Limiting mixed Hodge structures on the relative log de Rham cohomology groups of a projective semistable log smooth degeneration

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Abstract

We prove that the relative log de Rham cohomology groups of a projective semistable log smooth degeneration admit a natural limiting mixed Hodge structure. More precisely, we construct a family of increasing filtrations and a family of nilpotent endomorphisms on the relative log de Rham cohomology groups and show that they satisfy a part of good properties of a nilpotent orbit in several variables.

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1 Introduction

A morphism from a complex manifold to a polydisc is said to be semistable, if it is locally isomorphic to a product of semistable degenerations over the unit disc (cf. Example 3.5 and [11, Lemma 3.3]). The notion of semistable log smooth degeneration is an abstraction of the central fiber of a semistable...
morphism in the context of log geometry. Namely, a semistable log smooth degeneration is a log complex analytic space \((X, \mathcal{M}_X)\) over the log point \((*, \mathbb{N}^k)\) (i.e. a morphism of log complex analytic space \(f: (X, \mathcal{M}_X) \to (*, \mathbb{N}^k)\)), which is locally isomorphic to the central fiber of a semistable morphism to the \(k\)-dimensional polydisc in the category of log complex analytic spaces (cf. local description in 3.7). For the precise definition of semistable log smooth degeneration, see Definition 3.3.

One of the main results of this paper is the following.

**Theorem 1.1.** Let \(f: (X, \mathcal{M}_X) \to (*, \mathbb{N}^k)\) be a projective semistable log smooth degeneration. Then the relative log de Rham cohomology groups \(H^n(X, \Omega_{X/*}(\log(\mathcal{M}_X/\mathbb{N}^k)))\) admit a limiting mixed Hodge structure, whose Hodge filtration \(F\) is induced from the stupid filtration (filtration by in \([3, (1.4.7)]\)) on \(\Omega_{X/*}(\log(\mathcal{M}_X/\mathbb{N}^k))\).

Here, a \(\mathbb{Q}\)-mixed Hodge structure \(((V_\mathbb{Q}, W), (V_\mathbb{C}, W, F))\) is called a *limiting* mixed Hodge structure, if there exists a nilpotent endomorphism \(N\) of \(V_\mathbb{Q}\) with \(W = W(N)[k]\) for some \(k \in \mathbb{Z}\), where \(W(N)\) denotes the monodromy weight filtration of \(N\) (cf. [14, p. 90]). Theorem 1.1 is deduced from the following theorem:

**Theorem 1.2** (cf. Theorems 4.4 and 4.9). On \(H^n(X, \Omega_{X/*}(\log(\mathcal{M}_X/\mathbb{N}^k)))\), we can construct a finite increasing filtration \(L(I)\) for all \(I \subset \{1, 2, \ldots, k\}\) and nilpotent endomorphisms \(N_1, \ldots, N_k\) such that the following is satisfied:

1. (1.2.1) By setting \(L = L(\{1, 2, \ldots, k\})\), the triple \((H^n(X, \Omega_{X/*}(\log(\mathcal{M}_X/\mathbb{N}^k))), L[n], F)\) underlies a \(\mathbb{Q}\)-mixed Hodge structure.

2. (1.2.2) \(L(I)\) coincides with the monodromy weight filtration of the nilpotent endomorphism \(N_I(c_I) = \sum_{i \in I} c_i N_i\) for all \(c_I = (c_i)_{i \in I} \in (\mathbb{R}_{>0})^I\).

The case of \(I = \{1, 2, \ldots, k\}\) in (1.2.2) together with (1.2.1) implies Theorem 1.1. Moreover, (1.2.2) claims that the monodromy weight filtration of \(N_I(c_I)\) is independent of the choice of \(c_I \in (\mathbb{R}_{>0})^I\). The following theorem states the relation between the filtrations \(L\) and \(L(I)\).

**Theorem 1.3** (cf. Theorem 4.10). On \(H^n(X, \Omega_{X/*}(\log(\mathcal{M}_X/\mathbb{N}^k)))\), the filtration \(L\) is the monodromy weight filtration of \(N(c) = \sum_{i=1}^k c_i N_i\) relative to \(L(I)\) for all \(c = (c_i)_{i=1}^k \in (\mathbb{R}_{>0})^k\).

Theorems 1.2 and 1.3 are consequences of Theorems 4.4, 4.9 and 4.10. Theorem 4.4 follows directly from Theorem 4.3, which will be proved in Section 5. Theorems 4.9 and 4.10 will be proved together with Theorems 4.5 and 4.13 in Section 10. Theorem 4.5 claims the \(E_2\)-degeneracy of the spectral sequence associated to the filtration \(L(I)\). This is a generalization of the result on \(E_2\)-degeneracy for a projective semistable morphism in [8] and [11] to the case of a projective semistable log smooth degeneration. Theorem 4.13, which is a by-product of the proof of Theorems 4.9 and 4.10, states that the analogue of the hard Lefschetz theorem for \(H^*(X, \Omega_{X/*}(\log(\mathcal{M}_X/\mathbb{N}^k)))\) holds true. In [20], Y. Nakajima stated the log hard Lefschetz conjecture and proved it for a projective SNCL variety over the standard log point (cf. Conjecture 9.5 and Theorem 9.14 in [20]). Theorem 4.13 is the affirmative answer to an analogue of the log hard Lefschetz conjecture for a projective semistable log smooth degeneration.

1.4. This paper is partially motivated by Theorems I and I’ of Green and Griffiths [14]. Let \(X\) be a reduced complex analytic space, which is locally isomorphic to a product of normal crossing varieties as in (1.2) of [14]. Then Green and Griffiths claimed that a certain type of infinitesimal deformation of \(X\) (cf. p.100 and p.108 of [14]) canonically yields a polarized limiting mixed Hodge structure under
the appropriate projectivity assumption. In fact, the existence of a good infinitesimal deformation of $X$, which is assumed in [14] as above, implies that there exists a log structure $\mathcal{M}_X$ such that $(X, \mathcal{M}_X)$ becomes a semistable log smooth degeneration. Thus Theorem 1.1 above is an analogue of Theorems I and I’ of [14] in the context of log geometry. We note that the difference between Theorems I and I’ of [14] and Theorem 1.1 above is about polarization. We will return to this point later.

1.5. For the case of $k = 1$, a semistable log smooth degeneration is called a log deformation by Steenbrink in [25]. The relative log de Rham cohomology groups of a projective strict log deformation are thoroughly studied in [25], [12] and [10]. In particular, it is proved in [10] that they admit a natural polarized limiting mixed Hodge structure. Thus the result of this paper is a partial generalization of results in [25], [12] and [10] to a projective semistable log smooth degeneration.

Study of limiting mixed Hodge structures for a projective semistable degeneration over the unit disc originated from Steenbrink [24], in which he proved that such a morphism yields a natural limiting mixed Hodge structure on the relative log de Rham cohomology groups of the central fiber (cf. [4], [23], [16], [27]). His results were generalized by the author’s previous works [7], [8] and [11] to the case of a projective semistable morphism over a higher dimensional polydisc. The other motivation of this paper is to generalize these results to a projective semistable log smooth degeneration.

1.6. We briefly explain the outline of this paper. In Section 2, we fix notation and collect several preliminary definitions and results for the later use. In Section 3, we introduce the notion of a semistable log smooth degeneration. Hereafter, a semistable log smooth degeneration $f: (X, \mathcal{M}_X) \rightarrow (\ast, \mathbb{N}^k)$ is fixed. In 3.7, we give a local description of a semistable log smooth degeneration, which is constantly used throughout this paper. Some notation and results on Koszul complexes are briefly recalled in 3.14. In Definition 3.21, we construct $((A_{\mathbb{Q}}, L(I)), (A_{\mathbb{C}}, L(I), F), \alpha)$ consisting of a complex of $\mathbb{Q}$-sheaves $A_{\mathbb{Q}}$ equipped with an increasing filtration $L(I)$ and an decreasing filtration $F$, and a morphism of complexes of $\mathbb{C}$-sheaves $A_{\mathbb{C}}$ equipped with an increasing filtration $L(I)$ and a decreasing filtration $F$, and a morphism of complexes $\alpha: A_{\mathbb{Q}} \rightarrow A_{\mathbb{C}}$ preserving the filtrations $L(I)$ for all $I \subset \{1, 2, \ldots, k\}$. (We set $L = L(\{1, 2, \ldots, k\})$ as in (1.2.1).) These data play a central role in this paper. In fact, Lemma 3.26 states that $(A_{\mathbb{C}}, F)$ is filtered quasi-isomorphic to $(\Omega_{X/\mathbb{C}}(\log(\mathcal{M}_X/\mathbb{N}^k)), F)$. Therefore the filtered vector space $(H^r(\Omega_{X/\mathbb{C}}(\log(\mathcal{M}_X/\mathbb{N}^k))), F)$ is replaced by $(H^r(X, A_{\mathbb{C}}), F)$ in what follows (cf. Corollary 3.27). Section 4 is devoted to state the main results of this paper, Theorems 4.3, 4.4, 4.5, 4.9, 4.10, and 4.13. An endomorphism $\nu_i$ on $A_{\mathbb{Q}}$ and $A_{\mathbb{C}}$, which induces the nilpotent endomorphism $N_i$ in Theorem 1.2, is defined in Definition 4.6 for $i = 1, 2, \ldots, k$. In Section 5, Theorem 4.3 is proved. We first construct log complex manifolds $(X_r, \mathcal{M}_{X_r})$ in Definition 5.4. Then we define the residue morphism (5.5) for the log de Rham complex in Definition 5.12, and (5.11) for the Koszul complex in Definition 5.20 respectively. Once these residue morphisms are obtained, Theorem 4.3 is a consequence of the classical Hodge theory on $H^r(X_r, \mathcal{E}_r \otimes_{\mathbb{Z}} \mathbb{C})$, where $\mathcal{E}_r$ is a locally free $\mathbb{Z}$-module of rank one admitting a positive definite symmetric bilinear form (see Definition 5.8 for $\mathcal{E}_r$).

To prove the remaining theorems, we will apply a result on a polarized differential multi-graded Hodge-Lefschetz module to $V_{\mathbb{C}} = \bigoplus_{a, b} E_{1, a, b} (A_{\mathbb{C}}, L)$. (Precisely, we will apply Proposition 9.8 to the real form $V_{\mathbb{R}}$ of $V_{\mathbb{C}}$.) To prove that $V_{\mathbb{R}}$ is a polarized differential $\mathbb{Z} \oplus \mathbb{Z}^k$-graded Hodge-Lefschetz module, the most subtle point is to construct a polarization on $V_{\mathbb{R}}$. Apparently it looks possible to obtain such a polarization directly from the fact that $V_{\mathbb{C}}$ is expressed as a direct sum of cohomology groups $H^r(X_r, \mathcal{E}_r \otimes_{\mathbb{Z}} \mathbb{C})$. Actually we can obtain a bilinear form on $V_{\mathbb{C}}$ by using the polarization on $H^r(X_r, \mathcal{E}_r \otimes_{\mathbb{Z}} \mathbb{C})$ as in [16, (3.4)]. However, it is difficult to prove that the bilinear form obtained in this way is compatible with the morphism $d_1$ of $E_{1, r}$-terms of the spectral sequence $E_{1, r}^* (A_{\mathbb{C}}, L)$
The idea to avoid this difficulty is to construct a product on the filtered complex $(A_C, L)$, which induces the desired bilinear form on $V_C$. The fact that the bilinear form comes from a product on $(A_C, L)$ enables us to analyze the relation between this bilinear form and the morphism $d_1$ of the $E_1$-terms. To carry out this idea, we follow the arguments in [10] and adapt them to the case of a semistable log smooth degeneration. In Section 6, we construct a Čech type filtered complex $(\mathcal{C}(\Omega_{X*}(\log M_{X*})), \delta W)$ and a product on it. Moreover, under the assumption that $X$ is of pure dimension, we construct a morphism $\Theta: E^{1-k,2\dim X+2k}_1 \to \mathbb{C}$, where $E^{p,q}_1$ denotes the $E_1$-terms of the spectral sequence associated to the filtered complex $(\mathcal{R} \Gamma_c(X, \mathcal{C}(\Omega_{X*}(\log M_{X*}))), \delta W)$. In Section 7, a product $A_C \otimes_C A_C \to \mathcal{C}(\Omega_{X*}(\log M_{X*}))[k]$ is constructed by using the residue morphism on $A_C$ and the product on $\mathcal{C}(\Omega_{X*}(\log M_{X*}))$. This product induces a product on $E^{n,*}_1(A_C, L)$ with values in $E^{n,*}_1(\mathcal{C}(\Omega_{X*}(\log M_{X*})), \delta W)$. In Section 8, under the assumption that $X$ is projective and of pure dimension, a bilinear form on $V_C = \bigoplus_{a,b} E^{n,b}_1(A_C, L)$ is constructed as follows. For two elements of $V_C$, the product of these elements is contained in $\bigoplus_{p,q} E^{p,q}_1(C(\Omega_{X*}(\log M_{X*})), \delta W)$. Then taking its image by the projection to the direct summand $E^{1-k,2\dim X+2k}_1(\mathcal{C}(\Omega_{X*}(\log M_{X*})), \delta W)$, and evaluate it by the morphism $\Theta$ above with an appropriate sign. (For the precise definition, see Definition 8.13.)

Thus it is hoped that $\mathcal{H}^n(X, A_C)$ is the case, then we can prove that a projective semistable log smooth degeneration yields polarized log Hodge structures on the log point $(\ast, \mathbb{N}^k)$. This partially generalize the result in [13] to the case over a base with higher log rank.

1.7. We discuss about remaining problems. Compared to the results of Green and Griffiths mentioned in 1.4, the limiting mixed Hodge structure in Theorem 1.1 is expected to be polarized. To this end, we have to lift the polarization on $V_C$ to a bilinear form on $H^1(X, A_C)$. It seems possible to obtain such a lifting by following the arguments in [10].

Theorem 1.3 states the relation between the filtrations $L$ and $L(I)$. It is natural to consider the relation between $L(I)$ and $L(J)$ for $J \subset I \subset \{1, 2, \ldots, k\}$ as in the theory of nilpotent orbits in several variables (see e.g. [1]). Namely, $L(I)$ is expected to be the monodromy weight filtration of $N_I(c_I)$ relative to the filtration $L(J)$ for every $c_I \in (\mathbb{R}_{>0})^I$. Furthermore, Theorems 1.2 and 1.3 show that the limiting mixed Hodge structure $(H^n(X, A), L, F)$ equipped with the nilpotent endomorphisms $N_1, \ldots, N_k$ satisfy a part of good properties of a nilpotent orbit in several variables. Thus it is hoped that $(H^n(X, A_C), F, N_1, \ldots, N_k)$ generates a nilpotent orbit in $k$-variables. If this is the case, then we can prove that a projective semistable log smooth degeneration yields polarized log Hodge structures on the log point $(\ast, \mathbb{N}^k)$. This partially generalize the result in [13] to the case over a base with higher log rank.

Since a semistable log smooth degeneration is a special case of a log smooth degeneration defined in [9, Definition 4.3], it is already proved that the relative log de Rham cohomology groups of a projective semistable log smooth degeneration carries a $\mathbb{Q}$-mixed Hodge structure if all the irreducible components of $X$ are smooth. Although the two constructions, one in [9] and the other in this paper, are rather different, it should be proved that these two mixed Hodge structures are the same.

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2 Preliminaries

2.1. The cardinality of a finite set $A$ is denoted by $|A|$.

2.2. The set of the positive integers (resp. the positive real numbers) is denoted by $\mathbb{Z}_{>0}$ (resp. $\mathbb{R}_{>0}$).

2.3. For two sets $A$ and $B$, the set of all maps from $A$ to $B$ is denoted by $B^A$.

2.4. Let $A$ be a finite set. Then $\mathbb{Z}^A$ is a free $\mathbb{Z}$-module of rank $|A|$, whose canonical $\mathbb{Z}$-basis is denoted by $\{e_a\}_{a \in A}$. For a subset $B \subseteq A$, we have the canonical direct sum decomposition $\mathbb{Z}^A = \mathbb{Z}^B \oplus \mathbb{Z}^{A \setminus B}$, which induces the canonical surjection $\mathbb{Z}^A \twoheadrightarrow \mathbb{Z}^B$. For an element $q \in \mathbb{Z}^A$, its image by this canonical surjection is denoted by $q_B \in \mathbb{Z}^B$. We set $e = \sum_{a \in A} e_a \in \mathbb{Z}^A$. Then $e_B = \sum_{a \in B} e_a \in \mathbb{Z}^B$.

For $q = \sum_{a \in A} q_a e_a \in \mathbb{Z}^A$, we set $|q| = \sum_{a \in A} q_a \in \mathbb{Z}$. For the case of $A = \{1, 2, \ldots, k\}$, we use $\mathbb{Z}^k$ instead of $\mathbb{Z}^A$. As usual, we write $q = (q_1, q_2, \ldots, q_k)$ for $q = \sum_{i=1}^k q_i e_i \in \mathbb{Z}^k$.

A partial order $\geq$ on $\mathbb{Z}^A$ is defined by

$$q = \sum_{a \in A} q_a e_a \geq q' = \sum_{a \in A} q'_a e_a \iff q_a \geq q'_a \quad \text{for all } a \in A. \quad (2.1)$$

We set $\mathbb{Z}^A_q = \{ r \in \mathbb{Z}^A \mid r \geq q \}$ for $q \in \mathbb{Z}^A$. For the case of $q = 0$, we use $\mathbb{N}^A = \mathbb{Z}^A_{\geq 0}$. Then $\mathbb{N}^A$ is a monoid admitting a direct sum decomposition $\mathbb{N}^A = \bigoplus_{a \in A} \mathbb{N}e_a$ as monoids.

2.5. Let $\Lambda$ be a finite set. We sometimes use $\Delta, \mu, \nu, \ldots$ for subsets of $\Lambda$. The set of all subsets of $\Lambda$ is denoted by $S(\Lambda)$. For the case where a partition into disjoint union

$$\Lambda = \biguplus_{i=1}^k \Lambda_i \quad (2.2)$$

is given, we set

$$S_r(\Lambda) = \{ \Lambda \in S(\Lambda) \mid \Lambda \cap \Lambda_i = r_i \text{ for all } i = 1, 2, \ldots, k \}$$

for $r = (r_i)_{i=1}^k \in \mathbb{Z}^k$. Note that $|\Delta| = |r|$ for $\Delta \in S_r(\Lambda)$.

2.6. For a finite set $\Lambda$, we set $\varepsilon(\Lambda) = \bigwedge |\Lambda| \mathbb{Z}^\Lambda$, which is a free $\mathbb{Z}$-module of rank one. We note $\varepsilon(\emptyset) = \mathbb{Z}$ by definition. Moreover, we set $\bigwedge \mathbb{Z}^\Lambda = \bigoplus_{m \geq 0} \bigwedge^m \mathbb{Z}^\Lambda$. Then the equality

$$\bigwedge \mathbb{Z}^\Lambda = \bigoplus_{\Delta \in S(\Lambda)} \varepsilon(\Delta) \quad (2.3)$$

holds.

2.7 (Two products $\chi$ and $\varnothing$ on $\bigwedge \mathbb{Z}^\Lambda$). Let $\Lambda$ be a finite set. A morphism $\chi(\Lambda) \colon \bigwedge \mathbb{Z}^\Lambda \otimes \bigwedge \mathbb{Z}^\Lambda \twoheadrightarrow \bigwedge \mathbb{Z}^\Lambda$ is defined by $\chi(\Lambda)(v \otimes w) = v \wedge w$ for $v, w \in \bigwedge \mathbb{Z}^\Lambda$. Via the direct sum decomposition (2.3), the restriction of $\chi(\Lambda)$ on the direct summand $\varepsilon(\Delta) \otimes \varepsilon(\mu)$ induces an isomorphism

$$\chi(\Delta, \mu) : \varepsilon(\Delta) \otimes \varepsilon(\mu) \cong \varepsilon(\Delta \cup \mu) \quad (2.4)$$

if $\Delta \cap \mu = \emptyset$. Similarly, an isomorphism

$$e_\Lambda \wedge : \varepsilon(\mu) \longrightarrow \varepsilon(\{\lambda\} \cup \mu) \quad (2.5)$$

is defined by sending $v \in \varepsilon(\mu)$ to $e_\Lambda \wedge v \in \varepsilon(\{\lambda\} \cup \mu)$ for $\lambda \in \Lambda \setminus \mu$. 

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Now, we consider the case where Λ is equipped with a partition (2.2). For \( \frac{\lambda, \mu}{\Lambda} \subset \Lambda \) with \( |\Lambda \cap \mu \cap \Lambda| = 1 \) for all \( i = 1, 2, \ldots, k \), we set \( \{ \lambda_i \} = \frac{\lambda, \mu}{\Lambda} \cap \Lambda_i \) for each \( i \), and obtain an isomorphism \( (e_{\lambda_i} \wedge) \cdot (e_{\mu} \wedge)^{-1} : \varepsilon(\mu) \to \varepsilon(\mu \setminus \{ \lambda_1, \ldots, \lambda_k \}) \), where \( e_{\lambda_i} \wedge \) is the isomorphism (2.5). Then a morphism \( \chi(\frac{\lambda, \mu}{\Lambda}) : \varepsilon(\Lambda) \otimes \varepsilon(\mu) \to \bigwedge Z^\Lambda \) is defined by
\[
\chi(\frac{\lambda, \mu}{\Lambda}) = \chi(\frac{\lambda, \mu}{\Lambda} \setminus \{ \lambda_1, \lambda_2, \ldots, \lambda_k \}) \cdot (id \otimes (e_{\lambda_i} \wedge)^{-1} \cdots (e_{\lambda_k} \wedge)^{-1}).
\]
For the case where \( |\Lambda \cap \mu \cap \Lambda| \neq 1 \) for some \( i \in \{ 1, 2, \ldots, k \} \), we set \( \chi(\frac{\lambda, \mu}{\Lambda}) = 0 \) as a morphism from \( \varepsilon(\Lambda) \otimes \varepsilon(\mu) \) to \( \bigwedge Z^\Lambda \). Thus we obtain a morphism
\[
\chi(\Lambda) = \bigoplus \chi(\frac{\lambda, \mu}{\Lambda}) : \bigwedge Z^\Lambda \otimes \bigwedge Z^\Lambda \to \bigwedge Z^\Lambda
\]
via the direct sum decomposition (2.3). For \( v \in \Lambda^p Z^\Lambda, w \in \Lambda^q Z^\Lambda \), the equality
\[
\chi(\Lambda)(w \otimes v) = (-1)^{(p-k)(q-k)} \chi(\Lambda)(v \otimes w)
\]
(2.7) can be easily checked.

**Remark 2.8.** Let \( \Lambda \) be as above and \( \Gamma \subset \Lambda \). Then \( \Gamma \) has a partition \( \Gamma = \bigsqcup_{i=1}^k \Gamma_i \). For the morphisms \( \chi(\Lambda) \) and \( \chi(\Gamma) \) defined above, the diagram
\[
\begin{array}{ccc}
\bigwedge Z^\Lambda \otimes \bigwedge Z^\Lambda & \xrightarrow{\chi(\Lambda)} & \bigwedge Z^\Lambda \\
\downarrow & & \downarrow \\
\bigwedge Z^\Gamma \otimes \bigwedge Z^\Gamma & \xrightarrow{\chi(\Gamma)} & \bigwedge Z^\Gamma,
\end{array}
\]
is commutative, where the vertical arrows are the morphisms induced from the canonical surjection \( Z^\Lambda \to Z^\Gamma \) in 2.4.

**Finitely generated free monoids**

**Definition 2.9** (A finitely generated free monoid). In this paper, a monoid \( P \) is called a finitely generated free monoid if there exists an isomorphism of monoids \( P \simeq N^\Lambda \) for some finite set \( \Lambda \).

**Remark 2.10.** In the situation above, the finite set \( \Lambda \) is uniquely determined by \( P \) up to the unique isomorphism in the following sense. Let \( \Lambda \) and \( \Gamma \) be finite sets, and \( \xi_1 : P \xrightarrow{\simeq} N^\Lambda \) and \( \xi_2 : P \xrightarrow{\simeq} N^\Gamma \) isomorphisms of monoids. Then there exists a unique bijection \( \sigma : \Lambda \to \Gamma \) such that
\[
(\xi_2 \cdot \xi_1^{-1})(e_{\lambda}) = e_{\sigma(\lambda)} \quad \text{for all} \quad \lambda \in \Lambda.
\]

**Definition 2.11** (The canonical bilinear form on a finitely generated free monoid). Let \( P \) be a finitely generated free monoid. Fix an isomorphism \( \xi : P \xrightarrow{\simeq} N^\Lambda \) for a finite set \( \Lambda \). Then \( \xi \) induces an isomorphism \( \xi^{gp} : P^{gp} \simeq Z^\Lambda \). On \( Z^\Lambda \), there exists the canonical bilinear form (\( \cdot, \cdot \)) : \( Z^\Lambda \otimes Z Z^\Lambda \to Z \) defined by \( (e_{\lambda}, e_{\mu}) = 1 \) and \( (e_{\lambda}, e_{\mu}) = 0 \) for \( \lambda \neq \mu \). Via the isomorphism \( \xi^{gp} \) above, a symmetric bilinear form \( P^{gp} \otimes_Z P^{gp} \to Z \) is induced. By Remark 2.10 above, this bilinear form is independent of the isomorphism \( \xi \). This bilinear form \( P^{gp} \otimes_Z P^{gp} \to Z \) is called the canonical bilinear form associated to \( P \). Trivially, the induced bilinear form on \( R \otimes Z P^{gp} \) is symmetric and positive definite.

**Definition 2.12** (A semistable morphism to a finitely generated free monoid). Let \( \Lambda \) be a finite set. A morphism of monoids \( \varphi : N^k \to N^\Lambda \) is said to be semistable if there exists a partition \( \Lambda = \bigsqcup_{i=1}^k \Lambda_i \) such that \( \varphi(e_i) = \sum_{\lambda \in \Lambda_i} e_{\lambda} \) for all \( i = 1, 2, \ldots, k \). The partition \( \Lambda = \bigsqcup_{i=1}^k \Lambda_i \) is called the partition associated to \( \varphi \). More generally, a morphism of monoids \( \varphi : N^k \to P \) to a finitely generated free monoid \( P \) is said to be semistable if there exist a finite set \( \Lambda \) and an isomorphism \( \xi : P \xrightarrow{\simeq} N^\Lambda \) such that the composite \( \xi \cdot \varphi \) is semistable in the sense defined above.
Remark 2.13. For a semistable morphism \( \varphi : \mathbb{N}^k \rightarrow P \), the finite set \( \Lambda \) equipped with the partition \( \Lambda = \bigsqcup_{i=1}^{k} \Lambda_i \) above is uniquely determined by \( \varphi \) in the following sense. Let \( \Lambda \) and \( \Gamma \) be finite sets, and \( \xi_1 : P \overset{\cong}{\rightarrow} \mathbb{N}^\Lambda \) and \( \xi_2 : P \overset{\cong}{\rightarrow} \mathbb{N}^\Gamma \) isomorphisms such that \( \xi_1 \cdot \varphi \) and \( \xi_2 \cdot \varphi \) are semistable. Then the bijection \( \sigma : \Lambda \rightarrow \Gamma \) in Remark 2.10 preserves the partitions of \( \Lambda \) and \( \Gamma \) associated to \( \xi_1 \cdot \varphi \) and \( \xi_2 \cdot \varphi \) respectively.

Definition 2.14 (The direct sum decomposition associated to a semistable morphism). Let \( P \) be a finitely generated free monoid and \( \varphi : \mathbb{N}^k \rightarrow P \) a semistable morphism. Take a finite set \( \Lambda \), an isomorphism \( \xi : P \rightarrow \mathbb{N}^\Lambda \) and a partition \( \Lambda = \bigsqcup_{i=1}^{k} \Lambda_i \) associated to \( \xi \cdot \varphi \) as in Definition 2.12. Then a finitely generated free monoid \( P_i = \xi^{-1}(\mathbb{N}^{\Lambda_i}) \) is independent of the choice of \( \xi \) by the remark above. Thus we obtain a direct sum decomposition of monoids \( P = \bigoplus_{i=1}^{k} P_i \), called the decomposition associated to \( \varphi \).

