The minimal log discrepancies on a smooth surface in positive characteristic

Shihoko Ishii

Abstract

This paper shows that Mustaţă-Nakamura’s conjecture holds for pairs consisting of a smooth surface and a multiideal with a real exponent over the base field of positive characteristic. As corollaries, we obtain the ascending chain condition of the minimal log discrepancies and of the log canonical thresholds for those pairs. We also obtain finiteness of the set of the minimal log discrepancies of those pairs for a fixed real exponent.

1 Introduction

Two invariants, the minimal log discrepancies and the log canonical thresholds, for a singularity of a pair consisting of a variety and a multiideal with a real exponent play important roles in birational geometry. For example, the ascending chain condition (ACC, for short) of these invariants under certain conditions would give a significant step in a proof of the Minimal Model Problem (MMP, for short). ACC Conjecture for log canonical thresholds over the base field of characteristic 0 is proved in [2], while not yet over the base field of positive characteristic. On the other hand, ACC Conjecture for minimal log discrepancies is not proved even in characteristic 0.

Mustaţă and Nakamura ([10]) posed a conjecture, say Mustaţă-Nakamura’s conjecture (MN Conjecture, for short) and proved that it implies ACC Conjecture for the minimal log discrepancies for the base field of characteristic 0. Then, Kawakita ([8]) proved the converse also holds for dimension 3 in characteristic 0. In the same paper he proved ACC for dimension 2 in characteristic 0. Aside ACC, MN Conjecture plays important roles on basic properties of singularities (openness of good singularities, stability of good singularities under a deformation, etc., see, for example [3]).

In this paper, we focus on pairs \((A, a^e)\) consisting of a smooth surface \(A\) and a non-zero multiideal \(a^e = a_1^{e_1} \cdots a_s^{e_s}\) (\(a_i\)'s are non-zero coherent ideal sheaves on \(A\)) with an exponent \(e = \{e_1, \ldots, e_s\}\) \((e_i \in \mathbb{R}_{>0})\) on \(A\) defined over an algebraically closed base field \(k\) of arbitrary characteristic.

As our interest is in a smooth surface \(A\), we state the conjectures on smooth varieties, although the primary conjectures are stated under more general settings:

\footnote{The author is partially supported by JSPS 19K03428}
**Conjecture 1.1** (MN Conjecture). Let $A$ be a smooth variety of dimension $N$ defined over an algebraically closed field $k$ and let $0 \in A$ be a closed point. Given a finite subset $e \subset \mathbb{R}_{>0}$, there is a positive integer $\ell_{N,e}$ (depending on $N$ and $e$) such that for every multiideal $a^e$ on $A$ with the exponent $e$, there is a prime divisor $E$ that computes $\text{mld}(0; A, a^e)$ and satisfies $k_E \leq \ell_{N,e}$.

**Conjecture 1.2** (ACC Conjecture for mld). Let $A$, $N$, $e$ and $0$ be as above. For every fixed DCC set $J \subset \mathbb{R}_{>0}$, the set

$$\{ \text{mld}(0; A, a^e) \mid e \subset J, (A, a^e) \text{ is log canonical at } 0 \}$$

satisfies ascending chain condition (ACC).

We know the following relations between the two conjectures in characteristic 0.

**Proposition 1.3** ([10], [8]). Let $A$ be a smooth variety of dimension $N$ defined over an algebraically closed field of characteristic 0 and $0 \in A$ a closed point. For $N = 2$, the both MN Conjecture and ACC conjecture hold.

For general $N$, if the following (i) holds, then (ii) holds:

(i) MN Conjecture in the set of pairs $(A, a^e)$ holds for every finite set $e \subset \mathbb{R}_{>0}$;

(ii) ACC Conjecture in the set of pairs $(A, a^e)$ holds for every DCC set $J$.

When $N = 3$, the converse also holds.

When the base field $k$ is an algebraically closed field of characteristic 0, it is proved in [10] that MN Conjecture holds for the set of the pairs $(A_A^N, a^e)$ for arbitrary $N \geq 1$ with monomial ideals $a_i$, as well as the set of the pairs $(A, a^e)$ for a surface $A$ with an arbitrary non-zero multiideal $a^e$. Therefore, by Proposition 1.3, ACC Conjecture also holds for these classes in characteristic 0.

