Justification logic enjoys the strong finite model property

Thomas Studer

Abstract
We observe that justification logic enjoys a form the strong finite model property (sometimes also called small model property). Thus we obtain decidability proofs for justification logic that do not rely on Post’s theorem.

1 Introduction

Justification logics are a family of logics that that, like modal logics, can express knowledge or provability of propositions. However, instead of an implicit □-operator justification logics include explicit modalities of the form t: where t is a term representing a reason for an agent’s knowledge or a proof a proposition.

Artemov developed the first justification logic to provide intuitionistic logic with a classical provability semantics. Later Fitting introduced epistemic models for justification logic. In this semantics, justification terms represent evidence a very general sense. For instance, our belief in A may be justified by direct observation of A or by learning that a friend heard about A. This general reading of justification led to a big variety of epistemic justification logics for many different applications.

There are many known decidability results for justification logics, see, for instance, . However, many of these decidability proofs rely on completeness with respect to a recursively enumerable class of models and Post’s theorem.

In the present note we show that justification logic enjoys a form of the strong finite model property (which sometimes is called small model property) . Thus we obtain decidability proofs for justification logics that do not make use of Post’s theorem.

This note makes heavy use of .

2 Justification Logics

Justification terms are built from countably many constants c and countably many variables x according to the following grammar:

\[ t ::= c | x | (t \cdot t) | (t + t) | \neg t \]

We denote the set of terms by Tm. Formulae are built from countably many atomic propositions p according to the following grammar:

\[ F ::= p | \neg F | (F \to F) | t : F \]
Prop denotes the set of atomic propositions and Fm denotes the set of formulae.

The axioms of $J_{CS}$ consist of all instances of the following schemes:

A1 finitely many schemes axiomatizing classical propositional logic

A2 $t : (A \rightarrow B) \rightarrow (s : A \rightarrow t \cdot s : B)$

A3 $t : A \lor s : A \rightarrow t + s : A$

We will consider extension of $J_{CS}$ by the following axioms schemes.

(jd) $t : \bot \rightarrow \bot$

(jt) $t : A \rightarrow A$

(j4) $t : A \rightarrow !t : t : A$

A constant specification $CS$ for a logic $L$ is any subset

$CS \subseteq \{ c : A \mid c$ is a constant and $A$ is an axiom of $L \}.$

A constant specification $CS$ for a logic $L$ is called

1. axiomatically appropriate if for each axiom $A$ of $L$ there is a constant $c$ such that $c : A \in CS$

2. schematic if for each constant $c$ the set $\{ A \mid c : A \in CS \}$ consists of one or several (possibly zero) axiom schemes, i.e., every constant justifies certain axiom schemes.

For a constant specification $CS$ the deductive system $J_{CS}$ is the Hilbert system given by the axioms A1–A3 and by the rules modus ponens and axiom necessitation:

\[
\frac{A \quad A \rightarrow B}{B} \quad (\text{MP}), \quad \frac{c : A \in CS}{c : A} \quad (\text{AN!)} ,
\]

where $n \geq 0$. In the presence of the j4 axiom a simplified axiom necessitation rule can be used:

\[
\frac{c : A \in CS}{c : A} \quad (\text{AN}) .
\]

Table[1] defines the various logics we consider. We now present the semantics for these logics

Definition 1 (Evidence relation). Let $(W, R)$ be a Kripke frame, i.e., $W \neq \emptyset$ and $R \subseteq W \times W$, and $CS$ be a constant specification. An admissible evidence relation $E$ for a logic $L_{CS}$ is a subset of $\text{Tm} \times \text{Fm} \times W$ that satisfies the closure conditions:

1. if $(s, A, w) \in E$ or $(t, A, w) \in E$, then $(s + t, A, w) \in E$

2. if $(s, A \rightarrow B, w) \in E$ and $(t, A, w) \in E$, then $(s \cdot t, B, w) \in E$

Depending on whether or not the logic $L_{CS}$ contains the j4 axiom, the evidence function has to satisfy one of the following two sets of closure conditions. If $L_{CS}$ does not include the j4 axiom, then the additional requirement is:
Table 1: Deductive Systems

|       | A1 | A2 | A3 | jd | jt | j4 | MP | AN! | AN |
|-------|----|----|----|----|----|----|----|-----|-----|
| JCS   | ✓  | ✓  | ✓  | ✓  | ✓  | ✓  | ✓  | ✓   |
| JDcs  | ✓  | ✓  | ✓  | ✓  | ✓  | ✓  | ✓  |
| JTcs  | ✓  | ✓  | ✓  | ✓  | ✓  | ✓  |
| JD4cs | ✓  | ✓  | ✓  | ✓  | ✓  | ✓  |
| J4cs  | ✓  | ✓  | ✓  | ✓  | ✓  |
| LPcs  | ✓  | ✓  | ✓  | ✓  | ✓  |

