Formation of localized states in dryland vegetation: Bifurcation structure and stability

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We study theoretically the emergence of localized states of vegetation close to the onset of desertification. These states are formed through the locking of vegetation fronts, connecting a uniform vegetation state with a bare soil state, which occurs nearby the Maxwell point of the system. To study these structures we consider a universal model of vegetation dynamics in drylands, which has been obtained as the normal form for different vegetation models. Close to the Maxwell point localized gaps and spots of vegetation exist and undergo collapsed snaking. The presence of gaps strongly suggest that the ecosystem may undergo a recovering process. In contrast, the presence of spots may indicate that the ecosystem is close to desertification.

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I. INTRODUCTION

Localized structures, hereafter LSs, are a particular type of the more general dissipative states that emerge naturally in extended systems far from thermodynamic equilibrium [1,2]. LSs are not related with the intrinsic inhomogeneities of the system, but arise due to a double balance between nonlinearity and spatial coupling (e.g., diffusion) on one hand, and energy gain and dissipation on the other hand [3]. Their formation is usually related to the presence of bistability between two different stable states. In this context, a LS can be seen as a localized portion of a given domain embedded within a different one.

Examples of LSs can be found in a large variety of natural systems ranging from optics and material science to population dynamics and ecology [3–5]. In plant ecology, LSs have been observed in different contexts such as drylands [6–13] and marine sea-grass [14] ecosystems. In particular, in arid and semiarid regions, LSs can appear as spots [15–17], gaps [18–20], rings [21–23], and spirals [24], among others. Figures 1(a) and 1(b) show two examples of gaps and spots of vegetation, respectively.

These ecosystems are exposed to desertification processes which can take place through the slow advance of the barren state, i.e., front propagation [25], or abrupt collapse [26,27], and therefore, their study is very relevant.

One potential scenario for the formation of LSs is related to the presence of bistability between a uniform, and a spatially periodic (pattern) state emerging from a Turing instability [28]. In the context of semiarid ecosystems, spatially periodic patterns have been widely studied [6,10,29–41], and the formation of LSs in this context has been investigated [15,17,18,20]. Furthermore, the formation of LSs has been also analyzed in the presence of strong nonlocal coupling [16,19].

Another plausible bistable scenario for the formation of LSs is based on the locking of fronts that connect two different, but coexisting, uniform states. This mechanism is well understood and has been widely studied in different physical and natural systems [42–49]: near the Maxwell point of the system [50,51], and in the presence of spatial damped oscillations around the uniform states, fronts interact and lock at different separations, leading to the formation of LSs.

In semiarid ecosystems, the behavior of these vegetation fronts on either sides of the Maxwell point is closely related to the beginning of a gradual desertification process, and therefore its understanding is of great importance. Despite this relevance, only a few works focus on the formation of such types of states in dryland ecosystems, in particular, in the Gray-Scott model [52,53]. Here we present a detailed study of these LSs focusing on their origin, bifurcation structure, and stability.

The article is organized as follows. In Sec. II we introduce the model. Section III is devoted to the study of homogeneous or uniform states of the system, their spatiotemporal stability, and their spatial dynamics. Later, in Sec. IV we introduce the mechanism of front locking to describe the formation of LSs. In Sec. V, applying multiscale perturbation theory, we calculate small amplitude weakly nonlinear solutions near the main bifurcations of the uniform state. Following on from this, in Secs. VI and VII we build the bifurcation diagrams of the different LSs found analytically, and classify their regions of existence in the parameter space. Finally, in Sec. VIII we present a short discussion and the conclusions.

II. THE MODEL

Dryland vegetation ecosystems are a particular type of pattern-forming living systems. One characteristic of these systems is that the typical state variables, such as population
densities of organisms and biochemical reagents concentrations, cannot assume negative values.

In this context we study the reduced model

$$\partial_t A = \alpha A + \beta A^2 - A^3 + D\nabla^2 A - A\nabla^2 A - A\nabla^4 A, \quad (1)$$

where $\nabla^2 \equiv \partial_x^2 + \partial_y^2$, $A$ is a $\mathbb{R}^+$-valued scalar field, and $\alpha$, $\beta$, and $D$ are the three real control parameters of the system. In general, this model can be derived close to the onset of bistability where nonviable states undergo subcritical instabilities to viable states.

In the context of dryland vegetation, Eq. (1) was initially derived from the Lefever-Lejeune model [29] in the weak-gradient approximation [18,54], and just recently from the Gilad model [55–57]. Furthermore, similar models have been found in other living systems with Lotka-Volterra type of dynamics [58] and in sea-grass marine ecosystems [59]. Note that a similar equation has been also derived in nonliving systems [60].

In this work we analyze Eq. (1) in the framework of dryland vegetation, and we refer to the derivation obtained in Ref. [57]. In this context, $A$ is proportional to the biomass density, and is a positive quantity; $D > 0$ is proportional to the ratio between the biomass diffusion (lateral growth and seed dispersion) and the soil water diffusion; $\beta > 0$ quantifies the subcriticality of the uniform vegetation state and is related to the root-to-shoot ratio; and $\alpha$ measures the distance to the critical precipitation point where the bare state changes its stability.

Due to the complexity of this model, we focus on the study of Eq. (1) in one spatial dimension, and therefore $\nabla^2 \equiv \partial_x^2$. Furthermore, we consider an infinite domain for the analytical calculations, and large finite domains, with either periodic or Neumann boundary conditions, for the numerical computations.

III. THE HOMOGENEOUS SOLUTIONS AND THEIR LINEAR STABILITY

In this section we first introduce the homogeneous solutions of the system, we perform a spatiotemporal stability analysis, and we study their spatial dynamics. In this way we are able to identify the different stability regions, and the bifurcations from where small amplitude LSs may potentially arise.

A. Homogeneous steady states

In this subsection we consider the system without space, and we calculate their homogeneous solutions and their stability. The homogeneous steady state (HSS) solutions of the system satisfy

$$-A_r (A_r^2 - \beta A_r - \alpha) = 0, \quad (2)$$

and therefore consist of three branches of solutions: the trivial state $A_r = A_0 \equiv 0$, representing bare soil, and the two nontrivial state branches,

$$A^\pm = \frac{\beta \pm \sqrt{\beta^2 + 4\alpha}}{2}, \quad (3)$$

which represent uniform vegetation states. Note that, any solution $A_r$ should be zero or positive because negative biomass does not exist. It is easy to prove that $A_0 = 0$ is stable (unstable) when $\alpha < 0$ ($\alpha > 0$). It is also straightforward to obtain that $A^+$ is always stable, and $A^-$ is always unstable, when they are positive.

The branches $A^+$ and $A^-$ are separated by a fold or saddle-node (SN) point occurring at

$$A_r = \frac{\beta}{2}, \quad \alpha_r = -\frac{\beta^2}{4}. \quad (4)$$

Furthermore, the system undergoes a transcritical bifurcation at $\alpha = \alpha_p \equiv 0$.

In the $(\alpha, \beta)$-parameter space these bifurcations define the two lines shown in the phase diagram of Fig. 2(a). Slices of constant $\beta$ correspond to the bifurcation diagrams shown in Figs. 2(b) and 2(c), for $\beta = 0$, and $\beta = 2$, respectively. The point where the system changes from a subcritical to a supercritical bifurcation corresponds to the value $\beta = 0$.

At this stage we can identify three different regions that we label as I–III:

Region I: Only the bare soil $A_0$ exists and is stable. This region is spanned by parameter values below the fold line $\alpha_r$ when $\beta > 0$, and by $\alpha_p$ when $\beta = 0$.

Region II: The nontrivial states $A^+ > 0$, and $A^- > 0$ coexist with $A_0$. This region is spanned by $\alpha_r < \alpha < \alpha_p$ and $\beta > 0$. This is the bistability region [see light green area]. The solutions $A_0$ and $A^+$ are stable.

Region III: Only solutions $A_0$ and $A^+$ coexist, but now $A_0$ is always unstable. This region is defined by $\alpha > \alpha_p$.

In this work we are interested in the region of parameters where fronts connecting the bare soil and the homogeneous vegetation exist, i.e., the bistability region II. As we will see

FIG. 1. Localized structures in semiarid regions. In (a) gaps of vegetation in Namibia. In (b) spots of vegetation in the Atacama region in northern Chile. Photographs courtesy of C. Fernández-Oto.
periodic patterns emerge \[28\]. According to this principle two situations occur at a Turing instability (TI), from where \(\alpha\) the bare soil state is unstable, and corresponds to the dashed lines plotted in (b). The HSS solution is stable outside this region as shown with solid lines in panel (b). Panels (c, d) show the same type of diagrams but for \((\beta, D) = (2, 0.5)\). The TI occurs at the maximum of this curve and is signaled with a blue dot in panel (d).

