Smooth generalized interval exchange transformations with wandering intervals, from explicit derived from pseudo-Anosov maps

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Abstract
Starting from any pseudo-Anosov map $\phi$ on a surface of genus $g \geq 2$, we construct explicitly a family of Derived from pseudo-Anosov maps $f$ by adapting the construction of Smale’s derived from Anosov maps on the two-torus. This is done by perturbing $\phi$ at some fixed points. We first consider perturbations at every conical fixed point and then at regular fixed points. We establish the existence of a measure $\mu$, supported by the non-trivial unique minimal component of the stable foliation of $f$, with respect to which $f$ is mixing. In the process, we construct a uniquely ergodic generalized interval exchange transformation (GIET) with a wandering interval that is semi-conjugated to a self-similar interval exchange transformation. This GIET is obtained as the Poincaré map of a flow renormalized by $f$ which parametrizes stable foliation. When $f$ is $C^2$, the flow and the GIET are $C^1$.

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(Some figures may appear in colour only in the online journal)

1. Introduction
Since the work of Denjoy [1, 17], it is known that every $C^1$ diffeomorphism of the circle such that the logarithm of its derivative is a function of bounded variation has no wandering interval.
There is no analogous result concerning interval exchange transformations—IETs for short. An IET is a piece-wise translation bijection, with finitely many branches, of a given base interval, while a generalized IET—GIET for short—is a bijection of the interval which is a piece-wise increasing homeomorphism with finitely many branches. These transformations can be seen as generalizations of, respectively, rigid translations and diffeomorphisms of the circle. See for instance the surveys [15, 29, 31]. On the other hand, IET and GIET can also be seen as the first return map of a flow on a surface to an interval. This is the point of view we will adopt.

In fact, there are several counter-examples, including very smooth ones. In [21] Levitt found an example of non-uniquely ergodic affine IET—AIET for short—with wandering intervals. Latter, using Rauzy–Veech induction, Camelier and Gutierrez [7] exhibited a uniquely ergodic AIET with wandering intervals, semi-conjugated to a self-similar IET—i.e. an IET induced by the foliation of a pseudo-Anosov diffeomorphism. Then Bressaud et al [5] found a Galois type criterion on eigenvalues of a matrix associated to a self-similar IET in order to admit a semi-conjugated AIET with wandering intervals. Finally, Marmi et al [22] proved that almost every IET admits a semi-conjugated AIET with a wandering interval.

In this paper, we prove the following result using an explicit construction.

**Theorem 1.1.** For all self-similar IET $T_0$, there exists a $C^1$ GIET $T$ semi-conjugated to $T_0$ such that

1. $T$ has a unique minimal set $\Omega$. This set is a Cantor set and is an attractor for $T$ and $T^{-1}$,
2. $T$ is uniquely ergodic, of unique invariant measure $\nu$ supported by $\Omega$,
3. $T$ has wandering intervals.

Furthermore $T$ can be chosen to be the Poincaré map of a $C^1$ flow of a surface and to have any number of wandering intervals.

The above theorem will follow from proposition 4.3, theorems 4.4 and 4.8, and proposition 4.9.

The proof relies on a geometric construction initiated by Smale [26, section I.9]. More precisely, we built a transformation of $S_g$, the surface of genus $g$, by perturbing a pseudo-Anosov homeomorphism. We call it derived from pseudo-Anosov, as in [2, 23]. A pseudo-Anosov map on a surface $S_g$ of genus $g$ can be defined as an element of the homotopy class of a map preserving a flat metric on $S_g$ and locally given (in the natural coordinates associated to the half-translation structure) by the action of a diagonal and hyperbolic matrix of determinant 1. Furthermore, up to sign, the matrix is constant on $S_g \setminus \Sigma$, where $\Sigma$ denotes the set of conical points. Iterates of a pseudo-Anosov map are also pseudo-Anosov maps—see [20] for equivalent definitions.

The method used is similar to the one to pass from an Anosov map to a derived from Anosov diffeomorphism [9, 19, 26, 27] and has already appeared in the literature [2, 3, 23]. That is, we convert a fixed point of a pseudo-Anosov map, either regular or conical, into an attracting fixed point by a perturbation. In fact, in order to prove that the GIET in theorem 1.1 is piecewise $C^1$, we give an explicit construction of such a map by generalizing Coudène’s one for derived from Anosov maps.

For a large class of parameters, the family of maps obtained are derived from pseudo-Anosov and admits Axiom A attractors. It turns out that the stable manifolds of the constructed derived from pseudo-Anosov $f$ map can be parametrized by a $C^1$ flow $h_t$. This flow can be renormalized by $f$, in a similar fashion Giulietti–Liverani horocyclic flows are renormalized by an Anosov map [16]—see also [6] where a parabolic flow is renormalized by a partially hyperbolic map.
Theorem 1.2. For every Derived from pseudo-Anosov map $f$ constructed as in section 2.1, there exists a hyperbolic attractor $K$ and a flow $h_t$ on $S_g \setminus \Sigma$ such that

(a) $h_t$ is complete on $K$ and $\frac{d}{dt} |_{t=0} h_t |_K$ spans the stable foliation of $f$,
(b) $f \circ h_\lambda = h_t \circ f$, where $\lambda > 1$ is the expansion factor of the pseudo-Anosov homotopic to $f$,
(c) $h_t$ is uniquely ergodic, with unique invariant measure $\mu$ supported by $K$,
(d) $K$ is an attractor for future and past for $h_t$, on which $h_t$ is minimal.

The flow $h_t$ and the map $T$ from theorem 1.1 are related as follow:

(a) $h_t$ is the suspension flow of the GIET $T$,
(b) The unique invariant measures of $h_t$ and $T$ are related by $\mu = \nu \otimes \lambda$, where $\lambda$ is the Lebesgue measure. Also, $\mu$ is the SRB measure of $f$, for which $f$ is mixing—results for Axiom A attractors from [24] apply.

Furthermore, we can construct $f$ such that $h_t$ is $C^1$.

The above theorem will follow from proposition 3.11, theorem 4.1, corollary 4.10, lemma 4.11 and theorem 6.1.

1.1. Organization of the paper

The paper is organized as follow: section 2 is devoted to the construction of derived from pseudo-Anosov maps as in theorem 1.2. First we recall the basic ideas of the construction, already present in the works [2, 3, 23]. Then we give an explicit formulation for the map, generalizing Coudène’s construction of derived from Anosov map [9]. We prove that for appropriate parameters, the constructed map $f$ is indeed derived from pseudo-Anosov. Moreover, we prove the existence of an invariant compact set $K$, and show in theorem 2.9 that this set is hyperbolic by computing explicitly a vector field $v^s$ spanning the stable foliation—the unstable foliation coincide with the one of the initial pseudo-Anosov map $\varphi$. By construction, $v^s$ satisfies, for all $x$ in $K$, the relation

$$d_x f v^s(x) = \lambda^{-1} v^s(f(x)) \quad (1.1)$$

where $\lambda > 1$ is the expansion factor of $\varphi$.

Most of the work is carried in section 3. We prove in theorems 3.3 and 3.6 that $v^s$ can be extended over $S_g \setminus \Sigma$ into a Lipschitz continuous vector field satisfying (1.1). Under a stronger assumption occurring in the construction of $f$, we prove that the extension of $v^s$ is $C^1$ (theorem 3.7). We also prove that the homotopy between $\varphi$ and $f$ induces a homotopy between $v^s$ and the constant vector field spanning the stable foliation of $\varphi$ (theorem 3.9). By integration of $v^s$, we get a flow $h_t$ that satisfies

$$f \circ h_t(x) = h_{\lambda^{-1} t} \circ f(x) \quad (1.2)$$

because of (1.1), for all $x$ in $S_g \setminus \Sigma$ and $t$ in $\mathbb{R}$ whenever both sides are well defined. Using this commutation relation we deduce that $K$ is connected (theorem 3.13) and that $f: K \to K$ is topologically transitive (theorem 3.14). A similar commutation relation is used by Butterley–Simonelli [6] where a parabolic flow is renormalized by a partially hyperbolic map on some 3-dimensional manifold.

In section 4 we consider the GIET $T$ obtained as the Poincaré map of $h_t$ to some transversal interval. Using the homotopy between vector fields, we prove that $T$ follows the same full path as a self-similar IET $T_0$ during the Rauzy–Veech algorithm, hence $T$ is semi-conjugated to $T_0$. 

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by a result of Yoccoz [28, proposition 7]—this proves theorem 1.1. Unique ergodicity of \( h_t \) then follows by writing \( h_t \) as the suspension flow of \( T \). Because of the commutation relation (1.2) and the usual functional characterization of mixing, \( f \) is mixing with respect to the unique invariant measure of \( h_t \).

In section 5, we state the analogous results of sections 3 and 4 in the case where the perturbation of the pseudo-Anosov map done in section 2 is performed at a regular point instead of a conical point.

Finally, in the last section, using extensively Ruelle’s results [24] on the SRB measure of Axiom A attractors, we prove that \( \mu \) is the unique SRB measure of \( f^{-1} \) for a \( \mathcal{C}^2 \) perturbation, and that the correlations decrease exponentially fast for \( \mathcal{C}^1 \) observables. We also ask whether the result on Ruelle spectrum of linear pseudo-Anosov maps [13] extends to the present case, and if the asymptotic expansion of the ergodic integrals [14, corollary 1.5] applies for \( h_t \).

2. Derived from pseudo-Anosov map with smooth explicit foliations

The construction of derived from Anosov maps was initiated by Smale [26] by blowing up the stable manifold of a fixed point of an Anosov map. More precisely, an Anosov map is perturbed in such a way that some hyperbolic fixed point is turn into a sink (or a source). Derived from Anosov transformations are an example of Smale’s diffeomorphism. The adaptation of this procedure to the setting of pseudo-Anosov maps was already known since the earliest works on pseudo-Anosov transformations [11, 12]. General Smale diffeomorphisms of surfaces have been extensively studied by Bonatti and Langevin [3]. These maps can in fact be blown down into pseudo-Anosov ones as in [3, theorem 8.3.1] in the sense that in some neighbourhood of the non-wandering set, the map is semi-conjugated to a pseudo-Anosov transformation.

The non-wandering set of a derived from pseudo-Anosov transformation gives an example of a non trivial—different from a single periodic orbit—attractor on a surface. In fact, Barge and Martensen proved [2, theorem 1]—completing the work of [23]—that any expansive and transitive attractor, different from a single periodic orbit, comes from a derived from pseudo-Anosov transformation.

Here, since we focus on the smoothness of the stable foliation in order to derive a smooth GIET as in theorem 1.1, we give an explicit construction of derived from pseudo-Anosov maps, adapting Coudène’s one [9, chapter 9] to the setting of surface of genus larger than two.

In this section, we describe the explicit construction of a family of derived from pseudo-Anosov maps by perturbing a pseudo-Anosov transformation at each conical fixed points—a similar procedure for regular fixed point is performed in section 5. We prove that these maps are well defined, are homeomorphisms on \( S_g \) and \( \mathcal{C}^1 \) away from conical points, and that for a good choice of parameters, conical points are the only attractive fixed points. By connectedness of \( S_g \), the complement \( K \) of the union of basins of attraction is not empty. We prove in theorem 2.9 that \( K \) is hyperbolic by computing vector fields spanning the stable and the unstable foliations.

2.1. Perturbation of a pseudo-Anosov

Let \( \varphi \) be a pseudo-Anosov transformation on the Riemann surface \( S_g \) of genus \( g \). Therefore the invariant foliations of \( \varphi \) can be derived from a holomorphic quadratic differential \( q \) invariant by \( \varphi \). Up to consider a cover of order two in most cases, it is not too restrictive to assume that the quadratic differential is Abelian, in other words \( q = \omega \) so that the transition maps, of the half-translation structure induced by natural coordinates of \( \omega \), are translations. Up to multiplying
$\omega$ by a modulus one complex number, the horizontal and vertical foliations $\{\Re(\omega) = 0\}$ and $\{\Im(\omega) = 0\}$ are the invariant foliations of $\varphi$. Let $\lambda > 1$ denote the stretch factor of $\varphi$. This stretching is assumed to correspond to the horizontal measured foliation. The vertical measured foliation is stretch by a factor $\lambda^{-1}$. Let $\Sigma$ be the set of points where $\omega$ vanishes, and we call these points conical points. We now consider the flat structure induced by $\omega$ on $S_x \setminus \Sigma$, that is charts $z$ so that $\omega = dz$. In the neighbourhood of every conical point $\sigma \in \Sigma$, there exist an integer $n_\sigma > 1$, an open set and a chart $z$ on this set such that $\omega = z^{n_\sigma - 1} dz$. The angle around $\sigma$ is then $2\pi n_\sigma$.

Outside of these neighbourhoods of points of $\Sigma$, we set $f$ to be equal to $\varphi$. We now construct $f$ to be a perturbation of $\varphi$ around each $\sigma$ in $\Sigma$.

Let $\sigma$ be a conical point, $V_\sigma$ a neighbourhood of $\sigma$ and a chart $z$ on $V_\sigma$ so that $\omega = z^{n_\sigma - 1} dz$. Let $\xi$ be the branched cover at $\sigma$ associated to the chart $z, \xi : z \in z^{-1} V_\sigma \mapsto z^{n_\sigma} \in \xi(z^{-1}(V_\sigma)) \subset \mathbb{C}$. Let $(W_i)_{1 \leq i \leq 2n_\sigma}$ be a family of open sets of $\mathbb{C} \setminus \mathbb{R}_+$ such that all $\xi|_{W_i}$ are homeomorphisms. Up to replacing $\varphi$ by one of its power, we assume that every conical point is fixed by $\varphi$ and that $\varphi$ respects the leaves of the branched covers: $\varphi(W_i) \cap V_\sigma \subset W_i$ for all $i$.

We can define $f$ on the base of the branched cover in the exact same manner as Smale [26, section 1.9] does. In order to perform further analysis on the map, we give the following explicit formula that generalizes the one used in [9, chapter 9] and [8] in the case of the cat map on the two-torus.

For $z = x + iy \in \mathbb{C} \setminus \mathbb{R}_+$ in the image of $\xi$, we define $f$ as:

$$f(\xi|_{W_i}^{-1}(z)) := \xi|_{W_i}^{-1}\left((\lambda + \beta_\sigma k_\sigma(|z|/\alpha_\sigma))x + i\lambda^{-1}y\right),$$

for some $\alpha_\sigma > 0$, $\beta_\sigma \leq 0$ (we will be interested in the case $\beta < 1 - \lambda$) and with $|z| \leq \alpha_\sigma$ and where $k_\sigma : \mathbb{R} \to \mathbb{R}$ is an even unimodal map of class $C^1$, compactly supported in $[-1, 1]$ and such that $k_\sigma'$ is Lipschitz continuous, for example $k_\sigma(r) = (1 - r^2)^2 1_{[-1,1]}$. We do this perturbation at every conical point. We will see that such $f$ is well defined for small enough $\alpha_\sigma$.

When such a map $f$ is well defined, we will see in next section that interpolating $(\beta_\sigma)_{\sigma \in \Sigma}$ with 0 gives a homotopy between $f$ and $\varphi$. Therefore, $f$ is an example of derived from pseudo-Anosov transformation.

We give in figure 1 a heuristic representation, when $n_\sigma = 1$—which corresponds to the case treated by Smale in [26].

**Remark 2.1.** Because this construction generalizes Coudère’s one [9, chapter 9] on the two-torus, all results obtained in following sections have their counterparts in the two-torus case.

