PERIODIC ORBITS OF MAGNETIC FLOWS FOR WEAKLY EXACT UNBOUNDED FORMS AND FOR SPHERICAL MANIFOLDS

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Abstract. We show that for weakly exact magnetic flows with infinite Mañé critical value the action functional satisfies the Palais-Smale condition on the space of contractible loops with period bounded and bounded away from zero. This fills a gap in the proof of the existence of one closed contractible magnetic geodesic for almost every energy in [Mer10]. The gap has also been fixed independently by Will Merry and it will appear soon in an erratum by the author.

The idea used for the weakly exact case is then generalized to arbitrary magnetic flows to study critical sequences for the action 1-form. As a corollary we prove that if the second fundamental group of the base manifold is non-trivial, there exists one closed contractible magnetic geodesic for almost every energy.

1. Introduction

Let \((M,g)\) be a Riemannian manifold and let \(\sigma \in \Omega^2(M)\) be a closed 2-form. The Hamiltonian flow determined by the kinetic Hamiltonian

\[ H_{\text{kin}}(q,p) = \frac{1}{2} |p|^2 \]

and the twisted symplectic form \(\omega_\sigma := dp \wedge dq + \pi^* \sigma\) is called the magnetic flow of the pair \((g,\sigma)\). The reason for this terminology is that this flow can be thought of as modeling the motion of a particle of unit mass and charge under the effect of a magnetic field represented by the 2-form \(\sigma\).

In this paper we are interested in proving the existence of closed magnetic geodesics, namely of periodic orbits of the magnetic flow, on a given energy level \(H_{\text{kin}}^{-1}(k)\).

Our goal will be twofold. First, we assume that \(\sigma\) is weakly exact, namely that the lift \(\tilde{\sigma}\) of \(\sigma\) to the universal cover \(\tilde{M}\) is exact. In this case, one defines the Mañé critical value

\[ c(g,\sigma) := \inf_{[\partial \theta = \tilde{\sigma}]} \sup_{q \in \tilde{M}} \tilde{H}_{\text{kin}}(q,\theta_q) \in [0, +\infty] \]

where \(\tilde{H}_{\text{kin}}\) is the lift of \(H_{\text{kin}}\) to \(\tilde{M}\). It has been proven in [Mer10] the following

**Theorem A** (Merry, 2010). For almost every \(k \in (0, c(g,\sigma))\), the energy level set \(H_{\text{kin}}^{-1}(k)\) carries a contractible magnetic geodesic.

When \(\sigma\) is exact, respectively when \(c(g,\sigma) < +\infty\), the result was already shown by Contreras [Con06], respectively by Osuna [Osu05]. However, when \(\sigma\) is unbounded, namely \(c(g,\sigma) = +\infty\), the proof presented in [Mer10] contains a gap, since Theorem 5.1 stated therein does not hold. In Section 3 of the present paper we show how to fix the argument for the unbounded case through the crucial Lemma 3.5. We will say more about the proof further below in this introduction.

**Date:** December 3, 2014.

2010 Mathematics Subject Classification. 37J45, 58E05.

Key words and phrases. Dynamical systems, Periodic orbits, Symplectic geometry, Magnetic flows.

The second author was supported by a grant within the project SFB 878.
As we write this paper we have become aware that Will Merry has found independently a way of fixing the gap. More details will appear in an erratum by the author. We apologize with Will Merry for the lack of communication in this matter.

The second goal will be to generalize the existence result above when $\sigma$ is not weakly exact. In this case, $M$ is necessarily spherical, namely $\pi_2(M) \neq 0$. This topological information alone will suffice to give existence results on this kind of manifolds, for any magnetic form $\sigma$. In Section 3 we prove the following

**Theorem B.** Let $(M, g)$ be a Riemannian manifold with $\pi_2(M) \neq 0$ and $\sigma$ be a closed 2-form on $M$. For almost every $k > 0$ there exists a closed contractible magnetic geodesic of the magnetic flow defined by $(g, \sigma)$ with energy $k$.

For simply connected manifolds this result appears for the first time in [Koz85, Theorem 7], where it is claimed to hold for every $k > 0$. The author gives only a sketch of proof and does not take into account some crucial convergence problems, which are today only partially solved ([Con06], [Mer10], [Abb13]). These issues are also ultimately responsible for the fact that with our method we do not get a contractible periodic orbit for every energy (see further below in the introduction). Theorem [B] has concrete applications when $M = S^2$ and $\sigma$ is non-exact to the motion of rigid bodies (see [Koz85, Theorem 8] and [Nov82]).

If $\sigma \neq 0$ the two theorems above were already proved by Schlenk ([Sch06, Corollary 3.6]) for almost every $k$ in the range $(0, d_1(g, \sigma))$, where

$$d_1(g, \sigma) := \sup \left\{ k \in (0, +\infty) \mid \{ H_{\text{kin}}^{-1} \leq k \} \text{is stably displaceable in } (T^*M, \omega) \right\}.$$ 

It follows from results of Laudenbach-Sikorav ([LS94] and Polterovich [Pol95]) that $d_1(g, \sigma)$ is positive. Moreover, in the weakly exact case Merry shows in [Mer11, Theorem 1.1] that $d_1(g, \sigma) \leq c(g, \sigma)$. It is conjectured in [CFP10] that $d_1(g, \sigma) = c(g, \sigma)$. Such equality holds for non-exact forms on tori since in this case $c(g, \sigma) = +\infty$ and there exists a symplectomorphism between $(T^*\mathbb{T}^n, \omega_\sigma)$ and $(\mathbb{R}^2 \times W, \omega_{\text{std}} \oplus \omega)$ (where $(W, \omega)$ is some symplectic manifold) as shown in [GK99, Theorem 3.1].

In general, our methods yields existence results only for almost every $k$. However, when a particular $H_{\text{kin}}^{-1}(k)$ is stable [HZ94, page 122] we can upgrade such almost existence results to show that there exists a closed magnetic geodesic of energy $k$. This will be fruitful when $\sigma$ is a symplectic form and $M$ is a surface, since in this case low energy levels are stable. Such statement can be found in [Ben14a] for $M \neq \mathbb{T}^2$ (see also [FS07, Lemma 12.6] and [Pat09, Remark 2.2]). For $\mathbb{T}^2$, one just takes the global angular form associated to any section of $T^*\mathbb{T}^2$ as a stabilizing form. As a result, in this setting we get the existence of a contractible magnetic geodesics for every low energy (see Corollary 3.9). This is a weaker version of Assertion 3 in [Gin87], where better lower bounds on the number of periodic orbits are given. When $M \neq \mathbb{T}^2$ an alternative existence proof can be found in [Sch11, Sch12a, Sch12b]. The strength of this approach is that the energy range for which existence holds is characterized in terms of geometric quantities depending on $g$ and $\sigma$. This yields, for example, that if $g$ is a convex metric on $S^2$, every energy level carries two distinct magnetic geodesics.

A related interesting problem is to find the optimal lower bound for the number of contractible periodic orbits. If $M \neq S^2$, this lower bound is infinite for every low energy as proven in [LC06] and in [Hin09] for $M = \mathbb{T}^2$ and in [GGM14] for surfaces of higher genus. If $M = S^2$ there are either 2 or infinitely many magnetic geodesics and there is no known example where the first alternative holds [Ben14b].

When $M$ is a surface we can single out the class of oscillating magnetic form which is almost complementary to the class of symplectic ones. A form is oscillating if its density
with respect to the area form takes both positive and negative values. In this case there is a (non necessarily contractible) periodic orbit on every small energy level by [CMP04] or [Tab93]. When \( \sigma \) is in particular exact, the main theorem in [AMMP14] implies the existence of infinitely many (not necessarily contractible) closed magnetic geodesics for almost every low energy level. Such result has been generalized in [ABI4] for oscillating forms on surfaces of genus at least 2. Using a method similar to the one contained in Section 4 of the present paper, this statement can be extended to \( M = T^2 \). We address this problem in a forthcoming paper, where we hope to treat also the case of the 2-sphere.

Coming back to the discussion on stability we observe that if \( M \) has dimension bigger than 2, proving that low energy levels are stable when \( \sigma \) is symplectic is an open problem (see [CFP10, page 1773] and [CFP10, Proposition 6.19] for a proof in the homogeneous case). However, existence of contractible orbits for every low energy levels for symplectic \( \sigma \) still holds as Usher proves in [Ush09] using results of [GG09] (see also [Ker99] for multiplicity results, generalizing [Gin87], when \( \sigma \) is a Kähler form).

We now give an account of the tools we use to prove Theorem A and B. First, as we want to use stability, we will need to modify our Hamiltonian \( H_{\text{kin}} \) close to a given energy level (see [Abb13] and [MP10]). In order to do so, it will be crucial to have a theory which works for every Tonelli Hamiltonian and not only for \( H_{\text{kin}} \).

Therefore, let \( H : T^* M \to \mathbb{R} \) be a Tonelli Hamiltonian, \( L : T^* M \to TM \) the Legendre transform and \( L : TM \to \mathbb{R} \) the Fenchel dual of \( H \). If \( \sigma \) is weakly exact, we can define the Mañé critical value \( c(L, \sigma) \in \mathbb{R} \cup \{ +\infty \} \) similarly as we did before (see (3.5)). Finally denote by \( E : TM \to \mathbb{R} \) the energy function \( E := H \circ L^{-1} \) and define

\[
e_0(L) := \max_{q \in M} E(q,0).
\]

When \( H = H_{\text{kin}} \), then \( L_{\text{kin}} = E_{\text{kin}} = \frac{1}{2} |v|^2 / q \). In this case we have \( c(L_{\text{kin}}, \sigma) = c(g, \sigma) \) and \( e_0(L_{\text{kin}}) = 0 \).

To the pair \((L, \sigma)\) we can associate a flow on \( TM \) by gluing together the Euler-Lagrange flows of \( L + \vartheta_i \) on \( TU_i \), where \( \vartheta_i \) is a primitive of \( \sigma \) on \( U_i \), and \( \{ U_i \} \) is an open cover of \( M \). The flow defined by \((L, \sigma)\) is conjugated via \( L \) to the Hamiltonian flow of the pair \((H, \omega_\sigma)\), with \( \omega_\sigma \) twisted symplectic form defined as above.

The key ingredient of this paper is that the periodic orbits of the flow defined by \((L, \sigma)\) contained in \( E^{-1}(k) \) are in one to one correspondence with the zeros of the action 1-form

\[
\eta_k \in \Omega^1(\mathcal{M}), \quad \eta_k(x,T) := dS_k^L(x,T) + \int_0^1 \sigma_{x(t)}(\dot{x}(t),\cdot) \, dt
\]

where \( \mathcal{M} := H^1(T, M) \times \mathbb{R}^+ \) is the space of Sobolev loops in \( M \) with arbitrary period and \( S_k^L \) is the free-period Lagrangian action functional associated to \( L \).

In Section 2 we define the action 1-form rigorously and discuss its properties. In particular we will show that \( \eta_k \) is locally uniformly Lipschitz and, in a suitable sense, closed.

