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A new bijection relating $q$-Eulerian polynomials

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Abstract

On the set of permutations of a finite set, we construct a bijection which maps the 3-vector of statistics $(\text{maj} - \text{exc}, \text{des}, \text{exc})$ to a 3-vector $(\text{maj}_2, \text{des}_2, \text{inv}_2)$ associated with the $q$-Eulerian polynomials introduced by Shareshian and Wachs in *Chromatic quasisymmetric functions*, arXiv:1405.4269 (2014).

Keywords: $q$-Eulerian polynomials, descents, ascents, major index, exceedances, inversions.

Notations

For all pair of integers $(n, m)$ such that $n < m$, the set $\{n, n+1, \ldots, m\}$ is indifferently denoted by $[n, m], [n-1, m], [n, m+1]$ or $[n-1, m+1]$.

The set of positive integers $\{1, 2, 3, \ldots\}$ is denoted by $\mathbb{N}_{>0}$.

For all integer $n \in \mathbb{N}_{>0}$, we denote by $[n]$ the set $[1, n]$ and by $\mathfrak{S}_n$ the set of the permutations of $[n]$. By abuse of notation, we assimilate every $\sigma \in \mathfrak{S}_n$ with the word $\sigma(1)\sigma(2)\ldots\sigma(n)$.

If a set $S = \{n_1, n_2, \ldots, n_k\}$ of integers is such that $n_1 < n_2 < \ldots < n_k$, we sometimes use the notation $S = \{n_1 < n_2 < \ldots < n_k\}$.

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1. Introduction

Let $n$ be a positive integer and $\sigma \in \mathfrak{S}_n$. A descent (respectively exceedance point) of $\sigma$ is an integer $i \in [n-1]$ such that $\sigma(i) > \sigma(i+1)$ (resp. $\sigma(i) > i$). The set of descents (resp. exceedance points) of $\sigma$ is denoted by $\text{DES}(\sigma)$ (resp. $\text{EXC}(\sigma)$) and its cardinal by $\text{des}(\sigma)$ (resp. $\text{exc}(\sigma)$). The integers $\sigma(i)$ with $i \in \text{EXC}(\sigma)$ are called exceedance values of $\sigma$.

It is due to MacMahon [Mac15] and Riordan [Rio58] that

$$\sum_{\sigma \in \mathfrak{S}_n} t^{\text{des}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{exc}(\sigma)} = A_n(t)$$

where $A_n(t)$ is the $n$-th Eulerian polynomial [Eul55]. A statistic equidistributed with $\text{des}$ or $\text{exc}$ is said to be Eulerian. The statistic $\text{ides}$ defined by $\text{ides}(\sigma) = \text{des}(\sigma^{-1})$ obviously is Eulerian.

The major index of a permutation $\sigma \in \mathfrak{S}_n$ is defined as

$$\text{maj}(\sigma) = \sum_{i \in \text{DES}(\sigma)} i.$$ 

It is also due to MacMahon that

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}(\sigma)} = \prod_{i=1}^{n} \frac{1 - q^i}{1 - q}.$$ 

A statistic equidistributed with $\text{maj}$ is said to be Mahonian. Among Mahonian statistics is the statistic $\text{inv}$, defined by $\text{inv}(\sigma) = |\text{INV}(\sigma)|$ where $\text{INV}(\sigma)$ is the set of inversions of a permutation $\sigma \in \mathfrak{S}_n$, i.e. the pairs of integers $(i, j) \in [n]^2$ such that $i < j$ and $\sigma(i) > \sigma(j)$.

In [SW14], the authors consider analogous versions of the above statistics: let $\sigma \in \mathfrak{S}_n$, the set of 2-descents (respectively 2-inversions) of $\sigma$ is defined as

$$\text{DES}_2(\sigma) = \{i \in [n-1], \sigma(i) > \sigma(i+1) + 1\}$$

(resp.

$$\text{INV}_2(\sigma) = \{1 \leq i < j \leq n, \sigma(i) = \sigma(j) + 1\}$$

and its cardinal is denoted by $\text{des}_2(\sigma)$ (resp. $\text{inv}_2(\sigma)$).
It is easy to see that \( \text{inv}_2(\sigma) = \text{ides}(\sigma) \). The \( 2 \)-major index of \( \sigma \) is defined as

\[
\text{maj}_2(\sigma) = \sum_{i \in \text{DES}_2(\sigma)} i.
\]

By using quasisymmetric function techniques, the authors of [SW14] proved the equality

\[
\sum_{\sigma \in S_n} x^{\text{maj}_2(\sigma)} y^{\text{inv}_2(\sigma)} = \sum_{\sigma \in S_n} x^{\text{maj}(\sigma) - \text{exc}(\sigma)} y^{\text{exc}(\sigma)}.
\] (1)

Similarly, by using the same quasisymmetric function method as in [SW14], the authors of [HL12] proved the equality

\[
\sum_{\sigma \in S_n} x^{\text{amaj}_2(\sigma)} y^{\widetilde{\text{asc}_2}(\sigma) \cdot \text{ides}(\sigma)} = \sum_{\sigma \in S_n} x^{\text{maj}(\sigma) - \text{exc}(\sigma)} y^{\text{des}(\sigma) \cdot \text{exc}(\sigma)}
\] (2)

where \( \text{asc}_2(\sigma) \) is the number of \( 2 \)-ascents of a permutation \( \sigma \in S_n \), i.e. the elements of the set \( \text{ASC}_2(\sigma) = \{ i \in [n-1], \sigma(i) < \sigma(i+1) + 1 \} \), which rises the statistic \( \text{amaj}_2 \) defined by

\[
\text{amaj}_2(\sigma) = \sum_{i \in \text{ASC}_2(\sigma)} i,
\]

and where

\[
\widetilde{\text{asc}_2}(\sigma) = \begin{cases} 
\text{asc}_2(\sigma) & \text{if } \sigma(1) = 1, \\
\text{asc}_2(\sigma) + 1 & \text{if } \sigma(1) \neq 1.
\end{cases}
\]

**Definition 1.1.** Let \( \sigma \in S_n \). We consider the smallest \( 2 \)-descent \( d_2(\sigma) \) of \( \sigma \) such that \( \sigma(i) = i \) for all \( i \in [d_2(\sigma) - 1] \) (if there is no such \( 2 \)-descent, we define \( d_2(\sigma) \) as 0 and \( \sigma(0) \) as \( n + 1 \)).

Now, let \( d'_2(\sigma) > d_2(\sigma) \) be the smallest \( 2 \)-descent of \( \sigma \) greater than \( d_2(\sigma) \) (if there is no such \( 2 \)-descent, we define \( d'_2(\sigma) \) as \( n \)).

We define an inductive property \( \mathcal{P}(d_2(\sigma)) \) by:

1. \( \sigma(d_2(\sigma)) < \sigma(i) \) for all \( (i,j) \in \text{INV}_2(\sigma) \) such that \( d_2(\sigma) < i < d'_2(\sigma) \);
2. if \( (d'_2(\sigma), j) \in \text{INV}_2(\sigma) \) for some \( j \), then either \( \sigma(d_2(\sigma)) < \sigma(d'_2(\sigma)) \), or \( d'_2(\sigma) \) has the property \( \mathcal{P}(d'_2(\sigma)) \) (where the role of \( d_2(\sigma) \) is played by \( d'_2(\sigma) \) and that of \( d'_2(\sigma) \) by \( d'_2(\sigma) \) where \( d'_2(\sigma) > d_2(\sigma) \) is the smallest \( 2 \)-descent of \( \sigma \) greater than \( d'_2(\sigma) \), defined as \( n \) if there is no such \( 2 \)-descent).
This property is well-defined because \((n, j) \not\in \text{INV}_2(\sigma)\) for all \(j \in [n]\).

Finally, we define a statistic \(\widetilde{\text{des}}_2\) by:

\[
\widetilde{\text{des}}_2(\sigma) = \begin{cases} 
\text{des}_2(\sigma) & \text{if the property } P(d_2(\sigma)) \text{ is true}, \\
\text{des}_2(\sigma) + 1 & \text{otherwise}.
\end{cases}
\]

In the present paper, we prove the following theorem.

**Theorem 1.2.** There exists a bijection \(\varphi : \mathcal{S}_n \rightarrow \mathcal{S}_n\) such that

\((\text{maj}_2(\sigma), \widetilde{\text{des}}_2(\sigma), \text{inv}_2(\sigma)) = (\text{maj}(\varphi(\sigma)) - \text{exc}(\varphi(\sigma)), \text{des}(\varphi(\sigma)), \text{exc}(\varphi(\sigma))).\)

As a straight corollary of Theorem 1.2, we obtain the equality

\[
\sum_{\sigma \in \mathcal{S}_n} x^{\text{maj}_2(\sigma)} y^{\widetilde{\text{des}}_2(\sigma)} z^{\text{inv}_2(\sigma)} = \sum_{\sigma \in \mathcal{S}_n} x^{\text{maj}(\sigma) - \text{exc}(\sigma)} y^{\text{des}(\sigma)} z^{\text{exc}(\sigma)}
\]

which implies Equality (1).

