The theory of closed 1-forms, levels of quasiperiodic functions and transport phenomena in electron systems.

A.Ya. Maltsev¹, S.P. Novikov¹,²

¹ L.D. Landau Institute for Theoretical Physics of Russian Academy of Sciences
142432 Chernogolovka, pr. Ak. Semenova 1A
² V.A. Steklov Mathematical Institute of Russian Academy of Sciences
119991 Moscow, Gubkina str. 8

Abstract

The paper is devoted to the applications of the theory of dynamical systems to the theory of transport phenomena in metals in the presence of strong magnetic fields. More precisely, we consider the connection between the geometry of the trajectories of dynamical systems, arising at the Fermi surface in the presence of an external magnetic field, and the behavior of the conductivity tensor in a metal in the limit $\omega_B \tau \to \infty$. The paper contains a description of the history of the question and the investigation of special features of such behavior in the case of the appearance of trajectories of the most complex type on the Fermi surface of a metal.

1 Introduction.

This paper is devoted to one of the most important applications of the theory of dynamical systems on manifolds, namely, the description of transport phenomena in electron systems in condensed matter physics. The most important in this case is consideration of electron transport phenomena in the theory of normal metals, where arbitrarily complex dispersion laws that describe dynamics of electrons in a crystal lattice in the presence of external fields can arise. Thus, in particular, the application of a strong magnetic field generates an extremely nontrivial dynamics of electron states in a metal, determined by features of the dispersion relation (the dependence of the energy on the quasimomentum) for a given metal. The features of the corresponding dynamical system, in their own turn, determine the transport properties of the electron gas in the presence of strong magnetic fields, observed experimentally. The dynamical system, arising in the space of electron states in the presence of a magnetic field, can be considered here as a Hamiltonian system with a complex Hamiltonian $\epsilon(p)$ and a special Poisson bracket defined by applied magnetic field. Another important feature of this system is that its phase space represents a three-dimensional torus $\mathbb{T}^3$ but not Euclidean space. As can be shown, the theory of such systems is closely related both to the theory of foliations generated by 1-forms on manifolds, and with the theory of quasiperiodic functions on the plane. Let us note also here that the relationship between the geometry of the trajectories of these dynamical systems and electron transport properties in normal metals was first discovered by the school of I.M. Lifshitz in the 1950s.
The problem of the complete classification of the trajectories of dynamical systems, arising in the space of electron states in the presence of a magnetic field for an arbitrary dispersion law, was first set by S.P. Novikov in the early 1980s and then intensively studied in his topological school (S.P. Novikov, A.V. Zorich, S.P. Tsarev, I.A. Dynnikov). The study of this problem led to the appearance of a number of deep topological theorems that formed the foundation for the topological classification of different trajectories that may arise in systems of this type. In particular, an extremely important part of the classification was the description of geometric properties of stable nonclosed trajectories of such systems (A.V. Zorich, I.A. Dynnikov), which provides a basis for a description of stable nontrivial regimes of behavior of magneto-conductivity in normal metals. Another extremely important achievement was the discovery of two different types of unstable open trajectories in the described systems, which reveal much more complicated (chaotic) behavior in comparison with the stable trajectories (S.P. Tsarev, I.A. Dynnikov).

The appearance of new topological results for systems describing quasiclassical dynamics of electron states in crystals has played an important role in the further development of the theory of transport phenomena in metals in the presence of strong magnetic fields. Thus, the investigation of galvano-magnetic phenomena at the presence of stable open trajectories of these systems on the Fermi surface allowed to introduce new topological characteristics (topological quantum numbers) observed in the conductivity of normal metals (S.P. Novikov, A.Ya. Maltsev). At the same time, the study of galvano-magnetic phenomena in metals that allow the appearance of chaotic trajectories on the Fermi surface, makes it possible to observe special regimes in the behavior of the magneto-conductivity in strong magnetic fields, which were not considered before. In addition, it should also be noted that the topological description of different types of trajectories of systems describing the dynamics of electron states in crystals provides opportunities for a more accurate description of the behavior of the magneto-conductivity from the analytical point of view (A.Ya. Maltsev). Let us note also that the obtained results are applicable not only to galvano-magnetic phenomena but also to many other transport phenomena (electron thermal conductivity, etc.) in normal metals.

It must be said that the physical applications of the Novikov problem are not limited in reality only to the theory of normal metals, but are related also to transport phenomena in other systems. Among such systems one can particularly distinguish two-dimensional electron systems placed into artificially created quasiperiodic “superpotentials” and actively investigated in experimental physics at the present time. The Novikov problem can be considered here either as the problem of describing the geometry of trajectories of special dynamical systems (of different dimensions) or as a problem of describing the level lines of a quasiperiodic function on a plane with an arbitrary number of quasiperiods. Thus, in particular, the topological results obtained for systems that describe the quasiclassical dynamics of electron states in three-dimensional crystals also make it possible to describe in the semiclassical approximation all the main features of transport phenomena in a two-dimensional electron gas in potentials with three quasiperiods in strong magnetic fields (orthogonal to the plane of the sample). The study of the Novikov problem with a larger number of quasiperiods is in fact a rather complicated problem. Nevertheless, for the present moment there is a description of an important class of potentials with four quasiperiods whose non-closed level lines have a “regular form” (lie in a straight strip of a finite width and pass it through) and can be characterized by special topological numbers (S.P. Novikov, I.A. Dynnikov).

In this paper we will mainly discuss applications of the dynamical systems theory to the theory of galvano-magnetic phenomena in metals in sufficiently strong magnetic fields. More precisely, we will mainly consider the behavior of the conductivity tensor in the presence of chaotic trajectories
on the Fermi surface. Our results here will be based on a number of properties of such trajectories, established as a result of intensive study of the corresponding dynamical systems in the most recent time. In particular, we will show that the asymptotic Zorich - Kontsevich - Forni indices, defined for certain classes of dynamical systems on two-dimensional surfaces, have a direct relation to the behavior of the electric conductivity tensor in strong magnetic fields.

In the next chapter we will give a more detailed description of the connection between the theory of the galvano-magnetic phenomena in metals with the theory of dynamical systems on manifolds. In Chapter 3 we will consider the behavior of the tensor of electric conductivity in the presence of complex electron trajectories on the Fermi surface.

2 Theory of transport phenomena in metals and dynamical systems on manifolds.

One of the most important applications of the theory of closed 1-forms on two-dimensional surfaces is the investigation of the geometry of quasiclassical electron trajectories in metals with a complex Fermi surface in the presence of an external magnetic field. Such situation is due, first of all, to the peculiarities of the space of the electron states for a fixed conduction band, representing a three-dimensional torus $\mathbb{T}^3$ from the topological point of view.

More precisely, the electron states in a single crystal are described by solutions of the stationary Schrödinger equation

$$-\frac{\hbar^2}{2m} \Delta \psi + U(x,y,z) \psi = E \psi , \quad (2.1)$$

where the potential $U(r) = U(x,y,z)$ is a periodic function in $\mathbb{R}^3$ with three independent periods $l_1, l_2, l_3$:

$$U(r + l_1) \equiv U(r + l_2) \equiv U(r + l_3) \equiv U(r)$$

The physical states of electrons in a crystal are given by bounded solutions of the equation (2.1), which can be chosen in the form of the Bloch functions $\psi_p(r)$, satisfying the conditions

$$\psi_p(r + l_1) \equiv e^{i(p_1l_1)/\hbar} \psi_p(r) , \quad \psi_p(r + l_2) \equiv e^{i(p_2l_2)/\hbar} \psi_p(r) , \quad \psi_p(r + l_3) \equiv e^{i(p_3l_3)/\hbar} \psi_p(r)$$

The value $p = (p_1,p_2,p_3)$ is called the quasimomentum of an electron state and completely determines this state for a fixed allowed energy band.$^1$

It is easy to see that the quantity $p$ is defined modulo the vectors

$$m_1 a_1 + m_2 a_2 + m_3 a_3 , \quad m_1,m_2,m_3 \in \mathbb{Z} , \quad (2.2)$$

where the vectors $a_1, a_2, a_3$ are determined by the relations

$$a_1 = 2\pi\hbar \frac{l_2 \times l_3}{(l_1,l_2,l_3)} , \quad a_2 = 2\pi\hbar \frac{l_3 \times l_1}{(l_1,l_2,l_3)} , \quad a_3 = 2\pi\hbar \frac{l_1 \times l_2}{(l_1,l_2,l_3)}$$

$^1$In fact, real electron states have also a spin degeneracy, which doubles the number of states in a real crystal. Since for us this degeneracy does not play a significant role, we will not consider it here in detail.
The vectors \( \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \) give a basis of the so-called reciprocal lattice of a crystal that determines the geometry of the space of the electron states. The vectors (2.2) form a complete set of the reciprocal lattice vectors \( L^* \) in the \( \mathbf{p} \)-space.

