A Novel Hamiltonian Formulation of First Order Einstein-Hilbert Action: Connection with ADM, Diffeomorphism Invariance and Linearized Theory

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(Dated: April 7, 2009)

Abstract

A novel Dirac Hamiltonian formulation of the first order Einstein-Hilbert (EH) action, in which “algebraic” constraints are not solved to eliminate fields from the action at the Lagrangian level, has been shown to lead to an action and a constraint structure apparently distinct from the ADM action and the ADM constraint structure in that secondary first class constraints χ and χᵢ as well as tertiary first class constraints τ and τᵢ arise with an unusual Poisson Bracket (PB) algebra [24]. By canonical transformations of the fundamental fields we show how from the tertiary constraints τ and τᵢ one may derive the Hamiltonian and momentum constraints. Special attention is paid to the Hamiltonian formulation of the first order EH action in terms of the variables \(h = \sqrt{-g^{00}}\), \(h^i = \sqrt{-g^{0i}}\) and \(q^{ij} = -g^{0i}g^{0j} - g^{00}g^{ij}\) and their conjugate momenta employed in [19, 20]. It is shown that the variables \(h\) and \(h^i\) are left undetermined in the formalism. This fact is used for a proper gauge fixation of the secondary constraints χ and χᵢ and reduction to the Faddeev action [19, 20]. Considering invariances of the total action, the generator of the gauge transformations of the EH Lagrangian action is derived. Using this generator, the explicit form of the gauge invariance of the field \(h\) is obtained, by which the relation between the gauge functions and the descriptors of the diffeomorphism invariance is determined in order for the gauge transformations to correspond to diffeomorphism invariance. By linearizing the novel Hamiltonian formulation of [24], the Hamiltonian formulation of the first order action for the free spin two field [4, 24] is derived.

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I. INTRODUCTION

After the discovery of the Dirac constraint formalism \cite{2,15}, the Hamiltonian formulation of the EH action in second order form using the metric $g_{\mu\nu}$ as the configuration space fields was first attempted by Pirani and Schild \cite{37,38} and independently by Bergmann, Penfield, Schiller and Zatzkis \cite{10}, and was later formulated in a more convenient way by Dirac \cite{16,17}. Soon afterwards, the canonical formulation of the first order Einstein-Palatini action was considered by Arnowitt, Deser and Misner (ADM) \cite{4,5,6,7,8} starting from a geometrical, rather than an algebraic perspective. In achieving their formulation, ADM followed a procedure other than the Dirac constraint formalism, in which constraint equations are solved irrespective of their being first or second class \cite{21}. The result is derived using a set of variables possessing clear geometrical interpretation. It turns out that in both the Dirac and ADM formulations, the metric of the three-space and their conjugate momenta which are related to the extrinsic curvature of the three-space (subject to the Hamiltonian and momentum constraints), are sufficient for the description of the dynamics of general relativity, which is considered as the time evolution of spacelike surfaces. A characteristic of both formulations is that the manifest four dimensional general covariance is broken, which is to be expected by the choice of a particular time coordinate necessary for the Hamiltonian formulation. In early attempts, however, care was taken for a canonical formulation in terms of invariants \cite{10,37}, but this was soon overshadowed by abandoning such assumptions \cite{38}, and especially after Dirac’s triumphant results \cite{16,17}.

A key element in the ADM Hamiltonian formulation of the EH action in first order form is the “reduction” of the EH action by solving a combination of equations of motion which are independent of time derivatives (the algebraic constraints), thus eliminating a number of dynamical variables from the Lagrangian action. This algebraic manipulation, which brings the EH action in a Hamiltonian form \cite{7,20}, is done irrespective of whether the equations of motion which are solved are first or second class in the sense of the Dirac constraint formalism \cite{21}. Such a formalism has thus left untouched the question of what kind of a Hamiltonian formulation, with what characteristics, and potentially what differences, one would have obtained if one had used the Dirac constraint formalism in when casting the EH action in the Hamiltonian form. This task was recently undertaken in \cite{24}.

Here is a brief sketch of this paper. A summary of the ADM approach, in its original
formulation [4, 5, 6, 7, 8] and the formulation of Faddeev [19, 20], as well as an overview of the results of the novel Hamiltonian formulation of [24] are discussed in Section (II). In sections (III) and (IV) we explain how one may simplify the form of the constraints and the constraint algebra appearing in [24] by transforming the coordinates employed in [24] into the variables used by Faddeev [19, 20], ADM [5, 6, 7, 8] and Teitelboim [46, 47]. It is then shown how one may reduce these actions into the actions derived by Faddeev [19, 20] and ADM [5, 6, 7, 8] using the method of Faddeev and Jackiw [21]. Based on the equations of motion for \( h \) and \( h^i \), when \( h, h^i \) and \( q^{ij} \) are used as coordinates, tentative gauge constraints are suggested for reduction of the extended action into the Faddeev action [19, 20] in Section (V). Gauge invariance of this action is considered in Section (VI), where the generator of the gauge transformations of the total action is derived. Using this generator, the explicit form of the gauge transformation of the field \( h \) is obtained, and the relation between the gauge functions and the descriptors of the diffeomorphism invariance is determined for the gauge transformation to be a diffeomorphism. In Section (VII), the linearized form of the Hamiltonian formulation of the EH action of ref. [24], which is the Hamiltonian action corresponding to the first order spin two field Lagrangian action proposed in [4], is obtained. Concluding remarks are left to Section (VIII).

II. SUMMARY OF ADM APPROACH AND PREVIOUS RESULTS

ADM achieved their Hamiltonian formulation of the EH action by casting it in the form [5, 6, 7]

\[
\sqrt{-g} \, R \simeq \mathcal{L} (N, N_i, \gamma_{ij})
= - \gamma_{ij} \pi^{ij} + \frac{N}{\sqrt{\gamma}} \left( \gamma \, R + \frac{1}{2} (\pi_i)^2 - \pi^{ij} \pi_{ij} \right) + N_i \left( 2 \pi^{ij} \right)
- 2 \left[ \sqrt{\gamma} N_i + \left( \pi_{ij} - \frac{1}{2} \gamma_{ij} \pi_i \right) N_j \right]^i,
\]

where \( \gamma = \det(\gamma_{ij}) \),

\[
\pi^{ij} = \sqrt{-g} \left[ \Gamma^0_{mn} - \gamma_{mn} \Gamma^0_{pq} \gamma^{pq} \right] \gamma^{mi} \gamma^{nj},
\]

and

\[
N = (-g^{00})^{-1/2} \quad N_i = g_{0i}.
\]
\(N\) and \(N_i\) are treated as configuration fields instead of \(g_{00}\) and \(g_{0i}\), spanning the configuration space together with the metric of the 3-space \(\gamma_{ij}\). In eq. (1), \(R\) is the curvature scalar of the 3-space \(\gamma_{ij}\), and the vertical dash \(|\) denotes covariant derivative with respect to the 3-space \(\gamma_{ij}\) defined as usual. In particular, if \(S, T^{ij}\) and \(T^i_j\) are tensor densities of rank \(W\) we have

\[ S_{|k} = S_{,k} - W \Gamma^l_{lk} S, \]
\[ T^{ij}_{|k} = T^{ij}_{,k} + \Gamma^i_{kl} T^{lj} + \Gamma^j_{kl} T^{il} - W \Gamma^l_{lk} T^{ij}, \]
\[ T^i_j_{|k} = T^i_j_{,k} + \Gamma^i_{kl} T^l_j - \Gamma^l_{jk} T^i_l - W \Gamma^l_{lk} T^i_j. \]

The action of eq. (1) is obtained from the first order EH action by solving linear combinations of the equations of motion derived from this action, which are independent of the time derivative of fields (the constraint equations), for the components \(\Gamma^i_{jk}, \Gamma^i_{0k}\) and \(\Gamma^0_{0k}\) of the affine connections in terms of the lapse and shift functions \(N\) and \(N_i\), the metric fields of the 3-space \(\gamma_{ij}\) and the components \(\Gamma_{ij}^\mu\) of the affine connections, and by dropping some surface integrals. This is done without classifying the constraints. The notation \(\simeq\) rather than \(=\) is used in eq. (1) since the equality holds only if these solutions to the equations of motion are substituted into \(R\). (The components \(\Gamma_{00}^\mu\) of the affine connections disappear from the action when the solutions for the constraint equations are inserted and are not considered by ADM.)

The term appearing in the total divergence in eq. (1) is a covariant vector density of weight \(W = 1\). The total divergence may be dropped (as it is done below) if compact spaces are under consideration. From eq. (1), we see that \(\pi^{ij}\), which is a contravariant tensor density of weight \(W = 1\), is the momenta conjugate to \(\gamma_{ij}\). Therefore, the canonical Hamiltonian corresponding to the action of eq. (1) is given by

\[H_{ADM} = \int dx \left( N\mathcal{H} + N_i\mathcal{H}^i \right),\]

where

\[\mathcal{H} = \gamma^{-1/2} \left( \pi^{ij} \pi_{ij} - \frac{1}{2} (\pi^i_i)^2 \right) - \gamma^{1/2} R,\]
\[\mathcal{H}^i = -2 \pi^{ij}_{|j} = -2 \left( \pi^{ij}_{,j} + \Gamma^i_{jk} \pi^{jk} \right),\]

are called the “Hamiltonian” and “momentum” constraints respectively. The nomenclature becomes clear in the following way. Variation of the action with respect to the fields \(N\) and
which act as Lagrange multiplier fields, gives rise to the constraints

\[ \mathcal{H} \approx 0 \quad \mathcal{H}_i \approx 0. \]  

(10)

These constraints satisfy the algebra

\[
\{ \mathcal{H}(x), \mathcal{H}(y) \} = \left[ \gamma^{ij} \mathcal{H}_j(x) + \mathcal{H}_j(y) \right] \partial_i \delta(x - y),
\]

\[
\{ \mathcal{H}_i(x), \mathcal{H}(y) \} = \mathcal{H}(x) \partial_i \delta(x - y)
\]

\[
\{ \mathcal{H}_i(x), \mathcal{H}_j(y) \} = [\mathcal{H}_j(x) \partial_i + \mathcal{H}_i(y) \partial_j] \delta(x - y),
\]

which implies that the time change of the constraints \( \mathcal{H} \) and \( \mathcal{H}_i \) is ensured to weakly vanish when computed using the ADM Hamiltonian of eq. (7). The PBs of the ADM canonical coordinates \( \gamma_{ij} \) with the Hamiltonian and momentum constraints \( \mathcal{H} \) and \( \mathcal{H}_i \) have neat interpretations \[34, 43\]; namely,

\[
\left\{ \gamma_{ij}, \int dx N^k \mathcal{H}_k \right\} = \gamma_{ij,k} N^k + \gamma_{ik} N_j^k + \gamma_{jk} N_i^k
\]

\[
= N_{ij} + N_{ji}
\]

is nothing but the diffeomorphism invariance of the metric components \( \gamma_{ij} \) of the spacelike surfaces, and

\[
\left\{ \gamma_{ij}, \int dx N \mathcal{H} \right\} = 2 N \gamma^{-1/2} \left( \pi_{ij} - \frac{1}{2} \pi^j_i \gamma_{ij} \right),
\]

(12)

when set to zero, is the dynamical equation for the metric components \( \gamma_{ij} \) \[8\]; thus the nomenclature for the “Hamiltonian” and “momentum” constraints \( \mathcal{H} \) and \( \mathcal{H}_i \).

