ASYMPTOTIC BEHAVIORS OF SOLUTIONS TO A SIXTH-ORDER BOUSSINESQ EQUATION WITH LOGARITHMIC NONLINEARITY

HUAN ZHANG AND JUN ZHOU*

School of Mathematics and Statistics, Southwest University
Chongqing, 400715, China

(Communicated by Stefano Bianchini)

Abstract. To understand the characteristics of dynamical behavior especially the kinetic evolution for logarithmic nonlinearity, we aim to study a sixth-order Boussinesq equation with logarithmic nonlinearity in a bounded domain $\Omega \subset \mathbb{R}^n$ ($n \geq 1$ is an integer) with smooth boundary $\partial \Omega$, where the dispersive and the strong damping are taken into account. Based on the Faedo-Galerkin method, the logarithmic Sobolev inequality, and the potential well method, the main ingredient of this paper is to construct several conditions for initial data leading to the solution global existence or infinite time blow-up, and to study the polynomial decay and the exponential decay of the energy of the system.

1. Introduction. This paper deals with following the sixth-order Boussinesq equation with logarithmic nonlinearity:

$$\ddot{u} - a \Delta \dot{u} - 2b \Delta \dot{u} - \alpha \Delta^3 u + \beta \Delta^2 u - \Delta u + \Delta (u \log |u|) = 0, \quad (x, t) \in \Omega \times \mathbb{R}^+, \quad (1.1)$$

$$u(x, 0) = u_0(x), \quad \dot{u}(x, 0) = u_1(x), \quad x \in \Omega, \quad (1.2)$$

$$u|_{\partial \Omega} = 0, \quad \frac{\partial u}{\partial \nu}|_{\partial \Omega} = 0, \quad \Delta u|_{\partial \Omega} = 0, \quad t \in \mathbb{R}^+, \quad (1.3)$$

where the notations $\cdot'$ and $\cdot''$ denote the first derivative and second derivative with respect to time $t$ respectively; $\log |u| = \log_e |u|; \Omega$ is a bounded region in $\mathbb{R}^n$ ($n \geq 1$ is an integer) with smooth boundary $\partial \Omega$; $u = u(x, t)$ refers to the unknown function with $x = (x_1, x_2, ..., x_n) \in \Omega$ and $t \geq 0; \Delta$ is the n-dimensional Laplace operator; the parameters $a, b, \alpha, \beta$ satisfies

$$a > 0, \quad b \geq 0, \quad \alpha > 0, \quad \beta > -\alpha \lambda_1, \quad (1.4)$$

where $\lambda_1$ is the principal eigenvalue of $-\Delta$ in $\Omega$ with homogeneous Dirichlet boundary condition; initial values $u_0$ and $u_1$ belong to some Sobolev spaces to be specified below.

In 1872, Boussinesq [2] proposed the following equation

$$\ddot{u} + ru_{xxxx} - u_{xx} = s(u^2)_{xx} \quad (1.5)$$

to describe the propagation of small amplitude long waves on the surface of shallow water, the constants $r$ and $s$ depend on the depth of fluid and the characteristic

2020 Mathematics Subject Classification. Primary: 35B30; Secondary: 35B40, 35L30.

Key words and phrases. Sixth-order Boussinesq equation, logarithmic nonlinearity, global existence, infinite time blow-up.

* Corresponding author.
speed of long waves. Equation (1.5) can be used to describe many mathematical and physical phenomena, such as nonlinear beams, small oscillations of the nonlinear strings and the irrotational flows in a inviscid liquid in a uniform rectangular channel and so on (see [1, 15] and the references therein).

With regard to the generalization of model (1.5), we have to mention the following fourth-order Boussinesq equation

\[ \ddot{u} + \Delta^2 u - \Delta u - a\Delta \dot{u} - 2bg_\Delta \dot{u} + c\Delta^2 \ddot{u} + d\Delta^2 \dot{u} = \Delta f(u), \]  

If we neglect the effect of damp term, i.e. \( b = 0 \) and let \( a = b = c = d = 0 \), it becomes the classical Boussinesq equation (1.5). The relevant conclusions about the local well-posedness, global existence and nonexistence of solutions and finite time blow-up of solutions were obtained by means of a family of potential wells, Kato’s abstract theory and other skills (see [1, 14, 15, 18, 19, 37, 39] and the references therein).

But in real processes, because of the dissipation effect of this propagation of the wave, it is nature to take into consider the internal friction, i.e., \( b \neq 0 \) in (1.6) with viscous damping term. The Cauchy problem of (1.6) with \( n = 1, a = c = d = 0 \) and \( f(u) = u^2 \) was studied in [21, 24, 26, 27, 28], where the global well-posedness was considered. However, there was no result on the global nonexistence and finite time blow-up of solutions. For this reason, Xu [38] considered the case \( f(u) = \pm |u|^p \) or \( f(u) = \pm |u|^{p-1}u \), and showed the existence of global solutions and finite time blow-up solutions with suitable assumptions on the initial data. The Cauchy problem of (1.6) with \( n \geq 2, a = c = d = 0 \) was studied in [17, 25, 29, 35], where, with small initial data, the global well-posedness and asymptotic behavior of solutions were obtained by the semigroup method, the Green function method and the energy estimate. The Cauchy problem of (1.6) in \( \mathbb{R} \) and \( \mathbb{R}^n \) with \( c = d = 0 \) was studied in [33] and [22], respectively. By means of contraction mapping principle, the author got the existence and uniqueness of local and global solutions, and established the result of finite time blow-up by using concavity method. The Cauchy problem of (1.6) with \( a = c = 0 \) was studied by Godefroy in [8] and Wang and Su in [31, 32]. The authors considered the problem in \( \mathbb{R}^n \) and \( f(u) = \gamma |u|^{p-1}u, \gamma \in \mathbb{R} \setminus \{0\}, 1 < p < \infty \). By means of the contraction mapping principle, the authors proved the existence and uniqueness of local mild solutions in the phase space, and the continuation principle was obtained by establishing the time-space estimates of the corresponding Green operators. Furthermore, under some restriction on the initial data, the results on existence and uniqueness of global solutions and finite time blow-up of solutions were investigated.

All of these models above have been studied with algebraic forms of nonlinearity. It is well known that the logarithmic nonlinearity appears naturally in inflation cosmology and super symmetric field theories, quantum mechanics and nuclear physics (see [3, 5, 20] and references therein). The Boussinesq-type equations with logarithmic nonlinearity comes from the analysis of the Gaussian solitary wave solutions based on the so-called logarithmic-KdV equations (see [30]). In [11, 12], Hu, Zhang and Liu considered the initial boundary value problem of the following two Boussinesq-type equations with logarithmic nonlinearity

\[
\begin{aligned}
\ddot{u} - u_{xx} - \ddot{u}_{xx} + u_{xxxx} + \left( u \log |u|^k \right)_{xx} &= 0, & x \in \Omega, \ t > 0, \\
u(0, t) &= u(l, t) = 0, & u_{xx}(0, t) = u_{xx}(l, t) = 0, & t \geq 0, \\
u(x, 0) &= u_0(x), & \nu(x, 0) &= u_1(x), & x \in \Omega,
\end{aligned}
\]  

(1.7)
and
\[
\begin{aligned}
&\ddot{u} - u_{xx} + u_{xxxx} + \left( u_x \log |u_x|^k \right)_x = 0, \quad x \in \Omega, \ t > 0, \\
&u(0, t) = u(l, t) = 0, \ u_x(0, t) = u_x(l, t) = 0, \ t \geq 0, \\
&u(x, 0) = u_0(x), \ \dot{u}(x, 0) = u_1(x), \quad x \in \Omega.
\end{aligned}
\]  

(1.8)

Here, \( \Omega = (0, l) \) and \( k \geq 1 \) is a constant. By using potential well theory and logarithmic Sobolev inequality, the authors studied the local well-posedness and dynamical behaviors of solutions, such as global existence, exponential growth, etc.

In order to correct the bad numerical feature of the classical fourth-order Boussinesq equation as (1.5), the sixth-order Boussinesq equation was introduced. Christov et al. [6, 7] showed that, a way to make the fourth-order Boussinesq equation (1.5) mathematically correct is to retain the term containing the sixth-order spatial derivative in the approximation expansion. Recently, Wang [34] and Wang, Li, and Hu [36] considered the Cauchy problem of equation (1.1) in \( \mathbb{R}^n \) with the term \( \Delta (u \log |u|) \) replacing by \( -\Delta f(u) \) and \( f(u) = O(u^{1+\theta}) \), where \( \theta \) is a positive constant. Based on decay estimates of solutions to the corresponding linear equation, Wang [34] defined a solution space with time weighted norms, and showed the existence and asymptotic behavior of global solutions in the corresponding Sobolev spaces by the contraction mapping principle with the assumptions that the initial value is small enough; Wang, Li, and Hu [36] consider the case \( \theta = 1 \), based on the decay properties of the solutions operator in Morrey spaces and the contraction mapping principle, they showed the existence of a global solution, and studied the asymptotic behavior in Morrey spaces.

Inspired by the above works, the main purpose of this paper is to deal with the well-posedness and dynamical behaviors of solutions to problem (1.1), (1.2) and (1.3). The organizations of the remain parts of this paper are as follows. In Section 2, we introduce some notations and definitions used throughout this paper; In Section 3, we show the main results of the paper; In section 4, we give some preliminaries; In Section 5, we study the global well-posedness of the solutions; In Section 6, we obtain the energy decay estimates of the solutions; In Section 7, we study the infinite time blow-up of the solutions.

2. Notations and definitions. In this section, we introduce some notations and definitions used throughout this paper. We begin with the notations.

1. For \( 1 \leq p \leq \infty \), \( L^p = L^p(\Omega) \) denotes the usual Lebesgue space with the norm \( \| \cdot \|_{L^p} \); in particular, \( \| \cdot \| \triangleq \| \cdot \|_{L^2} \), and the inner product of \( L^2 \) is denoted by \( \langle \cdot, \cdot \rangle \).

2. The powers \( (-\Delta)^s \) of \( -\Delta \) for \( s \in \mathbb{R} \) in \( \Omega \) are denoted as (see, for example, [23, Section 2.2.1])
\[
(-\Delta)^s \phi \triangleq \sum_{k=1}^{\infty} \lambda_k^s e_k \phi, \quad s \in \mathbb{R},
\]  

(2.1)

where \( \{e_k(x)\}_{k=1}^{\infty} \) are the eigenfunctions of \( -\Delta \) in \( \Omega \) with homogeneous Dirichlet boundary condition, i.e.,
\[
\begin{aligned}
-\Delta e_k &= \lambda_k e_k, \quad x \in \Omega; \\
\|e_k\| &= 1, \ e_k|_{\partial \Omega} = 0,
\end{aligned}
\]  

(2.2)

where \( \lambda_k, \ k = 1, 2, \cdots, \) is the eigenvalues corresponding to \( e_k \), which satisfy \( 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \uparrow \infty \) as \( k \uparrow \infty \); and
\[
a_k = \int_{\Omega} \phi e_k \, dx.
\]
Especially, we have
\[ \lambda_1 = \inf_{\phi \in H^1_0, \|\phi\| \neq 0} \frac{\|\nabla \phi\|^2}{\|\phi\|^2}. \] (2.3)

3. \( H^2_0 = H^2_0(\Omega) \) stands for the classical Sobolev space, which is a Hilbert space with inner product \((\cdot, \cdot)_{H^2_0}\) defined by
\[ (\phi, \psi)_{H^2_0} \triangleq \int_{\Omega} (\phi \psi + \beta \nabla \phi \cdot \nabla \psi + \alpha \Delta \phi \Delta \psi) \, dx. \] (2.4)

By using \( \alpha > 0 \) and \( \beta > -\alpha \lambda_1 \) (see (1.4)), it follows from Poinca\'e\'s inequality that \((\cdot, \cdot)_{H^2_0}\) is indeed a inner product defined on \( H^2_0 \). Then the norm \( \| \cdot \|_{H^2_0} \) defined by
\[ \|\phi\|_{H^2_0} \triangleq \sqrt{(\phi, \phi)_{H^2_0}} = \sqrt{\|\nabla \phi\|^2 + \beta \|\nabla \phi\|^2 + \alpha \|\Delta \phi\|^2}. \] (2.5)

Especially, by (2.3), we have
\[ \|\phi\|^2_{H^2_0} \geq \beta \|\nabla \phi\|^2 + \alpha \|\Delta \phi\|^2 \geq (\beta + \alpha \lambda_1) \|\nabla \phi\|^2, \] (2.6)

and
\[ \|\phi\|^2_{H^2_0} \geq (1 + \beta \lambda_1 + \alpha \lambda^2_1) \|\phi\|^2. \] (2.7)

4. Let
\[ \mathcal{H} \triangleq \{ \phi \in L^2 : \|\phi\|_H < \infty \}, \] (2.8)

where
\[ \|\phi\|_H \triangleq \sqrt{a \|\phi\|^2 + \left\| (-\Delta)^{-\frac{1}{2}} \phi \right\|^2}; \] (2.9)

then by \( a > 0 \) (see (1.4)) and (2.1), \( \| \cdot \|_H \) is equivalent to \( \| \cdot \| \), in fact, since (note (2.1))
\[ \left\| (-\Delta)^{-\frac{1}{2}} \phi \right\|^2 = \sum_{k=1}^{\infty} \lambda^{-1}_k a_k^2 \leq \lambda^{-1}_1 \sum_{k=1}^{\infty} a_k^2 = \lambda^{-1}_1 \|\phi\|^2, \] (2.10)

we have
\[ a \|\phi\|^2 \leq \|\phi\|^2_{H^1} \leq (a + \lambda^{-1}_1) \|\phi\|^2; \] (2.11)

and then \( \mathcal{H} \) is a Hilbert space with the inner product \((\cdot, \cdot)_H\) defined by
\[ (\phi, \psi)_H \triangleq a(\phi, \psi) + \left( (-\Delta)^{-\frac{1}{2}} \phi, (-\Delta)^{-\frac{1}{2}} \psi \right). \] (2.12)

5. For two Banach spaces \( X \) and \( Y \), \( X \hookrightarrow Y \) means \( X \subset Y \) and the inclusion map of \( X \) into \( Y \) is continuous; \( (\cdot, \cdot)_{X', X} \) is used to denote the duality pairing between \( X \) and its dual space \( X' \).

6. The letter \( C \) is used to denote a constant whose value may change from line to line; and \( C(\cdot, \cdot) \) denotes positive constants depending on the quantities appearing in the parenthesis.

