Gelfand type problem for two phase porous media

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Abstract

We consider a generalization of the Gelfand problem arising in Frank-Kamenetskii theory of thermal explosion. This generalization is a natural extension of the Gelfand problem to two phase materials, where, in contrast to the classical Gelfand problem which utilizes single temperature approach, the state of the system is described by two different temperatures. We show that similar to the classical Gelfand problem the thermal explosion occurs exclusively due to the absence of stationary temperature distribution. We also show that the presence of inter-phase heat exchange delays a thermal explosion. Moreover, we prove that in the limit of infinite heat exchange between phases the problem of thermal explosion in two phase porous media reduces to the classical Gelfand problem with renormalized constants.

1 Introduction

Superlinear parabolic equations and systems of such equations serve as mathematical models of many nonlinear phenomena arising in natural sciences. It is well known that such models may often produce solutions that do not exist globally in time due to formation singularities. In particular, there are solutions which become infinite either somewhere or everywhere in the spatial domain in a finite time. Formation of such singularities is commonly referred to as blow up and has attracted considerable attention of scientists and engineers over past decades [6,9,16]. The classical problem in a theory of blow up for nonlinear parabolic equations, which is widely known in mathematical literature as a Gelfand problem, reads

\[
\begin{cases}
W_t - \Delta W = \Lambda g(W) & \text{in} \ (0, T) \times \Omega, \\
W = 0 & \text{on} \ \partial \Omega, \\
W(0, \cdot) = 0 & \text{in} \ \Omega,
\end{cases}
\]

(1.1)

where \( \Omega \subset \mathbb{R}^N \) is a smooth bounded domain, \( g : \mathbb{R} \to (0, \infty) \) is a \( C^1 \) convex non-decreasing function satisfying

\[
\int_{x_0}^{\infty} \frac{ds}{g(s)} < \infty \quad \text{for some} \quad x_0 \geq 0,
\]

(1.2)

and \( \Lambda > 0 \) is a parameter. This problem was originally introduced in a context of therm-diffusive combustion as a model of thermal explosion, the spontaneous development of rapid

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rates of heat release by chemical reactions in combustible mixtures and materials being initially in a non-reactive state \cite{17,20}. The model \eqref{1.1} describes an evolution of initially uniform temperature field $W$ which diffuses in space, increases in a bulk due to the heat release described by a reaction term $\Lambda g$ and is fixed on the boundary (cold boundary). The model \eqref{1.1} was derived by Frank-Kamenetskii \cite{8} as a short time asymptotic of a standard thermo-diffusive model and describes an initial stage of self-ignition of combustible mixture.

Depending on the parameters of this problem the solutions of \eqref{1.1} either blow up or exist globally. In a context of combustion the first case corresponds to successful initiation of combustion process whereas the second one corresponds to the ignition failure. Basic physical reasoning discussed in \cite{8,20} and formal (intermediate asymptotics) arguments of Barenblatt presented in \cite{10} suggest that blow up in model \eqref{1.1} occurs exclusively due to the absence of stationary solutions for this problem. That is the absence of stationary temperature distribution $w$ that solves, in a weak sense, the following time independent problem

\[
\begin{aligned}
-\Delta w &= \Lambda g(w), & w > 0 & \text{in } \Omega, \\
 w &= 0 & \text{on } \partial\Omega.
\end{aligned}
\]  

(1.3)

These formal arguments of \cite{10} were made rigorous in \cite{1}. The following theorem summarizes the main results regarding solutions of problems \eqref{1.1} and \eqref{1.3}, see \cite{1,3,Theorem 3.4.1} and further references therein.

**Theorem A.** Parabolic problem \eqref{1.1} has a classical global solution if and only if stationary problem \eqref{1.3} has a weak solution.

There exists $0 < \Lambda^* < \infty$ such that:

i) for $\Lambda > \Lambda^*$ problem \eqref{1.3} has no weak solutions,

ii) for $0 < \Lambda < \Lambda^*$ problem \eqref{1.3} has a minimal classical solution $w_\Lambda$,

iii) $w_\Lambda(x)$ is a monotone increasing functions of $\Lambda$, and for $\Lambda = \Lambda^*$ problem \eqref{1.3} has a weak solution $w^*$ defined by

\[
w^*(x) := \lim_{\Lambda \to \Lambda^*} w_\Lambda(x).
\]  

(1.4)

The statement of Theorem A, from a physical perspective, has a very clear interpretation. Indeed, the parameter $\Lambda$ can be understood as a scaling factor that reflects the size of the domain, which increases as $\Lambda$ increases. Thus, in relatively small domains the cold boundary suppresses intensive chemical reaction in the bulk which leads to a stationary temperature distribution, whereas when the size of the domain exceeds some critical value corresponding to $\Lambda^*$ the cooling on the boundary becomes insufficient to prevent chemical reaction inside the domain $\Omega$, which leads to thermal explosion.

The classical model \eqref{1.1} asserts that the process of combustion can be described using a unified single temperature approach. This assumption, which has rather wide range of validity, however, is not applicable in certain situations. For example in combustion of porous materials the difference of temperatures of gaseous and condensed phases can be substantial, which changes a combustion process \cite{17}. As a result the model describing self ignition of porous media has to be appropriately modified. Let us note that explosion in two phase materials has many technological applications ranging from ignition of metal nano-powders and solid rocket propellants to issues of safe storage of nuclear waste and industrial raw garbage \cite{4,17}.

In order to describe explosion in two phase materials one may adopt an approach of Frank-Kamenetskiii and make a standard reduction of governing equations describing combustion of
two phase porous materials. The conventional system of equation for the dynamics of two phase material are well known and we refer the reader to [14] for the details. Partial linearization of these equations incorporating Frank Kamenetskii transform [8,20] lead to a following system

\[
\begin{align*}
U_t - \Delta U &= \lambda g(U) + \nu(V - U), \\
\alpha V_t - d \Delta V &= \nu(U - V) & \text{in } (0,T) \times \Omega, \\
U &= V = 0 & \text{on } \partial \Omega, \\
U(0,\cdot) &= V(0,\cdot) = 0 & \text{in } \Omega,
\end{align*}
\]

(1.5)

here \(U(t,x)\) and \(V(t,x)\) are appropriately normalized temperatures of condensed (solid) and gaseous phases respectively and \(d > 0\) is a ratio of effective gaseous and thermal diffusivity, \(\nu > 0\) is inter-phase heat transfer coefficient and \(\alpha > 0\) is a parameter which depends on porosity and ratios of specific heats of the solid and gaseous phases. It is important to note that the model (1.5) is formally identical to the one describing formation of hot spots in transistors. In this case variables \(U\) and \(V\) can de interpreted as temperatures of electron gas and of the lattice respectively see e.g. [12].

As one may expect the behavior of solutions for the problem (1.5) depends crucially on existence of stationary solutions for the time independent problem

\[
\begin{align*}
-\Delta u &= \lambda g(u) + \nu(u - v), \\
-d \Delta v &= \nu(u - v) & \text{in } \Omega, \\
u, v > 0 & \text{ in } \Omega, \\
u = v &= 0 & \text{on } \partial \Omega.
\end{align*}
\]

(1.6)

The goal of this paper is to study dynamics of solutions for the problem (1.5) and and its stationary states described by (1.6). There are two main results of this paper. Our first result states that similar to the classical Gelfand problem blow up in system (1.5) is fully determined by solutions of problem (1.6). Namely the following holds.

**Theorem 1.1.** If elliptic problem (1.6) has a classical solution, then parabolic problem (1.5) has a global classical solution. If parabolic problem (1.5) has a global classical solution, then elliptic problem has a weak solution. Moreover, the global classical solution of (1.5) converges in \(L^1\)-norm to a minimal weak solution of (1.6) as \(t \to \infty\).

The precise definition of a minimal classical and weak solution of stationary problem (1.6) will be given later in Section 3. Here we note only that every classical solution is also a weak solution. On the other hand weak solutions may have singularities.

In a view of this result a detailed information on stationary solutions is needed. This is given by the following theorem.

