Recovery and the Data Processing Inequality for quasi-entropies

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Abstract

We prove a number of quantitative stability bounds for the cases of equality in Petz’s monotonicity theorem for quasi-relative entropies $S_f(\rho||\sigma)$ defined in terms of an operator monotone decreasing functions $f$; and in particular, the Rényi relative entropies. Included in our results are bounds in terms of the Petz recovery map, but we obtain more general results. The present treatment is entirely elementary and developed in the context of finite dimensional von Neumann algebras where the results are already non-trivial and of interest in quantum information theory.

1 Introduction

Quantum relative entropy was defined first by Umegaki in 1962 \cite{Umegaki1962} as a formal generalization of the classical relative entropy, also known as Kullback-Leibler divergence \cite{Kullback1951}. For two density matrices $\rho$ and $\sigma$ on a finite-dimensional Hilbert space $\mathcal{H}$, the quantum relative entropy of $\rho$ with respect to $\sigma$ is defined as

$$S(\rho||\sigma) = \text{Tr}(\rho \log \rho - \rho \log \sigma),$$

(1.1)

if the null space of $\sigma$ is contained in null space of $\rho$, and $+\infty$ otherwise. Quantum relative entropy can be viewed as a measure of distinguishability of two states in terms of large deviations theory for repeated measurements \cite{Csiszar1982, Hiai1994}. If two states are the same, the quantum relative entropy between them is zero, while the relative entropy between mutually singular states is infinity. The larger the value of the relative entropy, the easier it is to distinguish between two quantum states, and this is quantified by the results in \cite{Csiszar1982, Hiai1994}.

The relative entropy has a monotonicity property that is fundamental to information theory. In the classical case, this is relatively easy to prove, but in the non-commutative setting of quantum
information theory the result is much deeper. The quantum version of this monotonicity property is expressed by the inequality

$$S(\Phi(\rho)\|\Phi(\sigma)) \leq S(\rho\|\sigma)$$  \hspace{1cm} (1.2)$$

where $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, and $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is a completely positive trace preserving map (CPTP), also known as a quantum channel. The inequality (1.2) is the Data Processing Inequality (DPI). It was proved by Lindblad [20], building on work of Lieb and Ruskai [18]. The inequality states that the relative entropy can not increase after states pass through a noisy quantum channel (CPTP map); after passing through such a channel, the states become harder to distinguish.

Petz [22, 23] classified all states that lead to equality in DPI, showing that, for a given CPTP map $\Phi$, two states $\rho$ and $\sigma$ lead to equality in the monotonicity relation if and only if both $\rho$ and $\sigma$ are perfectly recovered by a map $R_\rho$, now known as a Petz recovery map. That is, equality holds in (1.2) if and only if $R_\rho(\Phi(\rho)) = \rho$ and $R_\rho(\Phi(\sigma)) = \sigma$. In a case of particular importance, the map $R_\rho$ has a very simple explicit form: Suppose that $\Phi : \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) \to \mathcal{B}(\mathcal{H}_2)$ is the partial trace $\text{Tr}_1$ over $\mathcal{H}_1$. In notation that will be useful in the more general setting discussed below, we write $\mathcal{M}$ for $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, $\mathcal{N}$ for $\mathcal{B}(\mathcal{H}_2)$, and for any density matrix $\tau \in \mathcal{M}$, we write $\tau_{\mathcal{N}}$ to denote $\Phi(\tau)$, which as explained below, may be regarded as a conditional expectation with respect to the normalized trace of $\tau$ given $\mathcal{N}$. Then for all $X \in \mathcal{B}(\mathcal{H}_2)$

$$R_\rho(X) = \rho^{1/2} (\rho_{\mathcal{N}}^{-1/2} X \rho_{\mathcal{N}}^{-1/2}) \rho^{1/2}.$$

(1.3)

It is evident from this formula that $R_\rho(\rho_{\mathcal{N}}) = \rho$, but that one cannot then expect, in general, that $R_\rho(\sigma_{\mathcal{N}}) = \sigma$. However, as Petz showed [23], this occurs if and only if $R_\sigma(\rho_{\mathcal{N}}) = \rho$. That is, the equality condition is symmetric in $\rho$ and $\sigma$.

These results motivated the following question: If the decrease of relative entropy after states pass through a quantum channel is small, how well can these states be recovered? Work on this question accelerated following a breakthrough result by Fawzi and Renner in 2015 [11]. They proved that if the strong subadditivity inequality (SSA) of Lieb and Ruskai (from which Lindblad derived his monotonicity theorem) is nearly saturated, then quantum Markov chain condition, known [12] to be necessary and sufficient for equality in SSA is also nearly satisfied, and they gave a precise quantitative version of this stability result. This result has numerous applications in quantum information theory, e.g. [24, 25, 31]. Further refinements of the monotonicity relation occurred later in, for example, [4, 7, 8, 15, 27, 28, 32, 33]. Most of these results involve a recovery channel in the lower bound, and even though it often derived from, or otherwise related to, the Petz recovery map, in no case is it the Petz map itself.

Very recently, Carlen and Vershynina [9] proved a sharp stability result for the DPI directly in terms of the Petz recovery map. This result shows, in quantitative terms, that if the decrease in the quantum relative entropy is small after two states $\rho$ and $\sigma$ pass a quantum channel, then the Petz recovery map $R_\rho$ approximately approximately recovers $\sigma$ as well as $\rho$. The precise statement involves the relative modular operator $\Delta_{\sigma, \rho}$ which for density matrices $\rho$ and $\sigma$ acting on $\mathcal{H}$, is the
operator action on the Hilbert space $\mathcal{B}(\mathcal{H})$ as follows:

$$\Delta_{\sigma,\rho}(X) = \sigma X \rho^{-1},$$

(1.4)

for all $X \in \mathcal{M}$ in the case that $\rho$ is invertible. This is the matricial version of an operator introduced in a more general von Neumann algebra context by Araki [3]. $\Delta_{\sigma,\rho}$ is evidently a positive operator on $\mathcal{M}$ (or $M_n(\mathbb{C})$) equipped with the Hilbert-Schmidt inner product $\langle X, Y \rangle_{HS} = \text{Tr}[X^*Y]$. When $\rho$ is not invertible, $\rho^{-1}$ should be interpreted as the pseudo-inverse of $\rho$: If $\Sigma(\rho)$ denotes the spectrum of $\rho$, and $P_{\lambda}$ denotes the corresponding spectral projection, the generalized inverse is $\rho^+ = \sum_{\lambda \in \Sigma(\rho) \setminus \{0\}} \lambda^{-1} P_{\lambda}$. It is easy to see that the eigenvalues of $\Delta_{\sigma,\rho}$ are of the form $\mu \lambda^{-1}$ where $\mu$ is an eigenvalue of $\sigma$ and $\lambda$ is a non-zero eigenvalue of $\rho$. (If $\phi$ and $\psi$ are corresponding eigenvectors, then $|\phi\rangle\langle\psi|$ is an eigenvector of $\Delta_{\sigma,\rho}$ with eigenvalue $\mu \lambda^{-1}$.) In particular, the operator norm of $\Delta_{\sigma,\rho}$ is given by

$$\|\Delta_{\sigma,\rho}\| = \|\sigma\|\|\rho^+\| = \max\{\mu : \mu \in \Sigma(\sigma)\} \max\{\lambda^{-1} : \lambda \in \Sigma(\rho) \setminus \{0\}\} < \infty.$$  

(1.5)

See the discussion below (1.7) for an explanation of why this formulation of what $\Delta_{\sigma,\rho}$ means in the degenerate case is the relevant one in this context.

For simplicity of notation, and consistency with standard usage, we generally write $\rho^{-1}$ for $\rho^+$, but wherever it appears in this paper, $\rho^{-1}$ is to be interpreted in this manner. This standard usage is non unreasonable: When there is an orthogonal projection $P$ such that $\text{Tr}[\rho P] = 0$ but $\text{Tr}[\sigma P] \neq 0$, there is an obvious simple test for distinguishing $\sigma$ from $\rho$ by making repeated observations corresponding to $P$.

