Have we already detected astrophysical symptoms of space-time noncommutativity?

Takashi Tamaki *, Tomohiro Harada,† Umpei Miyamoto,‡ and Takashi Torii §

Department of Physics, Waseda University, Ohkubo, Shinjuku, Tokyo 169-8555, Japan

Research Center for the Early Universe, University of Tokyo, Hongo, Bunkyo, Tokyo 113-0033, Japan

Advanced Research Institute for Science and Engineering, Waseda University, Ohkubo, Shinjuku, Tokyo 169-8555, Japan

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We discuss astrophysical implications of \( \kappa \)-Minkowski space-time, in which there appears space-time noncommutativity. We first derive a velocity formula for particles based on the motion of a wave packet. The result is that a massless particle moves at a constant speed as in the usual Minkowski space-time, which implies that an arrival time analysis by \( \gamma \)-rays from Markarian (Mk) 421 does not exclude space-time noncommutativity. Based on this observation, we analyze reaction processes in \( \kappa \)-Minkowski space-time which are related to the puzzling detections of extremely high-energy cosmic rays above the Greisen-Zatsepin-Kuzmin cutoff and of high-energy (\( \sim 20 \) TeV) \( \gamma \)-rays from Mk 501. In these analyses, we take into account the ambiguity of the momentum conservation law which can not be determined uniquely from a mathematical viewpoint. We find that peculiar types of momentum conservation law with some length scale of noncommutativity above a critical length scale can explain such puzzling detections. We also obtain stringent constraints on the length scale of noncommutativity and the freedom of momentum conservation law.

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I. INTRODUCTION AND OUTLINE

Recently, much attention has been paid to extremely high-energy cosmic rays (EHECRs) which have energies above that attained in any experimental apparatus on Earth [1,2]. It has been pointed out that these EHECRs provide an opportunity to investigate space-time properties on very short length scales or very high energy scales. The most striking feature is that some of these detections seem to be inconsistent with existing physics, in which such detections would be restricted by the Greisen-Zatsepin-Kuzmin (GZK) cutoff [3]. That is, if we consider the interaction between EHECRs and CMB photons, particles with energy \( \gtrsim 7 \times 10^{19} \) eV from distant sources cannot reach the Earth. There is also another anomalous phenomenon similar to this. That is detections of \( \gamma \)-rays above \( \sim 20 \) TeV from distant sources (\( \gtrsim 100 \) Mpc) reported in Refs. [4,5]. These \( \gamma \)-rays are expected to interact with infrared background (IRBG) photons and not to reach the Earth in a LI scenario [6]. In spite of exhaustive research, near sources which can explain such detections has not been found. Though there are many attempts explaining these anomalous phenomena, there is no absolute solution at present [7].

This could imply an encounter with new physics. Some authors argue that violation of Lorentz invariance (LI) might solve EHECRs above GZK cutoff [3,4]. LI violation might also explain detections of \( \gamma \)-rays above \( \sim 20 \) TeV. This possibility has been argued in Refs. [13,10].

One of the ways to introduce LI violation is to consider space-time noncommutativity with deformed LI, which has received attention in recent years since it naturally arises in the contexts of string/M theories [17,22]. It has also been argued that space-time uncertainty which comes from a fundamental string scale may be related to space-time noncommutativity [28].

Apart from string/M theories, space-time noncommutativity also arises as a result of deformation quantization [2]. Amelino-Camelia et al. [25,27] considered an interesting toy model called \( \kappa \)-Minkowski space-time where noncommutativity is introduced as \( [x^i, \tau] = i\lambda x^i \), where \( \lambda \) is a free length scale and the index \( i \) runs over 1, 2, 3 [29,31]. They
obtained a severe constraint on $\lambda$ through an arrival time analysis of signals from a $\gamma$-ray burst \[25,26\]. If we accept this scenario, there is no room for detectable symptoms such as anomalous threshold to explain EHECRs \[27\].

