TRICOMI TYPE BOUNDARY VALUE PROBLEM WITH INTEGRAL CONJUGATION CONDITION FOR A MIXED TYPE EQUATION WITH HILFER FRACTIONAL OPERATOR

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Abstract

In this work, The Tricomi type boundary problem with integral conjugation condition on the type-changing line for the mixed type equation with Hilfer fractional differential operator has been considered. Using method of integral equations, energy integral’s method, a unique solvability of the formulated problem has been proved.

MSC 2010: 35M10

Keywords: Mixed type equation; a boundary value problem; integral conjugation condition; Hilfer operator; method of integral equations.

1 Introduction

Fractional Calculus is developing intensively due to both practical [1] and theoretical importance [2]. Fractional analogues of essential equations such as diffusion, wave were studied involving different fractional differential and integral operators [3], [4], [5], [6]. We will omit a huge amount of works devoted to the studying of direct and inverse problems for partial differential equations (PDEs) with fractional order operators and note only studies closely related to the present topic. Several boundary value problems (BVPs) for mixed type equations with Riemann-Liouville fractional differential operator (FDO) were studied for unique solvability in works [7], [8], [9]. BVP with integral form conjugation conditions for PDEs with both Riemann-Liouville and Caputo FDOs were subject of series of investigations [10], [11], [12], [13], [14]. In these works, authors used an explicit solution of certain BVP for fractional diffusion equation studied by A.Pskhu [5].

For the first time, generalized Riemann-Liouville FDO which is also called as Hilfer FDO introduced by Hilfer [15]. The Cauchy and some BVPs for ODEs and PDEs with Hilfer FDO investigated by many authors, for instance [16], [4], [17], [18].

In this paper, we are aimed to study BVP with integral form conjugation condition in a mixed domain consisted of characteristic triangle and rectangle, for a mixed type PDE with diffusion equation involving Hilfer FDO.

2 Formulation of a problem and main functional relations

2.1 Formulation of a problem

Let us consider the following mixed type equation

\[ 0 = \begin{cases} k u_{xx} - D^{\alpha,\mu}_{0t} u, & t > 0 \\ u_{xx} - u_{tt}, & t < 0 \end{cases} \]

in a mixed domain \( \Omega = \Omega_1 \cup \Omega_2 \cup AB \). Here \( k = \text{const} > 0 \), \( \Omega_1 = \{(x,t) : 0 < x < l, 0 < t < T\} \), \( \Omega_2 = \{(x,t) : -t < x < t + l, -l/2 < t < 0\} \), \( AB = \{(x,t) : 0 < x < l, t = 0\} \),

\[ D^{\alpha,\mu}_{0t} f = I^{\mu(1-\alpha)}_{0t} \frac{d}{dt} I^{(1-\mu)(1-\alpha)}_{0t} f(t) \]

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is the Hilfer FDO of order $\alpha$ ($0 < \alpha \leq 1$) and of type $\mu$ ($0 \leq \mu \leq 1$) [15], where

$$I^\alpha_0 f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(z)dz \frac{1}{(t-z)^{1-\alpha}}$$

is the Riemann-Liouville fractional integral of order $\alpha$ ($\Re(\alpha) > 0$) [2].

**Tricomi type BVP** for Eq. (1) in $\Omega$ can be formulated as follows:

**Problem.** To find a function $u(x, t)$ which is continuous in $\Omega \setminus AB$, its Hilfer derivative is continuous in $\Omega_1$ and it has continuous second order partial derivatives in $\Omega_2$, and it satisfies Eq. (1) in $\Omega$ together with boundary conditions

$$u(0, t) = 0, \quad u(l, t) = 0, \quad 0 \leq t \leq T,$$

$$u(x/2, -x/2) = \psi(x), \quad 0 \leq x \leq l,$$

conjugation conditions on $AB$

$$\lim_{t \to +0} t^{(1-\mu)(1-\alpha)} u(x, t) = u(x, -0), \quad 0 \leq x \leq l,$$

$$\gamma_1 u_t(x, -0) + \gamma_2 \sum_{n=1}^\infty E_{\alpha, \alpha + \mu(1-\alpha)} (-k \lambda_n^2 t^{\alpha}) \sin(\lambda_n x) \sin(\lambda_n \xi) d\xi,$$

Here $\gamma_i$ ($i = 1, 3$) are real constants, $\psi(x)$, $P(\cdot, \cdot)$, $Q(\cdot, \cdot)$ are given functions such that $\psi(0) = 0$, $\sum_{i=1}^3 \gamma_i^2 > 0$.

