A GENERALIZED FERMAT-TORRICELLI TREE THAT HAS ACQUIRED A SUBCONSCIOUS ON A SURFACE

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Abstract. We study a generalized Fermat-Torricelli (S.FT) problem for infinitesimal geodesic triangles on a $C^2$ complete surface $M$ with variable Gaussian curvature $a < K < b$, for $a, b \in \mathbb{R}$, such that the intersection point (generalized Fermat-Torricelli point) of the three geodesics acquires a positive real number (subconscious).

The solution of the S.FT problem is a generalized Fermat-Torricelli tree with one node that has acquired a subconscious. This solution is based on a new variational method of the length of a geodesic arc with respect to arc length, which coincides with the first variational formula for geodesics on a surface with $K < 0$, or $0 < K < c$. The 'plasticity' solution of the inverse S.FT problem gives a connection of the absolute value of the Gaussian curvature $\|K(F)\|$ at the generalized Fermat-Torricelli point $F$ with the absolute value of the Aleksandrov curvature of the geodesic triangle by acquiring both of them the subconscious of the g.FT point.

1. Introduction

Let $\triangle A_1A_2A_3$ be a geodesic triangle on a $C^2$ complete surface $M$. We denote by $w_i$ a positive real number (weight), which corresponds to each vertex $A_i$, by $l_{A_i}(F)$, the geodesic distance from the vertex $A_i$, to the point $F_i$ for $i = 1, 2, 3$.

The weighted Fermat-Torricelli problem on a $C^2$ complete surface $M$ states that:

**Problem 1.** Find a point $F \in M$, such that:

$$f(F) = w_1l_{A_1}(F) + w_2l_{A_2}(F) + w_3l_{A_3}(F) \to \min.$$  

The inverse weighted Fermat-Torricelli problem on $M$ states that:

**Problem 2.** Given a point $F$ which belongs to the interior of $\triangle A_1A_2A_3$ on $M$, does there exist a unique set of positive weights $\{w_1, w_2, w_3\}$, such that

$$w_1 + w_2 + w_3 = c = \text{const},$$

for which $F$ minimizes

$$f(F) = w_1l_{A_1}(F) + w_2l_{A_2}(F) + w_3l_{A_3}(F).$$

The solutions w.r.t to the weighted Fermat-Torricelli problem and an inverse weighted Fermat-Torricelli problem for a $C^2$ complete surface with Gaussian curvature $0 < K < c$ or $K < 0$, has been given in [6], [7].

We mention the necessary and sufficient conditions to locate the weighted Fermat-Torricelli point at the interior of $\triangle A_1A_2A_3$ on a $C^2$ complete surface with Gaussian curvature $0 < K < c$ or $K < 0$:

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Proposition 1 (Floating Case). [6, 7] If \( P, Q \in \{ A_1, A_2, A_3 \} \) and \( \vec{U}_{PQ} \) is the unit tangent vector of the geodesic arc \( PQ \) at \( P \) and \( D \) is the domain of \( M \) bounded by \( \triangle A_1A_2A_3 \), then the following (I), (II), (III) conditions are equivalent:

(I) All the following inequalities are satisfied simultaneously:
\[
\left\| w_2\vec{U}_{A_1A_2} + w_3\vec{U}_{A_1A_3} \right\| > w_1, \tag{1.1}
\]
\[
\left\| w_1\vec{U}_{A_2A_1} + w_3\vec{U}_{A_2A_3} \right\| > w_2, \tag{1.2}
\]
\[
\left\| w_1\vec{U}_{A_3A_1} + w_2\vec{U}_{A_3A_2} \right\| > w_3. \tag{1.3}
\]

(II) The point \( F \) is an interior point of the triangle \( \triangle A_1A_2A_3 \) and does not belong to the geodesic arcs \( \gamma_{A_1A_2}, \gamma_{A_2A_3} \) and \( \gamma_{A_3A_1} \),

(III) \( w_1\vec{U}_{F A_1} + w_2\vec{U}_{F A_2} + w_3\vec{U}_{F A_3} = \vec{0} \).

The solution of the weighted Fermat-Torricelli problem is a weighted tree (Weighted Fermat-Torricelli tree or weighted Steiner tree). The derivative of the weighted length of weighted Fermat-Torricelli trees and weighted Steiner trees on a connected complete Riemannian manifold is calculated in [4] which is a generalization of the first variation formula for the length of geodesics w.r. to arc length ([5]). The weighted Fermat-Torricelli problem or weighted Steiner problem on a Riemannian manifold is a special case of a one-dimensional variational problem in which branching extremals are introduced in [3].