In Section 3 we focus on the weakly-exact case with special attention given to the unbounded case (i.e. when all the primitives of the lift of \( \sigma \) are unbounded). Under the weakly-exactness assumption the action 1-form \( \eta_k \) is exact on the space \( \mathcal{M}_0 \subseteq \mathcal{M} \) of contractible loops and a primitive \( S_k \) can be explicitly written. In Subsection 3.1 we prove Theorem 3.4 asserting that, even in the unbounded case, the functional \( S_k \) satisfies the Palais-Smale condition on subsets of \( \mathcal{M}_0 \) with periods bounded and bounded away from zero. This allows to use also in the unbounded case the same techniques that one uses in the bounded one (see [Mer10]). As a corollary in Subsection 3.2 we get the existence of one closed contractible periodic orbit for almost every energy in the range \((e_0(L), c(L, \sigma))\), in particular completing the proof of Theorem A. Finally, combining this result with the
stability properties of $H^{-1}_{\text{kin}}(k)$ on surfaces with positive genus, we show the existence of one closed contractible magnetic geodesic for every low energy level, when \( \sigma \) is symplectic.

In Section\[\text{4}\] we consider the general case and prove a version of Theorem\[\text{3}\] for arbitrary Tonelli Lagrangians, showing the existence of a contractible periodic orbit for spherical manifolds for almost every \( k \) in the range \((\epsilon_0(L), +\infty)\). We use the same strategy implemented in \[\text{Mer10}\] to show Theorem\[\text{A}\]. However, since \( \sigma \) might not be weakly-exact, an action functional whose differential is the action 1-form is in general not available. The existence of such functional was used in several stages of the proof of Theorem\[\text{A}\]. Hence, we now give a brief outline of Section\[\text{4}\] highlighting the alternative arguments we adopt to circumvent the lack of the action functional.

First, in Proposition\[\text{4.4}\] we show the existence of limit points of critical sequences (the natural replacement of Palais-Smale sequences in this context) for the action 1-form \( \eta_k \), provided the periods are bounded and bounded away from zero. This is the analogous of the first part of Theorem\[\text{3.4}\]. Here Lemma\[\text{3.5}\] is used to bypass the lack of the action and to find a uniform bound on the kinetic energy. Thanks to such bound the critical sequence eventually enters a local chart, where we know by Lemma\[\text{2.2}\] that \( \eta_k \) is exact. The rest of the argument follows as in the weakly exact case.

Second, under the assumption that \( \pi_2(M) \neq 0 \) we show the existence of a critical sequence with the properties above for almost every energy \( k \) in the range \((\epsilon_0(L), +\infty)\) as follows. We start by observing that, for \( \delta > 0 \) sufficiently small, the action 1-form \( \eta_k \) is exact on the set \( \mathcal{V}_\delta \) of loops \((x, T)\) such that the kinetic energy of \( x \) is less than \( \delta \). Indeed, there is a deformation retraction of \( \mathcal{V}_\delta \) onto the set of constant loops which yields a canonical choice of capping disks and hence a primitive \( S_k : \mathcal{V}_\delta \to \mathbb{R} \) of \( \eta_k \). Moreover, we have

\[
\inf_{\partial \mathcal{V}_\delta} S_k \geq \varepsilon_{k, \delta},
\]

for a suitable constant \( \varepsilon_{k, \delta} > 0 \). The existence of this local primitive suffices to prove the analogous of the second part of Theorem\[\text{3.4}\] and show that critical sequences with periods tending to zero enter eventually \( \mathcal{V}_\delta \) and have action \( S_k \) converging to zero (see Lemma\[\text{4.10}\]).

Consider now \((x_0, T_0) \in \mathcal{M}_0 \) constant loop such that \( S_k(x_0, T_0) < \varepsilon_{k, \delta}/4 \). To a non-zero element \( u \in \pi_2(M, x_0) \) we can associate a corresponding set of paths \( \mathcal{U} \in \pi_1(\mathcal{M}_0, (x_0, T_0)) \), where \( \mathcal{U} \) is any element in the preimage of \( u \) under the map \( \pi_1(\mathcal{M}_0, (x_0, T_0)) \to \pi_2(M, x_0) \) (see Definition\[\text{4.12}\]). We use the integration of \( \eta_k \) along the paths in \( \mathcal{U} \) to bypass the lack of action in this case and define the minimax value

\[
c^\mathcal{U}(k) := S_k(x_0, T_0) + \inf_{u \in \mathcal{U}} \sup_{s \in [0, 1]} \Delta S_k(u)(s),
\]

where

\[
\Delta S_k(u)(s) : [0, 1] \to \mathbb{R}, \quad \Delta S_k(u)(s) := \eta_k(u|_{[0, s]}) = \int_0^s u^* \eta_k.
\]

If \( \eta_k \) was actually exact, then \( c^\mathcal{U}(k) \) would be the minimax value considered in \[\text{Mer10}\]. In Lemma\[\text{4.14}\] we show that the function \( k \mapsto c^\mathcal{U}(k) \) is monotonically increasing and hence Lipschitz-continuous almost everywhere. The class \( \mathcal{U} \) is invariant under the normalized semi-flow defined by \( -\tau \eta_k \) and truncated below \( \varepsilon_{k, \delta}/2 \) inside \( \mathcal{V}_\delta \), since \( S_k(x_0, T_0) \) is strictly less than \( \varepsilon_{k, \delta}/2 \). We prove in Lemma\[\text{4.11}\] that such semi-flow is complete and, since all elements \( u \) of \( \mathcal{U} \) have the same starting point, it decreases the value of the function

\[
s \mapsto S_k(x_0, T_0) + \Delta S_k(u)(s)
\]

by Lemma\[\text{4.5}\]. In the crucial Proposition\[\text{4.17}\] we show the existence of critical sequences for \( \eta_k \) with periods bounded and bounded away from zero under the assumption that \( k > \epsilon_0(L) \) is a Lipschitz point for \( c^\mathcal{U}(\cdot) \). To ensure the boundedness of the periods we use the so-called Struwe monotonicity argument \[\text{Str90}\] (see also \[\text{Con06}, \text{Abb13}, \text{AMP13}\].
for other applications of the monotonicity argument). In this case we bypass the lack of a global action by using the well-known fact that the time-1 flow of \(-\frac{\partial}{\partial \tau_k}\) maps subsets of \(\mathcal{M}_0\) with bounded periods into subsets with bounded periods (see [Mer10], Lemma 5.7 and Lemma 1.7 below). To ensure that the periods are bounded away from zero, we show that the critical sequence can be taken to lie in the complement of \(\{S_k < \varepsilon_{k,\delta}/2\}\). To this purpose we use the fact that if \(s\) almost realizes the maximum of the function \(s \mapsto S_k(x_0, T_0) + \Delta S_k(u)(s)\), then \(u(s) \notin \{S_k < \varepsilon_{k,\delta}/2\}\). This is proved in Lemma 4.15 and uses the fact that since \(u \neq 0\), every \(u \in \Omega\) must intersect \(\partial V_\delta\) (see Lemma 4.13).

Since the set of points at which a monotone function is Lipschitz-continuous is a full-measure set, the version of Theorem B for arbitrary Tonelli Lagrangians follows.

Finally, using stability again, in Corollary 4.20 we prove that there exists a periodic orbit on \(M = S^2\) on every large energy level and, if \(\sigma\) is symplectic, on every low energy level.

2. Preliminaries

Let \((M, g)\) be a closed \(n\)-dimensional Riemannian manifold and let \(\sigma \in \Omega^2(M)\) be a closed 2-form on \(M\). Let \(L : TM \to \mathbb{R}\) be a smooth Tonelli Lagrangian. This means (see [ABb13]) that its restriction to each fiber \(T_qM\)

- is uniformly convex, namely
  
  \[d_{v_0}L > 0;\]

- has superlinear growth, namely
  
  \[
  \lim_{|v| \to +\infty} \frac{L(q, v)}{|v|} = +\infty.
  \]

If \(L\) is Tonelli, then we can define the Legendre transform \(\mathcal{L} : TM \to T^*M\), the Hamiltonian function \(H : T^*M \to \mathbb{R}\) as the Fenchel dual of \(L\) and the energy function \(E : TM \to \mathbb{R}\) by \(E := H \circ \mathcal{L}\). From the definition, we readily see that \(H\) and \(E\) are also Tonelli. This means, for example, that the energy \(E\) satisfies

\[
(2.1)\quad E_0 |v|^2_0 - E_1 \leq E(q, v), \quad \forall (q, v) \in TM,
\]

where \(|\cdot|\) is the norm induced by \(g\) and \(E_0 > 0, E_1 \in \mathbb{R}\) are suitable constants. This will be essential later on to prove the crucial Lemma 3.5. Finally, define the value

\[
e_0(L) := \max_{q \in M} E(q, 0).
\]

The pair \((L, \sigma)\) defines a flow on \(TM\) in the following way. Take an open cover \(\{U_i\}\) of \(M\) such that \(\sigma = d\vartheta_i\) on \(U_i \subset M\). The Lagrangian functions \(L + \vartheta_i : TU_i \to \mathbb{R}\) yield flows on each \(TU_i\) via the Euler-Lagrange equations. Such flows glue together since on \(U_i \cap U_j\), \(L + \vartheta_j - (L + \vartheta_i) = \vartheta_j - \vartheta_i\) is a closed form. We associate to \((L, \sigma)\) the global flow that we get on \(TM\) obtained by the gluing procedure.

We can associate to the pair \(H\) a flow on \(T^*M\) as well, that is the Hamiltonian flow of \(H\) with respect to the twisted symplectic form \(\omega_\sigma\) on \(T^*M\), which is defined as

\[
\omega_\sigma := dp \wedge dq + \pi^* \sigma.
\]

**Lemma 2.1.** The Hamiltonian flow associated to \((H, \omega_\sigma)\) is conjugated to the flow associated to \((L, \sigma)\) via the Legendre transform \(\mathcal{L}\).

**Proof.** By the gluing procedure above, it is enough to prove the statement when \(\sigma\) is exact. Thus, let \(\sigma = d\vartheta \in \Omega^1(M)\). If \(\bar{H}(q, p) = H(q, p - \vartheta_q)\), we readily see that the Hamiltonian flow of \(H\) with respect to \(\omega_{d\vartheta}\) is conjugated to the Hamiltonian flow of \(\bar{H}\) with respect to \(d\lambda\).
by the translation map \((q,p) \mapsto (q,p + \partial_q)\). If \(\mathcal{L} : TM \to T^*M\) is the Legendre transform associated with \(\bar{H}\) and \(\bar{L} : TM \to \mathbb{R}\) is the Fenchel dual of \(\bar{H}\), it suffices to show that
\[
(2.2) \quad \mathcal{L}(q,p) = \bar{L}(q,p + \partial_q) \quad \text{and} \quad \bar{L}(q,v) = L(q,v) + \partial_q(v).
\]
Indeed, we have
\[
v = d_p \bar{H}(q,p) = d_p H(q,p - \partial_q) \implies p = \partial_q + \mathcal{L}^{-1}(v)
\]
and the first identity in (2.2) follows. For the second identity we observe that
\[
\bar{L}(q,v) = \langle \partial_q + \mathcal{L}^{-1}(v), v \rangle - \bar{H}(q, \partial_q + \mathcal{L}^{-1}(v))
\]
\[
= \partial_q(v) + \langle \mathcal{L}^{-1}(v), v \rangle - H(q, \mathcal{L}^{-1}(v)) = \partial_q(v) + L(q,v).
\]

By the lemma trajectories of the Hamiltonian flow associated with \((H,\omega_B)\) contained in \(H^{-1}(k)\) correspond to trajectories of the flow defined by \((L,\sigma)\) contained in \(E^{-1}(k)\).

We aim to find closed orbits on a given energy level \(E^{-1}(k)\), using variational methods on a suitable space of loops \(\mathcal{M}\), which we now define.