When the base field $k$ is of positive characteristic, MN Conjecture holds for the set of pairs $(A_A^N, a^e)$ for arbitrary $N \geq 1$ and monomial ideals $a_i$ (see, Lemma 2.3 and [4, Corollary 1.10]). The proof of MN Conjecture for the set of pairs $(A, a^e)$ with surfaces $A$ in characteristic 0 in [10] uses generic smoothness, therefore the proof does not work directly for positive characteristic case. By making use of the result for monomial multiideal case and Kawakita’s result [7], we obtain the main result of this paper:

**Theorem 1.4.** Let $A$ be a smooth surface defined over an algebraically closed field of arbitrary characteristic and $0 \in A$ a closed point. Then, we obtain the following:

(i) MN Conjecture holds in the set of the pairs $(A, a^e)$ for multiideal $a^e$ with an exponent $e \subset \mathbb{R}_{>0}$. I.e., There exists a positive number $\ell_e$ (depending only on $e$) such that for every multiideal $a^e$ with the exponent $e$ there is a prime divisor $E$ over $A$ which computes $\text{mld}(0; A, a^e)$ and satisfies $k_E \leq \ell_e$.

(ii) Moreover, for every pair $(A, a^e)$, there is a monomial multiideal $a^e_\ast$ on $A_A^2$ such that

$$\text{mld}(0; A, a^e) = \text{mld}(0; A_A^2, a^e_\ast) = \text{mld}(0; A_A^2, \tilde{a}^e_\ast),$$

where $\tilde{a}^e_\ast$ is a divisorial multiideal on $A_A^2$. 

We refer to [7] for the details of this result.
where $\tilde{a}^e_i$ is the monomial multiideal on $\mathbb{A}^2_C$ whose ideals $\tilde{a}_{e_i}$'s are generated by the same monomial generators as of $a_{e_i}$'s.

On the way to prove the theorem, we generalize the result in [7] into the positive characteristic case (Lemma 2.2) and as its application we obtain the following:

**Theorem 1.5.** Let $A$ be a smooth surface over an algebraically closed base field $k$ of arbitrary characteristic and $a^e$ be a multiideal with a real exponent $e$ on $A$. Then, every exceptional prime divisor computing the log canonical threshold $\text{lct}(0; A, a^e)$ is obtained by a weighted blow up.

Moreover, for every pair $(A, a^e)$ over $k$ such that $\text{lct}(0; A, a^e)$ is computed by an exceptional divisor, there is a monomial multiideal $a^e_i$ on $\mathbb{A}^2_k$ such that

$$\text{lct}(0; A, a^e) = \text{lct}(0; \mathbb{A}^2_k, a^e_i) = \text{lct}(0; \mathbb{A}^2_C, \tilde{a}^e_i),$$

where $\tilde{a}^e_i$ is as in the theorem above.

In case the base field $k$ is of characteristic 0 and $a$ is a reduced principal ideal, the first statement is a result by Varčenco (for a proof, see [9, Theorem 6.40]).

As the relation between MN Conjecture and ACC Conjecture is not yet proved in positive characteristic case, (i) in Theorem 1.4 alone does not imply ACC Conjecture for concerned pairs. By making use also of (ii) in Theorem 1.4, we obtain the following ACC:

**Corollary 1.6.** Let $A$ be a smooth surface defined over an algebraically closed field of positive characteristic. ACC Conjecture in the set of pairs $(A, a^e)$ holds for every DCC set $J$.

**Corollary 1.7.** Let $A, 0 \in A$ and $J \subset \mathbb{R}_{>0}$ be as above Then, the set

$$\{\text{lct}(0; A, a^e) \mid e \subset J\}$$

satisfies ACC.

