Weak Bases of Boolean Co-Clones

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Abstract

Universal algebra and clone theory have proven to be a useful tool in the study of constraint satisfaction problems since the complexity, up to logspace reductions, is determined by the set of polymorphisms of the constraint language. For classifications where primitive positive definitions are unsuitable, such as size-preserving reductions, weaker closure operations may be necessary. In this article we consider strong partial clones which can be seen as a more fine-grained framework than Post’s lattice where each clone splits into an interval of strong partial clones. We investigate these intervals and give simple relational descriptions, weak bases, of the largest elements. The weak bases have a highly regular form and are in many cases easily relatable to the smallest members in the intervals, which suggests that the lattice of strong partial clones is considerably simpler than the full lattice of partial clones.

Keywords: Computational complexity, Clone theory, Boolean relations, Constraint satisfaction problems

1. Introduction

A set of functions is called a clone if it (1) is closed under composition of functions and (2) contain all projection functions of the form $p^n_i(x_1,\ldots,x_n) = x_i$. Dually, a set of relations $\Gamma$ is called a relational clone, or a co-clone, if it contains every relation $R$ definable through a primitive positive (p.p.) implementation of the form $R(x_1,\ldots,x_n) \equiv \exists y_1,\ldots,y_m . R_1(x_1) \land \ldots \land R_k(x_k)$, where each $R_i \in \Gamma \cup \{\vDash\}$ and each $x_i$ is a vector over $x_1,\ldots,x_n$, $y_1,\ldots,y_m$. In the case where $\Gamma$ is finite we say that it is a constraint language. For a set of functions $F$ and a set of relations $\Gamma$ we use $[F]$ to denote the smallest clone containing $F$ and $\langle \Gamma \rangle$ for the smallest co-clone containing $\Gamma$. If $\Gamma$ is a set of relations and $\text{IC}$ a co-clone such that $\langle \Gamma \rangle = \text{IC}$ then we say that $\Gamma$ is a base of $\text{IC}$. Ordering clones by set-inclusion yields a lattice structure which in the Boolean case is completely explicated and known as Post’s lattice due to Post’s seminal classification [11]. Essentially the lattice determines the expressive properties of all possible Boolean functions. Due to the Galois connection between clones and co-clones the lattice of Boolean co-clones is anti-isomorphic to Post’s lattice and therefore works as a complete classification of all Boolean languages. Simple bases for all Boolean co-clones minimal with respect to ar-
Figure 1: The lattice of Boolean co-clones. The co-clones which are covered by a single weak partial co-clone are coloured in grey.

ity of relations have been identified by Böhler et al. The lattice of Boolean co-clones is visualized in Figure 1. The complexity of various computational problems parameterized by constraint languages such as the constraint satisfaction problem (CSP) have been shown to be determined up to logspace reducibility by Post’s lattice. If one on the other hand is interested in complexity classifications based on reductions which preserves the exact complexity of problems, Post’s lattice falls short since even logspace reductions may introduce new variables which affects the running-time.

To remedy this a more fine-grained framework which further separates constraint languages based on their expressive properties is necessary. In Jonsson et al. the lattice of strong partial clones is demonstrated to have the required properties. Hence a classification of the lattice of strong partial clones similar to that of Post’s lattice would provide a powerful framework for studying exact complexity of CSP and related problems. We wish to emphasize that even though the lattice of partial clones is known to be uncountable the same does not necessarily hold for the lattice of strong partial clones. Ideally, for each clone \( C \), one would like to determine the interval of strong partial clones whose subset of total functions equal \( C \). The strong partial clones in this interval are said to cover \( C \). In Creignou et al. relational descriptions known as \emph{plain bases} of the smallest member of this interval is given. In this article we give simple relational descriptions known as \emph{weak bases} of the largest elements in these intervals. Our work builds on the result of Schnoor and Schnoor but differs in two important aspects: first, each weak base presented can in a natural sense be considered to be minimal; second, we present alternative proofs where Schnoor’s and Schnoor’s procedure results in relations which are exponentially larger than the bases given by Böhler et al. and Creignou et al., and are thus also able to cover the infinite chains in Post’s lattice. Due to the Galois connection between clones and co-clones the weak bases also constitutes the relations which in a precise sense results in the CSP problems with the lowest complexity. Hence the weak bases presented in Section 3 are closely connected to upper bounds of running times for all problems parameterized by constraint languages.
2. Preliminaries