Let \(H^1(\mathbb{T}, M)\) be the space of Sobolev loops with period 1 and let \(\mathbb{R}^+ := (0, +\infty)\). Every Sobolev loop \(\gamma : \mathbb{R}/T\mathbb{Z} \to M\) can be identified with the pair \((x,T) \in H^1(\mathbb{T}, M) \times \mathbb{R}^+\), where \(x(t) := \gamma(Tt)\). Conversely, from \((x,T) \in H^1(\mathbb{T}, M) \times \mathbb{R}^+\) we obtain a Sobolev loop \(\gamma : \mathbb{R}/T\mathbb{Z} \to M\) by \(\gamma(s) := x(s/T)\). To ease notation, we adopt the identification \(\gamma = (x,T)\) throughout the paper and, to avoid confusion, we denote with a dot the derivatives with respect to \(t\) and with a prime the derivatives with respect to \(s\). Hence, the space
\[
\mathcal{M} := H^1(\mathbb{T}, M) \times \mathbb{R}^+,
\]
can be interpreted as the space of Sobolev loops with arbitrary periods. We denote by \(\mathcal{M}_0\) the connected component of \(\mathcal{M}\) given by contractible loops and call \(\pi_{H^1} : \mathcal{M} \to H^1(\mathbb{T}, M)\) the projection on the first factor. For any Sobolev loop \(\gamma = (x,T)\) we define
\[
\ell(\gamma) := \int_0^T |\gamma'(s)| \, ds, \quad e(\gamma) := \int_0^T |\gamma'(s)|^2 \, ds
\]
as the \textit{length}, respectively the \textit{kinetic energy} of \(\gamma\). The space \(\mathcal{M}\) is a Hilbert manifold with the product metric \(g_{\mathcal{M}} = g_{H^1} + dT^2\). If \(\zeta_1, \zeta_2 \in T_x H^1(\mathbb{T}, M)\), then
\[
(2.3) \quad g_{H^1}(\zeta_1, \zeta_2)_x := \int_\mathbb{T} \left[ g_{x(t)}(\dot{\zeta}_1(t), \dot{\zeta}_2(t)) + g_{x(t)}(\dot{\zeta}_1(t), \dot{\zeta}_2(t)) \right] \, dt,
\]
where \(\dot{\zeta}\) denotes the covariant derivative of \(\zeta\) induced by \(g\). We use the shorthand
\[
g_{H^1}(\zeta, \zeta)_x := \|\zeta\|^2_2 + \|\dot{\zeta}\|^2_2.
\]
We have the splitting
\[
TM = TH^1(\mathbb{T}, M) \oplus \mathbb{R} \frac{\partial}{\partial T},
\]
where \(\frac{\partial}{\partial T}\) is the vector field on \(\mathcal{M}\) generating the semi-flow \(T_0 : (x,T) = (x,T_0 + T)\). Such vector field has norm 1 at every point and it is the dual with respect to \(g_{\mathcal{M}}\) of \(dT\).

The metric \(g_{\mathcal{M}}\) induces a distance \(d_{\mathcal{M}} : \mathcal{M} \times \mathcal{M} \to \mathbb{R}\) which is complete on all the subsets \(H^1(\mathbb{T}, M) \times \{T_-, +\infty\}\), where \(T_-\) is some positive number.

The manifold structure of \(\mathcal{M}\) also respects the product splitting. For every \(x \in H^1(\mathbb{T}, M)\) smooth, we define local charts centered at \(x\) as follows. Let \(B^n_\rho \subset \mathbb{R}^n\) be the open ball of radius \(\rho\) centered at the origin and consider a \textit{bi-bounded time-dependent chart} for \(M\)
\[
\psi : \mathbb{T} \times B^n_\rho \to M\ (\text{see } \text{AS09}). \quad \text{It is a smooth map such that for all } t \in \mathbb{T}:
\]
\[
\bullet \ \psi(t,0) = x(t);
\]
\[
\bullet \ \psi(t,\cdot) : B^n_\rho \to M\text{ is an embedding;}
\]
\[
\bullet \ \psi(t,\cdot)\text{ and its inverse have bounded } C^\infty\text{-norm.}
\]
For ease of notation, we write

$$\psi_t := \psi(t, \cdot), \quad \dot{\psi}_t := \frac{\partial}{\partial t} \psi(t, \cdot).$$

Notice that, for each $t \in \mathbb{T}$, $\dot{\psi}_t$ is a smooth section of $\psi^*_t(TM)$. Consider the maps

$$\iota : H^1(\mathbb{T}, B^n_\rho) \to H^1(\mathbb{T}, \mathbb{T} \times B^n_\rho), \quad \hat{\Psi}_{H^1} : H^1(\mathbb{T}, \mathbb{T} \times B^n_\rho) \to H^1(\mathbb{T}, M)$$

defined by

$$\iota(\xi)(t) := (t, \xi(t)), \quad \hat{\Psi}_{H^1}(\dot{\xi})(t) := \psi(\dot{\xi}(t)).$$

The required local chart around $x$ in $H^1(\mathbb{T}, M)$ is

$$\Psi_{H^1} := \hat{\Psi}_{H^1} \circ \iota : H^1(\mathbb{T}, B^n_\rho) \to H^1(\mathbb{T}, M), \quad \Psi_{H^1}(\xi)(t) := \psi(t)(\xi(t)).$$

To get a local chart at $(x, T)$, we just take

$$\Psi_{\mathcal{M}} := \Psi_{H^1} \times \text{Id}_{\mathbb{R}^+} : H^1(\mathbb{T}, B^n_\rho) \times \mathbb{R}^+ \to \mathcal{M}.$$

It is interesting to see how the tangent vector of a loop $\gamma = (x, T)$ is expressed using a local chart. Let $\beta : \mathbb{R}/TZ \to B^n_\rho$ be such that $\gamma(s) = \psi_{s/T}(\beta(s))$. We readily compute

$$\gamma'(s) = \frac{d}{ds} \left[ \psi_{s/T}(\beta(s)) \right](s) = \frac{\dot{\psi}_{s/T}(\beta(s))}{T} + d_{\beta(s)}\psi_{s/T}[\beta'(s)]$$

$$= \frac{\dot{\psi}_t(\xi(t))}{T} + d_{\xi(t)}\psi_t \left[ \frac{\dot{\xi}(t)}{T} \right],$$

where $t = s/T \in \mathbb{T}$ and $\xi(t) := \beta(Tt)$. If we define $D\psi : \mathbb{T} \times TB^n_\rho \times \mathbb{R}^+ \to TM$ by

$$D\psi(t, \xi, \dot{\xi}, T) := \left( \psi_t(\xi), \frac{\psi_t(\dot{\xi})}{T}, d_{\xi(t)}\psi_t \cdot \dot{\xi} \right),$$

then we tautologically have

(2.4) \hspace{1cm} \left( x, \frac{\dot{x}}{T} \right) = D\psi \left( t, \xi, \frac{\dot{\xi}}{T}, T \right).$$

The key ingredient of our discussion is that the periodic orbits of the flow defined by $(L, \sigma)$ on $E^{-1}(k)$ are in one to one correspondence with the zeros of a suitable 1-form $\eta_k \in \Omega^1(\mathcal{M})$. As a first step, we consider the free-period Lagrangian action functional $S^L_k : \mathcal{M} \to \mathbb{R}$ associated with $L$

(2.5) \hspace{1cm} S^L_k(x, T) := T \cdot \int_0^1 \left[ L \left( x(t), \frac{\dot{x}(t)}{T} \right) + k \right] dt.$$

As it is clear from its definition, $S^L_k$ is well-defined only if we further assume that $L$ grows at most quadratically along the fibers. This happens, for example, when $L$ is electromagnetic at infinity, namely when

$$L(q, v) = \frac{1}{2} |v|^2 + \alpha_q(v) - V(q)$$

outside a compact set, for some $\alpha \in \Omega^1(M)$ and $V : M \to \mathbb{R}$. This is not restrictive for our purposes, since given any Tonelli Lagrangian $L$, we can always modify it outside $E^{-1}(k)$ in order to make it electromagnetic at infinity.

We proceed to define $\eta_k$. When $\sigma$ is exact with primitive $\vartheta$, then $\eta_k := d\tilde{S}^L_k$ is simply the differential of the free-period Lagrangian action functional associated with the Tonelli Lagrangian $L_\vartheta(q, v) := L(q, v) + \vartheta_q(v)$. We refer to [Con06] Lemma 2.1 for the correspondence between zeros of $\eta_k$ and closed orbits in this case. A computation shows that

$$d\tilde{S}^L_k = dS^L_k + \pi_{H^1}^* \tau^\sigma,$$
Thus, Identity (2.9) is true if we set
\[ V \text{ for suitable } B \]
and hence we get
\[ \tau_x^{\sigma}[\xi] := \int_0^1 \sigma_{x(t)}(\dot{x}(t), \xi(t))dt \]
In particular \( \tau^{\sigma} \) does not depend on the particular choice of \( \vartheta \) and it is well-defined even if \( \sigma \) is not exact. Thus, in the general case we set
\[ (t, \xi) \]
\[ \eta_k := dS_k^T + \pi_{H^1}^{\tau^{\sigma}}. \]
We easily compute for future applications
\[ \eta_k := 1 + (2.12) \]
\[ S_k^{L^{\psi, \sigma}}(\xi, T) := T \cdot \int_0^1 \left[ L_{\psi, \sigma}(t, \xi(t), \frac{\dot{\xi}(t)}{T}, T) + k \right] dt. \]
Furthermore, for any \( T_+ > 0 \) and for any \( T \geq T_+ \) the family of functions \( L_{\psi, \sigma}(\cdot, \cdot, T) \) is Tonelli and quadratic at infinity uniformly in \( T \). Namely, there exist positive constants \( L_0, L_1, L_2 \) depending on \( T_+ \) such that for all \( (t, \xi(t)) \in T \times B_\rho \)
\[ \begin{align*}
& d\xi L(t, \xi, \zeta, T) \geq L_2; \\
& \lim_{|\zeta| \to \infty} \frac{L(t, \xi, \zeta, T)}{|\zeta|} = +\infty, \quad \text{uniformly in } T \in [T_+, +\infty); \\
& |d\xi L(t, \xi, \zeta, T)| \leq L_0(1 + |\zeta|^2), \quad |d\xi L(t, \xi, \zeta)| \leq L_1(1 + |\zeta|).
\end{align*} \]
Proof. Using (2.11), we write \( S_k^T \) in a local chart. If \( (\xi, T) \in H^1(T, B^n_\rho) \times \mathbb{R} \), then
\[ (\Psi_M S_k^T)(\xi, T) = T \int_0^1 \left[ (\psi L)(t, \xi(t), \frac{\dot{\xi}(t)}{T}, T) + k \right] dt, \]
where \( \psi L = L \circ D\psi \). Since \( H^2(T \times B_\rho) \cap \mathbb{R} = 0 \), there is \( \hat{\vartheta} \in \Omega^1(T \times B_\rho) \) with \( \psi^{\bullet} \sigma = d\hat{\vartheta} \) and hence we get
\[ \hat{\psi}_{H^1}^{\tau^{\sigma}} = \tau^{\psi^{\bullet} \sigma} = dS^{\hat{\vartheta}}, \]
where
\[ S^{\hat{\vartheta}} : H^1(T, T \times B^n_\rho) \to \mathbb{R}, \quad S^{\hat{\vartheta}}(\xi) := \int_0^1 \xi^* \hat{\vartheta}. \]
For every \( (t, \xi) \in T \times B_\rho^n \), we can write
\[ \hat{\vartheta}_{t(\xi)} = V_{\zeta}(\xi)dt + \vartheta_{t}^t \]
for suitable \( V_{\zeta} : B_\rho^n \to \mathbb{R} \) and \( \vartheta^t \in \Omega^1(B_\rho^n) \). Finally, \( \Psi_M^{\tau^{\sigma}} = d(S^{\hat{\vartheta}} \circ t) \) and
\[ S^{\hat{\vartheta}}(t(\xi)) = \int_0^1 \left[ V_{\zeta}(\xi(t)) + \vartheta_{t(\xi)}(\dot{\xi}(t)) \right] dt. \]
Thus, Identity (2.9) is true if we set
\[ L_{\psi, \sigma}(t, \xi, \zeta, T) := \psi L(t, \xi, \zeta, T) + \frac{V_{\zeta}(\xi)}{T} + \vartheta_{t}^t(\xi). \]
Since $L$ is Tonelli and $\psi$ bi-bounded, relations (2.10), (2.11) and (2.12) follow. □

**Corollary 2.3.** The 1-form $\eta_k$ is locally uniformly Lipschitz.

*Proof.* Given Lemma 2.2 we just adapt the computations in [AS09, Lemma 3.1(i)]. □

Since $\eta_k$ is locally the differential of a $C^1$-functional, the integral of $\eta_k$ over a closed path $u : T \to M$ depends only on the free-homotopy class of $u$. In this sense, we say that $\eta_k$ is *closed*, even if $\eta_k$ is in general not Fréchet differentiable.

