Difference Nullstellensatz

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Abstract

We prove different forms of Nullstellensatz for difference fields and absolutely flat simple difference rings. A difference ring is supposed to be a ring on which an arbitrary group is acting by ring automorphisms.

1 Introduction

The work is devoted to different forms of Nullstellensatz. First of all we note that a difference ring is a ring provided with an action of an arbitrary group. The second thing that should be noted is that we consider arbitrary many difference indeterminates.

In the first section 2 we give all necessary definitions and describe notation. It should be noted that a difference ring is a ring on which an arbitrary group is acting by ring automorphisms. Section 3 is devoted to discussion of different forms of Nullstellensatz. We give different conditions on difference rings, these conditions can be treated as forms of Nullstellensatz. We attend to the class of difference rings satisfying these statements. There are two different classes of difference rings under consideration. The first one is the class of all difference fields and the second one is the class of all absolutely flat simple difference rings (these rings are called pseudofields). In the following sections we discuss these two classes. Section 4 is devoted to the fields. It consists of two parts. In subsection 4.1 we discuss the case of finitely many difference indeterminates and in the subsection 4.2 we generalize our results to the case of infinitely many difference indeterminates. The main results are theorems 6 and 8. In the case of difference fields the results of section 4 can be improved. Namely, we are able to prove strong Nullstellensatz. Section 5 is devoted to such generalization. Theorem 10 of subsection 5.1 is devoted to the case of finitely many indeterminates and in theorem 12 of subsection 5.2 we consider infinitely many indeterminates. Section 6 is devoted to pseudofields on which infinite group is acting. The case of finite group is scrutinized in [4], where more strong results are obtained. In subsection 6.1 as in the case of fields we discuss difference polynomials in finitely many difference indeterminates. The case of pseudofields in contrast to the case of fields needs some technical preparation. The main result is theorem 19. The next subsection 6.2 is devoted to generalization of results about pseudofields to the case of infinitely many difference indeterminates. The section is completed by general theorem 22.
2 Terms and notation

Throughout the text the word ring means an associative commutative ring with an identity element. All homomorphisms preserve the identity element. A ring will be said to be a difference ring if there is a group \( \Sigma \) acting on the given ring by ring automorphisms. Difference homomorphism is a homomorphism commuting with the action of \( \Sigma \). We shall often replace the word difference by symbol \( \Sigma \). Simple difference ring is a difference ring with no nontrivial difference ideals. A difference ring will be called a pseudofield if it is an absolutely flat simple difference ring.

The set of all, radical, prime, maximal ideals of \( A \) will be denoted by \( \text{Id}_A, \text{Rad}_A, \text{Spec}_A, \text{Max}_A \), respectively. The set of all difference ideals of \( A \) will be denoted by \( \text{Id}_\Sigma A \). \( \text{Rad}_\Sigma A \) will denote the set of all radical difference ideals of \( A \).

Let us recall the definition of pseudoprime ideal. Let \( S \subseteq A \) be a multiplicatively closed subset and let \( q \) be a maximal \( \Sigma \)-ideal not meeting \( S \). In this case the ideal \( q \) will be called a pseudoprime ideal. The set of all pseudoprime ideals will be denoted by \( \text{PSpec}_A \) and will be called a pseudospectrum of \( A \).

If \( S \) is multiplicatively closed subset in \( A \), the fraction ring of \( A \) with respect to \( S \) will be denoted by \( S^{-1}A \). If \( S = \{ t^n \}_{n=0}^\infty \), then \( S^{-1}A \) will be denoted by \( A_t \). If \( p \) is a prime ideal of \( A \) and \( S = A \setminus p \), then \( S^{-1}A \) will be denoted by \( A_p \).

The residue field of prime ideal \( p \) is a field of fraction of \( A/p \) and is denoted by \( K(p) \). If \( A \) is an integral domain, the fraction field of \( A \) will be denoted by \( \text{Qt}(A) \).

For every subset \( X \) of a difference ring \( A \) the smallest difference ideal containing \( X \) will be denoted by \( [X] \) and the smallest perfect difference ideal containing \( X \) will be denoted by \( \{ X \} \) (see definition [2, chapter 2, sec. 2.3, def. 2.3.1]). The radical of ideal \( a \) will be denoted by \( \text{r}(a) \). The following inclusion always holds \( \text{r}([X]) \subseteq \{ X \} \).

Let \( f: A \to B \) be a homomorphism of rings, and let \( a \) and \( b \) be ideals of \( A \) and \( B \), respectively. We define the extension \( a^e \) of \( a \) to be the ideal \( f(a)B \) generated by \( f(a) \) in \( B \). The ideal \( f^*(b) = f^{-1}(b) \) is called the contraction of \( b \) and is denoted by \( b^c \). If a homomorphism \( f: A \to B \) is difference and ideals \( a \) and \( b \) are difference, then their extensions and contractions are difference.

