Mathematical modeling of dynamic stability of shell structures under large elastoplastic deformations

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Abstract. In this paper, a mathematical model of dynamic stability of a ring plate under the action of large elastoplastic strains is considered. Such problems are encountered in pressing, for example, by a pulse magnetic field, of ring plates supported on the inner contour. The obtained results show that the proposed mathematical model agrees well with experimental data.

1. Introduction
The manufacturing of thin-walled workpieces with bulging edges by traditional pressing methods leads to the loss of stability, requires a lot of hand work, and results in rather many defective pieces. The use of dynamic loading and, in particular, of pressing by a pulse magnetic field, permits significantly “hindering” the loss of stability of the workpiece, increasing the quality, and decreasing the cost of the workpieces. The problems of dynamic stability in the case of large displacements of plates and shells have been studied absolutely insufficiently and no physically justified criterion for the loss of stability has been proposed. In many operations of cold deformation, the stability is lost at the moment of a semi-finished workpiece shaping from a plane ring plate, and precisely this fact determined the choice of the computational model. Much attention has been paid to studying configurations under the action of momentary dynamic loads in [1–9]. At the same time, all numerous parameters of the dynamic loading, such as the rate of loading, the characteristics of the material dynamic behavior, etc., cannot be taken into account even in several papers.

2. Statement of the problem
We study the behavior of a ring plate supported on the inner contour under the action of highly intensive dynamic loads. The problem is solved under the assumption that the properties of the material are described by the theory of plastic flow and satisfy the Mises yield condition. It is required to construct a mathematical model of behavior of a ring plate under the action of intensive dynamic impact in a geometrically and physically nonlinear statement.

3. Method and construction of the solution
The problem is solved by the finite element method. As a finite element, we take a thin-walled axially symmetric finite element whose displacements in the normal, circular, and radial directions are approximated as polynomials in the radial distance and as Fourier series in the circular direction [5]. Let us consider a thin finite element (see figure 1). We
introduce the coordinate system $r, s, \theta$ related to the element. The angle $\varphi$ is approximated as $\varphi = \alpha_1 + \alpha_2 s + \alpha_3 s^2$, where $\alpha_1, \alpha_2, \alpha_3$ are the coefficients determined from the boundary conditions for $s = 0, l$:

$$\varphi_{s=0} = \varphi_m, \quad \varphi_{s=L} = \varphi_n, \quad \int_0^L \sin(\varphi - \varphi_L) \, ds = \int_0^L (\varphi - \varphi_L) \, ds = 0, \quad \int_0^L \cos(\varphi - \varphi_L) \, ds = L.$$

After several transformations, we obtain

$$\alpha_1 = \varphi_m, \quad \alpha_2 = \frac{6\varphi_L - 4\varphi_m - 2\varphi_n}{L}, \quad \alpha_3 = \frac{3m - 3\varphi_n - 6\varphi_L}{L^2},$$

$$\varphi_L = \arctan \frac{R_m - R_n}{z_m - z_n}, \quad R = R_m = \int_0^s \sin \varphi \, ds.$$

For the finite element shapes as a truncated cone, we have

$$\varphi = \varphi_L, \quad R = R_m + s \sin \varphi_L.$$

We assume that the field of displacements of the middle surface points inside a finite element has the form

$$
\begin{align*}
\left\{ \begin{array}{c}
u(s, \theta) \\ v(s, \theta) \\ w(s, \theta)
\end{array} \right\} &= \sum_{j=0}^{\infty} \begin{bmatrix}
\cos(j \theta) & 0 & 0 \\
0 & \sin(j \theta) & 0 \\
0 & 0 & \cos(j \theta)
\end{bmatrix} \begin{array}{c}
u_s \\ v_s \\ w_s
\end{array} \\
&+ \sum_{j=0}^{\infty} \begin{bmatrix}
\sin(j \theta) & 0 & 0 \\
0 & \cos(j \theta) & 0 \\
0 & 0 & \sin(j \theta)
\end{bmatrix} \begin{array}{c}
u_s \\ v_s \\ w_s
\end{array},
\end{align*}
$$

$$u(s, \theta) = \alpha_1 + \alpha_2 s, \quad v(s, \theta) = \alpha_7 + \alpha_8 s, \quad w(s, \theta) = \alpha_3 + \alpha_4 s + \alpha_5 s^2 + \alpha_6 s^3.$$