We see in the next section that $\eta_k$ is exact on $M_0$ if and only if $\sigma$ is weakly-exact and that $\eta_k$ is exact on the whole $M$ if the lift of $\sigma$ to the universal cover admits a bounded primitive, i.e. if $\sigma$ is a so called *bounded weakly-exact 2 form*.

As in the exact case, the zeros of $\eta_k$ can be shown to be precisely the periodic orbits we are seeking. We rephrase this statement in the following lemma.

**Lemma 2.4.** Let $(M, g)$ be a Riemannian manifold, let $\sigma$ be a closed 2-form on $M$ and let $L : T^*M \to \mathbb{R}$ be a Tonelli Lagrangian. A loop $\gamma = (x, T)$ is a zero of $\eta_k$ if and only if $(\gamma, \gamma') \subset TM$ is a periodic orbit of the flow defined by the pair $(L, \sigma)$ contained in $E^{-1}(k)$.

In view of this result, the aim of the following sections will be to show that the set of zeros of $\eta_k$ is non-empty.

### 3. The weakly-exact case

The contents of this section hold for any non-simply connected Riemannian manifold $(M, g)$, any weakly exact (i.e. with exact lift to the universal cover) 2-form $\sigma \in \Omega^2(M)$ and any Tonelli Lagrangian $L : TM \to \mathbb{R}$. We denote by $H : T^*M \to \mathbb{R}$ the Tonelli Hamiltonian given by the Fenchel dual of $L$. We assume that $\sigma$ is not itself exact, as otherwise everything would reduce to the contents of [Con06] and [Abb13].

With slight abuse of terminology we say that $\sigma$ is *bounded*, respectively *unbounded*, if the lift $\tilde{\sigma}$ of $\sigma$ to the universal cover admits a bounded primitive, respectively if all the primitives of $\tilde{\sigma}$ are unbounded. It is a well known result that, if $\sigma$ is not exact and $\pi_1(M)$ is amenable, then $\sigma$ is unbounded (cf. [Pat06, Corollary 5.4]).

In any case, on $M_0$ we can define a functional $S_k$, whose critical points are exactly the contractible periodic orbits of the flow defined by $(L, \sigma)$. In other words, the 1-form $\eta_k$ defined in (2.7) is exact on $M_0$, provided the 2-form $\sigma$ is weakly exact. Indeed, if $x : T \to M$ is a contractible loop and $X : D^2 \to M$ is any capping disc, then we define

$$
S_k(x, T) := T \int_0^1 \left[ L\left( x(t), \frac{\dot{x}(t)}{T} \right) + k \right] dt + \int_{D^2} X^* \sigma .
$$

The functional is well-defined since

$$
\int_{D^2} X^* \sigma
$$

does not depend on the choice of the capping disc $X$, because $\sigma|_{\pi_2(M)} = 0$. One can easily compute the differential of $S_k$ at a pair $(x, T)$ and show that

$$
(\eta_k)(x, T) = dS_k(x, T) .
$$

In particular, thanks to Equation (2.8) one has

$$
\frac{\partial}{\partial T} S_k(x, T) = (\eta_k)(x, T) \left[ \frac{\partial}{\partial T} \right] = \frac{1}{T} \int_0^T \left[ k - E(\gamma(s), \gamma'(s)) \right] ds ,
$$

where as usual $\gamma(s) := x(s/T)$ and $E$ is the energy associated to the Lagrangian $L$. Lemma 2.4 implies now the following
Lemma 3.1. Let \((M,g), \sigma, L\) be as above. A pair \((x,T) \in M_0\) is a critical point of \(S_k\) if and only if \(\gamma(s) := x(s/T)\) is a (contractible) periodic orbit of the flow defined by the pair \((L, \sigma)\) with energy \(E(\gamma, \gamma') = k\).

In the bounded case the 1-form \(\eta_k\) is actually exact on the whole \(\mathcal{M}\). We recall briefly the definition of the functional \(S_k\), referring to [Mer10] for the details. The crucial observation is that, if \(\sigma\) is bounded, then \(\sigma\) has zero integral on every 2-torus (see also [Pat06]), i.e.

\[
\int_{T^2} f^* \sigma = 0, \quad \text{for every} \ f : T^2 \to M \text{ smooth}.
\]

Fix now a connected component \(\mathcal{M}_\nu\) of \(\mathcal{M}\) different from \(\mathcal{M}_0\) and a reference loop \(x_\nu\) in that class. For any other element \((x,T) \in M_\nu\) we define

\[
S_k(x,T) := T \int_0^1 \left[ L\left(x(t), \frac{\dot{x}(t)}{T}\right) + k \right] dt + \int_{C(x)} \sigma,
\]

where \(C(x)\) is any cylinder connecting \(x\) to \(x_\nu\). The functional is well-defined since the integral of \(\sigma\) on each 2-torus vanishes. \(S_k\) does however depend on the reference loop we have chosen, but changing the reference loop only changes the functional by adding a constant. In particular all its geometric properties remain untouched.

It is an easy exercise to show that the differential of \(S_k\) coincides with \(\eta_k\); in particular Equation (3.2) still holds. Thanks to Corollary 2.3 \(S_k\) has locally Lipschitz continuous differential (see [AS09, Mer10] for the original proof in this setting).

Lemma 3.2. Let \((M,g), L\) be as above and suppose \(\sigma\) is bounded. A pair \((x,T) \in M_\nu\) is a critical point of \(S_k\) on \(M_\nu\) if and only if \(\gamma(s) := x(s/T)\) is a periodic orbit of the flow defined by \((L, \sigma)\) with energy \(E(\gamma, \gamma') = k\) and free-homotopy class \(\nu\).

We proceed now to the definition of the Mañé critical value which is relevant for the geometric properties of the functional \(S_k\).

Denote by \(\vartheta\) a primitive of \(\tilde{\sigma}\) on \(\tilde{M}\) and by \(\tilde{H}\) the lift of \(H\) to the universal cover; being the lift of a Tonelli Hamiltonian, \(\tilde{H}\) satisfies the chain of inequalities

\[
H_0 |p|_q^2 - H_1 \leq \tilde{H}(q,p) \leq H_0' |p|_q^2 + H_1', \quad \forall (q,p) \in \tilde{T}^* \tilde{M}
\]

for suitable constants \(H_0, H_0' > 0, H_1, H_1' \in \mathbb{R}\). We define the critical value

\[
c(L, \sigma) := \inf_{u \in C^\infty(\tilde{M})} \sup_{q \in \tilde{M}} \tilde{H}(q, d_q u - \vartheta_q).
\]

Observe that \(c(E_{\text{kin}}, \sigma) = c(g, \sigma)\) is precisely the Mañé critical value of the pair \((g, \sigma)\) (see [Mer10]); therefore \(c(E_{\text{kin}}, \sigma)\) is finite if and only if \(\sigma\) is bounded. From (3.4) we get

\[
H_0 \cdot c(g, \sigma) - H_1 \leq c(L, \sigma) \leq H_0' \cdot c(g, \sigma) + H_1';
\]

in particular \(c(L, \sigma)\) is finite if and only if \(\sigma\) is bounded, but in general the two critical values might be different. The proof of the following lemma is analogous to the one in [Abb13, Lemma 4.1] (see also [Mer10, Lemma 4.2]).

Lemma 3.3. The following statements hold:

1. Suppose \(c(L, \sigma) < \infty\). If \(k \geq c(L, \sigma)\), the functional \(S_k\) is bounded from below on every connected component of \(\mathcal{M}\). If instead \(k < c(L, \sigma)\), then \(S_k\) is unbounded from below on every connected component of \(\mathcal{M}\).
2. Suppose \(c(L, \sigma) = \infty\). Then \(S_k\) is unbounded from below on \(\mathcal{M}_0\) for every \(k \in \mathbb{R}\).
3.1. The Palais-Smale condition for the action functional. Recall that \((x_h, T_h) \subseteq \mathcal{M}\) is a Palais-Smale sequence for \(S_k\) at level \(c \in \mathbb{R}\) if
\[
S_k(x_h, T_h) \rightarrow c, \quad dS_k(x_h, T_h) \rightarrow 0.
\]

One says that the functional \(S_k\) satisfies the Palais-Smale condition at level \(c\) if any Palais-Smale sequence at level \(c\) admits a convergent subsequence; finally, \(S_k\) is said to satisfy the Palais-Smale condition if it satisfies the Palais-Smale condition at each level.

The relevance on Palais-Smale sequences relies on the fact that, if one is able to prove their existence and that a suitable subsequence converges, then one gets automatically the existence of a critical point of \(S_k\).

An easy adaptation of the proof of \cite{Mer10} Theorem 3.2] shows that in the bounded case (for instance whenever \(M\) is a high genus surface) the functional \(S_k\) defined in \((3.3)\) satisfies the Palais-Smale condition on subsets of \(\mathcal{M}\) with period bounded and bounded away from zero. Here we claim that the same holds even in the unbounded case. The idea behind this is the following: a Palais-Smale sequence \((x_h, T_h)\) with periods bounded from above must have uniformly bounded length; therefore, up to changing the lifts \(\tilde{x}_h\) we may assume that all the lifts are contained in a suitable compact subset \(K \subseteq \tilde{M}\). Now it is clear that the assertion follows from the bounded case, since on a compact subset any primitive \(\vartheta\) of \(\tilde{\sigma}\) is automatically bounded. This is the scheme of proof of the following theorem.

**Theorem 3.4.** Let \((M, g), L\) as above, \(\sigma\) be unbounded. If \((x_h, T_h) \subseteq \mathcal{M}_0\) is a Palais-Smale sequence for \(S_k\) defined as in \((3.1)\) such that the \(T_h\)'s are bounded from above, then:

1. If the \(T_h\)'s are bounded away from zero, then \((x_h, T_h)\) is compact in the \(H^1\)-topology.
2. If \(T_h \rightarrow 0\), then \(S_k(x_h, T_h) \rightarrow 0\).

