Canonical form of modular hyperbolas with an application to integer factorization

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Abstract

For a composite $n$ and an odd $c$ with $c$ not dividing $n$, the number of solutions to the equation $n + a \equiv b \mod c$ with $a, b$ quadratic residues modulus $c$ is calculated. We establish a direct relation with those modular solutions and the distances between points of a modular hyperbola. Furthermore, for certain composite moduli $c$, an asymptotic formula for quotients between the number of solutions and $c$ is provided. Finally, an algorithm for integer factorization using such solutions is presented.

Keywords — Modular Hyperbolas, Integer Factorization, Diophantine Equations

1 Introduction

For $n \in \mathbb{Z}$ and $p$ an odd prime, the set of integers

$\mathcal{H}_{n,p} = \{(x, y) \mid xy \equiv n \mod p, \ 0 \leq x, y < p\}$

defines a modular hyperbola of $n$ modulus $p$. If $p$ and $n$ are coprimes, the set $\mathcal{H}_{n,p}$ has $\phi(p) = p - 1$ points.

The above definition can be naturally extended to composite $c$ and the set $\mathcal{H}_{n,c}$. Regarding $\mathcal{H}_{n,c}$ as a subset of the Euclidean plane, the set of distances between points in the hyperbola is

$\mathcal{D}_{n,c} = \{|x - y| \mid x, y \in \mathcal{H}_{n,c}\}$

It is therefore not surprising to see that the set of distances and the set of solutions to the equation

$n + a \equiv b \mod c \quad a, b \in \mathbb{Z}^2 \cup \{0\}$ (1)

are related. The latter form, however, can be seen as more convenient for some applications, as shown in section 4. For the case of composite odd numbers, the number of solutions of (1) can be precisely calculated. As explained in section 3 the number of solutions to (1) is small with respect to $c$ for certain composite $c$, and that very fact can be directly used to build an integer factoring algorithm.

In usual notation, $(\frac{n}{p})$ stands for the Legendre symbol of $n$ modulus a prime $p$. In \cite{1} the following theorem is provided, that will become handy later on.

Theorem 1. Let $p$ be a odd prime and $\gcd(n,p) = 1$, then

$|\mathcal{D}_{n,p}| = \begin{cases} \frac{p-1}{4} + \left(\frac{n}{p}\right) & \text{if } p \equiv 1 \mod 4 \\ \frac{p-3}{4} + 1 & \text{if } p \equiv 3 \mod 4 \end{cases}$
For simplicity, let us consider only the region $A$. Lemma below characterizes the set $A$. That means there exists a correspondence between $u$ and $y$ symmetry axis $y$ symmetry lines $y$. There is a one-to-one correspondence between the points of $D$ and otherwise $|$. Lemma 1. Let $A_u$ be as above.

1. If $y$ is a root of $Y(Y-u) \equiv n \pmod{p}$ then $(y-u, y) \in A_u$
2. If $y$ is a root of $Y(Y+u) \equiv n \pmod{p}$ then $(y+u, y) \in A_u$
3. If $(x, y) \in A_u$ then $y$ is a root of the equations in 1 or 2.

Proof. The calculations are elementary and therefore omitted.

It is immediately clear that $|A_u| = 4$ provided that $4n+u^2 \neq 0 \pmod{p}$ and $u \neq 0$, and otherwise $|A_u| = 2$.

When dividing the square $[1, p-1]^2$ into four regions $R_1, \ldots, R_4$, according to the symmetry lines $y = x$ and $x + y = p$ not two points of $A_u$ belong to the same region, and that means there exists a correspondence between $u$ and the points in one region.

For simplicity, let us consider only the region

$$R_1 = \{(x, y) \in \mathcal{H}_{n,p} \mid 0 \leq x < p, 0 \leq y < \min(x, p-x)\}$$

and

$$\hat{R}_1 = \{(x, y) \in \mathcal{H}_{n,p} \mid 0 \leq x < p, 0 \leq y \leq \min(x, p-x)\}$$

Lemma 2. For $p$ an odd prime, let $n$ be an integer with $gcd(n, p) = 1$, $(\frac{2}{p}) = -1$ and $(\frac{n}{p}) = -1$. Then, there is a bijective correspondence between the points of $R_1$ and $D_{n,p}$.