Any two values of \( \mathbf{p} \), that differ by a reciprocal lattice vector, determine the same physical electron state (and the same solution of the equation (2.1)) for a fixed energy band. Consequently, we can state that the space of the electron states for a fixed energy band represents a three-dimensional torus \( \mathbb{T}^3 \):

\[
\mathbb{T}^3 = \mathbb{R}^3 / L^*,
\]

obtained from the \( \mathbf{p} \)-space by the factorization over the vectors of the reciprocal lattice.

The energy value \( E = \epsilon(\mathbf{p}) \) for a solution of the equation (2.1) in every band is completely defined by the value of \( \mathbf{p} \) and represents a smooth periodic function of \( \mathbf{p} \) with the periods \( \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \). It is easy to see that this function can also be regarded as a smooth function on the torus \( \mathbb{T}^3 \) defined above. In general case, we have an infinite number of allowed energy bands, so we have an infinite number of different smooth functions \( E = \epsilon_s(\mathbf{p}) \), \( s = 1, 2, \ldots \), that take bounded values

\[
\epsilon_s^{\text{min}} \leq \epsilon_s(\mathbf{p}) \leq \epsilon_s^{\text{max}}
\]

It must be said that in the case of a three-dimensional crystal the energy intervals \([\epsilon_s^{\text{min}}, \epsilon_s^{\text{max}}]\), in general, overlap with each other, so it may be more rigorous to talk about different branches \( \epsilon_s(\mathbf{p}) \) of the energy spectrum in a crystal.

The general picture of the electron structure in a normal metal can be represented as follows:

The electron gas is highly degenerate, so we can assume that all the electron states with energies less than \( \epsilon_F \) (the Fermi energy) are occupied and all the electron states with energies larger than \( \epsilon_F \) are empty.

The Fermi level \( \epsilon_F \) belongs to the interval \((\epsilon_s^{\text{min}}, \epsilon_s^{\text{max}})\), defined by one of the branches of the energy spectrum in the crystal (or several such intervals), so that the equation

\[
\epsilon_s(\mathbf{p}) = \epsilon_F
\]

defines a two-dimensional periodic surface in the \( \mathbf{p} \)-space. The surface defined by equation (2.3) is called the Fermi surface of a metal.

It is easy to see that under the condition

\[
\nabla \epsilon_s|_{\epsilon_F} \neq 0
\]

the equation (2.3) defines a smooth closed surface in \( \mathbb{T}^3 \) after the factorization over the reciprocal lattice vectors. We note here that we do not require that the surface, given by the equation (2.3) in \( \mathbb{T}^3 \), was connected. Thus, for many normal metals the Fermi surface consists of several connected components. For us, however, it will be important that different connected components of the Fermi surface do not intersect each other. Let us note that the last property, as a rule, is observed also in the case when the full Fermi surface is given by the union of the surfaces (2.3) for several branches of the energy spectrum. In the latter case, however, there can be exceptions to this rule due to presence of symmetries of a special type in crystal lattice. Here we will always assume that the Fermi surface is given by a set of smooth disjoint components embedded in \( \mathbb{T}^3 \).

With more precise consideration, taking into account the finite temperatures, the described above electron distribution with respect to the electron states must be replaced by a temperature-dependent
distribution function having the form:

\[ n(p) = \frac{1}{e^{(\epsilon(p)-\epsilon_F)/T} + 1} \quad (2.4) \]

In real metals, however, as a rule, we have the relation \( T \ll \epsilon_F \), so that the function \( (2.4) \) undergoes substantial changes only near the surface \( (2.3) \).

In the presence of external (constant) electric and magnetic fields the evolution of the electron states within one energy band can be described by a quasiclassical system describing the change of the quasimomentum \( p \) with time (see e.g. [1, 21, 42])

\[ \dot{p} = \frac{e}{c} [v_{gr} \times B] + eE = \frac{e}{c} [\nabla \epsilon(p) \times B] + eE \quad (2.5) \]

When describing galvanomagnetic phenomena in strong magnetic fields the electric field is assumed to be small, while the value of \( B \) must satisfy the condition of the geometric limit \( \omega_B \tau \gg 1 \), where \( \omega_B \) is the cyclotron frequency and \( \tau \) is the mean free time of electrons in the metal. The magnitude of the electric field can be considered in this case as a small correction giving a small perturbation of the system

\[ \dot{p} = \frac{e}{c} [\nabla \epsilon(p) \times B] \]

for the evolution of the electron states in the presence of an external magnetic field.

The system \( (2.5) \) is an analytically integrable system in the \( p \)-space whose trajectories are given by the intersections of surfaces of constant energy \( \epsilon(p) = \text{const} \) with planes, orthogonal to the magnetic field. It is not difficult to see, however, that this circumstance does not allow to obtain a direct description of the global geometry of trajectories of \( (2.5) \), since the \( p \)-space is not compact, while in the torus \( T^3 \) the energy \( \epsilon(p) \) gives the only single-valued integral of the system \( (2.5) \). The geometry of the trajectories of \( (2.5) \), thus, is determined by levels of the 1-form \( (dp, B) \), restricted to the compact two-dimensional surfaces \( \epsilon(p) = \text{const} \) in the torus \( T^3 \).

It is also easy to see that the system \( (2.5) \) preserves also the volume element \( d^3p \), which in fact is a consequence of the Hamiltonian property of this system with respect to the Poisson bracket

\[ \{p_1, p_2\} = \frac{e}{c} B^3, \quad \{p_2, p_3\} = \frac{e}{c} B^1, \quad \{p_3, p_1\} = \frac{e}{c} B^2 \]

As a consequence of this fact, the dynamical system \( (2.5) \) does not change the equilibrium electron distribution. Nevertheless, the behavior of trajectories of system \( (2.5) \) on the Fermi surface has a significant effect on the behavior of conductivity in strong magnetic fields, determined by the linear response of the electron system to a small electric field.

The influence of the geometry of trajectories of system \( (2.5) \) on the behavior of conductivity in strong magnetic fields was first discovered by the school of I.M. Lifshitz in the 1950s. Thus, in the work [22] an essentially different behavior of conductivity in the plane orthogonal to \( B \) in the case when the Fermi surface contains only closed trajectories (Fig. 1 a) and in the case when it contains open periodic trajectories in the \( p \)-space (Fig. 1 b) was indicated.

Let us agree here that the coordinates in the physical space are chosen in such a way that the axis \( z \) is directed along the magnetic field \( B \). In this case, the asymptotic behavior of the conductivity tensor in strong magnetic fields in the case of presence of closed trajectories only can be expressed by the formula

\[ \sigma^{ik} \approx \frac{ne^2\tau}{m^*} \left( \begin{array}{ccc} (\omega_B \tau)^{-2} & (\omega_B \tau)^{-1} & (\omega_B \tau)^{-1} \\ (\omega_B \tau)^{-1} & (\omega_B \tau)^{-2} & (\omega_B \tau)^{-1} \\ (\omega_B \tau)^{-1} & (\omega_B \tau)^{-1} & \ast \end{array} \right), \quad \omega_B \tau \to \infty \quad (2.6) \]
Figure 1: The Fermi surfaces, containing only closed (a) and open periodic trajectories (b) in the $p$-space.

The value $n$ represents here the concentration of the conductivity electrons and $m^*$ represents the effective mass of the electron in the crystal. The value $\omega_B$ is defined by the relation $\omega_B = eB/m^*c$. Here and everywhere we use the symbol $*$ to denote just some dimensionless constant of the order of unity.

