ADAPTED RANDOM PERTURBATIONS FOR NON-UNIFORMLY EXPANDING MAPS

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Abstract. We obtain stochastic stability of $C^2$ non-uniformly expanding one-dimensional endomorphisms, requiring only that the first hyperbolic time map be $L^p$-integrable for $p > 3$. We show that, under this condition (which depends only on the unperturbed dynamics), we can construct a random perturbation that preserves the original hyperbolic times of the unperturbed map and, therefore, to obtain non-uniform expansion for random orbits. This ensures that the first hyperbolic time map is uniformly integrable for all small enough noise levels, which is known to imply stochastic stability. The method enables us to obtain stochastic stability for a class of maps with infinitely many critical points. For higher dimensional endomorphisms, a similar result is obtained, but under stronger assumptions.

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1. Introduction

The main goal of Dynamical systems theory is the description of the typical behaviour of orbits as time goes to infinity, and to understand how this behaviour changes under small perturbations of the system.

Given a map \( f \) from a manifold \( M \) into itself, a central concept is that of physical measure, a \( f \)-invariant probability measure \( \mu \) whose ergodic basin

\[
B(\mu) = \left\{ x \in M : n^{-1} \sum_{j=0}^{n-1} \phi(f^j(x)) \longrightarrow \int \phi \, d\mu \right\} \quad (1)
\]

has positive volume or Lebesgue measure, which we write \( \lambda \) and take as the measure associated with any non-vanishing volume form on \( M \).

The stability of physical measures under small variations of the map allows for small errors along orbits not to disturb too much the long term behavior, as measured by asymptotic time averages of continuous functions along orbits. When considering practical systems we cannot avoid external noise, so every realistic mathematical model should exhibit these stability features to be able to deal with uncertainty about parameter values, observed initial states and even the specific mathematical formulation of the model itself.

We investigate, under the probabilistic point of view, which asymptotic properties of a dynamical system are preserved under random perturbation.

Random perturbations and their features were first studied in 1945 by Ulam and von Neumann, in [30]. The focus of this work are non-uniformly expanding transformations which were introduced by Alves-Bonatti-Viana in [4], and whose ergodic properties are now well established; see for instance [1, 2, 8, 7]. Here we show that the asymptotic behavior of these transformations is preserved when randomly perturbed in an adapted way to their first times of expansion, under a condition: that the first time of expansion is \( L^p \)-integrable with respect to Lebesgue measure; see next sections for precise definitions and statements.

The interest in this kind of stochastic stability condition lies in the fact that known conditions of stochastic stability for non-uniformly expanding maps are expressed as conditions on the random perturbations of the map and not solely on the original unperturbed dynamics. We mention the joint works with Alves [2] and Vasquez [3], and also the recent work by Alves and Vilarinho [8].

The uniformly hyperbolic case, studied by Kifer in [20, 19] (among others), is much simpler: uniformly hyperbolic systems are stochastically stable under a broad range of random perturbations without further conditions. Other cases with the same features, which we may say are “almost uniformly hyperbolic systems”, where studied in joint works with Tahzibi, in [12, 13].
Here, we present a sufficient condition for stochastic stability of non-uniformly expanding transformations that relies only on the dynamics of the unperturbed map, for a simple type of random perturbation that is adapted to the dynamics. This allows us to treat some exceptional cases.

Recently Shen [26] obtained stochastic stability for unimodal transformations under very weak assumptions, but does not cover the case of transformations with infinitely many critical points; and Shen together with van Strien in [27] obtained strong stochastic stability for the Manneville-Pomeaux family of intermittent maps, answering questions raised in [12].

Our method allows us to obtain stochastic stability for non-uniformly expanding endomorphisms having slow recurrence to the critical set, encompassing the family of infinite-modal applications presented in [24]. We also obtain stochastic stability (in the weak sense, see precise statements in the next sections) for intermittent maps but in a restricted interval of parameter values; see Section 2.3.

1.1. Setting and statement of results. We consider $M$ to be a $n$-torus, $\mathbb{T}^n = (S^1)^n$, for some $n \geq 1$ and $\lambda$ a normalized volume form in $\mathbb{T}^n$, which we call Lebesgue measure. This can be identified with the restriction of Lebesgue measure on $\mathbb{R}^n$ to the unit cube.

We write $d$ for the standard distance function on $\mathbb{T}^n$ in what follows and $\| \cdot \|$ for the standard Euclidean norm on $\mathbb{R}^n$ which can be identified with the tangent space at any point of $\mathbb{T}^n$.

We let $f : \mathbb{T}^n \to \mathbb{T}^n$ be a local $C^2$ diffeomorphism outside a non-degenerate critical set $C$, that is, $C = \{x \in M : \det Df(x) = 0\}$ and $f$ behaves as the power distance to $C$: there are constants $B > 1$ and $\beta > 0$ satisfying

(S1) $\frac{1}{B} \cdot d(x, C)^\beta \leq \frac{\|Df(x)\|}{\|\nu\|} \leq B \cdot d(x, C)^{-\beta}, \forall \nu \in T_x M$;

(S2) $|\log \|Df(x)^{-1}\| - \log \|Df(y)^{-1}\|| \leq B \frac{d(x, y)}{d(x, C)^\beta}$;

(S3) $|\log |\det Df(x)^{-1}\| - \log |\det Df(y)^{-1}\|| \leq B \frac{d(x, y)}{d(x, C)^\beta}$

for all $x, y \in M \setminus C$ with $(x, y) < \frac{1}{2}(x, C)$.

We say that $f$ is non-uniformly expanding if there is a constant $c > 0$ such that:

$$\limsup_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(j(x))^{-1}\| \leq -c < 0 \quad \text{for} \quad \lambda - \text{a.e.} \ x \in M. \quad (2)$$

We need to control the recurrence to the critical set in order to obtain nice ergodic properties. We say that $f$ has a slow recurrence to critical set if, for any given $\gamma > 0$ there exists $\delta > 0$ such that

$$\limsup_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} -\log d_\delta(f^j(x), C) \leq \gamma, \quad \text{for} \quad \lambda - \text{a.e.} \ x \in M, \quad (3)$$

where $d_\delta$ is the $\delta$-truncated distance to $C$, defined as $d_\delta(x, C) = d(x, C)$ if $d(x, C) < \delta$ and $d_\delta(x, C) = 1$ otherwise.
We recall the concept of physical measure. For any $f$-invariant probability measure $\mu$ we write $B(\mu)$ for the basin of $\mu$ as in (1). We say that a $f$-invariant measure $\mu$ is physical if its basin $B(\mu)$ has positive Lebesgue measure: $\lambda(B(\mu)) > 0$.

Roughly speaking, physical measures are those that can be “seen” by calculating the time average of the values of a continuous observable along the orbits the points on a subset with positive Lebesgue measure. Clearly Birkhoff’s Ergodic Theorem ensures that $\mu(B(\mu)) = 1$ whenever $\mu$ is $f$-ergodic. We note that every $f$-invariant ergodic probability measure $\mu$ which is also absolutely continuous with respect to Lebesgue measure, i.e. $\mu \ll \lambda$, is a physical measure.

The previous conditions on $f$ ensure that Lebesgue almost all points behave according to some physical measure.

**Theorem 1.1** (Theorem C, [4]). Let $f$ be $C^2$ diffeomorphism away from a non-degenerate critical set, which is also a non-uniformly expanding map whose orbits have slow recurrence to the critical set. Then there is a finite number of $f$-invariant absolutely continuous ergodic (physical) measures $\mu_1, \ldots, \mu_p$ whose basins cover a set of full measure, that is $\lambda(M \setminus (B(\mu_1) \cup \cdots \cup B(\mu_p))) = 0$. Moreover, each $f$-invariant absolutely continuous probability measure $\mu$ can be written as a convex linear combination the physical measures: there are $\alpha_1 = \alpha_1(\mu), \ldots, \alpha_p = \alpha_p(\mu) \geq 0$ such that $\sum \alpha_i = 1$ and $\mu = \sum \alpha_i \mu_i$.

**Remark 1.2.** Pinheiro [25] showed that the same conclusions of Theorem 1.1 can be obtained by replacing the of non-uniform expansion condition (2) by the weaker condition

$$\liminf_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^j(x))^{-1}\| \leq -c < 0, \quad \text{for } \lambda - \text{a.e. } x \in M.$$  

The proof of this fact involves showing that (4) implies (2). Therefore, all the arguments used in this paper remain valid in the more general setting of condition (4) replacing condition (2).

1.2. Random perturbations and stochastic stability. We let $B = B(0,1)$ denote the unitary ball centered at the origin 0 in $\mathbb{R}^n$, set $X = \overline{B}$ and $\mathcal{F} = \{f_t : M \to M; t \in X\}$ a parametrized family of maps. We write $f_t(x) = f(t, x), (t, x) \in X \times M$ and assume in what follows that $f_0 = f$ is a map in the setting of Theorem 1.1.

We consider also the family of probability measures $(\theta_\varepsilon)_{\varepsilon > 0}$ in $X$ given by the normalized restriction of Lebesgue measure to the $\varepsilon$-ball $B(0, \varepsilon)$ centered at 0 in $\mathbb{R}^n$, as follows

$$\theta_\varepsilon = \frac{\lambda|_{B(0,\varepsilon)}}{\lambda(B(0,\varepsilon))}. \quad (5)$$

This family is such that $\text{supp}(\theta_\varepsilon)_{\varepsilon > 0}$ is a nested family of compact and convex sets satisfying $\text{supp}(\theta_\varepsilon) \to 0$. Setting $\Omega = X^\mathbb{N}$ the space of sequences in $X$, the random iteration of $\mathcal{F}$ is defined by

$$f^{\omega}_{\bar{t}}(x) = (f_{t_n} \circ f_{t_{n-1}} \circ \cdots \circ f_{t_1})(x), \quad \bar{t} = (t_1, t_2, \ldots) \in \Omega, x \in M.$$
To define the notions of stationary and ergodic measure we consider the skew-product

\[ F : \Omega \times M \to \Omega \times M \]

\[ (\sigma(t), f_t(x)) \]

where \( \sigma : \Omega \to \Omega \) is a standard left shift, and the infinite product measure \( \theta^N \) on \( \Omega \), which is a probability measure on the Borel subsets of \( \Omega \) in the standard product topology.

From now on, for each \( \epsilon > 0 \), we refer to \( (f_\epsilon, \theta^N_\epsilon) \) as a random dynamical system with noise of level \( \epsilon \).

