A finite element approach for determining the dynamic behaviours of multiscale beams using the semi-continuum model

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Abstract. The dynamic mechanical behaviours of a multiscale beam with nanoscaled thickness are investigated. The thickness of the multiscale beam is divided to some discrete atomic layers, while the scales of length and width are within the classical continuum framework. Such a semi-continuum model is employed to measure the size dependence of the multiscale beam. Based on the governing equation of motion, the finite element expressions for transverse vibration are achieved through the weighted residual method. By using the Hermitian interpolation function, the structural mass matrix and stiffness matrix are obtained, and the system equation via finite element approach is got as well. Subsequently, the finite element analysis is carried out with one hundred equal length units. The results show that the natural frequencies of the multiscale beam are related to the relaxation coefficient of surface atoms. When the relaxation coefficient is more than 1, the fundamental frequency parameter is positively correlated with the number of atomic layers, while when the value is less than 1, it becomes negatively correlated. The lateral displacement and rotational angle of the first two modal functions are determined, and it is indicated that the phase of the angular variation is always half a period faster than the phase of the deflection variation.

1. Introduction

Recently, a lot of interest has been concentrated on mechanical properties of nanoscale structures. Nanoscale structures have been applied in the nano-electromechanical systems (NEMS), which can be precisely manufactured with the rapid development of nanotechnology. The characteristics of these devices are highly dependent on the properties of the ultra-thin beam-like elements. So it needs to be well characterized to control their functionality. What has been proved is that the classical continuum mechanics fails to explain the situation at nanoscale. For example, the stress at a crack tip is singular regardless of the load and it is hard to understand from physics for the existence of limited fatigue strength of each material. In order to suit for the development of nanotechnology, there should be some new approaches for the development of nanotechnology. As we know, Eringen proposed a new theoretical approach named non-local theory in the 1970s [1]. The theory assumes that the stress at a point in a domain to depend not only on the classical strain at that particular point, but also on the spatial integrals that represent the weighted averages of the classical strain contribution of all the other points in the body. A new and reasonable result was concluded via non-local theory [2] that the stress...
at the crack tip is nonsingularity. Therefore, the non-local theory is the focus of the current research in modeling the nano-structures.

According to the existence of approximation error using Green’s function under certain conditions, the original spatial integrals in the non-local relations [1] and the equivalent differential constitutive equation within a two-dimensional region proposed by Eringen [3], it is difficult to obtain the solutions of non-local problems mathematically. Subsequently, much work on nanoscale structures through non-local theory appeared, including lattice dispersion of vibration mechanics, elastic waves, wave propagation, dislocation mechanics, static deflection, fracture mechanics, surface tension fluids, etc. [4-19]. However, it is strange that recent prediction that natural frequency increases with the increase of the non-local small scale parameter [8-13] which is opposite to the conclusions concluded in some previous studies [4-7]. Although there exists some inconsistence, the non-local theory still attracts more and more researchers’ attention with increasing number of publications (e.g. see [14-19]).

This paper aims to judge and discuss these two different non-local results using the finite element based multiscale semi-continuum approach.

The semi-continuum model explains the discrete nature in thickness, which is different from the classical continuum mechanics because of the ultra thin structure. Sun and Zhang [20] presented the mechanical properties of ultra-thin structures firstly based on the semi-continuum model which can account the discrete nature in thickness direction. The research shows that the values of Young’s modulus and Poisson’s ratios strongly depend on the number of atomic layers. Subsequently, Li et al. [21] investigated the static analysis of ultra-thin beams based on the semi-continuum model, where the relaxation phenomenon [22, 23] is adopted to examine the transverse vibration of ultra-thin beams. Subsequently, Li et al. [24] and Shen and Li [25] further investigate the transverse vibration and bending stiffness by a refined semi-continuum model, in which it can conclude that relaxation coefficient results in two types of size dependence of the mechanical properties of multiscale beams. Similar with the work [21], we conclude that it is the relaxation coefficient that results in two opposite results in non-local theory.

However, unlike the previous studies, the finite element method with simpler computational procedure is employed in the present work. The results indicate effects of the relaxation properties and thickness on the frequency parameter directly. Additionally, besides the natural frequency, the multiscale semi-continuum based mode functions of the multiscale beam are determined. The work could provide useful reference for characterizing, designing and optimizing the small scale beams from a multiscale point of view.

