Klein geometries of high order

Ercüment H. Ortaçgil

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Abstract

Using irreducible representations of semi simple Lie algebras, we construct Klein pairs of arbitrarily high order. This allows us to view the representation theory of semi simple Lie algebras from the alternative perspective proposed in [O1].

1 Jet-filtrations on representations

Let \( \rho : g \to gl(V) \) be a representation of a (finite dimensional) Lie algebra \( g \). Suppose we have an ascending filtration of subspaces

\[
\{0\} \subsetneq V_1 \subsetneq V_2 \subsetneq \ldots \subsetneq V_k \subsetneq V
\]

satisfying

\[
V_i \subsetneq \rho(g)(V_i) \subset V_{i+1} \quad 1 \leq i \leq k
\]

where \( \rho(g)(V_i) \overset{def}{=} \{\rho(x)(v) \mid x \in g, v \in V_i\} \) and \( V_{k+1} = V \).

Definition 1 A filtration (1) satisfying (2) is a jet-filtration on the representation \( \rho \). The integer \( k \) is the length of the jet filtration (1).

We can refine (1): For instance, if there exists some \( V_i \subsetneq W_i \subsetneq V_{i+1} \) satisfying \( V_i \subsetneq \rho(V_i) \subset W_i \subsetneq \rho(g)(W_i) \subset V_{i+1} \), then we can insert \( W_i \) into (1) and obtain a finer jet-filtration. In particular, we define a maximally refined jet-filtration in the obvious way. In the opposite direction, we can omit some term(s) from (1) and the remaining terms define a coarser jet-filtration.

Definition 2 The jet-order of the representation \( \rho : g \to gl(V) \) is the maximum of the lengths of all (maximally refined) jet-filtrations (1).

Three natural questions:

Q1. What does (1) have to do with jets?
Q2. How do we construct jet-filtrations?
Q3. Why are jet-filtrations relevant?

Let us start with Q2.
2 Irreducible representations

Suppose that $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is irreducible. We choose some $0 \neq v \in V$ and define $V_1 \overset{\text{def}}{=} \text{Span}\{v\}$. If $V_1 = V$, we stop. If not, then $\rho(\mathfrak{g})(V_1) \not\subseteq V_1$ by irreducibility. In this case, we define $V_2 \overset{\text{def}}{=} \text{Span}\{\rho(\mathfrak{g})(V_1) \cup V_1\}$ and get $V_1 \not\subseteq \rho(\mathfrak{g})(V_1) \subset V_2$. If $V_2 = V$ we stop. If not, then $\rho(\mathfrak{g})(V_2) \not\subseteq V_2$ by irreducibility. In this case, we define $V_3 \overset{\text{def}}{=} \text{Span}\{\rho(\mathfrak{g})(V_2) \cup V_2\}$, and get $V_2 \not\subseteq V_3$ and $V_2 \not\subseteq \rho(\mathfrak{g})(V_2) \subset V_3$. Continuing this process, we finally get a jet-filtration (1) such that $V_{k-1} \not\subseteq \rho(\mathfrak{g})(V_k) = V$. Note that this jet-filtration is maximally refined by construction and its length depends on the vector $v$ we start with.

**Definition 3** A vector which maximizes the lengths of all such jet-filtrations is called a maximal vector of the irreducible representation $\rho$.

The above construction gives an algorithm for constructing jet-filtrations using an arbitrary irreducible representation. Since irreducible representations of semi simple Lie algebras are well known, it is instructive at this stage to look at a concrete example. So let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ with the well known basis

$$
\begin{align*}
eq \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ f = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \ h = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}
\end{align*}
$$

and $V = \mathbb{R}_k[x, y] = \text{polynomials in the variables } x, y \text{ of total degree } \leq k$. Now $\dim V = k+1$ and has the basis $v_1 = y^k, v_2 = y^{k-1}x, ..., v_k = yx^{k-1}, v_{k+1} = x^k$. We define the linear map $\rho : \mathfrak{sl}(2, \mathbb{R}) \to \text{End}(\mathbb{R}_k[x, y])$ by giving its values on this basis as

