Bargmann Invariants, Geometric Phases and Recursive Parametrization with Majorana Fermions

Rohan Pramanick*, Swarup Sangiri and Utpal Sarkar
Department of Physics, Indian Institute of Technology Kharagpur, Kharagpur 721302, India

Abstract

A generalized connection between the quantum mechanical Bargmann invariants and the geometric phases was established for the Dirac fermions. We extend that formalism for the Majorana fermions by defining proper quantum mechanical ray and Hilbert spaces. We then relate both the Dirac and Majorana type Bargmann invariants to the rephasing invariant measures of CP violation with the Majorana neutrinos, assuming that the neutrinos have lepton number violating Majorana masses. We then generalize the recursive parametrization for studying any unitary matrices to include the Majorana fermions, which could be useful for studying the neutrino mixing matrix.

1 Introduction

One of the most interesting problems of the standard model is to understand the origin of CP violation. It appears in different forms and was first observed in weak decays of the neutral K-mesons. CP violation is also needed to explain why there are more matter compared to antimatter in the universe [1].

CP violation has been studied extensively for the Dirac fermions [2, 3]. All known charged quarks and leptons are Dirac particles and their analysis does not have direct implications to the lepton or baryon number violating interactions, including the generation of matter asymmetry of the universe. We thus attempt to generalize some results for the Dirac fermions to models with Majorana fermions like the neutrinos.

Although we are yet to infer if there exists any Majorana fermion, many interesting aspects of the Majorana fermions have been pointed out, which may have far reaching consequences. In particle physics, the masses of the Majorana particles play a crucial role and can explain the smallness of the neutrino mass naturally [4]. The Majorana neutrino masses can also explain the baryon asymmetry of the universe [6] and predict the dark matter and resolve some issues of the dark energy.

Considering all these, we intend to study the CP phases for the Majorana fermions from a different angle. There have been some analysis of the Dirac fermions (quarks and leptons [5]) identifying their rephasing invariant measure of quantum mechanical CP phases to the quantum mechanical Bargmann invariants (BI) [7], which in turn, is

*rohanpramanick25@gmail.com
related to the classical geometric phases. Our main aim is to generalize this result for the Majorana fermions and apply our result to the CP violation in the leptonic sector with Majorana neutrinos.

The geometric phase was introduced \cite{9} in a cyclic adiabatic quantum mechanical system, where the dynamics is governed by the time-dependent evolution of the state vector in a Hilbert space. These geometric phases has been shown to be related to a family of quantum mechanical Bargmann invariants (BI). For a physical system, the state vectors represent Dirac fermions, and the BIs may be identified with rephasing invariant measures of CP violation. We shall generalize these results to the case when the state vectors represent both Dirac and Majorana fermions by defining the ray and the Hilbert spaces properly. This will introduce additional BIs representing CP violation arising from the Majorana phases and this, in turn, will relate the rephasing invariant measures of CP violation for both Dirac and Majorana fermions to the complete sets of Bargmann invariants.

We shall first demonstrate how one can define the quantum mechanical Bargmann variables for the Majorana fermions and relate them with the geometric phases after defining the proper quantum mechanical ray and Hilbert spaces for the Majorana fermions. We then construct the rephasing invariant measures for the Majorana fermions. As expected, compared to the Dirac fermions, there are more number of such CP violating measures arising due to the Majorana phases.

We shall then extend our analysis to study the CP violating invariants for the Majorana fermions in the formalism of recursive parametrization of unitary matrices. One can study the CP phases in the neutrino mixing matrix and the Majorana phases through the recursive parametrization of the unitary matrices. We shall extend these analyses and present explicit forms of the rephasing invariant quantities for a few examples when Majorana fermions are included.

2 Majorana Fermions

A Majorana fermion is the antiparticle of itself. It has two complex components or four real components. Two Majorana fermions may combine into a Dirac fermion, depending on the mass terms. The Lagrangian describing a Majorana fermion can be given as

\[ \mathcal{L}_M = \bar{\psi}_M i \gamma^\mu \partial_\mu \psi_M + m_M \bar{\psi}_M \psi_M^c \]

where \( \bar{\psi}_M = \psi_M^\dagger \gamma^0 \) and we work in the Weyl representation, where the \( \gamma \) matrices are defined as:

\[ \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix} \text{ with } \sigma^\mu = [I_2, \sigma_i]; \quad \bar{\sigma}^\mu = [I_2, -\sigma_i] \]

where \( I_2 \) is a \( 2 \times 2 \) unit matrix and \( \sigma^i \) are the Pauli matrices. In this basis, \( \gamma_5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \text{diag} (-I_2, I_2) \) is diagonal. Defining the charge conjugation as

\[ \psi^c = -i \gamma_2 \psi^* = -i \gamma_2 \gamma_0 \bar{\psi}^T, \]

the Majorana condition that the Majorana particles are their own antiparticles, can be written as

\[ \psi^c_M = \lambda^* \psi_M. \]
here $\lambda$ is a complex phase contributing to CP violation, and $|\lambda|^2 = 1$. Since the particles and antiparticles carry opposite quantum numbers or charges under any symmetry group, this condition implies violation of that quantum numbers or the charges. So, charged leptons or quarks cannot be Majorana particles. We shall thus work with the assumption that neutrinos are Majorana particles, while all other charged fermions are Dirac particles.

A Dirac fermion has eight real components (or equivalently, four complex components) and may be decomposed into two Majorana fermions. For example, we can consider the left-handed ($\psi_L$) and right-handed ($\psi_R$) components of a Dirac fermion ($\psi_D$) as two Majorana fermions ($\psi_{M1}$ and $\psi_{M2}$) as defined below:

$$\psi_{M1} = \psi_L + \lambda_1 \psi_R^c \quad \text{and} \quad \psi_{M2} = \psi_R + \lambda_2 \psi_L^c \quad \text{where} \quad \psi_L = \frac{(1-\gamma_5)}{2} \psi_D; \quad \psi_R = \frac{(1+\gamma_5)}{2} \psi_D; \quad \psi_R = \frac{(1+\gamma_5)}{2} \psi_D; \quad \psi_L = \frac{(1-\gamma_5)}{2} \psi_D.$$ 

and $\psi_R^c$ and $\psi_L^c$ are CP-conjugate states of $\psi_L$ and $\psi_R$, respectively.

