PHASE TRANSITION FOR EXTREMES OF A STOCHASTIC VOLATILITY MODEL WITH LONG-RANGE DEPENDENCE

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ABSTRACT. We consider a stochastic volatility model that both the volatility and innovation processes have power-law marginal distributions, with tail indices $\alpha, \alpha' > 0$, respectively. In addition, the volatility process is taken as the heavy-tailed Karlin model, a recently investigated model that has long-range dependence characterized by a memory parameter $\beta \in (0, 1)$. We establish extremal limit theorems for the empirical random sup-measures of the model, and reveal a phase transition: volatility-dominance regime $\alpha < \alpha' \beta$, innovation-dominance regime $\alpha > \alpha' \beta$, and critical regime $\alpha = \alpha' \beta$. The most intriguing case is the critical regime $\alpha = \alpha' \beta$, where the limit is the logistic random sup-measure. As for the proof, we actually establish the same phase-transition phenomena for the so-called Poisson–Karlin model with multiplicative noise defined on generic metric spaces, and apply a Poissonization method to establish the limit theorems for the volatility model as a consequence.

1. Introduction

The motivating example of this paper concerns the following model of stationary stochastic processes
\[
X_i = \sigma_i Z_i, \quad i \in \mathbb{N} := \{1, 2, \ldots\},
\]
where $\sigma := \{\sigma_i\}_{i \in \mathbb{N}}$ is a stationary sequence of random variables and $\{Z_i\}_{i \in \mathbb{N}}$ are i.i.d. copies of certain random variable $Z$, the two sequences being independent. This model is known in the literature as the stochastic volatility model, and its original application came from modeling and inference for financial time series data (log returns). The main difference, compared to well-investigated GARCH-like models in the literature, is the independence assumption between the innovation process $\{Z_i\}_{i \in \mathbb{N}}$ and the volatility process $\{\sigma_i\}_{i \in \mathbb{N}}$, which is appealing from a theoretical point of view. For earlier developments of stochastic volatility models, see [2, Part II] and in particular the two contributions by Davis and Mikosch. We shall draw some comparisons between some recent developments and our results in Remark 1.2 later.

We are mostly interested in characterizing scaling limits for the extremes of the stochastic volatility model from a theoretical point of view. It turned out that, despite its simple structure, it may exhibit an intriguing phase transition in terms of the limit random sup-measures. Here, we investigate the case that the volatility process $X = \{X_i\}_{i \in \mathbb{N}}$ has regularly-varying tails with index $\gamma > 0$ (i.e. $\mathbb{P}X_i(x) := \mathbb{P}(X_1 > x) \in \text{RV}_\gamma$). Our first assumption is that both $\sigma_1$ and $Z_1$ have regularly-varying tails, with index $\alpha > 0, \alpha' > 0$, respectively. Indeed, if we consider a single random variable $X_1 = \sigma_1 Z_1$, it is a well-known result due to Breiman [9] (see [22] for more references) that then $X_1$ has a regularly-varying tail with the dominant index of the two ($\gamma = \min\{\alpha, \alpha'\}$). (Strictly speaking, when say the tail of $\sigma_1$ dominates, we do not need to assume $Z_1$ to have a regularly-varying tail for $X_1$, but simply that $\mathbb{E}Z_1^{\alpha'+\epsilon} < \infty$. For the sake of simplicity we restrict our discussions here to the regularly-varying tails, while our main results later are proved under more general assumptions.)

Second, at the process level, we are interested in the case that the volatility process $\{\sigma_i\}_{i \in \mathbb{N}}$ is with long-range dependence, and we take the recently introduced heavy-tailed (power-law) Karlin model [12] [14] for $\sigma$. Heuristically, there is a memory parameter $\beta \in (0, 1)$ in the models of our interest, and by long-range dependence we mean that the scaling limit for extremes of $\sigma$ is of abnormal order, depending on $\beta$, compared to a sequence of i.i.d. random variables of the same marginal. Moreover, certain long-range clustering of
extremes appear in the limit [39]. This is in stark contrast to most time-series models investigated so far in the literature, which exhibit possibly local clustering of extremes (a.k.a. extremal clustering in the literature) in the limit. Local clustering is a feature of microscopic behaviors: it is usually quantified by the extremal index taking values from $(0, 1]$, interpreted as the reciprocal of the expected size of the extreme cluster (index equal to one meaning no clustering), and more precisely described by limit theorems for multivariate regular variations and tail processes. Extremes with local clustering have been extensively investigated in extreme-value theory (e.g., [12, 24] for the volatility sequence. The fact that this model exhibits long-range clustering of extremes was demonstrated in [13]. Consider a probability measure on $[0, 1]$, denoted throughout by $SM([0, 1])$, equipped with the sup-vague topology. (For the sake of simplicity, one may think of a random sup-measure appears as in (1.1) with $\{\beta, \ell\}$ has regularly-varying tail with index $\beta$, with the Karlin model (memory parameter $\ell$) below, for the exact assumption.) Let $\{Y_i\}_{i \in \mathbb{N}}$ be i.i.d. sampling from $\mathbb{N}$ according to $\mathbb{P}(Y_i = \ell) = p_\ell$. Let $\{\varepsilon_i\}_{i \in \mathbb{N}}$ be i.i.d. non-negative random variables with $\mathbb{F}_\varepsilon(x) \in RV_{-\infty}$, independent from $\{Y_i\}_{i \in \mathbb{N}}$, and set the Karlin model as

$$\sigma_i := \varepsilon_{Y_i}, \quad i \in \mathbb{N}.$$ 

The limit theorems for this model can be described in the language of convergence of random sup-measures [33, 44]. Let

$$\tilde{M}_n^\sigma(\cdot) := \max_{i=1, \ldots, n} \varepsilon_{Y_i}, \quad n \in \mathbb{N},$$

be the empirical random sup-measure of the Karlin model in the space of sup-measures on $[0, 1]$, denoted by $SM([0, 1])$, equipped with the sup-vague topology. (For the sake of simplicity, one may think of a random sup-measure $M(\cdot)$ as a set-indexed stochastic process $\{M(G)\}_{G \in \mathcal{G}}$ indexed by open sets.) In [13] it was shown that

$$(1.1) \quad \frac{1}{a_n} \tilde{M}_n^\sigma(\cdot) \Rightarrow M_{\alpha, \beta}(\cdot) := \sup_{\ell \in \mathbb{N}} \frac{1}{\Gamma_{\ell}^1/\alpha} \mathbb{I}(\mathcal{R}_{\beta, \ell} \cap \cdot \neq \emptyset) \quad \text{with some } a_n \in RV_{\beta/\alpha},$$

where $\{\Gamma_\ell\}_{\ell \in \mathbb{N}}$ are consecutive arrival times of a standard Poisson process, and $\{\mathcal{R}_{\beta, \ell}\}_{\ell \in \mathbb{N}}$ are i.i.d. random closed sets in $[0, 1]$. These random closed sets can be expressed explicitly as

$$\mathcal{R}_{\beta, \ell} := \bigcup_{i=1}^{Q_{\beta, \ell}} \{U_{\ell, i}\},$$

where $\{Q_{\beta, \ell}\}_{\ell \in \mathbb{N}}$ are i.i.d. copies of Sibuya random variables with parameter $\beta$, taking values in $\mathbb{N}$ (see (2.1) below), and $\{U_{\ell, i}\}_{i, \ell \in \mathbb{N}}$ are i.i.d. uniform random variables from $[0, 1]$.

An interpretation of (1.1) is as follows. First, at the boundary case $\beta = 1$ ($Q_\beta \Rightarrow 1$ as $\beta \uparrow 1$), the limit random sup-measure in (1.1) is simply the independently scattered $\alpha$-Fréchet random sup-measure with uniform control measure on $[0, 1]$, denoted throughout by

$$(1.2) \quad M_{\alpha, 1}^\sigma(\cdot) \equiv M_{\alpha, 1}(\cdot) \overset{d}{=} \sup_{\ell \in \mathbb{N}} \frac{1}{\Gamma_{\ell}^{1/\alpha}} \mathbb{I}(U_{\ell} \in \cdot),$$

where $1/\Gamma_{\ell}^{1/\alpha}$ represents the $\ell$-th largest order statistic and $U_{\ell}$ its location ($U_{\ell} \equiv U_{\ell, 1}$). For Karlin random sup-measure $M_{\alpha, \beta}$ with $\beta \in (0, 1)$, the $\ell$-th order statistic is again represented by $1/\Gamma_{\ell}^{1/\alpha}$, but it appears at multiple non-local locations (whence the notion of long-range clustering) represented by $\mathcal{R}_{\beta, \ell}$ (the cardinality $Q_{\beta, \ell}$ of $\mathcal{R}_{\beta, \ell}$ has regularly-varying tail with index $\beta$). This notion of long-range clustering of extremes is quite recent: another model of such a feature is investigated in [24, 44], where in the limit a similar random sup-measure appears as in (1.1) with $\mathcal{R}_{\beta, \ell}$ replaced by i.i.d. copies of randomly shifted $\beta$-stable regenerative sets.

We are interested in establishing corresponding limit theorems as in (1.1) for the (heavy-tailed) Karlin stochastic volatility model $X$, with the Karlin model (memory parameter $\beta \in (0, 1)$ and tail index $\alpha > 0$) in
Assume that \( f \) will be explained after the statement. Throughout we write \( d \) determined by the three parameters. For illustration purpose we give a simplified statement of the phase turns out that there are three different regimes in terms of the limit for the empirical random sup-measures of the volatility sequence, and the innovation process.

(i) The most intriguing regime is the critical regime. Here, \( S_{\beta} \) is a totally skewed \( \beta \)-stable random variable, independent from \( M_{\alpha} \). This random sup-measure \( S_{\beta} = M_{\alpha} \) is an \( \alpha \)-Fréchet logistic random sup-measure with parameter \( \beta \), corresponding to the so-called logistic model in multivariate extreme-value theory (see Remark 3.1). We do not have a simple explanation of the mechanism underlying the formulation of the extremes as in the two other regimes: the top order statistics of both volatility and innovation processes actually do not contribute. Our proof is by a pure analytical approach by computing the limit Laplace functional.

(ii) The volatility/signal-dominance regime corresponds to the case when the extremes of the stochastic volatility model are caused by the extremes of the innovation process, each multiplied by an independent copy from the volatility process. Here, \( M_{\alpha} \) is independent from \( (E_{\varepsilon} x_{\varepsilon}^{\alpha'})^{-1/\alpha'} \) (we write throughout \( E_{\varepsilon} (\cdot) \) for the conditional expectation with respect to \( \varepsilon := \sigma(\{\varepsilon_{\ell}\}_{\ell \in \mathbb{N}}) \)). This is the easiest case, once one realizes that conditionally on \( \varepsilon \), the random variables \( \{\varepsilon_{\ell}\}_{\ell \in \mathbb{N}} \) are i.i.d.. Then one immediately sees that, by Breiman’s Lemma (a key reference for us is Resnick [38, Proposition 7.5]), the above holds under the assumption \( E_{\varepsilon} x_{\varepsilon}^{\alpha'} + \epsilon < \infty \) almost surely for some \( \epsilon > 0 \), of which a sufficient condition is \( \alpha' < \alpha \).

(iii) The volatility/dominance regime follows from Breiman’s Lemma, an adaption of which then is applied to prove volatility-dominance regime. The main contribution of the paper is the limit theorem for the critical regime, where the proof is much more involved.
Moreover, all the random sup-measures that appear in the limit are of exchangeable nature, hence the phase transition shall be established for a more general model referred to as the *Poisson–Karlin model*, defined on a generic metric space instead of $\mathbb{R}_+$. The Poisson–Karlin model is needed in our earlier work establishing the Karlin random sup-measure [14] as the poissonized model, a key step in most of the analysis of the Karlin models since [24] (see also [17]). It also leads to some new families of stable set-indexed processes [16]. At the end, after a new Poissonization method that we developed, we establish a general theorem for the stochastic volatility model (see Theorem 4.1) from which Theorem 1.1 is a corollary.

We conclude the introduction with some remarks.

**Remark 1.1.** A main motivation behind our recent investigations on several variations of the Karlin model is to understand, for a stationary sequence of random variables, what types of stochastic processes/objects may arise in the presence of non-trivial dependence structure (long-range dependence). Here, the characterization of long-range dependence is by the corresponding central/extremal limit theorems, a modern approach that has become more and more popular [39]. The investigations of the Karlin stochastic volatility model turned out to be fruitful, as we reveal a phase-transition phenomenon, and in particular we are not aware of any extremal limit theorems for logistic random sup-measures (critical regime here) in the literature. On the other hand, the Karlin stochastic volatility model may not be the best to fit financial time series data. The empirical evidence of the power-law tails of the data has been well known. However, it has been extensively discussed in the literature that for such datasets extremes (tails) occur typically in local clusters (corresponding to extremal index in $(0,1)$) [3], and yet it can be argued that it is appropriate to apply stochastic volatility models with asymptotic tail-independence (no clustering) for modeling such datasets [11, 21]. From this point of view, the feature of long-range clustering makes the model not the most appealing. Nevertheless, in view of the recent result that (a variation of) the Karlin model serves as the counterpart of fractional Brownian motion as the simple random walk to the standard Brownian motion [13], it is yet to see whether the Karlin stochastic volatility model may find applications in other applied areas.

Some numerical simulations for Karlin stochastic volatility model are provided in Figure 1. The convergence for the critical regime seems very delicate to be visible. Some simulations for the limit logistic random sup-measures at the critical regime are provided in Figure 2.

**Remark 1.2.** Following the previous remark, most of the limit theorems regarding stochastic volatility models investigate the case that the extremes are asymptotically independent: the situations are nevertheless quite delicate, and could be elaborated further by either modeling the the so-called *coefficient of tail dependence* [21], or establishing conditional extreme-value distributions for the tail process [28]. The easiest
way to achieve asymptotic independence is to let the innovation process have the dominant tails, as most of the results in the literature on stochastic volatility models assume and so does our innovation-dominance regime. The only two references that we found where the volatility process has the dominant tail are Janssen and Drees [21] and Mikosch and Rezapour [31]. The former has no extremal clustering as mentioned above, and the second reference demonstrated, via several models, how local clustering may be inherited from the volatility process. As for volatility processes with long-range dependence, only a few notable references appeared recently; either they belong to the case that the innovation processes have the dominant tails [25, 27], or only their functional central limit theorems, no extremes, were studied [26]. We are unaware of any models that exhibit the similar long-range clustering or the phase transition as ours.

The paper is organized as follows. In Section 2 we review the Karlin random sup-measures, the Poisson–Karlin model, and prove that the empirical random sup-measures of the latter scales to the former. In Section 3 we state and prove our main theorems regarding the phase-transitions for Poisson–Karlin models with multiplicative noises. In Section 4 we prove the corresponding limit theorems for stochastic volatility models by a coupling method.

2. Karlin random sup-measures and Poisson–Karlin model

We review the Karlin random sup-measure and the Poisson–Karlin model, and prove that the empirical random sup-measures of the latter scale to the former, extending our earlier result in [14]. Standard references on random sup-measures and random closed sets are [32–34, 44]. We only recall a few facts.

We restrict to random sup-measures on a space $(E, \mathcal{E})$ taking values in $[0, \infty]$, and let $SM(E)$ denote the space of all such sup-measures. It is known that under the assumption that $E$ is locally compact and second Hausdorff countable and $\mathcal{E}$ its corresponding Borel $\sigma$-algebra, $SM(E)$ is separable and compact. Every $m \in SM(E)$ can be uniquely determined by its evaluations on open sets, and as a consequence when identifying random sup-measures in $SM(E)$ it suffices to restrict to their evaluations on open sets. In particular, when comparing two random sup-measures $M_1$ and $M_2$ on $E$, we shall write

$$M_1(\cdot) = M_2(\cdot) \text{ in short of } \{M_1(A)\}_{A \in \mathcal{A}} = \{M_2(A)\}_{A \in \mathcal{A}}$$

for some collection $\mathcal{A}$ of subsets of $E$ that form a probability-determining class, and the equalities above in practice shall be either equalities in the almost-sure sense or equalities for finite-dimensional distributions.

