On the Cauchy Problem for the Modified Korteweg–de Vries Equation with Steplike Finite-Gap Initial Data

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ON THE CAUCHY PROBLEM FOR THE MODIFIED KORTEWEG–DE VRIES EQUATION WITH STEPLIKE FINITE-GAP INITIAL DATA

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Abstract. We solve the Cauchy problem for the modified Korteweg–de Vries equation with steplike quasi-periodic, finite-gap initial conditions under the assumption that the perturbations have a given number of derivatives and moments finite.

1. Introduction

The purpose of the present paper is to investigate the Cauchy problem for the modified Korteweg–de Vries (mKdV) equation

\begin{equation}
    v_t(x,t) = -v_{xxx}(x,t) + 6v(x,t)^2v_x(x,t), \quad v(x,0) = v(x),
\end{equation}

(where subscripts denote partial derivatives as usual) for the case of steplike initial conditions \( v(x) \). More precisely, we will assume that \( v(x) \) is asymptotically close to (in general) different real-valued, quasi-periodic, finite-gap potentials \( u_\pm(x) \) in the sense that

\begin{equation}
    \pm \int_0^{\pm \infty} \left| \frac{d^n}{dx^n} (v(x) - u_\pm(x)) \right| (1 + |x|^{m_0}) \, dx < \infty, \quad 0 \leq n \leq n_0 + 1,
\end{equation}

for some positive integers \( m_0, n_0 \). Here by quasi-periodic, finite-gap potentials we mean algebro-geometric, quasi-periodic, finite-gap potentials which arise naturally as the stationary solutions of the mKdV hierarchy as discussed in \( \text{[8]} \). If \( \text{[13]} \) holds for all \( m_0, n_0 \) we will call it a Schwartz-type perturbation.

If \( u_\pm = 0 \) this problem is of course well understood, but for non-decaying initial conditions the only result we are aware of is the one by Kappeler, Perry, Shubin and, Topalov \( \text{[13]} \). In order to solve the Cauchy problem for the mKdV equation \( \text{[10]} \) with initial data satisfying \( \text{[10]} \) for suitable \( m_0, n_0 \), our main ingredient will be the corresponding result for the KdV equation \( \text{[3]} \), \( \text{[5]} \) combined with the Miura transform.

Next, let us state our main result. Denote by \( C^n(\mathbb{R}) \) the set of functions \( x \in \mathbb{R} \mapsto q(x) \in \mathbb{R} \) which have \( n \) continuous derivatives with respect to \( x \) and by \( C^n_\kappa(\mathbb{R}^2) \) the set of functions \( (x,t) \in \mathbb{R}^2 \mapsto q(x,t) \in \mathbb{R} \) which have \( n \) continuous derivatives with respect to \( x \) and \( k \) continuous derivatives with respect to \( t \).

**Theorem 1.1.** Let \( u_\pm(x,t) \) be two real-valued, quasi-periodic, finite-gap solutions of the mKdV equation corresponding to arbitrary quasi-periodic, finite-gap initial data \( u_\pm(x) = u_\pm(x,0) \). Let \( m_0 \geq 8 \) and \( n_0 \geq m_0 + 5 \) be fixed natural numbers.

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Suppose, that \( v(x) \in C^{n_0+1}(\mathbb{R}) \) is a real-valued function such that \( (2.1) \) holds. Then there exists a unique classical solution \( v(x,t) \in C^{\infty}_{1-\infty}(\mathbb{R}^2) \) of the initial-value problem for the mKdV equation \( (1.1) \) satisfying
\[
(1.3) \quad \pm \int_0^{\pm \infty} \left\| \frac{\partial^n}{x^n}(v(x,t) - u_\pm(x,t)) \right\| (1 + |x|^{m_0}) dx < \infty, \quad n \leq n_0 - m_0 - 1,
\]
for all \( t \in \mathbb{R} \).

