VIRTUAL LEGENDRIAN ISOTOPY

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Abstract. An elementary stabilization of a Legendrian link $L$ in the spherical cotangent bundle $ST^*M$ of a surface $M$ is a surgery that results in attaching a handle to $M$ along two discs away from the image in $M$ of the projection of the link $L$. A virtual Legendrian isotopy is a composition of stabilizations, destabilizations and Legendrian isotopies.

We study virtual Legendrian isotopy classes of Legendrian links and show that every such class contains a unique irreducible representative. In particular we get a solution to the following conjecture of Cahn, Levi and the first author: two Legendrian knots in $ST^*S^2$ that are isotopic as virtual Legendrian knots must be Legendrian isotopic in $ST^*S^2$.

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1. Introduction

Let $M$ be a closed oriented surface, possibly non-connected, and $L$ a Legendrian link in the total space of the spherical cotangent bundle $\pi : ST^*M \to M$ of $M$. An elementary stabilization of $L$ is a surgery that results in cutting out from $M$ two discs away from the image $\pi L$ of the projection of $L$ to $M$, and attaching a handle to $M$ along the created boundary components. The converse operation is called an elementary destabilization. More precisely, let $A$ be a simple connected closed curve in $M$ in the complement to $\pi L$. An elementary destabilization of $L$ along $A$ consists of cutting $M$ open along $A$ and then capping the resulting boundary circles.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{stabilization.png}
\caption{An elementary stabilization of a Legendrian curve in the spherical cotangent bundle of $\mathbb{R}^2$.}
\end{figure}

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A virtual Legendrian isotopy \([1]\) is a composition of elementary stabilizations, destabilizations, and Legendrian isotopies.

An elementary destabilization of a link is trivial if it chops off a sphere containing no components of \(L\). We say that a Legendrian link is irreducible if it does not allow non-trivial destabilizations.

Our main result is Theorem 1 which should be compared to the Kuperberg Theorem on virtual links, \([2]\), Theorem 1.

**Theorem 1.** Every virtual isotopy class of Legendrian links contains a unique irreducible representative. The irreducible representative can be obtained from any representative of the virtual Legendrian isotopy class by a composition of destabilizations and isotopies.

The second assertion of Theorem 1 is immediate. Indeed, for any Legendrian link in the given virtual Legendrian isotopy class, only finitely many consecutive non-trivial destabilizations are possible. Thus, after finitely many destabilizations we obtain an irreducible representative.

The main consequence of Theorem 1 is Corollary 2.

**Corollary 2.** Virtual Legendrian isotopy classes of irreducible Legendrian links in \(ST^*M\) of a surface \(M\) are in bijective correspondence with isotopy classes of irreducible Legendrian links in \(ST^*M\).

In particular we get the solutions to the following two Conjectures formulated by P. Cahn, A. Levi and the first author \([1]\), Conjecture 1.4, Conjecture 1.5].

**Conjecture 3.** Let \(K_1\) and \(K_2\) be two Legendrian knots in \(ST^*M\) that are isotopic as virtual Legendrian knots and suppose that \(M\) is the surface of smallest genus realizing knots in the virtual Legendrian isotopy class of \(K_1\) and \(K_2\). Then possibly after a contactomorphism of \(ST^*M\) \(K_1\) and \(K_2\) are Legendrian isotopic in \(ST^*M\).

**Conjecture 4.** Two Legendrian knots in \(ST^*S^2\) that are isotopic as virtual Legendrian knots must be Legendrian isotopic in \(ST^*S^2\).

In \([1]\), Conjecture 1.4] and \([1]\), page 5] a similar fact is also conjectured for virtual Legendrian knots in \(ST^*S^n, n \geq 3\) and \(ST^*\mathbb{R}^n, n \geq 2\). These conjectures are still open.

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2. **Proof of Theorem 1**

We say that two links \(L_1\) in \(ST^*M_1\) and \(L_2\) in \(ST^*M_2\) are descent-equivalent if after a composition of destabilizations and isotopies of \(L_1\) and \(L_2\) they become the same.
Suppose, contrary to the statement of Theorem 1, there is a Legendrian link \( L \) in \( ST^*M \) that has two different irreducible descendants. Then there are two Legendrian links \( L_1 \) and \( L_2 \) in \( ST^*M \), both isotopic to \( L \), and two simple connected closed connected curves \( A_1 \) and \( A_2 \) in \( M \) such that each \( A_i \) is disjoint from \( \pi L_i \), and such that the elementary destabilizations of \( L_1 \) along \( A_1 \) and of \( L_2 \) along \( A_2 \) are not descent-equivalent; in particular, both destabilizations are non-trivial.

