LOCAL MESHLESS DIFFERENTIAL QUADRATURE
COLLOCATION METHOD FOR TIME-FRACTIONAL PDES

IMTIAZ AHMAD*
Department of Mathematics
University of Swabi
Khyber Pakhtunkhwa 23430, Pakistan

SIRAJ-UL-ISLAM*
Department of Basic Sciences
University of Engineering and Technology
Peshawar, Pakistan

MEHNAZ
Department of Basic Sciences
University of Engineering and Technology
Peshawar, Pakistan

Shaheed Benazir Bhutto Women University
Peshawar, Pakistan

SAKHRI ZAMAN
Department of Basic Sciences
University of Engineering and Technology
Peshawar, Pakistan

Abstract. This paper is concerned with the numerical solution of time-
fractional partial differential equations (PDEs) via local meshless differential
quadrature collocation method (LMM) using radial basis functions (RBFs).
For the sake of comparison, global version of the meshless method is also con-
sidered. The meshless methods do not need mesh and approximate solution
on scattered and uniform nodes in the domain. The local and global meshless
procedures are used for spatial discretization. Caputo derivative is used in the
temporal direction for both the values of $\alpha \in (0, 1)$ and $\alpha \in (1, 2)$. To circum-
vent spurious oscillation caused by convection, an upwind technique is coupled
with the LMM. Numerical analysis is given to assess accuracy of the proposed
meshless method for one- and two-dimensional problems on rectangular and
non-rectangular domains.

1. Introduction. Fractional calculus has attracted significant interests in the field
of science and engineering in the last few years. The elementary knowledge of frac-
tional calculus can be found in [7]. Fractional differential equations contain deriva-
tives of any real or complex order, being considered as general form of differential

2010 Mathematics Subject Classification. Primary: 65M99, 35K55, 35K57; Secondary: 35R11.
Key words and phrases. Meshless method, radial basis functions, Caputo derivative, upwind
technique, Burgers' equation.

* Corresponding authors: imtiazkakakhil@gmail.com (Imtiaz Ahmad), siraj.islam@gmail.com
(Siraj-ul-Islam).
equations. Applications of such models appear in numerous real world problems in physics and applied mathematics [21, 17].

Recently, great efforts have been undertaken to develop approximate methods for fractional PDEs. Several techniques have been proposed for the solution of fractional PDEs such as homotopy analysis method [12], variational iteration method [18], Adomian decomposition method [24], fractional difference method [30, 6], reduced differential transform method [20], spectral method [14] and meshless methods [28, 16, 10].

Variety of meshless methods have been applied for solving different kind of PDE models arising in almost all disciplines of engineering. Meshless character is one of the most important reason for the rising demand of such type of methods. These methods eliminate the need to construct complicated meshes and are easily extendible to multi dimensional PDEs. Other reasons can be their implementation on complex geometries in uniform and non-uniform nodal settings [22, 27, 25, 1, 2].

In the case of shape parameter dependent global meshless method (GMM), selection of optimum value of the shape parameter value \( c \) and dense ill-conditioned matrix are counted as a major deficiencies. To ward off the effects of these deficiencies of the GMM, researchers suggested a counterpart local meshless method (LMM) [23, 26, 29]. In the recent literature, local meshless methods have been used for stable behaviour and better accuracy of complex PDE models (see [25, 26, 3, 4]).

Most of the numerical methods failed to capture flow dynamics in convection-dominated flows. Like other numerical methods, several strategies have been suggested to circumvent the numerical instabilities of the LMM [9, 15, 19, 22, 26, 3]. Following the idea of [22, 26, 3], the proposed LMM is combined with a technique based on local supported domain, called an upwind technique, in case of convection dominated PDE models. This technique has the ability to avoid mild spurious oscillatory solution. Two types of local supported domains are used i.e. central and upwind as shown in Figures 1-2.

