Fractal Behavior of the Shortest Path Between Two Lines in Percolation Systems

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Using Monte-Carlo simulations, we determine the scaling form for the probability distribution of the shortest path, ℓ, between two lines in a 3-dimensional percolation system at criticality: the two lines can have arbitrary positions, orientations and lengths. We find that the probability distributions can exhibit up to four distinct power law regimes (separated by cross-over regimes) with exponents depending on the relative orientations of the lines. We explain this rich fractal behavior with scaling arguments.

I. INTRODUCTION

There has been considerable recent activity analyzing $P(\ell|r)$, the probability distribution for the length of shortest path, $\ell$, between two points separated by Euclidean distance $r$ in a percolation system. This paper extends that work by determining the scaling form of the distribution of shortest paths between two lines of arbitrary position, relative orientation and lengths in 3-dimensional systems. These configurations are important because they much more accurately model the configurations used in oil recovery in which fluid is injected in one well (one of the lines in our configuration) and oil is recovered at a second well (the second line in our configuration); the wells may, in reality, be at arbitrary orientation and of different lengths.

The scaling form for the 2-points configuration in which the 2 points are located at $(L-r/2, L/2, L/2)$ and $(L+r/2, L/2, L/2)$ in a system of side $L$ has been found to be

$$ P(\ell|r) \sim \frac{1}{r^{d_{\min}}} \left( \frac{\ell}{r^{d_{\min}}} \right)^{-g_\ell} f_1 \left( \frac{\ell}{r^{d_{\min}}} \right) f_2 \left( \frac{\ell}{L^{d_{\min}}} \right), $$

where

$$ f_1(x) = e^{-ax^\phi} $$

and

$$ f_2(x) = e^{-bx^{\psi}}. $$

The exponents $g_\ell$, $d_{\min}$, $\phi$, and $\psi$ are universal and the constants $a$ and $b$ depend on lattice type. In 3D, the values of these exponents have previously been found to be $g_\ell = 2.3 \pm 0.1$, $d_{\min} = 1.39 \pm 0.05$, $\phi = 2.1 \pm 0.5$, and $\psi = 2.5 \pm 0.5$. The first stretched exponential function, $f_1$, reflects the fact that the shortest path must always be at least equal to the distance $r$ between the two points; the second stretched exponential function, $f_2$, reflects the fact that the lengths of the shortest paths are bounded because of the finite size, $L$, of the system.

We find that the scaling form for the 2-lines configuration has the same form as that found for the shortest path distribution between two points with the exceptions that: (i) the power law regime of the distribution as represented by the term $(\ell/\ell^{d_{\min}})^{-g_\ell}$ in Eq. (1) is replaced with up to four different power law regimes (separated by cross-over regimes) with exponents depending on the relative orientations of the lines and (ii) the Euclidean distance, $r$, in Eq. (1) between the two points is replaced by the shortest Euclidean distance between the two lines. The lengths of the lines affect the sizes of the power law regimes.

II. SIMULATIONS

Monte Carlo simulations were performed using the Leath method and growing clusters from 2 sets of seeds—one for each line. The length of the shortest path between the two lines is the sum of the chemical distances from each set of seed sites to the point where a cluster started at one set of seeds meets a cluster started from the other set of seeds. The cluster growth for a given realization is terminated when the two clusters meet. For parallel line configurations, in which the probability distributions decay rapidly, we use the method of Ref. 10 to obtain good statistics for shortest paths that have very low probabilities. We use the memory management technique described in 11 to perform simulations in which the growing clusters never hit a boundary of the system.

The clusters that are created and included in our analysis are of all sizes, not just the incipient infinite cluster.

III. NON-PARALLEL WELLS

A. Co-Planar Lines

1. Equal Length Symmetric Lines

We start by considering relatively simple configurations of the type shown in Fig. 1(a) in which the lines
are co-planar, of equal length and are positioned symmetrically. We will study configurations in which the lines are of unequal length [see Fig. 1(b)] and/or are not positioned symmetrically [see Fig. 1(c)] in the following sections. In all of these configurations, \( r \) is the shortest Euclidean distance between the two lines.

Figure 2 contains log-log plots of \( P(\ell|r) \), the shortest path distribution for \( r = 8 \) and various values of \( \theta \). We have chosen \( r = 8 \) so that the initial cutoff is present; for smaller values of \( r \), lattice effects destroy this initial cutoff. Since the focus of this paper is the power law regimes, not the initial or final cutoffs, in all later figures we will choose configurations with \( r = 1 \) so that the extent of the power law regimes is as long as possible. The exception to this will be cases in which \( \theta \) is very small where small \( r \) introduces other lattice effects.

