Inside the degenerate horizons of regular black holes

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The regularized stress-energy tensor of the quantized massive scalar, spinor and vector fields inside the degenerate horizon of the regular charged black hole in the (anti-)de Sitter universe is constructed and examined. It is shown that although the components of the stress-energy tensor are small in the vicinity of the black hole degenerate horizon and near the regular center, they are quite big in the intermediate region. The oscillatory character of the stress-energy tensor can be ascribed to various responses of the higher curvature terms to the changes of the metric inside the (degenerate) event horizon, especially in the region adjacent to the region described by the nearly flat metric potentials. Special emphasis is put on the stress-energy tensor in the geometries being the product of the constant curvature two-dimensional subspaces.

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I. INTRODUCTION

The class of the regular black holes, i.e., the black holes for which the curvature invariants are regular as $r \to 0$ has received much attention recently. The first constructions of such systems (in which singular interiors have been replaced by the regular cores) appeared in the mid-1960s in the works of Sakharow, Gliner and Bardeen [1–5]. Nowadays, quite a number of solutions of this type are known, supplemented by the no-go theorems which forbid their construction in certain circumstances [6]. One of the most interesting and intriguing solutions of this type has been proposed by Ayon-Beato and Garcia [7] and subsequently reinterpreted by Bronnikov [8]. It is a static and spherically-symmetric solution of the coupled system of nonlinear electrodynamics and gravity describing a class of the two-parameter regular black holes. For $r/M \gg 1$ and for small values of the ratio $|Q|/M$, where $Q$ is the (magnetic) charge and $M$ is the black hole mass, it closely resembles the Reissner-Nordström solution. Noticeable differences appear for the extreme

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and nearly extreme configurations. We will refer to this family of the exact solutions of the Einstein equations as the ABGB black holes.

Although the metric potentials of the ABGB black hole involve the hyperbolic functions of \( r \) it can be demonstrated that the radii of the event and the inner horizon can be expressed in terms of the real branches of the Lambert functions \([9, 10]\). This property together with the relative simplicity of the metric tensor have stimulated continued interest in the ABGB black holes \([11–15]\).

In this paper we shall analyze a class of the regular ABGB black holes in the asymptotically (anti-)de Sitter universe. It clarifies our earlier results presented in Ref. \([16]\) and extends them to the case of the negative cosmological constant. Subsequently, we construct the stress-energy tensor of the quantized massive scalar, spinor and vector field inside the extremal and ultraextremal horizon of the degenerate black hole. The calculations are carried out within the framework of the Schwinger-De Witt approximation \([9, 17–21]\). This approach is quite general as the sole criterion for its applicability is demanding that the ratio of the Compton length associated with the field and the characteristic radius of the curvature of the background geometry is small. In practice, it turns out that the reasonable results can be obtained for \( \mathcal{M} m > 2 \) \([22]\).

We shall explicitly demonstrate that important and interesting information regarding the stress-energy tensor can easily be obtained by studying the geometry of the closest vicinity of the degenerate horizon and the regular center. Indeed, near the regular center of the black hole the line element reduces to that of the (anti-)de Sitter, whereas near the degenerate event horizon the line element has a product form, where each part describes maximally symmetric two-dimensional subspace. This is a very fortunate feature of the problem as the stress-energy tensor is extremely complicated. On the other hand, the full stress-energy tensor inside the degenerate horizons reveals interesting features. Indeed, although the interior of the degenerate regular black hole in the asymptotically (anti)-de Sitter universe is described by simple functions, the components of the stress-energy tensor in the intermediate region are many orders of magnitude greater that the components at the center and in the neighborhood of the horizon. This behavior, although not quite unexpected, is totally different from what one usually encounters in the calculations of the quantum processes in the curved background, provided the calculations are carried out in the regions that are sufficiently distant from the central singularity. In the regular models the singularity is absent and the oscillatory character of the components of the stress-energy tensor is due to various response of the curvature terms constituting \( T^b_a \) to the changes of the metric inside the (degenerate) event horizon. The lengthy and complicated formulas describing the components of the stress-energy tensor of the quantized massive scalar, spinor and vector fields can be downloaded from the computer code.
Moreover, it should be noted that the general solution constructed here provides a natural setting for calculations presented in Ref. [24].

The paper is organized as follows: The regular ABGB black holes in the (anti-)de Sitter universe are introduced in the next section. In section III the geometries of the vicinity of the degenerate horizons are analyzed with the special emphasis put on the Bertotti-Robinson, Nariai, anti-Nariai and Plebanski-Hacyan solutions. The renormalized stress-energy tensor of the massive scalar, spinor and vector field inside the horizon of the regular ultraextremal black hole is constructed and examined in section IV. Throughout the paper a natural system of units is adopted. The sign convention is that of MTW [25].

