Exact controllability to nonnegative trajectory for a chemotaxis system

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Abstract

This paper studies the controllability for a Keller-Segel type chemotaxis model with singular sensitivity. Based on the Hopf-Cole transformation, a nonlinear parabolic system, which has first-order couplings, and the coupling coefficients are functions that depend on both time and space variables, is derived. Then, the controllability result is proved by a new global Carleman estimate for general coupled parabolic equations allowed to contain a convective term. Also, the global existence of nonnegative solution for the chemotaxis system is discussed.

Key Words. Exact controllability, chemotaxis, nonnegative trajectory, Carleman estimate

Mathematics Subject Classification (2020). 35K55, 93B05

1 Introduction and the main results

Chemotaxis is a biological process in which cells move toward a chemically more favorable environment, e.g., bacteria swim to places with high concentration of food molecules. In order to model the interaction between vascular endothelial cells (VEC) and vascular endothelial growth factor (VEGF), a Keller-Segel type chemotaxis model with logarithmic sensitivity was proposed in [31]:

\[
\begin{align*}
\tilde{u}_t &= \nabla \cdot (D \nabla \tilde{u} - \chi \tilde{u} \nabla \ln \tilde{c}), \quad (\tilde{x}, \tilde{t}) \in \tilde{Q}, \\
\tilde{c}_t &= \Delta \tilde{c} - \mu \tilde{u} \tilde{c}, \quad (\tilde{x}, \tilde{t}) \in \tilde{Q},
\end{align*}
\]

(1.1)

where \( \tilde{\Omega} \) is a bounded domain in \( \mathbb{R}^n \) and \( \tilde{Q} = \tilde{\Omega} \times (0, \tilde{T}) \) with \( \tilde{T} > 0 \). The functions \( \tilde{u} \) and \( \tilde{c} \) denote the density of VEC and concentration of VEGF, respectively. The parameter \( D > 0 \) is referred to as the diffusivity of VEC. The logarithmic sensitivity \( \ln \tilde{c} \) with the constant \( \chi > 0 \) indicates that cell chemotactic responding to VEGF follows the Weber-Fechner’s law, which has many important applications in biological modelings (see [11, 7, 14, 28]). The positive constant \( \mu \) measures the degradation rate of VEGF. The system (1.1), which plays a central role in illustrating the spreading of cancer cells to other tissues in cancer metastasis, also indicates that the population of VEC could aggregate over time at certain spatial locations, since it is driven against diffusion by...
the concentration gradient of VEGF at spatial locations where the chemical signal increases. This kind of aggregation may lead to the phenomenon of finite-time blow up.

It is obvious that the logarithmic sensitivity function \( \ln \tilde{c} \) is singular at \( \tilde{c} = 0 \). In order to overcome singularity, an effective approach is to apply the Hopf-Cole transformation as follows (see, for example, [30]):

\[
\tilde{v} = \nabla \ln \tilde{c} = \frac{\nabla \tilde{c}}{\tilde{c}},
\]

together with scalings

\[
t = \chi \mu D \tilde{t}, \quad x = \sqrt{\chi \mu D} \tilde{x}, \quad \tilde{v}(x,t) = -\sqrt{\mu} \sqrt{\chi} \nabla \tilde{v}(\tilde{x}, \tilde{t}), \quad \tilde{u}(x,t) = \tilde{u}(\tilde{x}, \tilde{t}).
\]

Then the system (1.1) is transformed into the following form:

\[
\begin{align*}
\hat{u}_t - \Delta \hat{u} &= \nabla \cdot (\hat{u} \hat{v}), & (x, t) \in Q, \\
\hat{v}_t - \frac{1}{D} \Delta \hat{v} &= \nabla (-|\hat{v}|^2 + \hat{u}) + \chi \omega h, & (x, t) \in Q,
\end{align*}
\]

where \( \Omega = \sqrt{\chi \mu D} \Omega \) and \( T = \frac{\chi \mu D T}{\tilde{D}} \). For the sake of simplicity, we take \( D = 1 \) in what follows. Here, \( Q = \Omega \times (0, T) \), where \( \Omega \subset \mathbb{R}^n \) \((1 \leq n \leq 3)\) is a bounded domain with smooth boundary \( \Gamma \). Let \( \Sigma = \Gamma \times (0, T) \) and \( T > 0 \).

To prevent the spread and metastasis of tumor cells, external intervention is essential. This urges us to study the controllability of the system (1.2). The controllability problem of chemotaxis models can be viewed as finding control strategies (such as the use of drug treatment) to make the concentration of chemical and the density of cells tend to the given substance concentration and cell density. In this paper, we will study the exact controllability of system (1.2) to a nonnegative trajectory defined by system (1.4).

Let \( \omega \) be a given nonempty open subset of \( \Omega \). Denote by \( \chi_\omega \) the characteristic function of \( \omega \). We will study the following controlled chemotaxis system:

\[
\begin{align*}
u_t - \Delta u &= \nabla \cdot (u \nabla \tilde{v}), & (x, t) \in Q, \\
v_t - \Delta v &= \nabla (-|v|^2 + u) + \chi_\omega h, & (x, t) \in Q, \\
u &= \overline{p}, \quad v = 0, & (x, t) \in \Sigma, \\
(u, v)(x, 0) &= (u_0, v_0)(x), & x \in \Omega,
\end{align*}
\]

where \( (u, v) \) is the state, \( h \) is the control function and \( \overline{p} \) is a positive constant. Obviously, the control acts on the chemical concentration equation.

In the last decades, there are many works addressing the qualitative theory of the solutions to chemotaxis models (see for example [25, 26, 27, 32, 36, 39, 41, 42] and the rich references therein). However, few results are known on the controllability of chemotaxis models, we refer to [11, 12, 23, 24]. The local null controllability for a chemotaxis system of parabolic-elliptic type was first considered in [23]. In [24], the authors proved the local exact controllability to a fixed trajectory for Keller-Segel model, where the control acts on the cell density equation. Moreover, the authors pointed out that the strategy to prove the controllability may not be applied to the case of the control acting on the chemical concentration equation. The main difficulty here is that one cannot obtain the observability estimate for the adjoint equation since one variable cannot be directly represented by the other in this case. The local controllability of the Keller-Segel system around a constant trajectory with the control acting on the component of the chemical was discussed in [11]. The controllability to a constant trajectory referring to the objective trajectory is the constant solution of parabolic-elliptic system, and the controllability to a fixed (non-constant)
trajectory denoting the objective trajectory is the solution of parabolic-parabolic system. Later, a controllability result for a chemotaxis-fluid model around some particular trajectories was studied in [12]. The chemotaxis models in these known results are completely different from our system (1.3). The strategies in these papers cannot be applied here directly since our system has first order nonlinear couplings. To our knowledge, there is no literature on the controllability to a nonnegative trajectory for systems considered in this paper.

For simplicity, we use notations $L^p(Q)$, $H^p(\Omega)$ and $W^{k,p}(\Omega)$ to denote the $n$ product spaces $L^p(Q)^n$, $H^p(\Omega)^n$ and $W^{k,p}(\Omega)^n$, respectively. Consider a free system without control function:

$$
\begin{align*}
\begin{cases}
\overline{u}_t - \Delta \overline{u} &= \nabla \cdot (\overline{u}\overline{v}), & (x,t) \in Q, \\
\nabla_t - \Delta \nabla &= -\nabla (|\nabla|^2) + \nabla \overline{u}, & (x,t) \in Q, \\
\pi = \overline{p}, & (x,t) \in \Sigma, \\
(\overline{u}, \overline{v})(x,0) &= (\overline{u}_0, \overline{v}_0)(x), & x \in \Omega,
\end{cases}
\end{align*}
$$

(1.4)

where $(\overline{u}_0, \overline{v}_0) \in H^4(\Omega) \times H^4(\Omega)$ satisfies

$$
\overline{u}_0 - \overline{p} \geq 0, \quad \overline{v}_0 \geq 0 \text{ and } ||\overline{u}_0 - \overline{p}||^2_{H^4(\Omega)} + ||\overline{v}_0||^2_{H^4(\Omega)} \leq \varepsilon
$$

for some constant $\varepsilon \in (0,1)$. Assume that $(\overline{u}, \overline{v})$ is a nonnegative trajectory of equation (1.4) associated to $(\overline{u}_0, \overline{v}_0)$ and $\overline{p}$. The existence and the nonnegativity of this kind of trajectories will be given in Section 2. The system (1.3) is said to be locally exactly controllable to the trajectory $(\overline{u}, \overline{v})$ at time $T$, if there is a neighborhood $\mathcal{O}$ of $(\overline{u}_0, \overline{v}_0)$ such that for any initial data $(u_0, v_0) \in \mathcal{O}$, there exists a control function $h$ with the corresponding solution $(u, v)$ of (1.3) satisfying

$$
u(x,T) = \overline{v}(x,T), \quad v(x,T) = \nabla(x,T), \text{ a.e. } x \in \Omega.
$$

We have the following main result for the system (1.3).

**Theorem 1.1** Let $r > n+2$, and $(\overline{u}, \overline{v})$ be the trajectory of system (1.4) corresponding to $(\overline{u}_0, \overline{v}_0) \in H^4(\Omega) \times H^4(\Omega)$, which satisfies

$$
||\overline{u}_0 - \overline{p}||^2_{H^4(\Omega)} + ||\overline{v}_0||^2_{H^4(\Omega)} \leq \varepsilon, \quad \text{and } \overline{u}_0 - \overline{p} \geq 0, \quad \overline{v}_0 \geq 0
$$

for some constant $\varepsilon \in (0,1)$. Then, there exists a constant $\delta > 0$, depending only on $n, \omega, \Omega$ and $T$, such that for any $(u_0, v_0) \in W^{2-\frac{2}{r},r}(\Omega) \times W^{2-\frac{2}{r},r}(\Omega)$ satisfying

$$
u_0 \geq 0, \quad \text{and } ||u_0 - \overline{u}_0||_{W^{2-\frac{2}{r},r}(\Omega)} + ||v_0 - \overline{v}_0||_{W^{2-\frac{2}{r},r}(\Omega)} \leq \delta,
$$

there is a control $h \in L^r(Q)$, with $\text{supp } h \subseteq \omega \times [0,T]$ and the system (1.3) satisfies

$$u(x,t) \geq 0 \text{ in } Q \text{ and } u(x,T) = \overline{u}(x,T), \quad v(x,T) = \nabla(x,T) \text{ in } \Omega.
$$

**Remark 1.1** In (1.4), the nonhomogeneous Dirichlet-Dirichlet boundary conditions ensure that the solution $\overline{u}$ has a positive lower bound. Due to the complexity of system (1.3), we need the positive lower bound result of the solution to derive the Carleman estimate. Therefore, the boundary condition $\overline{u} = \overline{p}$ is technical, and the strategy developed in this paper cannot be employed to the case of homogeneous Dirichlet-Dirichlet boundary conditions.
Remark 1.2 We assume that the initial data \((u_0, v_0)\) belongs to \(H^4(\Omega) \times H^4(\Omega)\) in Theorem 1.1. It is worth mentioning that this regularity of the initial values can be reduced to \(H^3(\Omega) \times H^3(\Omega)\) based on the regularizing effect of system (1.4), which has been proved in [40, Theorem 3.1]. This strategy has also been used in [13, Lemma 5]. It would be quite interesting to study the controllability for more general initial conditions. However, it seems that the method developed in this paper is not enough. We will explain this in Remark 3.3.

The study of controllability for coupled parabolic equations has attracted intensive attention in the past few years. In general, the controllability of coupled systems is more difficult than that of single equations. Some new phenomena may occur. For instance, the minimal time of control is required for the controllability of some coupled parabolic systems (see [16]).

There are many works addressing the controllability of parabolic systems with zero order coupling terms (see, e.g., [2, 3, 4, 5, 10, 11, 17, 21, 35] and the rich references therein). Concerning the case of first order coupling terms, we refer to [9, 16, 17, 18, 22, 33] and [37] for some known controllability results for coupled parabolic systems. In [22], the author investigated the case of first and second order coupling terms, and the coupling coefficients are constants or only dependent on time variable by means of the Carleman estimate approach. In [9] and [18], the controllability was obtained for some systems with time and space-varying coupling coefficients under some technical conditions. The one-dimensional results were given in [16] and [17]. Specifically, the main tool in [16] is the moment method and the coefficients only depend on space, while the authors of [17] used the fictitious control method to solve the case where the coefficients depend on the space and time variables. In [33], by means of the Lebeau-Robbiano strategy, the internal observability was established for the system with constant or time-dependent coupling terms. Recently, the work in [37] studied the case of constant coupling coefficients by an algebraic method.

Obviously, our system (1.3) is a nonlinear coupled parabolic system. The usual way to establish the controllability of a nonlinear system is to prove the controllability of the associated linearized system combined with the fixed point technique. The key point here is that one needs to establish the suitable observability inequality for the associated adjoint system (see [3, 13]). To achieve this goal, we shall employ a similar method as the one used in [22] to derive a new global Carleman estimate for general coupled parabolic equations. The main difficulty is the coefficients of the first order coupling terms involving the solutions of the free system (1.4), which depend on both time and space variables. Hence, it is technically more complicated and difficult to deal with this problem. Moreover, in order to establish the Carleman estimate for the adjoint system, we require that the coupling coefficients belong to \(W^{2,1}_\infty(Q)\). Accordingly, another key point of the proof is to show the global existence of nonnegative solution for system (1.4), and to establish the suitable regularity for the solutions, which has an independent interest even as a pure PDE problem. The main idea for obtaining the regularity of the solutions is to use the temporal derivatives of the solution to recover the spatial derivatives, since the information of the spatial derivatives of the solution is unknown on the boundary. At last, in order to ensure the application of the fixed point argument, we need to improve the regularity of the control, and the technique used to solve this problem is adapted from [8]. Certainly, the existence of first-order coupling terms also makes the proof of this problem more complicated.

The rest of this paper is organized as follows. In Section 2, we investigate the global existence of solutions for system (1.4). Section 3 is devoted to showing the null controllability of the linearized
system. Then Theorem 1.1 is proved in Section 4.

