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On a variation of the Erdős–Selfridge superelliptic curve

Sam Edis

Abstract

In a recent paper by Das, Laishram and Saradha, it was shown that if there exists a rational solution of $y^l = (x + 1) \cdots (x + i - 1)(x + i + 1) \cdots (x + k)$ for $i$ not too close to $k/2$ and $y \neq 0$, then $\log l < 3^k$. In this paper, we extend the number of terms that can be missing in the equation and remove the condition on $i$.

1. Introduction

The Erdős–Selfridge superelliptic curves are the following family of curves,

$$y^l = (x + 1) \cdots (x + k).$$

In [4], it is shown to not have any solutions in positive integers $x, y, k, l$ with $k, l \geq 2$. It has been conjectured by Sander [6] that for $l \geq 4$ there are no rational solutions to equation (1) with $y \neq 0$. In [1], for $k \geq 2$ a positive integer, there are at most finitely many solutions to (1) with $x$ and $y$ rational numbers, $l \geq 2$ an integer with $(k, l) \neq (2, 2)$ and $y \neq 0$. Further, it is shown that if $l$ is a prime, then all solutions satisfy $\log l < 3^k$.

In [2], by Das, Laishram and Saradha, they consider the following variation of the Erdős–Selfridge superelliptic curves,

$$y^l = (x + 1) \cdots (x + i - 1)(x + i + 1) \cdots (x + k),$$

for $k \geq 2$ an integer, $l$ a prime, $x$ and $y$ rational numbers and $i$ an integer strictly between 1 and $k$. Letting $q$ be the smallest prime greater than or equal to $k/2$, they show that if (2) holds and $2 \leq i \leq k - q$ or $q < i < k$ then $\log l < 3^k$. Further, they show that if (2) holds and $3 \leq k \leq 26$, then $\log l < 3^k$.

In this paper, we will further the results in [2] by removing the condition on $i$ and also extending the terms that can be missing from the equation. For $k \geq 2$ an integer, $l$ a prime, $i$ and $j$ integers $1 < i < j < k$ and $\epsilon_t \in \{0, 1\}$ for $i < t < j$, we call the following equation the Erdős–Selfridge curve with an incomplete block,

$$y^l = \prod_{t=1}^i (x + t) \prod_{t=i+1}^{j-1} (x + t)^{\epsilon_t} \prod_{t=j}^k (x + t).$$

We call a solution to (3) with $x$ and $y$ rational numbers and $y \neq 0$ a non-trivial rational solution. We note that the case $j - i = 2$ and $\epsilon_{i+1} = 0$ is the same as (2).

**Theorem 1.** If $(x, y)$ is a non-trivial rational solution to equation (3) for $k \geq 27$ and $j - i - 1 < k/18 - 1$, then $\log l < 3^k$. In particular, if $j - i = 2$, then $\log l < 3^k$ holds for $k \geq 3$. 
This will be proven by adjusting the proofs in [1, 2], by adding in new identities allowing us to consider prime numbers less than \( k/2 \) and using a more combinatorial approach.

We will also consider a variation of the Erdős–Selfridge superelliptic curve from which terms in the product have been removed without any specification of their location in the interval \([1, k]\).

**Theorem 2.** Letting \( 1 < t_1 < \ldots < t_L < k \) and \( S = \{1, \ldots, k\} \setminus \{t_1, \ldots, t_L\} \). If \( (x, y) \) is a non-trivial rational solution to

\[
y^l = \prod_{j \in S} (x + j),
\]

for \( k \geq 2 \) and \( L < 0.26 \sqrt{\frac{k}{\log k}} \), then \( \log l < 3^k \).

2. **Preliminaries**

We will assume throughout that \( l \) is prime and \( l > k - 1 \). We will first prove the existence of primes in the interval \([k/3, k/2]\). Following that we will look at the prime decomposition of the factors of equation (3).

