A new spherically-symmetric solution for a gravitational field is found in the conformally-unimodular metric. It is shown, that the surface of the black hole horizon in the standard Schwarzschild metric can be “squeezed” to a point by converting coordinates to the conformally-unimodular metric. In this new metric, there is no black hole horizon, while the naked singularity corresponds to a point massive particle. The reason for the study of this particular gauge (i.e., conformally-unimodular metric) is its relation to the vacuum energy problem. That aims to relate it to other physical phenomena (including black holes), and one could argue that they should be considered in this particular metric. That means the violation of the gauge invariance of the general theory of relativity. As a result, the nonsingular “eicheons”\footnote{The term “eicheon” refers to the fundamental work “Gravitation und Elektrizität” by Hermann Weyl where the concept of gauge field theory (Eichfeldtheorie) was invented for the first time (e.g., see \cite{1}). We would emphasize by this term the decisive role of the gauge conditions in our theory predicting an existence of extremely compact but nonsingular astrophysical objects. Moreover, the connotation with “Eichel” (that means an acorn in German) implies that “eicheon” can have an internal structure and a solid-like “surface.”} appear as the non-point compact static objects with different masses and structures. They are a final product of the stellar collapse, with the masses exceeding the Tolman-Oppenheimer-Volkoff limit.

I. INTRODUCTION

One of the most intriguing objects in the theory of general relativity (GR) is the “black hole” (BH) \cite{2,3}, which is a result of the collapse of astrophysical objects exceeding the Tolman-Oppenheimer-Volkoff limit.\footnote{Corresponding author}
with the masses exceeding the Tolman-Volkov-Oppenheimer (TVO) limit. The gravitational waves registered recently are considered as a result of the collision of massive BHs. Direct astrophysical observations also indicate the extremely compact supermassive objects in the galactic centers identifying with BH. However, such BH evidences should be considered with caution because they suggest only a presence of some compact massive astrophysical object possessing the BH properties for an external observer, but with the wholly unknown internal structure.

The strange properties of BH forced many researchers (including A. Einstein) to question the BH reality and consider these objects as a pathological artifact of GR. Several discouraging facts are well-known:

1) The first issue is the presence of BH singularity with an infinitely large density, which is physically questionable. In order to avoid a singular state, the different modifications of GR have been offered by taking into account torsion (see, for example, [9]); space-time curvature limitations [10]; or considering the gravitation as a physical tensor field which requires gauge invariance violation and non-zero graviton mass [11]; and, at least, development of quantum theories gravity, e.g., loop quantum gravity [12]. On the other hand, the BH singularity could be justified because it is “dressed,” i.e., surrounded by a horizon, making it invisible for an outside observer (the so-called “cosmic censorship” principle [13]).

2) The physical status of the “event horizon” itself could also raise the questions. However, from GR, it is merely a “one-sided membrane” (“no-return horizon”) for the free-falling observer. Nevertheless, a fact of the event horizon existence is doubted both from classical and quantum viewpoints. For example, the horizon formation relates to the stability of ultra-compact states of a substance [5]. The existence of such exotic stable phases (e.g., free-quark phase [14]) could explain the phenomenon of ultra-compact objects but with the size larger than the horizon. Then, the concept of the event horizon, as well as the unlimited gravitational collapse, are declared physically meaningless in the field formulation of gravity with a massive graviton [11, 15, 16]. However, the existence of ultra-compact objects,
which are finely larger than BH, is not disclaimed [17][18]. A quantum view on the horizon issue reveals an “information paradox,” and a ”non-cloning” of quantum states, as well as the thermodynamical problems [19–22].

Assuming the modernization of GR in its relation with the “no-BH”-hypothesis refers to the synthesis of gravity with quantum mechanics. That raises the question: what is the direction of such modernization? In this regard, one can recall the known statement by D.I. Blochintsev: “Number of facts is always enough, but fantasy is insufficient.” [23].