### 2.2 Main functional relations

Let us introduce a notation

$$\tau_1(x) = \lim_{t \to +0} t^{(1-\mu)(1-\alpha)} u(x, t), \quad 0 \leq x \leq l.$$

Solution of the Eq. (1) in $\Omega_1$ which satisfies conditions (2), (6) can be written as follows [11]:

$$u(x, t) = \frac{2}{l} \Gamma[\alpha + \mu(1-\alpha)] \sum_{n=1}^\infty \frac{E_{\alpha, \alpha + \mu(1-\alpha)} (-k \lambda_n^2 t^{\alpha}) \sin(\lambda_n x) \sin(\lambda_n \xi) d\xi}{\lambda_n^2 \lambda_n \xi},$$
where \( \lambda_n = \frac{n\pi}{l} \),

\[
E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)}
\]  

is a two-parameter Mittag-Leffler function \[19\].

Using representations (7) and (8), we evaluate \( t^{1-\alpha} (t^{(1-\mu)(1-\alpha)}u(x,t)) \):

\[
t^{1-\alpha} \left( \frac{2}{l} \Gamma[\alpha + \mu(1-\alpha)] \int_0^l \tau_1(\xi) \sum_{n=1}^{\infty} E_{\alpha,\alpha+\mu(1-\alpha)}(-k\lambda_n^2 t^{\alpha}) \sin(\lambda_n x) \sin(\lambda_n \xi) d\xi \right) =
\]

\[
\frac{2}{l} t^{1-\alpha} \Gamma[\alpha + \mu(1-\alpha)] \int_0^l \tau_1(\xi) \sum_{n=1}^{\infty} \frac{d}{dt} \left( \frac{1}{\Gamma(\alpha + \mu(1-\alpha))} + \frac{-k\lambda_n^2 t^{\alpha}}{\Gamma(2\alpha \mu(1-\alpha))} + \frac{(\frac{-k\lambda_n^2 t^{\alpha}}{\Gamma(3\alpha \mu(1-\alpha))})^2}{\Gamma(3\alpha \mu(1-\alpha))} + \ldots \right)
\times
\sin(\lambda_n x) \sin(\lambda_n \xi) d\xi
\]

We introduce another notation, namely

\[
\nu_1(x) = \lim_{t \to +0} t^{1-\alpha} \left( t^{(1-\mu)(1-\alpha)}u(x,t) \right)_t, \quad 0 < x < l.
\]  

Considering above-given evaluations from (7) we obtain the following functional relation on \( AB \) deduced from \( \Omega_1 \) as \( t \to +0 \):

\[
\nu_1(x) = \frac{k\alpha \Gamma[\alpha + \mu(1-\alpha)]}{\Gamma[2\alpha + \mu(1-\alpha)]} \tau_1''(x), \quad 0 < x < l.
\]  

Now we will establish another functional relation on \( AB \) which will be reduced from \( \Omega_2 \). For this aim, we use a solution of the following Cauchy problem:

\[
u_2(x) = \tau_2'(x) - 2\psi'(x), \quad 0 < x < l.
\]  

3 Existence and uniqueness results

3.1 Existence of a solution

Considering conjugation conditions (4), (5), from functional relations (10) and (12) we get

\[
\tau''_1(x) - A\tau'_1(x) = F_1(x), \quad 0 < x < l,
\]
where $A = \gamma_1 \Gamma[2\alpha + \mu(1 - \alpha)]/(k\alpha\Gamma[\alpha + \mu(1 - \alpha)])$.