**Lemma 3.** For all \( k \geq 22 \), there exists a prime \( p \) such that \( \frac{1}{3} k < p < \frac{9}{2} k \).

**Proof.** In [5], it is shown that there is always a prime between \( z \) and \((1 + \frac{1}{3})z\), for \( z \geq 25 \). Hence, for \( k \geq 75 \), the result now follows, and for the other \( k \), it follows from an explicit computation. \( \square \)

Following the work of Bennett and Siksek [1] and of Das, Laishram and Saradha [2], we write the coordinates \((x, y)\) as fractions in lowest common form, \( x = n/s \) and \( y = m/s' \) for \( m \neq 0 \), \( s \) and \( s' \) positive integers. From equation (3), we have

\[
m^l \equiv \frac{\prod_{i=1}^{i} (n + ts)}{s'^n} \prod_{i=1}^{j-1} (n + ts)^{\epsilon_i} \prod_{i=j}^{k} (n + ts) \quad \text{for} \quad s_0 = \sum \epsilon_i.
\]

As \( \gcd(n, s) = \gcd(m, s') = 1 \) and \( l \) is a prime greater than \( k - \sum \epsilon_i \), it follows there is a positive integer \( d \) such that \( s = d^k \) and \( s' = d^{k-\sum \epsilon_i} \).

Hence, equation (3) can be written as

\[
m^l = \prod_{i=1}^{i} (n + td^k) \prod_{i=1}^{j-1} (n + td^k)^{\epsilon_i} \prod_{i=j}^{k} (n + td^k),
\]

for \( m, n \) and \( d \) integers.

We now write each term in this product as

\[n + t_1 d^k = a_{t_1} x_{t_1}^k,\]

such that \( x_{t_1} \) is an integer and \( a_{t_1} \) is an \( 1 \)th power free integer. Let \( p \) be a prime that divides \( a_{t_1} \), then \( p \) must also divide \( a_{t_2} \) for some \( t_2 \), hence \( p \) divides \( (t_1 - t_2)d^k \). If \( p \) divides \( d \), then it must also divide \( n \), contradicting them being co-prime, hence \( p \) divides \( t_1 - t_2 \). It now follows that all prime factors of \( a_{t_1} \) are bounded above by \( k \).

We note here that the exact same reasoning applies to equation (4) giving the following equation,

\[
m^l = \prod_{i=1}^{k} (n + td^k)^{\epsilon_i}
\]

for \( \epsilon_i = 1 \) if \( t \in S \) and zero otherwise.
Lemma 4. For \( m, n \) and \( d \) solutions of equation (4) with \( L < 0.26 \sqrt{\frac{k}{\log k}} \) and \( k \geq 22 \), there exists a prime \( \frac{1}{2} k \leq p \leq \frac{1}{2} k \) that either divides \( d \) or divides \( m \).

Proof. We can assume that no prime \( p \) in the range \( \left[ \frac{k}{3}, \frac{k}{2} \right] \) divides \( d \), otherwise the result follows trivially. Such a prime must divide at least two and at most three of the terms \( n + td^t \) for \( t \in \{1, k\} \). If \( p \) does not divide \( m \), then there are at least 2 values of \( t \) such that \( \epsilon_t = 0 \). We will label these as \( i_p \) and \( i_p + p \). It is then clear that \( p \) is in the set of differences of the elements in \( \{t_1, \ldots, t_L\} \). It is easily seen that

\[
|\{ t_{i'} - t_{j'} : 1 \leq i' < j' \leq L \} | \leq L^2 - L + 1. \tag{8}
\]

It is then easily seen that if

\[
L^2 - L + 1 < \pi(k/2) - \pi(k/3), \tag{9}
\]

then there must be such a prime \( p \). For \( k < 181000 \), we can explicitly calculate using Magma, the following bound

\[
0.07 \frac{k}{\log(k)} < \pi(k/2) - \pi(k/3). \tag{10}
\]

For \( k \geq 181000 \), we use the following bounds in [3]

\[
\frac{x}{\log(x) - 1} < \pi(x) \quad \text{for} \quad x \geq 5393, \tag{11}
\]

and

\[
\pi(x) < \frac{x}{\log(x) - 1.1} \quad \text{for} \quad x \geq 60184. \tag{12}
\]

It is then simple algebraic manipulation to see that for \( k \geq 181000 \)

\[
0.17 \frac{k}{\log(k)} < \pi(k/2) - \pi(k/3). \tag{13}
\]

It is now seen that with \( L < 0.26 \sqrt{\frac{k}{\log k}} \), inequality (9) is true, completing the Lemma. \( \square \)

3. Fermat equation

In this section, we will attach a solution to a Fermat equation from a solution of (3) and (4). We will then use what is known about such equations to bound the exponent \( l \).

