Structure of the tensor product of two simple modules of quantum $GL_2$

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Abstract: In this article, we consider the tensor product of two simple modules of quantum $GL_2$ over a field of characteristic $p \neq 0$. We show that it can be expressed as a direct sum of indecomposable twisted tilting modules. This problem has been studied by Henke and Doty [1] for $SL_2$ and also later on for $SL_3$ by S R Doty, Chris Bowman and Stuart Martin, ([7, 8]).

Key words: Simple Modules, Tilting modules, Twisted tilting modules.

1 Preliminaries

In this section, we describe terminology and notation. We fix a field $k$ of characteristic $p \neq 0$. Let $q$ be a primitive $\ell$th root of unity in $k$. By a quantum $k$-group $G$, we mean a Hopf algebra $k[G]$ over $k$. We are concerned with the quantum $GL_n$ as introduced by R. Dipper and S. Donkin in [3].

Let $A_q(n)$ denote the $k$-algebra generated by $x_{ij}$, $1 \leq i, j \leq n$ subject to the relations:

(i) $x_{ir}x_{is} = x_{is}x_{ir}$ for all $1 \leq i, r, s \leq n$
(ii) $x_{is}x_{jr} = q^{-1}x_{jr}x_{is}$ for all $1 \leq i < j \leq n$ and $1 \leq r \leq s \leq n$

and finally $x_{js}x_{ir} = x_{ir}x_{js} + (q - 1)x_{is}x_{jr}$ for all $1 \leq i < j \leq n$. The algebra $A_q(n)$ is a bialgebra with the comodule structure maps $x_{ij} \mapsto \sum_{r=1}^{n} x_{ir} \otimes x_{rj}$ (comultiplication) and counit $\epsilon(x_{ij}) = \delta_{ij}$ (the kronecker delta). Let $S_n$ be the group of permutations on $\{1, 2, 3 \cdots, n\}$ and $d_q = \sum_{\sigma \in S_n} \text{sgn}(\sigma)x_{i_{1,\sigma}}x_{i_{2,2\sigma}}\cdots x_{i_{n,n\sigma}}$ the quantum determinant. The localisation $A_q(n)_{d_q}$ of $A_q(n)$ at $d_q$ (as $\{1, d_q, d_q^2, \cdots\}$ is a Ore set) is a Hopf algebra. By the quantum $GL_n(k)$ we mean the Hopf algebra $A_q(n)_{d_q}$. From now on we denote the quantum $GL_n$ over $k$ by $G$ and its corresponding Hopf algebra by $k[G]$. We also call the Hopf algebra $k[G]$ as the coordinate ring of $G$. We denote the coordinate functions of $k[G]$ by $c_{ij}$, $1 \leq i, j \leq n$. By a quantum subgroup $H$ of $G$ we mean the Hopf algebra $k[G]/I_H$ for some Hopf ideal $I_H$ of $k[G]$. We are concerned herewith two important quantum subgroups of $G$, namely a Borel subgroup of $G$ and torus contained in $B$. Let $I_B$ be the Hopf ideal of $k[G]$ generated by $\{c_{ij} \mid 1 \leq i < j \leq n\}$ and the corresponding quantum subgroup of $G$ is called Borel subgroup of...
$G$. The quantum subgroup $T$ with the corresponding Hopf ideal $I_T$ of $k[G]$ 
-generated by $\{c_{ij} \mid i \neq j\}$ is called torus. By definition, we have $T \subset B \subset G$. 
Let mod($H$) denote the category of finite dimensional $G$-modules, for $H = G, B, T$. We need to discuss some representation theory of quantum $GL_n(k)$ 
as we need in the sequel.

