Spatial coherence singularities of a vortex field in nonlinear media

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We study the spatial coherence properties of optical vortices generated by partially incoherent light in self-focusing nonlinear media. We reveal the existence of phase singularities in the spatial coherence function of a vortex field that can characterize the stable propagation of vortices through nonlinear media.

Vortices are known to occur in coherent systems having a zero intensity at the center and a well-defined phase associated with the circulation of momentum around the helix axis \( \mathbf{r} \). The last decade has seen a resurgence of interest in the study of optical vortices \( 1 \), owing in part to readily available computer-generated holographic techniques for creating vortices in laser beams.

If a vortex-carrying beam is partially incoherent, the phase front topology is not well defined, and statistics are required to quantify the phase. In the incoherent limit neither the helical phase nor the characteristic zero intensity at the vortex center can be observed. However, several recent studies have shed light on the question how phase singularities can be unveiled in incoherent light fields propagating in linear media \( 3, 4 \). In particular, Palacios et al. \( 4 \) used both experimental and numerical techniques to explore how a beam transmitted through a vortex phase mask changes as the transverse coherence length at the input of the mask varies. Assuming a quasi-monochromatic, statistically stationary light source and ignoring temporal coherence effects, they demonstrated that robust attributes of the vortex remain in the beam, most prominently in the form of a ring dislocation in the cross-correlation function.

Propagating in nonlinear coherent systems, optical vortices become unstable when the nonlinear medium is self-focusing, and this effect has been observed experimentally in different nonlinear systems \( 5 \). However, stable propagation of optical vortices in self-focusing nonlinear media has been recently demonstrated, both theoretically and experimentally, for the vortices created by self-trapping of partially incoherent light when the spatial incoherence of light exceeds a certain threshold \( 6 \).

The purpose of this Letter is twofold. First, we study numerically the spatial coherence function of a vortex beam propagating in a self-focusing nonlinear medium and reveal its importance for the study of singular beams in nonlinear media. Second, we provide a deeper physical insight into the effect of vortex stabilization by partially coherent light reported previously.

We consider a phase singularity carried by a single vortex beam. To simulate numerically the propagation of partially coherent light through a nonlinear medium, we use the coherence density approach \( 7 \) based on the fact that an incoherent light source can be thought of as a superposition of (infinitely) many coherent components \( E_j \) that are mutually incoherent, having slightly different propagation directions: 

\[
E_j(r,t) = \sum_j E_j(r) \exp(i k_{j} \cdot r) \exp(i \gamma_j(t)),
\]

where \( k_j = k(\alpha_j e_x + \beta_j e_y) \) is the transverse wave vector of the \( j \)-th component, having direction cosines \( \alpha_j \) and \( \beta_j \), \( r = x e_x + y e_y \), \( \gamma_j(t) \) is a random variable varying on the time scale of the coherence time of the light source, and \( k = 2\pi/\lambda \) is the wavenumber. The vortex is introduced via a phase mask at the input face \(( z = 0) \) of the medium. To avoid complexities that may arise from incoherent light sources having abrupt boundaries, we assume the source has a Gaussian profile 

\[
E_j(r) = \left( \frac{1}{\sqrt{\pi \theta_0}} e^{-(\alpha_j^2 + \beta_j^2)/\theta_0^2} \right)^{1/2} A(r).
\]

Here \( \theta_0 \) is a parameter that controls the beam’s coherence, \( A(r) = (r/w_0)^2 \exp(-r^2/\sigma^2) \exp(i \phi) \) is the vortex profile, and \( \phi \) \( \phi \) is the angular variable.