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1 Often in the rest of the paper, $x$ and $y$ might designate points in $S_x$. The notation $z = x + iy$ will only be used when clearly indicated.
2.2. Smoothness and range of parameters

In order to ensure that the explicit construction introduced above makes sense, we need to ensure that the open sets \( V_\sigma \) near each conical point do not overlap with one another, nor with themselves. This can be easily done by taking the parameter \( \alpha_\sigma \) small enough. We give a simple bound on their size by geometric considerations.

Let \( \text{Syst}_{s.c}(S_\varphi) = \inf \{ d(\sigma_1, \sigma_2) \mid \sigma_1, \sigma_2 \in \Sigma \} \), where \( d \) is the distance for the flat metric on \( S_\varphi \) associated to the invariant measured-foliation of \( \varphi \). Let \( \text{Syst}(S_\varphi) = \inf \{ l(\gamma) \mid \gamma \neq 0 \} \) in \( \pi_1(S_\varphi) \) be the smallest possible length of any non-trivial loop. Then define \( \delta_\Sigma = \min(\text{Syst}_{s.c}(S_\varphi), \text{Syst}(S_\varphi)) \).

Actually, it is a general fact that \( \text{Syst}_{s.c}(S_\varphi) \leq \text{Syst}(S_\varphi) \), and therefore \( \delta_\Sigma = \text{Syst}_{s.c}(S_\varphi) \). This can be seen from the fact that if \( \gamma \neq 0 \) in \( \pi_1(S_\varphi) \), we can assume that \( \gamma \) is a straight line. Translating \( \gamma \), we get that \( \gamma \) is contained in a cylinder. Thus, considering the maximal cylinder containing \( \gamma \), it must have a conical point in its boundary. In other words, there is a translation of \( \gamma \) that is a saddle connection.

**Proposition 2.2.** For all \( \beta_\sigma \in [-\lambda, 0] \) and all \( \alpha_\sigma < \delta_\Sigma / 2 \), \( f \) is a homeomorphism on \( S_\varphi \) and is a \( C^1 \) diffeomorphism on \( S_\varphi \setminus \Sigma \).

**Proof.** Clearly, \( f \) is continuous on \( S_\varphi \) and differentiable everywhere except on \( \Sigma \). The differential on \( S_\varphi \setminus \Sigma \) of \( f \) is invertible, hence \( f \) is a local homeomorphism on \( S_\varphi \setminus \Sigma \) and hence \( f(S_\varphi \setminus \Sigma) \) is open. In charts around points of \( \Sigma \), one can see that \( f \) is a local homeomorphism in a neighbourhood of \( \Sigma \). Hence \( f(S_\varphi) \) is open. Since \( S_\varphi \) is compact, \( f(S_\varphi) \) is closed. Hence \( f(S_\varphi) = S_\varphi \), because \( S_\varphi \) is connected. Therefore, \( f \) is a surjective local homeomorphism, hence \( f \) is a covering map. Since the pre-image of a point of \( \Sigma \) by \( f \) is itself, \( f \) is injective. 

By refining the range where the \( \beta_\sigma \) live, we can turn conical points into attractive fixed points.

**Proposition 2.3.** For \( \beta_\sigma \in [-\lambda, 1 - \lambda] \) and \( \alpha_\sigma < \delta_\Sigma / 2 \), \( \sigma \in \Sigma \) is an attractive fixed point for \( f \). Let \( U_\sigma \) be its basin of attraction. Then \( U_\sigma \) is an open set.

**Proof.** Fix some \( \sigma \in \Sigma \) and a \( W_i \). Then, in charts centred at \( \sigma \), the Jacobian matrix of \( f \) is

\[
(Jacf)(x,y) := \text{Jac } \circ f \circ \mathfrak{W}_x^{-1}(x,y) = \begin{pmatrix}
\lambda + \beta_\sigma k & \frac{|k|}{\alpha_\sigma} + \frac{\beta_\sigma}{\alpha_\sigma} k^2 \\
\frac{|k|}{\alpha_\sigma} & -\lambda^{-1}
\end{pmatrix},
\]

(2.1)

where \( z = x + iy \). Evaluating at \( (0,0) \), we get that \( \text{Jac} f(0,0) = \begin{pmatrix} \lambda + \beta_\sigma & 0 \\ 0 & -\lambda^{-1} \end{pmatrix} \). It is then a consequence of the Grobman–Hartman theorem that \( U_\sigma \) contains a ball centred at \( \sigma \). Finally, we have \( U_\sigma = \bigcup_{n \geq 0} f^{-n}(B(\sigma, \varepsilon)) \), for some small enough \( \varepsilon > 0 \).

Since basins of attraction \( U_\sigma \) are disjoint open sets and \( S_\varphi \) is connected, these basins are not an open cover. Therefore the complement of the union of basins is not empty. Define \( U_\Sigma := \bigcup_{\sigma \in \Sigma} U_\sigma \) and \( K := S_\varphi \setminus U_\Sigma \). These sets are clearly invariants by \( f \).

**Proposition 2.4.** If for some \( \sigma \in \Sigma \), \( \beta_\sigma \in [-\lambda, 1 - \lambda] \) and \( \alpha_\sigma < \delta_\Sigma / 2 \), then there exists a fixed hyperbolic point \( p^\sigma \), \( 1 \leq i \leq 2 n_\sigma \), on each horizontal ray starting at \( \sigma \). We number them by going counter-clockwise around \( \sigma \). All these points are at the same distance \( |p^\sigma| \) from \( \sigma \). Moreover \( B(\sigma, |p^\sigma|) \subset U_\sigma \).
Let \( \sigma, \beta, \alpha \) be as in the proposition. Let \( \gamma : [0, \alpha] \to S_k \) be a unit speed parametrization of a horizontal ray such that \( \gamma(0) = \sigma \). Hence, in charts, \( f(\gamma(t)) = ((\lambda + \beta k(t/\alpha)), 0) \). Let \( h : [0, \alpha] \to \mathbb{R} \) be the function \( h(t) = (\lambda + \beta k(t/\alpha))t \). Then \( h(\alpha) = \lambda \alpha > \alpha \), and \( h'(0) = \lambda + \beta \in [0, 1] \). Hence \( h \) has a fixed point in \( [0, \alpha] \). Call \( t_0 \) the smallest fixed point. This value does not depend on which horizontal ray starting from \( \sigma \) we consider. The point \( p = \gamma(t_0) \) is fixed by \( f \) and is hyperbolic: in the charts centred at \( \sigma \), the Jacobian matrix of \( f \) at \( p \) is
\[
(\text{Jac}) = \begin{pmatrix}
1 + \frac{\beta k(t_0)}{\alpha} & \frac{w}{\alpha} \\
0 & \frac{\lambda}{\alpha}
\end{pmatrix},
\]
where we used that \( \lambda + \beta k(t_0) = 1 \) because \( t_0 \) is a fixed point of \( h \). Now, since \( \frac{\beta k(t_0)}{\alpha} > 0 \), \( p \) is a hyperbolic fixed point.

By definition of \( t_0 \), we have \( \gamma([0, t_0]) \subset U_\sigma \). In the appropriate leaf of the branched-cover over \( \sigma \), we have \( z = (\lambda, \sigma) \) in coordinates. Hence,
\[
d(f(z), \sigma)^2 \leq (\lambda + \beta k(d(z, \sigma)/\alpha))^2 x^2 + \lambda^{-2}y^2 < (\lambda + \beta k(t_0/\alpha))^2 x^2 + \lambda^{-2}y^2 \leq x^2 + \lambda^{-2}y^2 \leq d(z, \sigma)^2.
\]
Hence, the function \( z \mapsto d(f(z), \sigma)/d(z, \sigma) \) is continuous and strictly bounded from above by \( 1 \) on the compact annulus \( \{z \in S_k \mid \varepsilon \leq d(z, \sigma) \leq t_0 - \varepsilon\} \). Therefore every orbit of point from the ball \( B(\sigma, t_0 - \varepsilon) \) ends up entering the ball \( B(\sigma, \varepsilon) \). Hence the claim.}

2.3. Invariant sets

Here we investigate the topological aspects of the invariant set \( K \). In particular we prove that it can be written as the union of the closure of some stable leaves of the hyperbolic fixed points \( p_i^\sigma \) and that it is a hyperbolic set.

We start by proving that the set \( U_{2n} \) is dense in \( S_k \), or equivalently that \( K \) is of empty interior. In order to do this we need the following lemma which is obtained by simply computing the differential of \( f \).

**Lemma 2.5.** Define \( q_i^\sigma \), as the \( 2n \) points at distance \( |p^\sigma| \) from \( \sigma \) on the vertical rays starting from \( \sigma \). Then for all \( x \in S_k \setminus \bigcup_{\sigma \in \Sigma} (B(\sigma, |p^\sigma|) \cup \{q_i^\sigma \mid 1 \leq i \leq 2n \}) \), \( f \) is a strict dilation in the horizontal direction.

**Proof.** Fix some \( \sigma \in \Sigma \). We want to prove that the upper-left coefficient in (2.1) is strictly larger than 1 on the claimed domain. Call \( a(x, y) \) this coefficient. For \( t > |p^\sigma| \), we have that \( (\lambda + \beta k(t/\alpha))t > t \). Then, for any \( \theta \in \mathbb{R} \), \( a(x \cos(\theta), x \sin(\theta)) = \lambda + \beta k(t/\alpha) + \frac{\beta}{\alpha}t \cos(\theta) \geq 1 \).

On the other hand, \( a(|p^\sigma| \cos(\theta), |p^\sigma| \sin(\theta)) = 1 + \frac{\beta}{\alpha}(|p^\sigma| \cos(\theta))^2 k(t/\alpha) = q_i^\sigma \) is equal to \( 1 \) if \( \cos(\theta) = 0 \) or \( \sin(\theta) \), \( (|p^\sigma| \sin(\theta)) = q_i^\sigma \) for some \( 1 \leq i \leq 2n \).

**Proposition 2.6.** For all \( z \in K \) and every \( \varepsilon > 0 \), every horizontal segment of length \( \varepsilon \) containing \( z \) in its interior crosses \( U_{2n} \). Hence \( U_{2n} \) is dense and \( K \) has empty interior.

**Proof.** By contradiction, let \( \gamma : [-\varepsilon, \varepsilon] \to S_k \) be a horizontal segment parametrized by arc length, containing some \( z \in K \) and such that \( \gamma([-\varepsilon, \varepsilon]) \cap U_{2n} = \emptyset \). Without loss of generality, we can assume that \( \gamma(0) = z \). Since \( U_{2n} \) is invariant by \( f \), we see that the existence of
some $-\varepsilon \leq t \leq \varepsilon$ such that $f^n(\gamma(t)) \in U_\Sigma$ is impossible. Hence $f^n(\gamma([-\varepsilon, \varepsilon])) \cap U_\Sigma = \emptyset$. By construction of $f$, the set $f^n(\gamma([-\varepsilon, \varepsilon]))$ is a horizontal segment, containing $f^n(z)$ in its interior and of length $l_n$. Since $f$ is a strict dilation in the vertical direction on the compact set $K$, there exists $l_n > 1$ such that $l_n \geq f_n$.

Let $\delta = \inf \{|p^\sigma| : \sigma \in \Sigma\}$ and since $K$ is compact and invariant by $f$ let $\gamma \in K$ be a subsequence limit of $(f^n(z))$. Let $n_k$ be an increasing sequence of integers such that $f^n(z)$ converges to $\gamma$ as $n_k \to \infty$. We know—see [10, corollary 14.15]—that the vertical leaf containing $\gamma$ is at least infinite in one direction and is dense in $S_k$. In particular, some sufficiently long section of this leaf, containing $\gamma$, is $\delta/4$-dense in $S_k$. Hence, for large enough $n_k$, the curve $f^n(\gamma([-\varepsilon, \varepsilon]))$ is sufficiently long and sufficiently close to the horizontal leaf containing $\gamma$ to be $\delta/2$-dense in $S_k$. In particular, there exists $-\varepsilon < t < \varepsilon$ such that $d(f^n(\gamma(t)), \sigma) < \delta$ for some $\sigma \in \Sigma$. This contradicts the fact that $B(\sigma, |p^\sigma|) \subset U_\Sigma$.

Recall definitions of strong stable and strong unstable leaves of $x \in S_k$ with respect to $f$

\[
W^s(x) = \{ y \in S_k : d(f^n(x), f^n(y)) \to 0 \text{ as } n \to +\infty \}, \\
W^u(x) = \{ y \in S_k : d(f^{-n}(x), f^{-n}(y)) \to 0 \text{ as } n \to +\infty \}.
\]

If $x$ is a fixed point of $f$, then these sets are invariant by $f$.

Here, these leaves at hyperbolic fixed points $p^\sigma_i$ enable to describe precisely the set $K$. We start by showing that the stable leaves can be seen as the horizontally accessible border of $U_\Sigma$—and are obviously contained in $K$. On the other hand, unstable leaves are dense.

**Proposition 2.7.** The stable and unstable leaves of the fixed point $p^\sigma_i$ satisfies the following assertions.

(a) If $x \in U_\sigma$ and $\gamma : [0, 1] \to S_k$ is a horizontal curve such that $\gamma(0) = x$, $\gamma([0, 1]) \subset U_\sigma$ and $\gamma(1) \notin U_\sigma$ then $\gamma(1)$ belongs to $\bigcup_{1 \leq i \leq 2n_\sigma} W^s(p^\sigma_i)$.

(b) For all $\sigma \in \Sigma$ and all $1 \leq i \leq n_\sigma$, the unstable leaf $W^u(p^\sigma_i)$ contains a full semi-infinite horizontal leaf. Hence $W^s(p^\sigma_i)$ is dense in $S_k$.

**Proof.** We begin with the first point. Let $\delta > 0$ be the length of the smallest side in the (finite) collection of rectangle neighbourhoods of points $p^\sigma_i$ given by the Grobman–Hartman theorem. For $n$ large enough, we find that $f^n(x)$ is $\delta/2$-close to some $\sigma \in \Sigma$. Once close to $\sigma$ by going horizontally along $f^n \circ \gamma$, the first time $f^n \circ \gamma$ intersects $K$ is at some point contained in one of the rectangle neighbourhood of some $p^\sigma_i$. Therefore this intersection point belongs to $\bigcup_{1 \leq i \leq 2n_\sigma} W^s(p^\sigma_i)$ and is attained at $f^n(\gamma(1))$. The result then follows from the invariance by $f$ of the stable leaves.

We now prove the second point. Let $\gamma : [0, +\infty[ \to S_k$ be a unit speed parametrization of the horizontal ray starting at $\sigma \in \Sigma$ and containing $p := p^\sigma_i$. In particular, $\gamma(0) = \sigma$ and $\gamma(|p^\sigma|) = p$.

By contradiction, assume there exists $t \geq |p^\sigma|$ such that $\gamma(t) \notin W^u(p)$. Let $t_0 = \inf \{t \geq |p^\sigma| : \gamma(t) \notin W^u(p)\}$.

---

2 The accessible border of an open set $U \subset X$ is the set of points of $\partial U$ that are the endpoints of a curve $\gamma : [0, 1] \to X$ contained in $U$ for $t \in [0, 1]$. The horizontally accessible border is obtained by assuming that the curves $\gamma$ are horizontal.

