QUINTIC THREEFOLDS AND FANO ELEVENFOLDS

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Abstract. The derived category of coherent sheaves on a general quintic threefold is a central object in mirror symmetry. We show that it can be embedded into the derived category of a certain Fano elevenfold. Our proof also generates related examples in different dimensions.

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1. INTRODUCTION

Fix a 10-dimensional vector space \( V \cong \mathbb{C}^{10} \). Consider the Grassmannian

\[
\text{Gr} := \text{Gr}(2, V) \subset \mathbb{P}(\wedge^2 V)
\]  

(1.1)

and the Pfaffian variety

\[
\text{Pf} := \text{Pf}_{10} = \{ [\omega] \in \mathbb{P}(\wedge^2 V^\vee) : \omega^{\wedge 5} = 0 \in \wedge^{10} V^\vee \} \subset \mathbb{P}(\wedge^2 V^\vee).
\]  

(1.2)

Notice that (1.2) is a quintic hypersurface in \( \mathbb{P}^{44} \), singular in codimension 5. It is the classical projective dual of (1.1). Now pick a 5-dimensional subspace

\[
\mathbb{C}^5 \cong U \subset \wedge^2 V
\]  

(1.3)

or equivalently a 40-dimensional subspace

\[
\mathbb{C}^{40} \cong U^\perp \subset \wedge^2 V.
\]  

We intersect (1.1) with \( \mathbb{P}(U^\perp) \cong \mathbb{P}^{39} \) and (1.2) with \( \mathbb{P}(U) \cong \mathbb{P}^4 \). This defines an 11-dimensional linear section of the Grassmannian

\[
Y_1 := \mathbb{P}(U^\perp) \cap \text{Gr}
\]  

(1.4)

and a quintic 3-fold

\[
Y_2 := \mathbb{P}(U) \cap \text{Pf}
\]  

(1.5)

respectively. For a generic choice of \( U \), both \( Y_1 \) and \( Y_2 \) are smooth. Conversely, Beauville [Be, Proposition 8.9] shows that the general smooth quintic threefold \( Y_2 \subset \mathbb{P}^4 \) arises in this way.\(^1\)

\(^1\) Though not uniquely. Different presentations of a given \( Y_2 \) as linear sections of \( \text{Pf} \) give rise to different dual Fano elevenfolds \( Y_1 \).
The moduli space of the Fano's $Y_1$ is a generically-finite cover of the moduli space of the quintics $Y_2$; in particular
\[ h^1(T_{Y_1}) = 101 = h^1(T_{Y_2}). \]
Moreover their cohomologies are as closely related as possible. By the Lefschetz hyperplane theorem, in degrees less than the middle, the cohomologies of $Y_1$ and $Y_2$ are the same as those of their ambient spaces Gr and $\mathbb{P}^5$ respectively. The same is true in degrees higher than the middle after a shift by twice the codimension. Finally in the middle degree, the nonzero pieces of the cohomologies have the same dimensions:
\[ h^{3,0} h^{2,1} h^{1,2} h^{0,3} = 1 \quad 101 \quad 101 \quad 1 \]
for $H^3(Y_2)$ and
\[ h^{7,4} h^{6,5} h^{5,6} h^{4,7} = 1 \quad 101 \quad 101 \quad 1 \]
for $H^{11}(Y_1)$. Our main result categorifies this relation.

**Theorem A.** There is a full and faithful embedding $D^b(Y_2) \hookrightarrow D^b(Y_1)$.

In fact this is a special case of a more general result, Theorem 2.8 below, which also covers some other interesting examples.

Theorem A should have various consequences when combined with mirror symmetry. In particular, the Fukaya categories of $Y_1$ and $Y_2$ should also be related after a rescaling of the Novikov parameter $q$, with the latter a summand of the former. Taking Hochschild cohomologies, we should find that the quantum cohomology ring $QH^*(Y_2)$ of $Y_2$ should be a summand of $QH^*(Y_1)$ after applying a rescaling of the quantum parameter $q$. Setting $q = 0$ would recover the embedding of the Hodge diamond of $Y_2$ into that of $Y_1$ alluded to above.

Theorem A would follow directly from Kuznetsov’s beautiful work on homological projective duality [K1, K3] if one could prove [K3, Conjecture 5] for $\text{Gr}(2,10)$. In short, Kuznetsov conjectures that (1.1) and (1.2) should be homologically projectively dual varieties [K1] once one replaces Pf with an appropriate categorical crepant resolution of its singularities (which has so far only been found in lower dimensions). This would imply a relation between the derived categories of the linear sections of Gr and of the orthogonal linear sections of Pf. In particular, for $\mathbb{P}(U)$ chosen to avoid the singularities of Pf, we would find that $D^b(Y_1)$ has a semi-orthogonal decomposition
\[ D^b(Y_1) = \langle A, A(1), \ldots, A(4), D^b(Y_2) \rangle, \tag{1.6} \]
where $A$ is the category generated by the exceptional collection
\[ \{ \text{Sym}^3 S, \text{Sym}^2 S, S, O \} \quad \text{on } \ Y_1 \]
and $S$ is the (restriction to $Y_1$ of the) universal subbundle on $\text{Gr}$. That is,
\[ D^b(Y_2) \cong \langle \text{Sym}^i S(j), \ 0 \leq i \leq 3, \ 0 \leq j \leq 4 \rangle \]
\[ = \left\{ E \in D^b(Y_2) : \text{RHom}_X(E, \text{Sym}^i S(j)) = 0 \text{ for all } 0 \leq i \leq 3, 0 \leq j \leq 4 \right\}. \]
Furthermore the inclusion $D^b(Y_1) \hookrightarrow D^b(Y_2)$ should be given\(^2\) by a Fourier-Mukai kernel $I_F$, the ideal sheaf of
\[ \Gamma := \left\{ (\phi, P) \in Pf \times Gr : \ker \phi \cap P \neq 0 \right\} \subset Y_1 \times Y_2. \]
Here $\ker \phi \subset V$ denotes the kernel of $\phi \in \Lambda^2 V^\vee$ when thought of as a (skew) linear map $V \to V^\vee$. The correspondence $\Gamma$ associates $\phi$ to the locus of 2-planes $P \subset V$ which intersect $\ker \phi$ nontrivially.

