The exploration of the non-perturbative regime of QCD, that is the low-energy portion of the hadron spectrum, requires the adoption of theoretical methods more frequently applied to other, more conventional, quantum many body systems, like the atomic nucleus, solid state systems, etc. In this work we have adopted, as a first step, the well-known BCS method to describe correlations between pairs of quarks and the associated ground state. Going beyond the BCS method would imply the inclusion of correlations by means of the TDA or RPA approximations. Since, we are interested in analyzing the role of constituent quark-pair correlations in the structure of hadrons we are restricted to the use of BCS as said before. The starting Hamiltonian is the effective Coulomb plus linear potential which we have used in previous calculations and performed a two-step approach, firstly by pre-diagonalizing it to built a single particle spectrum, and then, secondly, by applying the BCS transformations to it. Then, we have explored the resulting structure of the low energy meson spectra in terms of quasiparticle degrees of freedom. The dependence of the results upon the parameters which enter in the calculations is explored in detail, at the level of the quasiparticle mean-field approximation.

PACS numbers: 12.39.-x, 21.30.Fe, 21.60.Ev, 74.20.Fg
Keywords: QCD Hamiltonian, Coulomb gauge, BCS method, meson states

I. INTRODUCTION

QCD is the favored theory of strong interactions (see Refs. [1, 2]). However, due to properties like color confinement, its low energy regime becomes non-perturbative. In consequence, the description of the observed baryonic and mesonic spectrum cannot be attempted in terms of conventional, group symmetry based methods, without making extreme approximations [3–5]. This is a paradigmatic situation, e.g: the theory contains the correct degrees of freedom (quarks and gluons) but the correct description of the observed baryonic and mesonic states seemingly requires going beyond that. The effective Hamiltonian of Refs. [6–13], which is the QCD Hamiltonian in the Coulomb gauge, is a good starting point for the application of many body techniques [14, 15]. The feasibility of this approach was shown in previous publications Refs.[16–18]. A brief resume of the results so far obtained along this line is the following: In Refs.[17, 18], we have implemented the harmonic oscillator solutions as a trial basis. This first approach to treat the effective fermionic sector of the QCD Hamiltonian demonstrates the possibility to include an arbitrary number of radial and angular excitations, even for the exploration of hadronic excited states [18, 19]. We have extended our investigation of the motivated QCD Hamiltonian based on the framework of the Coulomb gauge, by taking as starting point the effective solutions of quarks and antiquarks, obtained by diagonalizing the Dirac term in the harmonic oscillator basis [17]. Then, the effects of a confining Coulomb plus linear interaction were taking into account by the implementation of well known many body techniques like the Tamm-Dancoff-Approximation (TDA) and the Random-Phase-Approximations (RPA). Both, the TDA and RPA described meson-like states as collective phonon solutions. In Refs.[18], it was shown that the lowest energy, meson-like solution of the RPA-method, describes a highly collective state as a superposition of particle-hole states. This picture may be completed by looking at correlations between
pairs of quarks which may result in a superfluid low energy regime. Therefore, the observed spectrum of hadrons could be interpreted in terms of quasiparticle excitations. This is the purpose of the present paper. We proceed gradually by treating the QCD Hamiltonian in the Coulomb gauge, to construct the effective quark spectrum. We will apply the BCS transformations and solve the corresponding equations, in order to get the relevant parameters of the model, that is the occupation factors and gaps. The dependence of the solutions upon the dimension of the basis and of it in terms of coupling schemes is presented and discussed in Section II and Section III, respectively. The results of the calculations are discussed in Section IV and the conclusions are drawn in Section V.

II. FROM QCD TO EFFECTIVE DEGREES OF FREEDOM.

The implementation of the BCS transformation in a fermionic or bosonic system requires, as a first step, the definition of a single particle field. In the case of QCD the literature is rich in the description of such fields [17, 20–29]. For this we have adopted the Coulomb plus linear QCD Hamiltonian, as it is described next

A. Coulomb gauge QCD Hamiltonian.

We start from the QCD Hamiltonian in its canonical Coulomb gauge representation [2, 6],

$$H^{QCD} = \int \left\{ \frac{1}{2} \left[ J^{-1} \Pi^{tr} \cdot J \Pi^{tr} + \mathcal{E} \cdot \mathcal{E} \right] - \overline{\psi} ( - i \gamma \cdot \nabla + m ) \psi - g \overline{\psi} \gamma \cdot A \psi \right\} dx$$

$$+ \frac{1}{2} g^2 \int \left\{ J^{-1} \rho^c (x) \langle c, x | \frac{1}{\mathcal{B}} (- \nabla^2) \frac{1}{\mathcal{B}} | c' y \rangle J \rho^c (y) dxdy \right\},$$

which has been widely studied in the past for the description of several properties of QCD at low energy [7–13, 27–29]. In [8, 18] a complete description of this Hamiltonian has been presented. The Hamiltonian of Eq. (1) includes the QCD Instantaneous color-Coulomb Interaction (QCD-IcCI) between color-charge-densities of quarks and gluons, last term in Eq. (1). At low energy the effects of dynamical gluons in the QCD-IcCI can be represented by the interaction

$$V(|x - y|) = - \frac{V_C}{|x - y|} + V_L |x - y|,$$

which is obtained from a self-consistent treatment of the interaction between color charge-densities [8, 9].

In [8, 18], the Hamiltonian of Eq. (1) was analyzed when it is restricted to the quark sector of the theory (no dynamical gluons), and an effective confining interaction $V(|x - y|)$ was used to describe the low energy interaction between color charge densities. We write

$$H^{QCD}_{eff} = \int \left\{ \psi^\dagger (x) (- i \alpha \cdot \nabla + \beta m) \psi (x) \right\} dx - \frac{1}{2} \int \rho_c (x) V(|x - y|) \rho^c (y) dxdy$$

$$= K + H_{Coul},$$

where $\rho^c (x) = \psi^\dagger (x) T^c \psi (x)$ is the quark and antiquark charge density. In Eq. (3), the first term is the kinetic energy, while the second term is the QCD-IcCI in its simplified form. The fermion field $\psi^\dagger (x)$, whose quantization is explained in [18], is expanded in terms of creation and annihilation operators in the harmonic oscillator basis

$$\psi^\dagger (x) = \sum_{\tau, N_{lN}, \sigma CF} R_{Nl}^*(x) Y^*_{lN_{\sigma CF}} (\hat{x}) \chi^\dagger_{\tau, N_{lN}, \sigma CF} q^\dagger_{\tau, N_{lN}, \sigma CF},$$

$$= \sum_{N_{lN}, \sigma CF} R_{Nl}^*(x) Y_{lN_{\sigma CF}} (\hat{x}) \chi^\dagger_{\sigma} \left( q^\dagger_{\frac{1}{2}, N_{lN}, \sigma CF} + q^\dagger_{-\frac{1}{2}, N_{lN}, \sigma CF} \right),$$

with $x = |x|$ and $R_{Nl}(x) = N_{lN} \exp (- \frac{B_{0l}^2}{2} x^2) L_{Nl}^{l+\frac{1}{2}} (B_{0l} x^2)$, where $L_{Nl}^{l+\frac{1}{2}}$ is an associated Laguerre polynomial and $(\sqrt{D_0})^{-1}$ is the oscillator length. The index $\tau$ denotes upper ($\tau = \frac{1}{2}$) and lower ($\tau = -\frac{1}{2}$) components of the Dirac spinors in the Dirac-Pauli representation of the Dirac matrices, and $\sigma, C, F$ denote spin, color and flavor intrinsic degrees of freedom, respectively. The harmonic oscillator basis is chosen, because it allows to obtain analytic expressions for the matrix elements of the interaction.
The diagonalization of the kinetic term is performed using the total spin \( J = l \pm \frac{1}{2} \) representation, for a given maximal number of quanta \( N = N_{\text{cut}} \), for which we introduce a general transformation to a basis of effective operators,

\[
q^\dagger_{\ell(N)} J M J C F = \sum_{\lambda \pi k} (\alpha^{J,T}_{\ell(N), \lambda \pi k})^* Q^\dagger_{\lambda \pi k J M J C F} \delta_{\pi, (-1)^{\frac{1}{2} - \tau + 1}}.
\]

The index \( \lambda = \pm \frac{1}{2} \) refers to the pseudo-spin components after the diagonalization of the kinetic energy term, and \( k \) runs over all orbital states after the diagonalization. The value \( \lambda = + \frac{1}{2} \) refers to positive energy states (effective quarks) and the value \( \lambda = - \frac{1}{2} \) to negative energy states (effective antiquarks), i.e. \( Q^\dagger_{\lambda \pi k J M J C F} \rightarrow b_{\pi k J C(Y, T), M J M C M T} \).

