QUANTUM AND THERMAL FLUCTUATIONS, UNCERTAINTY PRINCIPLE, DECOHERENCE AND CLASSICALITY

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Abstract

We scrutinize the commonly used criteria for classicality and examine their underlying issues. The two major issues we address here are that of decoherence and fluctuations. We borrow the insights gained in the study of the semiclassical limit of quantum cosmology to discuss the three criteria of classicality for a quantum closed system: adiabaticity, correlation and decoherence. We then use the Brownian model as a paradigm of quantum open systems to discuss the relation of quantum and thermal fluctuations and their role in the transition from quantum to classical. We derive the uncertainty relation at finite temperature. We study how the fluctuations of a quantum system evolve after it is brought in contact with a heat bath and analyse the decoherence and relaxation processes. From the effect of fluctuations on decoherence we show the relation between these two sets of criteria of classicality. Finally, we briefly comment on the issue of nonintegrability in quantum open systems.

\footnote{Invited Talk delivered by B. L. Hu at the Third Workshop on Quantum Nonintegrability, Drexel University, Philadelphia, May, 1992. To appear in Quantum Dynamics of Chaotic Systems, edited by J. M. Yuan, D. H. Feng and G. M. Zaslavsky (Gordon and Breach, Langhorne 1993)}
1 Criteria for Classicality

A quick sampling of textbooks of quantum mechanics and statistical mechanics reveals a variety of seemingly simple and straightforward criteria and conditions for classicality. For example, one can loosely associate:

1) $\hbar \to 0$
2) WKB approximation, which "gives the semiclassical limit"
3) Wigner function, which "behaves like the classical distribution function"
4) high temperature limit: "thermal=classical"
5) Uncertainty Principle: a system "becomes classical" when this is no longer obeyed
6) coherent states: the ‘closest’ to the classical
7) systems with large quantum number $n \to \infty$ (correspondance principle)
8) systems with large number of components $1/N \to 0$

Each of these conditions contains only some partial truth and when taken on face value can be very misleading. To understand the meaning of classicality it is important to examine the exact meaning of these criteria and the conditions of their validity.

We can divide the above conditions into four groups, according to the different issues behind these criteria:

a) quantum interference in 1) to 3),
b) quantum and thermal fluctuations in 1),4) and 5),
c) choice of special quantum states, and
d) meaning of the large n and N limit.

We will discuss only the first two groups of issues, using lessons we learned from quantum cosmology and the paradigm of quantum open systems. We will show that many of these criteria hold only under special conditions. They can approximately define the classical limit only when taken together in specific ways. We will also show the relation of the first two groups of criteria. Specifically, for issue a), decoherence is needed for the WKB Wigner function to show a peak indicating the correlation between the physical variables and their canonical conjugates which defines a classical trajectory in phase space. This clarifies the loose connection of WKB, Wigner function and classicality. For issue b), the time for thermal fluctuations to overtake quantum fluctuations is also the time of decoherence. But a decohered system is not necessarily classical. There is a quantum statistical regime in
between. One can derive an uncertainty principle for a quantum open system at finite temperature which interpolates between the (zero temperature) quantum Heisenberg relation and the high temperature result of classical statistical mechanics. This clarifies the sometimes vague notions of quantum, thermal and classical. The first set of issues was discussed in the context of quantum cosmology by the authors of [1]. We will only state the results without derivation. The second set of issues was clarified in a recent paper of ours [2], which the present report is based on. We will discuss these issues in the following sections. We end with a few brief comments on the main theme of this conference with the viewpoint espoused here, i.e., how one should perceive quantum and classical chaos, when quantum and classical physics are taken to be the physics of a closed versus an open system.