With the relative modular operator now introduced, we state the result of [9]. In the setting of finite dimensional von Neumann algebras, take the CPTP map $\Phi$ to be the tracial conditional expectation from one von Neumann algebra $\mathcal{M}$ onto a von Neumann sub-algebra $\mathcal{N}$. Then

$$S(\rho||\sigma) - S(\Phi(\rho)||\Phi(\sigma)) \geq \left(\frac{1}{8\pi}\right)^4 \|\Delta_{\sigma,\rho}\|^2 \|\mathcal{R}_\rho(\Phi(\sigma) - \sigma)\|^4.$$  

(1.6)

Another version, with a slightly more complicated constant, has the roles of $\rho$ and $\sigma$ can be reversed on the right-hand side.

The new results in this paper concern bounds of this sort for other relative entropies that have proven to be of interest in quantum information theory, and in particular, Rényi relative entropies. The Umegaki relative entropy is a particular case of a class of $f$-relative quasi-entropies, defined by Petz [21] as follows, for an operator monotone decreasing function $f$ on $(0, \infty)$ and invertible states

$$S_f(\rho||\sigma) = \text{Tr}(f(\Delta_{\sigma,\rho})\rho) = \langle \sqrt{\rho}, f(\Delta_{\sigma,\rho})\sqrt{\rho} \rangle_{HS}.$$  

(1.7)

In the next section we recall the relevant aspects of the theory of of operator monotone functions. For now, the key point to notice is the we only use functions of $\Delta_{\sigma,\rho}$ applied to $\sqrt{\rho}$, which is of course orthogonal in the Hilbert-Schmidt inner product to the projector onto the null space of $\rho$. For this reason, with the relative modular operator defined using the generalized inverse $\rho^+$ in place of $\rho^{-1}, (1.7)$ is always meaningful even when $\rho$ is degenerate, and moreover, taking $f = - \log$
this formula yields the Umegaki relative entropy. Likewise, in the definition of the Petz recovery map $\rho_N^{-1/2}$ should be read as $(\rho_N^{+})^{1/2}$.

Another important example, Rényi relative entropy, $S_\alpha(\rho\|\sigma) = \frac{1}{\alpha-1} \log \text{Tr}(\rho^\alpha \sigma^{1-\alpha})$ for $\alpha > 0$, $\alpha \neq 1$, involves the power $f$-relative quasi-entropy obtained by taking the power function as $f(t) = t^{1-\alpha}$. It is well-known that Rényi relative entropy satisfies the monotonicity relation when $\alpha \in [0,2]$.

For the class of $f$-relative quasi-entropies it was shown [17, 21, 29] that they satisfy the monotonicity relation for any CPTP map $\Phi$ and an operator convex function $f$

$$S_f(\Phi(\rho)\|\Phi(\sigma)) \leq S_f(\rho\|\sigma).$$

The equality condition in the monotonicity relation for a large class of functions is discussed in [13], where authors show that the monotonicity inequality is saturated if and only if Petz map recovers both states perfectly.

We investigate the question of almost-perfect recoverability for $f$-relative quasi-entropies, and prove analogs of (1.6) for them. For example, for the Rényi entropies $S_\alpha$, $\alpha \in (0,1)$, we prove in Corollary 6.2 that for any density matrices $\rho$ and $\sigma$ such that $\sigma_N$ is non-degenerate,

$$S_\alpha(\rho\|\sigma) - S_\alpha(\rho_N\|\sigma_N) \geq \frac{1}{1-\alpha} \log \left(1 + \tilde{K}_{\alpha,\rho,\sigma} \max\{\|\mathcal{R}_\rho(\sigma_N) - \sigma\|_1, \|\mathcal{R}_\rho(\rho_N) - \rho\|_1\}^{6-2\alpha}\right),$$

where $\tilde{K}_{\alpha,\rho,\sigma}$ is constant specified in the corollary. Dropping the requirement that $\sigma_N$ is non-degenerate, we have similar bound with $\max\{\|\mathcal{R}_\rho(\sigma_N) - \sigma\|_1, \|\mathcal{R}_\rho(\rho_N) - \rho\|_1\}$ on the right side replaced by $\|\mathcal{R}_\rho(\sigma_N) - \sigma\|_1$, and a different constant.

## 2 Operator monotone functions

A function $f : (a, b) \to \mathbb{R}$ is operator monotone if for any pair of self-adjoint operators $A$ and $B$ on some Hilbert space that have spectrum in $(a, b)$, $f(A) - f(B)$ is positive semidefinite whenever $A - B$ is positive semidefinite. We say that $f$ is operator monotone decreasing on $(a, b)$ in case $-f$ is operator monotone. (Traditional usage is asymmetric and gives preference to monotone increase.) A Pick function is a function $f$ that is analytic on the upper half plane and has a positive imaginary part. For an open interval $(a, b) \subset \mathbb{R}$, $\mathcal{P}_{(a,b)}$ denotes the class of Pick functions that may be analytically continued into the lower half plane across the interval $(a, b)$ by reflection. Thus any $f \in \mathcal{P}_{(a,b)}$ is real on $(a,b)$. Moreover, letting $u(x,y)$ and $v(x,y)$ denote the real and imaginary parts of $f$, since $\partial v(x,0)/\partial y \geq 0$, the Cauchy-Riemann equations say that $f'(x) = \partial u(x,0)/\partial x \geq 0$. In fact, much more is true: K. Löwner’s Theorem of 1934 states that $f$ is operator monotone on $(a, b)$ if and only if it is the restriction of a function $f \in \mathcal{P}_{(a,b)}$ to $(a,b)$.

A function $f$ is operator convex on the positive operators in case for all positive semidefinite operators $A$ and $B$, and all $\lambda$ in $(0,1)$, $(1-\lambda)f(A) + \lambda f(B) - f((1-\lambda)A + \lambda B)$ is positive semidefinite, and $f$ is operator concave in case $-f$ is operator convex. It turns out that every operator monotone function is operator concave [6, Theorem V.2.5]. Moreover, a function that
maps \((0, \infty)\) onto itself is operator monotone if and only if it is operator concave. Thus a function \(f\) on \((0, \infty)\) is operator convex and operator monotone decreasing if and only if \(-f \in \mathcal{P}(0,\infty)\). One example of an operator monotone decreasing function on \((0, \infty)\) is \(f_0(x) = -\log x\), and a closely related family of examples is given by \(f_\alpha(x) = -x^\alpha\), \(\alpha \in (0,1]\). The theory of Pick functions is reviewed in the next section. Most important for us is the canonical integral representation of Pick functions.

Every function \(f \in \mathcal{P}(0,\infty)\) has a canonical integral representation \([10]\, \text{Chapter II, Theorem I}\)

\[
f(x) = ax + b + \int_0^\infty \left(\frac{t}{t^2 + 1} - \frac{1}{t + x}\right) \, d\mu_f(t),
\]

where \(a \geq 0\), \(b \in \mathbb{R}\) and \(\mu\) is a positive measure on \((0,\infty)\) such that \(\int_0^\infty \frac{1}{t^2 + 1} \, d\mu_f(t) < \infty\). Conversely, every such function belongs to \(\mathcal{P}(0,\infty)\).

2.1 Remark. To facilitate comparison with related formulas that may be more commonly used in mathematical physics, we have changed the sign of \(t\) from what it is in \([10]\); this results in a corresponding change in (2.3) below.

There is a simple way to determine \(a\), \(b\) and \(\mu\) corresponding to \(f\). The following formulas \([10]\, \text{Chapter II, p. 24}\) are readily verified.

\[
a = \lim_{y \uparrow \infty} \frac{f(iy)}{iy} \quad \text{and} \quad b = \text{Re} \left( f(i) \right).
\]

Next, if one defines the monotone increasing function \(\mu(x) := \frac{1}{2} \mu(\{x\}) + \mu((-\infty, x))\), according to \([10]\, \text{Chapter II, Lemma 2}\) we have that

\[
\mu(x_1) - \mu(x_0) = \lim_{y \downarrow 0} \frac{1}{\pi} \int_{x_0}^{x_1} \text{Im} \, f(-x + iy) \, dx.
\]

2.2 Example. Let \(f(x) = \log x\). Then \(\text{Re} \left( \log(i) \right) = 0\) and \(\log(iy)/(iy) = \left( \log y + i\pi/2 \right)/(iy) \to 0\) as \(y \uparrow \infty\), so that \(a = b = 0\). It is clear from (2.3) that \(d\mu(x) = \frac{1}{\pi} \lim_{y \downarrow 0} \text{Im} \, \log(-x + iy) \, dx = dx\). Thus,

\[
\log x = \int_0^\infty \left( \frac{t}{t^2 + 1} - \frac{1}{t + x} \right) \, dt
\]

Since the integral is readily evaluated, thus verifying the formula, this proves that the logarithm function is in fact operator monotone.