We let $X(T) = \mathbb{Z}^n$ and $\mathbb{Z}X(T)$ be the group algebra with basis $\{e(\lambda) \mid \lambda \in \mathbb{X}(T)\}$ and multiplication is given by the rule $e(\lambda)e(\mu) = e(\lambda + \mu)$. By [7], 
for each $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{X}(T)$ we have the $T$-module $k_\lambda$, with the 
comodule structure map $a \mapsto a \otimes c_{11}^{\lambda_1}c_{22}^{\lambda_2}\cdots c_{nn}^{\lambda_n}$. Also the set $\{k_\lambda \mid \lambda \in \mathbb{X}(T)\}$ 
is a complete set of mutually non-isomorphic simple $G$-modules. We need to discuss some representation theory of quantum 
$GL_n(k)$ for each $\lambda \in X(T)$. For each $\lambda \in X(T)$, let $\nu(\lambda)$ be the sum of all submodules of $V$ isomorphic to $k_\lambda$. We denote the formal 
character $\chi(V)$ of $V$ by $\sum_{\lambda \in \mathbb{X}(T)} \nu(\lambda)e(\lambda)$. We say that an element $\lambda = 
(\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{X}(T)$ is a dominant weight if $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_n$. We denote 
the set of all dominant weights by $X^+(T)$.

We have a homomorphism $\phi : B \rightarrow T$, whose comorphism $\tilde{\phi} : k[T] \rightarrow \mathbb{K}B$ takes $c_{ii}$ to $c_{ii}$+ the defining ideal of $B$. Hence any $T$- module 
can be treated as a $B$-module via $\phi$. For each $\lambda \in \mathbb{X}(T)$, we have the induced 
module $\n(\lambda) = \text{Ind}_{G}^{\mathbb{K}G}k_\lambda$. The module $\n(\lambda)$ is simple if and only if $\lambda \in X^+(T)$. For 
any $\lambda \in X^+(T)$, $\n(\lambda)$ is called costandard modules. The socle $L(\lambda)$ of $\n(\lambda)$ 
is simple, for any $\lambda \in X^+(T)$. Moreover, the set $\{L(\lambda) \mid \lambda \in X^+(T)\}$ is a 
complete set mutually non-isomorphic simple $G$-modules. For $\lambda \in X^+(T)$, let $\n(\lambda) = \n(-w_0(\lambda))^*$, where $w_0$ is the longest element of $S_n$ and $\n(-w_0(\lambda))^*$ is the dual of $\n(-w_0(\lambda))$. The modules $\n(\lambda)$ are called standard/Weyl modules. For $V \in \text{mod}(G)$, we call a filtration $0 \subset V_0 \subset V_1 \subset \cdots \subset V_r \subset \cdots$ 
of submodules of $V$ is a good filtration, if $V = \bigcup_{i=1}^{\infty} V_i$, and for each $i \geq 1$, we have $V_i/V_{i-1}$ is either 0 or isomorphic to $\n(\lambda)$, for some $\lambda \in X^+(T)$. Similarly, if a module $V \in \text{mod}(G)$ has a filtration $(0) = V_0 \subset V_1 \subset \cdots \subset V_r \subset \cdots$ of submodules of $V$ such that the quotient $V_i/V_{i+1}$ is either (0) or isomorphic to $\n(\lambda)$ for some $\lambda \in X^+(T)$, then we say that $V$ has a standard filtration. A module $V \in \text{mod}(G)$ is said to be a tilting module if $V$ has a 
good filtration as well as a Weyl filtration.

By a theorem of C. M. Ringel [10], for each $\lambda \in X^+(T)$ we have a 
unique(upo isomorphism) indecomposable tilting module $\nu(\lambda)$ of highest 
weight $\lambda$. The set $\{\nu(\lambda) \mid \lambda \in X^+(T)\}$ is a full set of mutually non-isomorphic 
indecomposable tilting modules. These tilting modules are called partial tilting 
modules. And also any tilting module is a direct sum of partial tilting modules $\nu(\lambda)$, $\lambda \in X^+(T)$. To discuss representation theory of quantum 
$GL_n(k)$, we also need to recall some representation theory of general linear 
group $GL_n$ over $k$. 

