The anticommutator spin algebra, its representations and quantum group invariance

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March 27, 2022

Abstract

We define a 3-generator algebra obtained by replacing the commutators by anticommutators in the defining relations of the angular momentum algebra. We show that integer spin representations are in one to one correspondence with those of the angular momentum algebra. The half-integer spin representations, on the other hand, split into two representations of dimension \( j + \frac{1}{2} \). The anticommutator spin algebra is invariant under the action of the quantum group \( SO_q(3) \) with \( q = -1 \).

1 Introduction

The algebra of observables in quantum theory plays a fundamental role. When classical systems are quantized, their classical symmetry algebra acting on a set of physical observables, in simplest examples, remains the same. For some completely integrable non-linear models, consistent quantization requires that the classical symmetry group be replaced by a quantum group \( \mathfrak{g} \) via a deformation parameter \( q = 1 + O(h) \). In recent years quantum groups involving fermions have received widespread attention. These include deformed fermion algebras \( \mathfrak{g} \), spin chains \( \mathfrak{g} \) and Fermi gases \( \mathfrak{g} \). At the same time, some quantum systems, most notably fermionic quantum systems do not have any classical analogues. Nevertheless, fermions are perhaps the most important sector of quantum phenomena. Motivated by these considerations, we define a fermionic version of the angular momentum algebra by the relations
\[
\begin{align*}
\{J_1, J_2\} &= J_3 \\
\{J_2, J_3\} &= J_1 \\
\{J_3, J_1\} &= J_2
\end{align*}
\]

where \(J_1, J_2, J_3\) are hermitian generators of the algebra. We will name this algebra ACSA, the anticommutator spin algebra. In these expressions the curly bracket denotes the anticommutator

\[
\{A, B\} \equiv AB + BA
\]

so (1-3) should be taken as the definition of an associative algebra. This proposed algebra does not fall into the category of superalgebras in the sense of Berezin-Kac axioms. In particular, the algebra is consistent without grading and there are no (graded) Jacobi relations. As it is defined this algebra falls into the category of a (non-exceptional) Jordan algebra where the Jordan product is defined by:

\[
A \circ B \equiv \frac{1}{2} (AB + BA)
\]

A formal Jordan algebra, in addition to a commutative Jordan product, also satisfies \(A^2 \circ (B \circ A) = (A^2 \circ B) \circ A\). When the Jordan product is given in terms of an anticommutator this relation is automatically satisfied. Just as a Lie algebra where the Lie bracket as defined by the commutator leads to an enveloping associative algebra, a Jordan algebra defined in terms of the above product leads to an enveloping associative algebra which we consider as an algebra of observables.

The physical properties of this system turn out to be similar to those of the angular momentum algebra yet exhibit remarkable differences. Since the angular momentum algebra is used to describe various internal symmetries, ACSA could be relevant in describing those symmetries.

In section 2 we will show that ACSA is invariant under the action of the quantum group \(SO_q(3)\) with \(q = -1\). Here, \(SO_q(3)\) is defined as the quantum subgroup of \(SU_q(3)\) where each of the (non-commuting) matrix elements of the \(3\times3\) matrix is hermitian. We note that this defines a quantum group only for \(q = \pm 1\). For \(q = 1\) one has the real orthogonal group \(SO(3)\).

In section 3, we will construct all representations of ACSA and show that the representations can be labelled by a quantum number \(j\) corresponding to the eigenvalue of \(J_3\) whose absolute value is maximum. For integer \(j\), spectrum of \(J_3\) is given by \(j, j - 1, \ldots, -j\) whereas for half-integer \(j\) there are two representations. These two representations are such that for \(j = 2k \pm \frac{1}{2}\) spectrum of \(J_3\) is respectively given by \(j, j - 2, \ldots, \pm \frac{1}{2}\) and \(-j, j + 2, \ldots, \pm \frac{1}{2}\). Section 4 is reserved for conclusions and discussion.

