Quantum Information Theory of Entanglement

Nicolas J. Cerf∗ and Chris Adami†
California Institute of Technology

1 Introduction

The recent vigorous activity in the fields of quantum information processing (quantum computation) and quantum communication (quantum cryptography, teleportation, and superdense coding) has necessitated a better understanding of the relationship between classical and quantum variables [1, 2]. In classical physics, information processing and communication is best described by Shannon information theory [3], which succinctly associates information with random variables shared by two physical ensembles. Quantum information theory on the other hand is concerned with quantum bits (qubits) rather than bits, and the former obey quantum laws quite different from the classical physics of bits that we are used to. Most importantly, qubits can exist in quantum superpositions, a notion completely inaccessible to classical mechanics, or even classical thinking. To accommodate the relative phases in quantum superpositions, quantum information theory must be based on mathematical constructions which reflect these: the quantum mechanical density matrices. The central object of information theory, the entropy or uncertainty, has been introduced in quantum mechanics by von Neumann [4]

\[ S(\rho) = -\text{Tr} \rho \log \rho . \] (1)

Its relationship to the Boltzmann-Gibbs-Shannon entropy

\[ H(p) = -\sum_i p_i \log p_i \] (2)

becomes obvious when considering the von Neumann entropy of a mixture of orthogonal states. In this case, the density matrix \( \rho \) in (1) contains classical probabilities \( p_i \) on its diagonal, and \( S(\rho) \) equals (2). In general, however, quantum mechanical density matrices have off-diagonal terms, which, for pure states, reflect the relative quantum phases in superpositions.

In classical statistical physics, the concept of conditional and mutual probabilities has given rise to the definition of conditional and mutual entropies. These can be used to elegantly describe the trade-off between entropy and information in measurement, as well as the characteristics of a transmission channel. For example, for two ensembles \( A \) and \( B \), the measurement of (parts of the variables of) \( A \) by \( B \) is expressed by the equation for the entropies

\[ H(A) = H(A|B) + I(A : B) . \] (3)

Here, \( H(A|B) \) is the entropy of \( A \) after having measured those pieces that become correlated in \( B \) [thereby apparently reducing \( H(A) \) to \( H(A|B) \)], while \( I(A : B) \) is the information gained about \( A \) via the measurement of \( B \). As is well-known, \( H(A|B) \) and \( I(A : B) \) compensate each other, such that \( H(A) \) is unchanged, ensuring that the second law of thermodynamics is not violated in a measurement in spite of the decrease of \( H(A|B) \) [4]. Mathematically, \( H(A|B) \) is a conditional entropy, and is defined using the conditional probability \( p_{ij} \) and the joint probability \( p_{ij} \) describing random variables from ensembles \( A \) and \( B \):

\[ H(A|B) = -\sum_{ij} p_{ij} \log p_{ij} . \] (4)

The mutual information \( I(A : B) \), or mutual entropy \( H(A : B) \), on the other hand is defined via the mutual probability \( p_{i,j} = p_i p_j / p_{ij} \) as

\[ H(A : B) = -\sum_{ij} p_{ij} \log p_{i,j} . \] (5)

Simple relations such as \( p_{ij} = p_{i,j} p_i \) imply equations such as (6) and all the other usual relations of classical information theory. Curiously, a quantum information theory paralleling these constructions has up to recently never been attempted. Rather, a hybrid theory was constructed...
in which quantum probabilities (which mathematically describe bits instead of qubits) were inserted in the classical formulae given above (see, e.g., [13]). However, the wisdom of von Neumann ought not to be ignored. Below in Section 2 we show that a consistent quantum information theory can be developed that parallels the construction outlined above, while based entirely on matrices [3]. Only this theory allows for a proper description of quantum entanglement. In the third section, we analyse quantum measurement in this language and point out how the picture leads to a unitary and causal view of quantum measurement devoid of wavefunction collapse [3].