Definition 2.15 (The product \( \chi \) associated to a semistable morphism). Let \( P \) be a finitely generated free monoid and \( \varphi : \mathbb{N}^k \rightarrow P \) a semistable morphism. Take \( \xi : P \rightarrow \mathbb{N}^\Lambda \) and \( \Lambda = \bigsqcup_{i=1}^{k} \Lambda_i \) as above. Via the isomorphism \( \bigwedge \xi_{EP} : \bigwedge P_{EP} \overset{\cong}{\rightarrow} \bigwedge Z^\Lambda \), the morphism \( \chi(\Lambda) \) in (2.6) gives us a morphism \( \bigwedge P_{EP} \otimes \bigwedge P_{EP} \rightarrow \bigwedge P_{EP} \) which is independent of \( \xi \) by Remark 2.13. This morphism is denoted by \( \chi(\varphi) \).

Filtered complexes

Notation 2.16 (Finiteness for filtrations). Because we mainly use finite filtrations in this paper, we usually omit the adjective “finite” for filtrations.

Notation 2.17 (Spectral sequences). We follow the notation in [3, (1.3.1)] for the spectral sequence associated to a filtered complex. Let \( (K_1, F) \) and \( (K_2, F) \) be decreasingly filtered complexes. A morphism \( f : (K_1, F) \rightarrow (K_2, F) \) in the filtered derived category induces a morphism of spectral sequences \( E^{p,q}_{r}(K_1, F) \rightarrow E^{p,q}_{r}(K_2, F) \), denoted by \( E^{p,q}_{r}(f) \), for all \( p, q \) and for all \( r \) with \( 1 \leq r \leq \infty \). We often use \( E_{i}(f) \) instead of \( E^{i}_{i}(f) \) for short. The morphism \( E^{p,q}_{\infty}(f) \) coincides with \( Gr_{F}^{p} \text{H}^{p+q}(f) \) via the isomorphisms \( E^{p,q}_{\infty}(K_1, F) \simeq Gr_{F}^{p} \text{H}^{p+q}(K_1) \) for \( i = 1, 2 \).

2.18 (Tensor product of complexes). For two complexes \( K \) and \( L \), the differential of the complex \( K \otimes L \) is given by \( d = d \otimes 1 + (-1)^{p} id \otimes d \) on the direct summand \( K^{p} \otimes L^{q} \) of \( (K \otimes L)^{p+q} \). An identification \( K \otimes L \overset{\cong}{\rightarrow} L \otimes K \) is given by \( x \otimes y \mapsto (-1)^{pq} y \otimes x \) on \( K^{p} \otimes L^{q} \) as in [2, p. 11]. For \( a, b \in \mathbb{Z} \), an identification \( K[a] \otimes L[b] \overset{\cong}{\rightarrow} (K \otimes L)[a+b] \) is given by

\[
x \otimes y \mapsto (-1)^{pb} x \otimes y \quad (2.8)
\]
on \( K[a]^{p} \otimes L[b]^{q} = K^{p+a} \otimes L^{q+b} \) as in [2, (1.3.6)].

Definition 2.19. For two complexes \( K_1 \) and \( K_2 \), a morphism \( \text{H}^{a}(K_1) \otimes \text{H}^{b}(K_2) \rightarrow \text{H}^{a+b}(K_1 \otimes K_2) \) is canonically induced for all \( a, b \in \mathbb{Z} \). For a morphism of complexes \( f : K_1 \otimes K_2 \rightarrow K_3 \), the composite

\[
\text{H}^{a}(K_1) \otimes \text{H}^{b}(K_2) \rightarrow \text{H}^{a+b}(K_1 \otimes K_2) \overset{f}{\rightarrow} \text{H}^{a+b}(K_3)
\]
is denoted by \( \text{H}^{a+b}(f) \) in this paper. For the case where \( K_1, K_2, K_3 \) are complexes of abelian sheaves on a topological space \( X \), morphisms \( \text{H}^{a}(X, K_1) \otimes \text{H}^{b}(X, K_2) \rightarrow \text{H}^{a+b}(X, K_1 \otimes K_2) \) and \( \text{H}^{a,b}(X, f) : \text{H}^{a}(X, K_1) \otimes \text{H}^{b}(X, K_2) \rightarrow \text{H}^{a+b}(X, K_3) \) are defined similarly.
Definition 2.20 (Filtration on the tensor product). Let \((K_1, F)\) and \((K_2, F)\) be two decreasingly filtered complexes. A decreasing filtration \(F\) on \(K_1 \otimes K_2\) is defined by

\[
F^r(K_1^p \otimes K_2^q) = \sum_{a + b = r} \text{Image}(F^a K_1^p \otimes F^b K_2^q \to K_1^p \otimes K_2^q)
\]

for all \(p, q \in \mathbb{Z}\). There exists the canonical morphism \(\text{Gr}^a_{F} K_1 \otimes \text{Gr}^b_{F} K_2 \to \text{Gr}^{a+b}_{F}(K_1 \otimes K_2)\) for \(a, b \in \mathbb{Z}\). For a filtration of filtered complexes \(f: (K_1 \otimes K_2, F) \to (K_3, F)\), the composite

\[
\text{Gr}^a_{F} K_1 \otimes \text{Gr}^b_{F} K_2 \to \text{Gr}^{a+b}_{F}(K_1 \otimes K_2) \xrightarrow{\text{Gr}^{a+b}_{F} f} \text{Gr}^{a+b}_{F} K_3
\]

is denoted by \(\text{Gr}^{a,b}_{F} f\).

Definition 2.21. For two decreasingly filtered complex \((K_1, F)\) and \((K_2, F)\), a morphism

\[
\rho^{a,b,c,d}_r: E^{a,b}_r(K_1, F) \otimes E^{c,d}_r(K_2, F) \to E^{a+c,b+d}_r(K_1 \otimes K_2, F)
\]

is canonically induced for all \(0 \leq r \leq \infty\) and for all \(a, b, c, d \in \mathbb{Z}\). For the morphism \(d_r\) of \(E_r\)-terms, the equality

\[
d_r \cdot \rho^{a,b,c,d}_r = \rho^{a+r,b-r+1,c,d}_r \cdot (d_r \otimes \text{id}) + (-1)^{a+b} \rho^{a,b,c,r,d-r+1}_r \cdot (\text{id} \otimes d_r)
\]

holds on \(E^{a,b}_r(K_1, W) \otimes E^{c,d}_r(K_2, W)\) for all \(a, b, c, d \in \mathbb{Z}\). For a morphism \(f: (K_1 \otimes K_2, F) \to (K_3, F)\) in the filtered derived category, the composite

\[
E^{a,b}_r(K_1, F) \otimes E^{c,d}_r(K_2, F) \xrightarrow{\rho^{a,b,c,d}_r} E^{a+c,b+d}_r(K_1 \otimes K_2, F) \xrightarrow{E_r(f)} E^{a+c,b+d}_r(K_3, F)
\]

is simply denoted by \(E_r(f)\) for \(1 \leq r \leq \infty\) by abuse of notation.

2.22 (Gysin morphism for a bifiltered complex). Let \(F\) and \(G\) be two decreasing filtrations on a complex \(K\). The short exact sequence

\[
0 \to \text{Gr}^{a+1}_G K \to G^a K / G^{a+2} K \to \text{Gr}^a_G K \to 0
\]

defines a morphism

\[
\gamma_G: \text{Gr}^a_G K \to \text{Gr}^{a+1}_G K[1]
\]

in the derived category for all \(a \in \mathbb{Z}\). In fact, this morphism \(\gamma_G\) underlies a morphism

\[
\gamma_G: (\text{Gr}^a_G K, F) \to (\text{Gr}^{a+1}_G K[1], F),
\]

denoted by the same letter \(\gamma_G\), in the filtered derived category because

\[
0 \to \text{Gr}^{p}_F \text{Gr}^{a+1}_G K \to \text{Gr}^{p}_F(G^a K / G^{a+2} K) \to \text{Gr}^{p}_F \text{Gr}^a_G K \to 0
\]

is exact for all \(p\). Thus \(\gamma_G\) induces a morphism of spectral sequences

\[
E_r(\gamma_G): E^{p,q}_r(\text{Gr}^a_G K, F) \to E^{p,q+1}_r(\text{Gr}^{a+1}_G K, F)
\]

for \(1 \leq r \leq \infty\) by the identification \(E^{p,q}_r(\text{Gr}^{a+1}_G K[1], F) \simeq E^{p,q+1}_r(\text{Gr}^{a+1}_G K, F)\). Here we note that \(E_r(\gamma_G)\) is anti-commutative with \(d_r\) because the morphism \(d_r\) on \(E^{p,q}_r(\text{Gr}^{a+1}_G K[1], F)\) is identified with the morphism \(-d_r\) on \(E^{p,q+1}_r(\text{Gr}^{a+1}_G K, F)\).
2.23 (Convolution of two filtrations). Let $K$ be a complex. For two decreasing filtrations $F$ and $G$ on $K$, a decreasing filtration $F \ast G$ on $K$ is defined by

$$(F \ast G)^{p}K^{n} = \sum_{a+b=p} F^{a}K^{n} \cap G^{b}K^{n}$$

for all $n, p \in \mathbb{Z}$ as in [26, (1.4) Definition] and [17, Definition 1.3.1]. Then the canonical injection $F^{a}K \cap G^{b}K \hookrightarrow (F \ast G)^{a+b}K$ induces an isomorphism of complexes

$$\bigoplus_{a+b=p} \text{Gr}_{F}^{a}\text{Gr}_{G}^{b}K \xrightarrow{\sim} \text{Gr}_{F \ast G}^{p}K$$

for all $p$, under which we have the identification

$$\bigoplus_{a+b=p} \text{Gr}_{F}^{a}\text{Gr}_{G}^{b}K \xrightarrow{\sim} F^{k}\text{Gr}_{F \ast G}^{p}K$$

for all $k$. Thus we obtain identifications

$$E_{1}^{p,q}(K, F \ast G) \simeq H^{p+q}(\text{Gr}_{F \ast G}^{p}K) \simeq \bigoplus_{a+b=p} H^{p+q}(\text{Gr}_{F}^{a}\text{Gr}_{G}^{b}K) \simeq \bigoplus_{a+b=p} E_{1}^{a,b+q}(\text{Gr}_{G}^{b}K, F)$$

for all $p, q$, under which $F^{k}E_{1}^{p,q}(K, F \ast G)$ is identified with $\bigoplus_{a+b=p,a \geq k} E_{1}^{a,b+q}(\text{Gr}_{G}^{b}K, F)$. A morphism $d_{1}^{p}: E_{1}^{p,q}(K, F \ast G) \rightarrow E_{1}^{p+1,q}(K, F \ast G)$ is defined to be a direct sum of the morphisms of $E_{1}$-terms $E_{1}^{a,b+q}(\text{Gr}_{G}^{b}K, F) \rightarrow E_{1}^{a+1,b+q}(\text{Gr}_{G}^{b}K, F)$. Similarly, a morphism $d_{1}^{p+q}: E_{1}^{p,q}(K, F \ast G) \rightarrow E_{1}^{p+1,q}(K, F \ast G)$ is defined to be a direct sum of the morphisms $E_{1}(\gamma_{G}): E_{1}^{a,b+q}(\text{Gr}_{G}^{b}K, F) \rightarrow E_{1}^{a,b+q+1}(\text{Gr}_{G}^{b+1}K, F)$ in (2.10).

The following lemma is easily checked by definition.

**Lemma 2.24.** For the morphism of $E_{1}$-terms $d_{1}^{p}: E_{1}^{p,q}(K, F \ast G) \rightarrow E_{1}^{p+1,q}(K, F \ast G)$, the equality $d_{1} = d_{1}^{p} + d_{1}^{p+q}$ holds for all $p, q$.

**Notation 2.25** (Decreasing filtration and increasing filtration). A decreasing filtration $F$ induces an increasing filtration $W$ by $W_{m} = F^{-m}$ for all $m \in \mathbb{Z}$, and vice versa. We interchanges decreasing and increasing filtrations by this rule. For a decreasing filtration $F$, we use the notation $F[n] = F^{p+n}$. Hence, we use $W[n]_{m} = W_{m-n}$ for an increasing filtration $W$. Note that this notation for the shift of an increasing filtration coincides with the one in [3] and [5], and different from the one in [1].

**Log complex analytic spaces**

**Notation 2.26.** Let $(X, \mathcal{M}_{X})$ be a log complex analytic space. For an open subset $V \subset X$, the restriction $\mathcal{M}_{X}|_{V}$ is denoted by $\mathcal{M}_{V}$ for short. The monoid sheaf $\mathcal{M}_{X}/\mathcal{O}_{X}^{*}$ is denoted by $\overline{\mathcal{M}}_{X}$ as in [21]. The canonical morphism $\mathcal{M}_{X} \rightarrow \overline{\mathcal{M}}_{X}$ is denoted by $\pi_{X}$. The log de Rham complex of $(X, \mathcal{M}_{X})$ is denoted by $\Omega_{X}(\log \mathcal{M}_{X})$.

For an effective divisor $D$ on a complex manifold $X$, a log structure $\mathcal{M}_{X}(D)$ is defined by $\mathcal{M}_{X}(D) = j_{*}\mathcal{O}_{X \setminus D} \cap \mathcal{O}_{X}$, where $j: X \setminus D \rightarrow X$ is the open immersion (cf. [18, (1.5)]). For the case where $D$ is a normal crossing divisor on $X$, the log de Rham complex $\Omega_{X}(\log \mathcal{M}_{X}(D))$ coincides with the usual log de Rham complex $\Omega_{X}(\log D)$.
Notation 2.27. Let \( f: (X, \mathcal{M}_X) \rightarrow (Y, \mathcal{M}_Y) \) be a morphism of log complex analytic spaces. Then the morphism of monoid sheaves \( f^*: f^{-1}\mathcal{M}_Y \rightarrow \mathcal{M}_X \) induces a morphism of monoid sheaves \( f^*: f^{-1}\mathcal{M}_Y \rightarrow \mathcal{M}_X \) on \( X \), denoted by \( f^* \). Thus a morphism of monoids \( f^*: \mathcal{M}_Y, f(x) \rightarrow \mathcal{M}_{X,x} \) is induced for every \( x \in X \).

Notation 2.28. For two morphisms of log complex analytic spaces \((X, \mathcal{M}_X) \rightarrow (Z, \mathcal{M}_Z)\) and \((Y, \mathcal{M}_Y) \rightarrow (Z, \mathcal{M}_Z)\), we denote by \((X, \mathcal{M}_X) \times_{(Z, \mathcal{M}_Z)} (Y, \mathcal{M}_Y)\) the fiber product in the category of log complex analytic spaces.

Definition 2.29. Let \((X, \mathcal{M}_X)\) be a log complex analytic space. A monoid subsheaf \( \mathcal{N} \subset \mathcal{M}_X \) defines a log structure on \( X \) by restricting the structure morphism \( \mathcal{M}_X \rightarrow \mathcal{O}_X \) to \( \mathcal{N} \). Then the identity map of \( X \) induces a morphism of log complex analytic spaces \((X, \mathcal{M}_X) \rightarrow (X, \mathcal{N})\), which gives us the canonical morphism of the log de Rham complexes \( \Omega_X(\log \mathcal{N}) \rightarrow \Omega_X(\log \mathcal{M}_X) \). For \( m \in \mathbb{Z} \), an \( \mathcal{O}_X \)-submodule \( W(\mathcal{N})_m \Omega_X^m(\log \mathcal{M}_X) \) is defined by

\[
W(\mathcal{N})_m \Omega_X^m(\log \mathcal{M}_X) = \text{Image}(\Omega_X^{m-m}(\log \mathcal{N}) \otimes_{\mathcal{O}_X} \Omega_X^m(\log \mathcal{M}_X) \xrightarrow{\wedge} \Omega_X^m(\log \mathcal{M}_X)),
\]

where the morphism \( \wedge \) on the right hand side is induced from the wedge product on \( \Omega_X(\log \mathcal{M}_X) \). It is easy to see that \( W(\mathcal{N}) \) defines an increasing filtration on the complex \( \Omega_X(\log \mathcal{M}_X) \). By definition \( W(\mathcal{M}_X) \) is the trivial filtration. For the case of \( \mathcal{N} = \mathcal{O}_X^\times \), we use \( W \) instead of \( W(\mathcal{O}_X^\times) \).

3 Semistable log smooth degenerations

In this section, we first introduce the notion of a semistable log smooth degeneration. Then, we construct \(((A_Q, L(I), L), (A_C, L(I), L, F), \alpha)\), which is the object to be studied throughout this paper, for a semistable log smooth degeneration.

Notation 3.1. Let \( k \) be a positive integer. A pre-log structure \( \beta: \mathbb{N}^k \rightarrow \mathbb{C} \) over the point \( (\text{Spec} \mathbb{C})_\text{an} \) is given by \( \beta(0) = 1 \) and \( \beta(v) = 0 \) for \( v \in \mathbb{N}^k \setminus \{0\} \). The log structure associated to the pre-log structure \( \beta \) is \( \mathbb{C}^* \oplus \mathbb{N}^k \rightarrow \mathbb{C} \) sending \((a, v) \in \mathbb{C}^* \oplus \mathbb{N}^k \) to \( a\beta(v) \in \mathbb{C} \). The point equipped with this log structure is called the \( \mathbb{N}^k \)-log point and simply denoted by \((*, \mathbb{N}^k)\). The \( \mathbb{N} \)-log point \((*, \mathbb{N})\) is called the standard log point in \([25]\).

Notation 3.2. For a finitely generated monoid \( P \), the complex analytic space \((\text{Spec} \mathbb{C}[P])_\text{an} \) carries the log structure associated to the pre-log structure induced by the morphism \( P \rightarrow \mathbb{C}[P] \). This log complex analytic space is denoted by \((\text{Spec} \mathbb{C}[P])_\text{an}, P) \) for short. For a finite set \( \Lambda \), the log complex analytic space \((\text{Spec} \mathbb{C}[\mathbb{N}^\Lambda])_\text{an}, \mathbb{N}^\Lambda) \) is simply denoted by \((\mathbb{C}^\Lambda, \mathbb{N}^\Lambda) \). For the case of \( \Lambda = \{1, 2, \ldots, k\} \), we use \((\mathbb{C}^k, \mathbb{N}^k) \) instead of \((\mathbb{C}^\Lambda, \mathbb{N}^\Lambda) \). We have the canonical strict closed immersion \( \iota: (*, \mathbb{N}^k) \rightarrow (\mathbb{C}^k, \mathbb{N}^k) \), which sends the point \( * \) to the origin \( 0 \in \mathbb{C}^k \).

A morphism of finitely generated monoid \( h: Q \rightarrow P \) induces a morphism of log complex analytic spaces \((\text{Spec} \mathbb{C}[P])_\text{an}, P) \rightarrow ((\text{Spec} \mathbb{C}[Q])_\text{an}, Q) \) denoted by \( h \) throughout this paper.

Definition 3.3. Let \((X, \mathcal{M}_X)\) be an fs log complex analytic space. A morphism of log complex analytic spaces \( f: (X, \mathcal{M}_X) \rightarrow (*, \mathbb{N}^k) \) is called a semistable log smooth degeneration if the following three conditions are satisfied:

(3.3.1) \( f \) is log smooth.
(3.3.2) \( \mathcal{M}_{X,x} \) is a finitely generated free monoid for all \( x \in X \) (cf. Definition 2.9).
Moreover, a semistable log smooth degeneration \( f \) is said to be projective (resp. proper), if \( X \) is projective (resp. compact).

**Notation 3.4.** Let \( f: (X, \mathcal{M}_X) \to (\ast, \mathbb{N}^k) \) be a semistable log smooth degeneration. The relative log de Rham complex of \( f \) is denoted by \( \Omega_{X/k}(\log \mathcal{M}_X/\mathbb{N}^k) \) as in [18, (1.7)]. The image of \( e_i \in \Gamma(X, \mathbb{N}^k_X) \) by the morphism \( f^*: f^{-1}\mathbb{N}^k = \mathbb{N}^k_X \to \mathcal{M}_X \) is denoted by \( t_i \in \Gamma(X, \mathcal{M}_X) \) for \( i = 1, 2, \ldots, k \). This gives us a global section \( \text{dlog} t_i \in \Gamma(X, \Omega^1_X(\log \mathcal{M}_X)) \).

**Example 3.5.** Let \( \Delta^k \) be the \( k \)-dimensional polydisc with the coordinates \((t_1, \ldots, t_k)\) and \( g: X \to \Delta^k \) be a surjective morphism of complex manifolds. Let \( E \) be the divisor on \( \Delta^k \) defined by \( t_1 \cdots t_k \). Assume that \( D = f^*E \) is reduced simple normal crossing divisor on \( X \). Then \( X = g^{-1}(0) \to \{0\} \) underlies a semistable log smooth degeneration once we equip the log structures on \( X \) and \( \{0\} \) induced from \( \mathcal{M}_X(D) \) and \( \mathcal{M}_{\Delta^k}(E) \) respectively. Here we remark that the morphism \( g \) as above is called a semistable morphism in [11].

The following proposition shows that a semistable log smooth degeneration is locally isomorphic to the one obtained in the example above.

**Proposition 3.6.** Let \( f: (X, \mathcal{M}_X) \to (\ast, \mathbb{N}^k) \) be a semistable log smooth degeneration. For every \( x \in X \), there exist

\begin{align*}
\text{(3.6.1) } & \text{an open neighborhood } V \text{ of } x, \\
\text{(3.6.2) } & \text{a finite set } \Lambda, \\
\text{(3.6.3) } & \text{a semistable morphism of monoids } \varphi: \mathbb{N}^k \to \mathbb{N}^\Lambda, \text{ and} \\
\text{(3.6.4) } & \text{a commutative diagram of log complex analytic spaces}
\end{align*}

\[
\begin{array}{ccc}
(V, \mathcal{M}_V) & \xrightarrow{(\dagger)} & (\ast, \mathbb{N}^k) \\
\downarrow f|_V & & \downarrow \varphi \\
(\ast, \mathbb{N}^k) & \xrightarrow{(\ddagger)} & (\mathbb{C}^k, \mathbb{N}^k)
\end{array}
\]  

in which the morphism \((\dagger)\) on the top horizontal line is strict and log smooth.

Moreover, these data can be taken such that the composite of \((\dagger)\) and \((\ddagger)\) in (3.1) sends \( x \in V \) to the origin of \( \mathbb{C}^\Lambda \).

**Proof.** This is an analogue of Theorem 1.2.7 of [21] in the analytic context. By definition, there exist a finite set \( \Lambda \) and an isomorphism \( \xi: \overline{\mathcal{M}}_{X,x} \xrightarrow{\cong} \mathbb{N}^\Lambda \) such that \( \xi \cdot \overline{f}_x^* \) is semistable. On the other hand, \( \text{Ext}^1(G, \mathcal{O}_{X,x}^*) = 0 \) for any finitely generated abelian group \( G \) because \( \mathcal{O}_{X,x}^* \) is \( n \)-divisible for all \( n \in \mathbb{Z}_{>0} \). Then the proof is similar to the argument in [21].

### 3.7 (Local description of a semistable log smooth degeneration)

From the proposition above, we obtain a local description of a semistable log smooth degeneration \( f: (X, \mathcal{M}_X) \to (\ast, \mathbb{N}^k) \) as follows.

For any \( x \in X \), take the data in (3.6.1)–(3.6.4). Moreover, the partition associated to \( \varphi \) is denoted by \( \Lambda = \coprod_{i=1}^k \Lambda_i \). The morphism \((\dagger)\) in (3.6.4) is smooth in the usual sense because it is
strict and log smooth. Therefore by shrinking \( V \) sufficiently small, the morphism (\( \dagger \)) induces an strict open immersion

\[
(V, \mathcal{M}_V) \rightarrow (U, \mathcal{M}_U) = (\ast, \mathbb{N}^k) \times_{(\mathbb{C}^k, \mathbb{N}^k)} (\mathbb{C}^\Lambda, \mathbb{N}^\Lambda) \times (\mathbb{C}^l, \mathcal{O}_{C^l})
\]

for some \( l \in \mathbb{Z}_{>0} \). We may assume that the natural morphism \( (V, \mathcal{M}_V) \rightarrow (\mathbb{C}^\Lambda, \mathbb{N}^\Lambda) \times (\mathbb{C}^l, \mathcal{O}_{C^l}) \) sends \( x \in V \) to the origin of \( \mathbb{C}^\Lambda \times \mathbb{C}^l \) because of the latter part of Proposition 3.6. Such \((V, \mathcal{M}_V)\) (or \((U, \mathcal{M}_U)\)) is called a local model of \( f: (X, \mathcal{M}_X) \rightarrow (\ast, \mathbb{N}^k) \).

The coordinate function of \( \mathbb{C}^k \) corresponding to \( e_i \in \mathbb{N}^k \) is denoted by \( t_i \) for \( i = 1, 2, \ldots, k \). Then the log complex analytic space \((\mathbb{C}^k, \mathbb{N}^k)\) is the complex analytic space \( \mathbb{C}^k \) equipped with the log structure associated to the divisor \( \mathcal{E} = \{ t_1 t_2 \cdots t_k = 0 \} \). The coordinate function of \( \mathbb{C}^\Lambda \times \mathbb{C}^l \) corresponding to \( e_\lambda \in \mathbb{N}^\Lambda \) is denoted by \( x_\lambda \) and the divisor on \( \mathbb{C}^\Lambda \times \mathbb{C}^l \) defined by \( x_\lambda \) is denoted by \( D_\lambda \) for \( \lambda \in \Lambda \). For \( i = 1, 2, \ldots, k \) and \( I \subset \{ 1, 2, \ldots, k \} \), we set

\[
D_I = \sum_{\lambda \in \Lambda_i} D_\lambda \quad (i = 1, 2, \ldots, k), \quad D_I = \sum_{i \in I} D_i.
\]

We use \( D \) instead of \( D_{\{1, 2, \ldots, k\}} \). Then \((\mathbb{C}^\Lambda, \mathbb{N}^\Lambda) \times (\mathbb{C}^l, \mathcal{O}_{C^l}) = (\mathbb{C}^\Lambda \times \mathbb{C}^l, \mathcal{M}_{\mathbb{C}^\Lambda \times \mathbb{C}^l}(D))\) where \( \mathcal{M}_{\mathbb{C}^\Lambda \times \mathbb{C}^l}(D) \) denotes the log structure associated to the divisor \( D \) (cf. Notation 2.26). The composite of the projection \( \mathbb{C}^\Lambda \times \mathbb{C}^l \rightarrow \mathbb{C}^\Lambda \) and the morphism \( \tilde{\varphi}: \mathbb{C}^\Lambda \rightarrow \mathbb{C}^k \) coincides with the morphism given by \( \iota = \prod_{\lambda \in \Lambda_1} x_\lambda \) for \( i = 1, 2, \ldots, k \). Therefore

\[
U = \bigcap_{i=1}^k \{ \prod_{\lambda \in \Lambda_i} x_\lambda = 0 \} = \bigcap_{i=1}^k D_i
\]

by definition. Because \( \iota: (\ast, \mathbb{N}^k) \rightarrow (\mathbb{C}^k, \mathbb{N}^k) \) is strict, \( \mathcal{M}_U \) coincides with the pull-back of \( \mathcal{M}_{\mathbb{C}^\Lambda \times \mathbb{C}^l}(D) \) by the closed immersion \( U \hookrightarrow \mathbb{C}^\Lambda \times \mathbb{C}^l \). Then \( V \) is identified with an open neighborhood of \( 0 \) in \( U \).

**Remark 3.8.** Let \( f: (X, \mathcal{M}_X) \rightarrow (\ast, \mathbb{N}^k) \) be a semistable log smooth degeneration. Then the local description above shows that \( f \) is a log smooth degeneration defined in [9, Definition 4.3]. Moreover, we can see that the underlying complex analytic space \( X \) is locally isomorphic to a product of normal crossing varieties as in [14, (1.2)].

**Lemma 3.9.** For a semistable log smooth degeneration \( f: (X, \mathcal{M}_X) \rightarrow (\ast, \mathbb{N}^k) \), there exists a unique monoid sheaf \( \mathcal{M}(i)_X \) with \( \mathcal{O}_X^\times \subset \mathcal{M}(i)_X \subset \mathcal{M}_X \) for every \( i = 1, 2, \ldots, k \), such that the direct sum decomposition of \( \overline{\mathcal{M}}_{X,x} \) associated to \( \overline{f}_x \) (cf. Definition 2.14) is given by \( \overline{\mathcal{M}}_{X,x} = \bigoplus_{i=1}^k \overline{\mathcal{M}(i)}_{X,x} \) for all \( x \in X \).

**Proof.** The uniqueness is clear. Therefore we may assume that \((X, \mathcal{M}_X)\) is an open neighborhood of the origin of a local model \((U, \mathcal{M}_U)\) as in 3.7. Then the pull-back of the log structure \( \mathcal{M}_{\mathbb{C}^\Lambda \times \mathbb{C}^l}(D_i) \) by the closed immersion \( U \hookrightarrow \mathbb{C}^\Lambda \times \mathbb{C}^l \) gives us the desired monoid sheaf \( \mathcal{M}(i)_X \) for \( i = 1, 2, \ldots, k \).