**Theorem 1.2.** Let \(d > 0\). Then for every \(\nu > 0\) there exists \(0 < \lambda^*_\nu < \infty\) such that:

i) for \(\lambda > \lambda^{*}_\nu\) system (1.6) has no classical solutions,

ii) for \(0 < \lambda < \lambda^{*}_\nu\) system (1.6) has a minimal classical solution \((u_{\lambda,\nu}, v_{\lambda,\nu})\),

iii) for \(\nu > 0\) both \(u_{\lambda,\nu}(x)\) and \(v_{\lambda,\nu}(x)\) are monotone increasing functions of \(\lambda\) for every \(x \in \Omega\), and for \(\lambda = \lambda^{*}_\nu\) system (1.6) has a weak solution \((u^*_\nu, v^*_\nu)\) defined by

\[
u^*_\nu(x) := \lim_{\lambda \to \lambda^{*}_\nu} u_{\lambda,\nu}(x), \quad v^*_\nu(x) := \lim_{\lambda \to \lambda^{*}_\nu} v_{\lambda,\nu}(x) \quad (x \in \Omega).
\]

(1.7)
iv) \( \lambda^*_\nu \geq \Lambda^* \) and \( \lambda^*_\nu = \lambda^*(\nu) \) is a nondecreasing function of \( \nu > 0 \) having the following properties

\[
\lim_{\nu \to 0} \lambda^*(\nu) = \Lambda^*, \quad \lim_{\nu \to \infty} \lambda^*(\nu) = \Lambda^*(1 + d),
\]

where \( \Lambda^* \) is the critical value of the classical Gelfand problem \((1.3)\).

v) for \( \lambda < \lambda^*_\nu \), \( u_{\lambda,\nu}(x) \) is a non-increasing function of \( \nu \) for every \( x \in \Omega \). For \( \lambda < \Lambda^* \) and \( \nu \to 0 \) solution \( (u_{\lambda,\nu}, v_{\lambda,\nu}) \) converges uniformly to \( (u_0, 0) \), where \( u_0 \) is the minimal solution of

\[
\begin{align*}
-\Delta u_0 &= \lambda g(u_0), & u_0 > 0 & \text{in } \Omega, \\
 u_0 &= 0 & \text{on } \partial \Omega.
\end{align*}
\]

For \( \lambda < \Lambda^*(1 + d) \) and \( \nu \to \infty \) solution \( (u_{\lambda,\nu}, v_{\lambda,\nu}) \) converges uniformly to \( (u_\infty, u_\infty) \), where \( u_\infty \) is the minimal solution of

\[
\begin{align*}
-\Delta u_\infty &= \frac{\lambda}{1 + d} g(u_\infty), & u_\infty > 0 & \text{in } \Omega, \\
 u_\infty &= 0 & \text{on } \partial \Omega.
\end{align*}
\]

Remark 1.1. The limit weak solution \( (u^*_\nu, v^*_\nu) \), constructed in \((iii)\) might be either classical or singular. In the proof of part \((i)\) of Theorem \((1.2)\) we show that there exists \( \lambda^{**}_\nu > \lambda^*_\nu \) such that system \((1.6)\) has no weak solutions for \( \lambda > \lambda^{**}_\nu \). In the case of the single equation \((1.3)\) it is known that actually \( \lambda^{**} = \lambda^* \), see \((i)\) and \((ii)\) of Theorem A. One may expect that a similar result holds for the system considered in this paper. However, the proof of this fact is very delicate even in the case of a single equation, see [1, Theorem 3] and further discussion in [2].

We also want to point out that in the case of single equation \((1.3)\) weak solutions corresponding to \( \lambda = \lambda^* \) in most of the cases relevant to applications are in fact classical.

The Theorem \((1.2)\) proves that solutions for the problem \((1.6)\) behave similarly to solutions of the classical Gelfand problem and that the presence of the heat exchange increases the value of critical parameter \( \lambda^* \). These results are quite in line with the physical intuition behind this problem. What we found rather surprising is the limiting behavior of solutions of \((1.6)\) when \( \nu \to \infty \). Indeed, it is quite remarkable that the substantial heat exchange between the two phases reduces problem \((1.6)\) to the classical Gelfand problem with re-normalized parameters.

We also note that this observation in fact also justifies the use of single temperature model as effective models for two phase materials in this asymptotic regime.

The paper is organized as follows: in section 2 we give some basic heuristic arguments and present numerical examples which clarify the main results. Sections 3 and 4 are dedicated to the proof of Theorems \((1.2)\) and \((1.1)\) respectively.

## 2 Heuristic arguments and numerical example

In this section we would like to give some formal arguments and provide results of numerical simulations of problems \((1.5), (1.6)\) that clarify and illustrate results of Theorems \((1.1)\) and \((1.2)\).

Theorem \((1.1)\) basically states that the presence or absence of global solutions for problem \((1.5)\) is fully determined by the presence or absence of solutions for system \((1.6)\). Thus, the behavior of solutions for system \((1.5)\) is essentially similar to the behavior of solutions for single equation \((1.1)\). As it is well known, the dynamics of a system of parabolic equations, in general,
is substantially more complex than the one of a single equation. However, in our case the system \((1.1)\) is, at least formally, can be written as a gradient flow

\[
U_t = -\frac{\delta E}{\delta U}, \quad \alpha V_t = -\frac{\delta E}{\delta V}
\]

with the ”energy” functional defined as

\[
E(U,V) = \frac{1}{2} \int_{\Omega} \left[ |\nabla U|^2 + d|\nabla V|^2 + \nu(U-V)^2 + 2\lambda\Phi(U) \right] dx, \quad \Phi(U) = -\int_0^U g(s) ds. \tag{2.1}
\]

Moreover, the system \((1.3)\) is quasi-monotone (quasi-monotone nondecreasing in terminology of \([15]\)) and thus its classical solutions obey the component-wise parabolic comparison principle \([15, \text{Theorem 3.1, p. 393}]\) or \([18, \text{Theorem 3.4, p.130}]\). Thus the time evolution for solutions of \((1.5)\) is very much restricted and indeed expected to be similar to the one of a single equation. This situation is somewhat similar to the one considered in \([11]\) where self explosion in confined porous media was considered.

Now let us turn to Theorem 1.2. First we note that transition from existence to non-existence of solution in this problem is very similar to the one observed in the classical Gelfand problem. This is again occurs due to the presence of a component-wise comparison principle and the fact that system \((1.6)\) is the Euler-Lagrange equation of the functional

\[
E(u,v) = \frac{1}{2} \int_{\Omega} \left[ |\nabla u|^2 + d|\nabla v|^2 + \nu(u-v)^2 + 2\lambda\Phi(u) \right] dx, \quad \Phi(u) = -\int_0^u g(s) ds. \tag{2.2}
\]

In order to understand monotonicity with respect to parameter \(\nu\) it is convenient to rewrite system \((1.6)\) as a non-local equation. Combining the first and the second equations of the system \((1.6)\) we have

\[
[-\gamma\nu^{-1}\Delta + 1](u-v) = \lambda\gamma
\]

where \(\gamma = \frac{d}{1+d}\). Thus,

\[
u - v = \lambda\gamma^{-1}[-\gamma\nu^{-1}\Delta + 1]^{-1}g(u). \tag{2.5}
\]

Substituting this expression in the first equation of the system \((1.6)\) yields

\[
-\Delta u = \frac{\lambda}{1+d} \left(1 + d \left\{ 1 - [-\gamma\nu^{-1}\Delta + 1]^{-1} \right\} \right) g(u). \tag{2.6}
\]

As it is readily seen the operator in a curly brackets is an increasing function of \(\nu^{-1}\). This implies that the effective right hand side of this equation is a decreasing function of \(\nu\) and thus one may expect that \(u\) decreases as \(\nu\) increases. Moreover, in the limiting case \(\nu \to \infty\) the right hand side becomes essentially local. Let us also note that for sufficiently large \(\nu\) the components \(u\) and \(v\) of the system \((1.6)\) away from the boundary are related (at least formally) by a simple formula \(v = u - \nu^{-1}\lambda g(u)/(1+d) + o(\nu^{-1})\) which follows directly from \((2.5)\).

In order to illustrate statements of Theorems 1.1, 1.2 we performed numerical studies of a simple one dimensional versions of problems \((1.5), (1.6)\) with \(\Omega = (-1,1), d = \alpha = 1\) and \(g(u) = e^u\). Physically these two problems describe stationary temperature distributions and
evolution of temperature fields in plane-parallel vessel under assumption of Arrhenius chemical kinetics.

Let us first consider the stationary problem (1.6). Solution of this problem was obtained numerically using the conventional shooting method. The numerical study shows that, in a full agreement with the statement of Theorem 1.2, stationary temperature distribution exists only for values of scaling parameter $\lambda(\nu)$ which does not exceed some critical value $\lambda^*(\nu)$. Moreover, this critical value $\lambda^*(\nu)$ is an increasing function of the heat exchange parameter $\nu$ and has following asymptotic properties: $\lambda^*(0) = \Lambda^*$, where $\Lambda^*$ is a critical value of the classical Gelfand problem (1.3) (in the considered case $\Lambda^* \approx 0.88$, see e.g. [8]) and $\lambda^*(\infty) = (1 + d)\Lambda^* \approx 1.76$. The dependency of the critical value $\lambda^*(\nu)$ as a function of $\nu$ is shown on Figure 1. In addition for a fixed value of $\lambda < \lambda^*$ one can see that $u$ component describing the temperature of the solid phase is decreasing monotonically as $\nu$ increases, while the temperature of the gas phase ($v$ component) is bounded from above by $u$ and approaches to the latter from below as $\nu$ increases see Figure 2.