The Majorana fermions $\psi_{M1}$ and $\psi_{M2}$ satisfy the Majorana conditions:

$$\psi_{M1}^c = \lambda_1^* \psi_{M1} \quad \text{and} \quad \psi_{M2}^c = \lambda_2^* \psi_{M2}. \quad (4)$$

However, we can always remove an overall phase, so that the relative CP phases remain as independent phases.

We can now express the Majorana fields $\psi_{M1}$ and $\psi_{M2}$ in terms of complex two-component spinors $\eta$ and $\chi$ as:

$$\psi_L = \frac{1-\gamma_5}{2} \psi_M = \begin{pmatrix} \eta \\ -\bar{\eta} \end{pmatrix} \quad \text{and} \quad \psi_R = \frac{1+\gamma_5}{2} \psi_M = \begin{pmatrix} 0 \\ \bar{\chi} \end{pmatrix};$$

$$\psi_R^c = \begin{pmatrix} 0 \\ \bar{\eta} \end{pmatrix} \quad \text{and} \quad \psi_L^c = \begin{pmatrix} \chi \\ 0 \end{pmatrix}. \quad (5)$$

such that

$$\psi_{M1} = \begin{pmatrix} \eta \\ \lambda_1 \bar{\eta} \end{pmatrix}; \quad \psi_{M1}^c = \begin{pmatrix} \lambda_1^* \eta \\ \bar{\eta} \end{pmatrix}; \quad \psi_{M2} = \begin{pmatrix} \lambda_2 \chi \\ \bar{\chi} \end{pmatrix}; \quad \psi_{M2}^c = \begin{pmatrix} \chi \\ \lambda_2 \bar{\chi} \end{pmatrix}; \quad (6)$$

which satisfies the Majorana condition of equation (4).

Any Majorana field can be expressed in terms of the creation and the annihilation operators as

$$\psi_M(x) = \sum_{p,s} \sqrt{\frac{m_M}{2\epsilon}} \left( f_{ps} u_p e^{-ipx} + \lambda^* \tilde{f}_{ps}^\dagger v_p e^{ipx} \right), \quad (7)$$

where the energy of the Majorana fermion is $\epsilon = \sqrt{p^2 + m_M^2}$ and $m_M$ is the mass. The $u_{ps}$ and the $v_{ps}$ spinor operators satisfy the equations of motion

$$(\gamma_{\mu}p^\mu - m_M) u_p = 0; \quad (\gamma_{\mu}p^\mu + m_M) v_p = 0; \quad (8)$$

and $s$ is the spin.

To complete the discussion we shall define the density matrix for the Majorana fermions as $[10, 11]

$$\rho^M(\psi) = \psi^c (\psi^c)^\dagger = |\psi^c\rangle\langle\psi^c|, \quad (9)$$
which satisfies the equation of motion
\[
\frac{d\rho^M}{dt} = -i (H\rho^M - \rho^M H^\dagger).
\] (10)

This is a key ingredient in this analysis of Majorana fermions, which allows us to define the smooth parametrized curves \(C\) of unit vectors in the Hilbert space \(\mathcal{H}\) and the corresponding free geodesics. The Bargmann invariants are then expressed in terms of these free geodesics and the geometric phases can then be defined in terms of the Bargmann invariants.

The density matrices \(\rho_r\), corresponding to any unit vector \(\psi_r\) in the Hilbert space \(\mathcal{H}\) of Majorana fermion states, are images in the ray space \(\mathcal{R}\) and any two neighbouring density matrices (say, \(\rho_{r-1}\) to \(\rho_r\)) are connected by free geodesics in \(\mathcal{H}\). In case of a Dirac fermion or Dirac neutrino, the density matrices or the images of the unit vectors in \(\mathcal{H}\) onto \(\mathcal{R}\) is defined as
\[
\rho^D(\psi) = \psi \psi^\dagger = |\psi\rangle\langle\psi|.
\] (11)

Thus the Dirac density matrix does not correspond to any violation of charge or any conserved quantum number. The density matrix of the Majorana fermions would introduce the Majorana phases into the density matrix giving rise to new sources of CP violation. The Majorana density matrix will provide us with additional Bargmann invariants corresponding to the new Majorana CP phases \(\lambda\) that is defined in equation 3. The Majorana phases would disappear when the neutrinos dont have Majorana masses. Furthermore, if there is only one Majorana particle in any model, an overall phase transformation can remove it. We shall elaborate on these discussions in the next few sections.

3 Bargmann Invariants and Geometric Phase for Majorana Fermions

In this section we shall review the formalism of connecting the Bargmann invariants with the geometric phases and extend the earlier results by including the Majorana fermions, and hence, the Majorana phases. A connection between the Bargmann invariants and the geometric phases has been established rigorously for the Dirac fermions [12]. The basic structure of this formalism relies on the cyclic adiabatic quantum-mechanical evolution of the state vectors [13]. This has been further generalized to show that the geometrical phases can be related to a family of quantum-mechanical invariants which were proposed by Bargmann [7]. In this section we shall develop the connection between the geometrical phases and the Bargmann invariants for the Majorana particles and discuss how these studies can be extended to the CP violation in the lepton sector with Majorana neutrinos.

This analysis largely depends on the free geodesics in quantum-mechanical ray and Hilbert spaces, as the geometric phases vanish for these geodesics. It has been demonstrated that the generalization of free geodesics to the so-called null phase curves are more general in establishing a connection between the Bargmann invariants and the geometric phases. These null phase curves are a family of ray and Hilbert space curves, which includes the free geodesics and a large class of other curves, and establish a general connection between the Bargmann invariants and the geometric phases. However, for demonstrating the geometric phases of the Majorana fermions and its connection with the Bargmann invariants, we shall restrict our discussions to free geodesics only, with the understanding that these results are more general.
We start with a Hilbert space of some quantum system of both Dirac and Majorana particles $H$, and construct the associated ray space $R$ with the pure state density matrices. The dual space of $H$ will contain both particles and antiparticles for the Majorana fermions, but for the Dirac fermions the dual space of $H$ will contain only the particles. The density matrices for the Dirac fermions are defined in equation 11, where $\psi_a$ represents a vector in the Hilbert space $H$. The inner product of any two Dirac fermions would then be given by

$$\mathcal{I}^D = (\psi_a(s), \psi_b(s)) = \langle \psi_a(s) | \psi_b(s) \rangle.$$  \hspace{1cm} (12)

The key ingredient for studying a Majorana fermion in this formalism is to enhance the ray space with the pure state density matrices for the Majorana fermions defined by equation 9 in the previous section. For the Majorana fermions, the charge conjugation of $\psi$ satisfies the Majorana condition of equation 3 and as a result differs from the Dirac fermions by the Majorana phase $\lambda$. Thus the ray space curves for the Majorana fermions will differ from that of the Dirac fermions and this would modify the inner product of any two Hilbert space vectors.

