The direct coupling of light quarks to heavy di-quarks

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A B S T R A C T
In the limit \( m_Q > m_Q v_{\text{rel}} > m_Q v_{\text{rel}}^2 \gg \Lambda_{QCD} \) hadronic states with two heavy quarks \( Q \) should be describable by a version of HQET where the heavy quark is replaced by a di-quark degree of freedom. In this limit the di-quark is a small (compared with \( 1/\Lambda_{QCD} \)) color anti-triplet, bound primarily by a color Coulomb potential. The excited Coulombic states and color six states are much heavier than the color anti-triplet ground state. The low lying spectrum of hadrons containing two heavy quarks is then determined by the coupling of the light quarks and gluons with momentum of order \( \Lambda_{QCD} \) to this ground state di-quark. In this short paper we calculate the coefficient of leading local operator \( \left( 1/\Lambda^4_s \right) (\bar{q} \gamma^\mu q) \) that couples this color anti-triplet di-quark field \( S_\mu \) (with four-velocity \( \nu \)) directly to the light quark \( q \) in the low energy effective theory. It is \( \mathcal{O}(1/\rel \mu m_Q^2) \). While our work is mostly of pedagogical value we make an estimate of the contribution of this operator to the masses of \( \Xi_{bbq} \) baryon and \( T_{QQq} \) tetraquark using the non-relativistic constituent quark model.

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1. Introduction

The lowest lying \( \Xi_{QQq} \) baryons containing two heavy quarks are stable with respect to the strong interactions. For very heavy quarks \( Q \) the lowest lying \( T_{QQq} \) tetraquark states are also stable with respect to the strong interactions [1, 2]. The reason for this is quite simple. If \( m_Q \gg \Lambda_{QCD} \) then, when the heavy di-quark is in a color 3 configuration, because of the attractive one gluon color Coulombic potential the di-quark has a large binding energy (compared with \( \Lambda_{QCD} \) ) and a small size (compared with \( 1/\Lambda_{QCD} \) ). Strong decay of the lowest lying \( T_{QQq} \) tetraquark states to a baryon with two heavy quarks and an anti-nucleon (when \( q = u, d \) ) \( \Xi_{QQq} + N_{QQq} \), is kinematically forbidden since the final state has an additional \( q\bar{q} \) pair, which costs an additional \( \sim 600 \text{ MeV} \) of mass. Strong decay to two heavy mesons \( M_{Qq} + M_{Q\bar{q}} \) does not require an additional \( q\bar{q} \) pair but now the final state does not have the large color Coulombic binding energy proportional to \( m_Q \) that the tetraquark state does and so this channel is also kinematically forbidden.

In nature the heavy quarks that are long lived are the charm and bottom quarks and whether they are heavy enough for tetraquarks containing them to be stable with respect to the strong interactions is not certain but widely believed to be the case for the lowest lying tetraquarks with two bottom quarks. See [3–5] for recent support for this hypothesis. There are indications that this is not true for the tetraquarks with charm quarks from the previous mentioned studies and [6].

Effective field theory methods have been developed [7–11] to take advantage of the fact that for very heavy quarks \( Q \) the color anti-triplet di-quark \( Q \) has a size small compared with \( 1/\Lambda_{QCD} \). These are mostly discussed in the context of the \( Q \bar{Q} \) channel but [10, 11] focus on the \( Q \bar{Q} \) channel. In this paper we work in the limit \( m_Q > m_Q v_{\text{rel}} > m_Q v_{\text{rel}}^2 \gg \Lambda_{QCD} \) where at leading order the light quarks and gluons with momentum of order \( \Lambda_{QCD} \) (which we call \( \Lambda_{QCD} \) degrees of freedom) in hadrons containing this di-quark regard the di-quark as a point object.

By the sequence of inequalities \( m_Q > m_Q v_{\text{rel}} > m_Q v_{\text{rel}}^2 \gg \Lambda_{QCD} \) we mean that while we do treat \( v_{\text{rel}} \) as small compared with unity, \( \Lambda_{QCD}/m_Q \) is much smaller. Hence we will not treat logarithms of the relative velocity as small and resume them. In this case one can match full QCD directly onto an HQET like theory at a scale \( \mu \) which we take to be \( \mu = m_Q v_{\text{rel}} \).

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1 We often use a subscript to denote the flavor quantum numbers of a state, particle or field.