For our limit theorems in the space of random sup-measures, we shall write

$$M_n \Rightarrow M \text{ in short of } \{M_n(A)\}_{A \in \mathcal{A}} \Rightarrow \{M(A)\}_{A \in \mathcal{A}}$$

in $SM(E)$ for some convergence-determining class $\mathcal{A}$ of subsets from $\mathcal{E}$, where $\{M_n\}_{n \in \mathbb{N}}$ and $M$ are random sup-measures on $E$. Remark that the support of limit random sup-measures $M$ in this paper do not have
fixed points \((\mathcal{M}(\{x\}) = 0 \text{ almost surely for all } x \in E)\), and in this case the probability-determining and convergence-determining class coincide \(\text{[44, Section 12]}\). Assume in addition that \(E\) is a metric space, then both of the following are probability/convergence-determining classes \(\text{[44, Theorem 12.2]}\): let \(D\) be a countable dense set of \(E\) and \(B(x, r), x \in E, r > 0\), denote an open ball in \(E\),

\[
\mathcal{G}_0 := \left\{ B(x, r) : x \in D, r > 0, B(x, r) \in \mathcal{K} \right\},
\]

\[
\mathcal{K}_0 := \left\{ B(x, r) : x \in D, r > 0, B(x, r) \in \mathcal{K} \right\},
\]

where \(\mathcal{K} \equiv \mathcal{K}(E)\) is the set of compact subsets of \(E\).

Most of our random sup-measures are based on Poisson-point-processes. By writing

\[
\sum_i \delta_{\xi_i} \sim \text{PPP}(S, \mu),
\]

we mean that \(\{\xi_i\}_i\) are measurable enumerations of points from a Poisson point process on \(S\) with intensity measure \(\mu\). For any real-valued random variable \(W\), we write \(\bar{F}_W(x) = P(W > x)\).

2.1. **Karlin random sup-measures.** Throughout we fix a locally compact second countable Hausdorff metric space \((E, \mathcal{E})\), and a \(\sigma\)-finite measure \(\mu\) on it. Fix \(\alpha > 0\). A Karlin \(\alpha\)-Fréchet random sup-measure on \((E, \mathcal{E})\) with control measure \(\mu\) and parameter \(\beta \in (0, 1]\), denoted by \(\mathcal{M}_{\alpha, \beta}\) throughout, is a Choquet Fréchet random sup-measure with extremal coefficient functional \(\theta(\cdot) = \mu^\beta(\cdot)\). Since the law of a Choquet Fréchet sup-measure is uniquely determined by its marginal law over compact sets \(K\) of \(E\), the law of \(\mathcal{M}_{\alpha, \beta}\) is determined by

\[
P(\mathcal{M}_{\alpha, \beta}(K) \leq z) = \exp\left(-\mu^\beta(K)z^{-\alpha}\right), \text{ for all } K \in \mathcal{K}, z > 0.
\]

For limit theorems, it is more convenient to work with series representations. Since we are only concerned with the joint law of \(\mathcal{M}_{\alpha, \beta}\) evaluated at \(A_1, \ldots, A_d \in \mathcal{A} := \{A \in \mathcal{E} : \mu(A) < \infty\}\) for finite \(d \in \mathbb{N}\), without loss of generality we assume \(\mu(E) < \infty\). The advantage of working under this assumption is to have the following simple series representation.

Throughout we let \(Q_\beta\) denote a Sibuya random variable with parameter \(\beta \in (0, 1]\), which takes values from \(\mathbb{N}\) and has probability mass function \(\text{[42]}

\[
P(Q_\beta = k) = \frac{\beta \Gamma(k - \beta)}{\Gamma(1 - \beta)\Gamma(k + 1)}, \quad k \in \mathbb{N},
\]

Note that \(\mathbb{P}(Q_\beta = k) \sim (\beta/\Gamma(1 - \beta))k^{-1-\beta}\) as \(k \to \infty\). Equivalently, it is determined by \(\mathbb{E}z^{Q_\beta} = 1 - (1 - z)^\beta\) for \(|z| < 1\). Introduce

\[
\mathcal{R}_\beta := \bigcup_{i=1}^{Q_\beta} \{U_i\},
\]

where \(\{U_i\}_{i \in \mathbb{N}}\) are i.i.d. random element from \(E\) with law \(\overline{\mu} := \mu(\cdot)/\mu(E)\), independent from the Sibuya random variable \(Q_\beta\). Let \(P_\beta\) denote the law of \(\mathcal{R}_\beta\) on \(\mathcal{F}_0(E)\), the space of non-empty closed sets of \(E\). Consider

\[
\sum_{\ell=1}^{\infty} \delta_{(\Gamma_\ell, \mathcal{R}_\beta, \ell)} \sim \text{PPP} (\mathcal{R}_\beta \times \mathcal{F}_0(E), dxdP_\beta),
\]

where as a convention \(\{\Gamma_\ell\}_{\ell \in \mathbb{N}}\) are ordered in increasing order and \(\{\mathcal{R}_\beta, \ell\}_{\ell \in \mathbb{N}}\) can be viewed as i.i.d. marks.

**Proposition 2.1.** Assume that \(\mu(E) < \infty\). With the notation above,

\[
\mathcal{M}_{\alpha, \beta}(\cdot) \overset{d}{=} \mu^{\beta/\alpha}(E) \bigvee_{\ell=1}^{\infty} \frac{1}{\Gamma_\ell} \mathbb{1}_{\{\mathcal{R}_\beta, \ell \cap \cdot \neq \emptyset\}}, \text{ for all } \beta \in (0, 1].
\]
Proof. Let $M$ denote the random sup-measure on the right-hand side of (2.2), which is a Choquet α-Fréchet random sup-measure \cite[Theorem 4.4]{33}. So it suffices to compute the extremal coefficient functional:

$$
\mathbb{P}(M(K) \leq z) = \exp \left( -\mu^\beta(E)z^{-\alpha}\mathbb{P}\left( \bigcup_{i=1}^{Q_\beta} \left(U_i \cap K \neq \emptyset \right) \right) \right) = \exp \left( -\mu^\beta(E)z^{-\alpha}\mathbb{P}(1 - \mathbb{P}(U_1 \notin K)^{Q_\beta}) \right)
$$

$$
= \exp \left( -\mu^\beta(E)z^{-\alpha}\mathbb{P}(U_1 \in K)^\beta \right) = \exp \left( -\mu^\beta(K)z^{-\alpha}\right),
$$

as desired. \hfill \square

Remark 2.1. With $\beta = 1$, it is well known that the Choquet α-Fréchet random sup-measure with extremal coefficient functional $\mu(\cdot)$ is independently scattered: it has a representation

$$
\mathcal{M}_\alpha^{is} \equiv \mathcal{M}_{\alpha,1}(\cdot) \overset{d}{=} \sum_{i=1}^{\infty} \delta_{(\xi_i,U_i)} \in \text{SM}(E) \quad \text{with} \quad \sum_{i=1}^{\infty} \delta_{(\xi_i,U_i)} \sim \text{PPP}(\mathbb{R}_+ \times E, \alpha x^{-\alpha-1}dx\mu).
$$

Remark 2.2. In the case $\mu(E) = \infty$, the above representation is no longer valid. Another series representation is as follows. We first introduce a $\sigma$-finite measure on $\mathcal{F}(E)$, the space of closed sets on $E$. Let $\mathcal{N}^{(r)}(E)$ be a Poisson point process on $(E,E)$ with intensity measure $r \cdot \mu$, $r > 0$. Then its support, denoted by $\text{supp}\mathcal{N}^{(r)}$ (closed by definition) is a random closed set, and hence the law of $\mathcal{N}^{(r)}$ induces a probability measure on $\mathcal{F}(E)$, denoted by $\mathcal{L}_{\mu,r}$ (determined by $\mathcal{L}_{\mu,r}(\{F \in \mathcal{F}(E) : F \cap K \neq \emptyset\}) = 1 - e^{-\mu(K)r}$, $K \in \mathcal{K}(E)$). Then, introduce

$$
\sum_{i=1}^{\infty} \delta_{(\xi_i,U_i)} \sim \text{PPP}(\mathbb{R}_+ \times E, \alpha x^{-\alpha-1}dx\mu_\beta) \quad \text{with} \quad \mu_\beta(\cdot) := \frac{1}{\Gamma(1 - \beta)} \int_0^{\infty} \beta r^{-\beta-1}\mathcal{L}_{\mu,r}(\cdot)dr.
$$

We shall also consider

$$
\sum_{i=1}^{\infty} \delta_{(\xi_i,U_i)} \sim \text{PPP}(\mathbb{R}_+ \times \mathbb{R}_+, \alpha x^{-\alpha-1}dx\Gamma(1 - \beta)^{-1}\beta r^{-\beta-1}dr),
$$

and, given the above, conditionally independent Poisson point processes $\{\mathcal{N}_i^{(r_i)}\}_{i \in \mathbb{N}}$ on $(E,E)$ with intensity measure $r_i$ respectively. With the notations above,

$$
\mathcal{M}_{\alpha,\beta}(\cdot) \overset{d}{=} \sum_{i=1}^{\infty} \xi_i \mathbb{1}_{(\mathcal{R}_\beta \cap \cdot \neq \emptyset)} \overset{d}{=} \sum_{i=1}^{\infty} \xi_i \mathbb{1}_{\{\text{supp}\mathcal{N}_i^{(r_i)} \cap \cdot \neq \emptyset\}}, \quad \text{for all} \ \beta \in (0,1),
$$

as Choquet α-Fréchet random sup-measures on $(E,E)$. Indeed, the expressions in the middle and on the right-hand side of (2.3) are Choquet α-Fréchet random sup-measures \cite[Theorem 4.4]{33}. Therefore it suffices to compute the extremal coefficient functionals. Write $\mathcal{F}_K = \{F \in \mathcal{F}(E) : F \cap K \neq \emptyset\}$. Then,

$$
\mathbb{P}\left( \bigcup_{i=1}^{\infty} \xi_i \mathbb{1}_{(\mathcal{R}_\beta,i \cap K \neq \emptyset)} \leq z \right) = \exp \left( -z^{-\alpha}\mu_\beta(\mathcal{F}_K) \right) = \exp \left( -z^{-\alpha}\frac{1}{\Gamma(1 - \beta)} \int_0^{\infty} \beta r^{-\beta-1} \left( 1 - e^{-\mu(K)r} \right) dr \right)
$$

$$
= \exp \left( -z^{-\alpha}\mu_\beta(K) \right),
$$

where the expression after the second equality is the extremal coefficient functional for the right-hand side of (2.3).

2.2. Poisson–Karlin model and its scaling limit. We introduce the Poisson–Karlin model, of which the special case $(E,E) = (\mathbb{R}_+, B(\mathbb{R}_+))$ is the poissonized version of the model discussed in introduction. We shall then prove that the empirical random sup-measures of the Poisson–Karlin model converge in distribution to the Karlin random sup-measure.

From now on, we restrict ourselves to the case that

$$
\mu(E) = 1,
$$
and we have seen in this case,

\[(2.4) \quad M_{\alpha,\beta}(\cdot) \stackrel{d}{=} \sup_{\ell \in \mathbb{N}} \frac{1}{\Gamma(1/\alpha)} \mathbb{I}\{\mathcal{R}_{\beta,\ell} \cap \cdot \neq \emptyset\} \quad \text{with} \quad \sum_{\ell=1}^{\infty} \delta_{(\Gamma_{\ell},\mathcal{R}_{\beta,\ell})} \sim \text{PPP}(\mathbb{R}_+ \times \mathcal{F}_0(E), dx dP_{\mathcal{R}_\beta}). \]

Now for the Poisson–Karlin model, introduce the following families of random variables, and assume all three families are independent.

- Let \( \{U_i\}_{i \in \mathbb{N}} \) be i.i.d. random elements from \( E \) with law.
- Let \( \{\varepsilon_{\ell}\}_{\ell \in \mathbb{N}} \) be i.i.d. non-negative random variables.
- Let \( \{Y_i\}_{i \in \mathbb{N}} \) be i.i.d. \( \mathbb{N} \)-valued random variables.

Write

\[ p_k := P(Y_1 = k), k \in \mathbb{N} \quad \text{and} \quad \nu := \sum_{k \in \mathbb{N}} \delta_{1/p_k}. \]

The key assumptions for the Poisson–Karlin model, in addition to the above, are that

\[ F_{\varepsilon}(x) \in \text{RV}^{-\alpha} \]

\( \{p_k\}_{k \in \mathbb{N}} \) are non-increasing in \( k \) and that \( \nu \) satisfies

\[(2.5) \quad \nu(x) := \nu((0,x]) = \max \left\{ k \in \mathbb{N} : \frac{1}{p_k} \leq x \right\} = x^{\beta} L(x), \quad x > 0, \beta \in (0,1), \]

for a slowly varying function \( L \) at infinity.

Let \( N(\lambda) \) be another Poisson random variable with mean \( \lambda > 0 \), independent from the above. Then, by the Poisson–Karlin model we refer to the following point process

\[ N(\lambda) \sum_{i=1}^{N(\lambda)} \delta_{(\varepsilon_{Y_i}, U_i)} \]

and in this paper we are interested in its empirical random sup-measure defined as

\[ M_{\lambda}(\cdot) := \max_{i: U_i \in \cdot} \varepsilon_{Y_i}. \]

When considering limit theorems, without loss of generality we examine only \( \lambda = n \in \mathbb{N} \). The following result generalizes the main result of \([14]\). (See \([16]\) for how the Poisson–Karlin model leads to sum-stable random fields.) We let \( \mathfrak{M}_p(S) \) denote the space of Radon point measures on a topological space \( S \).

**Theorem 2.1.** Consider the Poisson–Karlin model. Assume that \( F_{\varepsilon}(x) \in \text{RV}_{-\alpha} \) with \( \alpha > 0 \), and \( \nu \) satisfies \((2.5)\). Consider

\[ R_{n,\ell} := \bigcup_{i=1,\ldots,N(n); Y_i=\ell} \{U_i\}, \quad \ell \in \mathbb{N}. \]

Then for \( \{a_n\}_{n \in \mathbb{N}} \) satisfying

\[ \lim_{n \to \infty} \Gamma(1-\beta)\nu(n) F_{\varepsilon}(a_n) = 1, \]

we have (with notations from \((2.4)\))

\[(2.6) \quad \sum_{\ell=1}^{\infty} \delta_{(\varepsilon_{a_n} R_{n,\ell})} \Rightarrow \sum_{\ell=1}^{\infty} \delta_{(1/\ell^{1/\alpha} R_{\beta,\ell})}, \]

as \( n \to \infty \) in \( \mathfrak{M}_p((0,\infty] \times \mathcal{F}_0(E)) \), and as a consequence,

\[ \frac{1}{a_n} M_n \Rightarrow M_{\alpha,\beta} \]

as \( n \to \infty \) in \( \text{SM}(E) \).

