REAL HYPERSURFACES OF COMPLEX AND QUATERNIONIC HYPERBOLIC SPACES

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ABSTRACT. We introduce curvature-adapted foliations of complex hyperbolic space and study some of their properties. Generalized pseudo-Einstein hypersurfaces of complex hyperbolic space are classified. Analogous results for curvature-adapted foliations of quaternionic hyperbolic space are also discussed.

1. Introduction

Curvature-adapted hypersurfaces are of great interest to geometers and have been the focus of much attention since the concept was introduced by d’Atri [8]. Into this class fall many known examples of hypersurfaces with constant principal curvatures, a subject intensively studied since the time of Levi-Civita and Segre. Generalizing many distinguished families of hypersurfaces (for example umbilic hypersurfaces), their geometry is adapted to that of the ambient space in a special way. The Riccati equation was used by Gray in his monograph [9] to develop foundational theorems about curvature-adapted submanifolds in complex space forms: here it is employed to tackle outstanding problems in complex and quaternionic hyperbolic spaces.

Throughout this paper, we define $M$ to be a connected hypersurface of a connected, simply connected rank one symmetric space $\overline{M}$, $\overline{R}$ to be the Riemannian curvature tensor of $\overline{M}$ and $\xi$ to be a unit normal vector of $M$ at $p \in M$. We give the Riemannian metrics of $\overline{M}$ the standard scaling, so that sectional curvatures lie between $\pm 1$ and $\pm 4$. The normal Jacobi operator

$$K_\xi := \overline{R}(\xi, \cdot)\xi \in \text{End}(T_pM)$$

of $M$ (with respect to $\xi$) describes the curvature of the ambient manifold $\overline{M}$ at $p$, whereas the shape operator $A_\xi$ of $M$ (with respect to $\xi$) describes the curvature of $M$ as a submanifold of $\overline{M}$ in direction $\xi$. Both of these are self-adjoint operators, and hence have eigendecompositions. $M$ is said to be curvature-adapted if these operators are simultaneously diagonalizable, that

Date: April 1st, 2012.
1991 Mathematics Subject Classification. Primary 53C40.

Key words and phrases. Curvature-adapted foliations, quaternionic and complex hyperbolic spaces, pseudo-Einstein hypersurfaces.
is if

\[ K_\xi \circ A_\xi = A_\xi \circ K_\xi \]

at every point \( p \in M \). This means that a common eigenbasis for \( K_\xi \) and \( A_\xi \) exists at every point, which will generically be denoted \( E \).

The geometry of curvature-adapted hypersurfaces in rank one symmetric spaces has been a fruitful field of study and there is a substantial body of literature concerned with their classification. In real space forms it is easy to see that every hypersurface is curvature-adapted. In non-flat complex space forms they coincide exactly with Hopf hypersurfaces. Examples abound; any tube around any complex submanifold of complex projective space is a Hopf hypersurface \([7]\). Recall that a Hopf hypersurface is defined by the property that the structure vector field \( -J_\xi \) is a principal curvature vector field. For a Hopf hypersurface with structure vector field \( U = -J_\xi \), denote the principal curvature function corresponding to the structure vector field by \( \alpha \). It is well-known that \( \alpha \) is constant for \( \mathbb{C}P^n \) \([17]\) and \( \mathbb{C}H^n \) \([12]\). An explicit classification for Hopf hypersurfaces has been achieved in \( \mathbb{C}P^n \) \([13]\) and \( \mathbb{C}H^n \) \([3]\) under the assumption of constant principal curvatures.

**Definition 1.1.** A singular Riemannian foliation \( \mathcal{F} \) of \( M \) is a decomposition of \( M \) into embedded submanifolds, called the leaves of the foliation \( L \), such that

1. \( T_p L = \{X_p, X \in \Xi_\mathcal{F}\} \) for every \( L \in \mathcal{F} \) and \( p \in L \), where \( \Xi_\mathcal{F} \) is the space of smooth vector fields on \( M \) everywhere tangent to the leaves of \( \mathcal{F} \), and
2. every geodesic is orthogonal to the leaves at all or none of its points.