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We write $\overline{G}$ for the algebraic group $GL_n(k)$. Let $y_{ij}$ be the coordinate functions for $\overline{G}$. Let $\overline{B}$ be a Borel subgroup of $\overline{G}$ containing a torus $\overline{T}$. Let $X^+(\overline{T})$ be the set of all dominants weights of $\overline{G}$. Let $\overline{F} : \overline{G} \longrightarrow \overline{G}$ be the Frobenius morphism (ordinary). By [3, 1.3.2 Corollary] we have the Frobenius morphism $F : G \longrightarrow \overline{G}$ whose comorphism takes $y_{ij}$ to $c_{ij}^j$. Hence any $\overline{G}$-module can be treated as $G$ module via $F$. In a paper, by Ann Henke and S R Doty [1], it is shown that the tensor product of two simple $SL_2$-modules can be decomposed as a direct sum of twisted tensor product of certain tilting modules. In this paper, we show that the tensor product of two simple quantum $GL_2$-modules is a direct sum of certain indecomposable modules. And each indecomposable summand is the tensor product of a certain tilting module with the Frobenius twist $(F)$ of twisted tilting module of $\overline{G}$. Our methods are tilting modules, Steinberg’s tensor product theorem for quantum groups, Clebsch-Gordan formula.

2 The Quantum $GL_2$.

In this section, we shall specialize to quantum $GL_2(k)$. Let $G$ be the quantum $GL_2(k)$. Let $X_1 = \{(a, b) \in X^+(T) \mid 0 \leq a - b \leq \ell - 1, a, b \geq 0\}$ and $\pi = \{(a, b) \in X^+(T) \mid a, b \geq 0, 0 \leq a - b \leq 2(\ell - 1)\}$. The set $\pi$ is a saturated subset of $X^+(T)$ in the sense that for $\lambda \in \pi$ and $\mu \in X^+(T)$, if $\mu \leq \lambda$ then $\mu \in \pi$. Therefore the category of finite dimensional $G$-modules $V$ whose composition factors belonging to $\{L(\lambda) \mid \lambda \in \pi\}$ is equivalent to the category of finite dimensional modules for the generalized $q$-schur algebra (see [9]). We have the following Lemma.

**Lemma 2.1**

(a) If $\lambda \in X_1$, then $L(\lambda) = \Delta(\lambda) = \nabla(\lambda) = T(\lambda)$.

(b) Let $\lambda \in \pi \setminus X_1$ and $\lambda = \ell + r$, $0 \leq r \leq \ell - 2$. Then the composition factors of $T(\lambda)$ are isomorphic to $L(\lambda)$ or $L(\lambda - (r + 1)(e_1 - e_2)$ and the character of $T(\lambda)$ is given by $\chi(\lambda) + \chi(\lambda - (r + 1)(e_1 - e_2)$.

**Proof:** (a) This follows by [7] or [11].

(b) We follow the arguments given in [7]. Let $\lambda \in \pi \setminus X_1$. By [8, 3.2], we know that $\nabla(\lambda)$ is uniserial. Let $\lambda = (a, b) \in \pi$, $a - b = \ell + r$, $0 \leq r \leq \ell - 2$ and $\mu = \lambda - (r + 1)(e_1 - e_2)$. Then by [7], the injective hull $I(\lambda)$ of $L(\lambda)$ in mod($\pi$) is $\nabla(\lambda)$. Now by [3], we have $(I(\lambda) : \nabla(\tau)) = [\nabla(\tau), L(\lambda)]$. As $\lambda \in \pi \setminus X_1$, $(I(\mu) : L(\lambda')) = 1$ if $\lambda' = \lambda - (r + 1)(e_1 - e_2)$ or $\lambda' = \lambda$. By [2], the block in mod($\pi$) containing $\lambda$ is $\{\lambda, \lambda - (r + 1)(e_1 - e_2)\}$. Therefore $T(\lambda)$ has $\nabla$-filtration with sections $\nabla(\lambda), \nabla(\lambda - (r + 1)(e_1 + e_2))$.

Let St = $(\ell - 1 + b, b)$ be the Steiberg weight and $\mu = (a - b - (\ell - 1), 0)$. 