The rest of this paper is organised as follows.

In Section 2, we introduce two graphical representations of a given permutation so as to highlight either the statistic \((\text{maj} - \text{exc}, \text{des}, \text{exc})\) or \((\text{maj}_2, \widetilde{\text{des}}_2, \text{inv}_2).\)

Practically speaking, the bijection \(\varphi\) of Theorem 1.2 will be defined by constructing one of the two graphical representations of \(\varphi(\sigma)\) for a given permutation \(\sigma \in \mathcal{S}_n.\)

We define \(\varphi\) in Section 3.

In Section 4, we prove that \(\varphi\) is bijective by constructing \(\varphi^{-1}.\)

### 2. Graphical representations

#### 2.1. Linear graph

Let \(\sigma \in \mathcal{S}_n.\) The linear graph of \(\sigma\) is a graph whose vertices are (from left to right) the integers \(\sigma(1), \sigma(2), \ldots, \sigma(n)\) aligned in a row, where every \(\sigma(k)\) (for \(k \in \text{DES}_2(\sigma)\)) is boxed, and where an arc of circle is drawn from \(\sigma(i)\) to \(\sigma(j)\) for every \((i, j) \in \text{INV}_2(\sigma)\).

For example, the permutation \(\sigma = 34251 \in \mathcal{S}_5\) (such that \((\text{maj}_2(\sigma), \widetilde{\text{des}}_2(\sigma), \text{inv}_2(\sigma)) = (6, 3, 2)\)) has the linear graph depicted in Figure 1.
2.2. Planar graph

Let \( \tau \in \mathfrak{S}_n \). The planar graph of \( \tau \) is a graph whose vertices are the integers 1, 2,..., \( n \), organized in ascending and descending slopes (the height of each vertex doesn’t matter) such that the \( i \)-th vertex (from left to right) is the integer \( \tau(i) \), and where every vertex \( \tau(i) \) with \( i \in \text{EXC}(\tau) \) is encircled.

For example, the permutation \( \tau = 32541 \in \mathfrak{S}_5 \) (such that \((\text{maj}(\tau) - \text{exc}(\tau), \text{des}(\tau), \text{exc}(\tau)) = (6, 3, 2)\)) has the planar graph depicted in Figure 2.

3. Definition of the map \( \varphi \) of Theorem 1.2

Let \( \sigma \in \mathfrak{S}_n \). We set \((r, s) = (\text{des}_2(\sigma), \text{inv}_2(\sigma))\), and

\[
\begin{align*}
\text{DES}_2(\sigma) &= \{d_k^2(\sigma), k \in [r]\}, \\
\text{INV}_2(\sigma) &= \{(i_l(\sigma), j_l(\sigma)), l \in [s]\}
\end{align*}
\]

with \( d_k^2(\sigma) < d_{k+1}^2(\sigma) \) for all \( k \) and \( i_l(\sigma) < i_{l+1}(\sigma) \) for all \( l \).

We intend to define \( \varphi(\sigma) \) by constructing its planar graph. To do so, we first construct (in Subsection 3.1) a graph \( G(\sigma) \) made of \( n \) circles or dots organized in ascending or descending slopes such that two consecutive vertices are necessarily in a same descending slope if the first vertex is a circle and the second vertex is a dot. Then, in Subsection 3.2 we label the vertices of this graph with the...
integers 1, 2, . . . , n in such a way that, if y_i is the label of the i-th vertex v_i(σ) (from left to right) of G(σ) for all i ∈ [n], then :

1. y_i < y_{i+1} if and only if v_i and v_{i+1} are in a same ascending slope;
2. y_i > i if and only if v_i is a circle.

The permutation τ = ϕ(σ) will then be defined as y_1y_2 . . . y_n, i.e. the permutation whose planar graph is the labelled graph G(σ).

With precision, we will obtain

τ(EXC(τ)) = {j_k(σ), k ∈ [s]}

(in particular exc(τ) = s = inv_2(σ)), and

DES(τ) = \begin{cases} 
\{d^k(σ), k ∈ [1, r]\} & \text{if des}_2(σ) = r, \\
\{d^k(σ), k ∈ [0, r]\} & \text{if des}_2(σ) = r + 1 
\end{cases}

for integers 0 ≤ d^0(σ) < d^1(σ) < . . . < d^r(σ) ≤ n (with d^0(σ) = 0 ⇔ des_2(σ) = des_2(σ)) defined by

\begin{align*}
d^k(σ) &= d^k_2(σ) + c_k(σ) \\
\end{align*}

(with d^0_2(σ) := 0) where (c_k(σ))_{k ∈ [0, r]} is a sequence defined in Subsection 3.1 such that ∑_k c_k(σ) = inv_2(σ) = exc(τ). Thus, we will obtain des(τ) = des_2(σ) and maj(τ) = maj_2(σ) + exc(τ).

3.1. Construction of the unlabelled graph G(σ)

We set (d^0_2(σ), σ(d^0_2(σ))) = (0, n + 1) and (d^{r+1}_2(σ), σ(n + 1)) = (n, 0).

For all k ∈ [r], we define the top t_k(σ) of the 2-descent d^k_2(σ) as

\begin{align*}
t_k(σ) &= \min\{d^l_2(σ), 1 ≤ l ≤ k, d^l_2(σ) = d^k_2(σ) - (k - l)\}, \quad (4)
\end{align*}

in other words t_k(σ) is the smallest 2-descent d^l_2(σ) such that the 2-descents d^l_2(σ), d^{l+1}_2(σ), . . . , d^k_2(σ) are consecutive integers.

The following algorithm provides a sequence (c^0_k(σ))_{k ∈ [0, r]} of nonnegative integers.
Algorithm 3.1. Let \( I_r(\sigma) = \text{INV}_2(\sigma) \). For \( k \) from \( r = \text{des}_2(\sigma) \) down to 0, we consider the set \( S_k(\sigma) \) of 2-inversions consecutive of length \( k \) maximal and whose elements (as integers). Then, \( \{i_{k_1}(\sigma), i_{k_2}(\sigma), \ldots, i_{k_m}(\sigma)\} \) such that:

1. \((i_{k_p}(\sigma), j_{k_p}(\sigma)) \in I_k(\sigma)\) for all \( p \in [m]; \)
2. \( t_k(\sigma) \leq i_{k_1}(\sigma) < i_{k_2}(\sigma) < \ldots < i_{k_m}(\sigma); \)
3. \( \sigma(i_{k_1}(\sigma)) < \sigma(i_{k_2}(\sigma)) < \ldots < \sigma(i_{k_m}(\sigma)). \)

The length of such a sequence is defined as \( l = \sum_{p=1}^{m} n_p \) where \( n_p \) is the number of consecutive 2-inversions whose beginning is \( i_{k_p} \), i.e. the maximal number \( n_p \) of 2-inversions \((i_{k_1}(\sigma), j_{k_1}(\sigma)), (i_{k_2}(\sigma), j_{k_2}(\sigma)), \ldots, (i_{k_n}(\sigma), j_{k_n}(\sigma))\) such that \( k_1^l = k_p \) and \( j_{k_p}(\sigma) = i_{k_p+1}(\sigma) \) for all \( i \). If \( I_k(\sigma) \neq \emptyset \), we consider the sequence \((i_{k_1}^{\max}(\sigma), i_{k_2}^{\max}(\sigma), \ldots, i_{k_m}^{\max}(\sigma)) \in I_k(\sigma)\) whose length \( l^{\max} = \sum_{p=1}^{m} n_p^{\max} \) is maximal and whose elements \( i_{k_1}^{\max}(\sigma), i_{k_2}^{\max}(\sigma), \ldots, i_{k_m}^{\max}(\sigma) \) also are maximal (as integers). Then,

- if \( I_k(\sigma) \neq \emptyset \), we set \( c^0_k(\sigma) = l^{\max} \) and
  \[
  I_{k-1}(\sigma) = I_k(\sigma) \setminus \{\bigcup_{p=1}^{m} \{(i_{k_1}^{\max}(\sigma), j_{k_1}^{\max}(\sigma)), i \in [n_p^{\max}]\}\};
  \]
- else we set \( c^0_k(\sigma) = 0 \) and \( I_{k-1}(\sigma) = I_k(\sigma) \).