**Remark 2.3.** It is known that the convergence of the point processes \((2.6)\) implies the convergence of the corresponding random sup-measures. See \([14]\) Theorem 4.2. So for all our limit theorems we only establish the point-process convergence.
Sketch of the proof. The proof is essentially the same as in [14, Theorem 4.1], where the case \((E,\mathcal{E}) = ([0,1],\mathcal{B}([0,1]))\) was considered. We only sketch the key steps shedding light on how the Sibuya distribution appears in the limit. We first introduce the following statistics.

\[
(2.7) \quad \begin{align*}
K_{n,\ell} &:= \sum_{i=1}^{N(n)} \mathbb{1}(Y_i = \ell), \quad K_n := \sum_{\ell=1}^{\infty} \mathbb{1}(K_{n,\ell} > 0), \quad J_{n,k} := \sum_{\ell=1}^{\infty} \mathbb{1}(K_{n,\ell} = k).
\end{align*}
\]

Note that the left-hand side of (2.6) is restricted to \(M_p((0,\infty] \times \mathcal{F}_0(E))\), so the points corresponding to those \(\ell\) such that \(R_{n,\ell} = \emptyset\) are not involved. Let \(\hat{L}_n\) denote the collection of all such \(\ell\). So \(|\hat{L}_n| = K_n\). Then we rewrite the left-hand side of (2.6) as

\[
\sum_{\ell \in \hat{L}_n} \delta(\varepsilon_{\ell,a_n,R_{n,\ell}}).
\]

Next, we order \(\{\varepsilon_{\ell}\}_{\ell \in \hat{L}_n}\) in decreasing order, and assume that there are no ties for the sake of simplicity. Let \(\{\hat{\ell}_{n,1},\ldots,\hat{\ell}_{n,K_n}\} = \hat{L}_n\) be the corresponding relabellings such that the reordering is \(\varepsilon_{\hat{\ell}_{n,1}} > \cdots > \varepsilon_{\hat{\ell}_{n,K_n}}\).

It is a standard argument to focus first on say the top \(r\) largest \(\varepsilon\), and then let \(r \to \infty\) eventually. We only elaborate the first part here, and fix \(r \in \mathbb{N}\). The goal is then to show that

\[
\sum_{i=1}^{r} \delta(\varepsilon_{\hat{\ell}_{n,i},a_n,R_{n,\hat{\ell}_{n,i}}}) \Rightarrow \sum_{i=1}^{r} \delta(\Gamma_{1/\alpha},\ldots,\Gamma_{r/\alpha}).
\]

To see the above holds, we first recall that \([17, 24]\)

\[
\lim_{n \to \infty} \frac{K_n}{\nu(n)} = \Gamma(1 - \beta) \text{ almost surely.}
\]

So, since \(K_n \mathcal{F}_\varepsilon(a_n) \sim 1\) almost surely, we have

\[
\frac{1}{a_n} (\varepsilon_{\hat{\ell}_{n,1}},\ldots,\varepsilon_{\hat{\ell}_{n,r}}) \Rightarrow (\Gamma_1^{-1/\alpha},\ldots,\Gamma_r^{-1/\alpha}),
\]

following from a well-known fact in extreme-value theory for i.i.d. random variables with power-law tails \([38]\), and it remains to show

\[
\left(R_{n,\hat{\ell}_{n,1}},\ldots,R_{n,\hat{\ell}_{n,r}}\right) \Rightarrow (R_{\beta,1},\ldots,R_{\beta,r}).
\]

In view of the representation of \(R_{n,\ell}\) and \(R_{\beta,\ell}\), it suffices to prove that, for \(\hat{Q}_{n,s} := |R_{n,\hat{\ell}_{n,s}}|\),

\[
\left(\hat{Q}_{n,1},\ldots,\hat{Q}_{n,r}\right) \Rightarrow (Q_{\beta,1},\ldots,Q_{\beta,r}).
\]

Since \(K_{n,\ell} = |R_{n,\ell}|\), the left-hand side corresponds to the law of sampling without replacement of \(r\) elements from \(K_n\) elements \(\{K_{n,\ell}\}_{\ell \in \mathbb{N}: K_{n,\ell} > 0}\), consisting of \(J_{n,k}\) of \(k\) for each \(k \in \mathbb{N}\). So we have, for \(r\) fixed,

\[
\mathbb{P} \left(\hat{Q}_{n,j} = k\right) = \frac{J_{n,k}}{K_n}, \quad k \in \mathbb{N}, \; j = 1,\ldots,r.
\]

Moreover, it is easy to show that \(\hat{Q}_{n,1},\ldots,\hat{Q}_{n,r}\) are asymptotically independent. Therefore, it remains to show that \(\hat{Q}_{n,1} \Rightarrow Q_{\beta}\), which is a well known fact for the Karlin model (a.k.a. the paintbox random partition \([36]\)).
3. Phase transitions for Poisson–Karlin model with multiplicative noise

We introduce multiplicative noise to the Poisson–Karlin model. Let \( \{Z_i\}_{i \in \mathbb{N}} \) be non-negative i.i.d. random variables. For a brief overview, we assume that \( F_Z(x) \in RV_{\beta/\alpha} \) for some \( \alpha > 0 \), and this condition might be relaxed or strengthened later. Assume furthermore that \( \{Z_i\}_{i \in \mathbb{N}} \) are independent from the Poisson–Karlin model introduced above. We are interested in the random sup-measure \( M_n \) defined by

\[
M_n(\cdot) := \sup_{i=1, \ldots, N(n)} \varepsilon_i Z_i.
\]

There are three different regimes for the scaling limits of \( M_n \) depending on the relation between \( \alpha \) and \( \beta/\alpha \).

(i) Signal-dominance regime: \( \alpha < \beta/\alpha \),

\[
\frac{1}{a_n} M_n(\cdot) \Rightarrow M_{\alpha, \beta, Z}(\cdot) := \max_{\ell=1}^{\infty} 1^{1/\alpha} \max_{i=1, \ldots, Q_{\ell, \epsilon}} Z_{\ell,i} \mathbb{1}_{\{U_{\ell,i} \leq \cdot\}} \quad \text{for some } a_n \in RV_{\beta/\alpha},
\]

in SM(\(E\)) (Theorem 3.2, Section 3.2).

(ii) Critical regime: \( \alpha = \beta/\alpha \)

\[
\frac{1}{b_n} M_n \Rightarrow S_{\beta}^{1/\alpha} \cdot M_{\alpha}^{\text{is}}, \quad \text{for some } b_n \in RV_{\beta/\alpha},
\]

where \( S_{\beta} \) is a totally skewed \( \beta \)-stable random variable independent from \( M_{\alpha}^{\text{is}} \) (Theorem 3.3, Section 3.3).

(iii) Noise-dominance regime: \( \alpha > \beta/\alpha \), with \( \varepsilon := \sigma(\{\varepsilon_\ell\}_{\ell \in \mathbb{N}}) \) and \( E_{\varepsilon}(\cdot) := E(\cdot \mid \varepsilon) \),

\[
\frac{1}{c_n} M_n \Rightarrow \left( E_{\varepsilon} \varepsilon'_{\varepsilon} \right)^{1/\alpha'} \cdot M_{\alpha}^{\text{is}} \quad \text{almost surely, conditionally on } \varepsilon, \text{for some } c_n \in RV_{1/\alpha'},
\]

in SM(\(E\)) (Theorem 3.1, Section 3.1). The above convergence is understood as the almost-sure weak convergence with respect to \( \mathcal{E} \). That is,

\[
\lim_{n \to \infty} E_{\varepsilon} f(c_n^{-1} M_n) = E_{\varepsilon} f \left( \left( E_{\varepsilon} \varepsilon'_{\varepsilon} \right)^{1/\alpha'} \cdot M_{\alpha}^{\text{is}} \right) \quad \text{almost surely},
\]

for all continuous and bounded functions \( f : \text{SM}(\mathcal{E}) \to \mathbb{R} \).

**Remark 3.1.** At the critical regime, the limit is known as the \((\alpha, \beta)\)-logistic random sup-measure on \((E, \mathcal{E})\) with control measure \( \mu \), denoted by \( M_{\alpha, \beta}^{\text{lo}} \). However, we are unaware of any example that \( M_{\alpha, \beta}^{\text{lo}} \) arises from the extremes of a stationary sequence. It is an \( \alpha \)-Fréchet random sup-measure, with an equivalent series representation as

\[
M_{\alpha, \beta}^{\text{lo}}(\cdot) \equiv \sup_{\ell \in \mathcal{E}} J_{\ell}^{1/\alpha} \cdot M_{\alpha, \ell}^{\text{is}}(\cdot),
\]

where

\[
\mathcal{J} := \sum_{\ell=1}^{\infty} \delta_{J_\ell} \sim \text{PPP}([0, \infty], \Gamma(1 - \beta)^{-1} \beta x^{-\beta - 1} dx),
\]

(corresponding to the jumps of a standard \( \beta \)-stable subordinator up to time 1; in particular \( S_{\beta} \equiv \sum_{\ell=1}^{\infty} J_\ell \)) and \( \{M_{\alpha, \ell}^{\text{is}}\}_{\ell \in \mathcal{E}} \) are i.i.d. copies of \( M_{\alpha}^{\text{is}} \), independent from \( \mathcal{J} \). Moreover,

\[
\mathbb{P}(M_{\alpha, \beta}^{\text{lo}}(A_i) \leq x_i, i = 1, \ldots, d) = \exp \left( - \left( \sum_{i=1}^{d} \frac{\mu(A_i)}{x_i^{\alpha'}} \right)^{\beta} \right), \quad x_1, \ldots, x_d > 0,
\]

for all disjoint \( A_i \in \mathcal{E} \), and the joint law is known as the multivariate logistic extreme-value distribution. This family of distributions was first considered by Gumbel (19) (see e.g. (15) for more references and some recent developments). A combinatorial structure underlying the logistic Fréchet random sup-measure was recently
pointed out in [43, Remark 3.5], where the name sub-max-stable was also used (in parallel to sub-stable processes [40]).

Arising at the critical regime, one may wonder what special properties the logistic random sup-measures enjoy. From (3.1), it is immediately seen that \( \mathcal{M}^{\alpha,\beta} \) is exchangeable in the sense that \( \{ \mathcal{M}^{\alpha,\beta}(A_i) \}_{i=1,\ldots,d} \) have the same joint law for all disjoint \( \{ A_i \}_{i=1,\ldots,d} \) with the same values \( \{ \mu(A_i) \}_{i=1,\ldots,d} \); when defined on \( \mathbb{R}^d \) with \( \mu \) being the Lebesgue measure, it is also translation-invariant and self-similar in the usual sense. We also mention the following relation that is close to (but not) an invariance property. For \( \gamma, \beta \in (0,1), \alpha > 0 \),

\[
S_\gamma^{1/\alpha} \cdot \mathcal{M}^{\alpha,\beta} \overset{d}{=} \mathcal{M}^{\alpha,\beta}\gamma,
\]

which follows from (3.1) by conditioning on \( S_\gamma \) first (the skewed \( \beta \)-stable random variable \( S_\gamma \) is independent from \( \mathcal{M}^{\alpha,\beta} \)). Some simulation examples are provided in Figure 2.

The proofs of each regime are ordered according to their difficulties for the rest of this section. Some further properties of the limit random sup-measures will also be developed. We only prove the corresponding point-process convergence in each case (see Remark 2.3). Throughout, we let \( C \) denote a strictly positive constant that may change from line to line.

3.1. Noise-dominance regime. The main theorem in this regime is the following.

**Theorem 3.1.** Assume that \( \overline{F}_Z(x) \in RV_{-\alpha'} \) and \( \mathbb{E} \varepsilon_Y^{\alpha'+\epsilon} < \infty \) for some \( \epsilon > 0 \). Then, for any sequence \( \{ c_n \}_{n \in \mathbb{N}} \) such that

\[
\lim_{n \to \infty} n \overline{F}_Z(c_n) = 1,
\]

conditionally on \( \varepsilon \),

\[
\sum_{i=1}^{N(n)} \delta_{(\varepsilon_Y Z_i, c_n U_i)} \Rightarrow \sum_{\ell=1}^{\infty} \delta_{(\varepsilon_Y \Gamma^{-1/\alpha'}_\varepsilon U_i)}
\]

in \( \mathcal{M}_p((0,\infty) \times E) \) almost surely. As a consequence, conditionally on \( \varepsilon \),

\[
\frac{1}{c_n} M_n \Rightarrow \left( \mathbb{E} \varepsilon_Y^{\alpha'} \right)^{1/\alpha'} \cdot \mathcal{M}^{\alpha,\beta}_\gamma
\]

in \( \text{SM}(E) \) almost surely.

Before proving the limit theorem, we first examine the limit random sup-measure.

**Lemma 3.1.** Assume that \( \mathbb{E} \varepsilon_Y^{\alpha'} < \infty \) almost surely. Then,

\[
(\mathbb{E} \varepsilon_Y^{\alpha'})^{1/\alpha'} \cdot \mathcal{M}^{\alpha,\beta}(\cdot) = \sup_{i \in \mathbb{N}} \frac{1}{\Gamma^{1/\alpha'}_\varepsilon \varepsilon_Y} \mathbb{1}_{(U_i, \cdot)}.
\]

**Proof.** Indeed, given \( \varepsilon \), \( \{ \varepsilon_Y \}_{i \in \mathbb{N}} \) are i.i.d. random variables, and the above follows from

\[
\sum_{\ell=1}^{\infty} \delta_{\varepsilon_Y^{-1/\alpha'} U_i} \overset{d}{=} \frac{1}{c_n} M_n \overset{d}{=} \sum_{i=1}^{\infty} \delta_{(\varepsilon_Y^{\alpha'})^{1/\alpha'} \varepsilon_Y^{-1/\alpha'}} \text{ almost surely.}
\]

Conditionally on \( \varepsilon \), the left-hand side is again a Poisson point process [38, Proposition 5.2], and it suffices to compute the intensity measure evaluated at the region \( (z, \infty) \), which equals

\[
\int_0^{\infty} \int_0^{\infty} \mathbb{1}_{\{ y > z \}} dF_{\varepsilon_Y}(x, dy) = z^{-\alpha'} \int_0^{\infty} x^{-\alpha'} dF_{\varepsilon_Y}(x) = z^{-\alpha'} \mathbb{E} \varepsilon_Y^{\alpha'} \text{ almost surely.}
\]

**Proof of Theorem 3.1.** We shall then work with the representation of the limit random sup-measure based on the left-hand side of (3.2). It suffices to prove the convergence of point processes. Since \( \overline{F}_Z(x) \in RV_{-\alpha'} \), it follows that [38, Theorem 5.3]

\[
\sum_{i=1}^{N(n)} \delta_{z_i/c_n} \Rightarrow \sum_{i=1}^{\infty} \delta_{\varepsilon_Y^{1/\alpha'}},
\]
whence, conditioning on \( \varepsilon \),

\[
(3.3) \quad \sum_{i=1}^{N(n)} \delta(z_{i,n}, \varepsilon Y_i, U_i) \Rightarrow \sum_{i=1}^{\infty} \delta(T^{-1/\alpha', \varepsilon} Y_i)
\]

in \( \mathcal{M}_p((0, \infty) \times (0, \infty) \times E) \), almost surely. The third coordinates can be viewed as i.i.d. marks and do not change in the limiting procedure, and hence can be omitted in the analysis. The goal is then to show that (3.3) implies

\[
(3.4) \quad \sum_{i=1}^{N(n)} \delta(z_{i,n}, \varepsilon Y_i, \varepsilon) \Rightarrow \sum_{i=1}^{\infty} \delta(T^{-1/\alpha' \varepsilon} Y_i)
\]
as \( n \to \infty \) in \( \mathcal{M}_p((0, \infty)) \), almost surely. Here \( N(n) \) is Poisson with parameter \( n \). Remark that if one replaces \( N(n) \) by \( n \) above, [35] Proposition 7.5] proves exactly that (3.3) implies (3.4), provided \( \mathbb{E} \varepsilon e^{\alpha' \varepsilon} < \infty \) almost surely for some \( \varepsilon > 0 \). Since \( N(n) \) is independent from the other random variables, the analysis here is essentially the same. We omit the details. \( \square \)

We conclude this section by elaborating on the conditions \( \mathbb{E} \varepsilon e^{\alpha' \varepsilon} < \infty, \varepsilon \geq 0 \). Note that in our limit theorem we need \( \varepsilon > 0 \), while for the limit random sup-measure to be finite almost surely, \( \varepsilon = 0 \) is sufficient (and this condition is also necessary). We say a function \( f \) is dominated by a function \( g \in RV_\gamma \) at infinity, if for all \( x \) large enough, \( f(x) \leq C g(x) \).