3 Using the flow $h_t$ introduced in section 3.5, one can bootstrap from proposition 2.7(i) and remove the assumption that $\gamma$ is horizontal.
We now show that \( t_0 > \| p^n \| \). Let \( h : t \mapsto (\lambda + \beta_n k(t/\alpha_\sigma))t \). By construction of \( f \), we have the relation \( f(\gamma(t)) = \gamma(h(t)) \) for every \( t \in [0, \alpha_\sigma] \), and hence \( f^{n}(\gamma(t)) = \gamma(h^n(t)) \) for all \( n \in \mathbb{Z} \). Now \( (h^{-1})^{\prime}\left(\| p^n \| \right) < 1 \), so for \( t \) close to \( \| p^n \| \), \( f^{-n}(\gamma(t)) \to p \) as \( n \) goes to infinity. Therefore \( t_0 > \| p^n \| \).

We now prove that \( \gamma(t_0) \) is a fixed point of \( f \). We know that \( f(\gamma(\| p^n \|, t_0)) = \gamma(\| p^n \|, s) \) for some \( s \). But \( f(\gamma(\| p^n \|, t_0)) \subset W^{su}(p) \). Hence \( s \lesssim t_0 \).

By contradiction, assume there exists \( \varepsilon > 0 \) such that \( s + \varepsilon < t_0 \). So \( \gamma(\| p^n \|, s + \varepsilon) \subset W^{su}(p) \), and so \( f^{-1} \circ \gamma((\| p^n \|, s + \varepsilon)) \subset W^{su}(p) \). However, \( f^{-1} \circ \gamma((\| p^n \|, s + \varepsilon)) = \gamma((\| p^n \|, t_0 + \varepsilon, \delta_\varepsilon)) \) for some \( \delta_\varepsilon > 0 \) since \( f \) is strictly preserving horizontal orientation. This contradicts the definition of \( t_0 \). Therefore \( s = t_0 \) and \( \gamma(t_0) \) is fixed by \( f \).

The point \( \gamma(t_0) \) cannot be in \( \Sigma \) nor be a \( p^n \), otherwise \( \gamma \) would connect two conical points, which is impossible. By computing the differential of \( f \) at \( \gamma(t_0) \), we see that \( \gamma(t_0) \) is a hyperbolic fixed point of \( f \) with a horizontal unstable leaf. Therefore there exist points whose iterates by \( f^{-1} \) converge to \( p \) and to \( \gamma(t_0) \neq p \).

These properties of stable and unstable leaves yield to the fact that the set \( K \) can be written as a finite union of closure of stable leaves. In fact, we have the following slightly stronger result.

**Proposition 2.8.** The compact set \( K \) can be written as a finite union of closed invariant sets as follow \( K = \bigcup_{\sigma \in \Sigma} \bigcup_{n} \left( W^{spi}(p_0) \cap W^{su}(p_0) \right) \).

**Proof.** Let \( x \in K \) and \( \varepsilon > 0 \). Let \( y \in U_{\varepsilon}x \) be in the same horizontal leaf as \( x \) and obtained by going leftward by a distance less than \( \varepsilon \). Since \( U_{\varepsilon}x = \bigcup_{\sigma} U_{\varepsilon}x \), there exists \( \sigma \in \Sigma \) such that \( y \in U_{\varepsilon}x \).

From the Grobman–Hartman theorem, for each \( 1 \leq i \leq 2n_\sigma \) there exists a neighbourhood of \( l_\sigma^{p^n} \) on which the dynamic of \( f \) is the same as the one of the differential of \( f \). Without loss of generality, we assume that these neighbourhoods are rectangles with vertical and horizontal sides and with centres the \( l_\sigma^{p^n} \). Up to replacing these rectangles by smaller ones, let \( \delta_\sigma \) be a common vertical size for these rectangles.

For \( n \geq 0 \) large enough, the point \( y \) lies in \( B(\sigma, \delta_\sigma/4) \). By construction and by the first point of proposition 2.7, we know that by going rightward from \( y \) we cross some \( W^{su}(p_0) \), for some \( 1 \leq i \leq 2n_\sigma \). Therefore, by going rightward from \( f^{-n}(y) \) we cross the rectangle of linearization associated with \( p_0^{n} \), and hence the stable leaf \( W^{sp}(p_0^{n}) \) at some point \( y' \).

Let \( \delta \) be the modulus of absolute continuity of \( f^{-n} \) associated with \( \varepsilon \). By density of the unstable leaf of \( p_0^{n} \), we can chose a point \( z \) such that \( d(f^{n}(z), f^{n}(y)) < \min(\delta, \delta_\sigma/4) \) so that by going rightward from \( f^{n}(z) \) we cross \( W^{su}(p_0^{n}) \) at some point \( z' \), at distance less than \( \delta \) from \( y' \). Finally, the point \( f^{-n}(z') \in W^{su}(p_0^{n}) \cap W^{su}(p_0^{n}) \) is at distance less than \( 3\varepsilon \) from \( x \).

Finally, we explicit stable and unstable foliations such that the set \( K \) is hyperbolic with respect to \( f \). To do this, we compute a vector field that is uniformly contracted by the differential of \( f \).

**Theorem 2.9.** The set \( K \) is hyperbolic. The invariant distributions are \( E^s(x) = \mathbb{R}e_h \) and \( E^u(x) = \mathbb{R}v(x) \), \( x \in K \), where

\[
 v(x) := e_r - \sum_{i,j \geq 0} \lambda^{-i} b(f^i(x)) \prod_{j=0}^{i} \frac{1}{a(f^j(x))} e_h, \tag{2.2}
\]

with \( a(x) := \langle d(f^i \cdot e_h, e_h) \rangle \) and \( b(x) := \langle d(f^i \cdot e_r, e_h) \rangle \). In particular, \( v \) satisfies \( dfv^\prime = \lambda^{-1} v \) on \( K \).
Proof. We will explicit the stable and the unstable directions of the splitting of the tangent space. Write the differential of \( f \) at \( x \in \mathcal{S}_K \smallsetminus \Sigma \) in the basis \((e_h, e_v)\)

\[
d_x f = \begin{pmatrix} a(x) & b(x) \\ 0 & \lambda^{-1} \end{pmatrix}.
\]

Therefore, for every positive integer \( n \), we have the following.

\[
d_x(f^n) = \prod_{i=0}^{n-1} (a(f^{i-1}(x)) \ b(f^{i-1}(x))) = \begin{pmatrix} a(f(x)) & b(f(x)) \\ 0 & \lambda^{-1} \end{pmatrix}^{n-1},
\]

\[
= \begin{pmatrix} A_n(x) & B_n(x) \\ 0 & \lambda^{-n} \end{pmatrix}.
\]

We have that \( A_n(x) = \prod_{i=0}^{n-1} a(f^i(x)) \). We compute \( B_n \) explicitly. This sequence satisfies a recursive formula, which can be solved

\[
B_{n+1} - a(f^n)B_n = \lambda^{-n}b(f^n),
\]

\[
B_{n+1}/A_{n+1} - B_n/A_n = \lambda^{-n}b(f^n)/A_{n+1},
\]

\[
B_n/A_n = \sum_{i=0}^{n-1} \lambda^{-i}b(f^i)/A_{i+1}.
\]

Finally we get

\[
B_n/A_n = \sum_{i=0}^{n-1} \lambda^{-i}b(f^i) \prod_{j=0}^{i} \frac{1}{a(f^j(x))}.
\]

We can now explicit the eigenvectors of \( d_x(f^n) \). The obvious one, associated with the eigenvalue \( A_n(x) \), is \( e_h \). The other one is

\[
v_u(x) = \begin{pmatrix} -B_n(x)/A_n(x) \\ 1 - \lambda^{-n}/A_n(x) \end{pmatrix} = \begin{pmatrix} \sum_{i=0}^{n-1} \lambda^{-i}b(f^i(x)) \prod_{j=0}^{i} \frac{1}{a(f^j(x))} \\ 1 - \prod_{i=0}^{n-1} \frac{1}{a(f^i(x))} \end{pmatrix}.
\]

We now study the convergence of the \( v_u \)'s as \( n \) goes to infinity. First, since \( a > 1 \), \( b \) are continuous functions over the compact set \( K \), there exist constants \( a^* \) and \( C \) such that \( a > a^* > 1 \) and \( |b| < C \). Therefore, the second coordinate converges to 1 as \( n \) goes to infinity. For the first coordinate, we have the uniform bound over \( K \)

\[
\sum_{i=0}^{n-1} \lambda^{-i}b(f^i(x)) \prod_{j=0}^{i} \frac{1}{a(f^j(x))} \leq C \sum_{i=0}^{n-1} (\lambda a^*)^{-i} \leq C \frac{\lambda a^*}{\lambda a^* - 1}.
\]

Hence, the series of continuous functions converges uniformly over \( K \) to a continuous function. Call \( v^* \) the limit of \( v_u \) as \( n \) goes to infinity.

A short computation shows that for all \( x \in K \), \( v^* \) satisfies \( d_x f \cdot v^*(x) = \lambda^{-1}v^*(f(x)) \). Finally, we get the following splitting of the tangent space at each \( x \) in \( K \), \( T_x S_x \cap T_x T_x = Rv^*(x) \oplus \mathbb{R}e_h, \) so that \( K \) is a hyperbolic set.
3. Smoothness of the stable foliation and renormalized flow

In this section we prove that $v^t$ can be extended to the whole set $S_g \setminus \Sigma$ of regular points, such that the extension is still uniformly contracted by the action of $f$ (1.1) and so that it is Lipschitz continuous. Under further assumption on the smoothness of $f$, we prove that $v^t$ is $C^1$. Furthermore, in view of the next section, we prove that $v^t$ depends continuously on the parameter $\beta$—occurring in the construction of $f$. This regularity property will be crucial in section 4. Since $v^t$ is Lipschitz continuous, it can be integrated into a continuous flow $h_t$ which enjoys the commutation relation (1.2) with $f$—in other words, $f$ renormalizes $h_t$. From the properties of $h_t$, we show that the set $K$ is connected, transverse to any vertical leaf, and that $f$ is transitive with respect to the trace topology on $K$.

3.1. Construction of a useful open cover of $S_g$

In order to proceed, we first need to construct an open cover of $S_g \setminus \Sigma$ such that $f$ satisfies some nice estimates on elements of this cover. This is done in the following proposition.

**Proposition 3.1.** For all $\varepsilon > 0$ small enough, there exist $\eta > 0$, $\delta > 0$, and an open cover $S_g = A_\eta \cup \bigcup_{\sigma \in \Sigma} B_{\sigma, \delta}$ such that $a > 1 + \eta$ on $A_\eta$ and $d(f(x), \sigma) < (1 - \delta)d(x, \sigma)$ on $B_{\sigma, \delta} \setminus \{\sigma\}$.

**Proof.** By continuity of $f$, there exists an $\varepsilon > 0$ such that

$$\{x \in V_\sigma \mid d(f(x), \sigma) < d(x, \sigma)\} \supseteq B(\sigma, |p^\sigma|) \cup \bigcup_{i=1}^{2n_\sigma} B(q^\sigma_i, \varepsilon) =: B^\varepsilon_\sigma,$$

for all $\sigma$, where $V_\Sigma = \bigcup_{\sigma \in \Sigma} V_\sigma$ is the open neighbourhood of $\Sigma$ on which $f \neq \varphi$.

Since $S_g \setminus \bigcup_{\sigma \in \Sigma} B^\varepsilon_\sigma$ is compact and $a > 1$ on it, there exists $\eta > 0$ such that $a > 1 + 2\eta$ on this compact set. Call $A_\eta = \{x \in S_g \mid a > 1 + \eta\}$. By construction, $S_g = A_\eta \cup \bigcup_{\sigma \in \Sigma} B^\varepsilon_\sigma$.

Since all $B^\varepsilon_\sigma$ are open sets, radial and centred on $\sigma$, we have $B^\varepsilon_\sigma = \bigcup_{n \geq 1} \left(1 - \frac{1}{n}\right)B^\varepsilon_{\sigma, n}$. Now, by compactness of $S_g$, there exists $n_0$ such that:

$$S_g = A_\eta \cup \bigcup_{\sigma \in \Sigma} \left(1 - \frac{1}{n_0}\right)B^\varepsilon_{\sigma, n_0}.$$

On a small open neighbourhood $W_\sigma$ of $\sigma$, by construction of $f$ we have that $d(f(x), \sigma)/d(x, \sigma) < C < 1$. Now, on the compact set $\left(1 - \frac{1}{n_0}\right)B^\varepsilon_{\sigma, n_0} \setminus W_\sigma$, the continuous function $d(f(x), \sigma)/d(x, \sigma)$ is positive and strictly bounded from above by 1. On the other hand, up to shrinking $W_\sigma$, the function $d(f(x), \sigma)/d(x, \sigma)$ is bounded on $W_\sigma \setminus \{\sigma\}$ by $\max(\lambda^{-1}, \lambda + \beta \sigma + \delta) < 1$, for some small $\delta > 0$. Hence, there exists $\delta > 0$, independent of $\sigma$, such that for all $x \in (1 - \frac{1}{n_0})B^\varepsilon_{\sigma, n_0} \setminus \{\sigma\}$, $d(f(x), \sigma) < (1 - \delta)d(x, \sigma)$. We then call $B_{\sigma, \delta} = (1 - \frac{1}{n_0})B^\varepsilon_{\sigma, n_0}$.

3.2. Lipschitz extension of $v^t$ to $S_g \setminus \Sigma$

Here we prove that the infinite sum in the definition of the vector field $v^t$ on $K$ does converge on all $S_g \setminus \Sigma$. This way we can define $v^t$ on $S_g \setminus \Sigma$. Furthermore, we prove that this extended vector field is Lipschitz continuous.

We proceed in two steps. First we show that $v^t$ is bounded and continuous on $S_g \setminus \Sigma$. To do this, we need a lemma which follows directly from computation of $df$. 

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Lemma 3.2. On each basin $U_\sigma$, the partial derivative $a = \langle df(e_h), e_h \rangle$ of $f$ is bounded from below by $\lambda + \beta_\sigma$.

The partial derivative $b = \langle df(e_v), e_v \rangle$ of $f$ is locally Lipschitz in some neighbourhood of $\Sigma$. Furthermore, by continuity we can set $b(\sigma) = 0$ for each $\sigma \in \Sigma$.

Theorem 3.3. If $\beta_\sigma \in [-\lambda + \lambda^2, -\lambda + 1]$ for all $\sigma$ in $\Sigma$, then the vector field $v^\sigma$ is bounded and continuous on $S_\sigma \setminus \Sigma$. Furthermore, by construction, the formula $df(v^\sigma) = \lambda^{-1}v^\sigma$ holds on $S_\sigma \setminus \Sigma$.

Proof. Call $s_i = \lambda^{-i}b \circ f^i \prod_{j=1}^i \frac{1}{\lambda^{2j}}$. Let $V$ be a neighbourhood of some $\sigma$ such that $b$ is Lipschitz on it and $f$ contracts by a factor $\max(\lambda^{-1}, \lambda + \beta_\sigma + \delta_\sigma) < 1$. Without loss of generality, we assume that $V$ is a ball centred at $\sigma$ of radius $\varepsilon$ and that $f(V) \subset V$. Since $U_\sigma = \bigcup_{N \geq 0} f^{-N}V$, for all $x \in U_\sigma$ there exist some $N = N(x)$ and an integer $n_\sigma$ which only depends on $V$, such that for all $n \geq N$, $f^n(x) \in V$, at most $n_\sigma$ points of the orbits fall into $B_{\sigma, \delta} \setminus V$ and the rest lives in $A_{\eta}$. Let $x \in U_\sigma$, $x \neq \sigma$. Since $U_\sigma = \bigcup_{n \geq 0} f^{-n}V$, let $N$ be the smallest integer such that $f^N(x) \in V$.