Without loss of generality we may assume that \( M \) has no naked sphere components and its genus is minimal (among those surfaces for which there exist \( L_1, L_2, A_1, A_2 \) as above); in particular, every link obtained from \( L \) by an elementary non-trivial destabilization has a unique descendent.

Our main tool is Theorem 9. In the hypothesis of Theorem 9 we require that the manifold \( M \) is distinct from a sphere, and that \( A_1 \) does not bound a disc. Let us show that these assumptions are not restrictive.

**Lemma 5.** The statement of Theorem 7 is true in the case where \( M \) is a sphere.

The proof is a particular case of the proof of the next Lemma.

**Lemma 6.** Suppose that \( A_1 \) bounds a disc. Then the destabilization of \( L_1 \) along \( A_1 \) is descent-equivalent to the destabilization of \( L_2 \) along \( A_2 \).

**Proof.** We will show that we can assume that the intersection of \( A_1 \) and \( A_2 \) is empty and hence destabilizations along \( A_1 \) and \( A_2 \) are descent equivalent. In more detail, we assume that \( \pi L_1 \) is located close to the center of the disk \( D \) bounded by \( A_1 \). If \( A_1 \) and \( A_2 \) intersect, take such a pair of curves with the minimal number of intersection points among those pairs of curves destabilizations along which are not descent equivalent.

We show that the number of intersection points may be further decreased yielding a contradiction. \( A_2 \) subdivides the disk \( D \) into regions, by induction at least two of these regions are bigons and one of them does not contain the center of \( D \) (with \( \pi L_1 \) in its small neighborhood). So the bigon does not contain any components of \( \pi L_1 \) and we can compress the arc \( \alpha_2 \) along this bigon in such a way that during the compression it does not intersect \( \pi L_1 \). If this bigon contains curves of \( \pi L_2 \) they will be pushed out through \( \alpha_1 \) during the compression. \( \square \)

We will need the following lemma.

**Lemma 7.** If \( L_1 = L_2 = L \), then the genus of \( M \) is not minimal.

**Proof.** The argument is similar to that by Greg Kuperberg. Namely, assume, contrary to the statement, that \( L_1 = L_2 \) and the genus of \( M \) is minimal.

It follows that the intersection \( A_1 \cap A_2 \) is non-empty; otherwise destabilizations of \( L \) along \( A_1 \) and \( A_2 \) are descent-equivalent. Without loss of
generality we may assume that \( A_1 \) and \( A_2 \) intersect in the minimal number of points among pairs of simple connected closed curves destabilizations along which are not descent-equivalent.

If the two curves \( A_1 \) and \( A_2 \) intersect at only one point, then take the boundary \( A_3 \) of a neighborhood of \( A_1 \cup A_2 \). Note that the destabilization along \( A_3 \) is not trivial; it chops off a naked torus. Destabilizations along \( A_1 \) and \( A_3 \) are descent equivalent since the curves are disjoint. Similarly for \( A_2 \) and \( A_3 \). Therefore destabilizations along \( A_1 \) and \( A_2 \) are descent-equivalent, contrary to the assumption.

Finally, suppose that \( A_1 \) and \( A_2 \) have at least two common points. Let \( D_1 \) be an interval in \( A_1 \) bounded by two intersection points and containing no other points of \( A_2 \). Compress \( A_2 \) along \( D_1 \), i.e., remove small arcs of \( A_2 \) intersecting \( A_1 \), and then join the two pairs of boundary points of \( A_2 \) by two new arcs parallel to \( D_1 \). Then \( A_2 \) turns into two new connected curves \( A'_2 \) and \( A''_2 \) in \( M \). The destabilization along at least one of these components, say \( A'_2 \), is non-trivial. Observe that the destabilizations of \( L_2 \) along \( A_2 \) and \( A'_2 \) are equivalent since both are disjoint from \( \pi L_2 \) and have no common points (after a small displacement of one of them along a vector field orthogonal to the curve). On the other hand, the curve \( A'_2 \) has less intersection points with \( A_1 \). Therefore destabilizations along \( A_1, A'_2 \) and \( A_2 \) are descent equivalent. This completes the proof.

To motivate the proof of Theorem 9 let us prove Lemma 8.

**Lemma 8.** Let \( M \) be a hyperbolic surface. Let \( L_1 \) and \( L_2 \) be two Legendrian links in \( ST^* M \) whose projections belong to an open disc \( D \subset M \). Then \( L_1 \) and \( L_2 \) are isotopic in \( ST^* M \) if and only if they are isotopic in \( ST^* D \).

