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Regularity properties of singular degenerate abstract differential equations and applications

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Abstract

Singular degenerate differential operator equations are studied. The uniform separability of boundary value problems for degenerate elliptic equation and optimal regularity properties of Cauchy problem for degenerate parabolic equation are obtained. These problems have a numerous applications which occur in fluid mechanics, environmental engineering and atmospheric dispersion of pollutants.

Key Words: differential-operator equations, Semigroups of operators, Banach-valued function spaces, separability, degenerate differential equations

1. Introduction, notations and background

In this work, boundary value problems (BVPs) for singular degenerate elliptic differential-operator equations (DOEs) and the Cauchy problem for degenerate abstract parabolic equation are considered. BVPs for DOEs have been studied extensively by many researchers (see e.g. [1 − 20] and the references therein). A comprehensive introduction to the DOEs and historical references may be found in [4] and [6]. The maximal regularity properties for differential operator equations have been investigated e.g. in [3 − 4, 8 − 17]. The main objective of the present paper is to discuss BVPs for the following singular degenerate DOE

\[-\sum_{k=1}^{n} x_k^{2\alpha_k} \frac{\partial^2 u}{\partial x_k^2} + x_k^{\alpha_k} A_k (x) \frac{\partial u}{\partial x_k} + Au = f (x), \]  

where $A, A_k$ are linear operators in a Banach space $E$.

Several conditions for the uniform separability and the resolvent estimate in $E$-valued $L_p$-spaces are given. Especially, it is proven that corresponding differential operator is $R$-positive and also is a generator of the analytic semigroup.

By using separability properties of elliptic problem (1.1), maximal regularity properties of Cauchy problem is derived for the singular degenerate parabolic equation

\[ \frac{\partial u}{\partial t} - \sum_{k=1}^{n} x_k^{2\alpha_k} \frac{\partial^2 u}{\partial x_k^2} + Au = f (t, x). \]  

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One of the important characteristics of these DOEs considered here are that the degeneration process takes place at different speeds on boundaries, in general. Unlike the regular degenerate equations, due to singularity of degeneration, boundary conditions only on undegenerate points are given. Therefore, only one boundary condition with respect to the given variable is given. Note that, maximal regularity properties for nonlinear DOEs are studied e.g. in [1, 7, 9 − 11, 15].

In application, the BVP for infinity system of singular degenerate partial differential equations and Wentzell-Robin type BVP for singular degenerate partial differential equations on cylindrical domain are studied.

Let we choose $E = L_2 (0, 1)$ and $A$ to be differential operator with generalized Wentzell-Robin boundary condition defined by

$$D (A) = \left\{ u \in W^2_2 (0, 1), \ B_j u = A u (j) + \sum_{i=0}^{1} \alpha_{ij} u^{(i)} (j) = 0, \ j = 0, 1 \right\},$$

$$A (x) u = a (x, y) u^{(2)} + b (x, y) u^{(1)} + c (x, y) u, \ \text{for all } x \in R^n,$$

where $\alpha_{ij}$ are complex numbers, $a (x, .)$, $b (x, .)$, $c (x, .)$ are complex-valued functions on $(0, 1)$ for all $x \in R^n$. Then, from (1.2) we get the following Wentzell-Robin type mixed problem for singular degenerate parabolic equation

$$\frac{\partial u}{\partial t} - \sum_{k=1}^{n} x_k^{2 \alpha_k} \frac{\partial^2 u}{\partial x_k^2} + a \frac{\partial^2 u}{\partial y^2} + b \frac{\partial u}{\partial y} + cu = f (x, y, t),$$

(1.3)

$$L_k u = 0, \ x \in G, \ y \in (0, 1), \ t \in (0, T)$$

$$B_j u = 0, \ j = 0, 1, \ \text{for } x \in R^n,$$

(1.4)

where $L_k$ are boundary conditions with respect $x \in G \subset R^n$ that will be defined in late. By virtue of Theorem 3.1 derived here, we obtain that problem (1.3) − (1.4) is maximal regular in $L_p (\tilde{\Omega})$, where $L_p (\tilde{\Omega})$ denotes the space of all $p$-summable complex-valued functions with mixed norm and

$$\tilde{\Omega} = G \times (0, T) \times (0, 1), \ \tilde{p} = (p, p_1, 2).$$

Note that, the regularity properties of Wentzell-Robin type BVP for elliptic equations were studied e.g. in [24, 25] and the references therein.

Let $\gamma = \gamma (x)$ be a positive measurable function on a domain $\Omega \subset R^n$. Here, $L_{p, \gamma} (\Omega; E)$ denote the space of strongly measurable $E$-valued functions that are defined on $\Omega$ with the norm

$$\| f \|_{L_{p, \gamma}} = \| f \|_{L_{p, \gamma}(\Omega; E)} = \left( \int \| f (x) \|_E^p \gamma (x) \, dx \right)^{\frac{1}{p}}, \ 1 \leq p < \infty.$$
The Banach space $E$ is called an $UMD$-space if the Hilbert operator $(Hf)(x) = \lim_{\varepsilon \to 0} \int \frac{f(y)}{|x-y|} dy$ is bounded in $L_p(R,E)$, $p \in (1, \infty)$ (see e.g. [21]). $UMD$ spaces include e.g. $L_p$, $l_p$ spaces and Lorentz spaces $L_{pq}$, $p, q \in (1, \infty)$.