The integral representation (2.4) is equivalent to the familiar representation

\[
\log x = \int_0^\infty \left( \frac{1}{t + 1} - \frac{1}{t + x} \right) \, dt
\]

since \(\int_0^\infty \left( \frac{1}{t + 1} - \frac{t}{t^2 + 1} \right) \, dt = 0\).
2.3 Example. Let $f_\alpha(x) = x^\alpha$, $\alpha \in (0,1)$. Then evidently $a = \lim_{y \to \infty} f_\alpha(iy)/(iy) = 0$. Next, $f_\alpha(i) = \cos(\alpha \pi/2) + i \sin(\alpha \pi/2)$, and hence $b = \sin(\alpha \pi/2)$. Finally, for $x > 0$, $\lim_{y \to 0} \Im f(-x + iy) = x^\alpha \sin(\alpha \pi)$ so that $d\mu(x) = \pi^{-1} \sin(\alpha \pi) x^{\alpha} dx$. This yields the representation

$$x^\alpha = \sin(\alpha \pi/2) + \frac{\sin(\alpha \pi)}{\pi} \int_0^\infty t^\alpha \left( \frac{t}{t^2 + 1} - \frac{1}{1 + t + x} \right) dt$$

(2.5)

Since the integral is readily evaluated, thus verifying the formula, this proves that the function $f_\alpha(x)$ is in fact operator monotone.

The integral representation (2.5) is equivalent to the familiar representation

$$x^\alpha = \frac{\sin \alpha \pi}{\pi} \int_0^\infty t^\alpha \left( \frac{t}{t + 1} - \frac{1}{1 + t + x} \right) dt$$

by the same sort of calculation made in the previous example. The merit of the slightly more complicated representation (2.5) lies in the simple relation between $f_\alpha$ and $a$, $b$ and $\mu$.

3 Petz’s proof of the DPI for quasi entropies

Petz [21] used the properties of operator monotone decreasing functions to generalize Umegaki’s relative entropy, producing a family of quasi-entropies that share with the original Umegaki relative entropy its fundamental monotonicity property, as we now explain.

Let $M$ be a finite dimensional von Neumann algebra, which we may regard, for some $n \in \mathbb{N}$, as a subalgebra of $M_n(\mathbb{C})$, the von Neumann algebra of $n \times n$ matrices over $\mathbb{C}$. Let $\rho$ and $\sigma$ be two density matrices on $M$. That is, $\rho$ and $\sigma$ are positive and have unit trace. We shall frequently refer to density matrices $\rho$ as states identifying $\rho$ with the positive linear functions $X \mapsto \Tr[\rho X]$ on $M$. As noted above, the Umegaki relative entropy of $\rho$ with respect to $\sigma$, $S(\rho||\sigma)$, can be written as

$$S(\rho||\sigma) = \Tr[-\log(\Delta_{\sigma,\rho})] = \langle \rho^{1/2}, -\log(\Delta_{\sigma,\rho})\rho^{1/2} \rangle_{HS}.$$  

(3.1)

Let $N$ be a von Neumann sub-algebra of $M$, and let $\mathcal{E}_\tau$ be orthogonal projection with respect to the Hilbert-Schmidt inner product from $M$ onto $N$. This turns out to be a conditional expectation in the sense of Umegaki [30]. (See [2] for a detailed and elementary discussion of this topic.)

Lindblad proved [19] a fundamental monotonicity property of the Umegaki relative entropy, namely that with $\rho_N := \mathcal{E}_\tau \rho$ and $\sigma_N := \mathcal{E}_\tau \sigma$,

$$S(\rho_N||\sigma_N) \leq S(\rho||\sigma).$$  

(3.2)

In a subsequent paper [20], he showed, using the Stinespring Dilation Theorem [26], that this readily implies his more general monotonicity property mentioned above, namely that

$$S(\Phi(\rho)||\Phi(\sigma)) \leq S(\rho||\sigma),$$  

(3.3)

for any CPTP map $\Phi$ from $M$ to $N$, where now it is no longer necessary that $N$ be a sub-algebra of $M$. However, the extension is simple, and the main result is contained in (3.2).
Given an operator monotone decreasing function $f$ on $(0, \infty)$, Petz defined the $f$-relative quasi-entropy (a.k.a. $f$-divergence) $S_f(\rho||\sigma)$ by

$$S_f(\rho||\sigma) = \text{Tr}[f(\Delta_{\sigma,\rho})\rho] = \langle \rho^{1/2}, f(\Delta_{\sigma,\rho})\rho^{1/2} \rangle_{\text{HS}} .$$

Comparing with (3.1), we see that the generalization consists of replacing the specific operator monotone decreasing function, $-\log$, with a general function of this type. Petz then proved [21 Theorem 4] that Lindblad’s monotonicity property holds in his more general setting. That is, for all such $f$,

$$S_f(\Phi(\rho)||\Phi(\sigma)) \leq S_f(\rho||\sigma).$$

Again, the essence of the matter is in the case of tracial conditional expectations, from which the general cases again follows, and our focus is on the inequality

$$S_f(\rho_N||\sigma_N) \leq S_f(\rho||\sigma) .$$

Various special cases of the quasi-entropies had been considered earlier, for example, the Rényi relative entropies are closely related to what one obtains from the choice $f_\alpha(x) := -x^\alpha$, $\alpha \in (0, 1)$, as we discuss below in detail.

In the rest of the paper, $\| \cdot \|_p, p \in [1, \infty)$ denotes that Schatten $p$-trace norm; i.e, $\|X\|_p$ is the $\ell_p$ norm of the vectors of singular values of $X$. We simply write $\|X\|$ to denote the operator norm of $X$; i.e., the largest singular value of $X$.

We shall show below that for any $\beta \in (0, 1)$ and a very broad class of operator monotone decreasing functions $f$, that depend on parameter $c > 0$, there is an explicitly computable constant $K$ depending only on $\|\rho^{-1}\|$, $\beta$, and $f$ such that for $\beta \leq 1/2$,

$$\|\sigma_N^{\beta} \rho_N^{-\beta} \rho^{1/2} - \sigma^{\beta} \rho^{1/2} \|_2 \leq K(S_f(\rho||\sigma) - S_f(\rho_N||\sigma_N))^{\frac{\beta(1-\beta)}{1+2c(1-\beta)}} ,$$

while for $\beta \geq 1/2$,

$$\|\sigma_N^{\beta} \rho_N^{-\beta} \rho^{1/2} - \sigma^{\beta} \rho^{1/2} \|_2 \leq K(S_f(\rho||\sigma) - S_f(\rho_N||\sigma_N))^{\frac{1-\beta}{1+2c(1-\beta)}} .$$

In fact, we prove a more general, if somewhat more complicated, result in Theorem 4.2.

The Petz recovery map $R_\rho$ is defined as follows: For all $X \in \mathcal{N}$,

$$R_\rho(X) = \rho^{1/2}(\rho_N^{-1/2} X \rho_N^{-1/2})\rho^{1/2} .$$

It is evident form this formula that $R_\rho$ is a CPTP map, and that $R_\rho(\rho_N) = \rho$, which is the reason for the term “recovery map”. See [9] for an extensive and self-contained discussion of this map and the closely related Accardi-Cecchini coarse graining operator [11 2]. In what follows we use inequalities of the type (3.7) and (3.8) to obtain bounds on

$$\max\{\|R_\rho(\rho_N)||\sigma_N\| - \sigma||_1 , \|R_\rho(\rho_N) - \rho||_1\}$$

and $\|\sigma_N^{\beta} \rho_N^{-\beta} - \sigma^{\beta} \rho^{-\beta}\|_2$ in terms of $S_f(\rho||\sigma) - S_f(\rho_N||\sigma_N)$. In particular, we shall see that for a broad class of operator monotone decreasing functions $f$, not only is it the case that any one of them vanishes if and only if the others all vanish, but we can quantitatively relate their sizes.
4 Stability of $f$-relative quasi-entropy

In this section we examine the monotonicity property (3.6) of $f$-relative quasi-entropy for a broad class of operator monotone decreasing functions $f$.

