On causality violation on a Kerr-de Sitter spacetime

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The causal properties of the family of Kerr-de Sitter spacetimes are analyzed and compared to those of the Kerr family. At first we show that a Kerr-de Sitter spacetime can be viewed as an assembly of Carter’s blocks i.e. four dimensional spacetime regions contained within Killing horizons or a Killing horizon and the asymptotic de Sitter region. From this perspective and leaving aside topological identifications, the causal properties of a Kerr de Sitter spacetime are determined by the causal properties of the individual Carter’s blocks viewed as spacetimes in their own right. We show that any Carter’s block is stably causal except for the blocks that contain the ring singularity. The latter are vicious sets: any two events within such block can be connected by a future (respectively past) directed timelike curve. This behavior is identical to the causal behavior of the Boyer-Lindquist blocks that contains the Kerr ring singularity which are also vicious sets. On the other hand, while for the case of a naked Kerr singularity the entire spacetime is a vicious set and thus closed timelike curves pass through any event including events lying in the asymptotic region, for the case of a Kerr-de Sitter spacetime the cosmological horizons imply that the asymptotic de Sitter region is causally well behaved. In that regard a positive cosmological constant appears to improve the causal behavior of the underlying spacetime.

I. INTRODUCTION

In this paper, we discuss causality violations exhibited by the family of Kerr-de Sitter spacetimes. This family has been discovered by Carter [1],[2] and besides the positive cosmological constant $\Lambda > 0$ it contains two real parameters $(M, a)$ where $M$ is the mass parameter while $a$ is the rotation parameter. Depending upon the values of $(\Lambda, M, a)$ a Kerr-de Sitter may describe a black hole asymptotically to a de Sitter spacetime [2],[3],[4] or extremal configurations where black hole horizons and cosmological horizons may coincide [5]. Since in a Kerr-de Sitter spacetime event horizons and cosmological horizons coexist, it is worthwhile to investigate whether these spacetimes admit causality violated regions i.e regions admitting closed causal curves. If such regions are admitted, can they lie in the asymptotic de Sitter like regions? Do Kerr-de Sitter spacetimes admit vicious causality violating regions and if such regions exist where do they lie? Can a causality violating region occur within cosmological horizons?

In this work, we address some of these issues and at first we discuss the global structure of a Kerr-de Sitter spacetime. We show that any Kerr-de Sitter spacetime can be thought as been assembled by a family of Carter’s blocks i.e. four dimensional spacetime regions enclosed between Killing horizons...
or a Killing horizon and the asymptotic region. The Carter-Penrose diagrams for an extended Kerr-de Sitter spacetime has a structure similar to the structure exhibited by the two dimensional rotation axis of a Kerr-de Sitter spacetime. Based on these diagrams and on the property that any Killing horizon is an achronal boundary, the causality properties of any four dimensional Kerr-de Sitter spacetime are determined. Their causality properties are identical to the causality properties of the individual Carter’s blocks. We show that any Carter’s block is causally well behaved and in fact is causally stable except for the blocks that contain the ring singularity. These blocks are vicious set in the sense defined by Carter [6]: Any two events within a block that contains the ring singularity can be connected by a future (resp. past) directed timelike curve. As long as the Carter’s time machine (CTM) i.e. the spacetime region where the axial Killing vector field becomes temporal, is non empty, we show that the CTM permits the generation of future (resp. past) directed timelike curves that joins any pair of events within the block that contains the ring singularity in other words a non empty CTM effectively destroys any aspect of causality.

Our analysis shows that the block that contains the ring singularity patterns the causal behavior of the Boyer-Lindquist block that contains the Kerr ring singularity which is also a vicious set. On the other hand, while for the case of a naked Kerr singularity the entire spacetime is a vicious set and thus through every event in this spacetime-including those events in the asymptotic region—passes a closed timelike curve, for a Kerr-de Sitter the asymptotic de Sitter region is causally well behaved. In that respect it seems that a positive cosmological constant appear to improve the causal behavior of the underlying spacetime.

The structure of the present paper is as follows. In the next section, the Kerr-de Sitter metrics are presented. Section III, contains a brief construction of the maximal analytical extension of the rotation axis of a Kerr-de Sitter spacetime while section IV discuss the global structure of a four dimensional Kerr-de Sitter. In section V we prove three propositions describing the causal properties of a Kerr-de Sitter spacetime. We finish the paper with a discussion of open problems while in an Appendix the reader is reminded of a few basic notions of causality theory.

II. SOME PROPERTIES OF THE KERR-DE SITTER SPACETIMES

In a local set of Boyer-Lindquist coordinates \((t, \varphi, r, \vartheta)\), a Kerr-de Sitter metric takes the form:

\[
g = -\frac{\Delta(r)}{I^2 \rho^2} [dt - asin^2 \vartheta d\varphi]^2 + \frac{\hat{\Delta}(\vartheta) \sin^2 \vartheta}{I^2 \rho^2} (adt - (r^2 + a^2)d\varphi)^2 + \frac{\rho^2}{\Delta(r)} dr^2 + \frac{\rho^2}{\hat{\Delta}(\vartheta)} d\vartheta^2
\]  

\[
\rho^2 := r^2 + a^2 \cos^2 \vartheta, \quad \Delta(r) := -\frac{1}{3} \Lambda r^2 (r^2 + a^2) + r^2 - 2Mr + a^2, \quad \hat{\Delta}(\vartheta) := 1 + \frac{1}{3} \Lambda a^2 \cos^2 \vartheta, \quad I := 1 + \frac{1}{3} \Lambda a^2,
\]

where above and here after always \(\Lambda > 0\). The \(t\)-coordinate takes its values over the real line, the angular coordinates \((\vartheta, \varphi)\) vary in the familiar range, while \(r\) is restricted to a suitable open set of the real line to be specified further bellow. The fields \(\xi_t = \frac{\partial}{\partial t}\) and \(\xi_\varphi = \frac{\partial}{\partial \varphi}\) are commuting Killing fields with the zeros of \(\xi_\varphi\) define the rotation axis. From \([1]\), it follows that the covariant components of \(g\) are:

\[
g_{tt} = -\frac{\Delta(r) - \hat{\Delta}(\vartheta) a^2 \sin^2 \vartheta}{I^2 \rho^2}, \quad g_{t\varphi} = \frac{\Delta(r) - \hat{\Delta}(\vartheta) (r^2 + a^2)}{I^2 \rho^2} asin^2 \vartheta,
\]

Kerr-de Sitter and in order to avoid confusion, we employ the term Carter’s blocks. Further ahead we define these blocks more precisely.
are consecutive roots of \( \Delta(r) = 0 \) and these are also coordinate singularities. Depending upon the values of \( (\Lambda, M, a) \), the quartic equation \( \Delta(r) = 0 \) may admit up to four distinct real roots exhibiting one of the following arrangements (see for instance the discussion in [5]):

a) all four roots are real and distinct arranged according to: \( r_1 < 0 < r_2 < r_3 < r_4 \),

b) all roots are real but \( r_2 \) is doubly degenerate ie \( r_1 < 0 < r_2 = r_3 < r_4 \),

c) all roots are real but \( r_4 \) is doubly degenerate ie \( r_1 < 0 < r_2 < r_3 = r_4 \),

d) the three positive roots coincide ie \( r_1 < 0 < r_2 = r_3 = r_4 \),

e) the equation \( \Delta(r) = 0 \) admits a pair of complex conjugates and a pair of real roots: \( r_1 < 0 < r_2 \).

