Anti–Field Formalism and Non–Abelian Duality

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Abstract

The act of implementing non-Abelian duality in two dimensional sigma models results unavoidable in an additional reducible symmetry. The Batalin–Vilkovisky formalism is employed to handle this new symmetry. Valuable lessons are learnt here with respect to non–Abelian duality. We emphasise, in particular, the effects of the ghost sector corresponding to this symmetry on non–Abelian duality.

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1. Introduction

Duality transformations have understandably brought about a surge of new interests in string theory. The importance of these transformations lies in their ability to connect seemingly different string backgrounds. This might shed some light on one of the longstanding problems in superstring theory, namely the non–uniqueness of the low energy physics expected from this theory. As it is well–known, the phenomenology predicted by superstring theory depends upon the way the extra six dimensions are compactified. Hence, if the spaces on which one carries out the compactification are related to each other by duality transformations, then their corresponding low energy physics should also be related. This is also the idea behind mirror symmetry [1] which might well be another manifestation of duality transformations [2, 3].

The duality transformation that concerns us here is the so–called T–duality [4]. This can be understood as canonical transformations on the phase space of a sigma model [5]. There is, however, a well defined procedure at the level of the Lagrangian which allows the construction of dual theories [6]. It consists in gauging an isometry group of a non–linear sigma model and at the same time restricting, by means of a Lagrange multiplier, the gauge field to be pure gauge. The integration over the gauge fields (without a kinetic term) leads to the dual theory.

The duality transformation is termed Abelian or non–Abelian depending on whether the isometry group is Abelian or not. Abelian duality has proved to be of crucial importance in string [7] and membrane [8] theories. On the other hand, its non–Abelian counterpart has not yet been fully exploited [9]. This is because non–Abelian duality is hampered by conceptual problems (such as global issues and the fact that carrying out the transformation twice does not lead to the original model). One of the issues in non–Abelian duality is the appearance, as explained below, of a new local symmetry in the formism [10].

It is the aim of this paper to deal with the quantisation of this new symmetry. The understanding of this symmetry is crucial to any possible exploitation ( and probably to the understanding of the other issues) of non–Abelian duality. We outline below the manifestation of this symmetry. As this symmetry is reducible we appeal to the Batalin–Vilkovisky formalism [11] for a rigourous treatement. The formalism is briefly summarised in section two. Our main result is that the dual theory depends on the ghost sector corresponding to this new symmetry and on its gauge fixing conditions. Let us therefore start by stating the problem.
Suppose that one has a two-dimensional theory described by an action $S(\phi)$ which is invariant under some global symmetry for the generic fields $\phi$. Let us also assume that the generators of this symmetry form a closed Lie algebra $G$. Furthermore it is also assumed that one can gauge these symmetry in an anomaly-free way. It is then straightforward to find the dual of this theory at the classical level. This is found by considering the gauge invariant action $\mathcal{I}(\phi, A, \Lambda) = S(\phi, A) + \int d^2x \text{tr} \left( \Lambda F \right)$.

(1)

Here $S(\phi, A)$ is the gauged version of $S(\phi)$. The gauge field $A_\mu$ takes value in the Lie algebra $G$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ is the corresponding field strength. The trace $\text{tr}$ is the invariant bi-linear form of the Lie algebra $G$ such that $\text{tr}(XY) = \eta_{ab}X^aY^b$.

The new field $\Lambda$ is a Lagrange multiplier which, at the classical level, imposes the constraints $F_{\mu\nu} = 0$. This is then solved by $A_\mu = g^{-1}\partial_\mu g$, where $g$ is an element in the Lie group corresponding to $G$. Recall now that $A_\mu$ and $\Lambda$ transform as

$$A_\mu \rightarrow hA_\mu h^{-1} - \partial_\mu hh^{-1}$$

$$\Lambda \rightarrow h\Lambda h^{-1}$$

(2)

where $h$ is the Lie algebra valued gauge function. Of course, the transformation of the generic field $\phi$ is also governed by this same function. Using this gauge freedom, we can choose a gauge such that $g = 1$. Hence, in this gauge, the gauge field vanishes and the action $\mathcal{I}(\phi, A, \Lambda)$ is classically equivalent to the original action $S(\phi)$.