The key fact indicating a possible path in the forest of the alternative gravity theories is the vacuum energy problem. In GR, any spatially uniform energy density (including that of zero-point fluctuations of the quantum fields) causes the expansion of the universe. Using the Planck level of UV-cutoff results in the Planckian vacuum energy density $\rho_{\text{vac}} \sim M_p^4$ [24], which leads to the universe expanding with the Planckian rate [25]. In this sense, the vacuum energy problem is an observational fact [26].

One of the possible solutions is to build a theory of gravity, allowing an arbitrarily reference level of energy density. One such theory has long been known. That is the unimodular gravity [27–32], which admits an arbitrary cosmological constant. However, under using of the comoving momentums cutoff, the vacuum energy density scales with time as radiation [26], but not as the cosmological constant.

Recently, another theory has been suggested [33], which considers the Friedman equation defined up to some arbitrary constant. This constant corresponds to the invisible radiation and, thus, can compensate the vacuum energy. In this case, one could ask why the $k$-cutoff of comoving momentums is used instead of, for instance, a cutoff of physical momentums related to $p = k/a$ ($a$ is the universe scale factor)? The answer could be that it is relatively simple to construct a theory with the $k$-cutoff, but it is challenging to introduce the $p$-cutoff fundamentally. For instance, merely considering gravity on a lattice gives rather fundamental theory with comoving momentums restricted by the period of a lattice.
The next noteworthy fact of GR is the absence of a vacuum state, which is invariant relative to the general transformation of coordinates. It indicates the violation of gauge invariance at a quantum level but one could assume that the gauge invariance should be broken at the classical level in GR, as well. In particular, the five-vector theory of gravity (FVT) assumes the gauge invariance violation in GR by constraining the class of all possible metrics in varying the standard Einstein-Hilbert action. A question arises, how the classical Schwarzschild solution looks in this class of metrics? The purpose of this work is to elucidate the nature of compact astrophysical objects in this limited class of conformally-unimodular metrics.

II. VIOLATION OF GAUGE INVARIANCE IN A FRAMEWORK OF FVT

The observational fact, that the bulk of vacuum energy density does not affect the expansion of the universe, points out a gravity theory, in which the reference level of energy density could be chosen arbitrarily. Such a theory arises if one varies the standard Einstein-Hilbert action over not all possible space-time metrics $g_{\mu\nu}$, but over some class of conformally-unimodular metrics

$$ds^2 \equiv g_{\mu\nu}dx^\mu dx^\nu = a^2 (1 - \partial_m P^m)^2 d\eta^2 - \gamma_{ij}(dx^i + N^i d\eta)(dx^j + N^j d\eta), \quad (1)$$

where $x^i = \{\eta, \mathbf{x}\}$, $\eta$ is conformal time, $\gamma_{ij}$ is a spatial metric, $a = \gamma^{1/6}$ is a locally defined scale factor, and $\gamma = \det \gamma_{ij}$. The spatial part of the interval (1) reads as

$$dl^2 \equiv \tilde{\gamma}_{ij}dx^i dx^j = a^2(\eta, \mathbf{x})\tilde{\gamma}_{ij}dx^i dx^j, \quad (2)$$

where $\tilde{\gamma}_{ij} = \gamma_{ij}/a^2$ is a matrix with the unit determinant.

The interval (1) is similar formally to the ADM one, but with the lapse function $N$ changed by the expression $1 - \partial_m P^m$, where $P^m$ is a three-dimensional (relatively rotations) vector, and $\partial_m$ is a conventional particular derivative.

As was found, most of the symmetries in nature are violated. The exception is the color symmetry of the quantum chromodynamics.

