BENDING TEICHMÜLLER SPACES AND CHARACTER VARIETIES

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ABSTRACT. We consider the mapping $b_L : \mathcal{T} \to \chi$ of the Fricke-Teichmüller space $\mathcal{T}$ into the $\text{PSL}_2\mathbb{C}$-character variety $\chi$ of the surface, obtained by holonomy representations of bent hyperbolic surfaces along a fixed measured lamination $L$. We prove that this mapping is an equivariant symplectic real-analytic embedding and, for most of the measured laminations, proper. Therefore $b_L$ is

• a reminiscent of an equivariant pleated surface $\mathbb{H}^2 \to \mathbb{H}^3$, and moreover

• an analogue of a Poincaré holonomy variety ($\mathfrak{sl}_2\mathbb{C}$-oper) in the Thurston parametrization of $\mathbb{C}P^1$-structures.

In addition, if the measured lamination $L$ is a weighted multiloop, we construct a complexification of $b_L$, using the product variety $\chi \times \chi$, by a new type of bending deformation, so that this complexification retains similar properties.

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1. Introduction

Thurston discovered the bent hyperbolic surfaces $\tau$ on the boundary of the convex core of a (geometrically finite) hyperbolic three-dimensional manifolds ([Thu81]). Indeed, the intrinsic metric of the convex surface is hyperbolic, and the surface is bent along a measured lamination, where the bending angles correspond to the transversal measure of the lamination. Such bent surfaces are particularly useful for capturing the global properties of the hyperbolic manifold.

Lifting the convex surface $\tau$ to the universal cover $\mathbb{H}^3$ of the hyperbolic manifold, we obtain an equivariant bending $\mathbb{H}^2 \to \mathbb{H}^3$ which preserves the (intrinsic) hyperbolic metric of the surface. Then, this bending map is equivariant via a holonomy representation of a surface group into $\text{PSL}_2\mathbb{C}$. Moreover, if $\tau$ is $\pi_1$-injective (equivalently incompressible) in the ambient hyperbolic 3-manifold, then the bending map $\mathbb{H}^2 \to \mathbb{H}^3$ is a proper embedding.

In this paper, utilizing the bending construction, moreover, in a new generalized manner, we construct similar equivariant geometry-preserving mappings, in fact, at the level of associated deformations spaces.

1.1. Holonomy varieties. Let $Y$ be a marked Riemann surface structure on a closed oriented surface $S$ of genus $g$ at least two. Let $\text{QD}(Y)$ be the space of the holomorphic quadratic differentials on $Y$, which is a complex vector space of dimension $3g - 3$. Then $\text{QD}(Y)$ is identified with the space $\mathcal{P}_Y$ of all $\mathbb{C}P^1$-structures on $Y$, and this correspondence yields the Schwarzian parameterization of $\mathbb{C}P^1$-structures (see [Dum09] for example).

Let

$$\text{Hol}: \mathcal{P} \to \chi$$

be the holonomy map from the deformation space $\mathcal{P}$ of all $\mathbb{C}P^1$-structures on $S$ to the the $\text{PSL}_2\mathbb{C}$-character varieties $\chi$ of $S$. Recall that the character variety $\chi$ is an affine algebraic variety, and it has Goldman’s complex symplectic structure invariant under the action of the mapping class group; see [Gol84]. Many interesting properties of this mapping, associated with the Schwarzian parametrization, have been discovered, and particularly the following holds.

**Theorem 1.1.** The restriction of the holonomy map to $\mathcal{P}_Y \cong \text{QD}(Y)$ is a proper Lagrangian complex-analytic embedding into $\chi$.

On the other hand, the entire holonomy map $\text{Hol}: \mathcal{P} \to \chi$ of $\mathbb{C}P^1$-structures is neither injective nor proper (see [Hej75]).
The injectivity of Theorem 1.1 is due to Poincaré [Poi84]. The properness is due to Kapovich [Kap95] (see [GKM00] for the full proof; see also [Dum17, Tan99]). The Lagrangian property is proven by Kawai [Kaw96].

By Theorem 1.1, for every marked Riemann surface structure $Y$, the vector space $QD(Y) \cong \mathbb{C}^{3g-3}$ is property embedded onto a half-dimensional smooth subvariety of $\chi$. We call this image, associated with the Schwarzian parametrization, the Poincaré holonomy variety of $Y$. In particular, the holonomy variety of $Y$ contains the Bers slice of $Y$ as a bounded pseudo-convex domain.

1.2. Real bending varieties. Recall that $\mathbb{CP}^1$ is the ideal boundary of the hyperbolic three-space $\mathbb{H}^3$, and the automorphism group $\text{PSL}_2 \mathbb{C}$ of $\mathbb{CP}^1$ is identified with the group of orientation preserving isometries of $\mathbb{H}^3$. Utilizing this correspondence in a sophisticated manner, Thurston gave another parametrization of $\mathcal{P}$, so that $\mathbb{CP}^1$-structures correspond to equivariant pleated surfaces in $\mathbb{H}^3$ (§3.1.1). In this paper, we first yield an analogue of Theorem 1.1 by specific slices in the Thurston parametrization of $\mathbb{CP}^1$-structures.

In fact, Tanigawa [Tan97], Wolf-Scannel [SW02], Dumas-Wolf [DW08] considered the $\mathbb{CP}^1$-structures with a fixed bending measured lamination and analyzed their conformal structures. In this paper, as in the holonomy variety, we instead consider the holonomy representation of those $\mathbb{CP}^1$-structures.

For a measured lamination $L$ on a hyperbolic surface $\tau$, we obtain an equivariant pleated surface in $\mathbb{H}^3$ by bending the universal cover of $\tau$, the hyperbolic plane $\mathbb{H}^2$ along the inverse-image $\tilde{L}$ of $L$ in $\mathbb{H}^2$, and the pleated surface $\tilde{\tau} \cong \mathbb{H}^2 \to \mathbb{H}^3$ is equivariant via a representation $\tilde{\pi}_1(S) \to \text{PSL}_2 \mathbb{C}$. (See §3.1 for details.) Let $\mathcal{T}$ be the space of marked hyperbolic structures on $S$, the Fricke-Teichmüller space; then $\mathcal{T}$ is diffeomorphic to $\mathbb{R}^{6g-6}$ as a smooth manifold. The Weil-Petersen form gives a symplectic structure on $\mathcal{T}$, and Goldman extended it to a complex-symplectic structure on $\chi$ ([Gol84]). For a measured lamination $L$ on $S$, let $b_L: \mathcal{T} \to \chi$ be the map taking $\tau \in \mathcal{T}$ to the holonomy representation $\pi_1(S) \to \text{PSL}_2 \mathbb{C}$ of the pleated surface given by $\tau$ and $L$.

This mapping is closely related to the Thurston parametrization of $\mathcal{P}$ (Theorem 3.1), and the following theorem is an analogue of Theorem 1.1 in the Thurston parametrization.
Theorem A (Theorems 4.1, 14.4, Lemma 3.2). Let \( L \) be an arbitrary measured lamination on \( S \). Then, the bending map \( b_L : \mathcal{T} \to \chi \) is a real-analytic symplectic embedding, and it is equivariant by the subgroup of the mapping class group \( \mathcal{G}_L \) of \( S \) preserving \( L \).

Moreover, \( b_L \) is proper if and only if \( L \) contains no periodic leaves of weight \( \pi \) modulo \( 2\pi \).

On the other hand, the conservation of the symplectic structure of \( \mathcal{T} \) by \( b_L \) resembles the conservation of the hyperbolic metric by the bending map \( \mathbb{H}^2 \to \mathbb{H}^3 \), and the equivariant property resembles that of the bending map. Moreover, by Theorem A, the real bending map \( b_L \) is a proper mapping for almost all measured laminations \( L \). In addition, for exceptional laminations, we explicitly characterize the non-properness in the Fenchel-Nielsen coordinates (Theorem 6.1).

The stabilizer \( \mathcal{G}_L \) can be a large subgroup and, on the other hand, can be the trivial subgroup of the mapping class group \( \text{MCG} \) depending on \( L \in \mathcal{ML} \) (Remark 3.3).

1.3. Complex bending varieties. Historically, a real analytic deformation determined by a measure lamination or a measured foliation (an equivalent object) has a significant complexification: A Teichmüller geodesic in the Teichmüller space \( \mathcal{T} \) is determined by a measured foliation on a Riemann surface, and its complexification is a Teichmüller disk in \( \mathcal{T} \). A measured lamination on a hyperbolic surface yields a real-analytic earthquake line in \( \mathcal{T} \) ([Thu86, Ker85]), and an earthquake disk is its complexification ([McM98]).

We aim to geometrically complexify the real-analytic embedding \( b_L : \mathcal{T} \to \chi \) in Theorem A, and obtain a complex-analytic mapping from a closed complex analytic variety. It is plausible that such complexifications of the real bending varieties \( \text{Im} b_L \) in a common analytic space will lead us to discover intersecting properties of the original real analytic varieties.

We first explain the domain of the complexified bending map. Given a representation \( \rho : \pi_1(S) \to \text{PSL}_2\mathbb{C} \), if a holonomy \( \rho(\ell) \in \text{PSL}_2\mathbb{C} \) along a loop \( \ell \) is either hyperbolic or elliptic, then one can certainly bend \( \rho \) along \( \ell \) as the axis of \( \rho(\ell) \) gives the axis of bending deformation. However, it is not clear if one can bend if \( \rho(\ell) \) is parabolic or the identity.

Thus, given a weighted oriented multiloop \( M \) on \( S \), we introduce an appropriate closed analytic set \( X_M \) consisting of certain (double) framed representations, so that the framing determines the bending axes even when the holonomy along some loops of \( M \) is trivial (§7).
In fact, this modification of $\chi$ essentially occurs only in a complex-analytic subvariety of $\chi$ disjoint from $\mathcal{T}$, so that the map forgetting the framing induces a finite-to-one covering map from $X_M$ to $\chi$ when specific subvarieties are removed from $X_M$ and $\chi$. (To be more precise, $X_M$ is a finite covering map of a blow-up of the complement of some subvariety of $\chi$.) In particular, there is a canonical embedding of the Fricke-Teichmüller space $\mathcal{T}$ into $X_M$ as a real-analytic smooth subvariety. In addition, we can pull back the complex symplectic structure on $\chi$ to $X_M$ minus a subvariety.

We next explain the target space. Notice that the Fricke-Teichmüller space $\mathcal{T}$ is a component of the real slice of the character variety $\chi$. Moreover, the real bending map $b_L: \mathcal{T} \to \chi$ is totally real in the complex affine variety $\chi$ (i.e. its tangent spaces contain no complex lines). Therefore, in order to obtain nontrivial complexifications and also to obtain different complexifications (as images) for different bending laminations, it is necessary to enlarge the ambient space.

When the PSL$_2\mathbb{C}$-Lie algebra $\mathfrak{psl}_2\mathbb{C}$ is regarded as a real Lie algebra, its complexification is isomorphic to $\mathfrak{psl}_2\mathbb{C} \oplus (\mathfrak{psl}_2\mathbb{C})^\ast$, where $\ast$ denotes the complex conjugate. Thus, for a representation $\rho: \pi_1(S) \to$ PSL$_2\mathbb{C}$, we consider the diagonal representation $\pi_1(S) \to$ PSL$_2\mathbb{C} \times$ PSL$_2\mathbb{C}$ twisted by conjugation, defined by $\gamma \mapsto (\rho(\gamma), \rho(\gamma)\ast)$. Then, given a representation framed along loops of $M$, we can appropriately bend it along the axes determined by their framings, where the bending happens in the space of representations into PSL$_2\mathbb{C} \times$ PSL$_2\mathbb{C}$. Then we obtain the complex bending map $B_M: X_M \to \chi \times \chi$. (See §8 for details.) Let

$$\Delta^\ast = \{(\rho_1, \rho_2): \pi_1(S) \to \chi \times \chi \mid \rho_1 = \rho_2^\ast\},$$

the anti-holomorphic diagonal in $\chi \times \chi$. Define $\psi: \chi \to \Delta^\ast \subset \chi \times \chi$ by $\rho \mapsto (\rho, \rho^\ast)$.

**Theorem B (Complexification).** Let $M$ be an oriented weighted multiloop on $S$. Then $B_M: X_M \to \chi \times \chi$ is a complex analytic mapping, such that

1. the restriction of $B_M$ of $\mathcal{T}$ is a real-analytically embeds into $\Delta^\ast$;  
2. $\psi \circ b_M: \mathcal{T} \to \chi \times \chi$ coincides with the restriction of $B_M$ to $\mathcal{T}$ (Figure 1);  
3. $B_M$ is complex-symplectic in the complement of a subvariety of $X_L$;  
4. $B_M$ is equivariant by the action of the subgroup of the mapping class group preserving $M$. 

Figure 1. The commutative diagram for the complexification $B_M$ of the real analytic bending map $b_M$.

(The complex-analyticity is proven in Theorem 11.1. For (1), see Proposition 12.1. For (2), see Proposition 12.1. For (3), see Theorem 14.5; For (4), see Lemma 8.2.) we remark that the removed subvariety in (3) consists of the framed representations such that at least one loop of $M$ has trivial holonomy.

The complex bending map $B_M$ is not proper or injective in general. However, $B_M$ is injective and proper “almost everywhere”: If an analytic subset is removed from the domain $X_M$ and a subvariety is removed from the target $\chi \times \chi$, then $B_M$ becomes injective and proper (Theorem 9.1, Theorem 10.1). Indeed, in certain cases, the complex bending map is genuinely proper.

Theorem C. If $\ell$ is a weighted oriented non-separating loop of weight not equal to $\pi$ modulo $2\pi$, then, the bending map $B_\ell: X_\ell \to \chi \times \chi$ is a proper mapping. (Theorem 13.1.)

Therefore, under the assumption of Theorem C, the image of $B_\ell$ is a closed analytic subvariety in $\chi \times \chi$ (complex bending variety). Moreover, it is plausible that such subvarieties are well-defined for all measured laminations on $S$:

Conjecture 1.2. For every measured lamination $L$ on $S$, let $\ell_i$ be a sequence of non-separating weighted loops converging to $L$. Then the closed analytic set $\text{Im } B_{\ell_i}$ converges to a closed analytic set in $\chi \times \chi$ as $i \to \infty$.

1.4. Outline of the paper. The preliminary section (§3) explains some basic notions for this paper. In particular, we recall that a measured lamination on a hyperbolic surface induces an equivariant locally convex pleated surface $\mathbb{H}^2 \to \mathbb{H}^3$, then we define the real bending map $b_L: \mathcal{T} \to \chi$ for a measured lamination. In §4, we show the injectivity of the real bending map. In §5, we prove the properness of the real bending map for most of the measured laminations $L$. On the other hand, in §6.1, for particular types of measured laminations, we characterize the non-properness of the bending map.
In §7, we introduce the space of representations double-framed along a weighted oriented multiloop $M$ on $S$ (the framed character variety $X_M$). Then, in §8, we define the complex bending map from the framed character variety $X_M$ to the product character variety $\chi \times \chi$. For the definition, a more general type of bending deformation is introduced. In fact, when a representation framed along $M$ is bent along $M$, accordingly, the hyperbolic space $\mathbb{H}^3$ is equivariantly “bent” inside the $\mathbb{H}^3 \times \mathbb{H}^3$ (§8.4). In §9, we show that the complex bending map is injective almost everywhere. In §10, we show the complex bending map is a proper mapping almost everywhere. In §11, using the “almost-everywhere” injectivity, we prove the analyticity of the complex bending map on the entire domain. In §12, we show that the complex bending map is a complexification of the real bending map. In §13, we show that the complex bending map is, indeed, genuinely a proper mapping when $M$ is a single non-separating loop of the weight not equal to $\pi$.