2 Semiclassical Limit: Adiabaticity, Correlation, and Decoherence

Our interest in the problem of quantum to classical transition stemmed from related problems in quantum cosmology. How did the classical world as we observe it today evolve from a wave function or density matrix of the universe [3]? An important transition point is the Planck time ($10^{-43} \text{sec}$). The belief is that only after this time can the world be described by semiclassical gravity, a theory where the spacetime is classical but the matter field is quantized. The issue of transition from quantum to classical spacetimes is complicated here by two additional factors unique to quantum cosmology (as an example of a quantum closed system described as a parametrized theory): there is no ‘external’ classical observer to carry out a measurement, and there is no ‘explicit’ time variable to define the evolution [4]. We shall try to comply with these conditions in this part of our discussion. It is easier to relax these conditions when there is a well-defined time, and when there is a reasonable system-environment separation. After five years of search and research, one can say that at least three conditions are necessary [5, 6, 7] for the appearance of classical behavior in spacetime: i) the adiabaticity condition, which ensures that the leading terms of a WKB expansion of the wave function dominate; ii) the strong correlation condition, by requiring a sharp peak in the reduced WKB Wigner function; and iii) the strong decoherence condi-
tion, by requiring the close diagonalization of the reduced density matrix. In this section we follow the summary in [8]:

2.1 WKB

The first condition to see the emergence of classical spacetime is that certain regions of superspace (the space of all three-geometries) has to admit oscillatory solutions to the wave equation, which in quantum cosmology is known as the Wheeler DeWitt equation. For a universe consisting of a gravitational sector described by coordinates \( r^a \) of the minisuperspace variables (for the Robertson-Walker universe, \( r \) is the scale factor \( a \)) and a matter-field sector whose coordinates are the amplitude functions of the normal modes \( f_k \) of, say, a scalar field \( \Phi \), the Wheeler-DeWitt equation for the wave function of the universe \( \Psi(r^a, f_k) \) can be written in the general form

\[
\left[ \frac{1}{2M} \nabla^2 + MV(r^a) - H_\Phi(r^a, \Phi) \right] \Psi(r^a) = 0 \tag{1}
\]

where \( M \) is a large parameter (in units of Planck mass squared), \( \nabla^2 \) is the Laplacian and \( V(r) \) is a potential function on the minisuperspace (which can contain contributions from spatial curvature, the cosmological constant and classical matter), and \( H_\Phi = \Sigma_k H_k \) is the sum of the hamiltonians for each mode of the matter field. [We can denote these variables schematically as \((r, f)\). Later, in the discussion of quantum open systems, we can think of them as the system and environment variables.] Classical limit exists only in those regions of superspace where one can write the wave function in the WKB form:

\[
\Psi(r, f) = e^{iMS(r)}C(r)\chi(r, f). \tag{2}
\]

The existence of a large parameter \( M \) in the theory which measures the discrepancy between the gravitational and the matter sector makes it possible to carry out an expansion (Born-Oppenheimer approximation) of all the functions \( S, C, \chi \) in powers of \( M^{-1} \), i.e.,

\[
S = S_0(r) + M^{-1}S_1(r) \ldots \tag{3}
\]

\(^2\)In the case of pure gravity, a large \( M \) expansion (Born-Oppenheimer) is equivalent to a small \( \hbar \) expansion. However, these two expansions are not equivalent when matter fields are included.
One obtains a set of interlinked equations for these functions of different orders. In particular, the order $M^0$ terms yield an equation for $S_0$ which is the Hamilton-Jacobi equation for the eikonal function:

$$\frac{1}{2} \left( \frac{dS_0}{dr} \right)^2 - V(r) = 0 \quad (4)$$

while the order $M^{-1}$ terms yield an equation for $S_1$ which gives the WKB prefactor.

In quantum gravity time is an emergent quantity believed to be well-defined only when the classical spacetime takes shape. In the WKB regime one can define such a time by

$$\frac{d}{dt} = \frac{dS_0}{dr} \frac{d}{dr} \quad (5)$$

In terms of this WKB time, the equation for the lowest order prefactor assumes the form of a Schrödinger equation

$$i \frac{d\chi_0}{dt} = H\Phi \chi_0 \quad (6)$$

Thus the WKB approximation renders the wave mechanics of the universe into a form like ordinary quantum mechanics.