B. Spatiotemporal stability analysis

The next step in our study is to analyze the linear stability of the HSS solutions \(A_s\) against general nonuniform perturbations \(\xi(x, t) = e^{\sigma t + i k x}\), or what is the same, to consider weakly modulated solutions of the form \(A_s(x, t) = A_s + \epsilon \xi(x, t)\), with \(|\epsilon| \ll 1\). Considering this ansatz and an infinite domain, the stability problem reduces to the study of the linear equation

\[ \partial_t \xi = \mathcal{L} \xi, \]

(5)

where \(\mathcal{L}\) is the linear operator

\[ \mathcal{L} \equiv \alpha + 2\beta A_s - 3A_s^2 + D \partial_x^2 - A_s \partial_t^2 - A_s A_t^2. \]

(6)

For this equation one obtains that the growth rate \(\sigma\) satisfies

\[ \sigma(k) = \alpha + 2\beta A_s - 3A_s^2 - Dk^2 + A_s k^2 - A_s A_t^2. \]

(7)

The HSS \(A_s\) is stable when \(\mathrm{Re}(\sigma(k)) < 0\) and unstable otherwise. For a finite value of \(k\), the transition between these two situations occurs at a Turing instability (TI), from where periodic patterns emerge \[28\]. According to this principle the bare soil state \(A_0\) is always stable (unstable) against homogeneous perturbations \((k = 0)\) for \(\alpha < 0\) \((\alpha > 0)\).

We analyze the stability of the uniform vegetation states \(A^+\) and \(A^-\) by means of the marginal stability curve \(A_s(k)\) [see Figs. 3(a) and 3(c)]. This curve defines the band of unstable wave numbers (i.e., unstable modes) and is composed by the two branches \(A_s^\pm(k)\), solutions of the equation

\[ 2A_s^2 + (k^4 - k^2 - \beta)A_s + Dk^2 = 0, \]

(8)

which is obtained by taking \(\sigma = 0\) in Eq. (7). Thus, \(A_s\) is unstable to perturbations with a fixed \(k\), if \(A_s\) lays inside the curve, i.e., \(A_s(\pm k) < A_s < A_s^+(k)\), and stable otherwise.

The maximum of this curve satisfies the condition \(d\sigma / dk|_{k_c} = 0\), what leads to the critical wave number

\[ k_c^2 = \frac{A_s - D}{2A_s}. \]

(9)

Note that, for \(k_c\) to be real, it is necessary that \(D < A_c\). Then, the TI corresponds to the real solution \(A_c\) of \(\sigma|_{k_c} = 0\), i.e., solution of the equation

\[ 8A_c^4 - (4\beta + 1)A_c^2 + 2DA_c - D^2 = 0, \]

(10)

which reads

\[ A_c = \frac{1}{24} \left[ 1 + 4\beta + l_1(\beta, D) \right] \left[ (1 + 4\beta)^2 - 48D \right], \]

(11)

with

\[ l_1 = \left[ 1 + 48\beta^2 + 64\beta^3 + \beta(12 - 288\delta)D - 72D \right] + 864D^2 + l_2^{1/3}. \]

and

\[ l_2 = 48\sqrt{3}D\sqrt{\delta}(1 + 4\beta^2)^2 - 72D + 2D(54D - 1). \]

Figures 3(a) and 3(b) show the marginal stability curve and the HSS solutions for \((\beta, D) = (2, 1.5)\). The maximum of this curve occurs at \(k = 0\), and therefore \(A^+\) is stable all the way until \(\mathrm{SN}_t\) [see solid line in Fig. 3(b)], while \(A^-\) is always unstable. Decreasing \(D\), the maximum of the curve shifts from...
where $k = 0$ to a finite value $k_c$, where the TI occurs. This is the situation shown in Figs. 3(c) and 3(d) for $(\beta, D) = (2, 0.5)$. Now, $A^+$ is stable for $A^+ > A_c$ and unstable otherwise.

The TI defines the manifold $\alpha_c = A_c^2 - \beta A_c$ in the parameter space, and region II can be subdivided as follows:

Region II$_1$: $A^+$ is unstable against nonuniform perturbations ($k \neq 0$). This region is defined by $\alpha_c < \alpha < \alpha_c$ when $\alpha_c$ exists.

Region II$_2$: $A^+$ is stable against nonuniform perturbations ($k \neq 0$). This region is defined by $\alpha > \alpha_c$ when $\alpha_c$ exists, or by $\alpha > \alpha_t$ when $\alpha_t$ does not exist.

C. Spatial stability of the uniform states

The steady states of the system Eq. (1) can be described in the framework of what is called spatial dynamics [61,62]. This technique consists in recasting the stationary version of Eq. (1) into a finite-dimensional dynamical system where the evolution in time is substituted by an evolution in space. In this context, an analogy between the solutions of the physical spatial system (determined by the spatial eigenvalues), and those of the new dynamical system is established such that a uniform state of Eq. (1) corresponds to a fixed or equilibrium point of the dynamical system; a LS biasymptotic to the HSS to a fixed point; and a front connecting two different uniform states is seen as a heteroclinic orbit between two different equilibria.

Thus, the linear dynamics near the different fixed points of the spatial system (determined by the spatial eigenvalues), and the bifurcations that may undergo are essential to understand the origin of the different dissipative states emerging in the system [61,62].

To calculate the spatial eigenvalues, it is enough to substitute $k = -i\lambda$ in Eq. (7) and find its roots, i.e., to solve

$$A_\alpha \lambda^4 + (A_x - D)\lambda^2 + 3A_x^2 - 2\beta A_x - \alpha = 0. \quad (12)$$

For the stable bare soil solution ($A_x = 0$), Eq. (12) becomes

$$D\lambda^2 + \alpha = 0, \quad (13)$$

which leads to the solution $\lambda_{\pm} = \pm\sqrt{-\alpha/D}$. For $\alpha < 0$ this solution shows that the bare soil cannot have any kind of oscillations, damped or not. This is in agreement with the fact that biomass cannot be negative.

For the uniform vegetation $A^+$, the solution of Eq. (12) becomes

$$\lambda = \pm\sqrt{\frac{|D - A^+| \pm \sqrt{\Delta_\lambda}}{2A^+}}, \quad (14)$$

where

$$\Delta_\lambda = (D - A^+)^2 + 4A^+(\alpha + 2\beta A^+ - 3A^+^2). \quad (15)$$

Note that the condition $\Delta_\lambda = 0$ leads to the same equation as Eq. (10), which defines the position of the TI at $A^+ = A_c$.

Depending on the control parameters of the system the spatial eigenvalues change as indicated in the $(\alpha, D)$—phase diagram shown in Fig. 4 for a fixed value of $\beta$. The transition between these configurations occurs through the TI and the SN bifurcations, in solid green and purple, respectively, and through the dashed purple line.

The line corresponding to the fold SN$_c$ is given by Eq. (4) and is constant for any value of $D$. Along this line the spatial eigenvalues read

$$\lambda_{1,2} = 0, \quad \lambda_{3,4} = \pm \sqrt{\frac{2D - \beta}{\beta}}. \quad (16)$$

if $2D - \beta > 0$, and

$$\lambda_{1,2} = 0, \quad \lambda_{3,4} = \pm i \sqrt{\frac{2D - \beta}{\beta}}, \quad (17)$$

if $2D - \beta < 0$. We refer to the fold as SN$_{c+}$ in the first configuration, and as SN$_{c-}$ in the second. The condition $\Delta_\lambda = 0$ defines the dashed and solid purple lines in Fig. 4. The solid line corresponds to the TI with the double pure imaginary eigenvalues

$$\lambda_{1,2} = i \sqrt{\frac{|D - A_c|}{2A_c}}, \quad \lambda_{3,4} = -i \sqrt{\frac{|D - A_c|}{2A_c}}, \quad (18)$$

while on the dashed purple line the eigenvalues are purely real,

$$\lambda_{1,2} = \sqrt{\frac{|D - A_c|}{2A_c}}, \quad \lambda_{3,4} = -\sqrt{\frac{|D - A_c|}{2A_c}}. \quad (19)$$

This line corresponds to a Belyakov-Devaney transition [61,62], and hereafter we label it as BD.

In Fig. 4 we observe that the configuration of the different eigenvalues is modified while crossing the previous lines. We can classify them in three main groups:
A: The four eigenvalues $\lambda$ are real. This region is located between SN$_1^+$ and the BD line. Trajectories approach or leave the uniform vegetation state monotonically with $x$.

B: The eigenvalues $\lambda$ consist in a quartet of complex numbers. This region is located in-between the TI and BD lines. Here the trajectories suffer a damped oscillatory dynamics in $x$ near the fixed point $A^+$.