Lastly, in §14, we prove the real is symplectic and the complex bending map is complex symplectic.

2. Acknowledgements

I thank Misha Kapovich, Bill Goldman, Hisashi Kasuya, and James Farre for their helpful conversations. The author is supported by the Grant-in-Aid for Scientific Research (20K03610).

3. Preliminaries

3.1. Bending deformation. ([Thu81], [EM87].) Thurston discovered that the boundary of the convex core of a hyperbolic-three manifold is a hyperbolic surface bent along a measured lamination ([Thu81]). More generally, one can bend a hyperbolic surface along an arbitrarily measured lamination and obtain a holonomy representation from the surface fundamental group into $\text{PSL}_2 \mathbb{C}$ as follows.

We shall first describe basic bending maps when the bending locus is a single loop. Let $\tau$ be a hyperbolic structure on $S$, and let $\ell$ be a geodesic loop on $\tau$ with weight $w \geq 0$. The union $\tilde{\ell}$ of all lifts of $\ell$ to the universal cover $\mathbb{H}^2$ of $\tau$ is a set of disjoint geodesics, each with weight $w$, and it is invariant under the deck transformation. We call the union $\tilde{\ell}$ the total lift of $\ell$.

Put the universal cover $\mathbb{H}^2$ in the three-dimensional hyperbolic space $\mathbb{H}^3$ as a totally geodesic hyperbolic plane. By this embedding, the isometric deck transformations of $\mathbb{H}^2$ extend to an isometric action on $\mathbb{H}^3$, and we obtain a representation of $\rho_\tau: \pi_1(S) \rightarrow \text{PSL}_2 \mathbb{C}$. Note that, as $S$ is oriented, the orientation of the universal cover $\mathbb{H}^2$ determines
a normal direction of the plane. Then we can bend $\mathbb{H}^2$ along every geodesic $\alpha$ of $\tilde{l}$ by angle $w$ so that the normal direction is in the exterior. Thus we obtain a bending map $\beta: \mathbb{H}^2 \to \mathbb{H}^3$, which is totally geodesic on every complement of $\mathbb{H}^2 \setminus \tilde{l}$. The map $\beta$ is unique up to an orientation-preserving isometry of $\mathbb{H}^3$. Moreover, $\beta$ is equivariant by its holonomy representation $\rho: \pi_1(S) \to \mathrm{PSL}_2 \mathbb{C}$. This $\rho$ is called a bending deformation of $\rho$. 

If $C_1, C_2$ are components of $\mathbb{H}^2 \setminus \tilde{l}$ such that $C_1, C_2$ are adjacent along a geodesic $\alpha$ of $\tilde{l}$. Let $G_1$ and $G_2$ be the subgroups of $\pi_1(S)$ which preserve $C_1$ and $C_2$, respectively. If $\beta$ is normalized so that $\beta_r = \beta$ on $C_1$, then the restriction of $\beta$ to $G_2$ is the conjugation of the restriction of $\rho_r$ to $G_2$ by the elliptic isometry with the axis $\alpha$ by angle $w$.

More generally, given an arbitrary measured lamination $L$ on $\tau$, we can take a sequence of weighted loops $\ell_i$ converging to $L$ as $i \to \infty$. For each $i$, let $\rho_i: \pi_1(S) \to \mathrm{PSL}_2 \mathbb{C}$ be the bending deformation of $\rho_r$ along $\ell_i$. Then $\rho_i$ converges to a representation $\pi_1(S) \to \mathrm{PSL}_2 \mathbb{C}$ as $i \to \infty$ if $\rho_i$ are appropriately normalized by $\mathrm{PSL}_2 \mathbb{C}$. This limit is the bending deformation of $\rho_r$ along $L$, and it is unique up to conjugation by an element of $\mathrm{PSL}_2 \mathbb{C}$.

3.1.1. Equivariant property of the real bending map. The equivariant property of $b_L: \pi_1(S) \to \mathrm{PSL}_2 \mathbb{C}$ in Theorem A can directly be proven from the definition of the bending map. Here we give a proof in a broader context.

A $\mathbb{CP}^1$-structure on $S$ is a $(\mathbb{CP}^1, \mathrm{PSL}_2 \mathbb{C})$-structure. That is, an atlas of charts mapping open subsets of $S$ into $\mathbb{CP}^1$ with translation maps in $\text{Aut}(\mathbb{CP}^1) = \mathrm{PSL}_2 \mathbb{C}$. (General references about $\mathbb{CP}^1$-structures are [Dum09, Kap01, Gol22]). Recall that $\mathbb{CP}^1$ is the ideal boundary of the hyperbolic space $\mathbb{H}^3$, and $\mathrm{PSL}_2 \mathbb{C}$ is the group of orientation-preserving isometrics of $\mathbb{H}^3$. Using equivariant bending maps described above, Thurston gave a parametrization of the deformation space $\mathcal{P}$ of $\mathbb{CP}^1$-structures by corresponding them with holonomy-equivariant pleated surfaces in $\mathbb{H}^3$.

**Theorem 3.1** (Thurston, [KP94, KT92]).

$$\mathcal{P} = \mathcal{T} \times \mathcal{ML}.$$ 

Then $b_L(\tau) = \text{Hol}(\tau, L)$ where $(\tau, L) \in \mathcal{T} \times \mathcal{ML}$ denote the $\mathbb{CP}^1$-structure in Thurston coordinates.

**Lemma 3.2.** For $L \in \mathcal{ML}$, let $\mathcal{G}_L$ be the subgroup of $\mathrm{MCG}$ which preserves $L$. Then, the real bending map $b_L: \mathcal{T} \to \chi$ is $\mathcal{G}_L$-equivariant.
Remark 3.3. If $L$ is a multiloop, then $\mathcal{G}_L$ contains the subgroup of $\text{MCG}$ generated by Dehn twists along loops not intersecting $L$ (but including the loops of $L$). On the other hand, for almost all $L$ in $ML$, $\mathcal{G}_L$ is the trivial group, since $\text{MCG}$ is a countable group.

Proof. The $\text{MCG}$-action on $\mathcal{P}$ is given by marking change and on $\chi$ by precomposing induces isomorphisms $\pi_1(S) \to \pi_1(S)$. Then the holonomy map $\text{Hol}: \mathcal{P} \to \chi$ is $\text{MCG}$-equivariant (see, for example, [Gol06]).

By the Thurston’s parametrization, For $\tau \in T$ and $h \in \text{MCG}$, $h(\tau, L) = (\tau, L)$.

Thus the desired equivariant property holds. \qed

3.2. Quasi-geodesics in the hyperbolic space. We first recall the definition of quasi-isometries. Let $(X, d_X), (Y, d_Y)$ be metric spaces, where $d_X, d_Y$ are the distance functions. Then, for $P, Q > 0$, a mapping $f: X \to Y$ is a $(P, Q)$-quasi-isometry if, for all $x_1, x_2 \in X$,

$$P^{-1}d_X(x_1, x_2) - Q < d_Y(f(x_1), f(x_2)) < P d_X(x_1, x_2) + Q.$$ 

In this section, we discuss certain conditions for a piecewise geodesic curve in $\mathbb{H}^3$ to be a quasi-geodesic.

3.2.1. Quasi-geodesics in $\mathbb{H}^3$. Let $c$ be a bi-infinite piecewise geodesic curve in $\mathbb{H}^3$. Let $s_i (i \in \mathbb{Z})$ be the maximal geodesic segments of $c$ indexed along $c$, so that $s_i$ and $s_{i+1}$ are adjacent geodesic segments for every $i \in \mathbb{Z}$ and $c = \bigcup_{i \in \mathbb{Z}} s_i$.

Lemma 3.4. For every $\epsilon > 0$, there are $R > 0$ and $\delta > 0$, such that if length $s_i > R$ for all $i \in \mathbb{Z}$ and the angle between arbitrary adjacent geodesic segment $s_i, s_{i+1}$ is at least $\pi - \delta$, then $c$ is a $(1 + \epsilon)$-bilipschitz embedding.

Proof. This lemma follows from [CEG87, I.4.2.10]. \qed

Proposition 3.5. If every $\epsilon > 0$ and $\epsilon' > 0$, there are $R > 0$ and $Q > 0$, such that if length $s_i > R$ for all $i \in \mathbb{Z}$ and the angle between arbitrary every pair of adjacent geodesic segments is at least $\epsilon'$, then $c$ is an $(1 + \epsilon, Q)$-quasi-isometric embedding.

Proof. For each $i \in \mathbb{Z}$, let $x_i$ be the common endpoint of $s_{i-1}$ and $s_i$, so that $x_i$ is a non-smooth point of $c$. Let $0 < r < R/2$. Let $x_i^-\overline{r}$ be the point on $s_{i-1}$ such that $d(x_i^-, x_i) = r$. Let $x_i^+\overline{r}$ be the point on $s_i$ such that $d(x_i, x_i^+\overline{r}) = r$. Then, we replace two geodesic segments $[x_i^-\overline{r}, x_i] \cup [x_i, x_i^+\overline{r}]$ of $c$ with the single geodesic segment $[x_i^-\overline{r}, x_i^+\overline{r}]$. Let $c_r$ be the piecewise geodesic in $\mathbb{H}^3$ obtained from $c$ by applying this replacement for every $i \in \mathbb{Z}$. 

By basic hyperbolic geometry, the following holds.

**Lemma 3.6.** For every $\delta > 0$, if $r > 0$ is sufficiently large, then the angle at every non-smooth point of $c_r$ is more than $\pi - \delta$.

Then Lemma 3.4 and Lemma 3.6 imply the proposition.

3.3. **Complex analytic geometry.** ([Gol84]) We recall a standard theorem about a complex analytic set.

**Theorem 3.7** (Removable Singularity Theorem; see for example [Tay02], §3.3.2). Let $Y$ be an analytic set. Let $A$ be a closed subset of $Y$ contained in a proper subvariety of $Y$. Suppose that $f: Y \setminus A \to \mathbb{C}$ is an analytic function which is bounded in a small neighborhood of every point in $A$. Then $f$ continuously extends to an analytic function on $Y$.

3.4. **Goldman’s symplectic form.** ([Gol84]) Let $\mathfrak{g}$ be the $\text{PSL}_2\mathbb{C}$-lie algebra. Then the adjoint representation $\text{Ad}: \text{PSL}_2\mathbb{C} \to \text{Aut}\mathfrak{g} \subset \text{GL}_3\mathbb{C}$ is induced by the conjugation of $\text{PSL}_2\mathbb{C}$ by $\text{PSL}_2\mathbb{C}$. By $\mathfrak{g}_{\text{Ad}_\rho}$, we regard $\mathfrak{g}$ as a $\pi_1(S)$-module via the composition of $\rho: \pi_1(S) \to \text{PSL}_2\mathbb{C}$. Then the Zariski tangent space of the representation variety $\mathcal{R}$ at $\rho \in \mathcal{R}$ is the vector space of 1-cocycles

$$Z^1(\pi_1(S); \mathfrak{g}_{\text{Ad}_\rho}) = \{u \in \mathfrak{g}^{\pi_1(S)} \mid u(xy) = u(x) + (\text{Ad}_\rho(x)) u(y)\}.$$

The subspace of 1-coboundaries $B^1(\pi_1(S); \mathfrak{g}_{\text{Ad}_\rho})$ consists of $u \in \mathfrak{g}^{\pi_1(S)}$, such that there is $u_0 \in \mathfrak{g}$ satisfying $u(x) = u_0 - \text{Ad}(\rho(x)) u_0$ for all $x \in \pi_1(S)$. Then the Zariski tangent space of $\chi$ at $\rho$ is the quotient vector space

$$H^1(\pi_1(S); \mathfrak{g}_{\text{Ad}_\rho}) = \frac{Z^1(\pi_1(S); \mathfrak{g}_{\text{Ad}_\rho})}{B^1(\pi_1(S); \mathfrak{g}_{\text{Ad}_\rho})}.$$
Let \( w(\rho) \) denote the bilinear form on the Zariski tangent space obtained by the composition

\[
H^1(\pi_1(S); \mathfrak{g}_{Ad\rho}) \times H^1(\pi_1(S); \mathfrak{g}_{Ad\rho}) \xrightarrow{\cup} H^2(\pi_1(S); \mathfrak{g}_{Ad\rho} \otimes \mathfrak{g}_{Ad\rho}) \cong H^2(\pi_1(S); \mathbb{C}) \cong \mathbb{C}.
\]

Here the first mapping is the cup product, and the second mapping is an isomorphism given by the coefficients pairing by the bilinear form \( B : \mathfrak{g}_{Ad\rho} \otimes \mathfrak{g}_{Ad\rho} \rightarrow \mathbb{C} \) given by \((A, B) \mapsto \text{tr} AB\). Goldman proved that \( w \) is a complex-symplectic form on \( \chi \), i.e. a non-degenerate closed holomorphic \((2,0)\)-form on the character variety \( \chi \); see [Gol84].

4. Injectivity of the real bending maps

Let \( \mathcal{ML} \) be the space of measured laminations on \( S \). Each pair \( (\tau, L) \in \mathcal{T} \times \mathcal{ML} \) induces an equivariant pleated surface \( \mathbb{H}^2 \rightarrow \mathbb{H}^3 \), unique up to \( \text{PSL}_2 \mathbb{C} \). Let \( b : \mathcal{T} \times \mathcal{ML} \rightarrow \chi \) be the holonomy map of the bending maps.

**Theorem 4.1.** Fix arbitrary \( L \in \mathcal{ML}(S) \). Then the restriction \( b \) to \( \mathcal{T} \times \{L\} \) is a real-analytically embedding. Moreover, this embedding is proper if and only if \( L \) contains no periodic leaf of weight \( \pi \) modulo \( 2\pi \).

Let \( b_L : \mathcal{T} \rightarrow \chi \) denote the restriction of \( b \) to \( \mathcal{T} \times \{L\} \). Given a representation \( \rho : \pi_1(S) \rightarrow \text{PSL}_2 \mathbb{C} \), geodesic lamination \( \lambda \) on \( S \) is realizable if there is a \( \rho \)-equivariant pleated surface \( \mathbb{H}^2 \rightarrow \mathbb{H}^3 \), such that its pleating loci contains \( \lambda \). Then, for \( L \in \mathcal{ML} \), let \( N = N_L \) be an open neighborhood of the Fuchsian space \( \mathcal{T} \) in the smooth part of \( \chi \) such that the underlying geodesic lamination \( |L| \) is realizable for all \( \rho \in \chi \). Then, \( b_L : \mathcal{T} \rightarrow \chi \) extends to the bending map \( \hat{b}_L : N_L \rightarrow \chi \) by bending cocycle ([Bon96]).

