Controllable non-Markovianity in phase relaxation

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Abstract

Recently remarkable progress in quantum technology has been witnessed. In view of this it is important to investigate an open quantum system as a model of such quantum devices. Quantum devices often require extreme conditions for the devices to operate: dynamics can be non-Markovian here. This observation necessitates us to investigate a non-Markovian open quantum system, both theoretically and experimentally. In this paper, we report two important results: (1) exact solution of a simple but non-trivial theoretical model and (2) demonstration of this model by NMR experiments, where non-Markovianity is continuously controllable. We observe qualitative agreement between theory and experiment.

1. Introduction

Quantum resources provide us with novel protocols in several fields in particular in quantum information processing, such as quantum communication, quantum computing and quantum sensing [1]. Many of such protocols have already been demonstrated in actual physical systems thanks to the advance of quantum technology. Since quantum devices suffer from environmental noise, it is important to investigate open quantum systems [2]. Quantum devices are often cooled down to very low temperature and/or high vacuum to make the devices work. In such situations, the system dynamics often shows non-Markovian behavior [3–7] while Markovian one is observed commonly in higher temperature regime. Therefore, it is necessary for us to investigate open quantum systems in various environments theoretically [8–10] and experimentally [11–14]. It is, however, generally difficult to experimentally control non-Markovianity of a system dynamics.

Recently, a simple model that showed time-homogeneous, time-inhomogeneous Markovian relaxations and non-Markovian relaxations was proposed in reference [15]. The system considered was composed of three subsystems, namely, system I (principal system), Markovian environment and system II inserted between system I and the environment. They analysed the dynamics of this system by solving the Gorini–Kossakowski–Lindblad–Sudarshan (GKLS) master equation [16, 17] analytically. They found that the characteristics of relaxation of system I was controlled by tuning parameters of the environment as well as coupling/decoupling system II with system I. Here decoupling is a common technique in NMR, where a coupling between nuclear species A and B is fully or partially nullified by applying a series of strong RF pulses in resonance with one of the species [18]. Rapid rotation of one of the nuclei effectively averages out the interaction. Moreover, they experimentally demonstrated the theoretical results with star-topology molecules in isotropic liquids by using NMR. Through their study, it was found that system II worked as a temporal storage of quantum information that was stored in system I and was dissipating into the Markovian environment.

The coupling between system I and system II in reference [15] was simply turned on and off by using decoupling. It is the purpose of this paper to further extend the model discussed in reference [15] by controlling the coupling strength between system I and system II. We analyse this model by solving the GKLS master equation [16, 17] analytically under some reasonable assumptions [19–21] and compare the
Figure 1. (a) System I interacts with the Markovian environment directly. (b) System I interacts with the Markovian environment indirectly through system II, which results in non-Markovian behaviour in system I. The arrow symbol in the coupling of systems I and II indicates the coupling strength is effectively variable.

2. Engineered environment: theory

It is well known that a quantum system relaxes exponentially if it interacts with an environment that has an infinitesimally short-time memory. This process is called ‘Markovian’. On the other hand, the relaxation is non-exponential when the system interacts with an environment with a long-time memory. In this case, information of the system temporarily stays in the surrounding environment before it totally dissipates. We call this process ‘non-Markovian’. There are many studies on Markovian and non-Markovian dynamics; in particular, non-Markovian dynamics is currently attracting much attention [2–7, 22, 23]. Non-Markovian dynamics often manifests itself at low temperature [3–7], small size environment, and/or strong coupling regime, for example.

We propose a theoretical model where non-Markovianity of the system dynamics is controlled by adjusting an external field. The first step is to construct an open system that shows non-Markovian dynamics. This is realised by employing the prescription proposed in references [15, 24] as depicted in figure 1. System I in figure 1(a) interacts with the environment with a very short-time memory and shows Markovian relaxation. In figure 1(b), system I is surrounded by system II, where two systems interact with each other with a fixed strength. While system II interacts with the Markovian environment, system I interacts with the environment only weakly. Hence the main contribution to the relaxation of system I comes through the interaction with system II. Relaxation of system I in this case can be non-Markovian. System II works as a temporal memory and quantum information escaped from system I is temporarily stored in system II before it totally dissipates into the environment. In other words, system I is in a composite environment (system II and the environment), which has a long-time memory. In the following, we consider a case in which system I is made of one qubit while system II is made of $n \geq 1$ identical qubits.

Let us illustrate how to control non-Markovianity before we present detailed calculations. As mentioned before, systems I and II interact with a fixed strength. However, the coupling strength can be effectively reduced by applying an external field that rotates qubits in system II so that the coupling is partially time-averaged. In the high-field limit, it is totally averaged out and system I suffers only from the Markovian environment. In this way, it is possible to interpolate between Markovian and non-Markovian regimes continuously.

2.1. Markovian environment

Let us consider the dynamics of system I of figure 1(a) composed of a single qubit, whose state is given by $\rho$. See also reference [15]. Dynamics of the qubit as an open quantum system is governed by the GKLS master equation.
equation [2, 16, 17],
\[
\frac{\text{d}\rho}{\text{d}t} = -i[H, \rho] + \mathcal{L}[\rho],
\]
(1)
where \(H\) is the Hamiltonian of system I and we call \(\mathcal{L}\) the Lindbladian, which represents the effect of the environment. We take \(H = 0\) here to simplify our analysis. We use the natural unit \(\hbar = 1\) throughout this paper. The Lindbladian for any completely positive semigroup has the following form [16, 17]:
\[
\mathcal{L}[\rho] := \sum_{i} \gamma_i (2L_i \rho L_i^\dagger - \{L_i L_i^\dagger, \rho\})
\]
(2)
where \(\{\gamma_i\}\) are positive constants. We consider the case where the environment randomly flips a qubit, in which the explicit form of \(\mathcal{L}\) is given by
\[
\mathcal{L}[\rho] := \sum_{\pm} \gamma_\pm \left( 2 \frac{\sigma_\pm \rho \sigma_\pm}{4} - \left\{ \frac{\sigma_\pm \sigma_\pm}{4}, \rho \right\} \right),
\]
(3)
where \(\sigma_\pm = (\sigma_x \pm i \sigma_y)/2\) and \(\sigma_\mu (\mu = x, y, z)\) are the Pauli matrices [2]. In this equation, \(\gamma_\pm\) represents the flip-flop (\(|\downarrow\rangle \leftrightarrow |\uparrow\rangle\)) rate of the qubit and we assume these rates are symmetric, namely \(\gamma_+ = \gamma_- := \gamma_1\).

Now GKLS equation is given by
\[
\frac{\text{d}\rho}{\text{d}t} = \sum_{\pm} \gamma_1 \left( 2 \frac{\sigma_\pm \rho \sigma_\pm}{4} - \left\{ \frac{\sigma_\pm \sigma_\pm}{4}, \rho \right\} \right).
\]
(4)

It is shown that equation (4) is solved exactly leading to exponential relaxation with a characteristic time \(2/\gamma_1\).