### 2.2 Correlations

To make predictions in a quantum closed system like the universe, one needs to develop a viewpoint beyond the Copenhagen interpretation of quantum mechanics which does not rely on the existence of a classical external observer or apparatus. It has been proposed that one can regard a strong peak in the wave function or a distribution constructed from it as predicting some correlations among the variables in the support of the peak. (In the absence of a strong peak no predictions can be made.) This proposal also avoids the use of probabilities conserved with respect to an external time. (See, however, [10])

Following this interpretive scheme one can propose the following criterion: a system can be regarded as behaving classically when the wave function predicts the existence of correlations between a physical variable and its
canonical conjugate such as coordinate and momentum. These correlations should be such that the classical equations of motion are satisfied.

A more direct way of implementing the correlation criterion \cite{11} is to look for peaks in a quantum phase space distribution function about such correlations. The closest quantum mechanical analog of a classical phase space probability distribution is the Wigner function \cite{12} which is essentially the Fourier transform of the density matrix [see, e.g., Eq.(5.3)]. Initially it was claimed that a calculation of the Wigner function for a WKB wave function gave a delta function peak around the correlations. Habib \cite{14} and Anderson \cite{15} using earlier results of Berry \cite{16} and Heller \cite{17} showed that this claim was incorrect. In fact the WKB Wigner function exhibits no peak around classical trajectories and is highly oscillatory. To seek a peak one needs to decohere the Wigner function. This brings in the next criterion.

2.3 Decoherence

In quantum physics one must assign a complex probability amplitude to each history. When combined with the superposition principle this implies the existence of quantum interference effects amongst alternative histories. These effects are nevertheless not seen at the classical level. Classical behavior in a system therefore requires some mechanism for the destruction of quantum interference. This property is commonly referred to as decoherence. As keys to understanding the emergence of classical behavior in quantum closed systems Griffiths \cite{18} and Omnes \cite{19} have proposed the concept of consistent histories, and Gell-Mann and Hartle \cite{20} have proposed the use of decoherence functional as a measure of the interference between histories. One can also use the concept of environment-induced decoherence in quantum open systems to address these issues. An open system is one which interacts with an environment whose information is not fully preserved. In coarse-graining

\[^3\text{The Wigner function can however qualify only as a quasiprobability distribution because in general } W(X, p) \text{ is not positive definite (it essentially reflects the fact that in quantum mechanics } X \text{ and } p \text{ are not simultaneously measurable). In spite of this property it can be used to obtain expectation values in quantum mechanics by integrating over phase space and in this respect presents a formulation of quantum mechanics exactly equivalent to that given by wave functions or density matrices. To interpret the Wigner function as a distribution function, we will only use wave functions for which it is indeed positive definite. For a more detailed discussion on the subtleties of Wigner functions in quantum cosmology, see [13].}\]
the environment one introduces certain statistical measures. Its averaged effect on the system appear as dissipation and decoherence in the system dynamics. Decoherence was also discussed in the context of measurement theory in quantum mechanics [21, 22, 23]. The connection between the density matrix approach and the decoherence functional approach is a current topic of investigation.

Therefore, apart from the existence of correlations, a second necessary requirement for classical behavior in certain variables is the lack of interference between alternative histories of those degrees of freedom implied by decoherence. Indeed these processes are interconnected [24]. Incorporating the effect of the environment by working with the corresponding reduced Wigner function rather than the pure state Wigner function, Habib and Laflamme [13] and Paz and Sinha [6] showed that such a peak indeed appears in the decohered WKB Wigner function which follows the classical trajectory. In addition, the environment will inevitably have a backreaction on the system (just as in the case of Brownian motion the environment produces a systematic damping in the motion of the Brownian particle. See [25, 26, 27, 28, 29, 30]). Paz and Sinha [6] showed that under the adiabaticity condition this backreaction indeed gives rise to the expected semiclassical Einstein equations [31].

