FUNCTIONAL EQUATION OF THE $p$-ADIC $L$-FUNCTION OF BIANCHI MODULAR FORMS

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Abstract. Let $K$ be an imaginary quadratic field with class number 1, in this paper we obtain the functional equation of the $p$-adic $L$-function of: (1) a small slope $p$-stabilisation of a Bianchi modular form, and (2) a critical slope $p$-stabilisation of a Base change Bianchi modular form that is $\Sigma$-smooth. To treat case (2) we use $p$-adic families of Bianchi modular forms.

1. Introduction

Fix $p$ a rational prime. Via the theory of overconvergent modular symbols, Pollack and Stevens (see [PS11] and [PS13], or for an exposition, [Pol14]) gave a method of constructing the $p$-adic $L$-function of a suitable rational modular form.

In [Wil17] was developed an analogue of the work of [PS11] for the case of Bianchi modular forms, that is, the case of automorphic forms for $GL_2$ over an imaginary quadratic field $K$. Let $F$ be a Bianchi modular form of level $\Gamma_0(n)$ with $(p)\mid n$, weight $(k,k)$ and with small slope, i.e., $v(\lambda_p) < (k+1)/e_p$ for all $p|p$, where $\lambda_p$ are the $U_p$-eigenvalues of $F$ and $e_p$ is the ramification index of $p$; in particular, in [Wil17] was constructed the $p$-adic $L$-function $L_p(F,\cdot)$ on $X(\mathbb{C}l_K(p^\infty))$, the two-dimensional rigid space of $p$-adic characters on the ray class group $\mathbb{C}l_K(p^\infty)$.

In this paper, $K$ has class number 1.

Our first result is the functional equation of $L_p(F,\cdot)$ where $F_p$ is a small slope Bianchi modular form obtained by successively stabilising at each different prime $p$ above $p$ a newform $F$.

A functional equation for the classical $L$-function attached to a Bianchi modular form is given in [JL70]. In Section 3, we recover the following reformulation:

Theorem 1.1. Let $F \in S_{(k,k)}(\Gamma_0(n))$ be a Bianchi newform and $\psi$ be a Hecke character of $K$ of conductor $\mathfrak{f}$ with $(n,\mathfrak{f}) = 1$ and infinity type $0 \leq (q,r) \leq (k,k)$, we have

$$
\Lambda(F,\psi) = \frac{-\epsilon(n)v|\psi|^k\tau(\psi)|_{\lambda_k}}{\psi(\nu)^{\psi}(\nu)^{-\psi(\psi^{-1})}}\Lambda(F,\psi^{-1})|_{\lambda_k}.
$$

where $n = (\nu)$, and $\epsilon(n) = \pm 1$ is the eigenvalue of $F$ for the Fricke involution $W_\nu$.

In theorem above, $\Lambda(F,\cdot)$ is the $L$-function of $F$ renormalised by Deligne's $\Gamma$-factors at infinity, $\psi = \prod q_{p} \psi_q$ with $\psi_q$ the restriction of $\psi$ to $K_q^*$, $\psi_\infty$ is the infinite part of $\psi$, $\tau(\cdot)$ is the Gauss sum of Section 1.2.3 in [Wil17] and $|_{\lambda_k}$ is the adelic norm.

Key words and phrases. Bianchi modular forms, $p$-adic $L$-function, functional equation, $p$-adic families of Bianchi modular forms.

1We expect our results follows for higher class number taking direct sum over class group (as in [Wil17]).
We then obtain the functional equation of \( L_p(F_{\mathfrak{p}}, -) \) in the small slope case:

**Theorem 1.2.** Let \( F_{\mathfrak{p}} \) be a small slope \( p \)-stabilisation of the newform \( F \) of weight \((k, k)\) and level \( \Gamma_0(n) \) with \( n = \nu \) and \( (p) \nmid n \). Then for all \( \kappa \in \mathfrak{X}_{1}(\mathfrak{C}_{\mathfrak{K}}(p^{\infty})) \), the distribution \( L_p(F_{\mathfrak{p}}, -) \) satisfies the following functional equation

\[
L_p(F_{\mathfrak{p}}, \kappa) = -\epsilon(n)N(n)^{k/2}\kappa(x_{-\nu,p})^{-1}L_p(F_{\mathfrak{p}}, \kappa^{-1}\sigma_p^{k,k}),
\]

where \( \epsilon(n) = \pm 1 \) is the eigenvalue of \( F \) for the Fricke involution \( W_n \), \( x_{-\nu,p} \) is the idele associated to \(-\nu\) defined in Remark 6.7, and \( \sigma_p^{k,k} \) is as in equation (6.7).

The proof uses three ingredients, namely: (a) the main theorem in [Wil17], i.e. the construction and interpolation of \( L_p(F, -) \) for a Bianchi modular form \( F \); (b) the complex functional equation obtained in Theorem 1.1 and (c) the work done in [Loe14] that uniquely determines \( L_p(F, -) \) by its values on the \( p \)-adic characters \( \psi_{p,fin} \) coming from a Hecke character \( \psi \) as in Theorem 1.1 with conductor \( \mathfrak{f}p^\infty \) (see Section 6.4 for the definition of \( \psi_{p,fin} \)), when \( F \) has small slope at every \( \mathfrak{p} \).

**Remark 1.3.** For Theorem 1.1 to hold, the level of \( F \) must be coprime with the conductor of \( \psi \). On the other hand for the \( p \)-adic setting of Theorem 1.2, \( p \) needs to be in the level and also its proof use Hecke characters \( \psi \) with conductor \( \mathfrak{f}p^\infty \) then the level and the conductor are not coprime. As a consequence we are forced to work first with Bianchi modular forms without level at all the primes above \( p \), and then successively stabilise at each prime \( \mathfrak{p} \mathfrak{p} \), consequently missing the new Bianchi modular forms at primes above \( p \).

The construction of the \( p \)-adic \( L \)-function in [Wil17] and consequently the functional equation in Theorem 1.2 depend of the small slope condition of the Bianchi modular form \( F \). It is natural to ask for the \( p \)-adic \( L \)-function when \( F \) does not have small slope, i.e. the critical slope case. In [BSWWE18] such function is constructed for a base change Bianchi modular form \( f|_{K} \) of a modular form \( f \) satisfying certain properties. We briefly describe the construction.

Fix \( f \in S_{k,2}(\Gamma_0(N)) \) with \( N \) divisible by \( p \) a finite slope eigenform, new or \( p \)-stabilised of a newform at \( p \), regular, without CM by \( K \), decent and \( \Sigma \)-smooth (for the precise definitions see Section 6.22). Let \( V_{\mathfrak{Q}} \) be a neighbourhood of \( f \) such that the weight map \( w \) is étale except possibly at \( f \). Then, after shrinking \( V_{\mathfrak{Q}} \), in [BSWWE18] was constructed the three-variable \( p \)-adic \( L \)-function

\[
L_p : V_{\mathfrak{Q}} \times \mathfrak{X}_{1}(\mathfrak{C}_{\mathfrak{K}}(p^{\infty})) \to \mathbb{C}_p
\]

attached to \( V_{\mathfrak{Q}} \). Now, suppose \( f|_{K} \) is \( \Sigma \)-smooth and has critical slope; then the missing \( p \)-adic \( L \)-function of \( f|_{K} \) is defined to be the specialisation \( L_p(f|_{K}, -) := L_p(x_f, -) \), where \( x_f \in V_{\mathfrak{Q}} \) is the point corresponding to \( f \).

Our second result is the functional equation of \( L_p(F_{\mathfrak{p}}, -) \) where \( F_{\mathfrak{p}} \) is a critical slope Bianchi modular form obtained by successively stabilising at each different prime \( \mathfrak{p} \) above \( p \) a newform \( F \) that is the \( \Sigma \)-smooth base-change of a modular form \( f \) to \( K \). For this we want to transfer the functional equation in Theorem 1.2 to \( L_p \), and then specialise to obtain a functional equation on the critical slope case.

In order to find such a functional equation for \( L_p \), we can suppose \( V_{\mathfrak{Q}} \) small enough to contain a set \( S \subset V_{\mathfrak{Q}} \) of Zariski-dense classical points \( y \) such that if we denote by \( f_y \) the corresponding eigenform, then the associated base change to \( K \), \( f_y|_{K} \), has level \( \Gamma_0(n) \) with \( (\frac{N}{(N,x_y)}) \mathcal{O}_K[n]\mathcal{O}_K, n = (\nu) \), is a \( p \)-stabilisation of a newform, has small slope and \( w(y) \equiv k \mod p - 1 \). We have
Theorem 1.4. For every \( y \in V_Q \) and all \( \kappa \in \mathcal{X} ( \text{Cl}_K ( p^\infty )) \) we have
\[
L_p ( y, \kappa ) = - \epsilon ( n ) w_{ T_M} ( N ( n ) )^{ k/2} ( N ( n ) )^{ w ( y ) / 2} \kappa ( x_{ - \nu, p} )^{- 1} L_p ( y, \kappa , w_{ T_M} ( )^{ w ( y ) } ),
\]
where \( w_{ T_M} \) and \( \langle \cdot \rangle \) are as in Definition 2.2.

Since the \( p \)-adic \( L \)-function of \( f_{ / K} \) is defined to be the specialisation \( L_p ( f_{ / K}, - ) := L_p ( x_f, - ) \), from Theorem 1.4 we obtain the following:

Corollary 1.5. Let \( \mathcal{F}_p \) be a critical slope \( p \)-stabilisation of \( \mathcal{F} \), the base change of \( f \in S_k ( \Gamma_0 ( N ) ) \), a newform of finite slope, regular, non CM, decent and \( \Sigma \)-smooth. Then for all \( \kappa \in \mathcal{X} ( \text{Cl}_K ( p^\infty )) \), the distribution \( L_p ( \mathcal{F}_p, - ) \) satisfies the following functional equation
\[
L_p ( \mathcal{F}_p, \kappa ) = - \epsilon ( n ) N ( n )^{ k/2} \kappa ( x_{ - \nu, p} )^{- 1} L_p ( \mathcal{F}_p, \kappa , \sigma^k_p ) .
\]

Acknowledgements

I would like to thank my PhD supervisor Daniel Barrera for suggesting this topic to me, as well as for the many conversations we’ve had on the subject. Thanks also to Chris Williams for helpful conversations about Bianchi modular forms. This work was funded by the National Agency for Research and Development (ANID)/Scholarship Program/BECA DOCTORADO NACIONAL/2018 - 21180506.