\(^2\)This inclusion differs from the one in (1.6) by some mutations and twists by line bundles.
Since we deliberately avoid the singularities of Pf the methods of [K1, K2, K3] are surely strong enough to prove Theorem A without finding the right categorical resolution of singularities of Pf. Here however we take a different approach, inspired by string theory.

In their paper [HT], Hori and Tong wrote down a non-abelian gauged linear sigma model (GLSM) that gave a physical explanation of the so-called ‘Pfaffian-Grassmannian’ derived equivalence between two particular Calabi-Yau threefolds. The paper [ADS] gave a mathematical treatment of Hori and Tong’s construction at the level of B-brane categories.

In this paper we take the techniques and results of [ADS] and apply them to a slightly more general GLSM. This gives us a more general result, Theorem 2.8, which says that we have a derived embedding between certain smooth linear sections of the Pfaffian variety and the dual smooth linear sections of a Grassmannian. Special cases then give the quintic threefold case of Theorem A, the Pfaffian-Grassmannian equivalence, and examples with $K3$ surfaces and Calabi-Yau 5-folds.

Although our terminology is different, our approach is intimately connected with homological projective duality; see [B+] for another situation in which HPD is realised via GLSMs.

This paper is based heavily on [ADS]; the only new technical ingredient is the work in Section 4 to show the vanishing of a Brauer class. Consequently we have made little attempt to make this paper self-contained, and we refer the reader to [ADS] for background, motivation, references and more detailed explanations.

Acknowledgements. We would like to thank Nick Addington and Will Donovan for allowing us to re-use the arguments of the paper [ADS]. We also thank Nick Addington for useful conversations and for computational help, and Ivan Smith for generous help with the Fukaya category and quantum cohomology. R.T. was partially supported by EPSRC programme grant EP/G06170X/1.

2. Geometric setup and statement of theorem

Fix vector spaces $V, U$ and $S$ of dimensions $n, k \leq \binom{n}{2}$ and 2 respectively, and consider

$$X = \left[ \left( \text{Hom}(S, V) \oplus (U \otimes \Lambda^2 S) \right) / \text{GL}(S) \right].$$

The square brackets indicate that we consider this as an Artin stack (rather than a scheme-theoretic or GIT quotient). We let $x$ and $p$ denote elements of $\text{Hom}(S, V)$ and $U \otimes \Lambda^2 S$ respectively. We have open substacks

$$\iota_1 : X_1 = \{ \text{rank } x = 2 \} \hookrightarrow X,$$

$$\iota_2 : X_2 = \{ p \neq 0 \} \hookrightarrow X.$$

The locus $X_1$ is a variety: the total space of the vector bundle

$$\mathcal{O}(-1) \otimes U \to \text{Gr}(2, V).$$

The locus $X_2$ is still an Artin stack; it is a bundle over $\mathbb{P}(U)$ with fibres

$$\left[ \text{Hom}(S, V) / \text{SL}(S) \right].$$  \hfill (2.1)

We can rephrase this: we let $\mathcal{P}$ be the stack

$$\mathcal{P} = \left[ (U \otimes \Lambda^2 S) \setminus \{ 0 \} / \text{GL}(S) \right],$$

which is a Zariski-locally trivial bundle of stacks over $\mathbb{P}(U)$, with fibre $\text{BSL}_2$. Then $X_2$ is a vector bundle over $\mathcal{P}$.
The loci $X_1$ and $X_2$ are the semi-stable loci for a positive or negative GIT stability condition, so one of the GIT quotients is $X_1$, and the other is the underlying scheme of $X_2$.

Now fix a surjective linear map\(^3\)

$$A: \Lambda^2 V \longrightarrow U^\vee.$$  

This defines an (invariant) function $W: \mathcal{X} \rightarrow \mathbb{C}$ by

$$W = p \circ A \circ \Lambda^2 x.$$  

We also use $W$ to denote the restriction of this function to $X_1$ and $X_2$. Finally we fix a $\mathbb{C}^*$ action (an “R-charge”) on $\mathcal{X}$ by giving $x$ weight zero and $p$ weight 2, so that both $X_1$ and $X_2$ are invariant and $W$ has weight 2. Given this data, the three pairs

$$(\mathcal{X}, W), \quad (X_1, W) \quad \text{and} \quad (X_2, W)$$

are all Landau-Ginzburg B-models, as defined in [Se], and we have restriction functors

$$D^b(X_1, W) \overset{\iota_1^*}{\longrightarrow} D^b(\mathcal{X}, W) \overset{\iota_2^*}{\longrightarrow} D^b(X_2, W)$$

between their categories of (global) matrix factorizations.