Scaling the lengths in the transverse directions to \( x_0 = 1 \mu m \) and the length in propagation direction to \( z_0 = 2 k x_0 \), where we chose \( k = 2\pi/(230 nm) \), the propagating field \( E_j(r,z) \) can be described by the nonlinear Schrödinger equation:

\[
i \frac{\partial E_j(r,z)}{\partial z} + \nabla_\perp^2 E_j(r,z) + \eta(r,z) E_j(r,z) = 0,
\]

where \( \eta(r,z) \) accounts for the nonlinear refractive index change in the material. We assume a photorefractive medium with a saturable nonlinearity having a response time much longer than the coherence time of the light source. In this case \( \eta \) depends on the time-integrated intensity. The intensity is given by

\[
I = \sum_j |E_j|^2.
\]

The nonlinear factor may thus be written in the form,

\[
\eta(r,z) = I(r,z)/(1 + s I(r,z)),
\]

where \( s \) is a saturation parameter. Whereas numerical solutions of Eq. \( 2 \) may be readily computed using the coherence density approach, we shall later adopt the equivalent multi-mode theory \( 8 \) to provide a physical basis for our findings. The numerical simulations are performed using a split-step method. In the examples shown below we use \( N = 1681 \) components, \( w_0 = 1.8, \sigma = 1.5 \) and \( s = 0.5 \). We chose \( \theta_0 = 0.64 \circ \).
Second-order coherence properties of the propagating field may be quantified by determining the the mutual coherence function $\Gamma(r_1, r_2; z) = \langle E^*(r_2, z, t) E(r_1, z, t) \rangle$, where the brackets represent an average over the net field $E(r, z, t) = \sum_{j=1}^{N} E_j(r, z) \exp(i\gamma_j(t))$. Again, we assume that the random phase factors $\gamma_j(t)$ vary on a much faster timescale than the response time of the medium. For the linear propagation, Palacios et al. [4] demonstrated that the phase singularities occur in the cross-correlation $\Gamma(-r, r)$ of an incoherent vortex, where the origin of the coordinate system is chosen to coincide with the vortex center.

![FIG. 1](image1.png)

**FIG. 1**: Contour plots of the intensity (left column) and the modulus of the cross-correlation (right column) of an incoherent vortex with $\theta_0 = 0.64\degree$. Contrary to the case of the linear propagation, there is a local intensity minimum in the beam's center. The cross-correlation, however, shows the same ring of phase singularities as predicted in the linear theory. The size is $35 \times 35 \mu m$.

In Fig. 1 we show the results for the nonlinear evolution of an incoherent vortex and the cross correlation. First, we notice that the beam intensity has a local minimum in the center of the vortex, even after propagating many diffraction lengths. This is contrary to the case of the linear propagation where a beam with the same degree of coherence $\theta_0$ has maximum intensity in the center of the vortex after only a few diffraction lengths. Also, if we had chosen to propagate an incoherent ring of light without topological charge instead of an incoherent vortex, we would also observe a maximum in the beam’s center. Thus, we can state that the coherence function of the vortex manifests itself in the intensity distribution of the light beam after propagating through a nonlinear medium. In fact, the intensity profile remains reminiscent of a vortex, even if the intensity does not quite drop to zero in the center of the beam.

Looking at the beam’s cross-correlation, we clearly observe a ring of phase singularities in the cross-correlation $\Gamma(-r, r)$. Thus, as the first result of our numerical studies we state that the phase singularities predicted for the incoherent vortices propagating in linear media also survive the propagation through a nonlinear medium. This is not self-evident, considering that in the nonlinear case the single components that form an incoherent light beam do interact, contrary to the linear case. A physically intuitive explanation how this ring of phase singularities develops under linear propagation is given in Ref. [4]. However, this issue becomes more complicated for the propagation in a nonlinear medium.

![FIG. 2](image2.png)

**FIG. 2**: An incoherent vortex soliton calculated using the modal theory and three modes with the topological charges $m = 0, 1$ and 3. (a) shows the profiles of the three components, (b) shows the total intensity of the soliton and (c) shows its cross-correlation $\Gamma(-r, r)$.