2 Global existence of the trajectory

In this section, we first prove the following well-posedness result for system (1.4) in order to guarantee global existence of the trajectory. Then, we will show that the global trajectory preserves the nonnegative property of the initial data. To the best of our knowledge, the global well-posedness of system (1.4) has not been studied in the literature. It is worth mentioning that the Dirichlet boundary value problem (1.4) is meaningful from the biological point of view, see for example [30, 32].

We have the following well-posedness result for (1.4).

**Theorem 2.1** If \((u_0, v_0) \in H^4(\Omega) \times H^4(\Omega)\) satisfies

\[
\|u_0 - p\|^2_{H^4(\Omega)} + \|v_0\|^2_{H^4(\Omega)} \leq \varepsilon \tag{2.1}
\]

for some constant \(\varepsilon \in (0, 1)\), then there exists a unique solution \((u, v)\) of (1.4) satisfying

\[
(u, v) \in C([0, T]; H^4(\Omega)) \times C([0, T]; H^4(\Omega)),
\]

\[
(u_t, v_t) \in C([0, T]; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)) \times C([0, T]; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)).
\]

Moreover, it holds that

\[
\|u - p\|^2_{L^\infty(0, T; H^4(\Omega))} + \|v\|^2_{L^\infty(0, T; H^4(\Omega))} \leq C\left(\|u_0 - p\|^2_{H^4(\Omega)} + \|v_0\|^2_{H^4(\Omega)}\right). \tag{2.2}
\]

Let \(\bar{w} = u - p\). We will start by studying the following system of \((\bar{w}, \nabla)\):

\[
\begin{align*}
\bar{w}_t - \Delta \bar{w} &= \nabla \cdot (\bar{w} \nabla) + p \nabla \cdot \nabla, & (x, t) \in Q, \\
\nabla_t - \Delta \nabla &= -\nabla (|\nabla|^2) + \nabla \bar{w}, & (x, t) \in Q, \\
\bar{w} = \nabla = 0, & (x, t) \in \Sigma, \\
(\bar{w}, \nabla)(x, 0) = (w_0, \nabla_0)(x) = (u_0 - p, \nabla_0)(x), & x \in \Omega.
\end{align*}
\]

The proof of Theorem 2.1 is based on the standard continuity argument. Hence, we first need to assume that there exists a small positive constant \(\delta_0 < 1\) satisfying

\[
\sup_{0 \leq t \leq T} \left(\|\bar{w}\|^2_{H^4(\Omega)} + \|\nabla\|^2_{H^4(\Omega)}\right) < \delta_0. \tag{2.4}
\]

In what follows, we will establish some a priori estimates to close (2.4).

**Lemma 2.1** Under the assumption (2.4), it holds that

\[
\sup_{0 \leq t \leq T} (\|\bar{w}\|^2_{H^4(\Omega)} + \|\nabla\|^2_{H^4(\Omega)}) + \int_0^T (\|\nabla \bar{w}\|^2_{H^1(\Omega)} + \|\nabla \nabla\|^2_{H^1(\Omega)}) dt \leq C(\|\bar{w}_0\|^2_{H^4(\Omega)} + \|\nabla_0\|^2_{H^4(\Omega)}). \tag{2.5}
\]

**Proof.** Multiplying (2.3) by \(\bar{w}\) and \(\nabla\) respectively, integrating over \(\Omega\), and using integration by parts, Hölder, Poincaré, Young and Cauchy inequalities, we have

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega (\bar{w}^2 + p \nabla^2) dx + \int_\Omega (|\nabla \bar{w}|^2 + \bar{p} |\nabla|^2) dx
\]
\[ \delta \]

**Corollary 2.1**

Then, taking the \( L^2 \) inner product of the first equation in (2.3) with \( \Delta \overline{w} \) and the second equation in (2.3) with \( \overline{p} \Delta \overline{v} \), we derive

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla \overline{w}|^2 dx + \int_{\Omega} |\nabla^2 \overline{w}|^2 dx \\
\leq C \| \nabla^2 \overline{w} \|_{L^2(\Omega)} \| \nabla \overline{w} \|_{L^2(\Omega)} + \| \nabla \overline{w} \|_{L^2(\Omega)} \| \nabla^2 \overline{w} \|_{L^2(\Omega)} + \overline{p} \| \nabla^2 \overline{w} \|_{L^2(\Omega)} \| \nabla^2 \overline{w} \|_{L^2(\Omega)} \\
\leq C \| \nabla^2 \overline{w} \|_{L^2(\Omega)} \| \nabla \overline{v} \|_{L^2(\Omega)} \| \nabla^2 \overline{w} \|_{L^2(\Omega)} + \| \nabla \overline{w} \|_{L^2(\Omega)} \| \nabla^2 \overline{w} \|_{L^2(\Omega)} + \overline{p} \| \nabla^2 \overline{w} \|_{L^2(\Omega)} \| \nabla^2 \overline{w} \|_{L^2(\Omega)} \\
\leq C \delta_0 (\| \nabla^2 \overline{w} \|_{L^2(\Omega)}^2 + \| \nabla \overline{w} \|_{L^2(\Omega)}^2 + \| \nabla \overline{v} \|_{L^2(\Omega)}^2 + \frac{\overline{p}^2}{3} \| \nabla \overline{w} \|_{L^2(\Omega)}^2 + \frac{3p}{4} \| \nabla^2 \overline{w} \|_{L^2(\Omega)}^2) \tag{2.6}
\]

and

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \overline{p} |\nabla \overline{v}|^2 dx + \int_{\Omega} \overline{p} |\nabla^2 \overline{v}|^2 dx \\
\leq C \| \nabla^2 \overline{v} \|_{L^2(\Omega)} \| \nabla \overline{v} \|_{L^2(\Omega)} + \| \nabla \overline{v} \|_{L^2(\Omega)} \| \nabla^2 \overline{v} \|_{L^2(\Omega)} + \frac{\overline{p}^2}{4} \| \nabla \overline{v} \|_{L^2(\Omega)}^2 + \frac{3p}{4} \| \nabla^2 \overline{v} \|_{L^2(\Omega)}^2. \tag{2.7}
\]

It follows from (2.7) and (2.8) with \( \delta_0 \) small enough that

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \left( |\nabla \overline{w}|^2 + \overline{p} |\nabla \overline{v}|^2 \right) dx + C_1 \int_{\Omega} (|\nabla^2 \overline{w}|^2 + \overline{p} |\nabla^2 \overline{v}|^2) dx \\
\leq C \delta_0 (\| \nabla \overline{w} \|_{L^2(\Omega)}^2 + \| \nabla \overline{v} \|_{L^2(\Omega)}^2 + \frac{\overline{p}^2}{3} \| \nabla \overline{w} \|_{L^2(\Omega)}^2 + \frac{p^2}{3} \| \nabla \overline{v} \|_{L^2(\Omega)}^2), \tag{2.9}
\]

where \( C_1 > 0 \).

Multiplying (2.6) by \( \overline{p} \), adding the resulting inequality with (2.9), and integrating over \([0,t]\), we obtain (2.5) for \( \delta_0 \) small enough. The proof is completed. □

Obviously, we can get the following corollary by applying Lemma 2.1 directly to system (2.3).

**Corollary 2.1**

Under the assumption (2.4), it holds that

\[
\int_0^T (\| \overline{w} \|^2_{L^2(\Omega)} + \| \overline{v} \|^2_{L^2(\Omega)}) dt \leq C (\| \overline{w}_0 \|^2_{H^1(\Omega)} + \| \overline{v}_0 \|^2_{H^1(\Omega)}). \tag{2.10}
\]

Next we shall turn to the estimation of higher order spatial derivatives of the solution. Because of the lack of information of the spatial derivatives of the solution on the boundary, we need to use temporal derivatives and system (2.3) to obtain bounds for the spatial derivatives.

**Lemma 2.2**

Under the assumption (2.4), it holds that

\[
\sup_{0 \leq t \leq T} (\| \overline{w}_t \|^2_{H^1(\Omega)} + \| \overline{v}_t \|^2_{H^1(\Omega)}) + \int_0^T (\| \nabla \overline{w}_t \|^2_{H^1(\Omega)} + \| \nabla \overline{v}_t \|^2_{H^1(\Omega)}) dt \\
\leq C (\| \overline{w}_0 \|^2_{H^3(\Omega)} + \| \overline{v}_0 \|^2_{H^3(\Omega)}). \tag{2.11}
\]

Proof. Differentiating the first equation of (2.3) with respect to $t$, multiplying the resulting equation by $\overline{w}_t$ and integrating over $\Omega$, we have
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla \overline{w}_t|^2 dx + \int_\Omega |\nabla \overline{w}_t|^2 dx = -\int_\Omega \nabla \overline{w}_t \cdot (\overline{w} \nabla \overline{w}_t) dx - \int_\Omega \nabla \overline{w}_t \cdot \nabla \overline{v}_t dx
\]
\[
\leq \frac{1}{2} \|\nabla \overline{w}_t\|_{L^2(\Omega)}^2 + C(\|\overline{w}_t\|_{L^2(\Omega)}^2 \|\nabla \overline{v}_t\|_{L^\infty(\Omega)}^2 + \|\nabla \overline{v}_t\|_{L^2(\Omega)}^2 \|\overline{w}\|_{L^\infty(\Omega)}^2 + \|\nabla \overline{v}_t\|_{L^2(\Omega)}^2),
\]
which implies
\[
\frac{d}{dt} \int_\Omega |\nabla \overline{w}_t|^2 dx + \int_\Omega |\nabla \overline{w}_t|^2 dx \leq C\delta_0(\|\overline{w}_t\|_{L^2(\Omega)}^2 + \|\nabla \overline{v}_t\|_{L^2(\Omega)}^2) + C\|\nabla \overline{v}_t\|_{L^2(\Omega)}^2.
\]
(2.12)
Moreover, it follows from equations (2.3) that for any $0 \leq t \leq T$,
\[
\|\overline{w}_t(t)\|_{L^2(\Omega)}^2 \leq C(1 + \|\nabla(t)\|_{H^2(\Omega)}^2)(\|\overline{w}_t(t)\|_{L^2(\Omega)}^2 + \|\nabla(t)\|_{H^1(\Omega)}^2)
\]
(2.13)
and
\[
\|\nabla(t)\|_{L^2(\Omega)}^2 \leq C(1 + \|\nabla(t)\|_{H^2(\Omega)}^2)(\|\overline{w}_t(t)\|_{L^2(\Omega)}^2 + \|\nabla(t)\|_{H^1(\Omega)}^2).
\]
(2.14)
Thus, integrating (2.12) over $[0, t]$ and using (2.10), we obtain
\[
\sup_{0 \leq t \leq T} \|\overline{w}_t\|_{L^2(\Omega)}^2 + \int_0^T \|\nabla \overline{w}_t\|_{L^2(\Omega)}^2 dt \leq \|\overline{w}_t(0)\|_{L^2(\Omega)}^2 + C \int_0^T (\|\overline{w}_t\|_{L^2(\Omega)}^2 + \|\nabla \overline{v}_t\|_{L^2(\Omega)}^2) dt
\]
(2.15)
\[
\leq C(\|\overline{w}_0\|_{H^2(\Omega)}^2 + \|\nabla_0\|_{H^2(\Omega)}^2),
\]
where we have used the fact from (2.13) that $\|\overline{w}_t(0)\|_{L^2(\Omega)}^2 \leq C(\|\overline{w}_0\|_{H^2(\Omega)}^2 + \|\nabla_0\|_{H^2(\Omega)}^2)$. Similarly, with the help of (2.10) and (2.14), for $\nabla$, it holds that
\[
\sup_{0 \leq t \leq T} \|\nabla(t)\|_{L^2(\Omega)}^2 + \int_0^T \|\nabla \nabla(t)\|_{L^2(\Omega)}^2 dt \leq C(\|\overline{w}_0\|_{H^2(\Omega)}^2 + \|\nabla_0\|_{H^2(\Omega)}^2).
\]
(2.16)
Next, we take $\frac{d}{dt}$ to (2.3), multiply the resulting equations by $\Delta \overline{w}_t$ and $\Delta \nabla(t)$ respectively and use integration by parts to derive
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega (|\nabla \overline{w}_t|^2 + |\nabla \nabla(t)|^2) dx + \int_\Omega (|\nabla^2 \overline{w}_t|^2 + |\nabla^2 \nabla(t)|^2) dx
\]
\[
= -\int_\Omega (\nabla \overline{w}_t) \cdot \Delta \overline{w}_t dx - \int_\Omega \nabla \overline{w}_t \cdot \Delta \overline{v}_t dx + \int_\Omega (\nabla \nabla(t)) \cdot \Delta \nabla(t) dx - \int_\Omega \nabla \overline{w}_t \cdot \Delta \nabla(t) dx
\]
\[
\leq \frac{1}{2} \int_\Omega (|\nabla^2 \overline{w}_t|^2 + |\nabla^2 \nabla(t)|^2) dx + C \left( \|\nabla\|_{L^\infty(\Omega)}^2 \int_\Omega (|\nabla \overline{w}_t|^2 + |\nabla \nabla(t)|^2) dx + \|\overline{w}\|_{L^\infty(\Omega)}^2 \int_\Omega |\nabla \overline{w}_t|^2 dx + \|\nabla\|_{L^\infty(\Omega)}^2 \int_\Omega |\nabla \overline{v}_t|^2 dx + \|\overline{w}_t\|_{L^\infty(\Omega)}^2 \int_\Omega |\nabla \overline{v}_t|^2 dx \right),
\]
which implies
\[
\frac{d}{dt} \int_\Omega (|\nabla \overline{w}_t|^2 + |\nabla \nabla(t)|^2) dx + \int_\Omega (|\nabla^2 \overline{w}_t|^2 + |\nabla^2 \nabla(t)|^2) dx
\]
\[
\|\nabla \varpi_t(t)\|_{L^2(\Omega)}^2 \leq C(1 + \|\nabla(t)\|_{H^3(\Omega)}^2)(\|\varpi(t)\|_{H^3(\Omega)}^2 + \|\nabla(t)\|_{H^2(\Omega)}^2) 
\]  
(2.18)

and

\[
\|\nabla \varphi(t)\|_{L^2(\Omega)}^2 \leq C(1 + \|\nabla(t)\|_{H^3(\Omega)}^2)(\|\varphi(t)\|_{H^3(\Omega)}^2 + \|\varpi(t)\|_{H^2(\Omega)}^2). 
\]  
(2.19)

Now, integrating (2.17) over \([0, t]\), by (2.10), (2.14), (2.16) and (2.18), we arrive at

\[
\sup_{0 \leq t \leq T} \left( \|\nabla \varpi(t)\|_{L^2(\Omega)}^2 + \|\nabla \varphi(t)\|_{L^2(\Omega)}^2 \right) + \int_0^T \left( \|\nabla^2 \varpi_t\|_{L^2(\Omega)}^2 + \|\nabla^2 \varphi_t\|_{L^2(\Omega)}^2 \right) dt 
\leq C(\|\varpi_0\|_{H^3(\Omega)}^2 + \|\varphi_0\|_{H^3(\Omega)}^2). 
\]  
(2.20)

This together with (2.15) and (2.16) yields (2.11). The proof is completed.

