Coarse-grained cosmological perturbation theory: stirring up the dust model

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We study the effect of coarse-graining the dynamics of a pressureless self-gravitating fluid (coarse-grained dust) in the context of cosmological perturbation theory, both in the Eulerian and Lagrangian framework. We obtain recursion relations for the Eulerian perturbation kernels of the coarse-grained dust model by relating them to those of the standard pressureless fluid model. The effect of the coarse-graining is illustrated by means of power and cross spectra for density and velocity that are computed up to 1-loop order. In particular, the large scale vorticity power spectrum that arises naturally from a mass-weighted velocity is derived from first principles. We find qualitatively good agreement of the magnitude, shape and spectral index of the vorticity power spectrum with recent measurements from N-body simulations and results from the effective field theory of large scale structure. To lay the ground for applications in the context of Lagrangian perturbation theory we finally describe how the kernels obtained in Eulerian space can be mapped to Lagrangian ones.

I. INTRODUCTION

Cold dark matter (CDM) and dark energy comprise 95% of the energy budget of the Universe and are mainly responsible for its current expansion rate and the observed large scale structure (LSS). The expansion history as well as LSS are thus key to our understanding of the fundamental laws of nature, such that increasingly larger efforts are spent to map them directly or indirectly through galaxy and lensing surveys and through the cosmic microwave background. Therefore accurate analytical models for the formation of LSS are indispensable to constrain cosmological parameters and to search for deviations from ΛCDM, the standard model of cosmology. In the ΛCDM model, dark energy is a cosmological constant Λ and CDM is a particle species that effectively interacts only gravitationally, or collisionless, and whose initial velocity distribution is cold, or fully described by a smooth gradient field.

Among analytical methods developed to describe the LSS formation, perturbative schemes based on the popular dust model [1] play an important part. The dust model describes self-gravitating collisionless cold dark matter (CDM) as a pressureless fluid which fulfills a coupled system of differential equations consisting of continuity, Euler and Poisson equation. These equations can be solved perturbatively – either in the Eulerian frame [2] where everything is expanded in terms of density and velocity or in the Lagrangian framework [3] where fluid-trajectories or displacements are considered. Those perturbative techniques provide satisfactory results within the linear regime of structure formation and resumming some classes of perturbative corrections [4–6] can enhance their range of applicability towards mildly nonlinear scales. However, they are condemned to break down eventually in the deeply nonlinear regime due to their inability to dynamically generate higher cumulants like velocity dispersion dynamically. Indeed, the dust model is a truncation of the infinite hierarchy for the cumulants of the phase-space distribution of particles which fulfills the Vlasov (or collisionless Boltzmann) equation. Truncating the hierarchy is only consistent as long as the particle trajectories are well described by a single coherent flow, called single-stream approximation, since as soon as multiple streams become relevant all higher cumulants are sourced dynamically, see [7].

To tackle this shortcoming several semi-analytical methods based on Effective Field Theory (EFT) both in the Eulerian [8–13] and Lagrangian framework [14, 15] have been developed. The strategy of EFT of LSS is to integrate out (or formally solve) the dynamics of the short-wavelength part in order to obtain closed-form equations of motion for the long-wavelength quantities. They describe the large scale physics in terms of an effective fluid that is treated perturbatively and characterized by several parameters arising from small scale physics. These parameters are not calculable within the EFT framework itself but have to be inferred from observations or N-body simulations, at least as long as no full theory describing the small scale physics is at hand. All formulations of EFT of LSS have an underlying coarse-graining approach in common but differ in the precise implementation of the cut-off, while some rely on sharp-k filtering, others employ smooth filters like spherical top-hat or Gaussian window functions. The coarse-graining procedure allows to separate long from short scale modes and handle the former perturbatively while regarding the latter as source terms for higher phase space cumulants like velocity dispersion.

In [16] a spatially coarse-grained description of a many-body gravitating system for the evolution of LSS has been studied and shown to lead to a fluid-like description which recovers the usual dust model when scales substantially larger than the coarse-graining scale are considered. It was noted that the corresponding hierarchy for the moments can in principle be closed by expressing the microscopic degrees of freedom through the macroscopic density and velocity. This requires the coarse-graining filter to be invertible which excludes sharp-k and top-hat filter but favors the Gaussian window which was considered. In [16] the Gaussian filter was Taylor-expanded in the filter length $\sigma_x$ up to leading order, called large-scale expansion. In this expansion the lowest order term was shown to automatically yield the dust model

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whereas the first order involves a correction proportional to the coarse-graining scale squared $\sigma^2_c$. It was demonstrated that this term gives rise to a velocity dispersion which enters into the Euler equation and it was argued that it leads to adhesive behavior, see also [17]. The method described therein was conjectured to allow for successive improvements over the aforementioned dust and adhesion models.

Our approach is based on the Schrödinger method (ScM) as described in [18], which is able to catch the fully-fledged N-body dynamics and incorporate higher cumulants like velocity dispersion which are relevant for multi-streaming. In the limit $\hbar \to 0$ the ScM constitutes a full resummation in the filter length $\sigma_c$ of the coarse-grained hydrodynamics described in [16]. We will restrict our attention to the mildly nonlinear scales, where shell crossing is not yet dominant. In this regime the limit $\hbar \to 0$ of the ScM reduces to the coarse-grained dust model which we will study perturbatively in analogy to the dust model. One shortcoming of the dust model is the absence of vorticity and inability to generate it dynamically. While it has been supposed in [7] that considering a mass-weighted velocity may account for large scale vorticity, we provide the first consistent implementation of this idea. We compare our result for the vorticity power spectrum to cosmological numerical simulations, see for example [7, 19] and the effective field theory of large scale structure [10]. To lay the foundation for applications of the coarse-grained dust model we describe how perturbative kernels in the Lagrangian framework can be obtained from those in Eulerian space.

Structure This paper is organized as follows: In Sec. II we review the phase space description of cold dark matter starting from the Vlasov equation on an expanding background and investigate the hierarchy of moments arising from the Vlasov equation. We then introduce the dust model as well as the coarse-grained dust model and determine the moments of the two different phase space distributions. In Sec. III we provide a review of the two standard schemes of perturbation theory, Eulerian as well as Lagrangian Perturbation Theory and explain their connection. Sec. IV is devoted to the derivation of the corresponding Eulerian perturbation kernels to determine the power and cross spectra for the coarse-grained dust model. Furthermore we describe how these kernels can be mapped to Lagrangian space. We conclude in Sec. V.

II. PHASE-SPACE DESCRIPTION OF COLD DARK MATTER

A. Vlasov equation

The dynamics of cold dark matter (CDM) can be conveniently described using a phase space distribution function $f(t, x, p)$ which contains all relevant information about the system. Imposing phase-space conservation one directly obtains the Vlasov (or collisionless Boltzmann) equation which governs the time evolution of the distribution function. This equation is supplemented by the Poisson equation which encodes gravitational interaction and causes the Vlasov equation to be nonlocal and nonlinear in the phase space distribution $f$. We use comoving coordinates $x$ with associated conjugate momentum $p = a^2 m \frac{dx}{dt}$, where $a$ is the scale factor satisfying the Friedmann equation of a $\Lambda$CDM or Einstein-de Sitter universe. Then the Vlasov-Poisson system reads

$$\partial_t f = -\frac{p}{a^2 m} \cdot \nabla_x f + m \nabla_t V \cdot \nabla_p f, \quad (1a)$$

$$= \left[ \frac{p^2}{2a^2 m} + mV(x) \right] \left( \nabla_x \nabla_p - \nabla_p \nabla_x \right) f, \quad (1b)$$

$$\Delta V = \frac{4\pi G \rho_0}{a} \left( \int d^3 p \ f - 1 \right), \quad (1c)$$

where $\rho_0$ is the (constant) comoving matter background density such that $f$ has a background value or spatial average value $\langle \int d^3 p \ f \rangle_{vol} = 1$. For convenience we will in general suppress the $t$-dependence of the distribution function.

B. Hierarchy of Moments

Since the phase space distribution function $f$ depends on seven variables – three each for position and momentum and one for time – it is more manageable to consider purely spatial distributions which characterize the system. This can be done by taking moments of the phase space distribution function with respect to momentum.