2. Model and solution

The governing equation for the transverse dynamics of a multiscale beam based on the semi-continuum model is given by [24]

\[ S_1 \frac{\partial^4 \ddot{w}}{\partial x^4} - \frac{\partial^2 \ddot{w}}{\partial x^2} + S_2 \frac{\partial^2 \ddot{w}}{\partial t^2} = 0 \]  

(1)

where

\[ S_1 = \frac{kBa^2N(N+1)(N+2)}{6(N-2+2r)TL^2} \left[ \frac{N-1}{6} + \frac{1}{r(r^2+2)} \right] \]

\[ S_2 = \frac{\rho B(N-2+2r)aL^2}{P^2T} \]  

(2)

in which \( k \) is the elastic coefficient, \( B \) is width, \( L \) is length, \( a \) is the crystal lattice parameter of the material, \( N \) is the number of atomic layers, \( r \) is relaxation coefficient, \( T \) is the initial axial tension, \( \rho \) is the mass density, \( \ddot{w} = \ddot{w}/L \) and \( \ddot{x} = \ddot{x}/L \) are the corresponding dimensionless displacement and axial

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coordinates, respectively, and $\bar{T} = t/P_e$ is the non-dimensional time in which $P_e$ is a characteristic time. Note that the nanoscaled thickness $H$ satisfies $H=(N-2+2r)a$.

![Figure 1. The finite element analysis sketch.](image)

The finite element method is applied to Eq. (1). Firstly, an analysis sketch is shown in figure 1, where $w$ is the transverse displacement, $\theta$ is the rotation angle. Because the beam is ultra-thin and the thickness is at nanoscale, the Euler beam element is utilized. Considering the weighted residual method, we can determine the following finite element format by using the integration by parts and dispersing the result, as

$$I = \sum_{i=1}^{n} \left( \int_{x_i}^{x_{i+1}} S_2 \frac{\partial^2 w}{\partial t^2} v dx + \int_{x_i}^{x_{i+1}} \frac{\partial^2 v}{\partial x^2} \frac{\partial^2 w}{\partial x^2} dx + \int_{x_i}^{x_{i+1}} \frac{\partial v}{\partial x} \frac{\partial w}{\partial x} dx \right) + \left[ S_1 v \frac{\partial^3 w}{\partial x^3} - S_1 \frac{\partial v}{\partial x} \frac{\partial^2 w}{\partial x^2} - \frac{\partial w}{\partial x} \right]_0^L = 0 \quad (3)$$

where $v$ is trial function and $n$ is the number of elements.

Using the Hermitian interpolation function as

$$H_1(x) = 1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3}; \quad H_2(x) = \frac{3x^2}{l^2} - \frac{2x^3}{l^3}; \quad H_3(x) = x - \frac{2x^2}{l} + \frac{x^3}{l^2}; \quad H_4(x) = -\frac{x^2}{l} + \frac{x^3}{l^2} \quad (4)$$

where $l$ is unit length.

Then unit mass matrix is generated and the stiffness matrices are as follows

$$[M] = \int_{0}^{l} S_2 [H]^T [H] dx$$
$$[H] = [H_1 \ H_2 \ H_3 \ H_4]$$
$$[K] = \int_{0}^{l} S_1 [B]^T [B] dx + \int_{0}^{l} [D]^T [D] dx$$
$$[B] = [H_1 \ H_2 \ H_3 \ H_4]$$
$$[D] = [H_1 \ H_2 \ H_3 \ H_4]$$

Integration into structure mass matrix and stiffness matrix, the system equation is as follows

$$[M] \{\ddot{x}\} + [K] \{x\} = 0 \quad (6)$$

3. Results and discussion

Using 100 equal length units in length, the first order frequency of clamped-clamped beam are determined and discussed as follows.
Figure 2. Fundamental frequency parameter versus the number of atomic layers with respect to different relaxation coefficients.

Firstly, it is shown from figure 2 that the fundamental frequency parameter may increase or decrease with an increase in the number of atomic layers, which is related to the value of relaxation coefficients. When the magnitude of the relaxation coefficient is less than 1, the fundamental frequency parameter decreases with increasing the number of atomic layers. On the other hand, when the relaxation coefficient is more than 1, the frequency parameter increases with increasing the number of atomic layers. It is indicated that there are two trends for the transverse vibration frequency of multiscale beams. Therefore, there are two modes of strengthening and weakening of the equivalent rigidity of the nanostructures as well. Furthermore, no matter which trend, when the number of atomic layers or thickness increases to a certain level, the fundamental frequency parameter with different relaxation coefficients tends to be a consistent. Therefore, the present study verifies the different non-local elasticity models again through a simple finite element approach.

Secondly, the first two mode functions of lateral displacement and rotational angle are as shown in figures 3 and 4, respectively, where \(N=50\), and \(r=0.6\) are utilized.

Figure 3. The first mode function.
From figures 3 and 4, we can see the vibration periods of the angle function and the deflection function become smaller with an increase of the mode number. Moreover, the amplitude increases and the initial phase angle changes obviously. It can be seen that the deflection frequency of the second mode increases to two times of the first mode, and the angular frequency of the second mode increases to one point five times of the first mode. In addition, angular variation is always half a period faster than that of the deflection for unit length.

