Bound and scattering states of extended Calogero model
with an additional PT invariant interaction

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Received XXX

Here we discuss two many-particle quantum systems, which are obtained by adding
some non-hermitian but PT (i.e. combined parity and time reversal) invariant interaction
to the Calogero model with and without confining potential. It is shown that the energy
eigenvalues are real for both of these quantum systems. For the case of extended Calogero
model with confining potential, we obtain discrete bound states satisfying generalised
exclusion statistics. On the other hand, the extended Calogero model without confining
term gives rise to scattering states with continuous spectrum. The scattering phase shift
for this case is determined through the exchange statistics parameter. We find that, unlike
the case of usual Calogero model, the exclusion and exchange statistics parameter differ
from each other in the presence of PT invariant interaction.

Key words: Calogero model, Bound and scattering states, Non-hermitian PT invariant
interactions

1 Introduction

Exactly solvable many particle quantum mechanical systems with long-range
interactions have recently attracted a lot of interest due to their close connection
with diverse subjects like fractional statistics, random matrix theory, level statistics
for disordered systems, Yangian algebra etc. The $A_{N−1}$ Calogero model (related to
$A_{N−1}$ Lie algebra) is the simplest example of such a dynamical model, containing
$N$ particles on a line and with Hamiltonian given by \( H \)

$$H = \frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \frac{\omega^2}{2} \sum_{j=1}^{N} x_j^2 + \frac{g}{2} \sum_{j<k} \frac{1}{(x_j - x_k)^2}, \tag{1}$$

where $g$ is the coupling constant associated with long-range interaction. One can
exactly solve this Calogero model and find out the complete set of energy eigenvalues
as

\[ E_{n_1,n_2,\ldots,n_N} = \frac{N\omega}{2} [1 + (N - 1)\nu] + \omega \sum_{j=1}^{N} n_j. \] (2)

Here \( n_j \)s are non-negative integer valued quantum numbers with \( n_j \leq n_{j+1} \) and \( \nu \) is a real positive parameter which is related to \( g \) as

\[ g = \nu^2 - \nu. \] (3)

It may be noted that, apart from a constant shift for all energy levels, the spectrum (2) coincides with that of \( N \) number of free bosonic oscillators. Furthermore, one can easily remove the above mentioned constant shift for all energy levels and express (2) exactly in the form of energy eigenvalues for free oscillators: \( E_{\tilde{n}_1,\tilde{n}_2,\ldots,\tilde{n}_N} = \frac{N\omega}{2} + \omega \sum_{j=1}^{N} \tilde{n}_j \), where \( \tilde{n}_j = n_j + \nu(j - 1) \) are quasi-excitation numbers. However it is evident that these \( \tilde{n}_j \)s are no longer integers and they satisfy a modified selection rule given by \( \tilde{n}_{j+1} - \tilde{n}_j \geq \nu \), which restricts the difference between the quasi-excitation numbers to be at least \( \nu \) apart. As a consequence, the Calogero model (1) provides a microscopic realization for generalised exclusion statistics (GES) (2) with \( \nu \) representing the corresponding GES parameter [3, 4, 5].

The Calogero model in absence of confining potential, i.e. setting \( \omega = 0 \) in eqn. (1), is also studied in Ref. [1]. Unlike the earlier case, the spectrum of this model is continuous and only scattering states occur. Due to such scattering, particle momentums in a outgoing \( N \)-particle plane wave get rearranged (reversely ordered) in terms of momentums in the incoming plane wave. The corresponding scattering phase shift is given by \( \theta_{sc} = \pi \nu \frac{N(N-1)}{2} \), which is simply \( \nu \pi \) times the total number of two-body exchanges that is needed for rearranging \( N \) particles in the reverse order. Thus it is natural to identify \( \nu \) as the exchange statistics parameter in this case [4]. It may be noted that this exchange statistics parameter coincides with the exclusion statistics parameter as defined earlier in the presence of confining potential.

