Inverse parabolic problems of determining functions with one spatial-component independence by Carleman estimate

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Abstract. For an initial-boundary value problem for a parabolic equation in the spatial variable \( x = (x_1, \ldots, x_n) \) and time \( t \), we consider an inverse problem of determining a coefficient which is independent of one spatial component \( x_n \) by extra lateral boundary data. We apply a Carleman estimate to prove a conditional stability estimate for the inverse problem. Also we prove similar results for the corresponding inverse source problem.

Key words. inverse source problem, inverse coefficient problem, Carleman estimates, stability

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1. Introduction and the main results

Let \( x = (x', x_n) \in \mathbb{R}^n \) with \( x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} \) be the spatial variable, \( t \) the time variable, and \( D \subset \mathbb{R}^{n-1} \) be a bounded domain with smooth boundary \( \partial D \), and \( \ell > 0 \) a constant. We set

\[ \Omega = D \times (0, \ell). \]

We note that \( \Omega \subset \mathbb{R}^n \) is a cylindrical domain of spatial variables \( x \in \mathbb{R}^n \).

We consider

\[ \partial_t v = \Delta v + p(x', t)v(x, t), \quad (x, t) \in \Omega \times (-\delta, \delta) \]  \hspace{1cm} (1.1)

with \( p \in L^\infty(D \times (-\delta, \delta)) \). For notational convenience, we choose \(-\delta < t < \delta\) as time interval with initial time \(-\delta\), not 0. Henceforth we denote \( \partial_{x_i} = \frac{\partial}{\partial x_i}, \partial_{x_i}\partial_{x_j} = \frac{\partial^2}{\partial x_i \partial x_j} \).
For a parabolic equation: problem of determining $p$ for example. The main purpose of this article is to establish the stability for the inverse coefficient problem of determining $p = p(x', t)$ by extra data $\partial_{x_n} v$ on a lateral subboundary $\Gamma \times (0, \ell) \times (-\delta, \delta)$, where $\Gamma \subset \partial D$ is a given subboundary.

For studying the inverse coefficient problem, we first consider an inverse source problem for a parabolic equation:

$\partial u(x', x_n, t) = \Delta u + p_0(x', t) u + R(x', x_n, t) f(x', t), \quad (x, t) \in \Omega \times (-\delta, \delta),

\partial u(x', 0, t) = \partial_{x_n} u(x', 0, t) = 0, \quad \partial x', t) \in D \times (-\delta, \delta),$

where $p_0 \in L^\infty(D \times (-\delta, \delta))$ is given.

In (1.2) we assume

$R \in C^1([0, \ell]; L^\infty(D \times (-\delta, \delta)))$

and

$R(x', 0, t) \neq 0, \quad x' \in \overline{D}, -\delta \leq t \leq \delta.$

Let $\Gamma \subset \partial D$ be an arbitrarily fixed non-empty relatively open subset. We arbitrarily choose a subdomain $D_0 \subset D$ satisfying

$\overline{D_0} \subset D \cup \Gamma, \quad \partial D_0 \cap \partial D$ is a non-empty relatively open subset of $\partial D,$

and $\partial D_0 \cap \partial D \subset \Gamma.$

Here and henceforth $\overline{D}$ denotes the closure of a set $D.$

The inverse source problem is formulated as follows: determine a factor $f(x', t)$ of the source term by $u, \nabla u$ on $\Gamma \times (0, \ell) \times (-\delta, \delta).$
where a positive constant $M > 0$ is arbitrarily fixed a priori bound. We set
\[ D(u) = \left( \int_{\Gamma \times (0, t) \times (-\delta, \delta)} (|\nabla_{x,t} \partial_x u|^2 + |\partial_x u|^2) d\sigma dt + \|\partial_x u\|_{L^2(-\delta, \delta; H^2(\Gamma \times (0, t)))}^2 \right)^{\frac{1}{2}}. \] (1.7)

We have

**Theorem 1.1.**

Let $u$ satisfy equation (1.2), $p_0 \in L^\infty(D \times (-\delta, \delta))$ and conditions (1.3), (1.4) and (1.6) hold true.

(i) For a given subdomain $D_0 \subset D$ satisfying (1.5), there exist some constants $0 < \delta_0 < \delta$, $\theta \in (0, 1)$ and $C > 0$ such that
\[ \|f\|_{L^2(D_0 \times (-\delta_0, \delta_0))} \leq C D(u)^\theta. \]

(ii) For any given $\delta_1 \in (0, \delta)$, there exist constants $C_1 > 0$, $\theta_1 \in (0, 1)$ and a subdomain $D_1 \subset D$ satisfying (1.5) such that
\[ \|f\|_{L^2(D_1 \times (-\delta_1, \delta_1))} \leq C_1 D(u)^{\theta_1}. \]

In terms of the function $d \in C^2(\overline{D})$ constructed in Lemma 2.1 in Section 2, we can rewrite Theorem 1.1 (i) as follows: If
\[ \delta_0 < \left( \frac{\min_{x' \in D_0} d(x')}{\max_{x' \in \overline{D}} d(x')} \right)^{\frac{1}{2}} \delta, \] (1.8)
then the conclusion of (i) holds.

