A Delayed Black and Scholes Formula II*†

By Mercedes Arriojas‡, Yaozhong Hu§, Salah-Eldin Mohammed¶, and Gyula Pap∥

January 25, 2022

Abstract

This article is a sequel to [A.H.M.P]. In [A.H.M.P], we develop an explicit formula for pricing European options when the underlying stock price follows a non-linear stochastic delay equation with fixed delays in the drift and diffusion terms. In this article, we look at models of the stock price described by stochastic functional differential equations with variable delays. We present a class of examples of stock dynamics with variable delays that permit an explicit form for the option pricing formula. As in [A.H.M.P], the market is complete with no arbitrage. This is achieved through the existence of an equivalent martingale measure. In subsequent work, the authors intend to test the models in [A.H.M.P] and the present article against real market data.

1 Stock price models with memory

In this section we present models of stock price dynamics that are described by stochastic functional differential equations (sfde’s). These models are feasible, in the sense that they admit unique solutions that are positive almost surely.

Consider a stock whose price $S(t)$ at time $t$ is given by the stochastic functional differential equation (sfde):

$$
\begin{align*}
\frac{dS(t)}{S(t)} &= f(t, S_t) dt + g(S(t - b))S(t) dW(t), & t \in [0, T] \\
S(t) &= \varphi(t), & t \in [-L, 0]
\end{align*}
$$

(1)

The above sfde lives on a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ satisfying the usual conditions. The constants $L$, $b$ and $T$ are positive with $L \geq b$. The space $C([-L, 0], \mathbb{R})$ of all continuous functions $\eta: [-L, 0] \to \mathbb{R}$ is a Banach space with the supremum norm

$$
\|\eta\| := \sup_{s \in [-L, 0]} |\eta(s)|.
$$

*Key Words and Phrases: stochastic functional differential equation, option pricing, Black and Scholes formula, equivalent martingale measure.

†AMS classification: 60H05, 60H07, 60H10, 91B28.

‡The research of this author is supported in part by the Center of Scientific and Human Development in Venezuela, and by Southern Illinois University.

§The research of this author is supported in part by the National Science Foundation under Grant No. DMS 0204613 and No. EPS-9874732, matching support from the State of Kansas and General Research Fund of the University of Kansas.

¶The research of this author is supported in part by NSF Grants DMS-9975462 and DMS-0203368, and by the University of Central Venezuela.

∥The research of this author is supported in part by a Fulbright fellowship.
The drift coefficient $f : [0, T] \times C([-L, 0], \mathbb{R}) \to \mathbb{R}$ is a given continuous functional, and $g : \mathbb{R} \to \mathbb{R}$ is continuous. The initial process $\varphi : \Omega \to C([-L, 0], \mathbb{R})$ is $\mathcal{F}_0$-measurable with respect to the Borel $\sigma$-algebra of $C([-L, 0], \mathbb{R})$. The process $W$ is a one-dimensional standard Brownian motion adapted to the filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$; and $S_t \in C([-L, 0], \mathbb{R})$ stands for the segment $S_t(s) := S(t+s)$, $s \in [-L, 0]$, $t \geq 0$. A general theory of existence and uniqueness of solutions to sfde’s is provided in [Mo2] and [Mo3]. However, the results in [Mo1] and [Mo2] do not cover Hypotheses (E) below.

Under the following hypotheses, we will demonstrate the feasibility of the model (1): That is, it has a unique pathwise solution such that $S(t) > 0$ almost surely for all $t \geq 0$ whenever $\varphi(t) > 0$ for all $t \in [-L, 0]$.

**Hypotheses (E).**

(i) There is a positive constant $L'$ such that
\[
|f(t, \eta)| \leq L'(1 + \|\eta\|)
\]
for all $(t, \eta) \in [0, T] \times C([-L, 0], \mathbb{R})$.

