Sorting Permutations with Fixed Pinnacle Set

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Abstract

We give a positive answer to a question raised by Davis et al. (Discrete Mathematics 341, 2018), concerning permutations with the same pinnacle set. Given \( \pi \in S_n \), a pinnacle of \( \pi \) is an element \( \pi_i \) (\( i \neq 1, n \)) such that \( \pi_{i-1} < \pi_i > \pi_{i+1} \). The question is: given \( \pi, \pi' \in S_n \) with the same pinnacle set \( S \), is there a sequence of operations that transforms \( \pi \) into \( \pi' \) such that all the intermediate permutations have pinnacle set \( S \)? We introduce balanced reversals, defined as reversals that do not modify the pinnacle set of the permutation to which they are applied. Then we show that \( \pi \) may be sorted by balanced reversals (i.e. transformed into a standard permutation \( \text{Id}_S \)), implying that \( \pi \) may be transformed into \( \pi' \) using at most \( 4n - 2 \min\{p, 3\} \) balanced reversals, where \( p = |S| \geq 1 \). In case \( p = 0 \), at most \( 2n - 1 \) balanced reversals are needed.

Keywords. permutation; reversal; pinnacle set

1 Introduction

In a permutation \( \pi = (\pi_1 \pi_2 \ldots \pi_n) \) from the symmetric group \( S_n \), a peak is an index \( i \neq 1, n \) such that \( \pi_{i-1} < \pi_i > \pi_{i+1} \), whereas a valley is an index \( i \neq 1, n \) such that \( \pi_{i-1} > \pi_i < \pi_{i+1} \). Descents and ascents respectively identify indices \( i \) such that \( \pi_i > \pi_{i+1} \) and \( \pi_i < \pi_{i+1} \).

Many studies have been devoted to the combinatorics of peaks, especially to enumeration and counting problems [1, 2, 3, 6, 9, 11] (and many others). They identify strong and elegant relationships between peaks or descents in permutations, on the one hand, and Fibonacci numbers, Eulerian numbers, chains in Eulerian posets, etc. on the other hand.

In [5], Davis et al. revive the point of view considered in [4], and propose to identify peaks by their values rather than by their positions. They call a pinnacle any element \( \pi_i \) with \( i \neq 1, n \) such that \( \pi_{i-1} < \pi_i > \pi_{i+1} \), and show that considering pinnacles instead of peaks changes the combinatorial considerations behind counting and enumerating permutations with a given peak/pinnacle set. They characterize the sets of integers that may be the pinnacle set of a permutation (so-called admissible pinnacle sets), count them, and propose bounds, involving the Stirling number, on the numbers of permutations with given pinnacle set.

They further ask several questions, one of which is considered in this paper:

**Question 4.2 in [5].** For a given admissible pinnacle set \( S \), is there a class of operations that one may apply to any \( \pi \in S_n \) whose pinnacle set is \( S \) to obtain any other permutation \( \pi' \in S_n \) with the same pinnacle set, and no other permutation?

This question is motivated by the search for similarities between pinnacles and peaks [5]. In this paper we give a positive answer to this question. More particularly, we identify a reduced set of reversals (the operation that reverses a block of a permutation) - called balanced reversals - which do not modify the set of pinnacles. Then we show that it is possible to transform any permutation with pinnacle set \( S \) into a canonical permutation of the same size with pinnacle set \( S \) by applying a sequence of at most \( 2n - \min\{p, 3\} \) balanced reversals. As the inverse transformation is always possible, this answers Question 4.2. above.

The paper is organized as follows. Section 2 contains the main definitions and notations. In Section 3 we identify balanced reversals and state the main results. Section 4 is devoted to the proof of...
our main theorem. This proof describes the algorithm allowing us to find the sequence of reversals transforming a given permutation into the canonical permutation. For the sake of completeness, we give in Section 5 the implementation details for an optimal running time of our algorithm. Section 6 is the conclusion.

2 Definitions and notations

Permutations $\pi, \pi', \pi''$ we use in the paper belong to the symmetric group $S_n$, for a given integer $n$. Elements $n + 1$ and $n + 2$ are artificially added at the beginning and respectively the end of each permutation, so that a permutation $\pi \in S_n$ is written as $\pi = (\pi_0 \pi_1 \pi_2 \ldots \pi_n \pi_{n+1})$ with $\pi_0 = n + 1$ and $\pi_{n+1} = n + 2$. For each $i > 0$, we define $\text{Prec}_\pi(\pi_i) = \pi_{i-1}$ and for each $i < n + 1$ we define $\text{Next}_\pi(\pi_i) = \pi_{i+1}$. The block of $\pi$ with endpoints $\pi_a$ and $\pi_b$ (where $a \leq b$) is defined as $(\pi_a \pi_{a+1} \ldots \pi_{b-1} \pi_b)$.

A pinnacle is any element $\pi_i$ with $i \neq 0, n + 1$ such that $\pi_{i-1} < \pi_i > \pi_{i+1}$. Similarly to pinnacles (whose indices are the peaks), we define the dells (whose indices are the valleys). A dell of $\pi$ is any element $\pi_i$ with $i \neq 0, n + 1$ such that $\pi_{i-1} > \pi_i < \pi_{i+1}$. The shape of the permutation $\pi$ is the permutation $B_\pi = (y_0 v_1 y_1 v_2 y_3 \ldots y_{p-1} v_{p+1} y_{p+1})$ where $v_1, \ldots, v_{p+1}$ are the dells of $\pi$, $y_1, \ldots, y_p$ are its pinnacles, whereas $y_0 = \pi_0 = n + 1$ and $y_{p+1} = \pi_{n+1} = n + 2$. The presence of the elements $n + 1$ and $n + 2$ at the beginning and the end of the permutation adds no pinnacle to the initial permutation, and ensures that dells $v_1, v_{p+1}$ exist.

Moreover, let $A_\pi(v_i, y_i)$ with $1 \leq i \leq p + 1$ be the set of elements in the block of $\pi$ with endpoints $v_i$ and $y_i$, which are neither a dell nor a pinnacle. Similarly let $D_\pi(y_i, v_{i+1})$ with $0 \leq i \leq p$ be the set of elements in the block of $\pi$ with endpoints $y_i$ and $v_{i+1}$, which are neither a dell nor a pinnacle. Sets $A_\pi()$ and $D_\pi()$ are respectively called ascending and descending sets of $\pi$. Note that $y_0$ and $y_n$ belong respectively to the leftmost descending and the rightmost ascending set. The dells and pinnacles belong to no such set.

Example 1. Consider $\pi = (11 8 \overline{6} \overline{7} \overline{4} 3 2 \overline{1} \overline{5} \overline{10} 9 12)$ from $S_{10}$ (thus $n = 10$) with elements $11$ and $12$ artificially added. Dells are underlined, pinnacles are overlined, $p = 2$. The shape is $B_\pi = (11 \overline{6} \overline{7} \overline{1} \overline{0} \overline{9} \overline{12})$. The ascending sets are $A_\pi(6, 7) = \emptyset$, $A_\pi(1, 10) = \{5\}$ and $A_\pi(9, 12) = \{12\}$, whereas the descending sets are $D_\pi(11, 6) = \{8, 11\}$, $D_\pi(7, 1) = \{2, 3, 4\}$ and $D_\pi(10, 9) = \emptyset$.

We define a canonical permutation according to [5]. Given a set $S = \{s_1, s_2, \ldots, s_d\}$ and an integer $n > 2d$, the canonical permutation $\text{Id}_S \in S_n$ with pinnacle set $S$ is the permutation built as follows: place the elements of $S$ in increasing order on positions $2, 4, \ldots, 2d$ respectively; then place the elements in $\{1, 2, \ldots, n\} \setminus S$ in increasing order on positions $1, 3, \ldots, 2d - 1, 2d + 1, \ldots, n$. With our convention, elements $n + 1$ and $n + 2$ are added at the beginning and respectively the end of $\text{Id}_S$.

Definition 1. Let $w_1, w_2$ be two elements of $\pi$, such that $w_1 = \pi_a, w_2 = \pi_b$ and $0 < a < b < p + 1$. The reversal $\rho(w_1, w_2)$ is the operation that transforms $\pi = (\pi_0 \ldots \pi_{a-1} \pi_a \pi_{a+1} \pi_{a+2} \ldots \pi_{n+1})$ into $\pi' = (\pi_0 \ldots \pi_{a-1} \pi_{b-1} \ldots \pi_{a+1} \pi_a \pi_{b+1} \ldots \pi_{n+1})$. Notation: $\pi' = \pi \cdot \rho(w_1, w_2)$.