Having reviewed the original Hamiltonian formulation of ADM \[4, 5, 6, 7, 8\], we note that for the Hamiltonian formulation one may choose to start with the EH action written in terms of the metric density \( h^{\mu\nu} = \sqrt{-g} g^{\mu\nu} \) and the affine connection \( \Gamma^\lambda_{\mu\nu} \) as independent fields,

\[
S = S \left( h^{\mu\nu}, \Gamma^\lambda_{\mu\nu} \right) = \int dx h^{\mu\nu} \left( \Gamma^\lambda_{\mu\nu,\lambda} - \Gamma^\lambda_{\lambda\mu,\nu} + \Gamma^\lambda_{\mu\nu,\lambda} + \Gamma^\lambda_{\sigma\mu,\nu} - \Gamma^\lambda_{\sigma\mu,\lambda} - \Gamma^\lambda_{\lambda\sigma,\nu} \right).
\]

(14)

This choice of variables is made in \[19, 20\]. An advantage of such a choice is that it eliminates the square root of the determinant of the metric of the 3-space in the final Hamiltonian formulation; terms including this factor appear in the Hamiltonian constraint of eq. (8). A similar set of variables have been employed in the novel Dirac Hamiltonian formulation of the first order EH action presented in the following subsection. A total divergence appears in
The EH action of eq. (14), after addition of a surface term, becomes

$$ S (h_{\mu\nu},\Gamma^\lambda_{\mu\nu}) = \int dx \left[ \Gamma^\sigma_{\nu\sigma} h^\mu_{\mu} - \Gamma^\sigma_{\nu\sigma} h^\mu_{\mu} + h^\mu_{\mu} (\Gamma^\lambda_{\mu\nu} \Gamma^\sigma_{\sigma\lambda} - \Gamma^\lambda_{\sigma\mu} \Gamma^\sigma_{\nu\lambda}) \right]. $$

Thirty of the equations of motion that arise from the action of eq. (15) are independent of time derivatives and can be written as

$$ h^i_{i,k} + \Gamma^i_{l\sigma} h^k_{\sigma k} + \Gamma^k_{l \sigma} h^i_{\sigma i} - h^i_{i \lambda} \Gamma^\lambda_{i\sigma} = 0, $$

$$ h^0_{0,k} + \Gamma^0_{l m} h_{m k} + \Gamma^k_{l m} h^0_{0 m} + \Gamma^k_{l 0} h^0_{0 m} - h^0_{0 k} \Gamma^m_{l m} = 0, $$

$$ h^0_{0,0} + 2 \Gamma^0_{l m} h^m_{0 0} + \Gamma^0_{l 0} h^0_{0 0} - h^0_{0 0} \Gamma^m_{l m} = 0. $$

These equations are used by Faddeev [19, 20] to eliminate the variables $\Gamma^l_{ik}, \Gamma^k_{i0}$ and $\Gamma^0_{i0}$ from the action of eq. (15). The reduced action is

$$ S_R = \int dx \left[ \Pi_{ik} \dot{q}^{ik} - \lambda^0 C_0 - \lambda^k C_k - \mathcal{H} \right], $$

where

$$ C_k = 2 \nabla_k (q^{il} \Pi_{il}) - 2 \nabla_l (q^{il} \Pi_{ik}), $$

$$ C_0 = q^{ik} q^{mn} (\Pi_{ik} \Pi_{mn} - \Pi_{im} \Pi_{kn}) + \gamma \mathcal{R}, $$

$$ \mathcal{H} = -C_0 - q^{ik}_{,ik}. $$

In the above equations $q^{ik} = h^0_{0 k} h^0_{0 k} - h^0_{0 0} h^i_{i k}$ is a contravariant metric density of weight $W = 2$, $\Pi_{ik} = \Gamma^0_{ik}/h^0_{0 0}$ is a covariant tensor density of weight $W = -1$, the fields $\lambda^0 = 1 + 1/h^0_{0 0}$ and $\lambda^k = h^0_{0 k}/h^0_{0 0}$ are Lagrange multiplier fields, $\nabla$ is the covariant derivative with respect to the metric $\gamma_{ik}$ of the three dimensional space as defined in eqs. (16), $\mathcal{R}$ is its scalar curvature and $\gamma = \det (\gamma_{jk})$. (Note that the quantities $q^{il} \Pi_{il}$ and $q^{il} \Pi_{ik}$ appearing in eq. (20) are scalar and mixed second rank tensor densities of weight $W = 1$ respectively.) The fields $\Gamma^\mu_{0 0}$ enter linearly in eq. (15) and disappear in the reduced action; they are no longer considered when counting degrees of freedom. At this stage the only dynamical fields are $q^{ik}$ and their conjugate momenta $\Pi_{ik}$. The fields $h^0_{0 0}$ and $h^0_{0 k}$ are taken to be non-dynamical in $S_R$ on the account of their appearing as Lagrange multiplier fields through $\lambda^0$ and $\lambda^i$. Variation of the action with respect to these Lagrange multipliers in turn results in the constraints
\( C_0 \approx 0 \) and \( C_k \approx 0 \). The PBs of these constraints are convenient to express in terms of the functionals \[ C(X) = \int C_k(x)X^k(x) \, dx , \] (23) \[ C_0(f) = \int C_0(x)f(x) \, dx , \] (24) and are \[ \{ C(X_1), C(X_2) \} = C (\{ X_1, X_2 \}) , \] (25) \[ \{ C(X), C_0(f) \} = C_0 (X f) , \] (26) \[ \{ C_0(f_1), C_0(f_2) \} = C (\{ f_1, f_2 \}) , \] (27) where \[ [X_1, X_2]^k = X^l_1 \partial_l X^k_2 - X^l_2 \partial_l X^k_1 , \] (28) \[ X f = X^l \partial_l f - f \partial_l X^l , \] (29) \[ [f_1, f_2]^k = q^{ik} (f_1 \partial_i f_2 - f_2 \partial_i f_1) . \] (30) where \( f(x) \) and \( X(x) \) are test functions and the PB of the fundamental fields is defined in the following way, \[ \{ \Pi_{ij}(x), q^{kl}(y) \} = \frac{1}{2} \left( \delta_i^k \delta_j^l + \delta_j^k \delta_i^l \right) \delta(x - y) . \] (31) \( C(X) \) is the generator of the three-dimensional coordinate transformations, and \( C_0(f) \) corresponds to the transformation of the first and second quadratic forms of the surface when it is deformed \[ 20 \]. Using the convention of eq. (31), the PBs of eqs. (25)-(27) can alternatively be written in the form \[ X_i \{ C_i, C_j \} X_j = (X^k_i X^k_{2,i} - X^k_2 X^k_{1,i}) C_j , \] (32) \[ f \{ C_0, C_i \} X^i = (f X^k_i - X^i f) C_0 , \] (33) \[ f_1 \{ C_0, C_0 \} f_2 = q^{ij} (f_1 f_{2,i} - f_2 f_{1,i}) C_j , \] (34) which makes it easier comparing the PBs of the constraints derived in \[ 19, 20 \], with the PBs of the tertiary first class constraints derived below.

The Hamiltonian of the first order EH action in terms of \( q^{ij} \) and \( \Pi_{ij} \) as canonical variables was formulated in \[ 19, 20 \], where the authors use the metric-connection formulation of
the first order EH action as the basis of their analysis. The same Hamiltonian has been independently formulated in [41] where the starting point is the first order EH action in terms of the vierbein $e^a_\mu$ and the connection $\omega^a_{\mu\nu}$. As in [19, 20], equations of motion are solved in [41] in order to eliminate fields from the action, compatible with the method of Faddeev and Jackiw [21].

A novel canonical formulation of the metric-connection formulation of the EH action in first order form using the Dirac constraint formalism [15, 18, 26, 28, 30, 42, 43] has been recently performed [24]. In this approach, only equations of motion which correspond to second class constraints are solved to eliminate fundamental fields from the action, and the algebraic equations of motion which correspond to first class constraints are used to generate constraints of higher order. The final form of the Hamiltonian action principle involves the fields $h, h^i, H^{ij}$ and their conjugate momenta $\omega, \omega_i, \omega_{ij}, \Omega$ and $\Omega_i$. The fields $h, h^i$ and $H^{ij}$ are $h = \sqrt{-g} g^{00}, h^i = \sqrt{-g} g^{0i}$ and $H^{ij} = \frac{h^i h^j}{h} - h^{ij}$, where $h^{ij} = \sqrt{-g} g^{ij}$ and $g = \text{det}(g_{\mu\nu})$. The momenta $\omega_{ij}$ are given as $\omega_{ij} = \Gamma^0_{ij}$. In terms of these fields, the Hamiltonian action principle reads as

$$S = \int dx \left[ \omega \dot{h} + \omega_i \dot{h}^i + \omega_{ij} \dot{H}^{ij} + \Omega \dot{i} + \Omega_i \dot{\xi}^i ight] - \mathcal{H}_c^0 - u \Omega - u^i \Omega_i - v \chi - v^i \chi_i - w \tau - w^i \tau_i \right]$$

where

$$\mathcal{H}_c^0 = h \omega^2 + h^i \omega_i - \frac{d-3}{4(d-2)} H^{ij} \omega_i \omega_j - 2 \frac{h^m}{h} H^{ij} \omega_m \omega_j - \frac{1}{h} H^{ik} H^{jl} \omega_k \omega_l$$

$$+ \frac{1}{h} h^i j h^j \omega_i - \frac{1}{h} h^i H^{jk} \omega_{jk} + \frac{1}{2(d-2)} H_{jk} H^{ik} H^{jm} \omega_m$$

$$- \frac{1}{h} h^i j h^j \omega_i + \frac{1}{2} H^{ij} j H_{jk} H^{ik} + \frac{1}{4} H^{ip} H_{kr} H^{kp} + \frac{1}{4(d-2)} H^{ip} H_{jk} H^{jk} H^{qr} H_{kr} H^{pq}$$

$$+ \frac{1}{d-1} \frac{1}{h} \left( \chi^2 - (2 h \omega + h^i \omega_i) \chi \right) - \bar{\xi} \chi_i - \frac{i}{d-1} \chi + B^i \Lambda_i + B^{ij} \Lambda_i \Lambda_j,$$

where $u, u^i, v, v^i, w$ and $w^i$ are Lagrange multiplier fields; $B^i$ and $B^{ij}$ are quantities that depend on the canonical variables $h, h^i, H^{ij}$ and their conjugate momenta $\omega, \omega_i$ and $\omega_{ij}$; $\Omega$ and $\Omega_i$ are primary first class constraints; $\chi \approx 0$ and $\chi_i \approx 0$ are secondary first class

\footnote{For the definition of the rest of the fields in terms of the metric $g^{\mu\nu}$ and the affine connection $\Gamma^a_{\mu\nu}$ see [24].}
constraints of secondary stage

\[ \chi = h_j^i + h \omega - H_j^k \omega_j^k , \]
\[ \chi_i = h_i - h \omega_i ; \tag{37} \]

and \( \tau \approx 0 \) and \( \tau_i \approx 0 \) are secondary first class constraints of tertiary stage,

\[ \tau = -H_i^j - (H_i^j \omega_j^i) - \frac{d-3}{4(d-2)} H_i^j \omega_j^i + \frac{1}{2(d-2)} H_{kl} H_{j}^{kl} H_i^j \omega_j^i \]
\[ - \frac{1}{h} H^{ik} H^{jl} (\omega_{jk} \omega_{ki} - \omega_{ik} \omega_{jl}) + \frac{1}{2} H_j^k H_{jl} H_{j}^{kl} + \frac{1}{4} H_j^i H_{kl,i} H_{j}^{kl} \]
\[ + \frac{1}{2(d-2)} H_{ij}^{kl} H_{mn} H_{j}^{mn} , \tag{39} \]

and

\[ \tau_i = h \left( \frac{1}{h} H_{pq}^{pq} \omega_{pq} \right)^{,i}_i + H_{pq}^{pq} \omega_{pq,i} - 2 \left( H_{pq}^{pq} \omega_{pq} \right)^{,i}_p . \tag{40} \]

The constraints \( \chi, \chi_i, \tau \) and \( \tau_i \) are first class and satisfy an unusual PB algebra as follows.

For the PB of \( \chi \) and \( \chi_i \) we have

\[ \{ \chi_i, \chi \} = \chi_i , \tag{41} \]

while

\[ \{ \chi_i, \chi_j \} = 0 = \{ \chi, \chi \} . \tag{42} \]

Also,

\[ \{ \chi, \tau_i \} = 0 , \tag{43} \]
\[ \{ \chi, \tau \} = \tau , \tag{44} \]
\[ \{ \chi_i, \tau \} = 0 , \tag{45} \]

and

\[ \{ \chi_i, \tau_j \} = 0 . \tag{46} \]

The PBs of the constraints \( \tau_i \) and \( \tau \) are nonlocal\(^2\), as

\[ f \{ \tau_i, \tau_j \} g = f(\partial_j f) \tau_i - f(\partial_i g) \tau_j , \tag{47} \]

\[ f \{ \tau, \tau \} g = (g \partial_i f - f \partial_i g) \frac{H_i^j}{h^2} (h \tau_j - H_{mn} \omega_{mn} \chi_j + 2 H_{mn} \omega_{mj} \chi_n) . \tag{48} \]

\(^2\) We use the short notation \( f \{ X, Y \} g \equiv \int \int dxdy f(x) \{ X(x), Y(y) \} g(y) . \)
where \( f \) and \( g \) are test functions. It may also be shown that the Hamiltonian of eq. (36) can be expressed in terms of the first class constraints \( \Omega, \Omega_i, \chi, \chi_i, \tau \) and \( \tau_i \),

\[
\mathcal{H} = \tau + \frac{h^i}{h} \tau_i + \frac{h^i h^i}{h} \chi_i - \frac{1}{h^2} h^j h^i \chi_i + \frac{d - 2}{d - 1} H^{kl} \omega_{kl} \chi - \frac{h^i}{h} \omega \chi_i
\]

(49)

\[
+ \frac{2}{h^2} h^k H^{ij} \omega_{ik} \chi_j + \frac{d - 2}{d - 1} \omega \chi + \frac{1}{d - 1} \frac{1}{h} h^i \chi_i - \frac{1}{d - 1} \frac{1}{h} \omega \chi_i
\]

\[
- \frac{t}{d - 1} \chi - \xi^i \chi_i + B^i \Lambda_i + B^{ij} \Lambda_i \Lambda_j .
\]

The (secondary) constraints \( \chi \) and \( \chi_i \), which have no counterpart in the ADM Hamiltonian formulation of the first order EH action, are seen to arise because of the consistency condition of vanishing of the primary constraints \( \Omega \) and \( \Omega_i \), which are the momenta conjugate to the fields \( \bar{t} \) and \( \bar{\xi} \), which are in turn related to the connections \( \Gamma^\lambda_{\mu\nu} \).