Generating functional for moments and cumulants The moments $M^{(n)}$ can be conveniently obtained from the generating functional $G[J]$ by taking functional derivatives. The cumulants $C^{(n)}$ provide an alternative yet equivalent description elucidating the prominent dust-model, the only known consistent truncation of the Vlasov hierarchy. The generating functional, moments and cumulants are given by

$$G[J] = \int d^3 p \ \exp \left\{ ip \cdot J (x, p) \right\}, \quad (2a)$$

$$M^{(n)}_{i_1 \cdots i_n} := \int d^3 p \ p_{i_1} \cdots p_{i_n} f = (-i)^n \frac{\partial^n G[J]}{\partial J_{i_1} \cdots \partial J_{i_n}} \bigg|_{J=0}, \quad (2b)$$

$$C^{(n)}_{i_1 \cdots i_n} := (-i)^n \frac{\partial^n \ln G[J]}{\partial J_{i_1} \cdots \partial J_{i_n}} \bigg|_{J=0}. \quad (2c)$$

Evolution equations The dynamics of the moments $M^{(n)}$ are encoded in the Vlasov equation (1a) and can be extracted easily

$$\partial_t M^{(n)}_{i_1 \cdots i_n} = -\frac{1}{a^2 m} \nabla_j M^{(n+1)}_{i_1 \cdots i_n \cdot j} - m \nabla_{i_1} V \cdot M^{(n-1)}_{i_2 \cdots i_n}, \quad (3)$$

where indices enclosed in round brackets imply symmetrization according to $a_{ij} = a_{ji} + a_{ji}$. Unfortunately, the time-evolution of the $n$-th moment depends in turn on the $n + 1$-th moment thereby constituting an infinite coupled hierarchy. The underlying hierarchy becomes much more transparent when expressed in terms of cumulants $C^{(n)}$

$$\partial_t C^{(n)}_{i_1 \cdots i_n} = -\frac{1}{a^2 m} \left\{ \nabla_j C^{(n+1)}_{i_1 \cdots i_n \cdot j} + \sum_{S \in P(i_1, \cdots, i_n)} C^{(n+1-|S|)}_{i_1 \cdots i_n} \cdot \nabla_j f^{(|S|)} \right\}$$

$$- \delta_{i_1} \cdot m \nabla_{i_1} V, \quad (4)$$
which can also be obtained from this coarse-grained Wigner
trolled manner and allows to compute them analytically. In the
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less fluid with density
∇
relevant. The corresponding distribution function is given by
soon as ‘shell-crossings’ occur and multiple streams become
as ‘single-stream’ indicating that this model breaks down as

strategy dispersion cannot arise. This regime is usually referred to
Schrödinger method (ScM) which provides an ansatz \( \bar{W} \)
we established an approach relying on (ii), namely the
matter described by the Vlasov hierarchy of cumulants. In

(ii) resorting to a special ansatz for the distribution function.

strategies for closing the hierarchy

We have established an approach relying on (ii), namely the

scM incorporates higher cumulants that approximately solve the Vlasov hierarchy in a con-
trolled manner and allows to compute them analytically.
In the following we will consider the coarse-grained dust model \( f_d \)
which can also be obtained from this coarse-grained Wigner
ansatz \( \bar{W} \) when sending \( h \rightarrow 0 \).

C. Dust model

Within the dust model CDM is described as a pressure-
less fluid with density \( n(x) \) and an irrotational fluid velocity
\( \nabla \phi(x) \) which remains single-valued at each point meaning
that particle trajectories are not allowed to cross and velocity
dispersion cannot arise. This regime is usually referred to
as ‘single-stream’ indicating that this model breaks down as
soon as ‘shell-crossings’ occur and multiple streams become
relevant. The corresponding distribution function is given by

\[ f_d(x, p) = n(x) \delta_D(p - \nabla \phi(x)). \]  

Moments and cumulants The generating functional for the
dust model where \( f_d \) was inserted in (2a) yields

\[ G_d[J] = n \exp \left[ i \nabla \phi \cdot J \right]. \]

The moments \( M^{(n)} \) and cumulants \( C^{(n)} \) are then given by

\[
\begin{align*}
M^{(0)} & = n, \\
M^{(1)}_i & = n \phi_i, \\
M^{(n=2)}_{i_1i_2} & = n \phi_{i_1} \phi_{i_2}, \\
C^{(0)} & = \ln n, \\
C^{(1)}_i & = \phi_i, \\
C^{(n=2)}_{i_1i_2} & = 0,
\end{align*}
\]

where the shorthand notation \( \phi_{i_1} \) := \( \nabla \phi \) has been used. All
cumulants of order two and higher vanish identically since

the exponent of the generating functional is linear in \( J \). This
simply shows that the dust model does not include multi-
streaming effects like velocity dispersion, which is encoded
in the second cumulant.

Evolution equations Due to the absence of higher cumu-
lants in the dust model, the Vlasov equation is equivalent to
its first two equations of the hierarchy of moments, the hydro-
dynamical system consisting of the continuity and Bernoulli
equation for a perfect pressureless fluid with density \( n \) and ve-
locity potential \( \phi/m \), which is supplemented by the Poisson
equation

\[
\begin{align*}
\partial_x n & = -\frac{1}{ma^2} \nabla \cdot (n \nabla \phi), \\
\partial_t \phi & = -\frac{1}{2a^2m}(\nabla \phi)^2 - mV, \\
\Delta V & = \frac{4\pi G \rho_0}{a}(n - 1).
\end{align*}
\]

If \( n \) and \( \phi \) fulfill these equations then all evolution equations
(3) of the higher moments are automatically satisfied. Finally
we note that by defining an irrotational velocity according to
\( u = \nabla \phi/m \) one can rewrite (8a) and (8b) in the following

The continuity and Euler equation given above are supple-
mented by the curl-free constraint

\( \nabla \times u = 0 \).

D. Coarse-grained dust model

The distribution function of the coarse-grained dust model
is defined as a smoothing of the dust probability distribution
(5) with a Gaussian filter of width \( \sigma_z \) and \( \sigma_p \) in \( x \) and \( p \) space,
respectively. For convenience we will adopt the shorthand op-
erator representation of the smoothing which can be easily ob-
tained by switching to Fourier space

\[
\tilde{f}_d = \int \frac{d^3 x'}{(2\pi \sigma_z \sigma_p)^3} \exp \left[ -\frac{(x' - x)^2}{2\sigma_z^2} - \frac{(p - p')^2}{2\sigma_p^2} \right] f_d(x', p'),
\]

\[
\tilde{f}_d = \exp \left[ \frac{1}{2} \sigma_z^2 \Delta_x + \frac{1}{2} \sigma_p^2 \Delta_p \right] f_d.
\]

As mentioned in the introduction the coarse-grained dust
model is a special case of the ScM presented in [18] which

one can obtain from the coarse-grained Wigner function \( \bar{f}_W \)
in the limit \( h \rightarrow 0 \) as long as no shell-crossing has occurred yet

\[ f_d(x, p) \big|_{h=0} = \tilde{f}_d(x, p). \]

If \( x_{\text{typ}} \) and \( p_{\text{typ}} \) are the (minimal) scales of interest we have to ensure that

