Continuous-discrete mathematical model for control of growth of a plant population in a given time period under a given budget

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Abstract: The paper discusses an economically viable way of controlling biomass of a vegetable plant population under the use of fertilizer in a given span of time under a given budget of expenditure. The span of time is divided in some suitable consecutive periods of equal duration p, called cohorts. The treatment is done on the available population of plant at the beginning of each cohort for a suitable time; the continuous dynamics in the change of biomass for this time period is governed by an ordinary differential equation involving total effort exerted in treating the initial population. Taking this improved value at the end of the time period as the initial value, the biomass of the population is allowed to move under its normal continuous dynamics given by Logistic growth equation for the rest of the time of that cohort. The final concentration of the biomass at the end of the first cohort is obtained by following the above two types of dynamics. This is also considered as the starting biomass for the next cohort. The same process adopted for the first cohort is repeated for calculating the improvement of the biomass in the second cohort and the whole process is repeated till the end of the final cohort is reached. Next an objective function is formed for the given span of time. This measures the net profit in getting improvement in the weight of the biomass less the cost involved in the process of improving the weight of the biomass for the given period of time. As the analysis is done in considering different cohorts at regular intervals of time, so it is a discrete model. As within each cohort, the dynamics takes place continuously, so it is a continuous model too. As a whole, the model is found to be a continuous-discrete model. Hence method of optimal control for continuous-discrete model is used to determine how the treatment at the starting of each cohort be adjusted, depending on the allocated budget, so that the total net profit is maximum.

Keywords: continuous-discrete mathematical model, cohort, logistic growth equation, cobb-douglas type of growth rate, myopic and non-myopic type of control

1. Introduction

For different places there are fixed seasons of the year for favourable growth of a vegetable plant. Naturally within that period optimal growth of the plant is desirable in order to get maximum profit by selling the final product. Obviously some artificial means like use of fertilizer and overall supervision of the cultivated area by employing trained persons also becomes necessary. Both the measures are to be applied in regular intervals of time, which are called cohorts.

Thus some sort of control measure is to be taken up in the agricultural field. Such control measures are called the total effort exerted. In this case the total effort exerted is measured in terms of total number of workers involved in the process and the amount of fertilizer used from time to time. Now the period of application is fixed, the amount of ex-

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penditure is also limited. So the whole execution is to be done in an optimal fashion to get maximum profit. The paper shows that by proper selection of total effort exerted at the beginning of each cohort it is possible to maximize the profit. The paper shows, for the first time, the use of the theory of optimization of continuous-discrete mathematical model in agricultural field. But the use of similar models in economics and fishery are already known. Such mathematical models are of two types. The first type of models are those for which the allied control problem may be solved without applying the general discrete optimal control technique and that the choice of control sequence normally involves most rapid or bang-bang approach to equilibrium. Such problems are termed as ‘separable’ by Spence et al. and ‘myopic’ by Heyman et al. Other type of problem is called ‘non-separable (non-myopic)’, where the corresponding control problem can be solved by the general method of discrete optimal control. Our control problem is of ‘myopic’ type. So we apply most rapid approach to equilibrium.

2. Formation of Continuous-discrete Mathematical Model

Let B_1(t) be the concentration of biomass of plant at time t. Let us assume that the given span of time is divided into m number of cohorts of equal duration P (cohort time). Let B_1(o) = B_{10} be the total biomass at the start of the treatment in the first cohort under the use of fertilizer. Let \( E_{10} \) be the total effort exerted at the beginning of the first cohort. Let B_1(t), the biomass in the first cohort grow under the general form of growth rate given by Cobb-Douglass type with the \( \alpha \) \_power of \( E_{10} \) and \( \beta \_power of B_1(t), 0 < \alpha < \beta , 0 < \beta < 1 \). Then the continuous dynamics of this process is given by

\[
\frac{dB_1}{dt} = \delta E_{10}^{\alpha} B_1(t)^{\beta}, \quad (\delta \text{ is a constant of proportionality}) \quad (1)
\]