A ring of difference polynomials \( A\{Y\} \) over \( A \) in variables \( Y \) is a polynomial ring \( A[\Sigma Y] \), where \( \Sigma \) is acting in natural way. Difference ring \( B \) will be called a difference \( A \)-algebra if there is a difference homomorphism \( A \to B \). It is clear that every difference \( A \)-algebra can be presented as a quotient ring of a polynomial ring over \( A \) in sufficiently many indeterminates.

Let us recall the notion of the ring \( F_B \). For an arbitrary commutative ring \( B \) the set of all functions from \( \Sigma \) to \( B \) will be denoted by \( F_B \), i.e. \( F_B = B^\Sigma \). As a commutative ring it coincides with \( \prod_{\sigma \in \Sigma} B \). The group \( \Sigma \) is acting on \( F_B \) by the following rule \( \sigma(f)(\tau) = f(\sigma^{-1}\tau) \). For every element \( \sigma \) of \( \Sigma \) there is a
substitution homomorphism
\[ \gamma_\sigma : F B \to B \]
\[ f \mapsto f(\sigma) \]

It is clear that \( \gamma_\tau(\sigma f) = \gamma_{\sigma^{-1}\tau}(f) \).

The following statement is proved in [4].

**Statement 1.** Let \( A \) be a difference ring and \( \varphi : A \to B \) be a homomorphism. Then for every \( \sigma \in \Sigma \) there exists a unique difference homomorphism \( \Phi_\sigma : A \to F B \) such that the following diagram is commutative

\[
\begin{array}{ccc}
F B & \xrightarrow{\Phi_\sigma} & B \\
\downarrow{\gamma_\sigma} & & \\
A & \xrightarrow{\varphi} & B \\
\end{array}
\]

3 Posing the problem

In this section we shall discuss different kinds of Nullstellensatz appeared in difference algebra and prepare some statements for further using.

First of all we consider the most interesting statement. We formulate a condition on an arbitrary difference ring \( A \).

\( (A1) \): For every natural \( n \) and every proper difference ideal \( a \subseteq A\{y_1, \ldots, y_n\} \) there is a common zero for \( a \) in \( A^n \).

Note that the condition \( (A1) \) implies the simplicity of a ring \( A \). Indeed, if \( a \) is a nontrivial difference ideal in \( A \), then ideal \( a\{y_1, \ldots, y_n\} \) has no common zeros in \( A^n \) and is nontrivial.

Consider another statement about difference ring \( A \).

\( (A2) \): For every natural number \( n \) and every subset \( E \subseteq A\{y_1, \ldots, y_n\} \) if for some ring \( B \) containing \( A \) there exists a common zero for \( E \) in \( B^n \), then there exists a common zero for \( E \) in \( A^n \).

Both these statements are equivalent.

**Statement 2.** Conditions \( (A1) \) and \( (A2) \) are equivalent.

*Proof.* Let us show that \( (A1) \Rightarrow (A2) \). Let for some subset

\[ E \subseteq A\{y_1, \ldots, y_n\}. \]

There is a ring \( B \) containing \( A \) such that there is a common zero for \( E \) in \( B^n \).

Consider a substitution homomorphism

\[ A\{y_1, \ldots, y_n\} \to B \]
for this common zero. Then its kernel is proper difference ideal containing \( E \). Therefore ideal \( [E] \) is proper difference ideal, then (\( A1 \)) implies that there is a common zero in \( A^n \).

(\( A2 \)) \( \Rightarrow \) (\( A1 \)). Conversely, let \( a \subseteq A\{y_1, \ldots, y_n\} \) be a proper difference ideal. Then there exists a common zero for \( a \) in \( A\{y_1, \ldots, y_n\}/a \). Then from (\( A2 \)) it follows that there is a common zero for \( a \) in \( A^n \).

We are interested in difference rings satisfying these equivalent conditions. But before, let us note that statement (\( A1 \)) depends on the class of all difference ideal. On default we consider all difference ideals in the ring of difference polynomials. If we shall consider only perfect difference ideals we obtain a little bit another statement. From the other hand, statement (\( A2 \)) depends on the class of all difference rings. We can change it by using only integral domains instead of all rings. Consider new statements.

(\( B1 \)): For every natural number \( n \) and every proper perfect difference ideal \( a \subseteq A\{y_1, \ldots, y_n\} \) there exists a common zero for \( a \) in \( A^n \).

(\( B2 \)): For every natural number \( n \) and every subset \( E \subseteq A\{y_1, \ldots, y_n\} \) if for some difference integral domain \( B \) containing \( A \) there is a common zero for \( E \) in \( B^n \) there is a common zero for \( E \) in \( A^n \).

(\( B2' \)): For every natural number \( n \) and every subset \( E \subseteq A\{y_1, \ldots, y_n\} \) if for some difference field \( B \) containing \( A \) there is a common zero for \( E \) in \( B^n \), then there is common zero for \( E \) in \( A^n \).

From statement (\( B2 \)) it follows that \( A \) is an integral domain. Note the following.