**Lemma 3.2.** For \( \alpha > \alpha' \beta \), assume the following assumptions:

(i) \( \overline{F}_z(x) \in RV_{-\alpha'} \),

(ii) \( \overline{F}_\varepsilon(x) \) is dominated by a function in \( RV_{-\alpha} \) at infinity,

(iii) \( \nu(x) \) (recall (2.5)) is dominated by a function in \( RV_\beta \) at infinity.

Then, \( \mathbb{E} \varepsilon e^{\alpha' \varepsilon} < \infty \) almost surely for all \( \varepsilon \in [0, \alpha/\beta - \alpha'] \).

**Proof.** By definition, \( \mathbb{E} \varepsilon e^{\alpha' \varepsilon} = \sum_{\ell=1}^{\infty} p\ell e^{\alpha' \varepsilon} \). The convergence of this series follows from the Kolmogorov’s three-series theorem. Indeed, first for any \( c > 0 \),

\[
\sum_{\ell=1}^{\infty} \mathbb{P}(p\ell e^{\alpha' \varepsilon} > c) = \sum_{\ell=1}^{\infty} \overline{F}_\varepsilon((c/p\ell)^{1/(\alpha' + \varepsilon)}) \leq C \sum_{\ell=1}^{\infty} p\ell^{\alpha/\alpha' \varepsilon} - \varepsilon_1
\]

for some small \( \varepsilon_1 > 0 \) by Potter’s bound. Assume that (2.5) holds, which is equivalent to that \( p\ell \in RV_{-1/\beta} \) as \( \ell \to \infty \), and hence the above is bounded by \( C \sum_{\ell=1}^{\infty} \ell^{-(1/\beta - \varepsilon_1)(\alpha/\alpha' + \varepsilon_1)} \) by Potter’s bound again. By the assumption \( \alpha' + \varepsilon < \alpha/\beta \), one can tune \( \varepsilon_1 > 0 \) small enough so that the power over \( \ell \) is strictly less than \(-1 \), and hence the series is finite. Next, choose \( \beta' \in (\beta, 1 \wedge \alpha/(\alpha' + \varepsilon)) \). Then, \( \sum_{\ell=1}^{\infty} p_\ell^{\beta'} < \infty \) as \( \beta' > \beta \) and \( \mathbb{E} e^{\alpha' \beta'} < \infty \) as \( (\alpha' + \varepsilon) \beta' < \alpha \). It then follows that

\[
\sum_{\ell=1}^{\infty} \mathbb{E} \left( p\ell e^{\alpha' \varepsilon} \mathbb{1}_{\{p\ell e^{\alpha' \varepsilon} \leq c\}} \right) \leq C \sum_{\ell=1}^{\infty} p\ell^{\beta'} \mathbb{E} e^{\alpha' \varepsilon} \beta' < \infty
\]

and \( \sum_{\ell=1}^{\infty} \mathbb{V}(p\ell e^{\alpha' \varepsilon} \mathbb{1}_{\{p\ell e^{\alpha' \varepsilon} \leq c\}}) < \infty \), where \( \mathbb{V} \) stands for the variance. The proof can be modified to prove the case that \( \nu(x) \) is dominated by a function in \( RV_\beta \). \( \square \)

**Remark 3.2.** Assume that \( \nu \) satisfies (2.5), \( \overline{F}_z(x) \in RV_{-\alpha} \) and \( \overline{F}_\varepsilon(x) \in RV_{-\alpha'}. \) The above says that if \( \alpha > \alpha' \beta \) then \( \mathbb{E} \varepsilon e^{\alpha' \varepsilon} < \infty \) almost surely. For this to hold at the boundary case when \( \alpha = \alpha' \beta \), a necessary and sufficient condition is that

\[
(3.5) \quad \mathbb{E} \nu \left( \epsilon^{\alpha'} \right) < \infty.
\]
In particular when \( \nu(n) \sim Cn^\beta \), the above is equivalent to \( \mathbb{E}x^{\alpha'/\beta} < \infty \). To see this, apply the three-series theorem to \( \mathbb{E}x^{\alpha'/\beta} = \sum_{\ell=1}^{\infty} p_{\ell}x^{\alpha'/\ell} \). The first series becomes
\[
(3.6) \quad \sum_{\ell=1}^{\infty} \mathbb{P}(p_{\ell}x^{\alpha'/\ell} > 1) = \int_{0}^{\infty} F_x(x^{1/\alpha'}) \nu(dx) = \int_{0}^{\infty} F_x(y) \nu(dy).
\]

Note that, by integration by parts,
\[
(3.7) \quad \int_{0}^{\alpha} F_x(y) \nu(dy) = F_x(a) \nu(a^{\alpha'}) + \int_{0}^{\alpha} \nu(y^{\alpha'}) dF_x(y).
\]

Observe also that 0 \( \leq \mathbb{F}_x(a) \nu(x^{\alpha'}) \leq \mathbb{E} \nu(x^{\alpha'}) \) for \( \alpha > 0 \) and that \( \int_{0}^{\alpha} \nu(y^{\alpha'}) dF_x(y) = \mathbb{E} \nu(x^{\alpha'}) \) as \( x \to \infty \) with \( \alpha' > \alpha \), we have that
\[
\sum_{\ell=1}^{\infty} \mathbb{E}(p_{\ell}x^{\alpha'/\ell} \mathbb{1}_{p_{\ell}x^{\alpha'/\ell} \leq 1}) \leq C \sum_{\ell=1}^{\infty} F_x(p_{\ell}^{-1}x^{\alpha'/\ell}) = C \int_{0}^{\infty} F_x(x^{1/\alpha'}) \nu(dx),
\]
the same upper bound as in (3.6). The third series can be treated similarly. Note that (3.7) also says that if \( \alpha < \alpha' \), then \( \mathbb{E}x^{\alpha'/\beta} = \infty \) almost surely.

3.2. Signal-dominance regime. Throughout we write
\[
(3.8) \quad Z_W = \max_{i=1,\ldots,W} Z_i,
\]
where \( W \) is an \( N \)-valued random variable (possibly a constant) that is assumed to be independent from \( \{Z_i\}_{i \in \mathbb{N}} \). The main theorem of this regime is the following.

**Theorem 3.2.** Assume that \( \mathbb{F}_x(x) \in RV_{-\alpha} \), and that
\[
(3.9) \quad \mathbb{E}Z_{Q_{\beta-\epsilon}}^{a+\epsilon} < \infty \text{ for some } \epsilon > 0.
\]

For any sequence \( \{a_n\}_{n \in \mathbb{N}} \) such that
\[
(3.10) \quad \lim_{n \to \infty} (1-\beta) \nu(n) \mathbb{F}_x(a_n) = 1,
\]
we have
\[
(3.11) \quad \sum_{i=1}^{N(n)} \delta_{(z_i, Z_i)} = \sum_{i=1}^{Q_{\beta-\epsilon}} \delta_{(Z_i)} = \sum_{i=1}^{Q_{\beta-\epsilon}} \delta_{(Z_i)} \mathbb{1}_{\{U_i, i \in A\}}
\]
as \( n \to \infty \) in \( \mathcal{M}_p((0, \infty) \times E) \). As a consequence,
\[
\frac{1}{a_n} M_n(\cdot) \Rightarrow M_{\alpha, \beta, Z}(\cdot) := \bigvee_{\ell=1}^{Q_{\beta-\epsilon}} \frac{1}{\Gamma_{\ell}} \max_{i=1,\ldots,Q_{\beta-\epsilon}} Z_{t,i} \mathbb{1}_{\{U_i, i \in A\}},
\]
as \( n \to \infty \) in \( \text{SM}(E) \).

Notice that it is straightforward to see that for \( M_{\alpha, \beta, Z} \) to be almost surely finite, a sufficient and necessary condition is \( \mathbb{E}Z_{Q_{\beta-\epsilon}}^{a+\epsilon} < \infty \). Indeed, for every open set \( A \subset E \), writing \( Z_{Q_{\beta-\epsilon}}(A) := \max_{i=1,\ldots,Q_{\beta-\epsilon}} U_{t,i} Z_i \),
\[
\mathbb{P}(M_{\alpha, \beta, Z}(A) \leq z) = \mathbb{P} \left( \sup_{\ell \geq 1} \frac{1}{\Gamma_{\ell}} \max_{i=1,\ldots,Q_{\beta-\epsilon}} Z_{t,i} \leq z \right)
= \exp \left( -\int_{0}^{z} \mathbb{F}_{Q_{\beta-\epsilon}}(A) x^{1/\alpha} \alpha^{-1} dx \right) = \exp \left( -z^{-\alpha} \mathbb{E}Z_{Q_{\beta-\epsilon}}^{a+\epsilon} \right),
\]
Again, the condition for $M_{\alpha, \beta, Z}$ to be finite almost surely (3.9) with $\epsilon = 0$ is strictly weaker than what is needed for the convergence. To see that (3.9) holds for $F_Z \in RV_{-\alpha'}$ with $\alpha < \alpha' \beta$, it suffices to pick $\epsilon > 0$ such that $\alpha + \epsilon < (\beta - \epsilon)\alpha'$. Indeed,

$$(3.12) \quad \mathbb{E}Z_{Q_{\beta, \epsilon}}^{\alpha+\epsilon} = C \int_0^{\infty} x^{\alpha+\epsilon-1} F_{Z_{Q_{\beta, \epsilon}}}(x) dx \leq C \left( 1 + \int_1^{\infty} x^{\alpha+\epsilon-1} x^{-(\beta-\epsilon)\alpha'} L_Z(x) dx \right)$$

for some slowly-varying function $L_Z$, where in the last step we used $F_{Z_{Q_{\beta}}}(x) = F_Z(x)$.\[\]

**Remark 3.3.** In view of (3.12), the condition $\mathbb{E}Z_{Q_{\beta, \epsilon}}^{\alpha+\epsilon} < \infty$ is slightly more restrict than $\mathbb{E}Z_{Q_{\beta}}^{\alpha} < \infty$. This is similar in spirit to the condition in Breiman’s Lemma: for non-negative independent random variables $X, Y$, $F_Y(x) \in RV_{-\alpha}$, for the limit theorem $\lim_{x \to \infty} F_{XY}(x)/F_Y(x) = \mathbb{E}X^\alpha$ to hold, one needs $\mathbb{E}X^{\alpha+\epsilon} < \infty$ for some $\epsilon > 0$.\[\]

**Proof of Theorem 3.2.** We focus on (3.11). Recall $K_{n, \ell}$ in (2.7). We have seen in Theorem 2.1 that

$$\sum_{i=1}^{N(n)} \delta_{(\varepsilon_Y/a_n, U_i)} \cdots \sum_{i=1}^{N(n)} \delta_{(\varepsilon_Y/a_n, U_i, i)} \Rightarrow \sum_{i=1}^{\infty} \sum_{\ell=1}^{N(n)} \delta_{(\varepsilon_Y/a_n, U_i, i)}$$

whence

$$(3.13) \quad \sum_{i=1}^{N(n)} \delta_{(\varepsilon_Y/a_n, Z_i, U_i)} \Rightarrow \sum_{i=1}^{\infty} \sum_{\ell=1}^{N(n)} \delta_{(\varepsilon_Y/a_n, Z_i, U_i, i)}$$

The third coordinates of the points can be viewed as i.i.d. marks and they do not change in the limit. So it suffices to focus on

$$\eta_n := \sum_{i=1}^{N(n)} \sum_{\ell=1}^{K_{n, \ell}} \delta_{(\varepsilon_Y Z_i/a_n)} \quad \text{and} \quad \eta := \sum_{i=1}^{\infty} \sum_{\ell=1}^{Q_{\beta, \ell}} \delta_{(\varepsilon_Y Z_i/a_n)}.$$

and prove

$$\eta_n \Rightarrow \eta \quad \text{in} \quad \mathcal{M}_p([0, \infty]).$$

Note that we cannot directly apply the product functional to (3.13) as $\{(x, y) \in (0, \infty] \times (0, \infty) : |xy| \geq 1\}$ is not compact in $(0, \infty] \times (0, \infty)$. The proof follows the approach of Resnick [38, Proposition 7.5]. Let $\delta \in (0, 1)$ and

$$\Lambda_\delta := \{ (x, y) \in (0, \infty] \times (0, \infty) : x \geq \delta, y \in [\delta, \delta^{-1}] \}.$$\[\]

It is a compact subset of $(0, \infty] \times (0, \infty)$ and by restriction,

$$\eta_n(\Lambda_\delta \cap \cdot) \Rightarrow \eta(\Lambda_\delta \cap \cdot) \quad \text{in} \quad \mathcal{M}_p([0, \infty]).$$

Since for any $c > 0$, $(x, y) \in \Lambda_\delta : |xy| \geq c$ is a compact subset of $\Lambda_\delta$, we can use the product functional to get

$$\eta_n, \delta := \sum_{i=1}^{N(n)} \delta_{\varepsilon_Y Z_i/a_n} 1_{\{xy/a_n \in \Lambda_\delta\}} \Rightarrow \eta_\delta := \sum_{i=1}^{\infty} \sum_{\ell=1}^{Q_{\beta, \ell}} \delta_{\varepsilon_Y Z_i/a_n} 1_{\{xy/a_n \in \Lambda_\delta\}},$$

as $n \to \infty$ in $\mathcal{M}_p([0, \infty])$. Further, $\eta_\delta \Rightarrow \eta$, as $\delta \downarrow 0$. To conclude, it remains to prove that for all positive continuous function $f$ with compact support in $(0, \infty]$ and all $\epsilon > 0$,

$$\lim_{\delta \downarrow 0} \limsup_{n \to \infty} \mathbb{P} \left( \sum_{i=1}^{N(n)} f(\varepsilon_Y Z_i/a_n) 1_{\{xy/a_n \in \Lambda_\delta\}} \geq \epsilon \right) = 0.$$\[\]

Fix such a function $f$ and a real $\kappa > 0$ such that $f \equiv 0$ on $(0, \kappa)$. It is sufficient to prove that

$$(3.14) \quad \lim_{\delta \downarrow 0} \limsup_{n \to \infty} \mathbb{P} \left( \bigcup_{i=1}^{N(n)} \{\varepsilon_Y Z_i/a_n > \kappa, (\varepsilon_Y/a_n, Z_i) \in \Lambda_\delta\} \right) = 0,$$
where \( \Lambda_\delta^c := ((0, \infty] \times (0, \infty)) \setminus \Lambda_\delta \). The proof of (3.14) is divided into 4 steps by writing \( \Lambda_\delta^c \) as the disjoint union of the sets

\[
\begin{align*}
A_{1,\delta} &:= \left\{ (x, y) \in (0, \infty] \times (0, \infty) : x < \delta, y < \delta^{-1/2} \right\}, \\
A_{2,\delta} &:= \left\{ (x, y) \in (0, \infty] \times (0, \infty) : x < \delta, y \geq \delta^{-1/2} \right\}, \\
A_{3,\delta} &:= \left\{ (x, y) \in (0, \infty] \times (0, \infty) : x \geq \delta, y < \delta \right\}, \\
A_{4,\delta} &:= \left\{ (x, y) \in (0, \infty] \times (0, \infty) : x \geq \delta, y > \delta^{-2} \right\}.
\end{align*}
\]

Write

\[E_{j,\delta,n} := P\left( \bigcup_{i=1}^{N(n)} \left\{ \varepsilon_{Y_i} Z_i / a_n > \kappa, (\varepsilon_{Y_i} / a_n, Z_i) \in A_{j,\delta} \right\} \right).\]

1) If \((\varepsilon_{Y_i} / a_n, Z_i) \in A_{1,\delta}\), then \(\varepsilon_{Y_i} Z_i / a_n < \delta^{1/2}\). Thus, when \(\delta^{1/2} \leq \kappa\), \(E_{1,\delta,n} = 0\) for all \(n \in \mathbb{N}\).

