MANY NON-EQUIVALENT REALIZATIONS
OF THE ASSOCIAHEDRON

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Abstract. We show that three systematic construction methods for the \(n\)-dimensional associahedron,
\(\circ\) as the secondary polytope of a convex \((n+3)\)-gon (by Gelfand–Kapranov–Zelevinsky),
\(\circ\) via cluster complexes of the root system \(A_n\) (by Chapoton–Fomin–Zelevinsky), and
\(\circ\) as Minkowski sums of simplices (by Postnikov)
produce substantially different realizations, independent of the choice of the parameters
for the constructions.

The cluster complex and the Minkowski sum realizations were generalized by Hohlweg–Lange to produce exponentially many distinct realizations, all of them with normal vectors in \(\{0, \pm 1\}^n\). We present another, even larger, exponential family, generalizing the cluster complex construction — and verify that this family is again disjoint from the previous ones, with one single exception: The Chapoton–Fomin–Zelevinsky associahedron appears in both exponential families.

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The first author is supported by DFG via the Research Training Group “Methods for Discrete Structures”; the second author is partially supported by the Spanish Ministry of Science through grant MTM2008-04699-C03-02; the third author is partially supported by DFG and by ERC Advanced Grant “SDModels” (agreement no. 247029). We are grateful to Anton Dochtermann, Vincent Pilaud, and in particular Carsten Lange for helpful discussions and comments.
1. Introduction

The $n$-dimensional associahedron is a simple polytope with $C_{n+1}$ (the Catalan number) vertices, corresponding to the triangulations of a convex $(n+3)$-gon, and $n(n+3)/2$ facets, in bijection with the diagonals of the $(n+3)$-gon. It appears in Dov Tamari’s unpublished 1951 thesis [32], and was described as a combinatorial object and realized as a cellular ball by Jim Stasheff in 1963 in his work on the associativity of $H$-spaces [30]. A realization as a polytope by John Milnor from the 1960s is lost; Huguet & Tamari claimed in 1978 that the associahedron can be realized as a convex polytope [18]. The first such construction, via an explicit inequality system, was provided in a manuscript by Mark Haiman from 1984 that remained unpublished, but is available as [15]. The first construction in print, which used stellar subdivisions in order to obtain the dual of the associahedron, is due to Carl Lee, from 1989 [20].

![Figure 1. The 3-dimensional associahedron, realized as the secondary polytope of a regular hexagon.](image)

Subsequently three systematic approaches were developed that produce realizations of the associahedra in more general frameworks and suggest generalizations:

- the associahedron as a secondary polytope due to Gelfand, Kapranov and Zelevinsky [13] [14] (see also [12] Chap. 7),
- the associahedron associated to the cluster complex of type $A_n$, conjectured by Fomin and Zelevinsky [11] and constructed by Chapoton, Fomin and Zelevinsky [6], and
- the associahedron as a Minkowski sum of simplices introduced by Postnikov in [24]. Essentially the same associahedron, but described much differently, had been constructed independently by Shnider and Sternberg [28], (compare Stasheff and Shnider [31] Appendix B), Loday [21], Rote, Santos and Streinu [26], and most recently Buchstaber [5]. Following [16] we reference it as the “Loday realization”, as Loday obtained explicit vertex coordinates that were used subsequently.

The last two approaches were generalized by Hohlweg and Lange [16] and by Santos [27], who showed that they are particular cases of exponentially many constructions of the associahedron. The Hohlweg–Lange construction produces roughly $2^{n-3}$ distinct realizations,
while the Santos construction produces about \( \frac{1}{2(n+3)} C_{n+1} \approx \frac{2^{2n+1}}{\sqrt{n}} \) different ones; exact counts are in Sections 4 and 5. The construction by Santos appears in print for the first time in this paper, so we prove in detail that it actually works. For the others we rely on the original papers for most of the details.

The goal of this paper is to compare the constructions, showing that they produce essentially different realizations for the associahedron. Let us explain what we exactly mean by different (see more details in Section 2). Since the associahedron is simple, its realizations form an open subset in the space of \( (n+3)^n \) tuples of half-spaces in \( \mathbb{R}^n \). Hence, classifying them by affine or projective equivalence does not seem the right thing to do. But most of the constructions of the associahedron (all the ones in this paper except for the secondary polytope construction) happen to have facet normals with very small integer coordinates. This suggests that one natural classification is by linear isomorphism of their normal fans or, as we call it, normal isomorphism.

The secondary polytope construction has a completely different flavor from the others. Coordinates for its vertices are computed from the actual coordinates of the \((n+3)\)-gon used, which can be arbitrary, and a continuous deformation of the polygon produces a continuous deformation of the associahedron obtained. The rest of the constructions are more combinatorial in nature, with no need to give coordinates for the polygon. This is apparent comparing Figures 1 and 2. The first one shows the secondary polytope of a regular hexagon, and the second shows (affine images of) other constructions of the 3-associahedron.

![Figure 2](image-url)

**Figure 2.** Four normally non-isomorphic realizations of the 3-dimensional associahedron. From left to right: The Postnikov associahedron (which is a special case of the Hohlweg–Lange associahedron), the Chapoton-Fomin-Zelevinsky associahedron (a special case of both Hohlweg–Lange and Santos) and the other two Santos associahedra. Since they all have three pairs of parallel facets, we draw them inscribed in a cube.

One way of pinning down this difference (and of testing, for example, whether two associahedra are normally isomorphic) is to look at which parallel facets arise, if any. We start doing this in Section 3 where we show that secondary polytope associahedra never have parallel facets (Theorem 3.5, but see Remark 3.6) while the Chapoton-Fomin-Zelevinsky and the Postnikov ones have \( n \) pairs of parallel facets each (Theorems 3.11 and 3.22).
In Sections 4 and 5 we present the families of realizations by Hohlweg–Lange and by Santos. The first one produces one \( n \)-associahedron for each sequence in \( \{+, -\}^{n-1} \). The second one constructs one \( n \)-associahedron from each triangulation of the \((n + 3)\)-gon. We call them associahedra of types I and II.

Apart of reviewing the two constructions, we show they both provide exponentially-many normally non-isomorphic realizations of the \( n \)-dimensional associahedron with the following common features:

- They all have \( n \) pairs of parallel facets.
- In the basis given by the normals to those \( n \) pairs, all facet normals have coordinates in \( \{0, \pm 1\} \).

For the Santos construction both properties follow from the definition, for Hohlweg–Lange we prove them in Sections 4.2 and 4.3. Put differently, all these constructions are (normally isomorphic to) polytopes obtained from the regular \( n \)-cube by cutting certain \( \binom{n}{2} \) faces according to specified rules; for example, the last example of Figure 2 cannot be obtained by cutting faces lexicographically; the three faces, edges in this case, need to be cut at exactly the same depth.

In Section 6 we put together results from the previous two sections, and show that there is a single associahedron that can be obtained both with the Hohlweg–Lange and the Santos construction, namely the one by Chapoton–Fomin–Zelevinsky.

We also note that Hohlweg–Lange–Thomas [17] provided a generalization of the Hohlweg–Lange construction to general finite Coxeter groups; Bergeron–Hohlweg–Lange–Thomas [2] have provided a classification of the Hohlweg–Lange–Thomas \( c \)-generalized associahedra in Coxeter group theoretic language up to isometry, and also up to normal isomorphism [2, Cor. 2.6]. For type \( A \), this specializes to a classification of the Hohlweg–Lange associahedra, which we obtain in Theorem 4.7 in a different, more combinatorial, setting. Besides the isometries of \( c \)-generalized associahedra presented in [2], normal isomorphisms of these polytopes are discussed earlier by Reading–Speyer [25] in the context of \( c \)-Cambrian fans. In particular, they obtained combinatorial isomorphisms of the normal fans, which are in general only piecewise-linear [25, Thm. 1.1 and Sec. 5].

One of the questions that remains is whether there is a common generalization of the Hohlweg–Lange and the Santos construction, which may perhaps produce even more examples of “combinatorial” associahedra. It has to be noted that the associahedron seems to be quite versatile as a polytope. For example, besides the four \( 3 \)-associahedra of Figure 2 we have found another four \( 3 \)-associahedra that arise by cutting three faces of a \( 3 \)-cube (see Figure 3). Do these admit a natural combinatorial interpretation as well?

2. Some preliminaries

We start by recalling the definition of an \( n \)-dimensional associahedron in terms of polyhedral subdivisions of an \((n + 3)\)-gon.

**Definition 2.1.** Let \( P_{n+3} \) be a convex \((n + 3)\)-gon, whose vertices we label cyclically with the symbols 1 through \( n + 3 \).
An *associahedron* $\text{Ass}_n$ is an $n$-dimensional simple polytope whose poset of non-empty faces is isomorphic to the poset of non-crossing sets of diagonals of $P_{n+3}$, ordered by reverse inclusion.

Equivalently, the poset of non-empty faces of the associahedron is isomorphic to the set of polyhedral subdivisions of $P_{n+3}$ (without new vertices), ordered by coarsening. The minimal elements (vertices of the associahedron) correspond to the *triangulations* of $P_{n+3}$.

For example, for the associahedron of dimension two we look at which diagonals of the pentagon cross each other. There are five diagonals, with five of the $\binom{5}{2}$ pairs of them crossing and the other five non-crossing. Thus, the poset of non-empty faces of the two-dimensional associahedron is isomorphic to the Hasse diagram of Figure 4, in which the five bottom elements correspond to the five triangulations of the pentagon and the top element corresponds to the “trivial” subdivision into a single cell, the pentagon itself.

![Figure 3. More 3-associahedra inscribed in a 3-cube. The 3-associahedron is the only simple 3-polytope with nine facets all of which are quadrilaterals or pentagons.](image)

![Figure 4. The Hasse diagram of the 2-dimensional associahedron.](image)

This is also the Hasse diagram of the poset of non-empty faces of a pentagon, so the 2-dimensional associahedron is a pentagon. Figure 4 shows the associahedra of dimensions 0, 1, and 2.