We can now write down an inner product of two Majorana fermions $\psi_i$ and $\psi_j$ in $H$ as

$$\mathcal{I}^M = (\psi_i(s), \psi_j(s)) = \langle \psi_i(s) | \psi_j(s) \rangle.$$  \hspace{1cm} (13)

For the Majorana fermions one can write both $\mathcal{I}^D$ and $\mathcal{I}^M$ type inner products, but for the Dirac fermions one can only write $\mathcal{I}^D$ type inner products.

If the geometric phase vanishes for any connected part of the ray space curves, then the inner product of two Hilbert space vectors along the lift of such ray space curves would also vanish. This condition on the inner product of the Hilbert space vectors is also valid for the free geodesics, which can also link the Bargmann invariants with the geometric phases. We shall now define the free geodesics in ray and Hilbert spaces, in which the geometric phase vanishes and then demonstrate the connection between the Bargmann invariants and geometric phases. We shall follow the formalism and notation of [12, 14].

Any smooth parametrized curves $C$ of unit vectors in $H$ may then be expressed as

$$C = \{ \psi(s) \in H \mid ||\psi(s)|| = 1, \ s_1 \leq s \leq s_2 \} \subset H.$$  \hspace{1cm} (14)

The projection of the Hilbert space to the ray space

$$\pi : H \rightarrow R.$$  

will then provide us the image $C_r$ in $R$:

$$\pi[C] = C_r \subset R,$$

where the image $C_r$ for the Majorana fermions follows from the definition of the pure state density matrices for the Majorana fermions:

$$C_r = \{ \rho^M(s) = \psi^c(s) \psi^c(s)^\dagger \mid s_1 \leq s \leq s_2 \}.$$  \hspace{1cm} (15)

Thus any curve of unit vectors $C$ in the Hilbert space $H$ ($C \subset H$) is a lift of the image $C_r$ in the ray space $R$ ($C_r \subset R$). It is apparent that the end points should satisfy the boundary condition that they are not orthonormal

$$(\psi^c(s_1), \psi^c(s_2)) \neq 0$$  \hspace{1cm} (16)
and $\psi(s), \psi^c(s), \rho^M(s)$ and $\rho^D(s)$ are smooth curves, satisfying certain smoothness conditions [14].

We shall now consider the horizontal lift of the curve $C_r^{(h)} \subset \mathcal{R}$, such that the vectors $\psi^{c(h)}(s)$ along this lift satisfy

$$\left( \psi^{c(h)}(s), \frac{d}{ds} \psi^{c(h)}(s) \right) = 0.$$  (17)

This immediately implies vanishing of the dynamical phase for any curve $C_r \subset \mathcal{R}$, along the horizontal lift:

$$\phi_{\text{dyn}}[C] = \text{Im} \int_{s_1}^{s_2} ds \left( \psi^{c(h)}(s), \frac{d}{ds} \psi^{c(h)}(s) \right) = 0.$$  (18)

The geometric phase is thus given by

$$\phi_g = \arg \left( \psi^c(s_1), \psi^c(s_2) \right).$$  (19)

One can then define the free geodesics in $\mathcal{H}$ and $\mathcal{R}$ and show that the geometric phase vanishes along the free geodesics [12]

$$\phi_g[\text{free geodesics } \in \mathcal{R}] = 0,$$  (20)

and relate the geometric phases to the Bargmann variables. A more general analysis utilizing the null phase curves can also establish these relations [12], but for our purpose we shall directly move to the final result.

Bargmann invariants (BIs) were developed for the Dirac fermions, and the relationship with the geometric phase utilized the definition of the inner product ($\mathcal{I}^D$) and the density matrices ($\rho^D$) for the Dirac fermions [7]. To include the Majorana particles and extend the applicability of the Bargmann invariants, the Hilbert space $\mathcal{H}$ for the Dirac fermions is enhanced to incorporate the antiparticles, and hence, the Majorana phases, as defined in equation [3]. The corresponding ray space is also modified by the definition [9] of the density matrix ($\rho^M$), and hence, the inner products involving the Majorana fermions ($\mathcal{I}^M$). Accordingly we can write down two types of the Bargmann invariants (BI) for the Majorana fermions, one ($\Delta^D$) containing only $\mathcal{I}^D$ type inner products, and the other ($\Delta^M$) containing both $\mathcal{I}^D$ and $\mathcal{I}^M$ type inner products.

We first present a BI with Dirac fermions. Since $\langle \psi_a | \psi_b \rangle = \langle \psi_b | \psi_a \rangle^*$, all second order BIs are real and the corresponding geometric phase vanishes. We thus present a third order BI with Dirac fermions:

$$\Delta^D_3(\psi_1, \psi_2, \psi_3) = (\psi_1, \psi_2) (\psi_2, \psi_3) (\psi_3, \psi_1) = \text{Tr} \left[ \rho^D(\psi_1) \rho^D(\psi_2) \rho^D(\psi_3) \right] = \text{Tr} \left[ (\psi_1 \psi_1^\dagger) (\psi_2 \psi_2^\dagger) (\psi_3 \psi_3^\dagger) \right].$$  (21)

This is an example of a third order Bargmann invariant defined with three mutually nonorthogonal vectors $\psi_i \in \mathcal{H}|i = 1, 2, 3$ and the ray space is defined by density matrices $\rho^D = \psi_i \psi_i^\dagger \in \mathcal{R}|i = 1, 2, 3$. Any fourth or higher order BIs may be reduced to third order BIs. It is straightforward to generalize this definition to $m$-th order Bargmann invariants

$$\Delta^D_m(\psi_1, \psi_2, \cdots, \psi_m) = (\psi_1, \psi_2, \psi_3, \cdots, (\psi_m, \psi_1) = \text{Tr} \left[ \rho^D(\psi_1) \rho^D(\psi_2) \cdots \rho^D(\psi_m) \right] = \text{Tr} \left[ \psi_1 \psi_1^\dagger \psi_2 \psi_2^\dagger \cdots \psi_m \psi_m^\dagger \right].$$  (22)
From the properties of the free geodesics (equation 20), we can write down the relationship between the Bargmann invariants and the geometric phases for an \( m \)-vertex closed loop \((P_m)\) as,
\[
\phi_g [P_m] = - \arg \Delta^D_m(\psi_1, \psi_2, \cdots, \psi_m),
\]
where \( P_m = m \)-vertex closed loop \( \in \mathcal{R} \),
\[
\text{with } \rho^D_1 \rightarrow \rho^D_2 \rightarrow \rho^D_3 \rightarrow \cdots \rightarrow \rho^D_m \rightarrow \rho^D_1,
\]
being connected by free geodesics.