$$
\rho(e) \overset{\text{def}}{=} \frac{\partial}{\partial y}, \quad \rho(f) \overset{\text{def}}{=} y \frac{\partial}{\partial x}, \quad \rho(h) \overset{\text{def}}{=} \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}
$$

and check by a straightforward computation that (4) gives a representation $\rho : \mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{gl}(\mathbb{R}_k[x, y])$. Now $\rho(e)v_1 = v_2, \rho(e)v_2 = v_3, ..., \rho(e)v_k = v_{k+1}$ and $\rho(e)v_{k+1} = 0$. We define $V_i = \text{Span}\{v_1, v_2, ..., v_i\}$ and get the filtration

$$
\{0\} \not\subseteq V_1 \not\subseteq V_2 \not\subseteq \ldots \not\subseteq V_k \not\subseteq V
$$

satisfying

$$
V_i \not\subseteq \rho(e)(V_i) \subset V_{i+1}
$$

The key fact now is that both $\rho(f)$ and $\rho(h)$ preserve the filtration (5) because $\rho(f)v_i = v_{i-1}$ and $\rho(h)v_i = \lambda_i v_i$ for some constant $\lambda_i$. Therefore we can replace $e$ in (6) by the whole $\mathfrak{sl}(2, \mathbb{R})$ and (5) becomes a jet-filtration on the irreducible representation $\rho$. Thus for any positive integer $k$, the above well known irreducible representation of $\mathfrak{sl}(2, \mathbb{R})$ of degree $k+1$ has the jet-filtration (5) with jet-order $k$ and has $v_1$ as a maximal vector.
It is well known that irreducible representations of semi simple Lie algebras are "made up" of the representations of $\mathfrak{sl}(2, \mathbb{R})$ (see, for instance, [H]). Without going into the technical details, we will state here the following proposition whose proof is now almost trivial for anyone familiar with the representation theory of semi simple Lie algebras.

**Proposition 4** Let $\mathfrak{g}$ be any semi simple Lie algebra and $k$ be any positive integer. Then there exists an irreducible representation of $\mathfrak{g}$ whose jet-order is greater than $k$.

We now come to Q1.

### 3 Klein geometries

Let $(\mathfrak{h}, \mathfrak{h}_0)$ be an infinitesimal and effective Klein geometry, i.e., $\mathfrak{h}$ is a finite dimensional Lie algebra, $\mathfrak{h}_0 \subset \mathfrak{h}$ is a subalgebra which does not contain any ideals of $\mathfrak{h}$ other than $\{0\}$ (see [O1], [O3] for details). Any such Klein geometry defines the filtration (called the Weissfeiler filtration in some works)

$$\mathfrak{h} \supsetneq \mathfrak{h}_0 \supsetneq \mathfrak{h}_1 \supsetneq \ldots \supsetneq \mathfrak{h}_m \supsetneq \mathfrak{h}_{m+1} = \{0\}$$ (7)

where $\mathfrak{h}_{i+1} \overset{def}{=} \{x \in \mathfrak{h}_i \mid [x, \mathfrak{h}] \subset \mathfrak{h}_i\}$, $0 \leq i \leq k$. We called the integer $m + 1$ the infinitesimal order of $(\mathfrak{h}, \mathfrak{h}_0)$ and denoted it by $\text{ord}(\mathfrak{h}, \mathfrak{h}_0)$. Now let $(G, H)$ be an effective Klein geometry inducing $(\mathfrak{h}, \mathfrak{h}_0)$, i.e., $G$ is a Lie group with Lie algebra $\mathfrak{g}$, $H \subset G$ a closed subgroup with Lie algebra $\mathfrak{h}$. Now $G$ acts on $G/H = M$ as a transitive transformation group with the stabilizer $H$. If $g(x) = y$, then the transformation $g \in G$ is locally determined near $x$ (sometimes globally on $M$!) by its $m$'th order jet at $x$ where $m = \text{ord}(\mathfrak{h}, \mathfrak{h}_0)$. Therefore, in order to answer Q1, we must relate Klein geometries to representations.