Example 2. With $S = \{7, 10\}$, the canonical permutation $\text{Id}_S \in S_{10}$ is $\text{Id}_S = (11 \overline{1} \overline{2} \overline{3} \overline{4} 5 \overline{6} \overline{8} 9 12)$, with shape $(11 \overline{1} \overline{2} \overline{3} \overline{10} \overline{9} \overline{12})$. Then $S$ has the same pinnacle set as $\pi$ in Example 1 but not the same dells and thus not the same shape. Applying $\rho(1, 10)$ to $\text{Id}_S$ yields the permutation $\text{Id}_S \cdot \rho(1, 10) = (11 10 \overline{2} \overline{7} \overline{1} \overline{3} 4 5 6 8 9 12)$. It may be noticed that the resulting permutation has pinnacle set $\{7\}$, showing that reversals may modify the pinnacle set.

Definition 2. Let $\pi \in S_n$. A reversal $\rho(w_1, w_2)$ is a balanced reversal for $\pi$ if $\pi$ and $\pi \cdot \rho(w_1, w_2)$ have the same pinnacle set.
Balanced reversals are characterized in the next section. In order to identify appropriate balanced reversals when needed, we make use of cutpoints. Let \( i \) be an integer with \( 1 \leq i \leq p + 1 \) and \( z \) be an element of \( \pi \) not belonging to \( \Pi_\pi(v_i, y_i) \), such that \( v_i < z < y_i \). The largest element \( e \) of \( \Pi_\pi(v_i, y_i) \cup \{v_i\} \) such that \( e < z \) is called the cutpoint of \( z \) on \( \Pi_\pi(v_i, y_i) \) and is denoted \( \text{cut}A_\pi(z, v_i, y_i) \). The similar definition holds for \( \Pi_\pi(y_i, v_{i+1}) \). Let \( i \) be an integer with \( 0 \leq i \leq p \) and \( z \) be an element of \( \pi \) not belonging to \( \Pi_\pi(y_i, v_{i+1}) \), such that \( v_{i+1} < z < y_i \). The largest element \( e \) of \( \Pi_\pi(y_i, v_{i+1}) \cup \{v_{i+1}\} \) such that \( e < z \) is called the cutpoint of \( z \) on \( \Pi_\pi(y_i, v_{i+1}) \) and is denoted \( \text{cut}D_\pi(z, y_i, v_{i+1}) \).

Finally, define the following problem:

**Balanced Sorting Problem**

**Input:** A permutation \( \pi \in S_n \) with pinnacle set \( S \).

**Question:** Is it possible to transform \( \pi \) into \( \text{Id}_S \in S_n \) using only balanced reversals?

The difficulty in solving this problem has mainly two origins: first, one cannot perform any wished reversal since a reversal is not necessarily balanced (see Example 2); and second, given a set \( S \) of pinnacles and a permutation \( \sigma \) of the elements in \( S \), it is possible that no permutation \( \pi \) of given size \( n \) and with pinnacle \( S \) exists that has the pinnacles in the order (from left to right) given by \( \sigma \).

**Example 3.** Let set \( S = \{3, 5, 7\} \). Then with \( n = 7 \) and \( \sigma = (3 \ 5 \ 7) \) we may find the permutation \( \pi = (8 \ 2 \ 3 \ 1 \ 7 \ 6 \ 4 \ 9) \), but with \( n = 7 \) and \( \sigma = (3 \ 7 \ 5) \) there is no permutation from \( S_n \) with pinnacles in this order.

Therefore, the **Balanced Sorting Problem** is a question of feasibility in the first place. The optimal sorting is proposed as an open problem in the conclusion.

### 3 Main results

Let \( \pi \in S_n \) be a permutation with pinnacle set \( S \) such that \( |S| = p \). The main result of the paper is the following one.

**Theorem 1.** There is a sequence \( R \) that solves the **Balanced Sorting Problem** on \( \pi \) using at most \( 2n - \min\{p, 3\} \) balanced reversals when \( p \geq 1 \), and at most \( 2n - 1 \) reversals when \( p = 0 \).

An answer to Question 4.2 is an immediate consequence of this theorem.

**Corollary 1.** Let \( \pi, \pi' \in S_n \) be two permutations with pinnacle set \( S \) such that \( |S| = p \). Then, when \( p \geq 1 \) there is a sequence \( T \) of at most \( 4n - 2 \min\{p, 3\} \) balanced reversals that transforms \( \pi \) into \( \pi' \), using only intermediate permutations with pinnacle set \( S \). When \( p = 0 \), \( T \) contains at most \( 4n - 2 \) balanced reversals.

**Proof.** Let \( R \) be the sequence of balanced reversals needed to sort \( \pi \) according to Theorem 1. Similarly, let \( R' = (\rho(w_1, w'_1), \rho(w_2, w'_2), \ldots, \rho(w_q, w'_q)) \) be the sequence of balanced reversals needed to sort \( \pi' \). Let \( T \) be the sequence made of \( R \) followed by the sequence \( \rho(w'_q, w_q), \rho(w'_{q-1}, w_{q-1}), \ldots, \rho(w'_1, w_1) \). Then \( T \) transforms \( \pi \) into \( \text{Id}_S \) and subsequently \( \text{Id}_S \) into \( \pi' \) using only balanced reversals. The definition of a balanced reversal guarantees that all the intermediate permutations have pinnacle set \( S \).

Recall that, by Definition 1, in a reversal \( \rho(w_1, w_2) \) the endpoints \( w_1 \) and \( w_2 \) are in this order from left to right on \( \pi \) and are distinct from \( y_0, y_{p+1} \). Depending on the position of \( w_1 \) and \( w_2 \) in \( \pi \), balanced reversals are of different types and imply different constraints, that need to be satisfied in order to
Table 1: Different types of balanced reversals. Each reversal is defined by constraints on $w_1$ and $w_2$, defining their places (middle column) and the relative orders required between some elements (rightmost column). Recall that $w_1$ and $w_2$ are in this order from left to right on $\pi$ and are distinct from $y_0, y_{p+1}$. Then $\text{Prec}_\pi(w_1)$ and $\text{Next}_\pi(w_2)$ always exist.

| Type | Positions of $w_1$, $w_2$ | Constraints |
|------|---------------------------|-------------|
| A.1  | $w_1 \in A_\pi(v_i, y_i)$, $w_2 \in D_\pi(y_i, v_{j+1}), i \leq j$ | if $\text{Next}_\pi(w_2) \neq v_{j+1}$ then $w_1 > \text{Next}_\pi(w_2)$ and if $\text{Prec}_\pi(w_1) \neq v_i$ then $w_2 > \text{Prec}_\pi(w_1)$ |
| A.2  | $w_1 = y_i, w_2 \in D_\pi(y_i, v_{j+1}), i \leq j$ | $w_1 > \text{Next}_\pi(w_2)$ |
| A.2s | $w_1 \in A_\pi(v_i, y_i), w_2 = y_i, i \leq j$ | $w_2 > \text{Prec}_\pi(w_1)$ |
| A.3  | $w_1 = v_i, w_2 \in D_\pi(y_i, v_{j+1}), i \leq j$ | if $\text{Next}_\pi(w_2) \neq v_{j+1}$ then $w_1 > \text{Next}_\pi(w_2)$ and if $\text{Prec}_\pi(w_1) = y_{i-1} \neq y_0$ then $w_2 < \text{Prec}_\pi(w_1)$ |
| A.3s | $w_1 \in A_\pi(v_i, y_i), w_2 = v_{j+1}, i \leq j$ | if $\text{Prec}_\pi(w_1) \neq v_i$ then $w_2 > \text{Prec}_\pi(w_1)$ and if $\text{Next}_\pi(w_2) = y_{j+1} \neq y_{p+1}$ then $w_1 < \text{Next}_\pi(w_2)$ |
| B.1  | $w_1 \in D_\pi(y_{i-1}, v_i), w_2 \in A_\pi(v_i, y_i), i \leq j$ | $w_1 < \text{Next}_\pi(w_2)$ and $w_2 < \text{Prec}_\pi(w_1)$ |
| B.2  | $w_1 = y_i, w_2 \in A_\pi(v_j, y_j), i < j$ | $\text{Prec}_\pi(w_1) = y_i, w_2 < \text{Prec}_\pi(w_1), \text{Next}_\pi(w_2) \neq y_j$ and $\text{Next}_\pi(w_2) < w_2$ |
| B.2s | $w_1 \in D_\pi(y_{i-1}, v_i), w_2 = y_j, i \leq j$ | $\text{Prec}_\pi(w_1) \neq y_{i-1}, \text{Prec}_\pi(w_1) < w_2, \text{Next}_\pi(w_2) = v_{j+1}$ and $w_1 < \text{Next}_\pi(w_2)$ |
| B.3  | $w_1 = v_i, w_2 \in A_\pi(v_j, y_j), i \leq j$ | $w_2 < \text{Prec}_\pi(w_1)$ and if $\text{Next}_\pi(w_2) = y_j \neq y_{p+1}$ then $w_1 < \text{Next}_\pi(w_2)$ |
| B.3s | $w_1 \in D_\pi(y_{i-1}, v_i), w_2 = v_j, i \leq j$ | $w_1 < \text{Next}_\pi(w_2)$ and if $\text{Prec}_\pi(w_1) = y_{i-1} \neq y_0$ then $w_2 < \text{Prec}_\pi(w_1)$ |
| C.1  | $w_1 = v_i, w_2 = y_j, i \leq j$ | $\text{Prec}_\pi(w_1) \neq y_{i-1}, w_2 > \text{Prec}_\pi(w_1)$ and $w_1 > \text{Next}_\pi(w_2)$ |
| C.1s | $w_1 = y_i, w_2 = v_j, i \leq j$ | $\text{Next}_\pi(w_2) \neq v_{j+1}, w_1 > \text{Next}_\pi(w_2)$ and $w_2 > \text{Prec}_\pi(w_1)$ |
| C.2  | $w_1 = v_i, w_2 = v_j, i < j$ | if $\text{Prec}_\pi(w_1) = y_{i-1} \neq y_0$ then $w_2 < \text{Prec}_\pi(w_1)$ and if $\text{Next}_\pi(w_2) = y_{j+1} \neq y_{p+1}$ then $w_1 < \text{Next}_\pi(w_2)$ |
| C.3  | $w_1 = y_i, w_2 = y_j, i < j$ | $w_1 > \text{Next}_\pi(w_2)$ and $w_2 > \text{Prec}_\pi(w_1)$ |

