Sequential Estimation of Network Cascades

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Abstract—We consider the problem of locating the source of a network cascade, given a noisy time-series of network data. We assume that at time zero, the cascade starts with one unknown vertex and spreads deterministically at each time step. The goal is to find a sequential estimation procedure for the source that outputs an estimate for the cascade source as fast as possible, subject to a bound on the estimation error. For general graphs that satisfy a symmetry property, we show that matrix sequential probability ratio tests (MSPRTs) are first-order asymptotically optimal up to a constant factor as the estimation error tends to zero. We apply our results to lattices and regular trees, and show that MSPRTs are asymptotically optimal for regular trees. We support our theoretical results with simulations.

Index Terms—Network cascade, sequential estimation, asymptotic optimality, hypothesis testing

I. INTRODUCTION

Network cascades refer to the phenomena where the behavior of an individual or a small group of individuals diffuses rapidly through a network. Such instabilities have been observed in a variety of practical scenarios, including the spread of epidemics in physical or geographical networks [1], [2], fake news in social networks [3], [4], and the propagation of viruses in computer networks [5], [6]. In each of these examples, the network cascade compromises the functionality of the network and it is therefore of paramount importance to locate the source of the cascade as fast as possible.

This problem poses several interesting challenges. On one hand, network cascades are typically not directly observable even if one can monitor the network in real time. In the example of an epidemic spreading through a network, an individual’s sickness could be caused by the epidemic or by exogenous factors (e.g., allergies). As the network is monitored over time, one may be able to distinguish between these possibilities at the cost of allowing the cascade to propagate further. Thus there is a fundamental tradeoff between the accuracy of the estimated cascade source and the amount of vertices affected by the cascade. How can we design algorithms that achieve the best possible tradeoff?

In this paper, we take the first steps towards formalizing and solving the challenges addressed above. We begin by formulating a new mathematical model for network cascades with noisy observations. We then study the problem of minimizing the expected run time of a sequential estimation algorithm for the cascade source subject to the estimation error being at most $\alpha$, for some $\alpha \in (0, 1)$. We show that simple algorithms based on cumulative log likelihood ratio statistics - specifically, matrix sequential probability ratio tests (MSPRTs) - are first-order asymptotically optimal up to constant factors as we send the estimation error to zero under general conditions on the network topology. In certain cases we can say more: the MSPRT we construct is first-order optimal in regular trees.

A. A model of network cascades with noisy observations

Let $G$ be a graph with vertex set $V$. We assume that the cascade starts from some vertex $v$, and spreads over time via the edges of the graph. To discuss the specifics, we first introduce some notation. We assume that time is discrete and is indexed by $t$, an element of the nonnegative integers. The cascade evolves via the following deterministic dynamics. At $t = 0$, $v$ is the unique vertex affected by the cascade. For any $t \geq 1$, a vertex $u$ is affected if and only if $u \in \mathcal{N}_v(t)$, where $\mathcal{N}_v(t)$ denotes the set of all vertices within distance $t$ from $v$ in the graph, with respect to the shortest path distance.

We assume that the system cannot observe the cascade, but can instead monitor the public states of vertices, which can be thought of as a noisy observation of the cascade. The public state of a vertex $u$ is given by $y_u(t) \in \mathbb{R}$, defined by

$$y_u(t) = \begin{cases} Q_0 & u \notin \mathcal{N}_v(t); \\ Q_1 & u \in \mathcal{N}_v(t), \end{cases}$$

where $Q_0, Q_1$ are two distinct mutually absolutely continuous probability measures over $\mathbb{R}$. We can think of $y_u(t) \sim Q_0$ as typical behavior and $y_u(t) \sim Q_1$ as anomalous behavior caused by the cascade.

B. Formulation as a sequential hypothesis testing problem

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a common probability space for all random objects. For each vertex $u$, let $H_u$ be the hypothesis that $u$ is the cascade source and let $\mathbb{P}_u := \mathbb{P}(\cdot | H_u)$ be the associated measure. Any algorithm for estimating the cascade source can be represented by a sequential test $(D, T)$, where $T$ is a (data dependent) stopping time and $D$ is a terminal decision rule that outputs a vertex in $V$ at the stopping time. Given some notion of estimation error, our goal is to characterize the behavior of the minimum expected stopping time for a sequential test with error at most $\alpha$. A typical notion of error in the hypothesis testing setting is the Type I error. Given the geometric nature of our hypotheses, we also specify a confidence radius $R$, so
that we do not count it as an error if \( D \in \mathcal{N}_v(R) \) when \( v \) is the cascade source. We can write the estimation error formally as \( \max_{u,v \in V : d(u,v) > R} \mathbb{P}_v(D = u) \).