**Definition 1.5** (Stationary measure). A measure \( \mu^\epsilon \) is a stationary measure for the random system \( (f_\epsilon, \theta^N_\epsilon) \) if

\[ \int \phi \, d\mu^\epsilon = \int \int \phi(f_t(x)) \, d\mu^\epsilon(x) \, d\theta_\epsilon(t), \quad \text{for all} \quad \phi \in C^0(M, \mathbb{R}). \]

**Remark 1.4.** If \( (\mu^\epsilon)_{\epsilon > 0} \) is a family of stationary measures having \( \mu^0 \) as a weak* accumulation point when \( \epsilon \searrow 0 \), then from (1.3) and the convergence of \( \text{supp} (\theta_\epsilon) \) to \( \{0\} \) it follows that \( \mu^0 \) must be invariant by \( f = f_0 \); see e.g. [2].

We say that \( \mu \) is a stationary measure if the measure \( \theta^N_\epsilon \times \mu \) is \( F \)-invariant. Moreover, we say that a stationary measure \( \mu \) is ergodic if \( \theta^N_\epsilon \times \mu \) is \( F \)-ergodic.

**Definition 1.5.** We say that \( f, \) in the setting of Theorem 1.1, is stochastically stable under the random perturbation given by \( (f_\epsilon, \theta^N_\epsilon)_{\epsilon > 0} \) if, for all weak* accumulation points \( \mu^0 \) of families \( (\mu^\epsilon)_{\epsilon > 0} \) of stationary measures for the random dynamical system \( (f_\epsilon, \theta^N_\epsilon) \) when \( \epsilon \searrow 0 \), we have that \( \mu^0 \) belongs to the closed convex hull of \( \{\mu_1, \ldots, \mu_p\} \). That is, for all such weak* accumulation points \( \mu^0 \) there are \( \alpha_1 = \alpha_1(\mu^0), \ldots, \alpha_p = \alpha_p(\mu^0) \geq 0 \) such that \( \sum \alpha_i = 1 \) and \( \mu^0 = \sum \alpha_i \mu_i \).

In this work we consider additive perturbations given by families of maps with the following form

\[ f_t(x) = f(x) + t\zeta(x) \]

where \( \zeta : M \to \mathbb{R}^+ \) is Borel measurable and locally constant at \( \lambda \)-almost every point.

**Remark 1.6.** For such additive perturbations we have \( Df_t(x) = Df(x) \) for all \( t \in \Omega \) and \( \lambda \)-a.e. \( x \in M \).

1.3. **First hyperbolic time map and adapted random perturbations.** The following is the fundamental concept in this work.

**Definition 1.7** (Hyperbolic time). Given \( \sigma < 1 \) and \( \delta > 0 \), we say that \( n \) is a \((\sigma, \delta)\)-hyperbolic time for \( x \in M \) if

\[ \prod_{j=n-k}^{n-1} \|Df(f^j(x))^{-1}\| \leq \sigma^k \quad \text{and} \quad d_\delta(f^{n-k}(x), C) \geq \sigma^{bk} \quad \text{for all} \quad 1 \leq k \leq n, \]

where \( b = \min\{1/2, 1/2\beta\} \) and \( \beta \) is the constant given in the non-degenerate conditions (S1)-(S2).
The notion of hyperbolic times was defined in [4]. To explain our Main Theorem we cite the following technical result.

**Lemma 1.8** (Lemma 5.4 in [4]). Let $f$ be a $C^2$ local diffeomorphism away from a non-degenerate critical set, which satisfies the non-uniform expansion condition (2) with $c = 3 \log \sigma$ for some $0 < \sigma < 1$ and also the slow recurrence condition (3).

Then there exist $\theta_0, \delta > 0$ depending on $\sigma$ and $f$, such that for $\lambda$-a.e. $x$ and each big enough $N \geq 1$, there are $(\sigma, \delta)$-hyperbolic times $1 \leq n_1 < \cdots < n_l \leq N$ for $x$ with $l \geq \theta_0 N$. Moreover, the hyperbolic times $n_i$ satisfy

$$\sum_{j=n_i-k}^{n_i-1} \log d_\delta(f^j(x), C) \geq bk \log \sigma, \quad \text{for all} \quad 0 \leq k \leq n_i, 1 \leq i \leq l.$$  

**Remark 1.9.** Let $\mathcal{G}$ the set of points $x \in M$ that have no hyperbolic time. Then $\lambda(\mathcal{G}) = 0$ after Lemma 1.8. Thus, if $x$ has only finitely many hyperbolic times, then some iterate of $x$ belongs to $\mathcal{G}$. Hence, the subset of points with finitely many hyperbolic times is contained in $\bigcup_{j \geq 0} f^{-j}(\mathcal{G})$.

Moreover, $\lambda(f^{-1}(\mathcal{G})) = 0$ because $f$ is a local diffeomorphism away from a critical/singular set with zero $\lambda$-measure. Therefore, $\lambda$-a.e. $x \in M$ has infinitely many hyperbolic times.

Hence, in our setting we have that Lebesgue almost every point has infinitely many $(\sigma, \delta)$-hyperbolic times. Thus we may define the map $h : M \to \mathbb{Z}^+$ such that for $\lambda$-a.e. point $x$ the positive integer $h(x)$ is the first hyperbolic time of $x$. We say $h$ is the first hyperbolic time map.

In our main theorem, we will see that is possible to randomly perturb a non-uniformly expanding map so that almost all randomly perturbed orbits have infinitely many hyperbolic times but also the same hyperbolic times as the non-perturbed map. We start with a one-dimensional version.

**Theorem A.** Let $f : \mathbb{T}^1 \to \mathbb{T}^1$ be a $C^2$ diffeomorphism away from a non-degenerate critical set, which is also a non-uniformly expanding map whose orbits have slow recurrence to the critical set. Let us assume that $f$ has a dense orbit and that the first hyperbolic time map is $L^p$-integrable for some $p > 3$, that is, $\int h(x)^p \, d\lambda(x) < \infty$.

Then $f$ is stochastically stable for a family of adapted random perturbations. More precisely, there exists $\zeta : \mathbb{T}^1 \to \mathbb{R}^+$ measurable and locally constant such that the family (7) generates a family of random perturbations $(f_c \theta_c^N)_{c > 0}$ for which $f$ is stochastically stable.

The same proof gives the following result for endomorphisms of compact manifolds in higher dimension, with a technical assumption on the rate of decay of the measure of sets of points with first hyperbolic time.

**Theorem B.** Let $f : \mathbb{T}^n \to \mathbb{T}^n$ be a $C^2$ diffeomorphism away from a non-degenerate critical set, which is also a non-uniformly expanding map whose orbits have slow recurrence to the critical set, where $n > 1$. If the first hyperbolic time map satisfies

$$\sum_{n \geq 1} \sum_{j=0}^{n-1} \lambda(f^j(h^{-1}(n))) < \infty,$$  

(9)
then $f$ is stochastically stable for a family of adapted random perturbations given by (7).

1.4. Comments and organization of the text. The method of proof relies on showing that the random adapted perturbation preserve hyperbolic times in such a way that the first hyperbolic time map of the random system is the same as the first hyperbolic time map of the original system. In this way, we can use the main result of [2] to prove (weak*) stochastic stability.

This construction of the adapted random perturbation depends on an assumption of integrability of the first hyperbolic time map for one-dimensional non-uniformly expanding maps. For higher dimensional maps, condition (9) is needed and apparently much difficult to check.

**Conjecture 1.** A non-uniformly expanding map having a sufficiently fast rate of decay of correlations satisfies the summability condition (9).

We presented the results using a uniform measure for $\theta$ but many simple generalizations are possible assuming only that $\theta << \lambda$ and $\text{supp}(\theta) \to \{0\}$.

We also avoided technical complexities by considering only maps on tori, on which it is clear how to make additive perturbations in the form (7). However, it is possible (although technically more involved) to make similar perturbations in any compact manifold, arguing along the lines of [2, Example 2]. We focus on additive perturbations on parellelizable manifolds to present the ideas in a simple form.

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2. Examples of Application

Theorem A ensures stochastic stability for any non-uniformly expanding map that has slow recurrence to the critical set with the first hyperbolic function in $L^p$ for $p > 3$. We present natural conditions on the speed of expansion that imply this integrability condition and use this to obtain examples where our results apply.

We note that, from slow recurrence to the critical set and non-uniform expansion, Lemma 1.8 ensures that for $c = -\log \sigma > 0$ and small $\gamma, \delta > 0$ the following values are well defined $\lambda$-a.e.

$$D(x) = \min \left\{ k \geq 1 : \frac{1}{n} \sum_{j=0}^{n-1} - \log \delta(f^j(x), C) \leq \gamma \quad \text{for all} \quad n \geq k \right\}; \quad \text{and}$$

$$E(x) = \min \left\{ k \geq 1 : \frac{1}{n} \sum_{j=0}^{n-1} \log \|D f(f^j(x))^{-1}\| \geq \frac{c}{3} \quad \text{for all} \quad n \geq k \right\}.$$

We combine these two estimates in the set

$$\Gamma_n = \{ x \in M : D(x) > n \quad \text{and} \quad E(x) > n \}.$$
We now observe that, trivially from the definitions, every point in \( \Gamma_n \) has a first \((\sigma,\delta)\)-hyperbolic time of at most \( n \), thus

\[
h^{-1}(\{n\}) \subset h^{-1}((1,2,\ldots,n)) \subset \Gamma_n.
\]

**Remark 2.1.** If for some constant \( \kappa > 0 \) and \( q > 4 \) we have \( \lambda(\Gamma_n) \leq \kappa n^{-q} \) for all sufficiently large \( n \), then \( h \in L^p(\lambda) \) for some \( p > 3 \), since for all small enough \( \epsilon > 0 \) we have

\[
\sum_{n>m} n^{q-1-\epsilon} \lambda(h^{-1}(\{n\})) \leq \kappa \sum_{n>m} n^{q-1-\epsilon} < \infty \text{ for some } m > 1.
\]

**2.1. Non-uniformly expanding maps with infinitely many critical points.** We now present the main motivating example of application of Theorem A: maps with infinite critical points. We consider the family \( f_t : S^1 \to S^1 \) from the work of Pacifico-Rovella-Viana [24]. This family is obtained from the map \( \hat{f} : [-\epsilon_1,\epsilon_1] \to [-1,1] \) given by

\[
\hat{f}(z) = \begin{cases} 
az^{\alpha} \sin(\beta \log(1/|z|)) & \text{if } z > 0 \\
-a|z|^\alpha \sin(\beta \log(1/|z|)) & \text{if } z < 0,
\end{cases}
\]

where \( a > 0, 0 < \alpha < 1, \beta > 0 \) and \( \epsilon_1 > 0 \), see Figure 1.