**Proposition 4.2.** For all \( L \in \mathcal{ML} \), \( \hat{b}_L : N_L \rightarrow \chi \) is injective.

**Proof.** As \( |L| \) is realizable on \( \text{Im} \hat{b}_L \), we have the unbending map \( \hat{b}_{-L} : \text{Im} \hat{b}_L \rightarrow \chi \) by \(-L\). Then, clearly, \( \hat{b}_{-L} \circ \hat{b}_L \) is the identity map on \( N_L \). Thus \( \hat{b}_L \) is injective. \( \square \)

**Proposition 4.3.** The injective map \( b_L : \mathcal{T} \rightarrow \chi \) is a real-analytic embedding.

**Proof.** (cf. [Ker85].) We regard \( \mathcal{T} \) as the Fricke space, i.e. the space of discrete faithful representations into \( \text{PSL}(2, \mathbb{R}) \) up to conjugation by \( \text{PSL}_2 \mathbb{R} \). Then, take a small open neighborhood \( N \) of \( \mathcal{T} \) whose closure is contained in \( N_L \).
If $L$ is a weighted multilooop, the bending map $b_L$ is holomorphic on $N$ as bending transforms the holonomy along a loop by some elliptic elements in a holomorphic manner. In general, pick a sequence of weighted multiloops $M_i$ converging to $L$ as $i \to \infty$. By the injectivity of Proposition 4.2, $b_{M_i} : N_{M_i} \to N_{M_i}$ is a holomorphic embedding. Then, the holomorphic embedding $b_{M_i}|N$ converges uniformly to $b_L|N$ uniformly on compacts as $i \to \infty$. Therefore $b_L|N$ is a holomorphic embedding.

Since $T$ is a real-analytic submanifold of $\chi$, thus $b_L|T$ is a real-analytic embedding. \hfill $\square$

5. Properness of the bending maps from the Teichmüller spaces

**Theorem 5.1.** Let $L \in \mathcal{ML}$. Then, the bending map $b_L : T \to \chi$ is proper if and only if $L$ contains no leaves of weight $\pi$ modulo $2\pi$.

First, we prove the sufficiency of the condition in Theorem 5.1.

**Lemma 5.2.** Fix $L \in \mathcal{ML}$ such that every closed leaf of $L$ contains no leaves of weight $\pi$ modulo $2\pi$. Let $M$ be the (possibly empty) sublamination of $L$ consisting of the periodic leaves of $L$. Then, for all $\nu, R > 0$, there are finitely many loops $\ell_1, \ldots, \ell_n$ on $S$ such that

- the lengths of $\ell_1, \ldots, \ell_n$ form length coordinates of $T$, and
- for each $i = 1, \ldots, n$,
  - the transversal measure $(L \setminus M)(\ell_i) < \nu$, and
  - $\ell_i$ intersects at most one leaf $m$ of $M$, and the intersection number is at most two.

**Proof.** For every $\delta > 0$, there is a pants decomposition $P = P_\delta$ (i.e. a maximal multilooop) on $S$ consisting of

- the loops of $M$,
- loops which are disjoint from $L$,
- loops $\ell$ with $L(\ell) < \delta$ (so that $\ell$ is a good approximation of a minimal irrational sublamination of $L$).

By the third condition, if $Q$ is a component of $S \setminus P$, and $\alpha$ is an arc on $Q$ with endpoints on $\partial Q$, then there is an isotopy of $\alpha$ keeping its endpoints on $\partial Q$ such that $L(\alpha) < 3\delta$. Therefore, if $\delta > 0$ is small enough, for each loop $m$ of $P$, we can take two loops $m_1, m_2$ such that

- $m_i$ intersects $m$ at a point or two, and it does not intersect any other loop of $P$, and
- $(L \setminus M)m_i < \nu$. 
Then we obtain a desired set of loops by adding such two loops for all loops of $M$. (For length coordinates of $T$, see [FM12, Theorem 10.7] for example.) □

Proof of the sufficiently of Theorem 5.1. For $\epsilon > 0$, let $\ell_1, \ldots, \ell_n$ be the set of loops given by Lemma 5.2. Let $\tau_i$ be a sequence in $T$ which leaves every compact. Then, for some $1 \leq k \leq n$, length $\ell_k \to \infty$ as $i \to \infty$ up to a subsequence.

Claim 5.3. For every $\epsilon > 0$, if $\delta > 0$ is sufficiently small, then

1. If $L(\ell_k) < \delta$, then $\beta \mid \ell_k$ is a $(1 + \epsilon)$-bilipschitz embedding for sufficiently large $i$, and
2. If $\ell_k$ intersects a loop $m$ of $M$, then $\beta \mid \ell_k$ is $(1 + \epsilon, q)$-quasi-isometric embedding for all sufficiently large $i$, where $q$ only depends on the weight of $m$.

Proof. (1) See [Bab10, Lemma 5.3], which was proved based on [CEG87, I.4.2.10].

(2) We straighten $\ell_k$ and $M$ on $\tau_i \in T$. From Lemma 5.2, $\ell$ intersects only one loop $m$ of $M$, and their intersection number is one or two. We thus assume that $\ell_k \cap m$ consists of two points $x_1, x_2$ — the proof when the intersection number is one is similar. Then $x_1$ and $x_2$ decompose $\ell_k$ into 2 geodesic segments $a_1$ and $a_2$. Since length $\tau_i \ell_k \to \infty$, the lengths of $a_1$ and $a_2$ both goes to $\infty$ as well. Let $\tilde{\ell}_k$ be the geodesic in $\mathbb{H}^2$ obtained by lifting $\ell_k$ to the universal cover. Let $\tilde{a}_j$ be a lift of $a_j$ to $\tilde{\ell}_k$, and let $\tilde{x}_j$ and $\tilde{x}_{j+1}$ be its endpoints. For every $\epsilon' > 0$, if $\nu > 0$, is sufficiently small, then $\beta \mid (\tilde{a}_j)$ is $\epsilon'$-close to the geodesic segment $[\beta \tilde{x}_j, \beta \tilde{x}_{j+1}]$ connecting its endpoints $\beta \tilde{x}_j$ and $\beta \tilde{x}_{j+1}$ in the Hausdorff metric. Since every periodic leaf of $L$ has weight not equal to $\pi$ modulo $2\pi$, there is $\omega > 0$ such that, for every periodic leaf $\ell$ of $L$, the distance from the weight of $\ell$ to the nearest odd multiple of $\pi$ is at least $\omega$. Therefore, if $\delta > 0$ is sufficiently small, then the angle between $[\beta_i x_j, \beta_i x_{j+1}]$ and $[\beta_i x_{j-1}, \beta_i x_j]$ at $x_j$ is at least $\omega/2$. Let $c_i$ be the piecewise geodesic in $\mathbb{H}^3$ which is a union of the geodesic segments $[\beta_i x_j, \beta_i x_{j+1}]$ over all lifts $\tilde{a}_1, \tilde{a}_2$ of $a_1, a_2$ to $\tilde{\ell}_k$. Then $c_i$ is $\epsilon'$-hausdorff close to $\beta_i \tilde{\ell}_k$. Therefore, by Proposition 3.5, we see that $c_i$ is a $(\epsilon, 1 + q)$-quasigeodesic. □

By this claim, for large $i$, the holonomy of $b_L \tau_i$ along $\ell_k$ is hyperbolic and its translation length diverges to $\infty$ as $i \to \infty$. Thus $b_L(\tau_i)$ leaves every compact in $\chi$. Thus we have proven the properness. 5.1
6. Characterization of non-properness

In this section, we explicitly describe how \( b_L : \mathcal{T} \rightarrow \chi \) is non-proper when the condition in Theorem 5.1 fails. Let \( L \) be a measured lamination on \( S \). Let \( m_1, \ldots, m_p \) be the periodic leaves of \( L \) which have weight \( \pi \) modulo \( 2\pi \). Then, set \( M = m_1 \sqcup \cdots \sqcup m_p \). Pick any pants decomposition \( P \) of \( S \) which contains \( m_1, \ldots, m_p \). Consider the Fenchel-Nielsen coordinates of \( T \) associated with \( P \). Recall that its length coordinates take values in \( \mathbb{R}_{>0} \) and its twist coordinates in \( \mathbb{R} \).

Theorem 6.1. Let \( \tau_i \) be a sequence \( \mathcal{T} \) which leaves every compact. Then \( b_L(\tau_i) \) converges in \( \chi \) if and only if

- \( \lim \text{length}_{\tau_i} m_j \rightarrow 0 \) for some \( j \in \{1, \ldots, p\} \) as \( i \rightarrow \infty \) (pinched), and
- the Fenchel-Nielsen coordinates of \( \tau_i \) w.r.t. \( P \) converge in their parameter spaces as \( i \rightarrow \infty \), except that the length parameters of the pinched loops go to zero.

We immediately have

Corollary 6.2. The image sequence \( b_L(\tau_i) \) is bounded in \( \chi \) if and only if

- \( \lim \text{length}_{\tau_i} m_j \rightarrow 0 \) as \( i \rightarrow \infty \) for, at least, one \( j \in \{1, \ldots, p\} \), and
- the Fenchel-Nielsen coordinates of \( \tau_i \) w.r.t. \( P \) are bounded in their parameter spaces as \( i \rightarrow \infty \), except that the length parameters of the pinched loops (but including the twisting parameters of the pinched loops).

Proof of Theorem 6.1. Let \( F \) be a component of \( S \setminus M \). Then \( b_L(\tau_i)|F \) converges in \( \chi(F) \) if and only if \( \lim \tau_i|F := \tau_i|\pi_1(F) \) converges.

Let \( E \) and \( F \) be adjacent components of \( \tilde{S} \setminus \tilde{M} \). Let \( \tilde{m} \) be the component of \( \tilde{M} \) separating \( E \) and \( F \), and let \( m \) be the loop of \( M \) which lifts to \( \tilde{m} \). Let \( \Gamma_E \) and \( \Gamma_F \) be the subgroups of \( \pi_1(S) \) preserving \( E \) and \( F \), respectively. Then \( E/\Gamma_E \) and \( F/\Gamma_F \) are the components of \( S \setminus M \); let \( S_E = E/\Gamma_E \) and \( S_F = F/\Gamma_F \).

Proposition 6.3. Let \( \tau_i \) be a sequence of \( \mathcal{T} \), such that the restriction of \( \tau_i \) to \( S_E \) and to \( S_F \) converge in their respective Teichmüller spaces as \( i \rightarrow \infty \). Pick, for each \( i \), a representative \( \xi_i : \pi_1(S) \rightarrow \text{PSL}_2\mathbb{C} \) of \( b_L(\tau_i) \in \chi \) so that \( \xi_i|\Gamma_E \) converges. Then, the restriction \( \xi_i|\Gamma_F \) converges if and only if the Fenchel-Nielsen twisting parameter along \( m \) converges as \( i \rightarrow \infty \).

Proof. For each \( i \), let \( \beta_i : \mathbb{H}^2 \rightarrow \mathbb{H}^3 \) be the bending map for \( (\tau_i, L) \) equivariant via \( \xi_i \), so that \( \beta_i \) converges on \( E \). Let \( M_{\tau_i} \) be the geodesic
Figure 3. The convergence of the twist coordinate under neck-pinching.

representative of \( M \) on \( \tau_i \), and let \( \tilde{M}_{\tau_i} \) be the total lift of \( M_{\tau_i} \) on \( \mathbb{H}^2 \). Let \( \tilde{m}_i \) be the component of \( \tilde{M}_{\tau_i} \) corresponding to \( \tilde{m} \). Let \( F_i, E_i \) be the region on \( \tilde{\tau}_i \setminus \tilde{M}_{\tau_i} \) corresponding to \( F \) and \( E \), respectively. For each \( i \), pick a geodesic ray \( r_i \) in \( F_i \) starting from \( \tilde{m}_i \) such that \( r_i \) is orthogonal to \( \tilde{m}_i \) and that \( r_i \) is does not intersect the total lift \( \tilde{L} \) of \( L \).

Let \( \nu \) be the unit tangent vector of \( r_i \) at the base point on \( \tilde{m}_i \). Since the weight of \( \tilde{m} \) is \( \pi \), \( d\beta_i(v) \) is tangent to \( \beta_i E_i \) at a point of \( \tilde{m} \) (Figure 3, Left).

First suppose that \( \lim_{i \to \infty} \text{length}_{\tau_i} m \) is positive. Then \( \xi_i|\Gamma_F \) converges if and only if \( \beta_i(r) \) converges, which is equivalent to saying the twisting parameter of \( m \) converges in \( \mathbb{R} \) as \( i \to \infty \).

Next suppose that \( \lim_{i \to \infty} \text{length}_{\tau_i} m \) is zero. Then the holonomy of \( m \) converges to a parabolic element not equal to the identity. Then \( \xi_i|\Gamma_F \) converges, if and only if \( \beta_i(r) \) converges to a geodesic starting from the parabolic fixed point. This is equivalent to saying the twisting parameter of \( m \) converges as \( i \to \infty \) (Figure 3, Right).

The theorem follow from Proposition 6.3 as follows: Suppose that \( b_L(\tau_i) \) converges as \( i \to \infty \). Then, the hyperbolic structure on every component of \( S \setminus M \) must converge. Thus, for each loop \( m \) of \( M \), \( \text{length}_{\tau_i} m \) limits to a non-negative number. By Proposition 6.3, as \( b_L(\tau_i) \) converges, the twist parameters along each loop of \( M \) converge. Since \( \tau_i \) leaves every compact, at least one loop of \( M \) must be pinched as \( i \to \infty \). Hence the two conditions hold.

To prove the other direction, suppose that the lengths of some loops of \( M \) limit to zero and all the other Fenchel-Nielsen coordinates with respect to \( P \) converge in the parameter space as \( i \to \infty \). Let \( M' \) be the sub-multiloop of \( M \) consisting of the loops whose lengths go to zero. Then, for each component \( F \) of \( S \setminus M' \), \( b_L(\tau_i)|\pi_1(F) \) converges as \( i \to \infty \). Therefore, by Proposition 6.3, \( b_L(\tau_i) \) converges. This completes the
7. Framed Character Varieties along Loops

We have analyzed the real analytic embedding $b_L : \mathcal{T} \to \chi$ defined for an arbitrary measured lamination $L \in \mathcal{ML}$. As $\mathcal{T}$ is regarded as the Fricke space, a component of the real slice of the character variety $\chi$, one can certainly extend $b_L$ to a holomorphic mapping from a neighborhood of $\mathcal{T}$ in $\chi$ to $\chi$. However, it does not extend to the entire character variety $\chi$ for multiple reasons. Thus, in this section, we modify the character variety $\chi$ and obtain a closed complex analytic set, which will be a domain of the complexification.

For a surface with punctures, Fock and Goncharov introduced a framing of a surface group representation ([FG06]). Their framing assigns a fixed point of peripheral holonomy around each puncture. In particular, the framing was useful to describe the deformation space of $\mathbb{C}P^1$-structures on a surface with punctures by their framed holonomy representations ([AB20, GM21, Bab]).