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In this limit the color six \( QQ \) configurations and Coulombic excitations above the lowest lying color anti-triplet di-quark state (described by principal quantum numbers \( n > 1 \)) are much heavier than the ground state and can be integrated out of the theory. Hence (in the single di-quark sector) one arrives at a theory like HQET \([12, 13]\) with Lagrange density.

\[
\mathcal{L} = S_i^\dagger iv^\mu D^\mu S_i - \frac{1}{4} G^{\mu
u \alpha \beta} G^\mu\nu_{\alpha \beta} + \sum_q q^\dagger (i\gamma^\mu D^\mu - m_q) q + \ldots \tag{1.1}
\]

Including the heavy quark fields\(^2\) the leading terms would not only have the familiar heavy quark spin-flavor symmetry \([14]\) but also an enlarged heavy quark and di-quark spin-flavor symmetry \([15–17]\).

\( \Lambda_{QCD} \) gluons coupling directly to the heavy di-quark already occur at leading order through the covariant derivative

\[
D = \partial - igT^A q^\dagger A^B
\]

where the bar denotes that the SU(3) generators are in the \( \mathbf{3} \) representation. Of course the \( \Lambda_{QCD} \) quarks \( q \) interact with the gluons, so even at leading order they interact with the di-quark. The purpose of this paper is to use the direct coupling of the \( \Lambda_{QCD} \) quarks \( q \) to the di-quark field \( S \) which occurs in the ellipses of eq. (1.1). We find that this operator is of the form, \( \langle S_i^\dagger S_j \rangle q^\dagger (i\gamma^\mu D^\mu q) \), and that its coefficient occurs at order \( \mathcal{O}(1/\alpha_s(m_Q v_{rel} m_\pi^2)) \) which is between \( \mathcal{O}(1/m_Q) \) and \( \mathcal{O}(1/m_\pi^2) \). The \( \mathcal{O}(1/\alpha_s(m_Q v_{rel})) \) arises because this term is suppressed by the di-quark size.\(^3\) We find it interesting to compute the coefficient of this term because it gives rise to dependence on the heavy quark mass and hence a breaking of heavy quark di-quark flavor symmetry that arises from the size of the di-quark system.\(^4\) The pattern and heavy quark mass dependence of its contribution to the breaking of heavy quark di-quark flavor symmetry is different from that of the heavy quark and di-quark kinetic terms that arise at \( \mathcal{O}(1/m_Q) \) (i.e., \( \Delta L_{kin} = \bar{Q}_v D^2 2 Q_v/(2 m_Q) + S_i^\dagger S_j D^2 S_j/(2 m_\pi) \)). For example, the term we are focusing on contributes to the \( \Xi_{QQ} \) baryon mass but not to the \( M_{QQ} \) meson mass, while \( \Delta L_{kin} \) contributes to both.

In nature the heavy quarks are the top, bottom and charm quarks. While the top is very heavy compared with the QCD scale, it is short lived and does not form hadronic bound states. That leaves the bottom and charm quarks. Dimensional analysis suggests that for neither of these quarks will approximations based on \( m_q, \alpha_s v_{rel} \gg \Lambda_{QCD} \) be valid and even predictions based on the condition \( m_q, \alpha_s v_{rel} \gg \Lambda_{QCD} \) are suspect although it is likely that in the bottom quark case that they have some utility. Hence we view our work as mostly of pedagogical value. Despite these cautionary remarks we will make an estimate of the importance of the operator \( \langle S_i^\dagger S_j \rangle (q^\dagger \gamma^\mu D^\mu q) \) to the masses of the lowest lying \( \Xi_{QQ} \) baryon and \( T_{QBB} \) tetraquark using the non-relativistic constituent quark model.