We now present an example of the third order BIs with Majorana fermions. These BIs are possible only when the Hilbert space \( \mathcal{H} \) includes the antiparticles \((\psi^c)\) and both the density matrices \( \rho^D \) and \( \rho^M \) appear in the definition of the BIs:
\[
\Delta^M_3(\psi_1, \psi^c_2, \psi^c_3) = (\psi_1, \psi_2^c) (\psi_2^c, \psi_3^c) (\psi_3^c, \psi_1) = \text{Tr} \left[ \rho^D(\psi_1) \rho^M(\psi_2^c) \rho^M(\psi_3^c) \right] = \text{Tr} \left[ \psi_1 \psi_1^\dagger \psi_2^c \psi_2^{c\dagger} \psi_3^c \psi_3^{c\dagger} \right],
\]
both \( \psi_i \) and \( \psi_i^c \) enter in the definition of the BIs and the Majorana phases enter in the definition of the BIs through the density matrices \( \rho^M(\psi_i^c) \). The consequences of the Majorana type BIs \((\Delta^M_i)\) including the Majorana phases will become clear when we shall relate them to the rephasing invariant measures of CP violation in the leptonic sector with Majorana neutrinos in the next section.

## 4 Majorana neutrinos and BI

The smallness of the neutrino mass can be explained in simple extensions of the standard model without any fine tuning or introducing arbitrarily small parameters, by considering the neutrinos to be Majorana fermions. Any information about CP violation in the leptonic sector with Majorana neutrinos are contained in the neutrino mass matrix and their charged current interactions. Without loss of generality we can work in a basis, in which the charged lepton mass matrix is diagonal, so that the complex phases in the neutrino masses and mixing matrix will determine the CP violation in any model.

Some of the complex phases in the neutrino mass and mixing matrices can be removed by the rephasing of the neutrinos, so it is a general practice to construct rephasing invariant measures to study the CP violation. In this section we shall demonstrate that these CP violating rephasing invariant measures are the Bargmann invariants with the Majorana neutrinos and are related to the geometric phases. In particular, we shall emphasize on the lepton number violating CP violating measures, which are the new Bargmann invariants with the Majorana fermions.

We begin with the charged current interactions of the neutrinos with the charged leptons and the neutrino mass matrix for the Majorana neutrinos:
\[
L_{\text{CC}} = - \frac{g}{\sqrt{2}} \sum_{\alpha = e, \mu, \tau} \bar{\nu}_\alpha L \gamma^\rho l_\alpha L^C W^\rho_\alpha ,
\]
\[
L_{\text{mass}} = \nu^T_{iL} C^{-1} \nu_{iL} = m_i \overline{\nu}_{iL^C} \nu_{iL},
\]
where \( \nu_{iL}, \ i = 1, 2, 3 \) are the three left-handed Majorana neutrinos. The corresponding neutrino mass matrix is diagonal with eigenvalues \( m_i \) and \( l_{\alpha L}, \ \alpha = e, \mu, \tau \) are the
weak charged lepton eigenstates, which are the states with diagonal charged lepton mass matrix. The Majorana neutrinos in this basis $\nu_{aL}$ are related to the physical neutrinos $\nu_{iL}$ by a unitary transformation \[5\] given by

$$\nu_{iL} = \sum_{\alpha = e, \mu, \tau} \left( U_{\alpha i}^* \nu_{\alpha L} + \lambda_i U_{\alpha i} \nu_c^R \right).$$

(26)

$U$ is the neutrino mixing matrix. Since the right-handed fermions are blind to the $SU(2)_L$ interactions, we shall be working with only the left-handed Majorana neutrinos. So we shall drop the index $L$.

The mixing matrix $U_{\alpha i}$ that relates the weak neutrino eigenstates $\nu_{\alpha}$ to the physical neutrino eigenstates $\nu_{i}$:

$$\left| \nu_i \right> = \sum_{\alpha = e, \mu, \tau} (U_{\alpha i}^* \left| \nu_{\alpha} \right> + \lambda_i U_{\alpha i} \left| \nu_c \right> ).$$

(27)

appears in the charged current interactions of the physical neutrinos $\nu_{i}$ with the physical charged leptons $l_{\alpha}$:

$$L_{CC} = -\frac{g}{\sqrt{2}} \sum_{\alpha = e, \mu, \tau} \bar{\nu}_{i} (U_{\alpha i}^T)_{\alpha \rho} \gamma^\rho l^-_{\alpha} W^+_{\rho} + H.c.$$

and it relates the neutrino mass matrix in this weak interaction basis to the physical neutrino mass matrix (diagonal) by

$$(U^T)_{\alpha i} M_{\alpha \beta} U_{\beta j} = \lambda_i^* M_{ij}^{\text{diag}}.$$

(28)

$\lambda_i$ are the Majorana phases defined by equation\[3\] so that the physical Majorana neutrinos $\nu_{i}$ also satisfy the Majorana condition

$$\nu_i^c = \lambda_i^* \nu_i.$$

Any complex phases in the mass matrix $M_{\alpha \beta}$ may be transferred to the mixing matrix $U_{\alpha i}$ by the rephasing of the physical neutrinos and the weak basis states of neutrinos, but the Majorana phases may not be removed independently. Rephasing the gauge basis and the mass basis of the neutrinos

$$\nu_{i} \rightarrow e^{-i\delta_i} \nu_{i} \quad \text{and} \quad \nu_{\alpha} \rightarrow e^{-i\delta_\alpha} \nu_{\alpha},$$

(29)

would then imply rephasing of the mixing matrix and the Majorana phase matrix as:

$$U_{\alpha i} \rightarrow e^{-i(\delta_\alpha - \delta_i)} U_{\alpha i}, \quad \lambda_i \rightarrow e^{-2i\delta_i} \lambda_i \quad \text{and} \quad \tilde{\lambda}_i \rightarrow e^{i\delta_i} \tilde{\lambda}_i.$$

(30)

where we defined $\lambda^* = \tilde{\lambda}^2$.