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Let $\lambda = \text{St} + \mu$. By (a), St and $L(\mu)$ are tilting. Hence by \cite{7} the module $\nabla(\text{St}) \otimes \nabla(\mu)$ is tilting and has highest weight $\lambda$. Therefore the module $T(\lambda)$ will be the block component of $\nabla(\text{St}) \otimes \nabla(\mu)$ of highest weight $\lambda$ \cite[Prop(1.2)]{4}. By \cite{7}, we note that two tilting modules are isomorphic if and only if they have same character. By Clebsch-Gordan formula, we have $\chi(\text{St})\chi(\mu) = \chi(\lambda) + (\lambda - (\epsilon_1 - \epsilon_2)) \cdots + \chi(\lambda - (r + 1)(\epsilon_1 - \epsilon_2))$. As any tilting module is a direct sum partial tilting modules, by putting altogether, we see that $\chi(\lambda) + \chi(\lambda - (r + 1)(\epsilon_1 - \epsilon_2))$ is the character of $T(\lambda)$. \hfill $\square$

Let $T = \{T(\lambda) \mid \lambda \in \pi\}$ and we call them as special tilting modules of quantum $GL_2$. By \cite[2.2]{5} any tilting module of $SL_2$ can be expressed as a direct sum of twisted tensor product of special tilting modules. This remark still holds for the general linear group $GL_2$ as well as quantum $GL_2$ also (see \cite{7}). In the case of quantum $GL_2(k)$, we call these modules as $(\mathcal{T})^F$-twisted tilting modules of $G$. In Theorem 3.2, we show that the tensor product $L \otimes L'$ of two simple module $L, L'$ of quantum $GL_2$ can expressed as a direct sum of $(\mathcal{T})^F$ twisted tilting module of $G$. All of these summands are indecomposable. In general, a direct summand of this expression need not tilting. But $L, L'$ are tilting if and only if every indecomposable summand is tilting. To prove this, it is enough consider the dominant weights of the form $(a, 0)$. We can see this as follows: let $\lambda = (a, b) \in X_1$ and $d_q = L(\epsilon_1 + \epsilon_2)$. We can write $\lambda = (a - b, 0) + b(\epsilon_1 + \epsilon_2)$. Then we have $L(\lambda) = d_q \otimes L(a-b, 0) = d_q L(a-b, 0)$.

Let $\lambda = (a, 0), \mu = (c, 0) \in X_1$. Then $a + b < \ell$ or $a + b = \ell + r, r \leq \ell - 2$. By Clebsch-Gordan formula, we have $\chi(\lambda)\chi(\mu) = \chi(a + b, 0) + \chi(a + b - 1, 1) + \cdots + \chi(a, b)$ or $\chi(\lambda)\chi(\mu) = \sum_{i=0}^{r+1} \chi(a + b - i(\epsilon_1 + \epsilon_2)$, if $a + b = \ell + r$. We let $W(\lambda, \mu) = \{(a + b, 0), (a + b - 1, 1), \cdots (a, b) - (r + 1)(\epsilon_1 + \epsilon_2)\}$ and $W_S(\lambda, \mu) = \{(c, d) \in W(\lambda, \mu) \mid c - d \geq \ell\}$. We need the following lemma.

**Lemma 2.2(2):** Let $(a, 0), (b, 0) \in X_1$ and $L = L(a, 0)$ and $L' = L(b, 0)$ be two simple modules of highest weights $(a, 0)$ and $(b, 0)$ respectively. Then $L \otimes L'$ is a tilting module and it is isomorphic to the direct sum of tilting modules $T(\lambda)$, $\lambda$ varies over $W(L, L') \setminus W_S(L, L')$.

**Proof:** We know that the tensor product of two tilting modules is a tilting module, therefore $L \otimes L'$ is a tilting module. Now by the Lemma 2.1, if $W(L, L') \subset X_1$, then $L \otimes L'$ is isomorphic to the direct sum $\bigoplus_{\gamma \in W(L, L')} T(\gamma)$. Suppose $W(L, L') \not\subset X_1$. If $(c, d) \in W_S(L, L')$, then as $L \otimes L'$ is tilting, $\chi(c, d) + \chi((c, d) - (c - d + 1)(\epsilon_1 + \epsilon_2))$ is the character of $T(c, d)$, by the Lemma 2.1. Hence $L \otimes L'$ is isomorphic to the direct sum $\bigoplus_{\gamma \in W(L, L') \setminus W_S(L, L')} T(\gamma)$. \hfill $\square$

**Lemma 2.3(3):** Let $(a, 0), (b, 0) \in X_1$ and let $L = L(a, 0)$ and $L' = (b, 0)$,
where \((a, 0), (b, 0) \in X_1\). Then the tilting module \(L \otimes L'\) is indecomposable
if and only if \(a = 0\) or \(((a, b) = (\ell - 1, 1)\).