**Proof.** Clearly if \( L_1, L_2 \) are isotopic in \( ST^* D \) then they are isotopic in \( ST^* M \), let us prove the other implication.

Let \( \gamma : \mathbb{R}^2 \to M \) denote the universal covering of \( M \). We may choose lifts \( L'_1 \) and \( L'_2 \) of \( L_1 \) and \( L_2 \) respectively so that the isotopy of \( L_1 \) to \( L_2 \) lifts to an isotopy of \( L'_1 \) to \( L'_2 \) in \( ST^* \mathbb{R}^2 \). Choose an arbitrary diffeomorphism \( \varphi : \mathbb{R}^2 \to D^2 \). It lifts to a contactomorphism \( \tilde{\varphi} \) of spherical cotangent bundles. Thus, we obtain a Legendrian isotopy of \( \tilde{\varphi}(L'_1) \) to \( \tilde{\varphi}(L'_2) \). It remains to show that \( L_1 \) admits a Legendrian isotopy to \( \tilde{\varphi}(L'_1) \) and \( L_2 \) admits a Legendrian isotopy to \( \tilde{\varphi}(L'_2) \).

We may assume that both \( L_1 \) and \( L_2 \) are links whose images with respect to \( \pi \) are located in a small neighborhood \( U \) of a point in \( D \). Furthermore we may choose \( \varphi \) so that the composition of a lift of \( D \) and \( \varphi \) is the identity map on \( U \) so that \( \tilde{\varphi}(L_i) = L_i, i = 1, 2 \). Then, it remains to show that for any link \( L \) in \( ST^* D \) and any lifts \( L' \) and \( L'' \) in \( ST^* \mathbb{R}^2 \), the link \( \tilde{\varphi}(L') \) admits a Legendrian isotopy to \( \tilde{\varphi}(L'') \). Choose a Legendrian isotopy \( \gamma \) from \( L' \) to \( L'' \) in \( ST^* \mathbb{R}^2 \). The desired Legendrian isotopy is \( \tilde{\varphi}(\gamma) \). □

**Theorem 9.** Let \( L_1 \) and \( L_2 \) be isotopic Legendrian links in the spherical cotangent bundle \( ST^* M \) of a connected closed surface \( M \neq S^2 \), and let \( A \)
be a simple connected closed curve in $M$ disjoint from $\pi L_1$ and $\pi L_2$. If $A$ breaks $M$ into two surfaces, suppose that $\pi L_1$ and $\pi L_2$ belong to the same path component of $M \setminus A$, and the other path component of $M \setminus A$ is distinct from the disc. Then there exists a Legendrian isotopy of $L_1$ to $L_2$ whose projection to $M$ avoids the curve $A$.

Before proving Theorem 9, let us construct an (in general, non-regular) covering of $M$ by a surface $\tilde{M}$ homeomorphic to the connected component of $M \setminus A$ which contains $\pi(L_1)$ and $\pi(L_2)$. In fact we will give three equivalent definitions, each has its advantage.

**Definition 10** (First definition). Choose a base point in $M$ in the path component of $M \setminus A$ that contains $\pi L_1$ and $\pi L_2$. We say that an element in the fundamental group $\pi_1 M$ avoids $A$ if it admits a representing curve that does not intersect $A$. The subset of elements in $\pi_1 M$ avoiding $A$ forms a subgroup. Let $\tilde{M} \to M$ be the covering corresponding to the subgroup of $\pi_1 M$ of elements avoiding $A$.

**Definition 11** (Second definition). Since $M$ is distinct from a sphere, it admits a universal covering $u: \mathbb{R}^2 \to M$. We choose a base point in $\mathbb{R}^2$ that projects to the base point in $M$. Then every point $x$ in the universal covering space can be identified with the pair of a point $y = u(x)$ and the homotopy class of the projection in $M$ of the curve in $\mathbb{R}^2$ from the base point to $x$. The manifold $\tilde{M}$ is the quotient of $\mathbb{R}^2$ by the relation that identifies $(y, \gamma_1)$ with $(y, \gamma_2)$ whenever $\gamma_1 \gamma_2^{-1}$ contain a loop that does not intersect $A$.

