Test Matter in a Spacetime with Nonmetricity

Yuval Ne’eman
Raymond and Beverly Sackler Faculty of Exact Sciences
Tel-Aviv University, Tel-Aviv, Israel 69978
and
Center for Particle Physics, University of Texas,
Austin, Texas 78712

Friedrich W. Hehl
Institute for Theoretical Physics, University of Cologne
D-50923 Köln, Germany

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Abstract

Examples in which spacetime might become non-Riemannian appear above Planck energies in string theory or, in the very early universe, in the inflationary model. The simplest such geometry is metric-affine geometry, in which nonmetricity appears as a field strength, side by side with curvature and torsion. In matter, the shear and dilation currents couple to nonmetricity, and they are its sources. After reviewing the equations of motion and the Noether identities, we study two recent vacuum solutions of the metric-affine gauge theory of gravity. We then use the values of the nonmetricity in these solutions to study the motion of the appropriate test-matter. As a Regge-trajectory like hadronic excitation band, the test matter is endowed with shear degrees of freedom and described by a world spinor.
1 The Case for Metric-Affine Gravity

Even though Einstein’s treatment of spacetime as a Riemannian manifold appears fully corroborated experimentally, there are several reasons to believe that the validity of such a description is limited to macroscopic structures and to the present cosmological era. Indications [1] from the only available finite perturbative treatment of quantum gravity – namely the theory of the quantum superstring – point to non-Riemannian features on the scale of the Planck length. On the other hand, recent advances in cosmogony, i.e. in the study of the early universe, as represented by the inflationary model, involve, in addition to the metric tensor, at the very least a scalar dilaton [2] induced by a Weyl geometry, i.e. again an essential departure from Riemannian metricity.

Allowing minimal departures from Riemannian geometry (i.e. from a \(V_4\) manifold) would consist in allowing torsion (i.e. a \(U_4\)) and nonmetricity (i.e. an \((L_4, g)\)). Andrzej Trautman, to whom this article is dedicated on the occasion of his 64th birthday, has made important contributions to the study of the first suggestion [3, 4], namely the possibility of a spacetime with torsion \(T^\alpha \neq 0\). In this work, we would like to sketch some of the features relating to the second possibility, namely to the assumption that spacetime is endowed with nonmetricity \(Q_{\alpha\beta} \neq 0\).

We have recently reviewed [6] the class of gravitational theories with such geometries, the Metric-Affine Gauge Theories of Gravity (or ‘metric-affine gravity’ MAG for short). As in any gauge theory, the geometrical fields of gravity are induced by matter currents. In Einsteinian gravity, it is the symmetric (Hilbert) energy-momentum current which acts as a source for the metric field and the Riemannian curvature. In MAG, we have, in addition, the spin current and the dilation plus shear currents inducing the torsion and nonmetricity fields, respectively. Both spin and dilation plus shear are components of the hypermomentum current, symmetric for dilation plus shear and antisymmetric for spin.

And yet, there is a rather profound difference between these two physical features. Special Relativity (SR) is synonymous with Poincaré invariance, which includes the Noether conservation of angular momentum. It would

\[\text{This is a ‘positive’ paraphrasing of the more conventional ‘negative’ assertion, namely that spacetime does not fulfill the Riemannian metricity constraint } Q_{\alpha\beta} = -D g_{\alpha\beta} = 0 \]
then be relatively straightforward, at the level of Relativistic Quantum Field Theory (RQFT), to constrain the kinematics so as to do away with the orbital part of angular momentum and thus obtain a conserved spin current. However, even this is unnecessary, since through its Pauli-Lubański realization, spin itself is related to the density of a Poincaré group invariant and is thus an ‘absolute’ of SR. The conservation of shear, on the other hand, is not a characteristic of SR and would require the homogeneous Lorentz group – or its double-covering group \( SL(2, C) = \overline{SO}(3, 1) \) – to be embedded in the larger \( \overline{SL}(4, R) \). Such switching from Poincaré to affine \( A(4, R) = R^4 \rtimes \overline{SL}(4, R) \) is, however, implied in our having given up the Riemannian metricity condition, since we have thereby also lost the presence of the pseudo-orthogonal group as the local symmetry of the tangent manifold, i.e. the local Lorentz frames and with them the Equivalence Principle, with the direct transition to SR. Indeed, this is the ‘meaning’ of our basic non-Riemannian ansatz, namely that we are studying phenomena and situations in which there is no conventional ‘flat’ SR limit – either in the small, when approaching Planck length, or in the early universe, during inflation, within Planck times from the ‘seeding’ vacuum fluctuation ‘event’. Presumably, it is then through a spontaneous breakdown of the local \( \overline{A}(4, R) \) symmetry below Planck energies, down to Poincaré invariance, that SR and the Riemannian metricity condition set in (see [7, 8] for such examples). Alternatively, we might be dealing with situations in which the dynamics have led to boundary conditions generating shear currents – quadrupolar pulsations of nuclear or hadron matter in the small [9], e.g., or the Obukhov-Tresguerres hyperfluid [10] in macroscopic configurations.