In this paper, we provide a new variational method to solve the weighted Fermat-Torricelli problem by assigning a positive number at the weighted Fermat-Torricelli point (g.FT that has acquired a subconscious) for infinitesimal geodesic triangles on a \( C^2 \) complete surface \( M \) with variable Gaussian curvature \( a < K < b \), for \( a, b \in \mathbb{R} \).

This variational method is based on the unified cosine law of Berg-Nikolaev given in [2] for the K-plane (Sphere \( S_k^2 \), Hyperbolic plane \( H_k^2 \) and Euclidean Plane \( \mathbb{R}^2 \)) and an assertion that the generalized Fermat-Torricelli point is located at three spherical regions or three hyperbolic regions or three plane regions or a combination of spherical, hyperbolic and plane regions with different constant curvatures. Thus, we may obtain a generalized Fermat-Torricelli tree on a Torus or a surfaces of revolution in \( \mathbb{R}^3 \), having elliptic points \( (K > 0) \) hyperbolic points \( (K < 0) \) and parabolic points \( K = 0 \).

2. The generalized Fermat-Torricelli (w.F-T) problem on a \( C^2 \) complete surface \( M \) with \( a < K < b \)

We denote by \( \triangle ABC \) an infinitesimal geodesic triangle on a surface \( M \), by \( w_R \) a positive real number (weight) which corresponds to each vertex \( R \), for \( R \in \{ A, B, C \} \) and by \( w_S \) is a positive real number (weight) which corresponds to an interior point \( F \) of \( \triangle ABC \).

The generalized Fermat-Torricelli problem with one node that has acquired unconscious (S.FT problem) states that:
Assume that we select weights \( w_A, w_B, w_C \), such that the g.FT point is located at the interior of \( \triangle ABC \).

**Problem 3.** Find the point \( F \in M \), that has acquired a subconscious \( w_S \) such that:

\[
f(F) = w_A l_A(F) + w_B l_B(F) + w_C l_C(F) \rightarrow \min.
\]

(2.1)

We denote by \( \varphi_Q \), the angle between the geodesic arcs \( \gamma_{RF} \) and \( \gamma_{SF} \) for \( Q, R, S \in \{A, B, C\} \) and \( Q \neq R \neq S \).

**Theorem 1.** If the g.F-T point \( F \) is an interior point of the infinitesimal geodesic triangle \( \triangle ABC \) (see figure 1), then each angle \( \varphi_Q, Q \in \{A, B, C\} \) can be expressed as a function of \( w_A, w_B \) and \( w_C \):

\[
\cos \varphi_Q = \frac{w_Q^2 - w_R^2 - w_S^2}{2w_R w_S},
\]

(2.2)

for every \( Q, R, S \in \{A, B, C\}, Q \neq R \neq S \).

**Proof.** Assume that \( \triangle ABF, \triangle BFC, \triangle AFC \) belong to a spherical, hyperbolic or planar region of constant Gaussian curvature \( k_1, k_2, k_3 \), for \( k_i \in \mathbb{R}, i = 1, 2, 3 \).

We set \( l_A(F') = l_A(F) + dl_A \) and \( l_B(F) = dl_B \).

We denote by

\[
\kappa_i = \begin{cases} 
\frac{\sqrt{K_i}}{i \sqrt{-K_i}} & \text{if } K_i > 0, \\
\frac{i}{\sqrt{-K_i}} & \text{if } K_i < 0.
\end{cases}
\]

The unified cosine law for \( \triangle ABF \) is given by:

\[
\cos(\kappa_3(l_A(F) + dl_A)) = \cos(\kappa_3 l_A(F)) \cos(\kappa_3 dl_B) + \sin(\kappa_3 l_A(F)) \sin(\kappa_3 dl_B) \cos(\varphi_C),
\]

(2.3)

or

\[
\cos(\kappa_3 l_A(F)) \cos(\kappa_3 dl_A) - \sin(\kappa_3 l_A(F)) \sin(\kappa_3 dl_A) = \cos(\kappa_3 dl_B) + \sin(\kappa_3 l_A(F)) \sin(\kappa_3 dl_B) \cos(\varphi_C),
\]

(2.4)

By applying Taylor’s formula, we obtain:

\[
\cos \kappa_3 dl_A = 1 + o((k_3 dl_A)^2),
\]

(2.5)

\[
\sin \kappa_3 dl_A = \kappa_3 dl_A + o((k_3 dl_A)^3),
\]