![Figure 1. Graph of the circle map \( f \).](image)

Maps \( \hat{f} \) as above have infinitely many critical points, of the form

\[
x_k = \hat{x} \exp(-k \pi/\beta) \text{ and } x_{-k} = -x_k \text{ for each large } k > 0
\]

where \( \hat{x} = \exp \left( -\frac{1}{\beta} \tan^{-1} \frac{\beta}{\pi} \right) > 0 \) is independent of \( k \). Let \( k_0 \geq 1 \) be the smallest integer such that \( x_k \) is defined for all \( |k| \geq k_0 \), and \( x_{k_0} \) is a local minimum.

We extend this expression to the whole circle \( S^1 = I/\{-1 \sim 1\} \), where \( I = [-1,1] \), in the following way. Let \( \tilde{f} \) be an orientation-preserving expanding map of \( S^1 \) such that \( \tilde{f}(0) = 0 \) and \( \tilde{f}' > \tilde{\sigma} \) for some constant \( \tilde{\sigma} >> 1 \). We define \( \epsilon = 2 \cdot x_{k_0}/(1 + e^{-\pi/\beta}) \), so that \( x_{k_0} \) is the middle point of the interval \((e^{-\pi/\beta}\epsilon, \epsilon)\) and fix two points \( x_{k_0} < \hat{y} < \tilde{y} < \epsilon \), with

\[
|\hat{f}'(\hat{y})| >> 1 \quad \text{and also} \quad 2 \cdot \frac{1-e^\epsilon}{1+e^{-\pi/\beta}} x_{k_0} > \hat{y} > x_{k_0},
\]
where $\tau$ is a small positive constant and we take $k_0 = k_0(\tau)$ sufficiently big (and $\epsilon$ small enough) in order that (12) holds. Then we take $f$ to be any smooth map on $S^1$ coinciding with $\hat{f}$ on $[-\hat{y}, \hat{y}]$, with $\tilde{f}$ on $S^1 \setminus [-\tilde{y}, \tilde{y}]$, and monotone on each interval $\pm[\hat{y}, \tilde{y}]$.

Finally let $f_t$ be the following one-parameter family of circle maps unfolding the dynamics of $f = f_0$

$$f_t(z) = \begin{cases} f(z) + t & \text{for } z \in (0, \epsilon) \\ f(z) - t & \text{for } z \in [-\epsilon, 0) \end{cases}$$

(13) for $t \in (-\epsilon, \epsilon)$. For $z \in S^1 \setminus [-\epsilon, \epsilon]$ we assume only that $|\frac{\partial}{\partial z} f_t(z)| \geq 2$.

From the works [24] together with [11], it is known that for a positive Lebesgue measure subset $P$ of parameters $t$ the map $f_t$ has a dense orbit, is non-uniformly expanding with slow recurrence to the critical set $C = \{0\} \cup \{x_k : |k| \geq k_0\}$, admits a unique absolutely continuous invariant probability measure $\mu_t$ and the corresponding tail set $\Gamma_t^n$ satisfies $\lambda(\Gamma_t^n) \leq Ce^{-\xi n}$ for some constants $C, \xi > 0$; see [24, Theorem A] and [11, Theorems A, B and C].

Hence, from Remark 2.1 we can apply Theorem A to each of these maps $f_t$.

**Corollary 2.2.** Given $t_0 \in P$, the map $f = f_{t_0}$ is stochastically stable for the adapted family of random perturbations $(f_t, \theta^N_{\epsilon})$ obtained according to Theorem A.

This is the first result on stochastic stability of one-dimensional maps with infinitely many critical points.

### 2.2. Non-uniformly expanding quadratic maps.

The quadratic family $f_a : [-1, 1] \to [-1, 1]$ given by $f_a = 1 - ax^2$ for $0 < a \leq 2$ provides a class of maps satisfying the hypothesis of Theorem A. Indeed, Jakobson [18] and Benedicks-Carleson [15] prove the existence of a physical measure for a positive Lebesgue measure subset of parameters $a \in (0, 2]$ for which $f_a$ is non-uniformly expanding with slow recurrence to the critical point; Young [31] and, more recently, Freitas [16] obtain exponential decay of the tail sets $\Gamma_n$. From Remark 2.1 we can apply Theorem A for all the maps in the positive Lebesgue measure subset of parameters found by Jacobson and Benedicks-Carleson, obtain stochastic stability for this class of maps. We note that strong stochastic stability was obtained for the same class in the work of Baladi-Viana [14].

### 2.3. Intermittent Maps.

Our results enables us also to deduce stochastic stability for a class of intermittent applications [22], where this property was obtained for maps $C^{1+\alpha}$ but with the condition that $\alpha \geq 1$; see [12]. Recently Shen, together with van Strien in [27], obtained strong stochastic stability for the Manneville-Pomeaux family of intermittent maps, answering the questions raised in [12].

Consider $\alpha > 0$ and the map $T_\alpha : [0, 1] \to [0, 1]$ given by:

$$T_\alpha(x) = \begin{cases} x + 2^\alpha x^{1+\alpha}, & \text{if } x \in [0, \frac{1}{2}) \\ x - 2^\alpha (1 - x)^{1+\alpha}, & \text{if } x \in [\frac{1}{2}, 1] \end{cases}.$$
This map is a $C^{1+\alpha}$ local diffeomorphism of $S^1 := [0, 1]/\{0 \sim 1\}$, so there are no critical points. The unique fixed point is 0 with $DT_\alpha(0) = 1$. If $\alpha \geq 1$, then the Dirac mass in zero $\delta_0$ is the unique physical probability measure and so the Lyapunov exponent in Lebesgue almost every point is zero; see [28]. But, for $0 < \alpha < 1$, there exists a unique absolutely continuous invariant probability $\mu$ which is physical and whose basin has full Lebesgue measure. To deduce stochastic stability for $\alpha$ in a subinterval of $(0, 1)$, we need some definitions and results.

Given a $T_\alpha$-invariant and ergodic probability measure $\mu$ and $\epsilon > 0$ we define the large deviation in time $n$ of the time average of the observable $\varphi$ from its spatial average as

$$LD_\mu(\varphi, \epsilon, n) = \mu \left\{ x : \left| \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) - \int \varphi d\mu \right| > \epsilon \right\}$$

We note that Birkhoff’s Ergodic Theorem ensures $\lim_{n \to \infty} LD_\mu(\varphi, \epsilon, n) = 0$ and the rate of this convergence is a relevant quantity.

Since $T_\alpha$ is a local diffeomorphism we have $\Gamma_n = \{ x \in S^1 : E(x) > n \}$ and this is naturally a deviation set for the time averages of $\varphi = \log |DT_\alpha|$: if $\mu_\alpha$ is the unique absolutely continuous $T_\alpha$-invariant probability, then the Lyapunov exponent $\lambda = \int \varphi d\mu > c$, where $c > 0$ is the constant in the definition of non-uniform expansion (12), and so for all large enough $n > 1$ and small enough $\epsilon > 0$

$$LD_\mu(\log |DT_\alpha(x)|, \epsilon, n) \geq \mu(\Gamma_n). \tag{14}$$

To estimate $\mu(\Gamma_n)$ we now relate $LD_\mu$ with the rate of decay of correlations. Let $\mathcal{B}_1, \mathcal{B}_2$ denote Banach spaces of real valued measurable functions defined on $M$. We denote the correlation of non-zero functions $\varphi \in \mathcal{B}_1$ and $\psi \in \mathcal{B}_2$ with respect to a measure $\mu$ as

$$\text{Cor}_\mu(\varphi, \psi) = \frac{1}{\|\varphi\|_{\mathcal{B}_1} \|\psi\|_{\mathcal{B}_2}} \left| \int \varphi \psi d\mu - \int \varphi d\mu \int \psi d\mu \right|.$$ 

We say that we have decay of correlations, with respect to the measure $\mu$, for observables in $\mathcal{B}_1$ against observables in $\mathcal{B}_2$ if, for every $\varphi \in \mathcal{B}_1$ and every $\psi \in \mathcal{B}_2$ we have

$$\text{Cor}_\mu(\varphi, \psi \circ f^n) \to 0. \quad n \to \infty.$$ 

The following result from [23] allows us to relate decay of correlations with large deviations; see also [6]. We say that a measure $\mu$ is $f$-non-singular if for all measurable sets $A$ such that $\mu(A) = 0$, then $\mu(f^{-1}(A)) = 0$.

**Theorem 2.3** (23,6). Let $f : M \to M$ preserve an ergodic probability measure $\mu$ with respect to which $f$ is non-singular. Let $\mathcal{B} \subset L^\infty(\mu)$ be a Banach space with norm $\| \cdot \|_{\mathcal{B}}$ and $\varphi \in \mathcal{B}$. Let $\beta > 0$ and suppose that there exists $\kappa > 0$ such that for all $\psi \in L^\infty(\mu)$ we have $\text{Cor}_\mu(\varphi, \psi \circ f^n) \leq \kappa \cdot n^{-\beta}$. Then, for every $\epsilon > 0$, there exists $C = C(\varphi, \epsilon) > 0$ such that $\lim_{n \to \infty} LD_\mu(\varphi, \epsilon, n) \leq C n^{-\beta}$.

We now observe that the absolutely continuous $T_\alpha$-invariant probability measure $\mu_\alpha$ is $f$-non-singular and that the following estimate for the rate of decay of correlations is known.
Theorem 2.4 (Theorem 4.1 in [21]). For all \( \psi \in L^\infty \) and \( \varphi \in C^1([0,1]) \) such that \( \int \varphi \, d\mu = 0 \) we have:
\[
\left| \int (\psi \circ T^n_\alpha) \cdot \varphi \, d\mu \right| \leq A(\|\varphi\|_{C^1}) \cdot \|\psi\|_\infty \cdot n^{1-1/\alpha} (\log n)^{1/\alpha},
\]
where \( A : \mathbb{R} \rightarrow \mathbb{R} \) is an affine map.

