A Poincaré Lemma for Sigma Models of AKSZ Type

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A Poincaré lemma for sigma models of AKSZ type

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\textbf{ABSTRACT.} For a sigma model of AKSZ-type with target space a $Q$-manifold, we show that the cohomology in the space of local functionals of the differential associated to the BV master action is locally isomorphic to the cohomology of $Q$ in target space. An analogous result is shown to hold for the cohomology in the space of functional multivectors. Applications in the context of the inverse problem of the calculus of variation for gauge systems are briefly discussed.

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1 Introduction

The Batalin-Vilkovisky formalism has originally been devised as a means to control gauge symmetries during perturbative quantization of systems with a complicated gauge algebra [1, 2, 3, 4, 5] (see e.g. [6, 7] for reviews). In this context, some questions of physical interest, such as the classification of divergences or anomalies arising during renormalization can be efficiently reformulated in terms of “local BRST cohomology”, i.e., the cohomology groups of the antifield-dependent BRST differential in the space of local functionals (see e.g. [10, 11] for reviews). On the classical level, these groups control the deformation theory for gauge systems and encode generalized global symmetries and conservations laws.

Even though the BV master action is usually constructed using an algorithmic procedure with input a classical action and a generating set of gauge symmetries, it is sometimes more natural to define a theory directly in terms of a master action. This is the case for instance for gauge field theories associated to BRST first quantized systems (see e.g. [12] for a review), such as open string field theory or higher spin gauge fields [13, 14, 15, 16].

Another class of models that falls into this category are AKSZ sigma models [17], for which the master action on the space of maps is directly constructed out of the geometrical data of the base and target manifolds. In the AKSZ case, the target space is a $QP$ manifold, a supermanifold equipped with a graded symplectic structure and a compatible homological vector field. We will restrict ourselves here to the case where the base space is $\Pi T^*X_0$, the tangent space to a manifold $X_0$ with shifted parity of the fibers equipped with the de Rham differential.

In the context of massless higher spin gauge fields, and in particular in the so-called unfolded formulation [18, 19, 20, 21], the focus is in a first stage on the equations of motion, whether they derive from an action principle or not. When translated in BRST language, this amounts to defining a theory through a differential which is not necessarily generated through the adjoint action of a master action in an appropriate antibracket. For the AKSZ construction, this means that one is mainly interested in the $Q$ structure and forgets about the $P$ structure. Such non-Lagrangian AKSZ-type sigma models are directly related to a BRST extended version of the non-linear unfolded formalism [22, 23].

In the same way as in the applications to soliton equations or quantum field theory, algebraic control on the space of maps for sigma models can be achieved in the context of the formal variational calculus, where derivatives of fields are considered as independent

\footnote{With perturbative quantum field theory in mind, one might be tempted to use the gauge fixed, on-shell nilpotent BRST differential. Why it is far more transparent to fix the gauge through a canonical transformation while keeping the antifields instead of reducing to a Lagrangian submanifold when taking locality into account is explained in [8, 9].}
coordinates on so-called jet-spaces and local functionals are quotients of horizontal $n$-forms modulo exact ones (see e.g. [24, 25, 26] for reviews).

The purpose of this paper is to show that, in a coordinate neighborhood, the local BRST cohomology for Lagrangian and non-Lagrangian AKSZ-type sigma models, and thus also for BRST extended unfolded models, is isomorphic to the $Q$-cohomology in target space.

The result that, locally, the field theoretic BRST cohomology is isomorphic to target space cohomology can be extended to the cohomology computed in the space of functional multivector. These cohomology groups become important in the non-Lagrangian setting. For instance, they control consistent deformations and global symmetries in this context. We also show that the cohomology for higher multivectors controls weak Poisson structures and their counterparts in the Lagrangian formalism known as Lagrange structures [27, 28]. They are therefore relevant for the inverse problem of the calculus of variations applied to gauge systems.

The drastic simplification of the field theoretic cohomology is not really surprising in view of the structure of the BRST differential for AKSZ sigma models. Nevertheless, the precise isomorphism that we have established gives concrete meaning to the notion of background independence for non-Lagrangian AKSZ-type sigma models.

More interesting is the global situation with non trivial topology. In a global approach, it is known how the cohomology of the bundle of target over base space is reflected in the cohomology of the variational bicomplex [29, 24]. What one then needs to analyze is how this latter cohomology affects the so-called descent equations that are used to compute the BRST cohomology in the space of local functionals from the cohomology in the space of horizontal forms. An elementary example of this interplay has been given in the context of Einstein gravity, where the target space has non trivial topology due to the determinant condition on the metric [30]. The appropriate framework to address this global question is likely to be the $C$-spectral sequence by Vinogradov (see [31, 32, 33, 34] and references therein). We plan to return to the global question elsewhere.

Let us end this introduction by briefly reviewing related literature.

The first AKSZ sigma model for which local BRST cohomology has been explicitly computed and shown to reduce to a cohomology problem in target space is Chern-Simons [35, 36, 37] (see also [38, 11]).

For general AKSZ sigma models, we have followed in our paper the general strategy proposed for BF theory in [39, 40] and reviewed in [10]. The proof in these papers is, however, incomplete as the contractible pairs have not been correctly identified. The correct identification has been discussed in details using Young diagrams in [41], although in a slightly different context: these authors considered the gauge part of models involving form-fields, whereas here we need to apply their results to the linear part of the antifield-
dependent BRST differential.

In the non-linear unfolded off-shell formalism, it has been shown in \cite{23} that the $Q$-cohomology in target space gives rise to interesting field theoretic invariants like actions and conserved charges (see also \cite{22}). From this perspective, what we have shown here is in some sense the inverse statement: all field theoretic invariants which can be represented as BRST cohomology classes in the space of local functionals in the fields and their space-time derivatives arise from $Q$-cohomology in the case of AKSZ type sigma models.

While completing this paper, we came across reference \cite{42}, where a reduction of the BV formalism for AKSZ sigma models to source cohomology is discussed along different lines. The assumptions underlying our main result amount to restricting the source cohomology to just the volume form. The result can then be understood as a concrete proof that, for AKSZ sigma models, this reduction does indeed occur in cohomology, or in other words, that there is a quasi-isomorphism between the classical field theoretic BV formalism and the classical BV formalism in target space. From a technical point of view, this concrete proof is possible because in the jet-space approach, there is precise algebraic control over the space of maps.

As said above, it would be interesting to extend this concrete proof to the case of non trivial topology. Concerning the quantum BV formalism, it is clear from renormalization theory that the coupling constants need to play an active role in a precise, non formal, definition of the field theoretic $\Delta$-operator. How this can done on the level of cohomology is discussed in \cite{43, 44}.