With these a priori estimates at hand, we are ready to close the assumption (2.4).

**Proof of Theorem 2.1.** It follows from the equations (2.3) and all the estimates above that

\[
\sup_{0 \leq t \leq T} \left( \|\nabla \varpi(t)\|_{H^3(\Omega)}^2 + \|\nabla \varphi(t)\|_{H^3(\Omega)}^2 \right) \leq C(\|\varpi_0\|_{H^3(\Omega)}^2 + \|\varphi_0\|_{H^3(\Omega)}^2) \leq C \varepsilon. 
\]  
(2.21)

If \( \varepsilon \) is suitably small such that \( C \varepsilon \leq \delta_0 \), by the standard continuity argument (see [34, 38]), the estimate (2.4) is closed. Notice that the local existence and uniqueness of the solution to the equations (2.3) can be established by using the classical theory of linear parabolic system (see, for example, [29, p.616]) combining with Schauder fixed point theorem. Thus, applying (2.4) and all a priori estimates, we extend the local solutions to be a global solution and the uniqueness of global solution in \( C([0, T]; H^3(\Omega)) \) is guaranteed by the uniqueness of local solution.

Thus, it only remains to establish the regularity in \( H^4 \) space. In view of equations (2.3), (2.18) and (2.19), it holds that for any \( 0 \leq t \leq T \),

\[
\|\nabla \varpi_{tt}(t)\|_{L^2(\Omega)}^2 \leq C(1 + \|\nabla(t)\|_{H^4(\Omega)}^2)(\|\varpi(t)\|_{H^4(\Omega)}^2 + \|\nabla(t)\|_{H^3(\Omega)}^2), 
\]  
(2.22)

\[
\|\nabla \varphi_{tt}(t)\|_{L^2(\Omega)}^2 \leq C(1 + \|\nabla(t)\|_{H^4(\Omega)}^2)(\|\varphi(t)\|_{H^4(\Omega)}^2 + \|\varpi(t)\|_{H^3(\Omega)}^2), 
\]  
(2.23)

and

\[
\|\nabla \varpi_{tt}(t)\|_{L^2(\Omega)}^2 \leq C \left( \|\nabla^2 \varpi_{tt}\|_{L^2(\Omega)}^2 + \|\varpi_{tt}\|_{H^1(\Omega)}^2 \|\nabla^2 \varpi(t)\|_{H^1(\Omega)}^2 + \|\varpi_t\|_{H^2(\Omega)}^2(1 + \|\varpi\|_{H^3(\Omega)}^2) \right), 
\]  
(2.24)

\[
\|\nabla \varphi_{tt}(t)\|_{L^2(\Omega)}^2 \leq C \left( \|\nabla^2 \varphi_{tt}\|_{L^2(\Omega)}^2 + \|\varphi_{tt}\|_{H^1(\Omega)}^2 \|\varphi(t)\|_{H^3(\Omega)}^2 + \|\varphi_t\|_{H^2(\Omega)}^2 \right). 
\]  
(2.25)

Differentiating the first equation of (2.3) with respect to \( t \) twice, and then multiplying the resulting equation by \( \varpi_{tt} \) and integrating over \( \Omega \), we have

\[
\frac{1}{2} \int \frac{d}{dt} \|\nabla \varpi_{tt}\|_{L^2(\Omega)}^2 dx + \int \nabla \varpi_{tt} \cdot \nabla \varphi_{tt} dx = -\int \nabla \varpi_{tt} \cdot \nabla \varphi dx - \int \nabla \varpi \cdot \nabla \varphi_{tt} dx 
\leq \frac{1}{2} \|\nabla \varpi_{tt}\|_{L^2(\Omega)}^2 + C(\|\varpi_{tt}\|_{L^2(\Omega)}^2 ||\varphi||_{L^\infty(\Omega)}^2 + ||\nabla \varphi_{tt}\|_{L^2(\Omega)}^2 ||\varphi||_{L^\infty(\Omega)}^2 + ||\varphi_t||_{L^2(\Omega)}^2 ||\varphi_t||_{L^\infty(\Omega)}^2 ||\varphi||_{L^\infty(\Omega)}^2 
\]
which together with (2.16) and (2.21) implies
\[
\frac{d}{dt} \int_\Omega |\nabla u|^2 \, dx + \int_\Omega |\nabla \nabla u|^2 \, dx \\
\leq C(\|u_t\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{H^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{H^2(\Omega)})
\]
\[
\leq C(\|v_0\|^2_{H^2(\Omega)} + \|\nabla v_0\|^2_{H^2(\Omega)})\left(\|u_t\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{H^2(\Omega)}\right) + C\|u_t\|^2_{L^2(\Omega)}. \quad (2.27)
\]

Then, integrating (2.27) over \([0, t]\), by (2.10), (2.15), (2.16), (2.20) and (2.22)-(2.25), we obtain
\[
\sup_{0 \leq t \leq T} \|\nabla u_t\|^2_{L^2(\Omega)} + \int_0^T \|\nabla \nabla u|^2 \, dt \leq C(\|u_0\|^2_{H^2(\Omega)} + \|\nabla u_0\|^2_{H^2(\Omega)}). \quad (2.28)
\]

Similarly, for \(\nabla\), it holds that
\[
\sup_{0 \leq t \leq T} \|\nabla u_t\|^2_{L^2(\Omega)} + \int_0^T \|\nabla \nabla u|^2 \, dt \leq C(\|v_0\|^2_{H^2(\Omega)} + \|\nabla v_0\|^2_{H^2(\Omega)}). \quad (2.29)
\]

Therefore, with the help of the equations (2.3), (2.21), (2.28) and (2.29), we deduce (2.2). This completes the proof of Theorem 2.1.

\[\Box\]

**Corollary 2.2** Assume that the conditions in Theorem 2.1 hold, and \(\overline{\nabla} - \underline{\nabla} \geq 0, \ n_0 \geq 0\). Then the solution of (1.4) satisfies
\[
\overline{\nabla} - \underline{\nabla} \geq 0, \ n \geq 0, \ \forall \ (x, t) \in Q.
\]

**Proof.** It follows from Theorem 2.1 that the global trajectory \((\overline{\nabla}, \nabla)\) is the classical solution of (1.4). Obviously, \((\overline{\nabla}, 0)\) can be regarded as a lower solution of (1.4). Thus, the conclusion of this corollary follows from the comparison principle immediately. \[\Box\]

### 3 Null controllability of the linearized system

Let \(y = u - \overline{\nabla}, \ z = \nabla - \underline{\nabla}, \ y_0 = u_0 - \overline{\nabla}_0\) and \(z_0 = \nabla_0 - \underline{\nabla}_0\). An easy computation shows that \((y, z)\) satisfies
\[
\begin{align*}
\begin{cases}
y_t - \Delta y = \nabla \cdot (y(z + \overline{\nabla})) + \nabla \cdot (\overline{\nabla}z), & (x, t) \in Q, \\
z_t - \Delta z = -\nabla(|z|^2 + 2\nabla \cdot z) + \nabla y + \chi_0h, & (x, t) \in Q, \\
y = 0, & (x, t) \in \Sigma, \\
(y, z)(x, 0) = (y_0, z_0)(x), & x \in \Omega.
\end{cases}
\end{align*}
\]

Obviously, the local exact controllability to the trajectory \((\overline{\nabla}, \nabla)\) for equations (1.3) is equivalent to the local null controllability of system (3.1).

For \(p \geq 2\), define the Banach space \(V^p\) by
\[
V^p := \{y : y \in L^p(0, T; W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)); y_t \in L^p(Q)\},
\]
and its natural norm \(\|\cdot\|_{V^p}\) by \(\|y\|_{V^p} = \|y\|_{L^p(0, T; W^{2,p}(\Omega))} + \|y_t\|_{L^p(Q)}\).
In this section, we consider the null controllability of the following linearized system of (3.1):

\[
\begin{align*}
    y_t - \Delta y &= \nabla \cdot (ay) + \nabla \cdot (Bz), \quad (x, t) \in Q, \\
    z_t - \Delta z &= -\nabla (b \cdot z) + \nabla y + \chi_\omega h, \quad (x, t) \in Q, \\
    y &= z = 0, \quad (x, t) \in \Sigma, \\
    (y, z)(x, 0) &= (y_0, z_0)(x), \quad x \in \Omega,
\end{align*}
\]

(3.2)

where $h$ is the control, $(y_0, z_0)$ is the given initial value, and

\[
an, b, \nabla a, \nabla b \in L^\infty(Q), \quad B, B_t, \nabla B, \Delta B \in L^\infty(Q) \quad \text{and} \quad B \text{ has a positive lower bound.}
\]

(3.3)

In fact, $a = \eta + \nabla$, $b = \eta + 2\nabla$ and $B = \pi$, where $\eta \in V^r$ is a known function, and $r > n + 2$.

Write

\[
M_1 = 1 + T\left(1 + \|\nabla \cdot a\|_{L^\infty(Q)} + \|\nabla \cdot b\|_{L^\infty(Q)} + \|a\|_{L^\infty(Q)}^2 + \|b\|_{L^\infty(Q)}^2 + \|B\|_{L^\infty(Q)}^2 + \|\nabla B\|_{L^\infty(Q)} \right),
\]

\[
M_2 = 1 + \|\nabla \cdot a\|_{L^\infty(Q)} + \|\nabla \cdot b\|_{L^\infty(Q)} + \|a\|_{L^\infty(Q)} + \|b\|_{L^\infty(Q)} + \|B\|_{L^\infty(Q)} + \|\nabla B\|_{L^\infty(Q)}.
\]

We have the following well-posedness result for system (3.2):

**Proposition 3.1** Assume that $a, b, B \in L^\infty(Q)$, $\nabla \cdot a, \nabla \cdot b, \nabla B \in L^\infty(Q)$, $y_0, z_0 \in W^{2, \frac{n}{2}}(\Omega) \cap H_0^1(\Omega)$, and $h \in L^p(\omega \times (0, T))$ with $p \geq 2$ being arbitrary. Then system (3.2) admits a unique strong solution $(y, z) \in V^p \times V^p$. Moreover, there exist positive constants $C = C(\Omega, n, p)$ and $k_1 = k_1(n)$ such that

\[
\|(y, z)\|_{V^p \times V^p} \leq c^{CM_1}M_2^{k_1}\left(\|(y_0, z_0)\|_{W^{2, \frac{n}{2}}(\Omega) \cap H_0^1(\Omega)} + \|h\|_{L^p(\omega \times (0, T))}\right).
\]

**Proof.** We split the proof into two steps.

**Step 1.** When $p = 2$, multiplying the first equation of (3.2) by $y$ and integrating it on $\Omega$, we get

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega y^2 dx + \int \nabla y \cdot \nabla ydx = \int a y \cdot \nabla y dx + \int \frac{1}{2} B z \cdot \nabla y + \int \frac{1}{2} y \cdot \nabla a dx
\]

\[
\leq \left(\|a\|_{L^\infty(\Omega)}^2 + \|\nabla \cdot a\|_{L^\infty(\Omega)}\right) \int \|y\|^2 dx + \frac{1}{2} \int \nabla y \cdot \nabla y dx + \int \|B\|_{L^\infty(\Omega)}^2 \int |z|^2 dx.
\]

Doing the same thing to the second equation of (3.2), we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega z^2 dx + \int \nabla z \cdot \nabla z dx = \int b z \cdot \nabla z dx + \frac{1}{4} \int \|\nabla z\|^2 dx + \frac{1}{4} \int \|\nabla y\|^2 dx + \int \chi_\omega h \cdot z dx
\]

\[
\leq C \left(1 + \|b\|_{L^\infty(\Omega)}^2\right) \int \|z\|^2 dx + \frac{1}{4} \int \|\nabla z\|^2 dx + \frac{1}{4} \int \|\nabla y\|^2 dx + \int \chi_\omega h^2 dx.
\]

Then,

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega \left(y^2 + z^2\right) dx + \int \left(\|\nabla y\|^2 + \|\nabla z\|^2\right) dx
\]

\[
\leq C \left(\|a\|_{L^\infty(\Omega)}^2 + \|\nabla \cdot a\|_{L^\infty(\Omega)}\right) \int \|y\|^2 dx + C \left(1 + \|b\|_{L^\infty(\Omega)}^2 + \|B\|_{L^\infty(\Omega)}^2\right) \int \|z\|^2 dx
\]

\[
+ C \int \|h\|^2 dx.
\]
By Gronwall’s inequality, we have
\[\int_\Omega \left( y^2(x, t) + z^2(x, t) \right) dx + \int_0^t \int_\Omega \left( |\nabla y|^2 + |\nabla z|^2 \right) dx dt \leq C e^{CM_1} \left( \int_\Omega (y_0^2 + z_0^2) dx + \int_0^T \int_\omega |h|^2 dx dt \right). \tag{3.5}\]

On the other hand, by the first equation of (3.2), we have
\[(y_t - \Delta y)^2 = \left( \nabla \cdot (ay) + \nabla \cdot (Bz) \right)^2.\]

Integrating the previous inequality on \(\Omega \times (0, t)\), we obtain
\[
\int_0^t \int_\Omega y_t^2 dx dt + \int_0^t \int_\Omega |\Delta y|^2 dx dt - 2 \int_0^t \int_\Omega y_t \Delta y dx dt
= \int_0^t \int_\Omega y_t^2 dx dt + \int_0^t \int_\Omega |\Delta y|^2 dx dt + \int_0^t \int_\Omega \frac{d}{dt} |\nabla y|^2 dx dt
= \int_0^t \int_\Omega \left( \nabla \cdot (ay) + \nabla \cdot (Bz) \right)^2 dx dt. \tag{3.6}
\]