guarantee that the reversal is balanced. Table 1 presents the different possible positions for $w_1$ and $w_2$, each defining a type. On the rightmost column are given the constraints that $w_1, w_2$ and their adjacent elements must fulfill in order to obtain a balanced reversal. For instance, reversal $\rho(w_1, w_2)$ of type A.1 is obtained when $w_1$ belongs to an ascending set of $\pi$ and $w_2$ belongs to a descending set of $\pi$. One further requires that the following constraints be verified: when $\text{Next}_\pi(w_2) \neq v_{j+1}$ we must have $w_1 > \text{Next}_\pi(w_2)$; when $\text{Prec}_\pi(w_1) \neq v_i$ we must have $w_2 > \text{Prec}_\pi(w_1)$.

The standard cases A.1 and B.1 are shown in Figure 1. The other cases are obtained from A.1 or B.1 when $w_1$ or $w_2$ or both of them are a pinnacle or a dell. Cases denoted A.x are obtained from case A.1 only, cases B.x are obtained from B.1 only and cases C.x are obtained from both A.1 and B.1. Symmetrical cases are identified by an “s”. We show below that these types form altogether the entire collection of balanced reversals.

**Proposition 1.** Reversal $\rho(w_1, w_2)$ is balanced iff it belongs to the collection of types in Table 1.

**Proof.** $\Rightarrow$: Several cases may appear.

If both $w_1$ and $w_2$ belong to ascending sets of the permutation, that is $w_1 \in A_\pi(v_i, y_i)$ and $w_2 \in A_\pi(v_j, y_j)$ with $i \leq j$, then when the reversal is performed $w_2$ or $\text{Prec}_\pi(w_1)$ becomes a new pinnacle, a contradiction. A similar reasoning holds when both $w_1$ and $w_2$ belong to descending sets of the permutation. So these cases cannot appear.

If one element among $w_1$ and $w_2$ belongs to an ascending set of the permutation, and the other one to a descending set of it, then it is easy to check that only the conditions in type A.1. or in type B.1. guarantee that no new pinnacle is added.

If exactly one element among $w_1$ and $w_2$ is a dell, then we necessarily have one of the types A.3, B.3, C.1 (or the symmetric ones) since any other condition creates a new pinnacle or removes an
Figure 1: Intuitive description of types A.1 (top) and B.1 (bottom) where elements are placed on ascending and descending regions according to their values (high or low). A consequence is that neighboring elements on the permutation are not always at equal distance on the horizontal axis. Elements $w_1$ and $w_2$ are drawn as grey circles, $\text{Prec}_\pi(w_1)$ is drawn as a white square and $\text{Next}_\pi(w_2)$ is drawn as a black square. a) Permutation $\pi$. b) Result once the reversal $\rho(w_1, w_2)$ is applied.

existing one.

If exactly one element among $w_1$ and $w_2$ is a pinnacle, then the other one is either a dell or belongs to an ascending or descending set of $\pi$. The former possibility necessarily leads to type C.1 (or the symmetric one). The latter possibility results into types A.2, A.2s, B.2 or B.2s.

If both $w_1$ and $w_2$ are pinnacles, or both are dells, then we must have the conditions in types C.2 or C.3 to preserve the pinnacle set.

"$\Leftarrow$": This part only requires to check, for each type, that the pinnacle set is not modified under the indicated conditions.

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4 Proof of Theorem

We assume below that $p \geq 1$ and postpone the case $p = 0$ to Remark at the end of the section.

In order to build the sequence $R$ required in Theorem we follow three steps:

1. Sort the $p$ pinnacles of $\pi$ in increasing order.

2. Place the wished dells (i.e. the dells of $\text{Id}_S$) as dells of $\pi$, in increasing order.
3. Move each element belonging to an ascending or descending set of $\pi$ on the rightside of the last dell of $\pi$, in increasing order.

The result of these three steps is $Id_{\mathcal{S}}$. Then $R$ is the sequence of all the balanced reversals performed during these three steps in order to transform $\pi$ into $Id_{\mathcal{S}}$.

**Remark 1.** Note that in the subsequent, when permutation $\pi$ is successively transformed using balanced reversals into some other permutation $\pi'$, the elements of $\pi'$ are identified both by their names in $\pi$, i.e. $\pi'_1, \pi'_2$ etc. and by their names in $\pi$, according to the needs. Once a given task is fulfilled by applying one or several balanced reversals, the resulting permutation is renamed as $\pi$, so that the following task begins with an initial permutation still denoted $\pi$.

### 4.1 Step 1: Sort the pinnacles

This is done by successively replacing the pinnacle $y_k$, for $k = 1, 2, \ldots, p - 1$, by the $k$-th lowest pinnacle without modifying the set of pinnacles. Then in the resulting permutation the highest pinnacle is necessarily $y_p$. The other elements are not constrained at this step. Algorithm 1 presents the balanced rotations to be performed, as identified in this subsection.

**Lemma 1.** There is a sequence of at most $p - 1$ balanced reversals that transforms $\pi$ with pinnacle $S$ into $\pi^*$ with pinnacle $S$ such that $y_1^*$ is the lowest pinnacle in $\pi^*$. Moreover, when $p \geq 3$, exactly one of the two following configurations occurs:

- *(X)* the sequence contains exactly $p - 1$ balanced reversals and $y_p^*$ is the highest pinnacle in $S$.
- *(Y)* the sequence contains at most $p - 2$ balanced reversals.

**Proof.** If $y_1$ is already the lowest pinnacle, then nothing is done. Assume now the lowest pinnacle is $y_i$ with $i \neq 1$. Several cases are possible, that we present below, before giving the algorithm.

**Case 1).** $y_i < v_1$ and for all $h > i$, we have $y_h < v_1$.

Then $v_{p+1} < y_p < v_1$ and the balanced reversal (type C.2) $\rho(v_1, v_{p+1})$ allows to obtain a permutation $\pi'$ with $v'_1 = v_p < v_1$. In $\pi'$:

- a) if $y_i = y'_1$, then we are done.
- b) if $y_i \neq y'_1$ and $y_i < v'_1$, then we have that $y_{p+1} = y_1 > v_1 > v'_1$ and we deduce that at least one pinnacle placed on the rightside of $y_i$ is larger than $v'_1$. Then $\pi'$ satisfies Case 2 below.
- c) if $y_i \neq y'_1$ and $y_i > v'_1$, then $\pi'$ satisfies Case 3 below.

**Case 2) $y_i < v_1$ and there is $h > i$ such that $y_h > v_1$.**

Then we assume w.l.o.g. that $h$ is the minimum index with this property. Then $v_h < v_1$, otherwise we also have $y_{h-1} > v_h > v_1$ which contradicts the choice of $h$. Now, $\rho(v_1, v_h)$ is a balanced reversal (type C.2) since $\text{Prec}_\pi(v_1)$ is not a pinnacle and even if it may happen that $\text{Next}_\pi(v_h) = y_h$ we have $y_h > v_1$. Once $\rho(v_1, v_h)$ is performed, the new first dell is $v''_1 = v_h$ and is smaller than $v_1$ as proved above. Let us call $\pi''$ this new permutation, whose elements satisfy $y''_t = y_t$ for some $t$, $y''_{h-1} = y_1, y''_h = y_h, y''_1 = y_{h-1}$. In $\pi''$:

- a) if $t = 1$, then we are done.
Algorithm 1 Permutation sorting by balanced reversals: Step 1

Input: A permutation $\pi \in S_n$ with pinnacle set $S$ of cardinality $p$.

