ON POLYA’S THEOREM IN SEVERAL COMPLEX VARIABLES

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Abstract. Let be a compact set in C, a function analytic in \( \overline{\mathbb{C}} \setminus K \) vanishing at \( \infty \). Let \( f(z) = \sum_{k=0}^{\infty} a_k z^{-k-1} \) be its Taylor expansion at \( \infty \), and \( H_s(f) = \det (a_{k+i})_{k,i=0}^{n} \) the sequence of Hankel determinants. The classical Polya inequality says that
\[
\limsup_{s \to \infty} |H_s(f)|^{1/s} \leq d(K),
\]
where \( d(K) \) is the transfinite diameter of \( K \). Goluzin has shown that for some class of compacta this inequality is sharp. We provide here a sharpness result for the multivariate analog of Polya’s inequality, considered by the second author in Math. USSR Sbornik, 25 (1975), 350-364.

1. Preliminaries and Introduction

We denote by \( A(\mathbb{C}^n)^* \) the dual space to the space \( A(\mathbb{C}^n) \) of all entire functions on \( \mathbb{C}^n \), equipped with the locally convex topology of locally uniform convergence in \( \mathbb{C}^n \). Following Hörmander ([11], Section 4.5), we call the elements of \( A(\mathbb{C}^n)^* \) analytic functionals.

Let \( \mathbb{Z}_+^n \) be the collection of all \( n \)-dimensional vectors with non-negative integer coordinates. For \( k = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n \) and \( z = (z_1, \ldots, z_n) \in \mathbb{C}^n \), let \( z^k = z_1^{k_1} \cdots z_n^{k_n} \) and \( |k| := k_1 + \cdots + k_n \) be the degree of the monomial \( z^k \). We consider the enumeration \( \{k(i)\}_{i \in \mathbb{N}} \) of the set \( \mathbb{Z}_+^n \) such that \( |k(i)| \leq |k(i+1)| \) and on each set \( \{k(i) = s\} \) the enumeration coincides with the lexicographic order relative to \( k_1, \ldots, k_n \). We will write \( s(i) := |k(i)| \). The number of multiindices of degree at most \( s \) is \( m_s := C_{s+n}^n \) and the number of those of degree exactly \( s \) is \( N_s = m_s - m_{s-1} = C_{s+n-1}^n, s \geq 1; N_0 = 1 \). Let \( l_s := \sum_{q=0}^{n} qN_q \) for \( s = 0, 1, 2, \ldots \).

Consider Vandermondians:
\[
V_1, \ldots, V_i := \det (e_{\alpha}(\zeta_\beta))_{\alpha,\beta=1}^{i}, \quad i \in \mathbb{N},
\]
where \( e_{\alpha}(z) := z^{k(\alpha)}, \alpha \in \mathbb{N} \) and \( (\zeta_\beta) \in K^i \).

For a compact set \( K \subset \mathbb{C}^n \), define "maximal Vandermondians":
\[
V_i := \sup \{|V_1, \ldots, V_i| : (\zeta) \in K^i\}, \quad i \in \mathbb{N}.
\]
Set \( d_s(K) := (V_{m_s})^{1/s} \). The transfinite diameter of \( K \) is the number:
\[
d(K) := \limsup_{s \to \infty} d_s(K).
\]

In the one-dimensional case, this notion was introduced by Fekete [7] for \( n = 1 \), and by Leja [13] for \( n \geq 2 \). That, in fact, the usual limit can be taken in [11] was proved in [7] for \( n = 1 \) and in [24] for \( n \geq 2 \).
The pluripotential Green function of a compact set $K \subset \mathbb{C}^n$ is defined as follows
\[ g_K(z) = \limsup_{\zeta \to z} \sup \{ u(\zeta) : u|_K \leq 0, \ u \in \mathcal{L}(\mathbb{C}^n) \}, \]
where $\mathcal{L}(\mathbb{C}^n)$ represents the Lelong class consisting of all functions $u \in Psh(\mathbb{C}^n)$ such that $u(\zeta) - \ln|\zeta|$ is bounded from above near infinity. Since $d(K) = d(\hat{K})$, with no loss of generality, we are going to consider polynomially convex compact sets. We will also consider the class of functions $\mathcal{L}^+(\mathbb{C}^n) := \{ u \in \mathcal{P}_{sh}(\mathbb{C}^n) : u(z) \geq \log^+|z| + C \}$.