Whenever possibility a) occurs, after suitable extensions, the resulting spacetime describes a black hole asymptotically de Sitter [2], [3], [4] while whenever the roots are degenerate, the resulting spacetime still describes an asymptotically de Sitter, black hole but now some of the horizons coincide. For instance, if the possibility b) occurs, the inner and outer black hole horizons coincide, for the case c) the outer black hole horizon coincides with the cosmological horizon while the case d) describes a super extreme black hole where inner, outer and one cosmological horizon coincide. Finally the possibility e) describes a pair of cosmological horizons enclosing the ring singularity.

Before we discuss the causality properties of Kerr-de Sitter spacetimes, we define more precisely the notion of a Carter’s block. A Carter’s block\(^4\) denoted hereafter as \((T_{(r_i,r_{i+1})},g)\), is a region of a Kerr-de Sitter spacetime covered by a single Boyer-Lindquist chart \((t,r,\vartheta,\varphi)\) so that the metric \( g \) is described by \([7]\) with the \( r \)-coordinate varying in the open interval\(^5\) \((r_i,r_{i+1})\) where \( r_i, r_{i+1} \) are consecutive roots of \( \Delta(r) = 0 \). The block \((T_{(r_4,\infty)},g)\) respectively \((T_{(-\infty,r_1)},g)\) define the two

\(^2\) Note the possibility that \( \Delta(r) = 0 \) admits two pairs of complex conjugates roots is not compatible with \( \Lambda > 0 \) and \( a^2 > 0 \).

\(^3\) Here, the term horizon should be understood as a Killing horizon.

\(^4\) See also footnote [1].

\(^5\) It should be mentioned that the rotation axis is also included in a Carter’s block.
asymptotic blocks while \((T_{r_1, r_2}, g)\) stands for the block that contains the ring singularity. As we shall see in the next sections, these blocks play an important role in determining the global structure of a Kerr-de Sitter spacetime.

For the purposes of the last section, we introduce the canonical vector fields:

\[
V = (r^2 + a^2) \frac{\partial}{\partial t} + a \frac{\partial}{\partial \varphi}, \quad W = \frac{\partial}{\partial \varphi} + asin^2 \vartheta \frac{\partial}{\partial t}
\]

that are well defined on any Carter's block. These fields combined with \([1]\) imply:

\[
\begin{align*}
g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) &= \frac{\rho^2}{\Delta(r)}, \quad g\left(\frac{\partial}{\partial \vartheta}, \frac{\partial}{\partial \vartheta}\right) = \frac{\rho^2}{\hat{\Delta}(\vartheta)}, \quad g(V, V) = -\frac{\rho^2 \Delta(r)}{I^2}, \quad g(W, W) = \frac{\rho^2 sin^2 \vartheta \hat{\Delta}(\vartheta)}{I^2} \\
g(V, \frac{\partial}{\partial r}) &= g(V, \frac{\partial}{\partial \vartheta}) = g(W, \frac{\partial}{\partial r}) = g(W, \frac{\partial}{\partial \vartheta}) = g(V, W) = 0
\end{align*}
\]

In addition, the gradient fields:

\[
L_t = \nabla^a t \frac{\partial}{\partial x^a}, \quad L_r = \nabla^a r \frac{\partial}{\partial x^a},
\]

satisfy

\[
g(L_t, L_t) = g'' = -\frac{I^2 [\hat{\Delta}(\vartheta)(r^2 + a^2)^2 - \Delta(r)a^2 sin^2 \vartheta]}{\rho^2 \hat{\Delta}(\vartheta) \Delta(r)}, \quad g(L_r, L_r) = \frac{\Delta(r)}{\rho^2}
\]

and formulas \([6, 9]\) will be useful latter on.

**III. ON THE MAXIMAL ANALYTICAL EXTENSION OF THE ROTATION AXIS**

In order to address the causality properties of the family of Kerr-de Sitter spacetimes, at first we discuss their global structure. We recall that in the pioneering analysis in ref.\([6\]), Carter obtained the maximal analytical extension of the Kerr metric guided largely by the structure of the rotation axis of the Kerr metric (see \([7\]) combined with a deep understanding of the behavior of causal geodesics on a Kerr background (actually he assumed a Kerr-Newman background and for details see \([8\])).

In this work\(^6\), we show that the maximal analytical extension of the rotation axis of a Kerr-de Sitter offers clues regarding the global structure of a Kerr-de Sitter spacetime and for this reason first we discuss briefly the maximal analytical extension of the rotation axis of a Kerr-de Sitter spacetime. Invariantly, the rotation axis is a two dimension closed, geodesic submanifold consisting of the zeros of the axial Killing field (see for instance discussion in \([8\])). Since the Boyer-Lindquist chart does not cover this submanifold, we introduce local coordinates \(x = sin \vartheta cos \varphi, y = cos \vartheta sin \varphi\)

\(^6\) We restrict our attention to the case where \(\Delta(r) = 0\) admits four distinct real roots arranged according to \(r_1 < 0 < r_2 < r_3 < r_4\). Once the structure of these spacetimes is understood, it is relatively easy to understand the structure of spacetimes where some of the roots of \(\Delta(r) = 0\) coincide.
so that the $x = y = 0$ defines the rotation axis. Carrying out this transformation, we find that relative to $(t, r, x, y)$ coordinates the induced metric on the axis takes the form (for details see [9]):

$$g_i = -f(r)dt^2 + \frac{dr^2}{f(r)}, \quad f(r) = \frac{\Delta(r)}{r^2 + a^2}, \quad (t, r) \in R \times (r_1, r_i)$$

(10)

where a factor of $I$ has been absorbed in the redefinition of the Killing time. For the case where $\Delta(r) = 0$ admits four distinct real roots $r_1 < 0 < r_2 < r_3 < r_4$, there exist five\(^7\) two dimensional spacetimes in (10) representing disconnected components of the rotation axis and by gluing\(^8\) them together we obtain the maximal analytical extension of the rotation axis.

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FIG. 1: The figure on the left represents the Carter-Penrose diagram for the two dimensional ingoing spacetime $(\Lambda^*, \gamma)$ introduced in the main text and the embedding of the five disconnected blocks in (10) are shown. Ingoing null geodesics are future pointing running from $-\infty$ to $\infty$ and thus are complete. By standard conventions these geodesics are considered as future pointing and thus time-orient $(\Lambda^*, \gamma)$. Incomplete outgoing null geodesics are also indicated. The figure on the right represents the Carter-Penrose diagram for the two dimensional outgoing spacetime $(^*\Lambda, \gamma)$. Here outgoing null geodesics are complete and future pointing running from $-\infty$ to $\infty$ while incomplete ingoing ones are also shown.

As discussed in section IV, these diagrams schematically represent the four dimensional spacetimes $(\Lambda^*, \gamma)$ and $(\Lambda^*, \gamma)$ The left figure represents the four dimensional Eddington-Finkelstein $(\Lambda^*, \gamma)$ while the right figure represents the four dimensional outgoing Eddington-Finkelstein $(\Lambda^*, \gamma)$. In such interpretation, the blocks are four dimensional Carter’s blocks, the null lines marked by $r_1, r_2$ etc represent Killing horizons and principal ingoing resp. outgoing null geodesics are also indicated.

For this procedure, we start from the two dimensional spacetime $(M_1, \gamma_1)$ (see footnote (6)) and introduce ingoing Eddington-Finkelstein coordinates $(v, \hat{r})$ via

$$dv = dt + \frac{dr}{f(r)}, \quad d\hat{r} = dr, \quad t \in (-\infty, \infty), \quad r \in (r_1, r_2).$$

so that

$$g_1 = -f(r)dv^2 + 2dvdr, \quad (v, r) \in R \times (r_1, r_2)$$

(11)

\(^7\) In the following, we denote by $(M_{-\infty}, \gamma_{-\infty})$, $(M_1, \gamma_1)$, $(M_2, \gamma_2)$, $(M_3, \gamma_3)$, $(M_4, \gamma_4)$ the family of the two dimensional spacetimes defined by [10] where in each of these spacetimes the coordinate $r$ takes its values respectively in the intervals: $(-\infty, r_1)$, $(r_1, r_2)$, $(r_2, r_3)$, $(r_3, r_4)$, $(r_4, \infty)$.

