BERRY-ESSEEN BOUNDS IN THE ENTROPIC
CENTRAL LIMIT THEOREM

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Abstract. Berry-Esseen-type bounds for total variation and relative entropy distances to the normal law are established for the sums of non-i.i.d. random variables.

1. Introduction

Let $X_1, \ldots, X_n$ be independent (not necessarily identically distributed) random variables with mean $\mathbb{E}X_k = 0$ and finite variances $\sigma_k^2 = \mathbb{E}X_k^2$ ($\sigma_k > 0$). Put $B_n = \sum_{k=1}^n \sigma_k^2$. Under additional moment assumptions, the normalized sum

$$S_n = \frac{X_1 + \cdots + X_n}{\sqrt{B_n}}$$

has approximately a standard normal distribution in a weak sense. Moreover, the closeness of the distribution function $F_n(x) = \mathbb{P}\{S_n \leq x\sqrt{B_n}\}$ to the standard normal distribution function

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy$$

has been studied intensively in terms of the so-called Lyapunov ratios

$$L_s = \frac{\sum_{k=1}^n \mathbb{E}|X_k|^s}{B_n^{s/2}}.$$

In particular, if all $X_k$ have finite third absolute moments, the classical Berry-Esseen theorem says that

$$\sup_x |F_n(x) - \Phi(x)| \leq CL_3,$$  \hspace{1cm} (1.1)

where $C$ is an absolute constant (cf. e.g. [E], [F], [Pe]).

One of the most remarkable features of (1.1) is that the number of summands does not explicitly appear in it, while in the i.i.d. case, that is, when $X_k$ have equal distributions, $L_3$ is of order $\frac{1}{\sqrt{n}}$, which is best possible for the Kolmogorov distance under the 3-rd moment condition.

In this paper we shall prove bounds for stronger distances between $F_n$ and $\Phi$, such as total variation $\|F_n - \Phi\|_{TV}$ and relative entropy $D(F_n||\Phi)$. However, these distances are clearly

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useless for example when all summands have discrete distributions. Therefore, some further assumptions are needed.

When estimating the error of normal approximation by means of these distances, it seems natural to require that every \( X_k \) has an absolutely continuous distribution. Even with this assumption we cannot exclude the case that our distances of \( S_n \) to the normal law may be growing when the distributions of \( X_k \) get near to discrete distributions. Thus we shall assume that the densities of \( X_k \) are bounded on a reasonably large part of the real line. This can be guaranteed quite naturally, for instance, by using the entropy functional, defined for a random variable \( X \) with density \( p(x) \) by

\[
h(X) = - \int_{-\infty}^{+\infty} p(x) \log p(x) \, dx.
\]

Once \( X \) has a finite second moment, the entropy is well-defined as a Lebesgue integral, although the value \( h(X) = -\infty \) is possible. Introduce a related functional

\[
D(X) = h(Z) - h(X) = \int_{-\infty}^{+\infty} p(x) \log \frac{p(x)}{\varphi_{a,\sigma}(x)} \, dx,
\]

where \( Z \) is a normal random variable with density \( \varphi_{a,\sigma} \) having the same mean \( a \) and variance \( \sigma^2 \) as \( X \). Note that this functional is affine invariant, that is, \( D(c_0 + c_1 X) = D(X) \), for all \( c_0 \in \mathbb{R}, c_1 \neq 0 \), and in this sense it does not depend neither on the mean or the variance of \( X \).

The quantity \( D(X) \), denoted also \( D(F_X || F_Z) \), where \( F_X \) and \( F_Z \) are the corresponding distributions of \( X \) and \( Z \), is known as the "entropic distance to normality or Gaussianity". It may be characterized as the shortest Kullback-Leibler distance from \( F_X \) to the class of all normal laws on the real line. In general, \( 0 \leq D(X) \leq +\infty \), and the equality \( D(X) = 0 \) is possible, when \( X \) is normal, only. Moreover, by Pinsker’s inequality, the entropic distance dominates the total variation in the sense that

\[
D(X) \geq \frac{1}{2} \| F_X - F_Z \|^2_{TV}.
\]

Thus, the size of \( D(X) \) provides a strong distance of \( F_X \) to normality, while finiteness of \( D(X) \) guarantees that \( F_X \) is separated from the class of discrete probability distributions. Using \( D \) for both purposes, one may obtain refinements of Berry-Esseen’s inequality (1.1) in terms of the total variation and the entropic distances to normality for the distributions \( F_n \).

**Theorem 1.1.** Let \( D \) be a non-negative real number. Assume that \( X_k \) have finite third absolute moments, and \( D(X_k) \leq D \) (1 \( \leq k \leq n \)). Then

\[
\| F_n - \Phi \|_{TV} \leq C L_3,
\]

(1.2)

where the constant \( C \) depends on \( D \), only.

In particular, if all \( X_k \) are equidistributed with \( \mathbb{E} X_1^2 = 1 \), we get

\[
\| F_n - \Phi \|_{TV} \leq \frac{C}{\sqrt{n}} \mathbb{E} |X_1|^3
\]

(1.3)

with a constant \( C \) depending on \( D(X_1) \), only. Although (1.2)-(1.3) seem to be new, related estimates in the i.i.d.-case were studied by many authors. For example, in the early 1960’s
Mamatov and Sirazhdinov [M-S] found an exact asymptotic \( \| F_n - \Phi \|_{TV} = \frac{c}{\sqrt{n}} + o(\frac{1}{\sqrt{n}}) \), where the constant \( c \) is proportional to \( |EX_1^3| \), and which holds under the assumption that the distribution of \( X_1 \) has a non-trivial absolutely continuous component (cf. also [Pr], [Se]).

Now, let us turn to the entropic distance to normality.

**Theorem 1.2.** Assume that \( X_k \) have finite fourth absolute moments, and that \( D(X_k) \leq D(1 \leq k \leq n) \). Then

\[
D(S_n) \leq CL_4,
\]

where \( C \) depends on \( D(X_1) \) only.

In (1.2) and (1.4) one may take \( C = e^{c(D+1)} \), where \( c \) is an absolute constant. Moreover, \( C \) can be chosen to be just a numerical constant, provided that \( D \) is not too large, namely, if \( D \leq c_0 \log \frac{1}{L_3} \) and \( D \leq c_0 \log \frac{1}{L_4} \), respectively (with \( c_0 > 0 \) absolute).

These Berry-Esseen-type estimates are consistent in view of the Pinsker-type inequality. In some sense, one may consider (1.4) as a stronger assertion than (1.2), which is indeed the case, when \( L_4 \) is of order \( L_3^2 \). (In general \( L_3^2 \leq L_4 \).)

In the i.i.d. case as in (1.3), the inequality (1.4) becomes

\[
D(S_n) \leq C \frac{EX_1^4}{n},
\]

where \( C \) depends on \( D(X_1) \) only. Thus, we obtain an error bound of order \( O(1/n) \) under the 4th moment assumption. Note that the property \( D(S_n) \to 0 \) always holds under the second moment assumption (with finite entropy of \( X_1 \)). This is the statement of the entropic central limit theorem, which is due to Barron [B]. Here, the convergence may have an arbitrarily slow rate. Nevertheless, the expected typical rate \( D(S_n) = O(\frac{1}{n}) \) was known to hold in some cases, for example, when \( X_1 \) has a distribution satisfying an integro-differential inequality of Poincaré-type. These results are due to Artstein, Ball, Barthe and Naor [A-B-B-N], and Barron and Johnson [B-J]; cf. also [J]. Recently, an exact asymptotic for \( D(S_n) \) has been studied in [B-C-G1]. If the entropy and the 4th moment of \( X_1 \) are finite, it was shown that

\[
D(S_n) = \frac{c}{n} + o\left( \frac{1}{n \log n} \right), \quad c = \frac{1}{12} (EX_1^3)^2.
\]

Moreover, with finite 3rd absolute moment (and infinite 4th moment) such a relation may not hold, and it may happen that \( D(S_n) \geq n^{-(1/2+\varepsilon)} \) for all \( n \) large enough with a given prescribed \( \varepsilon > 0 \). This holds, for example, when \( X_1 \) has density

\[
p(x) = \int_0^{+\infty} \frac{1}{\sigma \sqrt{2\pi}} e^{-x^2/2\sigma^2} dP(\sigma),
\]

where \( P \) is a probability measure on \((\frac{1}{e}, +\infty)\) with density \( \frac{dP(\sigma)}{d\sigma} = (\sigma \log \sigma)^{-4} \) for \( \sigma \geq e \) and with an arbitrary extension to the interval \( \frac{1}{e} < \sigma < e \) satisfying \( \int_{1/e}^{+\infty} \sigma^2 dP(\sigma) = 1 \).

Therefore, in the general non-i.i.d.-case, the Lyapunov coefficient \( L_3 \) cannot be taken as an appropriate quantity for bounding the error in Theorem 1.2, and \( L_4 \) seems more relevant. This is also suggested by the result of [A-B-B-N] for the weighted sums

\[
S_n = a_1 X_1 + \cdots + a_n X_n \quad (a_1^2 + \cdots + a_n^2 = 1)
\]
of i.i.d. random variables $X_k$, such that $\mathbb{E}X_1 = 0$ and $\mathbb{E}X_1^2 = 1$. Namely, it is proved there that

$$D(S_n) \leq \frac{L(a)}{c/2 + (1 - c/2)L(a)} D(X_1),$$

where $L(a) = a_1^4 + \cdots + a_n^4$ and $c \geq 0$ is an optimal constant in the Poincaré-type inequality $c \text{Var}(u(X_1)) \leq \mathbb{E}[u'(X_1)^2]$. But for the sequence $a_k X_k$ and $s = 4$, the corresponding Lyapunov coefficient is exactly $L_4 = L(a) \mathbb{E}X_1^4$. Therefore, when $c = c(X_1)$ is positive, (1.5) yields the estimate

$$D(S_n) \leq \frac{2D(X_1)}{c \mathbb{E}X_1^4} L_4,$$

which is of a similar nature as (1.4).

Another interesting feature of (1.4) is that it may be connected with transportation cost inequalities for the distributions $F_n$ of $S_n$ in terms of the quadratic Wasserstein distance $W_2$. For random variables $X$ and $Z$ with finite second moments and distributions $F_X$ and $F_Z$, this distance is defined by

$$W_2^2(F_X, F_Z) = \inf_{\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |x - y|^2 d\pi(x, y),$$

where the infimum is taken over all probability measures $\pi$ on the plane $\mathbb{R}^2$ with marginals $F_X$ and $F_Z$. The value $W_2^2(F_X, F_Z)$ is interpreted as the minimal expenses needed to transport $F_Z$ to $F_X$, provided that it costs $|x - y|^2$ to move any “particle” $x$ to any “particle” $y$.

The metric $W_2$ is of weak type in the sense that it can be used to metrize the weak convergence of probability distributions ([V]). Moreover, if $Z$ is standard normal and if $X$ has a density, $W_2(F_X, F_Z)$ may be bounded in terms of the relative entropy by virtue of Talagrand’s transportation inequality

$$W_2^2(F_X, F_Z) \leq 2D(F_X \| F_Z)$$

(cf. [T], or [B-G] for a different approach). If additionally $X$ has mean zero and unit variance, $D(F_X \| F_Z) = D(X)$. Hence, applying (1.6) with $X = S_n$, we get, by Theorem 1.2,

$$W_2(F_n, \Phi) \leq C \sqrt{L_4},$$

(1.7)

where $C$ depends on $D$. In fact, this inequality holds true with an absolute constant. This result is due to Rio [Ri], who also studied more general Wasserstein distances $W_r$, by relating them to Zolotarev’s “ideal” metrics. It has also been noticed in [Ri] that the 4-th moment condition is essential, so the Lyapunov’s ratio $L_4$ in (1.7) cannot be replaced with $L_3$ including the i.i.d.-case (like in Theorem 1.2).

The paper is organized according to the following plan.

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6. Entropic bounds for Cramer constants of characteristic functions.
2. General Bounds on Total Variation and Entropic Distance

Let a random variable $X$ have an absolutely continuous distribution $F$ with density $p(x)$ and finite first absolute moment. We do not require that it has mean zero and/or unit variance.

First, we recall an elementary bound for the total variation distance $\|F - \Phi\|_{TV}$ in terms of the characteristic function

$$f(t) = \mathbb{E} e^{itX} = \int_{-\infty}^{+\infty} e^{itx} p(x) \, dx \quad (t \in \mathbb{R}).$$

Introduce the characteristic function $g(t) = e^{-t^2/2}$ of the standard normal law.

In the sequel, we use the notation

$$\|u\|_2 = \left( \int_{-\infty}^{+\infty} |u(t)|^2 \, dt \right)^{1/2}$$

to denote the $L^2$-norm of a measurable complex-valued function $u$ on the real line (with respect to Lebesgue measure).

**Proposition 2.1.** We have

$$\|F - \Phi\|_{TV}^2 \leq \frac{1}{2} \|f - g\|_2^2 + \frac{1}{2} \|f' - g'\|_2^2. \quad (2.1)$$

This bound is standard (cf. e.g. [I-L], Lemma 1.3.1). In fact, the inequality (2.1) remains to hold for an arbitrary probability distribution (in place of $\Phi$) with finite first absolute moment and characteristic function $g$. However, the general case won’t be needed in the sequel.

Note that the assumption $\mathbb{E} |X| < +\infty$ guarantees that $f$ is continuously differentiable, so that the last integral in (2.1) makes sense.