2.2. Non-Markovian environment: \((1 + 1)\)-qubit case

We now introduce a theoretical model, in which non-Markovianity can be continuously controlled by an external field. First, we consider the simplest case where both systems I and II consist of a single qubit, which we call the \((1 + 1)\)-system. The system I qubit has an index 0 while the system II qubit has an index 1. The density matrix \(\rho^{(1)}\) of the total system is given by
\[
\rho^{(1)} = \frac{1}{2} \begin{pmatrix}
\rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\
\rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\
\rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\
\rho_{41} & \rho_{42} & \rho_{43} & \rho_{44}
\end{pmatrix}.
\]
(5)

Here the basis vectors are ordered as \(\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}\) with \(|ab\rangle = |a\rangle_0 \otimes |b\rangle_1, a, b \in \{0, 1\}\). Each qubit in this system is subject to the flip-flop noise independently. The Lindbladian in this case is given by
\[
\mathcal{L}[\rho^{(1)}] = \sum_{i=0,1} \sum_{\pm} \gamma_i \left( 2 \frac{\sigma_{\mu}^{(i)} \rho^{(1)} \sigma_{\mu}^{(i)}}{4} - \left\{ \frac{\sigma_{\mu}^{(i)} \sigma_{\mu}^{(i)}}{4}, \rho^{(1)} \right\} \right) := \sum_{i=0,1} \mathcal{L}^{(i)}[\rho^{(1)}],
\]
(6)
where \(\sigma_{\mu}^{(i)}\) is the \(\mu\)-component of the Pauli matrices acting non-trivially only on the \(i\)th qubit, i.e., \(\sigma_{\mu}^{(0)} = \mathbf{1}_{2 \times 2}\) and \(\sigma_{\mu}^{(1)} = \sigma_\mu \otimes \sigma_0\). The superscript \(0\) refers to the system I qubit while \(1\) refers to the system II qubit. Here \(\gamma_i\) is the flip-flop rate of the \(i\)th qubit, \(\gamma_0(\gamma_1)\) is also denoted \(\gamma_i(\gamma_{11})\) because \(\gamma_{11}\) is the flip-flop rate of the qubit in system I (II). Suppose the Hamiltonian of the total system is given by
\[
H^{(1)} = H^{(1)}_f + H^{(1)}_c, \quad H^{(1)}_f := J \frac{\sigma_{\mu}^{(0)} \cdot \sigma_{\mu}^{(1)}}{4}, \quad H^{(1)}_c := \omega_1 \frac{\sigma_{z}^{(1)}}{2}.
\]
(7)
In equation (7) and hereafter the symbol \(\cdot, \cdot\) denotes usual matrix multiplication. With this notation, we have \(\sigma_{\mu}^{(0)} \cdot \sigma_{\mu}^{(1)} = \sigma_\mu \otimes \sigma_\mu\), for example \(H^{(1)}_f\) in equation (7) is a qubit–qubit interaction with a constant strength \(J\), while \(H^{(1)}_c\) represents a controllable external field \(\omega_1\) coupled to the \(z\)-component of the system II qubit.
The dynamics of this system is governed by the GKLS master equation,

\[
\frac{d\rho^{(1)}}{dt} = -i[H^{(1)}, \rho^{(1)}] + L^{(1)}[\rho^{(1)}] + D^{(1)}[\rho^{(1)}] + L^{(1)}[\rho^{(1)}],
\]

where

\[
D^{(1)[\bullet]} := -i[H^{(1)}, \bullet] + L^{(1)}[\bullet].
\]

Let us write the density matrix \( \rho^{(1)} \) in the following form:

\[
\rho^{(1)} = \frac{\sigma_0}{2}, \quad \frac{A_1^{(1)} + A_2^{(1)}}{2} + \frac{\sigma_0}{2}, \quad \frac{A_1^{(1)} - A_2^{(1)}}{2} + \frac{\sigma_0}{2}, \quad B^{(1)} + \frac{\sigma_0}{2}, \quad (B^{(1)})^\dagger,
\]

where

\[
A_1^{(1)} := \sigma_0 \otimes \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}, \quad A_2^{(1)} := \sigma_0 \otimes \begin{pmatrix} \rho_{33} & \rho_{34} \\ \rho_{43} & \rho_{44} \end{pmatrix}, \quad B^{(1)} := \sigma_0 \otimes \begin{pmatrix} \rho_{13} & \rho_{14} \\ \rho_{23} & \rho_{24} \end{pmatrix}.
\]

We easily find that equation (8) is decomposed into the following three equations,

\[
\frac{dA_1^{(1)}}{dt} = f(A_1^{(1)}, A_2^{(1)}), \quad \frac{dA_2^{(1)}}{dt} = g(A_1^{(1)}, A_2^{(1)}), \quad \frac{dB^{(1)}}{dt} = h(B^{(1)}),
\]

where

\[
f(A_1^{(1)}, A_2^{(1)}) = -\gamma_1 \left( \frac{A_1^{(1)} - A_2^{(1)}}{2} \right) - i\gamma \left[ \frac{\sigma_0}{2}, A_1^{(1)} \right] - i\omega \left[ \frac{\sigma_0}{2}, A_1^{(1)} \right] + L^{(1)}[A_1^{(1)}],
\]

\[
g(A_1^{(1)}, A_2^{(1)}) = -\gamma_1 \left( \frac{A_1^{(1)} - A_2^{(1)}}{2} \right) + i\gamma \left[ \frac{\sigma_0}{2}, A_2^{(1)} \right] - i\omega \left[ \frac{\sigma_0}{2}, A_2^{(1)} \right] + L^{(1)}[A_2^{(1)}].
\]

The third equation in equation (11) will be extensively studied in the following. The dynamics of \((B^{(1)})^\dagger\) is simply obtained by Hermitian conjugation of that of \(B^{(1)}\). It should be emphasised that the dynamics of \(B^{(1)}\) is decoupled from those of \(A_1^{(1)}, A_2^{(1)}\) and \((B^{(1)})^\dagger\). Independence of the dynamics of \(B^{(1)}\) from \((B^{(1)})^\dagger\) is essential in our analysis. If we had \(dB^{(1)}/dt = h(B^{(1)})\), the dynamics would mix up the real and imaginary parts of \(B^{(1)}\).

Now we solve equation (11) with an appropriate initial condition. Suppose qubit 0 is polarized along the x axis and qubit 1 is uniformly mixed at \( t = 0 \);

\[
\rho^{(1)}(0) = \ket{+}\bra{+} \otimes \frac{1}{2} \sigma_0 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix},
\]

where \( \ket{+} = \frac{1}{\sqrt{2}}(\ket{0} + \ket{1}) \). This initial condition is rewritten as

\[
A_1^{(1)}(0) = A_2^{(1)}(0) = B^{(1)}(0) = (B^{(1)})^\dagger(0) = \sigma_0 \otimes \frac{1}{2} \sigma_0.
\]

Then it turns out that the first two equations in equation (11) have no dynamics:

\[
f(A_1^{(1)}, A_2^{(1)}) = g(A_1^{(1)}, A_2^{(1)}) = 0 \text{ at any } t.\] In other words, \(A_1^{(1)}\) and \(A_2^{(1)}\) are time-independent with this initial condition. As a result we only need to solve \(dB^{(1)}/dt = h(B^{(1)})\) to find the dynamics of the GKLS equation.