Such displacements $u(s, \theta), v(s, \theta), w(s, \theta)$ are chosen to ensure the continuity of displacements and their first derivatives in transition from one element to another and the smoothness of the interface between elements.

We express the parameters $\alpha$ in terms of displacements at nodal points

$$\{\alpha^j\} = [C]\{q^j\}, \quad \{q^j\} = (u^j_m, w^j_m, \beta^j_m, v^j_m, u^j_n, w^j_n, \beta^j_n, v^j_n),$$

$$\{\alpha^j\} = (\alpha_1, \alpha_2, \ldots, \alpha_8), \quad \beta^j = \frac{du^j}{ds} + u^j \frac{d\varphi}{ds}.$$
The displacements \( u(s, \theta), v(s, \theta), \) and \( w(s, \theta) \) expressed in terms of nodal displacements have the form

\[
\{u\} = [A]\{\alpha\}, \quad \{u\} = \{u(s), v(s), w(s)\}, \quad \{u\} = [B]\{q\}.
\]

To derive the mathematical model for studying the shell configurations made of homogeneous and multilayer composite materials under intensive momentary impacts with large displacements, we consider the Lagrange equations

\[
\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_k} + \frac{\partial U}{\partial q_k} = Q_k. \tag{2}
\]

Here \( T \) is the kinetic energy, \( U \) is the potential energy, \( Q \) are external loads, \( \dot{q} \) and \( q \) are the generalized velocities and displacements, and the \( k \) are the degrees of freedom.

We use the relations between strains and displacements for a moderate deflection

\[
\begin{align*}
\varepsilon_s &= e_s + \frac{e_{13}^2 + e_{23}^2}{2}, \quad \varepsilon_\theta = e_\theta + \frac{e_{23}^2 + e_{33}^2}{2}, \quad \varepsilon_{s\theta} = e_{s\theta} + \frac{e_{13}^2 e_{23}}{2}, \quad \chi_s = \frac{\partial e_{13}}{\partial s}, \\
\chi_\theta &= \frac{1}{r} \left( \frac{\partial e_{23}}{\partial \theta} + e_{13} \sin \varphi \right), \quad \chi_{s\theta} = \frac{1}{2} \left[ \frac{\partial e_{23}}{\partial s} + \frac{1}{r} \frac{\partial e_{13}}{\partial \theta} - e_{23} \sin \varphi \right] + \left( \frac{\partial e_\varphi}{\partial s} + \frac{\partial e_\varphi}{\partial r} \right) \sin \varphi, \\
e_s &= \frac{\partial u}{\partial s} - w \frac{\partial e_\varphi}{\partial s}, \quad e_\theta = \frac{1}{r} \left( u \sin \varphi + \frac{\partial v}{\partial \theta} + w \cos \varphi \right), \quad e_{s\theta} = \frac{1}{r} \left( \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial s} - v \sin \varphi \right), \\
e_{13} &= \left( \frac{\partial w}{\partial s} + u \frac{\partial e_\varphi}{\partial s} \right), \quad e_{23} = \frac{1}{r} \left( \frac{\partial w}{\partial \theta} - v \sin \varphi \right), \quad e_{33} = -\frac{1}{2r} \left( r \frac{\partial u}{\partial s} + v \sin \varphi - \frac{\partial u}{\partial \theta} \right), \\
\varphi_s &= -(w_s + u \sin \varphi), \quad \varphi_\theta = -\frac{w_\theta - v \cos \varphi}{r}.
\end{align*}
\]

In equations (3), we use the following notation: \( \varepsilon_s, \varepsilon_\theta, \) and \( \varepsilon_{s\theta} \) are the strains of the shell middle surface in the corresponding coordinates, \( \chi_s, \chi_\theta, \) and \( \chi_{s\theta} \) are the curvature strains, \( e_{13} \) and \( e_{23} \) are the angles of rotation about the coordinate lines, \( R \) is the shell radius, \( \varphi_s \) and \( \varphi_\theta \) are the angle of the meridian inclination to the shell axis and the angle in the circular direction, and \( e_{33} \) is the “torsion” of the middle surface.