This statement is proved for $\mathbb{R}$-Cartier divisors $a_i$ in characteristic $p > 5$ in [1, Theorem 1.10]. Note that in positive characteristic case, the ACC for ideals does not follow from ACC for Cartier divisors. Actually, the reason why for characteristic 0 we can reduce the problem into the problem for Cartier divisors is because the equality

$$\text{lct}(0; A, a^e) = \text{lct}(0; A, (f)^e)$$

holds for a general element $f_i \in a_i$ and $(f)^e = (f_1)^{e_1} \cdots (f_s)^{e_s}$ by virtue of Bertini’s theorem (generic smoothness) which does not hold in positive characteristic even for surfaces.

**Corollary 1.8.** Let $A$ be a smooth surface defined over an algebraically closed field of positive characteristic and $0 \in A$ a closed point. Then, for a fixed finite subset $e \subset \mathbb{R}_{>0}$, the set

$$\{\text{mld}(0; A, a^e) \mid a^e \text{ is a multiideal on } A\}$$

is a finite set and coincides with the following set

$$\{\text{mld}(0; \mathbb{A}^2_C, a^e) \mid a_i \text{ is a monomial ideal on } \mathbb{A}^2_C \text{ for every } i\}.$$
The structure of this paper is as follows: In the second section we give the definitions of two invariants and the proofs of the theorem and the corollaries. As we reduce the problem into that on the pairs \((A_k^2, \mathfrak{a}^e)\), the calculation \(\ell_e\) for a given \(e\) is a kind of combinatorics. We show some example of \(\ell_e\) for some \(e\).

**Acknowledgement.** The author expresses her hearty thanks to Kohsuke Shibata for his insightful comments which improves the paper. She also would like to thank Masayuki Kawakita, Lawrence Ein and Mircea Mustaţă for the useful discussions. A big part of these discussions was done during the author’s stay in MSRI (Program: Birational Geometry and Moduli Theory) and she is grateful for the support of MSRI. The author would like to thank the referee for useful comments to improve the paper.

## 2 Preliminaries and the proofs

**Definition 2.1.** For a prime divisor \(E\) over a non-singular variety \(A\), let \(\varphi : A' \rightarrow A\) be a proper birational morphism with normal \(A'\) such that \(E\) appears on \(A'\). Let \(k_E\) be the coefficient of the relative canonical divisor \(K_{A'/A}\) at \(E\) and \(\text{val}_E\) is the valuation defined by the divisor \(E\).

A log discrepancy of the pair \((A, \mathfrak{a}^e)\) is defined as

\[
a(E; A, \mathfrak{a}^e) := k_E - \sum_i e_i \text{val}_E(a_i) + 1
\]

and the minimal log discrepancy of the pair at a closed point 0 is defined as

\[
\text{mld}(0; A, \mathfrak{a}^e) := \inf\{a(E; A, \mathfrak{a}^e) \mid E \text{ prime divisor over } A \text{ with the center 0}\}.
\]

The log canonical threshold of the pair at a closed point 0 is defined as

\[
\text{lct}(0, A, \mathfrak{a}^e) := \inf\left\{\frac{k_E + 1}{\sum_i e_i \text{val}_E(a_i)} \mid E \text{ prime divisor over } A \text{ with the center containing 0}\right\}.
\]

**Lemma 2.2.** Let \(A\) be a smooth surface defined over an algebraically closed field \(k\) of characteristic \(p > 0\) and \(\mathfrak{a}^e\) a multiideal with an exponent \(e \subset \mathbb{R}_{>0}\). Then, the following hold:

1. Every prime divisor computing \(\text{mld}(0; A, \mathfrak{a}^e) \geq 0\) is obtained by a weighted blow up, and

2. There exists a prime divisor computing \(\text{mld}(0; A, \mathfrak{a}^e) = -\infty\) such that it is obtained by a weighted blow up.

**Proof.** For characteristic 0, the statements are proved by Kawakita in [7]. Note that the only point in the proof he uses the characteristic 0 is the “Inversion of Adjunction” of the form:

Let \(Q \in Y\) be a smooth point on a surface \(Y\) and \(F\) a smooth curve passing through \(Q\), then (a) a triple \((Y, F, \mathfrak{a}^e)\) is plt at \(Q\), if and only if (b) \((F, \mathfrak{a}^e\mathcal{O}_F)\) is klt at \(Q\).