In this section we introduce some basic notions from universal algebra and clone theory necessary for the construction of weak bases. If \( f \) is an \( n \)-ary function and \( R \) a relation with \( m \) tuples it is possible to extend \( f \) to operate over tuples from \( R \) as follows:

\[
f(t_1, \ldots, t_n) = (f(t_1[1], \ldots, t_n[1]), \ldots, f(t_1[m], \ldots, t_n[m])),
\]

where \( t_i[j] \) denotes the \( j \)-th argument of the tuple \( t_i \in R \). If \( R \) is closed under \( f \) we say that \( f \) preserves \( R \) or that \( f \) is a polymorphism of \( R \). For a set of functions \( F \) we define \( \text{Inv}(F) \) (often abbreviated as \( IF \)) to be the set of all relations preserved by all functions in \( F \). Dually we define \( \text{Pol}(\Gamma) \) for a set of relations \( \Gamma \) to be the set of polymorphisms to \( \Gamma \). It is easy to verify that \( \text{Pol}(\Gamma) \) always form clones and that \( \text{Inv}(F) \) always form co-clones. Moreover we have the Galois connection between clones and co-clones normally presented as:

**Theorem 1.** (3, 4, 7) Let \( \Gamma \) and \( \Delta \) be two sets of relations. Then \( \langle \Gamma \rangle \subseteq \langle \Delta \rangle \) if and only if \( \text{Pol}(\Delta) \subseteq \text{Pol}(\Gamma) \).

To extend these notions to the case of partial clones we need some additional notation. If \( R \) is an \( n \)-ary Boolean relation and \( \Gamma \) a constraint language we say that \( R \) has a quantifier-free primitive positive (q.p.p.) implementation in \( \Gamma \) if \( R(x_1, \ldots, x_n) \equiv R_1(x_1) \land \ldots \land R_k(x_k) \), where each \( R_i \in \Gamma \cup \{=\} \) and each \( x_i \) is a vector over \( x_1, \ldots, x_n \). We use \( \langle \Gamma \rangle_\exists \) to denote the smallest set of relations closed under q.p.p. definability. If \( IC = \langle IC \rangle_\exists \) then we say that \( IC \) is a weak partial co-clone. We use the term weak partial co-clone to avoid confusion with partial co-clones used in other contexts (see Chapter 20.3 in Lau [10]). To get a corresponding concept on the functional side we extend the previous definition of a polymorphism and say that a partial function \( f \) is a partial polymorphism to a relation \( R \) if \( R \) is closed under \( f \) for every sequence of tuples for which \( f \) is defined. A set of partial functions \( C \) is said to be a partial clone if it contain all projection functions and is closed under composition of functions. If \( C \) is a partial clone we say that it is strong if for every \( f \in C \), \( C \) also contain all partial subfunctions \( g \) of \( f \) which agrees with \( f \) for all values that they are defined. By \( \text{pPol}(\Gamma) \) we denote the set of partial polymorphisms to the set of relations \( \Gamma \). Obviously sets of the form \( \text{pPol}(\Gamma) \) always form strong partial clones and again we have a Galois connection between clones and co-clones.

**Theorem 2.** (3, 4, 12) Let \( \Gamma \) and \( \Delta \) be two sets of relations. Then \( \langle \Gamma \rangle_\exists \subseteq \langle \Delta \rangle_\exists \) if and only if \( \text{pPol}(\Delta) \subseteq \text{pPol}(\Gamma) \).