2 World Spinors as Matter Fields

The unavailability of local Lorentz frames poses no problem in the context of boson fields. The latter are conventionally represented by tensors, i.e. linear field representations of \( SL(4, R) \). These become world tensors in the transition from Special to General Relativity, i.e. nonlinear realisations of the group of local diffeomorphisms \( Diff(4, R) \), carried linearly through the \( SL(4, R)_H \) holonomic linear subgroup. In RQFT, the fact that tensor fields are built to carry the action of a group larger than that allowed by SR, is taken care of through subsidiary conditions, etc. Thus, the (symmetric)
metric tensor density’s 10 components $\sigma^{ij}$, as defined in GR through the action of $SL(4, R)$, e.g. through

$$\sigma^{ij} := 2(-g)^{-1/2} \delta L/\delta g_{ij}, \quad (1)$$

are a good example of a 10-dimensional $SL(4, R)$-irreducible multiplet then reducing under $SO(1,3)$ (or $SL(2, R)$) into $9 + 1$ – the segregation of the ‘1’ being assured through the removal of the trace, indeed a Lorentz scalar. In any case, boson fields are naturally constructed so as to be capable of carrying the action of $SL(4, R)$, instead of the Lorentz group, whether in a local frame or holonomically.

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This is not true of the conventional fermion fields we use to represent matter. These are spin $|J| = 1/2$ field representations of the double-covering of the Lorentz or Poincaré groups, i.e. of $Spin(1, 3) = SL(2, C)$ or of $R^4 \cong SL(2, C)$ and can only carry – at best – nonlinear realisations of $A(4, R)_H \subset Diff(4, R)$. Linear action can nevertheless be realised, through the use of infinite-component manifields, linear field representations of the double-covering of the linear, affine and diffeomorphism groups [11]. Such fields can be used in a Riemannian context and even in SR, as well as in our present metric-affine geometry. In the former, they are particularly suited for the description of hadrons and nuclei, composite objects displaying excitation bands [9]. These phenomenological features have no other description in the framework of an effective field theory. Fermionic hadrons or nuclei are then assigned to spinor manifields – world spinors in GR – and boson excitation bands to the related boson manifields (‘infinitensors’). As to the non-Riemannian scenarios of super-Planckian energies or of the early universe, spinor manifields enter naturally in the context of quantum superstring theory.

As field representations of $SL(4, R)$, world spinors can also be assigned to a local $SL(4, R)_A$ anholonomic frame, as well as serving holonomically and carrying the action of $Diff(4, R) \supset SL(4, R)_H$. The different spin levels in a world spinor are related by the

$$|\delta J| = 2 \quad (2)$$

spin-raising and spin-lowering action of the gravitational field. In its absence, i.e. in SR, world spinor manifields reduce to an (infinite) direct sum of Lorentz
spinor fields – a reduction similar in principle to that what happens to the $\sigma^{ij}$ tensor in our example above; moreover, the anholonomic spinor manifolds can be assigned to the more elegant multiplicity-free representations. World spinor manifolds, however, cannot stay in such representations; transvection of an anholonomic spinor manifold into a world spinor, using countable-infinite vielbeins, destroys the multiplicity-free feature (see [12], also Chapter 4 and Appendices C1-C6 in [6]), as exemplified by the Mickelsson equation [13].

