CORNER VIEW ON THE CROWN DOMAIN

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1. Introduction

This paper is about the crown domain, henceforth denoted by \( \Xi \), which is the canonical complexification of a Riemannian symmetric space \( X \) of the non-compact type. Specifically we are interested in the nature of the boundary of \( \Xi \) and eventually in good compactifications of \( \Xi \).

Let us begin with some possible definitions of \( \Xi \). Let us denote by \( G \) the connected component of the isometry group of \( X \). Then

\[ X = G/K \]

for \( K < G \) a maximal compact subgroup. Group-theoretically one can view \( X \) as the moduli space of all maximal compact subgroups of the semisimple group \( G \).

Before we advance let us recall some examples.

- For \( G = \text{PSl}(2, \mathbb{R}) \) it is custom to identify \( X \) with the upper half plane \( H = \{ z \in \mathbb{C} \mid \text{Im } z > 0 \} \). Let us refer to \([6]\) for a comprehensive discussion of the corresponding crown domain and the analysis thereon.
- For \( G = \text{PSl}(n, \mathbb{R}) \) one can view \( X \) as the space of positive definite unimodular matrices, i.e. \( X = \text{Sym}(n, \mathbb{R}) \) \( \det=1 \).
- For \( G = \text{PSp}(n, \mathbb{R}) \) one often realizes \( X \) as the Siegel domain \( V + i\Omega \) with \( V = \text{Sym}(n, \mathbb{R}) \) and \( \Omega \subset V \) the cone of positive definite matrices.

Next we discuss complexifications of \( X \). By a complexification we simply mean a connected complex manifold \( \Xi \) which contains \( X \) as a totally real submanifold. We wish to request that complexification respects symmetry, i.e. the action of \( G \) on \( X \) extends to \( \Xi \).

The natural candidate for a complexification seems to be \( X_C = G_C/K_C \) with \( G_C \) and \( K_C \) the universal complexifications of \( G \) and \( K \) respectively. We often call \( X_C \) the affine complexification of \( X \) as it is an affine variety. For example for \( X = \text{Sym}(n, \mathbb{R}) \) \( \det=1 \) one has \( X_C = \text{Sym}(n, \mathbb{C}) \) \( \det=1 \).

Let us be more demanding on our complexification and request that the Riemannian metric of \( X \) extends to a \( G \)-invariant metric on \( \Xi \) (see \([9]\), Sect. 4). This is now a severe restriction on \( \Xi \) as it forces the \( G \)-action on \( \Xi \) to be proper. For instance \( G \) does not act properly on \( X_C \) \(^1\) and this guides us to smaller \( G \)-domains in \( X_C \) where \( G \) acts properly. The study of proper \( G \)-actions on \( X_C \) began in \([7]\) and a complete classification of all maximal open \( G \)-neighborhoods of \( X \) in

\(^1\) The \( G \)-stabilizer of \( \text{diag}(i, -i) \in \text{Sym}(2, \mathbb{C}) \) \( \det=1 \) is \( \text{PSO}(1, 1) \) and not compact.
$X_C$ with proper action was obtained in [7]. In view of the results of [7] one can define the crown domain as the intersection of all maximal open $G$-neighborhoods of $X$ in $X_C$ with proper action ([7]).

Let us give another construction of $\Xi$. Denote by $TX$ the tangent bundle of $X$. This is a homogeneous vector bundle over $X$ on which $G$ acts properly. There is a natural $G$-map from $TX$ to $X_C$ (see [9], Sect. 4) and the crown domain corresponds to the maximal $G$-neighborhood of $X$ in $TX$ which embeds into $X_C$.

A third and perhaps preferred way to define the crown domain is as the universal domain for holomorphically extended orbit maps of unitary spherical representations of $G$ (see the introduction of [8]).

Let us turn now to the subject proper of this paper, the topological boundary $\partial \Xi$ of $\Xi$ in $X_C$. The boundary is a very complicated object and there is little hope to obtain an explicit description. However $\partial \Xi$ features some structure; for instance, in it one finds the distinguished boundary $\partial_d \Xi \subset \partial \Xi$, introduced in [3]. The distinguished boundary is some sort of Shilov boundary of $\Xi$ in the sense that it is the smallest closed subset in $\partial \Xi$ on which bounded plurisubharmonic functions on $\overline{\text{cl}(\Xi)}$ attend their maximum.