Both Dirac and Majorana neutrinos can have CP violation coming from the phases in the mixing matrix $U$. The simplest rephasing invariant combination with the mixing matrix can be defined as \[15\]

$$T_{\alpha i \beta j}^D = U_{\alpha i} U_{\beta j} U_{\alpha j}^* U_{\beta i}^*$$

(31)

While the rephasing invariant measure $T_{\alpha i \beta j}^D$ contains all the CP violating phases in the mixing matrix, it does not include the Majorana phases, and hence, any CP violation in
a lepton number violating interaction will not have any contribution from this measure $T^{\alpha\beta\delta}_{\alpha\beta\delta}$.

The simplest rephasing invariant measure containing the Majorana phases consists of two mixing matrix and the Majorana phase matrices $[15]$

$$s^{\alpha\beta}_{\alpha\beta} = U_{\alpha\beta}^* \lambda^*_i \lambda_j . \quad (32)$$

Although this rephasing invariant measure contains the Majorana phases, it may not appear in the probability or cross-section of any lepton number violating physical processes.

The rephasing invariant measure that contains the Majorana phase and also enter in the physical processes may be defined as $[17]$

$$T^{\alpha\beta\delta}_{\alpha\beta\delta} = U_{\alpha\beta} U_{\alpha\beta}^* U_{\alpha\beta}^* U_{\alpha\beta}^* \lambda_i \lambda^*_j \quad (33)$$

We shall now demonstrate that the measures $T^{\alpha\beta\delta}_{\alpha\beta\delta}$ and $T^{\alpha\beta\delta}_{\alpha\beta\delta}$ may be defined as Dirac and Majorana type Bargmann invariants and can be viewed as geometric phases.

We start with the state vectors $|\nu_i\rangle$, $|\nu_\alpha\rangle$ and $|\nu_\alpha^c\rangle$ in the Hilbert space $\mathcal{H}$. The state vector $|\nu_i\rangle$ satisfy the Majorana condition of equation $[3]$ that is, $\nu_i^c = \lambda_i \nu_i$ and can be expressed in terms of the state vectors $|\nu_\alpha\rangle$ and $|\nu_\alpha^c\rangle$ as given by equation $[27]$. We can then utilize the orthogonality conditions of the state vectors $|\nu_i\rangle$, $|\nu_\alpha\rangle$ and $|\nu_\alpha^c\rangle$

$$\langle \nu_a| \nu_b \rangle = (\nu_a, \nu_b) = \delta_{ab}$$

where $\nu_{a,b} \equiv \nu_{\alpha,\beta}$, $\nu_{i,j}$ or $\nu_{\alpha,\beta}$, and using equation $[27]$ express the inner products of the non-orthogonal state vectors as

$$\begin{align*}
(\nu_i, \nu_\alpha) &= U_{\alpha i} \\
(\nu_i, \nu_\alpha^c) &= \lambda^*_i U_{\alpha i}^* \quad (34)
\end{align*}$$

We now construct Bargmann invariants without Majorana phases and with only the mixing matrix, which is

$$\Delta^D_i(\nu_i, \nu_\alpha, \nu_j, \nu_\beta) = (\nu_i, \nu_\alpha)(\nu_\alpha, \nu_j)(\nu_j, \nu_\beta)(\nu_\beta, \nu_i)$$

$$\Delta^D_i(\nu_i, \nu_\alpha, \nu_j, \nu_\beta) = \text{Tr} \left[ \rho^D(\nu_i) \rho^D(\nu_\alpha) \rho^D(\nu_j) \rho^D(\nu_\beta) \right]$$

$$\Delta^D_i(\nu_i, \nu_\alpha, \nu_j, \nu_\beta) = \text{Tr} \left[ (\nu_i \nu_\alpha^\dagger)(\nu_\alpha \nu_j^\dagger)(\nu_j \nu_\beta^\dagger)(\nu_\beta \nu_i^\dagger) \right]$$

$$\Delta^D_i(\nu_i, \nu_\alpha, \nu_j, \nu_\beta) = U_{\alpha i} U_{\alpha j} \lambda_j U_{\beta i}^* \quad (35)$$

Similarly we can construct the Bargmann invariants with Majorana fermions, and hence, Majorana phases $\lambda_i$. It should contain both $\nu_\alpha$ and $\nu_\alpha^c$ and the simplest one is given by

$$\Delta^M_i(\nu_i, \nu_\alpha, \nu_j, \nu_\beta) = (\nu_i, \nu_\alpha)(\nu_\alpha, \nu_j)(\nu_j, \nu_\beta^c)(\nu_\beta^c, \nu_i)$$

$$\Delta^M_i(\nu_i, \nu_\alpha, \nu_j, \nu_\beta) = \text{Tr} \left[ \rho^M(\nu_i) \rho^D(\nu_\alpha) \rho^D(\nu_j) \rho^M(\nu_\beta^c) \right]$$

$$\Delta^M_i(\nu_i, \nu_\alpha, \nu_j, \nu_\beta) = \text{Tr} \left[ (\nu_i \nu_\alpha^\dagger)(\nu_\alpha \nu_j^\dagger)(\nu_j \nu_\beta^c)(\nu_\beta^c \nu_i^\dagger) \right]$$

$$\Delta^M_i(\nu_i, \nu_\alpha, \nu_j, \nu_\beta) = U_{\alpha i} U_{\alpha j} \lambda_j U_{\beta i}^* \lambda_i \quad (36)$$

This is the rephasing invariant measure of CP violation with Majorana fermions. It is clear from the form of $s_{\alpha ij}$ that it can enter in any Bargmann invariants, but it cannot be a
Bargmann invariant because it is not closed. The Bargmann invariants we constructed $\Delta_4^D$ and $\Delta_4^M$, are the conventional rephasing invariant measures $T_{\alpha i\beta j}^D$ and $T_{\alpha i\beta j}^M$, respectively. Moreover, the phases in $s_{\alpha ij}$ are related to these BIs $T_{\alpha i\beta j}^D$ and $T_{\alpha i\beta j}^M$

$$T_{\alpha i\beta j}^D = s_{\alpha ij} s_{\beta ij} \quad \text{and} \quad T_{\alpha i\beta j}^M = s_{\alpha ij} s_{\beta ij}$$