3 Geometrical Fields, Currents and Equations of Motion

We denote the frame field by

$$e_\alpha = e^i_\alpha \partial_i,$$

and the coframe field by

$$\vartheta^\beta = e_j^\beta dx^j.$$  

The $GL(4, R)$-covariant derivative for a tensor valued $p$-form is

$$D = d + \Gamma_\alpha^\beta \rho(L^\alpha_\beta) \wedge,$$  

where $\rho$ is the representation of $GL(4, R)$ and $L^\alpha_\beta$ are the generators; the connection one-form is $\Gamma_\alpha^\beta = \Gamma_{i\alpha}^\beta dx^i$. The nonmetricity is a one-form

$$Q_{\alpha\beta} := -Dg_{\alpha\beta},$$

the torsion and curvature are two-forms

$$T^\alpha := D\vartheta^\alpha,$$

$$R_\alpha^\beta := d\Gamma_\alpha^\beta - \Gamma_{\alpha\gamma}^\beta \wedge \Gamma_\gamma^\beta.$$  

The Weyl one-form

$$Q := (1/4) Q_\gamma^\gamma,$$

when subtracted from the nonmetricity, yields the traceless nonmetricity

$$\mathcal{Q}_{\alpha\beta} := Q_{\alpha\beta} - Q g_{\alpha\beta}.$$  

The Bianchi identities are

\[ DQ_{\alpha\beta} \equiv 2R_{(\alpha\beta)}, \]
\[ DT^{\alpha} \equiv R_{\gamma}^{\ \alpha} \wedge \partial^{\gamma}, \]
\[ DR_{\alpha}^{\ \beta} \equiv 0. \]

Physics-wise, \( Q_{\alpha\beta} \), \( T^{\alpha} \) and \( R_{\alpha}^{\ \beta} \) play the role of field strengths.

We now turn to the source-currents for the fields above. These will depend on the Lagrangian (\( \Psi \) is a matter manifold),

\[ L_{\text{tot}} = L_{\text{tot}}(g_{\alpha\beta}, dg_{\alpha\beta}, \vartheta^{\alpha}, \partial^{\alpha}, \Gamma_{\alpha}^{\ \beta}, d\Gamma_{\alpha}^{\ \beta}, \Psi, D\Psi), \]

which can be rewritten in a covariantized form as

\[ L_{\text{tot}} = L_{\text{tot}}(g_{\alpha\beta}, Q_{\alpha\beta}, \vartheta^{\alpha}, T^{\alpha}, R_{\alpha}^{\ \beta}, \Psi, D\Psi). \]

Separating the Lagrangian \( L_{\text{tot}} = V_{\text{MAG}} + L \) into geometrical \( V_{\text{MAG}} \) and matter \( L \) parts, the matter current three-forms are then given by the Euler-Lagrange functional derivatives (denoted by \( \delta \)) of the material piece \( L \). We have the canonical energy-momentum current

\[ \Sigma_{\alpha} := \delta L/\delta \vartheta^{\alpha} = \partial L/\partial \vartheta^{\alpha} + D(\partial L/\partial T^{\alpha}), \]

the hypermomentum current

\[ \Delta_{\alpha}^{\beta} := \delta L/\delta \Gamma_{\alpha}^{\ \beta} = (L_{\beta}^{\alpha} \Psi) \wedge (\partial L/\partial (D\Psi)) + 2g_{\alpha\gamma}(\partial L/\partial Q^{\alpha\gamma}) + \vartheta^{\alpha} \wedge (\partial L/\partial T^{\beta}) + D(\partial L/\partial R_{\alpha}^{\ \beta}), \]

and also a related ‘current’, which is a four-form, the (symmetric) metric energy-momentum, which we used in \([I]\) as an example of a tensor which reduces under SR, namely

\[ \sigma^{\alpha\beta} := 2\delta L/\delta g_{\alpha\beta} = 2\partial L/\partial g_{\alpha\beta} + 2D(\partial L/\partial Q_{\alpha\beta}). \]

Then the field equations turn out to be \([E]\)

\[ \delta L/\delta \Psi = 0 \quad (\text{matter}), \]
\[ DM^{\alpha\beta} - m^{\alpha\beta} = \sigma^{\alpha\beta} \quad (0\text{th}), \]
\[ DH_{\alpha} - E_{\alpha} = \Sigma_{\alpha} \quad (1\text{st}), \]
\[ DH^{\alpha}_{\ \beta} - E^{\alpha}_{\ \beta} = \Delta_{\alpha}^{\beta} \quad (2\text{nd}), \]
where we have used the canonical momenta (‘excitations’),
\[ M^{\alpha\beta} := -2\partial V_{\text{MAG}}/\partial Q_{\alpha\beta}, \]  
(23)
a (three-form) momentum conjugate to the metric field,
\[ H_\alpha := -\partial V_{\text{MAG}}/\partial T^\alpha, \]  
(24)
a (two-form) momentum conjugate to the coframe field and
\[ H^{\alpha\beta} := -\partial V_{\text{MAG}}/\partial R^\alpha_{\beta}, \]  
(25)
the (two-form) momentum conjugate to the \( GL(4, R) \)-connection.