We know from [3] and [8] that $\partial_d \Xi$ is a finite (and explicit) union of $G$-orbits, say

$$\partial_d \Xi = \mathcal{O}_1 \amalg \ldots \amalg \mathcal{O}_s.$$ 

From now on we shall identify each $\mathcal{O}_j$ with a homogeneous space: $G/H_j$. The main result of [3] was:

If $G/H_j$ is a symmetric space, then it is a non-compactly causal symmetric space. Moreover, every non-compactly causal symmetric space $Y = G/H$ appears in the distinguished boundary of the corresponding crown domain for $X = G/K$.

One aim of this paper is to understand this result better. To be more concise: what is the reason that precisely non-compactly causal (NCC) symmetric spaces appear in the boundary? As we will see, answering this question will eventually reveal the structure of $\partial \Xi$.

NCC-spaces are very special among all semisimple symmetric spaces. We recall their definition (see [5]). We assume the Lie algebra of $G$ to be simple and write $\mathfrak{q}$ for the tangent space of $Y$ at the standard base point $y_0 = H \in Y$. We note that $\mathfrak{q}$ is a linear $H$-module. Now, non-compactly causal means that $\mathfrak{q}$ admits an non-empty open $H$-invariant convex cone, say $C$, which is hyperbolic and does not contain any affine lines.
The theme of this paper is to view $\Xi$ from the corner point $y_0 \in Y$ and not as a thickening of $X$ as in the customary definitions from above. Now a slight precisioning of terms is necessary. As we saw, $\partial d\Xi$ might have several connected components. If this happens to be the case, then we shrink $\Xi$ to a $G$-domain $\Xi_H$ whose distinguished boundary is precisely $Y$, see \cite{4}.

For $C \subset \mathfrak{q}$ the minimal cone (see \cite{5}) we form in the tangent bundle $TY = G \times_H \mathfrak{q}$ the cone-subbundle

$$C = G \times_H C$$

and with that its boundary cone-bundle

$$\partial C = G \times_H \partial C.$$ 

In this context we ask the following

**Question:** Is there a $G$-equivariant, generically injective, proper continuous surjection $p : \partial C \to \partial F_H$?

In other words, we ask if there exists an equivariant "resolution" of the boundary in terms of the geometrically simple boundary cone bundle $\partial C$.

In this paper we give an affirmative answer to this question if $X$ is a Hermitian tube domain. In this simplified situation the crown domain is $\Xi = X \times X$ with $\overline{X}$ denoting $X$ but endowed with the opposite complex structure (i.e., if $X$ is already complex, then the crown is the complex double). On top of that $\partial d\Xi = Y$ is connected, i.e. $\Xi = \Xi_H$.

I wish to point out that the presented method of proof will not generalize. In order to advance one has to understand more about the structure of the minimal cone $C$; one might speculate that some sort of "$H \cap K$-invariant theory" for $C$ could be useful.

Let me pose two open problems:

**Problem 1:** For general $\Xi$, does $\partial \Xi_H$ admit a resolution as a cone bundle in the sense described above.

**Problem 2:** Construct $G$-equivariant compactification of $\Xi$, resp. $\Xi_H$.

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2. Main part

Let \( X = G/K \) be a Hermitian symmetric space of tube type. This means that there is an Euclidean (or formally real) Jordan algebra \( V \) with positive cone \( W \subset V \) such that

\[
X = V + iW \subset V_\mathbb{C}.
\]

The action of \( G \) is by fractional linear transformation and our choice of \( K \) is such it fixes the base point \( x_0 = ie \) with \( e \in V \) the identity element of the Jordan algebra.

It is no loss of generality if we henceforth restrict ourselves to the basic case of \( G = \text{Sp}(n, \mathbb{R}) \) – the more general case is obtained by using standard dictionary which can be found in text books, e.g. \([2]\).