The leaves of maximal dimension are called regular, otherwise they are singular. If every leaf is regular one recovers the traditional definition of a Riemannian foliation. Let \( \nu_p L \) denote the normal space to a leaf \( L \) at a point \( p \). A Riemannian foliation is said to admit sections (or to be polar) if there exists a complete immersed submanifold \( \Sigma \subset M \) such that, for all leaves \( L \in \mathcal{F} \), \( L \cap \Sigma \neq \emptyset \) and, for all points \( p \in \Sigma \), one has \( T_p \Sigma \subset \nu_p L \). Singular Riemannian foliations arise naturally in geometry; for example in the study of isometric group actions and Riemannian submersions.

**Definition 1.2.** We define a connected submanifold \( P \subset M \) to be curvature-adapted if, for every unit normal vector \( \xi \) at a point \( p \),

1. \( \overline{R}(\xi, X)\xi \in T_p P \) for \( X \in T_p P \), and
2. if \( K_\xi := \overline{R}|_{T_p P} \), then \( K_\xi \) and \( A_\xi \) commute, at all points \( p \in P \).

This generalizes the definition for hypersurfaces. Let \( \nu^1(P) \) be the unit sphere bundle. For \( t \in \mathbb{R}^+ \), set \( M_t := \{\exp(t\xi) : \xi \in \nu^1(P)\} \). This is the tube of radius \( t \) around \( P \). Gray’s Theorem (Theorem (2.1) here) states
that the tubes of sufficiently small radius around a curvature-adapted sub-
manifold \( P \subset M \) are curvature-adapted hypersurfaces. Motivated by this
result, let us make the following definition:

**Definition 1.3.** A curvature-adapted foliation \( F \) of \( M \) is a singular Rie-
mannian foliation of \( M \) whose regular leaves arise as the tubes around a
curvature-adapted submanifold \( P \in F \) (together with \( P \) if there are no
singular leaves).

The horospherical foliation of \( \mathbb{C}H^n \) is an example of a curvature-adapted
foliation without singular leaves. Recall, following Ivey and Ryan [10], [11], a
Hopf hypersurface of \( \mathbb{C}H^n \) is said to have *small Hopf curvature* if \( 0 < \alpha < 2 \).
They have constructed many examples of such hypersurfaces. Similarly one
defines a Hopf hypersurface of \( \mathbb{C}H^n \) to have large Hopf curvature if \( 2 < \alpha \).
Any tube around a complex submanifold gives a Hopf hypersurface with
large Hopf curvature. In the borderline case (i.e. \( \alpha^2 - 4 = 0 \)) one says that
the hypersurface is degenerate.

A curvature-adapted foliation \( F \) of \( \mathbb{C}H^n \) is said to be degenerate (resp.
non-degenerate) if a regular leaf \( M \in F \) is degenerate (resp. non-degenerate).
It follows from the Riccati equation that if \( M \) is a degenerate (resp. non-
degenerate) Hopf hypersurface, then so are all parallel hypersurfaces \( M_t \).

We shall show the following:

**Theorem 1.4.** A regular leaf of a curvature-adapted foliation \( F \) of \( \mathbb{C}H^n \)
has small Hopf curvature if, and only if, \( F \) is the set of all tubes around a
totally geodesic \( \mathbb{R}H^n \) together with \( \mathbb{R}H^n \).

A precisely analogous result also holds true for curvature-adapted folia-
tions of \( \mathbb{H}H^n \), as there are similar results on the spectral data of the shape-
operator known [4]. Some partial results for degenerate curvature-adapted
foliations of \( \mathbb{C}H^n \) and \( \mathbb{H}H^n \) are then given.

We present in Section 5 new proofs of the work of Okumura [21], and
Montiel-Romero [19], classifying real hypersurfaces of \( \mathbb{C}P^n \) and \( \mathbb{C}H^n \) whose
induced almost-contact structure \( P \) commutes with \( A_\xi \). Our approach, ex-
ploting the spectral data of the shape operator, is considerably shorter and
also works for the analogous problem in quaternionic space forms \( \mathbb{H}H^n \). This
unifies their work with that of Lyu-Perez-Suh [16].