To write down the dynamical equation of \(B^{(1)}\), we now evaluate the GKLS equation on \( \frac{\sigma_0}{2} \cdot B^{(1)} \). First note that \(L^{(0)}\) acts only on \(\sigma_+^{(0)}\) and gives just a scalar multiplication:

\[
L^{(0)} \left[ \frac{\sigma_+^{(0)}}{2} \cdot B^{(1)} \right] = -\left( \frac{\gamma_1}{2} \cdot \sigma_0^{(0)} \right) \cdot B^{(1)}.
\]

This implies that \(B^{(1)}\) is factorised as \(B^{(1)} = e^{-\gamma t/2} \tilde{B}^{(1)}\). The dynamics of \(\tilde{B}^{(1)}\) following from the GKLS equation (8) is written as

\[
\frac{d}{dt} \left[ \frac{\sigma_+^{(0)}}{2} \cdot \tilde{B}^{(1)} \right] = D^{(1)} \left[ \frac{\sigma_+^{(0)}}{2} \cdot \tilde{B}^{(1)} \right].
\]

The dynamics of \(B^{(1)}\) can be obtained by multiplying \(e^{-\gamma t/2}\) to \(\tilde{B}^{(1)}\) (or, \(B^{(1)}\) with \(\gamma_1 = 0\)). Therefore, we will consider the case when \(\gamma_1 = 0\) hereafter.
\( \vec{B}^{(1)} \) can be expanded as \( \vec{B}^{(1)} = \frac{1}{2} \sum_{i=x,y,z} b_i \sigma_i^{(1)} \), where

\[
\begin{align*}
    b_0 &= \rho_{13} + \rho_{24}, \\
    b_x &= \rho_{14} + \rho_{23}, \\
    b_y &= i(\rho_{14} - \rho_{23}), \\
    b_z &= \rho_{13} - \rho_{24}.
\end{align*}
\]

We now evaluate the right-hand side of equation (16). The action on each basis \( \frac{1}{2} \sigma_+^{(0)} \cdot (\sigma_-^{(1)}/2) \) of \( \frac{1}{2} \sigma_+^{(0)} \cdot \vec{B}^{(1)} \) is given as

\[
D^{(1)} \left[ \frac{\sigma_+^{(0)}}{2} \cdot \frac{\sigma_-^{(1)}}{2} \right] = -iJ \left( \frac{\sigma_+^{(0)}}{2} \cdot \frac{\sigma_-^{(1)}}{2} \right),
\]

\[
D^{(1)} \left[ \frac{\sigma_+^{(0)}}{2} \cdot \frac{\sigma_-^{(1)}}{2} \right] = -\frac{\gamma_{11}}{2} \left( \frac{\sigma_+^{(0)}}{2} \cdot \frac{\sigma_-^{(1)}}{2} \right),
\]

\[
D^{(1)} \left[ \frac{\sigma_+^{(0)}}{2} \cdot \frac{\sigma_-^{(1)}}{2} \right] = \omega_1 \left( \frac{\sigma_+^{(0)}}{2} \cdot \frac{\sigma_-^{(1)}}{2} \right) - \frac{\gamma_{11}}{2} \left( \frac{\sigma_+^{(0)}}{2} \cdot \frac{\sigma_-^{(1)}}{2} \right),
\]

\[
D^{(1)} \left[ \frac{\sigma_+^{(0)}}{2} \cdot \frac{\sigma_-^{(1)}}{2} \right] = -iJ \left( \frac{\sigma_+^{(0)}}{2} \cdot \frac{\sigma_-^{(1)}}{2} \right) - \gamma_{11} \left( \frac{\sigma_+^{(0)}}{2} \cdot \frac{\sigma_-^{(1)}}{2} \right) - \omega_1 \left( \frac{\sigma_+^{(0)}}{2} \cdot \frac{\sigma_-^{(1)}}{2} \right).
\]

We summarise the action of \( D^{(1)} \) on \( \frac{1}{2} \sigma_+^{(0)} \cdot \vec{B}^{(1)} \) as

\[
D^{(1)} \left[ \frac{\sigma_+^{(0)}}{2} \cdot \vec{B}^{(1)} \right] = \frac{\sigma_+^{(0)}}{2} \cdot \sum_{\nu,x,y,z} h_\nu (M_0)_{\nu\mu} \frac{\sigma_-^{(1)}}{2},
\]

where

\[
M_0 := \frac{1}{2} \begin{pmatrix}
    0 & x & y & z \\
    0 & 0 & 0 & -if \\
    0 & -\gamma_1 & 0 & 0 \\
    -if & 0 & -2\omega_1 & -2\gamma_1
\end{pmatrix}.
\]

Comparing the coefficients of each basis \( \frac{1}{2} \sigma_+^{(0)} \cdot (\sigma_-^{(1)}/2) \) in the left-hand side and the right-hand side of equation (16), we obtain the following differential equations for \( b_i \):

\[
\frac{d}{dt} \begin{pmatrix}
    b_0 \\
    b_x \\
    b_y \\
    b_z
\end{pmatrix} = M_0^T \begin{pmatrix}
    b_0 \\
    b_x \\
    b_y \\
    b_z
\end{pmatrix}.
\]

Note that the dynamics of \( b_z \) is totally decoupled from the other variables. Hereafter, we ignore the dynamics of \( b_z \) by employing the initial condition of equation (14) that is equivalent with \( b_0(0) = 1 \) and \( b_x(0) = b_y(0) = b_z(0) = 0 \). The remaining equations are concisely written in the following matrix form:

\[
\frac{d}{dt} \begin{pmatrix}
    b_0 \\
    b_x \\
    b_y
\end{pmatrix} = M \begin{pmatrix}
    b_0 \\
    b_x \\
    b_y
\end{pmatrix},
\]

where

\[
M := \frac{1}{2} \begin{pmatrix}
    0 & 0 & -if \\
    0 & -\gamma_1 & 2\omega_1 \\
    -if & -2\omega_1 & -2\gamma_1
\end{pmatrix}.
\]

This equation is analytically solvable since \( M \) is constant and its eigenvalues and eigenvectors are easily found (see appendix A).