For our specific choice, the Jordan algebra is \( V = \text{Sym}(n, \mathbb{R}) \) and \( W \subset V \) is the cone of positive definite symmetric matrices. The identity element \( e \) is \( I_n \), the \( n \times n \) identity matrix. The group \( G \) acts on \( X \) by standard fractional linear transformations:

\[
g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \text{ with appropriate } a, \ldots, d \in M(n, \mathbb{R}) \text{ acts as} \]

\[
g \cdot z = (az + b)(cz + d)^{-1} \quad (z \in X).
\]

The maximal compact subgroup \( K \) identifies with \( U(n) \) under the standard embedding

\[
U(n) \to G, \quad u + iv \mapsto \begin{pmatrix} u & v \\ -v & u \end{pmatrix} \quad (u, v \in M(n, \mathbb{R})).
\]

It is then clear that \( K = U(n) \) is the stabilizer of \( x_0 = iI_n \). In the sequel we consider \( V_\mathbb{C} \) as an affine piece of the projective variety \( \mathcal{L} \) of Lagrangians in \( \mathbb{C}^{2n} \); the embedding is given by

\[
V_\mathbb{C} \mapsto \mathcal{L}, \quad T \mapsto L_T := \{(T(v), v) \mid v \in \mathbb{C}^n\}.
\]

It is then clear that \( G_\mathbb{C} = \text{Sp}(n, \mathbb{C}) \) acts on \( \mathcal{L} \); in symbols:

\[
g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_\mathbb{C} \text{ with appropriate } a, \ldots, d \in M(n, \mathbb{C}) \text{ acts as}
\]

\[
g \cdot L = \{(av + bw, cv + dw) \mid (v, w) \in L\} \quad (L \in \mathcal{L}).
\]

The space \( \mathcal{L} \) is homogeneous under \( G_\mathbb{C} \). If we choose the base point

\[
x_0 \leftrightarrow L_0 = \{(iv, v) \mid v \in \mathbb{C}^n\},
\]

then the stabilizer of \( x_0 \) in \( G_\mathbb{C} \) is the Siegel parabolic

\[
S^+ = K_\mathbb{C} \ltimes P^+ \quad \text{and} \quad P^+ = \left\{ 1 + \begin{pmatrix} u & -iu \\ -iu & -u \end{pmatrix} \mid u \in V_\mathbb{C} \right\}.
\]
Thus we have
\[ \mathcal{L} = G_C \cdot L_0 \simeq G_C / S^+ . \]
Sometimes it is useful to take the conjugate base point \( \overline{x}_0 = -iI_n \).
Then the stabilizer of \( \overline{L}_0 \) in \( \mathcal{L} \) is the opposite Siegel parabolic
\[ S^- = K_C \ltimes P^- \quad \text{and} \quad P^- = \left\{ 1 + \begin{pmatrix} u & iu \\ iu & -u \end{pmatrix} \mid u \in V_C \right\} \]

and
\[ \mathcal{L} = G_C \cdot \overline{L}_0 \simeq G_C / S^- . \]
Next we come to the realization of the affine complexification \( X_C = G_C / K_C \). We consider the \( G_C \)-equivariant embedding
\[ X_C \rightarrow \mathcal{L} \times \mathcal{L}, \quad gK_C \mapsto (g \cdot L_0, g \cdot \overline{L}_0) . \]
It is not hard to see that
\[ X_C = \{(L, L') \in \mathcal{L} \times \mathcal{L} \mid L + L' = \mathbb{C}^{2n}\} , \]
i.e., \( X_C \) is the affine variety of pairs of transversal Lagrangians.
Set \( \overline{X} = V - iW \) and note that the map \( z \mapsto \overline{z} \) identifies \( X \) with \( \overline{X} \)
in a \( G \)-equivariant, but antiholomorphic manner.
Next we come to the subject matter, the crown domain of \( X \):
\[ \Xi = X \times \overline{X} \subset X_C. \]

Let us denote by \( \partial \Xi \) the topological boundary of \( \Xi \) in \( X_C \). The goal is to resolve \( \partial \Xi \) by a cone bundle over the affine symmetric space \( Y = G / H \) where \( H = \text{Gl}(n, \mathbb{R}) \) is the structure group of the Euclidean Jordan algebra \( V \).
We define an involution \( \tau \) on \( G \) by
\[ \tau(g) = I_{n,n}gI_{n,n} \quad \text{where} \quad I_{n,n} = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} . \]
The fixed point set of \( \tau \) is
\[ H = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in \text{Gl}(n, \mathbb{R}) \right\} = \text{Gl}(n, \mathbb{R}) . \]
We write \( \mathfrak{h} \) for the Lie algebra of \( H \) and denote by \( \tau \) as well the derived involution on \( \mathfrak{g} \). The \( \tau \)-eigenspace decomposition on \( \mathfrak{g} \) shall be denoted by
\[ \mathfrak{g} = \mathfrak{h} + \mathfrak{q} \quad \text{where} \quad \mathfrak{q} = \begin{pmatrix} 0 & V \\ V & 0 \end{pmatrix} . \]
Write \( q^+ = \begin{pmatrix} 0 & V \\ 0 & 0 \end{pmatrix} \) and \( q^- = \begin{pmatrix} 0 & 0 \\ V & 0 \end{pmatrix} \) and note that

\[
q = q^+ \oplus q^- 
\]

is the splitting of \( q \) into two inequivalent irreducible \( H \)-modules.