The function $g_K(z)$ is either plurisubharmonic in $\mathbb{C}^n$ or identically equal to $+\infty$. For more detail about the pluripotential Green function, we refer the reader to [12], [20] and [26].

The Monge-Ampere energy $E(u, v)$ of $u$ relative to $v$ for $u, v \in \mathcal{L}^+(\mathbb{C}^n)$ is defined as follows ([5], Section 5):
\[ E(u, v) := \int_{\mathbb{C}^n} (u - v) \sum_{j=0}^{n} (dd^c u)^j \wedge (dd^c v)^{n-j}. \]

Let $K$ be a compact set in $\mathbb{C}^n$. $A(K)$ represents the locally convex space of all germs of analytic functions on $K$, equipped with the countable inductive limit topology, i.e.,
\[ A(K) = \lim ind_{j \to \infty} A(D_j) \]
considered in regard to the inclusion of sets. $D_j$ are open sets such that $D_{j+1} \subset D_j$ for each $j \in \mathbb{N}$ and $K = \bigcap_{j=1}^{\infty} D_j$. Thus, in this setting, a sequence $\{ u_j \}$ of germs converges to a germ $u$ in this topology in case there exists an open neighbourhood $V \supset K$ and functions $g_j, g \in A(V)$ being the representatives of the germs $u_j, u$ respectively, such that $g_j$ converges uniformly to $g$ on any compact subset of $V$.

The Polya Theorem (Theorem 2.1) and its multivariate analog (Theorem 2.3), considered by the second author in [25], are discussed in Section 2. The sharpness result of the generalized Polya inequality (section 4) is based on the comparison of the expression (2.6) for Hankel-like determinants from [25] with the expression (2.9) for the transfinite diameter from Bloom and Levenberg [1]. The main result of this article (Theorem 4.7) says that, for real compact sets, the equality attains in the generalized Polya inequality (2.7) for some analytic functional $f^* \in A(K)^*$. This result seems to be new even in the one-dimensional case. Additionally, we introduce two sharpness properties for compact sets $K \subseteq \mathbb{C}^n$ and study the stability of these properties relative to the approximations from inside and outside (Proposition 4.3 and 4.4).

2. Polya’s Theorem

The following result is due to Polya [19].

**Theorem 2.1.** Let $K$ be a polynomially convex compact set in $\mathbb{C}$ and $f \in A(\mathbb{C} \setminus K)$ have the following expansion in a neighbourhood of $\infty$:
\[ f(z) = \sum_{k=0}^{\infty} \frac{a_k}{z^{k+1}}. \]
Let $A_s(f) := \det(a_{k+m})_{k,m=0}^{s-1}$, $s \in \mathbb{N}$, be a sequence of Hankel determinants composed from the coefficients of the expansion (2.7). Then,

$$D(f) := \limsup_{s \to \infty} |A_s(f)|^{1/s^2} \leq d(K).$$

A direct multivariate analog of the inequality (2.2) makes no sense, since there are functions analytic on the complement of $K$ but constants only. Schiffer and Siciak (21) proved some analog for the product of plane compact sets $K = K_1 \times K_2 \times \ldots \times K_n \subset \mathbb{C}^n$ and functions $f \in A((\mathbb{C} - K_1) \times \ldots \times (\mathbb{C} - K_n))$. Sheinov (22, 23) considered another analog of Polya’s inequality for a linearly convex compact set $K$, considering the Taylor expansion at the origin for functions analytic in the domain $D = K^*$ linearly convex adjoint (conjugate) to $K$ (projective complement of $K$ by Martineau [18]).