Example 3.2. Consider the permutation \( \sigma = 549321867 \in S_9 \), with \( \text{DES}_2(\sigma) = \{3, 7\} \) and \( I_2(\sigma) = \text{INV}_2(\sigma) = \{(1, 2), (2, 4), (3, 7), (4, 5), (5, 6), (7, 9)\} \). In Figure 3 are depicted the \( \text{des}_2(\sigma) + 1 = 3 \) steps \( k \in \{2, 1, 0\} \) (at each step, the 2-inversions of the maximal sequence are drawed in red then erased at the following step):

![Figure 3: Computation of \((c^0_k(\sigma))_{k \in [0, \text{des}_2(\sigma)]}\) for \( \sigma = 549321867 \in S_9 \).](image_url)
105 \bullet k = 2 : there is only one legit sequence \((i_{k_1}(\sigma)) = (7)\), whose length is 
l = n_1 = 1. We set \(c_2^0(\sigma) = 1\) and \(I_1(\sigma) = I_2(\sigma)\\{\{7, 9\}\}.

110 \bullet k = 1 : there are three legit sequences \((i_{k_1}(\sigma)) = (3)\) (whose length
is \(l = n_1 = 1\)) then \((i_{k_1}(\sigma)) = (4)\) (whose length is \(l = n_1 = 2\)) and
\((i_{k_1}(\sigma)) = (5)\) (whose length is \(l = n_1 = 1\)). The maximal sequence is the
second one, hence we set \(c_1^0(\sigma) = 2\) and \(I_0(\sigma) = I_1(\sigma)\\{(4, 5), (5, 6)\}\).

115 \bullet k = 0 : there are three legit sequences \((i_{k_1}(\sigma), i_{k_2}(\sigma)) = (1, 3)\) (whose
length is \(l = n_1 + n_2 = 2 + 1 = 3\)) then \((i_{k_1}(\sigma), i_{k_2}(\sigma)) = (2, 3)\) (whose
length is \(l = n_1 + n_2 = 2 + 1 = 3\)) and \((i_{k_1}(\sigma)) = (3)\) (whose length is
\(l = n_1 = 1\)). The maximal sequence is the first one, hence we set \(c_0^0(\sigma) = 3\)
and \(I_{-1}(\sigma) = I_0(\sigma)\\{(1, 2), (2, 4), (3, 7)\}\) = \(\emptyset\).

**Lemma 3.3.** The sum \(\sum_k c_k^0(\sigma)\) equals \(\text{inv}_2(\sigma)\) (i.e. \(L_{-1}(\sigma) = \emptyset\) and, for all
\(k \in [0, r] = [0, \text{des}_2(\sigma)]\), we have \(c_k^0(\sigma) \leq d_2^{k+1}(\sigma) - d_2^k(\sigma)\) with equality only if
\(c_{k+1}^0(\sigma) > 0\) (where \(c_{r+1}^0(\sigma)\) is defined as \(0\)).

**Proof.** With precision, we show by induction that, for all \(k \in \{\text{des}_2(\sigma), \ldots, 1, 0\}\),
the set \(I_{k-1}(\sigma)\) contains no 2-inversion \((i, j)\) such that \(d_2^k(\sigma) < i\). For \(k = 0\), it
will mean \(L_{-1}(\sigma) = \emptyset\) (recall that \(d_2^0(\sigma)\) has been defined as \(0\)).

\* If \(k = \text{des}_2(\sigma) = r\), the goal is to prove that \(c_r^0(\sigma) < n - d_2^r(\sigma)\). Suppose
there exists a sequence \((i_{k_1}(\sigma), i_{k_2}(\sigma), \ldots, i_{k_m}(\sigma))\) of length \(c_r^0(\sigma) \geq n - d_2^r(\sigma)\)
with \(t_r(\sigma) \leq i_{k_1}(\sigma) < i_{k_2}(\sigma) < \ldots < i_{k_m}(\sigma)\). In particular, there exist
\(c_r^0(\sigma) \geq n - d_2^r(\sigma)\) 2-inversions \((i, j)\) such that \(d_2^r(\sigma) < j\), which forces \(c_r^0(\sigma)\)
to equal \(n - d_2^r(\sigma)\) and every \(j > d_2^r(\sigma)\) to be the arrival of a 2-inversion \((i, j)\)
such that \(t_r(\sigma) \leq i\). In particular, this is true for \(j = d_2^r(\sigma) + 1\), which is
absurd because \(\sigma(i) \geq \sigma(d_2^r(\sigma)) > \sigma(d_2^r(\sigma) + 1)\) for all \(i \in [t_r(\sigma), d_2^r(\sigma)]\).
Therefore \(c_r^0(\sigma) < n - d_2^r(\sigma)\). Also, it is easy to see that every \(i > d_2^r(\sigma)\)
that is the beginning of a 2-inversion \((i, j)\) necessarily appears in the maximal
sequence \((i_{k^1_{max}}(\sigma), i_{k^2_{max}}(\sigma), \ldots, i_{k^m_{max}}(\sigma))\) whose length defines \(c_r^0(\sigma)\), hence
\((i, j) \notin I_{r-1}(\sigma)\).

\* Now, suppose that \(c_k^0(\sigma) \leq d_2^{k+1}(\sigma) - d_2^k(\sigma)\) for some \(k \in \{\text{des}_2(\sigma)\}\) with
equality only if \( c_{k+1}^0(\sigma) > 0 \), and that no 2-inversion \((i, j)\) with \( d_2^k(\sigma) < i \) belongs to \( I_{k-1}(\sigma) \).

If \( t_{k-1}(\sigma) = t_k(\sigma) \) (i.e., if \( d_2^{k-1}(\sigma) = d_2^k(\sigma) - 1 \)), since \( I_{k-1}(\sigma) \) does not contain any 2-inversion \((i, j)\) with \( d_2^k(\sigma) < i \), then \( c_{k-1}^0(\sigma) \leq 1 = d_2^k(\sigma) - d_2^{k-1}(\sigma) \). Moreover, if \( c_{k-1}^0(\sigma) = 1 \), then there exists a 2-inversion \((i, j)\) in \( I_{k-1}(\sigma) \subset I_k(\sigma) \) such that \( i \in [t_{k-1}(\sigma), d_2^k(\sigma)] \). Consequently \((i)\) was a legit sequence for the computation of \( c_k^0(\sigma) \) at the previous step (because \( t_k(\sigma) = t_{k-1}(\sigma) \)), which implies \( c_k^0(\sigma) \) equals at least the length of \((i)\). In particular \( c_k^0(\sigma) > 0 \).

Else, consider a sequence \((i_{k_1}(\sigma), i_{k_2}(\sigma), \ldots, i_{k_m}(\sigma))\) that fits the three conditions of Algorithm [3.1] at the step \( k - 1 \). In particular \( t_{k-1}(\sigma) \leq i_{k_1}(\sigma) \). Also \( i_{k_m}(\sigma) \leq d_2^k(\sigma) \) by hypothesis. Since \( \sigma(i_{k_p}(\sigma)) < \sigma(i_{k_{p+1}}(\sigma)) \) for all \( p \), and since \( \sigma(t_{k-1}(\sigma)) > \sigma(t_{k-1}(\sigma) + 1) > \ldots > \sigma(d_2^{k-1}(\sigma)) > \sigma(d_2^{k-1}(\sigma) + 1) \), then only one element of the set \([t_{k-1}(\sigma), d_2^{k-1}(\sigma) + 1]\) may equal \( i_{k_p}(\sigma) \) for some \( p \in [m] \).

Thus, the length \( l \) of the sequence verifies \( l \leq d_2^k(\sigma) - d_2^{k-1}(\sigma) \), with equality only if \( i_{k_m}(\sigma) = d_2^k(\sigma) \) (which implies \( c_k^0(\sigma) > 0 \) as in the previous paragraph).

In particular, this is true for \( l = c_{k-1}^0(\sigma) \).

Finally, as for \( k = \text{des}_2(\sigma) \), every \( i \in [d_2^{k-1}(\sigma) + 1, d_2^k(\sigma)] \) that is the beginning of a 2-inversion \((i, j)\) necessarily appears in the maximal sequence \((i_{k_m^{\text{max}}}(\sigma), i_{k_{m-1}^{\text{max}}}(\sigma), \ldots, i_{k_2^{\text{max}}}(\sigma))\) whose length defines \( c_k^0(\sigma) \), hence \((i, j) \notin I_{k-2}(\sigma)\).