In these papers, the velocity of particles was evaluated using a group velocity formula in the usual Minkowski space-time. Here, we derive a more realistic velocity formula based on the motion of a wave packet in $\kappa$-Minkowski space-time. With this formula, we find that the space-time noncommutativity does not affect the velocity of massless particles. Motivated by this observation, we analyze reaction processes which are related to both detections of EHECRs beyond the GZK cutoff and of $\sim 20$ TeV photons. In particular, we pay attention to the momentum conservation law which has some ambiguities in this model. We propose to determine the form of the momentum conservation law by deciding whether or not space-time noncommutativity is consistent with observations. In fact, we can exclude some forms of momentum conservation. Though our approach is purely kinematical, our result will provide a strong motivation to consider realistic model of space-time noncommutativity \[28\]. Throughout this paper, we use the units in which $c = \hbar = 1$.

II. $\kappa$-MINKOWSKI SPACE-TIME

We briefly review $\kappa$-Minkowski space-time. The basic commutation relations are

\[ [x^i, t] = i\lambda x^i, \quad [x^i, x^j] = 0, \]  

(1)

where the indices $i, j$ run 1, 2, 3. We can define differentiation, integration \[29\] and Fourier transformation in this space-time \[30\]. In order to define Fourier transformation consistently, the energy $E$ and the momentum $p = (p_1, p_2, p_3)$ of a particle form a non-Abelian group $G$ which can be written in a matrix form as,

\[ (E, p) := \begin{pmatrix} e^{\lambda E} p_1 & p_2 & p_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]  

(2)

Thus, if we denote the additive operator in $\kappa$-Minkowski space-time by $\hat{+}$ to distinguish from the conventional one, we can write as

\[ (E_1 + E_2, p_1 + p_2) := (E_1, p_1)(E_2, p_2) = (E_1 + E_2, p_1 + e^{\lambda E_1}p_2) \]  

(3)

Following Ref. \[26\], we describe a plane wave as

\[ \psi(E, p) = e^{i\mathbf{p} \cdot \mathbf{x}} e^{iEt}, \]  

(4)

and place the $t$ generator to the right of $x$ generator, i.e., $\psi(E, p) \neq e^{iEt}e^{\mathbf{p} \cdot \mathbf{x}}$. Then, the property

\[ \psi(E_1, p_1)(E_2, p_2) = \psi(p_1, E_1)\psi(p_2, E_2), \]  

(5)

is found. We can also define the wave in the reverse direction as

\[ \psi(E, p)^{-1} := e^{-i\mathbf{p}e^{-\lambda E} \mathbf{x}} e^{-iEt} = e^{-iEt}e^{-i\mathbf{p} \cdot \mathbf{x}}, \]  

(6)

which implies that $(E, p)^{-1}$ is an inversion of $(E, p)$.

Because of these noncommutative structures, modification of Poincaré invariance is required to describe physics in a covariant way \[31\]. The rotation and boost generators can be written as

\[ M_i = -\epsilon_{imn}p_m \frac{\partial}{\partial p_n}, \]

\[ N_i = p_i \frac{\partial}{\partial E} - \left( \frac{\lambda}{2}p^2 + \frac{1 - e^{2\lambda E}}{2\lambda} \right) \frac{\partial}{\partial p_i} + \lambda p_i p_j \frac{\partial}{\partial p_j}. \]  

(7)

(8)

Using \(8\), a finite boost transformation for the $i = 1$ direction can be obtained as \[32\]
\begin{equation}
\begin{split}
p_1 &= \frac{\tanh(\lambda m) \sinh \xi}{\lambda [1 - \tanh(\lambda m) \cosh \xi]}, \\
p_2 &= p_3 = 0, \\
E &= m + \frac{1}{\lambda} \ln \left( \frac{1 - \tanh(\lambda m)}{1 - \tanh(\lambda m) \cosh \xi} \right),
\end{split}
\end{equation}

where \( \xi \) is a boost parameter and we choose \( p = 0 \) and \( E = m \), i.e., \( m \) is a rest mass of the particle, for \( \xi = 0 \).