This is the story about how one obtains the semiclassical limit in quantum cosmology. The many subtleties in how these criteria are reached and applied contrast strongly with the simplistic conditions listed in 1) to 3). One has to be very careful in the interpretation and application of these conditions. The outstanding issue there is decoherence. Let us now turn to the second set of issues, concerning the role of quantum and thermal fluctuations in the transition from quantum to classical. We will derive the uncertainty principle at a finite temperature and use it to examine these effects. We will see that

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4In the context of quantum cosmology, even though in theory the universe is often regarded as containing everything and thus a closed system with no ‘external’ environment, in practice our observations are often restricted to a limited number of physical variables, such as the homogeneous modes of cosmological perturbations, the low energy limit of spacetime and field excitations, the causal domain within the particle horizon, etc. In this sense, only these ‘relevant’ variables constitute the physical ‘system’, and the rest of the universe are the irrelevant variables, serving as an ‘environment’ to one’s system of interest. This is how the basic paradigm of non-equilibrium statistical mechanics becomes relevant to issues in quantum cosmology [24].
the fluctuations are indeed what is responsible for the decoherence process.

3 Quantum, Thermal and Classical

The usual definition or demarkation of quantum, classical and thermal regimes are not always very precise. Oftentimes one hears the vague statement (condition 4) that high temperature regime gives the classical limit. How does thermal properties enter into the criteria of classicality. In particular, how do they relate to the first set of criteria discussed above, e.g., decoherence? What is the role of thermal fluctuations in the establishment of classical behavior in a quantum system? We will explore this issue using the model of quantum Brownian motion as a paradigm of a quantum open system. We view thermal fluctuations not as the activity of some additional ‘fundamental’ field (such as is assumed in, e.g., thermal field dynamics \cite{32}), but simply as statistical variations of the physical variables of a quantum system interacting with a stochastic source associated with the environmental variables, the exact microdynamics of the system and the environment obeying the laws of quantum mechanics. We show that under the conditions studied, the first group (decoherence) and second group (fluctuations) of criteria give equivalent results. The time the quantum system decoheres is also the time when thermal fluctuation overtakes quantum fluctuations [see Eq. (27)]. However, we warn that the post-decoherence regime should really not be called classical, as is customary in many quantum to classical transition studies. In fact, after the decoherence time, the system is aptly described by quantum statistical mechanics (QSM), indeed, non-equilibrium QSM, not classical. Only after the relaxation time can one use equilibrium QSM. Classical has still a long way to go— only when the fermions and bosons in the system can be approximated as distinguishable particles, usually at high temperatures when the Fermi-Dirac or Bose-Einstein statistics approaches the Maxwell-Boltzmann statistics, can the system be rightfully called classical. In this regard quantum carries two meanings, one refers to the interference effect and the other refers to spin-statistics effect.

A clean-cut problem where these issues manifest clearly is the uncertainty principle at finite temperature. We learn from quantum mechanics that a lower bound exists in the product of the variances of pairs of noncommutative
observables. Taking the coordinate $x$ and momentum $p$ as examples, the Heisenberg uncertainty principle states that, with $(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2$, the uncertainty function

$$U_{0}^{QM} = (\Delta x)^2(\Delta p)^2 \geq \frac{\hbar^2}{4} \quad (T = 0, \text{ quantum mechanics}). \quad (1)$$

In realistic conditions, however, quantum systems are often prepared and studied at finite temperatures where thermal fluctuations permeate. At high temperatures the equipartition theorem of classical statistical mechanics imparts for each degree of freedom an uncertainty of $kT/2$. Thus the uncertainty function for a one-dimensional particle approaches the limit

$$U_{T}^{MB} \approx \left(\frac{kT}{\hbar \Omega}\right)^2 \quad \text{(high } T, \text{ classical statistical mechanics)}, \quad (2)$$

where $\hbar \Omega$ is the energy of a normal mode with physical frequency $\Omega$. This result, obtained by assuming that the system obeys the Maxwell-Boltzmann distribution, is usually regarded as the classical limit. For a system of bosons in equilibrium at temperature $T$, the application of canonical ensemble gives the result in quantum statistical mechanics as

$$U_{T}^{BE} = \frac{\hbar^2}{4}[\coth(\frac{\hbar \Omega}{2kT})]^2 \quad \text{(all } T, \text{ quantum statistical mechanics)}, \quad (3)$$

which interpolates between the two results (3.1) and (3.2) at $T = 0$ and $T >> \hbar \Omega/k$. This result applies to a system already in equilibrium at temperature $T$.

