Inconsistences of a purported probability current in the Duffin-Kemmer-Petiau theory

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Abstract

The Duffin-Kemmer-Petiau (DKP) equation with a square step potential is used in a simple way with polymorphic purposes. It proves adequate to refuse a proposed new current that is currently interpreted as a probability current, to show that the Klein paradox does exist in the DKP theory and to revise other minor misconceptions diffused in the literature.
The first-order Duffin-Kemmer-Petiau (DKP) equation has experimented
a renewal of life due to the discovery of a new conserved four-vector current
density \([1]-[6]\), whose positive-definite time component would be a candidate
to a probability density, and as a bonus a hope for avoiding Klein’s paradox
for bosons \([6]\). The DKP equation for a boson minimally coupled to the
electromagnetic field is given by

\[
(i\beta^\mu D_\mu - m)\psi = 0
\]  

(1)

where the matrices \(\beta^\mu\) satisfy the algebra

\[
\beta^\mu \beta^\nu \beta^\lambda + \beta^\lambda \beta^\nu \beta^\mu = g^{\mu\nu} \beta^\lambda + g^{\lambda\nu} \beta^\mu
\]  

(2)

the covariant derivative is given by \(D_\mu = \partial_\mu + ieA_\mu\) and the metric tensor is \(g^{\mu\nu} = \text{diag} (1, -1, -1, -1)\). The second-order Klein-Gordon and Proca equations are obtained when one selects the spin-0 and spin-1 sectors of the DKP
theory. A well-known conserved four-current is given by

\[
J^\mu = \bar{\psi} \beta^\mu \psi
\]  

(3)

where the adjoint spinor \(\bar{\psi} = \psi^\dagger \eta^{00}\) with

\[
\eta^{\mu\nu} = \beta^\mu \beta^\nu + \beta^\nu \beta^\mu - g^{\mu\nu}
\]  

(4)

and \((\beta^\mu)^\dagger = \eta^{00} \beta^\mu \eta^{00}\). The time component of this current is not positive def-
finite but it may be interpreted as a charge density. An alleged new conserved
current, however, is written as \([1], [5]\]

\[
S^\mu = \bar{\psi} \eta^{\mu\nu} \psi u_\nu
\]  

(5)

Here \(u_\nu\) is the unity timelike four-velocity of the observer \((u^\nu u_\nu = 1)\). Since
\(S^0 = \psi^\dagger \psi \geq 0\) in the lab frame, it might be tempting to interpret this
alternative current as a probability current.

In the present work, the simple problem of scattering in a square step po-
tential, considered as a time-component of the electromagnetic four-potential,
is used to show not only that this new current leads to inconsistencies suffi-
cient enough to reject it as a true probability current, but also to show that
Klein’s paradox is absent in Ref. \([6]\) just because it was not searched for,
and that it is not necessary to refer to limiting cases of smooth potentials for
finding the appropriate boundary conditions for discontinuous potentials as
done in \([7]\).
Let us consider the one-dimensional time component of the static electromagnetic potential, so that the time-independent DKP equation can be written as
\[ \{ \beta^0 [E - eA_0(z)] + i\beta^3 \frac{d}{dz} - m \} \varphi(z) = 0 \] (6)
where the decomposition \( \psi(z, t) = \varphi(z) \exp(-iEt) \) has been used.

For the case of spin 0, we use the representation for the \( \beta^\mu \) matrices given by [8]
\[ \beta^0 = \begin{pmatrix} \theta \\ \vec{0}^T \\ 0 \end{pmatrix} \quad \text{and} \quad \beta^i = \begin{pmatrix} 0 & \rho_i \\ -\rho_i^T & 0 \end{pmatrix}, \quad i = 1, 2, 3 \] (7)
where
\[ \theta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]
\[ \rho_2 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \]

\( \vec{0} \), \( \vec{0} \) and \( 0 \) are 2x3, 2x2, and 3x3 zero matrices, respectively, while the superscript T designates matrix transposition. The five-component spinor can be written as \( \varphi^T = (\varphi_1, ..., \varphi_5) \) in such a way that the DKP equation decomposes into
\[ O_{KG} \varphi_1 = 0, \quad \varphi_2 = \frac{E - eA_0}{m} \varphi_1, \quad \varphi_3 = \varphi_4 = 0, \quad \varphi_5 = \frac{i}{m} \frac{d\varphi_1}{dz} \] (9)
where \( O_{KG} = -d^2/dz^2 + m^2 - (E - eA_0)^2 \). Using now the components of the spinor, the time and space components of the currents \( J^\mu \) and \( S^\mu \) can be written as
\[ J^0 = 2\text{Re}(\varphi_1 \varphi_2^*), \quad J^3 = -2\text{Re}(\varphi_1 \varphi_5^*) \]
\[ S^0 = |\varphi_1|^2 + |\varphi_2|^2 + |\varphi_5|^2, \quad S^3 = -2\text{Re}(\varphi_2 \varphi_5^*) \] (10)