2 Quantum Information Theory

Let us consider the information theoretic description of a composite quantum system $AB$. A straightforward quantum generalization of Eq. (3) suggests the definition

$$S(AB) = -\text{Tr}_{AB}[\rho_{AB} \log \rho_{AB}]$$

(6)

for the quantum conditional entropy. In order for such an expression to hold, we need to define the concept of a “conditional” density matrix,

$$\rho_{A|B} = \left[\rho_{AB}^{1/n} (1_A \otimes \rho_B)^{-1/n}\right]^n \quad n \to \infty ,$$

(7)

which is the analog of the conditional probability $p_{ij}$. Here, $1_A$ is the unity matrix in the Hilbert space for $A$, $\otimes$ stands for the tensor product in the joint Hilbert space, and

$$\rho_B = \text{Tr}_A[\rho_{AB}]$$

(8)

denotes a “marginal” density matrix, analogous to the marginal probability $p_j = \sum_i p_{ij}$. The peculiar form involving the infinite limit in the definition of the conditional density matrix (3) is a technical requirement due to the fact that joint and marginal density matrices do not commute in general. However, the definition implies that the standard relation

$$S(AB) = S(A) + S(B) - S(AB)$$

(9)

holds for the quantum entropies as long as $\rho_{AB}$ is expressed in a product basis. Despite the apparent similarity between the quantum definition for $S(AB)$ and the standard classical one for $H(A|B)$, dealing with matrices (rather than scalars) opens up a quantum realm for information theory exceeding the classical one. The crucial point is that, while $p_{ij}$ is a probability distribution in $i$ (i.e., $0 \leq p_{ij} \leq 1$), its quantum analog $\rho_{A|B}$ is not a density matrix. Indeed, it can have eigenvalues larger than one, and, consequently, the associated conditional entropy $S(AB)$ can be negative. This means that it is acceptable, in quantum information theory, to have $S(AB) < S(B)$, i.e., the entropy of the entire system $AB$ can be smaller than the entropy of one of its subparts $B$, a situation which is of course forbidden in classical information theory. This happens for example in the case of quantum entanglement between $A$ and $B$, and illustrates that only a matrix-based quantum formalism can consistently account for this.

Similarly, the quantum analog of the mutual entropy can be constructed, defining a “mutual” density matrix

$$\rho_{AB} = \left[(\rho_A \otimes \rho_B)^{1/n} \rho_{AB}^{-1/n}\right]^n \quad n \to \infty ,$$

(10)

the analog of the mutual probability $p_{ij}$. As previously, this definition implies the standard relation

$$S(A:B) = S(A) + S(B) - S(AB)$$

(11)

between the quantum entropies. Clearly, this definition extends the classical notion of correlation entropy $H(A:B)$ to the quantum notion of entanglement entropy $S(AB)$. Thus, $S(A:B)$ must be considered as a general definition for the entropy of entanglement, and applies to pure as well as mixed states [3]. This is in contrast with various definitions of the entropy of entanglement which can be found in the literature [3]. As the definition of entanglement entropy covers classical correlations also, $S(AB)$ is a general measure of correlations and “super-correlations” in physics. As we will see later, our formalism implies that classical correlation can be achieved through quantum entanglement, while the reverse is not true.

The relations between $S(A)$, $S(B)$, $S(AB)$, $S(A|B)$, $S(B|A)$, and $S(A:B)$ are conveniently summarized by an entropy diagram, as shown in Fig. 3. The important difference between classical and quantum entropy diagrams is that the basic inequalities relating the entropies are “weaker” in the quantum case, allowing for negative conditional entropies and “excessive” mutual entropies [3]. For example, the upper bound for the mutual entropy (which is directly related to the channel capacity) is

$$H(A:B) \leq \min[H(A), H(B)]$$

(12)

in classical information theory, while it is

$$S(A:B) \leq 2 \min[S(A), S(B)]$$

(13)

in quantum information theory. This means that a quantum channel has a capacity that can reach twice the classical upper bound [3]; this limit is reached for instance in the superdense coding scheme [3].

