Bosonization of 2D Fermions due to Spin and Statistical Magnetic Field Coupling and Possible Nature of Superconductivity and Pseudogap Phases Below $E_g$

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Abstract

A ground state energy variational calculation of anyon gas with Hamiltonian included the interaction of spins of particles with anyon vector potential induced, i.e. statistical, magnetic field exhibits exact cancelation of terms connected with fractional statistics. This leads to bosonization of anyons due to coupling of their spins with statistical magnetic field. We presume that at the dense gas fluctuations of effective spins destroy the coupling and bosons become anyons. At the assumption that pseudogap (PG) boundary is temperature independent and when anyons are fermions we use this model to interpret experimental phase diagrams of Tallon and Loram hole and electron doped High-$T_c$ superconductors below PG energy $E_g$ and find the qualitative and quantitative agreement. We do the hypothesis that phase transition (PT) of bosons into Bose-Einstein condensate is not of second order, but of first order, close to second one, PG regime is meta stable phase of bosons, and $E_g = 0$ is the critical point of this PT. Bosons undergo PT into fermions on PG boundary. Described in the literature non-Fermi quasi-particles might be related to bosons with effective spins.

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I. INTRODUCTION

In 2D systems the concept of anyons provides us with unique opportunity to introduce antisymmetric property of fermion wave function into Hamiltonian. It is assumed that particles are to be spinless. The background of physics of anyons, the Aharonov-Bohm effect, itself is interesting phenomenon. Related experimental investigation [1] has been recently performed. Anyons enable to explicitly investigate relationship of spins of particles and statistics. From standard courses of relativistic quantum theory (see, for example, Ref. [2]) it is well known that particles with integer number of $\hbar/2$ spins should obey a Fermi statistics. It seems, this connection between spins and statistics is strong. In this work, however, we show that the gas of 2D anyons or fermions with spin $\hbar/2$ can undergo unexpected changing of statistics and this occurs due to Zeeman interaction of spins with statistical magnetic field [3] produced by vector potential of anyons. The calculation of ground state energy expectation value will be carried out in the framework of variational approach with cut-off parameter regularization [4, 5], which we developed recently.

II. THEORY AND MATHEMATICAL TREATMENT

The Hamiltonian is

$$\hat{H} = \frac{1}{2M} \sum_{k=1}^{N} \left( \left( \vec{p}_k + \vec{A}_\nu(\vec{r}_k) \right)^2 + M^2 \omega_0^2 |\vec{r}_k|^2 \right) + \frac{1}{2} \sum_{k=1}^{N} \left[ V(\vec{r}_k) + \sum_{j \neq k}^{N} \frac{e^2}{|\vec{r}_{kj}|} \right]$$

for gas of $N$ anyons with mass $M$ and charge $e$, confined by 2D parabolic well, interacting through Coulomb repulsion potential, in the presence of uniform positive background $V(\vec{r}_k)$. Here $\vec{r}_k$ and $\vec{p}_k$ represent the position and momentum operators of the $k$th anyon in two space dimensions, $\vec{A}_\nu(\vec{r}_k) = \hbar \nu \sum_{j \neq k}^{N} \vec{e}_z \times \vec{r}_{kj} / |\vec{r}_{kj}|^2$ is the anyon gauge vector potential [7], $\vec{r}_{kj} = \vec{r}_k - \vec{r}_j$, and $\vec{e}_z$ is the unit vector normal to the 2D plane. The factor $\nu$ determines the fractional statistics of anyon: $\nu = 0$ (bosons) and $\nu = 1$ (fermions).
In the variational scheme \[4\] we minimize the expression

\[ E = \frac{\int \Psi_T^*(\vec{R}) \hat{H} \Psi_T(\vec{R}) \ d\vec{R}}{\int \Psi_T^*(\vec{R}) \Psi_T(\vec{R}) \ d\vec{R}}. \] (3)

Here \( \vec{R} = \{\vec{r}_1, \ldots, \vec{r}_N\} \) is the configuration space of the \( N \) anyons. When energies are expressed in units of \( \hbar \omega_0 = \hbar^2/(ML^2) \) and lengths in units of \( L \) the normalized trial wave function in the bosonic representation of anyons reads

\[ \Psi_T(\vec{R}) = \left(\frac{\alpha}{\pi}\right)^{N/2} \prod_{k=1}^{N} \exp \left(-\alpha \frac{(x_k^2 + y_k^2)}{2}\right). \] (4)

Here \( \alpha \) is variational parameter. The harmonic potential regularization \[5\] with tending number of particles \( N \) to infinity yields the ground state energy of infinite Coulomb anyon gas.