To conclude we investigate a related problem. It is well-known that an
Einstein manifold cannot be embedded isometrically as a hypersurface in
complex projective space \( \mathbb{C}P^n \). Kon [15] defined a hypersurface to be
*pseudo-Einstein* if there exist constants \( \rho \) and \( \sigma \) so that for any tangent
vector \( X \),

\[
SX = \rho X + \sigma \langle X, U \rangle U
\]

where \( S \) denotes the (1,1)-Ricci tensor and \( U = -J\xi \) denotes the structure
vector field. Kon classified such hypersurfaces under the assumption \( n \geq 3 \).
Montiel [18] derived a classification in complex hyperbolic space \( \mathbb{C}H^n \) under
the same assumption. A canonical generalization of this concept is to allow
ρ and σ to be non-constant smooth functions. Such hypersurfaces are called generalized pseudo-Einstein. Cecil and Ryan [7] showed that in \( \mathbb{C}P^n, n \geq 3 \) such hypersurfaces coincide precisely with the pseudo-Einstein hypersurfaces, i.e. \( \rho \) and \( \sigma \) must in fact be constant. In the survey paper of Niebergall and Ryan [20] the following open problem is listed:

- The Cecil-Ryan theorem shows us that the assumption that \( \sigma \) and \( \rho \) are constant is unnecessary in Kon’s work. Is the analogous statement true for complex hyperbolic space?

In other words, if \( M \subset \mathbb{C}H^n, n \geq 3 \) is a generalized pseudo-Einstein hypersurface, must it in fact be a pseudo-Einstein hypersurface? If this were true, Montiel’s work would classify such hypersurfaces in \( \mathbb{C}H^n, n \geq 3 \). Recently Kim and Ryan [14] have classified generalized pseudo-Einstein hypersurfaces in \( \mathbb{C}P^2 \) and Ivey and Ryan [11] classified such hypersurfaces in \( \mathbb{C}H^2 \), so the classification of generalized pseudo-Einstein hypersurfaces in complex space forms with \( n = 2 \) is settled. We establish:

**Theorem 1.5.** Let \( M \) be a generalized pseudo-Einstein hypersurface of complex hyperbolic space \( \mathbb{C}H^n, n \geq 3 \). Then \( M \) is congruent to an open part of:

1. a tube of radius \( r \in \mathbb{R}^+ \) around a totally geodesic \( \mathbb{C}H^k \subset \mathbb{C}H^n \), where \( k = 0 \) or \( n - 1 \), or
2. a horosphere.

**2. Curvature-adapted foliations**

For this section, we set our ambient manifold to be \( \overline{M} = \mathbb{C}H^n \). We denote the Kähler structure by \( J \) and the Levi-Civita connection by \( \nabla \). The curvature tensor \( \overline{R} \) is given as

\[
\overline{R}(X,Y)Z = -\langle Y,Z \rangle X + \langle X,Z \rangle Y \\
- \langle JY,Z \rangle JX + \langle JX,Z \rangle JY + 2\langle JX,Y \rangle JZ
\]

We first recall a theorem of Gray ([9], Theorem 6.14):

**Theorem 2.1.** Let \( P \subset \overline{M} \) be an embedded curvature-adapted submanifold. Then

- any tube around \( P \) is also curvature-adapted, and
- the common eigenspace \( E(C_\xi(t)) \) of the shape operator \( A_\xi(t) \) of the tube around \( P \) and the normal Jacobi operator \( K_\xi(t) \) may be chosen parallel along geodesics normal to \( P \).

Given a hypersurface \( M = M_0 \subset \overline{M} \), the parallel hypersurface \( M_a \) is defined by, for each point \( p \in M \), flowing by a distance \( a \) along the geodesic \( C_\xi(t) = \exp_p(t\xi) \) which passes through \( p \) with initial direction \( \xi \). The Riccati equation describes the evolution of the shape operator \( A_\xi(t) \) of nearby parallel hypersurfaces in terms of its derivative and the normal Jacobi operator.
$K_\xi$ along the geodesic $C_\xi(t)$: $p \in M$:
\begin{equation}
A'_\xi(t) = (A_\xi(t))^2 + K_\xi(t),
\end{equation}
with $A_\xi(0)$ the shape operator of $M_0$ at $p$. This equation relates the shape operator of nearby parallel hypersurfaces to $K_\xi$. Taking a curvature-adapted hypersurface simplifies this to a family of ordinary differential equations, namely
\[\lambda'_i(t) = (\lambda_i(t))^2 + \kappa_i,\]
for $i = 1, \ldots, \dim(M) - 1$, with initial conditions $\lambda_i(0)$ being the principal curvatures of $M_0$ at $p$. Here $\kappa_i = -1$ or $-2$. This suggests that the investigation of nearby parallel hypersurfaces to a given curvature-adapted hypersurface might be profitable. For the rest of this section we assume that $M$ is curvature-adapted.

