Fractional Spin for Quantum Hall Effect Quasiparticles

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We investigate the issue of whether quasiparticles in the fractional quantum Hall effect possess a fractional intrinsic spin. The presence of such a spin \( S \) is suggested by the spin-statistics relation \( S = \theta/2\pi \), with \( \theta \) being the statistical angle, and, on a sphere, is required for consistent quantization of one or more quasiparticles. By performing Berry-phase calculations for quasiparticles on a sphere we find that there are two terms, of different origin, that couple to the curvature and can be interpreted as parts of the quasiparticle spin. One, due to self-interaction, has the same value for both the quasihole and quasielectron, and fulfills the spin-statistics relation. The other is a kinematical effect and has opposite signs for the quasihole and quasielectron. The total spin thus agrees with a generalized spin-statistics theorem \( \frac{1}{2}(S_{qh} + S_{qe}) = \theta/2\pi \). On the plane, we do not find any corresponding terms.

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I. INTRODUCTION

The fractional quantum Hall effect (FQHE) reflects the existence of a family of strongly correlated states of two-dimensional electrons in a transverse magnetic field. Perhaps the most striking consequence of the strong correlation in these states is that they support quasiparticle excitations with fractional quantum numbers. Their fractional charge was discovered by Laughlin in his initial work on the problem \[1\] and subsequently Halperin \[2\], in his construction of the hierarchy, pointed out that they are most naturally assigned fractional statistics. These results were subsequently confirmed by Arovas, Schrieffer, and Wilczek \[3\] by a direct Berry-phase analysis.

In this communication, we are concerned with a third fractional quantum number for FQHE quasiparticles — a fractional intrinsic angular momentum or spin. Two remarks are immediately in order. First, we emphasize that the spin of the quasiparticles has, logically, nothing to do with the intrinsic spin \( \frac{1}{2} \) of the electron itself; it arises even for spinless electrons. (Consequently, in most of the subsequent analysis we restrict ourselves to the spinless problem and only take account of the, benign, complications of the true electron spins at the end.) Second, this fractional spin is evidently not an ordinary SU(2) spin, but rather an SO(2) angular momentum perpendicular to the two-dimensional surface the electrons inhabit.

There are two distinct reasons why one might expect to find such a fractional spin for the quasiparticles. The first has to do with a general expectation of a spin-statistics relation for fractional-statistics particles and goes back to Wilczek’s first papers about anyons, which he called fractional-spin particles \[4\]. In model calculations, the spin arises from the self-interaction of the anyon’s own charge and flux-tube and, while its value \( S \) is not quantized, it fulfills the spin-statistics relation \( S = \theta/2\pi \) for anyons with exchange statistics \( \theta \). For general non-relativistic systems there is no proof of this relation, particularly in a background magnetic field which might well introduce extra phases. There is, however, a fairly compelling argument for a generalized spin-statistics relation, which we now sketch.

Consider a cluster of one anyon \( a \) and one antianyon \( \bar{a} \). Since \( a \) and \( \bar{a} \) have opposite charge and opposite flux, their total flux and charge is zero. In the standard case, all the quantum numbers of this cluster are zero. The Berry phase for the rotation of this cluster by an angle of \( 2\pi \) around its center of mass is then unity, but, in the spirit of Thouless and Wu \[5\], it can also be calculated as \( \exp[\pi(-2\theta + 2\pi S_a + 2\pi S_{\bar{a}})] = 1 \), where \(-2\theta\) is the phase for taking the anyon around the antianyon (remember that...
the anyion has opposite “charge” and “flux” with respect to the anyon) and \( S_\alpha \) (\( S_\bar{\alpha} \)) is the anyon (antianyon) spin. For this to be valid, it is necessary that the part of the spin that is even under charge conjugation satisfies

\[
S_{\text{even}} \equiv \frac{1}{2} (S_\alpha + S_{\bar{\alpha}}) = \frac{\theta}{2\pi} \quad (\text{mod} \quad \frac{1}{2}), \quad (1.1)
\]

while the odd part, \( S_{\text{odd}} = \frac{1}{2} (S_\alpha - S_{\bar{\alpha}}) \), is unconstrained. This generalized spin-statistics relation should be valid in all systems with anyons and antianyons, but it is not clear that it is applicable to the FQHE case. Indeed, if we consider a quasihole-quasielectron pair on a plane, and rotate the relative coordinate \((z_\alpha - z_{\bar{\alpha}}) \) by \( 2\pi \), there is no spin contribution at all; the rotation of the cluster is not a rigid operation. One could instead try to rotate the whole system, but here the calculation is plagued by a divergent angular momentum \( \frac{1}{2} \), and there is not, to the best of our knowledge, any convincing argument yielding a spin in agreement with Eq. (1.1).

The second reason for an intrinsic spin involves consistency conditions on a system of fractionally charged anyons on curved and closed surfaces \( \mathcal{C} \). These conditions typically occur because, on a closed surface, one can view either the enclosed region to the left or to the right as the interior of a loop, and the corresponding geometric phases must agree up to a multiple of \( 2\pi \). In general, this leads to relations between the statistics \( \theta \), the spin \( S \), the fractional charge \( e^* \), the total magnetic flux through the surface \( N_\alpha \), and the total curvature of the surface. These types of arguments do apply to FQH systems on curved and closed surfaces. The study of such systems, begun by Haldane in his seminal study of the FQHE on a sphere \( \mathcal{S} \), has been argued by Wen \( \mathcal{W} \) to be a fundamental way of characterizing “topological” fluids of which the Hall fluids are an example.