In this paper, we introduce a certain framing along loops which assigns a pair of distinct fixed points of their holonomy. Such framings will be used to determine the axes for bending deformation even if the holonomy along loops is trivial.

7.1. Framing of Representations along a loop. For simplicity, we first discuss the modification in the case that the bending lamination is a single loop. Let $\mathcal{R}$ be the space of representations $\pi_1(S) \to \text{PSL}_2\mathbb{C}$ (without any equivalence relation). Then $\mathcal{R}$ is an affine algebraic variety: Namely, pick a presentation of the fundamental group $\pi_1(S)$, for instance

$$\pi_1(S) = \langle a_1, b_1, \ldots, a_g, b_g | \Pi_{i=1}^n [a_i, b_i] \rangle.$$

Since $\text{PSL}_2(\mathbb{C})$ embeds into $\text{GL}_3(\mathbb{C})$ by the adjoint representation, $\text{PSL}_2(\mathbb{C})$ is a complex affine Lie group sitting in $\mathbb{C}^9$. Then, by the embedding $\mathcal{R} \to (\mathbb{C}^9)^{2g}$ defined by

$$\rho \mapsto (\rho(a_1), \rho(b_1), \ldots, \rho(a_g), \rho(b_g)) \in (\mathbb{C}^9)^{2g},$$

$\mathcal{R}$ has an affine algebraic structure on cut by the equation corresponding to the relator $\Pi_{i=1}^n [a_i, b_i]$.

Let $\ell$ be an oriented simple closed curve on $S$. The orientation will later be used to determine the bending direction in §8.2 in order to define the complexification. Let $\Lambda_\ell$ be the set of elements in $\pi_1(S)$ whose free homotopy classes are the homotopy class of $\ell$ on $S$; clearly, elements in $\Lambda$ are conjugate to each other by elements in $\pi_1(S)$. 
Pick an element $\alpha \in \Gamma_\ell$. Let $\rho: \pi_1(S) \to \text{PSL}_2\mathbb{C}$ be a homomorphism. Suppose that $\rho(\alpha_\ell)$ is not a parabolic (but it can be the identity). Then, there is an ordered pair $(u, v)$ of distinct points $u, v$ on $\mathbb{C}P^1$ which are fixed by $\alpha_\ell$ pointwise. We can equivariantly extends a pair $(u, v)$ to pairs $(u_\gamma, v_\gamma)$ for all representatives $\gamma \in \Lambda_\ell$ so that $\gamma$ fixes $u_\gamma$ and $v_\gamma$ in $\mathbb{C}P^1$. Such an equivariant assignment $(u_\gamma, v_\gamma)_{\gamma \in \Lambda_\ell}$ of ordered fixed points of $\gamma$ is called a framing of $\rho$ along $\ell$. By abuse of notation, we denote this equivariant framing $\{(u_\gamma, v_\gamma)\}_{\gamma \in \Lambda_\ell}$ by $(u, v)$, since it is determined by the initial choice $(u, v)$ for $\alpha_\ell$. We call the triple $(\rho, u, v)$ a framed representation. In order to produce the equivariant bending axes (later), we utilize the equivariant framing. Let

$$R_\ell = \left\{ (\rho, u, v) \in \mathcal{R} \times (\mathbb{C}P^1)^2 \mid \rho(\alpha_\ell)u = u, \rho(\alpha_\ell)v = v, u \neq v \right\}.$$ 

Then $R_\ell$ is a closed analytic subset of $\mathcal{R} \times (\mathbb{C}P^1 \times \mathbb{C}P^1 \setminus D)$, where $D$ is the diagonal $\{(z, z) \mid z \in \mathbb{C}P^1\}$. Note that if $(\rho, u, v) \in R_\ell$, then $\rho(\alpha_\ell)$ can not be a parabolic element, since $u, v$ are distinct fixed points of $\rho(\alpha_\ell)$. On the other hand, $\rho(\alpha_\ell)$ can be the identity.

Let $\mathcal{G}_\ell$ be the subgroup of the mapping class group of $S$ which preserves the oriented loop $\ell$. Clearly, $\mathcal{G}_\ell$ acts on $R_\ell$ by marking change.

We now assume that the oriented loop $\ell$ has a weight $w$ in $\mathbb{R}_{>0}$. Suppose, first, that the weight of the oriented loop $\ell$ is not equal to $\pi$ modulo $2\pi$. Fix any complex number $w \in \mathbb{C}$ with $|w| > 1$. Then, given $(u, v) \in \mathbb{C}P^1 \times \mathbb{C}P^1$ with $u \neq w$, there is a unique hyperbolic element $\gamma_{u,v,w} \in \text{PSL}_2\mathbb{C}$, such that $u$ is the repelling fixed point, $v$ is the attracting fixed point of $\gamma_{u,v,w}$ and that $\gamma_{u,v,w}$ can be conjugated to the hyperbolic element $z \mapsto wz$ by an element of $\text{PSL}_2\mathbb{C}$. Clearly, this mapping $(u, v) \mapsto \gamma_{u,v,w}$ is a biholomorphic mapping onto its image. Then, $(\rho, u, v) \in R_\ell$ biholomorphically corresponds to a unique element $(\rho, \gamma_{u,v,w})$ of $\mathcal{R} \times \text{PSL}_2\mathbb{C}$. Thus $R_\ell \to \mathcal{R} \times \text{PSL}_2\mathbb{C}$ is a biholomorphic map onto its image. Since $\text{PSL}_2\mathbb{C} \cong \text{SO}_3(\mathbb{C}) \subset \mathbb{C}^9$, we see that $R_\ell$ is biholomorphic to a closed analytic set in a complex vector space of finite dimension. (It is closed, since if $(u, v) \in (\mathbb{C}P^1)^2 \setminus \Delta$ converges to a point in the diagonal $\Delta$, then $\gamma_{u,v,w}$ must leaves every compact subset of $\text{PSL}_2\mathbb{C}$.) Therefore $R_\ell$ is also a Stein space, as it is a closed analytic subset of a Stein space.

The theory of categorical quotients of Stein manifolds has been developed analogously to GIT-quotients affine algebraic varieties (see [Sno82]). We let $X_\ell$ be the categorical quotient (Stein quotient) $R_\ell \sslash \text{PSL}_2\mathbb{C}$, which is again Stein. In this quotient, two framed representations $(\rho_1, u_1, v_1)$ and $(\rho_2, u_2, v_2)$ in $R_\ell$ are identified if and only if every $\text{PSL}_2\mathbb{C}$-invariant analytic function $f$ on $R_\ell$ takes the same value at
(ρ₁, u₁, v₁) and (ρ₂, u₂, v₂); see [Sno82, §3]. We denote, by [ρ, u, v], the equivalence class of (ρ, u, v) in \(X_ℓ\).

Next suppose that ℓ has weight π modulo 2π. In this case, the ordering of the framing \((u, v)\) will not affect the complexified bending map, and thus we take a slightly stronger quotient. Then, let \(\gamma_{u,v}\) be the elliptic element of angle π with the axes connecting \(u\) and \(v\). Let \(R_ℓ/\mathbb{Z}_2\) be the quotient of \(R_ℓ\) by the \(\mathbb{Z}_2\)-action which switches the ordering of the framing, namely, given by \((ρ, u, v) → (ρ, v, u)\). Consider the map \(R_ℓ/\mathbb{Z}_2 → \mathcal{R} × \text{PSL}_2\mathbb{C}\) defined by \((ρ, u, v) → (ρ, γ_{u,v})\). Thus \(R_ℓ/\mathbb{Z}_2\) is biholomorphic to a closed analytic set in \(\mathcal{R} × \text{PSL}_2\mathbb{C}\). Similarly, we let \(X_ℓ\) be the stein quotient \((R_ℓ/\mathbb{Z}_2) // \text{PSL}_2\mathbb{C}\). The action of \(\mathfrak{g}_ℓ\) on \(R_ℓ\) descends to an action on \(X_ℓ\).

7.1.1. Coordinates for the quotient space of representations framed along a single loop. We defined the Stein space \(X_ℓ\) as a Stein quotient. In this section, we indeed realize \(X_ℓ\) as an analytic set in an affine space by identifying it with a subset of a \(\text{PSL}_2\mathbb{C}\)-character variety \(χ(π₁(S) * \mathbb{Z})\) of \(π₁(S) * \mathbb{Z}\). Recall that, for \((ρ, u, v) ∈ R_ℓ\), the element \(γ_{u,v,w} ∈ \text{PSL}_2\mathbb{C}\) is a certain hyperbolic element if the weight of the oriented loop ℓ is not equal to π modulo 2π and a certain elliptic element of angle π otherwise.

Given \((ρ, u, v) ∈ R_ℓ\), let \(\hat{ρ} = \hat{ρ}_{u,v,w}\) be the homomorphism from the free product \(π₁(S) * \mathbb{Z}\) to \(\text{PSL}_2\mathbb{C}\), such that every \(γ ∈ π₁(S)\) maps to \(ρ(γ)\) and \(1 ∈ \mathbb{Z}\) maps to \(γ_{u,v,w}\). Then, with respect to the \(\text{PSL}_2\mathbb{C}\)-action on \(R_ℓ\), we clearly have the following.

**Lemma 7.1.**

1. Suppose that the weight of ℓ is not equal to π modulo 2π. Then \((ρ₁, u₁, v₁)\) and \((ρ₂, u₂, v₂)\) are identified by an element of \(\text{PSL}_2\mathbb{C}\) if and only if \(\hat{ρ}_1\) and \(\hat{ρ}_2\) are conjugate by \(\text{PSL}_2\mathbb{C}\).

2. Suppose that the weight of ℓ is equal to π modulo 2π. Then \((ρ₁, u₁, v₁)\) and \((ρ₂, u₂, v₂)\) are identified by an element of \(\text{PSL}_2\mathbb{C} \times \mathbb{Z}_2\) if and only if \(\hat{ρ}_1\) and \(\hat{ρ}_2\) are conjugate conjugate by \(\text{PSL}_2\mathbb{C}\), where the \(\mathbb{Z}_2\)-action exchanges the ordering of the framing.

Let \(\hat{R}\) be the space of representations \(π₁(S) * \mathbb{Z} → \text{PSL}_2\mathbb{C}\). Then \(\hat{R}\) is an affine algebraic variety. Suppose that the weight of ℓ is not equal to π modulo 2π. We have seen that the mapping \(R_ℓ → \mathcal{R} × \text{PSL}_2\mathbb{C}\) is a biholomorphic map onto its image by the mapping \((ρ, u, v) → \hat{ρ}\). Let \(\hat{R}_ℓ\) be this image. Then \(\hat{R}_ℓ\) is the closed analytic subset in \(\hat{R}\) biholomorphic to \(R_ℓ\), and thus in particular it is Stein. Moreover, this
biholomorphism $R_\ell \to \hat{R}_\ell$ is equivariant with respect to the PSL$_2\mathbb{C}$-action. Thus the Stein space $X_\ell = R_\ell//\text{PSL}_2\mathbb{C}$ is biholomorphic to the subvariety $\hat{R}_\ell//\text{PSL}_2\mathbb{C}$ of $\chi(\pi_1(S) \ast \mathbb{Z})$.

A similar identification holds in the case when $\ell$ has weight $\pi$ modulo $2\pi$. The Stein space $R_\ell/\mathbb{Z}_2$ biholomorphically maps to its image, denoted by $\hat{R}_\ell$, in $\hat{R}$ by the mapping $(\rho, u, v) \mapsto \hat{\rho}$. Then $X_\ell = (R_\ell/\mathbb{Z}_2)//\text{PSL}_2\mathbb{C}$ is biholomorphic to the Stein space $\hat{R}_\ell//\text{PSL}_2\mathbb{C}$.

Let $\gamma \in \pi_1(S) \ast \mathbb{Z}$. Let $\text{tr}^2(\gamma)$ be the (polynomial) function on $\hat{R}_\ell$ defined by $(\rho, u, v) \mapsto \text{tr}^2(\rho(\gamma))$. Then $\text{tr}^2(\gamma)$ is a PSL$_2\mathbb{C}$-equivariant analytic function on $\hat{R}_\ell$. Then, by [HP04, Corollary 2.3], such trace square functions form coordinates of the Stein quotient $\hat{R}_\ell//\text{PSL}_2\mathbb{C}$, and they also form coordinates for $X_\ell(\cong \hat{R}_\ell//\text{PSL}_2\mathbb{C})$.

**Proposition 7.2.** There are finitely many elements $\gamma_1, \gamma_2, \ldots, \gamma_N$ in $\pi_1(S) \ast \mathbb{Z}$ such that the analytic mapping $\hat{R}_\ell \to \mathbb{C}^N$ given by $\text{tr}^2(\gamma_1), \ldots, \text{tr}^2(\gamma_N)$ induces an analytic embedding of $X_\ell$ into $\mathbb{C}^N$. Thus $\text{tr}^2(\gamma_1), \ldots, \text{tr}^2(\gamma_N)$ form a coordinate ring.

### 7.2. Representations framed along a multi-loop

In §7.1, we introduced the space of representations $\pi_1(S) \to \text{PSL}_2\mathbb{C}$ framed along a single (oriented) loop, constructed a quotient space by the PSL$_2\mathbb{C}$ action, and realized as an analytic subset of a complex affine space. In this section, we similarly consider the space of representations framed along a weighted multiloop, and then construct its Stein quotient by the action of PSL$_2\mathbb{C}$.

Let $m_1, \ldots, m_n$ be non-isotopic essential simple closed curves on $S$, and let $M$ be their union $m_1 \sqcup m_2 \sqcup \cdots \sqcup m_n$. Recall that $\mathcal{R}$ denotes the representation variety $\{\pi_1(S) \to \text{PSL}_2\mathbb{C}\}$. For each $i = 1, \ldots, n$, let $\Lambda_{m_i}$ denote the set of elements in $\pi_1(S)$ whose free homotopy classes are the homotopy class of $m_i$. Pick a representative $\alpha_i \in \Lambda_{m_i}$. Then, consider the space $R_M$ of tuples $(\rho, (u_i, v_i)_{i=1}^n) \in \mathcal{R} \times (\mathbb{C}P^1)^{2n}$ where

- $\rho \in \mathcal{R}$ is a homomorphism $\pi_1(S) \to \text{PSL}_2\mathbb{C}$, and
- $u_i, v_i \in \mathbb{C}P^1$ are different fixed points of $\rho(\alpha_i)$ for $i = 1, \ldots, n$.

As in the case of a single loop, $\rho(\alpha_i)$ are not parabolic elements (but can be the identity). Then $R_M$ is a closed analytic subvariety of $\mathcal{R} \times (\mathbb{C}P^1 \times \mathbb{C}P^1 \setminus \Delta)^n$, where $\Delta$ denotes the diagonal as before. Given $(\rho, (u_i, v_i)_{i=1}^n) \in R_M$, we can equivariantly extend $(u_i, v_i)$ to the pairs of fixed points for all representatives of $m_1, \ldots, m_n$ in $\pi_1(S)$. We call this extension a **framing** of $\rho$ along the multiloop $M$.