For a co-clone \( IC \) we define \( I(IC) = \{ ID \mid ID = \langle ID \rangle_\exists \text{ and } (ID) = IC \} \). In other words \( I(IC) \) is the interval of all weak partial co-clones occurring inside of \( IC \). Let \( I_\cup(IC) = \bigcap_{ID \in I(IC)} ID \). To be consistent with Schnoor’s and Schnoor’s notation which is defined in terms of clones instead of co-clones we also define \( I_\cup(C) = \bigcup_{ID \in I(IC)} \text{pPol}(ID) \). Obviously \( I_\cup(C) \) is the union of all strong partial clones covering \( C \), from which it follows that \( \text{pPol}(I_\cup(\langle IC \rangle)) = I_\cup(\langle C \rangle) \).

**Definition 3.** Let \( C \) be a clone. A constraint lan-
Due to the Galois connection between strong partial clones and weak partial co-clones a weak base for a co-clone $I_C$ therefore results in smallest element in $I(I_C)$. The following theorem is immediate from the definition and the fact that $pPol(I \cap (I_C)) = I \cup (C)$.

**Theorem 4** ([13]). Let $C$ be a clone and $\Gamma$ be a weak base of $I_C$. Then, for any base $\Gamma'$ of $C$, it holds that $\Gamma \subseteq \langle \Gamma' \rangle_{\not=}$. If $R$ is an $n$-ary relation with $m = |R|$ elements we let the matrix representation of $R$ be the $m \times n$-matrix containing the tuples of $R$ as rows stored in lexicographical order. Note that the ordering is only relevant to ensure that the representation is unambiguous. Given a natural number $n$ the $2^n$-ary relation $COLS^n$ is the relation which contains all natural numbers from $0$ to $2^n - 1$ as columns in the matrix representation. For any clone $C$ and relation $R$ we define $C(R)$ to be the relation $\bigcap_{R' \in I_C} R \subseteq R'$, i.e. the smallest extension of $R$ which is preserved under every function in $C$. For a relation $R$ we say that the co-clone $\langle R \rangle$ has core-size $s$ if there is a relation $R'$ such that $|R'| = s$ and $R = (Pol(R))(R')$. Minimal core-sizes for all Boolean co-clones have been identified by Schnoor [14]. We are now ready to state Schnoor’s and Schnoor’s [13] main result which effectively gives a weak base for any co-clone with a finite core-size.

**Theorem 5** ([13]). Let $C$ be a clone and $s$ be a core-size of $I_C$. Then the relation $C(COLS^n)$ is a weak base of $I_C$.

The disadvantage of the theorem is that relations of the form $C(COLS^n)$ have exponential arity with respect to the core-size. We therefore introduce another measurement of minimality which ensures that a given relation is indeed minimal with respect to cardinality. A relation $R$ is said to be irredundant if there are no duplicate rows in the matrix representation.

**Definition 6.** A relation $R$ is minimal if it is irredundant and there is no $R' \subset R$ such that $\langle R \rangle = \langle R' \rangle$.

Minimal weak bases have the property that they can be implemented without the use of the equality operator. If we let $\langle \cdot \rangle_{\not=} \exists$ denote the closure of q.p.p. definitions without equality we therefore get the following theorem.

**Theorem 7** ([13]). Let $C$ be a clone and $\Gamma$ be a minimal weak base of $I_C$. Then, for any base $\Gamma'$ of $C$, it holds that $\Gamma \subseteq \langle \Gamma' \rangle_{\not=}$. Hence minimal weak bases give the largest possible expressibility results and are applicable for problems where the equality operator is not permissible, e.g. counting CSP, where the number of solutions can be increased by an exponential factor [13].

3. Minimal weak bases of all Boolean co-clones

In this section we proceed by giving minimal weak bases for all Boolean co-clones with finite core-size. The results are presented in Table 1. Each line in the table consists of a co-clone, its minimal core-size and a minimal weak base. As con-
viation we use normal Boolean connectives to represent relations whenever this promotes readability. For example, \( x_1 \cdot x_2 \) denotes the relation \( \{(1,1)\} \) while \( x_1 \neq x_2 \) denotes the relation \( \{(0,1), (1,0)\} \). We use \( F \) for the relation \( \{(0)\} \) and \( T \) for the relation \( \{(1)\} \).