### 2. The ground state color \( \bar{3} \) di-quark

Di-quarks can be in either a color triplet or color six representation. The \( 3 \times 3 \rightarrow 1 \) channel is attractive and after tracing over the color the short range color Coulombic potential is,

\[
V(r) = \frac{2 \alpha_s}{3} \frac{1}{r} \tag{2.1}
\]

This is half as strong as the attractive potential in the \( 3 \times \bar{3} \rightarrow 1 \) channel, which is relevant for quarkonium. While it is a bit odd to consider the qualitative reason for a factor of two difference between the potentials in these two channels, one can be found in the large number of colors \( N_c \) limit \([19]\). Assuming (just for simplicity) that all the quark flavors are different in the large \( N_c \) limit the interpolating field for a \( \Xi_{QBB} \) baryon is (suppressing constants, and all indices except for flavor and color)

\[
\bar{q}_1 \bar{q}_2 q_3 \bar{q}_4 \bar{q}_5 q_6 \ldots \rightarrow \bar{q}_1 \bar{q}_2 \bar{q}_3 q_1 q_2 q_3 \ldots
\]

and the short range effective potential between the two heavy quarks in this state is,

\[
V(r) = -\frac{N_c + 1}{2 N_c} \frac{\alpha_s}{r} \tag{2.2}
\]

This is suppressed by a factor of \( 1/N_c \) in the large number of colors limit where \( N_c \alpha_s \) is held fixed as \( N_c \rightarrow \infty \). At large \( N_c \), the appropriate interpolating field for tetraquarks with two heavy quarks \( T_{QBB} \) is \( \bar{q}_1 \bar{q}_2 q_3 q_4 \) and eq. (2.2) still applies for the color Coulombic potential between the heavy quarks. On the other hand, for the case of \( \bar{3} \) quarkonium with \( N_c \) colors the appropriate interpolating field is \( Q \bar{Q} Q \bar{Q} \) and the color Coulombic potential is

\[
V(r) = -\frac{N_c^2 - \alpha_s}{2 N_c} \frac{1}{r} \tag{2.3}
\]

which does not vanish in the large \( N_c \) limit.

Returning to the real world where \( N_c = 3 \) the color \( \bar{3} \) di-quark states are twice the size of the color singlet quarkonium states and so the multipole expansion should be somewhat less reliable in the di-quark case.

The ground state di-quark has a spatial wave-function

\[
\phi(r) = \left( \frac{1}{\pi a_0^2} \right)^{1/2} e^{-r/a_0} \tag{2.4}
\]

where the Bohr radius is

\[
a_0 = \frac{3}{2 \alpha_s m_Q} \tag{2.5}
\]

and the reduced heavy quark mass is \( \mu_Q = m_Q (m_2 + m_3) / (m_Q + m_3) \).

In the case where the heavy quarks are the same flavor they must be in a spin-one state. When they are different there are degenerate spin-zero and spin-one cases. Spin is usually inert for our purposes in this paper and we will usually not keep track of those labels in our equations. Later we will need the square of the charge radius,

\[
\langle r^2 \rangle = \int d^3 r r^2 \left| \phi(r) \right|^2 = 3 a_0^2 = \frac{27}{4 \alpha_s^2 m_Q^2} \tag{2.6}
\]

Since we are not treating \( v_{rel} \) as very small the argument of the strong coupling can be taken to be either \( m_Q \) or \( m_Q v_{rel} \), in the equations given in this section. However the latter is physically more appropriate so we will use it for any quantitative estimates we make using these formulae going forward.

### 3. Matching

One can compute matching onto the effective HQET like di-quark effective field theory by computing an appropriate physical perturbative process in QCD. We will work in the leading logarithmic approximation, which means tree level matching and one-loop
renormalization group running. By physical we mean on-shell but not taking into account confinement. Since we are interested in local operators involving the light quarks and the di-quark field $S$ the appropriate process is light quark heavy di-quark elastic scattering, $q_f(k_f) + (Q_1 Q_2)_u \rightarrow (Q_1 Q_2)_u + q_f(k_f)$, as shown in Fig. 1. In the heavy quark limit the scattering must be elastic, $k_f^0 = k_i^0$ and we denote the three-momentum transfer by $k = k_i - k_f$ and $k = |k|$. We work in the rest frame of the di-quark.