The currents \( m^{\alpha\beta}, E_\alpha, E^{\alpha\beta} \) are respectively components of the metric energy-momentum, of the canonical energy-momentum and of the hyper-momentum currents, contributed by the gravitational fields themselves, in \( V_{\text{MAG}} \) – the so-called vacuum contributions.

Diffeomorphisms and \( GL(4, R) \) invariance yield two Noether identities [6] which, given in their ‘weak’ form, i.e. after the application of the matter equation of motion (19), become
\[ D\Sigma_\alpha = (e_\alpha^\beta T^\beta) \wedge \Sigma_\beta + (e_\alpha^\beta R^\gamma_\beta) \wedge \Delta^\beta_\gamma - (1/2)(e_\alpha^\beta Q^\sigma_\beta_\gamma) \sigma^{\beta\gamma}, \]  
(26)
\[ D\Delta^\beta_\alpha + \theta^\alpha \wedge \Sigma_\beta - g^\beta_\gamma \sigma^{\alpha\gamma} = 0. \]  
(27)

4 The OVETH Spherically-Symmetric Vacuum Solution

The search for exact solutions to the field equations of MAG is still in its infancy. First, Tresguerres [14] and, subsequently, Tucker and Wang [15] treated simplified situations, in which only gravitational dilation currents represented a departure from Riemannian geometry. Recently, a vacuum solution (i.e. with \( L = 0 \)) has been found by Obukhov et al. [16] (‘OVETH’), in which the selection of \( V_{\text{MAG}} \), however, is such as to provide for (gravitational) sources of shear, dilation and spin. Most recently, a static vacuum solution with axial symmetry was added to the set [17] (‘VTOH’).
In terms of irreducible components, in a metric-affine spacetime, the curvature has 11 pieces, the torsion 3 and the nonmetricity 4 (see App.B in [3]). A general quadratic Lagrangian (signature $-+++$) can thus be written as

$$
V_{\text{MAG}} = \frac{1}{2\kappa} \left[ -a_0 R^{\alpha\beta} \wedge \eta_{\alpha\beta} - 2\lambda \eta + T^\alpha \wedge \star \left( \sum_{I=1}^{3} a_I^{(I)} T^\alpha \right) + 2 \left( \sum_{I=2}^{4} c_I^{(I)} Q_{\alpha\beta} \right) \wedge \vartheta^\alpha \wedge \star T^\beta + Q_{\alpha\beta} \wedge \star \left( \sum_{I=1}^{4} b_I^{(I)} \bar{Q}^{\alpha\beta} \right) \right] - \frac{1}{2} R^{\alpha\beta} \wedge \star \left( \sum_{I=1}^{6} w_I^{(I)} W_{\alpha\beta} + \sum_{I=1}^{5} z_I^{(I)} Z_{\alpha\beta} \right). \tag{28}
$$

Here $\kappa := 2\pi \ell_{\text{Planck}}^2/(hc)$ is the gravitational and $\lambda$ the cosmological constant, $\eta$ is the volume four-form, $\eta_{\alpha\beta} := \star (\vartheta_\alpha \wedge \vartheta_\beta)$ and $a_{0-3}, b_{1-4}, c_{2-4}, w_{1-6}, z_{1-5}$ are dimensionless coupling constants. The antisymmetric and symmetric components of the curvature are denoted by $W_{\alpha\beta} := R_{[\alpha\beta]}$ and $Z_{\alpha\beta} := R_{(\alpha\beta)}$, respectively.