Combining this with (3.5), we have
\[
\int_0^t \int_\Omega y_t^2 dx dt + \int_0^t \int_\Omega |\Delta y|^2 dx dt + \int_0^t \int_\Omega |\nabla y(x, t)|^2 dx
= \int_0^t \int_\Omega \left( \nabla \cdot (ay) + \nabla \cdot (Bz) \right)^2 dx dt + \int_0^T \int_\Omega |\nabla y_0|^2 dx
\leq C e^{CM_1} M_2^{k_1} \left[ \int_\Omega (y_0^2 + z_0^2 + |\nabla y_0|^2) dx + \int_0^T \int_\omega |h|^2 dx dt \right], \tag{3.7}\]

where \(k_1 = k_1(n), C = C(n, \Omega, p)\). Similarly, we deal with the second equation of (3.2), which implies
\[
\int_0^t \int_\Omega z_t^2 dx dt + \int_0^t \int_\Omega |\Delta z|^2 dx dt + \int_0^t \int_\Omega |\nabla z(x, t)|^2 dx
\leq C e^{CM_1} M_2^{k_1} \left[ \int_\Omega (y_0^2 + z_0^2 + |\nabla z_0|^2) dx + \int_0^T \int_\omega |h|^2 dx dt \right]. \tag{3.7}\]

By (3.6) and (3.7), we have
\[
\int_0^t \int_\Omega \left( |y_t|^2 + |z_t|^2 \right) dx dt + \int_0^t \int_\Omega \left( |\Delta y|^2 + |\Delta z|^2 \right) dx dt + \int_\Omega \left( |\nabla y(x, t)|^2 + |\nabla z(x, t)|^2 \right) dx
\leq C e^{CM_1} M_2^{k_1} \left[ \int_\Omega (y_0^2 + z_0^2 + |\nabla y_0|^2 + |\nabla z_0|^2) dx + \int_0^T \int_\omega |h|^2 dx dt \right].
\]

**Step 2.** We consider the case \(p > 2\). We only show the case when \(n = 3\), since the proof is similar when \(n = 1\) or \(2\).

By Step 1, we know that the solution of (3.2) lies in \(V^2 \times V^2\), and
\[
\| (y, z) \|_{V^2 \times V^2} \leq C e^{CM_1} M_2^{k_1} \left( \| (y_0, z_0) \|_{W^{1,2}(\Omega) \times W^{1,2}(\Omega)} + \| h \|_{L^2(\omega \times (0, T))} \right). \tag{3.8}
\]
On the one hand, let \( f_1 = -z \nabla \cdot b - b \nabla \cdot z + \nabla y + \chi_\omega \cdot h \). Note that \( b, \nabla \cdot b \in L^\infty(Q) \), by Sobolev embedding theorem, we have
\[
\nabla y, b \nabla \cdot z \in L^2(0, T; L^p(\Omega)) \cap L^\infty(0, T; L^2(\Omega)),
\]
where \( q = \frac{2n}{n-2} \). Therefore, \( f_1 \in L^p(0, T; L^{p_1}(\Omega)) \), with \( p_1 = \min\{p, \frac{2np}{np-4}\} \), and
\[
\|f_1\|_{L^p(0, T; L^{p_1}(\Omega))} \leq C \left( 1 + \|b\|_{L^\infty(Q)} + \|\nabla \cdot b\|_{L^\infty(Q)} \right) \cdot \left( \|y\|_{V^2 \times V^2} + \|h\|_{L^p(\omega \times (0, T))} \right).
\] (3.9)

Then by Theorem 2.3 in [20], we obtain
\[
z \in L^p(0, T; W^{2,p_1}(\Omega)), \ z_t \in L^p(0, T; L^{p_1}(\Omega)),
\]
and
\[
\|z\|_{L^p(0, T; W^{2,p_1}(\Omega))} + \|z_t\|_{L^p(0, T; L^{p_1}(\Omega))} \leq C \left( \|f_1\|_{L^p(0, T; L^{p_1}(\Omega))} + \|z_0\|_{W^{2-\frac{2}{p},p}(\Omega)} \right),
\]
where \( C > 0 \) is a constant independent of \( T \). Combining this with (3.9), we get
\[
\|z\|_{L^p(0, T; W^{2,p_1}(\Omega))} + \|z_t\|_{L^p(0, T; L^{p_1}(\Omega))} 
\leq C \left( 1 + \|b\|_{L^\infty(Q)} + \|\nabla \cdot b\|_{L^\infty(Q)} \right) \left( \|y\|_{V^2 \times V^2} + \|h\|_{L^p(\omega \times (0, T))} + \|z_0\|_{W^{2-\frac{2}{p},p}(\Omega)} \right). \] (3.10)

On the other hand, take \( f_2 = \nabla \cdot ay + a \cdot \nabla y + \nabla B \cdot z + B \nabla \cdot z \). Similar to the estimate of \( f_1 \), by (3.10), we get \( f_2 \in L^p(0, T; L^{p_1}(\Omega)) \), and
\[
\|f_2\|_{L^p(0, T; L^{p_1}(\Omega))} \leq CM_2 \left( \|y\|_{V^2 \times V^2} + \|h\|_{L^p(\omega \times (0, T))} + \|z_0\|_{W^{2-\frac{2}{p},p}(\Omega)} \right), \] (3.11)

Applying Theorem 2.3 in [20] again to the solution \( y \), we deduce that
\[
y \in L^p(0, T; W^{2,p_1}(\Omega)), \ y_t \in L^p(0, T; L^{p_1}(\Omega))
\]
and
\[
\|y\|_{L^p(0, T; W^{2,p_1}(\Omega))} + \|y_t\|_{L^p(0, T; L^{p_1}(\Omega))} \leq C \left( \|f_2\|_{L^p(0, T; L^{p_1}(\Omega))} + \|y_0\|_{W^{2-\frac{2}{p},p}(\Omega)} \right).
\]
By (3.11), it follows that
\[
\|y\|_{L^p(0, T; W^{2,p_1}(\Omega))} + \|y_t\|_{L^p(0, T; L^{p_1}(\Omega))} 
\leq CM_2 \left( \|y\|_{V^2 \times V^2} + \|y_0\|_{W^{2-\frac{2}{p},p}(\Omega)} + \|z_0\|_{W^{2-\frac{2}{p},p}(\Omega)} + \|h\|_{L^p(\omega \times (0, T))} \right).
\]
If \( p \leq \frac{2np}{np-4} \), i.e., \( p \leq 2 + \frac{4}{n} \), this ends the proof. If \( p > \frac{2np}{np-4} \), the proof will be completed by repeating the above procedure for finitely many times.

The null controllability result for the equation (3.2) can be stated as follows.

**Theorem 3.1** Assume that the condition (3.3) holds and \( T > 0 \). Then there exists a function \( h \in L^2(Q) \) such that the associated solution \((y, z)\) of equations (3.2) satisfies
\[
y(x, T) = z(x, T) = 0, \ a.e. \ x \in \Omega.
\]
Moreover,
\[
\|h\|_{L^2(Q)} \leq C \left( \|y_0\|_{L^2(\Omega)} + \|z_0\|_{L^2(\Omega)} \right). \] (3.12)
We consider the following adjoint system of (3.2):

\[
\begin{align*}
-\varphi_t - \Delta \varphi + a \cdot \nabla \varphi &= -\nabla \cdot \psi, & (x, t) \in Q, \\
\psi_t + \Delta \psi + b \nabla \cdot \psi &= B \nabla \varphi, & (x, t) \in Q, \\
\varphi &= \psi = 0, & (x, t) \in \Sigma, \\
(\varphi, \psi)(x, T) &= (\varphi_0, \psi_0)(x), & x \in \Omega.
\end{align*}
\]

(3.13)

We derive a new global Carleman estimate for (3.13). Assume that \( \rho \in C^2(\bar{\Omega}) \) satisfies

\[
|\nabla \rho| \geq C > 0 \text{ in } \Omega \setminus \omega_0, \quad \rho > 0 \text{ in } \Omega, \quad \text{and } \rho = 0 \text{ on } \partial \Omega,
\]

where \( \omega_0 \neq \emptyset \) is an open subset of \( \omega \). Let \( \omega_1 \) be any fixed open subset of \( \omega \) such that \( \omega_0 \subseteq \omega_1 \) and \( \omega_1 \subseteq \omega \). Inspired by [22], we first introduce the weight functions. For any real number \( \lambda > 1 \) and \( s > 1 \), set

\[
\theta = e^t, \quad l = -s\phi, \quad \phi(x, t) = \exp\left\{ \frac{k(m+1)\lambda \rho \|\rho\|_{L^\infty(\Omega)}}{m(T-t)^m} \right\} - \exp\left\{ \lambda \rho \|\rho\|_{L^\infty(\Omega)} + \rho(x) \right\},
\]

where \( m > 3 \) and \( k > m \) are fixed. Put

\[
\xi(x, t) = \frac{\exp\{\lambda \rho \|\rho\|_{L^\infty(\Omega)} + \rho(x)\}}{m(T-t)^m},
\]

\[
\phi^*(t) = \max_{x \in \bar{\Omega}} \phi(x, t) = \phi(x, t)|_{\partial \Omega}, \quad \xi^*(t) = \min_{x \in \bar{\Omega}} \xi(x, t) = \xi(x, t)|_{\partial \Omega}.
\]

We have the following global Carleman estimate for (3.13).

**Theorem 3.2** Assume that the condition (3.3) holds. Then there exist \( \lambda_1, s_1 > 0 \) such that for all \( \lambda \geq \lambda_1, \ s \geq s_1 \), one can find a constant \( C > 0 \) such that the following inequality holds for the solutions of (3.13):

\[
s\lambda^2 \int_Q \theta^2 |\Delta \varphi|^2 dxdt + s^3 \lambda^4 \int_Q \theta^2 |\nabla \varphi|^2 dxdt + s^6 \lambda^8 \int_Q \theta^2 |\psi|^2 dxdt
\]

\[
\leq C(1 + \frac{T^{2m}}{s^8 \lambda^8}) \int_0^T \int_{\Omega} \theta^4 e^{2s\phi^*} \xi^8 |\psi|^2 dxdt.
\]

(3.15)

Before giving the proof of Theorem 3.2, we first recall a Carleman estimate for the parabolic equation with nonhomogeneous Neumann boundary conditions, which will be useful (see [19]).

**Lemma 3.1** Let \( \bar{y}_0 \in L^2(\Omega) \), \( h_1 \in L^2(Q) \), \( h_2 \in L^2(Q) \) and \( h_3 \in L^2(\Sigma) \). Then there is a constant \( C(\Omega, \omega_0) > 0 \), such that for any \( \lambda \geq C \) and \( s \geq C(T^{2m} + T^{2m-1}) \), any solution \( \bar{y} \in L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega)) \) of

\[
\begin{align*}
\bar{y}_t - \Delta \bar{y} &= h_1 + \nabla \cdot h_2, & (x, t) \in Q, \\
\partial_{\nu} \bar{y} + h_2 \cdot \nu &= h_3, & (x, t) \in \Sigma, \\
\bar{y}(x, 0) &= \bar{y}_0(x), & x \in \Omega
\end{align*}
\]

satisfies that

\[
s\lambda^2 \int_Q \theta^2 |\nabla \bar{y}|^2 dxdt + s^3 \lambda^4 \int_Q \theta^2 |\bar{y}|^2 dxdt
\]

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Applying Lemma 3.1 for \( \partial \phi \)

The main idea of this proof is borrowed from [22]. The proof will be divided into two steps.

**Step 1.** We first consider the parabolic equation satisfied by \( \nabla \varphi \), because \( B \nabla \varphi \) appears on the right hand side of the equation satisfied by \( \psi \). By (3.13), we know that \( \nabla \varphi \) satisfies

\[
-(\nabla \varphi)_t - \Delta (\nabla \varphi) + \nabla (a \cdot \nabla \varphi) = -\nabla (\nabla \cdot \psi) \quad \text{in} \quad Q.
\]

Set \( a = (a_1, a_2, ..., a_n) \), \( \psi = (\psi_1, \psi_2, ..., \psi_n) \), then by (3.16), it follows that \( \frac{\partial \varphi}{\partial x_i} \) satisfies

\[
-(\frac{\partial \varphi}{\partial x_i})_t - \Delta (\frac{\partial \varphi}{\partial x_i}) + \sum_{j=1}^{n} \left( a_j \frac{\partial^2 \varphi}{\partial x_i \partial x_j} + \frac{\partial a_j}{\partial x_i} \frac{\partial \varphi}{\partial x_j} \right) = -\sum_{j=1}^{n} \frac{\partial^2 \psi_j}{\partial x_i \partial x_j} = \nabla \cdot \frac{\partial \psi}{\partial x_i}.
\]

Applying Lemma 3.1 for \( \frac{\partial \varphi}{\partial x_i} \), here \( h_1 = -\sum_{j=1}^{n} \left( a_j \frac{\partial^2 \varphi}{\partial x_i \partial x_j} + \frac{\partial a_j}{\partial x_i} \frac{\partial \varphi}{\partial x_j} \right), \quad h_2 = -\frac{\partial \psi}{\partial x_i} \) and \( h_3 = \sum_{j=1}^{n} \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \nu_j - \sum_{j=1}^{n} \frac{\partial \psi}{\partial v} \nu_j \), we conclude that

\[
I(\nabla \varphi) := s\lambda^2 \int_{Q} \theta^2 \xi \left( \sum_{i=1}^{n} \left| \nabla \left( \frac{\partial \varphi}{\partial x_i} \right) \right|^2 \right) dxdt + s^3 \lambda^4 \int_{Q} \theta^2 \xi \left( \sum_{i=1}^{n} \left| \frac{\partial \varphi}{\partial x_i} \right|^2 \right) dxdt
\]

\[
\leq C \left[ s^3 \lambda^4 \int_{0}^{T} \int_{\omega_0} \theta^2 \xi \left( \sum_{i=1}^{n} \left| \frac{\partial \varphi}{\partial x_i} \right|^2 \right) dxdt + s^2 \lambda^2 \int_{Q} \theta^2 \xi \left( \sum_{i,j=1}^{n} \left| \frac{\partial \psi_j}{\partial x_i} \right|^2 \right) dxdt
\]

\[
+ s\lambda \int_{\Sigma} e^{-2s\phi^*} \xi^* \sum_{i=1}^{n} \left( \sum_{j=1}^{n} \left| \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \nu_j - \frac{\partial \psi_i}{\partial v} \nu_j \right| \right)^2 d\sigma dt
\]

\[
+ \int_{Q} \theta^2 \sum_{i=1}^{n} \left( \sum_{j=1}^{n} \left( a_j \frac{\partial^2 \varphi}{\partial x_i \partial x_j} + \frac{\partial a_j}{\partial x_i} \frac{\partial \varphi}{\partial x_j} \right) \right)^2 dxdt \right].
\]