We note that after the initial peak in each distribution, there is a power-law regime, the slope of which, \( g_{1}(\theta) \), increases with increasing \( \theta \). We will call this power law regime the “2-lines regime.” Simple scaling arguments imply that if the lengths of the lines are infinite, these 2-lines regimes would continue indefinitely. For finite line lengths, we would expect that, for large \( \ell \), the distributions would exhibit a crossover to a power law regime with the same exponent as that for a configuration with two points—because for large \( \ell \) the long paths travel far enough away from the lines that they appear to be points. In this regime, the power law exponent has the value of that of two points, 2.35. For the plots in Fig. 3 in order to see the power regimes clearly, we have chosen the lengths of the lines long enough that this crossover occurs after the maximum value of \( \ell \) in the plots.

In Fig. 3 we plot \( g_{1}(\theta) \) versus \( \theta \). The plot suggests that \( g_{1}(\theta) \) diverges as \( \theta \) goes to zero. We attempt to fit this function with a power of \( 1/\sin((\theta)/2) \) and find the best fit with the function

\[
f(\theta) = \frac{g_{1}(180^\circ)}{\sin(\theta/2)^{0.4}}.
\]

This form is not based on any fundamental theory; it simply has the properties that \( f(\theta = 180^\circ) = g_{1}(\theta = 180^\circ) \), it diverges as \( \theta \) goes to zero and it fits the intermediate points reasonably well.

The crossover between the 2-line regime and the 2-point regime is illustrated in Fig. 4, where in each panel we plot \( P(\ell|r) \) for fixed \( \theta \), and various values of \( W \). As expected, the larger the length of the lines, the higher the value of \( \ell \), the value of \( \ell \) at which the crossover occurs. Quantitative analysis of the crossover behavior is given in Section III.C.

### 2. Point-Line Configurations

We next study configurations in which one line has zero length (i.e., a point) and the other is a line of finite length. This is the extreme case of the configuration in which the two lines have different length. We will study the case where both lines have finite length in the next section. We in fact study the three configurations shown in Figs. 3(a)–(c). The plots of \( P(\ell|r) \) for these configurations are shown in Fig. 4. The plots have a power-law regime with exponent \(-1.75\) for the configurations of Fig. 3(a) and Fig. 3(c), and exponent \(-2.2\) for the configuration of Fig 3(b). We denote this regime the “point-line regime.” Fig. 4 shows the crossover from point-line behavior to 2-points behavior.

### 3. Unequal Length Symmetric Lines

We can now study configurations of the type shown in Fig. 1(b) in which the lines are of different lengths, \( W_{1} \) and \( W_{2} \). For such a configuration we would expect three power-law regimes: (i) for small \( \ell \) such that the two lines appear to be infinite, a 2-line regime, with slope dependent on \( \theta \), (ii) a point-line regime, with slope \(-1.75\), for values of \( \ell \) large enough that the shorter line appears to be a point, and (iii) a 2-point regime, for even larger values of \( \ell \) where both lines appear to be points. Plots of \( P(\ell|r) \) for such configurations are shown in Fig. 5 and are consistent with our expectations.

### 4. Complex Configurations (Unequal Length Non-Symmetric Lines)

The last of the co-planar configurations is of the type shown in Fig. 1(c). In general, based on the reasoning above, for configurations of this type we would expect \( P(\ell|r) \) to have four power-law regimes. For the configuration shown in Fig. 1(c), in which \( W_{1a} \ll W_{2} \ll W_{1b} \), the power law regimes would be as follows: (i) a power law regime corresponding to the angle \( \theta \) between the segments \( W_{1a} \) and \( W_{2} \), (ii) a power law regime corresponding to the angle \( \pi - \theta \) between segments \( W_{2} \) and \( W_{1b} \), (iii) a point-line power law regime entered when \( \ell \gg W_{1b} \), and (iv) the 2-points regime. Figure 6 is a plot of \( P(\ell|r) \) which shows this behavior.