The OVETH solution belongs to a somewhat simplified Lagrangian with

$$
w_I = 0, \quad z_1 = z_2 = z_3 = z_5 = 0, \tag{29}
$$
i.e. preserving in $V_{\text{MAG}}$ only one component $^{(4)}Z_{\alpha\beta} := R^{\gamma}_{\gamma \alpha \beta}/4$ from the symmetric part of the curvature, namely the trace, Weyl’s segmental curvature. In addition, the following constants won’t occur in the solution and can be put to zero:

$$
a_1 = a_3 = b_1 = b_2 = c_2 = 0. \tag{30}
$$

This leaves in $V_{\text{MAG}}$ terms involving two pieces of the nonmetricity, a shear

$$
^{(3)}Q_{\alpha\beta} = (4/9) \left( \bar{Q}_{\alpha}^{\alpha} \wedge \bar{Q}_{\beta} \right) \Lambda - (1/4) g_{\alpha\beta} \Lambda \tag{31}
$$

with

$$
\Lambda := \vartheta^\alpha \left( e^\beta \bar{Q}_{\alpha\beta} \right), \tag{32}
$$

and the dilation

$$
^{(4)}Q_{\alpha\beta} = Q^{\gamma}_{\gamma \alpha \beta}/4 = Q_{g\alpha\beta}. \tag{33}
$$

Torsion appearing in $V_{\text{MAG}}$ is restricted to its vector piece,

$$
^{(2)}T^\alpha = (1/3) \vartheta^\alpha \wedge T, \tag{34}
$$
with
\[ T := e_\beta | T^\beta. \] (35)

Taking polar Boyer-Lindquist coordinates \((t, r, \theta, \phi) \equiv (\hat{0}, 1, 2, 3)\) and a Schwarzschild type (static, spherically symmetric and Minkowski-orthonormal) coframe, with one unknown function \(f(r)\),
\[ \vartheta^0 = f \, dt, \quad \vartheta^1 = (1/f) \, dr, \quad \vartheta^2 = r \, d\theta, \quad \vartheta^3 = r \sin \theta \, d\phi, \] (36)
i.e. a metric
\[ ds^2 = -f^2 \, dt^2 + dr^2 / f^2 + r^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) = o_{\alpha\beta} \vartheta^\alpha \otimes \vartheta^\beta, \] (37)
where we have also used the local Minkowski metric \(o_{\alpha\beta} = \text{diag}(-1, 1, 1, 1)\),
then the three one-forms \(Q, \Lambda, T\) should have the structure
\[ Q = u(r) \, f \, dt, \quad \Lambda = v(r) \, f \, dt, \quad T = \tau(r) \, f \, dt. \] (38)
The exact solution is given by the functions
\[ f = \sqrt{1 - (2\kappa M/r) + (\lambda r^2 / 3a_0) + z_4 [\kappa (k_0 N)^2 / 2a_0 r^2]}, \] (39)
\[ u = k_0 N / fr, \quad v = k_1 N / fr, \quad \tau = k_2 N / fr, \] (40)
with \(M\) and \(N\) arbitrary integration constants and the couplings \(k_0, k_1, k_2\) given by combinations of the \(a_0, a_2, b_3, c_3, c_4\) dimensionless couplings in the Lagrangian. In addition, \(b_4\) is constrained by a condition relating it to the five other couplings [16]. The nonmetricity is thus given by
\[ Q^{\alpha\beta} = (1/r) \left[ k_0 N o^{\alpha\beta} + (4/9) k_1 N \left( \vartheta^{(\alpha} e^{\beta)} \right) - (1/4) o^{\alpha\beta} \right] \, dt \] (41)
and the torsion by
\[ T^\alpha = (k_2 N / 3r) \, \vartheta^\alpha \wedge dt. \] (42)
The integration constant \(M\) is the Schwarzschild mass, \(k_0 N\) a dilation, \(k_1 N\) a (traceless) shear and \(k_2 N\) a spin charge. For \(N = 0\) and \(a_0 = 1\), one recovers the Schwarzschild-deSitter solution in GR.

For our purposes, we note that the vacuum solution’s nonmetricity, which will couple to a test particle’s shear, in this spherically-symmetric case, represents a \(1/r\) potential.
5 The VTOH Axially-Symmetric Vacuum Solution

Still using the gravitational Lagrangian \([28]\), with the simplifications \([29\) and \([30]\), and the same polar coordinates, VTOH \([17]\) posit a Kerr type solution

\[
\begin{align*}
\vartheta^0 &= \left(\frac{A}{B}\right)^{1/2} \left( dt - j_0 \sin^2 \theta \, d\phi \right), \\
\vartheta^1 &= \left(\frac{A}{B}\right)^{-1/2} \, dr, \\
\vartheta^2 &= \left(\frac{B}{f}\right)^{1/2} \, d\theta, \\
\vartheta^3 &= \left(\frac{B}{f}\right)^{-1/2} \sin \theta \left[ -j_0 \, dt + \left( r^2 + j_0^2 \right) \, d\phi \right],
\end{align*}
\]