We assume the following properties for the graph \( G \).

**Assumption 1.** \( G \) has infinitely many vertices, is connected, and is locally finite (each vertex has finite degree).

This assumption allows us to ignore any boundary effects that would be present in finite graphs. Important examples that we study in this paper are lattices and regular trees. However, the infinite graph setting corresponds to testing infinitely many hypotheses, and it is unclear whether there exists a sequential test with small Type I error that will terminate in finite time. To remedy this situation, we will consider the behavior of sequential tests on finite restrictions of the graph. Formally, choose an arbitrary vertex \( v_0 \in V \), and let \( \{ V_n \}_{n \geq 1} \) be a sequence of subsets of vertices where \( V_n := \mathcal{N}_{v_0}(n) \). We define the class of sequential tests \( \Delta_G(n, R, \alpha) \), given by

\[
\left\{ (D, T) : \max_{u,v \in V_n} \mathbb{P}_v(D = u) \leq \frac{\alpha}{|V_n \setminus \mathcal{N}_v(R)|} \right\}.
\]

In particular if \( (D, T) \in \Delta_G(n, R, \alpha) \), then for every \( v \in V_n \),

\[
\mathbb{P}_v(D \notin \mathcal{N}_v(R)) \leq \sum_{u \in V_n \setminus \mathcal{N}_v(R)} \frac{\alpha}{|V_n \setminus \mathcal{N}_v(R)|} = \alpha.
\]

For a given \( G, R_n, \alpha \) (note that the radius \( R_n \) may depend on \( n \)), our goal is to characterize as \( n \to \infty \) the first order asymptotics of\(^2\)

\[
T^*(n, R_n, \alpha) := \min_{(D, T) \in \Delta_G(n, R_n, \alpha)} \max_{v \in V_n} \mathbb{E}_v[T].
\]

In general sequential multi-hypothesis testing problems, characterizing the optimal test for a fixed \( \alpha \) is intractable. We therefore study the asymptotics of \(^2\) first as \( n \to \infty \) then as \( \alpha \to 0 \). We consider confidence radii \( R_n \) that may be fixed with respect to all other parameters, or may grow with \( n \).

**C. Related work**

Although we are, to the best of our knowledge, the first to study this variant of the cascade source estimation problem, our work has close connections to several bodies of work. Shah and Zaman gave the first systematic study of estimating the source of a network cascade \(^7\), which spawned several follow-up works, see for example \(^9\)–\(^14\). In their setup, they assume that the network cascade evolves according to a probabilistic model, and that at some future time a snapshot of the cascade is perfectly observed. The goal is then to estimate the source, given only this snapshot. The problems we address surrounding network cascades are complementary to this approach, and are more appropriate for the setting where one may monitor the state of the network in real time.

\(^1\)The choice of \( V_n \) does not matter for our general results, but we may define \( V_n \) this way without loss of generality as the problem is only harder when all vertices are close to each other.

\(^2\)Throughout the paper, we use standard asymptotic notation, e.g., \( g(t) \sim h(t) \) if and only if \( \lim_{t \to \infty} g(t)/h(t) = 1 \).

Our work naturally falls under the growing body of literature on sequential detection and estimation in networks \(^15\)–\(^18\). Recently Zou, Veeravalli, Li and Towsley studied the problem of quickest detection of network cascades \(^15\)–\(^17\), which is similar in nature to our work but has fundamentally different goals and approaches. The objective of their work is to detect with minimum delay when a network cascade has affected \( \ell \) vertices in the network. They assume that the cascade dynamics are unknown, but one observes the time-series of public states in the network (the public states are generated in the same manner as our model). In formulating their notion of an optimal test, they consider the worst-case performance over the possible times where different nodes in the graph are affected by the cascade. A natural consequence of this formulation is that to analyze asymptotic properties of the optimal tests, one must send the time at which \( \ell \) vertices are affected by the cascade to infinity. On the other hand, in our model every vertex is affected by the cascade at some finite time, which is incompatible with the formulation of Zou et al. Thus our analysis can be viewed as a complementary perspective on the sequential analysis of cascades, where the underlying cascade dynamics are known but we instead study the asymptotics as the probability of error tends to zero. Interestingly, in our work, the work of Zou et al, as well as in other settings dealing with the sequential analysis of network data \(^18\), the authors prove asymptotic optimality of tests based on cumulative log-likelihood ratios.

