**Dilaton Chiral Perturbation Theory - Determining Mass and Decay Constant of Technidilaton on the Lattice**

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We propose a scale-invariant chiral perturbation theory of the pseudo-Nambu-Goldstone bosons of chiral symmetry (pion \(\pi\)) as well as the scale symmetry (dilaton \(\phi\)) for large \(N_f\) QCD. The resultant dilaton mass \(M_\phi\) reads \(M_\phi^2 = m_\phi^2 + \frac{(3 - \gamma_m)(1 + \gamma_m)}{4} \left(\frac{2N_fP_f^2}{f^2}\right) m_\pi^2 + \text{(chiral log corrections)}\), where \(m_\phi, m_\pi, \gamma_m, P_f\) and \(F_\phi\) are the dilaton mass in the chiral limit, the pion mass, the mass anomalous dimension, and the decay constants of \(\pi\) and \(\phi\), respectively. The chiral extrapolation of the lattice data, when plotted as \(M_\phi^2\) vs \(m_\pi^2\), then simultaneously determines \((m_\phi, F_\phi)\) of the technidilaton in walking technicolor with \(\gamma_m \simeq 1\). The chiral logarithmic corrections are explicitly given.

Since the Higgs boson was discovered at the LHC\[1\], the next stage of particle physics will be to elucidate the dynamical origin of the Higgs boson, whose mass and coupling are free parameters within the Standard Model. One theory beyond the standard model is Walking Technicolor, which, based on the approximately scale-invariant gauge dynamics, predicted a large anomalous dimension \(\gamma_m \simeq 1\) and a pseudo Nambu-Goldstone (NG) boson of the approximate scale invariance ("Technidilaton") as a light composite Higgs boson\[2\]. The technidilaton was actually shown to be consistent with current LHC data for the Higgs\[2,4\].

A strongly coupled dynamics, walking technicolor would need fully nonperturbative calculations in order to make reliable estimates of the properties of the technidilaton and other composite particles to be compared with the upcoming high statistics data at LHC. There has been much work on the lattice in search for walking technicolor\[5\]. Among others, the LatKMI Collaboration\[6\] observed a flavor-singlet scalar meson lighter than the "pion" (corresponding to the NG boson in the chirally broken phase) in \(N_f = 12\) QCD — a theory shown\[7\] to be consistent with the chirally unbroken (conformal) phase on the same lattice setting. Such a light scalar might be a bound state generated only in the presence of the explicit fermion mass \(m_f\) in the conformal phase. Still, it gives a good hint for the technidilaton signature in the walking theory, which should have a similar conformal dynamics, with the role of \(m_f\) instead played by the dynamical mass of the fermion generated by spontaneous chiral symmetry breaking.

Amazingly, LatKMI Collaboration also observed indications of a light flavor-singlet scalar with comparable mass to the pion in \(N_f = 8\) QCD\[8\] — a theory shown\[9\] to be walking, having both signals of spontaneous chiral symmetry breaking and a remnant of conformality. This should be a candidate for the technidilaton as a light composite Higgs boson in walking technicolor.

However, walking technicolor makes sense only for vanishing fermion mass, \(m_f \equiv 0\), and hence the technidilaton mass should be determined in the chiral limit. We would need an extrapolation formula for the dilaton mass in the same sense as the usual chiral perturbation theory (ChPT)\[10\] for the lattice data measured at nonzero \(m_f\) to be extrapolated to the chiral limit.

In this article we propose a scale-invariant ChPT (sChPT) for the use of chiral extrapolation of the lattice data on the dilaton and the pion in the presence of explicit mass of the fermion \(m_f\). It is a scale-invariant generalization of the usual ChPT\[10\], based on the nonlinear realization of chiral symmetry in a way to realize the symmetry structure of the underlying walking gauge theory.