Output: The permutation $\pi$ whose pinnacles have been placed in increasing order (Proposition 2)

1. $x \leftarrow \min\{y_1, \ldots, y_p\}$  // $x$ is the lowest pinnacle
2. let $y_i = x$  // $x$ has index $i$
3. if $(i \neq 1)$ and $y_i < v_1$ then
   4. $h \leftarrow \min\{h \mid h > i, y_h > v_1\} \cup \{0\}$  // $h = 0$ occurs when the first set is empty
   5. if $h = 0$ then
      6. $\pi \leftarrow \pi \cdot \rho(v_1, v_{p+1})$  // Case 1
   7. end if
   8. end if
9. while $x$ is not the leftmost pinnacle of $\pi$ do
   10. let $y_i = x$  // $x$ has index $i \neq 1$
   11. if $y_i < v_1$ then
      12. $h \leftarrow \min\{h \mid h > i, y_h > v_1\}$
      13. $\pi \leftarrow \pi \cdot \rho(v_1, v_h)$  // Case 2
   14. else
      15. $e \leftarrow \text{cutA}_\pi(y_i, v_1, y_1); \pi \leftarrow \pi \cdot \rho(\text{Next}_\pi(e), y_i)$  // Case 3
   16. end if
17. end while  // the lowest pinnacle is now placed in position $y_1$
18. for $k = 1$ to $p - 2$ do
   19. $x \leftarrow \min\{y_{k+1}, \ldots, y_p\}$; let $y_i = x$  // $x$ is the lowest remaining pinnacle
   20. if $t \neq k + 1$ then
      21. $e \leftarrow \text{cutA}_\pi(y_i, v_{k+1}, y_{k+1}); \pi \leftarrow \pi \cdot \rho(\text{Next}_\pi(e), y_i)$  // $x$ is now in position $y_{k+1}$
   22. end if
23. end for
24. Return $\pi$

b) if $t \neq 1$ and $y_i < v''_1$, then $\pi''$ satisfies Case 2 since $v''_1 = v_h < v_1 < y_1 = y''_{h-1}$, so there is at least one index as required in Case 2. The smaller such index, say $g$, satisfies $t < g < h$.

c) if $t \neq 1$ and $y_i > v''_1$, then $\pi''$ is in Case 3 below.

Condition $t < g < h$ in item b means that the recursivity we find here will end, as we show later (once Case 3 is presented).

Case 3) $y_i > v_1$.

Let $e = \text{cutA}_\pi(y_i, v_1, y_1)$. Then $\rho(\text{Next}_\pi(e), y_i)$ is a balanced reversal (type A.2s if $\text{Next}_\pi(e) \neq y_1$ and type C.3 otherwise) since $y_i > e$ and $\text{Next}_\pi(e) > y_i > \text{Next}_\pi(y_i)$, both by the definition of the cutpoint $e$. This reversal places $y_i$ as the leftmost pinnacle.

The algorithm consists in applying Case 1 if necessary, then Case 2 as long as the current permutation requires it (in item b of Case 2) and finally Case 3 if needed. It is presented in Algorithm 1. In order to compute the number of balanced reversals performed in the worst case, we denote:

- $\pi$ the initial permutation
- $\pi^0$ the permutation obtained at the end of Case 1, whether it is applied or not (so that $\pi^0 = \pi$ if not).
- $\pi^1, \ldots, \pi^m$ the $m$ successive permutations obtained using Case 2 ($m = 0$ if Case 2 is not applied).
• \( \pi^{m+1} \) the permutation obtained once Case 3 is applied, if it is applied.

As a consequence, if \( m > 0 \) then for \( 0 \leq q \leq m - 1 \), permutation \( \pi^q \) is transformed into permutation \( \pi^{q+1} \) using the balanced reversal \( \rho(v^0_1, v^q_i) \) of type C.2, as mentioned in Case 2 before. Here, \( h^q \) denotes the minimum index \( h \) computed in Case 2 for each \( \pi^q \), i.e. \( h^0 = h \) (see Case 2), \( h^1 = g \) (see item b in Case 2), and so on. Indices \( h^0, h^1, \ldots, h^{m-1} \) computed respectively in \( \pi^0, \pi^1, \ldots, \pi^{m-1} \) satisfy (see again Case 2 item b where we show that \( g < h \)):

(i) \( h^{m-1} < h^{m-2} < \ldots < h^1 < h^0 \)

(ii) \( y^q_{h^q}, y^q_{h^{q-1}}, \ldots, y^q_{h^1}, y^q_{h^0} \) are pinnacles in the permutation \( \pi^q \) for each \( q \) with \( 0 \leq q \leq m - 1 \), in this order from left to right. Moreover, each permutation \( \pi^q \) inherits the pinnacles of the previous permutation \( \pi^{q-1} \), that is, \( y^q_s = y^{q-1}_s \) for all \( 0 \leq s \leq q - 1 \) (meaning that the pinnacles as well as their indices in the permutation are the same).

(iii) \( h^q \) and \( v^q_i \) satisfy the conditions of Case 2 item b in \( \pi^q \) for each \( q \) with \( 0 \leq q \leq m - 1 \).

(iv) permutation \( \pi^m \) obtained when Case 2 does not apply any longer contains all the pinnacles \( y^m_{h^m}, y^m_{h^{m-1}}, \ldots, y^m_{h^1}, y^m_{h^0} \) built by the previous iterations, in this order from left to right.

The number of balanced reversals performed in this step strongly depends on the number \( m \) of reversals performed at worst in Case 2. By item (iv) above, \( y^m_{h^m}, y^m_{h^{m-1}}, \ldots, y^m_{h^1}, y^m_{h^0} \) are m pinnacles in the permutation \( \pi^m \) obtained when Case 2 does not apply any longer, in this order from left to right.

The number of such pinnacles (i.e. \( m \)) is upper bounded by \( p - 2 \), since (1) \( m \leq p \), and (2) when the \( m \)-th reversal is applied (to \( \pi^{m-1} \)), at least two pinnacles exist in the block to be reversed, the current leftmost pinnacle \( y^m_{1} \) and \( y_i \). They are distinct, otherwise no reversal is applied. Thus \( m \leq p - 2 \). Now:

• If \( m = p - 2 \), then the pinnacles of \( \pi^{m-1} \) are necessarily, in this order from left to right, \( y^{m-1}_1, y_i (= y^{m-1}_2), y^{m-1}_{h^{m-1}} (= y^{m-1}_3), \ldots, y^{m-1}_{h^0} (= y^{m-1}_p) \). The last reversal due to Case 2, i.e. \( \rho(v^{m-1}_1, v^{m-1}_{h^m}) \), places \( y_i \) as the leftmost pinnacle, and thus we are done. Item a in Case 2 applies, and no other reversal is needed. Then the total number of reversals applied in Step 1 is \( p - 1 \) when Case 1 applies and \( p - 2 \) otherwise. In the latter case, configuration (Y) in the lemma occurs. The former case is fixed using property (P) below.

• If \( m \leq p - 3 \), then we distinguish again several situations:
  - When \( m \leq p - 4 \), the total number of reversals applied in Step 1 is \( p - 3 \) when exactly one of Case 1 and Case 3 applies, and \( p - 2 \) when both Case 1 and Case 3 apply. Configuration (Y) then occurs.
  - When \( m = p - 3 \) and \( h^0 \neq p \), then as above the pinnacles of \( \pi^{m-1} \) must be \( y^{m-1}_1, y_i (= y^{m-1}_2), y^{m-1}_{h^{m-1}} (= y^{m-1}_3), \ldots, y^{m-1}_{h^0} (= y^{m-1}_p) \) and \( y^{m-1}_{p-1} \), where \( y^m_{p-1} \) is the rightmost pinnacle, that is never involved in the reversals. The last reversal due to Case 2, i.e. \( \rho(v^{m-1}_1, v^{m-1}_{h^m}) \), places \( y_i \) as the leftmost pinnacle, and thus we are done. Item a in Case 2 applies, and no other reversal is needed. Then the total number of reversals in step 1 is \( p - 2 \) when Case 1 applies and \( p - 3 \) otherwise, yielding configuration (Y) again.
  - When \( m = p - 3 \) and \( h^0 = p \), the total number of reversals applied in Step 1 is \( p - 2 \) when at most one of Case 1 and Case 3 applies (configuration (Y) again), and \( p - 1 \) when both Case 1 and Case 3 apply. The latter case is fixed using property (P) below.
We now finish the two unresolved cases, both of which occur when Case 1 applies and $y_p^m = y_{h_0}^m$ (recall that the pinnacles with indices $h^0, \ldots, h^{n-1}$ are inherited from one execution of Case 2 to the next one, by affirmation (ii) above).