At the classical level, the dual theory is obtained by keeping the Lagrange multiplier and eliminating instead the gauge fields by their equations of motion. We are supposing that the gauge fields appear quadratically at most and without derivatives in the gauged action $S(\phi, A)$. To get the right degrees of freedom in the dual theory a gauge fixing condition must be chosen.

The issues that concerns us in this paper are those necessary to implement the duality transformation at the quantum level. This is a well-known procedure if the Lie algebra $G$ is Abelian. However, if $G$ is non-Abelian then the matter must be considered carefully. This is mainly because the action now has another local symmetry which must be taken into account in the path integral. Due to the properties of the trace, the gauge invariant action $\mathcal{I}$ is also invariant under

$$\Lambda \rightarrow \Lambda + [\xi, F]$$

$$A_\mu \rightarrow A_\mu , \ \phi \rightarrow \phi ,$$

(3)
where $\xi$ is the new local gauge function corresponding to this extra symmetry. It should be noted that if the gauge function $\xi$ takes value in the centre (or maximal ideal) of the Lie algebra $\mathcal{G}$, then the transformation of $\Lambda$ vanishes; thus the new symmetry is reducible (i.e., not all the components of $\Lambda$ enter the transformation). This fact will have consequences, as we will see, on the Faddeev-Popov ghosts required to gauge fix this new symmetry. In the rest of the paper and for simplicity, we will consider only the case when $\mathcal{G}$ is semi-simple (that is, no maximal ideals are present in $\mathcal{G}$); hence the new transformation is reducible only when $\xi$ is proportional to $F$. In this case in the formalism of Batalin-Vilkovosky, which suitably deals with reducible symmetries, our symmetry is first-stage reducible. We will apply this formalism to quantise the new symmetry.

To obtain the dual theory, we have to perform the path integral over the $\phi$, $A_\mu$ and $\Lambda$ in the action (1). There are, therefore, two symmetries that one needs to gauge fix. The first one is the usual local gauge transformation in (2) and the second is the extra symmetry in (3). Since the two symmetries are completely independent and different in nature, it is therefore essential to keep one symmetry intact if the other is being fixed.

We choose first to fix the extra symmetry in (3) keeping the gauge symmetry in (2) intact. This is easily achieved if we choose a gauge fixing condition for the symmetry (3) which transforms covariantly with respect to the local gauge transformation (2).

We intend to employ the formalism of Batalin-Vilkovisky to quantise the new reducible theory, we will give the main ingredients of this formalism in what follows.

2. Review of the Batalin-Vilkovisky Formalism

The Batalin-Vilkovisky formalism manages theories with reducible symmetries. The Faddeev-Popov procedure is, in general, not sufficient for such theories. A simplistic use of the Becchi-Rouet-Stora-Tyutin (BRST) quantisation is also inappropriate in this case. We will give the essential tools of this formalism in that which follows.

Let $S$ be a classical action for some generic fields $\phi^i$, $i = 1, \ldots, n$ (fermionic or bosonic in nature). The equations of motion of this gauge action are assumed to possess at least one solution $\phi_0$. Let $m_0$ be the number of gauge parameters (fermionic and bosonic) of this gauge invariant action; hence $m_0$ Noether identities hold

$$\frac{\partial_s S}{\partial \phi^i} R^i_{\alpha_0} = 0 \quad \alpha_0 = 1, \ldots, m_0.$$  \hspace{1cm} (4)

$R^i_{\alpha_0} (\phi)$ are the generators of the gauge transformations and are supposed to be regular
functionals of the fields $\phi^i$. These transformations are written as $\delta \phi^i = R^i_{\alpha_0} \delta \theta^{\alpha_0}$, where $\theta^{\alpha_0}$ are the gauge parameters. We will denote by $\partial_r$ and $\partial_l$ the right and left functional derivatives, respectively. We also use the de Witt convention that summation over repeated indices includes an integration over spacetime.