We distinguish three cases:

(a) $i < N - n_\sigma$. Therefore $|s_i(x)| \leq \lambda^{-i} \left( \frac{1}{1 + \eta} \right)^{i+1} \max |b|$.

(b) $N - n_\sigma < i \leq N$. Hence $|s_i(x)| \leq \lambda^{-i} \left( \frac{1}{1 + \eta} \right)^{n_\sigma} \left( \frac{1}{1 + \beta} \right)^{i-(N-n_\sigma)} \max |b|$.

(c) $i = j + N > N$. We get $|s_i(x)| \leq \lambda^{-i+j+N} \left( \frac{1}{1 + \beta} \right)^{i+j} \max(\lambda^{-1}, \lambda + \beta_\sigma + \delta_\sigma)^j$.

Therefore, if $\lambda^{-2} < \lambda + \beta_\sigma$, then

$$
\sum_{i \geq 0} |s_i(x)| \leq \max |b| \frac{\lambda(1 + \eta)}{\lambda(1 + \eta) - 1} \left( 1 + \sum_{i=0}^{n_\sigma} \left( \frac{1}{\lambda + \beta} \right)^i \right) + \frac{\text{Lip}(b)\varepsilon}{1 - \max(\lambda^{-1}, \lambda + \beta_\sigma + \delta_\sigma)^N}
$$

which is uniform in $x$ on $U_\sigma$. Hence, the convergence is uniform on the compact subsets of $U_\sigma \setminus \{\sigma\}$ and $\sum s_i$ is continuous on $U_\sigma \setminus \{\sigma\}$, for all $\sigma \in \Sigma$.

We now show that this function defined on $U_\Sigma = \bigcup U_\sigma$ can be extended by continuity on $K$.

Call $u(x) = e_v = \sum_{i \geq 0} s_i(x)e_h$ the vector based at $x \in S_\sigma \setminus \Sigma$.

Let $x \in K$ and, by density of $U_\Sigma$ in $S_\sigma$, $(x_n)_n \in U_\Sigma^k$ such that $x_n \to x$ as $n$ goes to infinity. Since $(u(x_n))_n$ is bounded, up to extracting, the sequence converges to some $u_0$. Furthermore, by a diagonal argument and up to extracting, $u(f^k(x_n)) \to u_k$ for all $k \in \mathbb{Z}$ as $n$ goes to infinity.

Now, by construction of $u$, $d_u u = \lambda^{-1}u \circ f$. Hence, by continuity of $f$ and $d_u$, $d_u(f^k)(u_0) = \lambda^{-k}u_k$. We now show that $u_0 = v^\sigma(x)$. By hyperbolicity of $K$, there exist real numbers $x_v, x_u$ such that $u_0 = x_v v^\sigma(x) + x_u e_h$. Therefore, by hyperbolicity of $K$, $
\[ |x_u| = |x_u e_h|, \]
\[ = |d_{j,\tilde{f}}^k x_u e_h|, \]
\[ \leq C \left( \frac{1}{\sigma^2} \right)^k |d_{j,\tilde{f}}^k x_u e_h|, \]
\[ = C \left( \frac{1}{\sigma^2} \right)^k |d_{j,\tilde{f}}^k (u_0 - x_v v'(x))|, \]
\[ \leq C \left( \frac{1}{\sigma^2} \right)^k \lambda^{-k} (\sup |u| + x_s \sup |v'|), \]

which goes to zero as \( k \) goes to infinity. Hence \( u_0 = x_s v'(x) \). Now both \( u_0 \) and \( v'(x) \) have the same non-zero coordinate along \( e_i \) in the base \( (e_i, e_j) \). Hence \( u_0 = v'(x) \). Finally, \( u \) extends continuously on \( K \) by \( v' \). We call \( v' \) this vector field on \( S_k \setminus \Sigma \).

We can now present the proof of the Lipschitz continuity of \( v' \) on \( S_k \setminus \Sigma \). To this end, we need a few more estimates on the differential of \( f \) and on its coefficients.

**Lemma 3.4.** For all \( x \in S_k \setminus \Sigma \), the following estimate holds

\[ \frac{||d_{j,\tilde{f}}^m||}{A_n(x)} \leq 2 \max \left( 1, \frac{|B_n(x)| + \lambda^{-n}}{A_n(x)} \right). \]

In particular, \( ||d_{j,\tilde{f}}^m||/A_n \) is bounded on \( \bigcup_{i=0}^n f^{-i}A_n \). Furthermore, the bound \( B \) can be chosen independently of \( n \).

**Proof.** By a direct computation, for \( (u, v) := u e_h + v e_v \)

\[ ||d_{j,\tilde{f}}^m(u, v)||^2 = \langle A_n(x)u + B_n(x)v, v \rangle, \]
\[ = 4 \langle A_n(x)u, v \rangle^2 + (4 \lambda^{-n}) \langle B_n(x)w, v \rangle^2, \]
\[ \leq 4 \max (A_n(x)^2, B_n(x)^2 + \lambda^{-2n}) ||(u, v)||^2. \]

For \( x \in \bigcup_{i=0}^n f^{-i}A_n \), we know that \( \lambda^{-k}/A_k(x) < (\lambda(1 + \eta))^{-k} \) and that \( -B_n/A_n \) is the partial sum of \( \sum s_i \), hence uniformly bounded.

The following lemma is a direct consequence of the Lipschitz continuity of \( k' \) intervening in the construction of \( f \), and of lemma 3.2.

**Lemma 3.5.** The functions \( a \) and \( \frac{1}{a} \) are Lipschitz continuous on \( S_k \).

**Theorem 3.6.** If \( \beta_\sigma \in \sigma = -\lambda - \lambda^{-2}, -\lambda + 1 \) for all \( \sigma \) in \( \Sigma \), then the vector field \( v' \) is Lipschitz continuous on \( S_k \setminus \Sigma \).

**Proof.** Since all of the partial sums of \( \sum s_i \) are Lipschitz continuous, we give summable estimates of local Lipschitz constants. Let \( x \in U_\sigma \). Let \( V, N = N(x) \) and \( n \) as in the proof of theorem 3.3. Therefore \( U_\sigma = \bigcup_{i=0}^n f^{-i}V \). We use the notation \( \text{Lip}_g \) to indicate the local Lipschitz constant of a function \( g \) in at least one neighbourhood of \( x \).

Let \( \varepsilon > 0 \). On a small enough neighbourhood of \( x \), we have that \( \text{Lip}_g (f') \leq (1 + \varepsilon) ||d_{j,\tilde{f}}^m|| \) and \( \sup_{A_j} \leq (1 + \varepsilon) \frac{1}{A_j(x)} \) for all \( j \leq i \). We distinguish the following three cases:
(a) \( i \leq N - n_v \). We have directly that,

\[
\text{Lip}_x(s_i) \leq \lambda^{-i} \left( \text{Lip}(b) \text{Lip}(f^i) \sup_{A_N} \frac{1}{A_N} + \sup(b \circ f^i) \text{Lip}_{A_N}^{j} \sum_{j=0}^{i} \text{Lip}(f^i) \sup_{A_N} \frac{1}{A_N} \right),
\]

\[
\leq \lambda^{-i} B(1 + \varepsilon)^2 \left( \text{Lip}(b) + \sup |b| \text{Lip}_{A_N}^{j} \sup(a) \sum_{j=0}^{i} \left( \frac{1}{1 + \eta} \right)^j \right),
\]

\[
\leq C_{\perp, i, N, x} \lambda^{-i},
\]

where \( C_{\perp, i, N, x} \) stands for a constant independent of \( i, N \) and \( x \).

(b) \( N - n_v \leq i < N \). Up to multiplying some part of the above estimate by \( (\frac{1}{\lambda + \beta \sigma})^{n_v} \), we have:

\[
\text{Lip}_x(s_i) \leq C_{\perp, i, N, x} \lambda^{-i}.
\]

(c) \( i = l + N \geq N \). In this case, the following estimates hold:

\[
\text{Lip}_x(b \circ f^{l+N}) \frac{1}{A_{l+N}} \leq \text{Lip}(b) \text{Lip}(f^N) \sup_{A_N} \frac{1}{A_N} \text{Lip}_N(s_l) \sup_{A_N} \frac{A_N}{A_{l+N}},
\]

\[
\leq \text{Lip}(b)(1 + \varepsilon)^2 \left[ \frac{\|d_x f^N\|}{A_N(x)} \max(\lambda^{-l}, \lambda + \beta \sigma + \delta \sigma) l \left( \frac{1}{\lambda + \beta \sigma} \right)^l \right],
\]

\[
\leq C_{\perp, i, l, N} \max \left( \frac{\lambda^{-l}}{\lambda + \beta \sigma}, 1 + \frac{\delta \sigma}{\lambda + \beta \sigma} \right)^l.
\]

\[
\sup(b \circ f^{l+N}) \text{Lip}_x \frac{1}{A_{l+N}} \leq \varepsilon \max(\lambda^{-l}, \lambda + \beta \sigma + \delta \sigma)^l \text{Lip}_x^{1} \sup_{A_{l-1}} \frac{1}{A_{l-1}} \sup_{A_N} \frac{A_N}{A_{l+N}} \sum_{j=0}^{l-1} \text{Lip}(f^j) \sup_{A_N} \frac{A_N}{A_{l+N}}
\]

\[
\leq \varepsilon \max(\lambda^{-l}, \lambda + \beta \sigma + \delta \sigma)^l \text{Lip}_x^{1} \sup_{A_N} \frac{A_N}{A_{l+N}} \sum_{j=0}^{l-1} \left( \frac{1}{\lambda + \beta \sigma} \right)^{n_v} \max(\lambda^{-l}, \lambda + \beta \sigma + \delta \sigma)^l \text{C}_{\perp, i, l, N} \left( \frac{1}{\eta} + n_v \right)^{n_v}
\]

These two bounds are independent of \( N \), hence of \( x \).

By setting \( \beta \sigma \in [\lambda - \lambda^{-2}, -\lambda - 1] \), all the bounds on \( \text{Lip}_x(s_i) \) decay geometrically. Hence all partial sums of \( \sum s_i \) share a common Lipschitz constant near each point of \( U_{\Sigma} \), independent of the base-point.
We give now some estimates when \( x \in K \). Therefore \( f^n(x) \in A_n \) for all \( n \). The following estimate holds:

\[
\text{Lip}_x(s_i) \leq \lambda^{-i} \left( \text{Lip}(b) \text{ Lip}(f^i) \sup_{j=0}^i \frac{1}{A_j} \sup |b| \text{ Lip} \left( \frac{1}{a} \right) \sum_{j=0}^i \text{ Lip}(f^j) \sup_{j=0}^i \frac{1}{A_j} \right),
\]

\[
\leq \lambda^{-i} (1 + \varepsilon)^2 B \left( \text{Lip}(b) + \sup |b| \sup(a) \text{ Lip} \left( \frac{1}{a} \right) \sum_{j=0}^i \left( \frac{1}{1 + \eta} \right)^j \right),
\]

\[
\leq C_{\lambda, \eta} \lambda^{-i}.
\]

Finally, every partial sum of \( \sum s_i \) shares a common Lipschitz constant on \( S_g \setminus \Sigma \). Therefore \( v^\prime \) is Lipschitz continuous on \( S_g \setminus \Sigma \).

3.3. Differentiability of \( v^\prime \)

Here we prove that when the function \( k \) is \( C^2 \), the stable vector field \( v^\prime \) is \( C^1 \). In order to prove this result, we use similar computations as in the proof of theorem 3.6 and show that \( v^\prime \) is differentiable on every compact set of \( U_\Sigma \) and on \( K \). We then use the relation \( d_f v^\prime = \lambda^{-1} v^\prime \circ f \) (more precisely, the differential of this relation) in order to prove that there is a unique extension of \( d v^\prime \) from \( U_\Sigma \) to \( S_g \setminus \Sigma \), and it coincides with \( d v^\prime \) on \( K \).

**Theorem 3.7.** If the function \( k : \mathbb{R} \to \mathbb{R} \) in the construction of \( f \) is also \( C^2 \), then the vector field \( v^\prime \) and the flow \( h_t \) are \( C^1 \).

**Proof.** From the same estimates as in the proof of theorem 3.6, we get that the series of differentials \( \sum_{i \geq 0} d_{x_i} \) converges uniformly on \( K \) and on compact subsets of \( U_\Sigma \setminus \Sigma \). By uniform converge, \( v^\prime \) is therefore differentiable on \( K \) and on \( U_\Sigma \setminus \Sigma \), but we still need to prove that \( x \mapsto d_{x} v^\prime \) is continuous on \( S_g \setminus \Sigma \). To this end, we use the fact that \( v^\prime \) is uniformly contracted by \( f \).

By design, \( v^\prime \) satisfies the equality \( d_f v^\prime(x) = \lambda^{-1} v^\prime(f(x)) \) for all \( x \notin \Sigma \). Now, by differentiation, we get for all \( x \) in \( U_\Sigma \):

\[
d_f^2(v^\prime(x), \cdot) + d_f d_v v^\prime = \lambda^{-1} d_{f(x)} v^\prime d_x f.
\]

(3.1)

Let \( x \in K \) and \((x_n)_n\) be a sequence in \( U_\Sigma \) converging to \( x \) as \( n \) goes to infinity. By the Arzelà–Ascoli theorem, in order to prove that \((d_{x_n} v^\prime)_n\) converges to \( d_x v^\prime \), it is sufficient to prove that \((d_{x_n} v^\prime)_n\) has a unique subsequential limit.