II. REGULAR BLACK HOLES IN (ANTI-)DE SITTER UNIVERSE

The coupled system of equations describing the nonlinear electrodynamics and gravity considered in this paper can be constructed from the action

$$S = \frac{1}{16\pi} \int (R - 2\Lambda) \sqrt{-g} d^4x + S_m, \quad (1)$$

where

$$S_m = -\frac{1}{16\pi} \int \mathcal{L}(F) \sqrt{-g} d^4x \quad (2)$$

and $\mathcal{L}(F)$ is a functional of $F = F_{ab}F^{ab}$ with $\mathcal{L}(F) \to F$ as $F \to 0$. All symbols have their usual meaning and $\Lambda = \varepsilon l^2$. The parameter $\varepsilon$ may take one of the three values: 1 for the positive cosmological constant, −1 for the negative one and 0 if the cosmological term is absent.

The standard definition of the stress-energy tensor

$$T^{ab} = \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g_{ab}} S_m \quad (3)$$

leads to the following expression

$$T^b_a = \frac{1}{4\pi} \left( \frac{d\mathcal{L}(F)}{dF} F_{ca} F^{cb} - \frac{1}{4} \delta^b_a \mathcal{L}(F) \right) \quad (4)$$

whereas equation of the nonlinear electrodynamics can be written as

$$\nabla_a \left( \frac{d\mathcal{L}(F^{ab})}{dF} \right) \quad \text{and} \quad \nabla^a \star F^{ab} = 0, \quad (5)$$

where an asterix denotes, as usual, the Hodge dual. It is clear that the above equations reduce to their classical counterparts as $F \to 0$. 
Let us consider spherically-symmetric and static configuration described by the line element of the form
\[ ds^2 = -e^{2\psi(r)}f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\Omega^2, \] (6)
where \( f(r) \) and \( \psi(r) \) are unknown functions. Since the Lie derivative of the tensor \( F_{ab} \) with respect to the generators of the \( O(3) \) group vanish, the only (independent) components of \( F_{ab} \) compatible with the assumed symmetry are \( F_{01} \) and \( F_{23} \). Simple integration yields
\[ F_{23} = Q \sin \theta \] (7)
and
\[ r^2e^{-2\psi} \frac{d\mathcal{L}(F)}{dF} F_{10} = Q_e, \] (8)
where \( Q \) and \( Q_e \) are the integration constants interpreted as the magnetic and electric charge, respectively. In the latter we shall assume that the electric charge vanishes, and, consequently, \( F \) is given by
\[ F = \frac{2Q^2}{r^4}. \] (9)
With the substitution
\[ f(r) = 1 - \frac{2M(r)}{r} \] (10)
the time and the radial components of the Einstein field equations with the cosmological term
\[ G^b_a + \varepsilon l^2\delta^b_a = 8\pi T^b_a \] (11)
assume simple and transparent form
\[ -\frac{2}{r^2} \frac{dM}{dr} + \varepsilon l^2 = -\frac{1}{2}\mathcal{L}(F) \] (12)
and
\[ -\frac{2}{r^2} \frac{dM}{dr} + \frac{2}{r} \left(1 - \frac{2M}{r}\right) \frac{d\psi}{dr} + \varepsilon l^2 = -\frac{1}{2}\mathcal{L}(F) \] (13)
and can be formally integrated. The angular component of (11)
\[ \left(1 - \frac{2M}{r}\right) \left[ \frac{d^2\psi}{dr^2} + \left(\frac{d\psi}{dr}\right)^2 \right] - \frac{M}{r} \frac{d^2M}{dr^2} + \frac{1}{r} \left(\frac{M}{r} - 3\frac{dM}{dr} + 1\right) \frac{d\psi}{dr} + \varepsilon l^2 = 2\frac{d\mathcal{L}(F)}{dF} \frac{Q^2}{r^4} - \frac{1}{2}\mathcal{L}(F) \] (14)
is merely constraint equation. Further considerations require specification of the Lagrangian \( \mathcal{L}(F) \).

We demand that it should have proper asymptotic, i.e., in a weak field limit it should approach \( F \).

Following Ayón-Beato, García and Bronnikov let us chose it in the form

\[
\mathcal{L}(F) = F \left[ 1 - \tanh^2 \left( s \sqrt{\frac{Q^2 F}{2}} \right) \right],
\]

where

\[
s = \frac{|Q|}{2b},
\]

and the free parameter \( b \) will be adjusted to guarantee regularity at the center. Inserting Eq. (16) into (15) and making use of Eq. (9) one has

\[
8\pi T^t_t = 8\pi T^r_r = -\frac{Q^2}{r^4} \left( 1 - \tanh^2 \frac{Q^2}{2br} \right).
\]

Now the equations can easily be integrated in terms of the elementary functions:

\[
M(r) = C_1 - b \tanh \frac{Q^2}{2br} + \frac{\varepsilon l^2 r_+^3}{6}, \quad \psi(r) = C_2
\]

where \( C_1 \) and \( C_2 \) are the integration constant. Making use of the condition

\[
\psi(\infty) = 0
\]

gives \( C_2 = 0 \). The next step requires some prescience: let us assume that the solution describes black hole and its event horizon is located at \( r = r_+ \). The integration constant \( C_1 \) can be determined from the condition

\[
M(r_+) = \frac{r_+}{2}.
\]

The solution for \( M(r) \) can be written in the form

\[
M(r) = \frac{r_+}{2} + b \tanh \frac{Q^2}{2br} + \frac{\varepsilon l^2 r_+^3}{6} - b \tanh \frac{Q^2}{2br} + \frac{\varepsilon l^2 r_+^3}{6}.
\]