Associated to $M$ is the induced Levi-Civita connection $\nabla$, and we denote its curvature tensor by $R$. Let $TM$ denote the tangent bundle of $M$ and $\nu(M)$ the normal bundle. Set $U = -J\xi$ and let $\mathcal{D} := (RU)^\perp$ denote the maximal complex subbundle of $TM$.

The Riccati equation for hypersurfaces is equivalent to the Jacobi equation, where there are also well-established techniques to describe the principal curvatures of nearby parallel hypersurfaces. Recall that $J$ is said to be an $M$-Jacobi field if it is a non-zero Jacobi field along $C_\xi$ satisfying the initial conditions $J(0) \in T_pM$ and $J'(0) = -A_\xi(J(0))$. A focal point of $M$ along $C_\xi$ is given as $C_\xi(t_0)$ when $J(t_0) = 0$ for an $M$-Jacobi vector field. We refer the reader to [5], Chapter 8 for further details about $M$-Jacobi vector fields. Take $E_i(C_\xi(t))$ to be a basis for the $M$-Jacobi vector fields along $C_\xi$ from $M$. As $M$ is curvature-adapted, $E_i$ being a focal point is equivalent to the corresponding principal curvature function developing a singularity (i.e. one has $\lambda_i(t_0) = \infty$). Let us now show how this is the case.

Suppose that $F$ is a singular Riemannian foliation with a singular leaf $P$. We firstly explain how a regular leaf $M$ of $F$ and the singular leaf $P$ are related.

One can calculate the principal curvatures of the tube around $P$ using the same calculation as in Lemma 7.8 of [9]. The Riccati equation for the tube around a curvature-adapted submanifold $P$ simplifies to the following family of equations along $C_\xi(t)$, where $C_\xi(0) = p \in P$:
\[\lambda'_i(t) = \lambda_i^2(t) + \kappa_i,\]
for $i = 1, \ldots, \dim(M) - 1$, with initial conditions
\begin{enumerate}
\item $\lambda_i(0) = \lambda_i(p)$, for $i = 1, \ldots, \dim(P)$,
\item $\lambda_i(0) = -\infty$, for $i = \dim(P) + 1, \ldots, 2n - 1$.
\end{enumerate}

Now one can start with a regular leaf $M$ which we take to be the tube of radius $t_0$ around $P$. Then one can calculate the principal curvatures of the tube around $M$ along the same geodesic, but this time with initial conditions the point $q \in M$ and initial normal vector $\bar{\xi}$. Here $\bar{\xi}(0) = -\xi(t_0)$,
etc. Travelling back along $C_p(t)$ to the point $p \in P$, the vector fields in $E_i(t)$ whose corresponding principal curvature functions “focalize” at $P$ (i.e. the functions $\lambda_i(t)$ which become infinite at $p$) in the eigenbasis $E(C_p(t))$ are precisely the $M$-Jacobi vector fields which have a focal point at the point $p$. This follows from the equation $E'_i(t) = -\lambda_i(t)(E_i(t))$. We say $P$ is the focal leaf of $M$.

Let us state the following well-known result:

**Theorem 2.2.** [3] A hypersurface $M \subset \mathbb{C}H^n$ with constant principal curvatures is Hopf if and only if it is locally congruent to an open part of:

- a tube of radius $r \in \mathbb{R}^+$ around a totally geodesic $\mathbb{C}H^k \subset \mathbb{C}H^n$ for $k = 0, \ldots, n - 1$,

- a tube of radius $r \in \mathbb{R}^+$ around a totally geodesic $\mathbb{R}H^n \subset \mathbb{C}H^n$,

- a horosphere.

Let $\sigma_p(\mathcal{D})$ denote the spectrum of $A_\xi|_{\mathcal{D}_p}$. Set $T_\lambda = \ker(A_\xi|_{\mathcal{D}_p} - \lambda \text{id}_p)$ to be the eigenspace associated with $\lambda \in \sigma_p(\mathcal{D})$.