**Definition 12** (Third definition). Suppose that $A$ does not separate $M$. Since $M$ is either a torus or hyperbolic, there is an infinite covering $\mathbb{H} \to M$ (or $\mathbb{R}^2 \to M$), and we may assume that a lift $\tilde{A}$ of $A$ is a geodesic (every simple non-contractible curve on $M$ is isotopic to a unique simple geodesic). There is a monodromy action $\mathbb{Z}$ on $\mathbb{H}$ (or on $\mathbb{R}^2$) corresponding to the loop $A$; namely, we know that $M$ is the quotient of $\mathbb{H}$ (or of $\mathbb{R}^2$) by the action of $\pi_1 M$, and the mentioned monodromy action is the action by the subgroup generated by the loop $A$. It acts on the geodesic $\tilde{A}$ by translations. Attach $(\mathbb{H} \setminus \tilde{A})/\mathbb{Z}$ (or $(\mathbb{R}^2 \setminus \tilde{A})/\mathbb{Z}$) to $M \setminus A$ so that the projections of the two cylinders $(\mathbb{H} \setminus \tilde{A})/\mathbb{Z}$ (or of $(\mathbb{R}^2 \setminus \tilde{A})/\mathbb{Z}$) and of the manifold $M \setminus A$ to $M$ form an infinite covering $\tilde{M} \to M$; this is the desired covering.

Suppose now that $A$ separates $M$ into two components $M_1$ and $M_2$, where $M_1$ is the component containing the images of the projections of $L_1$ and $L_2$ to $M$. Again, take a covering $\mathbb{H} \to M$ (respectively $\mathbb{R}^2 \to M$) and cut $\mathbb{H}$ (respectively $\mathbb{R}^2$) along a lift $\tilde{A}$ of $A$. Attach one component of $(\mathbb{H} \setminus \tilde{A})/\mathbb{Z}$ (respectively of $(\mathbb{R}^2 \setminus \tilde{A})/\mathbb{Z}$) to $M_1$ so that their projections to $M$ form a desired covering $\tilde{M} \to M$.

**Remark 13.** If $A$ bounds a disc, i.e., the case that we exclude from the consideration, then the first and the second definitions result in the one sheet covering, while the third definition makes no sense since a lift of a contractible curve $A$ is not a geodesic.
Let $M, A$ be as in Theorem 9 and $\tilde{M} \to M$ be the covering from the definitions 10, 11, 12. If $A$ does not separate $M$, let $M_1$ denote $M \setminus A$. If $A$ does separate $M$, let $M_1$ denote the connected component of $M \setminus A$ that contains the projections of $L_1, L_2$.

**Lemma 14.** The surface $\tilde{M}$ is homeomorphic to $M_1$.

**Proof.** The statement of Lemma 14 immediately follows from Definition 12. Indeed, the manifold $\tilde{M}$ is obtained from $M_1$ by attaching one or two cylinders depending on whether $M_1$ has one or two ends.

To summarize we constructed a covering $\tilde{M} \to M$ by a surface homeomorphic to $M_1$.

**Proof of Theorem 9.** Since $\pi L_1$ is disjoint from $A$, it lifts to a simple (not connected if $L_1$ is a link) closed Legendrian curve $L'_1$ in $ST^*M_1 \subset ST^*\tilde{M}$. Furthermore, the Legendrian isotopy of $L_1$ to $L_2$ lifts to a Legendrian isotopy of $L'_1$ to $L'_2$ in $ST^*\tilde{M}$.

Let $U$ be a thin neighborhood of the boundary of $M_1$ disjoint from $\pi L_1$ and $\pi L_2$. Suppose that $L'_2$ belongs to the leaf $ST^*M_1 \subset ST^*\tilde{M}$. Then there is an isotopy of the identity map of $ST^*\tilde{M}$ relative to $ST^*(M_1 \setminus U)$ to a map with image in $ST^*M_1$ and that brings the isotopy of $L'_1$ to $L'_2$ into $ST^*M_1$. This isotopy of $\text{id}_{ST^*\tilde{M}}$ comes from a deformation retraction $\tilde{M} \to M_1$ fixing points in $M_1 \setminus U$.

Suppose now that $L'_2$ belongs to a leaf of the covering $ST^*\tilde{M} \to ST^*M$ distinct from the leaf $ST^*M_1$. In this case, a deformation retraction of $\tilde{M}$ to $M_1$ moves $L'_2$ and therefore the above argument does not work. During the isotopy of $L'_1$ to $L'_2$ we get that at a certain moment $L'_1$ leaves $ST^*M_1$ and in view of the deformation retraction, in this case we may assume that the projection of $L_1$ to $\tilde{M}$ belongs to the interior of $U \cap M_1$. (Here $U$ is a thin neighborhood of $\partial M_1$.) Similarly, by exchanging the roles of $L_1$ and $L_2$, we may assume that the projection of $L_2$ to $\tilde{M}$ belongs to the interior of $U \cap M_1$. Now we are in position to apply the isotopy argument. Again, the Legendrian isotopy of $L_1$ to $L_2$ lifts to a Legendrian isotopy of $L'_1$ to $L'_2$. The link $L'_2$ may belong to the leaf distinct from the leaf $ST^*M_1$. However, since its projection $L_2$ is in the interior of $U \cap M_1$, there is an isotopy that performs a parallel translation of $L'_2$ to the lift of $L_2$ in the leaf $ST^*M_1$. This case was considered before.

