MODELS FOR ANISOTROPIC SPHERICAL STELLAR SYSTEMS WITH A CENTRAL POINT MASS AND KEPLERIAN FALLOFF VELOCITY DISPERSIONS

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Received 1996 July 1; accepted 1996 September 3

ABSTRACT

We add to the lore of spherical stellar system models a two-parameter family with an anisotropic velocity dispersion and a central point mass ("black hole"). The ratio of tangential to radial dispersions is constant, and constitutes the first parameter, while each dispersion decreases with radius as $r^{-1/2}$. The second parameter is the ratio of the central point mass to the total mass. The Jeans equation is solved to give the density law in closed form: $\rho \propto (r/r_0)^{-\gamma}/[1 + (r/r_0)^{3-\gamma}]^2$, where $r_0$ is an arbitrary scale factor. The two parameters enter the density law only through their combination $\gamma$. At the suggestion of Tremaine, we also explore models with only the root-sum-square of the velocities having a Keplerian run, but with a variable anisotropy ratio. This gives rise to a more versatile class of models, with analytic expressions for the density law and the dispersion runs, which contain more than one radius-scale parameter.

Subject headings: celestial mechanics, stellar dynamics — galaxies: clusters: general — galaxies: kinematics and dynamics — globular clusters: general

1. INTRODUCTION

Model spherical stellar systems are useful in various contexts, as models of elliptical galaxies, star clusters, cores of galaxies, etc., and also as test beds for various theoretical ideas. To the library of classic models such as the Emden polytropes, isothermal spheres, and the King and Michie models (see, e.g., Binney & Tremaine 1987 and references therein) were added, in recent years, the models of Jaffe (1983), of Bertin & Stiavelli (1989), and of Hernquist (1990) and their generalizations to "eta" models by Dehnen (1993) and by Tremaine et al. (1994).

There are various routes to constructing such spherical models: One may start from a distribution function (DF) satisfying the Jeans theorem and see whether one gets a useful density and dispersion profiles. Alternatively, we may start by dictating constraints on the more directly observable density, $\rho(r)$, and the runs of tangential and radial velocity dispersions, $\sigma_t(r)$ and $\sigma_r(r)$, respectively. A necessary (but not sufficient) condition that there be an underlying stationary DF is that these three functions satisfy the Jeans equation

$$\frac{1}{\rho} \frac{d}{dr} \left( \frac{\sigma^2}{r^2} \right) + 2 \beta \frac{\sigma_t^2}{r^2} = -\frac{M(r)G}{r^2},$$

(1)

with

$$M(r) = 4\pi \int_0^r r'^2 \rho(r')dr',$$

(2)

and

$$\beta \equiv 1 - \frac{\sigma_r^2}{\sigma_t^2}.$$  

(3)

This is still only one functional constraint on the three functions, so to pinpoint a model we need two further functional constraints. For example, we can restrict ourselves to isotropic models and put $\sigma_t(r) = \sigma_r(r) = \sigma(r)$. Then we can start with a density profile and calculate $\sigma(r)$ from the Jeans equation, as in Tremaine et al. (1994). We can, alternatively, assume an equation of state—a relation between $\rho$ and $\sigma$—and solve the Jeans equation for both $\rho(r)$ and $\sigma(r)$ (as for gas polytropes). Or we may start with an assumed $\sigma(r)$ and solve the Jeans equation for $\rho(r)$ (as in the case of isothermal spheres).

If we want anisotropic models, we may start with a different constraint on $\sigma_t(r)$, $\sigma_r(r)$. For example, we may assume that their ratio is position independent (as we do in much of this paper), and then dictate either a density or a dispersion profile.

Such models are useful if at least some aspects of them can be given in simple, closed forms. In the end one has to ascertain that the model is underpinned by a positive DF satisfying the Jeans theorem, if one seeks to ensure that the model is a realization of at least one equilibrium system.