2. Bianchi modular forms

2.1. Notation.

Throughout this paper, we fix \( p \) a rational prime and take \( K \) to be an imaginary quadratic field with class number \( 1 \) and ring of integers \( \mathcal{O}_K \), let \( \delta = \sqrt{- D} \) (where \( - D \) is the discriminant of \( K \)) be a generator of the different ideal \( D \) of \( K \), \( n = (\nu) \) an ideal of \( \mathcal{O}_K \). At each prime \( q \) of \( K \), denote by \( K_q \) the completion of \( K \) with respect to \( q \), \( \mathcal{O}_q \) the ring of integers of \( K_q \) and fix a uniformiser \( \pi_q \) at \( q \). Denote the adele ring of \( K \) by \( K = K_\infty \times K_f ^{\mathbb{Q}} \) where \( K_\infty \) are the infinite adeles and \( K_f ^{\mathbb{Q}} \) are the finite adeles. Furthermore, define \( \mathcal{O}_K := \mathcal{O}_K \otimes \mathbb{Q} \mathbb{Z} \) to be the finite integral adeles. Let \( k \geq 0 \) be an integer and \( R \) be any ring, let \( V_{2k+2} ( R ) \) denote the space of homogeneous polynomials in two variables over \( R \) of degree \( 2k + 2 \). Note that \( V_{2k+2} ( \mathbb{C} ) \) is an irreducible complex right representation of \( SU_2 ( \mathbb{C} ) \), denote it by \( \rho_{2k+2} \).

For a general Hecke character \( \psi \) of \( K \), for each prime \( q \) of \( K \) we denote by \( \psi_q \) the restriction of \( \psi \) to \( K_q ^* \) and for an ideal \( I \subset \mathcal{O}_K \), we define \( \psi_I = \prod_{I \mid I} \psi_q ; \) we also write \( \psi_\infty \) for the restriction of \( \psi \) to the infinite ideles, and \( \psi_f \) for the restriction to the finite ideles.

2.2. Background.

Let \( \Omega_0 ( n ) = \{ ( a \ b \ c \ d ) \in \text{GL}_2 ( \mathcal{O}_K ) : c \in n \mathcal{O}_K \} \) and let \( \varphi \) be a Hecke character, with infinity type \(- k, - k \) and conductor \( c | n \). For \( u_f = ( a \ b \ c \ d ) \in \Omega_0 ( n ) \) we set \( \varphi_0 ( u_f ) = \varphi_0 ( d ) = \prod_{q | \delta} \varphi_q ( d_q ) \), with \( \varphi_q ( d_q ) \) trivial if \( q \nmid c \).

Definition 2.1. We say a function \( \Phi : \text{GL}_2 ( K_K ) \to V_{2k+2} ( \mathbb{C} ) \) is a cuspidal automorphic form over \( K \) of weight \( ( k, k ) \), level \( \Omega_0 ( n ) \) and central action \( \varphi \) if it satisfies:
(i) $\Phi$ is left-invariant under $GL_2(K)$;
(ii) $\Phi(zg) = \varphi(z)\Phi(g)$ for $z \in \mathbb{H}_K$ and $g \in GL_2(K)$, where $Z(G)$ denote the center of the group $G$;
(iii) $\Phi(u_\varphi) = \varphi(u_f)\Phi(g)\rho_{2k+2}(u_\infty)$ for $u = u_f \cdot u_\infty \in \Omega_0(n) \times SU_2(\mathbb{C})$;
(iv) $\Phi$ is an eigenfunction of the operator $\partial$, where $\partial/4$ denotes a component of the Casimir operator in the Lie algebra $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}$, and where we consider $\Phi(g_\infty g_f)$ as a function of $g_\infty \in GL_2(\mathbb{C})$.
(v) $\Phi$ satisfies the cuspidal condition that for all $g \in GL_2(\mathbb{A}_K)$, we have
\[ \int_{K\backslash \mathbb{A}_K} \Phi((b\ 0 \ 1/1\ b)g) \, du = 0 \]
where $du$ is the Lebesgue measure on $\mathbb{A}_K$.

The space of such functions will be denoted by $S_{(k,k)}(\Omega_0(n),\varphi)$.

**Remark 2.2.** 1) For conditions (ii) and (iii) to be compatible, we need that $\rho_{2k+2}$ agree with $\varphi$ on the center of $SU_2(\mathbb{C})$.

2) The cuspidal condition is natural; the value of the integral for a fixed $g$ corresponds to a constant Fourier coefficient.

3) The general definitions given above are already slightly tailored to work with cuspidal forms, that appear only at parallel weights, more generally, it is possible to define automorphic forms over $K$ of weight $(k_1,k_2)$, for distinct integers $k_1$ and $k_2$.

A cuspidal automorphic form $\Phi$ of weight $(k,k)$ and level $\Omega_0(n)$ descends to give a function $F: GL_2(\mathbb{C}) \rightarrow V_{2k+2}(\mathbb{C})$, via $F(g) := \Phi(g)$ for $g \in GL_2(\mathbb{C}) < GL_2(\mathbb{A}_K)$.

Let $H_3 := \{(z,t) : z \in \mathbb{C}, t \in \mathbb{R}_{>0}\}$ be the hyperbolic space and since $GL_2(\mathbb{C}) = Z(GL_2(\mathbb{C})) \cdot B \cdot SU_2(\mathbb{C})$, where $B = \{\left( \begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix} \right) : z \in \mathbb{C}, t \in \mathbb{R}_{>0}\} \cong H_3$. We can descend further using ii) and iii) in Definition 2.1 to obtain a function
\[ F: H_3 \rightarrow V_{2k+2}(\mathbb{C}), \]
\[ (z,t) \mapsto t^{-1}F(1 \ 0 
1 \ 1\ t). \]

**Remark 2.3.** There are two ways to define the function $F$ in the literature.

1) In accounts such as [Cre81], [CW94] and [Byg98], all of which deal predominantly with weight $(0,0)$ and trivial central action; such function is defined simply restricting to $B$.

2) In [Gha99] and [Wil17] such $F$ is defined as in (2.1), where they scaled the restriction by the factor $t^{-1}$ for all weight $(k,k)$.

For the two definitions above, the modularity condition satisfied by such $F$ resulting are different, but there is a clear bijection between the sets of functions that arise.

**Definition 2.4.** A function $F: H_3 \rightarrow V_{2k+2}(\mathbb{C})$ is a cuspidal Bianchi modular form if comes from $\Phi$, a cuspidal automorphic form over $K$ (in the sense of definition 2.1) by the descent described above.

Note that for $\gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_0(n) := \text{SL}_2(K) \cap \Omega_0(n)GL_2(\mathbb{C})$, a Bianchi modular form $F$ satisfies the following automorphic condition:
\[ F(\gamma \cdot (z,t)) = \varphi(d)^{-1}F(z,t)\rho_{2k+2}(J(\gamma; (z,t))), \]
where $\gamma \cdot (z,t)$ denotes the action of $GL_2(\mathbb{C})$ on $H_3$ and $J(\gamma; (z,t)) := \left( \begin{smallmatrix} cz+d \\ -ct \end{smallmatrix} \right)$. 

We denote by \( S_{(k,k)}(\Gamma_0(n), \varphi^{-1}) \) the space of cuspidal Bianchi modular forms \( \mathcal{F} \) that comes from cuspidal automorphic forms \( \Phi \in S_{(k,k)}(\Gamma_0(n), \varphi) \) and then they satisfy (2.2). Also denote by \( S_{(k,k)}(\Gamma_0(n)) \) when, in particular, \( \varphi = |\cdot|_{\mathbb{A}_K} \).

**Definition 2.5.** Let \( \gamma \in \text{GL}_2(\mathbb{C}) \) and let \( \mathcal{F} \in S_{(k,k)}(\Gamma_0(n), \varphi^{-1}) \), then define a new function \( \mathcal{F}|_{\gamma} \) by

\[
(\mathcal{F}|_{\gamma})(z, t) := |\text{det}(\gamma)|^{-k} \mathcal{F}(\gamma \cdot (z, t)) \rho_{2k+2}^{-1} \left( J(J(J(J(z, t)))) \right).
\]

**Remark 2.6.** Note \( \mathcal{F} \in S_{(k,k)}(\Gamma_0(n), \varphi^{-1}) \) satisfies:

1) \( \mathcal{F}|_{(0, 1)} = \Phi(g) \) for \( g \in \text{GL}_2(\mathbb{C}) \), and for \( g = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in B \) we obtain (2.1).

2) \( (\mathcal{F}|_{\gamma})(z, t) = \varphi_{\gamma}^{-1}(d) \mathcal{F}(z, t) \) for \( \gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_0(n) \).

### 2.3. Fourier-Whittaker expansions.

Let \( \Phi : \text{GL}_2(\mathbb{A}_K) \to V_{2k+2}(\mathbb{C}) \) be a cuspidal automorphic form of weight \((k, k)\). Then, from Theorem 6.1 in [Hid94], \( \Phi \) has a Fourier expansion of the form

\[
\Phi[\left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)] = \sum_{\alpha \in \mathbb{K}} c(\alpha \delta, \Phi) W(\alpha t_\infty) e_K(\alpha z) \text{ for } \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \text{GL}_2(\mathbb{A}_K),
\]

where:

i) the Fourier coefficient \( c(\cdot, \Phi) \) is a function on the fractional ideals of \( \mathbb{K} \), with \( c(I, \Phi) = 0 \) for \( I \) non-integral;

ii) \( c_K \) is an additive character of \( \mathbb{K}\backslash \mathbb{A}_K \) defined by

\[
e_K = \left( \prod_p \left( e_p \circ \text{Tr}_{\mathbb{K}_p/\mathbb{Q}_p} \right) \right) \circ \left( e_\infty \circ \text{Tr}_{\mathbb{C}/\mathbb{R}} \right),
\]

for

\[
e_p \left( \sum_j d_j p^j \right) = e^{-2\pi i \sum_{j \neq 0} d_j p^j} \text{ and } e_\infty(r) = e^{2\pi ir};
\]

iii) \( W : \mathbb{C}^* \to V_{2k+2}(\mathbb{C}) \) is the Whittaker function

\[
W(s) := \sum_{n=0}^{2k+2} \binom{2k+2}{n} \frac{2k+2}{n} \left( \frac{s}{i|s|} \right)^{k+1-n} K_{(k+1)}(4\pi|s|) X^{2k+2-n} Y^n,
\]

where \( K_n(x) \) is a modified Bessel function.

