The B-model on the A-model NS5-brane

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Abstract: We formulate the dynamics for the NS5-brane of the A-model, via the ‘maximal form’ method, which couples all background fields to the world-volume. This procedure provides the extra one- and five-forms fields of the extended Hitchin model. The generalised B-model emerges as the world-volume theory of this brane. The starting point of the construction is an embedding of the brane in a superspace geometry with 16 fermionic directions. The correspondence with Hitchin’s formulation in terms of pure spinors is explained.
1. Introduction

It has been conjectured that the topological string theories in six dimensions [1, 2, 3], the A-model and the B-model, are dual to each other by “S-duality”, where the two models are formulated on the same manifold. The duality demands the presence of various branes in the models [4, 5]. In a similar spirit, it has been demonstrated that the topological B-model is related to an extended Hitchin model [6], i.e., a model describing deformations of the generalised complex structure [7]. The extended Hitchin models contain field strengths that are even forms for the A-model and odd forms for the B-model, and in addition obey certain non-linear relations that most elegantly are expressed as pure spinor conditions. D-branes in topological string theory have been considered e.g. in refs. [8, 9, 10].

It has been argued that one way of obtaining the duality is to consider a space-filling brane in the A-model, an NS5-brane (argued to exist in [4]). The theory on this brane should provide the dynamics of the B-model. In ref. [11] (see also [12]) the topological M5-brane was directly dimensionally reduced to the A-model NS5-brane, and it was shown that the equation of motion for the (non-linearly) self-dual 2-form on the brane reduced to the Kodaira–Spencer equation, governing the deformations of the complex structure. It has also been demonstrated that the A-model is described by a space-filling brane in the B-model [13, 14].

Our previous results [15, 11] have been formulated in a framework where the topological models (topological string theories or M-theory [16, 17]) are embedded directly in space-time.
supersymmetric versions of string or M-theory, using a 6- or 7-dimensional supergravity with 16 real supercharges on manifolds with \( SU(3) \) or \( G_2 \) holonomy\(^1\). This formulation has the obvious advantage that the correspondence between supersymmetric branes in the topological theories and brane instantons in the full string or M-theory becomes direct.

The purpose of the present paper is to demonstrate how the string/5-brane duality naturally yields the extended Hitchin models and generalised complex geometry. Our earlier space-time supersymmetric methods are generalised to a setting where the full coupling of the NS5-brane to the background fields are included. This is done using the methods of refs. [19, 20, 21, 22, 23], where a world-volume gauge potential is introduced for every background field strength. In such a formalism, successfully applied to a variety of branes in string theory and M-theory, the number of world-volume degrees of freedom generically is too high, and has to be reduced by some self-duality relation consistent with the background couplings. Thanks to the correspondence between self-duality and pure spinors in \( d = 6 \), this approach will turn out to provide the correct fields to describe the extended Hitchin model. In addition to these fields, the NS5-brane theory will itself contain branes, namely the boundaries of A-model D-branes ending on the NS5-brane. These will provide the D-branes of the B-model.

The earlier description of the 5-brane in topological M-theory, and of the A-model NS5-brane, involves a non-linearly self-dual 2-form on the brane. This corresponds to the field content in the “unextended” Hitchin model [24], and was shown to produce Kodaira–Spencer theory for the deformation of the complex structure. In order to match the extended Hitchin B-model [6], the world-volume theory must include a scalar and a 4-form, and the entire field strength, which is composed by a 1-form, a 3-form and a 5-form, has to obey some relation that on one hand reduces to the above self-duality when the 1-form and 5-form vanish, on the other hand is equivalent to a pure spinor condition. Only in \( d = 6 \) does the number of components in a self-dual odd (or even) form match the number of components of a pure spinor of \( SO(d,d) \). The correspondence between the two parametrisations of the fields will be demonstrated.