The affine space \( Y = G/H \) admits (up to sign) a unique \( H \)-invariant convex open cone \( C \subset q \), containing no affine lines and consisting of hyperbolic elements. Explicitely:

\[
C = \begin{pmatrix} 0 & W \\ W & 0 \end{pmatrix} = W \oplus W \subset q^+ \oplus q^- .
\]

We form the cone bundle

\[
C = G \times_H C
\]

and note that there is a natural \( G \)-equivariant map

\[
P : G \times_H C \to \Xi, \quad [g, (y_1, y_2)] \mapsto g \cdot (iy_1, (iy_2)^{-1}).
\]

Let us verify that this map is in fact defined. For that one needs to check that for \( h \in H \) and \( y_1, y_2 \in W \), the elements \((h, y_1, y_2)\) and \((1, hy_1h^t, h^{-t}y_2h^{-1})\) have the same image. Indeed,

\[
h \cdot (iy_1, (iy_2)^{-1}) = (ihy_1h^t, h(iy_2)^{-1}h^t) = (ihy_1h^t, (ih^{-t}y_2h^{-1})^{-1})
\]

which was asserted.

**Lemma 2.1.** The map \( P : C \to \Xi \) is onto.

**Proof.** Write \( A \) for the group of diagonal matrices in \( G \) with positive entries. Note that the Lie algebra \( a \) of \( A \) is a maximal flat in \( \mathfrak{p} = \mathfrak{g} \cap \text{Sym}(2n, \mathbb{R}) \). In general, we know that \( \mathfrak{p} = \text{Ad}(K)\mathfrak{a} \). Furthermore, if \( W_d \) denotes the diagonal part of \( W \), then \( iW_d = A \cdot x_0 \). From \( G = KAK \) it now follows that for any two points \((z, w) \in X\) there exist a \( g \in G \) such that \( g \cdot (z, w) = (x_0, w') \) with \( w' \in iW_d \). As a consequence we obtain that

\[
\Xi = G \cdot (iW_d, -iI_n).
\]

Clearly the right hand side is contained in the image of \( P \) and this finishes the proof. \( \square \)

**Remark 2.2.** (a) The map \( P \) is not injective. We shall give two different arguments for this assertion, beginning with an abstract one. If \( P \) were injective, then \( P \) establishes an homeomorphism between \( \Xi \) and \( C = G \times_H C \). In particular \( \Xi \) is homotopy equivalent to \( Y = G/H \). But we know that \( \Xi \) is contractible; a contradiction.
More concretely for \( k \in K, k \neq 1 \), the elements \([k, (iI_n, -iI_n)] \neq [1, (iI_n, -iI_n)]\) have the same image in \( \Xi \). It should be remarked however, that the map is generically injective.

(b) As \( H \) acts properly on \( C \), it follows that \( G \) acts properly on the cone-bundle \( G \times_H C \). Further it is not hard to see that the map \( P \) is proper.

We need a more invariant formulation of the map \( P \). For that, note that the rational map \( V_C \to V_C, z \mapsto -z^{-1} \) belongs to \( K \). Its extension to \( L \), shall be denoted by \( s_0 \) and is given by

\[
s_0(L) = \{(-w, v) \in \mathbb{C}^{2n} | (v, w) \in L\}.
\]

Also, the anti-symplectic map \( V_C \to V_C, z \mapsto -z \) has a natural extension to \( L \) given by

\[
L \mapsto -L := \{(-v, w) \in \mathbb{C}^{2n} | (v, w) \in L\}.
\]

In this way, we can rewrite \( P \) as

\[
P : G \times_H C \to \Xi, \quad [g, (y_1, y_2)] \mapsto g \cdot (iy_1, -s_0(iy_2))
\]

and we see that \( P \) extends to a continuous map

\[
\hat{P} : G \times_H L \to L \times L, \quad [g, (y_1, y_2)] \mapsto g \cdot (iy_1, -s_0(iy_2)).
\]

We restrict \( \hat{P} \) to \( G \times_H \partial C \) and call this restriction \( p \). It is clear that \( \text{im } p \) is contained in the boundary of \( \Xi \) in \( L \times L \). But even more is true: the following proposition constitutes a \( G \)-equivariant “resolution” of \( \partial \Xi \).