If our cuspidal automorphic form, \( \Phi \), corresponds to a cuspidal Bianchi modular form \( \mathcal{F} \) on \( \mathcal{H}_3 \), then the Fourier expansion stated above descends to the following Fourier expansion of \( \mathcal{F} \) (see [Gha99]):

\[
\mathcal{F}(z, t) \left( \begin{smallmatrix} X \\ Y \end{smallmatrix} \right) = \sum_{n=0}^{2k+2} \mathcal{F}_n(z, t) X^{2k+2-n} Y^n,
\]

\[
(2.5) \quad \mathcal{F}_n(z, t) := t^{\binom{2k+2}{n}} \sum_{\alpha \in \mathbb{K}^*} \left[ c(\alpha \delta) \left( \frac{\alpha}{|\alpha|} \right)^{k+1-n} K_{n-k-1}(4\pi|\alpha| t) e^{2\pi i(\alpha z+\overline{\alpha z})} \right].
\]

Here to ease notation we have written \( c(\alpha \delta) \) instead \( c(\alpha \delta, \Phi) \).
2.4. Hecke operators.

As with classical modular forms, we can extend the action of $GL_2(\mathbb{C})$ on functions, given by \( (2.3) \), to the group ring of $GL_2(\mathbb{C})$; Hecke operators will be defined by particular elements on this group ring.

Let $q \not| n$ be a prime ideal of $\mathcal{O}_K$ generated by the fixed uniformiser $\pi_q$. Then for $\mathcal{F} \in S_{(k,j)}(\Gamma_0(n), \varphi_c^{-1})$ we define the Hecke operator.

\[
\mathcal{F} \mapsto \mathcal{F}|_{T_q} := |\pi_q|^{2k} \left[ \sum_{b \in (\mathcal{O}_K/q)^	imes} \mathcal{F}(\begin{pmatrix} 1 & b \\ 0 & \pi_q \end{pmatrix}) - \varphi_c(\pi_q)^{-1} \mathcal{F}(\begin{pmatrix} 1 & 0 \\ 0 & \pi_q \end{pmatrix}) \right].
\]

When $q|n$ we denote $T_q$ by $U_q$ and

\[
\mathcal{F}|_{U_q} := |\pi_q|^{2k} \sum_{b \in (\mathcal{O}_K/q)^	imes} \mathcal{F}(\begin{pmatrix} 1 & b \\ 0 & \pi_q \end{pmatrix}).
\]

We can similarly define Hecke operators for each ideal $I$ of $K$. Indeed, let $I = \prod_q q^r$ where $q^r$ exactly divides $I$, then the Hecke operator $T_I$ is totally determined by the Hecke operators $T_q$ for $qI$.

Remark 2.7. We define above the Hecke operators for Bianchi modular forms, but they are defined more generally on automorphic forms over $K$ by the action of double cosets.

In the same way as in the rational case (elliptic modular forms), for Bianchi modular forms of level $\Gamma_0(n)$ with $n = (\nu)$ there is a Fricke involution $W_n$ defined by

\[
(\mathcal{F}|_{W_n}) = |\nu|^{-k} \mathcal{F}(\begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix}).
\]

Lemma 2.8. Let $\mathcal{F} \in S_{(k,j)}(\Gamma_0(n), \varphi_c^{-1})$ then for $0 \leq n \leq 2k + 2$ we have

\[
(\mathcal{F}|_{W_n})|_{2k+2-n}(0,t) = t^{-2k-2}(-1)^n \nu^{n-k-1} |\nu|^{-n} \mathcal{F}|_{n}(0, \frac{1}{|\nu|t}).
\]

Proof. Note that if $\gamma = (\begin{pmatrix} 0 & 1 \\ \nu & 0 \end{pmatrix})$ then $\gamma \cdot (0,t) = (0,1/(|\nu|t))$ and

\[
\rho_{2k+2}^{-1}\left( J \left( \frac{\gamma}{\sqrt{\text{det}(\gamma)}}, (0,t) \right) \right) \left( \begin{array}{c} X \\ Y \end{array} \right)^{2k+2} = J \left( \frac{\gamma}{\sqrt{\text{det}(\gamma)}}, (0,t) \right)^{-1} \left( \begin{array}{c} X \\ Y \end{array} \right)^{2k+2} = \left( \begin{array}{cc} 0 & -\nu^{-1/2}t^{-1} \\ \nu^{-1/2}t^{-1} & 0 \end{array} \right) \left( \begin{array}{c} X \\ Y \end{array} \right)^{2k+2} = \left( \begin{array}{c} -\nu^{-1/2}t^{-1}X \\ \nu^{-1/2}t^{-1}X \end{array} \right)^{2k+2}
\]

where $\left( \begin{array}{c} X \\ Y \end{array} \right)^{2k+2} = (X^{2k+2}, X^{2k+1}Y, ..., X^{2k+2-n}Y^n, ..., XY^{2k+1}, Y^{2k+2})^t$.

Then,

\[
(\mathcal{F}|_{W_n})(0,t) \left( \begin{array}{c} X \\ Y \end{array} \right)^{2k+2} = \mathcal{F}(0,1/(|\nu|t)) \cdot \left( \begin{array}{c} -\nu^{-1/2}t^{-1}Y \\ \nu^{-1/2}t^{-1}X \end{array} \right)^{2k+2}.
\]
2.5. Twisted series of Bianchi modular forms.

Definition 2.9. Let $\Phi \in S_{(k,k)}(\Omega_0(n), \varphi)$ and $\psi$ be a finite order Hecke character of conductor $f$. Define the twisting operator $R(\psi)$ by

$$\Phi(R(\psi))(g) := \psi(\det(g)) \sum_{[a] \in (\mathbb{Z}/f\mathbb{Z})^*} \psi(a) \Phi(g(\begin{smallmatrix} a & \gamma \\ 0 & 1 \end{smallmatrix})), \quad g \in \text{GL}_2(\mathbb{A}_K).$$

Proposition 2.10. Let $\Phi \in S_{(k,k)}(\Omega_0(n), \varphi)$ where $\varphi$ has conductor $c$ and let $\psi$ be a finite order Hecke character of conductor $f$. Then $\Phi(R(\psi))(\Omega_0(m), \varphi \psi^2) = \Phi(S(\psi))(\Omega_0(m), \varphi \psi^2)$ where $m = n \cap f^2$.

Proof. See [Hid94, section 6], where is proved that $\Phi|R(\psi)$ is again an automorphic form of level $\Omega_0(m)$ and central action $\varphi \psi^2$; in our case the weight of $\Phi|R(\psi)$ is $(k, k)$ since the infinity type of $\psi$ is trivial. We then have after checking (v) in Definition 2.1 that $\Phi(R(\psi))(\Omega_0(m), \varphi \psi^2)$. □

Remark 2.11. Note that for $f = (f)$, if $\mathcal{F}$ is the descent of a cuspidal automorphic form $\Phi \in S_{(k,k)}(\Omega_0(n), \varphi)$ and we denote by $\mathcal{F}_\psi$ the descent of $\Phi|\psi$ then by equation (6.9) in [Hid94], we have

$$\mathcal{F}_\psi = \sum_{\nu \in \mathcal{O}_K} \psi(\nu) \mathcal{F}(\begin{smallmatrix} 1 & \nu \\ 0 & 1 \end{smallmatrix}).$$

Lemma 2.12. If $\mathcal{F} \in S_{(k,k)}(\Gamma_0(n), \varphi \psi^2)$ then $\mathcal{F}|_{W_n} \in S_{(k,k)}(\Gamma_0(n), \varphi \psi^n)$.

Proof. Note that $(0 1)$ normalizes the group $\Gamma_0(n)$, explicitly $(0 1)^\gamma = \gamma(0 1)$ where $\gamma' = (d \ b \ c \ a)$. Hence, for $\mathcal{F} \in S_{(k,k)}(\Gamma_0(n), \varphi \psi^2)$ and $\gamma \in \Gamma_0(n)$ we have

$$(\mathcal{F}|_{W_n}) \gamma = |\nu|^{k'} \mathcal{F}(\begin{smallmatrix} 0 & -1 \\ \nu & 0 \end{smallmatrix}) = |\nu|^{k'} \mathcal{F}(\begin{smallmatrix} 0 & -1 \\ \nu & 0 \end{smallmatrix}) \gamma, \quad \gamma \in \Gamma_0(n),$$

where in last equality we use that $ad \equiv 1 \mod n$. □

Proposition 2.13. Let $\mathcal{F} \in S_{(k,k)}(\Gamma_0(n), \varphi \psi^2)$ be a Bianchi modular form and let $\psi$ be a Hecke character of finite order and conductor $f$ with $(n, f) = 1$. Then

$$\mathcal{F}_\psi|_{W_n} = \varphi \psi^{-1} \mathcal{F}_\psi|_{W_n} = \varphi \psi^{-1} (\mathcal{F}|_{W_n}) \psi^{-1}.$$

for $(m) = m = n f^2 = (\nu)^2$, $m = \nu f^2$.

Proof. Since for any $\nu$ we have the identity $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ \nu & 0 \end{pmatrix} \begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix} = f \begin{pmatrix} 0 & -1 \\ \nu & 0 \end{pmatrix} \begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix}$ and choosing $\nu$ such that $\nu \equiv -1 \mod f$ to bring $f = \frac{-\nu}{1 + \nu f^2}$ into $\Gamma_0(n)$, then

$$\mathcal{F}_n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} W_n = |m|^k \mathcal{F}_n \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = |m|^k \mathcal{F}_n \begin{pmatrix} 0 & -1 \\ \nu & 0 \end{pmatrix} \begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} W_n = |m|^k |\mathcal{F}|_{W_n} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = |m|^k |\mathcal{F}|_{W_n} \begin{pmatrix} 0 & -1 \\ \nu & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



(2.6)
Where in the last equality we use that \( m = \nu f^2 \) and \( \mathcal{F}|_{W_n} \in S_{(k,k)}(\Gamma_0(n), \varphi) \) by Lemma [2.12]. Since we have that \( b \nu \equiv -1 \pmod{f} \), then \( \psi_f(b) = \psi_f(-\nu)^{-1} = (\nu)^{-1}. \)

Now multiplying (2.6) by the latter and summing over the reduced residue class of \((\mathcal{O}_K/f)\), using Remark [2.11] we obtain the result. 

Proposition 2.13 above is a generalization to the Bianchi setting of Theorem 7.5 in [Iwa97].

3. L-function

3.1. Definition of the L-function.

Let \( \psi \) be a Hecke character with conductor \( f \), for each ideal \( m = \prod_{q|m} q^n \), coprime to \( f \), we define \( \psi(m) = \prod_{q|m} \psi_q(\pi_q)^n \) with \( q = \langle \pi_q \rangle \) a prime ideal; and \( \psi(m) = 0 \) if \( m \) is not coprime to \( f \). In an abuse of notation, we write \( \psi \) for both the idelic Hecke character and the function it determines on ideals; will always be clear from the context which formulation we mean.