Some words on what is done in this paper, and what is not. We introduce superspace field strengths and Bianchi identities in the \( d = 6, N = (1,1) \) superspace relevant for the A-model. The Bianchi identities essentially follow from the basic Fierz identities, and imply (with some extra assumptions) the form of the modified Bianchi identities for the world-volume fields. A generic Ansatz is made for the action of the NS5-brane, and its exact form is determined. The calculations are quite involved, but are simplified to some extent by the observation that we can manifest an \( so(6) \oplus so(6) \) subalgebra of \( so(6,6) \). We find the explicit form of the action and of the self-duality relations, and demonstrate that the action is \( \kappa \)-symmetric (\( \kappa \)-symmetry and self-duality, as usual, go hand in hand—it is the presence of the former, seen as a chirality of the physical fermions, that allows for the decoupling and consistent chiral truncation in

\( ^1 \)The correspondence of this formulation with the relevant Hitchin models will be the subject of a forthcoming publication [18].
the bosonic sector expressed by the latter). The calculations rely on an intricate and very non-trivial interplay between the background supergeometry, non-linear self-duality of world-volume fields and κ-symmetry. What remains to be done, in order for the brane to describe a topological model, is to identify the BRST operator $Q$ as a singlet supersymmetry when holonomy is reduced to $SU(3)$, and consider the stability relation imposed on the brane by demanding that it is $Q$-invariant. This is the procedure performed in ref. [11] for the reduction of the topological M5-brane. We are confident that it will pose no problems, and provide the correct generalisation of the Kodaira–Spencer equation obtained there. Considering the simple form of projection matrix, we expect the calculation to be straightforward. The issue will be addressed in a forthcoming publication.

A note on index conventions: We use $a, b, \ldots$ for background Lorentz indices, and $i, j, \ldots$ for curved world-volume indices. Fermions transform in $(4 \oplus \bar{4}, 2)$ of $so(6) \oplus sl(2)$. The $sl(2)$ doublet index is denoted $I, J, \ldots$ and the 4 of $so(6)$ by a lower $A, B, \ldots$. A collective 8-dimensional Dirac spinor index for $4 \oplus \bar{4}$ is written $\alpha, \beta, \ldots$. When spinor indices for $so(6) \oplus so(6)$ are introduced, we use lower $A, B, \ldots$ for spinors in 4 under the first $so(6)$ and lower $A', B', \ldots$ for the second.

2. World-volume and background fields

Utilising the $d = 7$ Clifford algebra from [15], reduced to $d = 6$, we may write the target space Bianchi identities as Fierz identities and derive the following $\text{dim} = 0$ field strengths:

$$
\begin{cases}
F_{(2)} : & F_{\alpha I, \beta J} = 2\varepsilon_{IJ}(\gamma^7)_{\alpha\beta}, \\
F_{(4)} : & F_{ab, \alpha I, \beta J} = -2\varepsilon_{IJ}(\gamma_{ab})_{\alpha\beta}, \\
F_{(6)} : & F_{abcd, \alpha I, \beta J} = -2\varepsilon_{IJ}(\gamma_{abcd}^7)_{\alpha\beta}, \\
H_{(3)} : & H_{a, \alpha I, \beta J} = 2\varepsilon_{IJ}(\gamma_a^7)_{\alpha\beta}, \\
H_{(7)} : & H_{a b c d e, \alpha I, \beta J} = 2\varepsilon_{IJ}(\gamma_{a b c d e})_{\alpha\beta}, \\
T^{a}_{\alpha I, \beta J} : & = 2\varepsilon_{IJ}(\gamma^a)_{\alpha\beta}.
\end{cases}
$$

These field strengths obey the modified Bianchi identities

$$
\begin{align*}
&dF_{(2)} = 0, \\
&dF_{(4)} + F_{(2)} \wedge H_{(3)} = 0, \\
&dF_{(6)} + F_{(4)} \wedge H_{(3)} = 0,
\end{align*}
$$

$$
\begin{align*}
&dH_{(3)} = 0, \\
&dH_{(7)} - F_{(2)} \wedge F_{(6)} + \frac{1}{2} F_{(4)} \wedge F_{(4)} = 0.
\end{align*}
$$

