Investigation of a Family of Dynamic Systems with Reciprocal Polynomial Right Parts in a Poincare Circle

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Abstract

A paper describes methods and results of a fundamental study of some family of dynamic systems having reciprocal polynomial right parts, which is considered on the arithmetical (real) plane. One of the equations in these systems includes a cubic form in its right part, while the other one includes a square form. The goal was to find out all topologically different phase portraits possible for differential dynamic systems under consideration in a Poincare circle and outline close to coefficient criteria of them. A Poincare method of consecutive central and orthogonal mappings has been applied, and allowed to obtain more than 230 independent phase portraits. Each phase portrait has been described with a special table, every line of which corresponds to one invariant cell of the portrait and describes its boundary, as well as a source and a sink of its phase flow. All finite and infinitely remote singularities of considered dynamic systems were investigated.

1. Introduction.

A lot of phenomena and processes under investigation allow researchers to disregard statistical events and do not consider fluctuations. Under such conditions it is fruitful to use dynamic systems as mathematical models of such processes and phenomena. We may characterize a given dynamic system with the initial state and a law of its transformation into a different state. A phase space of a dynamic system is called a totality of its admissible states. Dynamic systems may be distinguished into the systems with the discrete time (the cascades) for the one side, and systems with the continuous time (the flows) for the other side. For cascades their behavior may be described with a sequence of states. For flows their state is defined per each time moment on a real (or an imaginary) axis.

An autonomous system of ordinary differential equations, which is defined in some domain and satisfying in that domain the conditions of the Cauchy theorem of existence and uniqueness of solutions, usually helps to describe the both types of dynamic systems. Under this approach equilibrium positions of dynamic systems correspond to singular points of differential equations, and closed phase curves of dynamic systems correspond to periodical solutions of differential equations.

A main goal of dynamic systems investigation is to find out, study and describe curves, defined by considered differential equations. This investigation process means splitting of a phase space of a given dynamic system into trajectories with the aim of the further study of their limit behavior, i.e. revealing (and classification) of the dynamic systems’ equilibrium positions, as well as indicating of its attracting and repulsive manifolds.

Any normal autonomous differential system of the second order with polynomial right parts on an extended real plane $\mathbb{R}_x^2$, allows full qualitative investigation, as J.H. Poincare has taught [1]. Some kinds of such systems were studied later, i.e. the quadratic ones [2], as well as the systems with nonzero linear terms, and the systems which contain nonlinear homogeneous terms of some odd
degrees, namely 3, 5, 7 [3], showing a center (or a focus) situated in a singular point O (0,0) [4], and some others.

In the present paper a family of dynamic systems is considered on a real plane of their phase variables \( x, y \)

\[
\frac{dx}{dt} = X(x, y), \quad \frac{dy}{dt} = Y(x, y),
\]

where \( X(x, y), Y(x, y) \) be reciprocal polynomials of \( x \) and \( y \), \( X \) be a cubic, while \( Y \) be a square form, for which \( X(0,1) > 0, Y(0,1) > 0 \). A goal is to study, reveal and describe in a Poincare circle all admissible for the Eq. (1) - systems topologically different phase portraits and show the criteria of the appearance of every phase portrait. A Poincare method of consecutive displays helps to achieve this goal: at first we use a central mapping of the plane \( x, y \) (from a center (0, 0, 1) of a Poincare sphere \( \Sigma \)), augmented with a line at infinity (i.e. \( \mathbb{R}^2 \) plane) onto a Poincare sphere \( \Sigma : x^2 + y^2 + z^2 = 1 \) with identified diametrically opposite points, and after that - the orthogonal mapping of a lower enclosed semi sphere of a Poincare sphere \( \Sigma \) to a Poincare circle \( \Omega : x^2 + y^2 \leq 1 \) with identified diametrically opposite points of its boundary \( \Gamma \) [1].

2. Main definitions and basic notations

\( \varphi(t,p), p = (x,y) \) - a fixed point := a solution (a motion) of an Eq. (1) – system with initial data \((0,p)\).

\( L_p : \varphi = \varphi(t,p), t \in I_{\text{max}}, \) a trajectory of a motion \( \varphi(t; p) \).