Note that there is no reason to require that the spinor and its derivative are continuous across finite discontinuities of the potential, as naively advocated in Ref. [7]. A little careful analysis reveals, though, that proper matching conditions follow from the differential equations obeyed by the spinor components, as they should be, avoiding in this manner the hard tasking of
recurring to the limit process of smooth potentials. Only the first component of the spinor satisfies the second-order Klein-Gordon equation, so that $\varphi_1$ and its first derivative are continuous even the potential suffers finite discontinuities. In this case of a discontinuous potential, $\varphi_2$ is discontinuous and so are $J^0$, $S^0$ and $S^3$. The discontinuity of $J^0$ does not matter if it is to be interpreted as a charge density. As for $S^\mu$, it is an obvious nonsense to interpret it as a probability current seeing that a probability density should always be continuous and that the probability flux should be uniform in a stationary regime. In this point we are faced with serious defects of $S^\mu$. Nevertheless, despite those unpleasant properties of $S^\mu$ we shall explore the scattering in a square step potential in order to clarify additional misapprehensions of the DKP theory.

The one-dimensional square step potential is expressed as

$$A_0 (z) = V_0 \theta (z)$$

where $\theta (z)$ denotes the Heaviside step function. For $z < 0$ the DKP equation has the solution

$$\varphi (z) = \varphi_+ e^{+ikz} + \varphi_- e^{-ikz}$$

where

$$\varphi^T_\pm = \frac{a_\pm}{\sqrt{2}} \left( 1, \frac{E}{m}, 0, 0, \pm \frac{k}{m} \right)$$

and $k = \sqrt{E^2 - m^2}$. For $|E| > m$, the solution expressed by (12) and (13) describes plane waves propagating on both directions of the Z-axis with group velocity $v_g = dE/dk$ equal to the classical velocity. If we choose particles inciding on the potential barrier ($E > m$), $\varphi_+ \exp(+ikz)$ will describe incident particles ($v_g = +k/E > 0$), whereas $\varphi_- \exp(-ikz)$ will describe reflected particles ($v_g = -k/E < 0$). The flux related to the standard current $J^\mu$, corresponding to $\varphi$ given by (12), is expressed as

$$J^3 = \frac{k}{m} (|a_+|^2 - |a_-|^2)$$

Note that the relation $J^3 = J^0 v_g$ maintains for the incident and reflected waves, since

$$J^0_\pm = \frac{E}{m} |a_\pm|^2$$

On the other hand, for $z > 0$ one should have $v_g \geq 0$ in such a way that the solution in this region of space describes an evanescent wave or a progressive
wave running away from the potential interface. The general solution has the form

$$\varphi_t(z) = (\varphi_t)_+ e^{+iqz} + (\varphi_t)_- e^{-iqz}$$  \hspace{1cm} (16)$$

where

$$T \pm = \frac{b_\pm}{\sqrt{2}} \left( 1, \frac{E - eV_0}{m}, 0, 0, \mp \frac{q}{m} \right)$$  \hspace{1cm} (17)$$

and $q = \sqrt{(E - eV_0)^2 - m^2}$. Due to the twofold possibility of signs for the energy of a stationary state, the solution involving $b_-$ cannot be ruled out a priori. As a matter of fact, this term may describe a progressive wave with negative energy and phase velocity $v_{ph} = |E|/q > 0$. One can readily envisage that three different classes of solutions can be segregated:

- **Class A.** For $eV_0 < E - m$ one has $q \in \mathbb{R}$, and the solution describing a plane wave propagating in the positive direction of the Z-axis with group velocity $v_g = q/(E - eV_0)$ is possible only if $b_- = 0$. In this case the components of the standard current are given by

$$J^0 = \frac{E - eV_0}{m} |b_+|^2, \quad J^3 = \frac{q}{m} |b_+|^2$$  \hspace{1cm} (18)$$

- **Class B.** For $E - m < eV_0 < E + m$ one has that $q = \pm i |q|$, and with $b_+ = 0$ describes an evanescent wave. The condition $b_- = 0$ is necessary for furnishing a finite current as $z \to \infty$. In this case

$$J^0 = \frac{E - eV_0}{m} e^{-2|q|z} |b_+|^2, \quad J^3 = 0$$  \hspace{1cm} (19)$$

- **Class C.** With $eV_0 > E + m$ it appears again the possibility of propagation in the positive direction of the Z-axis, now with $b_+ = 0$ and a group velocity given by $v_g = q/(eV_0 - E)$. The standard current takes the form

$$J^0 = \frac{E - eV_0}{m} |b_-|^2, \quad J^3 = -\frac{q}{m} |b_-|^2$$  \hspace{1cm} (20)$$

In this last class we meet a bizarre circumstance as long as both $J^0$ and $J^3$ are negative quantities. The maintenance of the relation $J^3 = J^0 v_g$, though, is a license to interpret the solution $(\varphi_t)_- \exp(-iqz)$ as describing the propagation, in the positive direction of the Z-axis, of
particles with electric charges of opposite sign to the incident particles. This interpretation is consistent if the particles moving in this region have energy $-E$ and are under the influence of a potential $-eV_0$. It means that, in fact, the progressive wave describes the propagation of antiparticles in the positive direction of the Z-axis.