C: The four eigenvalues $\lambda$ are imaginary. This configuration exists in-between the TI and the SN$_2^+$ lines. Here the uniform vegetation state is unstable to periodic patterns.

In Sec. IV we show that LSs can form through the locking of fronts when the dynamics near $A^+$ is described by the configuration B, and therefore, in the following we focus on this region. Furthermore, dynamical systems theory predicts that small amplitude LSs (i.e., homoclinic orbits) bifurcate from the TI and SN$_2^+$ [61,62]. In Sec. V we analytically obtain weakly nonlinear LS solutions near these points.

IV. VEGETATION FRONTS AND LOCKING

In this section we introduce the concept of front locking, also called pinning, as the mechanism responsible for the formation of LSs. In region IIb the system is bistable, i.e., the bare soil state $A_0$ and the uniform vegetation state $A^+$ coexist for the same range of parameters. Considering this situation, vegetation fronts may arise connecting $A_0$ with $A^+$ ($A_0 \rightarrow A^+$) or viceversa ($A^+ \rightarrow A_0$). In what follows we refer to the former front as $F^+$, and the later as $F^-$ [see Fig. 5(a)].

Either $F^+$ or $F^-$ drifts at a constant velocity $v$ that can be positive or negative depending on the control parameters of the system. These fronts are solutions of the stationary equation

$$\alpha A + \beta A^2 - A^3 + v \partial_x A + D \partial_x^2 A - A \partial_x^2 A = 0,$$

that we obtain writing Eq. (1) in the moving reference frame at constant velocity $v$ (i.e., considering the transformation $x \rightarrow x - vt$) and setting $\partial_t A = 0$.

Furthermore, to preserve the symmetry $x \rightarrow -x$ the velocity of the fronts $F^+$ and $F^-$ must have same modulus but opposite direction. We have to notice that this type of dynamics is not valid, for example, in slightly sloped topographies where the previous invariance is destroyed [63,64].

The threshold between these two situations is marked with a vertical pointed-dashed gray line [see Fig. 5(a)] and corresponds to the Maxwell point of the system $\alpha_M$ [45]. At the Maxwell point the velocity of the front cancels out, and the front changes the direction of propagation. To illustrate this phenomenon we perform direct numerical simulations using Eq. (1) and considering a finite domain with Neumann boundary conditions. On the left of $\alpha_M$ the bare soil state $A_0$
involves the uniform vegetation one $A^+$, as illustrated in the top panel of Fig. 5(b) for $F^-$ and $(\beta, D) = (2, 0.5)$. Such a front is called desertification front. In contrast, on the right of $\alpha_M$, $A^+$ invades $A_0$ [see the bottom panel in Fig. 5(b)], and we refer to this front as recovery front. Hence, the Maxwell point of the system appears to be of great importance to predict desertification. In Fig. 5(c) we show the modification of the front velocity with $\alpha$ for $(\beta, D) = (2, 0.5)$, which has been obtained through direct numerical simulations. For this set of parameters the Maxwell point is situated at $\alpha_M \approx -0.8962$, and is marked with a solid-blue vertical line.

The tails of the front can be described, asymptotically around $A^+$, by the ansatz

$$A(x) - A^+ \sim \cos(Kx)e^{-Qx},$$

(21)

where $Q$ and $K$ correspond to the real and imaginary parts of the spatial eigenvalue $\lambda$. In region $\mathbb{A}$ [see Fig. 4] $K = 0$, and the tails are monotonic around $A^+$, whereas in region $\mathbb{B}$, $K$ and $Q$ are different from zero, and the tails show damped oscillations in $x$ around the uniform state $A^+$. In infinite domains, and for a set of parameters in region $\mathbb{B}$, two fronts with opposite polarity (i.e., $F^+$ and $F^-$), and separated by a distance $\Delta$, experience an interaction described by

$$\partial_t \Delta = \varrho \cos(K\Delta)e^{-Q\Delta} + \eta,$$

(22)

where $\eta \sim \alpha - \alpha_M$, and therefore proportional to the separation from the Maxwell point, and $\varrho$ depends on the parameters of the system. Although the type of front interaction described here is generic, and well described by Eq. (22), its explicit derivation from Eq. (1) is not realizable [65], and we introduce it here for illustrating the mechanism of front locking. Note that, for some particular cases, this derivation can be carried out explicitly through perturbation analysis [16,42,45,46,66]. A detailed general derivation of such type of equations can be found in Ref. [66].

The presence of $K \neq 0$ is responsible for the oscillatory nature of the interaction which alternates attraction with repulsion, as shown in Fig. 5(d) for three different values of $\eta$. When $\partial_t \Delta = 0$, the fronts lock at different stationary separations $\Delta_s$ satisfying $\cos(K\Delta_s)e^{-Q\Delta_s} = 0$. At $\alpha = \alpha_M (\eta = 0)$ [see the bottom graph in Fig. 5(d)] the width of the LSs $\Delta_s$ is quantized $\Delta_s = \frac{\pi}{2K}(2n +1)$, with $n = 0, 1, 2, \ldots$ [42,45]. By increasing $n$ by one, an extra spatial oscillation or dip is nucleated in the LS. The stable (unstable) separation distances are marked with $\circ$ ($\ast$).

As soon as $\eta > 0$ the blue curve is shifted upwards (downwards if $\eta < 0$), and as a result the number of stationary intersections decreases (see middle and top graphs for $\eta = \eta_1$ and $\eta_2$). Thus, the separation from $\alpha_M$ implies the disappearance of wider LSs, until eventually even the single peak LS disappears. In the coming sections we will see that the interaction described by Eq. (22) is responsible for the bifurcation structure that LSs undergo.

V. WEAKLY NONLINEAR LOCALIZED STATES

In Sec. IV we have introduced the mechanism of front locking to explain the formation of high amplitude LSs of different widths. However, this mechanism does not reveal the origin of these structures from a bifurcation point of view. In infinite domains, normal form theory predicts the existence of small amplitude LSs emerging from the main local bifurcations that the HSS undergoes [61,62]. Here, we use multiscale perturbation theory (see Appendix) to compute weakly nonlinear steady solutions of Eq. (1) near the main bifurcations of interest: the transcritical bifurcation occurring at $\alpha_p$, the fold or SN bifurcation at $\alpha_t$, and the TI located at $\alpha_c$. In the neighborhood of such bifurcations, weakly nonlinear states are captured by the ansatz:

$$A(x) - A_t \sim \epsilon a(X)e^{ik_x} + c.c.,$$

where $\epsilon \ll 1$ measures the distance from the bifurcation, $k_c$ is the characteristic wave number of the marginal mode at the bifurcation ($k_c = 0$ for the fold and transcritical and $k_c \neq 0$ for the TI), and $a$ is the amplitude or envelope describing a modulation occurring at a larger scale $X = \epsilon x$, with the election of $\epsilon$ depending on the problem. In what follows we show the analytical expressions for small amplitude states about the different bifurcations of the system, and refer to Appendix for a detailed exposition of the analysis. We have to point out that the temporal stability of such asymptotic states can be estimated analytically as done in Refs. [67–69]. However, here, the temporal stability is calculated numerically (see Sec. VI).

A. Small amplitude spots around the transcritical bifurcation

In Appendix 1 we show that near $\alpha_p$, small amplitude spots of the form

$$A(x) = -\frac{3}{2\beta}sech^2\left(\frac{1}{\sqrt{D-x}}\right)$$

(23)

exist for negative values of $\alpha$. Figure 6(a) shows the profile of such structure in blue. The red dashed line corresponds to the exact numerical solution, that has been obtained using a Newton-Raphson solver and considering Eq. (23) as initial guess. The plot shows excellent agreement.

B. Small amplitude gaps around the fold bifurcation

Following a similar procedure, in Appendix 2 we show that in a neighborhood of $\alpha_t$ small amplitude gap LSs of the form

$$A(x) = \frac{\beta}{2} + \sqrt{\alpha - \alpha_t} - 3\sqrt{\alpha - \alpha_t}sech^2\left(\frac{1}{\sqrt{2D-\beta}(\alpha - \alpha_t)^{1/4}x}\right)$$

(24)

arise whenever $2D - \beta > 0$. This last condition ensures that $A^+$ is stable all the way until SN$_c$ at $\alpha_t$. Figure 6(b) shows in solid-blue the small amplitude gap LS, and in dashed-red the exact numerical solution obtained also from a Newton-Raphson algorithm. Like in the previous case, the plot shows very good agreement.
FIG. 6. Panel (a) shows the weakly nonlinear spot solution Eq. (23) using a solid blue line, and the exact numerical solution (dashed red line) obtained from a Newton-Raphson solver for \( D = 1.5 \) and \( \alpha = \alpha_{\gamma} - 0.01 \); panel (b) shows in blue the analytical gap LS (24) and the exact numerical one (dashed red) for \( D = 1.5 \) and \( \alpha = \alpha_{\gamma} + 0.0001 \); panel (c) shows the agreement for the \( \phi = 0 \) gap LS arising from the TI for \( D = 0.5 \) and \( \alpha = \alpha_{\gamma} + 0.00003 \); panel (d) is the same as in panel (c) but for a gap LS with \( \phi = \pi \). For every case we choose \( \beta = 2 \).