\( L^+_p(\cdot) := +(-) \cdot \) a semi trajectory of a trajectory \( L_p \).

\( O \)-curve of a system := its semi trajectory \( L^+_p(p \neq O, s \in \{+, -\}) \), adjoining to a point \( O \) under a condition that \( x(t) \to +\infty \).

\( O^+(-) \) curve of a system := its \( O \)-curve \( L^+_p^{(+)(-)} \).

\( O_{+(-)} \) curve of a system := its \( O \)-curve, adjoining to a point \( O \) from a domain \( x > 0 \) (\( x < 0 \)).

\( TO \)-curve of a system := its \( O \)-curve, which, being supplemented by a point \( O \), touches some ray in it.

A nodal bundle of \( NO \)-curves of a system := an open continuous family of its \( TO \)-curves \( L^+_p \),

where \( s \in \{+, -\} \) is a fixed index, \( p \in s, \): a simple open arc, \( L^+_p \cap s = \{p\} \).

A saddle bundle of \( SO \)-curves of a system, a separatrix of the point \( O := \) a fixed \( TO \)-curve, which isn’t included into some bundle of \( NO \)-curves of a system.

\( H, P, E \) - \( O \)-sectors of a system: a hyperbolic, a parabolic and an elliptical sector.

A topological type (T-type) of a singular point \( O \) of a system := a word \( A_o \) constructed with letters \( S, N \) (a word \( B_o \) constructed with letters \( H, P, E \), describes a circular order of bundles \( S, N \) of its \( O \)-curves (of its \( O \)-sectors \( H, P, E \) when traversing the point \( O \) in the \( \leftrightarrow \) direction (that mean counterclockwise), starting with some of them.

\[
P(u) := X(1,u) \equiv p_0 + p_1u + p_2u^2 + p_3u^3,
\]

\[
Q(u) := Y(1,u) \equiv a + bu + cu^2.
\]

Note 1.

Necessary to note for every Eq. (1) - system:

1) A topological type (T-type) of a singular point \( O \) in its form \( B_o \) can be easily constructed using its T-type in the form \( A_o \), and backwords (we need the both forms, see the Corollary 1 below);
2) The real roots of the polynomial \( P(u) \) (the polynomial \( Q(u) \)) are actually appear to be the angular coefficients of isoclines of the infinity (isoclines of a zero);

3) Writing out the system’s polynomials \( P(u), Q(u) \) real roots, all together or separately, let’s always number them in an ascending order.

3. The investigation of a topological type (the T – type) of a singular point \( O(0, 0) \)

A method of exceptional directions of a system in the point O is being used with the aim to investigate all \( O \)-curves and to split the totality of those curves into the saddle and nodal bundles \( S, N \) [1 - 3]. Taking these sources into consideration, we can see, that the equation of exceptional directions for the point O of the Eq. (1) - system has the form shown below:

\[
x F(x, y) = x(ax^2 + kxy + cy^2) = 0.
\]

The follows situations may take place for it:

1) when \( d = b^2 - 4ac > 0 \) the equation of exceptional directions will define the simple straight lines \( x = 0 \) and \( y = q_i x, \quad i = 1,2, \quad q_1 < q_2, \)

2) when \( d = 0 \) the exceptional directions’ equation will define the straight line \( x = 0 \) and the double straight line \( y = qx, q = \frac{-b}{2c}. \)

3) when \( d < 0 \) – this equation defines only one straight line \( x = 0. \)

**Theorem 1.**

Words \( A_O \) and \( B_O \) that define the T-type of a singular point \( O(0, 0) \) of the Eq. (1) - system:

1) Under the condition of \( d > 0 \) and depending on signs of values \( P(q_i), i = 1,2, \) have forms, indicated in a Table 1.

2) Under the condition of \( d = 0 \) and depending on signs of values \( q \) and \( P(q) \) - forms, indicated in a Table 2.