(P) If Case 1 applies and in Case 2 we have $h_0 = p$, then $y_p^m$ is the highest pinnacle in $\pi^m$.

Indeed, since Case 1 applies, in $\pi$ we have $y_t < v_1$, for all $t \geq i$. Thus in $\pi'$ (see Case 1) we have $y'_t < v'_{p+1} = v_1$ for all $u \leq s$, where $y'_s = y_t$. When $\pi'$ is renamed as $\pi^0$, we have:

$$y_u^0 < v'_{p+1} = v_1$$

for all $u \leq s$, where $y_u^0 = y_t$. (1)

Moreover, since Case 2 applies with $h^0 = p$, we have that:

$$y_p^0 > v_1^0 = v_{p+1}$$

(2)

$$y_r^0 < v_1^0$$

for all $r$ with $s \leq r < p$. (3)

By (1), $y_p^0 < v'_{p+1}$ and by definition $v'_{p+1} < y'_p$, thus $y_u^0 < y_p^0$ for all $u \leq s$, where $y_u^0 = y_t$. By (2) and (3), $y_p^0$ is the highest pinnacle in $\pi^0$. Due to affirmation (ii) above, $y_p^m$ is the highest pinnacle in $\pi^m$ and property (P) is proved.

Now, property (P) applies in each of the two unresolved cases and yield configuration (X). Lemma 1 is proved.

Once the lowest pinnacle is placed first, i.e. it is $y_1$, each of the other pinnacles is easily placed. The reasoning is by induction.

**Lemma 2.** Assume that $y_1, y_2, \ldots, y_k$ are the $k$ lowest pinnacles, with $k \geq 1$, and assume $y_t$ with $t \neq k + 1$ is the next lowest pinnacle. Then there is a balanced reversal allowing to replace $y_{k+1}$ with $y_t$, which does not modify the pinnacles $y_s$, with $s \in \{1, \ldots, k, t + 1, \ldots, p\}$.

**Proof.** We notice that $y_{k+1} > y_t > y_k > v_{k+1}$. With $e = cutA_{\pi}(y_t, v_{k+1}, y_{k+1})$, the reversal $\rho(Next_{\pi}(e), y_t)$ is balanced (type A.2s if $Next_{\pi}(e) \neq y_{k+1}$ and type C.3 otherwise) and moves $y_t$ at the sought place.

**Example 4.** Consider $\pi = (20 \ 16 \ 10 \ \underline{17} \ \underline{18} \ \underline{7} \ \underline{8} \ \underline{1} \ \underline{3} \ \underline{2} \ \underline{5} \ 4 \ \underline{13} \ \underline{12} \ 9 \ \underline{15} \ 14 \ 19 \ 21)$ Here, $n = 19$, $p = 7$, the dells are underlined and the pinnacles are overlined. Then $S = \{3, 5, 8, 11, 13, 15, 18\}$. The first and last elements are the bounds $y_0$ and $y_{n+1}$ that are artificially added. Table 2 indicates the reversals needed to achieve step 1, according to Algorithm 1.

Lemmas 1 and 2 allow to deduce the following result.

**Proposition 2.** There is a sequence $R_1$ of at most one reversal (when $p = 2$), and at most $2p - 4$ reversals (when $p \geq 3$) allowing to order the pinnacles of $\pi$ in increasing order.

**Proof.** When $p = 2$, Lemma 1 guarantees that at most $p - 1(= 1)$ reversals are needed.

For $p \geq 3$, in configuration (X) from Lemma 1 the leftmost pinnacle is already the highest one, so that in Step 18 of Algorithm 1 the last execution (for $k = p - 2$) will find the pinnacle $y_{p-1}$ already on its place (since $y_1, \ldots, y_{p-2}$ and $y_p$ are already correctly placed). Therefore, only $p - 3$ applications of Lemma 2 are required in this case. The total number of reversals is then $(p - 1) + (p - 3) = 2p - 4$.

For $p \geq 3$, in configuration (Y) from Lemma 1 we apply Lemma 2 for each $k$ in $\{1, 2, \ldots, p - 2\}$ (see Algorithm 1). The number of reversals is then at most $(p - 2) + (p - 2) = 2p - 4$. □
4.2 Step 2: Place the wished dells

Now we replace $v_1, v_2, \ldots, v_{p+1}$ respectively with the lowest, the second lowest etc. element which is not a pinnacle, in order to have $\pi$ the same dells as in $Id_g$. To this end, we need the following technical lemmas.

**Lemma 3.** Assume $v_i \leq v_q$ with $i \leq q$, such that $\text{Prec}_\pi(v_i)$ is not a pinnacle and satisfies $y_0 \neq \text{Prec}_\pi(v_i) < y_q$. Then there exist two balanced reversals transforming $\pi$ into $\pi''$ such that the only differences between $\pi$ and $\pi''$ are the following ones:

i) if $\text{Prec}_\pi(v_i) > v_q$, then $\text{Prec}_\pi(v_i)$ is moved immediately after $\text{cut}A_\pi(\text{Prec}_\pi(v_i), v_q, y_q)$, so that $\text{Prec}_\pi(v_i) \in A_{\pi''}(v_q, y_q)$.

ii) if $\text{Prec}_\pi(v_i) < v_q$, $\text{Prec}_\pi(v_i)$ is moved immediately after $v_q$ and becomes $v''_q$.

**Proof.** Let $u = \text{Prec}_\pi(v_i)$. See Algorithm 2.

i) If $u > v_q$ then let $e = \text{cut}A_\pi(u, v_q, y_q)$. Then $\rho(u, e)$ is a balanced reversal (type B.1 if $e \neq v_q$ or type B.3s otherwise), since $u < \text{Next}_\pi(e)$ and $e < u < \text{Prec}_\pi(u)$, both by the definition of the cutpoint $e$. When applied, this reversal yields a permutation $\pi'$ where $\pi'_i = v_q, e \in D_{\pi'}(y'_{i-1}, v'_i), \text{Prec}_{\pi'}(e) = \text{Prec}_\pi(u), v'_i = v_i$ and $\text{Next}_{\pi'}(v_i) = u$. Then $\rho(e, v_i)$ is a balanced reversal (type B.3s if $e \neq v_q$ and type C.2 otherwise) in $\pi'$ for we have $e < \text{Next}_{\pi'}(v_i) = u$ by the definition of the cutpoint $e$, and $v_i < u < \text{Prec}_\pi(u) = \text{Prec}_\pi(e)$. The resulting permutation $\pi''$ satisfies the conditions in the lemma.

ii) If $u < v_q$ then $\rho(v_i, v_q)$ is a balanced reversal (type C.2) since $\text{Prec}_\pi(v_i) \neq y_{i-1}$, and we have $v_i < u < v_q < \text{Next}_\pi(v_q)$ whether $\text{Next}_\pi(v_q) = y_q$ or not. In the permutation $\pi'$ resulting once $\rho(v_i, v_q)$ is applied, $v'_i = u$ (since $u < v_q$), $\text{Next}_{\pi'}(v'_i) = v_q, v'_i = v_i, \text{Next}_{\pi'}(v_i) = \text{Next}_\pi(v_q)$. Then $\rho(u, v_i)$ is a balanced reversal (type C.2) in $\pi'$. To see this, we need to show that $v_i < \text{Prec}_\pi(u)$ which is true since $\text{Prec}_\pi(u) > u > v_i$ in $\pi$, and that $u < \text{Next}_{\pi'}(v_i)$ which is also true since
Algorithm 2 ApplyLemma

Input: A permutation $\pi \in S_n$, pinnacles $v_i, v_q$ satisfying the hypothesis of Lemma 3
Output: The permutation $\pi''$ obtained according to Lemma 3

1: $u = \text{Prec}_\pi(v_i)$
2: if $u > v_q$ then
3: $e \leftarrow \text{cutA}_\pi(u, v_q, y_q); \pi' \leftarrow \pi \cdot \rho(u, e); \pi'' \leftarrow \pi' \cdot \rho(e, v_i);$ //case i)
4: else
5: $\pi' \leftarrow \pi \cdot \rho(v_i, v_q); \pi'' \leftarrow \pi' \cdot \rho(u, v_i)$q //case ii)
6: end if
7: Return $\pi''$

$\text{Next}_\pi(v_i) = \text{Next}_\pi(v_q) > v_q > u$. The permutation $\pi''$ obtained after the execution of the reversal $\rho(u, v_i)$ satisfies the conditions in the lemma.