For our purposes, the interesting questions have to do with the subtleties of quantizing the effective action for a collection of fractionally charged anyons \( \mathcal{A} \) such as the quasiparticles that are present away from the commensurate fillings. As an example, it cannot be true that the total Berry phase for one quasihole being transported around a loop on a sphere is simply the Aharonov-Bohm (AB) phase as it is on the plane. This is because the consistency relations demand that \( N_\phi e^* \) is an integer, which is not the case (see Sec. 11B). In an effective theory describing the quasiholes, there must thus be another geometric phase which together with the AB-phase gives a well-defined overall phase. Such a phase should come out of the calculation of the total Berry phase, which is always well-defined.

In this paper we perform a detailed calculation of the Berry phase for a closed loop for a single quasiparticle on a sphere. For the Laughlin fractions, with filling \( \nu = 1/m \) \((m \text{ odd integer})\), we find that the additional phase can be interpreted as coming from a spin of the quasielectron/quasihole with the value \( S = \pm \frac{1}{2} + 1/2m \). What one sees from our derivation is that the second, charge-conjugation-even, part is due to self-interaction, while the odd part is a kinematical effect due (heuristically) to the coupling of the cyclotron orbits to the curvature \( \mathcal{L} \).

Since the self-interaction part fulfills the standard spin-statistics relation \( S = \theta/2\pi \), the total spin fulfills relation (1.1). This is true also for the hierarchy states. Thus, we find that the general spin-statistics relation (1.1) is valid for FQH quasiparticles on a sphere, even though the cluster-argument for this relation does not apply. As discussed above, on the plane there is no spin or self-interaction contribution.

We emphasize that the above two motivations for a spin are logically independent, at least for non-relativistic systems. This becomes clear in a path-integral description, where the action can include terms that measure the self-linking of the world lines of the particles in \((2+1)\)-dimensional spacetime (corresponding to the first motivation) and terms that measure the curvature enclosed by their projection onto the 2-dimensional space (corresponding to the second motivation) \( \mathcal{L} \).

Our principal conclusions are 1) for quantum Hall systems there are no terms in the quasihole/electron actions proportional to self-linking 2) there are terms proportional to the curvature of the surface they inhabit, and 3) the coefficients of these terms, the spins of the quasiparticles, satisfy the generalized spin-statistics relation (1.1).

There has been previous work on this issue. Sondhi and Kivelson \( \mathcal{K} \) considered planar FQHE systems and investigated spins defined as integrals of angular momentum operators restricted to the region occupied by an isolated quasiparticle at rest but, correctly, failed to find one that satisfied that spin-statistics relation. Our current understanding is that given the lack of dynamical significance for a spin in planar systems (the absence of self-linking) there is no reason to expect a localized fractional angular momentum consistent with the statistics. Einarsson and Girvin \( \mathcal{G} \) considered topological definitions more in the spirit of this paper (albeit on the plane) which is, in a sense, the completion of their program. In their work on effective theories of the FQHE on curved surfaces \( \mathcal{C} \), Wen and Zee focussed on the anti-symmetric part of the quasiparticle spin (their orbital spin) though they also noted that an intrinsic spin ought to be present on general grounds. Li derived the

\[ ^2 \text{The notion of self-linking is somewhat ambiguous in the case of non-relativistic anyon theories. For a discussion of the path-integral formulation of the relativistic case, see, e.g., Ref. 12.} \]
existence of the spin term by demanding that the wavefunctions of hierarchical daughter states be rotationally symmetric \[ \ell = q \] on the sphere. A treatment of the spin terms in the effective theories was then presented by Lee and Wen \cite{15} in the bosonic Chern-Simons approach, but as they inferred the self-interaction piece directly from the Chern-Simons action, which has ultraviolet ambiguities \cite{14}, their derivation cannot be considered microscopic.

The paper is organized as follows. We begin by reviewing the formalism for the sphere and the general constraint on the values of the spin that it imposes. Next we describe the Berry-phase calculation on the sphere using the explicit wavefunction. We also explain why a previously reported attempt at this calculation by Li \cite{14} is incorrect. We then reproduce our results by a different route which generalizes readily to the hierarchy. We conclude with a summary and some remarks on the inclusion of the intrinsic spin of the electrons.

II. FQHE ON THE SPHERE

A. Formalism

We briefly review the formalism for the QHE on the sphere due to Haldane \cite{8,17}, largely because we use a different set of conventions. We consider a system of \( N_e \) electrons on a sphere of radius \( R = 1 \) that encloses a magnetic monopole. The strength of the monopole is quantized according to the Dirac condition which requires that the integrated flux across the sphere, which we take to be positive, \( N_\phi \equiv 2q = 4\pi B \) be an integer number of flux quanta \( \Phi_0 = \hbar c/e \equiv 1 \). The single-particle states form representations of the rotation group \( SU(2) \) with \( l = q, q + 1, \ldots \), where each value of \( l \) occurs precisely once and corresponds to one particular Landau level. The lowest Landau level (LLL) has \( l = q \) and contains \( 2q + 1 = N_\phi + 1 \) degenerate states, one more than for the same flux on the plane or torus.