Since we formulate our problem as a sequential hypothesis testing problem, our work naturally draws upon methods in this literature. In particular, we show that a family of MSPRTs, which are known to be asymptotically optimal as the Type I error tends to zero in the hypothesis testing setting \(^19\)–\(^21\), is asymptotically optimal for our problem as well. An important additional dimension that our analysis provides is how the performance of the MSPRTs scales with the number of hypotheses, which was not addressed in this literature.

**D. Organization of the paper**

In Section II we overview our results on the asymptotic upper and lower bounds for \( T^*(n, R_n, \alpha) \) in the general case. The proof of the lower bounds is deferred to Section III as it is more involved. In Section IV we apply our results to two families of sparse graphs: regular trees and lattices. We also provide simulations to support our theoretical results. Finally, we conclude in Section V.

II. ASYMPTOTIC BEHAVIOR OF OPTIMAL TESTS

Our goal is to understand how the first-order asymptotics\(^2\) depend on the number of vertices \( n \), the confidence radius \( R_n \), and the Type I error bound \( \alpha \). We study the problem for general graphs under some structural assumptions and derive asymptotic expressions for the expected value of the optimal stopping time. At the core of the analysis is a characterization of the rate of convergence of the log-likelihood ratios. It is well
known in the sequential hypothesis testing literature that tests based on log-likelihood ratios are optimal for distinguishing between two hypotheses \( \mathcal{H}_0 \) and \( \mathcal{H}_1 \) and are asymptotically optimal as the Type I error tends to 0 in the general multi-hypothesis testing problem \([19]-[21]\). Thus, to motivate our results, we begin by studying some basic properties of the log-likelihood ratios that arise from our problem structure.

Let \( \mathbb{P}_u \) and \( \mathbb{P}_w \) be the measures corresponding to the hypotheses \( \mathcal{H}_0 \) and \( \mathcal{H}_1 \). For a positive integer \( s \), let \( y(s) := \{y_w(s)\}_{w \in V} \). We are interested in studying the quantity

\[
Z_{vu}(t) := \sum_{s=0}^{t} \frac{d\mathbb{P}_u}{d\mathbb{P}_w}(y(s)).
\]

From the cascade dynamics defined in Section I-A, we can write the log-likelihood ratio \( \log \frac{d\mathbb{P}_u}{d\mathbb{P}_w}(y(s)) \) as

\[
\log \prod_{w \in N_u(s)} dQ_1(y_w(s)) / \prod_{w \in N_v(s)} dQ_0(y_w(s)) = \sum_{w \in N_u(s) \setminus N_v(s)} \log \frac{dQ_1(y_w(s))}{dQ_0(y_w(s))} + \sum_{w \in N_v(s) \setminus N_u(s)} \log \frac{dQ_0(y_w(s))}{dQ_1(y_w(s))}. \tag{3}
\]

Under \( \mathbb{P}_u \), all observed variables \( y_w(s) \) are independent, with distributions given by

\[
y_w(s) \sim \begin{cases} Q_0 & w \in N_u(s) \setminus N_v(s); \\ Q_1 & w \in N_v(s) \setminus N_u(s). \end{cases} \tag{4}
\]

To simplify the analysis we assume the following structural condition for the graph.

**Assumption 2.** For each \( u, v \in V \) and \( t \geq 0 \), \( N_u(t) \setminus N_v(t) \) is nonempty, and \( |N_u(t) \setminus N_v(t)| = |N_v(t) \setminus N_u(t)| \).

This assumption holds for a large class of graphs (e.g., vertex-transitive graphs such as regular trees and lattices).