The transformation coefficients \( \alpha^{J,T}_{\ell(N), \lambda \pi k} \) depend on the type of quarks, whether it is an up or down quark (equal masses are assumed) or a strange quark \( (m_{qu} < m_s) \), i.e. from now on we will distinguish flavor representations according to their flavor hypercharge and isospin \( (Y, T) \), with magnetic projection \( M_T \). For the transformation coefficients and the matrix elements of the kinetic energy term of the Hamiltonian only the dependence on the flavor isospin is given, because the flavor-hypercharge is fixed by \( T \).

The eigenvalue problem to be solved acquires the following form

\[
\sum_{\pi, N_{\ell_i}} \alpha^{J,T}_{\ell_1, (N_{\ell_1}), \lambda_1 \pi_1 k_1} R^{J,T}_{\ell_1, (N_{\ell_1}), \pi_2 (N_{\ell_2})} \alpha^{J,T}_{\ell_2, (N_{\ell_2}), \lambda_2 \pi_2 k_2} = \varepsilon_{\lambda_1 \lambda_2 \pi_1 \pi_2 \delta \kappa \kappa_1 \kappa_2},
\]

where we have taken the transformation coefficients of Eq. (6) to be real.

The correct identification of the effective quark and antiquark degrees of freedom is one of the most important steps in order to describe hadronic states. By implementing the harmonic oscillator basis to treat the fermionic sector of the QCD Hamiltonian, we have performed a prediagonalization and identified effective quarks and antiquarks as a linear combinations of the bare quarks and antiquarks. In terms of the effective quarks and antiquarks, the kinetic energy term acquires the following structure

\[
K = \sum_{k \pi \gamma} \varepsilon_{k \pi \gamma} \sum_{\mu} \left( b_{k \pi \gamma \mu}^\dagger b_{k \pi \gamma \mu} - d_{k \pi \gamma \mu}^\dagger^* d_{k \pi \gamma \mu} \right),
\]

where the creation and annihilation operators of the effective quarks and antiquarks are given by \( b_{k \pi \gamma \mu}^\dagger \), \( b_{k \pi \gamma \mu} \) and \( d_{k \pi \gamma \mu}^\dagger \), \( d_{k \pi \gamma \mu} \) respectively. The upper and lower indices indicate the principal number \( (k = 1, 2, \ldots) \) which run over all orbital states and the parity \( (\pi = \pm) \) while \( \gamma, \mu \) are short hand notation for the particle spin, color and flavor hypercharge and isospin representations \( \gamma = \{ J, C, (Y, T) \} \) and their magnetic projections \( \mu = \{ M_J, M_C, M_T \} \), respectively. The flavor hypercharge and isospin quantum numbers for quarks are given by \( (Y, T) = (\pm \frac{1}{2}, 0) \) and for antiquarks \( (Y, T) = (-\frac{1}{2}, 0, 0) \).

The QCD-IcCI term, in its simplified form \( (H_{\text{Coul}}) \), rewritten in terms of effective quarks and antiquarks operators is given by

\[
H_{\text{Coul}} = -\frac{1}{2} \sum_{L} \sum_{\lambda_1 \lambda_2} \sum_{\ell_1, \ell_2} V_{\{\lambda_1 \ell_1, \lambda_2 \ell_2\}} \left( \left[ F_{\lambda_1 \lambda_2 \ell_1, \ell_2} \delta_{\lambda_1 \lambda_2 \ell_1, \ell_2} \delta_{\gamma f_0} \right] \gamma_0^0 + \left[ G_{\lambda_1 \lambda_2 \ell_1, \ell_2} \delta_{\lambda_1 \lambda_2 \ell_1, \ell_2} \delta_{\gamma f_0} \right] \gamma_0^0 \right) \mu_0^0 + \left[ G_{\lambda_1 \lambda_2 \ell_1, \ell_2} \delta_{\lambda_1 \lambda_2 \ell_1, \ell_2} \delta_{\gamma f_0} \right] \gamma_0^0 \mu_0^0,
\]

where we have compacted the single particle orbital number, parity and irreps into the short-hand notation \( \ell_i = k_i \pi_i \gamma_i \), and use for the (flavorless) quantum numbers of the intermediate coupling in the interaction \( \gamma_{f_0} = \{ L(11)(0,0) \} \) and for their magnetic projections \( \mu_{f_0} = \{ M_L, M_C, 0 \} \). The conjugate representations satisfy \( \gamma_{f_0} = \gamma_{f_0}^* \) and \( \mu_{f_0} = \mu_{f_0}^* \). For the total couplings (upper index) and magnetic numbers (lower index) of the interaction, we have used \( \gamma_0 = \{ 0, 0, 0 \} \) and \( \mu_0 = \{ 0, 0 \} \) respectively. The operators \( F \) and \( G \) are given by

\[
\begin{align*}
F_{\lambda_1 \lambda_2 \ell_1, \ell_2} &= \frac{1}{2} \left\{ \delta_{\lambda_1 \lambda_2 \ell_1, \ell_2} \left[ b_{\ell_1} \otimes b_{\ell_2} \right] \gamma_{f_0}^\gamma_{f_0} \mu_{f_0}^\mu_{f_0} - \delta_{\lambda_1 \lambda_2 \ell_1, \ell_2} \left[ d_{\ell_1} \otimes d_{\ell_2} \right] \gamma_{f_0}^\gamma_{f_0} \mu_{f_0}^\mu_{f_0} \right\}, \\
G_{\lambda_1 \lambda_2 \ell_1, \ell_2} &= \frac{1}{2} \left\{ \delta_{\lambda_1 \lambda_2 \ell_1, \ell_2} \left[ d_{\ell_1} \otimes b_{\ell_2} \right] \gamma_{f_0}^\gamma_{f_0} \mu_{f_0}^\mu_{f_0} - \delta_{\lambda_1 \lambda_2 \ell_1, \ell_2} \left[ b_{\ell_1} \otimes d_{\ell_2} \right] \gamma_{f_0}^\gamma_{f_0} \mu_{f_0}^\mu_{f_0} \right\}.
\end{align*}
\]
In this basis, and using the above introduced states, the matrix elements of the interaction are given by

\[
V^L_{(\lambda_1, \pi_1, \lambda_3, \pi_3, \lambda_4, \pi_4, k_1, \lambda_2, \pi_2, k_2, \lambda_3, \pi_3, k_3, \lambda_4, \pi_4, k_4)} = \sum_{\tau_1, \tau_2, \tau_3, \tau_4} V^L_{(\tau_1, \tau_2, \tau_3, \tau_4)} \alpha^{\tau_1}_{\lambda_1, \pi_1, k_1} \alpha^{\tau_2}_{\lambda_2, \pi_2, k_2} \alpha^{\tau_3}_{\lambda_3, \pi_3, k_3} \alpha^{\tau_4}_{\lambda_4, \pi_4, k_4}
\]