Finally, let us illustrate dynamical features of the combustion process given by the theorem 1.1. For this reason let us consider a one dimensional problem (1.5) with all the parameters as
Figure 3: Solution of (1.5) with $\Omega = (-1,1)$ and $g(u) = e^u$, $d = 1$, $\alpha = 1$, $\nu = 5$ and $\lambda = 1.2$ (a) and $\lambda = 1.5$ (b) in the middle of an interval $\Omega$ where solution has its maximum value as long as it exists. $U(0,t)$ is solid line, $V(0,t)$ is dashed line.

above and for a fixed value of $\nu = 5$ and two values of $\lambda = 1.2$ and $\lambda = 1.5$, one of which is below critical $\lambda^*(5) \approx 1.468$ and the other is above critical. Figure 3 shows the time evolution of temperatures of gas and solid phases in the middle of the vessel $x = 0$ where both temperatures of the solid and the gas have its maximal values as long as the solution exists. As predicted by Theorem 1.1 in case of sub-critical $\lambda$ (Figure 3a) the solution, after some short transition period, approaches to its steady state, whereas for supercritical $\lambda$ (Figure 3b) the solution rapidly accelerates and becomes infinite (blows up) in finite time.

3 Stationary problem: Proof of Theorem 1.2

In this section we study solutions of the stationary system (1.6) and discuss their qualitative properties.

First, let us note that if $\lambda = 0$ then system (1.6) becomes linear and has a cooperative structure for all $d, \nu > 0$. Further, since $g(u)$ is a monotone non-decreasing function, (1.6) is a quasi-monotone non-decreasing nonlinear system in the sense of [15, Theorem 4.1, p.406] for every $d, \nu, \lambda > 0$ and thus can be studied using comparison type of arguments.

We start with definitions of weak solution of problem (1.6) as well as weak sub and super-solutions for this system.

Similarly to [1], we say $(u,v)$ is a weak solution of system (1.6) if $u, v \in L^1(\Omega)$, $g(u)\delta(x) \in L^1(\Omega)$, where $\delta(x) := \text{dist}(x, \partial \Omega)$, and

$$-\int_\Omega u\Delta \phi + \nu \int_\Omega u\phi - \nu \int_\Omega v\phi = \lambda \int_\Omega g(u)\phi,$$

$$-d \int_\Omega u\Delta \psi + \nu \int_\Omega u\psi - \nu \int_\Omega v\psi = 0, \quad \forall \phi, \psi \in C^2_0(\Omega). \quad (3.1)$$

Note that the assumption $\varphi \in C^2_0(\Omega)$ implies $|\varphi| \leq C\delta$ for some constant $C > 0$, so the integral on the right hand side of the 1st equation is well-defined. Note also that zero boundary data are encoded in this definition since we allow test functions $\phi, \psi$ which have a nontrivial normal derivative on the boundary.
We say \((u, v)\) is a classical solution of (1.6) if \((u, v)\) is a weak solution of (1.6) and in addition, \(u, v \in C^2(\Omega) \cap C_0(\overline{\Omega})\). As usual, \((u, v)\) is a sub or super-solution of system (1.6) if \(=\) above is replaced by \(\leq\) or \(\geq\), respectively, and in addition only non-negative test functions \(\phi\) and \(\psi\) are considered.

Given two pairs of functions \((u_1, v_1)\) and \((u_2, v_2)\) defined on \(\Omega\), we write \((u_1, v_1) \leq (u_2, v_2)\) provided that \(u_1(x) \leq u_2(x)\) and \(v_1(x) \leq v_2(x)\) for all \(x \in \Omega\). We say that \((u, v)\) is a minimal (super-) solution of system (1.6), if \((u, v)\) is a (super-) solution of (1.6) and \((u, v) \leq (\tilde{u}, \tilde{v})\) for every other super-solution \((\tilde{u}, \tilde{v})\) of (1.6).

Using these definitions we now can proceed to a proof of Theorem 1.2. The proof of parts \((i - iii)\) are relatively standard and can be viewed as an extension of similar results of [1–3] obtained for single equation, to the system of equations of considered class. For completeness we sketch the main steps of the proof of \((i - iii)\). The proofs of parts \((iv)\) and \((v)\) of Theorem 1.2 are new and will be given in details. We start with the proof of part \((i)\) of Theorem 1.2.

**Proof of part (i) of Theorem 1.2.** Until the proof of part \((iv)\) of Theorem 1.2 we assume that \(\nu > 0\) is fixed and when there is no ambiguity, drop the subscript \(\nu\) in the notations.

Let \(\mu_1 = \mu_1(-\Delta, \Omega) > 0\) and \(\phi_1 > 0\) be the principal eigenvalue and the corresponding principal eigenfunction of \(-\Delta\) in \(H^1_0(\Omega)\) with \(||\phi_1||_1 = 1\). Recall that since \(\Omega\) is smooth, \(c\delta \leq \phi_1 \leq C\delta\), cf. [3, Theorems 3.1.4 and 4.3.1].

Given \(\lambda > 0\), let \((u_{\lambda}, v_{\lambda})\) be a weak solution of (1.6). Testing (1.6) against \(\phi_1\), in the 2nd equation we obtain

\[
\mu_1 d \int_{\Omega} v_{\lambda} \phi_1 + \nu \int_{\Omega} v_{\lambda} \phi_1 - \nu \int_{\Omega} u_{\lambda} \phi_1 = 0, \tag{3.3}
\]
or

\[
\int_{\Omega} v_{\lambda} \phi_1 = \frac{\nu}{\nu + \mu_1 d} \int_{\Omega} u_{\lambda} \phi_1. \tag{3.4}
\]

Substituting into the 1st equation we then derive

\[
\mu_1 (1 + \kappa) \int_{\Omega} u_{\lambda} \phi_1 = \mu_1 \int_{\Omega} u_{\lambda} \phi_1 + \nu \int_{\Omega} u_{\lambda} \phi_1 - \nu \int_{\Omega} v_{\lambda} \phi_1 = \lambda \int_{\Omega} g(u_{\lambda}) \phi_1, \tag{3.5}
\]
where \(\kappa = \frac{\nu d}{\nu + \mu_1 d}\). Since \(g(0) > 0\) and \(g\) is convex, by assumption (1.2) there is a constant \(\eta > 0\) such that

\[
g(u) \geq \eta u \quad \text{for all} \quad u \geq 0. \tag{3.6}
\]

Then from (3.5) we conclude that

\[
\mu_1 (1 + \kappa) \int_{\Omega} u_{\lambda} \phi_1 = \lambda \int_{\Omega} g(u_{\lambda}) \phi_1 \geq \lambda \eta \int_{\Omega} u_{\lambda} \phi_1. \tag{3.7}
\]

This implies that for

\[
\lambda > \frac{\mu_1}{\eta} (1 + \kappa) \tag{3.8}
\]

system (1.6) has no weak solutions. \(\square\)
To prove part (ii) of Theorem 1.2 we need two following lemmas.

**Lemma 3.1.** Let \((\phi, \psi)\) be a solution of the following problem

\[
\begin{aligned}
-\Delta \phi + \nu \phi - \nu \psi &= f & \text{in } \Omega, \\
-d\Delta \psi + \nu \psi - \nu \phi &= 0 & \text{in } \Omega, \\
\phi &= \psi &= 0 & \text{on } \partial \Omega.
\end{aligned}
\] (3.9)

Then, for every \(f \in C(\bar{\Omega})\) system (3.9) has unique classical solution \((\phi, \psi)\). Moreover, \((\phi, \psi) \geq (0,0)\) provided that \(f \geq 0\). In addition, classical solutions of (3.9) satisfy a strong maximum principle, in the sense that \(f \geq 0\) and \(f \neq 0\) implies that for some \(c, C > 0\) it holds

\[c \delta(x) \leq \psi(x) < \phi(x) \leq C \delta(x) \quad \text{for all } x \in \Omega.\] (3.10)

**Proof.** The existence and uniqueness as well as the regularity and positivity properties for systems of type (3.9) follow from well known results of [5,7,19].