3) Under the condition of \( d < 0 \) a form: \( A_O = S_0 S^0, \quad B_O = HH: \) [4, 5]

**Table 1.** The topological type of a singular point \( O \) under the condition of \( d > 0 \) (\( r = 1,6 \)).

| \( r \) | \( P \) \( (q_1) \) | \( P \) \( (q_2) \) | \( A_O \) | \( B_O \) |
|------|----------------|------------------|--------|--------|
| 1, 4 | +             | +                | \( S_0 S^1 N^2 S^0 N^1 S^2 = S_0 S^1 N S^2 \) | \( PH^2 \) |
| 2    | -             | -                | \( S_0 N^1 S^2 S^0 S^1 N^2 = NS^2 S^0 S^1 \) | \( PH^2 \) |
| 3, 6 | -             | +                | \( S_0 N^1 S^2 S^0 S^1 S^2 = PEPH^4 \) | \( PEPH^4 \) |
| 5    | +             | -                | \( S_0 S^1 S^2 S^0 N^1 N^2 \) | \( H^4 PEP \) |
Table 2. The topological type of the singular point O (0, 0) if \( d = 0 \).

| \( q \) | \( P (q) \) | \( A_0 \) | \( B_0 \) |
|---|---|---|---|
| + | + | \( S_0S_+N_+S^+_0 \) | \( H^2P \) |
| - | - | \( S_0N_+S_+S^+_0 \) | \( PH^2 \) |
| + | - | \( S_0S^+_0N_-N_- \) | \( H^2P \) |
| - | + | \( S_0S^+_0N_+S_- \) | \( PH^2 \) |
| 0 | + | \( S_0S_+NS_- \) | \( H^2P \) |
| 0 | - | \( NS_+S^+_0S_- \) | \( PH^2 \) |

Note 2.
Commentaries for some new symbols used in the Theorem 1.
A symbol \( S_0 \) (a symbol \( S^0 \)) means a bundle \( S \), adjoining to the point \( O(0, 0) \) from the domain \( x > 0 \) along a semi axis \( x = 0, y < 0 \), while \( t \to +\infty \) (along a semi axis \( x = 0, y > 0 \), while \( t \to -\infty \)).

The lower sign index \( \ll + \) or \( \ll - \) of bundle \( S \), different from \( S_0 \) and \( S^0 \), indicates if the bundle consists of \( + \) curves or of \( - \) curves. Upper index 1 or 2 of a bundle indicates if its \( \ll + \) curves adjoining to the point \( O \) along a straight line \( y = q_1x \) or a line \( y = q_2x \).

In a Table 2 (see lines 5, 6) a bundle \( N \) was indicated without a lower sign index due to the fact that it contains \( \ll + \)-curves as well as \( \ll - \)-curves both.

Corollary 1.
The Theorem 1 (above) says us, that all described with the Eq. (1) differential dynamic systems cannot have limit cycles on the \( R^2_{x,y} \) plane.

Really, a singular point \( O (0,0) \) of an Eq. (1) – system could be surrounded with a possible limit cycle, and in such a case the Poincare index of this singular point must be equal to 1 [1]. But let’s take into consideration the formula of Bendixon for the index of a smooth dynamic system’s isolated singular point:

\[
I(O) = 1 + \frac{\varepsilon - h}{2},
\]

where \( \varepsilon \) (\( h \)) is a number of elliptical (hyperbolic) \( \ll + \)-sectors. The given formula of Bendixon and our Theorem 1 prove, that for the singular point \( O (0, 0) \) of the Eq. (1) – system Poincare index \( I(O) = 0 \).

Corollary 2.
There are eleven (different in the topological understanding) types admissible for the singular point \( O (0,0) \) of the Eq. (1) - system, and from their analysis we can conclude, that the singular point \( O (0, 0) \) has not more than four separatrices (actually 2, 3 or 4 ones).

4. The investigation of infinitely remote singular points (IR-points)
The method of the Poincare consecutive transformations, displays or mappings is fruitful for the investigation of the behavior of trajectories of the Eq. (1) – systems in a neighborhood of infinity [1].

Let’s now take into the consideration the first Poincare transformation

\[
x = \frac{1}{x}, \quad y = \frac{y}{z} \quad (u = \frac{y}{z}, \quad z = \frac{1}{x})
\]
which maps a phase plane $R^2_{x,y}$ of the Eq.(1) - system anambiguously onto a sphere $\Sigma$:

$$x^2 + y^2 + z^2 = 1$$

(where $z = -Z$) with the diametrically opposite points identified, which is called the Poincare sphere and is considered without its equator $E$, and an infinitely remote straight line of a plane $R^2_{x,y}$ it maps onto the equator $E$ of the Poincare sphere $\Sigma$, where the diametrically opposite points are identified as well.

