Analysing the impact of anisotropy pressure on tokamak equilibria

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Abstract
Neutral beam injection and ion cyclotron resonance heating induce pressure anisotropy. The axisymmetric plasma equilibrium code HELENA has been upgraded to include anisotropy and toroidal flow. We have studied, using both analytical and numerical methods, the determining factors for anisotropic equilibria and their impact on flux surfaces, the magnetic axis shift, and the displacement of pressure and density contours from the flux surface. With $p_\parallel/p_\perp \approx 1.5$, $p_\perp$ can vary by 20% on the $s = 0.5$ flux surface, in a MAST-like (MAST: Mega Amp Spherical Tokamak) equilibrium. We have also re-evaluated the widely applied approximation to the anisotropy in which $p^* = (p_\parallel + p_\perp)/2$, the average of the parallel and perpendicular pressures, is taken as the approximate isotropic pressure. We show that an isotropic reconstruction can yield a correct $p^*$ only by giving an incorrect $RB_A$. We find the reconstructions of the same MAST discharge with $p_\parallel/p_\perp \approx 1.25$ using isotropic and anisotropic models to have a 3% difference in a toroidal field and a 66% difference for a poloidal current.

Keywords: tokamak equilibrium, pressure anisotropy, equilibrium reconstruction

(Some figures may appear in colour only in the online journal)

1. Introduction
Auxiliary heatings, such as neutral beam injection (NBI) and ion cyclotron resonance heating (ICRH), are widely implemented in modern tokamaks. Unlike Ohmic heating, NBI and ICRH generate a large population of fast ions. The NBI induced energetic ions mainly come with a large energy parallel to the injection, while ICRH heats the ions resulting in higher velocities perpendicular to the magnetic field \[1\]. The distribution functions of these fast ions in phase space are thus distorted into anisotropic forms with $p_\perp \neq p_\parallel$, where $p_\perp (p_\parallel)$ refers to the total pressure of both the thermal and the fast population perpendicular (parallel) to the magnetic field. These heating methods also drive plasma rotation. The resulting magnitude of the anisotropy in a tokamak can be very large, according to recent studies. In JET, the anisotropy magnitude reaches $p_\perp/p_\parallel \approx 2.5$ \[2\] with ICRH. In MAST, the beam pressure reaches $p_\perp/p_\parallel \approx 1.7$ during NBI heating \[3\].

However, in the magnetohydrodynamic (MHD) description of plasma, pressure is assumed to be isotropic. Three questions are raised immediately. How is an anisotropic equilibrium different from an isotropic one? How accurate is the MHD model for anisotropic equilibria? How does the change in equilibrium affect the further study of a plasma (as regards features such as stability and transport)?

The theory of tokamak anisotropic equilibrium has been studied by many authors \[4–8\]. One basic result is that the two pressures $p_\parallel$, $p_\perp$ and the density $\rho$ are no longer flux functions \[9–11\]. At the same time, anisotropy could add to or reduce the magnetic axis outward shift (Shafranov shift \[12\]) \[10, 13, 14\]. The latter result has been confirmed by the numerical code FLOW \[15\]. Some authors also find the experimentally inferred equilibrium assuming that a single pressure and an anisotropic pressure can be quite different \[2, 3, 16\].

In this work, we address the first two questions with analytical and numerical approaches. We show how $p_\parallel$, $p_\perp$ and the ‘nonlinear’ part contribute separately to the force balance and deviate from flux functions. We also answer the second question of what problems a scalar pressure model will lead to in equilibrium reconstruction, and its dependence on the aspect ratio and the magnitude of the anisotropy.

This work is organized as follows. In section 2, the anisotropic and toroidal flow modified Grad–Shafranov equation that we use in our analytical and numerical study
is derived and presented. Section 3 briefly describes the numerical methods and the code HELENA+ATF. The features of an anisotropic equilibrium are studied in section 4. Section 5 evaluates the widely used MHD scalar pressure approximation to anisotropic pressure.

2. The Grad–Shafranov equation with anisotropic pressure and toroidal flow

2.1. The basic equations

Our assumptions of plasma equilibrium are based on guiding centre plasma theory (GCP) [6, 17] with the ideal MHD Ohm’s law. The basic equations are (in SI units)

\[ \rho (u \cdot \nabla u) + \nabla \cdot P = J \times B, \]
\[ \nabla \times B = \mu_0 J, \]
\[ \nabla \cdot B = 0, \]
\[ \nabla \times E = 0, \]
\[ E + u \times B = 0, \]
\[ P = p_\perp I + \frac{\Delta}{\mu_0} BB, \]  
\[ \Delta = \mu_0 \frac{p_\perp - p_\parallel}{B^2}, \]

where \( \rho \) is the mass density, \( u \) the single-fluid velocity, \( P \) the pressure tensor, \( J \) the current density, \( B \) the magnetic field, \( E \) the electric field, and \( \mu_0 \) the vacuum permeability constant. Equation (1) is the GCP force balance. Equations (2), (3) and (4) are Maxwell equations. Equation (5) is the ideal Ohm’s law. Equation (6) is the GCP assumption of anisotropic pressure, which assumes that the pressure tensor consists of two components, \( p_\perp \) and \( p_\parallel \), with \( I \) the identity tensor. The fast ion finite orbit width (FOW) effects are ignored in our fluid model. FOW effects can be important for tokamaks with fast ion heating, especially for tight aspect ratio tokamaks. For example, the fast ion orbit width can be as large as 20% of the minor radius in MAST with parallel on-axis beam. The inclusion of these effects in equilibrium requires a kinetic/gyrokinetic treatment of the fast ions (e.g. the inclusion in fast ion currents and thus the equilibrium, when the fast ion proportion is low [18, 19]).