Now we introduce in the Hamiltonian the term

\[ \frac{\hbar}{M} \sum_{k=1}^{N} \hat{s} \cdot \vec{b}_k, \] (5)

with statistical magnetic field \[3\]

\[ \vec{b}_k = -2\pi \hbar \nu \hat{e}_z \sum_{j(k\neq j)} \delta^{(2)}(\vec{r}_k - \vec{r}_j), \] (6)

which can be derived if calculates \( \vec{b}_k = \vec{\nabla} \times \vec{A}_\nu(\vec{r}_k) \) by using Eq. \[2\]. The sign in Eq. \[5\] is taken for electrons with charge \( e = -|e| \). For holes, with charge \( e = |e| \), we need to change a sign for \( \nu \) in Eqs. \[2\] and \[6\], then Eq. \[5\], as also the expectation value for energy, Eq. \[9\], (see below), will retain the sign.

For \( s_z = \hbar/2 \) and if we take into account that length unit is \( L \), then \( \delta^{(2)}(\vec{r}) \) should be replaced by \( \delta^{(2)}(\vec{r})/L^2 \), hence,

\[ \frac{\hbar}{M} \sum_{k=1}^{N} \hat{s} \cdot \vec{b}_k = -\pi \nu \frac{\hbar^2}{ML^2} \sum_{k,j(k\neq j)} \delta^{(2)}(\vec{r}_k - \vec{r}_j). \] (7)

The calculation of expectation value Eq. \[13\] when Hamiltonian is Eq. \[7\] and wave function \( \Psi_T(\vec{R}) \) is Eq. \[14\] gives (we omit a factor \( \hbar^2/(ML^2) \))

\[ -\pi \nu \sum_{k,j(k\neq j)} \int \Psi_T(\vec{R}) \delta^{(2)}(\vec{r}_k - \vec{r}_j) \Psi_T(\vec{R}) \ d\vec{R} \]

\[ = -\nu \alpha N(N - 1)/2. \] (8)
The total expectation value for energy, Eq. (3), including all terms of Eq. (1), is

\[ E = \frac{N\mathcal{N}}{2} + \frac{N}{2\alpha} + N\mathcal{M}^{1/2} - \frac{\nu\alpha N(N - 1)}{2}, \]  

(9)

where the term for \( \mathcal{M} \) is responsible for Coulomb interaction (see Refs. [4, 5]).

We did in [4] a cut-off parameter regularization and found \( \mathcal{N} = 1 + \nu(N - 1) \). This expression for \( \mathcal{N} \) successfully describes energy of confined anyons [4] with and without Coulomb interaction as well as one of infinite anyon gas [5] with Coulomb interaction. Substituting \( \mathcal{N} \) into energy \( E \), Eq. (9), we see the exact cancelation of terms with \( \nu \) factors. This result for energy can be obtained if we put \( \nu = 0 \), i.e., for case of bosons. As the energy of bosons is lower than one for anyons and fermions, there appears a coupling of spin with statistical magnetic field for every particle or bosonization of 2D anyons and fermions.

One can assume the fluctuations of spins coupled to magnetic field. Therefore, bosons with effective spins might look like as Fermi particles. However, fermions with different spins are independent [8]. Thus, the spins of bosons interact with each other and do not interact with spins of another fermions if they exist in the system. We introduce a some correlation length, inside of which spins of bosons interact with each other. For temperature \( T = 0 \) we denote it \( \xi_o \). The increase of fluctuations destroys the coupling, and bosons become the anyons or fermions. This occurs when the gain in the energy due to fluctuations of spins of bosons is equal to energy difference between the anyon (or Fermi) and Bose ground states.

The interaction of spins of bosons we bring in the form

\[ e^{-\frac{r_o}{\xi_o}} \sum_{k=1}^{N} \hat{s}_{k+\delta} \cdot \hat{s}_k. \]  

(10)

Here it was introduced a factor \( e^{-\frac{r_o}{\xi_o}} \) with \( r_o \) is being the mean distance between particles. For screened by magnetic field spins \( \xi_o \) is to be assumed phenomenological and taken from experiment.

We establish the explicit form of Eq. (10). The growth of boson spin fluctuations should cancel term, Eq. (5), in the Hamiltonian. Therefore, for dense \( (r_o < \xi_o) \) Bose gas there should be \( \hat{s}_{k+\delta} = -\hbar \vec{b}_k/M. \)

The Hamiltonian of bosonized infinite anyon Coulomb gas with interaction of spins has
a form

\[ \hat{H} = \frac{1}{2M} \sum_{k=1}^{N} \left[ \left( \vec{p}_k + \vec{A}_\nu(\vec{r}_k) \right)^2 + MV(\vec{r}_k) \right] + \frac{1}{2} \sum_{k,j \neq k}^{N} e^2 |\vec{r}_{kj}| + \frac{\hbar(1 - e^{-r_o/\xi_o})}{M} \sum_{k=1}^{N} \vec{s} \cdot \vec{b}_k. \]  \tag{11}