The ‘maximal form’ formalism relies on coupling background field strengths to the world volume of the object one wishes to model. In the current model we will do so for all background fields apart from the NS-NS 2-form potential $B_{(2)}$ since such a coupling would signal the possibility of ending strings on NS-branes; a non-existent scenario. The couplings are all
of the basic form \( f = da_{\text{worldvolume}} - A_{\text{background}} \) with additional terms introduced by the modified nature of the background Bianchi identities. These, together with the exclusion of \( B(2) \), completely fix the world-volume Bianchi identities

\[
\begin{align*}
\text{df}(1) &= -F(2), \\
\text{df}(3) &= -F(4) - f(1) \wedge H(3), \\
\text{df}(5) &= -F(6) - f(3) \wedge H(3),
\end{align*}
\]
(2.4)

\[
\begin{align*}
\text{dh}(2) &= -H(3), \\
\text{dh}(6) &= -H(7) - \frac{1}{2} f(1) \wedge F(6) + \frac{1}{2} f(3) \wedge F(4) - \frac{1}{2} f(5) \wedge F(2),
\end{align*}
\]
(2.5)

where the Bianchi identities for the RR-fields are neatly summarised by

\[
df + f \wedge H(3) + F = 0.
\]
(2.6)

Stipulating further that these field strengths should be gauge invariant (again with the exclusion of the background 2-form potential from fields coupled to the world-volume) fixes the field strengths in terms of field potentials:

\[
\begin{align*}
\text{f} &= da - A + a \wedge H(3), \\
\text{h}(2) &= db(1) - B(2), \\
\text{h}(6) &= db(5) - \frac{1}{2} a(0) \wedge F(6) + \frac{1}{2} a(2) \wedge F(4) - \frac{1}{2} a(4) \wedge F(2),
\end{align*}
\]
(2.7)

where we write

\[
\begin{align*}
a &= a(0) + a(2) + a(4), \\
f &= f(1) + f(3) + f(5)
\end{align*}
\]
(2.8)
as formal sums. Further on we will write this sum as a structure in components of the differential forms and endow it with a symmetry larger than \( so(6) \).

3. Action and self-duality

Maximal form models all share the same basic form of action:

\[
S = \int d^6 \xi \sqrt{g} \lambda \left( 1 + \Phi(f) + (\ast h(6))^2 \right),
\]
(3.1)

where \( g \) is the determinant of the world-volume metric, \( \lambda \) is a Lagrange multiplier and \( \Phi \) is an, as of yet unknown, polynomial in all the world-volume field strengths apart from the maximal form \( h(6) \). This action gives us the implicit equations of motion

\[
\begin{align*}
\lambda : 1 + \Phi(f) + (\ast h(6))^2 &= 0, \\
\Rightarrow \ast h(6) &= -i \sqrt{1 + \Phi} = -i N, \\
b_5 : d(\lambda \ast h(6)) &= 0, \\
a : -d(\lambda \ast q) + \lambda \ast q \wedge H(3) - \lambda(\ast h(6)) \pi F &= 0,
\end{align*}
\]
(3.2)
(3.3)
(3.4)
where \( q = \frac{\partial \Phi}{\partial f} \) and the minus sign has been chosen for the square root in (3.2). The operator \( \pi \) acts on some structure of differential forms, changing their sign based on form degree, and is defined by

\[
\pi(\omega(0) + \omega(2) + \omega(4) + \omega(6)) = \omega(0) - \omega(2) + \omega(4) - \omega(6),
\]

(3.5)

\[
\pi(\omega(1) + \omega(3) + \omega(5)) = -\omega(1) + \omega(3) - \omega(5).
\]

(3.6)

This action carries a large number of excess degrees of freedom as compared to the A-model NS5-brane itself. These are reduced by demanding an implicit non-linear self-duality relation, between the fields, that equates the equation(s) of motion for \( a \) with the Bianchi identity of \( f \). A straightforward comparison between (3.4) and (2.6), together with the use of (3.2) and (3.3), gives the following implicit self-duality:

\[
\star q = \pi(\star h(6)) f \implies i\pi \star q = N f,
\]

(3.7)

which in more explicit terms (for later use in the \( \kappa \)-variation) becomes

\[
q^i = -iN(\star f(5))^i,
\]

(3.8)

\[
q^{ijk} = iN(\star f(3))^{ijk},
\]

(3.9)

\[
q^{ijklmn} = -iN(\star f(1))^{ijklmn}.
\]

(3.10)