Lemma 4. Assume $v_i \leq v_q$, with $i \leq q$, such that $\text{Next}_\pi(v_i)$ is not a pinnacle and satisfies $\text{Next}_\pi(v_i) < y_q-1$. Then there exist two balanced reversals transforming $\pi$ into $\pi''$ such that the only differences between $\pi$ and $\pi''$ are the following ones:

i) if $\text{Next}_\pi(v_i) > v_q$, then $\text{Next}_\pi(v_i)$ is moved immediately before $\text{cutD}_\pi(\text{Next}_\pi(v_i), y_q-1, v_q)$, so that $Next_\pi(v_i) \in D_{\pi''}(y_q-1, v_q)$.

ii) if $\text{Next}_\pi(v_i) < v_q$, $\text{Prec}_\pi(v_i)$ is moved immediately before $v_q$ and becomes $v_q''$.

Proof. Let $u = \text{Next}_\pi(v_i)$. See Algorithm 3

i) If $u > v_q$ then let $e = \text{cutD}_\pi(u, y_q-1, v_q)$ et $f = \text{Prec}_\pi(e)$. Then $\rho(u, f)$ is a balanced reversal (type A.1 if $f \neq y_q-1$, type A.2s otherwise), since $u > e$ and $f > u > \text{Prec}_\pi(u)$, both by the definition of a cutpoint and whether $f = y_q-1$ or not. When applied, this reversal yields a permutation $\pi'$ where $v_i' = v_i, \text{Next}_\pi(v_i') = f$ with $f \in A_{\pi'}(v_i, y_q-1)$ or $f = y_q-1$, as well as $v_q' = v_q, y_{q-1}' = y_i$, and $u \in D_{\pi'}(y_i, v_q)$ with $\text{Next}_\pi(u) = e$ and $\text{Prec}_\pi(u) = \text{Next}_\pi(u)$. Let $t = \text{Next}_\pi(u)$. Then $t = \text{Prec}_\pi(u)$ and $\rho(f, t)$ is a balanced reversal in $\pi'$ (type A.1 if $f \neq y_i$ and $f \neq y_q-1$, type A.2 or A.2s if exactly one equality holds and type C.3 otherwise). Indeed, in all types but A.2 we need to show that $t > \text{Prec}_\pi(f)$ and this is true since $\text{Prec}_\pi(f) = v_i' = v_i < u < t$, because $v_i, u, t$ occur in this order on $A_{\pi'}(v_i, y_i)$. Moreover, types A.1, A.2 and C.3 require that $f > \text{Next}_\pi(t)$, which is true since $\text{Next}_\pi(t) = u$ and $u < f$ by the definition of the cutpoint $e$. The permutation $\pi''$ resulting once $\rho(f, t)$ is performed satisfies the conditions in the lemma.

ii) If $u < v_q$ then $\rho(u, \text{Prec}_\pi(v_q))$ is a balanced reversal (type A.1 if $\text{Prec}_\pi(v_q) \neq y_q-1$, type A.2s otherwise). Indeed, with the notation $s = \text{Prec}_\pi(v_q)$, in case that $s \neq y_q-1$ the reversal is of type A.1, and we have $\text{Next}_\pi(s) = v_q$ and $\text{Prec}_\pi(u) = v_i$, so that the conditions in type A.1 are trivially verified. If $s = y_q-1$ we have to check for type A.2s that $s > \text{Prec}_\pi(u)$, which is true as $s > v_q > u > v_i = \text{Prec}_\pi(u)$. In the permutation $\pi'$ resulting once $\rho(u, s)$ is applied, $v_i' = v_i, \text{Next}_\pi(v_i') = s \in A_{\pi'}(v_i, y_q-1)$ since $s > v_q > u > v_i, v_q' = u$, deduced because $u < v_q$, and $\text{Next}_{\pi'}(u) = v_q$. With $t = \text{Next}_\pi(u)$, we also have that $t = \text{Prec}_{\pi'}(u)$. Then, in $\pi'$, $\rho(s, t)$ is a balanced reversal (type A.1 if $s \neq y_i$ and $s \neq y_q-1$, type A.2 or A.2s if exactly one equality occurs, resp. type C.3 if both equalities occur). Type A.1 is trivially verified, types A.2 and A.2s are guaranteed by $s > v_q > u$ respectively $t = \text{Next}_\pi(u) > u = \text{Next}_\pi(v_i) > v_i$, whereas type C.3 is guaranteed by the latter two conditions together. Once this reversal is applied, the resulting permutation $\pi''$ satisfies the lemma.
Algorithm 3: ApplyLemma

Input: A permutation π ∈ Sn, pinnacles vi, vj satisfying the hypothesis of Lemma

Output: The permutation π" obtained according to Lemma

1: u = Nextπ(vi)
2: if u > vq then
3:   e ← cutDπ(u, yq−1, vq)
4:   π' ← π · ρ(u, Precπ(e)); π" ← π' · ρ(Precπ(e), Nextπ(u))
5: else
6:   π' ← π · ρ(u, Precπ(vq))
7:   π" ← π' · ρ(Precπ(vq), Nextπ(u))
8: end if
9: Return π"

We are now able to place the dells in IdS as dells of π, in increasing order from left to right according to the method described in Proposition below and its proof. Algorithm presents the approach. The continued example in Table illustrates it.

Proposition 3. Let π ∈ Sn be a permutation with pinnacles y1 < y2 < ... < yq. There is a sequence R2 of at most 2p + 2 balanced reversals that places the dells in IdS as dells of π, in increasing order from left to right, without modifying the pinnacles of π (nor their order).

Proof. Assume the lowest k < p + 1 dells (k = 0 is admitted here) are correctly placed as v1, v2, ..., vk respectively, and let w be the next lowest element in π which is not a pinnacle. Then w must replace vk+1. We have that w < yk since w < vk+1 < yk.

Then either w is adjacent to a dell among v1, v2, ..., vk, or w is itself a dell. In all the other cases, a smaller element would be found, contradicting the choice of w.

Case 1. w is adjacent to a dell

Let w = Precπ(vj) or w = Nextπ(vj) for some j with 1 ≤ j ≤ k. Then we use Lemma with i = j and q = k + 1. We have that vj < vk+1 by the minimality of the elements v1, v2, ..., vk. We also have w < yk as proved above, and thus w < vk+1 by the increasing order of the pinnacles. So the hypothesis of the appropriate Lemma is satisfied. As w < vk+1 by the minimality of w, item ii) of the lemma holds. Consequently, after two balanced reversals, π" is the same as π except that w has been removed from its place and has been placed before or after vk+1 (depending on which lemma is applied), thus becoming the dell v′′ k+1. Then we are done in this case.

Case 2. w is a dell

If w is already a dell, let w = vj with k + 1 < j ≤ p + 1. Then w < vk+1 by the minimality of w and vk+1 < yj since vk+1 < yk+1 ≤ yj. Let e = cutAπ(vk+1, vj, yj) and, if it exists, f = Nextπ(e). Then ρ(vk+1, e) is a balanced reversal (type B.3 if e ≠ vj and type C.2 otherwise) since e ≠ yj (vk+1 is an intermediate value among them), e < vk+1 < Precπ(vk+1) and vk+1 < f by the definition of the cutpoint e. In the permutation π resulting once ρ(vk+1, e) is applied, v′′ k+1 = vj, y′′ k+1 = yj−1, y′′ j−1 = yk+1, v′′ j = vk+1 and y′′ j′ = yj. If k + 1 = j − 1, then the order of the pinnacles does not change since the reversed block contains a unique pinnacle. In this case we are done. Otherwise, due to k + 1 < j − 1 we deduce that yk+1 < yj−1 and thus vj < yk+1 < yj−1. The cutpoint defined as e′ = cutAπ(yk+1, vj, yj−1) satisfies then the condition e′ ∈ {vj} ∪ Aπ(vj, yj−1). As a consequence, with f′ = Nextπ(e′) we have that ρ(f′, yk+1) is a balanced reversal (type A.2s if f′ ≠ yj−1, type C.3 otherwise). The required conditions are fulfilled since both types need yk+1 > Precπ(f′) and this is true by the definition of the cutpoint e′, which is Precπ(f′); and in type C.3 we moreover need f′ > Nextπ(yk+1) and this is true too by the definition of the cutpoint e′, since f′ > yk+1 > Nextπ(yk+1). The resulting permutation π" has v′′ k+1 = v′′′ j = vj and the pinnacles are in increasing order.
Algorithm 4 Permutation sorting by balanced reversals: Step 2

Input: A permutation $\pi \in S_n$ with pinnacle set $S$ of cardinality $p$. The pinnales of $\pi$ are increasingly ordered.