Let us evaluate the reduced density matrix of system I, \( \rho_1^{(1)} \) by tracing out system II with the initial condition of equation (14). After straightforward calculation, we obtain

\[
\rho_1^{(1)} = \text{Tr}_{II}(\rho_2^{(1)}) = \frac{1}{2} \begin{pmatrix}
    1 & e^{-\gamma_1/2}b_0(t) \\
    e^{-\gamma_1/2}b_0(t) & 1
\end{pmatrix}.
\]

The explicit form of \( b_0(t) \) is given in appendix A, where we also show that \( b_0(t) \) is real. Note that \( \rho_1^{(1)}(0) = |+\rangle\langle +| \) and \( \rho_1^{(1)}(\infty) = \sigma_0/2 \).
2.3. Non-Markovian environment: \((1+n)\)-qubit case

The above analysis is readily generalised to the case where system II consists of \(n\) identical qubits. We call this system the \((1+n)\)-system \([15]\). We consider a system in which the system I qubit interacts with all system II qubits with equal coupling strength \(J\) while the qubits in system II do not interact among themselves. Moreover, there is an external field \(\omega_1\) that couples equally with all the system II qubits. The Hamiltonian of this system is then given by

\[
H = \sum_{i=1}^{n} \left( H^{(i)}_I + H^{(i)}_w \right), \quad H^{(i)}_I := \frac{1}{2} \sigma_z^{(i)} \cdot \sigma_z^{(i)}, \quad H^{(i)}_w := \omega_i \frac{\sigma_z^{(i)}}{2},
\]

(24)

where \(\sigma_z^{(i)} = \sigma_0 \otimes \ldots \otimes \sigma_0 \otimes \sigma_i \otimes \sigma_0 \otimes \ldots \otimes \sigma_0\) acts nontrivially on the \(i\)th qubit. Here we assign an index 0 to the system I qubit while indices 1 to \(n\) to the system II qubits. The basis vectors are ordered as

\[
\{|00\ldots00\rangle, |00\ldots01\rangle, |00\ldots11\rangle, \ldots, |11\ldots10\rangle, |11\ldots11\rangle\},
\]

(25)

where \(|ab\ldotscd\rangle = |a\rangle_0 \otimes |b\rangle_1 \otimes \ldots \otimes |c\rangle_{\pi-1} \otimes |d\rangle_\pi\).

The Lindbladian which represents the flip-flop noise that acts on all qubits independently is

\[
\mathcal{L}[\rho] = \sum_{i=0}^{n} \sum_{\pm} \gamma_i \left( 2 \frac{\sigma_+^{(i)} \rho \sigma_-^{(i)}}{4} - \left\{ \frac{\sigma_+^{(i)} \sigma_-^{(i)}}{4}, \rho \right\} \right) := \sum_{i=0}^{n} \mathcal{L}^{(i)}[\rho].
\]

(26)

We assume from now on that the strength \(\gamma_i\) for all the qubits in system II are identical: 
\(\gamma_1 = \gamma_2 = \ldots = \gamma_n = \gamma_n\).

The dynamics of the density matrix \(\rho^{(n)}\) of systems I and II is described by

\[
\frac{d\rho^{(n)}}{dt} = -i[H, \rho^{(n)}] + \mathcal{L}[\rho^{(n)}] = \sum_{i=1}^{n} \mathcal{D}^{(i)}[\rho^{(n)}] + \mathcal{L}^{(0)}[\rho^{(n)}],
\]

(27)

Let us write \(\rho^{(n)}\) in the same form as the \((1+1)\)-case,

\[
\rho^{(n)} = \frac{1}{2} \left( \begin{array}{cc} \sigma_0^{(0)} \otimes A_1^{(n)} + A_2^{(n)} \otimes \sigma_0^{(0)} & A_1^{(n)} - A_2^{(n)} \otimes \frac{\sigma_0^{(0)}}{2} \otimes \sigma_0^{(0)} \\ A_1^{(n)} - A_2^{(n)} \otimes \frac{\sigma_0^{(0)}}{2} \otimes \sigma_0^{(0)} & \sigma_0^{(0)} \otimes B^{(n)} + \frac{\sigma_0^{(0)}}{2} \otimes (B^{(n)})^\dagger \end{array} \right).
\]

(28)

\(A_1^{(n)}, A_2^{(n)}\) and \(B^{(n)}\) respectively have matrix forms \(\sigma_0 \otimes A_1^{(n)}, \sigma_0 \otimes A_2^{(n)}\) and \(\sigma_0 \otimes B^{(n)}\) where \(A_1^{(n)}, A_2^{(n)}\) and \(B^{(n)}\) are \(2^n \times 2^n\) matrices. Equivalently, \(\rho^{(n)}\) can be represented by the following block matrix form:

\[
\rho^{(n)} = \begin{pmatrix}
A_1^{(n)} & B^{(n)} \\
(B^{(n)})^\dagger & A_2^{(n)}
\end{pmatrix}.
\]

(29)

We can find that the dynamics of \(B^{(n)}\) is decoupled from those of \(A_1^{(n)}, A_2^{(n)}\) and \((B^{(n)})^\dagger\) as in the \((1+1)\)-case. We are interested in the initial state

\[
\rho^{(n)}(0) = |+\rangle\langle+| \otimes \frac{1}{2} \sigma_0^{\otimes n} = \frac{1}{2^{n+1}} \begin{pmatrix} \sigma_0^{\otimes n} & \sigma_0^{\otimes n} \\ \sigma_0^{\otimes n} & \sigma_0^{\otimes n} \end{pmatrix},
\]

(30)

which is a generalisation of equation (13) for the \((1+1)\)-system to the \((1+n)\)-system. This initial condition in terms of \(A_1^{(n)}, A_2^{(n)}, B^{(n)}\) and \((B^{(n)})^\dagger\) is

\[
A_1^{(n)}(0) = A_2^{(n)}(0) = B^{(n)}(0) = (B^{(n)})^\dagger(0) = \sigma_0 \otimes \frac{1}{2^n} \sigma_0^{\otimes n}.
\]

(31)

It is easy to show \(A_1^{(n)}\) and \(A_2^{(n)}\) are time independent with this initial condition.

Since the action of \(\mathcal{L}^{(n)}\) gives just a scalar multiplication as mentioned previously, we find that the GKLS equation (27) can be rewritten as

\[
\frac{d}{dt} \left( \frac{\sigma_+^{(0)}}{2} \cdot \tilde{B}^{(n)}(t) \right) = \sum_{i=1}^{n} \mathcal{D}^{(i)} \left[ \frac{\sigma_+^{(0)}}{2} \cdot \tilde{B}^{(n)}(t) \right],
\]

(32)
where $\tilde{B}^{(n)}(t) := e^{-\gamma t/2} B^{(n)}(t)$. We write

$$
\tilde{B}^{(n)}(t) = \prod_{j=1}^{n} \varsigma^{(j)}(t),
$$

(33)

where

$$
\varsigma^{(i)} = \frac{1}{2} \sum_{\nu = x,y,z} b^{(i)}_\nu \sigma^{(i)}_\nu
$$

(34)

with $b^{(i)}_\nu \in \mathbb{C}$. Our initial condition gives $b^{(i)}_0(0) = 1$, $b^{(i)}_\nu(0) = \tilde{b}^{(i)}(0) = b^{(i)}_\nu(0) = 0$ where $1 \leq i \leq n$. It turns out that $b^{(i)}_\nu$ decouples from the dynamics of the other $b^{(j)}_\nu$’s and we can set $b^{(i)}_\nu(t) = 0$ with the given initial condition. It follows from equation (33) that the density matrix $\rho^{(n)}$ correctly reflects the symmetry under arbitrary permutation of $n$ qubits in system II and that there are no correlations among them.