**Remark 3.10.** We have \( \overline{\mathcal{M}}_X = \bigoplus_{i=1}^k \overline{\mathcal{M}(i)}_X \) by definition. Moreover, \( t_i \) in Notation 3.4 is contained in \( \Gamma(X, \mathcal{M}(i)_X) \), because \( \overline{f}_x(e_i) \in \overline{\mathcal{M}(i)}_{X,x} \) for all \( x \in X \).

**Definition 3.11.** For \( I \subset \{ 1, 2, \ldots, k \} \), a monoid subsheaf \( \mathcal{M}(I)_X \) of \( \mathcal{M}_X \) is defined by \( \mathcal{M}(I)_X = \pi_X^{-1}(\bigoplus_{i \in I} \overline{\mathcal{M}(i)}_X) \). We set \( \mathcal{M}(\emptyset)_X = \mathcal{O}_X^\times \).

**Definition 3.12.** Let \( I \subset \{ 1, 2, \ldots, k \} \). By setting \( J = \{ 1, 2, \ldots, k \} \setminus I \), the monoid sheaf \( \mathcal{M}(J)_X \) satisfies the condition \( \mathcal{O}_X^\times \subset \mathcal{M}(J)_X \subset \mathcal{M}_X \) and gives us the filtration \( W(\mathcal{M}(J)_X) \) on \( \Omega_X(\log \mathcal{M}_X) \).
as in Definition 2.29. We denote it by \( W(I) \) for short. By definition \( W(\{1, 2, \ldots, k\}) \) coincides with \( W \) in Definition 2.29. We use \( W(i) \) instead of \( W(\{i\}) \) for \( i = 1, 2, \ldots, k \). The properties

\[
d\log t_i \wedge W(I)_m \Omega_X^n(\log \mathcal{M}_X) \subset \begin{cases} W(I)_{m+1} \Omega_X^{n+1}(\log \mathcal{M}_X) & \text{if } i \in I \\ W(I)_m \Omega_X^{n+1}(\log \mathcal{M}_X) & \text{if } i \notin I \end{cases} \tag{3.2}
\]

can be easily seen from the fact \( t_i \in \Gamma(X, \mathcal{M}(i)_X) \).

3.13 (Local description of the log de Rham complex). We consider a local model \( (U, \mathcal{M}_U) \) and use the notation in 3.7. Then the log structure \( \mathcal{M}(I)_U \) coincides with the pull-back of \( \mathcal{M} \subset C^A \times C^I \). On the other hand

\[
\Omega^n_U(\log \mathcal{M}_U) \simeq \mathcal{O}_U \otimes_{\mathcal{O}_{C^A \times C^I}} \Omega^n_{C^A \times C^I}(\log D) \tag{3.3}
\]

for every \( n \) by Lemma (3.6)(2) of [19]. Via the identification above, \( W(I)_m \Omega^n_U(\log \mathcal{M}_U) \) coincides with the image of \( \mathcal{O}_U \otimes_{\mathcal{O}_{C^A \times C^I}} W(D) \otimes \mathcal{O}_{C^A \times C^I} \Omega^n_{C^A \times C^I}(\log D) \) in \( \mathcal{O}_U \otimes_{\mathcal{O}_{C^A \times C^I}} \Omega^n_{C^A \times C^I}(\log D) \) for all \( m \).

3.14. A rational structure on \( \Omega_X(\log \mathcal{M}_X) \) can be constructed by using the Koszul complex as in [25], [7], [9] and [22]. Here, we make a list of definitions and elementary properties about Koszul complexes, which will be used throughout this paper. The main reference is Sections 1 and 2 of [9]. Let \( (X, \mathcal{M}_X) \) be a log complex analytic space.

(3.14.1) A complex of \( \mathbb{Q} \)-sheaves \( \text{Kos}_X(\mathcal{M}_X) \) is defined in [9, (2.3)]. For the case of \( \mathcal{M}_X = \mathcal{M}_X(D) \) as in Notation 2.26, we use \( \text{Kos}_X(D) \) instead of \( \text{Kos}_X(\mathcal{M}_X(D)) \).

(3.14.2) For a morphism of log complex analytic spaces \( f: (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y) \), there exists the canonical morphism \( f^{-1} \text{Kos}_Y(\mathcal{M}_Y) \to \text{Kos}_X(\mathcal{M}_X) \) of the complexes of \( \mathbb{Q} \)-sheaves.

(3.14.3) A morphism of complexes of \( \mathbb{Q} \)-sheaves \( \psi_{(X, \mathcal{M}_X)}: \text{Kos}_X(\mathcal{M}_X) \to \Omega_X(\log \mathcal{M}_X) \) is defined in [9, (2.4)]. For the case of \( \mathcal{M}_X = \mathcal{M}_X(D) \), we use \( \psi_{(X, D)} \) instead of \( \psi_{(X, \mathcal{M}_X(D))} \). Moreover, we use \( \psi_X \) instead of \( \psi_{(X, \mathcal{M}_X)} \) if there is no danger of confusion.

(3.14.4) For the case of trivial log structure \( \mathcal{M}_X = \mathcal{O}_X \), there exists a quasi-isomorphism \( \mathbb{Q}_X \to \text{Kos}_X(\mathcal{O}_X^\ast) \) such that the diagram

\[
\begin{array}{ccc}
\mathbb{Q}_X & \longrightarrow & \text{Kos}_X(\mathcal{O}_X^\ast) \\
\downarrow & & \downarrow \psi_{(X, \mathcal{O}_X^\ast)} \\
\mathcal{O}_X & \longrightarrow & \Omega_X
\end{array}
\]

is commutative (cf. [10, Lemma 3.12]).

For a semistable log smooth degeneration \( f: (X, \mathcal{M}_X) \to (*, \mathbb{N}^k) \), we have the following:

(3.14.5) For every \( I \subset \{1, 2, \ldots, k\} \), a finite increasing filtration \( W(I) \) on \( \text{Kos}_X(\mathcal{M}_X) \) is defined as \( W(\mathcal{M}(J)^{\text{pr}}_{X, \mathbb{Q}}) \) in [9, Definition 1.8], where \( J = \{1, 2, \ldots, k\} \setminus I \).

(3.14.6) The morphism \( \psi_X \) in (3.14.3) preserves the filtration \( W(I) \).

(3.14.7) A morphism of complexes of \( \mathbb{Q} \)-sheaves \( t_i \wedge: \text{Kos}_X(\mathcal{M}_X) \to \text{Kos}_X(\mathcal{M}_X)[1] \) is defined in [9, (1.11)]. Then \( (t_i \wedge)(t_j \wedge) + (t_j \wedge)(t_i \wedge) = 0 \) for all \( i, j \in \{1, 2, \ldots, k\} \) (cf. [9, (3.29)]). For \( I \subset \{1, 2, \ldots, k\} \),

\[
(t_i \wedge)(W(I)_m \text{Kos}_X(\mathcal{M}_X)) \subset \begin{cases} W(I)_{m+1} \text{Kos}_X(\mathcal{M}_X)[1] & \text{if } i \in I \\ W(I)_m \text{Kos}_X(\mathcal{M}_X)[1] & \text{if } i \notin I \end{cases} \tag{3.4}
\]
for all \( m \) (cf. [9, (1.12)]). Moreover, the diagram
\[
\begin{array}{c}
\xymatrix{
\Kos_X(\mathcal{M}_X) \ar[r]^{t_i \wedge} & \Kos_X(\mathcal{M}_X)[1] \\
\psi_X \downarrow & \\
\Omega_X(\log \mathcal{M}_X) \ar[r]_{\dlog t_i \wedge} & \Omega_X(\log \mathcal{M}_X)[1]
}
\end{array}
\]
(3.5)
is commutative (cf. [10, 3.13]).

3.15 (The stalk of the Koszul complex). Here we look at \( \Kos_X(\mathcal{M}_X) \) and \( \psi_X \) stalkwise. It is enough to consider the origin \( x = 0 \) of a local model \( (U, \mathcal{M}_U) \) in 3.7. We use the same notation as in 3.7. In particular, the partition \( \Lambda = \bigsqcup_{i=1}^k \Lambda_i \) is given. The global sections \( x_\lambda \in \Gamma(U, \mathcal{M}_U) \) for all \( \lambda \in \Lambda \) gives us a decomposition \( \mathcal{M}_{U,x} = \mathcal{O}_{U,x}^r \oplus \mathbb{N}_x \) at the origin \( x \). Then
\[
\Kos_U(\mathcal{M}_U)^n_x \simeq \bigoplus_{a \in \mathbb{Z}} \bigwedge^a \mathbb{Z} \otimes \mathbb{Z} \Kos(\mathcal{O}_{U,x}^r)^{n-a} \simeq \bigoplus_{\lambda \in S(\Lambda)} \varepsilon(\lambda) \otimes \mathbb{Z} \Kos(\mathcal{O}_{U,x}^r)^{n-|\lambda|}
\]
(3.6)
for all \( n \in \mathbb{Z} \) by the definition of \( \Kos_U(\mathcal{M}_U) \) and by (2.3). Under the identification above,
\[
W(I)_m \Kos_U(\mathcal{M}_U)^n_x \simeq \bigoplus_{|\lambda| \leq m} \varepsilon(\lambda) \otimes \mathbb{Z} \Kos(\mathcal{O}_{U,x}^r)^{n-|\lambda|}
\]
(3.7)
for all \( m \), where \( \Lambda_I = \bigsqcup_{i \in I} \Lambda_i \). Therefore \( W(I)_m \Kos_U(\mathcal{M}_U)^n_x = \Kos_U(\mathcal{M}_U)^n_x \) if \( m \geq |\Lambda_I| \). Via the identification (3.6), the restriction of \( \psi_{U,x} \) on the direct summand \( \varepsilon(\lambda) \otimes \mathbb{Z} \Kos(\mathcal{O}_{U,x}^r)^{n-|\lambda|} \) is given by
\[
e^{\lambda_1 \wedge e^{\lambda_2} \wedge \cdots \wedge e^{\lambda_p} \otimes \eta} \mapsto (2\pi \sqrt{-1})^{-p} \dlog x^{\lambda_1} \wedge \dlog x^{\lambda_2} \wedge \cdots \wedge \dlog x^{\lambda_p} \wedge \psi_{(U,\mathcal{O}_U),x}(\eta),
\]
where \( p = |\lambda| \) and \( \lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_p\} \). Note that \( \psi_{(U,\mathcal{O}_U),x}(\eta) \in \Omega_{X,x}^{n-p} \).

Notation 3.16. Throughout this paper, the polynomial rings \( \mathbb{Q}[u_1, u_2, \ldots, u_k] \) and \( \mathbb{C}[u_1, u_2, \ldots, u_k] \) are simply denoted by \( \mathbb{Q}[u] \) and \( \mathbb{C}[u] \) respectively. We use the multi-index notation as usual.

Definition 3.17. We set \( d_0 = \text{id} \otimes d : \mathbb{C}[u] \otimes \mathbb{C} \Omega^n_X(\log \mathcal{M}_X) \longrightarrow \mathbb{C}[u] \otimes \mathbb{C} \Omega^{n+1}_X(\log \mathcal{M}_X) \) for all \( n \in \mathbb{Z} \). For \( i = 1, 2, \ldots, k \), a morphism \( d_i : \mathbb{C}[u] \otimes \mathbb{C} \Omega^n_X(\log \mathcal{M}_X) \longrightarrow \mathbb{C}[u] \otimes \mathbb{C} \Omega^{n+1}_X(\log \mathcal{M}_X) \) is defined by
\[
d_i(P \otimes \omega) = u_i P \otimes \dlog t_i \wedge \omega
\]
for \( P \in \mathbb{C}[u] \) and \( \omega \in \Omega^n_X(\log \mathcal{M}_X) \). Then these morphisms satisfy
\[
d_i d_j + d_j d_i = 0
\]
(3.8)
for all \( i, j \in \{0, 1, 2, \ldots, k\} \). Thus a complex \( \mathbb{C}[u] \otimes \mathbb{C} \Omega_X(\log \mathcal{M}_X) \) of \( \mathbb{C} \)-sheaves on \( X \) is obtained by setting \( d = \sum_{i=0}^k d_i \).

Definition 3.18. A decreasing filtration \( F \) on \( \mathbb{C}[u] \otimes \mathbb{C} \Omega_X(\log \mathcal{M}_X) \) is defined by
\[
F^p(\mathbb{C}[u] \otimes \mathbb{C} \Omega^n_X(\log \mathcal{M}_X)) = \bigoplus_{q \in \mathbb{N}^k} \mathbb{C} u^q \otimes \mathbb{C} F^{p+|q|+k} \Omega^n_X(\log \mathcal{M}_X)
\]
for all \( n, p \), where \( F \) denotes the stupid filtration on \( \Omega_X(\log \mathcal{M}_X) \). By definition, we have
\[
F^p(\mathbb{C}[u] \otimes \mathbb{C} \Omega^n_X(\log \mathcal{M}_X)) = \bigoplus_{|q| \leq n-p-k} \mathbb{C} u^q \otimes \mathbb{C} \Omega^n_X(\log \mathcal{M}_X)
\]
for all \(n,p\). For every \(I \subset \{1,2,\ldots,k\}\), increasing filtrations \(W(I)\) and \(L(I)\) on \(\mathbb{C}[u] \otimes \Omega_X(\log \mathcal{M}_X)\) are defined by

\[
W(I)_m(\mathbb{C}[u] \otimes \Omega_X^n(\log \mathcal{M}_X)) = \bigoplus_{q \in \mathbb{N}^k} \mathbb{C}u^q \otimes \mathbb{C} W(I)_{m+|q_i|+|I|} \Omega_X^n(\log \mathcal{M}_X)
\]

\[
L(I)_m(\mathbb{C}[u] \otimes \Omega_X^n(\log \mathcal{M}_X)) = \bigoplus_{q \in \mathbb{N}^k} \mathbb{C}u^q \otimes \mathbb{C} W(I)_{m+2|q_i|+|I|} \Omega_X^n(\log \mathcal{M}_X)
\] (3.9)

for all \(m,n\), where \(q_I \in \mathbb{N}^l\) denotes the image of \(q \in \mathbb{N}^k\) by the projection \(\mathbb{Z}^k \rightarrow \mathbb{Z}^l\). Actually, they define subcomplexes of \(\mathbb{C}[u] \otimes \Omega_X(\log \mathcal{M}_X)\) because of (3.2). We use \(W = W(\{1,2,\ldots,k\})\), \(L = L(\{1,2,\ldots,k\})\), \(W(i) = W(\{i\})\) and \(L(i) = L(\{i\})\) for short.

**Definition 3.19.** We set \(d_0 = \text{id} \otimes d: \mathbb{Q}[u] \otimes \text{Kos}_X(\mathcal{M}_X)^n \rightarrow \mathbb{Q}[u] \otimes \text{Kos}_X(\mathcal{M}_X)^{n+1}\). For \(i = 1,2,\ldots,k\), a morphism \(d_i: \mathbb{Q}[u] \otimes \text{Kos}_X(\mathcal{M}_X)^n \rightarrow \mathbb{Q}[u] \otimes \text{Kos}_X(\mathcal{M}_X)^{n+1}\) is defined by

\[
d_i(P \otimes \eta) = u_i P \otimes t_i \wedge \eta
\]

for \(P \in \mathbb{Q}[u]\) and \(\eta \in \text{Kos}_X(\mathcal{M}_X)^n\), where \(t_i \wedge\) is the morphism in (3.14.7). Then these morphisms satisfy the same equalities as (3.8). Thus a complex of \(\mathbb{Q}\)-sheaves \(\mathbb{Q}[u] \otimes \text{Kos}_X(\mathcal{M}_X)\) is obtained by setting \(d = \sum_{i=0}^k d_i\). For a subset \(I \subset \{1,2,\ldots,k\}\), increasing filtrations \(W(I)\) and \(L(I)\) on \(\mathbb{Q}[u] \otimes \text{Kos}_X(\mathcal{M}_X)\) are defined by

\[
W(I)_m(\mathbb{Q}[u] \otimes \text{Kos}_X(\mathcal{M}_X)^n) = \bigoplus_{q \in \mathbb{N}^k} \mathbb{Q}u^q \otimes \mathbb{Q} W(I)_{m+|q_i|+|I|} \text{Kos}_X(\mathcal{M}_X)^n
\]

\[
L(I)_m(\mathbb{Q}[u] \otimes \text{Kos}_X(\mathcal{M}_X)^n) = \bigoplus_{q \in \mathbb{N}^k} \mathbb{Q}u^q \otimes \mathbb{Q} W(I)_{m+2|q_i|+|I|} \text{Kos}_X(\mathcal{M}_X)^n
\]

for all \(m,n\). By (3.4), these are actually increasing filtrations on the complex \(\mathbb{Q}[u] \otimes \text{Kos}_X(\mathcal{M}_X)\).

We use \(W = W(\{1,2,\ldots,k\})\), \(L = L(\{1,2,\ldots,k\})\), \(W(i) = W(\{i\})\) and \(L(i) = L(\{i\})\) as in Definition 3.18.

**Definition 3.20.** A morphism of \(\mathbb{Q}\)-sheaves \(\alpha: \mathbb{Q}[u] \otimes \text{Kos}_X(\mathcal{M}_X)^n \rightarrow \mathbb{C}[u] \otimes \Omega_X^n(\log \mathcal{M}_X)\) is defined by

\[
\alpha(u^q \otimes \eta) = (2\pi \sqrt{-1})^{|q|+k} u^q \otimes \psi_X(\eta),
\]

which turns out to be a morphism of complexes by the commutativity of (3.5). The morphism \(\alpha\) preserves the filtrations \(W(I)\) and \(L(I)\) for any \(I \subset \{1,2,\ldots,k\}\) by (3.14.6).

**Definition 3.21.** Complexes of \(\mathbb{C}\)-sheaves \(A_C\) and of \(\mathbb{Q}\)-sheaves \(A_Q\) on \(X\) are defined by

\[
A_C = (\mathbb{C}[u] \otimes \Omega_X(\log \mathcal{M}_X)/\sum_{i=1}^k W(i)_{-1})[k],
\]

\[
A_Q = (\mathbb{Q}[u] \otimes \text{Kos}_X(\mathcal{M}_X)/\sum_{i=1}^k W(i)_{-1})[k].
\]

The filtrations on \(A_C\) and on \(A_Q\) induced by \(L(I)\) on \(\mathbb{C}[u] \otimes \Omega_X(\log \mathcal{M}_X)\) and on \(\mathbb{Q}[u] \otimes \text{Kos}_X(\mathcal{M}_X)\) are denoted by \(L(I)\) again. We use \(L = L(\{1,2,\ldots,k\})\) as before. The filtration on \(A_C\) induced by \(F\) on \(\mathbb{C}[u] \otimes \Omega_X(\log \mathcal{M}_X)\) is denoted by \(F\) again. The morphism \(\alpha\) in Definition 3.20 induces a morphism of complexes \(A_Q \rightarrow A_C\), which is denoted by the same letter \(\alpha\). The morphism \(\alpha\) preserves the filtrations \(L(I)\) for any \(I \subset \{1,2,\ldots,k\}\).
Remark 3.22. By definition, we have

\[ A^n_C = \bigoplus_{q \in \mathbb{N}^k} \mathbb{C}u^q \otimes_{\mathbb{C}} \left( \Omega_X^{n+k}(\log \mathcal{M}_X) / \sum_{i=1}^{k} W(i)_{q_i} \right) \cong \bigoplus_{q \in \mathbb{N}^k} \left( \Omega_X^{n+k}(\log \mathcal{M}_X) / \sum_{i=1}^{k} W(i)_{q_i} \right) \]  

(3.10)

for all \( n \). For the later use, we set

\[ (A^n_C)_q = \Omega_X^{n+k}(\log \mathcal{M}_X) / \sum_{i=1}^{k} W(i)_{q_i} \]

\[ L(I)_m(A^n_C)_q = W(I)_{m+2|q|+|I|} \left( \Omega_X^{n+k}(\log \mathcal{M}_X) / \sum_{i=1}^{k} W(i)_{q_i} \right) \]

for \( I \subset \{1,2,\ldots,k\}, q \in \mathbb{N}^k \) and \( m, n \in \mathbb{Z} \). Then we simply have

\[ A^n_C = \bigoplus_{q \in \mathbb{N}^k} \mathbb{C}u^q \otimes_{\mathbb{C}} (A^n_C)_q \]

\[ F^p A^n_C = \bigoplus_{|q| \leq n-p} \mathbb{C}u^q \otimes_{\mathbb{C}} (A^n_C)_q \]  

(3.11)

\[ L(I)_m A^n_C = \bigoplus_{q \in \mathbb{N}^k} \mathbb{C}u^q \otimes_{\mathbb{C}} L(I)_m(A^n_C)_q \]

for all \( m, n, p \). In the notation above, we use \( L \) instead of \( L(\{1,2,\ldots,k\}) \) as before.

If \( X \) is of finite dimension, then \( (A^n_C)_q \neq 0 \) implies \( |q| \leq n \leq \dim X \). Therefore \( F \) and \( L(I) \) on \( A_C \) are finite filtration for any \( I \subset \{1,2,\ldots,k\} \). Similarly, we can check that \( L(I) \) on \( A_Q \) is finite by using (3.7) if \( X \) is of finite dimension.

Assumption 3.23. In the remainder of this paper, we assume that \( X \) is of finite dimension.

Remark 3.24. For the case of \( k = 1 \), the bifiltered complex \((A_C, L, F)\) coincides with \((A^\bullet, L, F)\) in [25, (5.3)] except for the sign of the differentials. For the case where \( f: (X, \mathcal{M}_X) \to (\ast, \mathbb{N}^k) \) is the central fiber of a morphism \( g: \mathcal{X} \to \Delta^k \) as in Example 3.5, the complex \( A_C \) is isomorphic to the complex \( sB(g) \) defined in [11, Definition 4.3].

Definition 3.25. A morphism of \( \mathcal{O}_X \)-modules

\[ \text{dlog } t_1 \wedge \text{dlog } t_2 \wedge \cdots \wedge \text{dlog } t_k \wedge: \Omega_X^n(\log \mathcal{M}_X) \to \Omega_X^{n+k}(\log \mathcal{M}_X) \]

is defined by

\[ \Omega_X^n(\log \mathcal{M}_X) \ni \omega \mapsto \text{dlog } t_1 \wedge \text{dlog } t_2 \wedge \cdots \wedge \text{dlog } t_k \wedge \omega \in \Omega_X^{n+k}(\log \mathcal{M}_X) \]

for all \( n \). The composite of this morphism with the inclusion

\[ \Omega_X^{n+k}(\log \mathcal{M}_X) \cong \mathbb{C}u^0 \otimes_{\mathbb{C}} \Omega_X^{n+k}(\log \mathcal{M}_X) \to \mathbb{C}[u] \otimes_{\mathbb{C}} \Omega_X^{n+k}(\log \mathcal{M}_X) \]

and with the canonical surjection \( \mathbb{C}[u] \otimes_{\mathbb{C}} \Omega_X^{n+k}(\log \mathcal{M}_X) \to A^n_C \), defines a morphism of \( \mathcal{O}_X \)-modules

\[ \Omega_X^n(\log \mathcal{M}_X) \to A^n_C \]

(3.12)
is obtained. Moreover, this morphism factors through the canonical surjection \( \Omega_X(\log \mathcal{M}_X) \to \Omega_{X/\ast}(\log(\mathcal{M}_X/\mathbb{N}^k)) \). Thus we obtain a morphism of filtered complexes

\[
\theta: (\Omega_{X/\ast}(\log(\mathcal{M}_X/\mathbb{N}^k)), F) \to (A_{\mathbb{C}}, F),
\tag{3.13}
\]

where \( F \) on the left hand side denotes the stupid filtration on \( \Omega_{X/\ast}(\log(\mathcal{M}_X/\mathbb{N}^k)) \).

The following lemma shows that \((A_{\mathbb{C}}, F)\) is a substitute for \((\Omega_{X/\ast}(\log(\mathcal{M}_X/\mathbb{N}^k)), F)\).

**Lemma 3.26.** The morphism \( \theta \) is a filtered quasi-isomorphism.

**Proof.** We may work in the local situation as in 3.7. Then we obtain the conclusion by Lemma (3.6)(2) of [19] and by Corollary 4.13 of [11]. \( \square \)

**Corollary 3.27.** The morphism \( \theta \) induces an isomorphism of filtered \( \mathbb{C} \)-vector spaces

\[
(H^n(X, \Omega_{X/\ast}(\log(\mathcal{M}_X/\mathbb{N}^k)), F) \xrightarrow{\sim} (H^n(X, A_{\mathbb{C}}), F) \quad (3.14)
\]

for all \( n \in \mathbb{Z} \).

## 4 Main results

In this section, we state all the main results of this paper.

**Assumption 4.1.** The semistable log smooth degeneration \( f: (X, \mathcal{M}_X) \to (\ast, \mathbb{N}^k) \) is assumed to be projective throughout this section.

**Notation 4.2.** For a subset \( I \subset \{1, 2, \ldots, k\} \), we set

\[
(A, L(I), L, F) = ((A_{\mathbb{Q}}, L(I), L), (A_{\mathbb{C}}, L(I), L, F), \alpha),
\]

\[
(H^n(X, A), L(I), L, F) = ((H^n(X, A_{\mathbb{Q}}), L(I), L), (H^n(X, A_{\mathbb{C}}), L(I), L, F), H^n(X, \alpha))
\]

and

\[
(E^r_{p,q}(A, L(I)), L_{rec}, F_{rec}) = ((E^r_{p,q}(A_{\mathbb{Q}}, L(I)), L_{rec}), (E^r_{p,q}(A_{\mathbb{C}}, L(I)), L_{rec}, F_{rec}), E^r_{p,q}(\alpha)). \quad (4.1)
\]

For the case of \( I = \{1, 2, \ldots, k\} \), we omit \( L_{rec} \) in (4.1).

**Theorem 4.3.** For all \( I \subset \{1, 2, \ldots, k\} \), the quadruple \((A, L(I), L, F)\) is a filtered cohomological \( \mathbb{Q} \)-mixed Hodge complex on \( X \) in the sense of El Zein [5, 6.1.5].

The following is a direct consequence of the theorem above by [5, 6.1.8 Théorème].

**Theorem 4.4.** We have the following:

(4.4.1) \((H^n(X, A), L[n], F)\) is a mixed Hodge structure for all \( n \).

(4.4.2) The spectral sequence \( E^r_{p,q}(A, F) \) degenerates at \( E_1 \)-terms.

(4.4.3) The spectral sequence \( E^r_{p,q}(A, L) \) degenerates at \( E_2 \)-terms.

(4.4.4) \((E^r_{p,q}(A, L), F_{rec})\) is a \( \mathbb{Q} \)-Hodge structure of weight \( q \) for \( r = 1, 2 \), and the morphism of \( E_1 \)-terms \( d_1: E^p_{1,q}(A, L) \to E^{p+1,q}_{1}(A, L) \) is a morphism of \( \mathbb{Q} \)-Hodge structures.

Moreover, for \( I \subset \{1, 2, \ldots, k\} \), we have the following:
(4.4.5) The spectral sequence $E_r^{p,q}(\text{Gr}_m^{L(I)} A, L)$ degenerates at $E_2$-terms for all $m \in \mathbb{Z}$.

(4.4.6) $(E_r^{p,q}(A, L(I)), L_{rec}[p + q], F_{rec})$ is a $\mathbb{Q}$-mixed Hodge structure and the morphism of $E_r$-terms $\partial_r : E_r^{p,q}(A, L(I)) \rightarrow E_r^{p+r,q-r+1}(A, L(I))$ is a morphism of $\mathbb{Q}$-mixed Hodge structures for all $p, q, r$ with $1 \leq r \leq \infty$.

(4.4.7) $L(I)_m H^n(X, A(f))$ is a sub mixed Hodge structure of $H^n(X, A(f))$ for all $m$. Moreover, we have the canonical isomorphism of mixed Hodge structures

$$ (\text{Gr}_m^{L(I)} H^{p,q}(X, A), L[p + q], F) \simeq (E_\infty^{p,q}(A, L(I)), L_{rec}[p + q], F_{rec}) $$

for all $p, q$.

**Theorem 4.5.** For any $I \subset \{1, 2, \ldots, k\}$, the spectral sequence $E_r^{p,q}(A_C, L(I))$ degenerates at $E_2$-terms.