Lemma 5. For \( k \geq 27 \), assume that equation (3) has a non-trivial rational point \((x, y)\) for \( j - i - 1 = L < k/18 - 1 \) or \( L = 1 \), or equation (4) has a solution for \( L < 0.26 \sqrt{\frac{k}{\log k}} \). Then, there exists a prime \( \frac{1}{3} k \leq p \leq \frac{1}{2} k \) such that there are non-zero integers \( a, b, c, u, v, w \) satisfying

\[
au^l + bv^l + cw^l = 0 \tag{14}
\]

such that

1. \( a, b, c \) are \( l \)-th power free integers;
2. all prime factors of \( abc \) are less than or equal to \( k \);
3. \( p \nmid abc \);
4. \( p \) divides precisely one of \( u, v, w \).

Proof. We first deal with the case of equation (3). Let \( p \) be a prime as described and assume that \( p \nmid d \), then \( p \) must divide \( m \). This follows simply from the following, let \( j \) be a value in
[1,k] such that \( n + jd^\ell \equiv 0 \mod p \). Then, if \( p \nmid m \), it follows that \( j - p \leq 0 \) and \( j + p \geq k + 1 \), hence \( p \geq (k + 1)/2 \) contradicting our assumption on \( p \). It follows that \( p \) either divides \( d \) or divides exactly 1, 2 or 3 factors in the Erdős–Selfridge curve.

We first deal with \( p \mid d \), then it follows that \( p \nmid m \), so \( p \nmid a_i x_i^\ell \). Using (6) we see that

\[ d^\ell = a_i x_i^\ell - a_{i+1} x_{i+1}^\ell, \]

choosing a \( t \) such that \( \epsilon_t \) and \( \epsilon_{t+1} \) are non-zero that gives the desired result.

We now deal with the case that \( p \) divides exactly one factor, which we take to be \( n + td^\ell \).

We consider the identity,

\[ (n + td^\ell) - (n + t'd^\ell) = (t - t')d^\ell, \]

for \( t' \) a positive integer less than \( k + 1 \) such that \( |t' - t| < p \). Because \( L < p - 1 \), it follows that there exists such a \( t' \) such that \( (n + t'd^\ell) \) appears on the right-hand side of (5). As \( p \) must divide \( n + td^\ell \) to an \( l \)th power, applying (6), we then get an equation satisfying the Lemma, that is,

\[ a_i x_i^\ell - a_{i'} x_{i'}^\ell - (t' - t)d^\ell = 0. \]

We now consider the case that \( p \) divides exactly two factors, \( n + td^\ell \) and \( n + (t + p)d^\ell \). We consider a similar identity as before,

\[ (n + td^\ell)(n + (t + p)d^\ell) - (n + (t + \alpha)d^\ell)(n + (t + p - \alpha)d^\ell) = \alpha(\alpha - p)d^{2l}, \]

for \( \alpha \) a positive integer less than \( p \).

It is clear that for distinct \( \alpha \) and \( \alpha' \leq p/2 \), \( \{ t + \alpha, t + p - \alpha \} \cap \{ t + \alpha', t + p - \alpha' \} = \emptyset \).

Hence, as \( L < p/2 - 1 \), there exists \( \alpha \) such that both \( n + (t + \alpha)d^\ell \) and \( n + (t + p - \alpha)d^\ell \) appear as factors in (5). Hence, the result now follows from (6) and the same finishing argument as above.

We are left to deal with the case that \( p \) divides exactly three factors, \( n + td^\ell \), \( n + (t + p)d^\ell \) and \( n + (t + 2p)d^\ell \).