Hence, since \( \log |DT_\alpha(x)| \) is a bounded continuous function on \([0,1]\), there is a constant \( C > 0 \) such that
\[
\text{Cor}_\mu(\log |DT_\alpha(x)|, \psi \circ T^n_\alpha) < C n^{1-1/\alpha} \log n^{1/\alpha}.
\]

From Theorem 2.3 and relation (14) we deduce that, for every \( \delta > 0 \), we have a constant \( C_1 > 0 \) such that
\[
\mu(\Gamma_n) < C_1 \cdot n^{(1-1/(\alpha+\delta))}.
\] (15)

Since \( \mu \ll \lambda \), we have \( d\mu = h \, d\lambda \) with a density function \( h \) which, from [17, Theorem A], is bounded, strictly positive and, for a neighborhood \( I_0 \) of 0 there are constants \( R > 0 \) and \( \sigma_0 = \lim_{x \to 0} \sum_{x_1 \in T_\alpha^{-1}(x) \cap I_0} \frac{h(x_1)}{DT_\alpha(x_1)} \) such that \( |x^\alpha \cdot h(x) - \sigma_0| \leq R \cdot x^\alpha \). This enables us to find \( \kappa > 0 \) such that \( \lambda(\Gamma_n) \leq \kappa \mu(\Gamma_n) \) which, together with (15) provides a constant \( C > 0 \) such that for all small \( \delta > 0 \) and large \( n \)
\[
\lambda(\Gamma_n) < C \cdot n^{(1-1/(\alpha+\delta))}.
\]

We therefore have for \( p > 3 \), since \( \delta > 0 \) may be take arbitrarily small
\[
\sum_{n=1}^{\infty} n^p \cdot \lambda(\Gamma_n) < C \sum_{n=1}^{\infty} n^{p+1-1/(\alpha+\delta)} < \infty \quad \text{for all} \quad 0 < \alpha \leq \frac{1}{p+2}.
\]

Thus, for any \( p > 3 \), we get for \( 0 < \alpha < \frac{1}{5} \) the \( L^p \) integrability of the first hyperbolic time map with respect to \( \lambda \) and, from Theorem we obtain

**Corollary 2.5.** All intermitent maps \( T_\alpha \) with parameters \( 0 < \alpha < \frac{1}{5} \) are stochastically stable under adapted random perturbations.

### 3. Adapted random perturbations

Here we construct adapted random perturbations. These perturbations are constructed by an adequate choice of hyperbolic times along almost all orbits. Then we show that these specially chosen hyperbolic times are preserved under the adapted random perturbations in such a way that the random map is non-uniformly expanding and has slow-recurrence for random orbits. In addition, the hyperbolic times for a point \((\xi, x) \in \Omega\) under the adapted random perturbations are the same as the hyperbolic times of \( x \) for the unperturbed dynamics.

The only assumption is that the original unperturbed map admits a pair \((\sigma, \delta)\), with \( 0 < \delta, \sigma < 1 \), satisfying: the first \((\sigma, \delta)\)-hyperbolic time map \( h \) is defined \( \lambda \)-almost everywhere and \( h \) is \( L^p \)-integrable for some \( p > 3 \), i.e., \( \sum_{n=1}^{\infty} n^p \lambda(h^{-1}(n)) < \infty \).

In what follows we fix \((\sigma, \delta)\) as above and write hyperbolic time to mean \((\sigma, \delta)\)-hyperbolic time.
Definition 3.1. The adapted hyperbolic time of $x \in M \setminus C$ is the number
\[ H(x) := \sup \{ h(z) - l; x = f^l(z), z \in M \text{ and } l \geq 0 \} \]
where $h : M \to \mathbb{Z}^+$ is the first hyperbolic time function.

Note that $H(x)$ is a hyperbolic time for $x$. In fact, if $x = f^l(z)$ for $l \geq 1$ and some point $z$, and $h(z)$ is the first hyperbolic time of $z \in M$, then $h(z) - l$ is a hyperbolic time for $f^l(z) = x$. Moreover, it is clear that $H(x) \geq h(x)$ if $h(x)$ is finite.

To check that $H$ is finite almost everywhere, we note that
\[ H(x) \leq \sup \left\{ n \in \mathbb{Z}^+ : x \in \bigcup_{i=0}^{n-1} f^i(h^{-1}(n)) \right\}. \] (16)

Since for a one-dimensional map $f$ we have $|\det Df| = ||Df|| = |Df|$, then the assumption $h \in L^p(\lambda)$ with $p > 3$ implies (9) in the one-dimensional setting.

Lemma 3.2. Let $f$ be a non-uniformly expanding one-dimensional map having slow recurrence to the non-degenerate critical set. Let us assume that the first hyperbolic time map satisfies $h \in L^p(\lambda)$ for some $p > 3$. Then $\sum_{n=1}^{\infty} \sum_{j=1}^{n-1} \lambda(f^{-1}(h^{-1}(n))) < \infty$.

Proof. We follow [5, Section 3]. We note that if $n \geq 1$ is a $(\sigma, \delta)$-hyperbolic time, then $|\det Df^n(x)| \geq a_n = \sigma^{-n}$. Let $q(x) = \min\{k \geq 1 : |\det Df^k(x)| \geq a_k\}$. Then $q(x) \leq h(x)$ and so $q \in L^p(\lambda)$ if $h \in L^p(\lambda)$.

Let $W_n = \{x \in M : q(x) > n\}$. Then $W_n \subset \bigcup_{m>n}h^{-1}(m)$ and so we can find constants $\kappa, C > 0$ such that
\[ \lambda(W_n) \leq \sum_{m>n} \lambda(h^{-1}(m)) \leq \sum_{m>n} \frac{\kappa}{mp} \leq \frac{C}{np^{-1}}. \]

Hence there exists $\beta > 0$ and $N \in \mathbb{N}$ such that $b_n = n^\beta$ satisfies $b_n \leq \min\{a_n, \lambda(W_n)^{-\epsilon}\}$ for all $n \geq N$ and some $0 < \epsilon < \frac{p-3}{p-1}$. In addition, we clearly have $b_n b_k \geq b_{n+k}$ for all big enough $k, n \in \mathbb{N}$. In this setting, $U_n = \{x \in M : |\det Df^n(x)| \geq b_n\}$ is such that
- $U_{n+1} U_n$ has full Lebesgue measure, since $T_n = \{x \in M : n \text{ is a } (\sigma, \delta)-\text{hyperbolic time}\}$ satisfies $h^{-1}(n) \subset T_n \subset U_n$, and
- if $x \in U_n$ and $f^n(x) \in U_m$, then $x \in U_{n+m}$

(i.e., $(U_n)_{n \geq 1}$ is a concatenated collection as defined in [5]). In addition, letting $\hat{q}(x) = \min\{n \geq 1 : x \in U_n\}$, we have again $\hat{q}(x) \leq h(x)$ in general. However, if $f$ is one-dimensional, then we obtain equality $\hat{q}(x) = h(x)$.

The choices of $U_n$ and the sequence $b_n$ ensure that $\sum_{n \geq N} \sum_{j=0}^{n-1} \lambda(f^{j}(\hat{q}^{-1}(n))) < \infty$; see [5, Section 3]. Moreover, in the one-dimensional setting, this series coincides with the one in the statement of the lemma. \hfill \Box

Under this summability condition we obtain the following.

Lemma 3.3 (Lemma 2.1 in [5]). If (9) is true, then $H(x) < \infty$ to $\lambda$-almost every $x \in M$. 

\[ \blacksquare \]
Proof. For \( \lambda \)-almost every \( x \in M \) we consider the set \( \mathcal{K}(x) = \{ f^j(x) \}_{j=0}^{h(x)-1} \), which we call a chain. Suppose that for some \( z \in M \) we have that \( z \) belongs to infinitely many chains \( \mathcal{K}(x_j) = \{ x_j, f(x_j), \ldots, f^{s_j-1}(x_j) \} \) for \( j \geq 1 \) where \( s_j = h(x_j) \) is the first hyperbolic time for \( x_j \) and \( s_j \to \infty \).

Now for each \( j \geq 1 \) we take \( 1 \leq r_j < s_j \) such that \( z = f^{r_j}(x_j) \) and claim that \( \lim r_j = \infty \). Indeed, otherwise, taking a subsequence of \( r_j \), we can assume that there is \( N > 0 \) such that \( r_j < N, \forall k \geq 1 \). But this implies that \( x_j \in \cup_{i=1}^{N} f^{-i}(z) \), \( \forall j \geq 1 \) and so the number of elements of \( \cup_{i=1}^{N} f^{-i}(z) \) is finite: \( \#(\cup_{i=1}^{N} f^{-i}(z)) < \infty \). However we are assuming that the number of chains is infinite. This contradiction proves the claim.

Hence \( r_j \to \infty \) and \( z = f^{r_j}(x_j) \subset f^{r_j}(h^{-1}(s_j)) \) and so we get

\[
z \in \cup_{n \geq k} \cup_{j=0}^{n-1} f^{j}(h^{-1}(s_j)), \forall k \geq 0.
\]

Since \( \sum_{n \geq 1} \sum_{j=0}^{n-1} \lambda(f^{j}(h^{-1}(n))) < \infty \), we obtain \( \lambda(\cup_{n \geq k} \cup_{j=0}^{n-1} f^{j}(h^{-1}(n))) \to 0 \) as \( k \to \infty \). Then the set of points belonging to infinitely many chains has null Lebesgue measure. Finally, from relation (16) the proof of the lemma is complete. \( \square \)

Note that it is not possible ensure that \( H(f(x)) = H(x) - 1 \) in general, because \( x \) and \( f(x) \) can be in orbits of different points, namely \( z \neq w \) whose first hyperbolic times do not satisfy the relation \( h(w) = h(z) - 1 \). Then the adapted hyperbolic time for \( f(x) \) can be bigger than \( H(x) - 1 \). However, note that \( H(f(x)) \) can not be smaller than \( H(x) - 1 \) because \( x \) already has \( H(x) \) as hyperbolic time. In any case we have the following important monotonicity property of our choice of adapted hyperbolic time

\[
H(f(x)) \geq H(x) - 1.
\] (17)

Similarly we obtain \( H(f^{j}(x)) \geq H(x) - j \) for \( 0 \leq j < H(x) \) as long as \( H(x) \) is finite.

**Lemma 3.4** (Lemm 5.2 in [4]). Given \( \sigma < 1 \) and \( \delta > 0 \), there is \( \delta_1 > 0 \) such that if \( n \) is a \((\sigma, \delta)\)-hyperbolic time for \( x \in M \setminus C \) then there exits a neighborhood \( V_n(x) \) of \( x \) such that:

1. \( f^n \) maps \( V_n \) diffeomorphically into the ball of radius \( \delta_1 \) centered at \( f^n(x) \).
2. For all \( 1 \leq k < n \) and \( y, z \in V_n(x) \)

\[
dist(f^{n-k}(y), f^{n-k}(z)) \leq \sigma^{k/2} \cdot dist(f^{n}(y), f^{n}(z)).
\]

By the definition of hyperbolic time, if \( n \) is a \( \sigma \)-hyperbolic time for a point \( x \in M \), then there are neighborhoods \( V_{n-j} \subset B_{bh_1}(f^{j}(x)) \) of \( f^{j}(x) \) which are sent in time \( j \) diffeomorphically into the ball \( B_{bh_1}(f^{n}(x)) \) for all \( 0 \leq j \leq n \).

**Lemma 3.5.** In our setting, for \( \lambda \)-almost every \( x \), there exists an open neighborhood \( V_{H}(x) \) of \( x \) such that \( H | V_{H}(x) \) is constant.