Output: The permutation $\pi$ with pinnales still in increasing order, and whose dells have become equal to the dells of $Id_S$, in increasing order (Proposition 5).

1: for $k = 0$ to $p$ do
2: $w \leftarrow \min\{1, 2, \ldots, n\} - S - \{v_k, \ldots, v_k\}$ // $x$ is the lowest wished dell
3: if $\exists v_j$ such that $w = \text{Prec}_\pi(v_j)$ then
4: $\pi \leftarrow \text{ApplyLemma} \, \pi, v_j, v_k+1$) 
5: else
6: if $\exists v_j$ such that $w = \text{Next}_\pi(v_j)$ then
7: $\pi \leftarrow \text{ApplyLemma} \, \pi, v_j, v_k+1$ 
8: else
9: let $v_j = w$ // $w$ is a dell 
10: if $j \neq k + 1$ then
11: $e \leftarrow \text{cutA}_\pi(v_k+1, v_j, y_j); \pi' \leftarrow \pi \cdot \rho(v_k+1, e); \pi'' \leftarrow \pi'$
12: if $k + 1 \neq j - 1$ then
13: $e' \leftarrow \text{cutA}_\pi(y_k+1, v_j, y_j-1); \pi'' \leftarrow \pi' \cdot \rho(\text{Next}_\pi(e'), v_k+1);$
14: end if
15: $\pi \leftarrow \pi''$
16: end if
17: end if
18: end if
19: end for
20: Return $\pi$

Using the previous approach for each $k = 0, 1, \ldots, p$, we obtain a permutation still denoted $\pi$ whose pinnales are in increasing order and whose dells are identical to those of $Id_S$, and in increasing order. Each $k$ requires 0 or 2 balanced reversals, depending whether $v_{k+1}$ is already correct or not, so that at most $2p + 2$ reversals are performed. $\square$

Table 2 shows an example.

4.3 Step 3: Move the remaining elements towards the place they occupy in $Id_S$

It remains to move in $\pi$ the elements from each ascending and each descending set towards the end of the permutation.

Proposition 4. Let $\pi \in S_n$ be a permutation with pinnales $y_1 < y_2 \ldots < y_p$, and dells $v_1 < v_2 < \ldots < v_{p+1}$ which are the $p + 1$ lowest values in $\{1, 2, \ldots, n\} \setminus S$. There is a sequence $R_3$ of at most $2(n - 2p) - 1$ balanced reversals that transforms $\pi$ into $Id_S$.

Proof. This is done as follows. By hypothesis, $v_{p+1}$ is smaller than all the elements from each ascending and each descending set, since the dells are the smallest elements that are not pinnales.

a) As long as $\text{Next}_\pi(v_{p+1}) < y_p$, we use the following trick to move $\text{Next}_\pi(v_{p+1})$ towards $D_\pi(y_p, v_{p+1})$ without changing the rest of $\pi$. Let $\pi^{rev}$ be the permutation obtained from $\pi$ by reversing the whole $\pi$. Lemma 3(i) may be applied to $\pi^{rev}$ with $i = q = p + 1$ in order to move $\text{Prec}_\pi^{rev}(v_{p+1})$ towards $A_{\pi^{rev}}(v_{p+1}, y_p)$ using two balanced reversals (recall that $y_p > \text{Next}_\pi(v_{p+1}) = \text{Prec}_\pi^{rev}(v_{p+1})$ by the hypothesis above). Now, if we apply the balanced reversals with the same endpoints in $\pi$ (without reversing the whole permutation), we obtain that
Next_π(v_{p+1}) is moved towards D_π(y_p, v_{p+1}) without changing the rest of π. The resulting permutation is still called π and we continue. When the process is finished, each t ∈ A_π(v_{p+1}, y_{p+1}) satisfies t > y_p.

b) For each i ≤ p, as long as Next_π(v_i) ≠ y_i, use Lemma 4 to move Next_π(v_i), which is smaller than y_i and thus smaller than y_p, towards D_π(y_p, v_{p+1}), without changing the rest of π. More precisely, item i) in the lemma is used, by the minimality of v_{p+1}. When this step is finished, we have A_π(v_i, y_i) = ∅ for all i with 1 ≤ i ≤ p.

c) If Next_π(y_p) ≠ v_{p+1}, the reversal ρ(Next_π(y_p), v_{p+1}) is balanced (type B.3s), since Next_π(y_p) < y_p < Next_π(v_{p+1}) by the constructions in the two previous items above, and v_{p+1} < Prec_π(Next_π(y_p)) = y_p. Then, in the new permutation still denoted π, D_π(y_p, v_{p+1}) = ∅.

d) For each i ≤ p, as long as Prec_π(v_i) ≠ y_{i−1}, use Lemma 3 to move Prec_π(v_i) towards A_π(v_{p+1}, y_{p+1}) (item i) in the lemma) without changing the rest of π. When this step is finished, we have D_π(v_i, y_i) = ∅ for all i with 0 ≤ i ≤ p.

It is easy to see that the result of these transformations is Id_S. Indeed, item b) ensures that v_i immediately precedes y_i, for each pinnacle y_i, 1 ≤ i ≤ p. Once step b) is performed, the elements in D_π(y_p, v_{p+1}) are smaller than y_p whereas by item a) those in A_π(v_{p+1}, y_{p+1}) (if any) exceed y_p. The reversal in item c) only makes y_p adjacent to v_{p+1} by concatenating the elements in D_π(y_p, v_{p+1}) to those in A_π(v_{p+1}, y_{p+1}). Finally, item d) ensures that each y_i is adjacent to each v_{i+1} for 0 ≤ i ≤ p − 1, by successively inserting each element in D_π(y_i, v_{i+1}) into \{v_{p+1}\} ∪ A_π(v_{p+1}, y_{p+1}).

As a consequence, all elements but the p pinnacles and the p + 1 dells are possibly moved in items a), b) and d) using Lemma 3 or Lemma 4 thus performing two reversals per element. Since in item c) only one reversal is performed, the total number of reversals is at most 2(2n − p − (p + 1)) + 1 = 2(n − 2p) − 1.

See the example in Table 2.

Proof of Theorem 1. The sequence R obtained by concatenating the sequences R_1, R_2, R_3 issued from Propositions 2, 3 and respectively Proposition 4 transforms π into Id_S as shown by these propositions. The number of balanced reversals in R needs to identify three cases:

- when p = 1, Steps 1, 2, 3 respectively take at most 0, 2p + 2 and 2(n − 2p) − 1 reversals, so the total number is 0 + (2p + 2) + (2n − 4p − 1) = 2n − 2p + 1, so that with p = 1 we have 2n − 1 reversals.

- when p = 2, Steps 1, 2, 3 respectively take at most 1, 2p + 2 and 2(n − 2p) − 1 reversals, so the total number is 1 + (2p + 2) + (2n − 4p − 1) = 2n − 2p + 2, so that with p = 2 we have 2n − 2 reversals.

- when p ≥ 3, Steps 1, 2, 3 respectively take at most 2p − 4, 2p + 2 and 2(n − 2p) − 1 reversals, so the total number is (2p − 4) + (2p + 2) + (2n − 4p − 1) = 2n − 3 reversals.

The number of reversals is therefore bounded by 2n − min\{p, 3\} in all cases if p ≥ 1.

Remark 2. In the case where p = 0, Steps 1 and 2 in the algorithm are not performed. Step 3 reduces to items c) and d) which perform 1 and respectively at most 2(n − 1) reversals. The total number of reversals is thus upper bounded by 2n − 1 in this case.
Algorithm 5 Permutation sorting by balanced reversals: Step 3

Input: A permutation \( \pi \in S_n \) with pinnacle set \( S \) of cardinality \( p \). The pinnacles of \( \pi \) are increasingly ordered.

The dells of \( \pi \) are the same as those of \( \text{Id}_S \).

Output: The permutation \( \text{Id}_S \), obtained after placing into their correct places the elements of \( \pi \) not yet correctly placed (Proposition 4)

1: while \( \text{Next}_\pi(v_{p+1}) < y_p \) do
2: \( u \leftarrow \text{Next}_\pi(v_{p+1}); e \leftarrow \text{cutD}_\pi(u, y_p, v_{p+1}); \pi' \leftarrow \pi \cdot \rho(e, u); \pi'' \leftarrow \pi' \cdot \rho(v_{p+1}, e); \pi \leftarrow \pi'' \) //item a)
3: end while
4: for \( i = 1 \) to \( p \) do
5: \( \text{while} \ \text{Next}_\pi(v_i) \neq y_i \ \text{do} \)
6: \( \pi \leftarrow \text{ApplyLemma}_3(\pi, v_i, v_{p+1}) \) //item b)
7: end while
8: end for
9: if \( \text{Next}_\pi(y_p) \neq v_{p+1} \) then
10: \( \pi \leftarrow \pi \cdot \rho(\text{Next}_\pi(y_p), v_{p+1}) \) //item c)
11: end if
12: for \( i = 1 \) to \( p \) do
13: \( \text{while} \ \text{Prec}_\pi(v_i) \neq y_{i-1} \ \text{do} \)
14: \( \pi \leftarrow \text{ApplyLemma}_4(\pi, v_i, v_{p+1}) \) //item d)
15: end while
16: end for
17: Return \( \pi \)

5 Running time

The algorithm for transforming \( \pi \) in \( \text{Id}_S \) is the concatenation of Algorithms 1, 4 and 5 above, and is called Algorithm \textbf{BalancedSorting}. In this section we briefly show that \textbf{BalancedSorting} may be implemented in \( O(n \log n) \).