The theory consists of the pseudo-NG bosons of the chiral symmetry (pion \(\pi\), with mass \(m_\pi\)) as well as the scale symmetry (dilaton \(\phi\), with mass \(M_\phi\)), where both symmetries are broken spontaneously by the fermion-pair condensate, and also explicitly by both the fermion mass \(m_f\) and the nonperturbative scale anomaly (induced by the same fermion-pair condensate)\[11\]. We obtain a tree-level formula in \(M_\phi^2\) vs \(m_\pi^2\), Eq.\(8\), which can be plotted linearly in such a way that the intercept determines the chiral limit dilaton mass \(m_\phi^2\) (technidilaton mass), while its slope gives the technidilaton decay constant \(F_\phi\), defined as \(|\langle 0|D^\mu(0)|\phi(q)\rangle| = -iF_\phi q^\mu\), and hence \(|\langle 0|\partial_\mu D^\mu(0)|\phi(q)\rangle| = -F_\phi M_\phi^2\), where \(D^\mu(x)\) is the dilatation current. Based on the sChPT we also explicitly calculate one-loop corrections of the chiral logarithm, Eq.\(11\), which turn out to be negligibly small in current lattice simulations.

Let us start with the chiral/scale Ward-Takahashi (WT) identities for the axialvector \((J_{J}^{\mu})\)/dilatation \((D^\mu)\) currents in the underlying walking gauge theory with \(N_f\)-
femion fields ($\psi$):

$$
\theta_\mu = \partial_\mu D^\mu = \frac{\beta_{NP}(\alpha)}{4\alpha} G_{\mu\nu} + (1 + \gamma_m) N_{f m f} \bar{\psi} \psi, \\
\partial_\mu J^\mu_s = 2 m_f \bar{\psi} i \gamma_5 T^a \psi,
$$

where $T^a$ ($a = 1, \ldots, N_f^2 - 1$) are the $SU(N_f)$ generators, and $\beta_{NP}(\alpha)$ is the nonperturbative beta function for the nonperturbative running \[12\] of the gauge coupling $\alpha$ due to the mass scale $\Lambda_\chi$ dynamically generated by the spontaneous breaking of the chiral and scale symmetries through the condensate $\langle (\bar{\psi}\psi)_{\mu=\Lambda_\chi} \rangle \sim -\Lambda_\chi^3 \frac{\alpha}{4\alpha^3} G^2_{\mu\nu}$ is the nonperturbative trace (scale) anomaly defined as a part associated with the nonperturbative running and is induced solely by the chiral condensate with the scale $\Lambda_\chi$: $\langle (\bar{\psi}\psi)_{\mu=\Lambda_\chi} \rangle_{m_f=0} \sim -\Lambda_\chi^4$.

We now formulate the sChPT as to reproduce these WT identities. The building blocks $\varphi(x)$ to construct the sChPT are: $\varphi(x) = \{U(x), \chi(x), \mathcal{M}(x), S(x)\}$. $U(x) = e^{2i\pi(x) F_{\rho}}$, $\pi \equiv \pi^a T^a$, is the usual chiral field with the pion decay constant $F_{\pi}$, and $\chi(x) = e^{\phi(x)/F_{\phi}}$ with the dilaton field $\phi(x)$ and the decay constant $F_{\phi}$, $\mathcal{M}(x) = S(x)$ are fermion fields introduced so as to incorporate explicit breaking effects of the chiral and scale symmetries, respectively. Under the chiral $SU(N_f)_L \times SU(N_f)_R$ symmetry, these building blocks transform as $U(x) \rightarrow g_L \cdot U(x) \cdot g_R$, $\mathcal{M}(x) \rightarrow g_L \cdot \mathcal{M}(x) \cdot g_R$, $\chi(x) \rightarrow \chi(x)$ and $S(x) \rightarrow S(x)$ with $g_L, g_R \in SU(N_f)_L, R$. Under the scale symmetry they are infinitesimally transformed as $\delta U(x) = x_\nu \partial^\nu U(x)$, $\delta \mathcal{M}(x) = x_\nu \partial^\nu \mathcal{M}(x)$, $\delta \chi(x) = (1 + x_\nu \partial^\nu) \chi(x)$ and $\delta S(x) = (1 + x_\nu \partial^\nu) S(x)$, with scale dimensions $d_U = d_M = 0$, $d_\chi = d_\phi = 1$. The rule of chiral-order counting \[10\] is thus determined consistently with both the scale and chiral symmetries: $U \sim \chi \sim S \sim O(p^0)$, $\mathcal{M} \sim m_f \sim O(p^2)$, $\partial_\mu \sim m_{\pi} \sim M_{\phi} \sim O(p)$, where $m_{\pi}$ and $M_{\phi}$ are pion and dilaton masses arising from the vacuum expectation values of the spurious fields $\mathcal{M}$ and $S$, $\langle \mathcal{M} \rangle = m_{\pi}^2 \times 1_{N_f \times N_f}$, and $\langle S \rangle = 1$.