Statement 3. If \( A \) is a difference integral domain then statements (\( B1 \)), (\( B2 \)), and (\( B2' \)) are equivalent.

Proof. The equivalence between (\( B2 \)) and (\( B2' \)) follows from the fact that every difference integral domain can be embedded into difference field, namely, its field of fraction.

Let us show that (\( B1 \)) \( \Rightarrow \) (\( B2 \)). We need to repeat the proof of previous statement. Assume that for some set

\[
E \subseteq A\{y_1, \ldots, y_n\}
\]

there exists a difference integral domain \( B \) containing \( A \) such that there is a common zero for \( E \) in \( B^n \). Consider substitution homomorphism

\[
A\{y_1, \ldots, y_n\} \rightarrow B
\]

for this common zero. Then its kernel is a nontrivial perfect ideal containing \( E \). Therefore the ideal \( \{E\} \) is not trivial and from (\( B1 \)) it follows that there is a common zero in \( A^n \).
(B2)⇒(B1). Conversely, let \( t \subseteq A\{y_1, \ldots, y_n\} \) be a proper perfect difference ideal. Then there exists a maximal perfect ideal \( p \) containing \( t \). It is known that \( p \) is a prime ideal (the proof is analogous to \([2, \text{chapter 2, sec. 2.3, prop. 2.3.4}]\)). There is a common zero for \( t \) in an integral domain \( A\{y_1, \ldots, y_n\}/p \). Then from (B2) it follows that there is a common zero for \( t \) in \( A^n \).

In further text we are interested in conditions (A1) and (A2) in the class of all pseudofields and in conditions (B1), (B2), and (B2') in the class of all difference fields. The fields (pseudofields) satisfying these conditions will be called differently closed. We shall attract attention to the following two problems. The question of existence of differently closed fields (pseudofields) and the question whether difference field (pseudofield) can be embedded in to differently closed field (pseudofield). Additionally, we consider the case of infinitely many indeterminates.

4 Fields case

4.1 Finitely many indeterminates

The countable cardinal will be denoted by \( \omega \).

**Statement 4.** Let \( K \) be a field. Consider a polynomial ring \( K[Y] \), where \( Y \) is an arbitrary set of indeterminates. Then every ideal of \( K[Y] \) is generated by the set of cardinality not greater than \( |Y| \times \omega \).

**Proof.** If the set \( Y \) is finite, then every ideal is finitely generated and thus the statement is true. Now we suppose that \( Y \) is infinite set. Then \( |Y| \times \omega = |Y| \). Let the set of indeterminates \( Y = \{y_\alpha\} \) be well-ordered. Then by \( Y_\gamma \) we shall denote the following set \( \{y_\alpha \in Y \mid y_\alpha < y_\gamma\} \).

Consider subsets \( Y_0, Y_1, \ldots, Y_n, \ldots, Y_\omega \). Let \( a \subseteq K[Y_\omega] \) be an ideal, we set \( a_\alpha = a \cap K[Y_\alpha] \). It is clear that \( a = \bigcup_\alpha a_\alpha \). Every ideal \( a_\alpha \) is finitely generated. Consequently, their union is not more than countably generated. In other words, every ideal of \( K[Y_\omega] \) is not more than countably generated.

Using transfinite induction we shall prove that every ideal in \( K[Y_\gamma] \) is generated by the set of cardinality not greater than \( |\gamma| \). Consider the case of not limit ordinal. Let all ideals in \( K[Y_\alpha] \) are generated by the set of elements of cardinality not greater than \( |\alpha| \). Then the rings \( K[Y_\alpha] \) and \( K[Y_{\alpha+1}] \) are isomorphic to each other. Therefore every ideal in \( K[Y_{\alpha+1}] \) is generated by the set of elements of cardinality not greater than \( |\alpha| = |\alpha+1| \). Consider the case of limit ordinal. Let \( \gamma = \cup_{\alpha<\gamma} \alpha \), then \( Y_\gamma = \cup_{\alpha<\gamma} Y_\alpha \). For every ideal \( a \subseteq K[Y_\gamma] \) there is the ideal \( a_\alpha = a \cap K[Y_\alpha] \). By the induction hypothesis the ideals \( a_\alpha \) are generated by the sets \( E_\alpha \) such that \( |E_\alpha| \leq |\alpha| \). Then \( a \) is generated by the set \( E = \cup_{\alpha<\gamma} E_\alpha \). Using \([3, \text{chapter 8, sec. 8.3}]\) we get

\[
|E| \leq \bigcup_{\alpha<\gamma} |E_\alpha| \leq \bigcup_{\alpha<\gamma} |\alpha| \leq |\gamma| \times |\gamma| = |\gamma|
\]

As we can see the cardinality of \( E \) is not greater than \( |\gamma| \). \( \square \)
Statement 5. For every field $K$ in a polynomial ring $K\{y_1, \ldots, y_n\}$ every ideal generated by the set of cardinality not greater than $|\Sigma| \times \omega$.