To prove (3.10) we observe that combining the first and the second equations of (3.9) in a way identical to the one discussed in the section 2 (see Eq. (2.4)) we obtain

\[[-\gamma \nu^{-1} \Delta + 1](\phi - \psi) = \gamma f,\] (3.11)

where \(\gamma = \frac{d}{1+d} > 0\). By the strong maximum principle (cf. [3, Theorem 3.1.4]) applied to (3.11) we have \(\phi(x) - \psi(x) > c_1 \delta(x)\) for all \(x \in \Omega\), for some constant \(c_1\). Substituting this into the 2nd equation of the system (3.9) and using the strong maximum principle again we obtain the lower bound in (3.10), while from the 1st equation of (3.9) we derive the upper bound of (3.10) via [3, Theorem 4.3.1].

**Lemma 3.2.** Assume that for some \(\lambda_s > 0\) system (1.6) has a classical supersolution. Then (1.6) has a minimal classical solution \((u_\lambda, v_\lambda)\) for every \(0 < \lambda \leq \lambda_s\).

**Proof.** Let us first observe that if \((\bar{u}, \bar{v})\) is a classical super-solution of (1.6) for some \(\lambda_s > 0\), then \((\bar{u}, \bar{v})\) is also a classical super-solution of (1.6) for every \(0 < \lambda \leq \lambda_s\).

Next, given \(0 < \lambda \leq \lambda_s\), set \((\phi_0, \psi_0) = (0,0)\). For \(k \in \mathbb{N}\), recursively define \((\phi_k, \psi_k)\) as the unique positive solution of the linear system

\[
\begin{aligned}
-\Delta \phi_k + \nu \phi_k - \nu \psi_k &= \lambda g(\phi_{k-1}) & \text{in } \Omega, \\
-d\Delta \psi_k + \nu \psi_k - \nu \phi_k &= 0 & \text{in } \Omega, \\
\phi_k &= \psi_k &= 0 & \text{on } \partial \Omega.
\end{aligned}
\] (3.12)

By Lemma 3.1 it is clear that

\[0 \leq \phi_1(x) \leq \bar{u}(x), \quad 0 \leq \psi_1(x) \leq \bar{v}(x) \quad (x \in \Omega).\] (3.13)

Assume that for some \(k \in \mathbb{N}\) it holds

\[0 \leq \phi_{k-1}(x) \leq \phi_k(x) \leq \bar{u}(x), \quad 0 \leq \psi_{k-1}(x) < \psi_k(x) \leq \bar{v}(x) \quad (x \in \Omega).\] (3.14)

Then, taking into account monotonicity of \(g\), we obtain

\[
\begin{aligned}
-\Delta(\phi_{k+1} - \phi_k) + \nu(\phi_{k+1} - \phi_k) - \nu(\psi_{k+1} - \psi_k) &= \lambda(g(\phi_{k+1}) - g(\phi_k)) \geq 0 & \text{in } \Omega, \\
-d\Delta(\psi_{k+1} - \psi_k) + \nu(\psi_{k+1} - \psi_k) - \nu(\phi_{k+1} - \phi_k) &= 0 & \text{in } \Omega, \\
\phi_{k+1} - \phi_k &= \psi_{k+1} - \psi_k &= 0 & \text{on } \partial \Omega.
\end{aligned}
\] (3.15)
By Lemma 3.1 and the principle of mathematical induction we conclude that the sequence \((\phi_k, \psi_k)\) is monotone non-decreasing. Similarly, we deduce \((\phi_k, \psi_k)\) is uniformly bounded by \((\bar{u}, \bar{v})\), so that for all \(k \in \mathbb{N}\) it holds

\[
0 \leq \phi_k(x) \leq \phi_{k+1}(x) \leq \bar{u}(x), \quad 0 \leq \psi_k(x) \leq \psi_{k+1}(x) \leq \bar{v}(x) \quad (x \in \Omega). \tag{3.16}
\]

Therefore, the sequence \((\phi_k, \psi_k)\) converge pointwisely in \(\Omega\), and we denote

\[
u_k(x) := \lim_{k \to \infty} \phi_k(x) \leq \bar{u}(x), \quad v_k(x) := \lim_{k \to \infty} \psi_k(x) \leq \bar{v}(x) \quad (x \in \Omega). \tag{3.17}
\]

By the standard elliptic regularity (cf. [3, Proof of Theorem 3.3.3, Step 3]), \((\nu, v)\) is a classical solution of the nonlinear system \((1.6)\). Moreover, since the construction of \((\phi_k, \psi_k)\) does not depend on the specific choice of a super-solution \((\bar{u}, \bar{v})\), we conclude that \((\nu, v)\) is a minimal solution of \((1.6)\).

Now we turn to the proof of part (ii) of Theorem 1.2.

**Proof of part (ii) of Theorem 1.2.** Let \(\Lambda^*\) be the critical value of the classical Gelfand problem \((1.3)\). For \(0 < \Lambda < \Lambda^*\), let \(u_0 := w_\Lambda\) be the minimal classical solution of \((1.3)\). Since \(g\) is positive and monotone nondecreasing, it is clear that \((u_0, u_0)\) is a classical super-solution of the nonlinear system \((1.6)\) for every \(0 < \lambda \leq \Lambda\). It follows from Lemma 3.2 and upper bound (3.8) in the proof of part (i) of Theorem 1.2 that the set of \(\lambda > 0\) where \((1.6)\) has a minimal classical solution is a bounded, nonempty interval. Thus we defined

\[
\lambda^* := \sup\{\lambda > 0 : \text{\((1.6)\) has a minimal classical solution}\}. \tag{3.18}
\]

This completes the proof of part (ii) of Theorem 1.2.

As a next step we continue to the proof of part iii) of the main theorem.

**Proof of the claim iii) of Theorem 1.2.** Given \(0 < \lambda < \lambda^*\) and \(0 < \varepsilon < \lambda\), we observe that \((u_\lambda, v_\lambda)\) is a super-solution of \((1.6)\) with \(\lambda\) replaced by \(\lambda - \varepsilon\). Therefore, \((u_{\lambda-\varepsilon}, v_{\lambda-\varepsilon}) \leq (u_\lambda, v_\lambda)\). Let \(\phi := u_\lambda - u_{\lambda-\varepsilon}\), \(\psi := u_\lambda - u_{\lambda-\varepsilon}\). Then, taking into account the monotonicity of \(g\) we see that

\[
\begin{cases}
-\Delta \phi - \nu(\psi - \phi) = \lambda g(u_\lambda) - (\lambda - \varepsilon)g(u_{\lambda-\varepsilon}) > 0 \quad \text{in } \Omega, \\
-\Delta \psi - \nu(\phi - \psi) = 0 \quad \text{in } \Omega, \\
\phi = \psi = 0 \quad \text{on } \partial\Omega,
\end{cases} \tag{3.19}
\]

By the strong maximum principle of [3, Theorem 3.1.4] we conclude that for some \(c_\varepsilon > 0\) it holds

\[
u_\lambda(x) \geq u_{\lambda-\varepsilon}(x) + c_\varepsilon \delta(x), \quad v_\lambda(x) \geq v_{\lambda-\varepsilon}(x) + c_\varepsilon \delta(x) \quad (x \in \Omega). \tag{3.20}
\]

In particular, \(u_\lambda(x)\) and \(v_\lambda(x)\) are strictly monotone increasing functions of \(\lambda\), for every \(x \in \Omega\). Furthermore, since \(g(0) > 0\) and \(g\) is convex, by assumption \((1.2)\) there is a constant \(m_* > 0\) such that for all \(s \geq 0\),

\[
\frac{\lambda^*}{2} g(s) \geq \mu_1 (1 + \kappa) s - m_* \tag{3.21}
\]
Now, testing (1.6) against $\phi_1$ and using (3.4) and (3.5) we obtain
\[
\lambda \int_{\Omega} g(u_\lambda) \phi_1 = \mu_1 (1 + \kappa) \int_{\Omega} u_\lambda \phi_1 \leq \frac{\lambda^*}{2} \int_{\Omega} g(u_\lambda) \phi_1 + m^* \int \phi_1.
\] (3.22)

We conclude that
\[
\lim_{\lambda \to \lambda^*} \int_{\Omega} g(u_\lambda) \phi_1 \leq \frac{2m^*}{\lambda^*} \int \phi_1 < \infty.
\] (3.23)

Let $\zeta$ be unique positive solution of the following problem
\[
\begin{cases}
-\Delta \zeta = 1 & \text{in } \Omega, \\
\zeta = 0 & \text{on } \partial \Omega.
\end{cases}
\] (3.24)

Similarly to (3.2), by [3, Theorems 3.1.4 and 4.3.1] we conclude that
\[
c_\delta \leq \zeta \leq C_\delta.
\] (3.25)

Testing (1.6) with $\phi = \psi = \zeta$, we obtain
\[
\int_{\Omega} u_\lambda + \nu \int (u_\lambda - v_\lambda) \zeta = \lambda \int_{\Omega} g(u_\lambda) \zeta,
\]
\[
d \int_{\Omega} v_\lambda + \nu \int (v_\lambda - u_\lambda) \zeta = 0,
\] (3.26)

Adding these equations together and taking into account (3.2), (3.25) and (3.23), we have
\[
\int_{\Omega} (u_\lambda + dv_\lambda) = \lambda \int_{\Omega} g(u_\lambda) \zeta \leq c_1 \lambda \int_{\Omega} g(u_\lambda) \phi_1 < \infty.
\] (3.27)

In a view of positivity of $u_\lambda$ and $v_\lambda$ we conclude that $u_\lambda$ and $v_\lambda$ are bounded in $L^1(\Omega)$.

