Non-perturbative unitarity and fictitious ghosts in quantum gravity

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We discuss aspects of non-perturbative unitarity in quantum field theory. The additional ghost degrees of freedom arising in “truncations” of an effective action at a finite order in derivatives could be fictitious degrees of freedom. Their contributions to the fully-dressed propagator – the residues of the corresponding ghost-like poles – vanish once all operators compatible with the symmetry of the theory are included in the effective action. These “fake ghosts” do not indicate a violation of unitarity.

I. INTRODUCTION

A consistent and fundamental quantum theory of gravity has to be renormalizable and unitary. On the one hand, the Einstein-Hilbert action is perturbatively non-renormalizable, but unitary. On the other hand, the inclusion of terms with four derivatives in the gravitational action makes the theory renormalizable, but introduces a spin-2 ghost spoiling the perturbative unitarity of the theory [1].

It has been shown that some classes of non-local theories of gravity can be unitary [2, 3]. Specifically, when considering an exponential of entire functions, the propagator does not display any “extra” ghost poles: at a tree level, this type of non-local theories are unitary [4]. However, if these theories are considered to be non-local at a fundamental level, i.e., at the level of the bare theory, then quantum effects could generate infinitely many massive complex poles [5], leading to the presence of acausal effects on microscopic scales [6–8]. This type of violation of microcausality emerged in several approaches to quantum gravity, including the Hawking’s space-time foam [9], deWitt’s theory [10] and, recently, in the fakeons approach to quantum gravity [11]. Depending on the scale of the violation, these acausal effects could still be compatible with observations, thus making non-local gravity a viable approach to construct a renormalizable and unitary theory of quantum gravity. However, microscopic locality is one of the fundamental properties of quantum field theories. Is it possible to construct a unitary and renormalizable theory of quantum gravity whose fundamental (bare) action is local?

From the point of view of quantum field theory (QFT), starting from a local fundamental theory, it is the process of resumming quantum fluctuations (quantum loops) at all scales that generates non-localities at the level of the effective action. Indeed, it has been proposed [12] that quantum corrections may restore unitarity: the expectation is that interaction can make the spin-2 ghost of Stelle-gravity unstable [13], or even remove it from the spectrum of all possible asymptotic states. If interaction is able to remove the ghost from the Fock space of asymptotic states, then the unitarity of the theory is safe. As a lesson, important aspects of unitarity and stability are best discussed on the level of the quantum effective action, since all fluctuations effects are included. The second functional derivative of the effective action is the inverse propagator. A consistent propagator is an essential requirement for unitarity and stability.

A compelling proposal for a theory of quantum gravity based on local QFT is the asymptotic safety scenario for quantum gravity. According to the asymptotic-safety conjecture [14], a (non-perturbatively) renormalizable quantum theory of gravity can be constructed based on the existence of a suitable non-trivial fixed point of the renormalization group (RG) flow. The non-perturbative methods of the functional renormalization group (FRG) (see [15, 16] for recent reviews), based on Wilsonian idea of renormalization [17], indicate the existence of a fixed point in four dimensions - the Reuter fixed point. So far this has been seen in various truncations of the exact flow equations [18–44]. In the framework of the FRG, the effective action can in principle be derived by computing the flow of a RG-scale-dependent effective action $\Gamma_k$ containing all operators compatible with symmetry and field content of the theory. The quantum effective action is obtained in the limit of vanishing RG-scale, $k \to 0$, as in this limit all quantum fluctuations are integrated out. All relevant scattering amplitudes derived from the effective action at a tree level incorporate the effects of all quantum loops, i.e., they are fully-dressed quantities. Thus, the fully-quantum effective action allows in principle to assess whether or not a field theory is unitary [45]. In practical computations however, truncations of the theory space must be employed and the presence of a finite number of higher-derivative operators naturally lead to the generation of several poles in the graviton propagator. Due to fluctuation effects, the inverse propagator $D^{-1}(q^2)$ is not simply linear in $q^2$, $D^{-1}(q^2) = 2q^2 + m^2$. It is typically a non-trivial function, involving logarithms, for example, even for well behaved renormalizable and unitary QFTs. Any finite Taylor expansion of $D^{-1}(q^2)$ involving terms $\sim q^4$ or higher will lead to fictitious ghosts and tachyons that are artifacts of the truncation. In this letter we want to investigate the question whether the poles of the graviton propagator observed in truncations to the effective action constitute a real problem for the unitarity of the theory, or are rather artifacts of the truncation. This...
would give a physical answer to criticisms that asymptotically safe gravity is not unitary (see also the discussions in ref. [46] and in refs. [47, 48]).