\[
\sigma_z \ll x_{\text{typ}} \quad \text{and} \quad \sigma_p \ll p_{\text{typ}}.
\]
Moments and cumulants The generating functional for the coarse-grained dust model is given by
\[ G_d[J] = \exp\left(\frac{1}{2}\sigma_r^2\Delta - \frac{1}{2}\sigma_p^2 J^2\right) G_d[J]. \] (13)
From this expression the calculation for the moments \(\tilde{M}^{(n)}\) is straightforward and shows that the first two are given by a spatial coarse-graining of the dust moments (7a)
\[ \tilde{M}^{(0)} = \exp\left(\frac{1}{2}\sigma_r^2\Delta\right) M^{(0)} =: \tilde{n} , \quad \tilde{C}^{(0)} = \ln \tilde{n} \] (14a)
\[ \tilde{M}^{(1)} = \exp\left(\frac{1}{2}\sigma_r^2\Delta\right) M^{(1)} =: \tilde{n} \tilde{u} , \quad \tilde{C}^{(1)} = \tilde{m} \tilde{u} . \] (14b)
The macroscopic velocity \(\tilde{u}\) is the mass-weighted dust velocity which is obtained by smoothing the momentum field \(n\mathbf{u}\) and then dividing by the smoothed density field \(\tilde{n}\). This is precisely the definition commonly used in the EFT of LSS, compare [11, 21]. From a physical point of view \(\tilde{u}\) describes the center-of-mass velocity of the collection of particles inside a coarsening cell of diameter \(\sigma_r\) around \(x\).
Note that higher moments \(\tilde{M}^{(n>2)}\) are not simply given by the coarse-graining of \(M^{(n>2)}\) but receive an extra \(\sigma_p^2\)-term
\[ \tilde{M}^{(2)}_{ij} = \exp\left(\frac{1}{2}\sigma_r^2\Delta\right) M^{(2)}_{ij} + \sigma_r^2 M^{(0)} \delta_{ij} \] (14c)
\[ \tilde{M}^{(3)}_{ijk} = \exp\left(\frac{1}{2}\sigma_r^2\Delta\right) M^{(3)}_{ijk} + \sigma_r^2 M^{(1)} \delta_{ij} \] (14d)
The corresponding cumulants can be calculated from the previous results using
\[ \tilde{C}^{(2)}_{ij} = \sigma_p^2 \delta_{ij} + \frac{\partial x}{\partial \rho} \frac{\partial \rho}{\partial x} + \frac{\partial x}{\partial y} \frac{\partial y}{\partial x} \] (14e)
\[ \tilde{C}^{(3)}_{ijk} = \frac{\partial x}{\partial \rho} \frac{\partial \rho}{\partial x} + \frac{\partial x}{\partial y} \frac{\partial y}{\partial x} \] (14f)
with the shorthand notation \(\bar{\rho} = \sigma_p^2 \delta_{ij} \) [\(n\delta_{ij}\)] and \(\bar{\rho} = \sigma_p^2 \delta_{ij} \) [\(n\delta_{ij}\)]. We can observe that all higher moments are determined self-consistently from the lowest two, which are dynamical and represent the coarse-grained density \(\tilde{n}\) and mass-weighted velocity \(\tilde{u}\), respectively.
Evolution equations The dust equations (8) can be employed to obtain evolution equations for the first two moments \(\tilde{n} = \tilde{M}^{(0)}\) and \(\tilde{u} = \tilde{M}^{(1)}/(\tilde{n}\tilde{u})\) corresponding to coarse-grained density and mass-weighted velocity, respectively
\[ \partial_t \tilde{n} = -\frac{1}{a^2} \nabla \cdot (\tilde{n} \tilde{u}) , \] (15a)
\[ \partial_t (\tilde{n} \tilde{u}) = -\exp\left(\frac{1}{2}\sigma_r^2\Delta\right) \left\{ \frac{1}{a^2 m^2} \nabla \left[ n \delta_{ij} \frac{\partial x}{\partial y} \frac{\partial y}{\partial x} \right] + n \nabla \mathbf{V} \right\} . \] (15b)
Note that \(\sigma_r\) drops out and that one would obtain exactly the same evolution equations when taking moments of the coarse-grained Vlasov equation, see Eq.(12) in [18] and inserting the moments (14) for the coarse-grained dust ansatz \(\tilde{f}_d\), Eq.(10). This is because \(\tilde{f}_d\) fulfills the coarse-grained Vlasov equation before shell crossing. A specific feature of the Gaussian filter we employed here is that it can be inverted such that there exists a closed-form analogue of Eq.(15b) for the macroscopic quantities \(\tilde{n}\) and \(\tilde{u}\). This equation should be valid even after shell crossing. For details we refer the interested reader to [18]. Since the macroscopic velocity \(\tilde{u}\) is obtained from the dust velocity \(\mathbf{u} = \nabla \phi/m\) by mass-weighting, these equations are supplemented by the constraint
\[ m \tilde{n} \tilde{u} = \exp\left(\frac{1}{2}\sigma_r^2\Delta\right) (n \nabla \phi) . \] (15c)
which is the analogue of the curl-free constraint \(\mathbf{u} = \nabla \phi/m\) Eq.(9c) and enforces a very particular non-zero vorticity for \(\tilde{u}\). For practical applications, instead of solving the coarse-grained fluid equations (15) for \(\tilde{n}\) and \(\tilde{u}\) one can simply solve (8) for \(n\) and \(\phi\) and construct the cumulants of interest according to (14). Note that Eqs.(15) are naturally written in terms of the macroscopic momentum \(\tilde{j} \equiv \tilde{n} \tilde{u}\).

III. PERTURBATION THEORY FOR DUST

In this section we will shortly review the standard of perturbation theory for the dust model both in the Eulerian (SPT) and Lagrangian framework (LPT).

A. Eulerian Perturbation Theory

In Eulerian Perturbation Theory the quantities under consideration are the density contrast \(\delta(\tau, x)\), which is defined via the normalised number density field \(n/(\tau, x) = 1 + \delta(\tau, x)\) and the peculiar velocity field \(\mathbf{v}(\tau, x) = \mathbf{u}(\tau, x)/a(\tau)\) for conformal time \(\tau\) given by \(d\tau = a(\tau)d\tau\). Then the dust equations (9) can be recast in the following form
\[ \partial_\tau \delta + \nabla \cdot [(1 + \delta) \mathbf{v}] = 0 , \] (16a)
\[ \partial_\tau \mathbf{v} + \mathcal{H} \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla \mathbf{V} . \] (16b)
The two coupled evolution equations for a fluid modeling CDM are supplemented by the Poisson equation
\[ \Delta \mathbf{V} = \frac{4\pi G\rho_0}{a} \delta . \] (16c)
The velocity field \(v\) can be decomposed into velocity divergence \(\theta = \nabla \cdot v\) and vorticity \(\omega = \nabla \times v\) that vanishes identically due to (9). The corresponding equations can be obtained easily from (16b) when using the Poisson equation (16c)
\[ \partial_\tau \theta + \mathcal{H} \theta + \nabla \cdot [(\mathbf{v} \cdot \nabla) \mathbf{v}] = -\frac{1}{2} \mathcal{H}^2 \delta , \] (17a)
\[ \partial_\tau \mathbf{w} + \mathcal{H} \mathbf{w} + \nabla \times (\mathbf{v} \times \mathbf{w}) = 0 , \] (17b)
where we restricted ourselves to the Einstein-de Sitter (EdS) case. We see that if there is no initial vorticity then it cannot be generated. Since furthermore in linear perturbation theory any initial vorticity decays due to the expansion of the universe, it is natural and consistent to regard the velocity as a gradient field given by \(\mathbf{v} = \nabla \theta/\Delta\).
**Perturbative expansion**  The density contrast $\delta$ and the divergence of velocity $\theta = \nabla \cdot v = \Delta \delta / a m$ can be written in Fourier space and expanded in terms of the scale factor $a(\tau)$ for the fastest growing mode

\[
\delta(\tau, k) = \sum_{n=1}^{\infty} a^2(\tau) \delta_n(k), \quad (18a)
\]

\[
\theta(\tau, k) = \mathcal{H}(\tau) \sum_{n=1}^{\infty} a^2(\tau) \theta_n(k). \quad (18b)
\]

Although we describe the perturbative procedure here for the EdS case, one can easily translate the result to a general ACMD universe, as described in [2], by replacing $a \to D$ and $H \to fH$, where $D(\tau)$ is the linear growth function and $f(\tau) = d \ln D(\tau)/d \ln a(\tau)$ the linear growth rate. Although this replacement is not an exact it is a very good approximation [2].

We define the integral kernels $F_n$ and $G_n$ of $\delta$ and $\theta$ with

\[
F_n(p_1, \ldots, p_n) = \sum_{m=1}^{n-1} \frac{G_m(p_1, \ldots, p_m)}{(2n+3)(n-1)} \left[ (2n+1) \frac{k \cdot k_1}{k_1^2} F_{n-m}(p_{m+1}, \ldots, p_n) + \frac{k^2(k_1 \cdot k_2)}{k_1^2 k_2^2} G_{n-m}(p_{m+1}, \ldots, p_n) \right], \quad (20a)
\]

\[
G_n(p_1, \ldots, p_n) = \sum_{m=1}^{n-1} \frac{G_m(p_1, \ldots, p_m)}{(2n+3)(n-1)} \left[ 3 \frac{k \cdot k_1}{k_1^2} F_{n-m}(p_{m+1}, \ldots, p_n) + \frac{k^2(k_1 \cdot k_2)}{k_1^2 k_2^2} G_{n-m}(p_{m+1}, \ldots, p_n) \right], \quad (20b)
\]

where $k_1 := p_1 + \ldots + p_m$, $k_2 := p_{m+1} + \ldots + p_n$ and $k := k_1 + k_2$.

**B. Lagrangian Perturbation Theory**

In the Lagrangian scheme [2, 23] the quantity under consideration is the displacement field $\Psi(\tau, q)$ which maps initial particle or fluid element positions $q$ into their final Eulerian position $x(\tau)$

\[
x(\tau) = q + \Psi(\tau, q). \quad (21)
\]

The Jacobian $F_{ij}$ of the transformation from Eulerian to Lagrangian coordinates is given by

\[
F_{ij} = \frac{\partial x_i}{\partial q_j} = \delta_{ij} + \Psi_{ij}, \quad (22a)
\]

and has the following properties

\[
J_F = \det \left[ \delta_{ij} + \Psi_{ij} \right], \quad F_{ij}^{-1} = \frac{1}{2} J_F^{-1} \epsilon_{ilm} \epsilon_{ipq} F_{pl} F_{qm}, \quad (22b)
\]

where $\epsilon_{ijk}$ refers to the totally antisymmetric Levi-Civita symbol. Mass conservation and absence of decaying modes imply

\[
\delta_n(k) = \int \frac{d^3 p_1 \ldots d^3 p_n}{(2\pi)^{3(n-1)}} \delta_{D}(k - p_{1,m}) F_n(p_1, \ldots, p_n) \times \delta_1(p_1) \ldots \delta_1(p_n) \quad (18c)
\]

\[
\theta_n(k) = - \int \frac{d^3 p_1 \ldots d^3 p_n}{(2\pi)^{3(n-1)}} \delta_{D}(k - p_{1,m}) G_n(p_1, \ldots, p_n) \times \delta_1(p_1) \ldots \delta_1(p_n), \quad (18d)
\]

where $p_{1,m} := p_1 + \ldots + p_n$. Substituting the ansatizes (18) into the fluid equations (16a) and (17a) rewritten in Fourier space allows to separate the time dependence and obtain for $n > 1$\n
\[
n \delta_n(k) + \theta_n(k) = - \sum_{m=1}^{n-1} k_1^0 k_2^0 \delta_n(k) \partial_m(k) \theta_m(k), \quad (23a)
\]

\[
3 \delta_n(k) + (1 + 2n) \theta_n(k) = - \sum_{m=1}^{n-1} \frac{k^2(k_1 \cdot k_2)}{k_1^2 k_2^2} \theta_n(k) \partial_m(k), \quad (23b)
\]

where $k = k_1 + k_2$. Solving this system for $\delta_n$ and $\theta_n$, respectively we obtain recursion relations for $F_n$ and $G_n$ with starting values $F_1 = G_1 = 1$, compare [22] the following relation between the Jacobian determinant $J_F(q)$ and the density contrast

\[
\tilde{\rho}(1 + \delta(x)) d^3 x = \tilde{\rho} d^3 q \iff 1 + \delta = J_F^{-1}. \quad (23)
\]