Let $\mathbb{C}$ be the set of the complex numbers and

$S_\varphi = \{ \lambda; \; \lambda \in \mathbb{C}, \; |\arg \lambda| \leq \varphi \} \cup \{ 0 \}, 0 \leq \varphi < \pi$.

Let $E_1$ and $E_2$ be two Banach spaces. $L(E_1, E_2)$ denotes the space of bounded linear operators from $E_1$ into $E_2$. For $E_1 = E_2 = E$ it will be denoted by $L(E)$.

A linear operator $A$ is said to be $\varphi$-positive in a Banach space $E$ with bound $M > 0$ if $D(A)$ is dense on $E$ and $\left\| (A + \lambda I)^{-1} \right\|_{L(E)} \leq M (1 + |\lambda|)^{-1}$ for any $\lambda \in S_\varphi$, $0 \leq \varphi < \pi$, where $I$ is the identity operator in $E$. Sometimes $A + \lambda I$ will be written as $A + \lambda$ and will be denoted by $A_\lambda$. It is known [22, §1.15.1] that a positive operator $A$ has well-defined fractional powers $A^\theta$.

Let $E(A^\theta)$ denote the space $D(A^\theta)$ with norm

$$\|u\|_{E(A^\theta)} = \left( \|u\|^p + \|A^\theta u\|^p \right)^{\frac{1}{p}}, 1 \leq p < \infty, \; 0 < \theta < \infty.$$ 

Let $E_1$ and $E_2$ be two Banach spaces. Now $(E_1, E_2)_{\theta,p}$, $0 < \theta < 1, 1 \leq p \leq \infty$ will denote interpolation spaces obtained from $\{E_1, E_2\}$ by the $K$ method [22, §1.3.1].

Let $\mathbb{N}$ denote the set of natural numbers and $\{r_j\}$ is a sequence of independent symmetric $\{-1,1\}$-valued random variables on $[0,1]$. A set $K \subset L(E_1, E_2)$ is called $R$-bounded if there is a positive constant $C$ such that for all $T_1, T_2, \ldots, T_m \in K$ and $u_1, u_2, \ldots, u_m \in E_1$, $m \in \mathbb{N}$

$$\int_0^1 \left\| \sum_{j=1}^m r_j (y) T_j u_j \right\|_{E_2} dy \leq C \int_0^1 \left\| \sum_{j=1}^m r_j (y) u_j \right\|_{E_1} dy.$$ 

The smallest $C$ for which the above estimate holds is called a $R$-bound of the collection $K$ and denoted by $R(K)$.

The $\varphi$-positive operator $A$ is said to be $R$-positive in a Banach space $E$ if the set $L_A = \{ \xi (A + \xi I)^{-1}; \xi \in S_\varphi \}$, $0 \leq \varphi < \pi$ is $R$-bounded.

Assume $E_0$ and $E$ are two Banach spaces and $E_0$ is continuously and densely embedded into $E$ and $\Omega$ is a domain in $\mathbb{R}^n$. Let $\alpha_k = \alpha_k (x)$ be a positive measurable functions on $\Omega$. Consider the Sobolev-Lions type space $W_{p,\alpha}^m (\Omega; E_0, E)$, consisting of all functions $u \in L_p(\Omega; E_0)$ that have generalized derivatives $D^m_k u = \frac{\partial^m u}{\partial x_k^m} \in L_{p,\alpha_k} (\Omega; E)$ with the norm

$$\|u\|_{W_{p,\alpha}^m (\Omega; E_0, E)} = \|u\|_{L_p(\Omega; E_0)} + \sum_{k=1}^n \left\| \alpha_k^m \frac{\partial^m u}{\partial x_k^m} \right\|_{L_p(\Omega; E)} < \infty.$$
Let \( \chi = \chi (t) \) be a positive measurable function on \((0, a)\) and

\[
u ^ i (t) = \left( \chi (t) \frac{d}{dt} \right)^i u (t).
\]

Consider the following weighted abstract space

\[
W_{p, \chi}^{[m]} \left( 0, a; E_0, E \right) = \{ u ; u \in L_p \left( 0, a; E_0 \right), u^{[m]} \in L_p \left( 0, a; E \right), \}
\]

\[
\| u \|_{W_{p, \chi}^{[m]}} = \left\| u \right\|_{L_p(0,a;E_0)} + \left\| u^{[m]} \right\|_{L_p(0,a;E)} < \infty \right\}.
\]

\[
W_{p, \chi}^{m} \left( 0, a; E_0, E \right) = \{ u ; u \in L_{p, \chi} \left( 0, a; E_0 \right), u^{(m)} \in L_{p, \chi} \left( 0, a; E \right), \}
\]

\[
\| u \|_{W_{p, \chi}^{m}} = \left\| u \right\|_{L_{p, \chi}(0,a;E_0)} + \left\| u^{(m)} \right\|_{L_{p, \chi}(0,a;E)} < \infty \right\}.
\]

Let

\[
\alpha = (\alpha_1, \alpha_2, ..., \alpha_n), \quad D^{[\alpha]} = D_1^{[\alpha_1]} D_2^{[\alpha_2]} ... D_n^{[\alpha_n]}, \quad D_k^{[\alpha]} = \left( \gamma_k (x) \frac{\partial}{\partial x_k} \right)^i.
\]