\[
\delta_{\tau_1 \tau_2} \delta_{\tau_3 \tau_4} \delta_{\pi_1, \pi_1, \pi_1, \pi_1} \delta_{\lambda_2, \lambda_2, \lambda_2, \lambda_2} \delta_{\pi_2, \pi_2, \pi_2, \pi_2} \delta_{\lambda_3, \lambda_3, \lambda_3, \lambda_3} \delta_{\pi_3, \pi_3, \pi_3, \pi_3} \delta_{\lambda_4, \lambda_4, \lambda_4, \lambda_4} \delta_{\pi_4, \pi_4, \pi_4, \pi_4}
\]

\[
\times (-1)^{\tau_1 + \tau_2 + \tau_3 + \tau_4} \frac{\sqrt{2\tau_1 + 1}}{\sqrt{3}} \delta_{\tau_2 \tau_1} \delta_{\tau_3 \tau_2} \delta_{\tau_4 \tau_3} \delta_{\tau_4 \tau_1} (-1)^{\tau_1 + \tau_2 + \tau_3 + \tau_4} \frac{\sqrt{2\tau_3 + 1}}{\sqrt{3}} \delta_{\tau_4 \tau_3} \delta_{\tau_4 \tau_1} .
\]

The matrix elements in the harmonic oscillator basis \((V^L_{(\lambda_1, \pi_1, \lambda_3, \pi_3, \lambda_4, \pi_4, k_1, \lambda_2, \pi_2, k_2, \lambda_3, \pi_3, k_3, \lambda_4, \pi_4, k_4)})\) are analytic and actually easy to compute.

**B. Bogoliubov Transformation.**

Here, we apply a canonical transformation from the particle (antiparticle) quark basis to another basis known as the quasiparticle basis, in order to approximately diagonalize part of the QCD interaction. The method is well known in many-particle physics under the name *Bogoliubov Transformation* \(^{14, 15}\).

The use of the Bogoliubov transformations in fermionic and bosonic systems, to account for pairing-type of correlations and the consequences of it in terms of the building of ground state correlations, both in finite and continuous systems, has been documented in hundred (if not thousand) of papers since they have been proposed in the earlier 1950’s. A compilation of such a references, for the case of nuclear systems, may be found in \(^{31}\). In the case of hadronic physics the notions related to the use of pairs correlations can be found in the chapter 21 of the book by S. Weinberg \(^{1}\), where the concept of spontaneous symmetry breaking has been explicitly applied to interacting hadrons. Another relevant reference about the use of the BCS approach is \(^{31}\).

The transformation from the effective quark degrees of freedom to the quasiquark basis is done by means of the Bogoliubov transformations. Herewith we shall apply the transformations for each quark-flavor, separately. Thus, the reference (ground state) state will be referred to as a quark condensate with a definite flavor. In terms of the BCS solutions, which we are going to introduce below, this amounts to the construction of a set of parameters (occupation numbers and gap) for each flavor. Then, the Hamiltonian of Eq. \(^{33}\) is written in terms of quasi-quark operators and the standard conditions of the BCS theory are applied to it by asking the one-quasiparticle term to be diagonal, the pairs terms to vanish and by extracting from the constant terms the corresponding gaps. These steps are shown next. The creation and annihilation quasi-quark operators are written as

\[
B_{k_1, \pi_1, \gamma_1, \mu_1}^{\dagger} = u_{k_1, \pi_1, \gamma_1, \mu_1} b_{k_1, \pi_1, \gamma_1, \mu_1}^{\dagger} - v_{k_1, \pi_1, \gamma_1, \mu_1} d_{k_1, \pi_1, \gamma_1, \mu_1}
\]

and

\[
D_{k_1, \pi_1, \gamma_1, \mu_1}^{\dagger} = u_{k_1, \pi_1, \gamma_1, \mu_1} d_{k_1, \pi_1, \gamma_1, \mu_1}^{\dagger} + v_{k_1, \pi_1, \gamma_1, \mu_1} b_{k_1, \pi_1, \gamma_1, \mu_1}
\]

for the creation operators, and

\[
B_{k_1, \pi_1, \gamma_1, \mu_1} = u_{k_1, \pi_1, \gamma_1, \mu_1}^{*} b_{k_1, \pi_1, \gamma_1, \mu_1} - v_{k_1, \pi_1, \gamma_1, \mu_1}^{*} d_{k_1, \pi_1, \gamma_1, \mu_1}
\]

\[
D_{k_1, \pi_1, \gamma_1, \mu_1} = u_{k_1, \pi_1, \gamma_1, \mu_1}^{*} d_{k_1, \pi_1, \gamma_1, \mu_1} + v_{k_1, \pi_1, \gamma_1, \mu_1}^{*} b_{k_1, \pi_1, \gamma_1, \mu_1}
\]

for the annihilation ones, respectively. In the above equations we have used a short hand notation to denote the states, that is: \(\gamma_i = \{J_i C_i(Y_i, T_i)\}\) and \(\mu_i = \{M_i M_C M_T\}\). The coefficients \(u\) and \(v\) should be taken as real.

The inverse transformations are given by

\[
b_{k_1, \pi_1, \gamma_1, \mu_1}^{\dagger} = u_{k_1, \pi_1, \gamma_1, \mu_1}^{*} B_{k_1, \pi_1, \gamma_1, \mu_1}^{\dagger} + v_{k_1, \pi_1, \gamma_1, \mu_1}^{*} D_{k_1, \pi_1, \gamma_1, \mu_1}
\]

\[
d_{k_1, \pi_1, \gamma_1, \mu_1}^{\dagger} = u_{k_1, \pi_1, \gamma_1, \mu_1}^{*} D_{k_1, \pi_1, \gamma_1, \mu_1}^{\dagger} - v_{k_1, \pi_1, \gamma_1, \mu_1}^{*} B_{k_1, \pi_1, \gamma_1, \mu_1}
\]

A crucial step in the treatment, leading to the transformation between ordinary particles (in this case fermions like the quarks and antiquarks) to quasiparticles is the replacement of the ordinary vacuum \((0)\) by the BCS vacuum \((BCS)\), which amounts to a spontaneous symmetry breaking, which is expressed by the weak identity:

\[
\langle BCS | b_{k_1, \pi_1, \gamma_1, \mu_1}^{\dagger} d_{k_1, \pi_1, \gamma_1, \mu_1} | BCS \rangle \sim \Delta_k \neq 0
\]

where \(\Delta_k\) is the state (flavor) dependent gap. At the same time the following conditions:

\[
B_{k_1, \pi_1, \gamma_1, \mu_1} | BCS \rangle = 0
\]
should be obeyed. As an additional comment about the meaning of the BCS vacuum expectation value Eq. (14), it
is worth to mention that it plays the role of a mass, since the square of the gap will appear in the definition of
the quasiparticle energies. This is indeed the case of up and down quarks, because for these flavors the gap is non-zero.
For strange quarks the gap is always zero, as we shall show in Section IV.B.

The method implemented to determine the actual value of the parameters $u$ and $v$ of Eq. (13), for each flavor, is a
variational one, where the terms of the Hamiltonian to be varied are kept up to the forth power of these parameters.
The convergence of the solutions was tested as a function of the maximum number of quanta $N_{cut}$.