The relations \( OR^n \) and \( NAND^n \) are \( n \)-ary or and \( nand. \) \( EVEN^n \) is the \( n \)-ary relation which holds if the sum of its arguments is even, and conversely for \( ODD^n \). By \( R^{i,j} \) we denote the \( 3 \)-ary relation \( \{(0, 0, 1), (0, 1, 0), (1, 0, 0)\} \). If \( R \) is an \( n \)-ary relation we often use \( R_{m, \neq} \) to denote the \((n+m)\)-ary relation defined as \( R_{m, \neq}(x_1, \ldots, x_{n+m}) \equiv R(x_1, \ldots, x_n) \land (x_1 \neq x_{n+1}) \land \ldots \land (x_n \neq x_{n+m}) \). Variables are named \( x_1, \ldots, x_n \) or \( x \) except when they occur in \( F \) or \( T \) in which case they are named \( c_0 \) and \( c_1 \) respectively to explicate that they are in essence constant values.

For the co-clones \( IR_2, IM, ID, ID_1, IL, IL_0, IL_1, IL_2, IL_3, IV, IV_0, IE, IE_1, IN, IN_2, II, IO_0, II_1 \) and \( BR \), the result follows immediately from Theorem 5 the minimal core-sizes for each co-clone, and a suitable rearrangement of arguments. Through exhaustive search, i.e. by repeatedly removing redundant columns and tuples, one can verify that the bases are also minimal. This has been done by a computer program which is available upon request from the author. For the remaining co-clones the proof is divided into two parts. First, we prove that the weak base for every co-clone \( IC \) in \( IM_0, IM_1, IM_2, ID_2, IV_1, IV_2, IE_0 \) and \( IE_2 \), can be obtained by collapsing columns from \( C(COLS^*) \). Second, we prove that for every \( n \geq 2 \) there exists simple weak bases for the co-clones \( IS^n_0, IS^n_0, IS^n_1, IS^n_0 \) and their duals \( IS^n_1, IS^n_1, IS^n_1, IS^n_0 \).

To make the proofs more concise we introduce some admissible operations on relations which preserves the weak base property. Let \( R \) be an \( n \)-ary relation. Each rule is of the form \( R \xmapsto{(i=j)} R' \) and implies that \( \langle R' \rangle \subseteq \langle R \rangle \).

- \( R \xmapsto{(i=j)} R' \), \( 1 \leq i < j \leq n \),
  (Identify argument \( i \) with argument \( j \)),

- \( R \xmapsto{\pi(i_1, \ldots, i_n)} R' \), where \( \pi \) is the permutation \( \pi(j) = i_j, 1 \leq j \leq n, 1 \leq i_j \leq n \),
  (Swap arguments),

- \( R \xmapsto{irr} R' \),
  \( (R') \) is the irredundant core of \( R \).

**Lemma 8.** Let \( IC \) be a co-clone, \( R \) an \( n \)-ary weak base for \( IC \), and let \( R' \) be a Boolean relation such that \( R \xmapsto{\Rightarrow} R' \) for some rule \( \Rightarrow \). If \( R' \) is a base for \( IC \) then it is also a weak base for \( IC \).

**Proof.** We prove that \( \langle R \rangle \subseteq \langle R' \rangle \) which implies that \( I_{\cup}(\text{Pol}(R)) = I_{\cup}(\text{Pol}(R')) \) and that \( R' \) is a weak base for \( IC \). The first inclusion \( \langle R \rangle \subseteq \langle R' \rangle \) is obvious since \( R \) is a weak base by assumption. To prove that \( \langle R' \rangle \subseteq \langle R \rangle \) we show that \( R' \in \langle R \rangle \) by giving a q.p.p. implementation of \( R' \) with \( R \). There are two cases to consider. Either \( R \xmapsto{(i=j)} R' \), \( 1 \leq i < j \leq n \), in which case \( R' \) is the \((n-1)\)-ary relation defined as \( R'(x_1, \ldots, x_{n-1}) \equiv R(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n-1}) \), or \( \underbrace{R \xmapsto{\pi(i_1, \ldots, i_n)} R'}_{j-1} \equiv R(x_{\pi(1)}, \ldots, x_{\pi(n)}) \).