\section{AKSZ construction}

\subsection{BRST differential}

\textit{The construction of the field theoretic BRST differential on the space of maps from differentials in base and target space is briefly recalled.}

Consider two $Q$ manifolds, i.e., supermanifolds equipped with an odd nilpotent vector field \cite{45}. The first, called the base manifold, is denoted by $\mathcal{X}$. It is equipped with a grading $gh_{\mathcal{X}}$ and its odd nilpotent vector field is denoted by $d$, $gh_{\mathcal{X}}(d) = 1$. Furthermore, the existence of a volume form $d\mu$ preserved by $d$ is also assumed. As implied by the notation, the basic example for $\mathcal{X}$ is the odd tangent bundle $\Pi T\mathcal{X}_0$ to some manifold $\mathcal{X}_0$ which has a natural volume form and is equipped with the de Rham differential. We restrict ourselves to this case below. If $x^\mu$ and $\theta^\mu$ are coordinates on $\mathcal{X}$ and the fibres of $\Pi T\mathcal{X}_0$ respectively, the differential and the volume form are given explicitly by

$$d = \theta^\mu \frac{\partial}{\partial x^\mu}, \quad d\mu = dx^0 \ldots dx^{n-1} d\theta^{n-1} \ldots d\theta^0 \equiv d^n x d^n \theta, \quad n = \dim \mathcal{X}_0. \quad (2.1)$$
The second supermanifold, called the target manifold, is denoted by $\mathcal{M}$ and equipped with another degree $\text{gh}_{\mathcal{M}}$. The odd nilpotent vector field is denoted by $Q$ and $\text{gh}_{\mathcal{M}}(Q) = 1$.

Consider then the manifold of maps from $\mathcal{X}$ to $\mathcal{M}$. This space is naturally equipped with the total degree $\text{gh}(A) = \text{gh}_{\mathcal{M}}(A) + \text{gh}_{\mathcal{X}}(A)$ and an odd nilpotent vector field $s$, $\text{gh}(s) = 1$. Using local coordinates $x^\mu$, $\theta^\mu$ on $\mathcal{X}$ and $\Psi^A$ on $\mathcal{M}$ the expression for $s$ is given by

$$s = \int_{\mathcal{X}} d^n x d^n \theta \left[ d \Psi^A(x, \theta) + Q^A(\Psi(x, \theta)) \right] \frac{\delta}{\delta \Psi^A(x, \theta)}. \quad (2.2)$$

Vector field $s$ can be considered as the BRST differential of a field theory on $\mathcal{X}_0$ and the construction described above is the non-Lagrangian part of the AKSZ approach. Indeed, the field space, BRST differential, and ghost grading determine a gauge field theory for which the physical fields can be identified with those carrying ghost number zero, while the equations of motion, gauge symmetries, and higher structures of the gauge algebra are encoded in the BRST differential.

### 2.2 Bracket and BV master action

The construction of the field theoretic bracket on the space of maps from a target space (odd) Poisson bracket is recalled.

In the case where the target manifold $\mathcal{M}$ is in addition equipped with a compatible (odd) Poisson bracket $\{ \cdot, \cdot \}_{\mathcal{M}}$ and $Q = \{ S, \cdot \}_{\mathcal{M}}$ is generated by a master function $S$, i.e., a function satisfying the classical master equation $\frac{1}{2} \{ S, S \}_{\mathcal{M}} = 0$, one can construct a functional $S$ on the space of maps that can be interpreted either as the BV master action or the BRST charge of the BFV Hamiltonian approach of the field theory on $\mathcal{X}_0$.

More precisely, let $E^{AB} = \{ \Psi^A, \Psi^B \}_{\mathcal{M}}$ be an (odd) Poisson bivector for the bracket $\{ \cdot, \cdot \}_{\mathcal{M}}$. The associated bracket on the space of maps is given by

$$\{ F, G \} = (-1)^{(\text{gh}(F) + n)k} \int_{\mathcal{X}} d^n x d^n \theta \left( \frac{\delta R F}{\delta \Psi^A(x, \theta)} E^{AB}(\Psi(x, \theta)) \frac{\delta G}{\delta \Psi^B(x, \theta)} \right). \quad (2.3)$$

Here $F = F[\Psi], G = G[\Psi]$ are functionals on the space of maps. If the bracket on $\mathcal{M}$ carries Grassmann parity $\kappa$ and ghost number $k$, parity and ghost number of the functional Poisson bracket are given respectively by $\kappa + n \mod 2$ and $k + n$. There is a natural map from target space functions to functionals on the space of maps: given a target space function $f$ one defines

$$\mathcal{I}(f) = \int_{\mathcal{X}} \Psi^* f = \int_{\mathcal{X}} d^n x d^n \theta f(\Psi(x, \theta)), \quad (2.4)$$

2 More generally, one could of course consider the space of sections of a bundle over $\mathcal{X}$ with fibers isomorphic to $\mathcal{M}$.

3 See e.g. [46] for a review and [47] for further developments in the non Lagrangian context.

4 The latter identification for an odd $S$ of ghost number 1 was proposed in [48]. We follow the conventions from this reference for brackets and functional derivatives.
where \( \Psi^* f \) is the pull-back of \( f \) on \( \mathcal{M} \) to \( \mathcal{X} \) by the map \( \Psi \). The map \( \mathcal{I} \) is compatible with the differentials in the sense that
\[
\mathcal{I}(Qf) = s \mathcal{I}(f),
\]
when the map \( \Psi \) is of compact support. Moreover, \( \mathcal{I} \) is a homomorphism of graded Lie superalgebras
\[
\mathcal{I}(\{f, g\}_\mathcal{M}) = \int_\mathcal{X} \{\mathcal{I}(f), \mathcal{I}(g)\},
\]
provided one shifts by \( n = \text{dim} \mathcal{X}_0 \) the ghost number and the Grassmann parity for functions on \( \mathcal{M} \) in order to make \( \mathcal{I} \) compatible with the gradings.

If in addition the bracket \( \{\cdot, \cdot\}_\mathcal{M} \) is non degenerate, i.e., if \( \mathcal{M} \) is equipped with a symplectic structure \( E_{AB} \) determined by \( E_{AB}E^{BC} = \delta^C_A \) and a symplectic potential \( V_A \) can be defined through \( E_{AB} = (\partial_AV_B - (-1)^{|A||B|}\partial_BV_A)(-1)^{|B|(|E|+1)} \), the functional vector field induced by \( d \) is Hamiltonian. Combining this with the Hamiltonian induced by \( S \), one obtains the functional,
\[
S[\Psi] = \int d^n x d^n \theta \left[ (d \Psi^A(x, \theta)) V_A(\Psi(x, \theta)) + S(\Psi(x, \theta)) \right],
\]
so that \( s = \{S, \cdot\} \). Parity and ghost number of \( S \) are \( |S| = |S| - n \mod 2 \) and \( \text{gh}(S) = \text{gh}(S) - n \). In particular, if \( |S| = \text{gh}(S) = 0 \), functional \( S \) is to be interpreted as a BV master action, while if \( |S| = \text{gh}(S) = 1 \), functional \( S \) becomes the BRST charge of a field theory on \( \mathcal{X}_0 \).