4.1 Definition. A function $f \in P_{(0, \infty)}$ is regular in case the measure $\mu$ in the canonical integral representation of $f$ is absolutely continuous with respect to Lebesgue measure, and for each $S, T > 0$, there is a finite constant $C_{S,T}^f$ such that

$$dt \leq C_{S,T}^f d\mu(t)$$

(4.1)

on the interval $[1/S, T]$. An operator monotone decreasing function is regular if and only if $-f$ is regular.

The examples from the previous section show that the Pick functions $f_0(x) = \log(x)$ and $f_\alpha(x) = x^\alpha$, $\alpha \in (0, 1)$, are regular.

For every regular operator monotone decreasing function $f$, we produce a one parameter family of lower bounds on

$$S_f(\rho||\sigma) - S_f(\rho_N||\sigma_N)$$

(4.2)

in terms of

$$\|\sigma_N^\beta \rho_N^{-\beta} \rho^{1/2} - \sigma^\beta \rho^{1/2 - \beta}\|_2$$

(4.3)

for each $\beta \in (0, 1)$. The bounds will show, in particular, that the difference in (4.2) vanishes if and only if

$$\sigma_N^\beta \rho_N^{-\beta} = \sigma^\beta \rho^{-\beta}$$

(4.4)

for all $\beta \in (0, 1)$. Then, since for any strictly positive matrix $X$, $\beta \mapsto X^\beta$ is an entire analytic function, (4.4) is valid for all $\beta \in \mathbb{C}$. As we discuss later, this is closely related to a result of Petz, proved in a more general von Neumann algebra setting without assuming any finite dimensionality, but then of course, restricting to purely imaginary values of $\beta$, and expressing everything in terms of modular operators.

The main novelty of our work is that we prove a quantitative relation between the quantities in (4.2) and (4.3), and do not only concern ourselves with cases of equality. The reader who is familiar with the Tomita-Takesaki Theory will also see how to generalize a number of our results beyond the case in which $\mathcal{M}$ and $\mathcal{N}$ are finite dimensional, and we plan to return to this in later work. However, the results are new and interesting already in the present context, and it is therefore worthwhile to explain them in their simplest setting, which is in any case the main arena of quantum information theory.

4.2 Theorem. Let $\mathcal{N}$ be a von Neumann sub-algebra of a finite dimensional von Neumann algebra $\mathcal{M}$. Let $f$ be a regular operator monotone decreasing function, and $T > 0$.

1. For $\beta \leq 1/2$, define $T_L(\beta) := T$ and $T_R(\beta) := T^{\beta/(1-\beta)}$.

2. For $\beta \geq 1/2$, define $T_L(\beta) := T^{(1-\beta)/\beta}$ and $T_R(\beta) := T$. 
Define $C_{T,\beta}^f$ to be the least positive constant such that $dt \leq C_{T,\beta}^f \, d\mu_f(t)$ for $t \in [T_L(\beta)^{-1}, T_R(\beta)]$, noting that $C_{T,\beta}^f > 0$ since $f$ is regular. Then for all density matrices $\rho$ and $\sigma$ in $\mathcal{M}$, we have the following bounds (with $\rho_N^{-1}$ denoting the generalized inverse of $\rho_N$):

(1) for $\beta \leq 1/2$,

$$
\frac{\pi}{\sin \beta \pi} \| \sigma_{X}^{\beta} \rho_{N}^{-\beta} - \sigma^{\beta} \rho_{\sigma}^{1/2} \|_2 \\
\leq 2 \left( \frac{1}{\beta} + \| \Delta_{\sigma, \rho} \| \right) \frac{1}{T^\beta} + T^{1-\beta} \left( C_{T,\beta}^f \right)^{1/2} (S_f(\rho||\sigma) - S_f(\rho_N||\sigma_N))^{1/2}
$$

(4.5)

(2) for $\beta \geq 1/2$,

$$
\frac{\pi}{\sin \beta \pi} \| \sigma_{X}^{\beta} \rho_{N}^{-\beta} - \sigma^{\beta} \rho_{\sigma}^{1/2} \|_2 \\
\leq 2 \left( \frac{1}{\beta} + \| \Delta_{\sigma, \rho} \| \right) \frac{1}{T^\beta} + T^\beta \left( C_{T,\beta}^f \right)^{1/2} (S_f(\rho||\sigma) - S_f(\rho_N||\sigma_N))^{1/2}
$$

(4.6)

Note that for $\beta = 1/2$ both bounds in the theorem coincide. The proof of the theorem is given in Section 4.

Naturally, we wish to optimize in $T$, and shall do so for specific choices of $f$, so that $C_{T,\beta}^f$ is explicit. Note that in general, the function $T \mapsto C_{T,\beta}^f$ is, by construction, monotone non-decreasing. The right sides of (4.5) and (4.6) have the form

$$
\phi(T) := KT^{-k} + NT^n (C_{T,\beta}^f)^{1/2}
$$

(4.7)

for $K, N, k$ and $n > 0$. When $C_{T,\beta}^f$ grows like a power of $T$, as it will in examples discussed in the next section, we can absorb $(C_{T,\beta}^f)^{1/2}$ into the term $NT^n$, and then, after this reduction, the optimization is very simple. For later use we record the following simple lemma whose proof is elementary calculation.

4.3 Lemma. Let $K, N, k, n > 0$. Then the minimum of the function $KT^{-k} + NT^n$ on the interval $(0, \infty)$ is

$$
\left( \frac{1}{k} + \frac{1}{n} \right) (kK)^{-n}(nN)^{k/n}
$$

We then have the following Corollary of Theorem 4.

4.4 Corollary. Let $f$ be a regular operator monotone decreasing function, and $T > 0$.

(1) For $\beta \leq 1/2$, define $T_L(\beta) := T$ and $T_R(\beta) := T^{\beta/(1-\beta)}$.

(2) For $\beta \geq 1/2$, define $T_L(\beta) := T^{(1-\beta)/\beta}$ and $T_R(\beta) := T$.

Define $C_{T,\beta}^f$ to be the least positive constant such that $dt \leq C_{T,\beta}^f \, d\mu_f(t)$ for $t \in [T_L(\beta)^{-1}, T_R(\beta)]$. Suppose for some constant $c > 0$, there is a constant $C > 0$ so that $C_{T,\beta}^f \leq CT^{2c}$. Then there is an explicitly computable constant $K$ depending only on the smallest non-zero eigenvalue of $\rho$, $\beta$, and $\sigma$. Naturally, for earlier choices of $f$, so that $C_{T,\beta}^f$ is explicit.
C and c, such that, (with $\rho_N^{-1}$ denoting the generalized inverse of $\rho_N$),
(1) for $\beta \leq 1/2$,
\[
\|\sigma_N^\beta \rho_N^{-\beta} \rho^{1/2} - \sigma^\beta \rho^{1/2}\|_2 \leq K(S_f(\rho||\sigma) - S_f(\rho_N||\sigma_N))^{\frac{2(1-\beta)}{1+2(1-\beta)}}.
\] (4.8)
(2) for $\beta \geq 1/2$,
\[
\|\sigma_N^\beta \rho_N^{-\beta} \rho^{1/2} - \sigma^\beta \rho^{1/2}\|_2 \leq K(S_f(\rho||\sigma) - S_f(\rho_N||\sigma_N))^{\frac{1-\beta}{1+\beta}}.
\] (4.9)

Proof. For $\beta \leq 1/2$, Theorem 4.2 and the assumption on $C^f_{T,\beta}$ guarantee that the left side of (4.8) is bounded by an expression of the form $AT^{-a} + BT^b$ where $a = \beta$ and $b = \frac{1-2\beta+2\beta^2}{2(1-\beta)} + c$, and where $B$ is a multiple of $(S_f(\rho||\sigma) - S_f(\rho_N||\sigma_N))^{1/2}$. Then $a + b = c + \frac{1}{2(1-\beta)}$, so that $\frac{a}{a+b} = \frac{2\beta(1-\beta)}{1+2c(1-\beta)}$.