Proof. Let us note that the ring $K\{y_1, \ldots, y_n\}$ coincides with $K[\Sigma y_i]$. So the cardinality of the set of all indeterminates is not greater than $|\Sigma| \times n \leq |\Sigma| \times \omega$. Now the statement follows from statement 4.

Theorem 6. Every difference field can be embedded into a differentially closed field.

Proof. Construction. Let the field $K$ be given. The set of all differentially finitely generated integral domains over $K$ up to isomorphism will be denoted by $B$. Consider the following tensor product $\otimes_{B \in B} \text{Qt}(B)$ (all not denoted tensor products are taken over $K$). Let us construct a particular ordered set consisting of the following pairs

$\left( \otimes_{B \in B'} \text{Qt}(B), p_{B'} \right)$,

where $B' \subseteq B$ be some subset in $B$ and $p_{B'}$ be a prime difference ideal in $\otimes_{B \in B'} \text{Qt}(B)$. This set is particular ordered set with respect to the following order

$\left( \otimes_{B \in B'} \text{Qt}(B), p_{B'} \right) \leq \left( \otimes_{B \in B''} \text{Qt}(B), p_{B''} \right) \iff B' \subseteq B'', p_{B''} \cap \otimes_{B \in B'} \text{Qt}(B) = p_{B'}$.

It is easy to see that we can apply Zorn’s lemma. Consequently, there exists a maximal element. Let

$\left( \otimes_{B \in \mathcal{B}} \text{Qt}(B), p_{\mathcal{B}} \right)$

be a maximal element in this set. The residue field of the ideal $p_{\mathcal{B}}$ will be denoted by $K_1$. Now we can repeat this construction with the field $K_1$ instead of $K$. Using transfinite induction we are able to define the field $K_\alpha$ for every ordinal $\alpha$. Let $\kappa$ be the first cardinal greater than $|\Sigma| \times \omega$. Let the field $K_\kappa$ be denoted by $L$. We shall show that $L$ is the desired field.

Difference closeness. Let $t$ be a proper perfect ideal in the polynomial ring $L\{y_1, \ldots, y_n\}$. Then it is contained in some prime difference ideal $p$ (the proof is similar to [2] chapter 2, sec. 2.3, prop. 2.3.4)). The ring

$R = L\{y_1, \ldots, y_n\}/p$.

will be denoted by $R$. From statement 5 it follows that ideal $p$ is generated by the set of cardinality less then or equal to $|\Sigma| \times \omega$. Let $E$ denotes the set of generators of $p$, so $p[E]$, where $|E| \leq |\Sigma| \times \omega$. Since $|\Sigma| \times \omega < \kappa$, all elements of $E$ are defined over an intermediate subfield $K_\alpha$ with $\alpha < \kappa$.

The difference ring $K_\alpha\{y_1, \ldots, y_n\}/[E]$ will be denoted by $R_0$. Then

$R = L \otimes_{K_\alpha} R_0 = L \otimes_{K_{\alpha+1}} K_\alpha \otimes_{K_\alpha} R_0$. 6
Consequently, \( R_0 \) can be embedded to \( R \). Hence \( R_0 \) is an integral domain. Now consider the field \( K_{\alpha + 1} \). It is clear that \( K_{\alpha + 1} \otimes R_0 \) also can be embedded to \( R \), and hence is an integral domain.

Let us show that there exists a difference homomorphism from \( R_0 \) to \( K_{\alpha + 1} \). For that it suffices to find a difference subring in \( K_{\alpha + 1} \) isomorphic to \( R_0 \). Let \( B \) be a family of all differenly finitely generated integral domains over \( K_{\alpha} \). Then it can be presented as a union \( B = B' \sqcup B'' \), where

\[
K_{\alpha + 1} = \mathrm{Qt} \left( \left( \bigotimes_{B \in B'} \mathrm{Qt}(B) \right) / \mathfrak{p}_{B'} \right).
\]

Algebra \( R_0 \) belongs to \( B \) by the data. Let us show that \( R_0 \) belongs to \( B' \). Suppose that contrary holds, consider the ring \( \bigotimes_{B \in B'} \mathrm{Qt}(B) \otimes \mathrm{Qt}(R_0) \) and its ideal \( \mathfrak{p}_{B'} \otimes \mathrm{Qt}(R_0) \). Then the quotient ring

\[
\left( \bigotimes_{B \in B'} \mathrm{Qt}(B) \right) / \mathfrak{p}_{B'} \otimes \mathrm{Qt}(R_0)
\]

can be embedded to \( K_{\alpha + 1} \otimes \mathrm{Qt}(R_0) \). So we have a contradiction with maximality of family \( B' \). Consequently \( K_{\alpha + 1} \) contains \( \mathrm{Qt}(R_0) \) and hence \( R_0 \).