7. For any \( \phi \in H^2_0 \), the functional \( J \) and \( I \) are defined by
\[ J(\phi) = \frac{1}{2} \|\phi\|^2_{H^2_0} - \frac{1}{2} \int_{\Omega} \phi^2 \log |\phi| \, dx + \frac{1}{4} \|\phi\|^2, \] (2.13)
and
\[ I(\phi) = \|\phi\|^2_{H^2_0} - \int_{\Omega} \phi^2 \log |\phi| \, dx. \] (2.14)

Obviously,
\[ J(\phi) = \frac{1}{2} I(\phi) + \frac{1}{4} \|\phi\|^2. \] (2.15)
Note that \( \sup_{0<\tau<1} |\tau \log \tau| = e^{-1} \), \( H^2_0 \hookrightarrow L^{2+2\epsilon} \) for \( \epsilon > 0 \) small enough (with \( \Theta \) be the optimal Sobolev constant of \( H^2_0 \hookrightarrow L^{2+2\epsilon} \)), and \( \log \tau = \frac{1}{2} \log \tau^e \leq \frac{\epsilon}{2} \) for \( \tau \geq 1 \) and \( \epsilon > 0 \), it follows, for \( \epsilon > 0 \) small enough, that

\[
\left| \int_{\Omega} \phi^2 \log |\phi| \, dx \right| = \frac{1}{2} \int_{\Omega} \phi^2 \log \phi \, dx \leq \frac{1}{2} \int_{\{x \in \Omega : \phi(x) \leq 1\}} \phi^2 \log \phi \, dx + \frac{1}{2} \int_{\{x \in \Omega : \phi(x) \geq 1\}} \phi^2 \log \phi^2 \, dx
\]

\[
\leq \frac{1}{2} e^{-1}|\Omega| + \frac{1}{2\epsilon} \int_{\Omega} |\phi|^{2+2\epsilon} \, dx \leq \frac{1}{2} e^{-1}|\Omega| + \frac{\Theta^{2+2\epsilon}}{2\epsilon} ||\phi||_{H^2_0}^{2+2\epsilon}.
\]

Then, \( J \) and \( I \) are well-defined on \( H^2_0 \).

8. The constant \( d \) is defined by

\[
d = \inf \left\{ \sup_{\lambda \geq 0} J(\lambda \phi) : \phi \in H^2_0 \setminus \{0\} \right\}.
\]

By Lemma 4.4 below,

\[
d = \inf_{\phi \in \mathcal{N}} J(\phi) \geq \frac{1}{4} (2\pi (\beta + \alpha \lambda_1))^2 e^n > 0,
\]

where

\[
\mathcal{N} \triangleq \{ \phi \in H^2_0(\Omega) \setminus \{0\} : I(\phi) = 0 \}.
\]

Moreover, two sets \( \mathcal{W} \) and \( \mathcal{V} \) associate to \( J, I \), and \( d \) are defined by

\[
\mathcal{W} \triangleq \{ \phi \in H^2_0 : I(\phi) > 0, J(\phi) < d \} \cup \{0\},
\]

\[
\mathcal{V} \triangleq \{ \phi \in H^2_0 : I(\phi) < 0, J(\phi) < d \}.
\]

9. Let \( u \in C([0,T]; H^2_0) \cap C^1([0,T]; L^2) \cap C^2([0,T]; H^{-2}) \) be a maximal weak solution to the problem (1.1), (1.2) and (1.3) (see Definitions 2.1), where \( T \) denotes the maximal existence time. The total energy \( E(t) \) is defined by

\[
E(t) = \frac{1}{2} ||\dot{u}(t)||_{H^2}^2 + \frac{1}{2} ||u(t)||_{H^2}^2 - \frac{1}{2} \int_{\Omega} u^2(x,t) \log |u(x,t)| \, dx + \frac{1}{4} ||u(t)||^2, 0 \leq t < T.
\]

Obviously,

\[
E(t) = \frac{1}{2} ||\dot{u}(t)||_{H^2}^2 + J(u(t)).
\]

Next, we shall introduce the appropriate definition of a weak solution, which satisfies a certain variational equality. By [13, Lemma 1.7], for any

\[
u \in \left\{ \phi \in H^6(\Omega) : \phi|_{\partial \Omega} = \frac{\partial \phi}{\partial \nu}|_{\partial \Omega} = \Delta \phi|_{\partial \Omega} = 0 \right\},
\]

there holds

\[
(-\Delta)^{-1} \Delta^2 u = -\Delta u \text{ and } (-\Delta)^{-1} \Delta^3 u = -\Delta^2 u \text{ and } (-\Delta)^{-1} \Delta u = -u.
\]

Then applying the operator \((-\Delta)^{-1}\) to (1.1), we have

\[
(-\Delta)^{-1} \ddot{u} + a\dot{u} + u + 2b \dot{u} - \beta \Delta u + \alpha \Delta^2 u = u \log |u|.
\]
Therefore, the weak solution of (2.23) with the initial data (1.2) and boundary value condition (1.3) is said to be the weak solution of the problem (1.1), (1.2) and (1.3). This leads to the following definition.

**Definition 2.1.** Let $T > 0$ be fixed. Assume $u_0 \in H_0^2$ and $u_1 \in L^2$. A function $u \in C([0, T]; H_0^2) \cap C^1([0, T]; L^2) \cap C^2([0, T]; H^{-2})$ is said to be a weak solution to the problem (1.1), (1.2) and (1.3) over $[0, T]$, if and only if for any $t \in [0, T]$, it satisfies

\[
\langle (-\Delta)^{-\frac{1}{2}} \ddot{u}, (-\Delta)^{-\frac{1}{2}} \varphi \rangle_{H^{-2}, H_0^2} + a \langle \dot{u}, \varphi \rangle_{H^{-2}, H_0^2} + (u, \varphi) + \beta \langle \nabla u, \nabla \varphi \rangle + 2b(\ddot{u}, \varphi) + \alpha \langle \Delta u, \Delta \varphi \rangle - (u \log |u|, \varphi) = 0, \tag{2.24}
\]

for all test functions $\varphi \in C([0, T]; H_0^2)$, and

\[
u(x, 0) = u_0(x), \quad \dot{u}(x, 0) = u_1(x). \tag{2.25}\]

**Remark 1.** We remark all the terms of (2.24) are well defined. In fact, in view of Definition 2.1.

Let (1.3). This leads to the following definition.

Finally, since $(-\Delta)^{-\frac{1}{2}} u(t) \in H^{-2}(\Omega)$ for any $t \in [0, T]$, where in the last inequality of (2.26) we have used the following estimate (note (2.5) and (2.10))

\[
\left\| (-\Delta)^{-\frac{1}{2}} u \right\|_{H^{-2}} \leq \frac{1}{\sqrt{\lambda_1}} \left\| \ddot{u} \right\|_{H^{-2}}, \tag{2.27}
\]

Secondly, in view of $u(t) \in H_0^2$ for any $t \in [0, T]$, by similar proof as (2.16), one can show $(u \log |u|)(t) \in L^2$ for any $t \in [0, T]$. Finally, since $\varphi(t) \in H_0^2$ for any $t \in [0, T]$, in view of (2.27), one can see$(-\Delta)^{-\frac{1}{2}} \varphi(t) \in H_0^2$ for any $t \in [0, T]$.

3. **Main results.** In this section, we state the main results of this paper. Firstly, we give a comment to $E(0)$ and $I(u_0)$ by (2.21) and (2.14) as follows:

\[
E(0) = \frac{1}{2} \left\| u_0 \right\|_{H_0^2}^2 + \frac{1}{2} \left\| u_0 \right\|_{H^2_0}^2 - \frac{1}{2} \int_\Omega u_0^2 \log |u_0| dx + \frac{1}{4} \left\| u_0 \right\|_{H_0^2}^2,
\]

\[
I(u_0) = \left\| u_0 \right\|_{H_0^2}^2 - \frac{1}{2} \int_\Omega u_0^2 \log |u_0| dx.
\]

The first one is about global existence.
Theorem 3.1. (Global existence) Let (1.4) hold. Assume the initial data \((u_0, u_1) \in H_0^2 \times L^2\) such that \(E(0) \leq d\) and \(I(u_0) \geq 0\). Then the problem of (1.1), (1.2) and (1.3) admits a global weak solution \(u\), i.e.,
\[
    u \in C \left( [0, \infty); H_0^2 \right), \quad \dot{u} \in C \left( [0, \infty); L^2 \right), \quad \ddot{u} \in C \left( [0, \infty); H^{-2} \right),
\]
and (2.24) and (2.25) hold (see Definition 2.1). Moreover, \(u(t) \in W\) for \(0 \leq t < \infty\) and the solution satisfies the energy equality as follows:
\[
    E(t) + 2b \int_0^t \|\dot{u}(\tau)\|^2 d\tau = E(0), \quad \text{for all} \quad t \in [0, \infty). \tag{3.2}
\]

Theorem 3.2. (Energy decay estimates) Let (1.4) hold. Assume the initial data \((u_0, u_1) \in H_0^2 \times L^2\) such that \(E(0) \leq d\) and \(I(u_0) \geq 0\). Let \(u\) be global weak solution of the problem of (1.1), (1.2) and (1.3) got in Theorem 3.1.

(i) (Polynomial decay) If \(b > 0\), \(E(0) < d\), and \(I(u_0) \leq 0\), then
\[
    E(t) \lesssim \left\{ E(0) + \frac{1 + b \log d - \log E(0)}{2(\log d - \log E(0))} \left[ \kappa d + \frac{1}{2} \|u_0\|_{H^2}^2 + \frac{1}{2} \|u_1\|_{H^2}^2 + b \|u_0\|^2 \right] \right\} (1 + t)^{-1}
\]
\[
    + \left\{ \frac{d(a \lambda_1 + 1)(1 + 2(\log d - \log E(0)))}{4b \lambda_1 (\log d - \log E(0))} \left( 1 - e^{-\frac{a \lambda_1}{\lambda_1 + b}(1 + t)} \right) \right\} (1 + t)^{-1}, \tag{3.3}
\]
where \(E\) and \(d\) are defined in (2.21) and (2.17) respectively, and
\[
    \kappa \triangleq \text{max} \left\{ 1, \frac{2(a \lambda_1 + 1)}{\lambda_1} \right\}
\]
\[
    = \begin{cases} 
        1, & \text{if } \lambda_1 > 2 \text{ and } a \leq \frac{\lambda_1 - 2}{2 \lambda_1}; \\
        \frac{2(a \lambda_1 + 1)}{\lambda_1}, & \text{if } \lambda_1 \leq 2, \text{ or } \lambda_1 > 2 \text{ and } a > \frac{\lambda_1 - 2}{2 \lambda_1}.
    \end{cases} \tag{3.4}
\]

(ii) (Exponential decay) If \(b \geq 0\), \(E(0) \leq \min\{d, E_0\}\) for some \(\delta \in (0, 1)\), and \(I(u_0) \leq 0\), where
\[
    E_0 \triangleq \begin{cases} 
        \frac{1}{4} \left( \frac{(1 - \delta) \varepsilon}{(1 - 2 \varepsilon)^{3/2}} \right)^{1/2}, & \text{if } \sigma \leq 0; \\
        \frac{1}{4(1 + \sigma)} \left( \frac{(1 - 2 \varepsilon)^{3/2}}{(1 - \delta) \varepsilon} \right)^{1/2}, & \text{if } \sigma > 0,
    \end{cases} \tag{3.5}
\]
and
\[
    \sigma \triangleq \log(4d) - 1 - n \left[ 1 + \log \sqrt{\pi (\beta + \alpha \lambda_1)} \right], \tag{3.6}
\]
then
\[
    E(t) \leq \frac{E(0) + \varepsilon(u_0, u_1) + b \|u_0\|^2}{1 - \varepsilon} e^{-\frac{a \lambda_1}{\lambda_1 + b}(1 + t)}. \tag{3.7}
\]
Here,
\[
    \varepsilon \text{ is any positive constant satisfying}
\]
\[
    \varepsilon < \frac{1}{\vartheta}, \tag{3.9}
\]
\[
    \varepsilon \begin{cases} 
        \infty, & \text{if } n = 1, 2, 3, 4, \\
        \frac{8}{n - 4}, & \text{if } n = 5, 6, 7, \cdots,
    \end{cases} \tag{3.10}
\]
\[ \epsilon \leq \min \left\{ \frac{1}{2}, \sqrt{\frac{b\lambda_1}{1 + a\lambda_1}} + \frac{1}{16} - \frac{1}{4}, \delta \left( \frac{1}{1 + \beta\lambda_1 + \alpha\lambda_1^2} + 2 \right)^{-1} \right\}, \tag{3.11} \]

\( \Theta \) is the optimal Sobolev constant of \( H_0^2 \hookrightarrow L^{2+\epsilon} \) (note (3.10)), i.e.

\[ \Theta = \sup_{u \in H_0^2 \setminus \{0\}} \frac{\|u\|_{L^{2+\epsilon}}}{\|u\|_{H_0^2}}. \tag{3.12} \]

**Theorem 3.3.** (blow-up at \( \infty \)) Let (1.4) hold. Assume the initial data \((u_0, u_1) \in H_0^2 \times L^2\) such that \(E(0) < d\) and \(I(u_0) < 0\). Then the maximal existence time of solutions to problems (1.1), (1.2) and (1.3) is infinite and

\[ \lim_{t \uparrow \infty} \left( \|u(t)\|^2 + 2h \int_0^t \|u(\tau)\|^2 d\tau \right) = \infty. \]

4. **Preliminaries.** In this section, we introduce some lemmas, which will be used in below to prove the main results.

Firstly, we introduce [23, Section II, Theorem 4.1 and Lemma 4.1]. To this end, we firstly give some notations. Let \( V \) and \( H \) be two Hilbert spaces satisfy

1. \( V \subset H \subset V' \);
2. \( V \) is dense in \( H \).

Here \( V' \) is the dual space of \( V \). \( A \) is a linear continuous operator form \( V \) to \( V' \). Let \( a(\cdot, \cdot) \) be the bilinear continuous form on \( V \) associated to \( A \), i.e.,

\[ a(u, v) = \langle Au, v \rangle_{V', V}, \forall u, v \in V, \]

and \( a(\cdot, \cdot) \) satisfies

1. \( a(\cdot, \cdot) \) is symmetric, i.e., \( a(u, v) = a(v, u) \);
2. there exists two positive constants \( M_0 \) and \( \alpha_0 \) such that, for any \( u, v \in V \),

\[ |a(u, v)| \leq M_0 \|u\|_V \|v\|_V; \]

\[ a(u, u) \geq \alpha_0 \|u\|_V^2. \]

**Lemma 4.1.** [23, Section II, Lemma 4.1] We assume that \( u \) is such that

\[ u \in L^2(0, T; V), \quad \dot{u} \in L^2(0, T; H), \]

and

\[ \ddot{u} + Au \in L^2(0, T; H). \]

Then, after modification on a set of measure zero, \( u \) is continuous from \([0, T]\) into \( V \), \( \dot{u} \) is continuous from \([0, T]\) into \( H \) and, in the sense of distributions on \((0, T)\),

\[ (\ddot{u} + Au, \dot{u})_H = \frac{d}{dt} \left[ \|\dot{u}\|_H^2 + a(u, u) \right]. \]

Here \((\cdot, \cdot)_H\) is the inner product of \( H \) and \( \|\cdot\|_H^2 = (\cdot, \cdot)_H \).