3. if $c : A \in CS$ and $w \in W$, then $(\ldots ! c, \ldots ! c : c : A, w) \in E$

If $L_{CS}$ includes the j4 axiom, then the additional requirement is:

4. if $c : A \in CS$ and $w \in W$, then $(c, A, w) \in E$

5. if $(t, A, w) \in E$, then $(! t, t : A, w) \in E$

6. if $(t, A, w) \in E$ and $wRv$, then $(t, A, v) \in E$

If we drop condition 6, then we say $E$ is a t-evidence relation. Sometimes we use $E(s, A, w)$ for $(s, A, w) \in E$.

Definition 2 (Evidence bases).

1. An evidence base $B$ is a subset of $Tm \times Fm \times W$.
2. An evidence relation $E$ is based on $B$, if $B \subseteq E$.

The closure conditions in the definition of admissible evidence function give rise to a monotone operator. The minimal evidence relation based on $B$ is the least fixed point of that operator and thus always exists.

Definition 3 (Model). Let $CS$ be a constant specification. A Fitting model for a logic $L_{CS}$ is a quadruple $M = (W, R, E, \nu)$ where

- $(W, R)$ is a Kripke frame such that
  - if $L_{CS}$ includes the j4 axiom, then $R$ is transitive;
  - if $L_{CS}$ includes the jt axiom, then $R$ is reflexive;
  - if $L_{CS}$ includes the jd axiom, then $R$ is serial.
- $E$ is an admissible evidence relation for $L_{CS}$ over the frame $(W, R),$
- $\nu : \text{Prop} \rightarrow \mathcal{P}(W)$, called a valuation function.

Definition 4 (Satisfaction relation). The relation of formula $A$ being satisfied in a model $M = (W, R, E, \nu)$ at a world $w \in W$ is defined by induction on the structure of $A$ by

- $M, w \models p_i$ if and only if $w \in \nu(p_i)$
- $\models$ commutes with Boolean connectives
\[ \mathcal{M}, w \vdash t : B \text{ if and only if} \]
\[ \begin{aligned} 1) & \quad \mathcal{M}, v \vdash B \text{ for all } v \in W \text{ with } wRv \text{ and} \\
& \quad (t, B, w) \in E \end{aligned} \]

We say a formula \( A \) is valid in a model \( \mathcal{M} = (W, R, E, \nu) \) if for all \( w \in W \) we have \( \mathcal{M}, w \vdash A \). We say a formula \( A \) is valid for a logic \( L_{CS} \) if for all models \( \mathcal{M} \) for \( L_{CS} \) we have that \( A \) is valid in \( \mathcal{M} \).

The logics defined above are sound and complete (with a restriction in case of the logics containing the jd axiom). See [3, 9, 13] for the full proofs of the following results.

**Theorem 5** (Soundness). Let \( CS \) be a constant specification. If a formula \( A \) is derivable in a logic \( L_{CS} \), then \( A \) is valid for \( L_{CS} \).

**Theorem 6** (Completeness).

1. Let \( CS \) be a constant specification. If a formula \( A \) is not derivable in \( L_{CS} \in \{ J_{CS}, JT_{CS}, J4_{CS}, LP_{CS} \} \), then there exists a model \( \mathcal{M} \) for \( L_{CS} \) with \( \mathcal{M}, w \nvDash A \) for some world \( w \) in \( \mathcal{M} \).

2. Let \( CS \) be an axiomatically appropriate constant specification. If a formula \( A \) is not derivable in \( L_{CS} \in \{ JD_{CS}, JD4_{CS} \} \), then there exists a model \( \mathcal{M} \) for \( L_{CS} \) with \( \mathcal{M}, w \nvDash A \) for some world \( w \) in \( \mathcal{M} \).

3 **The Strong Finite Model Property and Decidability**

In this section we define and establish the strong finitary model property for many justification logics. As a corollary we get decidability proofs for these logics.

**Definition 7** (Finitary model). A model \( \mathcal{M} = (W, R, E, \nu) \) is called finitary if

1. \( W \) is finite,

2. there exists a finite base \( B \) such that \( E \) is the minimal evidence relation based on \( B \), and

3. the set \( \{(w, p) \in W \times \text{Prop} \mid w \in \nu(p)\} \) is finite.

If \( \mathcal{M} = (W, R, E, \nu) \) is a finitary model for \( L_{CS} \), then will sometimes specify this model by the tuple \( (W, R, B, \nu) \) where \( B \) is the finite base for \( E \).