Motivated by the case of gravity, and using quantum electrodynamics (QED) and Lee-Wick QED as working examples, we will discuss aspects of non-perturbative unitarity in QFT. It will be shown that the inclusion of quantum effects at all scales is crucial to assess unitarity of quantum field theories. We will also show with explicit examples that poles appearing in truncations of the effective action for a consistent QFT correspond to fake degrees of freedom of the theory: their residues are negative only if a few terms in a derivative expansion are considered, while increasing the truncation order the absolute value of the corresponding residues decreases and vanishes once all operators allowed by symmetry are included in the action. We will formulate criteria for a consistent graviton propagator and show the existence of functions $D(q^2)$ obeying these criteria.

II. NON-PERTURBATIVE ASPECTS OF UNITARITY IN QFT AND QUANTUM GRAVITY

Perturbative expansions in QFT are typically used as a tool to simplify computations. While this approach can work for field theories that are perturbative at all scales, it could give the wrong answer for theories where non-perturbative or all-orders effects are important.

An example of this behavior could concern, for instance, the renormalizability properties of field theories: according to [17], the renormalizability of QFTs and their UV-completeness are related to the existence of suitable fixed points of the corresponding renormalization group (RG) flows. In particular, if such fixed points are non-gaussian (i.e., not free), the theory might appear to be perturbatively non-renormalizable, while being non-perturbatively renormalizable, with its ultraviolet completion being defined by a non-gaussian fixed point of the RG flow.

In this section we point out some of the arguments related to the definition of unitarity and based on perturbation theory which could fail for non-perturbative field theories. We also highlight some subtle details and ambiguities that render the issue of unitarity in quantum gravity even more involved. In particular we discuss (apparent) issues with unitarity that can easily arise when employing the FRG to extract the effective action.

A. Non-perturbative optical theorem

The optical theorem itself, complemented by the full LSZ expansion of the $S$-matrix, do not rely on any perturbative expansion. The fully non-perturbative optical theorem can be represented diagrammatically as follows

$$2\text{Im}[T] = -2\text{Re} \sum_n T^\dagger \Gamma(\omega_n) T(\omega_n)$$

where $T$ is the transfer matrix. The sum in the right-hand-side runs over all possible intermediate states belonging to the space of states of the full (possibly non-perturbative) interacting theory. If there are no negative-norm states in the full theory, the space of asymptotic states is a Fock space and the sum over projectors in the right-hand-side defines the identity in the corresponding Fock space.

In the standard perturbative approach, the optical theorem (which follows from the condition that the $S$-matrix is unitary, $S^\dagger S = 1$) is translated into an infinite set of equalities, and unitarity has to be satisfied at each order in perturbation theory. However, as we will see, if a perturbative expansion breaks down, this would immediately lead to a(n apparent) violation of unitarity. In particular, a theory could violate unitarity at a perturbative level, while being non-perturbatively unitary.