**Lemma 4.2.** [23, Section II, Theorem 4.1] The abstract ordinary differential equation

\[ \begin{cases} \ddot{u} + \alpha\dot{u} + Au = f, & t \in (0, T); \\ u(0) = u_0, \\ \dot{u}(0) = u_1, \end{cases} \tag{4.1} \]

with \( \alpha \in \mathbb{R}, f \in L^2(0, T; H), u_0 \in V, u_1 \in H \) admits a unique solution \( u \) such that \( u \in C([0, T]; V) \cap C^1([0, T]; H) \).
Since \( \beta \) where Lemma 4.3. For any \( \phi \in H^2_0 \{0\} \) and any \( \omega > 0 \), we have the following logarithmic Sobolev inequality (see, for example, [4, 9, 10])

\[
2 \int_\Omega \phi^2 \log \left( \frac{|\phi|}{\|\phi\|} \right) \, dx + n(1 + \log \omega)\|\phi\|^2 \leq \frac{\omega^2}{2} \|\nabla \phi\|^2. \tag{4.2}
\]

Since \( \beta + \alpha \lambda_1 > 0 \) (see (1.4)), by (4.2) and (2.6), we have, for any \( \omega > 0 \),

\[
2 \int_\Omega \phi^2 \log \left( \frac{|\phi|}{\|\phi\|} \right) \, dx + n(1 + \log \omega)\|\phi\|^2 \leq \frac{\omega^2}{\pi (\beta + \alpha \lambda_1)} \|\phi\|^2_{H^2_0}. \tag{4.3}
\]

**Lemma 4.3.** For any \( \phi \in H^2_0 \{0\} \), we have

(i) \( \lim_{\lambda \downarrow 0} J(\lambda \phi) = 0 \), \( \lim_{\lambda \uparrow \infty} J(\lambda \phi) = -\infty \);

(ii) \( I(\lambda \phi) = \frac{1}{\lambda} \frac{d}{d\lambda} J(\lambda \phi) \) \( \begin{cases} > 0, & 0 < \lambda < \lambda^*; \\ = 0, & \lambda = \lambda^*; \\ < 0, & \lambda^* < \lambda < \infty, \end{cases} \tag{4.4} \)

where

\[
\lambda^* = \exp \left\{ \frac{\|\phi\|^2_{H^2_0} - \int_\Omega \phi^2 \log |\phi| \, dx}{\|\phi\|^2} \right\} = \exp \left\{ \frac{I(\phi)}{\|\phi\|^2} \right\}; \tag{4.5}
\]

(iii) \( J(\lambda \phi) \) is increasing for \( 0 < \lambda < \lambda^* \), and is decreasing for \( \lambda^* < \lambda < \infty \), and takes its maximum at \( \lambda = \lambda^* \),

where \( J \) and \( I \) are the functionals defined in (2.13) and (2.14) respectively.

**Proof.** (i) By the definition of \( J \) (see (2.13)), for any \( \lambda > 0 \), we have

\[
J(\lambda \phi) = \frac{1}{2} \lambda^2 \|\phi\|^2_{H^2_0} - \frac{1}{2} \lambda^2 \int_\Omega \phi^2 \log(\lambda|\phi|) \, dx + \frac{1}{4} \lambda^2 \|\phi\|^2
\]

\[
= \frac{1}{2} \lambda^2 \left( \|\phi\|^2_{H^2_0} + \frac{1}{2} \|\phi\|^2 - \int_\Omega \phi^2 \log |\phi| \, dx - \log \lambda \|\phi\|^2 \right). \tag{4.6}
\]

(i) follows directly from the above equality.

(ii) By the definition of \( I \) (see (2.14)),

\[
I(\lambda \phi) = \lambda^2 \|\phi\|^2_{H^2_0} - \lambda^2 \int_\Omega \phi^2 \log(\lambda|\phi|) \, dx = \lambda^2 \left( \|\phi\|^2_{H^2_0} - \int_\Omega \phi^2 \log |\phi| \, dx - \log \lambda \|\phi\|^2 \right)
\]

By (4.6), we know that

\[
\frac{d}{d\lambda} J(\lambda \phi) = \lambda \psi(\lambda) + \frac{1}{2} \lambda^2 \psi'(\lambda) = \lambda \psi(\lambda) - \frac{1}{2} \lambda \|\phi\|^2
\]

\[
= \lambda \left( \|\phi\|^2_{H^2_0} - \int_\Omega \phi^2 \log |\phi| \, dx - \log \lambda \|\phi\|^2 \right) \begin{cases} > 0, & \text{if } 0 < \lambda < \lambda^*; \\ = 0, & \text{if } \lambda = \lambda^*; \\ < 0, & \text{if } \lambda^* < \lambda < \infty. \end{cases}
\]

Then (ii) follows directly.

(iii) is a direct result of (ii). \qed
**Lemma 4.4.** Let $d$ be the constant defined in (2.17), i.e.,

$$d \triangleq \inf \left\{ \sup_{\lambda \geq 0} J(\lambda \phi) : \phi \in H^2_0 \setminus \{0\} \right\}.$$ 

Then $d = \inf_{\phi \in \mathcal{N}} J(\phi)$ and

$$d \geq \frac{1}{4} \left[ 2\pi (\beta + \alpha \lambda_1) \right]^2 e^n, \quad (4.7)$$

where $J$, $\mathcal{N}$, and $\lambda_1$ are defined in (2.13), (2.19) and (2.3) respectively.

**Proof.** Firstly, we prove $d = \inf_{\phi \in \mathcal{N}} J(\phi)$. For any $\phi \in H^2_0(\Omega) \setminus \{0\}$, by Lemma 4.3(iii),

$$\sup_{\lambda \geq 0} J(\lambda \phi) = J(\lambda^* \phi),$$

and $I(\lambda^* \phi) = 0$, where $\lambda^*$ is defined in (4.5). Then we have $\lambda^* \phi \in \mathcal{N}$ (see the definition of $\mathcal{N}$ in (2.19)), and then it follows from (2.17) that

$$d = \inf_{\phi \in H^2_0 \setminus \{0\}} \sup_{\lambda \geq 0} J(\lambda \phi) = \inf_{\phi \in \mathcal{N}} J(\lambda^* \phi). \quad (4.8)$$

On the other hand, for any $\phi \in \mathcal{N} \subset H^2_0 \setminus \{0\}$, we have $\lambda^* = 1$ (see the definition of $\lambda^*$ in (4.5)). Then

$$\inf_{\phi \in \mathcal{N}} J(\phi) = \inf_{\phi \in \mathcal{N}} J(\lambda^* \phi) = \inf_{\phi \in \mathcal{N}} \sup_{\lambda \geq 0} J(\lambda \phi) \geq \inf_{\phi \in H^2_0 \setminus \{0\}} J(\phi) = d.$$ \quad (4.9)

Therefore, (4.8), (4.9), we get

$$d = \inf_{\phi \in \mathcal{N}} J(\phi).$$

Now, we calculate the value of $d$. We obtain from (2.15), (4.8) that

$$d = \inf_{\phi \in \mathcal{N}} J(\phi) = \inf_{\phi \in \mathcal{N}} \left( \frac{1}{2} I(\phi) + \frac{1}{4} \|\phi\|^2 \right) = \frac{1}{4} \inf_{\phi \in \mathcal{N}} \|\phi\|^2 > 0. \quad (4.10)$$

For any $\phi \in \mathcal{N}$, it follows from the definition of $\mathcal{N}$ (see (2.19)) and (4.3) (with $\omega = \sqrt{2\pi (\beta + \alpha \lambda_1)}$) that

$$2 \|\phi\|^2_{H^2_0} = 2 \int_\Omega \phi^2 \log |\phi| dx$$

$$\leq \frac{\left( \sqrt{2\pi (\beta + \alpha \lambda_1)} \right)^2}{\pi (\beta + \alpha \lambda_1)} \|\phi\|^2_{H^2_0} + 2 \|\phi\|^2 \log \|\phi\| - n \left( 1 + \log \left( \sqrt{2\pi (\beta + \alpha \lambda_1)} \right) \right) \|\phi\|^2,$$

which implies

$$\|\phi\| \geq \exp \left( \frac{n}{2} \left( 1 + \log \left( \sqrt{2\pi (\beta + \alpha \lambda_1)} \right) \right) \right) = \left[ 2\pi (\beta + \alpha \lambda_1) \right]^\frac{n}{2} e^n.$$ 

Then by (4.10), we get (4.7). \qed

**Lemma 4.5.** For any $\phi \in H^2_0 \setminus \{0\}$. If $I(\phi) < 0$, then

$$I(\phi) < 2(J(\phi) - d). \quad (4.11)$$

where $I$, $J$, and $d$ are defined in (2.14), (2.13), and (2.17) respectively.
Proof. Let $\lambda^*$ be the constant defined in (4.5). Since $I(\phi) < 0$, by Lemma 4.3, we have $\lambda^* < 1$ and $\lambda^* \phi \in \mathcal{N}$. Then by Lemma 4.4 and (2.15), we obtain

$$d \leq J(\lambda^* \phi) = \frac{1}{2} I(\lambda^* \phi) + \frac{1}{4} (\lambda^*)^2 \| \phi \|^2$$

i.e., $(4.11)$ holds.

5. **Global well-posedness.** In this section, we will study the existence and uniqueness of global solution to problems (1.1), (1.2) and (1.3) by Faedo-Galerkin method.

**Proof of Theorem 3.1.** Step 1: existence of global solutions when $E(0) < d$ and $I(u_0) \geq 0$. By (2.22) and $E(0) < d$, we get $J(u_0) < d$. Then it follows from (2.18) and $I(u_0) \geq 0$ that $I(u_0) > 0$. Moreover, by (2.22) and (2.15), it follows $E(0) \geq J(u_0) = \frac{1}{2} I(u_0) + \frac{1}{2} \| u_0 \|^2 > 0$. So, in the following we assume

$$0 < E(0) < d \text{ and } I(u_0) > 0.$$  

(5.1)

Let $e_k(x)$ be the base function as mentioned in (2.2). Let

$$u_m(x, t) = \sum_{k=1}^{m} a_{km}(t) e_k(x),$$

(5.2)

satisfy (note $u_0 \in H_0^2$ and $u_1 \in L^2$), for $k = 1, \cdots, m,$

$$(-\Delta)^{-\frac{1}{2}} \ddot{u}_m(x, 0) + a u_m + 2b \dot{u}_m + \alpha \Delta^2 u_m - \beta \Delta u_m + u_m, e_k) = (u_m \log |u_m|, e_k),$$

(5.3)

$$u_m(x, 0) = u_{m0}(x) = \sum_{k=1}^{m} \rho_k e_k \to u_0 \text{ in } H_0^2, \text{ as } m \uparrow \infty,$$

(5.4)

$$\dot{u}_m(x, 0) = u_{m1}(x) = \sum_{k=1}^{m} \xi_k e_k \to u_1 \text{ in } L^2, \text{ as } m \uparrow \infty.$$  

(5.5)

By (2.10) and (5.5), we have

$$(-\Delta)^{-\frac{1}{2}} \ddot{u}_m(x, 0) \to (-\Delta)^{-\frac{1}{2}} u_1 \text{ in } L^2, \text{ as } m \uparrow \infty.$$  

(5.6)

Substituting (5.2) into (5.3), it follows that

$$\lambda_k^{-1} a_k + 2b \dot{a}_k + \alpha \Delta^2 a_k - \beta \Delta a_k + a_k \dot{a}_k = (u_m \log |u_m|, e_k)$$  

(5.7)

for $k = 1, 2, \cdots, m$. Substituting (5.2) into (5.4) and (5.5) it follows that

$$a_{km}(0) = \rho_k, \quad a_{km}(0) = \xi_k, \quad k = 1, 2, \cdots, m.$$  

(5.8)

According to the standard theory of ordinary differential equations , for each $m$, the problem (5.7) and (5.8) admits a solution $a_{km} \in C^2[0, t_m]$ for some $t_m > 0$.

Multiplying the both sides of (5.3) by $a_{km}(t)$ and summing up for $k = 1, 2, \cdots, m$, one has

$$(-\Delta)^{-\frac{1}{2}} \ddot{a}_m + a \dot{a}_m + 2b \dot{a}_m + \alpha \Delta^2 a_m - \beta \Delta a_m + a_m \cdot \sum_{k=1}^{m} a_{km}(t) e_k)$$

$$= (u_m \log |u_m|, \sum_{k=1}^{m} a_{km}(t) e_k),$$

i.e.,

$$(-\Delta)^{-\frac{1}{2}} \ddot{a}_m + a \dot{a}_m + 2b \dot{a}_m + \alpha \Delta^2 a_m - \beta \Delta a_m + a_m \dot{a}_m) = (u_m \log |u_m|, \dot{a}_m),$$
which implies
\[
\left( (-\Delta)^{-\frac{1}{2}} \tilde{u}_m, (-\Delta)^{-\frac{1}{2}} \tilde{u}_m + a(\tilde{u}_m, \tilde{u}_m) + 2b(\tilde{u}_m, \tilde{u}_m) \right.
+ a(\Delta u_m, \Delta \tilde{u}_m) + \beta(\nabla u_m, \nabla \tilde{u}_m) + (u_m, \tilde{u}_m) = (u_m \log |u_m|, \tilde{u}_m).
\]
Integrating the above equality from 0 and \(t\), one has
\[
\frac{1}{2} \left\| (-\Delta)^{-\frac{1}{2}} \tilde{u}_m(t) \right\|^2 + \frac{a}{2} \left\| \tilde{u}_m(t) \right\|^2 + 2b \int_0^t \left\| \tilde{u}_m(\tau) \right\|^2 d\tau
= \frac{1}{2} \left\| u_m(0) \right\|^2_{H^2}
+ \frac{\alpha}{2} \left\| \Delta u_m \right\|^2 + \frac{\beta}{2} \left\| \nabla u_m \right\|^2 + \frac{1}{2} \left\| u_m \right\|^2
- \frac{1}{2} \int_{\Omega} |u_m(x, t)|^2 \log |u_m(x, t)| dx + \frac{1}{4} \left\| u_m \right\|^2
\]
i.e.,
\[
\frac{1}{2} \left\| \tilde{u}_m(t) \right\|^2_{H^2} + \frac{1}{2} \left\| u_m(t) \right\|^2_{H^2} - \frac{1}{2} \int_{\Omega} |u_m(x, t)|^2 \log |u_m(x, t)| dx
+ \frac{1}{4} \left\| u_m(t) \right\|^2 + 2b \int_0^t \left\| \tilde{u}_m(\tau) \right\|^2 d\tau
= \frac{1}{2} \left\| u_m \right\|^2_{H^2} + \frac{1}{2} \left\| u_{m0} \right\|^2_{H^2}
- \frac{1}{2} \int_{\Omega} |u_{m0}(x)|^2 \log |u_{m0}(x)| dx + \frac{1}{4} \left\| u_{m0} \right\|^2. \tag{5.9}
\]
Then it follows
\[
E_m(t) + 2b \int_0^t \left\| \tilde{u}_m(\tau) \right\|^2 d\tau = E_m(0), \quad t \in [0, t_m], \tag{5.10}
\]
where (note the definition of \( J \) (see (2.13)))
\[
E_m(t) = \frac{1}{2} \left( \left\| \tilde{u}_m(t) \right\|^2_{H^2} + \left\| u_m(t) \right\|^2_{H^2} \right)
- \frac{1}{2} \int_{\Omega} |u_m(x, t)|^2 \log |u_m(x, t)| dx + \frac{1}{4} \left\| u_m \right\|^2
= \frac{1}{2} \left\| \tilde{u}_m(t) \right\|^2_{H^2} + J(u_m(t)), \tag{5.11}
\]
and
\[
E_m(0) = \frac{1}{2} \left( \left\| u_m \right\|^2_{H^2} + \left\| u_{m0} \right\|^2_{H^2} \right)
- \frac{1}{2} \int_{\Omega} |u_{m0}(x)|^2 \log |u_{m0}(x)| dx + \frac{1}{4} \left\| u_{m0} \right\|^2. \tag{5.12}
\]
By (5.4) and (5.5), it follows that
\[
\lim_{m \to \infty} E_m(0) = E(0) < d.
\] (5.13)

Then \(E_m(0) < d\) for \(m\) large enough.