(2.6)

\[
\cos \kappa_3 dl_B = 1 + o((k_3 dl_B)^2),
\]

(2.7)

and

\[
\sin \kappa_3 dl_B = \kappa_3 dl_B + o((k_3 dl_B)^3).
\]

(2.8)

By replacing (2.5), (2.6), (2.7), (2.8) in (2.4) and neglecting second order terms, we derive that:

\[
\frac{dl_A}{dl_B} = \cos(\pi - \varphi_C).
\]

(2.9)

The unified cosine law for \( \triangle CBF \) is given by:

\[
\cos(\kappa_1(l_C(F) + dl_C)) = \cos(\kappa_1 l_C(F)) \cos(\kappa_1 dl_B) + \sin(\kappa_1 l_C(F)) \sin(\kappa_1 dl_B) \cos(\varphi_A),
\]

(2.10)
By applying Taylor’s formula, we obtain:

\[ \cos \kappa_1 dl_C = 1 + o((k_1 dl_C)^2), \]  \hfill (2.11)

\[ \sin \kappa_1 dl_C = \kappa_1 dl_C + o((k_1 dl_C)^3), \]  \hfill (2.12)

\[ \cos \kappa_1 dl_B = 1 + o((k_1 dl_B)^2), \]  \hfill (2.13)

and

\[ \sin \kappa_1 dl_B = \kappa_1 dl_B + o((k_1 dl_B)^3). \]  \hfill (2.14)

Similarly, by replacing (2.11), (2.12), (2.13), (2.14) in (2.10) and neglecting second order terms, we derive that:

\[ \frac{dl_C}{dl_B} = \cos(\pi - \varphi_A). \]  \hfill (2.15)

By differentiating the objective function (2.1) w.r. to a parameter \( s \), we get:

\[ \frac{df}{ds} = w_A \frac{dl_A}{ds} + w_B \frac{dl_B}{ds} + w_C \frac{dl_C}{ds} \]  \hfill (2.16)

By setting \( s = -l_B \) and by replacing (2.9) and (2.15) in (2.16), we have:

\[ w_A + w_B \cos(\varphi_C + w_C \cos(\varphi_B)) = 0. \]  \hfill (2.17)

Similarly, by working cyclically and setting the parametrization \( s = -l_C \) and \( s = -l_A \), we derive:

\[ w_A \cos \varphi_C + w_B + w_C \cos \varphi_A = 0, \]  \hfill (2.18)

\[ w_A \cos \varphi_B + w_B \cos \varphi_A + w_C = 0, \]  \hfill (2.19)

and

\[ \varphi_A + \varphi_B + \varphi_C = 2\pi. \]

The solution of (2.17), (2.18) and (2.19) w.r. to \( \cos \varphi_Q \) yields (2.22).

Suppose that \( w_A, w_B, w_C \) are variables and \( \varphi_A, \varphi_B, \varphi_C \), are given. The solution of (2.17), (2.18) and (2.19) w.r. to \( w_A, w_B, w_C \) yields a positive answer w.r to the inverse weighted Fermat-Torricelli problem on \( M \):

**Proposition 2.** The solution of the inverse weighted Fermat-Torricelli problem on a surface \( M \) is given by:

\[ w_Q = \frac{\text{Constant}}{1 + \frac{\sin \varphi_R}{\sin \varphi_Q} + \frac{\sin \varphi_S}{\sin \varphi_Q}}, \]  \hfill (2.20)

for \( Q, R, S \in \{A, B, C\} \).

**Remark 1.** The solution of the inverse weighted Fermat-Torricelli problem on a \( C^2 \) complete surface with Gaussian curvature \( 0 < K < a \) or \( K < 0 \), has been derived in [6], [7].
The idea of assigning a residual weight (subconscious) at a weighted Fermat-Torricelli point (generalized Fermat-Torricelli point) is given in [9], by assuming that a weighted Fermat-Torricelli tree is a two way communication network and the weights $w_A, w_B, w_C$ are three small masses that may move through the branches of the weighted Fermat-Torricelli tree. By assuming mass flow continuity of this network, we obtain the generalized inverse weighted Fermat-Torricelli problem (inverse s.F.T problem).

The inverse s.F.T problem is the inverse weighted Fermat-Torricelli problem, such that the weighted Fermat-Torricelli point has acquired a subconscious $w_S$.