2) Let \(Y = \sigma(\{Y_i\}_{i \in \mathbb{N}})\), with respect to which \(K_{n,\ell}\) is measurable. We start by writing that

\[
E_{2,\delta,n} \leq P\left( \bigcup_{i=1}^{N(n)} \left\{ \varepsilon_{Y_i} Z_i / a_n > \kappa, Z_i > \delta^{-1/2} \right\} \right) \leq E\left( \sum_{\ell=1}^{\infty} P\left( \varepsilon_{1,\ell} Z_{K_{n,\ell}} / a_n > \kappa, Z_{K_{n,\ell}} > \delta^{-1/2} \mid Y \right) \right).
\]

\[
= \sum_{k=1}^{\infty} E_{n,k} P\left( \varepsilon_{k} Z_{k} > \kappa a_n, Z_{k} > \delta^{-1/2} \right) =: \overline{E}_{2,\delta,n}.
\]

The goal is to show that

\[\lim_{\delta \downarrow 0} \lim_{n \to \infty} \overline{E}_{2,\delta,n} = 0.\]

Introduce

\[\varphi_{n,k,\delta} := \frac{1}{F_{\varepsilon}(a_n)} P\left( \varepsilon_{k} Z_{k} / a_n > \kappa, Z_{k} > \delta^{-1/2} \right) = E\left( \frac{F_{\varepsilon}(\kappa a_n / Z_{k})}{F_{\varepsilon}(a_n)} I\{Z_{k} > \delta^{-1/2}\} \right).\]

Then \(p_{n,k} := E_{n,k} / E K_n, k \in \mathbb{N}\), yield a probability measure on \(\mathbb{N}\). Let \(Q_n\) be a random variable with such a law, independent from all other random variables. Then,

\[\overline{E}_{2,\delta,n} = E K_n \cdot F_{\varepsilon}(a_n) \cdot E \varphi_{n,Q_n,\delta}.\]

Recall that

\[\lim_{n \to \infty} \frac{E K_n}{\nu(n)} = \Gamma(1 - \beta) \quad \text{and} \quad \lim_{n \to \infty} E K_n \cdot F_{\varepsilon}(a_n) = 1,\]

where the second part follows from the first and our assumption on \(a_n\) in (3.10). We shall argue that

\[\lim_{n \to \infty} E \varphi_{n,Q_n,\delta} = \kappa^{-\alpha} E\left( Z_{Q_n}^\alpha I\{Z_{Q_n} > \delta^{-1/2}\} \right).\]

This and (3.17) shall then conclude the proof of (3.15). The almost-sure convergence of (3.18), in view of (3.16), is straightforward by regular-variation assumption on \(F_{\varepsilon}\) and the fact that \(Q_n \Rightarrow Q_\beta\). To show that expectation also converges, it suffices to prove uniform integrability. Namely, we shall show that for some \(\epsilon_1 > 0\),

\[E_{n,Q_n,\delta} := E\left( \left( \frac{F_{\varepsilon}(\kappa a_n / Z_{\hat{Q}_n})}{F_{\varepsilon}(a_n)} \right)^{1+\epsilon_1} I\{Z_{\hat{Q}_n} > \delta^{-1/2}\} \right) \leq C < \infty \text{ for all } n \in \mathbb{N}.\]
By Potter’s bound [7, Proposition 1.5.6], for some \( \alpha_+ \in (\alpha, \alpha') \) (depending on \( \epsilon_1 \), which can be arbitrarily small)

\[
\mathbb{E} \left( \frac{F_n(\kappa a_+ / \bar{Z}_k)}{F_n(a_n)} \right)^{1+\epsilon_1} 1 \{\bar{Z}_k > \delta - 1/2\} \leq C \bar{Z}_k^{\alpha_+} \quad \text{for all } n, k \in \mathbb{N},
\]

whence

\[
\mathbb{E}_{n, Q_n, \delta} \leq C \mathbb{E} \sum_{k=1}^{\infty} EJ_{n,k} \bar{Z}_k^{\alpha_+} = C \bar{Z}_Q^{\alpha_+}.
\]

We shall compare \( EJ_{n,k} / K_n \) with \( p_k^{(\beta)} \), and eventually show that

\[(3.20) \quad \mathbb{E} \bar{Z}_Q^{\alpha_+} \leq C \left( 1 + \mathbb{E} \bar{Z}_Q^{\alpha_+} \right)
\]

for some \( \beta_- \in (0, \beta) \). Then, under the assumption \( (3.9) \), we can pick \( \beta_- < \beta \) and \( \alpha_+ > \alpha \) so that the right-hand side above is finite, whence \( (3.19) \) and \( (3.15) \) follow. To show \( (3.20) \), introduce

\[(3.21) \quad F_{n,k}(x) := \left( \frac{n}{x} \right)^k e^{-n/x} \quad \text{and } f_{n,k}(y) := \frac{d}{dy} \left( F_{n,k} \left( \frac{n}{y} \right) \right) = (k-y)y^{k-1}e^{-y}.
\]

Then,

\[(3.22) \quad EJ_{n,k} = \frac{1}{k!} \int_0^\infty F_{n,k}(x) \nu(dx) = \frac{1}{k!} \int_0^\infty \frac{d}{dx} F_{n,k}(x) \nu(x)dx
\]

\[= \frac{\nu(n)}{k!} \int_0^{np_1} \frac{d}{dy} \left( F_{n,k} \left( \frac{n}{y} \right) \right) \frac{\nu(n/y)}{\nu(n)} dy = \frac{\nu(n)}{k!} \int_0^{np_1} f_{n,k}(y) - L(n/y) L(n) dy.
\]

Further,

\[(3.23) \quad \Gamma(1 - \beta) p_k^{(\beta)} = \frac{\beta \Gamma(k - \beta)}{k!} = \frac{1}{k!} \int_0^\infty F_{n,k} \left( \frac{n}{y} \right) \beta y^{-\beta - 1} dy = \frac{1}{k!} \int_0^\infty f_{n,k}(y) y^{-\beta} dy.
\]

Note that we cannot compare the two directly as \( f_{n,k} \) is not non-negative. Instead we write

\[
\frac{1}{\mathbb{E} K_n} \sum_{k=1}^{\infty} EJ_{n,k} \bar{Z}_k^{\alpha_+} = \frac{\nu(n)}{\mathbb{E} K_n} \int_0^{np_1} \sum_{k=1}^{\infty} \frac{(k-y)y^{k-1-\beta}}{k!} \bar{Z}_k^{\alpha_+} e^{-y} \frac{L(n/y)}{L(n)} dy,
\]

and deal with the integral over \([0,1]\) and \([1, np_1]\), respectively. First, using that \( \mathbb{E} \bar{Z}_k^{\alpha_+} \leq k \mathbb{E} Z_k^{\alpha_+} \),

\[(3.24) \quad \int_0^1 \sum_{k=1}^{\infty} \frac{(k-y)y^{k-1-\beta}}{k!} \bar{Z}_k^{\alpha_+} e^{-y} \frac{L(n/y)}{L(n)} dy \leq C \int_0^1 \sum_{k=1}^{\infty} \frac{ky^{k-1-\beta}}{(k-1)!} e^{-y} dy \leq C \int_0^1 y^{-\beta} dy \leq C,
\]

for some \( \beta_+ \in (\beta, 1) \), where in the first step we also applied Potter’s bound. Second, for the integral over \([1, np_1]\), we shall use the identity, for any increasing sequence of numbers \( \{D_k\}_{k \in \mathbb{N}} \),

\[(3.25) \quad \sum_{k=1}^{\infty} \frac{(k-y)y^k}{k!} D_k = \sum_{k=0}^{\infty} \frac{y^{k+1}}{k!} (D_{k+1} - D_k).
\]

Then

\[
\int_1^{\infty} \sum_{k=1}^{\infty} \frac{(k-y)y^{k-1-\beta}}{k!} \mathbb{E} Z_k^{\alpha_+} e^{-y} \frac{L(n/y)}{L(n)} dy \leq \int_1^{np_1} \sum_{k=1}^{\infty} \frac{y^{k-\beta}}{k!} (\mathbb{E} Z_{k+1}^{\alpha_+} - \mathbb{E} Z_k^{\alpha_+}) e^{-y} \frac{L(n/y)}{L(n)} dy
\]

\[\leq C \int_1^{np_1} \sum_{k=0}^{\infty} \frac{y^{k-\beta}}{k!} (\mathbb{E} Z_{k+1}^{\alpha_+} - \mathbb{E} Z_k^{\alpha_+}) e^{-y} dy
\]

\[= C \int_1^{np_1} \sum_{k=1}^{\infty} \frac{(k-y)y^{k-1-\beta}}{k!} \mathbb{E} Z_k^{\alpha_+} dy,
\]
for some $\beta_- < \beta$, where we applied \((3.25)\) in the first and the third steps, and Potter’s bound in the second. The last expression above is then bounded from above by $C \sum_{k=1}^{\infty} p_k^{(\beta_-)} E \tilde{Z}_{Q_{\beta_-}}^\alpha = C E \tilde{Z}_{Q_{\beta_-}}^\alpha$. Combined with \((3.24)\), we have shown \((3.20)\).

3) We have

$$E_{\delta, \alpha, n} \leq P \left( \bigcup_{i=1}^{N(n)} \{ \varepsilon_Y, Z_i > \kappa a_n, Z_i < \delta \} \right) \leq P \left( \bigcup_{i=1}^{N(n)} \{ \varepsilon_Y, Z_i > \kappa \delta^{-1} a_n \} \right)$$

$$\leq P \left( \bigcup_{i=1}^{K_n} \{ \varepsilon_Y, Z_i > \kappa \delta^{-1} a_n \} \right) \leq \mathbb{E}K_n \cdot \overline{F}_\varepsilon (\kappa \delta^{-1} a_n).$$

It then follows that $\limsup_{n \to \infty} E_{\delta, \alpha, n} \leq \kappa^{-\alpha} \delta^\alpha$ which converges to 0 as $\delta \to 0$.

4) This time,

$$E_{\delta, \alpha, n} \leq P \left( \bigcup_{i=1}^{N(n)} \{ \varepsilon_Y, Z_i > \delta a_n, Z_i > \delta^{-2} \} \right) \leq P \left( \bigcup_{i=1}^{N(n)} \{ \varepsilon_Y, Z_i > \delta a_n, \tilde{Z}_{K_n, \ell} > \delta^{-2} \} \right)$$

$$\leq \sum_{k=1}^{\infty} EJ_{n,k} \overline{F}_\varepsilon (\delta a_n) \overline{F}_{\tilde{Z}_{Q_{\beta}}} (\delta^{-2}) = \mathbb{E}K_n \cdot \overline{F}_\varepsilon (\delta a_n) \overline{F}_{\tilde{Z}_{Q_{\beta}}} (\delta^{-2}).$$

Then, $\limsup_{n \to \infty} E_{\delta, \alpha, n} \leq \delta^{-\alpha} \overline{F}_{\tilde{Z}_{Q_{\beta}}} (\delta^{-2}) \leq \delta^\alpha \mathbb{E} \tilde{Z}_{Q_{\beta}}^\alpha \to 0$ as $\delta \downarrow 0$. Thus, \((3.14)\) is established and the proposition is proved.

### 3.3. Critical regime

Here we assume $\alpha = \alpha' \beta$. We introduce the following technical assumptions before stating the main theorem in this regime. Recall our notation for $\tilde{Z}_W$ in \((3.8)\). In particular, $\overline{F}_{\tilde{Z}_{Q_{\beta}}} (x) = \overline{F}_Z (x)^\beta$, and in the subscript of $\overline{F}_\varepsilon \tilde{Z}_{Q_{\beta}}$ below $\varepsilon$ and $\tilde{Z}_{Q_{\beta}}$ are understood as independent.

**Assumption 3.1.**

(i) $\overline{F}_\varepsilon (x) \in RV_{-\alpha}$ and $\overline{F}_Z (x) \in RV_{-\alpha'}$.

(ii) $\varepsilon$ has a probability density function $x^{-\alpha-1} l_\varepsilon (x)$, that satisfies

$$\limsup_{x \to \infty} \sup_{y \in [x, x]} \frac{l_\varepsilon (y)}{1 \vee l_\varepsilon (x)^{\beta}} < \infty \text{ for some } x_\varepsilon \geq 0.$$

(iii) $\overline{F}_Z (x) = x^{-\alpha'} L_Z (x)$ with

$$\limsup_{x \to \infty} \sup_{y \in [x, x]} \frac{L_Z (y)}{1 \vee L_Z (x)} < \infty \text{ for some } x_Z \geq 0.$$

(iv) As $x \to \infty$,

$$\max \left\{ \overline{F}_\varepsilon \left( x (1 \wedge L_Z^{-1/\alpha'} (x)) \right), x^{-\alpha}, \overline{F}_Z (x)^\beta \right\} = o \left( \overline{F}_\varepsilon \tilde{Z}_{Q_{\beta}} (x) \right).$$

(v) $\nu (x) \sim C x^\beta$ for some constant $C \in (0, \infty)$ (i.e. $L (x)$ in \((2.5)\) has a limit in $(0, \infty)$ as $x \to \infty$).

**Theorem 3.3.** Under Assumption \ref{assumption3.1} with $\alpha = \alpha' \beta$ and $\{b_n\}_{n \in \mathbb{N}}$ satisfying

$$\lim_{n \to \infty} \Gamma (1 - \beta) \nu (n) \overline{F}_{\tilde{Z}_{Q_{\beta}}} (b_n) = 1,$$

we have

$$\sum_{i=1}^{N(n)} \delta (\varepsilon_Y, Z_i / b_n, U_i) \Rightarrow \sum_{i=1}^{\infty} \delta (s_{i, \beta}^1, y_i / b_i, U_i)$$
as $n \to \infty$ in $\mathfrak{M}_p((0,\infty))$, and

$$\frac{1}{b_n} M_n \to S_{\beta}^{1/\alpha'} \cdot \mathcal{M}_{\alpha'}^{\text{is}},$$

as $n \to \infty$ in $\text{SM}(E)$, where $S_{\beta}$ is a totally skewed $\beta$-stable random variable, independent from $\mathcal{M}_{\alpha'}^{\text{is}}$.