Explicit wavefunctions require a choice of gauge for the vector potential of the monopole and hence a choice of its Dirac strings. In the QHE literature it is conventional to follow Haldane in choosing

\[
A = -B \frac{\cos \theta}{r \sin \theta} \hat{\phi},
\]

(2.1)

which involves a string through each pole of the sphere. This is inconvenient for our purposes for it produces spurious additions to the Berry phase compared to the planar case. These spurious contributions arise because even the stereographically projected planar vector potential has a singularity at the origin. We therefore revert to a more traditional gauge choice, the “Dirac” gauge,

\[
A = B \frac{1 - \cos \theta}{r \sin \theta} \hat{\phi},
\]

(2.2)

with a single string through the south pole.

We also depart from the standard conventions by requiring that the LLL consist of holomorphic functions. As on the plane, this is accomplished either by reversing the sign of the magnetic field in Eq. (2.2) or equivalently, as in this paper, by considering a system of charge +e electrons (positrons)\([\bar{\psi}]) They then the LLL wavefunctions are conveniently written using the spinorial coordinates \( u = \cos(\theta/2), v = \sin(\theta/2) e^{i\varphi} \) as

\[
\psi_{qm} = \left[ \frac{2q + 1}{4\pi} \left( \frac{2q}{q + m} \right)^{1/2} \right] u^{q-m} v^{q+m},
\]

(2.3)

where \( m \) is the eigenvalue of the z-component of the angular momentum operators,

\[
- L_x = \frac{1}{2} (v \partial_u + u \partial_v), \quad L_y = \frac{i}{2} (v \partial_u - u \partial_v), \quad L_z = \frac{1}{2} (v \partial_v - u \partial_u).
\]

(2.4a)

(2.4b)

(2.4c)

Stereographic projection to the plane from the north pole is accomplished by the map \( (u, v) \rightarrow \pi/v = \tan(\theta/2) e^{i\varphi} \). The rotationally invariant \( (L = 0) \) many-particle wavefunctions,

\[
|m\rangle = \prod_{i<j} (u_i v_j - u_j v_i)^m,
\]

(2.5)

describe the ground state when \( N_\phi = m(N_e - 1) \), and are the generalizations to the sphere of the Laughlin wavefunctions for the fractions \( \nu = 1/m \). The shift in the relation between \( N_e \) and \( N_\phi \) with respect to the planar problem is important in what follows since it gives rise to the odd component of the spin. Finally, the one-quasihole state with the quasihole center given by the spinor coordinates \( (\alpha, \beta) \) is described by

\[
|\alpha, \beta\rangle = \prod_{j=1}^{N_e} (\beta u_j - \alpha v_j) |m\rangle,
\]

(2.6)

where \( N_\phi = m(N_e - 1) + 1 \), i.e., it is created by increasing the total flux by one quantum. These states have \( L = N_e/2 \), consistent with their fractional charge \( 1/3 \), and the angular momentum is maximally polarized along the outward-pointing axis through \( (\alpha, \beta) \). More generally, the flux-number relation for a system with \( N_{qh} \) quasiholes and \( N_{qe} \) quasielectrons is

\[
N_\phi = m(N_e - 1) + N_{qh} - N_{qe}.
\]

(2.7)

\( ^3 \)Since we already need a monopole this does not greatly add to the difficulty of realizing the system experimentally.
III. THE BERRY PHASE ON THE SPHERE

B. Aharonov-Bohm phase for a Quasiparticle

We can now precisely state the problem of consistently quantizing quasiparticle actions on the sphere. When geometric phases are considered on a sphere, there is an ambiguity in what is considered as being inside and outside a closed loop. More specifically, one may calculate two geometric phases $\gamma^I$ and $\gamma^{II}$ by considering the region to the left and the right (with opposite orientation of the curve and thereby an additional sign) as the interior of the loop. These phases must agree up to a multiple of 2$\pi$, i.e.,

$$\gamma(\Omega) = -\gamma(4\pi - \Omega) \pmod{2\pi}. \quad (2.8)$$

Suppose we have integrated out all the electrons so that we have an effective theory of one quasihole in a magnetic field. We now consider the Aharonov-Bohm phases for one charge-$e^*$ object being transported around a loop enclosing the solid angle $\Omega$ to the left, and $4\pi - \Omega$ to the right. These are

$$\gamma_{AB}(\Omega) = 2\pi e^* \Omega e \frac{N_\phi}{4\pi}, \quad (2.9a)$$

$$-\gamma_{AB}(4\pi - \Omega) = \gamma_{AB}(\Omega) - 2\pi e^* N_\phi, \quad (2.9b)$$

where $\frac{e^*}{4\pi} N_\phi$ is the enclosed magnetic flux. From the consistency relation $\gamma_{AB} + \gamma_{geom} = 0$ together with the quasihole charge $e^* = -e/m$, it must be that $N_\phi$ is a multiple of $m$. This does not agree with Eq. $(2.8)$ with $N_{qh} = 1$, $N_{qe} = 0$, for the Haldane-Laughlin wavefunction. Thus, in contrast to the plane, the actual total geometric phase of a one-quasihole system, must differ from the AB phase for one fractionally charged particle. This discrepancy originates in the sphere being curved and closed, as we will see below. What is the additional phase needed to make the total geometric phase well defined? A simple calculation shows that the phase

$$\gamma_{\text{Spin}} = \left(\frac{1}{2m} + \frac{n}{2}\right) \Omega, \quad (2.10)$$

with $n$ being an arbitrary integer, does the job.$^4$

The problem we have just discussed is a consequence of the fractional charge of the quasihole. If one instead studies the case of $m$ quasiholes, there is no problem with the fractional charge and the total flux, but instead the braid relations are inconsistent unless an additional phase is present. This phase should be proportional to the total curvature $\Omega_{\text{tot}} = 4\pi$, and one finds $\gamma_{\text{Spin}} = \left(\frac{1}{2m} + \frac{n}{2}\right) \Omega_{\text{tot}}$, (2.11)

so that both the fractional charge and the fractional statistics need the same spin-phase for consistency.