Let $Z$ be a standard normal random variable, with density $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. Consider the relative entropy

$$D(X||Z) = D(F||\Phi) = \int_{-\infty}^{+\infty} p(x) \log \frac{p(x)}{\varphi(x)} \, dx. \quad (2.2)$$

As a preliminary bound, we first derive:
Lemma 2.2. For all \( T \geq 0 \),
\[
D(X \| Z) \leq e^{-T^2/2} + \sqrt{2\pi} \int_{-T}^{T} (p(x) - \varphi(x))^2 e^{x^2/2} \, dx \\
+ \frac{1}{2} \int_{|x| \geq T} x^2 p(x) \, dx + \int_{|x| \geq T} p(x) \log p(x) \, dx.
\]

Proof. We split the integral in (2.2) into the two regions. For the interval \(|x| \leq T\), using the elementary inequality \( t \log t \leq (t-1) + (t-1)^2 \), \( t \geq 0 \), we have
\[
\int_{-T}^{T} p \log \frac{p}{\varphi} \varphi \, dx \leq \int_{-T}^{T} \left( \frac{p}{\varphi} - 1 \right) \varphi \, dx + \int_{-T}^{T} \left( \frac{p}{\varphi} - 1 \right)^2 \varphi \, dx \\
= \int_{|x| \geq T} (\varphi - p) \, dx + \int_{-T}^{T} \frac{(p - \varphi)^2}{\varphi} \, dx \\
= 2 (1 - \Phi(T)) - \int_{|x| \geq T} p(x) \, dx + \sqrt{2\pi} \int_{-T}^{T} (p(x) - \varphi(x))^2 e^{x^2/2} \, dx.
\]

For the second region, just write
\[
\int_{|x| \geq T} p(x) \log \frac{p(x)}{\varphi(x)} \, dx = \int_{|x| \geq T} p(x) \log p(x) \, dx \\
+ \log \sqrt{2\pi} \int_{|x| \geq T} p(x) \, dx + \frac{1}{2} \int_{|x| \geq T} x^2 p(x) \, dx.
\]

It remains to collect these relations and use \( \log \sqrt{2\pi} < 1 \) together with a well-known elementary inequality \( 1 - \Phi(T) \leq \frac{1}{2} e^{-T^2/2} \). Thus, Lemma 2.2 is proved.

Remark. If \( p \) is bounded by a constant \( M \), the estimate (2.3) yields
\[
D(X \| Z) \leq e^{-T^2/2} + \sqrt{2\pi} \int_{-T}^{T} (p(x) - \varphi(x))^2 e^{x^2/2} \, dx \\
+ \frac{1}{2} \int_{|x| \geq T} x^2 p(x) \, dx + \log M \int_{|x| \geq T} p(x) \, dx.
\]

This bound might be of interest in other applications, although it involves the maximum of the density. For our purposes, the important integral in (2.3), \( \int_{|x| \geq T} p(x) \log p(x) \, dx \), will be bounded in a different way and in terms of the characteristic functions, without involving the parameter \( M \).

3. Entropic Distance and Edgeworth-type Approximation

To estimate the integrals in (2.3) in terms of the characteristic functions like in Proposition 2.1, define
\[
\varphi_\alpha(x) = \varphi(x) \left( 1 + \alpha \frac{x^3 - 3x}{3!} \right),
\]
where $\alpha$ is a parameter. These functions appear with $\alpha$ proportional to $n^{-1/2}$ in the Edgeworth-type expansions up to order 3 for densities of the normalized sums $S_n = \frac{X_1 + \cdots + X_n}{\sqrt{B_n}}$ of i.i.d. summands. In the non-i.i.d. case such expansions hold as well with

$$\alpha = \frac{1}{B_n^{3/2}} \sum_{k=1}^{n} E X_k^3.$$ 

Note that every $\varphi_{\alpha}$ has the Fourier transform

$$g_{\alpha}(t) = \int_{-\infty}^{+\infty} e^{itx} \varphi_{\alpha}(x) \, dx = g(t) \left(1 + \alpha \frac{(it)^3}{3!}\right),$$

where $g(t) = e^{-t^2/2}$.

**Proposition 3.1.** Let $X$ be a random variable with $E |X|^3 < +\infty$. For all $\alpha \in \mathbb{R}$,

$$D(X||Z) \leq \alpha^2 + 4 \left(\|f - g_{\alpha}\|_2 + \|f''' - g'''\|_2\right), \tag{3.1}$$

where $Z$ is a standard normal random variable and $f$ is the characteristic function of $X$.

The assumption on the 3rd absolute moment is needed to insure that $f$ has first three continuous derivatives.

As a particular case, the inequality (3.1) is valid for $\alpha = 0$, as well. Then it becomes

$$D(X||Z) \leq 4 \left(\|f - g\|_2 + \|f''' - g'''\|_2\right),$$

which may be viewed as a full analog of Proposition 2.1. However, with properly chosen values of $\alpha$, (3.1) may provide a much better asymptotic approximation (especially when applying it to the sums of independent random variables).

**Proof.** We may assume that the characteristic function $f$ and its first three derivatives are square integrable, so that the right-hand side of (3.1) is finite. Note that in this case, $X$ has an absolutely continuous distribution with some density $p$.

We apply Lemma 2.2. Given $T \geq 0$ to be specified later on, let us start with the estimation of the last integral in (2.3). Define the even function $\tilde{p}(x) = p(x) + p(-x)$, so that $p \log p \leq p \log^+ \tilde{p}$ (where we use the notation $a^+ = \max\{a, 0\}$). Subtracting $\varphi_{\alpha}(x)$ from $p(x)$ and then adding, one can write

$$\int_{|x|\geq T} p(x) \log p(x) \, dx \leq \int_{|x|\geq T} p(x) \log^+ \tilde{p}(x) \, dx \leq \int_{-\infty}^{+\infty} |p(x) - \varphi_{\alpha}(x)| \log^+ \tilde{p}(x) \, dx + \int_{|x|\geq T} \varphi_{\alpha}(x) \log^+ \tilde{p}(x) \, dx.$$

But the function $\varphi_{\alpha} - \varphi$ is odd, so the last integral does not depend on $\alpha$ and is equal to

$$\int_{|x|\geq T} \varphi(x) \log^+ \tilde{p}(x) \, dx. \tag{3.2}$$

To estimate it from above, one may use Cauchy’s inequality together with the elementary bound $(\log^+ t)^2 \leq C t$, where the optimal constant $C$ is equal to $4e^{-2}$. Since $\int_{-\infty}^{+\infty} \tilde{p}(x) \, dx = 2$, we have
(3.2) does not exceed
\[
\left( \int_{|x| \geq T} \varphi(x)^2 \, dx \right)^{1/2} \left( \int_{|x| \geq T} \left( \log^+ \tilde{p}(x) \right)^2 \, dx \right)^{1/2} \leq \left( \int_{|x| \geq T} \varphi(x)^2 \, dx \right)^{1/2} \frac{2\sqrt{2}}{e}.
\]

On the other hand,
\[
\left( \int_{|x| \geq T} \varphi(x)^2 \, dx \right)^{1/2} = \left( \frac{1}{\sqrt{\pi}} \left( 1 - \Phi(T\sqrt{2}) \right) \right)^{1/2} \leq \frac{1}{\pi^{1/4}\sqrt{2}} e^{-T^2/2},
\]

where we applied the inequality $1 - \Phi(x) \leq \frac{1}{2} e^{-x^2/2}$ ($x \geq 0$). Thus, using \( \frac{2\sqrt{2}}{e} \cdot \frac{1}{\pi^{1/4}\sqrt{2}} < 1 \) to simplify the constant, we get
\[
\int_{|x| \geq T} p(x) \log p(x) \, dx \leq \int_{-\infty}^{+\infty} |p(x) - \varphi_\alpha(x)| \log^+ \tilde{p}(x) \, dx + e^{-T^2/2}.
\]

Here, again by the Cauchy inequality, the last integral does not exceed
\[
\frac{2\sqrt{2}}{e} \left( \int_{-\infty}^{+\infty} (p(x) - \varphi_\alpha(x))^2 \, dx \right)^{1/2} = \frac{2\sqrt{2}}{e} \cdot \frac{1}{\sqrt{2\pi}} \left( \int_{-\infty}^{+\infty} |f(t) - g_\alpha(t)|^2 \, dt \right)^{1/2},
\]

where we applied Plancherel’s formula. The constant in front of the last integral is smaller than $\frac{1}{4}$, so we arrive at the estimate
\[
\int_{|x| \geq T} p(x) \log p(x) \, dx \leq \frac{1}{2} \|f - g_\alpha\|_2 + e^{-T^2/2}. \tag{3.3}
\]

Now, let us turn to the pre-last integral in (2.3). Once more, subtracting $\varphi_\alpha(x)$ from $p(x)$ and then adding, one can write
\[
\int_{|x| \geq T} x^2 p(x) \, dx \leq \int_{-\infty}^{+\infty} x^2 |p(x) - \varphi_\alpha(x)| \, dx + \int_{|x| \geq T} x^2 \varphi_\alpha(x) \, dx.
\]

Since the function $\varphi_\alpha - \varphi$ is odd, the last integral is equal to
\[
\int_{|x| \geq T} x^2 \varphi_\alpha(x) \, dx = \frac{2}{\sqrt{2\pi}} \int_{T}^{+\infty} x^2 e^{-x^2/2} \, dx = 2(1 - \Phi(T)) + \frac{2}{\sqrt{2\pi}} T e^{-T^2/2}
\]
(by direct integration by parts). Hence, using $2(1 - \Phi(T)) \leq e^{-T^2/2}$ once more, we get
\[
\frac{1}{2} \int_{|x| \geq T} x^2 p(x) \, dx \leq \frac{1}{2} \int_{-\infty}^{+\infty} x^2 |p(x) - \varphi_\alpha(x)| \, dx
\]
\[
+ \frac{1}{2} e^{-T^2/2} + \frac{1}{\sqrt{2\pi}} T e^{-T^2/2}. \tag{3.4}
\]

In addition, by Cauchy’s inequality,
\[
\left( \int_{-\infty}^{+\infty} x^2 |p(x) - \varphi_\alpha(x)| \, dx \right)^2 \leq \int_{-\infty}^{+\infty} \frac{dx}{1 + x^2} \int_{-\infty}^{+\infty} (1 + x^2) x^4 (p(x) - \varphi_\alpha(x))^2 \, dx
\]
\[
= \pi \int_{-\infty}^{+\infty} (x^4 + x^6) (p(x) - \varphi_\alpha(x))^2 \, dx
\]
\[
\leq \pi \int_{-\infty}^{+\infty} (1 + 2x^6) (p(x) - \varphi_\alpha(x))^2 \, dx.
\]
But, by Plancherel’s formula,
\[
\int_{-\infty}^{+\infty} (p(x) - \varphi_\alpha(x))^2 \, dx = \frac{1}{2\pi} \| f - g_\alpha \|_2^2 \tag{3.5}
\]
\[
\int_{-\infty}^{+\infty} x^6 (p(x) - \varphi_\alpha(x))^2 \, dx = \frac{1}{2\pi} \| f''' - g'''_\alpha \|_2^2. \tag{3.6}
\]
Hence,
\[
\int_{-\infty}^{+\infty} x^2 |p(x) - \varphi_\alpha(x)| \, dx \leq \left( \frac{1}{2} \| f - g_\alpha \|_2^2 + \| f''' - g'''_\alpha \|_2^2 \right)^{1/2}
\leq \| f - g_\alpha \|_2 + \| f''' - g'''_\alpha \|_2,
\]
and from (3.4),
\[
\frac{1}{2} \int_{|x| \geq T} x^2 p(x) \, dx \leq \frac{1}{2} e^{-T^2/2} + \frac{1}{\sqrt{2\pi}} Te^{-T^2/2} + \frac{1}{2} \| f - g_\alpha \|_2 + \frac{1}{2} \| f''' - g'''_\alpha \|_2. \tag{3.7}
\]
Using the bounds (3.3) and (3.7) in the inequality (2.3), we therefore obtain that
\[
D(X\|Z) \leq \frac{5}{2} e^{-T^2/2} + \frac{1}{\sqrt{2\pi}} Te^{-T^2/2}
+ \sqrt{2\pi} \int_{-T}^{T} (p(x) - \varphi(x))^2 e^{x^2/2} \, dx + \| f - g_\alpha \|_2 + \| f''' - g'''_\alpha \|_2. \tag{3.8}
\]
Next, let us consider the integral in (3.8). First, writing
\[
p(x) - \varphi(x) = (p(x) - \varphi_\alpha(x)) + \alpha \frac{x^3 - 3x}{3!} \varphi(x)
\]
and applying an elementary inequality \((a + b)^2 \leq \frac{a^2}{t} + \frac{b^2}{T} (a, b \in \mathbb{R}, 0 < t < 1)\) with \(t = 1/6\), we get
\[
(p(x) - \varphi(x))^2 \leq \frac{6}{5} (p(x) - \varphi_\alpha(x))^2 + \alpha^2 \frac{(x^3 - 3x)^2}{6} \varphi(x)^2,
\]
or equivalently,
\[
(p(x) - \varphi(x))^2 e^{x^2/2} \leq \frac{6}{5} (p(x) - \varphi_\alpha(x))^2 e^{x^2/2} + \frac{1}{\sqrt{2\pi}} \alpha^2 \frac{(x^3 - 3x)^2}{6} \varphi(x).
\]
Integrating this inequality over the interval \([-T, T]\) and using \(E(Z^3 - 3Z)^2 = 6\), where \(Z \sim N(0, 1)\), we obtain
\[
\sqrt{2\pi} \int_{-T}^{T} (p(x) - \varphi(x))^2 e^{x^2/2} \, dx \leq \frac{6}{5} \sqrt{2\pi} \int_{-T}^{T} (p(x) - \varphi_\alpha(x))^2 e^{x^2/2} \, dx + \alpha^2.
\]
To estimate the last integral, first note that the function \(t \to e^{t^2}/(2 + t)\) is increasing for \(t \geq 0\). Hence, for all \(|x| \leq T\),
\[
e^{x^2/2} = \frac{e^{x^2/2}}{2 + x^2} (2 + x^2) \leq \frac{e^{T^2/2}}{2 + T^2} (3 + x^6),
\]
and thus, using (3.5)-(3.6),
\[ \int_{-T}^{T} (p(x) - \varphi_{\alpha}(x))^2 e^{x^2/2} \, dx \leq \frac{e^{T^2/2}}{2 + T^2} \int_{-T}^{T} (3 + x^6) (p(x) - \varphi_{\alpha}(x))^2 \, dx \leq \frac{3}{2\pi} \frac{e^{T^2/2}}{2 + T^2} (\|f - g_{\alpha}\|_2^2 + \|f'' - g'''_{\alpha}\|_2^2). \]