\(^8\) For this procedure, at first these two dimensional spacetimes are mapped conformally either into the interior of a diamond configuration or to a half diamond configuration (for details of this mapping see for instance [10], [11], [12]). The spacetimes in (10) defined on $(r_1, r_2), (r_2, r_3), (r_3, r_4)$ are mapped into the interior of a diamond configuration, while those defined on $(-\infty, r_1)$ resp. $(r_4, \infty)$ are mapped onto a half of a diamond configuration. Each of these five spacetimes can be time oriented so that for any block where $\Delta(r) > 0$, either the timelike Killing field $X = \frac{\partial}{\partial t}$ (or the alternative $X = -\frac{\partial}{\partial t}$) can be chosen to provide the future direction, while for any block with $\Delta(r) < 0$ the timelike field $X = -\frac{\partial}{\partial t}$ (or the alternative $X = \frac{\partial}{\partial t}$) provides the future direction.
where above and whenever there is no danger of confusion we write \( r \) instead of \( \hat{r} \). Since this \( g_1 \) is regular over the roots of \( \Delta(r) = 0 \) by making use of the function \( f(r) \) in \( (10) \), we extend \((M_1, g_1)\) by allowing the coordinates \((v, r)\) to run in \( R \times R \) and denote the extended spacetime by \((K\Lambda^*, g)\) with the extended metric \( g \) defined by:

\[
g = -f(r)dv^2 + 2dvdr, \quad (v, r) \in R \times R.
\]

The so defined \((K\Lambda^*, g)\) is referred to as a two dimensional ingoing Eddington-Finkelstein and has the property that \( v = \text{const}, -r \in (-\infty, \infty) \) represents the ingoing, complete family of radial null geodesics with \(-r \in (-\infty, \infty)\) acting as an affine parameter. This family has \( L = -\frac{\partial}{\partial r} \) as the tangent null vector field and is accustomed to consider \( L \) as been future pointing thus providing the global time orientation on \((K\Lambda^*, g)\).

It is not difficult to verify that any of the five two dimensional spacetimes in \( (10) \), can be isometrically embedded as open submanifolds within \((K\Lambda^*, g)\). For instance starting from \((M_2, g_2)\) we introduce ingoing Eddington-Finkelstein coordinates \((\hat{v}, \hat{r})\) via

\[
d\hat{v} = dt + \frac{dr}{|f(r)|}, \quad d\hat{r} = dr, \quad t \in (-\infty, \infty), \quad r \in (r_2, r_3)
\]

so that \( g_2 \) takes the form

\[
g_2 = |f(r)|dt^2 - \frac{1}{|f(r)|}dr^2 = |f(r)|d\hat{v}^2 - 2d\hat{v}d\hat{r}, \quad (\hat{v}, \hat{r}) \in R \times (r_2, r_3) \tag{12}
\]

and embed this \((M_2, g_2)\) within \((K\Lambda^*, g)\) via the map:

\[
\Phi : M_2 \to K\Lambda^* : (\hat{v}, \hat{r}) \to \Phi(\hat{v}, \hat{r}) = (v(\hat{v}, \hat{r}), r(\hat{v}, \hat{r})) = (-\hat{v}, \hat{r}). \tag{13}
\]

which clearly is a smooth isometry of \( M_2 \) onto \( \Phi(M_2) \). For the case of \((M_3, g_3)\) the isometry map \( \Phi \) has the same form as the one described in \( (13) \) with the only exception that \(-\hat{v} \) is replaced by \( \hat{v} \) and so on. In view of these embeddings, the conformal Carter-Penrose diagram for \((K\Lambda^*, g)\) has the form shown in the left diagram of Fig.1.

So far we have focused our attention on the ingoing family of null geodesics, however identical considerations hold for the outgoing family. For instance considering again \((M_1, g_1)\), we introduce outgoing Eddington-Finkelstein coordinates \((u, \hat{r})\) via:

\[
du = dt - \frac{dr}{f(r)}, \quad d\hat{r} = dr, \quad t \in (-\infty, \infty), \quad r \in (r_1, r_2)
\]

so that \( g \) takes the form:

\[
g_1 = -f(r)du^2 - 2dudr, \quad (u, r) \in R \times (r_1, r_2). \tag{14}
\]

Via identical reasoning as for those that lead us to \((K\Lambda^*, g)\), we introduce the outgoing Eddington-Finkelstein spacetime \((^*K\Lambda, g)\) with

\[
g = -f(r)du^2 - 2dudr, \quad (u, r) \in R \times R, \tag{15}
\]

which is regular over the entire domain of the radial coordinate \( r \). For this \((^*K\Lambda, g)\), the outgoing family of null geodesics is described by \( u = \text{const}, r \in (-\infty, \infty) \) with \( r \) acting as an affine parameter. This family has \( L = \frac{\partial}{\partial r} \) as the tangent null vector field taken to be future pointing and thus defining
the global time orientation on \( (KΛ, g) \). As for the case of \( (KΛ^*, g) \), the remaining blocks in \( (10) \) can be isometrically embedded as open submanifolds within \( (KΛ, g) \) so that the resulting Carter-Penrose diagram is the right diagram shown in Fig.1.

The final step leading to an extension of the rotation axis consists of gluing together the two diagrams in Fig.1 in a manner that the radial ingoing and outgoing null geodesics are complete. Here, some care is required so that the gluing procedure yields an extended spacetime admitting a consistent time orientation. One way to achieve this is to start from a copy of an ingoing Eddington-Finkelstein spacetime \( (KΛ^*, g) \) shown in Fig.1, and on a specific block introduce simultaneously outgoing Eddington-Finkelstein coordinates. Subsequently extend that block in the future direction by appending a part of the outgoing Eddington-Finkelstein spacetime and making sure that the resulting spacetime admits a consistent time orientation. Leaving details aside, the resulting Carter-Penrose diagram is shown in Fig.2.

In finishing this section, we mention that the employment of Eddington-Finkelstein coordinates as the means to construct the Carter-Penrose diagram shown in Fig.2, does not cover the vertex where four horizons meet. However, this deficiency can be removed by introducing Kruskal-like coordinates which are well defined provided the roots of \( f(r) = 0 \) are all simple roots. Here, we gloss over these details (they are discussed in [12],[13]), but we only mention that the extension shown in Fig.2 is a maximal analytical extension of the rotation axis. Maximality follows by verifying that any causal geodesics on this two dimensional spacetime is actually complete while the analytical nature of the extension follows from the analyticity of the function \( f(r) \) in \( (10) \).

IV. ON THE MAXIMAL ANALYTICAL EXTENSION OF A KERR-DE SITTER SPACETIME

Even though the construction of the maximal analytical extension of the rotation axis of a Kerr-de Sitter spacetime was a relatively easy task, the construction of the maximal analytical extension of the four dimension Kerr-de Sitter is not straightforward. An extension could be obtained by starting from a four dimensional Carter’s block and follow the same steps as those followed by Carter in extending the Kerr family (for details see [6]). However in this approach in order to address the maximality property of an extension a detailed understanding of the behavior of causal geodesics on a Kerr-de Sitter background is required. Although causal geodesics on a Kerr-de Sitter has been the subject of many investigations, these investigations restricted to particular families of geodesics, such as the family of equatorial ones [14], polar [15], spherical [16] or geodesics confined on a particular Carter’s block [17]. In a recent work [9], the completeness property of geodesics defined on an arbitrary Carter’s block has been addressed and evidences presented supporting the view that all causal geodesics defined initially on a Carter’s block can be extended as geodesics through Killing horizons so they become complete except for those geodesics that hit the ring singularity. Due to this incomplete understanding of causal geodesics on a Kerr-de Sitter background, bellow

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9 As far as we are aware, the maximal analytical extension of a four dimension Kerr-de Sitter has not been addressed in the literature before. Often and by analogy to what occurs for the Kerr case, the Carter-Penrose diagram representing the axis of a Kerr-de Sitter is interpreted as representing the maximal analytical extension of the four dimension Kerr de Sitter. Although likely this will be the case, we are not aware of any detailed work supporting this interpretation.

