Casimir Driven Evolution of the Universe

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Abstract

For a Friedman-Robertson-Walker space-time in which the only contribution to the stress-energy tensor comes from the renormalised zero-point energy (i.e. the Casimir energy) of the fundamental fields the evolution of the universe (the scale factor) depends upon whether the universe is open, flat or closed and upon which fundamental fields inhabit the space-time.

We calculate this “Casimir effect” using the heat kernel method, and the calculation is thus non-perturbative. We treat fields of spin $0, \frac{1}{2}, 1$ coupled to the gravitational background only. The heat kernels and/or zeta-functions for the various spins are related to that of a non-minimally coupled scalar field which is again related to that of the minimally coupled one. A WKB approximation is used in obtaining the radial part of that heat kernel.

The simulations of the resulting equations of motion seem to exclude the possibility of a closed universe, $K = +1$, as these turn out to have an overwhelming tendency towards a fast collapse – the details such as the rate of this collapse depends on the structure of the underlying quantum degrees of freedom; a non-minimal coupling to curvature accelerates the process. Only $K = -1$ and $K = 0$ will in general lead to macroscopic universes, and of these $K = -1$ seems to be more favourable.
The possibility of the scale factor being a concave rather than a convex function, as is the case for a $K = -1$ FRW-spacetime inhabited by a conformally coupled scalar field, potentially indicates that the problem of the large Hubble constant is non-existent as the age of the universe need not be less than or equal to the Hubble time. Note should be given to the fact, however, that we are not able to pursue the numerical study to really large times neither to do simulations for a full standard model.

1 Introduction

In a recent paper [1], we realized that the Casimir effect may drive inflation or otherwise strongly influence the evolution of space-time because the vacuum fluctuations couple to the (background) gravitational field. As one cannot escape the Casimir effect it may thus be that this property of the zero-point energy can be used for determining whether the universe is open, flat or closed. Or working the other way around, one may be able to use cosmological observations to tell us about the particle content in the very early universe.

In this paper the quantum fields reside in a Friedman-Robertson-Walker space-time. In section 2 we show how one in general calculates the Casimir effect of a free scalar field using the heat kernel method and continue to determine it explicitly in section 3 in the case of the Friedman-Robertson-Walker metric and it is noted that using this approach, the zero-point energy is almost automatically renormalised. The renormalised Casimir energy is then inserted in section 3.1 into the right hand side of the Einstein equations (it gives an effective action for the scale factor $a(t)$, the only degree of freedom) in order to obtain the equations of motion and the results of the corresponding simulations of the evolution of the space-time, perhaps relevant if the Higgs field is believed to dominate the early universe, are presented in section 3.2. Having established a procedure for determining the Casimir effect on the evolution of space-time we continue to consider the case of a non-minimally coupled scalar field (in section 4) and show how the Casimir energy is related to that of the minimally coupled scalar field. Then we proceed in sections 4.1 and 4.2 to present the equations of motion and the corresponding simulations for a conformally coupled scalar field. In section 5 the approach used
in section 4 is generalised to the case of vector bosons, with the problematic self-interaction being treated using a mean field approximation procedure - due to the complexity of the solution we have not been able to carry out the simulations, however. Finally in section 6, after some general considerations about how to link a fermionic operator to a scalar or bosonic one, we take on free (spin 1/2) fermions. For completeness, we have included a discussion of fermions interacting with gauge fields in an appendix (based on the mean field approach developed for pure Yang-Mills theory). Einstein equations and simulations for free fermions are presented in sections 6.1 and 6.2 Section 7 gives a few concluding remarks.

2 General Considerations concerning the calculation of the Casimir Effect for a Free Scalar Field

We need to determine the right hand side of the Einstein equations, the energy-momentum tensor. Now, the quantum expectation value, $\langle T_{\mu\nu} \rangle$, (the zero-point energy) is given in terms of the effective action $\Gamma$ as \[\langle T_{\mu\nu} \rangle = 2(-g)^{-1/2} \frac{\delta \Gamma}{\delta g^{\mu\nu}} \] (1)

which is just the quantum analogue of the classical relationship

$$T^{\text{class}}_{\mu\nu} = -g^{-1/2} \frac{\delta S}{\delta g^{\mu\nu}}$$

Another way of looking at this (the way chosen in [1]), is to use an analogy with classical statistical mechanics and introduce Helmholtz’ free energy, $F$, which is related to the partition function by

$$F = -\frac{1}{\beta} \ln Z$$ (2)

where $\beta$ is the inverse temperature and where the partition function is given by the functional integral

$$Z = \int e^{-S} \mathcal{D}\phi = e^{-\Gamma_{\text{eff}}}$$ (3)

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with $\Gamma_{\text{eff}}$ being the effective action, i.e.

$$\Gamma = \beta F = - \ln Z$$

Actually, this is an effective potential for the gravitational degrees of freedom, the full effective action for gravity being given by

$$\Gamma_{\text{full}}[g] = S_{\text{EH}}[g] + \Gamma[g]$$

but we will always take the Einstein-Hilbert action $S_{\text{EH}}$ to be understood, and simply refer to $\Gamma$ as the effective action. This slight abuse of notation should not cause any misunderstandings.

\*\*From Helmholtz’ free energy one can then, as in [1], proceed to find the internal energy and the pressures. It is because of this analogy with the free energy we refer to the entire scenario as Casimir driven evolution, the renormalised zero point energy being analogous to the well known Casimir effect in flat space.

Since the effective action is given by integrating out the matter degrees of freedom in the partition function, it gives the effective equation of motion for the remaining degree of freedom, namely the gravitational one, which for a Friedman-Robertson-Walker background reduces to an effective equation of motion for the scale factor $a(t)$.

Now, since we’re dealing with a free particle,

$$S = \frac{1}{2} \int d^4x \sqrt{-g} (g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - M^2 \phi^2)$$

albeit in a special geometry, this integral is merely a Gaussian. Thus we can perform it using standard techniques, arriving at the result

$$Z = (\det A)^{-1/2}$$

where the differential operator $A$ is given by

$$A \equiv \frac{\delta^2 S}{\delta \phi^2} = \Box - M^2$$

with $\Box$, the d’Alembertian of the curved space-time, given by [7]

$$\Box = \frac{1}{\sqrt{|g|}} \partial_\mu \left( g^{\mu\nu} \sqrt{|g|} \partial_\nu \right)$$
The effective action is thus seen to be given by a functional determinant:

$$\Gamma = - \ln \left( \text{det} \left( \Box - M^2 \right) \right)^{-1/2} \quad (10)$$

The determinant of an operator, $A$, is, by definition, the product of its eigenvalues

$$\text{det}(A) = \prod \lambda \quad (11)$$

The zeta-function is given by [2, 3]

$$\zeta_A(s) = \sum \lambda^{-s} = \text{Tr} \ A^{-s} \quad (12)$$

and it is related to $\text{det}(A)$ by the following equation

$$\frac{d\zeta_A}{ds} \bigg|_{s=0} = \sum \lambda - \ln \lambda \cdot \lambda^{-s} \bigg|_{s=0} = - \sum \ln \lambda = - \ln \prod \lambda = - \ln \text{det}(A) \quad (13)$$

Now (as shown below) the zeta-function, $\zeta_A$, can be determined from the heat kernel $G_A(x, x', \sigma)$ which is a function satisfying the heat kernel equation

$$A_\sigma G_A(x, x', \sigma) = - \frac{\partial}{\partial \sigma} G_A(x, x', \sigma) \quad (14)$$

subject to the boundary condition $G_A(x, x', 0) = \delta(x, x')$.

To see how $G_A(x, x', \sigma)$ is related to $\zeta_A$ note that if $A\phi_\lambda = \lambda \phi_\lambda$ then $G_A(x, x', \sigma) \equiv \sum \phi_\lambda(x) \phi_\lambda^*(x') e^{-\lambda \sigma}$ satisfies the heat kernel equation. This in turn makes it possible to show the following relationship between $G_A$ and $\zeta_A$

$$\int_0^\infty d\sigma \sigma^{s-1} \int \sqrt{g} d^4 x G_A(x, x, \sigma) = \int_0^\infty d\sigma \sigma^{s-1} \int \sqrt{g} d^4 x \sum |\phi_\lambda(x)|^2 e^{-\lambda \sigma}$$

$$= \sum \lambda \int_0^\infty d\sigma \sigma^{s-1} e^{-\lambda \sigma} \int \sqrt{g} d^4 x |\phi_\lambda(x)|^2$$

1If $A$ is the Laplace operator and the space is $S^1$ then $\zeta(s)$ becomes (proportional to) the Riemann zeta function, and if the space is $R^3$ and $A = \nabla^2$ the heat kernel equation becomes the usual heat equation.
where, with suitable normalisation, the last integral can be put equal to unity which gives us

$$\int_0^\infty d\sigma \sigma^{s-1} \int \sqrt{g} d^4 x G_A(x, x, \sigma) = \sum_\lambda \int_0^\infty d\sigma \sigma^{s-1} e^{-\lambda \sigma}$$

$$= \sum_\lambda \lambda^{-s} \int_0^\infty d(\lambda \sigma)(\lambda \sigma)^{s-1} e^{-\lambda \sigma}$$

$$= (\sum_\lambda \lambda^{-s}) \Gamma(s)$$

$$= \Gamma(s) \zeta_A(s)$$ \hspace{1cm} (15)

so that

$$\zeta_A(s) \equiv \frac{1}{\Gamma(s)} \int d\sigma \sigma^{s-1} \int G_A(x, x, \sigma) \sqrt{g} d^4 x$$

and so, finally, for the minimally coupled scalar case

$$\Gamma = \frac{1}{2} \int \sqrt{g} d^4 x \left. \frac{d}{ds} \right|_{s=0} \int_0^\infty d\sigma \frac{\sigma^{s-1}}{\Gamma(s)} G_{\Box - m^2}(x, x, \sigma)$$ \hspace{1cm} (16)

Note that the integral over $x$ is taken along the diagonal $x = x'$. Also note that the $\sigma$-integral is a standard one if one uses the spectral representation of the zeta-function, thus easy to calculate though it generally needs further regularisation (in the case of a minimally coupled field we use the spectral representation of the heat kernel (in a WKB approximation) while in other cases we use a hybrid expression, partially referring to eigenfunctions, partly a Taylor series generalising the usual Schwinger-DeWitt expansion [11]), including the exponential of the scalar curvature as shown by Jack and Parker, [13], to be needed.

Thus to calculate the zero-point energy of a scalar field one determines the scalar field operator (the d’Alembertian) in the relevant space-time, solve the corresponding heat kernel equation and subsequently calculates the zeta-function from which the generating functional and thus all relevant quantities (including the zero-point energy) can easily be calculated.