2 The invariance quantum group \(SO_q(3), q = -1\)

In order to find the invariance quantum group of this algebra, we transform the generators \(J_i\) to \(J'_i\) by:

\[
J'_i = \sum_j \alpha_{ij} J_j
\]

The matrix elements \(\alpha_{ij}\) are hermitian since \(J_i\)'s are hermitian and they commute with \(J_i\)'s but do not commute with each other. For the transformed
operators to obey the original relations, there should exist some conditions on the \( \alpha \)'s which define the invariance quantum group of the algebra. It is very convenient at this moment to switch to an index notation that encompasses all three defining relations of the algebra in one index equation. For the angular momentum algebra this is possible by defining the totally anti-symmetric rank 3 pseudo-tensor \( \epsilon_{ijk} \). A similar object for ACSA which we will call the fermionic Levi-Civita tensor, \( u_{ijk} \), is defined as:

\[
\begin{cases} 
1, & i \neq j \neq k, \\
0, & \text{otherwise.}
\end{cases}
\] (7)

Thus the defining relations (1-3) become:

\[
\{ J_i, J_j \} = \sum_k u_{ijk} J_k + 2\delta_{ij} J_i^2
\] (8)

The second term on the right is needed since when \( i = j \) the left-hand side becomes \( 2J_i \). When we apply the transformation (6) on this relation we get:

\[
\{ J'_k, J'_m \} = \sum_p u_{kmp} J'_p = \sum_{p, j} u_{kmp} \alpha_{pj} J_j \quad \text{for} \quad k \neq m.
\] (9)

However, substituting the transformation equations into the left-hand side, we have:

\[
\{ J'_k, J'_m \} = \sum_{i, j} \left( [\alpha_{ki}, \alpha_{mj}] J_i J_j + \alpha_{mj} \alpha_{ki} u_{ijm} J_n + 2\alpha_{mj} \alpha_{kj} J_j^2 \right)
\] (10)

These two equations yield the following relations among \( \alpha_{ij} \) when \( k \neq m \):

\[
\{ \alpha_{mj}, \alpha_{kj} \} = 0 \quad \text{for} \quad k \neq m
\] (11)

\[
[\alpha_{ki}, \alpha_{mj}] = 0 \quad \text{for} \quad i \neq j
\] (12)

\[
\sum_{ij} \alpha_{mj} \alpha_{ki} u_{ijm} = \sum_p u_{mkp} \alpha_{pn}
\] (13)

Now we will define the quantum group \( SO_q(3) \) and show that the relations above correspond to the case \( q = -1 \). The quantum group \( SO_q(3) \) can be defined as the quantum subgroup of \( SL_q(3, C) \) where an element is given by:

\[
A = \begin{pmatrix}
\alpha_{11} & \alpha_{12} & \alpha_{13} \\
\alpha_{21} & \alpha_{22} & \alpha_{23} \\
\alpha_{31} & \alpha_{32} & \alpha_{33}
\end{pmatrix}
\] (14)

where

\[
\alpha^*_{ij} = \alpha_{ij}
\] (15)

\[
A^T = A^{-1}
\] (16)

and

\[
\begin{pmatrix}
\alpha_{mj} \\
\alpha_{ki}
\end{pmatrix} \in GL_q(2) \quad \text{for} \quad k \neq m, i \neq j.
\] (17)

The quantum group \( SO_q(3) \) is equivalent to the quantum group \( SL_q(3, R) \cap SU_q(3) \). However one can show for \( SL_q(3) \) that \( q = e^{i\beta} \) for some
\( \beta \in R \) and similarly for \( SU_q(3) \) that \( q \in R \). Thus one finds that \( q = \pm 1 \) for \( SO_q(3) \). When \( q = 1 \) the quantum group becomes the usual \( SO(3) \) group; the interesting case is when \( q = -1 \) which, as we will show, is the invariance quantum group of ACSA.