The case when \( R' \) is the irredundant core of \( R \) follows trivially from this since \( R' \) can be obtained by identifying all variables that are equal.

**Lemma 9.** The bases for \( IM_0, IM_1, IM_2, ID_2, IV_1, IV_2, IE_0 \) and \( IE_2 \) in Table 1 are minimal weak bases.
Table 1: Weak bases for all Boolean co-clones with a finite base

| Co-clone | Core-size | Weak base |
|----------|-----------|-----------|
| IBF      | 1         | Eq(x_1, x_2) |
| IR_3     | 1         | F(c_0)     |
| IR_7     | 1         | T(c_1)     |
| IR_1     | 1         | F(c_0) ∨ T(c_1) |
| IM_0     | 2         | (x_1 ∨ x_2) ∧ F(c_0) |
| IM_1     | 2         | (x_1 ∨ x_2) ∧ T(c_1) |
| IM_2     | 3         | (x_1 ∨ x_2) ∧ F(c_0) ∨ T(c_1) |
| IS_{n, n} ≥ 2 | n | OR^n(x_1, ..., x_n) ∧ T(c_1) |
| IS_{n, n} ≥ 2 | n | OR^n(x_1, ..., x_n) ∧ F(c_0) ∧ T(c_1) |
| IS_{n, n} ≥ 2 | n | OR^n(x_1, ..., x_n) ∧ (x → x_1 ⋅⋅⋅ x_n) ∧ T(c_1) |
| IS_{n, n} ≥ 2 | max(3, n) | OR^n(x_1, ..., x_n) ∧ (x → x_1 ⋅⋅⋅ x_n) ∧ F(c_0) ∧ T(c_1) |
| IS_{n, n} ≥ 2 | n | NAND^n(x_1, ..., x_n) ∧ F(c_0) |
| IS_{n, n} ≥ 2 | n | NAND^n(x_1, ..., x_n) ∧ F(c_0) ∧ T(c_1) |
| IS_{n, n} ≥ 2 | n | NAND^n(x_1, ..., x_n) ∧ (x → x_1 ⋅⋅⋅ x_n) ∧ F(c_0) |
| IS_{n, n} ≥ 2 | max(3, n) | NAND^n(x_1, ..., x_n) ∧ (x → x_1 ⋅⋅⋅ x_n) ∧ F(c_0) ∧ T(c_1) |
| ID       | 1         | (x_1 ≠ x_2) |
| ID_1     | 2         | (x_1 ≠ x_2) ∧ F(c_0) ∧ T(c_1) |
| ID_2     | 3         | OR^2_{x_2}(x_1, x_2, x_3, x_4) ∧ F(c_0) ∧ T(c_1) |
| IL       | 2         | EVEN^4(x_1, x_2, x_3, x_4) |
| IL_1     | 2         | EVEN^3_2(x_1, x_2, x_3) ∧ F(c_0) |
| IL_2     | 2         | ODD^4(x_1, x_2, x_3) ∧ T(c_1) |
| IL_3     | 3         | EVEN^3_2(x_1, ..., x_6) ∧ F(c_0) ∧ T(c_1) |
| IL_{2}   | 3         | EVEN^3_2(x_1, ..., x_6) ∧ T(c_1) |
| IV       | 2         | (x_1 ↔ x_2, x_3) ∧ (x_2 ∨ x_3) → x_1 |
| IV_0     | 2         | (x_1 ↔ x_2, x_3) ∧ F(c_0) |
| IV_1     | 3         | (x_1 ↔ x_2, x_3) ∧ (x_2 ∨ x_3) → T(c_1) |
| IV_2     | 3         | (x_1 ↔ x_2, x_3) ∧ F(c_0) ∧ T(c_1) |
| IE       | 2         | (x_1 ↔ x_2 ∧ x_3 ∧ x_2 ∨ x_3 → x_4) |
| IE_0     | 3         | (x_1 ↔ x_2 ∧ x_3 ∧ x_2 ∨ x_3 → x_4) ∧ F(c_0) |
| IE_1     | 2         | (x_1 ↔ x_2 ∧ T(c_1)) |
| IE_2     | 3         | (x_1 ↔ x_2 ∧ T(c_1)) |
| IN       | 2         | EVEN^4(x_1, x_2, x_3, x_4) ∧ x_1 ∧ x_2 → x_2 ∧ x_3 |
| IN_1     | 3         | EVEN^4_2(x_1, ..., x_8) ∧ x_1 ∧ x_2 → x_2 ∧ x_3 |
| II       | 2         | (x_1 ↔ x_2 ∧ x_3 ∧ (x_1 ↔ x_2 ∧ x_3)) |
| II_0     | 2         | (x_1 ∨ x_2) ∧ (x_1 ∨ x_2) ∧ F(c_0) |
| II_2     | 2         | (x_1 ∨ x_2) ∧ (x_1 ∨ x_2) ∧ T(c_1) |
| BR       | 3         | R_{2^3}(x_1, ..., x_8) ∧ T(c_1) |
Proof. We consider each case in turn. For every co-clone IC we write $R_{IC}$ for the weak base from Table I and $R$, $R'$, ..., for intermediate relations in the derivation.