In the following section, we will show how the constraints \( \chi, \chi_i, \tau \) and \( \tau_i \) of eqs. (37), (38), (39) and (40) take a specially simple form when the coordinates \( H^{ij} \) are transformed to any set of coordinates that depend only on the metric \( \gamma_{ij} \) of the space-like surfaces \( t = \text{cons} \). Two of the best sets of coordinates that can be used to replace the fields \( H^{ij} \) are the coordinates \( q^{ij} \) used by Faddeev [19, 20] and the coordinates \( \gamma_{ij} \) used by ADM [4, 5, 6, 7, 8]. (These fields have been discussed in the previous chapter). In contrast to the variables \( H^{ij} \) introduced in the previous chapter, the Faddeev variables \( q^{ij} = h^{0i} h^{0j} - h^{00} h^{ij} \) (where \( h^{\mu\nu} = \sqrt{-g} g^{\mu\nu} \)) depend only on the components of the metric \( \gamma_{ij} \) of the spacelike surfaces, since

\[
q^{ij} \gamma_{jk} = \gamma \delta^i_k ,
\]

(50)

where \( \gamma = \det(\gamma_{ij}) \). As it will be seen in later sections, this has important simplifying implications on the form of the algebra of the PB of constraints and their dependence on the fundamental fields.

III. TRANSFORMING TO FADDEEV VARIABLES

In the ADM Hamiltonian formulation of Faddeev [19, 20], the canonical coordinates \( q^{ij} = h^{0i} h^{0j} - h^{00} h^{ij} \) and their conjugate momenta \( \Pi_{ij} \) are the dynamical variables in the
“Hamiltonian” and “momentum” constraints $C_0$ and $C_i$ of eqs. (20) and (21), and thus the only dynamical variables in the Hamiltonian formulation, subject to the constraints $C_0 \approx 0$ and $C_i \approx 0$. The fields $\lambda = 1 + 1/h^{00}$ and $\lambda^i = h^{0i}/h^{00}$ are non-dynamical and act as Lagrange multiplier fields. In transition from the variables $H^{ij}$ employed in the Dirac Hamiltonian action principle of the first order EH action of eq. (35) to the Faddeev variables $q^{ij}$,\(^3\)

\[ q^{ij} = h H^{ij}, \]  

one must be careful that the momenta $\omega$, $\omega_i$ and $\omega_{ij}$ must be transformed in such a way that

\[ \omega \delta h + \omega_i \delta h^i + \omega_{ij} \delta H^{ij} = \Pi \delta h + \Pi_i \delta h^i + \Pi_{ij} \delta q^{ij}, \]  

in order for the transformation to be canonical. This ensures preservation of the properties of canonical invariants and the canonical equations of motion. Eq. (52) in turn results in the transformations

\[ \omega = \Pi + \frac{1}{h} q^{ij} \Pi_{ij}, \quad \omega_i = \Pi_i, \quad \omega_{ij} = h \Pi_{ij} \]  

for the momenta. From eq. (53), one observes that since $\omega_{ij} = \Gamma^0_{ij}$, the momenta $\Pi_{ij}$ agree with their definition in [19, 20], i.e. $\Pi_{ij} = \Gamma^0_{ij}/h$. We note that the momentum corresponding to $h^i$ remains unchanged as the transformation of eq. (51) does not involve $h^i$. (This is also why the momenta $\Omega$ and $\Omega_i$ and their corresponding canonical coordinates $\bar{\xi}$ and $\bar{\xi}^i$ do not appear in eq. (52)). In terms of the new variables, the secondary first class constraints $\chi$ and $\chi_i$ of eqs. (37) and (38) remarkably transform into

\[ \tilde{\chi} = h^i_{,i} + h \Pi \]  
\[ \tilde{\chi}_i = h_{,i} - h \Pi_i \]  

respectively, while the tertiary first class constraint $\tau_i$ of eq. (40) transforms into

\[ \tilde{\tau}_i = (q^{mn} \Pi_{mn})_{,i} + q^{mn} \Pi_{mn,i} - 2 (q^{mn} \Pi_{mi})_{,n}. \]  

\(^3\) We will not transform the fields $h$ and $h^i$ to $\lambda$ and $\lambda^i$ in the following, and will only consider transformation of the fields $q^{ij}$. 

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Surprisingly, the tertiary first class constraint $\tau$ of eq. (39) splits into several terms, some of which depend on the secondary constraint $\tilde{\chi}$,

$$
\tau = \frac{1}{h^2} \tau - \frac{1}{2(d-2)} \frac{1}{h^2} g_{kl} q^{kl} i q^{ij} \tilde{\chi} - \frac{d-3}{4(d-2)} \frac{1}{h^2} q^{ij} \tilde{\chi} \tilde{\chi} + \frac{1}{h^3} h_{ij} q^{ij} \tilde{\chi} + \left( \frac{1}{h^2} q^{ij} \tilde{\chi} \right) .
$$

In eq. (57) we have

$$
\tilde{\tau} = -q^{ik} q^{jl} (\Pi_{jk} \Pi_{il} - \Pi_{ik} \Pi_{jl}) - q^{ij} + \frac{1}{2} q^{ik} q_{jl} q_{kl} + \frac{1}{4} q^{ij} q_{kl} i q^{kl} (58)
$$

According to eq. (57), we may take the constraint $\tilde{\tau}$ of eq. (58) to be the tertiary constraint arising from the consistency condition that the time change of the constraint $\tilde{\chi}$ must weakly vanish. The constraints $\tilde{\tau}$ and $\tilde{\tau}$ of eqs. (56,58) are indeed the constraints $C_i$ and $C_0$ of eqs. (20,21) in the Faddeev Hamiltonian formulation of the first order EH action.

It is seen from eqs. (54,56,58) that, when written in terms of the variables $h$, $h^i$, $q^{ij}$ and their conjugate momenta $\Pi$, $\Pi_i$ and $\Pi_{ij}$, the constraints $\tilde{\chi}$ and $\tilde{\chi}_i$ depend only on the variables $h$, $h^i$ and their conjugate momenta $\Pi$ and $\Pi_i$, while the constraints $\tilde{\tau}_i$ and $\tilde{\tau}$ depend exclusively on the canonical variables $q^{ij}$ and their conjugate momenta $\Pi_{ij}$. Thus, the variables $h$ and $h^i$ and their conjugate momenta $\Pi$ and $\Pi_i$ are seen to decouple from the variables $q^{ij}$ and their conjugate momenta $\Pi_{ij}$ in formation of the first class constraints.

Since the PB (as well as the DB, because it is defined in terms of the PB) is invariant under canonical transformations, we see that under the transformations of eqs. (51) and (53), the PBs of eqs. (41) and (42) imply that

$$
\{ \tilde{\chi}_i , \tilde{\chi} \} = \tilde{\chi}_i , \quad \{ \tilde{\chi}_i , \tilde{\chi}_j \} = 0 , \quad \{ \chi , \tilde{\chi} \} = 0 .
$$

There is a remarkable way of obtaining the algebra of the PBs of the new constraints $\tilde{\tau}_i$ and $\tilde{\tau}$ of eqs. (56) and (57) directly from the PBs of eqs. (47,48,49) of the constraints $\tau_i$ and $\tau$. We note that the constraint $\tilde{\tau}_i$ of eq. (56) can be obtained from the constraint $\tau_i$ of eq.
by substituting $h = 1$. Since $\tau_i$ does not depend on the momentum $\omega$ conjugate to $h$, the latter is passive in computing the PB of eq. (47), that is, since $\tau_i$ is independent of $\omega$, it makes no difference if we were to set $h = 1$ before or after the PB $\{\tau_i, \tau_j\}$ is computed. Therefore, we may set $h = 1$ in both sides of eq. (47) and conclude that

$$f \{\tilde{\tau}_i, \tilde{\tau}_j\} = gf, j\tilde{\tau}_i - fg, i\tilde{\tau}_j \quad (62)$$

since $\tilde{\tau}_i$ depends on $q^{ij}$ and $\Pi_{ij}$ in the same way that $\tau_i$ depends on $H^{ij}$ and $\omega_{ij}$ once we set $h = 1$ in $\tau_i$.

In a similar way, we may compute the PBs $\{\tilde{\tau}, \tilde{\tau}\}$ and $\{\tilde{\tau}_i, \tilde{\tau}\}$ from the PBs $\{\tau, \tau\}$ and $\{\tau_i, \tau\}$ of eqs. (48) and (49) without explicitly computing these PBs using the fundamental PBs among the new canonical variables. The constraint $\tau$ of eq. (39) reduces to $\tilde{\tau}$ of eq. (57) by substituting $h = 1$ and $\omega_i = 0$ in eq. (39). Since $\tau$ has no dependence on either the momenta $\omega$ conjugate to $h$ or the field $h^i$ conjugate to the momenta $\omega_i$, one may set $h = 1$ and $\omega_i = 0$ either before or after the PB $\{\tau, \tau\}$ of eq. (48) is computed, and obtain the same quantity. This implies that

$$f \{\tilde{\tau}, \tilde{\tau}\} = (gf, i - fg, i) q^{ij} \tilde{\tau}_j \quad (63)$$

In much the same way, one may set $h = 1$ and $\omega_i = 0$ in both sides of eq. (49), and conclude that

$$f \{\tilde{\tau}_i, \tilde{\tau}\} = (gf, i - fg, i) \tilde{\tau} \quad (64)$$

The PBs of the first class constraints $\tilde{\tau}_i$ and $\tilde{\tau}$ of eqs. (62), (63) and (64) are indeed identical to the PBs of eqs. (32), (34) and (33) of the ADM Hamiltonian formulation of Faddeev if we identify $\tilde{\tau}_i$ and $\tilde{\tau}$ of eqs. (56) and (58) with the constraints $C_i$ and $C_0$ of eqs. (20) and (21) derived by Faddeev, considering that in eq. (3.13) of [20] the fundamental PBs are defined as

$$\{\Pi_{ij}(x), q^{kl}(y)\} = \frac{1}{2} (\delta_i^k \delta_j^l + \delta_i^l \delta_j^k) \delta(x - y) \quad (65)$$

There must also be an appropriate identification of the corresponding fields and momenta; i.e., by replacing $H^{ij}$ and $\omega_{ij}$ with $q^{ij}$ and $\Pi_{ij}$ in the expression obtained.
In fact, an explicit calculation of the expressions of eqs. (20, 21) using eqs. (4-6) shows that

\[
C_k = (q^i \Pi_{ik})_{t, k} + q^i \Pi_{k, k} - 2 (q^i \Pi_{ik})_{t}
\]

\[
C_0 = q^{ik} q^{mn} (\Pi_{ik} \Pi_{mn} - \Pi_{im} \Pi_{kn}) - q^{ij} \Pi_{ij} + \frac{1}{2} q^{ik} q^{il} q^{, k}
\]

\[
+ \frac{1}{4} q^{ij} q_{kl, i} q_{k l, j} + \frac{1}{8} q^{ij} q_{k l, i} q_{k l, j} q_{mn} q_{, mn}^{, mn},
\]

(66)

which are the constraints \( \tilde{\tau}_i \) and \( \tilde{\tau} \) of eqs. (56, 58) when \( d = 4 \).

We now express the Hamiltonian of eq. (49) in terms of the new variables \( h, h^i, q^{ij} \) and their conjugate momenta. Under the transformations of eqs. (51) and (53), one obtains

\[
\mathcal{H}_c = \frac{1}{h} \tilde{\tau} + \frac{h^i}{h} \tilde{\tau}_i - \frac{1}{2(d - 2)} \frac{h^i}{h^2} \Pi_{ik} q^{kl} q^{,ij} \chi_j - \frac{d - 3}{4(d - 2)} \frac{h^i}{h^2} q^{ij} \chi_i \chi_j
\]

\[
+ \frac{h^i}{h^2} q^{ij} \chi_i - \frac{h^i}{h^2} \frac{h^j}{h^2} h^j \chi_i + 2 \frac{h^m}{h^2} \frac{q^{im} \Pi_l, \chi_l + h^i}{h^2} h^i \chi_i - \frac{h^i}{h} \Pi \chi_i
\]

\[
- \frac{h^i}{h^2} q^{mn} \Pi_{mn} \chi_i - \frac{1}{d - 1} \frac{h^i}{h} \Pi \chi_i + \frac{d - 2}{d - 1} \left( h^i \Pi \chi_i + 2 q^{ij} \Pi_{ij} \chi_i - h^l \chi_l \right)
\]

\[
- \frac{\tilde{t}}{d - 1} \chi - \xi^i \chi_i + \tilde{B}^i \tilde{\Lambda},
\]

(68)

after a surface term has been dropped. The Hamiltonian of eq. (68) contains the “Hamiltonian” and “momentum” constraints \( \tilde{\tau} = C_0 \) and \( \tilde{\tau}_i = C_i \) appearing in the Hamiltonian of the action of eq. (19) derived by Faddeev, but in addition it incorporates terms proportional to the secondary first class constraints \( \chi \) and \( \chi_i \), as well as the terms proportional to the primary first class constraints \( \Omega \) and \( \Omega_i \), which are present in \( \tilde{\Lambda} \).