Putting \( \varepsilon = \|f - g_{\alpha}\|_2 + \|f'' - g'''_{\alpha}\|_2 \), we get
\[ \sqrt{2\pi} \int_{-T}^{T} (p(x) - \varphi_{\alpha}(x))^2 e^{x^2/2} \, dx \leq \frac{18}{5\sqrt{2\pi}} \frac{e^{T^2/2}}{2 + T^2} \varepsilon^2 + \alpha^2. \]

Inserting this inequality in (3.8) leads to
\[ D(X\|Z) \leq \frac{5}{2} e^{-T^2/2} + \frac{1}{\sqrt{2\pi}} T e^{-T^2/2} + \frac{18}{5\sqrt{2\pi}} \frac{e^{T^2/2}}{2 + T^2} \varepsilon^2 + \varepsilon + \alpha^2. \quad (3.9) \]

It remains to optimize this bound over all \( T \geq 0 \). As before, consider the function \( \psi(t) = e^{t^2/(2 + t)} \). It is increasing for \( t \geq 0 \) with \( \psi(0) = \frac{1}{2} \). If \( 0 \leq \varepsilon \leq 2 \), define \( T = T_{\varepsilon} \) to be the (unique) solution to the equation
\[ \psi(T^2) = \frac{1}{\varepsilon}. \]

In this case,
\[ T e^{-T^2/2} \cdot \frac{1}{\varepsilon} = T e^{-T^2/2} \cdot \frac{e^{T^2/2}}{2 + T^2} \leq \frac{1}{2}, \]
so \( T e^{-T^2/2} \leq \frac{\varepsilon}{2} \). Furthermore, note that
\[ e^{-T^2/2} \cdot \frac{1}{\varepsilon} = e^{-T^2/2} \cdot \frac{e^{T^2/2}}{2 + T^2} \leq \frac{1}{2}, \]
so \( e^{-T^2/2} \leq \frac{\varepsilon}{2} \). Applying these bounds in (3.9), we arrive at
\[ D(X\|Z) \leq \frac{5\varepsilon}{4} + \frac{1}{\sqrt{2\pi}} \frac{\varepsilon}{2} + \frac{18}{5\sqrt{2\pi}} \varepsilon + \varepsilon + \alpha^2 \leq 4 \varepsilon + \alpha^2, \]
which is exactly the desired inequality (3.1).

In case \( \varepsilon \geq 2 \), let us return to (3.8) and apply it with \( T = 0 \). This yields
\[ D(X\|Z) \leq \frac{5}{2} + \varepsilon < 4 \varepsilon, \]
which is even better than (3.1). Thus, Proposition 3.1 is proved.

4. Quantile Density Decomposition

In order to effectively apply Propositions 2.1 and 3.1, one has to solve two different tasks. The first one is to estimate integrals such as
\[ \int_{-T}^{T} |f(t) - g_{\alpha}(t)|^2 \, dt, \quad \int_{-T}^{T} |f''(t) - g'''_{\alpha}(t)|^2 \, dt \]
over sufficiently large \( t \)-intervals with properly chosen values of the parameter \( \alpha \). When the characteristic function \( f \) has a multiplicative structure, i.e., corresponds to the sum of a
large number of small independent summands, this task can be attacked by using classical Edgeworth-type expansions (for characteristic functions). Such expansions are well-known including the non-i.i.d. case, and we consider one of them in Section 12.

The second task concerns an estimation of integrals such as

\[ \int_{|x| \geq T} |f(t)|^2 \, dt, \quad \int_{|x| \geq T} |f'''(t)|^2 \, dt, \]

which in general do not need to be small or even finite. The finiteness is guaranteed, for example, when \( f \) is the Fourier transform of a bounded density \( p \). For some purposes such as obtaining local limit theorems, it is therefore natural to restrict oneself to the case of bounded densities. For other purposes, such as an estimation of the total variation or relative entropy, the density \( p \) may slightly be modified, so that the new density, say \( \tilde{p} \), will be bounded, and at the same time will only slightly change the total variation distance or relative entropy with respect to the standard normal law.

To this aim, we shall use the so-called quantile density decomposition, based on the following elementary observation. (In fact, it is needed in case of bounded densities, as well.)

**Proposition 4.1.** Let \( X \) be a random variable with density \( p \). Given \( 0 < \kappa < 1 \), the real line can be partitioned into two Borel sets \( A_0, A_1 \) such that \( p(x) \leq p(y) \), for all \( x \in A_0, y \in A_1 \), and

\[ \int_{A_0} p(x) \, dx = \kappa, \quad \int_{A_1} p(x) \, dx = 1 - \kappa. \]

The argument is based on the continuity of the measure \( p(x) \, dx \) and is omitted.

Clearly, for some real number \( m_\kappa \) we get

\[ A_0 \subset \{ x \in \mathbb{R} : p(x) \leq m_\kappa \}, \quad A_1 \subset \{ x \in \mathbb{R} : p(x) \geq m_\kappa \}. \]

Here, \( m_\kappa \) represents a quantile (or one of the quantiles) for the function \( p \) viewed as a random variable on the probability space \((\mathbb{R}, p(x) \, dx)\). In other words, \( m_\kappa = m_\kappa(p(X)) \) is a quantile of order \( \kappa \) for the random variable \( p(X) \). If \( \kappa = \frac{1}{2} \), the index is usually omitted, and then \( m = m(p(X)) \) denotes a median of \( p(X) \).

**Definition 4.2.** Define the densities \( p_0 \) and \( p_1 \) to be the normalized restrictions of \( p \) to the sets \( A_0 \) and \( A_1 \), respectively. As a result, we have an equality

\[ p(x) = \kappa p_0(x) + (1 - \kappa) p_1(x), \quad (4.1) \]

which we call the quantile density decomposition for \( p \) (respectively – the median density decomposition, when \( \kappa = \frac{1}{2} \)).

Let us mention one obvious, but important property of the functionals \( m_\kappa(p(X)) \), assuming that \( X \) has a finite second moment.

**Proposition 4.3.** The functionals

\[ Q_\kappa(X) = m_\kappa(p(X)) \sqrt{\text{Var}(X)} \]

are affine invariant. That is, for all \( a \in \mathbb{R} \) and \( b \neq 0 \), \( Q_\kappa(a + bX) = Q_\kappa(X) \).
More precisely, one should either assume in the latter equality that the quantile \( m_\kappa(p(X)) \) is determined uniquely, or to use specific quantiles satisfying the relation \( m_\kappa(p_{a,b}(a+bX)) = |b|^{-1} m_\kappa(p(X)) \), where \( p_{a,b} \) denotes the density of the random variable \( a+bX \).

5. Properties of the Quantile Decomposition

In this section we establish basic properties of the quantile density decomposition. Although for purposes of Theorems 1.1-1.2 the median decomposition is sufficient, the general case is no more difficult (but may be used to provide more freedom especially for improving \( D \)-dependent constants).

First, let us bound from above the quantiles \( m_\kappa = m_\kappa(p(X)) \) in terms of the entropic distance to normality.

**Proposition 5.1.** Let \( X \) be a random variable with finite variance \( \sigma^2 \) (\( \sigma > 0 \)), having an absolutely continuous distribution, and let \( 0 < \kappa < 1 \). Then
\[
m_\kappa \leq \frac{1}{\sigma \sqrt{2\pi}} e^{(D(X)+1)\kappa}.\]

In particular,
\[
m \leq \frac{1}{\sigma \sqrt{2\pi}} e^{2D(X)+2}.
\]

**Proof.** By Proposition 4.3, we may assume that \( X \) has mean zero and variance one. Let \( A = \{x \in \mathbb{R} : p(x) \geq m_\kappa\} \). By the definition of the quantiles,
\[
\int_A p(x) \, dx \geq 1 - \kappa.
\]
Since \( p(x) \geq m_\kappa \) on the set \( A \), we have
\[
\int_{-\infty}^{+\infty} p(x) \log \left(1 + \frac{p(x)}{\varphi(x)}\right) \, dx \geq \int_A p(x) \log \left(1 + \frac{m_\kappa}{\varphi(x)}\right) \, dx 
\]
\[
\geq \int_A p(x) \log \frac{m_\kappa}{\varphi(x)} \, dx
\]
\[
= \log(m_\kappa \sqrt{2\pi}) \int_A p(x) \, dx + \frac{1}{2} \int_A x^2 p(x) \, dx
\]
\[
\geq (1 - \kappa) \log(m_\kappa \sqrt{2\pi}).
\]
On the other hand, using an elementary inequality \( t \log(1+t) - t \log t \leq 1 \) (\( t \geq 0 \), we get
\[
\int_{-\infty}^{+\infty} p(x) \log \left(1 + \frac{p(x)}{\varphi(x)}\right) \, dx = \int_{-\infty}^{+\infty} \frac{p(x)}{\varphi(x)} \log \left(1 + \frac{p(x)}{\varphi(x)}\right) \varphi(x) \, dx
\]
\[
\leq \int_{-\infty}^{+\infty} \frac{p(x)}{\varphi(x)} \log \frac{p(x)}{\varphi(x)} \varphi(x) \, dx + 1 = D(X) + 1.
\]
Hence, \( (1 - \kappa) \log(m_\kappa \sqrt{2\pi}) \leq D(X) + 1 \), and the proposition follows.
Now, let $V_0$ and $V_1$ be random variables with densities $p_0$ and $p_1$ from the quantile decomposition (4.1). They have means $a_j = \mathbb{E} V_j$ and variances $\sigma_j^2 = \text{Var}(V_j)$, connected by

$$\kappa a_0 + (1 - \kappa) a_1 = \mathbb{E} X,$$

and

$$\left(\kappa a_0^2 + (1 - \kappa) a_1^2\right) + \left(\kappa \sigma_0^2 + (1 - \kappa) \sigma_1^2\right) = \mathbb{E} X^2,$$

provided that $X$ has a finite second moment.

The next step is to prove upper bounds for the entropies of $V_0$ and $V_1$.

**Proposition 5.2.** If $X$ has mean zero and finite second moment, then

$$\kappa D(V_0) + (1 - \kappa) D(V_1) \leq D(X) - \kappa \log \kappa - (1 - \kappa) \log(1 - \kappa).$$

In particular, in case of the median decomposition,

$$D(V_0) + D(V_1) \leq 2D(X) + 2 \log 2.$$

**Proof.** Let $\text{Var}(X) = \sigma^2$ ($\sigma > 0$). We may assume that $D(X)$ is finite. By Definition 4.2,

$$-h(V_0) = \int_{-\infty}^{+\infty} p_0(x) \log p_0(x) \, dx$$

$$= \int_{A_0} (p(x)/\kappa) \log(p(x)/\kappa) \, dx = -\log \kappa + \frac{1}{\kappa} \int_{A_0} p(x) \log p(x) \, dx,$$

and similarly, $-h(V_1) = -\log(1 - \kappa) + \frac{1}{1 - \kappa} \int_{A_1} p(x) \log p(x) \, dx$. Adding the two equalities with weights, we get

$$-\kappa h(V_0) - (1 - \kappa) h(V_1) = -\kappa \log \kappa - (1 - \kappa) \log(1 - \kappa) - h(X). \tag{5.2}$$

Recall that

$$D(V_0) = h(Z_0) - h(V_0), \quad \text{where } Z_0 \sim N(a_0, \sigma_0^2),$$

$$D(V_1) = h(Z_1) - h(V_1), \quad \text{where } Z_1 \sim N(a_1, \sigma_1^2),$$

$$D(X) = h(Z) - h(X), \quad \text{where } Z \sim N(0, \sigma^2).$$

Hence, from (5.2),

$$\kappa D(V_0) + (1 - \kappa) D(V_1) = \kappa h(Z_0) + (1 - \kappa) h(Z_1)$$

$$-\kappa \log \kappa - (1 - \kappa) \log(1 - \kappa) + (D(X) - h(Z))$$

$$= \kappa \log(\sigma_0 \sqrt{2\pi e}) + (1 - \kappa) \log(\sigma_1 \sqrt{2\pi e})$$

$$-\kappa \log \kappa - (1 - \kappa) \log(1 - \kappa) + (D(X) - \log(\sigma \sqrt{2\pi e}))$$

$$= -\kappa \log \kappa - (1 - \kappa) \log(1 - \kappa) + D(X) + \frac{\sigma_0^{\kappa} \sigma_1^{1-\kappa}}{\sigma}.$$

Finally, by (5.1), and the arithmetic-geometric inequality,

$$\sigma_0^{2\kappa} \sigma_1^{2(1-\kappa)} \leq \kappa \sigma_0^2 + (1 - \kappa) \sigma_1^2 \leq \sigma^2,$$

so, $\frac{\sigma_0^{\kappa} \sigma_1^{1-\kappa}}{\sigma} \leq 1$. Proposition 5.2 is proved.
Note that bounds on $D(X)$ provide a quantitative measure of non-degeneracy of the distributions of $V_j$ via positivity of their variances $\sigma^2_j$.

**Proposition 5.3.** Let $X$ be a random variable with mean zero and variance $\sigma^2$ ($\sigma > 0$), having finite entropy. Then

$$\sigma_0 > \sigma e^{-(D(X)+4)/\kappa}, \quad \sigma_1 > \sigma e^{-(D(X)+4)/(1-\kappa)}.$$

**Proof.** By homogeneity with respect to $\sigma$, one may assume that $\sigma = 1$.

We modify the argument from the proof of Proposition 5.1. First note that

$$\log(\sigma_0 \sqrt{2\pi e}) = D(V_0) - \int_{-\infty}^{+\infty} p_0(x) \log p_0(x) \, dx$$

$$\geq - \int_{-\infty}^{+\infty} p_0(x) \log p_0(x) \, dx = - \int_{A_0} (p(x)/\kappa) \log(p(x)/\kappa) \, dx$$

$$= \log \kappa - \frac{1}{\kappa} \int_{A_0} p(x) \log p(x) \, dx,$$

where $A_0$ is a set from Definition 4.2.