The equation of motion for the Eulerian position $x$ is

\[
\frac{d^2 x}{d \tau^2} + \mathcal{H}(\tau) \frac{dx}{d \tau} = - \nabla V. \quad (24a)
\]

By taking the divergence of (24a) and using the Poisson equation (16e) as well as mass conservation (23) one obtains a closed equation for the displacement field $\Psi$

\[
J_F(\tau, q) \nabla \cdot \left[ \frac{d^2 \Psi}{d \tau^2} + \mathcal{H}(\tau) \frac{d \Psi}{d \tau} \right] = \rho [J_F(\tau, q) - 1], \quad (24b)
\]

supplemented by the constraint $\nabla \times v = 0$

\[
\epsilon_{ijk} (F_{im})^{-1} F'_{km} = 0. \quad (24c)
\]

The exact displacement field $\Psi(\tau, q)$ can be expanded in a series with spatial parts $\Psi^{(s)}(q)$ and temporal coefficients using the scale factor $a(\tau)$, concentrating on an EdS universe

\[
\Psi(\tau, q) = \sum_{n=1}^{\infty} a^s(\tau) \Psi^{(n)}(q). \quad (25)
\]
Again we express the different orders $\Psi^{(n)}$ in Fourier space with the help of perturbative kernels $L^{(n)}$ in terms of powers of the linear density field $\delta_1$

$$\Psi^{(n)}(k) = i \int \frac{d^3p_1 \ldots d^3p_n}{(2\pi)^{3(n-1)}} \delta_0(k - p_{1 \ldots n}) L^{(n)}(p_1, \ldots, p_n) \times \delta_1(p_1) \cdots \delta_1(p_n).$$

Note that we employ here a different notation for $L^{(n)}$ compared to Eq. (A2) in [6] such that when translating the results an additional prefactor $n!$ has to be taken into account. The vector valued kernels $L^{(n)}$ can be split into a longitudinal component $S^{(n)}$ and a transverse part $T^{(n)}$ according to

$$L^{(n)} = S^{(n)} + T^{(n)},$$

$$k \times S^{(n)}(p_1, \ldots, p_n) = 0,$$

$$k \cdot T^{(n)}(p_1, \ldots, p_n) = 0,$$

where $k := p_1 + \ldots + p_n$. Although a separation ansatz (25) with $a \rightarrow D$ on an ACDM background is not an exact solution to Eq. (24b) it is a good approximation to use the time independent kernels $L^{(n)}$ [2].

C. Mapping between Eulerian and Lagrangian picture

Note that LPT correctly recovers SPT when the exact relation between the density and displacement field, encoded in the continuity equation (23) with (22b), is expanded in the same manner. In lowest order LPT it is possible to keep this nonperturbative relation between $\Psi$ and the $\delta$, which is done in the Zel’dovich approximation (ZA) [24]. This allows to partially resum perturbation theory in a physically motivated way by combining $i$ an approximate law for the movement of particles from first order LPT (1LPT) with $ii$ a proper determination of the density within a small volume as the sum of all particles divided by this volume. Performing this proper counting is the resummation, giving it up leads to SPT. The generalization of the ZA to second order LPT (2LPT) is referred to as Post-Zel’dovich approximation (PZA), for which an exact translation from displacement to density field Eq. (23) would demand solving a non-Gaussian integral when computing correlation functions. Since this is analytically not tractable some approximation methods to the PZA, like integrated perturbation theory (iPT) [25] or Convolution Lagrangian perturbation theory (CLPT) [26] have been proposed that leave the relation, which is solved exactly in the ZA, at least partially resummed. It is because of those and many other advantages of the Lagrangian approach [27] that we also derive LPT for the coarse-grained dust model. In particular, for ZA it is known that a smoothing of the initial linear power spectrum improves ZA even further. This coarse-graining procedure is known as “truncated” ZA [3], but compared to our coarse-grained dust, only the initial conditions are smoothed while the dynamics is that of dust. We will explore CLPT, “truncated” CLPT both based on dust to a modified CLPT based on coarse-grained dust (cgCLPT) elsewhere.

In the following we will shortly describe the relation between the Eulerian framework that is based on the density contrast $\delta$ and the velocity divergence $\theta$ and the Lagrangian description which relies on the displacement field $\Psi$. For a more detailed study of the connection between the series in Lagrangian and Eulerian perturbation theory we refer to [23]. Rewriting the density contrast in Fourier space and employing mass conservation (23) gives

$$\delta(k) = \int d^3x \ e^{-ik \cdot x} \delta(x),$$

$$= \int d^3q \ e^{-ik \cdot q} \left(e^{-ik \cdot \Psi(q)} - 1\right).$$

Similarly, the divergence of the velocity $\theta = \nabla \cdot \mathbf{v}$ can be transformed to Lagrangian space by using $v = dx/d\tau = \Psi'(q)$ together with Eq. (25) and transforming $\nabla_x$ to $\nabla_q$ which leads to

$$\frac{\theta(x(q))}{\mathcal{H}} = F_{ij}^{-1} \left(a \Psi^{(1)}_{ji} + 2a^2 \Psi^{(2)}_{ji} + 3a^3 \Psi^{(3)}_{ji}(q)\right).$$

Using the explicit formula for $F_{ij}^{-1}$ (22b) we can obtain the Fourier transform of $\theta(q)$

$$\frac{\theta(k)}{\mathcal{H}} = \int d^3q \ e^{-i(k - q) \cdot k \cdot \Psi(q)} J_F \frac{\theta(x(q))}{\mathcal{H}}.$$

Note that, within the mapping from Eulerian to Lagrangian frame the absence of vorticity $\omega = 0$ implies a constraint (24c) for the transverse parts of $\Psi$, see [23].

IV. PERTURBATION THEORY FOR COARSE-GRAINED DUST

In order to obtain a solution to Eq. (15) for the coarse-grained quantities $\tilde{\delta}$ and $\tilde{\omega}$, we can solve the microscopic system (8) for $\delta = n - 1$ and $v = \nabla \theta / \Delta$ where $\theta = \Delta \phi / am$ and then simply coarse-grain the result according to

$$\tilde{\delta} = \exp \left(\frac{1}{2} \sigma_\delta^2 \Delta\right) \delta,$$

$$(1 + \tilde{\delta}) \tilde{\omega} = \exp \left(\frac{1}{2} \sigma_\omega^2 \Delta\right) [(1 + \delta)\omega].$$

As mentioned before, this procedure is possible as long as the solution space of the “microscopic” functions $\tilde{\delta}, \tilde{\theta}$ allows to invert the Gaussian smoothing operation; this it what justified the use of Eq. (15b) instead of the closed equation (45b) for the macroscopic quantities given in [18] in the limit $h \rightarrow 0$. However, at shell-crossing, where $\delta$ diverges at point-, line- or sheetlike structures a deconvolution is impossible. Therefore considering the coarse-grained dust case does not allow us to genuinely go beyond shell crossing. However, microscopic vorticity and velocity dispersion contribute to the true macroscopic vorticity $\tilde{\omega}$ and velocity dispersion $C_{ij}^{(2)}$. Those microscopic contributions simply add to the corresponding quantities of the coarse-grained dust model that arise without any microscopic origin [28]. Therefore one might hope that coarse-grained dust captures some aspects of the true macroscopic $\tilde{\omega}$ and $C_{ij}^{(2)}$. 
A. Eulerian kernels for density and velocity

We write the solution of the coarse-grained dust model (29) again as a perturbative series by separating time and momentum dependence,

\[ \bar{\delta}(\tau, k) = \sum_{n=1}^{\infty} a^n(\tau) \bar{\delta}_n(k), \quad (30a) \]

\[ \bar{v}(\tau, k) = \mathcal{H}(\tau) \sum_{n=1}^{\infty} a^n(\tau) \bar{v}_n(k). \quad (30b) \]