### B. Non-Coplanar Lines

For non-coplanar lines, for \( l \gg r \), the fact that the lines are not co-planar should be irrelevant; what is relevant is the effective angle between the lines. This angle is obtained by sliding the lines toward each other along the line of shortest Euclidean distance between the lines (without changing their orientations) until they touch; the lines are then co-planar and the angle between them determines the behavior of \( P(\ell|r) \). Figure 4 contains plots for two configurations which illustrate this: (i) two coplanar lines with \( r = 1 \), \( \theta = 90^\circ \), and \( W = 256 \), and (ii) the same configuration with the second line translated out of the plane by distance 8. We see that while
there is some difference in the plots for small \( l \), the slope of the 2-lines regime is the same for the two plots.

C. Scaling of the Crossover Between Power Law Regimes

We define the value of \( \ell \equiv \hat{\ell} \) at which \( P(\ell|r) \) crosses over from one power-law regime to another power-law regime as the value of \( \ell \) where straight lines fit to the power law regimes between which the crossover takes place, cross. In Eq. (3) the values of \( \ell \) at which the lower and upper cutoffs occur scale independently as \( r^{d_{\ell \min}} \) and \( L^{d_{\ell \min}} \), respectively. By extension, we would expect that all characteristic values of the distribution, including crossovers between different power-law regimes, would also scale as \( X^{d_{\ell \min}} \) where \( X \) is the length in the system which controls the crossover. Thus, in analogy with the scaling of the most probable value of \( \ell, \ell^* \),

\[
\ell^* = c r^{d_{\ell \min}}, \tag{5}
\]

we would, in fact, expect that the value of \( \ell, \hat{\ell} \) at which \( P(\ell|r) \) crosses over from 2-lines behavior to 2-points behavior scales as

\[
\hat{\ell} = c_1(\theta) r^{d_{\ell \min}}, \tag{6}
\]

where

\[
r_{\max} = r + 2W \sin(\theta/2) \tag{7}
\]

is the maximum Euclidean distance between the two lines and \( c_1(\theta) \) is a slowly varying function of \( \theta \). In order for a 2-lines regime to exist, the 2-lines regime cutoff \( \hat{\ell} \) must be greater than \( \ell^* \), the maximum value of the distribution. That is,

\[
c_1[r + 2W \sin(\theta/2)]^{d_{\ell \min}} > c r^{d_{\ell \min}}, \tag{8}
\]

which implies

\[
W > \frac{(c/c_1)^{1/d_{\ell \min}} - r}{2 \sin(\theta/2)} \tag{9}
\]

In Figs. (a), (b), and (c), the insets contain plots of \( \hat{\ell} \) vs. \( r_{\max} \). For \( \theta = 30^\circ \), the scaling exponent is consistent with Eq. (6) but for \( \theta = 20^\circ \) and \( 180^\circ \) the scaling exponent is 1.0 ± 0.1.

Using the same reasoning which led to Eq. (3), we would expect the crossover from point-line to 2-points behavior to scale as

\[
\hat{\ell} = c_2 W^{d_{\ell \min}}, \tag{10}
\]

because \( W \) is the length which controls this crossover; as seen in Fig. (b), the larger the value of \( W \), the larger the value of \( \ell \) at which the crossover from point-line to 2-points behavior occurs. However, as seen in the inset in Fig. (c), the crossover length scales with an exponent 1.0 ± 0.1 not \( d_{\ell \min} \).

Finally, we would expect that for different length lines, the crossover from 2-lines behavior to 2-point behavior would scale as

\[
\hat{\ell} = c_3(\theta) W_2^{d_{\ell \min}}, \quad (W_2 < W_1) \tag{11}
\]

because \( W_2 \) is the length which controls this crossover; as seen in Fig. (a), the larger the value of \( W_2 \), the larger the value of \( \ell \) at which the crossover occurs. Again, the insets in Fig. (c) indicate that the crossover scales with exponent 1.0 ± 0.1.

We cannot explain why sometimes the crossover scales with \( d_{\ell \min} \) and sometimes it scales with an exponent about 1. It is, of course, possible that corrections-to-scaling are strong and that we are not seeing the true asymptotic behavior of the scaling of the crossover. If this is the case, the question still remains as to why the corrections to scaling are strong in some configurations and not in others. This area is a subject for further study.