**Proof.** The subset \( Y \) of \( M \) of points having some hyperbolic time is such that \( \lambda(Y) = 1 \). Hence \( f^{-1}(Y) \) also has full \( \lambda \)-measure since \( f \) is a local diffeomorphism away from a critical/singular set with zero \( \lambda \)-measure. Therefore \( \lambda(\cap_{n \geq 1} (Y \cap f^{-n}(Y))) = 1 \) and we conclude that every point in the pre-orbit \( \cup_{n \geq 1} f^{-n}(\{x\}) \) of Lebesgue almost every point \( x \) has some hyperbolic time.
Let $X$ be the subset of $M$ such that $H(x) < \infty$ for all $x \in X$. We know that $\lambda(X) = 1$.

Let us now fix $x \in Y \cap X$. Hence we have $h(y) < \infty$ for every point $y$ in the pre-orbit of $x$ and, moreover, if $x = f^{k}(y)$ then $h(y) - k \leq H(x)$ by definition of $H(x)$.

It follows that the neighborhood $V_{h(y)}(y)$ of $y$ associated to the hyperbolic time $h(y)$ is such that $f^{k}(V_{h(y)}(y)) \supset V_{h(x)}(x)$, since $h(y) - k \leq H(x)$.

Therefore, for $x' \in V_{h(x)}(x) \subset f^{k}(V_{h(y)}(y))$ the inverse map $\varphi$ of $f^{k} \mid V_{h(y)}(y)$ is such that $\varphi(x') = y' \in V_{h(y)}(y)$. Thus $h(y') \leq h(y)$ (recall that $h(y')$ is the first hyperbolic time of $y'$ and $h(y)$ is already a hyperbolic time for $y'$). It follows that $h(y') - k \leq h(y) - k \leq H(x)$.

This argument is true of any element $y$ of the pre-orbit of $x$, whose neighborhood $V_{h(y)}(y)$ is sent by $f^{k}$ to a set covering $V_{h(x)}(x)$. Hence all pre-images of points $x' \in V_{h(x)}(x)$ respect the same inequality, that is, $H(x') \leq H(x)$. But the reverse inequality is also true by definition of $H$, since $x' \in V_{H(x)}(x)$ has $H(x)$ as an hyperbolic time. This completes the proof. \hfill \Box

**Remark 3.6.** We make the convention that $H(x) = 1$ wherever the supremum in Definition 3.7 is not finite.

**Remark 3.7.** Besides the obvious relation $H(x) \geq h(x)$ almost everywhere, we can say more in certain regions. Let us assume that $V$ is the largest open neighborhood of the critical set $C$ such that $|Df| (M \setminus V) > \sigma^{-}1$ and $V \cap f^{-1}(V) = \emptyset$. Then $H = h$ in $V$, since $h(x) \geq 2$ for almost all points $x \in V$ and all pre-orbits of $x$ have 1 as a first $\sigma$-hyperbolic time, which is smaller than $h(x) - 1$.

The above conditions on a neighborhood of the critical set are easily checked for non-uniformly expanding quadratic maps and, by [24, Section 4], this is also true for the infinite-modal family $f_{\mu}$ at every parameter of the positive Lebesgue measure subset $P$; see Section 2.

### 3.1. Preservation of hyperbolic times.

Now we show that hyperbolic times are preserved if we define a random perturbation adapted to the structure of hyperbolic times using $H$, as in (7) with $\zeta(x) = \xi e^{-nH(x)}$ for suitably chosen constants $\xi, \eta > 0$. We first define the notions of hyperbolic times and slow recurrence in our random setting.

#### 3.1.1. Random non-uniformly expanding maps and random slow recurrence.

We now define the analogous notions of non-uniform expansion and slow recurrence for random dynamical systems in our setting.

**Definition 3.8.** We say that a map is non-uniformly expanding map for random orbits if there exists a constant $c > 0$ such that for $e > 0$ sufficiently small and $\theta^{N}_{e} \times \lambda$-a.e. $(x, \omega)$ we have

$$\limsup_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^{j}(x))^{-1}\| \leq -c < 0.$$

**Definition 3.9.** We say that a random dynamical system $(f_{\omega}, \theta_{e})$ has slow recurrence to the critical set for random orbits if, for all small enough $\gamma > 0$, there exists $\delta > 0$ such that $\theta^{N}_{e} \times \lambda$-a.e. $(x, \omega)$ we have

$$\limsup_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} -\log d_{\delta}(f^{j}(x), C) \leq \gamma.$$

#### 3.1.2. Random hyperbolic times.

An definition of hyperbolic analogous to [17,27] can be made for the random system $(f_{i}, \theta_{e})$. 
Definition 3.10 (Random Hyperbolic Time). Given $\sigma \in (0, 1)$ and $\delta > 0$, we say that $n$ is a $(\sigma, \delta)$-hyperbolic time for a point $(\xi, x) \in \Omega \times M$ if:

$$\prod_{j=n-k}^{n-1} \|Df_{j,n,1}(f_{n,j}^{-1}(x))\| \leq \sigma^k \quad \text{and} \quad d_{\delta}(f_{n,k}^{-1}(x), C) \geq \sigma^{bk}, \quad \text{for all} \quad 1 \leq k \leq n.$$ 

Theorem 3.11. If $f$ is non-uniformly expanding with slow recurrence to the critical set in the interval or the circle, then for each $\delta > 0$ there is $\zeta : M \to \mathbb{R}^+$ measurable and locally constant such that the adapted random perturbation (7) satisfies: there exists $0 < \sigma < \delta < 1$ such that for $\lambda$-almost every point $x$ and all $t \in [-1/2, 1/2]^{\mathbb{N}}$ has $H(x)\sigma,\delta$-almost every point $x$ and all $t \in [-1/2, 1/2]^{\mathbb{N}}$ has $H(x)$ as $(\delta, \delta)$-hyperbolic time.

We assume that $f$ has a non-degenerate critical set $C$. We also assume without loss of generality in what follows that $B\delta^{1-\beta} \leq \log \sigma^{-1/2}$ and $\delta_1 = \frac{1}{2}\delta \leq \frac{1}{2}$, where $B, \beta > 0$ are given in the non-degeneracy conditions of $C$.

Remark 3.12. The same arguments and constructions presented in this section enable us to trivially obtain a version of Theorem 3.11 for the local diffeomorphism case, that is, the case where there are no critical (or singular) points: $C = \emptyset$.

Remark 3.13. Since by construction $h(x) \leq H(x)$, whenever $h(x)$ is finite, then we have for $\lambda$-a.e. $x$ that $V_n(x) \subset \tilde{V}_n(x)$ for all hyperbolic times $n$ of $x$ such that $h(x) \leq n \leq H(x)$.

Moreover, we have that the random orbit of $(\xi, x)$ has the same hyperbolic times $n$ of the unperturbed orbit of $x$ as long as $h(x) \leq n \leq H(x)$. In particular, the first hyperbolic time of $(\xi, x)$ is given by $h(x)$.

Lemma 3.14. There exists $\omega > \sigma^{-1/2}$ such that, if $n$ is a $(\sigma, \delta)$-hyperbolic time for $x$, then $\|Df^n(x)\| \leq \omega^n$.

Proof. Using the non-degenerate condition (S1) we get $\log \|Df(x)\| \leq \log B - \beta \log d(x, C)$. Hence, since $n$ is a hyperbolic time, we have from their construction that they satisfy (5) which implies

$$\log \|Df^n(x)\| \leq \sum_{j=0}^{n-1} \log \|Df^{j}(x)\| \leq n \log B - \beta \sum_{j=0}^{n-1} \log d(f^{j}(x), C)$$

$$\leq \log B^n + \beta \sum_{j=0}^{n-1} - \log d_{\delta}(f^{j}(x), C) + \beta \sum_{d(f^{j}(x), C) \geq \delta} - \log d(x, C)$$

$$\leq \log B^n + \beta \sum_{j=0}^{n-1} - \log d_{\delta}(f^{j}(x), C) + \beta (\log B + \beta(\epsilon - \log \delta))$$

and so $\|Df^n(x)\| \leq \omega^n$, where $\omega = \max(\log \beta + \beta(\epsilon - \log \delta), \sigma^{-1/2}).$ \hfill \Box

Lemma 3.15. If $n$ is a $(\sigma, \delta)$-hyperbolic time for $x$, then $B_{\delta_1,\omega^{-\beta}}(f^{j}(x)) \subset V_{n-j}(f^{j}(x)) \subset B_{\delta_1,\omega^{-\beta}}(f^{j}(x))$ for each $0 \leq j \leq n$.

This result is essential to show that to keep the hyperbolic time under perturbation all that we need is to maintain the random orbits within a certain distance to the unperturbed orbit during the iterated of the adapted hyperbolic time.
Proof of Lemma 3.15. We have $d_\delta(f^i(x), C) \geq \sigma^{b(n-i)}$ for all $0 \leq j \leq n$ and so either $d(f^i(x), C) \geq \sigma^{b(n-i)}$ with $f^i(x) \in B_\delta(C)$, or $d(f^i(x), C) \geq \delta$.

Hence for $y \in B_{\delta_1\sigma^{n-j}/2}(f^i(x))$ we have either

$$\frac{d(y, f^i(x))}{d(f^i(x), C)} \leq \delta_1 \sigma^{(1/2-b)(n-j)} \leq \frac{1}{2} \text{ or } \frac{d(y, f^i(x))}{d(f^i(x), C)} \leq \frac{\delta_1}{\delta} \sigma^{(n-j)/2} \leq \frac{1}{2} \text{ for } 0 \leq j \leq n \text{ (recall that } 0 < b \leq 1/2 \text{ from the definition of non-degenerate critical set).}$$

This enables us to use non-degeneracy conditions (S1) and (S2).

