Arities and aritizabilities of group, monoid and groupoid theories

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Abstract

We study applications of a general approach for arities and aritizabilities of theories to group and monoid theories. It is proved that a theory of a group $G$ is aritizable if and only if $G$ is finite. It is shown that this criterion does not hold for monoids/groupoids: there is an infinite monoid/groupoid having a binary theory.

Key words: elementary theory, arity, aritizability, group, monoid.

We continue to study arities of theories and of their expansions [1, 2]. An arity is a basic characteristic of complexity of a theory allowing to reduce all formulae to ones with boundedly many free variables. In the present paper a general approach for arities and aritizabilities of theories [1, 2] is applied to group theories [3]. It is proved that a theory of a group $G$ is aritizable if and only if $G$ is finite. It is shown that this criterion does not hold for monoids/groupoids: there is an infinite monoid/groupoid having a binary theory.

1 Preliminaries

Recall a series of notions related to arities and aritizabilities of theories.

Definition [4]. A theory $T$ is said to be $\Delta$-based, where $\Delta$ is some set of formulae without parameters, if any formula of $T$ is equivalent in $T$ to a Boolean combination of formulae in $\Delta$.

For $\Delta$-based theories $T$, it is also said that $T$ has quantifier elimination or quantifier reduction up to $\Delta$.

Definition [4, 5]. Let $\Delta$ be a set of formulae of a theory $T$, and $p(\bar{x})$ a type of $T$ lying in $S(T)$. The type $p(\bar{x})$ is said to be $\Delta$-based if $p(\bar{x})$ is isolated by a set of formulas $\varphi^\delta \in p$, where $\varphi \in \Delta$, $\delta \in \{0, 1\}$.

The following lemma, being a corollary of Compactness Theorem, noticed in [4].

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Lemma 1.1 A theory $T$ is $\Delta$-based if and only if, for any tuple $\bar{a}$ of any (some) weakly saturated model of $T$, the type $\text{tp}(\bar{a})$ is $\Delta$-based.

Definition [1]. An elementary theory $T$ is called unary, or 1-ary, if any $T$-formula $\varphi(\overline{x})$ is $T$-equivalent to a Boolean combination of $T$-formulas, each of which is of one free variable, and of formulas of form $x \approx y$.

For a natural number $n \geq 1$, a formula $\varphi(\overline{x})$ of a theory $T$ is called $n$-ary, or an $n$-formula, if $\varphi(\overline{x})$ is $T$-equivalent to a Boolean combination of $T$-formulas, each of which is of $n$ free variables.

For a natural number $n \geq 2$, an elementary theory $T$ is called $n$-ary, or an $n$-theory, if any $T$-formula $\varphi(\overline{x})$ is $n$-ary.

A theory $T$ is called binary if $T$ is 2-ary, it is called ternary if $T$ is 3-ary, etc.

We will admit the case $n = 0$ for $n$-formulae $\varphi(\overline{x})$. In such a case $\varphi(\overline{x})$ is just $T$-equivalent to a sentence $\forall\overline{x}\varphi(\overline{x})$.

If $T$ is a theory such that $T$ is $n$-ary and not $(n - 1)$-ary then the value $n$ is called the arity of $T$ and it is denoted by $\text{ar}(T)$. If $T$ does not have any arity we put $\text{ar}(T) = \infty$.

Similarly, for a formula $\varphi$ of a theory $T$ we denote by $\text{ar}_T(\varphi)$ the natural value $n$ if $\varphi$ is $n$-ary and not $(n - 1)$-ary. If $\varphi$ does not any arity we put $\text{ar}_T(\varphi) = \infty$.

If a theory $T$ is fixed we write $\text{ar}(\varphi)$ instead of $\text{ar}_T(\varphi)$.

Remark 1.2 [1] For the description of definable sets for models $M$ of $n$-theories it suffices describe links between definable sets $A$ and $B$ for $n$-formulas $\varphi(\overline{x})$ and $\psi(\overline{y})$, respectively, and definable sets $C$ and $D$ for $\varphi(\overline{x}) \land \psi(\overline{y})$ and $\varphi(\overline{x}) \lor \psi(\overline{y})$, respectively.