The case of an arbitrary compact set $K \subset \mathbb{C}^n$ was studied in [25]. It was suggested there, instead of analytic functions on some artificial “complement” of $K$, to deal with those analytic functionals in $A(\hat{K})$. We denote by $A_0(\{\infty^n\})$ the space of all analytic germs $f'$ at $\infty^n = (\infty, \infty, \ldots, \infty) \in \mathbb{C}^n$ with Taylor expansion of the form

$$f'(z) = \sum_{k \in \mathbb{Z}^n} \frac{a_k}{z^{k+I}}, \ I = (1, 1, \ldots, 1),$$

converging in some neighborhood of $\infty^n$.

**Lemma 2.2.** There is an isomorphism,

$$T : A(\mathbb{C}^n)^* \to A_0(\{\infty^n\}),$$

such that, for each $f^*$ and $f' = Tf^*$, we have

$$f^*(\varphi) = \langle \varphi, f' \rangle := \left(\frac{1}{2\pi i}\right)^n \int_{T^*_R} \varphi(\zeta) f'(\zeta) \ d\zeta, \ \varphi \in A(\mathbb{C}^n),$$

where

$$T^*_R := \{z = (z_\nu) \in \mathbb{C}^n : |z_\nu| = R, \ \nu = 1, \ldots, n\}, \ R = R(f^*).$$

**Proof.** See, e.g., [6], Chapter 3. □

Let us define, for every analytic functional $f^*$, a related sequence of multivariate Hankel-like determinants constructed from the coefficients of the expansion (2.3):

$$H_i = H_i(f^*) := \det(a_{k(\alpha)+k(\beta)})_{\alpha,\beta=1}^{i}, \ i \in \mathbb{N}$$

with $a_{k(\alpha)} := f^*(e_\alpha) = \langle e_\alpha, f' \rangle$, $\alpha \in \mathbb{N}$, $f' = Tf^*$. Now we are ready to formulate the general form of multivariate Polya’s inequality.

**Theorem 2.3.** Suppose that $K$ is a compact set in $\mathbb{C}^n$, $f^*$ is an analytic functional which has a continuous extension onto $A(K)$ and $f' = Tf^*$ is the corresponding analytic germ at $\infty^n$. Then for the determinants (2.6), the inequality holds:

$$D(f') := \limsup_{i \to \infty} |H_i(f^*)|^{\frac{1}{i}} \leq d(K).$$
It has been proved in [25] a bit weaker result with the outer transfinite diameter $\hat{d}(K)$ instead of $d(K)$, but later it was proved that $\hat{d}(K) = d(K)$ (see Proposition 3.1 below).

We send the reader for the proof of Theorem 2.3 to [25], Theorem 3. However we cite here the following equality, which is crucial there and will be essentially used in Section 4:

$$i! |H_i(f^*)| = |f^*_i(\ldots f^*_i(\ldots (f^*_i(V(\zeta^{(1)}, \zeta^{(2)}, \ldots, \zeta^{(i)}))^2) \ldots )|,$$

$i \in \mathbb{N}$, here the notation $f^*_i$ means that the functional $f^*$ is applied sequentially to a function of the variable $\zeta^{(j)}$ by keeping the other variables fixed.

**Remark 2.4.** The classical Polya’s Theorem (Theorem 2.1) is a particular case of Theorem 2.3 since, due to Gröthendieck-Köthe-Silva duality (see [10], [13], [24]), every $f \in A(C \setminus K)$ satisfying (2.1) in a neighborhood of $\infty$ represents a linear continuous functional $f^* \in A(K)^* \hookrightarrow A(C)^*$. Hereafter $\hookrightarrow$ denotes a linear continuous embedding.

Let $K \subset \mathbb{C}^n$ be a compact set, and $\mu$ be a bounded positive Borel measure on $K$. The pair $(K, \mu)$ is said to satisfy Bernstein-Markov inequality for holomorphic polynomials in $\mathbb{C}^n$ if, given $\varepsilon > 0$, there exists a constant $M = M(\varepsilon)$ such that for all polynomials $p_s$ of degree at most $s$

$$\|p_s\|_K \leq M(1 + \varepsilon)^s \|p_s\|_{L^2(\mu)}.$$

**Theorem 2.5.** (Bloom-Levenberg, [1]) Let $K \subset \mathbb{C}^n$ be a compact set, $\mu$ be a bounded positive Borel measure on $K$ and let $(K, \mu)$ satisfy Bernstein-Markov inequality. Then,

$$\lim_{s \to \infty} Z_s(K, \mu) = d(K),$$

where

$$Z_s(K, \mu) = \int_{K^{m_s(n)}} |V(\zeta^{(1)}, \ldots, \zeta^{(m_s(n))})|^2 d\mu(\zeta^{(1)}) d\mu(\ldots) d\mu(\zeta^{(m_s(n))}).$$