Now given some representation $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$, we define the operation $[\cdot , \cdot]$ on $\mathfrak{g} \times V$ by

$$[(g_1, v_1), (g_2, v_2)] \overset{def}{=} ([g_1, g_2], \rho(g_1)(v_2) - \rho(g_2)(v_1))$$ (8)

Clearly $[,]$ is bilinear and skew symmetric and a straightforward verification shows that it also satisfies the Jacobi identity. Therefore $(\mathfrak{g} \times V, [\cdot , \cdot])$ is a Lie algebra which we will denote by $\mathfrak{g} \times \rho V$. This construction is a special case of a more general one described in [J] (see pg. 17). We identify $\mathfrak{g}$ with its image $(\mathfrak{g}, 0) \subset \mathfrak{g} \times \rho V$ and identify $V$ with the abelian ideal $\{0, \mathfrak{g}\} \subset \mathfrak{g} \times \rho V$. From (8) we deduce the important formula

$$[\mathfrak{g}, V] = [(\mathfrak{g}, 0), (0, V)] = (0, \rho(\mathfrak{g})V) = \rho(\mathfrak{g})V$$ (9)

Now suppose that $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ is irreducible with the jet-filtration (5). To be consistent with the notation in (7), we rewrite the ascending filtration (5) as a descending filtration (writing $V_{k-i}$ for $V_i$)
\[ V \supsetneq V_0 \supsetneq V_1 \supsetneq \ldots \supsetneq V_k \supsetneq \{0\} \] (10)

We define
\[ \mathfrak{h} \overset{\text{def}}{=} g \times_{\rho} V \] (11)
\[ \mathfrak{h}_0 \overset{\text{def}}{=} V_0 \subset V \subset \mathfrak{h} \] (12)

and consider the Klein geometry \((\mathfrak{h}, \mathfrak{h}_0)\) which is effective by (9) and irreducibility of \(\rho\). By the definition of (7), we have \([\mathfrak{h}_{i+1}, \mathfrak{h}] \subset \mathfrak{h}_i\) and \(\mathfrak{h}_{i+1}\) is the largest subalgebra of \(\mathfrak{h}_i\) satisfying \([\mathfrak{h}_{i+1}, \mathfrak{h}] \subset \mathfrak{h}_i\). For \((\mathfrak{h}, \mathfrak{h}_0)\) defined by (11), (12), we have \([V_{i+1}, \mathfrak{h}] \subset V_i\) by (9) and therefore \(V_i \subset \mathfrak{h}_i\). It follows that \(m \geq k\). If (10) is maximally refined, we conclude \(V_i = \mathfrak{h}_i\) and \(m = k\). Hence we obtain

**Proposition 5** Let \(\rho : g \to \mathfrak{gl}(V)\) be an irreducible representation with some maximally refined jet-filtration (10) of length \(k\). Then the Klein geometry \((\mathfrak{h}, \mathfrak{h}_0)\) defined by (11), (12) is effective with \(ord(\mathfrak{h}, \mathfrak{h}_0) = k + 1\).

Proposition 5 gives a partial answer to the first fundamental problem of jet theory (FP1) posed in [O1]. It is easy to show that \([\mathfrak{h}_i, \mathfrak{h}_j] \subset \mathfrak{h}_{i+j}\) in (7) and therefore \(\mathfrak{h}_i\) is abelian if \(2i \geq m + 1 = ord(\mathfrak{h}, \mathfrak{h}_0)\), i.e., the "half tail" of the Weissfeiler filtration always consists of abelian ideals of \(\mathfrak{h}_0\). In the examples we construct by (11), (12), note that \(\mathfrak{h}_0\) is always abelian! It is easy to modify our construction by defining \(\mathfrak{h}_0 \overset{\text{def}}{=} r \times_{\rho} V_0\) (rather than by (12)) for some "carefully chosen" subalgebra \(r \subset g\) which will make \(\mathfrak{h}_0\) nonabelian (but solvable!) and still give the conclusion of Proposition 5. However, we do not know the degree of freedom we have for \(\mathfrak{h}_0\).