If $\overline{x} = \overline{y}$ then $C = A \cap B$ and $D = A \cup B$, i.e., conjunctions and disjunctions work as set-theoretic intersections and unions.

If $\overline{x}$ and $\overline{y}$ are disjoint then $C = A \times B$ and $D = (A + B)_M \overset{M}{=} \{ \overline{x}, \overline{y} \mid \overline{x} \in A$ and $\overline{y} \in B \}$, or $\overline{x} \in M$ and $\overline{y} \in B \}$, i.e., $C$ is the Cartesian product of $A$ and $B$, and $D$ is the (generalized) Cartesian sum of $A$ and $B$ in the model $M$.

If $\overline{x} \neq \overline{y}$, and $\overline{x}$ and $\overline{y}$ have common variables, then $C$ and $D$ are represented as a mixed product and a mixed sum, respectively, working partially as intersection and union, for common variables, and partially as Cartesian product and Cartesian sum, for disjoint variables.

If $\overline{x}$ and $\overline{y}$ consist of pairwise disjoint variables and $\overline{x} \subseteq \overline{y}$, $\overline{x} \neq \overline{y}$, then for any formula $\varphi(\overline{x})$ the set of solution of the formula $\varphi(\overline{x}) \land (\overline{y} \approx \overline{y})$ in $M$ is called a cylinder with respect to $M(\overline{x})$ and generated by the set of solutions $\varphi(M)$. In any case generating sets for cylinders coincide their projections, i.e., sets of solutions for formulas $\exists \overline{x}\varphi(\overline{x})$, where $\overline{x} \subseteq \overline{y}$.

Proposition 1.3 If $M$ is a $n$-element structure, for $n \in \omega$, then $\text{ar}(\text{Th}(M)) \leq n$.

Proposition 1.4 [1]. Any theory of a finite structure $M$ is unary-tizable, by an expansion using finitely many new binary language symbols.
Proposition 1.5 [1]. Any formula of a theory having finitely many solutions is unary-tizable.

Definition [2]. A theory $T$ is called almost $n$-ary if there are finitely many formulae $\varphi_1(\overline{x}), \ldots, \varphi_m(\overline{x})$ such that each $T$-formula is $T$-equivalent to a Boolean combination of $n$-formulae and formulae obtained by substitutions of free variables in $\varphi_1(\overline{x}), \ldots, \varphi_m(\overline{x})$.

In such a case we say that the formulae $\varphi_1(\overline{x}), \ldots, \varphi_m(\overline{x})$ witness that $T$ is almost $n$-ary.

Almost 1-ary theories are called almost unary, almost 2-ary theories are called almost binary, almost 3-ary theories are called almost ternary, etc.

A theory $T$ is called almost $n$-aritizable if some expansion $T'$ of $T$ is almost $n$-ary.

Almost 1-aritizable theories are called almost unary-tizable, almost 2-aritizable theories are called almost binarizable, almost 3-aritizable theories are called almost ternarizable, etc.

Assuming that the witnessing set $\{\varphi_1(\overline{x}), \ldots, \varphi_m(\overline{x})\}$ is minimal for the almost $n$-ary theory $T$ we have either $m = 0$ or $\ell(\overline{x}) > n$.

Thus we have two minimal characteristics witnessing the almost $n$-arity of $T$: $m$ and $\ell(\overline{x})$. The pair $(m, \ell(\overline{x}))$ is called the degree of the almost $n$-arity of $T$, or the aar-degree of $T$, denoted by $\text{deg}_{\text{aar}}(T)$. Here we assume that $n$ is minimal with almost $n$-arity of $T$, this $n$ is denoted by $\text{aar}(T)$. Clearly, $\text{aar}(T) \leq \text{ar}(T)$, and if $m = 0$, i.e., $n = \text{ar}(T) = \text{aar}(T)$ then it is supposed that $\ell(\overline{x}) = 0$, too.

We have $\text{aar}(T) \in \omega$ if and only if $\text{ar}(T) \in \omega$. So if $\text{ar}(T) = \infty$ then it is natural to put $\text{aar}(T) = \infty$.