We note that if $D_0$ is larger, that is, if we want to determine $f$ in a larger spatial subdomain $D_0$, then $\min_{x' \in D_0} d(x')$ is smaller, so that the time interval $(-\delta_0, \delta_0)$ when we can prove a stability estimate for the function $f$, becomes smaller. Moreover, since $d|_{\partial\Omega \setminus \Gamma} = 0$ by (2.1) stated in Section 2, the quantity $\min_{x' \in D_0} d(x')$ in (1.8) tends to 0 if $D_0$ approaches to $D$, that is, we cannot estimate $f$ in $D$ even if we choose any short time interval. In Theorem 1.1 (ii), as is seen by the proof below, when $\delta_1 < \delta$ is closer to $\delta$, the stability subdomain $D_1$ becomes smaller.

To sum up, Theorem 1.1 asserts a conditional stability estimate in determining a factor $f$ of the source term in a proper subdomain of the cylinder $D \times (-\delta, \delta)$, which holds conditionally with an a priori boundedness condition (1.6).

Theorem 1.1 (i) yields a uniqueness result only for $x' \in D$ at $t = 0$. More precisely we state
Corollary 1.1.
Let $u$ satisfy equation (1.2), $p_0 \in L^\infty(D \times (-\delta, \delta))$ and conditions (1.3), (1.4) and (1.6) hold true. If
\[
\partial_{x_n} u = \nabla_{x'} \partial_{x_n} u = 0 \quad \text{on } \Gamma \times (0, \ell) \times (-\delta, \delta),
\]
then
\[
f(x',0) = 0, \quad x' \in D.
\]

So far, we do not know the uniqueness in the whole spatial domain $D$ even on very small time interval.

Next we state the main result on the inverse coefficient problem. For $p = p(x',t)$ and $g_0, g_1 \in H^{2:2}(D \times (-\delta, \delta))$, let $v = v(p)(x,t)$ satisfy
\[
\begin{align*}
\partial_t v = & \Delta v + p(x',t)v(x,t), \quad (x,t) \in \Omega \times (-\delta, \delta), \\
\partial_{x_n}^k v(x',0,t) = & g_k(x',t), \quad x' \in D, \quad -\delta < t < \delta, \quad k = 0, 1.
\end{align*}
\] (1.9)

Fixing $g_0$ and $g_1$ in $D \times (-\delta, \delta)$, we estimate a coefficient $p(x',t)$ by data norm of the solutions defined by (1.7). We can change $g_0$ and $g_1$, but we fix them, which simplifies and does not affect the essence of the arguments.

For the statement of our main result, we introduce an admissible set of unknown coefficients $p(x',t)$. For arbitrarily fixed constant $M > 0$, we define an admissible set $\mathcal{P}$ by
\[
\mathcal{P} = \left\{ p(x',t); \|p\|_{L^\infty(D \times (-\delta, \delta))} \leq M, v(p), \partial_{x_n} v(p) \in H^{2:1}((\Omega \times (-\delta, \delta)) \cap L^\infty(\Omega \times (-\delta, \delta)), \right.

v(p) \text{ satisfies (1.9)} \}.
\]

We are ready to state a conditional stability estimate for the inverse coefficient problem.

Theorem 1.2.
(i) For a given subdomain $D_0 \subset D$ satisfying (1.5), there exist constants $\delta_0 \in (0, \delta)$, $\theta \in (0, 1)$, $\alpha_0 > 0$ and $C > 0$ such that
\[
\|p - q\|_{L^2(D_0 \times (-\delta, \delta))} \leq C \mathcal{D}(v(p) - v(q))^\theta
\]
for $p, q \in \mathcal{P}$ if
\[
\text{either} \quad |v(p)(x',0,t)| \geq \alpha_0 \quad \text{for all } (x',t) \in \overline{D_0} \times [-\delta, \delta]
\]
or
\[
|v(q)(x',0,t)| \geq \alpha_0 \quad \text{for all } (x',t) \in \overline{D_0} \times [-\delta, \delta].
\] (1.10)
(ii) For any given $\delta_1 \in (0, \delta)$, there exist constants $C_1 > 0$, $\theta_1 \in (0, 1)$ and a subdomain $D_1 \subset D$ satisfying (1.5) such that

$$\|p - q\|_{L^2(D_1 \times (-\delta_1, \delta_1))} \leq C_1 D(v(p) - v(q))^{\theta_1}$$

for $p, q \in P$ if (1.10) holds for $D_0 = D_1$.

An inverse problem of determining an $x_n$-independent factor of the source term, is considered in Beznoschenko [3, 4], Gaitan and Kian [6], Kian and Yamamoto [11]. Our proof is different from [3], [4], [11], and based on a Carleman estimate. The work [6] uses a technique similar to ours and establishes the stability in the whole domain $D \times (-\delta, \delta)$ with more data. The main machinery is a Carleman estimate which requires information of the trace and the normal derivative of estimated function on the whole boundary $\partial \Omega \times (-\delta, \delta)$. In this article, we apply a different Carleman estimate and prove conditional stability estimates in some subdomain of $D \times (-\delta, \delta)$.