The demand for continuity of $\varphi_1$ and $d\varphi_1/dz$ at $z = 0$ fixes the wave amplitudes in terms of the amplitude of the incident wave, viz.

\[
\frac{a_-}{a_+} = \begin{cases} \frac{k-q}{k+q} & \text{for the class A} \\ \frac{(k-i|q|)^2}{k^2+|q|^2} & \text{for the class B} \\ \frac{k+q}{k-q} & \text{for the class C} \end{cases}
\] (21)

\[
\frac{b_+}{a_+} = \begin{cases} \frac{2k}{k+q} & \text{for the class A} \\ \frac{2k(k-i|q|)}{k^2+|q|^2} & \text{for the class B} \\ 0 & \text{for the class C} \end{cases}
\] (22)

\[
\frac{b_-}{a_+} = \begin{cases} 0 & \text{for the class A} \\ 0 & \text{for the class B} \\ \frac{2k}{k-q} & \text{for the class C} \end{cases}
\] (23)

Now we focus attention on the calculation of the reflection ($R$) and transmission ($T$) coefficients. The reflection (transmission) coefficient is defined as the ratio of the reflected (transmitted) flux to the incident flux. Since $\partial J^0/\partial t = 0$ for stationary states, one has that $J^3$ is independent of $z$. This fact implies that

\[
R = \begin{cases} \left(\frac{k-q}{k+q}\right)^2 & \text{for the class A} \\ 1 & \text{for the class B} \\ \left(\frac{k+q}{k-q}\right)^2 & \text{for the class C} \end{cases}
\] (24)
For the class \(A\)

\[
T = \begin{cases} 
\frac{4kq}{(k+q)^2} & \text{for the class } A \\
0 & \text{for the class } B \\
-\frac{4kq}{(k-q)^2} & \text{for the class } C 
\end{cases}
\]  

(25)

For all the classes one has \(R + T = 1\) as should be expected for a conserved quantity. The class C presents \(R > 1\), the alluded Klein’s paradox, implying that more particles are reflected from the potential barrier than those incoming. Contrary to the assertion of Ghose et al. [6], Klein’s paradox there exists for bosons in the DKP theory. It must be so because, as seen before, the potential stimulates the production of antiparticles at \(z = 0\). Due to the charge conservation there is, in fact, the creation of particle-antiparticle pairs. Since the potential in \(z > 0\) is repulsive for particles they are necessarily reflected. From the previous discussion related to the classes B and C, one can realize that the threshold energy for the pair production is given by \(eV_0 = 2m\). The propagation of antiparticles inside the potential barrier can be interpreted as due to the fact that each antiparticle is under the influence of an effective potential given by \(-eV_0\). In this way, each antiparticle has an available energy (rest energy plus kinetic energy) given by \(eV_0 - E\), accordingly one conclude about the threshold energy. One can also say that there is an ascending step for particles and a descending step for antiparticles.

Note that the currents \(J^\mu\) and \(S^\mu\) are simply related by

\[
S^\mu = \frac{E - eA_0}{m} J^\mu 
\]  

(26)

in all the classes of solutions. In this manner, the conservation law \(\partial_\mu J^\mu = 0\) is not compatible with \(\partial_\mu S^\mu = 0\), at least for the case under investigation where \(\partial S^0 / \partial t = 0\) but \(S^3\) is not uniform. In order to understand the behaviour of \(S^\mu\) let us recall that the DKP equation can be recast into the Hamiltonian form [3], [10]

\[
i \frac{\partial \psi}{\partial t} = H \psi 
\]  

(27)

where

\[
H = i \left[ \beta^i, \beta^0 \right] D_i + eA_0 + m\beta^0 + i \frac{e}{2m} F_{\mu\nu} \left( \beta^\mu \beta^0 \beta^\nu + \beta^\mu g^{\nu0} \right) 
\]  

(28)

with the electromagnetic stress tensor \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\) (in [3] and [10] \(e\) was defined with the opposite sign). At this point is worthwhile to mention
that $H$ is not Hermitian, in opposition what was adverted in [3], since

$$\left( iF_{0i}\beta^i (\beta^0)^2 \right)^\dagger = - \left( iF_{0i}\beta^i (\beta^0)^2 \right) + iF_{0i}\beta^i \tag{29}$$