(37)

The Bargmann variables, $\Delta_4^D$ and $\Delta_4^M$, are defined on the ray space $\mathcal{R}$ and the points $\rho(\nu_i), \rho(\nu_\alpha), \rho(\nu_\delta)$ on $\mathcal{R}$ are non-orthogonal and pair wise linearly independent. These points are connected by geodesics, which form a closed loop in $\mathcal{R}$, representing a cyclic evolution in the state space. This establishes that the Bargmann invariants, and hence, the rephasing invariant measures, gives us the geometric phases

$$\phi_g = -\text{arg}(\Delta_4)$$

The Majorana nature of the neutrinos implies lepton number violation. So, $\Delta_4^M$ is the rephasing invariant measure of CP violation [17] that enters in the lepton number violating CP violating interactions like the neutrinoless double beta decays or $W^- W^- \rightarrow e^- e^-$. If we extend this analysis to include the right-handed neutrinos, then a similar CP violating measure with the right-handed neutrinos would appear in the lepton number violating CP asymmetry as in the models of leptogenesis [18].

5 Recursive Parametrization and rephasing invariants of unitary matrices

In the previous section we constructed the Bargmann invariants with Majorana neutrinos and demonstrated that the CP violation coming from the Majorana phase $\lambda_i$ and also the unitary mixing matrix $U_{\alpha \nu}$ can be presented in the form of rephasing invariant measures or equivalently as BIs. This establishes that the CP violating phases are geometric phases in the leptonic sector. In this section we shall extend this analysis to study the unitary mixing matrices in the framework of recursive parametrization and demonstrate how to include the Majorana phases in this formalism.

We shall first define the recursive parametrization of any unitary matrix, keeping in mind that we shall be applying this formalism to discuss the neutrino mixing matrix defined by equations [27] and [28]. Then we shall generalize the formalism to incorporate the Majorana phases, defined by equation [28]. For the unitary matrix without including the Majorana phases, we shall use the notation and conventions of reference [16].

Any $n \times n$ matrix $A_n \in U(n)$ can be uniquely decomposed [4] into $n$ block matrices given by

$$A_n = A_n(\zeta)A_{n-1}(\eta)A_{n-2}(\xi)\ldots A_4(\gamma)A_3(\beta)A_2(\alpha)A_1(\chi)$$

(38)

The elements $a_{jk}$ of $A_n(\zeta)$ matrix can be constructed in the following way

$$a_{jn} = \zeta_j; \quad j = 1, 2, 3...n$$

$$a_{j,j-1} = \frac{\rho_{j-1}}{\rho_j}; \quad j = 2, 3...n$$

$$a_{jk} = -\frac{\zeta_j \zeta_{k+1}^*}{\rho_k \rho_{k+1}}; \quad j \leq k \leq n - 1$$

$$a_{jk} = 0; \quad \forall \quad j \geq k + 2$$

(39)
\( \zeta \) is the unit vector forming the basis of the \( n \times n \) space implying the components \( \zeta_i \) to obey \( \sum_{i=1}^{n} \zeta_i^2 = 1 \) and \( \rho_j = \sqrt{\sum_{i=1}^{n} |\zeta_i|^2} \). Any \( A_m \in U(m) \) \( (m < n) \) is an unitary matrix with diagonal elements equal to 1 and all other elements equals to zero for trivial rows and columns. This method can also be used to construct \( SU(n) \) matrices by multiplying the first column of the obtained \( U(n) \) matrix with \( (-1)^{n-1} \sum_{i=1}^{n} \zeta_i^* \) assuming \( \zeta_1 \neq 0 \).

We shall now proceed to construct the rephasing invariants in this formalism. Once we have the required \( U(n) \) matrix, we consider the freedom to rephase the various states by multiplying \( U(n) \) matrix with the diagonal phase matrices

\[
U^\dagger \quad = \quad D(\theta) \quad U \quad D(\theta')
\]

where

\[
D(\theta) \quad = \quad \text{diag}(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}, ..., e^{i\theta_n}).
\]  

(40)

For the appropriate choice of the phases \( \theta \) and \( \theta' \), one can get explicit form of the rephasing invariants in this formalism. Before we consider explicit construction of such invariants, we shall demonstrate how these constructions differ for the Dirac and Majorana cases, so that we can demonstrate the Majorana phases for the groups \( SU(n) \) for \( n = 2, 3, 4 \) that we shall study.

So far we have considered the Hilbert space \( \mathcal{H} \) with only the Dirac neutrinos. The mass term for a Dirac neutrino can be written as

\[
\mathcal{L}_\text{mass}^D = m_D \bar{\nu}_L M^D \nu_R
\]

which can be diagonalized by a bi-unitary transformation

\[
U_L^\dagger M^D U_R = M^{\text{diag}}_\nu.
\]  

(41)

where \( U_L \) and \( U_R \) diagonalizes the matrices \( M^D \) \( M^D_U \) and \( M^D_M \), respectively.

If we make any phase transformation to the left-handed and the right-handed neutrinos by the matrices \( D(\theta) \) and \( D(\theta') \) then

\[
M^{\text{diag}}_\nu = U_L^\dagger M^D U_R
= U_L^\dagger D^{1}(\theta) \quad M^D_\nu \quad D(\theta') \quad U_R
= D^{1}(\theta') \quad U_L^\dagger D^{1}(\theta) \quad M^D_\nu \quad D(\theta') \quad U_R
= U_L^\dagger M^D_U D(\theta') \quad U_R.
\]  

(42)

Thus the phase transformation of the left-handed neutrinos can be represented by the transformation of \( U_L \) as

\[
U_L \rightarrow U'_L = D(\theta) \quad U_L \quad D(\theta').
\]  

(43)

Thus for the Dirac fermions, we have the freedom to make two sets of rephasing with the parameters \( \theta \) and \( \theta' \).

Since the right handed neutrinos \( \nu_R \) do not enter the \( SU(2)_L \times U(1)_Y \) charged current interactions, their transformation (rephasing of the matrix \( U_R \) by \( D ) \) will not affect our analysis.