Because of (8), the dispersion relation is altered as

\begin{equation}
\lambda^2 (e^{\lambda E} + e^{-\lambda E} - 2) - p^2 e^{-\lambda E} = K^2,
\end{equation}

where \( K \) is a constant with the dimension of mass. If we take a rest frame, this can be expressed as

\begin{equation}
\lambda^2 (e^{\lambda m} + e^{-\lambda m} - 2) = K^2.
\end{equation}

III. THE VELOCITY FORMULA

Here, we derive a new velocity formula which is one of the main results of this paper. The velocity of the particle in the usual Minkowski space-time is

\begin{equation}
v = \frac{dE}{dp}.
\end{equation}

If we apply this in \( \kappa \)-Minkowski space-time, \( |v| = e^{-\lambda E} \) is obtained for a massless particle, where we used Eq. (12). This formula, together with the data on \( \gamma \)-rays associated with Markarian (Mk) 421 in Ref. [33] leads to the constraint

\begin{equation}
|\lambda| \lesssim 10^{-33} \text{ meter} \quad [23,20,33].
\end{equation}

Since this discussion depends crucially on the form of Eq. (14), i.e., on what is the velocity, we reexamine the group velocity formula by forming a wave packet in \( \kappa \)-Minkowski space-time as a more realistic situation. For this purpose, we consider infinitesimal changes \( \Delta E \) and \( \Delta p \) in \( E \) and \( p \), respectively, as a result of adding \((\Delta E', \Delta p')\) as

\begin{equation}
(E \hat{+} \Delta E', p \hat{+} \Delta p') = (E + \Delta E, p + \Delta p).
\end{equation}

In this case, we can express \((\Delta E', \Delta p')\) as

\begin{equation}
(\Delta E', \Delta p') \cong (\Delta E, \frac{\Delta p}{e^{\lambda E}}),
\end{equation}

where we keep only terms in first order in \( \Delta E \) and \( \Delta p \). By using Eqs. (3) and (16), we make a wave packet as follows

\begin{equation}
I = \psi_{(E-\Delta E, p-\Delta p)} + \psi_{(E+\Delta E, p+\Delta p)} \\
= \psi_{(E, p)}[e^{-i\Delta E' \cdot \Delta p'} + \psi_{(E', p)}(\Delta E', \Delta p')] \\
= \psi_{(E, p)}[e^{-i\Delta E' \cdot x} e^{-i\Delta E' t} + e^{i\Delta p' \cdot x} e^{i\Delta E' t}] \\
\simeq 2e^{i\Delta p' \cdot x} e^{iEt} \cos \left[ \frac{\Delta p}{e^{\lambda E}} \left( x + \frac{e^{\lambda E} \Delta E t}{\Delta p} \right) \right].
\end{equation}

By considering \( |I|^2 \), the group velocity \( v_t \) can be written as

\begin{equation}
v_t := e^{\lambda E} \frac{dE}{dp}.
\end{equation}

We also consider a similar relation

\begin{equation}
(\Delta E', \Delta p' + p) = (E + \Delta E, p + \Delta p),
\end{equation}

which is different from Eq. (15) due to noncommutativity. In this case, the corresponding group velocity \( v_r \) is
Using (12) and (13), we obtain the important conclusion that \textit{massless particles move in a constant speed} \(|\mathbf{v}_r| = |\mathbf{v}_i| = 1\) \textit{as in the usual Minkowski space-time for arbitrary} \(\lambda\) \([33]\). Therefore, the argument in Ref. \[24\] does not apply. In this case, there appears the possibility that the large value of \(\lambda \gtrsim 10^{-33}\) may solve the puzzling problems of EHECRs above GZK cutoff and of \(\sim 20\) TeV photons simultaneously. We investigate this possibility next. However, we emphasize on the importance of the result \textit{not} because \(\kappa\)-Minkowski space-time can avoid the constraint \textit{but} because our result provides an opportunity to reconsider LI deformation models in general.

### IV. THRESHOLD ANOMALY

We first consider the two-body head-on collision of particles and subsequent creation of two particles \(1 + 2 \rightarrow 3 + 4\). We define the energy \(E_i\) and momentum \(\mathbf{p}_i\) of the \(i\)-th particle as those in the laboratory frame. We denote the rest mass of the \(i\)-th particle as \(m_i\). We also assume that \(m_2 = 0, m_3 \neq 0, m_4 \neq 0\) and \(\mathbf{p}_i = (p_i, 0, 0)\). In the usual Minkowski space-time, we use the dispersion relation

\[
E_i^2 - p_i^2 = m_i^2,
\]

and the energy momentum conservation law,

\[
p_1 + p_2 = p_3 + p_4, \quad E_1 + E_2 = E_3 + E_4,
\]

to obtain the threshold value of \(E_1\), which we denote by \(E_{\text{th},0}\). We assume that the resultant particles are at rest in the center-of-mass frame in the threshold reaction. In the laboratory frame, this means that the resultant particles move in the same speed, that is

\[
\frac{p_3}{m_3} = \frac{p_4}{m_4}.
\]