(43)

where \(A = A(r)\), \(B = B(r, \theta)\), \(f = f(\theta)\), and \(j_0\) is a constant. The three residual one-forms \(Q, T, \Lambda\) of Sec. 4 are now replaced by expressions involving three functions \(u(r, \theta)\), \(v(r, \theta)\), \(\tau(r, \theta)\) appearing in the third and fourth irreducible components of nonmetricity and in vector torsion,

\[
\begin{align*}
Q_{\alpha\beta} &= \left[ u(r, \theta) \, o_{\alpha\beta} + \left(\frac{4}{9}\right) \, v(r, \theta) \left( \vartheta_{(\alpha} e_{\beta)} \right) - \left(\frac{1}{4}\right) \, o_{\alpha\beta} \right] \vartheta^0, \\
T^\alpha &= (1/3) \, \tau(r, \theta) \, \vartheta^\alpha \wedge \vartheta^0,
\end{align*}
\]

(44) (45)

with the solutions,

\[
\begin{align*}
u &= k_0 Nr/(AB)^{1/2}, \\
v &= k_1 Nr/(AB)^{1/2}, \\
\tau &= k_2 Nr/(AB)^{1/2},
\end{align*}
\]

(46) (47) (48)

and

\[
\begin{align*}
A &= r^2 + j_0^2 - 2\kappa Mr - \left(\lambda/3a_0\right) r^2 \left( r^2 + j_0^2 \right) + z_4 \kappa(k_0 N)^2/(2a_0), \\
B &= r^2 + j_0^2 \cos^2 \theta, \\
f &= 1 + \left(\lambda/3a_0\right) j_0^2 \cos^2 \theta.
\end{align*}
\]

(49) (50) (51)

Here the \(k_0, k_1, k_2\) are functions of the couplings \(a_0, a_2, b_3, c_3, c_4\) (the same ones as in Sec. 4) and, again, the same constraint relates \(b_4\) to \(k_0, k_1, k_2, c_4\). Physically, \(M\) and \(j_0\) represent the Schwarzschild mass and the Kerr angular momentum. For vanishing \(j_0\), we recover the OVETH solution of Sec. 4.
6 The Test-Particle in the OVETH and VTOH Solutions

The Noether identities \([26,27]\) already provide important information with respect to the behaviour of test-matter in MAG. Obukhov \([18]\), generalizing a corresponding result \([19,20]\) from Riemann-Cartan spacetime, has recast equation (26) in the form (quantities with tilde denote the Riemannian parts),

\[
\tilde{D} \left[ \Sigma_\alpha + \Delta^{\beta\gamma} (e_\alpha \otimes \Gamma_{\beta\gamma}) \right] + \Delta^{\beta\gamma} \wedge (\mathcal{L}_{e_\alpha} \otimes \Gamma_{\beta\gamma}) = \tau^{\beta\gamma} \wedge \left( e_\alpha \otimes \tilde{R}_{\gamma\beta} \right),
\]

(52)

where \(\otimes\Gamma_{\beta\gamma} := \Gamma_{\beta\gamma} - \tilde{\Gamma}_{\beta\gamma}\) denotes the non-Riemannian part of the connection. The expression on the right hand side of (52) represents the Mathisson-Papapetrou force density of GR for matter with spin \(\tau^{\beta\gamma}\). For \(\Delta^{\beta\gamma} = 0\), the equation of motion becomes \(\tilde{D} \Sigma_\alpha = 0\), i.e. without dilation, shear and spin ‘charges’ the particle follows Riemannian geodesics, irrespective of the composition of V\(_{MAG}\). Thus, we have to use as test matter only configurations which carry dilation, shear or spin charges, whether macroscopic or at the quantum particle level. At the latter, the hadron Regge trajectories provide adequate test matter, as world spinors with shear.