**Definition 4.6.** A morphism $\nu_i : \mathbb{C}[u] \otimes \Omega_X^n (\log M_X) \rightarrow \mathbb{C}[u] \otimes \Omega_X^n (\log M_X)$ is defined by $\nu_i(P \otimes \omega) = u_i P \otimes \omega$ for $i = 1, 2, \ldots, k$. This defines a morphism of complexes $\nu_i$ for $i = 1, 2, \ldots, k$, which preserves the filtrations $W(I)$ and $L(I)$ for all $I \subset \{1, 2, \ldots, k\}$. In fact,

$$ \nu_i(W(I)_m (\mathbb{C}[u] \otimes \Omega_X^n (\log M_X))) \subset W(I)_{m-1} (\mathbb{C}[u] \otimes \Omega_X^n (\log M_X)) $$

$$ \nu_i(L(I)_m (\mathbb{C}[u] \otimes \Omega_X^n (\log M_X))) \subset L(I)_{m-2} (\mathbb{C}[u] \otimes \Omega_X^n (\log M_X)) $$

if $i \in I$. For the filtration $F$, we have

$$ \nu_i(F^p(\mathbb{C}[u] \otimes \Omega_X^n (\log M_X))) \subset F^{p-1}(\mathbb{C}[u] \otimes \Omega_X^n (\log M_X)) $$

for all $p \in \mathbb{Z}$. Therefore a morphism of complexes $\nu_i$ is induced for $i = 1, 2, \ldots, k$, which satisfies

$$ \nu_i(L(I)_m A_C) \subset \begin{cases} L(I)_{m-2} A_C & \text{if } i \in I \\ L(I)_{m} A_C & \text{otherwise,} \end{cases} $$

$$ \nu_i(F^p A_C) \subset F^{p-1} A_C $$

for all $I \subset \{1, 2, \ldots, k\}$ and $m, p \in \mathbb{Z}$. Similarly, a morphism $\nu_i : \mathbb{Q}[u] \otimes \text{Kos} \Omega_X^n (\log M_X)^n \rightarrow \mathbb{Q}[u] \otimes \text{Kos} \Omega_X^n (\log M_X)^n$ is defined by $\nu_i(P \otimes \eta) = u_i P \otimes \eta$, from which a morphism of complexes $\nu_i : A_Q \rightarrow A_Q$ is induced for $i = 1, 2, \ldots, k$. These morphisms satisfy the same properties as (4.2) for $A_Q$. We can easily check that the diagram

$$ \begin{array}{ccc} A_Q & \xrightarrow{\nu_i} & A_Q \\ \downarrow^\alpha & & \downarrow^\alpha \\ A_C & \xrightarrow{2\pi \sqrt{-1} \nu_i} & A_C \end{array} $$

is commutative. Because of $\nu_i(L_m A_C) \subset L_m A_C$ and because $L$ is finite, $\nu_i$ on $A_C$ is nilpotent for all $i$. By the same reason, $\nu_i$ on $A_Q$ is also nilpotent for all $i$. For $\mathfrak{c} = (c_i) \in \mathbb{C}^I$, a morphism of bifiltered complexes $\nu_i(\mathfrak{c}) : (A_C, L, F) \rightarrow (A_C, L[2], F[-1])$ is defined by $\nu_I(\mathfrak{c}) = \sum_{i \in I} c_i \nu_i$ for $I \subset \{1, 2, \ldots, k\}$. We use $\nu(\mathfrak{c})$ instead of $\nu_{\{1,2,\ldots,k\}}(\mathfrak{c})$ for $\mathfrak{c} = (c_i) \in \mathbb{C}^k$.

**Definition 4.7.** The morphism

$$ H^q(X, \nu_i) : (H^q(X, A_C), L, F) \rightarrow (H^q(X, A_C), L[2], F[-1]) $$

is denoted by $N_i$ for $i = 1, 2, \ldots, k$. Moreover we set $N_I(\mathfrak{c}_I) = \sum_{i \in I} c_i N_i$ for $\mathfrak{c}_I = (c_i)_{i \in I} \in \mathbb{C}^I$. We use $N(\mathfrak{c})$ instead of $N_{\{1,2,\ldots,k\}}(\mathfrak{c}_{\{1,2,\ldots,k\}})$. Since $\nu_i$ is nilpotent, so is $N_i$ for any $i$. Then $N_I(\mathfrak{c}_I)$ is also nilpotent for any $I \subset \{1, 2, \ldots, k\}$ and $\mathfrak{c}_I \in \mathbb{C}^I$. 

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Remark 4.8. For the case where $f: (X, \mathcal{M}_X) \rightarrow (*, \mathbb{N}^k)$ is the central fiber of a morphism $g: \mathcal{X} \rightarrow \Delta^k$ as in Example 3.5, we have $N_i = (2\pi e^{-1}) T_i$ for each $i = 1, 2, \ldots, k$, where $T_i$ denotes the monodromy automorphism around the coordinate hyperplane $\{t_i = 0\}$ (cf. [24, (2.21) Theorem and (4.22)], [11, Theorem 5.19]).

Theorem 4.9. For any $I \subset \{1, 2, \ldots, k\}$, $c_I \in (\mathbb{R}_{>0})^I$ and $l \in \mathbb{Z}_{>0}$, the morphism $N_I(c_I)^l$ induces an isomorphism

$$\Gr_l \mathcal{H}_q(X, A_C) \xrightarrow{\cong} \Gr_l \mathcal{H}^q(X, A_C)$$

for all $q \in \mathbb{Z}$.

Theorem 4.10. For any $I \subset \{1, 2, \ldots, k\}$, $c \in (\mathbb{R}_{>0})^k$ and $l \in \mathbb{Z}_{>0}$, the morphism $N(c)^l$ induces an isomorphism

$$\Gr_{l+m} \mathcal{H}_q(X, A_C) \xrightarrow{\cong} \Gr_{l+m} \mathcal{H}^q(X, A_C)$$

for all $m, q \in \mathbb{Z}$.

Definition 4.11. Let $\mathcal{L}$ be an invertible sheaf on $X$. The morphism $d\log: \mathcal{O}_X^*[-1] \rightarrow \Omega_X$ induces a morphism $H^2(X, d\log): H^1(X, \mathcal{O}_X^*) \rightarrow H^2(X, \Omega_X)$. We set $c(\mathcal{L}) = H^2(X, d\log)([\mathcal{L}]) \in H^2(X, \Omega_X)$ as in [3, (2.2.4)].

Definition 4.12. The wedge product on $\Omega_X(\log(\mathcal{M}_X/\mathbb{N}^k))$ induces the morphism of complexes of $\mathbb{C}$-sheaves $\Omega_{X,s}(\log(\mathcal{M}_X/\mathbb{N}^k)) \otimes_\mathbb{C} \Omega_X \rightarrow \Omega_{X,s}(\log(\mathcal{M}_X/\mathbb{N}^k))$. This morphism induces a morphism

$$H^a(X, \Omega_{X,s}(\log(\mathcal{M}_X/\mathbb{N}^k)) \otimes_\mathbb{C} H^b(X, \Omega_X) \rightarrow H^{a+b}(X, \Omega_{X,s}(\log(\mathcal{M}_X/\mathbb{N}^k))$$

as in Definition 2.19. For $\omega \in H^a(X, \Omega_{X,s}(\log(\mathcal{M}_X/\mathbb{N}^k)))$ and $\eta \in H^b(X, \Omega_X)$, the image of $\omega \otimes \eta$ by the morphism (4.4) is simply denoted by $\omega \cup \eta$ and called the cup product of $\omega$ and $\eta$. Thus the element $c(\mathcal{L}) \in H^2(X, \Omega_X)$ gives us a morphism $\cup c(\mathcal{L}): H^a(X, \Omega_{X,s}(\log(\mathcal{M}_X/\mathbb{N}^k))) \rightarrow H^{a+2}(X, \Omega_{X,s}(\log(\mathcal{M}_X/\mathbb{N}^k)))$ for all $a \in \mathbb{Z}$.

Theorem 4.13 (Log hard Lefschetz theorem). We assume that $X$ is of pure dimension in addition. For any ample invertible sheaf $\mathcal{L}$ on $X$, the morphism

$$(\cup c(\mathcal{L}))^i: H^{-i+\dim X}(X, \Omega_{X,s}(\log(\mathcal{M}_X/\mathbb{N}^k))) \rightarrow H^{i+\dim X}(X, \Omega_{X,s}(\log(\mathcal{M}_X/\mathbb{N}^k)))$$

is an isomorphism for all $i \in \mathbb{Z}_{>0}$.

5 Proof of Theorem 4.3

In this section, we will prove Theorem 4.3. To this end, we need to construct a residue isomorphism as in [3, (3.1.5)]. First, we introduce log complex manifolds $(X_r, \mathcal{M}_X)$ for all $r \in \mathbb{Z}_{>0}^k$, whose underlying complex manifolds $X_r$ are finite over $X$. Second, we define the residue morphism for the log de Rham complex $\Omega_X(\log\mathcal{M}_X)$ in Definition 5.12, and the residue morphism for the Koszul complex of $(X, \mathcal{M}_X)$ in Definition 5.20. In the construction of these two residue morphisms the log complex manifolds $(X_r, \mathcal{M}_X)$ above play a role of “target” spaces. Then, we will prove Theorem 4.3 at the end of this section.

Definition 5.1. A map $r_X: X \rightarrow \mathbb{Z}^k$ is defined by

$$r_X(x) = \left(\text{rank } M(i)^{\log}_{X,x}\right)_{i=1}^k \in \mathbb{Z}^k,$$

where $\mathcal{M}(i)_X$ is the monoid sheaf given in Lemma 3.9 for each $i = 1, 2, \ldots, k$. 

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Definition 5.2. For $r \in \mathbb{Z}_{\geq e}^k$, we set $\overline{X}_r = \{ x \in X \mid r_X(x) \geq r \}$, where $\geq$ is the partial order on $\mathbb{Z}^k$ defined in (2.1).

Lemma 5.3. $\overline{X}_r$ is a closed analytic subset of $X$.

Proof. Since the question is of local nature, we may assume that $(X, \mathcal{M}_X)$ is an open neighborhood of the origin of a local model $(U, \mathcal{M}_U)$ as in 3.7. Then we have

$$\overline{X}_r = X \cap \bigcup_{\Lambda \in S_r(A)} D[\Lambda],$$

(5.1)

where $D[\Lambda] = \bigcap_{\Delta \in \Lambda} D_{\Delta}$ for $\Lambda \subset A$.

Definition 5.4. For $r \in \mathbb{Z}_{\geq e}^k$, the normalization of the reduced complex analytic subspace $\overline{X}_r$ is denoted by $X_r$. The composite of the canonical morphism $X_r \rightarrow \overline{X}_r$ and the closed immersion $\overline{X}_r \hookrightarrow X$ is denoted by $a_r$. Then $a_r$ is a finite morphism. A log structure $\mathcal{M}_{X_r}$ on $X_r$ is defined by $\mathcal{M}_{X_r} = a_r^* \mathcal{M}_X$.

Lemma 5.5. For $r \in \mathbb{Z}_{\geq e}^k$, we have the following :

(5.5.1) $X_r$ is nonsingular.

(5.5.2) For any $x \in X_r$, there exist an open neighborhood $V$ of $x$ and a reduced simple normal crossing divisor $D_V$ on $V$, such that the log structure $\mathcal{M}_V$ is isomorphic to the log structure $\mathcal{M}_V(D_V) \oplus \mathbb{N}_V^{|r|}$, where the structure morphism $\alpha: \mathcal{M}_V(D_V) \oplus \mathbb{N}_V^{|r|} \rightarrow \mathcal{O}_V$ is given by

$$\alpha(f \oplus v) = \begin{cases} 0 & \text{if } v \neq 0, \\ f & \text{if } v = 0, \end{cases}$$

for $f \in \mathcal{M}_V(D_V) \subset \mathcal{O}_V$ and for $v \in \mathbb{N}_V^{|r|}$.

Proof. We may work on a local model $(U, \mathcal{M}_U)$ in 3.7. From (5.1) for $\overline{U}_r$, and from the equality $U \cap D[\Lambda] = D[\Lambda]$ for any $\Lambda \in S_r(A)$ with $r \geq e$, we obtain $U_r = \prod_{\Lambda \in S_r(A)} D[\Lambda]$. Thus $U_r$ is nonsingular and the restriction $a_r|_{D[\Lambda]}$ coincides with the canonical inclusion $D[\Lambda] \hookrightarrow U$. Moreover, the log structure $\mathcal{M}_{U_r}|_{D[\Lambda]}$ coincides with the pull-back of $\mathcal{M}_{\mathbb{C}^A \times \mathbb{C}^l}(D)$ by the closed immersion $D[\Lambda] \hookrightarrow \mathbb{C}^A \times \mathbb{C}^l$. By setting $D_V|_{D[\Lambda]} = \sum_{\Lambda \in A} D_\Lambda \cap D[\Lambda]$, the condition (5.5.2) is satisfied.

Remark 5.6. We note that the condition (5.5.2) is a special case of the condition (3.4.1) in [9].

Lemma 5.7. For any $r \in \mathbb{Z}_{\geq e}^k$, there exist a unique normal crossing divisor $D$ on $X_r$ and a unique inclusion $\mathcal{M}_{X_r}(D) \hookrightarrow \mathcal{M}_X$, satisfying the following:

(5.7.1) For any $x \in X_r$, there exists an open neighborhood $V$ of $x$ such that the inclusion $\mathcal{M}_{X_r}(D)|_V \hookrightarrow \mathcal{M}_X|_V$ induces an isomorphism $\mathcal{M}_{X_r}(D)|_V \oplus \mathbb{N}_V^{|r|} \simeq \mathcal{M}_X|_V$ as log structures.

Moreover $\mathcal{M}_{X_r}(D)^{gp}/\mathcal{M}_{X_r}(D)^{gp}$, which is a locally free $\mathbb{Z}$-module of rank $|r|$, carries a symmetric bilinear form with values in $\mathbb{Z}$, whose stalk at any $x \in X_r$ coincides with the canonical bilinear form associated to the finitely generated free monoid $(\mathcal{M}_{X_r}/\mathcal{M}_{X_r}(D))^e$ (cf. Definition 2.11).

Proof. By considering the connected components of $X_r$, we can easily deduce the conclusion form Lemmas 3.7 and 3.10 of [9].

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Definition 5.8. For \( r \in \mathbb{Z}_{\geq e}^{k} \), we denote by \( D_r \) the normal crossing divisor \( D \) in Lemma 5.7 on \( X_r \). We set \( L_r = \mathcal{M}^{gp}_{X_r}/\mathcal{M}_{X_r}(D)^{gp} \) and \( \varepsilon_r = \bigwedge^{|r|} L_r \), which are locally free \( \mathbb{Z} \)-modules of rank \( |r| \) and of rank one respectively. The symmetric bilinear form on \( L_r \) in Lemma 5.7 induces an isomorphism \( \varepsilon_r \otimes \varepsilon_r \to \mathbb{Z} \), which is denoted by \( \vartheta_r \).

Definition 5.9. For \( r \in \mathbb{Z}_{\geq e}^{k} \), the monoid subsheaf \( \mathcal{M}_{X_r}(D_r) \) defines an increasing filtration on \( \Omega_{X_r}(\log \mathcal{M}_{X_r}) \), as in Definition 2.29, which is denoted by \( \hat{W} \) for a while.

5.10. The morphism of abelian sheaves \( \bigwedge^m \text{dlog} : \bigwedge^m \mathcal{M}^{gp}_{X_r} \to \Omega^m_{X_r}(\log \mathcal{M}_{X_r}) \) induces a morphism of \( \mathcal{O}_{X_r} \)-modules \( \bigwedge^m \mathcal{M}^{gp}_{X_r} \otimes \Omega^n_{X_r}(\log D_r) \to \text{Gr}_m^{\hat{W}} \Omega^n_{X_r}(\log \mathcal{M}_{X_r}) \), which factors through the surjection \( \bigwedge^m \mathcal{M}^{gp}_{X_r} \otimes \Omega^n_{X_r}(\log D_r) \to \bigwedge^m L_r \otimes \Omega^n_{X_r}(\log D_r) \). Thus a morphism of \( \mathcal{O}_{X_r} \)-modules

\[
\bigwedge^m L_r \otimes \Omega^n_{X_r}(\log D_r) \to \text{Gr}_m^{\hat{W}} \Omega^n_{X_r}(\log \mathcal{M}_{X_r}) \tag{5.2}
\]

is obtained for all \( m, n \in \mathbb{Z} \).

Lemma 5.11. For \( r \in \mathbb{Z}_{\geq e}^{k} \), the morphism (5.2) gives us an isomorphism of \( \mathcal{O}_{X_r} \)-modules

\[
\bigwedge^m L_r \otimes \Omega^n_{X_r}(\log D_r) \to \text{Gr}_m^{\hat{W}} \Omega^n_{X_r}(\log \mathcal{M}_{X_r}) \tag{5.3}
\]

for all \( m, n \in \mathbb{Z} \). Therefore, \( \hat{W}_r \Omega_{X_r}(\log \mathcal{M}_{X_r}) = \Omega_{X_r}(\log \mathcal{M}_{X_r}) \).

Proof. Since the question is of local nature, we may assume that \( \mathcal{M}_{X_r} = \mathcal{M}_{X_r}(D_r) \oplus \mathbb{N}^{[r]}_{X_r} \) as in (5.7.1). Then we have \( \Omega^1_{X_r}(\log \mathcal{M}_{X_r}) \simeq (\mathcal{O}_{X_r} \otimes L_r) \oplus \Omega^1_{X_r}(\log D_r) \), from which we obtain the conclusion easily.

\[\square\]

Definition 5.12 (Residue morphism). Let \( r \in \mathbb{Z}_{\geq e}^{k} \). By composing the three morphisms, the surjection \( \Omega^m_{X_r}(\log \mathcal{M}_{X_r}) \to \text{Gr}_m^{\hat{W}} \Omega^m_{X_r}(\log \mathcal{M}_{X_r}) \), the inverse of the isomorphism (5.3) for \( m = |r| \) and the inclusion \( \varepsilon_r \otimes \Omega^n_{X_r}(\log D_r) \hookrightarrow \varepsilon_r \otimes \Omega^n_{X_r}(\log \mathcal{M}_{X_r}) \), we obtain a morphism of \( \mathcal{O}_{X_r} \)-modules \( \Omega^n_{X_r}(\log \mathcal{M}_{X_r}) \to \varepsilon_r \otimes \Omega^n_{X_r}(\log \mathcal{M}_{X_r}) \). Then we have a morphism of \( \mathcal{O}_{X_r} \)-modules

\[
(a_r)_{*} \Omega^n_{X_r}(\log \mathcal{M}_{X_r}) \to (a_r)_{*} (\varepsilon_r \otimes \Omega^n_{X_r}(\log \mathcal{M}_{X_r})) \tag{5.4}
\]
on \( \mathcal{O}_{X_r} \)-modules

\[
\text{Res}_r : \Omega^n_{X_r}(\log \mathcal{M}_{X_r}) \to (a_r)_{*} (\varepsilon_r \otimes \Omega^n_{X_r}(\log \mathcal{M}_{X_r})) \tag{5.5}
\]
is defined as the composite of the canonical morphism \( \Omega^n_{X_r}(\log \mathcal{M}_{X_r}) \to (a_r)_{*} \Omega^n_{X_r}(\log \mathcal{M}_{X_r}) \) and the morphism (5.4). It is easy to see that these morphisms form a morphism of complexes of \( \mathbb{C} \)-sheaves

\[
\text{Res}_r : \Omega_{X_r}(\log \mathcal{M}_{X_r}) \to (a_r)_{*} (\varepsilon_r \otimes \Omega_{X_r}(\log \mathcal{M}_{X_r})[-|r|])
\]

for all \( r \in \mathbb{Z}_{\geq e}^{k} \).

5.13 (Local description of the residue morphism). For \( r \in \mathbb{Z}_{\geq e}^{k} \), we describe \( \text{Res}_r \) locally. We may work on a local model \((U', \mathcal{M}_U)\). Then \( U_r = \bigcup_{\lambda \in S_r(\Lambda)} D_\lambda \) and \( D_r|_{D_\Delta} = D_\lambda \bigcap D_\lambda \) as in the proof of Lemma 5.5. For \( \lambda \in S_r(\Lambda) \), we set \( D_\lambda = \sum_{\lambda' \in \Delta} D_{\lambda'} \). Then we have

\[
(\prod_{\lambda \in \Lambda} \Omega^n_{\mathcal{C}_{\lambda \times \mathcal{C}}} (\log D) \subset W(D_\lambda)_{|\Delta} \bigcup \Omega^n_{\mathcal{C}_{\lambda \times \mathcal{C}}} (\log D))
\]
because $\Lambda \cap \Lambda_i \neq \emptyset$ for all $i = 1, 2, \ldots, k$. Therefore the Poincaré residue morphism for $W(D_{\Lambda})$

$$\text{Res}^{\Lambda} : \Omega^n_{\mathcal{C}^{\Lambda} \times C}(\log D) \to \varepsilon(\Lambda) \otimes_{\mathcal{O}} \Omega^{n-|\Lambda|}_{D_{\Lambda}}(\log D_{\Lambda} \cap D[\Lambda])$$

induces a morphism of $\mathcal{O}_U$-modules $\Omega^n_U(\log M_U) \to \varepsilon(\Lambda) \otimes_{\mathcal{O}} \Omega^{n-|\Lambda|}_{D_{\Lambda}}(\log D_{\Lambda} \cap D[\Lambda])$ by the identification (3.3). Composing with the canonical inclusion, we obtain a morphism

$$\text{Res}^{\Lambda} : \Omega^n_U(\log M_U) \to \varepsilon(\Lambda) \otimes_{\mathcal{O}} \Omega^{n-|\Lambda|}_{D_{\Lambda}}(\log M_{D[\Lambda]})$$

denoted by $\text{Res}^{\Lambda}$ again. Under the identification

$$(a_r)_* (\varepsilon_r \otimes_{\mathcal{O}} \Omega^{n-|r|}_{U_r}(\log M_{U_r})) \simeq \bigoplus_{\Lambda \in S(\Lambda)} (\varepsilon(\Lambda) \otimes_{\mathcal{O}} \Omega^{n-|\Lambda|}_{D_{\Lambda}}(\log M_{D[\Lambda]}),$$

the equality $\text{Res}_r = \sum_{\Lambda \in S_r(\Lambda)} \text{Res}^{\Lambda}$ can be easily checked.

**Lemma 5.14.** For $r \in \mathbb{Z}^k_{\geq 0}$ and $I \subset \{1, 2, \ldots, k\}$, we have

$$\text{Res}_r(W_m \Omega^n_X(\log M_X)) \subset (a_r)_* (\varepsilon_r \otimes_{\mathcal{O}} \Omega^{n-|r|}_{U_r}(\log M_{U_r})).$$

$$\text{Res}_r(W(I)_m \Omega^n_X(\log M_X)) = 0 \text{ if } |r| > m,$$

where $r_I$ is the image of $r \in \mathbb{Z}^k$ by the projection $\mathbb{Z}^k \to \mathbb{Z}^I$.

**Proof.** Easy from the local description above.

**5.15.** For $q, r \in \mathbb{Z}^k_{\geq 0}$ with $r \geq q + e$, the morphism $\text{Res}_r$ in (5.5) factors through the surjection $\Omega^n_X(\log M_X) \to \Omega^n_X(\log M_X)/\sum_{i=1}^k W(i)_q$, by Lemma 5.14. Thus a morphism of $\mathcal{O}_X$-modules

$$\text{Res}_r : \Omega^n_X(\log M_X)/\sum_{i=1}^k W(i)_q \to (a_r)_* (\varepsilon_r \otimes_{\mathcal{O}} \Omega^{n-|r|}_{X_r}(\log M_{X_r}))$$

(5.6)

is obtained, which is denoted by $\text{Res}_r$ again by abuse of notation.

**Lemma 5.16.** We have the isomorphism of complexes of $\mathbb{C}$-sheaves

$$\sum_{r \geq q + e \atop |r| = m} \text{Res}_r : \text{Gr}^W_m(\Omega_X(\log M_X)/\sum_{i=1}^k W(i)_q) \xrightarrow{\sim} \bigoplus_{r \geq q + e \atop |r| = m} (a_r)_* (\varepsilon_r \otimes_{\mathcal{O}} \Omega_{X_r})[-m]$$

for any $q \in \mathbb{N}^k$ and $m \in \mathbb{Z}$, under which $W(I)_m \text{Gr}^W_m(\Omega_X(\log M_X)/\sum_{i=1}^k W(i)_q)$ is identified with the direct sum of $(a_r)_* (\varepsilon_r \otimes_{\mathcal{O}} \Omega_{X_r})[-m]$ over the index set

$$\{r \in \mathbb{N}^k \mid r \geq q + e, |r| = m, |r_I| \leq l\} \quad (5.7)$$

for all $I \subset \{1, 2, \ldots, k\}$ and $l \in \mathbb{Z}$.

**Proof.** We may work on a local model $(U, \mathcal{M}_U)$ and use the same notation as in 3.7. In particular, the partition $\Lambda = \bigsqcup_{i=1}^k \Lambda_i$ is associated to the semistable morphism $\varphi$ in (3.6.3). Since

$$(\prod_{\lambda \in \Lambda_i} x_{\lambda} \cdot \Omega^n_{\mathcal{C}^{\Lambda} \times C}(\log D) \subset W(D_{\Lambda}) \Omega^n_{\mathcal{C}^{\Lambda} \times C}(\log D)$$

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for all $i = 1, 2, \ldots, k$, the identification (3.3) induces an isomorphism
\[
\Omega_{C^\lambda \times C^\lambda} (\log D)/ \sum_{i=1}^{k} W(D_i)_{q_i} \cong \Omega_U (\log \mathcal{M}_U)/ \sum_{i=1}^{k} W(i)_{q_i} \tag{5.8}
\]
for any $q \in \mathbb{N}^k$. Then the local description of $\text{Res}_r$ in 5.13 implies the conclusion from the usual Poincaré residue isomorphism as in [3, 3.1] (cf. [7, Section 3]).

The following corollary will be used in Section 10.

**Corollary 5.17.** For $I \subset \{1, 2, \ldots, k\}$, we have $L = L(I) \ast L(\{1, 2, \ldots, k\} \setminus I)$ on $A_C$.

**Proof.** We set $J = \{1, 2, \ldots, k\} \setminus I$. By the definition (3.9) of $L(I)$ on $A_C$, it suffices to prove the equality $W(I) \ast W(J) = W$ on $\Omega_X (\log \mathcal{M}_X)/ \sum_{i=1}^{k} W(i)_{q_i}$ for all $q \in \mathbb{N}^k$. From the isomorphism (5.8) and the equality $W(D_i) \ast W(D_j) = W(D)$ on $\Omega_{C^\lambda \times C^\lambda} (\log D)$ for a local situation in 3.7, we have $W_m \subset (W(I) \ast W(J))_m$ on $\Omega_X (\log \mathcal{M}_X)/ \sum_{i=1}^{k} W(i)_{q_i}$ for all $m$. By the lemma above, $W(I)_a \cap W(J)_b = 0$ on $\text{Gr}^W_0 (\Omega_X (\log \mathcal{M}_X)/ \sum_{i=1}^{k} W(i)_{q_i})$ if $a + b < m$. Therefore $W(I)_a \cap W(J)_b \subset W_{a+b}$ on $\Omega_X (\log \mathcal{M}_X)/ \sum_{i=1}^{k} W(i)_{q_i}$ for any $a, b \in \mathbb{Z}$. \qed

**Definition 5.18.** Let $r \in \mathbb{Z}_+^k$. Then the monoid subsheaf $\mathcal{M}_{X_r}(D_r)$ of $\mathcal{M}_{X_r}$ gives us an increasing filtration $W(\mathcal{M}_{X_r}(D_r))$ on $\text{Kos}_X(\mathcal{M}_X)$ as in [9, Definition 1.8]. This filtration is denoted by $\widetilde{\text{W}}$ for a while as in the case of the log de Rham complex.

**Lemma 5.19.** Let $r \in \mathbb{Z}_+^k$. There exists an isomorphism of complexes of $\mathbb{Q}$-sheaves
\[
\bigwedge^m L_r \otimes \text{Kos}_{X_r}(D_r)[-m] \xrightarrow{\sim} \text{Gr}^W_m \text{Kos}_{X_r}(\mathcal{M}_{X_r}) \tag{5.9}
\]
for all $m$. Therefore, $\widetilde{\text{W}}_{[r]} \text{Kos}_{X_r}(\mathcal{M}_{X_r}) = \text{Kos}_{X_r}(\mathcal{M}_{X_r})$.