### 3.1. Equation of the stress-strain relationship

We consider the geometrical and physical dependencies for multilayer shells in the reduced rigidity method.

In the plane stress state, the matrices of elastic coefficients for an orthotropic material whose orthotropy axes coincide with the coordinate axes have the form

\[
\{\sigma\} = [E]\{\varepsilon\}, \tag{4}
\]

where

\[
[E] = \begin{bmatrix}
Q_{11} & Q_{12} & 0 \\
Q_{21} & Q_{22} & 0 \\
0 & 0 & Q_{66}
\end{bmatrix}, \quad \{\varepsilon\}^T = \{\varepsilon_s, \varepsilon_\theta, \varepsilon_{s\theta}\}, \quad \{\sigma\}^T = \{\sigma_s, \sigma_\theta, \sigma_{s\theta}\},
\]

\[
Q_{11} = \frac{E_s}{1 - v_{s\theta} v_{\theta s}}, \quad Q_{12} = \frac{v_{s\theta} E_s}{1 - v_{s\theta} v_{\theta s}}, \quad Q_{21} = \frac{v_{s\theta} E_s}{1 - v_{s\theta} v_{\theta s}}, \quad Q_{22} = \frac{E_\theta}{1 - v_{s\theta} v_{\theta s}}, \quad Q_{66} = G_{66}.
\]

For the layer at the distance \( z \) from the middle surface, the strains become

\[
\{\varepsilon\} = \{\varepsilon^o\} + z\{\chi^o\},
\]

where \( \varepsilon^o, \varepsilon_\theta^o, \) and \( \varepsilon_{s\theta}^o \) are the strains of the shell middle surface at the distance \( z \) from the middle surface, and \( \chi_s^o, \chi_\theta^o, \) and \( \chi_{s\theta}^o \) are the curvature strains at the distance \( z \) from the middle surface.
where \( \{ \varepsilon^o \} \) are the strains of the middle surface and \( \{ \chi^o \} \) are the curvature variations.

After substitution of these relations, equation (4) becomes

\[
\{ \sigma \} = \{ \bar{Q} \} \{ \varepsilon^o \} + z \{ \bar{Q} \} \{ \chi^o \}.
\]

We take the relations

\[
\{ N \} = \int_{-h/2}^{h/2} \{ \sigma \} dz, \quad \{ N \}^T = (N_s, N_\theta, N_{s\theta}), \tag{5}
\]

\[
\{ M \} = \int_{-h/2}^{h/2} \{ \sigma \} z dz, \quad \{ M \}^T = (M_s, M_\theta, M_{s\theta}),
\]

where \( N \) are membrane forces and \( M \) are bending moments, into account and integrate expressions (5) over the shell thickness. Then the forces and moments can be represented as

\[
\left\{ \begin{array}{c} N \\ M \end{array} \right\} = [E] \left\{ \begin{array}{c} \varepsilon^o \\ \chi^o \end{array} \right\}, \quad [E] = \left[ \begin{array}{cc} [A] & [B] \\ [C] & [D] \end{array} \right], \tag{6}
\]

\[
[A] = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{21} & A_{22} & A_{26} \\ A_{61} & A_{62} & A_{66} \end{bmatrix}, \quad [B] = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{21} & B_{22} & B_{26} \\ B_{61} & B_{62} & B_{66} \end{bmatrix}, \quad [D] = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{21} & D_{22} & D_{26} \\ D_{61} & D_{62} & D_{66} \end{bmatrix},
\]

\[
\{ A_{ij}, B_{ij}, D_{ij} \} = \int_{-h/2}^{h/2} Q_{ij}(1, z, z^2) dz \quad (i, j = 1, 2, 6).
\]