In case of Majorana neutrinos, the mass term may be written as

\[
\mathcal{L}_\text{mass}^M = m_M \bar{\nu}_L M^M \nu_L.
\]  

(44)
This mass matrix $M^M$ is symmetric and may be diagonalized by only one matrix $U$, and hence, the same unitary matrix $U$ will diagonalize the neutrino mass matrix, and hence,

$$
\lambda_i M^{\text{diag}} = U^T M^M U \\
= U^T D(\theta) M^M D(\theta) U
$$

Thus rephasing of the Majorana fermions may be represented by

$$
U \rightarrow U' = D(\theta) U.
$$

Given the prescription for the rephasing of Dirac neutrinos by equation 43 and for the rephasing of the Majorana neutrinos by equation 46, we can now explicitly construct the rephasing invariants in this recursive parametrization formalism for both the Dirac and Majorana fermions, as demonstrated below for the groups $SU(n)$, $n = 2, 3, 4$.

### 5.1 Parametrization of $SU(2)$

The $SU(2)$ matrix obtained in the formulation of recursive parametrization is given by

$$
U = A_2(\alpha) = \left( \begin{array}{cc}
\alpha_2^* & \alpha_1 \\
-\alpha_1^* & \alpha_2
\end{array} \right)
$$

Choosing the diagonal phase matrix as $D(\theta) = \text{diag}(e^{i\theta_1}, e^{i(-\theta_1)})$ and rephasing according to equation 43 for the Dirac neutrinos, it is clear that the matrix remains unchanged i.e. $U' = U$. Whereas, rephasing for Majorana neutrinos according to equation 46 gives

$$
U' = D(\theta) U = \left( \begin{array}{cc}
\alpha_2^* e^{i\theta_1} & \alpha_1 e^{i\theta_1} \\
-\alpha_1^* e^{-i\theta_1} & \alpha_2 e^{-i\theta_1}
\end{array} \right)
$$

The action of rephasing changes the elements of the matrix as

$$
\alpha_1 \rightarrow \alpha_1' = \alpha_1 e^{i\theta_1}, \\
\alpha_2 \rightarrow \alpha_2' = \alpha_2 e^{-i\theta_1};
$$

which can be written in general as $\zeta_j \rightarrow \zeta_j' = \zeta_j e^{in_1\theta_1}$ and can be represented in a tabular form in table II.

| $\zeta_j$ | $\zeta_j'$ |
|---|---|
| $\alpha_1$ | $e^{i\theta_1}$ |
| $\alpha_2$ | $e^{-i\theta_1}$ |

$\zeta_j e^{in_1\theta_1}$

Table 1: Changed components of unit vectors after rephasing $SU(2)$

The number of independent elements of the unit vector (in this case $\vec{\alpha}$) decreases by one after rephasing due to the constraint $\alpha_2' = \alpha_1'^*$, leaving only one rephasing invariant quantity given by $(\alpha_1 \alpha_2)$. It is important to note that no such quantities are found for Dirac type rephasing which reflect the fact that only two generations of quark cannot produce CP violating pure Dirac phase whereas it is sufficient to produce CP violating majorana phase in the lepton sector with two generations of Majorana neutrinos.
5.2 Recursive parametrization of $SU(3)$

The recursive parametrization scheme\textsuperscript{39} can be easily extended to obtain $SU(3)$ matrices in the form given by

$$U = A_3(\beta)A_2(\alpha)$$

$$= \begin{pmatrix} -\beta_2^*\alpha_2^* + \beta_3^*\beta_1\alpha_1 & -\beta_2^*\alpha_1 + \beta_3^*\beta_1\alpha_2 \\ \beta_1^*\alpha_2^* + \beta_3^*\beta_2\alpha_1^* & \beta_1^*\alpha_1 - \beta_3^*\beta_2\alpha_2^* \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \sigma_2 = \sqrt{|\beta_1|^2 + |\beta_2|^2}.$$

Choosing the diagonal phase matrix as $D(\theta) = \text{diag}(e^{i(\theta_1+\theta_2)}, e^{i(-\theta_1+\theta_2)}, e^{i(-2\theta_2)})$ and after rephasing by Dirac type i.e. $U' \rightarrow U = D(\theta)UD(\theta')$, we get the changed components of unit vectors given in the table\textsuperscript{2}

| $\zeta_j \rightarrow \zeta'_j$ | $\zeta_j e^{in_1\theta_1+i2\theta_2+in'_1\theta_1'+in'_2\theta_2'}$ |
|---|---|
| $\alpha_1 \rightarrow \alpha_1'$ | $0$ | $+2$ | $-1$ | $-1$ |
| $\alpha_2 \rightarrow \alpha_2'$ | $0$ | $-2$ | $-1$ | $+1$ |
| $\beta_1 \rightarrow \beta_1'$ | $+1$ | $+1$ | $0$ | $-2$ |
| $\beta_2 \rightarrow \beta_2'$ | $-1$ | $+1$ | $0$ | $-2$ |
| $\beta_3 \rightarrow \beta_3'$ | $0$ | $-2$ | $0$ | $-2$ |

Table 2: Changed components of unit vectors after rephasing for $SU(3)$

It is interesting to note that the only quantity which is linear in every component of unit vectors (in this case $\bar{\alpha}$ and $\bar{\beta}$) and remains invariant after rephasing is given by $\alpha_1\alpha_2^*\beta_1^*\beta_2\beta_3$. This fact reflects that only one pure Dirac phase can occur in the mixing matrix for CP violation with three generations.

However, using Majorana type rephasing for the mixing matrix (i.e $U' \rightarrow U = D(\theta)U$) leads to more invariant quantities. Also the number of independent components of unit vectors reduces to three after rephasing due to the constraint $\alpha_2' = \alpha_1^{*} = \beta_3'$. The two smallest forms of rephasing invariants, each of which is linear in each component of unit vectors in this case are $(\alpha_1\alpha_2)$ and $(\beta_1\beta_2\beta_3)$.