To obtain formal solutions we proceed in the same manner as in standard perturbation theory, see III A. The general solution may be written in terms of Fourier kernels

\[ \bar{\delta}_n(k) = \int \frac{d^3p_1 \cdots d^3p_n}{(2\pi)^{3(n-1)}} \bar{\delta}_D(k - p_{1 \cdots n}) \bar{F}_n(p_1, \ldots, p_n) \times \bar{\delta}_1(p_1) \cdots \bar{\delta}_1(p_n), \quad (31a) \]

\[ \bar{v}_n(k) = i \int \frac{d^3p_1 \cdots d^3p_n}{(2\pi)^{3(n-1)}} \bar{\delta}_D(k - p_{1 \cdots n}) \bar{V}_n(p_1, \ldots, p_n) \times \bar{\delta}_1(p_1) \cdots \bar{\delta}_1(p_n). \quad (31b) \]

It is convenient to decompose the velocity \( \bar{v} \) into velocity divergence \( \theta := \nabla \cdot \bar{v} \) and vorticity \( \bar{\omega} := \nabla \times \bar{v} \) for which we also define Eulerian kernels according to

\[ \bar{\bar{\delta}}_n(k) = -\int \frac{d^3p_1 \cdots d^3p_n}{(2\pi)^{3(n-1)}} \bar{\delta}_D(k - p_{1 \cdots n}) \bar{\bar{G}}_n(p_1, \ldots, p_n) \times \bar{\delta}_1(p_1) \cdots \bar{\delta}_1(p_n), \quad (31c) \]

\[ \bar{\bar{w}}_n(k) = -\int \frac{d^3p_1 \cdots d^3p_n}{(2\pi)^{3(n-1)}} \bar{\delta}_D(k - p_{1 \cdots n}) \bar{\bar{W}}_n(p_1, \ldots, p_n) \times \bar{\delta}_1(p_1) \cdots \bar{\delta}_1(p_n). \quad (31d) \]

The corresponding kernels are related to those of velocity \( \bar{V}_n \) via \( \bar{\bar{G}}_n = k \cdot \bar{V}_n \) and \( \bar{\bar{W}}_n = k \times \bar{V}_n \).

Since the macroscopic density contrast \( \bar{\delta} \) is trivially related to the microscopic \( \delta \), see (29a) we have that

\[ \bar{F}_n = \exp \left( -\frac{1}{2} \sigma_T^2 \Delta^2 \right) F_n. \quad (32a) \]

where \( F_n \) are the SPT kernels from (20). Therefore, in Eulerian perturbation theory, the matter power spectrum for the coarse-grained dust model at any order is simply given by the coarse-graining of the dust power spectrum, see Eq. (35a).

In order to determine the coarse-grained velocity field we have to expand Eq. (29b) perturbatively in terms of the micro- and macroscopic fields we have solved for before which gives

\[ \bar{v}_n = \exp \left( \frac{1}{2} \sigma_T^2 \Delta \right) v_n + \sum_{m=1}^{n-1} \left[ \exp \left( \frac{1}{2} \sigma_T^2 \Delta \right) (\bar{\delta}_m v_{n-m} - \bar{\delta}_m \bar{v}_{n-m}) \right], \quad (32b) \]

where \( v_n = \nabla \theta_n / \Delta \). Note that the curly bracket in (32b) basically calculates the difference between the average of a product and the product of averages (this statement is exact at 2nd order). It is precisely this deviation that sources the vorticity \( \bar{\omega}_n = \nabla \times \bar{\delta}_n \) which becomes relevant at 2nd order. In the limit \( \sigma_T \rightarrow 0 \), this contribution vanishes identically at all orders such that the velocity remains a gradient field thereby recovering the standard SPT kernels from Eq. (20) for \( \sigma_T \rightarrow 0 \).

The kernels \( \bar{V}_n \) for the velocity \( \bar{v} \) can be read off from (32b)

\[ \bar{V}_n(p_1, \ldots, p_n) = \frac{k}{k^2} \exp \left( -\frac{1}{2} \sigma_T^2 k^2 \right) G_n \quad (32c) \]

\[ + \sum_{m=1}^{n-1} \left\{ \exp \left( -\frac{1}{2} \sigma_T^2 k^2 \right) \bar{F}_m \frac{k^2}{k^2} G_{n-m} - \bar{F}_m \bar{V}_{n-m} \right\}. \]

Note that the kernel \( \bar{G}_n \) of \( \bar{\delta} = \nabla \cdot \bar{v} \) is not simply given by the coarse-graining of the kernel \( G_n \) of \( \theta = \Delta \phi / \Delta m \) since the velocity is mass-weighted according to (29b) . However, at first order we recover a curl-free velocity \( \bar{v}_1 = \exp \left( \frac{1}{2} \sigma_T^2 \Delta \right) \nabla \theta / \Delta \) and \( \bar{\delta}_1 = -\delta_1(k) \).

B. Eulerian power and cross spectra

In order to check whether our new kernels give sound results in perturbation theory, we calculate here some power and cross spectra up to one-loop order and therefore up to third order in perturbation theory. It turns out that expressions, displayed in App. A are convergent and reduce to the correct results in the limit where \( \sigma_T \rightarrow 0 \). The most interesting result of these checks will be the power spectrum for the vorticity \( \bar{\omega} \), because it vanishes identically in the standard dust model. The power spectra \( P(k) \) corresponding to density \( \delta \), velocity divergence \( \theta \) and vorticity \( \omega \) are defined according to

\[ \langle \delta(k) \delta(k') \rangle = (2\pi)^3 \delta_D(k + k') P_{\delta\delta}(k), \quad (33a) \]

\[ \langle \theta(k) \theta(k') \rangle = (2\pi)^3 \delta_D(k + k') P_{\theta\theta}(k), \quad (33b) \]

\[ \langle \omega(k) \cdot \omega(k') \rangle = (2\pi)^3 \delta_D(k + k') P_{\omega\omega}(k). \quad (33c) \]

Furthermore we have the cross spectrum between density \( \delta \) and velocity divergence \( \theta \)

\[ \langle \delta(k) \theta(k') \rangle = (2\pi)^3 \delta_D(k + k') P_{\delta\theta}(k). \quad (33d) \]

The velocity power spectrum is defined accordingly

\[ \langle \omega(k) \cdot \omega(k') \rangle = (2\pi)^3 \delta_D(k + k') P_{\omega\omega}(k). \quad (34a) \]

Since \( v = (\nabla \theta - \nabla \times \bar{\omega}) / \Delta \) it can be easily obtained from the divergence \( \theta = \nabla \cdot v \) and vorticity \( \bar{\omega} = \nabla \times v \) power spectra

\[ k^2 P_{\omega\omega}(k) = P_{\theta\theta}(k) + P_{\omega\omega}(k). \quad (34b) \]

In the following we will derive the power and cross spectra up to one-loop order for the coarse-grained dust model (cgSPT), and compare it to both standard SPT as well as standard SPT with a different coarse-graining procedure (SPTcg) where only the linear input power spectrum \( P_L \) is smoothed. This is done merely to illustrate the effect of the coarse-graining on the perturbation kernels rather than to suggest an improvement of SPT. SPT is known to fail to converge as a perturbative series, see [29] and is less accurate in predicting the nonlinear density field than Lagrangian methods [27]. We will present the application of the Lagrangian kernels elsewhere.
Density power spectrum  For the density power spectrum the effect of the coarse-grained fluid equations (15) is simply to coarse-grain the power spectrum obtained from SPT according to

\[ P_{\delta\delta}(k) = P_{\delta}(k) = \exp(-\sigma_{\delta}^2 k^2) P_{\delta}(k). \]  

This result holds at any order in SPT and shows that, as expected, the smoothing becomes effective only at small scales \( k \geq 1/\sigma_{\delta} \). Note that, since the power spectrum is quadratic in \( \delta \) it gets smoothed with \( \sqrt{2}\sigma_{\delta} \) when \( \delta \) is coarse-grained on scale \( \sigma_{\delta} \). Therefore we will write in the following \( P_k := \exp(-\sigma_{\delta}^2 k^2) P(k) \) even if \( \delta(k) := \exp(-\frac{1}{2}\sigma_{\delta}^2 k^2) \delta(k) \). The resulting power spectrum depicted in Fig. 1 shows that power on small spatial scales corresponding to large \( k \) is suppressed due to the coarse-graining.

Velocity power spectrum  The effect of the coarse-graining onto the velocity power spectrum is more involved than for the density power spectrum since the cgSPT kernels for the mass-weighted velocity have to be evaluated according to Eq. (32c). At linear level the velocity kernel \( \tilde{V}_1 \) corresponds to a smoothing of the microscopic velocity divergence. Because of this, the linear power spectrum of macroscopic velocity divergence \( \bar{\theta} \) is just given by the coarse-grained SPT result and the vorticity vanishes identically

\[ P_{\bar{\theta}\bar{\theta},L}(k) = P_{\bar{\theta}\bar{\theta},L}(k), \]  

\[ P_{\bar{\theta}\bar{w},L}(k) = P_{\bar{w}\bar{w},L}(k) = 0. \]

Note that since \( \bar{\theta}_1(k) = -\bar{\sigma}_1(k) \) the linear velocity power spectrum is identical to the linear density power spectrum when expressed in the same units.