If the coefficients of the matrix (6) are constant in each layer of the package, then we integrate (6) to obtain

\[
A_{ij} = \sum_{k=1}^{n} Q_{ij}(h_k - h_{k-1}), \quad B_{ij} = \sum_{k=1}^{n} Q_{ij}(h_k^2 - h_{k-1}^2), \quad D_{ij} = \sum_{k=1}^{n} Q_{ij}(h_k^3 - h_{k-1}^3), \quad i, j = 1, 2, 6,
\]

where \( A_{ij}, B_{ij}, \) and \( D_{ij} \) are the membrane, bending-membrane, and bending rigidities.

The equations describing the behavior of shell configurations under dynamic loads were derived in detail by the author in paper “Mathematical modeling of shell configurations made of homogeneous and composite materials experiencing intensive short actions and large displacements” presented in this issue. The difference is that, in problems of large deformations, there is an additional term related to the accumulated strains.

3.2. Resolving equations

The mathematical model for studying configurations made of homogeneous and composite materials under the action of intensive momentary impacts in the case of large strains can be represented as

\[
[M] \{ \dot{q} \} + [K] \{ q \} = \{ Q \} + \{ \Delta Q \} - [K_G] \{ q \} - \{ Q^{nl} \} - \{ Q^P \}, \tag{7}
\]

where \( [M] = \frac{\partial}{\partial q} \iint (u^2 + i^2 + \dot{w}^2 + \dot{w}_s I_s + \dot{w}_\theta I_\theta) dA \) is the mass matrix, \( [K] = \partial U_1^{(2)}/\partial q = \frac{\partial}{\partial q} \iint \sigma^e \varepsilon^e \, dA \) is the rigidity matrix, \( \{ Q \} = \frac{\partial}{\partial q} \iint (P_u u + P_v v + P_w w) \, dA \) is the vector of external forces, \( \{ Q^{nl} \} = \partial U_3^{(3)}/\partial q = \frac{\partial}{\partial q} \iint (\varepsilon^e A \varepsilon^n + \varepsilon^n B X - \varepsilon^n A \varepsilon^n) \, dA \) is a geometrically nonlinear term, \( K_G = \frac{\partial}{\partial q} \iint (\sigma_\text{init} + \sigma^n) \varepsilon^n \, dA \) is the matrix of initial stresses, \( \{ \Delta Q \} = \frac{1}{2} \partial W^{(2)}/\partial q = \frac{1}{2} \frac{\partial}{\partial q} \iint [-P_u w(c_s + e_s) + P_v v e_{23} + P_w w e_{13}] \, dA \) is the variation in the potential of external forces.
in the case of nonconservative loads, and \( \{Q^{n}\} = \frac{\partial}{\partial t} \int (\sigma^{l} + \sigma^{n}) \, dA \) is the vector of accumulated plasticity. The matrix of elastoplastic strains is determined from the relation

\[
\{\sigma\} = \lbrack E_p \rbrack \{\varepsilon\}.
\]

Here \( \{\sigma\} = \{\sigma_s, \sigma_\theta, \sigma_z, \tau_{\theta\theta}, \tau_{\theta z}, \tau_{sz}\} \) is the column vector of stresses, \( \{\varepsilon\} = \{\varepsilon_s, \varepsilon_\theta, \varepsilon_z, \varepsilon_{\theta\theta}, \varepsilon_{\theta z}, \varepsilon_{sz}\} \) is the column vector of strains, and \( \lbrack E_p \rbrack = \left( \lbrack E \rbrack - \lbrack E \rbrack \{A\} \{A\}^T \lbrack E \rbrack \right) \left( H + \{A\}^T \lbrack E \rbrack \{A\} \right) \) is the elastoplastic modulus; \( \{A\}^T = 6\{\sigma_s - \sigma_o, \sigma_\theta - \sigma_o, \sigma_z - \sigma_o, 2\tau_{\theta\theta}, 2\tau_{\theta z}, 2\tau_{sz}\}, \sigma_o = (\sigma_s + \sigma_\theta + \sigma_z)/3 \),
\( H = EE^p/(E - E^p) \), where \( E \) and \( E^p \) are elastic and plastic modules, respectively. To pass to the region of plastic flow, we use the Mises yield condition