10 Even though, we believe that the results of ref. [9], i.e. the statement that all causal geodesics are complete except those that hit the ring singularity holds true, unfortunately more work is needed. For instance the completeness property of geodesics hitting the bifurcation spheres, or the completeness property of geodesics through the axis have to be worked out. These issues are under investigation and will be reported elsewhere [13].
FIG. 2: The Carter-Penrose diagram for the two dimensional rotation axis of a Kerr-de Sitter spacetime. In order to arrive at this diagram, we have added half-diamond configurations in the two diagrams of Fig.1 and kept extending in all directions. As we discussed in section IV, this diagram can represents schematically the structure of a four dimensional Kerr-de Sitter spacetime. When interpreted in that way, the red dashed lines remind the reader of the location of the ring singularity, while the curved blue line representing $r \to \pm \infty$ signify that these region are distinct. For this spacetime the causality violating regions is the union of the blocks containing the ring singularity. If the topology of the spacetime is altered by identifying for instance asymptotic regions, then the causality violating regions are altered as well.

we outline an extension of a Carter’s block by employing a formalism developed in ref. [8]. This formalism is based on the idea that two smooth manifolds $M$ and $N$ admitting two isometric open subsets $(U, V)$ with $U \subset M$ and $V \subset N$, can be glued along these subsets so that a new smooth manifold $Q$ is obtained. If $\mu : U \to V$ stands for the admitted isometry the resulting manifold $Q$ is defined as the quotient space $Q = (M \cup N) \setminus \sim$ under a suitable equivalence relation $\sim$ and for proofs regarding the manifold structure, smoothness, Hausdorff property of the resulting manifold $Q$ the reader is referred to ref. [8].

To see how this abstract setting applies to our problem, we begin with a block $(T_{(r_i, r_{i+1})}, g)$, and at first introduce ingoing Eddington-Finkelstein coordinates $(v, \varphi, r, \vartheta)$ via\\(^{11}\): \[dv = dt + \frac{I(r^2 + a^2)}{\Delta(r)} dr, \quad d\varphi = d\vartheta + \frac{Ia}{\Delta(r)} dr\] (16) so that $g$ in (1) takes the form:

\[g = -\frac{\Delta(r) - a^2 \dot{\Delta}(\vartheta) \sin^2 \vartheta}{I^2 \rho^2} dv^2 + \frac{2}{I} dv dr - 2 \frac{a}{I} \sin^2 \vartheta d\varphi dr - 2 \frac{\sin^2 \vartheta [(r^2 + a^2) \Delta(\vartheta) - \Delta(r)]}{I^2 \rho^2} dv d\varphi + \]

\\(^{11}\) These coordinates are based on the family of principal null congruence admitted by a Kerr-de Sitter metric. For an introduction to these congruences and their role for constructing the Eddington-Finkelstein charts see for instance section V of ref. [9].
which is regular metric across the zeros of $\Delta(r) = 0$. Allowing the coordinates $(v, r)$ to run over the entire real line, we obtain the ingoing Kerr-de Sitter spacetime $(K\Lambda^*, g)$ where clearly $(T_{(r_i, r_{i+1})}, g)$ is now an open submanifold. Moreover, if we define the map:

$$J^*: T_{(r_i, r_{i+1})} \to J^*(T_{(r_i, r_{i+1})}) \subset KL^*: (t, r, \vartheta, \varphi) \to J^*(t, r, \vartheta, \varphi) = (r, \vartheta, v(t, r), \tilde{\varphi}(\varphi, r))$$

with $(v(t, r), \tilde{\varphi}(\varphi, r))$ the functions that (16) assigns to $(t, \varphi)$, then this $J^*$ isometrically embeds the rest of the Carters block within $(KL^*, g)$. This embedding has the property that the interfaces $r = r_i$ becomes a Killing horizons and schematically these embeddings are shown by the left digram in Fig.1, where now the blocks should be viewed as a four dimensional region enclosed between Killing horizons or Killing horizons and the asymptotic de Sitter regions. The four dimensional spacetime $(KL^*, g)$ has the property that all ingoing principal null geodesics are complete but the corresponding outgoing ones fail to be so. In order to achieve completeness of the latter congruence, we introduce outgoing Eddington-Finkelstein coordinates via:

$$du = dt - \frac{I(r^2 + a^2)}{\Delta(r)} dr, \quad d\tilde{\varphi} = d\varphi - \frac{Ia}{\Delta(r)} dr$$

and relative to these coordinates, $g$ takes a form identical to that in (17), except that $(v, \tilde{\varphi})$ are replaced by $(u, \tilde{\varphi})$ and the signs in the cross terms $(dr du)$ and $(d\tilde{\varphi} dr)$ are now reversed. By allowing the coordinates $(u, \tilde{\varphi})$ to run over the entire real line we obtain the outgoing Kerr-de Sitter denoted by $(k\Lambda, g)$ and by the same reasoning as above the map:

$$^*J: T_{(r_i, r_{i+1})} \to ^*J(T_{(r_i, r_{i+1})}) \subset ^*KL: (t, r, \vartheta, \varphi) \to ^*J(t, r, \vartheta, \varphi) = (r, \vartheta, u(t, r), \tilde{\varphi}(\varphi, r))$$

with $(u(t, r), \tilde{\varphi}(t, \varphi))$ the coordinates that (19) assigns to $(t, \varphi)$, embeds isometrically any Carters block within $(k\Lambda, g)$. These embeddings are shown in the right diagram of Fig.1 where again the blocks are considered as four dimensional regions enclosed between Killing horizons.

As for the case of the rotation axis, the crucial next involves assembling the four dimensional ingoing $(KL^*, g)$ and outgoing $(k\Lambda, g)$ incomplete spacetimes and here the gluing process discussed in the section 1.4 of O.Neil’s book comes into play. For this, let $(KL^*, g)$ stands for $M$ and $(k\Lambda, g)$ for $N$ and moreover let $B$ stands for any of the $(T_{(r_i, r_{i+1})}, g)$ blocks. Clearly the open sets $B^* = J^*(T_{(r_i, r_{i+1})})$ of $(KL^*, g)$ and $^*B = ^*J(T_{(r_i, r_{i+1})})$ of $(k\Lambda, g)$ are naturally isometric via

$$\mu := ^*J(J^*)^{-1}: B^* \to ^*B: (r, \vartheta, v, \tilde{\varphi}) \to \mu(r, \vartheta, v, \tilde{\varphi}) = r, \vartheta, -2 \int^r \frac{I(r^2 + a^2)}{\Delta(r)} d\hat{r}, -2 \int^r \frac{Ia}{\Delta(r)} d\hat{r}.$$ 

It can be seen that this map is an isometry and provides the most important ingredient for the gluing process. Via this isometry, $(KL^*, g)$ and $(k\Lambda, g)$ can be glued along the isometric copies $U = B^*$ and $V = ^*B$. Although we leave many details to be discussed elsewhere, nevertheless
the resulting spacetime has complete outgoing respectively ingoing principal null geodesics and a schematic representation is shown in Fig. 2. (see the last paragraph in the caption accompanying Fig.2.).