As \(Y\) is a surface, (a) is equivalent to:

\[
\text{mld}(Q; Y, I_F\mathfrak{a}^e) > 0,
\]

(1)
where $I_F$ is the defining ideal of $F$ on $Y$. On the other hand, as $F$ is a curve, (b) is equivalent to:

$$\text{mld}(Q; F, a^e \mathcal{O}_F) > 0.$$ (2)

In positive characteristic the Inversion of Adjunction of the following form is proved in [5]:

$$\text{mld}(Q; Y, I_F a^e) = \text{mld}(Q; F, a^e \mathcal{O}_F),$$

which completes the equivalence of (a) and (b) for positive characteristic, and therefore completes the proof of the lemma for positive characteristic.

**Lemma 2.3.** Let $k$ be an algebraically closed field of arbitrary characteristic. Then, we obtain

(i) For every pair $(\mathbb{A}_k^N, a^e)$ with a monomial multiideal $a^e$, the multiideal on $\mathbb{A}_k^N$ generated by the same monomial generators as of $a^e$ is denoted by $\tilde{a}^e$. Then we have

$$\text{mld}(0; \mathbb{A}_k^N, a^e) = \text{mld}(0; \mathbb{A}_k^N, \tilde{a}^e).$$

(ii) MN Conjecture holds in the set of the pairs $(\mathbb{A}_k^N, a^e)$ for a monomial multiideal $a^e$ with an exponent $e \subset \mathbb{R}_{>0}$.

**Proof.** Denote the maximal ideals of the origins in $\mathbb{A}_k^N$ and $\mathbb{A}_k^N$ by $\mathfrak{m}_0$ and $\tilde{\mathfrak{m}}_0$, respectively. The statement (i) follows from the fact that the both pairs $(\mathbb{A}_k^N, a^e \mathfrak{m}_0)$ and $(\mathbb{A}_k^N, \tilde{a}^e \tilde{\mathfrak{m}}_0)$ have toric log resolutions of the singularities, such that the associated fans are the same and the valuation of the monomial ideals at the toric divisors corresponding to the same cone are the same. Therefore the minimal log discrepancies are the same and computed by toric divisors associated to the same cone. The statement (ii) is proved in [10, Theorem 5.1] for characteristic 0 and in [4, Corollary 1.10] for positive characteristic. Here, we show a more direct proof than [4, Corollary 1.10] by making use of the result of characteristic 0. By Kawakita’s result [6], the set of mld’s with a fixed exponent $e$ is finite, if the base field is of characteristic 0. By our statement (i), we also obtain that the set of mld’s for our pairs $(\mathbb{A}_k^N, a^e)$ with a fixed exponent $e$ is finite for the base field $k$ of positive characteristic. Then, in the same way as in [10, Theorem 5.1], we can prove the statement for positive characteristic. □

**Proof of Theorem 1.4.** Let $A$ be a smooth surface over the base field $k$ of arbitrary characteristic and fix a closed point $0 \in A$. Let $E$ be a prime divisor computing mld($0; A, a^e$) as in Lemma 2.2. Then, there are a regular system of parameters $x_1, x_2$ of $\mathcal{O}_{A,0}$ and a pair of positive integers $w_1, w_2$ such that the exceptional divisor $E$ obtained by the weighted blow up with respect to $x_1, x_2$ with weight $w_1, w_2$ computes the minimal log discrepancy mld($0; A, a^e$). Now we have a morphism

$$\rho : A \longrightarrow \mathbb{A}_k^2 = \text{Spec } k[x_1, x_2],$$

which is étale around the origin 0 and have the equalities

$$\hat{\mathcal{O}}_{A,0} = k[[x_1, x_2]] = \hat{\mathcal{O}}_{\mathbb{A}_k^2,0}.$$ 

Denote Spec $k[[x_1, x_2]]$ by $\hat{A}$. As there are natural bijections:

$$\left\{ \begin{array}{l} \text{prime divisors over } A \text{ with the center } 0 \\ \text{prime divisors over } \hat{A} \text{ with the center at the closed point } \end{array} \right\} \simeq \left\{ \begin{array}{l} \text{prime divisors over } \mathbb{A}_k^2 \text{ with the center } 0 \\ \text{prime divisors over } \mathbb{A}_k^2 \text{ with the center } 0 \end{array} \right\}.$$
we denote prime divisors in these classes by the same symbol if those divisors correspond to each other under the above bijections.