In this gauge, a space-time metric is presented as a product of a common multiplier by a 4-dimensional matrix with a determinant equal to -1, including a 3-dimensional spatial block with unit determinant.
The starting point is the standard Einstein-Hilbert action \[ S_{\text{grav}} = -\frac{M_p^2}{12} \int G\sqrt{-g} \, d^4x, \] (3)
where \( G = g^{\alpha\beta} \left( \Gamma^\gamma_{\alpha\nu} \Gamma^\nu_{\beta\rho} - \Gamma^\nu_{\alpha\beta} \Gamma^\nu_{\rho\mu} \right) \), and \( M_p = \sqrt{\frac{3}{4\pi G}} = 1.065 \times 10^{-8} \, \text{kg} \) is the reduced Planck mass. The variation of (3) over vectors \( P, N \) and 3-metric \( \gamma_{ij} \) leads to the FVT equations:

\[
\begin{align*}
\frac{\partial g^{\mu\nu}}{\partial \gamma_{ij}} \left( \frac{\partial G\sqrt{-g}}{\partial g^{\mu\nu}} - \partial_\lambda \frac{\partial (G\sqrt{-g})}{\partial (\partial_\lambda g^{\mu\nu})} - \frac{6}{M_p^2} T_{\mu\nu} \sqrt{-g} \right) &= 0, \\
\frac{\partial g^{\mu\nu}}{\partial N^i} \left( \frac{\partial (G\sqrt{-g})}{\partial g^{\mu\nu}} - \partial_\lambda \frac{\partial (G\sqrt{-g})}{\partial (\partial_\lambda g^{\mu\nu})} - \frac{6}{M_p^2} T_{\mu\nu} \sqrt{-g} \right) &= 0, \\
\frac{\partial g^{\mu\nu}}{\partial (\partial_j P^i)} \frac{\partial}{\partial x^j} \left( \frac{\partial (G\sqrt{-g})}{\partial g^{\mu\nu}} - \partial_\lambda \frac{\partial (G\sqrt{-g})}{\partial (\partial_\lambda g^{\mu\nu})} - \frac{6}{M_p^2} T_{\mu\nu} \sqrt{-g} \right) &= 0,
\end{align*}
\] (4)
where the energy momentum tensor \( T_{\mu\nu} = \frac{\delta S_m}{\delta g^{\mu\nu}} \) is introduced. The last equation is weaker than the corresponding Hamiltonian constraint of GR. On the other hand, the restrictions \( \nabla (\nabla \cdot P) = 0, \nabla (\nabla \cdot N) = 0 \) on the Lagrange multipliers arise in FVT. Tacking into account the gauge \( \nabla \cdot N = 0 \) provides the Hamiltonian constraint fulfillment up to some constant.

### III. A SPHERICALLY SYMMETRIC STATIC GRAVITATIONAL FIELD

The spherically symmetric metrics belonging to the class (1) reads as:

\[
ds^2 = a^2 (d\eta^2 - \tilde{\gamma}_{ij} dx^i dx^j) = e^{2\alpha} \left( d\eta^2 - e^{-2\lambda} (dx)^2 - (e^{4\lambda} - e^{-2\lambda}) (x dx)^2 / r^2 \right), \]
(5)
where \( r = |x|, a = \exp \alpha, \lambda \) are the functions of \( \eta, r \). The matrix \( \tilde{\gamma}_{ij} \) with the unit determinant is expressed through \( \lambda(\eta, r) \). Thus, for the spherically symmetric case, the equations (4) take the form

\[
\mathcal{H} = e^{2\alpha} \left( -\frac{1}{2} \partial_\alpha \gamma^2 + \frac{1}{2} \lambda^2 - \frac{e^{2\lambda}}{6r^2} + \frac{e^{2\alpha}}{M_p^2} - e^{-4\lambda} \left( \frac{1}{6r^2} - \frac{4}{3} \partial_\tau \alpha \partial_\tau \lambda + \frac{1}{6} \partial_\tau \alpha^2 + \frac{2\partial_\tau \alpha}{3r} + \frac{1}{3} \partial_{\tau, \alpha} + \frac{7}{6} \partial_\tau \lambda^2 - \frac{5\partial_\tau \lambda}{3r} - \frac{5}{3} \partial_{\tau, \tau} \lambda \right) \right) = \text{const},
\] (6)