IV. PARALLEL WELLS

A. Simple Configurations

As with non-parallel wells we first consider the simple configurations shown in Fig. (a) in which the parallel wells are of the same length. Figure (a) plots \( P(\ell|r) \) vs \( \ell \) for \( r = 1 \) and various \( W \). The initial decay of the plots increases with increasing \( W \) because the longer the wells, the lower the probability for long shortest paths. Eventually, all plots cross over to a power-law regime with slope consistent with that for two points. To see if this initial decay is a lattice effect, Fig. (b) plots of the scaled distributions \( r^{d_{\ell \min}} P(\ell|r^{d_{\ell \min}}W) \) for various \( r \) and \( W \) where the aspect ratio,

\[
R = \frac{W}{r}, \tag{12}
\]

is fixed at \( R = 32 \). Changing \( r \) and \( W \) but keeping \( R \) fixed results in scaling all lengths in the geometry by the same factor and the plots collapse as expected. Again, we note the initial strong decay of the distribution followed by a 2-point power-law regime. The good collapse for small \( x = 1/r^{d_{\ell \min}} \) indicates that the strong initial decay is not a lattice effect.

Because of the small values of \( \ell \) at which the crossover to the 2-point regime occurs it is difficult to differentiate between a power-law and (stretched) exponential decay. We will proceed as if the decay were either a power law with slope \( g(R) \) or equivalently an exponential with “effective slope” \( g(R) \).

One might argue as follows that the initial decay for power parallel lines must be exponential: Since the 2-lines regime of the probability distribution for a parallel
well configuration must always decay faster than the 2-lines regime of a configuration with small but non-zero \( \theta \) and since we believe \( g_\ell(\theta) \) goes to infinity as \( \theta \) goes to zero, the decay for parallel lines must be exponential (i.e., faster than any power law decay). This, however, need not be the case. In order for a 2-lines regime to exist, Eq. (1) must hold. So as we decrease \( \theta \), we must increase \( W \), increasing the aspect ratio \( R \), to maintain a 2-lines regime. But since the effective slope for parallel wells, \( g(R) \) increases with increasing \( R \), the decay can be a power law and still always have a greater slope than the configuration with small but non-zero \( \theta \).

**B. Complex Configurations**

The treatment of the more complex configurations shown in Figs. 1(b) and (c) follows that of non-parallel wells. \( P(\ell) \) for configurations of the type in Fig. 1(b) would contain an initial 2-line regime with slope \( g(R = W_2/r) \), a point-line regime, and finally a two-point regime. \( P(\ell) \) for configurations of the type in Fig. 1(c), with \( W_{1b} < W_2 < W_{1a} \) would contain an initial 2-line regime with slope \( g(R = W_{1b}/r) \), a 2-line regime with slope \( g_\ell(\theta = \pi) \), a point-line regime, and a 2-point regime.

**C. Quasi-Euclidean Regime**

When the length of the wells is very large and the distance between the wells is small the behavior of the most probable shortest path between the wells is the same as in a Euclidean space where \( p = 1 \) and all bonds are occupied. This can be seen in Fig. 4 in which we plot \( \ell^* \), the most probable value of the shortest path, versus \( r \) for various lengths \( W \). For long enough wells, there is a regime of \( r \) in which

\[
\ell^* = r, \quad (13)
\]

as one would expect in Euclidean space in which the shortest path is a straight line path of occupied bonds. As also seen in Fig. 4, for a given well length, as \( r \) increases, there is a value of \( r^* \), \( r^* \), at which the behavior crosses over to that of 3D percolation. We can develop a simple expression for \( r^* \) as follows: The probability that all bonds in a straight line path between two wells separated by distance \( r \) are occupied is \( p_c^r \). The probability that one or more bonds in the straight line path is not occupied is thus \( 1 - p_c^r \) and the probability that one or more bonds in the \( W \) straight line paths between the wells are not occupied is \( (1 - p_c^r)^W \). The probability that at least one straight line path has all bonds occupied is then

\[
P(r, W) = 1 - (1 - p_c^r)^W. \quad (14)
\]

The shortest path will exhibit Euclidean behavior, i.e., \( \ell^* = r \) when \( P(r, W) \) is of the order unity. Setting \( P(r^*, W) = a \) in Eq. (14), we find

\[
r^* = \frac{\ln(1 - a^{1/W})}{\ln p_c}. \quad (15)
\]

In Fig. 7 we plot the observed values of \( r^* \) and the values predicted by Eq. (15) with a value of \( a = 0.55 \) which gives the best fit to the observed values.