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To see how the problem posed in the last section is resolved, we will now calculate the Berry phase for transporting one quasihole around a closed loop on the sphere in two different ways. The first is a generalization of the Arovas-Schrieffer-Wilczek (ASW) method on the plane to the sphere. The second uses the mapping of the whole system to a spin-problem and Berry’s original calculation of the phase for twisting a spin around a closed loop. The answers are, of course, identical but the first has the virtue of separating the background and the self-interaction contributions from each other. The latter has the advantage of computational simplicity and confirms the former.

A. ASW calculation

We consider the transport of a quasihole at a fixed latitude on the sphere, $(\alpha, \beta) = (\cos \frac{\theta}{2}, \sin \frac{\pi}{2} e^{i\delta})$ for $t : 0 \rightarrow 2\pi$. The time derivative of the Berry phase for the wavefunction $(2.6)$ is

$$\frac{d\gamma_B}{dt} = i\langle \alpha, \beta | \frac{d}{dt} (\alpha, \beta) \rangle$$

$$= i\langle \alpha, \beta | \frac{d}{dt} \left\{ \sum_{n=1}^{N_e} \ln(\beta u_i - \alpha v_i) \right\} \rangle$$

$$= i\int d\Omega \frac{d}{dt} \ln(\beta u - \alpha v) \langle \alpha, \beta | \sum_{n=1}^{N_e} \delta(\Omega - \Omega_i) \rangle$$

$$= i\int d\Omega \frac{d}{dt} \ln(\beta u - \alpha v) \langle \rho_{\alpha, \beta}(u, v) \rangle. \quad (3.1)$$

The expectation value of the electron density can be expressed as $\langle \rho_{\alpha, \beta}(u, v) \rangle = \rho_0 + \delta \rho_{\alpha, \beta}(u, v)$, where $\rho_0$ is the constant density away from the quasihole and $\delta \rho_{\alpha, \beta}(u, v)$ is the density deviation in the vicinity of the quasihole itself. Because the sphere is a closed surface, the expulsion of $1/m$ electrons from the quasihole area is reflected in an increase in the average electron density, and thus

$$\rho_0 = \frac{N_e + \frac{1}{m}}{4\pi}. \quad (3.2)$$

Also, note that the normalization of $|\alpha, \beta\rangle$ is independent of $(\alpha, \beta)$ for one quasihole, and is fixed by the electron density.

Using the “rotational” symmetry in $t$, we can calculate the Berry phase as

$$\gamma_B = \int_0^{2\pi} dt \frac{d\gamma_B}{dt} = 2\pi \frac{d\gamma_B}{dt} \bigg|_{t=0}. \quad (3.3)$$

The logarithmic derivative in spherical coordinates is
\[
\frac{d}{dt} \ln(\beta u - \alpha v) = \frac{i \sin(\frac{\theta}{2}) \cos(\frac{\phi}{2}) e^{it}}{\sin(\frac{\theta}{2}) \cos(\frac{\phi}{2}) e^{it} - \cos(\frac{\theta}{2}) \sin(\frac{\phi}{2}) e^{it}}. 
\]  
(3.4)

Inserting its value at \( t = 0 \) in (3.4), we find that

\[
\frac{d\gamma_B}{dt} \bigg|_{t=0} = -\int_0^\pi d\theta \sin \theta \int_0^{2\pi} d\phi \frac{1}{1 - \cot(\frac{\theta}{2}) \tan(\frac{\phi}{2})} e^{i\varphi} \times [\rho_0 + \delta \rho_{0,t=0}(\theta, \varphi)]. 
\]  
(3.5)

We first calculate the contribution from the background density \( \rho_0 \). Performing the \( \varphi \)-integral, we obtain

\[
\frac{d\gamma_B^0}{dt} \bigg|_{t=0} = \pi \rho_0 \int_0^\pi d\theta \sin \theta [\text{sign}(\theta - \theta_0) - 1] = -\rho_0 \Omega, 
\]  
(3.6)

where \( \Omega = 2\pi[1 - \cos(\theta_0)] \) is the solid angle enclosed (to the left) by going around the latitude \( \theta_0 \). As on the plane, we therefore have

\[
\gamma_B^0 = -2\pi \langle n_e \rangle \Omega, 
\]  
(3.7)

where \( \langle n_e \rangle \Omega \) denotes the total number of enclosed electrons.

Using the relation \( N_e = (N_\phi - 1)/m + 1 \), we find that

\[
\gamma_B^0 = -2\pi \frac{1}{m} \frac{\Omega}{4\pi} N_\phi - \frac{\Omega}{2}, 
\]  
(3.8)

so that in addition to the Aharonov-Bohm part (2.9a), there is a part proportional to the enclosed curvature \( \Omega \). The curvature part can be traced back to the \(-1\) offset in the electron-flux relation \( m(N_e - 1) = N_\phi \). It is due to the coupling of the electron cyclotron motion to the curvature and contributes \( \frac{1}{2} \) to the total spin, consistent with relation (2.10).