The zeta-function of spin 1 gauge bosons can be determined by a generalisation of the procedure for the non-minimal scalar field while the zeta-function associated with fermions is directly related to that of the a non-minimally coupled scalar field. This in turn is related to the zeta-function of a minimally coupled scalar field, substantially reducing the amount of work involved in
determining the zero-point energy of these fields, as will be shown in the following sections.

3 Solving the Heat Kernel Equation for the Minimally Coupled Scalar Field

Having in principle established a procedure for determining the zero-point energy for a minimally coupled scalar field the next step is to find the heat kernel and subsequently the Casimir energy of the field explicitly. The heat kernels for higher spin, as well as for non-minimally coupled scalar fields, are going to be related to the result found in this section. Thus this calculation is a central part of the paper.

If one writes the Friedman-Robertson-Walker line-element with the scale factor $a = a(t)$ as

$$ds^2 = dt^2 - a^2(t)(d\chi^2 + f^2(\chi)(d\theta^2 + \sin^2(\theta)d\phi^2))$$  \hfill (17)

where

$$f(\chi) = \begin{cases} 
\sin \chi & \text{for } K = +1 \text{ (closed universe)} \\
\chi & \text{for } K = 0 \text{ (flat universe)} \\
\sinh \chi & \text{for } K = -1 \text{ (open universe)} 
\end{cases}$$ \hfill (18)

then the d’Alembertian becomes

$$\Box = \frac{1}{\sqrt{|g|}} \partial_\mu \left( g^{\mu\nu} \sqrt{|g|} \partial_\nu \right)$$ \hfill (19)

$$\Box = (\partial_t^2 + \frac{3}{a} \partial_t) - \frac{1}{a^2} \left( \partial_\chi^2 - 2 \partial_\chi \left( \frac{\partial_\chi f}{f} \partial_\chi \right) \right) - \frac{L^2}{a^2 f^2}$$ \hfill (20)

where $L$ is the usual angular momentum operator.

Even though the equation separates, the minimally coupled scalar heat kernel equation

$$\Box G_\Box(x, x', \sigma) = \left( (\partial_t^2 + \frac{3}{a} \partial_t) - \frac{1}{a^2} \left( \partial_\chi^2 - 2 \partial_\chi \left( \frac{\partial_\chi f}{f} \partial_\chi \right) \right) - \frac{L^2}{a^2 f^2} \right) G_\Box(x, x', \sigma)$$

$$= -\frac{\partial}{\partial \sigma} G_\Box(x, x', \sigma)$$ \hfill (21)
is rather hard to solve. Fortunately, the eigenfunctions, \( \psi_\lambda(x) \), of the spatial part of the d’Alembertian are already known to be \[7, 12\]

\[
\psi_\lambda(x) = \begin{cases} 
(2\pi)^{-\frac{3}{2}} e^{i\mathbf{k} \cdot \mathbf{x}} & \text{for } K = 0 \\
\Pi_{k,J}^{(\pm)}(\chi) Y_J^M(\theta, \phi) & \text{for } K = \pm 1
\end{cases}
\] (22)

where

\[-\infty < k_i < \infty ; \text{ where } k = |\mathbf{k}| \text{ for } K = 0\]

\[M = -J, -J + 1, \ldots, J; \left\{ \begin{array}{ll}
J = 0, 1, \ldots, k - 1 \quad & \text{for } K = 1 \\
J = 0, 1, \ldots \quad & \text{for } K = -1
\end{array} \right.\] (23)

and

\[
\Pi_{k,J}^{(-)}(\chi) = \left\{ \frac{1}{2} \pi k^2(k^2 + 1) \ldots (k^2 + (2 + J)^2) \right\}^{-\frac{1}{2}} \sinh \chi \left( \frac{d}{d \cosh \chi} \right)^{1+J} \cos(k\chi)
\]

\[
\Pi_{k,J}^{(+)}(\chi) = \left\{ \frac{1}{2} \pi (-k^2)(-k^2 + 1) \ldots (-k^2 + (2 + J)^2) \right\}^{-\frac{1}{2}}
\]

\[
\times i \sin \chi \left( \frac{d}{d \cos \chi} \right)^{1+J} \cos(k\chi)
\] (24)

as quoted by \[9\] so proceed by assuming the heat kernel of the d’Alembertian to be of the form

\[G \Box = \sum_\lambda \psi_\lambda(x) \psi_\lambda^*(x') T_\lambda(t, t', \sigma)\] (25)

which is a solution of the heat kernel equation provided that the set of functions \( T_\lambda(t, t', \sigma) \) solves

\[(\partial_t^2 + 3\frac{\dot{a}}{a} \partial_t)T_\lambda(t, t', \sigma) = -\partial_\sigma T_\lambda(t, t', \sigma)\] (26)

Writing

\[T_\lambda(t, t', \sigma) = a^{-3/2}(t) f_\lambda(t)a^{-3/2}(t') f_\lambda(t')e^{-\lambda \sigma}\] (27)

gives the following equation for the function \( f_\lambda \)

\[
\ddot{f}_\lambda = \left( \lambda + \frac{27}{4} \left( \frac{\dot{a}}{a} \right)^2 - \frac{3 \dot{a}}{2 a} - \lambda a^{-2} \right) f_\lambda \equiv h_\lambda(t) f_\lambda(t)\] (28)
with

\[ \lambda = k^2 - K \]  

being the eigenvalue of the spatial part of the d’Alembertian.

We have not been able to solve this equation explicitly (due to the fact that we have no \textit{a priori} knowledge about \( h_\lambda(t) \)). Note however the similarity with the stationary Schrödinger equation;

\[ \frac{2m}{\hbar}[E - V(x)]\psi(x) = \Delta \psi(x). \]

Thus if \( \frac{\dot{h}}{\sqrt{h}} \) is negligible it is sound to use the WKB-approximation giving

\[ f_\lambda = h_\lambda^{-\frac{1}{4}}(t)e^{\int_0^t dt' \sqrt{h_\lambda(t')}} \]  

One should note that this is the only approximation introduced so far.

Having obtained a set of functions on which to expand the heat kernel;

\[ G_\Box(x, x', \sigma) = \sum_\lambda \phi_\lambda(x)\phi^*_\lambda(x')e^{-\lambda \sigma} \]

\[ = \sum_\lambda \psi_\lambda(x)\psi^*_\lambda(x')T_\lambda(t, t', \sigma) \]

\[ = \sum_\lambda \psi_\lambda(x)\psi^*_\lambda(x')a^{-3/2}(t)f_\lambda(t)a^{-3/2}(t')f^*_\lambda(t')e^{-\lambda \sigma} \]

\[ = \sum_\lambda \psi_\lambda(x)\psi^*_\lambda(x')a^{-3/2}(t)h_\lambda^{-\frac{1}{4}}(t)e^{\int_0^t dt' \sqrt{h_\lambda(t')}} \times \]

\[ a^{-3/2}(t')h_\lambda^{-\frac{1}{4}}(t')e^{\int_0^t dt'' \sqrt{h_\lambda(t'')}}e^{-\lambda \sigma} \]  

the \( \zeta \)-function reads

\[ \zeta_\Box(s) = \frac{1}{\Gamma(s)} \sum_\lambda \int_0^\infty \int |\psi_\lambda(x)|^2 h^{-1/2}(t)e^{-2\int_0^t \sqrt{h_\lambda(t')}dt'} f^2(\chi)\sqrt{g}e^{\lambda \sigma} \sigma^{s-1}d^3x dt d\sigma \]

\[ = \frac{1}{\Gamma(s)} \sum_\lambda \int_0^\infty \int h^{-1/2}(t)e^{-2\int_0^t \sqrt{h_\lambda(t')}dt'} e^{-\lambda \sigma} \sigma^{s-1} dt d\sigma \]  

where we have used \( \sqrt{g} = a^3 f^2 \sin \theta \equiv a^3 \sqrt{g^{(3)}} \) and that the eigenfunctions of the spatial part of the d’Alembertian are orthonormal with respect to the measure \( \sqrt{g^{(3)}}d^3x = f^2(\chi)d\chi d\Omega \).

\[ ^2 \text{This approximation assumes that } \hbar \text{ and thus the scale factor } a \text{ is relatively stable and, in turn, tends to stabilise } a. \text{ Thus, when simulating the behaviour of the scale factor later on, the evolution of } a \text{ will probably be understated.} \]
3.1 The Einstein Equations for the Friedman-Robertson-Walker space-time with a minimally coupled scalar field

With the metric given by equation (17) the non-zero components of the Einstein tensor are \( (f' = \partial_\chi f) \)

\[
\begin{align*}
G_{00} &= -\frac{2ff'' + f'^2 - 3\dot{a}^2 f^2}{a^2 f^2} \\
G_{11} &= -\frac{f'^2 + (-2\ddot{a} - \dot{a}^2)f^2}{a^2 f^2} \\
G_{22} &= -\frac{f'' + (-2\ddot{a} - \dot{a}^2)f}{a^2 f} \\
G_{33} &= -\frac{f'' + (-2\ddot{a} - \dot{a}^2)f}{a^2 f}
\end{align*}
\] (33)

Inserting \( f \) from equation (19) it is explicitly seen that, incidentally, the (two) last components can be rewritten as

\[
G_{33} = G_{22} = \frac{2a\ddot{a} + \dot{a}^2 + K}{a^2}
\] (34)

leading to the following (fourth component of the) equation of motion for the scale factor \( a(t) \) (using the first or second component would have complicated things unnecessarily as \( t \) would no longer be the only variable)

\[
\frac{2a\ddot{a} + \dot{a}^2 + K}{a^2} = \langle T_{33} \rangle
\] (35)

One should note that while this may sound a bit arbitrary (involving a choice of using only \( G_{33} \)), it is actually the uniquely defined equation of motion which follows from the effective action. The full Einstein tensor is only listed in order to give an interpretation of the resulting equation, namely as the pressure coupled to \( G_{33} \).