To be exact, in order to apply the Arzelà–Ascoli theorem, we need the maps to have a compact domain. We address this problem by associating to any linear map \( l : \mathbb{R}^d \to \mathbb{R}^d \) its restriction to the unit sphere \( l : \mathbb{S}^{d-1} \to \mathbb{R}^d \), in addition with the closed condition

\[
||x + \theta y|| \cdot l \left( \frac{x + \theta y}{||x + \theta y||} \right) = l(x) + \theta l(y), \quad x, y \in \mathbb{S}^{d-1}, \theta \in \mathbb{R}.
\]

(3.2)

Now, any linear map can be built from a map on the sphere satisfying the condition (3.2). This one-to-one correspondence is enough to overcome the issue of non-compactness of the domain.
Let \( u_k \) be a subsequential limit of \((d_{x_k}v^\sigma)_\alpha\). Using (3.1) and the fact that \( f \) is \( C^2 \), we also get a subsequential limit \( u_{\hat{\rho}(i)} \) of \((d_{(x_i,\sigma)}v^\sigma)_\alpha\). By the same process, we get for all integer \( k \) a subsequential limit \( u_{\check{\rho}(i)} \) of \((d_{p(s_i)}v^\sigma)_\alpha\) so that

\[
d^2_{p(s_i)}(v^\sigma(x),\cdot) + d_{p(s_i)}fu_{\check{\rho}(i)} = \lambda^{-1}u_{\check{\rho}(i)}d_{p(s_i)}f.
\]

Taking the difference with (3.1) we get, after induction, that for all integer \( k \)

\[
d^k(d_{x\sigma}v - u_\sigma) = \lambda^{-k}(d_{p(s_i)} - u_{\check{\rho}(i)})d_x^k.
\]  

(3.3)

We now prove that the difference \( \alpha_0 := d_{x\sigma}v - u_\sigma \) is the zero map. First, notice that since \( v^\sigma(x) = e_\sigma - (\Sigma_\sigma x_\sigma(x))e_\sigma \), we must have \( \text{Im}(d_{x\sigma}v) \subset \mathbb{R}e_\sigma = E^\sigma(x) \), and, by taking limits, \( \text{Im}(u_\sigma) \subset E^\sigma(x) \). Therefore \( \text{Im}(\alpha_0) \subset E^\sigma(x) \). Since \( v^\sigma \) is Lipschitz continuous, the operators \( d_{p(s_i)}v^\sigma - u_{\check{\rho}(i)} \) are uniformly bounded. Therefore, from the hyperbolicity of \( K \) and the relation (3.3), we get that \( \alpha_0(\mathbb{R}v^\sigma(x)) \subset \mathbb{R}v^\sigma(x) \), and so \( \alpha_0(\alpha_0(x)) = 0 \). Since \( (v^\sigma(x), e_\sigma) \) is a basis of \( \mathbb{R}^2 \), there exists some real number \( \alpha \) such that \( \alpha_0(e_\sigma) = \alpha e_\sigma \). Applying (3.3) to \( e_\sigma \), we get that

\[
0 = (\alpha \text{Id} - \lambda^{-k}(d_{p(s_i)} - u_{\check{\rho}(i)})d_x^ke_\sigma.
\]

If \( \alpha \) is not zero, then for large enough value of \( k \) the map \((\text{Id} - \lambda^{-k}(d_{p(s_i)} - u_{\check{\rho}(i)}))d_x^ke_\sigma \) is invertible, hence a contradiction. Therefore \( \alpha = 0 \) and \( u_\sigma = d_{x\sigma}v \). Finally, we get that \( x \in U_{\Sigma} \setminus \Sigma \mapsto d_xv^\sigma \) extends continuously, in a unique fashion, to \( S_{\beta} \setminus \Sigma \).

\[
\blacksquare
\]

Remark 3.8. In the case when the surface is the torus \( \mathbb{T}^2 \), \( v^\sigma \) cannot be \( C^2 \); if so the induced flow would also be \( C^2 \), as well as its Poincaré map to a transverse circle. However this map is a Denjoy counterexample since it has a wandering interval, and is therefore at most \( C^1 \) with bounded-variation derivative. It is not clear whether this bound on the regularity of \( v^\sigma \) still holds for higher genus surfaces.

3.4. Continuity of \( v^\beta \) with respect to \( \beta \)

In the next section we prove that \( h_\beta \) is uniquely ergodic and that \( f \) is mixing with respect to the invariant measure of \( h_\beta \). To do so, we first prove that the family of vector fields \( v^\beta \) is smooth with respect to the amplitude parameter \( \beta \) in the definition of \( f \).

We will use the following notations. For all \( \beta = (\beta_\sigma)_{\sigma \in \Sigma} \), write \( f_\beta \) the function \( f \) with the amplitude parameter \( \beta \), and \( v^\beta_\sigma \) its corresponding vector field. We also assume the parameter \((\alpha_\sigma)_{\sigma \in \Sigma}\) to be fixed.

In this section, in order to simplify the notations, we only consider the case \#\Sigma = 1, hence the vector \( \beta \) has only one component. The general case leads to very similar computations where one must estimates the norm and Lipschitz constant on each \( U_\sigma, \sigma \in \Sigma_\beta = \{ \sigma \in \Sigma \mid \beta_\sigma < 1 - \lambda \} \), and on \( K_\beta = S_{\beta} \setminus U_{\Sigma_\beta} \), where \( U_{\Sigma_\beta} = \bigcup_{\sigma \in \Sigma_\beta} U_\sigma \). The analogous of the partition used in the proof of theorem 3.9 then has \( 2^{\#\Sigma} \) elements, but the argument used in order to deal with the boundaries of the elements of the partition remain the same.

More precisely, we prove the following theorem.

Theorem 3.9. The map \( \beta \in [-\lambda + \lambda^2, 0] \mapsto v^\beta_\sigma \) is continuous for the sup-norm. As a consequence, the function \( (x, \beta) \mapsto v^\beta_\sigma(x) \) is continuous on \((S_\beta \setminus \Sigma) \times [-\lambda + \lambda^{-2}, 0]\).

To show this continuity, we split the domain into three subsets. We will need the following lemma.
Lemma 3.10. For all $\beta$ in $]-\lambda + \lambda^{-2}, 0[$, the eigenspace of $(f_\beta)_* := (df_\beta)^{-1} U_{f_\beta}$ associated with the eigenvalue $\lambda$ is of dimension one when acting on the space of bounded and continuous vector fields on the tangent vector bundle of $S_\kappa \setminus \Sigma$, where $U_f$ stands for the Koopman operator of $f$.

Proof. Let $\beta \in ]-\lambda + \lambda^{-2}, 0]$. Let $w$ be a vector field in the eigenspace of $(f_\beta)_*$ associated with the eigenvalue $\lambda$. In other words, $w$ is such that $d_x f_\beta (w(x)) = \lambda^{-1} w(f_\beta(x))$, for all $x$. Now, since $v^i$ is continuous, non vanishing and transverse to $e_h$, there exist two functions $w_1$ and $w_2$ uniquely determined such that $w(x) = w_1(x) v^i(x) + w_2(x) e_h$ for all $x$. These two functions are bounded and continuous. Hence, we have,

$$d_x f_\beta (w_2(x)e_h) = a(x) w_2(x)e_h = d_x f_\beta (w(x) - w_1(x)v^i(x)),$$

$$= \lambda^{-1} (w(f_\beta(x)) - w_1(x)v^i(f_\beta(x)),$$

$$w(f_\beta(x)) = w_1(x)v^i(f_\beta(x)) + \lambda a(x) w_2(x)e_h.$$

Therefore, $w_1$ is invariant by $f_\beta$ and for all $i > 0$,

$$w_2(x) = \prod_{j=0}^{i-1} \frac{1}{\lambda a(f_\beta^j(x))} w_2(f_\beta^j(x)).$$

By continuity of $w_2$ and compactness of $S_\kappa$, $w_2$ is bounded. Now, we distinguish two cases in order to prove that $w_2 = 0$.

For $\beta_\sigma < 1 - \lambda$, there exists a fixed point $p^w_\sigma$, in $K$, whose unstable leaf is dense. Since at this point $a(p^w_\sigma) > 1$, by continuity of $a$, we get that $a > 1$ in a neighbourhood of $p^w_\sigma$, hence $1/(\lambda a) < \lambda^{-1} < 1$ and $w_2 = 0$.

For $1 - \lambda \leq \beta_\sigma \leq 0$, we know that the unstable leaf of $\sigma$ is dense in $S_\kappa$. By continuity on every leaf of the branched cover at $\sigma$, we can set $a(\sigma) = \lambda + \beta_\sigma \geq 1$. Hence, in a neighbourhood of $\sigma$, we get $1/(\lambda a) \leq \lambda^{-1} < 1$, hence $w_2 = 0$.

In order to prove that $w_1$ is constant, we also distinguish two cases.

For $\beta_\sigma < 1 - \lambda$, the unstable leaf of each $p^w_\beta$ is dense. Hence $w_1(x) = w_1(p^w_\beta)$ for all $x$. Hence the claim in this case.

For $1 - \lambda \leq \beta_\sigma \leq 0$, the unstable leaf of $\sigma$ is dense. Therefore, $w_1(x) = w_1(\sigma)$ for all $x$. Hence the claim.

Proof of theorem 3.9. We first prove that $||v^i_\beta - v^i_{\beta_0}||_\infty \xrightarrow{\beta \to \beta_0} 0$ for all $\beta_0$ in $]-\lambda + \lambda^{-2}, 1 - \lambda[$. From proofs of theorems 3.3 and 3.6, we can see that on a small enough neighbourhood $B_0$ of $\beta_0$, the vector fields $v^i_\beta$ are uniformly bounded, as well as their Lipschitz constants. By the Arzelà–Ascoli theorem, the set $\{v^i_\beta | \beta \in B_0\}$ is relatively compact. Take a sequence of $(\beta_n)_n$ converging to $\beta_0$, then every sub-sequential limit $w$ of $(v^i_{\beta_n})_n$ must satisfies $(f_{\beta_0})_w = w_\lambda$. By lemma 3.10, the space of such vector fields is one dimensional, hence there exists a constant $c$ such that $w = cv^i_{\beta_0}$. Since in the basis $(e_h, e_i)$ all the component of $v^i_\beta$ along $e_i$ is 1, we get that $c = 1$. Hence $v^i_{\beta_0}$ converges uniformly to $v^i_{\beta_0}$, and so for all sequences $(\beta_n)_n$. The rest of the claim follows directly by the triangle inequality and Lipschitz continuity.

We now prove that $||v^i_\beta - v^i_{\beta_0}||_\infty \xrightarrow{\beta \to \beta_0} 0$ for all $\beta_0 \in [1 - \lambda, 0]$. The same argument as in the case above holds. Indeed, for all $\beta \in [1 - \lambda, 0]$ we get

$$\sum_{i \geq 0} \left| \lambda^{-i} b_\beta \circ f_\beta^i \prod_{j=0}^i \frac{1}{a_\beta \circ f_\beta^j} \right| \leq \frac{1}{1 - \lambda} ||b_\beta||_\infty.$$
Hence $v^\prime_\beta$ is uniformly bounded for $\beta$ in a neighbourhood of $\beta_0$. Similarly, the following estimate on the Lipschitz constant holds for all $\varepsilon > 0$

$$\sum_{i \geq 0} \text{Lip}_x \left( \lambda^{-i} b_\beta \circ f^i_\beta \prod_{j=0}^{i} \frac{1}{a_\beta \circ f^j_\beta} \right) \leq (1 + \varepsilon)^2 \|v^\prime_\beta\|_{\infty} \sum_{i \geq 0} \lambda^{-i} \left( \text{Lip}(b_\beta) \right. \left. + i \text{Lip} \left( \frac{1}{a_\beta} \right) \|a_\beta\|_{\infty} \|b_\beta\|_{\infty} \right).$$

Finally, we prove that $\|v^\prime_\beta - v^\prime_{\lambda-\lambda}\|_{\infty} \rightarrow 0$ as $\beta \rightarrow 1 - \lambda$. Recall notations from proposition 3.1 and let $V$ be a neighbourhood of some $\sigma \in \Sigma$ as in the proof of theorem 3.3. Let $x \in f^{-N}(V) \cap U_\sigma$ and let $n(x)$ be the number of points in the orbit of $x$ that belong to $B_{\sigma, \delta} \setminus V$. Then $N - n(x) \geq 0$ and we have the following estimates depending on $i$:

(a) if $i \leq N - n(x)$, then $|s_i(x)| \leq \lambda^{-i} \left( \frac{1}{1 + \eta} \right)^{i+1} \sup |b|$.  

(b) if $N - n(x) < i \leq N$, then $|s_i(x)| \leq \lambda^{-i} \left( \frac{1}{1 + \eta} \right)^{n-\sigma(x) - 1} \sup |b|$ so that $|s_i(x)| \leq \sup |b| \lambda^{-i} \left( \frac{1}{1 + \eta} \right)^{i}$.  

(c) if $i = j + N > N$, then $|s_i(x)| \leq \lambda^{-(j+N)} \left( \frac{1}{(1 + \eta)} \right)^{j+N} \text{Lip}(b) \varepsilon \max(\lambda^{-1}, \lambda + \beta_\sigma + \delta_\sigma)^j$.

Therefore, $\sum_{i \geq 0} |s_i(x)| \leq \|b\|_{\infty} \left( \frac{\lambda \lambda^{-1} + \lambda(\lambda + \beta_\sigma) \lambda^{-1} + \varepsilon \text{Lip}(b)}{\frac{\lambda}{1 + \eta} \lambda^{-1} + \delta_\sigma} \right)$, and so for all $\varepsilon > 0$.

Hence, the family of vector fields $(v^\prime_\beta)_\beta$ is uniformly bounded on $S_\kappa$ and the bound can be chosen uniformly in $\beta$ for $\beta \in [1 - \lambda - \varepsilon, 1 - \lambda]$. However, the estimates we had on the Lipschitz constants are no longer good enough to apply the same argument as in above cases.

Let $x \in S_\kappa$ and $(x_n, \beta_n)$ be a sequence converging to $(x, 1 - \lambda)$ and such that $\beta_n < 1 - \lambda$ for all $n$. For $n$ large enough, the sequence $(v^\prime_{\beta_n}(x_n))_n$ is bounded and let $w(x)$ be a sub-sequential limit. Since for all $k \geq 0$, the sequence $(v^\prime_{\beta_k}(f^k_{\beta_k}(x_n)))_n$ is bounded, by a diagonal argument we can assume up to extracting that the sequences converge to some vectors $w(f^k_{\lambda-\lambda}(x))$. By continuity of $d_\beta f^k_\beta$ in $\beta$, we get that $d_\beta f^k_{\beta_n} w(x) = \lambda^{-k} w(f^k_{\lambda-\lambda}(x))$ for all $k$. By expressing vectors $w(f^k_{\lambda-\lambda}(x))$ in the basis $(v^\prime_{\lambda-\lambda}(x), e_\alpha)$, we see that $w(x) \in \mathbb{R}v^\prime_{\lambda-\lambda}(x)$. Since each vector of the form $v^\prime_{\beta_n}(f^k_{\beta_n}(x_n))$ has a component equal to 1 along $e_\alpha$ in the basis $(e_\alpha, e_\beta)$, we get $w(x) = v^\prime_{\lambda-\lambda}(x)$. Hence $(x, \beta) \rightarrow v^\prime_\beta(x)$ is continuous at $(x, (1 - \lambda)^-)$.  

Now, suppose that $\|v^\prime_\beta - v^\prime_{\lambda-\lambda}\|_{\infty}$ does not converge to zero as $\beta$ converges to $1 - \lambda$ from below. Then, there exists some positive $\varepsilon$ and sequences $(\beta_n)$ and $(x_n)$ such that $\lim_{n \to \infty} \beta_n = (1 - \lambda)^- \text{ and } \|v^\prime_{\beta_n}(x_n) - v^\prime_{\lambda-\lambda}(x_n)\| \geq \varepsilon$. Up to extracting, we can assume that $(x_n)_n$ converges to some $x$. Therefore $\|v^\prime_\beta(x_n) - v^\prime_{\lambda-\lambda}(x_n)\| \geq \varepsilon/2$ for large enough $n$. This contradicts the continuity of $(x, \beta) \rightarrow v^\prime_\beta(x)$ at $(x, (1 - \lambda)^-)$.  

The continuity of $(x, \beta) \rightarrow v^\prime_\beta(x)$ on $(S_\kappa \setminus \Sigma) \times [-\lambda + \lambda^2, 0]$ follows from

$$\|v^\prime_\beta(x) - v^\prime_{\beta_n}(x_0)\| \leq \|v^\prime_\beta - v^\prime_{\beta_n}\|_{\infty} + \|v^\prime_{\beta_n}(x) - v^\prime_{\beta_n}(x_0)\| \xrightarrow{(x, \beta) \rightarrow (x_0, \beta_0)} 0.$$  

$\square$
3.5. Renormalized flow and topological properties of K

Since $v'$ is Lipschitz continuous, we can integrate it into a flow $h_t$. Since some trajectories reaches in finite time conical points, for which $v'$ is not defined, this flow must be treated carefully. On the other hand, since $v'$ is uniformly contracted by the action of $f$, $h_t$ is renormalized by $f$. From this relationship between $f$ and $h_t$, we can deduce further topological properties about stable leaves and the set $K$. We first prove that for each fixed hyperbolic point $p_i$, its stable leaf coincides with the orbit by $h_t$ of this point. From this fact and proposition 2.8, we deduce that $K$ is transverse to any horizontal leaf. We then show that $K$ is in fact equal to the closure of the stable leaf of any hyperbolic fixed point $p_i$, hence $K$ is connected. Finally, we prove that $f$ is topologically transitive with respect to the trace topology of $S_k$ on $K$.