This approach was originally proposed in [17] as a method for constructing the BV formulation of topological sigma models. Further developments can be found in [49, 48, 50, 51, 52, 53, 54] and references therein.

2.3 Examples

Some standard and not so standard examples of AKSZ sigma models are briefly reviewed.

Chern-Simons theory

The first example of an AKSZ sigma model discussed in [17] is Chern-Simons theory. It corresponds to taking \( \mathcal{M} = \Pi G \) where \( G \) is a Lie algebra equipped with an invariant non degenerate metric \( g_{ab} \). This metric determines a non degenerate Poisson structure on \( \Pi G \). The AKSZ construction then gives the standard BV master action for the Chern-Simons theory provided one takes \( \mathcal{X}_0 \) to be a 3-dimensional manifold.
Note that the BRST differential is well defined for $X_0$ of any dimension and does not require an invariant bilinear form. Such a BRST differential describes the zero-curvature equations for a $G$-connection and its natural gauge symmetry.

**Poisson sigma model**

The Poisson sigma model \cite{55, 56} can also be formulated in the AKSZ framework \cite{49}. As a target space, one takes $\mathcal{M} = \Pi T^* N$, with $N$ a Poisson manifold. If $X^i, C_i$ are local coordinates on $\mathcal{M}$, the $QP$ structure is determined by

$$\{X^i, C_j\}_\mathcal{M} = \delta^i_j, \quad Q = \{S, \cdot\}_\mathcal{M}, \quad S = \frac{1}{2} C_i \alpha^{ij}(X) C_j,$$

where $\alpha^{ij} \partial_i \wedge \partial_j$ is a Poisson bivector. The homological vector field $Q$ defines the Poisson cohomology on $\mathcal{M}$. As a spacetime, one takes a 2 dimensional manifold $X_0$. The associated AKSZ master action is then the standard BV master action for the Poisson sigma model.

**BF theories**

For BF theories, the base space is $\mathcal{X} = \Pi T X_0$ where $X_0$ is an $n$-dimensional manifold. The target space is $\mathcal{M} = \Pi T^* (\Pi G)$ for even $n$ and $\mathcal{M} = T^* (\Pi G)$ for odd $n$ with its canonical odd (even) symplectic structure. Using the standard coordinates $c^a, b_a$ on $\mathcal{M}$, with $\text{gh}(e^a) = 1, \text{gh}(b_a) = n - 2$, the $QP$ structure is determined by

$$\{b_a, c^b\}_\mathcal{M} = \delta^b_a, \quad Q = \left\{\frac{1}{2} b_a f^{a}_{bc} c^b c^c, \cdot\right\}_\mathcal{M}.$$

(2.9)

Note that the bracket carries ghost number $1 - n$ so that it induces the standard BV antibracket on the space of maps. That the associated AKSZ master action is indeed the standard master action for non-abelian BF theory follows in particular from the fact that in ghost number zero, the field content consists of 1-forms and $n - 2$-forms.

**Hamiltonian BFV systems with vanishing Hamiltonian**

Let us take as $\mathcal{M}$ the extended phase space of the Hamiltonian BFV formulation of a first class constrained system \cite{57, 58, 59} (see also \cite{60}). Such a system is described by a phase space $\mathcal{M}$, with coordinates $\Psi^A$ and a symplectic structure with potential $V_A$, an associated non degenerate Poisson structure $\{\cdot, \cdot\}_\mathcal{M}$, a BRST charge $\Omega$ and a BRST invariant Hamiltonian $H$. The associated BV formulation is governed by a master action that can be directly constructed out of $\Omega$ and $H$ \cite{61, 62, 63, 64}. It was shown in \cite{48} that, in the case of vanishing Hamiltonian $H$, it is an AKSZ sigma model with target...
space the symplectic manifold $M$, target space differential generated by the BRST charge, $Q = \{\Omega, \cdot\}$ and base space $X_0$ a “time” line. The master action can be written as

$$S = \int dt d\theta \left[ (d \Psi A(t, \theta)) V_A(\Psi(t, \theta)) + \Omega(\Psi(t, \theta)) \right].$$

where $d = \theta \frac{\partial}{\partial t}$. From this point of view, a general AKSZ sigma model appears simply as a multi-dimensional generalization of this example.

The original BFV formulation has been constructed with quantization in mind. Another class of AKSZ-type sigma models can be associated with such quantum systems. More precisely, the target space $Q$-structure is determined by the BRST operator and the operator superalgebra of the quantum constrained system. The typical example is given by higher spin fields as background fields for a quantized scalar particle [65].

3 Generalities on jet-spaces and local BRST cohomology

3.1 Horizontal complex

The definition of local functions, the horizontal complex and of local functionals are recalled.

Consider a graded vector space $F$ with coordinates $z^\alpha$, $\alpha = 1, \ldots, m$. They include both “fields” and antifields. The $\mathbb{Z}$ grading is denoted by $gh$ (“ghost number”). For simplicity we assume here that the Grassmann parity, denoted by $|\cdot|$ is just $gh$ modulo 2. Consider further the space $X_0 \cong \mathbb{R}^n$ (“spacetime”) with coordinates $x^\mu$, and the jet-bundle associated to $F \times X_0 \xrightarrow{\pi} X_0$, with coordinates $x^\mu, z^\alpha_{(\mu)}$. Local functions are functions of $x^\mu$ and $z^\alpha_{(\mu)}$ that depend on the derivatives $z^\alpha_{(\mu)}$ up to some finite order, where $gh(x^\mu) = 0$, $gh(z^\alpha_{(\mu)}) = gh(z^\alpha)$ and $|z^\alpha_{(\mu)}| = |z^\alpha|$. The complex $\hat{\Omega}^{*,*}$ of horizontal forms $\omega = \omega(x, dx, [z])$ involves forms in $dx^\mu$ with coefficients that are local functions. As by now standard, we identify $dx^\mu$ with $\theta^\mu$ which are taken to anticommute with the odd elements among $z^\alpha_{(\mu)}$. The horizontal differential is $d_H = \theta^\mu \partial_\mu$ where the total derivative is defined by

$$\partial_\mu = \frac{\partial}{\partial x^\mu} + z^\alpha_{(\mu)} \frac{\partial}{\partial z^\alpha} + \cdots = \frac{\partial}{\partial x^\mu} + z^\alpha_{(\mu)} \frac{\partial}{\partial z^\alpha}.$$  

We assume that horizontal forms can be decomposed into field/antifield independent and dependent parts, $\omega = \omega(x, \theta, 0) + \tilde{\omega}(x, \theta, [z])$. The complex involving the latter is denoted by $\hat{\Omega}^{*,*}$. A standard result (see e.g. [24, 25, 26]) is that the cohomology of this

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5We follow the conventions of [24] for multi-indices and their summation. A summary can be found for instance in Appendix A of [66].
complex is trivial in form degrees less than \( n \),

\[
H^k(d_H, \hat{\Omega}) = 0, \text{ for } 0 \leq k < n.
\]  

(3.2)

The space of local functionals \( \hat{\mathcal{F}}^* \) is then defined as the quotient space \( \hat{\Omega}^{*,n} / d_H \hat{\Omega}^{*,n-1} \).