Let \( \Phi \) be an automorphic form, define the twist of the L-function of \( \Phi \) by \( \psi \) by

\[
L(\Phi, \psi, s) = \sum_{m \in \mathcal{O}_K} c(m, \Phi) \psi(m) N(m)^{-s},
\]

where \( c(\cdot, \Phi) \) are the Fourier coefficients of \( \Phi \).

**Remark 3.1.** In [Wei71], it is proved that the twisted L-function converges absolutely in some suitable right half-plane. The L-function can be written in terms of an integral formula, then via meromorphic continuation, this integral gives the definition of the L-function on all of \( \mathbb{C} \). In fact, a little more work shows that this function is an analytic continuation and the L-function is holomorphic on the whole complex plane.

Let \( \mathcal{F} \) be a Bianchi modular form corresponding to the automorphic form \( \Phi \), then

\[
L(\Phi, \psi, s) = L(\mathcal{F}, \psi, s) = w^{-1} \sum_{\alpha \in \mathcal{O}_K} c(\alpha \delta) \psi((\alpha \delta)) N((\alpha \delta))^{-s},
\]

where \( w = |\mathcal{O}_K| \) and \( c(\cdot) \) are the Fourier coefficients of \( \mathcal{F} \) as in (2.5).

It is convenient to think the twisted L-function as a function on Hecke characters instead as a complex function of one variable. Then we put

\[
L(\mathcal{F}, \psi) = L(\mathcal{F}, \psi, 1).
\]

We complete the L-function by adding the appropriate factors at infinity. If the infinity type of \( \psi \) is \( (q, r) \) then we define

\[
\Lambda(\mathcal{F}, \psi) := \frac{\Gamma(q+1)\Gamma(r+1)}{(2\pi i)^{q+1}(2\pi i)^{r+1}} L(\mathcal{F}, \psi).
\]

where \( \Gamma \) is the usual Gamma function. This is the L-function renormalised by Deligne’s Gamma-factors at infinity.

In [Will17] in Theorem 2.11 is proved:

**Theorem 3.2.** Let \( \mathcal{F} \in S_{(k,k)}(\Gamma_0(n)) \) then for a Hecke character \( \psi \) of conductor \( f = (f) \) and infinity type \( 0 \preceq (q, r) \preceq (k, k) \) we have

\[
\Lambda(\mathcal{F}, \psi) = \frac{(-1)^{k+q+r}2^{q+r}(f)}{Dw_{\tau}(\psi^{-1})} \sum_{b \in (\mathcal{O}_K/f)\times} \psi_1(b)c_{q,r}(b/f),
\]
Theorem 3.4. Let \( \mathcal{F} \in S_{(k,k)}(\Gamma_0(n)) \) be a newform with \( n = (\nu) \), then for a Hecke character \( \psi \) of conductor \( \mathfrak{f} = (f) \) with \((f,\nu) = 1\) and infinity type \( 0 \leq (q,r) \leq (k,k) \), we have

\[
\lambda(\mathcal{F}, \psi) = -\epsilon(n)|\nu|^k \tau(\psi) \cdot \frac{1}{|\Delta_K|} \lambda(\mathcal{F}, \psi^{-1}) \cdot \frac{1}{|\Delta_K|}.
\]

Proof. By Theorem 3.2 we know that for a Hecke character \( \psi \) of conductor \( \mathfrak{f} = (f) \) and infinity type \( 0 \leq (q,r) \leq (k,k) \), we have

\[
\lambda(\mathcal{F}, \psi) = \frac{(-1)^{k+q+r}2\psi_\infty(f)}{DwT(\psi^{-1})} \sum_{b \in (\mathcal{O}_K)f} \psi_f(b)c_{q,r}(b/f)
\]

\[
= \frac{(-1)^{k+q+r}2\psi_\infty(f)}{DwT(\psi^{-1})} \sum_{b \in (\mathcal{O}_K)f} \psi_f(b) \left[ \frac{2(-1)^{k+r+1}}{q^{k+q+r+1}} \int_0^\infty t^{q+r} F_{k+q-r+1}(b/f, t) dt \right]
\]

\[
= \frac{(-1)^{q+1}4\psi_\infty(f)}{DwT(\psi^{-1})(2k+2)} \int_0^\infty t^{q+r} \left[ \sum_{b \in (\mathcal{O}_K)f} \psi_f(b)F_{k+q-r+1}(b/f, t) \right] dt
\]

\[
= \frac{(-1)^{q+1}4\psi_\infty(f)}{DwT(\psi^{-1})(2k+2)} \int_0^\infty t^{q+r} \left[ \sum_{b \in (\mathcal{O}_K)f} \varphi_f(b)F_{k+q-r+1}(b/f, t) \right] dt,
\]

where in the last equality \( \varphi = \psi|\psi|^{-1} \), then \( \varphi_f = \psi_f \).

By Remark 2.11 since \( \varphi \) has finite order we have

\[
\sum_{b \in (\mathcal{O}_K)f} \varphi_f(b) F_{1/1} \begin{pmatrix} 1 & b/f \\ 0 & 1 \end{pmatrix} = F_{\varphi}.
\]
where $F_c$ is the descent of $\Phi| R(\psi) \in S_{(k,k)}(\Omega_0(m), \varphi \psi^2)$ with $m = (m)$ and $m = \nu f^2$.

Then

$$
\Lambda(F, \psi) = \frac{(-1)^{q+1}4\psi_{\infty}(f)}{DwT(\psi^{-1})(k_{2k+2}^{q+1})} \int_0^\infty t^{q-r-2} F_{k,q-r+1}(0, t) dt.
$$

Changing variable $t \rightarrow 1/(|m|t)$ we have

$$
\Lambda(F, \psi) = \frac{(-1)^{q+1}4\psi_{\infty}(f)m^{q-r-1}}{DwT(\psi^{-1})(k_{2k+2}^{q+1})} \int_0^\infty t^{q-r-2} F_{k,q-r+1}(0, 1/(|m|t)) dt.
$$

We know by Lemma 2.8 that

$$(F_c|W_m)_{k,q-r+1}(0, t) = t^{-2k-2}(-1)^{k+q-r+1}m^{q-r}|m|^{-k+q-r+1} \Lambda_{k,q-r+1}(0, t)$$

and replacing $F_{k,q-r+1}(0, 1/(|m|t))$ above we have

$$
\Lambda(F, \psi) = \frac{(-1)^{q+1}4\psi_{\infty}(f)m^{q-r-1}}{DwT(\psi^{-1})(k_{2k+2}^{q+1})} \int_0^\infty t^{q-r-2} F_{k,q-r+1}(0, t) dt.
$$

By Proposition 2.13 since $(\nu, f) = 1$ we have $F_c|W_m = \varphi f(-\nu)^{-1}(F|W_m)_c^{-1}$, also $F$ is a newform, then $F_c|W_m = \epsilon(n) \varphi f(-\nu)^{-1}F_{c^{-1}}$ and we have

$$
\Lambda(F, \psi) = \frac{(-1)^{k+r}(n)4\psi_{\infty}(f)m^{q-r+1}|m|^{k-2r}}{DwT(\psi^{-1})(k_{2k+2}^{q+1})} \varphi f(-\nu)^{-1} \int_0^\infty t^{2k-q-r} F_{k,q-r+1}(b/f, t) dt
$$

$$
= \frac{(-1)^{k+r}(n)4\psi_{\infty}(f)m^{q-r+1}|m|^{k-2r}}{DwT(\psi^{-1})(k_{2k+2}^{q+1})} \varphi f(-\nu)^{-1} \int_0^\infty t^{2k-q-r} \left[ \sum_{b \in (O_k)[f]} \varphi f^{-1}(b) F_{k,q-r+1}(b/f, t) \right] dt
$$

$$
= \frac{(-1)^{k+r}(n)4\psi_{\infty}(f)m^{q-r+1}|m|^{k-2r}}{DwT(\psi^{-1})(k_{2k+2}^{q+1})} \psi f(-\nu)^{-1} \int_0^\infty t^{2k-q-r} F_{k,q-r+1}(b/f, t) dt,
$$

where in the last equality we substituted $\varphi f^{-1} = \psi f$ and took the sum out of the integral.

Now note that the integral above is exactly the integral appearing on the coefficient $c_{k,q,k-r}(b/f)$, more explicitly

$$
c_{k,q,k-r}(b/f) = \frac{2(-1)^{k+r}(k-r+1)}{(k+r)^{(k+r-1)}} \int_0^\infty t^{k-r-1}(k-r) \frac{F_{k+r+1}(b/f, t)}{\psi f(-\nu)^{-1}} dt
$$

$$
= \frac{2(-1)^{k-r+1}}{(k-r+1)} \int_0^\infty t^{k-r-1} F_{k+r+1}(b/f, t) dt.
$$

And since $(k_{2k+2}^{q+1}) = (k_{2k+2}^{q+1})_{k,q-r+1}$, we have

$$
\Lambda(F, \psi) = \frac{(-1)^{k+1}4\psi_{\infty}(f)m^{q-r+1}|m|^{k-2r}}{DwT(\psi^{-1})(k_{2k+2}^{q+1})} \sum_{b \in (O_k)[f]} \psi f^{-1}(b) c_{k,q,k-r}(b/f).
$$
The Hecke character $\psi^{-1}|_{\mathbb{A}_K}$ has conductor $\mathfrak{f}$ and infinity type $(k - q, k - r)$ and we know by Theorem 3.2 that

$$
\Lambda(\mathcal{F}, \psi^{-1}|_{\mathbb{A}_K}) = \frac{(-1)^{k+q+r}2((\psi^{-1}|_{\mathbb{A}_K})_\infty(\mathfrak{f}))}{Dw(\psi^{-1}|_{\mathbb{A}_K})_{f}} \sum_{b \in (G_K \setminus f)^*} (\psi^{-1}|_{\mathbb{A}_K})_f(b)c_{k,q,k-r}(bf).$
$$