We also assume that \(p_2\) has an opposite sign against that of \(p_1\). If we neglect higher order terms in \(E_2\), then

\[
E_{\text{th},0} = \frac{(m_3 + m_4)^2 - m_1^2}{4E_2}.
\]

We also define the threshold value of \(p_1\) as \(p_{\text{th},0}\) which can be approximated as \(p_{\text{th},0} \sim E_{\text{th},0}\).

Next, we consider the same reaction in \(\kappa\)-Minkowski space-time. Eq. (21) is replaced by

\[
\lambda^{-2}(e^{\lambda E_i} + e^{-\lambda E_i} - 2) - (p_i)^2 e^{-\lambda E_i} = \lambda^{-2}(e^{\lambda m_i} + e^{-\lambda m_i} - 2).
\]

If we interpret the algebra in \(\kappa\)-Minkowski space-time faithfully, the energy momentum conservation law is expressed as

\[
(E_1, p_1)(E_2, p_2) = (E_3, p_3)(E_4, p_4).
\]

Even if it holds, one should note that we need a rule to distinguish two particles. If we consider the collision of two particles with \(A, B \in G\), respectively, does it correspond to \(AB, BA\) or anything else? At present, we have no way to determine it. Amelino-Camelia et al. \[24\] proposed to find the rule by experiments. Here, we introduce a phenomenological parameter \(a\), which controls the form of conservation law as follows:

\[
a(E_1, p_1)(E_2, p_2) + (1 - a)(E_2, p_2)(E_1, p_1)
\]
\[= a(E_3, p_3)(E_4, p_4) + (1 - a)(E_4, p_4)(E_3, p_3).
\]

As regards plausible values for \(a\), care must be taken. If we consider two particles of the same species, \(a = 1/2\) would be physically reasonable value, since if they have same energy and move opposite direction each other, they have zero total momentum only for this choice. In fact, the parameter \(a\) may be a function of physical quantities of two particles.
such as mass, charge and/or spin for two different species. Here, we use the same value of $a$ on the left and the right hand sides of (28) for convenience. Moreover, we restrict our attention to $0 \leq a \leq 1$ for clarity.

We also need to impose the condition that the resultant particles are at rest in the center-of-mass frame. To obtain a relation between momenta $p_3$ and $p_4$, we use the boost transformation (1). For the same value of $\xi$, we obtain

$$\frac{p_3}{\tanh(\lambda m_3)} = \frac{p_4}{\tanh(\lambda m_4)}. \quad (29)$$

We can solve $E_1$ as a function of $a$, $\lambda$, $m_1$, $m_3$, $m_4$ and $E_2$ by using (26), (28) and (29). We apply this result to two astrophysical cases.

A. Threshold anomaly for TeV $\gamma$-rays

Here, we consider the process $\gamma + \gamma \to e^+ + e^-$, which may occur when a $\gamma$-ray travels in the IRBG. In this case, $m_1 = 0$ and $m_3 = m_4 = m_e$, where $m_e$ is the electron mass. If we assume the existence of IRBG photons ($0.2 \lesssim E_2 \lesssim 5$ eV) then the threshold is $E_{th,0} \sim 1$ TeV in Minkowski space-time. Then, the reported detection of $\sim 20$ TeV photons from Mk501 ($\sim 150$ Mpc from the Earth) would be difficult to explain (4).

We summarize the equation for the threshold in $\kappa$-Minkowski space-time which is derived from (26), (28) and (29) as

$$AB = yx(yx + 1)^2 \sinh^2 \frac{\lambda m_e}{2}, \quad (30)$$

where

$$A := (1 - a)y^4 - (1 - 2a)y^2 - a, \quad B := ax^4 + (1 - 2a)x^2 + a - 1, \quad (31)$$

and $x := e^{\lambda E_1/2}$ and $y := e^{\lambda E_2/2}$. Since we are considering the collision of two particles of the same species, $a = 1/2$ would be physically reasonable. Note that, though we have $E_{th,0} \approx p_{th,0}$ for high energy particles in the usual Minkowski space-time, this is not the case in $\kappa$-Minkowski space-time.