We proceed to calculate the resulting effective action explicitly using equations (14,15,22):

\[
\Gamma = \frac{1}{2} \left. \frac{d\zeta_A}{ds} \right|_{s=0}
\] (36)
$$\frac{d}{ds}igg|_{s=0} \frac{1}{\Gamma(s)} \lambda^{-s} \sum_{\lambda} \Gamma(s) \int h_\lambda(t)^{-1/2} e^{-2 \int_0^t \sqrt{\tau} d\tau'} dt = -\frac{1}{2} \sum_{\lambda} \ln \lambda \int h_\lambda(t)^{-1/2} e^{-2 \int_0^t \sqrt{\tau} d\tau'} dt$$

(37)

with

$$h_\lambda = 3\dot{a}a^{-2} - \frac{3}{2} \ddot{a}a^{-1} + \lambda + a^{-2}\lambda$$

(38)

coming from the WKB-approximation. The right hand side of the equation of motion (35) for the scale factor $a(t)$, obtained by varying $\Gamma$ with respect to $a(t)$, thus reads

$$-\sum_{\lambda} \ln \lambda \left\{ h_\lambda^{-3/2} \left( -\frac{3}{2} \dddot{a}a^{-3} + \frac{3}{4} \dddot{a}a^{-2} - \lambda a^{-3} \right) + h_\lambda^{1/2} \int_0^t \frac{-6\dot{a}a^{-3} + \frac{3}{2} \dddot{a}a^{-2} - 2\lambda a^{-3}}{\sqrt{3\dot{a}a^{-2} - \frac{3}{2} \dddot{a}a^{-1} + \lambda + a^{-2}\lambda}} dt' \right\} e^{-2 \int_0^t \sqrt{\tau} d\tau'}$$

(39)

We note the appearance of an integral over time: Formally the system depends on its entire past. Even though this dependence makes simulations somewhat tedious, we have still carried out quite a few, and these will be discussed in the next subsection.

These equations differ from the ones found by Suen and Anderson, [14], due to the use of the WKB-approximation. Note by the way that it is the exponential coming from this approximation which ensures rapid convergence of the summation over $\lambda$, thereby making the numerical solution more feasible.

### 3.2 Simulating the Equations of Motion

In order to simulate the evolution of a Friedman-Robertson-Walker spacetime according to the above equation of motion for $a(t)$, we first have to discretise the time, i.e., to introduce a time-step $\delta t$. One immediate problem is the appearance on the right hand side (the free energy contribution) of $a(t)$ and its derivatives raised to various powers and exponentiated. Thus we cannot just insert $\dot{a}(t) = (a(t+\delta t) - a(t))/\delta t$, $\ddot{a}(t) = (a(t+\delta t) - 2a(t) + a(t-\delta t))/\delta t^2$ into the equation of motion and then find a recursion relation telling us how to compute $a(t+\delta t)$ from the knowledge of $a(t), a(t-\delta t)$, as we are not able to isolate $a(t+\delta t)$. Instead we will consider the right hand side as a source
term to be evaluated at \( t - \delta t \) whereas the left hand side, the Einstein tensor, is to be evaluated at time \( t \). One can either justify this by sheer necessity or argue that it is plausible that the disturbance caused by the vacuum fluctuations do not give rise to an instantaneous change of geometry. In the limit \( \delta t \to 0 \) this distinction, of course, disappears. With this, we can now isolate \( a(t + \delta t) \) as it only appears in the discretised version of the Einstein tensor.

As mentioned in the introduction this work was inspired by the simulations of the (space-time called a) hyper-spatial tube. Those simulations did show a number of qualitatively different types of behaviour, the behaviour being determined by the initial conditions. Thus we have varied the initial conditions which for \( a(t = 0) \), in units of the Planck length, were 3, 5 or 10 and for \( \dot{a}(t = 0) \), in units of the velocity of light, were 0.0, ±0.1, ±0.5 ± 1.0 and 1.7.

Figures 1a-c shows, for the various initial conditions, simulations for \( K = +1, 0 \) and \( -1 \) respectively. As can be seen, the behaviour of the scale factor is rather universal in the sense that for given \( K \) the initial conditions determine whether the universe will collapse or expand but once it expands there is little (qualitative) dependence on the initial conditions of the evolution.

We notice that for \( K = +1 \) there is a marked tendency to collapse, whereas for the two other cases \( K = -1, 0 \) we have a tendency to expansion. Due to the amount of computer time and storage required, we have unfortunately not been able to pursue these evolutions very far, and so extrapolations on the evolution for large \( t \) becomes somewhat speculative. However, to see if the behaviour was qualitatively stable at somewhat higher times we prepared a plot of the evolution for \( K = -1 \) up to times \( 10^3 T_{Planck} \), this did not give rise to any new behaviour and is therefore not included here. Also note that the expansion becomes polynomial at late times\(^3\) (powers 0.6–0.7 for \( K = 0 \) and \( \sim 0.8 \) for \( K = -1 \)).

The early paper by Fischetti, Hartle and Hu \([15]\) found a solution for \( K = 0 \) to be of the asymptotic form \( a(t) \sim \sqrt{t} \), which was corrected by Starobinsky \([16]\) a few years later where he found \( a(t) \sim a_1 t^{2/3}(1 + (2/3M t) \sin M(t - t_1)) \), which is consistent with our findings.

\(^3\)These are not observable velocities but geometrical ones and thus potentially can be larger than \( c \).

\(^4\)This is not obvious from the plots, but by ignoring the data from first \( 10 T_{Planck} \) one gets data that indeed are on straight lines in a doubly logarithmic plot.
It should be noted, however, that both of these papers had to introduce unknown constants, $k_2, k_3$ (in the notation of Starobinsky), which they then supposed to be both positive (Fischetti et al.) or $k_2 > 0, k_3 < 0$ (Starobinsky). These quantities should be determined from the summation over quantum degrees of freedom. Our approach has no such arbitrary constants since we start from first principles, and hence these constants take on definite values which apparently are consistent with Starobinsky’s results.

Moreover, both of these early papers take the energy momentum tensor to obey $p = \frac{1}{3}\rho$, which is not necessarily true for quantum corrections, and moreover do not violate any energy conditions as the true quantum field theoretical vacuum supposedly does. Since we use the Casimir energy density our energy momentum tensor generally violates the classical energy conditions, as one would expect a quantum vacuum to do.

The FRW-space-time is somewhat similar to that of the hyperspatial tube [1] (which is a cylinder in 5d, i.e in 4d it has the topology $R \times R \times S^2$). Therefore the universal behaviour as well as the slow expansion rate is rather surprising: In analogy with the case of the hyperspatial tube (in one of the five qualitatively different scenarios), we get expansion by first decreasing the size of the universe a bit; at some point, then, a repulsive force sets in, and the expansion starts. Looking at the data in a double logarithmic plot shows this clearly. But in the case of the hyperspatial tube this repulsive force ultimately gave rise to inflation-like expansions followed by polynomial growth (cf figure 5b of [1]), this is not seen here.

It is of course still possible that such a type of expansion can occur and that we simply missed it due to our choice of initial conditions\(^5\) (or that some other residing field(s) than those considered in this paper could turn on such behaviour).

### 4 The Non-Minimally Coupled Case

The heat kernel equation for a non-minimally coupled scalar field is

$$ (\Box + \xi R)G_{\Box + \xi R}(x, x', \sigma) = -\partial_\sigma G_{\Box + \xi R}(x, x', \sigma) $$

\(^5\)The calculations done in this paper are ‘fully dynamical’ while those of [1] are quasi-static. However, we would not expect that this is what causes the difference.
where \( R = 6[a\ddot{a} + \dot{a}^2 + K]/a^2 \) \((K = 0, \pm 1)\) is the curvature scalar. We know how to solve this for \( \xi = 0 \) (i.e. minimal coupling), and the solution to the generally coupled heat kernel equation is now assumed to be of the form

\[
G^\square_{+\xi R}(x, x', \sigma) = G^\square(x, x', \sigma) F_\xi(x, x', \sigma)
\]  

(41)

where \( G^\square \) is the heat kernel for the minimally coupled case \((\xi = 0)\). Inserted into the heat kernel equation this gives

\[
(\Box + (\nabla \ln(G^\square)) \cdot \nabla + \xi R) F_\xi = -\partial_\sigma F_\xi
\]

(42)

where \( \nabla \ln G^\square \cdot \nabla F_\xi \) is short-hand for

\[
\nabla \ln G^\square \cdot \nabla F_\xi \equiv \frac{1}{G^\square \sqrt{g}} \partial_\mu (\sqrt{g}g^{\mu\nu})(\partial_\nu (G^\square F_\xi) + 2g^{\mu\nu} \partial_\mu \ln G^\square \partial_\nu F_\xi
\]

(43)

By considering the \( \xi R \)-terms as a kind of mass term, one would expect \( F_\xi \) to be of the form \( \exp(\text{...}) \). We will write

\[
F_\xi = e^{T_\xi}
\]

(44)

and then Taylor expand \( T_\xi \) in powers of \( \sigma \), i.e.

\[
T_\xi = \sum_{n=0}^\infty \tau_n \sigma^n
\]

(45)

The boundary condition implies \( \tau_0 \equiv 0 \), and we can arrive at a recursion relation for the remaining coefficients by inserting (44) and (13) into (43). But before we can collect powers of \( \sigma \) we must also Taylor expand \( \nabla \ln G^\square \) as this also depends on \( \sigma \). Thus, we must write

\[
\nabla^a \ln G^\square = \sum_{n=0}^{\infty} G^a_n \sigma^n
\]

(46)

We then obtain

\[
-\sum n \sigma^{n-1} \tau_n = 2 \sum_{n,m} (G^a_n \nabla_a \tau_m) \sigma^{n+m} + \sum_{n,m} (\nabla^a \tau_n)(\nabla_a \tau_m) \sigma^{n+m} + \sum_n (\Box \tau_n) \sigma^n + \xi R
\]

(47)
\[ \tau_{n+1} = -\frac{1}{n+1} \left[ 2 \sum_{n'=0}^{n} \left( G_{n'}^a \nabla_a \tau_{n-n'} + (\nabla^a \tau_{n'}) (\nabla_a \tau_{n-n'}) + \Box_0 \tau_n \right) \right] \quad n \geq 1 \] (48)

which leads to the following first few coefficients:

\[ \begin{align*}
\tau_0 & \equiv 0 \\
\tau_1 & = -\xi R \\
\tau_2 & = \xi G_0^a \nabla_a R + \frac{1}{2} \xi \Box R \\
\tau_3 & = -\frac{2}{3} \xi G_0^a \nabla_a \left( G_0^b \nabla_b R + \frac{1}{2} \Box R \right) + \\
& \quad \frac{1}{3} \xi G_1^a \nabla_a R - \frac{1}{3} \xi^2 (\nabla R)^2 - \\
& \quad \frac{1}{3} \xi \Box \left( G_0^a \nabla_a R + \frac{1}{2} \Box R \right)
\end{align*} \]

and so on. We notice first of all that \( \tau_n \) contains higher and higher derivatives of \( R \) as \( n \) grows. We furthermore notice that, in accordance with our expectations, \( T_\xi = -\xi R \sigma + O(\sigma^2) \). Many years ago it was argued by Parker and coworkers that such a term should be present in a resummed heat-kernel, see e.g. [13], but probably due to technical difficulties this suggestion doesn’t seem to have been taken up by other authors. By exponentiating not just the scalar curvature but also the remaining corrections to the heat kernel of the d’Alembertian in the way we do it here, we effectively circumvent those technical difficulties as will be apparent from what follows below.