4. Conclusions
The dynamic behaviours of a multiscale beam including a nanoscaled thickness and two macroscale plane scales are studied using a semi-continuum model and the finite element approach. It is concluded that the multiscale semi-continuum based fundamental frequency parameter may increase or decrease with an increase in the number of atomic layers or the thickness, which depends on the magnitude of the specific relaxation coefficient. The semi-continuum results recover to the corresponding classical counterparts when the number of atomic layers increases continuously and the thickness gets a macroscale value. The first two mode functions of the lateral displacement and rotational angle are determined as well. The phase of the angular is always half a period more than that of the deflection. The natural frequency parameter and mode function can be easily determined using the semi-continuum model and the finite element approach. The results reported in this paper may be useful for the design and control of multiscale structures.

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References
[1] Eringen A C Linear theory of nonlocal elasticity and dispersion of plane waves 1972 Int. J. Eng Sci. 10 425-35
[2] Eringen A C and Kim B S Stress concentration at the tip of the crack 1974 Mech. Res. Comm. 1 233-7
[3] Eringen A C On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves 1983 J. Appl. Phys. 54 4703-10
[4] Wang Q, Zhou G Y and Lin K C Scale effect on wave propagation of double-walled carbon nanotubes 2006 Int. J. Solids Struct. 43 6071-84
[5] Wang Q and Varadan V K Vibration of carbon nanotubes studied using nonlocal continuum mechanics 2006 Smart Mater. Struct. 15 659-66
[6] Wang Q Wave propagation in carbon nanotubes via nonlocal continuum mechanics 2005 J.
Wang C M, Zhang Y Y and Kitipornchai S Vibration of initially stressed micro- and nanobeams 2007 Int. J. Struct. Stab. Dy. 7 555-70
[8] Lim C W On the truth of nanoscale for nanobeams based on nonlocal elastic stress field theory: equilibrium, governing equation and static deflection 2010 Appl. Math. Mech. 31 37-54
[9] Lim C W, Li C and Yu J L Dynamic behaviour of axially moving nanobeams based on nonlocal elasticity approach 2010 Acta. Mech. Sin. 26 755-65
[10] Lim C W A nanorod (or nanotube) with lower Young’s modulus is stiffer? Is not Young’s modulus a stiffness indicator? 2010 Sci. China-Phys. Mech. Astron. 53 712-24
[11] Lim C W and Yang Y New predictions of size-dependent nanoscale based on nonlocal elasticity for wave propagation in carbon nanotubes 2010 J. Comput. Theor. Nanosci. 7 988-95
[12] Lim C W, Niu J C and Yu Y M Nonlocal stress theory for buckling instability of nanotubes: new predictions on stiffness strengthening effects of nanoscales 2010 J. Comput. Theor. Nanosci. 7 2104-11
[13] Lim C W, Li C and Yu J L Analytical solutions for vibration of simply supported nonlocal nanobeams with an axial force 2011 Int. J. Struct. Stab. Dy. 11(2) 257-71
[14] Fernandez S J and Zara R Vibrations of Bernoulli-Euler beams using the two-phase nonlocal elasticity theory 2017 Int. J. Eng. Sci. 119 232-48
[15] Rajasekaran S and Khaniki H B Bending, buckling and vibration of small-scale tapered beams 2017 Int. J. Eng. Sci. 120 172-88
[16] Khaniki H B On vibrations of nanobeam systems 2018 Int. J. Eng. Sci. 124 85-103
[17] Hashemi S H and Khaniki H B Dynamic behavior of multi-layered viscoelastic nanobeam system embedded in a viscoelastic medium with a moving nanoparticle 2017 J. Mech. 33 559-75
[18] Khaniki H B and Hosseini-Hashemi S Buckling analysis of tapered nanobeams using nonlocal strain gradient theory and a generalized differential quadrature method 2017 Mater. Res. Express 4 065003
[19] Khaniki H B and Hosseini-Hashemi S Dynamic transverse vibration characteristics of nonuniform nonlocal strain gradient beams using the generalized differential quadrature method 2017 Eur. Phys. J. Plus 132 500
[20] Sun C T and Zhang H T Size-dependent elastic moduli of platelike nanomaterials 2003 J. Appl. Phys. 93 1212-18
[21] Li C, Zheng Z J, Yu J L and Lim C W Static analysis of ultra-thin beams based on a semi-continuum model 2011 Acta Mech. Sin. 27 713-19
[22] Sun C Q, Tay B K, Zeng X T, Li S, Chen T P, Zhou J, Bai H L and Jiang E Y Bond-order-bond-length-bond-strength (bond-OLS) correlation mechanism for the shape-and-size dependence of a nanosolid 2002 J. Phys.: Condens. Matter. 14 7781-95
[23] Guo J G and Zhao Y P The size-dependent elastic properties of nanocrystals with surface effects 2005 J. Appl. Phys. 98 074306
[24] Li C, Shen Q, Yao L Q and Li S Lateral bending vibration of nanoscale ultra-thin beams using a semi-continuum model 2015 J. Comput. Theor. Nanosci. 12 2507-14
[25] Shen J P, and Li C A semi-continuum-based bending analysis for extreme-thin micro/nano-beams and new proposal for nonlocal differential constitutions 2017 Compos. Struct. 172 210-20