The gauge symmetry is then reducible if there exists (at least on-shell) a set of $m_1$ zero-eigenvalue eigenvectors $Z^{\alpha_0}_{(1)\alpha_1}$ such that

$$R^i_{\alpha_0}Z^{\alpha_0}_{(1)\alpha_1}|_{\phi^0} = 0 \ , \ \alpha_1 = 1, \ldots, m_1 \ .$$

The symmetry is said to be first-stage reducible if the null vectors $Z^{\alpha_0}_{(1)\alpha_1}$ are independent. We will consider here only symmetries such as these.

The fields $\phi^i$ are part of a larger set of fields $\Phi^A$, $A = 1, \ldots, N$ (the rest of the fields being the different ghosts and some Lagrange multipliers necessary for gauge fixing). The Batalin-Vilkovisky formalism associates with each field $\Phi^A$ an anti-field $\Phi^*_A$ possessing opposite statistics. These anti-fields are just tools for constructing a BRST invariant action. If we denote by $\epsilon (\Phi^A) \equiv \epsilon_A$ the statistics of the field $\Phi^A$, then the fermion number of the anti-field is $\epsilon (\Phi^*_A) = \epsilon_A + 1 \mod 2$.

It is then guaranteed that there exists a BRST invariant quantum action $S(\Phi, \Phi^*)$ which satisfies the two requirements

$$S(\Phi, \Phi^*) |_{\Phi^* = 0} = S(\phi)$$

$$\langle S, S \rangle \equiv \frac{\partial_r S}{\partial \Phi^A} \frac{\partial_l S}{\partial \Phi^*_A} - \frac{\partial_r S}{\partial \Phi^*_A} \frac{\partial_l S}{\partial \Phi^A} = 0 \ ,$$

The first expression demands that one can retrieve the correct classical field theory. The second equation is what is known as the master equation and its solution will be our main concern.

The minimum number of fields contained within a first-stage reducible theory is the number of fields in $\Phi^A_{\text{min}} = \{ \phi^i, C^{\alpha_0}_{(0)}, C^{\alpha_0}_{(1)} \}$ plus $\Phi^*_{\text{min}}$. The fields $C^{\alpha_0}_{(0)}$ are assigned a ghost number equal to 1 and are the usual Faddeev-Popov ghosts, whilst $C^{\alpha_1}_{(1)}$ are the ghosts-for-ghosts fields and have ghost number equal to 2. Of course, the field $\phi^i$ has zero ghost number. The statistics of a field, or anti-field, is the sum of the statistics of its index and the absolute value of its ghost number. The first stage in constructing a BRST invariant theory is to associate an action $S(\Phi_{\text{min}}, \Phi^*_{\text{min}})$ with this minimum set of fields. This action can be expanded in powers of the anti-fields, where each term in the expansion has zero ghost number. The leading terms in this expansion are of the form

$$S(\Phi_{\text{min}}, \Phi^*_{\text{min}}) = S + \phi^i R^i_{\alpha_0} C^{\alpha_0}_{(0)} + C^{\alpha_0}_{(0)} \left[ Z^{\alpha_0}_{(1)\alpha_1} C^{\alpha_1}_{(1)} + T^{\alpha_0}_{\beta_0 \gamma_0} C^{\beta_0}_{(0)} C^{\gamma_0}_{(0)} \right]$$
There are no more terms in this expansion for the usual first-stage reducible theories.

The master equation then imposes the following conditions on the different coefficients in the above expansion

\[
\frac{\partial S}{\partial \phi^i} R_{\alpha_0}^{i \alpha_0} C^{\alpha_0}_{(0)} = 0 ,
\]

(8)

\[
R_{\alpha_0}^i Z_{(1) \beta_1}^{\alpha_0} C^{\beta_1}_{(1)} - 2 \frac{\partial S}{\partial \phi^j} B_{\beta_1}^{ji} C^{\beta_1}_{(1)} (-1)^{\epsilon_i} = 0 ,
\]