Consider the space \( W_{p, \gamma}^{[m]} \left( \Omega; E_0, E \right) \), consisting of all functions \( u \in L_p \left( \Omega; E_0 \right) \) that have generalized derivatives \( D_k^{[m]} u \in L_p \left( \Omega; E \right) \) with the norm

\[
\| u \|_{W_{p, \gamma}^{[m]} \left( \Omega; E_0, E \right)} = \left\| u \right\|_{L_p(\Omega;E_0)} + \sum_{k=1}^{n} \left\| D_k^{[m]} u \right\|_{L_p(\Omega;E)} < \infty.
\]

By reasoning as [13, Theorem 2.3] we obtain

**Theorem B.** Suppose the following conditions are satisfied:

1. \( E \) is an UMD space and \( A \) is an \( R \)-positive operator in \( E \);
2. \( \gamma = (\gamma_1, \gamma_2, ..., \gamma_n), \quad \gamma_k (x) = |x_k|^{\nu_k}, \nu_k > 1 \) and \( m \) is an integer, \( \nu = \frac{m}{m} \leq 1, 1 < p < \infty; \)
3. \( \Omega \subset R^n \) is a region such that there exists a bounded linear extension operator from \( W_{p, \gamma}^{[m]} \left( \Omega; E (A) \right) \) to \( W_{p, \gamma}^{[m]} \left( R^n; E (A) \right), \)
4. the embedding \( D^{[\alpha]} W_{p, \gamma}^{[m]} \left( \Omega; E (A) \right) \subset L_p \left( \Omega; E \left( A^{1-\nu-\mu} \right) \right) \) is continuous. Moreover for all \( h > 0 \) with \( h \leq h_0 < \infty \) and \( u \in W_{p, \gamma}^{[m]} \left( \Omega; E (A) \right) \) the following estimate holds

\[
\left\| D^{[\alpha]} u \right\|_{L_p(\Omega;A^{1-\nu-\mu})} \leq h^\mu \left\| u \right\|_{W_{p, \gamma}^{[m]} \left( \Omega; E (A) \right)} + h^{-1-\mu} \left\| u \right\|_{L_p(\Omega;E)}.
\]

2. Singular degenerate elliptic DOE
Consider the BVP for the following singular degenerate DOEs

\[- \sum_{k=1}^{n} \left[ x_{2}^{\alpha_{k}} \frac{\partial^{2}u}{\partial x_{k}^{2}} + x_{2}^{\alpha_{k}} A_{k}(x) \frac{\partial u}{\partial x_{k}} \right] + Au + \lambda u = f(x), \quad x \in G \tag{2.1} \]

\[L_{k}u = \sum_{i=0}^{m_{k}} \left[ \delta_{ki} u^{[i]}(a_{k}, x(k)) + \sum_{j=0}^{N_{k}} \nu_{kij} u^{[i]}(x_{kij}, x(k)) \right] = 0, \tag{2.2} \]

where \(x(k) \in G_{k}\) and

\[u^{[i]} = \left[ x_{2}^{\alpha_{k}} \frac{\partial^{i}}{\partial x_{k}^{i}} \right] u(x), \quad G = \prod_{k=1}^{n} (0, a_{k}), \quad G_{k} = \prod_{j \neq k} (0, a_{j}), \quad \delta_{kmk} \neq 0 \]

\[m_{k} \in \{0,1\}, \quad x(k) = (x_{1}, x_{2}, ..., x_{k-1}, x_{k+1}, ..., x_{n}), \quad j, k = 1, 2, ..., n; \]

\(\delta_{ki}, \nu_{kij}\) are complex numbers, \(\lambda\) is a complex parameter, \(A\) and \(A_{k}(x)\) are linear operators in a Banach space \(E\).

Let \(\alpha = (\alpha_{1}, \alpha_{2}, ..., \alpha_{n}), \gamma_{k}(x) = x_{2}^{\alpha_{k}}\). The main result is the following

**Theorem 2.1.** Assume the following conditions are satisfied:

1. \(E\) is an UMD space and \(A\) is a \(R\)-positive operator in \(E\);
2. \(1 + \frac{1}{p} < \alpha_{k} < \frac{p}{p-1}, \quad p \in (1, \infty), \quad \delta_{kmk} \neq 0 ; \)
3. for any \(\varepsilon > 0\), there is a positive \(C(\varepsilon)\) such that

\[\|A_{k}(y)u\| \leq \varepsilon \|u\|_{(E(A), E)}^{\frac{1}{2}} + C(\varepsilon) \|u\| \text{ for } u \in (E(A), E)^{2, \infty}. \]

Then, the problem \((2.1) - (2.2)\) has a unique solution \(u \in W_{p, \alpha}^{2}(G; E(A), E)\) for \(f \in L_{p}^{\infty}(G; E)\) and sufficiently large \(|\lambda|\) with \(|\arg \lambda| \leq \varphi\) and the following uniform coercive estimate holds

\[\sum_{k=1}^{n} \sum_{i=0}^{2} |\lambda|^{1-\frac{i}{2}} \|x_{2}^{\alpha_{k}} \frac{\partial^{i}u}{\partial x_{k}^{i}}\|_{L_{p}(G; E)} + \|Au\|_{L_{p}(G; E)} \leq M \|f\|_{L_{p}(G; E)}. \tag{2.3} \]

For proving the main theorem, consider at first the BVP for the singular degenerate ordinary DOE

\[-u^{[2]}(t) + (A + \lambda)u(t) = f, \quad t \in (0, a), \tag{2.4} \]

\[L_{1}u = \sum_{i=0}^{m} \left[ \delta_{i} u^{[i]}(a) + \sum_{j=1}^{N} \nu_{ij} u^{[i]}(t_{ij}) \right] = 0, \]

where \(u^{[i]} = \left( \frac{d^{i}}{dt^{i}} \right)^{\nu}, \quad \nu > 1, \quad m \in \{0,1\}; \quad \delta_{i}, \nu_{ij}\), are complex numbers and \(x_{ij} \in (0, a); \quad A\) is a possible unbounded operators in \(E\).