**Theorem 2.3.** (See [2], [6].)

1. Suppose $M \subset \mathbb{C}H^n$ is a degenerate Hopf hypersurface. Then $1 \in \sigma_p(\mathcal{D})$, for all $p \in M$ and $J T_\lambda(p) \subset T_1(p)$ for all $p \in M, \lambda \in \sigma_p(\mathcal{D}) \setminus \{1\}$.

2. Suppose $M \subset \mathbb{C}H^n$ is a Hopf hypersurface with small Hopf curvature. Then given $\lambda \in \sigma_p(\mathcal{D})$, an associated eigenvector $Y$ satisfies $A_\xi(J Y) = \lambda^* (J Y)$ where $\lambda^*$ satisfies the equation

$$0 = (2\lambda^* - \alpha)(2\lambda - \alpha) = \alpha^2 - 4.$$  

Choose $p \in M \subset \mathbb{C}H^n$, with corresponding section $C_\xi$ through a point $p$ and suppose $M$ is degenerate. Choose an orthonormal framing $E_i(t)$ along $C_\xi(t)$ such that $E_1(t) = -J\xi(t)$, and for $E_{2k}(t) \in T_\lambda(t)$, $k \geq 1$, we take $E_{2k+1}(t) = J E_{2k}(t) \in T_1(t)$. We may always choose such a framing by Theorem 2.3, and this framing is parallel along $C_\xi$ (i.e. $\nabla_\xi E_i = 0$). Similarly if $M$ has small Hopf curvature choose $E_{2k}(t) \in T_\lambda(t)$ and $E_{2k+1}(t) = J E_{2k}(t) \in T_\lambda(t)$, with $k \geq 1$.

In the above we have lightly abused notation by replacing $C_\xi(t)$ with $t$ for ease of exposition when the context is clear. We remark that for the case $M = \mathbb{H}H^n$ results which are direct analogues of the above results are to found in [4]. This allows one to prove precisely analogous results.

3. **Proof of Theorem 1.4**

Proof. The foliation induced by taking the tubes of radius $t > 0$ around a totally geodesic $\mathbb{R}H^n \subset \mathbb{C}H^n$, together with the focal set $\mathbb{R}H^n$, has small
Hopf curvature. This may be calculated using the \( M \)-Jacobi field theory outlined, and is carried out in [3].

Suppose conversely \( F \) satisfies the assumptions of the theorem and has a regular leaf \( M \) with small Hopf curvature. Pick a point \( p \in M \). Choose the framing \( E_i(t) \) along \( C_\xi(t) \) as explained in the last section. Then from Equation (2.2) we see precisely half the principal curvatures lying in \( \sigma_p(2\xi) \) must focalize. This is because small Hopf curvature implies that \((2\lambda_i - \alpha)(2\lambda_i^* - \alpha)\) is now negative, so one of these two terms is positive. In particular, we may assume \( \lambda_i > \frac{\alpha}{2} \) without loss of generality. Solving the corresponding Riccati equation yields

\[
\lambda_i(t) = \coth(\theta_i(p) - t),
\]

where \( 0 < \theta_i(p) < \infty \) is chosen so that \( \coth(\theta_i) = \lambda_i(p) \). This focalizes at distance \( t = \theta_i \). As \( F \) is a Riemannian foliation, the distance between this focal leaf and \( M \) is constant so this implies that \( \theta_i \) is independent of our choice of \( p \). Thus \( \lambda_i \) is locally constant and has constant multiplicity. Then Equation (2.2) implies \( \lambda_i^* \) is also constant. Thus it follows that \( M \) is a Hopf hypersurface with constant principal curvatures and focal set a totally real submanifold, and we are done by the main result of [3]. \( \square \)

4. DEGENERATE HOPF HYPERSURFACES

The class of degenerate hypersurfaces merits further investigation as the usual equations stemming from the Gauss-Codazzi-Ricci equations break down. For this reason the usual techniques do not work, and so many classification results for Hopf hypersurfaces assume non-degeneracy. Böning [6] observes that the horosphere is the only known degenerate Hopf hypersurface: the obvious question is if there are any more. Böning [6] also classified non-degenerate Hopf hypersurfaces with at most three principal curvatures in \( \mathbb{C}^n \), \( n \geq 3 \); they are all homogeneous. In recent work, Ivey and Ryan [11] have associated a degenerate Hopf hypersurface \( M^3 \) of \( \mathbb{C}H^2 \) to any contact curve in \( S^3 \) using exterior differential systems. Their construction actually associates a contact curve in \( S^3 \) to an adapted lift of \( M, M \subset SU(2,1) \). In the degenerate case \( M \) depends on one function of one variable. However they cannot explicitly compute the principal curvatures of \( M^3 \) from \( M \) and so it is still an open problem to construct a degenerate Hopf hypersurface of \( \mathbb{C}H^2 \) (or \( \mathbb{C}H^n \) in general) which is not the horosphere.