In particular, this theorem shows that the mKdV equation has a solution within the class of steplike Schwartz-type perturbations of finite-gap potentials:

**Corollary 1.2.** Let \( u_\pm(x,t) \) be two real-valued, quasi-periodic, finite-gap solutions of the mKdV equation corresponding to arbitrary quasi-periodic, finite-gap initial data \( u_\pm(x) = u_\pm(x,0) \). In addition, suppose, that \( v(x) \) is a steplike Schwartz-type perturbations of \( u_\pm(x) \). Then the solution \( v(x,t) \) of the initial-value problem for the mKdV equation \( (1.1) \) is a steplike Schwartz-type perturbations of \( u_\pm(x,t) \) for all \( t \in \mathbb{R} \).

For a unique continuation result within this class of solutions we refer to [4].

## 2. The KdV equation with steplike finite-gap initial data

As a preparation we recall some basic facts on the Cauchy problem for the KdV equation
\[
(2.1) \quad q_t(x,t) = -q_{xxx}(x,t) + 6q(x,t)q_x(x,t), \quad q(x,0) = q(x),
\]
for the case of steplike initial conditions \( q(x) \) from [3], [4]. More precisely, we will assume that \( q(x) \) is asymptotically close to (in general) different quasi-periodic, finite-gap potentials \( p_\pm(x) \) in the sense that
\[
(2.2) \quad \pm \int_0^{\pm \infty} \left| \frac{d^n}{dx^n}(q(x) - p_\pm(x)) \right| (1 + |x|^{m_0}) dx < \infty, \quad 0 \leq n \leq n_0,
\]
for some positive integers \( m_0, n_0 \). The main result reads

**Theorem 2.1 ([3]).** Let \( p_\pm(x,t) \) be two real-valued, quasi-periodic, finite-gap solutions of the KdV equation corresponding to arbitrary quasi-periodic, finite-gap initial data \( p_\pm(x) = p_\pm(x,0) \). Let \( m_0 \geq 8 \) and \( n_0 \geq m_0 + 5 \) be fixed natural numbers.

Suppose, that \( q(x) \in C^{m_0}(\mathbb{R}) \) is a real-valued function such that \( (2.1) \) holds. Then there exists a unique classical solution \( q(x,t) \in C^{\infty}_{1-\infty}(\mathbb{R}^2) \) of the initial-value problem for the KdV equation \( (2.1) \) satisfying
\[
(2.3) \quad \pm \int_0^{\pm \infty} \left\| \frac{\partial^n}{x^n}(q(x,t) - p_\pm(x,t)) \right\| (1 + |x|^{m_0}) dx < \infty, \quad n \leq n_0 - m_0 - 2,
\]
and
\[
(2.4) \quad \pm \int_0^{\pm \infty} \left\| \frac{\partial}{\partial t}(q(x,t) - p_\pm(x,t)) \right\| (1 + |x|^{m_0}) dx < \infty,
\]
for all \( t \in \mathbb{R} \).
In order to invert the Miura transform we will also need the solutions of the
associated Lax system.

Introduce the Lax operators corresponding to the finite-gap solutions
\(p_\pm(x, t)\),
\[
 L_\pm(t) = -\partial_x^2 + p_\pm(x, t), \\
 P_\pm(t) = -4\partial_x^3 + 6p_\pm(x, t)\partial_x + 3\partial_x p_\pm(x, t).
\]
(2.5)

Then the time dependent Baker–Akhiezer functions \(\hat{\psi}_\pm(\lambda, x, t)\) are the unique solutions of the Lax system \([11, 15]\)
\[
 L_\pm(t)\hat{\psi}_\pm = \lambda \hat{\psi}_\pm, \\
 \frac{\partial \hat{\psi}_\pm}{\partial t} = P_\pm(t)\hat{\psi}_\pm,
\]
(2.6)
which satisfy \(\hat{\psi}_\pm(\lambda, \cdot, t) \in L^2(0, \pm\infty)\) and are normalized according to \(\hat{\psi}_\pm(\lambda, 0, 0) = 1\). We will denote by \(\check{\psi}_\pm(\lambda, \cdot, t)\) the other branch which satisfies \(\check{\psi}_\pm(\lambda, \cdot, t) \in L^2(0, \mp\infty)\).