The goal of this paper is to compare different types of constructions of the associahedron, saying which ones produce equivalent polytopes, in a suitable sense. The following notion reflects the fact that the main constructions that we are going to discuss produce associahedra whose normal vectors have small integer coordinates, usually 0 or ±1. In these
constructions the normal fan of the associahedron can be considered canonical, while there
is still freedom in the right-hand sides of the inequalities. (See [33, Sec. 7.1] for a discussion
of fans and of normal fans.) This leads us to use the following notion of equivalence.

**Definition 2.2.** Two complete fans in real vector spaces $V$ and $V'$ of the same dimension
are **linearly isomorphic** if there is a linear isomorphism $V \rightarrow V'$ sending each cone of one to
a cone of the other. Two polytopes $P$ and $P'$ are **normally isomorphic** if they have linearly
isomorphic normal fans.

Normal isomorphism is weaker than the usual notion of normal equivalence, in which
the two polytopes $P$ and $P'$ are assumed embedded in the same space and their normal
fans are required to be exactly the same, not only linearly isomorphic.

The following lemma is very useful in order to prove (or disprove) that two associahedra
are normally isomorphic. It implies that all linear (or combinatorial, for that matter)
isomorphisms between associahedra come from isomorphisms between the $(n+3)$-gons
defining them.

**Lemma 2.3.** The automorphism group of the face lattice of the associahedron $\text{Ass}_n$ is the
dihedral group of the $(n+3)$-gon: All automorphisms are induced by symmetries of the
$(n+3)$-gon.

**Proof.** Suppose $\varphi$ is an automorphism of the face lattice of the associahedron $\text{Ass}_n$, and
let $D$ be the set of all diagonals of a convex $(n+3)$-gon. $\varphi$ induces a natural bijection

$$\tilde{\varphi} : D \rightarrow D$$

such that for any two diagonals $\delta, \delta' \in D$ we have:

$$\delta \text{ cross } \delta' \iff \tilde{\varphi}(\delta) \text{ cross } \tilde{\varphi}(\delta').$$

For a diagonal $\delta \in D$ denote by $\text{length}(\delta)$ the minimum between the lengths of the two
paths that connect the two end points of $\delta$ on the boundary of the $(n+3)$-gon. Then

$$\text{length}(\delta) = \text{length}(\tilde{\varphi}(\delta)).$$

The reason is that the length of $\delta$ is determined by the number of diagonals that cross $\delta$,
and this property is invariant under the map $\tilde{\varphi}$. 
Figure 6. The situation in the proof of Lemma 2.3.

Let $\delta_0$ be a diagonal of length 2, and $\varphi(\delta_0)$ its image under $\varphi$. The diagonals that cross $\delta_0$ have a common intersection vertex $v_0$; from this vertex we label these diagonals in clockwise direction by $\delta_1, \ldots, \delta_n$. Similarly, the diagonals that cross $\varphi(\delta_0)$ have a common intersection vertex $\varphi(v_0)$, and they are labeled by $\varphi(\delta_1), \ldots, \varphi(\delta_n)$. For any non-empty interval $I \subset [n]$ there is an unique diagonal $\delta_I$ that intersects the diagonal $\delta_i$ if and only if $i \in I$. Applying the map $\varphi$ we obtain diagonals $\varphi(\delta_I)$ that intersect $\varphi(\delta_i)$ if and only if $i \in I$. This task is possible only if the labelings $\varphi(\delta_1), \ldots, \varphi(\delta_n)$ appear in either clockwise or counterclockwise direction. From this, we deduce that $\varphi$ restricted to $\{\delta_1, \ldots, \delta_n\}$ is equivalent to a reflection-rotation map. Moreover, this map coincides with $\varphi$ for all other diagonals $\delta_I$. \qed

3. Three realizations of the associahedron

3.1. The Gelfand–Kapranov–Zelevinsky associahedron. The secondary polytope is an ingenious construction motivated by the theory of hypergeometric functions as developed by I.M. Gelfand, M. Kapranov and A. Zelevinsky [12]. In this section we recall the basic definitions and main results related to this topic, which yield in particular that the secondary polytope of any convex $(n+3)$-gon is an $n$-dimensional associahedron. For more detailed presentations we refer to [7, Sec. 5] and [33, Lect. 9]. All the subdivisions and triangulations of polytopes that appear in the following are understood to be without new vertices.

The secondary polytope construction.

**Definition 3.1 (GKZ vector/secondary polytope).** Let $Q$ be a $d$-dimensional convex polytope with $n + d + 1$ vertices. The **GKZ vector** $v(t) \in \mathbb{R}^{n+d+1}$ of a triangulation $t$ of $Q$ is

$$v(t) := \sum_{i=1}^{n+d+1} \text{vol}(\text{star}_t(i))e_i = \sum_{i=1}^{n+d+1} \sum_{\sigma \subseteq t : i \in \sigma} \text{vol}(\sigma)e_i$$

The **secondary polytope** of $Q$ is defined as

$$\Sigma(Q) := \text{conv}\{v(t) : t \text{ is a triangulation of } Q\}.$$
Theorem 3.2 (Gelfand–Kapranov–Zelevinsky [13]). Let $Q$ be a $d$-dimensional convex polytope with $m = n + d + 1$ vertices. Then the secondary polytope $\Sigma(Q)$ has the following properties:

(i) $\Sigma(Q)$ is an $n$-dimensional polytope.

(ii) The vertices of $\Sigma(Q)$ are in bijection with the regular triangulations of $Q$.

(iii) The faces of $\Sigma(Q)$ are in bijection with the regular subdivisions of $Q$.

(iv) The face lattice of $\Sigma(Q)$ is isomorphic to the lattice of regular subdivisions of $Q$, ordered by refinement.

The associahedron as the secondary polytope of a convex $(n+3)$-gon.

Definition 3.3. The Gelfand–Kapranov–Zelevinsky associahedron $\text{GKZ}_n(Q) \subset \mathbb{R}^{n+3}$ is defined as the $(n$-dimensional) secondary polytope of a convex $(n+3)$-gon $Q \subset \mathbb{R}^2$:

$\text{GKZ}_n(Q) := \Sigma(Q)$.

Corollary 3.4 ([13]). $\text{GKZ}_n(Q)$ is an $n$-dimensional associahedron.

There is one feature that distinguishes the associahedron as a secondary polytope from all the other constructions that we mention in this paper: the absence of parallel facets. This property, in particular, will imply that the GKZ–associahedra are not normally isomorphic to the associahedra produced by the other constructions:

Theorem 3.5. Let $Q$ be a convex $(n+3)$-gon. Then $\text{GKZ}_n(Q)$ has no parallel facets for $n \geq 2$.

Our proof is based on the understanding of the facet normals in secondary polytopes. Let $Q$ be an arbitrary $d$-polytope with $n+d+1$ vertices $\{q_1, \ldots, q_{n+d+1}\}$, so that $\text{GKZ}_n(Q)$ lives in $\mathbb{R}^{n+d+1}$, although it has dimension $n$. In the theory of secondary polytopes one thinks of each linear functional $\mathbb{R}^{n+d+1} \to \mathbb{R}$ as a function $\omega : \text{vertices}(Q) \to \mathbb{R}$ assigning a value $\omega(q_i)$ to each vertex $q_i$. In turn, to each triangulation $t$ of $Q$ (with no additional vertices) and any such $\omega$ one associates the function $g_{\omega,t}: Q \to \mathbb{R}$ which takes the value $\omega(q_i)$ at each $q_i$ and is affine linear on each simplex of $t$. That is, we use $t$ to piecewise linearly interpolate a function whose values $(\omega(q_1), \ldots, \omega(q_n))$ we know on the vertices of $Q$. The main result we need is the following equality for every $\omega$ and every triangulation $t$ (see, e.g., [7, Thm. 5.2.16]):

$$\langle \omega, v(t) \rangle = (d+1) \int_Q g_{\omega,t}(x)dx.$$  

In particular:

- If $\omega$ is affine-linear (that is, if the points $\{(q_1, \omega_1), \ldots, (q_{n+d+1}, \omega_{n+d+1})\} \subset \mathbb{R}^{n+d+1} \times \mathbb{R}$ lie in a hyperplane) then $\langle \omega, v(t) \rangle$ is the same for all $t$. Moreover, the converse is also true: The affine-linear $\omega$'s form the lineality space of the normal fan of $\text{GKZ}_n(Q)$.

- An $\omega$ lies in the linear cone of the (inner) normal fan of $\text{GKZ}_n(Q)$ corresponding to a certain triangulation $t$ (that is, $\langle \omega, v(t) \rangle \leq \langle \omega, v(t') \rangle$ for every other triangulation $t'$) if and only if the function $g_{\omega,t}$ is convex; that is to say, if its graph is a convex hypersurface.
Proof of Theorem 3.5. With the previous description in mind we can identify the facet normals of the secondary polytope of a polygon \( Q \). For this we use the correspondence:

\[
\text{vertices} \leftrightarrow \text{triangulations of } Q \\
\text{facets} \leftrightarrow \text{diagonals of } Q
\]

For a given diagonal \( \delta \) of \( Q \), denote by \( F_\delta \) the facet of \( \text{GKZ}_n(Q) \) corresponding to \( \delta \). The vector normal to \( F_\delta \) is not unique, since adding to any vector normal to \( F_\delta \) an affine-linear \( \omega_0 \) we get another one. One natural choice is

\[
\omega_\delta(q_i) := \text{dist}(q_i, l_\delta),
\]

where \( l_\delta \) is the line containing \( \delta \) and \( \text{dist}(\cdot, \cdot) \) is the Euclidean distance. Indeed, \( \omega_\delta \) lifts the vertices of \( Q \) on the same side of \( \delta \) to lie in a half-plane in \( \mathbb{R}^3 \), with both half-planes having \( \delta \) as their common intersection. That is, \( g_{\omega,t} \) is convex for every \( t \) that uses \( \delta \). But another choice of normal vector is better for our purposes: choose one side of \( l_\delta \) to be called positive and take

\[
\omega^+(\delta)(q_i) := \begin{cases} 
\text{dist}(q_i, l_\delta) & \text{if } q_i \in l^+_\delta \\
0 & \text{if } q_i \in l^-_\delta
\end{cases}.
\]

For the end-points of \( \delta \), which lie in both \( l^+_\delta \) and \( l^-_\delta \), there is no ambiguity since both definitions give the value 0. Again, \( \omega^+_{\delta} \) is a normal vector to \( F_\delta \) since it lifts points on either side of \( l_\delta \) to lie in a plane.