V. ON THE CAUSAL PROPERTIES OF KERR-DE SITTER SPACETIMES

The discussion of the previous section combined with the diagram of Fig.2, offers a view of the global structure of a Kerr-de Sitter for the case that the parameters \((M,a,\Lambda)\) have been chosen so that \(\Delta(r) = 0\) admits four distinct real roots\(^{13}\). In this section, we analyze the causality properties of the four dimensional spacetime shown in Fig.2 and at first we identify the structure of the Killing horizons. Starting from ingoing coordinates \((v,\tilde{\varphi}, r, \vartheta)\), the normal vector \(N\) of any \(r = \text{const}\) hypersurface, has the form:

\[
N = \hat{g}^{\mu\nu} \delta^r \frac{\partial}{\partial x^\mu} = \hat{g}^{rr} \frac{\partial}{\partial x^r}, \quad x^\mu = (v,\tilde{\varphi}, r, \vartheta)
\]  

(22)

where \(\hat{g}^{\mu\nu}\) stand for the contravariant components of \(g\) relative to the ingoing coordinates. Since

\[
g(N, N) = \hat{g}^{rr} = g^{rr} = \frac{\Delta(r)}{\rho^2}
\]  

(23)

it follows that the set \(r = r_i\) defines a null hypersurface\(^{14}\). For each real root \(r_i\) of \(\Delta(r) = 0\), we define the constants

\[
\Omega_i = -\frac{g(\xi_t,\xi_\varphi)}{g(\xi_\varphi,\xi_\varphi)} = \frac{a}{r_i^2 + a^2}.
\]  

(24)

and introduce the Killing fields

\[
\hat{\xi}_i = \xi_t + \Omega_i \xi_\varphi = \frac{\partial}{\partial v} + \Omega_i \frac{\partial}{\partial \tilde{\varphi}}
\]  

(25)

which becomes null precisely over the \(r = r_i\) hypersurfaces. A computation shows that

\[
\nabla^\mu [g(\hat{\xi}_i, \xi)] = -2k_i \hat{\xi}_i^\mu
\]  

(26)

which establish the Killing property of the \(r = r_i\) hypersurfaces. The coefficients \(k_i\) stand for the surface gravity\(^{15}\) of the \(r_i\) horizon and they are given by (see ref. [9])

\[
k_i = \frac{1}{2I} \frac{1}{r_i^2 + a^2} \frac{\partial \Delta(r)}{\partial r}
\]  

(27)

where the derivative of \(\Delta(r)\) is evaluated at \(r = r_i\). In the limit of \(\Lambda \to 0\) these \(k_i\) reduce to the surface gravity for the Killing horizons of the Kerr black hole (compare with the results in ref.[19]).

---

\(^{13}\) The maximality property of the diagram in Fig.2 ought to be work out in details and establishing this property it is not a trivial task. For the rest of this section we assume that the extension shown in Fig.2 is maximal and we discuss consequences of this assumption.

\(^{14}\) The term \(\frac{\Delta(r)}{\rho^2}\) is well defined over the entire domain of validity of the ingoing chart and this coupled with the fact that the left hand side of \((23)\) is an analytic function relative to ingoing coordinates shows that the claim is not based on Boyer-Lindqust coordinates. The latter have been used only as an intermediate step.

\(^{15}\) Our convention for the surface gravity follows the same conventions as those in Wald’s book ref.[19].
and moreover (27) shows that any Killing horizon corresponding to a double or higher multiplicity root of $\Delta(r) = 0$ is degenerate. Identical computations based on the outgoing $(u, \varphi, r, \vartheta)$ coordinates shows that the sets $r = r_i$ are null hypersurfaces and in fact are Killing horizons whose surface gravities $k_i$ are still described by (27).

The Killing horizons defined above play an important role in determining the causality violating region of any Kerr-de Sitter spacetime. Since any Killing horizon [8] is an achronal set (for properties of these sets see [18], [19], [8]) no timelike future directed curve meets a Killing horizon more than once. In turn this property imply that the causal properties of the extended Kerr-de Sitter are determined by the causal properties of the Carters blocks. However the causality properties of these blocks can be easily inferred and we begin by first proving the following proposition:

**Proposition 1** Any Carters block having the property that $\Delta(r) < 0$, is stably causal (see the Appendix for a brief discussion of the terms involved).

**Proof.** The formulas in (7) combined with the property $\Delta(r) < 0$, imply that the vector field $X = \frac{\partial}{\partial r}$ is timelike and nowhere vanishing within the block under consideration and thus it can time-orient the block. Moreover, the gradient $L_r = \nabla^a r \frac{\partial}{\partial x^a}$ satisfies $g(L_r, L_r) = \frac{\rho^2}{\Delta(r)} < 0$ and thus is timelike. Accordingly if $X = \frac{\partial}{\partial r}$ is chosen to define the future part of the light cone, then $\tau = -r$ serves as a time function while for the alternative choice i.e if $X = -\frac{\partial}{\partial r}$ specifies the future part of the light cone, then $\tau = r$ serves as a time function. For any choice, all conditions of the Theorem I cited in the Appendix are met and thus any Carters block subject to $\Delta(r) < 0$ is stably causal.

**Proposition 2** Any Carters block with $\Delta(r) > 0$ is stably causal, except for the block that contains the ring singularity.

**Proof.** Again from the formulas in (7), we have $g(V, V) = -\frac{\rho^2 \Delta(r)}{I^2}$, and thus the vector field $V$ time-orients the block under consideration (remember the block under consideration does not contain the ring singularity). In order to construct a time function, we appeal to the gradient field $L_t = \nabla^a t \frac{\partial}{\partial x^a}$ which satisfies:

$$g(L_t, L_t) = g^{tt} = -\frac{I^2 [\hat{\Delta}(\vartheta)(r^2 + a^2)^2 - \Delta(r)a^2 \sin^2 \vartheta]}{\rho^2 \hat{\Delta}(\vartheta) \Delta(r)}.$$

Moreover a computation of the numerator shows:

$$[\hat{\Delta}(\vartheta)(r^2 + a^2)^2 - \Delta(r)a^2 \sin^2 \vartheta] =$$

$$= (r^2 + a^2)(r^2 + a^2 \cos^2 \vartheta) + 2Mr^2a^2 \sin^2 \vartheta + \frac{\Lambda a^2}{3}(r^2 + a^2)[(r^2 + a^2)\cos^2 \vartheta + r^2 \sin^2 \vartheta]$$

and thus as long as $r > 0$, the right hand side is positive definite, which means that $L_t = \nabla^a t \frac{\partial}{\partial x^a}$ is everywhere timelike on any block where $\Delta(r) > 0$ and $r > 0$. In addition, from the formulas (4, 5) and (7) it we find the identity:

$$L_t = -\frac{I^2 (r^2 + a^2)}{\rho^2 \Delta(r)} V + \frac{I^2 a}{\rho^2 \hat{\Delta}(\vartheta)} W.$$

Since $W$ is spacelike, this identity shows that $\tau = t$ serves as a time function whenever $X = V$ defines the future part of the light cone, while for the choice that $X = -V$ defines the future part,
then $\tau = -t$ serves as a time function. In any case, the proof of the proposition is established by appealing to the theorem $I$ of the Appendix.

We now consider the block that contains the ring singularity. Even though on this block $\Delta(r) > 0$, since now $r$ can take negative values, the right hand side of (28) fails to be positive definite and thus the argument leading to the proof of the proposition 2 fails. Instead we have the following proposition:

**Proposition 3** The block that contains the ring singularity is totally vicious in the sense of Carter: Any two events $I, F$ within this block, can be connected by a future (resp. past) directed timelike curve lying entirely within the block.