Next we consider the contribution from \( \delta \rho \). On a plane this is known to vanish. This can be seen by repeating the calculation of Arovas et al. for a circular charge distribution centered around the quasihole coordinate \( z_0 \). Translating the center of the coordinate system to \( z_0 \) and employing the circular symmetry gives \( d(\delta \gamma_B)/dt = 0 \) \( \int \). On the sphere, this is not the case. In fact, the transport of the charge distribution couples to the curvature and induces an effective rotation of the quasihole around its center. This is a type of self-interaction where the smeared-out electron density partly goes around the smeared-out vorticity of the quasihole itself. To see how this works, we begin by considering a simple form for the quasihole profile, namely, we assume that \( \delta \rho_{0,t=0}(\theta, \varphi) \) is constant inside a circle of (geodesic) radius \( \beta \). The area is then \( A(\beta) = 2\pi(1 - \cos \beta) \), and to make the total charge equal to \(-1/m \), the density deviation is \( \delta \rho_0 = -1/mA(\beta) \). The second term of Eq. (3.5) then gives

\[
\gamma_B^\delta = \frac{2\pi}{mA(\beta)} \int_{\theta_0 - \beta}^{\theta_0 + \beta} d\theta \sin(\theta) \int_{-\varphi_1(\theta)}^{\varphi_1(\theta)} d\varphi \frac{1}{1 - \cot \frac{\theta}{2} \tan \frac{\varphi}{2} e^{i\varphi}}. 
\]  
(3.9)

The inner integral can be performed analytically, but yields an unwieldy integral in \( \theta \) which we have solved analytically for small \( \beta \) (see the Appendix), and numerically for various values of \( \theta_0 \) and \( \beta \), with \( \beta \leq \theta_0, \pi - \theta_0 \). The answer is in fact independent of \( \beta \) and is given by

\[
\gamma_B^\delta = \frac{\Omega}{2m}. 
\]  
(3.11)

Since the normalized contribution is independent of the radius \( \beta \), the density profile has no influence on the phase change, and (3.9) is valid for any circularly symmetric quasihole with charge \(-1/m \). The self-interaction part of the Berry phase is the sought “spin-statistics” piece of the spin phase (2.10).

A heuristic argument suggests the origin of this term. The basic idea is that on a curved surface, the adiabatic transport of the quasiparticle will effectively induce an adiabatic rotation of the quasiparticle around its center. Assuming ordinary parallel transport to be valid, the transport of the center around a loop enclosing the solid angle (= curvature) \( \Omega \) induces a rotation angle \( \Omega \). The geometric phase accumulated in this process can be used to define a spin \( S \), perpendicular to the surface, by Berry’s relation \([18]\)

\[
\gamma(\Omega) = -S \Omega. 
\]  
(3.12)

Now consider a \( 2\pi \) rotation of a quasiparticle. As a quasiparticle corresponds to a deviation in the charge distribution, there will be a self-interaction as the quasihole is rotated around its center. The outermost parts of the quasihole will move around the total density-rotation, whereas the inner parts only move around a small part of that deviation. If we introduce \( \rho_\perp \), a coordinate denoting the total quasiparticle weight inside a

\footnote{Note that we have integrated over the whole sphere, but that only the density to the left contributes. This is not true in the Haldane gauge \([2,3]\), but the total Berry phases (when including the \( \delta \rho \)-term) are the same in both cases.}

\footnote{There is a sign error in Eq. (10) in the ASW paper.}

\footnote{In fact, the charge distribution does not even have to be circularly symmetric.}
certain radius, we have from Eq. \((3.4)\) that the contribution from a thin cylindrical slice \(dp_\gamma\) of the quasihole is 
\[-2\pi(-\rho)(-d\rho_\gamma/\langle \frac{1}{m}\rangle)\]. The total phase is then obtained as the integral
\[
\gamma(2\pi) = 2\pi m \int_0^{1/m} \rho_- d\rho_- = \pi/m. \tag{3.13}
\]
For a general angle \(\Omega\), we have \(\gamma(\Omega) = \Omega/2m\) in accordance with \((2.10)\), and from \((3.12)\) we have that the self-interaction part of the spin is \(S_{-\uparrow} = -1/2m\).

Collecting terms, we find that the total Berry phase is
\[
\gamma_{\text{B}}^{\text{qh}} = \gamma_{\text{B}}^0 + \gamma_{\text{B}}^\delta = -2\pi \left( \frac{1}{m} \frac{\Omega}{4\pi} N_\phi + \left( \frac{1}{2} + \frac{1}{2m} \right) \Omega \right) \\
\equiv \gamma_{\text{AB}} + \gamma_{\text{Spin}}. \tag{3.14}
\]
From the last expression it is obvious that the quasihole has not only a fractional charge \(e^* = -e/m\), but also a total spin \(S = -\gamma_{\text{Spin}}/\Omega = \frac{1}{2} - 1/2m\), where the first term is due to kinematical effects, or equivalently to the shift in the \(N_c-N_\phi\) relation, and the second term is due to the self-interaction.

For the quasielectron, the charge and the kinematical terms change sign whereas the self-interaction term stays the same. (Note that the statistics and self-interaction of the quasiholes and quasielectrons have the same sign because they involve the charge and vorticity both of which change sign between them.) Consequently, we find,
\[
\gamma_{\text{B}}^{\text{qe}} = 2\pi \left( \frac{1}{m} \frac{\Omega}{4\pi} N_\phi + \left( \frac{1}{2} + \frac{1}{2m} \right) \Omega \right) \equiv \gamma_{\text{AB}} + \gamma_{\text{Spin}}. \tag{3.15}
\]

To summarize: from \((3.14)\) and \((3.17)\) it is clear that the spin of the quasiparticles has a charge-conjugation odd part that arises from the shift in the number-flux relation and an even part, arising from the self-interaction, that fulfills the generalized spin-statistics relation \((4.1)\).