We then calculate the action of $D^{(i)}$. Since $D^{(i)}$ acts only on the 0th and $i$th qubits, it suffices to consider the term $\frac{1}{2} \sigma^{(0)}_x \cdot \varsigma^{(i)}$ only. The action of $D^{(i)}$ on $\frac{1}{2} \sigma^{(0)}_x \cdot (\sigma^{(i)}_\nu / 2)$ is given as

$$
D^{(i)} \left[ \frac{\sigma^{(0)}_x}{2} , \varsigma^{(i)} \right] = -J \frac{1}{2} \left( \frac{\sigma^{(0)}_y}{2} \cdot \frac{\sigma^{(i)}_x}{2} \right),
$$

$$
D^{(i)} \left[ \frac{\sigma^{(0)}_x}{2} , \varsigma^{(i)} \right] = \omega_1 \left( \frac{\sigma^{(0)}_y}{2} \cdot \frac{\sigma^{(i)}_x}{2} \right) - \gamma \frac{1}{2} \left( \frac{\sigma^{(0)}_y}{2} \cdot \frac{\sigma^{(i)}_y}{2} \right),
$$

$$
D^{(i)} \left[ \frac{\sigma^{(0)}_x}{2} , \varsigma^{(i)} \right] = -J \frac{1}{2} \left( \frac{\sigma^{(0)}_y}{2} \cdot \frac{\sigma^{(i)}_y}{2} \right) - \gamma \frac{1}{2} \left( \frac{\sigma^{(0)}_y}{2} \cdot \frac{\sigma^{(i)}_x}{2} \right) - \omega_1 \left( \frac{\sigma^{(0)}_y}{2} \cdot \frac{\sigma^{(i)}_y}{2} \right).
$$

(35)

We summarise the action of $D^{(i)}$ on $\frac{1}{2} \sigma^{(0)}_x \cdot \varsigma^{(i)}$ as

$$
D^{(i)} \left[ \frac{\sigma^{(0)}_x}{2} , \varsigma^{(i)} \right] = \frac{\sigma^{(0)}_x}{2} \cdot \sum_{\nu,\mu = 0,y,z} b^{(i)}_\nu (M)_{\nu\mu} \frac{\sigma^{(i)}_\mu}{2},
$$

(36)

where

$$
M = \frac{1}{2} \begin{pmatrix} 0 & y & z \\ 0 & 0 & -iJ \\ -iJ & 2\omega_1 & -2\gamma_1 \end{pmatrix}
$$

(37)

which is the same $M$ introduced in the $(1 + 1)$-system. The dynamics of $\tilde{B}^{(n)}$ following from equation (32) is written as

$$
\frac{d}{dt} \left( \frac{\sigma^{(0)}_x}{2} \cdot \prod_{i=1}^{n} \varsigma^{(i)} \right) = \sum_{i=1}^{n} D^{(i)} \left[ \frac{\sigma^{(0)}_x}{2} , \prod_{i=1}^{n} \varsigma^{(i)} \right] = \frac{\sigma^{(0)}_x}{2} \cdot \left[ \sum_{\nu,\mu = 0,y,z} b^{(1)}_\nu (M)_{\nu\mu} \frac{\sigma^{(1)}_\mu}{2} \right] \varsigma^{(2)} \cdots \varsigma^{(i)} \cdots \varsigma^{(n)} + \cdots
$$

$$
+ \cdots + \frac{\sigma^{(0)}_x}{2} \cdot \left[ \varsigma^{(1)} \cdots \left( \sum_{\nu,\mu = 0,y,z} b^{(1)}_\nu (M)_{\nu\mu} \frac{\sigma^{(1)}_\mu}{2} \right) \cdots \varsigma^{(n)} \right]
$$

$$
+ \cdots + \frac{\sigma^{(0)}_x}{2} \cdot \left[ \varsigma^{(1)} \cdots \varsigma^{(i)} \cdots \left( \sum_{\nu,\mu = 0,y,z} b^{(1)}_\nu (M)_{\nu\mu} \frac{\sigma^{(1)}_\mu}{2} \right) \right].
$$

(38)

Comparing the coefficients of each basis in the left-hand side and the right-hand side, we obtain differential equations for each coefficient $b^{(1)}_\nu \ldots b^{(n)}_\nu$ as
\[
\frac{d}{dt}(b_{\nu}^{(1)} \ldots b_{\nu}^{(i)} \ldots b_{\nu}^{(n)}) = \left( \sum_{\mu=0,y,z} (M^T)_{\nu\mu} b_{\mu}^{(1)} \right) b_{\nu}^{(2)} \ldots b_{\nu}^{(i)} b_{\nu}^{(n)} + \ldots \\
+ b_{\nu}^{(1)} b_{\nu}^{(i)} \left( \sum_{\mu=0,y,z} (M^T)_{\nu\mu} b_{\mu}^{(i-1)} \right) b_{\nu}^{(i+1)} b_{\nu}^{(n)} + \ldots \\
+ b_{\nu}^{(1)} b_{\nu}^{(i)} b_{\nu}^{(i-1)} \left( \sum_{\mu=0,y,z} (M^T)_{\nu\mu} b_{\mu}^{(i-1)} \right) b_{\nu}^{(i+1)} b_{\nu}^{(n)}.
\]

We rewrite this equation as
\[
\frac{db_{\nu}^{(1)}}{dt} - \sum_{\mu=0,y,z} (M^T)_{\nu\mu} b_{\mu}^{(1)} b_{\nu}^{(2)} b_{\nu}^{(3)} + \ldots \\
+ b_{\nu}^{(1)} b_{\nu}^{(i)} b_{\nu}^{(i-1)} \left( \frac{db_{\nu}^{(i)}}{dt} - \sum_{\mu=0,y,z} (M^T)_{\nu\mu} b_{\mu}^{(i-1)} \right) b_{\nu}^{(i+1)} b_{\nu}^{(n)} + \ldots \\
+ b_{\nu}^{(1)} b_{\nu}^{(i)} b_{\nu}^{(i-1)} \left( \frac{db_{\nu}^{(i-1)}}{dt} - \sum_{\mu=0,y,z} (M^T)_{\nu\mu} b_{\mu}^{(i-1)} \right) b_{\nu}^{(i+1)} b_{\nu}^{(n)} = 0.
\]

We obtain the differential equations
\[
\frac{d}{dt} \begin{pmatrix} b_{\nu}^{(0)} \\ b_{\nu}^{(1)} \\ \vdots \\ b_{\nu}^{(n)} \end{pmatrix} = M^T \begin{pmatrix} b_{\nu}^{(0)} \\ b_{\nu}^{(1)} \\ \vdots \\ b_{\nu}^{(n)} \end{pmatrix}, \quad 1 \leq i \leq n.
\]

Note that these differential equations are the same as equation (21) in the \((1 + 1)\)-system. Moreover, the initial conditions are the same for all \(i\) and thus the dynamics is solvable for any \(n\) by employing \(b_{\nu}(t)\) \((\nu = 0, y, z)\) obtained for the \((1 + 1)\)-system. This solution is reasonable because the qubits in system II are identical.