For \( u, v \in V \) define the neighborhood difference function

\[
f_{vu}(t) := \sum_{s=0}^{t} |N_u(s) \setminus N_v(s)|.
\]

It is clear from (3) and (4) that \( \mathbb{E}_v[Z_{vu}(t)] = \bar{D}(Q_0, Q_1)f_{vu}(t) \), where \( \bar{D}(Q_0, Q_1) \) is the symmetrized Kullback-Leibler divergence between \( Q_0 \) and \( Q_1 \), given explicitly by

\[
\bar{D}(Q_0, Q_1) := \int \log \left( \frac{dQ_1}{dQ_0} \right) dQ_1 + \int \log \left( \frac{dQ_0}{dQ_1} \right) dQ_0.
\]

Furthermore, (3) shows that \( Z_{vu}(t) \) can be written as a sum of independent random variables. For any \( \epsilon > 0 \) a Chernoff bound yields

\[
\mathbb{P}_v \left( Z_{vu}(t) \geq (\bar{D}(Q_0, Q_1) + \epsilon) f_{vu}(t) \right) \leq e^{-f_{vu}(t)I^+(\epsilon)}; \tag{5}
\]

\[
\mathbb{P}_v \left( Z_{vu}(t) \leq (\bar{D}(Q_0, Q_1) - \epsilon) f_{vu}(t) \right) \leq e^{-f_{vu}(t)I^-(\epsilon)}. \tag{6}
\]

Above, \( I^+(\epsilon) \) and \( I^-(\epsilon) \) are the corresponding large-deviations rate functions, given by

\[
I^+(\epsilon) := \sup_{\lambda \geq 0} \left( \epsilon \lambda - \log \mathbb{E}_{X \sim Q_0, Y \sim Q_1} \left[ \left( \frac{dQ_1}{dQ_0}(X) \frac{dQ_0}{dQ_1}(Y) \right)^{\lambda} \right] \right); 
\]

\[
I^-(\epsilon) := \sup_{\lambda \leq 0} \left( \epsilon \lambda - \log \mathbb{E}_{X \sim Q_0, Y \sim Q_1} \left[ \left( \frac{dQ_1}{dQ_0}(X) \frac{dQ_0}{dQ_1}(Y) \right)^{\lambda} \right] \right).
\]

We are now ready to state our first main result, which establishes asymptotic lower bounds for \( T^*(n, R_n, \alpha) \). To simplify the notation in the theorem, for two functions \( g(n, \alpha) \) and \( h(n, \alpha) \), we write, whenever it is well-defined,

\[
g(n, \alpha) \gtrless h(n, \alpha) \iff \lim_{\alpha \to 0} \liminf_{n \to \infty} \frac{g(n, \alpha)}{h(n, \alpha)} \geq 1.
\]

The orderwise comparison operator \( \gtrless \) is analogously defined. We also write \( g(n, \alpha) \approx h(n, \alpha) \) if and only if \( g(n, \alpha) \gtrless h(n, \alpha) \) and \( g(n, \alpha) \lesssim h(n, \alpha) \).

**Theorem 1.** Let \( F_{vu} \) be the inverse function of \( f_{vu} \). If \( \min_{v \in V_n} \log |N_v(R_n)| \ll \log n \),

\[
T^*(n, R_n, \alpha) \gtrsim \max_{u,v \in V_n: d(u,v) > R_n} F_{vu} \left( \frac{\log n}{\bar{D}(Q_0, Q_1)} \right).
\]

If \( \min_{v \in V_n} |N_v(R_n)| \sim n^\gamma \) for some \( \gamma \in (0, 1) \),

\[
T^*(n, R_n, \alpha) \gtrsim \max_{u,v \in V_n: d(u,v) > R_n} F_{vu} \left( (1 - \gamma) \log n \right) \frac{\bar{D}(Q_0, Q_1)}{D(Q_0, Q_1)}.
\]

If \( \min_{v \in V_n} |N_v(R_n)|, \max_{v \in V_n} |V_n \setminus N_v(R_n)| \sim n \),

\[
T^*(n, R_n, \alpha) \gtrsim \max_{u,v \in V_n: d(u,v) > R_n} F_{vu} \left( \frac{\log n}{\bar{D}(Q_0, Q_1)} \right).
\]

We defer the proof to Section III as it is more involved. Interestingly, the asymptotic lower bound when \( |N_v(R_n)| \sim n \) does not scale with \( n \). We provide a heuristic reason why this is the case. Since \( |N_v(R_n)| \) encompasses a positive fraction of the total number of vertices, if our decision rule outputs a random vertex, this will be correct with positive power. This is not the case when \( R_n \) falls under one of the other regimes in Theorem 1. For more details, see Section III.