Let us produce a difference homomorphism from \( R \) to \( L \). We use the following decomposition

\[
R = L \otimes_{K_{\alpha}} R_0.
\]

To get the embedding of \( R \) to \( L \) we need to find the embeddings of \( L \) to \( L \) and of \( R_0 \) to \( L \) over \( K_{\alpha} \). For the field \( L \) we use the identity mapping and for the ring \( R_0 \) we use the mapping constructed above. Then the images of elements \( y_1, \ldots, y_n \) give us the desired common zero for \( \mathfrak{p} \) and thus for \( t \).

\[\square\]

4.2 Infinitely many indeterminates

Let \( K \) be a difference field, consider the ring of difference polynomials \( K\{Y\} \), where \( Y \) is an arbitrary set of indeterminates. The set \( K^Y \) we shall call an affine space. It is clear that for every point \( x \in K^Y \) and every difference polynomial \( f \in K\{Y\} \) there is a well-defined result of substitution \( f(x) \). For every set \( E \subseteq K\{Y\} \) we define the set \( V(E) \subseteq K^Y \) such that \( V(E) \) is the set of all common zeros for \( E \) in \( K^Y \). Conversely, for every set \( X \subseteq K^Y \) we define a difference ideal \( I(X) \) as the set of all difference polynomials vanishing on \( X \). Let \( \kappa \) be an infinite cardinal. We shall say that difference field is \( \kappa \)-differencly closed if for every set \( Y \) of cardinality less than or equal to \( \kappa \) and every proper perfect ideal \( t \subseteq K\{Y\} \) the set \( V(t) \) is not empty.

**Statement 7.** For every difference field \( K \) every ideal of \( K\{Y\} \) is generated by the set of cardinality less than or equal to \( |\Sigma \times Y| \).
Proof. The statement is an direct corollary of statement \[4\] \qed

**Theorem 8.** Every difference field can be embedded into a \(\kappa\)-differencly closed one.

*Proof.* We should repeat the proof of theorem \[6\]. Let \(\kappa_1\) be the cardinal following after \(|\Sigma \times Y|\). Then, using notation of theorem \[6\] we should use the field \(K_{\kappa_1}\) instead of \(L\). To prove that \(L\) is \(\kappa\)-differencly closed we only need the fact that every difference ideal of \(K\{Y\}\) is generated by the set of cardinality strictly less than \(\kappa_1\). But statement \[7\] guaranties that this fact holds. Now we should repeat the last part of the proof of theorem \[6\]. \qed

5 Strong form of Nullstellensatz

5.1 Finitely many indeterminates

Let \(K\) be differencly closed field. Then we are able to prove strong Nullstellensatz for such field. We shall introduce some notation. For every subset \(E \subseteq K\{y_1, \ldots, y_n\}\) the set of all common zeros of \(E\) will be denoted by \(V(E) \subseteq K^n\). Conversely, for every subset \(X \subseteq K^n\) the ideal of all polynomials vanishing on \(X\) will be denoted by \(I(X)\).

**Statement 9.** Let \(K\) be a differencly closed field. And let \(A\) be a differencly finitely generated algebra over \(K\) such that the zero ideal is only one perfect ideal of \(A\). Then \(A\) coincides with \(K\).

*Proof.* The algebra \(A\) can be presented as follows

\[
A = K\{y_1, \ldots, y_n\}/\mathfrak{p},
\]

where \(\mathfrak{p}\) is a maximal perfect ideal. From the definition the set \(V(\mathfrak{p})\) is not empty. Therefore there exists a point \(x \in K^n\) such that \(\mathfrak{p} \subseteq I(x)\). But the ideal \(I(x)\) is perfect. So, \(\mathfrak{p} = I(x)\). Consequently, \(A\) coincides with \(K\). \qed

**Theorem 10.** Let \(K\) be differencly closed field. Then for every perfect difference ideal \(\mathfrak{t} \subseteq K\{y_1, \ldots, y_n\}\) the following holds

\[
\mathfrak{t} = I(V(\mathfrak{t})).
\]

*Proof.* The inclusion \(\mathfrak{t} \subseteq I(V(\mathfrak{t}))\) is obvious. Let us show the other one. Consider difference algebra \(A = K\{y_1, \ldots, y_n\}/\mathfrak{t}\). Let \(x\) be an element not belonging to the ideal \(\mathfrak{t}\). Consider the set

\[
S = \{\sigma_1(t)^{k_1} \cdots \sigma_n(t)^{k_n} \mid \sigma_i \in \Sigma\}.
\]

Then this set is multiplicatively closed, stable under the action of group \(\Sigma\), and do not meet the ideal \(\mathfrak{t}\). Then the ring \(S^{-1}A\) is a nontrivial difference ring. From the definition it follows that \(S^{-1}A = A\{1/\mathfrak{t}\}\) is differencly finitely generated algebra over \(K\). Let \(\mathfrak{n}\) be a maximal perfect ideal in the ring \(S^{-1}A\).
Then from previous statement if follows that $S^{-1}A/n$ coincides with $K$. Let $m$ be a contraction of $n$ to $A$. Then $A/m$ coincides with $K$. The last one means that the images of the elements $y_1, \ldots, y_n$ define the point in $K^n$ such that $t$ is vanishing at this point but $t$ is not.