We shall first consider the chiral limit $m_f \rightarrow 0$. To the leading order $O(p^2)$ of sChPT, the chiral Lagrangian for the scale-invariant action is uniquely determined as \[13\]:

$$
\mathcal{L}^{\text{inv}}_{(2)} = \frac{F_{\phi}^2}{2} (\partial_\mu \chi)^2 + \frac{F_{\phi}^2}{4} \chi^2 \text{tr}[\partial_\mu U^\dagger \partial^\nu U].
$$

As noted above, even in the chiral limit, the scale symmetry is explicitly broken by the dynamical generation of the fermion mass itself in the underlying walking gauge theory ("hard-scale anomaly", or scale violation by the marginal operator) characteristic to the conformal phase transition \[12\]. Hence we have $4E = \langle \theta_\mu^2 \rangle_{m_f=0} = \frac{F_{\phi}}{4\alpha} \langle \theta_\mu^2 \rangle_{m_f=0} = \frac{F_{\phi}}{4\alpha} \langle \partial_\mu D^\mu \theta_\mu \rangle_{m_f=0} = -\frac{F_{\phi}^2 m_{\phi}^2}{8} < 0$ (Partially Conserved Dilatation Current (PCDC) relation) \[16\], where $m_{\phi}$ denotes the chiral-limit dilaton mass and we understand that the scale dimension of $\theta_\mu$ is equal to the canonical dimension, $d_\theta = 4$, for $m_f = 0$.

We may incorporate the corresponding explicit breaking terms, involving the spurion field $S$ to make the action formally scale-invariant \[17\]:

$$
\mathcal{L}^{(S)}_{(2)} = \frac{F_{\phi}^2}{4} m_{\phi}^2 \chi^4 \left( \log \frac{\chi}{\Lambda S} - \frac{1}{4} \right).
$$

This is a unique form having scale dimension four, which correctly reproduces the underlying nonperturbative scale anomaly $\frac{\beta_{NP}(\alpha)}{4\alpha^3} G^2_{\mu\nu}$ in the scale WT identity, Eq.\[1\], in the chiral limit $m_f \rightarrow 0$. In fact, when $\langle S \rangle = 1$, non-invariant term arises from $\log \chi$ to yield the scale anomaly $\langle \theta_\mu^2 \rangle = \langle \partial_\mu D^\mu \rangle = \langle \delta \mathcal{L}^{(S)}_{(2)\text{hard}} \rangle = -F_{\phi}^2 m_{\phi}^2 \chi^4$/4, in accord with the PCDC relation. The last factor $-1/4$ yields a correct vacuum energy $E = \langle \mathcal{L}^{(S)}_{(2)\text{hard}} \rangle = -F_{\phi}^2 m_{\phi}^2/16 = \langle \theta_\mu^2 \rangle/4$.

As discussed in Ref.\[18\], the explicit breaking terms due to the fermion current mass $m_f$ may also be introduced so as to reproduce the chiral WT identity in Eq.\[1\]:

$$
\mathcal{L}^{(S)}_{(2)\text{soft}} = \frac{F_{\phi}^2}{4} \left( \frac{\chi}{S} \right)^{3-\gamma_m} S^4 \text{tr}[M^4 U + U^\dagger M] - \frac{3-\gamma_m}{8} \frac{F_{\phi}^2}{\chi} \cdot \left( N_f \text{tr}[M^4 M] \right)^{1/2}.
$$

The factor $(3-\gamma_m)$ in the first line reflects the full dimension of the fermion bilinear operator $\bar{\psi}\psi$ in the underlying gauge theory. The scale-invariant term in line two, having no contributions to $\theta^2_x$, was introduced in the case without the hard-scale anomaly term $\mathcal{L}^{(S)}_{(2)\text{hard}}$ \[18\] in order to stabilize the dilaton potential so as to make the otherwise tachyonic dilaton mass term positive, $M_{\phi}^2 > 0$.