2 Arities and aritizabilities of group theories

Proposition 2.1 Let $\mathcal{M}$ be an infinite structure, $\varphi(x_1, \ldots, x_m)$ be a formula satisfying the following conditions:

1. for any elements $a_1, \ldots, a_{m-1} \in M$ each result $\varphi_i(x_i)$ of substitution of these elements to the formula $\varphi(x_1, \ldots, x_m)$ instead of $m-1$ free variables has positively many and finitely many solutions in $\mathcal{M}$;

2. for any $i \leq m$ the formula

$$\psi_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m) := \exists x_i \varphi(x_1, \ldots, x_m)$$

has cofinitely many solutions in $\mathcal{M}$.

Then $\varphi(x_1, \ldots, x_m)$ is not $n$-aritizable for any $n < m$.

Proof. Let $\overline{x} = \langle x_1, \ldots, x_m \rangle$. Assume on contrary that $\varphi(\overline{x})$ is $n$-arizable for some $n < m$. Thus $\varphi(\overline{x})$ can be represented, for some expansion $T'$ of $T = \text{Th}(\mathcal{M})$, by a positive Boolean combination $\chi(\overline{x})$ of $n$-formulae, written in a disjunctive normal form $\bigvee_{j} \bigwedge_{k} \theta_{j,k}(\overline{x}_{jk})$, $\ell(\overline{x}_{jk}) = n$. For each $j$ we denote by $\chi_j(\overline{x})$ the formula $\bigwedge_{k} \theta_{j,k}(\overline{x}_{jk})$. Thus the set $Z$ of solutions for $\varphi(x_1, \ldots, x_m)$
in the expansion $\mathcal{M}'$ of $\mathcal{M}$ is represented by Cartesian and mixed sums and products of the sets of solutions for $\theta_{j,k}(\mathfrak{T}_{jk})$, as well as by unions $Z_j$ of the sets of solutions for $\chi_j(\mathfrak{T})$. Now, as in (1), we substitute elements $a_1, \ldots, a_{m-1} \in M'$ into the formulae $\varphi(\mathfrak{T})$ and $\chi_j(\mathfrak{T})$. Since $Z = \bigcup_j Z_j$ and the formulae $\varphi_i(x_i)$ have finitely many solutions, then the correspondent results $\chi_{ij}(x_i)$ of substitutions for $\chi_j(\mathfrak{T})$ have finitely many solutions, too. Now by (1) and (2) we choose $j$ with positively many solutions for $\chi_{ij}(x_i)$ and infinitely many solutions for conjunctive members $\theta_{j,k}(\mathfrak{T}_{jk})$, producing the mixed product $Z_j$ such that varying the tuple $(a_1, \ldots, a_{m-1})$ copies of $\chi_{ij}(x_i)$ run an infinite set of solutions. Now we step-by-step show that intersections of sets of solutions for the formulae $\theta_{i,j,k}(x_i)$, the results of substitutions of $a_1, \ldots, a_{m-1}$ into $\theta_{j,k}(\mathfrak{T}_{jk})$, are infinite, using $l(\mathfrak{T}_{jk}) < l(\mathfrak{T})$. It contradicts the condition that $\varphi_i(x_i)$ has finitely many solutions in $\mathcal{M}$, as required.

**Remark 2.2** Notice that the conditions:

1. a formula $\varphi(x_1, \ldots, x_m)$ has co-infinitely many solutions in a structure $\mathcal{M}$,
   2. for any $i \leq m$ the formula
   
   $$\psi_i(x_1, \ldots, x_i-1, x_{i+1}, \ldots, x_m) := \exists x_i \varphi(x_1, \ldots, x_m)$$

has cofinitely many solutions in $\mathcal{M}$, can be realized by unary-tizable, even unary theories. Indeed, taking a structure $\mathcal{M}$ with a unary predicate $Q(x)$ with infinitely and co-finitely many solutions, say $Q = M \setminus \{a\}$ for some $a \in M$, the formula $\varphi(x, y) := Q(x) \land y \approx y$ has co-infinitely many solutions in $\mathcal{M}$, and both $\exists x \varphi(x, y)$ and $\exists y \varphi(x, y)$ have cofinitely many solutions in $\mathcal{M}$.