(9)

\[
\frac{\partial_i R_{\alpha_0}^{i \alpha_0} C^{\alpha_0}_{(0)}}{\partial \phi^j} R_{\beta_0}^{j \beta_0} C^{\beta_0}_{(0)} + R_{\alpha_0}^i T_{(1) \beta_0 \gamma_0}^{\alpha_0} C^{\gamma_0}_{(0)} C^{\beta_0}_{(0)} - 2 \frac{\partial S}{\partial \phi^j} E_{\beta_0}^{ji} C^{\gamma_0}_{(0)} C^{\beta_0}_{(0)} (-1)^{\epsilon_i} = 0 ,
\]

(10)

\[
\frac{\partial_i T_{(1) \beta_0 \gamma_0}^{\alpha_0} C^{\beta_0}_{(0)} C^{\gamma_0}_{(0)}}{\partial \phi^j} R_{\delta_0}^{j \delta_0} C^{\delta_0}_{(0)} + 2 T_{(1) \beta_0 \gamma_0}^{\alpha_0} C^{\gamma_0}_{(0)} C^{\delta_0}_{(0)} + Z_{(1) \beta_1}^{\alpha_0} F_{(1) \gamma_1}^{\beta_1} C^{\gamma_0}_{(0)} C^{\beta_0}_{(0)}
\]

\[
+ 2 \frac{\partial_i S}{\partial \phi^j} D_{\beta_0}^{ji} C^{\beta_0}_{(0)} C^{\gamma_0}_{(0)} C^{\beta_0}_{(0)} (-1)^{\epsilon_{\alpha_0}} = 0 ,
\]

(11)

\[
\frac{\partial_i Z_{(1) \beta_1}^{\alpha_0} C^{\beta_1}_{(1)} C^{\beta_0}_{(0)}}{\partial \phi^j} R_{\gamma_1}^{j \gamma_1} C^{\gamma_0}_{(0)} + 2 T_{(1) \beta_0 \gamma_0}^{\alpha_0} Z_{(1) \beta_1}^{\gamma_0} C^{\delta_0}_{(0)} + Z_{(1) \beta_1}^{\alpha_0} A_{(1) \gamma_1}^{\beta_1} C^{\gamma_0}_{(0)} C^{\beta_0}_{(0)}
\]

\[
+ 2 \frac{\partial_i S}{\partial \phi^j} C^{\gamma_0}_{(0)} C^{\beta_0}_{(1)} (1) (-1)^{\epsilon_{\alpha_0}} = 0 .
\]

(12)

Here \( \epsilon_i = \epsilon (\phi^i) \), whilst \( \epsilon_{\alpha_0} \) is the Grassmann parity of the gauge parameter.

The minimum sets of fields \( \Phi_{\text{min}} \) and of anti-fields \( \Phi_{\text{min}}^\star \) can be enlarged to include more fields and their corresponding anti-fields. The master equation implies that, if \( S (\Phi_{\text{min}}, \Phi_{\text{min}}^\star) \) is a solution, then

\[
S (\Phi, \Phi^\star) = S (\Phi_{\text{min}}, \Phi_{\text{min}}^\star) + C^{\alpha_0 \alpha_1}_{(0)} \Pi_{(0) \alpha_0} + C^{\beta_1 \beta_0}_{(1)} \Pi_{(1) \beta_0} + C^{\gamma_0 \gamma_1}_{(1) \beta_1} \Pi_{(1) \gamma_1} \]

is also a solution. The new fields may be employed in gauge fixing as we will see shortly, and are assigned the ghost numbers

\[
\text{gh} (\Pi_{(0) \alpha_0}) = \text{gh} (C^{\alpha_0}_{(1)}) = 0
\]

\[
\text{gh} (C^{\alpha_0}_{(0)}) = -\text{gh} (\bar{C}_{(0) \alpha_0}) = -\text{gh} (\Pi_{(1) \alpha_0}) = \text{gh} (\Pi_{(1) \alpha_0}^\star) = 1
\]

\[
\text{gh} (C^{\alpha_0}_{(1)}) = -\text{gh} (\bar{C}_{(1) \alpha_0}) = 2 .
\]

(14)

The fields with a star denote their corresponding anti-fields.