So the lemma is true by induction. \(\square\)

**Definition 3.4.** We define a graph \( \mathcal{G}^0(\sigma) \) made of circles and dots organised in ascending or descending slopes, by plotting:

- for all \( k \in [0, r] \), an ascending slope of \( c_k^0(\sigma) \) circles such that the first circle has abscissa \( d_2^k(\sigma) + 1 \) and the last circle has abscissa \( d_2^k(\sigma) + c_k^0(\sigma) \) (if \( c_k^0(\sigma) = 0 \), we plot nothing). All the abscissas are distinct because

\[
d_2^0(\sigma) + c_0 < d_2^1(\sigma) + c_1 < \ldots < d_2^r(\sigma) + c_r
\]

in view of Lemma [3.3]
• dots at the remaining \( n-s = n-\text{inv}_2(\sigma) \) abscissas from 1 to \( n \), in ascending and descending slopes with respect to the descents and ascents of the word \( \omega(\sigma) \) defined by

\[
\omega(\sigma) = \sigma(u_1(\sigma))\sigma(u_2(\sigma)) \ldots \sigma(u_{n-s}(\sigma))
\]

where

\[
\{u_1(\sigma) < u_2(\sigma) < \ldots < u_{n-s}(\sigma)\} = \mathcal{S}_n \setminus \{i_1(\sigma) < i_2(\sigma) < \ldots < i_s(\sigma)\}
\]

**Example 3.5.** The permutation \( \sigma_0 = 425736981 \in \mathcal{S}_9 \) (with \( \text{DES}_2(\sigma_0) = \{1, 4, 8\} \) and \( \text{INV}_2(\sigma_0) = \{(1, 5), (2, 9), (4, 6), (7, 8)\} \)), which yields the sequence \( (c^0_k(\sigma_0))_{k \in [0,3]} = (1, 1, 2, 0) \) (see Figure 4 where all the 2-inversions involved in the computation of a same \( c^0_k(\sigma_0) \) are drawed in a same color) and the word \( \omega(\sigma_0) = 53681 \), provides the unlabelled graph \( \mathcal{G}^0(\sigma_0) \) depicted in Figure 5.

\[\begin{array}{ccccccc}
4 & 2 & 5 & 7 & 3 & 6 & 9 \ 8 & 1
\end{array}\]

Figure 4: \( (c^0_k(\sigma_0))_{k \in [0,3]} = (1, 1, 2, 0) \).

\[\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9
\end{array}\]

Figure 5: Graph \( \mathcal{G}^0(\sigma_0) \).

The following lemma is easy.

**Lemma 3.6.** For all \( i \in [n] \), if the \( i \)-th vertex (from left to right) \( v^0_i(\sigma) \) of \( \mathcal{G}^0(\sigma) \) is a dot and if \( i \) is a descent of \( \mathcal{G}^0(\sigma) \) (i.e., if \( v^0_i(\sigma) \) and \( v^0_{i+1}(\sigma) \) are two dots in a same descending slope) whereas \( i \not\in \text{DES}_2(\sigma) \), let \( k_i \) such that

\[d^0_{k_i}(\sigma) + c^0_{k_i}(\sigma) < i < d^0_{k_i+1}(\sigma)\]
and let \( p \in [n - s] \) such that \( v^0_i(\sigma) \) is the \( p \)-th dot (from left to right) of \( G^0(\sigma) \).

Then:

1. \( u_p(\sigma) \) is the greatest integer \( u < d^1_2(\sigma) \) that is not the beginning of a 2-inversion of \( \sigma \);
2. \( u_{p+1}(\sigma) \) is the smallest integer \( u > d^0_2(\sigma) \) that is not a 2-descent or the beginning of a 2-inversion of \( \sigma \);
3. \( c^0_k(\sigma) > 0 \) for all \( k \) such that \( d^0_k(\sigma) < i < d^0_{k+1}(\sigma) \).

In particular \( c^0_{k+1}(\sigma) > 0 \).

Lemma 3.6 motivates the following definition.

**Definition 3.7.** For \( i \) from 1 to \( n - 1 \), let \( k_i \in [0, r] \) such that

\[
d^0_k(\sigma) + c^0_{k_i}(\sigma) < i < d^0_{k_{i+1}}(\sigma).
\]

If \( i \) fits the conditions of Lemma 3.6, then we define a sequence \((c^i_k(\sigma))_{k \in [0, r]}\) by

\[
\begin{align*}
c^i_{k_i}(\sigma) &= c^{i-1}_{k_i}(\sigma) + 1, \\
c^i_{k_i+1}(\sigma) &= c^{i-1}_{k_i+1}(\sigma) - 1, \\
c^i_k(\sigma) &= c^{i-1}_k(\sigma) \text{ for all } k \notin \{k_i, k_i + 1\}.
\end{align*}
\]

Else, we define \((c^i_k(\sigma))_{k \in [0, r]}\) as \((c^{i-1}_k(\sigma))_{k \in [0, r]}\).

The final sequence \((c^n_k(\sigma))_{k \in [0, r]}\) is denoted by

\[
(c_k(\sigma))_{k \in [0, r]}.
\]

By construction, and from Lemma 3.3, the sequence \((c_k(\sigma))_{k \in [0, r]}\) has the same properties as \((c^0_k(\sigma))_{k \in [0, r]}\) detailed in Lemma 3.3.

Consequently, we may define an unlabelled graph

\[
G(\sigma)
\]

by replacing \((c^0_k(\sigma))_{k \in [0, r]}\) with \((c_k(\sigma))_{k \in [0, r]}\) in Definition 3.4.
Example 3.8. In the graph $G^0(\sigma)$ depicted in Figure 5, where $\sigma_0 = 425736981 \in G_9$, we can see that the dot $v_3^0(\sigma_0)$ is a descent whereas $3 \not\in \text{DES}_2(\sigma_0)$, hence, from the sequence $(c_k(\sigma_0))_{k \in [0,3]} = (1,1,2,0)$, we compute $(c_k(\sigma_0))_{k \in [0,3]} = (1,2,1,0)$ and we obtain the graph $G(\sigma_0)$ depicted in Figure 6.

Let $v_1(\sigma), v_2(\sigma), \ldots, v_n(\sigma)$ be the $n$ vertices of $G(\sigma)$ from left to right.

By construction, the descents of the unlabelled graph $G(\sigma)$ (i.e., the integers $i \in [n-1]$ such that $v_i(\sigma)$ and $v_{i+1}(\sigma)$ are in a same descending slope) are the integers

$$d^k(\sigma) = d^k_2(\sigma) + c_k(\sigma)$$

for all $k \in [0, r]$.

3.2. Labelling of the graph $G(\sigma)$

3.2.1. Labelling of the circles

We intend to label the circles of $G(\sigma)$ with the integers

$$j_1(\sigma), j_2(\sigma), \ldots, j_s(\sigma).$$

Algorithm 3.9. For all $i \in [n]$, if the vertex $v_i(\sigma)$ is a circle (hence $i < n$), we label it first with the set

$$[i+1, n] \cap \{j_1(\sigma), j_2(\sigma), \ldots, j_s(\sigma)\}.$$ 

Afterwards, if a circle $v_i(\sigma)$ is found in a descending slope such that there exists a quantity of $a$ circles above $v_i(\sigma)$, and in an ascending slope such that there exists a quantity of $b$ circles above $v_i(\sigma)$, then we remove the $a+b$ greatest integers from the current label of $v_i(\sigma)$ (this set necessarily had at least $a+b+1$
elements) and the smallest integer from every of the $a + b$ labels of the $a + b$ circles above $v_i(\sigma)$ in the two related slopes. At the end of this step, if an integer $j_k(\sigma)$ appears in only one label of a circle $v_i(\sigma)$, then we replace the label of $v_i(\sigma)$ with $j_k(\sigma)$.

Finally, we replace every label that is still a set by the unique integer it may contain with respect to the order of the elements in the sequence

$$(j_1(\sigma), j_2(\sigma), \ldots, j_s(\sigma))$$

(from left to right).

**Example 3.10.** For $\sigma_0 = 425736981$ (see Figure 4) whose graph $G(\sigma_0)$ is depicted in Figure 6, we have $s = \text{inv}_2(\sigma) = 4$ and $\{j_1(\sigma_0), j_2(\sigma_0), j_3(\sigma_0), j_4(\sigma_0)\} = \{5, 6, 8, 9\}$, which provides first the graph labelled by sets depicted in Figure 7. Afterwards, since the circle $v_2(\sigma_0)$ is in a descending slope with $a = 1$ circle above it (the vertex $v_1(\sigma_0)$) and in an ascending slope with also $b = 1$ circle above it (the vertex $v_3(\sigma_0)$), then we remove the $a + b = 2$ integers 8 and 9 from its label, which becomes $\{5, 6\}$, and we remove 5 from the labels of $v_1(\sigma_0)$ and $v_3(\sigma_0)$. Also, since the label of $v_2(\sigma_0)$ is the only set that contains 5, then we label $v_2(\sigma_0)$ with 5 (see Figure 8). Finally, the sequence $(j_1(\sigma_0), j_2(\sigma_0), j_3(\sigma_0), j_4(\sigma_0)) = (5, 9, 6, 8)$ gives the order (from left to right) of apparition of the remaining integers 6, 8, 9 (see Figure 9).