Let

$$Y_1 \subset \text{Gr}(2, V)$$

be the zero locus of the section

$$A \circ \Lambda^2 x \in \Gamma (\text{Gr}(2, V), \mathcal{O}(1) \otimes U^\vee).$$

The critical locus of the function $W$ on $X_1$ always contains $Y_1$, and is equal to $Y_1$ if and only if $Y_1$ is a smooth codimension-$k$ complete intersection, i.e. if and only if the section (2.6) is transverse to the zero section. From now on we restrict to generic $A$ for which this is true.

By global Knörrer periodicity [Sh, Theorem 3.4], there is a canonical equivalence

$$D^b(Y_1) \overset{\sim}{\longrightarrow} D^b(X_1, W).$$

This describes the left hand side of (2.5). The right hand side is more complicated. Over $\mathbb{P}(U)$ we have a family of 2-forms on $V$ up to scale given to us by $A^\vee$ (2.3). The locus where these have rank $< n-1$ is a variety

$$Y_2 \subset \mathbb{P}(U),$$

the intersection of the Pfaffian variety $\text{Pf} \subset \mathbb{P}(\Lambda^2 V^\vee)$ of degenerate two-forms on $V$ with the linear subspace

$$A^\vee: \mathbb{P}(U) \hookrightarrow \mathbb{P}(\Lambda^2 V^\vee).$$

It follows that $Y_2$ is also the locus where the (degenerate) quadratic form $W$ on the fibres (2.1) of $X_2 \rightarrow \mathbb{P}(U)$ drops rank. In this situation there is a more complicated version of Knörrer periodicity; see Sections 4, 5 and [ADS]. There is also a corresponding Brauer class, but we show this vanishes in Section 4.

The singular locus of $\text{Pf}$ is a subvariety of codimension 6 inside $\mathbb{P}(\Lambda^2 V^\vee)$ when $n$ is even, and codimension 10 when $n$ is odd. Therefore if $n$ is even and $k \leq 6$, or $n$ is odd and $k \leq 10$, then for a generic choice of $A$ the variety $Y_2$ is smooth. (If $k$ is larger than these bounds then $Y_2$ will never be smooth.) Under these assumptions, we prove in Section 5 that $D^b(Y_2)$ embeds into a certain subcategory of $D^b(X_2, W)$.

Note that the variety $Y_1$ is Fano if $k < n$, Calabi-Yau if $k = n$, and general type if $k > n$. The canonical bundle of $Y_2$ is easy to calculate if $n$ is even: $Y_2$ is Fano for $k > n/2$, Calabi-Yau for $k = n/2$ and general type for $k < n/2$. When $n$ is odd

\(^3\)The dual $A^\vee: U \rightarrow \Lambda^2 V^\vee$ will later specialise to the injection (1.3) of the Introduction.
the calculation is a little harder, but the three cases occur when $k > n$, $k = n$ and $k < n$ respectively.

**Theorem 2.8.** Suppose that

(i) $k \leq \min(n, 10)$ if $n$ is odd, or
(ii) $k \leq \min(n/2, 6)$ if $n$ is even.

Assume also that $A$ is generic, so that both $Y_1$ and $Y_2$ are smooth. Then we have an admissible embedding

$$D^b(Y_2) \hookrightarrow D^b(Y_1).$$

Here admissible means that the embedding admits a right adjoint; it follows that $D^b(Y_1)$ has a semi-orthogonal decomposition whose final term is $D^b(Y_2)$; c.f. (1.6).

We have not attempted to compute the orthogonal piece, however. Setting $n = 10$, $k = 5$ gives Theorem A of the Introduction. The case $n = k = 7$ is the ‘Pfaffian-Grassmannian’ equivalence, which is the subject of [ADS].

Setting $n = 8$, $k = 4$ gives an embedding of the derived category of a Pfaffian quartic $K3$ into the derived category of a codimension-4 linear section of $\text{Gr}(2, 8)$. Note that the general quartic in $\mathbb{P}^3$ is Pfaffian [Be, Prop. 7.6].

Setting $n = k = 9$ we get a novel derived equivalence between Calabi-Yau 5-folds.