It can be demonstrated that for \( \varepsilon = -1 \) the first three terms in the right hand side of the above equation comprise the Abbott-Deser mass of the black hole:

\[
\mathcal{M}_{AD} = \frac{r_+}{2} + b \tanh \frac{Q^2}{2br} + \frac{l^2 r_+^3}{6}.
\]

On the other hand the \( \varepsilon = 1 \) case is slightly more subtle. Nevertheless, one can always refer to the horizon-defined mass \(^1\)

\[
\mathcal{M}_H = \frac{r_+}{2} + b \tanh \frac{Q^2}{2br} - \frac{\varepsilon l^2 r_+^3}{6}.
\]

\(^1\) In Ref. [16] the argumentation leading to relation of the integration constant to the black hole mass is erroneous. The resulting line element is, however, correct.
In the latter, for brevity, we shall denote both masses by a single symbol $\mathcal{M}$. Demanding the regularity of the line element as $r \to 0$ yields $b = C_1 \equiv \mathcal{M}$, and, consequently, the resulting line element has the form (20) with $\psi(r) = 0$ and

$$f(r) = 1 - \frac{2\mathcal{M}}{r} \left( 1 - \tanh \frac{Q^2}{2\mathcal{M}r} \right) - \frac{\varepsilon \ell^2 r^2}{3}. \quad (24)$$

It should be noted that with such a choice of $C_1$ and $b$ both (22) and (23) are consistent with (24), i.e., depending on the sign of the cosmological constant $f(r+) = 0$ is equivalent either to (22) or to (23). We shall call this solution the Ayón-Beato-García-Bronnikov-(anti-)de Sitter solution (ABGB-(a)dS). It could be easily shown that putting $Q = 0$ yields the Schwarzschild-de Sitter (Kottler) solution, whereas for $\Lambda = 0$ one gets the Ayón-Beato, García line element as reinterpreted by Bronnikov.

To study ABGB-(a)dS line element it is convenient to introduce the dimensionless quantities $x = r/\mathcal{M}$, $q = |Q|/\mathcal{M}$ and $\ell = l\mathcal{M}$. Here we shall concentrate on configurations with at least one horizon. First, let us observe that for small charges ($q \ll 1$) as well as at great distances from the black hole ($x \gg 1$) the ABGB-(a)dS solution closely resembles that of RN-(a)dS. Indeed, expanding the function $f(r)$ in powers of $q^2$ (or in powers of $x^{-1}$) one obtains

$$f = 1 - \frac{2\mathcal{M}}{r} + \frac{Q^2}{r^2} - \frac{\varepsilon \ell^2 r^2}{3} - \frac{Q^6}{12\mathcal{M}^2 r^4} + ... \quad (25)$$

Similarly, near the center one has

$$f \sim 1 - \frac{4\mathcal{M}}{r} \exp \left( \frac{-Q^2}{\mathcal{M}r} \right) - \frac{\varepsilon \ell^2 r^2}{3} \quad (26)$$

and the metric in the closest vicinity of $r = 0$ may by approximated by the (anti-)de Sitter line element as the second term in the right hand side of the approximation rapidly goes to zero. It should be noted that for a given $\mathcal{M}$ and $Q$ and small cosmological constant ($\ell^2 \ll 1$) some of the features of the ABGB-(a)dS geometry are to certain extend similar to that of the ABGB spacetime. Simple analysis shows that for $\varepsilon = 1$ there are, at most, three distinct positive roots of the equation $f(r) = 0$. One expects that the two of them are located closely to the inner and event horizons of the ABGB black hole, whereas the third one (absent in the ABGB geometry) is approximately located at $x_c \approx \sqrt{3}/\ell$ and interpreted as the cosmological horizon. For $\varepsilon = -1$ these similarities are even more transparent as there is no the cosmological horizon. Qualitative behavior of the function $f(r)$ with $\varepsilon > 0$ is displayed in Fig. 1.

Although it is impossible for $\varepsilon \neq 0$ to give exact solutions representing location of the horizons, one can easily solve

$$1 - \frac{2}{x} \left( 1 - \tanh \frac{Q^2}{2x} \right) - \frac{1}{3} \varepsilon \ell^2 x^2 = 0 \quad (27)$$
FIG. 1: The qualitative behavior of the function \( f(r) \) for \( r_- = r_+ < r_c, \ r_- < r_+ < r_c \) and \( r_c < r_+ = r_c \).

with respect to \( q \). Simple manipulations give

\[
q = \pm \sqrt{x \ln \frac{12 - 3x + \varepsilon \ell^2 x^3}{x(3 - \varepsilon \ell^2 x^2)}}. \tag{28}
\]

The function \( q(x) \) is plotted in Figs. 2 and 3, where each curve is labeled by the cosmological constant \( \varepsilon \ell \). The inner, event and cosmological horizon (denoted by \( x_- \), \( x_+ \) and \( x_c \), respectively) lie on intersections of the \( q \)-constant line and the curve drawn for a constant \( \ell \). For \( \varepsilon \leq 0 \) the outermost curve represent the special case of the ABGB black hole with the inner and event horizon expressed in terms of the real branches of the Lambert W function \([9, 10]\). The extrema of each curve represent degenerate configuration with \( x_- = x_+ \). Especially interesting is the solution describing the extreme ABGB black hole \([10, 26]\).