Now we assign a positive number (**weight**) to each loop of $M$. Let $p$ be the number of components $m_i$ of $M$, such that the weight of $m_i$ is $\pi$ modulo $2\pi$. Without loss of generality, we can assume $m_1, \ldots, m_n$
are the loops of $M$ with weight $\pi$ modulo $2\pi$. Then, $\mathbb{Z}_2^p$ acts biholomorphically on $R_M$ by switching the ordering of the fixed points of the framing along $m_1, \ldots, m_n$. Note that this $\mathbb{Z}_2^p$-action has no fixed points in $R_M$.

Fix a complex number $w \in \mathbb{C}$ with $|w| > 1$. As in §7.1.1, let $\gamma_{u_i,v_i,w} \in \text{PSL}_2\mathbb{C}$ be, if the weight of $m_i$ is $\pi$ modulo $2\pi$, then the elliptic element of angle $\pi$ whose axis is the geodesic connecting $u_i$ to $v_i$, and otherwise, the hyperbolic element with the repelling fixed point $u_i$ and the attracting fixed point $v_i$ such that $\gamma_{u_i,v_i,w}$ is conjugate to the dilation $z \mapsto wz$. Then, define the mapping $R_M \to \mathbb{R} \times (\text{PSL}_2\mathbb{C})^m$ by $(\rho, (u_i,v_i)_{i=1}^n) \mapsto (\rho, (\gamma_{u_i,v_i,w})_{i=1}^n)$. This mapping takes $R_M/\mathbb{Z}_2^p$ onto its image $\hat{R}_M$ biholomorphically. Thus $R_M/\mathbb{Z}_2^p$ is a closed analytic set in a finite-dimensional complex vector space. Therefore $R_M/\mathbb{Z}_2^p$ is Stein. The Lie group $\text{PSL}_2\mathbb{C}$ acts analytically on $R_M/\mathbb{Z}_2^p$, by conjugation on $\rho$. By this action, we obtain its Stein quotient $(R_M/\mathbb{Z}_2^p) \bowtie \text{PSL}_2\mathbb{C} =: X_M$. Thus $X_M$ is a Stein space.

The biholomorphic map $R_M/\mathbb{Z}_2^p \to \hat{R}_M$ is equivariant w.r.t. the $\text{PSL}_2\mathbb{C}$-action, $X_M$ is biholomorphic to the corresponding Stein quotient $\hat{R}_M \bowtie \text{PSL}_2\mathbb{C}$.

We denote, by $[\rho, (u_i,v_i)]$, the equivalence class of $(\rho, (u_i,v_i)) \in R_M$ in $X_M$. The subgroup $\mathcal{G}_M$ of $\text{MCG}$ acts on $R_M$, and descends to an action on $X_M$.

7.2.1. Coordinates of the quotient space of representations framed along a multiloop. Let $g_1, g_2, \ldots, g_n$ be a standard generating set of the free group $\mathbb{F}_n$ of rank $n$, so that there are no relators. Every $(\rho, (u_i,v_i)_{i=1}^n) \in \hat{R}_M$ corresponds to a unique representation $\pi_1(S) \ast \mathbb{F}_n \to \text{PSL}_2\mathbb{C}$ such that

- $\gamma \in \pi_1(S)$ maps to $\rho(\gamma)$, and
- $g_i$ maps to $\gamma_{u_i,v_i,w}$ for every $i = 1, \ldots, n$.

By this correspondence, $R_M$ analytically embed into the space of representations $\pi_1(S) \times \mathbb{F}_n \to \text{PSL}_2\mathbb{C}$. As in §7.1.1, by the quotient of the image $\mathcal{R}_M$ by $\text{PSL}_2\mathbb{C}$, [HP04, Corollary 2.3] yields the coordinate ring of $X_M \cong \mathcal{R}_M \bowtie \text{PSL}_2\mathbb{C}$.

**Proposition 7.3.** There are finitely many elements $\gamma_1, \gamma_2, \ldots, \gamma_N$ of $\pi_1(S)$ corresponding to simple closed curves, such that $\text{tr}^2(\gamma_1), \ldots, \text{tr}^2(\gamma_N)$ form a coordinate ring of $X_M$. 
8. Bending a surface group representation into $\text{PSL}_2\mathbb{C}$ inside the representation space into $\text{PSL}_2\mathbb{C} \times \text{PSL}_2\mathbb{C}$

Originally bending deformation equivariantly bends a totally geodesic $\mathbb{H}^2$ along a measured lamination ([Thu81, EM87]), so that bending is in one-direction and the bent $\mathbb{H}^2$ is locally convex. Moreover, one can extend it to an equivariant bending pleated surface along the pleated locus using bending cocycles ([Bon96]). In both cases, bending is done along (bi-infinite)geodesics in $\mathbb{H}^3$ which are embedded in the pleated surfaces.

In this section, we introduce a new type of bending of more topological equivariant surfaces in $\mathbb{H}^3$. Using such more general bending, define a complex-analytic bending map $X_M \to \chi \times \chi$ which complexifies the real-analytic bending map $T \to \chi$.

8.1. A complexification of the Lie group $\text{PSL}_2\mathbb{C}$ regarded as a real Lie group. We first recall a complexification of $\text{PSL}_2\mathbb{C}$ when regarded as a real Lie group.

**Proposition 8.1** (See Proposition 1.39 in [Zil] for example). **Regard $\text{psl}_2\mathbb{C}$ as a real Lie algebra. Then the complexification of the Lie algebra $\text{psl}_2\mathbb{C}$ is isomorphic to $\text{psl}_2\mathbb{C} \oplus (\text{psl}_2\mathbb{C})^*$ by the mapping given by** $(u, 0) \mapsto (u, Iu)$ and $(0, v) \mapsto (v, -Iv)$, where $(\text{psl}_2\mathbb{C})^*$ is the complex conjugate of $\text{psl}_2\mathbb{C}$ and $I$ is the complex multiplication of $\text{psl}_2\mathbb{C}$.

We regard $\text{PSL}_2\mathbb{C}$ as a real Lie group, and we complexify $\text{PSL}_2\mathbb{C}$ by $c: \text{PSL}_2\mathbb{C} \to \text{PSL}_2\mathbb{C} \times \text{PSL}_2\mathbb{C}$, so that it corresponds to Proposition 8.1. Then $c$ is holomorphic in the first factor and anti-holomorphic in the second factor. Thus $c$ is, in particular, a proper real-analytic embedding of $\text{PSL}(2, \mathbb{C})$ into the complex Lie group $\text{PSL}_2\mathbb{C} \times \text{PSL}_2\mathbb{C}$.

8.2. Bending framed representations. We first define a complex bending of representations framed along a single loop. Let $\ell$ be an oriented loop $\ell$ on $S$, and we fixed a weight $w > 0$ of $\ell$. Fix $\alpha \in \pi_1(S)$ representing of $\ell$. Let $[(\rho, u, v)] \in X_\ell$, where $\rho: \pi_1(S) \to \text{PSL}_2\mathbb{C}$ and $(u, v)$ is a pair of fixed points of $\rho(\alpha)$. Let $(\rho, \rho): \pi_1(S) \to \text{PSL}_2\mathbb{C} \times \text{PSL}_2\mathbb{C}$ denote the diagonal representation given by $\gamma \mapsto (\rho(\gamma), \rho(\gamma))$.

Recall that $(u, v)$ generates a $\rho$-equivariant farming $f$ along $\ell$ and $\Lambda_\ell$ denotes the subset of $\pi_1(S)$ corresponding to $\ell$. That is, for every
element $\gamma \in \Lambda_{\ell}$, an ordered pair $(u_{\gamma}, v_{\gamma}) \in \mathbb{CP}^1 \times \mathbb{CP}^1$ of different fixed points of $\rho(\gamma)$ is assigned $\rho$-equivariantly. Consider the oriented geodesic $g_{\gamma} = (u_{\gamma}, v_{\gamma})$ in $\mathbb{H}^3$ connecting $u_{\gamma}$ to $v_{\gamma}$ for all $\gamma \in \Lambda_{\ell}$. Those equivariant geodesics $\{g_{\gamma}\}$ will be the axes of the bending.

First, we coherently define the direction of the bending so that bending is continuously defined on $X_{\ell}$. Pick any $\rho$-equivariant piecewise smooth surface $\Sigma: \tilde{S} \to \mathbb{H}^3$. Then, as $S$ is oriented, at every smooth point $x$ of $\Sigma$, there are a normal direction of $\Sigma$ and the hyperbolic plane tangent to $\Sigma$ at $x$.

Let $\tilde{\ell}$ be the lift of $\ell$ to the universal cover $\tilde{S}$ invariant by $\gamma \in \Lambda_{\ell}$. Then, homotope $\Sigma$ in $\mathbb{H}^3$ equivariantly and piecewise smoothly so that

1. $\Sigma$ takes $\tilde{\ell}$ into the bi-infinite geodesic $(u, v)$, and
2. $\Sigma: \tilde{S} \to \mathbb{H}^3$ is a local embedding at every point $x \in \tilde{\ell} \setminus \tilde{V}$ for some an $\gamma$-invariant discrete set $\tilde{V}$ in $\tilde{\ell}$ (See Figure 4).

In particular, each component of $\tilde{\ell} \setminus \tilde{V}$ is embedded in $[u, v]$ by $\Sigma$. Moreover, as $\Sigma$ is $\rho$-equivariant, there is at least one segment $s$ of $\tilde{\ell} \setminus \tilde{V}$ whose direction matches the oriented geodesic $[u, v]$ by $\Sigma$. Then, the orientation of the surface near the segment determines the positive direction of the bending, so that the normal direction is the exterior; see Figure 5. In the special case that $\rho$ is Fuchsian, $\Sigma$ can be taken to be an equivariant embedding onto a totally geodesic hyperplane if the orientation of $\ell$ matches with the ordering of the framing.

The choice of the homotopy of $\Sigma$ will not affect the resulting bending map determined by the normal direction. If $\Sigma'$ is another $\rho$-equivariant piecewise smooth homotopy to $\Sigma$ satisfying (1) and (2), then there is a piecewise smooth homotopy between $\Sigma$ and $\Sigma'$ preserving the conditions and taking the normal vector on $\Sigma$ to the normal vector on $\Sigma'$.
Then, for every $\theta \in \mathbb{R}$, we can bend the $\rho$-equivariant surface $\Sigma: \pi_1(S) \to \text{PSL}_2\mathbb{C}$ along the equivariant axes $\{g_s\}$ by angle $\theta$ with respect to the positive bending direction defined above. As we bend $\Sigma$ in an equivariant manner, the bent surface $\tilde{S} \to \mathbb{H}^3$ remains equivariant via a unique representation; we denote the induced representation by $b_{\ell,\theta}(\rho, u, v): \pi_1(S) \to \text{PSL}_2\mathbb{C}$. We now define a complex bending map $B_{\ell}: X_\ell \to \chi \times \chi$ by $B_{\ell,w}(\rho, u, v) = (b_{\ell,w}(\rho, u, v), b_{\ell,-w}(\rho, u, v))$. Note that, in the fast factor and the second factor, the bending $\rho$ is equivariantly done along the same axes and by the same angle, but in the opposite directions (Figure 6).

The bent representation is well-defined up to conjugation by an element of $\text{PSL}_2\mathbb{C} \times \text{PSL}_2\mathbb{C}$, and thus $B_{\ell}(\rho, u, v) \in \chi \times \chi$ is well-defined. We remark that, if $\rho: \pi_1(S) \to \text{PSL}_2\mathbb{C}$ is Fuchsian, then the representation of $B_{\ell}(\rho, u, v)$ in the first factor $\chi$ is the complex conjugate of that in the second factor.

For an oriented weighted multiloop $M$ on $S$, we can similarly define the complex bending map $B_M: X_M \to \chi \times \chi$ as follows. Let $m_1, \ldots, m_n$ are the weighted loops of $M$. Pick $\gamma_i \in \Lambda_{m_i}$. Let $\tilde{m}_i$ be a $\gamma_i$ invariant lift
of \( m_i \) to the universal cover \( \tilde{S} \). Let \([\rho, (u_i, v_i)]_{i=1}^n \in X_M\), where \((u_i, v_i)\) be the fixed point of \( \rho(\gamma_i) \). Then the oriented geodesic \( g_i \) connecting \( u_i \) to \( v_i \), equivariantly extends to a system of bending axes corresponding to all lifts of \( m_i \) to \( \tilde{S} \). Find a \( \rho \)-equivariant piecewise smooth surface \( \Sigma: \tilde{S} \rightarrow \mathbb{H}^3 \) such that \( \tilde{m}_i \) maps to its corresponding axes \( g_i \), and there is an \( \gamma_i \) invariant discrete subset \( V_i \) of \( \tilde{m}_i \) such that \( \Sigma \) is a local embedding at every point on \( \tilde{m}_i \setminus V_i \). Then, there is a segment \( s_i \) of \( \tilde{m}_i \setminus V_i \) such that its orientation matches with the orientation of \( g_i = (u_i, v_i) \). The normal direction of \( \Sigma \) at a point in \( s_i \) determines the positive bending direction of \( \Sigma \) along \( g_i \). Similarly, one can show the positive bending direction does not depend on the choice of \( \Sigma \) by property-preserving homotopy between different choices of \( \Sigma \).

Let \( \theta_1, \ldots, \theta_n \) be real numbers. We can bend the \( \rho \)-equivariant surface \( \Sigma: \tilde{S} \rightarrow \mathbb{H}^3 \) along the geodesics \( g_1, \ldots, g_n \) and their orbit geodesics by angles \( \theta_1, \ldots, \theta_n \), respectively, in the positive bending direction defined above. Since we bend \( \Sigma \) in an equivariant manner, the new bent surface \( \Sigma^+: \tilde{S} \rightarrow \mathbb{H}^3 \) is also equivariant via a unique representation. We denote the bend representation by

\[
\tilde{b}(\rho, (u_i, v_i)_{i=1}^n) = b_M(\rho, (u_i, v_i)_{i=1}^n): \pi_1(S) \rightarrow \text{PSL}_2\mathbb{C}.
\]

Similarly, we can bend \( \Sigma \) along the same axes by the same angles but in opposite directions, and we obtain another bent surface \( \Sigma^-: \tilde{S} \rightarrow \mathbb{H}^3 \). Then \( \Sigma^- \) is also equivariant via a unique representation

\[
b_M(\rho, (u_i, v_i)_{i=1}^n) = b_M(\rho, (u_i, v_i)_{i=1}^n): \pi_1(S) \rightarrow \text{PSL}_2\mathbb{C}.
\]

By combining those two bending of framed representations, we obtain the bending map \( B_M: X_M \rightarrow \chi \times \chi \) by

\[
B_M(\rho, (u_i, v_i)_{i=1}^n) = (b_{(m_i, w_i)}(\rho, (u_i, v_i)_{i=1}^n), b_{(m_i, -w_i)}(\rho, (u_i, v_i)_{i=1}^n)).
\]

Then the mapping \( \tilde{S} \rightarrow \mathbb{H}^3 \times \mathbb{H}^3 \) defined by \( x \mapsto (\Sigma^+(x), \Sigma^-(x)) \) is equivariant via \( B_M(\rho, (u_i, v_i)_{i=1}^n): \pi_1(S) \rightarrow \text{PSL}_2 \times \text{PSL}_2\mathbb{C} \).