III. QUASIPARTICLE HAMILTONIAN

The terms of the transformed Hamiltonian are obtained by replacing in Eq. (9), the tensorial product of quarks
and antiquark operators by their quasiparticle expression. From Eq. (13) we get for the terms entering Eq. (9)

$$\left[ b_{q_1}^\dag \otimes d_{q_2} \right] \gamma_{f_0}$$

$$= \langle \gamma_1 \mu_1, \bar{\gamma}_2 \bar{\mu}_2 | \gamma_{f_0} \mu_{f_0} \rangle (-1)^{\gamma_2 - \mu_2} b_{k_1 \pi_1 \gamma_{1}} b_{k_2 \pi_2 \gamma_{2}}$$

$$= \langle \gamma_1 \mu_1, \bar{\gamma}_2 \bar{\mu}_2 | \gamma_{f_0} \mu_{f_0} \rangle (-1)^{\gamma_2 - \mu_2} \left( u_{k_1 \pi_1 \gamma_{1}}^* B_{k_1 \pi_1 \gamma_{1}} + v_{k_1 \pi_1 \gamma_{1}} D_{k_1 \pi_1 \gamma_{1}} \right)$$

$$\left[ d_{q_1} \otimes d_{q_2} \right] \gamma_{f_0}$$

$$= \langle \gamma_1 \mu_1, \bar{\gamma}_2 \bar{\mu}_2 | \gamma_{f_0} \mu_{f_0} \rangle (-1)^{\gamma_2 + \mu_2} d_{k_1 \pi_1 \gamma_{1}} d_{k_2 \pi_2 \gamma_{2}}$$

$$= \langle \gamma_1 \mu_1, \bar{\gamma}_2 \bar{\mu}_2 | \gamma_{f_0} \mu_{f_0} \rangle (-1)^{\gamma_2 + \mu_2} \left( u_{k_1 \pi_1 \gamma_{1}}^* D_{k_1 \pi_1 \gamma_{1}} + v_{k_1 \pi_1 \gamma_{1}}^* B_{k_1 \pi_1 \gamma_{1}} \right)$$

and similarly for the kinetic energy terms of Eq. (10). We have used the short-hand notation $(\gamma_1 \mu_1, \bar{\gamma}_2 \bar{\mu}_2 | \gamma_{f_0} \mu_{f_0})$ and $(-1)^{\gamma_2 \pm \mu_2}$ for the product of the spin, color and isospin-flavor Clebsch–Gordan coefficients and phases [18], respectively.

The next step in our derivation consists of taking normal order respect to the quasiparticle vacuum and collecting
the different contributions to the Hamiltonian with constant terms $H_{00}$, one creation-one annihilations terms $H_{11}$,
two-quasiparticle terms $H_{20}$ and $H_{02}$. The value of the gap is extracted, for each channel, by solving the set of BCS
equations (see Section III A) and the remanent of the transformed Hamiltonian may be treated, as explain before, in
the TDA or in the RPA basis [18] to describe correlations between pairs of quasiparticles [32]. The terms of the
transformed Hamiltonian, which are relevant to determine the extend of the superfluid correlations, are the following:

i) Constant term $H_{00}$

$$\hat{H}_{00} = \sum_{k \pi \gamma} \tilde{\varepsilon}^{k \pi \gamma} (2 \varepsilon^2_{k \pi \gamma} - 1) \Omega_{k \pi \gamma} - \sum_{\mu_1 \pi_1 \gamma_1} h_{00}(k_i, \pi_i, \gamma_i)$$ (17)
ii) One quasiparticle-term $H_{11}$

$$\hat{H}_{11} = \sum_{k\pi\gamma} \tilde{\epsilon}_{k\pi\gamma}\left(u_{k\pi\gamma}^2 - v_{k\pi\gamma}^2\right) (B_{k\pi\gamma}^\dagger \cdot B_{k\pi\gamma}^{k\pi\gamma} + D_{k\pi\gamma}^{ik\pi\gamma} \cdot D_{k\pi\gamma})$$

$$- \sum_{k\pi\gamma} \left\{ h_{11}(k_i, \pi_i, \gamma_i)B_{k\pi,\gamma}^\dagger \cdot B_{k\pi,\gamma}^{k,\pi,\gamma} + h_{11}(k_i, \pi_i, \gamma_i)D_{k\pi,\gamma}^{ik\pi,\gamma} \cdot D_{k\pi,\gamma}\right\}$$

$$- \sum_{k\pi\gamma} \left\{ h_{11}(k_i, \pi_i, \gamma_i)B_{k\pi,\gamma}^\dagger \cdot B_{k\pi,\gamma}^{k,\pi,\gamma} + h_{11}(k_i, \pi_i, \gamma_i)D_{k\pi,\gamma}^{ik\pi,\gamma} \cdot D_{k\pi,\gamma}\right\}$$

(18)

iii) Two-quasiparticle terms: $H_{20} + H_{02}$

$$H_{20} + H_{02} = \sum_{k\pi\gamma} 2\tilde{\epsilon}_{k\pi\gamma}u_{k\pi\gamma}v_{k\pi\gamma} (B_{k\pi,\gamma}^\dagger \cdot D_{k\pi,\gamma}^{ik\pi,\gamma} + D_{k\pi,\gamma} \cdot B_{k\pi,\gamma}^{k\pi,\gamma})$$

$$- \sum_{k\pi\gamma} \left\{ h_{20}(k_i, \pi_i, \gamma_i)B_{k\pi,\gamma}^\dagger \cdot D_{k\pi,\gamma}^{ik\pi,\gamma} + h_{02}(k_i, \pi_i, \gamma_i)D_{k\pi,\gamma}^{ik\pi,\gamma} \cdot B_{k\pi,\gamma}\right\}$$

$$- \sum_{k\pi\gamma} \left\{ h_{20}(k_i, \pi_i, \gamma_i)B_{k\pi,\gamma}^\dagger \cdot D_{k\pi,\gamma}^{ik\pi,\gamma} + h_{02}(k_i, \pi_i, \gamma_i)D_{k\pi,\gamma}^{ik\pi,\gamma} \cdot B_{k\pi,\gamma}\right\}$$

(19)

The explicit form of the coefficients $h_{ab}(k_i, \pi_i, \gamma_i)$ of the previous equations is given in the Appendix A, for each term of the transformed Hamiltonian.

A. BCS equations.

The terms $H_{11}$ and $H_{20} + H_{02}$ of Eqs. (18) and (19) can be ordered in terms of the following variables

$$X_{k_{1\gamma}} = u_{k_{1\gamma}}^2 - v_{k_{1\gamma}}^2$$

$$Y_{k_{1\gamma}} = 2u_{k_{1\gamma}}v_{k_{1\gamma}}$$

(20)

which depend only on the quasiparticle operator indices, $k_i$. Here, we are using a short-hand notation $k_i = k_{1\pi_i, \gamma_i}$. The interaction terms are also ordered in terms of the structures $(u_{k_2}^2 - v_{k_2}^2)$ and $(u_{k_2}v_{k_2})$, being $k_2$ an internal index. The parameters $u_k$ and $v_k$ are determined self-consistently.