Proof. For a given pair $(x, y) \in \mathcal{H}_{n,p}$, let $u = |x - y| \in D_{n,p}$ and define $A$ as the set that contains $(x, y)$ and all points that are symmetric to $(x, y)$

$$A = \{(x, y), (y, x), (p-x, p-y), (p-y, p-x)\}$$

Clearly, $A \subseteq A_u$, hence all the elements in $A$ are different, as implied by $gcd(n, p) = 1$, $(\frac{2}{p}) = -1$ and $(\frac{n}{p}) = -1$, so $A = A_u$.

If $u = 0$ and thus $(\frac{2}{p}) = 1$ then $A_u$ has only two points, $(x, x)$ and $(p-x, p-x)$, with $x$ the square root of $n$ modulus $p$ such that $x < p/2$. In that case, $(x, x)$ is the only element of $A_u$ in $R_1$.

If $(\frac{n}{p}) = 1$ and $y$ is the square root of $-n$ modulus $p$ with $y > p/2$, then for $u = p - 2y$ and we have that $A_u = \{(y, p-y), (p-y, y)\}$, so $(y, p-y)$ is the only point of $A_u$ in $R_1$.

The previous observations and the preceding Lemma shows that if $gcd(n, p) = 1$ there is a one-to-one correspondence between the points of $D_{n,p}$ and $\hat{R}_1$. 

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2 The canonical form of a modular hyperbola

The motivation for the following definition can be found in the Fermat method to factor a composite \( n \). In such case the factors of \( n \) can be obtained by looking at the solutions of the Diophantine equation

\[
 n + x^2 = y^2. \tag{2}
\]

If \( n \) is reduced modulus \( c \) for some \( c \) of modest size, the modular solutions of the equation can be obtained without much effort. It is enough (and naive) to try with half of the elements \( \bar{x} \in \mathbb{Z}_c \) and keep the pairs \( (\bar{x}^2, n + \bar{x}^2 \mod c) \) for every \( \bar{x} \) that satisfy \( n + \bar{x}^2 \mod c \in (\mathbb{Z}_c)^2 \cup \{0\} \). Furthermore, if \( x^*, y^* \) satisfies \( 2 \) then clearly \( x^* \) and \( y^* \) must be equivalent modulus \( c \) to one pair of solutions modulo \( c \). This very fact motivates the following definition, which is due to Scolnik [2].

**Definition 1.** A target for \( n \) is a triplet \((a, b, c)\) of non-negative integers with \( a, b \in (\mathbb{Z}_c)^2 \cup \{0\} \) holding \( n + a \equiv b \mod c \)

It will also be the case that \( a, b \) is a target for \( n \) modulus \( c \), meaning that \( (a, b, c) \) is a target for \( n \). In addition, if \( (a, b, c) \) is a target for \( n \) and \( a \equiv x^2 \mod c \), then for some \( 0 \leq \alpha \leq c - 1 \) with \( a^2 \equiv a \mod c \) there is an integer \( z \) such that

\[
 x^2 = (\alpha + cz)^2 \tag{3}
\]

Let \( T(n,c) \) denote the set of targets for \( n \) modulus \( c \), that is

\[
 T(n,c) = \{(a,b,c) \mid a,b \in (\mathbb{Z}_c)^2 \cup \{0\} \text{ and } n + a \equiv b \mod c\}
\]

and let \( \tau(n,c) \) be the number of elements in \( T(n,c) \). For an odd prime \( p \) that does not divide \( n \), it is expected that \( \tau(n,p) \) and \( \left| D_{n,p} \right| \) are equal.

Although the proof can be stated by elementary means, it is more convenient to use a special case of a Jacobi sum. To do so, we write \( N(x^2 = a) \) as the number of solutions of \( x^2 \equiv a \mod p \) with \( 0 \leq a \leq p - 1 \). Unless stated otherwise, from now on all elements should be in \( \mathbb{Z}_p \) hence the solutions to \( x^2 \equiv a \mod p \) should be understood as the elements of \( \mathbb{Z}_p \) that satisfy the former congruence.

The Lemma below is rather intuitive if one considers the canonical form of an hyperbola in the Euclidean space and the definition of \( \mathcal{H}_{n,p} \).