We are now ready to prove the theorem. If two diagonals \( \delta \) and \( \delta' \) of \( Q \) do not cross, then they can simultaneously be used in a triangulation. Hence, the corresponding facets \( F_\delta \) and \( F_{\delta'} \) meet, and they cannot be parallel. So, assume in what follows that \( \delta \) and \( \delta' \) are two crossing diagonals. Let \( \delta = qr \) and \( \delta' = qs \), with \( pqr \) being cyclically ordered along \( Q \). Since \( n \geq 2 \) there is at least another vertex \( a \) in \( Q \). Without loss of generality suppose \( a \) lies between \( s \) and \( p \). Now, we call negative the side of \( l_\delta \) and the side of \( l_{\delta'} \) containing \( a \), and consider the normal vectors \( \omega^+_{\delta} \) and \( \omega^+_{\delta'} \) as defined above. They take the following values on the five points of interest:

\[
\omega^+_{\delta}(a) = 0, \quad \omega^+_{\delta}(p) = 0, \quad \omega^+_{\delta}(q) > 0, \quad \omega^+_{\delta}(r) = 0, \quad \omega^+_{\delta}(s) = 0,
\]

\[
\omega^+_{\delta'}(a) = 0, \quad \omega^+_{\delta'}(p) = 0, \quad \omega^+_{\delta'}(q) = 0, \quad \omega^+_{\delta'}(r) > 0, \quad \omega^+_{\delta'}(s) = 0.
\]

Suppose that \( F_\delta \) and \( F_{\delta'} \) were parallel. This would imply that \( \delta \) and \( \delta' \) are linearly dependent or, more precisely, that there is a linear combination of them that gives an affine-linear \( \omega \) (in the lineality space of the normal fan). But any (non-trivial) linear combination \( \omega \) of \( \omega^+_{\delta} \) and \( \omega^+_{\delta'} \) necessarily takes the following values on our five points, which implies that \( \omega \) is not affine-linear:

\[
\omega(a) = 0, \quad \omega(p) = 0, \quad \omega(q) \neq 0, \quad \omega(r) \neq 0, \quad \omega(s) = 0. \quad \square
\]

Remark 3.6. The secondary polytope can be defined for any set of points \( \{q_1, \ldots, q_{n+3}\} \) in the plane, not necessarily the vertices of a convex polygon. In general this does not produce an associahedron, but there is a case in which it does: if the points are cyclically placed on the boundary of an \( m \)-gon with \( m \leq n + 3 \) in such a way that no four of
them lie on a boundary edge. By the arguments in the proof above, a necessary condition for the associahedron obtained to have parallel facets is that \( m \leq 4 \). For \( m = 4 \) we can obtain associahedra up to dimension 4 with exactly one pair of parallel facets (those corresponding to the main diagonals of the quadrilateral). For \( m = 3 \), we can obtain 2-dimensional associahedra with two pairs of parallel facets, and 3-dimensional associahedra with three pairs of parallel facets. The latter is obtained for six points \( \{p, q, r, a, b, c\} \) with \( p, q \) and \( r \) being the vertices of a triangle and \( a \in pq, b \in qr \) and \( c \in ps \) intermediate points in the three sides. The associahedron obtained has the following three pairs of parallel facets:

\[
F_{pq} \parallel F_{ar}, \quad F_{qr} \parallel F_{bs}, \quad F_{ps} \parallel F_{cq}.
\]

**Remark 3.7.** Rote, Santos and Streinu [26] introduce a polytope of pseudo-triangulations associated to each finite set \( A \) of \( m \) points (in general position) in the plane. This polytope lives in \( \mathbb{R}^{2m} \) and has dimension \( m+3+i \), where \( i \) is the number of points interior to \( \text{conv}(A) \). They show that for points in convex position their polytope is affinely isomorphic to the secondary polytope for the same point set. Their constructions use rigidity theoretic ideas: the edge-direction joining two neighboring triangulations \( t \) and \( t' \) is the vector of velocities of the (unique, modulo translation and rotation) infinitesimal flex of the embedded graph of \( t \cap t' \).

### 3.2. The Postnikov associahedron.

We now review two further realizations of the associahedron: one by Postnikov [24] and one by Rote–Santos–Streinu [26] (different from the one in Remark 3.7). The main goal of this section is to prove that these two constructions produce affinely equivalent results. As special cases of these constructions one obtains, respectively, the realizations by Loday [21] and Buchstaber [5], which turn out to be affinely equivalent as well.

#### 3.2.1. The Postnikov associahedron.

**Definition 3.8.** For any vector \( a = \{a_{ij} > 0 : 1 \leq i \leq j \leq n+1\} \) of positive parameters we define the **Postnikov associahedron** as the polytope

\[
\text{Post}_n(a) := \sum_{1 \leq i \leq j \leq n+1} a_{ij} \Delta_{[i,...,j]},
\]

where \( \Delta_{[i,...,j]} \) denotes the simplex \( \text{conv}\{e_i, e_{i+1}, \ldots, e_j\} \) in \( \mathbb{R}^{n+1} \).

**Proposition 3.9** (Postnikov [24, Sec. 8.2]). \( \text{Post}_n(a) \) is an \( n \)-dimensional associahedron. In particular, for \( a_{ij} \equiv 1 \) this yields the realization of Loday [21].

In terms of inequalities the Postnikov associahedron is given as follows.

**Lemma 3.10.**

\[
\text{Post}_n(a) = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{p<i<q} x_i \geq f_{p,q} \quad \text{for} \ 0 \leq p < q \leq n+2, \quad x_1 + \cdots + x_{n+1} = f_{0,n+2}\},
\]

where \( f_{p,q} = \sum_{p<i<j<q} a_{i,j} \).
The facet of $\text{Post}_n(a)$ determined by the hyperplane with right hand side parameter $f_{p,q}$ corresponds to the diagonal $pq$ of an $(n+3)$-gon with vertices labeled in counterclockwise direction from 0 to $n+2$. In particular:

**Theorem 3.11.** $\text{Post}_n(a)$ has exactly $n$ pairs of parallel facets. These correspond to the pairs of diagonals $(\{0,i+1\}, \{i,n+2\})$ for $1 \leq i \leq n$, as illustrated in Figure 8.

**Proof.** Two hyperplanes of the form $\sum_{i \in S_1} x_i \geq c_1$ and $\sum_{i \in S_2} x_i \geq c_2$ for $S_1, S_2 \subseteq [n+1]$, intersected with an affine hyperplane $x_1 + \cdots + x_{n+1} = c$ are parallel if and only if $S_1 \cup S_2 = [n+1]$ and $S_1 \cap S_2 = \emptyset$. Therefore two diagonals $pq$ and $rs$ correspond to parallel facets if and only if $pq = \{0,i+1\}$ and $qr = \{i,n+2\}$. □

**Figure 8.** Diagonals of the $(n+3)$-gon that correspond to the pairs of parallel facets of both $\text{Post}_n(a)$ and $\text{RSS}_n(g)$.

### 3.2.2. The Rote–Santos–Streinu associahedron

By “generalizing” the construction of Remark 3.7 to sets of points along a line, Rote, Santos and Streinu [26] obtain a second realization of the associahedron.

**Definition 3.12.** The *Rote–Santos–Streinu associahedron* is the polytope

$$\text{RSS}_n(g) = \{(y_0, y_1, \ldots, y_{n+1}) \in \mathbb{R}^{n+2} : y_j - y_i \geq g_{i,j} \text{ for } j > i, \ y_0 = 0, \ y_{n+1} = g_{0,n+1}\}.$$
where \( g = (g_{i,j})_{0 \leq i < j \leq n+1} \) is any vector with real coordinates satisfying

\[
\begin{align*}
g_{i,l} + g_{j,k} &> g_{i,k} + g_{j,l} \quad \text{for all } i < j \leq k < l, \\
g_{i,l} &> g_{i,k} + g_{k,l} \quad \text{for all } i < k < l.
\end{align*}
\]

**Proposition 3.13** (Rote–Santos–Streinu [26, Sec. 5.3]). *If the vector \( g \) satisfies the previous inequalities then \( \text{RSS}_n(g) \) is an \( n \)-dimensional associahedron.*

A particular example of valid parameters \( g \) is given by \( g_0: g_{i,j} = i(i-j) \). In this case we get the realization of the associahedron introduced by Buchstaber in [5, Lect. II Sec. 5].

![Diagram](image-url)

**Figure 9.** The Rote–Santos–Streinu associahedron \( \text{RSS}_2(g_0) \) with the coordinates of the vertices. This coincides with the realization of Buchstaber.

The facet of \( \text{RSS}_n(g) \) related to \( y_j - y_i \geq g_{i,j} \) corresponds to the diagonal \( \{i, j + 1\} \) of an \((n+3)\)-gon with vertices labeled in counterclockwise direction from 0 to \( n + 2 \). One can also see that with this specified combinatorics of the facets, the conditions on the vector \( g \) are also necessary for the proposition to hold.