Notice that the resulting ordered Hamiltonian displays terms of the type

$$\left(u_{k_2}^2 - v_{k_2}^2\right)X_{k_1}$$

$$\left(u_{k_2}^2 - v_{k_2}^2\right)Y_{k_1}$$

(21)

and also

$$\left(u_{k_2}v_{k_2}\right)X_{k_1}$$

$$\left(u_{k_2}v_{k_2}\right)Y_{k_1}$$

(22)

When the sum on the internal indices is performed explicitly this structure decouples as explained in the Appendix A, leading to the equations

$$\Sigma_{k_1}X_{k_1} + \Delta_{k_1}Y_{k_1} = E_{k_1}$$

(23)

$$-\Delta_{k_1}X_{k_1} + \Sigma_{k_1}Y_{k_1} = 0$$

(24)
where $\Sigma_{k_1}$, $\Delta_{k_1}$ and the quasiparticle energy $E_{k_1}$ are given by

$$
\Sigma_{k_1} = \varepsilon_{k_1} + \bar{V}_T^{\Sigma} \sum_{k_2, k_3} (u_{k_2}^2 - v_{k_2}^2)
$$

$$
\Delta_{k_1} = \bar{V}_T^{\Delta} \sum_{k_2, k_3} (u_{k_2} v_{k_2})
$$

$$
E_{k_1} = \sqrt{\Sigma_{k_1}^2 + \Delta_{k_1}^2}
$$

with

$$
\bar{V}_T^{\Sigma} = -\frac{1}{2} \sum_{L} \sum_{\lambda_i} \left( \frac{1}{2} \right) \sqrt{\frac{L + 1}{2L + 1}} \frac{(-1)^{L+J_2-J_1}}{2J_1 + 1} \times \left\{ \sum_{N_{l_1 l_2}} V_{(l_1 l_2)(l_3 l_4)\alpha_{l_1 l_2}^l_{l_3 l_4}} \frac{J_1, J_3}{J_1} \times \delta_{\tau_1 \tau_3} \delta_{\tau_1 \tau_4} \delta_{\tau_1 \tau_4} \right\}
$$

$$
\bar{V}_T^{\Delta} = -\frac{1}{2} \sum_{L} \sum_{\lambda_i} \left( \frac{1}{2} \right) \sqrt{\frac{L + 1}{2L + 1}} \frac{(-1)^{L+J_2-J_1}}{2J_1 + 1} \times \left\{ \sum_{N_{l_1 l_2}} V_{(l_1 l_2)(l_3 l_4)\alpha_{l_1 l_2}^l_{l_3 l_4}} \frac{J_1, J_3}{J_1} \times \delta_{\tau_1 \tau_3} \delta_{\tau_1 \tau_4} \delta_{\tau_1 \tau_4} \right\}
$$

where the summation on the internal indices is performed.

These non-linear equations (25) and (28) are then solved for each of the quark-flavors and they are known as state dependent BCS-equations [13, 14], because $\Sigma_{k_1}$ and $\Delta_{k_1}$ depend on the flavor.

The numerical analysis of the matrix elements shows that in the $H_{11}$ term, the matrix elements associated with the structures $(u_{k_2}^2 - v_{k_2}^2) Y_{k_1}$ and $(u_{k_2} v_{k_2}) X_{k_1}$ are very small compared to the matrix elements associated to the structures $(u_{k_2}^2 - v_{k_2}^2) X_{k_1}$ and $(u_{k_2} v_{k_2}) Y_{k_1}$. On the other hand, the numerical analysis of the matrix elements shows that in the $H_{22}$ and $H_{20}$ terms, the matrix elements associated with the structures $(u_{k_2}^2 - v_{k_2}^2) X_{k_1}$ and $(u_{k_2} v_{k_2}) Y_{k_1}$ are very small compared to the matrix elements associated to the structures $(u_{k_2} v_{k_2}) X_{k_1}$ and $(u_{k_2}^2 - v_{k_2}^2) Y_{k_1}$.

The procedure to obtain the solutions of Eqs. (25) and (26) consist of the variation of the quantities $u_{k_2}$ and $v_{k_2}$, such that the iteration stops when the correlation energy $E_{k_1}$ reaches stability.

After introducing these expressions we are in conditions to present and discuss the results of our calculations

**IV. RESULTS AND DISCUSSIONS.**

In this section we shall present the results of our calculations. We have started by: i) studying the effects associated to the renormalization of the parameters entering the definition of the interaction (Coulomb plus linear potential), and, ii) by comparing the spectra resulting from the diagonalization of the interaction with the quasiparticle spectrum, as a function of the cut-off parameter $(N_{cut})$ which gives the size of the radial basis. Next, we have constructed the spectra for meson-like states as uncorrelated two- quasi quark states, for different values of total isospin $T$. The theoretical spectra include states up to 2 GeV and they are compared to the experimental ones [15], for different
values of the angular momentum and parity $J^\pi$. The density of states is shown as a function of $N_{\text{cut}}$. It is worth mentioning that the main purpose of this work is to discuss how feasible is the application of the BCS formalism to treat the non-perturbative regime of QCD. The main aspect of the comparison between calculations and data will focus on the density of states, that is to say we shall investigate if the space of uncorrelated two-quasiparticle states is dense enough to establish connections with data.

A. Renormalization of the interaction and masses.

To absorb any dependence of the configurational space on the cutoff $N_{\text{cut}}$, we have to implement a renormalization procedure. A cutoff in the number of oscillator quanta $N_{\text{cut}}$ is introduced to perform the numerical calculations. Such a truncation is similar to a momentum cutoff regularization. However, instead of a continuous cutoff that truncates the integrals in momentum space, our cutoff is discrete and truncates the space of admissible oscillations. The aim of the renormalization procedure is to keep the eigenvalues of the BCS equations $E_k$ unchanged, and hence the masses of the physical states, invariant under changes of the cutoff $N_{\text{cut}}$. Unfortunately, the exact implementation of such a procedure is very difficult due to the nonlinear dependence of the eigenvalues on the parameters that appear in the Hamiltonian, the bare quark masses $m_u, d$ and $m_s$ and the couplings $V_C$ and $V_L$. Here, we present the renormalization results which, in fact, does succeed in keeping the low-energy meson-like spectrum approximately cutoff-independent.

We shall proceed by studying the dependence of the parameters which enter into the definition of the interaction, which are $V_C$ and $V_L$ of Eq. (2). Figs. 1 and 2 show the dependence of the parameters $V_C$ and $V_L$ upon the value of the cut-off, $N_{\text{cut}}$, of the radial basis. In doing so we have varied both $V_C$ and $V_L$ such that the resulting value of the gap remains constant at the level of approximately 0.2 GeV.

![Graph showing the dependence of the parameter $V_C$ upon the size of the radial basis ($N_{\text{cut}}$). The actual values are represented by dots, the line is to guide the eye. We are using natural units all through the text.](image-url)
FIG. 2: Renormalization of the linear interaction: dependence of the parameter $V_L$ upon the size of the radial basis ($N_{\text{cut}}$). The actual values are represented by dots, the line is to guide the eye.

B. Quasiparticle spectrum.

Proceeding in the same manner, with the couplings of the previous subsection, we have diagonalized the one quasiparticle sector of the Hamiltonian Eqs. 17–19 and solved the BCS equations 23 and 24. The results are shown in Figures 4 and 5.
FIG. 4: Prediagonalization energies (symbols) for $T = \frac{1}{2}$ states and Quasiparticle (solid lines) energies versus $N_{\text{cut}}$.

FIG. 5: Gap solutions for quarks up and down vs $N_{\text{cut}}$.

The same sort of results, for the case of the strange quarks, show that they are insensitive to pairing correlations and their gap is null, see Figure 6.
From the comparison between the energies obtained by diagonalizing the one particle sector of the Hamiltonian and those corresponding to the quasiparticles, we may conclude that in both cases the spectrum reaches a harmonic limit for large values of $N_{\text{cut}}$. The constancy of the gap for the up and down quarks is well illustrated by the results shown in Figures 4 and 5, where $\Delta \approx 0.2 \text{ GeV}$.

1. **Meson $B^D$ spectrum as a function of $N_{\text{cut}}$**

Meson states, of positive and negative parities, are described as two-quasiparticle states. In Figure 7, the spectrum of two-quasiparticles, for the subspace $T = 0, 1$, is shown as a function of $N_{\text{cut}}$. The density of states increases as the number of states in the basis increases. It is seen that the spacing of levels is not regular and that for some energies the spectrum becomes nearly degenerate, a feature which is also observed experimentally, as shown in Figures 5 and 10. The two-quasiparticle spectrum shows a breaking of accidental degeneracies for larger values of $N_{\text{cut}}$. The theoretical and experimental results, for the strange sector, are shown in Figures 9 and 10, respectively.