The Lagrangian for the scale- and chirally invariant action at leading order $O(p^2)$ is thus constructed from terms in Eqs.\[2\], \[3\], and \[4\]:

$$
\mathcal{L}_{(2)} = \mathcal{L}^{\text{inv}}_{(2)} + \mathcal{L}^{(S)}_{(2)\text{hard}} + \mathcal{L}^{(S)}_{(2)\text{soft}}.
$$

From this we finally read off the dilaton mass term $\phi^2$ as \[21\]:

$$
M_{\phi}^2 = m_{\phi}^2 + (1 + \gamma_m) (3 - \gamma_m) N_f F_{\pi}^2 m_{\phi}^2 / 2 F_{\phi}^2.
$$

Our result can also be derived directly from the underlying gauge theory through Eq.\[11\] as \[21\]:

$$
\langle 0 | \theta_\mu^2 | \phi \rangle = \langle 0 | \frac{\beta_{NP}(\alpha)}{4\alpha} G^2_{\mu\nu} | \phi \rangle (1 + \gamma_m) N_f m_f \langle 0 | \bar{\psi}\psi | \phi \rangle.
$$

We may further rewrite the dilaton mass, Eq.\[6\], in a form convenient for lattice simulations:

$$
M_{\phi}^2 = m_{\phi}^2 + s \cdot m_{\phi}^2, \\
s \equiv \frac{(3 - \gamma_m)(1 + \gamma_m)}{4} \cdot \frac{2N_f F_{\pi}^2}{F_{\phi}^2} \simeq \frac{2N_f F_{\pi}^2}{F_{\phi}^2} \equiv r, \tag{8}
$$
where the prefactor $(3 - \gamma_m)(1 + \gamma_m)/4 = 1 - (\delta/2)^2 \simeq 1 
(\delta \equiv 1 - \gamma_m; (\delta/2)^2 \ll 1)$ is very insensitive to the exact value of $\gamma_m$ as long as $\gamma_m \simeq 1$ in walking gauge theory.

This is our main result. It is useful for determining simultaneously the chiral limit values of both the mass $m_{\pi}$ and the decay constant $F_{\pi}$ of the flavor-singlet scalar meson as the technidilaton of walking technicolor on the lattice. Simultaneously fitting the intercept and the slope of a plot of $M_{\phi}^2$ vs $m_{\pi}^2$ from the lattice data would give $m_{\phi}^2$ (intercept) and $F_{\phi}$ through the slope parameter $s = r \equiv \frac{2N_f F_{\phi}^2}{F_{\pi}^2}$ [22]. For a given $N_f$ all the quantities $\gamma_m, F_{\pi}, F_{\phi}$ and $m_{\pi}$ in the expression of the slope parameter $s$ can be measured separately in lattice simulations on the same set up. Hence measuring $s$ would be a self-consistency check of the simulations as a dilaton observation, when compared with the value of $F_{\phi}$ determined by some other way. In Fig. 1 we present plots ($x,y) = (m_{\pi}^2, M_{\phi}^2)$ of mock-up data for general case $s \simeq r = (0.2, 0.5, 1.0)$ in the one-family model, $N_f = 8$ (4 weak-doublets) with $F_{\pi} = v_{EW}/\sqrt{2} \simeq 123$ GeV, by normalizing the masses to a chiral breaking scale $\Lambda_\chi = 4\pi F_{\pi}/\sqrt{N_f}$. The first number ($s = 0.2$) corresponds to a phenomenologically favorable value [3, 4], $F_{\phi} \simeq \sqrt{2N_f F_{\pi}/0.44} \simeq 1.1$ TeV, consistent with the current Higgs boson data at LHC. The third one ($s = 1.0$) is the holographic estimate in the large $N_c$ limit [4]. The second value ($s = 0.5$) is just a sample number in between. The close-up window on the top-left panel in the figure shows that the dilaton mass gets larger than $m_{\pi}$ when the ChPT expansion parameter $X \equiv m_{\phi}^2/\Lambda_\chi^2 = N_f m_{\phi}^2/(4\pi F_{\pi})^2 \lesssim 0.06(0.1)$ for $s = 0.2(0.5)$. Note also that for $s < 1$ there exists a crossing point where $M_{\phi}^2 < m_{\pi}^2$ changes to $M_{\phi}^2 > m_{\pi}^2$ near the chiral limit, as noted in Ref. [8].