$\text{IR}_0$: $R_0(\text{COLS}^3) \xrightarrow{(1=2)} R \xrightarrow{\text{irr}} R_{\text{IR}_0}$.

$\text{IR}_1$: $R_1(\text{COLS}^3) \xrightarrow{(1=2)} R \xrightarrow{\text{irr}} R_{\text{IR}_1}$.

$\text{IM}_0$: $M_0(\text{COLS}^3) \xrightarrow{(1=2)} R \xrightarrow{\text{irr}} R' \xrightarrow{\pi(3,1,2)} R_{\text{IM}_0}$.

$\text{IM}_1$: $M_1(\text{COLS}^3) \xrightarrow{(1=2)} R \xrightarrow{\text{irr}} R_{\text{IM}_1}$.

$\text{IM}_2$: $M_2(\text{COLS}^3) \xrightarrow{(1=2)} R \xrightarrow{(1=2)} R' \xrightarrow{(2=3)} R'' \xrightarrow{\pi(3,1,2,4)} R_{\text{IM}_2}$.

$\text{ID}_2$: $D_2(\text{COLS}^3)$.

$\text{IV}_1$: $V_1(\text{COLS}^3) \xrightarrow{(4=8)} R \xrightarrow{(2=4)} R' \xrightarrow{(3=6)} R'' \xrightarrow{\text{irr}} R''' \xrightarrow{\pi(4,2,3,1,5)} R_{\text{IV}_1}$.

$\text{IV}_2$: $V_2(\text{COLS}^3) \xrightarrow{(4=8)} R \xrightarrow{(2=4)} R' \xrightarrow{(3=6)} R'' \xrightarrow{\text{irr}} R''' \xrightarrow{\pi(4,2,3,1,5)} R_{\text{IV}_2}$.

$\text{IE}_0$: $E_0(\text{COLS}^3) \xrightarrow{(1=2)} R \xrightarrow{(1=2)} R' \xrightarrow{(1=2)} R'' \xrightarrow{\text{irr}} R''' \xrightarrow{\pi(5,1,2,3,4)} R_{\text{IE}_0}$.

$\text{IE}_2$: $E_2(\text{COLS}^3) \xrightarrow{(1=2)} R \xrightarrow{(1=2)} R' \xrightarrow{(1=2)} R'' \xrightarrow{\text{irr}} R''' \xrightarrow{\pi(4,1,2,3,5)} R_{\text{IE}_2}$.

It is not hard to see that every relation $R_{IC}$ is a base of IC. As in the previous cases the minimality of each weak base can be verified through exhaustive search. As an example consider

$$R_{\text{IE}_2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix}.