**Theorem 2.3** Let $\mathcal{G}$ be a $\emptyset$-definable subgroup in a structure $\mathcal{M}$. Then the following conditions are equivalent:

1. all formulae of $\text{Th}(\mathcal{M})$ defining subsets of finite Cartesian powers of $\mathcal{G}$ are $n$-aritizable for some fixed natural $n$, and produce $n$-aritizable $\text{Th}(\mathcal{G})$;
2. all formulae of $\text{Th}(\mathcal{M})$ defining subsets of finite Cartesian powers of $\mathcal{G}$ are unary-izable, and produce unary-izable $\text{Th}(\mathcal{G})$;
3. $\mathcal{G}$ is a finite group.

Proof. (1) $\Rightarrow$ (3). Let all formulae of $\text{Th}(\mathcal{M})$ defining subsets of Cartesian products of $\mathcal{G}$ are $n$-aritizable for some fixed natural $n$ and $\mathcal{G}$ be an infinite group. We consider the formula $\varphi(x_1, \ldots, x_n, y) := y \approx x_1 \cdot x_2 \cdots x_n$. It satisfies the conditions of Proposition 2.1 for the definable substructure $\mathcal{G}$. Indeed, substituting $n$ parameters $a_1, \ldots, a_n$ instead of any $n$ variables we obtain some unique solution for the rest variable: for $\varphi(a_1, \ldots, a_n, y)$ we have the solution $y = a_1 \cdot a_2 \cdots a_n$, and for $\varphi(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n)$, $x = a_{i-1} \cdot \cdots a_1 a_{n-1} \cdots a_i a_{i+1}$, and $a_{i+1}$. Besides, each formula $\exists x_i \varphi(x_1, \ldots, x_n, y)$, $\exists y \varphi(x_1, \ldots, x_n, y)$ produces the set $G^n$ of solutions. By Proposition 2.1 the formula $\varphi(x_1, \ldots, x_n, y)$ is not $n$-aritizable. Since $n$ is chosen arbitrary we obtain a contradiction with the assumption (1).
Theorem 2.4 For any group \( G \) the following conditions are equivalent:

1. \( \text{Th}(G) \) is aritizable,
2. \( \text{Th}(G) \) is almost \( n \)-aritizable for some \( n \),
3. \( \text{Th}(G) \) is unary-tizable,
4. \( \text{Th}(G) \) is almost unary-tizable,
5. \( \text{Th}(G) \) is \( n \)-ary for some \( n \in \omega \),
6. \( G \) is a finite group.

Proof. (1) \( \Leftrightarrow \) (2) holds, since if a theory \( T \) is aritizable then it is \( n \)-aritizable for some \( n \) and therefore almost \( n \)-aritizable. Conversely, if a theory \( T \) is almost \( n \)-aritizable for some \( n \) which is witnessed by a finite set \( \Phi \) of \( T \)-formulae, then \( T \) is \( m \)-aritizable, where \( m = \max\{n, l(\Phi)\} \), \( l(\Phi) \) is a maximal length \( l(\mathfrak{T}) \) of free variable tuple \( \mathfrak{T} \), for \( \varphi(\mathfrak{T}) \in \Phi \).

(1) \( \Leftrightarrow \) (3) \( \Leftrightarrow \) (6) follows by Theorem 2.3

(6) \( \Rightarrow \) (5) holds by Proposition 1.3

(3) \( \Rightarrow \) (4), (4) \( \Rightarrow \) (2), and (5) \( \Rightarrow \) (1) are obvious.

The theorem is proved.

Corollary 2.5 For any group \( G \) either \( \ar(\text{Th}(G)) \in \omega \), if \( G \) is finite, or \( \ar(\text{Th}(G)) = \infty \), if \( G \) is infinite.

3 Aritizable theories of monoids and groupoids

Arguments for Theorem 2.3 and for Corollary 2.4 confirm that monoid and groupoid theories correspondent to a theory of a group \( G \) are non-aritizable if \( G \) is infinite. At the same time the following example shows that Corollaries 2.4 and 2.5 do not hold for monoids.