These results agree with the errata of Wen and Zee [11], and with Lee and Wen [13] and also with Li [14]. The latter also attempted to directly compute the Berry phase for quasiparticles on the sphere. Unfortunately, this last agreement is fortuitous because Li incorrectly ignored the effect of the presence of the quasiparticles on the background density and ignored their self-interaction—two errors that happen to cancel each other.

Finally, we digress to consider what happens when one puts another (static) quasihole on the sphere. From the above considerations it should be clear that there are three effects. The first two are an increase of the total flux by one flux quantum and an increase in the background density \(\rho_0\) by \(\frac{1}{2m}\). These effects cancel each other in the sense that the background density expressed in terms of \(N_\phi\) is always \(\rho_0 = \frac{1}{17} (\frac{1}{m} - N_\phi + 1)\). The third effect is a lack of \(1/m\) of an electron, either to the left or to the right of the loop. If it is to the left, there will be a change in the enclosed number of electrons and from \((7)\) one has \(\Delta \gamma_B = 2\pi/m\), while if to the right, \(\Delta \gamma_B = 0\). Hence, \(\theta = S_{\text{even}}/2\pi\) and the generalized spin-statistics theorem \((4.3)\) is satisfied, including signs.

### B. Berry-type calculation

A more straightforward way of calculating the total Berry phase for the transport of a quasihole is to use the analogy with a spinor. As remarked earlier, the state with one quasihole (electron) at \((\alpha, \beta)\) has a total angular momentum \(L = N_c/2\), with a maximal (minimal) projection in the direction given by \((\alpha, \beta)\). We can then use Berry’s original calculation of the geometric phase for a spin being turned around a closed loop \(C\) [18]. If the spinor is always in an eigenstate with an eigenvalue \(S_{||}\) along the direction of the magnetic field and the direction of the field encloses a solid angle \(\Omega\), the geometric phase is given by Eq. \((3.12)\), \(\gamma(\Omega) = -S_{||} \Omega\). In our case, we have \(S_{||} = N_c/2\) for the quasihole, and \(S_{||} = -N_c/2\) for the quasielectron, giving a total geometric phase
\[
\gamma_{\text{B}}^{\text{qh}} = -\left(\frac{N_c}{2}\right) \Omega = -2\pi \left( \frac{1}{m} \frac{\Omega}{4\pi} N_\phi + \left( \frac{1}{2} + \frac{1}{2m} \right) \Omega \right), \tag{3.16a}
\]
\[
\gamma_{\text{B}}^{\text{qe}} = -\left(\frac{-N_c}{2}\right) \Omega = 2\pi \left( \frac{1}{m} \frac{\Omega}{4\pi} N_\phi + \left( \frac{1}{2} + \frac{1}{2m} \right) \Omega \right), \tag{3.16b}
\]
in agreement with our previous calculation. However, the present derivation does not afford much insight into the physics of the different terms. In any case, it should be reassuring for readers worried about our numerical solution of the self-interaction integral \((8.9)\).

### C. Hierarchy states

We can extend our calculation of the spin to states outside the Laughlin sequence. For illustrative purposes we will use the technique of the last section and restrict ourselves to the states in the principal Jain sequences [22]
\[
\nu = \frac{p}{2np + 1}. \tag{3.17}
\]
Following Jain, we view these states as consisting of \(p\) filled Landau levels of “composite fermions”, each constructed from one electron and \(2n\) flux quanta. For our purposes we need two facts. First, the physical system with \(N_c\) electrons and \(N_\phi\) flux quanta is related to a composite fermion system with an equal number \((N_c)\) of

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8 We note that in this interpretation the dynamics on the plane, unlike on the sphere, do not lead to rotations of the quasiparticle.
composite fermions and the remaining $2q^*$ flux quanta by

$$ N_\phi = 2q^* + 2n(N_e - 1); \quad (3.18) $$

their wavefunctions are related by multiplication by the Laughlin-Jastrow factor $\prod_j (z_i - z_j)^{2n}$. Second, the one-quasiparticle states of the physical system can be correctly counted by counting the corresponding states of the composite fermion system.

At $\nu = p/(2np + 1)$, the $p$ Landau levels accommodate

$$ N_e = \sum_{s=1}^{p} 2[q^* + (s - 1)] + 1 = 2pq^* + p^2 \quad (3.19) $$

composite fermions (electrons). The one-quasihole states of the physical system correspond to states with one composite fermion removed from the $p$th Landau level. As there are $2(q^* + p - 1) + 1$ of the latter states, it follows that the one-quasihole states form a multiplet with $L = q^* + p - 1$.