The above theorem states that, even in the unbounded case, the only Palais-Smale sequences which may cause troubles are the ones for which the periods are not bounded from above, provided they are not at level zero. The proof follows from the one in the bounded case (cfr. \cite{Mer10} Theorem 3.2]) together with the following

**Lemma 3.5.** In the hypotheses of the theorem above, there exist a compact subset \(K \subseteq \tilde{M}\) and, for every \(h \in \mathbb{N}\), suitable lifts \(\tilde{x}_h\) of \(x_h\) to \(\tilde{M}\) such that \(\tilde{x}_h(\mathbb{T}) \subseteq K\) for every \(h \in \mathbb{N}\).

**Proof.** Clearly it suffices to show that the \(x_h\)'s have uniformly bounded length, since then one can simply choose for every \(h \in \mathbb{N}\) a lift \(\tilde{x}_h\) of \(x_h\) in such a way that \(\tilde{x}_h(\mathbb{T}) \cap F \neq \emptyset\), where \(F \subseteq \tilde{M}\) is any fundamental domain for \(M\). Since \((x_h, T_h)\) is a Palais-Smale sequence we have in particular that
\[
\alpha_h := -\frac{\partial}{\partial T}S_k(x_h, T_h) = \frac{1}{T_h} \int_0^{T_h} \left[ E(\gamma_h(s), \gamma_h'(s)) - k \right] ds \rightarrow 0.
\]

Since \(L\) is a Tonelli Lagrangian, by \((2.1)\) we get that
\[
\alpha_h \geq \frac{E_0}{T_h} e(\gamma_h) - (E_1 + k)
\]
and hence
\[
e(\gamma_h) \leq \frac{T_h}{E_0} (\alpha_h + k + E_1)
\]
The Cauchy-Schwarz inequality implies now
\[
l(x_h)^2 \leq e(x_h) \leq T_h e(\gamma_h) = \frac{T_h^2}{E_0} (\alpha_h + k + E_1);
\]
since the \(T_h\)'s are bounded and the \(\alpha_h\)'s infinitesimal, we get the thesis.
3.2. Periodic orbits in the weakly-exact case. In this subsection we investigate the existence of periodic orbits for the flow defined by \((L, \sigma)\) filling the existing gap in the proof of Theorem \(\text{A}\) in the case \(L = L_{\text{kin}}\).

We then focus on the case of Riemannian surfaces with positive genus and show how one can use the stability of the energy level sets to prove the existence of one closed (contractible) magnetic geodesic for all sufficiently low energy, provided \(\sigma\) is symplectic.

We will not give any proof, since they can be obtained with small adaptations from the analogous given in \([\text{Abb13}]\) and \([\text{Mer10}]\). In fact, the only piece missing was Theorem 3.4, which allows us to use the same techniques also in the unbounded case.

Suppose firstly that \(\sigma\) is bounded. For supercritical energy values we have the following result. The proof for the case \(L = L_{\text{kin}}\) can be found in \([\text{Abb13}]\) or \([\text{Mer10}]\). When \(L\) is a general Tonelli Lagrangian, the proof is entirely analogous.

**Theorem 3.6.** For every \(k > c(L, \sigma)\) and every connected component \(M_1\) of \(M\) different from \(M_0\) there exists a global minimizer of \(S_k\) on \(M_1\). As a corollary, the flow defined by \((L, \sigma)\) has a periodic orbit with energy \(k > c(L, \sigma)\) in each non-trivial free homotopy class.

We recall the idea of the proof. By Lemma 3.3 the functional \(S_k\) is bounded from below on \(M_1\). Being \(k\) strictly larger than the critical value, the period is bounded on the sublevels of \(S_k\). Since \(M_1 \neq M_0\), the period is also bounded away from zero and, therefore, the functional \(S_k\) satisfies the Palais-Smale condition on \(M_1\) at every level. One gets now the desired global minimizer by considering a minimizing sequence.

For subcritical levels we do not distinguish between bounded and unbounded forms, since by Theorem 3.4 both cases can be treated in the same way.

**Theorem 3.7.** For almost every \(k \in (e_0(L), c(L, \sigma))\) there exists a closed contractible periodic orbit of the flow defined by \((L, \sigma)\) with energy \(k\) with positive \(S_k\)-action.

The “almost every” relies on the lack of the Palais-Smale condition for the functional \(S_k\); the Palais-Smale condition is then replaced by a monotonicity argument which allows to find Palais-Smale sequences with bounded period for almost every energy. The idea is completely analogous to the one explained in \([\text{Con06}]\) (in which the same result in the exact case is proven, see also \([\text{Abb13}]\) for a beautiful survey): for subcritical energies the functional \(S_k\) exhibits a mountain pass geometry, where the two valleys are represented by the set of constant loops and by the loops with negative action.

**Corollary 3.8.** If \(\pi_1(M)\) is amenable and \(\sigma\) is not exact, then for almost every \(k \in (e_0(L), +\infty)\) there exists a closed contractible periodic orbit of the flow defined by \((L, \sigma)\).

We show now how to use the stability of an energy level, say \(E^{-1}(k_*)\), to prove the existence of a periodic orbit with energy \(k_*\) for the flow defined by \((L, \sigma)\); for the proof we refer to \([\text{Abb13}]\) and references therein. Let \(H : T^*M \to \mathbb{R}\) be a Tonelli Hamiltonian such that the energy level \(H^{-1}(k_*)\) is stable. Then we can find another Tonelli Hamiltonian \(K \in C^\infty(T^*M)\) such that \(K^{-1}(1) = H^{-1}(k_*)\) and such that the dynamics on \(K^{-1}(1 \pm \delta)\) is conjugated to the dynamics on \(K^{-1}(1)\) for every \(\delta > 0\) sufficiently small. Since the Legendre transform of \(K\) is a Tonelli Lagrangian, Theorem 3.7 implies in particular that \(K^{-1}(1 \pm \delta)\) contains a periodic orbit for the flow defined by \((K, \omega_\sigma)\) for almost every \(\delta > 0\) sufficiently small and hence the same is true for \(K^{-1}(1)\), since by construction the dynamics on \(K^{-1}(1)\) and on \(K^{-1}(1 \pm \delta)\) are conjugated.

As a sample application of stability, one can reprove a weak version of a theorem in \([\text{Gin87}]\), without the multiplicity results.

**Corollary 3.9.** Suppose \((M, g)\) is a Riemannian surface with positive genus, \(\sigma \in \Omega^2(M)\) is a symplectic 2-form. Then the following hold:
(1) If $M = \mathbb{T}^2$ then for every $k > 0$ sufficiently small the energy level $H_{\text{kin}}^{-1}(k)$ is stable. As a corollary, there exists a closed magnetic geodesic with energy $k$.

(2) If $M \neq \mathbb{T}^2$ then for every $k > 0$ sufficiently small the energy level $H_{\text{kin}}^{-1}(k)$ is of contact type. As a corollary, there exists a closed magnetic geodesic with energy $k$.

In the next section we move to the study of contractible periodic orbits in the general case. Using the idea behind Lemma 3.5 in this context, we will exhibit critical sequences when $\pi_2(M) \neq 0$ and prove Theorem $\mathrm{B}$. 

4. THE GENERAL CASE

From now on let $\sigma$ be a closed 2-form, not necessarily weakly exact, and let $L : TM \to \mathbb{R}$ be a Tonelli Lagrangian with $H$ and $E$ the associated Hamiltonian and energy function. We are interested into finding closed orbits of the flow defined by the pair $(L, \sigma)$ with energy $k$ using variational methods as done in the previous sections.

Unfortunately, a functional whose critical points are exactly the desired closed orbits is not available anymore. However, as seen in Section 2 there exists a closed 1-form $\eta_k$ whose zeros correspond to such closed orbits. The mechanism we will use to construct such zeros is to look at the limit points of critical sequences for $\eta_k$, which are the generalization of Palais-Smale sequences to this setting.

**Definition 4.1.** We call $(x_h, T_h) \subset \mathcal{M}$ a critical sequence for $\eta_k$, if $|(\eta_k)_{(x_h, T_h)}|_\mathcal{M} \to 0$.

Since $\eta_k$ is a continuous 1-form, we see that the set of limit points of critical sequences coincides with the set of zeros of $\eta_k$. Therefore, the first step is to know under which hypotheses a critical sequence has a limit point. Clearly, if $T_h \to 0$ or $T_h \to \infty$, the limit points set is empty. The following theorem shows that the converse is also true.

**Theorem 4.2.** Let $(x_h, T_h)$ be a critical sequence with uniformly bounded periods. Then:

1. if $T_h$ tends to zero, then $e(x_h) = O(T_h^2)$;
2. if $T_h$ does not tend to zero, then $(x_h, T_h)$ has a converging subsequence.

The theorem will follow from the next lemma and the next proposition.

**Lemma 4.3.** If $(x_h, T_h)$ is a critical sequence, then there exists $C > 0$ such that

\[
e(x_h) \leq CT_h^2.
\]

As a consequence,

1. if $T_h$ is uniformly bounded from above, then $e(x_h)$ is uniformly bounded from above.
2. if $\lim T_h = 0$, then $\lim e(x_h) = 0$.

**Proof.** We just repeat the proof of Lemma 3.5 to obtain inequality (3.6), which clearly implies (4.1). Point (i) and Point (ii) now follows from (4.1).

**Proposition 4.4.** A critical sequence $(x_h, T_h)$ for $\eta_k$ such that $0 < T_\ast \leq T_h \leq T^\ast < \infty$ has a converging subsequence.

**Proof.** Since the period is bounded from above, we know by Lemma 4.3 that $e(x_h)$ is also bounded. Hence, the curves $x_h$ are uniformly 1/2-Hölder continuous. By the Ascoli-Arzelà theorem, up to a subsequence they converge uniformly to a continuous curve $x$. Thus, there exists a smooth curve $x_0 \in \overline{H^1(\mathbb{T}, M)}$ such that $x$ and, up to a subsequence, all the $x_h$ belong to the image of $\psi_{x_0}$. Let us define $\xi$ and $\xi_h$ as $x = \Psi_{H^1}(\xi)$ and $x_h = \Psi_{H^1}(\xi_h)$. Taking $T = 1$ in (2.4) and using the fact that $\psi_t \gamma$ is equivalent to the Euclidean metric on $\mathbb{R}^n$, we see that $\xi_h$ is a bounded sequence in $H^1(\mathbb{T}, B^m_{\rho})$. Hence $\xi \in H^1(\mathbb{T}, B^m_{\rho})$ and, up to a subsequence, $\xi_h$ converges $H^1$-weakly to $\xi$. Since $(x_h, T_h)$ is a critical sequence, we have

\[
o(1) = d(\xi, (T_h, 0))S_{k, \rho}^L(\xi, 0).
\]
Now, we use the uniform bounds contained in Lemma 2.2 and we argue as in the part of the proof of Lemma 5.3 in Abb13 after Equation (5.2). The thesis follows.

At this point we need a mechanism to produce critical sequences for \( \eta_k \). We will do this in the next subsections via a minimax method.

For the argument we look for a vector field on \( \mathcal{M} \) which generalizes the pseudo-gradient of \( S_k \) when \( \sigma \) is weakly exact. In the next subsection, we briefly discuss what are the properties of this vector field in the general case.

4.1. A generalized pseudo-gradient. We know that when the 1-form \( \tau^\sigma \) is non-exact, we cannot define a global primitive \( S_k \) on \( \mathcal{M} \). However, if \( u : [0, 1] \to \mathcal{M} \) is of class \( C^1 \), we can define the variation of \( \Delta S_k(u) : [0, 1] \to \mathbb{R} \) along the path by

\[
\Delta S_k(u)(s) := \eta_k(u[0, s]) = \int_0^s u^* \eta_k.
\]

Then, since \( \eta_k \) is closed, we extend the definition of \( \Delta S_k \) to any continuous path by uniform approximation with paths of class \( C^1 \).