For $y \in B_{\delta_1\sigma^{n-j}/2}(f^i(x))$ since $b\beta \leq 1/2$, the value of $B_{\delta_1\sigma^{n-j}/2}(f^i(x))$ is bounded above by either $B\delta_1\sigma^{(1/2-b)(n-j)/2}$ or $B\delta_1\sigma^{-\beta}(n-j)/2 = \frac{2}{b} \delta_1^{-\beta} \sigma^{(n-j)/2}$, and both are smaller than $\log \sigma^{-1/2}$. Thus from (S2) for all $y \in B_{\delta_1\sigma^{n-j}/2}(f^i(x))$

$$\sigma^{1/2} \|Df(f^i(x))^{-1}\| \leq \|Df(y)^{-1}\| \leq \sigma^{-1/2} \|Df(f^i(x))^{-1}\|.$$  \hspace{1cm} (18)

For $j = n - 1$ above, we get for every $y \in B_{\delta_1\sigma^{n-j}/2}(f^{n-1}(x))$

$$\sigma^{1/2} = \sigma^{-1/2} \|Df(f^{n-1}(x))^{-1}\| \geq \|Df(y)^{-1}\| \geq \sigma^{-1/2} \|Df(f^{n-1}(x))^{-1}\| \geq \sigma^{3/2}.$$  Hence, a smooth curve $\gamma$ from $f^n(x)$ to the boundary of $B_{\delta_1}(f^n(x))$ and inside this ball must be such that the unique curve $\tilde{\gamma}$ contained in $V_1(f^{n-1}(x))$ such that $f^{n-1}(x) \in \tilde{\gamma}$ and $f(\tilde{\gamma}) = \gamma$ satisfies $\sigma^{3/2} \delta_1 = \sigma^{3/2} \ell(\gamma) \leq \ell(\gamma) \leq \sigma^{1/2} \ell(\gamma) = \delta_1 \sigma^{1/2}$, where $\ell(\cdot)$ denotes the length of any smooth curve on $M$ and, recall, $f^{n-1} \mid V_{n-1}(f^{n-1}(x)) : V_{n-1}(f^{n-1}(x)) \to B_{\delta_1}(f^n(x))$ is a diffeomorphism for all $j = 0, \ldots, n-1$. Thus $B_{\delta_1\sigma^{n-j}/2}(f^{n-1}(x)) \supset V_1(f^{n-1}(x)) \supset B_{\delta_1\sigma^{3/2}}(f^{n-1}(x))$. In particular this shows that the statement of the Lemma is true for $n = 1$, since $\omega > \sigma^{-1/2}$.

Now we argue by induction assuming the Lemma to be true for all hyperbolic times up to some $n \geq 1$ and consider $x$ having $n + 1$ as a hyperbolic time. Then for each $1 \leq j < n$

$$B_{\delta_1\sigma^{n-j}}(f^i(x)) \subset V_{n-j}(f^i(x)) \subset B_{\delta_1\sigma^{n-j}/2}(f^i(x))$$

since $f(x)$ has $n$ as a hyperbolic time. For all $y \in V_{n+1}(x) \cap B_{\delta_1\sigma^{n+1}/2}(x)$ we have $f(y) \in V_1(f^n(x))$ and so by the induction assumption together with (18)

$$\|Df^{n+1}(y)^{-1}\| \leq \prod_{i=0}^{n} \|Df(f^i(y))^{-1}\| \leq \prod_{i=0}^{n} (\sigma^{-1/2} \|Df(f^i(x))^{-1}\|) \leq \sigma^{(n+1)/2}$$

Therefore, for any smooth curve $\gamma$ from $f^{n+1}(x)$ to the boundary of $B_{\delta_1}(f^{n+1}(x))$ and inside this ball we have that the unique curve $\tilde{\gamma}$ contained in $V_1(f^n(x)) \cap B_{\delta_1\sigma^{n+1}/2}(x)$ such that $x \in \tilde{\gamma}$ and $f^{n+1}(\tilde{\gamma}) = \gamma$ satisfies $\ell(\gamma) \leq \sigma^{(n+1)/2} \ell(\gamma) = \delta_1 \sigma^{(n+1)/2}$. Hence $V_{n+1}(x) \subset B_{\delta_1\omega^{n+1}/2}(x)$.

Finally, from Lemma 3.14 we obtain for the same curves $\gamma, \tilde{\gamma}$ as above $\ell(\gamma) = \delta_1 \sigma^{(n+1)/2} \leq \omega^{n+1} \ell(\tilde{\gamma})$, or $\ell(\tilde{\gamma}) \geq \omega^{-n-1} \ell(\gamma)$. Since this holds for any smooth curve $\gamma$ from $f^{n+1}(x)$ to the boundary of $B_{\delta_1}(f^{n+1}(x))$ and inside this ball, we conclude that $V_{n+1}(x)$ contains $B(x, \delta_1\omega^{n+1}/2)$.

This completes the inductive step and concludes the proof. \hfill $\Box$

Remark 3.16. From condition (S1) we obtain using the estimate (18)

$$|Df(y)^{-1}| \geq \sigma^{1/2} \|Df(f^i(x))^{-1}\| \geq \frac{\sigma^{1/2}}{B} d(f^i(x), C)^{b} \geq \frac{\sigma^{1/2}}{B} \sigma^{b(n-j)} \geq \frac{\sigma^{1/2}}{B} \sigma^{(n-j)/2}$$
because $b\beta \leq 1/2$. Then we arrive at

$$|Df(y)| \leq C\sigma^{-(n-\beta)/2}, \quad y \in V_{n-j}(f^j(x))$$

where $C = B\sigma^{-1/2}$, whenever $x$ has $n \geq 1$ as an hyperbolic time and $0 \leq j < n$.

**Proposition 3.17.** Let $f$ is a $C^2$ non-uniformly expanding endomorphism having slow recurrence to the critical set. There exist constants $\xi, \eta > 0$ such that for $\zeta(x) = \xi \omega^{-nH(x)/2}$ and the family $f_i(x) = f(x) + t \cdot \zeta(x)$, if $x$ is such that $H(x)$ is a hyperbolic time, then we have $f_i^j(x) \in V_{H(x)-j}(x)$ for all $0 \leq j \leq H(x)$ and each $t \in \Omega \subset [-1/2, 1/2]^N$.

In particular, $H(x)$ is a $(\delta, \delta)$-hyperbolic time for $(\bar{t}, x) \in \Omega \times M$ whenever $H(x) < \infty$, for a constant $0 < \sigma < \delta < 1$.

Moreover, if $\Omega \subset [-\epsilon_0, \epsilon_0]^N$ for some $0 < \epsilon_0 < 1/2$ and $H(x) < \infty$, then $f_i^j(x) \in B_{\epsilon_0 \delta \omega^{-nH(x)/2}}(f^j(x))$ for each $0 \leq j \leq H(x)$ and for all $\bar{t} \in \Omega$.

**Proof.** Let $\eta > 3/2$ be big enough such that $\max(C\sigma^{n-1/2}, \sigma^{2\eta}) < 1/2$, choose $\xi = \min(\delta_1/2, 1/2)$ and fix $\bar{t} = (t_1, t_2, \ldots) \in \Omega$. Then

$$|f_{i_1}(x) - f(x)| \leq |t_1 \zeta(x)| < \xi \omega^{-nH(x)/2} < \delta_1 \omega^{-(H(x)-1)}$$

and so $f_{i_1}(x) \in B_{\delta \omega^{-nH(x)-1}}(f^j(x)) \subset V_{H(x)-1}(f(x))$. Observe that there is $z \in V_{H(x)}(x)$ such that $f_{i_1}(x) = f(z)$ and so $H(f_{i_1}(x)) = H(f(z)) \geq H(z) - 1 = H(x) - 1$.

Now we argue by induction on $k$ and assume that for $1 \leq j \leq k < n - 1$ we have

1. $f_l^j(x) \in B_{\xi \omega^{-nH(x)}-2}(f^j(x)) \subset V_{H(x)-j}(f^j(x))$, and
2. $H(f_l^j(x)) \geq H(x) - j$.

It is easy to see that this is true for $k = 1$. For $j = k+1$ we get, for some $w \in B_{\delta \omega^{-nH(x)-k}}(f^k(x))$ in a segment between $f_l^k(x)$ and $f^k(x)$, according to Remark 3.16

$$|f_l^{k+1}(x) - f^{k+1}(x)| \leq |f_{i_{k+1}}(f_l^k(x)) - f(f^k(x))| + |f(f^k(x)) - f(f^k(x))|$$

$$\leq |t_{k+1} \zeta(f^k_l(x))| + |Df(w)| \cdot |f^k_l(x) - f^k(x)|$$

$$\leq \xi \omega^{-nH(f^k_l(x))} + C\sigma^{-(H(x)-k)/2} \cdot \xi \omega^{-n(H(x)-k)/2}$$

$$\leq \xi \omega^{-n(H(x)-k)/2} (1 + C\sigma^{-(H(x)-k)/2})$$

$$= \xi \omega^{-n(H(x)-k-1)/2} \omega^{-n(2(H(x)-k)-1)} (1 + C\sigma^{-(H(x)-k)/2})$$

$$\leq \xi \omega^{-n(H(x)-k-1)/2} (\omega^{-n(2(H(x)-k)-1/2)} + C\sigma^{(2\eta-1)(H(x)-k)})$$

$$\leq \xi \omega^{-n(H(x)-k-1)/2}.$$

The last inequality comes from the choice of $\eta$ and because $H(x) - k \geq 1$ and $\omega > \sigma^{-1/2} > \sigma^{-1}$. This proves that part (1) of the inductive step. Then there exists $z \in V_{H(x)}(x)$ such that $f^{k+1}(z) = f^{k+1}_l(x)$ and so $H(f^{k+1}(x)) = H(f^{k+1}(z)) = H(z) - (k + 1) = H(x) - (k + 1)$, completing the proof of the inductive step.
Now we check that $H(x)$ is still a hyperbolic time for $(t, x)$. This follows easily from the statement of Proposition 3.17 together with the estimate (18) and Remark 1.6. However we have to relax the constants: for $1 \leq k < H(x)$

$$\prod_{j=n-k}^{H(x)-1} |Df_{t_{j+1}}(f^j_{t}(x))^{-1}| = \prod_{j=n-k}^{H(x)-1} |Df(f^j_{t}(x))^{-1}| \leq \prod_{j=n-k}^{H(x)-1} (\sigma^{-1/2} |Df(f^j_{t}(x))^{-1}|) \leq \sigma^{k/2}$$

(19)

and

$$d(f^{H(x)-j}_{t}(x), C) \geq d(f^{H(x)-j}_{t}(x), C) - d(f^{H(x)-j}_{t}(x), f^{H(x)-j}_{t}(x)) \geq \sigma^{b_j} - \delta_1 \sigma^{j/2} \geq (1 - \delta_1) \sigma^{b_j}$$

(20)

whenever $d(f^{H(x)-j}_{t}(x), C) < \delta$. Hence $H(x)$ is a $(\delta, \delta)$-hyperbolic time, for some $\sigma < \hat{\sigma} < 1$ for all $x$ such that $H(x)$ is finite.

Up until now, the proof of was done with a fixed maximum size $1/2$ for the perturbation. If we consider $\Omega \subset [-\epsilon_0, \epsilon_0]^N$ with $0 < \epsilon_0 < 1/2$, then the size of $t \cdot \zeta(x)$ is reduced proportionally in all the previous estimates, so that we obtain the last part of the statement.

This concludes the proof of Theorem 3.11.