5.2 Infinitely many indeterminantes

Let $K$ be a difference field and let $Y$ be an arbitrary set. Consider the ring of difference polynomials $K\{Y\}$. As in previous sections we are given by $V(E) \subseteq K^Y$ and $I(X) \subseteq K\{Y\}$. And let $\kappa$ be an infinite cardinal.

Statement 11. Let $K$ be a $\kappa$-differencly closed field, and let $A$ be a difference $K$ algebra generated by the family $Y$ over $K$. Let $|Y| \leq \kappa$ and zero ideal of $A$ is only one perfect ideal of $A$. Then $A$ coincides with $K$.

Proof. The algebra $A$ can be presented in the following form

$$A = K\{Y\}/p,$$

where $p$ is a maximal perfect ideal. From the definitions it follows that $V(p)$ is not empty. Hence there exists a point $x \in K^Y$ such that $p \subseteq I(x)$. But $I(x)$ is a prefect ideal, so $p = I(x)$. And therefore the algebra $A$ coincides with $K$.

Theorem 12. Let $K$ be a $\kappa$-differencly closed field and let $Y$ be a set such that $|Y| \leq \kappa$. Then for every perfect ideal $t \subseteq K\{Y\}$ the following equality holds

$$t = I(V(t)).$$

Proof. The inclusion $t \subseteq I(V(t))$ is obvious. Let us show the other one. Consider difference algebra $A = K\{Y\}/t$. Let $x$ be an element not belonging to ideal $t$. Consider the set

$$S = \{ \sigma_1(t)^{k_1} \cdots \sigma_n(t)^{k_n} | \sigma_i \in \Sigma \}.$$

Then this set is multiplicatively closed, stable under the action of group $\Sigma$, and do not meet the ideal $t$. Then the ring $S^{-1}A$ is a nontrivial difference ring. From the definition it is clear that $S^{-1}A = A\{1/t\}$. Consequently, $S^{-1}A$ is generated over $K$ by the set $Y \cup \{1/t\}$ and we have $|Y \cup \{1/t\}| \leq |Y| + 1 \leq \kappa + 1 = \kappa$. Let $n$ be a maximal perfect ideal of the ring $S^{-1}A$. Then from previous statement it follows that $S^{-1}A/n$ coincides with $K$. Let $m$ be the contraction of $n$ to $A$. Then $A/m$ coincides with $K$ too. The last one means that the images of elements of $Y$ define a point in $K^Y$ such that all elements of $t$ are vanishing at this point but $t$ is not.

6 Pseudofields Case

From this moment we assume that the group $\Sigma$ is infinite. The case of finite group is scrutinized in [4].
6.1 Finitely many indeterminants

We shall prove some auxiliary statements before proving the main result.

**Statement 13.** Let $\kappa$ be a cardinal coinciding with number of all maximal filters on $\Sigma$. Then for every pseudofield $L$ we have $|\text{Spec } L| \leq \kappa$.

**Proof.** Let $m$ be a prime ideal of $L$, then the quotient field will be denoted by $K$. Consider corresponding homomorphism $L \to K$. Then from statement it follows that there exists a difference homomorphism $L \to F K$. Since $L$ is pseudofield, this homomorphism is injective. Moreover, since $L$ is absolutely flat, the mapping $\text{Spec } F K \to \text{Spec } L$ is surjective (see. [1, chapter 3, ex. 29 and ex. 30]). Consequently, $|\text{Spec } L| \leq |\text{Spec } F K|$. But all prime ideals of $F K$ are maximal. The set of all maximal ideals of $F K$ is described by maximal filters on $\Sigma$. So we have the desired result. \[\square\]

**Statement 14.** Every simple difference ring can be embedded into a pseudofield.

**Proof.** Let $R$ be a simple difference ring, and let $m$ be its maximal ideal. Then the residue field of this ideal will be denoted by $K$. Let $R \to K$ be corresponding homomorphism. Then from statement it follows that there exists a difference homomorphism $R \to F K$. The ring $F K$ is an absolutely flat difference ring. Consider a maximal difference ideal $n$ in $F K$. Then $L = F K/\mathfrak{n}$ is a pseudofield. The composition $R \to F K \to L$ give us the desired embedding. \[\square\]

**Statement 15.** Let $A$ be a pseudofield, $B$ and $C$ being nonzero algebras over $A$. Then the difference algebra $B \otimes_A C$ is nonzero.

**Proof.** Let $q$ be a prime ideal of $B$, then its contraction to $A$ will be denoted by $p$. Since $A$ is simple difference ring, corresponding homomorphism $A \to C$ is injective. Since $A$ is absolutely flat, the mapping $\text{Spec } C \to \text{Spec } A$ is surjective (see. [1, chapter 3, ex. 29 and ex. 30]). Therefore there exists a prime ideal $q'$ is $C$ contracting to $p$. Consider

$$(B \otimes C)_p = B_p \otimes C_p$$

We shall show that $B_p$ and $C_p$ are nonzero. For example we shall show it for $B$. Set $S = A \setminus \mathfrak{p}$ and $T = B \setminus \mathfrak{q}$. Then by the construction we have $S \subseteq T$ and $T$ does not contain the zero. Therefore $S$ does not contain the zero. Thus $B_p = S^{-1}B$ is nontrivial ring.