The projection from a representative \( \omega^{g,n} \in \hat{\Omega}^{g,n} \) to an element of the quotient space is often denoted by the integral sign,

\[
\hat{\mathcal{F}}^g \ni [\omega^{g,n}] = \int \omega^{g,n}.
\]

(3.3)

Euler Lagrange derivatives\(^6\) are defined by

\[
\frac{\delta \omega^{g,n}}{\delta z^\alpha} = \frac{\partial \omega^{g,n}}{\partial z^\alpha} - \partial_\mu \frac{\partial \omega^{g,n}}{\partial z^{\alpha}_\mu} + \cdots = (-)^{|\mu|} \partial_\mu \frac{\partial S^{\omega^{g,n}}}{\partial z^\alpha_{\mu}}.
\]

(3.4)

A crucial property is that

\[
\int \omega^{g,n} = 0 \iff \frac{\delta \omega^{g,n}}{\delta z^\alpha} = 0.
\]

(3.5)

### 3.2 BRST differential

The definition of the field theoretic BRST differential is given.

The BRST differential \( s \) is an odd, nilpotent evolutionary\(^7\) vector field, i.e., a vector field of the form

\[
s = \partial_\mu S^\alpha \frac{\partial S}{\partial z^\alpha_{\mu}}
\]

(3.6)

with \( S^\alpha \) local functions and \( gh(s) = 1, s^2 = 0 \). It follows that \([s, \partial_\mu] = 0 = [s, d_H] \), where the bracket denotes the graded commutator. For later purposes, note that an evolutionary vector field is entirely defined through its action on the undifferentiated fields, \( sz^\alpha = S^\alpha \), and the requirement that it commutes with the total derivative \( \partial_\mu \).

### 3.3 Local BRST cohomology

The definition of BRST cohomology in the space of local functionals is given and standard ways to compute it are sketched.

Several cohomology groups can then be considered. For instance, the cohomology of \( s \) in the space of local functions or in the space of horizontal forms. As mentioned in the introduction, especially interesting in view of applications in classical and quantum Lagrangian gauge field theories are the so-called local BRST cohomology groups.

\(^6\)Unless otherwise specified, all derivatives are left derivatives.

\(^7\)See e.g. \cite{24, 25, 26} for detailed discussions of vector fields on jet-bundles.
i.e., the cohomology of $s$ in the space of local functionals, $H^*(s, \hat{F})$. By definition of local functionals, $H^*(s, \hat{F}) \cong H^*|H, \hat{\Omega}\rangle$. The latter group is defined by

$$s\omega^{g,n} + d_H\omega^{g+1,n-1} = 0, \quad \omega^{g,n} \sim \omega^{g,n} + s\eta^{g-1,n} + d_H\eta^{g,n-1},$$

(3.7) with $\omega, \eta \in \hat{\Omega}$. Using (3.2), one then finds that $H^g|H, \hat{\Omega}\rangle \sim H^g|\tilde{s}, \hat{\Omega}\rangle$ where $\tilde{s} = s + d_H$ and the grading is the sum of the ghost number and the form degree. This statement summarizes the content of the so-called “descent equations” which provide a standard way to compute $H^*|s, \hat{F}\rangle$ out of $H^*|s, \hat{\Omega}\rangle$ (see e.g. [67, 11]).

As usual, actual computations can be considerably simplified by using appropriate spectral sequences. An example that we will use below and was used in [10] is the following. Consider for instance as grading the homogeneity in the fields/antifields, i.e., the eigenvalues of the operator $H = z^{\alpha}_{\nu} \frac{\partial S}{\partial z^\alpha_{\nu}}$. The space of field/antifield dependent forms decomposes as $\hat{\Omega}^{*,*} = \bigoplus_{h \geq 1} \hat{\Omega}^{*,*}_h$. If $s$ contains only terms of non negative grading, $s = \sum_{h \geq 0} s_h$, then the cohomology groups are controlled by $\bigoplus_{h \geq 1} H^*(s_0, \hat{\Omega}_h)$ and $\bigoplus_{h \geq 1} H^*(s_0, \hat{\mathcal{F}}_h)$ respectively.

More precisely, let $V$ be either one of the spaces $\hat{\Omega}$ or $\hat{\mathcal{F}}$. The cohomology $H(s, V)$ is easily shown to be determined by elements $[A_m] \in H(s_0, V_m)$ that can be completed,

$$A^m = A_m + B_{m+1} + B_{m+2} + \cdots, \quad sA^m = 0,$$

(3.8) for some $B_{m+1}, B_{m+2} \ldots$ with $B_i \in V_i$, where such an element is trivial if it is $s$-exact in $V$,

$$A^m \sim A^m + sC, \quad C \in V.$$

(3.9) More involved gradings than simply homogeneity might be more appropriate. We briefly comment on this below.

### 3.4 BRST cohomology for functional multivectors

*It is shown how the introduction of canonical momenta allows one to generalize local BRST cohomology to functional multivector fields.*

Another cohomology group that is usually considered is the commutator cohomology of $s$ in the space of evolutionary vector fields. The space of evolutionary vector fields is known to be isomorphic to the space of functional univectors (see e.g. [26]). More generally, one can then consider the cohomology of $s$ in the space of graded symmetric or skew-symmetric functional multivectors. Graded skew-symmetric functional multivectors equipped with a functional version of the Schouten-Nijenhuis bracket (also called BV antibracket) are well known and extensively used for studying the Hamiltonian structures of evolution equations. In the graded symmetric case, the bracket is a functional version of the canonical graded Poisson bracket (also called BFV Poisson bracket).
More precisely, for each field \( z^\alpha \) one introduces the “momenta” \( \pi_\alpha \) with \( |\pi_\alpha| = |z^\alpha| \) and \( \text{gh}(\pi_\alpha) = -\text{gh}(z^\alpha) \) in the graded symmetric case and the “antifields” \( z^*_\alpha \) with \( |z^*_\alpha| = |z^\alpha| + 1 \) and \( \text{gh}(z^*_\alpha) = -\text{gh}(z^\alpha) + 1 \), with the natural extensions for the derivatives of momenta and antifields. The horizontal complex is then extended to include either the momenta or the antifields and their derivatives. We introduce a subscript \( E \) to denote elements of the extended complex. A graded symmetric (skew-symmetric) functional \( k \)-vector is then a local functional of homogeneity \( k \) in the momenta (antifields) and their derivative. There is a map from functional multivectors \( \int \omega^{g;n}_E = \int d^n x \int f^g_E \) to evolutionary vector fields on the extended complex defined through

\[
\left\{ \int d^n x f^g_E , \cdot \right\}_E = -\partial_{(\mu)} \frac{\delta^R f^g_E}{\delta \pi_\alpha} \frac{\partial^S}{\partial z^\alpha(\mu)} + (-1)^{|\alpha|}(\pi_\alpha \leftrightarrow z^\alpha). \tag{3.10}
\]

Using multiple integrations by parts and (3.5), it is then easy to see that this map induces a well defined even (odd) graded Lie bracket in the space of functional multivectors.