Considering that the theoretical results have been obtained at the quasiparticle level, that is without including residual interactions between pairs of quasiparticles, the overall tendency of them follows that of the experiment. This is particularly true for the sector of medium and high energies. This is encouraging because the addition of the residual terms of the interaction between pairs of quasiparticles, when treated in the context of non-perturbative linearization methods, like the TDA and RPA approaches, could certainly improve the agreement, as it was the case of the schematic forces used in Ref. [18].

In Figure 7 we show the dependence of the calculated density of states as a function of the cutoff. The figure is not meant to be compared with data but rather show the gross features of the spectrum. It is observed that in the low energy portion of the spectrum, the states are arrange in groups of levels with gaps between the groups.
FIG. 7: Two-Quasiparticle meson-like spectrum, for pairs of up and down quasi-quarks, isospin $T = 0, 1$ states, vs $N_{cut}$.

FIG. 8: Experimental meson spectrum for isospin $T = 0, 1$, up to 2 Gev.
FIG. 9: Quasiparticle meson spectrum, for pairs of up/down and strange quasi-quarks, isospin $T = 1/2$ meson states, vs $N_{\text{cut}}$.

FIG. 10: Experimental meson spectrum for isospin $T = 1/2$, up to 2 Gev.

The overall features of the theoretical results may be summarized in the following.

(a) The effect of the Coulomb interaction is minor compared to the linear term whose dependence upon the dimension of the basis is larger.

(b) The calculated spectra show a sort of pairs-like grouping of levels, a behavior which seems to be confirmed by the experimental data.

(c) The experimental spectra show larger spacing between groups of levels, but this feature is also shown, although at a smaller scale, by the theoretical results.

(d) The theoretical spectra saturate, for larger values of the dimension of the basis.
V. CONCLUSIONS.

In this work we have treated the Coulomb plus linear QCD Hamiltonian by applying non-perturbative techniques which originate in other branches of physics. The procedure was based on the transformation from the quark to the quasiquark basis by applying BCS transformations. A renormalization of the mass and interaction parameters was performed. The stability of the results was tested by increasing the dimension of the radial basis used in the calculations. We have calculated the gaps and quasiparticle energies. The two-quasiparticle configurations, that is mesons and kaons like states, even at this level of approximation show features similar to those exhibited by the experiments. It is expected that going beyond the BCS approximation, by including interactions between pairs of quasiparticles, would allow for a more detailed correspondence between theoretical and experimental results, as done in preliminary studies using schematic models $^3$ $^4$ $^{18}$. Work is in progress about the use of the TDA and RPA methods in this context.

Acknowledgments

P.O.H. acknowledges financial support from PAPIIT-DGAPA (IN100421). O.C acknowledges the support of the CONICET and of the ANPCyT of Argentina (PIP-616).

Appendix A: Interaction Terms.

In this Appendix we are given the explicit expressions of the coefficients of the transformed Hamiltonian Eqs. (17-19).

To begin with, we write the coefficients of the constant term $H_{00}$ Eq. (17), which is:

$$
h_{00}(k_1, \pi_1, \gamma_1) = \frac{1}{2} \sum_{L} \sum_{\lambda_1} \left\{ \frac{i}{2} \right\} \sqrt{8} \sqrt{2L + 1} (-1)^{L-J_1} \times \left[ (\delta_{\pi_1 \pi_2}) (\delta_{\pi_3 \pi_4}) (\delta_{\lambda_1 \lambda_2}) (\delta_{\gamma_1 \gamma_2}) \right]$$

$$\times \left\{ \begin{align*}
&\times \left[ + (\delta_{\pi_1 \pi_2}) (\delta_{\pi_3 \pi_4}) (\delta_{\lambda_1 \lambda_2}) (\delta_{\gamma_1 \gamma_2}) \right] \\
&\times \left[ + (\delta_{\pi_1 \pi_2}) (\delta_{\pi_3 \pi_4}) (\delta_{\lambda_1 \lambda_2}) (\delta_{\gamma_1 \gamma_2}) \right]
\end{align*} \right\} \right\}

(A1)

with

$$\left\{ \begin{align*}
\left\{ \begin{align*}
&\times \delta_{\pi_1 \pi_2} \delta_{\pi_3 \pi_4} \delta_{\lambda_1 \lambda_2} \delta_{\gamma_1 \gamma_2} \delta_{\pi_1 \pi_2} \delta_{\pi_3 \pi_4} \delta_{\lambda_1 \lambda_2} \delta_{\gamma_1 \gamma_2} \\
&\times \delta_{\pi_1 \pi_2} \delta_{\pi_3 \pi_4} \delta_{\lambda_1 \lambda_2} \delta_{\gamma_1 \gamma_2} \delta_{\pi_1 \pi_2} \delta_{\pi_3 \pi_4} \delta_{\lambda_1 \lambda_2} \delta_{\gamma_1 \gamma_2}
\end{align*} \right\}
\right\}

(A2)

and where $\Omega_{k\pi\gamma} = \sum_{\mu} 1 = \sum_{m,j,c} 1$ is the degeneracy of the state $k\pi\gamma$. The $\delta_{\pm\pm\pm}$ terms are a short hand for

$$\delta_{\lambda_1 \lambda_2 \lambda_3 \lambda_4} \delta_{\gamma_1 \gamma_2 \gamma_3 \gamma_4}

(A3)$$
Similarly, the four terms which appear in the definition of \( \hat{H}_{11} \), Eq. (13) are: coefficient in front of \( B_{k_1}^I B_{k_2}^4 \)

\[
\begin{align*}
\frac{1}{2} \sum_{L} \sum_{\lambda_1} \left( \frac{1}{2} \right) \frac{\sqrt{8(2L+1)}}{9} \left\{ \left( \delta_{\lambda_1 \lambda_4} \right) \left( \delta_{k_2 k_3} \right) \left( \delta_{J_2 J_4} \right) \left( \delta_{T_2 T_3 T_4} \right) \left( \delta_{Y_2 Y_3 Y_4} \right) \right. \\
- \left(v_{k_2, \pi_2, \gamma_2}^2 \right) \times \left[ \left( \delta_{++-} \right) \left( \delta_{k_1, \gamma_1} \right) \left( \delta_{k_4, \gamma_1} \right) \right] \\
+ \left(v_{k_2, \pi_2, \gamma_2}^2 \right) \times \left[ \left( \delta_{--} \right) \left( \delta_{k_1, \gamma_1} \right) \left( \delta_{k_4, \gamma_1} \right) \right] \\
\left( \delta_{--} \right) \left( \delta_{k_1, \gamma_1} \right) \left( \delta_{k_4, \gamma_1} \right) \\
\right) \\
\end{align*}
\]

coefficient in front of \( D_{k_1}^{I_4} \) is

\[
\begin{align*}
\frac{1}{2} \sum_{L} \sum_{\lambda_1} \left( \frac{1}{2} \right) \frac{\sqrt{8(2L+1)}}{9} \left\{ \left( \delta_{\lambda_1 \lambda_4} \right) \left( \delta_{k_2 k_3} \right) \left( \delta_{J_2 J_4} \right) \left( \delta_{T_2 T_3 T_4} \right) \left( \delta_{Y_2 Y_3 Y_4} \right) \right. \\
- \left(v_{k_2, \pi_2, \gamma_2}^2 \right) \times \left[ \left( \delta_{++-} \right) \left( \delta_{k_1, \gamma_1} \right) \left( \delta_{k_4, \gamma_1} \right) \right] \\
+ \left(v_{k_2, \pi_2, \gamma_2}^2 \right) \times \left[ \left( \delta_{--} \right) \left( \delta_{k_1, \gamma_1} \right) \left( \delta_{k_4, \gamma_1} \right) \right] \\
\left( \delta_{--} \right) \left( \delta_{k_1, \gamma_1} \right) \left( \delta_{k_4, \gamma_1} \right) \\
\right) \\
\end{align*}
\]