**Proposition 3.11.** For all $x \in S_k \setminus \Sigma$ and $t$ for which $h_t(f(x))$ is well defined, $f$ and $h_t$ satisfy the relation,

$$f \circ h_t(x) = h_t \circ f(x).$$

The orbit $\{h_t(x)\}$ of any point $x$ in $K$ is well defined for all time $t$. Furthermore, for all $t \in \mathbb{R}$, $h_t(K) = K$.

**Proof.** Since $d(v'(x)) = \lambda^{-1}v'(f(x))$, for $x \in S_k \setminus \Sigma$, notice that,

$$\frac{d}{dt}(f \circ h_t(x)) = \frac{d}{dt}h_t(x) + \frac{d}{dt}f(h_t(x)) = \lambda v'(f(h_t(x))) = v'(f \circ h_t(x)).$$

In particular, $t \mapsto f(h_t(x))$ solves $\frac{d}{dt}g = v'(g)$ with initial condition $g(0) = f(x)$ at $t = 0$. By construction of the flow $h_t$, $t \mapsto h_t(f(x))$ solves the same differential equation, with the same initial condition. Therefore, by uniqueness of the solution (since $v'$ is Lipschitz) the two functions $t \mapsto f(h_t(x))$ and $t \mapsto h_t(f(x))$ must coincide where they are well defined. In other words, $f \circ h_t = h_t \circ f$ for all $t$ where the solution is defined.

Let $\mathcal{F} := S_k \setminus (\Sigma \cup \{x \in S_k \setminus \Sigma \mid \forall t \in \mathbb{R}, h_t(x) \text{ exists}\})$ be the set of points whose trajectory are not well defined for all time. We now prove that if $x \in \mathcal{F}$, then there exist $\sigma \in \Sigma$ and $t_0 \in \mathbb{R}$ such that $h_t(x) \to \sigma$ as $t$ tends to $t_0$. Indeed, by compactness of $S_k$, up to taking a sub-sequence $(t_n)_n$ that converges to $t_0$, the limit of $(h_{t_n}(x))_n$ exists. If this limit does not belong to $\Sigma$, we can extend the solution past $t_0$.

To prove that $h_t$ is complete when restricted to $K$, it suffices to prove that $K \subset \mathcal{F}^c$, or equivalently, that $\mathcal{F} \subset U_{\Sigma}$. By contradiction, let $x \in \mathcal{F} \cap K$. Let $t_0$ and $\sigma$ be as above. Hence, the smooth curves $f^\sigma \circ h(t) : t \in [0, t_0] \to S_k$ join $K$ to $\Sigma$ and their lengths are less than $\lambda^{-n}t_0||v'||_{\infty}$. This contradicts the fact that $d(K, \Sigma) > \min\{|\sigma| \mid \sigma \in \Sigma\} > 0$ by proposition 2.4.

Since $\mathcal{F} \cap K = \emptyset$, $h_t(x)$ is well defined for all $x \in K$ and all time $t$. Let $x \in K$. By contradiction, assume there exists $t_1$ such that $h_{t_1}(x) \in U$. Therefore $f^n(h_{t_1}(x))$ converges to some $\sigma$ as $n$ goes to infinity and the curves $f^n \circ h_t(x) : t \in [0, t_1] \to S_k$ joins $K$ to some arbitrarily close point to $\sigma$ for $n$ large enough. Since such a curve is of length at most $\lambda^{-n}t_1||v'||_{\infty}$, it contradicts $d(K, \Sigma) > 0$.

This commutation relation between $f$ and $h_t$ is a central argument throughout the rest of this article.

We can now deduce the announced topological properties of the invariant leaves and of $K$.

**Proposition 3.12.** For all $p_i$, we have the equality of sets $W^s(p_i) = h_{\mathbb{R}}(p_i)$. Also, the set $K$ is transverse to any horizontal leaf.
Proof. Let \( t \in \mathbb{R} \). Hence \( f^n(h_\lambda(p_\sigma^n)) = h_{\lambda-\gamma}(p_\sigma^n) \) converges to \( p_\sigma^n \) as \( n \) goes to infinity. Hence \( h_{\lambda}(p_\sigma^n) \subset W^u(p_\sigma^n) \). By the commutation relation between \( f \) and \( h_\lambda \), we get that \( h_{\lambda}(p_\sigma^n) \) is invariant by \( f \). In the linearization near \( p_\sigma^n \) given by the Grobman–Hartman theorem, the only invariant part by \( f \) corresponds to a small piece \( \gamma \) of the stable leaf of \( p_\sigma^n \). By invariance of \( h_{\lambda}(p_\sigma^n) \) by \( f \), we get \( \gamma \subset h_{\lambda}(p_\sigma^n) \). Finally, since \( W^s(p_\sigma^n) = \bigcup_{i \geq 0} f^{-i}(\gamma) \), we get \( h_{\lambda}(p_\sigma^n) = W^s(p_\sigma^n) \).

Since the convergence of the infinite sum defining \( \nu^f \) is uniform on \( K \), the horizontal component of the vector field \( \nu^f \) is continuous, hence bounded. Therefore, all the stable leaves \( W^s(p_\sigma^n) \) are transverse to any horizontal leaf. The result holds by taking the closure since slopes are bounded and by proposition 2.8.

**Theorem 3.13.** The set \( K \) is connected and it can be written as \( K = \overline{W^s(p_\sigma^n)} \), for any \( \sigma \in \Sigma \) and any \( 1 \leq i \leq 2 \).

**Proof.** Let \( \sigma_1, \sigma_2 \in \Sigma \) and \( i_1, i_2 \) be two integers. For simplicity, call \( p_1 = p_{i_1}^{\sigma_1} \) and \( p_2 = p_{i_2}^{\sigma_2} \). Let \( W_2 \) be the open set containing \( p_2 \) given by the Grobman–Hartman theorem—without loss of generality we assume it is a rectangle with horizontal and vertical sides. Since \( W^s(p_2) \) contains a dense horizontal leaf, and \( W^u(p_1) \) is transverse with all vertical leaves, the intersection \( W^u(p_2) \cap W^s(p_1) \) is non-empty. Let \( x \in W^u(p_2) \cap W^s(p_1) \) and let \( \gamma \) be a small connected piece of \( W^s(p_1) \) containing \( x \) in its interior. Then, for large enough \( n \geq 0 \), we see that \( f^{-n}(\gamma) \cap W_2 \) accumulates on \( W^u(p_2) \cap W_2 \). Therefore, \( W^u(p_2) \cap W_2 \subset \overline{W^u(p_2) \cap W^s(p_1)} \subset W^s(p_1) \). Since \( W^s(p_1) \) is invariant by the action of \( f \) and \( W^s(p_2) = \bigcup_{n \geq 0} f^{-n}(W^u(p_2) \cap W_2) \), we get the inclusion \( \overline{W^s(p_2)} \subset \overline{W^s(p_1)} \). Since the choice of \( p_1 \) and \( p_2 \) is arbitrary, the result follows from proposition 2.8.

**Theorem 3.14.** The function \( f : K \to K \) is transitive with respect to the trace topology of \( S_\Sigma \) on \( K \).

**Proof.** Let \( U_1 \) and \( U_2 \) be open sets in \( S_\Sigma \) that have non-empty intersection with \( K \). Let \( p_1 = p_{i_1}^{\sigma_1} \) and \( p_2 = p_{i_2}^{\sigma_2} \) for some \( \sigma_1, \sigma_2 \in \Sigma \) such that \( U_1 \cap (W^u(p_1) \cap W^s(p_1)) \neq \emptyset \) for \( i = 1, 2 \). Since \( W^u(p_2) \) is transverse with all the horizontal leaves, we can find a rectangle \( V_2 \) contained in \( U_2 \) whose sides are vertical and horizontal, such that \( W^u(p_2) \) crosses \( V_2 \) from side to side.

By density of \( W^u(p_1) \), there exists \( x_2 \in V_2 \cap W^u(p_1) \). Let \( W_1 \) be the open set of linearization near \( p_1 \)—without loss of generality, we can assume \( W_1 \) to be a rectangle with horizontal and vertical sides. For large enough \( n \geq 0 \), the set \( f^{-n}(V_2) \) crosses \( V_2 \) vertically.

Let \( x_1 \in U_1 \cap W^u(p_1) \) and \( \epsilon > 0 \) be such that the horizontal segment \( \gamma \) of length \( \epsilon \), containing \( x_1 \) in its interior, is contained in \( U_1 \). For all large enough \( m \geq 0 \), the line \( f^m(\gamma) \) crosses \( W_1 \) horizontally. Hence \( f^m(U_1) \cap f^{-n}(U_2) \neq \emptyset \). It easily follows from the transitivity of \( f \) and the closing lemma that periodic points of \( f \) are dense in \( K \). Therefore \( K \) is an Axiom A attractor in the sense of [24].

**Theorem 3.15.** If \( \Sigma \) is an open \( \epsilon \)-neighbourhood of \( \Sigma \) for some small enough \( \epsilon > 0 \), \( U := S_{\hat{\Sigma}} \setminus \Sigma \) and \( f^{-1} \) is \( C^2 \) away from \( \Sigma \), then \( K \) is an Axiom A attractor for \( f^{-1} : U \to U \).

4. The induced GIET

In this part we construct a GIET \( T \) as the Poincaré map of \( h_\lambda \) to some transversal segment, and we prove that it satisfies the conclusion of theorem 1.1. For the semi-conjugacy, it is sufficient to prove—thanks to a result by Yoccoz [28]—that \( T \) follows the same orbit as a self-similar IET when renormalized by the Rauzy–Veech algorithm. To do so, we construct multiple partitions into rectangles of \( S_{\hat{\Sigma}} \). Minimality and unique ergodicity of \( T \) then follow from the one of the
semi-conjugated self-similar IET. Since $h_t$ is the suspension flow over $T$, $h_t$ is also uniquely ergodic, of unique invariant measure $\mu$, supported by $K$. Because of the commutation relation (1.2) between $f$ and $h_t$, the measure $\mu$ is also invariant by $f$. We prove that $f$ is mixing with respect to $\mu$.

**Theorem 4.1.** The flow $h_t$ is uniquely ergodic. Furthermore the support of the invariant measure is $K$.

**Corollary 4.2.** The unique invariant measure $\mu$ of $h_t$ is also invariant by $f$, and $f$ is mixing with respect to $\mu$.

This theorem and its corollary are a restatement of theorem 1.2(iii).

**Proof of corollary 4.2.** Since $K$ is invariant by $f$ and by the flow $h_t$ and since $h_t$ is well defined for all $t$ on $K$, we have

$$f_*\mu = f_*((h_t)_*\mu) = (f\circ h_t)_*\mu = (h_{\lambda^{-t}})_*(f_*\mu).$$

Therefore the measure $f_*\mu$ is invariant by the flow $h_t$. By unique ergodicity of the flow, we must have $f_*\mu = \mu$.

Let $F \in L^2(\mu)$ be such that $\int F d\mu = 0$. We now prove that the sequence $(f \circ f^n)_n$ weakly converges to zero. By invariance of the measure, the sequence is bounded in the $L^2(\mu)$ norm. By the Banach–Alaoglu–Bourbaki theorem, this sequence lives in a weakly compact set. Let $F$ be a sub-sequential weak limit of $(f \circ f^n)_n$ and let $(n_k)_k$ be a strictly increasing sequence of integers such that $F \circ f^{n_k} \to F$ as $k \to \infty$. On the other hand,

$$||F \circ f^{n_k} \circ h_t - F \circ f^n||_{L^2} = ||F \circ h_{\lambda^{-n_k}} \circ f^{n_k} - F \circ f^n||_{L^2},$$

where the final limit follows from the density of continuous functions in $L^2(\mu)$. Now, $F \circ f^{n_k} \circ h_t - F \circ f^n$ converges weakly to $F \circ h_t - F$. The identification of the strong limit with the weak limit gives $F \circ h_t - F = 0$. By unique ergodicity of $(h_t)_t$, $F$ is constant. By integration, this constant is zero. Hence all the sub-sequential weak limit of $(f \circ f^n)_n$ are 0, which proves the mixing. \qed

In order to prove theorem 4.1, we heavily rely on the semi-conjugacy result from [28, proposition 7], more precisely if an IET and a GIET have the same combinatorial datum and follow a same full path in the Rauzy diagram—when renormalized by the Rauzy–Veech algorithm—then there exists a continuous, increasing and surjective function that semi-conjugates the two transformations.

4.1. Construction of a GIET and $h_t$ as its suspension flow

Recall some notation from section 2.1. Let $\varphi$ be the pseudo-Anosov map that we perturbed in order to get $f$. By construction, $\varphi$ fixes each conical point and each separatrix. Let $\sigma \in \Sigma$ be a conical point and $\gamma_0$ be a segment of a horizontal separatrix starting at $\sigma$. From a general property of the pseudo-Anosov maps, there exists a decomposition in rectangles $R_{\gamma_0} = (R_1, \ldots, R_2)$ of $S_\gamma$ such that (up to shortening $\gamma_0$) the bases of these rectangles form a partition of $\gamma_0$ (this is a particular case of the decomposition into rectangles and cylinders from [18, proposition 5.3.4]).
Denote by $\partial_1 \mathcal{R}_0$ (resp. $\partial_0 \mathcal{R}_0$) the vertical (resp. horizontal) components of $\bigcup_i \partial \mathcal{R}_i$. By construction, $\partial_0 \mathcal{R}_0 = \gamma_0$. Now, $\partial_1 \mathcal{R}_0$ is made of portions of trajectories for the vertical flow associated to $\varphi$ that connect a conical point to $\gamma_0$, but do not intersect $\gamma_0$ at some other previous time.

Since the family of vector fields $(x, \beta) \mapsto v_\beta^0(x)$ is continuous, we can deform by some homotopy $\mathcal{R}_0$ into $\mathcal{R}_\beta = (R_1^\beta, \ldots, R_{\ell}^\beta)$ while preserving the horizontal direction, where $\beta$ is the amplitude of the perturbations in the construction of $f$. In more details, the homotopy sends the portions of trajectories of the vertical flow that connect conical points to $\gamma_0$, to the portions of trajectories of $h$ which contain a conical point. Since the vector field $v_\beta^0$ has its vertical component constant equal to 1, these latter trajectories are the ones connecting conical points to $\gamma$, where $\gamma$ is a slightly longer or shorter copy of $\gamma_0$. Since any two trajectories do not intersect, these portions of trajectories of $h$ are still the shortest ones that connect conical points to $\gamma$.

Call $T$ (resp. $T_0$) the Poincaré first return map to $\gamma$ (resp $\gamma_0$) of $h$ (resp. of the unit speed vertical flow associated to $\varphi$). It is clear from the construction that $T_0$ is an IET and that $T$ is a GIET. Since for all $\beta$, the vertical component of $v_\beta^0$ is equal to one, both $T$ and $T_0$ have the same combinatorial data. Furthermore, by construction, $T$ and $T_0$ have the same path in the Rauzy-graph: fix an integer $n \geq 1$, then because of the continuity with respect to $\beta$, there exists a neighbourhood of $\beta_0$ such that all of the induced GIET have the same first $n$ steps in the Rauzy-Veech algorithm. Consider the maximal open set with this property, and let $\beta^*$ be a point in its boundary. We distinguish two cases: assume first that for $\beta^*$ the algorithm does not stop at the $n$th step. Then there is a neighbourhood of $\beta^*$ contradicting the maximality. In the remaining case, the algorithm stops at the $n$th step: the induced GIET must have a connection. This corresponds geometrically to a side of a rectangle of $\mathcal{R}_\beta$, connecting a conical point to another one: this is impossible since $f_{\beta^*}$ would contract this curve. Thus $T$ and $T_0$ have the same first $n$ steps in the Rauzy-Veech algorithm, for any $n$.