We offer here what we believe is a novel family of such models. We start by dictating the run of velocity dispersions: a constant ratio between the tangential and radial dispersions, while each of them decreases, in a Keplerian manner, as $r^{-1/2}$. It is clear that in order to obtain a bound system with a finite total mass the dispersions must decline at least that fast asymptotically (lest they outstrip the escape velocity, which decreases in this manner). In fact, for a fixed $\beta$ the dispersions must behave so asymptotically. Such a behavior is also "natural" for a stellar system near a central, dominant point mass. We then extend this family of models by assuming that only $(\sigma_r^2 + \sigma_t^2)^{1/2}$ is exactly Keplerian, but the anisotropy ratio is not constant.
The useful features of the models are simple, closed-form expressions for the density, accumulated mass, velocity dispersions, and potential, and the effortless inclusion of a central point mass.

We describe the constant-$\beta$ models in § 2, derive some additional properties in § 3 (including a partial discussion of underlying DFs), and explore a generalization in § 4.

2. THE MODELS

We substitute our assumed dispersion profile,

$$ (1 - \beta)^{-1} \sigma_i^2 = \sigma_r^2 = S/r , \quad (4) $$

in the Jeans equation (1), to obtain

$$ \frac{d \ln \rho}{d \ln r} + 2\beta - 1 = -m(r) , \quad (5) $$

where $m(r) \equiv M(r)G/S$ is the dimensionless accumulated mass ($\bar{M} \equiv SG^{-1}$ has the dimensions of mass and can be used to normalize all the masses in the problem), and $\beta$ is constant.

The addition of a point mass $M_0$ at the center is effected by subtracting from the right-hand side of equation (5) the constant

$$ m_0 \equiv M_0/\bar{M} = M_0 G/S . \quad (6) $$

We then obtain for the Jeans equation

$$ \frac{d \ln \rho}{d \ln r} + \gamma = -m(r) , \quad (7) $$

where

$$ \gamma \equiv 2\beta + m_0 - 1 , \quad (8) $$

and $m(r)$ includes only the accumulated mass of the stellar “gas,” excluding the point mass; we thus have $m(0) = 0$. Equation (7) can be solved analytically: We eliminate $\rho$ using

$$ \rho = \frac{\bar{M}}{4\pi r^2} m' , \quad (9) $$

to get an equation for $m(r)$ ($m'$ is the $r$ derivative of $m$):

$$ rm'' + (\gamma - 2)m' + mm' = 0 . \quad (10) $$

The left-hand side is the derivative of $rm' + (\gamma - 3)m + m^2/2$, which must then be a constant. Since $m(0) = 0$, this constant must be zero, so we get

$$ m^{-1}\left(3 - \gamma - \frac{m}{2}\right)^{\frac{1}{2}} m' = \frac{1}{r} . \quad (11) $$

For $\gamma \geq 3$, $m'$ is negative, which is unphysical. For $\gamma < 3$, the equation is straightforwardly integrated to yield the desired mass profile

$$ m(r) = 2(3 - \gamma) \frac{x^{3-\gamma}}{1 + x^{3-\gamma}} , \quad (12) $$

where $x = r/r_0$ and the scale radius, $r_0$, is an arbitrary integration constant; it equals the half-mass radius. This mass profile corresponds to the density profile

$$ \rho(r) = \frac{(3 - \gamma)^2 \bar{M}}{2\pi r_0^3} x^{-\gamma} \left(1 + x^{3-\gamma}\right)^2 . \quad (13) $$

Near the origin, the density behaves as $r^{-\gamma}$, while at infinity it behaves as $r^{3-\gamma}$.

We see from equation (12) that the total dimensionless mass is $m(\infty) = 2(3 - \gamma)$, or

$$ GM_g = 2(3 - \gamma)S = 4(2 - \beta)S - 2M_0 G . \quad (14) $$

$S$ is thus determined by the two masses: the total “gas” mass, $M_0$, and the central point mass, $M_0$, through

$$ S = [4(2 - \beta)]^{-1} G(M_g + 2M_0) . \quad (15) $$

This is the virial relation for the models. We shall see in § 3 that for $\gamma \geq 2$ the kinetic and potential energies of the model are infinite, but equation (15) still holds for all legitimate values of $\gamma < 3$, and follows directly from the structure equation.