The anti-fields are not physical fields and should be eliminated from the theory. This is achieved through the introduction of what is known as the gauge-fixing fermion \( \Psi (\Phi) \). This
is a functional of odd statistics and having a ghost number equal to \(-1\). The anti-fields in the full action \((13)\) are then replaced by

$$
\Phi^* = \frac{\partial \Psi}{\partial \Phi^A} .
$$

(15)

The functional \(\Psi\) has to satisfy certain conditions in order to make all the ghost propagators invertible. The simplest choice of functional \(\Psi\) for first-stage reducible theories takes the form

$$
\Psi (\Phi) = \bar{C}_{(0)\alpha_0} \chi_{\alpha_0} + \bar{C}_{(1)\beta_1} \Omega_{\alpha_0}^{\beta_1} C_{(0)}^{\alpha_0} + \bar{C}_{(0)\alpha_0} \Sigma_{\beta_1}^{\alpha_0} C_{(1)}^{\beta_1} ,
$$

(16)

where \(\chi_{\alpha_0} (\phi^i)\) is an admissible gauge condition for the classical fields \(\phi^i\). The matrices \(\Omega_{\alpha_0}^{\beta_1}\) and \(\Sigma_{\beta_1}^{\alpha_0}\) are some suitable maximal rank matrices which remove the degeneracy of the kinetic term of the ghosts \(C_{(0)}^{\alpha_0}\) and \(C_{(0)\alpha_0}\).

Note that the integration in the path integral over the \(\Pi\)'s of \((13)\) leads to three sets of gauge conditions. These conditions are in the form of \(\delta\)-functions. To obtain the usual quadratic gauge-fixing Lagrangian (the 't Hooft method), a linear term in the \(\Pi\)'s is added to \(\Psi\). In the simplest cases the following gauge fermion leads to a quadratic gauge-fixing Lagrangian

$$
\tilde{\Psi} = \Psi + \frac{1}{2} \left[ \bar{C}_{(0)\alpha_0} \Gamma_{\alpha_0}^{\alpha_0 \beta_0} \Pi_{(0)\beta_0} + \bar{C}_{(1)\alpha_1} \Theta_{\beta_1}^{\alpha_1} \Pi_{(1)\beta_1} - (-1)^{\epsilon_{\alpha_1}} \Pi_{(1)\alpha_1} \Theta_{\beta_1}^{\alpha_1} C_{(1)}^{\beta_1} \right] ,
$$

(17)

where \(\Psi\) is given in \((16)\) and \(\Gamma_{\alpha_0}^{\alpha_0 \beta_0}\) and \(\Theta_{\beta_1}^{\alpha_1}\) are some invertible matrices assumed to contain no derivatives. The integration over the \(\Pi\)'s will give Gaussian averages of gauge conditions instead of \(\delta\)-functions. This issue will be of considerable relevance when we consider non-Abelian duality in sigma models.

To end this brief review of the Batalin-Vilkovisky formalism, we provide a means to determine the BRST transformations of the different fields. A generic quantity \(P (\Phi, \Phi^*)\) having statistics \(\epsilon_P\), has a BRST transformation given by

$$
\delta P = (-1)^{\epsilon_P} (P, S) .
$$

(18)

This transformation is nilpotent \((\delta^2 P = 0)\) by virtue of the master equation satisfied by \(S\). This definition of the BRST transformation guarantees that \(S\) is, by construction, BRST invariant. The factor \((-1)^{\epsilon_P}\) has been chosen to enforce graded Leibniz rules for \(\delta\).

Upon elimination of the anti-fields through \((15)\), the action \(S (\Phi, \Phi^* = \frac{\partial S}{\partial \Phi})\) is still BRST invariant. In general, however, the nilpotency of the BRST transformation holds only when the equations of motion of the quantum action \(S (\Phi, \Phi^* = \frac{\partial S}{\partial \Phi})\) are used.