Since foliations associated to a pseudo-Anosov have no closed leaf (see [10]), it follows that $T_0$ has no connection, hence, by [28], the path of $T_0$ in the Rauzy graph is full and so $T_0$ and $T$ are semi-conjugated by a continuous, increasing and surjective function. Also, since $T_0$ has no connection, it is minimal.

We summarize all this in the following proposition.

**Proposition 4.3.** If $\sigma$ is a conical point, there exist two portions of a same horizontal separatrix (both containing $\sigma$) $\gamma_0$ and $\gamma$, and maps $T : \gamma \to \gamma$, $T_0 : \gamma_0 \to \gamma_0$ such that:

(a) $T_0$ is an IET and $T$ is a GIET,
(b) $T_0$ is the Poincaré first return map of the vertical flow associated to $\varphi$,
(c) $T$ is the Poincaré first return map of the flow $h$, associated to $f$,
(d) $T_0$ and $T$ have the same combinatorial data, and the same path in the Rauzy-graph,
(e) there exists a continuous, increasing and surjective function $h$ such that $h \circ T = T_0 \circ h$.

**4.2. Minimality of the flow on $K$**

In this part we prove that the map $T$—from which $h_t$ is the suspension flow—is minimal on its nonwandering set. To do so, we rely on the analysis carried out in [28]. From this, we deduce that the flow $h_t$ acts minimally on $K$—actually, we also prove that $K$ is an attractor for positive
and negative times. This property will be useful to prove that the support of the unique invariant measure of $h$, is $K$.

As in [28], define $S(\infty)$ as the union of the forward orbit of the discontinuity points of $T^{-1}$ and the backward orbits of the discontinuity points of $T$. Similarly, define $S_0(\infty)$ from the discontinuities of $T_0$ and $T_0^{-1}$. By construction, $h$ is an increasing bijection from $S(\infty)$ to $S_0(\infty)$.

Define $\Omega$ as the set of non-isolated points of $\overline{S(\infty)}$. Clearly, $\Omega$ is a closed set. We now prove that $T$ is minimal on $\Omega$.

**Theorem 4.4.** When restricted to the set $\Omega$, $T$ is minimal.

**Proof.** We first prove that there exists a decomposition of $\Omega$ in closed sets $\Omega = \Omega_+ \cup \Omega_-$ such that $T(\Omega_+) \subset \Omega_+$ and $T^{-1}(\Omega_-) \subset \Omega_-$.

Let $S(\infty)_+$ be the forward orbits by $T$ of the discontinuity points of $T^{-1}$ and similarly $S(\infty)_-$ be the set of the backward orbits by $T$ of the discontinuity points of $T$. By definition of $S(\infty)$, $S(\infty) = S(\infty)_- \cup S(\infty)_+$. Define $\Omega_\pm$ as the set of non-isolated points of $\overline{S(\infty)}_\pm$.

These sets satisfy the claim.

Let $x$ be a point of $\Omega$. Up to considering its backward orbit, we assume that $x \in \Omega_+$. We want to prove that $(T^n(x))_{n \geq 0}$ is dense in $\Omega$. By contradiction, let $U$ be an open set such that $U \cap \Omega \neq \emptyset$ and $T^n(x) \notin U \cap \Omega$ for all $n$. Since $\Omega_+$ is stable by the action of $T$, we can relax the last condition by $T^n(x) \notin U$ for all $n \geq 0$.

Since $U \cap \Omega \neq \emptyset$, $U \cap \Omega$ contains at least two different points of $S(\infty)$, therefore $h$ is not constant on $U$. Hence $h(U)$ has a non-empty interior. Finally, since the sequence $h \circ T^n(x) = T^n_0(h(x))$ avoids an open set and $T_0$ is minimal, we get a contradiction.

In order to prove that $\Omega$ is an attractor for both $T$ and $T^{-1}$, we need the following three technical lemmas.

**Lemma 4.5.** The function $h$ such that $h \circ T = T_0 \circ h$ is constant on the connected components of $\gamma \cap \overline{S(\infty)}$.

**Proof.** By contradiction, let $[j_-, j_+]$ be a connected component of $\gamma \cap \overline{S(\infty)}$ on which $h$ is not constant. Therefore $h(j_-) < h(j_+)$. By density of $S_0(\infty)$ in $\gamma_0$, there exist infinitely many points of $S_0(\infty)$ in the middle third segment of $[h(j_-), h(j_+)]$. Since $h : S(\infty) \to S_0(\infty)$ is a bijection, the image by $h^{-1}$ of all these points of $S_0(\infty)$ is relatively compact in $[j_-, j_+]$. Hence, there exist accumulation points of $S(\infty)$ in $[j_-, j_+]$, which is a contradiction.

**Lemma 4.6.** The connected components of $\gamma \cap \overline{S(\infty)}$ are permuted without cycle by $T$.

**Proof.** By construction of $S(\infty)$, $T$ and $T^{-1}$ are continuous on each connected component of $\gamma \cap \overline{S(\infty)}$. If $J$ is a connected component of $\gamma \cap \overline{S(\infty)}$, then it is easy to see that $T(J)$ is a subset of a connected component of $\gamma \cap \overline{S(\infty)}$. The same argument applied with $T^{-1}$ proves that the connected components are permuted by the action of $T$.

By contradiction, let $J$ be a connected component of $\gamma \cap \overline{S(\infty)}$ and $n > 0$ be such that $T^n(J) = J$. Therefore $h \circ T^n(J) = h(J) = \{x\}$ by the lemma 4.5. Now $h \circ T^n(J) = T_0^n(h(J))$. Therefore $x$ is a periodic point for $T_0$, which contradicts the minimality of $T_0$.

**Lemma 4.7.** The isolated points of $\overline{S(\infty)}$ are wandering points.

**Proof.** Let $x$ be an isolated point of $\overline{S(\infty)}$. Therefore there exists an open set $U$ such that $U \cap \overline{S(\infty)} = \{x\}$. Hence $U \setminus \{x\} = U_1 \cup U_2$ is included in the union of two connected components of $\gamma \cap \overline{S(\infty)}$, which are wandering sets by lemma 4.6. Therefore, $T^n(U \setminus \{x\}) \cap U \neq \emptyset$ for only finitely many values of $n$. Now, if $T^n(x) \in U$ then $T^n(x) = x$ and therefore $h(x)$ is a periodic
point of $T_0$ which is impossible. Finally, we proved that $T^n U \cap U \neq \emptyset$ for only finitely many values of $n$, in other words $x$ is a wandering point.

**Theorem 4.8.** For every point $x \in \gamma$ whose forward orbit is infinite, the $\omega$-limit set of $x$ satisfies $\omega(x) = \Omega$. The counterpart is true for infinite backward orbits and $\alpha$-limit sets. In other words, $\Omega$ is an attractor for the transformations $T$ and $T^{-1}$. Furthermore, $\Omega$ coincides with the non-wandering set $\Omega(T)$ of $T$.

**Proof.** We prove both inclusions. We start by showing that $\Omega \subset \omega(x)$. By contradiction, let $y \in \Omega$ such that $y \notin \omega(x)$. Since $\omega(x)$ is a closed set, there exists an open set $U$ containing $y$ such that $U \cap \Omega \neq \emptyset$ and $U \cap \omega(x) = \emptyset$. Therefore $T^n(x) \notin U$ for large enough $n$. Since $U \cap \Omega \neq \emptyset$, $U$ contains at least two distinct points of $\mathcal{S}(\infty)$. Since $h$ is one-to-one on $\mathcal{S}(\infty)$ and continuous on $\gamma$, the set $h(U)$ has a non-empty interior. Therefore the sequence $T^n(h(x)) = h \circ T^n(x)$ is dense in $\gamma_0$ (by minimality of $T_0$) and avoids the set of non-empty interior $h(U)$, hence a contradiction.

We now prove that $\Omega^c \subset \omega(x)^c$. Let $y$ be in $\Omega^c$. There are two cases. If $y \in \gamma \setminus \overline{\mathcal{S}(\infty)}$, then by lemma 4.6 $y$ is contained in a wandering interval: $y$ cannot be obtain as a limit point of an orbit by $T$, hence $y \notin \omega(x)$. Otherwise, $y$ is an isolated point of $\overline{\mathcal{S}(\infty)}$. By contradiction, $y \in \omega(x)$ implies that $y$ is a non-wandering point, which contradicts lemma 4.7. Hence $\omega(x) = \Omega$.

We now prove that $\Omega = \Omega(T)$. By minimality of $T$ when restricted to $\Omega$, we get $\Omega \subset \Omega(T)$. Since $T$ permutes the connected components of $\gamma \setminus \overline{\mathcal{S}(\infty)}$, all points of $\gamma \setminus \overline{\mathcal{S}(\infty)}$ are wandering points. Therefore $\Omega(T) \subset \overline{\mathcal{S}(\infty)}$. Finally, by the lemma 4.7 we can refined this last inclusion by $\Omega(T) \subset \Omega$.

**Proposition 4.9.** The sets $\Omega$ and $K$ are related by $\Omega = \gamma \cap K$.

**Proof.** Let $p = p_\gamma^x$ be in $\gamma \cap K$. We know that $h_\gamma(p)$ is dense in $K$, therefore $(T^n(p))_\mu$ is dense in $\gamma \cap K$. However, $\omega(x) = \Omega$ for all $x \in \gamma$, in particular for $x = p$. Hence $\Omega = \gamma \cap K$.

**Corollary 4.10.** When restricted to $K$, the flow $h_t$ is minimal. Furthermore, the set $K$ is an attractor for the flow $h_t$, for positive and negative times.

**Proof.** Let $u : \gamma \to \mathbb{R}$ be the function giving the first return time in $\gamma$. This function is bounded by some constant $C$. Clearly, we have the equality $h_\gamma(\Omega) = h_{[0,C]}(\Omega)$ and the left hand side is a closed set containing the orbit of $p = p_\gamma^x \in \gamma \cap K = \Omega$, hence $h_{[0,C]}(\Omega) = K$. This last equality proves the minimality of $(h_t)$, when restricted to $K$.

From $h_{[0,C]}(\Omega) = K$ and theorem 4.8, we obtain that every infinite forward trajectory of $h_t$ accumulates on $K$. Similarly, every infinite backward trajectory of $h_t$ accumulates on $K$.

4.3. **Proof of the unique ergodicity of $h_t$**

**Lemma 4.11.** In the coordinates of the suspension, every $h_t$-invariant measure $\mu$ must be of the form $d\mu(x,t) = Cd\nu(x) \, d\text{Leb}(t), \text{ for } x \in \gamma, \, 0 \leq t < u(x)$, some constant $C > 0$ and some measure $\nu$ on $\gamma$, where $u(x)$ is the time of first return to $\gamma$ of $x$ and $\text{Leb}$ is the Lebesgue measure.

**Proof.** Let $\tilde{\pi} : \gamma \times \mathbb{R} \to \mathbb{R}$ be a covering map. The lift of $h_t$ is simply the unit speed translation flow along the second coordinate. Let $\mu$ be an invariant measure for this flow. Let $\tilde{\mu}$ be a lift of $\mu$ to $\gamma \times \mathbb{R}$. Therefore $\tilde{\mu}$ is invariant by translation along the second coordinate. Hence $\tilde{\mu} = C \nu \otimes \text{Leb}$, where $\text{Leb}$ is the Lebesgue measure and $\nu(S) := \tilde{\mu}(S \times [0, \varepsilon])$ is a measure on $\gamma$, for some $\varepsilon > 0$. Taking back the projection by $\tilde{\pi}$, we get $d\mu(x,t) = C d\nu(x) \, d\text{Leb}(t)$, as long as $\varepsilon < \inf_x u(x)$. ☐
We can now prove the unique ergodicity of $h_t$.

**Proof of theorem 4.1.** Let $\mu$ be a measure invariant by the flow $h_t$. By lemma 4.11, we can find a constant $C$ and a measure $\nu$ on $\gamma$ such that $d\mu(x, t) = C \nu(x) \, d\text{Leb}(t)$. By applying Fubini’s theorem on sufficiently small rectangles, we obtain that $\nu$ is invariant by $T$.

Since the vertical foliation associated to a pseudo-Anosov map is uniquely ergodic—see [11, 12, Exposé 12]—it follows that $T_0$ is uniquely ergodic.

Now, $T$ and $T_0$ have the same path in the Rauzy-graph. By [28], $T$ is semi-conjugated to $T_0$ by some continuous monotonic function $h$. This function $h$ is bijective when restricted, up to a countable set of points, to the set of non-wandering points of $T$. Therefore $T$ is also uniquely ergodic, of invariant measure $\nu$.

Hence $h_t$ is uniquely ergodic, of invariant measure $\mu$.

We now prove that the support of $\mu$ is $K$. First, since $\text{supp}(\nu)$ is included in the set of non-wandering points of $T$, which is $\Omega$, and $\text{supp}(\nu)$ is a closed set invariant by $T$, by minimality of $T$ we get that $\text{supp}(\nu) = \Omega$. Now, by the factorization of $\mu$ and the fact that $h_0(\Omega) = K$, we get $\text{supp}(\mu) = K$.

We give in figure 4 a graphical representation of the set $K$ in the case of the fully explicit example outlined in the description of figure 2. More precisely, figures 2–4 are obtained by integrating numerically an approximation of the vector field $\nu'$ associated to the pseudo-Anosov map constructed in the appendix of [25]. The approximation of $\nu'$ is obtained by truncating the sum (2.2) defining $\nu'$. The integration is done with a forth order Runge–Kutta method. The GIET in figure 3 is obtained as the first return map of to $\gamma$ of many initial conditions in $\gamma$ of the numerically estimated solutions. For figure 4, we plot a numerically estimated solution with some initial data for $t \in [T/2, T]$, for some large $T > 0$ (the initial point is not so important because of corollary 4.10).

As a final remark for this section, we can perform a similar analysis by perturbing a pseudo-Anosov only at some conical points $\Sigma_0 \subseteq \Sigma$. The proofs are mostly the same by replacing...
Figure 3. Graphs of the induced IET and GIET induced respectively by the rectangle decompositions in figure 2—both flows are going “downward”.

Figure 4. Numerical representations of the set $K$ for a perturbation of a pseudo-Anosov homeomorphism on a genus two surface. Right: perturbation at the unique conical point. Left: perturbation at a regular fixed point (the centre of the octagon).

$\Sigma$ by $\Sigma_0$ and using arguments from the following section in order to deal with the points $\Sigma \setminus \Sigma_0 \subset K$.

5. Perturbation at a regular periodic point

Because of the following general property concerning pseudo-Anosov maps—see for example [10]—we can consider periodic points that are not conical points—they are regular points.
Proposition 5.1. If \( \varphi : S_\theta \to S_\theta \) is pseudo-Anosov, then the set of periodic points of \( \varphi \) is a dense subset of \( S_\theta \).