We note that the value $k_E$ is preserved under these bijections. Let $\tilde{a}_i \subset k[[x_1, x_2]]$ be the extension of the ideals $a_i \subset O_A$ by $\rho*$ and define $\tilde{a}^e := \tilde{a}_1^i \cdots \tilde{a}_n^e$. Let $\tilde{a}_i \subset k[[x_1, x_2]]$ be the ideal generated by all monomials appearing in the elements of $\tilde{a}_i$. As the ring $k[[x_1, x_2]]$ is Noetherian, $\tilde{a}_i$ is generated by finite number of monomials. Let those monomials generate an ideal $\tilde{a}^e$ of the ideals $\tilde{a}_1^i \cdots \tilde{a}_n^e$. By Lemma 2.3, there is $\ell_{2, e} \in \mathbb{N}$ depending only on 2 and $e$ such that there is a prime divisor $E'$ over $\tilde{a}_1^i \cdots \tilde{a}_n^e$ that computes $\text{mld}(0; A, a^e)$ and satisfying $k_{E'} \leq \ell_{2, e}$. Then, by (3) and (5) this divisor $E'$ also computes $\text{mld}(0; A, a^e)$, which completes the proof of (i).

The proof of (ii) is clear from (i) in Lemma 2.3 and the above proof for (i). □

**Proof of Theorem 1.5.** Let $A$ be a smooth surface over an algebraically closed field $k$ of arbitrary characteristic, $a^e$ a multiideal with a real exponent $e$ on $A$ and $0 \in A$ a closed point. Let $t = \text{lct}(0; A, a^e)$, then the pair $(A, (a^e)^t)$ is strictly log canonical. For the proof of Theorem, we may assume that there is an exceptional prime divisor $E$ computing the lct. Then, $E$ has the center at 0 and computes $\text{mld}(0; A, (a^e)^t) = 0$. By Lemma 2.2, it follows that $E$ is obtained by a weighted blow up. For the second statement, let $t = \text{lct}(0; A, a^e)$ and it is computed by an exceptional divisor, then as $A$ is a smooth surface, the center of the exceptional divisor is 0. Therefore it follows $\text{mld}(0; A, (a^e)^t) = 0$ and it is computed by the exceptional divisor. By the result (ii) in Theorem 1.4, we have a monomial multiideal $\tilde{a}^e$ with a real exponent $e$ on $\tilde{a}_1^i \cdots \tilde{a}_n^e$ such that

$$0 = \text{mld}(0; A, (a^e)^t) = \text{mld}(0; \tilde{a}_1^i (a^e)^t).$$

And the proof of Theorem 1.4 also gives an exceptional divisor computing $\text{mld}(0; \tilde{a}_1^i, (\tilde{a}^e)^t) = 0$. This shows that $t = \text{lct}(0; \tilde{a}_1^i, \tilde{a}^e)$. □

Note that there is not necessarily an “exceptional” prime divisor computing the log canonical threshold, although it is computed by some divisor (may not be exceptional). Theorem 1.5 states nothing for such a case. The following is an example:
Example 2.4. Let $a$ be an ideal defining a line $L$ in $\mathbb{A}^2_k$. Then, $\text{lct}(0; \mathbb{A}^2_k, a) = 1$ and the prime divisor computing $\text{lct}(0; \mathbb{A}^2_k, a)$ is $L$ and there is no exceptional divisor over $\mathbb{A}^2_k$ computing the $\text{lct}$, although there is a divisor computing the $\text{lct}$.