**Lemma 3.** For an integer \( n \) and an odd prime \( p \) with \( p \nmid n \), the number of solutions to \( n + x^2 \equiv y^2 \mod p \) is \( p - 1 \).

**Proof.** The number of solutions to \( n + x^2 \equiv y^2 \mod p \) can be written as

\[
 N(n + x^2 = y^2) = \sum_{n+a=b \mod p} N(x^2 = a)N(y^2 = b)
\]

It is easy to see that \( N(x^2 = a) = 1 + \left(\frac{a}{p}\right) \), so the previous equation becomes

\[
 \sum_{n+a=b \mod p} \left(1 + \left(\frac{a}{p}\right)\right)\left(1 + \left(\frac{b}{p}\right)\right).
\]

or

\[
 \sum_{a \in \mathbb{Z}_p} \left(\frac{a}{p}\right) + \sum_{b \in \mathbb{Z}_p} \left(\frac{b}{p}\right) + \sum_{n+a=b \mod p} \left(\frac{a}{p}\right)\left(\frac{b}{p}\right).
\]

It is well known that for a non-trivial character \( \chi \) over a field \( \mathbb{F}_p \) the sum \( \sum_{a \in \mathbb{F}_p} \chi(a) \) is 0. As the Legendre symbol is a character over \( \mathbb{Z}_p^* \), the second and third summations are 0.
On the other hand, if we rename $a = -na'$ and $b = nb'$ we get

$$\sum_{n+a \equiv b} \left( \frac{a}{p} \right) \left( \frac{b}{p} \right) = \sum_{a'+b' \equiv 1} \left( \frac{-na'}{p} \right) \left( \frac{nb'}{p} \right)$$

$$= \left( \frac{-n}{p} \right) \sum_{a'+b' \equiv 1} \left( \frac{a'}{p} \right) \left( \frac{b'}{p} \right)$$

$$= (-1)^{\frac{n-1}{2}} \sum_{a'+b' \equiv 1} \left( \frac{a'}{p} \right) \left( \frac{b'}{p} \right)$$

(the congruence is modulus $p$). This is a Jacobi sum $J(\chi, \chi)$ with the Legendre symbol as the character $\chi$. Furthermore, as the Legendre symbol is a character of order 2, meaning that $J(\chi, \chi) = J(\chi, \chi^{-1})$, the special case

$$J(\chi, \chi^{-1}) = -(-1)^{\frac{p-1}{4}}$$

holds (see Theorem 1 in chapter 8, section 3 of [4]). Finally,

$$N(n + x^2 = y^2) = p + (-1)^{\frac{n-1}{2}} \left( \frac{-1}{p} \right) = p - 1$$

\[\Box\]

**Theorem 2.** Let $p$ be an odd prime with $gcd(p, n) = 1$, then

$$\tau(n, p) = \begin{cases} \frac{n-1}{2} + \left( \frac{n}{p} \right) & \text{if } p \equiv 1 \mod 4 \\ \frac{n-1}{2} + 1 & \text{if } p \equiv 3 \mod 4 \end{cases}$$

hence $\tau(n, p) = \left| D_{n,p} \right|$.

**Proof.** The proof goes by stages. If $p \equiv 1 \mod 4$ and $\left( \frac{n}{p} \right) = -1$ then there is no target $(a, b, p)$ with $a = 0$ or $b = 0$, so every target $(a, b, p)$ gives four solutions to the equation $n + x^2 \equiv y^2 \mod p$. That is,

$$4\tau(n, p) = p - 1.$$ 

by Lemma [3].

If $p \equiv 1 \mod 4$ and $\left( \frac{n}{p} \right) = 1$ then also $\left( \frac{-n}{p} \right) = 1$, there is one target of the form $(0, b, p)$ and one target of the form $(a, 0, p)$ with $a \neq 0$ and $b \neq 0$, each one giving two solutions to $n + x^2 \equiv y^2 \mod p$ and the rest providing four solutions. It therefore means that

$$4(\tau(n, p) - 2) + 4 = p - 1.$$ 

Finally, if $p \equiv 3 \mod 4$ exactly one of the pairs $\{n, -n\}$ is a quadratic residue modulus $p$, meaning there is only one target $(a, b, p)$ of $n$ with $a = 0$ or $b = 0$. That target gives two solutions to $n + x^2 \equiv y^2 \mod p$, and each one of the rest gives four. So,

$$4(\tau(n, p) - 1) + 2 = p - 1.$$ 

\[\Box\]

**Remark 1.** If $s$ and $t$ are coprime, then $\tau(n, st) = \tau(n, t)\tau(n, s)$ following from the Chinese Remainder Theorem, so $\tau(n, c)$ is multiplicative as a function of $c$.