**V. RELATIONSHIP BETWEEN PARALLEL WELLS AND “CLOSE TO PARALLEL” WELLS**

For a given \( r \), we expect that a configuration with small but non-zero angle will have a power-law regime slope very close to the (effective) power-law regime slope of a configuration of parallel lines with the same \( W \). This at first leads to a seeming paradox: if we increase or decrease \( W \), but keep the angle of the non-parallel wells fixed, the slope of the 2-lines regime of the non-parallel well configuration doesn’t change as discussed in Section III.A.1. However, if we consider this configuration as a parallel configuration, changing \( W \) changes the aspect ratio which changes the power-law regime slope as discussed in Section IV.A. This seeming inconsistency is resolved as follows: on the one hand, for a 2-lines regime to exist, \( W \) must be at least as large as the value given by Eq. (1). If \( W \) is too small, there will be no 2-lines regime and both the parallel and small angle configurations will look like the configuration for 2 points. On the other hand, if \( W \) is increased, keeping \( r \) and \( \theta \) fixed, the greater the deviation from parallel lines and there is no reason why the parallel and small-angle configurations should have the same slopes in their power-law regimes.

Thus, only for the very small range of \( W \) for which the power-law regime exists and for which the configuration with small but non-zero \( \theta \) is “close to parallel” (i.e., the difference between the values of \( r \) and \( r_{\text{max}} \) is small) should the slopes of the parallel configuration and the configuration with small but non-zero \( \theta \) be equal. That is,

\[
g(W/r) \approx g_\ell(\theta), \quad (16)
\]

where \( g(R) \) is defined in Section IV.A.

**VI. DISCUSSION AND SUMMARY**

Motivated by the need to more realistically model the geometries found in oil recovery activities, we have determined the scaling form for the distribution of shortest paths between two lines in 3 dimensional percolation systems. Using simple scaling arguments we explained the rich fractal behavior of the shortest path in these systems. A number of open questions, however, remain:
(i) From first principles, can one develop an expression for \( g_\ell(\theta) \)? An exact expression for \( g_\ell \) for two points in 2-dimensions was obtained by Ziff \cite{Ziff99} using conformal invariance arguments. Possibly this approach could be extended to find \( g_\ell \) for point-line and 2-line configurations, at least in 2-dimensions.

(ii) How is the fact that the crossover from one power-law regime to another does not scale with the exponent \( d_{\min} \) explained? Is this simply an artifact of corrections-to-scaling which would disappear if we could simulate much larger systems or is the scaling of the crossover actually anomalous in certain configurations?

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FIG. 1. Example configurations of two non-parallel lines which are studied. (a) Simple configuration of lines of equal length. (b) Configuration of lines of unequal length ($W_1 > W_2$). (c) Configuration in which shortest distance between lines does not terminate at the ends of lines ($W_{a1} < W_2 < W_{1b}$).

FIG. 2. $P(\ell|r)$ vs $\ell$ for configuration of two lines of equal length with $r = 8$, $\theta = (\text{from bottom to top})$ $3^\circ$ (filled square), $6^\circ$, $12^\circ$, $20^\circ$, $40^\circ$, and $180^\circ$ (unfilled square). The corresponding well lengths $W$ are $4890$, $2445$, $1224$, $737$, $374$ and $128$, respectively. The plots are normalized such that the initial sections of plots are coincident.
FIG. 3. $g_r(\theta)$ vs $\theta$. The solid line is a plot of Eq. (4).
FIG. 4. $P(\ell|r)$ vs $\ell$ for configuration of two lines of equal length. (a) $r = 8$, $\theta = 3^\circ$, $W = (\text{from top to bottom})$ 38, 76, 152, 304, 1216, and 2432 (b) $r = 1$, $\theta = 29^\circ$, $W = (\text{from bottom to top})$ 8, 17, 33, 66, 132, 265, and 529, (c) $r = 1$, $\theta = 180^\circ$, $W = (\text{from bottom to top})$ 8, 16, 32, 64, and 128. For all plots, the larger the value of $W$, the larger the value of $\ell$ at which behavior changes from 2-lines behavior to 2-points behavior for which the slope is $-2.35$. The insets plot the crossover value, $\hat{\ell}$, vs. $r_{\text{max}}$. 
FIG. 5. Example configurations in which one line is of finite length $W$ and one is of zero length (i.e., a point). In all cases, the shortest distance between the point and the line is $r$.