It is obvious that the unique degenerate curvature-adapted foliation of \( K H^n \), \( K = \mathbb{C} \) or \( \mathbb{H} \) where the non-Hopf principal curvatures of the regular leaves satisfy \( |\lambda_i(p)| \geq 1 \) is the horospherical foliation, because from the Riccati equation one sees that each regular leaf must have constant principal curvatures and we have already remarked that such hypersurfaces are classified. We also have the following result:

**Proposition 4.1.** There is no degenerate curvature-adapted foliation of \( \mathbb{C}H^n \) with a unique minimal singular leaf.
Proof. In our notation, $P$ denotes the minimal singular leaf and $M$ a regular leaf. Fix $p \in M$ and consider the section $C_\xi$ passing through $p$, where $\xi$ points in the direction of $P$. Let $\theta_1$ denote the distance between $M$ and $P$ along $C_\xi$. As each regular leaf of $F$ is degenerate, the Hopf principal curvature $\alpha = 2$ is constant along $C_\xi$ and does not focalize at $P$. Let $\lambda_1 = \coth(\theta_1) \in \sigma_p(D)$ denote the principal curvature of the shape operator of $M$ which focalizes at $P$. Then by Theorem (2.3) $JT\lambda_1(0) \in T_1(0)$. This vector subspace is invariant under parallel translation along $C_\xi$ and so corresponds to a subspace of $T_{C_\xi(\theta_1)}P$. Any other principal curvature function $\lambda_i$, $i > 1$ of $X$ at $T_{C_\xi(\theta_1)}X$ has corresponding eigenspace $T_{\lambda_i}$, and from the known spectral data $JT\lambda_i \subset T_1$. But $|\lambda_i| < 1$ for $i > 1$ as there is a unique singular leaf. Otherwise it would follow from the Riccati equation that there is another solution of the shape operator of the regular leaves of the form $\lambda_2(t) = \coth(\theta_2 - t)$, $\theta_2 \neq \theta_1$ and so a second focal submanifold would exist. From this it follows that $P$ cannot be minimal. To see this, observe that at least half of the principal curvatures in $\sigma(D)$ are $+1$, so they outweigh the other principal curvatures which all have $|\lambda_i| < 1$ and so $P$ cannot be minimal. \[\square\]

Again there is an analogous result for $\mathbb{H}H^n$.

5. Applications to the study of almost-contact structures

In this section, $\overline{\mathbb{M}}$ is either $\mathbb{CP}^n$ or $\mathbb{CH}^n$, and $M \subset \overline{\mathbb{M}}$ is a hypersurface. Let $P$ denote the skew-symmetric $(1,1)$ tensor field on $M$ given by $JX = PX + \langle X, U \rangle \xi$. This is the induced almost contact structure on $TM$. By the Gauss formula and the Weingarten equation we obtain

$$\nabla_XU = PA_\xi(X).$$

Recall the well-known fact that the shape operator of a real hypersurface of $\mathbb{CH}^n$ or $\mathbb{CP}^n$ cannot vanish: in fact

$$\|\nabla A\|^2 \geq 4(n-1)$$

in the standard scaling. Equality in the above bound is achieved precisely by hypersurfaces whose shape operators and induced almost-contact structure commute;

$$A_\xi \circ P = P \circ A_\xi.$$

Thus classifying which hypersurfaces in non-flat complex space forms achieve equality is a natural question. We now outline a simplified proof of the work of Okumura [21] and Montiel and Romero [19], who answered this question.