The generalised B-model of ref. [6] carries an \( so(6,6) \) symmetry, which in this model is broken by the twisted duality operator \( i\pi \star \) to \( so(6) \oplus so(6) \). We would like to realise this symmetry, using the introduced fields, on the NS5-brane world-volume. Now, \( so(6,6) \rightarrow so(6) \oplus so(6) \) implies \( 32 \rightarrow (4,4) \oplus (4,4) \) and \( 32' \rightarrow (4,\bar{4}) \oplus (\bar{4},4) \), and further breaking to the diagonal \( so(6) \) turns 32 and \( 32' \) into odd and even (\( \pi \)-twisted) self-dual forms. Starting with the \( so(6) \)-fields given in (2.7) we may use a parametrisation of the \( \gamma \)-matrices in terms of \( \sigma \)-matrices (see Appendix A for further details) to construct the fields

\[
\begin{aligned}
 f_{AB} &= if_i \sigma^i_{AB} + \frac{1}{3} f_{ijk} \sigma^{ijk}_{AB} - if_{ijkmn} \sigma^{ijklmn}_{AB} \\
 &= 2k_i \sigma^i_{AB} + \frac{1}{3} k_{ijk} \sigma^{ijk}_{AB}, \\
 \tilde{f}_{AB} &= -if_i \bar{\sigma}^i_{AB} + \frac{1}{3} f_{ijk} \bar{\sigma}^{ijk}_{AB} + if_{ijkmn} \bar{\sigma}^{ijklmn}_{AB} \\
 &= 2k_i \bar{\sigma}^i_{AB} + \frac{1}{3} \bar{k}_{ijk} \bar{\sigma}^{ijk}_{AB},
\end{aligned}
\]

(3.11)

where we have defined

\[
\begin{aligned}
 k_i &= -\frac{1}{2}(f_i + i(\star f)_i) \\
 \bar{k}_i &= \frac{i}{2}(f_i - i(\star f)_i), \\
 f_{ijk} &= (k_{ijk} + \bar{k}_{ijk})
\end{aligned}
\]

(3.12)

\( k_{ijk} \) and \( \bar{k}_{ijk} \) being self-dual and anti-selfdual forms respectively (i.e. \( \star k_{ijk} = i k_{ijk} \) etc.). The fields \( f \) and \( \tilde{f} \) are twisted self-dual and anti-selfdual under \( i\pi \star \). The precise way \( f_{(1)} \) and \( f_{(5)} \) should accompany \( f_{(3)} \) to form these \( so(6) \oplus so(6) \)-covariant fields is not obvious; these are the actual combinations that turn out to be demanded by \( \kappa \)-symmetry.
\((4, 4) \oplus (\bar{4}, \bar{4})\) simply by discarding their parametrisation in terms of \(\sigma\)-matrices and writing \(f_{AB}'\) and \(\tilde{f}^{A'B'}\).

Using this new covariance we may proceed to solve the implicit self-duality \((3.7)\) through purely algebraic means. It is clear that the self-duality relations demand a term \(\text{tr}(f \tilde{f})\) in \(\Phi\), after which we assume that the remaining part of \(\Phi\) is \(so(6,6)\)-invariant (it is not \(a\ priori\) obvious that this has to be true, but the actual calculation shows that it is). Furthermore there is a single quartic invariant, and it has to be a sum of the terms \(\det f\), \(\det \tilde{f}\), \(\text{tr}(f \tilde{f} f \tilde{f})\) and \((\text{tr}(f \tilde{f}))^2\). Whereas the generators in \(so(6,6)\), outside of \(so(6) \oplus so(6)\), transform as \((6, 6)\) and act on \(f\), \(\tilde{f}\) as

\[
\delta_M f_{AB}' = M_{AC, B'D'} \tilde{f}^{D'C} = \frac{1}{2} \varepsilon_{B'D'E'F'} M_{AC, E'F'} \tilde{f}^{D'C},
\]

\[
\delta_M \tilde{f}^{A'B} = M^{BC, A'D'} f_{CD'} = \frac{1}{2} \varepsilon_{BCEF} M_{EF, A'D'} f_{CD'}.
\]