We remark that we need not any data on $D \times \{\ell\} \times (-\delta, \delta)$. Our approach is new for the inverse problem of determining functions which are independent of one component of the spatial variables. Our formulation of the inverse problem is for example motivated by the following. Choosing the $x_n$-axis along the depth, we would like to determine functions which are independent of the depth variable without any data on the bottom $x_n = \ell$ of the cylindrical domain, but only data on the surface $x_n = 0$ and the side boundary data which can be approximated by $\partial_{x_n} u(\xi_k, x_n, t)$, $0 < x_n < \ell$, $-\delta < t < \delta$ with fixed probe points $\xi_1, \ldots, \xi_N \in \Gamma \subset \partial D$. Our main stability results guarantee that such data becomes more accurate as the number $N$ of probe holes $\{(\xi_k, x_n) \in \mathbb{R}^n; 0 < x_n < \ell\}; k = 1, \ldots, N,$ increases.

More precisely for the proof, we adapt the approach by Bukhgeim and Klibanov [5] and Klibanov [12] which originally was to establish the unique determination and stability a $t$-independent source terms and coefficients of evolution equations. In our case we determine an $x_n$-independent unknown function. As for inverse problems by Carleman estimates, see Beilina and Klibanov [11], Bellassoued and Yamamoto [2], Imanuvilov and Yamamoto [8, 9], Klibanov and Timonov [13], Yamamoto [14] and the references therein. Moreover we can refer for example to Chapter 3, Section 3 in Isakov [10] as for related inverse problems of determining functions which are independent of one component of variables.

This paper is composed of three sections and one appendix. In Section 2, we establish a key Carleman estimate and in Section 3, we complete the proof of Theorems 1.1 and 1.2 and Corollary 1.1.
2. Key Carleman estimate

We set
\[ \Omega_\pm := D \times (-\ell, \ell), \quad Q_\pm := \Omega_\pm \times (-\delta, \delta). \]

We recall that \( D \subset \mathbb{R}^{n-1} \) is a bounded domain with smooth boundary \( \partial D \), and \( \Omega = D \times (0, \ell) \), \( Q = \Omega \times (-\delta, \delta) \).

We start the proof of Carleman estimate with construction of the weight function. We can prove (e.g., Imanuvilov [7]):

**Lemma 2.1.**

For a given subdomain \( D_0 \subset D \) satisfying (1.5), there exists function \( d \in C^2(\overline{D}) \) such that

\[
\begin{align*}
    d(x') &\geq 0 \quad \text{for } x' \in \overline{D}, \\
    d(x') &> 0 \quad \text{for } x' \in \overline{D_0}, \\
    d(x') &= 0 \quad \text{for } x' \in \partial D \setminus \Gamma, \\
    |\nabla d(x')| &> 0 \quad \text{for } x' \in \overline{D}.
\end{align*}
\]

**Proof of Lemma 2.1.**

For \( \Gamma \subset \partial D \), we choose a bounded domain \( E \) with smooth boundary such that

\[ D \subsetneq E, \quad \Gamma = \partial D \cap \overline{E}, \quad \partial D \setminus \Gamma \subset \partial E. \]

In particular, \( E \setminus \overline{D} \) contains some non-empty open subset. We note that \( E \) can be constructed as the interior of a union of \( \overline{D} \) and the closure of a non-empty domain \( \hat{D} \) satisfying \( \hat{D} \subset \mathbb{R}^3 \setminus \overline{D} \) and \( \partial \hat{D} \cap \partial D = \Gamma \).

We choose a domain \( \omega \) such that \( \omega \subset E \setminus \overline{D} \). Then, by [7, Lemma 1.2] (see also [8, Lemma 2.1]), we can find \( d \in C^2(\overline{E}) \) such that

\[ d > 0 \text{ in } E, \quad |\nabla d| > 0 \text{ on } \overline{E} \setminus \omega, \quad d = 0 \text{ on } \partial E. \]

This \( d \) is our desired function, and the proof of Lemma 2.1 is complete. □

We set
\[ \psi(x, t) = d(x') - \alpha x_n^2 - \beta t^2, \quad \varphi(x, t) = e^{\lambda \psi(x, t)} \]
where the positive constants \( \alpha, \beta \) are chosen later and \( \lambda \) is a sufficiently large constant.

**Lemma 2.2** (Carleman estimate).

Let \( p_0 \in L^\infty(D \times (-\delta, \delta)) \) be given. Then there exist constants \( C > 0 \) and \( s_0 > 0 \) such that

\[
\int _{Q_\pm} \left( \frac{1}{s} \sum _{i,j=1} ^n \left| \partial _{x_i} \partial _{x_j} u \right|^2 + s \left| \nabla u \right|^2 + s^3 \left| u \right|^2 \right) e^{2s \varphi} \, dx \, dt
\]
We make the even extension of functions $u$ for all $s > s_0$ and all $u \in H^{2,1}(Q_\pm) \cap H^1(-\delta, \delta; H^1(\Omega_\pm))$ satisfying $u \in L^2(-\delta, \delta; H^2(\partial \Omega_\pm))$. Here $s_0$ can be uniformly chosen if $\|p_0\|_{L^\infty(D \times (-\delta, \delta))}$ is bounded.