As in the case of the usual ChPT [10], chiral logarithmic corrections at the loop level would modify the chiral scaling of the dilaton mass formula in Eq.[8]. Since the dilaton remains massive in the chiral limit due to the nonperturbative scale anomaly, only the pion loop corrections become significant for the chiral scaling of the dilaton mass. Such chiral logarithmic corrections will be operative in the soft-pion region $m_{\pi} \lesssim M_{\phi}$ (corresponding to the region where ChPT is valid: $X = m_{\phi}^2/\Lambda^2 \lesssim 0.1$ in Fig.[1]). We shall compute the chiral logarithmic corrections coming from the pion loops arising from the vertices at the leading $O(p^4)$ Lagrangian Eq.[5]. Those corrections softly break the scale symmetry by the form $\sim (1, r) \cdot X \log X$ when the cutoff $\Lambda$ is identified with $\Lambda_\chi$, which will be renormalized by the soft-breaking $O(p^4)$ counterterms proportional to $m_{\pi}^2 \sim M_{\phi}^2$.

Using dimensional regularization [22] we thus find the $D = 4$ pole (logarithmically divergent) contributions to the terms in quadratic order of dilaton fields:

$$
\frac{1}{2} Z_{F_{\phi}} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{2} m_{\phi}^2 \phi^2
$$

(9)

where

$$
Z_{F_{\phi}} = 1 + r \cdot \frac{N_f^2 - 1}{2N_f^2} X \log \frac{\Lambda^2}{m_{\pi}^2},
$$

$$
\tilde{m}_{\phi}^2 = \left[ m_{\phi}^2 - r m_{\pi}^2 \cdot \frac{2(N_f^2 - 1)}{N_f^2} X \log \frac{\Lambda^2}{m_{\pi}^2} \right]
+ \left[ \frac{(3 - \gamma_m)(1 + \gamma_m)}{4} r m_{\pi}^2 Z_{F_{\phi}} Z_{\chi} \log \frac{\Lambda^2}{m_{\pi}^2} \right],
$$

with $Z_{(r=8, m_{\pi})} = 1 + \Gamma_i/N_f \cdot X \log (\Lambda^2/m_{\pi}^2)$ and $\Gamma_i = N_f/4$ and $\Gamma_m = -1/N_f$. After renormalizing the divergent parts at the renormalization scale $\mu$ [23] and defining the renormalized dilaton field $\phi_0 = \sqrt{Z_{F_{\phi}}} \phi$, we find the renormalized $\phi^2$ terms $\frac{1}{2} \partial_{\mu} \phi_0 \partial^{\mu} \phi_0 - \frac{1}{2} m_{\phi_0}^2 \phi_0^2$, with the dilaton mass including the chiral logarithmic corrections of $O(p^4)$:

$$
M_{\phi}^2 = m_{\phi}^2 \left[ 1 + r \cdot \frac{N_f^2 - 1}{2N_f^2} X \log \frac{m_{\phi}^2}{\mu^2} \right]
+ r m_{\pi}^2 \left[ \frac{2(N_f^2 - 1)}{N_f^2} X \log \frac{m_{\phi}^2}{\mu^2} \right]
+ s \cdot m_{\pi}^2 \left[ \frac{N_f^2 - 4}{4N_f^2} X \log \frac{m_{\phi}^2}{\mu^2} \right]
+ \text{(counterterms renormalized at $\mu$)}. \tag{10}
$$

We may assume that all the counterterms in Eq.(10) vanish at $\mu = \Lambda_\chi$, so that they are induced only by the pion loops in the sChPT. As a concrete example, we again consider the one-family model with $N_f = 8$ and $F_{\pi} = 123$ GeV, and take the factor $s \simeq r = 2N_f F_{\pi}^2/F_{\phi}^2 = 0.2, 1.0$ and the chiral-limit dilaton mass $m_{\phi} = 125$ GeV in the light of the LHC. In Fig. 2 we plot the chiral scaling behavior of the dilaton mass for a small pion mass region

FIG. 1: A plot of $M_{\phi}^2/\Lambda_\chi^2$ with respect to $m_{\pi}^2/\Lambda_\chi^2 (\equiv X)$ obtained from Eq.[8], with $N_f = 8$ and $F_{\pi} = 123$ GeV and the chiral-limit dilaton mass $m_{\phi} = 125$ GeV. The slope $s \simeq r \equiv \frac{2N_f F_{\phi}^2}{F_{\pi}^2}$ in Eq.[8] has been taken to be 0.2 (solid black), 0.5 (dashed black) and 1.0 (dotted black). The solid red line corresponds to $M_{\phi}^2 = m_{\phi}^2$. 