We are now at a stage where we can apply the Batalin-Vilkovisky formalism to theories of the form given in \((1)\).
3. Application of the Batalin-Vilkovisky Formalism

In order to become familiar with the general ideas of the anti-field formalism, let us start by quantising the action (1). We will deal with the symmetry (3) leaving the usual gauge symmetry (2) untouched throughout the procedure. This may be regarded as a preliminary exercise before one tackles more complicated cases.

The variation of this action with respect to \( \Lambda \) leads to the equation of motion

\[
F^a \equiv \epsilon^{\mu\nu} F^a_{\mu\nu} = 0 ,
\]

where we have written \( A^a_{\mu} = A^a_{\mu} T_a \), \( F_{\mu\nu} = F^a_{\mu\nu} T_a \) and \( \Lambda = \Lambda^a T_a \). The \( T_a \) are the generators of the Lie algebra \( G \) such that \( [T_a, T_b] = f^{c}_{ab} T_c \).

The set of classical fields is \( \phi^i = \{ \varphi, A^a_{\mu}, \Lambda^a \} \). The transformation we are dealing with is Abelian and closes off-shell; hence the structure constants \( T^a_{\alpha_0 \gamma_0} \) vanish. Let us now investigate which of the coefficients of the expansion (7) survive in this case.

The transformation (3) leads to \( R^i_{\alpha_0} \) which are nonzero only when the index \( i \) refers to the field \( \Lambda^a \)

\[
R^i_{\alpha_0}(x) = f^a_{bc} F^c (x) \delta (x - y) ,
\]

where the index \( i = \{ a, x \} \) and \( \alpha_0 = \{ b, y \} \). Due to the anti-symmetry of the structure constants \( f^a_{bc} \), the null vectors of \( R^i_{\alpha_0} \) are given by

\[
Z^{b(1)}_{c\alpha_0}(z) = F^b (y) \delta (y - z) ,
\]

where the index \( \beta_1 = \{ z \} \). It is clear that these null vectors are linearly independent off-shell.; hence this theory is said to be first-stage reducible. Since \( T^a_{\beta_0 \gamma_0} \), \( R^i_{\alpha_0} \) and \( Z^{c(1)}_{\alpha_0 \beta_1} \) do not depend on the field \( \Lambda^a \), a solution to the master equation is obtained by setting all the other coefficients in (7) to zero.

Hence, keeping the Batalin-Vilkovisky notation, we are left with

\[
S (\Phi_{\min}, \Phi^*_{\min}) = S (\phi) + \phi^s R^i_{\alpha_0} C^{\alpha_0} (0) + C^{\alpha_1} (1) \alpha_0 C^{\alpha_1}_{(1)} .
\]

The full quantum action is then written in the suggestive form

\[
S (\Phi, \Phi^*) = I (\phi, A, \Lambda) + S_{\text{ghost}} + S_{\text{gauge}}
\]

\[
S_{\text{ghost}} = \frac{\partial \Psi}{\partial \Lambda} R^i_{\alpha_0} C^{\alpha_0} (0) + \frac{\partial \Psi}{\partial C^{\alpha_0}_{(0)}} Z^{\alpha_0}_{(1)\alpha_1} C^{\alpha_1}_{(1)}
\]

\[
S_{\text{gauge}} = \frac{\partial \Psi}{\partial C^{\alpha_0}_{(0)}} \Pi^{(0)\alpha_0} + \frac{\partial \Psi}{\partial C^{\alpha_1}_{(1)\beta_1}} \Pi^{(1)\beta_1} + \frac{\partial \Psi}{\partial C^{\beta_1}_{(1)}} \Pi^{\beta_1}_{(1)} .
\]
The anti-fields have been eliminated using the gauge-fixing fermion $\Psi$.