The proof of Lemma 2.2 is based on a classical Carleman estimate and is given in Appendix for completeness.

3. PROOF OF MAIN RESULTS

3.1. Proof of Theorem 1.1 (i). We recall

$$u(x',0,t) = \partial_{x_n} u(x',0,t) = 0, \quad x' \in D, \quad -\delta < t < \delta$$  \hspace{1cm} (3.1)

and

$$\partial_t u = \Delta u + p_0(x',t) u + R(x',x_n,t) f(x',t), \quad x' \in D, \quad 0 < x < \ell, \quad -\delta < t < \delta.$$  \hspace{1cm} (3.2)

We make the even extension of functions $u$ and $R$ in the variable $x_n$:

$$u(x',x_n,t) \begin{cases} 
  u(x',x_n,t), & x_n \geq 0, \\
  u(x',-x_n,t), & x_n < 0,
\end{cases} \quad R(x',x_n,t) \begin{cases} 
  R(x',x_n,t), & x_n \geq 0, \\
  R(x',-x_n,t), & x_n < 0.
\end{cases}$$

By (3.1) we can verify

$$\partial_{x_n} u(x',x_n,t) = \begin{cases} 
  \partial_{x_n} u(x',x_n,t), & x_n \geq 0, \\
  -\partial_{x_n} u(x',-x_n,t), & x_n < 0,
\end{cases}$$

and

$$\partial_{x_n}^2 u(x',x_n,t) = \begin{cases} 
  \partial_{x_n}^2 u(x',x_n,t), & x_n \geq 0, \\
  \partial_{x_n}^2 u(x',-x_n,t), & x_n < 0.
\end{cases}$$

Hence we can prove that $\partial_{x_n}^3 u \in L^2(D \times (-\ell, \ell) \times (-\delta, \delta))$ and so $\partial_{x_n} u \in L^2(-\delta, \delta; H^2(D \times (-\ell, \ell)))$, and $\partial_{x_n} u \in H^1(-\delta, \delta; H^1(D \times (-\ell, \ell)))$. Moreover

$$\partial_{x_n} R(x',x_n,t) = \begin{cases} 
  \partial_{x_n} R(x',x_n,t), & x_n \geq 0, \\
  -\partial_{x_n} R(x',-x_n,t), & x_n < 0,
\end{cases}$$

and so $\partial_{x_n} R \in L^\infty(D \times (-\ell, \ell) \times (-\delta, \delta))$ by (1.3).

Thus (3.2) yields

$$\begin{cases} 
  \partial_t u = \Delta u + p_0 u + R(x',x_n,t) f(x',t), \quad x' \in D, \quad -\ell < x_n < \ell, \quad -\delta < t < \delta, \\
  u(x',0,t) = \partial_{x_n} u(x',0,t) = 0, \quad x' \in D, \quad -\delta < t < \delta.
\end{cases} \hspace{1cm} (3.3)$$

Set

$$y = \partial_{x_n} u.$$
Then $y$ satisfies
\[
\begin{aligned}
\partial_t y &= \Delta y + p_0 y + \partial_{x_n} R(x', x_n, t) f(x', t), \quad x' \in D, \; -\ell < x_n < \ell, \; -\delta < t < \delta, \\
y(x', 0, t) &= 0, \quad x' \in D, \; -\delta < t < \delta.
\end{aligned}
\] (3.4)