### 8.3. Equivariant property

**Lemma 8.2.** Let \( M \) be a weighted oriented multiloop on \( S \). Let \( G_M \) be the subgroup of the mapping class group \( \text{MCG}(S) \), which preserves \( M \). Then \( B_M: X_M \rightarrow \chi \times \chi \) is \( G_M \)-equivariant.

**Proof.** Recall that \( G_M \) acts on \( X_M \) by marking change. Then \( b_{(m_i, w_i)}: X_M \rightarrow \chi \) and \( b_{(m_i, -w_i)}: X_M \rightarrow \chi \) are both \( G_M \)-equivariant, since the action is marking change. Therefore \( B_M \) is also \( G_M \)-equivariant. \( \square \)
8.4. Support planes and spaces. For a marked hyperbolic surface \( \tau \) homeomorphic to \( S \) and a measured lamination \( \tilde{L} \) on \( \tau \), we have a \( \pi_1(S) \)-equivariant bending map \( \beta_{\tau,L} : \mathbb{H}^2 \to \mathbb{H}^3 \) which is “locally convex". Letting \( \tilde{L} \) be the \( \pi_1(S) \)-invariant measured lamination on the universal cover \( \mathbb{H}^2 \) of \( \tau \). Then, for each component \( P \) of \( \mathbb{H}^2 \setminus \tilde{L} \), the mapping \( \beta_{\tau,L} \) embeds \( P \) into a totally geodesic hyperbolic plane \( P \) in \( \mathbb{H}^3 \). Such a hyperbolic plane is a support plane for \( \beta_{\tau,L} \). (See [EM87] for more general support planes.) On the other hand, this equivariant system \( \{ H_P \}_P \) of totally geodesic hyperbolic planes, indexed by the components, determines the original bending map \( b_{\tau,L} : \mathbb{H}^2 \to \mathbb{H}^3 \) for all components \( P \).

In §8, we bend framed representations \( \eta = [\rho, (u_i, v_i)] \) in \( X_M \) along a weighted oriented multiloop \( \tilde{M} \) defined in , and obtained a representation \( \pi_1(S) \to \text{PSL}_2\mathbb{C} \times \text{PSL}_2\mathbb{C} \). As the symmetric space associated with \( \text{PSL}_2\mathbb{C} \times \text{PSL}_2\mathbb{C} \) is the product \( \mathbb{H}^3 \times \mathbb{H}^3 \), we consider a system of supporting hyperbolic three-spaces in the product \( \mathbb{H}^3 \times \mathbb{H}^3 \) as follows. For every component \( P \) of \( \tilde{S} \setminus \tilde{M} \), the restriction of \( \Sigma^+ \) to \( P \) coincides with the restriction of \( \Sigma^- \) to \( P \) composed with an element \( \gamma \) of \( \text{PSL}_3\mathbb{C} \). Therefore, the restriction of the surface \( (\Sigma^+, \Sigma^-) : \tilde{S} \to \mathbb{H}^3 \times \mathbb{H}^3 \) to \( P \) is contained in a totally geodesic copy \( H_P \) of \( \mathbb{H}^3 \) given by \( \{(x, \gamma x) \mid x \in \mathbb{H}^3 \} \subset \mathbb{H}^3 \times \mathbb{H}^3 \).

Hence, we obtain an equivariant collection of supporting hyperbolic 3-spaces \( H_P \) for all components \( P \) of \( \tilde{S} \setminus \tilde{M} \). We call this collection \( \{ H_P \}_P \) the (equivariant) bending support system of \( B_M \) at \( \eta \). Let \( G_P \) denote the subgroup of \( \pi_1(S) \) consisting of the elements preserving the \( P \). Then \( H_P \) is preserved by the restriction of the bend representation

\[
B_M(\rho, (u_i, v_i)_{i=1}^n) : \pi_1(S) \to \text{PSL}_2\mathbb{C} \times \text{PSL}_2\mathbb{C}
\]

to the subgroup \( G_P \).

Suppose that \( P \) and \( P' \) are adjacent components of \( \tilde{S} \setminus \tilde{M} \) across a lift \( \tilde{m} \) of a loop \( m \) of \( M \). Let \( w \) be the weight of \( m \). Then, in \( \mathbb{H}^3 \times \mathbb{H}^3 \), the support spaces \( H_P \) and \( H_{P'} \) intersect in a geodesic at angle \( w \) (complex bending axis), which corresponds to the bending axis in \( \mathbb{H}^3 \) induced by the framing in the definition of \( B_M \) (Figure 7). In particular, if the weight of \( m \) is a multiple of \( \pi \), then \( H_P = H_{P'} \). Indeed, for an elliptic element \( e \in \text{PSL}_2\mathbb{C} \) with rotation angle \( \pi \), we have

\[
\{(x, x) \in \mathbb{H}^3 \times \mathbb{H}^3 \mid x \in \mathbb{H}^3 \} = \{(ex, e^{-1}x) \in \mathbb{H}^3 \times \mathbb{H}^3 \mid x \in \mathbb{H}^3 \}.
\]

Definition 8.3. Let \( \xi : \pi_1(S) \to \text{PSL}_2\mathbb{C} \times \text{PSL}_2\mathbb{C} \) be a representation. A support system of \( \xi \) with respect to \( M \) is an equivariant collection of totally geodesic hyperbolic planes \( H_P \) for all components \( P \) of \( \tilde{S} \setminus \tilde{M} \).
Figure 7. The intersection angle $w$ of totally geodesic copies $H_P, H_{P'}$ of $\mathbb{H}^3$ in $\mathbb{H}^3 \times \mathbb{H}^3$.

such that the restriction of $\xi$ to $G_P$ preserves $H_P$ for all components $P$ of $S \setminus \tilde{M}$.

In general, a representation $\pi_1(S) \to \text{PSL}_2\mathbb{C} \times \text{PSL}_2\mathbb{C}$ may have no support system or many support systems. On the other hand, we will prove that the support system is uniquely determined by $B_M(\rho, (u_i, v_i)_{i=1}^n)$ in most cases; see Lemma 9.2.

9. Complex bending maps are almost injective

In this section, we prove the injectivity of the complex bending map $B_M: X_M \to \chi \times \chi$ when restricted to the complement of certain subvarieties.

Let $M$ be an oriented weighted multiloop on $S$, and let $n$ be the number of the loops of $M$. Let $X^p_M$ be the subset of $X_M$ consisting of the framed representations $(\rho, (u_i, v_i)_{i=1}^n)$ such that $\text{tr}^2 \rho(m) = 4$ for, at least, one loop $m$ of $M$, i.e. $\rho(m)$ is either a parabolic element or the identity. As it is an algebraic equation, $X^p_M$ is an analytic subvariety of $X_M$.

Let $X^r_M$ be the subset consisting of the framed representations $(\rho, u, v)$ such that, for some component $F$ of $S \setminus M$, the restriction of $\rho$ to $\pi_1(F)$ is weakly reducible, i.e. the image is, up to a finite index, reducible. The set of weakly reducible representations $\pi_1(S) \to \text{PSL}_2\mathbb{C}$ is contained in a subvariety of $\chi$. Thus $X^r_M$ is also an analytic subset of $X_M$. We prove that the injectivity of the complex bending map holds in the complement of those analytic subsets.

**Theorem 9.1.** Let $M$ be a weighted oriented multiloop on $S$. Then, the complex bending map $B_M: X_M \to \chi \times \chi$ is injective on $X_M \setminus (X^p_M \cup X^r_M)$.

We first show a uniqueness of the support systems of the complex bending.
Lemma 9.2. Let $\eta \in X_M \setminus (X_M^p \cup X_M^r)$. Fix a representative $\xi: \pi_1(S) \to PSL_2\mathbb{C} \times PSL_2\mathbb{C}$ of $B_M(\eta)$. Let $P$ be a component of $S \setminus M$. Then, the support space $H_P$ of $\xi$ is the unique totally geodesic copy of $\mathbb{H}^3$ in $\mathbb{H}^3 \times \mathbb{H}^3$ which contains the bending axes of the boundary components of $P$.

Proof. As $\eta \notin X_M^p$, the bending axes of the boundary components of $P$ are uniquely determined by $\xi$. Let $G_P$ be the subgroup of $\pi_1(S)$ preserving $P$. Then, as $\eta|G_P$ is strongly irreducible (i.e. not weakly reducible), there is a unique totally geodesic copy of $\mathbb{H}^3$ in $\mathbb{H}^3 \times \mathbb{H}^3$, containing those bending axes. \hfill \Box

Lemma 9.2 immediately implies the following.

Corollary 9.3. Suppose that $\eta_1, \eta_2 \in X_M \setminus (X_M^p \cup X_M^r)$ satisfy $B_M(\eta_1) = B_M(\eta_2) \in \chi \times \chi$. Let $\xi: \pi_1(S) \to PSL_2\mathbb{C} \times PSL_2\mathbb{C}$ be a representative of $B_M(\eta_1) = B_M(\eta_2)$. Then, the $\xi$-equivariant bending support system of $B_M$ at $\eta_1$ equivariantly coincides with that at $\eta_2$.

Proof of Theorem 9.1. Suppose that $\eta_1, \eta_2 \in X_M \setminus (X_M^p \cup X_M^r)$ map to the same representation in $\chi \times \chi$ by $B_M$. Then, let $\xi: \pi_1(S) \to PSL_2\mathbb{C} \times PSL_2\mathbb{C}$ be a representative of their image.

By Corollary 9.3, the support system of the bending of $\eta_1$ equivariantly coincides with that of $\eta_2$. Therefore $\eta_1$ and $\eta_2$ are obtained by unbending $\xi$ exactly in the same manner; hence $\eta_1 = \eta_2$. \hfill 9.1

9.1. A non-injective example. We shall see, in an example, the non-injectivity of a complex bending map. Let $m$ be a separating loop on $S$ with some positive weight. Pick a connected subsurface $F$ of $S$ bounded by $m$. Fix a homomorphism $\rho: \pi_1(S) \to PSL_2\mathbb{C}$ such that $\rho|\pi_1(F)$ is the trivial representation. Then, as $\rho(m)$ is in particular the identity, any pair $(u, v) \in \mathbb{CP}^1 \times \mathbb{CP}^1$ is a framing of $\rho$ along $m$.

Lemma 9.4. Fix an arbitrary orientation of $m$ and an arbitrary weight on $m$. Then $B_m(\rho, (u, v)) = (\rho, \rho) \in \chi \times \chi$ for all framings $(u, v)$ along $m$. In particular, $B_m$ is not injective.

Proof. Pick a loop $\ell$ on $S$ which essentially intersects $m$ exactly in two points (see Figure 8). We can assume, without loss of generality, that the base point of $\pi_1(S)$ is on $m$. Let $\gamma$ be an element of $\pi_1(S)$ corresponding to $\ell$. Then homotope $\ell$ so that $\ell$ is a composition of a loop $\ell_1$ on $S \setminus F$ and a loop $\ell_2$ on $F$. Since $\rho|\pi_1(F)$ is trivial, we have $B_m(\rho, (u, v)) = (\rho, \rho)$. We can take a generating set of $\pi_1(S)$ consisting of loops in $S \setminus F$ and loops in $F$. Therefore $B_m(\rho, (u, v)) = (\rho, \rho)$ in $\chi \times \chi$. \hfill \Box
In this section, we prove the properness of the complex bending map, similarly to the injectivity in §9, in the complement of certain proper subvarieties. Similarly to $X^p_M$, we let $\chi^p_M$ be the subvariety of the $\text{PSL}_2\mathbb{C}$-character variety $\chi$ consisting of representations $\xi: \pi_1(S) \to \text{PSL}_2\mathbb{C} \times \text{PSL}_2\mathbb{C}$ such that, for at least one loop $m$ of $M$, its holonomy $\xi(m)$ is parabolic (possibly the identity) in each factor of $\text{PSL}_2\mathbb{C} \times \text{PSL}_2\mathbb{C}$ (equivalently in at least, one factor). Similarly to $X^r_M$, we let $\chi^r_M$ be the subvariety of $\chi$ consisting of representations $\xi: \pi_1(S) \to \text{PSL}_2\mathbb{C} \times \text{PSL}_2\mathbb{C}$ such that, for at least one component $F$ of $S \setminus M$, $\xi|_F$ is weakly reducible in each factor (equivalently, in one factor).

Theorem 10.1. The restriction of $B_M$ to $X_M \setminus (X^p_M \cup X^r_M)$ is a proper mapping to $(\chi \setminus (\chi^p_M \cap \chi^r_M))^2$.

Proof. Let $\eta_i \in X_M \setminus (X^p_M \cup X^r_M)$ be a sequence such that $B_M(\eta_i)$ converges to a representation in $(\chi \setminus (\chi^p_M \cap \chi^r_M))^2$ as $i \to \infty$. It suffices to show that $\eta_i$ also converges in $X_M \setminus (X^p_M \cup X^r_M)$.

Pick a representative $\xi_i: \pi_1(S) \to \text{PSL}_2\mathbb{C} \times \text{PSL}_2\mathbb{C}$ of $B_M(\eta_i)$ so that $\xi_i$ converges to $\xi_\infty: \pi_1(S) \to \text{PSL}_2\mathbb{C} \times \text{PSL}_2\mathbb{C}$, so that its equivalence class $[\xi_\infty]$ is in $(\chi \setminus (\chi^p_M \cap \chi^r_M))^2$. Let $\{H_{i,P}\}$ be the $\xi_i$-equivariant bending support system of the complex bending of $\eta_i$ along $M$, where $P$ varies over all connected components of $S \setminus M$. By the hypothesis, the restriction of $\xi_\infty$ to each component of $S \setminus M$ is strongly irreducible. Therefore, by Lemma 9.2, the $\xi_i$-equivariant support system $\{H_{i,P}\}$ converges to a unique support system $\{H_P\}$ of $\xi_\infty$ as $i \to \infty$.

We also show that the bending axes also converge.

Claim 10.2. The $\xi_i$-equivariant axes system for bending $\eta_i$ along $M$ in $\mathbb{H}^3 \times \mathbb{H}^3$ converges to a $\xi$-equivariant axis system as $i \to \infty$.

Proof. Let $m$ be a loop of $M$, and let $\tilde{m}$ be a component of $\tilde{M}$ which descends to $m$. Let $\alpha \in \pi_1(S)$ denote the element preserving $\tilde{m}$ such that the free homotopy class $\alpha$ is $m$. Let $P, Q$ denote the adjacent components of $\tilde{S} \setminus \tilde{M}$ separated by $\tilde{m}$. Then $H_{i,P} \cap H_{i,Q}$ is the complex
bending axis $g_{i,\tilde{m}}$ for $\tilde{m}$ in $\mathbb{H}^3 \times \mathbb{H}^3$, and also the axis of $\xi_i(\alpha)$. The angle of the intersection of $H_{i,P}$ and $H_{i,Q}$ along the axis is equal to the weight of $m$. As $\xi_i(m)$ converges to a non-parabolic element $\xi(m)$, the axis $H_{i,P} \cap H_{i,Q}$ converges to the axis of $\xi(\alpha)$ as $i \to \infty$. \hfill $\square$

For each $i = 1, 2, \ldots$, let $\{g_{i,\tilde{m}}\}$ denote the $\xi_i$-equivariant bending axis system in $\mathbb{H}^3 \times \mathbb{H}^3$ of $B_M$ at $\eta_i$. Note that $\eta_i$ is obtained by unbending $\xi_i$ along the axes $g_{i,\tilde{m}}$ by the angles given by the weights $M$. By the convergence, similarly unbending the limit $\xi$ in $(\chi \setminus (\chi_M^p \cup \chi_M^r))^2$ along the limit bending axis system by the angles given by $M$, we obtain the limit of $\eta_i$ as $i \to \infty$. As $\xi$ is in $(\chi \setminus (\chi_M^p \cup \chi_M^r))^2$, thus $\lim_{i \to \infty} \eta_i$ is contained in $X_M \setminus (X_M^p \cup X_M^r)$. \hfill 10.1

11. Analyticity of complex bending maps

**Theorem 11.1.** For every weighted oriented multiloop $M$ on $S$, the bending map $B_M: X_M \to \chi \times \chi$ is complex analytic.