Next, we claim \(u_m(t) \in \mathcal{W}\) for \(0 \leq t \leq t_m\) and \(m\) large enough, where the set \(\mathcal{W}\) is defined in (2.20). Firstly, by (5.11) and (5.10), we have \(J(u_m(t)) \leq E_m(t) \leq E_m(0) < d\) for all \(t \in [0, t_m]\) and \(m\) large enough. Since \(I(u_0) > 0\), in view of (5.4), we get \(I(u_{m0}) > 0\) for \(m\) large enough. Then \(u_{m0} \in \mathcal{W}\). So, if the claim is not true, by the definition of \(\mathcal{W}\), there is a \(t_0 \in (0, t_m]\) such that \(u_m(t_0) \neq 0\) and \(I(u_m(t)) > 0\) for all \(t \in [0, t_0]\) and \(m\) large enough, then \(u_m(t_0) \in \mathcal{N}\) (see the definition of \(\mathcal{N}\) in (2.19)). By the definition of \(d\) (see (2.18)), (5.11), and (5.10) we get
\[
d \leq J(u_m(t_0)) \leq E_m(t_0) \leq E_m(0) < d,
\]
a contradiction. Therefore, the claim is true, i.e., \(u_m(\cdot) \in \mathcal{W}\) for \(0 \leq t \leq t_m\) and \(m\) large enough.

By using (2.15) and \(u_m(\cdot, t) \in \mathcal{W}\) for \(0 \leq t \leq t_m\), we obtain that
\[
d > J(u_m(t)) = \frac{1}{2} I(u_m(t)) + \frac{1}{4} \|u_m(t)\|^2, \quad t \in [0, t_m],
\] (5.14)

that is
\[
\|u_m(t)\|^2 < 4d, \quad t \in [0, t_m].
\] (5.15)

Using the definition of \(I\) (see (2.14)) and the logarithmic Sobolev inequality (see (4.3)), we obtain, for any \(\omega > 0\),
\[
\|u_m(t)\|_{H^2_0}^2 = 2I(u_m(t)) + 2\int_\Omega u_m^2(x, t) \log |u_m(x, t)| dx - \|u_m(t)\|_{H^2_0}^2 \\
\leq 2I(u_m(t)) + [2 \log \|u_m(t)\| - n(1 + \log \omega)] \|u_m(t)\|^2 \\
+ \frac{\omega^2}{\pi(\beta + \alpha\lambda_1)} \|u_m(t)\|_{H^2_0}^2 - \|u_m(t)\|_{H^2_0}^2.
\] (5.16)

By taking \(\omega = \sqrt{\pi(\beta + \alpha\lambda_1)}\) in the above inequality, it follows from (2.15), (5.15), and (5.14) that
\[
\|u_m(t)\|_{H^2_0}^2 \leq 4J(u_m(t)) + \left[2 \log \|u_m(t)\| - 1 - n(1 + \log \sqrt{\pi(\beta + \alpha\lambda_1)})\right] \|u_m(t)\|^2 \\
< 4J(u_m(t)) + \left[\log(4d) - 1 - n(1 + \log \sqrt{\pi(\beta + \alpha\lambda_1)})\right] \|u_m(t)\|^2 \\
\triangleq 4J(u_m(t)) + \sigma \|u_m(t)\|^2 \\
< C_d \triangleq \begin{cases} 
4d, & \sigma < 0; \\
4d + 4d\sigma, & \sigma \geq 0, 
\end{cases} \quad t \in [0, t_m],
\] (5.17)

where
\[
\sigma = \log(4d) - 1 - n\left[1 + \log \sqrt{\pi(\beta + \alpha\lambda_1)}\right].
\] (5.18)

Then, by (5.17) and (2.16), we get that
\[
\int_\Omega |u_m(x, t)|^2 \log |u_m(x, t)| dx \leq \frac{1}{2} e^{-1} |\Omega| + \frac{\Theta^{2+2\epsilon}}{2\epsilon} C_d^{1+\epsilon}, \quad t \in [0, t_m].
\] (5.19)
Using (5.19), (5.9), we obtain that
\[
\frac{1}{2} (\| \dot{u}_m(t) \|_{H^2}^2 + \| u_m(t) \|_{H^6_0}^2) + \frac{1}{4} \| u_m(t) \|_2^2 + 2b \int_0^t \| u_m(\tau) \|_2^2 d\tau \\
\leq A + \frac{1}{4} \left[ e^{-1} |\Omega| + \frac{\Theta^{2+2\epsilon}}{\epsilon} C_{d+\epsilon}^1 \right], \quad t \in [0, t_m],
\]
where
\[
A = \frac{1}{2} (\| u_{m1} \|_{H^4}^2 + \| u_{m0} \|_{H^6_0}^2) - \frac{1}{2} \int_\Omega |u_{m0}(x)|^2 \log |u_{m0}(x)| dx + \frac{1}{4} \| u_{m0} \|_2^2.
\]
Then, the uniformly boundedness of the norms \( \| \dot{u}_m(t) \|_{H^4} \) and \( \| u_m(t) \|_{H^6_0} \) on \([0, t_m]\) is obtained. Using the Bessel inequality, we have \( \sum_{k=1}^m |a_{km}(t)|^2 \) and \( \sum_{k=1}^m |\dot{a}_{km}(t)|^2 \) are uniformly bounded on \([0, t_m]\), which, together with (5.7), implies \( \sum_{k=1}^m |\dot{a}_{km}(t)|^2 \) is uniformly bounded on \([0, t_m]\), and then \( a_{km}(t) \) can be extended to \([0, \infty)\) for each \( m \).

Now we pass to the limit \( m \uparrow \infty \). Note (2.11), the estimate of (5.20) indicates that a subsequence of \( \{ u_m \} \), which we still denote by \( \{ u_m \} \), can be extracted such that, for any \( \bar{T} \in (0, \infty) \),
\[
u_m \rightarrow u \text{ weakly-star in } L^\infty (0, \bar{T}; H^2_0),
\]
(5.21)
\[
u_m \rightarrow u \text{ weakly in } L^2 (0, \bar{T}; H^2_0),
\]
(5.22)
\[
\dot{u}_m \rightarrow v \text{ weakly-star in } L^\infty (0, \infty; L^2),
\]
(5.23)
\[
\dot{u}_m \rightarrow v \text{ weakly in } L^2 (0, \bar{T}; L^2),
\]
(5.24)
and \( \dot{u} = v \) (see [16]).

Since the embedding
\[
\{ \phi \in L^2 (0, \bar{T}; H^2_0), \; \phi_t \in L^2 (0, \bar{T}; L^2) \} \hookrightarrow L^2 (0, \bar{T}; L^2)
\]
is compact (see [40, Theorem 3.1.1]), it follows from (5.22) and (5.24) that
\[
u_m \rightarrow u \text{ strongly in } L^2 (0, \bar{T}; L^2),
\]
(5.25)
for any \( \bar{T} \in (0, \infty) \). So, extracting a further subsequence if necessary, \( u_m(x, t) \rightarrow u(x, t) \) a.e. \( x \in \Omega \) and \( t \in [0, \infty) \); hence,
\[
\int_\Omega (u_m(x, t) \log |u_m(x, t)|) dx \rightarrow \int_\Omega u(x, t) \log |u(x, t)| dx \quad \text{for a.e. } x \in \Omega \text{ and } t \in [0, \infty).
\]
(5.26)

On the other hand, similar to (2.16), let \( \Theta \) be the optimal Sobolev constant of \( H^2_0 \hookrightarrow L^{2+2\epsilon} \), we have (note (5.17))
\[
\int_\Omega (u_m(x, t) \log |u_m(x, t)|)^2 dx \\
= \int_{\{x \in \Omega : |u_m(x, t)| \leq 1\}} (u_m(x, t) \log |u_m(x, t)|)^2 dx \\
+ \int_{\{x \in \Omega : |u_m(x, t)| > 1\}} (u_m(x, t) \log |u_m(x, t)|)^2 dx
\]
\[
\leq e^{-2}|\Omega| + \frac{1}{e^2} \int_{x \in \Omega: |u_m(x,t)| > 1} |u_m(x,t)|^{2+2\epsilon} dx \leq e^{-2}|\Omega| + \frac{\theta^{2+2\epsilon}}{e^2} \|u_m(t)\|^{2+2\epsilon}
\]
\[
\leq e^{-2}|\Omega| + \frac{\theta^{2+2\epsilon}}{e^2} C_d^{1+\epsilon}, \quad 0 \leq t < \infty,
\]
which, together with (5.26), implies
\[
u_m \log |\nu_m| \to u \log |u| \text{ weakly-star in } L^\infty(0, \infty; L^2).
\]
(5.27)

For \( k = 1, 2, \ldots, m \), it follows from the definition of \( \| \cdot \|_{H_0^2} \) (see (2.5)) and (2.2) that
\[
\|e_k\|_{H_0^2}^2 = \|e_k\|^2 + \beta \|\nabla e_k\|^2 + \alpha \|\Delta e_k\|^2 = 1 + \beta \lambda_k + \alpha \lambda_k^2 \geq 1 + \beta \lambda_1 + \alpha \lambda_1^2.
\]
Let \( \varphi_m \equiv ((-\Delta)^{-1} + aI)u_m = (I + a\Delta)(-\Delta)^{-1}u_m \), then it follows from (5.3), the definition of \( u_m \) (see (5.2)), (5.20), and (5.27) that
\[
\|\varphi_m(t)\|_{H^{-2}} = \sup_{k=1,2,\ldots,m} \frac{\langle \varphi_m, e_k \rangle_{H^{-2},H_0^2}}{\|e_k\|_{H_0^2}} 
\]
\[
\leq \frac{1}{1 + \beta \lambda_1 + \alpha \lambda_1^2} \sup_{k=1,2,\ldots,m} \left( -2b\hat{u}_m - \beta \Delta u_m + u_m - u_m \log |u_m|, e_k \right) - \alpha (\Delta u_m, \Delta e_k) 
\]
\[
\leq C \left( \|\hat{u}_m(t)\| + \|u_m(t)\|_{H_0^2} + \|u_m(t) \log |u_m(t)|\| \right) < \infty,
\]
for any \( t \in [0, \infty) \), i.e., \( \varphi_m \in L^\infty(0, \infty; H^{-2}) \). Note \((-\Delta)(I - a\Delta)^{-1}\) is a bounded liner operator from \( H^{-2}(\Omega) \) to itself, then
\[
\bar{u}_m = (-\Delta)(I - a\Delta)^{-1}\varphi_m \in L^\infty(0, \infty; H^{-2}),
\]
and then a subsequence of \( \{u_m\} \), which we still denote by \( \{u_m\} \), can be extracted such that
\[
\bar{u}_m \to \bar{u} \text{ weakly-star in } L^\infty(0, \infty; H^{-2}).
\]
(5.28)

Moreover, noting that \((-\Delta)^{-\frac{1}{2}}\) is a bounded linear operator, by (2.26) and (5.28), we have
\[
(-\Delta)^{-\frac{1}{2}} \bar{u}_m \to (-\Delta)^{-\frac{1}{2}} \bar{u} \text{ weakly-star in } L^\infty(0, \infty; H^{-2}).
\]
(5.29)

For any \( \psi \in C[0, \infty) \), \( \varphi_k \equiv \psi(t)e_k \in C([0, \infty); H_0^2) \), multiplying both sides of (5.3) by \( \psi(t) \) we get
\[
((-\Delta)^{-1}\bar{u}_m + a\bar{u}_m + 2b\bar{u}_m + \alpha \Delta^2 u_m - \beta \Delta u_m + u_m, \varphi_k) = (u_m \log |u_m|, \varphi_k),
\]
which implies
\[
((-\Delta)^{-\frac{1}{2}} \bar{u}_m, (-\Delta)^{-\frac{1}{2}} \varphi_k)_{H^{-2},H_0^2} + a(\bar{u}_m, \varphi_k)_{H^{-2},H_0^2} + (u_m, \varphi_k) + \beta(\nabla u_m, \nabla \varphi_k) + 2b(\bar{u}_m, \varphi_k) + \alpha(\Delta u_m, \Delta \varphi_k) - (u_m \log |u_m|, \varphi_k) = 0.
\]
Then, in view of (5.21), (5.23), (5.27), (5.28) and (5.29), taking \( m \uparrow \infty \) in the above equality, we get, for a.e. \( t \in [0, \infty) \) and \( k = 1, 2, \ldots, \)
\[
\left((-\Delta)^{-\frac{1}{2}} \bar{u}, (-\Delta)^{-\frac{1}{2}} \varphi_k\right)_{H^{-2},H_0^2} + a(\bar{u}, \varphi_k)_{H^{-2},H_0^2} + (u, \varphi_k) + \beta(\nabla u, \nabla \varphi_k) + 2b(\bar{u}, \varphi_k) + \alpha(\Delta u, \Delta \varphi_k) - (u \log |u|, \varphi_k) = 0.
\]

It is obvious the above equality holds for any linear combination of the \( \varphi_k \)'s. Then, (2.24) holds.
Then we have 1. \( \omega \) for any \( \tilde{\omega} \) of (5.23), (5.27), (5.28), and (5.29). Let \( V = H_0^2 \), \( H = L^2 \) and \( A = \frac{1}{a}(I - \beta \Delta + \alpha \Delta^2) \). Then \( V \subset H \), \( V \) is dense in \( H \), and \( V \subset H \subset V' = H^{-2} \). The bilinear form \( \omega(\cdot, \cdot) \) associate to \( A \) is

\[
\omega(u, v) = (Au, v)_{V'}, V = \frac{1}{a} \int_\Omega (uv + \beta \nabla u \cdot \nabla v + \alpha \Delta u \Delta v) \, dx = \frac{1}{a} (u, v)_{H_0^2}, \forall u, v \in V.
\]

Then we have 1. \( \omega(u, v) = \omega(v, u) \); 2. and

\[
|\omega(u, v)| = \frac{1}{a} \|(u, v)_{H_0^2}\| \leq \frac{1}{a} \|u\|_V \|v\|_V, \\
\omega(u, u) = \frac{1}{a} (u, u)_{H_0^2} = \frac{1}{a} \|u\|_V^2.
\]

By using the above notations, (5.30) can be written as the following abstract ordinary differential equation

\[
\begin{cases}
\ddot{u} + Au = F \in L^2(0, T; H); \\
u_0 = u_0 \in V, \dot{u}(0) = u_1 \in H.
\end{cases}
\]

By Lemma 4.2 and arbitrariness of \( \tilde{T} \),

\[
u \in C([0, \infty); V) \cap C^1([0, \infty); H) = C([0, \infty); H_0^2) \cap C^1([0, \infty; L^2]).
\]

Then (3.1) holds. By the construction of \( u \) (note \( u_m \in W \)), one can easily see \( u \in \bar{W} \) for \( 0 \leq t < \infty \).