![Figure 7](image7.png) ![Figure 8](image8.png) ![Figure 9](image9.png)

**3.2.2. Labelling of the dots**

Let

$$\{p_1(\sigma) < p_2(\sigma) < \ldots < p_{n-s}(\sigma)\} = [n] \backslash \bigcup_{k=0}^{r} [d^k_2(\sigma), d^k(\sigma)].$$
We intend to label the dots \( \{v_{p_i}(\sigma), i \in [n-s]\} \) of \( G(\sigma) \) with the elements of
\[ \{1 = e_1(\sigma) < e_2(\sigma) < \ldots < e_{n-s}(\sigma)\} = [n]\setminus \{j_1(\sigma), j_2(\sigma), \ldots, j_s(\sigma)\}. \]

**Algorithm 3.11.**

1. For all \( k \in [n-s] \), we label first the dot \( v_{p_k}(\sigma) \) with the set
\[ [\min(p_k(\sigma), u_k(\sigma))] \cap ([n]\setminus \{j_1(\sigma), j_2(\sigma), \ldots, j_s(\sigma)\}) \]
where \( u_1(\sigma), u_2(\sigma), \ldots, u_{n-s}(\sigma) \) are the integers introduced in [5].

2. Afterwards, similarly as for the labelling of the circles, if a dot \( v_i(\sigma) \) is found in a descending slope such that \( a \) dots are above \( v_i(\sigma) \), and in an ascending slope such that \( b \) dots are above \( v_i(\sigma) \), then we remove the \( a+b \) greatest integers from the current label of \( v_i(\sigma) \) and the smallest integer from every of the \( a+b \) labels of the dots above \( v_i(\sigma) \) in the two related slopes. At the end of this step, if an integer \( l \) appears in only one label of a dot \( v_i(\sigma) \), then we replace the label of \( v_i(\sigma) \) with \( l \).

3. Finally, for \( k \) from 1 to \( n-s \), let
\[ w^k_1(\sigma) < w^k_2(\sigma) < \ldots < w^k_{q_k(\sigma)}(\sigma) \]
such that
\[ \{p_{w^k_i(\sigma)}(\sigma), i\} = \{p_i(\sigma), e_k(\sigma) \text{ appear in the label of } p_i(\sigma)\}, \]
and let \( i(k) \in [q_k(\sigma)] \) such that
\[ \sigma(\{u_{w^k_{i(k)}(\sigma)}(\sigma)\}) = \min\{\sigma(\{u_{w^k_i(\sigma)}(\sigma)\}) \mid i \in [q_k(\sigma)]\}. \]

Then, we replace the label of the dot \( p_{w^k_{i(k)}(\sigma)}(\sigma) \) with the integer \( e_k(\sigma) \) and we erase \( e_k(\sigma) \) from any other label (and if an integer \( l \) appears in only one label of a dot \( v_i(\sigma) \), then we replace the label of \( v_i(\sigma) \) with \( l \)).

**Example 3.12.** For \( \sigma_0 = 425736981 \) whose graph \( G(\sigma_0) \) has its circles labelled in Figure 9, the sequence \((u_1(\sigma_0), u_2(\sigma_0), u_3(\sigma_0), u_4(\sigma_0), u_5(\sigma_0)) = (3, 5, 6, 8, 9)\) provides first the graph labelled by sets depicted in Figure 10. The rest of the algorithm goes from \( k = 1 \) to \( n-s = 9-4 = 5 \).
Figure 10

Figure 11

Figure 12: Labelled graph $G(\sigma_0)$.

• $k = 1$: in Figure 10 the integer $e_1(\sigma_0) = 1$ appears in the labels of the dots $v_{p_1(\sigma_0)}(\sigma_0) = v_4(\sigma_0)$, $v_{p_2(\sigma_0)}(\sigma_0) = v_6(\sigma_0)$ and $v_{p_5(\sigma_0)}(\sigma_0) = v_9(\sigma_0)$, so, from

$$(\sigma_0(u_1(\sigma_0)), \sigma_0(u_2(\sigma_0)), \sigma_0(u_5(\sigma_0))) = (5, 3, 1),$$

we label the dot $v_{p_5(\sigma_0)}(\sigma_0) = v_9(\sigma_0)$ with the integer $e_1(\sigma_0) = 1$ and we erase 1 from any other label, and since the integer 4 now only appears in the label of the dot $v_7(\sigma_0)$, then we label $v_7(\sigma_0)$ with 4 (see Figure 11).

• $k = 2$: in Figure 11 the integer $e_2(\sigma_0) = 2$ appears in the labels of the dots $v_{p_1(\sigma_0)}(\sigma_0) = v_4(\sigma_0)$ and $v_{p_2(\sigma_0)}(\sigma_0) = v_6(\sigma_0)$ so, from

$$(\sigma_0(u_1(\sigma_0)), \sigma_0(u_2(\sigma_0))) = (5, 3),$$

we label the dot $v_{p_2(\sigma_0)}(\sigma_0) = v_6(\sigma_0)$ with the integer $e_2(\sigma_0) = 2$ and we erase 2 from any other label, which provides the graph labelled by integers depicted in Figure 12.

• The three steps $k = 3, 4, 5$ change nothing because every dot of $G(\sigma_0)$ is already labelled by an integer at the end of the previous step. So the final version of the labelled graph $G(\sigma_0)$ is the one depicted in Figure 12.
3.3. Definition of $\varphi(\sigma)$

By construction of the labelled graph $G(\sigma)$, the word $y_1y_2\ldots y_n$ (where the integer $y_i$ is the label of the vertex $v_i(\sigma)$ for all $i$) obviously is a permutation of the set $[n]$, whose planar graph is $G(\sigma)$.

We define $\varphi(\sigma) \in \mathfrak{S}_n$ as this permutation.

For the example $\sigma_0 = 425736981 \in \mathfrak{S}_9$ whose labelled graph $G(\sigma_0)$ is depicted in Figure 12, we obtain $\varphi(\sigma_0) = 956382471 \in \mathfrak{S}_9$.

In general, by construction of $\tau = \varphi(\sigma) \in \mathfrak{S}_n$, we have

$$\tau(\text{EXC}(\tau)) = \{j_k(\sigma), k \in [\text{inv}_2(\sigma)]\}$$

(7)

and

$$\text{DES}(\tau) = \begin{cases} 
\{d^k(\sigma), k \in [1, \text{des}_2(\sigma)]\} & \text{if } c_0(\sigma) = 0 (\leftrightarrow d^0(\sigma) = 0), \\
\{d^k(\sigma), k \in [0, \text{des}_2(\sigma)]\} & \text{otherwise}.
\end{cases}$$

Equality (7) provides

$$\text{exc}(\tau) = \text{inv}_2(\sigma).$$

By $d^k(\sigma) = d^k_2(\sigma) + c_k(\sigma)$ for all $k$, Equality (8) provides

$$\text{maj}(\tau) = \text{maj}_2(\sigma) + \sum_{k \geq 0} c_k(\sigma),$$

and by definition of $(c_k(\sigma))_k$ and Lemma 3.3 we have $\sum_{k \geq 0} c_k(\sigma) = \sum_{k \geq 0} c^0_k(\sigma) = \text{inv}_2(\sigma) = \text{exc}(\tau)$ hence

$$\text{maj}(\tau) - \text{exc}(\tau) = \text{maj}_2(\sigma).$$

Finally, it is easy to see that $\text{des}_2(\sigma) = \text{des}_2(\tau)$ if and only if $c_0(\sigma) = 0$, so Equality (8) also provides

$$\text{des}(\tau) = \text{des}_2(\sigma).$$

As a conclusion, we obtain

$$(\text{maj}(\tau) - \text{exc}(\tau), \text{des}(\tau), \text{exc}(\tau)) = (\text{maj}_2(\sigma), \text{des}_2(\sigma), \text{inv}_2(\sigma))$$

as required by Theorem 1.2.
4. Construction of $\varphi^{-1}$

To end the proof of Theorem 1.2 it remains to show that $\varphi : \mathfrak{S}_n \rightarrow \mathfrak{S}_n$ is surjective. Let $\tau \in \mathfrak{S}_n$. We introduce integers $r \geq 0$, $s = \text{exc}(\tau)$, and

$$0 \leq d^{0,\tau} < d^{1,\tau} < \ldots < d^{r,\tau} < n$$

such that

$$\text{DES}(\tau) = \{d^{k,\tau}, k \in [0, r]\} \cap \mathbb{N}_{>0},$$

$$d^{0,\tau} = 0 \iff \tau(1) = 1.$$  

In particular $\text{des}(\tau) = \begin{cases} 1 & \text{if } \tau(1) = 1, \\
1 + 1 & \text{otherwise.} \end{cases}$

For all $k \in [0, r]$, we define

$$c_k^{\tau} = \text{EXC}(\tau) \cap [d^{k-1,\tau}, d^{k,\tau}] \text{ (with } d^{-1,\tau} := 0),$$

$$d_k^{\tau} = d^{k,\tau} - c_k^{\tau}.$$ We have

$$0 = d_2^{0,\tau} < d_2^{1,\tau} < \ldots < d_2^{r,\tau} < n$$

and similarly as Formula 4 we define

$$t_k^{\tau} = \min\{d_2^{l,\tau}, 1 \leq l \leq k, d_2^{l,\tau} = d_2^{k,\tau} - (k - l)\} \quad (9)$$

for all $k \in [r]$.