We should recall that, to estimate the energy of primary particles, we calculate the sum of energy of all secondary particles. Since energy is conserved in the usual sense even in $\kappa$-Minkowski space-time, we do not need to take into account the effect of space-time noncommutativity to estimate the energy of primary particles. Thus, the observation of $\sim 20$ TeV photons in usual Minkowski space-time has the same meaning also in $\kappa$-Minkowski space-time. On the other hand, the usual sum of momenta of all secondary particles does not coincide with the momentum of the primary particle in this space-time. Therefore, if $p_{th}$ could be evaluated independently of the observation of $E_{th}$, it might become important to extract information about space-time noncommutativity through the detection of violation of the usual momentum conservation. We exhibit properties of both the energy and the momentum from this reason.

We first show the dependence of $E_{th}$ and $p_{th}$ on $\lambda > 0$ in Figs. 3 (a) and (b), respectively. For simplicity, $E_2$ is chosen as $E_2 = 1$ eV for IRBG photons. For $a = 0$, $E_{th}$ and $p_{th}$ increase with $\lambda$, compared with the same quantities in Minkowski space-time. In particular, $E_{th}$ and $p_{th}$ diverge for $\lambda := \lambda_c \sim 4$ TeV$^{-1}$. That is, the universe is entirely transparent for $\lambda > \lambda_c$. For $a = 1/2$ and 1, $p_{th}$ increases with $\lambda$, though $E_{th}$ decreases.

For all $a$, a first-order correction in $\lambda$ arises for $p_{th}$. If we expand $p_{th}$ as $p_{th} = \sum_{k=0}^{\infty} \frac{p_{th, k}}{k!} \lambda^k$, the first-order coefficient $p_{th, 1}$ is written as

$$p_{th, 1} = p_{th, 0} \left[ p_{th, 0} (1 - a) + E_2 \left( a - \frac{1}{2} \right) \right]. \quad (33)$$

On the other hand, the first-order correction in $\lambda$ for $E_{th}$, which we denote $E_{th, 1}$, is written as

$$E_{th, 1} = E_{th, 0} (E_{th, 0} - E_2) \left( \frac{1}{2} - a \right). \quad (34)$$

Thus, it disappears for $a = 1/2$.

The reason why $E_{th}$ and $p_{th}$ disappear for $a = 0$ above $\lambda_c \sim 4$ TeV$^{-1}$ can be understood as follows. For $\lambda E_{th} \gg 1$, and $\lambda m_e$, $\lambda E_2 \ll 1$, we can approximate eq. (30) as
\begin{align}
(1 + \lambda E_2)x &\approx \lambda E_2 - \frac{\lambda^2 m_e^2}{2} \quad \text{for } a = 0, \\
\alpha E_2 x &\approx \frac{\lambda m_e^2}{4} \quad \text{for } a \neq 0.
\end{align}

In this range of approximation, since \( \lambda E_{th,0} = \lambda m_e^2/E_2 \) is expected to be larger than 1, eq. (35) has no real solution \( E_{th} \), while a real solution \( E_{th} \) exists for \( a \neq 0 \). This means that \( \lambda_e \) for \( a = 0 \) is characterized by \( 1/E_{th,0} \sim 1 \) TeV\(^{-1} \).

For \( \lambda E_{th}, \lambda m_e, \lambda E_2 \gg 1 \), we can summarize the results as follows. For \( a = 0 \), eq. (35) is approximated as \( y^4 x^2 \sim (xy)^3 e^{\lambda m_e}/4 \), which yields \( E_1 = -2m_e + E_2 < 0 \). This contradicts the first assumption. So a solution does not exist. In a similar way, we can show that \( E_1 \) approaches \( 2m_e + E_2 \) and \( 2m_e - E_2 \) for \( a = 1 \) and for \( a \neq 0,1 \), respectively.