In order to estimate the last integral, put $r(x) = e^{-a^2x^2/2}$ with parameter $a > 0$. Using the property $r(x) \leq 1$ and once more the inequality $t \log(1 + t) \leq t \log t + 1$ ($t \geq 0$), we get

$$\int_{A_0} p(x) \log p(x) \, dx \leq \int_{-\infty}^{+\infty} p(x) \log \left(1 + \frac{p(x)}{r(x)}\right) \, dx$$

$$= \int_{-\infty}^{+\infty} \frac{p(x)}{r(x)} \log \left(1 + \frac{p(x)}{r(x)}\right) r(x) \, dx$$

$$\leq \int_{-\infty}^{+\infty} \left[ \frac{p(x)}{r(x)} \log \frac{p(x)}{r(x)} + 1 \right] r(x) \, dx$$

$$= \int_{-\infty}^{+\infty} p(x) \log p(x) \, dx + \frac{a^2}{2} \int_{-\infty}^{+\infty} p(x) x^2 \, dx + \int_{-\infty}^{+\infty} r(x) \, dx$$

$$= D(X) - \log(\sqrt{2\pi e}) + \left(\frac{a^2}{2} + \frac{1}{a} \sqrt{2\pi}\right).$$

The right-hand side is minimized for $a = (2\pi)^{1/6}$ in which case we obtain that

$$\int_{A_0} p(x) \log p(x) \, dx \leq D(X) - \log(\sqrt{2\pi e}) + \frac{3}{2} (2\pi)^{1/3} < D(X) + 1.35.$$

Together with (5.3), the above estimate yields

$$\log(\sigma_0 \sqrt{2\pi e}) > \log \kappa - \frac{1}{\kappa} (D(X) + 1.35).$$

But $\log(\sqrt{2\pi e}) \sim 1.42 < \frac{1.42}{\kappa}$, so $\log \sigma_0 > \log \kappa - \frac{1}{\kappa} (D(X) + 2.77)$, or equivalently,

$$\sigma_0 > \kappa e^{-(D(X)+2.77)/\kappa}.$$
Finally, using $\kappa > e^{-1/\kappa}$, the above estimate may be simplified to
\[ \sigma_0 > e^{-(D(X)+3.77)/\kappa}, \]
which gives the first estimate on $\sigma_0$. The second estimate for $\sigma_1$ is similar.

Thus, Proposition 5.3 is proved. Note that in case of the median decomposition, it becomes
\[ \sigma_0 > c\sigma e^{-2D(X)}, \quad \sigma_1 > c\sigma e^{-2D(X)}, \]
where $c$ is a positive absolute constant. One may take $c = e^{-8}$, for example.

6. Entropic Bounds for Cramer constants of Characteristic Functions

If a random variable $X$ has an absolutely continuous distribution with density, say $p$, then, by the Riemann-Lebesgue theorem, its characteristic function
\[ f(t) = \mathbb{E} e^{itX} = \int_{-\infty}^{\infty} e^{itx} p(x) \, dx \quad (t \in \mathbb{R}) \]
satisfies $f(t) \to 0$, as $t \to \infty$. Hence, for all $T > 0$,
\[ \delta_X(T) = \sup_{|t| \geq T} |f(t)| < 1. \]

An important problem is how to quantify this separation property (that is, separation from 1) by giving explicit upper bounds on the quantity $\delta_X(T)$, sometimes called Cramer constant. (At least $\delta_X(T) < 1$ is referred to as Cramer’s condition (C)). This problem arises naturally in local limit theorems for densities of the sums of non-identically distributed independent summands. Furthermore, it appears in the study of bounds and rates of convergence in the central limit theorem for strong metrics including the total variation and relative entropy. For our purposes, it is desirable to bound $\delta_X(T)$ explicitly in terms of the entropy of $X$ or, what is more relevant, in terms of the entropic distance to normality $D(X)$. Thus, this quantity controls separation of the distribution of $X$ from the class of discrete measures on the line.

A preliminary answer may be given in terms of the variance $\sigma^2 = \text{Var}(X)$, when it is finite, and in cases where the density $p$ is uniformly bounded.

**Proposition 6.1.** Assume $p(x) \leq M$ a.e. Then, for all $t$ real,
\[ |f(t)| \leq 1 - c \frac{\min\{1, \sigma^2 t^2\}}{M^2 \sigma^2}, \quad (6.1) \]
where $c > 0$ is an absolute constant.

In a slightly different form, this bound was obtained in the mid 1960’s by Statulevičius [St]. He also considered more complicated quantities reflecting the behavior of the density $p$ on non-overlapping intervals of the real line.

The inequality (6.1) can be generalized by involving non-bounded densities, but then $M$ should be replaced by other quantities such as quantiles $m_\kappa = m_\kappa(p(X))$ of the random variable $p(X)$. One can also remove any assumption on the moments of $X$ by replacing the standard deviation by the quantiles of the random variable $X - X'$, where $X'$ is an independent copy of $X$. We refer to [B-C-G2] for details, where the following bound is derived.
Proposition 6.2. Let $X$ be a random variable with finite variance $\sigma^2$ and finite entropy. Then, for all $t$ real,
\[
|f(t)| \leq 1 - c \min\{1, \sigma^2 t^2\} e^{-4D(X)},
\]
where $c > 0$ is an absolute constant.

At the expense of a worse constant in the exponent, this bound can be derived directly from (6.1) by combining it with Propositions 5.1 and 5.3.

Indeed, we may assume that $\mathbf{E}X = 0$. Let $V_0$ and $V_1$ be random variables with densities $p_0$ and $p_1$ from the median decomposition (4.1), that is, for $\kappa = \frac{1}{2}$, and denote by $f_0$ and $f_1$ the corresponding characteristic functions, so that $f = \frac{1}{2} f_0 + \frac{1}{2} f_1$. Hence, for all $t$,
\[
|f(t)| \leq \frac{1}{2} |f_0(t)| + \frac{1}{2}.
\]
(6.3)

Since $p_0$ is bounded – more precisely, $p_0(x) \leq m = m(p(X))$, one can apply Proposition 6.1 to the random variable $V_0$ with $M = m$. Then (6.1) and (6.3) give
\[
|f(t)| \leq 1 - c \frac{\min\{1, \sigma_0^2 t^2\}}{m^2 \sigma_0^2},
\]
where $\sigma_0^2 = \text{Var}(V_0)$ and $c > 0$ is an absolute constant. Note that $\sigma_0^2 \leq 2\sigma^2$, according to (5.1).

Now, by Proposition 5.1,
\[
m^2 \sigma_0^2 \leq 2m^2 \sigma^2 \leq \frac{1}{\pi} e^{4D(X)+4}.
\]

Hence,
\[
|f(t)| \leq 1 - c_1 \min\{1, \sigma_0^2 t^2\} e^{-4D(X)}.
\]

Finally, by Propositions 5.3, $\sigma_0^2 > c_2 \sigma^2 e^{-4D(X)}$, so
\[
|f(t)| \leq 1 - c_3 \min\{1, \sigma^2 t^2\} e^{-8D(X)}
\]
with some absolute constants $c_j > 0$.

7. Repacking of Summands

We now consider a sequence of independent (not necessarily identically distributed) random variables $X_1, \ldots, X_n$ and their sum $S_n = X_1 + \cdots + X_n$. Let $\mathbf{E}X_k = 0$, $\mathbf{E}X_k^2 = \sigma_k^2$ ($\sigma_k > 0$). One may always assume without loss of generality that $\sigma_1^2 + \cdots + \sigma_n^2 = 1$, so that $\text{Var}(S_n) = 1$.

In addition, all $X_k$ are assumed to have absolutely continuous distributions, having finite entropies in each place, where the functional $D$ is used.

To study integrability properties of the characteristic function $f_n$ of $S_n$ (more precisely – of its slightly modified variants $\tilde{f}_n$), it will be more convenient to work with a different representation,
\[
S_n = V_1 + \cdots + V_N,
\]
where the new independent summands represent appropriate partial sums of the $X_t$ resulting in almost equal variances, such that at the same time the number of blocks, $N$, is still reasonably
large. Such a representation may be introduced just by taking
\[ V_k = \sum_{n_{k-1}<l\leq n_k} X_l, \tag{7.1} \]
where \( n_0 = 0 \) and \( n_k = \max\{ l \leq n : \sigma_1^2 + \cdots + \sigma_l^2 \leq \frac{k}{N} \} \).

The number of new summands is restricted in terms of the parameter
\[ \sigma = \max_l \sigma_l \]
which in general may be an arbitrary real number between \( \frac{1}{\sqrt{n}} \) and 1.

**Lemma 7.1.** If \( N \leq \frac{1}{2\sigma^2} \), then for each \( k = 1, \ldots, N \),
\[ \frac{1}{2N} < \text{Var}(V_k) < \frac{2}{N}. \tag{7.2} \]

**Proof.** If \( n_1 = n \), then necessarily \( N = 1 \) and \( V_1 = S_n \), so (7.2) holds immediately.

If \( n_1 < n \), then, by the definition, \( \text{Var}(V_1) \leq \frac{1}{N} \) and \( \text{Var}(V_1 + X_{n_1+1}) \geq \frac{1}{2N} \). The latter implies \( \text{Var}(V_1) > \frac{1}{N} - \sigma^2 \geq \frac{1}{2N} \), thus proving (7.2) for \( k = 1 \).

Now, let \( 2 \leq k \leq N \). Again by the definition, \( \text{Var}(S_{nk}) \leq \frac{k}{N} \) and \( \text{Var}(S_{nk+1}) > \frac{k-1}{N} \). The latter implies \( \text{Var}(S_{nk+1}) > \frac{k-1}{N} - \sigma^2 \). Combining the two bounds, we get
\[ \text{Var}(V_k) = \text{Var}(S_{nk}) - \text{Var}(S_{nk-1}) \leq \frac{k}{N} - \left( \frac{k-1}{N} - \sigma^2 \right) = \frac{1}{N} + \sigma^2 < \frac{2}{N}. \]

On the other hand,
\[ \text{Var}(V_k) > \left( \frac{k}{N} - \sigma^2 \right) - \frac{k-1}{N} = \frac{1}{N} - \sigma^2 \geq \frac{1}{2N}. \]

Lemma 7.1 is proved.

Thus, to obtain the property (7.2), it seems suggestive to take \( N = \lfloor \frac{1}{2\sigma^2} \rfloor \) (the integer part). However, this choice is not used in the proof of Theorems 1.1-1.2, since we need to express \( N \) as a suitable function of Lyapunov’s coefficients.

As another useful property of the representation (7.1), let us mention the following.

**Lemma 7.2.** If \( \max_{l \leq n} D(X_l) \leq D \), then \( \max_{k \leq N} D(V_k) \leq D \), as well.

This is due to the general bound \( D(X + Y) \leq \max\{ D(X), D(Y) \} \), which holds for arbitrary independent random variables with finite second moments and absolutely continuous distributions. It can easily be derived, for example, from the entropy power inequality
\[ e^{2h(X+Y)} \geq e^{2h(X)} + e^{2h(Y)}, \]

cf. [C-D-T].

Now, let \( \rho_k \) denote density of the random variable \( V_k \). For each \( \rho_k \), one may consider a median density decomposition
\[ \rho_k(x) = \frac{1}{2} \rho_{k0}(x) + \frac{1}{2} \rho_{k1}(x) \tag{7.3} \]
in accordance with Definition 4.2 for the parameter \( \kappa = \frac{1}{2} \).

In particular, \( \rho_0(x) \leq m \), where \( m = m(\rho_k(V_k)) \) is a median of the random variable \( \rho_k(V_k) \). Note that by Proposition 5.1 with \( X = V_k \) and Lemmas 7.1-7.2, if \( \max_{j \leq n} D(X_j) \leq D \), we immediately obtain that

\[
m(\rho_k(V_k)) \leq \frac{1}{v_k \sqrt{2\pi}} e^{2D+2} \leq \sqrt{N} e^{2D+2},
\]

(7.4)

where \( v_k = \sqrt{\text{Var}(V_k)} \).

Let \( V_{kj} \) be random variables with densities \( \rho_{kj} \) and characteristic functions

\[
\hat{\rho}_{kj}(t) = \mathbb{E} e^{itV_{kj}} = \int_{-\infty}^{+\infty} e^{itx} \rho_{kj}(x) \, dx; \quad j = 0, 1.
\]

We collect their basic properties in the following lemma.

**Lemma 7.3.** Assume that \( N \leq \frac{1}{2\sigma^2} \) and \( \max_{l \leq n} D(X_l) \leq D \). For all \( k \leq N \) and \( j = 0, 1 \),

a) \( D(V_{kj}) \leq 2D + 2 \),

b) \( \text{Var}(V_{kj}) > \frac{1}{2N} e^{-4(D+4)} \),

c) \( |\hat{\rho}_{kj}(t)| \leq 1 - c e^{-12D} \) for all \( |t| \geq \sqrt{N} \) with an absolute constant \( c > 0 \).

**Proof.** The first assertion follows from Lemma 7.2 and Proposition 5.2 applied with \( X = V_k \). For the second one, combine Proposition 5.3 with \( X = V_k \) and Lemmas 7.1-7.2 to get

\[
v_{kj} > v_k e^{-2(D(V_k)+4)} \geq v_k e^{-2(D+4)} \geq \frac{1}{\sqrt{2N}} e^{-2(D+4)},
\]

where \( v_{kj}^2 = \text{Var}(V_{kj}) \) (\( v_{kj} > 0 \)). For the assertion in c), combine Proposition 6.2 for \( X = V_{kj} \) and the previous steps, which give

\[
|\hat{\rho}_{kj}(t)| \leq 1 - c \min \{1, v_{kj}^2 t^2 \} e^{-4D(V_k)} \\
\leq 1 - c \min \{1, t^2/(2N) \} e^{-4(D+4)} e^{-4(2D+2)} \\
\leq 1 - c' \min \{1, t^2/N \} e^{-12D}
\]

with some absolute constants \( c, c' > 0 \).