With an axisymmetric cylindrical coordinate system \((R, Z, \varphi)\) and equation (3), \( B \) is written as

\[ B = \nabla \Psi \times \nabla \varphi + RB \varphi \nabla \varphi, \]

where \( \Psi \) is the poloidal magnetic flux and \( B \) the toroidal magnetic field. The current densities in the toroidal and poloidal directions can be deduced from Ampere’s law (equation (2)):

\[ \mu_0 J_\varphi = -R \nabla \cdot \frac{\nabla \Psi}{R^2}, \]
\[ \mu_0 J_\varphi = \nabla (RB \varphi) \times \nabla \varphi. \]

If only the toroidal part of the flow is important, with \( \nabla \times (u \times B) = 0 \) from equations (4) and (5), the form of \( u \) becomes

\[ u = \Omega (\Psi) R^2 \nabla \varphi, \]

in which \( \Omega \) is the toroidal angular velocity and a flux function for zero resistivity.

Two different forms of toroidal flow and anisotropic modified Grad–Shafranov equations (modified GSE) [5, 12] can be derived from the above equations using different variables. The pressure form of the GSE has pressure as a function of three variables \((R, B, \Psi)\): \( p_{\parallel,\perp} = p_{\parallel,\perp}(R, B, \Psi) \) [8, 9, 14, 17, 20]. The enthalpy form uses \( \rho \) as a variable instead of \( R \), which means that \( p_{\parallel,\perp} = p_{\parallel,\perp}(\rho, B, \Psi) \) [10, 15, 20].

2.2. The Grad–Shafranov equation in the pressure form

To obtain the modified GSE in the pressure form, the momentum equation is rearranged into the following form, as mentioned by many authors (for example [4, 5, 10, 11, 15, 20]):

\[ \mu_0 \nabla p_\parallel = \Delta \nabla B^2 \]
\[ + \nabla \times [(1 - \Delta)B] \times B + \mu_0 \rho \Omega^2 R \nabla \cdot \nabla R. \]

Substituting \( p_\parallel = p_\parallel(R, B, \Psi) \) into equation (10), the components of equation (10) in the \( \nabla \varphi, \nabla B, \nabla R \) and \( \nabla \Psi \) directions are respectively given as

\[ F(\Psi) \equiv R B \varphi(1 - \Delta), \]
\[ \frac{\partial p_\parallel}{\partial B} \varphi, R = \frac{\Delta B}{\mu_0}, \]
\[ \frac{\partial p_\parallel}{\partial R} \varphi, B = \rho R \Omega^2, \]
\[ \nabla \cdot \Delta \nabla \Psi = -\frac{F F'}{R^2} \]
\[ + \frac{1 - \Delta}{R} \nabla R = -\frac{\mu_0}{R} \frac{\partial p_\parallel}{\partial \Psi} \].

We note that \( F = R B \varphi(1 - \Delta) \), rather than \( R B \varphi \), becomes a flux function. The restrictions for \( p_\parallel(R, B, \Psi) \) are equations (12) and (13); these also guarantee that the parallel force balance (multiplying equation (10) by \( B \)) is satisfied. In the limit of no toroidal flow, equation (12) can also be deduced from the parallel force balance. Finally, equation (14) is the modified GSE for anisotropic and toroidally rotating systems.

2.3. The Grad–Shafranov equation in the enthalpy form

A detailed derivation of the enthalpy form of the modified GSE can be found in [10, 15, 20]. Starting from the energy conservation equation, the relationships between the enthalpy \( W(\rho, B, \Psi) \) and plasma pressures, as well as the rotation, are derived. A new flux function \( H \), which is defined as

\[ H(\Psi) = W(\rho, B, \Psi) - \frac{1}{2} \Omega^2 R^2, \]

is inferred from these relationships.

In order to close the set of equations, a certain equation of state is needed. In our work, the bi-Maxwellian distribution model is chosen. This is the simplest distribution function that will capture anisotropy. The two pressures are now products of the plasma density and the parallel and perpendicular
temperatures, and the thermal closure chosen is that the parallel temperature is a flux function:
\[ p_i(\rho, B, \Psi) = \rho T_\parallel(\Psi), \quad p_\perp(\rho, B, \Psi) = \rho T_\perp(B, \Psi). \] (16)

The two temperatures \( T_\parallel \) and \( T_\perp \) are in units of energy per mass. Inserting the bi-Maxwellian assumptions yields expressions for \( W(\rho, B, \Psi) \) and \( T_\perp(B, \Psi) \) [10, 20], written as
\[ W(\rho, B, \Psi) = T_\parallel \ln \frac{T_\parallel \rho}{T_\parallel \rho_0}, \quad \rho = \rho_0 \frac{T_\parallel}{T_\parallel}, \quad \rho = \exp \left( \frac{H + \frac{1}{2} R^2 \Omega^2}{T_\parallel} \right), \] (17)

\[ T_\parallel = T_\parallel(\Psi), \quad T_\perp = \frac{T_\parallel B}{|B - T_\parallel \Theta(\Psi)|}. \] (18)

with \( \rho_0 \) a constant and a new flux function \( \Theta \) indicating the magnitude of the anisotropy.

Considering the \( \nabla \psi \) direction of equation (10) will give the enthalpy form of the modified GSE:
\[ \nabla \cdot (1 - \Delta) \nabla \psi = -\frac{F F'}{(1 - \Delta R^2)} \]
\[ -\mu_0 \rho \left[ T_\parallel H' + R^2 \Omega \right] \left( -\frac{\partial W}{\partial \Psi} \right)_{\rho, B}, \] (19)

with \( F \) defined by equation (11). The system is specified by five functions, \( \{T_\parallel, H, \Omega, F, \Theta\} \), of \( \psi \) and the boundary conditions on \( \psi \).