The Eq. (1) - system using such a mapping we transform into another differential system, which in the Poincare coordinates $u, z$ after a time change $d\tau = -z^2 dt$ looks like

$$\frac{du}{d\tau} = P(u)u - Q(u)z, \quad \frac{dz}{d\tau} = P(u)z,$$

where $P(u) \equiv X(1,u)$ and $Q(u) \equiv Y(1,u)$ are reciprocal polynomial forms. This new system is determined on the whole Poincare sphere $\Sigma$, including the equator of the sphere, and on the whole $(u,z) - plane \alpha'$, which is tangent to a sphere $\Sigma$ at a point $C = (1,0,0)$. Further we are going to study the obtained system on a plane $R^2_{u,z}$ and the results we project onto the enclosed Poincare circle $\Omega$ sequentially mapping at first a plane $R^2_{u,z}$ onto the sphere $\Sigma$ from its center, and after that mapping its lower semi sphere $\mathcal{H}$ onto the Poincare circle $\Omega$, (that means onto a closed unit circle of a plane $R^2_{u,z}$) using the orthogonal mapping.

For the newly obtained system the axis $z = 0$ appears to be the invariant one (consisting of this new differential system’s trajectories). On this axis are situated its singular points $Q_i(u_i,0), i = 0, m$, where $u_i, i = 1, m$ are all real roots of the polynomial $P(u)$, and $u_0 = 0$; the same time may exist $u_0 \in \{1, ..., m\}; u_{i_0} \neq 0$. We further call such singular points the IR-points of the 1-st kind.

Now the second Poincare transformation

$$x = \frac{y}{z}, \quad y = \frac{1}{z} \left( \nu = \frac{u}{y}, \quad z = \frac{1}{y} \right)$$

also displays a phase plane $R^2_{x,y}$ anambiguously onto a sphere $\Sigma$ with the identified diametrically opposite points, taken without it’s equator, and every Eq.(1) - system transforms into a system, which in the coordinates $\tau, \nu, z$ looks like:

$$\frac{d\nu}{d\tau} = -X(\nu, 1) + Y(\nu, 1)\nu z, \quad \frac{dz}{d\tau} = Y(\nu, 1)z^2.$$

Such a (last) system is being determined on a whole Poincare sphere $\Sigma$, and on the whole $(\nu, z) - plane \alpha$, which is tangent to a sphere $\Sigma$ at a point $D = (0, 1, 0)$ [1]. A set $z = 0$ is invariant for it. On this set are situated this system’s singular points $(v_0, 0)$, where $v_0$ is any real root of the polynomial $X(\nu, 1) \equiv p_2 + p_3 v + p_4 v^2 + p_5 v^3$. Naturally it would be to consider such singular points as the IR-points of the 2-st kind for the Eq.(1) – system, but every this point, for which $v_0 \neq 0$, coincides obviously to one of the IR-points of the 1-st kind, i.e. to the point $\left(\frac{1}{v_0}, 0\right)$.

While $v_0 = 0$ isn’t a root of the polynomial $X(x, 1)$, due to the fact that $X(0, 1) = p_3 \neq 0$ for the Eq. (1) – system. Thus, we obtain the

**Corollary 3.**

Among the infinitely remote singular points of an Eq. (1) – system we find only the IR-points of the 1-st kind.

Using the orthogonal projection of an enclosed lower semi sphere $\mathcal{H}$ of a sphere $\Sigma$ onto a plane $x, y$, its open part $H$ appears to be one-to-one displayed onto the open Poincare circle $\Omega$, while its
boundary $E$ (an equator of the Poincare sphere $\Sigma$) appears to be displayed onto the boundary of the circle $\Gamma = \partial \Omega$. Eigen trajectories of an Eq. (1) - system (the singular point $O(0,0)$ among them) are being displayed into a circle $\Omega$ (and fill it). The infinitely remote trajectories (the IR-points among them) are displayed on a boundary $\Gamma$ of a circle $\Omega$ (and fill it).