Although the coherence density approach can be used to simulate the propagation of partially incoherent light with arbitrary precision, it is of little use when it comes to finding an explanation for the results obtained from the simulations. More physical insight into the problem can be obtained by using the modal theory of incoherent solitons [5]. According to the modal theory, the incoherent solitons can be regarded as an incoherent superposition of guided modes of the waveguide induced by the total light intensity. Since the incoherent vortices that we are dealing with induce circularly symmetric waveguides, the guided modes we have to consider are also circularly symmetric. To explain our numerical findings, we construct a partially incoherent vortex soliton using circularly symmetric beams with topological charges $m = 0, 1$ and 2:

$$E(r) = \sum_{m=0}^{3} E_m(r) \exp(im\phi) \exp(i\gamma_m(t)).$$

This can be done using a standard relaxation technique [1]. A more precise modelling of incoherent solitons would require more modes. Here, we restrict ourselves to three modes only, assuming that for a partially incoherent vortex the $m = 1$ component should be dominant and that the next strongest components should be those with topological charge $m' = m \pm 1$, i.e. $m' = 0, 2$. Indeed, we find that the main features of incoherent vortex solitons can be explained qualitatively using only these three modes.

The relative intensity of the $m = 0$ beam and the $m = 2$ vortex as compared to the $m = 1$ vortex control the light coherence. However, in order to assure that the total topological charge of the light $m_{tot} = \text{Im}\{\int E^*(r) \times \nabla E \, dr\} e_z / \int I \, dr$ is equal to one, we have to chose the $m = 0$ and $m = 2$ components of equal intensity. In order to check whether this simple approach
FIG. 3: The intensity (left column) and the cross-correlation (right column) of the far field. The effects of the nonlinearity on the intensity distribution can be clearly seen, whereas the cross-correlation maintains more or less the structure one would expect in the case of linear propagation.

yields the results that agree at least qualitatively with the full numerical model, we calculate an incoherent vortex soliton using the modal theory. The resulting shape of the vortex components, the total intensity, and cross-correlation $\Gamma(-\mathbf{r}, \mathbf{r})$ are shown in Fig. 2. Comparing Fig. 1 and Fig. 2, we notice the presence of two similar features: the local minimum of the intensity in the center of the beam, and the ring-like structure of the cross-correlation. Hence, these two phenomena can be explained by considering a simple modal representation of the incoherent vortex consisting of only three modes with the topological charges $m = 0$, 1 and 2.

First, the local minimum in the center of the beam can be explained by the fact that the waveguide induced by the $m = 1$ and $m = 2$ components affects the $m = 0$ mode in such a way, that it also develops a local intensity minimum in its center, a fact well known from the vortex-mode vector solitons. Second, the ring-like structure of the cross-correlation comes from the different radial extent of the single components. As is known from the physics of vortex-mode vector solitons, the $m = 0$ component has the smallest radial extent, whereas the $m = 1$ and $m = 2$ components have larger radii. Hence the cross-correlation given by $\Gamma(-\mathbf{r}, \mathbf{r}) = \sum_{m,m'=0}^{3} \langle E^*_{m'}(-\mathbf{r})E_{m}(\mathbf{r}) \rangle = \sum_{m=0}^{3} E^*_{m}(-\mathbf{r})E_{m}(\mathbf{r})$, is dominated for small $\mathbf{r}$ by the auto-correlated $m = 0$ component, whereas the $m = 1$ component dominates for larger $\mathbf{r}$. For even larger $\mathbf{r}$, the $m = 2$ component can also come into play which can eventually result in a second ring of auto-correlation.

Returning to the more precise, yet numerically more demanding coherence density approach, we show in Fig. 3 the situation in the far field. All parameters are identical to those used in Fig. 1. In the far field as well we observe a ring-like structure of the cross-correlation function $\Gamma(-\mathbf{f}, \mathbf{f})$, where $\mathbf{f}$ stands for the spatial coordinates in the far field. The intensity distribution in the far field can also show a local minimum in the center of the beam, contrary to what one would obtain if the vortex was propagating through a linear medium [3], and also in contrast to the result we would obtain if we were propagating a light beam without topological charge. This emphasizes the importance of the interaction between the beam coherence function and the nonlinearity.

In conclusion, we have shown that the phase singularities in the spatial coherence function found earlier in the linear optics can survive the propagation through nonlinear media when the singular beam creates an incoherent vortex soliton. Our results emphasize the importance of the spatial coherence function in the studies of the propagation of incoherent singular beams. Not only the phase structure, but also the intensity distribution strongly depends on the initial form of the coherence function of the light beam as it enters a nonlinear medium.

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