B. Asymptotic states and vacuum

From a perturbative point of view, asymptotic states are constructed as free-particles states, and defined as excitations over the free-vacuum $|0\rangle$ of the (non-interacting) theory. The in- and out-states must thus be well-separated at asymptotic times, such that interaction can be neglected: in this limit the Heisenberg fields are assumed to become free fields. When the particles approach each other they start interacting and this interaction is governed by the full Hamiltonian $H = H_0 + \lambda H_{\text{int}}$, with $\lambda \ll 1$ to guarantee that interaction is just a small correction to the free Hamiltonian (this is equivalent to say that the couplings appearing in $H_{\text{int}}$ are small). When in a theory interaction or self-interaction is always present (e.g., when the bare theory is not free), asymptotic states and initial propagation should be defined using the fully-interacting theory [49–51]. This means that in/out states should be eigenstates (stable particles or bound states) of the fully non-perturbative Hamiltonian and should be defined as excitations over the (non-perturbative) vacuum of the full theory $|\Omega\rangle$.

In the case of gravity, it is not even obvious that the Minkowski spacetime is the true vacuum of the theory. Even in the simple case of quadratic gravity, at least in its conformally-reduced version, the dominant configuration in the gravitational path integral could correspond to a complicated “kinetic condensate” [52, 53], rather than a “simple” flat spacetime. While this result depends on the structure of the full theory, it is important to keep in mind that a proper definition of an $S$-matrix requires the knowledge of the asymptotic states [54] about the true vacuum of the theory, and that the latter might be non-trivial in the case of gravity.
C. Effective actions, scattering amplitudes, non-perturbative unitarity and truncations

The quantum effective action $\Gamma_0$ encodes the effects of all quantum loops and is the generator of 1PI Green functions. Thus, all scattering amplitudes (propagators and vertexes) computed at a tree level using $\Gamma_0$ are fully-dressed, i.e., they already contain the effects of all quantum loops. These are given by the functional derivatives of the effective action:

$$\langle f|S|i \rangle \propto \langle \Omega|T \{ \phi(x_1) \ldots \phi(x_n) \} |\Omega\rangle_{(c)} = \left[ \frac{\delta^n \Gamma_0[\phi]}{\delta \phi(x_1) \ldots \delta \phi(x_n)} \right]_{\phi=0},$$

where $\phi(x_i)$ are fully interacting quantum fields and $|\Omega\rangle$ is the vacuum of the fully interacting theory. As the quantum effective action includes all (perturbative or non-perturbative) effects of quantum loops at all momentum scales, it can be used to verify unitarity in both perturbative and (strongly or weakly) non-perturbative QFTs.

The effective action $\Gamma_0$ can be obtained either by solving the functional integral of a theory or via the FRG equation [55]

$$k \partial_k \Gamma_k = \frac{1}{2} \text{Str} \left\{ \left( \Gamma_k^{(2)} + R_k \right)^{-1} k \partial_k R_k \right\}.$$  \hspace{1cm} (2)

Here $R_k$ is a regulator function and $\Gamma_k^{(2)}$ denotes the second functional derivative of the effective average action $\Gamma_k$. The latter is a RG-scale-dependent effective action, which results from the integration of fluctuating modes with momenta $p \in (k, \infty)$. The quantum effective action is thus obtained as the limit $k \to 0$ of $\Gamma_k$ and is expected to be non-local (even when starting from a local bare or microscopic action, $S_{\text{bare}} = S_{\infty}$) due to the integration of quantum fluctuations at all scales.

The FRG turned out to be a powerful tool to study the (non-perturbative) renormalizability of field theories and explore their implications for infrared physics. Nevertheless, one of the drawbacks of the FRG is the practical necessity to “truncate” the theory space, i.e., to use a truncated (derivative or vertex) expansion of $\Gamma_k$, in order to solve Eq. (2) and derive $\Gamma_0$. While in the case of field theories which are perturbative at all scales it might be sufficient to consider only operators with positive or zero mass dimension, in general $\Gamma_k$ should contain all possible operators allowed by symmetry. In the case of gravity, this means that $\Gamma_k$ should contain all operators compatible with diffeomorphism invariance. While the truncated-FRG computations still allow to explore the existence of fixed points of the RG flow, it is clear that once a truncated derivative expansion for the effective action $\Gamma_k$ is employed, the propagator will automatically display additional (ghost or tachyon or tachyonic ghost) poles, which could just be an artifact of the truncation, rather than a problem for the theory. In particular, this could be the case for the ghost of Stelle gravity.