**Remark 2.6.** In [2] (Proposition 3.4 and Corollary 3.5), the same authors proved that for any compact set $K \subseteq \mathbb{C}^n$, there exists a measure $\mu \in \mathcal{M}(K)$ such that $(K, \mu)$ satisfies Bernstein-Markov property.

### 3. Stability of Transfinite Diameter

The following proposition provides the stability of transfinite diameter of a compact set in $\mathbb{C}^n$ approximated from outside.

**Proposition 3.1.** (V.A. Znamenskii [29, 30], Levenberg [15]) Let $K$ be a compact set in $\mathbb{C}^n$ and $\{K_j\}$ a sequence of compact sets such that $K_{j+1} \subseteq K_j$ for all $j \in \mathbb{N}$ and $K = \bigcap_{j=1}^\infty K_j$. Then,

$$\hat{d}(K) := \lim_{j \to \infty} d(K_j) = d(K).$$

In this section, we prove a stability property of transfinite diameter relative to the approximation from inside. The following is an easy consequence of Lemma 6.5 in [4]:
Lemma 3.2. Suppose that $K$ is a non-pluripolar compact set in $\mathbb{C}^n$, and $\{K_j\}$ is a sequence of non-pluripolar compact sets such that $K_j \subseteq K_{j+1} \subseteq K$, $j \in \mathbb{N}$ and for $L := \bigcup_{j=1}^{\infty} K_j$, we have
\[
(3.1) \quad \int_{K \setminus L} (dd^c g_K)^n = 0.
\]
Then
\[
\lim_{j \to \infty} g_{K_j}(z) = g_K(z), \quad z \in \mathbb{C}^n.
\]

Theorem 3.3. Under the conditions of Lemma 3.2, we have,
\[
\lim_{j \to \infty} d(K_j) = d(K).
\]

Proof. We will use the unweighted energy version of Rumely’s formula (see e.g., Theorem 5.1 of [16], or Section 9.1 of [5]). Since, by Lemma 3.2, $g_{K_j} \downarrow g_K$, applying the remark after Lemma 3.5 in [16], one obtains
\[
- \ln d(K_j) = \frac{1}{n(2\pi)^n} \mathcal{E}(g_{K_j}, g_T) \uparrow \frac{1}{n(2\pi)^n} \mathcal{E}(g_K, g_T) = - \ln d(K), \text{ as } j \to \infty,
\]
where $T$ is the unit torus in $\mathbb{C}^n$. □

4. Sharpness of Polya’s Inequality

The following Theorem is proved by Goluzin in [8] (see also [9], Section 11).

Theorem 4.1. For functions which are analytic in an infinite domain $B$ with boundary $K$ consisting of a finite number of closed Jordan curves and having the expansion
\[
(4.1) \quad f(z) = \sum_{k=1}^{\infty} \frac{a_k}{z^k},
\]
in a neighborhood of $z = \infty$, the inequality $D(f) = \limsup_{s \to \infty} |A_s(f)|^{1/s} \leq d(K)$ given by Theorem 2.1 is sharp.

Another way of expressing Theorem 4.1 is, for a compact set $K \subseteq \mathbb{C}^n$
\[
(4.2) \quad d(K) = \sup \{ D(f) : f \in A(\mathbb{C} \setminus K) \},
\]
if the boundary $\partial K$ consists of a finite number of closed Jordan curves.

Definition 4.2. Let $K$ be a polynomially convex compact set in $\mathbb{C}^n$. $K$ is said to satisfy the sharpness property in Polya inequality, shortly denoted as $K \in (SP)$, if
\[
d(K) = \sup \{ D(f^*) : f^* = T(f^*), f^* \in A(K)^* \}.
\]
We say that $K$ has a strong sharpness property in Polya inequality, denoted by $K \in (SSP)$, if there exists a $f^* \in A(K)^*$ such that
\[
D(f^*) = d(K)
\]
for $f^* = T(f^*)$, where $T$ is defined as in Lemma 2.2.