Observe that \( \Delta S_k(u)(0) = 0 \) and if \( u \) takes values in a region where \( \eta_k \) is exact with primitive \( S_k \), then there holds

\[
\Delta S_k(u)(s) = S_k(u(s)) - S_k(u(0)).
\]

The next lemma describes how \( \Delta S_k \) changes under deformation of paths with the first endpoint fixed. The proof follows from the fact that \( \eta_k \) is a closed form.

**Lemma 4.5.** Let \( R \) be a positive real number and suppose that we have a homotopy of paths \( u : [0, R] \times [0, 1] \to \mathcal{M} \). Denote by \( u_s := u(r, \cdot) \) and \( u^s := u(\cdot, s) \) the paths in \( \mathcal{M} \) obtained keeping one of the variables fixed. If \( u^0 \) is constant, then for every \( s \in [0, 1] \)

\[
\Delta S_k(u_R)(s) = \Delta S_k(u_0)(s) + \Delta S_k(u^s)(R).
\]

We now proceed to define a generalized pseudo-gradient. First we consider the vector field \( -\sharp \eta_k \), where \( \sharp \) is the duality between \( T\mathcal{M} \) and \( T^*\mathcal{M} \) given by the metric \( g_{\mathcal{M}} \). By Corollary 2.3 \( -\eta_k \) is locally uniformly Lipschitz and hence we have local existence and uniqueness for solutions of the associated Cauchy problem. However, since \( |\eta_k|_{\mathcal{M}} \) is not bounded on \( \mathcal{M} \), solutions of the Cauchy problem could escape to infinity in finite time, thus having a maximal interval of definition of finite length. To avoid this problem we consider the normalized vector field

\[
X_k := \frac{-\sharp \eta_k}{\sqrt{1 + |\eta_k|^2_{\mathcal{M}}}}.
\]

We define \( \Phi^k \) as the semi-flow of \( X_k \) on \( \mathcal{M} \) generated by the maximal solutions

\[
u_{(x,T)} : [0, R_{(x,T)}) \longrightarrow \mathcal{M}
\]

of the Cauchy problem with initial condition \( u(0) = (x, T) \). Here \( (x, T) \) is some element in \( \mathcal{M} \) and \( R_{(x,T)} \) is some number in \( \mathbb{R}^+ \cup \{+\infty\} \). By definition, we have

\[
\Phi^k_{x,T}(x, T) = u_{(x,T)}(r) = (x(r), T(r)).
\]

We say that \( \Phi^k \) is complete, if \( R_{(x,T)} = +\infty \) for all \( (x, T) \). We will see in the next proposition that the only source of incompleteness for \( \Phi^k \) is that \( r \mapsto T(r) \) has 0 as a limit point.

**Proposition 4.6.** Let \( u : [0, R) \to \mathcal{M} \) be a maximal flow line of \( \Phi^k \). If \( R < +\infty \), then

\[
\liminf_{r \to R} T(r) = 0.
\]
In this case there exists a constant $C > 0$ and a sequence $r_h \to R$ such that

$$T(r_h) \to 0 \quad \text{and} \quad e(x(r_h)) \leq CT(r_h)^2.$$ 

Proof. Suppose that $R < +\infty$ and assume by contradiction that $T(r) \geq T_*$ for every $r \in [0, R)$. Observe that

$$\left| \frac{d}{dr} u \right|_{\mathcal{M}} = \left| \frac{-\eta_k}{\sqrt{1 + |\eta_k|^2_{\mathcal{M}}}} \right| = \frac{|\eta_k|_{\mathcal{M}}}{\sqrt{1 + |\eta_k|^2_{\mathcal{M}}}} < 1.$$ 

Since the derivative of $u$ is bounded by the above inequality and $H^1(T, \mathcal{M}) \times [T_*, +\infty)$ is complete, there exists the limit

$$u_* := \lim_{r \to R} u(r).$$ 

As $X_k$ is locally uniformly Lipschitz, there exists a neighbourhood $\mathcal{U}$ of $u_*$, such that the solutions to the Cauchy problem with initial data in $\mathcal{U}$ all exist in a small fixed interval $[0, r_u]$. This yields a contradiction as soon as $u(r) \in \mathcal{U}$ and $R - r < r_u$.

Suppose now that (4.6) holds. In this case there exists a sequence $r_h \to R$ such that

$$T(r_h) \to 0 \quad \text{and} \quad \frac{dT}{dr}(r_h) \leq 0.$$ 

Using (2.1), we find

$$0 \geq \frac{dT}{dr}(r_h) = -\eta_k(u(r_h)) \left( \frac{\partial}{\partial T} \right)$$

$$= \int_0^1 E\left( x(r_h), \frac{\dot{x}(r_h)}{T} \right) dt - k$$

$$\geq E_0 \frac{e(x(r_h))}{T(r_h)^2} - E_1 - k,$$ 

which gives the required bound for the energy. \qed

The previous proposition shows that the only source of non-completeness of the semi-flow are trajectories that go closer and closer to the subset of constant loops. In particular, this yields that $\Phi^k$ is positively complete on $\mathcal{M} \setminus \mathcal{M}_0$. The case of $\mathcal{M}_0$ is more delicate and requires to take a deeper look to $\eta_k$ close to the set of constant loops. We will perform such study for $k > e_0(L)$ obtaining two outcomes. First, $\eta_k$ admits a positive primitive $S_k$ on the subset of paths with small energy and such function has an interesting geometry (which will be exploited for the minimax method). Second, we will improve Proposition 4.6 and show that on the flow lines with finite maximal interval of definition $S_k \to 0$.

We conclude this subsection proving a lemma relating the $1/2$-Hölder norm of the flow-lines of $\Phi^k$, that will be used in the proof of Proposition 4.17.

**Lemma 4.7.** If $u : [0, R] \to \mathcal{M}$ is a flow line of $\Phi^k$, then

$$-\Delta S_k(u)(R) \geq \frac{d_{\mathcal{M}}(u(R), u(0))^2}{R}.$$ 

In particular,

$$-\Delta S_k(u)(R) \geq \frac{(T(R) - T(0))^2}{R}.$$ 

(4.7)
Proof. We just compute
\[
-\Delta S_k(u)(R) = - \int_0^R \eta_k \left( \frac{du}{dr} \right)^2 dr \geq \int_0^R \frac{du}{dr} \left| \frac{du}{dr} \right|_\mathcal{M}^2 \geq \frac{1}{R} \left( \int_0^R \left| \frac{du}{dr} \right|_\mathcal{M}^2 \right) \geq \frac{d_M(u(R), u(0))^2}{R},
\]
where in the last but one inequality we used the Cauchy-Schwarz inequality. To obtain \((4.7)\) we just observe that \(|T(R) - T(0)| \leq d_M(u(R), u(0))\) as \(d_M\) is a product distance. \(\square\)

4.2. The action 1-form on loops with small kinetic energy. Inside \(\mathcal{M}_0\) we single out the submanifold of constant loops
\[
M_0 := M \times \mathbb{R}^+.
\]
For every \(\delta > 0\) consider the neighbourhood \(\mathcal{V}_\delta\) of \(M_0\) given by
\[
(4.8) \quad \mathcal{V}_\delta := \{ (x, T) \mid e(x) < \delta \}.
\]
If \(\delta\) is sufficiently small, then there is a deformation retraction of \(\mathcal{V}_\delta\) onto \(M_0\) which fixes the period. Such deformation can be obtained for example by considering the negative gradient flow of the function \(e\) on \(H^1(T, M)\). The deformation yields a capping disc \(D_x\) for \(x\) and, hence, an explicit primitive \(S_k\) for \(e\) on \(\mathcal{V}_\delta\) defined by
\[
S_k(x, T) := S_k^L(x, T) + \int_{D_x} \sigma.
\]

Lemma 4.8. If \(\delta\) is sufficiently small, there exists \(\Theta_0 > 0\) such that, for every \((x, T) \in \mathcal{V}_\delta,\)
\[
(4.9) \quad \left| \int_{D_x} \sigma \right| \leq \Theta_0 l(x)^2
\]
and we have the estimate
\[
(4.10) \quad S_k(x, T) \leq B \frac{e(x)}{T} + (B + k)T + \Theta_0 l(x)^2,
\]
where \(B\) is a suitable positive constant.

Proof. Equation \((4.9)\) is exactly Lemma 7.1 in \[Abb13\]. Since \(L\) is a Tonelli Lagrangian electromagnetic at infinity we can find \(B > 0\) such that
\[
L(q, v) \leq B(1 + |v|^2), \quad \forall (q, v) \in TM
\]
and we readily compute
\[
S_k(x, T) = T \int_0^1 \left[ L\left( x(t), \frac{\dot{x}(t)}{T} \right) + k \right] dt + \int_{D_x} \sigma
\leq T \int_0^1 \left[ B \frac{\left| \dot{x}(t) \right|^2}{T^2} + B + k \right] dt + \Theta_0 l(x)^2
\leq B \frac{e(x)}{T} + (B + k)T + \Theta_0 l(x)^2. \quad \square
\]

If we want some more information on the behaviour of the function \(S_k\), we have to restrict the range of energies we consider. Indeed, notice that on \(M_0\) the function \(S_k\) reduces to
\[
(4.11) \quad S_k(x, T) = T[L(x, 0) + k] = T[k - E(x, 0)].
\]
Hence, if \(k > e_0(L)\), then for every \(x \in M\) the function \(T \mapsto S_k(x, T)\) is increasing in \(T\) and tends to zero as \(T\) goes to 0. Moreover, in the same energy range we have a positive lower bound for \(S_k\) on \(\partial \mathcal{V}_\delta\) as the following lemma shows.
Lemma 4.9. If $k > e_0(L)$, there exists $\delta_k > 0$ such that for every $\delta \in (0, \delta_k)$ there exists $\varepsilon_{k,\delta} > 0$ such that
\[
\inf_{\partial V_\delta} S_k \geq \varepsilon_{k,\delta}.
\]
In particular, $S_k > 0$ on $V_\delta$.

Proof. The last inequality in the proof of Lemma 7.2 in [Abb13] implies that, for $\delta$ sufficiently small, every $(x, T) \in V_\delta$ satisfies
\[
S_k(x, T) \geq l(x) \sqrt{L_2(k - e_0(L))} - \Theta_1 l(x)^2,
\]
for suitable $L_2, \Theta_1 > 0$. Putting together this estimate with Inequality (4.9), we get
\[
S_k(x, T) \geq l(x) \sqrt{L_2(k - e_0(L))} - (\Theta_0 + \Theta_1) l(x)^2,
\]
which is positive if $l(x)$ is small and positive. The thesis follows.

We define the sublevel sets
\[
W'_\delta := \{ S_k < \varepsilon_{k,\delta}/4 \}, \quad W_\delta := \{ S_k < \varepsilon_{k,\delta}/2 \}.
\]
Thanks to the previous lemma, $W_\delta \cap \partial V_\delta = \emptyset$ and by (4.11) there is $(x_0, T_0)$ in $M_0 \cap W'_\delta$. These sets play an important role as far as critical sequences and flow lines are concerned.

Lemma 4.10. If $k > e_0(L)$, then for every $\delta > 0$ sufficiently small the following hold:

i) if $(x_h, T_h)$ is a critical sequence for $\eta_k$ in $M_0$ such that $T_h \to 0$, then $(x_h, T_h) \in W'_\delta$ for every $h$ sufficiently large;

ii) the semi-flow $\Phi^k$ is positively complete on $M_0 \setminus W'_\delta$.