coefficient in front of \( B_{k_1}^I B_{k_2}^2 \)

\[
\begin{align*}
\frac{1}{2} \sum_{L} \sum_{\lambda_1} \left( \frac{1}{2} \right) \frac{\sqrt{8(2L+1)}}{9} \left\{ \left( \delta_{\lambda_1 \lambda_4} \right) \left( \delta_{k_2 k_3} \right) \left( \delta_{J_2 J_4} \right) \left( \delta_{T_2 T_3 T_4} \right) \left( \delta_{Y_2 Y_3 Y_4} \right) \right. \\
- \left(v_{k_2, \pi_2, \gamma_2}^2 \right) \times \left[ \left( \delta_{++-} \right) \left( \delta_{k_1, \gamma_1} \right) \left( \delta_{k_4, \gamma_1} \right) \right] \\
+ \left(v_{k_2, \pi_2, \gamma_2}^2 \right) \times \left[ \left( \delta_{--} \right) \left( \delta_{k_1, \gamma_1} \right) \left( \delta_{k_4, \gamma_1} \right) \right] \\
\left( \delta_{--} \right) \left( \delta_{k_1, \gamma_1} \right) \left( \delta_{k_4, \gamma_1} \right) \\
\right) \\
\end{align*}
\]

(A4)

(A5)

(A6)
coefficient in front of $D^{jk_2}D_{k_3}$

$$h_{11}(k_i, \pi_i, \gamma_i)$$

$$= \frac{1}{2} \sum_{L} \sum_{\lambda_i} \left( \frac{1}{2} \right) \frac{\sqrt{8(2L + 1)}}{9} \left( -1 \right)^{L+J_z-J_x} \left\{ \left( \delta_{\pi_2,\pi_3} \right) \left( \delta_{k_2,k_3} \delta_{\pi_i,\pi_x} \right) \left( \delta_{J_1, J_4} \delta_{J_2, J_3} \right) \left( \delta_{T_2, T_1} \delta_{T_2, T_3} \delta_{T_3, T_4} \right) \left( \delta_{Y_5, Y_3} \delta_{Y_5, Y_4} \right) \right\}$$

$$\times \left\{ \left( u_{k_2,\pi_2,\gamma_2} \right) \left[ \left( \delta_{\pi_2,\pi_3} \right) \left( \delta_{k_2,k_3} \delta_{\pi_i,\pi_x} \right) \left( \delta_{J_1, J_4} \delta_{J_2, J_3} \right) \left( \delta_{T_2, T_1} \delta_{T_2, T_3} \delta_{T_3, T_4} \right) \left( \delta_{Y_5, Y_3} \delta_{Y_5, Y_4} \right) \right\}$$

$$\times \left\{ \left( u_{k_2,\pi_2,\gamma_2} \right) \left[ \left( \delta_{\pi_2,\pi_3} \right) \left( \delta_{k_2,k_3} \delta_{\pi_i,\pi_x} \right) \left( \delta_{J_1, J_4} \delta_{J_2, J_3} \right) \left( \delta_{T_2, T_1} \delta_{T_2, T_3} \delta_{T_3, T_4} \right) \left( \delta_{Y_5, Y_3} \delta_{Y_5, Y_4} \right) \right\}$$

Finally, the terms which appear in $\bar{H}_2 + \bar{H}_2$, Eq. [19], are written as: the coefficient in front of $B_{k_1}^{1}D^{jk_1}$

$$h_{20}(k_i, \pi_i, \gamma_i)$$

$$= \frac{1}{2} \sum_{L} \sum_{\lambda_i} \left( \frac{1}{2} \right) \frac{\sqrt{8(2L + 1)}}{9} \left( -1 \right)^{L+J_z-J_x} \left\{ \left( \delta_{\pi_1,\pi_4} \right) \left( \delta_{k_2,k_3} \delta_{\pi_i,\pi_x} \right) \left( \delta_{J_2, J_3} \delta_{J_1, J_4} \right) \left( \delta_{T_2, T_1} \delta_{T_2, T_3} \delta_{T_3, T_4} \right) \left( \delta_{Y_5, Y_2} \delta_{Y_5, Y_3} \delta_{Y_5, Y_4} \right) \right\}$$

coefficient in front of $D_{k_1}B^{k_4}$

$$h_{02}(k_i, \pi_i, \gamma_i)$$

$$= \frac{1}{2} \sum_{L} \sum_{\lambda_i} \left( \frac{1}{2} \right) \frac{\sqrt{8(2L + 1)}}{9} \left( -1 \right)^{L+J_z-J_x} \left\{ \left( \delta_{\pi_1,\pi_4} \right) \left( \delta_{k_2,k_3} \delta_{\pi_i,\pi_x} \right) \left( \delta_{J_2, J_3} \delta_{J_1, J_4} \right) \left( \delta_{T_2, T_1} \delta_{T_2, T_3} \delta_{T_3, T_4} \right) \left( \delta_{Y_5, Y_2} \delta_{Y_5, Y_3} \delta_{Y_5, Y_4} \right) \right\}$$

(A7)

(A8)

(A9)
coefficient in front of $B_{k_3}^t D_{1k_2}^{t}$

$$h_{20}(k_1, \pi_1, \gamma_1) = \frac{1}{2} \sum_{\lambda_i} \left( \frac{1}{2} \right) \frac{\sqrt{8(2L+1)}}{9} \frac{\sqrt{-L+J_2-J_1}}{2J_1+1} \left\{ (\delta_{\pi_2 \pi_3}) (\delta_{k_1 \pi_1 \delta_{\pi_1 \pi_4}})(\delta_{J_1 J_2 J_3})(\delta_{T_2 T_1} \delta_{T_4 T_3} \delta_{T_1 T_4})(\delta_{Y_2 Y_1} \delta_{Y_4 Y_3} \delta_{Y_1 Y_4}) \right\} \times \left\{ (u_{k_1 \pi_1 \gamma_1})^2 \times \left[ (\delta_{--}) (v_{k_2 \pi_3 \gamma_3} v_{k_3 \pi_3 \gamma_3}) - (\delta_{---}) (v_{k_2 \pi_3 \gamma_3} v_{k_3 \pi_3 \gamma_3}) \right] + \left( \delta_{++--} \right) (v_{k_2 \pi_3 \gamma_3} v_{k_3 \pi_3 \gamma_3}) \right\} - \left\{ (v_{k_1 \pi_1 \gamma_1})^2 \times \left[ (\delta_{++--} \left( v_{k_2 \pi_3 \gamma_3} v_{k_3 \pi_3 \gamma_3} \right) \right] + \left( \delta_{++--} \right) (v_{k_2 \pi_3 \gamma_3} v_{k_3 \pi_3 \gamma_3}) \right\} + \left( u_{k_1 \pi_1 \gamma_1} v_{k_1 \pi_1 \gamma_1} \right) \times \left[ \left( \delta_{++--} \left( v_{k_2 \pi_3 \gamma_3} v_{k_3 \pi_3 \gamma_3} \right) \right] + \left( \delta_{++--} \right) (v_{k_2 \pi_3 \gamma_3} v_{k_3 \pi_3 \gamma_3}) \right\} \right\}, \tag{A10}
$$