Example 3.1 Let \( \mathcal{M} \) be an infinite monoid with the unit \( e \) and the rule

\[ \forall x, y(x \neq e \land y \neq e \rightarrow x \cdot y = e). \]

The formula \( \varphi(x, y, z) := (x \cdot y = z) \), defining the multiplication in \( \mathcal{M} \), is \( \text{Th}(\mathcal{M}) \)-equivalent to the following Boolean combination of binary formulae:

\[ (x = e \land y = z) \lor (y = e \land x = z) \lor (x \neq e \land y \neq e \land z = e), \]

witnessing that \( \text{Th}(\mathcal{M}) \) is unary and implying that \( \text{Th}(\mathcal{M}) \) is unary-tizable.
Remark 3.2 Similarly Example 3.1, Corollaries 2.4 and 2.5 fail for monoids \( \mathcal{M} \) with finite \( R(\mathcal{M}) = \{ a \cdot b \mid a, b \in M, a \neq e, b \neq e \} \), and for related groupoids.

Indeed, since \( R(\mathcal{M}) \) is finite consisting of elements \( c_1, \ldots, c_m \), there is an expansion \( \mathcal{M}' \) of \( \mathcal{M} \) by finitely many disjoint binary predicates \( D_i = \{ \langle a, b \rangle \mid a \neq e, b \neq e, a \cdot b = c_i \} \), and unary single predicates \( R_i = \{ c_i \} \), \( i \leq m \), with \( D_1 \cup \ldots \cup D_m = (\mathcal{M} \setminus \{ a \})^2 \). The formula \( \varphi(x, y, z) := (x \cdot y = z) \), defining the multiplication in \( \mathcal{M} \), is \( \text{Th}(\mathcal{M}) \)-equivalent to the following Boolean combination of binary formulae:

\[
(x = e \land y = z) \lor (y = e \land x = z) \lor \left( x \neq e \land y \neq e \land \left( \bigvee_{i=1}^{m} (D_i(x, y) \land R_i(z)) \right) \right).
\]

witnessing that \( \text{Th}(\mathcal{M}') \) is binary and implying that \( \text{Th}(\mathcal{M}) \) is binarizable, with \( \text{aar}(\text{Th}(\mathcal{M})) = 1 \).

The same effect of binarizability is satisfied for groupoids \( \mathcal{M} \) with finite \( R(\mathcal{M}) := M \cdot M \).

In view of Remark 3.2 the following assertion holds:

**Proposition 3.3** Any monoid (groupoid) \( \mathcal{M} \) with finite \( R(\mathcal{M}) \) is binarizable with \( \text{aar}(\text{Th}(\mathcal{M})) = 1 \).

Arguments for Remark 3.2 show that Proposition 3.3 can be spread for each algebra \( \mathcal{A} = \langle A; f_1^{n_1}, \ldots, f_k^{n_k} \rangle \) with finite \( R(\mathcal{A}) = \bigcup_{i=1}^{k} f_i(A, A, \ldots, A) \). In such a case we replace binary relations \( D_i \) by \( n_j \)-ary relations \( D_{i,j}^{n_j} = \{ \langle a_1, \ldots, a_{n_j} \rangle \mid f_j(a_1, \ldots, a_{n_j}) = c_i \} \), \( 1 \leq j \leq k \).

**Proposition 3.4** Any algebra \( \mathcal{A} \) in a finite language and with finite \( R(\mathcal{A}) \) is aritizable and \( \text{aar}(\text{Th}(\mathcal{A})) = 1 \).

**Remark 3.5** Since arities and aritizabilities are preserved under disjoint unions and compositions of theories \[1, 2\] Proposition 3.4 admits natural generalizations for theories in infinite languages.

### 4 Conclusion

We considered possibilities for arities of group, monoid and groupoid theories. It is shown that both \( n \)-ary and aritizable group theories are exactly theories of finite groups. This property fails for theories of monoids, theories of groupoids, and theories of some universal algebras. A natural question arises on a characterization of arities and aritizabilities of theories of various algebras.
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