Proof of Corollary 1.6. Let $A$ be a smooth surface over an algebraically closed field $k$ of positive characteristic and $0$ a closed point. For every fixed DCC set $J \subset \mathbb{R}_{>0}$, given a sequence:

$$\text{mld}(0; A, a^{e(1)}) < \text{mld}(0; A, a^{e(2)}) < \cdots$$

such that $e(i) \subset J$ for all $i = 1, 2, \ldots$. Then, by (ii) of Theorem 1.4, this sequence coincides with

$$\text{mld}(0; \mathbb{A}^2_{\mathbb{C}}, \tilde{a}^{e(1)}) < \text{mld}(0; \mathbb{A}^2_{\mathbb{C}}, \tilde{a}^{e(2)}) < \cdots,$$

where $\tilde{a}^{e(i)}$'s are monomial multiideals on $\mathbb{A}^2_{\mathbb{C}}$. As ACC holds on the pairs over $\mathbb{C}$, we obtain that the sequence stops at a finite stage. □

Proof of Corollary 1.7. Let $A$ be a smooth surface over an algebraically closed field $k$ of positive characteristic and $0$ a closed point. For every fixed DCC set $J \subset \mathbb{R}_{>0}$, given a sequence:

$$\text{lct}(0; A, a^{e(1)}) < \text{lct}(0; A, a^{e(2)}) < \cdots$$

such that $e(i) \subset J$ for all $i = 1, 2, \ldots$. Here, we may assume that all $\text{lct}$ are computed by exceptional divisors, because the sequence of $\text{lct}$'s computed by non-exceptional divisors has ascending chain condition. Now, by the second statement of Corollary 1.5, we obtain the ascending chain of $\text{lct}$'s of pairs over $\mathbb{C}$. Apply the result of ACC for characteristic 0 ([2]) to obtain the sequence stops at a finite stage. □

Proof of Corollary 1.8. The first statement follows in the same way as in the proof of Corollary 1.6 by using (ii) of Theorm 1.4. The second statement follows from the proof of the Theorem 1.4. □

As we reduce the problem into the one on the pairs of monomial ideals on $\mathbb{A}^2_k$, we can calculate $\ell_e$ for a given $e$ by combinatorics.

Example 2.5. First of all, note that $\text{mld}(0; \mathbb{A}^2_k, a^e)$ and a toric divisor computing it are determined by $e$ and the Newton polygons $\Gamma(a_i)$ ($i = 1, \ldots, s$) (for the definition of the Newton polygon, the reader can refer to [3, Definition 5.2]). Actually, we have

$$\text{mld}(0; \mathbb{A}^2_k, a^e) = \inf \left\{ \langle p, 1 \rangle - \sum_i e_i \langle p, \Gamma(a_i) \rangle \right\},$$

where $p = (p_1, p_2)$ runs whole in $\mathbb{N}^2$ in the right hand side and $\langle p, \Gamma(a_i) \rangle := \min \{ \langle p, q \rangle \mid q \in \Gamma(a_i) \}$. A toric divisor computing the $\text{mld} \geq 0$ is a divisor $E_p$, where $p$ attains the infimum in the right hand side. In the following, we will estimate $\ell_e$ for a given $e$. Note that we check only toric divisors and obtain $\ell_e$, therefore it may not be optimal, i.e., there may be non-toric divisor $E$ computing the $\text{mld}$ with smaller $k_E$.

For every $e$, if $a_i = \mathcal{O}_{\mathbb{A}^2_k}$ ($i = 1, \ldots, s$), then $\text{mld}(0; \mathbb{A}^2_k, a^e) = 2$ and computed by $E_1$, where $1 = (1, 1)$. So, in the following, we exclude this trivial case.
1. Case \#e = 1. Let \( e = \{ e_1 \} \) (\( e_1 \in \mathbb{R}_{>0} \)).

(a) When \( e_1 > 2 \), it follows that \( a(E_1; \mathbb{A}^2, a^e) = 2 - e_1 < 0 \), therefore it is obvious that \( \text{mld}(0; \mathbb{A}^2, a^e) = -\infty \) and it is computed by \( E_1 \). Therefore, \( \ell_e = 1 \).