**Proof.** Recall that $X_M^p$ is the subvariety of the complex-analytic variety $X_M$ consisting of representations such that at least one loop of $M$ is parabolic, and also that $X_M^r$ is the subset of $X_M$ consisting of representations $\eta$ such that the restriction of $\eta$ to a component of $S \setminus M$ is weakly reducible. We have shown that the restriction of $B_M$ to $X_M \setminus (X_M^p \cup X_M^r)$ is injective (Theorem 9.1). We first prove the assertion of Theorem 11.1 for almost everywhere.

**Lemma 11.2.** The restriction of $B_M$ to $X_M \setminus (X_M^p \cup X_M^r)$ is complex analytic.

**Proof.** Recall that $R_M$ is the space of representations framed along $M$, and that $R_M//\text{PSL}_2\mathbb{C} = X_M$. Let $R_M^p$ be the subset of $R_M$ consisting of framed representations, such that, at least, one loop of $M$ is parabolic (or the identity). Let $\eta = (\rho, (u_i, v_i)_{i=1}^n)$ be an arbitrary framed representation in $R_M \setminus (R_M^p \cup R_M^r)$, where $n$ is the number of the loops of $M$. As the closed subvariety $R_M^p \cup R_M^r$ is $\text{PSL}_2\mathbb{C}$-invariant, we can take a $\text{PSL}_2\mathbb{C}$-invariant open neighborhood $U$ of $\eta$ in $R_M \setminus (R_M^p \cup R_M^r)$. Then, for every framed representation $\zeta \in U$, the stabilizer of $\zeta$ in $\text{PSL}_2\mathbb{C}$ is a discrete group, since $\zeta$ is not in $R_M^r$. Thus, if we take $U$ appropriately, $U$ is holomorphically a product of $\text{PSL}_2\mathbb{C}$ and an open disk $D$. Let $W$ be the image of $U$ in $X_M$. Then, we can biholomorphically identify $W$ in $X_M$ with $D$ in $U$ and take a holomorphic section $s: W \to U$.

Pick any component of $Q$ of $\tilde{S} \setminus \tilde{M}$, where $\tilde{M}$ is the inverse image of $M$ in $\tilde{S}$. Let $G_Q$ be the stabilizer of $Q$ in $\pi_1(S)$. By $\mathbb{C}$-bending along $M$ (normalizing so that the restriction to $G_Q$ unchanged), we obtain a
holomorphic mapping \( s(W) \to (\mathcal{R} \setminus \mathcal{R}_M^p \cup \mathcal{R}_M^r)^2 \) which is a lift of the restriction of \( B_M \) to \( W \). Then, for every \( \zeta \in s(W) \), its image by this mapping is a pair of strongly irreducible representations in \( \mathcal{R} \). Since \( W \) is isomorphic to \( s(W) \) and the quotient map from \( \mathcal{R} \times \mathcal{R} \) to \( \chi \times \chi \) is algebraic, the analyticity of \( B_W \) is isomorphic to \( \mathcal{R} \). The mapping is a pair of strongly irreducible representations in \( \mathcal{R} \). Since \( W \) is strongly irreducible, the analyticity of \( B_W \) \( \to (\mathcal{R} \setminus \mathcal{R}_M^p \cup \mathcal{R}_M^r)^2 \) implies the analyticity of \( B_M \) at the equivalent class of \( \eta \) in \( X_M \).

By Lemma 11.2, \( X_M \setminus (\mathcal{X}_M^p \cup \mathcal{X}_M^r) \to (\chi \setminus \chi_M^p \cup \chi_M^r) \times (\chi \setminus \chi_M^p \cup \chi_M^r) \) is an injective analytic mapping. Since \( \mathcal{X}_M^p \cup \mathcal{X}_M^r \) is an analytic subvariety of \( X_M \), by the removable singularity theorem (Theorem 3.7), the mapping \( B_M : X_M \to \chi \times \chi \) is analytic.

12. The real-bending map sits in the complex-bending map

In this section, we observe that the complex-analytic bending map \( B_M : X_M \to \chi \times \chi \) is a natural extension of the real-analytic bending map \( b_M : \mathcal{T} \to \chi \). Recall that \( \Delta^* \) is the twisted diagonal \( \{(\rho, \rho^*) \mid \rho \in \chi\} \) and \( \psi : \chi \to \Delta^* \subset \chi \times \chi \) is the embedding given by \( \rho \mapsto (\rho, \rho^*) \).

The forgetful map \( X_M \to \chi \) restricts to an analytic covering map \( X_M \setminus \chi_M^p \to \chi \setminus \chi_M^p \) of degree \( 2^n \), where \( n \) is the number of the loops of \( M \). As the base surface \( S \) is oriented, we let \( \mathcal{T} \) be the Teichmüller space of \( S \) is identified with a unique component of the real slice of \( \chi \). Then, by the choice of framings, there are \( 2^n \) ways to lift the Fricke-Teichmüller space \( \mathcal{T} \) to \( X_M \). Given an oriented weighted \( M \) on \( S \), there is a unique lift of \( \mathcal{T} \) to \( X_M \) such that, for each loop \( m \) of \( M \), the ordering of the fixed points of the framing along \( m \) coincides with the orientation of \( M \). Let \( \iota_M : \mathcal{T} \to X_M \) be this real-analytic embedding.

**Proposition 12.1.** Let \( M \) be a weighted oriented multiloop on \( S \). Then, the restriction of \( B_M \) to \( \mathcal{T} \) is a real-analytic embedding into the twisted diagonal \( \Delta^* \) of \( \chi \times \chi \), such that \( B_M \circ \iota_M \) coincides with \( \psi \circ b_M : \mathcal{T} \to \chi \times \chi \).

**Proof.** Let \( b_M^* : \mathcal{T} \to \chi \) denote the complex conjugate of \( b_M : \mathcal{T} \to \chi \), i.e. the Fuchsian representation \( \rho : \pi_1(S) \to \text{PSL}_2\mathbb{R} \) maps to the mapping taking \( \gamma \in \pi_1(S) \) to \( (b_M(\rho)(\gamma))^* \in \text{PSL}_2\mathbb{C} \). When applying the complex bending \( B_M \), a representation into \( \text{PSL}_2\mathbb{C} \) is bent in opposite directions in the first and the second factor of \( \chi \times \chi \) (§8.2). Therefore, when applying \( B_M \) to a Fuchsian representation, the representation in the second factor is the complex conjugate of the representation in the first factor. Therefore \( B_M \circ \iota_M(\rho) \) is \( (b_M(\rho), b_M^*(\rho)) \) for \( \rho \in \mathcal{T} \), as desired. The analyticity of the was already proven in Theorem 11.1. \( \square \)
13. Properness of the complex bending map along a non-separating loop

**Theorem 13.1.** Let $\ell$ be an oriented non-separating loop with weight not equal to $\pi$ modulo $2\pi$. Then, the complex bending map $B_{\ell}: X_{\ell} \to \chi \times \chi$ is proper.

**Corollary 13.2.** The image of $B_{\ell}$ is a closed analytic set in $\chi \times \chi$.

Pick $\theta \in (0, \pi)$. Let

$$E_\theta = \{ (\gamma, e) \in \text{PSL}_2 \mathbb{C} \times \text{PSL}_2 \mathbb{C} \mid e \text{ is elliptic of rotation angle } \theta \}.$$ 

Clearly, for every $(\gamma, e) \in E_\theta$, $tr^2 e$ is a fixed constant in $(0, 4)$ only depending on $\theta$. Thus $E_\theta$ is a smooth affine algebraic variety. Then $\text{PSL}_2 \mathbb{C}$ acts on $E_\theta$ by conjugating both parameters $\gamma$ and $e$ simultaneously. Let $E_\theta$ be the GIT-quotient $E_\theta \sslash \text{PSL}_2 \mathbb{C}$. Then $E_\theta$ is an affine algebraic variety. Then the following holds.

**Lemma 13.3.** The analytic mapping $E_\theta \sslash \text{PSL}_2 \mathbb{C} \to \mathbb{C}^2$ defined by $\phi: (\gamma, e) \mapsto (tr^2 \gamma, tr^2 \gamma e)$ is a proper mapping.

**Proof.** The map $\text{PSL}_2 \mathbb{C} \times \text{SL}_2 \mathbb{C} \sslash \text{SL}_2 \mathbb{C} \to \mathbb{C}^2$ given by $(\gamma, e) \mapsto (tr \gamma, tr e, tr \gamma e)$ is a biholomorphic map (see for example, [Gol09]).

Let $(\alpha_i, e_i)$ be a sequence in $E_\theta \subset \text{PSL}_2 \mathbb{C} \times \text{PSL}_2 \mathbb{C} \sslash \text{PSL}_2 \mathbb{C}$ which leaves every compact as $i \to \infty$. Pick any lift $(\tilde{\alpha}_i, \tilde{e}_i) \in \text{SL}_2 \mathbb{C} \times \text{SL}_2 \mathbb{C} \sslash \text{SL}_2 \mathbb{C}$ of $(\alpha_i, e_i)$ for each $i$. Then $(\tilde{\alpha}_i, \tilde{e}_i)$ also leaves every compact as $i \to \infty$.

By a basic trace identity, we have $tr \tilde{\alpha}_i \tilde{e}_i + tr \tilde{\alpha}_i \tilde{e}_i^{-1} = tr \tilde{\alpha}_i tr \tilde{e}_i$. Therefore, since $tr \tilde{e}_i$ is a fixed non-zero constant, up to a subsequence, either $tr \tilde{\alpha}_i$ or $tr \tilde{\alpha}_i \tilde{e}_i$ diverges to infinity as $i \to \infty$. Thus the image $\phi(\alpha_i, e_i)$ leaves every compact in $\mathbb{C}^2$ as $i \to \infty$. \hfill \square

Pick a generating set $\{ \gamma_1, \ldots, \gamma_{2g} \}$ of $\pi_1(S)$ such that $\gamma_1, \ldots, \gamma_{2g}$ correspond to loops on $S$ intersecting $\ell$ exactly once. Let $\eta_i = [\rho_i, (u_i, v_i)] \in X_{\ell}$ be a sequence which leaves every compact in $X_{\ell}$.

Let $w(\ell)$ denote the weight of $\ell$, and let $e_i \in \text{PSL}_2 \mathbb{C}$ be the elliptic element by angle $w(\ell)$ along the geodesic from $u_i$ to $v_i$. Then we can normalize $(\rho_i, (u_i, v_i))$, by an element of $\text{PSL}_2 \mathbb{C}$, so that $e_i \in \text{PSL}_2 \mathbb{C}$ is independent on $i$. Let $e$ be the independent elliptic element in $\text{PSL}_2 \mathbb{C}$.

As $\eta_i$ leaves every compact and $\gamma_1, \ldots, \gamma_n$ form a generating set of $\pi_1(S)$, then there is $k \in \{1, \ldots, n\}$ such that, up to a subsequence, $\rho_i(\gamma_k)$ leaves every compact as $i \to \infty$. Then, since $\gamma_k$ intersect $\ell$ exactly at once, by the properness of Lemma 13.3, the image $B_{\ell}(\eta_i)\gamma_k$ also leaves every compact as $i \to \infty$. This immediately implies the properness of $B_{\ell}$. 
14. SYMPLECTIC PROPERTY

In this section, we prove the symplectic property of the bending maps. Complex Fenchel-Nielsen coordinates on the quasi-Fuchsian space are introduced by [Kou94] and [Tan94], and the coordinates holomorphically extend to most part of the character variety $\chi$. We explicitly explain the subset of $\chi$ where the complex Fenchel-Nielsen coordinates are defined.

Let $M$ be a maximal multiloop on $S$. Then $M$ contains $3g-3$ loops, where $g$ is the genus of $S$. Let $\chi^h_M$ be the (Euclidean) open full-measure subset of $\chi$ consisting of $\rho: \pi_1(S) \to \text{PSL}_2\mathbb{C}$ such that

- all loops of $M$ are hyperbolic, and
- for each component $P$ of $S \setminus M$, the restriction of $\rho$ to $\pi_1(P)$ is irreducible.

Pick (real) Fenchel-Nielsen coordinates on the Teichmüller-Fricker space $\mathcal{T}$ with respect to $M$ (see [FM12] for example). Let $\mathbb{C}^+ = \{z \in \mathbb{C} \mid \text{Re} z > 0\}$. For each $\rho: \pi_1(S) \to \text{PSL}_2\mathbb{C}$ in $\chi^h_M$, let $\ell_i \in \mathbb{C}^+/2\pi\mathbb{Z}$ be the complex translation length of $\rho(m_i)$: When we $\ell_i = x_i + Iy_i$ in real and imaginary coordinates, $x_i \in \mathbb{R}_{>0}$ is the (real) translation length and the $y_i \in \mathbb{R}$ is the rotation angle of the screw motion of the hyperbolic element $\rho(m_i)$.

Clearly, for real representations $\pi_1(S) \to \text{PSL}_2\mathbb{R}$, their length parameters $\ell_1, \ldots, \ell_{3g-3}$ are all real numbers. Let $\tau_i \in \mathbb{C}/2\pi\mathbb{Z}$ be the twist coordinate along $\ell_i$ which complexifies the Fenchel-Nielsen twist coordinate, so that the imaginary direction is the direction of bending deformation (where $I$ denotes the imaginary unite). Similarly, for real representations $\pi_1(S) \to \text{PSL}_2\mathbb{R}$, their twist parameters $\tau_1, \ldots, \tau_{3g-3}$ are all real numbers.

**Lemma 14.1.** Then $\chi^h_M$ is a (Zariski) open dense subset of $\chi$ and biholomorphic to $(\mathbb{C}^+/2\pi\mathbb{Z})^{3g-3} \bigoplus (\mathbb{C}/2\pi\mathbb{Z})^{3g-3}$ by $(\ell_1, \ldots, \ell_{3g-3}, \tau_1, \ldots, \tau_{3g-3})$.

**Proof.** The mapping $\chi^h_M \to (\mathbb{C}^+/2\pi\mathbb{Z})^{3g-3} \bigoplus (\mathbb{C}/2\pi\mathbb{Z})^{3g-3}$ is a holomorphic mapping, as the coordinates are given by traces of loops.