The Hamiltonian action principle for the Hamiltonian of eq. (68), therefore, takes the form

\[
S = \int dx \left[ \Pi \dot{h} + \Pi_i \dot{h}^i + \Pi_{ij} \dot{q}^{ij} + \Omega \dot{\chi} + \Omega_i \dot{\xi}^i
\]

\[
- \mathcal{H}_c - u \Omega - u^i \Omega_i - v \chi - v^i \chi_i - w \tilde{\tau} - w^i \tilde{\tau}_i \right]
\]

(69)

where \( \mathcal{H}_c \) is given by eq. (68) and \( \chi, \chi_i, \tilde{\tau}_i \) and \( \tilde{\tau} \) are given by eqs. (54), (55), (56) and (57). In contrast with the Faddeev action of eq. (19), we see that in the action of eq. (69), besides the fields \( q^{ij} \) and \( \Pi_{ij} \), the fields \( h, h^i, \tilde{\tau}, \tilde{\Lambda} \) and their corresponding momenta \( \Pi, \Pi_i, \Omega \) and \( \Omega_i \) appear to be dynamical. However, these fields are subject to more constraints, namely, \( \Omega \approx 0, \Omega_i \approx 0, \chi \approx 0, \chi_i \approx 0, \tilde{\tau}_i \approx 0 \) and \( \tilde{\tau} \approx 0 \), so that the number of degrees of freedom turns out to be counted the same as that of the ADM.
One may apply the reduction method of Faddeev and Jackiw [21] to the action of eq. (69), in which one considers all the canonical variables and Lagrange multipliers in the action at the same footing as fields. Equations of motion for the fields $u$, $u^i$, $v$ and $v^i$ result in $\Omega = 0$, $\Omega_i = 0$, $\bar{\chi} = 0$ and $\bar{\chi}_i = 0$. These equations may be solved for the fields $\Omega$, $\Omega_i$, $\Pi$ and $\Pi_i$. Upon substituting these solutions into the action of eq. (69) all terms coming from the Hamiltonian of eq. (68) vanish except for the terms proportional to $\bar{\tau}$ and $\bar{\tau}_i$, and the kinetic term becomes

$$
\Pi \dot{h} + \Pi_i \dot{h}^i + \Pi_{ij} \dot{q}^{ij} = \left( \frac{h_i h^i}{h} \right)_0 - \left( \frac{\dot{h} h^i}{h} \right)_i + \Pi_{ij} \dot{q}^{ij}.
$$

(70)

The first two terms on the right hand side being total derivatives may be dropped from the action of eq. (69), which would now take the form

$$
S = \int dx \left[ \Pi_{ij} \dot{q}^{ij} - \frac{1}{h} \bar{\tau} - \frac{h^i}{h} \bar{\tau}_i - w \bar{\tau} - w^i \bar{\tau}_i \right],
$$

(71)

which is the Faddeev version of the ADM action of eq. (19).

In the context of the Dirac constraint formalism, however, the first class constraints $\Omega$, $\Omega_i$, $\bar{\chi}$ and $\bar{\chi}_i$ may be solved only if appropriate gauge fixing conditions for all first class constraints are assumed [30, 43]. Together with the first class constraints $\Omega$, $\Omega_i$, $\chi$ and $\chi_i$, their gauge constraints may then be turned into strong equations while the PB is replaced with the appropriate DB. These equations may then be solved in order to eliminate fields from the action of eq. (69).

The introduction of gauge fixing conditions for the action of eq. (69), however, requires a knowledge of the gauge transformations of this action beforehand [43]. To obtain the generator of the gauge transformations all first class constraints $\Omega$, $\Omega_i$, $\bar{\chi}$, $\bar{\chi}_i$, $\bar{\tau}$ and $\bar{\tau}_i$ are required [13, 18, 30]. Once a set of admissible gauge constraints are assumed and the first class constraints $\Omega$, $\Omega_i$, $\bar{\chi}$ and $\bar{\chi}_i$ are turned into second class, they no longer act as generators of gauge transformations. Therefore, gauge fixing of the action of eq. (69) will result in losing some information about the generator of the gauge transformations of this action. The situation is similar to the gauge fixing of the “algebraic” constraint $e^0 \approx 0$ ($e^0$ is the momentum conjugate to $A_0$, the temporal component of $A_\mu$) in the Hamiltonian formulation of Maxwell gauge fields by using the gauge constraint $A_0 \approx 0$, and subsequent loss of the generator of the gauge transformation for $A_0$. (See [43] for a discussion of the canonical formulation of the Maxwell gauge fields.)
IV. TRANSFORMING TO ADM VARIABLES

In the original formulation of ADM, the EH action to start with is written in terms of the covariant components of the metric $\gamma_{ij}$ of the spacelike surfaces characterized by a time coordinate $t = \text{cons.}$, and the components $N$ and $N_i$ of the lapse and shift functions defined in terms of the metric $g_{\mu \nu}$ of the four dimensional embedding space in eq. (3). In the action of eq. (69), a transformation from the variables $q^{ij}$ to $\gamma_{ij}$ using eq. (50),

$$q^{ij} = \gamma^{ij}, \quad \gamma = \det \gamma_{ij},$$  \hspace{1cm} (72)

must be accompanied by appropriate transformations of the momenta $\Pi_{ij}$ conjugate to $q^{ij}$ to the momenta $\pi^{ij}$ conjugate to $\gamma^{ij}$, so that

$$\Pi_{ij} \delta q^{ij} = \pi^{ij} \delta \gamma_{ij}. \hspace{1cm} (73)$$

This implies that the momenta should be transformed in the following way,

$$\Pi_{ij} = -\gamma^{-1} \left( \gamma_{ia} \gamma_{jb} - \frac{1}{d-2} \gamma_{ij} \gamma_{ab} \right) \pi^{ab}. \hspace{1cm} (74)$$

Once again, one may directly check that if $\Pi_{ij} = \Gamma_{ij}^0/h$, as defined in eqs. (19, 20) and eq. (53), then the momenta $\pi^{ij}$ defined in eq. (74) are the same as the ADM momenta $\pi^{ij}$ given in eq. (2). Under the canonical transformations of eqs. (72) and (74), the constraints $\tilde{\chi}$ and $\tilde{\chi}_i$ of eqs. (54) and (55) remain unchanged. The momentum constraint $\tilde{\tau}_i$ of eq. (56), however, transforms to

$$\tilde{\tau}_i = -\pi^{ab} \gamma_{ab,i} + 2 \left( \gamma_{ib} \pi^{ab} \right)_{,a} \hspace{1cm} (75)$$

$$= -H_i,$$

where the ADM momentum constraint $H^i$ is given by eq. (9). Also, from eq. (58) we find that

$$\tilde{\tau} = -\left( \pi^{ij} \pi_{ij} - \frac{1}{d-2} \left( \pi^l \right)^2 \right) + \gamma \mathcal{R} \hspace{1cm} (76)$$

$$= -\gamma^{1/2} H,$$

where the ADM Hamiltonian constraint $H$ is given by

$$H = \gamma^{-1/2} \left( \pi^{ij} \pi_{ij} - \frac{1}{d-2} \left( \pi^l \right)^2 \right) - \gamma^{1/2} \mathcal{R}. \hspace{1cm} (77)$$
Therefore, in terms of $h, h^i$ and $\gamma_{ij}$, the action of eq. (69) becomes

\[
S = \int dx \left[ \Pi \dot{h} + \Pi_i \dot{h}^i + \pi^{ij} \dot{\gamma}_{ij} + \Omega \dot{\bar{t}} + \Omega_i \dot{\bar{\xi}}_i \right. \\
- \mathcal{H}_c' - u \Omega - u^i \Omega_i - v \bar{X} - v^i \bar{X}_i - w \bar{\tau} - w^i \bar{\tau}_i, \tag{78}
\]

where $\mathcal{H}_c'$ is the Hamiltonian of eq. (68) transformed under eqs. (72) and (74),

\[
\mathcal{H}_c' = - \frac{1}{\hbar} \gamma^{1/2} \mathcal{H} + \frac{h^i}{\hbar} \mathcal{H}_i + \frac{1}{2\hbar^2} \gamma \gamma^{kl} \gamma^{ij} \bar{X}_i \bar{X}_j \\
+ \frac{h_{ij}}{\hbar^3} \gamma \gamma^{ij} \bar{X}_i - \frac{h^i}{\hbar^2} \gamma \bar{X}_i - 2 \frac{h_m}{\hbar^2} \gamma^{mj} \pi^{ij} \bar{X}_i + \frac{1}{d-1} \frac{h^i}{\hbar^2} \gamma^{ab} \pi^{ab} \bar{X}_i + \frac{h^i}{\hbar^2} h_{ij} \bar{X}_j \\
- \frac{1}{d-1} h^i \Pi \bar{X}_i - \frac{1}{d-1} \frac{h^i}{\hbar} \Pi \bar{X} + \frac{d-1}{d-1} \left( h \Pi \bar{X} - h^{ij} \bar{X}_i \right) + \frac{2}{d-1} \frac{1}{\hbar} \gamma^{ab} \pi^{ab} \bar{X} \\
- \frac{d-1}{d-1} \bar{X} - \bar{\xi}^i \bar{X}_i + \bar{B}^i \bar{\Lambda}_i + \bar{B}^{ij} \bar{\Lambda}_i \bar{\Lambda}_j. \tag{79}
\]

We have thus achieved a Hamiltonian formulation of the EH action in terms of the variables $h, h^i, \gamma_{ij}$ and their corresponding momenta $\Pi, \Pi_i$ and $\pi^{ij}$. The ADM Hamiltonian constraint $\mathcal{H}$ appears with a coefficient $\gamma^{1/2}$. Such a factor can be combined with the field $h$ in the action of eq. (78) in order to introduce the lapse and shift functions $N$ and $N^i$ and their conjugate momenta “as canonical variables”. In terms of the metric $g_{\mu\nu}$, the lapse and shift functions $N$ and $N^i$ are defined as\(^5\)

\[
g^{00} = - \frac{1}{N^2}, \quad g^{0i} = \frac{N^i}{N^2}. \tag{80}
\]

Consequently, in terms of the metric $\gamma_{ij}$ of the spacelike surfaces and the variables $h = \sqrt{-g} g^{00}$ and $h^i = \sqrt{-g} g^{0i}$ we have

\[
h = - \gamma^{1/2} \frac{1}{N}, \quad h^{0i} = \gamma^{1/2} \frac{N^i}{N}. \tag{81}
\]

As eqs. (81) depend on the metric $\gamma_{ij}$, we must require that the momenta $\Pi, \Pi_i$ and $\pi^{ij}$ conjugate to $h, h^i$ and $\gamma_{ij}$ transform to the canonical momenta $p, p_i$ and $p^{ij}$ conjugate to $N$ and $N^i$ and $\gamma_{ij}$ in such a way that

\[
\Pi \delta h + \Pi_i \delta h^i + \pi^{ij} \delta \gamma_{ij} = p \delta N + p_i \delta N^i + p^{ij} \delta \gamma_{ij}. \tag{82}
\]

\(^5\) We note that it makes difference whether we use $N^i$ or its “covariant” component $N_i = \gamma_{ij} N^j$ as the canonical variable.
This implies that

\[ \Pi = \frac{1}{\sqrt{\gamma}} N (Np + N^i p_i) \, , \quad (83) \]
\[ \Pi_i = \frac{1}{\sqrt{\gamma}} N p_i \, , \quad (84) \]
\[ \pi^{ij} = p^{ij} + \frac{1}{2} \gamma^{ij} N p \, . \quad (85) \]

The momenta \( p^{ij} \) defined in eq. (85) are not the same as the ADM momenta \( \pi^{ij} \) defined in eq. (2). Under the canonical transformations of eqs. (81) and (83-85), the constraints \( \tilde{\chi}_i \) and \( \tilde{\chi} \) transform to

\[ \tilde{\chi}_i = - \left( \frac{\sqrt{\gamma}}{N} \right)_{,i} + p_i \, , \quad (86) \]
\[ \tilde{\chi} = - \left( - \left( \frac{\sqrt{\gamma}}{N} \right)_{,i} + p_i \right) N^i + \left( \frac{\sqrt{\gamma}}{N^2} N^i_p - p \right) N \, ; \quad (87) \]

and for the constraints \( \tilde{\tau}_i \) and \( \tilde{\tau} \) one finds that

\[ \frac{h^i}{h} \tilde{\tau}_i = N^i \tilde{\mathcal{H}}_i = N^i \left( \mathcal{H}_i + \frac{1}{2} N p \gamma^{ab} \chi_{ab,i} - (Np)_{,i} \right) \, , \quad (88) \]
\[ \frac{1}{h} \tilde{\tau} = N \tilde{\mathcal{H}} = N \left( \mathcal{H} - \frac{d - 1}{4(d - 2)} \frac{1}{\sqrt{\gamma}} (Np)^2 - \frac{1}{d - 2} \frac{N}{\sqrt{\gamma}} p \gamma^{ab} p^{ab} \right) \, , \quad (89) \]

where

\[ \mathcal{H}_i = - \left( - p^{ab} \gamma_{ab,i} + 2 \chi_{ab,i} \right) \, , \quad (90) \]
\[ \mathcal{H} = \gamma^{-1/2} \left( p^{ij} p_{ij} - \frac{1}{d - 2} \left( p^i \right)^2 \right) - \gamma^{1/2} \mathcal{R} \, . \quad (91) \]

The canonical transformations of the variables \( h \) and \( h^i \) to the variables \( N \) and \( N^i \) result in the dependence of the constraints \( \tilde{\chi}_i \) and \( \tilde{\chi} \) on the metric \( \gamma_{ij} \) of the spacelike surfaces, and in the constraints \( \tilde{\tau}_i \) and \( \tilde{\tau} \) receiving contributions from the fields \( N \) and \( N^i \) and their conjugate momenta \( p \) and \( p_i \).