As in the case of the usual ChPT [10], chiral logarithmic corrections at the loop level would modify the chiral scaling of the dilaton mass formula in Eq.[8].
FIG. 2: A plot of $M^2/\Lambda^2$ with respect to $m^2/\Lambda^2 (= \lambda)$ including the chiral logarithmic corrections in Eq. (10) for the one-family model with $N_f = 8$, $F_\pi = 123$ GeV, the chiral-limit mass $m_\phi = 125$ GeV and the factor $s \approx r = 2N_f F_\pi^2 / F_\pi^2 = (0.2, 1.0)$ (solid black and blue curves). The leading-order scalings in Eq. (9) are also depicted for $s = 0.2$ and 1.0 by dashed black and blue lines, respectively. The solid red line corresponds to $M^2 = m^2$.

$\lambda \equiv m^2/\Lambda^2 \lesssim 0.1$, including the chiral logarithmic corrections from the pions at the one-loop level for $s = 0.2$ and 1.0 (solid black and blue curves). Also plotted is the leading-order formula in Eq. (8) (dashed black and blue curves). The figure implies that the chiral logarithmic effect may be appreciable for the soft-pion mass region. However, such chiral logarithmic effects are negligibly small for the current status of $N_f = 8$ QCD on the lattice, where simulations have been performed for a larger pion mass region $3 \lesssim \lambda \lesssim 5$.

In conclusion, we have established a scale-invariant chiral perturbation theory (sChPT) for the pseudo-NG bosons, the pion ($\pi$) and the dilaton ($\phi$), which will be useful in its own right in various situations. It is straightforward to include the vector mesons into this framework via hidden local symmetry. As its prominent consequence we obtained a formula relating the masses $M^2$ vs $m^2$, Eq. (8) (tree), or Eq. (10) (one-loop), which we believe plays a vital role for making chiral extrapolations of lattice data of the flavor-singlet scalar meson, thereby obtaining the mass ($m_\phi$) and decay constant ($F_\phi$) of the technidilaton as a composite Higgs boson in walking technicolor.

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[Note added] After submission of this article, the LatKMI Collaboration published a paper (25) (follow-up of [8]) finding a light flavor-singlet scalar in $N_f = 8$ QCD, with the data analyzed based on Eq. (9) to be roughly consistent with 125 GeV Higgs as the technidilaton.

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[13] The subtracted $\theta_\mu$ in Eq. (11) is defined as $\theta_\mu = \theta_\mu^{\text{full}} - \langle \theta_\mu^{\text{perturbation}} \rangle$, where the usual (perturbative) scale anomaly, $\langle \theta_\mu^{\text{perturbation}} \rangle$, is characterized by the intrinsic scale $\Lambda_{QCD}$, a scale responsible for the asymptotically free (perturbative) running of the coupling in the ultraviolet region. Hence $\langle \theta_\mu \rangle$ is non-zero only in the broken phase, $\langle \theta_\mu \rangle \sim -\Lambda^4$, in sharp contrast to the usual QCD, where $\Lambda_{QCD} \sim \Lambda$. For details see e.g., the third reference in [11].
[14] The Lagrangian is constructed uniquely (up to total derivatives) by the requirement that the action $S[\varphi(x)] \equiv \int d^4x L[\varphi(x)]$ be invariant under the scale (and also chiral) transformation $\delta \varphi(x) = (d_\mu + x_\mu \partial^\mu) \varphi(x)$, $\delta x_\mu = -x_\mu$. This implies that $\delta S[\varphi(x)] = \int (d^4x (d^4x L + d^4x \delta L) = \int d^4x (-4L + d \lambda L) = 0$, namely the scale dimension of $L$ must be 4, $d_L$. 

\[ \text{arXiv:1304.4882.} \]
Note that the second term comes from the pole contribution. Thus we have $dO(0)=F_0 \cdot \langle \phi |O| \rangle$. Similarly, the Lagrangians, Eqs. (5), as well as Eq. (2), is unique in a way to the fermion mass $m_f$, and hence does not exist in the chiral limit Lagrangian Eq. (2) with $m_f \to 0$. See Ref. [18] for details. Thus our Lagrangian, Eq. (5), as well as Eq. (2), is unique at order $O(p^0)$.

A similar formula was also discussed in a completely different context, i.e., hadron physics, which we believe has no approximate scale symmetry and is irrelevant to our discussions. See e.g., J. Ellis, Nucl. Phys. B 222, 478 (1970); R. J. Crewther and L. C. Tunstall, arXiv:1203.1321 [hep-ph].