Boundary conditions \([2]\) yield
\[
\tau_1(0) = 0, \quad \tau_1(l) = 0. \tag{14}
\]

Solution of \((13)-(14)\) can be written as \([21]\)
\[
\tau_1(x) = \int_0^l G_0(x, \xi) F_1(\xi) d\xi, \tag{15}
\]

where
\[
F_1(x) = \frac{\Gamma[2\alpha + \mu(1 - \alpha)]}{k\alpha\Gamma[\alpha + \mu(1 - \alpha)]} \left\{ \gamma_2 \int_0^x \tau_1'(z) P(x, z) dz + \gamma_3 \int_0^x \tau_1'(z) Q(x, z) dz - 2\gamma_1 \psi'(x) - 2\gamma_2 \int_0^x \psi'(z) P(x, z) dz - 2\gamma_2 \int_0^x \psi'(z) Q(x, z) dz \right\}, \tag{16}
\]
\[
G_0(x, \xi) = \frac{1}{A \left[ e^{Ax} - e^{A(x-l)} \right]} \left\{ \frac{1 (1 - e^{A\xi}) (1 - e^{A(x-l)})}{1 - e^{A(\xi-l)}} (1 - e^{Ax}), \right. 0 \leq \xi \leq x, \left. \frac{1 (1 - e^{A(\xi-l)}) (1 - e^{Ax})}{1 - e^{A(\xi-l)}} x \leq \xi \leq l. \right.
\]

Substituting \((16)\) into \((15)\), after integration by parts, we will get
\[
\tau_1(x) - \int_0^l \tau_1(\xi) K(x, \xi) d\xi = F_2(x), \quad 0 \leq x \leq l, \tag{17}
\]

where
\[
K(x, \xi) = \frac{\Gamma[2\alpha + \mu(1 - \alpha)]}{k\alpha\Gamma[\alpha + \mu(1 - \alpha)]} \times
\]
\[
\left\{ G_0(x, \xi) \left[ \gamma_2 P(\xi, x) - \gamma_3 Q(\xi, x) - \int_0^l \left[ \gamma_2 \frac{\partial P(z, \xi)}{\partial z} + \gamma_3 \frac{\partial Q(z, \xi)}{\partial z} \right] G(x, z) dz \right\}
\]
\[
F_2(x) = \frac{-2\Gamma[2\alpha + \mu(1 - \alpha)]}{k\alpha\Gamma[\alpha + \mu(1 - \alpha)]} \int_0^l G_0(x, \xi) \left[ \gamma_1 \psi(\xi) + \gamma_2 \int_0^\xi \psi'(z) P(\xi, z) dz + \gamma_3 \int_0^\xi \psi'(z) Q(\xi, z) dz \right] d\xi.
\]

If $K(x, \xi)$ is continuous or has a weak singularity and $F_2(x)$ is continuously differentiable, then a solution of the second kind Fredholm integral equation \((17)\) can be represented via resolvent-kernel \([22]\)
\[
\tau_1(x) = F_2(x) + \int_0^l F_2(\xi) R(x, \xi) d\xi, \tag{18}
\]

where $R(x, \xi)$ is a resolvent-kernel of $K(x, \xi)$.

We have reduced considered problem to the second kind Fredholm integral equation, which is solvable under the certain conditions to the given data, but it might be not unique. Hence, we have to prove a uniqueness of the formulated problem separately.

### 3.2 A uniqueness of the solution

We multiply equality \([10]\) by $\tau_1(x)$ and integrate along $AB$:
\[
\int_0^l \tau_1(x)\tau_1''(x) dx - \frac{\Gamma[2\alpha + \mu(1 - \alpha)]}{k\alpha\Gamma[\alpha + \mu(1 - \alpha)]} \int_0^l \tau_1(x)\nu_1(x) dx = 0.
\]
Considering $\int_0^l \tau_1(x) \tau_1''(x) dx = -\int_0^l (\tau_1'(x))^2 dx$, we deduce

$$\frac{\Gamma[2\alpha + \mu(1 - \alpha)]}{k\alpha\Gamma[\alpha + \mu(1 - \alpha)]} \int_0^l \tau_1(x) \nu_1(x) dx + \int_0^l (\tau_1'(x))^2 dx = 0. \quad (19)$$

Let us first consider the following integral

$$I = \int_0^l \tau_1(x) \nu_1(x) dx, \quad (20)$$

where $\tau_1(x)$ and $\nu_1(x)$ are defined by (6) and (9), respectively.