Since $A_p$ is a field and algebras $B_p$ and $C_p$ are nontrivial, considered localization is nontrivial too. So the initial tensor product is nonzero. \[\square\]

We shall say that particulary ordered set $S$ is $\kappa$-noetherian (where $\kappa$ is a cardinal) if every strictly ascending chain of elements of $S$ is of cardinality less than or equal to $\kappa$.

**Statement 16.** Let $S_\alpha$ be a family of particulary ordered sets and for every $\alpha$ the set $S_\alpha$ is $\kappa$-noetherian. Then the particulary ordered set $\prod_{\alpha \in A} S_\alpha$ is $|A| \times \kappa$-noetherian.
Proof. Let \{x_\gamma\} be a strictly ascending chain of \(S = \prod_{\alpha \in A} S_\alpha\). Let \(\pi_\alpha : S \rightarrow S_\alpha\) be a projection onto the corresponding factor. Define the set

\[T_\alpha = \{ s \in S_\alpha \mid \exists x_\gamma : s = \pi_\alpha(x_\gamma) \}.\]

From the fact that \(S_\alpha\) is \(\kappa\)-noetherian it follows that cardinality of \(T_\alpha\) is less than or equal to \(\kappa\). Set \(X = \bigsqcup_\alpha T_\alpha\). For every element \(x_\gamma\) we define the subset \(X_\gamma\) in \(X\) by the following rule

\[X_\gamma = \bigsqcup_\alpha \{ s \in T_\alpha \mid s \leq \pi_\alpha(x_\gamma) \}.\]

It is clear that the chain of \(X_\gamma\) is a strictly ascending chain. So cardinality of this chain is not greater than cardinality of \(X\) coinciding with \(|A| \times \kappa\).

**Statement 17.** Let \(\kappa\) be a cardinal coinciding with number of all maximal filters on \(\Sigma\). Then for every pseudofield \(K\) the set \(\text{Rad}^K K\{y_1, \ldots, y_n\}\) is \(|\Sigma| \times \kappa\)-noetherian.

**Proof.** We shall show more general statement. Namely, we shall show that \(\text{Rad} K\{y_1, \ldots, y_n\}\) is \(|\Sigma| \times \kappa\)-noetherian. Let \(m_\alpha\) be all prime ideals of \(K\), the corresponding residue fields will be denoted by \(F_\alpha\). Consider the rings

\[R_\alpha = R_{m_\alpha} = F_\alpha[\theta y_1, \ldots].\]

Then for every radical ideal \(t\) we define the family of ideals \(\{(t)_{m_\alpha}\}\). This family is an element of the particular ordered set

\[\prod_\alpha \text{Rad}(R_\alpha).\]

It is clear that this mapping preserve the order of elements, in other words from inclusion \(t \subseteq t'\) it follows that for every \(\alpha\) we have \((t)_{m_\alpha} \subseteq (t')_{m_\alpha}\). Note that this mapping is injective. Indeed, let \(t \neq t'\). Then, for example, there exists an element \(x \in t \setminus t'\). Consequently there is a prime ideal \(p\) containing \(t'\) and not containing \(x\). Then \((t')_p \neq (1)\) but \((t)_p = (1)\).

Hence the particular ordered set \(\text{Rad} K\{y_1, \ldots, y_n\}\) can be embedded into \(\prod_\alpha \text{Rad}(R_\alpha)\). The set \(\text{Id} R_\alpha\) is \(\Sigma\)-noetherian (see statement 14). Therefore its subset \(\text{Rad} R_\alpha\) is \(\Sigma\)-noetherian too. Then the desired result follows from the previous statement and statement 13.

**Corollary 18.** Let \(\kappa\) be a cardinal coinciding with number of all maximal filters on \(\Sigma\). Then for every pseudofield \(K\) every radical difference ideal of the ring \(K\{y_1, \ldots, y_n\}\) can be presented as \(r([E])\), where \(|E| \leq |\Sigma| \times \kappa\).

**Theorem 19.** Every pseudofield can be embedded into differently closed one.

**Proof. Construction.** Let \(A\) be a pseudofield. Consider the family of all simple differently finitely generated algebras \(\{B_\alpha\}\) over \(A\) up to isomorphism. Define the ring \(R = \otimes_\alpha B_\alpha\) (where the tensor product is taken over \(A\)). From
It follows that $R$ is a nonzero difference ring. Let $m$ be a maximal difference ideal in $R$. Then from statement 14 it follows that the ring $R/m$ can be embedded into a pseudofield $A_1$. Using transfinite induction, we obtain the rings $A_\alpha$ for every ordinal $\alpha$. Let $\kappa_1$ be the first cardinal greater than $|\Sigma| \times \kappa$.