Removing three rows results in a relation in IR₂ while removing two rows from $R_{\text{IE}_2}$ results in a relation in ID₁. Removing the first row results in a relation which generates BR and is hence no longer included in IE₂, removing the second or third row gives a relation in IM₂, and removing the fourth row gives a relation in IS₁₀. Hence there is no relation $R' \subset R_{\text{IE}_2}$ such that $(R') = \text{IE}_2$ by which it follows that $R_{\text{IE}_2}$ is a minimal weak base.\hfill \Box

We now turn our attention towards the infinite parts of Post’s lattice. In the sequel we sometimes represent relations by formulas in conjunctive normal form. If $x = x_1, \ldots, x_n$ we use $\phi(x)$ to denote a formula with $n$ free variables. If $\phi = C_1 \land \ldots \land C_m$ is a formula with $m$ clauses we say that $C_i$ is a prime implicate of $\phi$ if $\phi$ does not entail any proper subclause of $C_i$. A formula $\phi$ is said to be prime if all of its clauses are prime implicates. Obviously any finite Boolean relation is representable by a prime formula. If $R$ is an $n$-ary Boolean relation we can therefore prove that $R \in (\Gamma)^\sharp$ by showing that $R(x_1, \ldots, x_n)$ can be expressed as a conjunction $\phi_1(y_1) \land \ldots \land \phi_k(y_k)$, where each $y_i$ is a vector over $x_1, \ldots, x_n$ and each $\phi_i$ is a prime formula representation of a relation in $\Gamma$. This is advantageous since relations in $\text{IS}_n^0$, $\text{IS}_n^1$, $\text{IS}_n^2$, $\text{IS}_n^3$, $\text{IS}_n^0$, $\text{IS}_n^1$, $\text{IS}_n^2$, $\text{IS}_n^3$ and $\text{IS}_n^0$ are representable by prime implicative hitting set-bounded (IHSB) formulas [6].

We let $\text{IHSB}^+\text{n}$ be the set of formulas of the form $(x_1 \lor \ldots \lor x_m), 1 \leq m \leq n, (\neg x_1), (\neg x_1 \lor x_2)$, and dually for $\text{IHSB}^-\text{n}$. To avoid repetition we only present the full proof for $\text{IS}_n^0$. The proofs for the other cases follow through similar arguments.

Lemma 10. The relation $R_{\text{IS}_n^0}(x_1, \ldots, x_n, x, c_0, c_1) \equiv \text{OR}(x_1, \ldots, x_n) \land (x \rightarrow x_1 \cdots x_n) \land F(c_0) \land T(c_1)$ is a minimal weak base for $\text{IS}_n^0$.

Proof. Let $\Gamma$ be a constraint language such that $(\Gamma) = \text{IS}_n^0$. Since $\Gamma$ is finite we can without loss of
generality restrict the proof to a single relation $R$ defined to be the cartesian product of all relations in $\Gamma$. We must prove that $R_{\text{IS}_{00}^n} \in \langle R \rangle_3$. By Creignou et al.\cite{creignou2002} we know that $R$ can be expressed as an IHSB+$^n$ formula $\phi(y_1, \ldots, y_m)$.

We first implement $F(c_0)$ by identifying every variable $y_i$ occurring in a negative clause $(-y_i)$ to $c_0$. Note that there must exist at least one negative unary clause since otherwise $\langle R \rangle = \text{IS}_{01}^n$. Then, for any implicative clause $(-y_i \lor c_0)$ which also entails $(-c_0 \lor y_i)$ we identify $y_i$ with $c_0$. For any remaining clause we identify all unbound variable with $c_1$. Since there must exist at least one positive prime clause this correctly implements $T(c_1)$.