To investigate properties for \( \lambda < 0 \), we replace \( \lambda \) with \( -\lambda \). In eq. (36), this corresponds to the replacement \( x \rightarrow 1/x \) and \( y \rightarrow 1/y \). We find that eq. (36) becomes identical if \( a \) is also replaced by \( (1 - a) \).

Thus, the case \( a \ll 1, \lambda \gtrsim 4 \) TeV\(^{-1} \), and the case \( 1 - a \ll 1, -\lambda \gtrsim 4 \) TeV\(^{-1} \) remain as candidate solutions for \( \sim 20 \) TeV photons. On the other hand, we can exclude \( a = O(1) \) and \( \lambda \gtrsim 10 \) TeV\(^{-1} \), or \( 1 - a = O(1) \) and \( -\lambda \gtrsim 10 \) TeV\(^{-1} \) from the present experimental data.

\section*{B. Threshold anomaly for GZK cutoff}

Here, we consider the interaction of ultra high energy protons with CMB photons (\( \sim 10^{-3} \)eV) which results in a pair production \( p + \gamma \rightarrow p + \pi_0 \). In this case, \( m_1 = m_3 = m_p \) and \( m_4 = m_{\pi} \), where \( m_p \) and \( m_{\pi} \) are the proton mass and the pion mass, respectively. Because of \( E_{th,0} \sim 7 \times 10^{18} \) eV, it is difficult for EHECRs above \( E_{th,0} \) to reach the Earth from cosmologically distant sources.

We solve (28), (29) and (30) numerically. We show the dependence of \( E_{th} \) and \( p_{th} \) for \( \lambda > 0 \), in Figs. 2 (a) and (b), respectively. \( E_2 \) is chosen as \( E_2 = 10^{-3} \) eV for CMB photons. Compared with Fig. 1, we find that the qualitative features for small \( \lambda \) are quite similar, i.e., \( E_{th}/E_{th,0} > 1 \) for \( a = 0 \) and \( E_{th}/E_{th,0} < 1 \) for \( a = 1 \), while \( p_{th}/p_{th,0} > 1 \) for all cases.

However, we find qualitative differences from Fig. 1 for \( \lambda \gtrsim 10^{-8} \) TeV\(^{-1} \). The threshold disappears for \( \lambda > \lambda_c \sim 2 \times 10^{-8} \) TeV\(^{-1} \) in the \( a = 0 \) case, which can be explained as in the \( \gamma \) -ray case since \( \lambda_c \) coincides approximately with \( 1/E_{th,0} \). For the case \( a = 1/2 \) and \( 1 \), \( E_{th}/E_{th,0} \) increases with \( \lambda(\gtrsim 3 \times 10^{-8} \) TeV\(^{-1} \) and disappears for \( \lambda \gtrsim 5 \times 10^{-8} \) TeV\(^{-1} \), unlike the \( \gamma \) -ray case.

In this case, there is no simple symmetry about \( \lambda \rightarrow -\lambda \), as found in the previous case. Thus, we also show the dependence of \( E_{th} \) and \( p_{th} \) for \( \lambda < 0 \), in Figs. 3 (a) and (b), respectively. We have a crucial difference from the \( \gamma \) -ray case even for \( \lambda \ll -10^{-9} \) TeV\(^{-1} \). For \( a = 1 \), there is a value \( E_{th2} \) over which the reaction does not occur. We denote \( E_{th2} \) by a dot-dashed line. \( E_{th2} \) diverges as \( \lambda \rightarrow -0 \) and merges with \( E_{th} \) at \( \lambda = \lambda_c \sim -7 \times 10^{-9} \) TeV\(^{-1} \).

We find that the behavior for small \( |\lambda| \) is that \( E_{th}/E_{th,0} > 1 \) for \( a = 1 \) and \( E_{th}/E_{th,0} < 1 \) for \( a = 0 \) as in the \( \gamma \) -ray case for \( \lambda < 0 \). For \( a = 1/2 \) and \( 1 \), \( E_{th} \) decreases with \( \lambda(\lesssim -5 \times 10^{-8} \) TeV\(^{-1} \)).