![Figure 1. Schematics of central local supported domain in 2D geometry for \( n_i = 5 \).](image)

Current research work is devoted to use, the local meshless method for numerical solution of 1D time-fractional Burgers’ model equation [11, 13], 2D time-fractional diffusion model equation [28], 2D time-fractional Burgers’ model equation [5]. The space derivatives are approximated by the local and global meshless procedure using the multiquadric and the Gaussian radial basis functions whereas time-fractional
derivative is approximated by using Caputo definition. In case of convection-dominated time-fractional models, the LMM is coupled with a stencil based upwind technique. Both rectangular and non-rectangular geometries are considered in numerical experiments.

2. Time-fractional models. Consider the 1D nonlinear time-fractional Burgers’ equation (TFBE) [11, 13] which is given as
\[
\frac{\partial^{\alpha} V(x,t)}{\partial t^{\alpha}} + V(x,t) \frac{\partial V(x,t)}{\partial x} - \frac{1}{Re} \frac{\partial^2 V(x,t)}{\partial x^2} = 0, \quad a \leq x \leq b, \quad 0 < \alpha \leq 1, \quad t > 0,
\]
with the following initial and boundary conditions
\[
V(x,0) = V_0, \\
B_1(a,t) = f_1(t), \quad B_2(b,t) = f_2(t), \quad t > 0,
\]
where Re is known as Reynolds number.

The 2D linear time-fractional diffusion equation (TFDE) [28] is
\[
\frac{\partial^{\alpha} V(x,t)}{\partial t^{\alpha}} - \nabla V(x,t) = f(x,t), \quad x \in \Omega \subset \mathbb{R}^2, \quad 1 < \alpha \leq 2, \quad t > 0,
\]
with initial and boundary conditions
\[
V(x,0) = V_0, \\
B_3(x,t) = f_3(x,t), \quad x \in \partial \Omega, \quad t > 0.
\]
The 2D nonlinear time-fractional Burgers’ equation (TFBE) [5] is
\[
\frac{\partial^{\alpha} V(x,t)}{\partial t^{\alpha}} - \frac{1}{Re} \nabla V(x,t) + V(x,t) \Delta V(x,t) = 0, \quad x \in \Omega \subset \mathbb{R}^2, \quad 0 < \alpha \leq 1, \quad t > 0,
\]
with initial and boundary conditions
\[
V(x,0) = V_0, \\
B_4(x,t) = f_4(x,t), \quad x \in \partial \Omega, \quad t > 0,
\]
where Re is known as Reynolds number.
3. Formulation of the LMM. The LMM [26] is extended to the time-fractional Burgers’ and diffusion models. The derivatives of $V(x, t)$ are approximated at the centers $x_i$ by function values at a set of nodes in the neighborhood of $x_i$, \{$x_{i1}, x_{i2}, x_{i3}, \ldots, x_{in}$\} $\subset$ \{$x_1, x_2, \ldots, x_N$\}, $n_i \ll N^n$, where $i = 1, 2, \ldots, N$. For 1D case, when $n = 1$ then $x = x$ and for 2D case, when $n = 2$ then $x = (y, z)$. Now in 1D case, we have

$$V^{(m)}(x_i) \approx \sum_{k=1}^{n_i} \lambda_k^{(m)} V(x_{ik}), \quad i = 1, 2, \ldots, N. \quad (4)$$

To find the corresponding coefficient $\lambda^{(m)}_k$, radial basis function $\psi(||x - x_i||)$ can be substituted in equation (4) as follows

$$\psi^{(m)}(||x_i - x_l||) = \sum_{k=1}^{n_i} \lambda^{(m)}_{ik} \psi(||x_{ik} - x_l||), \quad l = i1, i2, \ldots, in_i, \quad (5)$$

where $\psi(||x_{ik} - x_l||) = \sqrt{1 + (c||x_{ik} - x_l||)^2}$ and $\psi(||x_{ik} - x_l||) = \exp(-||x_{ik} - x_l||^2/c^2)$ in case of multiquadric (MQ) and Gaussian (GA) RBFs respectively.