For anyon Coulomb gas with density parameter \( r_s > 2 \), where \( r_s = r_o \) in Bohr radius \( a_B \) units, the approximate ground state energy per particle expressed in \( \text{Ry} \), Rydberg energy units, and at \( \nu = 1 \) with 15% accuracy represents Quantum Monte Carlo result of Tanatar and Ceperley for 2D spin polarized electrons. Here

\[ E_0 \approx -\frac{c_{WC}^{2/3} f^{2/3}(\nu, r_s)}{r_s^{4/3}} + \frac{7\nu f^{4/3}(\nu, r_s)}{3c_{WC}^{2/3} r_s^{8/3}}. \]  \tag{12}

In Eqs. (12) and (13) \( c_{WC} = 3.2903 \) and \( c_{BL} = 1.2934 \).

The expectation value calculation of energy with Hamiltonian, Eq. (11), gives analogous expression for gas of bosonized anyons, however, one needs to replace in it \( \nu \) by \( \nu e^{-r_s/\xi_o} \) and now \( \xi_o \) is in \( a_B \) units.

### III. COMPARISON WITH EXPERIMENTAL PHASE DIAGRAMS

Microscopical mechanism of High-\( T_c \) superconductivity (HTSC), as also the symmetry of its condensate wave function, are challenging problems of condensed matter physics. Some aspects of the theory, related to these topics, have been considered in papers. We apply the model for clarification of experimental phase diagrams of hole and electron doped cuprates proposed recently in Ref. and Refs. We consider the bosonized fermions with \( \nu = 1 \). To become fermions bosons should overcome the energy difference

\[ \Delta_B^{o} = \frac{7(1 - e^{-r_s/\xi_o}) f^{4/3}(0, r_s)}{3c_{WC}^{2/3} r_s^{8/3}}, \]  \tag{14}

the gap of superconductivity. Our approach, as in 5, corresponds to spinless or fully spin polarized fermions. One needs to have deal with normal, i.e., no spin polarized electron
liquid. However, the accuracy of our calculations is lower than the difference of Tanatar and Ceperley data for ground state energy for these both phases of electrons.

We express $\Delta_B^0$ as function of density of dopants $n_s = 1/(\pi r_s^2)$ and connect $n_s$ with $p$, fractional part of doped hole or electron per atom $Cu$. At big values of $r_s$ or small $n_s$ one can neglect the exponential factor in Eq. (14) and $\Delta_B^0 \sim n_s \sim p$. At small $r_s$ or big $n_s$, without this factor, $\Delta_B^0$ would have $\Delta_B^0 \sim n_s^{2/3} \sim p^{2/3}$, but $e^{-r_s/\xi_o}$ suppresses this dependence to zero and the law of it depends from function $\xi_o(p)$. For this limit of $r_s$ we assume that $\Delta_B^0(p)$ coincides with experimental dependence $E_g(p)$. Extrapolating this asymptotic expression of $\Delta_B^0(p)$ to small values of $p$ and equating it to $E_g(p)$ one finds the empirical dependence $\xi_o(p)$. For it $\xi_o(p) \sim 1/E_g(p)$.

To be sure in the correctness of our 2D density of holes, we express it in $cm^{-2}$ units and compare with experimental one. For elementary structural cell of almost all cuprate $ab$ planes $a = 3.81\,\text{Å}$ and $b = 3.89\,\text{Å}$. Assuming that it has one atom of $Cu$, the density is $n_{ab} = N_{ab} \cdot p \, cm^{-2}$, where $N_{ab}$ is number of elementary cells per $1 cm^2$ square. One obtains $n_{ab} = 6.7472 \cdot p \cdot 10^{14} \, cm^{-2}$. We find experimental density $n_{ab}^{exp}$ from paper [15] for Y-123 ($YBa_2Cu_3O_7$) compound at optimal doping concentration of holes $p \approx 0.16$. It is $n_{ab}^{exp} = 0.9137 \cdot 10^{14} \, cm^{-2}$. Our optimal doping value for $n_{ab}$ is $1.0795 \cdot 10^{14} \, cm^{-2}$. From experiment [15] also leads important information about approximate equality of total density of holes to density of HTCS carriers. A comparison of this value for $n_{ab}$ with experimental concentration [16] of carriers for Fractional Quantum Hall Effect (FQHE) $n_{FQHE} \sim 10^{11} \div 10^{12} \, cm^{-2}$ shows importance for HTCS, as also for FQHE [6], of long-range, not screened, Coulomb potential interaction between particles. The screened potential is supposed to be as justification [17] for the treatments based on the Hubbard model.