Expanding the heavy quark spinors to zeroth order in three-momenta we find that the amplitude $A$ for this process is

$$\mathcal{A} \simeq \frac{g^2}{2k^2} (2m_{Q_1} + 2m_{Q_2}) \bar{u}(k_f) T^A_{\beta \gamma} \gamma^0 u(k_i) \left(-T^A\right)_{\alpha \alpha'}$$

$$\times \int \frac{d^3p}{(2\pi)^3} \left[ \phi^*(p - \frac{m_{Q_1} + m_{Q_2}}{k_i}) \right] \phi(p). \quad (3.1)$$

Here $\phi$ is the Fourier transform of the spatial wave-function for the di-quark state. Expanding eq. (3.1) in $k$ the term at zeroth order corresponds to the leading order Lagrangian in eq. (1.1) where the light quark scatters off the anti-triplet charge of the di-quark without resolving its size. The term linear order in $k$ vanishes because the ground state wave-function $\phi$ is s-wave. The term quadratic order in $k$ is

$$\mathcal{A}_2 \simeq \frac{g^2}{2} (2m_{Q_1} + 2m_{Q_2}) \bar{u}(k_f) T^A_{\beta \gamma} \gamma^0 u(k_i) \left(-T^A\right)_{\alpha \alpha'}$$

$$\times \left(-\frac{k^2}{6}\right)^{m_{Q_1} + m_{Q_2}} \frac{m_{Q_1} + m_{Q_2}}{(m_{Q_1} + m_{Q_2})^2} \pi. \quad (3.2)$$

where the subscript 2 denotes that we have expanded to quadratic order in $k$.

Matching onto the effective theory we find the contribution (generalizing to arbitrary di-quark four-velocity $v$) to its Lagrange density

$$\mathcal{L} = C \left( \bar{S}_v \bar{T}_A S_v \right) \sum_q \left( \bar{q} T^A \gamma^\mu \gamma^\mu \mu q \right) \quad (3.3)$$

where $\bar{T} = -(T^A)^T$ are the $SU(3)$ generators in the anti-triplet representation and the coefficient

$$C = \frac{\pi \alpha_s (r^2)}{3} \left( \frac{m_{Q_1}^2 + m_{Q_2}^2}{m_{Q_1}^2 m_{Q_2}^2} \right) = \frac{9\pi}{4\alpha_s} \left( \frac{m_{Q_1}^2 + m_{Q_2}^2}{m_{Q_1}^2 m_{Q_2}^2} \right) \quad (3.4)$$

The charge radius arises from the non-zero size of the di-quark as so we feel it is appropriate to view the coefficient in eq. (3.4) as evaluated at the subtraction point $\mu = m_{Q} v_{\text{rel}}$ even though as mentioned earlier we will not be keeping track of factors of $v_{\text{rel}}$ in logarithms.

The cancellation of the gluon propagator’s $1/k^2$ by the factor of $k^2$ from expanding the wave-functions has the same origin as in Feynman diagrams for weak decays and can be thought of as arising from an application of the equations of motion [20]. Finally we remove the product of $SU(3)$ generators using the identity (recall $T^A = (-T^A)^T$),

$$T^A_{\alpha \beta} \gamma_{\mu \nu} = -(1/6)\delta_{\alpha \beta} \delta_{\mu \nu} + (1/2)\delta_{\alpha \nu} \delta_{\beta \mu}$$

and write the effective Lagrangian as

$$\mathcal{L} = C_1 O_1 + C_2 O_2 \quad (3.5)$$

where

$$O_1 = \left( \bar{S}_{v\alpha} S_{v\alpha} \right) \sum_q (\bar{q}_\beta \gamma^\mu \gamma^\mu \mu q_\beta) \quad (3.6)$$

and

$$C_1 = C / 6, \quad C_2 = -C / 2. \quad (3.7)$$

4. Running

Although we are not keeping track of logarithms of $v_{\text{rel}}$ we do want to sum logs of the ratio $\Lambda_{\text{QCD}}/m_{Q}$ using the renormalization group. The values of the coefficients $C_{1,2}$ in eq. (3.7) are interpreted as evaluated at a subtraction point $\mu \sim m_{Q} v_{\text{rel}}$. To scale down to a lower value of the subtraction point, we need the anomalous dimension matrix for the operators $O_{1,2}$ calculated in the leading order effective Lagrange density displayed explicitly in eq. (1.1). The subtraction point dependence of the operators $O_{1,2}$ is given by the renormalization group equations,

$$\mu \frac{d}{d\mu} O_{\mu} = -\gamma_{\mu} O_{\mu} \quad (4.1)$$

This anomalous dimension matrix $\gamma$ is computed from the one loop diagrams Feynman diagrams in Fig. 2. Using dimensional regularization with minimal subtraction we find that

$$\gamma(g) = \frac{g^2}{16\pi^2} \left( 3 - \frac{4}{3} n_q - 9 + \frac{4}{3} n_q \right) \quad (4.2)$$

where $n_q$ is the number of light quark flavors.