(ii) For each integer $n > 0$, there is a positive constant $L_n$ such that
\[
|f(t, \eta^1) - f(t, \eta^2)| \leq L_n \|\eta^1 - \eta^2\|
\]
for all $(t, \eta^i) \in [0, T] \times C([-L, 0], \mathbb{R})$ with $\|\eta^i\| \leq n$, $i=1,2$.

(iii) $f(t, \eta) > 0$ for all $(t, \eta) \in [0, T] \times C([-L, 0], \mathbb{R}^+)$.

(iv) $g : \mathbb{R} \to \mathbb{R}$ is continuous.

(v) $a$ and $b$ are positive constants.

**Theorem 1** Assume Hypotheses (E). Then the sfde (1) has a pathwise unique solution $S$ for a given $\mathcal{F}_0$-measurable initial process $\varphi : \Omega \to C([-L, 0], \mathbb{R})$. Furthermore, if $\varphi(t) \geq 0$ for all $t \in [-L, 0]$ a.s., then $S(t) \geq 0$ for all $t \geq 0$ a.s.. If in addition $\varphi(0) > 0$ a.s., then $S(t) > 0$ for all $t \geq 0$ a.s..

**Proof.**

First let $t \in [0, b]$ and let $\varphi(t) \geq 0$ a.s. for all $t \in [-L, 0]$. Then (1) becomes
\[
\begin{align*}
\frac{dS(t)}{S(0)} &= f(t, S_t) \, dt + g(\varphi(t - b)) S(t) \, dW(t), \quad t \in [0, b] \\
S(0) &= \varphi(0).
\end{align*}
\]

(2)

Define the martingale
\[
M(t) := \int_0^t g(\varphi(u - b)) \, dW(u), \quad t \in [0, b].
\]

Then $S$ solves the stochastic functional differential equation (sfde)
\[
\begin{align*}
\frac{dS(t)}{S_0} &= f(t, S_t) \, dt + S(t) \, dM(t), \quad t \in [0, b] \\
S_0 &= \varphi.
\end{align*}
\]

(3)

Define the process $\psi : [-L, b] \times \Omega \to \mathbb{R}$ as follows: $\psi|_{[0, b]}$ is the solution of the linear sode
\[
\begin{align*}
\frac{d\psi(t)}{\psi(0)} &= \psi(t) \, dM(t), \quad t \in [0, b] \\
\psi(0) &= 1.
\end{align*}
\]

(4)
and for all $t \in [-L,0]$, set $\psi(t) = 1$.

Define the random process $y$ to be the unique solution of the random fde

$$\begin{align*}
y'(t) &= \psi(t)^{-1}f(t,\psi_t \cdot y_t), \quad t \in [0,b] \\
y_0 &= \varphi.
\end{align*} \tag{5}$$

Observe that the above fde admits a unique global solution $y$ by virtue of the linear growth hypothesis (E)(i) and the Lipschitz condition (E)(ii).

Denote by $[M,M]$ the quadratic variation of $M$. Then, from (4), it follows that

$$\psi(t) = \exp\{M(t) - \frac{1}{2}[M,M](t)\} > 0$$

for all $t \in [0,b]$.

Define the process $\tilde{S}$ by $\tilde{S}(t) := \psi(t)y(t)$ for $t \in [-L,b]$. Then by the product rule, it follows that

$$\begin{align*}
d\tilde{S}(t) &= f(t,\tilde{S}_t)dt + \tilde{S}(t)dM(t), \quad t \in [0,b] \\
\tilde{S}_0 &= \varphi.
\end{align*} \tag{6}$$

Comparing (3) and (6), it follows by uniqueness that $P$-a.s., $S(t) = \tilde{S}(t)$ for all $t \in [0,b]$. Now suppose that $\varphi(t) \geq 0$ a.s. for all $t \in [-L,0]$. Then using (5) and the monotonicity Hypothesis (E)(iii), it follows that $y(t) \geq 0$ a.s. for all $t \in [0,l]$. If in addition $\varphi(0) > 0$ a.s., then it also follows from (5) that $y(t) > 0$ for all $t \in [0,b]$ a.s. Hence $S(t) = \tilde{S}(t) > 0$ for all $t \in [0,b]$ a.s.