### 8. Decomposition of Convolutions

Starting from the representation \( S_n = V_1 + \cdots + V_N \) with the summands defined in (7.1), one can write the density of \( S_N \) as the convolution

\[
p_n = \rho_1 * \cdots * \rho_N,
\]

where \( \rho_k \) denotes the density of \( V_k \). Moreover, a direct application of the median decomposition (7.3) leads to the representation

\[
p_n = 2^{-N} \sum (\rho_{10}^{\delta_1} * \rho_{11}^{1-\delta_1}) * \cdots * (\rho_{N0}^{\delta_N} * \rho_{N1}^{1-\delta_N}),
\]

where the summation is carried out over all \( 2^N \) sequences \( \delta_k \) with values 0 and 1.
Let an integer number $m_0 \geq 0$ be given (For our purposes, one may take $m_0 = 3$). For $N \geq m_0 + 1$, we split the above sum into the two parts, so that

$$p_n = q_{n0} + q_{n1},$$

where

$$q_{n0} = 2^{-N} \sum_{\delta_1 + \cdots + \delta_N > m_0} (\rho_{10}^{\delta_1} \ast \rho_{11}^{1-\delta_1}) \ast \cdots \ast (\rho_{N0}^{\delta_N} \ast \rho_{N1}^{1-\delta_N}),$$

$$q_{n1} = 2^{-N} \sum_{\delta_1 + \cdots + \delta_N \leq m_0} (\rho_{10}^{\delta_1} \ast \rho_{11}^{1-\delta_1}) \ast \cdots \ast (\rho_{N0}^{\delta_N} \ast \rho_{N1}^{1-\delta_N}).$$

Put

$$\varepsilon_n = \int_{-\infty}^{+\infty} q_{n1}(x) \, dx = 2^{-N} \sum_{k=0}^{m_0} \frac{N!}{k! (N-k)!}.$$  

One can easily see that

$$\varepsilon_n \leq 2^{-(N-1)} N^{m_0}.$$  \hspace{1cm} (8.1)

**Definition 8.1.** Put

$$\tilde{p}_n(x) = p_{n0}(x) = \frac{1}{1 - \varepsilon_n} q_{n0}(x),$$  

and similarly $p_{n1}(x) = \frac{1}{\varepsilon_n} q_{n1}(x)$. Thus, we get the decomposition

$$p_n(x) = (1 - \varepsilon_n)p_{n0}(x) + \varepsilon_n p_{n1}(x).$$  \hspace{1cm} (8.3)

Accordingly, introduce the associated characteristic functions

$$\tilde{f}_n(t) = f_{n0}(t) = \int_{-\infty}^{+\infty} e^{itx} \tilde{p}_n(x) \, dx, \quad f_n(t) = \int_{-\infty}^{+\infty} e^{itx} p_n(x) \, dx.$$ 

The probability densities $\tilde{p}_n(x) = p_{n0}(x)$ are bounded and provide a strong approximation for $p_n(x)$. Indeed, from (8.3) it follows that

$$|\tilde{p}_n(x) - p_n(x)| = \varepsilon_n |p_{n0}(x) - p_{n1}(x)|$$  \hspace{1cm} (8.4)

which together with the bound (8.1) immediately implies:

**Proposition 8.2.** For all $n \geq N \geq m_0 + 1$,

$$\int_{-\infty}^{+\infty} |\tilde{p}_n(x) - p_n(x)| \, dx \leq 2^{-(N-2)} N^{m_0}.$$ 

In particular, the corresponding characteristic functions satisfy, for all $t \in \mathbb{R}$,

$$|\tilde{f}_n(t) - f_n(t)| \leq 2^{-(N-2)} N^{m_0}.$$

We need a similar inequality for derivatives of characteristic functions. To this aim, we shall use absolute moments $\mathbb{E} |X_k|^s$ and the associated Lyapunov ratios

$$L_s = \sum_{k=1}^{n} \mathbb{E} |X_k|^s \quad (s \geq 2).$$
Let $V_{kj} \ (1 \leq k \leq N, \ j = 0, 1)$ be independent random variables with respective densities $\rho_{kj}$ from the median decomposition (7.3) for the random variables $V_k$. For each sequence $\delta = (\delta_k)_{1 \leq k \leq N}$ with values 0 and 1, the convolution

$$\rho^{(\delta)} = (\rho_{10}^{\delta_1} \ast \rho_{11}^{1-\delta_1}) \ast \cdots \ast (\rho_{N0}^{\delta_N} \ast \rho_{N1}^{1-\delta_N})$$

represents the density of the sum

$$S(\delta) = \sum_{k=1}^{N} \delta_k V_{k0} + (1 - \delta_k) V_{k1}.$$ 

If all moments $\mathbb{E}|X_k|^s$ are finite, (7.3) yields

$$\mathbb{E}|V_k|^s = \frac{1}{2} \mathbb{E}|V_{k0}|^s + \frac{1}{2} \mathbb{E}|V_{k1}|^s. \quad (8.5)$$

Hence, for the $L^s$-norm $\|S(\delta)\|_s = (\mathbb{E}|S(\delta)|^s)^{1/s}$, using the Minkowski inequality, we have

$$\|S(\delta)\|_s \leq \sum_{k=1}^{N} \|\delta_k V_{k0} + (1 - \delta_k) V_{k1}\|_s$$

$$\leq \sum_{k=1}^{N} (\|\delta_k V_{k0}\|_s + (1 - \|\delta_k V_{k1}\|_s) \leq 2^{1/s} \sum_{k=1}^{N} \|V_k\|_s,$$

where (8.5) was used in the last step. But

$$\frac{1}{N} \sum_{k=1}^{N} \|V_k\|_s = \frac{1}{N} \sum_{k=1}^{N} (\mathbb{E}|V_k|^s)^{1/s} \leq \left(\frac{1}{N} \sum_{k=1}^{N} \mathbb{E}|V_k|^s\right)^{1/s},$$

so

$$\mathbb{E}|S(\delta)|^s \leq 2N^{s-1} \sum_{k=1}^{N} \mathbb{E}|V_k|^s \leq 2N^s \mathbb{E}|S_n|^s,$$

where we used $\mathbb{E}|V_k|^s \leq \mathbb{E}|S_n|^s$ (due to Jensen’s inequality).

Write $\mathbb{E}|S(\delta)|^s = \int_{-\infty}^{\infty} |x|^s \rho^{(\delta)}(x) \, dx$. Recalling the definition of $q_{nj}$ and $\varepsilon_n$, we get

$$\int_{-\infty}^{+\infty} |x|^s q_{n0}(x) \, dx = 2^{-N} \sum_{\delta_1 + \cdots + \delta_N > m_0} \mathbb{E}|S(\delta)|^s \leq 2 \mathbb{E}|S_n|^s (1 - \varepsilon_n) N^s,$$

$$\int_{-\infty}^{+\infty} |x|^s q_{n1}(x) \, dx = 2^{-N} \sum_{\delta_1 + \cdots + \delta_N \leq m_0} \mathbb{E}|S(\delta)|^s \leq 2 \mathbb{E}|S_n|^s \varepsilon_n N^s.$$

Hence, by the definition of $p_{n0}$,

$$\int_{-\infty}^{+\infty} |x|^s p_{n0}(x) \, dx \leq 2 \mathbb{E}|S_n|^s N^s,$$

and similarly for $p_{n1}$. But, from (8.4),

$$|x|^s (\tilde{p}_n(x) - p_n(x)) \leq \varepsilon_n |x|^s (p_{n0}(x) + p_{n1}(x)),$$
so, applying (8.1),
\[ \int_{-\infty}^{+\infty} |x|^s |\tilde{p}_n(x) - p_n(x)| \, dx \leq E|S_n|^s 2^{-(N-3)} N^{m_0+s}. \]

On the other hand, Rosenthal’s inequality (cf. e.g. [Ro], [P-U]) gives
\[ E|S_n|^s \leq C_s \left( 1 + \sum_{j=1}^{n} E|X_j|^s \right) = C_s (1 + L_s), \quad s \geq 2, \tag{8.6} \]

with some constants \( C_s \), depending on \( s \), only (where the assumption \( E S_n^2 = 1 \) is used). Note that in case \( 1 \leq s \leq 2 \), there is also an obvious bound \( E|S_n|^s \leq 1 \).

One may summarize, using the constant \( C_s \) in Rosenthal’s inequality (8.6).

**Proposition 8.3.** Assume that \( L_s \) is finite \((s \geq 2)\). For all \( n \geq N \geq m_0 + 1 \),
\[ \int_{-\infty}^{+\infty} |x|^s |\tilde{p}_n(x) - p_n(x)| \, dx \leq C_s (1 + L_s) 2^{-(N-3)} N^{m_0+s}. \]

In particular, if \( s \) is an integer, the \( s \)-th derivative of the corresponding characteristic functions satisfies, for all \( t \) real,
\[ |\tilde{f}_n^{(s)}(t) - f_n^{(s)}(t)| \leq C_s (1 + L_s) 2^{-(N-3)} N^{m_0+s}. \]

For \( s = 1 \) and \( s = 2 \), it is better to use \( E|S_n| \leq 1 \) and \( ES_n^2 = 1 \) instead of (8.6). For \( s = 3 \), Rosenthal’s inequality can be shown to hold with constant \( C_3 = 2 \). Hence, we obtain:

**Corollary 8.4.** For all \( n \geq N \geq m_0 + 1 \) and \( t \in \mathbb{R} \),
\[ |\tilde{f}_n^{(s)}(t) - f_n^{(s)}(t)| \leq 2^{-(N-3)} N^{m_0+s} \quad (s = 1, 2). \]

Moreover, if \( L_3 \) is finite,
\[ |\tilde{f}_n^{(m)}(t) - f_n^{(m)}(t)| \leq (1 + L_3) 2^{-(N-4)} N^{m_0+3}. \]

9. **Entropic Approximation of \( p_n \) by \( \tilde{p}_n \)**

As before, let \( X_1, \ldots, X_n \) be independent random variables with \( E X_k = 0 \), \( EX_k^2 = \sigma_k^2 \) \((\sigma_k > 0)\), such that \( \sigma_1^2 + \cdots + \sigma_n^2 = 1 \). Moreover, let \( X_k \) have absolutely continuous distributions with finite entropies, and let \( p_n \) denote the density of the sum
\[ S_n = X_1 + \cdots + X_n. \]

Put \( \sigma^2 = \max_k \sigma_k^2 \).

The next step is to extend the assertion of Propositions 8.2-8.3 to relative entropies, with respect to the standard normal distribution on the real line with density
\[ \phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}. \]
Thus put
\[ D_n = \int p_n(x) \log \frac{p_n(x)}{\varphi(x)} \, dx, \quad \tilde{D}_n = \int \tilde{p}_n(x) \log \frac{\tilde{p}_n(x)}{\varphi(x)} \, dx. \]

Recall that the modified densities \( \tilde{p}_n \) are constructed in Definition 8.1 with arbitrary integers \( 0 \leq m_0 < N \leq n \) on the basis of the representation (7.1), based on the independent random variables \( V_k \) and the median decomposition (7.3) for the densities \( \rho_k \) of \( V_k \).

**Proposition 9.1.** Let \( D = \max_k D(X_k) \). Given that \( m_0 + 1 \leq N \leq \frac{1}{2\sigma^2} \), we have
\[ |\tilde{D}_n - D_n| < 2^{-(N-6)}N^{m_0+1}(D+1). \] (9.1)

We shall use a few elementary properties of the convex function \( L(u) = u \log u \) \((u \geq 0)\).

**Lemma 9.2.** For all \( u, v \geq 0 \) and \( 0 \leq \varepsilon \leq 1 \),
\[ a) \quad L((1-\varepsilon)u + \varepsilon v) \leq (1-\varepsilon)L(u) + \varepsilon L(v); \]
\[ b) \quad L((1-\varepsilon)u + \varepsilon v) \geq (1-\varepsilon)L(u) + \varepsilon L(v) + uL(1-\varepsilon) + vL(\varepsilon). \]

**Proof of Proposition 9.1.** Define
\[ D_{nj} = \int p_{nj}(x) \log \frac{p_{nj}(x)}{\varphi(x)} \, dx \quad (j = 0, 1), \]
so that \( \tilde{D}_n = D_{n0} \), where the densities \( p_{nj} \) have been defined in (8.2)-(8.3).

By Lemma 9.2 a), \( D_n \leq (1-\varepsilon_n)D_{n0} + \varepsilon_n D_{n1} \). On the other hand, by Lemma 9.2 b),
\[ D_n \geq ((1-\varepsilon_n)D_{n0} + \varepsilon_n D_{n1}) + \varepsilon_n \log \varepsilon_n + (1-\varepsilon_n) \log(1-\varepsilon_n). \]

The two estimates give
\[ |\tilde{D}_n - D_n| \leq \varepsilon_n(D_{n0} + D_{n1}) - \varepsilon_n \log \varepsilon_n - (1-\varepsilon_n) \log(1-\varepsilon_n). \] (9.2)

Hence, we need to give appropriate bounds on both \( D_{n0} \) and \( D_{n1} \).