Note that $\langle 0|\theta_\mu^0|\phi\rangle_{m_f \neq 0} = \langle 0|\partial_\mu D_\mu^0|\phi\rangle_{m_f \neq 0} = -F_2 M_2^0 \langle 0|\partial_\mu D_\mu^0|\phi\rangle_{m_f = 0} = -F_2 m_2^0$, while the second term on the RHS of Eq. (5) comes from the scale-WT identity (see [18]): $\langle 0|m_f \psi \bar{\psi} |\phi\rangle = (3 - \gamma_m) 2m_f \langle \psi \bar{\psi} \rangle/(2F_0)$, when combined with the chiral-WT identity (Gell-Mann-Oakes-Renner relation): $2m_f \langle \psi \bar{\psi}\rangle \delta_{\alpha\beta} = \langle iQ_3^a 2m_f \bar{\psi} \gamma_\alpha T^a \psi \rangle = F_2 \langle 0|\partial_\mu D_\mu^0 |\phi\rangle = -F_2 m_2^0 \delta_{\alpha\beta}$, where $Q_3^a = \int d^4x J_3^a(x)$. (Note that $\theta_\mu^0$ in Eq. (1) need not be a subtracted one in this derivation, since the constant subtraction $\langle \theta_\mu^0|_{\text{perturbation}} |\rangle$ sandwiched by $|0\rangle$ and $|\phi\rangle$ is zero.)

Note that $r$ has no explicit $N_f$-independence, since $F_r^2$ is associated with the flavor-singlet operator having a sum of $N_f$-flavor contributions, namely $F_r^2$ is proportional to $N_f$, while $F_r^2$ is not.

One might suspect that typical regularization schemes such as dimensional regularization with a scale $\mu \sim \Lambda_\chi = 4 \pi F_\pi / \sqrt{N_f}$ would explicitly break the scale symmetry even in the chiral limit $m_f \sim m_2^0 \to 0$. However, such hard-breaking terms correspond to the nonperturbative scale anomaly due to the dynamical generation of the fermion mass of order $O(\Lambda_\chi)$, and should be absorbed by redefining the nonperturbative contributions to the scale anomaly: $\theta_\mu^0 = \theta_\mu^0|_{\text{full}} - \langle \theta_\mu^0|_{\text{perturbation}} \rangle$. Thus such a scale-breaking by the regularization of order $\Lambda_\chi$ is irrelevant to our chiral log effects which are the far infrared physics of near massless pion loops. We can thus safely perform the loop calculation by focusing only on the soft-scale breaking terms arising as the soft-pion effects.

The logarithmically divergent parts can be absorbed by introducing the $O(p^3)$ counterterms in the chiral and scale invariant form along with the spurion fields $\mathcal{M}$ and $S$:

$$L_{\text{counterterm}}^{(4)} = L_4 \theta \nabla^\mu U \theta^\mu U \nabla \mathcal{M} \cdot U + U^\dagger \mathcal{M} \cdot U \cdot S^2 + L_5 \partial_\mu U \theta^\mu U \nabla \mathcal{M} \cdot U + U^\dagger \mathcal{M} \cdot U \cdot S^2 + L_6 \nabla \mathcal{M} \cdot U + U^\dagger \mathcal{M} \cdot U \cdot S^2 + L_8 \nabla \mathcal{M} \cdot U + U^\dagger \mathcal{M} \cdot U \cdot S^2 + H_2 \nabla \mathcal{M} \cdot U + U^\dagger \mathcal{M} \cdot U \cdot (S^2 - 1) \cdot S^4 + L_6 \nabla \mathcal{M} \cdot U + U^\dagger \mathcal{M} \cdot U \cdot (S^2 - 1) \cdot S^4 + H_2 \nabla \mathcal{M} \cdot U + U^\dagger \mathcal{M} \cdot U \cdot (S^2 - 1) \cdot S^4,$$

where the chiral coefficients $L_{(=4,5,6,8)}$ and $H_2$ absorb the divergences in $Z_{F_\pi}$ and $Z_{m_2}$ in Eq. (9) which get the same renormalization effects as those in the usual chiral perturbation theory [10], while the others $L_{(=4,6,8)}$ and $H_2$ renormalize the log terms having $F_\pi^2$ in $Z_{F_\pi}$ and $m_2^2$.