Let us evaluate the reduced density matrix \(\rho_{1}^{(n)}\) of system I, by tracing out system II. Note that \(\text{Tr}(\varsigma_{i}^{(i)}) = b_{0}\) since Pauli matrices are traceless. We obtain
\[
\rho_{1}^{(n)}(t) := \text{Tr}_{II} \rho_{1}^{(n)} = \frac{\sigma_{0}^{(0)}}{2} \cdot \text{Tr} \left( \prod_{i=1}^{n} \frac{\sigma_{0}^{(i)}}{2} \right) e^{-\gamma t/2} \left[ \frac{\sigma_{+}^{(0)}}{2} \cdot \text{Tr} \left( \prod_{i=1}^{n} \varsigma_{i}^{(i)} \right) + \text{h.c.} \right] \\
= \frac{\sigma_{0}^{(0)}}{2} + e^{-\gamma t/2} \left[ \frac{\sigma_{+}^{(0)}}{2} (b_{0}(t))^{n} + \text{h.c.} \right] \\
= \frac{1}{2} \left( e^{-\gamma t/2} (b_{0}(t))^{n} \right)
\]

We find that the effect of the direct coupling of the Markovian environment with system I, shown as the factor of \(e^{-\gamma t/2}\), is well separated from those through system II, which is the origin of the non-Markovian dynamics of system I. We introduce
\[
\beta_{\nu}(t) := e^{-\gamma t/2} (b_{0}(t))^{n}
\]
for later convenience.

One might think that our model is a trivial extension of one introduced in reference [15] since the only difference is the existence of the external field \(\omega_{1}\). However, \(H_{1}^{(i)}\) does not commute with \(H_{2}^{(i)}\), which makes our solution highly non-trivial compared to that obtained in reference [15]. Moreover, the dynamics of our model is controlled by a \(3 \times 3\) matrix, namely equation (37), while the corresponding one in reference [15] is \(2 \times 2\). As a result, this model shows drastically different behaviours from the previous one. It is possible to control non-Markovianity of the dynamics continuously by changing the external field strength \(\omega_{1}\) in equation (37) as will be shown in section 3.

### 2.4. Non-Markovianity measure

We will discuss control of non-Markovianity of dynamics by manipulating an external field in section 3. To this end, let us first introduce a measure \(\mathcal{N}\) to quantify non-Markovianity of dynamics. We employ the
measure proposed in [25], which is based on the concept of information backflow from the environment. This measure is described in terms of the trace distance \( D[\rho(t), \rho'(t)] = \text{Tr}(\rho(t) - \rho'(t))/2 \) between two states \( \rho \) and \( \rho' \) of a system of interest. Note that the environmental degrees of freedom are traced out here. \( \mathcal{N} \) introduced in [25] is defined as

\[
\mathcal{N} := \max_{\rho(0), \rho'(0)} \int_{\Omega_+} \frac{dD[\rho(t), \rho'(t)]}{dt} dt,
\]

where \( \Omega_+ := \{ t \in [0, \infty) : dD[\rho(t), \rho'(t)]/dt \geq 0 \} \) is a disjoint union of many intervals in general.

In this study, let us restrict the maximisation in \( \mathcal{N} \) with respect to the initial system I states written as

\[
\rho^{(n)}(t = 0, \theta) = \frac{1}{2} \left( \begin{array}{cc} \mathrm{e}^{-i\theta} & \mathrm{e}^{i\theta} \\ 1 & 1 \end{array} \right), \quad \theta \in \mathbb{R}.
\]

Note that \( \rho^{(n)}(t = 0, \theta) \) corresponds the following initial state of system I and II since the initial state of system II is fixed to \( \prod_{i=1}^{n} \frac{\sigma_i^0}{2} \),

\[
\rho^{(n)}(t = 0, \theta) = \frac{1}{2} \left( \begin{array}{cc} \mathrm{e}^{-i\theta} & \mathrm{e}^{i\theta} \\ 1 & 1 \end{array} \right) \otimes \left( \prod_{i=1}^{n} \frac{\sigma_i^{(0)}}{2} \right).
\]

The dynamics of the reduced density matrix of system I starting from the above initial state can be written as

\[
\rho_1^{(n)}(t, \theta) = \frac{1}{2} \left( \begin{array}{cc} 1 & \mathrm{e}^{i\theta} \beta_n(t) \\ \mathrm{e}^{-i\theta} \beta_n(t) & 1 \end{array} \right).
\]

Thus, we calculate the trace distance between any two states initially written as equation (45):

\[
D[\rho_1^{(n)}(t, \theta_1), \rho_1^{(n)}(t, \theta_2)] = |\beta_n(t) \sin \left( \frac{\theta_1 - \theta_2}{2} \right)|.
\]

A pair of pure states in system I with antipodal initial Bloch vectors, \( \rho_1^{(n)}(0, \theta) \) and \( \rho_1^{(n)}(0, \theta + \pi) \), gives the maximum value of the integrand \( dD[\rho_1^{(n)}(t, \theta), \rho_1^{(n)}(t, \theta + \pi)]/dt \) at any \( t > 0 \). Thus, \( \mathcal{N} \) is rewritten as

\[
\mathcal{N} = \int_{\Omega_+} \frac{dD[\rho_1^{(n)}(t, \theta), \rho_1^{(n)}(t, \theta + \pi)]}{dt} dt = \int_{\Omega_+} \frac{d|\beta_n(t)|}{dt} dt.
\]

We evaluate \( \mathcal{N} \) in section 3 and compare it with that obtained by NMR experiment.

3. Non-Markovianity control: experiment

3.1. Experimental setup and Hamiltonian

In section 2, we conducted theoretical analysis of a fictitious system that is made of one-qubit system I, identical \( n \)-qubit system II and Markovian environment. In this section, we map this model to a molecular system that can be realised in liquid-state NMR. We briefly introduce this system to make our work self-contained. See reference [15] for further details.

In NMR, a spin-1/2 nucleus is identified with a qubit. Under a strong magnetic field, the nucleus has a well-defined spin-up (spin-down) state that corresponds to \(|0\rangle \rangle \langle \langle 1|\rangle\rangle \) qubit state. The strong magnetic field produces the Zeeman energy terms \( \omega_0^{(0)} \sigma_i^{(0)}/2 + \omega_0^{(1)} \sum_{i=1}^{n} \omega_i^{(0)} \langle \langle 1| \rangle\rangle \) in the Hamiltonian, where \( \omega_0^{(0)} \) and \( \omega_0^{(1)} \) are the products of respective nuclear gyromagnetic ratio and the magnetic field strength. However, these terms can be eliminated by employing the double rotating frames with the angular velocities \( \omega_i^{(0)} = \omega_0^{(0)} \) and \( \omega_i^{(1)} = \omega_0^{(1)} \), respectively, see equation (50) below. We take a star-topology molecule for systems I and II, in which system I is the central nucleus while system II is formed by the surrounding nuclei, see figure 2. We consider a molecule in which the nucleus of system I and nuclei of system II belong to different nuclear species while all nuclei in system II are identical. Because of the symmetry of system II, a nucleus of system I interacts with each nucleus of system II with equal strength \( J \). Interactions among nuclei of system II effectively vanish because of symmetry and motional narrowing [18]. In addition, an external radio frequency (RF) magnetic field is applied on the molecule. If the RF frequency is equal to the Larmor
frequency of the spins in system II, it acts as a static external field for the spins in system II, while it has no
effect on the spin in system I in the rotating frame of respective nuclei. As a result, the Hamiltonian of
system I and II is approximated by

\[ H = J \sum_{i=1}^{n} \sigma^{(0)}_i \cdot \sigma^{(i)}_s + \omega \sum_{i=1}^{n} \sigma^{(i)}_s. \]  

which reproduces equation (24). Here \( J \) is the common coupling strength between the system I spin and the
system II spins while \( \omega \) is a measure of the RF magnetic field amplitude.