Since $u_\lambda(x)$ and $v_\lambda(x)$ are both increasing in $\lambda$, we conclude that $(u_\lambda, v_\lambda)$ converge to $(u^*, v^*)$ in $L^1(\Omega)$, and $g(u_\lambda)$ converge to $g(u_\nu)$ in $L^1(\Omega, \delta(x) dx)$. Similarly to [1, Lemma 5], it follows that $(u^*, v^*)$ is a weak solution of (1.6) with $\lambda = \lambda^*$. \(\square\)

We now proceed to the proof of the final two parts of Theorem 1.2. First, we will study monotonicity properties of the minimal solutions $u_{\lambda, \nu}$ and $v_{\lambda, \nu}$ with respect to $\nu$, then we consider the limiting behavior of the solutions as $\nu \to 0$, and finally limiting behavior of the solutions as $\nu \to \infty$.

The following lemma establishes monotonicity of $u$ component with respect to parameter $\nu$.

**Lemma 3.3.** Let $\lambda^* = \lambda^*(\nu)$ is a nondecreasing function of $\nu > 0$. Moreover, for $0 < \nu < \bar{\nu}$ and $\lambda < \lambda^*_\nu$, let $u_{\lambda, \nu}$ and $u_{\lambda, \bar{\nu}}$ be the first components of the minimal solutions of problem (1.6).

Then $u_{\lambda, \nu} \geq u_{\lambda, \bar{\nu}}$ in $\Omega$.

**Proof.** Recall that the minimal solution $(u_{\lambda, \nu}, v_{\lambda, \nu})$ was constructed by iterations given by system (3.12). Let $(\phi_k, \psi_k)$ and $(\bar{\phi}_k, \bar{\psi}_k)$ be solutions of (3.12) corresponding to $(\lambda, \nu)$ and $(\lambda, \bar{\nu})$ respectively. We are going to show that $\nu < \bar{\nu}$ implies that $\phi_k > \bar{\phi}_k$ in $\Omega$ for each $k$. As a result, sequences converging point-wise to solutions $u_{\lambda, \nu}$ and $u_{\lambda, \bar{\nu}}$ are ordered. Hence, the limits are ordered.
Subtracting the first and the second equations of the system (3.12) we observe that

\[- \Delta (\phi_k - \psi_k) + \nu \left(1 + \frac{d}{d^\nu}\right) (\phi_k - \psi_k) = \lambda g(\phi_{k-1}) \text{ in } \Omega, \tag{3.28}\]

which implies that

\[\phi_k > \psi_k \text{ in } \Omega, \tag{3.29}\]

for each \(k\). By adding the first and the second equations of the system (3.12) we also have that

\[- \Delta (\phi_k + \psi_k) = \lambda g(\phi_{k-1}) \text{ in } \Omega, \tag{3.30}\]

for each \(k\) and \(\nu\). Needles to say that identical equations hold for \(\tilde{\phi}_k\) and \(\tilde{\psi}_k\).

Now let us show that \(\phi_1 > \tilde{\phi}_1\). Indeed since \(\phi_0 = \tilde{\phi}_0 = 0\) in \(\Omega\), we have from (3.30)

\[- \Delta (\phi_1 + d\psi_1) = - \Delta (\tilde{\phi}_1 + d\tilde{\psi}_1) = \lambda g(0) \text{ in } \Omega, \tag{3.31}\]

and therefore

\[\phi_1 + d\psi_1 = \tilde{\phi}_1 + d\tilde{\psi}_1 \text{ in } \Omega. \tag{3.32}\]

Taking difference of the first equations of the system (3.12) for \(\nu\) and \(\tilde{\nu}\) after some algebra we obtain

\[- \Delta (\phi_1 - \tilde{\phi}_1) + \nu(\phi_1 - \tilde{\phi}_1) - \nu(\psi_1 - \tilde{\psi}_1) = (\tilde{\nu} - \nu)(\tilde{\phi}_1 - \tilde{\psi}_1) \text{ in } \Omega. \tag{3.33}\]

Using (3.32) and (3.29) we obtain from (3.33)

\[- \Delta (\phi_1 - \tilde{\phi}_1) + \nu \left(1 + \frac{d}{d^\nu}\right) (\phi_1 - \tilde{\phi}_1) = (\tilde{\nu} - \nu)(\tilde{\phi}_1 - \tilde{\psi}_1) > 0 \text{ in } \Omega, \tag{3.34}\]

and thus

\[\phi_1 > \tilde{\phi}_1 \text{ in } \Omega. \tag{3.35}\]

Let us show now that \(\phi_k > \tilde{\phi}_k\) provided \(\phi_{k-1} > \tilde{\phi}_{k-1}\). Indeed, assume that

\[\phi_{k-1} > \tilde{\phi}_{k-1} \text{ in } \Omega, \tag{3.36}\]

then by (3.30) we have

\[- \Delta (\phi_k + d\psi_k) = \lambda g(\phi_{k-1}), \quad - \Delta (\tilde{\phi}_k + d\tilde{\psi}_k) = \lambda g(\tilde{\phi}_{k-1}) \text{ in } \Omega. \tag{3.37}\]

Using the fact that \(g\) is increasing and assumption (3.36) we have

\[g(\phi_{k-1}) > g(\tilde{\phi}_{k-1}), \tag{3.38}\]

which together with (3.37) gives

\[\phi_k + d\psi_k > \tilde{\phi}_k + d\tilde{\psi}_k \text{ in } \Omega. \tag{3.39}\]
Combining first equations of the system (3.12) for \( \nu \) and \( \tilde{\nu} \) we have

\[
- \Delta (\phi_k - \tilde{\phi}_k) + \nu (\phi_k - \tilde{\phi}_k) - \nu (\psi_k - \tilde{\psi}_k) = \\
\lambda (g(\phi_{k-1}) - g(\tilde{\phi}_{k-1})) + (\tilde{\nu} - \nu)(\tilde{\phi}_k - \tilde{\psi}_k) > 0 \quad \text{in} \ \Omega. \tag{3.40}
\]

Note that positivity of the right hand side of the above equation follows from (3.38) and (3.29).

Using (3.39) from (3.40) we have

\[
- \Delta (\phi_k - \tilde{\phi}_k) + \nu \left(1 + \frac{d}{d-1}\right) (\phi_k - \tilde{\phi}_k) > 0 \quad \text{in} \ \Omega, \tag{3.41}
\]

that yields

\[
\phi_k > \tilde{\phi}_k \quad \text{in} \ \Omega. \tag{3.42}
\]

In a view of (3.35) an inequality (3.42) holds for each \( k \geq 1 \). Since \( \phi_k \to u \) and \( \tilde{\phi}_k \to \tilde{u} \) we conclude that

\[
u_{\lambda,\nu} \geq \tilde{\nu}_{\lambda,\tilde{\nu}} \quad \text{in} \ \Omega. \tag{3.43}
\]

By construction, it follows that \( \lambda^*_{\nu} = \lambda^*(\nu) \) is a nondecreasing function of \( \nu > 0 \), which completes the proof.

Let us now consider the limiting behavior of the system (1.6) as \( \nu \to 0 \).

**Proposition 3.1.** For \( \lambda < \Lambda^* \) and \( \nu \to 0 \), the minimal solution \((u_{\lambda,\nu}, v_{\lambda,\nu})\) converges uniformly to \((u_0, 0)\), where \( u_0 \) is the minimal solution of (1.9).