**Theorem 3.14.** \( \text{RSS}_n(g) \) has exactly \( n \) pairs of parallel facets. They correspond to the pairs of diagonals \( \{(0, i+1), (i, n+2)\} \) for \( 1 \leq i \leq n \), as illustrated in Figure 8.

Rote, Santos and Streinu stated in [26, Sec. 5.3] that \( \text{RSS}_n(g) \) is not affinely equivalent to neither the associahedron as a secondary polytope nor the associahedron from the cluster complex of type A. Next we prove that \( \text{RSS}_n(g) \) is affinely isomorphic to \( \text{Post}_n(a) \). Furthermore, we prove, in Corollary 4.8 and Theorem 6.1, that these two polytopes are not normally isomorphic to the associahedron as a secondary polytope or the associahedron from the cluster complex of type A.

### 3.2.3. Affine equivalence.

**Theorem 3.15.** Let \( \varphi \) be the affine transformation

\[
\varphi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n, \quad (x_1, \ldots, x_{n+1}) \rightarrow (y_1, \ldots, y_n)
\]

defined by \( y_k = \sum_{i=1}^k (x_i - i) \). Then \( \varphi \) maps \( \text{Post}_n(a) \) bijectively to \( \text{RSS}_n(g) \), for \( g \) given by \( g_{i,j} = \frac{(i+j+1)(j-i)}{2} = f_{i,j+1}(a) \). In particular, \( \varphi \) maps the Loday associahedron \( \text{Post}_n(1) \) to the Buchstaber associahedron \( \text{RSS}_n(g_0) \).
Proof.

\[
y_j - y_i \geq g_{i,j} (x_i + 1 + \cdots + x_j) + ((i + 1) + \cdots + j) \geq g_{i,j} x_i + 1 + \cdots + x_j \geq g_{i,j} - \frac{(i+j+1)(j-i)}{2}.
\]

\[\square\]

**Corollary 3.16** (Minkowski sum decomposition of \(\text{RSS}_n(g)\)). Every Rote–Santos–Streinu associahedron can be written as

\[\text{RSS}_n(g) = \sum_{1 \leq i \leq j \leq n} b_{i,j} \tilde{\Delta}_{i,j},\]

for certain \((b_{i,j})\) with \(b_{i,j} > 0\) whenever \(i < j\), and \(b_{i,i}\) possibly negative. Here \(\tilde{\Delta}_{i,j} = \text{conv}\{u_i, u_{i+1}, \ldots, u_j\}\) and \(u_i = (0, \ldots, 0, 1, \ldots, 1) \in \mathbb{R}^n\) is a 0/1-vector with \(i\) zeros.

3.3. The Chapoton–Fomin–Zelevinsky associahedron.

3.3.1. The associahedron associated to a cluster complex. Cluster complexes are combinatorial objects that arose in the theory of cluster algebras [9] [10] initiated by Fomin and Zelevinsky. They correspond to the normal fans of polytopes known as generalized associahedra because the particular case of type \(A_n\) yields to the classical associahedron. This polytope was constructed by Chapoton, Fomin and Zelevinsky in [6]. We refer to [11], [8] and [6] for more detailed presentations.

3.3.2. The cluster complex of type \(A_n\). The root system of type \(A_n\) is the set \(\Phi := \Phi(A_n) = \{e_i - e_j, 1 \leq i \neq j \leq n + 1\} \subset \mathbb{R}^{n+1}\). The simple roots of type \(A_n\) are the elements of the set \(\Pi = \{\alpha_i = e_i - e_{i+1}, i \in [n]\}\), the set of positive roots is \(\Phi_0 = \{e_i - e_j : i < j\}\), and the set of almost positive roots is \(\Phi_{\geq -1} := \Phi_0 \cup -\Pi\).

There is a natural correspondence between the set \(\Phi_{\geq -1}\) and the diagonals of the \((n+3)\)-gon \(P_{n+3}\): We identify the negative simple roots \(-\alpha_i\) with the diagonals on the snake of \(P_{n+3}\) illustrated in Figure 10.

![Figure 10. Snake and negative roots of type \(A_n\).](image)

Each positive root is a consecutive sum

\[\alpha_{ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j, \quad 1 \leq i \leq j \leq n,\]

and thus is identified with the unique diagonal of \(P_{n+3}\) crossing the (consecutive) diagonals that correspond to \(-\alpha_i, -\alpha_{i+1}, \ldots, -\alpha_j\).
Definition 3.17 (Cluster complex of type $A_n$). Two roots $\alpha$ and $\beta$ in $\Phi_{\geq -1}$ are compatible if their corresponding diagonals do not cross. The cluster complex $\Delta(\Phi)$ of type $A_n$ is the clique complex of the compatibility relation on $\Phi_{\geq -1}$, i.e., the complex whose simplices correspond to the sets of almost positive roots that are pairwise compatible. Maximal simplices of $\Delta(\Phi)$ are called clusters.

In this case, the cluster complex satisfies the following correspondence, which is dual to the complex of the associahedron:

- Vertices $\longleftrightarrow$ diagonals of a convex $(n + 3)$-gon
- Simplices $\longleftrightarrow$ polyhedral subdivisions of the $(n + 3)$-gon (viewed as collections of non-crossing diagonals)
- Maximal simplices $\longleftrightarrow$ triangulations of the $(n + 3)$-gon (viewed as collections of $n$ non-crossing diagonals)

Theorem 3.18 ([11, Thms. 1.8, 1.10]). The simplicial cones $\mathbb{R}_{\geq 0}C$ generated by all clusters $C$ of type $A_n$ form a complete simplicial fan in the ambient space

$$\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1 + \cdots + x_{n+1} = 0\}.$$

Theorem 3.19 ([6, Thm. 1.4]). The simplicial fan in Theorem 3.18 is the normal fan of a simple $n$-dimensional polytope $P$.

Theorem 3.18 is the case of type $A_n$ of [11, Thm. 1.10]. It allows us to think of the cluster complex as the complex of a complete simplicial fan. Theorem 3.19 was conjectured by Fomin and Zelevinsky [11, Conj. 1.12] and subsequently proved by Chapoton, Fomin, and Zelevinsky [6]. For an explicit description by inequalities see [6, Cor. 1.9]. These two theorems are special cases of Theorems 5.1 and 5.2, proved in Section 5.

3.3.3. The Chapoton–Fomin–Zelevinsky associahedron $\text{CFZ}_n(A_n)$.

Definition 3.20. The Chapoton–Fomin–Zelevinsky associahedron $\text{CFZ}_n(A_n)$ is any polytope whose normal fan is the fan with maximal cones $\mathbb{R}_{\geq 0}C$ generated by all clusters $C$ of type $A_n$.

Proposition 3.21 ([11, 6]). $\text{CFZ}_n(A_n)$ is an $n$-dimensional associahedron.

A polytopal realization of the associahedron $\text{CFZ}_2(A_2)$ is illustrated in Figure 11; note how the facet normals correspond to the almost positive roots of $A_2$.

Theorem 3.22. $\text{CFZ}_n(A_n)$ has exactly $n$ pairs of parallel facets. These correspond to the pairs of roots $\{\alpha_i, -\alpha_i\}$, for $i = 1, \ldots, n$, or, equivalently, to the pairs of diagonals $\{\alpha_i, -\alpha_i\}$ as indicated in Figure 12.

4. Exponentially many realizations, by Hohlweg–Lange

4.1. The Hohlweg–Lange construction. In this section we give a short description of the first, “type I”, exponential family of realizations of the associahedron, as obtained by Hohlweg and Lange in [16]. We prove that the number of normally non-isomorphic
realizations obtained this way is equal to the number of sequences \{+,−\}^{n−1} modulo
reflection and reversal. This number is equal to \(2^{n−3} + 2^{\lfloor \frac{n−3}{2} \rfloor}\) for \(n \geq 3\) (see [29, Sequence
A005418]).

Let \(\sigma \in \{+,−\}^{n−1}\) be a sequence of signs on the edges of an horizontal path on \(n\)
nodes. We identify \(n + 3\) vertices \(\{0, 1, \ldots, n + 1, n + 2\}\) with the signs of the sequence
\(\bar{\sigma} = \{+,−,\sigma,−,+\}\), and place them in convex position from left to right so that all
positive vertices are above the horizontal path, and all negative vertices are below it. These vertices form a convex \((n + 3)\)-gon that we call \(P_{n+3}(\sigma)\). Figure 4.1 illustrates the example \(P_{7}(\{+,−,+,+\})\), where \(n = 4\).

**Definition 4.1.** For a diagonal \(ij\) (\(i < j\)) of \(P_{n+3}(\sigma)\), we denote by \(R_{ij}(\sigma)\) the set of
vertices strictly below it. We define the set \(S_{ij}(\sigma)\) as the result of replacing 0 by \(i\) in \(R_{ij}(\sigma)\)
if 0 \(\in R_{ij}(\sigma)\), and replacing \(n + 2\) by \(j\) if \(n + 2 \in R_{ij}(\sigma)\).

The **Hohlweg–Lange associahedron** \(\text{Ass}_{n}^{I}(\sigma)\) is the polytope
\[
\text{Ass}_{n}^{I}(\sigma) = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i \in S_{\delta}(\sigma)} x_i \geq \frac{1}{2}|S_{\delta}(\sigma)|(|S_{\delta}(\sigma)| + 1) \text{ for all diagonals } \delta, \\
x_1 + \cdots + x_{n+1} = \frac{(n+1)(n+2)}{2}\}.
\]
Remark 4.2. If in $\tilde{\sigma} = \{+, -, \sigma, -, +\}$ we interchange the first two signs and/or the last two signs, the sets $S_\delta(\sigma)$ do not change and the construction will produce the same associahedron $\text{Ass}_n(\sigma)$.

Proposition 4.3 ([16, Thm. 1.1]). $\text{Ass}_n^1(\sigma)$ is an $n$-dimensional associahedron.