**Proposition 2.3.** \( \text{im } p \subset \partial \Xi \) and the \( G \)-equivariant map

\[
p : G \times_H \partial C \to \partial \Xi, \quad [g, (y_1, y_2)] \mapsto g \cdot (iy_1, -s_0(iy_2))
\]

is onto and proper.

**Proof.** We first show that \( \text{im } p \subset \partial \Xi \). This means that \( \text{im } p \subset X_C \). In fact, from Lemma 2.1 and the definition of \( p \) it follows that \( \text{im } p \) is contained in the closure of \( \Xi \) in \( L \times L \) and does not intersect \( \Xi \).

Let us now show that \( \text{im } p \subset X_C \). First note that

\[
(2.1) \quad \partial C = W \times \partial W \sqcup \partial W \times \partial W \sqcup \partial W \times W.
\]

Thus the assertion will certainly follow if we verify the following slightly stronger statement: for \( y_1, y_2 \in \text{cl}(W) \) the Lagrangians

\[
L_1 = \{(iy_1 v, v) | v \in \mathbb{C}^n\} \quad \text{and} \quad L_2 = \{(w, iy_2 w) | w \in \mathbb{C}^n\}
\]
are transversal. We use the structure group $H$ to bring $y_1$ in normal form

$$y_1 = \text{diag}(1, \ldots, 1, 0, \ldots, 0).$$

Thus $(iy_1v, v) = (w, iy_2w)$ for some $v, w \in \mathbb{C}^n$ means explicitly that

$$(iv_1, iv_2, \ldots, iv_p, 0, \ldots, 0; v_1, \ldots, v_n) = (w_1, \ldots, w_n; iy_2(w)).$$

We conclude that $w_{p+1} = \ldots = w_n = 0$. If $p = 0$, then we are finished. So let us assume that $p > 0$. But then

$$y_2 = \begin{pmatrix} -I_p & * \\ * & * \end{pmatrix},$$

and this contradicts the fact that $y_2$ is positive semi-definite.

We turn our attention to the onto-ness of $p$. For that note that the closure $\text{cl}(X)$ in $\mathcal{L}$ equals the geodesic compactification. As a result $\partial X = K \cdot (i\partial W_d) = K \cdot (i\partial W)$. Likewise $\partial X = K \cdot (-i\partial W)$. Observe that

$$\partial \Xi = [X \times \partial X \amalg \partial X \times X \amalg \partial X \times X] \cap X_C.$$  

We first show that $X \times \partial X \subset \text{im} \ p$, even more precisely $p(G \times_H (W \times \partial W)) = X \times \partial X$. In fact,

$$X \times \partial X = G \cdot (iI_n, K \cdot i\partial W) = G \cdot (iI_n, i\partial W)$$

and the claim is implied by (2.1). In the manner one verifies that $\partial X \times X \subset \text{im} \ p$.

In order to conclude the proof it is now enough to show that $p$ is proper. This is because proper maps are closed and we have already seen that $\text{im} \ p$ contains the dense piece $X \times \partial X \amalg \partial X \times X \subset \partial \Xi$. Now to see that $p$ is proper, it is enough to show that inverse images of compact subsets in $[\partial X \times \partial X] \cap X_C$ are compact. For the other pieces in $\partial \Xi$ this is more or less automatic: Use that $G$ acts properly on $X$, resp. $\overline{X}$ which implies that $G$ acts properly on $X \times \partial X$ resp. $\partial X \times \overline{X}$; likewise $G$ acts properly on $G \times_H (W \times \partial W)$ and $G \times_H (\partial W \times W)$. Thus we are about to show that preimages of compacta in $[\partial X \times \partial X] \cap X_C$ are again compact. But this is more or less immediate from transversality; I allow myself to skip the details.

\begin{remark}
For $n = 1$ the map $p$ is in fact a homeomorphism which we showed in [8]. If $n > 1$, the map $p$ fails to be injective by the same computational reason shown in the preceding remark. However,
\end{remark}
we emphasize that the map is generically injective and that $p|_{\partial C}$ is injective.

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