Construction of Kostka matrix at the level of bases

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Abstract. We develop a method of construction of transformation matrix between two bases of the model of Heisenberg magnet. The first one is a natural basis of magnetic configurations while the second is adjusted to the irreducible basis of the duality of Weyl. Proposed method allows us to calculate each matrix element separately, so it does not depend on the dimension of the system. Calculation of a matrix elements is given by ladder construction of consecutive letters of magnetic configurations along the well known Robinson-Schensted algorithm. In this way we obtain a graph with vertices given by Gelfand patterns and edges labelled by insertion algorithm. This graph allows us to read off all Clebsch-Gordan coefficients for a unitary group $U(n)$ and then to calculate the matrix element.

1. Magnetic configurations
We consider a one dimensional Heisenberg magnet consisting of $N$ nodes, each with single node spin $s$. A natural state of the magnet is given by ascribing the spin projection to each node of the magnet. Mathematically, this is formulated in terms of two sets, the set of $N$ nodes of the crystal, $\tilde{N} = \{j = 1, 2, ..., N\}$-the alphabet of nodes, and the set of the single node states, $\tilde{n} = \{i = 1, 2, ..., n\}$-the alphabet of spins ($n = 2s + 1$).

In this language, a basis state of the magnet is a mapping $f: \tilde{N} \to \tilde{n}$, which can be presented as

$$|f\rangle = |f(1)f(2)\cdots f(N)\rangle \quad f(j) \in \tilde{n}, j \in \tilde{N}.$$  \hfill (1)

We call this state a magnetic configuration, or a word of length $N$ in the alphabet of spins. The set of all magnetic configurations $\tilde{n}^{\tilde{N}} = \{f: \tilde{N} \to \tilde{n}\}$ forms a basis

$$b_f = \{|f\rangle, \ | f \in \tilde{n}^{\tilde{N}}\}$$  \hfill (2)

which spans the Hilbert space $\mathcal{H} = \mathbb{C} \otimes b_f$ of the model.

2. The duality of Weyl
On the other hand, the space $\mathcal{H}$ is a scene of two dual actions: $A : \Sigma_N \times \mathcal{H} \to \mathcal{H}$ and $B : U(n) \times \mathcal{H} \to \mathcal{H}$, the symmetric and unitary group, respectively. These actions decompose into irreps

$$A = \sum_{\lambda \in \mathcal{D}_{U(N,n)}} m(A, \Delta^\lambda) \Delta^\lambda, \quad B = \sum_{\lambda \in \mathcal{D}_U(N,n)} m(B, D^\lambda) D^\lambda$$

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where appropriate multiplicities, by the duality of Weyl [1] satisfy the following relations
\[ m(A, \Delta^\lambda) = \dim D^\lambda, \quad m(B, D^\lambda) = \dim \Delta^\lambda, \]
\( \lambda \) belongs to the set of all partitions of the integer \( N \) into not more than \( n \) parts.

In this way, the duality of Weyl decomposes the whole space \( \mathcal{H} \) of quantum states of the composite system into sectors \( \mathcal{H}^\lambda \)

\[ \mathcal{H} = \sum_{\lambda \in D_W(N,n)} \mathcal{H}^\lambda. \]  

Each such sector is spanned by the vectors with fixed shape \( \lambda \) from the set

\[ b_{irr} = \{ |\lambda t y\rangle | \lambda \in D_W(N,n), \ t \in SWT(\lambda, \tilde{n}), \ y \in SYT(\lambda, \tilde{N}) \}, \]

where \( t \) denotes a standard Weyl tableau of the shape \( \lambda \) on the alphabet of spins \( \tilde{n} \) while \( y \) a standard Young tableau of shape \( \lambda \) on the alphabet of nodes \( \tilde{N} \).

### 3. The Kostka matrix at the level of bases

As a result from the two previous sections we have two physical bases, the basis of magnetic configurations (2) and the basis of the duality of Weyl (4). There exists a combinatorial bijection between these two sets (known as the Robinson-Schensted algorithm [2, 3]), which ascribes a triad \( |\lambda t y\rangle \) to each magnetic configuration \( |f\rangle \)

\[ b_f \xrightarrow{RS} b_{irr}. \]  

Since these two sets form physical bases, there is a very interesting question, how to find a unitary transformation between them.

Let us consider the action of the group \( \Sigma_N \) on the set \( \tilde{n}^\tilde{N} \) of all magnetic configurations. This action decomposes the set of all magnetic configurations into orbits \( O_\mu \). Such an orbit carries the transitive representation \( R^{\Sigma_N; \Sigma^\mu} \) with the stabiliser \( \Sigma^\mu \) being a Young subgroup. This transitive representation decomposes into irreps of \( \Sigma^\mu \)

\[ R^{\Sigma_N; \Sigma^\mu} \cong \sum_{\lambda \geq \mu} K_{\lambda \mu} \Delta^\lambda \]  

where \( K_{\lambda \mu} \) denotes Kostka numbers [4] and the sum runs over all partition \( \lambda \) greater than or equal to \( \mu \) in the dominance order. Decomposition (6) can be written at the level of bases in a form

\[ |\mu \lambda t y\rangle = \sum_{f \in O_\mu} \langle f|\mu \lambda t y\rangle \cdot |f\rangle. \]  

where the element \( \langle f|\mu \lambda t y\rangle \) denotes the matrix element at row \( |f\rangle \) and column \( |\lambda t y\rangle \), \( \mu \) labels the whole matrix. The question is how to compute this matrix element, or in other words, the probability amplitude of transition from the initial state \( |f\rangle \) to the final \( |\mu \lambda t y\rangle \).