Recently, theoretical investigations on different nonhermitian Hamiltonians have received a major boost because many such systems, whenever they are invariant under combined parity and time reversal (PT) symmetry, lead to either real or pairs of complex conjugate energy eigenvalues [6, 7, 8, 9, 10]. Such property of energy eigenvalues in nonhermitian PT invariant systems can be related to the pseudo-hermiticity [9] or anti-unitary symmetry [10] of the corresponding Hamiltonians in a general way and to the ODE/IM correspondence for some special cases [11]. However, as concrete examples of PT symmetric quantum mechanics, the Hamiltonians of only one particle in one space dimension have been usually considered in the literature so far. Therefore it should be interesting to consider nonhermitian but PT invariant Hamiltonian for \( N \)-particle system in one space dimension which remain invariant under the \( PT \) transformation [12]

\[ i \rightarrow -i, \quad x_j \rightarrow -x_j, \quad p_j \rightarrow p_j, \] (4)
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where \( j \in [1, 2, \ldots, N] \), and \( x_j \) \( (p_j \equiv -i \frac{\partial}{\partial x_j}) \) denotes the coordinate (momentum) operator of the \( j \)-th particle. In particular, one may construct an extension of Calogero model with or without confining term by adding to it some nonhermitian but PT invariant interaction, and enquire whether such extended model would lead to real spectrum.

The aim of the present article is to shed some light on the above mentioned issue for some special cases, where the PT invariant extension of the Calogero model can be solved exactly. In Sec. 2 of this article we consider such a PT invariant extension of \( A_{N-1} \) Calogero model \cite{[5, 12]} and show that, within a certain range of the related parameters, this extended Calogero model yields real energy eigenvalues obeying GES. In Sec. 3 we consider PT invariant extension of Calogero Model without confining potential and calculate the corresponding scattering phase shift \cite{[13]}. Section 4 is the concluding section.

2 Bound states of extended Calogero model with confining interaction

Let us consider a nonhermitian but PT invariant extension of the Hamiltonian \( \mathcal{H} \) as

\[
\mathcal{H} = H + \delta \sum_{j \neq k} \frac{1}{x_j - x_k} \frac{\partial}{\partial x_j},
\]

where \( \delta \) is a real parameter. It may be noted that, Calogero models and their distinguishable variants have been solved recently by mapping them to a system of free oscillators \cite{[14]}. With the aim of solving the extended Calogero model \cite{[5]} by similar method, we assume that (justification for this assumption will be given later) the corresponding ground state wave function is given by

\[
\psi_{gr} = e^{-\frac{1}{2} \sum_{j=1}^{N} x_j^2} \prod_{j<k} (x_j - x_k)^{\nu},
\]

where \( \nu \) is a real positive number which is related to the coupling constants \( g \) and \( \delta \) as

\[
g = \nu^2 - \nu(1 + 2\delta).
\]

Now if we use the expression \cite{[6]} for a similarity transformation to the Hamiltonian \( \mathcal{H} \), it reduces to an ‘effective Hamiltonian’ of the form

\[
\mathcal{H}' = \psi_{gr}^{-1} \mathcal{H} \psi_{gr} = S^- + \omega S^3 + E_{gr},
\]

where the Lassalle operator \( (S^-) \) and Euler operator \( (S^3) \) are given by

\[
S^- = -\frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} - (\nu - \delta) \sum_{j \neq k} \frac{1}{x_j - x_k} \frac{\partial}{\partial x_j}, \quad S^3 = \sum_{j=1}^{N} x_j \frac{\partial}{\partial x_j},
\]

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and

$$E_{gr} = \frac{N\omega}{2} [1 + (N - 1)(\nu - \delta)].$$  \hspace{1cm} (10)$$

It is easy to see that the Lassalle operator and Euler operator, as defined in eqn. (9), satisfy the simple commutation relation: \([S^3, S^-] = -2S^-\). Using therefore the well-known Baker-Hausdorff transformation we can remove the \(S^-\) part of the effective Hamiltonian \(H'\) and through some additional similarity transformations reduce it finally to the free oscillator model \(5\)

$$H_{free} = S^{-1} (H' - E_{gr}) S = -\frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \frac{\omega^2}{2} \sum_{j=1}^{N} x_j^2 - \omega N, \hspace{1cm} (11)$$

where \(S = e^{\frac{\omega}{4} \sum_j x_j^2} e^{\frac{\nu}{2} \sum_j \frac{x_j^2}{4}}\). As a consequence of these similarity transformations, nonsingular eigenfunctions of the extended Calogero model \(5\) can be obtained from the eigenfunctions of free oscillators as

$$\psi_{n_1, n_2, \ldots, n_N} = \psi_{gr} S \Lambda^+ \left\{ \prod_{j=1}^{N} e^{-\frac{\omega}{4} \sum_j x_j^2} H_{n_j}(x_j) \right\}, \hspace{1cm} (12)$$

where \(H_{n_j}(x_j)\) denotes the Hermite polynomials of order \(n_j\) and \(\Lambda^+\) projects the distinguishable many-particle wave functions to the bosonic part of the Hilbert space by completely symmetrising all coordinates. Evidently, the eigenfunctions \(12\) will be mutually independent if the excitation numbers \(n_j\)'s obey the bosonic selection rule: \(n_{j+1} \geq n_j\). The eigenvalues of the Hamiltonian \(10\) corresponding to the states \(12\) will naturally be given by

$$E_{n_1, n_2, \ldots, n_N} = E_{gr} + \omega \sum_{j=1}^{N} n_j = \frac{N\omega}{2} [1 + (N - 1)(\nu - \delta)] + \omega \sum_{j=1}^{N} n_j. \hspace{1cm} (13)$$