C. Small amplitude gaps around the Turing bifurcation

In Appendix 3 we perform the weakly nonlinear analysis about the Turing bifurcation at \( \alpha_{\gamma} \). In this case, small amplitude stationary gap periodic patterns of the form

\[
A(x) = A_c + (\alpha - \alpha_c) \bar{A}_2 + 2 \sqrt{\frac{\alpha - \alpha_c}{-c_3}} \cos(k_c x + \phi),
\]

(25)

arise from the TI, where \( \alpha_c, \bar{A}_2, \) and \( c_3 \) depend on the parameters of the system [see Appendix 3], \( k_c \) is the wave number associated with the critical pattern emerging from the TI [see Eq. (9)], and \( \phi \) is an arbitrary phase. For the range of parameters explored in this work \( c_3 \) is always negative, and therefore, the periodic pattern arises subcritically.

In this situation, small amplitude gap LSs of the form

\[
\begin{align*}
A(x) &= A_c + (\alpha - \alpha_c) \bar{A}_2 \\
&\quad - 2 \left( \frac{2(\alpha - \alpha_c)}{-c_3} \right) \text{sech} \left( \sqrt{\frac{\alpha - \alpha_c}{-c_2}} x \right) \cos(k_c x + \phi)
\end{align*}
\]

(26)

emerge, together with the subcritical pattern, from the TI if \( c_2 < 0 \), what occurs whenever \( 2D - \beta < 0 \). Figure 11 in Appendix 3 shows the dependence of \( c_2 \) and \( c_3 \) with \( D \) for \( \beta = 2 \).

The phase \( \phi \) remains arbitrary within the asymptotic theory. However, expansions beyond all orders show that two specific values of this phase are selected, namely \( \phi = 0 \), and \( \phi = \pi \) [70]. Figures 6(c) and 6(d) show in blue the analytical solution Eq. (26) for \( \phi = 0 \) and \( \phi = \pi \), respectively, and in dashed red the exact numerical solutions. The overlap shows very good agreement between both calculations. Note that the \( \phi = 0 \) state has a minimum at \( x = 0 \), whereas the \( \phi = \pi \) state has a maximum.

VI. BIFURCATION STRUCTURE OF LOCALIZED STATES: COLLAPSED SNAKING

In this section we study the bifurcation structure of the different dissipative LSs arising in this system. To do so we apply numerical continuation algorithms based on a Newton-Raphson solver which allows us to track the steady states of the system in their different control parameters [71]. In this way, we are able to study how the different LSs appearing in the system are organized in terms of bifurcation diagrams. For these calculations we consider a finite domain of length \( L = 100 \) and periodic boundary conditions.

In Sec. V we have calculated analytically weakly nonlinear solutions corresponding to different types of pulses (spots or gaps) which are only valid in the neighborhood of the local bifurcations of the uniform state. Starting from these analytical solutions we are able to study the bifurcation structure of these states. This structure is summarized in the \((\alpha, D)\)-phase diagram shown in Fig. 7, where we plot the main bifurcation lines of the system for \( \beta = 2 \). Figures 8(a) and 8(b) show the bifurcation diagrams for \( D = 1.5 \) and \( D = 0.5 \), respectively, and correspond to the two horizontal gray lines shown in Fig. 7. In these diagrams we plot the \( L^2 \)-norm

\[
||A||^2 = \frac{1}{L} \int_{-L/2}^{L/2} A(x)^2 dx
\]

as a function of the parameter \( \alpha \).
FIG. 8. Bifurcation diagrams for $\beta = 2$ and two different values of $D$: $D = 1.5$ in panel (a), and $D = 0.5$ in panel (b). Stable (unstable) solution branches are marked with solid (pointed-dashed lines). The green dots correspond to the solutions shown in subpanels (i)–(xii). The red lines correspond to the gap pattern branches arising from the TI. The subpanel in (a) shows a close-up view of the collapsed snaking around $\alpha_M$. The green line represents the collapsed snaking associated with the localized gaps for $\varphi = 0$, whereas the orange one shows partially the diagram corresponding to $\varphi = \pi$. The bottom subpanel in (b) shows a close-up view of the diagram around $SN_{p1}^{-}$, and the top panel shows the subcritical birth of the gap pattern, and the two families of localized gap states, from the TI at $\alpha_c$. The vertical gray dashed lines mark the Maxwell point of the system. Here we consider a spatial domain of length $L = 100$ and periodic boundary conditions.

For $D = 1.5$ (see point-dashed gray line in Fig. 7) the situation is like the one depicted in Fig. 8(a). In a neighborhood of the transcritical bifurcation $\alpha_p$ a spot LS of the form Eq. (23) exists. This solution is temporally unstable all the way until $SN_{1}^{-}$, and increases its amplitude while decreasing $\alpha$ [see Fig. 8(a), profile (i)].

When an analytical expression for a LS is known, its temporal stability can be computed analytically as has been done by different authors [67–69,72–74]. Here, however, we only have access to the LS solutions numerically, and therefore, their stability can only be determined by solving numerically the eigenvalue problem

$$ L\psi = \sigma \psi, $$

(27)

where $L$ is the linear operator Eq. (6) evaluated at a given LS field $A$, and $\sigma$ and $\psi$ are the eigenvalues and eigenvectors associated with $L$ at a given set of parameters.

Once the fold $SN_{1}^{-}$ is passed, the spot state becomes stable [see Fig. 8(a), panel (ii)] and conserves the stability until $SN_{2}^{-}$, where it becomes unstable once more. Proceeding up in the diagram (i.e., increasing $||A||^2$) the LS broadens [see Fig. 8(a), profiles (iii) and (iv)] as a result of the nucleation of new dips (i.e., spatial oscillations) at every $SN_{i}^{-}$ [see close-up view in Fig. 8(a)]. In the meanwhile, the solution branches suffer a sequence of exponentially decaying oscillations around the Maxwell point $\alpha_M$, and the different solutions gain and lose stability through the saddle-node bifurcations $SN_{l}^{-}, SN_{r}^{-}$. This type of bifurcation structure is known as collapsed snaking, and has been studied in detail in different systems [49,75–77]. At this stage [see Fig. 8(a), profile (iv)] we can see that the LS is formed by the locking of two fronts $F^{+}$ and $F^{-}$, as described by Eq. (22).

In Sec. V, we have also obtained the analytical expression Eq. (24) for a gap LS in the neighborhood of $SN_{1}^{+}$. Tracking numerically this state to larger values of $\alpha$, the LS deepens [see Fig. 8(a), profile (vi)] until reaching its maximal depth at $\alpha_M$. Along this branch, the gap is always temporally unstable. In finite domains the spot and gap types of LSs are interconnected as shown in the bifurcation diagram of Fig. 8(a). Furthermore, we have to mention that in the
presence of periodic boundary conditions, one cannot properly discern between both types of states. Indeed, a spot LS [e.g., Fig. 8(a), state (v)] can be seen as gap by just applying a translation of \( L/2 \). However, in real two-dimensional landscapes periodic boundary conditions make little sense, and spots and gaps are two different, and well defined, structures.

Figure 7 shows how the main bifurcation lines of the system vary with the diffusion \( D \). These lines correspond to the Maxwell point \( \alpha_M \) (solid red line), the folds SN\(_{g}^L\) (solid green line), the folds of the spot structures SN\(_{f,1}^L\), the fold of the localized gap states SN\(_{g}^L\) (solid orange line), and the TI \( \alpha_c \) (purple solid line).

Decreasing \( D \), the TI emerges from the fold SN\(_{g}^L\) at \( D = D_q \equiv \beta/2 = 1 \). Below this point (i.e., \( D < D_q \)), \( A^+ \) is unstable between TI and SN\(_{g}^L\), and this unstable region increases whereas decreasing \( D \). For \( D > D_q \) the TI fades away, and the uniform vegetation state \( A^- \) is stable everywhere, as shown in Fig. 8(a).

Decreasing \( D \) below \( D_q \) the region of existence of the spot LSs becomes wider, and every branch of solutions widens. This situation corresponds to the diagram depicted in Fig. 8(b) for \( D = 0.5 \). While in Fig. 8(a) the solution branches of the LSs collapse rapidly to \( \alpha_M \) as increasing \( |A| \), in Fig. 8(b) the collapse is much slower, and therefore, branches of wider structures persist. Examples of these type of LSs are shown in Fig. 8, panels (vii)–(x).