Our purpose here is to study the corresponding non-equilibrium problem. At time $t_0$ we put the system in contact with a heat bath at temperature $T$ and follow its time evolution. We want to see how the uncertainty function $U_T(t)$ changes from the initial quantum fluctuation-dominated condition to a later thermal fluctuation-dominated condition. By comparing this result with the decoherence studies reported in Sec. 2 where two characteristic times—the decoherence time $t_{\text{dec}}$ and the relaxation time $t_{\text{rel}}$—are defined, one can apply the physics of these two processes involved to examine the issues stated above, i.e., 1) the relation between quantum and thermal fluctuations; and 2) their role in quantum to classical transition.
4 Noise and Decoherence, Fluctuation and Dissipation

We use a simple model of a quantum open system to examine these issues. Consider a collection of coupled harmonic oscillators where one is distinguished as the system of interest and the rest as bath. We use the influence functional method to incorporate the statistical effect of the bath on the system. As the microdynamics is explicit in this approach, one can study how the result depends on the properties of the bath and the system-bath interaction. This model has been studied extensively, so we will only present the main results. (See, e.g., [33] for details.)

Our system is a Brownian particle with mass $M$ and natural frequency $\Omega$. The environment is modeled by a set of $n$ harmonic oscillators with mass $m_n$ and natural frequency $\omega_n$. The particle is coupled linearly to the $n$th oscillator with strength $C_n$. The action of the combined system and environment is

$$S[\mathbf{x}, \mathbf{q}] = S[\mathbf{x}] + S[\mathbf{q}] + S_{\text{int}}[\mathbf{x}, \mathbf{q}]$$

where $\mathbf{x}$ and $\mathbf{q}_n$ are the coordinates of the particle and the oscillators respectively, and $\Omega_0$ is the (bare) frequency of the particle. We are interested in how the environment affects the system in some averaged manner. The quantity containing this information is the reduced density matrix of the system $\rho_r(x, x')$ obtained from the full density operator of the system and environment $\rho(x, q; x', q')$ by tracing out the environmental degrees of freedom $(q, q')$

$$\rho_r(x, x', t) = \int_{-\infty}^{+\infty} dq \int_{-\infty}^{+\infty} dq' \rho(x, q; x', q', t) \delta(q - q') \quad (2)$$

The reduced density matrix evolves under the action of the propagator $J_r(x, x', t \mid x_i, x'_i, 0)$ in the following way:

$$\rho_r(x, x', t) = \int_{-\infty}^{+\infty} dx_i \int_{-\infty}^{+\infty} dx'_i J_r(x, x', t \mid x_i, x_i', 0) \rho_r(x_i, x_i', 0) \quad (3)$$
In general, this is a very complicated expression since the evolution operator \( J_r \) depends on the initial state. If we assume that at a given time \( t = 0 \) the system and the environment are uncorrelated, i.e. that
\[
\hat{\rho}(t = 0) = \hat{\rho}_s \times \hat{\rho}_e,
\] (4)
then the evolution operator for the reduced density matrix can be written as
\[
J(x_f, x'_f, t \mid x_i, x'_i, 0) = \int_{x_i}^{x_f} Dx \int_{x'_i}^{x'_f} Dx' \exp \left\{ \frac{i}{\hbar} \left\{ S[x] - S[x'] \right\} \right\} F[x, x']
\] (5)
where \( F[x, x'] \) is the Feynman-Vernon influence functional. Assuming (4.4) and that the environment is initially in thermal equilibrium at a temperature \( T = \beta^{-1} \) the influence functional for (4.1) can be computed exactly. The result is well known [25, 26]:
\[
F[x, x'] = \exp \left\{ -\frac{i}{\hbar} \int_0^t ds_1 \int_0^{s_1} ds_2 \left[ x(s_1) - x'(s_1) \right] \eta(s_1 - s_2) \left[ x(s_2) + x'(s_2) \right] \right. \\
- \left. \frac{1}{\hbar} \int_0^t ds_1 \int_0^{s_1} ds_2 \left[ x(s_1) - x'(s_1) \right] \nu(s_1 - s_2) \left[ x(s_2) - x'(s_2) \right] \right\}
\] (6)
The non-local kernels \( \eta \) and \( \nu \) are the dissipation and noise kernels defined respectively as
\[
\nu(s) = \int_0^{+\infty} d\omega \ I(\omega) \coth \left( \frac{\hbar \omega}{2kT} \right) \cos \omega s
\] (7)
and
\[
\eta(s) = \frac{d}{ds} \gamma(s)
\] (8)
where
\[
\gamma(s) = \int_0^{+\infty} d\omega \ \frac{I(\omega)}{\omega} \cos \omega s.
\] (9)
Here \( I(\omega) \) is the spectral density function of the environment,
\[
I(\omega) = \sum_n \delta(\omega - \omega_n) \frac{C_n^2}{2m_n \omega_n}.
\] (10)
An environment is classified as ohmic \( I(\omega) \sim \omega \), supra-ohmic \( I(\omega) \sim \omega^n, n > 1 \) or sub-ohmic \( n < 1 \). The most studied ohmic case corresponds to an environment which induces a dissipative force linear in the velocity of the system. An example which we have studied has spectral density given by

\[
I(\omega) = M\gamma_0\omega^2 e^{-\omega^2/\Lambda^2} \tag{11}
\]

where \( \tilde{\omega} \) is a frequency scale usually taken to be the cut-off frequency \( \Lambda \).

The most general environment gives rise to nonlocal dissipation and colored noises. (We refer the reader to Ref [30] for a discussion of the generalized fluctuation-dissipation relation and the time scales for the relevant processes.) The propagator for a general environment has been calculated before [29]:

\[
J(x_f, x', t | x_i, x_i', 0) = Z_0(t) \exp \left\{ \frac{i}{\hbar} \left[ [\dot{u}_1(0)X_i + \dot{u}_2(0)X_f]y_i - [\dot{u}_1(t)X_i + \dot{u}_2(t)X_f]y_f \right] + \frac{1}{\hbar} \left\{ a_{11}(t)y_i^2 + [a_{12}(t) + a_{21}(t)]y_i y_f + a_{22}(t)y_f^2 \right\} \right\}
\]

where \( X = (x+x')/2 \) and \( y = x' - x \), \( u_a(s) \) are elementary functions obtained as solutions of differential equations involving the dissipation kernel, and the coefficients \( a_{ij}(t) \) are obtained from integrals involving the noise kernel.

5 Uncertainty Principle at Finite Temperature

We now consider a Brownian oscillator with an initial wave function

\[
\psi(x_i, 0) = N_0 e^{-x_i^2/\sigma^2} \tag{1}
\]

where \( \sigma \) is the initial spread of the Gaussian packet. One can calculate \( \rho_r(x_f, x'_f, t) \) by performing the Gaussian integrals over \( x_i \) and \( x_i' \) and get

\[
\rho(x_f, x'_f, t) = \left[ Z_0(t)N_0^2 \frac{\pi}{\sqrt{\text{det} H}} \right] \exp \left\{ -\frac{1}{2} X^T Q^{-1} X \right\} \tag{2}
\]

The prefactor (terms within the square bracket) \( \tilde{N}_0(t) \) depends only on time. Here \( X = (X, y)^T \) and \( Q_{ij}(t) \) is a \( 2 \times 2 \) matrix whose elements are given in [3].
To calculate the averages of observables, it is convenient to use the Wigner function defined as

\[ W(X, p, t) = \int dy e^{i y p/\hbar} \rho(X - y/2, X + y/2, t), \tag{3} \]