Finally, we will prove the identity (3.2). By Lemma 4.1, we get

\[
(a\dot{u} + (I - \beta \Delta + \alpha \Delta^2)u, \dot{u})
\]

\[
= a(\ddot{u}_t + Au, \dot{u}) = \frac{a}{2} \frac{d}{dt} (\|\dot{u}(t)\|^2 + \omega(u, u)) = \frac{1}{2} \frac{d}{dt} \left( a\|\dot{u}(t)\|^2 + \|u(t)\|^2_{H_0^2} \right). \quad (5.31)
\]

Then multiplying both sides of (5.30) by \( \dot{u} \) and then integrating over \( \Omega \times [0, t] \), we get

\[
\frac{1}{2} \int_0^t \frac{d}{d\tau} \left( a\|\dot{u}(\tau)\|^2 + \|u(\tau)\|^2_{H_0^2} \right)
\]

\[
= \int_0^t (-\Delta)^{-1} \dot{u} + u \log |u| - 2b\dot{u}, \dot{u} \, d\tau
\]

\[
= \int_0^t \frac{d}{d\tau} \left[ -\frac{1}{2} \|(-\Delta)^{1/2} u_\tau(\tau)\|^2 + \frac{1}{2} \int_\Omega |u(\tau)|^2 \log |u(\tau)| \, dx - \frac{1}{4} \|u(\tau)\|^2 \right] - 2b \int_0^t \|\dot{u}(\tau)\|^2 \, d\tau,
\]

\[
\frac{d}{d\tau}\left( a\|\dot{u}(\tau)\|^2 + \|u(\tau)\|^2_{H_0^2} \right) = \int_0^t \frac{d}{d\tau} \left[ -\frac{1}{2} \|(-\Delta)^{1/2} u_\tau(\tau)\|^2 + \frac{1}{2} \int_\Omega |u(\tau)|^2 \log |u(\tau)| \, dx - \frac{1}{4} \|u(\tau)\|^2 \right] - 2b \int_0^t \|\dot{u}(\tau)\|^2 \, d\tau.
\]
i.e.,
\[
\int_0^t \frac{d}{d\tau} \left[ \frac{1}{2} \left( a \|\dot{u}(\tau)\|^2 + \left\|(-\Delta)^{\frac{1}{2}} \dot{u}(\tau)\right\|^2 + \|u(\tau)\|^2_{H_0^2} \right) \right] - \frac{1}{2} \int_{\Omega} |u(\tau)|^2 \log |u(\tau)| \, dx + \frac{1}{4} \|u(\tau)\|^2 \right] + 2b \int_0^t \|u_\tau(\tau)\|^2 d\tau = 0.
\]
Note the definition of \( E(t) \) in (2.21), the above equality can be written as
\[
\int_0^t \dot{E}(\tau) d\tau + 2b \int_0^t \|u_\tau(\tau)\|^2 d\tau = 0,
\]
which implies (3.2) holds.

**Step 2: the existence of Global solutions when \( E(0) = d \) and \( I(u_0) \geq 0 \).** Without loss of generality we assume \( \|u_0\| \neq 0 \), since the problem obviously admits a global solution \( u = u_0 \) when \( \|u_0\| = 0 \). Moreover, if \( I(u_0) = 0 \), then \( u_0 \in N \). By (2.17), we have \( d \leq J(u_0) \). On the other hand, by (2.22), we have \( d = E(0) = \frac{1}{2} \|u_1\|^2_{H^2} + J(u_0) > J(u_0) \), which is absurd. So in the following we only need to consider the case
\[
E(u_0) = d \text{ and } I(u_0) > 0.
\]
We pick up a sequence \( \{\lambda_m\}_{m=1}^{\infty} \) such that \( 0 < \lambda_m < 1, m = 1, 2, \ldots \), and \( \lambda_m \uparrow 1 \) as \( m \uparrow \infty \). Let \( u_{0m}(x) = \lambda_m u_0(x), u_{1m}(x) = \lambda_m u_1(x) \). Note \( (u_0, u_1) \in H_0^2 \times L^2 \), it is obvious that
\[
u_{0m} \rightarrow u_0 \text{ in } H_0^2 \text{ as } m \uparrow \infty \text{ and } u_{1m} \rightarrow u_1 \text{ in } L^2 \text{ as } m \uparrow \infty.
\]
By Lemma 4.3, there exists a constant \( \lambda^* > 1 \) (see (4.5)) such that \( I(\lambda^* u_0) = 0 \) and \( J(\lambda u_0) \) is increasing (decreasing) in \([0, \lambda^*] \) ([\( \lambda^*, \infty \) ]). Then, (note (2.17) (2.14),
and \( 0 < \lambda_m < 1 < \lambda^* \),
\[
E_m(0) = \frac{1}{2} \|u_{1m}\|^2_{H^2} + J(u_{0m}) = \frac{1}{2} \lambda_m^2 \|u_1\|^2_{H^2} + J(\lambda_m u_0)
\leq \frac{1}{2} \|u_1\|^2_{H^2} + J(u_0) = E(0) = d,
\]
and
\[
I(u_{m0}) = I(\lambda_m u_0) = \lambda_m^2 \|u_0\|^2_{H^2} - \lambda_m^2 \int_{\Omega} u_0^2 \log |\lambda_m u_0| \, dx
\geq \lambda_m^2 \|u_0\|^2_{H^2} - \lambda_m \log |\lambda_m| \|u_0\|^2 - \lambda_m^2 \int_{\Omega} u_0^2 \log |u_0| \, dx
\geq \lambda_m^2 \|u_0\|^2_{H^2} - \lambda_m^2 \int_{\Omega} u_0^2 \log |u_0| \, dx = \lambda_m^2 \left( \|u_0\|^2_{H^2} - \int_{\Omega} u_0^2 \log |u_0| \, dx \right)
\geq \lambda_m^2 I(u_0) > 0.
\]
Consider problem (1.1) with the boundary condition (1.3) and the initial conditions
\[
u(x, 0) = u_{0m}(x), \quad \dot{u}(x, 0) = u_{1m}(x).
\]
Then, it follows from the arguments of Step 1 that for each $m$ the problem (1.1), (1.3), (5.35) admits a global weak solution $u_m \in C([0, \infty); H_0^2)$ with $\dot{u}_m \in C([0, \infty); L^2)$, and
\[
\left\langle (-\Delta)^{-\frac{1}{2}} \dot{u}_m, (-\Delta)^{-\frac{1}{2}} \varphi \right\rangle_{H^{-2, H_0^2}} + a \langle \ddot{u}_m, \varphi \rangle_{H^{-2, H_0^2}} + (u_m, \varphi) + \beta(\nabla u_m, \nabla \varphi) + 2b(\dot{u}_m, \varphi) + \alpha(\Delta u_m, \Delta \varphi) + (u_m \log |u_m|, \varphi) = 0,
\]
holds for any $\varphi \in C([0, \infty); H_0^2)$. The remain proof is similar to Step 1.

6. Energy decay estimates. In this section, we study the polynomial or exponential energy decay estimates of weak solutions to problems (1.1), (1.2) and (1.3) under some suitable conditions, i.e., give the prove of Theorem 3.2.

Proof of Theorem 3.2. Step 1: proof of (i). In this step, we assume (1.4) holds, and $b > 0$, $E(0) < d$, and $I(u_0) \leq 0$. By Theorem 3.1, the problem of (1.1), (1.2) and (1.3) admits a global weak solution, and by the proof of Theorem 3.1, $E(t) < d$ and $I(u(t)) \geq 0$ for all $t \geq 0$. In view of the energy equality (3.2),
\[
\dot{E}(t) = -2b\|\dot{u}(t)\|^2 \leq 0, \ 0 \leq t < \infty. \quad (6.1)
\]
Hence it holds
\[
\frac{d}{dt}[(1 + t)E(t)] = E(t) + (1 + t)\dot{E}(t) \leq E(t). \quad (6.2)
\]
Integrating the above inequality form 0 to $t$, we have (note $E(t) = \frac{1}{2}\|\dot{u}(t)\|_{H^2}^2 + \frac{1}{4}I(u(t)) + \frac{1}{4}\|u(t)\|^2$),
\[
(1 + t)E(t) \leq E(0) + \int_0^t E(\tau)d\tau
\]
\[
= E(0) + \frac{1}{2} \int_0^t \|\ddot{u}(\tau)\|^2_{H^2}d\tau + \frac{1}{2} \int_0^t I(u(\tau))d\tau + \frac{1}{4} \int_0^t \|u(\tau)\|^2d\tau. \quad (6.3)
\]
By (4.5), we get
\[
\|u(t)\|^2 = \frac{I(u(t))}{\log \lambda^*}, \quad (6.4)
\]
where
\[
\lambda^* = \exp \left\{ \frac{\|u(t)\|^2_{H^2_0} - \int_{\Omega} u^2(x, t) \log |u(x, t)|dx}{\|u(t)\|^2} \right\} \geq 1,
\]
where we have used $I(u(t)) \geq 0$ in the above inequality, and it holds $\lambda^* u(t) \in \mathcal{N}$. Then, by Lemma 4.4,
\[
d = \inf_{\varphi \in \mathcal{N}} J(\varphi) \leq J(\lambda^* u(t))
\]
\[
= \frac{1}{2}(\lambda^*)^2\|u(t)\|^2_{H^2_0} - \frac{1}{2}(\lambda^*)^2 \int_{\Omega} u^2(x, t) \log |\lambda^* u(x, t)|dx + \frac{1}{4}(\lambda^*)^2\|u(t)\|^2
\]
\[
= (\lambda^*)^2 J(u(t)) - \frac{1}{2}(\lambda^*)^2 \log \lambda^* \|u(t)\|^2 \leq (\lambda^*)^2 J(u(t)), \quad (6.5)
\]
which, together with $J(u(t)) \leq E(t) \leq E(0)$ (see (2.22) and (3.2)), implies
\[
(\lambda^*)^2 \geq \frac{d}{J(u(t))} \geq \frac{d}{E(0)}.
\]
By (6.4) and (6.6), we get
\[ \|u(t)\|^2 \leq \frac{2I(u(t))}{\log(\lambda^*)^2} \leq \frac{2I(u(t))}{\log d - \log E(0)}. \]

Then it follows from (6.3) that
\[ (1 + t)E(t) \leq E(0) + \frac{1}{2} \int_0^t \|\dot{u}(\tau)\|_{H_0^2}^2 d\tau + \frac{1 + \log d - \log E(0)}{2(\log d - \log E(0))} \int_0^t I(u(\tau))d\tau. \tag{6.7} \]

Next we estimate the terms on the right-hand side of (6.7).

Firstly, we consider the term \( \int_0^t \|\dot{u}(\tau)\|_{H_0^2}^2 d\tau \). By the energy equality (3.2), \( I(u(t)) \geq 0 \), and (2.11), we have
\[ d > E(0) = E(t) + 2b \int_0^t \|u_\tau\|^2 d\tau = \frac{1}{2} \|\dot{u}(t)\|_{H_0^2}^2 + \frac{1}{2} \|\dot{u}(t)\|_{H_0^2}^2 \leq \frac{1}{2} \|\dot{u}(t)\|_{H_0^2}^2 + 2b \int_0^t \|\dot{u}(\tau)\|_{H_0^2}^2 d\tau \]
\[ \geq \frac{1}{2} \|\dot{u}(t)\|_{H_0^2}^2 + \frac{1}{4} \|u(t)\|^2 + 2b \int_0^t \|\dot{u}(\tau)\|_{H_0^2}^2 d\tau \geq \frac{1}{2} \|\dot{u}(t)\|_{H_0^2}^2 + \frac{2b\lambda_1}{a\lambda_1 + 1} \int_0^t \|\dot{u}(\tau)\|_{H_0^2}^2 d\tau, \tag{6.8} \]
which implies
\[ \int_0^t \|\dot{u}(\tau)\|_{H_0^2}^2 d\tau < 2d \int_0^t e^{-\frac{2b\lambda_1}{a\lambda_1 + 1}(t-s)} ds = \frac{d(a\lambda_1 + 1)}{2b\lambda_1} \left(1 - e^{-\frac{2b\lambda_1}{a\lambda_1 + 1}t}\right). \tag{6.9} \]

Moreover, since \( d > \frac{1}{2} \|\dot{u}(t)\|_{H_0^2}^2 + \frac{1}{4} \|u(t)\|^2 \) (see (6.8)), it follows from (2.11) that
\[ \frac{1}{2} \left( \|\dot{u}(t)\|_{H_0^2}^2 + \|u(t)\|^2 \right) \leq \frac{1}{2} \|\dot{u}(t)\|_{H_0^2}^2 + \frac{a\lambda_1 + 1}{2\lambda_1} \|u(t)\|^2 \]
\[ \leq \kappa \left( \frac{1}{2} \|\dot{u}(t)\|_{H_0^2}^2 + \frac{1}{4} \|u(t)\|^2 \right) < \kappa d, \tag{6.10} \]
where \( \kappa \) is the constant defined in (3.4), i.e.,
\[ \kappa \triangleq \max \left\{ 1, \frac{2(a\lambda_1 + 1)}{\lambda_1} \right\} = \begin{cases} 1, & \text{if } \lambda_1 > 2 \text{ and } a \leq \frac{\lambda_1 - 2}{2\lambda_1}, \\ 2(a\lambda_1 + 1), & \text{if } \lambda_1 \leq 2, \text{ or } \lambda_1 > 2 \text{ and } a > \frac{\lambda_1 - 2}{2\lambda_1}. \end{cases} \]

Secondly, we consider the term \( \int_0^t I(u(\tau))d\tau \). By (2.24),
\[ \int_0^t I(u(\tau))d\tau = \int_0^t \langle \|u(\tau)\|^2_{H_0^2} - \int_0^\Omega u^2(x, \tau) \log |u(x, \tau)| dx d\tau \]
\[ = - \int_0^t \left\langle (-\Delta)^{-\frac{1}{2}} \dot{u}(\tau), (-\Delta)^{-\frac{1}{2}} u(\tau) \right\rangle_{H^{-2}, H_0^2} d\tau \]
\[ - a \int_0^t \left\langle \dot{u}(\tau), u(\tau) \right\rangle_{H^{-2}, H_0^2} d\tau - 2b \int_0^t \langle \dot{u}(\tau), u(\tau) \rangle d\tau, \tag{6.11} \]
where

\[- \int_{0}^{t} \left\langle (-\Delta)^{-\frac{1}{2}} \dot{u}(\tau), (-\Delta)^{-\frac{1}{2}} u(\tau) \right\rangle_{H^{-\mu, H^{\mu}_{0}}} d\tau\]

\[= - \int_{0}^{t} \frac{d}{d\tau} \left\langle (-\Delta)^{-\frac{1}{2}} \dot{u}(\tau), (-\Delta)^{-\frac{1}{2}} u(\tau) \right\rangle + \int_{0}^{t} \|(-\Delta)^{-\frac{1}{2}} \dot{u}(\tau)\|^{2} d\tau\]

\[= - \left\langle (-\Delta)^{-\frac{1}{2}} \dot{u}(t), (-\Delta)^{-\frac{1}{2}} u(t) \right\rangle + \left\langle (-\Delta)^{-\frac{1}{2}} u_{1}(t), (-\Delta)^{-\frac{1}{2}} u_{0} \right\rangle + \int_{0}^{t} \|(-\Delta)^{-\frac{1}{2}} \dot{u}(\tau)\|^{2} d\tau\]

\[\leq \frac{1}{2} \|(-\Delta)^{-\frac{1}{2}} u(t)\|^{2} + \frac{1}{2} \|(-\Delta)^{-\frac{1}{2}} \dot{u}(t)\|^{2}\]

\[+ \frac{1}{2} \|(-\Delta)^{-\frac{1}{2}} u_{0}\|^{2} + \frac{1}{2} \|(-\Delta)^{-\frac{1}{2}} u_{1}\|^{2} + \int_{0}^{t} \|(-\Delta)^{-\frac{1}{2}} \dot{u}(\tau)\|^{2} d\tau, \quad (6.12)\]

and

\[- \int_{0}^{t} \left\langle \ddot{u}(\tau), u(\tau) \right\rangle_{H^{-\mu, H^{\mu}_{0}}} d\tau\]

\[= - \int_{0}^{t} \frac{d}{d\tau} \left\langle \ddot{u}(\tau), u(\tau) \right\rangle d\tau + \int_{0}^{t} \|\ddot{u}(\tau)\|^{2} d\tau = - \langle \ddot{u}(t), u(t) \rangle + \langle u_{0}, u_{1} \rangle + \int_{0}^{t} \|\ddot{u}(\tau)\|^{2} d\tau\]

\[\leq \frac{1}{2} \|u(t)\|^{2} + \frac{1}{2} \|\dot{u}(t)\|^{2} + \frac{1}{2} \|u_{0}\|^{2} + \frac{1}{2} \|u_{1}\|^{2} + \int_{0}^{t} \|\ddot{u}(\tau)\|^{2} d\tau, \quad (6.13)\]

and

\[- 2 \int_{0}^{t} \langle \ddot{u}(\tau), u(\tau) \rangle d\tau = - \int_{0}^{t} \frac{d}{d\tau} \|u(\tau)\|^{2} = - \|u(t)\|^{2} + \|u_{0}\|^{2} \leq \|u_{0}\|^{2}. \quad (6.14)\]