**Proof.** The proof of this proposition is long. At first we show that there is a non empty region in this block where the axial Killing field $\xi_\phi$ becomes timelike ie $g(\xi_\phi, \xi_\phi) < 0$. This region, defines the Carter’s time machine\textsuperscript{16} denoted by CTM via:

$$CTM = \{(t, r, \vartheta, \phi) \, | \, g(\xi_\phi, \xi_\phi) < 0\}.$$  \hspace{1cm} (29)

As long as this CTM is non empty, we prove that for any two events $I = (t_0, r_0, \vartheta_0, \phi_0)$ and $F = (t_f, r_f, \vartheta_f, \phi_f)$ within this block can be joined by a piecewise smooth, future (resp. past) directed timelike curve. In the proof that follows, we restrict attention to the construction of the future directed timelike curve, and this curve consists of three components: the first and third component joins $I$ respectively $F$ to a point on the equatorial plane of the CTM, while the intermediate piece consists of a future directed timelike curve that joins the two points lying on the equatorial plane of the CTM. The first and the last component are actually independent of the CTM, but the intermediate one depends crucially upon the non empty property of the CTM.

We begin by noting that in the block under consideration $g(V, V) = -\frac{\rho^2 \Delta(r)}{T^2}$ and thus the vector field $V$ time-orients the tangent space for any event on this block (points on the ring singularity are not considered as part of the spacetime). Moreover the axial Killing field satisfies:

$$g(\xi_\phi, \xi_\phi) = g_{\phi\phi} = \frac{\sin^2 \vartheta}{T^2 \rho^2} [\hat{\Delta}(\vartheta) (r^2 + a^2)^2 - \Delta(r) a^2 \sin^2 \vartheta]$$  \hspace{1cm} (30)

and upon using (28) we find:

$$g(\xi_\phi, \xi_\phi) = \frac{\sin^2 \vartheta}{T^2 \rho^2} \left[ (r^2 + a^2)(r^2 + a^2 \cos^2 \vartheta) + 2Ma^2 \sin^2 \vartheta + \frac{\Lambda a^2}{3} (r^2 + a^2) (r^2 + a^2 \cos^2 \vartheta + r^2 \sin^2 \vartheta) \right]$$  \hspace{1cm} (31)

Since in the block under consideration $r$ can take negative values, the term in the square bracket can be negative. Indeed by evaluating the right hand side on the $\vartheta = \frac{\pi}{2}$ equatorial plane, we find

$$g(\xi_\phi, \xi_\phi) = \frac{1}{T^2} [(r^2 + a^2)(1 + \frac{\Lambda a^2}{3} + \frac{2Ma^2}{r})]$$  \hspace{1cm} (32)

\textsuperscript{16} In the present context, a time machine is a spacetime region that can generate closed timelike curves passing through any point in the spacetime under consideration. Here the region defined in (29) acts as a time machine for the block that contains the ring singularity.
and thus for sufficiently small negative \( r \), \( g(\xi, \xi) < 0 \) i.e. \( \xi \) becomes timelike. By continuity arguments the CTM defined in (29) is a non empty spacetime region. Since the orbits of \( \xi \) are closed curves around the rotation axis, therefore near the ring singularity and for \( r < 0 \), causality violations take place in the sense that at any event \( q \) so that \( g(\xi, \xi) < 0 \) there exist a closed timelike curve through \( q \). However the non empty property of the CTM has far reaching consequences upon the causality properties of the block. The entire block is a vicious set in the sense of Carter [6]: any event on the block containing the ring singularity can be connected to any other event within the same block by a future (resp. past) directed timelike curve.

In order to prove this highly counterintuitive property, let two arbitrary events \( I = (t_i, r_i, \vartheta_i, \varphi_i) \) and \( F = (t_f, r_f, \vartheta_f, \varphi_f) \) within the block that contains the ring singularity. At first we construct a future directed timelike curve that joins \( I \) to a point on the equatorial plane of the CTM. For this, let on the \( \{(r, \vartheta)\} \)-plane the smooth non intersecting curve \( \gamma(\lambda) = (r(\lambda), \vartheta(\lambda)), \lambda \in [0, 1] \) that joins \((r_i, \vartheta_i) = (r(0), \vartheta(0))\) to a point \((r, \frac{\pi}{2}) = (r(1), \vartheta(1))\) on the equatorial plane of the CTM. Its tangent vector \( \dot{\gamma} \) satisfies

\[
g(\dot{\gamma}, \dot{\gamma}) = \rho(\lambda)^2 \left[ \frac{(\dot{r}(\lambda))^2}{\Delta(r(\lambda))} + \frac{(\dot{\vartheta}(\lambda))^2}{\Delta(\vartheta(\lambda))} \right], \quad \lambda \in [0, 1]
\]  

and the smoothness of \( \gamma \) combined with the compactness of the domain \([0, 1]\) imply that the right hand side is bounded on \([0, 1]\). Utilizing the integral curves of the vector field \( \dot{\gamma} = (\gamma, \varphi, Aa\lambda, t_i + At(\lambda)) \) we define a new curve:

\[
\dot{\gamma}(\lambda) = (\gamma(\lambda), \varphi, + Aa\lambda, t_i + At(\lambda)), \quad \lambda \in [0, 1]
\]  

where \( t(\lambda) \) satisfies \( \dot{t}(\lambda) = r(\lambda) + a^2 \) and \( A > 0 \) is a constant. The tangent vector \( \dot{\gamma} \) satisfies

\[
\dot{\gamma} = \dot{\gamma} + AV, \quad g(\dot{\gamma}, \dot{\gamma}) = g(\dot{\gamma}, \dot{\gamma}) - \frac{A^2 \Delta(r) \rho^2}{I^2}, \quad g(\dot{\gamma}, V) = Ag(V, V) < 0
\]  

and thus by choosing \( A \) sufficiently large, \( \dot{\gamma} \) is timelike and future pointing joining \( I = (t_i, r_i, \vartheta_i, \varphi_i) \) to the event \((t_i + At(1), r, \frac{\pi}{2}, \varphi + Aa)\) on the equatorial plane of the CTM.

By interchanging \( I = (t_i, r_i, \vartheta_i, \varphi_i) \) for \( F = (t_f, r_f, \vartheta_f, \varphi_f) \), we consider the curve

\[
\dot{\gamma}_1(\lambda) = (\gamma_1(\lambda), \varphi_f - Aa\lambda, t_f - At(\lambda)), \quad \lambda \in [0, 1], \quad A > 0
\]  

where now \( \gamma_1(\lambda) = (r_1(\lambda), \vartheta_1(\lambda)) \) satisfies: \((r_1(0), \vartheta_1(0)) = (r_f, \vartheta_f), (r_1(1), \vartheta_1(1)) = (r, \frac{\pi}{2})\). Clearly \( \dot{\gamma}_1 \) is timelike but it is past directed joining \( F = (t_f, r_f, \vartheta_f, \varphi_f) \) to the point \((t_f + At(1), r_1, \frac{\pi}{2}, \varphi_f + Aa)\) lying on the equatorial plane of the CTM. For later use note, that by reversing the parametrization in (36) the resulting \( \dot{\gamma}_1 \) defines a future pointing timelike curve joining \((t_f + At(1), r_1, \frac{\pi}{2}, \varphi_f + Aa)\) to \( F = (t_f, r_f, \vartheta_f, \varphi_f) \).