A basis of operators that are multiplicatively renormalized are $O_1$ and $O_− = O_1 - 3O_2$ which is the linear combination of $O_{1,2}$ that we matched onto at the scale $m_{Q} v_{\text{rel}}$. The operator $O_−$ has anomalous dimension,

$$\gamma_−(g) = \frac{g^2}{16\pi^2} \left( -9 + \frac{4}{3} n_q \right) \quad (4.3)$$

Combining this with the results of the previous section we arrive at

$$\Delta L = \left( \frac{3\pi}{8\alpha_s (m_{Q} v)} \left( \frac{m_{Q_2}^2 + m_{Q_1}^2}{m_{Q_2}^2 m_{Q_1}^2} \right) \right)$$

$$\times \left[ \left( \frac{m_{Q} v}{\alpha_s (\mu)} \left( \frac{2 + 3 n_q}{11 - \frac{4}{3} n_q} \right) \right) O_− (\mu) \right] \quad (4.4)$$
\[
O_\sim = \left( S_{\nu\alpha}S_{\nu\alpha} \right) \sum_q \left( \bar{q}_{\beta} \gamma^\mu \nu_{\mu \alpha} q_{\beta} \right) - 3 \left( S_{\nu\alpha}S_{\nu\alpha} \right) \sum_q \left( \bar{q}_{\alpha} \gamma^\mu \nu_{\mu \beta} q_{\beta} \right). \tag{4.5}
\]

The two equations above are the main results of this paper.

Including logarithmic corrections to scale the effective Lagrangian down from the scale \(m_Q v_{\text{rel}}\) is just not of academic interest. As an example of how it can matter consider the color magnetic moment term that arises in the matching from expanding the heavy quark spinors to leading order in the gluon momentum. It gives rise in the rest frame of the di-quark (when the two heavy quarks composing the di-quark are identical) to the term.\(^5\)

\[
\Delta L = \frac{1}{4m_Q^2} \left[ \frac{\alpha_s(m_Q v_{\text{rel}})}{\alpha_s(\mu)} \right] \left( \frac{\sqrt{2}}{\pi} \right) \left( S^T \gamma^\mu S \right) g^A_{\text{color}} \tag{4.6}
\]

where \(S\) is the di-quark spin vector. If we had not evaluated the strong coupling at \(\mu\) but rather at the matching scale the anomalous dimension of the operator would be large and effectively bring the scale the coupling is evaluated at to \(\mu\). What we have seen in this section is that a large anomalous scaling like this does not occur for the local operator \(O_\sim\).

5. A non-relativistic constituent quark model estimate

We can get a rough idea about how large the contribution of eq. (4.4) is to the mass of the \(\Xi_{bbq}\) baryon by making a non-relativistic quark model estimate of the matrix element of \(O_\sim\) which presumably we should view as reasonable for a subtraction point \(\mu\) around the QCD scale. In the non relativistic constituent quark model color is included through a color factor and then the light quarks are viewed as non-relativistic quasi-particles bound by some potential. By relating various physical quantities in the model an estimate can be made independent of the particular potential that binds the constituent quarks in a hadron. The estimate we make in this section is similar in spirit to using the vacuum insertion approximation [23] for the \(K - \bar{K}\) matrix element of the four-quark operator \(\langle \bar{q} \gamma^\mu (1 - \gamma_5) q \rangle \langle \bar{q} \gamma^\mu (1 - \gamma_5) q \rangle\).

The color configuration for the \(\Xi_{bbq}\) is \((1/\sqrt{2})S_{\alpha}q_{\alpha}\). In the non-relativistic quark model, for a \(\Xi_{bbq}\) at rest, we find the \(O_\sim\) expectation value to be

\[
\int d^3x O_\sim(x) = -8 \int d^3x n_S(x)n_q(x) = -8|\phi_q(0)|^2, \tag{5.1}
\]

where \(n_S(x) = \delta^3(x)\) is the number density of di-quarks, \(n_q(x) = |\phi_q(x)|^2\) is the number density of light quarks \(q\), \(\phi_q\) is the wave function of \(q\), and \(-8\) is the color factor. Using heavy quark symmetry \(\phi_q(0)\) is related to the \(B\)-meson decay constant,

\[
\phi(0) = \frac{f_B}{2\alpha_s(m_{B})} \left[ \frac{\alpha_s(m_{B})}{\alpha_s(\mu)} \right]^{\frac{6}{5}}. \tag{5.2}
\]