Similarly, for a solution \(q(x, t)\) of the KdV equation as in Theorem 2.1
define the Lax operators \(L(t)\) and \(P(t)\) as in (2.5) but with \(q(x, t)\) in place of \(p_\pm(x, t)\).

Lemma 2.2. Let \(q(x, t)\) be a solution of the KdV equation as in Theorem 2.1.
Then there exist unique solutions of the Lax system
\[
 L(t)\hat{\phi}_\pm = \lambda \hat{\phi}_\pm, \\
 \frac{\partial \hat{\phi}_\pm}{\partial t} = P(t)\hat{\phi}_\pm,
\]
(2.7)
which satisfy \(\hat{\phi}_\pm(\lambda, \cdot, t) \in L^2(0, \pm\infty)\) and are normalized according to
\[
 \hat{\phi}_\pm(\lambda, x, t) = \hat{\phi}_\pm(\lambda, x, t)(1 + o(1)) \quad \text{as} \quad x \to \infty.
\]
(2.8)
Moreover, we have
\[
 \hat{\phi}_\pm(\lambda, x, t) > 0 \quad \text{for} \quad \lambda \leq \inf \sigma(L(t)),
\]
(2.9)
where \(\sigma(L(t)) = \sigma(L(0))\) denotes the spectrum of the operator \(L(t)\) in \(L^2(\mathbb{R})\).

Proof. The first part follows from [7, Lemma 5.1]. To see recall that the Weyl solutions of \(L(t)\phi = \lambda \phi\) have no zeros for \(\lambda < \inf \sigma(L(t))\) and thus \(\hat{\phi}_\pm(\lambda, x, t) > 0\) for \(\lambda < \inf \sigma(L(t))\) since the same is true for \(\hat{\psi}_\pm(\lambda, x, t)\). Moreover, by continuity we obtain \(\hat{\phi}_\pm(\lambda, x, t) \geq 0\) for \(\lambda \leq \inf \sigma(L(t))\) and since (nonzero) solutions of a second order equation can only have first order zeros, we obtain .

The solutions \(\hat{\phi}_\pm(\lambda, x, t)\) can also be represented with the help of the transformation operators as
\[
 \hat{\phi}_\pm(\lambda, x, t) = \hat{\psi}_\pm(\lambda, x, t) \pm \int_x^{\pm\infty} K_\pm(x, y, t)\hat{\psi}_\pm(\lambda, y, t)dy,
\]
(2.10)
where \(K_\pm(x, y, t)\) are real-valued functions that satisfy
\[
 K_\pm(x, x, t) = \pm \frac{1}{2} \int_x^{\pm\infty} (q(y, t) - p_\pm(y, t))dy.
\]
(2.11)
Moreover, as a consequence of \((A.15)\), the following estimate is valid
\[
\left| \frac{\partial^{n+l}}{\partial x^n \partial y^l} K_\pm(x, y, t) \right| \leq C_\pm(x, t) \left( Q_\pm(x + y, t) \right) + \sum_{j=0}^{n+l-1} \left| \frac{\partial^j}{\partial x^j} \left( q\left(\frac{x + y}{2}, t\right) - p_\pm\left(\frac{x + y}{2}, t\right) \right) \right|, 
\]
for \(\pm y > \pm x\), where \(C_\pm(x, t) = C_{n,l,\pm}(x, t)\) are continuous positive functions decaying as \(x \to \pm \infty\) and
\[
Q_\pm(x, t) := \pm \int_0^{\pm \infty} \left| q(y, t) - p_\pm(y, t) \right| dy.
\]
Finally we recall, that for \(\lambda \leq \inf \sigma(L(t))\) the equation \(L(t)\phi = \lambda\phi\) has two minimal positive (also known as principal or recessive) solutions which are uniquely determined (up to a multiple) by the requirement
\[
\pm \int_0^{\pm \infty} \frac{dx}{\phi_\pm(\lambda, x)^2} = \infty.
\]
For \(\lambda = \inf \sigma(L(t))\) the two minimal positive solutions could be linearly dependent and the \(L(t) - \lambda\) is called critical in this case (and subcritical otherwise). And positive solution can be written as a linear combination of the two minimal positive solutions and in the critical case there is only one positive solution up to multiples. We refer to (e.g.) \([1,2]\) for further details.