3.2. Asymptotic rates of expansion and recurrence on random orbits. As a consequence of preservation of hyperbolic times, we have the following uniform estimates for the asymptotic rate of non-uniform expansion and slow recurrence for random orbits, i.e., the estimates we obtain do not depend on the perturbation as long as the perturbation is small enough.

**Proposition 3.18.** If $f$ is a non-uniformly expanding map with slow recurrence to the critical set having a first hyperbolic time map $L^p$-integrable for some $p > 3$ then there is $\epsilon_0 \in (0, 1/2)$ such that, for all $0 < \epsilon < \epsilon_0$, for $\lambda$-almost every point $x$ and for all $t \in [r, r]$, we have the bound

$$\liminf_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} - \log d_{b_j}(f^j_{t}(x), C) < 2\epsilon$$

and also

$$\liminf_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log ||Df(f^j_{t}(x))^{-1}|| \leq \frac{1}{2} \log \sigma.$$

**Proof.** The last limit inferior is clear: since we have infinitely many hyperbolic times $H(x)$ for $\lambda$-almost every $x$, we also have infinitely many hyperbolic times $H(x)$ for $\lambda$-almost every $x$ and every $t \in [r, r]$. Hence from (19) we obtain infinitely many hyperbolic times $n_1 = H(x), n_2 = n_1 + H(f^{n_1}_{t}(x)), n_3 = n_2 + H(f^{n_2}_{t}(x)), \ldots$ along the random orbit of $x$ with the average rate $\frac{1}{2} \log \sigma$, which implies the stated bound for the limit inferior.

For the limit inferior of slow approximation, we use (20) to write for all $0 \leq j < H(x)$

$$\frac{d(f^j_{t}(x), C)}{d(f^{j}(x), C)} \geq 1 - \frac{d(f^j_{t}(x), f^{j}(x))}{d(f^{j}(x), C)} \geq 1 - r\sigma^{(1/2-b)(H(x)-j)}.$$

(21)
From the definition of \( d_\delta \) we can write, since \( 0 < r < 1/2 \) and \( H(x) \) is a hyperbolic time
\[
\sum_{j=0}^{H(x)-1} - \log d_\delta(f^j(x), C) \leq \sum_{j=0}^{H(x)-1} - \log(1 - r \sigma^{(1/2-b)(H(x)-j)}) + \sum_{j=0}^{H(x)-1} - \log d_\delta(f^j(x), C)
\]
\[
\leq \sum_{j=0}^{H(x)-1} 2r \sigma^{(1/2-b)(H(x)-j)} + \epsilon n = \frac{2r \sigma^{1/2-b}}{1 - \sigma^{1/2-b}} + \epsilon n \leq 2\epsilon n
\]
if we take \( 0 < r < \epsilon_0 < 1/2 \) small enough.

The bound on the limit inferior follows again from the existence of infinitely many hyperbolic times along the orbit of \((\underline{t},x)\) for \( \lambda \)-almost every \( x \) and all \( \underline{t} \in [-r,r]^N \).

\[ \square \]

4. **Uniqueness of absolutely continuous stationary measure**

As a consequence of the choice of the adapted perturbations from Theorem 3.11 and the family \((\theta_\epsilon)_{\epsilon>0}\) of probability measures in (5), we obtain the following.

**Theorem 4.1.** For each sufficiently small \( \epsilon > 0 \) in the choice of \( \zeta \) in the construction of an adapted random perturbation from (7) as in Theorem 3.11 there exists a unique absolutely continuous and ergodic stationary measure for the random dynamical system \((f_{\underline{t}}, \theta_\epsilon^N)\).

Consider the measure \((f_\epsilon), \theta_\epsilon^N\) which is the push-forward of the measure \( \theta_\epsilon^N \) by \( f_\epsilon : M \to M \) for a fixed \( t \in \text{supp} \theta_\epsilon \), where we write \( f_\epsilon(\underline{t}) \) for \( f_\epsilon(x) \). We first mention a simple way to ensure the existence of a stationary measure for \((f_{\underline{t}}, \theta_\epsilon^N)\).

**Lemma 4.2.** For each sufficiently small \( \epsilon > 0 \) in the choice of \( \zeta \) in the construction of an adapted random perturbation from (7) as in Theorem 3.11 and for \( x \in M \) fixed, each weak* accumulation point of the sequence \( \mu_n^\epsilon(x) = \frac{1}{n} \sum_{j=1}^{n} (f^j_{\underline{t}}), \theta_\epsilon^N \) is a stationary measure.

**Proof.** Let \( \mu^\epsilon \) be a weak* accumulation point of the sequence \((\mu_n^\epsilon(x))_n\). For each continuous \( \phi : M \to \mathbb{R} \), we have by the Dominated Convergence Theorem
\[
\int \int \phi(f_{\epsilon}(y))d\mu^\epsilon(y)\theta_\epsilon(t) = \lim_{k \to +\infty} \int \int \phi(f_{\epsilon}(y))d \left( \frac{1}{n_k} \sum_{j=1}^{n_k} (f^j_{\underline{t}}), \theta_\epsilon^N \right) d\theta_\epsilon(t)
\]
\[
= \lim_{k \to +\infty} \frac{1}{n_k} \sum_{j=1}^{n_k} \int \int \phi(f_{\epsilon}(f^j_{\underline{t}}(x)))d\theta_\epsilon^N(t)d\theta_\epsilon(t). \quad (22)
\]

By definition of the perturbed iteration and of the infinite product \( \theta_\epsilon^N \), and because \( \mu_n^\epsilon(x) \xrightarrow{n_k \to +\infty} \mu^\epsilon \) in the weak* topology, the limit in (22) equals
\[
\lim_{k \to +\infty} \frac{1}{n_k} \sum_{j=1}^{n_k} \int \phi(f^{j+1}_{\underline{t}}(x))d\theta_\epsilon^N(t) = \int \phi d\mu^\epsilon.
\]
Hence \( \int \int \phi(f_{\epsilon}(y))d\mu^\epsilon(y)\theta_\epsilon(t) = \int \phi d\mu^\epsilon \) and \( \mu^\epsilon \) is a stationary measure. \[ \square \]
4.1. Absolutely continuity and support with nonempty interior. We now show that each stationary measure is absolutely continuous with respect to Lebesgue measure $\lambda$ (a volume form) in $M$.

**Proposition 4.3.** We have $(f_\epsilon)_* \theta^N_\epsilon << \lambda$ for all $x \in M$.

We recall that from [1,6] we have that $H$ is never zero on $M$, and so $\zeta(x) \neq 0$ for all $x \in M$.

**Proof.** In fact, consider $x \in M$ some ball in $M$ which (we assume is a parallelizable manifold, e.g. an interval, the circle or a $n$-torus). We have

\[
(f_\epsilon)_* \theta^N_\epsilon(A) = \theta^N_\epsilon[ t : f_\epsilon(x) \in A ] \\
= \theta^N_\epsilon[ t ; f(\tilde{x}) + t_1 \cdot \zeta(\tilde{x}) \in A ] \\
= \theta_\epsilon[ t_1 ; t_1 \in \frac{A-f(\tilde{x})}{\zeta(\tilde{x})} ] \\
= \frac{1}{\lambda(B_\epsilon(0))} \cdot \lambda( \frac{A-f(\tilde{x})}{\zeta(\tilde{x})} \cap B_\epsilon(0) ) \\
= \frac{1}{\lambda(B_\epsilon(0))} \cdot \lambda( (A-f(\tilde{x})) \cap B_\epsilon(0) )
\]

which shows that, if $\lambda(A) = 0$, then $(f_\epsilon)_* (\theta^N_\epsilon)(A) = 0$. \hfill $\square$

We observe that $B_\epsilon(0) \ni t \mapsto f_\epsilon(x) \in M$ is continuous for each fixed $x \in M$. We also note that, since the space $C^0(M, \mathbb{R})$ of continuous functions is dense in the space $L^1(\mu^\epsilon)$ of Borel integrable functions with respect to $\mu^\epsilon$, with the $L^1$-norm, then the stationary condition in Definition [1,3] holds also for any $\mu$-integrable $\phi : M \mapsto \mathbb{R}$.

**Lemma 4.4.** Every stationary probability measure $\mu^\epsilon$ is absolutely continuous with respect to $\lambda$.

**Proof.** From the above observation that the relation in Definition [1,3] is true for all integrable functions, we have that for any Borel measurable subset $B \subset M$

\[
\mu^\epsilon(B) = \int \chi_B d\mu^\epsilon = \int \int \chi_B \circ f_\epsilon(y) d\mu^\epsilon(y) d\theta_\epsilon(t) = \int (f_\epsilon)_* \theta^N_\epsilon(B) d\mu^\epsilon(y)
\]

and if $\lambda(B) = 0$, then we obtain $\mu^\epsilon(B) = 0$ from Proposition 4.3. \hfill $\square$

From this we are able to show that the support of any stationary measure has non-empty interior. Let $\mu^\epsilon$ be a stationary measure and let us write $S = \text{supp}(\mu^\epsilon)$. Using again that the relation in Definition [1,3] holds for $\mu^\epsilon$-integrable functions

\[
1 = \int \chi_S(y) d\mu^\epsilon(y) = \int \int \chi_S(f_\epsilon(y)) d\mu^\epsilon(y) d\theta_\epsilon(t) = \int \int \chi_S(f_\epsilon(y)) d\theta_\epsilon(t) d\mu^\epsilon(y)
\]

we conclude (since $0 \leq \chi_S \leq 1$) that $\int \chi_S(f_\epsilon(y)) d\theta_\epsilon(t) = 1$ for $\mu^\epsilon$-a.e. $y$. Therefore we get $\chi_S(f_\epsilon(y)) = 1$, that is, $f_\epsilon(y) \in S$ for $\theta_\epsilon$-a.e. $t$ and $\mu^\epsilon$-a.e. $y$.

In particular, $f_\epsilon(y) \in S$ for $t$ is a dense subset $D$ of $B_\epsilon(0) = \text{supp}(\theta_\epsilon)$ by definition of $\theta_\epsilon$. In addition, since $B_\epsilon(0) \ni t \mapsto f_\epsilon(y) \in M$ is continuous, we also have $f_\epsilon(D)$ is dense in
\( f(y)(B_\epsilon(0)) \) and so the closed set \( S \) contains \( B_{\zeta(y)}(f(y)) \), the closure of \( f(y)(D) \). We obtain that \( f(y) \in S \) for all \( t \in B_\epsilon(0) \) and \( \mu^\epsilon \)-a.e. \( y \).

From the definition of \( f(y) \) in (7), we see that the image of \( f(y)(B_\epsilon(0)) \) is the ball around \( f(y) \) with radius \( \zeta(y) \neq 0 \). Hence \( S \) has non-empty interior, as claimed.