(3.17)

Notice that \( \frac{1}{\xi} \leq CT^{2m} \), and by the definition of \( \xi \), it follows that

\[
\int_{Q} \theta^2 \sum_{i=1}^{n} \left( \sum_{j=1}^{n} \left( a_j \frac{\partial^2 \varphi}{\partial x_i \partial x_j} + \frac{\partial a_j}{\partial x_i} \frac{\partial \varphi}{\partial x_j} \right) \right)^2 dxdt
\]

\[
\leq C \left( \| a \|_{L^\infty(Q)} + \| \nabla a \|_{L^\infty(Q)} \right) \left[ T^{2m} \int_{Q} \theta^2 \xi \sum_{i=1}^{n} \left| \nabla \left( \frac{\partial \varphi}{\partial x_i} \right) \right|^2 dxdt
\]

\[
+ T^{6m} \int_{Q} \theta^2 \xi \sum_{i=1}^{n} \left| \frac{\partial \varphi}{\partial x_i} \right|^2 dxdt \right].
\]

(3.18)

We next estimate \( s\lambda \int_{\Sigma} e^{-2s\phi^*} \xi^* \sum_{i=1}^{n} \left( \sum_{j=1}^{n} \left| \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \nu_j - \frac{\partial \psi_i}{\partial v} \nu_j \right| \right)^2 d\sigma dt \). To do this, take

\[
\rho(t) = \frac{1}{2} - \frac{1}{m} \lambda e^{-s\phi^*(t)}(\xi^*) \frac{1}{2} - \frac{1}{m}(t)
\]

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and define $\varphi^* = \rho(t)\varphi$, then $\varphi^*$ will be the solution of

$$
\begin{cases}
-\varphi^*_{tt} - \Delta \varphi^* + a \cdot \nabla \varphi^* = -\rho \nabla \cdot \psi - \rho_t \varphi, & (x, t) \in Q, \\
\varphi^* = 0, & (x, t) \in \Sigma, \\
\varphi^*(x, T) = 0, & x \in \Omega.
\end{cases}
$$

(3.19)

It is easy to check that $-\rho \nabla \cdot \psi - \rho_t \varphi \in L^2(0, T; H^1(\Omega))$, then $\varphi^* \in L^2(0, T; H^3(\Omega)) \cap H^1(0, T; H^1(\Omega))$, and

$$
\|\varphi^*\|_{L^2(0,T,H^3(\Omega))}^2 + \|\varphi^*\|_{L^2(0,T;H^1(\Omega))}^2 \leq C \left( \|\rho_t \varphi\|_{L^2(0,T;H^1(\Omega))}^2 + \|\rho \nabla \cdot \psi\|_{L^2(0,T;H^1(\Omega))}^2 \right). 
$$

(3.20)

Moreover,

$$
\|\rho_t \varphi\|_{L^2(0,T;H^1(\Omega))}^2 \leq C \int_Q \rho_t^2 \sum_{i=1}^n \left| \frac{\partial \varphi}{\partial x_i} \right|^2 \, dx \, dt \leq CT s^2 \cdot \frac{1}{\lambda} \int_Q e^{-2s\varphi^*} \sum_{i=1}^n \left| \frac{\partial \varphi}{\partial x_i} \right|^2 \, dx \, dt 
$$

$$
\leq C s^2 \lambda \int_Q e^{-2s\varphi^*} \sum_{i=1}^n \left| \frac{\partial \varphi}{\partial x_i} \right|^2 \, dx \, dt \leq CI(\nabla \varphi) 
$$

(3.21)

for $s \geq CT_m$. By (3.20) and (3.21), we get

$$
\|\varphi^*\|_{L^2(0,T,H^3(\Omega))}^2 = s^{1-\frac{2}{m}} \lambda^2 \int_0^T e^{-2s\varphi^*} (\rho_{\xi^*})^{2-\frac{2}{m}} \|\varphi\|_{H^3(\Omega)}^2 \, dt 
$$

$$
\leq C \left( I(\nabla \varphi) + \|\rho \nabla \cdot \psi\|_{L^2(0,T;H^1(\Omega))}^2 \right). 
$$

(3.22)

From this, using the integration by parts, we conclude that

$$
\sum_{i=1}^n \left| \frac{\partial \varphi}{\partial x_i} \right|^2 \, dx \, dt \leq C \left( I(\nabla \varphi) + \|\rho \nabla \cdot \psi\|_{L^2(0,T;H^1(\Omega))}^2 \right). 
$$

(3.23)

Combining (3.22) with (3.23) yields

$$
\sum_{i=1}^n \left| \frac{\partial \varphi}{\partial x_i} \right|^2 \, dx \, dt \leq C \left( I(\nabla \varphi) + \|\rho \nabla \cdot \psi\|_{L^2(0,T;H^1(\Omega))}^2 \right). 
$$

(3.24)

In addition, by the trace theorem,

$$
\int_\Omega \sum_{i,j=1}^n \left| \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \right|^2 \, dx \, dt \leq C s \lambda \int_0^T e^{-2s\varphi^*} \|\nabla \varphi\|_{H^2(\Omega)}^2 \, dt. 
$$

(3.25)

By (3.17), (3.18), (3.21) and (3.25), noting that $m > 3$, we deduce that there exists a constant $C > 0$ such that, for any $s \geq \max\{CT_m, CT_m\}$ and $\lambda \geq C \left( \|a\|_{L^\infty(Q)} + \|\nabla a\|_{L^\infty(Q)} \right)$, it holds that

$$
I(\nabla \varphi) \leq C \left[ \sum_{i=1}^n \left| \frac{\partial^2 \varphi}{\partial x_i^2} \right|^2 \, dx \, dt + s^2 \lambda^2 \int_Q e^{-2s\varphi^*} \sum_{i,j=1}^n \left| \frac{\partial \varphi}{\partial x_i} \right|^2 \, dx \, dt \right].
$$

(3.26)
**Step 2.** Let us consider the equation satisfied by \( \psi \). Write \( \mathbf{b} = (b_1, b_2, \ldots, b_n)^T \) and

\[
J(\psi) := s^6 \lambda^8 \int_Q \theta^2 \xi^6 |\psi|^2 dxdt + s^4 \lambda^4 \int_Q \theta^2 \xi^4 |\nabla \psi|^2 dxdt + s^2 \lambda^2 \int_Q \theta^2 \xi^2 |\Delta \psi|^2 dxdt.
\]

By (3.13), applying the classical Carleman estimate of the parabolic operator with the right-hand side in \( L^2(Q) \) for \( \psi \), we see that

\[
J(\psi) \leq C \left( s^6 \lambda^8 \int_0^T \int_{\omega_0} \theta^2 \xi^6 |\psi|^2 dxdt + s^4 \lambda^4 \|B\|_{L^\infty(Q)}^2 \int_Q \theta^2 \xi^2 |\nabla \psi|^2 dxdt \right). \tag{3.27}
\]

Multiplying (3.26) by \( 1 + \|B\|_{L^\infty(Q)}^2 \), and adding (3.27) to it, we conclude that, for any \( s \geq \max \{CT^{2m}, CT^m\} \) and \( \lambda \geq C \left( \|\mathbf{a}\|_{L^\infty(Q)} + \|\nabla \mathbf{a}\|_{L^\infty(Q)} + \|B\|_{L^\infty(Q)} \right) \), it holds that

\[
I(\nabla \varphi) + J(\psi) \leq C \left[ s^3 \lambda^4 \int_0^T \int_{\omega_0} \theta^2 \xi^3 |\nabla \varphi|^2 dxdt + s^2 \lambda^2 \int_0^T e^{-2s\phi^*} \xi^* \|\nabla \psi\|_{H^1(\Omega)}^2 dt 
+ s^6 \lambda^8 \int_0^T \int_{\omega_0} \theta^2 \xi^6 |\psi|^2 dxdt \right]. \tag{3.28}
\]

We next claim that the second term in the right hand side of (3.28) can be absorbed by the left hand side. To this end, we set \( \zeta(t) = s^2 \lambda e^{-s\phi^*} (\xi^*)^2 \) and \( \Psi = \zeta(t) \psi \). Then \( \Psi \) satisfies

\[
\begin{cases}
\Psi_t + \Delta \Psi + b \nabla \cdot \Psi = \xi B \nabla \varphi + \zeta \psi, & (x, t) \in Q, \\
\Psi = 0, & (x, t) \in \Sigma, \\
\Psi(x, T) = 0, & x \in \Omega.
\end{cases}
\]

By a simple calculation, we have \( |\zeta(t)| \leq T s^2 \lambda e^{-s\phi^*} (\xi^*)^2 \) and \( \zeta B \nabla \varphi + \zeta \psi \in L^2(Q) \), then \( \Psi \in L^2(0, T; H^2(\Omega)) \) and

\[
\|\Psi\|^2_{L^2(0, T; H^2(\Omega))} = s^2 \lambda^2 \int_0^T e^{-2s\phi^*} \xi^* \|\psi\|^2_{H^2(\Omega)} dt 
\leq C \left( s^2 \lambda^2 \|B\|_{L^\infty(Q)} \int_Q e^{-2s\phi^*} \xi^* |\nabla \varphi|^2 dxdt + T^2 s^3 \lambda^2 \int_Q e^{-2s\phi^*} (\xi^*)^3 \|\psi\|^2 dxdt \right) 
\leq C \left( I(\nabla \varphi) + J(\psi) \right) \tag{3.29}
\]

for any \( \lambda \geq C \|B\|_{L^\infty(Q)} \). By (3.28) and (3.29), we deduce that

\[
I(\nabla \varphi) + J(\psi) + \|\Psi\|^2_{L^2(0, T; H^2(\Omega))} 
\leq C \left( s^3 \lambda^4 \int_0^T \int_{\omega_0} \theta^2 \xi^3 |\nabla \varphi|^2 dxdt + s^6 \lambda^8 \int_0^T \int_{\omega_0} \theta^2 \xi^6 |\psi|^2 dxdt \right) \tag{3.30}
\]

for any \( s \geq \max \{CT^{2m}, CT^m\} \) and \( \lambda \geq C \left( \|\mathbf{a}\|_{L^\infty(Q)} + \|\nabla \mathbf{a}\|_{L^\infty(Q)} + \|B\|_{L^\infty(Q)} \right) \).

We proceed to show that the first term in the right hand side of (3.30) can be also eliminated. Notice that

\[
\psi_t + \Delta \psi + b \nabla \cdot \psi = B \nabla \varphi \text{ in } \omega_0 \times (0, T).
\]
Then, we have

$$s^3 \lambda^4 \int_0^T \int_\omega \rho \theta^2 \xi^3 |B \nabla \varphi|^2 dx dt = s^3 \lambda^4 \int_0^T \int_\omega \rho \theta^2 \xi^3 B \nabla \varphi \left( \psi_t + \Delta \psi + b \nabla \cdot \psi \right) dx dt$$

$$= -s^3 \lambda^4 \int_0^T \int_\omega \rho \theta^2 \xi^3 \psi_t B \nabla \varphi \cdot \psi dx dt + s^3 \lambda^4 \int_0^T \int_\omega \Delta(\rho \theta^2 \xi^3) B \nabla \varphi \cdot \psi dx dt$$

$$+ 2s^3 \lambda^4 \int_0^T \int_\omega \nabla(\rho \theta^2 \xi^3) \nabla(B \nabla \varphi) \cdot \psi dx dt - s^3 \lambda^4 \int_0^T \int_\omega \nabla(\rho \theta^2 \xi^3) B \nabla \varphi \cdot b \cdot \psi dx dt$$

$$+ s^3 \lambda^4 \int_0^T \int_\omega \rho \theta^2 \xi^3 \psi \left( \Delta(B \nabla \varphi) - (B \nabla \varphi)_t - \nabla(B \nabla \varphi \cdot b) \right) dx dt. \quad (3.31)$$

An easy verification shows that

$$(\theta^2 \xi^3)_t \leq C \lambda \theta^2 \xi^4 + \frac{1}{\lambda^2}, \quad \nabla(\theta^2 \xi^3) \leq C s \lambda \theta^2 \xi^4 \quad \text{and} \quad \Delta(\theta^2 \xi^3) \leq C s^3 \lambda^2 \theta^2 \xi^5.$$

Applying these, and Young’s inequality, we get

$$-s^3 \lambda^4 \int_0^T \int_\omega \rho \theta^2 \xi^3 \psi_t B \nabla \varphi \cdot \psi dx dt \leq C \lambda \theta^2 \xi^4 + \frac{1}{\lambda^2} \int_\omega B \nabla \varphi \cdot \psi dx dt$$

$$\leq C \lambda \theta^2 \xi^4 + \frac{1}{\lambda^2} \int_\omega B \nabla \varphi \cdot \psi dx dt$$

$$\leq C \lambda \theta^2 \xi^4 + \frac{1}{\lambda^2} \int_\omega B \nabla \varphi \cdot \psi dx dt$$

$$\leq C \lambda \theta^2 \xi^4 + \frac{1}{\lambda^2} \int_\omega B \nabla \varphi \cdot \psi dx dt$$

$$\leq C \lambda \theta^2 \xi^4 + \frac{1}{\lambda^2} \int_\omega B \nabla \varphi \cdot \psi dx dt$$

$$\leq C \lambda \theta^2 \xi^4 + \frac{1}{\lambda^2} \int_\omega B \nabla \varphi \cdot \psi dx dt$$

and

$$\int_0^T \int_\omega \nabla(\rho \theta^2 \xi^3) \nabla(B \nabla \varphi) \cdot \psi dx dt \leq C \lambda \theta^2 \xi^4 \int_\omega |\nabla B \cdot \nabla \varphi + B \nabla(\nabla \varphi)| \psi dx dt$$

$$\leq C \lambda \theta^2 \xi^4 \int_\omega |\nabla B \cdot \nabla \varphi + B \nabla(\nabla \varphi)| \psi dx dt$$

$$\leq C \lambda \theta^2 \xi^4 \int_\omega |\nabla B \cdot \nabla \varphi + B \nabla(\nabla \varphi)| \psi dx dt$$

$$\leq C \lambda \theta^2 \xi^4 \int_\omega |\nabla B \cdot \nabla \varphi + B \nabla(\nabla \varphi)| \psi dx dt$$

By (3.13), we have

$$-(\nabla \varphi)_t = \Delta(\nabla \varphi) - \nabla a \cdot \nabla \varphi - a \cdot \nabla(\nabla \varphi) - \nabla(\nabla \cdot \psi). \quad (3.36)$$

Then, we have

$$\Delta(B \nabla \varphi) - (B \nabla \varphi)_t - \nabla(B \nabla \varphi \cdot b)$$

$$= (\Delta B - B_t - \nabla B \cdot b - B \nabla b - B \nabla a) \nabla \varphi$$
+ (2\nabla B - Ba - Bb) \cdot \nabla (\nabla \varphi) + 2B \Delta (\nabla \varphi) - B \nabla (\nabla \cdot \psi). 