The pressure form of the modified GSE (equation (14)), when closed with equation (18), is equivalent to the enthalpy form of the modified GSE. The enthalpy form of the modified GSE with the bi-Maxwellian assumption is numerically solved. We have used the pressure form of the modified GSE to explore the physics of anisotropic plasma.

3. The numerical scheme

On the basis of the modified GSE given by equation (19), we altered and updated the axisymmetric plasma equilibrium code HELENA [21] to its anisotropy and toroidal flow version HELENA+ATF. Since the internal physical assumptions and equations are completely changed, we have rewritten most of its matrix element calculations and post-processing, but have retained subroutines for isoparametric meshing. HELENA+ATF uses the same isoparametric bicubic Hermite elements as HELENA [21, 22].

Equation (19) is solved in its weak form. That is, with the spatial discretization in [21] and [22], the PDE system is transformed into a linear algebra problem by integrating the two sides after they are multiplied by each Hermite element. Here, a Picard iteration is used to solve the system. The flux functions and \( \Delta \) for the \( n \)th iteration are used to calculate the flux surfaces \( \psi(R, Z) \) of the \((n+1)\)th iteration.

If \( p_\parallel > p_\perp \), \( 1 - \Delta \) can go from positive to negative. In this case, the shear Alfvén wave becomes purely growing [23], labelled as the firehose instability. On the other hand, if \( p_\parallel < p_\perp \), the mirror instability may occur, with the non-oscillating mode becoming unstable [23]. The firehose and mirror stability criteria given by [5, 24] are
\[ 1 - \Delta > 0, \] (20)

which guarantee equation (19) being elliptic all of the time [10, 17]. These criteria are also sufficient conditions for the solvability (see appendix A) of the four interdependent variables \( p_i, p_\perp, B \) and \( \Delta \) (equation (6) (7) and (11)). In this work, we only discuss equilibria within these stability criteria. With the bi-Maxwellian equation (18), the stability criteria are written as
\[ \frac{3 \beta_E + 2 \sqrt{3(3 \beta_E + 2)^2 + 12 \beta_E}}{6 \beta_E} > \frac{p_\perp}{p_\parallel} > \frac{3 \beta_E - 2}{3 \beta_E + 4}. \] (22)

with \( \beta_E = \mu_0(4 p_\perp/3 + 2 p_\parallel/3)/B^2 \) the local ratio of the kinetic energy to the magnetic energy. Even for a tokamak with \( \beta_E = 0.4 \), we still have the upper limit three and a lower limit below zero. Therefore, these stability criteria are satisfied in most scenarios, although the mirror instability criterion may be approached in high \( \beta \) tokamaks with strong ICRH or perpendicular NBI heating.

In order to benchmark the force balance convergence of HELENA+ATF, we consider a test case with constant \( F \) and \( \psi \) profiles, a linear \( T_\parallel \) profile (\( \sim -\psi \)), and quadratic \( H \) and \( \Omega^2 \) profiles (\( \sim (1 - \psi)^2 \)). The plasma boundary is set to have elongation \( \kappa = 1.2 \), triangularity \( \delta = 0.2 \) and inverse aspect ratio \( \epsilon = 0.3 \). In anisotropic test cases, \( p_\parallel/p_\perp = 3.5 \) on the axis, while in test cases with toroidal flow, \( \Omega^2/T_\parallel = 0.5 \) on the axis.

Figure 1 shows the average force balance error for all grid cells and the maximum force balance error in four test cases. The force balance error decreases logarithmically as the grid resolution increases. To explain the difference between figures 1(a) and (b), we mention that the force balance error is close to zero near the core but reaches its maximum at the boundary. This is not only because the grid is more concentrated at the core, but also because a sharp boundary approaching an X point or triangular point will cause numerical degrading with a singular Jacobian.

Once the equilibrium is computed, HELENA+ATF also provides high precision coordinate information for stability codes. The solution of the modified GSE is mapped into the straight field line coordinate \( (s, \theta, \varphi) \), which is defined as
\[ s = \sqrt{\psi}/\psi_0, \quad \vartheta(\theta) = F(\psi)/q \int_\psi R(1 - \Delta)|\nabla \psi|, \] (23)

where \( q \) is defined as
\[ q(\psi) = \frac{F(\psi)}{2\pi} \int_\psi \frac{dl}{R(1 - \Delta)|\nabla \psi|}. \] (24)

The metric coefficients \( g^{ij} \) and Jacobian \( J \) can then be calculated.

4. The features of anisotropic equilibria

There are three major effects of anisotropic pressure that we can infer from our model and equation (19):

(i) \( p_\perp \) and \( p_\parallel \) contribute separately to the toroidal current;
(ii) the term, ‘1 − Δ′ inside the LHS operator will modulate
the poloidal flux and form a new ‘nonlinear current’;
(iii) pressure and density contours no longer lie on surfaces of
constant poloidal flux.