We employed Tetramethylsilane (TMS, \( \text{C}_4\text{H}_{12}\text{Si} \)) as such a molecule in our experiment. A TMS molecule
is a star-topology molecule that corresponds to the \((1, 12)\) system (figure 2). The central nucleus of \( ^{29}\text{Si} \) acts
as system I while surrounding 12 hydrogen nuclei form system II. Molecules are solved in acetone-d6 that is
isotropic [18]. The spin flip-flop rates \( \gamma_I \) and \( \gamma_{II} \) can be controlled by adding some magnetic impurities into
the sample solution, see references [15, 26–28]. Although we did not intentionally add the magnetic
impurities into the solvent in our experiment, oxygen molecules in the solvent act as the magnetic
impurities. In NMR experiments, we observe free induction decay signals (FID’s hereinafter) that represent
the relaxation of the expectation value of \( \sigma_x \) and \( \sigma_y \) (strictly speaking, it is an ensemble average over many
TMS molecules). In our model, this relaxation is described with \( \beta_n(t) \), see equations (42) and (43). To compare the theoretical and experimental results, we first measured \( \gamma_I \) by fitting the decoupling
(Markovian) limit of the experimental data with a function \( e^{-\gamma_I t/2} \). We independently measured
\( T_1 \) of H nuclei with a standard NMR technique called the inverse-recovery method to evaluate
\( \gamma_{II} = 1/T_1 \). We will use \( (\gamma_I, \gamma_{II}) = (0.41, 0.20) \) rad s\(^{-1} \) thus obtained, as listed in reference [15].

Now the dynamics of the total system including the environment is described by the GKLS
equation (27) and our theoretical analysis developed in section 2 is straightforwardly applicable to the
molecular system.

3.2. FID signals

Figure 3 shows the theoretical and experimental FID’s. The right panels are the normalized experimental
FID’s while the left panels show the theoretical ones. We plot the real and imaginary parts of the normalised
FID’s. Our theoretical analysis in appendix A shows that the imaginary part identically vanishes under the
initial condition we employed, see equation (A6). Experimental data is shown in the right panels. We
obtained smooth curves by averaging experimental data with respect to a small time-window to remove
random noise which is assumed to have vanishing average. This data will be used to evaluate
non-Markovianity measure in the next subsection.

Clearly, theoretical calculations well reproduce the experimental FID’s in both the Markovian
(decoupling) and non-Markovian (\( \omega_1 = 0 \) rad s\(^{-1} \)) limits, as discussed in reference [15]. The peaks in the
top-left panel are smaller than \( e^{-\gamma_I t/2} \) which implies that the information stored in system I can flow into
the environment through system II. In other words, the information can escape into the environment even
when \( \gamma_1 = 0 \). In the intermediate region (\( \omega_1 = 2\pi \times 21 \) rad s\(^{-1} \) case in figure 3), we can see that our
theoretical dynamics qualitatively agrees with the experimental data. However, there is a quantitative
difference between theory and experiment. The observed decay is faster than the theoretical prediction.
Here ‘decay’ has two meanings; decay of short wave-length ripples and decay of average amplitude. The
difference in decay of ripples is attributed to spatial inhomogeneity of \( \omega_1 \) [29]. The sample was sealed in an
NMR test tube with finite size and \( \omega_1 \)'s are slightly different for TMS molecules at various positions in the
tube. An observed FID signal is a result of ensemble average over a macroscopic number of TMS molecules.

![Figure 2](image)
Figure 3. Left three panels show the theoretical results while the right ones show the corresponding experimental data. The red (black) curves represent the real (imaginary) parts of normalised FID's. In theoretical dynamics, we employed the parameters \((\gamma_1, \gamma_2, J) = (0.41, 0.20, 2\pi \times 6.6)\) rad s\(^{-1}\) [15], \(\omega_1\) is shown on the top-right of each panel. Note that the imaginary part of FID signals vanishes identically, see equation (A6). The blue curves in the right panels are obtained by averaging experimental data with respect to a small time-window. The dotted curve in the top-left panel is \(e^{-\gamma_1 t/2}\), which corresponds to the direct influence of the Markovian environment on system I. The green curve in the middle-left panel is obtained by taking spatial inhomogeneity of \(\omega_1\) into account.

that have dynamics corresponding to local \(\omega_1\), which leads to destructive interference and faster decay of ripples. The middle-left panel in figure 3 shows the FID signals with (green curve) and without (red curve) spatial inhomogeneity. Here we assume the spatial inhomogeneity is described by the Gaussian distribution [29]:

\[
f(\omega'_1) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(\omega'_1 - \omega_c)^2}{2\sigma^2}\right),
\]

where \(\omega_c = 2\pi \times 21\) rad s\(^{-1}\) is \(\omega_1\) at the centre position. The variance \(\sigma\) is taken to be \(0.05 \times \omega_c\), which was employed in reference [29] where the same NMR spectrometer was used. As easily seen, the ripples with inhomogeneity decays much faster than that without inhomogeneity. By taking spatial inhomogeneity into account in this way, we show that the theoretical behaviour of ripples well approximates the experimental result. However, the faster decay of the average amplitude in the experimental data cannot be explained by spatial inhomogeneity of \(\omega_1\). It remains to our future study to account for the difference.

For the non-Markovian limit, such spatial inhomogeneity of \(\omega_1\) is not relevant since an external RF magnetic field does not exist. The Markovian (decoupling) limit is now achieved by WALTZ-16, a decoupling pulse sequence robust against spatial inhomogeneity of \(\omega_1\) [30]. This is the reason why theory well reproduces the experimental observation in both limits. We note en passant that the WALTZ-16 sequence is given by
where $R$ and $R$ stand for pulse sequences $1_x, 2_x, 3_x$ and $1_{-x}, 2_{x}, 3_{-x}$, respectively. Here $n_{\pm x}$ is a pulse generating a rotation about $\pm x$ axis by an angle $n\pi/2$.

### 3.3. Engineering Non-Markovianity

Let us study how non-Markovianity measure $\mathcal{N}$ changes as a function of $\omega_1$ in our theory and experiment. We evaluate $\mathcal{N}$ from the analytical solution of $\beta_{\omega_1}(t)$ as shown in figure 4. The upper limit of the integration (49) is taken to be $t = 50 \text{ s} \sim 20/\gamma_1$ instead of $t \to \infty$, which is sufficiently large compared to the time scale $2/\gamma_1$ of the dynamics.