Proof. We prove the first statement. By Inequality (4.11) in Lemma 4.3 there exists $C > 0$ such that $e(x_h) \leq CT_h^2$. Therefore, $(x_h, T_h) \in V_\delta$ for $h$ big enough and we can use Inequality (4.10) to obtain
\[
S_k(x_h, T_h) \leq B \frac{CT^2_h}{T_h} + (B + k)T_h + \Theta_0 CT^2_h,
\]
which goes to zero as $h$ goes to infinity.

We prove the second statement. Let $(x, T) \in M_0 \setminus W'_\delta$ and let $[0, R(x, T))$ be the maximal interval of definition of the flow line
\[
r \mapsto (x(r), T(r)) := \Phi^k(x, T).
\]
Suppose that $R(x, T) < +\infty$. By Proposition 4.6 there exists $r_h \to R$ such that $T(r_h) \to 0$ and $e(x(r_h)) \leq CT(r_h)$.

By the same argument as in the proof of the first statement, we have $(x_h, T_h) \in W'_\delta$, for large $h$. This completes the proof, as $W'_\delta$ is positively $\Phi^k$-invariant.

Thanks to the previous lemma, in order to make $\Phi^k$ positively complete, we stop the flow lines entering $W'_\delta$. Consider a smooth cut-off function $\kappa_\delta : \mathbb{R}^+ \to [0, 1]$ such that
\[
\kappa_\delta^{-1}(0) = (0, \varepsilon_{k,\delta}/4], \quad \kappa_\delta^{-1}(1) = [\varepsilon_{k,\delta}/2, +\infty).
\]
We use this function to define $\hat{\kappa}_\delta : M_0 \to [0, 1]$ as follows:
\[
\hat{\kappa}_\delta = \begin{cases} 
1 & \text{on } M_0 \setminus V_\delta, \\
\kappa_\delta \circ S_k & \text{on } V_\delta.
\end{cases}
\]
Finally, define the vector field $X_{k,\delta} := \hat{\kappa}_\delta X_k$ and denote its semi-flow by $\Phi^{k,\delta}$.

Lemma 4.11. The time-dependent semi-flow $\Phi^{k,\delta}$ is complete on $M_0$. 

Proof. The flow $\Phi^{k,\delta}$ has the same flow lines as $\Phi^k$, possibly traveled at a lower speed. Hence, if a flow-line does not intersect $W^\prime_\delta$, it is defined for all positive times by Lemma 4.10. On the other hand, if it does intersect $W^\prime_\delta$, it is eventually constant since $X_{k,\delta} = 0$ on $W^\prime_\delta$ and, therefore, the trajectory is defined for all positive times.

Summarizing, the non-completeness of the semi-flow $\Phi^k$ can be overcome by truncating it near the manifold of constant loop, namely by multiplying the vector field $X_k$ by a cut-off function $\kappa_\delta$, whose role is to make the flow-lines constant if the energy is sufficiently small. This will not be restrictive for our purposes, thanks to Lemma 4.13 below.

4.3. The minimax class for spherical manifolds. In this subsection we are going to see that, by Equation (4.11) and Lemma 4.9 the 1-form $\eta_k$ exhibits a mountain pass geometry on some space of closed paths $\mathcal{U} \in C^0(\mathbb{T}, M_0)$, when $k > e_0(L)$. The set $\mathcal{U}$ must enjoy the following two properties:

- is invariant under $\Phi^{k,\delta}$;
- all its elements are based at $(x_0, T_0) \in M_0 \cap W'_\delta$ and intersect $\partial V_\delta$.

We construct $\mathcal{U}$ under the hypothesis that $\pi_2(M) \neq 0$. In Remark 4.18 below we observe that a class with the same properties of $\mathcal{U}$ can also be constructed when $\sigma$ is weakly exact and $k \in (e_0(L), c(L, \sigma))$. In particular, the discussion contained in this subsection applies to that case as well and yields an alternative proof to Theorem 3.7 and Theorem A.

Thus, let $(x_0, T_0) \in M_0$ be a constant loop such that $S_k(x_0, T_0) < \varepsilon_{k,\delta}/4$ and consider the group homomorphism

$$j : \pi_1(H^1(\mathbb{T}, M), x_0) \to \pi_2(M, x_0), \quad j[x] := \left[ S^2_x \right],$$

where the sphere $S^2_x$ is defined as follows. We identify a closed path $x : [0, 1] \to H^1(\mathbb{T}, M)$ based at $x_0$ with a cylinder in $M$ whose ends are mapped to the point $x_0$. Quotienting each of the two ends to a point we get a sphere $S^2_x$ (with north and south pole identified).

We claim that the map $j$ is surjective. Indeed, let $\Theta : S^2 \to M$ be any continuous map such that $\Theta(N) = x_0$ and denote by $x_1 := \Theta(S)$, where $N$ and $S$ denote respectively the north and the south pole of $S^2$. Then $\Theta$ can be written as the quotient of a map

$$\overline{\Theta} : [0, 1] \times \mathbb{T} \to M, \quad \overline{\Theta}([0] \times \mathbb{T}) = x_0, \quad \overline{\Theta}([1] \times \mathbb{T}) = x_1.$$

Since $M$ is path-connected, we can find a path $y$ connecting $x_1$ to $x_0$ and then consider the (constant) cylinder $\Theta_y$ defined in the obvious way. The glued cylinder $\Theta_y \# \overline{\Theta}$ passes to the quotient and yields a map which is homotopic to $\Theta$ and lies in the image of $j$.

Notice that different choices of the path $y$ connecting $x_1$ to $x_0$ may lead to different classes in $\pi_1(H^1(\mathbb{T}, M), x_0)$. This is no surprise since in general $\pi_1(H^1(\mathbb{T}, M), x_0)$ is not isomorphic to $\pi_2(M, x_0)$. However, the choice of $y$ is in some sense “canonical” if $M$ is simply connected. In this case we actually have an isomorphism

$$\pi_1(H^1(\mathbb{T}, M), x_0) \cong \pi_2(M, x_0).$$

Definition 4.12. Given a non-zero element $u \in \pi_2(M, x_0)$ we say that a continuous loop $(x, T) : \mathbb{T} \to M_0$ based at $(x_0, T_0)$ belongs to $\mathcal{U}$ if and only if $j[x] = u$.

As $(x_0, T_0) \in W'_\delta$, $\mathcal{U}$ is invariant under the action of $\Phi^{k,\delta}$ on $C^0(\mathbb{T}, M_0)$. It remains to show that all elements of $\mathcal{U}$ must intersect $\partial V_\delta$.

Lemma 4.13. If $u \in \mathcal{U}$, then there exists $s \in [0, 1]$ such that $u(s) \in \partial V_\delta$.

Proof. We argue by contradiction and suppose that $u(T) \subset V_\delta$. Let $u = (x, T)$ and let $[x]$ be the class of $x$ in $\pi_1(H^1(\mathbb{T}, M), x_0)$. Since $V_\delta$ retracts onto $M_0$ fixing the period, we see at once that $[x]$ is in the image of the group homomorphism

$$i_* : \pi_1(M, x_0) \to \pi_1(H^1(\mathbb{T}, M), x_0)$$
induced by the inclusion of constant loops \( i : M \to H^1(\mathbb{T}, M) \), i.e. \([x] = i_*[y]\) for some \([y] \in \pi_1(M, x_0)\). Observe now that \(ji_*[y] = 0\), as it is represented by a quotiented cylinder \([0, 1] \times \mathbb{T} \to M\) which is constant in the second variable and therefore contractible in \(\pi_2(M, x_0)\). Hence we would have
\[
 u = j[x] = ji_*[y] = 0,
\]
which is clearly a contradiction.

Now that we have defined a suitable class \( \mathcal{U} \) starting from a non-zero element \( u \in \pi_2(M) \), we use it to define a minimax value \( c^u \) using the variation \( \Delta S_k \) along paths. For each \( u \in \mathcal{U} \), we consider the function \( S_k(u) : [0, 1] \to \mathbb{R} \) defined by
\[
(4.13) \quad S_k(u)(s) := S_k(x_0, T_0) + \Delta S_k(u)(s)
\]
and then set
\[
(4.14) \quad c^u(k) := \inf_{u \in \mathcal{U}} \max_{s \in [0, 1]} S_k(u)(s).
\]
In the next lemma we prove a crucial monotonicity property for the function \( c^u : \mathbb{R} \to \mathbb{R} \).

**Lemma 4.14.** If \( k_1 < k_2 \), then for every \( u \in \mathcal{U} \), we have
\[
(4.15) \quad S_{k_1}(u)(s) < S_{k_2}(u)(s), \quad \forall s \in [0, 1]
\]
and
\[
(4.16) \quad c^u(k_1) \leq c^u(k_2).
\]

**Proof.** We have \( \eta_{k_2} - \eta_{k_1} = (k_2 - k_1) \cdot dT \). Integrating this equality along \( u|_{[0,s]} \), we get
\[
(4.17) \quad \Delta S_{k_2}(u)(s) - \Delta S_{k_1}(u)(s) = (k_2 - k_1)(T(1) - T_0).
\]
Then, Inequality (4.15) follows since
\[
(k_2 - k_1)(T(1) - T_0) > -(k_2 - k_1)T_0.
\]
Inequality (4.16) follows from Inequality (4.15) and the fact that we are taking the inf-\(\max\) over the same set.

We now use the family \( \mathcal{U} \) to construct a critical sequence with period bounded from below. If \( k \) is a Lipschitz point for the function \( c^u \), then we can also establish a bound from above for the period using Struwe’s monotonicity argument [Str90] (see also [Con06, Abb13, AMP13, AMMP14, AB14] for other applications of Struwe’s argument). Before proving such result we need the following preparatory lemma.

**Lemma 4.15.** Fix \( k > e_0(L) \) and \( u \in \mathcal{U} \). If \( s_* \in [0, 1] \) is such that
\[
(4.18) \quad S_k(u)(s_*) > \max_{s \in [0, 1]} S_k(u)(s) - \epsilon_{k,\delta}/2,
\]
then \( u(s_*) \notin \mathcal{W}_\delta \).

**Proof.** If \( s_* \in \mathcal{W}_\delta \), then there exists an \( s_{**} \) bigger (or smaller) than \( s_* \) such that \( u(s_{**}) \in \partial \mathcal{V}_\delta \) and \( u([s_*, s_{**}]) \subseteq \mathcal{V}_\delta \) (or \( u((s_**, s_*]) \subseteq \mathcal{V}_\delta \)) by Lemma 4.13. From Equation (4.13), we get
\[
\Delta S_k(u)(s_{**}) - \Delta S_k(u)(s_*) = S_k(u(s_{**})) - S_k(u(s_*)) > \epsilon_{k,\delta} - \epsilon_{k,\delta}/2 = \epsilon_{k,\delta}/2.
\]
This implies that
\[
\max_{s \in [0, 1]} S_k(u)(s) \geq S_k(u)(s_{**}) > S_k(u)(s_*) + \epsilon_{k,\delta}/2,
\]
in contradiction with the assumption (4.18). \( \square \)
Remark 4.16. The same idea used for the above lemma can be employed to give an estimate from below on $c^u(k)$ for $k > e_0(L)$. Call $s_-$ and $s_+$ the smallest and the biggest elements in $[0, 1]$ such that $u(s) \in \partial \mathcal{V}_\delta$. Then,

$$S_k(u)(s_-) = S_k(u(s_-)) \geq \varepsilon_{k, \delta}.$$ 

To give an estimate at $s_+$ we observe that

$$S_k(u)(1) = S_k(x_0, T_0) + \int_0^1 u^\ast \eta_k = S_k(x_0, T_0) + \sigma(u).$$

Hence,

$$S_k(u)(s_+) = S_k(u)(1) + (S_k(u(s_+)) - S_k(u(1))) = \sigma(u) + S_k(u(s_+)) \geq \sigma(u) + \varepsilon_{k, \delta}.$$ 

These two lower bounds together imply that

$$c^u(k) \geq \max \{0, \sigma(u)\} + \varepsilon_{k, \delta}. \tag{4.19}$$

Proposition 4.17. If $k > e_0(L)$, there exists a critical sequence $(x_h, T_h) \subseteq \mathcal{M}_0$ for $\eta_k$ with periods bounded away from zero. Moreover, if $k$ is a Lipschitz point for the function $c^u$, we can choose such sequence to have bounded periods.