**Remark 3.4.** The first part of the assumption says that $F_\varepsilon(x), F_{\varepsilon \tilde{Z}_{Q, \beta}}(x) \in RV_{-\alpha}$, which then implies that $F_\varepsilon \tilde{Z}_{Q, \beta}(x) \in RV_{-\alpha}$; if in addition $\mathbb{E} \varepsilon = \infty$ and $\mathbb{E} Z_{Q, \beta} = \infty$, then $F_\varepsilon(x), F_{\varepsilon \tilde{Z}_{Q, \beta}}(x) = o(F_{\varepsilon \tilde{Z}_{Q, \beta}}(x))$ (see e.g. [22]). The latter is slightly weaker than the assumption (3.28), which, in the presence of (3.26) and (3.27), is simplified as follows:

$$\begin{cases}
F_\varepsilon(x(1 + L_Z^{-1/\alpha'}(x))) = o(F_{\varepsilon \tilde{Z}_{Q, \beta}}(x)), & \text{if } \lim_{x \to \infty} l_\varepsilon(x) > 0, \\
F_{\varepsilon}(x)^\beta = o(F_{\varepsilon \tilde{Z}_{Q, \beta}}(x)), & \text{if } \lim_{x \to \infty} l_\varepsilon(x) = 0, \lim_{x \to \infty} L_Z(x) > 0, \\
x^{-\alpha} = o(F_{\varepsilon \tilde{Z}_{Q, \beta}}(x)), & \text{if } \lim_{x \to \infty} l_\varepsilon(x) = 0, \lim_{x \to \infty} L_Z(x) = 0.
\end{cases}$$

Indeed, it suffices to express

$$F_\varepsilon(x(1 + L_Z^{-1/\alpha'}(x))) \sim \max \{x^{-\alpha}, F_{\varepsilon}(x)^\beta \}, \quad \alpha l_\varepsilon(x(1 + L_Z^{-1/\alpha'}(x))).$$

**Remark 3.5.** Assume $\nu(x) \sim C x^\beta$. If both $F_\varepsilon(x) \in RV_{-\alpha}$ and $F_{\varepsilon Z} \in RV_{-\alpha'}$ are asymptotically power laws (so that $l_\varepsilon, L_Z$ each has a limit in $(0,\infty)$), then (3.26), (3.27) and (3.28) hold, and more precisely we have, for some constants $C_1, C_2, C_3 > 0$,

$$F_\varepsilon(x) \sim C_1 F_{\varepsilon Z}(x)^\beta = C_1 F_{\varepsilon \tilde{Z}_{Q, \beta}}(x) \sim C_2 x^{-\alpha} \quad \text{and} \quad F_{\varepsilon \tilde{Z}_{Q, \beta}}(x) \sim C_3 x^{-\alpha} \log x.$$

For our proof, the assumption $\nu(x) \sim C x^\beta$ cannot be relaxed. Assumption (3.26) relaxes the asymptotic power-law behavior of the density. (A similar common applies to (3.27) and $L_Z$.) In the case $l_\varepsilon(x) \to \infty$, (3.28), in addition restricts $l_\varepsilon$ from increasing too fast. As an example, in the special case $L_Z(x) = l_\varepsilon(x)^{1/\beta}$, (3.28) becomes

$$F_{\varepsilon}(x l_\varepsilon^{-1/\alpha'}(x)) \sim \alpha^{-1} F_{\varepsilon}(x l_\varepsilon^{-1/\alpha'}(x)) = o(F_{\varepsilon}(x)), \quad \varepsilon' \text{ is an independent copy of } \varepsilon \text{ and }$$

$$\lim_{x \to \infty} \frac{L_\gamma(x l_\varepsilon^\beta(x))}{L_\gamma(x)} = \begin{cases} 1 & \text{if } \gamma \in (0,1/2), \\
\exp(\theta \gamma) & \text{if } \gamma = 1/2.\end{cases}$$

So, with $\gamma \in (0,1/2)$, $F_{\varepsilon}(x l_\varepsilon^{-1/\alpha'}(x)) \sim C F_{\varepsilon}(x l_\varepsilon(x))$. On the other hand, we have $L_\gamma(y) L_\gamma(x/y) \leq C L_\gamma^2(x)$ for all $y \in (0,x), x > 1$ (since $a^\gamma + b^\gamma \leq 2^{1-\gamma}(a+b)^\gamma$ for $a,b > 0, \gamma \in (0,1)$), and hence

$$F_{\varepsilon l_\varepsilon(x)} \leq \int_1^x y^{-\alpha-1} l_\varepsilon(y) F_{\varepsilon}(y) dy + F_{\varepsilon}(x) \leq C x^{-\alpha} \int_1^x y^{-1} l_\varepsilon(y) l_\varepsilon(x/y) dy + F_{\varepsilon}(x) \leq C F_{\varepsilon l_\varepsilon(x)} \left( x^{1-\alpha} \log x + l_\varepsilon(x) \right) = o(F_{\varepsilon}(x l_\varepsilon^{-1/\alpha'}(x))).$$

**Remark 3.6.** For the proof, we proceed by computing the Laplace functional instead of checking the widely applicable condition due to Kallenberg ([23] Theorem 4.1.8, [37] Theorem 3.23), which consists of checking the convergence of probabilities in the form of $\mathbb{P}(\xi_n((a,\infty]) = 0)$ and of expectations $\mathbb{E} \xi_n([a,\infty])$. The reason that this method does not apply here is that in the limit, we have $\mathbb{E} \xi([a,\infty]) = \infty$, violating one of the assumptions.

Introduce the point process on $(0,\infty]$,

$$\eta_n := \sum_{i=1}^{N(n)} \delta_{z_i} \cdot \zeta_i / b_n \quad \text{and} \quad \eta := \sum_{i=1}^{\infty} \delta_{\gamma_i} \gamma_i^{-1/\alpha'}. \quad (3.31)$$
Again we omit the variables $U$ for the locations. Let $f$ be a continuous non-negative function with compact support in $[\kappa, \infty]$, $\kappa > 0$. Write $\eta_n(f) = \int f \, d\eta_n$ and similarly for $\eta(f)$. The goal is to show

$$\lim_{n \to \infty} \mathbb{E} e^{-\eta_n(f)} = \mathbb{E} e^{-\eta(f)} = \exp(-\mathcal{C}_{\alpha, \beta}(f)) \quad \text{with} \quad \mathcal{C}_{\alpha, \beta}(f) = \left( \int_0^\infty (1 - e^{-f(v)}) \alpha' v^{-\alpha' - 1} \, dv \right)^\beta.$$

We have,

$$\mathbb{E} e^{-\eta_n(f)} = \mathbb{E} \left( \prod_{i=1}^{N(n)} \exp(-f(\varepsilon_i, Z_{i}/b_n)) \right) = \mathbb{E} \left( \exp \left( -\sum_{i=1}^{k} f(\varepsilon_i, Z_{i}/b_n) \right) \right) = \mathbb{E} \left( \prod_{k=1}^{\infty} \prod_{i \in K_n, \ell = k} \exp \left( -\sum_{i=1}^{k} f(\varepsilon_i, Z_{i}/b_n) \right) \right).$$

Therefore, recalling that $J_{n,k} = \sum_{i=1}^{\infty} 1_{\{K_n, \ell = k\}}$ and writing that

$$\psi_{n,k} \equiv \psi_{n,k}(f) := \mathbb{E} \exp \left( -\sum_{i=1}^{k} f(\varepsilon_i, Z_{i}/b_n) \right),$$

we infer that

$$\mathbb{E} e^{-\eta_n(f)} = \mathbb{E} \left( \prod_{k=1}^{\infty} \left( \mathbb{E} \exp \left( -\sum_{i=1}^{k} f(\varepsilon_i, Z_{i}/b_n) \right) \right)^{J_{n,k}} \right) = \mathbb{E} \exp \left( \sum_{k=1}^{\infty} J_{n,k} \log \psi_{n,k} \right).$$

The proof proceeds by a series of approximations. Consider

$$\check{\Psi}_n(f) := -\sum_{k=1}^{\infty} J_{n,k} \log \psi_{n,k},$$

$$\check{\Psi}_n(f) := \sum_{k=1}^{\infty} J_{n,k}(1 - \psi_{n,k}),$$

$$\check{\Psi}_n(f) := \sum_{k=1}^{\infty} \mathbb{E} J_{n,k}(1 - \psi_{n,k}),$$

$$\Psi_n(f) := \Gamma(1 - \beta) \nu(n) \sum_{k=1}^{\infty} p_k^{(\beta)}(1 - \psi_{n,k}).$$

Heuristically, the approximation makes sense as for every $k$ fixed, $\psi_{n,k} \to 1$ and hence $\log \psi_{n,k} \sim \psi_{n,k} - 1$, whereas $J_{n,k}$, $\mathbb{E} J_{n,k}$ and $\Gamma(1 - \beta) \nu(n) p_k^{(\beta)}$ are asymptotically equivalent (recall expressions of the last two in (3.22) and (3.23)). The uniform control in $k$ of these equivalences, in an appropriate sense, turned out to be quite involved.

We start with the relatively easy part that $\lim_{n \to \infty} \mathbb{E} e^{-\Psi_n(f)} = \mathbb{E} e^{-\eta(f)}$, as the following lemma shows. Note that here we need slightly weaker assumptions on $\varepsilon$ and $Z$ than Assumption 3.1 (see Remark 3.4).

**Lemma 3.3.** For $\eta$ given as in (3.31),

$$\mathbb{E} e^{-\eta(f)} = e^{-\mathcal{C}_{\alpha, \beta}(f)}.$$

If $\nu(x) \sim C x^\beta$ for some $C \in (0, \infty)$, $\overline{F}_\varepsilon(x) \in RV_{-\alpha}$, $\overline{F}_Z(x) \in RV_{-\alpha'}$, and $\overline{F}_\varepsilon(x) = o(\overline{F}_{\varepsilon \overline{Q}_\beta}(x))$, then with $b_n$ as in (3.29),

$$\lim_{n \to \infty} \Psi_n(f) = \mathcal{C}_{\alpha, \beta}(f).$$
Proof. Conditionally on $S_\beta$, express points from $\eta$ that are in the intervals $[\kappa, \infty]$ as $\sum_{i=1}^{N_*} \delta_{U_i^{-1/\alpha'}}$; then $N_*$ is Poisson distributed with parameter $S_\beta \kappa^{-\alpha'}$, and $\{U_i\}_{i \in \mathbb{N}}$ are i.i.d. random variables uniformly distributed over $(0, 1)$. So,

$$Ee^{-\eta(f)} = E \left( E \exp \left( - \sum_{i=1}^{N_*} f(U_i^{-1/\alpha'}) \right) \bigg| S_\beta \right) = E \exp \left( \frac{S_\beta}{\kappa^\alpha} \left( EE^{-f(U_i^{-1/\alpha'})} - 1 \right) \right)$$

$$= \exp \left( - \left( \frac{1}{\kappa^\alpha} \int_1^\infty \left( 1 - e^{-f(u\kappa)} \right)^\alpha u^{-\alpha' - 1} du \right)^\beta \right) = e^{-\epsilon_\alpha, \beta(f)}.$$ 

For the second part, we start by writing

$$\sum_{k=1}^{\infty} p_k^{(\beta)} (1 - \psi_{n,k}) = \sum_{k=1}^{\infty} p_k^{(\beta)} \left( 1 - E \left( e^{-f(z/\kappa)} \bigg| \varepsilon \right) \right)^k = 1 - E \left( e^{-f(z/\kappa)} \bigg| \varepsilon \right)^{Q_\beta},$$

where $Q_\beta$ a Sibuya random variable ($\mathbb{P}(Q_\beta = k) = p_k^{(\beta)}$), independent of all the rest. Using $E_{z}^{Q_\beta} = 1 - (1 - z)^\beta$ for $z \in [0, 1]$, we get

$$\sum_{k=1}^{\infty} p_k^{(\beta)} (1 - \psi_{n,k}) = E \left( 1 - E \left( e^{-f(z/\kappa)} \bigg| \varepsilon \right) \right)^\beta = \int_0^\infty \left( 1 - E e^{-f(z/\kappa)} \right)^\beta dF_\varepsilon(x).$$

Introduce $Z_{n,x}$ as a random variable with law determined by

$$\mathbb{P}(Z_{n,x} > y) = P \left( Z > \frac{\kappa b_n}{x} \cdot y \bigg| Z > \frac{\kappa b_n}{x} \right), \quad y \geq 1,$$

and

$$a_{n,x}(f) := 1 - EE^{-f(kZ_{n,x})}.$$ 

So we have (recalling that $f$ is supported over $[\kappa, \infty]$)

$$1 - EE^{-f(z/\kappa)} = a_{n,x}(f) \cdot \mathcal{F}(k b_n / x).$$ 

It follows from $\mathcal{F}(z(x) \in RV_{-\alpha'}$ that, for every $x > 0$ fixed,

$$\lim_{n \to \infty} a_{n,x}(f) = \int_1^\infty (1 - e^{-f(v)\alpha'})^\alpha v^{-\alpha'-1}dv = \kappa' \int_0^\infty (1 - e^{-f(v)\alpha'})^\alpha v^{-\alpha'-1}dv = (\kappa'^\alpha \mathcal{E}_{\alpha, \beta}(f))^{1/\beta},$$

and for all $\epsilon > 0$ we can take $d_\epsilon > 0$ small enough so that

$$\limsup_{n \to \infty} \sup_{x \in [0, d_\epsilon b_n]} \left| a_{n,x}^{(\beta)}(f) - \kappa'^\alpha \mathcal{E}_{\alpha, \beta}(f) \right| \leq \epsilon.$$

We then have

$$\left| \int_0^\infty \frac{F_{\varepsilon Q_\beta} (k b_n / x) a_{n,x}^{(\beta)}(f) dF_\varepsilon(x) - \kappa'^\alpha \mathcal{E}_{\alpha, \beta}(f) \bar{F}_{\varepsilon Q_\beta} (k b_n)}{F_{\varepsilon Q_\beta} (k b_n / x) \left( a_{n,x}^{(\beta)}(f) - \kappa'^\alpha \mathcal{E}_{\alpha, \beta}(f) \right) dF_\varepsilon(x) + (1 + \kappa'^\alpha \mathcal{E}_{\alpha, \beta}(f)) \bar{F}_{\varepsilon}(d_\epsilon b_n)} \right| \leq \int_0^{d_\epsilon b_n} \frac{F_{\varepsilon Q_\beta} (k b_n / x) a_{n,x}^{(\beta)}(f) dF_\varepsilon(x)}{a_{n,x}^{(\beta)}(f) - \kappa'^\alpha \mathcal{E}_{\alpha, \beta}(f) \bar{F}_{\varepsilon Q_\beta} (k b_n) + (1 + \kappa'^\alpha \mathcal{E}_{\alpha, \beta}(f)) \bar{F}_{\varepsilon}(d_\epsilon b_n)}.$$ 

The first term on the right-hand side is bounded by, for $n$ large enough, $2\varepsilon \bar{F}_{\varepsilon Q_\beta} (k b_n)$, and the second by $C F_{\varepsilon}(b_n) = o(\bar{F}_{\varepsilon Q_\beta} (b_n))$. Since $\epsilon > 0$ can be arbitrarily small, the above implies that

$$\int_0^\infty \frac{F_{\varepsilon Q_\beta} (k b_n / x) a_{n,x}^{(\beta)}(f) dF_\varepsilon(x) \sim \kappa'^\alpha \mathcal{E}_{\alpha, \beta}(f) \bar{F}_{\varepsilon Q_\beta} (k b_n) \sim \mathcal{E}_{\alpha, \beta}(f) \bar{F}_{\varepsilon}(k b_n).$$
To sum up,

\[ \Psi_n(f) = \Gamma(1 - \beta) \nu(n) \int_0^\infty \left(1 - E e^{-f(xZ_n/b_n)} \right)^\beta dF(x) = \Gamma(1 - \beta) \nu(n) \int_0^\infty \mathcal{F}_{Z^{\alpha\beta}}(\kappa b_n/x) a_\alpha^{\beta, \nu} dF(x) \sim \mathcal{F}_{F}\mathcal{F}_{\epsilon} \mathcal{F}_{Z^{\alpha\beta}}(b_n). \]

The desired result now follows from (3.29).

The hard part of the proof lies in approximating \( \tilde{\Psi}_n \) by \( \Psi_n \), where we shall need a very fine control of \( 1 - \psi_{n,k} \). For this purpose, introduce

\[ \bar{b}_n := b_n \left(1 - \mathcal{L}^{-1/\alpha'}(b_n) \right) \quad \text{and} \quad \mathcal{F}_{\epsilon}^{\alpha}(\bar{b}_n) := \mathcal{F}_{\epsilon}(\bar{b}_n) \vee \bar{b}_n^{\alpha}. \]

The key of the analysis is the following Lemma 3.4.