The two masses determine the parameter $m_0$ through $m_0 = 4(2 - \beta)M_0/(M_g + 2M_0)$, and $\gamma$ through

$$ \gamma = 2\beta - 1 + 4(2 - \beta) \frac{M_0/M_g}{1 + 2M_0/M_g} , \quad (16) $$
In the limit of test-particle “gas,” $M_g \to 0$, with $\beta$ fixed, we have $\gamma \to 3$, so the model becomes meaningless. A meaningful test-particle model is obtained if we let $\beta \to -\infty$ (circular orbits) in such a way that $\gamma < 3$; $\beta M_g$ is then fixed at $(\gamma - 3) M_o / 2$. Then $S \to 0$, but $2 \sigma^2 r \to -2 \beta S \to GM_o$, and we get a model with test particles on Keplerian circular orbits around a point mass, the like of which can be built with any density distribution. If we admix test particles on noncircular orbits, the Keplerian fall-off which we require cannot be maintained.

In the limit $\beta \to -\infty$, but with $M_g$ and $M_o$ kept constant, we have $\gamma \to -\infty$, and the sphere becomes an infinitely thin shell at an arbitrary radius $r_0$, with the particles moving on circular orbits: a self-gravitating sphere cannot consist of particles on circular orbits with Keplerian velocities unless they are all at the same radius.

For $\gamma = 2$, one obtains the density distribution of the Jaffe model (Jaffe 1983) (which is the same as the “eta” model of Tremaine et al. 1994, and Dehnen 1993 with $\eta = 1$). This value of $\gamma$ can be obtained only with a central mass, since the maximum value of $\gamma$ for $m_0 = 0$ is 1. In the isotropic case it corresponds to $M_o / M_g = 3 / 2$.

3. GENERAL PROPERTIES

3.1. Energies

We now consider the potential run of the model. The point mass makes the usual contribution to the potential; that of the stellar “gas,” $\varphi_g$, can be straightforwardly integrated from the expression

$$\frac{d\varphi_g}{dr} = -\frac{GM_g}{r^2} - \frac{GM_g}{r_0^2} \frac{x^1 - \gamma}{1 + x^{\gamma - 3}}$$

(17)

to give

$$\varphi_g(r) = -\frac{GM_g}{r} {}_2F_1[1, (3 - \gamma)^{-1}; (4 - \gamma)(3 - \gamma)^{-1}; \left(\frac{r}{r_0}\right)^{(3 - \gamma)}]$$

(18)

where $\_2F_1$ is the hypergeometric function (see, e.g., Gradshteyn & Ryzhik 1980). As $\_2F_1(a, b; c; 0) = 1$ we have asymptotically $\varphi_g(r) \to -GM_g / r$, as expected.

As to the behavior of $\varphi_g$ at the origin, we use the behavior of $\_2F_1$ at large values of its argument to deduce that, for $\gamma < 2$,

$$\varphi_g(0) = -\frac{GM_g}{r_0} \frac{2 - \gamma}{3 - \gamma} I\left(\frac{4 - \gamma}{3 - \gamma}\right).$$

(19)

For $\gamma = 2$, $\varphi_g$ diverges logarithmically at the origin, and for $2 < \gamma < 3$, $\varphi_g$ diverges as $r^{2 - \gamma}$.

Some special cases: For $\gamma = 1$, one needs $\_2F_1(1, 1/2; 3/2; -x^{-2}) = x \arctan(1/x)$ to get

$$\varphi_g(r) = -\frac{GM_g}{r_0} \arctan\left(\frac{r_0}{r}\right).$$

(20)

For $\gamma = 2$, $\_2F_1(1, 1; 2; -x^{-1}) = x \ln(1 + 1/x)$, yielding the known expression for the Jaffe model, $\varphi_g(r) = (GM_g / r_0) \ln[r/(r + r_0)]$.