Proposition 4.4 ([16, Remarks 1.2 and 4.3]). $\text{Ass}_n^1(\{-, -, \cdots, -\})$ produces the Postnikov (Loday) associahedron $\text{Post}_n^1(1)$, and $\text{Ass}_n^1(\{+, -, +, -\})$ is normally isomorphic to the Chapoton–Fomin–Zelevinsky associahedron $\text{CFZ}_n(A_n)$.

Proof. For the first part we note that for $\sigma = \{-, -, \cdots, -\}$, the set $S_{p,q}(\sigma)$ of a diagonal $pq$ is given by $S_{p,q}(\sigma) = \{i : p < i < q\}$, and that the description of $\text{Ass}_n^1(\sigma)$ coincides with that of $\text{Post}_n^1(a)$ in Lemma 3.10 for $a = 1$. For the second part let $\sigma = \{+, -, +, -\}$. We write $S_\delta$ instead of $S_\delta(\sigma)$ for simplicity, and denote by $I_S \in \mathbb{R}^{n+1}$ the 0/1 vector with ones in the positions of a set $S \subseteq [n+1]$. The snake triangulation is given by the set of diagonals of the form $i, i+1$, for $1 \leq i \leq n$ (in the case where $n,n+1$ is not a diagonal we interchange vertices $n+1$ and $n+2$; this doesn’t change the associahedron we get, see Remark 4.2). We denote by $-\alpha_i = I_{S_{i,i+1}}$ the normal vector associated to the diagonal $i, i+1$, and by $n_{i,j} = I_{S_{i-1,j+2}}$ ($i \leq j$) the normal vector associated to the diagonal crossing $\{-\alpha_i, -\alpha_2, \ldots, -\alpha_j\}$. We need to prove that $n_{i,j} \equiv \alpha_i + \alpha_{i+1} + \cdots + \alpha_j \mod (1, \ldots, 1)$.

The reason is that our polytope lies in an affine hyperplane orthogonal to the vector $(1, \ldots, 1)$, and so we must consider the normal vectors modulo $(1, \ldots, 1)$. To this end, note that $n_{i,i} = \alpha_i + (1, \ldots, 1)$ and $n_{i,j+1} = \begin{cases} n_{i,j} + (1, \ldots, 1) + \alpha_{j+1} & \text{if } j \text{ is odd,} \\ n_{i,j} + \alpha_{j+1} & \text{if } j \text{ is even.} \end{cases}$

Remark 4.5. The Postnikov associahedron was defined as a Minkowski sum of certain faces $\Delta_S$ of the standard simplex $\Delta_{[n+1]}$. The question arises whether such Minkowski sum
descriptions exist for Ass\(_n^1(\sigma)\) in general. A partial answer is as follows. Postnikov \[24\] introduced generalized permutahedra as the polytopes with facet normals contained in those of the standard permutahedron such that the collection of right hand side parameters of the defining inequalities belongs to the deformation cone of the standard permutahedron. This includes all the Minkowski sums \(\sum_{S \subseteq [n+1]} a_S \Delta_S\) for which the coefficients \(a_S\) are non-negative. Ardila et al. \[1\] have shown that by dropping the deformation cone condition every polytope of the resulting family admits a (unique) expression as a Minkowski sum and difference of faces of the standard simplex. These decompositions, for the case of Ass\(_n^1(\sigma)\), are studied by Lange in \[19\]. A different decomposition arises from the work of Pilaud and Santos \[23\], who show that the associahedra Ass\(_n^1(\sigma)\) are the “brick polytopes” of certain sorting networks. As such, they admit a decomposition as the Minkowski sum of the \(\binom{n}{2}\) polytopes associated to the individual “bricks”. However, these summands need not be simplices.

4.2. Parallel facets.

**Theorem 4.6.** Ass\(_n^1(\sigma)\) has exactly \(n\) pairs of parallel facets. They correspond to the diagonals of the quadrilaterals with vertices \(\{i, j, j+1, k\}\) for \(j = 1, \ldots, n\), where

\[
i = \max\{0 \leq r < j : \text{sign}(r) \cdot \text{sign}(j) = -\}
\]

\[
k = \min\{j + 1 < r \leq n + 2 : \text{sign}(r) \cdot \text{sign}(j + 1) = -\}
\]

**Proof.** Two diagonals \(\delta\) and \(\delta'\) correspond to two parallel facets of Ass\(_n^1(\sigma)\) if and only if the sets \(S_\delta\) and \(S_{\delta'}\) satisfy \(S_\delta \cup S_{\delta'} = [n+1]\) and \(S_\delta \cap S_{\delta'} = \emptyset\). These two properties hold if and only if \(\delta\) and \(\delta'\) are the diagonals of the quadrilateral \(\{i, j, j+1, k\}\) for \(j = 1, \ldots, n\), and \(i\) and \(k\) satisfying the conditions of the theorem. \(\square\)

Associated to a sequence \(\sigma\) we define two operations, reflection and reversal. The reflection of \(\sigma\) is the sequence \(-\sigma\), and the reversal \(\sigma^t\) is the result of reversing the order of the signs in \(\sigma\).

**Theorem 4.7.** Let \(\sigma_1, \sigma_2 \in \{+,-\}^{n-1}\). Then the two realizations Ass\(_n^1(\sigma_1)\) and Ass\(_n^1(\sigma_2)\) are normally isomorphic if and only if \(\sigma_2\) can be obtained from \(\sigma_1\) by reflections and reversals.

**Proof.** Suppose there is a linear isomorphism between the normal fans of Ass\(_n^1(\sigma_1)\) and Ass\(_n^1(\sigma_2)\). It induces an automorphism of the face lattice of the associahedron that, by Lemma \[2,3\] corresponds to a certain reflection-rotation of the polygon. We denote this reflection-rotation by \(\varphi : P_{n+3}(\sigma_1) \to P_{n+3}(\sigma_2)\). Any linear isomorphism of the normal fans preserves the property of a pair of facets being parallel, so \(\varphi\) maps the “parallel” pairs of diagonals of \(P_{n+3}(\sigma_1)\), to the “parallel” pairs of diagonals of \(P_{n+3}(\sigma_2)\). Furthermore, for both realizations there are exactly four diagonals that cross at least one diagonal of every parallel pair; they are \(\{0, n+1\}, \{0, n+2\}, \{1, n+1\}\) and \(\{1, n+2\}\). The set of these four diagonals is also preserved under \(\varphi\). This is possible only if \(\varphi\) is a reflection-rotation that corresponds to a composition of reflections and reversals of the sequence \(\tilde{\sigma}_1 = \{+, -, \sigma_1, -, +\}\).
It remains to prove that $\text{Ass}_n^I(\sigma)$ is normally-isomorphic to both $\text{Ass}_n^I(-\sigma)$ and $\text{Ass}_n^I(\sigma^t)$. The isomorphism between the normal fans of $\text{Ass}_n^I(\sigma)$ and $\text{Ass}_n^I(-\sigma)$ is given by multiplication by $-1$, since $S_\delta(-\sigma) = [n] - S_\delta(\sigma)$. The isomorphism between the normal fans of $\text{Ass}_n^I(\sigma)$ and $\text{Ass}_n^I(\sigma^t)$ is given by the permutation of coordinates $\tau(i) = n + 1 - i$, as $S_\delta(\sigma^t) = \tau(S_\delta(\sigma))$.

\textbf{Corollary 4.8.} The Postnikov associahedron is not normally isomorphic to the Chapoton–Fomin–Zelevinsky associahedron.

\textit{Proof.} The Postnikov associahedron is produced by the sequence $\sigma_1 = \{-,-,\ldots,-\}$, and the Chapoton–Fomin–Zelevinsky associahedron is normally isomorphic to the one produced by the sequence $\sigma_2 = \{+,-,+,\ldots\}$. The two sequences are not equivalent under reflections and reversals. \qed

\subsection{Facet vectors}

We now show that $\text{Ass}_n^I(\sigma)$ can (modulo normal isomorphism) be embedded in $\mathbb{R}^n$ so that its facet normals are a subset of $\{0, -1, +1\}^n$ and contain the $n$ standard basis vectors and their negatives among them. That is, it can be obtained from a cube by cutting certain faces, as in Figures 2 and 3.

Obviously, the basis vectors and their negatives will correspond to the $n$ pairs of parallel facets that we identified in Theorem 4.6. Each such pair consists of a diagonal with positive slope and one with negative slope. We choose as “positive basis vector” the one with positive slope, which can be characterized as follows:

\textbf{Lemma 4.9.} Let $\{i, j, j + 1, k\}$ for $j = 1, \ldots, n$ be as in Theorem 4.6. Let
\[
    a := \max\{0 \leq r \leq j : \text{sign}(r) = -\}, \\
    b := \min\{j + 1 \leq r \leq n + 2 : \text{sign}(r) = +\}.
\]

Then $ab$ is one of the diagonals of the quadrilateral with vertices $\{i, j, j + 1, k\}$ and it has positive slope.

\textit{Proof.} By construction, $\{i, j, j + 1, k\}$ has two positive points and two negative points ($i$ and $j$ have opposite sign, as have $j + 1$ and $k$). Our definition of $a$ and $b$ is equivalent to: $a$ is the negative point in $\{i, j\}$ and $b$ is the positive point in $\{j + 1, k\}$. \qed

As customary, we call \textit{characteristic vector} of a set $S \subset [n + 1]$ the vector in $\{0, 1\}^{n+1}$ with 1’s in the coordinates of the elements of $S$. We denote it $e_S$. In particular, the $i$-th standard basis vector is $e_i = e_{\{i\}}$.

For each $j = 1, \ldots, n$, let $X_j = e_{S_{a,b}(\sigma)}$, where $a$ and $b$ are as in Lemma 4.9 and $S_{a,b}(\sigma)$ is from Definition 4.1. Then $X_j$ is normal to the facet of $\text{Ass}_n^I(\sigma)$ corresponding to the diagonal $ab$, one of the facets in the $j$-th parallel pair. By convention, let $X_{n+1} = e_\emptyset = (0, \ldots, 0)$ and $X_0 = e_{[n+1]} = (1, \ldots, 1)$.