To be able to actually evaluate these coefficients, we need to know \( G_n^a \). One easily sees (by inserting the spectral decomposition of the heat kernel into equation (46) and putting \( \sigma = 0 \)) that

\[ G_0^a = \nabla^a \ln \sum_{\lambda} \psi_{\lambda}(x) \psi^*_{\lambda}(x') \] (49)

where \( \psi_{\lambda} \) denotes the eigenfunctions of \( \Box \). Now, this sum is actually a sum over projections when \( x = x' \). As the eigenfunctions are supposedly complete this means that the sum equals unity, whereby

\[ G_0^a = 0 \] (50)

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The next coefficient is found by differentiating once with respect to $\sigma$ and then putting $\sigma$ to zero. It is

$$G_1^a = \nabla^a \left( \frac{\sum \lambda \psi_\lambda(x) \psi^*_\lambda(x')}{\sum \lambda \psi_\lambda(x) \psi^*_\lambda(x')} \right)$$  \hspace{1cm} (51)$$

Again, along the diagonal $x = x'$ the denominator is unity. The numerator is actually a spectral decomposition of the d’Alembertian, and hence, inside an integral (such as that appearing in the definition of the $\zeta$-function), we can substitute $\nabla^a \Box$ for $G_1^a$ (possibly up to boundary terms, which we will simply throw away).

From a physical point of view, it is reasonable to only include first and second derivatives of the scalar curvature $R$; higher derivatives would be higher order quantum effects of the gravitational background, and we are only treating a semi-classical approximation to quantum gravity. With higher order derivatives of $R$ discarded we thus have

$$\tau_3 \approx -\frac{1}{3} \xi^2 (\nabla R)^2$$ \hspace{1cm} (52)

$$\tau_n \approx 0 \hspace{0.5cm} n \geq 4$$  \hspace{1cm} (53)$$

leading finally to the formula

$$G_{\Box + \xi R}(x, x, \sigma) \approx G_{\Box}(x, x, \sigma) e^{-\xi R_\sigma + \frac{1}{2} \xi \Box R_\sigma - \frac{1}{8} \xi^2 (\nabla R)^2 \sigma^3}$$ \hspace{1cm} (54)$$

A very useful formula indeed, allowing us to express the heat kernel of the non-minimally coupled scalar field in terms of that of the much simpler minimally coupled field. Also, in physical terms, one can think of the first term in the exponent as due to the background field and the next two terms as due to higher order corrections of the gravitational field. Thus, as general relativity is only applicable to the 1 loop level, it is probably beyond the theoretical framework employed in this paper to expand the exponent to even higher order.

The equation (54) is actually much more general than we need for the Friedman-Robertson-Walker geometry, as in this case the curvature depends only upon time. In any case, the trick of writing $F_\xi$ as an exponential and the way we were able to eliminate certain derivatives was what allowed us to be able to find this result generalising the old result by Jack and Parker [13].
4.1 Effective Action and the Einstein Equations in Friedman-Robertson-Walker Space-Time for the Non-Minimally Coupled Scalar Field

In order to find the free energy of a non-minimally coupled scalar field, we must first obtain the $\zeta$-function, i.e. we must perform an integration over the fictitious parameter $\sigma$. Using the formula derived above, the task is to evaluate the following integral

$$\zeta_{\Box + \xi R} = \frac{1}{\Gamma(s)} \int_0^\infty \sigma^{s-1} \int G_\Box(x, x, \sigma) e^{-\xi R \sigma + \frac{1}{2} \xi \Box R \sigma^2 - \frac{1}{4} \xi^2 (\nabla R)^2 \sigma^3} \, d^4 x \, d\sigma$$

$$= \frac{1}{\Gamma(s)} \int_0^\infty \sigma^{s-1} \sum \int |\psi_\lambda(x)|^2 h_\lambda^{-1/2}(t) e^{-2 \int_0^t \sqrt{h_\lambda(t')} \, dt'} \times e^{-(\xi R + \lambda) \sigma + \frac{1}{2} \xi \Box R \sigma^2 - \frac{1}{4} \xi^2 (\nabla R)^2 \sigma^3} \, d^4 x \, d\sigma$$

$$= \frac{1}{\Gamma(s)} \int_0^\infty \sigma^{s-1} \sum \int h_\lambda^{-1/2}(t) e^{-2 \int_0^t \sqrt{h_\lambda(t')} \, dt'} - (\xi R + \lambda) \sigma + \frac{1}{2} \xi \Box R \sigma^2 - \frac{1}{4} \xi^2 R^2 \sigma^3 \, d^4 x \, d\sigma$$

where we have used the fact that $R$ only depends on the time $t$ and not on any of the other coordinates together with the orthonormality of $\psi_\lambda(x)$ to carry out the integrals over $\chi$ and the angles. Unfortunately we have not been able to perform this $\sigma$-integration explicitly, so instead we have to make do with an approximation. The expression appearing in the exponential has an easy interpretation: the first term is the flat space-time contribution, the second is the non-minimal coupling term while the remaining two terms are first loop quantum corrections to the gravitational background. It thus makes sense to only expand these latter terms to the first order, whereby equation (55) becomes

$$\zeta_{\Box + \xi R} \approx \frac{1}{\Gamma(s)} \sum \int dt \int_0^\infty d\sigma \sigma^{s-1} e^{-(\xi R + \lambda) \sigma} \left( 1 + \frac{1}{2} \xi \Box R \sigma^2 - \frac{1}{3} \xi^2 \Box R \sigma^3 \right) h_\lambda^{-1/2} e^{-2 \int_0^t \sqrt{h_\lambda(t')} \, dt'}$$

$$= \sum \int h_\lambda^{-1/2} e^{-2 \int_0^t \sqrt{h_\lambda(t')} \, dt'} \left( (\xi R + \lambda)^{-s} + \frac{1}{2} \xi \Box R (s + 1) (\xi R + \lambda)^{-s-2}ight.$$

$$\left. - \frac{1}{3} \xi^2 \Box R^2 (s + 1)(s + 2)(\xi R + \lambda)^{-s-3} \right) \, dt$$

$$\text{(56)}$$

Whence the effective action becomes

$$\Gamma_{\text{non-minimal}} = \zeta'(0) \text{ (57)}$$
\[ -\sum_\lambda \int h_{\lambda}^{-1/2} e^{-2 \int_0^t \sqrt{\sigma} \, dt'} \left( -\ln(\xi R + \lambda) + \frac{1}{2} \xi R^2(\xi R + \lambda)^{-2} \right) \left( -\int_0^t \frac{h_{\lambda}^{-1/2} \partial h_{\lambda}}{\partial a} \, dt' \right) \left( -\ln(\xi R + \lambda) + \frac{1}{2} \xi R^2(\xi R + \lambda)^{-2} \right) \]

where the curvature scalar and its derivatives are found to be

\[ R = 6a^{-2}(a\ddot{a} + \dot{a}^2 + K) \]
\[ \dot{R} = -2a^{-1}\dot{a}R + 6a^{-2}(3\dot{a}\ddot{a} + aa^{(3)}) \]
\[ \ddot{R} = (6a^{-2}\dddot{a}^2 - 2a^{-1}\dot{a})R - 24a^{-3}\dot{a}(3\dot{a}\ddot{a} + aa^{(3)}) + 6a^{-2}(3\dddot{a}^2 + 4\dot{a}a^{(3)} + aa^{(4)}) \]

here \( a^{(3)} \) and \( a^{(4)} \) denotes the third and fourth derivative of \( a \) with respect to time. With this the pressure, which once more is to enter the Einstein equations as \( \langle T_{33} \rangle \), becomes

\[ p = \frac{1}{a^2} \sum_\lambda \left[ \left( -\frac{1}{2} h_{\lambda}^{-3/2} \frac{\partial h_{\lambda}}{\partial a} - \int_0^t h_{\lambda}^{-1/2} \frac{\partial h_{\lambda}}{\partial a} \, dt' \right) \left( -\ln(\xi R + \lambda) + \frac{1}{2} \xi R^2(\xi R + \lambda)^{-2} \right) \right. \]
\[ + \frac{1}{2} \xi R^2(\xi R + \lambda)^{-2} - \frac{2}{3} \xi^2 \dot{R}^2(\xi R + \lambda)^{-3} \]
\[ + h_{\lambda}^{-1/2} \left( -(\xi R + \lambda)^{-1} \frac{\partial R}{\partial a} + \frac{1}{2} \xi \frac{\partial \dot{R}}{\partial a}(\xi R + \lambda)^{-2} \right) \]
\[ - \xi^2 \dot{R}(\xi R + \lambda)^{-3} \frac{\partial R}{\partial a} - \frac{4}{3} \xi^2 \dot{R} \frac{\partial \ddot{R}}{\partial a}(\xi R + \lambda)^{-3} \]
\[ + 2\xi^3 \ddot{R}^2(\xi R + \lambda)^{-4} \left( \frac{\partial R}{\partial a} \right) \left( -\int_0^t \sqrt{\sigma} \, dt' \right) e^{-2 \int_0^t \sqrt{\sigma} \, dt'} \]

The differentiations with respect to the scale factor appearing in this expression are

\[ \frac{\partial R}{\partial a} = -2a^{-1}R + 6a^{-2}\ddot{a} \]
\[ \frac{\partial \dot{R}}{\partial a} = 6a^{-2}\ddot{a}R - 12a^{-3}\dddot{a}\dddot{a} \]
\[ \frac{\partial \ddot{R}}{\partial a} = -(12a^{-3}\dddot{a}^2 - 2a^{-2}\dddot{a})R + (6a^{-2}\dddot{a}^2 - 2a^{-1}\dddot{a})(-2a^{-1}R + 6a^{-2}\dddot{a}) \]
\[ + 72a^{-4}a(3\dot{a}\ddot{a} + aa^{(3)}) - 12a^{-3}(3\dddot{a}^2 + 4\dot{a}a^{(3)} + aa^{(4)}) \]
\[ - 24a^{-3}aa^{(3)} + 6a^{-2}a^{(4)} \]
and
\[ \frac{\partial h}{\partial a} = -6 \dot{a} a^{-3} + \frac{3}{2} a^{-2} \ddot{a} - 2a^{-3} \lambda \]  
(66)

All this is to be inserted into the Einstein equations.

\[ a^{-2}(2\ddot{a}a + \dot{a}^2 + K) = p \]  
(67)

The reader will no doubt appreciate that these equations are not presented in their discretised form.

Again, as with the minimal coupled case, this result differs from the equations of motion found by Suen and Anderson [14] due to the use of the WKB-approximation, which speeds up convergence. But this time, there is also a discrepancy due to the presence of an \( e^{-\xi R_a} \) term in the heat kernel, which, as demonstrated elsewhere, should be present [13].