**Proof.** By Proposition 1.10 of [9]. \qed

**Definition 5.20** (Residue morphism for the Koszul complex). Let $r \in \mathbb{Z}_+^k$. By composing the three morphisms, the surjection $\text{Kos}_X(\mathcal{M}_{X_r}) \twoheadrightarrow \text{Gr}^W_0 \text{Kos}_X(\mathcal{M}_{X_r})$, the inverse of the isomorphism (5.9) for $m = |r|$ and the inclusion $\varepsilon_r \otimes \mathbb{Z} \text{Kos}_{X_r}(D_r)[-|r|] \hookrightarrow \varepsilon_r \otimes \mathbb{Z} \text{Kos}_{X_r}(\mathcal{M}_{X_r})[-|r|]$, we obtain a morphism of complexes of $\mathbb{Q}$-sheaves $\text{Kos}_{X_r}(\mathcal{M}_{X_r}) \twoheadrightarrow \varepsilon_r \otimes \mathbb{Z} \text{Kos}_{X_r}(\mathcal{M}_{X_r})[-|r|]$. Then we have a morphism of complexes of $\mathbb{Q}$-sheaves
\[
(a_r)_* \text{Kos}_{X_r}(\mathcal{M}_{X_r}) \twoheadrightarrow (a_r)_* (\varepsilon_r \otimes \mathbb{Z} \text{Kos}_{X_r}(\mathcal{M}_{X_r}))[-|r|] \tag{5.10}
\]
on $X$. A morphism of complexes of $\mathbb{Q}$-sheaves
\[
\text{Res}_{X_r}^\mathbb{Q} : \text{Kos}_X(\mathcal{M}_X) \twoheadrightarrow (a_r)_* (\varepsilon_r \otimes \mathbb{Z} \text{Kos}_{X_r}(\mathcal{M}_{X_r}))[-|r|] \tag{5.11}
\]
is defined as the composite of the canonical morphism $\text{Kos}_X(\mathcal{M}_X) \twoheadrightarrow (a_r)_*, \text{Kos}_{X_r}(\mathcal{M}_{X_r})$ and the morphism (5.10).

5.21 (The stalk of the residue morphism $\text{Res}_{X_r}^\mathbb{Q}$). Now we describe $\text{Res}_{X_r}^\mathbb{Q}$ stalkwise. We may work at the origin $x = 0$ of a local model $(U, \mathcal{M}_U)$ in 3.7. Similarly to (3.6), we have
\[
(a_r)_* (\varepsilon_r \otimes \mathbb{Z} \text{Kos}_{X_r}(\mathcal{M}_{X_r})[-|r|])_x \cong \bigoplus_{\mu \in S_r(\Lambda)} \bigoplus_{\nu \in S(\Lambda)} \varepsilon(\mu) \otimes \varepsilon(\nu) \otimes \mathbb{Z} \text{Kos}(O^*_{D_{[|r|]}^{|\nu|}} x)^{n-|r|-|\mu|}
\]
for all \( n \in \mathbb{Z} \). Via the identification (3.6),

\[
\text{Res}_{r,x}^Q(\varepsilon(\Delta) \otimes_{\mathbb{Z}} \text{Kos}(\mathcal{O}_{U,x}^\ast)^{n-|\Delta|}) \subset \bigoplus_{\mu \in S_r(\Lambda)} \varepsilon(\mu) \otimes_{\mathbb{Z}} \varepsilon(\Delta \setminus \mu) \otimes_{\mathbb{Z}} \text{Kos}(\mathcal{O}_{D[\mu],x}^\ast)^{n-|\Delta|}
\]

and the restriction of \( \text{Res}_{r,x}^Q \) on the direct summand \( \varepsilon(\Delta) \otimes_{\mathbb{Z}} \text{Kos}(\mathcal{O}_{U,x}^\ast)^{n-|\Delta|} \) is identified with

\[
\sum_{\mu \in S_r(\Lambda)} \chi(\mu, \Delta \setminus \mu)^{-1} \otimes \text{Kos}(a[\mu]_x^\ast), \quad (5.12)
\]

where \( \chi(\mu, \Delta \setminus \mu) \) is the isomorphism (2.4) and \( \text{Kos}(a[\mu]_x^\ast) \) is the induced morphism from the canonical morphism \( a[\mu]_x^\ast : \mathcal{O}_{U,x}^\ast \rightarrow \mathcal{O}_{D[\mu],x}^\ast \) for the closed immersion \( a[\mu] : D[\mu] \hookrightarrow U \).

**Lemma 5.22.** For \( r \in \mathbb{Z}_{\geq e}^k \) and \( I \subset \{1, 2, \ldots, k\} \), we have

\[
\text{Res}_{r}^Q(W_{i_r} \text{Kos}_X(\mathcal{M}_X)) \subset (a_{i_r})_*(\varepsilon_r \otimes_{\mathbb{Z}} \text{Kos}_{X_{r}}(\mathcal{O}_{X_{r}}^\ast))[-|r|]
\]

\[
\text{Res}_{r}^Q(W(I)_m \text{Kos}_X(\mathcal{M}_X)) = 0 \quad \text{if } |r_i| > m.
\]

**Proof.** We may work stalkwise. Under the identification (3.6), \( W(I)_m \text{Kos}_X(\mathcal{M}_X)_x^n \) is identified with

\[
\bigoplus_{\Delta \in S(\Lambda)} \varepsilon(\Delta) \otimes_{\mathbb{Z}} \text{Kos}_U(\mathcal{O}_U^\ast)_x^{n-|\Delta|},
\]

where \( \Lambda_I = \coprod_{i \in I} \Lambda_i \). Then we can easily check the conclusions. \( \square \)

**Lemma 5.23.** We have the quasi-isomorphism

\[
\sum_{r \geq q+e \atop |r|=m} \text{Res}_{r}^Q : Gr_m^W(\text{Kos}_X(\mathcal{M}_X)) / \sum_{i=1}^k W(i)_{|q_i|} \longrightarrow \bigoplus_{r \geq q+e \atop |r|=m} (a_{i_r})_*(\varepsilon_r \otimes_{\mathbb{Z}} \text{Kos}_{X_{r}}(\mathcal{O}_{X_{r}}^\ast))[-m]
\]

for any \( q \in \mathbb{N}^k \) and \( m \in \mathbb{Z} \). Similarly, we have the quasi-isomorphism

\[
\sum \text{Res}_{r}^Q : W(I)_l \text{Gr}_m^W(\text{Kos}_X(\mathcal{M}_X)) / \sum_{i=1}^k W(i)_{|q_i|} \longrightarrow \bigoplus (a_{i_l})_*(\varepsilon_r \otimes_{\mathbb{Z}} \text{Kos}_{X_{r}}(\mathcal{O}_{X_{r}}^\ast))[-m]
\]

for all \( I \subset \{1, 2, \ldots, k\} \) and \( l \in \mathbb{Z} \), where the sum and the direct sum are taken over the same index set as (5.7).

**Proof.** We may work stalkwise as in 5.21. Note that the morphism \( \text{Kos}(a[\mu]_x^\ast) : \text{Kos}(\mathcal{O}_{U,x}^\ast) \rightarrow \text{Kos}(\mathcal{O}_{D[\mu],x}^\ast) \) in (5.12) is a quasi-isomorphism because both sides are canonically quasi-isomorphic to \( \mathbb{Q} \) by [9, Corollary 1.15]. Then the conclusions follows from the local description in 5.21. \( \square \)

**Lemma 5.24.** For \( r \in \mathbb{Z}_{\geq e}^k \), the diagram

\[
\begin{array}{ccc}
\text{Kos}_X(\mathcal{M}_X) & \xrightarrow{\text{Res}_{r}^Q} & (a_{i_r})_*(\varepsilon_r \otimes_{\mathbb{Z}} \text{Kos}_{X_{r}}(\mathcal{M}_{X_{r}}))[-|r|] \\
\psi_X & \downarrow & |(a_{i_r})(\text{id} \otimes 2\pi \sqrt{-1})^{-|r|}\psi_{X_{r}}|[-|r|] \\
\Omega_X(\log \mathcal{M}_X) & \xrightarrow{\text{Res}_{r}} & (a_{i_r})_*(\varepsilon_r \otimes_{\mathbb{Z}} \Omega_{X_{r}}(\log \mathcal{M}_{X_{r}}))[-|r|]
\end{array}
\]

is commutative, where \( \psi_X \) and \( \psi_{X_{r}} \) are the morphisms in (3.14.3).
Proof. The commutativity of the diagram

\[
\begin{array}{ccc}
\bigwedge^m L_r \otimes_{\mathcal{O}} \text{Kos}_{X_r}(D_r)[-m] & \xrightarrow{\sim} & \text{Gr}_m \text{Kos}_{X_r}(\mathcal{M}_{X_r}) \\
id \otimes (2\pi \sqrt{-1})^{-m} \psi_{(X_r, D_r)}[-m] & & \text{Gr}_m \psi_{(X_r, \mathcal{M}_{X_r})}
\end{array}
\]

\[
\bigwedge^m L_r \otimes_{\mathcal{O}} \Omega_{X_r}(\log D_r)[-m] \longrightarrow \text{Gr}_m \Omega_{X_r}(\log \mathcal{M}_{X_r})
\]

can be easily checked from the definition in [9, (2.4)]. The conclusion follows from the case of \(m = |r|\).

Proof of Theorem 4.3. We use the notation in Remark 3.22 for short. Because

\[d_i(L_m A^n_Q) \subset L_{m-1} A^{n+1}_Q, \quad d_i(L_m A^n_C) \subset L_{m-1} A^{n+1}_C\]

for all \(i = 1, 2, \ldots, k\) and \(m, n \in \mathbb{Z}\), we have

\[
\text{Gr}_m L_a A_Q = \bigoplus_{q \in \mathbb{N}^k} \mathbb{Q} u^q \otimes_C \text{Gr}_m W_{m+2|q|+k}(\text{Kos}_X(\mathcal{M}_X))/\sum_{i=1}^k W(i, q) [k]
\]

\[
\text{Gr}_m L_a A_C = \bigoplus_{q \in \mathbb{N}^k} \mathbb{C} u^q \otimes_C \text{Gr}_m W_{m+2|q|+k}(\Omega_X(\log \mathcal{M}_X))/\sum_{i=1}^k W(i, q) [k]
\]
as complexes. Therefore, by Lemmas 5.16 and 5.23, and by the canonical isomorphisms

\[
\text{Gr}_m L_b A_Q / L(I)_a A_Q \simeq L(I)_b \text{Gr}_m L_a A_Q / L(I)_a A_Q
\]

\[
\text{Gr}_m L_b A_C / L(I)_a A_C \simeq L(I)_b \text{Gr}_m L_a A_C / L(I)_a A_C
\]

we have a quasi-isomorphism

\[
\text{Gr}_m L_b A_Q / L(I)_a A_Q \longrightarrow \bigoplus_{q \in \mathbb{N}^k} \mathbb{Q} u^q \otimes_C (\varepsilon_r \otimes \text{Kos}_{X_r}(\mathcal{O}_{X_r}^n))[-m - 2|q|]
\]

and an isomorphism

\[
\text{Gr}_m L_b A_C / L(I)_a A_C \xrightarrow{\sim} \bigoplus_{q \in \mathbb{N}^k} \mathbb{C} u^q \otimes_C (\varepsilon_r \otimes \Omega_{X_r}^n)[-m - 2|q|],
\]

under which the morphism induced by \(\alpha\) is identified with

\[
\bigoplus (\alpha)_*(\id \otimes (2\pi \sqrt{-1})^{-m-|q|} \psi_{(X_r, \mathcal{O}_{X_r}^n)})[-m - 2|q|]
\]

by Lemma 5.24, where the direct sums (5.13)–(5.15) are taken over the index set

\[
\{(q, r) \in \mathbb{N}^k \times \mathbb{N}^k \mid r \geq q + e, |r| = m + 2|q| + k, a < |r| - 2|q|, a \leq |I| \leq b\}.
\]

From (3.11) and Lemma 5.16, we have

\[
F^p \text{Gr}_m L_b A^n_Q / L(I)_a A_C = \bigoplus_{|q| \leq n-p} \mathbb{C} u^q \otimes_C \text{Gr}_m L_b(A^n_Q)_{q} / L(I)_a(A^n_C)_{q}
\]

\[
\simeq \bigoplus_{|q| \leq n-p} \mathbb{C} u^q \otimes_C (L(I)_b \text{Gr}_m(A^n_C)_{q} / L(I)_a \text{Gr}_m(A^n_C)_{q})
\]

\[
\simeq \bigoplus_{|q| \leq n-p} \mathbb{C} u^q \otimes_C (\varepsilon_r \otimes \Omega_{X_r}^{n-m-2|q|}).
\]
where the direct sum in the last term is taken over the index set
\[ \{(q, r) \in \mathbb{N}^k \times \mathbb{N}^k \mid r \geq q + e, |q| \leq n - p, |r| = m + 2|q| + k, a < |r| - 2|q| - |I| \leq b \}. \]
Therefore the isomorphisms (5.14) induces an isomorphism of filtered complexes
\[ \left( \text{Gr}_m^L (L(I)_{bA_C} / L(I)_{aA_C}, F) \rightarrow \bigoplus \mathbb{C}u^q \otimes_{\mathbb{C}} ((a_r)_*(\varepsilon_r \otimes_{\mathbb{Q}} \Omega_{X_r})[-m - 2|q|], F[-m - |q|]) \right), \tag{5.17} \]
where \( F \) on the right hand side is the stupid filtration on \( \varepsilon_r \otimes_{\mathbb{Z}} \Omega_{X_r} \) and the index set of the direct sum on the right hand side is the same as (5.16). Because \( \varepsilon_r \) admits a positive definite symmetric bilinear form \( \theta_r \) as in Definition 5.8,
\[ ((L(I)_{bA_Q} / L(I)_{aA_Q}, L), (L(I)_{bA_C} / L(I)_{aA_C}, L, F), \alpha) \]
is a \( \mathbb{Q} \)-cohomological mixed Hodge complex on \( X \) by [3, (2.2.2)].

\[ \text{Remark 5.25.} \] The assumption for \( f: (X, \mathcal{M}_X) \rightarrow (\ast, \mathbb{N}^k) \) being projective in Theorem 4.3 can be relaxed to the assumptions that \( f: (X, \mathcal{M}_X) \rightarrow (\ast, \mathbb{N}^k) \) is proper and that \( X_r \) is Kähler for all \( r \in \mathbb{Z}^k_{\geq e} \).

\[ \text{Remark 5.26.} \] By taking \( a \) sufficiently small and \( b \) sufficiently large in (5.17), we have the isomorphism of filtered complexes
\[ \left( \text{Gr}_m^L A_C, F \rightarrow \bigoplus \mathbb{C}u^q \otimes_{\mathbb{C}} ((a_r)_*(\varepsilon_r \otimes_{\mathbb{Q}} \Omega_{X_r})[-m - 2|q|], F[-m - |q|]) \right), \tag{5.18} \]
for all \( m \in \mathbb{Z} \), where the direct sum on the right hand side is taken over the index set
\[ \{(r, q) \in \mathbb{N}^k \times \mathbb{N}^k \mid r \geq q + e, |r| = m + 2|q| + k \}. \tag{5.19} \]
Similarly, we have the quasi-isomorphism
\[ \text{Gr}_m^L A_Q \rightarrow \bigoplus \mathbb{Q}u^q \otimes_{\mathbb{Q}} (a_r)_*(\varepsilon_r \otimes_{\mathbb{Q}} \Omega_{X_r})[-m - 2|q|] \]
for all \( m \in \mathbb{Z} \), where the index set of the direct sum is the same as (5.19).

\section{A complex \( \mathcal{C}(\Omega_{X_\bullet} (\log \mathcal{M}_{X_\bullet})) \)}

In this section, we first construct a Čech type filtered complex \( \mathcal{C}(\Omega_{X_\bullet} (\log \mathcal{M}_{X_\bullet}), \delta W) \) and a product on it. Because the family of complex manifolds \( \{X_r\}_{r \in \mathbb{Z}^k_{\geq e}} \) does not admit a simplicial (or cubical) structure in general, it is not possible to apply the arguments in [10, Section 2]. Thus the construction in this section requires some other tasks, in which the log structures on \( X \) and on \( X_r \) play essential roles. Second, we construct a kind of “trace map” \( E_1^{-k,2 \dim X + 2k} \rightarrow \mathbb{C} \), where \( E_1^{p,q} \) denotes the \( E_1 \)-terms of the spectral sequence associated to \( (R\Gamma(X, \mathcal{C}(\Omega_{X_\bullet} (\log \mathcal{M}_{X_\bullet}))), \delta W) \) in this section. The construction of this map is similar to and slightly simplified from the one in [10, Definition 7.7].

\[ \text{Definition 6.1.} \] For \( r \in \mathbb{N}^k \), we set
\[ \bigwedge^{\otimes r} \mathcal{M}_X^p = \bigwedge^{r_1} \mathcal{M}(1)_X^p \otimes_{\mathbb{Z}} \bigwedge^{r_2} \mathcal{M}(2)_X^p \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \bigwedge^{r_k} \mathcal{M}(k)_X^p, \]
which is regraded as a subsheaf of \( \bigwedge^{\otimes |r|} \mathcal{M}_X^p \) by the inclusion
\[ \bigwedge^{\otimes r} \mathcal{M}_X^p \owns v_1 \otimes v_2 \otimes \cdots \otimes v_k \mapsto v_1 \wedge v_2 \wedge \cdots \wedge v_k \in \bigwedge^{\otimes |r|} \mathcal{M}_X^p. \]
6.2. For \( r \in \mathbb{Z}_{\geq e}^k \), the canonical morphism \( a_r^{-1}\overline{\mathcal{M}}_X^{\text{sp}} \to \mathcal{M}_X \) induces morphisms \( a_r^{-1}\overline{\mathcal{M}}_X^{\text{sp}} \to \mathcal{M}_{Xr}/\mathcal{M}_{Xr}(D_r)^{\text{sp}} = L_r \) and \( a_r^{-1}\bigwedge^{[r]}\overline{\mathcal{M}}_X^{\text{sp}} \to \bigwedge^{[r]}L_r = \varepsilon_r \). Thus we obtain a morphism of \( \mathbb{Z} \)-sheaves

\[
\bigwedge^{[r]}\overline{\mathcal{M}}_X^{\text{sp}} \to (a_r)_*\varepsilon_r
\]

for all \( r \in \mathbb{Z}_{\geq e}^k \).

Lemma 6.3. By restricting the morphism (6.1) to \( \bigwedge^r \overline{\mathcal{M}}_X^{\text{sp}} \), we obtain an isomorphism

\[
\bigwedge^r \overline{\mathcal{M}}_X^{\text{sp}} \to (a_r)_*\varepsilon_r
\]

for any \( r \in \mathbb{Z}_{\geq e}^k \).

Proof. We may work stalkwise. Then the local description in 3.7 implies the conclusion easily. \( \square \)

Definition 6.4. The image of \( t_i \in \Gamma(X, \mathcal{M}_X) \) by the projection \( \mathcal{M}_X \to \overline{\mathcal{M}}_X \) is denoted by \( \overline{t}_i \in \Gamma(\overline{X}, \overline{\mathcal{M}}_X) \) for \( i = 1, 2, \ldots, k \). Then a morphism \( \overline{t}_i \wedge \cdot \): \( \bigwedge \overline{\mathcal{M}}_X \to \bigwedge \overline{\mathcal{M}}_X \) is defined by sending \( \nu \) to \( \overline{t}_i \wedge \nu \). Because \( \overline{t}_i \in \Gamma(\overline{X}, \overline{\mathcal{M}}(i)_X) \) as in Remark 3.10, the morphism \( \overline{t}_i \wedge \cdot \) induces a morphism \( \bigwedge^r \overline{\mathcal{M}}_X^{\text{sp}} \to \bigwedge^{\overline{r}+e_r} \overline{\mathcal{M}}_X^{\text{sp}} \) for every \( r \in \mathbb{N}^k \). Thus we obtain a morphism

\[
\delta_i: (a_r)_*\varepsilon_r \to (a_{r+e_i})_*\varepsilon_{r+e_i}
\]

via the isomorphism (6.2) for \( r \in \mathbb{Z}_{\geq e}^k \) and for \( i = 1, 2, \ldots, k \). Trivially the equalities

\[
\delta_i^2 = 0, \quad \delta_i \delta_j + \delta_j \delta_i = 0
\]

hold for all \( i, j \in \{1, 2, \ldots, k\} \).

Lemma 6.5. For any \( r \in \mathbb{Z}_{\geq e}^k \), the canonical morphism

\[
(a_r)_*\varepsilon_r \otimes_{\mathbb{Z}} \Omega^n_X(\log \mathcal{M}_X) \to (a_r)_*(\varepsilon_r \otimes_{\mathbb{Z}} \Omega^n_{Xr}(\log \mathcal{M}_{Xr}))
\]

is surjective.

Proof. It suffices to consider the stalks at the origin \( x = 0 \) of a local model \( (U, \mathcal{M}_U) \) in 3.7. Then

\[
((a_r)_*\varepsilon_r \otimes_{\mathbb{Z}} \Omega^n_U(\log \mathcal{M}_U))_x = \bigoplus_{\Delta \in S_r(\Lambda)} \varepsilon(\Delta) \otimes_{\mathbb{Z}} \Omega^n_U(\log \mathcal{M}_U)_x
\]

\[
(a_r)_*(\varepsilon_r \otimes_{\mathbb{Z}} \Omega^n_{Ux}(\log \mathcal{M}_{Ux}))_x = \bigoplus_{\Delta \in S_r(\Lambda)} \varepsilon(\Delta) \otimes_{\mathbb{Z}} \Omega^n_{D\Delta}(\log \mathcal{M}_{D\Delta})_x
\]

and the stalk of the morphism (6.4) is the direct sum of \( \mathbb{L} \otimes a[\Lambda]^r \) over all \( \Lambda \in S_r(\Lambda) \), where \( a[\Lambda]^r: \Omega^n_U(\log \mathcal{M}_U)_x \to \Omega^n_{D\Delta}(\log \mathcal{M}_{D\Delta})_x \) is the surjection induced from the closed immersion \( a[\Lambda]: D[\Delta] \hookrightarrow U \). \( \square \)

Lemma 6.6. For all \( i = 1, 2, \ldots, k \), the composite

\[
(a_r)_*\varepsilon_r \otimes_{\mathbb{Z}} \Omega^n_X(\log \mathcal{M}_X) \xrightarrow{\delta_i \otimes \text{id}} (a_{r+e_i})_*(\varepsilon_{r+e_i} \otimes_{\mathbb{Z}} \Omega^n_{Xr+e_i}(\log \mathcal{M}_{Xr+e_i}))
\]

factors through the surjection (6.4).
Proof. It is enough to consider the stalk of the morphism (6.7) at the origin \( x = 0 \) of a local model \((U, \mathcal{M}_U)\) as above. Under (6.5) for \( r \) and (6.6) for \( r + e_i \), the stalk of (6.7) at \( x \) is the direct sum of \( \sum_{\lambda \in A \setminus (\Delta \setminus \Lambda)} (e_{\lambda \Lambda}) \otimes a_{\Delta \cup \{ \lambda \}}^{\ast} \) for all \( \lambda \in S_r(\Lambda) \). Because \( a_{\Delta \cup \{ \lambda \}}: D[\Delta \cup \{ \lambda \}] \hookrightarrow U \) factors as \( D[\Delta \cup \{ \lambda \}] \hookrightarrow D[\Lambda] \hookrightarrow U \), we obtain the conclusion. \( \square \)

Definition 6.7. By the lemma above, a morphism of \( \mathcal{O}_X \)-modules

\[
(a_{r})_{\ast}(\varepsilon_{r} \otimes_{\mathbb{Z}} \Omega_{X_{r}}^{n}(\log M_{X_{r}})) \rightarrow (a_{r+e_{i}})_{\ast}(\varepsilon_{r+e_{i}} \otimes_{\mathbb{Z}} \Omega_{X_{r+e_{i}}}^{n}(\log M_{X_{r+e_{i}}}))
\]

is induced from (6.7) for every \( i = 1, 2, \ldots, k \). This morphism is denoted by \( \delta_i \), again by abuse of notation. Then the same equalities as (6.3) hold trivially.

Remark 6.8. We look at the stalk of the morphism (6.8). As in the proof of Lemma 6.6, it suffices to consider the stalk at the origin \( x = 0 \) of a local model \((U, \mathcal{M}_U)\). Under (6.6) for \( r \) and \( r + e_i \), the stalk of (6.8) at \( x \) is the direct sum of \( \sum_{\lambda \in A \setminus (\Delta \setminus \Lambda)} (e_{\lambda \Lambda}) \otimes (-)[D[\Delta \cup \{ \lambda \}]] \) for all \( \lambda \in S_r(\Lambda) \), where \((-)[D[\Delta \cup \{ \lambda \}]]\) denotes the restriction morphism from \( D[\Lambda] \) to \( D[\Delta \cup \{ \lambda \}] \). Therefore the equality

\[
\delta_i \cdot (a_{r})_{\ast}(\text{id} \otimes d) = (a_{r+e_{i}})_{\ast}(\text{id} \otimes d) \cdot \delta_i
\]

holds for all \( i = 1, 2, \ldots, k \).

Definition 6.9. An \( \mathcal{O}_X \)-module \( \mathcal{C}(\Omega_{X_{r}}(\log M_{X_{r}}))^n \) and a morphism of \( \mathcal{O}_X \)-modules

\[
d: \mathcal{C}(\Omega_{X_{r}}(\log M_{X_{r}}))^n \rightarrow \mathcal{C}(\Omega_{X_{r}}(\log M_{X_{r}}))^{n+1}
\]

are defined by

\[
\mathcal{C}(\Omega_{X_{r}}(\log M_{X_{r}}))^n = \bigoplus_{r \in \mathbb{Z}_{\geq 0}^k} (a_{r})_{\ast}(\varepsilon_{r} \otimes_{\mathbb{Z}} \Omega_{X_{r}}^{n-|r|+k}(\log M_{X_{r}}))
\]

and

\[
d = \bigoplus_{r \in \mathbb{Z}_{\geq 0}^k} ((-1)^{|r|} \cdot (-1) \cdot (a_{r})_{\ast}(\text{id} \otimes d) + \sum_{i=1}^{k} \delta_i),
\]

where \( d \) in the right hand side is the differential of the complex \( \Omega_{X_{r}}(\log M_{X_{r}}) \). From (6.3) and (6.9), the equality \( d^2 = 0 \) can be easily checked. Thus the complex \( \mathcal{C}(\Omega_{X_{r}}(\log M_{X_{r}})) \) of \( \mathcal{C} \)-sheaves on \( X \) is obtained. By setting

\[
(\delta W)_m \mathcal{C}(\Omega_{X_{r}}(\log M_{X_{r}}))^n = \bigoplus_{r \in \mathbb{Z}_{\geq 0}^k} (a_{r})_{\ast}(\varepsilon_{r} \otimes_{\mathbb{Z}} W_{m+|r|} \cdot \Omega_{X_{r}}^{n-|r|+k}(\log M_{X_{r}}))
\]

for \( m, n \in \mathbb{Z} \), we obtain an increasing filtration \( \delta W \) on the complex \( \mathcal{C}(\Omega_{X_{r}}(\log M_{X_{r}})) \). We have

\[
\text{Gr}_{m}^{\delta W} \mathcal{C}(\Omega_{X_{r}}(\log M_{X_{r}})) = \bigoplus_{r \in \mathbb{Z}_{\geq 0}^k} (a_{r})_{\ast}(\varepsilon_{r} \otimes_{\mathbb{Z}} \text{Gr}_{m+|r|}^{W} \Omega_{X_{r}}(\log M_{X_{r}}))[-|r| + k]
\]

as complexes for all \( m \in \mathbb{Z} \), because \( \delta_i(\mathcal{C}(\Omega_{X_{r}}(\log M_{X_{r}}))) \subset (\delta W)_{m-1} \mathcal{C}(\Omega_{X_{r}}(\log M_{X_{r}})) \) for \( i = 1, 2, \ldots, k \) by the description of \( \delta_i \) in Remark 6.8.

Next task is to construct a product on the complex \( \mathcal{C}(\Omega_{X_{r}}(\log M_{X_{r}})) \) as in [10, Section 2].