The BRST differential \( s \) itself is then the evolutionary vector field generated by the univector

\[
\Omega_0 = -\int d^n x S^\alpha \pi_\alpha, \quad \frac{1}{2} \{ \Omega_0, \Omega_0 \}_E = 0, \quad \text{gh}(\Omega_0) = 1. \tag{3.11}
\]

The action of the BRST differential in the space of functional multivectors is then simply the adjoint action of \( \Omega_0 \)

\[
s_E \int \omega^{g;n}_E = \left\{ \Omega_0, \int \omega^{g;n}_E \right\}_E. \tag{3.12}
\]

In the space of functional univectors, this action is isomorphic to the commutator action of \( s \) in the space of evolutionary vector fields.

Given a functional \( k \)-vector represented by a local functional \( V_E \) it determines a well defined graded-symmetric \( k \)-multilinear operation on the space of local functionals of the original (non-extended) complex in a standard way. This can be expressed using the so-called derived bracket \( \mathcal{D} \) if \( F_1, \ldots, F_k \) are local functionals of the non-extended complex, identified as \( \pi \)-independent functionals of the extended complex, then

\[
V(F_1, \ldots, F_k) = \frac{1}{k!} \left\{ \cdots \{ V_E, F_1 \}_E, F_2 \right\}_E, \ldots F_k \right\}_E. \tag{3.13}
\]

This operation is well-defined on local functionals and gives a local functional of the non-extended complex as can be easily seen by counting homogeneity in \( \pi \).

For instance, a functional bivector \( \Omega_1 \) of unit parity and unit ghost number satisfying

\[
\frac{1}{2} \{ \Omega_1, \Omega_1 \}_E = 0 \quad \text{corresponds to a functional antibracket}. \quad \text{The cocycle condition } s_E \Omega_1 = 0 \quad \text{then means that the bracket is } s \text{-invariant, i.e., that } s \text{ differentiates the antibracket.}
\]

\[\text{We write down the formulas explicitly only for the symmetric case. The skew-symmetric case can be obtained by substituting } \pi_\alpha \text{ with } z^*_\alpha \text{ and changing the sign-factors appropriately.}
\]

\[\text{See e.g. } \text{[68]} \text{ for more details on derived brackets.}\]
4 Local BRST cohomology for AKSZ-type sigma models

4.1 Cohomology of space-time part

The cohomology of the space-time part of the BRST differential for AKSZ-type sigma models is derived by using results available in the literature.

The coefficients of $\Psi^A(x, \theta)$ in an expansion as series in $\theta^\mu$ constitute the field/antifield content of AKSZ-type sigma models.

$$z^\alpha \equiv (\Psi^A, \Psi^A_{\mu_1 \ldots \mu_k}, \ldots, \Psi^A_{\mu_1 \ldots \mu_k \ldots \mu_n}),$$

(4.1)

The $z^\alpha$ thus consist of “formfields”, a set of fields/antifields which are completely skew-symmetric in the spacetime indices, $\Psi^A_{\mu_1 \ldots \mu_k} = \Psi^A_{[\mu_1 \ldots \mu_k]}$ and contain all possible form degrees, $k = 0, \ldots, n$. Furthermore, we assign $\text{gh}(\Psi^A_{\mu_1 \ldots \mu_k}) = \text{gh}(\Psi^A) - k$. In the jet-space context, we can introduce an object analogous to the map $\Psi^A(x, \theta)$, the “complete ladder fields” in the terminology of [69, 10],

$$\tilde{\Psi}^A = \sum_{k=0}^{n} \Psi^A_k, \quad \Psi^A_k = \frac{1}{k!} \Psi^A_{\mu_1 \ldots \mu_k} \theta^{\mu_1} \ldots \theta^{\mu_k},$$

(4.2)

where $\Psi^A_0 \equiv \Psi^A$.

We refer to the first term in the BRST differential (2.2) involving the de Rham differential $d$ as the spacetime part and denote it by $s_0$. When translated in the jet-space context, we have $s_0 \tilde{\Psi}^A = -d_H \tilde{\Psi}^A$, or, more explicitly,

$$s_0 \Psi^A = 0, \quad s_0 \Psi^A_{\mu_1 \ldots \mu_k} = -(-)^{A+k-1} k \partial_{[\mu_1} \Psi^A_{\mu_2 \ldots \mu_k]},$$

(4.3)

which can be summarized by

$$\tilde{s}_0 \tilde{\Psi}^A = 0.$$  

(4.4)

As discussed in detail in the proof of theorem 3.1 of [41], the idea is to decompose the form fields and their derivatives $\partial_{\nu_1} \ldots \partial_{\nu_m} \Psi^A_{\mu_1 \ldots \mu_k}$ into irreducible tensors under the general linear group $GL(n)$. One then finds that all the field variables form contractible pairs except for the undifferentiated $\Psi^A$. Compared to the situation considered in [41], no curvatures remain because the last formfield $\Psi^A_{\mu_1 \ldots \mu_n}$ is of maximal degree $n$. The cohomology $H(s_0, \Omega)$ can thus be described by functions of $\Psi^A, x^\mu, \theta^\mu$ alone,

$$H(s_0, \Omega) \cong \{ \lambda(x, \theta, \Psi^A) \},$$

(4.5)

where $\lambda(x, \theta, \Psi^A)$ contains no field independent terms, $\lambda(x, \theta, 0) = 0$.

---

10 We use round (square) brackets to denote normalized (skew)-symmetrization.

11 To simplify notations in this section, we redefine the BRST differential by an overall factor $(-1)^n$ and change the sign of the term in $s$ involving $d$. This can be achieved by the transformation $\theta^\mu \rightarrow -\theta^\mu$. 

The analysis of the descent equations is then standard. In the present case, it is presented in [69, 10, 41] (see also section 14 of [11]). Using the ordinary Poincaré lemma on the base space, one finds

\[ H(s_0, \hat{F}) \cong \{ \nu(\tilde{\Psi}^A)|_n \}, \]  

(4.6)

where \( \nu \) denotes a polynomial in its arguments without constant term, while \( |_n \) means that one should restrict oneself to the form of top degree \( n \) in an expansion according to form degree. This cohomology is isomorphic to the cohomology of \( s_0 \) in the space \( \hat{\Omega}^{*,0}_{x=0} \) of \( x \)-independent zero forms

\[ H(s_0, \hat{F}) \cong H(s_0, \hat{\Omega}^{*,0}_{x=0}), \]  

(4.7)

since \( H(s_0, \hat{\Omega}^{*,0}_{x=0}) \cong \{ \nu(\Psi^A) \}. \)

### 4.2 Cohomology of complete differential

The main result that the field theoretic BRST cohomology in the space of local functionals is locally isomorphic to the target space cohomology is derived.