Finally we obtain

$$
\Lambda(\mathcal{F}, \psi) = \frac{(-1)^{k+1}c(n)2\psi_\infty(\mathfrak{f})m^{q+r}|m|^{k-2r}}{Dw(\psi^{-1})\psi_f(-\nu)} \frac{(-1)^{k+q+r}2\psi^{-1}_\infty(\mathfrak{f})|\mathfrak{f}|^{2k}^{-1}}{Dw(\psi^{-1}|_{\mathbb{A}_K})_{f}} \Lambda(\mathcal{F}, \psi^{-1}|_{\mathbb{A}_K}).
$$

where we used that

$$
\psi_\infty(\mathfrak{f})^2\psi^{-1}_\infty(\mathfrak{f})^{2k} = \psi^{-1}_\infty(\mathfrak{f})^2\psi^{2k}|m|^{k-2r} = \psi^{-1}_\infty(\mathfrak{f})^{2k}(\nu)^{2k} = \nu^{k-2r} = (\nu)^{k-2r}.
$$

and $(-1)^{q+r}\psi^{-1}_\infty(\nu) = \psi^{-1}_\infty(-\nu).$ \hfill \(\square\)

**Remark 3.5.** Theorem 3.3 is not a new result, but rather a reformulation of a classical result in [JL70].

### 3.3. L-function of a p-stabilisation

Let $\mathfrak{p} = (\pi_\mathfrak{p})$ be a prime ideal of $\mathcal{O}_K$ such that $\mathfrak{p}(\mathfrak{p})$, and let $\mathcal{F}$ be a Bianchi modular eigenform of level $\Gamma_0(n)$ and weight $(k, k)$ with $\mathfrak{p} \mid n$, let $\Lambda_p$ denote the $T_p$ eigenvalue of $\mathcal{F}$, and let $\alpha_p$ and $\beta_p$ denote the roots of the Hecke polynomial $X^2 - \lambda_p X + N(p)^{k+1}$. We define the $p$-stabilisations of $\mathcal{F}$ to be

$$
\mathcal{F}^{\alpha_p}(z, t) := \mathcal{F}(z, t) - \beta_p \mathcal{G}(z, t), \quad \mathcal{F}^{\beta_p}(z, t) := \mathcal{F}(z, t) - \alpha_p \mathcal{G}(z, t).
$$

Where

$$
\mathcal{G}(z, t) = \frac{|\pi_\mathfrak{p}|^{-2} \mathcal{F}(\pi_\mathfrak{p})}{0 1} (z, t).
$$

The form $\mathcal{F}^{\alpha_p}$ (resp. $\mathcal{F}^{\beta_p}$) is an eigenform of level $\Gamma_0(n\mathfrak{p})$ and weight $(k, k)$ with $U_p$ eigenvalue $\alpha_p$ (resp. $\beta_p$).

The $L$-function of a Bianchi modular form $\mathcal{F}$ of level coprime to a prime $\mathfrak{p}$ and the $L$-function of a $p$-stabilisation $\mathcal{F}^{\alpha_p}$ are related, in fact, if we define for a Hecke character $\chi$ of conductor $\mathfrak{f}$ and $\lambda \in \mathbb{C}^*$ the factor

$$
Z^\lambda_p(\chi) := \begin{cases}
1 - \lambda^{-1}\chi(\mathfrak{p})^{-1} : \mathfrak{p} \mid \mathfrak{f}, \\
1 : \text{otherwise},
\end{cases}
$$

we have the following:
Lemma 3.6. Let \( \psi \) be a Hecke character with conductor \( \mathfrak{f} \). We have for \( \lambda \in \{ \alpha_p, \beta_p \} \)

\[
\Lambda(\mathcal{F}^\lambda, \psi) = Z_p^\lambda(\psi^{-1} | \lambda|_{\mathcal{K}}) \Lambda(\mathcal{F}, \psi),
\]

Proof. Note that \( \mathcal{G}_n \) has the following Fourier expansion

\[
\mathcal{G}_n(z, t) = |\pi_p|^{-1} \mathcal{F}_n(\pi_{p^2} z, |\pi_p| t) \left( \frac{\pi_p}{|\pi_p|} \right)^{k+1-n} t^{2k+2} n \sum_{\alpha \in K^*} c(\alpha \delta) \left( \frac{\alpha |\pi_p|}{\pi_p} \right)^{k+1-n} K_{n-1} (4\pi |\alpha |_{\mathcal{K}})^2 2\pi i (\alpha z + \alpha \pi_p x) \right],
\]

where \( c(\alpha \delta) \) are the Fourier coefficients of \( \mathcal{F} \) as in (2.5). If \( c'(\cdot) \) is the Fourier coefficient of \( \mathcal{F}^{\alpha_p} \), then

\[
c'(\alpha \delta) = c(\alpha \delta) - \beta_p c \left( \frac{\alpha \delta}{\pi_p} \right) = c(\alpha \delta) - \frac{N(p)^{k+1}}{\alpha_p} \frac{\alpha \delta}{\pi_p}
\]

where \( c \left( \frac{\alpha \delta}{\pi_p} \right) = 0 \) if \( \pi_p \nmid \alpha \).

Multiplying by \( \psi \), we obtain

\[
c'(\alpha \delta) \psi((\alpha \delta)) = c(\alpha \delta) \psi((\alpha \delta)) - \frac{N(p)^{k+1}}{\alpha_p} \frac{\alpha \delta}{\pi_p} \psi((\alpha \delta))
\]

\[
= \begin{cases} 
  c(\alpha \delta) \psi((\alpha \delta)) - \frac{\psi(p) N(p)^{k+1}}{\alpha_p} \frac{\alpha \delta}{\pi_p} \psi((\alpha \delta)) & \text{if } p \nmid \mathfrak{f} \\
  c(\alpha \delta) \psi((\alpha \delta)) & \text{if } p | \mathfrak{f}.
\end{cases}
\]

Then

\[
L(\mathcal{F}^{\alpha_p}, \psi) = \begin{cases} 
  \left( 1 - \frac{\psi(p) N(p)^{k}}{\alpha_p} \right) L(\mathcal{F}, \psi) & \text{if } p \nmid \mathfrak{f} \\
  L(\mathcal{F}, \psi) & \text{if } p | \mathfrak{f}.
\end{cases}
\]

Noting that

\[
\psi(p) N(p)^{k} = (\psi^{-1}(p) N(p)^{-k})^{-1} = (\psi^{-1}(p)|_{\mathcal{K}})^{-1}.
\]

Where \( x_p \) is the idele associated to \( p \). We have

\[
L(\mathcal{F}^{\alpha_p}, \psi) = Z_p^{\alpha_p} (\psi^{-1} | \lambda|_{\mathcal{K}}) L(\mathcal{F}, \psi)
\]

obtaining the result for the \( \Lambda \)-function of \( \mathcal{F}^{\alpha_p} \); for \( \mathcal{F}^{\beta_p} \) is analogous. \( \square \)

Remark 3.7. Depending of the behavior of \( p \), there exist four or two \( p \)-stabilisations \( \mathcal{F}_p \). Note that the level of \( \mathcal{F}_p \) is \( \Gamma_0(\mathfrak{n} \mathfrak{p} \mathfrak{f}) \) if \( p \) splits as \( \mathfrak{f} \mathfrak{p} \), and \( \Gamma_0(\mathfrak{n} \mathfrak{p}) \) if \( p \) ramifies as \( \mathfrak{p}^2 \) or remains inert as \( \mathfrak{p} \).

4. Modular symbols

In this section we introduce Bianchi modular symbols. These are algebraic analogues of Bianchi modular forms that are easier to study \( p \)-adically.

Let \( \Delta_0 := \text{Div}^0(\mathbb{P}^1(K)) \) denote the space of “paths between cusps” in \( \mathcal{H}_3 \), and let \( V \) be any right \( \text{SL}_2(K) \)-module. For a subgroup \( \Gamma \subset \text{SL}_2(K) \), denote the space of \( V \)-valued modular symbols for \( \Gamma \) to be the space

\[
\text{Symb}_\Gamma(V) := \text{Hom}_\Gamma(\Delta_0, V)
\]

of functions satisfying the \( \Gamma \)-invariance property that \( (\phi | \gamma) (D) := (\phi | \gamma D) | \gamma = \phi(D) \)
where \( \Gamma \) acts on the cusps by \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot r = \frac{ar + b}{cr + d} \).
For a ring $R$ recall the definition of $V_k(R)$ on section (2), then define $V_{k,k}(R) := V_k(R) \otimes_R V_k(R)$.

Note that we can identify $V_{k,k}(\mathbb{C})$ with the space of polynomials that are homogeneous of degree $k$ in two variables $X,Y$ and homogeneous of degree $k$ in two further variables $\overline{X}, \overline{Y}$.

This space has a natural left action of $GL_2(\mathbb{C})^2$ induced by the action of $GL_2(\mathbb{C})$ on each factor by

$$
\gamma \cdot P \left( \frac{X}{Y}, \frac{X}{Y} \right) = P \left( \frac{dX + bY}{cX + aY}, \frac{dX + bY}{cX + aY} \right), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
$$

**Remark 4.1.** This induces a right action on the dual space $V_{k,k}^*(\mathbb{C}) := \text{Hom}(V_{k,k}(\mathbb{C}), \mathbb{C})$, also note that, in particular we obtain an action of $\Gamma_0(n)$ on $V_{k,k}^*(\mathbb{C})$.

**Definition 4.2.** The space of Bianchi modular symbols of parallel weight $(k,k)$ and level $\Gamma_0(n)$ is defined to be the space

$$\text{Symb}_{\Gamma_0(n)}(V_{k,k}^*(\mathbb{C})) := \text{Hom}_{\Gamma_0(n)}(\Delta_0, V_{k,k}^*(\mathbb{C})).$$

We can also define Hecke operators on the space of modular symbols, as in section (5), where the Hecke operators allow us to endow the space of Bianchi modular forms with additional structure.