Using forward steps of length $b$, it is easy to see that $S(t) > 0$ a.s. for all $t \geq 0$. ◦

Remark.

Another feasible model for the stock price is obtained by considering the sfde

$$\begin{align*}
dS(t) &= h(t,S^{t-s})S(t)dt + g(S(t-b))S(t)dW(t), \quad t \in [0,T], \\
S(t) &= \varphi(t), \quad t \in [-L,0].
\end{align*} \tag{2'}$$

with $S^{t-s} := S(t \wedge s)$, $t,s \in [-L,T]$, and $h : [0,T] \times C([-L,T],\mathbb{R}) \to \mathbb{R}$ is a continuous functional. Theorem 1 holds for the above model if Hypotheses (E) hold with E(iii) replaced by the following monotonicity condition:

(E)(iii)' For each $\xi \in C([-L,T],\mathbb{R})$ with $\xi(t) \geq 0$ for all $t \in [-L,T]$, one has $h(t,\xi) \geq 0$ for all $t \in [0,T]$.

The proof is analogous to that of Theorem 1.

2 A stock price model with variable delay

In this section, we give an alternative model for the stock price dynamics with variable delay. In this case we are able to develop a Black-Scholes formula for the option price (cf. [B.S], [Me1]).

Throughout this section, suppose $h$ is a given fixed positive number. Denote $[t] := kh$ if $kh \leq t < (k+1)h$.

Consider a market consisting of a riskless asset $\xi$ with a variable (deterministic) continuous rate of return $\lambda$, and a stock $S$ satisfying the following equations

$$\begin{align*}
d\xi(t) &= \lambda(t)\xi(t)dt \\
dS(t) &= f(t,S([t]))S(t)dt + g(t,S([t]))S(t)dW(t)
\end{align*} \tag{7}$$
for \( t \in (0, T] \), with initial conditions \( \xi(0) = 1 \) and \( S(0) > 0 \). The above model lives on a probability space \((\Omega, \mathcal{F}, P)\) with a filtration \((\mathcal{F}_t)_{0 \leq t \leq T}\) satisfying the usual conditions, and a standard one-dimensional Brownian motion \( W \) adapted to the filtration. Suppose \( f : [0, T] \times \mathbb{R} \to \mathbb{R} \) is a continuous function. Assume further that \( g : [0, T] \times \mathbb{R} \to \mathbb{R} \) is continuous and \( g(t, v) \neq 0 \) for all \((t, v) \in [0, T] \times \mathbb{R} \).

Under the above conditions, this model is feasible: That is \( S(t) > 0 \) a.s. for all \( t > 0 \). This follows by an argument similar to the proof of Theorem 1, Section 2 in [A.H.M.P]. Details are left to the reader.

Next, we will establish the completeness of the market \( \{\xi(t), S(t) : t \in [0, T]\} \) and the no-arbitrage property, following the approach in Section 3 in [A.H.M.P].

For \( t \in [k h, (k + 1) h] \), the solution of the second equation in (7) is given by

\[
S(t) = S(kh) \exp \left( \int_{kh}^{t} g(s, S(kh)) \, dW(s) + \int_{kh}^{t} f(s, S(kh)) \, ds - \frac{1}{2} \int_{kh}^{t} g(s, S(kh))^2 \, ds \right). \tag{8}
\]

As in Section 3 in [A.H.M.P], let

\[
\tilde{S}(t) := \frac{S(t)}{\xi(t)} = S(t) e^{-\int_{0}^{t} \lambda(s) \, ds}, \quad t \in [0, T],
\]

be the discounted stock price process. Again by Itô’s formula, we obtain

\[
d\tilde{S}(t) = \frac{1}{\xi(t)} dS(t) + S(t) \left( -\frac{\lambda(t)}{\xi(t)} \right) dt
\]

\[
= \tilde{S}(t) \left[ \{ f(t, S([t])) - \lambda(t) \} dt + g(t, S([t])) dW(t) \right].
\]