In Fig. 3b, we show the entropy diagram corresponding to three limiting cases of a composite system of two dichotomic variables (e.g., 2 qubits): independent variables (case I), classically correlated variables (case II), and quantum entangled variables (case III). In all three cases, each subsystem taken separately is in a mixed state of entropy $S(A) = S(B) = 1$ bit. Cases I and II correspond to
classical situations (which can of course be described in our quantum information theory as well), while case III is a purely quantum situation which violates the bounds of classical information theory. Let us focus on case III, since cases I and II are standard. This case corresponds to an EPR pair characterized by the pure state

\[ |\psi_{AB}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) , \]

and, accordingly, it is associated with a vanishing combined entropy \( S(AB) = 0 \). Defining \( \rho_{AB} = |\psi_{AB}\rangle \langle \psi_{AB}| \) as the density matrix of the joint system, we see that subpart \( A \) (or \( B \)) has the marginal density matrix

\[ \rho_A = \text{Tr}_B[\rho_{AB}] = \frac{1}{2}(|00\rangle \langle 00| + |11\rangle \langle 11|) , \]

and is therefore in a mixed state of positive entropy. This purely quantum situation corresponds to the unusual entropy diagram \((-1, 2, -1)\) shown in Fig. 2b. That the EPR situation cannot be described classically is immediately apparent when considering the associated density matrices. The joint density matrix can be written in basis \{00, 01, 10, 11\} as

\[ \rho_{AB} = \begin{pmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{pmatrix} . \]  

Using the marginal density matrices

\[ \rho_A = \rho_B = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} , \]

we obtain for the conditional density matrix

\[ \rho_{A|B} = \rho_{AB}(1_A \otimes \rho_B)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} . \]

Plugging (17) and (18) into definition (16) immediately yields \( S(A|B) = -1 \). This is a direct consequence of the fact that \( \rho_{A|B} \) has one “unclassical” (> 1) eigenvalue.

It is thus misleading to describe an EPR-pair (or any of the Bell states) as a correlated state within classical information theory, since negative conditional entropies are crucial to its description. Still, classical correlations [with entropy diagram \((0, 1, 0)\)] emerge when observing an entangled EPR pair. Indeed, after measuring \( A \), the outcome of the measurement of \( B \) is known with 100% certainty. The key to this paradox lies in the correct description of the measurement process. Anticipating the next section, let us just mention that the observation of quantum entangled states such as an EPR pair gives rise to classical correlations between the two measurement devices, thereby creating the confusion between entanglement and correlation.

The above quantum information theory turns out to be extremely useful to describe \( n \)-body quantum systems, and it is a key to understand the creation of classical correlations from quantum entanglement. Let us for example consider a 3-body system \( ABC \) in an EPR-triplet state

\[ |\psi_{ABC}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) . \]

As it is a pure (entangled) state, the combined entropy is \( S(ABC) = 0 \). The corresponding ternary entropy diagram of \( ABC \) is shown in Fig. 2a. Note that the zero mutual entropy \( S(A : B : C) \) in the center of the diagram is generic to any fully entangled three-body system. When tracing over the degree of freedom associated with \( C \), say, the resulting marginal density matrix for subsystem \( AB \) is

\[ \rho_{AB} = \text{Tr}_C[\rho_{ABC}] = \frac{1}{2}(|00\rangle \langle 00| + |11\rangle \langle 11|) , \]

corresponding to a classically correlated system (case II). As the density matrix fully characterize a quantum system, subsystem \( AB \) (unconditional on \( C \), i.e., ignoring the existence of \( C \)) is in this case physically indistinguishable from a statistical ensemble prepared with an equal number of \(|00\rangle \) and \(|11\rangle \) states. Thus, \( A \) and \( B \) are genuinely

\[ ^1 \text{Although we use the term “EPR state” for the wavefunction} (14), \text{this state is in fact one of the Bell states, which are a generalization of the EPR singlet state.} \]

\[ ^2 \text{Note that for EPR pairs, joint and marginal density matrices commute, simplifying definitions (17) and (18).} \]
correlated if C is ignored. The “tracing over” operation depicted in Fig. 2b illustrates this creation of classical correlation from quantum entanglement. This feature will be central to the measurement process.

In summary, the EPR-triplet entails quantum entanglement between any part, e.g., C, and the rest of the system AB. The subsystem AB unconditional on C has a positive entropy \( S(AB) \) of 1 bit, and is indistinguishable from a classical correlated mixture. On the other hand, the entropy of C conditional on AB, \( S(C|AB) \), is negative and equal to -1 bit, thereby counterbalancing \( S(AB) \) to yield a vanishing combined entropy

\[
S(ABC) = S(AB) + S(C|AB) = 0.
\] (21)

The above can be extended in a straightforward manner to \( n \)-body composite systems.