**Difference closeness.** Let $b$ be a difference ideal in $L\{y_1, \ldots, y_n\}$. Then it is contained in a maximal difference ideal $t$. Denote the algebra $L\{y_1, \ldots, y_n\}/t$ by $B$.

From corollary 18 it follows that the ideal $t$ can be presented as $r[E]$, where $|E| \leq |\Sigma| \times \kappa$. Denote the ideal $[E]$ by $a$ and the ring $L\{y_1, \ldots, y_n\}/a$ by $B'$. It is clear that the nilradical of $B'$ is the unique pseudoprime ideal of $B'$. Since cardinality of $E$ is less than $\kappa_1$, there exists an intermediate pseudofield $A_\gamma$ containing all coefficients of all elements of $E$. Then

$$B' = L \otimes_{A_\gamma} \{y_1, \ldots, y_n\}/[E].$$

But there is a common zero for $E$ in $A_{\gamma+1} \subseteq L$. Therefore there exists a difference homomorphism

$$A_{\gamma}\{y_1, \ldots, y_n\}/[E] \to L.$$  

Consequently, $B'$ can be mapped into $L$. Since $L$ is pseudofield, the kernel of the last homomorphism is pseudoprime. Hence there is a difference homomorphism $B \to L$. The images of elements $y_1, \ldots, y_n$ define a common zero for $b$ in $L^n$. \hfill \square

### 6.2 Infinitely many indeterminants

Let $K$ be a pseudofield, consider the ring of difference polynomials $K\{Y\}$, where $Y$ is an arbitrary set of indeterminates. The set $K^Y$ will be called an affine space.

As in the situation above we have a well-defined substitution homomorphism $K\{Y\} \to K$ by the rule $f \mapsto f(x)$, where $x \in K^Y$. Additionally, there are two mappings $E \mapsto V(E)$ and $X \mapsto I(X)$, where $E, I(X) \subseteq K\{Y\}$ and $V(E), X \subseteq K^Y$. Let $\kappa$ be an infinite cardinal. We shall say that pseudofield $K$ is $\kappa$-differencly closed if for every set $Y$ of cardinality less than or equal to $\kappa$ and every proper difference ideal $a \subseteq K\{Y\}$ the set $V(a)$ is not empty.

**Statement 20.** Let $\kappa$ be a cardinal coinciding with number of all maximal filters on $\Sigma$. Then for every pseudofield $K$ the set $\text{Rad}^\Sigma K\{Y\}$ is $|\Sigma \times Y| \times \kappa$-noetherian.

**Proof.** Let us show that the set $\text{Rad} K\{Y\}$ is $|\Sigma \times Y| \times \kappa$-noetherian. Let $m_\alpha$ be all primes ideals of $K$. The corresponding residue fields will be denoted by $F_\alpha$. Consider the ring

$$R_\alpha = R_{m_\alpha} = F_\alpha[\theta y, \ldots].$$
Then every radical ideal \( t \) will define the family of ideals \( \{ (t)_{m_{\alpha}} \} \). This family is an element of the particular ordered set
\[
\prod_{\alpha} \text{Rad}(R_{\alpha}).
\]

It is clear that this mapping preserve the inclusions. In other words from the inclusion \( t \subseteq t' \) it follows that for every \( \alpha \) we have \( (t)_{m_{\alpha}} \subseteq (t')_{m_{\alpha}} \). We know that this mapping is injective (proof is similar to proof of statement 17).

Therefore particular ordered set \( \text{Rad} K \{ \Sigma \times Y \} \) can be embedded into the set \( \prod_{\alpha} \text{Rad}(R_{\alpha}) \). The set \( \text{Id}_{\alpha} \) is \( |\Sigma \times Y| \)-noetherian, consequently, its subset \( \text{Rad} R_{\alpha} \) is \( |\Sigma \times Y| \)-noetherian too. Then the desired result follows from statements 16 and 13.

**Corollary 21.** Let \( \kappa \) be a cardinal coinciding with number of all maximal filters on \( \Sigma \). Then for every pseudofield \( K \) every radical difference ideal of \( K \{ \Sigma \times Y \} \) can be presented as \( r([E]) \), where \( |E| \leq |\Sigma \times Y| \times \kappa \).

**Theorem 22.** Every pseudofield can be embedded into \( \kappa \)-differencly closed one.

**Proof.** We need to repeat the proof of theorem 19 using previous corollary instead of corollary 18.

**References**

[1] M.F. Atiyah, I.G. Macdonald. Introduction to commutative algebra. Addison-Wesley. 1969.

[2] A Levin. Difference algebra. Springer, 2008.

[3] B. Poizat, M. Klein. A course in model theory: an introduction to contemporary mathematical logic. Springer, 2000

[4] D. Trushin. Difference Nullstellensatz in the case of finite group.