Effects (i) and (ii) will be explained in section 4.1, and
effect (iii) in section 4.2. In this section, flow is turned
off unless otherwise specified. We choose profiles that
represent the general shape and trend of the EFIT-TENSOR
reconstructed profiles with the TRANSP [25] constraint of
(MAST discharge #18696 at 290 ms [20]. They are
chosen to be \(q_0\) unchanged. Finally, with equilibrium D we examine
an equilibrium with triangularity \(\epsilon\). For these profiles, we examine four equilibrium
configurations. Equilibrium A is guided by a MAST-like
boundary with triangularity \(\delta = 0.4\), elongation \(\kappa = 1.7\) and
inverse aspect ratio \(\epsilon = 0.7\). The anisotropy of this case is
to leave \(q_0\) unchanged. With equilibrium C we examine the isotropic limit: \(\Theta_0\) is set to zero, and \(F_0\) adjusted to leave \(q_0\) unchanged. Finally, with equilibrium D we examine the impact of toroidal flow, with \(\Omega^2 \sim (1 − \Psi^3)\), such that the ion thermal Mach number \(M_{\text{th}}\) peaks at 0.7 on the axis and vanishes at the edge, where \(M_{\text{th}} = \nu_t/\sqrt{k_B T_i/m_i}\) and \(T_i\) is the ion temperature. This is the typical upper limit for toroidal flow in MAST [26]. In all cases, the anisotropy peaks at the core, due to the flat \(\Theta\) profile that we have chosen. Table 1 shows the parameters of these equilibria.

### 4.1. Toroidal current decomposition

In a cylindrical plasma with straight field lines and infinite
length, the perpendicular force balance is determined by \(p_\perp\). In
tokamak, there is a \(p_\parallel\) contribution [14] to the perpendicular
force balance. If the flow is negligible, we can rewrite equation
(14) and decompose \(J_\parallel\) as

\[
\mu_0 J_\parallel = \mu_0 R \sin^2 \alpha \left( \frac{\partial p_\parallel}{\partial \Psi} \right)_B + \mu_0 R \cos^2 \alpha \left( \frac{\partial p_\perp}{\partial \Psi} \right)_B \left( \frac{1 − \Delta}{2R} \frac{\partial B_\parallel}{\partial \Psi} \right)_B, \tag{26}
\]

where \(\alpha\) is the field pitch angle, i.e. \(\tan \alpha \equiv B_\parallel/B_\perp\), with \(B_\parallel\)
the poloidal magnetic field. The flux surface is determined by
\(J_\parallel\) through equation (8). The four contributing terms, \(J_{\text{pl}}, J_{\text{nl}}, J_{\text{it}}\) and \(J_{\text{ai}}\), are identified here. This equation shows that the balance of \(J_{\text{pl}}\) and \(J_{\text{nl}}\) is determined by the pitch angle \(\alpha\).

Figure 2 shows the decomposition of \(J_{\text{nl}}\) along the mid-plane for equilibria A and B. These two equilibria have similar profiles and their major difference is in the aspect ratio. In both cases, \(J_{\text{nl}}\) is dominated by \(J_{\text{pl}}\) and \(J_{\text{nl}}, which are roughly equal. The \(J_{\text{pl}}\) component is zero on the magnetic axis, which is consistent with \(\sin^2 \alpha = B_\perp^2/B_\parallel^2\) and \(B_\parallel = 0\) on the axis. For a low \(\beta\) plasma, \(\sin^2 \alpha = B_\perp^2/B_\parallel^2 \sim \epsilon^2/q^2\). We would thus expect, and observe, an increasing contribution from \(J_{\text{pl}}\) with increasing \(\epsilon\). For \(\epsilon = 0.7\), \(J_{\text{pl}}\) peaks at 20% on the low field side. Therefore, if the contribution of \(p_\parallel\) is ignored, or, in other words, attributed to \(p_\perp\), the current profile, and thus the \(q\) profile, will be changed up to 10% with \(p_\parallel/p_\perp \approx 1.5\).

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Shaded areas with different grey levels indicate different components. The maximum of indicate different magnitudes of anisotropy on the axis.

Figure 2. Contribution of each component to \( J_\phi \) across the mid-plane in (a) equilibrium A with \( \epsilon = 0.7 \), (b) equilibrium B with \( \epsilon = 0.3 \). The result stresses that for analytic working different markers indicate different magnitudes of anisotropy on the axis.

The contribution of the nonlinear current \( J_\phi \) to the total toroidal current \( J_\phi \) in per cent as a function of \( \Delta \). Different markers indicate different magnitudes of anisotropy on the axis.

Figure 3. The contribution of the nonlinear current \( J_\phi \) to the total toroidal current \( J_\phi \) in per cent as a function of \( \Delta \). Different markers indicate different magnitudes of anisotropy on the axis.

The result shows that \( \langle p_\perp \rangle \) is much better than \( \langle p^* \rangle \) for retaining global parameters, if \( \langle \rho \rangle \) and \( \langle RB_\rho \rangle \) are also unchanged.

4.2. Deviation from the flux function

4.2.1. The impact on the pressure and density. It is clear that with the isotropic assumption \( p_\parallel = p_\perp = p \) and the static assumption, we have \( \nabla p \cdot B = 0 \), which means that pressure is a flux function. But now with the additional term \( \Delta BB \) in equation (6), the two pressures and the density are not flux functions. This subsection will focus on their variation over a certain flux surface.