Let

\[
\tilde{S}(t) := \int_{0}^{t} \{ f(u, S([u])) - \lambda(u) \} \, du + \int_{0}^{t} g(u, S([u])) \, dW(u), \quad t \in [0, T].
\]

Then

\[
d\tilde{S}(t) = \tilde{S}(t) d\tilde{S}(t), \quad 0 < t < T, \tag{9}
\]

and

\[
\tilde{S}(t) = S(0) + \int_{0}^{t} \tilde{S}(u) d\tilde{S}(u), \quad t \in [0, T]. \tag{10}
\]

Define the stochastic process

\[
\Sigma(u) := \frac{\{ f(u, S([u])) - \lambda(u) \}}{g(u, S([u]))}, \quad u \in [0, T].
\]

It is clear that \( \Sigma(u) \) is \( \mathcal{F}_{[u]}^{\tilde{S}} \)-measurable for each \( u \in [0, T] \). Furthermore, by a backward conditioning argument using steps of length \( h \), the reader may check that

\[
E_{P}(\rho_T) = 1
\]

where

\[
\rho_T := \exp \left\{ -\int_{0}^{T} \left\{ \frac{f(u, S([u])) - \lambda(u)}{g(u, S([u]))} \right\} dW(u) - \frac{1}{2} \int_{0}^{T} \left| \frac{f(u, S([u])) - \lambda(u)}{g(u, S([u]))} \right|^2 \, du \right\}.
\]

(See the argument in Section 3 in [A.H.M.P] following the statement of Theorem 2.) Hence the Girsanov theorem ([K.K], Theorem 5.5) applies, and it follows that the process

\[
\tilde{W}(t) := W(t) + \int_{0}^{t} \left\{ \frac{f(u, S([u])) - \lambda(u)}{g(u, S([u]))} \right\} du, \quad t \in [0, T],
\]

is a standard one-dimensional Brownian motion adapted to \( \mathcal{F} \).
is a standard Wiener process under the probability measure $Q$ defined by $dQ := \varrho_t \, dP$. Using (9) and the definitions of $\bar{S}$ and $\tilde{W}$, it is easy to see that

$$d\bar{S}(t) = \bar{S}(t)g(t, S([t])) \, d\tilde{W}(t), \quad t \in [0, T].$$

This implies that $\bar{S}$ is a $Q$-martingale, and hence the market $\{\xi(t), S(t) : t \in [0, T]\}$ has the no-arbitrage property ([K.K], Theorem 7.1).

We now establish the completeness of the market $\{\xi(t), S(t) : t \in [0, T]\}$. To do so, let $X$ be any contingent claim, viz. an integrable $F_T$-measurable non-negative random variable. Define the process

$$M(t) := E_Q\left( \frac{X}{\xi(T)} \bigg| F_t^S \right) = E_Q\left( \frac{X}{\xi(T)} \bigg| F_t^{\tilde{W}} \right), \quad t \in [0, T].$$

Then $M(t), t \in [0, T]$, is an $(F_t^{\tilde{W}})$-adapted $Q$-martingale. Hence, by the martingale representation theorem ([K.K], Theorem 9.4), there exists an $(F_t^{\tilde{W}})$-predictable process $h_1(t), t \in [0, T]$, such that

$$\int_0^T h_1(u)^2 \, du < \infty \quad \text{a.s.,}$$

and

$$M(t) = E_Q\left( \frac{X}{\xi(T)} \right) + \int_0^t h_1(u) \, d\tilde{W}(u), \quad t \in [0, T].$$

Define

$$\pi_S(t) := \frac{h_1(t)}{S(t)g(t, S([t]))}, \quad \pi_\xi(t) := M(t) - \pi_S(t)\bar{S}(t), \quad t \in [0, T].$$

