The $R_\infty$-property for Chevalley groups of types $B_l, C_l, D_l$ over integral domains

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Abstract

We prove that Chevalley groups of the classical series $B_l, C_l, D_l$ over an integral domain of zero characteristic, which has periodic automorphism group, possess the $R_\infty$-property.

1 Introduction

Let $G$ be a group and $\varphi$ be an automorphism of $G$. Elements $x, y$ of the group $G$ are said to be (twisted) $\varphi$-conjugated ($x \sim_\varphi y$) if there exists an element $z \in G$ such that $x = zy\varphi(z^{-1})$. The relation of $\varphi$-conjugacy is an equivalence relation and it divides the group into $\varphi$-conjugacy classes. The number $R(\varphi)$ of these classes is called the Reidemeister number of the automorphism $\varphi$. If $R(\varphi)$ is infinite for any automorphism $\varphi$, then $G$ is said to possess the $R_\infty$-property.

The problem of determining groups which possess the $R_\infty$-property was formulated by A. Fel’shtyn and R. Hill [1]. One of the first general results in this area was obtained by A. Fel’shtyn, G. Levitt and M. Lustig, they proved that non-elementary Gromov hyperbolic groups possess the $R_\infty$-property [2, 3]. Another extensive result was established by A. Fel’shtyn and E. Troitsky, they proved that any non-amenable residually finite finitely generated group possesses the $R_\infty$-property [4]. This wide class of groups contains a lot of finitely generated linear groups, in particular, general linear groups $GL_n(\mathbb{Z})$, special linear groups $SL_n(\mathbb{Z})$, symplectic groups $Sp_{2n}(\mathbb{Z})$. In the paper [5] the author considered some infinitely generated linear groups. In particular, it was proved, that any Chevalley group (of normal type) over an algebraically closed field $F$ of zero characteristic possesses the $R_\infty$-property if the transcendence degree of the field $F$ over $\mathbb{Q}$ is finite.

In this paper we study the $R_\infty$-property for Chevalley groups of the classical series $B_l, C_l, D_l$ over integral domains which are not necessarily fields. The main result of the paper is the following

1The work is supported by Russian Science Foundation (project 14-21-00065).
**Theorem 1.** Let $G$ be a Chevalley group of type $B_l, C_l$ or $D_l$ over a local integral domain $R$ of zero characteristic. If the automorphism group of the ring $R$ is periodic, then $G$ possesses the $R_\infty$ property.

In the paper [7] similar result was proved for Chevalley groups of type $A_l$, therefore we do not consider the case of root system $A_l$ in the present paper.

The localization $\mathbb{Z}_{p\mathbb{Z}}$ of the ring of integers $\mathbb{Z}$ by the ideal $p\mathbb{Z}$ is a local integral domain of characteristic zero with the trivial automorphism group and therefore it satisfies the conditions of the theorem.

The condition that the ring $R$ has characteristic zero is essential. It follows from the result of R. Steinberg [6, Theorem 10.1] which says that for any connected linear algebraic group over an algebraically closed field of non-zero characteristic, there always exists an automorphism $\varphi$ for which $R(\varphi) = 1$.

At present, there are no examples of integral domains of characteristic zero such that Chevalley groups over these domains do not possess the $R_\infty$-property. The author believes that it is possible to discard the condition that the automorphism group of the ring $R$ is periodic. The result [5, Theorem 1] gives a lot of examples of fields of characteristic zero with non-periodic automorphism group such that Chevalley groups over these fields possess the $R_\infty$-property.

E. Jabara studied groups which do not possess the $R_\infty$-property. In particular, he proved that any residually finite group which admits an automorphism $\varphi$ of prime order with $R(\varphi) < \infty$ is virtually nilpotent.

**2 Preliminaries**

We use classical notation. Symbols $I_n$ and $O_{n\times m}$ mean the identity $n \times n$ matrix and the $n \times m$ matrix with zero entries, respectively. If $A$ an $n \times n$ matrix and $B$ an $m \times m$ matrix, then the symbol $A \oplus B$ denotes the direct sum of the matrices $A$ and $B$, i.e. the block-diagonal $(m+n) \times (m+n)$ matrix

$$
\begin{pmatrix}
A & O_{n\times m} \\
O_{m\times n} & B
\end{pmatrix}.
$$

It is obvious that for a pair of $n \times n$ matrices $A_1, A_2$ and for a pair of $m \times m$ matrices $B_1, B_2$ we have $(A_1 \oplus B_1)(A_2 \oplus B_2) = A_1A_2 \oplus B_1B_2$, $(A_1 \oplus B_1)^{-1} = A_1^{-1} \oplus B_1^{-1}$.