Matrix form of equation (5) is

$$\begin{bmatrix}
\psi^{(m)}_{i1}(x_i) \\
\psi^{(m)}_{i2}(x_i) \\
\vdots \\
\psi^{(m)}_{in_i}(x_i)
\end{bmatrix} = 
\begin{bmatrix}
\psi_{i1}(x_{i1}) & \psi_{i2}(x_{i1}) & \cdots & \psi_{in_i}(x_{i1}) \\
\psi_{i1}(x_{i2}) & \psi_{i2}(x_{i2}) & \cdots & \psi_{in_i}(x_{i2}) \\
\vdots & \vdots & \ddots & \vdots \\
\psi_{i1}(x_{in_i}) & \psi_{i2}(x_{in_i}) & \cdots & \psi_{in_i}(x_{in_i})
\end{bmatrix} \begin{bmatrix}
\lambda^{(m)}_{i1} \\
\lambda^{(m)}_{i2} \\
\vdots \\
\lambda^{(m)}_{in_i}
\end{bmatrix}, \quad (6)$$

where

$$\psi_l(x_k) = \psi(||x_k - x_l||), \quad l = i1, i2, \ldots, in_i,$$

for each $k = i1, i2, \ldots, in_i$. In matrix notation, equation (6) can be written as

$$\psi^{(m)}_{ni} = A_{ni} \lambda^{(m)}_{ni}, \quad (7)$$

where

$$\psi^{(m)}_{ni} = \begin{bmatrix}
\psi^{(m)}_{i1}(x_i) \\
\psi^{(m)}_{i2}(x_i) \\
\vdots \\
\psi^{(m)}_{in_i}(x_i)
\end{bmatrix}^T,$$

$$A_{ni} = 
\begin{bmatrix}
\psi_{i1}(x_{i1}) & \psi_{i2}(x_{i1}) & \cdots & \psi_{in_i}(x_{i1}) \\
\psi_{i1}(x_{i2}) & \psi_{i2}(x_{i2}) & \cdots & \psi_{in_i}(x_{i2}) \\
\vdots & \vdots & \ddots & \vdots \\
\psi_{i1}(x_{in_i}) & \psi_{i2}(x_{in_i}) & \cdots & \psi_{in_i}(x_{in_i})
\end{bmatrix},$$

$$\lambda^{(m)}_{ni} = \begin{bmatrix}
\lambda^{(m)}_{i1} \\
\lambda^{(m)}_{i2} \\
\vdots \\
\lambda^{(m)}_{in_i}
\end{bmatrix}^T.$$

From equation (7), we obtain

$$\lambda^{(m)}_{ni} = A_{ni}^{-1} \psi^{(m)}_{ni}. \quad (8)$$

From equation (4) and equation (8), we get

$$V^{(m)}(x_i) = (\lambda^{(m)}_{ni})^T V_{ni},$$

where

$$V_{ni} = [V(x_{i1}), V(x_{i2}), \ldots, V(x_{in_i})]^T.$$
3.2. Discretization of time derivatives.

We approximate the first-order derivative involved in the time fractional term $\partial V^\alpha / \partial t$ by using Caputo derivative. In this paper, two cases $0 < \alpha < 1$ and $1 < \alpha < 2$ are considered. The Caputo fractional derivative for $\alpha \in (0, 1)$ is

$$\frac{\partial^\alpha V(x, t)}{\partial t^\alpha} = \int_0^t \frac{1}{\Gamma(1-\alpha)} \left( \frac{\partial V(x, \xi)}{\partial \xi} (t - \xi)^{1-\alpha} \right) d\xi, \quad 0 < \alpha < 1$$

Consider $N + 1$ equally spaced time levels $t_0, t_1, \ldots, t_N$ in the interval $[0, t]$, such that $t_n = n\tau$, $n = 0, 1, 2, \ldots, N$, $\tau$ being the time step and finite difference scheme is used to approximate the first-order derivative involved in the time fractional term

$$\frac{\partial^\alpha V(x, t_{n+1})}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n} \left( \frac{k+1}{\Gamma(1-\alpha)} \right) \left( t_{n+1} - t_k \right)^{1-\alpha} d\xi,$$

$$= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n} \left( \frac{k+1}{\Gamma(1-\alpha)} \right) \left( t_{n+1} - t_k \right)^{1-\alpha} d\xi,$$

$$= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n} \left( \frac{k+1}{\Gamma(1-\alpha)} \right) \left( t_{n+1} - t_k \right)^{1-\alpha} d\xi.$$