The total kinetic energy of a model sphere is infinite for $\gamma \geq 2$, and for $\gamma < 2$ it is

$$E_k = \frac{SM_g}{r_0} \frac{3 - 2\beta}{2} I\left(\frac{2 - \gamma}{3 - \gamma}\right) \frac{4 - \gamma}{3 - \gamma}.$$

(21)

The potential energy of the stellar “gas” is, for $\gamma < 2$,

$$E_p = -\frac{GM_g^2}{2r_0} \frac{2 - \gamma}{3 - \gamma} I\left(\frac{4 - \gamma}{3 - \gamma}\right) \left(\frac{2 - \gamma}{3 - \gamma} + \frac{2M_o}{M_g}\right).$$

(22)

By equation (15) they satisfy $E_k = -E_p / 2$.

3.2. Projected Properties

If we define the integrals

$$I(\mu, \nu; \lambda) = \int_0^\infty x^{-\mu} \left(x^2 - \lambda^2\right)^{-1/2} \frac{dx}{(1 + x^4)^{\nu}},$$

(23)

then the projected surface density $\Sigma(a)$, at a projected radius $a = \lambda r_0$, is given by

$$\Sigma(a) = \frac{3 - \gamma}{2\pi} \frac{M_g}{r_0} I(\mu - 1, 3 - \gamma; \lambda).$$

(24)

The line-of-sight (projected) velocity dispersion, $\sigma^2_z(a)$, is

$$\sigma^2_z(a) = \frac{S}{r_0} \frac{I(\mu, 3 - \gamma; \lambda)}{I(\mu - 1, 3 - \gamma; \lambda)} - \beta \lambda^2 I(\mu + 2, 3 - \gamma; \lambda).$$

(25)
At large projected radii \((a \gg r_0)\) we can write

\[
\Sigma(a) \approx \frac{3 - \gamma}{4\pi^{1/2}} \frac{M_\odot}{r_0^{1/2}} \frac{\Gamma((5 - \gamma)/2)}{\Gamma((6 - \gamma)/2)} a^{\gamma - 5},
\]

and the line-of-sight dispersion

\[
\sigma^2(a) \approx \frac{\Gamma^2((6 - \gamma)/2)}{\Gamma((7 - \gamma)/2)\Gamma((5 - \gamma)/2)} \left(1 - \beta \frac{(6 - \gamma)}{7 - \gamma}\right) \frac{S}{a}.
\]

As examples, for \(\gamma = 1\) we get asymptotically

\[
\sigma^2(a) \approx \frac{9\pi}{32} \left(1 - \frac{5\beta}{6}\right) \frac{S}{a},
\]

for \(\gamma = 2\)

\[
\sigma^2(a) \approx \frac{8}{3\pi} \left(1 - \frac{4\beta}{5}\right) \frac{S}{a},
\]

and for \(\gamma \approx 3\)

\[
\sigma^2(a) \approx \frac{\pi}{4} \left(1 - \frac{3\beta}{4}\right) \frac{S}{a}.
\]

Near the origin \((a \ll r_0)\) we can write, when \(\gamma > 1\),

\[
\Sigma(a) \approx \frac{3 - \gamma}{4\pi^{1/2}} \frac{M_\odot}{r_0^{1/2}} \frac{\Gamma((\gamma - 1)/2)}{\Gamma(\gamma/2)} a^{\gamma - 1},
\]

and

\[
\sigma^2(a) \approx \frac{\Gamma^2(\gamma/2)}{\Gamma((\gamma + 1)/2)\Gamma((\gamma - 1)/2)} \left(1 - \beta \frac{\gamma}{\gamma + 1}\right) \frac{S}{a}.
\]