In the case of the presence of open periodic trajectories on the Fermi surface it is convenient to choose the $x$-axis in the direction of the periodic trajectories in the $p$-space. Note, that the quasiclassical trajectories in the $x$-space have such form that their projections onto the plane orthogonal to $B$ can be obtained from the trajectories in the $p$-space by rotation to 90°. The asymptotic behavior of the conductivity tensor in strong magnetic fields is expressed in this case by the formula

$$\sigma^{ik} \simeq \frac{ne^2\tau}{m^*} \begin{pmatrix} (\omega_B\tau)^{-2} & (\omega_B\tau)^{-1} & (\omega_B\tau)^{-1} \\ (\omega_B\tau)^{-1} & * & * \\ (\omega_B\tau)^{-1} & * & * \end{pmatrix}, \quad \omega_B\tau \to \infty$$

As can be easily seen, in the second case the conductivity is characterized by a strong anisotropy in the plane orthogonal to $B$, which allows to determine the mean direction of the open trajectories experimentally.

In the works [23, 24] open trajectories of more general form on the Fermi surfaces of different shapes were considered. The trajectories studied in [23, 24] are not periodic, nevertheless, they also have a mean direction in $p$-space, which also leads to a strong anisotropy of the conductivity tensor in the plane, orthogonal to $B$.

The works [25, 26, 27], and also the book [28], give a review of a wide range of questions of the electron theory of metals, and, in particular, of questions related to the geometry of open trajectories of the system (2.5), considered in that period. We would also like to refer here to the work [20], containing a return to this class of problems, which also includes consideration of aspects that arose in a later period.

The general problem of describing the geometry of open trajectories of the system (2.5) with
an arbitrary dispersion law $\epsilon(p)$ was set by S.P. Novikov (35) and was actively explored in his topological school in recent decades. The detailed study of the system (2.5) led to the full classification of possible types of trajectories of this system, which allowed also to describe all possible types of asymptotic regimes of behavior of electric conductivity in strong magnetic fields.

The most important part of the classification of open trajectories of system (2.5) is a description of stable open trajectories of this system that have remarkable topological properties. These properties of open trajectories of system (2.5) are determined by the topology of the carriers of such trajectories on the level surfaces $\epsilon(p) = \text{const}$ (in particular, on the Fermi surface), which can be defined as connected components of such a surface, obtained after removing of all closed trajectories of the system (2.5). We would like to note here that the main results in this direction were obtained in the works [43, 12] which provide a basis for the description of stable open trajectories of the system (2.5).

In particular, in the work [43] the following important statement about open trajectories of the system (2.5) on generic Fermi surfaces was proved:

For any rational direction of $B$ there exists a neighborhood of this direction such that for any direction of $B$ from this neighborhood the carriers of open trajectories of the system (2.5) on the level surface $\epsilon(p) = \epsilon_0$ (if they exist) represent two-dimensional tori $T^2$ (possibly with holes cut out), embedded in $T^3$.

In the work [12] a similar assertion was proved for open trajectories of the system (2.5), which are stable with respect to variations of the energy level $\epsilon_0$. Thus, according to [12], if open trajectories of the system (2.5) exist at energy levels $\epsilon(p) = \text{const}$ in some neighborhood of the energy value $\epsilon_0$, then the carriers of such trajectories are also two-dimensional tori $T^2$ (possibly with holes cut out), embedded in the torus $T^3$.

The covering carriers of open trajectories in the $p$-space represent periodically deformed integral (i.e., generated by two vectors of the reciprocal lattice) planes with flat holes, orthogonal to $B$ (Fig. 2). The corresponding open trajectories are given in this case by intersections of such planes with planes orthogonal to the magnetic field and represent quasiperiodic curves in the $p$-space. In both above cases for generic directions of $B$, the two-dimensional tori $T^2$, and also their coverings, are locally stable with respect to small rotations of the direction of $B$ and with respect to small variations of the value $\epsilon_0$.

Based on the above description of stable open trajectories of system (2.5), we can specify the following properties of such trajectories which play an important role in the theory of galvanomagnetic phenomena in metals:

1) Every stable open trajectory in $p$-space lies in a straight strip of a finite width in the plane, orthogonal to $B$, passing through it ([11]) (Fig. 3);

2) The mean direction of the open trajectories in the $p$-space is given by the intersection of the plane, orthogonal to $B$, with some integral plane, which is the same for a fixed “Stability Zone” in the space of directions of $B$.

Let us note here that the property (1) was originally formulated by S.P. Novikov in the form of a conjecture, which, thus, was proved for stable open trajectories in the works [43, 11] and [12]. Note also that, as follows from [43] and [12], it is enough to require the stability of trajectories with respect to either small rotations of the direction of $B$ or to small variations of the energy value $\epsilon_0$.

Properties (1) and (2) of stable (regular) open trajectories of the system (2.5) are extremely important in the theory of galvanomagnetic phenomena in normal metals and served as a basis for
Figure 2: A carrier of topologically regular open trajectories in $p$-space.

the introduction in [36] of important topological characteristics of the electron spectrum in a metal observable in the study of the electric conductivity. Thus, the presence of the property (1) for stable open trajectories leads to strong anisotropy of conductivity in the plane orthogonal to $B$, observable experimentally in the limit $\omega_B \tau \gg 1$. The direction of the greatest suppression of conductivity (in $x$-space) coincides with the mean direction of the open trajectories in $p$-space and is, therefore, observable experimentally. The property (2), together with the property of local stability of carriers of open trajectories, allows us to determine the homological class of the embedding $T^2 \subset T^3$, given by an irreducible triple of integers $(m_1, m_2, m_3)$. The numbers $(m_1, m_2, m_3)$ were named in [36] the topological quantum numbers observable in the conductivity of normal metals and represent (together with the form of the corresponding Stability Zone in the space of directions of $B$) important characteristics of the electron spectrum in a metal.

Let us note here that the topological quantum numbers characterize geometric properties of the electric conductivity tensor and are well-observable quantities in the experiment. At the same time, determination of the shape of a Stability Zone may require more precise experimental studies, in particular, the experimentally observable Stability Zone may differ from the exact mathematical Stability Zone. We must also say, that the analytic properties of the conductivity tensor in the experimentally observed Stability Zone can also be quite nontrivial (see e.g. [33, 34]).

It is interesting to note that the theorem of [43] admits a generalization to the case of functions on the plane with four quasiperiods. In this formulation trajectories of the system (2.5) in the $p$-space can be considered as the level lines of a function on the plane having three quasiperiods, i.e. the function obtained by restriction of a periodic function in three-dimensional space on a plane embedded in $\mathbb{R}^3$. Similarly, one can consider the level lines of functions on the plane having four quasiperiods, i.e. functions obtained by restriction of a 4-periodic function on the plane embedded in $\mathbb{R}^4$. It is natural to call here the plane integral if it is generated by two independent periods of the corresponding periodic function in $\mathbb{R}^4$. As can be shown (see [38, 19]), for noncompact level lines of generic quasiperiodic functions on the plane with four quasiperiods the following assertion is true:
Figure 3: A topologically regular trajectory lying in a strip of a finite width and passing through it.

For any integral embedding of a two-dimensional plane in a four-dimensional space there is an open neighborhood in the space of the directions of embedding $\mathbb{R}^2 \subset \mathbb{R}^4$, such that all the noncompact level lines of the corresponding quasiperiodic functions on the plane

1) lie in straight strips of finite widths, passing through them;

2) have a mean direction defined by the intersection of the embedded plane with some integral three-dimensional plane in $\mathbb{R}^4$, which is constant for this neighborhood.

Let us also note here that the description of the geometry of the level lines of quasiperiodic functions on the plane can play important role in the theory of transport phenomena in two-dimensional electron systems in the presence of an external “super-potential”, studied in modern experiments (see e.g. [32]).

For a description of the general picture that appears on the angular diagram, which describes the conductivity behavior as a function of the direction of $\mathbf{B}$, it is convenient to use the general description of behavior of open trajectories of (2.5), arising at all energy levels $\epsilon(p) = \text{const}$ for a fixed energy band ([15]).