The Fig. 1 displays pseudogap (PG) boundary energy $E_g$ (Fig. 11 from paper [11]), HTCS gap energy $\Delta_o = 4K_B T_c$, which was evaluated by empirical formula $T_c = T_{c,max}[1 - 82.6(p - 0.16)^2]$ with $T_{c,max} = 95 \, K$ for $Bi_{2212}$ ($Bi_2Sr_2CaCu_2O_{8+\delta}$) compound, and HTCS gap energy calculated from Eq. (14) as function of $p$. As we see our $\Delta_B^0$ has the same magnitude as experimental gap, but is qualitatively different from generally accepted "dome" like temperature-concentration phase diagram. However, it is in accordance with Fig. 10 of paper [11] of Tallon and Loram and their conclusion that PG energy $E_g$ up to $p_c \approx 0.19$ separates Bose-Einstein condensate into regions, where density of Cooper pairs is small and big (weak and strong superconductivity).
FIG. 1: The experimental PG $E_g$, HTCS gap $\Delta_o = 4K_B T_c$ (experiment for hole doped Bi - 2212 compound), and calculated from formula Eq. (14) HTCS gap for bosons $\Delta_o^B$ energies in Kelvin temperature (K) units as function of concentration of holes $p$.

For phase diagram data of electron doped cuprates we use Ref. [12] for NCCO ($Nd_{2-x}Ce_xCuO_4$) and Ref. [13] for PCCO ($Pr_{2-x}Ce_xCuO_4$). It was shown that $E_g/(k_B T^*) \approx 10$ for NCCO in $0.05 \leq p \leq 0.10$ and $E_g/(k_B T^*) \approx 11$ for PCCO in $0.11 \leq p \leq 0.17$, therefore, we assume $E_g/(k_B T^*) \approx 10$ for entire interval of $p$ and interpolate $E_g(p)$ with dependence $E_g = -3916.67p + 675.83$. For experimental HTCS gap we also assume $\Delta_o/(k_B T_c) \approx 10$. Fig. 2 shows the $p$ dependence of experimental $E_g$, $\Delta_o = 10k_B T_c$ and $\Delta_o^B$ calculated from Eq. (14) by using the above spacing constants of a and $b$ for elementary structural cell. Comparing with Fig. 1, we see the same qualitative and quantitative result. More obvious is extension of our $\Delta_o^B$ to small values of $p$, while experimental $\Delta_o$ starts with $p = 0.13$. However, absolute values of both HTCS gaps are nearly equal.

The spin correlation length $\xi_o$ sharply increases when $p$ approaches $p_c$, which might mean the vicinity of phase transition (PT). Fig. 6 of Tallon and Loram paper shows that experimental short-range Anti- Ferromagnetic (AF) correlations scale like $E_g(p)$ dependence and vanish at $p_c$. The presumption about correlations of spins of bosons would mean that there would exist the competing of these correlations with AF correlations when increasing of correlation length $\xi_o$ leads to revealing the Fermi like spin correlations of bosons in the
FIG. 2: The experimental PG $E_g$, HTCS gap $\Delta_o = 10K_B T_c$ (experiments for electron doped NCCO and PCCO compounds), and calculated from formula Eq. $\Delta^B_o$ HTCS gap for bosons energies in Kelvin temperature (K) units as function of concentration of electrons $p$.

extended region of sample and thus, assuming that these spins now interact with ones of fermions, to suppressing of short-range AF correlations inside of this region. The definition of competing of HTCS and AF phases, which is widely used in the literature, in this case would find the natural interpretation. Only two phases, which have the relative nature, can compete with each other. AF magnetic phase and based on Cooper pairs HTCS phase can not coexist, at least, for high, as for HTCS, temperatures, because they are antagonistic. One would expect the Cooper pairing in right side from $p_c$, where the magnetic correlations are not strong as AF correlations.

We do the hypothesis that due to vicinity of structural PT to HTCS state in the cuprates for concentration of holes below optimal the induced by it mechanical strain would strengthen a quadratic striction and therefore, the PT into Bose-Einstein condensate is not a second order, as it should be, but first order, close to second one. The PG regime would correspond to meta stable phase of bosons. At boundary $E_g$ bosons finally undergo PT into fermions. The critical concentration $p_c$, at which $E_g = 0$, would be considered as critical point of first order PT. In general, the gas of doped holes and electrons in the PG region might be considered as mixture of Coulomb interacting single particle bosons and fermions. In the deep underdoped regime and low temperatures dominate bosons, close to
$E_g$ dominate fermions.

IV. SUMMARY

We have shown that interaction of spins $s_z = \hbar/2$ of anyons or fermions with statistical magnetic field leads to bosonization. For fermions, as the particular case of anyons, we have applied our model for the possible clarification of phase diagrams of HTCS of hole and electron doped cuprates for region below pseudogap $E_g$ and found qualitative and quantitative agreement with experiment.

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