\footnote{Three dimensional spatial metric tensor can be written as the three-vectors triad. Thus 5-vectors appear in theory.}
\[ P = e^{2\alpha} \left( (-\partial_t \alpha (\alpha' + 2\lambda') - \partial_r \lambda' + \partial_r \alpha' - (3/r - 3\partial_r \lambda) \lambda') \right) = 0, \quad (7) \]

\[ \alpha'' + \alpha'^2 + \lambda'^2 = e^{-4\lambda} \left[ -4\partial_r \alpha \partial_r \lambda + \partial_r \alpha^2 + \frac{2\partial_r \alpha}{r} + \partial_r \partial_r \alpha + \frac{7}{3} \partial_r \lambda^2 - \frac{10}{3r} \partial_r \lambda - \frac{2}{3} \partial_r \partial_r \lambda + \frac{1}{3r^2} (1 - e^{6\lambda}) \right] + \frac{e^{2\alpha}}{M^2_p} (3p - \rho), \quad (8) \]

\[ \lambda'' + 2\alpha' \lambda' = \frac{2}{3} e^{-4\lambda} \left[ -\partial_r \alpha \partial_r \lambda - \partial_r \alpha^2 + \partial_r \partial_r \alpha + \partial_r \lambda^2 - \frac{1}{2} \partial_r \partial_r \lambda - \frac{1}{r} \partial_r \alpha \right. \]
\[ \left. - \frac{1}{r} \partial_r \lambda + \frac{1}{2r^2} (e^{6\lambda} - 1) \right], \quad (9) \]

where prime denotes differentiation over \( \eta \). Eq. (6) is the Hamiltonian constraint, but it includes an arbitrary constant now. If this constant equals zero, one returns to GR. Eq. (7) follows from the momentum constraint. The expressions (8), (9) are the equations of motion.

Differentiation of the constraints over time \( \eta \) results in the following equations

\[ \mathcal{H}' = \frac{1}{3r^2} \partial_r \left( e^{-4\lambda} r^2 P \right), \quad (10) \]

\[ P' = \partial_r \mathcal{H}, \quad (11) \]

which are satisfied if the equations of motion (8), (9) are fulfilled, and, besides, the following equations for the energy density and pressure are enforced:

\[ \rho' + 3(p + \rho) \alpha' = 0, \quad \partial_r p + (p + \rho) \partial_r \alpha = 0. \quad (12) \]

In GR, the equations (12) arise from the Bianchi identities resulting in \( D^\mu T_{\mu\nu} = 0 \). In the FVT case, the relations (12) arise from the requirement of the constraints conservation in the time (10), (11). Generally, the equations (10), (11) satisfy not only \( \mathcal{H} = 0, P = 0 \), as in GR, but weaker conditions \( \mathcal{H} = \text{const}, P = 0 \), as one can see from Eq. (6). A constant on the right hand side of Eq. (6) compensates the bulk of the vacuum energy, and, after the compensation (if it is exact), the equations become the same as in GR. All this take a place in the conformo-unimodular metric (5), in which we will find the Schwarzschild solution, assuming the time derivatives, as well as pressure and density equal to zero in Eqs. (6–9). Expressing the derivatives
∂_{r,r} \lambda, \partial_{r,r} \alpha \text{ from Eqs. (8), (9)} \text{ and substituting them into (6) under the } \text{const } = 0, \text{ one finds}

\begin{equation}
- 3r^2 \left( \frac{d \alpha}{dr} \right)^2 + 4r \frac{d \alpha}{dr} \left( r \frac{d \lambda}{dr} - 1 \right) - \left( r \frac{d \lambda}{dr} - 1 \right)^2 + e^{6\lambda} = 0. \tag{13}
\end{equation}