The full BRST differential for AKSZ-type sigma models acts according to

\[ \tilde{s}\tilde{\Psi}^A = Q^A(\tilde{\Psi}), \]  

(4.8)

In particular, in form degree zero, one finds that the BRST differential acts on the fields \( \Psi^A \) according to

\[ s\Psi^A = Q^A(\Psi), \]  

(4.9)

and is thus entirely determined through the homological vector field \( Q \) in the target space \( \mathcal{M} \).

We now assume that there is a grading \( H \) according to which (i) the space of horizontal forms is bounded from below, (ii) the spacetime part is in lowest degree 0 and all other terms are of higher degree, \( s = \sum_{h \geq 0} s_h \) with \( s_0 = -d_H \). This is the case for instance for Chern-Simons theory when the grading is simply homogeneity in the fields/antifields.

Introducing the notation \( s^R = \sum_{h \geq 1} s_h \) we have \( s^R\Psi^A = Q^A \) since \( s_0\Psi^A = 0 \). The action of the BRST differential on the remaining fields contained in \( \Psi^A_k, k \geq 1 \) is then determined by the same \( Q^A \) by expanding (4.8) according to higher form degrees, \( s^R\Psi^A_k = Q^A(\tilde{\Psi})|_k \) and taking into account (4.10). Explicitly

\[ s^R\Psi^A_1 = \Psi^B_1 \frac{\partial Q^A}{\partial \Psi^B}(\Psi), \]  

(4.10)

\[ s^R\Psi^A_2 = \Psi^B_2 \frac{\partial Q^A}{\partial \Psi^B}(\Psi) + \frac{1}{2} \Psi^B_1 \Psi^B_1 \frac{\partial Q^A}{\partial \Psi^B_2 \partial \Psi^B_1}(\Psi), \]  

(4.11)

;
It then follows that $H^{g+n}(\tilde{s}, \tilde{\Omega}) \cong H^{g}(s, \tilde{\Omega}^x_{x=0})$ because $\tilde{s}$ acts on $\tilde{\Psi}^A$ in exactly the same way as $s$ acts on $\Psi^A$, so that the problem of completion and non triviality of the completed cocycles as discussed at the end of section 3.3 is exactly the same in both spaces. Furthermore, on the one hand $H^{g}(s, \hat{\mathcal{F}}) \cong H^{g+n}(\tilde{s}, \hat{\Omega})$ and on the other hand $H^{g}(s, \hat{\Omega}^x_{x=0}) \cong H^{g}(Q)$.

We have thus shown

**Proposition 4.1.** The local BRST cohomology $H(s, \hat{\mathcal{F}})$ is locally isomorphic to the target space cohomology $H(Q)$,

$$H^{g}(s, \hat{\mathcal{F}}) \cong H^{g-n}(Q). \quad (4.12)$$

where

$$H^{g}(Q) \ni [\Theta^{\alpha}_{\beta}(\Psi^A)] \longleftrightarrow [\Theta^{\alpha}_{\beta}(\tilde{\Psi}^A)|_n] \in H^{g-n}(s, \hat{\mathcal{F}}), \quad (4.13)$$

with $[\Theta^{\alpha}_{\beta}(\Psi^A)]$ denoting representatives of $H^{g}(Q)$.

**Remarks:**

(i) By identifying integrals of functions evaluated for maps of compact support with the algebraic version of local functionals, one can consider the map $I$ defined in (2.4) as a map from functions on target space to local functionals. The proposition can then be reformulated by the statement that $I$, for AKSZ-type sigma models, is locally an isomorphism in cohomology or, more precisely, a quasi-isomorphism of differential complexes in the case without bracket and a quasi-isomorphism of differential graded Lie algebras in the case with bracket.

(ii) In the case of the 1-dimensional AKSZ sigma models associated with Hamiltonian BFV systems with vanishing Hamiltonian, Proposition 4.1 states that the Poisson algebra of Hamiltonian BRST cohomology and the antibracket algebra of Lagrangian BV cohomology in the space of local functionals are locally isomorphic, as originally derived in [70].

(iii) Because of the expansion in homogeneity, the cohomology of $Q$ is computed in the space of formal power series without constant terms. Depending on the problem at hand, other gradings or spaces might be more relevant. Suppose for instance that one would like to establish a similar result for $H(Q)$ with $Q$ acting on the space of smooth functions on $M$ and that one would like to allow for a linear part in $Q$. This can be done by choosing as grading the operator counting the number of derivatives of the fields, i.e., the eigenvalues of the operator $N_{\Theta} = |\mu| z^\alpha_{(\mu)} \frac{\partial}{\partial z^\alpha_{(\mu)}} = z^\alpha_{(\mu)} \frac{\partial}{\partial z^\alpha_{(\mu)}} + 2 z^\alpha_{(\mu)} |\mu| + \ldots$, where $|\mu|$ denotes the number of indices contained in the multi-index $\mu$. In this case, the assumptions imply that $s = s_0 + s_1$ where $s_1 = -d_H$ is now the spacetime part, while $s_0$ is determined by formulas (4.9) and (4.8). By a similar reasoning, one can then prove the isomorphism (4.12) for smooth functions on $M$, with $Q$ involving a linear term, if one
restricts oneself from the outset to the space of local functions that have a degree $N_\partial$ that is bounded from above.

(iv) The BRST extension \cite{47, 22, 71} of the unfolded linear equations\cite{72, 18, 73, 23} developed originally in the context of higher spin gauge fields is almost of the above form. Indeed, in this case, there are also only complete ladder fields but instead of \eqref{4.8}, one has more generally

\begin{equation}
\tilde{s}\tilde{\Psi}^A = Q_i^i(x^\mu, \theta^\nu, \Psi_{\mu_1, \ldots, \mu_k}^A).
\end{equation}

In some cases, this BRST differential can be seen as the linearization of some nonlinear AKSZ differential around a particular solution that brings the explicit dependence on $x, \theta$ \cite{65}.

To highest order 1 in $N_\partial$, the isomorphism \eqref{4.12} then still holds, but the problem of completion and non triviality is now much more involved. It would be interesting to compute the BRST cohomology in the space of local functionals for this more general case along these lines. We plan to return to this question elsewhere.

\section{4.3 Cohomology for functional multivectors}

The isomorphism of field theoretic cohomology with target space cohomology is extended to the case of the BRST differential acting in the space of functional multivectors by showing that the latter is again of AKSZ-type.