### 3. Grade-restriction windows

Recall that $n = \dim V$, and let us set $L = \frac{n-1}{2}$ for $n$ odd and $L = \frac{n}{2}$ for $n$ even. Let $S$ be the following set of representations of $\text{GL}(S)$:

$$S = \{ \text{Sym}^l S^\vee \otimes (\det S^\vee)^m : l \in [0, L], m \in [0, n) \}$$

if $n$ is odd, or

$$S = \{ \text{Sym}^l S^\vee \otimes (\det S^\vee)^m : l \in [0, L-2], m \in [0, n) \text{ or } l = L-1, m \in [0, \frac{n}{2}) \}$$

if $n$ is even. Each representation induces a vector bundle on $X$ which we denote by the same letters. On restriction to $\text{Gr}(2, V) \subset X$, we get the full strong Lefschetz exceptional collection found by Kuznetsov [K2]. This set $S$ is adapted to the ‘Grassmannian side’ of our set-up; for the ‘Pfaffian side’ we consider the set

$$T = \{ \text{Sym}^l S^\vee \otimes (\det S^\vee)^m : l \in [0, L], m \in [0, k) \},$$

where $k = \dim U$ as before. Notice that

$$T \subset S \text{ if and only if } k \leq n \text{ for } n \text{ odd, or } k \leq \frac{n}{2} \text{ for } n \text{ even.}$$

(There is also a ‘reverse’ numerical condition that implies that $S \subset T$, but this is less useful to us.) We let

$$G_1 = \langle S \rangle \text{ and } G_2 = \langle T \rangle \subset D^b(X)$$

be the subcategories of $D^b(X)$ generated by $S$ and $T$, i.e. the closures of $S$ and $T$ under mapping cones and shifts (but not direct summands). We also let

$$G_1^W \text{ and } G_2^W \subset D^b(X, W)$$

be the subcategories consisting of objects that are (homotopy-equivalent to) matrix factorizations whose underlying vector bundles are direct sums of shifts of the bundles appearing in $S$ and $T$ respectively.

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4By Serre duality the existence of a left adjoint is equivalent to the existence of a right adjoint.
Proposition 3.3. The restriction functors
\[ i_2^* : \mathcal{G}_2 \longrightarrow D^b(X_2) \quad \text{and} \quad i_1^* : \mathcal{G}_1^W \longrightarrow D^b(X_1, W) \]
are both equivalences, and the restriction functors
\[ i_2^* : \mathcal{G}_2 \longrightarrow D^b(X_2) \quad \text{and} \quad i_1^* : \mathcal{G}_1^W \longrightarrow D^b(X_1, W) \]
are both embeddings.

Proof. The statements without \( W \) are proved by the same argument as for [ADS, Proposition 4.1]; restriction to \( X_1 \) or \( X_2 \) does not create any higher Ext groups between the respective sets of vector bundles, and the restriction of \( \mathcal{S} \) generates \( D^b(X_1) \). The only additional ingredient we need is the appropriate generalisation of [ADS, Lemma 4.4] that certain Ext groups vanish on \( \text{Gr}(2, V) \), and this is a routine calculation.

The statements with \( W \) follow from the statements without \( W \), just as in [ADS, Proposition 4.8]. \( \square \)

Under the numerical condition in (3.2) we have that
\[ \mathcal{G}_2 \subset \mathcal{G}_1 \quad \text{and} \quad \mathcal{G}_2^W \subset \mathcal{G}_1^W. \]  
(3.4)

Proposition 3.5. The restriction functors
\[ i_2^* : \mathcal{G}_2 \longrightarrow D^b(X_2) \quad \text{and} \quad i_1^* : \mathcal{G}_1^W \longrightarrow D^b(X_1, W) \]
land in the subcategories \( i_2^*(\mathcal{G}_2) \) and \( i_1^*(\mathcal{G}_1^W) \) respectively. Given the numerical condition in (3.2), these functors are the right adjoints to the inclusions (3.4).

Proof. Consider first the statements without \( W \). On \( \mathcal{X} \) replace \( \mathcal{O}_{(p=0)} \) by its Koszul resolution. The resulting sheaves \( \wedge^*(U \otimes \wedge^2 \mathcal{S})^\vee \) are all sums of sheaves \( \mathcal{O}(k) \). Restricting to \( X_2 \) the complex becomes acyclic, giving the corresponding relation in \( D(X_2, W) \). This relation, tensored with any \( \mathcal{O}(i) \), shows that any line bundle \( \mathcal{O}(k) \) lies in \( i_2^*(\mathcal{G}_2) \).

Similarly, tensoring the acyclic complex with \( \text{Sym}^i \mathcal{S}^\vee \) shows that \( \text{Sym}^i \mathcal{S}^\vee(m) \) lies in \( i_2^*(\mathcal{G}_2) \) for \( l, m \) in the window defining \( \mathcal{S} \). Therefore \( i_2^*(\mathcal{G}_1) \subset i_2^*(\mathcal{G}_2) \).

The statement that \( i_2^* \) is the right adjoint to the inclusion \( \mathcal{G}_2 \subset \mathcal{G}_1 \) can be checked on the generators, so we need to know that if \( E \in \mathcal{T} \) and \( F \in \mathcal{S} \) then
\[ \text{RHom}_{\mathcal{X}}(E, F) = \text{RHom}_{\mathcal{X}_2}(i_2^*E, i_2^*F). \]
This is proved by checking that the necessary higher Ext groups on \( X_2 \) vanish, just as in the proof of Proposition 3.3.

As before, the statements with \( W \) follow from the statements without \( W \) by the techniques of [ADS, Proposition 4.8]. \( \square \)

We define
\[ \mathbb{BBr}(X_2, W) \subset D^b(X_2, W) \]
to be the image of \( \mathcal{G}_2^W \); it is hopefully the category of B-branes in some associated SQFT.