According to [15], open trajectories of system (2.5) for a fixed direction of $\mathbf{B}$ always arise either in a connected energy interval $\epsilon_1(\mathbf{B}) \leq \epsilon \leq \epsilon_2(\mathbf{B})$, or only at one energy level $\epsilon = \epsilon_1(\mathbf{B}) = \epsilon_2(\mathbf{B})$. In the first case, the open trajectories of (2.5) have topologically regular form described above and the corresponding direction of $\mathbf{B}$ belongs in generic case to some Stability Zone defined for the entire energy spectrum $\epsilon(p)$. The corresponding Stability Zones represent regions with piecewise smooth boundaries on the unit sphere $S^2$ and form an everywhere dense set in the space of directions of $\mathbf{B}$. As shown in [15], the unit sphere $S^2$ can consist either of one Stability Zone for a certain
type of the spectrum \( \epsilon(p) \), or contain infinitely many such Zones. Within each Stability Zone the mean direction of regular open trajectories of the system (2.5) in the \( p \) - space is given by the intersection of the plane orthogonal to \( B \) with some integral plane, which is the same for a given Zone. We also note here that the practical calculation of the boundaries of the Stability Zones and the corresponding topological quantum numbers represents a non-trivial computational problem based on serious topological methods (see, e.g. [6]).

In the case of the existence of open trajectories of system (2.5) just at one energy level such trajectories can be characterized by both topologically regular and more complex chaotic behavior.

Coming back to the theory of galvanomagnetic phenomena in metals, we must now fix an energy level in the conduction band, given by the Fermi energy, and consider the behavior of the trajectories of system (2.5) only at this level. It is not difficult to see here that in this case open trajectories are present at the Fermi level either if the Fermi energy belongs to the interval \([\epsilon_1(p), \epsilon_2(p)]\) or coincides with a single energy level containing open trajectories for a given direction of \( B \). As a consequence, the Stability Zones for a fixed Fermi energy do not form anymore an everywhere dense set on the unit sphere \( S^2 \) and in general we have also regions, corresponding to the presence of only closed trajectories on the Fermi surface, at the angular diagram. In addition to the Stability Zones and the regions corresponding to the presence of only closed trajectories, the angular diagram can contain also special directions of \( B \), corresponding to chaotic behavior of open trajectories of system (2.5) on the Fermi surface.

The above picture gives a general description of the typical angular diagram for the magneto-conductivity of a metal with a complex Fermi surface. It must be said that in the general case the angular diagram can have some additional features for special directions of the magnetic field. The most detailed consideration of various opportunities for open trajectories that arise for a general dispersion law, can be found in [15]. A detailed consideration of transport phenomena connected with different behavior of the open trajectories of system (2.5) was presented in the papers [37, 30, 31]. Let us also note here that the angular diagram associated with the full dispersion law is possibly also available for the experimental research in semiconductor structures in very strong magnetic fields (see [14]).

3 The chaotic trajectories and the conductivity behavior in strong magnetic fields.

We want to stop now on a more detailed description of chaotic behavior of the trajectories of the system (2.5), which is also possible in the case of complex Fermi surfaces. As we have already said above, such trajectories can exist only on one energy level for a given direction of \( B \). We note at once that chaotic trajectories can naturally be divided into two different types (Tsarev type and Dynnikov type) corresponding to different “degrees of irrationality” of the direction of the magnetic field. Thus, the trajectories of the Tsarev type arise in the case of “partially irrational” directions of \( B \), such that the plane, orthogonal to \( B \), contains a reciprocal lattice vector. The Tsarev trajectory ([11]) was the first example of a trajectory characterized by explicit chaotic behavior on the Fermi surface in \( T^3 \). In the planes, orthogonal to \( B \) in the covering \( p \) - space, trajectories of the Tsarev type can not be enclosed in a straight line of a finite width and, from this point of view, also are not topologically regular. Nevertheless, the trajectories of the Tsarev type possess asymptotic direction
in planes orthogonal to $\mathbf{B}$, which also leads to strong anisotropy of the conductivity tensor in strong magnetic fields. It can be shown (see [13]) that the last property always takes place for chaotic trajectories in the case of “partially irrational” directions of $\mathbf{B}$.

The chaotic trajectories of Dynnikov type arise only for directions of $\mathbf{B}$ having maximal irrationality (the plane orthogonal to $\mathbf{B}$ does not contain reciprocal lattice vectors). The trajectories of this type are characterized by an explicit chaotic behavior both on the Fermi surface in $\mathbb{T}^3$ and in the covering $\mathbf{p}$ - space (Fig. 4). As a rule, each carrier of such trajectories represents a surface of genus 3 embedded in $\mathbb{T}^3$ with the covering of the “maximal rank” in the $\mathbf{p}$ - space. More precisely, the carriers of such trajectories (obtained after removal of closed trajectories from the Fermi surface) form closed surfaces of genus 3 (or more) after gluing the holes, arising after the removal of the cylinders of closed trajectories, by flat discs, orthogonal to $\mathbf{B}$. Let us also note that compact piecewise smooth surfaces obtained in this way can be regularized by an arbitrarily small perturbation to smooth surfaces, embedded in $\mathbb{T}^3$. It is easy to see also that the corresponding regularization of the system (2.5) does not change geometric properties of open trajectories in $\mathbf{p}$ - space on a large scale ([15]).

The set of directions of the magnetic field corresponding to the appearance of chaotic trajectories of the Dynnikov type at any of the energy levels represents a rather complex set (of the Cantor type)
on the angular diagram $S^2$. According to the conjecture of S.P. Novikov, the Hausdorff dimension of its subset on $S^2$, corresponding to a fixed Fermi surface of general position, is strictly less than 1. Let us also note here that different properties of the trajectories of this type are actively investigated in modern works (see e.g. [2, 3, 4, 5, 7, 8, 9, 10, 17, 18, 39, 40, 44]). The chaotic trajectories arising for directions of $B$ of maximal irrationality are the most complex and lead to the most nontrivial regimes in conductivity behavior in strong magnetic fields.

The behavior of the electric conductivity in strong magnetic fields in the presence of chaotic trajectories of Dynnikov type was considered in [29] and is characterized by significant differences from the conductivity behavior in the presence of only closed or topologically regular trajectories of system (2.5). One of the main differences in the conductivity behavior in this case is the suppression of the conductivity along the direction of the magnetic field for $\omega B \tau \gg 1$. Another important feature of the conductivity behavior in the presence of trajectories of this type is the appearance of fractional powers of the parameter $\omega B \tau$ in the asymptotics of the components of the conductivity tensor in the limit $\omega B \tau \rightarrow \infty$.

It should be noted that the introduction of fractional powers of the parameter $\omega B \tau$ in [29] in the description of the conductivity tensor was based on a special property of chaotic open trajectories (self-similarity) constructed in [13]. This property is connected with the behavior of these trajectories on a large scale and is represented by the existence of two directions in the plane orthogonal to $B$ and the corresponding scale coefficients $\lambda_1$, $\lambda_2$, such that after stretching of the plane along these directions with coefficients $\lambda_1$, $\lambda_2$ the new trajectories obtained can be reduced to the initial form by a finite deformation in the plane. The presence of such a remarkable property for the trajectories constructed in [13], leads, in particular, to the appearance of two fractional parameters observable in the asymptotics of the components of the conductivity tensor in the plane orthogonal to $B$. It must be said, however, that such a remarkable property is not observed in the general case for chaotic trajectories arising for directions of $B$ of maximal irrationality.

Here we want to show, however, that the appearance of fractional powers in the asymptotics of the components of the electric conductivity tensor is related in fact to a more general phenomenon in the theory of dynamical systems and should, apparently, have a general character in the presence of open trajectories of this type on the Fermi surface.