In the world spinor equation, when written anholonomically, the \(GL(4, R)\) Lie-algebra-valued connection \(\Gamma_{\alpha}^{\beta}[\rho(L_{\alpha}^\beta)]_{N}^{M}\), acting on the component \(\Psi_{N}(x)\), parallels the action of the same \(GL(4, R)\) generators (symmetric \(g_{\gamma(\alpha} L_{\beta)}^\gamma\) for shear and dilations) in the expression for the original Noether current of hypermomentum for the matter Lagrangian, \(\Delta^{\alpha\beta} = [(L_{\alpha\beta}^N)_{N}^{M}\Psi^N] \wedge [\partial L / \partial (D\Psi)^M] + \cdots\), where they enter in writing the variation of the matter field. This is just a reflection of the universality of gauge couplings, in which gauge fields are coupled to conserved currents.

An identity (Eq.(3.10.8) in Ref. \([6]\)) expresses the components of the connection one-form as a linear combination of the components of the Christoffel symbol (of the first kind), the object of anholonomity \(C^\alpha := d\theta^\alpha\), the torsion \(T^\alpha\) and the nonmetricity \(Q^{\alpha\beta}\),

\[
\Gamma_{\gamma\alpha\beta} = (1/2) \left[ \partial_{(\gamma} g_{\beta)\alpha} + C_{(\gamma\beta)\alpha} - T_{(\gamma\beta)\alpha} + Q_{(\gamma\beta)\alpha} \right],
\]

(53)

where the \(\{}\) are Schouten braces \([21]\). It is through this replacement that we get in the matter equation \((19)\) the action of the nonmetricity field. This can also be rewritten as:

\[
\Gamma_{\alpha\beta} = [V_4\text{-terms}] + [U_4\text{-terms}] + (1/2) Q_{\alpha\beta} + (e_{[\alpha} Q_{\beta]\gamma) \vartheta^\gamma .
\]

(54)
Turning now to the two vacuum solutions with nonmetricity, and leaving out dilations ($k_0 = 0$), we find, with $G := (1/9) k_1 N$,

$$
^{(3)} Q_{\alpha\beta} = 4G r (AB)^{-1/2} (\bar{\psi} (e_{1} \alpha) - (1/4) o_{\alpha\beta}) \bar{\psi}^0, \tag{55}
$$

or

$$
(\text{(3)} Q)_{\alpha\beta} = \frac{G r}{(AB)^{1/2}} \begin{pmatrix}
-3\partial^0 & 2\partial^1 & 2\partial^2 & 2\partial^3 \\
2\partial^1 & -\bar{\partial}^0 & 0 & 0 \\
2\partial^2 & 0 & -\bar{\partial}^0 & 0 \\
2\partial^3 & 0 & 0 & -\bar{\partial}^0
\end{pmatrix}. \tag{56}
$$

Particularly simple are the diagonal elements,

$$
^{(3)} Q_{00}/3 = ^{(3)} Q_{11} = ^{(3)} Q_{22} = ^{(3)} Q_{33} = -G r \left( dt - j_0 \sin^2 \theta d\phi \right) / \left( r^2 + j_0^2 \cos^2 \theta \right), \tag{57}
$$

which reduce to the spherically symmetric $-G dt/r$ for $j_0 = 0$. This is then the static potential entering, via (54), the world spinor equation, a $1/r$ potential with a centrifugal cut-off at small $r$.

Applying this potential to a Dirac or Bargmann-Wigner equation, written for any component in the ‘flat’ equation (4.5.1) of Ref. [6], we get in the spherically-symmetric case, a hydrogen-like relativistic spectrum, thereby superimposed on every state in the diagonal, with multiplicities growing with the Bargmann-Wigner spin value. The resulting world spinor is thus much more populated, but not yet in the off-diagonal sectors of Fig.3 in Ref. [6].

In the axial-symmetric case, $j_0 \neq 0$, and we thus have, in addition, the $|\delta J| = 2$ action, reaching into the off-diagonal sectors and partially filling them.

The energy-spectrum will follow. The generic world spinor corresponds to $\mathfrak{S}A(4, R)$ representations in class IIA (see Appendix C5 of Ref. [6]), i.e. with no kinematical constraint on the mass spectrum. Let us take, for instance, a (hadronic) linear $M^2 = aJ_0 + b$ as our free world spinor. We shall now have, in the simplest (lowest state $J = 1/2$) case, a superposition of the (Dirac-relativistic) hydrogen-like gravitational excitations due to nonmetricity, onto the linear spectrum of the flat limit (‘free’) manifold. Next, we would have to solve the Bargmann-Wigner equation for $J = 5/2$ in this hydrogen-like potential, etc..
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