If we assume the numerical condition from (3.2), then putting together Knörrer periodicity (2.7) with Propositions 3.3 and 3.5 shows that (3.4) gives an embedding
\[ \mathbb{BBr}(X_2, W) \hookrightarrow D^b(Y_1) \]  
(3.6)
as a right-admissible subcategory. To prove Theorem 2.8, it remains to show that \( D^b(Y_2) \) embeds as a right-admissible subcategory of \( \mathbb{BBr}(X_2, W) \).
4. Quadratic bundles arising from symplectic bundles

Given a vector bundle equipped with an everywhere non-degenerate quadratic form, Knörrer periodicity implies that the category of matrix factorizations on the total space of the bundle is equivalent to the derived category of the base space, once we twist the latter by a Brauer class. If the bundle admits a Lagrangian subbundle $L$ then the Brauer class vanishes, and the sky-scraper sheaf $\mathcal{O}_L$ can be used to construct an equivalence between the matrix factorization category and the ordinary derived category of the base. However, the existence of $L$ is a rather stronger condition than the vanishing of the Brauer class.

In this section we describe, for quadratic vector bundles of a particular type, an alternative construction which proves the vanishing of the Brauer class and provides the equivalence between the two categories.

4.1. Cleanly intersecting submanifolds of $\{W = 0\}$. Before discussing any quadratic vector bundles we make a rather general observation. Let $(X, W)$ be any Landau-Ginzburg B-model, and let $A, B \subset \{W = 0\} \subset X$ be submanifolds of the zero locus of $W$. Assume that $A$ and $B$ intersect cleanly, so $A \cap B$ is also a submanifold, and we have an excess normal bundle $E = \frac{T_X}{T_A + T_B}$ on $A \cap B$.

Let $r$ denote the rank of $E$, and let $a$ be the codimension of $A \subset X$. Then in the ordinary derived category $D^b(X)$, a standard computation with the Koszul resolution gives the Ext sheaves between $\mathcal{O}_A$ and $\mathcal{O}_B$ as

$$\text{Ext}^i(\mathcal{O}_A, \mathcal{O}_B) = \wedge^{a-i} E^\vee \otimes \det N_{A/X}, \quad a - r \leq i \leq a,$$

(4.1) and zero otherwise.

Since $A$ and $B$ lie in $\{W = 0\}$, the sheaves $\mathcal{O}_A$ and $\mathcal{O}_B$ define objects in $D^b(X, W)$. By a minor extension of the argument in [ASS, §A.4], there is a spectral sequence computing the local sheaf of morphisms between them in $D^b(X, W)$, whose 2nd page consists of the sheaves (4.1) and whose differential is given by wedging with the section $^5$ $dW : \mathcal{O}_{A \cap B} \longrightarrow E^\vee$.

Suppose that this section of $E^\vee$ is transverse to 0 with zero locus $Z$ (which is therefore a component of the critical locus of $W$). Then by the 3rd page only one term remains:

$$\text{Ext}^{a-r}_{D^b(X, W)}(\mathcal{O}_A, \mathcal{O}_B) = \mathcal{O}_Z \otimes \det E^\vee \otimes \det N_{A/X}.$$

Thus the spectral sequence collapses to give

$$R\text{Hom}_{D^b(X, W)}(\mathcal{O}_A, \mathcal{O}_B) = \mathcal{O}_Z \otimes K_{A \cap B} \otimes K_B^{-1}[\dim A \cap B - \dim B].$$

(4.2)

Here $K_{A \cap B}$ and $K_B$ denote the canonical bundles, and we have used $a - r = \dim B - \dim A \cap B$.

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$^5$This section is well-defined, since $W$ vanishes along $A$ and $B$. 

4.2. Another version of Knörrer periodicity. Let $S$ and $V$ be two symplectic vector spaces. Let $\theta_S \in \wedge^2 S$ be the Poisson bivector on $S$ and $\Omega_V$ be the symplectic form on $V$. Then the vector space $\text{Hom}(S, V)$ carries a non-degenerate quadratic form

$$W: x \mapsto \langle \Omega_V, \wedge^2 x(\theta_S) \rangle. \quad (4.3)$$

By Knörrer periodicity, the category $D^b(\text{Hom}(S, V), W)$ is equivalent to the derived category of a point $D^b(\text{pt})$, non-canonically. An equivalence is specified by any exceptional object that generates the category. One option is to choose a Lagrangian $L \subset V$ and take the skyscraper sheaf of the corresponding maximally-isotropic subspace:

$$M := \text{Hom}(S, L) \subset \text{Hom}(S, V). \quad (4.4)$$

This gives an equivalence

$$\mathcal{R}\text{Hom}(\mathcal{O}_M, \cdot): D^b(\text{Hom}(S, V), W) \xrightarrow{\sim} D^b(\text{pt})$$

sending $\mathcal{O}_M$ to $\mathcal{O}_{\text{pt}}$. Our next result says that in this situation there is a more canonical generator, independent of any choices, and hence equivariant with respect to both $\text{Sp}(S)$ and $\text{Sp}(V)$.