III. NON-PERTURBATIVE UNITARITY IN ONE-LOOP QED AND LEE-WICK QED

Motivated by these arguments, we use QED as a working example and show how these fictitious ghost poles can appear in an artificially-truncated version of the theory and how they are dynamically removed once all operators allowed by symmetry are taken into account.

A. QED effective action and propagator

Effective actions are typically non-local. We can assume the quadratic part of the QED-effective action to take the form

$$\Gamma_0^{\text{QED}}[A_{\mu}] = -\frac{1}{4} \int d^4x \{ F_{\mu\nu}P(\Box)F^{\mu\nu} \}. \hspace{1cm} (3)$$

The latter has to be complemented with a gauge fixing term

$$S_{\text{gf}} = -\frac{1}{2e^2} \int d^4x \{ \partial_{\mu}A^\mu Q(\Box) \partial_{\nu}A^\nu \}, \hspace{1cm} (4)$$

and the corresponding propagator reads

$$\Delta_{\alpha\beta}(q^2) = \frac{i}{q^2 P(q^2)} \left\{ \eta_{\alpha\beta} - \left( 1 - \xi \frac{P(q^2)}{Q(q^2)} \right) \frac{q_{\alpha}q_{\beta}}{q^2} \right\}. \hspace{1cm} (5)$$

In what follows we will fix $\xi = 0$. We now need to specify the form of $P(q^2)$. Following [56–58], at one loop this function reads

$$P(q^2) = 1 - \frac{\alpha}{3\pi} \log \left( \frac{-q^2 + m_{\text{th}}^2}{m_{\text{th}}^2} \right), \hspace{1cm} (6)$$

where $\alpha$ is the fine structure constant and $m_{\text{th}}$ is a threshold mass, $m_{\text{th}}^2 = 4m^2$, with $m$ being the mass of the degree of freedom integrated out to obtain the one-loop effective action, typically the electron mass. Due to the presence of the logarithm, there is a branch cut singularity, corresponding to the production of particles. The scalar part of the propagator $D(q^2) = q^{-2}P^{-1}(q^2)$, with the function $P(q^2)$ given in Eq. (6), has no poles in the regime where the theory is valid, i.e., for momenta $q^2 \gtrsim q_L^2$, with $q_L^2 \sim -10^{600} m_{\text{th}}^2$, being the Landau pole.

Taking (3) as a toy model for the quadratic part of the full QED effective action, with $P(q^2)$ given in Eq. (6), we will perform truncations of the Taylor expansion of $P(q^2)$, and we will study the content of the space of asymptotic states as function of the truncation order.

B. Polology of the one-loop effective action and derivative expansions

Although effective actions are generally non-local, when expanding the effective action to get a low-energy
is thereby important to understand in detail the origin of the fictitious pole and to come up with conditions to understand a priori, namely, without knowing the form of the fully-quantum effective action $P$, whether a pole is a genuine or fake degree of freedom of the theory, i.e., a pole appearing in the full theory or a truncation artifact, respectively.

In the case at hand, the appearance of the additional ghost is due to the convergence properties of the logarithm at $z = -1$ and, in particular, to the fact that the logarithm has a finite radius of convergence, $|z| < 1$. We can understand how the fictitious pole is generated by visualizing the behavior of $P_N(z)$ for increasing values of $N$. This is shown in Fig. 2. As we see from the figure, when $N$ is even the function $P_N(z)$ diverges positively at $z = -1$. The other hand, when $N$ is odd, $P_N(z)$ diverges negatively and crosses the $z$-axis, thus generating a pole in the propagator. In particular, as the truncation order $N$ is increased, the position of the pole approaches the boundary of domain of convergence of the logarithm. In the limit $N \to \infty$, $P_N(z = -1)$ converges to a finite value and therefore in this limit (equivalent to say, when the action is not truncated) the fictitious pole disappears.