The perturbative analysis of Sec. V shows that localized small amplitude gap solutions of the form Eq. (26) arise from the TI, together with a subcritical periodic gap pattern [see Eq. (25)], whenever \( c_2 \) and \( c_3 \) are both negative. This branching behavior around the TI is shown in detail in the top close-up view of Fig. 8(b), where the solution branch of the gap pattern is shown in red, while the branches of localized gaps associated with \( \varphi = 0 \) and \( \varphi = \pi \) are plotted in green and orange, respectively.

The localized gap states associated with \( \varphi = 0 \) are then unstable between \( \alpha_c \) and SN\(_{g}^L\), and increase their amplitude as approaching SN\(_{g}^L\). An example of this gap state is plotted in Fig. 8, panel (xii). Once SN\(_{g}^L\) is crossed, the gap LS becomes stable [see Fig. 8(b), panel (xi)], continuing stable until reaching \( \alpha_M \). In contrast to the localized gap plotted in Fig. 8, panel (vi), the gap LS shown here possesses oscillatory tails around \( A^- \). As proceeding down in the diagram, the localized gap [see Fig. 8(b), panel (xi)] broadens and, in a periodic domain, eventually becomes the spot state shown in Fig. 8, panel (x).

The subcritical (unstable) gap pattern emerges from the TI with a wavelength \( 2\pi/k_c \approx 12.37 \), and increases its amplitude as approaching SN\(_{g}^L\), where it becomes stable. The stability is then preserved until reaching SN\(_{f,1}^L\), where the branch of patterns folds back and eventually connects with \( A^- \) [see red branches in Fig. 8(b)]. This pattern persists for \( D > D_q \), although becomes unstable and emerges from a \( A^- \) as shown in Fig. 8(a). Note that the subcritical gap pattern may be responsible for the appearance of a homoclinic snaking scenario as already reported in other works on this model [25]. Furthermore, the transition between the collapsed and the homoclinic snaking structures may be related to the presence of isolas of localized patterns as described in Ref. [53]. However, the confirmation of this scenario requires further investigation.

The bifurcation diagram associated with the \( \varphi = \pi \) states is shown in orange in Fig. 9, where for comparison we also add the green diagram corresponding to \( \varphi = 0 \). The modification that the \( \varphi = \pi \) state undergoes along the diagram is shown in profiles (i)–(iv) in Fig. 9. The unstable gap LS arising from the TI stabilizes in SN\(_{g}^L\), becoming the state shown in Fig. 9(i). Decreasing the norm, the two peaks reach \( A_0 \), and two fronts form, leading to the solution shown in Fig. 9(ii). Decreasing \( |A| \) even further the fronts move apart from the center of the domain (i.e., \( x = 0 \)), and the previous state turns into two nonidentical spots like those shown in Fig. 9(iii). This process repeats [see Fig. 9(iv)] until the solution branch terminates near the SN\(_{g}^L\), where it connects with a family of two identical spots [see purple line], like the one shown in Fig. 9(v). Proceeding down, these last states terminate at the transcritical bifurcation at \( \alpha_p \). Proceeding up, each individual spot undergoes collapsed snaking leading to wider states [see Fig. 9(vi)] until reaching \( A^+ \). Note that this scenario is very much alike the one described in [49], and that a similar process may occur for \( D > D_q \).

The collapsed snaking structure is a consequence of the damped oscillatory interaction experienced by the two fronts [see Eq. (22)], and can be understood from the sketch shown in Fig. 5(d). At the Maxwell point (\( \eta = 0 \)) a number of stable and unstable LSs form at the stationary front separations \( \Delta^I \). Stable (unstable) LSs in Fig. 5(d) then correspond to a set of
points on top of the stable (unstable) branches of solutions at $\alpha_M$ in the collapsed snaking diagrams of Fig. 8. As the parameter $\alpha$ separates from $\alpha_M$, the branches of wider LSs, both stable and unstable, start to disappear in a sequence of fold bifurcations, and only narrow LSs survive. This scenario corresponds to the situation shown in Fig. 5(d) for $\eta = \eta_1$, where the number of intersections of $\partial_z \Delta$ with zero decreases, and with it, the number of LSs. In this framework, the fold bifurcations of the collapsed snaking diagram take place when the extrema of $\partial_z \Delta$ become tangent to zero. Increasing $\alpha$ even further, less and less intersections occur [see Fig. 5(d) for $\eta = \eta_2$] until eventually the last fold $SN^r_1$ is passed and the single spot destroyed.

The type of LSs studied here can be extremely useful for predicting the onset of a desertification process. Desertification is related to the presence of the so called desertification fronts occurring for $\alpha < \alpha_M$. Hence, determining how far is the ecosystem from the Maxwell point is quite relevant. The phase diagram shown in Fig. 7 indicates that the region of existence of spot and gap LSs is quite localized around $\alpha_M$. As an example, the presence of the kind of gap LSs studied here indicates that the ecosystem is in the recovery regime. In other words, the presence of single gaps may indicate that a full vegetated state is stable and a flat front is always increasing the biomass of the ecosystem. However, the presence of spots which come from a collapsed snaking is not strictly related with one side of the Maxwell point, but it usually indicates that the ecosystem is in the desertification region. Indeed, considering $D = 0.5$, approximately the 90% of the spots are found in the desertification region, while the 10% belongs to the recovery one.

VII. LOCALIZED STRUCTURES IN THE $(\alpha, \beta)$-PARAMETER SPACE

In previous sections we have fixed the root-to-shoot ratio to $\beta = 2$ and studied how the different types of localized gaps and spots of vegetation, and their bifurcation structure, are modified when changing the diffusion $D$. In this section we explore how the previous scenario changes when the root-to-shoot ratio $\beta$ is varied with fixed $D = 0.1$. Figure 10 shows the phase diagram in the $(\alpha, \beta)$-parameter space, where the main bifurcation lines of the system are plotted.

Region II, i.e., the region of bistability between the bare soil and the uniform vegetation state, shrinks by decreasing $\beta$, and with it, the region of existence of spots, limited by the solid-blue lines ($SN^r_1$ and $SN^l_1$) also shrinks (see light blue area). Furthermore, as proceeding down in $\beta$ the different folds of the spot $SN^l_1$ approach each other and disappear in a cascade of cusp bifurcations at the Maxwell point (not shown here).

In contrast, by increasing $\beta$ the uniform vegetation state becomes more and more subcritical and the region of existence of the spot states widens. This result is closely related with the fact that larger root-to-shoot ratios allow plants to uptake more water from the soil, which in turn can improve their adaptation, and enlarge their stability region, as it was already shown in the context of patterns [41].

In most of this region, the spots undergo collapsed snaking. An example of such snaking is shown in the inset of Fig. 10 for the fixed value $\beta = 4$ (see pointed-dashed gray line). Labels (i) and (ii) correspond to the spot profiles plotted in the subpanels below the phase diagram. For very low values of $\beta$ the region of bistability shrinks, and consequently the region of fronts becomes small and not relevant under small variations in rainfall.

The region between $\alpha_M$ (solid-red line) and the fold $SN^g_1$ (solid-orange line), where gap LSs exist, undergoes a similar behavior: it widens with increasing subcriticality and shrinks otherwise [see light orange area].

In essence, these results show that the LSs studied in this work are robust and persist in a wide range of the parameters of the ecosystem model.

VIII. DISCUSSION AND CONCLUSIONS

In this work we have presented a detailed study of the formation and bifurcation structure of localized vegetation spots and gaps arising in semiarid regions close to the desertification onset, and therefore close to the Maxwell point of the system. To perform this analysis we have focused on the reduced model Eq. (1) in one spatial dimension, that has been derived from different models in plant ecology [15,18,54,57]. However, the results presented here can be relevant in the context of different pattern-forming living systems having...
nonviable states that undergo subcritical instabilities to viable states.

Applying multiscale perturbation theory we have found that small amplitude spots arise from the transcritical bifurcation at \( \alpha_p \equiv 0 \). This state increases its amplitude and eventually undergoes collapsed snaking: the spot solution branches experience a sequence of exponentially decaying oscillations around the Maxwell point of the system. As a result, the localized spots, now formed by the locking of two vegetation fronts of different polarity, increase their width. Indeed the collapsed snaking is a direct consequence of the interaction of fronts as described by Eq. (22). Due to this bifurcation structure, it is much easier to find spots with a single peak than wider structures, which accumulate at parameter values very close to the Maxwell point.