To obtain a solution of the equations (8), (9), (13), let us make the following substitution

\begin{equation}
\lambda = \alpha + \ln \left( \left( 1 - e^{2\alpha} \right) \frac{r}{r_g} \right), \tag{14}
\end{equation}

where the Schwarzschild radius is introduced for the sake of dimensionless of the expressions under logarithm. As a result, Eq. (13) takes the form:

\begin{equation}
r^4 e^{4\alpha} \left( e^{2\alpha} - 1 \right)^8 - 4 \left( \frac{d \alpha}{dr} \right)^2 r_g^6 = 0 \tag{15}
\end{equation}

and has the solution

\begin{equation}
\alpha(r) = \ln \left( f^{-1} \left( \frac{r^3 - r_0^3}{6r_g^3} \right) \right), \tag{16}
\end{equation}

where \( f^{-1} \) is the inverse function of

\begin{equation}
f(a) = 2 \ln \left( \frac{a^2}{1 - a^2} \right) + \frac{30a^4 - 12a^6 - 22a^2 + 3}{6a^2(a^2 - 1)^3} \tag{17}
\end{equation}

and \( r_0 \) is an integration constant. The function \( f(a) \), which maps an interval \((0, 1)\) into \( \mathbb{R} \), is mutually single-valued function shown in Fig. 1. Using (16), (17) and the
rules of the differentiation of the inverse function allows calculating
\[ \frac{d\alpha}{dr} = \frac{r^2}{2r_g f'(f^{-1})(y) f''(f^{-1})(y))} = \frac{r^2 f^{-1}(y)^2 (f^{-1}(y)^2 - 1)^4}{2r_g^3} = \frac{8e^{6\alpha} r^2 \sinh^4 \alpha}{r_g^3}, \]
where \( y = \frac{r^4-r_0^4}{6r_g^4} \). Similar calculations give
\[ \frac{d\lambda}{dr} = \frac{8e^{6\alpha} r^2 \sinh^4 \alpha (\coth \alpha + 2)}{r_g^3} + \frac{1}{r}, \]
\[ \frac{d^2\alpha}{dr^2} = \frac{16 e^{6\alpha} r \sinh^4 \alpha (8e^{6\alpha} r^3 \sinh^3 \alpha (3 \sinh \alpha + 2 \cosh \alpha) + r_g^3)}{r_g^6}, \]
\[ \frac{d^2\lambda}{dr^2} = \frac{64e^{12\alpha} r^4 \sinh^6 \alpha (7 \sinh (2\alpha) + 8 \cosh (2\alpha) - 5)}{r_g^6} - \frac{1}{r^2} + \frac{16e^{6\alpha} r \sinh^4 \alpha (\coth \alpha + 2)}{r_g^3}. \]
Substitution of Eqs. (18), (19), (20), (21) into Eqs. (6), (7), (8), (9) demonstrates that the last are satisfied at \( p = \rho = 0 \), and \( \text{const} = 0 \). Thus, Eqs. (14), (16), (17) are the exact spherically-symmetric static solution of the Einstein equations in vacuum. From the physical viewpoint, it appears that \( \text{const} \) in Eq. (5) compensates a vacuum energy of quantum fields.

The function \( \alpha \) is not singular everywhere, as it is shown in Fig. 2 (a), whereas the function \( \lambda \), describing the deviation of conformally-unimodular metric geometry from the Schwarzschild one, is singular only at \( r = 0 \). That means that the horizon is absent in the conformally-unimodular metric, and the real non-point compact astrophysical objects could have an arbitrary size in this metric is not restricted by the Schwarzschild radius.