### 4.2 Performing the Simulations

We have carried out the same discretisation procedure as for the minimally coupled case and have simulated the evolution of the various Friedmann-Robertson-Walker geometries for the case of conformal coupling \( \xi = \frac{1}{6} \). The results are shown in figures 2a-c.

We see that this conformal coupling to the gravitational background makes the tendencies already inherent in the minimally coupled case much more significant. For \( K = +1 \), the closed universe, the tendency towards collapse is now even more pronounced, in fact all the chosen values gave rise to a collapse, whereas the expansion in the open and flat universes is seen to be faster.

Again, for large times, we get power-law behaviour (assuming no surprises for really large times), giving powers \( \sim 0.6 \) for \( K = 0 \) and 1.5 for \( K = -1 \). Note that this means that for a vacuous FRW-geometry with \( K = -1 \) and with a conformal scalar field as its only inhabitant, the age of the universe is no longer bounded by its Hubble time. Also note that for \( K = 0 \) our result deviates only little from that of Starobinsky (who finds a power of \( 2/3 \)) [10].

Let us conclude this section by comparing the evolution for the minimally and the conformally coupled scalar field cases, see figure 3, to gain a little intuition about the influence of the coupling to the background.

It is seen that the coupling to the background tends to speed up expansion/collapse and also that the onset of expansion/collapse occurs earlier.
Also some initial conditions that for $K = 0$ (seemingly) give rise to expansion in the minimally coupled case gives rise to collapse in the conformal case and vice versa some initial conditions that for $K = -1$ give rise to collapse in the minimally coupled case gives rise to expansion in the conformal case.

5 Relating the Casimir Effect of (Spin 1) Abelian Gauge Bosons to That of a Scalar Field

In order to be able to actually carry out the computations for higher spins as well (and thereby eventually being able to handle realistic field theories such as the standard models or various GUTs), we want to derive some relationships between the heat kernels, and thus the $\zeta$-functions, for vector bosons and Dirac fermions on the one hand to that of the scalar field case on the other. As the case of gauge bosons carries some resemblance to that of the non-minimally coupled scalar field we first consider that case.

We can derive an expression for the effective action for a spin-1 field, even in the non-Abelian case, by a simple extension of the technique used for the non-minimally coupled scalar field, but we need a way to handle the self-interaction (the non-linear, non-Abelian terms in the field strength tensor), and we will introduce a way of determining meanfields that allows us to handle such non-linearities (consequently, we could then also handle, say, $\phi^4$ theory in this manner).

The self-interaction is treated by a mean field approximation – the mean field itself being expressible in terms of heat kernels. Due to the implicit complexity of the solution we have not been able to carry out the simulations for vector fields, however. But it is important to emphasise that this trick of using a mean field coupled to the way we compute heat kernels for non-minimally coupled scalar fields, actually allows us to find an expression for the heat kernel of a Yang-Mills theory, and consequently also the effective action of non-Abelian theories.

In order to keep the notation as simple as possible we have left out the ghost contribution coming from the over-counting of degrees of freedom in the functional integral. In Lorentz gauge, the ghost contribution is simply
-2 times the effective action for a minimally coupled scalar-field. It is thus straightforward to include this correction at the end.

When considering a Yang-Mills field in a curved space-time then, in order to obtain the field strength tensor the naive guess is to replace the derivatives of the Minkowski space field strength tensor with covariant derivatives which is not correct as this leads to a non-gauge covariant expression (even though, accidentally, it gives the right answer in the case of abelian fields and no background torsion). Instead proceed by considering the full theory of Dirac fermions interacting minimally with the gauge fields as well as with the gravitational field. In order to preserve local gauge and Lorentz covariance construct a covariant derivative of the form

\[ D_m = e_m^\mu (\partial_\mu + \frac{i}{2} \omega_{\mu}^{pq}(x)X_{pq} + igA_{\mu}^a(x)T_a) \] (68)

where \( e_m^\mu \) is the vierbein (local base vectors of a freely falling observer), \( \omega_{\mu}^{pq}(x) \) is the spin connection being the gravitational analogue of the gauge field \( A_{\mu}^a(x) \) and \( X_{pq} \) the corresponding Lorentz group \( (SO(3,1)) \) generators analogous to the gauge group generators \( T_a \). Greek indices refer to curvilinear coordinates while Latin indices refer to local Lorentz coordinates (of a freely falling observer) and are also used for gauge group indices (it should be clear from context: in general we will use small Latin letters from the beginning of the alphabet to denote gauge indices, whilst reserving letters from the last half of the alphabet for use as Lorentz indices).

As in flat space field theory the gauge field strength tensor \( F_{mn}^a \) is obtained from the commutator of the covariant derivatives

\[ [D_m, D_n] = S_{mn}^q(x)D_q + \frac{i}{2} R_{mn}^{pq}(x)X_{pq} + iF_{mn}^aT_a \] (69)

(where \( S_{mn}^q(x) \) is the torsion and \( R_{mn}^{pq}(x) \) is the Riemann curvature tensor). Using the commutation relations

\[ [\partial_\mu, \partial_\nu] = 0 \] , alternatively \[ [\partial_m, \partial_n] = (\partial_m e_\mu^n - \partial_n e_\mu^m) e_\mu^p \partial_p \] (70)
\[ [T_a, T_b] = i f_{abc} T^c \] (71)
\[ [X_{mn}, X_{pq}] = -i \eta_{mp}X_{nq} + i \eta_{np}X_{mq} - i \eta_{mq}X_{np} - i \eta_{mq}X_{np} \] (72)

(\( \eta_{mn} \) being the flat metric) the normalisations

\[ Tr(T_a T_b) = \frac{1}{2} \delta_{ab} \] (73)
\[ Tr(X_{mn} X_{pq}) = -16(\eta_{mp} \eta_{nq} - \eta_{mq} \eta_{np}) \] (74)
and the action of the $SO(3,1)$ generators upon the latin index $m$

$$X_{pq} e_m^\rho = i \eta_{pm} e_q^\rho - i \eta_{qm} e_p^\rho \quad (75)$$

one obtains (after a lengthy calculation) the field strength tensor

$$F_{mn}^a = e_m^\mu e_n^\nu (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + igf_{bc}^a A_\mu^b A_\nu^c) \quad (76)$$

The generating functional of the full theory of a fermion interacting with a non-abelian gauge field is

$$Z = \int \mathcal{D}A_\mu \int D\bar{\psi} D\psi e^{S_{\text{gaugefield}} + S_{\text{fermion}}} \quad (77)$$

$$= \int \mathcal{D}A_\mu \int D\bar{\psi} D\psi e^{-\frac{1}{4} \int F_{mn}^a F^{mn}_a dx^\mu + i \int \bar{\psi} \gamma^m D_m \psi dx^\mu} \quad (78)$$

Referring back to equation (78) we see that the fermion part of the action contain reference to the gauge field so that one a priori cannot carry out the two integrations independently. One could probably treat the fermion action as a source term when doing the gauge field integration and then subsequently do the fermion integration. Instead however, we are going to make a mean field approximation to $A_\mu$ in the fermionic integral, as well as in the higher order terms of the bosonic integral (see later). Thus we can consider the bosonic and the fermionic parts independently (the meanfield in the fermionic path integral is just a function of space-time variables).

The bosonic part of the generating functional then becomes:

$$Z = \int \mathcal{D}A_\mu e^{-\frac{i}{4} \int F_{mn}^a F^{mn}_a dx^\mu}$$

$$= \int \mathcal{D}A_\mu e^{-\frac{g^2}{4} \int d^4x \left[ \partial_\mu A_\nu^b T_b \partial_\nu A_\mu^c T_c - \partial_\mu A_\nu^b T_b \partial_\nu A_\mu^c T_c - g f_{def} \partial_\mu A_\nu^b T_b A_\mu^c A_\rho^d T_f + g f_{def} \partial_\mu A_\nu^b T_b A_\mu^c A_\rho^d T_f - g f_{def} A_\mu^b A_\rho^d T_f \partial_\nu A_\nu^c T_e + g f_{def} A_\mu^b A_\rho^d T_f \partial_\nu A_\nu^c T_e + \frac{1}{2} g^2 f_{def} A_\mu^b T_c A_\nu^c A_\rho^d A_\nu^e T_f \right] \eta^m \eta^n} \quad (79)$$
Using the commutation relation (70) and the normalisation (74) this can be rewritten as

\[
\int \mathcal{D}A_{\mu} \exp \left( -\frac{g^2}{4} \int d^4 x \left[ A_n^a (\partial_m e^m_n - \partial_n e^m_\mu) e_\mu^p \partial_p + \partial^n \partial_m A^m_a - A^m_a \partial_m \partial^m A^a_n + \frac{1}{2} g^2 f_{abc} f_{def} A^a_m A^b_m A^n_{ne} \right] \right) \]

(80)

where the last part of the first term can be eliminated by applying the Lorentz condition, \( \partial_m A^m_a = 0 \). Now make the following mean field approximation to the path integral:

\[
Z = \int \mathcal{D}A_{\mu} \exp \left( -\frac{g^2}{4} \int d^4 x \left[ A_n^a (\partial_m e^m_n - \partial_n e^m_\mu) e_\mu^p \partial_p A^m_a - A^m_a \partial_m \partial^m A^a_n + \frac{1}{2} g^2 f_{abc} f_{def} A^a_m A^b_m A^n_{ne} \right] \right) \]

(81)

### 5.1 Solving the Heat Kernel Equation

The full heat kernel equation then is

\[
\frac{g^2}{4} \left[ \delta_b^a \delta^m_p \partial_p \partial^p - \delta_b^a (\partial_r e^{m\mu} - \partial^m e^\mu_r) e_\mu^p \partial_p - (g f_{bc}^a (\partial_r A^{mc} - \partial^m A^c_r)) \right] C^{n(a)}_{m(b)}(x, x', \sigma) = -\partial_\sigma G^{m(a)}_{n(b)}(x, x', \sigma) \]

(82)

or, in short hand notation

\[
\frac{g^2}{4} \left[ \delta_b^a \delta^m_k \Box_0 - \delta_b^a \mathcal{E}^{mp} \partial_p - f_{k(b)}^m (\langle A \rangle) \right] C^{n(a)}_{m(b)}(x, x', \sigma) = -\partial_\sigma G^{m(a)}_{n(b)}(x, x', \sigma) \]

(83)

with

\[
\mathcal{E}^{mp} = (\partial_n e^{m\mu} - \partial^m e^\mu_n) e_\mu^p \]

(84)

Now, in order to be able to make use of the WKB solution from the scalar case, we must replace the “flat” d’Alembertian \( \Box_0 \equiv \partial_\mu \partial^\mu \), by the proper