Considering (5) and (12) at $\psi(x) \equiv 0$, after integration by parts we get

$$I = \int_0^l \tau_1^2(x) [\gamma_2 P(x,x) + \gamma_3 Q(x,x)] dx - \int_0^l \tau_1(x) \left[ \gamma_2 \int_0^x \tau_1(z) \frac{\partial P(x,z)}{\partial z} - \gamma_2 \int_x^l \tau_1(z) \frac{\partial Q(x,z)}{\partial z} \right] dx. \quad (21)$$

Suppose that

$$\frac{\partial P(x,z)}{\partial z} = -P_1(x)P_1(z), \quad \frac{\partial Q(x,z)}{\partial z} = -Q_1(x)Q_1(z), \quad (22)$$

then from \((21)\) it follows that

$$I = \int_0^l \tau_1^2(x) [\gamma_2 P(x,x) + \gamma_3 Q(x,x)] dx + \frac{\gamma_2 \Phi_1^2(l)}{2} + \frac{\gamma_3 \Phi_2^2(0)}{2}, \quad (23)$$

where

$$\Phi_1(x) = \int_0^x \tau_1(z) P_1(z) dz, \quad \Phi_2(x) = \int_x^l \tau_1(z) Q_1(z) dz.$$

If we suppose that

$$\gamma_2 \geq 0, \quad \gamma_3 \geq 0, \quad P(x,x) \geq 0, \quad Q(x,x) \geq 0, \quad (24)$$

from \((23)\) we will get $I \geq 0$.

Since $\Gamma(z) > 0$ for all $z > 0$, then for $k > 0$ from \((19)\) we will have $\tau_1(x) \equiv 0$. Further, considering solution of the first BVP for Eq. \((1)\) in $\Omega_1$, we will get $u(x,t) \equiv 0$ in $\Omega_1$. Due to \((4)\), one can easily deduce that $u(x,t) \equiv 0$ in $\Omega$.

Finally, we are able now formulate our result as the following

**Theorem.** If $\psi(x), P(\cdot,\cdot), Q(\cdot,\cdot)$ are continuous and continuously differentiable in their domain, and conditions \((22), (24), \gamma_1 \geq 0, k > 0\) are fulfilled, then formulated problem has a unique solution represented as follows

$$u(x,t) = \theta(t) \frac{2}{\Gamma[\alpha + \mu(1 - \alpha)]} \int_0^l \left[ F_2(\xi) + \int_0^l F_2(\eta) R(\xi,\eta) d\eta \right] t^{(1-\mu)(\alpha-1)} \sum_{n=1}^{\infty} E_{\alpha,\alpha+\mu(1-\alpha)} \left( -k\lambda_n^2 t^\alpha \right) \times$$

$$\times \sin(\lambda_n x) \sin(\lambda_n \xi) d\xi + \frac{\theta(t)}{2} \int_0^l \left[ F_2(x + t) - F_2(x - t) + \int_0^l F_2(\xi) [R(x + t,\xi) + R(x - t,\xi)] d\xi + \frac{x + t}{x - t} F_2(z) - 2\psi(z) + \int_0^l [F_2(\eta) - 2\psi(\eta)] R(z,\eta) d\eta \right] d\xi,$$
where $\theta(t) = 1$ for $t \geq 0$ and $\theta(t) = 0$ for $t < 0$.

**Remark.** Functions $P(x,t) = \sin x \cos t$ and $Q(x,t) = e^{-x}(1 + e^{-t})$ satisfy all conditions imposed in Theorem if $l \leq \pi/2$.

**References**

1. Uchaikin V. V. Fractional Derivatives for Physicists and Engineers. Vol. I: Background and Theory. Vol. II: Applications. Nonlinear Physical Science. Heidelberg: Springer and Higher Education Press, 2012

2. Kilbas A. A., Srivastava H. M., Trujillo J. J. Theory and Applications of Fractional Differential Equations, volume 204. North-Holland Mathematics Studies. Amsterdam: Elsevier, 2006.