For example, for \(\gamma = 2\),

\[
\sigma^2(a) \approx \frac{2}{\pi} \left(1 - \frac{2\beta}{3}\right) \frac{S}{a}.
\]

For \(\gamma \approx 3\) the behaviors near the origin and at large radii are the same. We see that unlike the density and surface-density runs, the projected-dispersion run depends on both \(\beta\) and \(\gamma\). For \(\gamma > 1, \sigma_0\) declines as \(r^{-1/2}\) at both ends; the proportionality constant is, in general, different, but may be the same for special choices of \(\beta\) and \(\gamma\) (for example, for \(\gamma = 2, \beta = 5/6\)).

### 3.3. Underlying Distribution Functions

We have not been able to ascertain that all the models discussed here have underlying nonnegative DFs that satisfy the Jeans theorem. For the isotropic case, \(\beta = 0\), one can try to search for a DF that depends on the particle energy alone. The procedure is then straightforward. First, we note from equation (4-139) in Binney & Tremaine (1987) that for this limited class of DFs the density must not increase with radius anywhere. This necessary condition excludes the (isotropic) models with \(\gamma < 0\), for which \(\rho\) increases near the origin. The DF is uniquely determined by the density distribution through the Eddington relation (eq. [4-140b] of Binney & Tremaine 1987)

\[
f(\epsilon) = \frac{1}{8^{1/2}\pi^2} \left[ \int_0^\infty d^2 \rho \frac{d\Psi}{\sqrt{\epsilon - \Psi}} + \frac{1}{\sqrt{\epsilon}} \left( \frac{d\rho}{d\Psi} \right)_{\Psi=\infty} \right].
\]

Here \(\epsilon = -E\), where \(E\) is the energy, \(\Psi = -\phi\), and \(\rho\) is viewed as a function of \(\Psi\) as both are functions of \(r\). The value \(\Psi = 0\) occurs at radial infinity, where \(\rho \propto r^{-\beta}\) decreases faster than \(\Psi \propto r^{-1}\); thus, the second term in equation (34) vanishes. If we change the integration variable to \(r\), equation (34) becomes

\[
f(\epsilon) = \frac{1}{8^{1/2}\pi} \int_{\Psi^{-1}(\epsilon)}^\infty \frac{d^2 \rho}{d\Psi^2} \frac{d\Psi}{\sqrt{\epsilon - \Psi}} \frac{dr}{\sqrt{\epsilon - \Psi}}.
\]

One can show that

\[
-\frac{d^2 \rho}{d\Psi^2} \frac{d\Psi}{dr} = \rho \frac{m + \gamma}{m + \gamma + 1} \left[ \gamma - 1 + m \frac{m^2 + m(2\gamma + 3/2) + (\gamma - 1)(\gamma + 3)}{(m + \gamma)(m + \gamma + 1)} \right].
\]
Thus, for \( \gamma \geq 1 \) the integrand is nonnegative, and the DF is always positive. For \( \gamma < 1 \), the integrand becomes negative for small enough radii, and the DF becomes negative for high enough energies. Such models do not have legitimate DFs that depend only on \( E \).

The two limiting models discussed at the end of §2 clearly also have legitimate underlying DFs.

4. A GENERALIZATION

S. Tremaine (1996, private communication) has pointed out to us that the crucial step in our derivation, that of once integrating the Jeans equation (going from eq. [10] to eq. [11]), does not require that \( \sigma_r \) and \( \sigma_t \) be separately Keplerian; it is enough that \((\sigma_r^2 + \sigma_t^2)^{1/2}\) is. Making only this relaxed assumption permits us to explore a larger family of models, which we now proceed to do.