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6.10. For every \( x \in X \), a morphism \( \chi(f_x) : \bigwedge \mathcal{M}_{X,x}^{\op} \otimes \mathbb{Z} \bigwedge \mathcal{M}_{X,x}^{\op} \to \bigwedge \mathcal{M}_{X,x}^{\op} \) is induced from the semistable morphism \( f_x : N^k \to \mathcal{M}_{X,x} \) as in Definition 2.15. It is easy to see that \( \chi(f_x)(\bigwedge^r \mathcal{M}_{X,x}^{\op} \otimes \bigwedge^s \mathcal{M}_{X,x}^{\op}) \subset \bigwedge^{r+s-e} \mathcal{M}_{X,x}^{\op} \) for all \( r, s \in \mathbb{N}_{\geq} \).

Lemma 6.11. There exists a unique morphism \( \chi : \bigwedge \mathcal{M}_{X}^{\op} \otimes \bigwedge \mathcal{M}_{X}^{\op} \to \bigwedge \mathcal{M}_{X}^{\op} \) such that \( \chi_x = \chi(f_x) \) for all \( x \in X \).

Proof. Since the uniqueness is trivial, it suffices to check the existence locally. Thus we may work over a local model \((U, \mathcal{M}_U)\) in 3.7. We note that there exists a chart \( N^0_U \to \mathcal{M}_U \) which induces a surjection \( \mathcal{Z}_U^A \to \mathcal{M}_U^{\op} \). For \( x \in U \), we set \( \Lambda_x = \{ \lambda \in \Lambda \mid x_\lambda \notin \mathcal{O}_{U,x} \} \), where \( x_\lambda \) is the coordinate function corresponding to \( \lambda \in \Lambda \) as in 3.7. The chart \( N^0_U \to \mathcal{M}_U \) induces the identification \( N^{\Lambda_x} \to \mathcal{M}_{U,x} \) for all \( x \in U \). On the other hand, the partition \( \Lambda = \coprod_{i=1}^k \Lambda_i \) as in 3.7 induces a morphism \( \chi(\Lambda) : \bigwedge \mathcal{Z}^{\op} \otimes \bigwedge \mathcal{Z} \to \bigwedge \mathcal{Z} \). By Remark 2.8, the diagram

\[
\begin{array}{ccc}
\bigwedge \mathcal{Z}^{\op} \otimes \bigwedge \mathcal{Z} & \xrightarrow{\chi(\Lambda)} & \bigwedge \mathcal{Z} \\
\downarrow & & \downarrow \\
\bigwedge \mathcal{Z}_{\Lambda} \otimes \bigwedge \mathcal{Z}_{\Lambda} & \xrightarrow{\chi(\Lambda_x)} & \bigwedge \mathcal{Z}_{\Lambda_x} \\
\simeq \downarrow & & \simeq \\
\bigwedge \mathcal{M}_{U,x}^{\op} \otimes \bigwedge \mathcal{M}_{U,x}^{\op} & \xrightarrow{\chi(f_x)} & \bigwedge \mathcal{M}_{U,x}^{\op}
\end{array}
\]

is commutative for all \( x \in U \). Therefore the composite

\[
\bigwedge \mathcal{Z}_U^{\op} \otimes \bigwedge \mathcal{Z}_U \xrightarrow{\chi(\Lambda)} \bigwedge \mathcal{Z}_U^{\op} \to \bigwedge \mathcal{M}_U^{\op}
\]

factors through the surjection \( \bigwedge \mathcal{Z}_U^{\op} \otimes \bigwedge \mathcal{Z}_U \to \bigwedge \mathcal{M}_U^{\op} \otimes \bigwedge \mathcal{M}_U^{\op} \) and induces the morphism \( \bigwedge \mathcal{M}_U^{\op} \otimes \bigwedge \mathcal{M}_U \to \bigwedge \mathcal{M}_U^{\op} \) as desired.

Definition 6.12. The restriction of \( \chi \) to \( \bigwedge \mathcal{M}_X^{\op} \otimes \bigwedge \mathcal{M}_X^{\op} \) gives us a morphism \( \bigwedge \mathcal{M}_X^{\op} \otimes \bigwedge \mathcal{M}_X^{\op} \to \bigwedge^{r+s-e} \mathcal{M}_X^{\op} \) by definition. Therefore, the morphism \( \chi \) induces a morphism

\[
(a_r)_* \varepsilon_r \otimes (a_s)_* \varepsilon_s \to (a_{r+s-e})_* \varepsilon_{r+s-e}
\]

via the isomorphism (6.2). It is also denoted by \( \chi \) by abuse of the notation.

Remark 6.13. The equalities

\[
\chi \cdot (\delta_i \otimes \text{id}) = (-1)^{|i|-k} \chi \cdot (\text{id} \otimes \delta_i) = \delta_i \cdot \chi
\]

can be easily checked for \( i = 1, 2, \ldots, k \).

Lemma 6.14. For \( r, s \in \mathbb{Z}_{\geq} \) and for \( p, q \in \mathbb{Z} \), we define a morphism of \( \mathcal{O}_X \)-modules

\[
((a_r)_* \varepsilon_r \otimes \mathcal{O}_X^p (\log \mathcal{M}_X)) \otimes_C ((a_s)_* \varepsilon_s \otimes \mathcal{O}_X^q (\log \mathcal{M}_X))
\]

\[
\to (a_{r+s-e})_* (\varepsilon_{r+s-e} \otimes \mathcal{O}_X^{p+q}_{X_{r+s-e}} (\log \mathcal{M}_{X_{r+s-e}}))
\]

as the composite of the three morphisms, the isomorphism

\[
((a_r)_* \varepsilon_r \otimes \mathcal{O}_X^p (\log \mathcal{M}_X)) \otimes_C ((a_s)_* \varepsilon_s \otimes \mathcal{O}_X^q (\log \mathcal{M}_X))
\]

\[
\simeq ((a_r)_* \varepsilon_r \otimes (a_s)_* \varepsilon_s) \otimes_C (\mathcal{O}_X^p (\log \mathcal{M}_X) \otimes_C \mathcal{O}_X^q (\log \mathcal{M}_X))
\]
exchanging the middle terms, the morphism $\nabla \otimes \wedge$, and the surjection (6.4) for $r + s = e \in \mathbb{Z}_{\geq e}$. Then this morphism factors through the surjection

$$((a_r)_*(\varepsilon_r \otimes \Omega^p_{X_r}(\log \mathcal{M}_X))) \otimes_\mathbb{C} ((a_s)_*(\varepsilon_s \otimes \Omega^q_{X_s}(\log \mathcal{M}_X))) \to (a_{r+s-e})_*(\varepsilon_{r+s-e} \otimes \Omega^{p+q}_{X_{r+s-e}}(\log \mathcal{M}_{X_{r+s-e}}))$$

induced from the surjections (6.4) for $r$ and $s$.

Proof. Similar to the proof of Lemmas 6.6 and 6.11. □

**Definition 6.15.** From the lemma above, the morphism (6.12) induces a morphism

$$(a_r)_*(\varepsilon_r \otimes \Omega^p_{X_r}(\log \mathcal{M}_X))) \otimes_\mathbb{C} ((a_s)_*(\varepsilon_s \otimes \Omega^q_{X_s}(\log \mathcal{M}_X))) \to (a_{r+s-e})_*(\varepsilon_{r+s-e} \otimes \Omega^{p+q}_{X_{r+s-e}}(\log \mathcal{M}_{X_{r+s-e}}))$$

for $r, s \in \mathbb{Z}_{\geq e}$ and for $p, q \in \mathbb{Z}$, which is denoted by $\nabla \otimes \wedge$ by abuse of the notation.

**Definition 6.16.** A morphism of $\mathbb{C}$-sheaves

$$\tau: \mathcal{C}(\Omega_{X_r}(\log \mathcal{M}_{X_r}))^p \otimes_\mathbb{C} \mathcal{C}(\Omega_{X_s}(\log \mathcal{M}_{X_s}))^q \to \mathcal{C}(\Omega_{X_r}(\log \mathcal{M}_{X_r}))^{p+q}$$

for $p, q \in \mathbb{Z}$ is defined by

$$\tau = (-1)^{(|s|-k)(p-|r|+k)} \prod_{i=1}^k \frac{(r_i - 1)!(s_i - 1)!}{(r_i + s_i - 1)!} \nabla \otimes \wedge$$

on the direct summand

$$(a_r)_*(\varepsilon_r \otimes \Omega^{p-|r|+k}_{X_r}(\log \mathcal{M}_{X_r})) \otimes_\mathbb{C} (a_s)_*(\varepsilon_s \otimes \Omega^{q-|s|+k}_{X_s}(\log \mathcal{M}_{X_s}))$$

of $\mathcal{C}(\Omega_{X_r}(\log \mathcal{M}_{X_r}))^p \otimes_\mathbb{C} \mathcal{C}(\Omega_{X_s}(\log \mathcal{M}_{X_s}))^q$. We can check that these morphisms define a morphism of complexes of $\mathbb{C}$-sheaves

$$\tau: \mathcal{C}(\Omega_{X_r}(\log \mathcal{M}_{X_r})) \otimes_\mathbb{C} \mathcal{C}(\Omega_{X_s}(\log \mathcal{M}_{X_s})) \to \mathcal{C}(\Omega_{X_r}(\log \mathcal{M}_{X_r}))$$

by the direct computation using (6.11). The inclusion

$$\tau(\delta W_a \mathcal{C}(\Omega_{X_r}(\log \mathcal{M}_{X_r})) \otimes_\mathbb{C} \delta W_b \mathcal{C}(\Omega_{X_s}(\log \mathcal{M}_{X_s})) \subset \delta W_{a+b} \mathcal{C}(\Omega_{X_r}(\log \mathcal{M}_{X_r}))$$

for all $a, b \in \mathbb{Z}$ can be easily checked from the definition above.

**Remark 6.17.** Direct computation using (2.7) shows that $\tau$ is compatible with the isomorphism

$$\mathcal{C}(\Omega_{X_r}(\log \mathcal{M}_{X_r})) \otimes_\mathbb{C} \mathcal{C}(\Omega_{X_s}(\log \mathcal{M}_{X_s})) \cong \mathcal{C}(\Omega_{X_r}(\log \mathcal{M}_{X_r})) \otimes_\mathbb{C} \mathcal{C}(\Omega_{X_s}(\log \mathcal{M}_{X_s}))$$

exchanging the left and right hand sides defined in 2.18.

**Assumption 6.18.** In the remainder of this section, $X$ is assumed to be of pure dimension.

**6.19.** We consider the $E_1$-terms of the spectral sequence

$$E_1^{p,q}(R\Gamma_c(X, \mathcal{C}(\Omega_{X_r}(\log \mathcal{M}_{X_r}))), \delta W).$$

By (6.10), we have

$$E_1^{p,q}(R\Gamma_c(X, \mathcal{C}(\Omega_{X_r}(\log \mathcal{M}_{X_r}))), \delta W) \cong \bigoplus_{r \in \mathbb{Z}_{\geq e}} H^{p+q-|r|+k}(X_r, \varepsilon_r \otimes \text{Gr}^{W}_{-p+|r|-k} \Omega_{X_r}(\log \mathcal{M}_{X_r}))$$

(6.14)

for all $p, q$ because $a_r$ is a finite morphism.
The following lemma is a special case of [9, Lemma 3.23].

Lemma 6.20. For \( r \in \mathbb{Z}_{\geq e}^k \), there exists an isomorphism of complexes of \( \mathbb{C} \)-sheaves

\[
\bigoplus_{l=0}^{m} (i_1)_* (i_l^{-1} (\varepsilon_r \otimes_{\mathbb{Z}} L_r) \otimes_{\mathbb{Z}} \varepsilon^l \otimes_{\mathbb{Z}} \Omega_{\widetilde{D}_r} [-m]) \overset{\sim}{\longrightarrow} \varepsilon_r \otimes_{\mathbb{Z}} \operatorname{Gr}_m^W \Omega_{X_r} (\log \mathcal{M}_{X_r})
\]  

(6.15)

where \( \widetilde{D}_r \), \( i_l \) and \( \varepsilon^l \) are defined in [3, (3.1.4)].

6.21. The composite of the inverse of (6.15) for \( m = |r| \), the projection for \( l = 0 \) and \( \vartheta_r \otimes \text{id} \) gives us a morphism

\[
(\alpha_r)_* (\varepsilon_r \otimes_{\mathbb{Z}} \operatorname{Gr}_r^W \Omega_{X_r} (\log \mathcal{M}_{X_r})) \longrightarrow \Omega_{X_r} [-|r|].
\]

Then a morphism

\[
(\alpha_r)_* (\varepsilon_r \otimes_{\mathbb{Z}} \operatorname{Gr}_r^W \Omega_{X_r} (\log \mathcal{M}_{X_r})) \longrightarrow (\alpha_r)_* \Omega_{X_r} [-|r|]
\]

is obtained. By taking the direct sum for all \( r \in \mathbb{Z}_{\geq e}^k \), a morphism

\[
\operatorname{Gr}_k^{\mathbb{R} W} C (\Omega_{X_r} (\log \mathcal{M}_{X_r})) \longrightarrow \bigoplus_{r \in \mathbb{Z}_{\geq e}^k} (\alpha_r)_* \Omega_{X_r} [-2|r| + k]
\]

(6.16)

is obtained by (6.10). Similarly, the composite of the inverse of (6.15) for \( m = |r| + 1 \), the projection for \( l = 1 \) and \( i^{-1} (\vartheta_r \otimes \text{id}) \) induces a morphism

\[
\operatorname{Gr}_{k+1}^{\mathbb{R} W} C (\Omega_{X_r} (\log \mathcal{M}_{X_r})) \longrightarrow \bigoplus_{r \in \mathbb{Z}_{\geq e}^k} (\alpha_r \cdot i)_* (\varepsilon \otimes_{\mathbb{Z}} \Omega_{\widetilde{D}_r} [-2|r| + k - 1])
\]

(6.17)

by (6.10) again, where we use \( \widetilde{D}_r, i \) and \( \varepsilon \) instead of \( \widetilde{D}_r, i_1 \) and \( \varepsilon^1 \) in (6.15) respectively.

Lemma 6.22. We have the isomorphisms

\[
E_{-1}^{-k, 2 \dim X + 2k} (R \Gamma_c (X, C (\Omega_{X_r} (\log \mathcal{M}_{X_r}))), \delta W) \simeq \bigoplus_{r \in \mathbb{Z}_{\geq e}^k} H_{-1}^{2 \dim X_r} (X_r, \Omega_{X_r})
\]

(6.18)

\[
E_{-1}^{-k-1, 2 \dim X + 2k} (R \Gamma_c (X, C (\Omega_{X_r} (\log \mathcal{M}_{X_r}))), \delta W) \simeq \bigoplus_{r \in \mathbb{Z}_{\geq e}^k} H_{-1}^{2 \dim X_r - 2} (\widetilde{D}_r, \varepsilon \otimes_{\mathbb{Z}} \Omega_{\widetilde{D}_r})
\]

(6.19)

induced from the morphisms (6.16) and (6.17) respectively.

Proof. Combining (6.14) and (6.15), we have

\[
E_{-1}^{-k-1, 2 \dim X + 2k} (R \Gamma_c (X, C (\Omega_{X_r} (\log \mathcal{M}_{X_r}))), \delta W)
\]

\[
\simeq \bigoplus_{r \in \mathbb{Z}_{\geq e}^k} \bigoplus_{l=0}^{|r|+1} H_{-1}^{2 \dim X_r - 2|r| + 2k - 2} (\widetilde{D}_r, i_l^{-1} (\varepsilon_r \otimes_{\mathbb{Z}} \bigwedge L_r) \otimes_{\mathbb{Z}} \varepsilon^l \otimes \Omega_{\widetilde{D}_r} [-l]).
\]

Because the inequalities \( 2 \dim X - 2|r| + 2k - 2 \leq 2 \dim \widetilde{D}_r = 2 \dim X - 2|r| + 2k - 2l \) implies \( l \leq 1 \), we obtain (6.19) from the equality rank \( L_r = |r| \). Similar argument shows (6.18).

Lemma 6.23. Let \( X = \bigcup_{a \in A} V_a \) be an open covering of \( X \). Then the canonical morphism

\[
\bigoplus_{a \in A} E_{-1}^{-k-1, 2 \dim X + 2k} (R \Gamma_c (V_a, C (\Omega_{X_a} (\log \mathcal{M}_{X_a})))), \delta W)
\]

\[
\longrightarrow E_{-1}^{-k-1, 2 \dim X + 2k} (R \Gamma_c (X, C (\Omega_{X_r} (\log \mathcal{M}_{X_r})))), \delta W)
\]

(6.20)

is surjective.
**Proof.** By the isomorphisms (6.19) for $X$ and $V_\alpha$, the morphism (6.20) is identified with the direct sum of the morphisms

$$
\bigoplus_{\alpha \in A} \mathcal{H}^{2 \dim D_r(i^{-1}(V_\alpha), \varepsilon \otimes \Omega_{D_r})} \longrightarrow \mathcal{H}^{2 \dim D_r(\widetilde{D}_r, \varepsilon \otimes \Omega_{D_r})},
$$

(6.21)

induced from the surjection $\bigoplus_{\alpha \in A} (j_\alpha)_! (\varepsilon \otimes \Omega_{D_r})|_{i^{-1}(V_\alpha)} \longrightarrow \varepsilon \otimes \Omega_{\widetilde{D}_r}$, where $j_\alpha$ denotes the open immersion $i^{-1}(V_\alpha) \hookrightarrow D_r$ for every $\alpha \in A$. Therefore the morphism (6.21) is surjective. \hfill $\Box$

**Definition 6.24.** We set $\varepsilon(a) = (-1)^{a(a-1)/2}$ for $a \in \mathbb{Z}$ as in [16, (3.3)] and [22, I-14].

**Definition 6.25.** A morphism $\Theta: E_1^{k,2 \dim X+2k}(R\Gamma_c(X, C(\Omega_{X*}(\log \mathcal{M}_{X*}))), \delta W) \longrightarrow \mathbb{C}$ is defined by

$$
\Theta = \bigoplus_{r \in \mathbb{Z}_{\geq e}^k} \varepsilon(|r| - k)(2\pi \sqrt{-1})^{e-k} \int_{X_r} \bigoplus_{r \in \mathbb{Z}_{\geq e}^k} \mathcal{H}^{2 \dim X_r(X_r, \Omega_{X_r})} \longrightarrow \mathbb{C}
$$

via the isomorphism (6.18).

**Lemma 6.26.** $\Theta \cdot d_1 = 0$, where $d_1$ is the morphism of $E_1$-terms of the spectral sequence (6.13).

**Proof.** In this proof, we use $E_1^{p,q}$ instead of $E_1^{p,q}(R\Gamma_c(X, C(\Omega_{X*}(\log \mathcal{M}_{X*}))), \delta W)$ for short. Since Lemma 6.23 enables us to compute $\Theta \cdot d_1$ locally, we may work on a local model $(U, \mathcal{M}_U)$ in 3.7. We use the notation in 3.7. In addition, we fix an total order on $\Lambda$ and identify $\varepsilon(\Lambda)$ with $\mathbb{Z}$ by fixing the base $e_{\lambda_1} \wedge e_{\lambda_2} \wedge \cdots \wedge e_{\lambda_m}$ for $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ with $\lambda_1 < \lambda_2 < \cdots < \lambda_m$. Then

$$
E_1^{k,2 \dim U+2k} \simeq \bigoplus_{r \in \mathbb{Z}_{\geq e}^k} \mathcal{H}^{2 \dim D[\lambda]}(D[\lambda], \Omega_{D[\lambda]})
$$

and

$$
E_1^{k-1,2 \dim U+2k} \simeq \bigoplus_{r \in \mathbb{Z}_{\geq e}^k} \mathcal{H}^{2 \dim D[\lambda]-2}(D[\lambda] \cap D_\lambda, \Omega_{D[\lambda]\cap D_\lambda})
$$

by (6.18) and (6.19). We fix $r \in \mathbb{Z}_{\geq e}^k, \lambda \in S_r(\Lambda)$ and $\lambda \in \Lambda \setminus \lambda$. For any $\omega \in \mathcal{H}^{2 \dim D[\lambda]-2}(D[\lambda] \cap D_\lambda, \Omega_{D[\lambda]\cap D_\lambda})$

we have

$$
d_1(\omega) = (-1)^{|r|-k+j}\gamma(\omega) + (-1)^j\omega
$$

$$
eq \mathcal{H}^{2 \dim D[\lambda]}(D[\lambda], \Omega_{D[\lambda]}) \oplus \mathcal{H}^{2 \dim D[\lambda]-2}(D[\lambda] \cap D_\lambda, \Omega_{D[\lambda]\cap D_\lambda})
$$

as in the proof of [10, Lemma 7.10], where $\gamma$ denotes the Gysin morphism for $D[\lambda] \cap D_\lambda$ in $D[\lambda]$ as in [10, 4.2] and where $\Lambda \cup \{\lambda\} = \{\lambda_0, \lambda_1, \ldots, \lambda = \lambda_j, \ldots, \lambda_m\}$ with $\lambda_0 < \lambda_1 < \cdots < \lambda_m$. Then

$$
\Theta(d_1(\omega)) = (-1)^{|r|-k+j}\varepsilon(|r| - k)(2\pi \sqrt{-1})^{e-k} \int_{D[\lambda]} \gamma(\omega)
$$

$$
+ (-1)^j\varepsilon(|r| - k + 1)(2\pi \sqrt{-1})^{e-k+1} \int_{D[\lambda]\cap D_\lambda} \omega
$$

$$
= (-1)^{|r|-k+j}\varepsilon(|r| - k)(2\pi \sqrt{-1})^{e-k} \int_{D[\lambda]} \gamma(\omega) + (2\pi \sqrt{-1}) \int_{D[\lambda]\cap D_\lambda} \omega
$$

$$
= 0
$$

by $\varepsilon(a + 1) = (-1)^a \varepsilon(a)$ and by [15, §2 (b)]. \hfill $\Box$
7 Products

In this section, we construct two products; one is the morphism $A_C \otimes_C \Omega_X \to A_C$ in Definition 7.1 and the other $A_C \otimes_A A_C \to C(\Omega_X, (\log \mathcal{M}_X))[k]$ in Definition 7.6. The construction of the first one is straightforward. To define the second, we use the morphism $\tau$ on $C(\Omega_X, (\log \mathcal{M}_X))$.

**Definition 7.1.** Morphisms of $\mathbb{C}$-sheaves given by

$$(\mathbb{C}[u] \otimes_C \Omega_X^p(\log \mathcal{M}_X)) \otimes_C \Omega_X^q \ni (P \otimes \omega) \otimes \eta \mapsto P \otimes \omega \wedge \eta \in \mathbb{C}[u] \otimes_C \Omega_X^{p+q}(\log \mathcal{M}_X)$$

for all $p, q$ define a morphism of complexes $(\mathbb{C}[u] \otimes_C \Omega_X(\log \mathcal{M}_X)) \otimes_C \Omega_X \to \mathbb{C}[u] \otimes_C \Omega_X(\log \mathcal{M}_X)$, which sends $W(I)_m(\mathbb{C}[u] \otimes_C \Omega_X(\log \mathcal{M}_X)) \otimes_C \Omega_X$ and $L(I)_m(\mathbb{C}[u] \otimes_C \Omega_X(\log \mathcal{M}_X)) \otimes_C \Omega_X$ to $W(I)_m(\mathbb{C}[u] \otimes_C \Omega_X(\log \mathcal{M}_X))$ and $L(I)_m(\mathbb{C}[u] \otimes_C \Omega_X(\log \mathcal{M}_X))$ for all $I \subset \{1, 2, \ldots, k\}$ and $m \in \mathbb{Z}$ respectively. Thus a morphism of complexes

$$\overline{\Psi} : A_C \otimes_C \Omega_X \to A_C \quad (7.1)$$

is induced. This morphism satisfies $\overline{\Psi}(L(I)_mA_C \otimes_C \Omega_X) \subset L(I)_mA_C$ for all $I \subset \{1, 2, \ldots, k\}$ and $m \in \mathbb{Z}$. The morphism $Gr^L_{m} A_C \otimes_C \Omega_X \to Gr^L_{m} A_C$ induced from $\overline{\Psi}$ is denoted by $Gr^L_{m} \overline{\Psi}$. The morphism $\overline{\Psi}$ induces a morphism

$$H^{a,b}(X, \overline{\Psi}) : H^a(X, A_C) \otimes_C H^b(X, \Omega_X) \to H^{a+b}(X, A_C)$$

as in Definition 2.19. For $\omega \in H^a(X, A_C)$ and $\eta \in H^b(X, \Omega_X)$, the element $H^{a,b}(X, \overline{\Psi})(\omega \otimes \eta) \in H^{a+b}(X, A_C)$ is simply denoted by $\omega \cup \eta$ and called the cup product of $\omega$ and $\eta$.

**Remark 7.2.** It is trivial that the diagram

$$\begin{array}{ccc}
\Omega_{X/\ast}(\log(\mathcal{M}_X/N^\ast)) \otimes_C \Omega_X & \longrightarrow & \Omega_{X/\ast}(\log(\mathcal{M}_X/N^\ast)) \\
\otimes \downarrow & & \downarrow \theta \\
A_C \otimes_C \Omega_X & \longrightarrow & A_C \\
\overline{\Psi} & & \\
\end{array}$$

is commutative, where the top horizontal arrow is the morphism defined in Definition 4.12, and $\theta$ is the morphism defined in (3.13). Therefore the morphism $\cup c(\mathcal{L})$ in (4.5) is identified with the morphism

$$\cup c(\mathcal{L}) : H^a(X, A_C) \to H^{a+2}(X, A_C)$$

via the isomorphisms (3.14) induced by $\theta$.

The following lemma computes $Gr^L_{m} \overline{\Psi}$ via the isomorphism (5.18).

**Lemma 7.3.** For $q, r \in N^k$ satisfying the conditions in (5.19), and for

$$u^q \otimes v \otimes \omega \in (a_r)_* (\varepsilon_r \otimes \Omega_{X/r}^{\rho-m-2|q|}), \quad \eta \in \Omega_X^k,$$

the image of $(u^q \otimes v \otimes \omega) \otimes \eta$ by the morphism $Gr^L_{m} \overline{\Psi}$ via the identification (5.18) is

$$u^q \otimes v \otimes \omega \wedge (a_r)^* \eta \in (a_r)_* (\varepsilon_r \otimes \Omega_{X/r}^{\rho+q-m-2|q|}).$$

**Proof.** By the direct computation in the local situation 3.7. \hfill \Box
Definition 7.4. A morphism of \( \mathcal{O}_X \)-modules

\[
C(\text{dlog} t_i \wedge): C(\Omega^*_X, (\log \mathcal{M}_X))^n \to C(\Omega^*_X, (\log \mathcal{M}_X))^{n+1}
\]

is defined by \( C(\text{dlog} t_i \wedge) = \bigoplus_{r \in \mathbb{Z}_{\geq 0}} (-1)^{r-1}k(a_r)_*(\text{id} \otimes \text{dlog} t_i \wedge) \) for all \( n \). Then these morphisms define a morphism of filtered complexes

\[
C(\text{dlog} t_i \wedge): (C(\Omega^*_X, (\log \mathcal{M}_X)), \delta W) \to (C(\Omega^*_X, (\log \mathcal{M}_X))[1], \delta W[-1]).
\]

The equality \( C(\text{dlog} t_i \wedge) \cdot C(\text{dlog} t_j \wedge) + C(\text{dlog} t_j \wedge) \cdot C(\text{dlog} t_i \wedge) = 0 \) holds for all \( i, j \in \{1, 2, \ldots, k\} \).

Definition 7.5. For \( q \in \mathbb{N}^k \), a morphism of \( \mathcal{O}_X \)-modules

\[
\text{Res}_{q+e}: \Omega^*_X \to C(\Omega^*_X, (\log \mathcal{M}_X))^n
\]

is obtained as the composite of the morphism \( \text{Res}_{q+e} \) in (5.6) and the inclusion \( (a_{q+e})_* (\varepsilon_{q+e} \otimes \Omega^*_X) \to C(\Omega^*_X, (\log \mathcal{M}_X))^n \). By the identification in (3.10), a morphism of \( \mathcal{O}_X \)-modules

\[
\text{Res} = \bigoplus_{q \in \mathbb{N}^k} (-1)^{k|q|} \text{Res}_{q+e}: A^n_C \to C(\Omega^*_X, (\log \mathcal{M}_X))^n
\]

is defined for all \( n \in \mathbb{Z} \). Lemma 5.14 implies \( \text{Res}(L_m A_C) \subset (\delta W)_m C(\Omega^*_X, (\log \mathcal{M}_X))^n \) for all \( m \in \mathbb{Z} \).