Now we will specify the weight function $\psi(x, t) := d(x') - \alpha x_n^2 - \beta t^2$ for the Carleman estimate Lemma 2.2. We set
\[
d_0 := \min_{x' \in D_0} d(x'), \quad d_1 := \max_{x' \in D} d(x').
\]
By Lemma 2.1, we see $d_0 > 0$. We choose $\delta_0 > 0$ such that
\[
\delta_0 < \left( \frac{d_0}{d_1} \right)^{\frac{1}{2}} \delta.
\] (3.5)
We note that $0 < \delta_0 < \delta$. Then, since (3.5) yields
\[
\frac{d_1 - d_0}{\delta^2 - \delta_0^2} < \frac{d_0}{\delta_0^2},
\]
we can choose $\beta > 0$ such that
\[
\frac{d_1 - d_0}{\delta^2 - \delta_0^2} < \beta < \frac{d_0}{\delta_0^2}.
\] (3.6)
Finally choose $\alpha > 0$ sufficiently large such that
\[
d_1 - d_0 + \beta \delta_0^2 < \alpha \ell^2.
\] (3.7)
Then inequalities (3.5) - (3.7) imply
\[
\left\{ \begin{array}{l}
d_1 - \beta \delta_0^2 < d_0 - \beta \delta_0^2, \\
0 < d_0 - \beta \delta_0^2, \\
d_1 - \alpha \ell^2 < d_0 - \beta \delta_0^2.
\end{array} \right.
\] (3.8)
Here we recall
\[
\psi(x, t) = d(x') - \alpha x_n^2 - \beta t^2, \quad \varphi(x, t) = e^{\lambda \psi(x, t)}.
\]
Inequalities (3.8) imply
\[
\max_{x' \in D_0, -d_0 \leq t \leq d_0} \psi(x', 0, t) = d_0 - \beta \delta_0^2.
\]
Therefore
\[
\sigma_1 := \max_{x' \in \Omega} \varphi(x, \delta), \quad \max_{x' \in \partial D \setminus \Gamma, \; -\ell \leq x_n \leq \ell, \; -\delta \leq t \leq \delta} \psi(x, \delta), \quad \max_{x' \in D, \; -\delta \leq t \leq \delta} \psi(x, \pm \ell, t)
\]
\[
< \sigma_0 := \min_{x' \in D_0, -d_0 \leq t \leq d_0} \varphi(x', 0, t).
\] (3.9)
Next in terms of (3.9), we estimate the integral
\[
\int_{D \times (-\delta, \delta)} |\partial_{x_n} y(x', 0, t)|^2 e^{2e\varphi(x', 0, t)} dx' dt
\]
We apply Lemma 2.2 to system (3.4), and we obtain

By (3.9) and (1.6), we have

Here we have

\[ \int_{\partial} \] and \[ \parallel \]

Since \[ \partial_{x_{n}}^{2}u = \partial_{x_{n}}y \], we rewrite the first term in estimate (3.10) to have

We apply Lemma 2.2 to system (3.4), and we obtain

By (3.9) and (1.6), we have

\[ \int_{\partial} \] and \[ \parallel \]
by (1.6), (1.7) and (3.9). Moreover,
\[
\|y e^{s\varphi}\|_{L^2(-\delta,\delta;H^2(\partial\Omega_\pm))} \leq C s^4 \int_\gamma \left( \sum_{i,j=1}^n |\partial_{x_i} \partial_{x_j} y|^2 + |\nabla y|^2 + |y|^2 \right) e^{2s\varphi(x,t)} \, d\sigma,
\]
where \( \gamma = \Gamma \times (-\ell, \ell) \) or \((\partial D \setminus \Gamma) \times (-\ell, \ell) \) or \( D \times \{\ell\} \) or \( D \times \{-\ell\} \). Therefore, again using (1.6), (1.7) and (3.9), we obtain
\[
\|y e^{s\varphi}\|_{L^2(-\delta,\delta;H^2(\partial\Omega_\pm))} \leq C s^4 e^{C s^2 D(u)^2} + C s^4 e^{2s_1 M^2} \tag{3.14}
\]
for all large \( s > 0 \). Since
\[
\int_{\Omega_\pm} \left( |\nabla y(x,\delta)|^2 + |y(x,\delta)|^2 + |\nabla y(x,-\delta)|^2 + |y(x,-\delta)|^2 \right) e^{2s\varphi(x,\delta)} \, dx
\]
\[
\leq CM^2 e^{2s_1}
\]
by (1.6) and (3.9), we substitute (3.13) and (3.14) into (3.12), and reach
\[
\int_{Q_\pm} \left( \frac{1}{s} |\partial_{x_n}^2 y|^2 + s |\partial_{x_n} y|^2 \right) e^{2s\varphi} \, dx \, dt \tag{3.15}
\]
\[
\leq C \int_{Q_\pm} |f(x',t)|^2 e^{2s\varphi(x',x_n,t)} \, dx' \, dx_n \, dt + C s^4 e^{C s D(u)^2} + C s^4 e^{2s_1 M^2}
\]
for all large \( s > 0 \).

From (3.11) and (3.15), we have
\[
\int_{D \times (-\delta,\delta)} |\partial_{x_n}^2 u(x',0,t)|^2 e^{2s\varphi(x',0,t)} \, dx' \, dt
\]
\[
\leq C \int_{D \times (-\delta,\delta)} |f(x',t)|^2 e^{2s\varphi(x',0,t)} \left( \int_{-\ell}^\ell e^{2s(\varphi(x',x_n,t)-\varphi(x',0,t))} \, dx_n \right) \, dx' \, dt
\]
\[
\quad + C s^4 e^{C s D(u)^2} + C s^4 e^{2s_1 M^2} \tag{3.16}
\]
Since \( f(x',t) R(x',0,t) = -\partial_{x_n}^2 u(x',0,t) \) for \( x' \in D \) and \(-\delta < t < \delta \) by (3.3), from (3.16) and (1.4), we obtain
\[
\int_{D \times (-\delta,\delta)} |f(x',t)|^2 e^{2s\varphi(x',0,t)} \, dx' \, dt \leq C \int_{D \times (-\delta,\delta)} |\partial_{x_n}^2 u(x',0,t)|^2 e^{2s\varphi(x',0,t)} \, dx' \, dt
\]
\[
\leq C \int_{D \times (-\delta,\delta)} |f(x',t)|^2 e^{2s\varphi(x',0,t)} \, dx' \, dt \left( \int_{-\ell}^\ell e^{2s(\varphi(x',x_n,t)-\varphi(x',0,t))} \, dx_n \right) \, dx' \, dt
\]
\[+Cs^4 e^{Cs} \mathcal{D}(u)^2 + Cs^4 e^{2s \sigma_1} M^2.\]