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\(^6\) The quantity \( \mathcal{E}^{mp}_{mn} \) is the structure coefficients of the algebra of “flat” derivatives \( \partial_m \). Thus it is therefore not surprising that it occurs here.
d’Alembertian operator on a scalar field, $\Box$. From the definition (3) with $g_{\mu\nu} = \eta_{ab} e_a^\mu e_b^\nu$ and $e = \sqrt{g}$ being the vierbein determinant, it follows that

$$\Box g = \Box - \frac{1}{e} \partial_\mu (ee^\mu p) \partial^p$$

(85)

Inserting this into (83) we get

$$\left[ \delta_b^a \delta^m_k \Box - \delta_b^a \tilde{\mathcal{E}}_{k}^{mp} \partial_p - f_{k(b)}^{m(a)} (\langle A \rangle) \right] G_{n(b)}^{k(a)}(x,x',\bar{\sigma}) = -\partial_\sigma G_{n(b)}^{m(a)}(x,x',\bar{\sigma})$$

(86)

where we have also rescaled $\sigma \to \bar{\sigma} = g^2 \sigma / 4$, in order to get rid of the overall factor of $\frac{g^2}{4}$ in (83) and also introduced $\tilde{\mathcal{E}}$ defined by

$$\tilde{\mathcal{E}}_{k}^{mp} = \mathcal{E}_{k}^{mp} + \frac{1}{e} \partial_\mu (ee^\mu p \delta_k^m)$$

(87)

The first order term is eliminated, as before, by the substitution

$$G = \tilde{G} e^{\frac{1}{2}\int F_{n}^{mp}dx}$$

(88)

where $G$ is a tensor but the factors on the right side of the equation independently seen are not, because the exponent is not. This leads to the following equation

$$\left[ \delta_b^a \delta^m_k \Box + \frac{1}{2} \partial_p \tilde{\mathcal{E}}_{n}^{mp} + \frac{1}{4} \delta_k^m \tilde{\mathcal{E}}_{np} + f_{r(b)}^{m(a)} (\langle A \rangle) \right] G_{n(c)}^{r(b)}(x,x',\sigma) = -\partial_\sigma G_{n(c)}^{m(a)}(x,x',\sigma)$$

(89)

which is of the form

$$\delta_c \left[ \delta_r^m \Box + O_{r(c)}^{m(a)} \right] G_{n(b)}^{r(c)}(x,x',\bar{\sigma}) = -\partial_\sigma G_{n(b)}^{m(c)}(x,x',\bar{\sigma})$$

(90)

where $O = O(\langle A(x) \rangle, x)$ is a matrix-valued function of $x$. The heat kernel becomes (once more) a matrix-valued function which we can assume to be of the form

$$G_{n(b)}^{m(a)}(x,x',\bar{\sigma}) = G_\Box(x,x',\bar{\sigma})(G(x,x',\sigma))^{m(a)}$$

(91)

where $G_\Box$ denotes the heat-kernel of $\Box$ and $T$ is some matrix $(T_{n(b)}^{m(a)})$. Inserting this expression for the heat-kernel into the heat equation we arrive at an equation for $T_{n(b)}^{m(a)}$

$$\Box T_{n(b)}^{m(a)} + (\partial_p T_{k(c)}^{m(a)}) (\partial^p T_{n(b)}^{k(c)}) - 2\partial_\sigma \ln (G_\Box) \partial^p T_{n(b)}^{m(a)} + O_{n(b)}^{m(a)} = -\frac{\partial}{\partial \bar{\sigma}} T_{n(b)}^{m(a)}$$

(92)
where a summation over repeated indices is understood. The third term on
the left hand side vanishes along the diagonal \( x = x' \) by the argument given
in section 4. We will furthermore write \( T_{n(b)}^{m(a)} \) as a Taylor series

\[
T_{n(b)}^{m(a)}(x, \bar{\sigma}) = \sum_{\nu=0}^{\infty} \tau_{n(b)}^{(\nu)m(a)}(x)\bar{\sigma}^\nu
\]

(93)

This leads to a recursion relation for the coefficients \( \tau_{n(b)}^{(\nu)m(a)} \)

\[
\Box \tau_{n(b)}^{(\nu)m(a)} + \sum_{\nu' = 0}^{\nu} (\partial_p \tau_{n(b)}^{(\nu-\nu')m(a)}) (\partial_p \tau_{n(b)}^{(\nu')k(c)}) = -(\nu + 1) \tau_{n(b)}^{(\nu)m(a)}
\]

(94)

or, as \( \partial_p = e^\mu_p \partial_\mu \) and \( e^\mu_p \) is the inverse of \( e_\mu^p \):

\[
\Box \tau_{n(b)}^{(\nu)m(a)} + \sum_{\nu' = 0}^{\nu} (\partial_p \tau_{n(b)}^{(\nu-\nu')m(a)}) (\partial_p \tau_{n(b)}^{(\nu')k(c)}) = -(\nu + 1) \tau_{n(b)}^{(\nu)m(a)}
\]

(95)

Due to the boundary condition one has

\[
\partial_p \tau_{n(b)}^{(0)m(a)} = \frac{1}{2} \partial_p E^{mp}
\]

(96)

and because of this and equation (92), the next coefficient becomes

\[
\tau_{n(b)}^{(1)m(a)} = \delta_b^a \partial_p E^{mp} - \frac{1}{2} \delta_b^a \eta_{pq} E^{mp} E^{kq} + f_{n(b)}^{m(a)} (\langle A \rangle)
\]

(97)

and subsequently, cf equation (94), the next few coefficients are seen to be

\[
\tau_{n(b)}^{(2)m(a)} = -\frac{1}{2} \delta_b^a \Box \partial_p E^{mp}
\]

\[
+ \frac{1}{4} \delta_b^a (\partial_q \partial_p E^{mp} E^{kq})
\]

\[
+ \frac{1}{4} \delta_b^a (\partial_q \partial_p E^{mp} E^{kq})
\]

\[
- \frac{1}{8} \delta_b^a \partial_q (E^{mp} E^{kq} \eta_{pr})
\]

\[
+ \frac{1}{4} \delta_b^a E^{mp} \partial_q \partial_p (E^{kq})
\]

(25)
\[ -\frac{1}{8} \delta^a \mathcal{E}_k \partial_q \mathcal{E}_n \mathcal{E}_o \eta_{lp} \]
\[ -\frac{1}{2} \Box f_{m(b)} (\langle A \rangle) \]
\[ -\frac{1}{2} \partial_p f_{m(b)} (\langle A \rangle) \mathcal{E}_n^k \]
\[ -\frac{1}{2} \mathcal{E}_n^m p \partial_p f_{n(b)} (\langle A \rangle) \]
\[ \approx -\frac{1}{2} \Box f_{m(a)} (\langle A \rangle) \]
\[ -\frac{1}{2} \partial_p f_{m(a)} (\langle A \rangle) \mathcal{E}_n^k \]
\[ -\frac{1}{2} \mathcal{E}_n^m p \partial_p f^{k(a)}_{n(b)} \] (98)

and

\[ \tau_{m(a) n(b)}^{(3)} = -\frac{1}{3} \left\{ \Box \left[ -\frac{1}{2} \partial_p f_{m(c)}^{k(a)} \mathcal{E}_n^l \right] \right. \]
\[ -\frac{1}{2} \mathcal{E}_n^k \partial_p f_{m(c)} \]
\[ -\frac{1}{2} \partial_p \left[ -\frac{1}{2} \partial_q f_{m(c)}^{k(a)} (\langle A \rangle) \cdot \mathcal{E}_n^l \right] \mathcal{E}_n^q \]
\[ -\frac{1}{2} \mathcal{E}_n^m p \partial_p f_{n(c)}^{k(a)} \]
\[ + \partial_p f_{k(b)} \partial_q f_{n(c)}^{k(a)} \]
\[ -\frac{1}{4} \partial_p \Box f_{k(b)} \mathcal{E}_n^l \]
\[ -\frac{1}{4} \mathcal{E}_n^m p \partial_p \Box f_{n(b)} \]
\[ + \partial_q f_{k(b)} \partial_q \partial_p \mathcal{E}_n^k \]
\[ + \partial_q \partial_p \mathcal{E}_n^m p \partial_p f_{n(b)} \]
\[ -\frac{1}{2} \left( \partial_q \mathcal{E}_n^m \right)^2 \partial_q f_{n(b)}^{k(a)} \]
\[ -\frac{1}{2} \partial_q f_{k(b)} \left( \partial_q \mathcal{E}_n^k \right)^2 \}
\[ + \text{terms of even higher order} \]
\[ \approx \partial_p f_{m(c)} \partial_q f_{n(c)}^{k(a)} \] (99)
and so forth.
For computational purposes one should, as indicated above, keep only the
terms that seem most important when doing simulations, i.e., the most im-
portant pure curvature term, the most important pure gauge field terms and
the most important mixing/gauge field-curvature coupling terms. Thus we
arrive at the following expression for the heat kernel:

\[ G_{m(a)}^{n(b)}(x, x, \bar{\sigma}) = G_{\Box}(x, x, \bar{\sigma}) \left( e^{\tau(x) \bar{\sigma} - \frac{1}{2} \tau^{(1)} x^2 - \frac{1}{3} \tau^{(3)} x^3} \right)^{m(a)}_{n(b)} \]  

(100)

This formula then expresses the heat kernel for a vector boson in a curved
space-time in terms of that of a scalar field in flat space-time and a matrix-
valued function depending on the curvature.
One could go on to develop a similar formula for the heat kernel for a spin
boson then, instead of a matrix-valued function, we would then get a rank
2 tensor-valued function.

5.2 Determining the Mean Field
In the above we substituted mean fields for the Yang-Mills field in the non-
abelian terms of the action. We now proceed to determine these mean fields.

\footnote{Note, however, that by discarding terms that are essentially higher and higher order
derivatives of the curvature one might lose a large contribution when close to singularities.
Also note the discussion at the end of this subsection.}
By definition the gauge mean field squared is

$$\langle A^a_m(x)A^b_n(x') \rangle \equiv \frac{\int A^a_m(x)A^b_n(x')e^{iS}DA}{\int e^{iS}DA} \quad (101)$$

where $S$ denotes the appropriate action. We want to carry out this functional integral. In order to do this, we note that in $n$-dimensional Euclidean space the following holds (see appendix for derivation)

$$\langle x_i x_j \rangle \equiv \frac{\int x_i x_j e^{-x^tMx}dx}{\int e^{-x^tMx}dx} = \frac{1}{2} \delta_{ij}(M^{-1})_{ii}$$

where $M$ is some symmetric matrix. This result is independent of the dimensionality $n$ and can hence be carried over to the infinite dimensional Hilbert space $L^2$. Thus, in the continuum case where we have $M \sim \frac{\delta^2 S}{\delta A^a_m(x)\delta A^b_n(x')}$, we have by inference

$$\langle A^a_m(x)A^b_n(x') \rangle = \frac{1}{2} \left( \frac{\delta^2 S}{\delta A^a_m(x)\delta A^b_n(x')} \right)^{-1} \equiv \frac{1}{2} \delta^{ab} \eta_{mn} \delta(x,x') D(x,x') \quad (102)$$

with $D$ being the Green’s function – (the inverse of the matrix $M = \frac{\delta^2 S}{\delta A^2}$).