According to the founder of the qualitative theory of differential equations J.H. Poincare, we name the first type of them the trajectories of an Eq. (1) - system in the Poincare circle $\Omega$, and the second type we name the trajectories of an Eq. (1) - system on the Poincare circle boundary $\Gamma$.

To each IR-point $O_i(\mu_i,0)$, of the Eq. (1) – system, $i \in \{1, ..., m\}$, do correspond two diametrically opposite points situated on the boundary $\Gamma$ of the Poincare circle:

$$O_i^+(\mu_i,0), O_i^-(\mu_i,0) \in \Gamma^{(+)} := \Gamma_{|u_i \geq 0 (u_i < 0)}.$$  

$\forall \ i \ \in \{1, ..., m\}$ for the point $O_i^+(\mu_i,0)$ we’ll introduce the following notations.

1) The $O_i^+(\mu_i,0)$-curve be a semi trajectory of the Eq. (1) – system in $\Omega$, starting in an ordinary point $p \in \Omega$ and adjacent to a point $O_i^{+(-)}$.

2) The notation of bundles $N, S$, adjacent to a point $O_i^+(\mu_i,0)$ from the circle $\Omega$, be similar to the notation introduced for the point $O(0,0)$.

3) The notation of a word $A_i^+(A_i^-)$ constructed with letters $S, N$, which is fixing an order of bundles of $O_i^+(\mu_i,0)$-curves at a semi circumvention of the point $O_i^+(\mu_i,0)$ in the Poincare circle $\Omega$ (in the direction of increasing of $u$).

We describe the topological type of a point $O_i^+(\mu_i,0)$ with a word $A_i^+(A_i^-)$, and the topological type of a point $O_i$ with words $A_i^\pm$.

The topological types of the IR-points $O_i^\pm(0,0)$ of Eq. (1) – systems are being described with the **Theorem 2.**

Let a number $\mu_i = 0$ be a root of the multiplicity $k \in \{0, ..., 3\}$ of a polynomial form $P(\mu)$ of an Eq. (1) - system. Then words $A_i^\pm$, which determine topological types (T-types) of IR-points $O_i^\pm(0,0)$ of this system, depending on the value of $k$ and a sign of a number $ap_i$ (where $a$ and $p_i$ are coefficients of the system), have the forms indicated in the Table 3 [5].

| $k$ | $ap_i$ | $A_0^+$ | $A_0^-$ |
|-----|--------|---------|---------|
| 0   | 0      | $N$     | $N$     |
| 0, 2| (+)    | $N_-(N_+)$ | $N_-(N_+)$ |
| 1, 3| (+)    | $N_N_-(\emptyset)$ | $\emptyset(N_-N_+)$ |

**Corollary 4.**

The IR-points $O_0^\pm$ of an Eq. (1) – system have no separatrices.

The topological types (the T-types) of the IR-points $O_i(\mu_i,0) \neq O_0(0,0), i = 1, m$, of an Eq. (1) – systems are described with the **Theorem 3.**

Let a real number $u_i(\neq 0)$ be a root of the multiplicity $k_i \in \{1, 2, 3\}$ of a polynomial form $P(\mu)$ of an Eq. (1) - system. Under these conditions a value $g_i = P^{(k_i)}(\mu_i)Q(\mu_i) \neq 0$ and words $A_i^\pm$, which determine the T-types of the IR-points $O_i^{\pm}(\mu_i,0)$, depending on the value of $k_i$ and signs of numbers $u_i$ and $g_i$, have forms from the Table 4 [5].
Corollary 5.

The Theorems 2 and 3 mean, that for the IR-points of an Eq. (1) – systems a finite number (namely 13) of different topological types may take place. The process of the thorough investigation of those T-types allows to conclude, that the IR-points of an Eq. (1) - system have precisely \( m \) separatrices: one separatrix for every singular point \( Q_i (u_i, 0), \ i = 1, m \).

Note 3.

For the Tables 3 and 4 a lower sign index ' – ' or ' + ' of every bundle \( N \) or \( S \) indicates if the bundle adjusts to the point \( Q_i^+ (u_i, 0) \) from the side \( u > u_i \) or from the side \( u < u_i \) of the isocline \( u = u_i \).