If $P \circ A_\xi = A_\xi \circ P$ then it is easy to see $M$ is Hopf. We need now the fact that, for $\overline{\mathbb{M}} = \mathbb{CP}^n$ there is an exact analogue of the framing along $C_\xi(t)$ and, as shown in [6], Equation (2.2) also holds for this framing. Suppose that $M$ is non-degenerate (the degenerate case is analogous, using Theorem
Then given $\lambda \in \sigma_p(\mathfrak{D})$, Equation (2.2) together with the equation $P \circ A_\xi = A_\xi \circ P$ implies $\lambda = \lambda^*$. Thus

$$(2\lambda - \frac{\alpha}{2})^2 = \alpha^2 \pm 4,$$

so $\lambda$ is constant. Hence $M$ is a Hopf hypersurface with constant principal curvatures of a non-flat complex space form, and a case-by-case check shows that $M$ is isometric to one of

1. a tube of radius $t$ around a totally geodesic $\mathbb{C}P^k \subset \mathbb{C}P^n$, $0 \leq k \leq n - 1$, $0 < t < \frac{\pi}{2}$,
2. a tube of radius $t$ around a totally geodesic $\mathbb{C}H^k \subset \mathbb{C}H^n$, $0 \leq k \leq n - 1$, $0 < t < \infty$,
3. a horosphere of $\mathbb{C}H^n$.

We note that in [16] the authors investigated in $\mathbb{H}H^n$ an analogous problem to that studied by Romero and Montiel in $\mathbb{C}H^n$. Again, every real hypersurface $M \subset \mathbb{H}H^n$ satisfies a similar bound. They prove that equality in the bound is achieved precisely by hypersurfaces satisfying

$$A_\xi \circ P_i = P_i \circ A_\xi,$$

$i = 1, 2, 3$. Here $P_i$ is the restriction of $J_i$ to $TM$, where $J_i$, $i = 1, 2, 3$ denotes a local section of the quaternionic-Kähler structure. It is easy to see such hypersurfaces are curvature-adapted. Then a long calculation shows that the only such hypersurfaces are horospheres and tubes over totally geodesic $\mathbb{H}H^k, 0 \leq k < n$. Just as in the case of $\mathbb{C}H^n$ one may instead apply the analogous spectral data contained in [4], Theorem 4.18 to avoid this calculation and shorten their proof.

6. Generalized pseudo-Einstein hypersurfaces

We come now to the complete classification of generalized pseudo-Einstein hypersurfaces of $\mathbb{C}H^n$ (Theorem 1.5).

Proof. From the Gauss Equation one calculates that for a real hypersurface $M \subset \mathbb{C}H^n$, one has

$$SX = -(2n + 1)X + 3\langle X, U \rangle U + mA_\xi(X) - A_\xi^2(X),$$

for $X \in TM$, where $m = tr(A_\xi)$ denotes the mean curvature of $M$.

As has been outlined in the introduction, the only remaining case is that where $M \subset \mathbb{C}H^n$ satisfies the assumptions of the theorem with $n \geq 3$. The proof of Proposition 5.2, 5.3 and 5.4 in [17] goes through (adjusting for the sign of the curvature tensor in the non-compact case), so $M$ is a Hopf hypersurface with at most three principal curvatures. By Bönig's theorem we may assume $M$ is degenerate. Then by the spectral data given in Theorem (2.3) the principal curvatures are 2 with eigenspace $U$, 1 with eigenspace $T_1$, and $\lambda$ with eigenspace $T_\lambda$ and moreover $JT_\lambda \subset T_1$. Set $k = \dim(T_\lambda)$: this must be locally constant. It is standard theory [20] to
show the restriction of $A_\xi$ to $\mathcal{D}$ has eigenvalues given as the root of the equation

$$A_\xi^2 - mA_\xi + (2n + 1 + \rho)Id = 0.$$ 

Hence

$$m = 2 + k(\lambda) + (2n - 2 - k) = \lambda + 1.$$  

Since $n \geq 3$, $k > 1$ and so one obtains that $\lambda$ is locally constant. Hence $M$ has constant principal curvatures, and we are done. \hfill \Box

P. Ryan has a proof of this result \cite{Ryan} which is independent of Böning’s work. This involves a detailed calculation of the Ricci tensor of $\mathbb{C}H^n$ and an investigation of the various possibilities for the dimensions of $T_\lambda$ and $T_1$.

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Acknowledgements This work was completed as part of a Ph.D. under the supervision of Professor Jürgen Berndt at University College Cork, Ireland. The author would like to thank him for his advice, Pat Ryan for stimulating discussions, and especially the referee, whose comments and suggestions greatly improved this paper. This work was supported by a postgraduate fellowship awarded by the Irish Research Council for Science, Engineering and Technology.

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