At one-loop level the different contributions to the total velocity kernel \( \tilde{V}_2 \) according to Eq. (32c) have been evaluated explicitly in Appendix A. As can be seen in the Fig. 2, the effect of the dynamical coarse-graining (cgSPT) for the velocity power spectrum differs from coarse-graining the initial conditions in SPT (SPTcg). Most notably, our coarse-graining procedure determining the mass-weighted velocity \( \bar{\nu} \) introduces a nonzero vorticity \( \bar{\omega} = \nabla \times \bar{\nu} \) which manifests itself from second order on. The vorticity marginally affects the velocity power spectrum at one-loop order via its contribution \( P_{\bar{w}\bar{w},22} \). However, this contribution present in cgSPT is a fundamental difference to SPT where vorticity cannot be sourced when it is zero initially, see Eq. (17b). The corresponding expression can be computed easily from the recursion relations for the kernels of velocity given in (32c) and reads

\[ P_{\bar{w}\bar{w},22}(k) = 2 \int \frac{d^3 p}{(2\pi)^3} |\hat{W}_2^{(2)}(p,k-p)|^2 P_L(p) P_L(k-p) = \frac{k^3}{2\pi^2} \int_0^\infty dr \int_{-1}^1 dx \frac{\hat{P}_L(k)r}{P_L(k \sqrt{1 - 2rx + r^2})} \times \left(1 - x^2\right) \left(1 - 2rx\right)^2 \left(4\sigma_{\delta}^2 x^2 - 1\right)^2. \]

Interestingly, the only effect of increasing the coarse-graining scale is to cause vorticity to become relevant at larger length scales whereas the shape of the vorticity power spectrum remains unchanged and the slope seems to be universal. In [7, 30] it has been suggested that the basic features of the vorticity power spectrum can be understood when assuming that the vorticity in regions which underwent shell-crossing is induced by mass-weighting the single-stream velocities. However, they used as estimate \( \bar{\omega} \sim \nabla \times [(1 + \delta)\nu]/(1 + \delta) \) where, in contrast, no real mass-weighting has been performed. Instead, they considered the vorticity of the momentum \( j = (1 + \delta)\nu \) and afterwards divided by \((1 + \delta)\). In turn, our approach based on coarse-graining automatically implements this idea correctly and yields a vorticity according to \( \bar{\omega} = \nabla \times \bar{\nu} = \nabla \times \left[ (1 + \delta)\nu/(1 + \delta) \right] \). Since in [7] there was no prediction for the amplitude of the vorticity power spectrum we can only compare the spectral index \( n_{\omega} := d \ln P_{\omega\omega}/d \ln k \) which is depicted in Fig. 3.

Our results agree with predictions made in EFT of LSS.
[10], which give $P_{ww} \propto \left(k/k_{\text{NL}}\right)^{n_w}$ with

$$
n_w = \begin{cases} 
4 & \text{for } k \leq 0.1, \\
3.6 & \text{for } 0.1 \leq k \leq 0.3, \quad (38) \\
2.8 & \text{for } 0.3 \leq k \leq 0.6 
\end{cases}
$$

where $k_{\text{NL}}$ is the nonlinear scale in EFT of LSS [10]. Furthermore we can clearly see that the spectral index of the vorticity caused by a mass-weighted velocity differs significantly from the estimation $\omega \sim \nabla \times [(1 + \delta\bar{u})/(1 + \bar{\delta})]$ made in [7, 30], which is the solid blue wiggly line Fig. 3 denoted by $\omega_{\text{ww}}$.

Cross spectrum The cross spectrum can be determined in the same way as the power spectra by employing the kernels for the density contrast and velocity divergence obtained before. At lowest order the cross spectrum of cgSPT is trivially related to the SPT cross spectrum

$$
P_{\theta\theta,\xi}(k) = P_{\theta\theta,\xi}(k). \quad (39)
$$

which is again just a scaled version of the density power spectrum. Beyond linear order there is another contribution besides the trivial smoothing which slightly affects the cross spectrum, namely the effect of the modified recursion relation for the kernel $G_\eta$ of $\bar{\theta}_n$ according to (32c) which differs from the smoothing of the kernel $G_n$ of $\theta_n$. The cross spectrum at 1-loop order, whose explicit expression can be found in Appendix A, is shown in Fig. 4.

Comparison to N-body simulations We can compare our analytical results to power and cross spectra obtained from cosmological numerical simulations, as given in [7] and [19]. Note that within both works a different Fourier convention has been employed. Therefore we show our power spectra divided by $(2\pi)^3$ in order to allow for comparison which reveals good qualitative agreement. Note also that our $\theta(k)$ is dimensionless because we factored out $\mathcal{H}a^2$ in (18b), while [19] measure $\theta$ in km/(s Mpc/h).

In [7] it was noted that the vorticity power spectrum shows significant sensitivity on the mass resolution which was confirmed by [19], compare Fig. 3 in [7] and Fig. 12 in [19]. Similarly, a strong dependence on the smoothing scale shows up in the vorticity power spectrum for cgSPT, see Fig. 2. The main effect of an increasing coarse-graining scale is to shift the wavenumber at which vorticity becomes relevant to smaller values corresponding to larger length scales. However, the spectral index $n_w$ is a rather universal feature of the vorticity power spectrum and was determined in [19] as a function of $k$. Its asymptotic values were found to agree with $P_{ww} \propto k^{5/2}$ on large scales and $P_{ww} \propto k^{-3/2}$ on small scales. Note that, due to the coarse-graining in our formalism small spatial scales corresponding to large $k$ are not accessible. On intermediate scales we find reasonable agreement for the spectral index $n_w$ predicted by cgSPT, shown as dashed lines in Fig. 3, with N-body simulations, compare Fig. 14 in [19]. The spectral index $n_w$ obtained from estimating the vorticity according to [7, 30] shows a qualitatively different behavior, see the solid blue line in Fig. 3.

C. Lagrangian displacement kernels

Well before the onset of strong non-linearity and shell crossing the dynamics of a perfect pressureless fluid qualitatively resembles coarse-grained hydrodynamics. Therefore the Zel’dovich approximation based on the dust fluid, which in the mildly nonlinear regime has proven quite successful [3, 27], should retain its applicability even in the coarse-grained dust model. We will thus consider the effect of the coarse-graining onto LPT which also paves the way to base the Post-Zel’dovich approximation or resummation schemes like iPT [25] and CLPT [26] on the coarse-grained dust model.

We perform a transformation to Lagrangian coordinates $q$ in which the coarse-grained fluid positions are given by the old Eulerian coordinates $x = q + \Psi$, where $\Psi(q, \tau)$ are the integral lines of $\dot{\psi}(q, \tau) = \partial_{\tau} \Psi(q, \tau)$ emanating at $q$. When

---

1 At 2nd order this power spectrum is equivalent to the power spectrum of vector metric perturbations $\omega$ as calculated in Eq. 4.9c in [31].

---

![FIG. 3. Spectral index $n_w$ of the vorticity power spectrum $P_{ww} \propto (k/k_{\text{NL}})^{n_w}$ as function of wavenumber $k$.](image1)