Then, by (6.11), (6.12), (6.13), (6.14), (6.10), (6.9), we obtain

\[\int_{0}^{t} I(u(\tau)) d\tau \leq \frac{1}{2} \|u(t)\|^{2}_{H^{\mu}} + \frac{1}{2} \|\dot{u}(t)\|^{2}_{H^{\mu}} + \frac{1}{2} \|u_{0}\|^{2}_{H^{\mu}} + \frac{1}{2} \|u_{1}\|^{2}_{H^{\mu}} + \int_{0}^{t} \|\ddot{u}(\tau)\|^{2} d\tau + 2b\|u_{0}\|^{2}\]

\[\leq \kappa d \frac{d(a\lambda_{1} + 1)}{2b\lambda_{1}} \left(1 - e^{-\frac{8b\lambda_{1}}{\pi^{2}d^{2} t}} t\right) + \frac{1}{2} \|u_{0}\|^{2}_{H^{\mu}} + \frac{1}{2} \|u_{1}\|^{2}_{H^{\mu}} + b\|u_{0}\|^{2}. \quad (6.15)\]

In view of (6.7), (6.9), and (6.15), we get

\[(1 + t) E(t) \leq E(0) + \frac{d(a\lambda_{1} + 1)}{4b\lambda_{1}} \left(1 - e^{-\frac{8b\lambda_{1}}{\pi^{2}d^{2} t}} t\right) + \frac{1}{2} \|u_{0}\|^{2}_{H^{\mu}} + \frac{1}{2} \|u_{1}\|^{2}_{H^{\mu}} + b\|u_{0}\|^{2}\]

\[+ \frac{1}{2} \|u_{0}\|^{2}_{H^{\mu}} + \frac{1}{2} \|u_{1}\|^{2}_{H^{\mu}} + b\|u_{0}\|^{2}\]
\[ E(0) + \frac{1 + \log d - \log E(0)}{2(\log d - \log E(0))} \left[ \kappa d + \frac{1}{2} \|u_0\|_H^2 + \frac{1}{2} \|u_1\|^2_H + b\|u_0\|^2 \right] \\
+ \frac{d(a\lambda_1 + 1)(1 + 2(\log d - \log E(0)))}{4b\lambda_1(\log d - \log E(0))} \left( 1 - e^{-\frac{a\lambda_1}{\kappa}t} \right), \]

i.e., (3.3) is true.

**Step 2: proof of (ii).** In this step, we assume (1.4) holds, \( E(0) \leq \min\{d, \mathcal{E}_0\} \) \((\mathcal{E}_0 \) is defined in (3.5)), and \( I(u_0) \leq 0 \). By Theorem 3.1, problem (1.1), (1.2) and (1.3) admits a global weak solution \( u(t) \) and \( u(t) \in \mathbb{W} \) for \( 0 \leq t < \infty \) (which implies \( I(u(t)) \geq 0 \) for \( 0 \leq t < \infty \)).

First of all, we define

\[ l(t) \triangleq (u(t), \dot{u}(t))_\mathcal{H} + b\|u(t)\|^2 \]

\[ = a(u(t), \dot{u}(t)) + \left( (-\Delta)^{-\frac{1}{2}} u(t), (-\Delta)^{-\frac{1}{2}} \dot{u}(t) \right) + b\|u(t)\|^2, \quad 0 \leq t < \infty. \quad (6.16) \]

Since \( u \in C([0, \infty); H^2_0 \cap C^1([0, \infty); L^2) \cap C^2([0, \infty); H^{-2}) \), \( \dot{l}(t) \) exists, and by (2.24)

\[ \dot{l}(t) = a\|\dot{u}(t)\|^2 + a(\dot{u}(t), u(t))_{H^{-2}, H^2} + \|(-\Delta)^{-\frac{1}{2}} \dot{u}(t)\|^2 \]

\[ + \left( (-\Delta)^{-\frac{1}{2}} \ddot{u}(t), (-\Delta)^{-\frac{1}{2}} u(t) \right)_{H^{-2}, H^2} + 2b(u(t), \dot{u}(t)) \]

\[ = \|\dot{u}(t)\|^2_H - \|u(t)\|^2_{H^2} + \int_\Omega u^2(x,t) \log |u(x,t)| dx. \quad (6.17) \]

We let

\[ L(t) \triangleq E(t) + \varepsilon l(t), \quad (6.18) \]

where \( \varepsilon > 0 \) is a small enough constant to be determined later.

By the (2.22), (2.15) and \( I(u(t)) \geq 0 \), we know that

\[ E(t) = \frac{1}{2} \|\dot{u}(t)\|^2_H + \frac{1}{4} I(u(t)) + \frac{1}{2} \|u(t)\|^2 \geq \frac{1}{2} \|\dot{u}(t)\|^2_H + \frac{1}{4} \|u(t)\|^2. \quad (6.19) \]

By Cauchy-Schwartz’s inequality, (2.11), it holds

\[ |l(t)| \leq \|u(t)\|_\mathcal{H}\|\dot{u}(t)\|_\mathcal{H}^2 + b\|u(t)\|^2 \leq \frac{1}{2} \left( \|u(t)\|_\mathcal{H}^2 + \|\dot{u}(t)\|_\mathcal{H}^2 \right) + b\|u(t)\|^2 \]

\[ \leq \frac{1}{2} \left( a + \lambda_1^{-1} \right) \|u(t)\|^2 + \frac{1}{2} \|\dot{u}(t)\|^2_H + b\|u(t)\|^2 \]

\[ \leq \frac{1}{2} \left( a + 2b + \lambda_1^{-1} \right) \|u(t)\|^2 + \frac{1}{2} \|\dot{u}(t)\|^2_H = \left( 2 \left( a + 2b + \lambda_1^{-1} \right) \right) \frac{1}{4} \|u\|^2 + \frac{1}{2} \|\dot{u}\|^2_H \]

\[ \leq \max \left\{ 1, 2(a + 2b + \lambda_1^{-1}) \right\} E(t). \]

Choosing \( \varepsilon \) small enough such that (3.9) holds, i.e.,

\[ \varepsilon \rho < 1, \quad (6.20) \]

then by (6.18), we have

\[ (1 - \varepsilon \rho)E(t) \leq L(t) \leq (1 + \varepsilon \rho)E(t), \quad 0 \leq t < \infty. \quad (6.21) \]

By (6.1), (6.17), and (2.11),

\[ \dot{L}(t) = \dot{E}(t) + \varepsilon \dot{l}(t) \]
By the definition of $E(t)$ (see (2.21)), we have
\[
0 = -4\varepsilon^2 E(t) + 2\varepsilon^2 \|\dot{u}(t)\|_{\mathcal{H}_t}^2 + 2\varepsilon^2 \|u(t)\|_{\mathcal{H}_0}^2
\]
\[
- 2\varepsilon^2 \int_{\Omega} u^2(x, t) \log |u(x, t)| dx + \varepsilon^2 \|u(t)\|^2.
\]  
(6.23)

By adding (6.23) to (6.22), and by using (2.7), we obtain
\[
\dot{L}(t) \leq -4\varepsilon^2 E(t) + \left(2\varepsilon^2 + \varepsilon - \frac{2b\lambda_1}{1 + a\lambda_1}\right) \|\dot{u}(t)\|_{\mathcal{H}_t}^2 + \left(2\varepsilon^2 - \varepsilon\right) \|u(t)\|_{\mathcal{H}_0}^2
\]
\[
+ (\varepsilon - 2\varepsilon^2) \int_{\Omega} u^2(x, t) \log |u(x, t)| dx + \varepsilon^2 \|u(t)\|^2
\]
\[
\leq -4\varepsilon^2 E(t) + \left(2\varepsilon^2 + \varepsilon - \frac{2b\lambda_1}{1 + a\lambda_1}\right) \|\dot{u}(t)\|_{\mathcal{H}_t}^2 + \left(\frac{\varepsilon^2}{1 + \beta\lambda_1 + \alpha\lambda_1^2} + 2\varepsilon^2 - \varepsilon\right) \|u(t)\|_{\mathcal{H}_0}^2
\]
\[
+ (\varepsilon - 2\varepsilon^2) \int_{\Omega} u^2(x, t) \log |u(x, t)| dx.
\]  
(6.24)

Now, we estimate the term of \(\int_{\Omega} u^2(x, t) \log |u(x, t)| dx\). Using the fact \(\theta^{-\varepsilon} \log \theta \leq (\varepsilon\theta)^{-1}\) for any \(\theta, \varepsilon > 0\), and the conditions of Theorem 3.2 (i.e., (3.10))
\[
\int_{\Omega} u^2(x, t) \log |u(x, t)| dx
\]
\[
= \int_{\Omega} |u(x, t)|^{2+\varepsilon} |u(x, t)|^{-\varepsilon} \log |u(x, t)| dx \leq (\varepsilon\theta)^{-1} \|u(t)\|_{L^{2+\varepsilon}}^{2+\varepsilon}
\]
\[
\leq \frac{\Theta^{2+\varepsilon}}{\varepsilon} \|u(t)\|_{\mathcal{H}_t}^{2+\varepsilon} \|u(t)\|_{\mathcal{H}_0} \|u(t)\|_{H^2}.
\]  
(6.25)

By using the energy equality (3.2), (2.22), (2.15), and \(I(u(t)) \geq 0\), we obtain
\[
E(0) \geq E(t) \geq J(u(t)) = \frac{1}{2} I(u(t)) + \frac{1}{4} \|u(t)\|^2 \geq \frac{1}{4} \|u(t)\|^2,
\]  
(6.26)

which yields
\[
\|u(t)\|^2 \leq 4E(0).
\]  
(6.27)

By (6.26) and (6.27), similar to (5.17), we have
\[
\|u(t)\|_{\mathcal{H}_0} \leq 4E(0) + \sigma \|u(t)\|^2 \leq \begin{cases} 4E(0), & \sigma \leq 0, \\ 4(1 + \sigma)E(0), & \sigma > 0, \end{cases}
\]  
(6.28)

where
\[
\sigma = \log(4d) - 1 - n \left[1 + \log \sqrt{\pi(\beta + \alpha\lambda_1)}\right].
(i) When \( \sigma \leq 0 \), it holds \( \| u(t) \|_{H_0^2} \leq (4E(0))^{\frac{7}{2}} \) (see (6.28)), then, by (6.25), we have
\[
\int_{\Omega} u^2(x,t) \log |u(x,t)| dx \leq \Theta^{2+\epsilon} \frac{4E(0)^{\frac{7}{2}}}{\epsilon \epsilon} \| u(t) \|^2_{H_0^2}. \tag{6.29}
\]
Then if \( \epsilon \) is small enough such that \( \epsilon \leq 1/2 \) (see (3.11)), by (6.29) and (6.24), we derive that
\[
\dot{L}(t) \leq -4\epsilon^2 E(t) + \left(2\epsilon^2 + \epsilon - \frac{2b\lambda_1}{1 + a\lambda_1}\right) \| \dot{u}(t) \|^2_{H_0^2} \]
\[
+ \left( \frac{\epsilon^2}{1 + \beta\lambda_1 + \alpha\lambda_1^2} + 2\epsilon^2 - \epsilon + \epsilon(1 - 2\epsilon) \frac{\Theta^{2+\epsilon}}{\epsilon \epsilon} (4E(0))^{\frac{7}{2}} \right) \| u(t) \|^2_{H_0^2} \]
\[
= -4\epsilon^2 E(t) + \left(2\epsilon^2 + \epsilon - \frac{2b\lambda_1}{1 + a\lambda_1}\right) \| \dot{u}(t) \|^2_{H_0^2} \]
\[
+ \epsilon \left( \left( \frac{1}{1 + \beta\lambda_1 + \alpha\lambda_1^2} + 2 \right) \epsilon - 1 + (1 - 2\epsilon) \frac{\Theta^{2+\epsilon}}{\epsilon \epsilon} (4E(0))^{\frac{7}{2}} \right) \| u(t) \|^2_{H_0^2}. \tag{6.30}
\]
If \( \epsilon > 0 \) is small enough such that
\[
2\epsilon^2 + \epsilon - \frac{2b\lambda_1}{1 + a\lambda_1} \leq 0, \tag{6.31}
\]
and
\[
\left( \frac{1}{1 + \beta\lambda_1 + \alpha\lambda_1^2} + 2 \right) \epsilon \leq \delta, \tag{6.32}
\]
for some \( \delta \in (0, 1) \), i.e. (see (3.11)),
\[
\epsilon \leq \min \left\{ \sqrt{\frac{b\lambda_1}{1 + a\lambda_1}} + \frac{1}{16} - \frac{1}{4}, \delta \left( \frac{1}{1 + \beta\lambda_1 + \alpha\lambda_1^2} + 2 \right)^{-1} \right\}, \tag{6.33}
\]
it follows from (6.30) that
\[
\dot{L}(t) \leq -4\epsilon^2 E(t) + \epsilon \left( \delta - 1 + (1 - 2\epsilon) \frac{\Theta^{2+\epsilon}}{\epsilon \epsilon} (4E(0))^{\frac{7}{2}} \right) \| u(t) \|^2_{H_0^2}. \tag{6.34}
\]
So, if \( E(0) \) is small enough such that
\[
\delta - 1 + (1 - 2\epsilon) \frac{\Theta^{2+\epsilon}}{\epsilon \epsilon} (4E(0))^{\frac{7}{2}} \leq 0,
\]
i.e. (see (3.5)),
\[
E(0) \leq \frac{1}{4} \left( \frac{(1 - \delta)\epsilon \epsilon}{(1 - 2\epsilon)\Theta^{2+\epsilon}} \right)^{\frac{7}{2}},
\]
we get from (6.34) and (6.21) that
\[
\dot{L}(t) \leq -4\epsilon^2 E(t) \leq -\frac{4\epsilon^2}{1 + \varrho \epsilon} L(t), \; 0 \leq t < \infty. \tag{6.35}
\]
By Gronwall’s inequality and (6.35), then we obtain that
\[
L(t) \leq L(0) e^{-\frac{4\epsilon^2}{1 + \varrho \epsilon} t}. \tag{6.36}
\]
Using (6.21), (6.36), (6.18), and (6.16), we get
\[
E(t) \leq \frac{1}{1 - \varrho \epsilon} L(t) \leq \frac{L(0)}{1 - \varrho \epsilon} e^{-\frac{4\epsilon^2}{1 + \varrho \epsilon} t} = \frac{E(0) + \varrho l(0)}{1 - \varrho \epsilon} e^{-\frac{4\epsilon^2}{1 + \varrho \epsilon} t}.
\]
Lemma 7.1. Similar to Theorem 3.1, the following lemma can be proved by standard Galerkin method.

\[ E(0) + \varepsilon (u_0, u_1)_H + b\varepsilon \|u_0\|^2 \leq \frac{4\varepsilon^2}{1 - \overline{\varepsilon}} e^{-\frac{4\varepsilon^2 t}{1 - \overline{\varepsilon}}} , \]  

(6.37)
i.e., (3.7) is true.