Furthermore, $\tau$ can be calculated for moduli that are powers of odd primes.

**Theorem 3.** Let $p$ be an odd prime, $n$ an integer with $gcd(p, n) = 1$ and $k > 0$. 

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1. If \((a,b,p^k)\) is a target for \(n\) with \(a \neq 0\) and \(b \neq 0\), then there are \(p\) targets \((a',b',p^{k+1})\) of \(n\) with \(a \equiv a' \mod p^k\) and \(b \equiv b' \mod p^k\).

2. If \((0,b,p^k)\) is a target for \(n\), the number of targets \((a',b',p^{k+1})\) for \(n\) with \(0 \equiv a' \mod p^k\) and \(b \equiv b' \mod p^k\) is
   
   \[ \begin{align} & (a) \ |(\mathbb{Z}_p)^2| + 1 \text{ for } k \text{ even.} \\ & (b) \ 1 \text{ for } k \text{ odd.} \end{align} \]

3. If \((a,0,p^k)\) is a target for \(n\), the number of targets \((a',b',p^{k+1})\) for \(n\) with \(a \equiv a' \mod p^k\) and \(0 \equiv b' \mod p^k\) is
   
   \[ \begin{align} & (a) \ |(\mathbb{Z}_p)^2| + 1 \text{ for } k \text{ even.} \\ & (b) \ 1 \text{ for } k \text{ odd.} \end{align} \]

**Proof.**

1. If \(g \neq 0 \mod p\) is a quadratic residue modulus \(p^k\), then \(g + tp^k\) is a quadratic residue modulus \(p^{k+1}\) for \(0 \leq t \leq p\). The set
   
   \[ \{(a + tp^k, b + t'p^k, p^{k+1}) \mid t = (n + a - b)/p^k + t, \ 0 \leq t < 0\} \]

   is a set of targets for \(n\).

2. If \(k\) is even, \(tp^k\) is a quadratic residue modulus \(p^{k+1}\) provided \(t \in (\mathbb{Z}_p^2) \cup \{0\}\). Besides, \(b + t'p^k\) is quadratic residue of \(p^{k+1}\) for \(0 \leq t' < p\). The set
   
   \[ \{(tp^k, b + t'p^k, p^{k+1}) \mid t' = (n - b)/p^k + t, \ 0 \leq t < 0\} \]

   is a set of targets for \(n\).

   If \(k\) is odd, \(tp^k\) is a quadratic residue of \(p^{k+1}\) only for \(t = 0\).

3. Analogous to the previous case.

For brevity, let \(s_p(n) = (1 + (\frac{n}{p}))/2 + (1 + (\frac{-n}{p})))/2\).

**Theorem 4.** If \(p\) is an odd prime and \(p \mid n\), then for \(k > 1\)

\[
\tau(n, p^{k+1}) = \begin{cases} 
\lfloor \tau(n, p^k) - s_p(n) \rfloor + s_p(-n)(|(\mathbb{Z}_p)^2| + 1) & \text{if } k \text{ is even} \\
\lfloor \tau(n, p^k) - s_p(-n) \rfloor + s_p(n) & \text{if } k \text{ is odd} 
\end{cases}
\]

**Proof.** Follows immediately from Theorem 3.

From the theorems 2, 3 and Remark 1 the number of targets for \(n\) modulus an odd composite \(c\) whose prime factors are known can be easily calculated.

### 3 Relation between targets and \(D_{n,c}\)

As the number of solutions to \(n + x^2 \equiv y^2 \mod p\) and the number of points in \(\mathcal{H}_{n,p}\) are the same, it can be easily seen that the targets for \(n\) modulus \(p\) and the set of distances between points of a modular hyperbola \(\mathcal{H}_{n,p}\) are related.