Making use of filtrations for justification logics, we obtain the following theorem [8].

**Lemma 8** (Completeness w.r.t. finitary models).

1. Let \( L_{CS} \in \{ J_{CS}, JT_{CS}, J4_{CS}, LP_{CS} \} \) and \( CS \) be a constant specification for \( L \). If a formula \( A \) is not derivable in \( L_{CS} \), then there exists a finitary model \( \mathcal{M} \) for \( L_{CS} \) with \( \mathcal{M}, w \nvDash A \) for some world \( w \) in \( \mathcal{M} \).

2. Let \( L_{CS} \in \{ JD_{CS}, JD4_{CS} \} \) and \( CS \) be an axiomatically appropriate constant specification for \( L \). If a formula \( A \) is not derivable in \( L_{CS} \), then there exists a finitary model \( \mathcal{M} \) for \( L_{CS} \) with \( \mathcal{M}, w \nvDash A \) for some world \( w \) in \( \mathcal{M} \).
Definition 9.

1. Let \( A \) be a formula. We denote the length of \( A \) (i.e. the number symbols in \( A \)) by \( |A| \).

2. Let \( \Gamma \) be a set. We denote the cardinality of \( \Gamma \) (i.e. the number of elements of \( \Gamma \)) by \( |\Gamma| \).

Definition 10 (Strong finitary model property). A justification logic \( L_{CS} \) has the strong finitary model property if there are computable functions \( f, g, h \) such that for any formula \( A \) that is not satisfiable, there exists a finitary model \( M = (W, R, B, \nu) \) for \( L_{CS} \) with

1. \( M, w \not\models A \) for some \( w \in W \)
2. \( |W| \leq f(|A|) \)
3. \( |B| \leq g(|A|) \)
4. \( |\nu| \leq h(|A|) \)

Given the proof of Lemma 8 in [8] it is easy to see that we can effectively compute bounds on the size of the finitary model. Thus we get the strong finitary model property as a corollary of Lemma 8.

Corollary 11 (Strong finitary model property).

1. Let \( L_{CS} \in \{J_{CS}, JT_{CS}, J_{4CS}, LP_{CS}\} \) and \( CS \) be a constant specification for \( L \). Then \( L_{CS} \) has the strong finitary model property.

2. Let \( L_{CS} \in \{JD_{CS}, JD_{4CS}\} \) and \( CS \) be an axiomatically appropriate constant specification for \( L \). Then \( L_{CS} \) has the strong finitary model property.

For a proof of the following lemma see [11, Lemma 4.4.6].

Lemma 12. Let \( CS \) be a decidable schematic constant specification and \( L_{CS} \in \{J_{CS}, JD_{CS}, J_{4CS}, JT_{CS}, J_{4CS}, LP_{CS}\} \). Let \( M = (W, R, E, \nu) \) be a finitary model for \( L_{CS} \). Then the relation \( M, w \models A \) between worlds \( w \in W \) and formulae \( A \) is decidable.

Corollary 13 (Decidability).

1. Any justification logic in \( \{J_{CS}, JT_{CS}, J_{4CS}, LP_{CS}\} \) with a decidable schematic \( CS \) is decidable.

2. Any justification logic in \( \{JD_{CS}, JD_{4CS}\} \) with a decidable, schematic and axiomatically appropriate \( CS \) is decidable.

Proof. Let \( L_{CS} \) be one of the above justification logics. Given a formula \( A \) we can generate all finitary models \( M = (W, R, B, \nu) \) for \( L_{CS} \) with

1. \( |W| \leq f(|A|) \)
2. \( |B| \leq g(|A|) \)
3. \( |\nu| \leq h(|A|) \)
for the functions \( f, g, h \) from Definition \([10]\). Note that we can decide whether a structure \( \mathcal{M} = (W, R, B, \nu) \) is a model for \( \mathcal{L}_{CS} \) since the required conditions on the accessibility relation, some combination of transitivity, reflexivity, and seriality can be effectively verified.

By Lemma \([12]\) we can decide for each of these finitary models, whether \( \mathcal{M}, w \vDash A \) for all \( w \in W \).

Making use of Corollary \([11]\) we know that if \( A \) is not \( \mathcal{L}_{CS} \)-satisfiable, then the above procedure will generate a finitary model \( \mathcal{M} = (W, R, B, \nu) \) such that \( \mathcal{M}, w \not\vDash A \) for some \( w \in W \). Therefore, we conclude that satisfiability for \( \mathcal{L}_{CS} \) is decidable.

\section{Conclusion}

We observed that justification logic enjoys a form of the strong finite model property (sometimes also called small model property). Thus we obtain decidability proofs for justification logics that do not rely on Post’s theorem.

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