The quantum average of an observable, e.g., \( x^n \), with respect to a pure state is given by

\[ \langle x^n \rangle_0 = \int dx x^n \rho(x, x, t) = \int dx \int \frac{dp}{2\pi\hbar} x^n W(X, p, t) \tag{4} \]

and

\[ \langle p^n \rangle_0 = \int dx \int \frac{dp}{2\pi\hbar} p^n W(X, p, t) \tag{5} \]

Similar relations exist between \( \rho_r \) and \( W_r \). The averages with respect to a mixed state now weighted by \( \rho_r \) or \( W_r \) have both quantum and thermal contributions. We get

\[ \langle x^2 \rangle_T = \frac{1}{Q_{11}(t)} \tag{6} \]

and

\[ \langle p^2 \rangle_T = \hbar^2 \det Q(t) \frac{Q_{11}(t)}{Q_{11}(t)} \tag{7} \]

From them, with \( (\Delta x)^2_T = \langle x^2 \rangle_T - \langle x \rangle_T^2 \) and \( (\Delta p)^2_T = \langle p^2 \rangle_T - \langle p \rangle_T^2 \),

\[ U_T(t) = (\Delta x)^2_T(\Delta p)^2_T = \hbar^2 \det Q(t) \frac{Q_{11}(t)}{(Q_{11}(t))^2}, = \hbar^2 \frac{Q_{22}(t) Q_{11}(t) - Q_{12}(t) Q_{21}(t)}{Q_{11}(t) Q_{11}(t)} \tag{8} \]

The exact result was obtained in [2] by solving the equations for the \( u_i(s) \) functions and the \( a_{ij}(t) \) coefficients numerically.

For an ohmic environment \( \gamma(t) = 2\gamma_0 \delta(t) \), and \( a_{ij}(t), u_i(t) \) are simple harmonic and exponential functions. Assume a quantum minimum-uncertainty initial state where \( \frac{\hbar}{2\pi\gamma_0} = 1 \) (which is also the ground state of a harmonic oscillator), we get for weak couplings (small \( \gamma_0 \)) at all temperatures,

\[ U_T(t) = \frac{\hbar^2}{4} \left[ e^{-\gamma_0 t} + \coth(\frac{\hbar \Omega}{2kT})(1 - e^{-\gamma_0 t}) \right]^2. \tag{9} \]

This is a simple, clean and intuitively clear result. We see that there are two factors at play here: time and temperature. Time is measured in units of the
relaxation time proportional to $t_{rel} = \gamma_0^{-1}$, and temperature is measured with reference to the ground state energy $\hbar\Omega/2$ of the system. At $t = 0$, when the initial uncorrelated conditions (4.4) is assumed valid, $U_T(0) = \hbar^2/4$, which is the Heisenberg relation (3.1). At $t \approx \gamma_0^{-1}$ the system begins to equilibrate with the bath. At very long time ($t >> \gamma_0^{-1}$), $U_T(t)$ approaches $U_T^{BE}$ [(3.3)] at finite temperature, or $U_T^{BM}$ [(3.2)] at high temperature. That means the system (the Brownian particle) approaches an equilibrium quantum statistical system.

Now call $z = \frac{\hbar\Omega}{2kT}$. At zero temperature, coth $z = 1$ and $U_T(t) = U_0^{QM}$ as in (3.1) at all times, as expected. At high temperature (coth $z \approx 1/z$) and at short times ($t << \gamma_0^{-1}$) this simplifies to

$$U_T(t) = \frac{\hbar^2}{4 \left[ 1 + \left( \frac{2kT}{\hbar\Omega} - 1 \right) \gamma_0 t + O(t^2) \right]}$$

For a finite temperature ohmic bath (with weak coupling), there always exist a time ($t > \gamma_0^{-1}$ and $e^{-\gamma_0 t} << \coth z$) such that

$$U_T(t) = \left( \frac{kT}{\Omega_0} \right)^2$$

which is the classical limit $U_T^{MB}$ [(3.2)].