**Theorem 5.** Let \(p\) be an odd prime with \(\gcd(n,p) = 1\), \(\left(\frac{n}{p}\right) = -1\) and \(\left(\frac{-n}{p}\right) = -1\). Then, there is a correspondence between the elements of \(T_{n,p}\) and \(D_{n,p}\).
Proof. For simplicity, instead of $R_1$ consider
\[ R_4 = \{(x, y) \in H_{n,p} \mid 0 \leq y < p, 0 \leq x < \min(y, p-y)\} \]
resulting from a reflection of $R_1$. As $\gcd(n, p) = 1$, $\left(\frac{a}{p}\right) = -1$ and $\left(\frac{-a}{p}\right) = -1$, there are no targets of the form $(b, b, p)$ or $(a, 0, p)$ and there are no points of $H_{n,p}$ in the boundaries of $R_4$.

There is indeed a correspondence between the elements of $T_{n,p}$ and the elements of $R_4$ as it can be seen by the mapping from $R_4$ to $T_{n,p}$, given by
\[ (x, y) \mapsto (2^{-2}(x - y)^2, 2^{-2}(x + y)^2, c) = (a, b, c) \]
and the mapping from $T_{n,p}$ to $R_4$, given by
\[ (a, b, c) \mapsto (x, y) \]
such that $A_u \cap R_4 = \{(x, y)\}$ with $u = 2\alpha$, $\alpha^2 \equiv a \mod p$ and $\alpha < p/2$ (by the proof of Lemma 2, this map is well-defined). From Lemma 2, the result follows.

Besides their algorithmic implications, theorems 6 and 7 provide bounds that show the asymptotic behavior of $\tau$ with respect to $c$ for certain composite moduli.

In what follows, $p$ stands for an odd prime. The symbols $O$ and $\Theta$ are used, as customary, to describe asymptotic bounds. A simple result (following from one of Mertens’ Theorems; see Corollary 2.10 in [3] and Theorem 5.13 in [5]) and that will become useful later is
\[ \prod_{p \leq B} \frac{p+1}{p} = \Theta(\log B). \]
As usual, $\pi$ is used for the prime-counting function, so $\pi(B)$ is the number of primes up to $B$.

**Theorem 6.** If $c = \prod_{p \leq B} p$ and for all $p \leq B$ with $p \equiv 1 \mod 4$ holds $\left(\frac{a}{p}\right) = -1$ then
\[ \frac{\tau(n, c)}{c} = O\left(\frac{4^{-\pi(B)}}{\log B}\right) \]

**Proof.** By Remark
\[ \tau(n, c) = \prod_{p \leq B} \tau(n, p) \]
and from the definition of $\tau$ and the assumptions of the above Theorem, it follows that $\tau(n, p) \leq (p+1)/4$ for all $p \leq B$. Therefore,
\[ \tau(n, c) = \prod_{p \leq B} \tau(n, p) \leq 4^{-\pi(B)} \prod_{p \leq B} p + 1. \]
Dividing by $c$ and applying $\prod_{p \leq B} \frac{p+1}{p} = \Theta(\log B)$ the desired result immediately follows.

**Theorem 7.** If $c = \prod_{p \leq B} p$ then
\[ \frac{1}{c} \prod_{2 < p \leq B} \left(\tau(n, p) - \left(\frac{n}{p}\right)\right) = O\left(\frac{4^{-\pi(B)}}{\log B}\right) \]

**Proof.** Note that $\tau(n, p) - \left(\frac{n}{p}\right) \leq (p+1)/4$. The proof is identical as the one given above.
4 An application to factoring

We now want to briefly sketch an algorithm for integer factorization using targets. The complexity analysis of the algorithm is extensive and will be presented in a forthcoming paper; hence we exhibit here the basic idea. Although the algorithm is not of practical use nowadays, given that its running time is exponential (roughly $O(n^{1/3})$), there are many subexponential algorithms that were born from previous exponential ones, and besides that, it shows one possible way for using the elements in $D_{n,c}$ to factor $n$.

Let $n = pq$ be composite with $p, q$ primes with $|p - q|$ not too big, so that if $x^*, y^*$ are solutions to $n + x^2 = y^2$ then $|x^*| < n^{1/2}$.