The term $\partial V(x, \xi_k) / \partial \xi$ is approximated as follows

$$\frac{\partial V(x, \xi_k)}{\partial \xi} = \frac{V(x, \xi_{k+1}) - V(x, \xi_k)}{\xi} + O(\tau).$$
Then
\[
\frac{∂^α V(x, t_{n+1})}{∂t^α} \approx \frac{1}{Γ(1-α)} \sum_{k=0}^{n} \frac{V(x, t_{k+1}) - V(x, t_k)}{τ} \int_{kτ}^{(k+1)τ} (t_{k+1} - ξ)^{-α} dζ,
\]
\[
= \frac{1}{Γ(1-α)} \sum_{k=0}^{n} \frac{V(x, t_{n+1-k}) - V(x, t_{n-k})}{τ} \int_{kτ}^{(k+1)τ} (t_{k+1} - ξ)^{-α} dζ,
\]
\[
= \left\{ \begin{array}{ll}
\frac{τ^{-α}}{Γ(2-α)} (V_{n+1} - V_n) + \frac{τ^{-α}}{Γ(2-α)} \sum_{k=1}^{n} (V_{n+1-k} - V_{n-k}) \\
\frac{(k+1)^{1-α} - k^{1-α}, n \geq 1}
\end{array} \right.
\]
\[
\approx \left\{ \begin{array}{ll}
a_0 (V_{n+1} - V_n) + a_0 \sum_{k=1}^{n} b_k (V_{n+1-k} - V_{n-k}), n \geq 1 \\
a_0 (V_1 - V_0), n = 0.
\end{array} \right.
\]

Letting \(a_0 = \frac{τ^{-α}}{Γ(2-α)}\) and \(b_k = (k+1)^{1-α} - k^{1-α}, k = 0, 1, \ldots, n\), we can write the above equation in more precise form as
\[
\frac{∂^α V(x, t_{n+1})}{∂t^α} \approx \left\{ \begin{array}{ll}
a_0 (V_{n+1} - V_n) + a_0 \sum_{k=1}^{n} b_k (V_{n+1-k} - V_{n-k}), n \geq 1 \\
a_0 (V_1 - V_0), n = 0.
\end{array} \right.
\]

The Caputo fractional derivative for \(α \in (1, 2)\) is defined by
\[
\frac{∂^α V(x, t)}{∂t^α} = \left\{ \begin{array}{ll}
\frac{1}{Γ(2-α)} \int_{0}^{t} \frac{∂^2 V(x, ξ)}{∂ξ^2} (t - ξ)^{1-α} dξ, & 1 < α < 2 \\
\frac{∂V^2(x, t)}{∂t^2}, & α = 2.
\end{array} \right.
\]

Like the previous case, the approximate fame work can be constructed as
\[
\frac{∂^α V(x, t_{n+1})}{∂t^α} = \frac{1}{Γ(2-α)} \int_{0}^{t_{n+1}} \frac{∂^2 V(x, ξ)}{∂ξ^2} (t_{n+1} - ξ)^{1-α} dξ,
\]
which leads to
\[
\frac{∂^α V(x, t_{n+1})}{∂t^α} \approx \frac{1}{Γ(2-α)} \sum_{k=0}^{n} \int_{kτ}^{(k+1)τ} \frac{∂^2 V(x, ξ_k)}{∂ξ^2} (t_{k+1} - ξ)^{1-α} dξ.
\]

The term \(\frac{∂^2 V(x, ξ_k)}{∂ξ^2}\) can be approximated as
\[
\frac{∂^2 V(x, ξ_k)}{∂ξ^2} = \frac{V(x, ξ_{k+1}) - 2V(x, ξ_k) + V(x, ξ_{k-1})}{ξ^2} + O(τ^2).
\]