The orthogonal group $O_l(R, f)$, which preserves a quadratic form $f$, and the symplectic group $Sp_{2l}(R)$ over a ring $R$ are defined by the formulas

$$
Sp_{2l}(R) = \left\{ A \in GL_{2l}(R) \mid A \begin{pmatrix}
O_{l\times l} & I_l \\
-I_l & O_{l\times l}
\end{pmatrix} A^T = \begin{pmatrix}
O_{l\times l} & I_l \\
-I_l & O_{l\times l}
\end{pmatrix}\right\},
$$

$$
O_l(R, f) = \left\{ A \in GL_l(R) \mid A[f]A^T = [f]\right\},
$$

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where \([f]\) is a matrix of the quadratic form \(f\) and \(T\) denotes transpose. We denote by \(\Omega_l(R,f)\) the derived subgroup of \(O_l(R,f)\). Factoring groups \(\text{Sp}_{2l}(R)\), \(O_l(R,f)\) and \(\Omega_l(R,f)\) by their center we obtain the corresponding projective groups \(\text{PSp}_{2l}(R)\), \(\text{PO}_l(R,f)\) and \(\text{P}\Omega_l(R,f)\).

The following proposition about the number of twisted conjugacy classes in a group and in a quotient group was proved in [9, Lemmas 2.1, 2.2].

**Proposition 1.** Let 
\[ 1 \rightarrow N \rightarrow G \rightarrow A \rightarrow 1 \]
be a short exact sequence of groups, and \(N\) be a characteristic subgroup of \(G\).

a. If \(A\) possesses the \(R_\infty\)-property, then \(G\) possesses the \(R_\infty\)-property.

b. If \(N\) is a finite group and \(G\) possesses the \(R_\infty\)-property, then \(A\) possesses the \(R_\infty\)-property.

The following proposition about the connection between the Reidemeister number of the automorphism \(\varphi\) and the automorphism \(\varphi_H\), where \(\varphi_H\) is an inner automorphism induced by the element \(H\), can be found in [10, Corollary 3.2].

**Proposition 2.** Let \(\varphi, \varphi_H\) be an automorphism and an inner automorphism of the group \(G\), respectively. Then \(R(\varphi\varphi_H) = R(\varphi)\).

An associative and commutative ring \(R\) is said to be an integral domain if it contains the unit element \(1\) and it has no zero devisors. The following simple proposition of ring theory can be found in [7, Lemma 1].

**Proposition 3.** Let \(K\) be an integral domain and \(M\) be an infinite subset of \(K\). Then for any polynomial \(f\) of non-zero degree the set \(P = \{f(a) : a \in M\}\) is infinite.

3 Proof of the main result

**Theorem 1.** Let \(G\) be a Chevalley group of type \(B_l, C_l\) or \(D_l\) over a local integral domain \(R\) of zero characteristic. If the automorphism group of the ring \(R\) is periodic, then \(G\) possesses the \(R_\infty\)-property.

**Proof.** We separately consider all the types of root systems.

Case 1. The root system has the type \(C_l\). Since the quotient group \(G/Z(G)\) is isomorphic to the elementary Chevalley group \(C_l(R)\) [11, §12.1], then by the proposition [11(a)] it is sufficient to prove that the group \(C_l(R)\) possesses the \(R_\infty\)-property.

The group \(C_l(R)\) is known to be isomorphic to the projective symplectic group \(\text{PSp}_{2l}(R)\) over the ring \(R\) [11, §11.3]. Since the center of the group \(\text{Sp}_{2l}(R)\) is finite,
then by the proposition \((\text{IV})\) we can consider \(G = \text{Sp}_{2l}(R)\) and prove that this group possesses the \(R_\infty\)-property.