### 3 Quantum Measurement

According to von Neumann [10], a consistent description of the measurement process must involve the interaction between the observed quantum system and a quantum measurement device. Such a view is in contrast with the Copenhagen interpretation of quantum mechanics (see, e.g., [1]), which states that the measurement is the non-causal process of projecting the wave-function, which results from the interaction with a classical apparatus. A classical apparatus is defined as one where the “pointer” variables take on definite values, and which therefore cannot reflect quantum superpositions. For 70 years, the Copenhagen interpretation has never failed in predicting a single experimental fact, which certainly has helped in cementing its reputation [1]. On the other hand, if the foundations of quantum mechanics are believed to be solid, it cannot be denied that measurement is not an abstract non-causal operation acting on wave functions, but rather a genuine interaction between two physical quantum systems: the observed system Q and the measurement device, or the ancilla A. This is the essence of the von Neumann theory of measurement.

Assume then that a quantum system is initially in state

\[
|Q\rangle = \sum_i \alpha_i |a_i\rangle
\] (22)

expressed in the basis \( \{|a_i\rangle\} \) of eigenvectors of an arbitrary observable (the one that we are measuring). Then, the von Neumann measurement is described by the unitary transformation that evolves the initial state of the joint system \( |Q,A\rangle = |Q,0\rangle \) into the state

\[
|QA\rangle = \sum_i \alpha_i |a_i, i\rangle
\] (23)

with \( \{|i\rangle\} \) denoting the eigenstates of the ancilla. Such a transformation was interpreted by von Neumann as inducing correlations between the system Q and the ancilla A. Indeed, if \( |Q\rangle \) is initially in one of the eigenstates \( |a_i\rangle \) (i.e., if it is not in a superposition), the “pointer” in A that previously pointed to zero now points to the eigenvector \( |i\rangle \) which labels outcome \( i \), suggesting that a measurement has been performed.

Now, a basic problem occurs if the initial state of Q is a superposition, as in Eq. (22), that is, if Q is not in an eigenstate of the considered observable. Then, according to Eq. (23), the apparatus apparently points to a superposition of \( i \)'s, a fact which obviously contradicts our everyday-life experience. In classical physics, a variable has, at any time, a definite value that can be recorded. Experiments show that a quantum measurement is probabilistic in nature, that is one of the possible outcomes (drawn from a probability distribution) becomes factual; in other words, a quantum superposition evolves into a mixed state. This apparent necessity led von Neumann to introduce an ad hoc, non-unitary, second stage of the measurement, called observation. In this stage, the measurement is “observed”, and a collapse occurs in order to yield a classical result from a quantum superposition. The central point in the resolution of the measurement paradox presented in Ref. 3 is that, in general, the state described by Eq. (23) is entangled, not just correlated. Entangled states have very peculiar properties, as emphasized earlier. For example, it has been shown that an arbitrary quantum state can not be cloned precisely because of the entanglement between the system Q and the ancilla A. If the system is in a state belonging to a set of orthogonal states, on the other hand, a faithful copy of the quantum state can be obtained applying von Neumann measurement. As a consequence it appears that an
arbitrary state (one which is not one of the eigenstates of the observable considered) can not be measured without creating entanglement.

In Ref. [5], we have shown that unitary evolution [such as Eq. (23)] can be reconciled with the creation of randomness in the measurement process if it is recognized that the creation of entanglement (rather than correlation) is generic to a quantum measurement, and if this entanglement is properly described in quantum information theory. This reconciliation is brought about by a redescription of the second stage of measurement, the observation. In order to observe the measurement, a macroscopic system involving generally a large number of degrees of freedom has to interact with Q. In Eq. (23), Q has interacted with a single degree of freedom of the ancilla A (first stage of the measurement), which led to an entangled state. As emphasized before, the creation of an entangled state does not mean that a measurement has been performed, since our (classical) perception of a measurement is intrinsically related to the existence of (classical) correlations. In order for classical correlations to emerge, a third degree of freedom (another ancilla A′) has to be involved. Now, iterating the von Neumann measurement, A′ interacts with AQ so that the resulting state of the combined system is

\[ |QAA′⟩ = \sum_i α_i |a_i, i, i⟩, \]