For \( \varepsilon > 0 \) Eq. (27) has, in general, three positive roots, which can merge leading to various interesting configurations. Indeed, for special choices of the parameters one can have a configuration with a degenerate and a nondegenerate horizon or one triply degenerate horizon. The first configuration is characterized by \( r_- < r_+ < r_c \) whereas the second configuration contains two subclasses depending on which horizons do merge. As the degenerate horizons are located at simultaneous zeros of \( f(r) \) and \( f'(r) \) they differ by a sign of the second derivative of \( f \). The first of degenerate configurations, called cold black hole, is characterized by \( f''(r_+) > 0 \) and \( r_- = r_+ < r_c \).

On the other hand, the configuration characterized by \( f''(r_+) < 0 \) and \( r_- < r_+ = r_c \), is usually referred to as the charged Nariai black hole. Finally, for the triply degenerate horizon characterized by \( r_- = r_+ = r_c \) occurs for \( f''(r_+) = 0 \). This configuration is characterized by \( q_{\text{crit}} = 1.1082 \).
$x_{\text{crit}} = 1.34657$ and $\ell_{\text{crit}} = 0.496$. Note that regardless of the sign of the cosmological constant the radial coordinate of the inner horizon, $x_-$, for $|q| \lesssim 0.9$ is weakly influenced by $\ell$ and is close to its $\ell = 0$ value.

FIG. 2: The charge $q$ plotted as function of $x$. Each curve is labeled by the negative cosmological constant. The curves are drawn for $\ell = 0.05i$ ($i = 0, ..., 20$). The outermost curve represents the ABGB black hole. The extrema of the curves represent degenerate configurations.

FIG. 3: The charge $q$ plotted as function of $x$. Each curve is labeled by positive cosmological constant. The curves are drawn for $\ell = 0.05i$ ($i = 0, ..., 20$). The innermost curve represents the ABGB black hole. The extrema of the curves represent degenerate configurations.

For $\varepsilon > 1$ one can single out the lukewarm black hole, for which the surface gravity of the event
horizon equals the surface gravity of the cosmological horizon. Since the Hawking temperatures of the horizons are equal, one can construct a regular thermal state. The lukewarm Reissner-Nordström-de Sitter black holes have been extensively studied in Refs. [27–34].

The Penrose diagrams representing a two-dimensional \( t - r \) section of the conformally transformed ABGB-(a)dS geometry are in many respects similar to the analogous diagrams constructed for the Reissner-Nordström-(anti-)de Sitter black holes [35–38] with central singularity replaced by a regular region. Indeed, for \( \varepsilon = 1 \) one has a two-dimensional infinite carpet, which can be obtained by vertical and horizontal translations of the simple diagram representing two black holes in asymptotically de Sitter universe. For \( \varepsilon = 0 \) the Penrose diagram is similar to the diagram drawn for the Reißner-Nordström black hole. Finally, for \( \varepsilon = -1 \) the diagram is still similar to the Reissner-Nordström case with the conformal null infinity replaced by the anti-de Sitter infinity.

III. THE GEOMETRIES OF THE CLOSEST VICINITY OF THE DEGENERATE HORIZONS

The curvature scalar for the line element (6) with \( \psi(r) = 0 \) is given by

\[
R = -\frac{d^2 f}{dr^2} - \frac{4}{r^2} \frac{df}{dr} - \frac{2f}{r^2} + \frac{2}{r^2}.
\] (29)

At the degenerate horizon both \( f(r_+) \) and \( f'(r_+) \) vanish and the Ricci scalar there is given by

\[
R = -\frac{d^2 f}{dr^2} \bigg|_{r_+} + \frac{2}{r_+^2}
\] (30)

whereas at the degenerate horizon of ultraextemal black hole \( R = 2/r_+^2 \). This behavior of the curvature scalar is a manifestation of the fact that the geometry of the closest vicinity of the degenerate horizons when expanded into a whole manifold is a direct product of the two maximally symmetric 2-dimensional subspaces. It can be demonstrated as follows: for the nearly extreme configuration the function \( f \) can be approximated by a parabola

\[
f(r) = \frac{1}{2} f''(x_d) (r - r_1)(r - r_2),
\] (31)

where \( x_d \) represents a degenerate horizon, and \( r_1 \) and \( r_2 \) denote a pair of close horizons, i.e., either \( r_- \) and \( r_+ \) or \( r_+ \) and \( r_c \). For \( f''(x_d) > 0 \) one can introduce new coordinates \( t = 2T/(\Delta f''(x_d)) \) and \( r = r_d + \Delta \cosh y \) and take the limit \( \Delta \to 0 \) to obtain

\[
ds^2 = \frac{2}{f''(x_d)} \left( -\sinh^2 y \, dT^2 + dy^2 \right) + r_d^2 \, d\Omega^2.
\] (32)
Similarly, for \( f''(r_d) < 0 \) the transformation given by \( t = 2T/(\Delta f''(r_d)) \) and \( r = r_d + \Delta \cos y \) yields
\[
d s^2 = \frac{2}{f''(r_d)} \left( \sin^2 y \, dt^2 - dy^2 \right) + r_d^2 \, d\Omega^2. \tag{33}
\]