**Proof:** If \(a = 0\) then \(L \otimes L\) is isomorphic to \(L'\) and it is indecomposable by
the Lemma 2.1. In case of \((a, b) = (\ell - 1, 1)\), the composition factors of
\(L(\ell - 1, 0) \otimes L(1, 0)\) are \(L(\ell, 0)\) and \(L(\ell - (\epsilon_1 + \epsilon_2))\). The modules \(L(\ell - 1, 0)\)
and \(L(1, 0)\) are tilting and hence \(L(\ell - 1, 0) \otimes L(1, 0)\) is also tilting. Now by
Lemma 1.2, the character of \(L(\ell - 1, 0) \otimes L(1, 0)\) is \(\chi(\ell, 0) + \chi(\ell - (\epsilon_1 + \epsilon_2))\)
and this is same as the character of \(T(\ell, 0)\) and hence it is indecomposable
tilting, as required.

Conversely, suppose \(L \otimes L'\) is tilting. Suppose \((a, b) \neq (\ell - 1, 1)\). First
we consider the case \(a = \ell - 1\) and \(b > 1\). In this case we have \(a + b = \ell + r, 1 \leq r \leq \ell - 2\).
By Clebsch-Gordan formula, \(\sum_{i=1}^{r+1} \chi(a + b - i(\epsilon_1 + \epsilon_2))\) is the
character of \(L \otimes L'\). From this we can conclude that the indecomposable
tilting modules \(T(\ell + 1, 0)\) and \(T(\ell - 1, 1)\) appear as direct summands in the
decomposition of \(L \otimes L'\) as direct sum of indecomposable tilting modules, which
is contradiction. Similarly, we can conclude that \(L(a, 0) \otimes L(b, 0)\) cannot be
indecomposable if \(b = 1\) and \(a < \ell - 1\) \(\Box\)

Example (i) For \(\ell = 5\), we have \(\ell - 1 = 4\). Let \(a = 4\) and \(b = 2\). Then we
have \(L(4, 0) \otimes L(2, 0) = T(6, 0) \oplus T(5, 1)\).
(ii) Suppose \(\ell = 5\). Let \(a = 3\) and \(b = 1\). Then \(L(3, 0) \otimes L(1, 0) = T(4, 0) \oplus T(3, 1)\).

For any \(\lambda \in X_1\) and \(\mu \in X^+\), by the Steinberg’s tensor product theorem [See
[7]], we have \(L(\lambda) \otimes L(\mu)^F \simeq L(\lambda + \ell \mu)\), where \(F\) is the Frobenius (quantum)
morphism. Given any \(a \in \mathbb{N}^+\), we can write \(a = \tau + \ell(\sum_{i=0}^{p-1} a_i p^i)\),
\(0 \leq \tau \leq \ell - 1\) and \(0 \leq a_i \leq p - 1\), uniquely. We call this as the
\((\ell, p)\) expansion of \(a\). Now for \(\lambda = (a, 0)\) and \(\mu = (\ell \sum_{i=0}^{p-1} a_i p^i, 0)\),
we have \(L(a, 0) \simeq L(\tau, 0) \otimes L(\sum_{i=0}^{p-1} a_i p^i, 0)\). Since \(0 \leq a_i \leq p - 1\), we have
\(L(a_i, 0) = T(a_i, 0)\) for all \(i\) and similarly \(L(\tau, 0) = T(\tau, 0)\) as \(0 \leq \tau \leq \ell - 1\).
Hence we have the following Lemma.