### 4.2. Every stationary measure is ergodic with full support.

Now we use that the unperturbed transformation \( f \) has a dense orbit. Let \( \mu^\epsilon \) be a stationary probability measure. We have already shown that the support \( S \) of \( \mu^\epsilon \) has non-empty interior and that \( S \) is almost invariant.

**Lemma 4.5.** Let \( (f, \theta^\epsilon) \) be a random dynamical system such that the unperturbed map \( f = f_0 \) is a local diffeomorphism outside a \( \lambda \)-measure zero set, has a dense positive orbit and the parameter \( 0 \) belongs to the support of \( \theta^\epsilon \). Then \( \mu^\epsilon \) has full support: \( S = \text{supp}(\mu^\epsilon) = M \).

**Proof.** Let \( S_0 \subset S \) be such that \( \mu^\epsilon(S \setminus S_0) = 0 \) and \( f(S_0) \subset S \) for all \( t \in B_\epsilon(0) \) – this was proved in the previous subsection. Hence we also have \( \lambda(S \setminus S_0) = 0 \) and so \( \overline{S_0} = S \).

We have that \( f \) is locally a diffeomorphism outside a critical set \( C \) with \( \lambda \)-measure zero. Then \( \lambda(f(S \setminus S_0)) = 0 \) and, because \( f(S) \setminus f(S_0) \subset f(S \setminus S_0) \), we get \( \lambda(f(S) \setminus f(S_0)) = 0 \).

Thus \( f(S) = f(S_0) \subseteq f(S_0) \subseteq \overline{S} = S \), and \( S \) is a positively \( f \)-invariant subset.

We also know that the interior of \( S \) is non-empty. Let \( w \in M \) have a positive dense \( f \)-orbit. Then there exists \( n > 1 \) such that \( f^n(w) \) interior to \( S \) and so \( M = \omega_f(x) \subset \overline{S} = S \subset M \).

To show ergodicity of any stationary measure, we need some known auxiliary results already obtained for maps with hyperbolic times for random orbits, as stated below.

The first result gives properties of random hyperbolic times similar to those of Lemma 3.4.

**Proposition 4.6** (Proposition 2.6 and Corollary 2.7 in [2]). There exist \( \delta_1, C_1 > 0 \) such that, if \( n \) is a \((\sigma, \delta)\)-hyperbolic time for \((f, x) \in \Omega \times M \), then there exists a neighborhood \( V_n(f, x) \) of \( x \) in \( M \) such that:

1. \( f_n^\epsilon \) maps \( V_n(f, x) \) diffeomorphically onto the ball of radius \( \delta_1 \) centered at \( f_n^\epsilon(x) \);
2. \( d(f_k^\epsilon(y), f_k^\epsilon(z)) \leq C_1 \cdot d(f_k^\epsilon(y), f_k^\epsilon(z)) \) for all \( 1 \leq k \leq n \) and \( y, z \in V_k(f, x) \);
3. \( C_1 \leq \frac{|\det Df_k^\epsilon(y)|}{|\det Df_k^\epsilon(z)|} \leq C_1 \) for all \( y, z \in V_n(f, x) \).

The next result says that every non-trivial positively invariant subset for random non-uniformly expanding dynamical system must contain a ball of a definite size.

**Definition 4.7** (Random positively invariant set). We say that a subset \( A \subset M \) is random positively invariant if, for \( \mu^\epsilon \)-almost every \( x \in A \), we have that \( f_t(x) \in A \) for \( \theta^\epsilon \)-almost every \( t \).

We note that if \( A \) is random positively invariant and \( \lambda(A) > 0 \), then the closure of its Lebesgue density points \( A^+ \) is also random positively invariant, since \( A \) is dense in \( A^+ \).
Proposition 4.8 (Proposition 2.13 in [8]). For $\delta_1$ given by previous proposition, given any random positively invariant set $A \subset M$ with $\mu^\epsilon(A) > 0$, there is a ball of radius $\delta_1/4$ such that $\lambda(B \setminus A^+) = 0$.

The following is well-known from the theory of Markov chains.

Lemma 4.9 (Lemma 8.2 in [9]). The normalized restriction of a stationary measure to a random positively invariant set is a stationary measure.

Now we can prove that each stationary probability measure $\mu^\epsilon$ for our random dynamical systems is ergodic. Arguing by contradiction, let us assume that $\mu^\epsilon$ is not ergodic.

Hence, there are random (positively) invariant sets $S_1$ and $S_2 = M \setminus S_1$ such that both have $\mu^\epsilon$-positive measure. From Proposition 4.8 both sets contain a $\delta_1/4$-ball. Thus there exist $n_1, n_2 > 1$ such that $f^{n_1}(w) \in S_1$ and $f^{n_2}(w) \in S_2$, where $w$ is a point with dense positive $f$-orbit. Therefore, $\overline{S}_1 = M = \overline{S}_2$ which is a contradiction.

5. Stochastic stability

Now we combine the results of the previous sections to prove our main Theorem A. We use the same strategy as [2] taking advantage of the uniformity of the first hyperbolic time with respect to the adapted random perturbations. Indeed, from the previous constructions and from Remark 3.13, we have that there exist $0 < \sigma, \delta < 1$ such that

$$\hat{h} : \Omega \times M \to M, \quad (\xi, x) \mapsto \inf \{k \geq 1 : k \text{ is a } (\sigma, \delta) - \text{hyperbolic time for } (\xi, x)\}$$

satisfies $\hat{h}(\xi, x) = \hat{h}(\xi, x) = h(x) \leq H(x)$ for all $\xi \in \text{supp } \theta^N_\epsilon$ for $\lambda$-a.e. $x \in M$, where $0$ is the constant sequence equal to zero and $h(x)$ denotes the first hyperbolic time map associated to the unperturbed dynamics of $f$, as defined in Section 3.

Hence, if we assume that $h \in L^p(\lambda)$ for some $p > 3$, then we have also that the series

$$||\hat{h}||_1 = \int \hat{h} \, d(\theta^N_\epsilon \times \lambda) = \sum_{k=0}^{\infty} k \cdot (\theta^N_\epsilon \times \lambda)(\{(\xi, x) : \hat{h}(\xi, x) = k\})$$

has uniform $L^1$-tail, that is, the series in the right hand side of (23) converges uniformly to $||\hat{h}||_1$ (as a series of functions of the variable $\epsilon$).

Remark 5.1. For this argument it is enough that we assume $h \in L^1(\lambda)$, as long as $\hat{h}(\cdot, x) = h(x)$ for $\lambda$-a.e. $x \in M$ is established.

Now we can follow the same arguments as in [2, Section 5]. We sketch them here for the convenience of the reader. Since there exists a unique ergodic absolutely continuous stationary measure $\mu^\epsilon$ for all small enough $\epsilon > 0$, we have that

$$\mu^\epsilon_n = \frac{1}{n} \sum_{j=0}^{n-1} \int (f^j_\epsilon)_* \lambda \, d\theta^N_\epsilon(\xi).$$
converges in the weak* topology to \( \mu^0 \) as \( n \to +\infty \). We define for each \( \xi \in \Omega^N \) and \( n \geq 1 \)
\[
H_n(\xi) = \{ x \in B(\mu^\epsilon) : n \text{ is a } (\sigma, \delta)\text{-hyperbolic time for } (t, x) \}, \quad \text{and} \\
H_n^*(\xi) = \{ x \in B(\mu^0) : n \text{ is the first } (\sigma, \delta)\text{-hyperbolic time for } (t, x) \}.
\]
Here \( H_n^*(\xi) \) is the set of points \( x \) for which \( \hat{h}(t, x) = n \). For \( n, k \geq 1 \) we define \( R_{n,k}(\xi) \) as the set of points \( x \) for which \( n \) is a \((\sigma, \delta)\)-hyperbolic time and \( n + k \) is the first \((\sigma, \delta)\)-hyperbolic time after \( n \), that is
\[
R_{n,k}(\xi) = \{ x \in H_n(\xi) : f_n^j(x) \in H_k^*(\sigma^n \xi) \},
\]
where \( \sigma : \Omega \cup \mathcal{R} \) is the left shift map. Now using the measures
\[
v_n^\epsilon = \int (f_n^\epsilon)_* (\lambda \mid H_n(\xi)) \, d\theta_N^N(\xi) \quad \text{and} \quad \eta_n^\epsilon = \sum_{k=1}^{\infty} \sum_{j=1}^{k-1} \int (f_n^{\epsilon+j})_* (\lambda \mid R_{n,k}(\xi)) \, d\theta_N^N(\xi),
\]
we obtain the bound \( \mu_n^\epsilon \leq \frac{1}{n} \sum_{j=0}^{n-1} (v_j^\epsilon + \eta_j^\epsilon) \). The bounded distortion property of hyperbolic times provides the following.

**Proposition 5.2.** [2, Proposition 5.2] There is a constant \( C_2 > 0 \) such that for every \( n \geq 0 \) and \( \xi \in \Omega \) we have \( \frac{d\mu^\epsilon}{d\lambda}(f_n^\epsilon)_*(\lambda \mid H_n(\xi)) \leq C_2 \).

Hence we have \( \frac{d\mu^\epsilon}{d\lambda} \leq C_2 \) for every \( n \geq 0 \) and small \( \epsilon > 0 \). We now control the density of the measures \( \eta_n^\epsilon \) so that we ensure the absolute continuity of the weak* accumulation point of \( \mu^\epsilon \) when \( \epsilon \searrow 0 \).

**Proposition 5.3.** [2, Proposition 5.3] Given \( \zeta > 0 \), there is \( C_3(\zeta) > 0 \) such that for every \( n \geq 0 \) and \( \epsilon > 0 \) we may bound \( \eta_n^\epsilon \) by the sum of two measures \( \eta_n^\epsilon \leq \omega^\epsilon + \rho^\epsilon \) satisfying \( \frac{d\omega^\epsilon}{d\lambda} \leq C_3(\zeta) \) and \( \rho^\epsilon(M) < \zeta \).

It follows from Propositions 5.2 and 5.3 that the weak* accumulation points \( \mu^0 \) of \( \mu^\epsilon \) when \( \epsilon \searrow 0 \) cannot have singular part, and so are absolutely continuous with respect to \( \lambda \). Moreover, from Remark 1.4 we have that the weak* accumulation points \( \mu^0 \) of a family of stationary measures are always \( f \)-invariant measures.

From the properties of non-uniformly expanding maps stated in Theorem 1.1 we conclude that \( \mu^0 \) is a convex linear combination of finitely many physical measures of \( f \). This proves stochastic stability under adapted random perturbations.

In our setting, where \( f \) is transitive, we have a unique physical measure \( \mu \) for \( f \), thus \( \mu^0 = \mu \).

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