The general Jeans equation (1), in the presence of a central point mass \( M_0 \), can be brought to the form
\[
(r^2 M' \sigma_r^2)' - [2r(\sigma_r^2 + \sigma_t^2) - GM_0]M' + \frac{G}{2} (M^2)' = 0 ,
\]
where the primes denote derivatives with respect to \( r \). Now assume only that
\[
\sigma_r^2 + \sigma_t^2 = \hat{S} r^{-1} ,
\]
and, as before, define the dimensionless quantities \( \hat{m}_0 = GM_0/\hat{S}, \hat{m}(r) = GM(r)/\hat{S} \), and also
\[
s(r) \equiv \frac{\sigma_r^2(r)r}{\hat{S}} = \frac{1}{2 - \beta(r)} .
\]

Then the Jeans equation becomes
\[
(rs \hat{m}') - \eta \hat{m}' + \frac{1}{2} (\hat{m}^2)' = 0 ,
\]
where \( \eta \equiv 2 - \hat{m}_0 \). As explained in §1, having imposed only one functional condition on the three functions \( \rho, \sigma_r, \) and \( \sigma_t \), we have yet to specify another in order to pinpoint a model. This can be a dictation of the run of the anisotropy parameter, as we shall do below by dictating \( s(r) \). Note that \( 0 \leq s(r) \leq 1 \), where \( s = 0, 1 \) for purely tangential and purely radial dispersion, respectively.

As before, the left-hand side of equation (40) is a derivative, and readily integrated once. As \( s \) is bounded, the integration constant vanishes, and we get
\[
rs \hat{m}' - \eta \hat{m}' + \frac{1}{2} \hat{m}^2 = 0 .
\]

Values of \( \eta \leq 0 \) are nonphysical, since they give a negative \( M' \). For \( \eta > 0 \), the integration of equation (41) gives the mass profile
\[
\hat{m}(r) = \frac{2 \eta e^{sX(r)}}{1 + e^{sX(r)}} ,
\]
where
\[
X(r) = \int_r^{r_0} \frac{dr}{rs'(r)} ,
\]
and \( r_0 \) is some arbitrary radius that will appear as a scale in the density law. The integration constant in equation (43) is chosen so that \( r_0 \) is the half-mass radius. Since \( s \) cannot exceed unity, \( X(r) \) must diverge (at least logarithmically) for \( r \rightarrow \infty \), and so \( \hat{m}(\infty) = 2\eta \), yielding the virial relation
\[
4\hat{S} = G(M_0 + 2M_0) .
\]
Thus, \( \eta \) is determined by
\[
\eta = \frac{2M_g}{M_0 + 2M_0} .
\]

The results for the constant-\( \beta \) case are reproduced when we note that in this case \( S \) and \( \hat{S} \) are related by \( \hat{S} = (2 - \beta)S \), so \( \hat{m} = m/(2 - \beta), \eta = (3 - \gamma)/(2 - \beta) \), etc.

The resulting density run is
\[
\rho(r) = \frac{M_g}{2\pi r^3 s(r)} \frac{\eta e^{sX}}{(1 + e^{sX})^2} .
\]

At large \( r \), where \( X(r) \) is positive and diverges, the asymptotic form of the density is
\[
\rho(r \rightarrow \infty) \approx \frac{\eta M_g}{2\pi r^3 s(r)} e^{-sX} ;
\]
near the origin, where $X(r)$ also diverges but is negative,

$$\rho(r \to 0) \approx \frac{\eta M_g}{2\pi r^3 s(r)} e^{\eta X}.$$ \hfill (48)

One can try different forms of $s(r)$, subject to $0 \leq s(r) \leq 1$, and see whether any interesting mass distributions result. Some general points first: If we depart from the constant-$\beta$ models discussed above, a new radius scale must be introduced by the choice of $s$, besides the scale, $r_0$, introduced through the integration constant. This is because $s$ must be of the form $s(r/a)$.

Near the origin the behavior is like that of the constant-$\beta$ models with $\beta = \beta(0)$. Thus, if $s(0) \neq 0$ [\(\beta(0) \neq -\infty\)], $X(r)$ diverges logarithmically (and is negative) at the origin, and we get from equations (43) and (48) a power-law density behavior: $\rho \propto r^{-(3+\eta(0))}$. If $s(0) = 0$, $X(r)$ diverges faster than logarithmically, and $\rho$ vanishes nonanalytically there [as in the constant $s(r) = 0$ case, where the "gas" is concentrated in an infinitely thin shell].