\textbf{Theorem 4.10.} For every $S \subset [n+1]$, the characteristic vector of $S$ is a linear combination of $\{X_0, \ldots, X_{n+1}\}$ with coefficients in $\{0, +1, -1\}$. 
Proof. Since
\[ e_S = \sum_{j \in S} e_j, \]
the statement follows from the formula
\[ e_j = X_{j-1} - X_j, \quad \forall j \in [n], \]
which we prove distinguishing the case of \( j \) being positive or negative (the cases \( j = 1 \) and \( j = n + 1 \) need separate treatment, but the formula holds for them too). Let \( a \) and \( b \) be as in Lemma 4.9 and let \( a' \) and \( b' \) be the same, but computed for \( j - 1 \) instead of \( j \). That is, let \( X_{j-1} \) be the characteristic vector of \( S_{a', \nu} \). If \( j \) is positive, then \( a = a', b' = j \) and \( b \) is the next positive point after \( j \). If \( j \) is negative, then \( b = b', a = j \) and \( a' \) is the previous negative point before \( j \).

Definition 4.10 says that the characteristic vector of \( S_{\delta}(\sigma) \) is a normal vector to the facet of \( \text{Ass}_n^I(\sigma) \) corresponding to a certain diagonal \( \delta \). Since \( \text{Ass}_n^I(\sigma) \) is not full-dimensional, the normal to each facet is not unique. Others are obtained adding multiples of \( e_{|n+1|} = (1, \ldots, 1) \) to it. Put differently, the normal fan of \( \text{Ass}_n^I(\sigma) \) lives naturally in \((\mathbb{R}^{n+1})^*/\langle X_0 \rangle\).

For the basis \( \{X_1, \ldots, X_n\} \) in this space, Theorem 4.10 yields the following.

Corollary 4.11. With respect to the basis \( \{X_1, \ldots, X_n\} \), the normal vectors of \( \text{Ass}_n^I(\sigma) \) are all in \( \{0, +1, -1\}^n \) and include the \( 2n \) vectors \( \{\pm X_1, \ldots, \pm X_n\} \).

5. Catalan many realizations, by Santos

In this section we describe a generalization of the Chapoton–Fomin–Zelevinsky construction of the associahedron (Section 3.3), originally presented at a conference in 2004 [27]. We prove that the number of normally non-isomorphic realizations obtained this way, our “type II exponential family”, is equal to the number of triangulations of an \((n + 3)\)-gon modulo reflections and rotations. Interest in this number goes back to Motzkin (1948) [22]. An explicit formula for it is
\[
\frac{1}{2(n+3)} C_{n+1} + \frac{1}{4} C_{(n+1)/2} + \frac{1}{2} C_{\lfloor(n+1)/2 \rfloor} + \frac{1}{3} C_{n/3},
\]
where \( C_n = \frac{1}{n+1} \binom{2n}{n} \) for \( n \in \mathbb{Z} \) and \( C_n = 0 \) otherwise [29] Sequence A000207.

Let \( \alpha_1, \ldots, \alpha_n \) denote a linear basis of an \( n \)-dimensional real vector space \( V \cong \mathbb{R}^n \), and let \( T_0 \) be a certain triangulation of the \((n + 3)\)-gon, fixed once and for all throughout the construction. We call \( T_0 \) the seed triangulation. The CFZ associahedron will arise as the special case where \( V = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum x_i = 0\} \), \( \alpha_i = e_i - e_{i+1} \), and \( T_0 \) is the snake triangulation of Figure 10.

Let \( \{\delta_1, \ldots, \delta_n\} \) denote the \( n \) diagonals present in the seed triangulation \( T_0 \). To each diagonal \( pq \) out of the \( \frac{n(n+3)}{2} \) possible diagonals of the \( n \)-gon we associate a vector \( v_{pq} \) as follows:

1. If \( pq = \delta_i \) for some \( i \) (that is, if \( pq \) is used in \( T_0 \)) then let \( v_{pq} = -\alpha_i \).
If $pq \not\in T_0$ then let $v_{pq} := \sum_{\delta_i \text{ crosses } pq} \alpha_i$.

As a running example, consider the triangulation $\{123, 345, 156, 135\}$ of a hexagon with its vertices labelled cyclically. Let $\delta_1 = 13$, $\delta_2 = 35$ and $\delta_3 = 15$. Written with respect to the basis $\{\alpha_1, \alpha_2, \alpha_3\}$ the nine vectors $v_{pq}$ that we get are as follows (see Figure 14):

$$
v_{13} = -\alpha_1 = (-1, 0, 0), \quad v_{35} = -\alpha_2 = (0, -1, 0), \quad v_{15} = -\alpha_3 = (0, 0, -1),
$$
$$
v_{25} = \alpha_1 = (1, 0, 0), \quad v_{14} = \alpha_2 = (0, 1, 0), \quad v_{36} = \alpha_3 = (0, 0, 1),
$$
$$
v_{46} = \alpha_2 + \alpha_3 = (0, 1, 1), \quad v_{26} = \alpha_1 + \alpha_3 = (1, 0, 1), \quad v_{24} = \alpha_1 + \alpha_2 = (1, 1, 0).
$$

With a slight abuse of notation, for each subset of diagonals of the polygon we denote with the same symbol the set of diagonals and the set of vectors associated with them. For example, $\mathbb{R}_{\geq 0} T_0 = \mathbb{R}_{\geq 0} \{-\alpha_1, \ldots, -\alpha_n\}$ is the negative orthant in $V$ (with respect to the basis $[\alpha_i]$). More generally, for each triangulation $T$ of the $(n+3)$-gon consider the cone $\mathbb{R}_{\geq 0} T$. We claim the following generalizations of Theorems 3.18 and 3.19, and Proposition 3.21:

**Theorem 5.1.** The simplicial cones $\mathbb{R}_{\geq 0} T$ generated by all triangulations $T$ of the $(n+3)$-gon form a complete simplicial fan $\mathcal{F}_{T_0}$ in the ambient space $V$.

**Theorem 5.2.** This fan $\mathcal{F}_{T_0}$ is the normal fan of an $n$-dimensional associahedron.

### 5.1. Proof of Theorem 5.1

The statement follows from the following two claims:

1. $\mathbb{R}_{\geq 0} T_0$ is a simplicial cone and is the only cone in $\mathcal{F}_{T_0}$ that intersects (the interior of) the negative orthant.
2. If $T_1$ and $T_2$ are two triangulations that differ by a flip, let $v_1 \in T_1$ and $v_2 \in T_2$ be the diagonals removed and inserted by the flip. That is, $T_1 \setminus T_2 = \{v_1\}$ and $T_2 \setminus T_1 = \{v_2\}$.

Then there is a linear dependence in $T_1 \cup T_2$ which has coefficients of the same sign (and different from zero) in the elements $v_1$ and $v_2$.

The first assertion is obvious, and the second one is Lemma 5.3 below. Before proving it let us argue why these two assertions imply Theorem 5.1. Suppose that we have two triangulations $T_1$ and $T_2$ related by a flip as in the second assertion, and suppose that we
already know that one of them, say $T_1$, spans a full-dimensional cone (that is, we know that $T_1$ considered as a set of vectors is independent). Then assertion (2) implies that $T_2$ spans a full-dimensional cone as well and that $\mathbb{R}_{\geq 0}T_1$ and $\mathbb{R}_{\geq 0}T_2$ lie in opposite sides of their common facet $\mathbb{R}_{\geq 0}(T_1 \cap T_2)$. This, together with the fact that there is some part of $V$ covered by exactly one cone (which is why we need assertion (1)) implies that we have a complete fan. (See, for example, [7, Cor. 4.5.20], where assertion (2) is a special case of “property (ICoP)” and assertion (1) a special case of “property (IPP)”.)

**Lemma 5.3.** Let $T_1$ and $T_2$ be two triangulations that differ by a flip, and let $v_1$ and $v_2$ be the diagonals removed and inserted by the flip from $T_1$ to $T_2$, respectively (that is, $T_1 \setminus T_2 = \{v_1\}$ and $T_2 \setminus T_1 = \{v_2\}$). Then there is a linear dependence in $T_1 \cup T_2$ which has coefficients of the same sign in the elements $v_1$ and $v_2$.

**Proof.** Let $p$, $q$, $r$ and $s$ be the four points involved by the two diagonals $v_1$ and $v_2$, in cyclic order. That is, the diagonals removed and inserted are $pr$ and $qs$. We claim that one (and exactly one) of the following things occurs (see Figure 15):

(a) There is a diagonal in the seed triangulation $T_0$ that crosses two opposite edges of the quadrilateral $pqrs$.

(b) One of $pr$ and $qs$ is used in the seed triangulation $T_0$.

(c) There is a triangle $abc$ in $T_0$ with a vertex in $pqrs$ and the opposite edge crossing two sides of $pqrs$ (that is, without loss of generality $p = a$ and $bc$ crosses both $qr$ and $rs$).

(d) There is a triangle $abc$ in $T_0$ with an edge in common with $pqrs$ and with the other two edges of the triangle crossing the opposite edge of the quadrilateral (that is, without loss of generality, $p = a$, $q = b$ and $rs$ crosses both $ac$ and $bc$).