FIG. 6. $P(\ell|r)$ vs $\ell$ for configuration of a point and a line with $r = 1$ and $W = 128$. From top to bottom, the plots are for the configurations shown in Figs. 5(a), (c), and (b) respectively. We see that the slopes in configurations where the point is closest to the end of the line [Fig. 5(a) and (c)] are the same (with some initial difference) and they are different from the slope in the configurations in which the point is closest to the middle of the line (Fig. 5(b)).
FIG. 7. $P(\ell|r)$ vs $\ell$ for configuration of a point and a line in which the point is closest to the end of the line with $r = 1$ and $W = (\text{from bottom to top})$ 2, 4, 8, 16, 32, 64, and 128. For all plots, the larger the value of $W$, the larger the value of $\ell \sim \hat{\ell}$ at which the behavior changes from point-line behavior to 2-points behavior. The inset plots $\hat{\ell}$ vs. $W$. 
FIG. 8. $P(\ell|r)$ vs $\ell$ for configurations of two lines of different lengths with $r = 1$ and $W_1 = 128$. (a) $\theta = 7^\circ$, $W_2 = (\text{from top to bottom})$ 16, 32, and 64. Three power law regimes can be seen: the 2-lines regime, the point-line regime and the 2 points regime. (b) $\theta = 180^\circ$, $W_2 = (\text{from bottom to top})$ 4, 8, 16, 32, 64, and 128. Only the first two power law regimes can be seen: the 2-lines regime and the point-line regime (the 2-points regime would require even larger values of $\ell$).
FIG. 9. $P(\ell|r)$ vs $\ell$ for configurations of two lines of different lengths which “overlap” [see Fig. 1(c)] with $\theta = 7^\circ$, $r = 1$, $W_{1a} = 32$, $W_{1b} = 128$, and $W_2 = 256$. Four power law regimes are present: the 2-line ($\theta = 7^\circ$) regime (slope $-3.0$), the 2-line ($\theta = 180^\circ - 7^\circ$) regime (slope $-1.2$), the point-line regime (slope $-1.75$), and the 2-points regime (slope $-2.35$).
FIG. 10. $P(\ell|r)$ vs $\ell$ for configurations of two lines of equal length. The co-planar configuration has $r = 1$, $\theta = 90^\circ$, and $W = 256$ and the lines are co-planar. The non-coplanar configuration is obtained from the co-planar configuration by moving one of the lines a distance $8$ perpendicular to the plane defined by the coplanar lines. One sees that for large $\ell$, the power law regimes of the two plots have the same exponent.

FIG. 11. Example configurations for parallel wells. (a) Simple configuration of wells of equal length. (b) Configuration of wells of unequal length ($W_1 > W_2$). (c) Configuration in which shortest line between end of one well does not terminate at end of other well ($W_{a1} < W_2 < W_{1b}$).
FIG. 12. $P(\ell | r)$ vs $\ell$ for configurations of two parallel lines of equal length with $r = 1$ and $W = (\text{from top to bottom})$ 0 (two points), 4, 8, 16, and 32. The slopes of the power law regimes of the plots for all configurations is the same but the initial decay of the plots increases with increasing $W$. 
FIG. 13. $P(\ell|r)$ vs $\ell$ for configurations of two parallel lines of equal length with $(W,r) =$ (from top to bottom) (32,1), (64,2), (96,3), (128,4), (160,5), and (196,6) (b) plots of (a) scaled with the variable $x = \ell/r^{d_{\text{min}}}$. The plots in (b) collapse nicely as would be expected since they all have the same aspect ratio, $W/r$. The good collapse for small $x$ indicates that the small $x$ behavior is not a lattice effect.
FIG. 14. Most probable $\ell$ vs $r$ for configurations of two parallel lines of equal length. (a) $W = 16$, $r = 1, 2, 4, 8,$ and $14$. (b) $W = 32$, $r = 1, 2, 3, 4, 8,$ and $16$. (c) $W = 64$, $r = 2, 4, 5, 6, 7, 8, 16,$ and $32$. (d) Combined plot of (a), (b) and (c). The upper and lower dashed lines have slope $d_{\text{min}}$ (1.374) and 1.0, respectively. The larger the value of $W$, the larger the value of $r$ at which scaling crosses over from Euclidean behavior to fractal behavior.
FIG. 15. Value of $r$ at which behavior changes from Euclidean to fractal, $r^*$, $W$. The dashed line is a prediction of Eq. (15).