Combining these results, neglecting the renormalization group running and setting \(m_0 = m_B\), we have that the contribution of the Lagrange density in eq. (4.4) to the \(\Xi_{bbq}\) mass, \(\Delta m_{\Xi_{bbq}}\), is estimated to be

\[
\Delta m_{\Xi_{bbq}} \simeq \frac{\pi}{2\alpha_s(m_{B}) m_{B}} \sim 30\text{ MeV}. \tag{5.3}
\]

Here we used \(f_B \approx 190\text{ MeV}\) and \(\alpha_s(m_{B}) \simeq 0.35\) for the numerical result. The numerical result in eq. (5.3) above is only a little smaller than a typical order \(\alpha_s^2/m_0^2\) contribution to the \(\Xi_{bbq}\) mass. This should not be particularly surprising given that the Bohr radius for such a color Coulombic bound state is \(a_0 = 3/(\alpha_s(m_B)m_B) \sim 1/(600\text{ MeV})\).

The color configuration for the \(T_{bbq}\) tetraquark is \((1/\sqrt{5})e^{i\epsilon_{\alpha\beta\gamma}} \bar{S}_{\alpha}q_{\beta}\), \(\epsilon\) are spin flavor labels on the light quarks that we have suppressed. The non-relativistic constituent quark model, for a \(T_{bbq}\) at rest expectation value, we find that

\[
\int d^3x O_\sim(x) = 4 \int d^3x n_S(x)(-n_q(x)) = -4n_q(0). \tag{5.4}
\]

The color factor is \(-1/2\) what it was for the baryon case and there is an additional minus sign because \(\bar{q} \gamma^\mu q = n_q - n_{\bar{q}}\). Since \(T_{bbq}\) contains two antiquarks (while \(\Xi_{bbq}\) contains a single quark) a contribution of about 30 MeV to the mass of the \(T_{bbq}\) tetraquark from this term is a reasonable estimate. Note that if the contribution of \(O_\sim\) to the mass of the \(T_{Qq\bar{q}}\) tetraquark and \(\Xi_{Qq\bar{q}}\) baryon are the same then \(\Delta L\) in eq. (4.4) does not correct the leading order sum rule [4], \(m_{Qq\bar{q}} - m_{Qq\bar{q}} = m_{Qq\bar{q}} - m_{Qq\bar{q}}\).

Of course there are additional contributions to the masses of the \(\Xi_{bbq}\) and \(T_{bbq}\) hadrons from the leading terms explicitly displayed in eq. (11) and the familiar (from HQET) terms of order \(1/m_Q\). However these do not arise from the size of the heavy di-quark and have a different pattern of contributions to the masses of hadrons containing one heavy quark or di-quark and a different dependence on the heavy quark mass.
6. Why do we take $m_Q v_{rel}^2 \gg \Lambda_{QCD}$

This paper is about the effective field theory for the ground state anti-triplet di-quark and the direct coupling of light $\Lambda_{QCD}$ quarks to the ground state di-quark degrees of freedom in that effective HQET like theory. If we did not take $m_Q v_{rel}^2 \gg \Lambda_{QCD}$ then such an effective field theory would not be appropriate. One could still write an effective theory \cite{17,24} for the lowest lying baryons (or tetraquarks) containing two heavy quarks interacting with low momentum photons and pions, or an effective theory containing the possible di-quark configurations (pNRQCD) but matching the latter to an effective field theory just containing the lowest lying di-quark configuration and the $\Lambda_{QCD}$ gluon and light quark degrees of freedom would not be justified.