If the aspect ratio is large, which means that the variation of the magnetic field on a flux surface, \( (B_{\text{max}} - B_{\text{min}})/B \) is small, we can Taylor expand \( p_\parallel \) about \( B_0 = B(R_0) \), with \( R_0 \) the major radius of the magnetic axis. We use equation (12) to substitute for the partial derivative and derive the difference \( \Delta p_\parallel = p_{\parallel,\text{out}} - p_{\parallel,\text{in}} \), where the subscript ‘out’ denotes the most outward point and the subscript ‘in’ denotes the most inward point on a flux surface. Generally \( B \approx B_0 R_0/(R_0 + r \cos \theta) \) on a flux surface, in which \( r \) is minor radius of a certain flux surface and \( \theta \) the poloidal angle. Combining these, we obtain

\[
\frac{\Delta p_\parallel}{p_\parallel} \approx \frac{2r}{R_0} \left( \frac{p_{\perp} - p_{\parallel}}{p_{\parallel}} \right)_{R=R_0} .
\]

We note here that to reach equation (27), we do not need any kinetic assumptions. Similarly, an expansion of \( \rho \) and \( p_\perp \) about \( B_0 \), using equations (16), (17) and (18), yields the difference of \( \rho \) and \( p_\perp \) on a flux surface:

\[
\frac{\Delta \rho}{\rho} \approx \frac{2r}{R_0} \left( \frac{p_{\perp} - p_{\parallel}}{p_{\parallel}} \right)_{R=R_0} ,
\]

\[
\frac{\Delta p_{\perp}}{p_{\perp}} \approx \frac{4r}{R_0} \left( \frac{p_{\perp} - p_{\parallel}}{p_{\parallel}} \right)_{R=R_0} ,
\]

where the meanings of \( \Delta \rho \) and \( \Delta p_{\perp} \) are similar to that of \( \Delta p_\parallel \). Equations (27) and (28) indicate the linear dependence of \( \rho \) and \( p_\perp \)’s non-flux-function effect on the magnitude of the anisotropy and \( \epsilon \). These equations also give the direction of contour shift. If \( p_\perp > p_\parallel \) (\( p_\perp < p_\parallel \)), the shift of the pressure and density contours with respect to the flux surfaces is outward (inward), which can be compared to previous findings [9, 10].
Figure 4. The changes of global parameters: Shafranov shift ($\Delta_\xi$), $q_0$, $l_i$ (equation (32)), total flux, total current, due to the changing magnitude of the anisotropy, based on equilibrium C, if the following quantities are held unchanged for each flux surface: (a) $\langle p_\perp \rangle$, $\langle \rho \rangle$ and $\langle RB\phi \rangle$, (b) $\langle p^* \rangle$, $\langle \rho \rangle$ and $\langle RB\phi \rangle$. For instance, the change of $\Delta_\xi$ is in the form of $(\Delta_\xi_{\text{anis}} - \Delta_\xi_{\text{iso}}) / \Delta_\xi_{\text{iso}} \times 100\%$.

We also study the non-flux-function effect numerically. In figure 5, we plot $p_\parallel$ and $p_\perp$ on different flux surfaces for equilibrium A. Moving outwards from the core, the anisotropy decreases and reaches $p_\perp \approx p_\parallel$ at the boundary, while $r/R_0$ increases from zero to its maximum at the boundary. The competition between these two factors makes the difference peak at $s = 0.5$, with $\Delta p_\parallel / p_\parallel \approx 10\%$ and $\Delta p_\perp / p_\perp \approx 20\%$. This figure demonstrates the deviation of the profiles from a function of flux in a single equilibrium. Figure 6 shows the maximum in $\Delta \rho / \rho$ as a function of $\epsilon$ and $\xi$, scanning about the isotropic equilibrium C. Inspection clarifies that the change of density on a flux surface is almost linear with the aspect ratio and anisotropy. Similar behaviour is found for $\Delta p_\parallel / p_\parallel$ and $\Delta p_\perp / p_\perp$. Thus, the results of equations (27) and (28) can be extrapolated to tight aspect ratio tokamaks.

To demonstrate the magnitude of the non-flux-function effect, we compare the pressure profiles from anisotropic equilibrium A to those for flowing isotropic equilibrium D. Figure 7 shows the pressure profiles on flux surfaces for equilibrium D. The pressure difference peaks at 7% at $s = 0.4$, which is comparable to the difference in $p_\parallel$ for static anisotropic equilibrium A. For equilibrium A, the pressure difference in $p_\perp$ is larger than that for equilibrium D.

4.2.2. The impact on the Shafranov shift. Using methods from [12–14, 22], for large aspect ratio ($\epsilon = a/R_0 \ll 1$), low $\beta$ ($\beta \sim \epsilon^2$) plasma, we have, to zeroth order in $\epsilon$, the modified GSE

$$\frac{d}{dr} \left( \mu_0 \langle p_\perp \rangle + \frac{1}{2} B^2 \phi_0 \right) + \frac{B^2 \phi_0}{r} \frac{d}{dr} (\hat{r} B \phi) = 0. \quad (29)$$
Replacing \( p_1 \) by \( p \) will return us to the original isotropic and static case. This also confirms our result that the flux surface is mostly decided by \( p_1 \) in the large aspect ratio scenario.

The next order contribution, \( O(\epsilon) \), along with the bi-Maxwellian relationships of equation (18), yields the formulation of the Shafranov shift:

\[
\Delta_{\phi}^i(\hat{\rho}) = -\frac{1}{\hat{\rho} R_0 B_{\psi 0}} \int_0^\hat{\rho} \hat{\rho} \, d\hat{\rho} \\
\times \left\{ 2\hat{\mu}_0 \left[ P_{\perp} \left[ 1 + \left( \frac{p_{||} - p_{\perp}}{2p_{\perp}} \right) \right] \right] \right\} + B_{\psi 0} \right). \tag{30}
\]

This result is same as those of [13, 14], the variables \( R_0 \) and \( \langle p_{\perp} \rangle \) are related through equation (29), and are independent of \( p_1 \). The anisotropy and flow contribute to the Shafranov shift only through \( p_1 \) and \( M_{\psi 0}^2 \), and their effect is to scale \( p_{\perp} \). An example of how anisotropy influences the Shafranov shift is provided in figure 4(a), where \( \langle p_{\perp} \rangle \) and \( \langle R B_{\psi} \rangle \) are fixed. The figure shows that \( p_1 > p_{\perp} \) (\( p_1 < p_{\perp} \)) indicates more (less) Shafranov shift, and the magnitude of this change is linear in \( \xi \).