We intend to construct a graph $H(\tau)$ which is the linear graph of permutation $\sigma \in \mathfrak{S}_n$ such that $\varphi(\sigma) = \tau$.

4.1. Skeleton of the graph $H(\tau)$

We consider a graph $H(\tau)$ whose vertices $v_1^{\tau}, v_2^{\tau}, \ldots, v_n^{\tau}$ (from left to right) are $n$ dots, aligned in a row, among which we box the $d_2^{k,\tau}$-th vertex $v_{d_2^{k,\tau}}^{\tau}$ for all $k \in [r]$. We also draw the end of an arc of circle above every vertex $v_j^{\tau}$ such that $j = \tau(i)$ for some $i \in \text{EXC}(\tau)$. 17
For the example $\tau_0 = 956382471 \in \mathcal{S}_9$ (whose planar graph is depicted in Figure 12), we have $r = \text{des}(\tau_0) - 1 = 3$ and

$$(c^\tau_k)_{k \in [0,3]} = (1, 2, 1, 0),$$

$$(d^\tau_k)_{k \in [0,3]} = (1 - 1, 3 - 2, 5 - 1, 8 - 0) = (0, 1, 4, 8),$$

$\tau_0(\text{EXC}(\tau_0)) = \{5, 6, 8, 9\},$ and we obtain the graph $H(\tau_0)$ depicted in Figure 13.

![Incomplete graph $H(\tau_0)$.](image)

In general, by definition of $\varphi(\sigma)$ for all $\sigma \in \mathcal{S}_n$, if $\varphi(\sigma) = \tau$, then $r = \text{des}_2(\sigma)$ and $d^2_k(\sigma)$ (respectively $c_k(\sigma), d^k(\sigma), t_k(\sigma)$) equals $d^k_{\tau_2}(\tau)$ (resp. $c^\tau_k, d^{\tau_k}, t^\tau_k$) for all $k \in [0, r]$ and $\{j_i(\sigma), l \in \text{inv}_{2}(\sigma)\} = \tau(\text{EXC}(\tau))$. Consequently, the linear graph of $\sigma$ necessarily have the same skeleton as that of $H(\tau)$.

The following lemma is easy.

**Lemma 4.1.** If $\tau = \varphi(\sigma)$ for some $\sigma \in \mathcal{S}_n$, then :

1. If $j = \tau(l)$ with $l \in \text{EXC}(\tau)$ such that $l \in [d^k_{\tau_2}, d^{k\tau}]$, and if $(i, j) \in \text{INV}_2(\sigma)$, then $t^\tau_i \leq i$.
2. A pair $(i, i + 1)$ cannot be a 2-inversion of $\sigma$ if $i \in \text{DES}_2(\sigma)$ (⇔ if the vertex $v^\tau_i$ of $H(\tau)$ is boxed).
3. For all pair $(l, l') \in \text{EXC}(\tau)^2$, if the labels of the two circles $v_l(\sigma)$ and $v_{l'}(\sigma)$ can be exchanged without modifying the skeleton of $G(\sigma)$, let $i$ and $i'$ such that $(i, l) \in \text{INV}_2(\sigma)$ and $(i', l') \in \text{INV}_2(\sigma)$, then $i < i' \iff l < l'$.

Consequently, in order to construct the linear graph of a permutation $\sigma \in \mathcal{S}_n$ such that $\tau = \varphi(\sigma)$ from $H(\tau)$, it is necessary to extend the arcs of circles of $H(\tau)$ to reflect the three facts of Lemma 4.1. When a vertex is necessarily the beginning of an arc of circle, we draw the beginning of an arc of circle above it.
When there is only one vertex \( v^\tau_i \) that can be the beginning of an arc of circle, we complete the latter by making it start from \( v^\tau_i \).

**Example 4.2.** For \( \tau_0 = 956382471 \in \mathfrak{S}_9 \), the graph \( \mathcal{H}(\tau_0) \) becomes as depicted in Figure 14. Note that the arc of circle ending at \( v^\tau_6 \) cannot begin at \( v^\tau_5 \) because

![Figure 14: Incomplete graph \( \mathcal{H}(\tau_0) \).](image)

otherwise, from the third point of Lemma 4.1 and since \( (6, 8) = (\tau_0(l), \tau_0(l')) \) with \( 3 = l < l' = 5 \), it would force the arc of circle ending at \( v^\tau_8 \) to begin at \( v^\tau_{i'} \) with \( 6 \leq i' \), which is absurd because a permutation \( \sigma \in \mathfrak{S}_9 \) whose linear graph would be of the kind \( \mathcal{H}(\tau_0) \) would have \( c_2(\sigma) = 2 \neq 1 = c_2^\tau_0 \). Also, still in view of the third point of Lemma 4.1 and since \( \tau_0^{-1}(9) < \tau_0^{-1}(6) \), the arc of circle ending at \( v^\tau_9 \) must start before the arc of circle ending at \( v^\tau_6 \), hence the configuration of \( \mathcal{H}(\tau_0) \) in Figure 14.

The following two facts are obvious.

**Facts 4.3.** If \( \tau = \varphi(\sigma) \) for some \( \sigma \in \mathfrak{S}_n \), then:

1. A vertex \( v^\tau_i \) of \( \mathcal{H}(\tau) \) is boxed if and only if \( i \in \text{DES}_2(\sigma) \). In that case, in particular \( i \) is a descent of \( \sigma \).
2. If a pair \((i, i + 1)\) is not a 2-descent of \( \sigma \) and if \( v^\tau_i \) is not boxed, then \( i \) is an ascent of \( \sigma \), i.e. \( \sigma(i) < \sigma(i + 1) \).

To reflect Facts 4.3, we draw an ascending arrow (respectively a descending arrow) between the vertices \( v^\tau_i \) and \( v^\tau_{i+1} \) of \( \mathcal{H}(\tau) \) whenever it is known that \( \sigma(i) < \sigma(i + 1) \) (resp. \( \sigma(i) > \sigma(i + 1) \)) for all \( \sigma \in \mathfrak{S}_n \) such that \( \varphi(\sigma) = \tau \).

For the example \( \tau_0 = 956382471 \in \mathfrak{S}_9 \), the graph \( \mathcal{H}(\tau_0) \) becomes as depicted in Figure 15. Note that it is not known yet if there is an ascending or descending arrow between \( v^\tau_7 \) and \( v^\tau_8 \).
4.2. Completion and labelling of $\mathcal{H}(\tau)$

The following lemma is analogous to the third point of Lemma 4.1 for the dots instead of the circles and follows straightly from the definition of $\varphi(\sigma)$ for all $\sigma \in S_n$.

**Lemma 4.4.** Let $\sigma \in S_n$ such that $\varphi(\sigma) = \tau$. For all pair $(l, l') \in ([n] \setminus \text{EXC}(\tau))^2$,

if the labels of the two dots $v_l(\sigma)$ and $v_{l'}(\sigma)$ can be exchanged without modifying the skeleton of $\mathcal{G}(\sigma)$, let $k$ and $k'$ such that $l = p_k(\sigma)$ and $l' = p_{k'}(\sigma)$, then $\tau(l) < \tau(l') \iff \sigma(u_k(\sigma)) < \sigma(u_{k'}(\sigma))$.