The next step in determining the full quantum action is to construct the gauge-fixing fermion $\Psi$. As mentioned earlier, we would like to gauge fix the transformation (3) without breaking the usual gauge symmetry in (2). This can be achieved by choosing a gauge fixing condition which transforms covariantly under (2). A gauge fixing condition which has this property is given by

$$\chi^a = f^a_{bc} \Lambda^b F^c .$$  \tag{24}

This is a set of $[\text{dim} G - \text{rank} G]$ equations which are compatible with the transformation (3). The gauge fermion then takes the form

$$\Psi = \int d^2 x \left[ \bar{C}'(0)_a f^a_{bc} \Lambda^b F^c + \bar{C}'(1) \Omega_a C^a_{(0)} + \bar{C}(0)_a \Sigma^a C(1) \right] .$$  \tag{25}

Under the gauge transformations (3), the ghost fields are obviously required to transform in the adjoint representation of $G$. The matrices $\Omega^a_{\alpha \alpha}$ and $\Sigma^a_{\alpha \alpha}$ are chosen such that the gauge covariance (3) is maintained. These matrices are also assumed to be independent of the Lagrange multiplier field, $\Lambda$.

The ghost action is therefore given by

$$S_{\text{ghost}} = \int d^2 x \left[ \bar{C}(0)_a f^a_{bc} \Lambda^b F^c + \bar{C}(1) \Omega_a C^a_{(0)} + \bar{C}(0)_a \Sigma^a C(1) \right] .$$  \tag{26}

It is clear that $S_{\text{ghost}}$ is invariant under

$$C^a_{(0)} \longrightarrow C^a_{(0)} + \alpha F^a$$

$$\bar{C}'(0)_a \longrightarrow \bar{C}'(0)_a + \bar{\alpha} \eta_{ab} F^b$$  \tag{27}

where $\alpha$ and $\bar{\alpha}$ are two local Grassmanian parameters. In this sense the ghost action is degenerate (that is, the gauge fixing did not remove all the symmetries of our theory). It is the role of the gauge fixing Lagrangian to remove all the degeneracies.

The integration over the $\Pi$’s in $S_{\text{gauge}}$ leads to three conditions

$$f^a_{bc} \Lambda^b F^c + \Sigma^a C'_{(1)} = 0 , \quad \Omega_a C^a_{(0)} = 0 , \quad \bar{C}_a \Sigma = 0 .$$  \tag{28}

The first condition fixes the gauge transformation in (3) and eliminates $C'_{(1)}$. Multiplication by $\eta_{ad} F^d$ of the first equation yields $\eta_{ad} F^d \Sigma^a C'_{(1)} = 0$. This is sufficient to eliminate $C'_{(1)}$ provided that $\eta_{ad} F^d \Sigma^a$ does not vanish identically. The remaining two conditions fix the ghost transformation mentioned in (27). We found that the two matrices

$$\Omega_a = \eta_{ab} F^b , \quad \Sigma^a = F^a$$  \tag{29}
satisfy all the above mentioned requirements.

In this way we have constructed a BRST invariant quantum theory. If one wishes to eliminate the anti-fields using the gauge fermion $\Psi$ then the BRST transformations are given by

$$
\delta \Psi A = (-1)^{\epsilon_A} \frac{\partial S}{\partial \Psi^A} \bigg|_{\Psi^\ast - \frac{\partial S}{\partial \Psi^\ast}}
$$

It is then a simple matter to write down the BRST transformations for the fields

$$
\begin{align*}
\delta \Psi A &= f^{a}_{bc} F^c C_{(0)}^b \\
\delta \Psi C_{(0)}^a &= - F^a C_{(1)} \\
\delta \Psi C_{(0)}^a &= - \Pi_{(0)}^a \\
\delta \Psi C_{(1)}^a &= \Pi_{(1)}^a \\
\delta \Psi C_{(1)}' &= \delta \Psi \Pi_{(0)}^a = \delta \Psi \Pi_{(1)}^a = \delta \Psi \Pi_{(1)}' = 0.
\end{align*}
$$

It then follows that the BRST transformations are nilpotent.