Let $\text{LGr}(S)$ denote the Lagrangian Grassmannian of $S$, and let

$$\text{LGr}(S) \overset{\pi_1}{\leftarrow} \text{LGr}(S) \times \text{Hom}(S, V) \overset{\pi_2}{\rightarrow} \text{Hom}(S, V)$$

denote the projections onto the two factors. The vector bundle $\pi_1$ carries a family of non-degenerate quadratic forms $\pi_1^* W$ and a natural maximally-isotropic subbundle

$$N := \text{Hom}(S/\Lambda, V), \quad (4.5)$$

where $\Lambda \rightarrow \text{LGr}(S)$ is the tautological Lagrangian subbundle of $S$. The skyscraper sheaf $\mathcal{O}_N$ is an object of $D^b(\text{LGr}(S) \times \text{Hom}(S, V), \pi_2^* W)$.

**Proposition 4.6.** The object

$$E := \mathcal{R}\pi_2^* \left( \mathcal{O}_N \otimes (\text{det } \Lambda)^{-\frac{1}{2}\dim V} \right) \in D^b(\text{Hom}(S, V), W) \quad (4.7)$$

is exceptional and generates the category.

**Proof.** Choose a Lagrangian $L \subset V$, giving a maximally-isotropic subspace $M \subset \text{Hom}(S, V)$ as in (4.4). Let $\tilde{M} = \text{LGr}(S) \times M$ be the corresponding maximally-isotropic subbundle of $\text{LGr}(S) \times \text{Hom}(S, V)$. The functor

$$\mathcal{R}\pi_1^* \mathcal{R}\text{Hom}(\mathcal{O}_{\tilde{M}}, \cdot): D^b(\text{LGr}(S) \times \text{Hom}(S, V), \pi_2^* W) \rightarrow D^b(\text{LGr}(S))$$

is an equivalence by the simplest version of Knörrer periodicity in families. Moreover the square

$$\begin{array}{ccc}
D^b(\text{LGr}(S) \times \text{Hom}(S, V), \pi_2^* W) & \xrightarrow{\mathcal{R}\pi_1^* \mathcal{R}\text{Hom}(\mathcal{O}_{\tilde{M}}, \cdot)} & D^b(\text{LGr}(S)) \\
\mathcal{R}\pi_2^* \downarrow & & \downarrow \mathcal{R}\Gamma \\
D^b(\text{Hom}(S, V), W) & \xrightarrow{\mathcal{R}\text{Hom}(\mathcal{O}_M, \cdot)} & D^b(\text{pt})
\end{array} \quad (4.8)$$

commutes by the projection formula.

Now we take our tautological maximal isotropic subbundle $N$ (4.5) and compute

$$\mathcal{R}\text{Hom}(\mathcal{O}_{\tilde{M}}, \mathcal{O}_N)$$

in $D^b(\text{LGr}(S) \times \text{Hom}(S, V), \pi_2^* W)$, using the analysis from Section 4.1. The submanifolds $\tilde{M}$ and $N$ intersect cleanly along the subbundle

$$\tilde{M} \cap N = \text{Hom}(S/\Lambda, L)$$
with excess normal bundle $E = \text{Hom}(\Lambda, V/L)$

over $\widetilde{M} \cap N$. We compute that

$$dW|_x = \theta_S \circ x^\vee \circ \Omega_V \in \text{Hom}(V, S) = \text{Hom}(S, V)^\vee,$$

where we consider $\theta_S$ and $\Omega_V$ as skew elements of $\text{Hom}(S^\vee, S)$ and $\text{Hom}(V, V^\vee)$ respectively. Therefore at a point $(\Lambda, x)$ of $\widetilde{M} \cap N$ the derivative of $\pi_2^* W$ is the map

$$\theta_S \circ x^\vee \circ \Omega_V \in \text{Hom}(L, S/\Lambda) = \text{Hom}(V/L^\vee, \Lambda^\vee),$$

where the last isomorphism follows from the Lagrangian property of $L$ and $\Lambda$. This lies in the fibre of $E^\vee$ over $(\Lambda, x)$. The resulting section of $E^\vee$ has zero locus $\{x = 0\} = \text{LGr}(S)$, so it is transverse to the zero section and we may apply (4.2).

By an elementary calculation

$$K_{\widetilde{M} \cap N} \otimes K_N^{-1} = (\det \Lambda)^{-\frac{1}{2} \dim V} \otimes (\det L)^{-\frac{1}{2} \dim S},$$

so (4.2) gives

$$R\text{Hom}(\mathcal{O}_{\widetilde{M}}, \mathcal{O}_N) = \mathcal{O}_{\text{LGr}(S)} \otimes (\det \Lambda)^{-\frac{1}{2} \dim V} \otimes (\det L)^{-\frac{1}{2} \dim S} \left[ - \frac{1}{2} \dim S \cdot \dim V \right].$$

Consequently, the upper arrow in the square (4.8) takes $\mathcal{O}_N \otimes (\det \Lambda)^{-\frac{1}{2} \dim V}$ to a shift of $\mathcal{O}_{\text{LGr}(S)}$. Therefore going the other way round the square shows that $\mathcal{E}$ is taken by the lower arrow to the same shift of $\mathcal{O}_{\text{pt}}$. In particular, $\mathcal{E}$ is isomorphic to a shift of $\mathcal{O}_M$ in the category $D^b(\text{Hom}(S, V), W)$. \hfill \Box

**Remark 4.9.** The same proof obviously gives another generator

$$\mathcal{E}' = R\pi_2^* \left( \mathcal{O}_N \otimes (\det \Lambda)^{-\frac{1}{2} \dim V} \otimes \pi_1^* K_{\text{LGr}(S)} \right).$$