We now prove that two arbitrary events \( A \) and \( B \) on the equatorial plane of the CTM can be joined by a future directed timelike curve. To prove this, let us first consider the special events

\[\text{\footnotesize{[\ref{footnote:CTM}] In this section, by the term equatorial plane of the CTM we mean the collection of events coordinatized by: } (t, r, \frac{\pi}{2}, \varphi) \text{ with } -\infty < t < \infty, \varphi \text{ varying in the usual range while a negative } r \text{ is subject to the restriction: } g(\xi, \xi)_{\varphi - \frac{\pi}{2}} < 0.\]
\( A = (t_0, r_0, \frac{\pi}{2}, \varphi_0) \) and \( B = (t_0, \dot{r}_f, \frac{\pi}{2}, \dot{\varphi}_f) \) on this plane and let the curve:

\[
\dot{\gamma}_2(\lambda) = (t_0, r(\lambda), \frac{\pi}{2}, \varphi_0 + (\dot{\varphi}_f - \varphi_0 + 2\pi n)\lambda), \quad \lambda \in [0, 1]
\]  

(37)

with \( r(\lambda) \) a smooth function obeying: \( r(0) = r_0, r(1) = r_f \) and \( n \) an arbitrary positive integer. For this curve the tangent vector \( \dot{\gamma}_2 \) satisfies

\[
g(\dot{\gamma}_2, \dot{\gamma}_2) = \rho(\lambda)^2 \left( \frac{\dot{r}(\lambda))^2}{\Delta(r(\lambda))} + (\dot{\varphi}_f - \varphi_0 + 2\pi n)^2 g(\xi_\varphi, \xi_\varphi), \quad g(\dot{\gamma}_2, V) = (\dot{\varphi}_f - \varphi_0 + 2\pi n)g(\varphi, \varphi) < 0. \]  

(38)

and since \( g(\xi_\varphi, \xi_\varphi) < 0 \) within the CTM, by choosing \( n \) sufficiently large \( \dot{\gamma} \) is timelike and future directed joining \( A = (t_0, r_0, \frac{\pi}{2}, \varphi_0) \) to the event \( B = (t_0, \dot{r}_f, \frac{\pi}{2}, \dot{\varphi}_f) \).

For the general case where \( A = (t_0, r_0, \frac{\pi}{2}, \varphi_0) \), and \( B = (\dot{t}_f, \dot{r}_f, \frac{\pi}{2}, \dot{\varphi}_f) \) with \( T = \dot{t}_f - t_0 \), we consider the curve \( \dot{\gamma}(\lambda) \) in (37) that joins \( A = (t_0, r_0, \frac{\pi}{2}, \varphi_0) \) to the event \( (t_0, \dot{r}_f, \frac{\pi}{2}, \dot{\varphi}_f) \) but also introduce two new curves

\[
\delta_\epsilon(\lambda) = (t_0 + \epsilon \lambda, r_{\dot{f}}, \frac{\pi}{2}, \dot{\varphi}_f - b \lambda), \quad \lambda \in [0, T], \quad \epsilon = \pm 1
\]  

(39)

which joins \( (t_0, \dot{r}_f, \frac{\pi}{2}, \dot{\varphi}_f) \) to \( B = (t_0 + \epsilon T, r_{\dot{f}}, \frac{\pi}{2}, \dot{\varphi}_f) \) provide we take \( b = \frac{2\pi n}{T} \) where \( n \) a non zero integer. For these curves tangent vector \( \delta_\epsilon = \epsilon \frac{\partial}{\partial \tau} - b \frac{\partial}{\partial \varphi} \) satisfies:

\[
g(\dot{\delta}_\epsilon, \dot{\epsilon}) = \epsilon^2 g(\xi_\tau, \xi_\tau) - 2\epsilon b g(\xi_\tau, \xi_\varphi) + b^2 g(\xi_\varphi, \xi_\varphi)
\]

(40)

\[
g(\dot{\delta}_\epsilon, V) = \epsilon(r^2 + a^2)g(\xi_\tau, \xi_\tau) + \epsilon a g(\xi_\varphi, \xi_\varphi) - b(r^2 + a^2)g(\xi_\tau, \xi_\varphi) + a g(\xi_\varphi, \xi_\varphi)].
\]  

(41)

From (40), it is seen that by taking \( b^2 \) large enough, \( \delta_\epsilon \) is timelike irrespectively of the value of \( \epsilon \). Moreover working out the right hand side of the (41) by evaluating (23) on the equatorial plane we find

\[
g(\dot{\delta}_\epsilon, V) = -\frac{\Delta(r)}{T^2}(\epsilon + ab)
\]  

(42)

and since on \( \Delta(r) > 0 \), therefore the curve \( \delta_1 \) which joins \( A = (t_0, r_0, \frac{\pi}{2}, \varphi_0) \), to \( B = (\dot{t}_f, \dot{r}_f, \frac{\pi}{2}, \dot{\varphi}_f) \) with \( t_f = T + t_0 \), is timelike and future directed. On the other hand \( \delta_{-1} \) which joins\(^{18} \) \( A = (t_0, r_0, \frac{\pi}{2}, \varphi_0) \), to \( B = (\dot{t}_f, \dot{r}_f, \frac{\pi}{2}, \dot{\varphi}_f) \) with \( t_f = t_0 - T \), is timelike and future directed provide we choose \( b > a^{-1} \).

Summarizing, the so far analysis shows that any two events \( A = (t_0, r_0, \frac{\pi}{2}, \varphi_0) \) and \( B = (\dot{t}_f, \dot{r}_f, \frac{\pi}{2}, \dot{\varphi}_f) \) on the equatorial plane of the CTM and irrespectively whether \( T = \dot{t}_f - t_0 \) is positive, negative or zero, always can be joined by a future directed timelike curve lying within the equatorial plane of the CTM.

Clearly this conclusion holds for the particular case where \( A \) and \( B \) are chosen so that \( A = (t_i + At(1), r, \frac{\pi}{2}, \varphi_i + Aa) \) and \( B = (t_f + At(1), r, \frac{\pi}{2}, \varphi_f + Aa) \). These two events can be

\(^{18} \) It is worth pointing out here an important difference between the curves \( \delta_{-1} \) introduced above. While both are timelike and future pointing note that \( \delta_1(t) > 0 \) implying that \( t \) increases along \( \delta_1 \) while for the case of \( \delta_{-1} \) we have \( \delta_{-1}(t) < 0 \) it the coordinate \( t \) decreases as one moves along \( \delta_{-1} \). It is this property of the curve \( \delta_{-1} \) which is responsible for traveling backward in time. An observer following \( \delta_{-1} \) while moves towards to the future, as a consequence of \( \delta_{-1}(t) < 0 \) the value of the Boyer-Lindquist \( t \) steadily reduces.
joined by a timelike future directed curve lying on the CTM and this conclusion almost proves the proposition. Indeed starting from the event \( I = (t_0, r_0, \vartheta_0, \varphi_0) \), the future directed timelike curve in (34) joins \( I \) to the event \( A = (t_i + At(1), r, \frac{\pi}{2}, \varphi_i + Aa) \) on the equatorial plane of the CTM while the timelike and future directed curve \( \hat{\gamma}_2 \) in (37) combined with one of the timelike and future directed curves \( \delta_1 \) or \( \delta_{-1} \) connects \( A = (t_i + At(1), r, \frac{\pi}{2}, \varphi_i + Aa) \) to \( B = (t_f - At(1), r_1, \frac{\pi}{2}, \varphi_f - Aa) \). Finally, the future directed timelike \( \hat{\gamma}_1 \) in (36) (with reversed parametrization) connects this \( B \) to the event \( F = (t_f, r_f, \vartheta_f, \varphi_f) \). Thus the non empty property of the CTM enable us to connect the arbitrary events \( I \) and \( F \) by a (piecewise) timelike, future directed curve starting from \( I \) and terminating at \( F \).