Let \( t_\varepsilon \) be the total time for which the process continues and let \( I_{10} \) be the total improvement for \( B_{10} \) during the time \( t_\varepsilon \). By solving (1) we get

\[
(I_{10})^{\alpha} = \delta (1 - \beta) E_{10}^{\alpha} t_\varepsilon \quad (2)
\]

Let \( R_{10} = B_{10} + I_{10} \) be the total concentration of biomass at the end of time \( t_\varepsilon \) of the first cohort. Obviously \( R_{10} \) depends on the choice of \( \alpha, \beta, t_\varepsilon, E_{10} \). Now \( R_{10} \) is governed by its own dynamics for the rest of the time \( P - t_\varepsilon \) of the first cohort. This dynamics is expressed by the Logistic equation

\[
\frac{dx}{dt} = rx(1 - x / K), x(0) = R_{10} \quad (3)
\]

\( r \) is the intrinsic growth rate of plant population and \( K \) is its carrying capacity.

\[
x = F_1(R_{10}), F_1(R_{10}) = \frac{K}{1 + e^{-r(P - t)}} C_1, C_1 = \frac{KR_{10}}{K - R_{10}} \quad (4)
\]

Solving (3) we get

This is the value of the concentration of biomass at the end of the first cohort.

This is also the value of the biomass at the start of the second cohort. So we write it as

\[
B_2(0) = F_1(R_{10}), \quad (5)
\]

Now at this stage second phase of treatment starts with the biomass \( B_2(0) \) and continues for time \( t_\varepsilon \). There is no necessity that same \( t_\varepsilon \) is to be taken. But it is better to use same \( t_\varepsilon \) to avoid complications of calculations. Now the most important point to be noted is that same effort is not workable for the dynamics of the next cohort time \( P \), as concentration of the plant population has already increased. In fact, \( E_{20} > E_{10} \). So replacing \( E_{10} \) by \( E_{20} \) and writing equation similar to (1), we get the net improvement \( I_{20} \) at the next phase for time \( t_\varepsilon \) as

\[
(I_{20})^{\alpha} = \delta (1 - \beta) E_{20}^{\alpha} t_\varepsilon \quad (6)
\]
Let \( R_{20} = B_2(o) + I_{20} \) be the total concentration of biomass at the end of time \( t_e \) of the first cohort. As usual, \( R_{20} \) runs of its own dynamics according to (3) for the time period \( P. t_e \) and results in giving the expression for \( B_3(o) \), the starting biomass for the third cohort given by

\[
B_3(0) = F_2(R_{20}) = \frac{K}{1 + e^{-r(P-t_e)}C_2}, \quad C_2 = \frac{KR_{20}}{K - R_{20}} \tag{7}
\]

The whole process continues similarly. In this way we get \( B_{m+1}(0) \), the biomass obtained at the end of the \( m^{th} \) cohort given by

\[
B_{m+1}(0) = F_m(R_m0) = \frac{K}{1 + e^{-r(P-t_m)}C_m}, \quad C_m = \frac{KR_{m0}}{K - R_{m0}} \tag{8}
\]

This is also the starting biomass for the \((m+1)^{th}\) cohort.

The whole process cited above explains the continuous-discrete dynamics for the given span of time. Had there been no updating at the start of each cohort, the whole process would have been governed by a discrete dynamics. The updating at regular intervals of time has made the process governed by a continuous-discrete dynamics.

3. Formation of the Objective Function for the Continuous-discrete Model

As our program is to update the plant Biomass in an economically viable way, so at the start of each \( k^{th} \) cohort, when the concentration of Biomass is \( B_k(0) \), the updating \( I_{0k} \) is to be so adjusted that the total profit out of improvement in the population of plant Biomass less the corresponding cost incurred is maximum in that cohort. If \( c \) is the cost per unit effort, then the total cost for the first updating is

\[
\int_0^{t_e} cE_{10} \, dt
\]

We first express this cost in terms of \( C(B_1) \), the cost per unit Biomass treated for the first cohort. We note that \( cE_{10} \) amount of cost was needed for the Biomass

\[
\delta E_{10}^\alpha B_1^\beta
\]