A straightforward calculation shows that the quartic invariant is

\[
R(f, \tilde{f}) = \det f + \det \tilde{f} + \frac{1}{2} \text{tr}(f \tilde{f} f \tilde{f}) - \frac{1}{4} (\text{tr}(f \tilde{f}))^2.
\]

The Ansatz, now a linear combination of \(\text{tr}(f \tilde{f})\) and \(R(f, \tilde{f})\), is established to be \(\Phi = -\frac{1}{8} \text{tr}(f \tilde{f}) - \frac{1}{16} R(f, \tilde{f})\), i.e.

\[
\Phi = -\frac{1}{8} \text{tr}(f \tilde{f}) - \frac{1}{8} \left( \det f + \det \tilde{f} + \frac{1}{2} \text{tr}(f \tilde{f} f \tilde{f}) - \frac{1}{4} (\text{tr}(f \tilde{f}))^2 \right).
\]

This implies the non-linear self-duality relations

\[
-N f = -f - \frac{1}{8} \left( (\det \tilde{f}) \tilde{f}^{-1} + f \tilde{f} \tilde{f} - \frac{1}{2} f \text{tr}(f \tilde{f}) \right),
\]

\[
N \tilde{f} = -\tilde{f} - \frac{1}{8} \left( (\det f) f^{-1} + \tilde{f} \tilde{f} \tilde{f} - \frac{1}{2} \tilde{f} \text{tr}(f \tilde{f}) \right).
\]

Tracing the first of these equations with \(\tilde{f}\), the second with \(f\) and adding them together gives \(R(f, \tilde{f}) = -4 \text{tr}(f \tilde{f})\), so that \(N = \sqrt{1 + \Phi} = \sqrt{1 - \frac{1}{16} \text{tr}(f \tilde{f})}\). The equations then give

\[
f \tilde{f} = \frac{1}{4} \text{tr}(f \tilde{f}) = 4(1 - N^2) I,\]

and finally \(\det f = -16(1 - N)(1 + N)^3\) and \(\det \tilde{f} = -16(1 - N)^3(1 + N)\). The two equations \((3.16)\) contain the same information, i.e. are consistent.

Especially noteworthy among these expressions is the product \(f \tilde{f}\) which, by multiplying with one of the matrix inverses results in the explicit non-linear self-duality relation:

\[
\tilde{f} = 4(1 - N^2)f^{-1}.
\]

With this expression for the non-linear self-duality at hand, along with our definitions, we may produce a dictionary, Table 1, for the translation of objects of \(so(6) \oplus so(6)\) and \(so(6)\).
$so(6) \oplus so(6)$ & $so(6)$
$f_{AB}$ & $2k_i \sigma^i_{\text{AB}} + \frac{1}{7}k_{ijk}\sigma^{ijk}_{\text{AB}}$
$f^AB$ & $2k_i \sigma^i_{\text{AB}} + \frac{1}{7}k_{ijk}\sigma^{ijk}_{\text{AB}}$
$\det(f)$ & $(k^2)^2 - 8\sigma^{ij}k_i k_j + \frac{8}{3}\sigma^{ij}f^{ij}$
$\det(\tilde{f})$ & $(k^2)^2 - 8\sigma^{ij}k_i k_j + \frac{8}{3}\sigma^{ij}f^{ij}$
$\tilde{f} = 4(1 - N^2)f^{-1}$ & $\begin{cases} k_i = \frac{1}{1+N^2} \left(k^2 k_i + r_{ij}k^j \right) \\ k_{ijk} = -\frac{1}{(1+N)^2} \left( r_{[i}k_{jk]l} + 6k_{[i}k^d_{jk]l} - k^2k_{ijk} \right) \end{cases}$

| Table 1: Dictionary for translation between $so(6) \oplus so(6)$ and $so(6)$ |

4. $\kappa$-symmetry

The explicit action given by (3.15), and the associated duality relation, must be compatible with $\kappa$-symmetry. In fact $\kappa$-symmetry uniquely determines the form of self-duality on the world-volume and can thus be used to solve the problem which was solved using algebraic means in the previous section (such an approach is found e.g. in ref. [19]). Here we will progress conventionally, beginning with the derivation of $\kappa$-variations for our world-volume fields as dictated by the background fields and gauge invariance. These variations are then inserted into the constraint (1.5-order formalism) originating in the equation of motion for the Lagrange multiplier (i.e., everything within the parenthesis) from the implicit action (3.1). A solution, $\kappa$, is then calculated which requires self-duality and associated relations.