Consider the strategy $\{(\pi_\xi(t), \pi_S(t)) : t \in [0, T]\}$ which consists of holding $\pi_S(t)$ units of the stock and $\pi_\xi(t)$ units of the bond at time $t$. The value of the portfolio at any time $t \in [0, T]$ is given by

$$V(t) := \pi_\xi(t)\xi(t) + \pi_S(t)S(t) = \xi(t)M(t).$$

Furthermore,

$$dV(t) = \xi(t)dM(t) + M(t)d\xi(t) = \pi_\xi(t)d\xi(t) + \pi_S(t)dS(t), \quad t \in (0, T].$$

Consequently, $\{(\pi_\xi(t), \pi_S(t)) : t \in [0, T]\}$ is a self-financing strategy. Clearly $V(T) = \xi(T)M(T) = X$. Thus the contingent claim $X$ is attainable. This shows that the market $\{\xi(t), S(t) : t \in [0, T]\}$ is complete.

Moreover, in order for the augmented market $\{\xi(t), S(t), X : t \in [0, T]\}$ to satisfy the no-arbitrage property, the price of the claim $X$ must be

$$V(t) = \frac{\xi(t)}{\xi(T)} E_Q(X | F_t^S)$$

at each $t \in [0, T]$ a.s. See, e.g., [B.R] or Theorem 9.2 in [K.K].

The above discussion may be summarized in the following formula for the fair price $V(t)$ of an option on the stock whose evolution is described by the SDE (7).

**Theorem 2** Suppose that the stock price $S$ is given by the SDE (7), where $S(0) > 0$ and $g$ satisfies Hypothesis (B). Let $T$ be the maturity time of an option (contingent claim) on the stock with payoff function $X$, i.e., $X$ is an $F_T^S$-measurable non-negative integrable random variable. Then at any time $t \in [0, T]$, the fair price $V(t)$ of the option is given by the formula

$$V(t) = E_Q(X | F_t^S)e^{-\int_t^T \lambda(s) \, ds}, \quad (11)$$
where $Q$ denotes the probability measure on $(\Omega, \mathcal{F})$ defined by $dQ := q_t \, dP$ with

$$q_t := \exp \left\{ - \int_0^t \frac{\{f(u, S([u])) - \lambda(u)\}}{g(u, S([u]))} \, dW(u) - \frac{1}{2} \int_0^t \left| \frac{f(u, S([u])) - \lambda(u)}{g(u, S([u]))} \right|^2 \, du \right\}$$

for $t \in [0, T]$. The measure $Q$ is a martingale measure and the market is complete.

Moreover, there is an adapted and square integrable process $h_1(t)$, $t \in [0, T]$, such that

$$E_Q \left( \frac{X}{\xi(T)} \big| \mathcal{F}_t^S \right) = E_Q \left( \frac{X}{\xi(T)} \right) + \int_0^t h_1(u) \, d\tilde{W}(u), \quad t \in [0, T],$$

where

$$\tilde{W}(t) := W(t) + \int_0^t \frac{\{f(u, S([u])) - \lambda(u)\}}{g(u, S([u]))} \, du, \quad t \in [0, T].$$

The hedging strategy is given by

$$\pi_S(t) := \frac{h_1(t)}{\tilde{S}(t)g(t, S([t]))}, \quad \pi_\xi(t) := M(t) - \pi_S(t)\tilde{S}(t), \quad t \in [0, T]. \quad (12)$$

The following result gives a Black-Scholes-type formula for the value of a European option on the stock at any time prior to maturity ([B.S], [Me1], [H.R]).

**Theorem 3** Assume the conditions of Theorem 2. Let $V(t)$ be the fair price of a European call option written on the stock $S$ with exercise price $K$ and maturity time $T$. Then for all $t \in [T - [T], T]$, $V(t)$ is given by

$$V(t) = S(t)\Phi(\beta_+(t)) - K\Phi(\beta_-(t))e^{-\int_0^T \lambda(s)ds}, \quad (13)$$

where

$$\beta_\pm(t) := \frac{\log S(t) - \int_0^T \left( \lambda(u) \pm \frac{1}{2} g(u, S([u]))^2 \right) du}{\sqrt{\int_0^T g(u, S([u]))^2 du}},$$

and $\Phi$ is the standard normal distribution function.