![FIG. 4. Comparison between cross spectrum for SPT and cgSPT in 1st (lin) and 2nd (1-loop) order perturbation theory.](image2)
we define analogously to (22)
\[ F_{ij} = \frac{\partial x_i}{\partial q_j} = \delta_{ij} + \bar{P}_{ij} , \quad J_F = \det \left[ \delta_{ij} + \bar{P}_{ij} \right], \quad (40) \]
we obtain the same relation between \( \bar{\delta} \) and \( \bar{\Psi} \) as it was the case for dust (23)
\[ 1 + \bar{\delta} = J_F^{-1}, \quad (41) \]
since \( \bar{\delta} \) and \( \bar{\nu} \) fulfill the continuity equation (15a). Therefore we have in Fourier space analogously to (28)
\[ \bar{\delta}(k) = \int d^3 q \ e^{-i k \cdot q} \left( e^{-i k \cdot \bar{\Psi}(q)} - 1 \right). \quad (42) \]
Next, we expand the displacement field \( \bar{\Psi}(\tau, k) \) perturbatively according to (25)
\[ \bar{\Psi}(\tau, q) = \sum_{n=1}^{\infty} \bar{a}^n(\tau, \bar{\Psi}(q)). \quad (43) \]
Then we express the different orders \( \bar{\Psi}^{(n)} \) with the help of perturbative kernels \( \bar{L}^{(n)} \) which are defined analogously to (26)
\[ \bar{\Psi}^{(n)}(k) = i \int \frac{d^3 p_1 \ldots d^3 p_n}{(2\pi)^{3(n-1)}} \delta_\Omega(k-p_{1-n}) \bar{L}^{(n)}(p_1, \ldots, p_n) \times \bar{\delta}_1(p_1) \cdots \bar{\delta}_1(p_n). \quad (44) \]
\[ \bar{\Psi}^{(1)}(p_1, p_2) = \begin{cases} 2 k \times \bar{T}^{(2)}(p_1, p_2), \\ 3 k \times \bar{T}^{(3)} + \frac{1}{3} \left[ \begin{array}{c} (k_1 \times k_2) \cdot \bar{S}^{(1)} + \bar{S}^{(2)} + \bar{T}^{(2)} \\ 2 k_2 \times \bar{T}^{(2)} k \cdot \bar{S}^{(1)} - 2 \bar{S}^{(1)} k_1 \cdot (k_2 \times \bar{T}^{(2)}) \end{array} \right] \end{cases} \quad (46b) \]
For the sake of brevity we suppress the functional dependencies on the right hand side. They can be easily restored by attaching each \( \bar{S}^{(n)} \) or \( \bar{T}^{(n)} \) a dependence on \( (p_i, \ldots, p_{n+1}) \) in ascending order beginning with \( i = 1 \) from the left, for example \( \bar{S}^{(1)} \cdot \bar{T}^{(2)} : = \bar{S}^{(1)}(p_1) \cdot \bar{T}^{(2)}(p_2, p_3) \). Note that we defined
\[ \bar{W}^{(n)}(p_1, \ldots, p_n) := \frac{1}{n!} \sum_{\sigma \in S_n} \bar{W}^{(n)}(p_{\sigma(1)}, \ldots, p_{\sigma(n)}), \]
where the sum goes over all \( n! \) permutations \( \sigma \) of \( n \) indices. Analogous to the case of dust where \( F_n \) and \( S^{(n)}, T^{(n)} \) are related via Eqs. (28a) and (24c), see Eqs. (6.9) and (B1)-(B3) in [23], we obtain from Eqs. (42) and (45c)
\[ F_1^{(1)}(p_1) = k \cdot \bar{S}^{(1)}, \quad (47a) \]
\[ F_2^{(1)}(p_1, p_2) = k \cdot \bar{S}^{(2)} + \frac{1}{2} \left( k \cdot \bar{S}^{(1)} \right) \left( k \cdot \bar{S}^{(1)} \right), \quad (47b) \]
\[ F_3^{(1)}(p_1, p_2, p_3) = k \cdot \bar{S}^{(3)} + \frac{1}{6} \left( k \cdot \bar{S}^{(1)} \right) \left( k \cdot \bar{S}^{(1)} \right) \left( k \cdot \bar{S}^{(1)} \right) + \frac{1}{3} \left( k \cdot \bar{S}^{(1)} \right) \left( k \cdot \bar{S}^{(2)} \right). \quad (47c) \]
Since \( k \cdot \bar{T} = 0 \) we have \( k \times (k \times \bar{T}) = -k^2 \bar{T} \) which allows to invert (46) for \( \bar{T} \). Therefore we can determine the longitudinal
\( S^{(0)} \) and transverse \( T^{(0)} \) kernels of the displacement field from the Eulerian kernels for density \( \bar{\delta} \) and vorticity \( \bar{\omega} \)

\[
S^{(1)}(p_1) = \exp\left(-\frac{1}{2} \sigma^2 p^2\right) \frac{p_1}{p_1^2}, \quad T^{(1)}(p_1) = 0, \tag{48a}
\]

\[
S^{(2)}(p_1, p_2) = \frac{p_1^2 p_2^2}{p_1^2} \left( \bar{\delta}_2 - \frac{1}{2} \left( p_{12} \cdot S^{(1)} \right) \left( p_{12} \cdot S^{(1)} \right) \right), \quad T^{(2)}(p_1, p_2) = \frac{1}{2} \frac{1}{p_1^2} p_{12} \times \bar{W}_s^{(2)}(p_1, p_2), \tag{48b}
\]

\[
S^{(3)}(p_1, p_2, p_3) = \frac{p_{123}}{p_{123}^2} \left( \bar{\delta}_3 - \frac{1}{6} \left( p_{123} \cdot S^{(1)} \right) \left( p_{123} \cdot S^{(1)} \right) - \frac{1}{3} \left( p_{123} \cdot S^{(1)} \right) \left( p_{123} \cdot \left[ S^{(2)} + T^{(2)} \right] \right) \right), \tag{48c}
\]

\[
T^{(3)}(p_1, p_2, p_3) = \frac{1}{3} \frac{p_{123}}{p_{123}^2} \left( \bar{W}_s^{(3)} + \frac{1}{3} \left[ p_1 \times p_{23} \left( S^{(1)} + \left[ S^{(2)} + T^{(2)} \right] \right) + 2 p_{23} \times \bar{W}_s^{(2)}(p_1, p_2, p_3) \right] + \text{cyclic permutation of } (p_1, p_2, p_3) \right). \]

Note that our kernels correctly reproduce to the standard dust kernels, given in [23] in the limit \( \sigma_\epsilon \to 0 \).

V. CONCLUSION AND OUTLOOK

In order to model collisionless selfgravitating matter we considered a coarse-grained dust fluid, which is in turn a good model for cold dark matter in the single-stream regime and entire halos on large scales. We studied it perturbatively in the Eulerian frame and derived recursion relations Eqs. (32) for the Fourier kernels of the coarse-grained density contrast \( \bar{\delta} \) and the mass-weighted velocity \( \bar{v} \). Those recursive expressions are given in terms of the standard perturbation kernels for the microscopic density contrast \( \delta \) and the velocity divergence \( \theta \) of a pressureless (dust) fluid.

We computed the corresponding power and cross spectra of the coarse-grained density contrast \( \bar{\delta} \) and the mass-weighted velocity \( \bar{v} \) up to 1-loop order perturbation theory and compared them to the standard dust case. Our study revealed that in the coarse-grained dynamics a vorticity \( \bar{\omega} = \nabla \times \bar{v} \) is generated dynamically which becomes manifest already at 1-loop order in the power spectrum Eq. (37). The magnitude, shape and spectral index of the analytically predicted vorticity power spectrum, see Fig. 2, exhibits qualitatively good agreement with recent measurements from N-body simulations [19]. Furthermore the scale dependence of the spectral index, shown in Fig. 3, resembles the observations made in the context of EFT of LSS [10].

Finally, we explained how the perturbation kernels of the displacement field \( \bar{\Psi} \) in the Lagrangian framework can be obtained from the Eulerian kernels of the density contrast \( \bar{\delta} \) and the vorticity \( \bar{\omega} \). To pave the way for applications in the context of LPT we gave explicit expressions for the displacement kernels up to third order in Eqs. (48). Based on these results we will determine the impact of the coarse-graining onto the correlation function of halos by extending CLPT [26], an approximation to the Post-Zeldovich approximation, in a further investigation. Therein we will especially study how redshift space distortions affect the correlation function by means of the Gaussian streaming model [33] and its possible extensions.

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\[ P_{\mathbf{w}_0, \mathbf{z}_2}(k) = \left[ (2\pi)^3 \delta_D(0) \right]^{-1} \langle \mathbf{w}_2(k) \cdot \mathbf{w}_2(-k) \rangle . \] (A1a)

\[ = \frac{1}{2\pi^3} \int_0^\infty \int_{-1}^0 \int_{-1}^0 \frac{1}{(2\pi)^9} \delta_D(p_1 + p_2 - k) \delta_D(\bar{p}_1 + \bar{p}_2 + k) \mathbf{W}_2(p_1, p_2) \cdot \mathbf{W}_2(\bar{p}_1, \bar{p}_2)(\delta_1(p_1)\delta_1(p_2)\delta_1(\bar{p}_1)\delta_1(\bar{p}_2)) \] (A1b)

\[ \begin{align*}
&= \frac{k^3}{2\pi} \int_0^\infty \int_{-1}^0 \int_{-1}^0 \frac{1}{(2\pi)^9} \delta_D(p_1 + p_2 - k) \delta_D(\bar{p}_1 + \bar{p}_2 + k) \\
&\quad \frac{1}{2\pi^3} \int_0^\infty \int_{-1}^0 \int_{-1}^0 \frac{1}{(2\pi)^9} \delta_D(p_1 + p_2 - k) \delta_D(\bar{p}_1 + \bar{p}_2 + k) \\
&= 2 \frac{k^3}{(2\pi)^9} \int_0^\infty \int_{-1}^0 \int_{-1}^0 \frac{1}{(2\pi)^9} \delta_D(p_1 + p_2 - k) \delta_D(\bar{p}_1 + \bar{p}_2 + k) \\
&\quad \mathbf{W}_2(p_1, p_2) \cdot \mathbf{W}_2(\bar{p}_1, \bar{p}_2)(\delta_1(p_1)\delta_1(p_2)\delta_1(\bar{p}_1)\delta_1(\bar{p}_2)) \end{align*} \] (A1c)