Let \(T\) be a variable and \(y\) be an element of the ring \(R\). Denote by the symbols \(X(T)\) and \(Y(y)\) the following \(2l \times 2l\) matrices

\[
X(T) = \begin{pmatrix} T \oplus I_{l-1} & I_l \\ -I_l & O_{l \times l} \end{pmatrix} \quad Y(y) = \begin{pmatrix} I_l & O_{l \times l} \\ yI_l & O_{l \times l} \end{pmatrix}.
\]

Let \(Z_y(T)\) be the product of \(X(T)\) and \(Y(y)\).

\[
Z_y(T) = X(T)Y(y) = \begin{pmatrix} T \oplus I_{l-1} & yI_l \\ -I_l & O_{l \times l} \end{pmatrix}
\]

By direct calculations we have that for every element \(x\) of the ring \(R\) the matrix \(X(x)\) belongs to \(G = \text{Sp}_{2l}(R)\).

Let us prove the following auxiliary statement:

For any positive integer \(k\) and for every elements \(y_1, \ldots, y_k\) of the ring \(R\) the matrix \(Z_{y_1}(T) \cdots Z_{y_k}(T)\) has the form

\[
\begin{pmatrix} f_k(T) \oplus a_k I_{l-1} & g_k(T) \oplus b_k I_{l-1} \\ h_k(T) \oplus c_k I_{l-1} & p_k(T) \oplus d_k I_{l-1} \end{pmatrix},
\]

where \(a_k, b_k, c_k, d_k\) are elements of the ring \(R\) and \(f_k, g_k, h_k, p_k\) are polynomials with coefficients from the ring \(R\) such that the degree of \(f_k\) is equal to \(k\) and degrees of polynomials \(g_k, h_k, p_k\) are less than \(k\).

To prove this statement we use induction on the parameter \(k\). If \(k = 1\), then the statement is obvious. Suppose that the statement holds for the number \(k - 1\), i.e. the following equality holds

\[
Z_{y_1}(T) \cdots Z_{y_{k-1}}(T) = \begin{pmatrix} f_{k-1} \oplus a_{k-1} I_{l-1} & g_{k-1} \oplus b_{k-1} I_{l-1} \\ h_{k-1} \oplus c_{k-1} I_{l-1} & p_{k-1} \oplus d_{k-1} I_{l-1} \end{pmatrix},
\]

where degree of the polynomial \(f_{k-1}\) is equal to \(k - 1\) and degrees of the polynomials \(g_{k-1}, h_{k-1}, p_{k-1}\) are equal to \(n, m, r < k - 1\), respectively. Then we have

\[
Z_{y_1}(T) \cdots Z_{y_k}(T) = \begin{pmatrix} f_{k-1} \oplus a_{k-1} I_{l-1} & g_{k-1} \oplus b_{k-1} I_{l-1} \\ h_{k-1} \oplus c_{k-1} I_{l-1} & p_{k-1} \oplus d_{k-1} I_{l-1} \end{pmatrix} \begin{pmatrix} T \oplus I_{l-1} & y_k I_l \\ -I_l & O_{l \times l} \end{pmatrix} = \begin{pmatrix} (Tf_{k-1} - g_{k-1}) \oplus (a - b) I_{l-1} & y_k f_{k-1} \oplus y_k a I_{l-1} \\ (Th_{k-1} - p_{k-1}) \oplus (c - d) I_{l-1} & y_k h_{k-1} \oplus y_k c I_{l-1} \end{pmatrix}.
\]

Let us look at degrees of the resulting polynomials. A polynomial in the position \((1, 1)\) has the degree \(k - 1 + 1 = k\); a polynomial in the position \((1, l + 1)\) has the degree \(k - 1 < k\); the degree of a polynomial in the position \((l + 1, 1)\) is less than or
equal to \( \max\{\deg(Th(T)), \deg(p(T))\} = \max\{m+1, r\} < \max\{k-1+1, k-1\} = k; \)

and the degree of a polynomial in the position \((l+1, l+1)\) is equal to \(m < k-1 < k.\)

The auxiliary statement is proved. In particular for any positive integer \(k\) and for every elements \(y_1, \ldots, y_k\) of the ring \(R\) the trace of the matrix \(Z_{y_1}(T) \ldots Z_{y_k}(T)\) is a polynomial of degree \(k\) with coefficient from the ring \(R.\)