Let $\mathfrak{q} \nmid n$ be a prime ideal of $O_K$ generated by the fixed uniformiser $\pi_\mathfrak{q}$. Then for $\phi \in \text{Symb}_{\Gamma_0(n)}(V_{k,k}^*(\mathbb{C}))$ we define the Hecke operator

$$
\phi \mapsto (\phi|_{U_\mathfrak{q}}) := \sum_{b \in (O_K/\mathfrak{q})^*} \phi|_{\begin{pmatrix} 1 & b \\ 0 & \pi_\mathfrak{q} \end{pmatrix}} - \varphi_\mathfrak{q}(\pi_\mathfrak{q})^{-1} \phi|_{\begin{pmatrix} \pi_\mathfrak{q} & 0 \\ 0 & 1 \end{pmatrix}}.
$$

When $\mathfrak{q} | n$ we denote $T_\mathfrak{q}$ by $U_\mathfrak{q}$ and

$$(\phi|_{U_\mathfrak{q}}) := \sum_{b \in (O_K/\mathfrak{q})^*} \phi|_{\begin{pmatrix} 1 & b \\ 0 & \pi_\mathfrak{q} \end{pmatrix}}.$$

We can define Hecke operators over Bianchi modular symbols for each ideal $I$ of $K$ in the analogous way as for Bianchi modular forms, then the Hecke algebra acts on $\text{Symb}_{\Gamma_0(n)}(V_{k,k}^*(\mathbb{C}))$ and by sections 3 and 8 in [Hid94] and [Har87], we have:

**Proposition 4.3.** There is a Hecke-equivariant injection

$$S_{(k,k)}(\Gamma_0(n)) \to \text{Symb}_{\Gamma_0(n)}(V_{k,k}^*(\mathbb{C})), \quad \mathcal{F} \mapsto \phi_{\mathcal{F}}.$$

Let $X^{k-q}Y^q\overline{X}^{k-r}\overline{Y}^r \in V_{k,k}^*(\mathbb{C})$ be the dual of $X^{k-q}Y^q\overline{X}^{k-r}\overline{Y}^r$. We can explicitly describe the modular symbol attached to $\mathcal{F}$ at generating divisors as

$$\phi_{\mathcal{F}}(\{a\} - \{\infty\}) = \sum_{q,r=0}^k c_{q,r}(a)(\gamma - a\mathcal{X})^{k-q}X^q(\overline{\gamma} - \overline{a}\overline{\mathcal{X}})^{k-r}\overline{X}^r;$$

for $a \in K$, where $c_{q,r}(a)$ is defined in Theorem 3.2 (see Proposition 2.9 in [Wil17]).

**Remark 4.4.** The coefficient $c_{q,r}(a)$ establish a link between values of a modular symbol and critical $L$-values of the Bianchi modular form, this link is the key to the interpolation property satisfied by the $p$-adic $L$-function.

**Proposition 4.5.** Let $\mathcal{F}$ be a cuspidal Bianchi modular form in $S_{(k,k)}(\Gamma_0(n))$.

(i) The element $\phi_{\mathcal{F}} \in \text{Symb}_{\Gamma_0(n)}(V_{k,k}^*(\mathbb{C}))$ associated to $\mathcal{F}$ above is equivariant with respect to the Hecke operators.
noting that the source is well-defined since $\Gamma_0$ of the specialisation map corresponding classical modular symbol.

We say (Control theorem, corollaries 5.9 and 6.13 in [Wil17]), takes values in $V^*_k,\overline{E}$ for some number field $E$.

5. Overconvergent modular symbols

Part (ii) of Proposition allows us to see the modular symbol $\phi$ as having values in $V^*_k,\overline{L}$ for a sufficiently large $p$-adic field $L$. For suitable level groups, one can then replace this space of polynomials with a space of $p$-adic distributions and obtain the so called overconvergent modular symbols.

**Definition 5.1.** Let $A(L)$ denote the space of locally analytic functions on $O_K \otimes \mathbb{Z} \mathbb{Z}_p$ defined over $L$. We equip this space with a weight $(k,k)$-action of the semigroup

$$\Sigma_0(p) := \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in M_2(O_K \otimes \mathbb{Z}_p) : p|c, a \in (O_K \otimes \mathbb{Z}_p)^*, ad-bc \neq 0 \right\}$$

by setting

$$\gamma \cdot \zeta(z) = (a + cz)^k \zeta \left( \frac{b + dz}{a + cz} \right).$$

Denote $A_{k,k}(L)$ the space $A(L)$ equipped with the action above. Let $D_{k,k}(L) := \text{Hom}_{\mathbb{C}}(A_{k,k}(L), L)$ denote the space of locally analytic distributions on $O_K \otimes \mathbb{Z} \mathbb{Z}_p$ defined over $L$, equipped with a weight $(k,k)$ right action of $\Sigma_0(p)$ given by $\mu|\zeta(\zeta) = \mu(\gamma \cdot \zeta)$.

**Remark 5.2.** When $p$ ramifies as $p^2$ in $K$, we can consider instead $\Sigma_0(p)$, the larger group coming from condition $c \in \pi_p O_K \otimes \mathbb{Z} \mathbb{Z}_p$ for $\pi_p$ the fixed uniformiser at $p$.

For $\Gamma \subset \Sigma_0(p)$, define the space of overconvergent modular symbols of weight $(k,k)$ and level $\Gamma$ to be $\text{Symb}_{\Gamma}(D_{k,k}(L))$.

There is a natural map $D_{k,k}(L) \to V^*_k,\overline{L}$ given by dualising the inclusion of $V^*_k,\overline{L}$ into $\text{Sym}_k,\overline{L}(L)$. When $(p)\mathfrak{n}$ for $p$ split in $K$ or when $\mathfrak{p}|\mathfrak{n}$ for $p$ inert or ramified this map induces a specialisation map

$$\rho : \text{Symb}_{\Gamma_0}(D_{k,k}(L)) \to \text{Symb}_{\Gamma_0}(V^*_k,\overline{L}),$$

noting that the source is well-defined since $\Gamma_0(\mathfrak{n}) \subset \Sigma_0(p)$.

**Theorem 5.3.** (Control theorem, corollaries 5.9 and 6.13 in [Wil17]). For each prime $\mathfrak{p}$ above $p$, let $\lambda_p \in L^\times$. If $v(\lambda_p) < (k+1)/e_p$ for all $\mathfrak{p}|p$, then the restriction of the specialisation map

$$\rho : \text{Symb}_{\Gamma_0}(D_{k,k}(L)) \to \text{Symb}_{\Gamma_0}(V^*_k,\overline{L}),$$

to the simultaneous $\lambda_p$-eigenspaces of the $U_p$ operators is an isomorphism. Here recall that $e_p$ is the ramification index of $\mathfrak{p}|p$.

**Definition 5.4.** If $\mathcal{F} \in S_{k,k}(\Gamma_0(\mathfrak{n}))$ is an eigenform with eigenvalues $\lambda_p$ for $\mathfrak{p}|p$, we say $\mathcal{F}$ has small slope if $v(\lambda_p) < (k+1)/e_p$ for all $\mathfrak{p}|p$. We say $\mathcal{F}$ has critical slope if it has no small slope.

Thus if $\mathcal{F}$ has small slope, using the above control theorem, we get an associated overconvergent modular symbol $\Psi_{\mathcal{F}} \in \text{Symb}_{\Gamma_0}(D_{k,k}(L))$ by lifting the corresponding classical modular symbol.
6. **p-adic L-function**

6.1. Constructing the **p-adic L-function**.

The **p-adic L-function** of a Bianchi modular form \( \mathcal{F} \) is defined as the locally analytic distribution \( L_p(\mathcal{F}, -) \) on \( \mathcal{X}(\text{Cl}_K(p^\infty)) \), the two-dimensional rigid space of **p-adic** characters on \( \text{Cl}_K(p^\infty) := K^\times/\mathbb{A}_K^\times \otimes \mathcal{O}_p^\times \) that interpolates the classical \( L \)-values of \( \mathcal{F} \) and satisfy certain growth properties. We recall its key properties.

Recall that there is a bijection between algebraic Hecke characters of conductor dividing \( p^\infty \) and locally algebraic characters of \( \text{Cl}_K(p^\infty) \) such that if \( \psi \) corresponds to \( \psi_{p-\text{fin}} \), both are equal when we restrict to the adeles away from the infinite place and the primes above \( p \).

If \( \psi \) is an algebraic Hecke character of \( K \) of conductor \( \mathfrak{f}|p^\infty \) and infinity type \( (q, r) \), then, fixing an isomorphism \( \psi_{p-\text{fin}} \) and the primes above \( p \) all \( \mathfrak{p} \), we associate to \( \psi \) a \( K^\times \)-invariant function

\[
\psi_{p-\text{fin}}(x) : (\mathbb{A}_K^\times)_{\mathfrak{f}} \longrightarrow \mathbb{C}, \quad \psi_{p-\text{fin}}(x) := \psi_1(x) \sigma_p^{\alpha}(x),
\]

where

\[
(6.1) \quad \sigma_p^{\alpha}(x) := \begin{cases} x^q x^r & : p \text{ splits as } p\mathfrak{p}, \\ x^q x^r & : p \text{ inert or ramified.} \end{cases}
\]

**Remark 6.1.** For \( \alpha \in K^\times \), we have \( \psi_{p-\text{fin}}(x_{\alpha, p}) = (\psi_{p-\text{fin}})(\alpha) = \psi(\alpha)\alpha^\sigma_{\mathfrak{p}} \) for \( (x_{\alpha, p})_p = \alpha \) when \( p(\mathfrak{f}) \) and \( (x_{\alpha, p})_p = 1 \) otherwise.

We now can construct the **p-adic L-function** of a small slope eigenform \( \mathcal{F} \in S_{(k,k)}(\Gamma_0(n)) \). First, associate to \( \mathcal{F} \) a classical Bianchi eigensymbol \( \phi_F \) with coefficients in a **p-adic** field \( L \), and lift it to its corresponding unique overconvergent Bianchi eigensymbol \( \Psi \). Define the **p-adic** L-function of \( \mathcal{F} \) as the locally analytic distribution \( L_p(\mathcal{F}, -) \) on \( \mathcal{X}(\text{Cl}_K(p^\infty)) \) by

\[
L_p(\mathcal{F}, -) := \Psi([0] - \{\infty\})|_{(\mathfrak{f} | \mathfrak{p} \mathbb{Z}_p)^\times}.
\]

Then, \( L_p(\mathcal{F}, -) \) satisfies the interpolation and admissibility properties desired (see definitions 5.10, 6.14 in [Wil17]) and we obtain the main theorem (Theorem 7.4) in op. cit. for class number 1:

**Theorem 6.2.** Let \( \mathcal{F} \) be a cuspidal Bianchi modular eigenform of weight \( (k,k) \) and level \( \Gamma_0(n) \), where \( (p)|n \), with \( U_p \)-eigenvalues \( \lambda_p \), where \( v(\lambda_p) < (k+1)/e_p \) for all \( p \not| n \). Let \( \Omega_F \) be a complex period as in (5.2). Then there exists a locally analytic distribution \( L_p(\mathcal{F}, -) \) on \( \mathcal{X}(\text{Cl}_K(p^\infty)) \) such that for any Hecke character of \( K \) of conductor \( \mathfrak{f}|(p^\infty) \) and infinity type \( 0 \leq (q, r) \leq (k, k) \), we have

\[
(6.2) \quad L_p(\mathcal{F}, \psi_{p-\text{fin}}) = \left( \prod_{\mathfrak{p}|p} Z_p^{\lambda_p}(\psi) \right) \left[ \frac{Dw(\psi^{-1})}{(-1)^{k+q+r}2\Lambda_F \Omega_F} \right] \Lambda(\mathcal{F}, \psi),
\]

with \( Z_p^{\lambda_p}(\psi) \) as in (5.4).