\[ P_{\mathbf{w}_0, \mathbf{z}_2}(k) = 0 \] (A1d)

\[ P_{\mathbf{w}_0, \mathbf{w}_2}(k) = \left[ (2\pi)^3 \delta_D(0) \right]^{-1} \langle \mathbf{w}_2(k) \cdot \mathbf{w}_2(-k) \rangle . \] (A2a)

\[ = 2 \frac{k^3}{(2\pi)^9} \int_0^\infty \int_{-1}^0 \int_{-1}^0 \frac{1}{(2\pi)^9} \delta_D(p_1 + p_2 - k) \delta_D(\bar{p}_1 + \bar{p}_2 + k) \\
&\quad \mathbf{G}_2(p_1, p_2)G_2^*(\bar{p}_1, \bar{p}_2)(\delta_1(p_1)\delta_1(p_2)\delta_1(\bar{p}_1)\delta_1(\bar{p}_2)) \] (A2b)

\[ = \frac{k^3}{2\pi} \int_0^\infty \int_{-1}^0 \int_{-1}^0 \frac{1}{(2\pi)^9} \delta_D(p_1 + p_2 - k) \delta_D(\bar{p}_1 + \bar{p}_2 + k) \\
&\quad \mathbf{G}_2(p_1, p_2)G_2^*(\bar{p}_1, \bar{p}_2)(\delta_1(p_1)\delta_1(p_2)\delta_1(\bar{p}_1)\delta_1(\bar{p}_2)) \] (A2c)

\[ P_{\mathbf{w}_0, \mathbf{w}_2}(k) = \frac{k^3}{2\pi} \int_0^\infty \int_{-1}^0 \int_{-1}^0 \frac{1}{(2\pi)^9} \delta_D(p_1 + p_2 - k) \delta_D(\bar{p}_1 + \bar{p}_2 + k) \\
&\quad \mathbf{G}_2(p_1, p_2)G_2^*(\bar{p}_1, \bar{p}_2)(\delta_1(p_1)\delta_1(p_2)\delta_1(\bar{p}_1)\delta_1(\bar{p}_2)) \] (A2d)

**Appendix A: Power spectra**

1. **Velocity divergence and vorticity**
\[ P_{\delta \theta,13}(k) = \frac{k^3}{2\pi^2} \cdot \frac{1}{168} P_L(k) \int_0^\infty dr P_L(kr) \left\{ \frac{12}{r^2} - 82 + 4r^2 - 6r^4 - 3 \left( r^2 - 1 \right)^2 \log \left[ \frac{r-1}{r+1} \right] \right\} \]

\[ P_{\delta \theta,13}(k) = 6P_L(k) \int \frac{d^3 p}{(2\pi)^3} \tilde{G}^{(2)}_2(p-k)P_L(k) \]

\[ P_{\delta \theta,13}(k) = \frac{k^3}{2\pi^2} \cdot \frac{1}{168} P_L(k) \int_0^\infty dr P_L(kr) \left\{ \frac{12}{r^2} - 82 + 4r^2 - 6r^4 - 3 \left( r^2 - 1 \right)^2 \log \left[ \frac{r-1}{r+1} \right] \right\} \]

\[ P_{\delta \theta,22}(k) = [(2\pi)^3 \delta_D(0)]^{-1} \langle \tilde{\delta}_2(k) \tilde{\theta}_2(-k) \rangle \]

\[ P_{\delta \theta,22}(k) = -\int \frac{d^3 p}{(2\pi)^3} F_2^{(1)}(p-k)G_2^{(1)}(p-k-P_L(k)P_L(k)) \]

\[ P_{\delta \theta,22}(k) = \frac{k^3}{2\pi^2} \int_0^\infty dr \int_0^1 dx \left\{ \frac{10r \log(1-r)}{196 (r^2 - 2rx + 1)^2} \right\} \]

\[ P_{\delta \theta,22}(k) = \frac{k^3}{2\pi^2} \int_0^\infty dr \int_1^0 dx \left\{ \frac{10r \log(1-r)}{196 (r^2 - 2rx + 1)^2} \right\} \]

\[ P_{\delta \theta,22}(k) = \frac{k^3}{2\pi^2} \int_0^\infty dr \int_0^1 dx \left\{ \frac{10r \log(1-r)}{196 (r^2 - 2rx + 1)^2} \right\} \]

\[ P_{\delta \theta,22}(k) = \frac{k^3}{2\pi^2} \int_0^\infty dr \int_0^1 dx \left\{ \frac{10r \log(1-r)}{196 (r^2 - 2rx + 1)^2} \right\} \]

\[ P_{\delta \theta,22}(k) = \frac{k^3}{2\pi^2} \int_0^\infty dr \int_0^1 dx \left\{ \frac{10r \log(1-r)}{196 (r^2 - 2rx + 1)^2} \right\} \]

\[ P_{\delta \theta,22}(k) = \frac{k^3}{2\pi^2} \int_0^\infty dr \int_0^1 dx \left\{ \frac{10r \log(1-r)}{196 (r^2 - 2rx + 1)^2} \right\} \]

\[ P_{\delta \theta,22}(k) = \frac{k^3}{2\pi^2} \int_0^\infty dr \int_0^1 dx \left\{ \frac{10r \log(1-r)}{196 (r^2 - 2rx + 1)^2} \right\} \]


\[ P_{0h,13}(k) = [(2\pi)^3\delta_D(0)]^{-1} \left( \langle \delta_1(k)\tilde{\theta}_3(-k) \rangle + \langle \delta_3(k)\tilde{\theta}_1(-k) \rangle \right) \tag{A5a} \]

\[
= - \int \frac{d^3p_1 \, d^3p_2 \, d^3p_3 \, d^3\tilde{p}_1}{(2\pi)^9} \frac{\delta_D(p_1 + p_2 + p_3 - k)\delta_D(\bar{p}_1 + k)}{\delta_D(0)} \times \\
\times \left[ \tilde{G}_3^{(e)}(p_1, p_2, p_3)F_1(\bar{p}_1) + F_3^{(e)}(p_1, p_2, p_3)\tilde{G}_1(\bar{p}_1) \right] \langle \delta_1(p_1)\delta_1(p_2)\delta_1(\bar{p}_3) \rangle 
\]

\[
= -3P_L(k) \int \frac{d^3p}{(2\pi)^3} \frac{\delta_D(\bar{p} - p, -k)}{\delta_D(0)} \left[ F(\bar{p}_1)\tilde{G}_3^{(e)}(p, -p, -k) + F_3^{(e)}(p, -p, \bar{p})\tilde{G}_1(\bar{p}) \right] \tag{A5b} \]

\[
= -\frac{k^3}{4\pi^2} P_L(k) \int_0^\infty dr \, P_L(kr) e^{-\sigma_i^2 kr} \frac{3 \left( r^2 - 1 \right)}{504r^3} \left\{ 9r^2 e^{\frac{3}{2} \sigma_i^2 kr} \left[ \frac{1}{2} \sigma_i^2 k^2 (r - 1)^2 \right] - e^{\frac{1}{2} \sigma_i^2 k^2 (r + 1)^2} \right\} \\
\left( 7r^2 + 2 \right) 2 \log \left| \frac{r - 1}{r + 1} \right| e^{\sigma_i^2 kr(r+1)} + 108r^5 e^{\sigma_i^2 kr} - 8r \left( 21r^6 - 50r^4 + 79r^2 - 6 \right) e^{\sigma_i^2 kr(r+1)} \tag{A5c} \]

\[
+ \frac{18r^2(r^2 - 1)}{\sigma_i^2 k^2} \left[ e^{2\sigma_i^2 kr} (3r^2 + 6r - 25) - (3r^2 - 6r - 25) \right] \\
+ \frac{36r}{\sigma_i^2 k^2} \left[ (3r^3 - 12r^2 - 19r - 28) - e^{2\sigma_i^2 kr} (3r^3 + 12r^2 - 19r + 28) \right] \\
+ \frac{144}{\sigma_i^2 k^6} \left[ e^{2\sigma_i^2 kr} (3r^2 - 12r + 7) - (3r^2 + 12r + 7) \right] \\
+ \frac{1728}{\sigma_i^2 k^6} \left( e^{2\sigma_i^2 kr} - 1 \right) \right) \right\} 
\]

\[
P_{6h,13}(k) = -\frac{k^3}{2\pi^2} \frac{1}{504} P_L(k) \int_0^\infty dr \, P_L(kr) \left\{ \frac{24}{r^2} - 202 + 56r^2 - 30r^4 - \frac{3}{r} \left( r^2 - 1 \right)^3 (5r^2 + 4) \log \left| \frac{r - 1}{r + 1} \right| \right\} \tag{A5d} \]

In the limit \( \sigma_i \to 0 \) the kernels reduce to the standard SPT result given in Eqs. (A25)-(A26) in [35], note however that our convention for \( \theta_h \) is different compared to [35] such that the power spectra have flipped signs.