Once again, we may apply the method of Faddeev and Jackiw to the action of eq. (78) after the fields \( h \) and \( h^i \) are canonically transformed to \( N \) and \( N^i \) according to eqs. (81,83,84,85). The equations of motion of the fields \( u, u^i, v \) and \( v^i \) result in \( \Omega = 0, \Omega_i = 0, \tilde{\chi} = 0 \) and \( \tilde{\chi}_i = 0 \), where \( \tilde{\chi}_i \) and \( \tilde{\chi} \) are given by eqs. (86,87). We may then solve these constraints for \( \Omega, \Omega_i, p \) and \( p_i \) and insert their solutions in the action, and in particular in
The kinetic part of the action then transforms to
\[ p \dot{N} + p_i \dot{N}^i + p^{ij} \dot{\gamma}_{ij} = \left( N^i \left( \frac{\gamma^{1/2}}{N} \right)_i \right)_{,0} \left( N^i \left( \frac{\gamma^{1/2}}{N} \right)_i \right)_{,0} + p^{ij} \dot{\gamma}_{ij} + \frac{1}{N^i} N^i \dot{\gamma}^{1/2} . \] (92)

The first two terms on the right hand side may be dropped from the action since they are total derivatives. The appropriate Darboux transformation \[21\] associated with the reduced kinetic term is
\[ \tilde{p}^{ij} = p^{ij} + \frac{\sqrt{\gamma}}{2N} N^i \dot{\gamma}^{ij} , \] (93)
by which the kinetic term takes the standard form
\[ p^{ij} \dot{\gamma}_{ij} + \frac{1}{N^i} N^i \dot{\gamma}^{1/2} = \tilde{p}^{ij} \dot{\gamma}_{ij} . \] (94)

The momenta \( \tilde{p}^{ij} \) defined in eq. (93) are the same as the ADM momenta defined in eqs. (2,74). Upon transforming the action under the transformations of eq. (93), it is seen that \( \tilde{\mathcal{H}}_i \) and \( \tilde{\mathcal{H}} \) of eqs. (88,89) transform into the ADM momentum and Hamiltonian constraints \( \mathcal{H}_i \) and \( \mathcal{H} \) of eqs. (75) and (77). The reduced action is therefore
\[ S = \int dx \left[ \tilde{p}^{ij} \dot{\gamma}_{ij} - N \tilde{\mathcal{H}} - N^i \tilde{\mathcal{H}}_i - w \tilde{\tau} - w^i \tilde{\tau}_i \right] , \] (95)
which is the ADM action upon a redefinition of the Lagrange multipliers \( w \) and \( w_i \).

Instead of introducing the lapse and shift functions \( N \) and \( N^i \) in the action of eq. (78), one may choose the most natural choice of coordinates that avoid mixing of the canonical fields in formation of the constraints, i.e. the “densitized” lapse function
\[ \alpha = N \gamma^{-1/2} \] (96)
and the shift functions \( \alpha^i \), which are defined as in the ADM approach.\(^6\) From eq. (81) one then has,
\[ h = -\frac{1}{\alpha} , \quad h^i = \frac{\alpha^i}{\alpha} \] (97)
and consequently
\[ \Pi = \alpha (\alpha \pi + \alpha^i \pi_i) , \quad \Pi_i = \alpha \pi_i . \] (98)

\(^6\) We note that \( \alpha = \lambda^0 - 1 \) and \( \alpha^i = \lambda^i \), where \( \lambda^0 \) and \( \lambda^i \) are the Lagrange multipliers appearing in eq. (19).
where \( \pi \) and \( \pi_i \) are the momenta conjugate to \( \alpha \) and \( \alpha_i \). We see that, in contrast with eqs. (83)-(85), the fields \( \gamma_{ij} \) and their conjugate momenta do not enter the transformations of eqs. (98). The constraints \( \tilde{\chi} \) and \( \tilde{\chi}_i \) of eqs. (54) and (55) then transform into

\[
\tilde{\chi} = \alpha \left( \frac{\alpha_i}{\alpha^2} - \pi \right) - \alpha^i \left( - \frac{1}{\alpha} \right) + \pi_i, \tag{99}\n\]

\[
\tilde{\chi}_i = - \left( \frac{1}{\alpha} \right) + \pi_i, \tag{100}\n\]

which depend only on a subset of the canonical variables; \( \alpha, \alpha^i \) and their conjugate momenta \( \pi \) and \( \pi_i \).\(^7\) The constraints \( \tilde{\tau} \) and \( \tilde{\tau}_i \) are seen to depend only on the rest of the canonical coordinates, i.e. \( \gamma_{ij} \) and their conjugate momenta \( \pi_{ij} \), and they remain intact under the transformations of eqs. (97) and (98). We thus introduce the quantities

\[
\tilde{\mathcal{H}} = - \tilde{\tau} = \left( \pi_{ij} \pi^{ij} - \frac{1}{d-2} \left( \pi^{l} \right)^2 \right) - \gamma \mathcal{R}, \tag{101}\n\]

\[
\tilde{\mathcal{H}}_i = - \tilde{\tau}_i = \pi^{ab} \gamma_{ab,i} - 2 \left( \gamma_{ib} \pi^{ab} \right)_\omega, \tag{102}\n\]

and for the action of eq. (78) we obtain

\[
S = \int dx \left[ \pi \dot{\alpha} + \pi_i \dot{\alpha}^i + \pi_{ij} \dot{\gamma}_{ij} + \Omega \dot{\tilde{\mathcal{H}}} + \Omega_i \dot{\tilde{\mathcal{H}}}_i \right. \tag{103}\n\]

\[
- \mathcal{H}'' - u \Omega - u^i \Omega_i - v \tilde{\chi} - v^i \tilde{\chi}_i - w \tilde{\mathcal{H}} - w^i \tilde{\mathcal{H}}_i \right],
\]

where

\[
\mathcal{H}'' = \alpha \mathcal{H} + \alpha^i \mathcal{H}_i - \frac{1}{2} \left( \alpha^2 \gamma \right)_{,i} \gamma^{ij} \tilde{\chi}_j - \frac{d-3}{4(d-2)} \alpha^2 \gamma^j \gamma^{ij} \tilde{\chi}_i \tag{104}\n\]

\[
+ \frac{1}{\alpha} \alpha^i \alpha^j \alpha_{,i} \tilde{\chi}_j - 2 \alpha \alpha^j \gamma_{jk} \pi^{ij} \tilde{\chi}_i + \frac{1}{d-1} \alpha \alpha^j \gamma_{jk} \pi^{jk} \tilde{\chi}_i + \frac{1}{d-1} \alpha^i \alpha^j \pi^{ij} \tilde{\chi}_i \n\]

\[
+ \alpha^2 \alpha^i \pi \tilde{\chi}_i + \alpha \alpha^i \alpha^j \pi_{ij} \tilde{\chi}_j + \alpha \alpha^i \pi \tilde{\chi}_i - \frac{2}{d-1} \alpha \gamma_{jk} \pi^{jk} \tilde{\chi} + \frac{d-2}{d-1} \left( \alpha^2 \pi + \alpha^2 \right) \tilde{\chi}.
\]

By applying the reduction method of Faddeev and Jackiw [21] to the action of eq. (103), one obtains

\[
S = \int dx \left[ \pi^{ij} \dot{\gamma}_{ij} - \alpha \mathcal{H} - \alpha^i \mathcal{H}_i - w \mathcal{H} - w^i \mathcal{H}_i \right], \tag{105}\n\]

\(7\) We note that at this stage the constraints \( \tilde{\chi} \approx 0 \) and \( \tilde{\chi}_i \approx 0 \) of eqs. (99) and (100) might be replaced with the constraints \( \phi_i \approx 0 \) and \( \phi \approx 0 \) through \( \tilde{\chi}_i = - \phi_i \) and \( \tilde{\chi} = \alpha \phi + \alpha^i \phi_i \), where \( \phi_i = - \frac{\alpha_i}{\alpha^2} - \pi_i \) and \( \phi = \frac{\alpha^2}{\alpha^2} - \pi \), however, since such an identification does not show to be particularly illuminating, we won’t pursue it at this stage.
upon dropping surface terms. This variant of the ADM action has been used by Teitelboim
[46, 47] and Ashtekar [9] in quantum gravity, and by York et al. in numerical relativity
[1, 14]. Since the constraints \( \mathcal{H}_i \) and \( \bar{\mathcal{H}} \) are derived from the constraints \( \tilde{\tau}_i \) and \( \bar{\tau} \) of eqs. (56) and (57) under the canonical transformations of eqs. (72) and (74) as in eqs. (101) and
(102), the algebra of the PB of these constraints is

\[
\begin{align*}
  f\{ \mathcal{H}_i, \mathcal{H}_j \} g &= g f_{,i} \mathcal{H}_i - f g_{,i} \mathcal{H}_j, \\
  f\{ \bar{\mathcal{H}}, \bar{\mathcal{H}} \} g &= (g f_{,i} - f g_{,i}) \gamma^{ij} \mathcal{H}_j, \\
  f\{ \mathcal{H}_i, \bar{\mathcal{H}} \} g &= (g f_{,i} - f g_{,i}) \bar{\mathcal{H}},
\end{align*}
\]

according to eqs. (62), (63) and (64), consistent with the constraint algebra given in [46, 47].
(Here \( f \) and \( g \) are test functions.)

V. TENTATIVE GAUGE CONSTRAINTS

Together with a set of “admissible” gauge constraints, one may put the first class con-
straints of the extended action (which are now second class) strongly equal to zero and solve
them in order to eliminate the redundant degrees of freedom from the action and introduce
the DB. Meanwhile, \textit{all} the gauge freedom of the Lagrangian action is fixed.

We now consider gauge fixing conditions for the action of eq. (69). A study of the
equations of motion of the extended action of eq. (69) is illuminating in the nature and role
of the canonical variables employed in this action. If we are only interested in the equations
of motion derived from this action we may then rewrite it as

\[
S = \int dx \left[ \Omega \dot{\bar{t}} + \Omega_i \dot{\bar{\xi}}^i + \Pi \bar{h} + \Pi_i \bar{h}^i + \Pi_{ij} \dot{q}^{ij} \\
- \bar{u} \Omega - \bar{u}^i \Omega_i - \bar{v} \bar{X} - \bar{v}^i \bar{X}_i - \bar{w} \bar{\tau} - \bar{w}^i \bar{\tau}_i \right],
\]

where we have shifted the Lagrange multipliers by adding to them the coefficients of the
constraints appearing in the Hamiltonian \( \mathcal{H}_c \) of eq. (68). The equations of motion for \( \bar{u}, \bar{u}^i, \bar{t} \) and \( \bar{\xi}^i \) are trivially satisfied while the equations of motion for \( \Omega \) and \( \Omega_i \) show that \( \bar{t} \) and \( \bar{\xi}^i \)
are undetermined,

\[
\begin{align*}
  \dot{\bar{t}} &\approx \bar{u}, \\
  \dot{\bar{\xi}}^i &\approx \bar{u}^i.
\end{align*}
\]
Therefore, tentative gauge constraints for the primary first class constraints

\[ \Omega \approx 0, \tag{111} \]
\[ \Omega_i \approx 0, \]

could be of the form

\[ \bar{t} - C_t(x) \approx 0, \tag{112} \]
\[ \bar{\xi}^i - C_{\xi^i}(x) \approx 0, \]

respectively, where \( C_t \) and \( C_{\xi^i} \) are arbitrary functions. The constraints of eqs. (111,112) form a minimal set of second class constraints and may thus be turned into strong equations. The DB of the rest of the canonical variables remains their PB. We now prove that much like \( \bar{t} \) and \( \bar{\xi}^i \), the fields \( h \) and \( h^i \) are left undetermined by the equations of motion. By extremizing the action of eq. (109), the equations of motion corresponding to \( \Pi \), \( \Pi_i \), \( h \), \( h^i \), \( \bar{v} \) and \( \bar{v}^i \) are

\[ \frac{\delta S}{\delta \Pi} = \dot{h} - \bar{v} h = 0, \tag{113} \]
\[ \frac{\delta S}{\delta \Pi_i} = \dot{h}^i + \bar{v} h^i = 0, \tag{114} \]
\[ \frac{\delta S}{\delta h} = - \dot{\Pi} - \bar{v} \Pi + \bar{v}^i \Pi_i + \bar{v}^i, = 0, \tag{115} \]
\[ \frac{\delta S}{\delta h^i} = - \dot{\Pi}_i + \bar{v}, i = 0, \tag{116} \]
\[ \frac{\delta S}{\delta \bar{v}} = h^i, + h \Pi = 0, \tag{117} \]
\[ \frac{\delta S}{\delta \bar{v}^i} = h^i, - h \Pi = 0. \tag{118} \]

In obtaining eqs. (113)-(116) we have used the constraint equations (117,118). Since the Lagrange multipliers \( \bar{v} \) and \( \bar{v}^i \) are arbitrary, the fields \( h \) and \( h^i \) can take the values of any arbitrary functions \( C_h(x) \) and \( C_{h^i}(x) \), as justified below. Suppose the latter is true, that is, \( h = C_h(x) \) and \( h^i = C_{h^i}(x) \). Eqs. (113), (114), (117) and (118) may be solved for \( \bar{v}, \bar{v}^i \), \( \Pi \) and \( \Pi_i \) in order to express them in terms of \( C_h(x) \) and \( C_{h^i}(x) \). Upon substituting these solutions into eqs. (115) and (116) they result in trivial identities.