\[ P_N(z) = 1 + \frac{\alpha}{3\pi} \sum_{n=1}^{N} \frac{z^n}{n} \]

The first term of this expansion reproduces classical electrodynamics. Although the fully-dressed propagator (5) with $P(q^2)$ given by (6) has a unique pole at $q^2 = 0$, the function $P_N(z)$ can show additional real and complex-conjugate zeros. In the case at hand, when $N$ is odd the function $P_N(z)$ shows to have a zero at $z \simeq -1$, i.e., $q^2 = -m^2$, corresponding to a stable tachyonic ghost and entailing an apparent violation of unitarity. In addition, the function $P_N(z)$ has several complex-conjugate poles, as shown in Fig. 1. The fact that the ghost is also a tachyon and the fact that it appears only for $N$ odd depends on the numeric factors in the effective action. The fact that it is a ghost, i.e., that it comes with negative residue, comes instead from generic properties of polynomials.

The presence of the tachyonic ghost leads to an apparent violation of unitarity: while this ghost does not appear in the full theory (3), it does if one performs a perturbative expansion of the effective action. As we started from a toy model for the full effective action and we performed a derivative expansion afterwards, it is easy to realize that the tachyonic ghost at $z \simeq -1$ is a truncation artifact. In general however the form of the fully-quantum effective action is not known a priori. It is thereby important to understand in detail the origin of this additional degree of freedom and to come up with conditions to understand a priori, namely, without knowing the form of the fully-quantum effective action $P$, whether a pole is a genuine or fake degree of freedom of the theory, i.e., a pole appearing in the full theory or a truncation artifact, respectively.

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1 This argument is not restricted to a logarithmic effective action. If the effective action contains a $P(q^2)$ with one or more branch cut singularities, then there will be fictitious poles approaching the boundaries of the domain of convergence of the function $P$ [59]. If instead the integration of quantum fluctuations in the path integral leads to an effective action defined by an entire non-local function, there is no branch-cut singularity and the fake poles slowly move to infinity [59].
Figure 3. Residue of the truncated propagator evaluated at the fictitious tachyonic ghost pole, as function of $N$. As the truncation order is increased the residue approaches zero, thus making the corresponding fake degree of freedom “confined”.

The full (exact) form of the effective action is not known a priori and, especially within the framework of the FRG, it is often necessary to work within a truncation. It is then a key question, if one can decide a priori whether a pole corresponds to a genuine or fake degree of freedom of the theory. A possible answer lies in its residue. In fact, it turns out that the residue of the propagator at the fake pole decreases by increasing the truncation order $N$ and vanishes in the limit $N \to \infty$, as shown in Fig. 3. The reason is that in this limit there is no well-defined particle associated with this pole, and therefore, it cannot give any contribution to the fully-dressed propagator. Interestingly, this is the same mechanism realized in the quasi-particle approach to a Bose-Einstein condensate with impurities [60], the impurity being the equivalent of the fake ghost pole in QFT.

As a second instructive example, let us analyze the function

$$P(q^2) = 1 + \frac{\alpha}{3\pi} \log \left( \frac{-q^2 + m_{th}^2}{m_{th}^2} \right) - \frac{q^2}{M^2},$$

(8)

which corresponds to a Lee-Wick model for QED, with a coupling whose sign is opposite with respect to the standard one. In this case $P(q^2)$ has a real pole on the principal branch of the logarithm, corresponding to a stable, massive ghost. An expansion of $P(q^2)$ about $q^2 = 0$ will generate again a tachyonic ghost and several complex-conjugate poles, but there will also be a stable, massive ghost for all $N$ (cf. Fig. 4) which lies well within the domain of convergence of the logarithm and does not approach the boundary of its domain of convergence for increasing values of $N$. This is in fact the ghost appearing in the “full theory”, and for this reason the corresponding residue is expected to stay negative as $N$ is increased. This is indeed the case, as shown in Fig. 5: while the residue of the fictitious tachyonic ghost approaches zero for large $N$, the residue of the ghost present in the “full theory” quickly stabilizes to a constant negative value.