**Figure 15.** The four cases in the proof of Lemma 5.3.

To prove that one of the four things occurs we argue as follows. It is well-known that in any triangulation of a $k$-gon one can “contract a boundary edge” to get a triangulation of a $(k - 1)$-gon. Doing that in all the boundary edges of the seed triangulation $T_0$ except those incident to either $p$, $q$, $r$ or $s$ we get a triangulation $\tilde{T}_0$ of a polygon $\tilde{P}$ with at most eight vertices: the four vertices $p$, $q$, $r$ and $s$ and at most one extra vertex between each two of them. We embed $\tilde{P}$ having as vertex a subset of the vertices of a regular octagon, with $pqrs$ forming a square. We now look at the position of the center of the octagon $\tilde{P}$ with respect to the triangulation $\tilde{T}_0$: If it lies in the interior of an edge, then this edge is a diameter of the octagon and we are in cases (a) or (b). If it lies in the interior of a triangle of $\tilde{T}_0$, then we are in cases (c) or (d). See Figure 15 again.
Now we show explicitly the linear dependences involved in $T_1 \cup T_2$ in each case.

(a) Suppose $T_0$ has a diagonal crossing $pq$ and $rs$. Then

$$v_{pr} + v_{qs} = v_{pq} + v_{rs},$$

(1)

because every diagonal of $T_0$ intersecting the two (respectively, one; respectively none) of $pr$ and $qs$ intersects also the two (respectively, one; respectively none) of $pq$ and $rs$.

(b) If $T_0$ contains the diagonal $pr$, let $a$ and $b$ be vertices joined to $pr$ in $T_0$, with $a$ on the side of $q$ and $b$ on the side of $s$. We define the following vectors $w_a$ and $w_b$:

- $w_a$ equals 0, $v_{pq}$ or $v_{qr}$ depending on whether $a$ equals $q$, lies between $p$ and $q$, or lies between $q$ and $r$.
- $w_b$ equals 0, $v_{ps}$ or $v_{rs}$ depending on whether $a$ equals $s$, lies between $p$ and $s$, or lies between $s$ and $r$.

We claim that in the nine cases we have the equality

$$v_{pr} + v_{qs} = w_a + w_b.$$  

(2)

This is so because $v_{pr} + v_{qs}$ now equals the sum of the $\alpha_i$'s corresponding to the diagonals of $T_0 \setminus \{pr\}$ crossing $qs$, and we have that:

- The diagonals of $T_0$ crossing $qs$ in the $q$-side of $pr$ are none, the same as those crossing $pq$, or the same as those crossing $qr$ in the three cases of the definition of $w_a$, and
- The diagonals of $T_0$ crossing $qs$ in the $s$-side of $pr$ are none, the same as those crossing $ps$, or the same as those crossing $rs$ in the three cases of the definition of $w_b$.

(c) If $T_0$ contains a triangle $pbc$ with $bc$ crossing both $qr$ and $rs$ then we have the equality

$$2v_{pr} + v_{qs} = v_{qr} + v_{rs},$$

(3)

because in this case the diagonals of $T_0$ crossing $pr$ are the same as those crossing both $qr$ and $rs$, while the ones crossing $qs$ are those crossing one, but not both, of $qr$ and $rs$.

(d) If $T_0$ contains a triangle $pqc$ with $rs$ crossing both $pc$ and $qc$ then we have the equality

$$v_{pr} + v_{qs} = v_{rs},$$

(4)

because the diagonals of $T_0$ crossing $rs$ are the same as those crossing $pr$ and the same as those crossing $qs$.

Observe that when $T_0$ is a snake triangulation (the CFZ case) or, more generally, when the dual tree of $T_0$ is a path, cases (c) and (d) do not occur.

5.2. Proof of Theorem 5.2. To prove that $F_{T_0}$ is the normal fan of a polytope we use the following characterization.

Lemma 5.4. Let $F$ be a complete simplicial fan in a real vector space $V$ and let $A$ be the set of generators of $F$ (more precisely, $A$ has one generator of each ray of $F$). Then the following conditions are equivalent:

1. $F$ is the normal fan of a polytope.
There is a map $\omega: A \to \mathbb{R}_{>0}$ such that for every pair $(C_1, C_2)$ of maximal adjacent cones of $F$ the following happens: Let $\lambda: A \to \mathbb{R}$ be the (unique, up to a scalar multiple) linear dependence with support in $C_1 \cup C_2$, with its sign chosen so that $\lambda$ is positive in the generators of $C_1 \setminus C_2$ and $C_2 \setminus C_1$. Then the scalar product $\lambda \cdot \omega = \sum_v \lambda(v) \omega(v)$ is strictly positive.

**Proof.** One short proof of the lemma is that both conditions are equivalent to "$F$ is a regular triangulation of the vector configuration $A$" [7]. But let us show a more explicit proof of the implication from (2) to (1), which is the one we need. What we are going to show is that if such an $\omega$ exists and if we consider the set of points

$$\tilde{A} := \{ \frac{v}{\omega(v)} : v \in A \},$$

then the convex hull of $\tilde{A}$ is a simplicial polytope with the same face lattice as the complete fan $F$. (We think of $A$ as points in an affine space, rather than as vectors in a vector space.) Hence $F$ is the central fan of $\text{conv}(\tilde{A})$, which coincides with the normal fan of the polytope polar to $\text{conv}(\tilde{A})$.

To show the claim on $\text{conv}(\tilde{A})$ we argue as follows. Consider the simplicial complex $\Delta$ with vertex set $\tilde{A}$ obtained by embedding the face lattice of $F$ in it. That is, for each cone $C$ of $F$ we consider the simplex with vertex set in $\tilde{A}$ corresponding to the generators of $C$. Since $F$ is a complete fan and since the elements of $\tilde{A}$ are generators for its rays (they are positive scalings of the elements of $A$), $\Delta$ is the boundary of a star-shaped polyhedron with the origin in its kernel. The only thing left to be shown is that this polyhedron is strictly convex, that is, that for any two adjacent maximal simplices $\sigma_1$ and $\sigma_2$ the origin lies in the same side of $\sigma_1 \setminus \sigma_2$ (or, equivalently, in the same side of $\sigma_2 \setminus \sigma_1$).

Equivalently, if we understand $\sigma_1$ and $\sigma_2$ as subsets of $\tilde{A}$, we have to show that the unique affine dependence between the points $\{O\} \cup \sigma_1 \cup \sigma_2$ has opposite sign in $O$ than in $\sigma_1$ and $\sigma_2$.

Now the proof is easy. The coefficients in the linear dependence among the vectors in $\sigma_1 \cup \sigma_2$ are the vector

$$(\lambda(v) \omega(v))_{v \in A}.$$  

To turn this into an affine dependence of points involving the origin we simply need to give the origin the coefficient $-\sum_v \lambda(v) \omega(v)$ which is, by hypothesis, negative. $\square$

So, in the light of Lemma 5.4, to prove Theorem 5.2 we simply need to choose weights $\omega_{ij}$ for the diagonals of the polygon with the property that, for each of the linear dependences exhibited in equations (1), (2), (3), and (4), the equation $\sum_{ij} \omega_{ij} \lambda_{ij} > 0$ holds.

As a first approximation, let $\omega_{ij} = 2$ if $ij$ is in $T_0$ and $\omega_{ij} = 1$ otherwise. This is good enough for equations (3) and (4) in which all the $\omega$'s in the dependence are 1 and the sum of the coefficients in the left-hand side is greater than in the right-hand side. It also works for equations (2), in which we have

$$\omega_{pr} = 2, \quad \omega_{qs} = 1, \quad \lambda_{pr} = 1, \quad \lambda_{qs} = 1,$$
so that the sum $\sum_{ij} \omega_{ij} \lambda_{ij}$ for the left-hand side is three, while that of the right-hand side can be 0, $-1$ or $-2$ depending on the cases for the points $a$ and $b$.

The only (weak) failure is that in equation (1) we have $\lambda_{pr} = 1$, $\lambda_{qs} = 1$, $\lambda_{pq} = -1$, $\lambda_{rs} = -1$ and all the $\omega$’s are 1, so we get $\sum_{ij} \omega_{ij} \lambda_{ij} = 0$. We solve this by slightly perturbing the $\omega$’s. A slight perturbation will not change the correct signs we got for equations (2), (3), and (4).

For example, for each $ij$ not in $T_0$ change $\omega_{ij}$ to $\omega_{ij} = 1 + \varepsilon g_{ij}$ for a sufficiently small $\varepsilon > 0$ and for a vector $(g_{ij})_{ij}$ satisfying $g_{ik} + g_{jl} > \max\{g_{ij} + g_{kl}, g_{il} + g_{jk}\}$ for all $i,j,k,l$, $1 \leq i < j < k < l \leq n + 3$.

This holds (for example) for $g_{ij} := (j - i)(n + 3 + i - j)$.

5.3. **Distinct seed triangulations produce distinct realizations.** Let $\text{Ass}^H_n(T)$ denote the $n$-dimensional associahedron obtained with the construction of the previous section starting with a certain triangulation $T$. (This is a slight abuse of notation, since the associahedron depends also in the weight vector $\omega$ that gives the right-hand sides for an inequality definition of our associahedron. Put differently, by $\text{Ass}^H_n(T)$ we denote the normal fan rather than the associahedron itself.) We want to classify the associahedra $\text{Ass}^H_n(T)$ by normal isomorphism.

In principle, it looks like we have as many associahedra as there are triangulations (that is, Catalan-many) but that is not the case because, clearly, changing $T$ by a rotation or a reflection does not change the associahedron obtained. The question is whether this is the only operation that preserves $\text{Ass}^H_n(T)$, modulo normal isomorphism. The answer is yes, as we show below.

**Lemma 5.5.** $\text{Ass}^H_n(T_0)$ has exactly $n$ pairs of parallel facets, each pair consisting of (the facet of) one diagonal in $T_0$ and the diagonal obtained from it by a flip in $T_0$.

**Proof.** As always, a necessary condition for the facets corresponding to two diagonals to be parallel is that the diagonals cross; if the diagonals do not cross, they are present in some common triangulation which implies the corresponding facets intersect.