In particular, Lemma \([4,2]\) implies that for \(\lambda \leq \inf \sigma(L(t))\) the solutions \(\hat{\phi}_\pm(\lambda, x, t)\) are the two minimal positive solutions of \(L(t)\phi = \lambda\phi\) and thus any positive solution of this equation is a multiple of
\[
\hat{\phi}_\sigma(\lambda, x, t) = \frac{1 + \sigma}{2} \hat{\phi}_+(\lambda, x, t) + \frac{1 - \sigma}{2} \hat{\phi}_-(\lambda, x, t), \quad \sigma \in [-1, 1].
\]
Finally, we also recall the following uniqueness result.

**Theorem 2.3** \((3)\). Let \(p_\pm(x, t)\) be two real-valued, quasi-periodic, finite-gap solutions of the KdV equation corresponding to arbitrary quasi-periodic, finite-gap initial data \(p_\pm(x, 0) = p_\pm(x, 0)\). Suppose \(q(x, t)\) is a solution of the KdV Cauchy problem satisfying
\[
\pm \int_0^{\pm \infty} \left( |q(x, t) - p_\pm(x, t)| + \left| \frac{\partial}{\partial t} (q(x, t) - p_\pm(x, t)) \right| \right) (1 + x^2) dx < \infty,
\]
then \(q(x, t)\) is unique within this class of solutions.

### 3. The Miura Transformation

Our key ingredient will be the Miura transform \([1,2]\) and its inversion (see also \([7, 8, 9, 10]\) and the references therein). Let \(v(x, t)\) be a (classical) solution of the mKdV equation
\[
v_t(x, t) = -v_{xxx}(x, t) + 6v(x, t)^2v_x(x, t).
\]
More precisely we will assume that
\[
v_t, v_x, \ldots, v_{xxxx}, \quad \text{and} \quad v_{xt}
\]
exist and are continuous.
Moreover, either $0$ is the lowest band edge of $\sigma$ which proves the uniqueness part of Theorem 1.1.

The KdV equation. Moreover, it is well-known (see also [7], [8]), that all quasi-periodic, finite-gap solutions of the KdV equation (see also [7], [8]), that all quasi-periodic, finite-gap solutions of the KdV equation. Moreover, it is well-known (see also [7], [8]), that all quasi-periodic, finite-gap solutions of the mKdV equation and their relation to the quasi-periodic, finite-gap solutions of the mKdV equation. Moreover, it is well-known (see also [7], [8]), that all quasi-periodic, finite-gap solutions of the mKdV equation and their relation to the quasi-periodic, finite-gap solutions of the mKdV equation.

(4.2) $\psi(\alpha, 0, x, t) = \psi(0, x, t)$ which is a solution of the mKdV equation. Moreover, it is well-known (see also [7], [8]), that all quasi-periodic, finite-gap solutions of the mKdV equation and their relation to the quasi-periodic, finite-gap solutions of the mKdV equation.

In this section we want to briefly look at quasi-periodic, finite-gap solutions of the mKdV equation and their relation to the quasi-periodic, finite-gap solutions of the KdV equation (see also [7], [8]).

Let $u_\pm(x, t)$ be quasi-periodic, finite-gap solutions of the mKdV equation. Fix a number $j = 0$ or $j = 1$ for the Miura transformation. Then

(4.1) $p_{\pm,j}(x, t) = u_{\pm}(x, t)^2 + (-1)^j u_{\pm,x}(x, t)$

are quasi-periodic, finite-gap solutions of the KdV equation. Moreover, it is well-known (see, for example, [9]), that $\inf \sigma(L_{\pm,j}(t)) \geq 0$, where $L_{\pm,j}(t)$ is defined by (4.1). Therefore, a positive solution $\psi_{\pm,j}(0, x, t)$ and $\tilde{\psi}_{\pm,j}(0, x, t)$ corresponding to $p_{\pm,j}(x, t)$, that is,

(4.2) $\psi_{\pm,j}(x, t) = (1 - \alpha_{\pm,j}(t)) \psi_{\pm,j}(0, x, t) + \alpha_{\pm,j}(t) \tilde{\psi}_{\pm,j}(0, x, t)$.