The next natural step in characterizing the first-order asymptotics of (2) is to establish an upper bound on the optimal expected stopping time. We do so by considering families of matrix sequential probability ratio tests (MSPRTs), which are known to be asymptotically optimal in a broad class of multi-hypothesis testing problems \([19]-[21]\). Define the stopping time

\[
T_v := \min \left\{ t \geq 0 : \min_{w \in V_n \setminus N_v(R_n)} Z_{vu}(t) \geq \log \frac{n}{\alpha} \right\},
\]

and define \( T_n := \min_{v \in V_n} T_v \), \( D_n := \arg \min_{v \in V_n} T_v \). It is simple to verify that \( (D_n, T_n) \in \Delta_G(n, R_n, \alpha) \). For any vertex \( u \in V_n \setminus N_v(R_n) \),

\[
\mathbb{P}_v (D_n = u) \leq \mathbb{P}_v (T_u < \infty) \leq \mathbb{E}_u \left[ 1 \{ T_u < \infty \} e^{-Z_{vu}(T_u)} \right] = e^{-\log n/\alpha \mathbb{E}_u \left[ 1 \{ T_u < \infty \} e^{-Z_{uv}(T_u)} \right]}. 
\]
Since \( Z_{vu}(T_u) \geq \log n/\alpha \) by the definition of \( T_u \), we have an upper bound of \( \alpha/n \). The following theorem gives us an upper bound for the expected stopping time of this MSPRT as \( n \to \infty \).

**Theorem 2.** Let \( \alpha \in (0, 1) \) be fixed. There exists a constant \( c \in (0, 1) \) depending only on \( Q_0 \) and \( Q_1 \) such that for every \( v \in V_n \),

\[
\mathbb{E}_v[T_n] \leq \max_{u \in V_n \setminus \mathcal{N}_v(R_n)} F_{vu} \left( \frac{\log n}{c \cdot \bar{D}(Q_0, Q_1)} \right) (1 + o_n(1)),
\]

where \( o_n(1) \to 0 \) as \( n \to \infty \).

**Proof.** We begin by upper bounding \( \mathbb{P}_v(T_v > t) \). We can write

\[
\mathbb{P}_v(T_v > t) \leq \mathbb{P}_v \left( \min_{u \in V_n \setminus \mathcal{N}_v(R_n)} Z_{vu}(t) < \log n/\alpha \right)
\]

\[
\leq \sum_{u \in V_n \setminus \mathcal{N}_v(R_n)} \mathbb{P}_v \left( Z_{vu}(t) < \log n/\alpha \right). \tag{7}
\]

Fix \( \epsilon > 0 \) and suppose that, for all \( u \in V_n \setminus \mathcal{N}_v(R_n) \),

\[
\log n/\alpha \leq (\bar{D}(Q_0, Q_1) - \epsilon) f_{vu}(t).
\]

Using (9), we can bound the summation by

\[
\log n/\alpha \leq (\bar{D}(Q_0, Q_1) - \epsilon) f_{vu}(t). \tag{7}
\]

Writing \( \mathbb{P}_v[T_v] = \sum_{t=0}^{\infty} \mathbb{P}_v(T_v > t) \) and applying (7), we can bound \( \mathbb{E}_v[T_v] \) by

\[
t_{n,\epsilon} + \sum_{t=t_{n,\epsilon}+1}^{\infty} e^{-\log n/\alpha} = t_{n,\epsilon} + o_n(1),
\]

where \( o_n(1) \to 0 \) as \( n \to \infty \). The statement of the theorem then follows from noting that \( \log n/\alpha \sim \log n \) as \( n \to \infty \), and by choosing \( \epsilon > 0 \) such that \( \bar{D}(Q_0, Q_1) - \epsilon = I^-(\epsilon) \).