**Lemma 3.4.** Under Assumption 3.1, there exists a constant \( C > 0 \) such that for all \( n \) large enough,

\[ 1 - \psi_{n,k} \leq \left[C \left(k^{\beta} \mathcal{F}_{\epsilon}^{\alpha}(\bar{b}_n) + k \mathcal{F}_{\epsilon}(b_n) \right) \right] \wedge 1 \quad \text{for all} \quad k \in \mathbb{N}. \]

**Proof.** We have, by (3.32),

\[ 1 - \psi_{n,k} = \int_0^\infty 1 - \left(E e^{-f(xZ_n/b_n)} \right)^k dF(x) = \int_0^\infty 1 - \left(1 - a_{n,x}(f) \mathcal{F}_{Z}(\kappa b_n/x) \right)^k dF(x). \]

Pick \( r = 1 \wedge (\kappa/x \mathcal{L}) \) (recall (3.27) for \( x \mathcal{L} \)). Then, the integration over \( x > r \) is bounded from above by \( \mathcal{F}_{\epsilon}(r \bar{b}_n) \sim C r^{-\alpha} \mathcal{F}_{\epsilon}(\bar{b}_n) \), and this term can be bounded by, for another constant \( C \) large enough, \( C k^{\beta} \mathcal{F}_{\epsilon}(\bar{b}_n) \leq C k^{\beta} \mathcal{F}_{\epsilon}(\bar{b}_n) \) for all \( k \in \mathbb{N} \). Therefore it suffices to show that integration over \( x \in [0, \bar{r}_n] \) is of the desired order. Recall \( x \) from (3.26).

Then,

\[ \int_{[x, \bar{r}_n]} 1 - \left(1 - a_{n,x}(f) \mathcal{F}_{Z}(\kappa b_n/x) \right)^k dF(x) \leq C k \mathcal{F}_{\epsilon}(\kappa b_n/x) \leq C k \mathcal{F}_{\epsilon}(b_n). \]

For the interval \([x, \bar{r}_n]\), we observe that by our choice of \( x \) in (3.27) for \( n \) large enough

\[ \sup_{x \in [x, \bar{r}_n]} L_{\mathcal{L}}(\kappa b_n/x) = \sup_{x \in [x, \bar{r}_n]} L_{\mathcal{L}}(x) \leq C (1 \vee L_{\mathcal{L}}(b_n)), \]

and thus for all \( x \in [x, \bar{r}_n] \), \( a_{n,x}(f) \mathcal{F}_{Z}(\kappa b_n/x) \leq \tilde{a}(x/\bar{b}_n)^{\alpha'} \) for some constant \( \tilde{a} > 0 \). Introduce

\[ u = (x/\bar{b}_n)^{\alpha'} \quad \text{with} \quad \tilde{b}_n := \tilde{a}^{-1/\alpha'} \bar{b}_n. \]

We then arrive at (we also need \( n \) large enough so that \( \tilde{a}(x/\bar{b}_n)^{\alpha'} < 1 \)),

\[ \int_{[x, \bar{r}_n]} 1 - \left(1 - a_{n,x}(f) \mathcal{F}_{Z}(\kappa b_n/x) \right)^k dF(x) \leq \int_{[x, \bar{r}_n]} 1 - \left(1 - \tilde{a}(x/\bar{b}_n)^{\alpha'} \right)^k dF(x) \]

\[ \leq C \int_{[x, \bar{r}_n]} (1 - (1 - u)^k) \cdot dF \left( u^{1/\alpha'} \tilde{b}_n \right). \]

Write \( dF(x) = f_z(x) dx = x^{-\alpha-1} l_z(x) dx \). By (3.26), we have, for \( u \) in the domain of the integral above,

\[ f_z \left( u^{1/\alpha'} \tilde{b}_n \right) d \left( u^{1/\alpha'} \tilde{b}_n \right) \leq C u^{-\beta-1/\alpha} \mathcal{L}^{-1} \left( l_z(\tilde{b}_n) \vee 1 \right) du. \]

So (3.35) is bounded from above by, uniformly for all \( k \in \mathbb{N} \) and \( n \) large enough,

\[ C \left( \mathcal{F}_{\epsilon}(\bar{b}_n) \vee \bar{b}_n^{\alpha} \right) \int_0^1 (1 - (1 - u)^k) u^{-\beta-1} du = C \mathcal{F}_{\epsilon}(b_n) \left( \frac{1}{\beta} + kB(k, 1-\beta) \right) \leq C \mathcal{F}_{\epsilon}(b_n) k^\beta, \]

where \( B(x, y) = \Gamma(x) \Gamma(y)/\Gamma(x + y) \) is the beta function.

\qed
Proof of Theorem 3.3. We know that $\mathbb{E}e^{-\Psi_n(f)} = \mathbb{E}e^{\tilde{\Psi}_n(f)}$ and that $\lim_{n \to \infty} e^{-\Psi_n(f)} = e^{-c_{\alpha, \beta}(f)}$ (Lemma 3.3). So, to prove Theorem 3.3, it is sufficient to prove that $\tilde{\Psi}_n(f) - \Psi_n(f) \to 0$ in probability for every fixed $f$, and we drop the dependence on $f$ from here on (note that $|e^{-\Psi_n} - e^{-\tilde{\Psi}_n}| \leq 2 + |1 - e^{\Psi_n - \tilde{\Psi}_n}|$).

We shall prove successively that $\tilde{\Psi}_n - \Psi_n \to 0$, $\Psi_n - \tilde{\Psi}_n \to 0$ in probability, and $\tilde{\Psi}_n - \Psi_n \to 0$ in probability.

Notice that Assumption 3.1 and the choice of $b_n, \tilde{b}_n$ (see (3.29), (3.33)) imply that (see Remark 3.4)

\begin{equation}
\lim_{n \to \infty} \nu(n) F^+_f(b_n) = 0 \quad \text{and} \quad \lim_{n \to \infty} \nu(n) F^-_f(b_n) = 0.
\end{equation}

These and (3.34) in Lemma 3.4 play a crucial role in the sequel.

(i) We first show that

\[ \Psi_n - \Psi_n = \sum_{k=1}^{\infty} \left( \mathbb{E}J_{n,k} - \nu(n) \Gamma(1 - \beta)p_k^{(\beta)} \right) \left( 1 - \psi_{n,k} \right) \to 0. \]

Recall $\mathbb{E}J_{n,k}$ and $p_k^{(\beta)}$ in (3.22) and (3.23):

\[ \mathbb{E}J_{n,k} = \frac{\nu(n)}{k!} \int_0^{np_k} f_{n,k}(y) y^{-\beta} \frac{L(n/y)}{L(n)} dy \quad \text{and} \quad \Gamma(1 - \beta)p_k^{(\beta)} = \frac{\nu(n)}{k!} \int_0^{\infty} f_{n,k}(y) y^{-\beta} dy, \]

with $f_{n,k}(y) = (k - y)y^{k-1}e^{-y}$.

For every $\epsilon > 0$, let $A_\epsilon \subset (0, p_1)$ be such that

\begin{equation}
\limsup_{n \to \infty} \sup_{y \in [0, nA_\epsilon]} \left| \frac{L(n/y)}{L(n)} - 1 \right| < \epsilon.
\end{equation}

We then write,

\[
\Psi_n - \Psi_n = \nu(n) \sum_{k=1}^{\infty} \frac{1}{k!} \int_0^{nA_\epsilon} f_{n,k}(y) y^{-\beta} \left( \frac{L(n/y)}{L(n)} - 1 \right) dy \cdot (1 - \psi_{n,k}) \\
+ \nu(n) \sum_{k=1}^{\infty} \frac{1}{k!} \int_{nA_\epsilon}^{np_k} f_{n,k}(y) y^{-\beta} \left( \frac{L(n/y)}{L(n)} - 1 \right) dy \cdot (1 - \psi_{n,k}) \\
- \nu(n) \sum_{k=1}^{\infty} \frac{1}{k!} \int_{np_k}^{\infty} f_{n,k}(y) y^{-\beta} dy \cdot (1 - \psi_{n,k}) =: I_{n,1}^\epsilon + I_{n,2}^\epsilon - I_{n,3}.
\]

We shall show that

\begin{equation}
\limsup_{n \to \infty} \frac{|I_{n,1}^\epsilon|}{\Psi_n} \leq \frac{\epsilon}{\Gamma(1 - \beta)} \quad \text{and} \quad \limsup_{n \to \infty} (|I_{n,2}^\epsilon| + |I_{n,3}|) = 0, \quad \text{for all } \epsilon > 0.
\end{equation}

We first deal with $I_{n,3}$. Introduce

\[ I_{n,3}(p) := \nu(n) \sum_{k=1}^{\infty} \frac{1}{k!} \int_{np}^{\infty} f_{n,k}(y) y^{-\beta} dy \cdot (1 - \psi_{n,k}), \quad p > 0. \]

So in (3.38) $I_{n,3} = I_{n,3}(p_1)$. Recall the definitions of $f_{n,k}$ and $F_{n,k}$ in (3.21) and observe that, by integration by part,

\[
\frac{1}{k!} \int_{np}^{\infty} f_{n,k}(y) y^{-\beta} dy = \frac{1}{k!} F_{n,k} \left( \frac{n}{y} \right) \bigg|_np + \frac{1}{k!} \beta \int_{np}^{\infty} F_{n,k} \left( \frac{n}{y} \right) y^{-\beta-1} dy \\
= \frac{1}{k!} \beta \int_{np}^{\infty} y^{k-1-\beta} e^{-y} dy - \frac{(np)^{k-\beta}}{k!} e^{-np} \\
= \Gamma(1 - \beta)p_k^{(\beta)} \mathbb{P}(\gamma_{k-\beta} > np) - \frac{(np)^{k-\beta}}{k!} e^{-np},
\]
where $\gamma_{k-\beta}$ is a random variable of Gamma distribution with parameter $k - \beta$. Thus,

$$\tag{3.39} |I_{n,3}(p)| \leq C \nu(n) \sum_{k=1}^{\infty} p_k^{(\beta)} \mathbb{P}(\gamma_{k-\beta} > np)(1 - \psi_{n,k}) + e^{-np} \nu(n) \sum_{k=1}^{\infty} \frac{(np)^{k-\beta}}{k!}(1 - \psi_{n,k}).$$

We deal with the two series separately. First, recalling (3.36), one can find a sequence of integers $\ell_n$ such that

$$\ell_n \to \infty, \quad \nu(n)\mathcal{F}_x^* (\overline{b}_n) \ell_n \to 0 \quad \text{and} \quad n\mathcal{F}_Z(\overline{b}_n) \ell_n^{2-\beta} \to 0.$$

Then, applying Markov inequality $\mathbb{P}(\gamma_{k-\beta} > np) \leq ((k - \beta)/np) \land 1$ to $k \leq n\ell_n$ and $k > n\ell_n$ respectively, we have

$$\nu(n) \sum_{k=1}^{n\ell_n} p_k^{(\beta)} \mathbb{P}(\gamma_{k-\beta} > np)(1 - \psi_{n,k}) \leq C \nu(n) \sum_{k=1}^{n\ell_n} p_k^{(\beta)}(k^\beta \mathcal{F}_x^* (\overline{b}_n) + k\mathcal{F}_Z(\overline{b}_n)) + C \nu(n) \sum_{k=n\ell_n+1}^{\infty} p_k^{(\beta)} \leq C \nu(n)\mathcal{F}_x^* (\overline{b}_n) \ell_n + C \nu(n)\mathcal{F}_Z(\overline{b}_n)(n\ell_n)^{2-\beta} + C \nu(n)(n\ell_n)^{-\beta},$$

where in the last step above we use the fact that $p_k^{(\beta)} \sim C k^{-\beta-1}$ (recall Sibuya distribution (2.1)) and Karamata theorem. By our assumption on $\ell_n$ we have shown that the first series in (3.39) goes to zero. The second series in (3.39) can be bounded by, using (3.34),

$$\tag{3.40} C \nu(n)e^{-np} \sum_{k=1}^{\infty} \frac{(np)^{k-\beta}}{k!} \left( k^\beta \mathcal{F}_x^* (\overline{b}_n) + k\mathcal{F}_Z(\overline{b}_n) \right)$$

$$\leq C \nu(n)\mathcal{F}_x^* (\overline{b}_n)e^{-np} \sum_{k=0}^{\infty} \frac{(np)^{k-1-\beta}}{\Gamma(k + 2 - \beta)} + C \nu(n)\mathcal{F}_Z(\overline{b}_n)(np)^{1-\beta} \leq C \nu(n)\mathcal{F}_x^* (\overline{b}_n) + Cn\mathcal{F}_Z(\overline{b}_n),$$

where in the second inequality, the first term is bounded by the following estimate on Mittag–Leffler function (e.g. [18 Eq.(6)])

$$E_{1,2-\beta}(y) = \sum_{k=0}^{\infty} \frac{y^k}{\Gamma(k + 2 - \beta)} \leq Cy^{\beta-1}e^y \text{ for all } y \geq 1,$$

and the second term by the fact $\nu(n) \leq Cn^{\beta}$. By (3.36), (3.40) tends to zero, and hence $I_{n,3}(p) \to 0$ for all $p > 0$.

Now we deal with $I_{n,1}^\epsilon$. Note that $f_{n,k}(y)$ changes sign at $y = k$ so we proceed with caution. First we write, by (3.25),

$$I_{n,1}^\epsilon = \nu(n) \int_0^{nA_\epsilon} \sum_{k=0}^{\infty} \frac{y^{k-\beta}e^{-y}}{k!} (\psi_{n,k} - \psi_{n,k+1}) \left( \frac{L(n/y)}{L(n)} - 1 \right) dy,$$

and recall that $\psi_{n,k} - \psi_{n,k+1} > 0$. Then, for $n$ large enough, thanks to (3.37),

$$|I_{n,1}^\epsilon| \leq \nu(n) \int_0^{nA_\epsilon} \sum_{k=0}^{\infty} \frac{y^{k-\beta}e^{-y}}{k!} (\psi_{n,k} - \psi_{n,k+1}) \left| \frac{L(n/y)}{L(n)} - 1 \right| dy$$

$$\leq 2\epsilon \cdot \nu(n) \int_0^{nA_\epsilon} \sum_{k=0}^{\infty} \frac{y^{k-\beta}e^{-y}}{k!} (\psi_{n,k} - \psi_{n,k+1}) dy$$

$$= 2\epsilon \cdot \nu(n) \int_0^{nA_\epsilon} \frac{f_{n,k}(y)y^{\beta}}{k!} dy \cdot (1 - \psi_{n,k}) = 2\epsilon \frac{\Psi_n}{\Gamma(1-\beta)} - 2\epsilon \cdot I_{n,3}(A_\epsilon),$$

where in the third step (3.25) is applied again. We have seen that $|I_{n,3}(A_\epsilon)| \to 0$. This proves the first part of (3.38).
It remains to deal with $I_{n,2}^*$. By the same trick on $I_{n,1}^*$ above using (3.25) twice, but this time combined with $\limsup_{n\to\infty} \sup_{y\in[nA_n,np]} |L(n/y)/L(n) - 1| \leq C$ (which cannot be arbitrarily small, but is finite under our assumption on $\nu$), we have that
\[
|I_{n,2}^*| \leq \nu(n) \int_{nA_n}^{np} \sum_{k=0}^{\infty} \frac{y^{k-\beta} e^{-y}}{k!} (\psi_{n,k} - \psi_{n,k+1}) |L(n/y)/L(n) - 1| \, dy
\leq C\nu(n) \int_{nA_n}^{np} \sum_{k=0}^{\infty} \frac{y^{k-\beta} e^{-y}}{k!} (\psi_{n,k} - \psi_{n,k+1}) \, dy
= C(I_{n,3}(A_\epsilon) - I_{n,3}(p_1)) \leq C(|I_{n,3}(A_\epsilon) + |I_{n,3}(p_1)|) \to 0.
\]
This completes the proof of $\bar{\Psi}_n - \Psi_n \to 0$.