Now, the ascending and descending arrows between the vertices of $\mathcal{H}(\tau)$ introduced earlier, and Lemma 4.4 induce a partial order on the set $\{v^\tau_i, i \in [n]\}$:

**Definition 4.5.** We define a partial order $\succ$ on $\{v^\tau_i, i \in [n]\}$ by:

- $v^\tau_i \prec v^\tau_{i+1}$ (resp. $v^\tau_i \succ v^\tau_{i+1}$) if there exists an ascending (resp. descending) arrow between $v^\tau_i$ and $v^\tau_{i+1}$;
- $v^\tau_i \succ v^\tau_j$ (with $i < j$) if there exists an arc of circle from $v^\tau_i$ to $v^\tau_j$;
- if two vertices $v^\tau_i$ and $v^\tau_j$ are known to be respectively the $k$-th and $k'$-th vertices of $\mathcal{H}(\tau)$ that cannot be the beginning of a complete arc of circle, let $l$ and $l'$ be respectively the $k$-th and $k'$-th non-exceedance point of $\tau$ (from left to right), if $(l, l')$ fits the conditions of Lemma 4.4 then we set $v^\tau_i \prec v^\tau_j$ (resp. $v^\tau_i \succ v^\tau_j$) if $\tau(l) < \tau(l')$ (resp. $\tau(l) > \tau(l')$).

**Example 4.6.** For the example $\tau_0 = 956382471$, according to the first point of Definition 4.5, the arrows of Figure 15 provide

$$v^\tau_1 \prec v^\tau_2 \prec v^\tau_3 \prec v^\tau_4 \succ v^\tau_5 \prec v^\tau_6 \prec v^\tau_7.$$
and
\[ v_8^{r_0} \succ v_9^{r_0}. \]

**Definition 4.7.** A vertex \( v_i^\tau \) of \( H(\tau) \) is said to be minimal on a subset \( S \subset [n] \) if \( v_i^\tau \not\succ v_j^\tau \) for all \( j \in S \).

Let
\[ 1 = e_1^\tau < e_2^\tau < \ldots < e_{n-s}^\tau \]
be the non-exceedance values of \( \tau \) (i.e., the labels of the dots of the planar graph of \( \tau \)).

**Algorithm 4.8.** Let \( S = [n] \) and \( l = 1 \). While the vertices \( \{v_i^\tau, i \in [n]\} \) have not all been labelled with the elements of \([n]\), apply the following algorithm.

1. If there exists a unique minimal vertex \( v_i^\tau \) of \( \tau \) on \( S \), we label it with \( l \), then we set \( l := l + 1 \) and \( S := S \setminus \{v_i^\tau\} \). Afterwards,
   
   (a) If \( v_i^\tau \) is the ending of an arc of circle starting from a vertex \( v_j^\tau \), then we label \( v_j^\tau \) with the integer \( l \) and we set \( l := l + 1 \) and \( S := S \setminus \{v_j^\tau\} \).
   
   (b) If \( v_i^\tau \) is the arrival of an incomplete arc of circle (in particular \( i = \tau(l) \) for some \( l \in \text{EXC}(\tau) \)), we intend to complete the arc by making it start from a vertex \( v_j^\tau \) for some integer \( j \in [t_k^\tau, j[l] \) (where \( l \in [d_k^{r,\tau}, d^{r,\tau}] \)) in view of the first point of Lemma 4.1. We choose \( v_j^\tau \) as the rightest minimal vertex on \([t_k^\tau, j[ \cap S \) from which it may start in view of the third point of Lemma 4.1 and we label this vertex \( v_j^\tau \) with the integer \( l \). Then we set \( l := l + 1 \) and \( S := S \setminus \{v_j^\tau\} \).

   Now, if there exists an arc of circle from \( v_j^\tau \) (for some \( j \)) to \( v_i^\tau \), we apply steps (a),(b) and (c) to the vertex \( v_j^\tau \) in place of \( v_i^\tau \).

2. Otherwise, let \( k \geq 0 \) be the number of vertices \( v_i^\tau \) that have already been labelled and that are not the beginning of arcs of circles. Let
\[ l_1 < l_2 < \ldots < l_q \]
be the integers \( l \in [n] \) such that \( l \geq \tau(l) \geq e_{k+1}^\tau \) and such that we can exchange the labels of dots \( \tau(l) \) and \( e_{k+1}^\tau \) in the planar graph of \( \tau \) without
modifying the skeleton of the graph. It is easy to see that \( q \) is precisely the number of minimal vertices of \( \tau \) on \( S \). Let \( l_{i_{k+1}} = \tau^{-1}(e_{k+1}^\tau) \) and let \( v_j^\tau \) be the \( i_{k+1} \)-th minimal vertex (from left to right) on \( S \). We label \( v_j^\tau \) with \( l \), then we set \( l := l + 1 \) and \( S := S \setminus \{v_j^\tau\} \), and we apply steps 1.(a), (b) and (c) to \( v_j^\tau \) instead of \( v_j^\tau \).

By construction, the labelled graph \( H(\tau) \) is the linear graph of a permutation \( \sigma \in \mathfrak{S}_n \) such that

\[
\text{DES}_2(\sigma) = \{d^\tau_{k+1}, k \in [r]\}
\]

and

\[
\{j_l(\sigma), l \in [\text{inv}_2(\sigma)]\} = \tau(\text{EXC}(\tau)).
\]

**Example 4.9.** Consider \( \tau_0 = 956382471 \in \mathfrak{S}_9 \) whose unlabelled and incomplete graph \( H(\tau_0) \) is depicted in Figure 15.

- As stated in Example 4.6, the minimal vertices of \( \tau_0 \) on \( S = [9] \) are \( (v_2^{\tau_0}, v_5^{\tau_0}, v_9^{\tau_0}) \). Following step 2 of Algorithm 4.8, \( k = 0 \) and the integers \( l \in [9] \) such that \( \tau_0(l) \geq e_{k+1}^{\tau_0} = 1 \) and such that the labels of dots \( \tau_0(l) \) can be exchanged with 1 in the planar graph of \( \tau_0 \) (see Figure 12) are \( (l_1, l_2, l_3) = (4, 6, 9) \). By \( \tau_0^{-1}(1) = 9 = l_3 \), we label the third minimal vertex on \([9]\), i.e. the vertex \( v_9^{\tau_0} \), with the integer \( l = 1 \).

Afterwards, following step 1.(b), since \( v_9^{\tau_0} \) is the arrival of an incomplete arc of circle starting from a vertex \( v_j^{\tau_0} \) with \( 1 = t_j^{\tau_0} \leq j \), and with \( j < 5 \) because that arc of circle must begin before the arc of circle ending at \( v_6^{\tau_0} \) in view of Fact 3 of Lemma 4.1, we complete that arc of circle by making it start from the unique minimal vertex \( v_j^{\tau_0} \) on \([1,5]\), i.e. \( j = 2 \), and we label \( v_2^{\tau_0} \) with the integer \( l = 2 \) (see Figure 16). Note that as from now we know that the arc of circle ending at \( v_5^{\tau_0} \) necessarily begins at \( v_1^{\tau_0} \), because otherwise \( v_1^{\tau_0} \), being the beginning of an arc of circle, would be the beginning of the arc of circle ending at \( v_6^{\tau_0} \), which is absurd in view of Fact 3 of Lemma 4.1 because \( \tau_0^{-1}(9) < \tau^{-1}(6) \), so we complete that arc of circle by making it start from \( v_1^{\tau_0} \), which has been depicted in Figure 16.
We now have $S = [9] \setminus \{2, 9\}$ and $l = 3$.

- From Figure 16, the minimal vertices on $S = [9] \setminus \{2, 9\}$ are $(v_{3}^{\tau_{0}}, v_{5}^{\tau_{0}})$. Following step 2 of Algorithm 4.8, $k = 1$ and the integers $l \in [9]$ such that $l \geq \tau_{0}(l) \geq e_{k+1}^{\tau_{0}} = 2$ and such that the labels of dots $\tau_{0}(l)$ can be exchanged with 2 in the planar graph of $\tau_{0}$ (see Figure 12) are $(l_{1}, l_{2}) = (4, 6)$. By $\tau_{0}^{-1}(2) = 6 = l_{2}$, we label the second minimal vertex on $S$, i.e. the vertex $v_{5}^{\tau_{0}}$, with the integer $l = 3$.

Afterwards, following step 1.(a), since $v_{5}^{\tau_{0}}$ is the arrival of the arc of circle starting from the vertex $v_{1}^{\tau_{0}}$, we label $v_{1}^{\tau_{0}}$ with the integer $l = 4$ (see Figure 17).

We now have $S = [9] \setminus \{1, 2, 5, 9\}$ and $l = 5$.