Finally, for the ultraextremal black hole, putting \( y = \eta \sqrt{f''(r_d)}/2 \) and taking limit \( f''(r_d) \to 0 \) one obtains
\[
d s^2 = -\eta^2 dt^2 + d\eta^2 + r_d^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right). \tag{34}
\]

Although there are no commonly accepted names for line elements (32-34) it seems that the convention of Podolsky and Griffiths [39] is the most appropriate in this regard. Consequently we shall address to (32) and (33) as the Bertotti-Robinson and Nariai line element, respectively, even though the modulus of the curvature radii of the 2-dimensional maximally symmetric subspaces are nonequal. The line element (34) will be addressed to as the Plebański-Hacyan solution. All the cases considered above are tabulated in Table I. It should be noted that in general, the family of geometries being the product of the constant curvature spaces is richer and allows for two-dimensional Euclidean (\( E^2 \)) and hyperbolic (\( H^2 \)) spaces.

| \( \varepsilon \) | \( f'(r_+) \) | \( f''(r_+) \) | topology       |
|-----------------|--------------|--------------|----------------|
| 1 0             | > 0          | AdS\(_2\) × S\(_2\) |
| 1 0             | < 0          | dS\(_2\) × S\(_2\) |
| 1 0             | 0            | M\(_2\) × S\(_2\) |
| -1 0            | > 0          | AdS\(_2\) × S\(_2\) |
| 0 0             | > 0          | AdS\(_2\) × S\(_2\) |

TABLE I: The geometries of the closest vicinity of the degenerate event horizon. \( \varepsilon \) gives the sign of the cosmological constant. The configurations with \( f''(r_+) > 0 \) are described by the Bertotti-Robinson line element, the configurations with \( f''(r_+) < 0 \) are described by the Nariai line element. The configuration with vanishing second derivative is described by the Plebański-Hacyan geometry.

IV. RENORMALIZED STRESS-ENERGY TENSOR

The approximate stress-energy tensor employed here is constructed from the one-loop effective action
\[
W_{\text{ren}}^{(1)} = \frac{1}{192\pi^2 m^2} \int d^4 y \sqrt{g} \left( \alpha_1^{(s)} R \Box R + \alpha_2^{(s)} R_{ab} \Box R^{ab} + \alpha_3^{(s)} R^3 + \alpha_4^{(s)} R R_{ab} R^{ab} + \alpha_5^{(s)} R R_{abcd} R^{abcd} + \alpha_6^{(s)} R_{ab} R_{c}^{a b} R_{c}^{a b} + \alpha_7^{(s)} R_{ab} R_{cd}^{a b} + \alpha_8^{(s)} R_{ab} R_{abcd}^{a b} + \alpha_9^{(s)} R_{a b}^{c d} R_{ch}^{c d} + \alpha_{10}^{(s)} R_{a b}^{d e} R_{c d}^{e f} \right), \tag{35}
\]
where $m$ is the mass of the field and the numerical coefficients depending on the spin of the field are given in a Table II. The tensor

$$
T_{ab} = \frac{2}{\sqrt{g}} \frac{\delta W^{(1)}_{ren}}{\delta g_{ab}}
$$

is known to yield reasonable results so long the length of the Compton wave associated with the field is smaller than the characteristic radius of curvature of the background geometry. This is satisfied in a number of physically interesting cases and allows for weak temporary changes. The most general expression constructed by functional differentiations of the effective action with respect to the metric tensor has been constructed in Refs. [9, 21], to which the reader is addressed for computational details. The approximate stress-energy tensor consists of almost 100 purely geometric terms constructed from the curvature tensor and its covariant derivatives. For $N$ fields $\psi_i$ of spin $s$ characterized by (possibly various) masses $m_i$, the one-loop effective action is still of the form (35) with

$$
\frac{1}{m^2} \rightarrow \sum_{i=1}^N \frac{1}{m_i^2}.
$$

Consequently, the quantum effects can be made arbitrary large simply by taking a large number of fields into account. For the quantized massive scalar field with the arbitrary curvature coupling in the spacetime of spherically-symmetric and asymptotically flat, static black holes the tensor (36) coincides with the tensor constructed in Ref. [40].