Such an \( \epsilon \) always exists, since \( \bar{D}(Q_0, Q_1) - \epsilon \) decreases from \( \bar{D}(Q_0, Q_1) \) to \( 0 \) as \( \epsilon \) ranges from 0 to \( \bar{D}(Q_0, Q_1) \), and \( I^-(\epsilon) \) is a continuous, increasing function that takes the value 0 at \( \epsilon = 0 \).

**Proof.** For any positive integer \( K \), we can write

\[
\mathbb{P}_v \left( \max_{t \leq L} Z_{vu}(t) \geq (\bar{D}(Q_0, Q_1) + \epsilon) f_{vu}(L) \right)
\]

\[
\leq \mathbb{P}_v \left( \max_{t \leq K} Z_{vu}(t) \geq (\bar{D}(Q_0, Q_1) + \epsilon) f_{vu}(L) \right)
\]

\[
+ \mathbb{P}_v \left( \max_{K < t \leq L} Z_{vu}(t) \geq (\bar{D}(Q_0, Q_1) + \epsilon) f_{vu}(L) \right).
\]

The first term tends to 0 as \( L \to \infty \) since \( \max_{t \leq K} Z_{vu}(t) \) is almost surely finite and does not depend on \( L \). Since \( f_{vu}(t) \) is an increasing function, we can bound the second term by

\[
\mathbb{P}_v \left( \max_{K < t \leq L} \left[ Z_{vu}(t) f_{vu}(t) - \bar{D}(Q_0, Q_1) \right] \geq \epsilon \right). \tag{8}
\]

Using (5) and a union bound, we can bound (8) by

\[
\sum_{t=K+1}^{\infty} e^{-f_{vu}(t) I^+(\epsilon)} < \infty.
\]

This bound does not depend on \( L \), so we can first take \( L \to \infty \) in (8) and then \( K \to \infty \) to obtain the desired result.

**Proof of Theorem 7** Fix any \( (D, T) \in \Delta_G(n, R_n, \alpha) \) as well as a vertex \( v \in V_n \). Define the event \( \Omega_{v,L} := \{ D \in \mathcal{N}_v(R_n) \} \cap \{ T \leq L \} \), where \( L \) is a positive integer to be chosen later. By a change of measure,

\[
\mathbb{P}_u(D \in \mathcal{N}_v(R_n)) = \mathbb{E}_u [\mathbb{1}_{\{D \in \mathcal{N}_v(R_n)\}} \exp(-Z_{vu}(T))] \cdot \mathbb{P}_u(D \in \mathcal{N}_v(R_n)) \geq \mathbb{E}_v [\mathbb{1}_{\{D \in \mathcal{N}_v(R_n)\}} \exp(-Z_{vu}(T))] \geq e^{-B} \mathbb{P}_v \left( \mathbb{1}_{\{D \in \mathcal{N}_v(R_n)\}} \right) \cdot \mathbb{P}_v \left( \mathbb{1}_{\{D \in \mathcal{N}_v(R_n)\}} \leq B \right)
\]

Noting that \( \mathbb{P}_v(\Omega_{v,L}) \geq \mathbb{P}_v(D \in \mathcal{N}_v(R_n)) - \mathbb{P}_v(T > L) \) and substituting this in place of \( \mathbb{P}_v(\Omega_{v,L}) \) gives

\[
\mathbb{P}_v(T > L) \geq \mathbb{P}_v(D \in \mathcal{N}_v(R_n)) - \mathbb{P}_v(T > L)
\]

From (1), \( \mathbb{P}_v(D \in \mathcal{N}_v(R_n)) \geq 1 - \alpha \) and \( \mathbb{P}_u(D \in \mathcal{N}_v(R_n)) \leq \alpha \mathcal{N}_v(R_n) \) for \( u \in V_n \setminus \mathcal{N}_v(R_n) \). It follows that

\[
\mathbb{P}_v(T > L) \geq (1 - \alpha) \cdot \mathbb{P}_v(D \in \mathcal{N}_v(R_n)) - \mathbb{P}_v \left( \max_{t \leq L} Z_{vu}(t) \geq B \right). \tag{9}
\]

Let \( \epsilon > 0 \), and set

\[
L := \frac{1 - \epsilon}{\bar{D}(Q_0, Q_1)} + \frac{\log |V_n \setminus \mathcal{N}_v(R_n)| - \alpha |\mathcal{N}_v(R_n)|}{\log |V_n \setminus \mathcal{N}_v(R_n)|}.
\]

\[
B := (1 - \epsilon) \log \frac{|V_n \setminus \mathcal{N}_v(R_n)|}{\alpha |\mathcal{N}_v(R_n)|}.
\]

**III. PROOF OF THEOREM 1**

In this section we prove Theorem 1 which provides asymptotic lower bounds for \( T^*(n, R_n, \alpha) \). We first prove the following lemma about the tail behavior of likelihood ratios.