It is worth noting that, for the purpose of obtaining real eigenvalues \(13\) as well as nonsingular eigenfunctions \(12\) at the limit \(x_i \to x_j\), \(\nu\) should be taken as a real positive parameter. Due to eqn. (17), this condition restricts the ranges of coupling constants \(g\) and \(\delta\) as (i) \(\delta > -\frac{1}{2}\), \(0 > g > -\frac{1}{2}\); and (ii) \(g > 0\) with arbitrary value of \(\delta\). Thus the energy eigenvalues \(13\) of the PT invariant Hamiltonian \(4\) would be real within the above mentioned ranges of the coupling constants. Furthermore, it is evident that for all \(n_j = 0\), the energy \(E_{n_1, n_2, \ldots, n_N}\) attains its minimum value \(E_{gr}\). At the same time, as can be easily seen from eqn. (12), the corresponding eigenfunction reduces to \(\psi_{gr} \psi_{gr}\).

To explore the GES in the case of PT invariant model \(4\), we observe that eqn. (13) can be rewritten \(5\) exactly in the form of energy spectrum for \(N\) free oscillators as

$$E_{n_1, n_2, \ldots, n_N} = \frac{N\omega}{2} + \omega \sum_{j=1}^{N} \bar{n}_j, \hspace{1cm} (14)$$
where $\bar{n}_j = n_j + (\nu - \delta)(j - 1)$. These quasi-excitation numbers ($\bar{n}_j$) evidently satisfy a modified selection rule: $\bar{n}_{j+1} - \bar{n}_j \geq \nu - \delta$. Since the minimum difference between two consecutive $\bar{n}_j$s is given by

$$\bar{\nu} = \nu - \delta,$$

the spectrum of extended Calogero model \(5\) satisfies GES with parameter $\bar{\nu}$.

Since both $A_{N-1}$ Calogero model \(4\) and its nonhermitian extension \(5\) can be solved by mapping them to a system of free harmonic oscillators, it is natural to enquire whether these models are directly related through some similarity transformation. Investigating along this line, we find that

$$\Gamma^{-1}H\Gamma = H' = \frac{1}{2} \sum_{j} p_j^2 + \frac{1}{2} \omega^2 \sum_{j} x_j^2 + g' \sum_{j \neq k} \frac{1}{(x_j - x_k)^2},$$

where $\Gamma = \prod_{j<k} (x_j - x_k)\delta$, and $H'$ denotes the Hamiltonian of $A_{N-1}$ Calogero model with ‘renormalised’ coupling constant given by $g' = g + \delta(1 + \delta)$. However, due to the above mentioned similarity transformation, singular eigenfunctions of the usual Calogero model \(4\) can generate nonsingular eigenfunctions of the extended Calogero model \(5\) in some region of the parameter space \([12, 15]\). As a result, the spectrum of extended Calogero model differs qualitatively from the spectrum of the original Calogero model and leads to a negative value of the GES parameter \(15\) in such region of parameter space.

### 3 Scattering states of extended Calogero model without confining interaction

Here our aim is to study the PT invariant extension of $A_{N-1}$ Calogero model in the absence of confining interaction. Putting $\omega = 0$ in $\mathcal{H} [5]$, we explicitly obtain the Hamiltonian of such extended Calogero model as

$$\mathcal{H}_0 = -\frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \frac{g}{2} \sum_{j \neq k} \frac{1}{(x_j - x_k)^2} + \delta \sum_{j \neq k} \frac{1}{(x_j - x_k)} \frac{\partial}{\partial x_j}.$$

Following [1], we try to solve the eigenvalue problem corresponding to the above Hamiltonian within a sector of configuration space corresponding to a definite ordering of particles like $x_1 \geq x_2 \geq \cdots \geq x_N$. We find that the solutions of the eigenvalue equation $\mathcal{H}_0 \psi = p^2 \psi$, where $p$ is real and positive, are given by \([13]\)

$$\psi = \prod_{j<k} (x_j - x_k)^{\nu} P_{k,q}(x)r^{-b} J_b(pr).$$

Here the ‘radial’ coordinate $r$ is defined as: $r^2 = \frac{1}{N} \sum_{i<j} (x_i - x_j)^2$, $b$ is given by $b = k + \left(\frac{N-3}{2}\right)^{\nu} + \frac{1}{2} N(N-1)\bar{\nu}$, and the parameters $\nu$, $\bar{\nu}$ are defined exactly in the