**Proof.** Let \( \lambda < \Lambda_* \) and let \( u_0 \) be the minimal classical solution of Gelfand problem (1.9). Then \((u_0, u_0)\) is a supersolution of (1.6) and in particular, for all \( \nu > 0 \) holds

\[
(u_{\lambda,\nu}, v_{\lambda,\nu}) \leq (u_0, u_0). \tag{3.44}
\]

The minimal solution \( v_{\lambda,\nu} \) of the second equation of (1.6) can be represented as

\[
v_{\lambda,\nu} = \nu [-\Delta + \nu]^{-1} u_{\lambda,\nu}. \tag{3.45}
\]

Since \( u_{\lambda,\nu} \) is uniformly bounded in \( L^\infty(\Omega) \) and \([-\Delta + \nu]^{-1}\) is a bounded operator from \( L^\infty(\Omega) \) into \( L^\infty(\Omega) \), we have

\[
0 < v_{\lambda,\nu} \leq \nu C \quad \text{in} \ \Omega, \tag{3.46}
\]

for some \( C > 0 \) independent of \( \nu \), that is

\[
\|v_{\lambda,\nu}\|_{L^\infty} \to 0 \quad \text{as} \quad \nu \to 0. \tag{3.47}
\]

Next, since \( g \) is of class \( C^1 \), combining first equations of (1.6) and (1.9) and setting \( w_{\nu} := u_0 - u_{\lambda,\nu} \) we obtain

\[
- \Delta w_{\nu} = \lambda (g(u_0) - g(u_{\lambda,\nu})) + \nu (u_{\lambda,\nu} - v_{\lambda,\nu}) = g'(\xi) w_{\nu} + \nu (u_{\lambda,\nu} - v_{\lambda,\nu}) \tag{3.48}
\]
where $\xi_\nu \in L^\infty(\Omega)$ satisfy
\begin{equation}
\nu u_{\lambda,\nu} \leq \xi_\nu \leq u_0 \quad \text{in } \Omega.
\end{equation}

Since $u_0$ is a minimal solution of corresponding Gelfand problem \(1.9\), it is stable \(1\) in the sense that the operator $-\Delta - \lambda g'(u_0)$ is invertible in $L^2(\Omega)$. In view of \(3.49\), the operator $-\Delta - \lambda g'(\xi)$ is also invertible in $L^2(\Omega)$. This allows to rewrite \(3.48\) as follows
\begin{equation}
w_\nu = \nu[-\Delta - \lambda g'(\xi)]^{-1}(u_{\lambda,\nu} - v_{\lambda,\nu}).
\end{equation}

Since $[-\Delta - \lambda g'(\xi)]^{-1}$ is bounded from $L^\infty(\Omega)$ into $L^\infty(\Omega)$, while $u_{\lambda,\nu}$ and $v_{\lambda,\nu}$ are uniformly bounded in $L^\infty(\Omega)$, we conclude that
\begin{equation}
0 < w_\nu \leq C \nu \quad \text{in } \Omega.
\end{equation}

Therefore, $\|w_\nu\|_{L^\infty} \to 0$ as $\nu \to 0$ and the assertion follows. \(\square\)

We now give several lemmas needed to study the behavior of solution for the problem \(1.6\) in the limit of $\nu \to \infty$. To shorten the notation, we denote $K_\nu := d \{ 1 - \nu[-\gamma \Delta + \nu]^{-1} \}$, \(3.52\)
where $\gamma = \frac{d}{1 + d}$. Clearly, $K_\nu$ is a bounded linear operator in $C(\Omega)$ and in $L^p(\Omega)$, for any $1 \leq p \leq \infty$. Similarly to the derivation of \(2.6\), we see that if $(u, v)$ is a classical solution of \(1.6\) then $u$ is a classical solution of the nonlocal equation
\begin{equation}
-\Delta u = \frac{\lambda}{1 + d}(1 + K_\nu)g(u) = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega.
\end{equation}

and
\begin{equation}
v = u - \lambda \gamma[-\gamma \Delta + \nu]^{-1}g(u).
\end{equation}

We present two standard results about the properties of the operator $K_\nu$.

**Lemma 3.4.** For all $\nu > 0$, $K_\nu$ is a positive operator in $L^\infty(\Omega)$, i.e. for every $f \in L^\infty(\Omega)$, $f \geq 0$ implies $K_\nu f \geq 0$. Moreover, $\|K_\nu\|_{L^2 \to L^2} = 1$ and $K_\nu$ strongly converges to zero as $\nu \to \infty$, i.e. for every $f \in L^2(\Omega)$, $\lim_{\nu \to \infty} \|K_\nu f\|_{L^2} = 0$.

**Proof.** Observe that the resolvent operator $[-\gamma \Delta + \nu]^{-1}$ is well-defined in $L^2(\Omega)$ for all $\nu > 0$, and by spectral theorem,
\begin{equation}
\|\nu[-\gamma \Delta + \nu]^{-1}\|_{L^2 \to L^2} = \nu(\gamma \mu_1 + \nu)^{-1} < 1,
\end{equation}
that is $\nu[-\gamma \Delta + \nu]^{-1}$ is a contraction in $L^2(\Omega)$. Then by \(13\) Proposition 1.3, the family $[-\gamma \Delta + \nu]^{-1}$ is a strongly continuous contraction resolvent, that is
\begin{equation}
\|f - \nu[-\gamma \Delta + \nu]^{-1}f\|_{L^2} \to 0 \quad \text{for every } f \in L^2(\Omega).
\end{equation}

Moreover, by \(13\) Definition 4.1 and Chapter 2.1, $[-\gamma \Delta + \nu]^{-1}$ is sub-Markovian, that is for all $f \in L^2(\Omega)$,
\begin{equation}
0 \leq f \leq 1 \quad \text{implies} \quad 0 \leq \nu[-\gamma \Delta + \nu]^{-1}f \leq 1.
\end{equation}

Since $K_\nu = I - \nu[-\gamma \Delta + \nu]^{-1}$, we conclude that $K_\nu$ is a positive operator in $L^\infty(\Omega)$ for all $\nu > 0$, and that $\|K_\nu f\|_{L^2} \to 0$ for every $f \in L^2(\Omega)$. \(\square\)
Lemma 3.5. Let $\lambda < \lambda_{\nu_0}$ for some $\nu_0 > 0$. Then $\mu_1(-\Delta - \frac{\lambda}{1+d} g'(u_{\lambda,\nu})) > 0$ for all $\nu > \nu_0$.

Proof. Let $(u_{\lambda,\nu}, v_{\lambda,\nu})$ be the minimal positive solution of (1.6). Consider the eigenvalue problem for the linearized system

$$
\begin{cases}
-\Delta \phi + (\nu - g'(u_{\lambda,\nu})) \phi - \nu \psi = \mu \phi & \text{in } \Omega, \\
-d\Delta \psi + \nu \psi - \nu \phi = \mu \psi & \text{in } \Omega, \\
\phi = \psi = 0 & \text{on } \partial \Omega.
\end{cases}
$$

(3.58)

It is known that system (3.58) admits the principal eigenvalue $\tilde{\mu}_{\nu,1}$, the corresponding eigenfunction $(\phi, \psi)$ can be chosen positive and $\phi, \psi \in C^2(\Omega) \cap C_0(\Omega)$, see [19, Theorem 1.1].

Rearranging system (3.58) as in Section 2 we see that $\tilde{\mu}_{\nu,1}$ and $\phi > 0$ satisfy the nonlocal equation

$$
-\Delta \phi - \frac{\lambda}{1+d} (1 + K_{\nu}) g'(u_{\lambda,\nu}) \phi = \tilde{\mu}_{\nu,1} \phi & \text{in } \Omega, \quad \phi = 0 & \text{on } \partial \Omega.
$$

(3.59)

Similarly to [3, Proposition 3.4.4], assume that $\tilde{\mu}_{\nu,1} < 0$. Given $\varepsilon > 0$, we compute

$$
-\Delta (u_{\lambda,\nu} - \varepsilon \phi) - \frac{\lambda}{1+d} (1 + K_{\nu}) g(u_{\lambda,\nu} - \varepsilon \phi) \\
= \frac{\lambda}{1+d} (1 + K_{\nu}) g(u_{\lambda,\nu}) - \varepsilon \frac{\lambda}{1+d} (1 + K_{\nu}) g(u_{\lambda,\nu}) \phi - \tilde{\mu}_{\nu,1} \varepsilon \phi - \frac{\lambda}{1+d} (1 + K_{\nu}) g(u_{\lambda,\nu} - \varepsilon \phi) \\
= -\frac{\lambda}{1+d} (1 + K_{\nu}) (g(u_{\lambda,\nu} - \varepsilon \phi) - g(u_{\lambda,\nu}) + \varepsilon g'(u_{\lambda,\nu}) \phi) - \tilde{\mu}_{\nu,1} \varepsilon \phi,
$$

(3.60)

where

$$
(1 + K_{\nu}) (g(u_{\lambda,\nu} - \varepsilon \phi) - g(u_{\lambda,\nu}) + \varepsilon g'(u_{\lambda,\nu}) \phi) = o(\varepsilon \phi),
$$

(3.61)

since $g$ is $C^1$, $u_{\lambda,\nu}, \phi \in L^\infty(\Omega)$ and $1 + K_{\nu}$ is bounded in $L^\infty(\Omega)$. Since $\tilde{\mu}_{\nu,1} < 0$, we deduce that

$$
-\Delta (u_{\lambda,\nu} - \varepsilon \phi) - \frac{\lambda}{1+d} (1 + K_{\nu}) g(u_{\lambda,\nu} - \varepsilon \phi) \geq 0
$$

(3.62)

for all sufficiently small $\varepsilon > 0$.