There results that

$$\partial_\mu S^\mu = i\psi^\dagger \left( H - H^\dagger \right) \psi = -\frac{e}{m} F_{0i} \psi^\dagger \left[ \beta^i, (\beta^0)^2 \right] \psi = \frac{e}{m} F_{0i} J^i \tag{30}$$

This result clearly shows that the electromagnetic coupling induces a source term in the current $S^\mu$. It is curious that the source term is due to the non-Hermitian piece of the anomalous term in (28). Now, coming back to the square step potential (11), one can write

$$\partial_\mu S^\mu = -\frac{eV_0}{m} \delta(z) J^3(z) \tag{31}$$

in such a way that the jumping of $S^3$ at $z = 0$ reads

$$S^3(0^+) - S^3(0^-) = -\frac{eV_0}{m} J^3(0) \tag{32}$$

a result in perfect agreement with (26).

For short, the DKP equation with a square step potential is a test ground to refuse $S^\mu$ as a probability current as well as to show that Klein’s paradox is alive and well in the DKP theory (we have talking about the spin-0 sector of the DKP theory but the state of affairs is not different for the spin-1 sector as one can see in Appendix A).

### Appendix A

For the case of spin 1, the $\beta^\mu$ matrices are [9]

$$\beta^0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \beta^i = \begin{pmatrix} 0 & \bar{\beta}^i & e_i & 0 \\ 0 & 0 & 0 & -is_i \\ 0 & 0 & 0 & 0 \\ -e_i & 0 & 0 & -is_i \end{pmatrix} \tag{33}$$

where $s_i$ are the $3 \times 3$ spin-1 matrices $(s_i)_{jk} = -i\varepsilon_{ijk}$, $e_i$ are the $1 \times 3$ matrices $(e_i)_{1j} = \delta_{ij}$ and $\bar{\beta} = (0 \ 0 \ 0)$, while $I$ and $0$ designate the $3 \times 3$ unit and zero matrices, respectively. With the spinor written as $\varphi^T = (\varphi_1, ..., \varphi_{10})$, ...
and defining $\Psi^T = (\varphi_2, \varphi_3, \varphi_7)$, $\Phi^T = (\varphi_5, \varphi_6, \varphi_4)$ and $\Theta^T = (\varphi_9, -\varphi_8, \varphi_1)$ as done in Ref. [7], the DKP equation [9] can be expressed in terms of the following equations

$$O_{KG} \Psi = 0, \quad \Phi = \frac{E - eV_0}{m} \Psi, \quad \Theta = \frac{i}{m} \frac{d\Psi}{dz}, \quad \varphi_{10} = 0$$

(34)

Now the components of the four-currents are given by

$$J^0 = 2\Re (\varphi_2 \varphi_5^* + \varphi_3 \varphi_6^* + \varphi_4 \varphi_7^*), \quad J^3 = -2\Re (\varphi_1 \varphi_7^* + \varphi_2 \varphi_9^* - \varphi_3 \varphi_8^*)$$

(35)

$$S^0 = \sum_{i=1}^9 |\varphi_i|^2, \quad S^3 = -2\Re (\varphi_1 \varphi_4^* + \varphi_5 \varphi_9^* - \varphi_6 \varphi_8^*)$$

A discontinuous potential makes $\varphi_4$, $\varphi_5$ and $\varphi_6$ discontinuous, and as an immediate consequence $J^0$, $S^0$ and $S^3$ are also discontinuous. The plane wave solutions for the potential given by (11) in the region $z < 0$ can be written as

$$\varphi (z) = \varphi_+ e^{ikz} + \varphi_- e^{-ikz}$$

(36)

where

$$\varphi^T_\pm = \left( \mp \frac{k}{m} m \pm \alpha_\pm, \beta_\pm, \frac{E}{m} \alpha_\pm, \frac{E}{m} \beta_\pm, \gamma_\pm, \pm \frac{k}{m} \beta_\pm, \mp \frac{k}{m} \alpha_\pm, 0 \right)$$

(37)

and $\alpha_\pm$, $\beta_\pm$, and $\gamma_\pm$ are arbitrary amplitudes. Defining

$$a_\pm = \sqrt{2(|\alpha_\pm|^2 + |\beta_\pm|^2 + |\gamma_\pm|^2)}$$

(38)

it follows that the components of the current can be written in the same form as (14) and (15). A similar procedure for the region $z > 0$ allows one to obtain the results (21)-(25). Even though $\varphi_2$, $\varphi_3$ and $\varphi_7$ obey the Klein-Gordon equation there is no reason why they have the same amplitudes, as assumed in Ref. [7]. As a matter of fact, a nontrivial spinor with only three nonvanishing components would be possible.

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