Consider the case $\dim S = 2$. Then $\Lambda \to \text{LGr}(S)$ is just $\mathcal{O}(-1) \to \mathbb{P}(S)$. The image of the map $\pi_2|_N$ is exactly

$$\{\text{rank } x \leq 1\} \subset \text{Hom}(S, V)$$

(4.10)

and both $\mathcal{E}$ and $\mathcal{E}'$ are supported on this locus. Furthermore, when $\dim V = 4$, we get the skyscraper sheaf on this locus:

$$\mathcal{E}' = R\pi_2^* \mathcal{O}_N \left( \frac{1}{2} \dim V - 2 \right) = R\pi_2^* \mathcal{O}_N = \mathcal{O}_{\{\text{rank } x \leq 1\}}.$$

That this sheaf is an exceptional generator of the category was observed in [ADS].

Since $\mathcal{E}$ is canonical, Proposition 4.6 works in families. Let $S$ and $V$ be symplectic vector bundles over a base $B$, or even vector bundles carrying symplectic forms only up to scale.\footnote{By this we mean a section of $\Lambda^2 V^\vee \otimes L$ for some line bundle $L$, such that the induced map $V \to V^\vee \otimes L$ is an isomorphism.} Then $\text{Hom}(S, V) \xrightarrow{\Delta B} B$ carries a fibrewise non-degenerate quadratic form $W$ up to scale, given by the formula (4.3). We form the bundle

$$\pi_2 : \text{LGr}(S) \times_Y \text{Hom}(S, V) \longrightarrow \text{Hom}(S, V)$$

carrying its tautological subbundle $\Lambda \subset S$. With this we can define the maximally-isotropic subbundle $N := \text{Hom}(S/\Lambda, V)$ of $\text{LGr}(S) \times_Y \text{Hom}(S, V) \to \text{LGr}(S)$, and

$$\mathcal{E}_B := R\pi_2^* \left( \mathcal{O}_N \otimes (\det \Lambda)^{-\frac{1}{2} \text{rank } V} \right).$$

(4.11)

This is the global analogue of the object (4.7). Zariski-locally, there is also a version of the object $\mathcal{O}_M$ of (4.4). The symplectic group is special, so $V$ is Zariski-locally trivial. Therefore, replacing $B$ by an open subset we may assume that $V$ is trivial. Hence it admits a trivial Lagrangian subbundle $L$, defining a maximally-isotropic subbundle $M \subset \text{Hom}(S, V)$ by the formula (4.4).
The proof of Proposition 4.6 now applies verbatim: $R_p RHom(O_M, E_B)$ is a shift of a line bundle on our shrunken $B$. Since $D^b(Hom(S, V), W)$ is generated over $D^b(B)$ by $O_M$, this shows that $E_B$ and $O_M$ are isomorphic up to a shift and a twist by a line bundle. That is, once we shrink $B$ to ensure that $V$ is trivial, we get the following isomorphism in $D^b(Hom(S, V), W)$:

$$E_B \cong O_M \otimes (\det L)^{-\frac{1}{2}\text{rank }S} \cdot -\frac{1}{4}\text{rank }S \cdot \text{rank }V]. \quad (4.12)$$

It follows that for any $B$ the two Fourier-Mukai functors

$$D^b(B) \xrightarrow{E_B \otimes p^*(\cdot)} D^b(Hom(S, V), W) \xleftarrow{R_p RHom(E_B, \cdot)}$$

are mutual inverses; i.e. the natural adjunction map from their composition to the structure sheaf of the diagonal is a quasi-isomorphism. Again this can be checked locally, where it follows from (4.12) and the corresponding result for $O_M$. Thus (4.13) gives an equivalence with zero Brauer class.

5. Pfaffian side

In this section we construct an embedding of $D^b(Y_2)$ into $\mathbb{BBr}(X_2, W)$. The method is the one employed in [ADS, §5] supplemented with the construction from Section 4 to produce a global Fourier-Mukai kernel.

We let $\pi$ denote the composition of the projections

$$X_2 \rightarrow P \rightarrow \mathbb{P}(U)$$

of (2.2). The first map is a vector bundle over $P$ with fibre $Hom(S, V)$; the second is a bundle of stacks $BSL_2$. The map $A^\vee: U \rightarrow \bigwedge^2 V^\vee$ of (2.3) defines a section of $\bigwedge^2 V^\vee(1)$ over $\mathbb{P}(U)$ – i.e. a family of 2-forms on the $n$-dimensional vector space $V$, defined up to scale. The variety $Y_2 \subset \mathbb{P}(U)$ is the locus where this family drops in rank, either from $n$ to $n-2$ (if $n$ is even) or from $n-1$ to $n-3$ (if $n$ is odd).

Let $K \rightarrow Y_2$ be the kernel of the family of 2-forms:

$$0 \rightarrow K \rightarrow V \xrightarrow{A} V^\vee(1).$$

It is a subbundle of the trivial bundle $V \times Y_2$ of rank 2 or 3. Dividing out by $K$ gives a quotient bundle

$$q: V \rightarrow \tilde{V} := V/K.$$

The family of forms $A$ descends to give a family $\tilde{A} \in \Gamma(\bigwedge^2 \tilde{V}^\vee(1))$ of symplectic forms (up to scale) on the vector bundle $\tilde{V} \rightarrow Y_2$.