(ii) Next, we prove
\[
\bar{\Psi}_n - \bar{\Psi}_n = \sum_{k=1}^{\infty} (J_{n,k} - EJ_{n,k})(1 - \psi_{n,k}) \xrightarrow{p} 0.
\]
Introduce $N_\ell(n) := \sum_{i=1}^{N(n)} 1_{\{Y_i = \ell\}}$. So $\{N_\ell(n)\}_{\ell\in\mathbb{N}}$ are independent Poisson random variables. Recall that $J_{n,k} = \sum_{\ell=1}^{\infty} 1_{\{N_\ell(n) = k\}}$. By independence,
\[
E(J_{n,k}J_{n,k'}) = \sum_{\ell=1}^{\infty} P(N_\ell(n) = k)P(N_\ell(n) = k') \leq EJ_{n,k}EJ_{n,k'} \text{ for all } k \neq k'.
\]
It follows that $\forall (\bar{\Psi}_n - \bar{\Psi}_n) \leq \sum_{k=1}^{\infty} \forall J_{n,k} \cdot (1 - \psi_{n,k})^2 \leq \sum_{k=1}^{\infty} EJ_{n,k} \cdot (1 - \psi_{n,k})^2$. Noticing that $J_{n,k} = 0$ when $k > N(n)$ and then using (3.34) and Cauchy–Schwarz inequality, we infer
\[
(3.41) \quad V(\bar{\Psi}_n - \bar{\Psi}_n) \leq \mathbb{E} \left( \bar{\Psi}_n \max_{k=1,\ldots,N(n)} (1 - \psi_{n,k}) \right) \leq \|\bar{\Psi}_n\|_2 \|1 - \psi_{n,N(n)}\|_2.
\]
Observe that $\|\bar{\Psi}_n\|_2^2 = V(\bar{\Psi}_n + (E\bar{\Psi}_n))^2 \leq V(\bar{\Psi}_n + \bar{\Psi}_n^2)$. Since $\bar{\Psi}_n$ has a finite limit, (3.41) is bounded from above by
\[
C \left\|1 - \psi_{n,N(n)}\right\|_2 \leq C \left\|N(n)^{1/2} \tilde{b}_n + N(n)\tilde{F}(n)\right\|_2 \to 0,
\]
as a consequence of (3.36) and the fact that $N(n)^{1/2}$ is of order $\nu(n) \sim Cn^{\beta}$ as $N(n)/n \xrightarrow{L^2} 1$. Therefore we have proved that $\bar{\Psi}_n - \bar{\Psi}_n \to 0$ in $L^2$.

(iii) It remains to prove that $\hat{\Psi}_n - \Psi_n \xrightarrow{p} 0$. Using that $J_{n,k} = 0$ when $k > N(n)$ and that $|\log(x) + (1 - x)|/(1 - x) \leq (1 - x)/x$ for $x \in (0, 1)$, for $n$ large enough, we get
\[
\left|\hat{\Psi}_n - \Psi_n\right| = \sum_{k=1}^{\infty} J_{n,k} (\log \psi_{n,k} + 1 - \psi_{n,k}) \leq \sum_{k=1}^{\infty} J_{n,k} \cdot (1 - \psi_{n,k}) \frac{1 - \psi_{n,k}}{\psi_{n,k}}
\leq \left|\Psi_n\right| \max_{k=1,\ldots,N(n)} \frac{1 - \psi_{n,k}}{\psi_{n,k}} = \left|\Psi_n\right| \frac{1 - \psi_{n,N(n)}}{\psi_{n,N(n)}}.
\]
Since we have seen that $\bar{\Psi}_n \xrightarrow{p} \lim_{n \to \infty} \Psi_n \in (0, \infty)$, by (3.42), we infer that $\hat{\Psi}_n - \Psi_n \to 0$ in probability. □

4. Extremal limit theorems for Karlin stochastic volatility model

We now apply Section 4 to Karlin stochastic volatility model discussed in introduction. Let $\{\varepsilon_i, Y_i, Z_i\}_{i \in \mathbb{N}}$ be as in the Poisson–Karlin model. Then, the Karlin stochastic volatility model is the stationary sequence defined as
\[
X_i := \varepsilon_i Z_i.
\]
We are interested in the empirical random sup-measure defined as
\[
\widehat{M}_n(\cdot) := \max_{i=1,\ldots,n} \varepsilon_i Z_i \text{ in SM([0, 1])}.
\]
Theorem 4.1. Assume $\alpha, \alpha' > 0$ and $\beta \in (0, 1)$.

(i) (Volatility-dominance regime) If $\tilde{F}_\alpha(x) \in RV_{-\alpha}$, $\nu$ satisfies \((2.5)\), and $E\tilde{Z}_{\alpha,+}^{n+\epsilon} < \infty$ for some $\epsilon > 0$, then for $\{a_n\}_{n \in \mathbb{N}}$ such that $\lim_{n \to \infty} \Gamma(1 - \beta)\nu(n)\tilde{F}_\alpha(a_n) = 1$,

$$\frac{1}{a_n} \tilde{M}_n \Rightarrow \mathcal{M}_{\alpha, \beta, Z}.$$ 

(ii) (Critical regime) If $\alpha = \alpha'\beta$, $\tilde{F}_\alpha(x) \in RV_{-\alpha}$, $\tilde{F}_{\alpha'}(x) \in RV_{-\alpha'}$ and Assumption \((3.1)\) holds, then for $\{b_n\}_{n \in \mathbb{N}}$ such that $\lim_{n \to \infty} \Gamma(1 - \beta)\nu(n)\tilde{F}_{\alpha'}(b_n) = 1$,

$$\frac{1}{b_n} \tilde{M}_n \Rightarrow S^1_{\beta} \cdot \mathcal{M}^{is}_{\alpha'}.$$ 

(iii) (Innovation-dominance regime) If $\tilde{F}_{\alpha'}(x) \in RV_{-\alpha'}$ and $E\tilde{e}^{\alpha'+\epsilon} < \infty$ almost surely for some $\epsilon > 0$, then for $\{c_n\}_{n \in \mathbb{N}}$ such that $\lim_{n \to \infty} n\tilde{F}_\alpha(c_n) = 1$,

$$\frac{1}{c_n} \tilde{M}_n \Rightarrow \left(E\tilde{e}^{\alpha'}\right)^{1/\alpha'} \cdot \mathcal{M}^{is}_{\alpha'}.$$

In all three cases, the convergence in distribution is in $S\mathcal{M}([0,1])$. See Section \((3)\) for details on the limit random sup-measures $((E, \mathcal{E}, \mu) = ((0,1), \mathcal{B}((0,1)), \text{Leb}))$. The Poissonization method actually allows to prove the corresponding convergence of point processes (the statements of which we do not include for the sake of simplicity).

Remark 4.1. Our Poissonization method is different from the one applied for the original Karlin model \((17, 23)\), which is essentially a time-change lemma \((6)\) that depends crucially on the fact that $\mathbb{R}$ and $\mathbb{N}$ are ordered. Our method is geometry free in the sense that it can be adapted to other situations where the time-change lemma does not apply. For example, one may consider the $\mathbb{N}^d$-extension of the problem: let $\{e_i\}_{i \in \mathbb{N}}$ be as before, $Y$ and $Z$ be i.i.d. indexed by $i \in \mathbb{N}^d$, all assumed to be independent, and

$$\tilde{M}_n(i) := \max_{i \in \{1, \ldots, n\} \times \mathbb{N}^d} \varepsilon_i Z_i.$$ 

Theorem 4.1 can be extended to this model with some obvious changes, and our Poissonization method applies to this model too with little extra effort. We omit the details.

4.1. A Poissonization method. Consider the point process of the Karlin stochastic volatility model

$$\tilde{\xi}_n := \sum_{i=1}^{n} \delta_{(\varepsilon_i Z_i / r_n, i/n)},$$

where $r_n = a_n, b_n$ or $c_n$ depending on the regime. Our method is unified for all three different regimes and we do not write the rate $r_n$ explicitly. A natural Poissonization of $\tilde{\xi}_n$ would be

$$\xi_n := \sum_{i=1}^{N(n)} \delta_{(\varepsilon_i Z_i / r_n, i)},$$

which is the same point-process investigated before, with i.i.d. uniform random variables $\{U_i\}_{i \in \mathbb{N}}$. We have seen in Section \((3)\) that

$$\xi_n \Rightarrow \xi$$

in $\mathcal{M}_p([0, \infty) \times (0, 1))$, where $\xi$ is the Poisson point process underlying the random sup-measure in the corresponding regime. This was actually achieved by computing, for $f$ a continuous function on $[0, \infty) \times [0, 1]$ with compact support,

$$\lim_{n \to \infty} E e^{-\xi_n(f)} = E e^{-\xi(f)} = e^{-\xi_{\alpha, \beta, \gamma}(f)},$$

with

$$\xi_{\alpha, \beta, \gamma}(f) = \left(\int_0^\infty \int_0^1 (1 - e^{-\gamma(v, u)})^{\alpha'} v^{-\alpha' - 1} dv du\right)^{1/\beta}.$$
(We now include the location variables $U$ and hence $f$ and $\mathcal{E}_{\alpha,\beta}(f)$ are modified accordingly.)

Consider $f$ in the form

\begin{equation}
    f(x, u) = \sum_{j=1}^{d} \theta_j \mathbb{I}_{\{x \in (g_j, h_j), u \in (s_j, t_j)\}}, \quad c_j > 0, 0 < g_j < h_j, 0 \leq s_j < t_j \leq 1.
\end{equation}

Let $\delta > 0$ denote a tuning parameter. Our Poissonization method is summarized by the following lemma.

**Lemma 4.1.** For $f$ as above and $\delta > 0$, there exist point processes $\xi_{n,\delta,-}, \xi_{n,\delta,+}$ in $\mathfrak{M}_p((0, \infty) \times [0, 1])$ such that

\begin{equation}
    \lim_{n \to \infty} \mathbb{P}\left(\xi_{n,\delta,-}(f) \leq \hat{\xi}_n(f) \leq \xi_{n,\delta,+}(f)\right) = 1,
\end{equation}

and moreover, there exist constants $c_{\alpha,\beta,+}(f)$ such that

\begin{equation}
    \lim_{\delta \to 0} \mathbb{E}e^{-\xi_{n,\delta,+}(f)} = e^{-c_{\alpha,\beta,+}(f)} \quad \text{and} \quad \lim_{\delta \to 0} c_{\alpha,\beta,+}(f) = c_{\alpha,\beta,+}(f).
\end{equation}

**Proof of Theorem 4.1.** By the first part of the lemma above and the limit theorem for the Poissonized model, we have

\[ e^{-c_{\alpha,\beta,+}(f)} = \lim_{n \to \infty} \mathbb{E}e^{-\xi_{n,\delta,+}(f)} \leq \liminf_{n \to \infty} \mathbb{E}e^{-\hat{\xi}_n(f)} \leq \limsup_{n \to \infty} \mathbb{E}e^{-\xi_{n,\delta,-}(f)} \leq \lim_{n \to \infty} \mathbb{E}e^{-\xi_{n,\delta,-}(f)} = e^{-c_{\alpha,\beta,+}(f)}. \]

The second part of Lemma 4.1 then entails, letting $\delta$ decrease to zero, that inequality in the middle above is actually an equality, and hence the desired convergence of Laplace functional for $f$ as a step function in (4.2). The convergence for general continuous $f$ in (4.1) follows by a standard approximation argument. □

**Proof of Lemma 4.1.** To start with, assume in addition that all $A_j := (s_j, t_j), j = 1, \ldots, d$ are disjoint. Introduce

\[ n_j := \sum_{i=1}^{n} \mathbb{I}_{\{i/n \in A_j\}}. \]

Then,

\begin{equation}
    \hat{\xi}_n(f) = \sum_{j=1}^{d} \mathbb{I}_{\{i \in [0,1]\}} \mathbb{I}_{\{i/n \in A_j\}} \sum_{i=1}^{n} \mathbb{I}_{\{\varepsilon_{Y_{j,i},Z_{j,i}}/b_n \in (g_j, h_j)\}},
\end{equation}

where $\varepsilon$ is as before, $Y_{j,i}$ and $Z_{j,i}$ are i.i.d. copies of $Y$ and $Z$, respectively, and independent from $\{\varepsilon_{\ell}\}_{\ell \in \mathbb{N}}$ (but $\{\varepsilon_{Y_{j,i},Z_{j,i}}\}_{j,i}$ are dependent).

Now we introduce, for every $\delta \in (0,1)$,

\[ \xi_{n,\delta,\pm} = \sum_{i=1}^{d} \delta_{\{\varepsilon_{Y_{j,i},Z_{j,i}}/r_n, U_{i}\}}, \]

where $\{\varepsilon_{Y_{j,i},Z_{j,i}}/r_n, U_{i}\}_{i \in \mathbb{N}}$ are as in the Poisson–Karlin model, independent from the Poisson random variable $N((1 \pm \delta)n)$ (with mean $(1 \pm \delta)n$). The above is interpreted as the law of $\xi_{n,\delta,+}$ and $\xi_{n,\delta,-}$ separately. We shall first derive for each of $\xi_{n,\delta,\pm}(f)$ a similar representation as (4.5) in (4.6) below, and then explain the coupling. Set

\[ N_{n,\delta,\pm}(j) := \xi_{n,\delta,\pm}(0, \infty) \times A_j, j = 1, \ldots, d. \]

Since $\{A_j\}_{j=1,\ldots,d}$ are disjoint, $\{N_{n,\delta,\pm}(j)\}_{j=1,\ldots,d}$ are independent Poisson random variables with parameters $(1 + \delta)n|A_j|/(1 - \delta)n|A_j|$ resp. We hence arrive at

\begin{equation}
    \xi_{n,\delta,\pm}(f) = \sum_{j=1}^{d} \mathbb{I}_{\{N_{n,\delta,\pm}(j)\}} \mathbb{I}_{\{\varepsilon_{Y_{j,i},Z_{j,i}}/r_n \in (g_j, h_j)\}}.
\end{equation}

Now we explain the coupling of $\hat{\xi}_n(f), \xi_{n,\delta,+}(f)$ and $\xi_{n,\delta,-}(f)$. In view of (4.5) and (4.6), we assume naturally that the three random summations share the same $\{\varepsilon_{\ell}\}_{\ell \in \mathbb{N}}, \{Y_{j,i}, Z_{j,i}\}_{j=1,\ldots,d, i \in \mathbb{N}}$, and that these random variables are independent from $\{N_{n,\delta,\pm}(j)\}_{j=1,\ldots,d}$. It is also natural to assume that $\xi_{n,\delta,+}$ and $\xi_{n,\delta,-}$...
are coupled in the sense that the latter is obtained from the former by a standard thinning procedure (with probability \((1 - \delta)/(1 + \delta)\) to keep independently each point from the former), which leads to \(N_{n,\delta,-}(j) \leq N_{n,\delta,+}(j)\) almost surely for all \(j\). Therefore it remains to show
\[
\lim_{n \to \infty} \mathbb{P}(N_{n,\delta,-}(j) \leq n_j \leq N_{n,\delta,+}(j) \text{ for all } j = 1, \ldots, d) = 1.
\]

But, since \(n_j = \#(nA_j \cap \mathbb{Z}) \sim n|A_j|\), the above follows immediately from the concentration of Poisson random variables \(N_{n,\delta,\pm}(j)\) around \((1 \pm \delta)n|A_j|\), and the probability approaches one exponentially fast as \(n \to \infty\). Combining (4.5), (4.6) and (4.7) yields (4.3). The part (4.4) follows from our result in the previous section.

\[\square\]

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