The simplified form of the fractional derivative becomes
\[
\frac{∂^α V(x, t_{n+1})}{∂t^α} \approx a_0 (V_{n+1} - 2V^n + V_{n-1}) + a_0 \sum_{k=1}^{n} b_k (V_{n+1-k} - 2V^n - V_{n-k}), n \geq 0.
\]

Letting \(a_0 = \frac{τ^{-α}}{Γ(3-α)}\) and \(b_k = (k+1)^{2-α} - k^{2-α}, k = 0, 1, \ldots, n\), one can write the above equation in a more precise form as
\[
\frac{∂^α V(x, t_{n+1})}{∂t^α} \approx \left\{ \begin{array}{ll}
a_0 (V_{n+1} - 2V^n + V_{n-1}) + a_0 \sum_{k=1}^{n} b_k (V_{n+1-k} - 2V^n - V_{n-k}) \\
+ 2a_0 b_0 (V_1 - V_0 - τ \frac{∂V(x, 0)}{∂t}), & n \geq 1 \\
2a_0 (V_1 - V_0 - τ \frac{∂V(x, 0)}{∂t}), & n = 0.
\end{array} \right.
\]
Table 1. Comparison of the LMM with different local sub-domain $n_i$ and the method reported in [8] for Test Problem 1.

| Time | Max. abs. error [8] | LMM ($L_\infty$) $n_i = 3$ | LMM ($L_\infty$) $n_i = 5$ | LMM ($L_\infty$) $n_i = 7$ | LMM ($L_\infty$) $n_i = 9$ | LMM ($L_\infty$) $n_i = 11$ |
|-------|---------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
| $t=1$ | 4.632e-03           | 3.6104e-04                  | 1.6807e-05                  | 6.5724e-06                  | 6.4933e-06                  | 6.4933e-06                  |
| $t=2$ | 5.267e-03           | 6.3774e-04                  | 2.5918e-05                  | 1.0749e-05                  | 1.0749e-05                  | 1.0749e-05                  |
| $t=2.5$ | 5.569e-03          | 7.8362e-04                  | 2.8962e-05                  | 1.2321e-05                  | 1.2321e-05                  | 1.2321e-05                  |
| $t=3$ | 5.857e-03           | 9.2358e-04                  | 3.3891e-05                  | 1.3640e-05                  | 1.3640e-05                  | 1.3640e-05                  |

4. Numerical experiments. This section is dedicated to the numerical results of the one- and two-dimensional nonlinear time-dependent fractional Burgers’ equation as well as two-dimensional linear time-dependent fractional diffusion equation using the proposed local meshless method. Multiquadric and Gaussian RBFs are used for space discretization in all numerical simulations. For 1D case, the local support domain of size $n_i = 3$ and in 2D case the local support domain of size $n_i = 5$ are used in all the numerical experiments. The LMM coupled with an upwind technique is used in the case of convection-dominated time-fractional PDEs is the major addition. Accuracy of the meshless method is measured through absolute error norm ($L_{abs}$), maximum error error ($L_{\infty}$) and average absolute error norm ($Ave.L_{abs}$) which are defined as

\[
L_{abs} = |\hat{V} - V|,
L_{\infty} = \max(L_{abs}),
Ave.L_{abs} = \frac{1}{N \times M} \sum_{i=1}^{M} \sum_{j=1}^{N} |\hat{V}(x_j, t_i) - V(x_j, t_i)|,
\]

where $\hat{V}$ is exact solution and $V$ is the approximate solution.

**Test Problem 1.** Consider the 1D TFBE (1) as a first test problem with initial and boundary conditions taken form the exact solution

\[
V(x, t) = \frac{\mu + \sigma + (\sigma - \mu) \exp[\text{Re} \mu(x - \sigma t - \lambda)]}{1 + \exp[\text{Re} \mu(x - \sigma t - \lambda)]},
\]

In Table 1, numerical results in the form of maximum absolute error $L_{\infty}$ norm are obtained by the LMM for Test Problem 1 using MQ RBF and the value of shape parameter in this case is taken $c = 1$, $N = 60$, time step size $\tau = 0.0002$, $\alpha = 1$, $\sigma = 0.4$, $\mu = 0.3$, $\text{Re} = 10$ and $\lambda = 0.8$. Table 1 shows that the numerical results of the LMM are more accurate than the method reported in [8].