The major difference between this calculation and other proofs of $\kappa$-symmetry for maximal form models is complexity. Given the relatively large number of fields this is expected; less expected is the lack of an apparent avenues for simplification. The $\kappa$-variation does not exhibit the covariance of $f_{AB'}, f^{A'B}$ which arises only at the level of solution. Here, we give only an outline of this lengthy calculation.

We use superspace conventions, meaning derivatives act from the right, in which the $\kappa$-variations of our world-volume fields become:

$$
\begin{align*}
\delta_\kappa f_i &= -i_\kappa F^{(2)}, \\
\delta_\kappa f_{ijk} &= -i_\kappa F^{(4)} - f_{(1)} \wedge i_\kappa H^{(3)}, \\
\delta_\kappa f_{ijklm} &= -i_\kappa F^{(6)} - f_{(3)} \wedge i_\kappa H^{(3)}, \\
\delta_\kappa h_{ijklmn} &= -i_\kappa H^{(7)} - \frac{1}{2}f_{(1)} \wedge i_\kappa F^{(6)} + \frac{1}{2}f_{(3)} \wedge i_\kappa F^{(4)} - \frac{1}{2}f_{(5)} \wedge i_\kappa F^{(2)}, \\
\delta_\kappa g_{ij} &= 4E_{(i,|\alpha I|}(\gamma_{j}))^{aI}_\beta J^\kappa I^\beta J.
\end{align*}
$$

The constraint whose invariance we wish to prove,

$$
\Psi = 1 + \Phi(f) + (\ast h^{(6)})^2,
$$

(4.2)
transforms as:
\[
\delta_\kappa \Psi = q^i \delta_\kappa f_i - \frac{1}{2} f^i q^j \delta_\kappa g_{ij} \\
+ \frac{1}{3!} q^{ijklm} \delta_\kappa f_{ijklm} - \frac{1}{2} \frac{1}{3!} q^i g_{kl} \delta_\kappa g_{ij} \\
+ \frac{1}{6} h^{ijklm} \delta_\kappa h_{ijklm} - \frac{1}{3} h^i h^{ijklm} \delta_\kappa g_{ij},
\] (4.3)
and produces a result on the form
\[
\delta_\kappa \Psi = E_{\gamma,\alpha} (M^\gamma)^{\alpha \beta} \kappa^\beta.
\] (4.4)

The matrix \(M\) is a complex expression of field strengths and \(\sigma\)-matrices (or, as this calculation can be performed in the representation 8 of \(\text{so}(6)\), \(\gamma\)-matrices) whereas \(\kappa\) exhibits the covariance of \(f, \tilde{f}\):
\[
\kappa = \frac{1}{2} \left[ \begin{array}{cc}
\frac{1}{2} A_B & 0 \\
0 & \frac{1}{2} A_B
\end{array} \right] + \frac{1}{N} \left[ \begin{array}{cc}
\frac{1}{2} A_B & \frac{i}{2} \tilde{f}_{A B} \\
\frac{i}{2} f_{A B} & -\frac{1}{2} A_B
\end{array} \right] \zeta.
\] (4.5)

As usual, \(\Gamma^2 = 1\). This is seen by using the self-duality from the previous section in the form
\(f \tilde{f} = \frac{1}{2} (N^2 - 1) \mathbb{1}\), which implies that \(\frac{1}{2} (1 \pm \Gamma)\) are projection matrices. The actual form of \(\Gamma\) was actually guessed using input from the case where the field strength only contains a 3-form \([19, 11]\), together with the observation that it should respect \(\text{so}(6) \oplus \text{so}(6)\). This solution renders \(\delta_\kappa \Psi = 0\), modulo terms that vanish when the self-duality constraints are imposed, and thus the model is \(\kappa\)-symmetric.