Equations (11) and (12) are easily shown to be satisfied by the matrix \( A \in SO_q(-1)(3) \) by recognizing that the quantities involved belong to a submatrix that is an element of \( GL_q(-1)(2) \), as in equation (17). For a general matrix \( M \in GL_q(2) \) where:

\[
M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

we have the relations:

\[
ac = qca
\]

\[
ad - qbc = da - q^{-1}cb
\]

\[
bc = cb
\]

The relation (18) implies that \( \alpha_{mj} \alpha_{kj} = (-1)\alpha_{kj} \alpha_{mj} \), which proves equation (11) is satisfied, and the relations (19) and (20) show \( ad = da \) for \( q = -1 \) which implies \( \alpha_{ki} \alpha_{mj} = \alpha_{mj} \alpha_{ki} \) thus proving that equation (12) is satisfied by the elements of an \( SO_q(-1)(3) \) matrix.

It is a little harder to show that equation (13) is satisfied by elements of \( SO_q=-1(3) \) matrices. However, if one writes out the indices of the equation, one finds that equation (13) implies that each matrix element is equal to the \( GL_q=-1(2) \)-determinant of its minor. This fact is indeed satisfied by \( SO_q=-1(3) \) matrices since \( \det A = 1 \) and \( A^{-1T} = A \), one can show that \( A = Co(A) \) which itself means that every element is equal to the determinant of its minor. Note that since \( q = -1 \), the cofactor of an element is always equal to the minor without any alternation of signs. This type of determinant with no alternation of signs is also called a permanent.

Thus, we have found that the invariance quantum group of ACSA is the quantum group \( SO_q(3) \) with \( q = -1 \). Strictly speaking, the ACSA is a module of the \( q \)-deformed \( SO(3) \) quantum algebra with \( q = -1 \). It is very interesting to note that the invariance group of the angular momentum algebra is also \( SO_q(3) \) but with \( q = 1 \).

### 3 Representations

The Anticommutator Spin Algebra is defined by the relations (13). In order to find the representations of this algebra we define the operators:

\[
J_+ = J_1 + J_2 \tag{21}
\]

\[
J_- = J_1 - J_2 \tag{22}
\]

\[
J^2 = J_1^2 + J_2^2 + J_3^2 \tag{23}
\]

which obey the following relations:

\[
\{J_+, J_3\} = J_3 \tag{24}
\]

\[
\{J_-, J_3\} = -J_3 \tag{25}
\]

\[
J^2_+ = J^2 - J_3^2 + J_3 \tag{26}
\]

\[
J^2_- = J^2 - J_3^2 - J_3 \tag{27}
\]
Furthermore, it can easily be shown that $J^2$ is central in the algebra, i.e., that it commutes with all the elements of the algebra. For this reason, we can label the states in our representation with the eigenvalues of $J^2$ and $J_3$:

$$J^2 \mid \lambda, \mu \rangle = \lambda \mid \lambda, \mu \rangle$$  \hspace{1cm} (28)  
$$J_3 \mid \lambda, \mu \rangle = \mu \mid \lambda, \mu \rangle$$  \hspace{1cm} (29)  

The action of $J_+$ and $J_-$ on the states such defined is easily shown to be:

$$J_+ \mid \lambda, \mu \rangle = f(\lambda, \mu) \mid \lambda, -\mu + 1 \rangle$$  \hspace{1cm} (30)  
$$J_- \mid \lambda, \mu \rangle = g(\lambda, \mu) \mid \lambda, -\mu - 1 \rangle$$  \hspace{1cm} (31)  

It is enough to look at the norm of the states $J_+ \mid \lambda, \mu \rangle$ and $J_- \mid \lambda, \mu \rangle$ to find $f(\lambda, \mu)$ and $g(\lambda, \mu)$. Thus:

$$\langle \lambda, \mu \mid J^2_+ \mid \lambda, \mu \rangle = |f(\lambda, \mu)|^2$$  \hspace{1cm} (32)  
$$\langle \lambda, \mu \mid J^2 - J_3^2 + J_3 \mid \lambda, \mu \rangle = |f(\lambda, \mu)|^2$$  \hspace{1cm} (33)  
$$\lambda - \mu^2 + \mu = |f(\lambda, \mu)|^2$$  \hspace{1cm} (34)  
$$f(\lambda, \mu) = \sqrt{\lambda - \mu^2 + \mu}$$  \hspace{1cm} (35)  