**Test Problem 2.** Consider 1D TFBE (1) with initial and boundary conditions given as

\[
V(x, 0) = \sin(\pi x), V(a, t) = V(b, t) = 0.
\]

The exact solution of this problem is not known.

Figure 3 shows the numerical results of the GMM for $t = 1$ and $N = 60$. One can see that the GMM can handle the Burgers’ equation reasonably well for Reynolds number $\text{Re} = 300$, but when $\text{Re}$ reaches 400 the method gets unstable.
4-5 show numerical results of the LMM for $t = 1$ and $N = 60$. It is clear from Figure 4 (left) the LMM can handle the Burgers’ equation only up to $Re = 100$, but when the Reynolds number reached to $Re = 200$ the method gets unstable. To stabilize the proposed LMM, we coupled it with a stabilized technique called an upwind technique and the results are shown in Figure 4 (right) and Figure 5. It is clear from the figures that the LMM coupled with the upwind technique produces stable results for large values of the Reynolds number i.e. up to $Re = 10^{17}$.
Table 2. Ave.$L_{abs}$ error norms of the LMM for Test Problem 3.

| N  | 5   | 10  | 15  | 20  |
|----|-----|-----|-----|-----|
| $\alpha = 1.5$ | 2.7911e-03 | 3.3331e-04 | 6.3412e-05 | 1.1145e-05 |
| $\alpha = 1.8$ | 2.8124e-03 | 3.5870e-04 | 8.4685e-05 | 2.7917e-05 |

Table 3. Numerical results of the LMM and the method reported in [28] for Test Problem 3.

| $\tau$ | Ave.$L_{abs}$ [28] | Ave.$L_{abs}$ [28] |
|--------|------------------|------------------|
| 1/10   | 4.3826e-02       | 1.2550e-02       |
| 1/20   | 6.6277e-03       | 1.0696e-02       |
| 1/30   | 4.5292e-03       | 7.2811e-03       |
| 1/40   | 3.4518e-03       | 5.5407e-03       |
| 1/50   | 2.7951e-03       | 4.4822e-03       |
| 1/60   | 2.3526e-03       | 3.2548e-03       |

Table 4. Ave.$L_{abs}$ of the LMM for Test Problem 3.

| $\alpha$ | Regular nodes | Chebyshev nodes |
|----------|---------------|------------------|
|          | Explicit CN | Implicit | Explicit CN | Implicit |
| 1.5      | 8.0527e-05   | 2.2050e-04 | 4.1275e-04 | 2.4177e-04 |
| 1.6      | 8.1357e-05   | 2.2263e-04 | 4.1881e-04 | 2.3932e-04 |
| 1.7      | 8.2287e-05   | 2.2326e-04 | 4.2312e-04 | 2.3691e-04 |
| 1.8      | 8.3535e-05   | 2.2109e-04 | 4.2679e-04 | 2.3405e-04 |

Test Problem 3. Consider 2D TFDE (2) with initial condition

\[ V(y, z, 0) = 10y^2z^2(1-y)(1-z), \]

\[ V_t(y, z, 0) = 30y^2z^2(1-y)(1-z). \]

where the source function is

\[ f(y, z, t) = 60y^2z^2(1-y)(1-z) \left( \frac{t^{3-\alpha}}{\Gamma(4-\alpha)} + \frac{t^{2-\alpha}}{\Gamma(3-\alpha)} \right) - 20 \left( (1-3y)z^2(1-z) + (1-3z)y^2(1-y) \right)(t+1)^3. \]

The exact solution is

\[ V(y, z, t) = 10y^2z^2(1-y)(1-z)(t+1)^3. \]