The foregoing observation suggests that tentative gauge constraints corresponding to the secondary first class constraints

\[ \bar{\chi} \equiv h^i, + h \Pi \approx 0, \tag{119} \]
\[ \bar{\chi}_i \equiv h^i, - h \Pi_i \approx 0, \]
and compatible with the equations of motion could be of the form

\[ h - C_h(x) \approx 0, \]
\[ h^i - C_{h^i}(x) \approx 0, \]

where \( C_h \) and \( C_{h^i} \) are arbitrary functions. Once again, the constraints of eqs. (119, 120) form a minimal set of second class constraints which may be turned into strong equations. Once the solutions of these equations are inserted into the action of eq. (69) it is reduced to

\[ S = \int dx \left[ \Pi_{ij} \dot{q}^{ij} - \bar{w} \tilde{\tau} - \bar{w}^i \tilde{\tau}_i \right], \]

upon dropping an irrelevant surface term. (The redefined Lagrange multipliers \( \bar{w} \) and \( \bar{w}^i \) are arbitrary and can depend on \( q^{ij} \) and \( \Pi_{ij} \)). Since the constraints of eqs. (119, 120) do not involve \( q^{ij} \) and \( \Pi_{ij} \), the PB of these variables remains unchanged upon solving the constraints of eqs. (119, 120) and introducing the DB. We note that the functions \( C_\xi, C_{h^i}, C_h \) and \( C_{h^i} \) can depend on \( q^{ij} \) and \( \Pi_{ij} \) without violating any of the arguments and conclusions made above, since under such an assumption the constraints of eqs. (119, 120) are proven to be of special form as follows. If

\[ \{ \theta_s \} = \{ h - C_h(\gamma_{ij}, \pi^{ij}), h^k - C_{h^k}(\gamma_{ij}, \pi^{ij}), \tilde{\chi}, \tilde{\chi}_k \}, \]

we have

\[ \{ \theta, \theta \}^{-1} \approx (1/C_h)^2 \begin{pmatrix} 0 & 0 & C_h & 0 \\ 0 & 0 & 0 & -\frac{1}{d-1} \delta^i_j C_h \\ -C_h & 0 & \{ C_h, C_h \} & \{ C_{h^i}, C_h \} \\ 0 & \frac{1}{d-1} \delta^i_j C_h & \{ C_h, C_{h^i} \} & \{ C_{h^k}, C_{h^l} \} \end{pmatrix}, \]

which implies that the DB of \( q^{ij} \) and \( \Pi_{ij} \) remains equal to their PB upon turning the first class constraints \( \tilde{\chi} \) and \( \tilde{\chi}_i \) and their corresponding gauge constraints into strong equations, thanks to the constraints \( \tilde{\chi} \) and \( \tilde{\chi}_i \) not depending on \( q^{ij} \) and \( \Pi_{ij} \). The action of eq. (121), therefore, is identical with the Faddeev action of eq. (19) upon appropriate gauge fixation.

The gauge constraints of eqs. (112) and (120) are not in general admissible for arbitrary functions \( C_\xi, C_{h^i}, C_h \) and \( C_{h^i} \), since they can not be achieved from an arbitrary configuration of the fields \( \tilde{\xi}, \bar{\xi}^i, h \) and \( h^i \) by a diffeomorphism invariance transformation. In principle, one needs to consider the gauge constraints corresponding to the tertiary constraints \( \bar{\tau} \).
and $\bar{t}_i$ along with the gauge constraints of eqs. (112) and (120), and choose appropriate functions $C_{\bar{t}}, C_{\bar{\xi}_i}, C_h$ and $C_{h^i}$ in such a way that the gauge constraints altogether are achieved by diffeomorphism invariance transformations, while in this process the gauge functions are completely fixed upon assuming appropriate behavior of the gauge functions on the boundaries.

More insight about eqs. (113-118) and the role of the fields $h, h^i, \Pi$ and $\Pi_i$ in the action of eq. (69) can be gained in the following way. We may add a surface term of the form

$$S = -\left(\frac{h' h^i}{h}\right)_0 + \left(\frac{\dot{h} h^i}{h}\right)_j$$

(123)

to the kinetic part of the action of eq. (69) and write it as

$$\Pi \dot{h} + \Pi_i \dot{h}^i + \Pi_{ij} q^{ij} + S = \bar{\Pi} \dot{h} + \bar{\Pi}_i \dot{h}^i + \bar{\Pi}_{ij} q^{ij}$$

(124)

where

$$\bar{\Pi} = \frac{1}{h} \tilde{\chi},$$

$$\bar{\Pi}_i = -\frac{1}{h} \tilde{\chi}_i,$$

(125)

(126)

with $\tilde{\chi}$ and $\tilde{\chi}_i$ given by eqs. (54,55). In particular

$$\{\bar{\Pi}, \bar{\Pi}\} = \{\bar{\Pi}_i, \bar{\Pi}_i\} = \{\bar{\Pi}_i, \bar{\Pi}_j\} = 0$$

(127)

according to eqs. (59,61). We may therefore observe that $\bar{\Pi}$ and $\bar{\Pi}_i$ are the momenta conjugate to $h$ and $h^i$, and write the action of eq. (69) as

$$S = \int dx \left[ \bar{\Pi} \dot{h} + \bar{\Pi}_i \dot{h}^i + \bar{\Pi}_{ij} \dot{q}^{ij} + \Omega \ddot{t} + \Omega_i \ddot{\xi}^i + \bar{\mathcal{H}}_c - u \Omega - u^i \Omega_i - v \bar{\Pi} - v^i \bar{\Pi}_i - w \bar{t} - w^i \bar{t}_i \right]$$

(128)

where now

$$\mathcal{H}_c = \frac{1}{h} \tilde{t} + \frac{h' h^i}{h} \tilde{t}_i + \frac{1}{2(d-2)} \frac{1}{h} q_{kl} q^{kl} q^{ij} \bar{\Pi}_j - \frac{d-3}{4(d-2)} q^{ij} \bar{\Pi}_i \bar{\Pi}_j$$

$$- \frac{h' h^i}{h^2} q^{ij} \bar{\Pi}_i + \frac{h^i}{h} h_j \bar{\Pi}_j - 2 \frac{h^m}{h^2} q^{ij} \Pi_m \bar{\Pi}_i + \frac{h^i}{h} h_j \bar{\Pi} + \left( \bar{\Pi} - \frac{1}{h} \bar{h}^i \right) h^i \bar{\Pi}_i$$

$$+ \frac{h^i}{h} q^{mn} \Pi_{mn} \bar{\Pi}_i - \frac{h^l}{d-1} \left( \frac{1}{h} h^l \bar{\Pi}_l + \bar{\Pi}_l \right) \bar{\Pi} + \frac{d-2}{d-1} (h \bar{\Pi} + 2 q^{ij} \Pi_{ij} - 2 h^i \bar{\Pi}_i) \bar{\Pi}$$

$$- \frac{\bar{\ell}}{d-1} h \bar{\Pi} + \bar{\xi}^i (h \bar{\Pi}_i) + \bar{B}^i \bar{\Lambda}_i + \bar{B}^{ij} \bar{\Lambda}_i \bar{\Lambda}_j,$$

(129)
Written in this form, it is explicitly seen that the fields \( h \) and \( h^i \) act as Lagrange multiplier fields, much in the same way as the fields \( \bar{\ell} \) and \( \bar{\xi}^i \) are Lagrange multipliers. Such a simplification of the action is reminiscent of Dirac’s simplification of the Hamiltonian formulation of the second order EH action by addition of the following surface terms to the EH Lagrangian

\[
\left[ (\sqrt{-g} g^{00})_{,\nu} g^{0\nu} \right]_{,0} - \left[ (\sqrt{-g} g^{00})_{,0} g^{0\nu} \right]_{,\nu},
\]

resulting in the primary constraints taking the simple form

\[
p^{\mu 0} = 0 ;
\]

in contrast with the second order Hamiltonian formulation of Pirani and Schild in which the EH action is considered without these surface terms, and the primary constraints are of the more complicated form \( p^{\mu 0} = p^{\mu 0}(\bar{q}, \bar{p}) \), with \( \bar{q} \) and \( \bar{p} \) being other canonical variables. (The two approaches have been compared and contrasted in [23].) The surface terms of eq. (130) indeed reduce to the surface terms of eq. (123).

Since the gauge constraints of eqs. (112) and (120) are canonical, one may use them in order to fix the gauge freedom of the actions of eqs. (78) and (103) if they are transformed under the associated canonical transformations. In the case of the action of eq. (69) when written in terms of \( N, N^i, \gamma_{ij} \) and their conjugate momenta \( p, p_i \) and \( p^{ij} \) defined in eqs. (83)-(85), a reduction to the ADM action is not quite immediate. In particular, since the constraints \( \tilde{\chi} \) and \( \tilde{\chi}_i \) of eqs. (86)-(87) depend on \( \gamma_{ij} \), we expect the PB of \( \gamma_{ij} \) and \( \pi^{ij} \) to be altered upon solving the constraints \( \tilde{\chi} \) and \( \tilde{\chi}_i \) and introducing the DB if the gauge constraints \( C_h \) and \( C_{hi} \) depend on \( \gamma_{ij} \) and \( p^{ij} \). For the specific class of admissible gauge constraints in which \( N \) and \( N^i \) are constant (\( N = 1 \) and \( N^i = 0 \) for instance) a reduction to a “gauge-fixed” ADM action is seen to easily be realized. A more straightforward reduction to the ADM action might be possible if we assume that the gauge constraints also depend on the momenta \( p \) and \( p_i \).

VI. GAUGE TRANSFORMATIONS

When written in terms of \( q^{ij} \) or \( \gamma_{ij} \), the problem of determining the gauge transformations of the first order EH Lagrangian action from the first class constraints generated in the
Hamiltonian formulation transforms into a more manageable task than when one works with the formalism in which $H^{ij}$ is used. This simplification occurs mainly because in terms of the former variables constraints of different stage depend on different sets of the canonical variables, as explained in previous sections. In this section we consider the action of eq. (69) (which is a functional of $h$, $h^i$, $q^{ij}$ and their conjugate momenta) and derive the explicit form of the generator of the gauge transformations of the fields $h$, $h^i$, $q^{ij}$, $\Pi$, $\Pi_i$ and $\Pi_{ij}$. The gauge transformations of $\bar{t}$ and $\bar{\xi}^i$ which act as Lagrange multipliers are given by separate equations which are necessary for the action to remain invariant under the gauge transformations. This is done using a method very similar to the method of HTZ [30]. In this approach one directly considers gauge transformations of the total action instead of the gauge transformations that leave the extended action invariant [43]. Using the generator thus obtained, we explicitly evaluate the gauge transformation of the field $h = \sqrt{-g} g^{00}$ assuming the gauge functions corresponding to the tertiary constraints to be independent of the canonical variables, and show that a field dependent redefinition of the gauge functions is necessary in order for this transformation to correspond to the usual diffeomorphism invariance, which is given by (19)

$$\delta h^{\mu\nu} = -(h^{\mu\nu} \eta^\lambda), \lambda + h^{\mu\lambda} \eta^\nu, \lambda + h^{\nu\lambda} \eta^\mu, \lambda,$$  \hspace{1cm} (132)

for the fields $h^{\mu\nu} = \sqrt{-g} g^{\mu\nu}$, where $\eta^\mu$ are arbitrary descriptors [11].