Note that $\mathcal{N}$ does not decrease monotonically in this theoretical curve. There is a dip in the small $\omega_1$ region. One may wonder why such a dip appears. We could intuitively guess that the non-Markovianity measure $\mathcal{N}$ would monotonically decrease as $\omega_1$ increases. However, we should point out that $\omega_1$ does not control the non-Markovianity directly but it controls $\mathcal{N}$ indirectly by modulating the effective coupling between systems I and II. This indirect nature makes the behaviour of $\mathcal{N}$ more complicated. The measure is basically determined by the competition of two time-scales $2\pi/J$ and $2\pi/\omega_1$ involved in the dynamics. Non-trivial behaviour such as dip may be observed when these time scales are comparable. This fact is also mathematically justified by examining insets in figure 4, which plots $\beta_{\omega_1}(t)$, the FID signal of the (1 + 12)-system. We also plot $\beta_+^2(t)$ for comparison. The magnitude of the signal is suppressed as a power of $n$ in the vicinity of $t$ satisfying $\beta_+^2(t) = 0$ (inset (a) in figure 4). The time intervals with such suppressed signals hardly contribute to $\mathcal{N}$. While $\omega_1$ increases, the oscillation centre of $\beta_+^2(t)$ is gradually lifted up. Suppression occurs prominently when the lower end of the oscillation is located around zero (inset (b)); thus $\mathcal{N}$ first decreases near $\omega_1 \sim 0$ and hits the minimum. After $\beta_+^2(t)$ is lifted up totally above zero, $n$ rather enhances the non-Markovianity since the oscillation is amplified according to the power of $n$ (inset (c)). This causes the dip shown in figure 4. In the remaining region, $\mathcal{N}$ monotonically decreases since the oscillation gradually disappears (inset (d)).

Non-Markovianity $\mathcal{N}$ obtained from experimental data is shown in figure 5. We employ the smoothing introduced in the previous subsection to suppress noise in experiment when evaluating $\mathcal{N}$. We also give the theoretical curve of $\mathcal{N}$ with spatial inhomogeneity of $\omega_1$. We take a sufficiently long time domain of integration in equation (44). We see that the dip and broad peak in the experimental result are qualitatively reproduced by the theoretical calculation. We should emphasise that the theoretical curve of $\mathcal{N}$ is obtained without any fitting parameters. We found again that the theoretical $\mathcal{N}$ deviates from the experimental one when $2\pi/J$ and $2\pi/\omega_1$ are comparable.
4. Summary

We have proposed an open-system model of which dynamics can be continuously tuned from non-Markovian to Markovian by changing an external field. The model consists of system I that is the principal system of interest, system II surrounding system I, and Markovian environment. We have shown that the dynamics of this model can be solved analytically with a reasonable initial condition. We compared our theoretical results with the experimental data.

We have shown that the results of the theoretical model qualitatively agree with the experimental results. In particular, the transition from Markovian to non-Markovian dynamics is well reproduced theoretically. Then we evaluated non-Markovianity of our model by introducing a non-Markovianity measure \( \mathcal{N} \) based on the trace distance. Our model is expected to serve to understand non-Markovian open systems.

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Appendix A. Exact solution of equation (21)

Here we will show the exact form of \( b_0(t) \) by solving equation (21). To do this, it is enough to find the eigenvalues and eigenvectors of \( M' := 2M^T \) defined as

\[
M' = \begin{pmatrix}
0 & 0 & -i\lambda_1 \\
0 & -\gamma_{II} & -2\omega_1 \\
-i\lambda_1 & 2\omega_1 & -2\gamma_{II}
\end{pmatrix}.
\]

The eigenvalues of \( M' \) are

\[
\lambda_1 = -\gamma_{II} - \frac{D}{C} + \frac{C}{3}, \quad \lambda_2 = -\gamma_{II} + \left(\frac{1 + \sqrt{3}i}{2}\right) \frac{D}{C} - \left(\frac{1 - \sqrt{3}i}{2}\right) \frac{C}{3}, \quad \lambda_3 = \lambda_2^*.
\]

\[
C = \left(54\gamma_{II}\omega_1^2 + 3\sqrt{3}\left(108\gamma_{II}^2\omega_1^4 + D^2\right)^{\frac{1}{2}}\right)^{\frac{1}{3}}, \quad D = f^2 - \gamma_{II}^2 + 4\omega_1^2.
\]

Note that \( C \) is always real with our parameters \((\gamma_{II}, f) = (0.41, 0.20, 2\pi \times 6.6) \) rad s\(^{-1}\). The corresponding (unnormalised) eigenvectors are given as

\[
\vec{v}_1 = \begin{pmatrix}
-2i\omega_1 \\
\frac{1}{f} \frac{\lambda_1}{\lambda_1 + \gamma_{II}} \\
\frac{i}{f} \frac{\lambda_1}{\lambda_1 + \gamma_{II}}
\end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix}
-2i\omega_1 \\
\frac{1}{f} \frac{\lambda_2}{\lambda_2 + \gamma_{II}} \\
\frac{i}{f} \frac{\lambda_2}{\lambda_2 + \gamma_{II}}
\end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix}
\frac{1}{f} \frac{\lambda_3}{\lambda_3 + \gamma_{II}} \\
-2i\omega_1 \\
\frac{i}{f} \frac{\lambda_3}{\lambda_3 + \gamma_{II}}
\end{pmatrix}.
\]
By using the above eigenvalues and eigenvectors, the solution is written as

\[
\begin{pmatrix}
    b_0 \\
    b_y \\
    b_z
\end{pmatrix}
= \sum_{i=1,2,3} u_i \bar{v}_i \exp(\lambda_i t/2)
\]  

(A4)

where \(\{u_i\}_{i=1,2,3}\) are constant parameters determined by the initial condition. When assigning the initial values \((b_0(0), b_y(0), b_z(0)) = (1, 0, 0)\), we obtain

\[
\begin{align*}
    u_1 &= \frac{\vert \lambda_2 \vert^2 (\lambda_1 + \gamma_I)}{\gamma_I (\lambda_1 - \lambda_2^R)^2 + (\lambda_2^I)^2}, \\
    u_2 &= \frac{\lambda_2^I \gamma_I - \lambda_1 (\vert \lambda_2 \vert^2 + 2 \lambda_2^R \gamma_I)}{2 \gamma_I ((\lambda_1 - \lambda_2^R)^2 + (\lambda_2^I)^2)} + \frac{i \lambda_1 (\lambda_1 - \lambda_2^R)(\vert \lambda_2 \vert^2 + \lambda_2^R \gamma_I) + \lambda_1 (\lambda_2^I)^2 \gamma_I}{2 \lambda_2^I \gamma_I ((\lambda_1 - \lambda_2^R)^2 + (\lambda_2^I)^2)}, \\
    u_3 &= u_2^\dagger,
\end{align*}
\]  

(A5)

where we introduce the real (imaginary) part of \(\lambda_2^i\): \(\lambda_2 = \lambda_2^R + i\lambda_2^I, \lambda_2^I, \lambda_2^I \in \mathbb{R}\). Thus, the explicit form of \(b_0(t)\) is

\[
\begin{align*}
    b_0(t) &= u_1 \exp(\lambda_1 t/2) + u_2 \exp(\lambda_2 t/2) + u_3 \exp(\lambda_3 t/2) \\
    &= u_1 \exp(\lambda_1 t/2) + 2 \exp(\lambda_2^R t/2) \left( u_2^R \cos(\lambda_2^I t/2) - u_2^I \sin(\lambda_2^I t/2) \right),
\end{align*}
\]  

(A6)

where \(u_2^R\) (or \(u_2^I\)) is a real (imaginary) part of \(u_2\). Note that \(b_0(t)\) is always real with our parameters.

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