Table 2 shows the numerical results obtained by the LMM using Crank-Nicolson (CN) for different nodal points $N$, $\tau = 0.001$, $t = 1$, shape parameter value $c = 1$ and local support domain $n_i = 15$. The table shows that good accuracy has been achieved for $\alpha = 1.5$ and $\alpha = 1.8$. Numerical results of the proposed LMM using Crank-Nicolson are compared in Table 3 for $N = 81$, $n_i = 11$ and different values of $\tau$. It can be observed from the table that the LMM is more accurate than the method reported in [28]. In Table 4 we have compared numerical result obtained by the LMM (Explicit, Implicit and Crank-Nicolson) for different values of $\alpha$ using regular and Chebyshev nodal points. The results are obtained in terms of Ave.$L_{abs}$ error norms, $N = 20$, $n_i = 11$ and $t = 1$. The table shows that the results of explicit
method using regular nodes is comparatively better than the other methods for small time step.

The numerical results of the 2D TFDE (2) on non-rectangular domains are shown in Figures 6-10 for Test Problem 3 using GA RBF. The numerical results shown in Figures 6-10 are obtained corresponding to the values $\tau = 0.001$, $t = 1$ and $\alpha = 1.5$. These figures show accuracy of the LMM in non-rectangular domains in term of absolute error $L_{abs}$ norm for Test Problem 3.

**Figure 6.** Computational domain and absolute error for Test Problem 3.

**Figure 7.** Computational domain and absolute error for Test Problem 3.

**Figure 8.** Computational domain and absolute error for Test Problem 3.
Test Problem 4. Consider 2D TFBE (3) with initial and boundary conditions
\[ V(y, z, 0) = (y^2 - 1)^2(z^2 - 1)^2, \]
\[ V(y, z, t) = 0, \quad (y, z) \in \partial\Omega. \]

The exact solution to this problem is unknown.

Numerical results of the GMM for \( N = 50, \tau = 0.0005, t = 1 \) and \( \alpha = 0.75 \)
are shown in Figure 11. It can be observed from the figure that the method gives
accurate results for Reynolds number \( Re = 200 \) but when it reaches to \( Re = 300 \)
the method gets unstable. The numerical results of the LMM for \( N = 50, c = 1 \)
Figure 12. Numerical solution of the LMM for \( Re = 100 \) (left) and \( Re = 150 \) (right) for Test Problem 4.

Figure 13. Results of the LMM using upwind technique for \( Re = 150 \) (left) and \( Re = 1000 \) (right) for Test Problem 4.

Figure 14. Results of the LMM using upwind technique for \( Re = 10^{10} \) (left) and \( Re = 10^{17} \) (right) for Test Problem 4.

\( \tau = 0.0005, t = 1 \) and \( \alpha = 0.75 \) are shown in Figures 12-14. The method can capture the solution for Reynolds number \( Re = 100 \) but when \( Re = 150 \), the method failed to compute accurate result, as shown in Figure 12 (right). To stabilize the method in case of convection dominated Burgers’ equation model, the LMM is coupled with an upwind technique and stable results are obtained for high Reynolds number up to \( Re = 10^{17} \) as shown in Figures 13-14. Comparison of CPU time (in seconds) of the local and global meshless methods are shown in Figure 15 by taking \( \tau = 0.0005 \) and \( t = 1 \). It is clear from this figure that the LMM is more efficient than the GMM.
5. Conclusion. In the present work, a local meshless method based on radial basis functions is applied to one- and two-dimensional time-dependent fractional PDE models. The time derivative is defined and simplified in Caputo sense and the scheme is constructed for $0 < \alpha < 1$ and $1 < \alpha < 2$. To check accuracy and efficacy of the proposed scheme different test problems have been considered on both rectangular and non-rectangular domains. Results of the local meshless method are compared with exact/approximate solutions reported in the existing literature. The stable results (in the case of high Reynolds number) of the LMM combined with upwind technique strongly supported the advantage of the LMM over other conventional methods. The LMM has been found to be a flexible interpolation method with respect to accuracy and well-conditioned coefficient matrix.

Acknowledgment. The authors are grateful to the anonymous reviewers for their useful suggestions for improvement of the manuscript.