More precisely, we will show here that the fractional powers describing the asymptotic behavior of the components of the symmetric part of the conductivity tensor in the plane, orthogonal to $B$, in the limit of strong magnetic fields have a direct relation to the Zorich - Kontsevich - Forni indices, defined for dynamical systems on compact surfaces (see [44, 45, 46, 47, 48, 49]). Our consideration here will be based on the description of these characteristics, presented in the paper [48], and we will assume for simplicity that the carriers of open trajectories on the Fermi surface have genus 3. Let us note here, that the results of [48] are formulated as taking place for “almost all” foliations generated by closed 1-forms on compact surfaces. In particular, these results are valid for the level lines of $1$ - forms obtained from constant $1$ - forms for almost any embedding of a two-dimensional surface $M^2_g \rightarrow \mathbb{T}^N$ into a torus $\mathbb{T}^N$ of sufficiently large dimension. For the embeddings $M^2_g \rightarrow \mathbb{T}^3$, in general, the conditions that lead to the existence of the Zorich - Kontsevich - Forni indices for chaotic trajectories on the surface $M^2_g$, require additional justification. As an example of such a justification, we can point the paper [2], where the construction and investigation of chaotic trajectories on the Fermi surface of a sufficiently general form was carried out. We shall consider here the properties of the electric conductivity tensor in the limit $\omega B \tau \rightarrow \infty$ in the presence of chaotic trajectories on the Fermi surface, corresponding to certain Zorich - Kontsevich - Forni indices. We shall use
here the simplest kinetic model of the conductivity in crystals and represent the key points of the corresponding derivation on the “physical” level of rigorousness. Nevertheless, the general form of the relations represented here remains true also after moving to more accurate physical models that take into account all the features of the electron kinetics in real conductors. For greater convenience, we can assume that we always deal with a regularized foliation on a smooth surface $M_2^3$ of genus 3 whose open fibers have the same geometric properties as the open trajectories of the system \((2.5)\).

Following [48], we will now assume that the foliation generated by the 1-form \((dp, B)\) on the regularized carrier of open trajectories, has the following properties:

Consider a layer (level line) in general position and fix on it a starting point $P_0$. Fix on the same layer a point $P_1$, in which this layer comes close to $P_0$ after passing a sufficiently large path along the carrier of open trajectories. Connect the points $P_0$ and $P_1$ by a short segment and define, thus, a closed cycle on the carrier of open trajectories. Let us denote the homology class of the resulting cycle by $c_{P_0}(l)$, where $l$ is the length of the corresponding segment of the layer in some metric.

There is a flag of subspaces $V_1 \subset V_2 \subset V_3 \subset V \subset H_1(M_3^2; \mathbb{R})$, such that:

1) For any such layer $\gamma$ and any point $P_0 \in \gamma$

$$\lim_{l \to \infty} \frac{c_{P_0}(l)}{l} = c,$$

where the non-zero asymptotic cycle $c \in H_1(M_3^2; \mathbb{R})$ is proportional to the Poincare cycle and generates the subspace $V_1$.

2) For any linear form $\phi \in Ann(V_j) \subset H_1(M_3^2; \mathbb{R})$, $\phi \notin Ann(V_{j+1})$

$$\limsup_{l \to \infty} \frac{\log |\langle \phi, c_{P_0}(l) \rangle|}{\log l} = \nu_{j+1}, \quad j = 1, 2$$

3) For any $\phi \in Ann(V) \subset H_1(M_3^2; \mathbb{R})$, $||\phi|| = 1$

$$|\langle \phi, c_{P_0}(l) \rangle| \leq \text{const},$$

where the constant is determined only by foliation.

4) The subspace $V \subset H_1(M_3^2; \mathbb{R})$ is Lagrangian in homology, where the symplectic structure is determined by the intersection form.

5) Convergence to all the above limits is uniform in $\gamma$ and $P_0 \in \gamma$, i.e. depends only on $l$.

Let us note that in generic situation we have in this case $1 > \nu_2 > \nu_3 > 0$, $\dim V_2 = 2$, $\dim V_3 = \dim V = 3$.

Note also that the description presented in [48] is based on a sequence of representations of the corresponding minimal component (the carrier of chaotic trajectories) as a union of a fixed number of classes (layers) of the cycles described above, whose lengths tend to infinity. Cycles of one class are almost identical for each of these representations and cover a finite area of the corresponding minimal component.

For the description of the properties of the trajectories of system \((2.5)\) in $p$-space which we need let us consider the map in homology, induced by the embedding $M_3^2 \subset T^3$. It is not difficult
to see that the image of each space $V_j$ belongs to the two-dimensional space defined by the plane orthogonal to $B$. Besides that, in examples of chaotic trajectories of Dynnikov type there is no linear growth of the deviation from the point $P_0$ with increasing of the length of the trajectory in the plane orthogonal to $B$, which means that the image of the asymptotic cycle $c$ is equal to zero. The image of the space $V_2$ is one-dimensional and determines a selected direction in the plane, orthogonal to $B$, along which the mean deviation of a trajectory grows faster with its length ($\sim l^{\nu_2}$), than in the orthogonal direction. In the generic case we suppose also that the image of the space $V_3$ is two-dimensional and is given by the plane orthogonal to $B$. It is not hard to see that the 1-forms $dp_x$ and $dp_y$ give in this case a necessary basis of 1-forms, to which the statement (2) formulated above can be applied. Therefore, we can choose a coordinate system $(x,y,z)$ in such a way that for some frame sequences of the values $l_k$ we will have the relations

$$|\Delta p_x(l)| \approx p_F \left( \frac{l}{p_F} \right)^{\nu_2}, \quad |\Delta p_y(l)| \approx p_F \left( \frac{l}{p_F} \right)^{\nu_3}, \quad (3.2)$$

for the corresponding deviations of the trajectory along the coordinates $p_x$ and $p_y$ when passing a part of the above approximate cycles on the Fermi surface. The value $p_F$ represents here the Fermi momentum, approximately equal to the size of the Brillouin zone in the p-space.

Let us explain here what we really mean by the relations given above. The existence of the relations (3.1) means the existence for any $\delta > 0$ of certain frame sequences $\{l'_k\}, \{l''_k\}$, such that we have the relations

$$A'_k p_F \left( \frac{l'_k}{p_F} \right)^{\nu_2 - \delta} < |\Delta p_x(l)| \leq B'_k p_F \left( \frac{l'_k}{p_F} \right)^{\nu_2}, \quad (3.3)$$

$$A''_k p_F \left( \frac{l''_k}{p_F} \right)^{\nu_3 - \delta} < |\Delta p_x(l)| \leq B''_k p_F \left( \frac{l''_k}{p_F} \right)^{\nu_3}, \quad (3.4)$$

where the sequences $\{A'_k\}, \{A''_k\}, \{B'_k\}, \{B''_k\}$ are bounded by any (arbitrarily small) power of the values $l_k$.

In the experimental measurement of the quantities $\nu_j$, however, we can consider the value $\delta$ lying within the experimental error and not distinguish the powers $\nu_j$ and $\nu_j - \delta$. Thus, under the relations (3.2) we mean actually the relations (3.3) - (3.4), which we will write here in this abbreviated form.

At the same time, for all the cycles described above, having lengths of the order of $l$, we can write the estimations

$$|\Delta p_x(l)| \leq p_F \left( \frac{l}{p_F} \right)^{\nu_2}, \quad |\Delta p_y(l)| \leq p_F \left( \frac{l}{p_F} \right)^{\nu_3}$$

We note here that the cycles, corresponding to the estimate (3.2), sweep out a finite area on the full Fermi surface for each reference value $l$, which can represent arbitrary fraction of the full area of the Fermi surface given by any number from 0 to 1 for each of these values. It can be noted here that the latter circumstance is also unimportant in the experimental determination of the quantities $\nu_j$ due to the so-called “logarithmic insensitivity” of these quantities to the size of the area covered by the corresponding sections of the trajectories.
Directly from the system (2.5), we can then write (for the same reference values of $l$) the estimations
\[ \left| \oint v_{gr}^x(t) \, dt \right| \simeq \frac{c\rho F}{eB} \left( \frac{l}{p_F} \right)^{\nu_3}, \quad \left| \oint v_{gr}^y(t) \, dt \right| \simeq \frac{c\rho F}{eB} \left( \frac{l}{p_F} \right)^{\nu_2} \] (3.5)
for the same part of the described cycles and
\[ \left| \oint v_{gr}^x(t) \, dt \right| \leq \frac{c\rho F}{eB} \left( \frac{l}{p_F} \right)^{\nu_3}, \quad \left| \oint v_{gr}^y(t) \, dt \right| \leq \frac{c\rho F}{eB} \left( \frac{l}{p_F} \right)^{\nu_2} \] for the rest of the cycles.