In order to discuss the cohomology in the space of graded symmetric (skew-symmetric) functional multivectors, one introduces the conjugate momenta $\pi_{\mu_1, \ldots, \mu_k}^A$ (antifields $\Psi_0^{*\mu_1, \ldots, \mu_k}^A$) and considers the functional

\begin{equation}
\Omega_0 = -\int d^n x \sum_{k=0}^n \frac{1}{k!} s\Psi^A_{\mu_1, \ldots, \mu_k} \pi_{\mu_1, \ldots, \mu_k}^A.
\end{equation}

In the AKSZ setting, it is then natural to combine the momenta (antifields) into superfields $\tilde{\Pi}_A$ in such a way that \eqref{4.15} takes the form

\begin{equation}
\Omega_0 = -\int d^n x d^n \theta s\Psi^A \tilde{\Pi}_A = \int d^n x d^n \theta \left[ -d_H \Psi^A \tilde{\Pi}_A + Q_A(\Psi) \tilde{\Pi}_A \right].
\end{equation}

Explicitly, one has

\begin{equation}
\tilde{\Pi}_A = \sum_{k=0}^n \Pi_{A}^{n-k}, \quad \Pi_{A}^{n-k} = \frac{(-1)^n k!}{n! (n-k)!} \Psi_{\mu_1, \ldots, \mu_k, \nu_{k+1}, \ldots, \nu_n}^A \epsilon_{\mu_1, \ldots, \mu_k, \nu_{k+1}, \ldots, \nu_n} \theta_{\nu_{k+1}} \cdots \theta_{\nu_n},
\end{equation}

where $|A|$ is a shortcut for $|\Psi^A|$. For the adjoint action of $\Omega_0$ one then finds

\begin{equation}
s\tilde{\Psi} = -d_H \tilde{\Psi} + Q^A(\tilde{\Psi}), \quad s\tilde{\Pi}_A = -d_H \tilde{\Pi}_A - (-1)^{|A|} \frac{\partial Q_B}{\partial \Psi^B}(\tilde{\Psi}) \tilde{\Pi}_B.
\end{equation}
The BRST differential (4.18) is then again of AKSZ-type. The associated target space is given by the (odd) cotangent bundle \((\Pi)T^*M\), with canonical (odd) Poisson structure
\[
\{\Pi_B, \Psi^A\}_{(\Pi)T^*M} = -\delta_B^A, \quad \text{gh}(\Pi_A) = -\text{gh}(\Psi^A) + n,
\] (4.19)
and homological vector field
\[
Q_E = \{ -Q^A\Pi_A, \cdot \}_{(\Pi)T^*M} = Q^A \frac{\partial}{\partial \Psi^A} - (-1)^{|A|} \frac{\partial Q^B}{\partial \Psi^A} \Pi_B \frac{\partial}{\partial \Pi_A} .
\] (4.20)
Furthermore, in terms of the map \(\Pi_A(x, \theta)\), the canonical Poisson bracket can be identified with a Poisson bracket of the form (2.3).

It then follows from Proposition 4.1 that the BRST cohomology in the space of functional multivectors is locally isomorphic to the cohomology of \(Q_E\), or, more precisely, that the map \(\mathcal{I}_E\), sending the target space multivectors to functional multivectors according to
\[
\mathcal{I}_E : f_E \mapsto \int_X d^n x d^n \theta f_E ,
\] (4.21)
where \(f_E \in \mathcal{A}_E\), with \(\mathcal{A}_E\) denoting the space of functions on the target space \((\Pi)T^*M\), is locally a quasi-isomorphism of differential graded Lie algebras.

Finally, as an illustration, we note that expression (2.3) of the bracket on the space of maps can be interpreted as an explicit realization of the map \(\mathcal{I}_E\) for (skew)-symmetric 2-vectors, when identifying algebraic local functionals with integrals of target space functions evaluated for maps of compact support.

5 Functional multivectors, symmetries and generalized Poisson structures

5.1 Applications of BRST cohomology for functional multivectors

It is pointed out that BRST cohomology in the space of functional multivectors is relevant for classifying global symmetries of the equations of motion and weak Poisson or Lagrange structures.

Consider a gauge theory for which a Lagrangian does not exist or is not (yet) specified. Such a theory can still be described in terms of a BRST differential \(s\) that is not necessarily generated by a master action in an appropriate antibracket. As pointed out in (4.17), consistent deformations of such theories are then controlled by the adjoint cohomology of \(s\) in the space of evolutionary vector fields, or in other words, by \(H^1_1(s_E, \mathcal{F}_E)\).

The identification of the cohomology groups controlling global symmetries requires some care. To fix ideas, let us first consider the case of partial differential equations of
motion of Cauchy-Kovalevskaya type, as considered for instance in \cite{25, 26}, for which \( s \) reduces to the so-called Koszul resolution \cite{74, 75, 76} of the surface defined by the equations in jet-space. In this case, it is straightforward to check that \( H^0_1(s_E, \mathcal{F}_E) \) coincides with the usual definition of equivalence classes of symmetries of the equations of motion, i.e., evolutionary vector fields that leave the equations of motion invariant quotiented by such vector fields that vanish when the equations of motion hold.

In the case of variational equations of motions, with non trivial relations between the equations and their derivatives ("Noether identities"), there is a well defined concept of a proper solution to the BV master equation. In this case, \( H^0_1(s_E, \mathcal{F}_E) \) is given by equivalence classes of equations of motion symmetries quotiented not only by the ones vanishing when the equations of motion hold, but in addition by all non trivial gauge symmetries (see e.g. \cite{11}).

In the non variational case, \( H^0_1(s_E, \mathcal{F}_E) \) is again given by the quotient space of equations of motion symmetries modulo evolutionary vector fields that vanish when the equations of motion hold and modulo the non trivial gauge symmetries encoded in \( s \). The question is then whether the latter include all the non trivial gauge symmetries, which in turn hinges on an appropriate non-variational version of properness for the BRST differential \( s \). The precise definition of this concept is beyond the scope of the present work \cite{12}.

In what follows we simply assume that all the gauge symmetries and reducibility relations between the equations and between the gauge generators are accounted for by the BRST differential.

Given a BRST differential \( s \), another natural question is whether the gauge theory determined by \( s \) admits a Lagrangian or Hamiltonian description. In the case of non-gauge systems, this question is known as the inverse problem of the calculus of variations. In the BRST context, it translates into the question of existence of a generator for \( s \) in an appropriate bracket that is usually assumed non degenerate. One can distinguish two cases. In the Lagrangian or BV case, the bracket \( \{ \cdot, \cdot \} \) for the fields \( z^a \) is an odd graded bracket of ghost number 1, while the canonical generator, the solution of the BV master equation, is even of ghost number 0. In the Hamiltonian BFV case, the Poisson bracket is even of ghost number 0, while the canonical generator, the BRST charge, is odd of ghost number 1. When \( s \) is proper in the sense discussed above, this question is equivalent to the question whether the equations it encodes are variational in the Lagrangian case or whether the constraints it describes are first class ("co-isotropic") in the Hamiltonian case.

Let us for definiteness restrict ourselves to the case of an odd bracket on the space of

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\(^{12}\)In the finite-dimensional setting an appropriate notion of properness was proposed in \cite{28}. Its generalization to the local field theory setting is not entirely straightforward.