To illustrate this let us consider the case where the two heavy quarks are different flavors. Then expanding eq. (3.1) to linear order in $k$ we match onto an electric dipole transition operator \cite{10,25,26} taking the lowest lying $(n = 2) L = 1$ color anti-triplet di-quark $S_j$ to the lowest lying $(n = 1) \ L = 0$ di-quark field $S$ we have been considering. In the rest frame of the di-quarks,

$$\Delta L = \frac{1}{2\sqrt{3}} \left( \frac{m_{Q2} - m_{Q1}}{m_{Q2} + m_{Q1}} \right) S_j^\dagger TA SgE^{ij}_\text{color} \langle r \rangle_{\text{trans}} + \text{h.c.} \ (6.1)$$

where the transition charge radius is

$$\langle r \rangle_{\text{trans}} = \int_0^\infty dr_1^3 R_{2,1}(r) R_{1,0}(r) = \sqrt{\frac{2}{3}} \frac{128}{81} a_0. \ (6.2)$$

Eq. (6.1) contributes to the mass of a $\Sigma_{Q1,Q2}$ baryon at second order in $\Delta L$, an amount of order $\Delta m_{\Sigma_{Q1,Q2}} \sim \Lambda_{QCD}/(\alpha_s(m_Q v_{rel})m_Q^2)$. Here the strong coupling $g$ in eq. (6.1) is evaluated at the subtraction point (i.e., near the QCD scale) and not the matching scale since we know from HQET that there is a large anomalous dimension that makes this appropriate. Recall that the contribution from the matrix element of $\cdots$ estimated in the previous section (see eq. (5.3)) is of order $\Delta m_{\Sigma_{Q1,Q2}} \sim \Lambda_{QCD}/(\alpha_s(m_Q v_{rel})m_Q^2)$. So the impact on the $\Sigma_{Q1,Q2}$ mass from eq. (6.1) at second order in perturbation theory is suppressed by a factor of $\Lambda_{QCD}/(\alpha_s(m_Q v_{rel})m_Q^2)$ which is $1/(m_{Q2} < \Lambda_{QCD}/m_Q v_{rel})$ when compared with the contribution of $\cdots$. This contribution and the contribution of other excited di-quark states (including the color six continuum and color anti-triplet scattering states) would not be suppressed if we did not work in the limit $m_Q v_{rel}^2 \gg \Lambda_{QCD}$.

7. Concluding remarks

In this paper we have computed the leading direct coupling of the quarks that have momenta of order $\Lambda_{QCD}$ to the effective color anti-triplet di-quark degree of freedom $S$ assuming the hierarchy of scales, $m_Q > m_{Q2} v_{rel} > m_Q v_{rel}^2 \gg \Lambda_{QCD}$. In the effective HQET like theory for di-quarks this comes from the operator $S_j^\dagger S_j \ (\bar{q} \gamma^\mu q)$ which corresponds in the baryon $\Sigma_{Q1,Q2}$ to a repulsive delta function potential between the heavy di-quark and the light quarks and in the $\Sigma_{Q1,Q2}$ a repulsive delta function potential between the heavy di-quark and the light anti-quarks. It arises from the finite size of the di-quark and

\footnote{Including numerical factors in the ground state di-quark color Coulombic binding energy we need $\alpha_s(m_Q v^2/m_Q^2) \gg \Lambda_{QCD}$ when the two heavy quarks are the same.}

\footnote{We take the two heavy quarks to be in the spin-zero configuration so the total spin of the initial di-quark is one and the final di-quark is zero.}

has a coefficient $O(1/\alpha_s(m_{Q2} v_{rel})m_Q^2)$. Its coefficient is anomalously large because the factor of $1/\alpha_s(m_{Q2} v_{rel})$ originates from $g(m_{Q2} v_{rel})^2/\alpha_s(m_{Q2} v_{rel})^2$ which gives an additional $4\pi r$ when written in terms of color fine structure constant.\footnote{Hence higher order terms in the multipole expansion that arise from expanding eq. (3.1) to higher orders in $k$ will not be even more enhanced.} We estimated, using the non-relativistic quark model, that this term would contribute around 30 MeV to the mass of tetraquarks and baryons containing two bottom quarks. It gives rise to the leading violation of heavy di-quark, di-quark flavor symmetry arising from the finite size of the di-quark.

If the stability (with respect to the strong interactions) of tetraquarks containing two heavy bottom quarks is firmly established then it will still be interesting to study other aspects of their physical properties. For example, will they correspond more to the small (compared with $1/\Lambda_{QCD}$) di-quark picture or to a di-meson molecule. The latter is possible since the long range potential from one pion exchange is attractive in some channels and capable of giving rise to two meson bound states \cite{2}. Perhaps tetraquarks that contain two heavy bottom quarks and are stable with respect to the strong interactions will lie between these two extremes.

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