So, let $pr$ and $qs$ be two crossing diagonals. Since $\text{Ass}^H_n(T)$ is full-dimensional, their facets are parallel only if $v_{pr}$ and $v_{qs}$ are linearly dependent. By definition of the vectors $v_{ij}$ this only happens when $\{v_{pr}, v_{qs}\} = \{\pm \alpha_i\}$ for some $i$, which is the case of the statement. □

**Lemma 5.6.** Let $Q$ be an $(n + 3)$-gon, with $n \geq 2$. For each triangulation $T$ of $Q$ let $B_T$ denote the set consisting of the $n$ diagonals in $T$ plus the $n$ diagonals that can be introduced by a single flip from $T$. Then for every $T_1 \neq T_2$ we have $B_{T_1} \neq B_{T_2}$.

**Proof.** Suppose that $T_1$ and $T_2$ had $B_{T_1} = B_{T_2}$. We claim that $T_2$ is obtained from $T_1$ by a set of “parallel flips”. That is, by choosing a certain subset of diagonals of $T_1$ such that no two of them are incident to the same triangle and flipping them simultaneously. This is so because every diagonal $pr$ in $T_2$ but not in $T_1$ intersects a single diagonal $qs$ of $T_1$. If
many non-equivalent realizations of the associahedron

\begin{quote}

If \( pqr \) and \( prs \) were not triangles in \( T_2 \), then let \( a \) be a vertex joined to \( pr \) in \( T_2 \), different from \( q \) or \( s \). One of \( pa \) and \( ra \) intersects the diagonal \( qs \) of \( T_1 \) and one of the edges \( pq \), \( qr \), \( rs \) and \( pr \) of \( T_1 \).

Once we have proved this for \( T_2 \), the statement is obvious. For every \( T_2 \) different from \( T_1 \) but with all its diagonals in \( B_{T_1} \), there is a diagonal that we can flip to get one that is not in \( B_{T_1} \) (same argument, let \( pr \) be a diagonal in \( T_2 \) but not in \( T_1 \); let \( pq \), \( qr \), \( rs \) and \( pr \) be the other sides of the two triangles of \( T_2 \) containing \( pq \). Flipping any of them, say \( pq \), gives a diagonal that crosses \( pq \) and \( qs \), which are both in \( T_1 \)).

\end{quote}

\begin{corollary}

Let \( T_1 \) and \( T_2 \) be two triangulations of a convex \((n + 3)\)-gon. Then \( \text{Ass}^\Pi_n(T_1) \) and \( \text{Ass}^\Pi_n(T_2) \) are normally isomorphic if and only if \( T_1 \) and \( T_2 \) are equivalent under rotation-reflection.

\end{corollary}

\begin{proof}

If \( T_1 \) and \( T_2 \) are equivalent under rotation-reflection then the resulting associahedra are clearly the same. Now suppose that \( \text{Ass}^\Pi_n(T_1) \) and \( \text{Ass}^\Pi_n(T_2) \) are normally isomorphic. By Lemma 2.3 the automorphism of the associahedron face lattice induced by the isomorphism corresponds to a rotation-reflection of the polygon. Now, normal isomorphism preserves the property of a pair of facets being parallel, so using the previous lemma we get that this rotation-reflection sends \( T_1 \) to \( T_2 \).

\end{proof}

However, the same is not true if we only look at the set of normal vectors of \( \text{Ass}^\Pi_n(T) \):

\begin{proposition}

Let \( T_1 \) and \( T_2 \) be two triangulations of the \((n + 3)\)-gon. Let \( A(T_1) \) and \( A(T_2) \) be the sets of normal vectors of \( \text{Ass}^\Pi_n(T_1) \) and \( \text{Ass}^\Pi_n(T_2) \). Then \( A(T_1) \) and \( A(T_2) \) are linearly equivalent if, and only if, \( T_1 \) and \( T_2 \) have isomorphic dual trees.

\end{proposition}

\begin{proof}

Let \( T \) be the dual tree of a triangulation \( T \). Observe that the edges of \( T \) correspond bijectively to the inner diagonals in \( T \). Moreover, the diagonals of the polygon not used in \( T \) correspond bijectively to the possible paths in \( T \). More precisely: for every pair of nodes of \( T \), the two corresponding triangles of \( T \) have the property that one edge of each triangle “see each other”. Let \( p \) and \( q \) be the vertices of the two triangles opposite (equivalently, not incident) to those two edges. Then the diagonals of \( T \) crossed by \( pq \) correspond to the path in \( T \) joining the two nodes.

This means that, if we label the edges of \( T \) with the numbers 1 through \( n \) in the same manner as we labelled the diagonals of \( T \) we have that

\[ A(T) = \{-\alpha_i : i \in [n]\} \cup \{\sum_{i \in p} \alpha_i : p \text{ is a path in } T\} \]

In particular, \( A(T) \) can be recovered knowing only \( T \) as an abstract graph. For the converse, observe that if two trees are not isomorphic then there is no bijection between their edges that sends paths to paths. For example, knowing only the sets of edges that form paths we can identify the (stars of) vertices of the tree as the sets of edges such that every two of them form a path.

In particular, this gives us exponentially many ways of embedding the associahedron of dimension \( n \) with facet normals in the root system of \( A_n \):
Corollary 5.9. Let $T_0$ be a triangulation whose dual tree is a path. Let its diagonals be numbered from 1 to $n$ in the order they appear in the path. Then, taking $\alpha_i = e_{i+1} - e_i$, we have that $A(T_0)$ is the set of almost positive roots in the root system $A_n$.

The number of normally non-isomorphic classes of associahedra, for which the dual tree of the seed triangulation $T_0$ is a path, is equal to the number of sequences $\{+,-\}^{n-1}$ modulo reflection and reversal.

It is surprising that the number of realizations that we get in this way is exactly the same as we got in the previous section. Nevertheless, we prove in Theorem 6.2 that the two sets of realizations are (almost) disjoint.

6. How many associahedra?

We have presented several constructions of the associahedron. We call associahedra of types I and II the associahedra $\text{Ass}^I_n(\sigma)$ and $\text{Ass}^I_n(T)$ studied in the previous two sections. Associahedra of type I include the Postnikov (or Rote–Santos–Streinu, or Loday, or Buchstaber) associahedron, and both types I and II include the Chapoton–Fomin–Zelevinsky associahedron. They all have pairs of parallel facets while the secondary polytope on an $n$-gon (according to Section 3.1) does not. This implies that:

**Theorem 6.1.** The associahedron as a secondary polytope is never normally isomorphic to any associahedron of type I or type II. In particular, it is not normally isomorphic to the Postnikov associahedron or the Chapoton–Fomin–Zelevinsky associahedron.

Both types I and II produce exponentially many normally non-isomorphic realizations. The number of normally non-equivalent associahedra of type I is asymptotically $2^{n-3}$, while for type II is asymptotically $2^{2n+1}/\sqrt{\pi n^5}$. Explicit computations up to dimension 15 are given in Table 1.

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|-----|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|
| $\text{Ass}^I_n$ | 1 | 1 | 2 | 3 | 6 | 10 | 20 | 36 | 72 | 136 | 272 | 528 | 1056 | 2080 | 4160 |
| $\text{Ass}^II_n$ | 1 | 1 | 1 | 3 | 4 | 12 | 27 | 82 | 228 | 733 | 2282 | 7528 | 24834 | 83898 | 285357 | 983244 |

Table 1. The number of normally non-isomorphic realizations of the associahedron of types I and II up to dimension 15.

Surprisingly, the realizations of types I and II are (almost) disjoint:

**Theorem 6.2.** The only associahedron that is normally isomorphic to both one of type I and one of type II is the Chapoton–Fomin–Zelevinsky associahedron.

**Proof.** Suppose that a sequence $\sigma \in \{+,-\}^{n-1}$ and a triangulation $T$ produce normally isomorphic associahedra $\text{Ass}^I_n(\sigma)$ and $\text{Ass}^I_n(T)$. The induced automorphism between the face lattice of these two associahedra comes from a reflection-rotation map on the $(n+3)$-gon, by Lemma 2.3, so there is no loss of generality in assuming that this reflection-rotation is the identity.
Denote by $B_\sigma$ and $B_T$ the $2n$ diagonals corresponding to the $n$ pairs of parallel facets in both constructions respectively. The diagonals of $B_T$ consist of the diagonals of $T$ together with its flips. Since normal isomorphisms preserve pairs of parallel facets, $B_T = B_\sigma$.

We consider the $(n+3)$-gon drawn in the Hohlweg–Lange fashion (with vertices placed along two $x$-monotone chains, the positive and the negative one, placed in the $x$-order indicated by $\sigma$). The crucial property we use is that $B_\sigma$ contains only diagonals between vertices of opposite signs. Knowing this we conclude:

- Every triangle in $T$ contains a boundary edge in one of the chains. (That is, the dual tree of $T$ is a path). Suppose, in the contrary, that $T$ has a triangle $pqr$ with no boundary edge. Then the three diagonals $pq$, $pr$ and $qr$ lie in $B_T = B_\sigma$. This is impossible since at least two of $p$, $q$ and $r$ must have the same sign.

- The third vertex of each triangle is in the opposite chain. (That is, the dual path of $T$ separates the two chains). Otherwise the three vertices of a certain triangle lie in the same chain. This is impossible, because (at least) one of the three edges of each triangle is a diagonal, hence it is in $B_\sigma$.

- No two consecutive boundary edges in one chain are joined to the same vertex in the opposite chain. (That is, the dual tree of $T$ alternates left and right turns). Otherwise, let $abp$ and $bcp$ be two triangles in $T$ with $ab$ and $bc$ consecutive boundary edges in one of the chains. Then the flip in $bp$ inserts the edge $ac$, so that $ac \in B_\sigma$. This is impossible, since $a$ and $c$ are in the same chain.

These three properties imply that $T$ is the snake triangulation, so $\text{Ass}_{n}^\Pi(T)$ is the Chopoton–Fomin–Zelevinsky associahedron. $\square$

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