Moreover, either 0 is the lowest band edge of $\sigma(L_{\pm,j})$, in which case $\psi_{\pm,j}(0, x, t) = \tilde{\psi}_{\pm,j}(0, x, t)$ and $\alpha_{\pm,j}(t)$ drops out, or 0 is below the spectrum $\sigma(L_{\pm,j})$, in which case we must have $\alpha_{\pm,j}(t) = 0$ or $\alpha_{\pm,j}(t) = 1$ (since otherwise 0 would be an eigenvalue of operator, corresponding to the potential $u_{\pm}(x, t)^2 - (-1)^j u_{\pm,x}(x, t)$).

Since the converse is also true, all quasi-periodic, finite-gap solutions of the mKdV equation arise in this way from quasi-periodic, finite-gap solutions of the KdV equation.

Moreover, by virtue of Theorem [4], we can already show the following result which proves the uniqueness part of Theorem [1].
Theorem 4.1. Let \( u_\pm(x, t) \) be quasi-periodic, finite-gap solutions of the mKdV equation and \( v(x, t) \) a solution of the Cauchy problem for the mKdV equation as above such that \( q_0(x, t) \) (or \( q_2(x, t) \)) satisfies \( q_0 \neq 0 \). Then \( v(x, t) \) is unique within this class.

Proof. Let \( v(x, t) \) and \( \tilde{v}(x, t) \) be two solutions corresponding to the same initial condition \( v(x, 0) = \tilde{v}(x, 0) = v(x) \). Then, by uniqueness for KdV, \( q_0(x, t) = \tilde{v}(x, t)^2 + \tilde{v}_x(x, t) \). Moreover, \( \phi_0(x, t) \) and \( \tilde{\phi}_0(x, t) \) defined by \( \phi_0 \) both solve \( \phi_0 \) and coincide for \( t = 0 \). Hence they are equal by \( \phi_0 \) Lem. 2.4] and so are \( v(x, t) \) and \( \tilde{v}(x, t) \).

5. Proof of the main theorem

Let \( u_\pm(x, t) \) be two quasi-periodic, finite-gap solutions of the mKdV equation and suppose \( v(x, t) \) is a (classical) solution of the mKdV equation. Then

\[
q_j(x, t) = v(x, t)^2 + (-1)^j v_x(x, t)
\]

is a classical solution of the KdV equation and \( p_{\pm,j}(x, t) \), defined by \( \psi_\pm \) are quasi-periodic, finite-gap solutions of the KdV equation. Choose numbers \( j_\pm \in \{0,1\} \) for the Miura transform such that (compare \( \psi_\pm \))

\[
\psi_\pm(x, t) = \psi_{\pm,j_\pm}(0, x, t)
\]

(5.2)

\[
= \exp \left( (-1)^{j_\pm} \int_0^x u_\pm(y, t)dy + (-1)^{j_\pm} \int_0^t (2u_\pm(0, s)^3 - u_{\pm,xx}(0, s)ds \right)
\]

and thus

\[
\frac{\partial}{\partial x} \psi_\pm(x, t) = (-1)^{j_\pm} u_\pm(x, t) \psi_\pm(x, t),
\]

which is possible by the considerations from the last section.

Lemma 5.1. Let \( u_+(x, t) \) and \( v(x, t) \) be as introduced above such that

\[
\int_0^\infty \left( |v(x, t) - u_+(x, t)| + |v_t(x, t) - u_+(x, t)| \right) dx < \infty.
\]

Then

\[
\phi_+(x, t) := \psi_+(x, t) \exp \left( (-1)^{j_+}+1 \int_x^\infty (v(y, t) - u_+(y, t))dy \right)
\]

is a minimal positive solutions of \( (-\partial_x^2 + q_j(x, t))\phi = 0 \). Moreover,

\[
\frac{\partial}{\partial x} \phi_+(x, t) = (-1)^{j_+} v(x, t) \phi_+(x, t),
\]