### 5. The performance of the isotropic model in the reconstruction of anisotropic systems

In this section we examine the implications of the choice of model in equilibrium reconstruction. The global invariants obtained by integrating the momentum conservation provide a useful starting point. Following this procedure, Cooper and Lao [27, 28] reached the following relationship between global parameters for large aspect ratio tokamaks (equation (12) of [28]):

\[
\frac{1}{2} (\beta_{p\perp} + \beta_{p\parallel}) + W_{\perp} + \frac{l_i}{2} = \frac{S_1}{4} + \frac{S_2}{4} (1 + \frac{R_e}{R_0}), \tag{31}
\]

with \( R_0 \) the major radius, \( R_e \) a volume dependent constant and

\[
\beta_{p\parallel} = \frac{2\mu_0 p_0}{B_{p0}^2}, \quad \beta_{p\perp} = \frac{2\mu_0 p_{\perp}}{B_{p\perp}^2}, \quad W_{\perp} = \frac{2\mu_0 p_{\perp}^2}{B_{p\perp}^2}, \quad l_i = \frac{B_{p\perp}^2}{B_{p0}^2}, \tag{32}
\]

in which \( B_{p\perp} \) is the average poloidal field at the boundary and \( \mu \) is the rotation velocity. The term \( \beta_{p\parallel} \) is the parallel poloidal beta, \( \beta_{p\perp} \) is the perpendicular poloidal beta, \( W_{\perp} \) is the rotation poloidal beta and \( l_i \) is the internal inductance. In this section, we consider static equilibria for which \( W_{\perp} = 0 \). The constants \( S_1 \) and \( S_2 \) are integrals of external fields and currents and therefore can be measured [29]. For a given set of data from magnetic probes, \( S_1 \) and \( S_2 \) are exactly determined. Equation (31) provides a good measurement of the fit for reconstructions using both anisotropic models and the MHD model with the \( p = p^* = (p_1 + p_{\perp})/2 \) approximation and \( \beta = (\beta_{p\parallel} + \beta_{p\perp})/2 \) (ideal MHD). This is the historical reason for using \( p^* \) as the approximate scalar pressure. The section intends to answer the question of, if the plasma is anisotropic and we still reconstruct using ideal MHD, how good the reconstructed profiles are, compared to using an anisotropic model.

#### 5.1. The model dependence in the equilibrium reconstruction

The impact of different models on the inferred pressure and current gradient profiles can be examined by comparison of the toroidal current profile at large aspect ratio. For the ideal MHD model, the GSE gives

\[
\mu_0 R_j_{\psi MHD} = \mu_0 R^2 p_{\psi MHD}(\Psi) + F_{\psi MHD} F'_{\psi MHD}(\Psi). \tag{33}
\]

Here we have added the subscript MHD to tag those functions with an ideal MHD model. A similar functional form can be written for the toroidal current using an anisotropy modified MHD model. At large aspect ratio, the approximations \( R \approx R_0 + p \cos \theta \) and \( B \approx B_0/R_0 \) can be applied. We also take the \( \Psi \) derivative on both sides of equation (12), and use it to substitute for the cross derivative in the Taylor expansion of \( \partial p_1 / \partial \Psi \) about \( B_0 \). If the flow is negligible, the right-hand side of the modified GSE (14) can thus be rearranged into

\[
\mu_0 R J_{\psi m} \approx \mu_0 R^2 P_{m m} + \frac{1}{2} \left( F_m F'_m + \mu_0 R_0^2 \int_{\Omega_m} \frac{p'_{|| m} - p'_{\perp m}}{2} \right) + O \left( \frac{r^2}{R^2} \right), \tag{34}
\]

where we have similarly added the subscript \( m \) to tag the functions with the anisotropy modified MHD model. The functions \( P_{m m} \), \( p_{|| m} \) and \( p_{\perp m} \) are those quantities on the flux surface at point \( R = R_0 \). The higher order terms are written as \( O(r^2/R^2) \).

Provided that internal current profile information (such as the MSE) is available, \( J_{\psi MHD} = J_{\psi m} \) in any reconstruction: the current profile is unique. To \( O(r/R) \), the right-hand sides of equations (40) and (41) have the same variables and functional dependence on \( R^2 \), that is, an \( R^2 \) flux surface varying part and a flux surface invariant part. On equating these two parts, reconstructions using different models but the same data yield

\[
P_{\psi MHD} = P_{\psi m}, \tag{35}
\]

\[
F_{\psi MHD} F'_{\psi MHD} = F_m F'_m + \mu_0 R_0^2 \int_{\Omega_m} \frac{p'_{|| m} - p'_{\perp m}}{2} \tag{36}
\]

Consequently the inferred pressure profile will be identical to the usual \( p^* \) approximation, but the toroidal flux functions, and thus the poloidal current profiles, will be different in the GSE and the modified GSE models. This is consistent with figure 4(b), which shows that the plasma cannot preserve its global parameters if we fix both \( \langle p^* \rangle \) and \( \langle R B_{\psi} \rangle \) but vary the anisotropy.