Given a pair of pants $P$, the irreducible representations $\pi_1(P)$ are algebraically parametrized by the holonomy traces of the three boundary components of $P$ ([Vog89] [Fri96]; see also [Gol09]). Now let $P$ be a component of $S \setminus M$. Then $\rho \in \chi^h_M$, the $\rho|\pi_1(P)$ is parametrized by the complex length coordinates of the boundary components of $P$.

For a loop $m_i$ of $M$, let $F$ be the component of $S \setminus (M \setminus \ell)$ which contains $M$. Then the representation on $\pi_1(F) \to \text{PSL}_2\mathbb{C}$ is determined by the twisting parameter $\tau_i$ of $m_i$ and the length parameters.
ℓ_i of m_i and the boundary loops of F. We see that the mapping is biholomorphic.

Due to Platis [Pla01] and Goldman [Gol04], the complex Fenchel-Nielsen coordinates yield Darboux coordinates for Goldman’s complex symplectic structure.

\[ w_G = \sum_{i=1}^{3g-3} d\ell_{m_i} \wedge dt_{m_i}. \]

(see Loustau [Lou15] for details). To be concrete and self-contained, we first explain the Darboux coordinates on \( \chi^h_M \).

**Lemma 14.2.** Let \( M = m_1 \sqcup m_2 \sqcup \cdots \sqcup m_{3g-3} \) be a maximal multiloop on \( S \). Then \( w_G = \sum_{i=1}^{3g-3} d\ell_{m_i} \wedge dt_{m_i} \) on \( \chi^h_M \).

**Proof.** The symplectic structure \( w_G \) is a complex symplectic structure, so that the 2-form changes holomorphically in \( \chi \). On the Fricke-Teichmüller space, \( w_G|_T \) is given by \( \sum d\ell_{m_i} \wedge dt_{m_i} \). Therefore, since the complex Fenchel-Nielsen coordinates are holomorphic coordinates (Lemma 14.1), \( w_G = \sum d\ell_{m_i} \wedge dt_{m_i} \) on \( \chi^h_M \).

Then this Darboux coordinates on \( \chi^h_M \) gives the symplectic property of the real bending map.

**Proposition 14.3.** If \( M \) is a weighted multiloop on \( S \), then \( b_M : T \to \chi \) is a symplectic embedding.

**Proof.** As \( M \) may not be maximal, we pick a maximal multiloop \( M' \) on \( S \) containing \( M \). Set \( m_1, m_2, \ldots, m_{3g-3} \) to be the loops of \( M' \). Let \( w_1, w_2, \ldots, w_{3g-3} \in \mathbb{R}_{\geq 0} \) be the weight of the loops of \( M' \) (so that, if \( \ell_i \) is not a loop of the original multiloop \( M \), then \( w_i = 0 \)). The Teichmüller-Fricke space \( T \) is a component of the real slice of \( \chi^h_M \). In the Darboux coordinates of Lemma 14.2, the real bending map \( b_M : T \to \chi \) extends to \( \hat{b}_M : \chi^h_M \to \chi^h_M \) by the translation

\[ (\ell_1, \ldots, \ell_{3g-3}, \tau_1, \ldots, \tau_{3g-3}) \mapsto (\ell_1, \ldots, \ell_{3g-3} + w_1 I, \ldots, \tau_{3g-3} + w_{3g-3} I). \]

As it is a translation in the Darboux coordinates, \( b_M : T \to \chi \) is clearly a symplectic embedding.

By the limiting argument, all real bending maps are symplectic.

**Theorem 14.4.** For every \( L \in \mathcal{ML} \), \( b_L : T \to \chi \) is a symplectic embedding w.r.t. Goldman’s symplectic structure.

**Proof.** Let \( \ell_i \) be a sequence of weighted loops which converges to \( L \) in \( \mathcal{ML} \) as \( i \to \infty \). (Recall that \( b_{\ell_i} : T \to \chi \) is a real analytic embedding.) For each \( \tau \in T \), the tangent space of \( b_{\ell_i} \) at \( \tau \) converges to the tangent space of \( b_L \) at \( \tau \). By Proposition 14.3, \( b_{\ell_i} : T \to \chi \) is a symplectic
embedding for each $i = 1, 2, \ldots$. Therefore, by the continuity of the symplectic structure $w_G$, the limit $b_L$ is also symplectic at $\tau$. $\square$

14.1. **Symplectic property for complex bending map.** As $X_M \setminus X_M^p \to \chi \setminus \chi_M^p$ is an analytic covering map, $X_M \setminus X_M^p$ has a pull-back symplectic structure.

A representation $\rho: \pi_1(S) \to \text{PSL}_2\mathbb{C}$ is **reductive**, if the Zariski-closure of the image $\text{Im} \rho \subset \text{PSL}_2\mathbb{C}$ is reductive. (That is, the maximal normal unipotent subgroup of $\text{Im} \rho$ is the trivial group.) Then a representation $\rho: \pi_1(S) \to \text{PSL}_2\mathbb{C}$ is non-reductive, if and only if $\text{Im} \rho$ is conjugate to a subgroup consisting of upper triangular matrices which contains at least one (non-identity) parabolic element. Let $X_M^r$ be the set of framed representations $\eta = [\rho, (u_i, v_i)]$ of $X_M$ such that $\rho$ is a reductive representation other than the trivial representation.

**Theorem 14.5.** The restriction of $B_M$ to $X_M^r \setminus X_M^p$ is a complex-symplectic map.

**Proof.** We show that the restriction of $b_M^\pm: X_M^r \to \chi$ is symplectic on $\chi_M^h$.

Let $\eta = [\rho, (u_i, v_i)]$ be a framed representation of $X_M^r \setminus X_M^p$, so that $\rho: \pi_1(S) \to \text{PSL}_2\mathbb{C}$ is in the representation variety $\mathcal{R}$. Note that the fixed point set of $\rho$ in $\mathbb{C}P^1$ is either the empty set or two points on $\mathbb{C}P^1$ as $\rho \notin \mathcal{R}_M^r$. Thus $\rho$ is reductive. Therefore, the tangent cone and the Zariski tangent space of $\mathcal{R}$ coincide at $\rho$ (see Goldman [Gol85, Theorem 3]). As $\rho$ is reductive, $\rho$ has a closed $\text{PSL}_2\mathbb{C}$-orbit. Therefore, we can apply Luna’s slice theorem ([ByBCM02] Theorem 15.5), and the neighborhood of $[\rho]$ is $\chi$ is isomorphically embedded in a finite quotient of an algebraic slice through $\rho$ in $X_M^p$. Thus the Zariski tangent space of $\chi$ at the projection $[\rho] \in \chi$ also coincides with the tangent cone of $\chi$ at $[\rho]$.

Recall that $\chi_M^h$ is an open dense subset of $\chi$. Therefore, the Zariski tangent space changes continuously in $\rho$, and the symplectic property of $b_M^\pm$ on $\chi_M^h$, given by Lemma 14.2, continuously extends to the entire $X_M^r \setminus X_M^p$. Hence, as $\eta \mapsto (b_M^+(\eta), b_M^-(\eta))$, the restriction of $B_M$ to $X_M^r \setminus X_M^p$ is symplectic. (Figure 9.) $\square$

**References**

[AB20] Dylan G. L. Allegretti and Tom Bridgeland. The monodromy of meromorphic projective structures. Trans. Amer. Math. Soc., 373(9):6321–6367, 2020.

[Bab] Shinpei Baba. Neck-pinching of CP$^1$-structures in the PSL(2, C)-character variety. Preprint, arXiv:1907.00092.
\[
X^p \setminus X^p \xrightarrow{b_M} \chi \setminus \chi^p \subset \chi
\]

\[
B_M \xrightarrow{b_M} \left[ b_M \times b_M^+ \right]
\]

\[
\chi \times \chi
\]

**Figure 9.** A local commutative diagram for the complexification of the real bending map.

[Bab10] Shinpei Baba. A Schottky decomposition theorem for complex projective structures. *Geom. Topol.*, 14(1):117–151, 2010.

[Bon96] Francis Bonahon. Shearing hyperbolic surfaces, bending pleated surfaces and Thurston’s symplectic form. *Ann. Fac. Sci. Toulouse Math. (6)*, 5(2):233–297, 1996.

[ByBCM02] A. Białynicki Birula, J. B. Carrell, and W. M. McGovern. *Algebraic quotients. Torus actions and cohomology. The adjoint representation and the adjoint action*, volume 131 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2002. Invariant Theory and Algebraic Transformation Groups, II.

[CEG87] R. D. Canary, D. B. A. Epstein, and P. Green. Notes on notes of Thurston. In *Analytical and geometric aspects of hyperbolic space (Coventry/Durham, 1984)*, volume 111 of *London Math. Soc. Lecture Note Ser.*, pages 3–92. Cambridge Univ. Press, Cambridge, 1987.

[Dum09] David Dumas. Complex projective structures. In *Handbook of Teichmüller theory. Vol. II*, volume 13 of *IRMA Lect. Math. Theor. Phys.*, pages 455–508. Eur. Math. Soc., Zürich, 2009.

[Dum17] David Dumas. Holonomy limits of complex projective structures. *Adv. Math.*, 315:427–473, 2017.

[DW08] David Dumas and Michael Wolf. Projective structures, grafting and measured laminations. *Geom. Topol.*, 12(1):351–386, 2008.

[EM87] D. B. A. Epstein and A. Marden. Convex hulls in hyperbolic space, a theorem of Sullivan, and measured pleated surfaces. In *Analytical and geometric aspects of hyperbolic space (Coventry/Durham, 1984)*, volume 111 of *London Math. Soc. Lecture Note Ser.*, pages 113–253. Cambridge Univ. Press, Cambridge, 1987.

[FG06] Vladimir Fock and Alexander Goncharov. Moduli spaces of local systems and higher Teichmüller theory. *Publ. Math. Inst. Hautes Études Sci.*, (103):1–211, 2006.

[FM12] Benson Farb and Dan Margalit. *A primer on mapping class groups*, volume 49 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 2012.

[Fri96] R. Fricke. Ueber die theorie der automorphen modulgruppen. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, 1896:91–101, 1896.

[GKM00] Daniel Gallo, Michael Kapovich, and Albert Marden. The monodromy groups of Schwarzian equations on closed Riemann surfaces. *Ann. of Math. (2)*, 151(2):625–704, 2000.
Subhojoy Gupta and Mahan Mj. Meromorphic projective structures, grafting and the monodromy map. *Adv. Math.*, 383:107673, 49, 2021.

William M. Goldman. The symplectic nature of fundamental groups of surfaces. *Adv. in Math.*, 54(2):200–225, 1984.

William M. Goldman. Representations of fundamental groups of surfaces. In *Geometry and topology (College Park, Md., 1983/84)*, volume 1167 of *Lecture Notes in Math.*, pages 95–117. Springer, Berlin, 1985.

William M. Goldman. The complex-symplectic geometry of $\text{SL}(2,\mathbb{C})$-characters over surfaces. In *Algebraic groups and arithmetic*, pages 375–407. Tata Inst. Fund. Res., Mumbai, 2004.

William M. Goldman. Mapping class group dynamics on surface group representations. In *Problems on mapping class groups and related topics*, volume 74 of *Proc. Sympos. Pure Math.*, pages 189–214. Amer. Math. Soc., Providence, RI, 2006.

William M. Goldman. Trace coordinates on Fricke spaces of some simple hyperbolic surfaces. In *Handbook of Teichmüller theory. Vol. II*, volume 13 of *IRMA Lect. Math. Theor. Phys.*, pages 611–684. Eur. Math. Soc., Zürich, 2009.

William M. Goldman. *Geometric structures on manifolds*, volume 227 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2022 ©2022.

[Hej75] Dennis A. Hejhal. Monodromy groups and linearly polymorphic functions. *Acta Math.*, 135(1):1–55, 1975.

Michael Heusener and Joan Porti. The variety of characters in $\text{PSL}_2(\mathbb{C})$. *Bol. Soc. Mat. Mexicana (3)*, 10(Special Issue):221–237, 2004.

Michael Kapovich. On monodromy of complex projective structures. *Invent. Math.*, 119(2):243–265, 1995.

Michael Kapovich. *Hyperbolic manifolds and discrete groups*, volume 183 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 2001.

Shingo Kawai. The symplectic nature of the space of projective connections on Riemann surfaces. *Math. Ann.*, 305(1):161–182, 1996.

Steven P. Kerckhoff. Earthquakes are analytic. *Comment. Math. Helv.*, 60(1):17–30, 1985.

Christos Kourouniotis. Complex length coordinates for quasi-Fuchsian groups. *Mathematika*, 41(1):173–188, 1994.

Ravi S. Kulkarni and Ulrich Pinkall. A canonical metric for Möbius structures and its applications. *Math. Z.*, 216(1):89–129, 1994.

Yoshinobu Kamishima and Ser P. Tan. Deformation spaces on geometric structures. In *Aspects of low-dimensional manifolds*, volume 20 of *Adv. Stud. Pure Math.*, pages 263–299. Kinokuniya, Tokyo, 1992.

Brice Loustau. The complex symplectic geometry of the deformation space of complex projective structures. *Geom. Topol.*, 19(3):1737–1775, 2015.

Curtis T. McMullen. Complex earthquakes and Teichmüller theory. *J. Amer. Math. Soc.*, 11(2):283–320, 1998.
[Pla01] Ioannis D. Platis. Complex symplectic geometry of quasi-Fuchsian space. Geom. Dedicata, 87(1-3):17–34, 2001.

[Poi84] H. Poincaré. Sur les groupes des équations linéaires. Acta Math., 4(1):201–312, 1884.

[Sno82] Dennis M. Snow. Reductive group actions on Stein spaces. Math. Ann., 259(1):79–97, 1982.

[SW02] Kevin P. Scannell and Michael Wolf. The grafting map of Teichmüller space. J. Amer. Math. Soc., 15(4):893–927, 2002.

[Tan94] Ser Peow Tan. Complex Fenchel-Nielsen coordinates for quasi-Fuchsian structures. Internat. J. Math., 5(2):239–251, 1994.

[Tan97] Harumi Tanigawa. Grafting, harmonic maps and projective structures on surfaces. J. Differential Geom., 47(3):399–419, 1997.

[Tan99] Harumi Tanigawa. Divergence of projective structures and lengths of measured laminations. Duke Math. J., 98(2):209–215, 1999.

[Tay02] Joseph L. Taylor. Several complex variables with connections to algebraic geometry and Lie groups, volume 46 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2002.

[Thu81] William P Thurston. The geometry and topology of three-manifolds. Princeton University Lecture Notes, 1978-1981.

[Thu86] William P. Thurston. Earthquakes in two-dimensional hyperbolic geometry. In Low-dimensional topology and Kleinian groups (Coventry/Durham, 1984), volume 112 of London Math. Soc. Lecture Note Ser., pages 91–112. Cambridge Univ. Press, Cambridge, 1986.

[Vog89] H. Vogt. Sur les invariants fondamentaux des équations différentielles linéaires du second ordre. Ann. Sci. École Norm. Sup. (3), 6:3–71, 1889.

[Zil] Wolfgang Ziller. Lie groups. representation theory, and symplectic spaces. Lecture notes.

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