To prove that the group \(G = \text{Sp}_{2l}(R)\) possesses the \(R_\infty\)-property we consider an arbitrary automorphism \(\varphi\) of the group \(G\) and prove that \(R(\varphi) = \infty.\) In the papers \([12, 13]\) it is proved that \(\varphi\) acts by the rule

\[\varphi : A \mapsto H_1 H_2 \delta(A) H_2^{-1} H_1^{-1},\]

where \(\delta\) is an automorphism which is induced by the automorphism \(\delta\) of the ring \(R\)

\[\delta : A = (a_{ij}) \mapsto (\delta(a_{ij}))\]

the matrix \(H_1\) belongs to \(G\) and the matrix \(H_2\) has the form

\[H_2 = \begin{pmatrix}
I_l & O_{l \times l} \\
O_{l \times l} & \beta I_l
\end{pmatrix} = Y(\beta)
\]

for a certain invertible element \(\beta\) of the ring \(R.\) By the proposition \(2\) we can consider that \(\varphi\) acts by the rule

\[\varphi : A \mapsto H_2 \delta(A) H_2^{-1} \]

Since an automorphism group of the ring \(R\) is periodic, then there exists a number \(k\) such that \(\delta^k = \text{id}.\) Let \(\psi\) be the following function

\[\psi(T) = \text{tr} \left( Z_{\delta(\beta)(T)} Z_{\delta(\beta^k-1)(T)} \ldots Z_{\delta(\beta)}(T) \right),\]

which is a polynomial of the degree \(k\) (as we already noted in the auxiliary statement). By the proposition \(3\) there exists an infinite set of integers \(a_1, a_2, \ldots \in \mathbb{Z} \subseteq R\) such that \(\psi(a_i) \neq \psi(a_j)\) for \(i \neq j.\)

Consider the set of matrices \(A_1, A_2, \ldots,\) where \(A_i = X(a_i),\) and suppose that \(R(\varphi) < \infty.\) Then there exist two numbers \(i \neq j\) such that \(A_i \sim_{\varphi} A_j,\) i. e. for a certain matrix \(D\) the following equality holds

\[A_i = D A_j \varphi(D^{-1}) = D A_j H_2 \delta(D^{-1}) H_2^{-1} .\]

If we multiply this equality by the matrix \(H_2\) we have

\[Z_{\delta}(a_i) = A_i H_2 = D A_j H_2 \delta(D^{-1}) = D Z_{\delta}(a_j) \delta(D^{-1})
\]

since \(A_i H_2 = X(a_i) Y(\beta) = Z_{\delta}(a_i).\)
Since $\delta$ is an automorphism of the ring $R$, it acts identically on the subring of integers and therefore $\delta(Z_\beta(a_i)) = Z_\delta(\beta)(a_i)$. Since $\delta^k = id$, acting by degrees of the automorphism $\delta$ on the equality (1) we have the following system of equalities

$$
Z_\beta(a_i) = DZ_\beta(a_j)\delta(D^{-1}),$
$$
Z_\delta(\beta)(a_i) = \delta(D)Z_\delta(\beta)(a_j)\delta^2(D^{-1}),$
$$
\vdots \quad \vdots \quad \vdots$
$$
Z_{\delta^{k-1}}(\beta)(a_i) = \delta^{k-1}(D)Z_{\delta^{k-1}}(\beta)(a_j)D^{-1}.
$$

If we multiply all of these equalities we conclude that

$$
Z_\beta(a_i)Z_\delta(\beta)(a_i)\ldots Z_{\delta^{k-1}}(\beta)(a_i) = DZ_\beta(a_j)Z_\delta(\beta)(a_j)\ldots Z_{\delta^{k-1}}(\beta)(a_j)D^{-1},
$$
i.e. the matrices $Z_\beta(a_i)Z_\delta(\beta)(a_i)\ldots Z_{\delta^{k-1}}(\beta)(a_i)$ and $Z_\beta(a_j)Z_\delta(\beta)(a_j)\ldots Z_{\delta^{k-1}}(\beta)(a_j)$ are conjugated. Therefore, their traces are the same and $\psi(a_i) = \psi(a_j)$. It contradicts to the choice of the elements $a_1, a_2, \ldots$ Then the matrices $A_i$ and $A_j$ can not be $\varphi$-conjugated and therefore $R(\varphi) = \infty$.

Case 2. The root system has the type $D_l$. By the arguments of the case 1 it is sufficient to prove that the elementary Chevalley group $D_l(R)$ possesses the $R_{\infty}$-property.