\(^{13}\)In the space of local functionals, these brackets are of the Gelfand-Dickey-Dorfman type \cite{77}, \cite{25}, see also \cite{70} for the current context.
fields and hence to the Lagrangian BV picture. The question of existence of a Lagrangian for a gauge theory can be addressed using the notion of Lagrange structure \[28\], which is the Lagrangian counterpart of a possibly weak and degenerate Poisson structure of the Hamiltonian formalism \[27\]. In the BRST theory terms the Lagrange structure can be represented \[28\] as a strong homotopy Schouten algebra structure (see e.g. \[68, 78\]).

In local field theory, such a structure can be defined as a collection of \(n\)-ary functional multivectors satisfying appropriate compatibility conditions including, in particular, the Jacobi identity for the bracket induced in \(s_E\)-cohomology. More concretely, it can be defined as a deformation of \(\Omega_0\) by terms of higher order in \(\pi_\alpha, \Omega = \Omega_0 + \Omega_1 + \Omega_2 + \ldots\) with \(\text{gh}(\Omega) = 1\), where \(\Omega_k\) denotes a local functional that is homogeneous of degree \(k + 1\) in \(\pi_\alpha\) and their derivatives. As usual, the compatibility conditions are combined into the master equation

\[ \begin{align*}
1/2 \{ \Omega, \Omega \}_E &= 0 \iff \\
\frac{1}{2} \{ \Omega_1, \Omega_1 \}_E + s_E \Omega_2 &= 0, \\
\{ \Omega_1, \Omega_2 \}_E + s_E \Omega_3 &= 0, \\
&\vdots
\end{align*} \tag{5.1} \]

Two such deformations \(\Omega\) and \(\Omega'\) are considered equivalent if there exists a local functional \(\Xi = \sum_{k \geq 1} \Xi_k\) such that \(\Omega' = \exp \{ \Omega_0, \cdot \}_E \Xi\), where \(\Xi_k\) is homogeneous of degree \(k + 1\) in \(\pi_\alpha\).

In particular, non trivial first order Lagrange structures are controlled by \(H_1^1(s_E, \hat{F}_E)\), the cohomology of \(s_E\) in the space of functional bivectors of ghost number \(1\), while the second equation on the right of (5.1) encodes the Jacobi identity satisfied in BRST cohomology.

In the standard deformation approach for gauge theories \[30, 31, 32\], it is crucial to take due care of locality since otherwise the deformation theory is trivial in the sense that all first order deformations extend to complete deformations. This is also true for Lagrange or weak and degenerate Poisson structures. More precisely, the classification result in \[28\] stating that all the Lagrange structures are trivial in the finite dimensional case will not generally hold once field theoretic locality is taken into account.

As defined above, a Lagrange structure is an equivalence class \([\Omega]\) of deformations of \(\Omega_0\) in the space of functional multivectors. This is consistent with the point of view adopted in \[47\] that being Lagrangian or not is a property of equivalence classes of equations of motion under addition/elimination of generalized auxiliary fields, because generalized auxiliary fields correspond to contractible pairs for \(s_E\).

\[14\] This is the local field theory version of the master equation considered in \[28, 27\]. Master equations of this type have first appeared in \[29\].
5.2 Consequences for AKSZ-type sigma models

As a direct consequence of the main result, the classification of Lagrange or weak Poisson structures simplifies to a target space problem for AKSZ-type sigma models. As we have seen, locally, the BRST cohomology of AKSZ-type sigma models originates from the target space cohomology, both for standard local functionals and for functional multivectors. It then follows from standard deformation theory arguments that Lagrange or weak Poisson structures for these models can be entirely discussed in the target space, or in other words, that one can consistently get rid of the space-time derivatives and of the higher forms in the Lagrange/Poisson structure of these models.

Indeed, for $\Omega$ satisfying (5.1), the term $\Omega_1$ quadratic in $\pi_\alpha$ and their derivatives can for instance be written as $\Omega_1 = \mathcal{I}_E(\omega_1) + s_E \Xi_1$ for some $\omega_1 \in \mathcal{A}_E$ and $\Xi_1 \in \tilde{\mathcal{F}}_E$. By exponentiating the transformation generated by $\Xi_1$ one arrives at an equivalent $\Omega$ with the same $\Omega_0$ but $\Omega_1 = \mathcal{I}_E(\omega)$. At the next order, one finds $s_E \Omega_2 + \frac{1}{2} \mathcal{I}_E \{\omega_1, \omega_1\}_{(II)T^*M} = 0$. A standard reasoning involving contractible pairs implies that $\Omega_2 = \mathcal{I}_E(\omega_2) + s_E \Xi_2$ with $Q_E \omega_2 + \frac{1}{2} \{\omega_1, \omega_1\}_{(II)T^*M} = 0$ and again, through exponentiation, one arrives at an equivalent $\Omega$ such that $\Omega_2 = \mathcal{I}(\omega_2)$. Going on in this way for the higher orders, one ends up with an equivalent $\Omega$ of the form

$$\Omega = \Omega_0 + \mathcal{I}_E(\omega) = \Omega_0 + \int_X d^n x d^n \theta (\omega_1 + \omega_2 + \ldots).$$

(5.2)

Here $\omega_k \in \mathcal{A}_E$ is a polynomial of order $k + 1$ in the momenta $\pi_A$. It follows that $\omega = -Q^A \pi_A + \sum_{k \geq 1} \omega_k$ satisfies the target space master equation

$$\frac{1}{2} \{\omega, \omega\}_{(II)T^*M} = 0.$$  

(5.3)

Finally, for the sake of illustration, let us consider Chern-Simons theory based on a simple Lie algebra. In this case, the target space of the extended model is the $(n|n)$-dimensional supermanifold $M = \Pi T^*G$ with coordinates $c^a$, $gh(c^a) = 1$ and $\pi_a$, $gh(\pi_a) = 2$ and QP structure determined by

$$\{\pi_a, c^b\}_M = -\delta^b_a, \quad Q_E = -\left\{\frac{1}{2} c^a c^b f_{ab} \pi_c, \cdot\right\}_M.$$  

(5.4)

The cohomology of $Q_E$ can be identified with the Lie algebra cohomology of $G$ with coefficients in polynomials in $\pi_a$.

For a simple Lie algebra $G$, this cohomology is given by the algebra generated by the primitive elements, which are at least cubic in $c^a$, tensored with the invariant polynomials in $\pi_a$. The cohomology in the space of elements linear in $\pi_a$ is empty and one concludes that the Chern-Simons theory is rigid and does not have nontrivial symmetries at the level of the equations of motion.
The cohomology in the space of elements quadratic in $\pi_a$ is given by the invariants $g^{ab}\pi_a\pi_b$, with $g^{ab}$ the inverse of the Killing form, tensored with the algebra of primitive elements. First order Lagrange structures are classified by ghost number one local functionals of the extended model that are quadratic in the $\pi$. Using the isomorphism, these can be represented by ghost number 4 target space functions quadratic in $\pi_a$, i.e., by $g^{ab}\pi_a\pi_b$. There is thus only one non trivial cohomology class and hence only one non-trivial first order Lagrange structure. It trivially extends to higher orders and obviously coincides with the usual one, i.e., the standard BV antibracket of the AKSZ description.

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