In the Table 3, line 1, a bundle \( N \) hasn’t a lower sign index, due to the reason that the detailed study reveals: it contains \( Q_i^- \)-curves (\( Q_i^- \)-curves) in every domain \( |u| > 0 \) [5].

5. Conclusions

The present paper describes the results and new methods of the fundamental original investigation.

Its key goal was to reveal, depict and describe all topologically different phase portraits in a Poincare circle, possible for the numerical subfamilies of a broad initial family of the polynomial differential dynamic systems. All those portraits were eventually constructed in the two ways - in a graphical and descriptive forms [6, 7].

The finite as well as the infinitely remote singular points of systems were totally investigated.

The additional solved problem of this work was to develop and apply some new, useful and effective research methods [8-11].

6. Recommendations

 Thankfully to the new research methods mentioned above the work could be useful for applied studies of dynamic systems of the second order with polynomial right parts. The work is considered to be interesting for students and postgraduates as well as for advanced researchers. Some special applications are indicated in papers [11 – 16].

References

[1] Andronov A A, Leontovich E A, Gordon I I and Mayer A G 1973 Qualitative theory of second-order dynamical systems (New York: Wiley)

[2] Andreev A F and Andreeva I A 1997 On limit and separatrix cycles of a certain quasiquadratic system Differential Equations 33:5 702 – 703

[3] Andreev A F and Andreeva I A 2007 Local study of a family of planar cubic systems Vestnik St. Petersburg University Mathematics Mechanics Astronomy 1:2 11-16

[4] Andreev A F., Andreeva I A, Detchenyya L V, Makovetskaya T V and Sadovskii A P 2017 Nilpotent Centers of Cubic Systems Differential Equations 53:8 1003 - 1008

[5] Andreev A F and Andreeva I A 2007 Phase flows of one family of cubic systems in a Poincare circle I Differential Equations and Control [Electronic Journal 4 17-26

Http://www.math.spbu.ru/user/diffjournal.
[6] Andreev A F and Andreeva I A 2008 Phase flows of one family of cubic systems in a Poincare circle II *Differential Equations and Control [Electronic Journal]* 1 1-13
Http://www.math.spbu.ru/user/diffjournal.

[7] Andreev A F and Andreeva I A 2008 Phase flows of one family of cubic systems in a Poincare circle III *Differential Equations and Control [Electronic Journal]* 3 39-54
Http://www.math.spbu.ru/user/diffjournal.

[8] Andreev A F and Andreeva I A 2009 Phase flows of one family of cubic systems in a Poincare circle. IV, *Differential Equations and Control. [Electronic Journal]* 4 181-213
Http://www.math.spbu.ru/user/diffjournal.

[9] Andreev A F and Andreeva I A 2017 Investigation of a Family of Cubic Dynamic Systems. *Vibroengineering Procedia* 15 88 – 93

[10] Andreev A F and Andreeva I A 2018 On a Behavior of Trajectories of a Certain Family of Cubic Dynamic Systems in a Poincare Circle *IOP Journal of Physics Conference Series* 1141

[11] Kuzkin V A and Krivtsov A M 2017 Fast and slow thermal processes in harmonic scalar lattices *Journal of Physics Condenced Matter* 29:50 14.

[12] Krivtsov A M and Kuzkin V A 2017 Enhanced vector-based model for elastic bonds in solids *Letters on Matherials* 7:4 455 – 458

[13] Krivtsov A M 2019 The Ballistic Heat Equation for a One-Dimensional Harmonic Crystal *Dynamical Processes in Generalized Continua and Structures* 103 345 – 358

[14] Murachev A S Krivtsov A M and Tsvetkov D V 2019 Thermal echo in a one-dimensional harmonic crystal *IOP Science* 31:9

[15] Krivtsov A M Sokolov A A Muller W H Freidin A B 2018 One-dimensional heat conduction and entropy production *Advances in Mechanics of Microstructured Media and Structures* 87 197 – 214

[16] Murachev A S Tsvetkov D V Galimov E M and Krivtsov A M 2018 Numerical Simulation of Circumsolar Ring Evolution *Advances in Mechanics of Microstructured Media and Structures* 87 251 – 262