Using the definition of $K_{\nu}$, we then conclude that

$$
-\Delta (u_{\lambda,\nu} - \varepsilon \phi) + \nu (u_{\lambda,\nu} - \varepsilon \phi - v_{\lambda,\nu}) - \lambda g(u_{\lambda,\nu} - \varepsilon \phi) \\
= -\Delta (u_{\lambda,\nu} - \varepsilon \phi) - \frac{\lambda}{1+d} (1 + K_{\nu}) g(u_{\lambda,\nu} - \varepsilon \phi) \geq 0 & \text{in } \Omega
$$

(3.63)

and further, we note that

$$
-\Delta v_{\lambda,\nu} + \nu (v_{\lambda,\nu} - u_{\lambda,\nu} + \varepsilon \phi) = \varepsilon \phi \geq 0,
$$

(3.64)

for all sufficiently small $\varepsilon > 0$. This means that $(u_{\lambda,\nu} - \varepsilon \phi, v_{\lambda,\nu})$ is a supersolution of system (1.6).

By Lemma 3.2 we conclude that system (1.6) admits a solution $(\bar{u}_{\lambda,\nu}, \bar{v}_{\lambda,\nu})$ with $0 < \bar{u}_{\lambda,\nu} < u_{\lambda,\nu}$. But this contradicts to the minimality of $(u_{\lambda,\nu}, v_{\lambda,\nu})$. Hence $\tilde{\mu}_{\nu,1} \geq 0$.

Now observe that

$$
\int_{\Omega} K_{\nu} g'(u_{\lambda,\nu}) \phi^2 \geq g'(0) \sigma_1(\nu) \int_{\Omega} \phi^2,
$$

(3.65)

where $\sigma_1(\nu) = d(1 - (\nu^{-1} \mu_1(-\Delta) + 1)^{-1}) > 0$ is the smallest eigenvalue of $K_{\nu}$ in $L^2(\Omega)$. This implies that $\mu_1(-\Delta - g'(u_{\nu})) > \tilde{\mu}_{\nu,1} \geq 0$. \qed
Using results presented above we can now describe the limiting behavior of solution for the problem (1.6).

**Proposition 3.2.** For $\lambda < \Lambda^*(1 + d)$ and $\nu \to \infty$, the minimal solution $(u_{\lambda,\nu}, v_{\lambda,\nu})$ converges uniformly to $(u_\infty, u_\infty)$, where $u_\infty$ is the minimal solution of (1.10).

**Proof.** Using representation (3.53), we see that

$$ -\Delta u_{\lambda,\nu} = \frac{\lambda}{d+1} g(u_{\lambda,\nu}) + \frac{\lambda}{d+1} K_{\nu} g(u_{\lambda,\nu}) \geq \frac{\lambda}{d+1} g(u_{\lambda,\nu}). $$  \hfill (3.66)

In view of positivity of $K_{\nu}$ (Lemma 3.4), we conclude that $u_{\lambda,\nu}$ is a supersolution of (1.10). Since $u_\infty$ is the minimal solution of (1.10), we see that

$$ u_{\lambda,\nu} \geq u_\infty. $$  \hfill (3.67)

By Lemma 3.3, $u_\nu$ is monotone decreasing as $\nu \to \infty$, so for a $\tilde{\nu} > 0$ and all $\nu \in [\tilde{\nu}, \infty)$ we have

$$ u_{\lambda,\nu} \geq u_{\lambda,\tilde{\nu}} \geq u_\infty. $$  \hfill (3.68)

In particular, $\{u_{\lambda,\nu}\}_{\nu \geq \tilde{\nu}}$ and $\{g(u_{\lambda,\nu})\}_{\nu \geq \tilde{\nu}}$ are bounded in $L^2(\Omega)$, for every $p \in [1, \infty]$. Since $K_{\nu}$ is bounded and $(-\Delta)^{-1}$ is compact in $L^p(\Omega)$ for every $p \in (1, \infty)$, and (3.66) can be rewritten as

$$ u_{\lambda,\nu} = (-\Delta)^{-1} \left( \frac{\lambda}{d+1} g(u_{\lambda,\nu}) + \frac{\lambda}{d+1} K_{\nu} g(u_{\lambda,\nu}) \right), $$  \hfill (3.69)

we conclude that for a sequence $\nu_n \to \infty$, $u_{\lambda,\nu_n}$ converges to a limit $\tilde{u}_\infty$. Moreover, since $u_{\lambda,\nu}(x)$ is monotone decreasing in $x$ (Lemma 3.3), we conclude that

$$ \lim_{\nu \to \infty} u_{\lambda,\nu} = \tilde{u}_\infty. $$

Using strong convergence of $K_{\nu}$ to zero (Lemma 3.4), we conclude that

$$ \|K_{\nu} g(u_{\lambda,\nu})\|_{L^2} \leq \|K_{\nu} g(\tilde{u}_\infty)\|_{L^2} + \|K_{\nu}\|_{L^2 \to L^2} \|g(u_{\lambda,\nu}) - g(\tilde{u}_\infty)\|_{L^2} \to 0, $$  \hfill (3.70)

as $\nu \to \infty$.

Combining (3.53) and (1.10) and setting $w_{\nu} := u_{\lambda,\nu} - u_\infty$ we obtain

$$ -\Delta w_{\nu} = \frac{\lambda}{d+1} \left( g(u_{\lambda,\nu}) - g(u_\infty) \right) + \frac{\lambda}{d+1} K_{\nu} g(u_{\lambda,\nu}) = \frac{\lambda}{d+1} g'(\xi_{\nu}) w_{\nu} + \frac{\lambda}{d+1} K_{\nu} g(u_{\lambda,\nu}) $$  \hfill (3.71)

where $\xi_{\nu} \in L^\infty(\Omega)$ and

$$ u_\infty \leq \xi_{\nu} \leq u_{\lambda,\nu}. $$  \hfill (3.72)

By Lemma 3.5, the operator $-\Delta - \frac{1}{d+1} g'(u_{\lambda,\nu})$ is invertible in $L^2(\Omega)$. Then, in view of (3.72), the operator $-\Delta - \frac{1}{d+1} g'(\xi_{\nu})$ is also invertible in $L^2(\Omega)$. This allows to rewrite (3.71) as follows

$$ w_{\nu} = \frac{\lambda}{d+1} \left( -\Delta - \frac{1}{d+1} g'(\xi_{\nu}) \right)^{-1} K_{\nu} g(u_{\lambda,\nu}) $$  \hfill (3.73)
In view of (3.70) and since the operator \([-\Delta - \frac{\lambda}{1+\nu} g'(\xi)]^{-1}\) is bounded in \(L^2(\Omega)\), we conclude that \(\|w_\nu\|_{L^2} \to 0\) as \(\nu \to \infty\) and thus \(\|u_{\lambda,\nu} - u_\infty\|_{L^2} \to 0\) as \(\nu \to \infty\). In particular, this implies that \(\tilde{u}_\infty = u_\infty\).

Further, using the standard bootstrap argument we improve the convergence to conclude that \(\|w_\nu\|_{L^\infty} \to 0\) as \(\nu \to \infty\), and thus \(\|u_{\lambda,\nu} - u_\infty\|_{L^\infty} \to 0\) as \(\nu \to \infty\).

Finally, by (3.74),

\[
v_{\lambda,\nu} = u_{\lambda,\nu} - \lambda \gamma [-\gamma \Delta + \nu]^{-1} g(u_{\lambda,\nu}). \tag{3.74}
\]

Since \([-\gamma \Delta + \nu]^{-1}\) is bounded as operator from \(L^\infty(\Omega)\) into \(L^\infty(\Omega)\), we also conclude that \(\|v_{\lambda,\nu} - u_{\lambda,\nu}\|_{L^\infty} \to 0\) as \(\nu \to \infty\).

We are now in a position to complete the proof of Theorem 1.2.

Proof of part (iv) and (v) of Theorem 1.2. The claim follows immediately from Lemma 3.3 and Propositions 3.1, 3.2.