So we have

\[
C(B_1) = \frac{cE_{10}}{\delta E_{10}^\alpha B_1^\beta} = A_1 B_1(t)^{-\beta}, \quad A_1 = (c/\delta)E_{10}^{1-\alpha}
\]

Assuming that solution of (1) also gives \( t \) as a function of \( B_1 \), we write

\[
\int_0^{t_e} cE_{10} \, dt = \int_{B_0}^{B_0+t_e} C(B_1) \, dB_1 = \int_{B_0}^{B_0+t_e} C(B_1) \, dB_1
\]

Thus

\[
\int_0^{t_e} cE_{10} \, dt = \int_{B_0}^{B_0+t_e} C(B_1) \, dB_1
\]

Again if \( p \) is the projected profit due to the improvement of the growth of the Biomass, then the total projected profit during the first cohort due to the improvement \( I_{0} \) is given by

\[
\pi(B_{10}, R_{10}) = \int_{B_0}^{B_0+t_e} [p - C(B_1)] \, dB_1
\]

Thus the net profit for the whole process is found to be

\[
\sum_{k=1}^{m} \pi(B_{k0}, R_{k0}) = \sum_{k=1}^{m} \int_{B_{k0}}^{B_{k0}+R_k} [p - C(B_k)] \, dB_k
\]

\[
= \sum_{k=1}^{m} \int_{B_{k0}}^{B_{k0}+R_k} [p - C(B_k)] \, dB_k + \sum_{k=1}^{m} \int_{X_{k0}}^{B_{k0}+R_k} [p - C(B_k)] \, dB_k
\]

where \( X_{k0} \) denotes the amount of concentration of population Biomass in the \( k^{th} \) cohort, for which the projected
profit equals the corresponding cost. Such a concentration is called a Bionomic equilibrium point. Further for \( k = 1, 1, 2, \ldots, m, X_{k0} \) is different for each cohort, but they have definitely fixed values.

\[
\phi(R_{k0}) = \int_{X_{k0}}^{R_{k0}} [p - C(B_k)]d(B_k)
\]

If we now write

\[
\sum_{k=1}^{m} \pi(B_{k0}R_{k0}) = -\sum_{k=1}^{m} \phi(B_{k0}) + \sum_{k=1}^{m} \phi(R_{k0}) = -\phi(B_{10}) - \sum_{k=2}^{m} \phi(B_{k0}) + \sum_{k=1}^{m} \phi(R_{k0})
\]

\[
= -\phi(B_{10}) - \sum_{k=1}^{m-1} \phi(F_{k-1}(R_{k-10})) + \sum_{k=1}^{m} \phi(R_{k0})
\]

\[
= -\phi(B_{10}) - \sum_{k=1}^{m-1} \phi(F_{k}(R_{k0})) + \sum_{k=1}^{m} \phi(R_{k0}) + \phi(R_{m0})
\]

\[
= \sum_{k=1}^{m-1} [\phi(R_{k0}) - \phi(F_{k}(R_{k0})) + \phi(R_{m0}) - \phi(B_{10})] = \sum_{k=1}^{m-1} W(R_{k0}) + \phi(R_{m0}) - \phi(B_{10}),
\]

where

\[
W(R_{k0}) = \phi(R_{k0}) - \phi(F_{k}(R_{k0}))
\]

Thus the objective function is given by

\[
\sum_{k=1}^{m} \pi(B_{k0}R_{k0}) = \sum_{k=1}^{m-1} W(R_{k0}) + \phi(R_{m0}) - \phi(B_{10}), \text{ where } W(R_{k0}) = \phi(R_{k0}) - \phi(F_{k}(R_{k0}))
\]

(11)

4. Optimal Criteria for the Continuous-discrete Model

\[
\phi(B_{10}) = \int_{X_{k0}}^{R_{k0}} [p - C(B_k)]d(B_k)
\]

We see that \( X_{k0} \) are fixed values for each \( k \). So \( \phi(B_{10}) \) depends only on \( B_{10} \). But \( B_{10} \) is given as the Biomass to start with. So \( \phi(B_{10}) \) does not contribute anything in framing the optimal criteria for (9).