This proves (4.8). For $\beta \geq 1/2$, we have $a = 1 - \beta$ and $b = \beta + c$ so that $\frac{a}{a+b} = \frac{1-\beta}{1+c}$, and this leads to (4.9). $\square$

Since $\frac{1-2\beta+2\beta^2}{2(1-\beta)} = \beta$ at $\beta = 1/2$, the two bounds provided by Theorem 4.2 and by Corollary 4.4 coincide for this value of $\beta$. The case in which $\beta = 1/2$ is particularly important. In this case, the quantity $\|\sigma_N^\beta \rho_N^{-\beta} \rho^{1/2} - \sigma^\beta \rho^{1/2}\|_2$ reduces to $\|\sigma_N^{1/2} \rho_N^{-1/2} \rho^{1/2} - \sigma^{1/2}\|_2$. This quantity may be bounded below in terms of the Petz recovery map $\mathcal{R}_\rho$, defined in (3.9). It was shown in [9] after Lemma 2.2] that the following bound holds
\[
\|\mathcal{R}_\rho(\sigma_N) - \sigma\|_1 \leq 2\|\sigma_N^{1/2} \rho_N^{-1/2} \rho^{1/2} - \sigma^{1/2}\|_2.
\] (4.10)

Exchanging $\rho$ and $\sigma$,
\[
\|\mathcal{R}_\sigma(\rho_N) - \rho\|_1 \leq 2\|\sigma_N^{1/2} \rho_N^{-1/2} \rho^{1/2} - \rho\|_2 \\
\leq 2\|\rho_N^{1/2} \sigma_N^{-1/2} \rho^{1/2} - \sigma^{1/2}\|_2 \\
\leq 2\|\rho_N^{1/2} \sigma_N^{-1/2} (\sigma - \sigma_N^{1/2} \rho_N^{-1/2} \rho^{1/2})\|_2 \\
\leq 2\|\rho_N\|^{1/2} \|\sigma_N^{-1/2}\|^{1/2} \|\sigma^{1/2} \rho_N^{-1/2} \rho^{1/2} - \sigma^{1/2}\|_2 \\
\leq 2\|\rho\|^{1/2} \|\sigma^{-1}\|^{1/2} \|\sigma^{1/2} \rho_N^{-1/2} \rho^{1/2} - \sigma^{1/2}\|_2.
\] (4.11)

(4.12)

The last inequality follows from the fact that the spectra of $\sigma_N$ and $\rho_N$ lie in the convex hulls of the spectra of $\sigma$ and $\rho$ respectively, and it might even be the case that $\sigma_N$ is invertible when $\sigma$ is not. Thus the inequality (4.12) is weaker than (4.11), though it may be useful to have a bound in terms of $\rho$ and $\sigma$ themselves.

This brings us to our second corollary:

4.5 Corollary. Let $\mathcal{N}$ be a von Neumann sub-algebra of a finite dimensional von Neumann algebra $\mathcal{M}$. Let $f$ be a regular operator monotone decreasing function, $T > 0$, define $C^f_{T,1/2}$ to be the least positive constant such that $dt \leq C^f_{T,1/2} \mu_f(t)$ for $t \in [T^{-1}, T]$. Suppose for some constant $c > 0$, there is a constant $C > 0$ so that $C^f_{T,1/2} \leq CT^{2c}$. Then for all density matrices $\rho$, $\sigma$ with $\sigma_N$ invertible, there is an explicitly computable constant $K$ depending only on the smallest non-zero eigenvalue of $\rho$, $\|\sigma_N^{-1}\|$, $\beta$, $C$ and $c$, such that
\[
\max\{\|\mathcal{R}_\rho(\sigma_N) - \sigma\|_1 , \|\mathcal{R}_\sigma(\rho_N) - \rho\|_1\} \leq K(S_f(\rho||\sigma) - S_f(\rho_N||\sigma_N))^{\frac{1}{1+c}}.
\] (4.13)
Dropping the assumption that $\sigma_N$ is invertible, the bound becomes
\[
\|R_\rho(\sigma_N) - \sigma\|_1 \leq K(S_f(\rho||\sigma) - S_f(\rho_N||\sigma_N))^{\frac{1}{1+\epsilon}}. \tag{4.14}
\]
where now $K$ depends only on the smallest non-zero eigenvalue of $\rho$, $\beta$, $C$ and $c$.

Consequently, $S_f(\rho||\sigma) = S_f(\rho_N||\sigma_N)$ if and only if both $\sigma = R_\rho(\sigma_N)$ and $\rho = R_\sigma(\rho_N)$. Conversely, if either of these equations is valid, say $\sigma = R_\rho(\sigma_N)$, then by Petz’s monotonicity theorem,
\[
S_f(\rho||\sigma) = S_f(R_\rho(\rho_N)||R_\rho(\sigma_N)) \leq S_f(\rho_N||\sigma_N) \leq S_f(\rho||\sigma),
\]
and then $S_f(\rho||\sigma) = S_f(\rho_N||\sigma_N)$, implying that the other equation, $\rho = R_\sigma(\rho_N)$, is also valid. This symmetry may be seen from a detailed analysis of the solution set of the Petz equation $\gamma = R_\sigma(\gamma_N)$; see for example [9], and it was proved by Petz [22] through another more complicated argument. However, it is worth noting that this symmetry, valid when $\sigma_N$ is invertible, is an immediate consequence of Corollary 4.5.

We obtain other interesting information for values of $\beta$ other than $\beta = 1/2$. Reasoning as above, note that
\[
\|\sigma_N^\beta \rho_N^{-\beta} - \sigma^\beta \rho^{-\beta}\|_2 = \|(\sigma_N^\beta \rho_N^{-\beta} - \sigma^\beta \rho_1^{1/2})\rho^{-1/2}\|_2 \leq \|\sigma_N^\beta \rho_N^{-\beta} - \sigma^\beta \rho_1^{1/2}\|_2 \quad (4.15)
\]
Thus absorbing the factor of $\|\rho^{-1/2}\|_2$, into the constant $K$, Corollary 4.4 can be restated with $\|\sigma_N^\beta \rho_N^{-\beta} - \sigma^\beta \rho^{-\beta}\|_2$ on the left sides of (4.18) and (4.19) in place of $\|\sigma_N^\beta \rho_N^{-\beta} - \sigma^\beta \rho_1^{1/2}\|_2$. We then conclude, arguing as above, $S_f(\rho||\sigma) - S_f(\rho_N||\sigma_N) = 0$ if and only if
\[
(\sigma_N)^\beta(\rho_N)^{-\beta} = \sigma^\beta \rho^{-\beta}
\]
for all $\beta \in (0, 1)$, and then, since for any positive matrix $X$, $\beta \mapsto X^\beta$ is an entire analytic function, this identity holds for all $\beta \in \mathbb{C}$.

5 Examples

In this section we apply the previous result, Theorem 4.2, to two particular cases: the logarithmic and the power functions.

5.1 Logarithmic function

In the previous section, let us take $f(x) = -\log(x)$, then from Example 2.2 we have $d\mu_{\log}(t) = dt$, and $C_{I,\beta} = 1$. The corresponding quasi-relative entropy is the Umegaki relative entropy (1.1).

5.1 Corollary. For $\beta < 1/2$, the Umegaki relative entropy satisfies
\[
S(\rho||\sigma) - S(\rho_N||\sigma_N) \geq K_{\beta}^f(\rho, \sigma)(\sigma_N)^\beta(\rho_N)^{-\beta}\rho_1^{1/2} - \sigma^\beta \rho_1^{1/2}\|_2^{\frac{1}{1-\beta}}, \tag{5.1}
\]

where $K^L_\alpha (\rho , \sigma ) = (\frac{\pi (1-2\alpha + 2\beta^2)\beta}{\sin (\beta \pi )})^{\frac{1}{1-\beta}} (1 + \frac{\beta}{1-\beta} \|\Delta_{\sigma , \rho}\|)^{-\frac{1-2\alpha + 2\beta^2}{\alpha (1-\beta)}} 2^{\frac{1-2\alpha + 2\beta^2}{\alpha (1-\beta)}} \left( \frac{1-2\alpha + 2\beta^2}{2(1-\beta)} \right)^{-2}$.