Let us compare the solution (14), (16) with the canonical Schwarzschild one which is
\[ ds^2 = (1 - r_g/R)dt^2 - (1 - r_g/R)^{-1}dR^2 - R^2(d\theta^2 + \sin^2 \theta d\phi^2). \]
For this aim, we rewrite the interval (5) in the spherical coordinates
\[ ds^2 = e^{2\alpha} (d\eta^2 - dr^2 e^{4\lambda} - e^{-2\lambda} r^2 (d\theta^2 + \sin^2 \theta d\phi^2)) \].
FIG. 2. (a) "Gravitational potentials" describing the metric (5). Solid and dashed lines correspond to $\alpha(r)$, and $\lambda(r)$, respectively. The dashed-doted line is the Newtonian potential $\varphi = -\frac{r_g}{2r}$. (b) Coordinate transformation $R(r)$ mapping the metric (5), (23) to the canonical Schwarzschild form (22) for the different integration constant values $r_0$ in the expression (16). Solid and dashed lines correspond to $r_0 = 0$ and $r_0 = 3r_g$, respectively. The level of $R = r_g$ is marked by the gray horizontal line.

FIG. 3. "Squeezing" of the BHs of Schwarzschild horizons into the nodes with point masses.

The solutions (22) and (23) of the Einstein equations should be interrelated by the transformation of coordinates $t = \eta, R = R(r)$, which gives another way to deduce Eqs. (14), (15). Actually, equating the coefficients at $dt^2 = d\eta^2$, and $d\theta^2 \sin^2 \theta d\phi^2$
as well as the radial terms in the intervals \[ \text{(22)}, \ (23) \] gives the equations
\[
1 - \frac{r_g}{R} = e^{2\alpha}, \tag{24}
\]
\[
R^2 = r^2 e^{-2\lambda + 2\alpha}, \tag{25}
\]
\[
(1 - \frac{r_g}{R})^{-1} \left( \frac{dR}{dr} \right)^2 = e^{4\lambda + 2\alpha}. \tag{26}
\]
The relations \[ \text{(24)}, \ (25) \] result in the expressions \[ \text{(14)} \] and \( R(r) = r_g (1 - e^{2\alpha})^{-1} \), which give \[ \text{(15)} \] after substitution in \[ \text{(26)}. \] As is shown in Fig. 2 (b), the solutions \[ \text{(14)}, \ (16) \] describe only a part of the external space \( r_g^+ \) of the Schwarzschild solution by virtue of \( \lim_{r \to 0} R(r) \geq r_g \). Fig. 3 illustrates this fact in the following way. Let there is a space filled with the Schwarzschild BHs. Then, by inverse coordinates transformation having the form \( r(R) \) in the vicinity of each hole, one can squeeze holes into the nodes \( r = 0 \) and consider that point particle is placed in each node. The space-time obtained in such a way will represent a single causally connected region.

In principle, using the Dirac delta function and writing the density energy in Eq. \[ \text{(6)} \] as \( \rho(x) = e^{-3\alpha} \delta^{(3)}(x) \), one could consider the solutions \[ \text{(14)}, \ (15) \] as corresponding to the \( \delta \)-source, but such a consideration is rather formal because the equations of gravity are nonlinear, whereas the product of generalized functions cannot be defined correctly. Some additional definition of the structure of the Dirac delta function is required to overcome this difficulty. For instance, one could consider a physical model of delta-function in the form of a sphere of constant density, with the radius approaching zero along with the density tending to infinity.

### IV. COMPACT OBJECTS OF THE CONSTANT DENSITY

#### A. Uniform compact object in the Schwarzschild metric

The well-known Tolman-Oppenheimer-Volkov equation (TOV) \[ \text{[5]} \], which defines the maximal mass of a stable neutron star, written in the Schwarzschild type metric
\[
ds^2 = B(R)dt^2 - A(R)dR^2 - R^2d\Omega, \tag{27}
\]
reads as:
\[
p'(R) = -\frac{3}{4\pi M_p^2 R^2} M(R) \rho(R) \left( 1 + 4\pi R^3 p(R) \right) \left( 1 + \frac{\rho(R)}{\rho(R')} \right) \left( 1 - \frac{3 M(R)}{2\pi M_p^2 R} \right)^{-1},
\]
where the function \( M(R) = 4\pi \int_0^R \rho(R') R'^2 dR' \).