**Corollary 2.4** Suppose \(a \in \mathbb{Z}^+\) and \(a = \tau + \ell(\sum_{i=0}^{p-1} a_i p^i)\) be the \((\ell, p)\) expansion
of \(a\). Then \(L(a, 0) = L(\tau) \otimes (\otimes_{i=0}^{p-1} L(a_i)^F)\). \(\Box\)
3 Structure of the tensor product of Simple Modules

In this section, we show that the tensor product of two simple $G$-modules can be expressed as a finite direct sum of $F$-twist of $\overline{F}$-twisted tensor product of special tilting modules. As we discussed above, we prove this for simple $G$-modules of the form $L(a,0)$, $a \in \mathbb{Z}^+$. Let $X_T = \{(a,b) | 0 \leq a - b \leq p - 1\}$ and $\overline{x} = \{(a,b) \in X^+(\overline{T}) | 0 \leq a - b \leq 2(p-1)\}$. First we prove the following Lemma:

**Lemma 3.1** For $M \in \text{mod}(G)$ and $N \in \text{mod}(G)$, we have $M \otimes N^F \simeq N^F \otimes M$.

**Proof:** Let $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_m\}$ be bases for $M$ and $N$ respectively. We shall show that $M \otimes N^F \simeq N^F \otimes M$. Let $\psi_M : M \rightarrow M \otimes k[G]$, $\psi(x_i) = \sum_r x_r \otimes f_{ri}$ (resp. $\psi_N(y_j) = \sum_s y_s \otimes g_{sj}$) be the structure map for $M$ (resp. for $N$). Let $\varphi : M \otimes N^F \rightarrow N^F \otimes M$ be the map defined as $\varphi(x \otimes y) = y \otimes x$.

Now consider $(\varphi \otimes 1)(\psi(x_i \otimes y_j)) = (\varphi \otimes 1)(\sum x_r \otimes y_s \otimes f_{ri}g_{sj}) = \sum x_r \otimes y_s \otimes x_i \otimes f_{ri}g_{sj}$. The module $N^F$ is a $G$-module with the structure map given by $\psi_{N^F}(y_j) = \sum_s y_s \otimes F(g_{sj})$. By [3], we have $F(f)$ is in the center, for any $f \in k[G]$.

Let $z_j$ be $y_j$ when we viewed as an element of $N^F$. Hence $\psi(\varphi \otimes 1)(x_i \otimes z_j) = \sum z_j \otimes x_r \otimes F(g_{sj})f_{ri} = \sum z_j \otimes x_r \otimes F(g_{sj})f_{ri}$. On the otherhand, we have, $(\varphi \otimes 1)\psi(x_i \otimes z_j) = (\varphi \otimes 1)(\sum x_r \otimes z_s \otimes f_{ri}F(g_{sj})) = \sum z_j \otimes x_r \otimes F(g_{sj})f_{ri}$. Hence $M \otimes N^F \simeq N^F \otimes M$. \hfill \Box

**Theorem 3.2** Let $a, b \in \mathbb{Z}^+$. Then the tensor product $L(a,0) \otimes L(b,0)$ can be expressed as a finite direct sum of indecomposable modules of the form

$$T(\lambda_{-1}) \otimes (\overline{T}(\lambda_0) \otimes \overline{T}(\lambda_1)^F \otimes \cdots \otimes \overline{T}(\lambda_r)^F)^F$$

where $\lambda_{-1} \in X_1$ and $\lambda_i \in \overline{x}$, for $0 \leq i \leq r$.

**Proof:** Let $a = \tau_a + \ell(\sum_{i=0}^m a_ip^i)$ and $b = \tau_b + \ell(\sum_{j=0}^n b_jp^j)$ be $(\ell, p)$ expansion of $a$ and $b$ respectively. Without loss of generality we can assume that $m = n$. By the the Steinberg’s tensor product theorem, we have $L(a,0) = L(\tau_a,0) \otimes (\otimes_{i=0}^m \overline{T}(a_i,0)^F)^F$ and $L(b,0) = L(\tau_b,0) \otimes (\otimes_{i=0}^n \overline{T}(b_i,0)^F)^F$. By the Lemma 3.1, we have $L(a,0) \otimes L(b,0) = (L(\tau_a,0) \otimes L(\tau_b,0)) \otimes (\otimes_{i=0}^n ((L(a_i,0) \otimes L(b_i,0))^F)^F.$
Assume that $\tau_a \geq \tau_b$. By Clebsch-Gordan formula, the character of $L(\tau_a, 0) \otimes L(\tau_b, 0)$ is given by $\chi(\tau_a + \tau_b, 0) + \chi(\tau_a + \tau_b - 1, 1) + \cdots + \chi(\tau_a + \tau_b, 0)$. Now as $(\tau_a, \tau_b) \in X_1$, by Lemma 2.2, we have $L(\tau_a, 0) \otimes L(\tau_b, 0)$ is a direct sum of tilting modules $T(\lambda)$, where $\lambda \in W(L(\tau_a, 0), L(\tau_b, 0)) \setminus W_S(L(\tau_a, 0), L(\tau_b, 0))$. Similarly, as the weights $(a_i, 0), (b_i, 0) \in X_1$, we can express $\overline{L}(a_i, 0) \otimes \overline{L}(b_i, 0)$ as a direct sum of tilting modules $\overline{T}(\mu)$, where $\mu$ varies over $\overline{W}(L(a_i, 0), L(b_i, 0))$, for all $0 \leq i \leq n$.