| $s = 0$ | $s = 1/2$ | $s = 1$ |
|--------|-----------|--------|
| $\alpha_1^{(s)}$ | $\frac{2}{5} \xi^2 - \frac{1}{5} \xi + \frac{1}{56}$ | $\frac{3}{280}$ | $\frac{27}{280}$ |
| $\alpha_2^{(s)}$ | $\frac{1}{140}$ | $\frac{1}{10}$ | $\frac{9}{10}$ |
| $\alpha_3^{(s)}$ | $\left(\frac{1}{5} - \xi\right)^3$ | $\frac{1}{840}$ | $\frac{5}{12}$ |
| $\alpha_4^{(s)}$ | $\frac{1}{30} \left(\frac{1}{5} - \xi\right)$ | $\frac{1}{180}$ | $\frac{31}{60}$ |
| $\alpha_5^{(s)}$ | $\frac{1}{30} \left(\frac{1}{5} - \xi\right)$ | $-\frac{7}{1440}$ | $-\frac{5}{12}$ |
| $\alpha_6^{(s)}$ | $-\frac{8}{315}$ | $-\frac{25}{756}$ | $-\frac{52}{63}$ |
| $\alpha_7^{(s)}$ | $\frac{2}{315}$ | $\frac{47}{1260}$ | $-\frac{19}{105}$ |
| $\alpha_8^{(s)}$ | $\frac{1}{1260}$ | $\frac{19}{1260}$ | $\frac{61}{1260}$ |
| $\alpha_9^{(s)}$ | $\frac{17}{7560}$ | $-\frac{29}{7560}$ | $-\frac{67}{2520}$ |
| $\alpha_{10}^{(s)}$ | $-\frac{1}{270}$ | $-\frac{1}{108}$ | $\frac{1}{18}$ |

**TABLE II:** The coefficients $\alpha_i^{(s)}$ for the massive scalar with arbitrary curvature coupling $\xi$, spinor, and vector field.
Equipped with the results of the previous section one can easily calculate the renormalized stress-energy tensor of the massive quantized fields in the geometries, tabulated in Table I. Making use of the general formulas presented in Refs. [9, 21] one can show that in the vicinity of the degenerate horizon the components of the stress-energy tensor of the massive scalar, spinor and vector field for the line element \( (5) \) with \( \psi(r) = 0 \) reduce to

\[
\bar{T}^{(q)t} = \bar{T}^{(q)r} = s_1^{(i)} \frac{f''(r_d)}{2r^2} + s_2^{(i)} (f''(r_d))^3 + 4s_2^{(i)} \frac{1}{r_0^6},
\]

(38)

and

\[
\bar{T}^{(q)\theta} = \bar{T}^{(q)\phi} = -s_1^{(i)} \frac{f''(r_d)}{r^2} - \frac{1}{2} s_2^{(i)} (f''(r_d))^3 - 8s_2^{(i)} \frac{1}{r_0^6},
\]

(39)

where \( \bar{T}^{(q)b} = 96\pi^2 m^2 T^{(q)b}_a \) and the coefficients \( s^{(j)}_i \) for scalar, spinor and vector fields are listed in Table II. Inspection of the stress-energy tensor shows that it depends only on the curvature radii

| \( s^{(j)}_i \) | \( j = 0 \) | \( j = \frac{1}{2} \) | \( j = 1 \) |
|----------------|-----------|-------------|-------------|
| \( i = 1 \)   | \(-\frac{1}{20} + \frac{8}{75} \xi - 3\xi^2 + 6\xi^3\) | \(\frac{1}{120}\) | \(\frac{1}{10}\) |
| \( i = 2 \)   | \(\frac{1}{100} - \frac{1}{40} \xi + \frac{1}{2} \xi^2 - \xi^3\) | \(\frac{1}{180}\) | \(-\frac{1}{10}\) |

TABLE III: The spin-dependent numerical coefficients standing in front of the geometric terms in Eqs. (38) and (39).

of the maximally symmetric subspaces, as expected. The type of the field enters the equations through spin-dependent numerical coefficients. This result can easily be generalized to all spaces with symmetric 2-dimensional subspaces. The general line element describing nine possibilities of product manifolds (six of which is physical) can be written in the compact form

\[
\frac{ds^2}{dx^2} = -\frac{2dudv}{(1 - \frac{1}{2} \varepsilon_1 uv b^{-2})^2} + \frac{2d\zeta d\bar{\zeta}}{(1 + \frac{1}{2} \varepsilon_2 \zeta b^{-2})^2}
\]

(40)

where \( a \) and \( b \) are related to the Gaussian curvature \( K_1 = \varepsilon_1 a^{-2} \) and \( K_1 = \varepsilon_2 b^{-2} \), respectively, and both \( \varepsilon_1 \) and \( \varepsilon_2 \) can take three values -1, 0, 1. All physical geometries are displayed in Table IV [39].

The nonzero components of the renormalized stress-energy tensor for all six geometries can be written compactly as

\[
\bar{T}^u = \bar{T}^v = -8 \varepsilon_1^2 \varepsilon_2 c_3 - \frac{2(2 \varepsilon_1 b^6 + \varepsilon_1^2 b^4 \varepsilon_2 a^2 - \varepsilon_2 a^6)}{b^4 a^6} \frac{(2 c_3 + c_4 + 2 c_5)}{b^4 a^6}
\]

\[
- \frac{(2 \varepsilon_1 b^6 - \varepsilon_2 a^6)}{b^4 a^6} (c_6 + c_7 + 2 c_8 + 4 c_9)
\]

(41)
and

\[ T^\theta_\theta = \bar{T}^\phi_\phi = -8 \frac{\varepsilon_2^2 \varepsilon_1 c_3 + 2 \left( \varepsilon_1 b^6 - \varepsilon_2^2 \varepsilon_1 \right) a^2 b^2 - 2 \varepsilon_2 a^6}{b^6 a^6} (2 c_3 + c_4 + 2 c_5) \]

\[ \quad + \frac{\left( \varepsilon_1 b^6 - 2 \varepsilon_2 a^6 \right) \left( c_6 + c_7 + 2 c_8 + 4 c_9 \right)}{b^6 a^6}, \]  