(ii) When \( \sigma > 0 \), it holds \( \|u\|^2_{H^2_0} \leq 4(1 + \sigma)E(0) \) (see (6.28)), that is

\[ \|u\|^2_{H^2_0} \leq |4(1 + \sigma)E(0)|^{\frac{1}{2}} , \]

(6.38)

By similar argument as (i), one can show if \( \varepsilon > 0 \) is small enough such that \( \varepsilon \leq 1/2 \) and (6.33) holds, and (see (3.5))

\[ E(0) \leq \frac{1}{4(1 + \sigma)} \left( \frac{(1 - \delta)\varepsilon}{(1 - 2\varepsilon)^2 + \sigma} \right)^{\frac{1}{2}} , \]

then (6.37) is true. The proof of Theorem 3.2 is complete. \( \square \)

7. Blow-up at \( \infty \). In this section, we study the infinite time blow-up results. Similar to Theorem 3.1, the following lemma can be proved by standard Galerkin method.

Lemma 7.1. Let (1.4) hold. For any \((u_0, u_1) \in H^2_0 \times L^2\), there exists a constant \( T_{u_0, u_1} > 0 \) depending only on \( \|u_0\|_{H^2_0} \) and \( \|u_1\| \) such that problem (1.1), (1.2) and (1.3) admits a weak solution on \([0, T_{u_0, u_1}]\), and the existence interval \([0, T_{u_0, u_1}]\) of the solution can be extended maximally to \([0, T]\).

Proof of Theorem 3.3. The proof will be divided two steps:

Step 1: The maximal existence time of solutions can be extended to infinite. Let \( u = u(t) \) be a weak solution of problem (1.1), (1.2) and (1.3) with \((u_0, u_1) \in H^2_0 \times L^2\) satisfying \( E(0) < d \) and \( I(u_0) < 0 \), and we assume that \( T \) is the maximal existence time of \( u \). To complete the proof of this step, we only need to show that there is a continuous function \( \kappa(t) : [0, \infty) \to [0, \infty) \) such that \( \kappa(t) < \infty \) for all \( t \in [0, \infty) \) and

\[ \|\dot{u}(t)\|^2 + \|u(t)\|^2_{H^2_0} \leq \kappa(t), \quad t \in [0, T) . \]  

(7.1)

In fact, we may claim that \( T = \infty \) if (7.1) holds. Indeed, if \( T < \infty \), it follows from (7.1) that

\[ \|\dot{u}(t)\|^2 + \|u(t)\|^2_{H^2_0} \leq \mathcal{M} := \max_{0 \leq t \leq T} \kappa(t) < \infty , \quad t \in [0, T) . \]

Then by \( u \in C\left([0, T); H^2_0(\Omega)\right) \cap C^1\left([0, T); L^2(\Omega)\right) \), we get \( u : [0, T) \to H^2_0(\Omega) \) and \( \dot{u} : [0, T) \to L^2(\Omega) \) are both uniformly continuous. So the limits \( \|u(T)\|_{H^2_0} := \lim_{t \uparrow T} \|u(t)\|_{H^2_0} \) and \( \|\dot{u}(T)\| := \lim_{t \uparrow T} \|\dot{u}(t)\| \) exist, and \((u(T), \dot{u}(T)) \in H^2_0(\Omega) \times L^2(\Omega) \). Taking \((u(T), \dot{u}(T))\) as the initial value, by Lemma 7.1, one can show problem (1.1), (1.2) and (1.3) admits a weak solution on \([0, T + T_{u(T), \dot{u}(T)}]\), which contradicts that \( T \) is the maximal existence time.

Since \( E(0) < d \), then by (2.22) and (3.2), we have

\[ J(u(t)) \leq E(t) \leq E(0) < d , \quad t \in [0, T) . \]  

(7.2)

We claim

\[ I(u(t)) < 0 , \quad t \in [0, T) . \]  

(7.3)
In fact, if the claim is not true, then, in view of $I(u(0)) = I(u_0) < 0$, there must exist a $t_1 \in [0, \infty)$ such that $I(u(t_1)) = 0$ and $I(u(t)) < 0$ for all $t \in [0, t_1)$. By logarithmic Sobolev inequality (4.3), for any $t \in [0, t_1)$ and any $\omega > 0$,

$$\|u(t)\|_{H^2_0}^2 < \int_{\Omega} u^2(t) \log |u(t)| \, dx$$

$$\leq \frac{\omega^2}{2\pi(\beta + \alpha \lambda_1)} \|u(t)\|_{H^2_0}^2 - n \left(1 + \log \omega\right) \|u(t)\|^2 + \log \|u(t)\| \|u(t)\|^2.$$ 

By taking $\omega = \sqrt{2\pi(\beta + \alpha \lambda_1)}$ in the above inequality, we get

$$\|u(t)\| > \sigma_0 \triangleq \exp \left(\frac{n}{2} \left(1 + \log \left(\sqrt{2\pi(\beta + \alpha \lambda_1)}\right)\right)\right) = \left[2\pi(\beta + \alpha \lambda_1)\right]^{\frac{n}{2}} e^{\frac{n}{2}}, \quad 0 \leq t < t_1.$$

Then we have $\|u(t_1)\| = \lim_{t \uparrow t_1} \|u(t)\| \geq \sigma_0$, which, together with $I(u(t_1)) = 0$ implies $u(t_1) \in \mathcal{N}$ (see (2.19)). So, by (2.18), $J(u(t_1)) \geq d$, which contradicts (7.2). So, the claim is true.

Next, we prove (7.1). Firstly, we define a functional as follows:

$$G(t) = \|u(t)\|_{H^2_0}^2 + 2b \int_0^t \|u(\tau)\|^2 \, d\tau$$

$$= a \|u(t)\|^2 + \|(-\Delta)^{-\frac{3}{2}} u(t)\|^2 + 2b \int_0^t \|u(\tau)\|^2 \, d\tau, \quad 0 \leq t < T. \quad (7.4)$$

Since $u \in C([0, T); H^2_0) \cap C^1([0, T); \mathcal{L}^2)$, $G(t) \in C^1[0, T)$, and by Cauchy-Schwartz’s inequality, we have

$$G(t) = 2a(u(t), \check{u}(t)) + 2 \left(\|(-\Delta)^{-\frac{3}{2}} u(t), (-\Delta)^{-\frac{3}{2}} \check{u}(t)\|^2 \right) + 2b\|u(t)\|^2$$

$$\leq a \left(\|u(t)\|^2 + \|\check{u}(t)\|^2\right) + \left(\|(-\Delta)^{-\frac{3}{2}} u(t)\|^2 + \|(-\Delta)^{-\frac{3}{2}} \check{u}(t)\|^2 + 2b\|u(t)\|^2$$

$$= \|u(t)\|_{H^2}^2 + \|\check{u}(t)\|_{H^2}^2 + 2b\|u(t)\|^2. \quad (7.5)$$

For the functional $G(t)$ defined above, we consider the following two cases:

Case 1: there exists a $t_0 \in [0, T)$ such that for all $t \in [t_0, T)$, we have $G(t) \geq \Theta$;

Case 2: for all $t_1 \in [0, T)$, there exists $t_2 \in [t_1, T)$ such that $G(t_2) < \Theta$, where

$$\Theta = \frac{1}{a} \exp \left[-n \left(1 + \log \left(\frac{\pi(\beta + \alpha \lambda_1)}{2}\right)\right)\right]. \quad (6.6)$$

(i) Let’s first consider the Case 2. If Case 2 holds, we take $t_1 = t_{1m} = T - \frac{1}{m}$ with sufficiently large $m \in \mathbb{Z}^+$, by the assumption of Case 2, there exists a $t_{2m} \in (t_{1m}, T)$ such that $G(t_{2m}) < \Theta$. Since $G(t) \in C([0, T)$, we have

$$\lim_{t \uparrow T} G(t) = \lim_{m \uparrow \infty} G(t_{2m}) \leq \Theta.$$

So by the continuity of $G(t)$ again, there exists $\tilde{t} \in [0, T)$ and a constant $\tilde{k} \geq \Theta$ such that $G(t) \leq \tilde{k}$ for all $t \in [\tilde{t}, T)$. By the definition of $G(t)$, we obtain (note $u \in C([0, T); \mathcal{H})$

$$\|u(t)\|_{H^2}^2 \leq \tilde{k} \triangleq \max \left\{\max_{0 \leq t \leq \tilde{t}} \|u(t)\|_{H^2}^2, \tilde{k}\right\}, \quad t \in [0, T). \quad (7.7)$$
By (2.11), we know
\[ \|u(t)\|^2 \leq \frac{1}{a} \|u(t)\|^2_{H_a^2} \leq \frac{k}{a} < 1 + \frac{k}{a}, \quad t \in [0, T). \] (7.8)

Due to (2.21) and the logarithmic Sobolev inequality (4.3), we have (note the function $\tau \log \tau$ is increasing for $\tau \geq 1$), for any $\omega > 0$,

\[
E(t) = \frac{1}{2} \|\dot{u}(t)\|^2_{H^2} + \frac{1}{2} \|u(t)\|^2_{H_a^2} - \frac{1}{2} \int_\Omega u^2(x, t) \log |u(x, t)| \, dx + \frac{1}{4} \|u(t)\|^2
\]

\[
\geq \frac{1}{2} \|\dot{u}(t)\|^2_{H^2} + \frac{1}{2} \|u(t)\|^2_{H_a^2} - \frac{1}{4} \|u(t)\|^2 \log \|u(t)\|^2
\]

\[
+ \frac{n(1 + \log a^2)}{4} \|u(t)\|^2 - \frac{\omega^2}{4\pi(\beta + \alpha \lambda_1)} \|u(t)\|^2_{H_a^2}
\]

\[
= \frac{1}{2} \|\dot{u}(t)\|^2_{H^2} + \left[ \frac{1}{2} - \frac{\omega^2}{4\pi(\beta + \alpha \lambda_1)} \right] \|u(t)\|^2_{H_a^2} - \frac{1}{4} \|u(t)\|^2 \log \|u(t)\|^2
\]

\[
+ \frac{n(1 + \log a^2)}{4} \|u(t)\|^2
\]

\[
> \frac{1}{2} \|\dot{u}(t)\|^2_{H^2} + \left[ \frac{1}{2} - \frac{\omega^2}{4\pi(\beta + \alpha \lambda_1)} \right] \|u(t)\|^2_{H_a^2} - \frac{a/4^4}{a} \log \frac{a+k}{a} + \frac{n(1 + \log \sqrt{\pi(\beta + \alpha \lambda_1)})}{4} \|u(t)\|^2
\]

By taking $\omega = \sqrt{\pi(\beta + \alpha \lambda_1)}$ in the above inequality, in view of (7.8) and (2.11), we get

\[
E(t) > \frac{1}{2} \|\dot{u}(t)\|^2_{H^2} + \frac{1}{4} \|u(t)\|^2_{H_a^2} - \frac{a+k}{4a} \log \frac{a+k}{a} + \frac{n(1 + \log \sqrt{\pi(\beta + \alpha \lambda_1)})}{4} \|u(t)\|^2
\]

\[
\geq \frac{1}{2a} \|\dot{u}(t)\|^2_{H^2} + \frac{1}{4} \|u(t)\|^2_{H_a^2}
\]

(7.9)

\[
+ \left\{ \begin{array}{ll}
\frac{a+k}{4a} \log \frac{a+k}{a}, & \text{if } \pi(\beta + \alpha \lambda_1) \geq e^{-2}; \\
\frac{a+k}{4a} \log \frac{a+k}{a} + \frac{n(1 + \log \sqrt{\pi(\beta + \alpha \lambda_1)})}{4a} k, & \text{if } \pi(\beta + \alpha \lambda_1) < e^{-2},
\end{array} \right.
\]

which, together with $E(t)$ is decreasing with respect to $t$ (see (3.2)) and $E(0) < d$, that

\[
\frac{1}{2a} \|\dot{u}(t)\|^2_{H^2} + \frac{1}{4} \|u(t)\|^2_{H_a^2}
\]

\[
< M \overset{d+}{=} \text{d+} + \left\{ \begin{array}{ll}
\frac{a+k}{4a} \log \frac{a+k}{a}, & \text{if } \pi(\beta + \alpha \lambda_1) \geq e^{-2}; \\
\frac{a+k}{4a} \log \frac{a+k}{a} - \frac{n(1 + \log \sqrt{\pi(\beta + \alpha \lambda_1)})}{4a} k, & \text{if } \pi(\beta + \alpha \lambda_1) < e^{-2}.
\end{array} \right.
\]

Then (7.1) holds with $\kappa(t) = \max\{2a, 4\} M$ for $0 \leq t < \infty$.

(ii) Let case 1 hold, i.e., there exists a $t_0 \in [0, T)$ such that $G(t) \geq 1$ for all $t \in [t_0, T)$.