The distribution \( L_p(\mathcal{F}, -) \) is \( (h_p)|_{\mathfrak{p}|p} \)-admissible, where \( h_p = v_p(\lambda_p) \), and hence is unique.

**Remark 6.3.** 1) There is a slight error in [Wil17], where the term \( Z_p^{\lambda_p} \) is incorrect in the case where \( p|f \).

2) When \( p \) ramifies as \( p^2 \), in the Theorem above it suffices \( p|n \) instead \( (p)|n \).

6.2. **Functional equation of the p-adic L-function.**
6.2.1. The small slope case.

In this section we obtain the functional equation of the $p$-adic $L$-function of a small slope $p$-stabilisation of a cuspidal Bianchi modular eigenform.

Let $F_p$ be a Bianchi modular form obtained by successively stabilising at each different prime $p$ above $p$ a newform $F \in S_{(k,k)}(\Gamma_0(n))$, with $n = (\nu)$ prime to $(p)$. Recall that $F$ is an eigenform for the Fricke involution $W_n$, with $F|_{W_n} = \epsilon(n)F$ with $\epsilon(n) = \pm 1$.

**Lemma 6.4.** For any Hecke character $\psi$ of conductor $f = (f)$ with $(f,\nu) = 1$ and infinity type $0 \preceq (q,r) \preceq (k,k)$ we have

$$
\left( \prod_{p|\ell} Z_p^{\psi}(\psi) \right) \Lambda(F_p, \psi) = \varepsilon(F, \psi) \left( \prod_{p|\ell} Z_p^{\psi_1}(\psi_1^{-1} \cdot \frac{k}{|k|_{K}}) \right) \Lambda(F_p, \psi^{-1} \cdot \frac{k}{|k|_{K}}),
$$

where $\varepsilon(F, \psi) = \left[ \frac{-\epsilon(n)\psi^{+}\tau(\psi_{|\ell}^{+})}{\psi_{1-\nu}\psi_{1-\nu}(\psi^{-1})} \right]$ and $\alpha_p$ are the $U_p$-eigenvalues of $F_p$ for each $p|\ell$.

**Proof.** By Theorem 3.4 we have

$$
\Lambda(F, \psi) = \varepsilon(F, \psi) \Lambda(F, \psi^{-1} \cdot \frac{k}{|k|_{K}})
$$

Now let $p$ be a prime over $p$, if we define $F^{\psi_p}$ as $p$-stabilisation of $F$, we obtain by Lemma 6.4 the following relations between the $\Lambda$-function of $F^{\psi_p}$ and $F$

$$
\Lambda(F^{\psi_p}, \psi) = Z_p^{\psi}(\psi^{-1} \cdot \frac{k}{|k|_{K}}) \Lambda(F, \psi),
$$

$$
\Lambda(F^{\psi_p}, \psi^{-1} \cdot \frac{k}{|k|_{K}}) = Z_p^{\psi}(\psi) \Lambda(F, \psi^{-1} \cdot \frac{k}{|k|_{K}}).
$$

Putting 6.3, 6.4 and 6.5 together

$$
Z_p^{\psi}(\psi) \Lambda(F^{\psi_p}, \psi) = Z_p^{\psi}(\psi) Z_p^{\psi}(\psi^{-1} \cdot \frac{k}{|k|_{K}}) \Lambda(F, \psi)
$$

$$
= Z_p^{\psi}(\psi) Z_p^{\psi}(\psi^{-1} \cdot \frac{k}{|k|_{K}}) \varepsilon(F, \psi) \Lambda(F, \psi^{-1} \cdot \frac{k}{|k|_{K}})
$$

$$
= \varepsilon(F, \psi) Z_p^{\psi}(\psi^{-1} \cdot \frac{k}{|k|_{K}}) \Lambda(F^{\psi_p}, \psi^{-1} \cdot \frac{k}{|k|_{K}}).
$$

Note that if $p$ is inert or ramified we are done and $F_p = F^{\psi_p}$. If $p$ split we have to do one more stabilisation, let $\overline{p}$ be the other prime above $p$.

If we define $F^{\psi_p,\overline{p}}$ as the $\overline{p}$-stabilisation of $F^{\psi_p}$ and doing the same process above, we obtain

$$
\left( \prod_{p|\ell} Z_p^{\psi}(\psi) \right) \Lambda(F^{\psi_p,\overline{p}}, \psi) = \varepsilon(F, \psi) \left( \prod_{p|\ell} Z_p^{\psi_1}(\psi_1^{-1} \cdot \frac{k}{|k|_{K}}) \right) \Lambda(F^{\psi_p,\overline{p}}, \psi^{-1} \cdot \frac{k}{|k|_{K}}).
$$

Putting $F_p = F^{\psi_p,\overline{p}}$ when $p$ split we obtain the result. \qed

**Proposition 6.5.** If $F_p$ has small slope, then for any Hecke character $\psi$ of conductor $f|(p^{\infty})$ with $f = (f)$ and infinity type $0 \preceq (q,r) \preceq (k,k)$, the distribution $L_p(F_p, \psi)$ satisfies

$$
L_p(F_p, \psi_{\psi_{\infty}}) = -\epsilon(n)N(n)^{k/2} \psi^{-1}_{\psi_{\infty}}(x_{-\nu}) L_p(F_p, \psi^{-1}_{\psi_{\infty}} \sigma_p^{k,k}),
$$

where $x_{-\nu}$ is the idele associated to $-\nu$ defined in Remark 6.1 and $\sigma_p^{k,k}$ as in equation (6.1).
Proof. By Theorem 6.2 we have the following interpolations

\[(6.6)\quad L_p(\mathcal{F}_p, \psi_{p-fin}) = \left(\prod_{p\mid p} \mathbb{Z}_p^G(\psi)\right)\left[\frac{Dw\pi(\psi^{-1})}{(-1)^{k+q+r}2\lambda_2\Omega_F}\right] \Lambda(\mathcal{F}_p, \psi)\]

\[(6.7)\quad L_p(\mathcal{F}_p, (\psi^{-1}|_{\mathfrak{a}_K})_{p-fin}) = \left(\prod_{p\mid p} \mathbb{Z}_p^G(\psi^{-1}|_{\mathfrak{a}_K})\right)\left[\frac{Dw\pi(\psi|_{\mathfrak{a}_K})}{(-1)^{k+q+r}2\lambda_2\Omega_F}\right] \Lambda(\mathcal{F}_p, \psi^{-1}|_{\mathfrak{a}_K}).\]

By (6.6), Lemma 6.4 and (6.7) we have

\[
L_p(\mathcal{F}_p, \psi_{p-fin}) = \left(\prod_{p\mid p} \mathbb{Z}_p^G(\psi)\right)\left[\frac{Dw\pi(\psi^{-1})}{(-1)^{k+q+r}2\lambda_2\Omega_F}\right] \Lambda(\mathcal{F}_p, \psi) = \left[\frac{Dw\pi(\psi^{-1})}{(-1)^{k+q+r}2\lambda_2\Omega_F}\right] E(\mathcal{F}, \psi) \left(\prod_{p\mid p} \mathbb{Z}_p^G(\psi^{-1}|_{\mathfrak{a}_K})\right) \Lambda(\mathcal{F}_p, \psi^{-1}|_{\mathfrak{a}_K})
\]

\[
= \left[\frac{Dw\pi(\psi^{-1})}{(-1)^{k+q+r}2\lambda_2\Omega_F}\right] E(\mathcal{F}, \psi) \left[\frac{Dw\pi(\psi|_{\mathfrak{a}_K})}{(-1)^{k+q+r}2\lambda_2\Omega_F}\right] L_p(\mathcal{F}_p, (\psi^{-1}|_{\mathfrak{a}_K})_{p-fin})
\]

\[
= E(\mathcal{F}, \psi) \tau(\psi^{-1}) \tau(\psi|_{\mathfrak{a}_K})^{-1} L_p(\mathcal{F}_p, (\psi^{-1}|_{\mathfrak{a}_K})_{p-fin}) = -\epsilon(n)\nu^k\psi^{-1}_f(-\nu)\psi^{-1}_{\psi^{\infty}}(-\nu) L_p(\mathcal{F}_p, (\psi^{-1}|_{\mathfrak{a}_K})_{p-fin}).
\]

By Remark 6.1 we have

\[
\psi^{-1}_f(-\nu)\psi^{-1}_{\psi^{\infty}}(-\nu) = \psi^{-1}_{p-fin}(\nu)\psi^{-1}_{p-fin}(-\nu) = \psi^{-1}_{p-fin}(x_{-\nu})
\]

and noting that for a finite idele \(x\) we have

\[
(\psi^{-1}|_{\mathfrak{a}_K})_{p-fin}(x) = \psi^{-1}_{p-fin}(x)(\mathfrak{a}_K)_{p-fin}(x) = \psi^{-1}_{p-fin}(x)\sigma(\mathfrak{a}_K)_{p-fin}(x) = (\psi^{-1}_{p-fin}\sigma(\mathfrak{a}_K)_{p-fin}(x),
\]

we obtain the result. \(\square\)

**Theorem 6.6.** For \(\mathcal{F}_p\) as above with small slope, the distribution \(L_p(\mathcal{F}_p, -)\) satisfies the following functional equation

\[
L_p(\mathcal{F}_p, \kappa) = -\epsilon(n)\kappa^{k/2}(x_{-\nu})^{-1} L_p(\mathcal{F}_p, \kappa^{-1}\sigma k k),
\]

for all \(\kappa \in \mathcal{X}(\text{Cl}_K(p^{\infty}))\).

**Proof.** Define a new distribution \(L'_p(\mathcal{F}_p, -)\) by

\[
L'_p(\mathcal{F}_p, \kappa) := L_p(\mathcal{F}_p, \kappa) + \epsilon(n)\kappa^{k/2}(x_{-\nu})^{-1} L_p(\mathcal{F}_p, \kappa^{-1}\sigma k k),
\]

for any \(\kappa \in \mathcal{X}(\text{Cl}_K(p^{\infty}))\).

Since \(\nu_p(\alpha_p) < (k + 1)/\epsilon_p\) for all \(p\mid p\), the distribution \(L_p(\mathcal{F}_p, -)\) is \((h_p)_{p\mid p}\)-admissible, where \(h_p = \nu_p(\alpha_p)\). Then \(L'_p(\mathcal{F}_p, -)\) is \((h_p)_{p\mid p}\)-admissible.