For $\beta \geq 1/2$, the Umegaki relative entropy satisfies

$$ S(\rho \| \sigma) - S(\rho_N \| \sigma_N) \geq K^U_\beta (\rho , \sigma) \| (\sigma_N)^\beta (\rho_N)^{-\beta} - \sigma^\beta \rho^{\frac{1}{\beta} - \beta} \|_2^2 , \quad (5.2) $$

where $K^U_\beta (\rho , \sigma) = (\frac{\pi (1-\beta)}{\sin (\beta \pi )})^{\frac{1}{1-\beta}} (1 + \|\Delta_{\sigma , \rho}\|)^{-\frac{2\beta}{1-\beta}} 2^{-\frac{2\beta}{1-\beta}} \beta^{-2}$. In particular, taking $\beta = 1/2$, we obtain an inequality very close to [2]

$$ S(\rho \| \sigma) - S(\rho_N \| \sigma_N) \geq \left( \frac{\pi}{4} \right)^4 (1 + \|\Delta_{\sigma , \rho}\|)^{-2} \| (\sigma_N)^{1/2} (\rho_N)^{-1/2} - \sigma^{1/2} \|_2^4 . \quad (5.3) $$

Proof. The proof follows directly from Theorem 4.2 and Lemma 4.3

\[\square\]

### 5.2 Power function

Another interesting example of $f$-relative quasi-entropy is given by the power function. These types of quasi-entropies appear in the definition of the Rényi entropy, which will be discussed in the next section. Let $p_\alpha (t) = -t^\alpha$ for $\alpha \in (0 , 1)$. From Example 2.3 we have that $d\mu_\alpha (x) = \pi^{-1} \sin (\alpha \pi) x^\alpha dx$. Therefore, the power-relative quasi-entropy $S_{p_\alpha} = -\text{Tr}(\rho^{1-\alpha} \sigma^\alpha)$ satisfies the following corollary.

### 5.2 Corollary. For $\alpha \in (0 , 1)$ the power-relative quasi-entropy satisfies: for $\beta < 1/2$,

$$ S_{p_\alpha} (\rho \| \sigma) - S_{p_\alpha} (\rho_N \| \sigma_N) \geq K^L_{\alpha , \beta} (\rho , \sigma) \| (\sigma_N)^\beta (\rho_N)^{-\beta} - \sigma^\beta \rho^{\frac{1}{\beta} - \beta} \|_2^{\frac{1-2\alpha + \alpha (1-\beta)}{2(1-\beta)}} , \quad (5.4) $$

where $K^L_{\alpha , \beta} (\rho , \sigma)$ is defined in (5.7) below;

for $\beta \geq 1/2$,

$$ S_{p_\alpha} (\rho \| \sigma) - S_{p_\alpha} (\rho_N \| \sigma_N) \geq K^U_{\alpha , \beta} (\rho , \sigma) \| (\sigma_N)^\beta (\rho_N)^{-\beta} - \sigma^\beta \rho^{\frac{1}{\beta} - \beta} \|_2^{\frac{2(1-\beta) + \alpha (1-\beta)}{1-\beta^2}} , \quad (5.5) $$

where $K^U_{\alpha , \beta} (\rho , \sigma)$ is defined in (5.8) below.

In particular, for $\beta = 1/2$

$$ S_{p_\alpha} (\rho \| \sigma) - S_{p_\alpha} (\rho_N \| \sigma_N) \geq K^U_{\alpha} (\rho , \sigma) \| (\sigma_N)^{1/2} (\rho_N)^{-1/2} - \sigma^{1/2} \|_2^{4+2\alpha} . \quad (5.6) $$

Here $K^U_\alpha (\rho , \sigma) = K^U_{\alpha , 1/2} (\rho , \sigma) = K^L_{\alpha , 1/2} (\rho , \sigma)$.

Proof. For $\beta < 1/2$, $C^\alpha_{T , \beta} \leq \frac{\pi}{\sin (\alpha \pi)} T^\alpha$. Then from Theorem 4.2 we have

$$ \frac{\pi}{\sin \beta \pi} \| (\sigma_N)^\beta (\rho_N)^{-\beta} - \sigma^\beta \rho^{\frac{1}{\beta} - \beta} \|_2 \leq 2 \left( \frac{1}{\beta} + \|\Delta_{\sigma , \rho}\| \right) \frac{1}{T^\beta} + T^{\frac{1-2\alpha + 2\beta^2}{\alpha (1-\beta)}} \left( \frac{\pi}{\sin (\alpha \pi)} \right)^{1/2} (S_f (\rho \| \sigma) - S_f (\rho_N \| \sigma_N))^{1/2} . $$
Using Lemma 4.3 we obtain

\[ S_{p_a}(\rho \| \sigma) - S_{p_a}(\rho_N \| \sigma_N) \geq K^L_{\alpha,\beta}(\rho, \sigma)(\sigma_N)^{\beta}(\rho_N)^{-\beta} \rho^{1/2} - \sigma^\beta \rho^\frac{1}{2} - \beta 2^{1 + \alpha(1 - \beta)} \left\| \frac{1}{2} \right\|_{2}^{\beta(1 - \beta)}, \]

for

\[ K^L_{\alpha,\beta}(\rho, \sigma) = 1 + \frac{\beta}{1 - \beta} \left\| \Delta_{\sigma,\rho} \right\| - \frac{\alpha(1 - \beta) + 2\beta + \beta^2}{\beta(1 - \beta)} 2^{\beta(1 - \beta)} \left( \frac{1}{1 + \alpha(1 - \beta)} \right) \sin(\alpha \pi) \sin(\beta \pi) \]

\[ \frac{1}{2} \left( \frac{\pi \beta(\alpha(1 - \beta) + 1 - 2\beta + 2\beta^2)}{(1 + \alpha(1 - \beta)) \sin(\pi \beta)} \right)^{1/2}. \]

For \( \beta \geq 1/2 \), we have \( C_{T,\beta}^{p_a} \leq \frac{\pi}{\sin(\alpha \pi)} T^{\alpha(1 - \beta)/\beta} \). Therefore,

\[ \frac{\pi}{\sin(\beta \pi)} \left\| (\sigma_N)^{\beta}(\rho_N)^{-\beta} \rho^{1/2} - \sigma^\beta \rho^\frac{1}{2} \right\|_2 \]

\[ \leq 2 \left( \frac{1}{\beta} + \left\| \Delta_{\sigma,\rho} \right\| \right) T^{\alpha(1 - \beta)/\beta} \left( \frac{\pi}{\sin(\alpha \pi)} \right)^{1/2} \left( S_{p_a}(\rho \| \sigma) - S_{p_a}(\rho_N \| \sigma_N) \right)^{1/2}. \]

Similarly to the previous case, using Lemma 4.3 we obtain

\[ S_{p_a}(\rho \| \sigma) - S_{p_a}(\rho_N \| \sigma_N) \geq K^U_{\alpha,\beta}(\rho, \sigma)(\sigma_N)^{\beta}(\rho_N)^{-\beta} \rho^{1/2} - \sigma^\beta \rho^\frac{1}{2} - \beta 2^{1 + \alpha(1 - \beta)} \left\| \frac{2^{\beta + \alpha(1 - \beta)}}{2^{\beta(1 - \beta)}} \right\|_{2}^{\beta(1 - \beta)}, \]

for

\[ K^U_{\alpha,\beta}(\rho, \sigma) = \left( \frac{1 - \beta}{\beta} + \left\| \Delta_{\sigma,\rho} \right\| \right)^{-\frac{2\beta + \alpha(1 - \beta)}{\beta(1 - \beta)}} 2^{-\frac{2\beta^2 + \alpha(1 - \beta)}{\beta(1 - \beta)}} \sin(\alpha \pi) \sin(\beta \pi) \]

\[ \frac{1}{2} \left( \frac{\pi (1 - \beta) (2\beta^2 + \alpha(1 - \beta))}{(2\beta + \alpha(1 - \beta)) \sin(\beta \pi)} \right)^{2^{\beta + \alpha(1 - \beta)}} \left( \frac{2\beta^2 + \alpha(1 - \beta)}{2^\beta} \right)^{-2}. \]