To complete the proof of the proposition, we need show that the events \( I \) and \( F \) can be connected by a timelike but past directed curve. The proof of this claim can proceed along the same lines as for the case of the future curve but here we follow a short cut that avoids this long procedure. The existence of a timelike past directed curve joining \( I \) to \( F \) can be inferred by noting that by interchanging \( I \) by \( F \), the previous analysis shows the existence of a future directed timelike curve from \( F \) to \( I \). Hence by a parametrization reversal, this future directed curve becomes a past directed timelike curve connecting \( I \) to \( F \) and this conclusion completes the proof of the proposition.

In the limit that \( \Lambda \to 0 \), our analysis shows that for the Kerr spacetime the Boyer-Lindquist block that contains the ring singularity is also a vicious set i.e. we arrive at the well known results obtained by Carter (see [6]). Carter arrived at this conclusion by appealing to the properties of the two dim transitive Abelian isometry group acting on the background Kerr (or Kerr-Newman) spacetime. Even though his method likely can cover the case of Kerr-de Sitter, in this work, we have chosen an alternative proof which even though pedestrian, nevertheless makes clear the role of the CTM in destroying any element of causality. Our approach is along the lines of a proof outlined in ref. [8] although in the present work the background is different than the one in ref. [8] and we use different representation of the (highly non unique) family of causal curves that joining the events under consideration. Also Chrusciel in [20] discuss qualitatively properties of the CTM for the case of Kerr background. Our detail proof of the last proposition aims to point out that even a tiny non empty CTM converts the entire block to a vicious set where any notion of causality is lost. Throughout any event on this block, the CTM generates a closed timelike through this event. This last point it is not stressed enough in the current literature.

VI. DISCUSSION

In this work, the causality properties of the family of Kerr-de Sitter spacetimes have been worked out and the main conclusions are summarized in the three propositions proved in the previous section. Even though our emphasis has been focused on the family of the Kerr-de Sitter spacetimes describing a black hole enclosed within a pair of cosmological horizons, the three propositions quoted in the last section remain valid whenever the equation \( \Delta = 0 \) admits double roots of roots of higher multiplicity. For instance, for the case where \( \Delta = 0 \) admits only two real roots \( r_1 < 0 < r_2 \) i.e. the underlying spacetime describes a singular ring enclosed within a pair of cosmological horizons. For this spacetime, the region between the cosmological horizons is a vicious set while the asymptotic de Sitter like regions are causally well behaved. This behavior is to be contrasted to the case of a Kerr spacetime describing a naked singularity where the asymptotic region fails to be causally well behaved.
In summary the causality violating regions in a Kerr-de Sitter is the disjoined union of the Carter's blocks that contain the ring singularity. It should be stressed however that this conclusion assumes that the global structure of the underlying spacetime is the one shown in Fig.2. If however, one of the \( r \rightarrow -\infty \) asymptotic region is to be identified to an \( r \rightarrow \infty \) region (see Fig.2 and comments in the caption of this figure) then the change in the connectivity properties of the underlying manifold leads to the appearance of closed causal curves through the asymptotic regions. These causality violations are of the same nature as those encountered whenever different asymptotic regions of a Kerr \([7]\) or a Reissner-Nordstrom spacetime \([21]\) are identified. As pointed out by Carter \([6]\), the causality violating curves are not homotopic reducible to a point and thus they can be eliminated by moving to a suitable covering spacetime manifold (see discussion in \([6]\)).

However, the causality violations occurring in a Kerr-de Sitter spacetime within the blocks that contain the ring singularity is of different nature since the causality violating curves cannot be removed by moving to a suitable covering space. This type of causality violations also occurs for the Kerr or Kerr-Newman family of spacetimes and furthermore there exist other solutions of Einsteins equations that exhibit the same behavior with the best known example provided by the Godel's solution \([22]\). The solution admits closed timelike curves that are homotopic reducible to a point and thus in the Godel universe a non trivially causality violation takes place (for an introduction to Godel's solution see for instance \([18]\) while for a review of spacetimes exhibiting non trivial causality violations see \([26]\). The ref. \([27]\) discuss implications of closed causal curves, time travel and time machines.).

For the moment, there is not a consensus of how to treat spacetimes exhibiting non trivial causality violations. For instance Hawking in \([28]\) sought a dynamical reasoning to eliminate closed timelike curves and towards this direction he introduced the Chronology Protection Conjecture stating: The laws of physics do not allow the appearance of closed timelike curves. However other authors notably Thorne and collaborators\(^2\) take different attitude towards causality violating spacetimes. Rather consider them as an anomaly, they take the view point that it is prudent to investigate thoroughly their consequences. For instance, in \([31]\) it is argued that closed timelike curves may be generated by matter satisfying the weak energy condition a situation to be contrasted with the assumptions underlying the Chronology Protection Conjecture. Irrespective however of the attitude towards spacetimes violating causality, clearly it is helpful to have a good supply of exact solutions of Einsteins equations exhibiting causality violating regions. This work added to this compartment another family of such solutions namely the family of Kerr-de Sitter spacetimes.

**VII. APPENDIX**

In this Appendix, we remind the reader of a few basic notions of causality theory (a more elaborate discussion can be found in \([18], [19]\)). We recall that any physically relevant spacetime \((M, g)\) besides the standard requirements that \(M\) be a smooth, connected, Hausdorff and paracompact is further

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\(^{19}\) Although the Godel solution \([22]\) seems to be the best known example of a spacetime violating causality, chronologically it is not the first constructed solution of Einstein equations that exhibits this property. In 1937, van Stockum \([23]\) published a solution of Einstein equations with a source a rapidly rotating, infinitely long, dust cylinder and has shown this spacetime admits closed timelike curves. A re-examination of the causality properties of the van Stockum solution has been presented in \([24]\).

\(^{20}\) In \([29]\), it is asked whether the laws of physics permit the creation of wormholes in a universe whose spatial sections initially are simply connected. If the laws indeed allow the formation of wormholes, then appearance of closed timelike curves (and also violation of weak energy condition) is unavoidable. For a proof of the former property see \([28]\), while for the latter see \([29]\).
required to be time orientable and causally well behaved. Causally well behaved means that \((M, g)\) is minimally causal (resp. chronological) according to the definition:

**Definition 1** A time orientable spacetime \((M, g)\) is said to be causal (resp. chronological), if admits no causal (resp. timelike) closed curves.

The absence of closed timelike curves from any physically relevant \((M, g)\) is required to be a stable property in the sense that any small perturbations of the background metric \(g\) does not lead to the appearance of closed causal curves. This additional requirement leads to notion of stable causality according to the definition:

**Definition 2** A time orientable spacetime \((M, g)\) is stably causal, if there exists a continuous timelike vector field \(t\) such that the spacetime \((M, \hat{g})\) with \(\hat{g} = g - \hat{t} \otimes \hat{t}\), possesses no closed timelike curves, (here the covector \(\hat{t}\) is defined by: \(\hat{t} = g(t, )\)).

A very useful criterion guaranteeing that a given \((M, g)\) is stably causal is expressed by the following theorem [18],[19]:

**Theorem 1** A time orientable \((M, g)\) is stably causal, if and only if there exists a differentiable function \(\tau\) (often referred as time function), such that \(\nabla \tau\) is a past directed timelike vector field.

Clearly if \((M, g)\) admits a function \(\tau : M \rightarrow R\) with these properties, then no closed timelike curves can occur, since for any future directed timelike curve \(\gamma\) with tangent vector field \(X\), the inequality \(0 < g(X, \nabla \tau) = X(\tau)\) implies that \(\tau\) is strictly increasing along this \(\gamma\). Therefore under the hypothesis of the theorem, there exist no closed timelike curves in this \((M, g)\). The proof of the converse is more involved but it can be found in [18],[19].

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