If $T > h$ and $t < T - [T]$, then

$$V(t) = e^{\int_0^T \lambda(s)ds} \int_0^T \left\{ H \left( S(T - [T]), -\frac{1}{2} \int_{T-[T]}^T g(u, S([u]))^2 du, \right. \right.$$  

$$\left. \int_{T-[T]}^T g(u, S([u]))^2 du \right\} \bigg| \mathcal{F}_t \bigg) \quad (14)$$

where $H$ is given by

$$H(x, m, \sigma^2) := xe^{m+\sigma^2/2} \Phi(\alpha_1(x, m, \sigma)) - K\Phi(\alpha_2(x, m, \sigma))e^{-\int_0^T \lambda(s)ds},$$

and

$$\alpha_1(x, m, \sigma) := \frac{1}{\sigma} \left[ \log \left( \frac{x}{K} \right) + \int_0^T \lambda(s)ds + m + \sigma^2 \right],$$

$$\alpha_2(x, m, \sigma) := \frac{1}{\sigma} \left[ \log \left( \frac{x}{K} \right) + \int_0^T \lambda(s)ds + m \right],$$

for $\sigma, x \in \mathbb{R}^+$, $m \in \mathbb{R}$.

The hedging strategy is given by

$$\pi_S(t) = \Phi(\beta_+(t)), \quad \pi_\xi(t) = -K\Phi(\beta_-(t))e^{-\int_0^T \lambda(s)ds}, \quad t \in [T - [T], T].$$
Acknowledgments.

The authors are very grateful to R. Kuske and B. Øksendal for very useful suggestions and corrections to earlier versions of the manuscript. The authors also acknowledge helpful comments and discussions with Saul Jacka.

References

[A.H.M.P] Arriojas, M., Hu, Y., Mohammed, S.-E.A. and Pap, G., A delayed Black and Scholes formula I, *preprint*, pp. 10.

[B.R] Baxter, M. and Rennie, A., *Financial Calculus*, Cambridge University Press (1996).

[B.S] Black, F. and Scholes, M., The pricing of options and corporate liabilities, *Journal of Political Economy* 81 (May-June 1973), 637-654.

[H.R] Hobson, D., and Rogers, L. C. G., Complete markets with stochastic volatility, *Math. Finance* 8 (1998), 27–48.

[K.K] Kallianpur, G. and Karandikar R. J., *Introduction to Option Pricing Theory*, Birkhäuser, Boston-Basel-Berlin (2000).

[Me1] Merton, R. C., Theory of rational option pricing, *Bell Journal of Economics and Management Science* 4, Spring (1973), 141-183.

[Mo1] Mohammed, S.-E. A., *Stochastic Functional Differential Equations*. Pitman 99 (1984).

[Mo2] Mohammed, S.-E. A., Stochastic differential systems with memory: Theory, examples and applications. In “Stochastic Analysis”, Decreusefond L. Gjerde J., Øksendal B., Ustunel A.S., edit., *Progress in Probability* 42, Birkhauser (1998), 1-77.

Mercedes Arriojas
Department of Mathematics
University Central of Venezuela
Caracas
Venezuela
e-mail: marrioja@euler.ciens.ucv.ve

Yaozhong Hu
Department of Mathematics
University of Kansas
405 Snow Hall
Lawrence
KS 66045-2142
e-mail: hu@math.ukans.edu

Salah-Eldin A. Mohammed
Department of Mathematics
Southern Illinois University at Carbondale
Carbondale
Illinois 62901, U.S.A.
e-mail: salah@sfde.math.siu.edu
web-site: [http://sfde.math.siu.edu](http://sfde.math.siu.edu)

Gyula Pap
Institute of Mathematics and Informatics