### 4. The algorithm

We know from elementary Quantum Mechanics [5] that the probability amplitude for the transition of an electron, from a source \( s \) through a sequence of walls with slits in them to the detector \( x \), is given by the formula

\[ \langle x|s \rangle = \sum_{\text{all paths from } s \text{ to } x} \prod_{\text{all parts of a path}} A_{\text{a part of a path}} \]  

\[ \]
where $A_k$ part of a path denotes the probability amplitude of transition through a part of a given path. We will show in the sequel that a method of computation of Kostka matrix is very similar in ideology to this picture of interference.

A general sketch of algorithm of computation of the element $\langle \lambda ty|f \rangle$ of the Kostka matrix goes as follows

(i) Read the sequence of partitions

$$\lambda_{RS} = \{\lambda_1, \lambda_12, \ldots, \lambda_12\ldots N = \lambda\}$$

from the reverse RS algorithm (details are given in [6])

(ii) Using Schensted insertion [7] construct a graph $\Gamma$ of Gelfand patterns [8] adjusted to the sequence $\lambda_{RS}$

(iii) From the graph $\Gamma$ we can read the amplitude as

$$\langle \lambda ty|f \rangle = \sum_{\text{all paths from top to bottom of the graph}} \prod_{\text{all edges } j \text{ of the one path of the graph}} \left[ \frac{\lambda_{1..j-1}}{t_{1..j-1}} \{1\} \frac{\lambda_{1..j}}{f(j)} \frac{t_{1..j}}{t_{1..j}} \right]$$

where the element in the rectangular bracket denotes the Clebsch-Gordan coefficient for the unitary group $U(n)$.

4.1. Construction of the graph $\Gamma$

The graph $\Gamma$ is a combinatorial construction whose vertices are given by Gelfand patterns and edges are described by Schensted insertion in terms of Gelfand patterns. The Schensted insertion relies on a very simple rule, if we take a subtriangle of the Gelfand pattern of the form $a \quad b \quad c$

where $c$ is this element which has to be increased by one due to insertion algorithm, then if ($c > a$) $a = a + 1$; else $b := b + 1$. To construct vertices of the graph $\Gamma$ we insert to an empty Gelfand pattern succesive letters $f(i)$, $i \in N$ from a magnetic configuration $|f\rangle$, according to Schensted insertion, in such a way, that after inserting a letter $f(j) \in |f\rangle$ the first row of Gelfand pattern should be equal to partition $\lambda_{1..j} \in \lambda_{RS}$.

5. An example

Let us consider an example of calculation of the matrix element $\langle (3, 1), (1, 3, 2, 1) | (1, 3, 2, 1) \rangle$.

First from the reverse Robinson-Schensted algorithm applied to tableaux $(\begin{array}{c}1 & 2 & 3 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{array})$, $(\begin{array}{c}1 & 2 & 3 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{array})$ we obtain a sequence $\lambda_{RS} = \{\lambda_1 = (100), \lambda_12 = (200), \lambda_123 = (210), \lambda_1234 = (310) = \lambda\}$. Next using Schensted insertion we construct a graph $\Gamma$ given in Figure 1.

From such a graph we can read the matrix element of the form

$$\langle (3, 1), (1, 3, 2, 1) | (1, 3, 2, 1) \rangle = \sum_{\text{two paths}} \left[ \frac{\lambda_1(1) \lambda_12}{f(1) f(2) t_{12}} \right] \left[ \frac{\lambda_12 (1) \lambda_123}{f(3) t_{123}} \right] \left[ \frac{\lambda_123 (1) \lambda}{f(4) t_{123}} \right] =$$

$$(\begin{array}{c}1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{array}) + \left[ \begin{array}{c}1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{array} \right] \left[ \begin{array}{c}1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{array} \right] \left[ \begin{array}{c}1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{array} \right] \left[ \begin{array}{c}1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{array} \right] \left[ \begin{array}{c}1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{array} \right] \left[ \begin{array}{c}1 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \end{array} \right] \left[ \begin{array}{c}1 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \end{array} \right] \left[ \begin{array}{c}1 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \end{array} \right] \left[ \begin{array}{c}1 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \end{array} \right]$$

These Clebsch-Gordan coefficients can be calculated using a technique called Pattern Calculus [9, 10] by interpreting them in terms of fundamental tensor operators [11].
\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}
\]

\[
\lambda_1 = (1, 0, 0)
\]

\[
\begin{pmatrix}
1 & 0 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
\end{pmatrix}
\]

\[
\lambda_{12} = (2, 0, 0)
\]

\[
\begin{pmatrix}
2 & 1 & 0 \\
2 & 1 & 0 \\
1 & 1 & 0 \\
\end{pmatrix}
\]

\[
\lambda_{123} = (2, 1, 0)
\]

\[
\begin{pmatrix}
3 & 2 & 1 \\
3 & 2 & 1 \\
2 & 1 & 0 \\
\end{pmatrix}
\]

\[
\lambda_{1234} = (3, 1, 0)
\]

Figure 1. Graf Γ for matrix element $\langle (3, 1), (1, 3, 2, 1) \rangle$. Here the vertices are given by Gelfand patterns while edges are labeled by Schensted insertion. In second column we have a sequence of partitions given by the reverse Robinson-Schensted algorithm. We see that during insertion of first two letters 1 and 3 we have only one possibility of insertion, while during insertion of third letter 2 we have two possibilities what can be interpreted as combinatorial counterpart of quantum interference. It is worth to observe that after insertion of the last letter 1 two Gelfand patterns converge to one, which is in combinatorial correspondence with the Weyl tableau $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}$.

6. Final remarks and conclusions
I have shown that a method of calculation of Kostka matrix at the level of bases can be understood in terms of quantum interference and Feynman diagrams. This method allows us to construct a wave packet of magnetic configurations which have a strictly defined symmetry of the permutation and unitary group along the duality of Weyl.

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