Let $L = L(\tau_a, 0)$, $L' = L(\tau_b, 0)$ and $\overline{L}_i = \overline{L}(a_i, 0)$, $\overline{L}_i = \overline{L}(b_i, 0)$, and $I = W(L, L') \setminus W_S(L, L')$ and $I_i = W(L_i, L'_i) \setminus W_S(L_i, L'_i)$, for $0 \leq i \leq n$. Then we have $L \otimes L' = \oplus_{\lambda \in I} T(\lambda)$ and $\overline{L}_i \otimes \overline{L}_i = \oplus_{\mu \in I_i} \overline{T}(\mu)$, for $0 \leq i \leq n$. Thus we have,

$$L(a, 0) \otimes L(b, 0) = (\oplus_{\lambda \in I} T(\lambda)) \otimes (\otimes_{i=0}^n (\oplus_{\mu \in I_i} \overline{T}(\mu)) F)$$

By interchanging tensor product with direct sums, we can express $L(a, 0) \otimes L(b, 0)$ can be expressed as a direct sum of modules of the form

$$T(\lambda_{-1}) \otimes (\overline{T}(\lambda_0) \otimes \overline{T}(\lambda_1) F \otimes \cdots \otimes \overline{T}(\lambda_r) F)$$

By $[q$-Schur], we see that each term in the sum is indecomposable.

Let $\lambda = (\lambda_{-1}, \lambda_0, \lambda_1, \cdots \lambda_n)$, where $\lambda_{-1} \in \pi$ and $\lambda_i \in \overline{\pi}$. Let $\lambda_i = (a_{i1}, a_{i2})$, for $-1 \leq i \leq r$. Let $M(\overline{\lambda}) = T(\lambda_{-1}) \otimes (\otimes_{i=0}^n \overline{T}(\lambda_i) F)$. $\square$

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**References**

[1] Ann Henke and S. R. Doty *Decomposition of tensor product of modular irreducibles for SL$_2$*, Quarterly Jurnal of Mathematics, 56, (2005), 189-207.

[2] A. G. Cox *On Some applications of infinitesimal methods to quantum groups and related algebra*, Ph.D. thesis, University of London, 1997.

[3] R. Dipper and S. Donkin *Quantum GL$_n$*, Proc. London Mathematical society (3), 63 (1991), 165-211.

[4] S. Donkin *The Blocks of a semisimple algebraic Group*, Journal of Algebra, Vol. 67, No. 1 November 1980.
[5] S. Donkin *On tilting modules for algebraic groups*, Math. Zeit. 212 (1993), 39-60.

[6] S. Donkin, *standard homological properties for quantum $GL_n$*, J. Algebra, 181, (1996), 235-266.

[7] S. Donkin *The $q$-Scur Algebra*, Cmbridge University Press, London Mathematical Society Lecture Note Series. 253, 1998.

[8] S R Doty, Chris Bowman and Stuart Martin, *Decomposition of tensor product of modular irreducibles for $SL_3$*, Journal of Algebra 9, (2011), 177-219.

[9] S R Doty, Chris Bowman and Stuart Martin, *Decomposition of tensor product of modular irreducibles for $SL_3$, the $p \geq 5$ case*, Int. Electron, Journal of Algebra, 17, (2018), 105-138.

[10] C. M. Ringel, *The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences*, Math. Zeit. 208 (1991), 209-225.

[11] L. Thamas, *The subcomodule structure of quantum symmetric powers*, Bull. Australian Math. Soc. 50 (1994), 29-39.