4 Parabolic problem: proof of Theorem 1.1

In this section we present a proof of Theorem 1.1. This theorem can be viewed as an extension of the result obtained in [11] for a system which describes thermal ignition in one-phase confined materials which in turn is an extension of the result for classical Gelfand problem given in [1].

In order to proceed we will need a following lemma.

Lemma 4.1. Let \(U, V\) be a global classical solution of (1.5). Then, \((U_t, V_t) \geq 0\) in \(\Omega\).

Proof. Differentiating system (1.5) with respect to time and setting \(\xi = U_t, \eta = V_t\) we have:

\[
\begin{aligned}
\begin{cases}
\xi_t - \Delta \xi = \lambda g'(U) \xi + \nu(\eta - \xi), \\
\alpha \eta_t - d \Delta \eta = \nu(\xi - \eta) \\
\xi = \eta = 0 \\
\xi(0, \cdot), \eta(0, \cdot) \geq 0
\end{cases}
\end{aligned}
\tag{4.1}
\]

The linear system (4.1) is quasi-monotone and thus component-wise comparison principle holds [15]. Since \(\xi = \eta = 0\) is a sub-solution we have \(\xi, \eta \geq 0\) for all \(t \geq 0\) in \(\Omega\).

Now we turn to a proof of Theorem 1.1.

Proof of Theorem 1.1. First we claim that if (1.6) has a classical solution, then (1.5) has a global solution. This follows directly from the fact that the system (1.5) is quasi-monotone and thus a comparison principle holds component-wise [15].

Now let us show that existence of global solution for problem (1.5) imply existence of weak solution for (1.6).

Let us first note that by lemma 4.1 \(U, V, U_t, V_t \geq 0\) for all \(x \in \Omega\) and \(t \geq 0\), so that solutions of the problem (1.5) are non-negative and non-decreasing.
Next, observe that for each \( \phi, \psi \in C^2(\bar{\Omega}) \) with \( \phi = \psi = 0 \) on \( \partial \Omega \) we have

\[
\frac{d}{dt} \int_{\Omega} U\phi + \int_{\Omega} U(-\Delta \phi) = \lambda \int_{\Omega} g(U)\phi + \nu \int_{\Omega} (V - U)\phi, \\
\alpha \frac{d}{dt} \int_{\Omega} V\psi + d \int_{\Omega} V(-\Delta \psi) = \nu \int_{\Omega} (U - V)\psi.
\] (4.2)

As before, let \( \mu_1 > 0 \) be the principal Dirichlet eigenvalue of \(-\Delta\) in \( \Omega \) and \( \phi_1 > 0 \) be the corresponding eigenfunction, with \( ||\phi||_1 = 1 \). Setting \( \phi = \psi = \phi_1 \) in (4.2) we have

\[
\frac{d}{dt} \int_{\Omega} U\phi_1 + \mu_1 \int_{\Omega} U\phi_1 = \lambda \int_{\Omega} g(U)\phi_1 + \nu \int_{\Omega} (V - U)\phi_1, \\
\alpha \frac{d}{dt} \int_{\Omega} V\phi_1 + \mu_1 d \int_{\Omega} V\phi_1 = \nu \int_{\Omega} (U - V)\phi_1.
\] (4.3)

We first claim that \( \int_{\Omega} U\phi_1 \) and \( \int_{\Omega} V\phi_1 \) are uniformly bounded in time.

From first equation of (4.3), non negativity of \( V, \phi_1 \) and Jensen’s inequality we have

\[
\frac{d}{dt} \int_{\Omega} U\phi_1 + (\mu_1 + \nu) \int_{\Omega} U\phi_1 \\
= \lambda \int_{\Omega} g(U)\phi_1 + \nu \int_{\Omega} V\phi_1 = \lambda \int_{\Omega} g(U)\phi_1 \geq \lambda g\left( \int_{\Omega} U\phi_1 \right) \geq \lambda g\left( \int_{\Omega} U\phi_1 \right). \] (4.4)

By assumption (1.2) we have \( g'(s) \to \infty \) as \( s \to \infty \). Therefore, there is a constant \( M_1 > 0 \) such that

\[
\lambda g(s) - (\mu_1 + \nu)s \geq \frac{\lambda}{2} g(s), \quad \text{for} \quad s \geq M_1. \] (4.5)

Now assume that \( \int_{\Omega} U\phi_1 = M_1 \) at \( t = t_0 \), then for \( t \geq t_0 \) we have

\[
\frac{d}{dt} \int_{\Omega} U\phi_1 \geq \frac{\lambda}{2} g\left( \int_{\Omega} U\phi_1 \right) \] (4.6)

which contradicts (1.2) and thus

\[
\int_{\Omega} U\phi_1 \leq M_1. \] (4.7)

In a view of (4.7) we have from the second equation of (4.3) that

\[
\alpha \frac{d}{dt} \int_{\Omega} V\phi_1 + (\mu_1 d + \nu) \int_{\Omega} V\phi_1 = \nu \int_{\Omega} U\phi_1 \leq \nu M_1 \] (4.8)

which immediately imply that

\[
\int_{\Omega} V\phi_1 \leq M_2 \] (4.9)
for some constant $M_2 > 0$. Finally integrating (4.3) on $(t, t+1)$ and taking into account that $g(U)$ and $U$ are non-decreasing we have

$$\lambda \int_\Omega g(U(t))\phi_1 \leq \int_t^{t+1} \int_\Omega g(U)\phi_1 \leq \int_\Omega U(t)\phi_1 + (\mu_1 + \nu) \int_t^{t+1} \int_\Omega U\phi_1 \leq (1 + \mu_1 + \nu)M_1 \quad (4.10)$$

and thus

$$\sup_{t>0} \int_\Omega g(U)\phi_1 \leq \frac{1 + \mu_1 + \nu}{\lambda} M_1. \quad (4.11)$$

Now let us show that both $U$ and $V$ are bounded in $L^1$ uniformly in time. Let $\zeta$ be the solution of (3.24). Observe that estimates (4.7), (4.9) and (4.11) imply that

$$\int_\Omega U\zeta, \quad \int_\Omega V\zeta, \quad \sup_{t>0} \int_\Omega g(U)\zeta \leq M_3 \quad (4.12)$$

for some constant $M_3$ independent of time. Next setting in (4.2) $\phi = \psi = \zeta$ and integrating the result on $(t, t+1)$ we have

$$\int_\Omega U \leq \int_t^{t+1} \int_\Omega U = \int_\Omega U(t, \cdot)\zeta - \int_\Omega U(t + 1, \cdot)\zeta + \lambda \int_t^{t+1} \int_\Omega g(U)\zeta + \nu \int_t^{t+1} (V - U)\zeta \leq M_4,$$

$$d \int_\Omega V \leq d \int_t^{t+1} \int_\Omega V = \alpha \left( \int_\Omega V(t, \cdot)\zeta - \int_\Omega V(t + 1, \cdot)\zeta \right) + \nu \int_t^{t+1} (U - V)\zeta \leq M_5. \quad (4.13)$$

The first and the last inequalities in both equations of (4.13) hold since $U$ and $V$ are non-decreasing functions of time (lemma 4.1) and by (4.12) respectively. Thus,

$$\sup_{t>0} ||U(t)||_{L^1(\Omega)} \leq M_4, \quad \sup_{t>0} ||V(t)||_{L^1(\Omega)} \leq M_5. \quad (4.14)$$

From (4.14) and monotone convergence theorem we deduce that $U$ and $V$ have a limit $u, v$ in $L^1(\Omega)$. Moreover by (4.10) we have that $g(U)$ converges to $g(u)$ in $L^1(\Omega, \delta(x)dx)$ as $t \to \infty$. Integrating (4.2) on $(t, t+1)$ we have

$$\int_\Omega U\phi|_{t}^{t+1} + \int_\Omega U(-\Delta \phi) = \lambda \int_\Omega g(U)\phi + \nu \int_\Omega (V - U)\phi,$$

$$\alpha \int_\Omega V\psi|_{t}^{t+1} + d \int_\Omega V(-\Delta \psi) = \nu \int_\Omega (U - V)\psi. \quad (4.15)$$

Finally letting $t \to \infty$ we have

$$\int_\Omega u(-\Delta \phi) = \lambda \int_\Omega g(u)\phi + \nu \int_\Omega (u - v)\phi,$$

$$d \int_\Omega v(-\Delta \psi) = \nu \int_\Omega (u - v)\psi. \quad (4.16)$$
Therefore, the $L^1$-limit of $U$ and $V$ as $t \to \infty$ is a weak solution of (1.6), as defined in (3.1). Let us also note that $U$ and $V$ are non-decreasing in time. In view of the parabolic comparison principle for (1.5) we conclude that the limit of $U$ and $V$ is a minimal weak solution of elliptic problem (1.6).

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