(b) When \( 1 < e_1 \leq 2 \), either \( \text{mld}(0; \mathbb{A}^2, a^e) = 2 - e_1 \) or \( \text{mld}(0; \mathbb{A}^2, a^e) = -\infty \), the first case is computed by \( E_1 \) and in the second case an upper bound of minimal \( k_E \) such that \( E \) computes the mld is \( \left\lceil \frac{1}{e_1 - 1} \right\rceil + 1 \). In this case \( E = E_p \) (\( p = \left( \left\lceil \frac{1}{e_1 - 1} \right\rceil + 1, 1 \right) \)) and the ideal \( a \) is generated by \( x \). Therefore, we obtain \( \ell_e = \left\lceil \frac{1}{e_1 - 1} \right\rceil + 1 \).

This is proved as follows:

**Case 1** \( \text{mult}_a \geq 2 \).

In this case the Newton polygon \( \Gamma(a) \subset \bar{\Gamma} \), where \( \bar{\Gamma} \) is the convex hull of

\[
((2, 0) + \mathbb{R}_{\geq 0}^2) \cup ((0, 2) + \mathbb{R}_{\geq 0}^2).
\]

As \( e_1 > 1 \), we have \( e_1 \Gamma(a) \subset e_1 \bar{\Gamma} \) and \( 1 \not\in e_1 \bar{\Gamma} \) which implies

\[ a(E_1; \mathbb{A}^2, a^e) = 2 - e_1 \langle 1, \Gamma(a) \rangle < 2 - e_1 \langle 1, \bar{\Gamma} \rangle < 0. \]

Therefore, in this case \( \text{mld} = -\infty \) and it is computed by \( E_1 \).

**Case 2** \( \text{mult}_a = 1 \).

In this case, the Newton polygon is either the convex hull of

\[
((1, 0) + \mathbb{R}_{\geq 0}^2) \cup ((0, 1) + \mathbb{R}_{\geq 0}^2)
\]

or the convex hull of

\[
((1, 0) + \mathbb{Z}^2_{\geq 0}) \quad \text{or} \quad ((0, 1) + \mathbb{Z}^2_{\geq 0}).
\]

In the case (6), \( a(E_p; \mathbb{A}^2, a^e) = \langle p, 1 \rangle - e_1 \langle p, \Gamma(a) \rangle \) and it is minimized by \( p = 1 \). It says that \( E_1 \) computes \( \text{mld} = 2 - e_1 \).

In the case (7) we can only prove the first case, as these two are symmetric. As \( e_1 > 1 \), we have \( 1 \not\in e_1 \Gamma(a) \) which means \( \text{mld} = -\infty \). An integer vector \( p \) such that \( a(E_p; \mathbb{A}^2, a^e) = \langle p, 1 \rangle - e_1 \langle p, \Gamma(a) \rangle < 0 \) minimizes \( \langle p, 1 \rangle = p_1 + p_2 \) is

\[ p_1 = \left\lceil \frac{1}{e_1 - 1} \right\rceil \quad \text{and} \quad p_2 = 1, \]

where \( p_1 \) and \( p_2 \) are the coordinates of \( p \).

(c) When \( e_1 = 1 \), \( \text{mld}(0; \mathbb{A}^2, a^e) = 1 \) or 0 or \( -\infty \) and is computed by \( E_1 \) for the former two cases. For the case \( \text{mld}(0; \mathbb{A}^2, a^e) = -\infty \) an upper bound of minimal \( k_E \) such that \( E \) computes the mld is 4. In this case \( E = E_{(3,2)} \) and the ideal \( a \) is generated by \( x^2 \) and \( y^3 \). Therefore, we obtain \( \ell_{(1)} = 4 \). The proof is similar to the previous case by divided into the cases according to the multiplicity of \( a \), so we omit the proof.