Now, given an operator $A$, a simple relationship exists between its heat kernel and its corresponding Green’s function, namely

$$D(x,x') = -\int_0^\infty G_A(x,x',\sigma)d\sigma \quad (103)$$

---

8 Wherever the mean value of a any odd power of the gauge fields occur, we substitute $\sqrt{\langle A^2 \rangle}$ for $\langle A \rangle$. Of course, for $\langle A \rangle \neq 0$, this corresponds to spontaneous symmetry breaking of gauge symmetry (at least in the meanfield approximation) due to the fact that virtual particles interact with gravity while propagating (see e.g. [8]). A couple of things have to be said about this fact. Firstly, it has been noted that the effective action need not be gauge invariant beyond the 1-loop level [9]. Secondly, we do indeed prefer to see the symmetry breaking as an indication that a more complete theory would have to have some ‘mixing’ of gauge transformations and coordinate ones, i.e., one should only be able consider gauge transformations independently of local Lorentz ones as a ‘corner’ of the theory. Actually, at the higher level one might well find that masses occur naturally in the theory, as witnessed by the condensate formation seen in the mean field approximation. Thirdly, it is most satisfying that the fact that an observer in curved space do see an (Casimir) energy density is related to the fact that mean fields do attain finite, non-zero values in curved space vacuum.

9 One should note that this gives an interpretation of the corresponding $\zeta$-function at $s = 1$, since we get $\langle x^2 \rangle = \frac{1}{2} \zeta_M(1)$.
We can derive this formula by writing the Green’s function (i.e., the left inverse of the operator, $AD(x,x') = \delta(x,x')$) in terms of the eigenfunctions $\psi_\lambda$ of $A$:

$$D(x,x') = \sum_{\lambda} \frac{1}{\lambda} \psi^*_\lambda(x') \psi_\lambda(x)$$

and then using

$$\frac{1}{\lambda} = - \int_0^\infty e^{-\lambda \sigma} d\sigma$$

and the spectral representation of the heat kernel to obtain (103). In order to obtain the mean field values $\langle A^a_m A^b_n \rangle$ we use, to a first approximation, that part of the action which only contains the abelian terms (i.e., excluding the terms involving the structure constants, $f_{abc}$, of the Lie algebra). If one desires a better approximation, the procedure of this paragraph may be repeated using the full path integral with the first mean field approximation substituted for the gauge fields. This scheme can then be iterated until the desired accuracy is obtained. Towards the end of this subsection, we will describe the changes being made in the appropriate expressions during this iteration.

The heat kernel equation is as equation (89) but with $f_{r(b)}^{m(a)} \equiv 0$. The solution is then (still in matrix notation) to the chosen order

$$\tilde{G}(x, x', \bar{\sigma}) = G_{\square_a}(x, x', \sigma) e^{-A^{a} + \frac{1}{2}B^{a2} - \frac{1}{3}C^{a3}}$$

with (using (94)) the coefficients given by (to the order chosen)$^{10}$

$$A^m_n = -\partial_p E^m_{np} + \frac{1}{2} E^m_{kp} E^k_{np}$$
$$B^m_n = \Box A^m_n$$
$$C^m_n = (\partial_p A^m_k)(\partial_p A^k_n)$$

For the mean field we then get

$$\langle A^a_m(x) A^b_n(x') \rangle = \lim_{s \to 0} \frac{1}{2} \delta^{ab} \eta_{mn} \delta(x - x') \int_0^\infty d\sigma \sigma^{s-2} G(x,x,\sigma)$$

$^{10}$Even though the matrices $A,B,C$ do not a priori commute, there is no problem here to the chosen order of accuracy. It is conceivable, though, that one encounters difficulties when iterating the scheme for determining the meanfield because the gauge field dependent terms are not symmetric.
\[
\begin{align*}
&= \lim_{s \to 0} \frac{1}{2} \delta^{ab} \eta_{mn} \delta(x - x') \int_0^\infty d\sigma \sigma^{s-2} e^{-A \sigma + \frac{1}{2} B \sigma^2 - \frac{1}{3} C \sigma^3} \\
&\approx (4\pi)^{-2} \frac{1}{2} \delta^{ab} \eta_{mn} \delta(x - x') \lim_{s \to 0} \int_0^\infty \sigma^{s-2} e^{-A \sigma} \left(1 + \frac{1}{2} B \sigma^2 - \frac{1}{3} C \sigma^3\right) d\sigma \\
&= \frac{1}{32\pi^2} \delta^{ab} \eta_{mn} \delta(x - x') \\
&\quad \times \lim_{s \to 0} \left(A^{-(s-1)} \Gamma(s - 1) + \frac{1}{2} BA^{-(s+1)} \Gamma(s + 1) - \frac{1}{3} CA^{-(s+2)} \Gamma(s + 2)\right)
\end{align*}
\]

where we have used that \( A \) is related to the curvature, and hence we can make do with only including the first terms in the Taylor series defining \( \exp(\frac{1}{2} B \sigma^2 - \frac{1}{3} C \sigma^3) \), just like we did for the non-minimally coupled scalar field in an earlier section. It turns out, however, that the resulting integral is divergent in our case (as \( s \to 0 \) we get a term involving \( \Gamma(-1) \), since we do not have a \( 1/\Gamma(s) \) factor outside). We regularise this singularity simply by using the principal value of the meromorphic function \( \Gamma(z) \) at \( z = -1 \) (see [3]). Laurent expanding we get
\[
\Gamma(z) = \frac{-1}{z + 1} + (\gamma - 1) - \frac{1}{12}(z + 1)(6\gamma^2 - 12\gamma + \pi^2 + 12) + ...
\]

Since the principal part of a meromorphic function at a pole is just the finite part, we finally obtain
\[
\langle A^a_m(x) A^b_n(x) \rangle_{\text{reg}} = -\delta^{ab} \left([A(\gamma - 1) + \frac{1}{2} BA^{-1} - \frac{1}{3} CA^{-2}]\right)_{mn} \quad (110)
\]
\[
= -\delta^{ab} \left((A_{mp}(\gamma - 1) + \frac{1}{2} B_{mk}(A^{-1})^k_p - \frac{1}{3} C_{mk}(A^{-1})^k_q (A^{-1})^q_p)\right) \quad (111)
\]

Now, upon iteration of this we must obviously include the mean field in this. We then take the above result for \( \langle A^a_m A^b_n \rangle_{\text{reg}} \) and insert it into the action for the Yang-Mills field as an extra term. The heat kernel for this new action is then found (the mean fields merely corresponds to an extra curvature term) by the same methods as above, from which one gets a better approximation to the mean fields, and so on.
5.3 Using the method on other problems

A note on the generality of the above calculation of the mean field is in order: We removed the flat space d’Alembertian \( \Box_0 \) by introducing the curved space d’Alembertian \( \Box \) through equation (85). This was done to mimic the case of the non-minimally coupled scalar field and is possible because we know \( G_\Box \) from the section of the minimally coupled scalar field. However, for other problems one might not know \( G_\Box \). It should thus be remarked that had we continued using \( \Box_0 \) instead of \( \Box \) the functional form of the heat kernel equation (90) would have been unchanged and the derivations proceed as before. One can then take advantage of the fact that the heat kernel of the ‘flat’ space-time d’Alembertian, \( \Box_0 \), is known to be (see e.g. [1, 3, 8])

\[
G_{\Box_0}(x, x', \sigma) = \frac{1}{(4\pi \bar{\sigma})^2} e^{-\frac{\Delta(x, x')^2}{4\bar{\sigma}}} \tag{112}
\]

in four dimensions. Here \( \Delta(x, x') \) is the geodesic distance between the points \( x \) and \( x' \), i.e. \( \Delta = \int_{x'}^{x} ds \), which in Cartesian coordinates would be simply \( |x - x'| \). The exact form is not needed, only the fact that \( G_{\Box_0} \) taken along the diagonal \( x = x' \) is independent of \( x \). One should also note that one can only use freely falling coordinates for as well \( x \) as \( x' \) when these are sufficiently close to each other, i.e., both within some sufficiently small neighbourhood. But as we would only be interested, ultimately, in the limit \( x' \to x \), this approximation would be justified.

6 Heat Kernel Equation and Zeta-Function for Spin 1/2 Fermions

In the fermionic case the heat kernel equation becomes

\[
\nabla G_{\nabla}(x, x', \sigma) = -\frac{\partial}{\partial \sigma} G_{\nabla}(x, x', \sigma) \tag{113}
\]

\footnote{Actually, the formula in [3] is for Cartesian coordinates. Here we have brought it on a covariant form. This form is the simplest possible form for the heat kernel compatible with the equivalence principle.}
Now assume that we know the eigenfunctions of $\nabla$ i.e. $\nabla(\psi_\lambda)_\alpha = \lambda(\psi_\lambda)_\alpha$ and guess that the heat kernel has the following form
$$G_\nabla(x, x', \sigma) = \sum_\lambda e^{\alpha\beta}(\psi_\lambda(x'))^*_\alpha(\psi_\lambda(x))_\beta e^{g(\lambda)\sigma}$$
(114)

which inserted into the heat kernel equation gives
$$\nabla G_\nabla(x, x', \sigma) = \sum_\lambda \nabla e^{\alpha\beta}(\psi_\lambda(x'))^*_\alpha(\psi_\lambda(x))_\beta e^{g(\lambda)\sigma} = -\frac{\partial}{\partial \sigma} G_\nabla(x, x', \sigma)$$
(115)

provided that $[\nabla, e^{\alpha\beta}] = 0$ and $g(\lambda) = -\lambda$.