We denote by $w_R$ a mass flow which is transferred from $R$ to $F$ for $R \in \{A, B\}$, by $w_S$ a residual weight which remains at $F$ and by $w_C$ a mass flow which is transferred from $F$ to $C$, by $\tilde{w}_R$ a mass flow which is transferred from $F$ to $R$, $R \in \{A, B\}$, and by $\tilde{w}_S$ a residual weight which remains at $F$ and by $\tilde{w}_C$ a mass flow which is transferred from $C$ to $F$.

The following equations are derived by this mass flow along the infinitesimal geodesic arcs $AF, BF, CF$:

\[ w_A + w_B = w_C + w_S \]  \hspace{1cm} (2.21)

and

\[ \tilde{w}_A + \tilde{w}_B + \tilde{w}_S = \tilde{w}_C. \]  \hspace{1cm} (2.22)

By taking into account (2.21) and (2.22) and by setting $\bar{w}_S = w_S - \tilde{w}_S$, we get:

\[ \bar{w}_A + \bar{w}_B = \bar{w}_C + \bar{w}_S \]  \hspace{1cm} (2.23)

such that:

\[ \bar{w}_A + \bar{w}_B + \bar{w}_C = c > 0, \]  \hspace{1cm} (2.24)

**Problem 4.** Given a point $F$ which belongs to the interior of the infinitesimal geodesic triangle $\triangle ABC$ on $M$, does there exist a unique set of positive weights $\bar{w}_R$, such that

\[ \bar{w}_A + \bar{w}_B + \bar{w}_C = c = \text{const}, \]  \hspace{1cm} (2.25)

for which $F$ minimizes

\[ f(F) = w_A l_A(F) + w_B l_B(F) + w_C l_C(F), \]

\[ f(F) = \tilde{w}_A l_A(F) + \tilde{w}_B l_B(F) + \tilde{w}_C l_C(F), \]

\[ f(F) = \bar{w}_A l_A(F) + \bar{w}_B l_B(F) + \bar{w}_C l_C(F), \]

under the condition for the weights:

\[ \bar{w}_i + \bar{w}_j = \bar{w}_S + \bar{w}_k \]  \hspace{1cm} (2.27)

for $i, j, k \in A, B, C$ and $i \neq j \neq k$. 

Theorem 2. Given the g.FT point $F$ to be an interior point of the triangle $\triangle ABC$ with the vertices lie on three geodesic arcs that meet at $F$ and from the two given values of $\varphi_B, \varphi_C$, the positive real weights $\bar{w}_R$ given by the formulas

$$\bar{w}_A = \left( \frac{\sin(\varphi_B + \varphi_C)}{\sin \varphi_C} \right) \frac{c - \bar{w}_S}{2}, \quad (2.28)$$

$$\bar{w}_B = \left( \frac{\sin \varphi_B}{\sin \varphi_C} \right) \frac{c - \bar{w}_S}{2}, \quad (2.29)$$

and

$$\bar{w}_C = \frac{c - \bar{w}_S}{2} \quad (2.30)$$
give a negative answer w.r. to the inverse s.FT problem on $M$.

Remark 2. Theorem 2 is proved in [9] for the case of $\mathbb{R}^2$.

We conclude with an evolutionary scheme of infinitesimal geodesic triangles, which connects the subconscious of a weighted Fermat-Torricelli tree with the Aleksandrov curvature of a geodesic triangle $(\mathbb{R}(2))$.

Phase 1 At time zero, we assume that a point $F$ in $\mathbb{R}^3$ tends to split in three directions. It acquires a subconscious which equals with the absolute value of the Gaussian curvature $\|K(F)\|$ and predefines the surface with Gaussian curvature $K$, on which these three geodesic arcs will move (Weighted Fermat-Torricelli tree).

Phase 2 After time $t$, the subconscious quantity is increased and the value of the Aleksandrov curvature of the infinitesimal geodesic triangle $T$ is reached:

$$\bar{w}_S = \|K(T)\| = \| \angle A + \angle B + \angle C - \pi \|.$$

The following equations determine the values of $\bar{w}_A$, $\bar{w}_B$ and $\bar{w}_C$:

$$\bar{w}_A = \left( \frac{\sin(\varphi_B + \varphi_C)}{\sin \varphi_C} \right) \frac{1 - \bar{w}_S}{2},$$

$$\bar{w}_B = \left( \frac{\sin \varphi_B}{\sin \varphi_C} \right) \frac{1 - \bar{w}_S}{2},$$

and

$$\bar{w}_C = \frac{1 - \bar{w}_S}{2}.$$

Phase 2 gives a plasticity solution of the inverse s.FT problem that has acquired a subconscious $\bar{w}_S = \|K(\triangle ABC)\|$. 

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