}), f_2(u - \frac{1}{2}))\) is a solution to equation (9.3). Let \(u - \frac{1}{2} = u_1, t - \frac{1}{2} = t_1\). Then
\[
(H_k f_i) (t) = \int_0^1 K \left( t - \frac{1}{2}, u - \frac{1}{2}; k \right) f_i^k \left( u - \frac{1}{2} \right) du = \int_0^1 K(t_1, u_1; k) f_i^k(u_1) du_1 =
\]
\[
= \int_0^1 [1 + K_1(t_1, u_1; k) + K_2(t_1, u_1; k)] f_i^k(u_1) du_1.
\]
It’s easy to see that
\[
K_2(t_1, -u_1; k) = -K_2(t_1, u_1; k), \quad f_i(u_1) = f_i(-u_1), \quad i \in \{1, 2\}.
\]
Hence
\[
K_2(t_1, -u_1; k) f_i(u_1) = -K_2(t_1, u_1; k) f_i(u_1) \Rightarrow \int_{-\frac{1}{2}}^{\frac{1}{2}} K_2(t_1, u_1; k) f_i(u_1) du_1 = 0.
\]
Thus
\[
(H_k f_i) \left( t - \frac{1}{2} \right) = (H_k f_i) (t_1) = \int_{-\frac{1}{2}}^{\frac{1}{2}} [1 + K_1(t_1, u_1; k)] f_i^k(u_1) du_1. \tag{9.4}
\]
Case: \(i = 1\)
\[
(H_k f_1) \left( t - \frac{1}{2} \right) = (H_k f_1) (t_1) = \int_{-\frac{1}{2}}^{\frac{1}{2}} [1 + K_1(t_1, u_1; k)] \left( \sqrt[6]{u_1^2 + \frac{1}{2}} \right)^k du_1 =
\]
\[
= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( 6u_1^2 \right)^{\frac{1}{2}} du_1 + \int_{-\frac{1}{2}}^{\frac{1}{2}} K_1(t_1, u_1; k) \left( 6u_1^2 + \frac{1}{2} \right) du_1 =
\]
\[
1 + 6 \int_{-\frac{1}{2}}^{\frac{1}{2}} K_1(t_1, u_1; k) u_1^2 du_1 + \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} K_1(t_1, u_1; k) du_1 =
\]
By (9.2) we get
\[
= 1 + 6 \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[ \psi_1(u_1) \left( \sqrt[6]{20t_1^4 + \frac{3}{4}} - 1 \right) + \psi_2(u_1) \left( \sqrt[6]{6t_1^2 + \frac{1}{2} - 1} \right) \right] u_1^2 du_1 =
\]
\[
= 1 + 6 \left( \sqrt[6]{20t_1^4 + \frac{3}{4}} - 1 \right) \int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_1(u_1) u_1^2 du_1 + 6 \left( \sqrt[6]{6t_1^2 + \frac{1}{2} - 1} \right) \int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_2(u_1) u_1^2 du_1.
\]
By Lemma 9.2
\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_1(u_1)u_1^2 du_1 = \frac{1}{6}, \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_2(u_1)u_1^2 du_1 = 0. \tag{9.5}
\]
By (9.2) and (9.5) we obtain
\[
(H_k f_1) \left( t - \frac{1}{2} \right) = 1 + \left( \sqrt{20t^4 + \frac{3}{4}} - 1 \right) = \sqrt{20t^4 + \frac{3}{4}} = f_2(t_1) = f_2 \left( t - \frac{1}{2} \right).
\]
Case: $i = 2$
\[
(H_k f_2) \left( t - \frac{1}{2} \right) = (H_k f_2) (t_1) = \int_{-\frac{1}{2}}^{\frac{1}{2}} [1 + K_1(t_1, u_1; k)] f_2^k(u_1) du_1 =
\]
\[
= \int_{-\frac{1}{2}}^{\frac{1}{2}} [1 + K_1(t_1, u_1; k)] (20u_1^4 + \frac{1}{2}) du_1 =
\]
\[
= \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} K_1(t_1, u_1; k) du_1 + 20 \int_{-\frac{1}{2}}^{\frac{1}{2}} K_1(t_1, u_1; k) u_1^2 du_1 + \int_{-\frac{1}{2}}^{\frac{1}{2}} (20u_1^4 + \frac{1}{2}) du_1 =
\]
By (9.2)
\[
= 1 + 20 \int_{-\frac{1}{2}}^{\frac{1}{2}} K_1(t_1, u_1; k) u_1^2 du_1 =
\]
\[
= 1 + 20 \left( \sqrt{20t^4 + \frac{3}{4}} - 1 \right) \int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_1(u_1) u_1^4 du_1 + 20 \left( \sqrt{6t^2 + \frac{1}{2}} - 1 \right) \int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_2(u_1) u_1^4 du_1.
\]
By Lemma 9.2
\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_1(u_1) u_1^4 du_1 = 0, \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_2(u_1) u_1^4 du_1 = \frac{1}{20}.
\]
Then
\[
(H_k f_2) \left( t - \frac{1}{2} \right) = 1 + \left( \sqrt{6t^2 + \frac{1}{2}} - 1 \right) = \sqrt{6t^2 + \frac{1}{2}} = f_1(t_1) = f_1 \left( t - \frac{1}{2} \right).
\]
By symmetry of $(f_1, f_2)$ we have $(f_2, f_1)$ is also solution to equation (9.3).