- From Figure 17, the minimal vertices on $S = \{3, 4, 6, 7, 8\}$ are $(v_{3}^{\tau_{0}}, v_{6}^{\tau_{0}})$. Following step 2 of Algorithm 4.8, $k = 2$ and the integers $l \in [9]$ such that $l \geq \tau_{0}(l) \geq e_{k+1}^{\tau_{0}} = 3$ and such that the labels of dots $\tau_{0}(l)$ can be exchanged with 3 in the planar graph of $\tau_{0}$ (see Figure 12) are $(l_{1}, l_{2}) = (4, 7)$. By $\tau_{0}^{-1}(3) = 4 = l_{1}$, we label the first minimal vertex on $S$, i.e. the vertex $v_{3}^{\tau_{0}}$, with the integer $l = 5$ (see Figure 18). Note that as from now we know that the arc of circle ending at $v_{6}^{\tau_{0}}$ necessarily begins at $v_{4}^{\tau_{0}}$. 

Figure 17: Beginning of the labelling of $H(\tau_{0})$. 

Figure 16: Beginning of the labelling of $H(\tau_{0})$. 

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since it is the only vertex left it may start from. Consequently, the arc of circle ending at \( v_{n}^{\tau_{0}} \) necessarily starts from \( v_{6}^{\tau_{0}} \), which is prevented by Definition 4.5 because we cannot have \( v_{8}^{\tau_{0}} \prec v_{6}^{\tau_{0}} \prec v_{7}^{\tau_{0}} \prec v_{8}^{\tau_{0}} \). The two latter remarks are taken into account in Figure 18.

Figure 18: Beginning of the labelling of \( \mathcal{H}(\tau_{0}) \).

We now have \( S = \{4, 6, 7, 8\} \) and \( l = 6 \).

- From Figure 18 there is only one minimal vertex on \( S = \{4, 6, 7, 8\} \), i.e. the vertex \( v_{6}^{\tau_{0}} \). Following step 1 of Algorithm 4.8 we label \( v_{6}^{\tau_{0}} \) with \( l = 6 \).

Afterwards, following step 1(a), since \( v_{6}^{\tau_{0}} \) is the arrival of the arc of circle starting from the vertex \( v_{4}^{\tau_{0}} \), we label \( v_{4}^{\tau_{0}} \) with the integer \( l = 7 \) (see Figure 19).

Figure 19: Beginning of the labelling of \( \mathcal{H}(\tau_{0}) \).

We now have \( S = \{7, 8\} \) and \( l = 8 \).

- From Figure 19 there is only one minimal vertex on \( S = \{7, 8\} \), i.e. the vertex \( v_{6}^{\tau_{0}} \). Following step 1 of Algorithm 4.8 we label \( v_{6}^{\tau_{0}} \) with \( l = 8 \).

Afterwards, following step 1(a), since \( v_{6}^{\tau_{0}} \) is the arrival of the arc of circle starting from the vertex \( v_{7}^{\tau_{0}} \), we label \( v_{7}^{\tau_{0}} \) with the integer \( l = 9 \) (see Figure 20).
As a conclusion, the graph $H(\tau_0)$ is the linear graph of the permutation $\sigma_0 = 425736981 \in \mathfrak{S}_9$, which is mapped to $\tau_0$ by $\varphi$.

**Proposition 4.10.** We have $\varphi(\sigma) = \tau$, hence $\varphi$ is bijective.

**Proof.** By construction, for all $k \in [0, \text{des}_2(\sigma)] = [0, r]$,

$$d_k^0(\sigma) = d^k_{\tau} - c_k^\tau,$$

$$c_k(\sigma) = c_k^\tau,$$

$$d_k(\sigma) = d_k^0(\sigma) + c_k(\sigma) = d^k_{\tau} + c_k^\tau = d^k_{\tau},$$

so $G(\sigma)$ has the same skeleton as the planar graph of $\tau$, i.e. $\text{DES}(\varphi(\sigma)) = \text{DES}(\tau)$ and $\text{EXC}(\varphi(\sigma)) = \text{EXC}(\tau)$.

The labels of the circles of $G(\sigma)$ are the elements of

$$\{j_l(\sigma), l \in [s]\} = \tau(\text{EXC}(\tau)),$$

and by construction of $\sigma$, every pair $(l, l') \in \text{EXC}(\tau)^2$ such that we can exchange the labels $\tau(l)$ and $\tau(l')$ in the planar graph of $\tau$ is such that

$$i < i' \iff l < l'$$

where $(i, \tau(l))$ and $(i, \tau(l'))$ are the two corresponding 2-inversions of $\sigma$. Consequently, by definition of $\varphi(\sigma)$, the labels of the circles of $G(\sigma)$ appear in the same order as in the planar graph of $\tau$ (i.e. $\varphi(\sigma)(i) = \tau(i)$ for all $i \in \text{EXC}(\varphi(\sigma)) = \text{EXC}(\tau)$).

As a consequence, the dots of $G(\sigma)$ and the planar graph of $\tau$ are labelled by the elements

$$1 = e_1(\sigma) = e_1^\tau < e_2(\sigma) = e_2^\tau < \ldots < e_{n-s}(\sigma) = e_{n-s}^\tau.$$
As for the labels of the circles, to show that the above integers appear in the same order among the labels of \( G(\sigma) \) and the planar graph of \( \tau \), it suffices to prove that
\[
\varphi(\sigma)^{-1}(e^\tau_i) < \varphi(\sigma)^{-1}(e^\tau_j) \iff \tau^{-1}(e^\tau_i) < \tau^{-1}(e^\tau_j)
\]
for all pair \((i, j)\) such that we can exchange the labels \( e^\tau_i \) and \( e^\tau_j \) in the planar graph of \( \tau \) (hence in \( G(\sigma) \) since the two graphs have the same skeleton). This is guaranteed by Definition 4.5 because the vertices \( v^\tau_i \) that are not the beginning of an arc of circle correspond with the labels of the dots of the planar graph of \( \tau \).

As a conclusion, the planar graph of \( \tau \) is in fact \( G(\sigma) \), i.e. \( \tau = \varphi(\sigma) \).

5. Open problem

In view of Formula (2) and Theorem 1.2, it is natural to look for a bijection \( S_n \to S_n \) that maps \((\text{maj}_2, \text{des}_2, \text{inv}_2)\) to \((\text{amaj}_2, \text{asc}_2, \text{ides})\).

Recall that \( \text{ides} = \text{des}_2 \) and that for a permutation \( \sigma \in S_n \), the equality \( \text{des}_2(\sigma) = \text{des}_2(\tau) \) is equivalent to \( \varphi(\sigma)(1) = 1 \), which is similar to the equivalence \( \text{asc}_2(\tau) = \text{asc}_2(\tau) \iff \tau(1) = 1 \) for all \( \tau \in S_n \).

Note that if \( \text{DES}_2(\sigma) = \bigsqcup_{p=1}^r [i_p, j_p] \) with \( j_p + 1 < i_{p+1} \) for all \( p \), the permutation \( \pi = \rho_1 \circ \rho_2 \circ \ldots \circ \rho_r \circ \sigma \), where \( \rho_p \) is the \((j_p - i_p + 2)\)-cycle
\[
\begin{pmatrix}
i_p & i_p + 1 & i_p + 2 & \ldots & j_p & j_p + 1 \\
\sigma(j_p + 1) & \sigma(j_p) & \sigma(j_p - 1) & \ldots & \sigma(i_p + 1) & \sigma(i_p)
\end{pmatrix}
\]
for all \( p \), is such that \( \text{DES}_2(\sigma) \subset \text{ASC}_2(\pi) \) and \( \text{INV}_2(\sigma) = \text{INV}_2(\pi) \). One can try to get rid of the eventual unwanted 2-ascents \( i \in \text{ASC}_2(\pi) \setminus \text{DES}_2(\sigma) \) by composing \( \pi \) with adequate permutations.

References

[Eul55] L. Euler, Institutiones calculi differentialis cum eius usu in analysi finito- rum ac Doctrina serierum, Academiae Imperialis Scientiarum Petropolitanae, St. Petersbourg, 1755.
[HL12] T. Hance and N. Li, An Eulerian permutation statistic and generalizations, (2012), arXiv:1208.3063.

[LZ15] Z. Lin and J. Zeng, The $\gamma$-positivity of basic Eulerian polynomials via group actions, (2015), arXiv:1411.3397.

[Mac15] P. A. MacMahon, Combinatory Analysis, volume 1 and 2, Cambridge Univ. Press, Cambridge, 1915.

[Rio58] J. Riordan, An Introduction to Combinatorial Analysis, J.Wiley, New York, 1958.

[SW14] J. Shareshian and M. L. Wachs, Chromatic quasisymmetric functions, (2014), arXiv:1405.4629.