As it was suggested by theorems 6 and 7, the idea is to let $c' \cdot c = p_1 \ldots p_m$ be the product of the first consecutive odd primes (with $c' = p_1 \ldots p_r$, $0 < r < m$) and observe that, for some pair of targets $(a, b, c)$ and $(a', b', c')$ it holds that

$$x^* = a + tc = a' + uc'$$

for integers $t, u$. Clearly then, if $(a, b, c)$ is a right target for $n$ for some $\alpha$ square root of $a$ modulus $c$, then there is an integer $z$ bounded by $|z| < n^{1/2}/c$ such that

$$x^2 = (\alpha + cz)^2.$$  

But also note that $(\alpha + cz)^2 \equiv a' \mod c'$, hence if $z_1, \ldots, z_l$ are the solutions of $(\alpha + cz)^2 \equiv a' \mod c'$ with $0 \leq z_i \leq c'$, in which case

$$x^2 = (\alpha + cz_i + c \cdot c' \cdot k)^2$$

for some $i = 1, \ldots, l$, with $k$ an integer bounded by $|k| \leq \lceil n^{1/2}/(c' \cdot c) \rceil$. The search is conducted over $k$, and as $m$ grows the bound gets narrowed. An appropriate choice is to take $r = \lfloor m/2 \rfloor$ and $c \cdot c'$ as the maximum product of consecutive odd primes that satisfy

$$n^{1/2}/(c' \cdot c) \geq 1.$$  

The complete procedure is detailed as Algorithm 1.
Algorithm 1 Factoring integers using targets

Require: \(n\), a composite to factor. \(m\) a positive integer such that \(p_1 \ldots p_m\) is the maximum product of odd primes with \(n^{1/2}/(p_1 \ldots p_m) \geq 1\).

1: Choose \(0 < r < m\), let \(c' \leftarrow p_1 \ldots p_r\), \(c \leftarrow p_{r+1} \ldots p_m\).
2: \(z_{\text{max}} \leftarrow \lceil n^{1/2}/c \rceil\)
3: \(k_{\text{max}} \leftarrow \lceil n^{1/2}/(c \cdot c') \rceil\)
4: \(k_{\text{min}} \leftarrow -k_{\text{max}}\)
5: for every target \((a_i, b_i, c)\) do
6: Let \(\{\alpha_{i,1}, \ldots, \alpha_{i,k_i}\}\) be the square roots of \(a_i\) modulus \(c\).
7: for \(j = 1, \ldots, k_i\) do
8: \(t_{i,j} \leftarrow (\alpha_{i,1} - \alpha_{i,j})(c^{-1} \mod c')\) \{Precalculations\}
9: \(\delta_{i,j} \leftarrow \alpha_{i,j} + c t_{i,j}\)
10: end for
11: \(\theta_i \leftarrow (c^{-1} \mod c')(\alpha_{i,1} - \alpha_{1,1})\)
12: end for
13: Let \(P(z) = (\alpha_{1,1} - cz)^2\).
14: \(i \leftarrow 0\)
15: for \(\ell = 1, \ldots, \tau(n, c')\) do
16: Take \((a'_\ell, b'_\ell, c')\), the target number \(\ell\) of \(n\) modulus \(c'\)
17: for \(z = 0, \ldots, c' - 1\) do
18: if \((P(z) - a'_\ell) \mod c' = 0\) then
19: \(z_i \leftarrow z, i \leftarrow i + 1\) \{Initial dots\}
20: end if
21: end for
22: \(u \leftarrow i - 1\)
23: end for
24: for \(s = 1, \ldots, \tau(n, c)\) do
25: for \(t = 1, \ldots, k_i\) do
26: Define \(x_{i,j} = \delta_{s,t} + c(\theta_s + c' \cdot j + z_i)\) \{Search for a solution\}
27: \(f \leftarrow \prod_{t=0}^{n} \prod_{j=k_{\text{min}}}^{k_{\text{max}}} (x_{i,j} + n)^{1/2} \mod n\)
28: \(g \leftarrow |\gcd(f, n)|\)
29: if \(1 < g < n\) then
30: \(\text{return} \ (g, n/g)\)
31: end if
32: end for
33: end for

The algorithm was implemented in Python. After conducting several tests with small numbers (up to 30 digits) it became evident that the number of iterations is near \(O(\log p_{m} n^{1/3})\) (all the operations can be bounded by the cost of multiplying integers of size at most \(\log n\)).

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