We point out the following identity,

\[
(n + td^\ell)(n + (t + p)d^\ell)(n + (t + 2p)d^\ell) - (n + (t + \alpha)d^\ell)(n + (t + p + \alpha)d^\ell)(n + (t + 2p - 2\alpha)d^\ell) = 3\alpha(\alpha - p) \left( n + \left( t + \frac{2(p + \alpha)}{3} \right) d^\ell \right) d^{2l},
\]

defined for \( \alpha \) a positive integer less than \( p \) with \( \alpha \equiv -p \mod 3 \). For \( \alpha \) and \( \alpha' \) positive integers either less than \( p/2 \), then

\[
\left\{ t + \alpha, t + \frac{2(p + \alpha)}{3}, t + p + \alpha, t + 2p - 2\alpha \right\} \cap \left\{ t + \alpha', t + \frac{2(p + \alpha')}{3}, t + p + \alpha', t + 2p - 2\alpha' \right\} = \emptyset.
\]

This follows from some simple inequalities and calculations mod 3. Hence, it follows that there are more than \( \frac{k}{6} - 1 \) distinct values of \( \alpha \) with \( \alpha \equiv -p \mod 3 \), such that the terms in (15) involving \( \alpha \) do not coincide. So, we see that we have more choices of \( \alpha \) than terms deleted, hence at least one \( \alpha \) will give us such an equation with all terms defined. We note that as \( k \geq 26 \), there will always be a prime greater than or equal to 13 in the permitted interval, meaning we can always take \( L = 1 \) for these values of \( k \).

In the case of equation (4), we first apply Lemma 4, then follow the above argument identically. \( \square \)
It is worth noting that in the third case there is also the following identity,
\[(n + t d^l)(n + (t + p)d^l)(n + (t + 2p)d^l) - (n + (t + 2\alpha)d^l)(n + (t + p - \alpha)d^l)(n + (t + 2p - \alpha)d^l)\]
\[= 3\alpha(\alpha - p) \left(n + \left(t + \frac{4p - 2\alpha}{3}\right)d^l\right) d^{2l},\]  
(16)
defined for \(\alpha\) a positive integer less than \(p\) with \(\alpha \equiv -p \pmod{3}\). In specific cases of a fixed \(L\), the use of (15) and (16) together can give specific values of \(\alpha\) removing the need for combinatorial arguments.

We now state a Lemma which follows from [1].

**Lemma 6.** If \(a, b, c, u, v, w\) are non-zero integers satisfying
\[au^l + bv^l + cw^l = 0,\]  
(17)
\(k\) is a fixed integer and \(\frac{1}{3}k \leq p \leq \frac{1}{2}k\) is a prime such that

1. \(a, b, c\) are \(l\)th power free integers;
2. all prime factors of \(abc\) are less than or equal to \(k\);
3. \(p \nmid abc\);
4. \(p\) divides precisely one of \(u, v, w\);
5. \(l > k\) is prime.

Then, \(\log l \leq \frac{(N' + 1)}{6} \log(\sqrt{p} + 1)\), where \(N' = 2^l \text{Rad}_2(abc)\) and \(\text{Rad}_2(n)\) denotes the product of all primes dividing \(n\), apart from 2.

**Proof.** This follows immediately from [1, p.4].

**Remark 1.** It is then a routine calculation, as in [1], using
\[\sum_{q \leq k \text{ prime}} \log q < 1.000081k,\]
from [7] and \(k \geq 26\) to conclude that
\[\log l < 3^k.\]

**Proof of Theorem 1.** For \(k \geq 27\), this follows immediately by applying Lemma 5, Lemma 6 and the remark above. We now finish with the case of \(L = 1\) and \(k \leq 26\). If \(\epsilon_i + 1 = 1\), then this follows from [1]. If however \(\epsilon_{i+1} = 0\) and \(k \leq 26\), then this is covered by [2].

**Proof of Theorem 2.** For \(k \geq 27\), this follows identically to above, if \(k < 27\), then it is clear that \(L = 0\) and so follows from [1].

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