To this aim, as before, let \( V_{kj} \) \((1 \leq k \leq N, j = 0, 1)\) be independent random variables with respective densities \( \rho_{kj} \) from the median decomposition (7.3) for \( V_k \). By Definition 4.2, we have the identity (5.1), which for \( V_k \) reads
\[ v_k^2 = (\frac{1}{2}a_{k0}^2 + \frac{1}{2}a_{k1}^2) + \left( \frac{1}{2}v_{k0}^2 + \frac{1}{2}v_{k1}^2 \right), \]
where \( a_{kj} = \text{EV}_{kj}, v_{kj}^2 = \text{Var}(V_{kj}) \) and \( v_k^2 = \text{Var}(V_k) \). Using Lemma 7.1, this implies
\[ v_{k0}^2 \leq 2v_k^2 \leq \frac{4}{N}, \quad v_{k1}^2 \leq 2v_k^2 \leq \frac{4}{N}. \] (9.3)

As in the previous section, for each sequence \( \delta = (\delta_k)_{1 \leq k \leq N} \) with values 0 and 1, consider the convolution
\[ \rho^{(\delta)} = (\rho_{10}^{\delta_1} * \rho_{11}^{1-\delta_1}) * \cdots * (\rho_{N0}^{\delta_N} * \rho_{N1}^{1-\delta_N}), \]
i.e., the densities of the random variables

\[ S(\delta) = \sum_{k=1}^{N} \delta_k V_{k0} + (1 - \delta_k)V_{k1}. \]

By convexity of the function \( u \log u \),

\[ D_{n1} \leq \frac{1}{\varepsilon_n} 2^{-N} \sum_{\delta_1 + \cdots + \delta_N \leq m_0} \int_{-\infty}^{+\infty} \rho(\delta)(x) \log \frac{\rho(\delta)(x)}{\varphi(x)} dx, \quad (9.4) \]

\[ D_{n0} \leq \frac{1}{1 - \varepsilon_n} 2^{-N} \sum_{\delta_1 + \cdots + \delta_N > m_0} \int_{-\infty}^{+\infty} \rho(\delta)(x) \log \frac{\rho(\delta)(x)}{\varphi(x)} dx. \quad (9.5) \]

Furthermore, if \( S \) denotes a random variable with variance \( v^2 \) \((v > 0)\) having density \( \rho \), and if \( Z \) is a standard normal random variable, the relative entropy of \( S \) with respect to \( Z \) is connected with the entropic distance to normality \( D(S) \) by the simple formula

\[ D(S||Z) = \int \rho(x) \log \frac{\rho(x)}{\varphi(x)} dx = D(S) + \log \frac{1}{v} + \frac{ES^2 - 1}{2}. \quad (9.6) \]

In the case \( S = S(\delta) \), applying Lemma 7.3 \( b \), we have

\[ v^2 = \sum_{k=1}^{N} \left[ \delta_k v_{k0}^2 + (1 - \delta_k) v_{k1}^2 \right] \geq \frac{1}{2} e^{-4(D + 4)}, \]

hence

\[ \log \frac{1}{v} \leq 2D + 9. \quad (9.7) \]

In addition, arguing as in the proof of Proposition 8.2, specialized to the particular case \( s = 2 \), and applying (9.3), we get

\[ \|S(\delta)\|_2 \leq \sum_{k=1}^{N} \|\delta_k V_{k0} + (1 - \delta_k)V_{k1}\|_2 \]

\[ \leq \sum_{k=1}^{N} (\delta_k \|V_{k0}\|_2 + (1 - \delta_k) \|V_{k1}\|_2) \leq \sqrt{2} \sum_{k=1}^{N} v_k \leq 2\sqrt{N}. \]

Hence, \( ES(\delta)^2 \leq 4N \). Combining this estimate with (9.7), we get that

\[ \log \frac{1}{v} + \frac{ES(\delta)^2 - 1}{2} \leq (2D + 9) + 2N. \]

Consequently, if we apply this bound in (9.6) with \( S = S(\delta) \), we obtain

\[ D(S(\delta)||Z) \leq D(S(\delta)) + (2D + 9) + 2N. \quad (9.8) \]

The remaining term, \( D(S(\delta)) \), can be estimated by virtue of the same general inequality \( D(X+Y) \leq \max\{D(X), D(Y)\} \) mentioned before. This bound can be applied to all summands of \( S(\delta) \), which together with Lemma 7.3 \( a \) gives

\[ D(S(\delta)) \leq \max_{1 \leq k \leq N} \max\{D(V_{k0}), D(V_{k1})\} \leq 2D + 2. \]
Applying this result in (9.8), we arrive at
\[
\int_{-\infty}^{+\infty} \rho^{(\delta)}(x) \log \frac{\rho^{(\delta)}(x)}{\varphi(x)} \, dx = D(S(\delta)\|Z) \leq 4D + 11 + 2N.
\]

Finally, by (9.4)-(9.5), we have similar bounds for \( D_{n0} \) and \( D_{n1} \), namely,
\[
D_{n0} \leq 4D + 11 + 2N, \quad D_{n1} \leq 4D + 11 + 2N.
\]

Having obtained these estimates, we are prepared to return to (9.2), which thus gives
\[
|\tilde{D}_n - D_n| \leq 2\varepsilon_n (4D + 11 + 2N) + \varepsilon_n \log \frac{1}{\varepsilon_n} + (1 - \varepsilon_n) \log \frac{1}{1 - \varepsilon_n}. \tag{9.9}
\]

To simplify this bound, consider the function \( H(\varepsilon) = \varepsilon \log \frac{1}{\varepsilon} + (1 - \varepsilon) \log \frac{1}{1 - \varepsilon} \), which is defined for \( 0 \leq \varepsilon \leq 1 \), is concave and symmetric about the point \( \frac{1}{2} \), where it attains its maximum \( H\left(\frac{1}{2}\right) = \log 2 \). Recall (8.1), that is, \( \varepsilon_n \leq d_n = 2^{-(N-1)} N^{m_0} \).

If \( d_n \geq \frac{1}{2} \), then
\[
H(\varepsilon_n) \leq \log 2 \leq 2d_n = 2^{-(N-2)} N^{m_0}. \tag{9.10}
\]

Note that
\[
\log \frac{1}{d_n} = m_0 \log \frac{1}{N} + (N - 1) \log 2 < N.
\]

Hence, in the other case \( d_n \leq \frac{1}{2} \), we have
\[
H(\varepsilon_n) \leq H(d_n) \leq 2d_n \log \frac{1}{d_n} \leq 2^{-(N-2)} N^{m_0+1}. \tag{9.11}
\]

Comparing (9.10) and (9.11), we see that they can be combined to the following estimate
\[
H(\varepsilon_n) \leq 2^{-(N-2)} N^{m_0+1},
\]
which is valid regardless of whether \( d_n \) is greater or smaller than \( \frac{1}{2} \).

Using this estimate in (9.9), we finally get
\[
|\tilde{D}_n - D_n| \leq 2^{-(N-2)} N^{m_0} (4D + 11 + 2N) + 2^{-(N-2)} N^{m_0+1}
\]
\[
= 2^{-(N-2)} N^{m_0} (4D + 11 + 3N).
\]

Since \( 4D + 11 + 3N < 2^N (D + 1) \), we arrive at the desired inequality (9.1).

Thus, Proposition 9.1 is proved.

10. Integrability of Characteristic Functions \( \tilde{f}_n \) and their Derivatives

Now we turn to the question of quantitative bounds for the modified characteristic functions \( \tilde{f}_n \) in terms of the maximal entropic distance to normality
\[
D = \max_{k \leq n} D(X_k).
\]

Again, let \( X_1, \ldots, X_n \) be independent random variables with \( \mathbf{E}X_k = 0, \mathbf{E}X_k^2 = \sigma_k^2 (\sigma_k > 0) \), such that \( \sigma_1^2 + \cdots + \sigma_n^2 = 1 \). Moreover, all \( X_k \) are assumed to have absolutely continuous distributions with finite entropies.

We assume that the modified density \( \tilde{p}_n \) and its characteristic function \( \tilde{f}_n \) have been constructed for arbitrary integers \( m_0 + 1 \leq N \leq n \). Put \( \sigma = \max_k \sigma_k \).
Proposition 10.1. If \( m_0 \geq 1 \) and \( m_0 + 1 \leq N \leq \frac{1}{2\pi} \), then
\[
\int_{|t|\geq\sqrt{N}} |\tilde{f}_n(t)|^2 \, dt \leq C\sqrt{N} \, e^{-CN}
\] (10.1)
with some positive constants \( C \) and \( c \), depending on \( D \), only.

In fact, one can choose the constants to be of the form \( C = e^{2D+4} \) and \( c = c_0 e^{-12D} \), where \( c_0 \) is a positive absolute factor.

**Proof.** Consider any convolution
\[
\rho = (\rho_{10}^{\delta_1} * \rho_{11}^{1-\delta_1}) * \cdots * (\rho_{N0}^{\delta_N} * \rho_{N1}^{1-\delta_N})
\]
participating in the definition of \( q_{n0} \), that is, with \( \delta_1 + \cdots + \delta_N > m_0 \). It has the Fourier transform
\[
\hat{\rho}(t) = \int_{-\infty}^{+\infty} e^{itx} \rho(x) \, dx = \prod_{k=1}^{N} \hat{\rho}_{k0}(t)^{\delta_k} \hat{\rho}_{k1}(t)^{1-\delta_k},
\] (10.2)
where \( \hat{\rho}_{kj} \) denote the characteristic functions of the random variable \( V_{kj} \) from the median decomposition \((4.1)\) with \( X = V_k \) \((1 \leq k \leq N, j = 0, 1)\). In every such convolution there are at least \( m_0 + 1 \) terms \( \rho_{k0} \) for which \( \delta_k = 1 \). For definiteness, let \( k = N \) be one of them, so that \( \delta_N = 1 \). Then, we may write
\[
\hat{\rho}(t) = \hat{\rho}_{N0}(t) \prod_{k=1}^{N-1} \hat{\rho}_{k0}(t)^{\delta_k} \hat{\rho}_{k1}(t)^{1-\delta_k}.
\] (10.3)

By Lemma 7.3 c), for all \( |t| \geq \sqrt{N} \),
\[
|\hat{\rho}_{kj}(t)| \leq \exp \left\{ -c_0 e^{-12D} \right\}
\] (10.4)
with some absolute constant \( c_0 > 0 \). Inserting this in (10.3) and using \( N \geq 2 \) leads to
\[
|\hat{\rho}(t)|^2 \leq A |\hat{\rho}_{N0}(t)|^2, \quad A = \exp \left\{ -c_0 e^{-12D} N \right\},
\] (10.5)
where \( c_0 > 0 \) is a different absolute constant.

Now, integrate (10.5) over the region \( |t| \geq \sqrt{N} \) and use Plancherel’s formula. Applying the property \( \rho_{N0}(x) \leq m = m(\rho_N(V_N)) \), we get
\[
\int_{|t|\geq\sqrt{N}} |\hat{\rho}(t)|^2 \, dt \leq A \int_{-\infty}^{+\infty} |\hat{\rho}_{N0}(t)|^2 \, dt = 2\pi A \int_{-\infty}^{+\infty} \rho_{N0}(x)^2 \, dx \leq 2\pi A \, m.
\] (10.6)
But, as noted in (7.4), we have \( m \leq e^{2D+2\sqrt{N}} \), so together with \( 2\pi < e^2 \) (10.6) gives the desired bound
\[
\int_{|t|\geq\sqrt{N}} |\hat{\rho}(t)|^2 \, dt \leq e^{2D+4\sqrt{N}} e^{-cN} \quad \left( c = c_0 e^{-12D} \right)
\]
for \( \hat{\rho} \). But \( \tilde{f}_n \) is a finite convex combination of such functions, so (10.1) immediately follows. Thus Proposition 10.1 is proved.

Next, we shall extend Propositions 10.1 to the derivatives of \( \tilde{f}_n \), which are needed up to order \( s = 3 \) in case of finite 4-th moments of \( X_k \). Assume that \( s \geq 1 \) is an arbitrary integer.
Hence, using also (10.8), we get
\[
\text{sequence } \delta_{c} \text{ with some absolute constant }
\]
any admissible fixed sequence \( u \).

But, by the decomposition (7.3) and Jensen’s inequality, \( N \delta \) combination of such characteristic functions over all sequences \( \delta = (\delta_{1}, \ldots, \delta_{N}) \) such that \( \delta_{1} + \cdots + \delta_{N} \geq m_{0} + 1 \). Hence, it will be sufficient to derive an estimate, such as (10.1), for any admissible fixed sequence \( \delta \).

Put
\[
u_{k} = \delta_{k} \frac{1}{k} \cdot \hat{\rho}_{k0} \hat{\rho}_{k1} \quad (1 \leq k \leq N),
\]
which is the characteristic function of the random variable \( \delta_{k} V_{k0} + (1 - \delta_{k}) V_{k1} \).

Thus, \( \hat{\rho} = \prod_{k=1}^{N} u_{k} \). For the \( s \)-th derivative of the product we write a general polynomial formula
\[
\hat{\rho}^{(s)}(t) = \sum_{(s_{1}, \ldots, s_{N})} \frac{s!}{s_{1}! \cdots s_{N}!} u_{1}^{(s_{1})} \cdots u_{N}^{(s_{N})},
\]
where the summation runs over all integer numbers \( s_{1}, \ldots, s_{N} \geq 0 \), such that \( s_{1} + \cdots + s_{N} = s \).