\[
\phi(R_{k0}) = \int_{X_{k0}}^{R_{k0}} [p - C(B_k)]d(B_k)
\]

also depends only on \( R_{k0} \), for each \( k = 1, 2, \ldots, m-1 \). Again \( R_{k0} \) results in the improvement of \( I_{k0} \). But \( I_{k0} \) depends on the initial biomass \( B_{00} \) of the \( k^{th} \) cohort and the effort \( E_{k0} \) to be applied at the starting of the \( k^{th} \) cohort. Again \( B_{00} \) is obtained in the process of reaching the \( k^{th} \) cohort starting from the initial biomass \( B_{10} \). So there is no special choice for \( B_{10} \). But \( E_{0} \) can be chosen from the given budget to ensure optimal \( R_{k0} \). Lastly

\[
\phi(R_{m0}) = \int_{X_{m0}}^{R_{m0}} [p - C(B_k)]d(B_k)
\]

also depends only on \( R_{m0} \). But the optimal choice is fixed for \( R_{m0} \), as this depends on \( E_{m0} \), which in its turn, is dependable on the available budget, which is left after being spent at the beginning of earlier \( k \) cohorts, \( k = 1, 2, \ldots, m-1 \). So if the problem is to maximize the total profit coming from the whole process as given by (9), then practically the maximum value of the profit depends on the choice of \( R_{k0} \), \( i = 1, 2, \ldots, m-1 \). Thus the optimal criterion is reduced to the following: Find out the sequence \( \{R_{k0}^{*}\} \) for

\[
\sum_{k=1}^{m} \pi(B_{k0}R_{k0}) = \sum_{k=1}^{m-1} W(R_{k0}) + W(R_{k0}) = \phi(R_{k0}) - \phi(F_{k}(R_{k0}))
\]

which \( k = 1 \), \( 2, \ldots, m-1 \). Thus the

5. Solution to the Optimal Control Problem

Theorem 1

Let the continuous-discrete dynamics for control of plant population Biomass be given by equations (1)–(8). Let the

\[
\sum_{k=1}^{m} \pi(B_{k0}R_{k0}) = \sum_{k=1}^{m-1} W(R_{k0}) + W(R_{k0}) = \phi(R_{k0}) - \phi(F_{k}(R_{k0}))
\]
objective function and the objective criterion be given by (11) and (12) respectively. Let \( E_{k0}^* \) be the maximum effort applied at the start of the \( k \)th cohort. Then the profit given by \( \sum_{k=1}^{m} \pi(R_{k0}) \) is maximum if

\[
\sum_{k=1}^{m-1} W(R_{k0}) = \sum_{k=1}^{m-1} [\phi(R_{k0}) - \phi(F_k(R_{k0})]
\]

is maximum, the condition for which is that the optimal choice of \( \{R_{k0}\}, k=1, 2, \ldots, m-1 \) would be \( \{R_{k0}^*\} \), where \( \{R_{k0}^*\} \) satisfies the following relation

\[
p - A_k^* (R_{k0}^*)^{-\beta} = K - \beta K^-(1 + \exp(DC_k^* R_{k0}^*))^\beta + KD \exp(DC_k^*) \left( \frac{K}{(K - R_{k0}^*) (1 + \exp(DC_k^*))} \right)^2 = 0 \quad (13)
\]

\[
C_k^* = \frac{K R_{k0}^*}{K - R_{k0}^*}, D = -r(p - t_e), A_k^* = \frac{c}{\delta} E_{k0}^* (R_{k0}^*)^{-\beta}
\]

Proof:

\[
\sum_{k=1}^{m} \pi(R_{k0})
\]

is maximum if each of \( W(R_{k0}) \) is maximum at \( R_{k0}^* \), \( k = 1, 2, \ldots, m-1 \). The necessary condition is

\[
W'(R_{k0}) = 0, \forall R_{k0} = R_{k0}^*, k = 1, 2, \ldots, m-1
\]

On simplification of (14), we get (13) as the optimal criteria.

6. Conclusion

It is shown mathematically that the whole programme of treatment can be so carried out that under a given time and under a given budget, it runs in an economically viable way. This is really needed in every real situation. This shows the usefulness of our present approach.

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