Here we have
\[
e^{2s(\varphi(x',x_n,t)-\varphi(x',0,t))} = e^{2s(\lambda(d(x')-\beta^2) - \lambda \alpha x_n^2 - 1)}
\]
\[= e^{-2s(e^{\lambda(d(x')-\beta^2)}(1-e^{-\lambda \alpha x_n^2}) - 1)} \leq e^{-2sc_0(1-e^{-\lambda \alpha x_n^2})},\]

where
\[c_0 := \min_{x' \in D, \delta \leq t \leq \delta} e^{\lambda(d(x')-\beta^2)}.
\]

Therefore the Lebesgue theorem yields
\[
\int_{-\ell}^{\ell} e^{2s(\varphi(x',x_n,t)-\varphi(x',0,t))} dx_n \leq \int_{-\ell}^{\ell} e^{-2sc_0(1-e^{-\lambda \alpha x_n^2})} dx_n = o(1) \quad \text{as } s \to \infty.
\]

Hence
\[
\int_{D \times (-\delta, \delta)} |f(x', t)|^2 e^{2s \varphi(x',0,t)} dx' dt
\]
\[= o(1) \int_{D \times (-\delta, \delta)} |f(x', t)|^2 e^{2s \varphi(x',0,t)} dx' dt + Cs^4 e^{Cs} \mathcal{D}(u)^2 + Cs^4 e^{2s \sigma_1} M^2.
\]

Choosing the parameter \(s > 0\) large, we can absorb the first term on the right-hand side into the left-hand side, and
\[
\int_{D \times (-\delta, \delta)} |f(x', t)|^2 e^{2s \varphi(x',0,t)} dx' dt \leq Cs^4 e^{Cs} \mathcal{D}(u)^2 + Cs^4 e^{2s \sigma_1} M^2
\]

for all sufficiently large \(s > 0\). Shrinking \(D \times (-\delta, \delta)\) to \(D_0 \times (-\delta_0, \delta_0)\) in the integral on the left-hand side and applying the definition of \(\sigma_0\) in (3.9), we obtain
\[
\|f\|^2_{L^2(D_0 \times (-\delta_0, \delta_0))} e^{2s \sigma_0} \leq Cs^4 e^{Cs} \mathcal{D}(u)^2 + Cs^4 e^{2s \sigma_1} M^2,
\]

that is,
\[
\|f\|^2_{L^2(D_0 \times (-\delta_0, \delta_0))} \leq Cs^4 e^{Cs} \mathcal{D}(u)^2 + Cs^4 e^{-2s(\sigma_0-\sigma_1)} M^2
\]

for all \(s > s_0\): some constant. By \(\sigma_0 - \sigma_1 > 0\), we see that \(\sup_{s \geq 0} s^4 e^{-s(\sigma_0-\sigma_1)} < \infty\), and choosing a constant \(C_1 > 0\) satisfying \(s^4 e^{Cs} \leq e^{C_1 s}\) for all \(s \geq 0\), we have
\[
\|f\|^2_{L^2(D_0 \times (-\delta_0, \delta_0))} \leq Ce^{C_1 s} \mathcal{D}(u)^2 + C_2 e^{-s(\sigma_0-\sigma_1)} M^2
\]

for all \(s \geq s_0\). Replacing \(C\) by \(Ce^{C_1 s_0}\) and changing \(s\) into \(s + s_0\) with \(s \geq 0\), we obtain
\[
\|f\|^2_{L^2(D_0 \times (-\delta_0, \delta_0))} \leq Ce^{C_1 s} \mathcal{D}(u)^2 + C_2 e^{-s(\sigma_0-\sigma_1)} M^2
\]

(3.17) for all \(s \geq 0\). We choose \(s > 0\) in order that the right-hand side of (3.17) is small and consider the two cases separately.

**Case 1:** \(M > \mathcal{D}(u)\):

We choose \(s > 0\) such that
\[
e^{C_1 s} \mathcal{D}(u)^2 = e^{-s(\sigma_0-\sigma_1)} M^2,\]

that is,
\[
s = \frac{2}{C_1 + \sigma_0 - \sigma_1} \log \frac{M}{\mathcal{D}(u)} > 0.
\]
Substituting this value of $s$ into (3.17), we reach
\[ \| f \|_{L^2(D_0 \times (-\delta_0, \delta_0))}^2 \leq 2M^{2(1-\theta)}D(u)^{2\theta}, \]
where $\theta = \frac{\sigma_0 - \sigma_1}{C_1 + \sigma_0 - \sigma_1} \in (0, 1)$.