Plugging in \( \beta = 1/2 \) in the last inequality we obtain the bound claimed in the corollary.

\[ \square \]

6 Rényi entropy

For \( \alpha \in (0, 1) \) Rényi entropy is be defined as

\[ S_{\alpha}(\rho \| \sigma) := \frac{1}{\alpha - 1} \log \text{Tr}(\rho^\alpha \sigma^{1-\alpha}) = \frac{1}{\alpha - 1} \log \left( -S_{p_{1-\alpha}}(\rho \| \sigma) \right), \]

where \( S_{p_{1-\alpha}}(\rho \| \sigma) \) is the power quasi entropy of the previous section with power \( 1 - \alpha \). Notice that for all \( \alpha \in (0, 1) \), \( \rho \) and \( \sigma \),

\[ 0 < -S_{p_{1-\alpha}}(\rho \| \sigma) \leq 1 \]

(6.1)

where the later inequality follows from Hölder’s inequality.
6.1 Theorem. For any $\alpha \in (0,1)$, any density matrix $\rho$ and any non-degenerate density matrix $\sigma$,

$$S_\alpha(\rho||\sigma) - S_\alpha(\rho_N||\sigma_N) \geq \frac{1}{1 - \alpha} \log \left( 1 + K_{1-\alpha}^U(\rho,\sigma) \right) \left( (\sigma_N)^{1/2}(\rho_N)^{-1/2} - 1 \right)^{6-2\alpha} \cdot (6.2)$$

where $K_{1-\alpha}^U(\rho,\sigma)$ is given in Corollary 5.2.

6.2 Corollary. For any $\alpha \in (0,1)$, suppose $\rho$ and $\sigma$ are such that Corollary 4.2 is satisfied. Then

$$S_\alpha(\rho||\sigma) - S_\alpha(\rho_N||\sigma_N) \geq \frac{1}{1 - \alpha} \log \left( 1 + \frac{1}{2} ||\rho||^{-1/2} ||\sigma||^{-1/2} K_{1-\alpha}^U(\rho,\sigma) \max\{||R_\rho(\sigma_N) - \sigma||_1, ||R_\sigma(\rho_N) - \rho||_1\}^{6-2\alpha} \right),$$

(6.3)

where $K_{1-\alpha}^U(\rho,\sigma)$ is given in Corollary 5.2.

6.3 Remark. An equivalent formulation of (6.3) is of course that

$$\max\{||R_\rho(\sigma_N) - \sigma||_1, ||R_\sigma(\rho_N) - \rho||_1\}^{6-2\alpha} \leq \frac{2||\rho||^{1/2}||\sigma||^{-1/2}}{K_{1-\alpha}^U(\rho,\sigma)} \left( \exp[(1 - \alpha)(S_\alpha(\rho||\sigma) - S_\alpha(\rho_N||\sigma_N))] - 1 \right) \quad (6.4)$$

We are of course most interested in the case in which $S_\alpha(\rho||\sigma) - S_\alpha(\rho_N||\sigma_N)$ is small. Since for any $r > 0$, $e^{(1-\alpha)x} - 1 \leq (x/r)(e^{(1-\alpha)r} - 1)$, one can simplify the bound if one knows that $S_\alpha(\rho||\sigma) - S_\alpha(\rho_N||\sigma_N)$ is small.

Proof of Corollary 6.2. This is immediate from Theorem 6.1 and (4.10) and (4.12).

Proof of Theorem 6.1.

$$S_\alpha(\rho||\sigma) - S_\alpha(\rho_N||\sigma_N) = \frac{1}{\alpha - 1} \log \frac{S_{p_{1-\alpha}}(\rho||\sigma)}{S_{p_{1-\alpha}}(\rho_N||\sigma_N)}$$

$$= \frac{1}{\alpha - 1} \log \frac{S_{p_{1-\alpha}}(\rho_N||\sigma_N)}{S_{p_{1-\alpha}}(\rho||\sigma)}$$

$$= \frac{1}{\alpha - 1} \log \left( 1 + \frac{S_{p_{1-\alpha}}(\rho||\sigma) - S_{p_{1-\alpha}}(\rho_N||\sigma_N)}{-S_{p_{1-\alpha}}(\rho||\sigma)} \right)$$

$$\geq \frac{1}{\alpha - 1} \log \left( 1 + S_{p_{1-\alpha}}(\rho||\sigma) - S_{p_{1-\alpha}}(\rho_N||\sigma_N) \right),$$

where in the last line we have used (6.1) and the monotonicity of the logarithm. Making one more use of this monotonicity, we now simply apply the bound (5.6) from the previous section with $1 - \alpha$ in place of $\alpha$.

Note that at the end of the proof of this theorem, we could use Corollary 5.2 for any $\beta$ instead of $1/2$ to obtain a stronger lower bound as a one-parameter family.
7 Proof of Theorem 4.2

From the definition of the quantum $f$-relative entropy, we construct the following family of relative entropies: for each $t > 0$, the function $x \mapsto (t + x)^{-1}$ is operator convex, and so define a one parameter family of quasi relative entropies by

$$S(t)(\rho||\sigma) = \text{Tr} \left[ (t + \Delta_{\sigma,\rho})^{-1} \rho \right].$$

(7.1)

For an operator monotone decreasing function $f$ (which implies that $f$ is operator convex), according to the integral representation (2.1) the $f$-relative quasi-entropy $S_f$ can be written as

$$S_f(\rho||\sigma) = -a - b + \int_0^{\infty} \left( S(t)(\rho||\sigma) - \frac{t}{t^2 + 1} \right) d\mu_f(t),$$

for $a \geq 0$ and $b \in \mathbb{R}$.

For an orthogonal projection $E_{\tau}$ from the von Neumann algebra $M$ to the sub-algebra $N$ denote the processed states as $\rho_N := E_{\tau} \rho$ and $\sigma_N := E_{\tau} \sigma$. The monotonicity inequality (3.2) holds, and we are interested in the lower bound on the relative entropy difference. From the above integral representation, it is clear that the difference between relative entropies can be written in terms of the $S(t)$-family,

$$S_f(\rho||\sigma) - S_f(\rho_N||\sigma_N) = \int_0^{\infty} \left( S(t)(\rho||\sigma) - S(t)(\rho_N||\sigma_N) \right) d\mu_f(t).$$

Proof. From [9, Lemma 2.1] we have

$$S(t)(\rho||\sigma) - S(t)(\rho_N||\sigma_N) \geq t\| w_t \|^2,$$

where

$$w_t := U(t1 + \Delta_{\sigma_N,\rho_N})^{-1/2} (t1 + \Delta_{\sigma,\rho})^{-1/2} - (t1 + \Delta_{\sigma,\rho})^{-1/2},$$

with the operator $U$ being the mapping $H := (M, \langle \cdot, \cdot \rangle_{HS})$ to itself defined as

$$U(X) = E_{\tau}(X) \rho_N^{-1/2} \rho^{1/2}.$$

Notice that for all $X \in N$, $U(X) = X \rho_N^{-1/2} \rho^{1/2}$. On account of this identity,

$$-w_t = U[t^{-1}1 - (t1 + \Delta_{\sigma_N,\rho_N})^{-1}](\rho_N)^{1/2} - [t^{-1}1 - (t1 + \Delta_{\sigma,\rho})^{-1}] \rho^{1/2}.$$

Therefore, since $U$ is a contraction,

$$\| w_t \| \leq \|[t^{-1}1 - (t1 + \Delta_{\sigma_N,\rho_N})^{-1}]\| \rho_N^{1/2} \| + \|[t^{-1}1 - (t1 + \Delta_{\sigma,\rho})^{-1}]\| \rho^{1/2} \|.$$