Since $\langle R \rangle = \text{IS}_{00}^n$ there is at least one $n$-ary prime clause of the form $(y_1 \lor \ldots \lor y_n)$ in $\phi$. We can therefore implement $\text{OR}(x_1, \ldots, x_n)$ with $\phi(y_1, \ldots, y_m)$ by first identifying $y_1, \ldots, y_n$ and $x_1, \ldots, x_n$. Let the resulting formula be $\phi'$. Note that $\phi'$ might still contain unbound variables. In the subsequent formula we use $x_i$, $1 \leq i \leq n$, to denote a variable in $x_1, \ldots, x_n$ and $y_j$, $n + 1 \leq j \leq m$, to denote a variable in $y_{n+1}, \ldots, y_m$. Hence we need to replace each $y_j$ still occurring in $\phi'$ with $x_i$, $c_0, c_1$ or $x$. For every implicative clause $C$ in $\phi'$ there are four cases to consider:

1. $C = (-x_i \lor x'_i)$
2. $C = (-x_i \lor y_j)$
3. $C = (-y_j \lor y'_j)$
4. $C = (-y_j \lor x_i)$

The first case is impossible since $(x_1 \lor \ldots \lor x_n)$ was assumed to be prime. This also implies that the clauses $(-x_i \lor y_j)$ and $(-y_j \lor x'_i)$ cannot occur simultaneously in the formula. For the second case we identify $y_j$ with $c_1$. For the third case we identify both $y_j$ and $y'_j$ with $x$. For the fourth case we identify $y_j$ with $c_0$. As can be verified the resulting formula implements $\text{OR}(x_1, \ldots, x_n)$.

In order to implement $(-x \lor x_1 \land \ldots \land x_n)$ we need to ensure that $-x \lor x_i$ for all $1 \leq i \leq n$. Since $\langle R \rangle = \text{IS}_{00}^n$ its prime formula representation $\phi$ must contain a prime clause of the form $(-y_j \lor y'_j)$ where $\phi$ does not entail $(-y_j \lor y'_j)$ to implement $(-x \lor x_i)$ we therefore identify $y_j$ with $x$ and $y'_j$ with $x_i$. In the subsequent formula there are three implicative cases to consider:

1. $C = (-x \lor y_j)$
2. $C = (-x_i \lor y_j)$
3. $C = (-y_j \lor x_i)$

In the first case we identify $y_j$ with $x_i$, in the second case we identify $y_j$ with $c_1$, and in the third case we identify $y_j$ with $x$. For any remaining positive clause we identify each unbound variable to $c_1$, and for any remaining negative unary clause $(-y_j)$ we identify $y_j$ with $c_0$. If we repeat the procedure for all $1 \leq i \leq n$ we see that $(-x \lor x_1 \land \ldots \land x_n)$. All resulting formulas now only contain variables from $x_1, \ldots, x_n, x, c_0, c_1$, and hence the implementation is indeed a q.p.p. implementation.

One can also prove that $R_{\text{IS}_{00}^n}$ is a base of $\text{IS}_{00}^n$ by giving an explicit p.p. definition of the base given by Böhler et al.\cite{boehler2001}. As for the minimality we simply note that removing any tuple from $R_{\text{IS}_{00}^n}$ results in a relation which is no longer a base of $\text{IS}_{00}^n$.

Due to the duality of $\text{IS}_{00}^n, \text{IS}_{02}^n, \text{IS}_{01}^n, \text{IS}_{00}^n$ with $\text{IS}_{12}^n, \text{IS}_{11}^n, \text{IS}_{10}^n$ we skip the latter proofs and instead refer to Lemma\cite{boehler2001}. We have thus proved the main result of the paper.
Theorem 11. The relations in Table 1 are minimal weak bases.

4. Conclusions and future work

We have determined minimal weak bases for all Boolean co-clones with a finite base. Below are some topics relevant for future pursuits.

The lattice of strong partial clones. Since the weak and plain base of a co-clone $IC$ constitute the smallest and largest weak partial co-clone occurring inside of $IC$ it would be interesting to determine the full interval of weak partial co-clones between the weak base and the plain base. Especially one would like to determine whether these intervals are finite, countably infinite or equal to the continuum.

Exact complexity of constraint problems. Each weak base effectively determines the constraint problem with the lowest complexity in a given co-clone. Example applications which follows from the categorization in this article include the easiest NP-complete Boolean CSP(·) problem in Jonsson et al. [9] which is simply the weak base of BR without constant columns. Are there other problems besides Boolean CSP(·) which admits a single easiest problem?

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