(3.37)

From this, we get

\begin{align*}
s^3 \lambda^4 \int_0^T \int_\omega & \vartheta^2 \xi^3 \psi \left( \Delta (B \nabla \varphi) - (B \nabla \varphi)_t - \nabla (B \nabla \varphi \cdot b) \right) dx dt \\
& = s^3 \lambda^4 \int_0^T \int_\omega \vartheta^2 \xi^3 \psi (\Delta B - B_t - \nabla B \cdot b - B \nabla b - B \nabla a) \nabla \varphi dx dt \\
& + s^3 \lambda^4 \int_0^T \int_\omega \vartheta^2 \xi^3 \psi (2 \nabla B - Ba - Bb) \cdot \nabla (\nabla \varphi) dx dt \\
& + 2s^3 \lambda^4 \int_0^T \int_\omega \vartheta^2 \xi^3 \psi B \Delta (\nabla \varphi) dx dt - s^3 \lambda^4 \int_0^T \int_\omega \vartheta^2 \xi^3 \psi B \nabla (\nabla \cdot \psi) dx dt \\
& := I_1 + I_2 + I_3 + I_4. 
\end{align*}

(3.38)

In what follows, let $C_0$ denote a constant dependent on $\|B\|_{L^\infty(Q)}$, $\|B_t\|_{L^\infty(Q)}$, $\|\nabla B\|_{L^\infty(Q)}$, $\|\Delta B\|_{L^\infty(Q)}$, $\|b\|_{L^\infty(Q)}$, $\|\nabla b\|_{L^\infty(Q)}$, $\|a\|_{L^\infty(Q)}$ and $\|\nabla a\|_{L^\infty(Q)}$, which may vary from line to line. By Young’s inequality, we obtain

$$I_1 \leq \varepsilon_0 C_0 I(\nabla \varphi) + C(\varepsilon_0) s^3 \lambda^4 \int_0^T \int_\omega \vartheta^2 \xi^3 |\psi|^2 dx dt,$$

(3.39)

$$I_2 \leq \varepsilon_0 C_0 I(\nabla \varphi) + C(\varepsilon_0) s^5 \lambda^6 \int_0^T \int_\omega \vartheta^2 \xi^5 |\psi|^2 dx dt,$$

(3.40)

$$I_3 = -2s^3 \lambda^4 \int_0^T \int_\omega \left( \nabla (\vartheta^2 \xi^3) \cdot \psi B \Delta \varphi + \vartheta^2 \xi^3 \nabla \cdot \psi B \Delta \varphi + \vartheta^2 \xi^3 \psi \cdot \nabla B \Delta \varphi \right) dx dt,$$

$$\leq C(\varepsilon_0) s^{7} \lambda^{8} \int_0^T \int_\omega \vartheta^2 \xi^7 |\psi|^2 dx dt + C(\varepsilon_0) s^5 \lambda^6 \int_0^T \int_\omega \vartheta^2 \xi^5 |\nabla \psi|^2 dx dt \\
+ \varepsilon_0 C_0 I(\nabla \varphi),$$

(3.41)

$$I_4 \leq \varepsilon_0 C_0 s^2 \lambda^2 \int_0^T \int_\omega e^{-2s\phi^+} \xi^* |\nabla (\nabla \cdot \psi)|^2 dx dt + C(\varepsilon_0) s^5 \lambda^6 \int_0^T \int_\omega \vartheta^2 \xi^6 \frac{\xi^6}{\xi^6} |\psi|^2 dx dt,$$

$$\leq \varepsilon_0 C_0 \|\Psi\|^2_{L^2(0,T;H^2(\Omega))} + C(\varepsilon_0) T^{2m} s^{5} \lambda^6 \int_0^T \int_\omega \vartheta^2 e^{-4s\phi^+2s\phi^+} \xi^7 |\psi|^2 dx dt.$$

(3.42)

By (3.35) and (3.42), we conclude that

$$s^3 \lambda^4 \int_0^T \int_\omega \vartheta^2 \xi^3 \psi \left( \Delta (B \nabla \varphi) - (B \nabla \varphi)_t - \nabla (B \nabla \varphi \cdot b) \right) dx dt \\
\leq \varepsilon_0 C_0 \left( I(\nabla \varphi) + \|\Psi\|^2_{L^2(0,T;H^2(\Omega))} \right) + C(\varepsilon_0) T^{2m} s^{7} \lambda^{8} \int_0^T \int_\omega \vartheta^2 e^{-4s\phi^+2s\phi^+} \xi^7 |\psi|^2 dx dt \\
+ C(\varepsilon_0) s^{5} \lambda^6 \int_0^T \int_\omega \vartheta^2 \xi^5 |\nabla \psi|^2 dx dt.$$

(3.43)

Moreover, we have

$$s^5 \lambda^6 \int_0^T \int_\omega \vartheta^2 \xi^5 |\nabla \psi|^2 dx dt = s^5 \lambda^6 \int_0^T \int_\omega \left( \nabla (\vartheta^2 \xi^5) \nabla \psi + \vartheta^2 \xi^5 \nabla (\nabla \psi) \right) dx dt$$
\[ \xi \text{ completes the proof.} \]

By Cauchy inequality, we have
\[ \varepsilon \leq C(\varepsilon) + C(\varepsilon) s^8 \lambda^8 \int_0^T \int_\omega \theta^2 \xi^8 |\psi|^2 dxdt. \quad (3.44) \]

Combining (3.43) with (3.44) yields
\[ s^3 \lambda^4 \int_0^T \int_\omega \theta^2 \xi^3 |\nabla \varphi|^2 dxdt \leq C s^3 \lambda^4 \int_0^T \int_\omega \theta^2 \xi^3 |B \nabla \varphi|^2 dxdt \]
\[ \leq C s^3 \lambda^4 \int_0^T \int_\omega \theta^2 \xi^3 |B \nabla \varphi|^2 dxdt \leq C \varepsilon \left( I(\nabla \varphi) + J(\psi) + \|\Psi\|^2_{L^2(0,T;H^2(\Omega))} \right) \]
\[ + C(\varepsilon)(1 + T^{2m}) s^8 \lambda^8 \int_0^T \int_\omega \theta^2 e^{-4s \phi + 2s \phi^*} \xi^8 |\psi|^2 dxdt. \quad (3.45) \]

Note that \( B \) has a positive lower bound, by (3.31)-(3.35) and (3.45), we obtain
\[ s^3 \lambda^4 \int_0^T \int_\omega \theta^2 \xi^3 |\nabla \varphi|^2 dxdt \leq C s^3 \lambda^4 \int_0^T \int_\omega \theta^2 \xi^3 |B \nabla \varphi|^2 dxdt \]
\[ \leq C s^3 \lambda^4 \int_0^T \int_\omega \theta^2 \xi^3 |B \nabla \varphi|^2 dxdt \leq C \varepsilon \left( I(\nabla \varphi) + J(\psi) + \|\Psi\|^2_{L^2(0,T;H^2(\Omega))} \right) \]
\[ + C(\varepsilon)(1 + T^{2m}) s^8 \lambda^8 \int_0^T \int_\omega \theta^2 e^{-4s \phi + 2s \phi^*} \xi^8 |\psi|^2 dxdt. \quad (3.46) \]

Substituting (3.46) into (3.30), and choosing \( \varepsilon \) small enough, we conclude that
\[ I(\nabla \varphi) + J(\psi) + \|\Psi\|^2_{L^2(0,T;H^2(\Omega))} \leq C(1 + T^{2m}) s^8 \lambda^8 \int_0^T \int_\omega \theta^2 e^{-4s \phi + 2s \phi^*} \xi^8 |\psi|^2 dxdt, \quad (3.47) \]
which completes the proof. \( \square \)

By the classical fact, the statement of Theorem 3.1 will be obtained once we prove the lemma below, which can be obtained by Theorem 3.2.

**Lemma 3.2** Assume that the condition (3.3) holds. Then there exists a constant \( C = C(\Omega, \omega, T) > 0 \) independent of \( (\varphi_0, \psi_0) \) such that
\[ \| \varphi \|_{L^2(\Omega)} + \| \psi \|_{L^2(\Omega)} \leq C \| \theta^2 e^{s \phi^*} \xi^4 \psi \|_{L^2(\omega \times (0,T))}, \quad \forall (\varphi_0, \psi_0) \in L^2(\Omega) \times L^2(\Omega). \quad (3.48) \]

**Proof.** Multiplying (3.13) by \( \varphi \) and \( \psi \) respectively, integrating over \( \Omega \), and using integration by parts, we have
\[ \frac{1}{2} \frac{d}{dt} \int_\Omega (\varphi^2 + \psi^2) dx - \int_\Omega (|\nabla \varphi|^2 + |\nabla \psi|^2) dx = \int_\Omega (B \nabla \varphi \cdot \psi + \nabla \cdot \psi \varphi - b \nabla \cdot \psi \psi + a \cdot \nabla \varphi \varphi) dx. \]

By Cauchy inequality, we have
\[ \frac{1}{2} \frac{d}{dt} \int_\Omega (\varphi^2 + \psi^2) dx - \int_\Omega (|\nabla \varphi|^2 + |\nabla \psi|^2) dx \geq -C \varepsilon(1 + |B|_{L^\infty(\Omega)} + |b|_{L^\infty(\Omega)} + |a|_{L^\infty(\Omega)}) \int_\Omega (\varphi^2 + \psi^2) dx - \varepsilon \int_\Omega (|\nabla \varphi|^2 + |\nabla \psi|^2) dx. \]

Taking \( \varepsilon \) small enough, we obtain
\[ \frac{1}{2} \frac{d}{dt} \int_\Omega (\varphi^2 + \psi^2) dx \geq -C(1 + |B|_{L^\infty(\Omega)} + |b|_{L^\infty(\Omega)} + |a|_{L^\infty(\Omega)}) \int_\Omega (\varphi^2 + \psi^2) dx. \]
By Gronwall’s inequality, we get
\[
\int_{\Omega} (\varphi^2(x,0) + \psi^2(x,0)) \, dx \leq C \int_{\Omega} (\varphi^2(x,t) + \psi^2(x,t)) \, dx.
\]
Integrating this inequality on \((T/4, 3T/4)\), we have
\[
\|\varphi|_{t=0}\|_{L^2(\Omega)}^2 + \|\psi|_{t=0}\|_{L^2(\Omega)}^2 \leq C \int_{T/4}^{3T/4} \int_{\Omega} (\varphi^2 + \psi^2) \, dx \, dt.
\]  
(3.49)

Moreover, by Poincaré’s inequality and the definitions of \(\theta\) and \(\xi\), we have
\[
s^3 \lambda^4 \int_{Q} \theta^2 \xi^3 |\nabla \varphi|^2 \, dx \, dt + s^6 \lambda^8 \int_{Q} \theta^2 \xi^6 |\psi|^2 \, dx \, dt \geq Cs^3 \lambda^4 \int_{T/4}^{3T/4} \int_{\Omega} \varphi^2 \, dx \, dt + Cs^6 \lambda^8 \int_{T/4}^{3T/4} \int_{\Omega} \psi^2 \, dx \, dt.
\]  
(3.50)

By (3.49), (3.50) and Theorem 3.2, we deduce (3.48). This completes the proof of Lemma 3.2.

Remark 3.1 Note that it is a technical condition that \(B\) has a positive lower bound in Theorem 3.2, which plays a critical role in the proof of Theorem 3.2. Indeed, in order to establish the global Carleman estimate for (3.13), we need to derive a local estimate for \(\nabla \varphi\) (see the first term in the right hand side of (3.30)). However, this local estimate for \(\nabla \varphi\) is obtained by estimating \(B \nabla \varphi\) because \(B \nabla \varphi\) appears in the equation satisfied by \(\psi\). Therefore, we require that \(B\) has a positive lower bound, then (3.46) and (3.47) hold.

Remark 3.2 Notice that in the proof of Theorem 3.2, we require that the coefficients in system (3.13) satisfy \(a, b, \nabla a, \nabla b \in L^\infty(Q), B, B_t, \nabla B, \Delta B \in L^\infty(Q)\), by the relationship between \(a, b, B\) and \(\nabla, \overline{\omega}\) (see (4.1)) and the embedding theorem, we need to establish the regularity of \((\overline{u}, \overline{v})\), which actually lies in \(C([0,T]; H^4(\Omega))\) with smallness on \(H^3\) norm of the initial data.

Next, we establish the null controllability of system (3.2) with a control function in \(L^r(Q)\), where \(r > n + 2\).