(42)

where, for typographical reasons, we put \( c_i = c_i^{(1)} \). It can be shown that the tensor (41) and (42) calculated for the Bertotti-Robinson, Nariai and Plebański-Hacyan \( (M_2 \times S^2) \) geometries reduces to (38) and (39). Indeed, simple manipulations give

\[ a^2 = \frac{2 \varepsilon_3}{|f''(r_+)|} \]  

\[ \text{and} \quad b = r_+, \]

(43)

where \( \varepsilon_3 = -1 \) for the Nariai geometry and \( \varepsilon_3 = 1 \) for the Bertotti-Robinson geometry.

Before going further let us calculate the renormalized stress-energy tensor in the closest vicinity of the regular center of the ABGB-(a)dS black hole. One expects, that the result will depend on solely on the cosmological constant as the de Sitter and Anti-de Sitter spacetimes are maximally symmetric. Making use of the general expressions one has

\[ \bar{T}_{2}^{(i)l} = -\frac{1}{9} \varepsilon_3 \varepsilon_2 \left( 144 c_3 + 36 c_4 + 24 c_5 + 9 c_6 + 9 c_7 + 6 c_8 + 4 c_9 + 2 c_{10} \right) \text{diag}[1, 1, 1, 1]_2, \]

(44)

where the expression in parentheses reduces to

\[ \frac{1}{315} \left( 185 - 3654 \xi + 22680 \xi^2 - 45360 \xi^3 \right) \]

(45)

for the massive scalars,

\[ -\frac{31}{1260} \]

(46)

for massive spinors and

\[ -\frac{5}{21} \]

(47)
for vectors. One expects that it is the stress-energy tensor of the quantized massive fields in the closest vicinity of the black hole center.

Now we shall construct the stress-energy tensor of the quantized massive fields inside the extremal black hole. The calculations may by thought of as generalization of the analogous calculation carried out in Ref. [41]. There are, however, notable differences: The geometry under consideration is regular and static.

To analyze the general stress-energy tensor in the black hole interior it is helpful to introduce two functions

\[ \beta(r) = 1 - \tanh\left(\frac{Q^2}{2Mr}\right) \quad \text{and} \quad \omega(r) = 1 + \tanh\left(\frac{Q^2}{2Mr}\right) \]

It could be easily checked that their derivatives can be expressed in terms of themselves and powers of \( r^{-1} \). The stress-energy tensor expressed in terms of the dimensionless \( x, q \) and \( \ell \) has the general structure

\[
\bar{T}_a^{(i)b} = \frac{1}{M^6} \sum_{n=1}^{15} \sum_{p=1}^{6} \sum_{s=1}^{6} \sum_{t=0}^{2} \sum_{u=1}^{8} \alpha_{npstu}^{(i)} b \frac{q^{2s} \ell^{2t}}{x^n} \omega^p(x) \beta^u(x) + \bar{T}_a^{(i)b}, \tag{48}
\]

where the first term in the right hand side vanishes as \( r \rightarrow 0 \) and the second approaches the stress-energy tensor of the quantized massive field in the (anti-)deSitter spacetime. The difference between the radial and time components of the stress-energy tensor factors as

\[
\bar{T}_r^{(i)r} - \bar{T}_t^{(i)t} = g_{tt} F(r), \tag{49}
\]

where \( F(r) \) is a regular function with \( F(0) = 0 \) and hence the stress-energy tensor is regular in the physical sense.

The general form of the stress-energy tensor is rather complicated, and, for obvious reasons, it will not be presented here. (The components of the tensor \( \bar{T}_a^{(i)b} \) for scalar, spinor and vector fields are available from the computer code repository [23].) Instead, we shall analyze the components of the tensor numerically. First, let us consider the most interesting case of the ultraextremal and regular black hole. Such a configuration is characterized by \( q = q_{\text{crit}} \) and \( \ell = \ell_{\text{crit}} \). The triply degenerated horizon is located at \( x = x_{\text{crit}} \). The run of the (rescaled) components of the tensor of the quantized conformally coupled massive scalar field is displayed in Figs. 4-6.

For the massive spinor and vector field the key features of the renormalized tensor are similar. Inspection of the figures shows that although the components of the stress-energy tensor are small in the vicinity of the degenerate horizon and near the black hole center, they are quite big in the region \( 0.03 < x < 0.25 \). Indeed, the ratio \( |T_a^{(i)b}|_{\text{max}} / |T_a^{(i)b}|_{x=\text{crit}} \sim 10^7 \) and \( |T_a^{(i)b}|_{\text{max}} / |T_a^{(i)b}|_{x=0} \sim 10^9 \). Of course, this behavior is not unexpected and can be ascribed to the particular form of the \( f(r) \) function in that region. Simple consideration shows that the leading terms are those with the lowest power of \( \beta \) and highest of \( x \) and the oscillatory character of the components of the tensor is
FIG. 4: The rescaled components of the stress-energy tensor of the conformally coupled massive scalar field in the vicinity of the ultraextremal horizon of the regular black hole. Top to bottom the curves represent the radial, time and angular components, respectively. [$\mu = M^6 m^2 / 10^2$].