(24)

where the eigenstates of A′ are also denoted by |i⟩ for simplicity. The state so created is pure [S(QAA′) = 0], akin to an EPR-triplet since the system has undergone only unitary transformations from a pure initial state |Q, 0, 0⟩. The central point is that, considering the state of the entire ancilla AA′ unconditionally on system Q yields a mixed state

\[ ρ_{AA′} = \text{Tr}_Q[ρ_{QAA′}] = \sum_i |α_i|^2 |i, i⟩⟨i, i| \]

(25)
describing maximal correlation between A and A′. The second stage consists in observing this classical correlation (that extends, in practice, to the 10^{23} particles which constitute the macroscopic measurement device). Paradoxically, it is the physical state of the ancilla which contains the outcome of the measurement, whereas the quantum state Q itself must be ignored. This crucial point is easily overlooked, since intuition dictates that performing a measurement means somehow “observing the state of Q”. Rather, a measurement is constructed such as to infer the state of Q from that of the ancilla—but ignoring Q itself.

The correlations (in AA′) which emerge from the fact that a part (Q) of an entangled state (QAA′) is ignored give rise to the classical idea of a measurement. Randomness is generated, and the quantum entropy of the ancilla (unconditional on Q)

\[ S(AA′) = H[p_i] \quad \text{with} \quad p_i = |α_i|^2 \]

(26)
is thus thought to represent the “physical” entropy of Q. This happens to be the classical entropy associated with the probability distribution \( p_i = |α_i|^2 \), i.e., the probabilities predicted by quantum mechanics. Thus the unconditional entropy of the ancilla is equal to the entropy of Q predicted in orthodox quantum mechanics (which involves the projection of the wave function). Still, the entropy of Q conditional on AA′ is negative, and exactly compensates \( S(AA′) \) to allow for a vanishing entropy for the joint system. This then explains how measurement can be probabilistic in nature, while at the same time being described by a unitary process (which does not permit the evolution of pure into mixed states).

The illusion of a wave function collapse is also crucial in the physics of sequential measurements. If a second ancilla B (in general, also a large number of degrees of freedom) interacts with Q in order to measure the same observable (after a first measurement involving ancilla A), the result is an EPR-ntplet (consisting of all the degrees of freedom of A, B, and the measured quantum state Q). To simplify, let us consider two ancillary variables A and B (and neglect their amplification). Then, the final quantum state after the sequential measurement is

\[ |QAB⟩ = \sum_i α_i |a_i, i, i⟩ \]

(27)
illustrating clearly that the state of A and B (unconditional on Q) are classically correlated just as described earlier. This is the basic consistency requirement for two consecutive measurements of the same variable. The standard assertion of orthodox quantum mechanics is that, after the first measurement, the wave function of Q is projected on |a_i⟩, the observed eigenstate, so that any following measurement yields the same value i without any remaining uncertainty since the state of Q is now |a_i⟩. As we just showed [Eq. (23)], such a classical correlation between the outcome of two measurements actually involves no collapse; rather, it is due to the fact that one considers only part of an entangled system.

In order to conclude this section, let us consider the measurement of an EPR pair to clarify how quantum entanglement can have the appearance of classical correlation. Let us prepare a composite system Q_1Q_2 in the EPR-entangled state

\[ |Q_1Q_2⟩ = \frac{1}{\sqrt{2}}(|↑⟩|↓⟩ + |↓⟩|↑⟩) \]

(28)
and separate the two members at remote locations in space. At each location, the system (Q_1 or Q_2) is measured by interacting with an ancilla (A_1 or A_2), following the same procedure as before. In brief, each system (Q_1 or Q_2) becomes entangled with its corresponding ancilla, resulting in the entangled state

\[ |Q_1Q_2A_1A_2⟩ = \frac{1}{\sqrt{2}}(|↑↑11⟩ + |↓↓00⟩) \]