These results indicate that it might be possible to determine the nature of the ghosts appearing in truncations of the effective action, by studying their residues as functions of the truncation order $N$. In particular, this might have important implications for the case of gravity, to understand whether the ghost of Stelle theory is a true ghost or a truncation artifact.
IV. GOOD PROPAGATORS

Based on the requirements of unitarity, causality, and the possibility of performing an analytic Wick rotation connecting the Euclidean and Lorentzian theories, a fully-dressed propagator should have:

- No complex poles in the first and third quadrants of the physical sheet of the $q_0$-complex plane (neither complex-conjugate poles, nor ghosts)
- No essential singularities at infinity
- Positive-definite spectral density

A propagator with these properties can arise from a consistent theory which is valid to infinitely short distances. For a valid QFT of gravity based on asymptotic safety, such a propagator should have:

- A fully-dressed propagator should have:
  - A connection connecting the Euclidean and Lorentzian theories,
  - The possibility of performing an analytic Wick rotation
  - The presence of a branch cut at $q^2 = m_{th}^2$.

If $\alpha$ is taken to be negative, then $\text{Re}(P(z = x + iy)) \geq 1$ $\forall (x, y) \in \mathbb{R}^2$. This implies that for $\alpha < 0$ the function $P(z)$ can never be zero, i.e., $D(q^2)$ cannot have any pole beyond the massless one (cf. Fig. 6). On the other hand, if $\alpha > 0$, there might be both ghost-like poles or complex-conjugate poles. The pole structure as function of $\alpha$ is shown in Fig. 7. In order to avoid ghosts and maintain unitarity, we thus require the coupling $\alpha$ to be negative.

The fact that for $\alpha < 0$ the real part of $P(z)$ can never be zero has an important implication: the are no additional stable (ghost) degrees of freedom and the two branch cuts are not associated to a (ghost) resonance as in [13, 61], rather to multi-particle states, which are produced for $|p^2| > m_{th}^2$. In particular, the spectral density $\rho(p^2) = -\pi^{-1} \text{Im}(D(q^2 + i\epsilon))$ is positive-definite for $\alpha < 0$, as shown in Fig. 8.

In our example, the key to avoid stable/unstable, standard or tachyonic ghosts, is the presence of two symmetric branch cuts$^2$ on the real axis, at $|\text{Re}(z)| \geq 1$, i.e., for $|q^2| \geq m_{th}^2$. The pole structure of the propagator (9)

\[ iD(q^2) = \frac{i}{q^2} \left( 1 + \frac{m_{th}^2}{q^2} \tanh \left( \frac{q^2}{m_{th}} \right) \right), \quad \alpha < 0 \]

(9)

where $m_{th}^2$ is a mass scale. The coupling $\alpha$ must be negative in order to avoid ghost or complex-conjugate poles. This fact can be easily seen from the form of the real part of the function $P(z) = z^{-2} D^{-1}(z)$, with $z = m_{th}^2 q^2$. It reads

\[ \text{Re}(P(z = x + iy)) = \frac{x}{4 x^2 + y^2} \log \left( \frac{(1 - x)^2 + y^2}{(1 + x)^2 + y^2} \right) \]

(10)

+ $\frac{\alpha}{2} \left( \frac{x}{x^2 + y^2} \right) \left( \text{arg}(1 - x - iy) - \text{arg}(1 + x + iy) \right)$.