In [Loc14] it is proved that a distribution \((h_p)_{p\mid p}\)-admissible like \(L'_p(\mathcal{F}_p, -)\) is uniquely determined by its values on the \(p\)-adic characters \(\psi_{p-fin} \in \mathcal{X}(\text{Cl}_K(p^{\infty}))\) that arise from Hecke characters \(\psi\) of conductor \(\mathfrak{f}(p^{\infty})\) and infinity type \(0 \leq (q, r) \leq (k, k)\). By Proposition 6.3, we have that \(L'_p(\mathcal{F}_p, \psi_{p-fin}) = 0\) for all \(\psi_{p-fin}\), then \(L'_p(\mathcal{F}_p, -) = 0\) and the functional equation of \(L_p(\mathcal{F}_p, -)\) follows. \(\square\)
Remark 6.7. In the case when \( p \) split, the property that \( L'_p(\mathcal{F}_p, -) \) is uniquely determined by its values on the \( p \)-adic characters \( \psi_{p^{-\infty}} \) is proved in Theorem 3.11 in [Loe14] in the case where \( v_p(\alpha_p) < 1 \) for \( p \) and \( \mathfrak{F} \), which he assumes merely for simplicity. For a more detailed example of the general situation in the one variable case, see [Co10].

Example: Suppose \( p \) splits in \( K \) as \( \mathfrak{p}\mathfrak{p} \). Let \( \mathcal{F} \) be a newform with weight \((k, k)\) and level \( \mathfrak{n} \) prime to \( p \) with \( \lambda_p = \mathfrak{P}_p = 0 \). Then the Hecke polynomials at \( p \) and \( \mathfrak{F} \) coincide, and their roots \( \alpha, \beta \) both have \( p \)-adic valuation \((k+1)/2\). Assuming \( \alpha \neq \beta \) there are four choices of stabilisations of level \((p)\mathfrak{n} \) and each is small slope, giving rise to four \( p \)-adic \( L \)-functions attached to \( \mathcal{F} \), each one satisfying the corresponding \( p \)-adic functional equation of Theorem 6.6.

6.2.2. The critical slope case.

The construction of the \( p \)-adic \( L \)-function in [Wei17] and consequently the functional equation in Theorem 6.6 depend of the small slope of the Bianchi modular form \( \mathcal{F} \). In this section we generalize the functional equation of Theorem 6.6 for a base change Bianchi modular form, in particular, making no assumption about the slope. To this end we use the three-variable \( p \)-adic \( L \)-function constructed in [BSWWE18] that specialises to \( L_p(f_{j/K}, -) \), the \( p \)-adic \( L \)-function of a base change Bianchi modular form \( f_{j/K} \).

We first recall briefly the definitions and construction of such \( p \)-adic \( L \)-function. Let \( N \) be divisible by \( p \). Fix \( f \in S_{k+2}(\Gamma_0(N)) \) such that:

1. (finite slope eigenform) \( f \) is an eigenform, and \( U_p f = \lambda_p f \) with \( \lambda_p \neq 0 \);
2. (\( p \)-stabilised newform) \( f \) is new or the \( p \)-stabilisation of a newform \( f_{new} \) of level prime to \( p \);
3. (regular) if \( f \) is the \( p \)-stabilisation of \( f_{new} \), then \( \alpha_p \neq \beta_p \), and if \( p \) is inert, \( \alpha_p(f_{new}) \neq 0 \);
4. (non CM) \( f \) does not have CM by \( K \);
5. \( f \) is decent (see Definition 5.5 in [BSWWE18]).
6. \( f_{j/K} \) is \( \Sigma \)-smooth (see Definition 5.12 in [BSWWE18]).

There exists a neighbourhood \( \mathcal{V}_q \) of \( f \) in the Coleman-Mazur eigencurve such that the weight map \( w \) is étale except possibly at \( f \).

Up to shrinking \( \mathcal{V}_q \) there exists a unique \emph{rigid-analytic} function

\[
\mathcal{L}_p : \mathcal{V}_q \times \mathfrak{X}(\text{Cl}_K(p^\infty)) \to L,
\]

for sufficiently large \( L \subset \mathcal{Q}_{\mathfrak{p}} \), such that for any classical point \( y \in \mathcal{V}_q(L) \) with small slope base-change \( f_{y/K} \) we have \( \mathcal{L}_p(y, -) = c_y L_p(f_{y/K}, -) \), where \( c_y \in L^* \) is a \( p \)-adic period at \( y \) and \( L_p(f_{y/K}, -) \) is the \( p \)-adic \( L \)-function of \( f_{y/K} \) of Theorem 6.2.

Note that in [BSWWE18], \( \mathcal{L}_p \) depends of \( \phi \), a finite order Hecke character of \( K \) of conductor prime to \( p\mathcal{O}_K \), and is denoted by \( \mathcal{L}_p^\phi \). Here we take \( \phi \) to be trivial.

Remark 6.8. Shrinking \( \mathcal{V}_q \) we can suppose that there exists a Zariski-dense set \( S \subset \mathcal{V}_q \) such that for all classical point \( y \in S \) we have

(a) the base-change \( f_{y/K}^{N} \) is obtained by successively stabilising at each different prime \( p \) above \( p \) a newform \( f_{y/K} \in S_{(k,k)}(\Gamma_0(\mathfrak{n})) \), with \( \mathfrak{n} = (\nu) \) prime to \( (p) \) satisfying \( \left( \frac{N}{(N,\mathfrak{d})} \right) \mathcal{O}_K[\mathfrak{n}]N\mathcal{O}_K \);

(b) the weight of \( f_y \) is such that \( w(y) \equiv k \mod{p-1} \).
Where condition (b) comes as we are working in one of the \((p - 1)\) discs in the weight space.

For purposes of \(p\)-adic variation of the weight we have to give meaning to \(p\)-adic exponents.

### Definition 6.9.
Let \(p\mid p\) and \(s \in \mathcal{O}_p\), define the function \(\cdot^s := \exp(s \cdot \log_p(\cdot))\) in \(\mathcal{O}_p^\times\), where \(\log_p\) denotes the \(p\)-adic logarithm and \(\cdot\) is the projection of \(\mathcal{O}_p^\times\) to \(1 + p^{r_p}\mathcal{O}_p\) for \(r_p\) the smallest positive integer such that the usual \(p\)-adic exponential map converges on \(p^{r_p}\mathcal{O}_p\). Define by \(\cdot^s = \prod_p \cdot^{s_p}\) with \(s_p = (s_p)_p \in \mathcal{O}_K \otimes \mathbb{Z}_p \cong \prod_p \mathcal{O}_p\) the corresponding function in \((\mathcal{O}_K \otimes \mathbb{Z}_p)^\times\). Let \(w_{\text{TM}, p} : \mathcal{O}_p^\times \to (\mathcal{O}_p/p^{r_p})^\times \subset \mathcal{O}_p^\times\) denote the Teichmüller character at \(p\), so that for \(z \in \mathcal{O}_p^\times\), we have \(z = w_{\text{TM}, p}(z)\). Also let \(w_{\text{TM}} := \prod_p w_{\text{TM}, p}\) be the corresponding character of \((\mathcal{O}_K \otimes \mathbb{Z}_p)^\times\).

Recall the definition of \(\sigma_k^k(x)\) in equation \([6.1]\) and note that, for example, for \(x \in (\mathbb{A}_K)^f\) we have \(\sigma_k^k(x) = [x_p w_{\text{TM}}(x_p)]^k\), where \(x_p = (x_p)_p\).

### Theorem 6.10.
Let \(V_Q\) as in Remark \([6.8]\) then for every \(y \in V_Q\) of weight \(w(y)\), and all \(\kappa \in \mathfrak{X}(\text{Cl}_K(p^{\infty}))\) we have
\[
L_p(y, \kappa) = -\epsilon(n) w_{\text{TM}}(N(n))^{k/2}(N(n))^{w(y)/2} \kappa(x_{-\nu, p})^{-1} L_p(y, \kappa^{-1} w_{\text{TM}}(\cdot)^{w(y)}),
\]
where \(\epsilon(n) = \pm 1\) is the eigenvalue of \(f_{\nu/K}\) for the Fricke involution \(W_n, x_{-\nu, p}\) is the idele associated to \(-\nu\) in \([6.7]\).

**Proof.** Consider the set \(S \subset V_Q\) as in Remark \([6.8]\) for every \(y \in S\) of weight \(w(y) = k_y\), we have by Theorem \([6.6]\) that for all \(\kappa \in \mathfrak{X}(\text{Cl}_K(p^{\infty}))\), the distribution \(L_p(f_{\nu/K})\) satisfies
\[
L_p(f_{\nu/K}, \kappa) = -\epsilon(n) N(n)^{k/2} \kappa(x_{-\nu, p})^{-1} L_p(f_{\nu/K}, \kappa^{-1} \sigma_p^k, k_y),
\]
in particular multiplying both sides by the \(p\)-adic period \(c_y\), we have
\[
L_p(y, \kappa) = -\epsilon(n) N(n)^{w(y)/2} \kappa(x_{-\nu, p})^{-1} L_p(y, \kappa^{-1} \sigma_p^{w(y), w(y)})
= -\epsilon(n) [w_{\text{TM}}(N(n))\left(N(n))^{w(y)/2} \kappa(x_{-\nu, p})^{-1} L_p(y, \kappa^{-1}\{w_{\text{TM}}(\cdot)^{w(y)}\}),
\]
noting that in the second equality \(w_{\text{TM}}(N(n))\) and \(\{N(n)\}\) are well defined because \(p \nmid n\) for all \(p\). Since we suppose \(w(y) = k\) \((mod\ p - 1)\) for every classical point \(y \in S\) and such points are Zariski-dense on \(V_Q\), then the functional equation hold for every \(y \in V_Q\).

### Corollary 6.11.
Let \(\mathcal{F}_p\) be as in Section \([6.2.4]\) suppose that \(\mathcal{F}_p\) has critical slope and \(\mathcal{F}\) is the base change of \(f \in S_\Delta(\Gamma_0(N))\), a newform of finite slope, regular, non CM, decent and \(\Sigma\)-smooth. Then for all \(\kappa \in \mathfrak{X}(\text{Cl}_K(p^{\infty}))\), the distribution \(L_p(\mathcal{F}_p, \kappa)\) satisfies the following functional equation
\[
L_p(\mathcal{F}_p, \kappa) = -\epsilon(n) N(n)^{k/2} \kappa(x_{-\nu, p})^{-1} L_p(\mathcal{F}_p, \kappa^{-1} \sigma_p^k, k_y),
\]

**Proof.** Specialise the functional equation in Theorem \([6.10]\) at the \(p\)-stabilisation of \(f\) whose corresponding base change to \(K\) is \(\mathcal{F}_p\).

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