With these pieces of information, it is straightforward to write down the Berry phase. Noting that Eq. (3.18) holds for the quasihole states, which have one fewer electron (i.e., $N_e = 2pq^* + p^2 - 1$), we find

$$ \gamma_{\text{B}}^{\text{qh}} = -L\Omega = -\frac{2\pi}{2np + 1} \frac{\Omega}{4\pi} N_\phi - \left(\frac{2n - np^2}{2np + 1} + p - 1\right) \Omega. \quad (3.20) $$

Similarly, quasielectrons are created by adding one composite fermion to the $(p + 1)$st Landau level and form an $L = q^* + p$ multiplet, which leads to the Berry phase

$$ \gamma_{\text{B}}^{\text{qe}} = +L\Omega = +\frac{2\pi}{2np + 1} \frac{\Omega}{4\pi} N_\phi + \left(\frac{-np^2}{2np + 1} + p\right) \Omega. \quad (3.21) $$

As a check, one can put $p = 1$ and $2n + 1 = m$ to see that the above results agree with the previous results for the Laughlin states. In the general case, we confirm from (3.20) and (3.21) that the quasiparticles have a charge $|e^*/e| = 1/(2np + 1)$, and we conclude that they have an intrinsic spin

$$ S_{\text{even}} = -\frac{1}{2} \frac{2n(p - 1) + 1}{2np + 1}, \quad (3.22) $$

in agreement with the generalized spin-statistics relation.

---

9 We can also consider vacancies in the lower Landau levels which correspond to vortices in the lower condensates in the standard hierarchy construction. However, these are prima facie higher energy excitations and, in fact, do not form a well-defined band in numerical studies.

10 This can be verified, for instance, from the expression for the statistics at a general filling factor given by Su [2]: at $\nu = r/s$, $-\theta/\pi = (n_1^r s - n_2^s s)/s \pmod{2}$, where $n_1 + n_2 s = 1$. For the principal Jain states, $r = p$, $s = 2np + 1$, whence $n_1 = -2n$, $n_2 = 1$ and $\theta/\pi = -2n - 2n(p - 1) + 1)/(2np + 1) \pmod{2}$.

11 As a result the quasiparticle spins are modified but the generalized spin-statistics relation still holds. The monopole polarization can effectively be described by reducing the total flux by unity [2]. For the Haldane-Laughlin states, this gives $N_\phi - 1 = m(N_e - 1) + N_{\text{qh}} - N_{\text{qe}}$, and an additional odd piece to the spin $\pm 1/(2m)$. This is the same result as one would get if one naively assumed that there was a deficiency (excess) of $1/m$ electron spin. Since this part is odd under charge conjugation, it does not violate the generalized spin-statistics relation.

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IV. DISCUSSION

Let us briefly recapitulate our analysis. We began by noting that fractional-statistics particles may in general possess a fractional spin, related to their statistics by the generalized spin-statistics relation, that couples either to the self-linking of their world lines or to the curvature of the surface they inhabit or both. For the quasiparticles in the FQHE we observed that a coupling to the curvature was essential for consistent quantization. We calculated the Berry phase for a single quasiparticle on the sphere and found two terms that couple to the curvature: a term odd under charge conjugation (the orbital spin) that comes from the flux-number shift and an even term (the intrinsic spin) that is due to the self-interaction of the density profile of the quasiparticle and agrees with the spin-statistics relation. On the plane, the orbital spin vanishes trivially, while the self-interaction also vanishes implying that there is no self-linking contribution to the quasiparticle actions. Consequently, we believe, in disagreement with [13], that the quasiparticle spin has no dynamical consequences for planar systems.

In the foregoing we have assumed that we were dealing with spinless electrons, i.e., we ignored their physical SU(2) spin. For QH systems whose ground states and excitations are fully polarized this is perfectly legitimate as the corresponding wavefunctions factorize into an orbital part and a trivial spin part (all spins up) which does not contribute to any of the Berry phases. We remind the reader that in studying QH systems on the sphere, it is standard practice to translate the orbital effects of the transverse magnetic field on the plane into the action of a monopole field while leaving the Zeeman piece of the Hamiltonian intact. The alternative, coupling the electron spins to the monopole field as well, introduces a coupling of the intrinsic electron spin to the curvature which corresponds to a spurious spin-orbit interaction in the planar problem [7].

The simplest non-trivial case involves fully polarized ground states but spin-reversed quasiparticles. Here it is...
not hard to see that the reversed spins contribute to the odd part of the quasiparticle spin without affecting the even part. In the framework of Section 11.1 B, we are interested in the magnitude of $L$ for the quasiparticle states; as the location of the quasiparticle is registered only by the expectation values of $L$, and since $L$ commutes with the electron spin $S_{c}$, the Berry phase for a closed loop is still $-L\Omega$. As $L$ is decreased from its value for a fully polarized quasiparticle by the number of reversed spins $r$ [22], the conclusion follows. For more general cases, where the ground state itself is partially polarized or unpolarized, the spin enters in a more fundamental way into the dynamics, and the analysis is more involved. While we have not investigated these in detail, we believe that our analysis in this paper generalizes straightforwardly.

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APPENDIX: THE SELF-INTERACTION INTEGRAL

In this appendix we derive the result of Eq. (E) in the limiting case of a small quasiparticle. We have that the self-interaction part of the Berry phase is

\[
\gamma_{B}^\delta = \frac{1}{m(1 - \cos \beta)} \int_{-\beta}^{\beta} d\alpha \sin(\theta_{0} + \alpha) \int_{-\varphi_{1}(\alpha)}^{\varphi_{1}(\alpha)} \frac{d\varphi}{1 - B(\alpha) e^{i\varphi}}
\]

\[
= \frac{2}{m(1 - \cos \beta)} \int_{0}^{\beta} d\alpha \left( \sin(\theta_{0} - \alpha) \times \right.
\]

\[
\left\{ \varphi_{1}(-\alpha) + \arctan \left[ \frac{\sin \varphi_{1}(-\alpha)}{B^{-1}(-\alpha) - \cos \varphi_{1}(-\alpha)} \right] \right.
\]

\[- \sin(\theta_{0} + \alpha) \arctan \left[ \frac{\sin \varphi_{1}(\alpha)}{B^{1}(\alpha) - \cos \varphi_{1}(\alpha)} \right] \right),
\]

where

\[
B(\alpha) = \frac{1 + \cot \frac{\theta_{0}}{2} \tan \frac{\alpha}{2}}{1 - \cot \frac{\theta_{0}}{2} \tan \frac{\alpha}{2}},
\]

\[
\cos \varphi_{1} = \frac{\cos \beta - \cos \theta_{0} \cos(\theta_{0} + \alpha)}{\sin \theta_{0} \sin(\theta_{0} + \alpha)}.
\]