(5.6)

\[
\frac{\partial}{\partial t} \phi_+(x, t) = ((-1)^{j_+} 2q_j(x, t)v(x, t) - q_{j_+,x}(x, t))\phi_+(x, t).
\]

(5.7)

Proof. First of all note that \( \psi_+(x, t) = \psi_{+,j_+}(0, x, t) \) is the minimal positive solutions of \( L_{+,j_+} \psi = 0 \) and by our choice of \( j_+ \) we have \( \psi_+ \) from which \( \phi_+ \) is immediate. Similarly, \( \phi_+ \) follows after a straightforward computation.

Now we are ready to proof our main theorem: We begin with the initial condition \( v(x) \) and define

\[
q(x) = v(x)^2 + (-1)^{j_+} v_x(x).
\]

(5.8)
By our assumptions (1.2) we infer that \( q(x) \) satisfies (4.4). Hence, by Theorem 1.1 there is a corresponding solution \( q(x,t) \) of the KdV equation and by Lemma 2.2 associated solution \( \hat{\varphi}_+(\lambda, x, t) := \hat{\varphi}_{+,j_+}(\lambda, x, t) \).

Recall (5.2) and define \( \varphi_+(x) \) by

\[
\varphi_+(x) := \psi_+(x, 0) \exp \left( -1)^{j_+ + 1} \int_x^\infty (v(y) - u_+(y, 0)) dy \right)
\]

which, by Lemma 5.1 is a minimal positive solution of \( L(0) \). Moreover, since

\[
\varphi_+(x) = \psi_+(x, 0)(1 + o(1)) \quad \text{as} \quad x \to \infty
\]

we conclude

\[
\varphi_+(x) = \hat{\varphi}_{+,j_+}(0, x, 0).
\]

Consequently

\[
v(x, t) = (-1)^{j_+} \frac{\partial}{\partial x} \log \hat{\varphi}_{+,j_+}(0, x, t)
\]

is a solution of the mKdV equation which satisfies the initial condition

\[
v(x, 0) = (-1)^{j_+} \frac{\partial}{\partial x} \log \hat{\varphi}_{+,j_+}(0, x, 0) = (-1)^{j_+} \frac{\partial}{\partial x} \log \varphi_+(x) = v(x)
\]

as required.

To see (1.3) set \( \varphi_+(x, t) := \hat{\varphi}_{+,j_+}(0, x, t) \) and observe that from (5.11)

\[
\frac{\varphi_+(x, t)}{\psi_+(x, t)} = 1 + \int_x^\infty K_+(x, y, t) \frac{\psi_+(y, t)}{\psi_+(x, t)} \psi_+(x, t) dy,
\]

and thus

\[
\frac{1}{2} \leq \frac{\varphi_+(x, t)}{\psi_+(x, t)} \leq 2
\]

for \( x > x_0(t) \). Moreover, differentiating (4.4) we obtain

\[
v(x, t) - u_+(x, t) = \frac{\partial}{\partial x} \log \frac{\varphi_+(x, t)}{\psi_+(x, t)}
\]

\[
= \frac{\psi_+(x, t)}{\varphi_+(x, t)} \left( -K_+(x, x, t) + \int_x^\infty \left( K_+, K(x, y, t) - u_+(x, t) K(x, y, t) \right) \frac{\psi_+(y, t)}{\psi_+(x, t)} dy \right)
\]

which implies

\[
|v(x, t) - u_+(x, t)| \leq C_+(t) \left( Q_+(2x, t) + \int_x^\infty Q_+(x + y, t) dy \right).
\]

The higher derivatives then follow in a similar fashion using

\[
\frac{\partial}{\partial x} \left( v(x, t) - u_+(x, t) \right) = q(x, t) - p_+(x, t) - \left( \frac{\varphi_-, \varphi_+(x, t)}{\varphi_+(x, t)} \right)^2 + \left( \frac{\psi_-, \psi_+(x, t)}{\psi_+(x, t)} \right)^2.
\]

This shows (1.3) for the plus sign. To see it for the minus sign, repeat the argument with \( j_- \).
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