To conclude this section, we would like to express our belief that the modern representation theory of Lie algebras can be reconstructed, possibly with some new insights and perspectives, as a special case of the theory of Klein geometries.

We now come to **Q3** (which is partially answered by the above paragraph).

## 4 Stiffening of symmetries and integrability

Any introductory book on differential geometry starts by defining the tangent space, vector fields, tensor fields, differential forms...etc. These are all first order objects, i.e., a transformation acts on these objects by its first order derivatives. One is naturally lead to ask the following admittedly vague question:

**Q:** Do the "higher order derivatives" play any role in the global structure of smooth manifolds?

If the answer to **Q** is affirmative, then "higher order derivatives" are surely quite relevant since understanding the structures of manifolds is a fundamental problem in differential geometry. We do not know of any attempt in the literature to answer **Q** other than the striking unpublished preprint [O1].

4
Now suppose some Lie group $G$ (rather, a pseudogroup of finite order) acts transitively on a smooth manifold $M$. This situation expresses the fact that $M$ is "symmetric" in some particular way. If some proper subgroup $G' \subset G$ also acts transitively on $M$, i.e., if the action of the transformation group $G$ can be stiffened to the transformations of $G'$, we can speculate that $M$ is "more symmetric" in the same way. As an extreme, if $G'' \subset G' \subset G$ acts simply transitively, then $M$ is a Lie group, the most symmetric object from this standpoint. However, if we allow the transformation group to be "too large" like $\text{Diff}(M)$, then all manifolds become symmetric since $\text{Diff}(M)$ acts transitively on $M$.

Now the key fact: For some fixed $M$, if a Lie group $G$, whose dimension is much greater than the dimension of $M$, acts transitively and effectively on $M$, in which case the dimension of the stabilizer $H$ must be large too since $\dim G/H = \dim G - \dim H = \dim M$, then the Klein geometry $(G, H)$ must have high order as defined above, i.e., we need higher order derivatives of the transformations of $G$ to determine them. The reason is that $H$ injects into the $k$'th order jet group $G_k(n)$ where $n = \dim M$. For instance, if $k = 1$, then we must have $\dim H \leq \dim G_1(n) = n^2$. If $\dim H \nless \dim G_1(n)$, then we need at least second order derivatives to inject $H$ into $G_2(n)$, if $\dim H \nless \dim G_2(n)$, then we need at least third order derivatives... etc. In short, "large groups act transitively and effectively on small spaces with large order". Now suppose $(G, H)$ has order $k+1$ and the action of $G$ can not be stiffened to a proper subgroup of order $k$. This indicates an obstruction coming from $k$'th order derivatives and the problem is, of course, to formulate this obstruction in precise mathematical terms. Now $(G, H)$ defines a flat pre-homogeneous geometry (PHG) of order $k+1$ ([O1], [O2]) which will "contain" many PHG's of order $k$ and none of these PHG's can integrate, i.e., have vanishing curvature, for otherwise they would give a stiffening.

Remarkably, the stiffening problem has a purely algebraic counterpart: If $(G, H)$ stiffens $(G', H')$, then $(\mathfrak{g}, \mathfrak{h}) \preceq (\mathfrak{g}', \mathfrak{h}')$, i.e., $\mathfrak{g} + \mathfrak{h}' = \mathfrak{g}'$, $\mathfrak{h} = \mathfrak{g} \cap \mathfrak{h}'$ (see [O1], pg. 131 for details). Note that $\dim \mathfrak{g} - \dim \mathfrak{h} = \dim \mathfrak{g}' - \dim \mathfrak{h}'$. Now $\preceq$ is a partial order on Klein geometries and the problem is to find the minimal (also maximal!) elements.

The above speculative scenario has been one of the motivations for us to become interested in Klein geometries and geometric structures of high order and we hope it may be inspiring also for some others.

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Ercüment H. Ortaçgil
ortacgile@gmail.com