These simple expressions are revealing in several aspects: Note that in the expression for short time behavior (5.10) the first term is the ubiquitous quantum fluctuation contribution, the second term is the thermal contribution, which depends on the initial spread and increases with increasing dissipation and temperature.

The time when thermal fluctuations overtake quantum fluctuations is when the second term in the square bracket becomes larger than unity which occurs at (the temperature is higher than the ground state energy by assumption)

$$t_1 = \frac{\hbar\Omega_0}{2\gamma_0 kT}$$

This is indeed the decoherence time scale $t_{dec}$ [2], the time when the off-diagonal components of the reduced density matrix diminishes to zero. The second time scale is the relaxation time scale, $t_{rel} = \gamma_0^{-1}$, when the particle reaches equilibrium with the environment. After this, for ohmic and subohmic environments the uncertainty relation takes on the Bose-Einstein form (3.3). At high temperatures the system reaches the Maxwell-Boltzmann
limit and the uncertainty relation takes on the classical form (3.2). For weak coupling and supraohmic environments at low temperature, the highly non-local frequency response makes it difficult for the system to settle down. The decoherence time scale is longer, and the relaxation can even be incomplete. This is the regime where one expects to find more intricate behavior in the interplay of quantum and thermal effects.

6 Chaos in Quantum Open Systems

In this talk we have only partially addressed half of the theme of this conference, i.e., the relation of quantum and classical. The other part on non-integrability is not touched upon, because we don’t know enough about this important and fascinating subject. In light of what we have discussed above, i.e., viewing classical behavior as the result of decoherence on a quantum system, it is nonetheless interesting to raise a few general questions, naive as they may be to the experts:

A direct question to ask is nonintegrability in both quantum and classical open systems: i.e., How would the introduction of an environment alter the intrinsically chaotic or nonchaotic character in the dynamics of a system? Is chaotic behavior likely to be enhanced or quenched? Some aspects of this problem have been studies as chaos in systems with noise [35]. Usually if the system’s nonlinear dynamics shows chaotic behavior, the presence of an environment which adds a stochastic force is expected to modify the chaotic behavior only quantitatively. If the coupling of the system with the environment is strong and nonlinear, qualitative changes in the system’s dynamics are likely. Although structures and forms are usually discussed in the context of classical dissipative systems, it would be very gratifying if one can understand the rules from which such macroscopic structures [36] in the classical world emerge from the microscopic dynamics of a quantum system [37, 38]. Nonlinear dissipative dynamics in a quantum open system may play an important role in this issue.

An indirect, and perhaps conceptually more subtle problem is the following: If we view classical behavior as a macroscopic phenomenon emergent from a quantum open system, will it provide us with some insight to resolve the apparent puzzle that there are chaotic behavior in the classical equation of motion but usually not in the associated quantum equation of motion? Is
it easier to account for nonintegrability in an effective theory than a fundamental theory? In the viewpoint we presented above, while a quantum closed system is completely describable by a fundamental theory; vis., quantum mechanics, the corresponding classical theory is only an effective theory in the sense that it has incorporated the backreaction effect of the environment, which is subjected to coarse-graining approximations. (In such a process usually a simple dynamical equation is transformed into an integro-differential equation of motion). Thus classical dynamics is a limiting form of an effective quantum theory. One can study how under these statistical procedures (coarse-graining, decoherence and backreaction) an integrable quantum system becomes a classical chaotic system. Conversely, a more difficult problem is to take a classical model which exhibits chaotic behavior and determine if it can be obtained through certain reduction schemes as the macroscopic limit of an associated set of integrable quantum models. This is, of course, a many-to-one relation, depending on the choice of schemes. The nonintegrability of the classical system could then be seen as an outcome both of the original quantum dynamics and the quantum to classical reduction scheme.

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