Now we’ll prove $(g_1(u - \frac{1}{2}), g_2(u - \frac{1}{2}))$ is a solution to equation (9.3).
For $i \in \{1, 2\}$
\[
(H_k g_i) \left( t - \frac{1}{2} \right) = \int_0^1 \tilde{K} \left( t - \frac{1}{2}, u - \frac{1}{2}; k \right) g_i^k \left( u - \frac{1}{2} \right) du = \int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{K}(t_1, u_1; k) (1 + u_1^{2i+1}) du_1,
\]
where $u_1 = u - \frac{1}{2}$, $t_1 = t - \frac{1}{2}$. Then
\[
(H_k g_i) \left( t - \frac{1}{2} \right) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{K}(t_1, u_1; k) du_1 + \int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{K}(t_1, u_1; k) u_1^{2i+1} du_1 =
\]

By Lemma 9.3

\[ (H_k g_i) \left( t - \frac{1}{2} \right) = 1 + \int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{K}(t, u_1; k) u_1^{2i+1} du_1 = \]

\[ = 1 + \int_{-\frac{1}{2}}^{\frac{1}{2}} [1 + K_1(t_1, u_1; k) + K_2(t_1, u_1; k)] u_1^{2i+1} du_1. \]

Hence

\[ (H_k g_i) \left( t - \frac{1}{2} \right) = 1 + \int_{-\frac{1}{2}}^{\frac{1}{2}} [1 + K_1(t_1, u_1; k)] u_1^{2i+1} du_1 + \int_{-\frac{1}{2}}^{\frac{1}{2}} K_2(t_1, u_1; k) u_1^{2i+1} du_1 \] (9.6)

One can easily check that

\[ K_1(t_1, -u_1; k) = K_1(t_1, u_1; k) \Rightarrow K_1(t_1, -u_1; k)(-u_1^{2i+1}) = -K_1(t_1, u_1; k) u_1^{2i+1}, \]

then

\[ \int_{-\frac{1}{2}}^{\frac{1}{2}} K_1(t_1, u_1; k) u_1^{2i+1} du_1 = 0, \quad i \in \{1, 2\} \] (9.7)

By (9.6) and (9.7) we obtain

\[ (H_k g_i) \left( t - \frac{1}{2} \right) = 1 + \int_{-\frac{1}{2}}^{\frac{1}{2}} u_1^{2i+1} du_1 + \int_{-\frac{1}{2}}^{\frac{1}{2}} K_1(t_1, u_1; k) u_1^{2i+1} du_1 + \int_{-\frac{1}{2}}^{\frac{1}{2}} K_2(t_1, u_1; k) u_1^{2i+1} du_1 = \]

\[ = 1 + \int_{-\frac{1}{2}}^{\frac{1}{2}} K_2(t_1, u_1; k) u_1^{2i+1} du_1 = \]

\[ = 1 + \left( \sqrt{t_1^3 + 1} - 1 \right) \int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_3(u_1) u_1^{2i+1} du_1 + \left( \sqrt{t_1^5 + 1} - 1 \right) \int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_4(u_1) u_1^{2i+1} du_1. \]

Case: \( i = 1 \)

\[ (H_k g_1) \left( t - \frac{1}{2} \right) = 1 + \left( \sqrt{t_1^3 + 1} - 1 \right) \int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_3(u_1) u_1^3 du_1 + \left( \sqrt{t_1^5 + 1} - 1 \right) \int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_4(u_1) u_1^3 du_1 \]

By Lemma 9.2

\[ \int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_3(u_1) u_1^3 du_1 = 0, \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_4(u_1) u_1^3 du_1 = 1. \]

Then

\[ (H_k g_1) \left( t - \frac{1}{2} \right) = 1 + \left( \sqrt{1 + t_1^3} - 1 \right) = \sqrt{1 + t_1^5} = \sqrt{1 + \left( t - \frac{1}{2} \right)^5} = g_2 \left( t - \frac{1}{2} \right). \]

Case: \( i = 2 \)

\[ (H_k g_1) \left( t - \frac{1}{2} \right) = 1 + \left( \sqrt{t_1^3 + 1} - 1 \right) \int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_3(u_1) u_1^5 du_1 + \left( \sqrt{t_1^5 + 1} - 1 \right) \int_{-\frac{1}{2}}^{\frac{1}{2}} \psi_4(u_1) u_1^5 du_1. \]
By Lemma 9.2 we get

\[ (H_k g_1) \left( t - \frac{1}{2} \right) = 1 + \left( \sqrt[3]{t_1^2 + 1} - 1 \right) = \sqrt[3]{t_1^2 + 1} = g_1 \left( t - \frac{1}{2} \right). \]

Thus we have proved

\[ (H_k g_1) \left( t - \frac{1}{2} \right) = g_2 \left( t - \frac{1}{2} \right), \quad (H_k g_2) \left( t - \frac{1}{2} \right) = g_1 \left( t - \frac{1}{2} \right). \]

**Theorem 9.5** Let \( k \geq k_0 \). The model

\[ H(\sigma) = -\frac{1}{\beta} \sum_{<x,y>} \ln \tilde{K} \left( \sigma(x) - \frac{1}{2}, \sigma(y) - \frac{1}{2}; k \right) \]

on the Cayley tree \( \Gamma^k \) has at least four periodic Gibbs measures.

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