**Remark 2.1.** Let

\[\tau = - \int_{t}^{a} z^{-\nu}dz, \quad t = [a^{1-\nu} - (\nu - 1)\tau]^{\nu}. \tag{2.5} \]
Under the substitution (2.5) the spaces \( L_p(0, a; E), W^{[2]}_{p, \nu}(0, a; E(A), E) \) are mapped isomorphically onto weighted spaces

\[ L_{p, \tilde{\nu}}(\infty, 0; E), W^2_{p, \tilde{\nu}}(\infty, 0; E(A), E), \]

respectively, where \( \tilde{\nu} = \nu(t(\tau)) \). Moreover, under the substitution (2.5) the problem (2.1) – (2.2) is transformed into the following non degenerate problem

\[ -u^{(2)}(\tau) + Au(\tau) = f(\tau), \quad L_1 u = \sum_{i=0}^{m} \left[ \delta_i u^{(i)}(a) + \sum_{j=1}^{N} \nu_{ij} u^{(i)}(\tau_{ij}) \right] = 0 \]

considered in the weighted space \( L_{p, \tilde{\nu}}(\infty; 0; E) \).

In a similar way as in [13, Theorem 4.1] and [11, Lemma 3.2], we obtain

**Proposition 2.1.** Let the following conditions be satisfied:

1. \( E \) is a UMD space Banach space and \( A \) is an \( R \)-positive in \( E \);
2. \( 1 + \frac{1}{p} < \nu < \frac{p-1}{2}, 1 < p < \infty, \delta_m \neq 0. \)

Then, the problem (2.4) has a unique solution \( u \in W^{[2]}_{p, \nu}(0, a; E(A), E) \) for all \( f \in L_p(0, a; E) \), for \( |\arg \lambda| \leq \varphi \) with sufficiently large \( |\lambda| \) and the uniform coercive estimate holds

\[ \sum_{i=0}^{2} |\lambda|^{1/2} \left\| u^{(i)} \right\|_{L_p(0, a; E)} + \| Au \|_{L_p(0, a; E)} \leq C \| f \|_{L_p(0, a; E)}. \]

**Proof.** Consider the transformed problem (2.6). Since the operator \( A \) generates an analytic semigroups, by reasoning as in [4, Lemma 5.3.2/1] we fined the representation of solution of this problem. Then by using the properties of positive operator \( A \), the estimates of analytic semigroups and integral operators in weighted spaces \( L_{p, \tilde{\nu}}(\infty; 0; E) \) we obtain the assertion.

Consider the operator \( B \) generated by problem (2.4), i.e.

\[ D(B) = W^{[2]}_{p, \nu}(0, a; E(A), E, L_1), Bu = -u^{[2]} + Au. \]

In a similar way as in [11, Theorem 3.1] we obtain

**Proposition 2.2.** Suppose all conditions of Proposition 2.1 are satisfied. Then, the operator \( B \) is \( R \)-positive in \( L_p(0, a; E) \).

Proposition 2.1 implies that the operator \( B \) is positive in \( L_p(0, a; E) \) and also is a generator of an analytic semigroup. Consider the principal part of the problem (2.4), i.e. consider the problem (2.6).

**Proposition 2.3.** Assume all conditions of the Proposition 2.1 are satisfied. Then, the problem (2.6) has a unique solution \( u \in W^2_{p, \nu}(0, a; E(A), E) \) for all \( f \in L_p(0, a; E) \), \( |\arg \lambda| \leq \varphi \) and sufficiently large \( |\lambda| \). Moreover, the uniform coercive estimate holds

\[ \sum_{i=0}^{2} |\lambda|^{1/2} \left\| \tau^{\nu} u^{(i)} \right\|_{L_p(0, a; E)} + \| Au \|_{L_p(0, a; E)} \leq C \| f \|_{L_p(0, a; E)}. \]
Proof. Since $\nu > 1$, by Theorem B and Remark 2.1 for all $\varepsilon > 0$ there is a continuous function $C(\varepsilon)$ such that

$$\left\| \nu x^{\nu-1}u^{[1]} \right\|_{L_p(0,a;E)} \leq \varepsilon \|u\|_{W^{2,1}_{\nu}(0,a;E)} + C(\varepsilon) \|u\|_{L_p(0,a;E)}.$$ (2.8)

Then, in view of (2.6), (2.7) and due to positivity of operator $B$ we have the following estimate

$$\left\| \nu x^{\nu-1}u^{[1]} \right\|_{L_p(0,a;E)} \leq \varepsilon \|Bu\|_{L_p(0,a;E)}.$$ (2.9)

Since $-x^{2\nu}u^{(2)} = -u^{[2]} + \nu x^{\nu-1}u^{[1]}$, the assertion is obtained from Proposition 2.1 and estimate (2.9).