The operations that need to be efficiently implemented, \textit{i.e.} in \( O(\log n) \) each, are easily identified.

- In Algorithm 1, successively computing the first, second etc. minimum of the initial set \( S \) of pinnacles (lines 1 and 19), computing \( \text{cutA}_\pi \) for a given element and a given pinnacle (lines 15 and 21), finding the leftmost pinnacle larger than a given value and with index larger than a given value (lines 4 and 12), computing \( \text{Next}_\pi(x) \) for a given element \( x \) (lines 15 and 21), performing the reversal between two given elements (lines 15 and 21).

- In addition, in Algorithm 4, computing the first, second etc. minimum of the set of wished dells (line 2), computing \( \text{cutD}_\pi \) for a given element and a given pinnacle (line 7), computing \( \text{Prec}_\pi(x) \) for a given element \( x \) (lines 3, 4, 7), deciding whether a given element is adjacent to a dell (lines 3 and 6).

- Algorithm 5 has no supplementary requirements.

These operations require to combine several efficient data structures that allow both a rapid access to the information and an efficient update. In particular, performing a reversal is a sensitive issue since it modifies the places of many elements, and - in the precise case we study here - swaps ascending and descending sets of the reversed block. In order to avoid recording all these changes one by one, the solutions proposed in literature in order to perform a (not necessarily balanced) reversal use three types of approaches. They are due to Kaplan and Verbin [8] (needs \( O(\sqrt{n \log n}) \) time to perform a reversal whose endpoints are known), Han [7] (needs \( O(\sqrt{n}) \) time for the same task) and Rusu [10] (needs \( O(\log n) \) time for the same task). The latter one, that we choose for efficiency reasons, uses
so-called \textit{log-lists}. Log-lists may be assimilated, with a view to simplification, to double-linked lists in which a collection of operations may be performed in $O(\log n)$. The ones we are interested in here are: given (a pointer on) $x$, compute (the pointers on) $\text{Prec}_\pi(x)$ and $\text{Next}_\pi(x)$; change the sign (positive or negative) of all the elements of a sublist; perform a reversal (and update the structure).

The data structure we propose for the implementation of our algorithm combines log-lists, binary search trees (BSTs, for short), arrays and pointers. The shape of the permutation $\pi$ is stored in a log-list $L$. For each pinnacle $y_i$ in the shape, two pointers $\text{toA}$ and $\text{toD}$ go towards the roots of two BSTs. BST $\text{toA}$ (respectively $\text{toD}$) contains the elements in $A_\pi(y_i, v_i)$ (respectively in $D_\pi(y_i, v_{i+1})$) represented as pairs $(\pi_a, a)$. The order between pairs is defined as the standard (increasing) order between their left values (the elements). This implies that the pairs are also ordered according to the increasing order of their right values (the indices) in $\text{toA}$, respectively according to the decreasing order of their right values (the indices) in $\text{toD}$. Both of them, elements and indices, are used by the algorithm. An array $P$ of pointers contains, in this order, the pointers to the pinnacles in $L$ in increasing order of the pinnacles, followed by pointers to the wished dells (the dells of $I_d$), in increasing order of the dells. The wished dells belong either to $L$ or to one of the $2p$ BSTs pointed by the pointers $\text{toA}$, $\text{toD}$ of each pinnacle.

With this data structure, we have:

- All the operations except finding the leftmost pinnacle larger than a given value and with index larger than a given value (lines 4 and 12 of Algorithm 1) take $O(\log n)$ time each. This is quite easy, given the abovementioned properties of a log-list and those of a BST (searching a value in a BST, cutting a BST, merging two BSTs take $O(\log n)$ time). It is important to notice that a reversal is performed on the shape of the permutation (thus in the log-list $L$), but: 1) it cuts either $\text{toA}$ or $\text{toD}$ at each endpoint of the reversed block (except when the endpoint is a pinnacle or a dell) and recombines the resulting BSTs (also modifying the dells at the endpoints if needed) once the reversal is performed, both in $O(\log n)$ time; 2) it swaps the roles of $\text{toA}$ and $\text{toD}$ for each pinnacle situated strictly inside the reversed block, and thus each reversal must be followed by a sign change (in $O(\log n)$ time) in the reversed block of $L$ (whose meaning is that when the value of the pinnacle is positive, $\text{toA}$ and $\text{toD}$ correspond respectively to the ascending and descending sets neighboring the pinnacle, whereas when the value is negative their roles are swapped).

- The operation of finding the leftmost pinnacle larger than a given value and with index larger than a given value (lines 4 and 12 of Algorithm 1) cannot be implemented in $O(\log n)$ time per operation. Instead, it may be shown that all these operations (in lines 4 and 12 of Algorithm 1) may be implemented altogether in $O(n \log n)$ time.

To this end, one has to remember that in Lemma 1 when Case 2 (tested in lines 4 and 12) is applied to $y_i$, one finds a first index $h > i$ such that $v_1 < y_h$. Then we reverse the block with endpoints $v_1$ and $y_h$. Again as proved in Lemma 1 the values $h^k$ computed similarly with $h$ in the next executions of the \textit{while} loop (lines 9-17 in Algorithm 1) belong to the block of $\pi$ situated between $v_1$ and $y_h$, and become smaller at each execution. That means each reversal due to Case 2 reverses a proper prefix of the block resulting after the latest reversal. Then it is sufficient to perform in $O(n \log n)$ time a pre-treatment of the block of the (initial) permutation $\pi$ with endpoints $v_1$ and $y_h$ (once $h$ is known). This pre-treatment affects to each dell $v_j$ with $j < i$ a pointer to the leftmost pinnacle $y_{h'}$ such that $y_{h'} > v_j$ and $i < h' < h$, and symmetrically affects to each dell $v_j$ with $j > i$ a pointer to the rightmost pinnacle $y_{h'}$ such that $y_{h'} > v_j$ and $h' < i$. By the previous considerations, the reversals performed by the algorithm do not affect the meaning of the pointers affected to the dells $v_j$ obtained during the execution of Case 2. These pointers, computed independently from the data structure presented above, may be stored
as additional information in the cells representing the dells in the log-list. They allow to find the pinnacles $y_{h^j}$ in $O(1)$ time each, and thus the dells $v_h$ needed by Algorithm 1 (line 13) in $O(\log n)$ time each.

6 Conclusion

We have shown in this paper that the Balanced Sorting problem has a solution using at most $2n - \min\{p, 3\}$ balanced reversals (when $p \geq 1$), for each permutation of $n$ elements. This is an upper bound, but many permutations may be sorted using less balanced reversals. Then we can ask:

**Question 1.** (Minimum Balanced Sorting) Given $\pi \in S_n$ with pinnacle set $S$, find the minimum number of balanced reversals needed to transform $\pi$ into $Id_S$.

A similar question may be asked when transforming a permutation $\pi$ into a permutation $\pi'$ with the same pinnacle set:

**Question 2.** (Minimum Balanced Transformation) Given $\pi, \pi' \in S_n$ with the same pinnacle set $S$, find the minimum number of balanced reversals needed to transform $\pi$ into $\pi'$.

This question is different from the previous one, since the identity permutation is not necessarily an intermediate permutation in the transformation of $\pi$ into $\pi'$.

A related problem is raised by Example 3. Even when a set is admissible as the pinnacle set of a permutation, the order of the pinnacles in the permutation is important, since some orders may be impossible to respect. Identifying these orders could be a way to better target the balanced reversals yielding a minimum sorting.

**Question 3.** Let $S$ be an admissible pinnacle set. Give a characterization of the total orders $\sigma$ on $S$ such that a permutation $\pi \in S_n$ exists whose sequence of pinnacles read from left to right is exactly $\sigma$.

Note that Davis et al. [5] study the number of permutations on $n$ elements with a given admissible set $S$, and give recursive formulas for it. This is a related question, which involves however supplementary combinatorial aspects related to the elements not belonging to $S$.

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