(29)
for the entire system. Note that an ancilla in state $|1\rangle$ means that a spin-up has been measured, and conversely. As previously, we describe the ancilla with just one internal variable, even though in practice it must be thought of as consisting of a large number of correlated ones. The important point here is that, despite the fact that $Q_1$ and $Q_2$ were initially in an entangled state (distinct from a classically correlated state), the state of the two ancilla unconditional on $Q_1$ and $Q_2$ is a mixed (classically correlated) state

$$\rho_{A_1A_2} = \frac{1}{2} (|00\rangle\langle00| + |11\rangle\langle11|) \quad (30)$$

This implies that the ancilia are correlated: the corresponding entropy diagram $(0,1,0)$ clearly shows that, after observing $A_1$, for instance, the state of $A_2$ can be inferred without any uncertainty. However, this must not be attributed to the existence of classical correlation between $Q_1$ and $Q_2$: rather it is the act of measuring which gives rise to this appearance.

4 Conclusions

In conclusion, we have shown that quantum entanglement can be consistently described using the notion of negative conditional entropy, an essential feature of a quantum information theory built entirely on density matrices. Negative quantum entropy can be traced back to “conditional” density matrices which admit eigenvalues larger than unity. A straightforward definition of mutual quantum entropy, or entropy of entanglement, can also be constructed using a “mutual” density matrix. This quantum formalism gives rise to new results, such as the violation of well-known bounds in classical information theory, and clarifies in which sense quantum entanglement can induce classical correlation. This last point allows for a fully consistent unitary description of quantum measurement, devoid of any assumption of wave-function collapse, which, at the same time, accounts for the creation of entropy (random numbers) in the measurement outcome. From a more general point of view, the quantum theory of entanglement reveals that quantum statistical mechanics is qualitatively very different from classical statistical mechanics, even though most of the formulae are identical. As quantum entanglement is a central feature of quantum computation, we believe that the present formalism will shed new light on decoherence (entanglement with an environment), as well as the error-correcting codes being devised to counteract it.

We would like to thank Hans Bethe for many enlightening discussions. This work was supported in part by the National Science Foundation Grant Nos. PHY91-15574 and PHY94-12818

References

[1] BENNETT, Charles H., “Quantum information and computation”, Physics Today 48(10) (1995), 24.

[2] BENNETT, Charles H. et al., “Concentrating Partial Entanglement by Local Operations”, Phys. Rev. A 53 (1996) 2046; BENNETT, Charles H. et al., “Purification of Noisy Entanglement and Faithful Teleportation via Noisy Channels”, Phys. Rev. Lett. 76 (1996), 722; BENNETT, Charles H. et al., “Mixed State Entanglement and Quantum Error Correction”, preprint quant-ph/9604024.

[3] BENNETT, Charles H. and Stephen J. WIESENR, Phys. Rev. Lett. 69 (1992), 2881.

[4] CERF, Nicolas, and Chris ADAMI, “Negative entropy and information in quantum mechanics”, preprint quant-ph/9512022.

[5] CERF, Nicolas, and Chris ADAMI, “Quantum mechanics of measurement”, preprint quant-ph/9605002.

[6] CERF, Nicolas, and Chris ADAMI, “Quantum information theory” in preparation.

[7] DI VINCENZO, David P., “Quantum Computation”, Science 270 (1995), 255.

[8] LANDAUER, Rolf, “Irreversibility and Heat Generation in the Computing Process”, IBM J. Res. Dev. 3 (1961), 113; BENNETT, Charles H., “The Thermodynamics of Computation – a Review”, Int. J. Theor. Phys. 21 (1982), 305.

[9] SHANNON, Claude, and W. WEAVER, The Mathematical Theory of Communication, University of Illinois Press, Urbana (1949).

[10] VON NEUMANN, Johann, Mathematische Grundlagen der Quantenmechanik, Springer Verlag, Berlin (1932).

[11] WHEELER, John A. and Wojciech H. ZUREK (eds.), Quantum Theory and Measurement, Princeton University Press (1983).

[12] WOOTTERS William K. and Wojciech H. ZUREK, “A single quantum cannot be cloned”, Nature 299 (1982), 802; DIEKS, Dennis, “Communication by EPR devices”, Phys. Lett. 92A (1982), 271.

[13] ZUREK, Wojciech H. (ed.), Complexity, Entropy and the Physics of Information, Santa Fe Institute Studies in the Sciences of Complexity Vol. VIII, Addison-Wesley (1990).