**Case 2:** $M \leq D(u)$:

Setting $s = 0$ in (3.17), we directly obtain
\[ \| f \|_{L^2(D_0 \times (-\delta_0, \delta_0))}^2 \leq (C + C_2)D(u)^2. \]

By the definitions (1.6) and (1.7) of $M$ and $D(u)$, we see $D(u) \leq CM$, and so
\[ \| f \|_{L^2(D_0 \times (-\delta_0, \delta_0))}^2 \leq (C + C_2)D(u)^{2\theta}M^{2(1-\theta)}. \]

Thus the proof of Theorem 1.1 (i) is complete.

**3.2. Proof of Theorem 1.1 (ii).** We fix an arbitrary point $x_0'$ from an interior of the set $\Gamma$. For sufficiently small $\varepsilon > 0$, we choose
\[ \tilde{D} = D \cap \{ x' \in \mathbb{R}^{n-1}; |x' - x_0'| < 2\varepsilon \}, \quad D_1 = D \cap \{ x' \in \mathbb{R}^{n-1}; |x' - x_0'| < \varepsilon \} \]
such that $\tilde{D} \cap \partial D \subseteq \Gamma$. Then, for small $\varepsilon > 0$, replacing $D_0$ and $D$ respectively by $D_1$ and $\tilde{D}$, in terms of Lemma 2.1 we can construct $d \in C^2(\tilde{D})$ satisfying (2.1) with $\tilde{D}$ replacing $D$. Let $\delta_1 > 0$ be chosen arbitrarily such that $0 < \delta_1 < \delta$. Then for sufficiently small $\varepsilon > 0$, we can verify
\[ \left( \frac{\delta_1}{\delta} \right)^2 < \frac{\min_{x' \in D_1} d(x')}{\max_{x' \in D_0} d(x')} < 1. \]

This is possible because
\[ \lim_{\varepsilon \to 0} \frac{\min_{x' \in D_1} d(x')}{\max_{x' \in D_0} d(x')} = 1 \]
and $\frac{\delta_1}{\delta} < 1$. Replacing $D_0$ and $D$ by $D_1$ and $\tilde{D}$ respectively, we apply (3.18) instead of (3.5), and argue similarly to the proof of Theorem 1.1 (i), so that the proof of Theorem 1.1 (ii) is complete.

**3.3. Proof of Corollary 1.1.** By Theorem 1.1 (i) and its proof in Section 3.1, for arbitrary subdomain $D_0 \subset D$ satisfying (1.4), we see that $f = 0$ in $D_0 \times (-\delta_0, \delta_0)$ where $\delta_0 > 0$ satisfies (1.8). Therefore the trace theorem yields $f = 0$ in $D_0 \times \{0\}$. Since $D_0 \subset D$ can be chosen arbitrarily provided that (1.5) holds, we see that $f = 0$ in $D \times \{0\}$. Thus the proof of the corollary is complete.

**Proof of Theorem 1.2.** The theorem follows directly from Theorem 1.1. Indeed, setting $u := v(p) - v(q)$, $p_0 := p$, $R := v(q)$ and $f := p - q$ and taking the difference between the two corresponding equations with $v(p)$ and $v(q)$, we reduce Theorem 1.2 to Theorem 1.1.
APPENDIX. PROOF OF LEMMA 2.2

The following inequality is a standard Carleman estimate: There exist constants $C > 0$ and $s_0 > 0$ such that

$$
\int_{Q_\pm} \left( \frac{1}{s} |\Delta u|^2 + s |\nabla u|^2 + s^3 |u|^2 \right) e^{2s\varphi} \, dx \, dt \\
\leq C \left( \int_{Q_\pm} |\partial_t u - \Delta u - p_0 u|^2 e^{2s\varphi} \, dx \, dt + s^3 \int_{\partial\Omega_\pm \times (-\delta, \delta)} (|\nabla x,u|^2 + |u|^2) e^{2s\varphi} \, d\sigma \, dt \\
+ s^3 \int_{\Omega_\pm} (|\nabla u(x,\delta)|^2 + |u(x,\delta)|^2 + |\nabla u(x,-\delta)|^2 + |u(x,-\delta)|^2) e^{2s\varphi(x,\delta)} \, dx \right) =: J
$$

for all $s > s_0$. See e.g., [2], [14] as for the proof.

We recall that $\Omega_\pm = D \times (-\ell, \ell)$. Henceforth $\nu = (\nu_1, ..., \nu_n)$ denotes the unit outward normal vector to $\partial\Omega_\pm$. Then we prove

**Lemma 1.**

Let $w \in H^2(\Omega_\pm)$ satisfy $w \in H^2(\partial\Omega_\pm)$. Then

$$
\sum_{i,j=1}^n \|\partial_{x_i} \partial_{x_j} w\|^2_{L^2(\Omega_\pm)} + \sum_{i,j=1}^n \int_{\partial\Omega_\pm} \partial_{x_i} w((\partial_{x_j} \partial_{x_j} w)\nu_i - (\partial_{x_i} \partial_{x_j} w)\nu_j) \, d\sigma = \|\Delta w\|^2_{L^2(\Omega_\pm)}.
$$

In particular,

$$
\sum_{i,j=1}^n \|\partial_{x_i} \partial_{x_j} w\|^2_{L^2(\Omega_\pm)} \leq C (\|\Delta w\|^2_{L^2(\Omega_\pm)} + \|w\|^2_{H^2(\partial\Omega_\pm)}),
$$

where the constant $C > 0$ depends only on $\Omega_\pm$.