Coming back to the theory of galvanomagnetic phenomena in metals, we must consider the kinetic equation for the distribution function $f(p, t)$
\[ f_t + \frac{e}{c} \sum_{k=1}^{3} [\nabla \epsilon(p) \times B]^k \frac{\partial f}{\partial p^k} + e \sum_{k=1}^{3} E^k \frac{\partial f}{\partial p^k} = I[f](p, t) \] (3.6)
in the presence of external electric and magnetic fields. We will be interested here in stationary solutions of the equation (3.6), so that we can put in fact $f(p, t) = f(p)$. The functional $I[f]$ is the collision integral responsible for the relaxation of perturbations of the distribution function to its equilibrium values $f_0(p)$. In particular, its properties also play a decisive role for the magnitude of the constant response of an electron system to a constant external influence. In the linear approximation in $E$ we can write the linearized equation for the linear response $f_1(p)$:
\[ -\frac{e}{c} \sum_{k=1}^{3} [\nabla \epsilon(p) \times B]^k \frac{\partial f_1}{\partial p^k} + e \sum_{k=1}^{3} E^k \frac{\partial f_0}{\partial p^k} = \left[ \hat{L}_{f_0} \cdot f_1 \right](p), \] (3.7)
where $\hat{L}_{f_0}$ represents the linearization of the functional $I[f](p)$ on the corresponding function $f_0$. In the so-called $\tau$-approximation, the right-hand side of the system (3.7) can be replaced by the expression $-f_1(p)/\tau$, where $\tau$ plays the role of a characteristic relaxation time (electron mean free time). Let us note here that a more complicated form of the linearized collision integral does not change in fact the results represented below.

It is easy to see that in the $\tau$-approximation the equations on $f_1(p)$ represent ordinary differential equations written along trajectories of the system (2.5). It can also be seen that the physical solutions of the corresponding equations are concentrated near the Fermi surface due to the corresponding behavior of the quantities
\[ \frac{\partial f_0}{\partial p^k} = \frac{\partial f_0}{\partial \epsilon} v_{gr}^k(p) \]
As a consequence of this fact, the expression for the conductivity (taking into account the spin degeneracy) is actually determined by the behavior of the solutions of (3.7) on the Fermi surface and can be represented in the following general form
\[ \sigma^{ik}(B) = \frac{2ec}{B} \int_{s_F} \frac{dp_z}{(2\pi\hbar)^3} v_{gr}^i(p_z, s) \int_{-\infty}^{s} v_{gr}^k(p_z, s') e^{\frac{\epsilon(s'-s)}{eBr}} \, ds' \] (3.8)
where $s = teB/c$ is the Hamiltonian parameter along the trajectories of system (2.5).
The contribution of the open trajectories to the conductivity tensor $\Delta \sigma^{ik}$ is given here by the restriction of the integral in (3.8) to the set of carriers of open trajectories $\tilde{S}_F$ instead of the whole Fermi surface, so that we can write:

$$\Delta \sigma^{ik}(B) = \frac{2e \hbar}{B} \int_{\tilde{S}_F} \frac{dp_z ds}{(2\pi \hbar)^3} v^i_{gr}(p_z, s) \int_s^s v^k_{gr}(p_z, s') e^{\frac{cl(s'-s)}{eB\tau}} ds'$$

(3.9)

As we have said, we shall assume here that the set of carriers of open trajectories on the Fermi surface is represented by a single connected component of genus 3.

For the contribution of the open trajectories on the Fermi surface to the symmetric part of the conductivity tensor it is not difficult to obtain after some calculations the following expression

$$\Delta s^{ik}(B) = 2 e^2 \tau \int_{\tilde{S}_F} \langle v^i_{gr} \rangle_B \langle v^k_{gr} \rangle_B \frac{dp_z ds}{(2\pi \hbar)^3}$$

(3.10)

where

$$\langle v^i_{gr} \rangle_B (p_z, s) \equiv \frac{c}{eB\tau} \int_{-\infty}^s v^i_{gr} (p_z, s') e^{\frac{cl(s'-s)}{eB\tau}} ds'$$

(3.11)

To estimate the values (3.11), we can now use the relations (3.5) if we use the approximation

$$\langle v^i_{gr} \rangle_B (p_z, s) \simeq \frac{c}{eB\tau} \int_{s-eB\tau/c}^s v^i_{gr} (p_z, s') ds'$$

(3.12)

on the corresponding trajectories of the system (2.5).

It should be noted that between the Hamiltonian parameter $s$ and the trajectory length $l$ there may actually be differences due to the presence of singular points of the system (2.5) on the Fermi surface. In our case, however, when calculating the conductivity containing contributions from all trajectories, this difference will not play a significant role in the main order. In this order, we can thus use the estimate $s \simeq m^* l/p_F$ as the common for the trajectories of the system (2.5) and write the approximations

$$\langle v^i_{gr} \rangle_B (p_z, s) \simeq \frac{c}{eB\tau} \int_{s-eB\tau/c}^s v^i_{gr} (p_z, s') \frac{l}{m^*} dt(l)$$

Analyzing the behavior of the trajectories in the plane orthogonal to $B$, one can in fact see that for the corresponding reference values $\omega_B \tau = l/p_F$ we can now use (using “logarithmic insensitivity”) the estimations

$$\langle v^x_{gr} \rangle_B (p_z, s) \simeq \frac{p_F}{m^*} (\omega_B \tau)^{\nu_1-1}$$

$$\langle v^y_{gr} \rangle_B (p_z, s) \simeq \frac{p_F}{m^*} (\omega_B \tau)^{\nu_2-1}$$

on some finite area of the Fermi surface and

$$\langle v^x_{gr} \rangle_B (p_z, s) \leq \frac{p_F}{m^*} (\omega_B \tau)^{\nu_1-1}$$

$$\langle v^y_{gr} \rangle_B (p_z, s) \leq \frac{p_F}{m^*} (\omega_B \tau)^{\nu_2-1}$$

on the remaining area.
The above estimations permit us to write the relations

\[ \Delta s^{xx}(B) = \Delta \sigma^{xx}(B) \simeq \frac{ne^2 \tau}{m^*} (\omega_B \tau)^{2\nu_3 - 2} , \]

\[ \Delta s^{yy}(B) = \Delta \sigma^{yy}(B) \simeq \frac{ne^2 \tau}{m^*} (\omega_B \tau)^{2\nu_2 - 2} \] (3.13)

for the same frame sequences.

We note here that the estimations (3.13) do not represent the principal term in any asymptotic expansion of the quantities \( \sigma^{xx}(B) \) and \( \sigma^{yy}(B) \), but only characterize some “general trend” in the behavior of these quantities, which in the general case is accompanied by some fluctuations in their values as functions of \( \omega_B \tau \) (Fig. 5).

Strictly speaking, we also have here just the relations

\[ \lim \sup_{\omega_B \tau \to \infty} \frac{\log |\Delta s^{xx}(B)|}{\log \omega_B \tau} = \nu_2 + \nu_3 - 2 , \]

\[ \lim \sup_{\omega_B \tau \to \infty} \frac{\log |\Delta s^{yy}(B)|}{\log \omega_B \tau} = \nu_2 - 2 \]

Nevertheless, the regimes (3.13) are directly related to the asymptotic behavior of \( \sigma^{xx}(B) \) and \( \sigma^{yy}(B) \), since the fluctuations in their magnitudes are actually limited by a number of additional conditions. In particular, \( \sigma^{xx}(B) \) and \( \sigma^{yy}(B) \) are monotone (decreasing) functions of \( B \), which imposes significant restrictions on the deviation of their values from the general trend. We also note here that the replacement of the estimate (3.12) with a more accurate definition (3.11) (or using the exact collision integral), in general, smoothes such oscillations while maintaining the presence of a common trend (3.13).

Similarly, we can write the relations

\[ \lim \sup_{\omega_B \tau \to \infty} \frac{\log |\Delta s^{xy}(B)|}{\log \omega_B \tau} \leq \nu_2 + \nu_3 - 2 , \]

which allow us to write

\[ |\Delta s^{xy}(B)| \leq (\omega_B \tau)^{\nu_2 + \nu_3 - 2} \]

as an estimate for the common trend for these quantities.

The described above contribution of chaotic open trajectories to conductivity should be in general added with the contribution of short closed trajectories, which usually also present on the Fermi surface. From the formula (2.6) it is easy to see, however, that they give a vanishingly small contribution in the plane orthogonal to \( B \), as compared to the one described above, in the limit \( \omega_B \tau \to \infty \).