If $K$ is a pluripolar compact set in $\mathbb{C}^n$, then $K \in (SSP)$ by the result of Levenberg-Taylor ([17]) which says that $d(K) = 0$ if and only if $K$ is pluripolar. From now on, we only consider non-pluripolar compact sets.
Proposition 4.3. Let $K$ be a compact set in $\mathbb{C}^n$, $\{K_i\}$ a sequence of compact sets with $K = \bigcap_{i=1}^{\infty} K_i$. Assume $K_i \in (SP)$ for all $i \in \mathbb{N}$. Then there exists a sequence of analytic functionals $\{f_i^*\}$ such that $f_i^* \in A(K_i)^*$ for each $i \in \mathbb{N}$ and
\[
(4.3) \quad \lim_{i \to \infty} D(f_i^*) = d(K).
\]

Proof. By Definition 4.2, for each $i \in \mathbb{N}$, there exists $f_i^* \in A(K_i)^*$ with $f_i' = T(f_i^*)$ such that $d(K_i) \leq D(f_i') + \frac{1}{i}$. Theorem 2.3 gives $D(f_i') \leq d(K_i)$. By using Proposition 3.1 we have
\[
d(K) = \lim_{i \to \infty} d(K_i) \leq \lim_{i \to \infty} D(f_i') \leq \lim_{i \to \infty} d(K_i) = d(K),
\]
which gives the limit (4.3). \qed

As seen from Proposition 4.3, $(SP)$ is not preserved under the approximation from outside, however, for an approximation from inside, we have the stability of the property $(SP)$:

Proposition 4.4. Let the conditions of Lemma 3.2 be given. Suppose further that $K_i \in (SP)$ for all $i \in \mathbb{N}$. Then $K \in (SP)$.

Proof. Proof is almost the same as the proof of Proposition 4.3 except we only use Theorem 3.3 instead of Proposition 3.1 in the end, hence we have the following:
\[
d(K) \leq \lim_{i \to \infty} D(f_i') = \sup \{D(f_i') : i \in \mathbb{N}\} \leq d(K),
\]
which concludes that $d(K) = \sup \{D(f_i') : i \in \mathbb{N}\}$ and so $K \in (SP)$ by Definition 4.1. \qed

For an arbitrary compact set in $\mathbb{C}$, a following sharpness statement, which is weaker than $(SP)$, is derived easily from Goluzin’s result above.

Proposition 4.5. Let $K$ be a compact set in $\mathbb{C}$, $\{K_i\}$ a sequence of compact sets with the properties $K_{i+1} \subseteq K_i$ for all $i \in \mathbb{N}$, $K = \bigcap_{i=1}^{\infty} K_i$. Then there exists a sequence of functions $f_i \in A(\mathbb{C} \setminus K_i)$ such that
\[
(4.4) \quad \lim_{i \to \infty} D(f_i) = d(K).
\]

Proof. For each $i \in \mathbb{N}$, we can find a compact set $L_i$ whose boundary consists of a finite number of closed analytic Jordan curves so that $K_{i+1} \subseteq L_i \subseteq K_i$ holds. By the result of Goluzin, there exists $f_i \in A(\mathbb{C} \setminus L_i)$ such that $d(L_i) < D(f_i) + \frac{1}{i}$. Since $f_i \in A(\mathbb{C} \setminus K_i)$ holds, we get by Theorem 2.1 $D(f_i) \leq d(K_i)$. Hence by using Proposition 3.1 we obtain the following
\[
d(K) = \lim_{i \to \infty} d(L_i) \leq \lim_{i \to \infty} D(f_i) \leq \lim_{i \to \infty} d(K_i) = d(K),
\]
which gives the desired limit (4.4). \qed

Let $K$ be a compact set in $\mathbb{C}^n$, and $J : A(K) \to C(K)$ the natural restriction homomorphism. $AC(K)$ is the Banach space obtained as the completion of the set $J(A(K))$ in the space $C(K)$ with respect to the uniform norm.