Since on account of the non-negativity of the modular operator, $0 \leq t^{-1}1 - (t1 + \Delta_{\sigma_N,\rho_N})^{-1} \leq t^{-1}1$, with the analogous estimate valid with $\Delta_{\sigma,\rho}$ in place of $\Delta_{\sigma_N,\rho_N}$, Therefore,

$$\| w_t \| \leq 2t^{-1}.$$

(7.2)
Now using the integral representation of the power function from Example 2.3 (recall that $\beta \in (0, 1)$)
\[
X^\beta = \frac{\sin \beta \pi}{\pi} \int_0^\infty t^\beta \left(\frac{1}{t} - \frac{1}{t + X}\right) dt,
\]
and $U(\rho_N)^{1/2} = \rho^{1/2}$, we conclude that
\[
U(\Delta_{\sigma_N,\rho_N})^\beta(\rho_N)^{1/2} - (\Delta_{\sigma,\rho})^\beta \rho^{1/2} = -\frac{\sin \beta \pi}{\pi} \int_0^\infty t^\beta w_t dt.
\] (7.3)
On the other hand,
\[
U(\Delta_{\sigma_N,\rho_N})^\beta(\rho_N)^{1/2} - (\Delta_{\sigma,\rho})^\beta \rho^{1/2} = \sigma_N^\beta \rho_N^{1/2} - \sigma^\beta \rho^{1/2}.
\] (7.4)
Combining the last two equalities (7.3) and (7.4), and taking the Hilbert space norm associated with $H$, for any $T_L, T_R > 0$,
\[
\| (\sigma_N)^\beta(\rho_N)^{1/2} - \sigma^\beta \rho^{1/2} \|_2 \leq \frac{\sin \beta \pi}{\pi} \left\| \int_0^\infty t^\beta w_t dt \right\|_2.
\]
(7.5)
Let us look at these three terms separately. The first term can be bounded using (7.2):
\[
\int_0^{1/T_L} t^\beta \|w_t\|_2 dt \leq 2 \int_0^{1/T_L} t^{\beta-1} dt = \frac{2}{\beta} T_L^{\beta}.
\] (7.6)
The third term in (7.5) can be bounded the following way: For any positive operator $X > 0$,
\[
t^\beta \left(\frac{1}{t} - \frac{1}{t + X}\right) \leq t^\beta \left(\frac{1}{t} - \frac{1}{t + \|X\|}\right) 1 = \frac{t^{\beta-1} \|X\|}{(t + \|X\|)} 1,
\]
and hence
\[
\int_T^\infty t^\beta \left(\frac{1}{t} - \frac{1}{t + X}\right) dt \leq \|X\|^\beta \left(\int_T^{\infty} \frac{t^{\beta-1} dt}{1 + t}\right) 1 \leq \frac{\|X\|}{(1 - \beta)T^{1-\beta}} 1.
\]
Since spectra of $\sigma_N$ and $\rho_N$ lie in the convex hulls of the spectra of $\sigma$ and $\rho$ respectively, it follows that $\|\Delta_{\sigma_N,\rho_N}\| \leq \|\Delta_{\sigma,\rho}\|$. Therefore, recalling the definition of $w_t$, we obtain
\[
\left\| \int_{T_R}^\infty t^\beta w_t dt \right\|_2 \leq \frac{2\|\Delta_{\sigma,\rho}\|}{(1 - \beta) T_R^{1-\beta}}.
\] (7.7)
The second term can be bounded using Cauchy-Schwartz inequality and the equivalence of measures on the finite interval, i.e. there is a constant $C_{T_L,T_R}^f$ such that $dt \leq C_{T_L,T_R}^f d\mu_f(t)$ for
Case 1: $\beta \leq 1/2$. 

\[
\left( \int_{1/T_L}^{T_R} t^\beta \| w_t \|_2 dt \right)^2 \leq T_R \int_{1/T_L}^{T_R} t^{2\beta} \| w_t \|_2^2 dt \\
\leq T_R T_L^{1-2\beta} \int_{1/T_L}^{T_R} t \| w_t \|_2^2 dt \\
\leq T_R T_L^{1-2\beta} \int_{1/T_L}^{T_R} S(t) (\rho \| | \sigma) - S(t)(\rho_N \| | \sigma_N) dt \\
\leq T_R T_L^{1-2\beta} C_{T_L,T_R}^f \int_{1/T_L}^{T_R} S(t) (\rho \| | \sigma) - S(t)(\rho_N \| | \sigma_N) d\mu_f(t) \\
\leq T_R T_L^{1-2\beta} C_{T_L,T_R}^f (S_f(\rho \| | \sigma) - S_f(\rho_N \| | \sigma_N)) .
\]

(7.8)

Therefore, combining (7.5), (7.6), (7.7), and (7.8) we have

\[
\frac{\pi}{\sin \beta \pi} \| (\sigma_N)^\beta (\rho_N)^{-\beta} \rho^{1/2} - \sigma^\beta \rho^{1/2-\beta} \|_2 \leq \frac{2}{\beta T_L^\beta} + \frac{2\| \Delta_{\sigma,\rho} \|}{(1-\beta) T_R^{1-\beta}} \\
+ T_R^{1/2} T_L^{1/2-\beta} \left( C_{T_L,T_R}^f \right)^{1/2} (S_f(\rho \| | \sigma) - S_f(\rho_N \| | \sigma_N))^{1/2} .
\]

Taking $T_L := T$ and $T_R := T^{(1-\beta)}$ we obtain

\[
\frac{\pi}{\sin \beta \pi} \| (\sigma_N)^\beta (\rho_N)^{-\beta} \rho^{1/2} - \sigma^\beta \rho^{1/2-\beta} \|_2 \\
\leq 2 \left( \frac{1}{\beta} \cdot \frac{\| \Delta_{\sigma,\rho} \|}{1-\beta} \right) \frac{1}{T^\beta} + T_R^{1/2} T_L^{1/2-\beta} \left( C_{T_L,T_R}^f \right)^{1/2} (S_f(\rho \| | \sigma) - S_f(\rho_N \| | \sigma_N))^{1/2} .
\]

Case 2: $\beta > 1/2$. 

\[
\left( \int_{1/T_L}^{T_R} t^\beta \| w_t \|_2 dt \right)^2 \leq T_R \int_{1/T_L}^{T_R} t^{2\beta} \| w_t \|_2^2 dt \\
\leq T_R T_L^{2\beta} \int_{1/T_L}^{T_R} t \| w_t \|_2^2 dt \\
\leq T_R^{2\beta} \int_{1/T_L}^{T_R} (S(t) (\rho \| | \sigma) - S(t)(\rho_N \| | \sigma_N)) dt \\
\leq T_R^{2\beta} C_{T_L,T_R}^f \int_{1/T_L}^{T} (S(t) (\rho \| | \sigma) - S(t)(\rho_N \| | \sigma_N)) d\mu_f(t) \\
\leq T_R^{2\beta} C_{T_L,T_R}^f (S_f(\rho \| | \sigma) - S_f(\rho_N \| | \sigma_N)) .
\]

(7.9)

Therefore, combining (7.5), (7.6), (7.7), and (7.9) we have

\[
\frac{\pi}{\sin \beta \pi} \| (\sigma_N)^\beta (\rho_N)^{-\beta} \rho^{1/2} - \sigma^\beta \rho^{1/2-\beta} \|_2 \leq \frac{2}{\beta T_L^\beta} + \frac{2\| \Delta_{\sigma,\rho} \|}{(1-\beta) T_R^{1-\beta}} \\
+ T_R^{3} \left( C_{T_L,T_R}^f \right)^{1/2} (S_f(\rho \| | \sigma) - S_f(\rho_N \| | \sigma_N))^{1/2} .
\]
Taking $T_L := T^{(1-\beta)/\beta}$ and $T_R := T$ we obtain
\[
\frac{\pi}{\sin \beta \pi} \| (\sigma_N)^{\beta} (\rho_N)^{-\beta} \rho^{1/2} - \sigma^{\beta} \rho^{1/2} \|_2 \leq 2 \left( \frac{1}{\beta} + \| \Delta_{\sigma,\rho} \| \right) \frac{1}{T^{1-\beta}} + T^\beta \left( C_{T,\beta}^{\beta} \right)^{1/2} \left( S_f(\rho \parallel \sigma) - S_f(\rho_N \parallel \sigma_N) \right)^{1/2} .
\]

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