It is well known that $D_l(R) \cong \Omega_{2l}(R, f_D)$ [11, §11.3], where the matrix of the quadratic form $f_D$ has the form

$$
[f_D] = \begin{pmatrix}
O_{l \times l} & I_l \\
I_l & O_{l \times l}
\end{pmatrix}.
$$

Since the center of the group $\Omega_{2l}(R, f_D)$ is finite, then by the proposition [11b] we can consider $G = \Omega_{2l}(R, f_D)$ and prove the $R_{\infty}$-property for the group $\Omega_{2l}(R, f_D)$.

Let $T$ be a variable and $X(T), Y(T)$ be the following matrices

$$
X(T) = \begin{pmatrix}
1 & T \\
0 & 1
\end{pmatrix} \oplus I_{l-2} \oplus \begin{pmatrix}
1 & 0 \\
-T & 1
\end{pmatrix} \oplus I_{l-2},$
$$
$$
Y(T) = \begin{pmatrix}
T & 1 \\
-1 & 0
\end{pmatrix} \oplus I_{l-2} \oplus \begin{pmatrix}
0 & 1 \\
-1 & T
\end{pmatrix} \oplus I_{l-2}.
$$

Let $Z(T)$ be the commutator of $X(T)$ and $Y(T)$

$$
Z(T) = [X(T), Y(T)] = \begin{pmatrix}
T^2 + 1 & -T \\
-T & 1
\end{pmatrix} \oplus I_{l-2} \oplus \begin{pmatrix}
1 & T \\
T & T^2 + 1
\end{pmatrix} \oplus I_{l-2}.
$$

By direct calculations we see that for any element $x$ of the ring $R$ the matrices $X(x), Y(x)$ belong to $O_{2l}(R, f_D)$ and therefore $Z(x)$ belongs to $\Omega_{2l}(R, f_D)$.
Let us show that for every positive integer \( k \) the trace of the matrix \( Z(T)^k \) is a non-constants integral polynomial. To do it we prove more general result: For any positive integer \( k \) the matrix \( Z(T)^k \) has the form

\[
\begin{pmatrix}
 f_k(T) & g_k(T) \\
 h_k(T) & p_k(T)
\end{pmatrix} \oplus I_{l-2} \oplus \begin{pmatrix}
 p_k(T) & -h_k(T) \\
 -g_k(T) & f_k(T)
\end{pmatrix} \oplus I_{l-2},
\]

where \( f_k \) is a polynomial of degree \( 2k \), and \( g_k, h_k, p_k \) a polynomials of degrees which are less than \( 2k \).

We use induction on the parameter \( k \). The basis of induction \( (k = 1) \) is obvious. Suppose that this statement holds for the number \( k - 1 \), i.e., for certain integral polynomials \( f_{k-1}, g_{k-1}, h_{k-1}, p_{k-1} \) the following equality holds

\[
Z(T)^{k-1} = \begin{pmatrix}
 f_{k-1} & g_{k-1} \\
 h_{k-1} & h_{k-1}
\end{pmatrix} \oplus I_{l-2} \oplus \begin{pmatrix}
 p_{k-1} & -h_{k-1} \\
 -g_{k-1} & f_{k-1}
\end{pmatrix} \oplus I_{l-2},
\]

where degree of the polynomial \( f_{k-1} \) is equal to \( 2(k - 1) \) and degrees of the polynomials \( g_{k-1}, h_{k-1}, p_{k-1} \) are equal to \( n, m, r < 2(k - 1) \), respectively. Then the matrix \( Z(T)^k = Z(T)^{k-1}Z(T) \) has the form

\[
\begin{pmatrix}
 (T^2 + 1)f_{k-1} - Tg_{k-1} & -Tf_{k-1} + g_{k-1} \\
 (T^2 + 1)h_{k-1} - Tp_{k-1} & -Th_{k-1} + p_{k-1}
\end{pmatrix} \oplus I_{l-2} \oplus \begin{pmatrix}
 -Tf_{k-1} + h_{k-1} & -(T^2 + 1)h_{k-1} + Tp_{k-1} \\
 Tf_{k-1} - g_{k-1} & (T^2 + 1)f_{k-1} - Tg_{k-1}
\end{pmatrix} \oplus I_{l-2}
\]