Definition 7.6. A morphism of \( \mathbb{C} \)-sheaves \( \Psi: A^p_C \otimes \mathbb{C} A^q_A \to C(\Omega^*_X, (\log \mathcal{M}_X))^{p+q} \) is defined by

\[
\Psi = \tau \cdot (\text{Res} \otimes \text{Res}) \text{ for all } p, q.
\]

Moreover, we set

\[
\tilde{\Psi} = C(\text{dlog} t_1 \wedge) \cdots C(\text{dlog} t_k \wedge) \cdot \Psi: A^p_C \otimes \mathbb{C} A^q_C \to C(\Omega^*_X, (\log \mathcal{M}_X))^{p+q+k}
\]

for all \( p, q \).

Lemma 7.7. The morphism \( \tilde{\Psi} \) gives us a morphism of filtered complexes

\[
\tilde{\Psi}: (A_C \otimes \mathbb{C} A_C, L) \to (C(\Omega^*_X, (\log \mathcal{M}_X))[k], (\delta W)[-k]),
\]

where the filtration \( L \) on \( A_C \otimes \mathbb{C} A_C \) is defined as in Definition 2.20.

Proof. By definition, we clearly have \( \tilde{\Psi}(L_m A_C \otimes \mathbb{C} A_C)^n \subset \delta W_{m+k} C(\Omega^*_X, (\log \mathcal{M}_X))^{n+k} \) for all \( m, n \in \mathbb{Z} \). The following lemma implies that \( \tilde{\Psi} \) is a morphism of complexes. \( \square \)

Lemma 7.8. For \( r \in \mathbb{Z}^k_{\geq 0} \),

\[
\text{Res}_{r+i, e_i} \cdot \text{dlog} t_i \wedge = (-1)^{|r|+1} (a_{r+i, e_i})_* (\text{id} \otimes \text{dlog} t_i \wedge) \cdot \text{Res}_{r+i, e_i} + \delta_i \cdot \text{Res}_r
\]

for all \( i = 1, 2, \ldots, k \).

Proof. We may work in the local situation 3.7. Then the proof is similar to [10, Lemma 3.9]. \( \square \)

Remark 7.9. The composite of the canonical morphism \( \Omega_X \to \Omega_X (\log \mathcal{M}_X) \) and the morphism (3.12) gives us a morphism of complexes \( \Omega_X \to A_C \). We can easily check that the diagram

\[
\begin{array}{c}
A^p_C \otimes \Omega^q_X \xrightarrow{\tilde{\Psi}} A^{p+q}_C \\
\downarrow \quad \quad \quad \downarrow \text{Res} \\
A^p_C \otimes \mathbb{C} A^q_A \xrightarrow{\Psi} C(\Omega^*_X, (\log \mathcal{M}_X))^{p+q}
\end{array}
\]

is commutative, where the left vertical arrow is the tensor product of the identity and the morphism \( \Omega_X \to A_C \) above. However, we will not use this commutativity in this paper.
**Remark 7.10.** The morphism $\Omega_{X/\alpha}(\log(M_X/N^k)) \otimes \Omega_{X/\alpha}(\log(M_X/N^k)) \longrightarrow \Omega_{X/\alpha}(\log(M_X/N^k))$ defined by taking the wedge product is compatible with $\Psi$, that is, the diagram

$$\Omega_{X/\alpha}(\log(M_X/N^k)) \otimes \Omega_{X/\alpha}(\log(M_X/N^k)) \longrightarrow \Omega_{X/\alpha}(\log(M_X/N^k))$$

is commutative, where $\theta$ is the morphism defined in (3.13). Here we omit the proof because this fact is not needed later.

**Remark 7.11.** By Remark 6.17, $\Psi$ is compatible with the isomorphism $A_C \otimes_C A_C \simeq A_C \otimes_C A_C$ exchanging the left and right hand sides defined in 2.18.

**7.12.** Here, we compute the morphism

$$\text{Gr}^L_a A_C \otimes_C \text{Gr}^L_a A_C \longrightarrow \text{Gr}^W_a C(\Omega_{X*}(\log M_X))[k] \longrightarrow \bigoplus_{r \in \mathbb{Z}_e^k} (a_r)_* \Omega_{X*}[-2|r| + 2k] \quad (7.2)$$

given by the composition of Gr$_{a,a}$ and the morphism (6.16) shifted by $k$.

**Lemma 7.13.** For $(q, r), (q', r') \in \mathbb{N}^k \times \mathbb{N}^k$ satisfying the conditions in (5.19) for $m = -a$ and for $m = a$ respectively, the restriction of the morphism (7.2) on the direct summand

$$C u^q \otimes_C (a_r)_*(\varepsilon_r \otimes \Omega_{X*})[a - 2|q|] \otimes_C (C u^{q'} \otimes_C (a_r)_*((\varepsilon_{r'} \otimes \Omega_{X_{r'}}))[a - 2|q'|]$$

via the isomorphism (5.18) is zero unless $r = r' = q + q' + e$. For the case of $r = r' = q + q' + e$, it coincides with the composite of the following five morphisms of complexes; the isomorphism

$$C u^q \otimes_C (a_r)_*(\varepsilon_r \otimes \Omega_{X*})[a - 2|q|] \otimes_C C u^{q'} \otimes_C (a_r)_*((\varepsilon_{r'} \otimes \Omega_{X_{r'}}))[a - 2|q'|] \simeq ((a_r)_*(\varepsilon_r \otimes \Omega_{X*}) \otimes_C (a_r)_*((\varepsilon_{r'} \otimes \Omega_{X_{r'}}))[a - 2|q| + 2k]$$

given by (2.8), the canonical morphism shifted by $-2|r| + 2k$

$$(a_r)_*((\varepsilon_r \otimes \Omega_{X*}) \otimes_C (a_r)_*((\varepsilon_{r'} \otimes \Omega_{X_{r'}}))[a - 2|q| + 2k] \longrightarrow (a_r)_*((\varepsilon_r \otimes \Omega_{X*}) \otimes_C (a_r)_*((\varepsilon_{r'} \otimes \Omega_{X_{r'}}))[a - 2|q| + 2k],$$

the isomorphism

$$(a_r)_*((\varepsilon_r \otimes \Omega_{X*}) \otimes_C (\varepsilon_{r'} \otimes \Omega_{X_{r'}}))[a - 2|q| + 2k] \simeq (a_r)_*(\varepsilon_r \otimes \varepsilon_{r'} \otimes \Omega_{X*} \otimes C \Omega_{X_{r'}})[a - 2|q| + 2k]$$

induced by exchanging the middle terms, the morphism

$$(-1)^{|q|}(a_r)_*(\theta_r \otimes \wedge)[-2|r| + 2k]$$

: $$(a_r)_*(\varepsilon_r \otimes \varepsilon_{r'} \otimes \Omega_{X*} \otimes C \Omega_{X_{r'}})[a - 2|q| + 2k] \longrightarrow (a_r)_* \Omega_{X*}[-2|r| + 2k],$$

and the inclusion $(a_r)_* \Omega_{X*}[-2|r| + 2k], \simeq \bigoplus_{s \in \mathbb{Z}_e^k} (a_s)_* \Omega_{X*}[-2|s| + 2k]$.

**Proof.** Since the question is of local nature, we can apply the same argument as in the proof of [10, Lemma 6.13].
8 A bilinear form on $V_C$

In this section, we define a bilinear form on $V_C = \bigoplus_{a,b} E_1^{a,b}(A_C, L)$ by using the product $\tilde{\Psi}$ and the morphism $\Theta$ constructed in Sections 7 and 6 respectively. Then we can check that $E_1(A_C, L)$ satisfies the conditions to be a polarized differential $\mathbb{Z} \oplus \mathbb{Z}^k$-graded Hodge-Lefschetz module, which will be introduced in the next section.

**Assumption 8.1.** In this section, the semistable log smooth degeneration $f : (X, M_X) \longrightarrow (\ast, \mathbb{N}^k)$ is assumed to be projective and $X$ to be of pure dimension.

**Definition 8.2.** A finite dimensional filtered $\mathbb{C}$-vector space $(V_C, F)$ is defined by

$$(V_C, F) = \bigoplus_{a,b \in \mathbb{Z}} (E_1^{a,b}(A_C, L), F).$$

The direct sum of the morphisms $d_1$ of the $E^1$-terms gives us an endomorphism of $(V_C, F)$ denoted by the same letter $d_1$. Moreover, we set

$$V_Q = \text{Image}(\bigoplus_{a,b \in \mathbb{Z}} E_1^{a,b}(\alpha) : \bigoplus_{a,b \in \mathbb{Z}} E_1^{a,b}(A_Q, L) \longrightarrow \bigoplus_{a,b \in \mathbb{Z}} E_1^{a,b}(A_C, L) = V_C),$$

which is a finite dimensional $\mathbb{Q}$-subspace of $V_C$ with the property $\mathbb{C} \otimes \mathbb{Q} V_Q = V_C$. By definition, $V_Q$ is preserved by $d_1$.

**8.3.** By (5.18),

$$(V_C, F) \simeq \bigoplus \mathbb{C} u^q \otimes \mathbb{C} (H^j(X_r, \varepsilon_r \otimes \Omega_{X_r}), F[-|r| + |q| + k])$$

where the direct sum is taken over the index set

$$\{(q, r, j) \in \mathbb{N}^k \times \mathbb{N}^k \times \mathbb{Z} \mid r \geq q + e\}$$

and the filtration $F$ on the right hand side is the usual Hodge filtration on $H^j(X_r, \varepsilon_r \otimes \Omega_{X_r})$. In particular,

$$V_C = \bigoplus_{r \in \mathbb{N}^k, j \in \mathbb{Z}} \mathbb{C}[u]/(u_1^{r_1}, \ldots, u_k^{r_k}) \otimes \mathbb{C} H^j(X_r, \varepsilon_r \otimes \Omega_{X_r}),$$

(8.2)

as $\mathbb{C}$-vector spaces.

**Definition 8.4.** Under the identification (8.1), a filtered $\mathbb{C}$-subspace $(V_C^{j_0, j}, F)$ of $(V_C, F)$ is defined by

$$(V_C^{j_0, j}, F) \simeq \bigoplus_{-r + 2q + e = j} \mathbb{C} u^q \otimes \mathbb{C} (H^{j_0 + \dim X - |r| + k}(X_r, \varepsilon_r \otimes \Omega_{X_r}), F[-|r| + |q| + k])$$

(8.3)

for $j_0 \in \mathbb{Z}$ and for $j \in \mathbb{Z}^k$. Moreover, a $\mathbb{Q}$-subspace $V_Q^{j_0, j}$ of $V_Q$ is defined by $V_Q^{j_0, j} = V_Q \cap V_C^{j_0, j}$.

**Remark 8.5.** By definition, we have

$$(V_C, F) = \bigoplus_{j_0 \in \mathbb{Z}, j \in \mathbb{Z}^k} (V_C^{j_0, j}, F), \quad (E_1^{a,b}(A_C, L), F) = \bigoplus_{|j| = a} (V_C^{a+b-\dim X, j}, F).$$

(8.4)
Moreover, \((V_{\mathbb{Q}}^{j_0,j}, (V_{\mathbb{Q}}^{j_0,j}, F))\) is a \(\mathbb{Q}\)-Hodge structure of weight \(j_0 - |j| + \dim X\). In fact, we have an identification as \(\mathbb{Q}\)-Hodge structures

\[
V_{\mathbb{Q}}^{j_0,j} \simeq \bigoplus_{-r+2q+e=j} \mathbb{Q}u^q \otimes_{\mathbb{Q}} H^{n+\dim X-|r|+k}(X_r, \varepsilon_r \otimes \mathbb{Q})(-|r| + |q| + k) \quad (8.5)
\]

by the canonical quasi-isomorphism \(\mathbb{Q} \simeq \text{Kos}_{X_r}(O_{X_r})\) as in (3.14.4) (cf. [9, Corollary 1.15]) and by (5.15), where \((-|r| + |q| + k)\) stands for the Tate twist as usual.

**Remark 8.6.** Let \(I \subset \{1,2,\ldots, k\}\). The filtration \(L(I)\) on \(A_C\) induces the filtration \(L(I)\) on \(E_{i}^{a,b}(A_C, L)\). Thus we obtain a filtration \(L(I)\) on \(V_C\). We have \(L(I)_i V_C = \bigoplus_{|j| \geq -l} V_{\mathbb{Q}}^{j_0,j}\) by Lemma 5.16. By definition \(d_1\) preserves \(L(I)\) for all \(I\).

**Lemma 8.7.** There exist the unique endomorphisms \(d'_i\) of \((V_C, F)\) for \(i = 1,2,\ldots, k\) such that

\[
d_1 = \sum_{i=1}^{k} d'_i \quad (8.7.1)
\]

\[
d'_i(V_{\mathbb{Q}}^{j_0,j}) \subset V_{\mathbb{Q}}^{j_0+j+e_i} \quad \text{for all } j_0 \in \mathbb{Z} \text{ and } j \in \mathbb{Z}^k. \quad (8.7.2)
\]

They satisfy \(d'_i d'_i + d'_j d'_j = 0\) for all \(i, j \in \{1,2,\ldots, k\}\). Moreover, they preserve the subspace \(V_Q\) and \(d'_i : V_{\mathbb{Q}}^{j_0,j} \rightarrow V_{\mathbb{Q}}^{j_0+j+e_i}\) is a morphism of \(\mathbb{Q}\)-Hodge structures for all \(i \in \{1,2,\ldots, k\}\).

**Proof.** Since \(d_1\) preserves the filtration \(L(I)\) for all \(I\), we have \(d_1(V_{\mathbb{Q}}^{j_0,j}) \subset \bigoplus_{|j'| = |j| + 1, j' \geq j} V_{\mathbb{Q}}^{j_0+j'}\) by the second equality of (8.4). The conditions \(j' \geq j\) and \(|j'| = |j| + 1\) imply \(j' = j + e_i\) for some \(i \in \{1,2,\ldots, k\}\), Thus we obtain the unique morphisms \(d'_i\) satisfying (8.7.1) and (8.7.2). Because \(d_1\) preserves \(V_Q\) and \(F\), then so does \(d'_i\) for each \(i \in \{1,2,\ldots, k\}\). Therefore \(d'_i : V_{\mathbb{Q}}^{j_0,j} \rightarrow V_{\mathbb{Q}}^{j_0+j+e_i}\) is a morphism of \(\mathbb{Q}\)-Hodge structures. The equality \(d_1^2 = 0\) implies \(d'_i d'_j + d'_j d'_i = 0\) for all \(i, j \in \{1,2,\ldots, k\}\).

**Definition 8.8.** The morphism \(\nu_i : (A_C, L, F) \rightarrow (A_C, L[2], F[-1])\) induces a morphism

\[
E_r(\nu_i) : (E_{i}^{a,b}(A_C, L), F) \rightarrow (E_{r}^{a+2k-2}(A_C, L), F[-1])
\]

for \(i = 1,2,\ldots, k\). By taking direct sum for all \(a, b \in \mathbb{Z}\), we obtain \(E_1(\nu_i) : (V_C, F) \rightarrow (V_C, F[-1])\). We set \(l_i = (2\pi \sqrt{-1})E_1(\nu_i)\) for \(i = 1,2,\ldots, k\).

**Lemma 8.9.** The following holds:

\[
l_i(V_Q) \subset V_Q \quad \text{for all } i \in \{1,2,\ldots, k\}. \quad (8.9.1)
\]

\[
l_i(V_{\mathbb{C}}^{j_0,j}) \subset V_{\mathbb{C}}^{j_0+j+2e_i} \quad \text{for all } i \in \{1,2,\ldots, k\}. \quad (8.9.2)
\]

\[
l_i : (V_{\mathbb{Q}}^{j_0,j}, (V_{\mathbb{Q}}^{j_0,j}, F)) \rightarrow (V_{\mathbb{C}}^{j_0+j+2e_i}, (V_{\mathbb{C}}^{j_0+j+2e_i}, F[-1])) \quad \text{is a morphism of } \mathbb{Q}\text{-Hodge structures}. \quad (8.9.3)
\]

\[
l_i d'_j = d'_j l_i \quad \text{for all } i,j \in \{1,2,\ldots, k\}. \quad (8.9.4)
\]

\[
l_i l_j = l_i l_j \quad \text{for all } i,j \in \{1,2,\ldots, k\}. \quad (8.9.5)
\]

\[
l_i j_0 \in \mathbb{Z}^k \quad \text{with } j_i > 0, \text{ and } j_0 \in \mathbb{Z}, \text{ the morphism } l_i : V_{\mathbb{Q}}^{j_0-j} \rightarrow V_{\mathbb{Q}}^{j_0-j-2j_0-e_i} \text{ is an isomorphism}. \quad (8.9.6)
\]

**Proof.** (8.9.1) follows from (4.3). By definition, \(E_1(\nu_i)\) is identified with \(\bigoplus (u_i) \otimes \text{id}\) via the isomorphism (8.2). where \((u_i)\) denotes the morphism defined by the multiplication by \(u_i\) in \(\mathbb{C}[u]\). Therefore we obtain (8.9.2), (8.9.3), (8.9.5) and (8.9.6). Since \(E_1(\nu_i)\) commutes with \(d_1\) by definition, we obtain (8.9.4).

\[
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\]
Notation 8.10. We take an ample invertible sheaf $\mathcal{L}$ on $X$. Then the cohomology class $c(\mathcal{L}) \in H^2(X, \Omega_X)$ is defined in 4.11. For any $r \in \mathbb{Z}_{\geq 0}$, we set $L_r = a_r^* \mathcal{L}$, which is an ample invertible sheaf on $X_r$ because $a_r : X_r \to X$ is finite. Moreover the usual first Chern class $c_1(L_r) \in H^2(X_r, \mathbb{Z})$ is sent to $-2(2\pi \sqrt{-1})^{-1}a_r^*c(\mathcal{L})$ by the morphism induced from $\mathbb{Z} \hookrightarrow \mathbb{C} \simeq \Omega_{X_r}$ as in [3, (2.2.5)]. We usually identify $c_1(L_r)$ and $-2(2\pi \sqrt{-1})^{-1}a_r^*c(\mathcal{L})$ in $H^2(X_r, \Omega_{X_r})$.

Definition 8.11. The morphism $\tilde{\Psi}$ in (7.1) induces a morphism

$$E_r(\tilde{\Psi}) : E^a_b(A_C, L) \otimes_{\mathbb{C}} H^d(X, \Omega_X) \to E^{a,b+d}_r(A_C, L)$$

as in Definition 2.21, where $\Omega_X$ is equipped with the trivial filtration. By using $c(\mathcal{L}) \in H^2(X, \Omega_X)$ above, a morphism $l_0 : E^a_b(A_C, L) \to E^{a,b+d}_r(A_C, L)$ is defined by $l_0(\omega) = -(2\pi \sqrt{-1})^{-1}E_1(\tilde{\Psi})(\omega \otimes c(\mathcal{L}))$ for $\omega \in E^{a,b}_r(A_C, L)$.

Lemma 8.12. We have the following:

(8.12.1) $l_0(V_0) \subseteq V_0$.
(8.12.2) $l_0(V^j_0 \cap V^j_0) \subseteq V^j_0 + 2j$ for all $p, j_0 \in \mathbb{Z}$, $j \in \mathbb{Z}^k$.
(8.12.3) $l_0 : (V^j_0, (V^j_0, F)) \to (V^j_0 + 2j, (V^j_0 + 2j, F[1]))$ is a morphism of $\mathbb{Q}$-Hodge structures.
(8.12.4) $l_0d_j^i = d^i_0l_0$ for all $i \in \{1, 2, \ldots, k\}$.
(8.12.5) $l_0d_j = d^i_j$ for all $i \in \{1, 2, \ldots, k\}$.
(8.12.6) $l_0^0 : V^j_0 \to V^j_0$ is an isomorphism for all $j_0 \in \mathbb{Z}_{>0}$ and $j \in \mathbb{Z}^k$.

Proof. Under the identification (8.2), $l_0$ is identified with $\text{id} \otimes \cup c_1(L_r)$ by Lemma 7.3, where $\cup$ denotes the cup product induced from the morphism $\varepsilon_r \otimes_{\Omega_{X_r}} \Omega_{X_r} \to \varepsilon_r \otimes_{\Omega_{X_r}} \Omega_{X_r}$. Then (8.12.2) and (8.12.5) are trivial. The commutativity of $l_0$ with $d_1$ by (2.9) together with (8.12.2) implies (8.12.4). Because $c_1(L_r) \in H^2(X_r, \mathbb{Z})$, we obtain (8.12.1) via the identification (8.5). From the Hodge theory for $H^*(X_r, \mathbb{C}) \otimes_{\mathbb{Z}} \Omega_{X_r}$ we obtain (8.12.3) and (8.12.6). Here we note that dim $X_r = \text{dim } X - |r| + k$.

Definition 8.13. As in Definition 2.21, the morphism $\tilde{\Psi}$ induces a morphism

$$E_r(\tilde{\Psi}) : E^a_b(A_C, L) \otimes_{\mathbb{C}} E^c_d(A_C, L) \to E^{a+c-k,b+d+2k}_r(C(\Omega_{X_r}(\log \mathcal{M}_{X_r})), \delta W)$$

because $E^p_q(C(\Omega_{X_r}(\log \mathcal{M}_{X_r}))[k], (\delta W)[-k]) = E^{p-k,q+2k}_r(C(\Omega_{X_r}(\log \mathcal{M}_{X_r})), \delta W)$. Then we define a morphism $S : E^{a,b}_r(A_C, L) \otimes_{\mathbb{C}} E^c_d(A_C, L) \to \mathbb{C}$ by

$$S = \begin{cases} 
\epsilon(-a - b) \Theta \cdot E_1(\tilde{\Psi}) & \text{if } a + c = 0 \text{ and } b + d = 2 \text{ dim } X \\
0 & \text{otherwise},
\end{cases}$$

where $\Theta$ is the morphism defined in Definition 6.25 and $\epsilon(-a - b)$ is given in Definition 6.24. Then $S$ induces a bilinear form $V_C \otimes V_C \to \mathbb{C}$, which is denoted by the same letter $S$.

Lemma 8.14. $S \cdot (d_1 \otimes \text{id}) = S \cdot (\text{id} \otimes d_1)$ on $E^{a,b}_r(A_C, L) \otimes_{\mathbb{C}} E^{c,d}_r(A_C, L)$

Proof. By definition, we may assume $a + c = -1, b + d = 2 \text{ dim } X$. From (2.9), we have

$$E_1(\tilde{\Psi}) \cdot (d_1 \otimes \text{id}) + (-1)^{a+b}E_1(\tilde{\Psi}) \cdot (\text{id} \otimes d_1) = (-1)^d d_1 \cdot E_1(\tilde{\Psi}),$$

where $d_1$ on the right hand side is the morphism of $E_1$-terms for $C(\Omega_{X_r}(\log \mathcal{M}_{X_r}), \delta W)$. Because $\Theta \cdot d_1 = 0$ by Lemma 6.26, the conclusion is obtained from $\epsilon(-a - b) = (-1)^{a+b+1} \epsilon(-a - b - 1)$. 

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Lemma 8.15. The restriction of \( S \) to the direct summand
\[
(Cu^q \otimes_C H^j(X_r, \varepsilon_r \otimes \Omega_{X_r})) \otimes_C (Cu^{q'} \otimes_C H^{j'}(X_{r'}, \varepsilon_{r'} \otimes \Omega_{X_{r'}}))
\]
(8.6)
of \( V_C \otimes_C V_C \) via the identification (8.2) is zero unless \( r = r' = q + q' + e \) and \( j + j' = 2 \dim X_r \). For the case of \( r = r' = q + q' + e \) and \( j + j' = 2 \dim X_r \), the restriction of \( S \) to the direct summand (8.6) coincides with
\[
(-1)^{q(q'-e)}(2\pi \sqrt{-1})^{r-k} \int_{X_r} \cdot H^{j'j}(X_r, \vartheta_r \otimes \wedge),
\]
where \( \vartheta_r \otimes \wedge \) denotes the composite
\[
(\varepsilon_r \otimes \Omega_{X_r}) \otimes_C (\varepsilon_r \otimes \varepsilon_r) \otimes_C (\Omega_{X_r} \otimes C \Omega_{X_r}) \xrightarrow{\vartheta_r \otimes \wedge} \Omega_{X_r}
\]
by abuse of notation.

Proof. Note that \( Cu^q \otimes_C H^j(X_r, \varepsilon_r \otimes \Omega_{X_r}) \subset E^{a,b}_1(A_C, L) \) for \( a = 2q - |r| + k \) and \( b = j - 2a + 2|q| = j - a + |r| - k \) as in the second equality in (8.4). From Lemma 7.13, we obtain the conclusion because \((-1)^{a+b+2}\varepsilon(-a - b)(|r| - k) = (-1)^{(|r| - k)(j + |r| - k)}\varepsilon(-j - |r| + k)\varepsilon(|r| - k) = \varepsilon(-j)\). \( \square \)

Corollary 8.16. \( S \) is \((-1)\dim X\)-symmetric.

Lemma 8.17. We have the following:

(8.17.1) \( S(V_C^{q,j} \otimes_C V_C^{q',j'}) = 0 \) unless \( j_0 + j'_0 = 0 \) and \( j + j' = 0 \).

(8.17.2) \( S \cdot (l_i \otimes \id) + S \cdot (\id \otimes l_i) = 0 \) for all \( i \in \{1, 2, \ldots, k\} \).

(8.17.3) \( S \cdot (l_0 \otimes \id) + S \cdot (\id \otimes l_0) = 0 \).

(8.17.4) \( S \cdot (d_i' \otimes \id) = S \cdot (\id \otimes d_i') \) for all \( i \in \{1, 2, \ldots, k\} \).

(8.17.5) \( S(F^pV_C \otimes_C F^qV_C) = 0 \) if \( p + q > \dim X \).

Proof. For (8.17.1), it suffices to consider the cases of
\[
\begin{align*}
\varepsilon_r - 2q - j + e, & \quad \varepsilon_r - 2q' - j' + e, \\
\varepsilon_r - j_0 - |r| + \dim X + k + j'_0 - |r| + \dim X + k = 2(\dim X - |r| + k)
\end{align*}
\]
in (8.3), by Lemma 8.15. Then these equalities imply \( j + j' = 0 \) and \( j_0 + j'_0 = 0 \). Since \( E_1(\nu_i) \) is identified with the morphism \( \bigoplus(u_i \cdot \id) \) under the isomorphism (8.2), we can easily check (8.17.2) from Lemma 8.15. Similarly, (8.12.2) implies (8.17.3) by \( \varepsilon(-j - 2) = -\varepsilon(-j) \). The equality \( S \cdot (d_1 \otimes \id) = S \cdot (\id \otimes d_1) \) in Lemma 8.14 combined with (8.17.1) implies (8.17.4). We can easily check (8.17.5) by (8.3) and Lemma 8.15. \( \square \)

Lemma 8.18. \( S(V_Q \otimes V_Q) \subset \Q \).

Proof. Under the isomorphism (8.1), \( V_Q \) is identified with \( \bigoplus \Q u^q \otimes \Q H^j(X_r, \varepsilon_r \otimes \Omega \Q)(|r| + |q| + k) \) by (8.5). Therefore Lemma 8.15 implies the conclusion. \( \square \)

Definition 18.9. For \( j_0 \in \N \) and \( j \in \N^k \), we set
\[
V_{C,0}^{-j_0,-j} = V_C^{-j_0,-j} \cap \bigcap_{i=0}^k \ker(l_i^{j_i+1}), \quad V_{Q,0}^{-j_0,-j} = V_Q \cap V_{C,0}^{-j_0,-j}.
\]
Then, together with the induced filtration \( F \) on \( V_{C,0}^{-j_0,-j} \), the data \( (V_{C,0}^{-j_0,-j}, (V_{C,0}^{-j_0,-j}, F)) \) is a \( \Q \)-Hodge structure of weight \(-j_0 + |j| + \dim X\).
Lemma 8.20. The bilinear form \( S \cdot (\id \otimes C^{l_0}_{l_1} \ldots l_k) \) on \( V_{Q,0}^{-j_0,-j} \) is symmetric and positive definite, where \( C \) denotes the Weil operator of a \( \mathbb{Q} \)-Hodge structure \( (V_{Q,0}^{-j_0,-j}, (V_{C,0}^{-j_0,-j}, F)) \).