**Lemma 1.** For any \( u, v \in V \),

\[
\lim_{L \to \infty} \mathbb{P}_v \left( \max_{t \leq L} Z_{vu}(t) \geq (\bar{D}(Q_0, Q_1) + \epsilon) f_{vu}(L) \right) = 0.
\]
Then by Lemma 1, \( P_v (\max_{t \leq L} Z_{vu}(t) \geq B) \) goes to 0 as \( n \to \infty \) and \( \alpha \to 0 \). Plugging in these values to (9) gives the following lower bound on \( P_v (T > L) \):

\[
1 - \alpha - \left( \frac{\alpha |N_v(R_n)|}{|N_v(R_n)|} \right) - P_v \left( \max_{t \leq L} Z_{vu}(t) \geq B \right).
\]

Take \( n \to \infty \) and then \( \alpha \to 0 \) to obtain

\[
\lim_{\alpha \to 0} \lim_{n \to \infty} P_v \left( T > F_{vu} \left( 1 - \epsilon, \frac{\log |N_v(R_n)|}{\alpha |N_v(R_n)|} \right) \right) \geq 1.
\]

Following the same steps as Lemma 2.1 in [19], we see that as we send \( n \to \infty \), \( \alpha \to 0 \) and \( \epsilon \to 0 \) in that order we obtain

\[
\lim_{\alpha \to 0} \lim_{n \to \infty} \max_{u,v \in V, \delta(u,v) > R_n} T^*(n, R_n, \alpha) \geq 1.
\]

To conclude the proof we note that the expression

\[
\log \left( \frac{|N_v(R_n)|}{|N_v(R_n)|} \right) = \log |N_v(R_n)| - \log |N_v(R_n)|
\]

can be simplified for different values of the radius \( R_n \). If \( \log |N_v(R_n)| < \log |N_v(R_n)| = n \), then \( \log |N_v(R_n)| \sim n \log n \). If \( |N_v(R_n)| \sim n^\gamma \) for some \( \gamma \in (0, 1) \), then \( \log |N_v(R_n)| \sim (1 - \gamma) \log n \). Finally, if \( |N_v(R_n)|, |N_v(R_n)| \sim n \), then \( \log |N_v(R_n)| \sim \log n \).

**IV. APPLICATIONS TO REGULAR TREES AND LATTICES**

In \( k \)-regular graphs, the following result establishes the asymptotic optimality of the MSPRT \( (D_n, T_n) \) introduced in Section III. The results of the corollary are supported by numerical simulations in Figure 1.

**Corollary 1.** Let \( G \) be the infinite \( k \)-regular tree. If \( R_n \ll \log n \), then the MSPRT is asymptotically optimal, and

\[
T^*(n, R_n, \alpha) \approx n, \alpha \log \log n \over \log k.
\]

**Proof.** For distinct \( u, v \in V \) we can write

\[
f_{vu}(t) = \sum_{s=0}^{t} |N_v(s) \setminus N_u(s)|
\]

\[
= \sum_{s=0}^{t} (|N_v(s)| - |N_v(s) \cap N_u(s)|).
\]

(10)

We remark that it suffices to compute the asymptotic behavior of \( f_{vu} \) and \( F_{vu} \) to apply Theorems 1 and 2 to lattices. From straightforward computations that \( \sum_{s=0}^{t} |N_v(s)| \sim k^{t+1} \cdot k^t \).