**Remark.** We can relax the norm $\|w\|_{H^2(\partial\Omega_\pm)}$ by $\|w\|_{H^3(\partial\Omega_\pm)}$, but this choice of the norm of Dirichlet data is sufficient for the proof of the main theorems.

**Proof.**

By the density argument, it suffices to assume that $w \in C^\infty(\overline{\Omega_\pm})$. Then the integration by parts yield

$$
\int_{\Omega_\pm} |\Delta w|^2 \, dx = \sum_{i,j=1}^n \int_{\Omega_\pm} (\partial_{x_i} \partial_{x_j} w)(\partial_{x_j} \partial_{x_j} w) \, dx \\
= - \sum_{i,j=1}^n \int_{\Omega_\pm} (\partial_{x_i} w)(\partial_{x_j} \partial_{x_j} w) \, dx + \sum_{i,j=1}^n \int_{\partial\Omega_\pm} (\partial_{x_i} w)(\partial_{x_j} \partial_{x_j} w)\nu_i \, d\sigma \\
= \sum_{i,j=1}^n \int_{\Omega_\pm} |\partial_{x_i} \partial_{x_j} w|^2 \, dx + \sum_{i,j=1}^n \int_{\partial\Omega_\pm} \partial_{x_i} w((\partial_{x_j} \partial_{x_j} w)\nu_i - (\partial_{x_i} \partial_{x_j} w)\nu_j) \, d\sigma.
$$

Thus the proof of Lemma 1 is complete. ■
We arbitrarily fix $t \in (-\delta, \delta)$. Direct calculations yield
\[
\Delta(u(x, t)e^{s\varphi(x,t)}) = (\Delta u)e^{s\varphi} + 2s(\nabla u \cdot \nabla \varphi)e^{s\varphi} + (s^2|\nabla \varphi|^2 + s\Delta \varphi)ue^{s\varphi}, \quad x \in \Omega_\pm.
\]
Applying Lemma 1 to $u(\cdot, t)e^{s\varphi(\cdot,t)}$ and integrating over $t \in (-\delta, \delta)$, we have
\[
\frac{1}{s}\sum_{i,j=1}^n \|\partial_{x_i}\partial_{x_j}(ue^{s\varphi})\|^2_{L^2(Q_\pm)} \leq C \left( \frac{1}{s}\| (\Delta u)e^{s\varphi}\|^2_{L^2(Q_\pm)} + s\| (\nabla u)e^{s\varphi}\|^2_{L^2(Q_\pm)} + s^3\| u\|^2_{L^2(Q_\pm)} \right) + \frac{C}{s}\| ue^{s\varphi}\|^2_{L^2(-\delta, \delta; H^2(\partial \Omega_\pm))}.
\]
Applying (1), we obtain
\[
\frac{1}{s}\sum_{i,j=1}^n \|\partial_{x_i}\partial_{x_j}(ue^{s\varphi})\|^2_{L^2(Q_\pm)} \leq CJ + \frac{C}{s}\| ue^{s\varphi}\|^2_{L^2(-\delta, \delta; H^2(\partial \Omega_\pm))}. \tag{2}
\]
Since
\[
\partial_{x_i}\partial_{x_j}(ue^{s\varphi}) = (\partial_{x_i}\partial_{x_j}u)e^{s\varphi} + s(\partial_{x_i}u)\partial_{x_j}\varphi + (\partial_{x_j}u)\partial_{x_i}\varphi + (s^2(\partial_{x_i}\varphi)\partial_{x_j}\varphi + s(\partial_{x_i}\partial_{x_j}\varphi))e^{s\varphi}u,
\]
again using (1) and (2), we have
\[
\frac{1}{s}\sum_{i,j=1}^n \| (\partial_{x_i}\partial_{x_j}u)e^{s\varphi}\|^2_{L^2(Q_\pm)} \leq C \left( \frac{1}{s}\sum_{i,j=1}^n \|\partial_{x_i}\partial_{x_j}(ue^{s\varphi})\|^2_{L^2(Q_\pm)} + s\| (\nabla u)e^{s\varphi}\|^2_{L^2(Q_\pm)} + s^3\| u\|^2_{L^2(Q_\pm)} \right) \leq CJ + \frac{C}{s}\| ue^{s\varphi}\|^2_{L^2(-\delta, \delta; H^2(\partial \Omega_\pm))}.
\]
Integrating over $t \in (-\delta, \delta)$, we complete the proof of Lemma 2.2. \(\blacksquare\)

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