Let \( \theta \in S_\theta \setminus \Sigma \) be a periodic point of \( \varphi \) that is not a conical point. Up to considering a power of \( \varphi \), we assume that \( \theta \) is a fixed point.

In this part, we present that a very similar analysis can be done when a pseudo-Anosov map is perturbed at a fixed point that is regular instead of conical.

5.1. Definition of the perturbation

We can proceed to the same type of perturbation as described in section 2.1 at a regular fixed point \( \theta \), except that it is much easier to define since \( \theta \) is not a conical point and we do not have to deal with branched cover.

Write \( \varphi(x + iy) = \lambda x + i\lambda^{-1}y \) in some local chart centred at \( \theta \). In these coordinates, define

\[
    f(x + iy) := \left( \lambda + \beta k \left( \frac{\sqrt{x^2 + y^2}}{\alpha} \right) \right) x + i\lambda^{-1}y,
\]

for some \( \beta \in ]-\lambda,0] \) (we are interested in the case \( \beta \in ]-\lambda, -\lambda + 1] \)), \( 0 < \alpha < \delta_\theta \), where \( \delta_\theta := \min \{ \frac{1}{2} \text{Syst}(S_\theta), \inf \{ d(\theta, \sigma) \mid \sigma \in \Sigma \} \} \) and \( k : \mathbb{R} \to \mathbb{R} \) is an even unimodal function of class \( C^1 \), compactly supported in \([ -1, 1] \) such that \( k' \) is Lipschitz continuous, for example \( k(r) = (1 - r^2)^2 \, 1_{[-1,1]}(r) \). Set \( f = \varphi \) elsewhere. Actually, this perturbation corresponds to a ‘fake’ conical point \( \sigma = \theta \), that is where \( n_\sigma = 1 \) (in section 2 we assumed that \( n_\sigma > 1 \) for all \( \sigma \in \Sigma \)).

5.2. Differences of this case

With the change on the range of the parameter \( \alpha \), analogues of Propositions 2.2 to 2.8 hold where \( \sigma \in \Sigma \) is replaced by \( \theta \). The proof for these results are formally the same.

That is, for \( \beta \in ]-\lambda, 0] \) and \( \alpha \in ]0, \delta_\theta[ \), \( f \) is a homeomorphism on \( S_\theta \) and a \( C^1 \) diffeomorphism on \( S_\theta \setminus \Sigma \). Restricting further the range of \( \beta \) to be \( ]-\lambda, 1 - \lambda] \), \( \theta \) is an attractive fixed point of \( f \). Call \( U_\theta \) its basin of attraction (which is open), and define \( K = S_\theta \setminus U_\theta \). In this case, we get that on the horizontal leaf containing \( \theta \), there are two hyperbolic fixed points, \( p_1 \) and \( p_2 \), at the same distance \( \vert p \vert > 0 \) from \( \theta \). Furthermore \( B(\theta, \vert p \vert) \subset U_\theta \). Also, \( U_\theta \) is dense in \( S_\theta \), its accessible border is \( W^{a}(p_1) \cup W^{a}(p_2) \) and the compact set \( K \) is equal to \( W^{a}(p_1) \cup W^{a}(p_2) \).

The first difference with the case treated in the previous sections is that here \( \Sigma \subset K \). In particular, \( K \) can no longer be a hyperbolic set since \( f \) is not differentiable on points of \( \Sigma \). Nonetheless, one can still construct a vector field \( \nu' \) on \( K \setminus \Sigma \) such that \( d(f(\nu'(x))) = \lambda^{-1} \nu'(f(x)) \), \( x \in K \setminus \Sigma \), as in theorem 2.9 (in particular, strictly speaking, \( f \) is not Axiom A since \( K \setminus \Sigma \) is not a compact set). Results from the first half of section 3 do not change, as well as there proofs. More precisely, \( \nu' \) can be extended into a Lipschitz (or \( C^1 \)) vector field on \( S_\theta \setminus \Sigma \), such that \( (x, \beta) \in (S_\theta \setminus \Sigma) \times ]-\lambda - \lambda^{-2}, 0[ \to \nu_\beta(x) \) is continuous.

The second difference concerns proposition 3.11. The commutation relation between \( f \) and \( h_s \) still holds, but \( K \) is no longer \( h_s \)-invariant—this is due to the fact that each branch of stable manifold of each \( \sigma \in \Sigma \) belongs to \( K \) and is parametrized by \( h_s \), thus some trajectories in \( K \) end in finite time.

Nonetheless, analogue of proposition 3.12 and theorems 3.13 and 3.14 hold. More precisely, only the statement of theorem 3.14 has to be modified; \( f : K \setminus \Sigma \to K \setminus \Sigma \) is topologically transitive for the trace topology of \( S_\theta \) on \( K \setminus \Sigma \). Proofs do not change.
Concerning section 4, since the study of GIET and IET already deal with the singularities of the flows at \( \Sigma \), the results concerning the map \( T \) and \( T_0 \) have analogous counterparts. Here the map segments \( \gamma \) and \( \gamma_0 \) are contained in the horizontal leaf containing \( \theta \). In particular there is a subset \( \Omega \subset \gamma \), attractor for positive and negative iterates of \( T \), such that \( T|_{\Omega} \) is minimal, and \( T \) is uniquely ergodic of unique invariant measure \( \nu \) supported by \( \Omega \). The relation \( \Omega = \gamma \cap K \) still holds with change in the proof. Since the measure \( \nu \) has no atoms—otherwise, \( T_0 \) would have periodic orbits, contradicting the minimality of \( T_0 \)—we get that \( \nu(\Omega) = \nu(\Omega \setminus S(\infty)) = 1 \). Furthermore, since \( \Omega \setminus S(\infty) = \gamma \cap (K \setminus \bigcup_{i \in \{1,2\}} W^{\pm}(p_i)) \), and \( K \setminus \bigcup_{i \in \{1,2\}} W^{\pm}(p_i) \) is \( h_f \)-invariant, we get that \( \mu = C \nu \otimes \text{Leb} \) is a \( h_f \)-invariant probability measure (for some \( C > 0 \), with support \( K \setminus \bigcup_{i \in \{1,2\}} W^{\pm}(p_i) = K \).

Finally, \( h_f \) is uniquely ergodic, and \( f \) is mixing with respect to \( \mu \), as in the previous section.

6. The measure \( \mu \)

In this last section, using extensively Bowen and Ruelle’s work [4, 24], we prove that \( \mu \) is the unique SRB-like measure of \( f^{-1} \), and that correlations decrease exponentially fast for \( C^1 \) observables compactly supported away from \( \Sigma \), in the case where \( f \) is constructed as in section 2. Finally, using the maximizing property associated with SRB measure, we compute the entropy of \( f \) with respect to \( \mu \). We also ask whether the result on the Ruelle spectrum of a linear pseudo-Anosov by Faure et al [13], and the asymptotic expansion for ergodic integral of the Giulietti–Liverani flow proved by Forni [14], can be adapted to the settings of the present paper.

We used the term ‘SRB-like’ instead of just ‘SRB’ because SRB measure are only defined for \( C^2 \) (or \( C^{1+\alpha} \)) diffeomorphisms, but the above map \( f \) is only continuous at conical points. Nonetheless, we show that \( \mu \) is the unique SRB measure associated to \( f^{-1}|_{\Sigma \setminus \Sigma} \) and that the usual definitions of SRB measure extend to \( f^{-1} \). We will therefore refer to SRB measure in the rest of this section instead of ‘SRB-like’ measure.

For now on, we assume that \( f \) is a \( C^2 \) diffeomorphism away from \( \Sigma \), which can be achieved by choosing a \( C^2 \) bump function \( k \). Such a bump function \( k \) is also assumed to be \( C^2 \).

6.1. SRB measure and entropy of \( f^{-1} \)

Sinai–Ruelle–Bowen measures are particular invariant measures of \( C^2 \) transformations. See [30] for a survey about these measures and which dynamical systems have them.

The problem here is that \( f \) and \( f^{-1} \) are smooth only away from conical points, where they are only continuous. Still, \( S_\theta \setminus \Sigma \) is an invariant set on which \( f^{-1} \) is a \( C^2 \) diffeomorphism. Furthermore, \( K \) is an Axiom A attractor for \( f^{-1} \), in the sense that \( K \) is locally maximal, \( f^{-1}|_{K} \) is uniformly hyperbolic and \( f^{-1}|_{K} \) is topologically transitive. Notice that \( K \) is connected.

By [24, Theorem 1.5], there exists a unique SRB measure \( \mu_K \) supported by \( K \), maximizing \( h_\nu(f^{-1}|_{\Sigma \setminus \Sigma}) + \nu(-\log \det df^{-1}|_{E^f}) \)—and the maximum is equal to 0.

Theorem 6.1. If \( W \) is a curve of finite length contained in \( W^{\pm}(p) \) and containing \( p \), where \( p \) is some hyperbolic fixed point \( p_0 \) of \( f \), and \( \nu_W \) is a measure on \( W \) with bounded Radon–Nikodym derivative with respect to the measure induce by the Riemann metric on \( W \), then

\[
\mu = \lim_{n \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f^n_* \nu_W.
\]

In particular, \( \mu = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f^n_* \nu_W \) and according to [30], \( \mu \) is a SRB measure for \( f^{-1}|_{\Sigma \setminus \Sigma} \). Therefore, by uniqueness, \( \mu = \mu_K \).
Proof. Let \( W \subset \tilde{W} \subset W^s(\nu) \) be a strictly longer curve than \( W \). Let \( \tilde{\nu} \) be the measure on \( \tilde{W} \) induced by the Riemann metric. By assumption there exists a bounded function \( \rho \geq 0 \) such that \( dW = \rho d\tilde{\nu} \). If needed, \( \rho \) is implicitly extended by 0.

Since \( k \) is assumed to be \( C^2 \), by theorem 3.7, \( h_t \) is a \( C^1 \) flow. Therefore, for small enough \( t \), \((h_t)_*\nu_w\) is supported by \( \tilde{W} \) and

\[
d((h_t)_*\nu_w) = \rho \frac{\text{Jac} h_t}{\text{Jac} h_{-t}} d\tilde{\nu},
\]

where \( \text{Jac} h_t \) is the Jacobian determinant of the time \( t \) of the flow. Therefore, if \( \varphi \) is a continuous function on \( S_g \), then for all small enough \( t \),

\[
|((h_t)_* (f^{-n}\nu_w)) - (f^{-n}\nu_w)|(_{f^{-1}|_{S_g}})(\varphi) = |f^{-n}( (h_{-t})_* \nu_w - \nu_w))|(_{f^{-1}|_{S_g}})(\varphi) 
\leq |\varphi|_{\infty} \int_{\tilde{W}} \left| \frac{\rho}{\text{Jac} h_{-t}} \circ h_{-t} - \rho \right| d\tilde{\nu},
\]

which converges, by dominated converge, to zero as \( n \) goes to infinity. Therefore, all subsequential limits of \( f^{-n}\nu_w \) are \( h_t \)-invariant. By unique ergodicity of \( h_t \), all subsequential limits of \( f^{-n}\nu_w \) must coincide with \( \mu \). Therefore \( f^{-n}\nu_w \) converges to \( \mu \).\( \square \)

We can now compute the entropy of \( f \) with respect to \( \mu \).

**Theorem 6.2.** The entropy \( h_\mu(f) \) with respect to \( \mu \) is equal to \( \log(\lambda) \).

**Proof.** It follows from the fact that \( df^\nu = \lambda^{-1} \nu \circ f \) that \( df^{-1}|_{E^s} \) is constant equal to \( \lambda \) on \( K \). Therefore \( h_\mu(f^{-1}|_{S_{g\cup\Sigma}}) = \log(\lambda) \). Now, since \( \Sigma \cap K = \emptyset \), we get that \( h_\mu(f) = h_\mu(f^{-1}) = \log(\lambda) \).\( \square \)

Finally, remark that since the nonwandering set of \( f \) is \( K \cup \Sigma \) and since we can extend by continuity \( df^{-1}|_{E^s} \) at each \( \sigma \) in \( \Sigma \) by \( \lambda^{-1}(\lambda + \beta_\sigma) < 1 \), the measure \( \mu \) is still the unique measure maximizing \( h_\mu(f^{-1}) + \nu(-\log \text{det} df^{-1}|_{E^s}) \) for \( \nu \) ranging over the set of \( f \)-invariant measures.

6.2. Bernoulli and exponential mixing

Using the careful analysis over Markov partition done by Ruelle in [24], we are able to deduce that \((f, \mu)\) is isomorphic to a Bernoulli shift and that the correlations decrease exponentially fast for \( C^1 \) observables supported away from \( \Sigma \).

**Theorem 6.3.** The system \((f, \mu)\) is isomorphic to a Bernoulli shift.

**Theorem 6.4.** There exist constants \( 0 < \theta < 1 \) and \( C > 0 \) such that for all \( C^1 \) observables \( \varphi \) and \( \psi \) compactly supported away from \( \Sigma \),

\[
|\mu(\varphi \circ f^{-n} \psi) - \mu(\varphi) \mu(\psi)| < C||\varphi||_{C^1}|\psi||_{C^1} \theta^{n}, \quad \forall n \geq 0.
\]

The proofs of these two theorems directly follows from [24, Theorem 1.5].

6.3. What about the Ruelle spectrum?

In [13], Faure, Gouëzel and Lanneau proved that for any orientation preserving linear pseudo-Anosov map \( \varphi \) on a surface \( S_g \) of genus \( g \), the Ruelle spectrum can be computed explicitly. More precisely, if \( \lambda > 1 \) is the expansion factor of \( \varphi \) and \( \lambda^{-1}, \lambda, \mu_1, \ldots, \mu_{2g-2} \) is the spectrum of \( \varphi^* \) – where \( \varphi^* \) is the natural action of \( \varphi \) on the first space of cohomology \( H^1(S_g) \) – then the
Ruelle spectrum of \( \varphi \) for \( C^\infty_c(S \setminus \Sigma) \) observables is \( \{ \lambda^{-n} \mu_i \mid 1 \leq i \leq 2g - 2, n \geq 1 \} \). Furthermore, the multiplicity of \( \lambda^{-n} \mu_i \) is \( n \). In order to prove this result, the authors first show that \( \lambda^{-n} \mu_i \) are indeed Ruelle resonances and then that there are no other Ruelle resonances.

Since \( f \) is, by construction, homotopic to such linear pseudo-Anosov map \( \varphi \), the action on the cohomology is the same. One might expect that the Ruelle spectrum of \((f, \mu)\) is the same as the one of \( \varphi \), up to a few modifications.

The key ingredients in the first part of [13]—where it is proved that \( \lambda^{-n} \mu_i \) are Ruelle resonances—are the smoothness of the invariant foliations and the uniform contraction of the stable foliation. This particularities remain true in the case of the perturbation \( f \). The argument then should carry over to the case of the specific derived from pseudo-Anosov maps studied in this paper.

However, the second part of [13]—where it is proved that Ruelle resonances must be of the form \( \lambda^{-n} \mu_i \)—relies on many geometric considerations and also on the uniform dilation of the unstable foliation. Unfortunately this last assumption fails, by construction, in the case of \( f \).

6.4. What about deviation from Ergodic Integral?

In [14, corollary 1.5], Forni proved an asymptotic expansion for the ergodic integrals of the Giulietti–Liverani flow [16] on surface of genus \( g \geq 2 \). Because of all the common properties between the flow \( h_t \) studied in the present paper and the Giulietti–Liverani flow, it seems reasonable that a similar formula should holds. However, it is not clear whether Forni’s proof can be adapted in this setting.

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