5.3 Recursive parametrization of $SU(4)$

The procedure can be extended easily to four generations of leptons to get $SU(4)$ mixing matrix in the form of $U = A_4(\gamma)A_3(\beta)A_2(\alpha)$. We choose the diagonal phase matrix for Dirac type rephasing to be $D(\theta) = \text{diag}(e^{i(\theta_1+\theta_2+\theta_3)}, e^{i(-\theta_1+\theta_2+\theta_3)}, e^{i(-2\theta_2+\theta_3)}, e^{i(-3\theta_1)})$. The set of changes in the components of the unit vectors after rephasing in the form $U \rightarrow U' = D(\theta)UD(\theta')$ is given in table\textsuperscript{3}.
Table 3: Changed components of unit vectors after rephasing for SU(4)

There are three pure Dirac phases appearing in the mixing matrix for four generations and are related to the invariant quantities which are given as \((\alpha_1 \alpha_2^* \beta_1 \beta_2^* \beta_3 \beta_4^*)\), \((\beta_2 \beta_3^* \gamma_3^* \gamma_4^*)\) and \((\beta_1 \beta_2^* \gamma_1^* \gamma_3^* \gamma_4^*)\).

On the other hand, Majorana type rephasing constraints the number of independent components of unit vectors by \(\alpha_1^* = \alpha_2^* = \beta_3 = \beta_4^* = \gamma_3^*\). The three invariant quantities of smallest forms are given by \((\alpha_1 \alpha_2)\), \((\beta_1 \beta_2 \beta_3)\) and \((\gamma_1 \gamma_2 \gamma_3 \gamma_4)\).

It is quite evident that rephasing invariant quantities are structurally different due to the additional constraints appearing only in Majorana type rephasing. All the invariant quantities are of lower order and linear in every component of unit vectors required to construct the unitary matrix recursively. Higher order invariants can be constructed out of these lower order invariants. For \(n\) generations of leptons \([15]\), the number of CP violating pure Dirac phases in the mixing matrix is \((n-1)(n-2)/2\), whereas the number of pure Majorana phases is \((n-1)\) with a total number of \(n(n-1)/2\) phases. It is important to note that the number of Dirac and Majorana phases are exactly equal to the number of rephasing invariants for the Dirac type and Majorana type rephasing respectively. The parametrization fails if the first component of \(\vec{\zeta}\) (in equation \[38\]) vanishes. In such a case \([19]\), a different recursive approach produces different form of unitary matrix which upon rephasing changes the components of the unit vector in a different way, however, the number of rephasing invariants and the forms remain unchanged.

6 Summary

Bargmann invariants have been shown to connect the rephasing invariant quantities with the geometric phase for the Dirac fermions. We extend this analysis to include the Majorana fermions and show how to relate the Majorana phases with the Bargmann invariants, by defining proper quantum mechanical ray and Hilbert spaces for the Majorana fermions. As an explicit example, we included Majorana neutrinos in the leptonic sector and constructed the Bargmann invariants for both the Dirac and the Majorana neutrinos.
This allows us to interpret the CP violating phases of Dirac and Majorana fermions to a geometric phase. We then explained how to incorporate Majorana phases in a recursive parametrization of unitary matrices with explicit examples of SU(n), n = 2,3,4.

References

[1] A D Sakharov, Pis'ma Zh. Eksp Teor. Fiz. 5 (1967) 32,

[2] C. Jarlskog, Phys. Rev. Lett. 55, 1039 (1985).

[3] O.W. Greenberg, Phys. Rev. D 32, 1841 (1985); D. Wu, Phys. Rev. 33, 860 (1986); L. Dunietz, O.W. Greenberg, and D. Wu, Phys. Rev. Lett. 55, 2935 (1985).

[4] P. Minkowski, Phys. Lett. B 67, 421 (1977); T. Yanagida, in Proc. of the Workshop on Unified Theory and the Baryon Number of the Universe, ed. O. Sawada and A. Sugamoto (KEK, Tsukuba, 1979), p. 95; M. Gell-Mann, P. Ramond, and R. Slansky, in Supergravity, ed. F. van Nieuwenhuizen and D. Freedman (North Holland, Amsterdam, 1979), p. 315; S.L.Glashow, in Quarks and Leptons, ed. M. Lévyet al.(Plenum, New York, 1980), p. 707; R.N. Mohapatra and G. Senjanović, Phys. Rev. Lett. 44, 912 (1980); J. Schechter and J.W.F. Valle, Phys. Rev. D 22, 2227 (1980).

[5] S. M. Bilenky and S. T. Petcov, Rev. Mod. Phys. 59 (1987), 671 [erratum: Rev. Mod. Phys. 61 (1989), 169; erratum: Rev. Mod. Phys. 60 (1988), 575-575

[6] M. Fukugita and T. Yanagida, Phys. Lett. B174, 45 (1986).

[7] V. Bargmann, J. Math. Phys. 5, 862 (1964).

[8] P. Patel, M.Sc. Thesis, IIT Kharagpur (2013); S. Chaturvedi, V. Gupta, G. Sanchez-Colon and N. Mukunda, Rev. Mex. Fis. 57, 146 (2011); e-Print: 1006.4863 [hep-ph] H. Fanchiotti, C.A. Garcia Canal and V. Vento, arXiv:1705.08127;

[9] M.V. Berry, Proc. Roy. Soc. Lond. A 392, 45 (1984).

[10] M. Flanz, E.A. Paschos and U. Sarkar, Phys. Lett. B 345, 248 (1995).

[11] J.E. Ellis, J.S. Hagelin, D.V. Nanopoulos, and M. Srednicki, Nucl. Phys. A 241, 381 (1992);
       J.E. Ellis, N.E. Mavromatos, and D.V. Dimopoulos, Phys. Lett. B 293, 142 (1992).

[12] E.M. Rabei, Arvind, N. Mukunda and R. Simon, Phys. Rev. A 60, 3397 (1999).

[13] N. Mukunda and R. Simon, Ann. Phys. 228, 205 (1993).

[14] N. Mukunda, Arvind, E. Ercolessi, G. Marmo, G. Morandi and R. Simon, Phys. Rev. A 67, 042114 (2003).

[15] J.F. Nieves and P.B. Pal, Phys. Rev. D 36, 315 (1987); Phys. Rev. D 64, 076005 (2001).

[16] N. Mukunda, Arvind, S. Chaturvedi and R. Simon, Phys. Rev. A 65, 012102 (2003).
[17] A. Acker, H. Kikuchi, E. Ma and U. Sarkar, Phys. Rev. D 48, 5006 (1993).
[18] P.J. O’Donnell and U. Sarkar, Phys. Rev. D 52, 1720 (1995).
[19] S. Chaturvedi and N. Mukunda, Int. J. Mod. Phys. A 16, 1481 (2001).