Although the projection on \(\kappa\) is manifestly \(\text{so}(6) \oplus \text{so}(6)\)-covariant, neither the action nor the \(\kappa\)-variations of the fields are. Thus, the only local symmetry we can rely on in the calculation is \(\text{so}(6)\). This makes the number of terms to check quite large, and the calculation becomes long and cumbersome. We have not been able to find a more efficient way than to decompose \(M\) and \(\Gamma\) in terms of 6-dimensional \(\gamma\)-matrices, using the explicit parametrisation of \(f\) and \(\tilde{f}\), as well as the less covariant fields occurring in the \(\kappa\)-variation \([13]\), in terms of the vector \(k_i\) and the linearly self-dual \(k_{ijk}\) given in Table 1 together with the projected \(\kappa\) parameter of eq. \((4.5)\), and check all terms.

5. Pure spinors and self-dual odd forms

When we have actions with the complete set of fields coupling to background tensors as above, the world-volume degrees of freedom are reduced by some self-duality condition. This will also be the case on the NS5-brane of the A-model, where it will be natural to describe the generalised complex structure in terms of self-duality.

Six dimensions is special, this is the only case where purity of a spinor can be expressed as (non-linear) self-duality of an even/odd form. The counting is trivial, both a pure spinor and
a self-dual even/odd form has 16 independent components. The rest of the correspondence can be seen as follows.

Let us first examine the case where \( \varphi \) is a 3-form. The purity condition says that \( \iota_v \varphi = 0 = \varphi \wedge \eta \) (the \( \text{Spin}(6, 6) \) \( \gamma \)-matrices are \( \Gamma_i \cdot \varphi = \iota_i \varphi, \ \Gamma^i \cdot \varphi = \varphi \wedge e^i \)), for a 3-dimensional subspace of vectors \( v \) and a three-dimensional subspace of 1-forms \( \eta \) fulfilling \( \iota_v \eta = 0 \), i.e., \( \Gamma(V) \cdot \varphi = 0 \) for an isotropic subspace of \( V \)'s. Given a linearly self-dual 3-form \( h \) with \( i \star h = h \), and \( r_{ij} = \frac{1}{2} h_{ijkl} h_{jkl} \), we can construct projection matrices \( P_{\pm} = \frac{1}{16} (\pm 16 + N) \), where \( \pm \rho \) are the roots to \( \rho^2 = \frac{1}{6} \text{tr} r^2 \), as long as the quartic invariant \( \text{tr} r^2 \) is non-zero. We can then write \( \varphi = P_+ h \) (as usual, it suffices to act on one index, since \( r_{i[j} h_{k]l} = r_{i[j} h_{k]l} \)). This 3-form \( \varphi \) is annihilated by \( \iota_v \) and by \( \cdot \wedge \eta \) for \( v = P_- w, \ \eta = P_+ \nu \).

We can think of \( P_{\pm} \) as projections on holomorphic and anti-holomorphic components, adapted to a complex structure (not yet generalised) defined by \( h \). This makes the rest, the inclusion of a 1-form and a 5-form, somewhat less technical. Denote the 3-form above by \( \Omega \), and think about it as a holomorphic 3-form. The deformation from \( \varphi = \Omega \) to a \( \varphi \) containing a 1-form and a 5-form can be achieved by a transformation in \( \text{Spin}(6, 6) \times \mathbb{C} \). It is however easier to find \( \varphi \) by trial and error. It turns out that

\[
\varphi = \xi + \Omega + \iota_\xi \bar{\Omega} \wedge \xi + \star \bar{\zeta},
\]

where \( \xi \) is a holomorphic 1-form and \( \bar{\zeta} \) an anti-holomorphic vector density. This pure spinor only has form degrees \( (1,0), (3,0), (1,2) \) and \( (3,2) \) with respect to the complex structure defined by \( \Omega \). It is annihilated by an isotropic subspace, \( \iota_v \varphi + \varphi \wedge \eta = 0 \), with

\[
v = \bar{w} - \star (\bar{\Omega} \wedge \xi \wedge \nu),
\]

\[
\eta = \nu - \iota_\bar{w} \iota_\xi \bar{\Omega},
\]

where \( \bar{w} \) is an anti-holomorphic vector and \( \nu \) a holomorphic 1-form density. We have defined duality (without raising or lowering indices) with \( \varepsilon \), and \( \Omega_{abc} = \varepsilon_{abc} \).