Proof. Since \( E_1(\nu_s) \) is identified with the morphism \( \bigoplus (u_t) \otimes \id \) via the isomorphism (8.2), the equality as \( \mathbb{Q} \)-Hodge structures

\[
V_{Q,0}^{-j_0,-j} = \mathbb{Q} u^0 \otimes_{\mathbb{Q}} H^{-j_0,|j|+\dim X} (X_{j_0+\epsilon}, \varepsilon_{j_0+\epsilon} \otimes_{\mathbb{Z}} \mathbb{Q})(-|j|) \cap \ker (\pi_{Q}^{l_{j_0}+1})
\]
can be easily seen. We note that \( l_1^0 \ldots l_k^0 \) is identified with the multiplication by \( u^0 \otimes (2\pi \sqrt{-1})^{j_0} \).

Then Lemma 8.15 and the classical Hodge theory on \( X_{j_0+\epsilon} \) imply the conclusion because \( l_0 \) is identified with the cup product \( \cup (2\pi \sqrt{-1}) C_{1} (L_{j_0+\epsilon}) \) on \( H^* (X_{j_0+\epsilon}, \varepsilon_{j_0+\epsilon} \otimes_{\mathbb{Z}} \mathbb{Q}) \) by Lemma 7.3. \( \square \)

Remark 8.21. In fact, we can check that the bilinear form \( (2\pi \sqrt{-1})^{j_0-|j|+\dim X} S \cdot (\id \otimes l_0^{l_1} \ldots l_k^0) \) is a polarization of the \( \mathbb{Q} \)-Hodge structure \( V_{Q,0}^{-j_0,-j} \) in the sense of Deligne [3, Définition (2.1.15)].

9 Multi-graded Hodge-Lefschetz modules

In this section, we introduce the notion of a multi-graded Hodge-Lefschetz modules, which slightly generalize the notion of a bigraded Hodge-Lefschetz module in [16, Section 4] (cf. [23, Section 4] and [22, 11.3.2]). Then, we prove Proposition 9.8, which is a key tool for the proofs of Theorems 4.9, 4.10 and 4.13 in Section 10.

Definition 9.1. Let \( A \) be a finite set. A \( \mathbb{Z}^{A} \)-graded Lefschetz module \( (V, \{ l_a \}_{a \in A}) \) consists of a finite dimensional \( \mathbb{Z}^{A} \)-graded \( \mathbb{R} \)-vector space \( V = \bigoplus_{j \in \mathbb{Z}^{A}} V^j \) and a family of endomorphisms \( l_a \) of \( V \) satisfying the following conditions:

\[
\begin{align*}
(9.1.1) \quad & l_a l_b = l_b l_a \quad \text{for all } a, b \in A. \\
(9.1.2) \quad & l_a (V^j) \subset V^{j+2e_a} \quad \text{for all } a \in A. \\
(9.1.3) \quad & \text{For all } a \in A, \text{ the morphism } l_a^0 : V^{-j} \to V^{-j+2e_a} \text{ is an isomorphism for all } j = \sum a \in A j_a e_a \in \mathbb{Z}^{A} \text{ with } j_a > 0.
\end{align*}
\]

A \( \mathbb{Z}^{A} \)-graded Lefschetz module \( (V, \{ l_a \}_{a \in A}) \) is called a \( \mathbb{Z}^{A} \)-graded Hodge-Lefschetz module if \( V^j \) is an \( \mathbb{R} \)-Hodge structure of certain weight and \( l_a : V^j \to V^{j+2e_a} \) is a morphism of \( \mathbb{R} \)-Hodge structures of certain type (cf. [15, (1.2) Definition], [5, 1.2.9]) for all \( j \in \mathbb{Z}^{A} \) and \( a \in A \). We set \( V_{Q,0}^{-j} = V^{-j} \cap \bigcap_{a \in A} \ker (l_a^{l_0+1}) \) for \( j \in \mathbb{N}^{A} \). Then \( V_{Q,0}^{-j} \) is a sub \( \mathbb{R} \)-Hodge structure of \( V^{-j} \). Taking direct sum of the Weil operator of \( V^j \) for all \( j \in \mathbb{Z}^{A} \), we obtain an endomorphism \( C \) of \( V \).

Remark 9.2. As in [16, (4.1)], the \( \mathbb{Z}^{A} \)-graded Lefschetz modules correspond bijectively to the finite dimensional representations of \( SL(2, \mathbb{R})^{A} \simeq SL(2, \mathbb{R})^{4|A} \). We set

\[
w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL(2, \mathbb{R}).
\]

Moreover, \( w_A \in SL(2, \mathbb{R})^{A} \) is the image of \( w \) by the diagonal map \( SL(2, \mathbb{R}) \hookrightarrow SL(2, \mathbb{R})^{A} \).

Definition 9.3. For a \( \mathbb{Z}^{A} \)-graded Hodge-Lefschetz module \( (V, \{ l_a \}_{a \in A}) \), a polarization is an \( \mathbb{R} \)-linear map \( S : V \otimes_{\mathbb{R}} V \to \mathbb{R} \) satisfying the following conditions:

\[
\begin{align*}
(9.3.1) \quad & S(V^j \otimes \mathbb{R} V^{j'}) = 0 \quad \text{if } j + j' \neq 0. \\
(9.3.2) \quad & S : V^{-j} \otimes_{\mathbb{R}} V^j \to \mathbb{R} \quad \text{is a morphism of } \mathbb{R} \text{-Hodge structures of certain type.}
\end{align*}
\]
(9.3.3) $S \cdot (l_a \otimes \text{id}) + S \cdot (\text{id} \otimes l_a) = 0$ for all $a \in A$.

(9.3.4) The bilinear form $S \cdot (\text{id} \otimes C \prod_{a \in A} l_a^n)$ on $V_0^{-j}$ is symmetric and positive definite for all $j = \sum_{a \in A} j_a e_a \in \mathbb{N}^A$.

**Remark 9.4.** Under the conditions (9.3.1) and (9.3.3), the condition (9.3.4) is equivalent to the condition that the bilinear form on $V$ defined by $S(x \otimes C w_A y)$ is symmetric and positive definite. We can check this equivalence by computation similar to [16, (4.3) Proposition]. Note that $C$ commutes with the action of $w_A$. In fact, $C$ commutes with the action of $SL(2, \mathbb{R})^A$ because $C$ preserves the $Z^A$-grading of $V$ and commutes with $l_a$ for all $a \in A$.

Next, we define the notion of a differential of a polarized $Z^A$-graded Hodge-Lefschetz module. Because one distinguished component of $Z^A$ plays a special role for the notion of a differential, we replace $Z^A$ by $Z \oplus Z^A$ in the definition below.

**Definition 9.5.** A differential of a polarized $Z \oplus Z^A$-graded Hodge-Lefschetz module

$$(V = \bigoplus_{j_0 \in Z, j \in Z^A} V^{j_0 j}, \{l_0, \{l_a\}_{a \in A}\}, S),$$

is a family of $\mathbb{R}$-linear maps $d_a : V \to V$ for $a \in A$ satisfying the following conditions:

(9.5.1) $d_a(V^{j_0 j}) \subset V^{j_0 + 1 j + e_a}$ for $a \in A$.

(9.5.2) $d_a : V^{j_0 j} \to V^{j_0 + 1 j + e_a}$ is a morphism of $\mathbb{R}$-Hodge structures of certain type.

(9.5.3) $d_a d_b + d_b d_a = 0$ for all $a, b \in A$.

(9.5.4) $d_a l_0 = l_0 d_a$ and $d_a l_b = l_b d_a$ for all $a, b \in A$.

(9.5.5) $S \cdot (d_a \otimes \text{id}) = S \cdot (\text{id} \otimes d_a)$ for all $a \in A$.

**Remark 9.6.** For the case of $|A| = 1$, a polarized differential $Z \oplus Z$-graded Hodge-Lefschetz module is nothing but a polarized differential bigraded Hodge-Lefschetz module in [16, Section 4].

**Definition 9.7.** Let $(V, \{l_0, \{l_a\}_{a \in A}\}, S, \{d_a\}_{a \in A})$ be a polarized differential $Z \oplus Z^A$-graded Hodge-Lefschetz module. For $B \subset A$ and for $c = \sum_{a \in B} c_a e_a \in \mathbb{R}^B$, we set $d_B = \sum_{a \in B} d_a$ and $l_B(c) = \sum_{a \in B} c_a l_a$. Then $d_B^2 = 0$ by (9.5.3). Moreover, by setting

$$V^{j_0 j_1 j_2} = \bigoplus_{|j_B| = j_1, j_{A \setminus B} = j_2} V^{j_0 j_2},$$

for $j_0, j_1 \in Z$ and $j_2 \in Z^{A \setminus B}$, we have $V = \bigoplus V^{j_0 j_2}$, where the direct sum is taken over all $j_0, j_1 \in Z, j_2 \in Z^{A \setminus B}$. Then $d_B(V^{j_0 j_1 j_2}) \subset V^{j_0 + 1 j_1 + 1 j_2}$ and $H(V, d_B) = \text{Ker}(d) / \text{Image}(d)$ carries the natural direct sum decomposition

$$H(V, d_B) = \bigoplus_{j_0, j_1, j_2} H(V, d_B)^{j_0, j_1, j_2},$$

by setting

$$H(V, d_B)^{j_0, j_1, j_2} = V^{j_0, j_1, j_2} \cap \text{Ker}(d_B) / V^{j_0, j_1, j_2} \cap \text{Image}(d_B).$$

Because of (9.5.3)–(9.5.5), the morphisms $l_0, l_B(c), l_a, d_a$ ($a \in A \setminus B$) and $S$ commute with $d_B$ and descend to $H(V, d_B)$, denoted by the same letters. We set $d_1 = 0$ on $H(V, d_B)$. 

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Proposition 9.8. Equipped with the direct sum decomposition (9.2),

\[(H(V,d_B),\{l_0,l_B(c),\{l_a\}_{a\in A\setminus B}, S, \{d_1, \{d_a\}_{a\in A\setminus B}\})\]

is a polarized differential \(\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}^{A\setminus B}\)-graded Hodge-Lefschetz module if \(c_a > 0\) for all \(a \in B\).

Proof. Because \(c_a > 0\), the condition (9.3.4) is satisfied for \(\{c_a l_a\}_{a \in B} \cup \{l_a\}_{a \in A\setminus B}\). Therefore we may assume \(c = e_B\) by replacing \(l_a\) with \(c_a l_a\).

First, we treat the case of \(A = B\). In this case, the \(\mathbb{Z} \oplus \mathbb{Z}\)-grading (9.1) for \(B = A\) corresponds to the representation of \(SL(2,\mathbb{R}) \times SL(2,\mathbb{R})\) induced from the inclusion \(SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \hookrightarrow SL(2,\mathbb{R}) \times SL(2,\mathbb{R})^{A}\), where the first factor is the identity of \(SL(2,\mathbb{R})\) and the second factor \(SL(2,\mathbb{R}) \hookrightarrow SL(2,\mathbb{R})^{A}\) is the diagonal map. Then the action of \((w, w) \in SL(2,\mathbb{R}) \times SL(2,\mathbb{R})\) on \(V\) is the same as the action of \((w, w_A) \in SL(2,\mathbb{R}) \times SL(2,\mathbb{R})^{A}\). Therefore \((V = \bigoplus V_{j_0,j}, \{l_0, l_A\}, S, d_A)\) is a polarized differential bigraded Hodge-Lefschetz module. By applying [16, (4.5) Théorème], we obtain (9.1.3) and (9.3.4) for \(H(V,d_A)\).

Next, we treat the general case. Note that \(l_a\) on \(H(V,d_B)\) for \(a \in A\setminus B\) trivially satisfies the condition (9.1.3). Moreover, the endomorphism \(C\) commutes with \(d_B\) and descends to \(H(V,d_B)\), which coincides with the endomorphism \(C\) of \(H(V,d_B)\). The \(\mathbb{Z} \oplus \mathbb{Z}^{B}\)-grading \(V = \bigoplus_{j_0,j} (\bigoplus_{j_B = j} V_{j_0,j})\) gives us a \(\mathbb{Z} \oplus \mathbb{Z}^B\)-graded Lefschetz module \((V,\{l_0,\{l_a\}_{a \in A\setminus B}\})\), which corresponds to the representation of \(SL(2,\mathbb{R}) \times SL(2,\mathbb{R})^{B}\) induced from the injection defined by \(SL(2,\mathbb{R}) \times SL(2,\mathbb{R})^{B} \ni (g_0, g) \mapsto (g_0, g, \text{id}) \in SL(2,\mathbb{R}) \times SL(2,\mathbb{R})^{B} \times SL(2,\mathbb{R})^{A\setminus B}\). By using \(w' = (\text{id}, \text{id}, w_{A\setminus B}) \in SL(2,\mathbb{R}) \times SL(2,\mathbb{R})^{B} \times SL(2,\mathbb{R})^{A\setminus B}\), we set \(S_B(x \otimes y) = S(x \otimes w'y)\) for \(x, y \in V\). Then the bilinear form \(S_B\) satisfies the condition (9.3.4) as in Remark 9.4 and \((V,\{l_0,\{l_a\}_{a \in A\setminus B}\}, S_B, \{d_a\}_{a \in B})\) is a polarized differential \(\mathbb{Z} \oplus \mathbb{Z}^B\)-graded Hodge-Lefschetz module. Therefore \((H(V,d_B),\{l_0, l_B(e_B)\}, S_B)\) is a polarized bigraded Hodge-Lefschetz module as proved above. Thus \(l_0, l_A\) satisfy the condition (9.1.3) for \(H(V,d_B)\) equipped with the direct sum decomposition (9.2). Because \(S(x \otimes C(w, w_A)y) = S_B(x \otimes C(w, w_B)y)\), the bilinear form \(S\) on \(H(V,d_B)\) satisfies the desired condition (9.3.4). \(\square\)

10 Proof of Theorems 4.5, 4.9, 4.10 and 4.13

First, we prove the following lemma, which slightly generalize Lemma 3.17 of [6].

Lemma 10.1. Let \(((A_Q, W^I, W), (A_C, W^I, W, F), \alpha)\) be a filtered \(\mathbb{Q}\)-mixed Hodge complex and \(\nu : A_C \to A_C\) a morphism of complexes preserving the filtration \(W^I\) and satisfying the condition \(\nu(W_m A_C) \subset W_{m-2} A_C\) for all \(m\). If the morphism \(H^n(G_{\nu W}^n A_C)\) induces an isomorphism \(\text{Gr}_{l}^{W_{-m}} H^n(G_{\nu W}^n A_C) \cong \text{Gr}_{l-m}^{W_{-m}} H^n(G_{\nu W}^n A_C)\) for all \(l \in \mathbb{Z}_{>0}\) and \(m, n \in \mathbb{Z}\), then we have the following:

10.1.1 The spectral sequence \(E^p_q(A_C, W^I)\) degenerates at \(E_2\)-terms.

10.1.2 The morphism \(H^n(\nu)^I\) induces an isomorphism \(\text{Gr}_{l+m}^W H^n(A_C) \cong \text{Gr}_{l+m}^W H^n(A_C)\) for all \(l \in \mathbb{Z}_{>0}\) and \(m, n \in \mathbb{Z}\).
Proof. In this proof, we write $E_{p,q} = E_{p,q}(A_C, W')$ for short. The morphism of $\mathbb{E}_r$-terms $d_r: E_{r}^{p,q} \rightarrow E_{r}^{p+r,q-r+1}$ is strictly compatible with $W_{rec}$ on the left hand side and $W_{rec}[1]$ on the right by [5, 6.1.8 Théorèm]. On the other hand, the morphism $\nu$ induces a morphism of the spectral sequences $E_{r}(\nu): E_{r}^{p,q} \rightarrow E_{r}^{p,q}$. Via the identification $E_{1}^{p,q} \simeq H^{p+q}(Gr_{W'}^{p} A_C)$, the assumption implies that $E_{1}(\nu)^l$ induces an isomorphism $Gr_{W'}^{l}[p] E_{1}^{p,q} \xrightarrow{\sim} Gr_{W'}^{l}[p] E_{1}^{p,q}$ for all $l \in \mathbb{Z}_{>0}$ and $p, q \in \mathbb{Z}$. Then the strictness of $d_1$ above implies that $E_{2}(\nu)^l$ induces an isomorphism $Gr_{W_{rec}}^{l}[p] E_{2}^{p,q} \xrightarrow{\sim} Gr_{W_{rec}}^{l}[p] E_{2}^{p,q}$ for all $l \in \mathbb{Z}_{>0}$, that is, $W_{rec}[p]$ is the monodromy weight filtration of $E_{2}(\nu)$ on $E_{2}^{p,q}$ for all $p, q \in \mathbb{Z}$. Because $d_2$ commutes with $E_{2}(\nu)$, the monodromy weight filtration of $E_{2}(\nu)$ is preserved by $d_2$. Namely, $d_2: E_{2}^{p,q} \rightarrow E_{2}^{p+2,q-1}$ preserves $W_{rec}[p]$ on the left hand side and $W_{rec}[p+2]$ on the right. Therefore

$$d_2(W_{rec}[m]E_{2}^{p,q}) = d_2(W_{rec}[p]mE_{2}^{p,q}) \subset (W_{rec}[p+2]m+pE_{2}^{p+2,q-1} = (W_{rec}[1])m-1E_{2}^{p+2,q-1}$$

for all $m \in \mathbb{Z}$. Thus we obtain $d_2 = 0$ on $E_{2}^{p,q}$ for all $p, q \in \mathbb{Z}$, because of the strict compatibility of $d_2$ with $W_{rec}$ on $E_{2}^{p,q}$ and $W_{rec}[1]$ on $E_{2}^{p+2,q-1}$. Repeating this procedure inductively, we obtain $d_r = 0$ for all $r \geq 2$. Once (10.1.1) is obtained, (10.1.2) follows from Lemma 3.17 of [6].

Proof of Theorems 4.5, 4.9, 4.10 and 4.13. A semistable log smooth degeneration $f: (X, \mathcal{M}_X) \rightarrow (*, \mathbb{N}^k)$ is assumed to be projective. Moreover, we may assume that $X$ is of pure dimension by considering the connected components. We fix $I \subset \{1, 2, \ldots, k\}$ and $c = (c_i)_{i=1}^{k} \in (\mathbb{R}_{>0})^k$, and set $J = \{1, 2, \ldots, k\} \setminus I$. Morphisms of complexes $\nu(c), \nu_{J}(c, J): \mathcal{A}_C \rightarrow \mathcal{A}_C$ are defined by $\nu(c) = \sum_{i=1}^{k} c_i \nu_i$ and $\nu_{J}(c, J) = \sum_{i \in J} c_i \nu_i$. Recall that these morphisms induces $N(c)$ and $N_{J}(c, J)$ on $H^{*}(X, \mathcal{A}_C)$.

Let $V_Q$ and $V_{Q, j=0}^{j=0}$ be as in Definitions 8.2 and 8.4. We set $V_R = \mathbb{R} \otimes V_Q$ and $V_{j=0}^{j=0} = \mathbb{R} \otimes V_{Q, j=0}^{j=0}$. Then $V_C = \mathbb{C} \otimes R V_R$ and $V_{C, j=0}^{j=0} = \mathbb{C} \otimes R V_{R, j=0}^{j=0}$. Then we obtain

$$(V_R = \bigoplus_{j_0 \in \mathbb{Z}} V_{j_0}^{j=0})(\{l_0, \{l_i\}_{i=1}^{k}, S, \{d'_i\}_{i=1}^{k}),$$

which is a polarized differential $\mathbb{Z} \oplus \mathbb{Z}^{k}$-graded Hodge-Lefschetz module by Lemmas 8.7, 8.9, 8.12, 8.17, 8.18 and 8.20. We set

$$V_{R, j=1, j_2}^{j_0, j, j_2} = \bigoplus_{|j_1|=j_1, |j_2|=j_2} V_{C, j=1, j_2}^{j_0, j, j_2}, \quad V_{C, j=1, j_2}^{j_0, j, j_2}, \quad d'_j = \sum_{i \in J} d'_i, \quad d''_i = \sum_{i \in I} d''_i,$$

and $l_I(c_I) = \sum_{i \in I} c_i l_i$ for $I \subset \{1, 2, \ldots, k\}$. We use $l(c)$ instead of $l_{\{1,2,\ldots,k\}}(c)$. Then we have

$$V_C = \bigoplus_{j_0, j_1, j_2 \in \mathbb{Z}} V_{C, j_0, j_1, j_2}^{j_0, j, j_2}, \quad d''_j(V_{C, j_0, j_1, j_2}) \subset V_{C, j_0+1, j_1+1, j_2}, \quad d'_j(V_{C, j_0, j_1, j_2}) \subset V_{C, j_0+1, j_1, j_2+1}$$

and $d_1 = d'_j + d''_i$ by definition.

Since $L(I) \ast L(J) = L$ on $A_C$ by Corollary 5.17, we have the identifications

$$\bigoplus_{p+q=a}^{\oplus} E_{1}^{p,q+b}(Gr_{W'}^{l[I]} A_C, L(J)) \simeq E_{1,a}^{b}(A_C, L) \simeq \bigoplus_{|j|=a}^{\oplus} V_{C}^{a+\dim X,j}$$

for all $a, b$, which induces

$$\bigoplus_{p+q=a, p \geq -l}^{\oplus} E_{1}^{p,q+b}(Gr_{W'}^{l[I]} A_C, L(J)) \simeq L(J)_{l}E_{1,a}^{b}(A_C, L) \simeq \bigoplus_{|j|=a, |j| \geq -l}^{\oplus} V_{C}^{a+\dim X,j}.$$
for all \(l\) as in 2.23 and Remark 8.6. Therefore

\[
V_C^{j_0,j_1,j_2} \simeq E_1^{j_1,j_0-j_1+\dim X}(\text{Gr}_{-j_2}^L A_C, L(J)) \tag{10.1}
\]

for all \(j_0, j_1, j_2 \in \mathbb{Z}\). We denote the morphism of \(E_1\)-terms of \(E^{p,q}_r(\text{Gr}_m^L A_C, L(J))\) by \(\tilde{d}_1\) for a while. Then we obtain \(\tilde{d}_1: V_C^{j_0,j_1,j_2} \rightarrow V_C^{j_0+1,j_1+1,j_2}\) via the identification (10.1). On the other hand, the morphism

\[
\gamma: (\text{Gr}_m^L A_C, L(J)) \rightarrow (\text{Gr}_{m-1}^L A_C[1], L(J))
\]

in the filtered derived category as in 2.22 induces a morphism \(E_1(\gamma): V_C^{j_0,j_1,j_2} \rightarrow V_C^{j_0+1,j_1,j_2+1}\) via the identification (10.1). Because \(d_1 = \tilde{d}_1 + E_1(\gamma)\) by Lemma 2.24, \(d_1 = d'_j\) and \(E_1(\gamma) = d''_j\). Since \(L(J) = L[-m]\) on \(\text{Gr}_m^L A_C\) by Corollary 5.17, we have

\[
\bigoplus_{|j'| = j_2} \mathbb{C} \otimes \mathbb{R} H(V, d'_j)^{j_0,j_1,j'} \simeq V_C^{j_0,j_1,j_2} \cap \ker(d'_j)/V_C^{j_0,j_1,j_2} \cap \text{image}(d'_j)
\]

\[
\simeq E_2^{j_1,j_0-j_1+\dim X}(\text{Gr}_{-j_2}^L A_C, L(J)) \tag{10.2}
\]

\[
\simeq \text{Gr}_{-j_1}^L H^{j_0+\dim X}(X, \text{Gr}_{-j_2}^L A_C)
\]

from (9.3), the equality \(d'_j = \tilde{d}_1\) and the \(E_2\)-degeneracy (4.4.5). In particular,

\[
\mathbb{C} \otimes \mathbb{R} H(V, d_j)^{j_0,j_1} \simeq \text{Gr}_{-j_1}^L H^{j_0+\dim X}(X, A_C) \tag{10.3}
\]

as the case of \(I = \emptyset\). In the identification (10.2), the morphism \(\text{id} \otimes d'_j\) on the first term \(\bigoplus \mathbb{C} \otimes \mathbb{R} H(V, d'_j)^{j_0,j_1,j'}\) is identified with \(\text{Gr}_{-j_1}^L H^{j_0+\dim X}(X, \gamma)\) on the last term. Moreover, under the identification \(H^{j_0+\dim X}(X, \text{Gr}_{-j_2}^L A_C) \simeq E_2^{j_1,j_0-j_1+\dim X}(A_C, L(I))\), the morphism \(H^{j_0+\dim X}(X, \gamma)\) is identified with the morphism of \(E_1\)-terms of the spectral sequence \(E^{p,q}_r(A_C, L(I))\). By (4.4.6), the morphism of \(E_1\)-terms \(E_1^{j_2,j_0-j_2+\dim X}(A_C, L(I)) \rightarrow E_1^{j_2+1,j_0-j_2+\dim X}(A_C, L(I))\) is strictly compatible with \(L[j_2]\) and \(L[j_2+1]\), we obtain the identification

\[
\mathbb{C} \otimes \mathbb{R} H(H(V, d'_j), d'_j)^{j_0,j_1,j_2} \simeq \text{Gr}_{-j_1}^L E_2^{j_2,j_0-j_2+\dim X}(A_C, L(J)) \tag{10.4}
\]

for all \(j_0, j_1, j_2 \in \mathbb{Z}\).

In the identifications (10.2)–(10.4),

\[\langle H(V, d'_j), \{l_0, l_j(c_j), \{l_i\}_{i \in I}\}, S, \{d_1 = 0, \{d'_i\}_{i \in I}\} \rangle\]

is a \(\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}'\)-graded Hodge-Lefschetz module,

\[\langle H(V, d_1), l_0, l(c), S \rangle\]

is a \(\mathbb{Z} \oplus \mathbb{Z}\)-graded Hodge-Lefschetz module, and further,

\[\langle H(H(V, d'_j), d'_j), \{l_0, l_j(c_j), l_1(c_i)\}, S \rangle\]

is a \(\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}\)-graded Hodge-Lefschetz module by Proposition 9.8.

Under the identification (10.3), the morphism \(l_0\) is identified with the morphism induced by \((2\pi \sqrt{-1})/(\cup c(L))\). Therefore \((\cup c(L))^{i}\) induces an isomorphism

\[\text{Gr}_i^L H^{-i+\dim X}(X, A_C) \simeq \text{Gr}_i^L H^{i+\dim X}(X, A_C)\]
for all $i \in \mathbb{Z}_{>0}$ and $l \in \mathbb{Z}$. Hence we obtain Theorem 4.13.

Under the identification (10.2), $l_J(c_J)$ on the first term $\bigoplus \mathbb{C} \otimes \mathbb{R} \mathbb{H}((V \mathbb{R}, d'_J))^{j_0, j \cdot J'}$ is identified with the morphism induced from $(2 \pi \sqrt{-1})^{h_0 + \dim X} (X, \Gr^{L(I)}_{j_2} \nu_J(c_J))$. Because $\Gr^{L(I)}_{m} \nu_J(c_J) = \Gr^{L(I)}_{m} \nu(c)$ on $\Gr^{L(I)}_{m} A_C$, the morphism $H^j(X, \Gr^{L(I)}_{m} \nu(c))^i$ induces an isomorphism

$$
\text{Gr}^{L[-m]}_l H^j(X, \Gr^{L(I)}_{m} A_C) \xrightarrow{\sim} \text{Gr}^{L[-m]}_{l-1} H^j(X, \Gr^{L(I)}_{m} A_C)
$$

for all $l \in \mathbb{Z}_{>0}$ and $m \in \mathbb{Z}$. Therefore we obtain Theorems 4.5 and 4.10 by Lemma 10.1.

Under the identification (10.4), the morphism $l_I(c_I)$ is identified with $\Gr^{L(2)}_{I_2} E_2(\nu_I(c_I))$. Moreover, $L(J) = L[-m]$ on $\Gr^{L(I)}_{m} A_C$ implies $L(J) = L[p]$ on $E_1^{p,q}(A_C, L)$ and $L(J) = L[p]$ on $E_2^{p,q}(A_C, L(I))$. Then $\Gr^{L(J)}_{m} E_2(\nu_I(c_I))^i$ induces an isomorphism

$$
\text{Gr}^{L(J)}_{m} E_2^{l-i+l}(A_C, L(I)) \xrightarrow{\sim} \text{Gr}^{L(J)}_{m} E_2^{l-i-l}(A_C, L(I))
$$

for all $l \in \mathbb{Z}_{>0}$ and $i, m \in \mathbb{Z}$. Thus $E_2(\nu_I(c_I))^i$ induces an isomorphism

$$
E_2^{l-i+l}(A_C, L(I)) \xrightarrow{\sim} E_2^{l-i-l}(A_C, L(I))
$$

for all $l \in \mathbb{Z}_{>0}$ and $i \in \mathbb{Z}$. Therefore we obtain Theorem 4.9 by the $E_2$-degeneracy in Theorem 4.5. $\Box$

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