Finally, we would like to investigate a point which is relevant to non-Abelian duality. This concerns the addition of linear terms in the $\Pi$’s to the gauge fermion $\Psi$. In this case the new gauge fermion takes the form

$$
\tilde{\Psi} = \Psi + \frac{1}{2} \int d^2 x \left[ \tilde{C}_{(0)}^a \Gamma^{ab} \Pi_{(0)}^b + \tilde{C}_{(1)}^a \Theta \Pi_{(1)}^b - \Pi_{(1)}^a \Theta C_{(1)}^a \right],
$$

where $\Psi$ is the gauge fermion given in (25). In order to maintain covariance under (2), a simple choice for the two matrices $\Gamma^{ab}$ and $\Theta^{a}$ is

$$
\Gamma^{ab} = n \eta^{ab}, \quad \Theta = m,
$$

where $\eta^{ab}$ is the inverse of $\eta_{ab}$ and $n$ and $m$ are two constant parameters.

The integration over the $\Pi$’s results in the quadratic gauge-breaking Lagrangian

$$
S_{\text{gauge}} = \int d^2 x \left[ - \frac{1}{2n} \left( f^{a}_{bc} A^b C^c \right) \eta_{ad} \left( f^{d}_{rs} A^r F^s \right) - \frac{1}{m} \tilde{C}_{(0)}^a F^a \eta_{bc} F^b C_{(0)}^c \right] - \frac{1}{2n} \tilde{C}_{(1)}^a F^a \Pi_{(1)}^b C_{(1)}^b.
$$

This is the usual Gaussian gauge fixing Lagrangian. The first term removes the gauge freedom of the original action while the second term removes the degeneracy of the ghost Lagrangian (26). The last term is required for BRST invariance and is a characteristic of the anti-field formalism.

This completes the quantisation of the new symmetry (3). Let us now list the consequences of our work on non-Abelian duality. We will, however, leave the detailed investigation to a forthcoming publication [12].
4. Conclusions

We have shown in this paper that the procedure by which non–Abelian duality is implemented in sigma models naturally leads to the presence of a reducible symmetry. We have dealt with this symmetry using the Batalin–Vilkovisky formalism. This unavoidably introduces new fields into the theory. Some of these fields are bosonic in nature ($C^{(1)}$, $\bar{C}^{(1)}$ and $C^{(1)}_{\prime}$) and could play a rôle similar to that of the Lagrange multiplier $\Lambda$. This is further investigated in [12].

In order to proceed further in the determination of the dual theory one must carry out an integration over the gauge fields in the full action (23). However, this is no more straightforward as this action includes terms quadratic in the field strength of the gauge fields. This fact is worsened if we consider the gauge fermion $\tilde{\Psi}$ instead of $\Psi$. The integration over the gauge fields would lead to a dual theory containing non–local terms. The latter can no longer be interpreted as a sigma model corresponding to a string background. This issue, in fact, is particularly specific to our choice of gauge fixing condition which contains the field strength. It is possible to find a gauge breaking term which does not contain any gauge fields. These types of gauges are reported in [12] and involve only the sigma model fields $\varphi$ and the Lagrange multiplier $\Lambda$.

In this paper we have started by quantising the symmetry (3) keeping manifest the usual gauge symmetry (2). It is then natural to address the following question: could we have started the other way around? That is, to quantise first the symmetry in (2). This is an important issue which is also investigated in the forthcoming paper [12]. Let us simply mention that there are two ways in which to gauge fix the symmetry (4). The first is, for instance, to choose a standard gauge of the Landau type $\partial^\mu A^a_\mu = 0$. This could be solved by setting $A^a_\mu = \epsilon_{\mu\nu}^a \partial^\nu \lambda^a$ and leads to a non–vanishing field strength. Therefore, this type of gauge fixing does not break the new symmetry in (3). The second type of gauge fixing is a non–standard one and involves setting some fields ($\varphi$ and $\Lambda$) to zero. In general, however, this gauge automatically breaks the new symmetry in (3). This is the type of gauge fixing which has been considered in the literature on non–Abelian duality.

Another direction of research concerns a class of sigma models, identified in [13], which possess a symmetry of the type considered in this paper. It would be interesting to explore their quantisation à la Batalin–Vilkovisky. This is a quite involved programme as the Noether currents $R^a_{\alpha\alpha_0}$ and the null vectors $Z^{a_0}_{(1)\alpha_1}$ are field dependent, and the gauge algebra closes only on–shell. We will report on the work in progress in [12].
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