The solution can be parametrised in terms of a linearly (twisted) self-dual form \( h \), which up to rescaling is the component \( f \) of the previous sections. One convenient choice is

\[
f = 4(1 + N) h, \quad (5.4)
\]

\[
\tilde{f} = -16(1 + N) R(h, 0) h^{-1}, \quad (5.5)
\]

with \( R(h, 0) = \text{det} h = -\frac{1}{16} \frac{1 - N}{1 + N} \). Given a linearly self-dual \( h \), one can construct a pure spinor \((\varphi, \tilde{\varphi})\). Taking

\[
\varphi = c(N) h, \quad (5.6)
\]

\[
\tilde{\varphi} = \pm c(N) \sqrt{R(h, 0)} h^{-1}, \quad (5.7)
\]

for some function \( c \) defines a pure odd form \((\varphi, \tilde{\varphi})\) with \( R(\varphi, \tilde{\varphi}) = 0 \). Note that the anti-selfdual part of the pure spinor scales as \( h \), while the anti-selfdual part of the field strength above starts with a term scaling as \( h^3 \). The relations above can of course be used to obtain a direct relation between the field strength and the pure spinor.
A. $\sigma$-matrix algebra

In lifting so(6)-covariant expressions to so(6) $\oplus$ so(6) we have used a parametrisation of the Clifford algebra

$$\{\gamma^{a}, \gamma^{b}\} = -2\delta^{ab},$$  \hspace{1cm} (A.1)

in terms of $\sigma$-matrices,

$$\left(\gamma^{a}\right)_{\alpha}^{\beta} = \begin{bmatrix} 0 & \sigma^{aAB} \\ \sigma^{aAB} & 0 \end{bmatrix},$$  \hspace{1cm} (A.2)

that reduces this algebra to:

$$\sigma^{(a} \sigma^{b)} = -2\delta^{ab}.$$  \hspace{1cm} (A.3)

We also define $\gamma^{7}$ which transforms symmetric matrices to anti-symmetric ones in our calculations,

$$\left(\gamma^{7}\right)_{\alpha}^{\beta} = \frac{1}{6!} \varepsilon_{abcdef} \left(\gamma^{abcdef}\right)_{\alpha}^{\beta} = -\left(\gamma^{1} \gamma^{2} \gamma^{3} \gamma^{4} \gamma^{5} \gamma^{6}\right)_{\alpha}^{\beta} = \begin{bmatrix} i\frac{A_{B}}{2} & 0 \\ 0 & -i\frac{B_{A}}{2} \end{bmatrix}.$$  \hspace{1cm} (A.4)

These matrices are explicitly defined by

$$\left\{ \begin{array}{ll} \sigma^{1,2,3} = \frac{1}{\sqrt{2}} (A_{x} + B_{x}) \; , \; x = 1, 2, 3 \\ \sigma^{4,5,6} = \frac{1}{\sqrt{2}} (A_{x} - B_{x}) \; , \; x = 1, 2, 3 \end{array} \right.$$  \hspace{1cm} (A.5)

with the following basis:

$$A_{1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad B_{1} = \begin{bmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix}.$$

$$A_{2} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad B_{2} = \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix}.$$  \hspace{1cm} (A.6)

$$A_{3} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \quad B_{3} = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}.$$

The dual, or complex conjugate, matrices are given calculated by:

$$\bar{\sigma}^{aAB} = (\sigma^{*})^{aAB} = \frac{1}{2} \varepsilon^{ABCD} \sigma^{c}_{CD}.$$  \hspace{1cm} (A.7)
The $\sigma$-matrices dualise as:

\[
\begin{align*}
\star\sigma^{(1)} &= i\sigma^{(5)} \\
\star\sigma^{(3)} &= -i\sigma^{(3)} \\
\star\sigma^{(5)} &= i\sigma^{(1)} \\
\star\sigma^{(2)} &= -i\sigma^{(4)} \\
\star\sigma^{(4)} &= i\sigma^{(2)} \\
\star F_{A}^{B} &= i\sigma^{(6)} A^{B}
\end{align*}
\]

(A.8)

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