\[
\sqrt{\frac{0.5}{3} [ (\sigma_s - \sigma_\theta)^2 + (\sigma_\theta - \sigma_z)^2 + (\sigma_z - \sigma_s)^2 ] + 6(\tau_{\theta\theta}^2 + \tau_{\theta z}^2 + \tau_{sz}^2)} \gg \sigma_T.
\]

In formulas (7), we used the notation: the index “l” denotes the linear component, the index “n” denotes the nonlinear component, and index “init” denotes the initial stress (prestress), and the dot over a symbol denotes the differentiation with respect to time.

In the expression for the strain energy, we preserve the nonlinear terms up to the fourth order inclusively.

The process of obtaining exact solutions of nonlinear equations of shell and structure dynamics, especially for complex multilayer configurations, where the number of equations depends on the number of layers, encounters great mathematical difficulties. Therefore, such problems (differential equations with variable coefficients) are usually solved numerically.

In this method, the nonlinear terms are placed in the right-hand sides of equilibrium equations and are considered as additional generalized forces calculated from the values of generalized coordinates obtained at the preceding step of loading [1–9]. The convergence is usually improved by using iteration and extrapolation methods.

The equilibrium equations describing the nonlinear dynamic reaction, which are derived from the Lagrange equations (1), can be applied both to linear and nonlinear systems provided that the terms characterizing the strain energy and the work are expressed in terms of generalized coordinates, their time-derivatives, and variations.

Equations (7) were integrated by the Runge-Kutta method with automatic choice of the loading step with a prescribed accuracy under the following initial conditions:

\[
\hat{q} = 0, \quad q = q_o \quad \text{for} \quad t = 0,
\]

where \( \hat{q} = 0 \) is the initial velocity, \( q = q_o \) is the initial deflection, and \( t \) is the time.

The nonlinear terms were calculated for each harmonic from the values of generalized coordinates for the load increment at the preceding loading step.

At the first loading step, it was assumed that the system is linear and the nonlinearities are equal to zero; for the second loading increment, the linear interpolation was used:

\[
q_j = 2q_{j-1} - q_{j-2}.
\]

After the second and subsequent loading steps, the quadratic extrapolation was used:

\[
q_j = \left( 1 + \frac{3}{2}d + \frac{1}{2}d^2 \right) q_{j-1} - (2d + d^2)q_{j-2} + \frac{1}{2}(d + d^2)q_{j-3},
\]

where \( q_j \) are the extrapolated displacements, \( q_{j-1}, q_{j-2}, q_{j-3} \) are the known displacements at the preceding loading steps, and \( d \) is an extrapolation parameter. In the present paper, \( d = 1 \), which corresponds to the case of quadratic extrapolation.
The shape of the loss of stability of the ring plate was taken as the initial deflection $q_o$. The maximal value (amplitude) of the initial deflection was $10^{-8}$ mm. In the calculations, the first three terms of the Fourier series (1) were preserved and the loading increment (workpiece conic punch) was set to be equal to $1 \times 10^{-5}$. In the further calculations, the program itself determined the step necessary to ensure the prescribed accuracy. The loss of accuracy at each step did not exceed $\Delta X = 1 \times 10^{-5}$. The solution with verified relative accuracy $\Delta X = 1 \times 10^{-7}$ was obtained to control the convergence of the solution. In both cases, the results were practically the same for the displacements. The discrepancy was at most 0.2%. The solution was stable until the relative deflection attained the value 0.1–0.2 for the time increment $\Delta t = 1 \times 10^{-6}$ s. To ensure the stability of the solution at the moment of sharp increase in the relative deflection, the increment was decreases to $\Delta t = 1 \times 10^{-7}$ s, and the time step was equal to $\Delta t = 1 \times 10^{-8}$ s in the supercritical domain. Figure 2 illustrates the variations in the ring plate shape under the action of rapidly increasing normal load.