3. Luchko Y., Gorenflo R. An operational method for solving fractional differential equations with the Caputo derivatives. *Acta Mathematica Vietnamica*, 24, 1999, pp.207-233.

4. Sandev T., Metzler R., Tomovski Ž. Fractional diffusion equation with a generalized Riemann-Liouville time fractional derivative. *J. Phys. A: Math. Theor.* 44, 2011, 255203 (21pp)

5. Pskhu A. V. Partial Differential Equations of Fractional Order (In Russian). Moscow: Nauka, 2005.

6. Al-Musalhi F., Al-Salti N., Karimov E. T. Initial boundary value problems for a fractional differential equation with hyper-Bessel operator. *Fract. Calc. Appl. Anal.* 21(1), 2018, p.200-219

7. Kilbas A. A., Repin O. A. An analog of the Tricomi problem for a mixed type equation with a partial fractional derivative. *Fract. Calc. Appl. Anal.* 13(1), 2010, p.69-84

8. Gekkieva S. Kh. A boundary value problem for the generalized transfer equation with a fractional derivative in a semi-infinite domain (In Russian). *Izv. Kabardino-Balkarsk. Nauchnogo Tsentra RAN* 1(8), 2002, pp.6-8.

9. Kadirkulov B. J. Boundary problems for mixed parabolic-hyperbolic equations with two lines of changing type and fractional derivative. *Electronic Journal of Differential Equations*, 2014(57), 2014, pp. 1-7.

10. Akhatov J. S., Karimov E. T. A boundary problem with integral gluing condition for a parabolic-hyperbolic equation involving the Caputo fractional derivative. *Electronic Journal of Differential Equations*, 2014(14), 2014, pp.1-6.

11. Berdyshev A. S., Cabada A., Karimov E. T. On a non-local boundary problem for a parabolic-hyperbolic equation involving Riemann-Liouville fractional differential operator. *Nonlinear Analysis*, 75, 2012, pp.3268-3273.

12. Agarwal P., Berdyshev A. S., Karimov E. T. Solvability of a non-local problem with integral transmitting condition for mixed type equation with Caputo fractional derivative. *Results in Mathematics*. 71(3), 2017, pp. 1235-1257

13. Karimov E. T., Berdyshev A. S., Rakhmatullaeva N. A. Unique solvability of a non-local problem for mixed-type equation with fractional derivative. *Mathematical Methods in the Applied Sciences*. 40(8), 2017, pp.2994-2999

14. Abdullaev O. Kh., Sadarangani K. Non-local problems with integral gluing condition for loaded mixed type equations involving the Caputo fractional derivative. *Electronic Journal of Differential Equations*, 2016(164), 2016, pp. 1-10.
15. Hilfer R. Applications of Fractional Calculus in Physics. Singapore: World Scientific, 2000.

16. Hilfer R., Luchko Y., Tomovski Ž. Operational method for the solution of fractional differential equations with generalized Riemann-Liouville fractional derivatives. Fract. Calc. Appl. Anal. 12(3), 2009, pp.299-318

17. Furati Kh. M., Iyiola O. S., Kirane M. An inverse problem for a generalized fractional diffusion. Applied Mathematics and Computation, 249, 2014, pp.24-31

18. Malik S., Aziz S. An inverse source problem for a two parameter anomalous diffusion equation with nonlocal boundary conditions. Computers and Mathematics with Applications, 73(12), 2017, pp.2548-2560.

19. Podlubny I. Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution, Mathematics in Science and Engineering. Vol. 198. San Diego: Academic Press, 1999

20. Sobolev S. L. Partial Differential Equations of Mathematical Physics. Pergamon Press LTD, 1964.

21. Dzhuraev T. D., Sopuev A., Mamazhanov M. Boundary Value Problems for Equations of Parabolic-Hyperbolic Type Tashkent: FAN, 1986.

22. Tricomi F. G. Integral Equations, New York: Dover Publ., 1985.