Consider the operator $S$ generated by problem (2.6), i.e.

$$D(S) = W^2_{p,a}(0,a;E), \quad Su = -x^{2\nu}u^{(2)} + Au.$$ (2.10)

Result 2.1. Suppose all conditions of Proposition 2.1 are satisfied. Then, the operator $S$ is $R$-positive in $L_p(0,a;E)$.

The assertion is obtained from Proposition 2.2 and the estimate (2.9).

Consider now the principal part of the problem (2.1) – (2.2) with constant coefficients, i.e.

$$-\sum_{k=1}^{n} \left( \sum_{i=0}^{2} x_k^{2\alpha_i} \frac{\partial^2 u}{\partial x_k^{i}} \right) + Au + \lambda u = f(x), \quad L_k u = 0, \quad k = 1, 2, ..., n$$ (2.10)

Proposition 2.4. Assume $E$ is a UMD space and $A$ is an $R$-positive operator in $E$. Let $1 + \frac{1}{p} < \alpha_k < \frac{1}{2}$, $p \in (1, \infty)$.

Then problem (2.10) has a unique solution $u \in W^2_{p,a}(G;E)$ for $f \in L_p(G;E)$ and sufficiently large $|\lambda|$ with $|\arg \lambda| \leq \varphi$ and the uniform coercive estimate holds

$$\sum_{k=1}^{n} \sum_{i=0}^{2} |\lambda|^{1-i} \left\| x_k^{i} \frac{\partial^i u}{\partial x_k^{i}} \right\|_{L_p(G;E)} + \|Au\|_{L_p(G;E)} \leq M \|f\|_{L_p(G;E)}.$$ (2.11)

Proof. Consider first all of the problem (2.10) for $n = 2$ i.e.

$$-\sum_{k=1}^{2} x_k^{2\alpha} \frac{\partial^2 u}{\partial x_k^{2}} + Au + \lambda u = f(x_1, x_2), \quad L_k u = 0, \quad k = 1, 2.$$ (2.12)

Since

$$L_p(0,a_2;L_p(0,a_1;E)) = L_p((0,a_1) \times; E)$$

then the BVP (2.12) can be expressed as.
By virtue of [1, Theorem 4.5.2], \( F = L_p(0, a_1; E) \in UMD \) provided \( E \in UMD, p \in (1, \infty) \). Then, by virtue of [21], \( L_p(0, a_1; E) \) is the space satisfying the multiplier condition. By Result 2.1 the operator \( S \), by virtue of Proposition 2.3 we get that, for \( f \), \( \lambda \)

The estimate of type (2.11) implies that the operator \( S \) has a bounded inverse from \( L_p(G; E) \) to \( W_{p,\alpha}^2(G; E(\lambda), E) \), i.e. for all \( f \in L_p(G; E), \lambda \in S(\varphi) \) with sufficiently large \( |\lambda| \) the estimate holds

\[
\left\| (Q_0 + \lambda)^{-1} f \right\|_{W_{p,\alpha}^2(G; E(\lambda), E)} \leq C \left\| f \right\|_{L_p(G; E)}.
\]

Moreover, by virtue of Theorem B and in view of assumption (3), for all \( \varepsilon > 0 \) there is a continuous function \( C(\varepsilon) \) such that

\[
\sum_{k=1}^{n} \left\| x_k^{\alpha_k} A_k u \right\|_{L_p(G; E)} \leq \varepsilon \left\| u \right\|_{W_{p,\alpha}^2(G; E(\lambda), E)} + C(\varepsilon) \left\| u \right\|_{L_p(G; E)}.
\]

From the above estimates we obtain that there is a positive number \( \delta < 1 \) such that

\[
\left\| Q_1 u \right\|_{L_p(G; E)} < \delta \left\| (Q_0 + \lambda) u \right\|_{L_p(G; E)}
\]

for all \( u \in W_{p,\alpha}^2(G; E(\lambda), E) \), where

\[
Q_1 u = \sum_{k=1}^{n} x_k^{\alpha_k} A_k (x) \frac{\partial u}{\partial x_k}.
\]

Let \( Q \) denote differential operator generated by problem (2.1) - (2.2) for \( \lambda = 0 \). It is clear that

\[
(Q + \lambda) = \left[ I + Q_1 (Q_0 + \lambda)^{-1} \right] (Q_0 + \lambda).
\]
Therefore, we obtain that the operator \((Q + \lambda)^{-1}\) is bounded from \(L_p(G; E)\) to \(W_{p,0}^2(G; E(A)\), and the estimate \((2.11)\) is satisfied.

Let \(B = L(L_p(G; E))\). We get the following result from Theorem 2.1:

**Result 2.2.** Theorem 2.1 implies that differential operator \(Q\) has a resolvent \((Q + \lambda)^{-1}\) for \(|\arg \lambda| \leq \varphi\) and the following estimate holds

\[
\sum_{k=1}^{n} \sum_{i=0}^{2} |\lambda|^{1-\frac{i}{2}} \left\| x_k^{10} \partial^i \left( Q + \lambda \right)^{-1} \right\|_B + \left\| A (Q + \lambda)^{-1} \right\|_B \leq M.
\]

**Proposition 2.5.** Assume \(E\) is a UMD space and \(A\) is a \(R\) positive in \(E\). Let \(1 + \frac{1}{p} < \alpha_k < \frac{p-1}{p}, 1 < p < \infty, \delta_m \neq 0,\) then, the operator \(Q_0\) is \(R\)-positive in \(L_p(G; E)\).