It has been shown that for most relevant field theories one may drop fields (and their corresponding momenta) that act as Lagrange multipliers from the total Hamiltonian without loss of the gauge transformations if after the elimination the Lagrange multipliers are identified with the eliminated coordinates [35]. We therefore rewrite the total action corresponding to the extended action of eq. (69) as

$$S_T = \int dx \left[ \Pi \dot{h} + \Pi_i \dot{h}^i + \Pi_{ij} \dot{q}^{ij} - \mathcal{H}_T \right],$$ \hspace{1cm} (133)

where

$$\mathcal{H}_T = \frac{1}{h^4} \bar{t} \frac{\tau + h^i}{h} \tau_i - t \chi - \xi^i \chi_i,$$ \hspace{1cm} (134)

$$t = \frac{\bar{t}}{d - 1} - \frac{h_i}{h^2} h^i, \frac{1}{d - 1} h^i \Pi_i \frac{d - 3}{d - 2} \frac{1}{h} (h \Pi + 2 q^{ij} \Pi_{ij} - h^i \Pi),$$ \hspace{1cm} (135)

$$\xi^i = \frac{\xi^i}{d - 2} - \frac{1}{h^2} q_{kl} q^{ij} + \frac{d - 3}{4(d - 2)} \frac{1}{h^2} q^{ij} \dot{\chi}_j - \frac{h_i}{h^3} q^{ij},$$ \hspace{1cm} (136)

$$+ \frac{h_i}{h^2} h_i - \frac{2}{h^2} q^{im} \Pi_{im} + \frac{h_i}{h} \Pi + \frac{h_i}{h^2} q^{mn} \Pi_{mn}.$$
(Note that we have dropped the tilde from the constraints of eqs. (54-55,56,58).) The usefulness of the redefinitions of eqs. (135-136) lies in that the terms other than $\bar{t}$ and $\bar{\xi}^i$ which are included in $t$ and $\xi^i$ do not contribute to the gauge transformations of the fields $h, h^i, q^{ij}$ and their conjugate momenta $\Pi, \Pi_i$ and $\Pi_{ij}$ but to the gauge transformations of $\bar{t}$ and $\bar{\xi}^i$ which now explicitly appear as the Lagrange multiplier fields. We emphasize that the actual dependence of $t$ and $\xi^i$ of eqs. (135-136) on the canonical variables is quite important for obtaining the gauge transformations of $\bar{t}$ and $\bar{\xi}^i$.

In contrast with the first and second order formulations of the free spin two field actions considered in [24, 25], in which the structure functions were constant, we need to consider a more general formalism when dealing with the gauge transformations of the full EH action, where we need to consider the structure functions to be field dependent. The most general form of the generator $G$ of the gauge transformations of the total action of eq. (133) is

$$G = \int dx \left( \bar{\mu} \chi + \bar{\mu}^i \chi_i + \mu \tau + \mu^i \tau_i \right) ,$$

(137)

where the gauge functions $\mu$ and $\mu^i$ corresponding to the tertiary constraints $\tau$ and $\tau_i$ are arbitrary functions depending on spacetime as well as the canonical variables, and the functions $\bar{\mu}$ and $\bar{\mu}^i$ are arbitrary functions of spacetime and the canonical variables which satisfy a set of differential equations that arise by requiring the invariance of the total action.\(^8\)

Using eq. (137) we may show that

$$\delta H_T = - \left( \bar{\delta} \chi + \bar{\delta}^i \chi_i + \delta \tau + \delta^i \tau_i \right)$$

(138)

where

$$\bar{\delta} \chi = \int dx' dx \left[ \chi \left( \left\{ t, \chi \right\} \bar{\mu} + \left\{ t, \chi_i \right\} \bar{\mu}^i + \left\{ t, \tau \right\} \mu + \left\{ t, \tau_i \right\} \mu^i \right) - \left\{ \mathcal{H}_T, \bar{\mu} \right\} \chi \right]$$

(139)

$$\bar{\delta}^i \chi_i = \int dx' dx \left[ \chi_i \left( \left\{ \xi^i, \chi \right\} \bar{\mu} + \left\{ \xi^i, \chi_j \right\} \bar{\mu}^j + \left\{ \xi^i, \tau \right\} \mu + \left\{ \xi^i, \tau_j \right\} \mu^j \right) - \left\{ \mathcal{H}_T, \bar{\mu}^i \right\} \chi_i \right]$$

(140)

\(^8\) We consider the special case where the gauge functions do not depend on the Lagrange multiplier fields.
$$
\delta \tau = \int dx \left[ \left( \frac{1}{h} \bar{\mu} + \frac{1}{h} \mu^i_j - \left( \frac{h^i}{h} \right)_j \mu^i - \left( \frac{h^i}{h} \right)_i \mu + \frac{h^i}{h} \mu, i - \left\{ H_T, \mu \right\} \right) \tau \right] \tag{141}
$$

$$
\delta^i \tau_i = \int dx \left[ \left( \bar{\mu}^i - q^i_j \left( \frac{1}{h} \right)_j \mu + \frac{1}{h} q^i_j \mu_j + \frac{h^i}{h} \bar{\mu} - \left( \frac{h^i}{h} \right)_j \mu^j + \frac{h^j}{h} \mu^i_j \right. \right.
\left. - \left\{ H_T, \mu^i \right\} \right] \tau_i \right]. \tag{142}
$$

Since

$$
\delta \int dx \left( \Pi \dot{h} + \Pi_i \dot{h}^i + \Pi_{ij} \dot{q}^{ij} \right) = \int dx \left( \bar{\mu}, \chi + \bar{\mu}^i_{,i} \chi^i + \mu, \tau + \mu^i_{,i} \tau_i \right), \tag{143}
$$

where the partial derivative with respect to time is denoted by a $t$ index, we then have

$$
\delta S_T = \int dx \left[ \left( \mu, \partial_{,i} + \frac{\delta t}{\delta - 1} \right) \chi + \left( \bar{\mu}^i_{,i} + \delta^i_{,i} + \delta \xi^i \right) \chi^i \right.
\left. + \left( \mu_{,i} + \delta^i \right) \tau + \left( \mu^i_{,i} + \delta \right) \tau_i \right], \tag{144}
$$

where we have symbolically written $\delta \chi = \delta \cdot \chi$, etc. to indicate that the integral signs have been dropped after all PBs have been evaluated and the derivatives over the constraints $\chi$, $\chi^i$, $\tau$ and $\tau_i$ have been removed by addition of appropriate surface terms. If we require the total action of eq. (133) to be invariant under the gauge transformations of eq. (137) we have $\delta S_T = 0$, which is satisfied only if the coefficients of the constraints $\chi$, $\chi^i$, $\tau$ and $\tau_i$ are set equal to zero. By a choice of the gauge functions $\mu$ and $\mu^i$ corresponding to the tertiary constraints $\tau$ and $\tau_i$, we may determine the gauge functions $\bar{\mu}$ and $\bar{\mu}^i$ corresponding to the secondary constraints $\chi$ and $\chi^i$ by setting the coefficients of $\tau$ and $\tau_i$ in eq. (144) equal to zero. In particular, we note that according to eqs. (141, 142) these are simple algebraic equations for the gauge functions $\bar{\mu}$ and $\bar{\mu}^i$. Vanishing of the coefficients of the constraints $\chi$ and $\chi^i$ in eq. (144), on the other hand, provides with the gauge transformations of the Lagrange multipliers $\bar{\ell}$ and $\bar{\xi}^i$.

Let us choose the gauge functions $\mu$ and $\mu^i$ to depend only on spacetime and not on the canonical variables,

$$
\mu = \mu(x) \quad \mu^i = \mu^i(x). \tag{145}
$$
This choice is not necessary in principle, and one may choose any arbitrary functions that depend on the canonical variables as well. Setting the coefficients of the constraints \( \tau \) and \( \tau_i \) in eq. (144) equal to zero and solving for \( \bar{\mu} \) and \( \bar{\mu}^i \) using \( \mu \) and \( \mu^i \) of eq. (145) gives,

\[
\bar{\mu} = -h \left( \mu + \frac{1}{h} \mu^i_j - \left( \frac{1}{h} \right)_j \mu^j - \left( \frac{h^j}{h} \right) \mu + \frac{h^j}{h} \mu_i \right) \tag{146}
\]

\[
\bar{\mu}^i = -\dot{\mu}^i + h^j \dot{\mu} + q^{ij} \left( \frac{1}{h} \right)_j \mu - \frac{1}{h} q^{ij} \mu_j + \left( \frac{h^j}{h} \right) \mu^j - \frac{h^j}{h} \mu^j_i + \frac{h^i}{h} \mu^j \tag{147}
\]

where in obtaining eq. (147) we have used eq. (146). The generator of gauge transformations \( G \) is therefore given by eq. (137), with \( \mu, \mu^i, \bar{\mu} \) and \( \bar{\mu}^i \) given by eqs. (145,147). Using this generator we may find the gauge transformations of \( h, h^i, q^{ij}, \Pi, \Pi_i \) and \( \Pi_{ij} \). The gauge transformation for \( h \) is thus

\[
\delta h = \{ h, G \} = -h^2 \dot{\mu} - h \bar{\mu}^i_j - h \bar{\mu}^i - h^j \bar{\mu}_i^j \mu + h h^j_i \mu_i - h h^i_j \mu^j .
\]

This is identical with the diffeomorphism invariance transformation of \( h \) given by eq. (132) if we substitute

\[
\eta^0 = -h \mu, \tag{149}
\]

\[
\eta^i = \mu^i - h^i \mu, \tag{150}
\]

for the descriptor \( \eta^\mu \) in eq. (132).

The gauge transformations of the fields \( h^i, q^{ij}, \Pi, \Pi_i \) and \( \Pi_{ij} \) can be determined using the gauge generator \( G \) of eq. (137). One may thus easily observe that by the dependence of the constraints \( \chi \) and \( \chi_i \) on the derivatives of \( h \) and \( h^i \) the gauge transformations for \( \Pi \) and \( \Pi_i \) involve first order derivatives of \( \bar{\mu} \) and \( \bar{\mu}^i \) and thus second order derivatives of \( \mu \) and \( \mu^i \). Also, since \( \tau \) depends on second order derivatives of the fields \( q^{ij} \), we see how second order derivatives of the gauge functions \( \mu \) and \( \mu^i \) enter the gauge transformations of \( \Pi_{ij} \). The gauge transformations for the Lagrange multiplier fields \( \bar{\ell} \) and \( \bar{\xi}^i \) on the other hand are obtained by requiring that the coefficients of the constraints \( \chi \) and \( \chi_i \) in eq. (144) vanish, which according to eqs. (144,146,147) involve second order derivatives of the gauge functions \( \mu \) and \( \mu^i \). The existence of second order derivatives of the gauge functions \( \mu \) and
\( \mu^i \) is expected for the gauge invariance of the fields \( \Pi, \Pi_i, \Pi_{ij}, \bar{t} \) and \( \bar{\xi}^i \) produced by the gauge generator \( G \) of eq. (137) to coincide with their diffeomorphism invariance, which is found by the diffeomorphism invariance of the Christoffel symbols [19, 20].

We have verified that if we had used the action of eq. (69) instead of the action of eq. (133) for evaluation of the gauge transformations, we would have obtained gauge symmetries which differed from the gauge symmetries obtained above by trivial equations of motion symmetries. Such symmetries have been discussed in [30].

VII. LINEARIZED THEORY

The Linearized theory of the novel Hamiltonian formulation of the extended EH action of eq. (35) can be obtained by linearizing the fields \( h, h^i \) and \( H^{ij} \) around the metric of the flat spacetime,

\[
h^{\mu\nu} = \eta^{\mu\nu} + \tilde{h}^{\mu\nu},
\]

where the signature of the metric of the flat spacetime is \( \eta^{\mu\nu} = (-,+,+,\ldots,+), \) and we have ignored terms of higher order in \( \tilde{h}^{\mu\nu} \). This implies that, in particular,

\[
h = -1 + \tilde{h}, \quad h^i = \tilde{h}^i, \quad H^{ij} = -\delta^{ij} - \tilde{h}^{ij}, \quad H_{ij} = -\delta_{ij} + \tilde{h}_{ij}
\]

if we keep terms linear in the perturbation fields only. Under the expansion of eq. (152), the fundamental PBs

\[
\{h, \omega\} = 1, \quad \{h^i, \omega_j\} = \delta^i_j, \quad \{H^{ij}, \omega_{kl}\} = \frac{1}{2} \left( \delta^i_k \delta^j_l + \delta^i_l \delta^j_k \right),
\]

transform into

\[
\{\tilde{h}, \omega\} = 1, \quad \{\tilde{h}^i, \omega_j\} = \delta^i_j, \quad \{\tilde{h}^{ij}, -\omega_{kl}\} = \frac{1}{2} \left( \delta^i_k \delta^j_l + \delta^i_l \delta^j_k \right),
\]

showing that the fields \( \omega, \omega_i \) and \( -\omega_{ij} \) act as the momenta conjugate to the perturbation fields \( \tilde{h}, \tilde{h}^i \) and \( \tilde{h}^{ij} \). Keeping only terms in the EH Hamiltonian action of eq. (35) which are bilinear in the fields and Lagrange multipliers, and by defining

\[
\tilde{\omega}_{ij} = -\omega_{ij}
\]
we obtain
\[ S = \int dx \left[ \omega \dot{h} + \omega_i \dot{h}^i + \bar{\omega}_{ij} \dot{h}^{ij} + \Omega \dot{\bar{\epsilon}} + \Omega_i \dot{\tilde{\xi}}^i 
- \bar{H}_c^0 - u \Omega - u^i \Omega_i - v \chi' - v^i \chi'_i - w \tau' - w^i \tau'_i \right], \]

where
\[
\bar{H}_c^0 = -\frac{d-2}{d-1} \omega^2 + \frac{d-3}{4(d-2)} \omega_i \omega_i + \xi^k (\dot{h}_k + \omega_k) + \frac{t}{d-1} (\dot{h}_k^i - \omega - \bar{\omega}_u)
\]
\[+ \left( \bar{\omega}_{ij} \bar{\omega}_{ij} - \frac{1}{d-1} \bar{\omega}_{ii} \bar{\omega}_{jj} - 2 \bar{\omega}_{ij} \dot{h}_j^i + \frac{2}{d-1} \bar{\omega}_{kk} \dot{h}_k^j + \dot{h}_j^i \dot{h}_i^j - \frac{1}{d-1} \dot{h}_k^k \dot{h}_l^l \right) \]
\[- \left( \frac{1}{2(d-2)} \dot{h}_{ij}^i \omega_j + \frac{1}{2} \dot{h}_{i}^{jk} \dot{h}^{ik} + \frac{1}{4(d-2)} \dot{h}_{ij}^m \dot{h}_{ij}^n - \frac{1}{4} \dot{h}_{ij}^m \dot{h}_{ij}^m \right), \]

and
\[
\chi' = \dot{h}_k^k - \omega - \bar{\omega}_{kk}, \quad (158)
\]
\[
\chi'_i = \dot{h}_i^i + \omega_i, \quad (159)
\]
\[
\tau' = \dot{h}_{ij,ij} + \omega_{i,i}, \quad (160)
\]
\[
\tau'_i = 2 \bar{\omega}_{kk,i} - 2 \bar{\omega}_{ki,k}. \quad (161)
\]