FIG. 5: The rescaled components of the stress-energy tensor of the conformally coupled massive scalar field inside the ultraextremal regular black hole. Top to bottom at $x = 0, 1$ the curves represent the time, angular and radial components of the stress-energy tensor. [$\mu = M^6 m^2 / 10^2$].

due to the competition between various such terms in the region adjacent to the region described by the nearly flat metric potentials. Fig. 5 reveals existence of the layers of negative energy-density. This behavior raises two questions. First, is it permissible to use the semiclassical approximation inside the degenerate horizon, and second, whether the perturbative approach is legitimate in this region. The first question have been addressed to earlier, and the affirmative answer can be
given provided the Compton length associated with the quantized field is much smaller that the characteristic radius of the curvature of the background geometry. The second question is more subtle as the perturbative approach usually involves asymptotic series and to answer it we employed the general tensor calculated form the action functional constructed from the coincidence limit of the Hadamard-DeWitt coefficient \( [a_4] \) as presented in Ref. [42] and checked if these terms comprise small correction to the main approximation. We shall omit the details of the rather lengthy and not particularly illuminating calculations that we have carried out with the aid of the computer algebra. Once again the answer is affirmative, but now, it is necessary to carefully examine the leading terms of the approximation, and, for a given black hole mass, to determine the minimal allowable mass of the field.

Finally, let us compare the stress-energy tensor of the quantized massive fields with the analogous tensor calculated in the ultraextremal Reissner-Nordström-de Sitter black hole. The ultraextremal configuration is characterized by

\[
\mathcal{M} = \frac{2}{3} r_+, \quad Q^2 = \frac{1}{2} r_+^2, \quad l^2 = \frac{1}{2} r_-^2.
\]

and the line element is given by (6) with \( \psi(r) = 0 \) and

\[
f(r) = -\frac{r^2}{6r_+^2} \left(1 - \frac{r_+}{r}\right)^3 \left(1 + \frac{3r_+}{r}\right).
\]
FIG. 7: The rescaled components of the stress-energy tensor of the conformally coupled massive scalar field inside the ultraextremal Reissner-Nordström-deSitter black hole. Top to bottom at $x = 0, 1$ the curves represent the time and angular components of the stress-energy tensor, respectively [$\mu = M^6m^2/10^2$].

After some algebra one has

$$\bar{T}^{(i)b}_{a} = \sum_{n=4}^{12} [\beta_n]_a^b \frac{t^{n-6}}{r^n} + \bar{T}^{(i)b}_{a},$$  \hspace{1cm} (52)$$

with $[\beta_n]_a^b = 0$, where the coefficients $\beta$ depend on the spin of the field and (for the massive scalar field) the coupling constant $\xi$. The tensor $\bar{T}^{(i)b}_{a}$ is given by (44) with the critical $l$ (See Eq. 50).
Since the stress-energy tensor diverges as \( r \to 0 \) it is expected that its applicability is severely limited: It gives reasonable results in the region close to the event horizon. The components of the tensor are available from \([23]\).

V. CONCLUSIONS AND DISCUSSION

In this paper we analyzed a three-parameter class of static and spherically symmetric regular black holes in the (anti-)de Sitter universe with the special emphasis put on their horizon structure and the geometry inside the degenerate horizons. Each configuration is uniquely specified by the mass, the (magnetic) charge and the cosmological constant. The cosmological constant is treated on the same footing as other fundamental constants.

We have studied the regularized stress-energy tensor of the quantized massive spinor, scalar and vector fields in a large mass limit within the framework of the Schwinger-DeWitt formalism inside the event horizon of the (ultra)extremal regular black hole. It is shown that the stress-energy tensor exhibits oscillatory behaviour with the layers of the negative energy-density and the amplitudes in the region \( 0.03 < x < 0.25 \) are a few orders of magnitude greater than its components near the event horizon or the regular center. It can be demonstrated (making use of the computer codes \([22]\)) that this behavior persists for a wide range of ABGB(a-)dS parameters. This is an interesting and important behaviour, especially in the context of the back-reaction of the quantized fields upon the black hole geometry. On the other hand, the stress-energy tensor in the vicinity of the event horizon, depending on which horizons do merge, coincides with the tensor calculated in the Bertotti-Robinson, Nariai and Plebański-Hacyan spacetime. For completeness we have also calculated the stress-energy tensor for six physical product geometries. Similarly, near the regular center the stress-energy tensor can be approximated by the tensor calculated in the (anti-)de Sitter spacetime.

The ultraextremal ABGB-dS black hole provides a natural setting for analysis of the degenerate horizon under the influence of the quantized fields. In view of the results obtained in Refs. \([24, 43–45]\) it seems that it would be possible to construct the self-consistent semiclassical analog of the classical ultraextremal ABGB-dS black hole. Similarly, the important problem of the characteristics of the quantized fields in the Schwarzschild black hole in the asymptotically anti-de Sitter geometries has not been touched upon in this paper. It would be interesting to supplement and
extend the discussion of Ref. [46, 47]. We intend to return to this group of problems elsewhere.

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