At tight aspect ratio, we should consider the \( O(r^2/R^2) \) contribution to the modified GSE, with the second term in the Taylor expansion of equations (16), (17) and (18). The result is

\[
f \left( \frac{O \left( \frac{r^2}{R^2} \right)}{\frac{r^2}{R^2}} \right) = -\mu_0 (p_{||} - p_{\perp}) \left( 1 + \frac{P_{\perp 0}}{P_{|| 0}} \right) \frac{r^2}{R_0^2} \cos^2 \theta \]
\[
+ O \left( \frac{r^3}{R^3} \right). \tag{37}
\]

Due to the \( \cos^2 \theta \) dependence, it is not possible to resolve \( J_{\psi} \) into two MHD flux functions, as was done for the zeroth and first order parts of equation (34). Equation (37) reveals
the dependence of the higher order term on the product of \((p_{\parallel 0} - p_{\perp 0})/p_{\parallel 0}\) and \(r^2/R^2\). Thus, for tight aspect ratio tokamaks with large anisotropy, the reconstructed \(J_\varphi\) and \(q\) profiles formed by the two flux functions may be distorted, in comparison to the results from anisotropic reconstruction.

5.2. The equilibrium reconstruction of a MAST discharge

We here study a pair of reconstructions from a single discharge. The example is from the EFIT-TENSOR reconstruction for the MAST \((\epsilon \approx 0.7)\) discharge \#18696 at 290 ms, using an anisotropic model and an isotropic model. For this discharge, MSE data are not available. The constraints that we used are magnetic probes, total currents and pressures from TRANSP. These constraints are identical in the two reconstructions, except that for the anisotropic reconstruction, \(p_{\parallel}\) and \(p_{\perp}\) are constrained to the TRANSP \(p_{\parallel}\) and \(p_{\perp}\) respectively, while for the isotropic reconstruction, the isotropic pressure is constrained to \(p^* = (p_{\parallel} + p_{\perp})/2\).

In this discharge, the NBI is parallel and we have \(p_{\parallel}/p_{\perp} \approx 1.25\) on the magnetic axis, as shown in figure 8(a). We can see from figure 8(b) that the two reconstructions give \(J_\varphi\) profiles that are almost the same, with a small difference in the core region. We also notice that these two reconstructions give slightly different boundaries, causing the differences of the \(q\) and \(J_\varphi\) profiles on the low field side. Both inference differences arise because the EFIT-TENSOR reconstruction is not constrained by a \(J_\varphi\) profile. Despite these differences, the \(q\) profiles as functions of the flux are found to be nearly identical in the two cases. In our previous work [3], we recorded a 15% lift in \(q_0\) due to the anisotropy, which is not observed here. The reason is that in [3], the two equilibria with/without anisotropy had fixed profiles, not fixed external constraints on the equilibrium, as studied here. In addition, the anisotropy modelled in [3] was \(p_{\perp}/p_{\parallel} = 1.7\) (only the beam pressure was considered).

As predicted by equation (36), the MHD reconstructed toroidal field is underestimated in comparison to the anisotropic reconstruction. This prediction is verified by figure 8(c), showing that \(RB_\varphi\) is underestimated by 3% at the core. When looking at the \(J_p\) profiles of the two cases in figure 8(d), we discover a large discrepancy near the core region, which peaks at \(R = 0.9\) m with isotropic \(J_p\) only 1/3 of its anisotropic reconstruction value. The difference can be explained by equation (36). Since the two models yield different \(RB_\varphi + J_p\) differs, through \(\mu_0 J_p = |\nabla RB_\varphi|/R\) from equation (8). In this case the maximum contribution of the \(O(r^2/R^2)\) term to the total current is 1.5%, so the higher order contribution can be ignored.

5.3. The implications of using MHD to reconstruct anisotropic plasma

On the basis of the above findings, if single-pressure MHD is used to reconstruct a purely anisotropic plasma, the following...
four problems will occur, according to the aspect ratio and the magnitude of the anisotropy.

(i) The poloidal current is different.
This problem is demonstrated in section 5.1 and 5.2 and occurs when the variation of the F profile is comparable to the variation of \( p_1 - p_\perp \) across the flux surfaces.

(ii) The anisotropic profiles are not flux functions.
In MHD, \( p_1, R B_p \) and \( \rho \) are flux functions. As shown in section 4.2.1, they deviate from flux functions. According to equations (27), (28) and figure 5, this problem increases linearly with \( \epsilon \) and \( \xi \).

(iii) Force balance is only satisfied to \( O(r/R) \) with two flux functions.
For tight aspect ratio and large anisotropy, we should take into account terms \( O(r^2/R^2) \) in the modified GSE. It is not possible to decompose the \( J_\phi \) profile into the combination of two flux functions, as we demonstrated in section 5.2.
If MHD reconstruction is used, the reconstructed \( J_\phi \) profile formed from two flux functions may be distorted. Inspection of equation (37) reveals that this problem is a linear function of \( \epsilon^2 \xi \).

(iv) The nonlinear current \( J_{a1} \) is important at high \( \beta \) and large anisotropy.
In section 4.1, we showed that \( J_{a1} \) is proportional to \( \Delta \). The ideal MHD reconstruction neglects \( J_{a1} \), which might have an impact on the accuracy of the reconstructed \( J_\phi \) profile and the \( q \) profile for a plasma with high \( \beta \) and large anisotropy.

To illustrate the problems in \( \epsilon - \xi \) space, we have sketched regimes where each problem might occur. The corresponding contours are shown in figure 9, consisting of four regions with a different numbers of problems. The lower boundaries are: for problem (i), \( |\xi| = 0.05 \) which represents a 5% difference between \( p_1 \) and \( p_\perp \) on average; for problem (ii), \( |\Delta \rho|/\rho = 5\% \) calculated from figure 6, taking the average of \( \xi > 0 \) and \( \xi < 0 \); for problem (iii), the maximum contribution of the \( O(r^2/R^2) \) term to \( J_\phi \) equals 5\%, which is obtained by scanning around equilibrium C. The projection of problem (iv) is not meaningful in \( \epsilon - \xi \) space, as it is a function of \( \Delta \), and thus \( \beta \) and \( \xi \), not \( \epsilon \).