The action of eq. (156), with the Hamiltonian of eq. (157) and the first class constraints of eqs. (158)-(161), indeed coincide with the extended action principle for the free spin two field theory on a flat spacetime in first order form as developed in [24]. The tertiary constraints \( \tau' \) and \( \tau'_i \) in fact contribute to the generator of the linearized diffeomorphism transformation of the “linerized” affine connections \( \Gamma^\lambda_{\mu\nu} \) as found in [24].

**VIII. SUMMARY AND CONCLUSION**

A major distinction between the Dirac Hamiltonian formulation of the first order EH action as performed in [24] and the ADM Hamiltonian formulation of the same action [4, 5, 6, 7, 8, 19, 20] is that in the latter all “algebraic” constraints are solved in order to eliminate a number of fundamental fields from the action at the Lagrangian level, while in the analysis of [24] only those algebraic constraints which are second class (in the sense of the Dirac constraint formalism) are used to eliminate fundamental fields; first class “algebraic constraints” are treated according to the Dirac constraint formalism. This results
in the appearance of tertiary first class constraints, and an unusual PB algebra of first class constraints apparently different from the ADM algebra of the Hamiltonian and momentum constraints $\mathcal{H}$ and $\mathcal{H}_i$. Therefore, it is very important to compare the results of this novel Hamiltonian formulation with the usual ADM formulation of the first order EH action. Such a comparison remains obscure however, especially because of the different choices of the canonical variables made in these formulations.

The connection between this Hamiltonian formulation and the Faddeev and ADM formulations was considered in this chapter, first using the method of Faddeev and Jackiw [3, 21], and then by proposing tentative gauge constraints for the reduction of the formalism in the context of the Dirac constraints method [30, 43]. At first, the variables $(h, h^i, H^{ij})$ employed in [24] were canonically transformed to $(h, h^i, q^{ij}), (h, h^i, \gamma_{ij}), (N, N^i, \gamma_{ij})$ and $(\alpha, \alpha^i, \gamma_{ij})$. Upon the first set of transformations, the tertiary constraint $\tau$ of eq. (39) splits into several terms as in eq. (57), some of which depend on the secondary constraints. Therefore, the new choice of the tertiary constraint $\tilde{\tau}$ of eq. (58) is made possible and a great simplification of the algebra of constraints occurs, as in eqs. (59-64). The secondary constraints commute with the tertiary constraints as a result, and the tertiary constraints coincide with the Hamiltonian and momentum constraints $C_0$ and $C_i$ of eqs. (66,67) of the Faddeev formulation [19, 20]. The successive canonical transformations mentioned above were performed considering the new tertiary constraint $\tilde{\tau}$ rather than $\tau$ as the tertiary constraint arising from the secondary constraint $\tilde{\chi}$. A choice of $(h, h^i, q^{ij}), (h, h^i, \gamma_{ij})$ or $(\alpha, \alpha^i, \gamma_{ij})$ was demonstrated to be preferred to a choice of $(h, h^i, H^{ij})$ or $(N, N^i, \gamma_{ij})$ as coordinates of the Hamiltonian formulation, since the constraints take a especially simple form when expressed in terms of the former sets of variables; the secondary first class constraints depend only on the variables which are absent in the tertiary constraints, and vice versa. This not only simplifies the task of determining the gauge transformations produced by the first class constraints, but also reveals the unimportant role of the subset of canonical variables $(h, h^i)$ or $(\alpha, \alpha^i)$ in the formalism. More importantly, gauge fixing of the extended Hamiltonian action becomes more transparent when the former sets of variables are used.

Considering the equations of motion arising from the Hamiltonian EH action when written in terms of $(h, h^i, q^{ij})$, we observe that there are no dynamical restrictions on the fields $h$ and $h^i$, and thus they may be considered as Lagrange multiplier fields when multiplied into the tertiary constraints $\tau$ and $\tau_i$. This was illustrated in an alternative way by adding surface
terms to the action and transforming the secondary constraints $\bar{\chi}$ and $\bar{\chi}_i$ into the momenta conjugate to $h$ and $h^i$. The necessary surface terms are equal to the surface terms added to the second order EH action by Dirac [16] in order to facilitate the task of a Hamiltonian formulation of this action.

When $(h, h^i, q^{ij})$ are used as coordinates, the gauge transformation of the field $h$ generated by the first class constraints coincides with the diffeomorphism invariance transformation of this field if the descriptor of the diffeomorphism invariance has the particular dependence on the canonical variables and the gauge functions of eqs. (149,150). Though our results correspond only to the case where the gauge functions associated with the tertiary constraints do not depend on the canonical variables, we expect that this feature is valid under more general assumptions. The relationship between the gauge generator and the descriptor of the diffeomorphism invariance has been considered in [11, 39, 40]. Although we have only determined the explicit form of the diffeomorphism invariance of $h$ in this chapter, it is possible to find the gauge transformations of all other fields from the formalism developed in the foregoing sections, thus the gauge transformations of the Christoffel symbols, as briefly pointed out. In the ADM approach, however, one needs to make use of the equations of motion for the Christoffel symbols in order to determine their gauge invariance.

It is interesting to investigate if the Dirac quantization of the above Hamiltonian formulations, in which first class constraints act as operators, would produce results other than quantization of the ADM action in which “reduction” is done before quantization. The importance of this issue has been discussed in [3, 32].

IX. ACKNOWLEDGMENTS

The author would like to thank K. Kargar, I. Khavkin and D.G.C. McKeon and colleagues from the University of Western Ontario for the enjoyment of numerous discussions. An unfinished collaboration with Prof. McKeon on the gauge invariance of the action of eq. (35) was helpful.
[1] A. Abrahams, A. Anderson, Y. Choquet-Bruhat, J.W. York, *Proceedings of the 18th Texas Symposium on Relativistic Astrophysics*, World Scientific, Singapore (1998).

[2] J.L. Anderson, P.G. Bergmann, *Phys. Rev.* **83**, 1018 (1951).

[3] J. Antonio-Garcia and J.M. Pons, *Int. J. Mod. Phys.* **A12**, 451 (1997).

[4] R. Arnowitt and S. Deser, *Phys. Rev.* **113**, 745 (1959).

[5] R. Arnowitt, S. Deser and C.W. Misner, *Phys. Rev.* **116**, 1322 (1959).

[6] R. Arnowitt, S. Deser and C.W. Misner, *Phys. Rev.* **117**, 1595 (1960).

[7] R. Arnowitt, S. Deser and C.W. Misner, *J. Math. Phys.* **1**, 434 (1960).

[8] R. Arnowitt, S. Deser and C.W. Misner, in *Gravitation: An Introduction to Modern Research* (L. Witten, ed., Wiley, NY, 1962).

[9] A. Ashtekar, *Phys. Rev.* **D36**, 1587 (1987).

[10] P.G. Bergmann, R. Penfield, R. Schiller and H. Zatzkis, *Phys. Rev.* **80**, 81 (1950).

[11] P.G. Bergmann and A. Komar, *Int. J. Theor. Phys.* **5**, 15 (1972).

[12] M. Carmeli, *Classical Fields: General Relativity and Gauge Theories* (John Wiley and Sons, 1982).

[13] L. Castellani, *Ann. Phys.* (NY) **143**, 357 (1982).

[14] Y. Choquet-Bruhat, J.W. York and A. Anderson, gr-qc/9802027

[15] P.A.M. Dirac, *Can. J. Math.* **2**, 129 (1950).

[16] P.A.M. Dirac, *Proc. Roy. Soc. (London)* **A246**, 333 (1958).

[17] P.A.M. Dirac, *Phys. Rev.* **114**, 924 (1959).

[18] P.A.M. Dirac, *Lectures on Quantum Mechanics* (Dover, Mineola, 2001).

[19] L.D. Faddeev and V.N. Popov, *Sov. Phys. Usp.* **16**, 777 (1975).

[20] L.D. Faddeev, *Sov. Phys. Usp.* **25**, 130 (1982).

[21] L.D. Faddeev and R. Jackiw, *Phys. Rev. Lett.* **60**, 1692 (1988).

[22] M. Fierz and W. Pauli, Proc. R. Soc. **A73**, 211 (1939).

[23] A.M Frolov, N. Kiriushcheva and S.V. Kuzmin, arXiv:0809.1198v1

[24] R.N. Ghalati and D. G. C. McKeon gr-qc 07112543.

[25] R.N. Ghalati gr-qc 0803.3651

[26] D.M. Gitman and I.V. Tyutin, *Quantization of fields with constraints*, Springer-Verlag, 1990.
[27] H. Goldstein, *Classical Mechanics* (Second Edition) (Addison-Wesley, Reading) (1980).

[28] A. Hanson, T. Regge and C. Teitelboim, *Constrained Hamiltonian Systems* Roma, Accademia Nazionale Dei Lincei, 1976.

[29] M. Henneaux, C. Teitelboim and J. Zanelli, *Nucl.Phys.* B332, 169 (1990).

[30] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems* (Princeton U. Press, Princeton, 1992).

[31] N. Kiriushcheva, S.V. Kuzmin, arxiv 0809.0097.

[32] K. Kuchar, *Phys. Rev. D* 35, 596 (1987).

[33] C. Lanczos, *The Variational Principles of Mechanics* (U. of Toronto Press, Toronto) (1970).

[34] C.W. Misner, K.S. Thorne and J.A. Wheeler, *Gravitation* (Freeman Press, San Francisco, 1971).

[35] A Wipf, *Canonical gravity: from classical to quantum: proceedings of the 117th WE Heraeus Seminar held at Bad Homref, Germany, 13-17 September, J. Ehlers and H. Friedrich (Eds.) Springer-Verlag* (1994) ;V. Mukhanov and A. Wipf *On the symmetries of Hamiltonian systems*, preprint ETH-TH/94-04.

[36] P. Mukherjee and A. Saha, hep-th 0705.4358

[37] F.A.E. Pirani, A. Schild, *Phys. Rev.* 79 986 (1950).

[38] F.A.E. Pirani, A. Schild and S. Skinner, *Phys. Rev.* 87, 452 (1952).

[39] J.M Pons, D.C Salisbury, *Phys. Rev. D* 71, 124012 (2005).

[40] D.C. Salisbury and K. Sundermeyer, *Phys. Rev. D* 27, 740, 1983.

[41] J. Schwinger, *Phys. Rev.* 130, 1253 (1963); *Phys. Rev.* 132, 1317 (1963).

[42] E.C.G. Sudarshan and N. Mukunda, *Classical Dynamics, A Modern Perspective* (John Wiley and Sons, 1974).

[43] K. Sundermeyer, *Constrained Dynamics* (Springer-Verlag, Berlin, 1982).

[44] C. Teitelboim, *Ann. Phys. (NP)* 79, 542 (1973).

[45] C. Teitelboim, *Phys. Rev. Lett.* 38, 1108 (1977).

[46] C. Teitelboim, *Phys. Rev.* D25, 3159 (1982).

[47] C. Teitelboim, *Phys. Rev.* D28, 297 (1983).

[48] R.M. Wald, *General Relativity* (U. of Chicago Press, Chicago, 1971).

[49] S. Weinberg, *Gravitation and Cosmology* (John Wiley and Sons, Inc., 1972)