Fix such a sequence \( s_{1}, \ldots, s_{N} \). Note that it contains at most \( s \) non-zero terms. The sequence \( \delta = (\delta_{1}, \ldots, \delta_{N}) \) defining \( \rho \) satisfies \( \delta_{1} + \cdots + \delta_{N} \geq m_{0} + 1 \). Hence, in the row \( u_{1}^{(s_{1})}, \ldots, u_{N}^{(s_{N})} \) there are at least \( m_{0} + 1 \) terms corresponding to \( \delta_{k} = 1 \). Therefore, if \( m_{0} \geq s \), there is at least one index, say \( k \), for which \( \delta_{k} = 1 \) and in addition \( s_{k} = 0 \). For definiteness, let \( k = N \), so that
\[
\psi \equiv u_{1}^{(s_{1})} \cdots u_{N}^{(s_{N})} = \hat{\rho}_{N0} u_{1}^{(s_{1})} \cdots u_{N-1}^{(s_{N-1})}.
\]

If \( s_{k} > 0 \), then
\[
|u_{k}^{(s_{k})}(t)| \leq E|\delta_{k} V_{k0} + (1 - \delta_{k}) V_{k1}|^{s_{k}} \leq \max\{E|V_{k0}|^{s_{k}}, E|V_{k1}|^{s_{k}}\}.
\]
But, by the decomposition (7.3) and Jensen’s inequality,
\[
\frac{1}{2} E|V_{k0}|^{s_{k}} + \frac{1}{2} E|V_{k1}|^{s_{k}} = E|V_{k}|^{s_{k}} \leq E|S_{n}|^{s_{k}},
\]
so \( |u_{k}^{(s_{k})}(t)| \leq 2E|S_{n}|^{s_{k}} \). Hence,
\[
\prod_{s_{k}>0} |u_{k}^{(s_{k})}(t)| \leq 2^{s} \prod_{s_{k}>0} E|S_{n}|^{s_{k}} \leq 2^{s} \prod_{s_{k}>0} (E|S_{n}|^{s_{k}})^{s_{k}/s} = 2^{s} E|S_{n}|^{s}.
\]

When \( s_{k} = 0 \), we apply the estimate (10.4) on Cramer’s constants, which may be used in (10.7). Note that (10.4) is fulfilled for at least \( (N - 1) - (s - 1) \geq N - m_{0} \) indices \( k \leq N - 1 \). Hence, using also (10.8), we get
\[
|\psi(t)| \leq C |\hat{\rho}_{N0}(t)| \exp \left\{ -c_{0}(N - m_{0}) e^{-12D} \right\}, \quad C = 2^{s} E|S_{n}|^{s}.
\]

In case \( N \geq 2m_{0} \), one may simplify this bound by writing \( N - m_{0} \geq \frac{N}{2} \). In addition, since the sum of the multinomial coefficients in the representation of \( \hat{\rho}^{(s)} \) is equal to \( N^{s} \), and using Jensen’s inequality for the quadratic function, we arrive at
\[
|\hat{\rho}^{(s)}(t)|^{2} \leq A |\hat{\rho}_{N0}(t)|^{2}, \quad A = C N^{s} \exp \left\{ -c_{0}e^{-12D}N \right\},
\]
with some absolute constant \( c_{0} > 0 \). It remains to integrate this inequality like in (10.6) over the region \( |t| \geq \sqrt{N} \) and apply the estimate (7.4). As a result, we obtain
\[
\int_{|t| \geq \sqrt{N}} |\hat{\rho}^{(s)}(t)|^{2} dt \leq A e^{2D+4} \sqrt{N}.
\]
Since $\tilde{f}_n$ is a convex combination of the functions $\tilde{\rho}^{(s)}$, a similar inequality holds for $\tilde{f}_n(t)$, as well. That is,

$$\int_{|t| \geq \sqrt{N}} |\tilde{f}_n^{(s)}(t)|^2 \, dt \leq 2^s \mathbf{E} |S_n|^s e^{2D+4} \exp \left\{ -c_0 e^{-12D} N \right\} N^{s+1/2}.$$  

For $s = 1$ and $s = 2$, we have $\mathbf{E} |S_n|^s \leq 1$, while for $s \geq 3$, one may use Rosenthal’s inequality (8.6). In particular, for $s = 3$ it gives $\mathbf{E} |S_n|^3 \leq 2(1 + L_3)$.

Summarizing the results obtained so far, we have:

**Proposition 10.2.** Let $m_0 \geq 3$ and $2m_0 \leq N \leq \frac{1}{27\pi^2}$. Then

$$\int_{|t| \geq \sqrt{N}} |\tilde{f}_n^{(s)}(t)|^2 \, dt \leq CN^{s+1/2} e^{-cN} \quad (s = 1, 2)$$  

with positive constants $C$ and $c$, depending on $D$, only. Moreover, if $L_s$ is finite, $s \geq 3$ integer, and $m_0 \geq s$, then

$$\int_{|t| \geq \sqrt{N}} |\tilde{f}_n^{(s)}(t)|^2 \, dt \leq C \cdot C_s (1 + L_s) N^{s+1/2} e^{-cN}.$$  

Here, the constants $C = e^{2D+4}$ and $c = c_0 e^{-12D}$ are of the same form as in Proposition 10.1, and $C_s$ is a constant in Rosenthal’s inequality (8.6). In particular, for $s = 3$, we arrive at

$$\int_{|t| \geq \sqrt{N}} |\tilde{f}_n^{(s)}(t)|^2 \, dt \leq C(1 + L_3) N^{7/2} e^{-cN}. \quad (10.10)$$

Note also that, for $s = 0$, (10.9) is true, as well, and returns us to Proposition 10.1.

### 11. Proof of Theorem 1.1 and its Refinement

We are now ready to complete the proof of Theorems 1.1-1.2 and emphasize some of their refinements. Thus, let $X_1, \ldots, X_n$ be independent random variables with mean zero and finite third absolute moments, having finite entropies, and such that the sum $S_n = X_1 + \cdots + X_n$ has variance $\text{Var}(S_n) = 1$.

Our main quantities are the Lyapunov coefficient

$$L_3 = \sum_{k=1}^n \mathbf{E} |X_k|^3$$

and the maximal entropic distance to normality $D = \max_k D(X_k)$.

To bound the total variation distance $\|F_n - \Phi\|_{TV}$ from the distribution $F_n$ of $S_n$ to the standard normal law $\Phi$, one may apply the general bound (2.1) of Proposition 2.1. However, it is only applicable when the characteristic function $f_n$ of $S_n$ and its derivative are square integrable. But even in the case that, for example, each density $p_n$ of $S_n$ is bounded individually, we still could not properly bound the maximum of the convolutions of these densities explicitly in terms of $D$ and $L_3$. That is why, we are forced to consider modified forms of $p_n$.

Thus, consider these modifications $\tilde{p}_n$ together with their Fourier transforms $\tilde{f}_n$ described in Definition 8.1. By the triangle inequality,

$$\|F_n - \Phi\|_{TV} \leq \|\tilde{F}_n - \Phi\|_{TV} + \|\tilde{F}_n - F_n\|_{TV}, \quad (11.1)$$
where \( \tilde{F}_n \) denotes the distribution with density \( \tilde{p}_n \).

In the construction of \( \tilde{p}_n \) it suffices to take the values \( m_0 = 3 \) and \( 6 \leq N \leq \frac{1}{2\sigma^2} \). Then, by Proposition 8.2,

\[
\|\tilde{F}_n - F_n\|_{TV} = \int_{-\infty}^{+\infty} |\tilde{p}_n(x) - p_n(x)| \, dx \leq 2^{-(N-2)} N^3. \tag{11.2}
\]

This gives a sufficiently good bound on the last term in (11.1), if \( N \) is sufficiently large.

The first term on the right-hand side of (11.1) can be bounded by virtue of (2.1), which gives

\[
\|\tilde{F}_n - \Phi\|_{TV}^2 \leq \frac{1}{2} \|\tilde{f}_n - g\|_2^2 + \frac{1}{2} \|f_n - g\|_{2}^2, \tag{11.3}
\]

where \( g(t) = e^{-t^2/2} \). To estimate the \( L^2 \)-norms, first write

\[
\frac{1}{2} \|\tilde{f}_n - g\|_2^2 \leq \frac{1}{2} \int_{|t| \leq \sqrt{N}} |\tilde{f}_n(t) - g(t)|^2 \, dt
\]

\[
+ \int_{|t| > \sqrt{N}} |\tilde{f}_n(t)|^2 \, dt + \int_{|t| > \sqrt{N}} g(t)^2 \, dt.
\]

Since \( |\tilde{f}_n(t) - f_n(t)| \leq 2^{-(N-2)} N^3 \), we have

\[
\frac{1}{2} \int_{|t| \leq \sqrt{N}} |\tilde{f}_n(t) - g(t)|^2 \, dt \leq \int_{|t| \leq \sqrt{N}} |\tilde{f}_n(t) - f_n(t)|^2 \, dt + \int_{|t| \leq \sqrt{N}} |f_n(t) - g(t)|^2 \, dt \leq \int_{|t| \leq \sqrt{N}} |f_n(t) - g(t)|^2 \, dt + 2^{-(2N-5)} N^{7/2}. \tag{11.4}
\]

In addition, by Proposition 10.1,

\[
\int_{|t| \geq \sqrt{N}} |\tilde{f}_n(t)|^2 \, dt \leq C \sqrt{N} e^{-cN} \tag{11.5}
\]

with \( C = e^{2D+4} \) and \( c = c_0 e^{-12D} \), where \( c_0 \) is an absolute positive constant.

Using a well-known bound \( 1 - \Phi(x) \leq \frac{1}{2} \varphi(x) \) \( (x > 0) \), we easily get \( \int_{|t| > \sqrt{N}} g(t)^2 \, dt < e^{-N} \).

Together with (11.4)-(11.5), and since one may always assume that \( c_0 \leq \frac{1}{2} \), the latter gives

\[
\frac{1}{2} \|\tilde{f}_n - g\|_2^2 \leq \int_{|t| \leq \sqrt{N}} |f_n(t) - g(t)|^2 \, dt + C \sqrt{N} e^{-cN} \tag{11.6}
\]

with \( D \)-dependent constants \( C = C_0 e^{2D} \) and \( c = c_0 e^{-12D} \) (where \( C_0 \) and \( c_0 \) are numerical).

A similar analysis based on the application of Proposition 8.3 (cf. Corollary 8.4) and Proposition 10.2 with \( s = 1 \) leads to an analogous estimate

\[
\frac{1}{2} \|\tilde{f}_n' - g'\|_2^2 \leq \int_{|t| \leq \sqrt{N}} |f_n'(t) - g'(t)|^2 \, dt + C N^{3/2} e^{-cN}.
\]

Together with (11.6) it may be applied in (11.3), and then we get

\[
\|\tilde{F}_n - \Phi\|_{TV}^2 \leq \int_{|t| \leq \sqrt{N}} |f_n(t) - g(t)|^2 \, dt
\]

\[
+ \int_{|t| \leq \sqrt{N}} |f_n'(t) - g'(t)|^2 \, dt + C N^{3/2} e^{-cN}.
\]
It is time to appeal to the classical theorem on the approximation of \( f_n \) by the characteristic function of the standard normal law, cf. e.g. [R-RR].

**Lemma 11.1.** Assume \( L_3 \leq 1 \). Up to an absolute constant \( A \), in the interval \(|t| \leq L_3^{-1/3} \) we have

\[
|f_n(t) - g(t)| \leq AL_3 e^{-t^2/4},
\]

and similarly for the first three derivatives of \( f_n - g \).

In fact, the above inequality holds in the larger interval \(|t| \leq 1/(4L_3) \). But this will not be needed for the present formulation of Theorem 1.1.

Thus, if in addition to the original condition \( 6 \leq N \leq \frac{1}{2\pi^2} \), we require that \( \sqrt{N} \leq L_3^{-1/3} \), Lemma 11.1 may be applied, and we get

\[
\|\tilde{F}_n - \Phi\|_{TV} \leq AL_3 + CN^{3/2} e^{-cN}.
\]

Using this together with (11.2) in (11.1), we arrive at

\[
\|F_n - \Phi\|_{TV} \leq AL_3 + CN^{3/2} e^{-cN}, \tag{11.7}
\]

where \( A \) is some positive absolute constant, while \( C = C_0 e^{2D} \) and \( c = c_0 e^{-12D} \), as before.

**Proof of Theorem 1.1.** To finish the argument, we may take \( N = \left[ \frac{1}{2} L_3^{-2/3} \right] \), so that \( \sqrt{N} \leq L_3^{-1/3} \). In view of the elementary bound \( \sigma \leq L_3^{1/3} \), the condition \( N \leq \frac{1}{2\pi^2} \) is fulfilled, as well. Finally, the condition \( N \geq 6 \) just restricts us to smaller values of \( L_3 \), and, for example, \( L_3 \leq \frac{1}{64} \) would work. Indeed, in this case, \( \frac{1}{2} L_3^{-2/3} \geq 8 \), so \( N \geq 8 \).

Thus, if \( L_3 \leq \frac{1}{64} \), then (11.7) holds true. But since \( N \geq \frac{1}{6} L_3^{-2/3} \), the last term in (11.7) is dominated by any power of \( L_3 \) (up to constants). For example, using \( e^x \geq c_1 x^3 \) \((x \geq 0)\), we get

\[
N^{3/2} e^{-cN} \leq \frac{1}{c_1 c^3} N^{-3/2} \leq \frac{8}{c_1 c^3} L_3 = \frac{8}{c_1 c_0} e^{36D} L_3.
\]

Hence, (11.7) implies

\[
\|F_n - \Phi\|_{TV} \leq CL_3, \tag{11.8}
\]

with \( C = C_0 e^{C_1 D} \), where \( C_0, C_1 \) are positive numerical constants.

Finally, if \( L_3 > \frac{1}{64} \), (11.8) automatically holds with \( C = 128 \).

Thus, Theorem 1.1 is proved.

Note, however, that the inequality (11.7) contains more information in comparison with Theorem 1.1. Again assume, as above, that \( L_3 \leq \frac{1}{64} \) and take \( N = \left[ \frac{1}{4} L_3^{-2/3} \right] \). If \( D \leq \frac{1}{24} \log \frac{1}{L_3} \), then \( cN \geq c_0 L_3^{1/2} \cdot \frac{1}{4} L_3^{-2/3} = c_0 L_3^{-1/6} \) and \( C = C_0 e^{2D} \leq C_0 L_3^{-1/12} \). Hence,

\[
CN^{3/2} e^{-cN} \leq C_0 L_3^{-1/12} \cdot L_3^{-1} \cdot e^{-c_0 L_3^{-1/6}} \leq C_0 L_3
\]

with some absolute constant \( C_0' \). As a result, (11.7) yields \( \|F_n - \Phi\|_{TV} \leq (A + C_0') L_3 \). If \( L_3 > \frac{1}{64} \), (11.8) holds with \( C = 128 \), and we arrive at:
Theorem 11.2. Assume that independent random variables \( X_k \) have mean zero and finite third absolute moments. If they satisfy \( D(X_k) \leq c \log \frac{1}{L_3} \) \((1 \leq k \leq n)\), then

\[
\|F_n - \Phi\|_{TV} \leq CL_3, \tag{11.9}
\]

where \( C \) and \( c \) are positive absolute constants. (One may take \( c = \frac{1}{24} \)).