Localized vegetation gaps emerge from the uniform vegetation state at two different periods depending on the value of \( D \). For \( D > D_p \), they arise unstable from the uniform vegetation fold \( \text{SN}_v \), and in a periodic domain, connect back with the spot states at \( \alpha_M \). In contrast, for \( D < D_p \) the situation is rather different. In this case, localized gaps arise together with a periodic vegetation pattern from a Turing instability. The localized gaps arise initially unstable but stabilize in the fold \( \text{SN}_v \). As before, these states connect back with the spots at \( \alpha_M \). In principle, these gap states could undergo homoclinic snaking. However, for the regime of parameters explored here, such structure has not been found.

We have also classified the different types of LSs, both spots and gaps, in two phase diagrams in the \((\alpha, D)\)- and \((\alpha, \beta)\)-parameter space. For a constant \( \alpha \), the \((\alpha, D)\)-phase diagram shows how, the region of existence of both types of LSs widens as \( D \) is decreased, and shrinks otherwise. Fixing \( D \), the \((\alpha, \beta)\)-phase diagram now shows how the region of existence enlarges with increasing \( \beta \), and therefore with the subcriticality of the uniform vegetation state. The enlargement of the LSs stability region may be related with the improvement of plants capacity for uptaking water for large root-to-shoot ratios, as it is the case in the context of vegetation patterns [41].

The LSs presented here are robust, and persist for a large range of parameters. These states arise in the proximity of the Maxwell point of the system, which signals the threshold between desertification and recovering processes. The presence of gap LSs in an ecosystem strongly suggests that the system is in a recovery region, as they only exist \( \alpha > \alpha_M \). Therefore, any flat front expands and covers the bare soil with vegetation. In contrast, spots exist, in approximately a 90\%, for \( \alpha < \alpha_M \), and therefore their presence may indicate that the system is in the desertification region. Thus, any flat front connecting the bare soil with homogeneous vegetation may end up, with large probability, in a completely unproductive ecosystem.

We plan to study the dynamics and bifurcation structure of spots and gaps in two spatial dimensions. In this context the interaction of vegetation fronts is much more complex due to the effect of the front curvature and the presence of front instabilities which are absent in one spatial dimension [57,78–80].

Another interesting line of research is to understand how localized spots and gaps evolve under perturbations of the ecosystem, such as periods of weak droughts. In this context, close to \( \alpha_M \) one could expect that isolated gaps which are temporally perturbed slightly below \( \alpha_M \) (e.g., a weak drought) may trigger gradual desertification, as the system is brought momentarily to the desertification region. In contrast, as the region of stability of spots is larger, weak droughts usually do not generate gradual desertification, whereas strong droughts may imply abrupt desertification.

The ultimate hope is that studies of this kind will prove useful for understanding the dynamics and stability of LSs in pattern forming systems, in particular in the context of plant ecology.

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Appendix: Weakly nonlinear analysis

In this Appendix, we calculate stationary weakly nonlinear dissipative structures using multiple scale perturbation theory near the main bifurcations of the HSS of the system, namely, the transcritical, fold, and Turing bifurcations.

To study these types of solutions we consider the ansatz

\[
A(x, t) = A_x + u(x), \quad (A1)
\]

to decouple Eq. (1) in the equation for the homogeneous state \( A_x \),

\[
-A_x^3 + \beta A_x^2 + \alpha A_x = 0, \quad (A2)
\]

and the stationary equation for the space-dependent component \( u(x) \), namely,

\[
(\mathcal{L} + \mathcal{N})u = 0, \quad (A3)
\]

where \( \mathcal{L} \) and \( \mathcal{N} \) are the linear and nonlinear operators,

\[
\mathcal{L} \equiv \alpha + 2\beta A_x - 3A_x^2 + D\partial_x^2 - A_x\partial_x^2 - A_x^3 , \quad (A4a)
\]

\[
\mathcal{N} \equiv -u^2 + (\beta - 3A_x)u - u\partial_x^2 - u\partial_x^4 . \quad (A4b)
\]

Following Refs. [49,76,81], we fix the value of both \( D \) and \( \beta \), consider \( \alpha \) as the bifurcation parameter, and for each case, we introduce appropriate asymptotic expansions for \( A_x, u, \) and \( x \) in terms of \( \epsilon \). In what follows we show the detail calculation for each of the cases considered in this manuscript.

1. Weakly nonlinear states near the transcritical bifurcation

The transcritical bifurcation occurs at \( A_x = A_p \equiv 0 \) at the parameter value \( \alpha_p \), and the solution of the system reduces to \( A(x, t) = u(x, t) \). In this particular case the linear and nonlinear operators read

\[
\mathcal{L} \equiv \alpha + D\partial_x^2, \quad (A5a)
\]

\[
\mathcal{N} \equiv -u^2 + \beta u - u\partial_x^2 - u\partial_x^4 . \quad (A5b)
\]

In this case, an appropriate asymptotic expansion for the control parameter \( \alpha \), as a function of the expansion parameter

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\( \epsilon, \) is \( \alpha = \alpha_p + \epsilon \delta, \) whereas the space dependent variable \( u \) can be expanded as
\[
  u(X) = \epsilon u_1(X) + \epsilon^2 u_2(X) + \epsilon^3 u_3(X) + \cdots, \tag{A6}
\]
where any variable depends on the long-scale variable \( X \equiv \sqrt{\epsilon} x. \) In this way the differential operator becomes \( \partial_X^2 = \epsilon \partial_X^2. \)

With these considerations, the linear operator expands as \( \mathcal{L} = \mathcal{L}_0 + \epsilon \mathcal{L}_1 + O(\epsilon^2), \) with
\[
  \mathcal{L}_0 = \alpha_p, \quad \mathcal{L}_1 = \delta + D \partial_X^2, \tag{A7}
\]
while the nonlinear operator develops as \( \mathcal{N} = \epsilon \mathcal{N}_1 + O(\epsilon^2), \) with
\[
  \mathcal{N}_1 = \beta u_1. \tag{A8}
\]

Inserting the previous expansions in Eq. (A3), and matching the different terms at the same order in \( \epsilon \) we get the next two equations at order \( \epsilon \) and \( \epsilon^2: \)
\[
  \begin{align*}
  O(\epsilon) : & \quad \mathcal{L}_0 u_1 = 0, \tag{A9a} \\
  O(\epsilon^2) : & \quad \mathcal{L}_0 u_2 + (\mathcal{L}_1 + \mathcal{N}_1) u_1 = 0. \tag{A9b}
\end{align*}
\]
The solvability condition at \( O(\epsilon) \) implies that \( \alpha_p = 0 \) for a solution \( u_1 = a(X). \)

At \( O(\epsilon^2), \) Eq. (A9b) reduces to
\[
  (\mathcal{L}_1 + \mathcal{N}_1) u_1 = 0, \tag{A10}
\]
which leads to the amplitude equation
\[
  c_1 \partial_X^2 a(X) + c_2 a(X) + c_3 a(X)^2 = 0, \tag{A11}
\]
with
\[
  c_1 = D, \quad c_2 = \delta, \quad c_3 = \beta. \tag{A12}
\]
This amplitude equation has two homogeneous solutions,
\[
a(\delta + \beta a) = 0, \tag{A13}
\]
as corresponds to a transcritical bifurcation, and furthermore, supports pulse solutions of the form
\[
a(X) = -\frac{3}{2} \frac{c_2}{c_3} \text{sech}^2 \left( \frac{1}{2} \sqrt{-\frac{c_2}{c_1}} X \right), \tag{A14}
\]
i.e.,
\[
a(X) = -\frac{3}{2} \frac{\delta}{\beta} \text{sech}^2 \left( \frac{1}{2} \sqrt{-\frac{\delta}{D}} X \right), \tag{A15}
\]
Thus, we conclude that, in a neighborhood of \( \alpha_p, \) small amplitude spots of the form
\[
  A(x) = -\frac{3}{2} \frac{\alpha}{\beta} \text{sech}^2 \left( \frac{1}{2} \sqrt{-\frac{\alpha}{D}} x \right) + O(\alpha^2), \tag{A16}
\]
exist for negative values of \( \alpha. \)

2. Weakly nonlinear states near the fold bifurcation

Here we perform weakly nonlinear analysis about the fold bifurcation of the uniform state \( \text{SN}_1, \) occurring at \( \lambda_r. \) In this case, a proper asymptotic expansion for the control parameter \( \alpha, \) and the uniform and space-dependent variables \( A_\lambda \) and \( u, \) about the fold reads
\[
  \begin{align*}
  \alpha &= \alpha_r + \delta \epsilon^2, \tag{A17a} \\
  A_\lambda &= A_r + \epsilon A_1 + \epsilon^2 A_2 + \cdots, \tag{A17b} \\
  u(X) &= \epsilon u_1(X) + \epsilon^2 u_2(X) + \cdots. \tag{A17c}
\end{align*}
\]
where the space dependent variables are functions of the long-scale \( X = \sqrt{\epsilon} x. \) In what follows we first solve the homogeneous problem Eq. (A2) and later Eq. (A3).