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(10)

\[ + \frac{\alpha}{2} \left( \frac{x}{x^2 + y^2} \right) \left( \text{arg}(1 - x - iy) - \text{arg}(1 + x + iy) \right). \]

If $\alpha$ is taken to be negative, then $\text{Re}(P(z = x + iy)) \geq 1$ $\forall (x, y) \in \mathbb{R}^2$. This implies that for $\alpha < 0$ the function $P(z)$ can never be zero, i.e., $D(q^2)$ cannot have any pole beyond the massless one (cf. Fig. 6). On the other hand, if $\alpha > 0$, there might be both ghost-like poles or complex-conjugate poles. The pole structure as function of $\alpha$ is shown in Fig. 7. In order to avoid ghosts and maintain unitarity, we thus require the coupling $\alpha$ to be negative.

The fact that for $\alpha < 0$ the real part of $P(z)$ can never be zero has an important implication: the are no additional stable (ghost) degrees of freedom and the two branch cuts are not associated to a (ghost) resonance as in [13, 61], rather to multi-particle states, which are produced for $|p^2| > m_{th}^2$. In particular, the spectral density $\rho(p^2) = -\pi^{-1} \text{Im}(D(q^2 + i\epsilon))$ is positive-definite for $\alpha < 0$, as shown in Fig. 8.

\[ \text{Re}(P(x + i\epsilon)) = \frac{x}{4 x^2 + y^2} \log \left( \frac{(1 - x)^2 + y^2}{(1 + x)^2 + y^2} \right) \]

(10)

\[ + \frac{\alpha}{2} \left( \frac{x}{x^2 + y^2} \right) \left( \text{arg}(1 - x - iy) - \text{arg}(1 + x + iy) \right). \]
theory is thus causal in the sense of QFT, i.e., the commutators/anticommutators of operators at spacelike-separated points are zero [62]. The presence of these multiparticle states at $p^2 < 0$ instead indicates that there might be instabilities, namely, a spontaneous symmetry breaking associated with a second-order phase transition [63].
Figure 9. Pole structure of the propagator (9) for $\alpha < 0$. In the complex $q^2$-plane (figure on the left panel), there are two branch cuts, but no additional poles beyond the massless one. The figure on the right panel shows the pole structure of the full propagator $D(q_0^2 - q^2 + i\epsilon)$ in the complex $q_0$-plane, with $\epsilon > 0$. For $\alpha < 0$ there is no obstruction towards performing an analytic Wick rotation from Lorentzian to Euclidean and vice versa. The theory is thus causal and Wick-rotatable. Both figures have been produced using $\alpha = -0.4$. In the second figure $\epsilon$ has been set to $\epsilon = 1$ in order to make the effect of the Feynmann prescription on the branch cuts visible.

within truncations of the effective action (fake ghosts) and ghosts which also appear in the full, untruncated theory. While for the latter the residue remains always negative, in the former the residue is negative but its absolute value decreases with the truncation order and vanishes once all operators allowed by symmetry are included in the effective action. These fake ghosts disappear from the spectrum of asymptotic states of the theory. Interestingly, this mechanism is very similar to the one encountered in the context of Bose-Einstein condensates with impurities [60]. Our results lead us to the conjecture that, even not knowing the form of the fully-quantum effective action, it might be possible to determine the nature of an apparent ghost-pole just by tracing the behavior of the corresponding residue as function of the truncation order: if its residue is negative and stays negative for any value of the truncation order, then the pole corresponds to a genuine degree of freedom of the model and indicates a lack of unitarity. If instead the residue decreases with the truncation order and tends to zero when a sufficiently large number of terms is included in the action, it is likely that it corresponds to a fake ghost, i.e., a fictitious degree of freedom generated by the truncation of the theory space. It will be interesting to see if high order derivative expansion for quantum gravity can be used for an investigation in that direction.

Finally, within FRG there is no need to limit oneself to a derivative expansion. Alternatively, one may employ numerical approximations to arbitrary functions of momentum for the inverse graviton propagator. If more knowledge about families of good propagators becomes available, one could also taylor a truncation based on the flow of parameters characterizing such families. Unitarity would then be guaranteed if a fixed point is found within such a truncation.

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