(1) to the fact that $|B(\alpha)| < 1$ in the former region, while $|B(\alpha)| > 1$ in the latter region.)

In the limit $\beta \ll 1$, we can expand to obtain

\[
\varphi_{1}(\alpha) = \sqrt{\beta^{2} - \alpha^{2}} \csc(\theta_{0}) \left[ 1 - \frac{1}{2} \alpha \cot(\theta_{0}) \right] + O(\alpha^{4}, \beta^{4}),
\]

and

\[
B(\alpha) = 1 + \alpha \csc \theta_{0} + \frac{1}{2} \alpha^{2} \sec^{2} \frac{\theta_{0}}{2} + O(\alpha^{3}), \quad (A5a)
\]

\[
B^{-1}(-\alpha) = 1 + \alpha \csc \theta_{0} + \frac{1}{2} \alpha^{2} \sec^{2} \frac{\theta_{0}}{2} + O(\alpha^{3}). \quad (A5b)
\]

Note that since $\alpha$ runs from $-\beta$ to $\beta$, an expansion in powers of $\alpha$ is tantamount to an expansion in powers of $\beta$. Thus, for our purposes, $\alpha$ and $\beta$ may be regarded as of the same degree of smallness. From these results, we derive

\[
\sin \varphi_{1}(\alpha) = \frac{\sin \varphi_{1}(-\alpha)}{B(\alpha) - \cos \varphi_{1}(\alpha)} \sin \varphi_{1}(-\alpha) = \left[ 1 - \frac{\beta^{2}}{2\alpha} \csc \theta_{0} + \ldots \right],
\]

which, when inserted into Eq. (A4) together with an evaluation to first order in $\alpha$ and $\beta$ and the change of variables $\alpha = \beta \sin(\omega)$, leave us with elementary integrals yielding $\gamma_{B}^\delta = \Omega/2m$, as claimed in Eq. (3.11).

[1] R. B. Laughlin, Phys. Rev. Lett. 50 (1983) 1395.
[2] B. I. Halperin, Phys. Rev. Lett. 52 (1984) 1583.
[3] D. P. Arovas, J. R. Schrieffer, and F. Wilczek, Phys. Rev. Lett. 53 (1984) 722.
[4] F. Wilczek, Phys. Rev. Lett. 48 (1982) 1144; ibid. 49 (1982) 957.
[5] D. J. Thouless and Y. Wu, Phys. Rev. B 31 (1985) 1191.
[6] S. L. Sondhi and S. A. Kivelson, Phys. Rev. B 46 (1992) 13319.
[7] T. Einarsson, Mod. Phys. Lett. B 5 (1991) 675; D. Li, ibid. 7 (1993) 1103.
[8] F. D. M. Haldane, Phys. Rev. Lett. 51 (1983) 605.
[9] X. G. Wen, Phys. Rev. B 40 (1989) 7387.
[10] A. P. Balachandran, T. Einarsson, T. R. Govindarajan, and R. Ramachandran, Mod. Phys. Lett. A 6 (1991) 2801.
[11] X.-G. Wen and A. Zee, Phys. Rev. Lett. 69 (1992) 953; (E) 3000.
[12] T. H. Hansson, A. Karlhede, and J. Grundberg, Acta Physica Polonica B 22 (1991) 17.
[13] T. Einarsson and S. M. Girvin, Bull. Am. Phys. Soc. 37 (1992) 164.
[14] D. Li, Phys. Rev. A 169 (1992) 82.
[15] D.-H. Lee and X.-G. Wen, Phys. Rev. B 49 (1994) 11066.
[16] A. M. Polyakov, Mod. Phys. Lett. A 3 (1988) 325; E. Witten, Comm. Math. Phys. 121 (1989) 1351.
[17] G. Fano, F. Ortolani, and E. Colombo, Phys. Rev. B 34 (1986) 2670.
[18] M. V. Berry, Proc. Roy. Soc. London Ser. A 392 (1984) 45.
[19] F. D. M. Haldane and E. H. Rezayi, Phys. Rev. Lett. 54 (1985) 237.
[20] F. D. M. Haldane (unpublished); B. Blok and X. G. Wen, Phys. Rev. B 43 (1991) 8337.
[21] T. H. Hansson, M. Sporre, and J. M. Leinaas, Mod. Phys. Lett. A 6 (1991) 45.
[22] For a review of Jain's approach, see, J. K. Jain, Adv. Phys. 41 (1992) 105.
[23] W. P. Su, Phys. Rev. B 34 (1986) 1031.
[24] A. P. Polychronakos, Phys. Lett. B 241 (1990) 37.
[25] E. H. Rezayi, Phys. Rev. B 36 (1987) 5454.