We can see that in the presence of the Dynnikov chaotic trajectories, the conductivity in the plane orthogonal to \( B \) also reveals anisotropy, although it is not expressed so much as in the case of stable open trajectories. We also note that the picture described above does not change significantly if we assume that the minimal components (carriers) containing chaotic trajectories have genus greater than 3 and the corresponding dynamics are also described by some Zorich - Kontsevich - Forni indices.

Indeed, in this case, we must similarly assume existence of a flag of subspaces

\[ V_1 \subset V_2 \subset \ldots \subset V_g \subset V \subset H_1(M_g^2; \mathbb{R}) , \]
Figure 5: The common trend and the actual behavior of the quantities $\sigma^{xx}(B)$ and $\sigma^{yy}(B)$ in the limit $\omega_B \tau \to \infty$.

possessing the properties (1)-(5) presented above, and we can also assume in the generic case $1 > \nu_2 > \nu_3 > \ldots \nu_g > 0$, $\dim V_2 = 2$, $\dim V_3 = 3$, $\ldots$, $\dim V_g = g$. Generically, we can now assume that the image of $V_2$ under the embedding $M_g^2 \to \mathbb{T}^3$ is one-dimensional and the images of $V_3$, $\ldots$, $V_g$ coincide with the plane orthogonal to $B$. Repeating exactly the arguments given above, it is easy to see that the asymptotic behavior of the components of the symmetric part of the conductivity tensor in the plane orthogonal to $B$ is now determined by the largest indices $\nu_2$ and $\nu_3$ and the image of the space $V_2$ under the embedding $M_g^2 \to \mathbb{T}^3$.

Let us make here one more remark about chaotic trajectories of Dyornikov type and the properties of the electric conductivity tensor in strong magnetic fields. As it can in fact be shown, the chaotic trajectories arising for directions of $B$ of the maximal irrationality can in some sense be divided into two different classes (see [16, 40, 17, 18]). Namely, let us consider a fixed plane in $p$-space containing chaotic trajectories and ask the question: how many connected components (i.e. how many different chaotic trajectories) is contained in a given plane? It turns out that only two situations are possible:

1) Almost all planes, orthogonal to $B$, contain only one chaotic trajectory;
2) Almost all planes, orthogonal to $B$, contain infinitely many chaotic trajectories.

It is not difficult to see that the belonging of a dynamic system to one of the above classes should definitely affect the geometric properties of the corresponding chaotic trajectories on large scales. Thus, in the first case the trajectory densely sweeps out the whole plane and it is possible to estimate the area, covered by a segment of the trajectory of length $l$, by the value of $l$. In the second case, each of the chaotic trajectories sweeps the plane much less densely, leaving large areas for other trajectories in the same plane. In the first case, therefore, we can expect the presence of the special relation $\nu_2 + \nu_3 = 1$ for the corresponding Zorich - Kontsevich - Forni indices, while
in the second case one should expect inequality \( \nu_2 + \nu_3 > 1 \). Coming back to the asymptotics of the components of the conductivity tensor, we can see that the second case corresponds to a slower decreasing of the conductivity in different directions in the plane, orthogonal to \( \mathbf{B} \), as the magnetic field increases.

In conclusion, let us make one more remark about the behavior of the tensor of electric conductivity in strong magnetic fields in the presence of chaotic trajectories on the Fermi surface. In a sense, the reasoning we have given is of a theoretical nature and the question of experimental investigation of the corresponding regimes must be studied separately. In particular, we can ask about applicability of the physical model of the electron transport used above and possible corrections to the regimes, described by us, in real crystals. Here we would like to indicate the main of the quantum corrections to the quasiclassical consideration presented above, which substantially affects the conductivity behavior in strong magnetic fields. Namely, we would like to mention the phenomenon of (intraband) magnetic breakdown, which should be observed on chaotic trajectories in a sufficiently strong magnetic field.

The magnetic breakdown is a quantum phenomenon that allows an electron to “jump” from one section of a quasiclassical trajectory to another preserving the value \( p_z \) if the corresponding sections of the trajectory approach sufficiently close to each other in the \( \mathbf{p} \) - space (Fig. [4]). The probability of a quantum jump is higher, the closer to each other the corresponding sections of the trajectory and the larger the value of \( B \). In the limit of small distances between the sections or very large values of \( B \) the corresponding probability tends to \( 1/2 \). Arguing very approximately, for a given value of \( B \) we can introduce the characteristic distance \( \delta p(B) \), such that the magnetic breakdown can be considered improbable at a distance between the sections \( \Delta p > \delta p(B) \) and quite probable for \( \Delta p < \delta p(B) \). We consider here very schematically the effect of intraband breakdown on the behavior of the components of \( \sigma^{ik}(B) \) in our case.

It is easy to see that, because the carriers of chaotic trajectories necessarily contain saddle singular points, on each of the Dynnikov chaotic trajectories we have sections that are arbitrarily close to other parts of the trajectory (or sections of other trajectories) separated by segments of trajectories of different lengths. For a given value of the magnetic field, we can introduce the characteristic length \( l_B^{(1)} \) of the trajectory in the \( \mathbf{p} \) - space that separates two sections that approach at the distance \( \leq \delta p(B) \) to other sections of the trajectory (\( l_B^{(1)} \) decreases with the growth of \( B \)). At the same time, we can introduce the characteristic length \( l_B^{(2)} \) passed by the electron along a trajectory in the \( \mathbf{p} \) - space between two acts of scattering by impurities (\( l_B^{(2)} \) \( \sim \) \( B \tau \)). It is easy to see that the intraband magnetic breakdown has small effect on the quasiclassical regimes of conductivity that we describe, if we have the condition \( l_B^{(1)} \gg l_B^{(2)} \). At the same time, under the condition \( l_B^{(1)} \leq l_B^{(2)} \) the magnetic breakdown should have a significant effect on the conductivity behavior. We note that the quantities \( l_B^{(1)} \) and \( l_B^{(2)} \) depend on features of the electron spectrum and should be evaluated separately in each specific case. It is not difficult to see here that for a rough estimate of the effect of the magnetic breakdown on the regimes described by us it is possible to use the change of the time \( \tau \) to the effective electron mean free time \( \tau_{eff}(B) \), using the formula

\[
\tau_{eff}^{-1} = \tau^{-1} + \tau_{(1)}^{-1},
\]

where \( \tau_{(1)}(B) \sim l_B^{(1)}/B \) plays the role of the characteristic time of the electron motion over the length \( \simeq l_B^{(1)} \) along the trajectory. We note here that there is also a sufficiently large number of more complicated quantum corrections to the conductivity, but they play smaller role in our situation.
Figure 6: The phenomenon of magnetic breakdown on a chaotic trajectory in the $p$-space.

References

[1] A.A. Abrikosov., Fundamentals of the Theory of Metals., Elsevier Science & Technology, Oxford, United Kingdom, 1988.

[2] A. Avila, P. Hubert, A. Skripchenko., Diffusion for chaotic plane sections of 3-periodic surfaces., Inventiones mathematicae, October 2016, Volume 206, Issue 1, pp 109-146.

[3] A. Avila, P. Hubert, A. Skripchenko., On the Hausdorff dimension of the Rauzy gasket., Bulletin de la societe mathematique de France, 2016, 144 (3), pp. 539 - 568.

[4] R. De Leo., Existence and measure of ergodic leaves in Novikov’s problem on the semiclassical motion of an electron., Russian Math. Surveys 55:1 (2000), 166-168.

[5] R. De Leo., Characterization of the set of “ergodic directions” in Novikov’s problem of quasi-electron orbits in normal metals., Russian Math. Surveys 58:5 (2003), 1042-1043.
[6] R. De Leo., First-principles generation of stereographic maps for high-field magnetoresistance in normal metals: An application to Au and Ag., *Physica B: Condensed Matter* **362** (1-4) (2005), 62-75.

[7] R. De Leo., Topology of plane sections of periodic polyhedra with an application to the Truncated Octahedron., *Experimental Mathematics* **15**:1 (2006), 109-124.

[8] R. De Leo, I.A. Dynnikov., An example of a fractal set of plane directions having chaotic intersections with a fixed 3-periodic surface., *Russian Math. Surveys* **62**:5 (2007), 990-992.

[9] R. De Leo, I.A. Dynnikov., Geometry of plane sections of the infinite regular skew polyhedron \( \{4, 6 | 4\}\), *Geom. Dedicata* **138**:1 (2009), 51-67.