a. Solution of the uniform problem

Inserting the asymptotic expansion Eq. (A17b) in Eq. (A2) we derive the following set of linear equations for the uniform state:
\[
  \begin{align*}
  O(\epsilon^0) : & \quad ( -A^2_r + \beta A_r + \alpha_r ) A_r = 0, \tag{A18a} \\
  O(\epsilon^1) : & \quad \mathcal{L}_0 A_1 \equiv ( -3A^2_r + 2\beta A_r + \alpha_r ) A_1 = 0, \tag{A18b} \\
  O(\epsilon^2) : & \quad \mathcal{L}_0 A_2 - 3A_1 A_r^2 + \beta A_r^2 + \delta A_r = 0. \tag{A18c}
\end{align*}
\]
The solutions at \( O(\epsilon^0) \) gives
\[
  \alpha_r = A^2_r - \beta A_r. \tag{A19}
\]
The equation at \( O(\epsilon^1) \) has a nontrivial solution (i.e., \( A_1 \neq 0 \)) if \( \mathcal{L}_0 = 0, \) from where one obtains
\[
  \alpha_r = 3A^2_r - 2\beta A_r. \tag{A20}
\]
From conditions Eqs. (A19) and (A20) one finally gets the value of \( A \) at the fold,
\[
  A_r = \frac{\beta}{2}. \tag{A21}
\]
The solution \( A_1 \) is obtained from the solvability condition at \( O(\epsilon^2), \) namely,
\[
  -3A_1 A_r^2 + \beta A_r^2 + \delta A_r = 0, \tag{A22}
\]
from where one obtains
\[
  A_1 = \pm \sqrt{\frac{\delta A_r}{3A_r - \beta}} = \pm \sqrt{\delta}. \tag{A23}
\]

b. Solution of the space-dependent problem

Considering the asymptotic expansion Eq. (A17c), the linear operator Eq. (A4a) becomes \( \mathcal{L} = \mathcal{L}_0 + \epsilon \mathcal{L}_1 + O(\epsilon^2), \) with \( \mathcal{L}_0 \) defined previously in Eq. (A18b), and
\[
  \mathcal{L}_1 = 2\beta A_1 - 6A_1 A_1 + (D - \delta) \partial_X^2. \tag{A24}
\]
whereas the nonlinear operator expands as \( \mathcal{N} = \epsilon \mathcal{N}_1 + O(\epsilon^2), \) with
\[
  \mathcal{N}_1 = (\beta - 3A_r) u_1. \tag{A25}
\]
The insertion of the previous expansions in Eq. (A3) yields to
the set of equations
\[ O(\epsilon^1) : \mathcal{L}_0 u_1 = 0, \quad (A26a) \]
\[ O(\epsilon^2) : \mathcal{L}_0 u_2 + (\mathcal{L}_1 + N_1) u_1 = 0. \quad (A26b) \]

From the equation at \(O(\epsilon^1)\) one gets that the nontrivial solutions must be proportional to \(A_1\), and therefore we can write
\[ u_1 = A_1 a(X) = \sqrt{\delta} a(X). \quad (A27) \]

We look for pulse solutions biasymptotic to the top homogeneous branch \(A^+\), and then we choose \(A_1 = \sqrt{\delta}\), instead of \(A_1 = -\sqrt{\delta}\) which corresponds to \(A^-\).

Finally, at \(O(\epsilon^2)\) the solvability condition imposes
\[ (\mathcal{L}_1 + N_1) u_1 = 0, \quad (A28) \]
where
\[ \mathcal{L}_1 + N_1 = \frac{\beta \sqrt{\delta}}{2} \left[ 2D - \beta \frac{\partial^2}{\sqrt{\delta} \beta} \partial_X^2 - 2 - a(X) \right]. \quad (A29) \]

which finally leads to the amplitude equation
\[ c_1 \partial_X^2 a(X) + c_2 a(X) + c_3 a(X)^2 = 0, \quad (A30) \]
which has the same form than Eq. (A11), with
\[ c_1 = \frac{2D - \beta}{\sqrt{\delta} \beta}, \quad c_2 = -2, \quad c_3 = -1. \quad (A31) \]

The pulse solution
\[ a(X) = -\frac{3c_2}{2c_3} \text{sech}^2 \left( \frac{1}{2} \sqrt{\frac{c_2}{c_1}} X \right), \quad (A32) \]
yields, in this case, to
\[ a(x) = -3 \text{sech}^2 \left( \frac{1}{2} \sqrt{\frac{2\beta}{2D - \beta}} (\alpha - \alpha_c)^{1/4} x \right). \quad (A33) \]
which exists providing that \(2D - \beta > 0\).

Hence, a weakly nonlinear gap localized solution of the form
\[ A(x) = \frac{\beta}{2} + \sqrt{\alpha - \alpha_c} [1 + a(x)] \quad (A34) \]
arises from the fold \(\alpha_c\) if \(2D - \beta > 0\).

3. Weakly nonlinear states near the Turing bifurcation

Considering that for a fixed value of \(\beta\) and \(D\) the Turing bifurcation occurs at a given point \((\alpha, A_c) = (\alpha_c, A_c)\), an appropriate asymptotic expansion in terms of \(\epsilon\) for the different variable reads
\[ \alpha = \alpha_c + \delta \epsilon^2, \quad (A35a) \]
\[ A_c = A_c + \epsilon^2 A_2 + \cdots, \quad (A35b) \]
\[ u(x, X) = \epsilon u_1(x, X) + \epsilon^2 u_2(x, X) + \epsilon^3 u_3(x, X) + \cdots. \quad (A35c) \]

where the space-dependent variables are functions of the long scale \(X \equiv \epsilon x\) and \(x\).

\[ a. \text{ Solution of the uniform problem} \]
Considering the asymptotic expansion Eq. (A35b) we derive the following hierarchical equations for the homogeneous state
\[ O(\epsilon^0) : \left( - A_c^2 + \beta A_c + \alpha_c \right) A_c = 0, \quad (A36a) \]
\[ O(\epsilon^2) : \mathcal{M}_0 A_2 + \delta A_c = 0, \quad (A36b) \]
with
\[ \mathcal{M}_0 = -3A_c^2 + 2\beta A_c + \alpha_c. \quad (A36c) \]
The solutions at \(O(\epsilon^0)\) are
\[ \alpha_c = A_c^2 - \beta A_c, \quad (A37) \]
and \(A_c = 0\).

The equation at \(O(\epsilon^2)\), leads to the solution
\[ A_2 = -\delta \tilde{A}_2 = -\delta A_c \frac{\mathcal{M}_0}{\mathcal{M}_0}. \quad (A38) \]

\[ b. \text{ Solution of the space-dependent problem} \]

The expansion Eq. (A35c) for space-dependent state implies \(L = L_0 + \epsilon L_1 + \epsilon^2 L_2 + \cdots\), with
\[ L_0 = \mathcal{M}_0 + (D - A_c) \partial_X^2 - A_c \partial_X^4, \quad (A39a) \]
\[ L_1 = 2(D - A_c) \partial_X \partial_{\partial_X} - 4A_c \partial_X^3, \quad (A39b) \]
\[ L_2 = (1 - 2\beta A_c + 6A_c A_2) \delta + (D - A_c) \partial_X^2 - A_c (\partial_X^2 + \partial_{\partial_X}^2) - 6A_c \partial_X^2 \partial_X^2 \partial_{\partial_X}^2, \quad (A39c) \]

and \(N = \mathcal{N}_1 + \epsilon^2 \mathcal{N}_2 + \cdots\) for the nonlinear operator, where
\[ \mathcal{N}_1 = (\beta - 3A_c) u_1 - u_1 \left( \partial_X^2 + \partial_{\partial_X}^2 \right), \quad (A40a) \]
\[ \mathcal{N}_2 = -u_1^2 + (\beta - 3A_c) u_2 - u_2 \left( \partial_X^2 + \partial_{\partial_X}^2 \right) - 2u_1 (\partial_X \partial_{\partial_X} + 2 \partial_{\partial_X} \partial_{\partial_{\partial_X}}). \quad (A40b) \]

The insertion of the previous expansion in Eq. (A3) yields to the set of equations
\[ O(\epsilon^1) : \mathcal{L}_0 u_1 = 0, \quad (A41a) \]
\[ O(\epsilon^2) : \mathcal{L}_0 u_2 + (\mathcal{L}_1 + N_1) u_1 = 0, \quad (A41b) \]
\[ O(\epsilon^3) : \mathcal{L}_0 u_3 + (\mathcal{L}_1 + N_1) u_2 + (\mathcal{L}_2 + N_2) u_1 = 0. \quad (A41c) \]

To solve the \(O(\epsilon^1)\) equation we consider the ansatz
\[ u_1(x, X) = a(X) e^{ik_c x} + c.c., \quad (A42) \]
from where we can derive the solvability condition
\[ A_c k^2_c + (D - A_c) k^2_c - 2\beta A_c - \alpha_c + 3A_c^2 = 0, \quad (A43) \]
as it was already done in Sec. IV.