Thus, the \( a \ll 1 \) case for \( \lambda \gtrsim 2 \times 10^{-8} \) TeV\(^{-1} \), the \( a = O(1) \) case for \( \lambda \gtrsim 5 \times 10^{-8} \) TeV\(^{-1} \) and the \( (1 - a) \ll 1 \) case for \( \lambda \lesssim -7 \times 10^{-9} \) TeV\(^{-1} \) remains as candidate explanations for detections of super GZK events. For \( (1 - a) = O(1) \), we can exclude \( \lambda \lesssim -10^{-7} \) TeV\(^{-1} \).

\section*{V. CONCLUSION AND DISCUSSION}

We have first considered a velocity formula to describe the particle motion based on the motion of a wave packet in \( \kappa \)-Minkowski space-time. In this formula, space-time noncommutativity does not affect the motion of a massless particle. Thus, an arrival time analysis of \( \gamma \) -ray bursts in Refs. 28-33 does not exclude space-time noncommutativity in this model. Since this feature had not been discussed so far, it should be stressed and is one of our main conclusions here.

Based on this consideration, we have obtained threshold values for reactions \( \gamma + \gamma \rightarrow e^+ + e^- \) and \( p + \gamma \rightarrow p + \pi_0 \) in \( \kappa \)-Minkowski space-time and analyzed their relevance to the puzzling observations of \(~20\) TeV photons and EHECRs above the GZK cutoff, introducing a parameter \( a \) to take into account the ambiguity of the momentum conservation law.

In the TeV \( \gamma \) -ray case, though \( a = 1/2 \) is favorable in the physical context, only \( a \ll 1 \) for \( \lambda \gtrsim 4 \) TeV\(^{-1} \), or \( (1 - a) \ll 1 \) for \( \lambda \lesssim -4 \) TeV\(^{-1} \) appear able to explain the detections of \( \gamma \) -rays above \(~20\) TeV. The possibilities \( a = O(1) \) for \( \lambda \gtrsim 10 \) TeV\(^{-1} \), or \( (1 - a) = O(1) \) for \( \lambda \lesssim -10 \) TeV\(^{-1} \), are excluded.
In the EHECR case, we cannot assign definite values to a, because it may depend on, e.g., masses and/or charges of two particles. The possibilities $a \ll 1$ and $\lambda \gtrsim 2 \times 10^{-8} \text{ TeV}^{-1}$, or $a = O(1)$ and $\lambda \gtrsim 5 \times 10^{-8} \text{ TeV}^{-1}$ remain viable. We can exclude cases in which $(1 - a) = O(1)$ and $\lambda < \sim -10^{-7} \text{ TeV}^{-1}$.

Thus, $a \ll 1$ for $\lambda > \sim 4 \text{ TeV}^{-1}$ or $(1 - a) \ll 1$ for $\lambda < \sim 4 \text{ TeV}^{-1}$ appear able to explain both phenomena. Our results are important because they suggest that extremely high-energy particles might be expected in realistic models with space-time noncommutativity. If this is the case, then we might have already detected symptoms of the space-time noncommutativity.

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FIG. 1. Threshold anomaly for TeV-γ rays for λ > 0. (a) λ-Eth, (b) λ-pth are plotted for E2 = 1 eV. For a = 0.5 and 1, Eth decreases with λ increases for λ > 1 TeV⁻¹, while pth monotonically increases. The a ≪ 0 case is only desirable to explain ∼20 TeV photons. It is noted that Eth is invariant under the transformation λ → −λ and a → (1 − a).
FIG. 2. Threshold anomaly for GZK cutoff for $\lambda > 0$. (a) $\lambda$-$E_{th}$, (b) $\lambda$-$p_{th}$ are plotted for $E_2 = 10^{-3}$ eV. Though qualitative features for small $\lambda$ are similar to those of Fig. 1, they show drastic difference from Fig. 1 for $\lambda \gtrsim 3 \times 10^{-8}$ TeV$^{-1}$. 
FIG. 3. Threshold anomaly for GZK cutoff for $\lambda < 0$. (a) $\lambda\cdot E_{th}$, (b) $\lambda\cdot p_{th}$ are plotted for $E_2 = 10^{-3}$ eV and $\lambda < 0$. Unlike the case $\lambda > 0$, the threshold vanishes only the $a = 1$ case for $\lambda \lesssim -7 \times 10^{-9}$ TeV$^{-1}$. 