**Proof.** By reasoning as in the proof of Theorem 2.1, we get that problem \((2.10)\) can be expressed as the problem \((2.13)\). By virtue of Result 2.1 the operator \(S\) is \(R\)-positive, then by applying again the Result 2.1 to problem \((2.13)\) in \(L_p(0, a_2; E)\), \(F = L_p(0, a_1; E)\) and by continuing it \(n\) time we obtain that the operator \(Q_0\) is \(R\)-positive in \(L_p(G; E)\).

**Remark 2.2.** Note that, by using the techniques similar to those applied in Theorem 2.1 we obtain the same results for differential-operator equations of the arbitrary order.

### 3. Cauchy problem for singular degenerate parabolic equation

Consider the mixed problem for singular degenerate parabolic DOE equation

\[
\frac{\partial u}{\partial t} - \sum_{k=1}^{n} x_k^{2\alpha_k} \frac{\partial^2 u}{\partial x_k^2} + Au = f(t, x), \ x \in G, \ t \in R_+, \quad (3.1)
\]

\[
\sum_{i=0}^{m_k} \delta_{ki} u_i^j(t, a_k, x(k)) = 0, \ x(k) \in G_k, \quad (3.2)
\]

\[
u(0, x) = 0,
\]

where \(\delta_{ki}\) are complex numbers, \(G = \bigcap_{k=1}^{n} (0, a_k), \ G_k = \bigcap_{j \neq k} (0, a_j), j = 1, 2, ..., n, x(k) = (x_1, x_2, ..., x_{k-1}, x_{k+1}, ..., x_n), A\) is a possible unbounded operator in a Banach space \(E\) and \(u = u(t, x)\) is a solution of the problem \((3.1) - (3.2))\).

For \(p = (p, p_1), \Delta_+ = R_+ \times G, L_p(\Delta_+; E)\) will be denoted the space of all \(E\)-valued \(p\)-sumable functions with mixed norm i.e. the space of all measurable functions \(f\) defined on \(\Delta_+\), for which

\[
\|f\|_{L_p, a(\Delta_+)} = \left( \int_{R_+} \left( \int_{G} \|f(x)\|^p \alpha(x) \ dx \right)^{\frac{1}{p}} \ dt \right)^{\frac{1}{p}} < \infty.
\]
Analogously, $W^m_{p,a}(\Delta_+, E(A), E)$ denotes the Sobolev space with corresponding mixed norm (see [23, § for scalar case).

**Theorem 3.1.** Assume all conditions of Theorem 2.1 hold for $\varphi > \frac{\pi}{2}$. Then, for all $f \in L_p(\Delta_+; E)$ and sufficiently large $d > 0$ problem (11) − (12) has a unique solution belonging to $W^{1,2}_{p,a}(\Delta_+; E(A), E)$ and the following coercive estimate holds

$$
\|\frac{\partial u}{\partial t}\|_{L_p(G^+; E)} + \sum_{k=1}^n 2\alpha_k \|\frac{\partial^2 u}{\partial x_k^2}\|_{L_p(G^+; E)} + \|Au\|_{L_p(G^+; E)} \leq C \|f\|_{L_p(G^+; E)}.
$$

**Proof.** The problem (3.1) − (3.2) can be expressed as the following Cauchy problem

$$
\frac{du}{dt} + Q_0 u(t) = f(t), \quad u(0) = 0.
$$

(3.3)

Preposition 2.4 implies that the operator $Q_0$ is $R$-positive in $F = L_p(G; E)$. By [22, §1.14], $Q_0$ is a generator of an analytic semigroup in $F$. Then, by virtue of [16, Theorem 4.2], we obtain that for all $f \in L_{p_1}(R_+; F)$ and sufficiently large $d > 0$, problem (15) has a unique solution belonging to $W^{1,1}_{p_1}(R_+; D(Q_0), F)$ and the estimate holds

$$
\|\frac{du}{dt}\|_{L_{p_1}(R_+; F)} + \|Qu\|_{L_{p_1}(R_+; F)} \leq C \|f\|_{L_{p_1}(R_+; F)}.
$$

Since $L_{p_1}(R_+; F) = L_p(\Delta_+; E)$, by Theorem 2.1, we have

$$
\|Q_0u\|_{L_{p_1}(R_+; F)} = D(Q_0).
$$

These relations and the above estimate prove the hypothesis to be true.

**4. Singular degenerate boundary value problems for infinite systems of equations**

Consider the infinite system of BVPs

$$
\begin{align*}
-x^{2\alpha} \frac{\partial^2 u_m}{\partial x^2} - x^{2\beta} \frac{\partial^2 u_m}{\partial y^2} + d_m u_m + \sum_{j=1}^\infty x^\alpha a_{mj}(x, y) \frac{\partial u_j}{\partial x} \\
+ \sum_{j=1}^\infty y^{2\beta} b_{mj}(x, y) \frac{\partial u_j}{\partial y} + \lambda u = f_m(x, y), \quad L_1u = 0, \quad L_2u = 0,
\end{align*}
$$

(4.1)

where $L_k$ are defined by (2.2) and $x, y \in G = (0, a) \times (0, b)$.

$$
D = \{d_m\}, \quad d_m > 0, \quad u = \{u_m\}, \quad Du = \{d_m u_m\}, \quad m = 1, 2, \ldots,
$$
From Theorem 2.1, we obtain