Let $G(t)$ be the functional defined in (7.4). By (7.5), (2.22), $E(t) \leq E(0) < d$ (see (3.2)), Lemma 4.5, the definition of $I(u)$ (see (2.14)), logarithmic Sobolev inequality (4.3), the definition of $G(t)$ (see (7.4)), and (2.11), we obtain, for any
constant $A \geq 1$, any constant $\omega > 0$, and
\[
B = \begin{cases} 
 \frac{2A}{a}, & \text{if } 0 < a < 1; \\
2A, & \text{if } a \geq 1,
\end{cases}
\]
that (note $G(t) \geq 1$ for $t_0 \leq t < T$)
\[
BG(t) \log G(t) - \hat{G}(t) \
\geq BG(t) \log G(t) - (\|u(t)\|^2_{H^2_t} + A\|\dot{u}(t)\|^2_{H^2_t} + 2b\|u(t)\|^2)
\]
\[
= BG(t) \log G(t) - \|u(t)\|^2_{H^2_t} + 2A(J(u(t)) - E(t)) - 2b\|u(t)\|^2 
\geq BG(t) \log G(t) - \|u(t)\|^2_{H^2_t} + 2A(J(u(t)) - d) - 2b\|u(t)\|^2
\]
\[
> BG(t) \log G(t) + AI(u(t)) - \|u(t)\|^2_{H^2_t} - 2b\|u(t)\|^2
\]
\[
= BG(t) \log G(t) + A\left(\|u(t)\|^2_{H^2_t} - \int_0^t \|u^2(x, t) \log |u(x, t)| dx \right) - \|u(t)\|^2_{H^2_t} - 2b\|u(t)\|^2
\]
\[
= \omega^2 \frac{2A}{a} \|u(t)\|^2_{\Omega^2} - \|u(t)\|^2_{H^2_t} - 2b\|u(t)\|^2
\]
\[
= 2A\|u(t)\|^2 \log G(t) + 2bB \int_0^t \|u(t)\|^2 dr \log G(t) + A\left[1 - \frac{\omega^2}{\frac{2A}{a}}\right] \|u(t)\|^2_{H^2_t}
\]
\[
\geq 2A\|u(t)\|^2 \log G(t) + \left[1 - \frac{\omega^2}{\frac{2A}{a}}\right] \|u(t)\|^2_{H^2_t} - A\|u(t)\|^2 \log G(t)
\]
\[
\geq 2A\|u(t)\|^2 \log G(t) - A\log a\|u(t)\|^2, \quad t_0 \leq t < T.
\]
By taking $\omega = \sqrt{2A(\beta + \alpha \lambda_1)}$ in the above inequality and noting that
\[
A\|u(t)\|^2 \log G(t) - A\log a\|u(t)\|^2, \quad t_0 \leq t < T,
\]
it follows
\[
BG(t) \log G(t) - \hat{G}(t) \
\geq 2A\|u(t)\|^2 \log G(t) - A\|u(t)\|^2 \log G(t) + A\log a\|u(t)\|^2 
\]
\[
+ \left[1 - \frac{\omega^2}{\frac{2A}{a}}\right] \|u(t)\|^2 - \left(\alpha + \lambda_1^{-1}\right) \|u(t)\|^2 - 2b\|u(t)\|^2
\]
\[
= \|u(t)\|^2 \left[A\log(aG(t)) + \left[1 - \frac{\omega^2}{\frac{2A}{a}}\right] \|u(t)\|^2 - \left(\alpha + \lambda_1^{-1}\right) \|u(t)\|^2 - 2b\right]. \quad (7.10)
\]
Since $G(t) \geq \Theta$ for $t \in [t_0, T)$, by (7.6),
\[
\log(aG(t)) \geq -\frac{n(1 + \log \sqrt{\pi(\beta + \alpha \lambda_1)})}{2} + 1,
\]
then we get from (7.10) that
\[
BG(t) \log G(t) - \dot{G}(t) \geq \|u(t)\|^2 \left[ A - (a + \lambda^{-1}_1) - 2b \right],
\]
which implies
\[
BG(t) \log G(t) - \dot{G}(t) \geq 0, \quad t \in [t_0, T), \tag{7.11}
\]
if we take
\[
A = \max \{1, (a + \lambda^{-1}_1) + 2b \}.
\]
That is,
\[
B \log G(t) \geq \frac{\dot{G}(t)}{G(t)} = \frac{d}{dt} \left[ \log G(t) \right], \quad t \in [t_0, T).
\]
By Gronwall’s inequality
\[
\log G(t) \leq e^{B(t-t_0)} \log G(t_0), \quad t \in [t_0, T),
\]
i.e.,
\[
G(t) \leq G(t_0)^{e^{B(t-t_0)}}, \quad t \in [t_0, T).
\]
Then it follows from (2.11) and the definition of $G(t)$ (see (7.4))
\[
\|u(t)\|^2 \leq \frac{1}{a} \|u(t)\|^2_{H^2_t} \leq \frac{1}{a} G(t_0)^e^{B(t-t_0)} < \frac{1}{a} G(t_0)^e^{B(t-t_0)} + 1, \quad t \in [t_0, T).
\]
In view of the above estimate, similar to (7.9), we obtain
\[
E(t) > \frac{1}{2a} \|\dot{u}(t)\|^2 + \frac{1}{4} \|u(t)\|^2_{H^2_t} - \frac{1}{4} \left( \frac{1}{a} G(t_0)^e^{B(t-t_0)} + 1 \right) \log \left( \frac{1}{a} G(t_0)^e^{B(t-t_0)} + 1 \right)
\]
\[
\geq \frac{1}{2a} \|\dot{u}(t)\|^2 + \frac{1}{4} \|u(t)\|^2_{H^2_t} - \frac{1}{4} \left( \frac{1}{a} G(t_0)^e^{B(t-t_0)} + 1 \right) \log \left( \frac{1}{a} G(t_0)^e^{B(t-t_0)} + 1 \right), \quad \text{if } \pi(\beta + \alpha \lambda_1) \geq e^{-2};
\]
\[
\left\{ \begin{array}{l}
\frac{1}{4} \left( \frac{1}{a} G(t_0)^e^{B(t-t_0)} + 1 \right) \log \left( \frac{1}{a} G(t_0)^e^{B(t-t_0)} + 1 \right), \\
\frac{1}{4} \left( \frac{1}{a} G(t_0)^e^{B(t-t_0)} + 1 \right) \log \left( \frac{1}{a} G(t_0)^e^{B(t-t_0)} + 1 \right) + \frac{n(1 + \log \sqrt{\pi(\beta + \alpha \lambda_1)})}{4a} G(t_0)^e^{B(t-t_0)},
\end{array} \right.
\]
which, together with $E(t)$ is decreasing with respect to $t$ (see (3.2)) and $E(0) < d$, that, for $t_0 \leq t < T$,
\[
\frac{1}{2a} \|\dot{u}(t)\|^2 + \frac{1}{4} \|u(t)\|^2_{H^2_t} < \bar{M}
\]
\[
\geq d + \left\{ \begin{array}{l}
\frac{1}{4} \left( \frac{1}{a} G(t_0)^e^{B(t-t_0)} + 1 \right) \log \left( \frac{1}{a} G(t_0)^e^{B(t-t_0)} + 1 \right), \\
\frac{1}{4} \left( \frac{1}{a} G(t_0)^e^{B(t-t_0)} + 1 \right) \log \left( \frac{1}{a} G(t_0)^e^{B(t-t_0)} + 1 \right) + \frac{n(1 + \log \sqrt{\pi(\beta + \alpha \lambda_1)})}{4a} G(t_0)^e^{B(t-t_0)},
\end{array} \right.
\]
\[
\text{if } \pi(\beta + \alpha \lambda_1) \geq e^{-2};
\]
\[
\text{if } \pi(\beta + \alpha \lambda_1) < e^{-2}.
\]
Then (7.1) holds with
\[
\kappa(t) = \begin{cases} 
\max_{0 \leq t \leq t_0} \left( \| \ddot{u}(t) \|^2 + \| u(t) \|^2_{H^2_0} \right), & \text{if } 0 \leq t < t_0; \\
\max \{2a, 4\} \bar{M}, & \text{if } t_0 \leq t < \infty.
\end{cases}
\]

Step 2: The solutions of the problem (1.1), (1.2) and (1.3) will blows up at \( \infty \). Let \( G(t) \) be the functional defined in (7.4). By \( u \in C([0, \infty); H^2_0) \cap C^1([0, \infty); L^2) \cap C^2([0, \infty); H^{-2}) \), (7.5), (2.24) (with \( \varphi = u \), (2.5), and (2.14), we obtain
\[
\dot{G}(t) = 2a\| \ddot{u}(t) \|^2 + 2a\langle \ddot{u}(t), u(t) \rangle_{H^{-2}, H^2_0} + 2 \left\| (-\Delta)^{-\frac{1}{2}} \ddot{u}(t) \right\|^2_{H^{-2}, H^2_0} + 4b\langle \ddot{u}(t), u(t) \rangle_{H^{-2}, H^2_0} = 2\| \ddot{u}(t) \|^2_{H^2_0} - 2\| u(t) \|^2_{H^2_0} + 2 \int_\Omega u^2(x, t) \log |u(x, t)| dx = 2\| \ddot{u}(t) \|^2_{H^2_0} - 2I(u(t)).
\]
Since \( J(\ddot{u}(t)) \leq E(t) \leq E(0) < d \), by Lemma 4.5, we get
\[
\dot{G}(t) = 2\| \ddot{u}(t) \|^2_{H^2_0} - 2I(u(t)) \geq -2I(u(t)) \geq 4(d - J(u(t))) \geq 4(d - E(0)) \geq C_0 > 0, \quad t \geq 0,
\]
that is
\[
\dot{G}(t) \geq C_0 t + \dot{G}(0),
\]
and then
\[
G(t) \geq G(0) + \dot{G}(0)t + \frac{1}{2}C_0 t^2,
\]
i.e.
\[
G(t) \geq \left\| u_0 \right\|^2_{H} + \left( 2\langle u_0, u_1 \rangle_{H} + 2b\| u_0 \|^2 \right) t + \frac{C_0}{2} t^2, \quad t \geq 0. \tag{7.13}
\]
Note the definition of \( G(t) \) in (7.4), \( u \) blows up at \( \infty \) in the sense of
\[
\lim_{t \uparrow \infty} \left( \| u(t) \|^2_{H} + 2b \int_0^t \| u(\tau) \|^2 d\tau \right) = \infty.
\]

Acknowledgments. We would like to thank the reviewers for their valuable comments and suggestions, which improve the initial version of the paper significantly.

REFERENCES

[1] J. L. Bona and R. L. Sachs, Global existence of smooth solutions and stability of solitary waves for a generalized Boussinesq equation, Commun. Math. Phys., 118 (1988), 15–29.
[2] J. Boussinesq, Théorie des ondes et des remous qui se propagent le long d’un canal rectangulaire horizontal, en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond, J. Math. Pures Appl., 17 (1872), 55–108.
[3] T. Cazenave and A. Haraux, Équations d’évolution avec non linéarité logarithmique, Ann. Fac. Sci. Toulouse Math., 2 (1980), 21–51.
[4] H. Chen, P. Luo and G. W. Liu, Global solution and blow-up of a semilinear heat equation with logarithmic nonlinearity, J. Math. Anal. Appl., 422 (2015), 84–98.
[5] H. Chen and S. Y. Tian, Initial boundary value problem for a class of semilinear pseudo-parabolic equations with logarithmic nonlinearity, J. Differ. Equ., 258 (2015), 4424–4442.
[6] C. I. Christov, G. A. Maugin and M. G. Velarde, Well-posed boussinesq paradigm with purely spatial higher-order derivatives, Physical Review E Statistical Physics Plasmas Fluids and Related Interdisciplinary Topics, 54 (1996), 3621–3638.
[7] C. I. Christov, G. A. Maugin and A. V. Porubov, On boussinesq paradigm in nonlinear wave propagation, Comptes Rendus Mécanique, 335 (2007), 521–535.
[8] A. Ddé Godefroy, Existence, decay and blow-up for solutions to the sixth-order generalized Boussinesq equation, *Discrete Contin. Dyn. S.*, **35** (2015), 117–137.

[9] L. C. Evans, Graduate studies in mathematics, in *Partial Differ. Equ.*, Am. Math. Soc., 1998.

[10] L. Gross, Logarithmic Sobolev inequalities, *Amer. J. Math.*, **97** (1975), 1061–1083.

[11] Q. Y. Hu and H. W. Zhang, Initial boundary value problem for generalized logarithmic Boussinesq equation, *Math. Methods Appl. Sci.*, **40** (2017), 3687–3697.

[12] Q. Y. Hu, H. W. Zhang and G. W. Liu, Global existence and exponential growth of solution for the logarithmic Boussinesq-type equation, *J. Math. Anal. Appl.*, **436** (2016), 990–1001.

[13] V. Komornik, *Exact Controllability and Stabilization: the Multiplier Method*, Wiley Chichester, 1994.

[14] Q. Lin, Y. H. Wu and R. Loxton, On the Cauchy problem for a generalized Boussinesq equation, *J. Math. Anal. Appl.*, **353** (2009), 186–195.

[15] F. Linares, Global existence of small solutions for a generalized Boussinesq equation, *J. Differ. Equ.*, **106** (1993), 257–293.

[16] J. L. Lions, Quelques méthodes de Résolution des Problemes aux Limites Nonlinéaires, 1969.

[17] M. Liu and W. K. Wang, Global existence and pointwise estimates of solutions for the multidimensional generalized Boussinesq-type equation, *Commun. Pure Appl. Anal.*, **13** (2014), 1203–1222.

[18] Y. C. Liu and R. Z. Xu, Global existence and blow up of solutions for Cauchy problem of generalized Boussinesq equation, *Phys. D.*, **237** (2008), 721–731.

[19] Y. Liu, Instability and blow-up of solutions to a generalized Boussinesq equation, *SIAM J. Math. Anal.*, **26** (1995), 1527–1546.

[20] L. W. Ma and Z. B. Fang, Energy decay estimates and infinite blow-up phenomena for a strongly damped semilinear wave equation with logarithmic nonlinear source, *Math. Methods Appl. Sci.*, **41** (2018), 2639–2653.

[21] R. L. Pego and M. I. Weinstein, Eigenvalues, and instabilities of solitary waves, *Philos. Trans. Roy. Soc. London Ser. A.*, **340** (1992), 47–94.

[22] N. Polat and A. Erta¸s, Existence and blow-up of solution of Cauchy problem for the generalized damped Boussinesq equation, *J. Math. Anal. Appl.*, **349** (2009), 10–20.

[23] R. Temam, *Applied Mathematical Sciences*, Springer-Verlag, New York, second edition, 1997.

[24] M. Tsutsumi and T. Matabashi, On the Cauchy problem for the Boussinesq type equation, *Math. Japon.*, **36** (1991), 371–379.

[25] V. Varlamov, Existence and uniqueness of a solution to the Cauchy problem for the damped Boussinesq equation, *Math. Methods Appl. Sci.*, **19** (1996), 639–649.

[26] V. Varlamov, On the Cauchy problem for the damped Boussinesq equation, *Differ. Integral Equ.*, **9** (1996), 619–634.

[27] V. V. Varlamov, On spatially periodic solutions of the damped Boussinesq equation, *Differ. Integral Equ.*, **10** (1997), 1197–1211.

[28] V. V. Varlamov, On the initial-boundary value problem for the damped Boussinesq equation, *Discrete Contin. Dyn. S.*, **4** (1998), 451–444.

[29] V. V. Varlamov, Asymptotic behavior of solutions of the damped Boussinesq equation in two space dimensions, *Int. J. Math. Sci.*, **22** (1999), 131–145.

[30] A. M. Wazwaz, Gaussian solitary waves for the logarithmic Boussinesq equation and the logarithmic regularized Boussinesq equation, *Ocean Eng.*, **94** (2015), 111–115.

[31] S. B. Wang and X. Su, Global existence and nonexistence of the initial-boundary value problem for the dissipative Boussinesq equation, *Nonlinear Anal.*, **134** (2016), 164–188.

[32] S. B. Wang and X. Su, The Cauchy problem for the dissipative Boussinesq equation, *Nonlinear Anal. Real World Appl.*, **45** (2019), 116–141.

[33] Y. Wang, Existence and blow-up of solutions for the sixth-order damped Boussinesq equation, *Bull. Iranian Math. Soc.*, **43** (2017), 1057–1071.

[34] Y. X. Wang, Existence and asymptotic behavior of solutions to the generalized damped Boussinesq equation, *Electron. J. Differ. Equ.*, **96** (2012), 11 pp.

[35] Y. X. Wang, Asymptotic decay estimate of solutions to the generalized damped Bq equation, *J. Inequal. Appl.*, **323** (2013), 12 pp.

[36] Y. Z. Xu, Y. S. Li and Q. H. Hu, Asymptotic behavior of the sixth-order Boussinesq equation with fourth-order dispersion term, *Electron J. Differ. Equ.*, **161** (2018), 14 pp.

[37] R. Z. Xu, Cauchy problem of generalized Boussinesq equation with combined power-type nonlinearities, *Math. Methods Appl. Sci.*, **34** (2011), 2318–2328.
[38] R. Z. Xu, Y. B. Luo, J. H. Shen and S. B. Huang, Global existence and blow up for damped generalized Boussinesq equation, Acta Math. Appl. Sin. Engl. Ser., 33 (2017), 251–262.

[39] R. Y. Xue, Local and global existence of solutions for the Cauchy problem of a generalized Boussinesq equation, J. Math. Anal. Appl., 316 (2006), 307–327.

[40] S. M. Zheng, Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, Chapman & Hall/CRC, Boca Raton, FL, 2004.

Received August 2020; revised January 2021.

E-mail address: 532216902@qq.com
E-mail address: jzhou@swu.edu.cn