At large radii, if $s(\infty) \neq 0$, $X$ diverges logarithmically, and $\rho$ has a power-law behavior: $\rho \propto r^{-(3+\eta(\infty))}$. If $s(\infty) = 0$, $X$ diverges faster, and $\rho$ vanishes faster than a power (e.g., as an exponential of a power).

Of the many models that can be produced, we consider one class in more detail: Take $s(r)$ to vary monotonically between the values $v$ at $r = 0$ and $\tau$ at infinity as

$$s(r) = v + \tau(r/a)^d \frac{1}{1 + (r/a)^d}. \hfill (49)$$

Here $d$ is a power that controls the sharpness of the transition of $s$ from $v$ to $\tau$, and, when either vanishes, $d$ measures the speed with which $s$ does $d \to -d$ is tantamount to $v \to \tau$. We take $v \neq 0$ ($v = 0$ can be treated by changing the sign of $d$); then, for $\tau \neq 0$, the integral in equation (43) for $X$ gives

$$e^{\eta X} = \frac{x^q(v + \tau x^d)^{\eta/d}}{x_{v}^q(v + \tau x_{v}^d)^{\eta/d}}, \hfill (50)$$

where $x \equiv r/a$, $x_0 \equiv r_0/a$, $\alpha \equiv \eta/v$, and $\delta \equiv \eta/\tau - \eta/v$. The mass profile is thus

$$m(r) = \frac{2\eta x^q(v + \tau x^d)^{\eta/d}}{x_{v}^q(v + \tau x_{v}^d)^{\eta/d} + x_{v}^q(v + \tau x_{v}^d)^{\eta/d}}, \hfill (51)$$

and evinces the expected appearance of two distinct scale lengths. The density law is obtained by substituting equation (50) in equation (46):

$$\rho(r) = \frac{\eta M_g}{2\pi r^3 s(r)} \frac{x^q(v + \tau x^d)^{\eta/d} x_{v}^q(v + \tau x_{v}^d)^{\eta/d}}{[x_{v}^q(v + \tau x_{v}^d)^{\eta/d} + x_{v}^q(v + \tau x_{v}^d)^{\eta/d}]^2} \hfill (52)$$

The density is a power law in radius at both ends with a power $-(3 - \eta/v)$ at the origin, and $-(3 + \eta/\tau)$ at infinity.

The case $\tau = 0$ requires special treatment: here one finds

$$e^{\eta X} = \left(\frac{x}{x_0}\right)^{\alpha} e^{(\alpha/d)(x - x_0)}, \hfill (53)$$

giving rise to a density run that decreases exponentially at large $r$:

$$\rho(r) \propto r^{-(2 + \alpha)} e^{-(\alpha/d)(r/a)}, \hfill (54)$$

for $r \to \infty$.

We thus get a class of models that are more flexible as regards the density run, but they contain two scale lengths instead of one, and are certainly less wieldy.

I thank Scott Tremaine for useful comments and suggestions.

REFERENCES

Bertin, G., & Stiavelli, M. 1989, ApJ, 338, 723
Binney, J., & Tremaine, S. 1987, Galactic Dynamics (Princeton: Princeton Univ. Press)
Dehnen, W. 1993, MNRAS, 265, 250
Gradshetyn, I. S., & Ryzhik, I. M. 1980, Tables of Integrals, Series, and Products (San Diego: Academic)
Hernquist, L. 1990, ApJ, 356, 359
Jaffe, W. 1983, MNRAS, 202, 995
Tremaine, S., Richstone, D. O., Byun, Y., Dressler, A., Faber, S. M., Grillmair, C., Kormendy, J., & Lauer, T. R. 1994, AJ, 107, 634