**Theorem 4.1.** Assume \( p \in (1, \infty), 1 < \alpha, \beta < p - 1, a_{mj}, b_{mj} \in L_{\infty}(G). \) For \( 0 < \mu \leq \frac{1}{2}, 0 < \nu < 1 \) and for all \( x, y \in G \)

\[
\sup_{m} \sum_{j=1}^{\infty} a_{mj}(x, y) d_j (\frac{4}{\mu} - \nu) < M, \quad \sup_{m} \sum_{j=1}^{\infty} b_{mj}(x, y) d_j (1 - \nu).
\]

Then, for all \( f(x, y) = \{f_m(x, y)\}_1^\infty \in L_p((G); l_q), \) \( p, q \in (1, \infty), |\arg \lambda| \leq \varphi, 0 \leq \varphi < \pi \) and for sufficiently large \( |\lambda| \), problem (4.1) has a unique solution \( u = \{u_m(x, y)\}_1^\infty \) that belongs to space \( W^{2}_{p, \alpha, \beta}((G), l_q(D), l_q) \) and

\[
\sum_{j=0}^{2} |\lambda|^{1-\frac{j}{2}} \left[ \left\| x^{2\alpha} \frac{\partial^j u}{\partial x^j} \right\|_{L_p(G;l_q)} + \left\| y^{2\beta} \frac{\partial^j u}{\partial y^j} \right\|_{L_p(G;l_q)} \right]
\]

\[+ \|Du\|_{L_p(G;l_q)} \leq M \|f\|_{L_p(G;l_q)}.
\]

5. Wentzell-Robin type mixed problem for degenerate parabolic equation

Consider the problem

\[
\frac{\partial u}{\partial t} - \sum_{k=1}^{n} x_k 2^{\alpha_k} \frac{\partial^2 u}{\partial x_k^2} + a \frac{\partial^2 u}{\partial y^2} + b \frac{\partial u}{\partial y} + cu = f(x, y, t), \quad (5.1)
\]

\[
L_k u = 0, B_j u = 0, j = 0, 1, t \in (0, T), x \in \sigma, y \in (0, 1), \quad (5.2)
\]

\[
u(x, y, 0) = 0, x \in G, y \in (0, 1), \quad (5.3)
\]

where \( a = a(x, y, t), a_1 = a_1(x, y, t), b_1 = b_1(x, y, t), c = c(x, y, t) \) are complex-valued functions on \( \tilde{\Omega} = G \times (0, 1) \times (0, T). \) For \( \tilde{\mathbf{p}} = (p_1, 2) \) and \( L_{\tilde{\mathbf{p}}} (\tilde{\Omega}) \) will denote the space of all \( \tilde{\mathbf{p}} \)-summable scalar-valued functions with mixed norm. Analogously, \( W^{2,1}_{\tilde{\mathbf{p}}, \alpha, \beta} (\tilde{\Omega}) \) denotes the Sobolev space with corresponding mixed norm, i.e., \( W^{2,1}_{\tilde{\mathbf{p}}, \alpha, \beta} (\tilde{\Omega}) \) denotes the space of all functions \( u \in L_{\tilde{\mathbf{p}}} (\tilde{\Omega}) \) possessing the derivatives \( \frac{\partial u}{\partial t}, x^{2\alpha} \frac{\partial^2 u}{\partial x^2}, y^{2\beta} \frac{\partial^2 u}{\partial y^2} \in L_{\tilde{\mathbf{p}}} (\tilde{\Omega}) \) with the norm

\[
\|u\|_{W^{2,1}_{\tilde{\mathbf{p}}, \alpha, \beta}(\tilde{\Omega})} = \|u\|_{L_{\tilde{\mathbf{p}}}(\tilde{\Omega})} + \left\| \frac{\partial u}{\partial t} \right\|_{L_{\tilde{\mathbf{p}}}(\tilde{\Omega})} + \left\| x^{2\alpha} \frac{\partial^2 u}{\partial x^2} \right\|_{L_{\tilde{\mathbf{p}}}(\tilde{\Omega})} + \left\| y^{2\beta} \frac{\partial^2 u}{\partial y^2} \right\|_{L_{\tilde{\mathbf{p}}}(\tilde{\Omega})}.
\]
Condition 5.1 Assume:
(1) $1 < \alpha, \beta < p - 1$, $p, p_1 \in (1, \infty)$;
(2) $a_1(x,.,t) \in W^1_{2\infty}(0,1)$, $a_1(x,.,t) \geq \delta > 0$, $b_1(x,.,t), c(x,.,t) \in L_{\infty}(0,1)$ for a.e. $x \in G$, $t \in (0,T)$;
(3) $b(.,y,t), c(.,y,t) \in C(\bar{G})$ for $y \in (0,b)$ and $t \in (0,T)$.