Next, we compute \( \sum_{s=0}^{t} |N_v(s) \cap N_u(s)| \). Without loss of generality, suppose that \( d(u, v) = r \) is even. Then there is a unique vertex \( w \) such that \( d(w, v) = d(u, w) = r/2 \), so \( |N_v(s) \cap N_u(s)| = |N_w(s - r/2)| \). It follows that

\[
f_{vu}(t) \sim \frac{k^2}{(k-1)^2} \left( k^t - k^{t-r/2} \right) = \frac{k^2}{(k-1)^2} (1 - k^{r/2}) k^t.
\]

The inverse function may be computed up to first-order terms as \( F_{vu}(z) \sim \frac{\lambda \log n}{R_n} \). Plugging in the expression for \( F_{vu} \) into Theorems 1 and 2 and considering only the first-order terms yields the desired result.

**Corollary 2.** Let \( G \) be the infinite line graph (equivalently, the one-dimensional lattice). If \( R_n \ll \log n \), then the MSPRT \( (D_n, T_n) \) is asymptotically optimal up to a constant factor depending on the distributions \( Q_0 \) and \( Q_1 \). Let \( c \) be the constant defined in the statement of Theorem 2. If \( R_n \ll (\log n)^{1/2} \),

\[
\log n \over R_n \cdot D(Q_0, Q_1) \leq n, \alpha \log n \over c \cdot R_n \cdot D(Q_0, Q_1).
\]

If \( R_n \gg (\log n)^{1/2} \) and \( \log R_n \ll \log n \),

\[
\sqrt{\log n \over D(Q_0, Q_1)} \leq n, \alpha \log n \over c \cdot D(Q_0, Q_1).
\]

If \( R_n \sim n^\gamma \) for \( \gamma \in (0, 1) \),

\[
\sqrt{\frac{(1 - \gamma) \log n}{D(Q_0, Q_1)}} \leq n, \alpha \log n \over c \cdot D(Q_0, Q_1).
\]

**Proof.** It is easy to see that for any \( v \in V \) and \( s \geq 2s + 1 \), \( |N_v(s)| = 2s + 1 \). Next, fix \( u, v \in V \) and assume without loss of generality that \( d(u, v) = r \) is even. Thus \( |N_v(r/2) \cap N_u(r/2)| = 1 \), so for \( s \geq r/2 \), \( |N_v(s) \cap N_u(s)| = |N_w(s - r/2)| = 2s + 1 \), where \( w \) is the unique vertex in \( N_v(r/2) \cap N_u(r/2) \). We can then compute

\[
f_{vu}(t) = \begin{cases} (t+1)^2 & \text{if } t < \frac{r}{2}; \\ \frac{1}{2} (2t^2 - 2t) & \text{if } t \geq \frac{r}{2} \end{cases}.
\]

(11)

First suppose \( R_n \ll (\log n)^{1/2} \). Then by (11), it holds for any constant \( \lambda > 0 \) that

\[
\max_{u,v} \max_{d(u,v) > R_n} F_{vu}(\lambda \log n) \sim \frac{\lambda \log n}{R_n}.
\]
The desired result then follows from Theorems 1 and 2. By Theorems 1 and 2 in the general case.

The most important open question is if we can say anything in the general case. There are several avenues for future work.

An interesting difference between the behavior of the optimal stopping time on lattices and on regular trees is the dependence on the confidence radius. This is illustrated via numerical results in Figure 2.

Computing $F_{vu}$ and $f_{vu}$ even to a first-order approximation in higher-dimensional lattices requires more involved combinatorial arguments. However, the analysis of the one-dimensional case gives us a strong intuition of what we should expect in higher dimensions. The volume of the $\ell_1$ ball of radius $s$ in $k$-dimensional euclidean space is on the order of $s^k$, so for $t < \frac{d(u,v)}{2}$, $f_{vu}(t)$ should be on the order of $t^{k+1}$. Following the same arguments in the proof of Corollary 2 would imply that $T^*(n, R_n, \alpha)$ will increase as $(\log n)^{1+\gamma}$ when $R_n \gg (\log n)^{1+\gamma}$.

V. CONCLUSION

In this paper, we studied the problem of estimating the source of a network cascade given noisy time-series data. We found that if $\min_{v \in V_n} [N(v, R_n)] \ll n$, the MSPRT is asymptotically optimal as $\alpha \to 0$ in the case of regular trees, and is asymptotically optimal up to a constant factor in the general case. There are several avenues for future work. The most important open question is if we can say anything about non-asymptotic optimality. Another useful challenge is to close the gap between the upper and lower bounds given by Theorems 1 and 2 in the general case.

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