Now consider the vector bundle

$$\text{Hom}(S, \tilde{V})$$

on the stack $P|_{Y_2}$. Since $S$ is 2-dimensional, this carries an associated family of non-degenerate quadratic forms (up to scale) given by the formula (4.3). Via $q$ this pulls back to a family of degenerate quadratic forms on the bundle $\text{Hom}(S, V) = X_2|_{Y_2}$; this is precisely the restriction of the function $W$ (2.4).

We now apply the method of Section 4 to the symplectic bundles $S, \tilde{V}$ over the base $B = P|_{Y_2}$ to give an object $E \in D^b(\text{Hom}(S, \tilde{V}), W)$ by the formula (4.11). Pulling up to $\text{Hom}(S, V)$ and pushing forward into $X_2$ gives an object

$$j_*q^*E \in D^b(X_2, W),$$
where \( j : X_2|_{Y_2} \hookrightarrow X_2 \) denotes the inclusion map. We claim that

\[
O_{Y_2} \xrightarrow{id} R\pi_* R\mathcal{H}om_{D^b(X_2, W)}(j_* q^* \mathcal{E}, j_* q^* \mathcal{E})
\]  

(5.1)

is a quasi-isomorphism. Again, we can check this locally on \( Y_2 \).

We proceed as at the end of Section 4. Even though our base \( B = \mathcal{P}|_{Y_2} \) is a stack rather than a scheme, the bundle \( \hat{V} \) is pulled back from the scheme \( Y_2 \). Therefore we can use the same Zariski-locally-trivial argument. We replace \( \mathcal{P}(U) \) by an open subset, thus shrinking \( X_2 \) and \( Y_2 \) by base change. We may then assume \( \hat{V} \) is trivial and pick a trivial Lagrangian subbundle \( L \subset \hat{V} \). This defines a maximal isotropic subbundle \( M \subset \text{Hom}(S, \hat{V}) \) by the formula (4.4), and we get the isomorphism (4.12). That is \( \mathcal{E} \) is isomorphic to \( O_M \) up to a shift and a twist by a line bundle. In particular (now that we have shrunk \( X_2 \) and \( Y_2 \) to produce an \( M \) we get an isomorphism between

\[
j_* q^* \mathcal{E} \quad \text{and} \quad j_* O_{q^{-1}(M)}
\]

in \( D^b(X_2, W) \) up to a shift and a twist.

A key result of [ADS, Proposition 5.3 and Remark 5.13] was that when such an \( M \) exists we have

\[
O_{Y_2} \xrightarrow{id} R\pi_* R\mathcal{H}om_{D^b(X_2, W)}(j_* O_{q^{-1}M}, j_* O_{q^{-1}M})
\]

is a quasi-isomorphism. Therefore (5.1) is also a quasi-isomorphism over our open set, and hence also globally.

Using \( j_* q^* \mathcal{E} \) as a Fourier-Mukai kernel, we consider the functor

\[
F : D^b(Y_2) \rightarrow D^b(X_2, W),
\]

\[
\mathcal{F} \mapsto j_* (\pi^* \mathcal{F} \otimes q^* \mathcal{E}).
\]

We will see in Proposition 5.2 below that \( j_* (\pi^* \mathcal{F} \otimes q^* \mathcal{E}) \) really lies in \( D^b(X_2, W) \) - i.e. that it is quasi-isomorphic to a curved complex of vector bundles rather than just sheaves. This functor has a right adjoint

\[
F^R : \mathcal{G} \mapsto R\pi_* R\mathcal{H}om(j_* q^* \mathcal{E}, \mathcal{G})
\]

and (5.1) says that \( F^R \circ F \) is the identity.

Therefore \( F \) embeds \( D^b(Y_2) \) as a right-admissible subcategory of \( D^b(X_2, W) \). To conclude the proof of Theorem 2.8 we need only show the following.

**Proposition 5.2.** The image of the functor \( F \) is contained in the subcategory

\[
\text{BBr}(X_2, W) \subset D^b(X_2, W).
\]

**Proof.** Recall that \( \mathcal{E} \) is supported on the locus \( \{ \text{rank } x \leq 1 \} \subset \text{Hom}(S, \hat{V}) \) (4.10). By pushing-down a twist of the Koszul resolution of \( \mathcal{O}_V \) we can get a free resolution of \( \mathcal{E} \) by bundles of the form

\[
\wedge^a \hat{V}^\vee \otimes \text{Sym}^b S^\vee \otimes (\det S)^e;
\]

see for example [Ei, §A2.6]. Furthermore, the symmetric powers of \( S^\vee \) that occur lie in the range \( b \leq \frac{1}{2} \text{rank } \hat{V} \). This is precisely the range of symmetric powers included in our set \( \mathcal{T} \) (3.1), since \( \text{rank } \hat{V} \) is either \( n - 3 \) (if \( n \) is odd) or \( n - 2 \) (if \( n \) is even). Now the argument proceeds exactly as in [ADS, Proposition 5.8].

It is plausible that \( F \) is actually an equivalence between \( D^b(Y_2) \) and \( \text{BBr}(X_2, W) \).
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