In this section, we present the following result:

**Theorem 5.1.** Suppose the Condition 5.1 hold. Then, for $f \in L^p_{\bar{\Omega}}(\bar{\Omega};E)$ problem (5.1) – (5.3) has a unique solution $u$ belonging to $W^{2,1}_{p,a,\beta}(\bar{\Omega})$ and the following coercive estimate holds

$$
\|\frac{\partial u}{\partial t}\|_{L^p(\bar{\Omega};E)} + \|x^{2\alpha} \frac{\partial^2 u}{\partial x^2}\|_{L^p(\bar{\Omega})} + \|y^{2\beta} \frac{\partial^2 u}{\partial y^2}\|_{L^p(\bar{\Omega})} + \|Au\|_{L^p(G_1,E)} \leq C \|f\|_{L^p(\bar{\Omega};E)}.
$$

**Proof.** Let $E = L_2(0,1)$. It is known [10] that $L_2(0,1)$ is an $UMD$ space. Consider the operator $A$ defined by

$$
D(A) = W^2_{2,2}(0,1;B_ju = 0), \quad Au = a_1 \frac{\partial^2 u}{\partial y^2} + b_1 \frac{\partial u}{\partial y} + cu.
$$

Therefore, the problem (5.1) – (5.3) can be rewritten in the form of (3.1) – (3.2), where $u(x) = u(x,.)$, $f(x) = f(x,.)$ are functions with values in $E = L_2(0,1)$. By virtue of [24, 25] the operator $A$ generates analytic semigroup in $L_2(0,b)$. Then in view of Hill-Yosida theorem (see e.g. [22, § 1.13]) this operator is positive in $L_2(0,b)$. Since all uniform bounded set in Hilbert space is $R$-bounded (see [4]), i.e. we get that the operator $A$ is $R$-positive in $L_2(0,b)$. Then from Theorem 3.1 we obtain the assertion.

**References**

1. H. Amann, Linear and quasi-linear equations,1, Birkhauser, Basel 1995.
2. S. Yakubov and Ya. Yakubov, Differential-operator Equations. Ordinary and Partial Differential Equations, Chapman and Hall /CRC, Boca Raton, 2000.
3. Krein S. G., Linear differential equations in Banach space, American Mathematical Society, Providence, 1971.
4. Lunardi A., Analytic semigroups and optimal regularity in parabolic problems, Birkhauser, 2003.
5. Dore C. and Yakubov S., Semigroup estimates and non coercive boundary value problems, Semigroup Forum 60 (2000), 93-121.
6. Denk R., Hieber M., Prüss J., $R$-boundedness, Fourier multipliers and problems of elliptic and parabolic type, Mem. Amer. Math. Soc. 166 (2003), n.788.

7. Shakhmurov V. B., Linear and nonlinear abstract equations with parameters, Nonlinear Anal-Theor., 2010, v. 73, 2383-2397.

8. Shakhmurov V. B., V., Nonlinear abstract boundary value problems in vector-valued function spaces and applications, Nonlinear Anal-Theor., v. 67(3) 2006, 745-762.

9. Shakhmurov V. B, Shahmurova A., Nonlinear abstract boundary value problems atmospheric dispersion of pollutants, Nonlinear Anal-Real., v.11 (2) 2010, 932-951.

10. Shakhmurov V. B., Coercive boundary value problems for regular degenerate differential-operator equations, J. Math. Anal. Appl., 292 (2), (2004), 605-620.

11. Shakhmurov V. B., Degenerate differential operators with parameters, Abstr. Appl. Anal., 2007, v. 2006, 1-27.

12. Shakhmurov V. B., Separable anisotropic differential operators and applications, J. Math. Anal. Appl. 2006, 327(2), 1182-1201.

13. Shakhmurov V. B., Linear and nonlinear abstract equations with parameters, Nonlinear Anal-Theor., 2010, v. 73, 2383-239.

14. Agarwal R., O’Regan, D., Shakhmurov V. B., Separable anisotropic differential operators in weighted abstract spaces and applications, J. Math. Anal. Appl. 2008, 338, 970-983.

15. Ashyralyev A., Claudio Cuevas and Piskarev S., “On well-posedness of difference schemes for abstract elliptic problems in spaces”, Numer. Func. Anal.Opt., v. 29, No. 1-2, Jan. 2008, 43-65.

16. Weis L, Operator-valued Fourier multiplier theorems and maximal $L_p$ regularity, Math. Ann. 319, (2001), 735-758.

17. Favini A., Shakhmurov V., Yakubov Y., Regular boundary value problems for complete second order elliptic differential-operator equations in UMD Banach spaces, Semigroup Forum, v. 79 (1), 2009.

18. Shahmurov R., On strong solutions of a Robin problem modeling heat conduction in materials with corroded boundary, Nonlinear Anal. Real World Appl., 2011,13(1), 441-451.

19. Shahmurov R., Solution of the Dirichlet and Neumann problems for a modified Helmholtz equation in Besov spaces on an annualls, J. Differential Equations, 2010, 249(3), 526-550.
20. Lions J. L and Peetre J., Sur une classe d’espaces d’interpolation, Inst. Hautes Etudes Sci. Publ. Math., 19(1964), 5-68.

21. Burkholder D. L., A geometrical conditions that implies the existence certain singular integral of Banach space-valued functions, Proc. conf. Harmonic analysis in honor of Antonu Zigmund, Chicago, 1981, Wads Worth, Belmont, (1983), 270-286.

22. Triebel H., ”Interpolation theory, Function spaces, Differential operators.”, North-Holland, Amsterdam, 1978.

23. Besov, O. V., P. Ilin, V. P., Nikolskii, S. M., Integral representations of functions and embedding theorems, Nauka, Moscow, 1975.

24. Favini A., Goldstein G. R. , Goldstein J. A. and Romanelli S., Degenerate second order differential operators generating analytic semigroups in $L_p$ and $W^{1,p}$, Math. Nachr. 238 (2002), 78 – 102.

25. Keyantuo V., Lizama, C., Maximal regularity for a class of integro-differential equations with infinite delay in Banach spaces, Studia Math. 168 (2005), 25-50.