(d) When \( e_1 < 1 \), for a smaller \( e_1 \) we have more cases to be checked to get \( \ell_e \). As an example, we consider \( e = \{1/2\} \), then we have \( \text{mld}(0; \mathbb{A}^2, a^e) = 3/2 \) or 1 or 1/2 or 0 or \( -\infty \) and it is computed by \( E_1 \) for the former four cases. For the case \( \text{mld}(0; \mathbb{A}^2, a^e) = -\infty \) an upper bound of the minimal \( k_E \) such that \( E \) computes the mld is 9. In this case \( E = E_{(7,3)} \) and the ideal \( a \) is generated by \( x^3 \) and \( y^7 \). Therefore, we obtain \( \ell_{(1/2)} = 9 \).
An example for non rational $e_1$ is as follows:
Let $e_1 = 2/\pi$, where $\pi$ is the circular constant. Then, the value of $\mld(0; \mathbb{A}^2_\mathbb{R}, a^{e_1}) = 2 - 2/\pi$ or $2 - 4/\pi$ or $2 - 6/\pi$ or $-\infty$. And an upper bound of the minimal $k_E$ such that $E$ computes the $\mld$ is 6. In this case $E = E_{(4,3)}$ and the ideal $a$ is generated by $x^3$ and $y^4$. Therefore, we obtain $\ell_{(2/\pi)} = 6$.

2. Case $#e = 2$. We consider $e = (1, 1/2)$. In this case, the possible values of $\mld(0; \mathbb{A}^2_\mathbb{R}, a^{e_1})$ are $3/2$ or $1/2$ or $0$ or $-\infty$. An upper bound of minimal $k_{E_p}$ such that $E_p$ computes $\mld(0; \mathbb{A}^2_\mathbb{R}, a^{e_1})$ is 9 and in this case $E = E_{(7,3)}$, $a_1 = \mathcal{O}_{\mathbb{A}^2}$ and $a_2$ is generated by $x^3$ and $y^7$.

**Remark 2.6.** If $e' \subseteq e \subseteq \mathbb{R}_{>0}$, then we have $\ell_{e'} \leq \ell_e$, where we assume that $\ell_{e'}$ and $\ell_e$ are optimal. This is because every $a^{e'}$ can be written as $b^e$ by $b_i = a_i$ for $e_i \in e'$ and $b_i = \mathcal{O}_{\mathbb{A}^2}$ otherwise.

**References**

1. C. Birkar, *Existence of flips and minimal models for 3-folds in char $p$*, Ann. Scient. L’École Norm. Sup. 49, (2016), 169–212.

2. C. Hacon, J. McKernan and C. Xu, *ACC for log canonical thresholds*, Ann. Math., 180, (2014), 523–571.

3. S. Ishii, *Finite determination conjecture for Mather-Jacobian minimal log discrepancies and its applications*, Europ. J. Math., 4, (2018), 1433–1475.

4. S. Ishii, *Inversion of modulo $p$ reduction and a partial descent from characteristic 0 to positive characteristic*, preprint 2018, ArXiv: 1808.10155, to appear in the Proceedings of JARCS VII.

5. S. Ishii and A. Reguera, *Singularities in arbitrary characteristic via jet schemes*, Math. Zeitschrift, 275, Issue 3-4, (2013), 1255–1274.

6. M. Kawakita, *Discreteness of log discrepancies over log canonical triples on a fixed pair*, J. Algebraic Geom. 23 (2014), no. 4, 765–774.

7. M. Kawakita, *Divisors computing the minimal log discrepancy on a smooth surface*, Mathematical Proceedings of the Cambridge Philosophical Society 163, No. 1, (2017),187–192.

8. M. Kawakita, *On equivalent conjectures for minimal log discrepancies on smooth threefolds*, arXiv:1803.02539.

9. J. Kollár, K. Smith and A. Corti, *Rational and Nearly Rational Varieties*, Cambridge Studies in Advanced Math., 92 (2002), 235 pages.

10. M. Mustaţă and Y. Nakamura, *A boundedness conjecture for minimal log discrepancies on a fixed germ*, AMS Contemporary Mathematics, 712, (2018), 287–306.

YMSC, Tsinghua University, Beijing/ Graduate School of Math. Sci., the University of Tokyo, shihokoishii@mac.com/ shihoko@ms.u-tokyo.ac.jp