[10] Roberto De Leo., A survey on quasiperiodic topology., [arXiv:1711.01716](https://arxiv.org/abs/1711.01716)

[11] I.A. Dynnikov., Proof of S.P. Novikov's conjecture for the case of small perturbations of rational magnetic fields., *Russian Math. Surveys* **47**:3 (1992), 172-173.

[12] I.A. Dynnikov., Proof of S.P. Novikov’s conjecture on the semiclassical motion of an electron., *Math. Notes* **53**:5 (1993), 495-501.

[13] I.A. Dynnikov., Semiclassical motion of the electron. A proof of the Novikov conjecture in general position and counterexamples., Solitons, geometry, and topology: on the crossroad, Amer. Math. Soc. Transl. Ser. 2, 179, Amer. Math. Soc., Providence, RI, 1997, 45-73.

[14] I.A. Dynnikov, A.Ya. Maltsev., Topological characteristics of electronic spectra of single crystals., *Journal of Experimental and Theoretical Physics* **85**:1 (1997), 205-208.

[15] I.A. Dynnikov., The geometry of stability regions in Novikov’s problem on the semiclassical motion of an electron., *Russian Math. Surveys* **54**:1 (1999), 21-59.

[16] I.A. Dynnikov., Interval identification systems and plane sections of 3-periodic surfaces., *Proceedings of the Steklov Institute of Mathematics* **263**:1 (2008), 65-77.

[17] I. Dynnikov, A. Skripchenko., On typical leaves of a measured foliated 2-complex of thin type., Topology, Geometry, Integrable Systems, and Mathematical Physics: Novikov’s Seminar 2012-2014, Advances in the Mathematical Sciences., Amer. Math. Soc. Transl. Ser. 2, 234, eds. V.M. Buchstaber, B.A. Dubrovin, I.M. Krichever, Amer. Math. Soc., Providence, RI, 2014, 173-200, arXiv: 1309.4884

[18] I. Dynnikov, A. Skripchenko., Symmetric band complexes of thin type and chaotic sections which are not actually chaotic., *Trans. Moscow Math. Soc.*, Vol. 76, no. 2, 2015, 287-308.

[19] I.A. Dynnikov, S.P. Novikov., Topology of quasi-periodic functions on the plane., *Russian Math. Surveys* **60**:1 (2005), 1-26.

[20] M.I. Kaganov, V.G. Peschansky., Galvano-magnetic phenomena today and forty years ago., *Physics Reports* **372** (2002), 445-487.

[21] C. Kittel., Quantum Theory of Solids., Wiley, 1963.
[22] I.M. Lifshitz, M. Ya. Azbel, M. I. Kaganov. The Theory of Galvanomagnetic Effects in Metals., Sov. Phys. JETP 4:1 (1957), 41.

[23] I.M. Lifshitz, V. G. Peschansky., Galvanomagnetic characteristics of metals with open Fermi surfaces., Sov. Phys. JETP 8:5 (1959), 875.

[24] I.M. Lifshitz, V. G. Peschansky., Galvanomagnetic characteristics of metals with open Fermi surfaces. II., Sov. Phys. JETP 11:1 (1960), 137.

[25] I.M. Lifshitz, M. I. Kaganov., Some problems of the electron theory of metals I. Classical and quantum mechanics of electrons in metals., Sov. Phys. Usp. 2:6 (1960), 831-835.

[26] I.M. Lifshitz, M. I. Kaganov., Some problems of the electron theory of metals II. Statistical mechanics and thermodynamics of electrons in metals., Sov. Phys. Usp. 5:6 (1963), 878-907.

[27] I.M. Lifshitz, M. I. Kaganov., Some problems of the electron theory of metals III. Kinetic properties of electrons in metals., Sov. Phys. Usp. 8:6 (1966), 805-851.

[28] I.M. Lifshitz, M. Ya. Azbel, M. I. Kaganov., Electron Theory of Metals. Moscow, Nauka, 1971. Translated: New York: Consultants Bureau, 1973.

[29] A.Y. Maltsev., Anomalous behavior of the electrical conductivity tensor in strong magnetic fields., Journal of Experimental and Theoretical Physics 85 (5) (1997), 934-942.

[30] A.Ya. Maltsev, S.P. Novikov., Quasiperiodic functions and Dynamical Systems in Quantum Solid State Physics., Bulletin of Braz. Math. Society, New Series 34:1 (2003), 171-210.

[31] A.Ya. Maltsev, S.P. Novikov., Dynamical Systems, Topology and Conductivity in Normal Metals in strong magnetic fields., Journal of Statistical Physics 115:(1-2) (2004), 31-46.

[32] A.Ya. Maltsev., Quasiperiodic functions theory and the superlattice potentials for a two-dimensional electron gas., Journal of Mathematical Physics 45:3 (2004), 1128-1149.

[33] A.Ya. Maltsev., On the Analytical Properties of the Magneto-Conductivity in the Case of Presence of Stable Open Electron Trajectories on a Complex Fermi Surface., Journal of Experimental and Theoretical Physics 124 (5) (2017), 805-831.

[34] A.Ya. Maltsev., Oscillation phenomena and experimental determination of exact mathematical stability zones for magneto-conductivity in metals having complicated Fermi surfaces., Journal of Experimental and Theoretical Physics 125:5 (2017), 896-905.

[35] S.P. Novikov., The Hamiltonian formalism and a many-valued analogue of Morse theory., Russian Math. Surveys 37 (5) (1982), 1-56.

[36] S.P. Novikov, A.Y. Maltsev., Topological quantum characteristics observed in the investigation of the conductivity in normal metals., JETP Letters 63 (10) (1996), 855-860.

[37] S.P. Novikov, A.Y. Maltsev., Topological phenomena in normal metals., Physics-Uspekhi 41:3 (1998), 231-239.
[38] S.P. Novikov., Levels of quasiperiodic functions on a plane, and Hamiltonian systems., *Russian Math. Surveys* **54** (5) (1999), 1031-1032.

[39] A. Skripchenko., Symmetric interval identification systems of order three., *Discrete Contin. Dyn. Sys.* **32:2** (2012), 643-656.

[40] A. Skripchenko., On connectedness of chaotic sections of some 3-periodic surfaces., *Ann. Glob. Anal. Geom.* **43** (2013), 253-271.

[41] S.P. Tsarev. Private communication. (1992-93).

[42] J.M. Ziman., Principles of the Theory of Solids., Cambridge University Press 1972.

[43] A.V. Zorich., A problem of Novikov on the semiclassical motion of an electron in a uniform almost rational magnetic field., *Russian Math. Surveys* **39** (5) (1984), 287-288.

[44] A.V. Zorich., Asymptotic Flag of an Orientable Measured Foliation on a Surface., Proc. “Geometric Study of Foliations”., (Tokyo, November 1993), ed. T.Mizutani et al. Singapore: World Scientific Pb. Co., 479-498 (1994).

[45] A.V. Zorich., Finite Gauss measure on the space of interval exchange transformations. Lyapunov exponents., *Annales de l’Institut Fourier* **46:2**, (1996), 325-370.

[46] Anton Zorich., On hyperplane sections of periodic surfaces., Solitons, Geometry, and Topology: On the Crossroad, V. M. Buchstaber and S. P. Novikov (eds.), Translations of the AMS, Ser. 2, vol. **179**, AMS, Providence, RI (1997), 173-189.

[47] Anton Zorich., Deviation for interval exchange transformations., *Ergodic Theory and Dynamical Systems* **17**, (1997), 1477-1499.

[48] Anton Zorich., How do the leaves of closed 1-form wind around a surface., “Pseudoperiodic Topology”, V.I.Arnold, M.Kontsevich, A.Zorich (eds.), Translations of the AMS, Ser. 2, vol. 197, AMS, Providence, RI, 1999, 135-178.

[49] Anton Zorich., Flat surfaces., in collect. “Frontiers in Number Theory, Physics and Geometry. Vol. 1: On random matrices, zeta functions and dynamical systems”; Ecole de physique des Houches, France, March 9-21 2003, P. Cartier; B. Julia; P. Moussa; P. Vanhove (Editors), Springer-Verlag, Berlin, 2006, 439-586.