REFERENCES

[1] I. Ahmad, M. Riaz, M. Ayaz, M. Arif, S. Islam and P. Kumam, Numerical simulation of partial differential equations via local meshless method, *Symmetry*, 11 (2019), 257 pp.
[2] I. Ahmad, M. Ahsan, Zaheer-ud-Din, M. Ahmad and P. Kumam, An efficient local formulation for time-dependent PDEs, *Mathematics*, 7 (2019), 216 pp.
[3] I. Ahmad, Siraj-ul-Islam and A. Q. M. Khaliq, Local RBF method for multi-dimensional partial differential equations, *Comput. Math. Appl.*, 74 (2017), 292–324.
[4] I. Ahmad, M. Ahsan, I. Hussain, P. Kumam and W. Kumam, Numerical simulation of PDEs by local meshless differential quadrature collocation method, *Symmetry*, 11 (2019), 394 pp.
[5] W. Cao, Q. Xu, Qiuwu and Z. Zheng, Solution of two-dimensional time-fractional Burgers’ equation with high and low Reynolds numbers, *Advances in Difference Equations*, (2017), 338, 14 pp.
[6] S. Chen, F. Liu, P. Zhuang and V. Anh, Finite difference approximations for the fractional Fokker-Planck equation, *Appl. Math. Model.*, 33 (2009), 256–273.
[7] K. Diethelm, *The Analysis of Fractional Differential Equations*, An application-oriented exposition using differential operators of Caputo type, Lecture Notes in Mathematics, 2004. Springer-Verlag, Berlin, 2010.
[8] T. S. El-Danaf and A. R. Hadhoud, Parametric spline functions for the solution of the one time fractional Burgers’ equation, *Appl. Math. Model.*, 36 (2012), 4557–4564.
[9] Y. T. Gu and G. R. Liu, Meshless techniques for convection dominated problems, *Comput. Mech.*, 38 (2006), 171–182.
[10] V. R. Hosseini, E. Shivanian and W. Chen, Local radial point interpolation (MLRPI) method for solving time fractional diffusion-wave equation with damping, *J. Comput. Phys.*, 312 (2016), 307–332.
[11] M. Inc, The approximate and exact solutions of the space-and time-fractional Burgers’ equations with initial conditions by variational iteration method, *J. Math. Anal. Appl.*, **345** (2008), 476–484.

[12] H. Jafari and S. Seifi, Solving a system of nonlinear fractional partial differential equations using homotopy analysis method, *Commun. Nonlinear Sci. Numer. Simul.*, **14** (2009), 1962–1969.

[13] D. Li, C. Zhang and M. Ran, A linear finite difference scheme for generalized time fractional Burgers’ equation, *Appl. Math. Model.*, **40** (2016), 6069–6081.

[14] Y. Lin and C. Xu, Finite difference/spectral approximations for the time-fractional diffusion equation, *J. Comput. Phys.*, **225** (2007), 1533–1552.

[15] M. D. Ortigueira, The fractional quantum derivative and its integral representations, *Commun. Nonlinear Sci. Numer. Simul.*, **15** (2010), 956–962.

[16] A. Mohebbi, M. Abbaszadeh and M. Dehghan, The use of a meshless technique based on collocation and radial basis functions for solving the time fractional nonlinear Schrödinger equation arising in quantum mechanics, *Eng. Anal. Bound. Elem.*, **37** (2013), 475–485.

[17] E. Scalas, R. Gorenflo and F. Mainardi, Fractional calculus and continuous-time finance, *Physica A*, **284** (2000), 376–384.

[18] P. Thounthong, M. N. Khan, I. Hussain, I. Ahmad and P. Kumam, Symmetric radial basis function method for simulation of elliptic partial differential equations, *Mathematics*, **6** (2018), 327 pp.

[19] J. Y. Yang, Y. M. Zhao, N. Liu, W. P. Bu, T. L. Xu and Y. F. Tang, An implicit MLS meshless method for 2-D time dependent fractional diffusion–wave equation, *Appl. Math. Model.*, **39** (2015), 1229–1240.