and similarly, $g(\lambda, \mu) = \sqrt{\lambda - \mu^2 - \mu}$. These coefficients must be real due to the fact that $J_+$ and $J_-$ are Hermitian operators. This constraint imposes the following conditions on $\lambda$ and $\mu$:

$$\lambda - \mu^2 + \mu \geq 0$$  \hspace{1cm} (36)  
$$\lambda - \mu^2 - \mu \geq 0$$  \hspace{1cm} (37)  

which can be satisfied by letting $\lambda = j(j + 1)$ for some $j$ with:

$$j \geq \mu \geq -j$$  \hspace{1cm} (38)  

Note that equation (31) shows that the action of $J_+$ is composed of a reflection which changes sign of $\mu$, the eigenvalue of $J_3$, followed by raising by one unit. Similarly, equation (32) shows that $J_-$ reflects and lowers. Thus the highest state $\mu = j$ is annihilated by $J_-$ and "lowered" by $J_+$. Applying $J_+$ or $J_-$ twice to any state gives back the same state due to relations (26) and (27). Thus starting from the highest state we apply $J_-$ and $J_+$ alternately to get the spectrum:

$$j, -j + 1, j - 2, -j + 3, ...$$  \hspace{1cm} (39)  

This sequence ends so as to satisfy equation (38) only for integer or half-integer $j$. For integer $j$, it terminates, after an even number of steps, at $-j$ and visits every integer in between only once. For half-integer $j = 2k \pm \frac{1}{2}$ it ends at $j = \pm \frac{1}{2}$ having visited only half the states with $\mu$ half-odd integer between $j$ and $-j$. The rest of the states cannot be reached from these but are obtained by starting from the $\mu = -j$ state and applying $J_-$ and $J_+$ alternately; starting with $J_-$. We now give a few examples:

- For $j = 2$ the states follow the sequence:

$$\mu = 2, -1, 0, 1, -2$$  \hspace{1cm} (40)  

• For $j = \frac{3}{2}$ there exist two irreducible representations one with:

$$\mu = \frac{3}{2}, -\frac{1}{2},$$

and the other with:

$$\mu = -\frac{3}{2}, \frac{1}{2}.$$

• For $j = \frac{5}{2}$ the two representations are given by:

$$\mu = \frac{5}{2}, \frac{3}{2}, \frac{1}{2},$$

and by:

$$\mu = -\frac{5}{2}, \frac{3}{2}, -\frac{1}{2}.$$

4 Conclusions

The Anticommutator Spin Algebra, which is a special Jordan algebra, described in this paper has many implications. The first of these is the fact that this algebra is a consistent fermionic algebra which is not a superalgebra. For possible physical applications the right-hand side of the defining relations (1 - 3) must also be supplied with an $\hbar$. In a superalgebra approach where the $J_i$ are regarded as odd operators, the $\hbar$ on the right-hand side should also be regarded as an operator anticommuting with the $J_i$. These models [13, 14] result from the quantization of the odd Poisson bracket. In our approach however, the concept of grading and therefore an underlying Poisson bracket formalism does not exist. In particular, there is no Jacobi identity. Nevertheless, the associative algebra we consider is consistent with quantum mechanics where physical observables correspond to hermitian operators and their eigenvalues to possible results of physical measurement of these observables. It is for this reason that ACSA suggests a new kind of statistics which, we believe, will be useful in physics.

The second implication is the important role of quantum groups in mathematical physics. As we have shown in this paper, the invariance group of ACSA turns out to be a quantum group. Given the fact that ACSA is very similar to normal spin algebra and that the invariance group of spin algebra plays an important role in physics, the invariance quantum group of ACSA, $SO_{q=-1}(3)$, becomes a prime example of how central quantum groups have become in mathematical physics. It is also interesting to note that more algebras like ACSA can be constructed where the commutators of the original Lie algebra are turned into anticommutators and that such algebras might also have invariance quantum groups that is the same as the invariance group of the original Lie algebra with $q = -1$. This possibility is open to investigation in a more general framework.

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