The process of solving the problem of large displacements and strains splits into several stages. As the deflection becomes equal to half the plate thickness in the initial state, the mass and rigidity matrices and the vector of generalized external forces are recalculated with regard to the attained level of the stress strain state and the changed geometry of the plate. The calculations continue till the moment of the loss of stability of the plate. As the criterion for the loss of stability, we take the load at which a nonzero harmonic is suddenly excited. The obtained solutions were used to calculate the dynamic processes of flanging, drawing without pressing, and drawing with conic punch. A good agreement between the theoretical and experimental results was observed.

3.3. Experiment
The experiment was performed by using a magnetic pulse facility of energy capacity 30 kJ. The magnetic field pressure on the workpiece varied in time according to the damped sine law with maximal amplitude equal to $P_{\text{max}} = 75$ MPa and duration of $t = (40–50) \times 10^{-6}$ s. The workpiece deformation mechanism was investigated by using the high-speed filming with frequency varying from 83.330 to 5000 frames per second. The magnetic field topography was determined by the inductance transducers. The maximal pressure was attained at $t = 20 \mu s$. A ring plate of diameter $d = 190$ mm and thickness $h = 1.5$ mm deformed into a ring matrix ($R = 190$ mm) was studied. Initially, the plate was pressed to the matrix by the force $P = 14.7$ kN. The material was the aluminum alloy Amg6M ($\sigma_s = 3.46 \times 10^8$ N/m$^2$, $\sigma_\theta = 715 \times 10^8$ N/m$^2$, the hardening modulus $3g = 4.35 \times 10^8$ N/m$^2$).

The plastic deformation starts at the matrix radius and propagates towards the grinding center as an extension wave and bending and shear waves. As a wave arrives, the speed of the...
pole part increases sharply, which often results in the fracture. For sufficiently thin workpieces \((h/R = 0.02)\), the maximal strains and the fracture arise at the center independently of the fixation practice. In thicker rigidly fixed workpieces, the fracture occurs near the fixation. Any workpieces are similarly fractured under the action of a momentary powerful pulse.

The comparative experiments on static deformation of workpieces under the pressure of resin showed that the pulse loading permits increasing the limit degree of strain before the fracture by \((50–90)\%\). The degree of strain before the loss of stability in operations of deep drawing was increased by \((15–20)\%\) compared with the static one, and in the operations of flanging, by \(50\%\).

In the experiments, where the influence of the pulse value on the moment of the loss of stability was investigated, a ring workpiece of diameter \(D = 100\) mm and thickness \(h = 1.2\) mm manufactured from the D16M material \((\sigma_s = 3.0 \times 10^8\) N/m², \(\sigma_\theta = 700 \times 10^8\) N/m², the hardening modulus \(3g = 5.55 \times 10^8\) N/m²) was considered. The central part of diameter \(d = 45\) mm of the ring workpiece was supported by a steel punch. The dynamics of the workpiece motion depending on the pulse value and shape and the moment of the loss of stability were obtained by decoding the cinegrams. An analysis of the performed experiments showed that the plate limit degree of strain before the loss of stability increases by \((15–40)\%\) compared with the static loading.

As a rule, the discrepancy between all experimental data and the numerical data obtained by solving the problems did not exceed \((10–20)\%\).

Conclusion
A mathematical model for studying the shell configurations made of homogeneous and composite materials under static and dynamic loads was derived in a geometrically and physically nonlinear statement. The main result of this study is the possibility of controlling the process of loss of stability of thin-walled plates and shells experiencing large elastoplastic deformations under the conditions of dynamic loading.

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