coefficient in front of $D_{k_3} B^{k_2}$

$$h_{02}(k_1, \pi_1, \gamma_1) = \frac{1}{2} \sum_{\lambda_i} \left( \frac{1}{2} \right) \frac{\sqrt{8(2L+1)}}{9} \frac{\sqrt{-L+J_2-J_1}}{2J_1+1} \left\{ (\delta_{\pi_2 \pi_3}) (\delta_{k_1 \pi_1 \delta_{\pi_1 \pi_4}})(\delta_{J_1 J_2 J_3})(\delta_{T_2 T_1} \delta_{T_4 T_3} \delta_{T_1 T_4})(\delta_{Y_2 Y_1} \delta_{Y_4 Y_3} \delta_{Y_1 Y_4}) \right\} \times \left\{ (u_{k_1 \pi_1 \gamma_1})^2 \times \left[ (\delta_{--}) (v_{k_2 \pi_3 \gamma_3} v_{k_3 \pi_3 \gamma_3}) + (\delta_{---}) (v_{k_2 \pi_3 \gamma_3} v_{k_3 \pi_3 \gamma_3}) \right] \right\} - \left\{ (v_{k_1 \pi_1 \gamma_1})^2 \times \left[ (\delta_{++--} \left( v_{k_2 \pi_3 \gamma_3} v_{k_3 \pi_3 \gamma_3} \right) \right] + \left( \delta_{++--} \right) (v_{k_2 \pi_3 \gamma_3} v_{k_3 \pi_3 \gamma_3}) \right\} + \left( u_{k_1 \pi_1 \gamma_1} v_{k_1 \pi_1 \gamma_1} \right) \times \left[ \left( \delta_{++--} \left( v_{k_2 \pi_3 \gamma_3} v_{k_3 \pi_3 \gamma_3} \right) \right] + \left( \delta_{++--} \right) (v_{k_2 \pi_3 \gamma_3} v_{k_3 \pi_3 \gamma_3}) \right\} \right\}. \tag{A11}
$$

In the above equations the $\{ \}$ is the same in Eq. A2.

These matrix elements contain different combinations of $\delta_{++--}$-terms (see Eq. A3), and the product of the transformation coefficients $\prod_{L} \alpha_{\gamma_i}^{J_i} (N_{L}, \lambda_i, \pi_i, k_i)$ of Eq. A2. Therefore, the matrix elements retain the information of the effective quark and antiquark degrees of freedom, as well as the associated symmetry properties. Because the prediagonalization requires a numerical procedure, which depends on the value of $N_{cut}$, i.e. the dimension of the configurational space, we have verified the stability of the results by changing $N_{cut}$ in the range $N_{cut} \leq 11$.

The numerical analysis shows that the following relations hold

$$\delta_{1234} = \delta_{4321}, \tag{A12}$$

and in the case of the substitution of quark for antiquark

$$\delta_{++++} = \delta_{---} \tag{A13} $$

are exact symmetries.

However, when changing quark for antiquark but with the same parity, the following relations

$$\delta_{++++} \approx \delta_{---} \tag{A14}$$
and

\[
\delta_{--+} \approx \delta_{+-+} \quad (A15)
\]

are approximately fulfilled with a maximal deviation of the order of 4% with respect to average value \( \frac{\delta_{++-} + \delta_{----}}{2} \) and \( \frac{\delta_{++-} + \delta_{----}}{2} \), respectively. This last symmetry is not a particle-antiparticle symmetry, because for that also the parity has to be changed. Nevertheless, these deviations decreases as the maximal number of quanta \( N_{\text{cut}} \) increases. In some cases the symmetry is restored up to 0.1%.

[1] S. Weinberg, *The Quantum Theory of Fields* (Vol. II, Cambridge University Press, 1996).
[2] T. D. Lee, *Particle Physics and Introduction to Field Theory* (Harwood Academic Publishers, New York, 1981).
[3] T. Yépez-Martínez, O. Civitarese and P. O. Hess, Int. J. Mod. Phys. E 25, 1650067 (2016).
[4] T. Yépez-Martínez, O. Civitarese and P. O. Hess, Int. J. Mod. Phys. E 26, 1750012 (2017).
[5] U. I. Ramirez-Soto, O. A. Rico-Trejo, T. Yépez-Martínez, P. O. Hess, A. Weber and O. Civitarese, J. Phys. G: Nucl. Part. Phys. 48, 085013 (2021).
[6] N. H. Christ and T. D. Lee, Phys. Rev. D 22, 939 (1980).
[7] A. Szczepaniak, E. S. Swanson, C. R. Ji and S. R. Cotanch, Phys. Rev. Lett. 76, 2011 (1996).
[8] A. P. Szczepaniak and E. Swanson, Phys. Rev. D 65, 025012 (2001).
[9] C. Feuchter and H. Reinhardt, Phys. Rev. D 70, 105021 (2004).
[10] H. Reinhardt, D. R. Campagnari and A. P. Szczepaniak, Phys. Rev. D 84, 045006 (2011).
[11] T. Yépez-Martínez, A. P. Szczepaniak and H. Reinhardt, Phys. Rev. D 86, 076010 (2012).
[12] J. Greensite and A. P. Szczepaniak, Phys. Rev. D 91, 034503.
[13] J. Greensite and A. P. Szczepaniak, Phys. Rev. D 93, 074506.
[14] P. Ring and P. Schuck, *The Nuclear Many Body Problem* (Springer, Heidelberg, 1980).
[15] A. L. Fetter and J. D. Walecka, *Quantum Theory of Many-Particle Systems* (Dover, New York, 2003).
[16] P. O. Hess and A. P. Szczepaniak, Phys. Rev. C 73, 025201 (2006).
[17] T. Yépez-Martínez, P. O. Hess, A. P. Szczepaniak and O. Civitarese, Phys. Rev. C 81, 045204 (2010).
[18] D. A. Amor-Quiroz, T. Yépez-Martínez, P. O. Hess, O. Civitarese, and A. Weber, Int. J. Mod. Phys. E 26, 1750082 (2017).
[19] P. Bicudo, M. Cardoso, F. J. Llanes-Estrada and T. V. Cauteren, Phys. Rev. D 94, 054006 (2016).
[20] J. R. Finger and J. E. Mandula, Nucl. Phys. B199, 168 (1982).
[21] S. L. Adler and A. C. Davis, Nucl. Phys. B244, 469 (1984).
[22] A. Le Yaouanc, L. Oliver, O. Pêne, and J.-C. Raynal, Phys. Rev. D 29, 1233 (1984).
[23] P. J. de A. Bicudo and J. E. F. T. Ribeiro, Phys. Rev. D 42, 1611 (1990).
[24] F. J. Llanes-Estrada and S. R. Cotanch, Phys. Rev. Lett. 84, 1102 (2000).
[25] F. J. Llanes-Estrada and S. R. Cotanch, Nucl. Phys. A 697, 303 (2002).
[26] A. V. Nefediev, J. E. F. T. Ribeiro, and A. P. Szczepaniak, JETP Lett. 87, 271 (2008).
[27] D. Zwanziger, Phys. Rev. Lett. 90, 102001 (2003).
[28] J. Greensite and S. Olejnik, Phys. Rev. D 67, 094503 (2003).
[29] P. Guo, A. P. Szczepaniak, G. Galat’ a, A. Vassallo, and E. Santopinto, Phys. Rev. D 78, 056003 (2008).
[30] Edited by R. A. Broglia and V. Zelevinsky, Fifty years of nuclear BCS (World Scientific, 2020).
[31] D. R. Bes and G. G. Dusuel, Nucl. Phys. A 135 1 (1969).
[32] T. Yépez-Martínez, P. O. Hess and O. Civitarese, work in progress to be published.
[33] P. A. Zyla et al. (Particle Data Group), Prog. Theor. Exp. Phys. 2020, 083C01 (2020).