12. Proof of Theorem 1.2 and its Refinement

In the proof of Theorem 1.2, we apply the general bound (3.1) of Proposition 3.1 to the modified densities \( \tilde{p}_n \) constructed under the same constraints \( m_0 = 3 \) and \( 6 \leq N \leq \frac{1}{12\sigma^2} \), as in the proof of Theorem 1.1. It then gives

\[
\tilde{D}_n \leq \alpha^2 + 4 \left( \| \tilde{f}_n - g_\alpha \|_2 + \| (\tilde{f}_n)' - g_\alpha' \|_2 \right),
\]

where \( \tilde{D}_n \) is the relative entropy of \( \tilde{F}_n \) with respect to \( \Phi \) and

\[
g_\alpha(t) = g(t) \left( 1 + \alpha \frac{(it)^3}{3!} \right), \quad \alpha = \sum_{k=1}^{n} E X_k^3.
\]

As we know from Proposition 9.1, \( \tilde{D}_n \) provides a good approximation for the entropic distance \( D_n = D(S_n) \), namely

\[
|\tilde{D}_n - D_n| < 2^{-(N-6)} N^4 (D + 1).
\]

Hence,

\[
D_n \leq \alpha^2 + 4 \left( \| \tilde{f}_n - g_\alpha \|_2 + \| (\tilde{f}_n)' - g_\alpha' \|_2 \right) + 2^{-(N-6)} N^4 (D + 1). \tag{12.1}
\]

On the other hand, the closeness of \( f_n \) and \( g_\alpha \) on relatively large intervals is provided by:

Lemma 12.1. Assume \( L_4 \leq 1 \). Up to an absolute constant \( A \), in the interval \(|t| \leq L_4^{-1/6} \) we have

\[
|f_n(t) - g_\alpha(t)| \leq AL_4 e^{-t^2/4}, \tag{12.2}
\]

and similarly for the first four derivatives of \( f_n - g_\alpha \).

Again, we refer to [BR-R], where one can find several variants of such bounds. We also use the following elementary relations, cf. e.g. [Pe].

Lemma 12.2. \( \alpha^2 \leq L_3^3 \leq L_4 \).

Now, assume that \( L_4 \leq 1 \). To estimate the \( L^2 \)-norms in (12.1), again write

\[
\|f_n(t) - g_\alpha(t)\|_2 \leq \int_{|t| \leq \sqrt{N}} |f_n(t) - g_\alpha(t)|^2 dt \\
+ 2 \int_{|t| > \sqrt{N}} |f_n(t)|^2 dt + 2 \int_{|t| > \sqrt{N}} |g_\alpha(t)|^2 dt. \tag{12.3}
\]
Using $|\tilde{f}_n(t) - f_n(t)| \leq 2^{-(N-2)} N^3$ and the inequality (12.2) with $|t| \leq \sqrt{N} \leq L_4^{-1/6}$, we have
\[
\int_{|t| \leq \sqrt{N}} |\tilde{f}_n(t) - g_\alpha(t)|^2 dt \leq 2 \int_{|t| \leq \sqrt{N}} |\tilde{f}_n(t) - f_n(t)|^2 dt + 2 \int_{|t| \leq \sqrt{N}} |f_n(t) - g_\alpha(t)|^2 dt
\]
\[
\leq AL_4^2 + 2^{-(2N-5)} N^{7/2}
\]
(12.4) with some absolute constant $A$.

The middle integral on the right-hand side of (12.3) has been already estimated in (11.5). In addition, using $\theta^g(t) \leq 6^3/e^3$, we have
\[
|g_\alpha(t)|^2 = g(t)^2 \left(1 + \alpha^2 \frac{t^6}{36}\right) < (1 + \alpha^2) g(t) \leq 2 g(t),
\]
where we applied Lemma 12.2 together with the assumption $L_4 \leq 1$ (so that $|\alpha| \leq 1$). Hence,
\[
\int_{|t| > \sqrt{N}} |g_\alpha(t)|^2 dt < 2 \int_{|t| > \sqrt{N}} e^{-t^2/2} dt < 2 e^{-N/2}.
\]

One may combine this bound with (11.5) and (12.4), and then (12.3) gives
\[
||\tilde{f}_n - g_\alpha||_2^2 \leq AL_4^2 + 2^{-(2N-5)} N^{7/2} + C\sqrt{N} e^{-cN} + 4 e^{-N/2}
\]
with $C = e^{2D+4}$ and $c = c_0 e^{-12D}$ as in (11.5), where $c_0$ is an absolute positive constant. Since one may always choose $c_0 \leq \frac{1}{2}$, the above inequality may be simplified as
\[
||\tilde{f}_n - g_\alpha||_2 \leq AL_4 + CN^{1/4} e^{-cN}
\]
with some absolute constant $A$ and $D$-dependent constants $C = C_0 e^{2D}$ and $c = c_0 e^{-12D}$.

By a similar analysis based on the application of Corollary 8.4 and Proposition 10.2 with $s = 3$ (cf. (10.10)), we also have an analogous estimate
\[
||\tilde{f}_n''' - g_\alpha'''||_2 \leq AL_4 + CN^{7/4} e^{-cN}.
\]
Hence, (12.1) together with Lemma 12.2 yields
\[
D_n \leq AL_4 + CN^{7/4} e^{-cN},
\]
(12.5) where $A$ is absolute, and $C = C_0 e^{2D}$ and $c = c_0 e^{-12D}$, as before. The obtained estimate holds true, as long as $6 \leq N \leq \frac{1}{2x^3}$ and $\sqrt{N} \leq L_4^{-1/6}$ with $L_4 \leq 1$.

**Proof of Theorem 1.2.** The last condition, $\sqrt{N} \leq L_4^{-1/6}$, is satisfied for $N = \lfloor \frac{1}{2} L_4^{-1/3} \rfloor$. Then, by the elementary bound $\sigma \leq L_4^{1/4}$, we also have $N \leq \frac{1}{2x^3}$. The condition $N \geq 6$ restricts us to smaller values of $L_4$. If, for example, $L_4 \leq 4^{-6}$, we have $\frac{1}{2} L_4^{-1/3} \geq 8$ and thus $N \geq 8$.

Thus, if $L_4 \leq 4^{-6}$, then (12.5) holds true. But, since $N \geq \frac{1}{4} L_4^{-1/3}$, the last term in (12.5) is dominated by any power of $L_4$. In particular, using $e^x \geq c_1 x^5$ ($x \geq 0$), we get
\[
N^2 e^{-cN} \leq \frac{1}{c_1 c^3} N^{-3} \leq \frac{4^5}{c_1 c^3} L_4 = \frac{4^5}{c_1 c_0} e^{60D} L_4.
\]
Hence, (12.5) yields
\[
D_n \leq CL_4
\]
(12.6)
with \( C = C_1 e^{2D} e^{60D} = C_1 e^{62D} \), where \( C_1 \) is an absolute constant.

Finally, for \( L_4 > 4^{-6} \), one may use the relation \( D_n \leq D \) (according to the entropy power inequality), which shows that (12.6) holds with \( C = 4^6D \).

Thus, Theorem 1.2 is proved.

Now, again assume, as above, that \( L_4 \leq 4^{-6} \) and take \( N = \left\lfloor \frac{1}{2} L_4^{-1/3} \right\rfloor \). If \( D \leq \frac{1}{48} \log \frac{1}{L_4} \), then \( cN \geq c_0 L_4^{1/4} \frac{1}{2} L_4^{-1/3} = c_0 L_4^{-1/12} \) and \( C = C_0 e^{2D} \leq C_0 L_4^{-1/24} \). Hence,

\[
CN^{7/4} e^{-cN} \leq C_0 L_4^{-1/24} \cdot L_4^{-7/12} \exp \left\{ -c_0 L_4^{-1/12} \right\} \leq C_0' L_4
\]

with some absolute constant \( C_0' \). As a result, (12.5) yields \( D_n \leq (A + C_0') L_4 \). If \( L_4 > 4^{-6} \), (12.6) holds with \( C = 4^6 \), and we arrive at another variant of Theorem 1.2.

**Theorem 12.3.** Assume that independent random variables \( X_k \) have mean zero and finite fourth absolute moments. If they satisfy \( D(X_k) \leq c \log \frac{1}{L_4} \) \((1 \leq k \leq n)\), then

\[
D(S_n) \leq C L_4,
\]

where \( C \) and \( c \) are certain positive absolute constants. (One may take \( c = 1/48 \).)

Let us illustrate this result in the scheme of weighted sums

\[
S_n = a_1 X_1 + \cdots + a_n X_n
\]

of independent identically distributed random variables \( X_k \), such that \( \mathbb{E}X_1 = 0, \quad \mathbb{E}X_1^2 = 1 \), and with coefficients such that \( a_1^2 + \cdots + a_n^2 = 1 \). In this case \( L_4 = \mathbb{E}X_1^4 \sum_{k=1}^n a_k^4 \), so Theorem 12.3 is applicable, when the last sum is sufficiently small.

**Corollary 12.4.** Assume that \( X_1 \) has density with finite entropy, and let \( \mathbb{E}X_1^4 < +\infty \). If the coefficients satisfy

\[
\sum_{k=1}^n a_k^4 \leq \frac{1}{\mathbb{E}X_1^4} e^{-cD(X_1)},
\]

then

\[
D(S_n) \leq C \mathbb{E}X_1^4 \sum_{k=1}^n a_k^4,
\]

where \( C \) and \( c \) are positive absolute constants. (One may take \( c = 48 \).)

For example, in case of equal coefficients, so that \( S_n = \frac{X_1 + \cdots + X_n}{\sqrt{n}} \), the conclusion becomes

\[
D(S_n) \leq \frac{C}{n} \mathbb{E}X_1^4,
\]

for all \( n \geq n_1 \),

which holds true with an absolute constant \( C \) and \( n_1 = e^{48D(X_1)} \mathbb{E}X_1^4 \).
13. The Case of Bounded Densities

In this Section we give a few remarks about Theorems 1.1-1.2 for the case, where the densities of summands $X_k$ are bounded.

First, let us note that, if a random variable $X$ has an absolutely continuous distribution with a bounded density $p(x) \leq M$, where $M$ is a constant, and if the variance $\sigma^2 = \text{Var}(X)$ is finite ($\sigma > 0$), then $X$ has finite entropy, and moreover,

$$D(X) \leq \log (M\sigma\sqrt{2\pi e}).$$

(13.1)

Indeed, if $Z$ is a standard normal random variable, and assuming (without loss of generality) that $\sigma = 1$, we have

$$D(X) = h(Z) - h(X) = \log (\sqrt{2\pi e}) + \int_{-\infty}^{+\infty} p(x) \log p(x) \, dx,$$

which immediately implies (13.1).

It is worthwhile also noticing that, similarly to $D$, the functional $X \to M\sigma$ is affine invariant, where $M = \text{ess sup}_x p(x)$. Therefore, $M\sigma$ does not depend neither on the mean or the variance of $X$. In addition, one always has $M\sigma \geq \frac{1}{\sqrt{12}}$, and the equality is achieved only for $X$ which is uniformly distributed in a finite interval of the real line. (Without proof this lower bound is already mentioned in [St].)

Using (13.1), Theorems 1.1 and 1.2 admit formulations involving maximum of densities. In the statement below, let $(X_k)_{1 \leq k \leq n}$ be independent random variables with mean zero and variances $\sigma_k^2 = \text{Var}(X_k) (\sigma_k > 0)$, such that $\sum_{k=1}^{n} \sigma_k^2 = 1$. Let $F_n$ be the distribution function of the sum $S_n = X_1 + \cdots + X_n$.

**Corollary 13.1.** Assume that every $X_k$ has density bounded by $M_k$. If $\max_k M_k \sigma_k \leq \tilde{D}$, then

$$\|F_n - \Phi\|_{\text{TV}} \leq CL_3,$$

(13.2)

where the constant $C$ depends on $\tilde{D}$, only. Moreover,

$$D(S_n) \leq CL_4.$$

(13.3)

Moreover, one may take $C = C_0 \tilde{D}^c$ with some positive absolute constants $C_0$ and $c$.

In particular, consider the weighted sums

$$S_n = a_1 X_1 + \cdots + a_n X_n$$

of independent identically distributed random variables $X_k$, such that $\mathbb{E} X_1 = 0, \mathbb{E} X_1^2 = 1$, and with coefficients satisfying $a_1^2 + \cdots + a_n^2 = 1$. If $X_1$ has density, bounded by $M$, (13.2)-(13.3) yield respectively

$$\|F_n - \Phi\|_{\text{TV}} \leq C_M \mathbb{E} |X_1|^3 \sum_{k=1}^{n} |a_k|^3, \quad D(S_n) \leq C_M \mathbb{E} X_1^4 \sum_{k=1}^{n} a_k^4,$$

where $C_M$ depends on $M$, only. (One may take $C_M = C_0 M^c$).
Moreover, in the i.i.d. case, where $S_n = \frac{X_1 + \ldots + X_n}{\sqrt{n}}$, the last bound may also be written with an absolute constant $C$, i.e.,

$$D(S_n) \leq \frac{C}{n} \mathbb{E}X_1^4,$$

for all $n \geq n_1$.

Here one may take $n_1 = (M\sqrt{2\pi e})^{48} \mathbb{E}X_1^4$.

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