We have identified the #18696 MAST equilibrium and our equilibrium A and B in these contours. Also, \( p_\perp/p_1 \approx 2.5 \) was found in a JET discharge (\( \epsilon \approx 0.3 \)) during ICRH heating [2]. The parameter \( |\xi| \), if assumed to reach a third of its maximum local value, is 0.3. Problem (ii) is significant in this case, with maximum \( \Delta p_1/p_1 \approx 17\% \). Recent unpublished MAST data suggest the existence of discharges with \( |\xi| > 0.3 \), and thus encounter Problems (i)–(iii). We will include the study of this discharge in our later publications. Finally, Problem (iv) appears for discharges with relatively high \( \beta \). To date, we have not identified a discharge with \( \Delta > 5\% \) in MAST. However, a \( > 40\% \) volume average \( \beta \) is observed in NSTX discharges with strong parallel injection [30]. Also, the beam power will increase to 7.5 MW in the MAST upgrade [31], providing the possibility of triggering Problem (iv) and enriching our study in the future.

\section{Conclusion}

The impact of pressure anisotropy on plasma equilibrium is studied analytically and numerically. To achieve the latter, we have extended the fixed boundary equilibrium and mapping code HELENA to include toroidal flow and anisotropy (HELENA+ATF). We decompose the toroidal current into contributions from both pressures, the toroidal field and the nonlinear part and find the dependence of \( J_{\rho 1} \) on the ratio \( B_\rho^2/B^2 \). We find a dominant role of \( J_{\rho 1} \) over \( J_{\phi 1} \) in the anisotropy and toroidal flow modulated Grad–Shafranov equation for large aspect ratio tokamaks. However in a MAST-like equilibrium, the \( J_{\rho 1} \) contribution can reach 20\% of the total current with \( \epsilon = 0.7 \) and \( p_1/p_\perp \approx 1.5 \), which should not be ignored. The impact of this is a 10\% change in the current profile, and thus the \( q \) profile, with corresponding implications for plasma stability. The nonlinear current \( J_{a1} \) is proportional to \( \Delta \), and should not be neglected when anisotropy appears in a high \( \beta \) plasma. We have also found that the deviation of the profiles from flux functions is of the order of \( \epsilon |p_1 - p_\perp|/p_\perp \), showing a larger contour shift with tighter aspect ratio and larger anisotropy.

Motivated by this analysis, we find that, depending upon the aspect ratio and the magnitude of the anisotropy, the following problems may be encountered when the ideal MHD model with \( p^* = (p_1 + p_\perp)/2 \) is used to reconstruct an anisotropic plasma. First, the poloidal current is different. This occurs when the variation of the F profile is comparable to the variation of \( p_1 - p_\perp \) across the flux surfaces. Second, the anisotropy profiles are not flux functions; their difference on a flux surface increases linearly with the magnitude of the anisotropy and \( \epsilon \). Third, the \( O(r^2/R^2) \) contribution to \( J_\phi \) is not considered. This may distort the \( J_\phi \) and \( q \) profiles for tight aspect ratio tokamaks with large anisotropy. Finally, the nonlinear current is neglected, degrading the accuracy of the result for a plasma with high \( \beta \) and large anisotropy.

In future work, we plan to study the impact of anisotropy on the magnetic configurations, for a range of experimental
discharges and machines, in order to address this problem empirically. We also plan to study the effect of anisotropy on plasma stability.

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Appendix. The solvability of $p_\parallel, p_\perp, B$ and $\Delta$

Here, we demonstrate that equations (20) and (21) are a set of sufficient conditions for the four interdependent variables $p_\parallel, p_\perp, B$ and $\Delta$ (equation (6), (7) and (11)) to have one and only one root.

The $n$th Picard iteration gives $\Psi_n(R,Z)$ and thus $B_{p,n} = [\nabla \Psi_n] / R$. To calculate the magnetic field $B$ after the $n$th iteration at a certain grid point: $B_n(R,Z)$, the following equations need to be solved for unknown $B_n$, with known $\Psi_n, B_{p,n}$ and $R$:

$$B_n^2 = \frac{F^2(\Psi_n)}{(1 - \Delta_n)^2 R^2} + B_{p,n}^2, \quad (A.1)$$

$$\Delta_n = \frac{\mu_0 [p_\parallel(\Psi_n, B_n, R) - p_\perp(\Psi_n, B_n, R)]}{B_n^2}, \quad (A.2)$$

Rearranging equation (A.2) and taking the derivative leads to

$$g(B_n) = (B_n^2 - B_{p,n}^2)(1 - \Delta_n)^2 - \frac{F^2(\Psi_n)}{R^2} = 0, \quad (A.3)$$

$$g'(B_n) = 2B_n(1 - \Delta_n) \times \left[ (1 - \frac{B_n^2}{B_{p,n}^2}) \left( 1 + \frac{\mu_0}{B_n} \frac{\partial p_\perp(\Psi_n, B_n, R)}{\partial B_n} \right) + \frac{B_{p,n}^2}{B_n^2}(1 - \Delta_n) \right]. \quad (A.4)$$

With equations (20) and (21), and $B > B_p$, we have $g'(B_n) > 0$. Therefore $g(B_n)$ is monotonically increasing from $B_{p,n}$ to $+\infty$. Providing that $g(B_{p,n}) < 0$ and $g(+\infty) \rightarrow +\infty$, equation (A.3) should have one and only one root in region $[B_{p,n}, +\infty)$.

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