At \(O(\epsilon^2)\), the solvability condition is obtained by projecting on the subspace defined by the null eigenvector of the self-adjoint operator. To obtain this condition we first define
the scalar product

$$\langle f|g \rangle = \frac{1}{l} \int_{-l/2}^{l/2} f(x)g(x)dx,$$

where $l = 2\pi/k_c$. With this definition, $\mathcal{L}_0$ is self-adjoint (i.e., $\mathcal{L}_0 = \mathcal{L}_0^\dagger$), and the null eigenspace is spanned by the two null eigenvectors $w = \{e^{ik_c x}, e^{-ik_c x}\}$, such that $\mathcal{L}_0^\dagger w = 0$.

To calculate this condition we first write

$$\langle f|\mathcal{L}_1 + \mathcal{N}_1 |u_1 \rangle = f_0\|a\|^2 + f_1 e^{i\phi} ae^{ik_c x} + f_2 a^2 e^{2ik_c x} + \text{c.c.}, \quad (A44)$$

where

$$f_0 = 2(\beta - 3A_c + k_c^2 - k_0^2), \quad (A45a)$$
$$f_1 = 2k_c(D + 2k_0^2A_c - A_c), \quad (A45b)$$
$$f_2 = f_0/2. \quad (A45c)$$

The solvability condition then implies

$$\langle w|\mathcal{L}_1 + \mathcal{N}_1 |u_1 \rangle = 0, \quad (A46)$$

which leads to $f_1 = 0$, or equivalently to the nontrivial critical wave number

$$k_c^2 = \frac{A_c - D}{2A_c}. \quad (A47)$$

Once this condition is satisfied we can solve Eq. (A41b) considering the ansatz

$$u_2(x, X) = W_0\|a\|^2 + W_1 i\phi a e^{ik_c x} + W_2 a^2 e^{2ik_c x} + \text{c.c.}, \quad (A48)$$

and matching the coefficients with the same element of the base $\{e^{i(k_c x)}\}$. Thus, we obtain

$$W_0 = -f_0/M_0, \quad (A49)$$
$$W_1 = \frac{-f_1}{M_0 + (A_c - D)k_c^2 - A_c k_0^2} = 0, \quad (A50)$$

which follows from the solvability condition Eq. (A46), and

$$W_2 = \frac{-f_2}{M_0 + 4(A_c - D)k_c^2 - 16A_c k_0^2}. \quad (A51)$$

Finally, the solvability condition at $O(e^3)$ leads to an equation describing the amplitude of the Turing mode $a(X)$. The second term in Eq. (A41c) becomes

$$\langle f|\mathcal{L}_1 + \mathcal{N}_1 |u_2 \rangle = g_0 + g_1 e^{ik_c x} + g_2 e^{2ik_c x} + g_3 e^{3ik_c x} + \text{c.c.}, \quad (A52)$$

with

$$g_1(X) = g_1'(X) a(X)^2 a(X)$$
$$= ([\beta - 3A_c](W_0 + W_2) + (4k_c^2 - 16k_0^2)W_2]$$
$$\times |a(X)|^2 a(X), \quad (A53)$$

whereas the third term becomes

$$\langle f|\mathcal{L}_2 + \mathcal{N}_2 |u_1 \rangle = h_0 + h_1 e^{ik_c x} + h_2 e^{2ik_c x} + h_3 e^{3ik_c x} + \text{c.c.}, \quad (A54)$$

with

$$h_1 = h_1' \delta a(X) + h_1' a^2(X) + h_1' |a(X)|^2 a(X). \quad (A55)$$

FIG. 11. Coefficients $c_3$ in panel (a) and $c_2$ in panel (b) as a function of $D$ for $\beta = 2$. The vertical orange dashed line represents the condition $D = \beta/2$. Thus, $c_2$ is negative for $D < \beta/2$ and positive otherwise. For the range of parameter considered in this work $c_3$ is always negative.

and

$$h_0' = 1 + 2\beta \tilde{A}_2 - 6A_c \tilde{A}_2 + (k_c^2 - k_0^2)\tilde{A}_2, \quad (A56a)$$
$$h_1' = D - A_c + 6A_c k_c^2, \quad (A56b)$$
$$h_1' = (\beta - 3A_c)(W_0 + W_2) + (k_c^2 - k_0^2)(W_0 + W_2) - 3. \quad (A56c)$$

The solvability condition

$$\langle w|\mathcal{L}_1 + \mathcal{N}_1 |u_2 \rangle + \langle w|\mathcal{L}_2 + \mathcal{N}_2 |u_1 \rangle = 0, \quad (A57)$$

then yields to the amplitude equation

$$\delta a(X) + c_2 \partial_X^2 a(X) + c_3 |a(X)|^2 a(X) = 0, \quad (A58)$$

with

$$c_2 = h_1'/h_0', \quad (A59a)$$
$$c_3 = \left(g_1' + h_1'/h_1'\right). \quad (A59b)$$

Due to the complex form of Eq. (11), the simplification of the coefficients $c_1$ and $c_2$ to a simple expression of $\beta$ and $D$ is not possible. However, using symbolic software we find that $c_2$ cancels out exactly at $D = D_0 = \beta/2$ (see Sec. III), being negative ($c_2 < 0$) for $D < \beta/2$ and positive otherwise.

We can solve Eq. (A58) by considering the ansatz $a(X) = \psi(X)e^{i\phi}$. If $\psi(X) \equiv \psi$, then the amplitude equation reduces to

$$\psi(\delta + c_3 \psi^2) = 0, \quad (A60)$$

which implies the solutions $\psi = 0$ and $\psi = \sqrt{-\delta/c_3}$. This last solution corresponds to a periodic gap pattern of the form

$$A(x) = A_c + (\alpha - \alpha_c)\tilde{A}_2 + 2\sqrt{\frac{\alpha - \alpha_c}{c_3}} \cos(k_c x + \phi), \quad (A61)$$

that arises sub- or supercritical depending on the sign of the coefficient $c_3$: if $c_3 < 0$, then the pattern arises...
subcritically and supercritically otherwise. Figure 11(a) shows $c_3$ as a function of $D$ for $\beta = 2$. As we can observe, this coefficient is negative for any value of $D$, what means that the periodic pattern is born subcritically from the TI. For the range of $\beta$ studied in this manuscript $c_3$ is always negative.

In the subcritical regime, moreover, Eq. (A58) also has a localized solution of the form

$$\psi(X) = \sqrt{-\frac{2\alpha}{\epsilon_3}} \text{sech}\left(\sqrt{-\frac{\delta}{\epsilon_2}} X\right),$$

(A62)

which corresponds to the state

$$A(x) = A_c + (\alpha - \alpha_c)\hat{A}_2 - 2\sqrt{2(\alpha - \alpha_c) - c_3} \text{sech}\left(\sqrt{\frac{\alpha - \alpha_c}{-c_2}} x\right) \cos(k_c x + \phi).$$

(A63)

This state arises subcritically from the TI together with the gap pattern Eq. (A61), and exists whenever $c_3$ is negative. In Fig. 11(b) we plot $c_2$ as a function of $D$ for $\beta = 2$. $c_2$ cancels out at exactly $D = \beta / 2$, as is marked with vertical orange dashed line. Therefore, gap LSs of the form Eq. (A63) exist whenever $D < \beta / 2$.

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In principle, the derivation of Eq. (22) from Eq. (1) can be done applying perturbation theory about the Maxwell point \(\alpha_M\), where the fronts \(F^+\) and \(F^-\) are stationary. For \(\alpha = \alpha_M + \epsilon\), with \(|\epsilon| \ll 1\), these two fronts can move and interact. To study the interaction one can consider the solution ansatz \(A(x,t) = F^+ [x - \Delta(t)] + F^- [x + \Delta(t)] - A^\mp + \phi(x, \Delta(t))\), where \(\phi\) is a correction, \(\Delta\) is the separation between the front cores, and \(\epsilon \sim \phi \sim \Delta\). Inserting this ansatz in Eq. (1), and applying some approximations related with the overlapping of the fronts, one can obtain Eq. (22) as the solvability condition at first order in \(\epsilon\). However, to do so the asymptotic behavior of the front, and the nullvector of the linear operator \(L\) [see Eq. (6)] evaluated at \(F^+\) are required. In our case, no analytical front solution is known, and moreover the linear operator is not self-adjoint. For these reasons the derivation of Eq. (22) is intractable.
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