A polynomial in the position \((1, 1)\) of this matrix has the degree \( 2(k - 1) + 2 = 2k \); a polynomial in the position \((1, 2)\) has the degree \( 2(k - 1) + 1 = 2k - 1 < 2k \); the degree of a polynomial in the position \((2, 1)\) is less than or equal to

\[
\max\{\deg((T^2 + 1)h_{k-1}), \deg(Tp_{k-1})\} = \max\{m + 2, r + 1\} < \max\{2k, 2k - 1\} = 2k;
\]

and the degree of a polynomial in the position \((2, 2)\) is less than or equal to

\[
\max\{\deg(Tf_{k-1}, \deg(p_{k-1}))\} = \max\{m + 1, r\} < \max\{2k - 1, 2k - 2\} = 2k - 1.
\]

The auxiliary statement is proved. As a corollary we have that for every positive integer \( k \) the function \( \psi_k(T) = tr(Z(T)^k) \) is a non-constant integral polynomial.

To prove that the group \( G = \Omega_{2d}(R, f_D) \) possesses the \( R_\infty \)-property we consider an arbitrary automorphism \( \varphi \) of the group \( G \) and prove that \( R(\varphi) = \infty \). In the papers [13][15] it is proved that there exist

1. An inner automorphism \( \varphi_H \)

\[
\varphi_H : A \mapsto HAH^{-1}
\]
2. A central automorphism $\Gamma$

$$\Gamma : A \mapsto \gamma(A)A,$$

where $\gamma$ is a homomorphism from the group $G$ into its center $Z(G)$.

3. A ring automorphism $\delta$

$$\delta : A = (a_{ij}) \mapsto (\delta(a_{ij})),$$

where $\delta$ is an automorphism of the ring $R$ such that $\phi = \phi_H \Gamma \delta$.

By the proposition 2 we can consider $\phi = \Gamma \delta$.

Since an automorphism group of the ring $R$ is periodic, there exists such a number $k$ that $\delta^k = id$. By the proposition 3 there exists an infinite set of elements $a_1, a_2, \ldots \in \mathbb{Z} \subseteq R$ such that $(\psi_k(a_i))^2 \neq (\psi_k(a_j))^2$ for $i \neq j$.

Consider the set of matrices $A_1 = Z(a_1), A_2 = Z(a_2), \ldots$ and suppose that $R(\varphi) < \infty$. Then there are two $\varphi$-conjugated matrices in the set $A_1, A_2, \ldots$, i.e. for some indexes $i \neq j$ and for some matrix $D \in G$ the following equality holds

$$A_i = D A_j \varphi(D^{-1}) = D A_j \Gamma \delta(D^{-1}) = D A_j C_1 \delta(D^{-1}),$$

where the matrix $C_1$ belongs to $Z(G)$.

Since the matrices $A_i, A_j$ have integer coefficients and the automorphism $\delta$ acts identically on the subring of integers, then $\delta(A_i) = A_i, \delta(A_j) = A_j$. Acting by degrees of the automorphism $\delta$ on the equality (2) we have the following system of equalities:

$$A_i = D A_j C_1 \delta(D^{-1}),$$
$$A_i = \delta(D) A_j C_2 \delta^2(D^{-1}),$$
$$\vdots$$
$$A_i = \delta^{k-1}(D) A_j C_k D^{-1}.$$}

If we multiply all of this equalities denoting $C = C_1 C_2 \ldots C_k$, then we have

$$A_i^k = D C A_j^k D^{-1},$$

i.e. the matrices $A_i^k$ and $C A_j^k$ are conjugated and therefore $tr(A_i^k) = tr(C A_j^k)$.

Since $C \in Z(\Omega_{2l}(R, f_D)) = \{\pm I_{2l}\}$, we have

$$\psi_k(a_i) = tr(A_i^k) = \pm tr(A_j^k) = \pm \psi_k(a_j).$$

It contradicts to the choice of the elements $a_1, a_2, \ldots$
Case 3. The root system has the type $B_l$. The elementary Chevalley group $B_l(R)$ is isomorphic to the group $\Omega^{2l+1}(R, f_B)$ [11], where the matrix of the quadratic forms $f_B$ has the following form

$$[f_B] = 1 \oplus \begin{pmatrix} O_{l \times l} & I_l \\ I_l & O_{l \times l} \end{pmatrix}.$$  

Using this fact, the proof of the case 3 literally repeats the proof of the case 2 after changing the matrix $Z(T)$ by the matrix $1 \oplus Z(T)$, and using the result [16] (instead of [14, 15]) about the automorphism group of the Chevalley groups of the type $B_l$. Theorem is proved.

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