Proposition 3.2 Assume that the condition (3.3) holds. Let \(r > n + 2\), \(y_0, z_0 \in W^{2-\frac{2}{r}, r}(\Omega) \cap H^1_0(\Omega)\). Then one can find a control \(h \in L^r(Q)\) supported in \(\omega \times [0,T]\) such that the solution \((y, z) \in V^r \times V^r\) of system (3.2) satisfies
\[
y(x,T) = z(x,T) = 0 \text{ a.e. in } \Omega.
\]  
(3.51)

Moreover,
\[
\|h\|_{L^r(Q)} \leq C e^{CM_1} M_2^{k_1} (\|y_0\|_{L^2(\Omega)} + \|z_0\|_{L^2(\Omega)}).
\]  
(3.52)

We give the following lemma, which will be needed in the proof of Proposition 3.2. Let \(m_0, \gamma \geq 1\). Consider the following Banach space
\[
X^{m_0, \gamma}(Q) := L^\infty(0,T; L^{m_0}(\Omega)) \cap L^\gamma(0,T; W^{1,\gamma}(\Omega)),
\]  
(3.53)

equipped with the norm \(\|v\|_{X^{m_0, \gamma}(Q)} = \text{esssup}_{0 < t < T} \|v(\cdot, t)\|_{L^m_0(\Omega)} + \|Dv\|_{L^\gamma(Q)}\).
Lemma 3.3 ([15], Proposition 3.2] There exists a constant $C > 0$ depending only upon $n, \gamma$ and $m_0$ such that for every $v \in X^{m_0,\gamma}(Q)$, it holds that
\[
\|v\|_{L^q(Q)} \leq C \left(1 + \frac{T}{|\Omega|} \frac{1}{n^{(\gamma-m_0)+m_0\gamma}}\right)^{\frac{1}{q}} \|v\|_{X^{m_0,\gamma}(Q)},
\]
where $q = \gamma \frac{n + m_0}{n}$.

Proof of Proposition 3.2. For any given $\epsilon > 0$, we consider the following optimal control problem:
\[
(P_\epsilon): \text{Min}\left\{ \frac{1}{2} \int_Q e^{4s\phi - 2s\phi^*} \xi - 8 |h|^2 dxdt + \frac{1}{2\epsilon} \int_\Omega y^2(x,T)dx + \frac{1}{2\epsilon} \int_\Omega \|z(x,T)|^2 dx, \text{ subject to } (3.2) \right\}.
\]
By the standard variational method, we know that for any $\epsilon > 0$, the problem $(P_\epsilon)$ has a unique solution $(y_\epsilon, z_\epsilon, h_\epsilon)$ and
\[
h_\epsilon = \chi e^{-4s\phi+2s\phi^*} \xi \psi_\epsilon,
\]
where $(\varphi_\epsilon, \psi_\epsilon)$ satisfies
\[
\begin{align*}
-\varphi_\epsilon - \Delta \varphi_\epsilon + a \cdot \nabla \varphi_\epsilon &= -\nabla \cdot \psi_\epsilon, & (x, t) \in Q, \\
\psi_\epsilon + \Delta \psi_\epsilon + b \nabla \cdot \psi_\epsilon &= B \nabla \varphi_\epsilon, & (x, t) \in Q, \\
\varphi_\epsilon &= 0, & (x, t) \in \Sigma, \\
(\varphi_\epsilon, \psi_\epsilon)(x, T) &= -\frac{1}{\epsilon}(y_\epsilon(x, T), z_\epsilon(x, T)), & x \in \Omega,
\end{align*}
\]
where $(y_\epsilon, z_\epsilon)$ is the solution of (3.2) associated to $(y_0, z_0)$ and $h^4$.
Multiplying the first (resp. second) equation of (3.56) by $y_\epsilon$ (resp. $z_\epsilon$) and integrating it on $Q$, by the boundary conditions of (3.2) and (3.56), we have
\[
\int_\Omega y_\epsilon(x, T) \varphi_\epsilon(x, T) dx - \int_\Omega y_\epsilon(x, 0) \varphi_\epsilon(x, 0) dx + \int_\Omega \psi_\epsilon(x, T) z_\epsilon(x, T) dx - \int_\Omega \psi_\epsilon(x, 0) z_\epsilon(x, 0) dx = \int_\Omega \chi e^{4s\phi+2s\phi^*} \xi \psi_\epsilon dx dt.
\]
From (3.55), (3.56) and Lemma 3.2 we obtain
\[
\frac{1}{\epsilon} \int_\Omega y_\epsilon^2(x, T) dx + \frac{1}{\epsilon} \int_\Omega z_\epsilon^2(x, T) dx + \int_\Omega \chi e^{-4s\phi+2s\phi^*} \xi^8 \psi_\epsilon^2 dx dt
\]
\[
= - \int_\Omega y_0(x) \varphi_\epsilon(x, 0) dx - \int_\Omega z_0(x) \psi_\epsilon(x, 0) dx
\]
\[
\leq C \left(\|y_0\|_{L^2(\Omega)} + \|z_0\|_{L^2(\Omega)}\right) \left(\psi_\epsilon e^{-2s\phi+2s\phi^*} \xi^4\right)_{L^2(\omega \times (0,T))},
\]
which implies that
\[
\frac{1}{\epsilon} \int_\Omega y_\epsilon^2(x, T) dx + \frac{1}{\epsilon} \int_\Omega z_\epsilon^2(x, T) dx + \int_\Omega \chi e^{-4s\phi+2s\phi^*} \xi^8 \psi_\epsilon^2 dx dt
\]
\[
\leq C \left(\|y_0\|_{L^2(\Omega)}^2 + \|z_0\|_{L^2(\Omega)}^2\right),
\]
\[\text{(3.57)}\]
which implies that \(\{h_\epsilon\}\) is a family of “approximate” control, because \(y_\epsilon(x, T) \to 0\) in \(L^2(\Omega)\), \(z_\epsilon(x, T) \to 0\) in \(L^2(\Omega)\) as \(\epsilon \to 0\).

Next, we prove that \(h_\epsilon \in L^r(Q)\) with \(r > n + 2\). Let \(\tau > 0\) and \(\{\tau_k\}_{k \in \mathbb{N}}\) be an increasing sequence such that \(0 < \tau_k < \tau < \frac{\tau}{2}\). Set

\[
\Phi^k_\epsilon = e^{-(s+\tau_k)\phi^*(t)}(\xi^*)^8(t)\psi_\epsilon, \quad \Psi^k_\epsilon = e^{-(s+\tau_k)\phi^*(t)}(\xi^*)^8(t)\varphi_\epsilon.
\]

On one hand, it is easy to prove that \(\Phi^k_\epsilon\) satisfies

\[
\begin{cases}
\Phi^k_\epsilon + \Delta \Phi^k_\epsilon + b \nabla \cdot \Phi^k_\epsilon \\
= e^{-(s+\tau_k)\phi^*}(\xi^*)^8 \psi_\epsilon + e^{-(s+\tau_k)\phi^*}(\xi^*)^8 B \nabla \varphi_\epsilon := \Phi^k_\epsilon, \quad (x, t) \in Q, \\
\Phi^k_\epsilon = 0, \quad (x, t) \in \Sigma; \\
\Phi^k_\epsilon(x, 0) = \Phi^k_\epsilon(x, T) = 0, \quad x \in \Omega.
\end{cases}
\]

Then

\[
\|g^1_\epsilon\|^2 \leq 2 \int_Q \left| e^{-(s+\tau_1)\phi^*}(\xi^*)^8 \right|^2 \psi^2 \, dx \, dt + 2 \int_Q \left| e^{-(s+\tau_1)\phi^*}(\xi^*)^8 \right|^2 B^2 |\nabla \varphi^*_\epsilon|^2 \, dx \, dt
= I_1 + I_2.
\]

Since

\[
\left| e^{-(s+\tau_1)\phi^*}(\xi^*)^8 \right| = |-(s + \tau_1)\phi^* e^{-(s+\tau_1)\phi^*}(\xi^*)^8 + 8e^{-(s+\tau_1)\phi^*}(\xi^*)^8| \leq (s + \tau_1)C(T)(\xi^*)^{9 + \frac{2}{m}} e^{-(s+\tau_1)\phi^*} + C(T)e^{-(s+\tau_1)\phi^*}(\xi^*)^{8 + \frac{1}{m}},
\]

we obtain

\[
I_1 \leq C \int_Q s^2(\xi^*)^{2(0 + \frac{1}{m})} e^{-2(s+\tau_1)\phi^*} \psi^2 \, dx \, dt + C \int_Q e^{-2(s+\tau_1)\phi^*}(\xi^*)^{2(8 + \frac{1}{m})} \psi^2 \, dx \, dt
\leq C \int_Q e^{-2s\phi^*} \xi^6 \psi^2 \, dx \, dt,
\]

since \((\xi^*)^{12 + \frac{2}{m}} e^{-2\tau_1\phi^*} \leq 1\) and \((\xi^*)^{10 + \frac{2}{m}} e^{-2\tau_1\phi^*} \leq 1\).

Similarly, we have

\[
I_2 \leq C(\|B\|_{L^\infty(Q)}) \int_Q e^{-2s\phi^*} \xi^3 |\nabla \varphi^*_\epsilon|^2 \, dx \, dt.
\]

Therefore, by Theorem 3.3 and (3.54), it follows that

\[
\|g^1_\epsilon\|^2 \leq C \int_0^T \int_\omega e^{-4s\phi + 2s\phi^*} \xi^8 \psi^2 \, dx \, dt \leq C \left(\|y_0\|^2_{L^2(\Omega)} + \|z_0\|^2_{L^2(\Omega)}\right).
\]

By Proposition 3.1, we have \(\Phi^k_\epsilon \in V^2\). Moreover,

\[
\|\Phi^k_\epsilon\|^2 \leq e^{CM_1} M^k_2 \|g^1_\epsilon\|^2 \leq e^{CM_1} M^k_2 \left(\|y_0\|^2_{L^2(\Omega)} + \|z_0\|^2_{L^2(\Omega)}\right).
\]

By the embedding theorem, \(V^2 = W^{2,1}_r(Q) \hookrightarrow L^{s_1}(Q)\) for \(s_1 = \left\{ \begin{array}{ll} \frac{2(n+2)}{n-2}, & n > 2, \\ \text{any constant } \kappa > 1, & n \leq 2. \end{array} \right. \)

Then,

\[
\|\Phi^k_\epsilon\|^2 \leq e^{CM_1} M^k_2 \left(\|y_0\|^2_{L^2(\Omega)} + \|z_0\|^2_{L^2(\Omega)}\right).
\]
On the other hand, it is easy to check that $\Psi^k_\varepsilon$ satisfies

$$
\begin{aligned}
-\Psi^k_{\varepsilon,t} - \Delta \Psi^k_{\varepsilon} + a \cdot \nabla \Psi^k_{\varepsilon} = - \left[ e^{-(s+\tau_2)\phi^*}(\xi^*)^8 \right] \varphi_{\varepsilon} - \left[ e^{-(s+\tau_2)\phi^*}(\xi^*)^8 \right] \nabla \cdot \psi_{\varepsilon} := f^k_{\varepsilon}, \\
(\varepsilon, t) \in Q,
\end{aligned}
$$

(3.61)

Moreover,

$$
\Psi^k_{\varepsilon}(x,0) = \Psi^k_{\varepsilon}(x,T) = 0,
$$

Next, we prove $f^1_{\varepsilon} \in L^2(Q)$. Similarly, by (3.57), Theorem 3.2 (or (3.47)) and Poincaré’s inequality, we deduce

$$
\int_Q |f^1_{\varepsilon}|^2 dxdt \leq C s^3 \lambda^4 \int_Q e^{-2s\phi} \xi^3 |\nabla \varphi_{\varepsilon}|^2 dxdt + s^4 \lambda^6 \int_Q e^{-2s\phi} \xi^4 |\nabla \cdot \psi_{\varepsilon}|^2 dxdt
$$

$$
\leq C s^8 \lambda^8 \int_0^T \int_\omega e^{-4s\phi + 8s\phi \varepsilon^2} |\psi_{\varepsilon}|^2 dxdt \leq C \left( \|y_0\|_{L^2(\Omega)}^2 + \|z_0\|_{L^2(\Omega)}^2 \right).
$$

By Proposition 3.1, we know $\Psi^1_{\varepsilon} \in V^2$. Moreover,

$$
\|\Psi^1_{\varepsilon}\|_{V^2}^2 \leq e^{CM_1 M_2 k_1} \|f^1_{\varepsilon}\|_{L^2(Q)}^2 \leq e^{CM_1 M_2 k_1} \left( \|y_0\|_{L^2(\Omega)}^2 + \|z_0\|_{L^2(\Omega)}^2 \right).
$$

By the embedding theorem, $V^2 \hookrightarrow L^{s_1}(Q)$, then

$$
\|\Psi^1_{\varepsilon}\|_{L^{s_1}(Q)}^2 \leq e^{CM_1 M_2 k_1} \left( \|y_0\|_{L^2(\Omega)}^2 + \|z_0\|_{L^2(\Omega)}^2 \right).
$$

In what follows, we give the estimates of $\Phi^2_{\varepsilon}$ and $f^2_{\varepsilon}$, respectively. By (3.58) and (3.59), we have

$$
\Phi^2_{\varepsilon} = \left[ e^{-(s+\tau_2)\phi^*}(\xi^*)^8 \right] \nabla \cdot \psi_{\varepsilon} = \left[ e^{-(s+\tau_2)\phi^*}(\xi^*)^8 \right] \nabla \cdot \Phi^1_{\varepsilon} + e^{(\tau_1 - \tau_2)\phi^*} \nabla \Phi^1_{\varepsilon}.
$$

(3.62)

Notice that, by (3.61) and Proposition 3.1, we have $\nabla \Psi^1_{\varepsilon} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega))$. Taking $m_0 = 2, \gamma = 2$ in (3.53), by Lemma 3.3 we deduce that $\nabla \Psi^1_{\varepsilon} \in L^{q_1}(Q)$, where $q_1 = \frac{2(n+2)}{n} > 2$, and

$$
\|\nabla \Psi^1_{\varepsilon}\|_{L^{q_1}(Q)}^2 \leq C \|\nabla \Psi^1_{\varepsilon}\|_{X^{2,2}(Q)}^2 \leq C e^{CM_1 M_2 k_1} \left( \|y_0\|_{L^2(\Omega)}^2 + \|z_0\|_{L^2(\Omega)}^2 \right).
$$

(3.63)

Moreover,

$$
\left[ e^{-(s+\tau_2)\phi^*}(\xi^*)^8 \right] e^{(\tau_1 - \tau_2)\phi^*} \xi^* \leq C e^{(\tau_1 - \tau_2)\phi^*} \xi^* \frac{1}{m} \text{ and } \Phi^1_{\varepsilon} \in L^{s_1}(Q),
$$