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same way as in section 2. Moreover, \( J_b(pr) \) denotes the Bessel function and \( P_{k,q}(x) \)'s are translationally invariant, symmetric, \( k \)-th order homogeneous polynomials satisfying the differential equations

\[
\sum_{j=1}^{N} \frac{\partial^2 P_{k,q}(x)}{\partial x_j^2} + \nu \sum_{j \neq k} \frac{1}{(x_j - x_k)} \left( \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_k} \right) P_{k,q}(x) = 0. \tag{19}
\]

Note that the index \( q \) in \( P_{k,q}(x) \) can take any integral value ranging from 1 to \( g(N,k) \), where \( g(N,k) \) is the number of independent polynomials which satisfy eqn. (19) for a given \( N \) and \( k \).\[1\]

It is evident that, within the same range of coupling constants for which \( H \) \[5\] yields discrete bound states with real energy eigenvalues, \( H_0 \) \[17\] yields continuum scattering states with real energy eigenvalues. Due to (18), the most general eigenfunction for \( H_0 \) with eigenvalue \( p^2 \) can be written as

\[
\psi = \prod_{j<k} (x_j - x_k)^\nu \sum_{k=0}^{\infty} \sum_{q=1}^{g(N,k)} C_{kq} r^{-k} J_b(pr) P_{k,q}(x), \tag{20}
\]

where \( C_{kq} \)'s are some arbitrary constants. To discuss scattering, we need only the asymptotic behaviour of the wavefunction (20) when all particles are far apart from each other. Hence, using the asymptotic properties of Bessel function at \( r \to \infty \) limit, one can write \( \psi \) (20) as

\[
\psi \sim \psi_+ + \psi_- \tag{21}
\]

where

\[
\psi_\pm = (2\pi pr)^{-\frac{1}{2}} \prod_{j<k} (x_j - x_k)^\nu r^{-A} \sum_{k=0}^{\infty} \sum_{q=1}^{g(N,k)} C_{kq} r^{-k} P_{k,q}(x) e^{\pm i (b + \frac{1}{2}) \frac{\pi}{2} \pm ipr}.
\]

By choosing the coefficients \( C_{kq} \) in a proper way, the incoming wavefunction \( (\psi_+) \) can be expressed in the form of a plane wave like

\[
\psi_+ = C \exp \left[ i \sum_{j=1}^{N} p_j x_j \right], \tag{22}
\]

where \( p_j \leq p_{j+1}, \quad p^2 = \sum_{j=1}^{N} p_j^2 \) and \( \sum_{j=1}^{N} p_j = 0 \). Then, by following the approach of \[1\], it can be shown that the outgoing wavefunction \( (\psi_-) \) takes the form \[13\]

\[
\psi_- = C e^{-i\pi \nu \frac{N(N-1)}{2}} \exp \left[ i \sum_{j=1}^{N} x_j p_{N+1-j} \right]. \tag{23}
\]

Comparing (22) with (23), we find that the momentums of incoming plane wave gets rearranged (reversely ordered) in the scattering process and the corresponding phase shift is given by \( \pi \nu \frac{N(N-1)}{2} \). Thus \( \nu \) can be identified with the exchange statistics parameter associated with this phase shift.
4 Conclusion

Here we have constructed two exactly solvable many-particle quantum systems by adding some nonhermitian but PT invariant interaction to the $A_{N-1}$ Calogero model with and without confining potential. It is shown that the energy eigenvalues are real for both of these quantum systems in some region of the parameter space. The exclusion statistics parameter for the case of extended Calogero model with confining potential is determined through the allowed energy levels of discrete bound states. On the other hand, the exchange statistics parameter for the case of extended Calogero model without confining term is determined through the scattering phase shift of plane waves. Surprisingly we find that, in contrary to the case of original Calogero model, the exclusion and exchange statistics parameters derived in the above mentioned way differ from each other in the presence of PT invariant interaction. As a future study, it might be interesting to find out the inner product for which the eigenstates of extended Calogero models would satisfy the orthonormality property and completeness relation.

Acknowledgments

One of the authors (BBM) would like to thank Prof. M. Znojil for kind invitation and hospitality during the ‘1st International Workshop on Pseudo-Hermitian Hamiltonians in Quantum Physics’, Prague, June 16-17, 2003.

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