Proof. We treat only the case in which $k$ is a Lipschitz point, since the case in which we do not require an upper bound on the periods is simpler and uses the same ideas.

By assumption there exists $A > 0$ such that for every $k' \geq k$ sufficiently near $k$

$$c^u(k') - c^u(k) \leq A (k' - k).$$

Consider a sequence $k_m \downarrow k$ and denote by $\lambda_m := k_m - k \downarrow 0$. Clearly we may suppose the inequality above to hold for every $m$. Take a corresponding $u_m \in \mathfrak{U}$ such that

$$\max_{s \in [0, 1]} S_k(u_m)(s) < c^u(k_m) + \lambda_m.$$ 

By the definition of $c^u(k)$, the subset of all $s \in [0, 1]$ such that

$$S_k(u_m)(s) > c^u(k) - \lambda_m$$

is non-empty. If $s$ belongs to such subset, we have

$$T_m(s) = \frac{S_k(u_m)(s) - S_k(u_m)(s)}{\lambda_m} \leq \frac{c^u(k_m) + \lambda_m - (c^u(k) - \lambda_m)}{\lambda_m} \leq A + 2.$$ 

Together with Inequality (4.15), this implies

$$S_k(u_m)(s) \in \left(c^u(k) - \lambda_m, c^u(k) + (A + 1)\lambda_m \right). \tag{4.20}$$

Summing up, for every $m \in \mathbb{N}$ and every $s \in [0, 1]$ either

- $S_k(u_m)(s) \leq c^u(k) - \lambda_m$,

or

- $S_k(u_m)(s) \in \left(c^u(k) - \lambda_m, c^u(k) + (A + 1)\lambda_m \right)$ and $T_m(s) < A + 2$.

Consider now for every $r \in [0, 1]$ the path $u^r_m \in \mathfrak{U}$ given by

$$u^r_m(s) := \Phi^r_{\sigma, \delta}(u_m(s)).$$

Equation (4.4) in Lemma 4.5 implies that the map

$$r \mapsto S_k(u^r_m)(s)$$

is decreasing. Combining this fact with Inequality (4.20), we get

$$\max_{s \in [0, 1]} S_k(u^r_m)(s) < c^u(k) + (A + 1)\lambda_m, \quad \forall r \in [0, 1] \tag{4.21}.$$
Theorem 4.19. Suppose function is Lipschitz almost everywhere, we get the following generalization of Theorem B. Then,

\[ \Phi^T \]

A minor modification of the arguments of this section can be used to treat the weakly exact case. If \( \sigma \) is weakly exact and \( k \in (e_0(L), c(L, \sigma)) \), then we have a global action \( S_k \) and we define \( \Lambda^- \) as the set of all paths \( u \) such that \( u(0) = (x_0, T_0) \) and \( S_k(u(1)) < 0 \). Then, \( \Lambda^- \) enjoys the same properties of the spherical class \( \Lambda \) defined above.

Combining the previous proposition with Proposition 4.4 and the fact that a monotone function is Lipschitz almost everywhere, we get the following generalization of Theorem B.

Theorem 4.19. Suppose \((M, g)\) is a Riemannian manifold with \( \pi_2(M) \neq 0 \). Consider a Tonelli Lagrangian \( L : T^*M \to \mathbb{R} \) and a closed form \( \sigma \in \Omega^2(M) \). For almost every \( k \in (e_0(L), +\infty) \), there exists a contractible periodic orbit for the system \((L, \sigma)\) on \( E^{-1}(k) \).

In the particular case \( M = S^2 \) Theorem 4.19 together with the stability property for energy levels implies the following result about the existence of closed magnetic geodesics, which extends Corollary 3.9 to the case of the 2-sphere. Both statements below are already known (see [Gin96, Theorem 3.13] and [Gin87, Assertion 3], respectively).
Corollary 4.20. Let \((S^2, g)\) be a Riemannian 2-sphere and \(\sigma \in \Omega^2(S^2)\). Then:

1. If \(k > 0\) is sufficiently large, then the energy level \(E^{-1}_{\text{kin}}(k)\) is of contact-type. Hence there exists a closed magnetic geodesic with energy \(k\).
2. Suppose \(\sigma\) is symplectic. If \(k > 0\) is sufficiently small then the energy level \(E^{-1}_{\text{kin}}(k)\) is of contact-type. Hence there exists a closed magnetic geodesic with energy \(k\).

References

[AB14] L. Asselle and G. Benedetti, Infinitely many periodic orbits of non-exact oscillating magnetic flows on surfaces with genus at least two for almost every low energy level, arXiv:1405.0415, 2014.

[Abb13] A. Abbondandolo, Lectures on the free period Lagrangian action functional, J. Fixed Point Theory Appl. 13 (2013), no. 2, 397–430.

[AMMP14] A. Abbondandolo, L. Macarini, M. Mazzucchelli, and G. P. Paternain, Infinitely many periodic orbits of exact magnetic flows on surfaces for almost every subcritical energy level, preprint, arXiv:1404.7611, 2014.

[AMP13] A. Abbondandolo, L. Macarini, and G. P. Paternain, On the existence of three closed magnetic geodesics for subcritical energies, to appear in Comm. Math. Helv., arXiv:1305.1871, 2013.

[AS09] A. Abbondandolo and M. Schwarz, A smooth pseudo-gradient for the Lagrangian action functional, Adv. Nonlinear Stud. 9 (2009), no. 4, 597–623.

[Ben14a] G. Benedetti, The contact property for magnetic flows on surfaces, Ph.D. thesis, University of Cambridge, 2014.

[Ben14b] _, The contact property for symplectic magnetic fields on \(S^2\), Ergodic Theory Dynam. Systems (2014), published online. doi:10.1017/etds.2014.82

[CMP04] G. Contreras, L. Macarini, and G. P. Paternain, Periodic orbits for exact magnetic flows on surfaces, Int. Math. Res. Not. (2004), no. 8, 361–387.

[Con06] G. Contreras, The Palais-Smale condition on contact type energy levels for convex Lagrangian systems, Calc. Var. Partial Differential Equations 27 (2006), no. 3, 321–395.

[FS07] U. Frauenfelder and F. Schlenk, Hamiltonian dynamics on convex symplectic manifolds, Israel J. Math. 159 (2007), 1–56.

[GGM14] V. L. Ginzburg, B. Gürel, and L. Macarini, On the Conley conjecture for Reeb flows, preprint, arXiv:1407.1773v1, 2014.

[Gin87] V. L. Ginzburg, New generalizations of Poincaré’s geometric theorem, Funktsional. Anal. i Prilozhen. 21 (1987), no. 2, 16–22, 96.

[Gin96] _, On closed trajectories of a charge in a magnetic field. An application of symplectic geometry, Contact and symplectic geometry (Cambridge, 1994), Publ. Newton Inst., vol. 8, Cambridge Univ. Press, Cambridge, 1996, pp. 131–148.

[GK99] V. L. Ginzburg and E. Kerman, Periodic orbits in magnetic fields in dimensions greater than two, Geometry and topology in dynamics (Winston-Salem, NC, 1998/San Antonio, TX, 1999), Contemp. Math., vol. 246, Amer. Math. Soc., Providence, RI, 1999, pp. 113–121.

[Hin09] N. Hingston, Subharmonic solutions of Hamiltonian equations on tori, Ann. of Math. (2) 170 (2009), no. 2, 529–560.

[HZ94] H. Hofer and E. Zehnder, Symplectic invariants and Hamiltonian dynamics, Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks], Birkhäuser Verlag, Basel, 1994.

[Ker99] E. Kerman, Periodic orbits of Hamiltonian flows near symplectic critical submanifolds, Internat. Math. Res. Notices (1999), no. 17, 953–969.

[Koz85] V. V. Kozlov, Calculus of variations in the large and classical mechanics, Uspekhi Mat. Nauk 40 (1985), no. 2(242), 33–60, 237.

[LC06] P. Le Calvez, Periodic orbits of Hamiltonian homeomorphisms of surfaces, Duke Math. J. 133 (2006), no. 1, 125–184.

[LS94] F. Laudenbach and J.-C. Sikorav, Hamiltonian disjunction and limits of Lagrangian submanifolds, Internat. Math. Res. Notices (1994), no. 4, 161 ff., approx. 8 pp. (electronic).

[Mer10] W. J. Merry, Closed orbits of a charge in a weakly exact magnetic field, Pacific J. Math. 247 (2010), no. 1, 189–212.
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[Mer11] ________, On the Rabinowitz Floer homology of twisted cotangent bundles, Calc. Var. Partial Differential Equations 42 (2011), no. 3-4, 355–404.

[MP10] L. Macarini and G. P. Paternain, On the stability of Mañé critical hypersurfaces, Calc. Var. Partial Differential Equations 39 (2010), no. 3-4, 579–591.

[Nov82] S. P. Novikov, The Hamiltonian formalism and a multivalued analogue of Morse theory, Uspekhi Mat. Nauk 37 (1982), no. 5(227), 3–49, 248.

[Osu05] O. Osuna, Periodic orbits of weakly exact magnetic flows, preprint, 2005.

[Pat06] G. P. Paternain, Magnetic rigidity of horocycle flows, Pacific J. Math. 225 (2006), no. 2, 301–323.

[Pat09] ________, Helicity and the Mañé critical value, Algebr. Geom. Topol. 9 (2009), no. 3, 1413–1422.

[Pol95] L. Polterovich, An obstacle to non-Lagrangian intersections, The Floer memorial volume, Progr. Math., vol. 133, Birkhäuser, Basel, 1995, pp. 575–586.

[Sch06] F. Schlenk, Applications of Hofer’s geometry to Hamiltonian dynamics, Comment. Math. Helv. 81 (2006), no. 1, 105–121.

[Sch11] M. Schneider, Closed magnetic geodesics on $S^2$, J. Differential Geom. 87 (2011), no. 2, 343–388.

[Sch12a] ________, Alexandrov-embedded closed magnetic geodesics on $S^2$, Ergodic Theory Dynam. Systems 32 (2012), no. 4, 1471–1480.

[Sch12b] ________, Closed magnetic geodesics on closed hyperbolic Riemann surfaces, Proc. Lond. Math. Soc. (3) 105 (2012), no. 2, 424–446.

[Str90] M. Struwe, Existence of periodic solutions of Hamiltonian systems on almost every energy surface, Bol. Soc. Brasil. Mat. (N.S.) 20 (1990), no. 2, 49–58.

[Ta93] I. A. Taĭmanov, Closed non-self-intersecting extremals of multivalued functionals, Siberian Math. J. 33 (1993), no. 4, 686–692.

[Ush09] M. Usher, Floer homology in disk bundles and symplectically twisted geodesic flows, J. Mod. Dyn. 3 (2009), no. 1, 61–101.

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