(3.64)

where we choose $s_1 = q_1 = \frac{2(n+2)}{n}$ when $n = 1$ or 2, and $s_1 = \frac{2(n+2)}{n-2} > q_1$ when $n = 3$. Therefore, By (3.62) and (3.64), we get that

$$
\|\Phi^2_{\varepsilon}\|_{L^{q_1}(Q)}^2 \leq C e^{CM_1 M_2 k_1} \left( \|y_0\|_{L^2(\Omega)}^2 + \|z_0\|_{L^2(\Omega)}^2 \right).
$$

(3.65)

Again, by (3.59) and Proposition 3.1, we see that $\Phi^2_{\varepsilon} \in V^{q_1}$. Moreover,

$$
\|\Phi^2_{\varepsilon}\|_{V^{q_1}}^2 \leq C e^{CM_1 M_2 k_1} \|\Phi^2_{\varepsilon}\|_{L^{q_1}(Q)}^2 \leq C e^{CM_1 M_2 k_1} \left( \|y_0\|_{L^2(\Omega)}^2 + \|z_0\|_{L^2(\Omega)}^2 \right).
$$

(3.66)
By the embedding theorem, $V^{q_1} \hookrightarrow L^{s_2}(Q)$, where
\[
s_2 = \begin{cases}
    \frac{q_1(n+2)}{n+2-2q_1}, & n + 2 - 2q_1 > 0, \\
    \text{any constant } \kappa > 1, & n + 2 - 2q_1 \leq 0.
\end{cases}
\]
Hence,
\[
\|\phi_\epsilon^2\|^2_{L^2(Q)} \leq C e^{C M_1 M_2 k_1} \left( \|y_0\|^2_{L^2(\Omega)} + \|z_0\|^2_{L^2(\Omega)} \right). 
\tag{3.66}
\]
In addition, by (3.58) and (3.61), we arrive at
\[
f_\epsilon^2 = -[e^{-(s+\tau_2)\phi^*}(\xi^*)^8]e^{(s+\tau_1)\phi^*(\xi^*)^{-8}}\psi_{\epsilon}^1 - e^{(\tau_1-\tau_2)\phi^*} \nabla \cdot \phi_\epsilon^1.
\]
Similar to (3.63), we can prove that $\nabla \cdot \phi_\epsilon^1 \in L^{q_1}(Q)$. Combining with $\psi_{\epsilon}^1 \in L^{s_1}(Q)$, we have $f_\epsilon^2 \in L^{q_1}(Q)$. Using Proposition 3.1 again, we deduce that $\psi_{\epsilon}^2 \in V^{q_1}$.

By the embedding theorem, it follows that
\[
\|\psi_{\epsilon}^2\|^2_{L^2(Q)} \leq C \|\psi_{\epsilon}^2\|^2_{V^{q_1}} \leq C e^{C M_1 M_2 k_1} \left( \|y_0\|^2_{L^2(\Omega)} + \|z_0\|^2_{L^2(\Omega)} \right). 
\tag{3.67}
\]
Similarly, since $\nabla \psi_{\epsilon}^2 \in L^{\infty}(0,T; L^2(\Omega)) \cap L^{q_1}(0,T; W^{1,q_1}(\Omega))$, by Lemma 3.3, we take $m_0 = 2$, $\gamma = q_1$, it follows that
\[
\nabla \psi_{\epsilon}^2 \in L^{q_2}(Q), \text{ where } q_2 = \frac{q_1(n+2)}{n}. 
\tag{3.68}
\]
Combining with $\phi_\epsilon^2 \in L^{s_2}(Q)$, we have $\Phi_{\epsilon}^3 \in L^{s_2}(Q)$ and
\[
\|\Phi_{\epsilon}^3\|^2_{L^2(\Omega)} \leq C e^{C M_1 M_2 k_1} \left( \|y_0\|^2_{L^2(\Omega)} + \|z_0\|^2_{L^2(\Omega)} \right),
\]
where we take $s_2 = q_2$, since $n + 2 - 2q_1 < 0$. By (3.59), it follows that
\[
\Phi_{\epsilon}^3 \in V^{q_2} \hookrightarrow L^{s_1}(Q), \text{ where } s_3 = \begin{cases}
    \frac{q_2(n+2)}{n+2-2q_2}, & n + 2 - 2q_2 > 0, \\
    \text{any constant } \kappa > 1, & n + 2 - 2q_2 \leq 0.
\end{cases}
\]

Moreover, similar to (3.63), we can show that $\nabla \cdot \Phi_{\epsilon}^2 \in L^{q_2}(Q)$. By (3.67), $\psi_{\epsilon}^2 \in L^{s_2}(Q)$. Then, $f_\epsilon^2 \in L^{q_2}(Q)$. By (3.61), $\Phi_{\epsilon}^3 \in V^{q_2} \hookrightarrow L^{s_1}(Q)$.

Repeating the above procedure, since $q_{N+1} - q_N = q_N \left( \frac{q_N + 2}{n} - 1 \right) = q_N \cdot \frac{2}{n} > 0$, there exists a $N^* \in \mathbb{N}$ such that
\[
\Phi_{\epsilon}^{N^*} \in L^{q_N^*}(Q), \quad \psi_{\epsilon}^{N^*} \in L^{q_N^*}(Q), \text{ where } q_{N^*} > n + 2.
\]

By (3.55),
\[
h_\epsilon = \chi_\omega e^{-4s\phi + 2s\phi^*} \xi^8 \psi_{\epsilon} = \chi_\omega e^{-4s\phi + 2s\phi^*} \xi^8 e^{(s+\tau_{N^*})\phi^*}(\xi^*)^{-8} \Phi_{\epsilon}^{N^*}.
\]
Since $\tau_{N^*} < \frac{s}{2}$, one has $e^{-4s\phi + 2s\phi^*} \xi^8 e^{(s+\tau_{N^*})\phi^*}(\xi^*)^{-8} \leq C$. Hence, $h_\epsilon \in L^r(\Omega)$, where $r > n + 2$. Moreover,
\[
\|h_{\epsilon}\|^r_{L^r(\Omega)} \leq C e^{C M_1 M_2 k_1} \left( \|y_0\|^2_{L^2(\Omega)} + \|z_0\|^2_{L^2(\Omega)} \right). 
\tag{3.69}
\]
Letting $\epsilon \to 0$, by (3.69) and (3.57), we conclude that there exists a control $h \in L^r(\Omega)$ such that the solution of (3.2) satisfies $y(x, T) = z(x, T) = 0$ in $\Omega$. Moreover,
\[
\|h\|^r_{L^r(\Omega)} \leq C e^{C M_1 M_2 k_1} \left( \|y_0\|^2_{L^2(\Omega)} + \|z_0\|^2_{L^2(\Omega)} \right),
\]
which is the desired conclusion.

□
4 The Proof of main result

Proof of Theorem 4.1 Set $K = \{ \eta \in V^r \mid \| \eta \|_{V^r} \leq 1 \}$. For any $\eta \in K$, we consider the following linearized system:

$$
\begin{align*}
    y_t - \Delta y &= \nabla \cdot (a_\eta y) + \nabla \cdot (Bz), \quad (x, t) \in Q, \\
    z_t - \Delta z &= -\nabla (b_\eta \cdot z) + \nabla y + \chi_{\omega} h, \quad (x, t) \in Q, \\
    y &= z = 0, \quad (x, t) \in \Sigma, \\
    (y, z)(x, 0) &= (y_0, z_0)(x), \quad x \in \Omega,
\end{align*}
$$

(4.1)

where $a_\eta = \eta + \nabla$, $b_\eta = \eta + 2\nabla$ and $B = \nabla$. Define

$$
\Lambda(\eta) = \{ z \in V^r \mid \exists h \in L'(Q) \text{ and a constant } C > 0 \text{ such that the solution of (4.1)} \\
\text{corresponding to } \eta \text{ and } h \text{ satisfies (3.51) and (3.52)} \}.
$$

Obviously, $K$ is a nonempty convex subset of $V^r$. By Proposition 3.2, we know that $\Lambda(\eta)$ is a nonempty convex subset of $V^r$.

Next, we prove that $\Lambda(\eta)$ is a compact subset of $V^r$. By Proposition 3.1 and (4.2), we get

$$
\| z \|_{V^r} \leq e^{CM_1 M_2^{1/2}} \left( \| (y_0, z_0) \|_{W^{2,2} r (\Omega) \times W^{2,2} r (\Omega)} + \| y_0 \|_{L^2(\Omega)} + \| z_0 \|_{L^2(\Omega)} \right).
$$

(4.2)

Therefore, $\| z \|_{V^r}$ is bounded. Note that, when $r > n + 2$, $V^r \hookrightarrow C^{1+\alpha, \frac{n+2}{r}}_\alpha (Q)$, here $\alpha = 1 - \frac{n+2}{r}$. Applying the Arzela-Ascoli Theorem, we can obtain that $\Lambda(\eta)$ is a compact subset of $V^r$.

Further, we show that $\Lambda$ is upper semi-continuous. For this, let $\{ \eta_n \}_{n=1}^\infty \subset K$ such that $\eta_n \rightarrow \eta$ in $K$, and set $z_n \in \Lambda(\eta_n)$. By the definition of $\Lambda(\eta_n)$, there exists $h_n \in L'(Q)$ such that the solution $(y_n, z_n)$ of (4.1) satisfies (3.51) and (3.52). By Proposition 3.1 we have

$$
\| y_n \|_{V^r} + \| z_n \|_{V^r} \leq e^{CM_1 M_2^{1/2}} \left( \| (y_0, z_0) \|_{W^{2,2} r (\Omega) \times W^{2,2} r (\Omega)} + \| y_0 \|_{L^2(\Omega)} + \| z_0 \|_{L^2(\Omega)} \right).
$$

Hence, there exist $h \in L'(Q)$, $y, z \in V^r$, and the subsequences of $\{ h_n \}, \{ y_n \}, \{ z_n \}$ (still denoted by themselves), such that

$$
h_n \rightarrow h \text{ in } L'(Q), \quad y_n \rightarrow y \text{ in } V^r, \quad z_n \rightarrow z \text{ in } V^r.
$$

(4.3)

Then $(y, z)$ is the solution of (4.1) corresponding to $\eta$ and $h$. Take $Y_n = y_n - y$, $Z_n = z_n - z$, and $H_n = \chi_{\omega}(h_n - h)$. Then $(Y_n, Z_n)$ satisfies

$$
\begin{align*}
    Y_{n,t} - \Delta Y_n &= \nabla \cdot [a_{\eta_n} Y_n + (a_{\eta_n} - a_\eta) y] + \nabla \cdot (B Z_n), \quad (x, t) \in Q, \\
    Z_{n,t} - \Delta Z_n &= -\nabla (b_{\eta_n} \cdot Z_n + (b_{\eta_n} - b_\eta) \cdot z) + \nabla Y_n + H_n, \quad (x, t) \in Q, \\
    Y_n &= Z_n = 0, \quad (x, t) \in \Sigma, \\
    (Y_n, Z_n)(x, 0) &= (0, 0), \quad x \in \Omega.
\end{align*}
$$

(4.4)

Moreover, an easy computation shows that

$$
\| Y_n(\cdot, t) \|_{L^2(\Omega)}^2 + \| \nabla Y_n(\cdot, t) \|_{L^2(\Omega)}^2 + \| Z_n(\cdot, t) \|_{L^2(\Omega)}^2 \leq e^{M_2 T} \left( \int_\Omega H_n \cdot Z_n dx + \int_\Omega |\eta_n - \eta|^2 (|y|^2 + |z|^2) dx \right).
$$

(4.5)

By (4.3), it follows that

$$
Z_n \rightarrow 0 \text{ in } L^{r_0}(Q), \quad \text{where } r_0 = \frac{r}{r - 1}.
$$

(4.6)
By (4.5) and (4.6), we have

\[ \| Y_n(\cdot, t) \|_{L^2(\Omega)}^2 \to 0, \quad \| Z_n(\cdot, t) \|_{L^2(\Omega)}^2 \to 0, \quad \forall \, t \in [0, T], \]

and notice that \( y_n(x, T) = z_n(x, T) = 0 \) in \( \Omega \). Hence, \( y(x, T) = z(x, T) = 0 \) in \( \Omega \), i.e., \( z \in \Lambda(\eta) \).

At last, we claim that \( \Lambda(\eta) \subset K \). Indeed,

\[
\| y \|_{L^\infty(Q)} + \| z \|_{V^r(Q)} \leq e^{CM_1} M_2^{k_1} \left( \| (y_0, z_0) \|_{W^{2,2}_r(\Omega) \times W^{2,2}_r(\Omega)} + \| y_0 \|_{L^2(\Omega)} + \| z_0 \|_{L^2(\Omega)} \right) \\
\leq C \| (y_0, z_0) \|_{W^{2,2}_r(\Omega) \times W^{2,2}_r(\Omega)}.
\]

Therefore, there exists a constant \( \delta > 0 \) such that, if \( \| (y_0, z_0) \|_{W^{2,2}_r(\Omega) \times W^{2,2}_r(\Omega)} < \delta \), we have

\[
\| y \|_{L^\infty(Q)} + \| z \|_{V^r(Q)} \leq C \| (y_0, z_0) \|_{W^{2,2}_r(\Omega) \times W^{2,2}_r(\Omega)} \leq \min\{1, p\}. \tag{4.7}
\]

Thus, by the Kakutani’s fixed point theorem, there exists \( z \in K \) such that \( z \in \Lambda(z) \). Moreover, by Corollary 2.2, \( \overline{u} \geq \underline{p} > 0 \). Therefore, \( u = y + \overline{u} \geq 0 \), which proves Theorem 1.1.

Acknowledgement

The authors would like to thank the referees for valuable comments and suggestions. Tao is partially supported by the National Science Foundation of China under grant 11971320 and Guangdong Basic and Applied Basic Research Foundation under grant 2020A1515010530. Zhang is partially supported by the National Science Foundation of China under grants 12001094, 12001087 and 11971179, and Fundamental Research Funds for the Central Universities under grant 2412020QD027.

Declarations

Conflict of interest The authors certify that they have no interest directly or indirectly related to the work submitted for publication.

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