To establish the link with the scalar case make exactly the same exercise for the operator $\nabla^2$ i.e. assume that the heat kernel for $\nabla^2$ is
$$G_{\nabla^2}(x, x', \sigma) = \sum_\lambda e^{\alpha\beta}(\psi_\lambda(x'))^*_\alpha(\psi_\lambda(x))_\beta e^{h(\lambda)\sigma}$$
(116)

and insert this into the heat kernel equation to get (using $\nabla(\psi_\lambda)_\alpha = \lambda(\psi_\lambda)_\alpha$ twice)
$$\nabla^2 G_{\nabla^2}(x, x', \sigma) = \sum_\lambda \lambda^2 e^{\alpha\beta}(\psi_\lambda(x'))^*_\alpha(\psi_\lambda(x))_\beta e^{h(\lambda)\sigma}$$
$$= -\frac{\partial}{\partial \sigma} G_{\nabla^2}(x, x', \sigma)$$
(117)

If one choose to normalise the spinor eigenfunctions as follows
$$\sum_\lambda \int \sqrt{g} d^4 x \sqrt{g} d^4 x' e^{\alpha\beta}(\psi_\lambda(x'))^*_\alpha(\psi_\lambda(x))_\beta \delta(x - x') = \delta(\lambda - \lambda')$$
(118)

(if $e^{\alpha\beta}$ is chosen to be the Kronecker delta $\delta^{\alpha\beta}$ then this is the ordinary normalisation) then the zeta-functions are related as follows (cf equation (115));
$$\zeta_{\nabla^2}(s) = \frac{1}{\Gamma(s)} \int d\sigma \sigma^{s-1} \sum_\lambda \int G_{\nabla^2}(x, x, \sigma) \sqrt{g} d^4 x$$
$$= \sum_\lambda \frac{1}{\Gamma(s)} \int d\sigma \sigma^{s-1} e^{-\lambda^2 \sigma}$$
$$= \sum_\lambda \lambda^{-2s}$$
$$= \zeta_{\nabla}(2s)$$
(119)

---

12 choosing another normalisation trivially alters equation (117).
6.1 Heat Kernel Equation and Zeta-Function for Free Spin 1/2 Fermions

Now note that for a free fermion field the covariant derivative is

$$\nabla = \gamma^m e_m^\mu (\partial_\mu + \frac{i}{2} \omega_\mu^{pq}(x) X_{pq})$$  \hspace{1cm} (120)

Representing the $SO(3,1)$ generators in terms of the sigma matrices, $\sigma_{pq} = \frac{i}{4} [\gamma_p, \gamma_q]$, one obtains for the derivative squared

$$\nabla^2 = (\Box + \xi_f R) \cdot 1_4$$  \hspace{1cm} (121)

(where $1_4$ is the four dimensional unit matrix, and $\xi_f = \frac{1}{8}$) establishing the link between the scalar and the fermion cases if one remembers to include a factor of 4 (one for each spinor component) in the Dirac case (corresponding to taking the trace over the unit matrix). Thus we find

$$\zeta_{\nabla}(s) = \zeta_{\nabla^2}(\frac{1}{2}s) = 4^{\zeta_{\text{scalar}}}(s)$$  \hspace{1cm} (122)

This is our final result for free Dirac spinors. We have now arrived at formulas expressing the heat kernels of free higher spin fields in terms of that of the scalar field. We notice, however, that in the fermion case we arrive at a non-minimally coupled scalar field equation. The above result is the result we will use when performing the simulations. The case of interacting fields is discussed in the appendices.

6.2 Performing the Simulations

Due to the simple relationship between the $\zeta$-functions for a non-minimally coupled scalar field and the Dirac field, the simulations of evolution of Friedman-Robertson-Walker geometries due to vacuum fluctuations of fermionic fields can be carried out quite easily. As the functional integral giving the free energy is now over Grassmannian variables, the sign of the energy will be different (we will get the determinant raised to the power of plus one half and not minus one half as for bosons), due to the abovementioned simple relationship we will furthermore get a factor four (one for each spinor component) and a factor $1/2$ from the argument of the $\zeta$-function.
These changes, however, are easily performed and we can carry out the simulations with very little change to the program for non-minimally coupled scalar fields. We ran the simulations with an initial value of the scale factor fixed at $5L_{\text{Planck}}$. The results are, for $K = 0, \pm 1$ plotted in figure 4. Note that all $K = +1$ universes either collapse or seem to be on the verge of collapse and that the evolution for $K = 0$ and $K = -1$ tends to converge. Again, this latter part of the expansion is polynomial in time, the corresponding power is $\sim 0.7$. Also note that rather contrary to what one might expect due to the fact that the pressure has the opposite sign of that of the case of a (conformal or scalar) scalar field (cf formulas (7) and the corresponding one for Grassmann fields for which the power is positive), the behaviour is not qualitatively very different in the two cases. Actually, the fermionic case looks rather similar to the minimally coupled scalar field.

7 Conclusion

We have derived rather simple expressions for the effective actions for quantum fields of spin zero, one-half and one in a Friedman-Robertson-Walker background, thereby obtaining the equations of motion governing the gravitational degrees of freedom i.e. the scale factor $a(t)$ of a vacuous FRW universe. Or, in other words, the Einstein equation now describes the coupling of the zero-point energy (or pressure), the so-called Casimir energy, to the Einstein tensor, i.e. we consider the evolution of space-time including back-reaction of gravity upon itself as mediated through virtual particles of (other) quantum fields. This is not the whole picture (direct gravitational self-interaction is of course not included) but probably is as far seeing gravitation as a background field (first quantisation) can bring us.

Looking at the plots from the simulations we see first of all that, irrespective of whether the residing field is minimally or conformally coupled scalar field or a Dirac fermion field, the closed universe $K = +1$ seems to be ruled out, as the simulations show a marked tendency to collapse in this case.

For $K = 0$ there is a tendency towards expansion according to a power law with power $\sim 0.6 - 0.7$, whatever the residing field.

For $K = -1$ however, the nature of the residing field is rather important: In all cases there is a marked tendency, at large times, towards expansion according to some power law, the power being $\sim 0.7$ for fermions, $\sim 0.8$ for
a minimally coupled scalar field and $\sim 1.5$ for a conformally coupled scalar field.

This indicates two things: First the evolution of even a vacuous space-time may depend crucially on its (virtual) content, i.e. on the residing fields. Secondly, the quantum field content might conspire to give expansion according to power laws with powers greater than 1. Thus, the age of a FRW-universe need not be less than its Hubble-time. Or stated differently; the local Hubble expansion could be larger than the one observed in the early universe (at large redshifts).

It should be noted that this gravitational back-reaction through quantum fields cannot be done away with, it should be included even in the simplest of models that attempts to describe the real universe. We failed to make simulations for the case where all fields of the standard model resides in the FRW-geometry so we abstain from guessing at the evolution of ”a real” FRW universe. But we do think that the age-problem of the universe (the problem of the large Hubble constant) is non-existent.

A The Zeta-Function for Spin 1/2 Fermions Minimally Coupled to a Gauge Field

For real world purposes one should consider the case where the fermion field couples to a gauge boson field. The gauge invariant and space covariant derivative is

$$D_m = \gamma^p \omega_{pq} (x) X_{pq} + ig A^a \gamma^p (x) T_a$$

(123)

Using the representation (??) and the relations

$$\gamma^p \gamma^q = \eta^{pq} + 2i \sigma^{pq}$$

$$\{ \gamma^m, \sigma^{pq} \} = i \eta^{mp} \gamma q - \eta^{mq} \gamma p$$

$$\{ \gamma^p \sigma_{mn}, \gamma^q \} = 2\gamma^{pq} \sigma_{mn} + \delta^q_m \delta^p_n - 2\delta^q_m \sigma^p_n - i \delta^q_m \delta^p_n + 2 \delta^q_m \sigma^p_n$$

(124) (125) (126)

one obtains, for the derivative squared

$$D^2 = \square + \xi f R + g^2 \gamma^p \gamma^q A^a \gamma^b T_a T_b + g \gamma^p \gamma^q ( \partial_q A^a_p ) T_a$$

(127)
\[
+ \frac{i}{2} g \gamma^m \sigma_{pq} \gamma^p \omega\mu e\gamma^m A^b T_b + \frac{i}{2} g \gamma^\mu \gamma^{m'} \sigma_{p'q'} A^a T_a \omega^{p'q'} e^\nu
\]
\[
= \Box + \xi J R + 2g \sigma^{pq} \epsilon_{pq} T_a + g^2 \eta \rho q \xi A^a T_a + g \xi J A^a T_a
\]
\[
+ \frac{i}{2} g \gamma^a T_a \left( \sigma_{pq} \omega^{pq} \epsilon\gamma^x A^a + i \omega^m \epsilon\gamma^m A^a - c\gamma^m A^a + d\gamma^m A^a - 4 \omega^m \epsilon\gamma^m A^a \right)
\]
\[
\equiv \Box + \xi J R + 2g \sigma^{pq} \epsilon_{pq} T_a + g^2 \eta \rho q \xi A^a T_a + G(A) \quad (127)
\]

To make this expression manageable note that putting
\[
G(A) = 0 \quad (128)
\]
is an allowed gauge condition. To show this one needs to demonstrate that 
\[
\det \left( \frac{\delta G}{\delta \omega} \right) \neq 0
\]
which is easily done. Perform an infinitesimal gauge transformation \( \delta A_a = D_m \omega^a \) (where \( \omega^a \) is the arbitrary function appearing in the transformation rule for fermions \( \psi \rightarrow \exp(ig\omega^a T_a) \psi \) in \( G \) to get
\[
\frac{\delta G}{\delta \omega^a} \equiv \eta^{mn} \partial_m D^a + \text{function}
\]
Since Lorentz gauge is an allowed gauge condition, we know that \( \det \partial_m D^m \neq 0 \), and adding a function cannot change this fact. Hence \( G(A) = 0 \) is good gauge condition.

Using the mean field approximation described in appendix A, the gauge field dependent terms simply becomes a function which is in principle no different from the \( \xi J R \) term to which it, for calculational purposes, can be added.

**B Calculation of \( \langle x_i x_j \rangle \)**

We want to derive an expression for \( \langle x_i x_j \rangle \) where \( x_i, x_j \) are components of an \( N \)-dimensional vector, and where
\[
\langle x_i x_j \rangle = \frac{\int x_i x_j e^{-x^t M x} d^N x}{\int e^{-x^t M x} d^N x}
\]
with \( M \) some symmetrical \( N \times N \)-matrix.

First note that the integral vanishes whenever \( i \neq j \). Secondly, that since \( M \) is symmetrical we can diagonalise it in which case the problem reduces to
the $N = 1$ case.
For $N = 1$ we have

$$\langle x^2 \rangle = \frac{1}{2} M^{-1} \quad (129)$$

For arbitrary $N$ we then have

$$\langle x_i x_j \rangle = \frac{1}{2} \delta_{ij} (M^{-1})_{ii} \quad (130)$$

where no summation over $i$ is to be performed. This is the needed result.

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Figure 1: Evolution of the scale-factor for: a) $K = +1$, b) $K = 0$, c) $K = -1$ with a minimally coupled scalar field.

Figure 2: Evolution of the scale factor for: a) $K = +1$, b) $K = 0$ and c) $K = -1$ with a conformally coupled scalar field.

Figure 3: Evolution of scale factor for the cases of the residing field being a minimally coupled (solid line) or a conformally coupled (dotted line): a) $K = +1$, b) $K = -1$.

Figure 4: Evolution of the scale factor for: a) $K = +1$, b) $K = 0$, c) $K = -1$ with a fermion field.