Lemma 4.6. Let $K$ be an infinite polynomially convex compact set in $\mathbb{C}^n$. Then, for each bounded Borel measure $\mu \in M(K)$, there exists an analytic functional $f^* \in A(K)^* \to A(\mathbb{C}^n)^*$ and a corresponding analytic germ $f' = Tf^*$ such that
\[
(4.5) \quad f^*(f) = \int_K f(\zeta) d\mu(\zeta),
\]
for every \( f \in A(\mathbb{C}^n) \).

Proof. The dense embedding \( A(K) \hookrightarrow AC(K) \) implies, for the dual spaces, the following embedding: \( AC(K)^* \hookrightarrow A(K)^* \). Since \( AC(K) \) is a closed subspace of \( C(K) \), every bounded Borel measure \( \mu \in \mathcal{M}(K) \) defines a linear continuous functional \( F^* \in AC(K)^* \) such that

\[
F^*(f) = \int_{\mathbb{C}^n} f(\zeta) d\mu(\zeta)
\]

for every \( f \in AC(K) \). Then, the restriction \( f^* = F^*|_{A(K)} \) belongs to \( A(K)^* \). By Lemma \([2.2]\) since \( A(K)^* \hookrightarrow A(\mathbb{C}^n)^* \), there is \( f' \in A_0(\{\infty^\infty\}) \) such that

\[
f^*(f) = \left< f, f' \right> = \left( \frac{1}{2\pi i} \right)^n \int_{\mathbb{C}^n} f(\zeta) f'(\zeta) \, d\zeta, \quad f \in A(\mathbb{C}^n),
\]

where \( T^n_R \) is defined as in \([2.5]\), and \( R \) is sufficiently large. \( \square \)

Now we show that, for any real compact set in \( \mathbb{C}^n \), the equality in the estimate \([2.7]\) is attained at some \( f^* \in A(K)^* \).

**Theorem 4.7.** Let \( K \subseteq \mathbb{R}^n \subseteq \mathbb{C}^n \) be a compact set. Then \( K \in (SSP) \).

Proof. By Theorem \([2.5]\) and Remark \([2.6]\) there exists a measure \( \mu \in \mathcal{M}(K) \) such that \( (K, \mu) \) satisfies the Bernstein-Markov inequality. Let \( f^* \) be an analytic functional corresponding to \( \mu \) by Lemma \([4.6]\). Initially, we show that \( Z_s(K, \mu) = m_s(n) ! |H_{m_s(n)}(f^*)| \), where \( Z_s(K, \mu) \) and \( H_{m_s(n)}(f^*) \) are defined in Section 2. Indeed, considering the relation \([2.3]\) gives:

\[
m_s(n) ! |H_{m_s(n)}(f^*)| = |f_{s(m_s(n))}^*| \ldots (f_{s(1)}^*([V(\zeta^{(1)}, \ldots, \zeta^{(m_s(n))})]^2 \ldots)].
\]

Since \( K \) is a real subset and so \([V(\zeta^{(1)}, \ldots, \zeta^{(m_s(n))})]^2 \) is nonnegative, by iterating \([4.5]\) \( m_s(n) \) times, the righthand side of \([4.6]\) becomes:

\[
\int_K \ldots \int_K |V(\zeta^{(1)}, \ldots, \zeta^{(m_s(n))})|^2 \, d\mu(\zeta^{(1)}) \ldots d\mu(\zeta^{(m_s(n))}),
\]

which is equal to \( Z_s(K, \mu) \). Since \( (m_s(n)! \frac{1}{m_s(n)!}) \to 1 \) as \( s \to \infty \), we have, by Theorem \([2.5]\)

\[
d(K) = \lim_{s \to \infty} Z_s(K, \mu) \frac{1}{m_s(n)!} = \lim_{s \to \infty} \, \frac{1}{m_s(n)!} |H_{m_s(n)}(f^*)| \frac{1}{m_s(n)!} = D(f^*).
\]

\( \square \)

**Problem.** Characterize compact sets in \( \mathbb{C}^n \) such that either \((SP)\) or \((SSP)\) holds.

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