**Corollary 15.** Assume that $M \neq S^2$ and $A_1, A_2$ are not contractible. If $\pi L_2$ does not intersect $A_1$, or $\pi L_1$ does not intersect $A_2$, then the destabilization of $L_1$ along $A_1$ is descent-equivalent to the destabilization of $L_2$ along $A_2$.

**Proof.** Suppose that $\pi L_2$ does not intersect $A_1$. To simplify notation let us assume that $L$ is a Legendrian knot; the case where $L$ is a link is similar. If $L_1$ and $L_2$ belong to the same component of $M \setminus A_1$, then by Theorem 9 we may assume that $L_1 = L_2$; this case has been treated in Lemma 7.
Suppose now that $A_1$ separates the surface into two components, and that $L_1$ and $L_2$ belong to different path components of $M \setminus A_1$. In this case the argument in the proof of Theorem 9 shows that we may assume that $\pi L_1$ belongs to a neighborhood of $A_1$. After the destabilization along $A_1$ we obtain a curve over a neighborhood of a point. Thus $L_1$ is descent equivalent to a curve in $ST^*S^2$. For this reason, by a sequence of stabilizations and destabilizations (that split off or add $ST^*S^2$ with $L_1$ on it) we can move $L_1$ in $M$ to the other path component of $M \setminus A_1$ that contains $L_2$. Thus, we may assume that $L_1$ and $L_2$ belong to the same path component of $M \setminus A$ and apply Theorem 9.

Recall that we assumed that $M$ is not a sphere and there is a Legendrian link $L$ represented by a link $L_1$ and $L_2$ and there are two simple closed connected curves $A_1$ and $A_2$ that are not null-homotopic such that the destabilization of $L_1$ along $A_1$ is not descent-equivalent to the destabilization of $L_2$ along $A_2$. Furthermore, we may assume that $A_1$ and $A_2$ are geodesics. Indeed, there exists an ambient isotopy $\varphi_t$, with $t \in [0,1]$, of the surface $M$ that takes $A_1$ into a geodesic. The ambient isotopy of the surface lifts to an isotopy $\tilde{\varphi}_t$ of the spherical cotangent bundle of $M$. Clearly, the destabilization of the Legendrian link $\varphi_1 L_1$ along $\varphi_1 A_1$ is descent-equivalent to the destabilization of $L_1$ along $A_1$. Thus we may assume that $A_1$ is a geodesic. Similarly, we may find an isotopy $\psi_t$ and its lift $\tilde{\psi}_t$ such that $\psi_1 A_2$ is a geodesic, and the destabilization of $L_2$ along $A_2$ is descent-equivalent to the destabilization of $\tilde{\psi}_1 L_2$ along $\psi_1 A_2$. If we replace now the original pairs $(L_1, A_1)$ and $(L_2, A_2)$ with the new pairs $(\tilde{\varphi}_1 L_1, \varphi_1 A_1)$ and $(\tilde{\psi}_1 L_2, \psi_1 A_2)$, then we obtain an example as the original one but with an additional property that the destabilizations are performed along geodesics.

**Lemma 16.** Suppose that $A_1, A_2$ is not null-homotopic and the surface $M$ is distinct from a sphere. Then there is an isotopy of $L_1$ whose projection does not intersect $A_1$ and that takes $L_1$ to a curve whose projections is disjoint from $A_2$. In other words, we may assume that the projection of $L_1$ is disjoint from both $A_1$ and $A_2$.

By this Lemma assuming that $M \neq S^2$ and $A_1, A_2$ are not contractible, the conditions of Corollary 15 are always satisfied.

**Proof.** As in the argument of the proof of Theorem 9, consider a covering $\tilde{M} \to M$ by a surface $\tilde{M}$ homeomorphic to $M_1$. Take the lift of an isotopy from $L_1$ to $L_2$ to a covering isotopy from $L_1'$ to $L_2'$ in $ST^*\tilde{M}$. A crucial observation is that the inverse image $A_2'$ of $A_2$ in $\tilde{M}$ consists of disjoint geodesics. The parts of these geodesics over cylinders attached to $M_1$ are easy to visualize. There is an isotopy of $\tilde{M}$ to $M_1$ that at each moment of time takes the geodesics of $A_2'$ to themselves. This isotopy takes $L_1$ to a curve disjoint both from $A_1$ and $A_2$. $\square$
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