Although an ideal incompressible fluid seemed to be not existing in nature, an approximation of constant density \[38\] allows describing the general features of the compact physical objects. In this case \( M(R) = \frac{4\pi}{3} \rho R^3 \), the solution of Eq. \[28\] takes the form
\[
p(R) = \rho \frac{\sqrt{M_p^2 - 2 \rho R^2} - \sqrt{M_p^2 - 2 \rho R_f^2}}{3 \sqrt{M_p^2 - 2 \rho R_f^2} - \sqrt{M_p^2 - 2 \rho R_f^2}},
\]
where \( R_f \) is the radius of an object. As it is seen from the formula \[29\], pressure turns to infinity at \( R = \sqrt{4M_p^2/\rho - 9R_f^2} \), that points to some limitations on the size of the object. A condition of pressure finiteness yields \( 4M_p^2/\rho < 9R_f^2 \), i.e., the size of an object has to be \( R_f > \frac{9}{8} R_g \), where \( r_g = \frac{3m}{2\pi M_p^2}, m = M(R_f) = \frac{4\pi \rho R_f^3}{3} \).

**B. Shell compact object in the Schwarzschild metric**

Let us consider a more complex model of astrophysical object consisting of two immiscible and incompressible liquids with the densities \( \rho_1 \) and \( \rho_2 \). It is the simplest prototype for the neutron star with a non-uniform internal structure \[39\].

Then, the function \( M(R) \) is written as
\[
M(R) = \frac{4\pi}{3} \begin{cases} 
\rho_1 R^3, & R < R_i, \\
\rho_2 (R^3 - R_i^3) + \rho_1 R_i^3, & R_i < R < R_f, \\
\rho_2 (R_f^3 - R_i^3) + \rho_1 R_i^3, & R > R_f.
\end{cases}
\]

When \( \rho_1 \) is close to zero and \( \rho_2 = \rho \), the function \[30\] becomes
\[
M(R) = \frac{4\pi \rho}{3} \begin{cases} 
0, & R < R_i, \\
R_i^3 - R_i^3, & R_i < R < R_f, \\
R_f^3 - R_i^3, & R > R_f.
\end{cases}
\]
The analytical solution of (28) for pressure with this $M(R)$ is still cumbersome, however calculation shows softer condition for the pressure finiteness, which is shown in Fig. 4. For a sufficiently thin shell $R_f$ approaches to $r_g$.

![Graph](image)

**FIG. 4.** Minimum possible outer radius $R_f = R_{\text{min}}$ in dependence on the thickness of a shell in the Schwarzschild metric [27].

### C. Compact object in the conformally-unimodular metric

#### 1. Object of a star class

Modern observations of ultra-compact BH-like objects, formed as a result of collapse of massive stars, give the maximum estimation of their masses of order of $m = 15 \div 36 \, m_\odot$ [40, 41]. Let us consider the constant density objects in the metric (27) related by the coordinate transformation $R(r) = \exp (\alpha(r) - \lambda(r))$ with the metrics (14). A quantity $r_f$ denoting boundary of a matter corresponds to $R_f = R(r_f)$ in the conformally-unimodular metric metric of (23), while $R_i = R(0)$. Because the horizon is absent in this metric, nothing prevents $r_f$ to be smaller then $r_g$. The functions $\alpha(r), \lambda(r), p(r)$ are defined by the equations (6)-(9) within a sphere occupied by matter. The initial conditions at $r = r_f$ are given by linkage with the Schwarzschild solution (14), (16).
After solving of Eqs. (6) - (9), the mass of an object can be recovered

\[ m = \frac{4\pi}{3} \rho \left( R_f^3 - R_i^3 \right) = 4\pi \rho \int_0^{r_f} \frac{d\alpha}{dr} \left( r \frac{d\alpha}{dr} - r \frac{d\lambda}{dr} + 1 \right) r^2 dr, \quad (32) \]

which determines the Schwarzschild radius \( r_g = \frac{3}{2\pi} \frac{m}{M_\text{p}^2} \), appearing in the formulas (14) and (16).

Let us first discuss compact objects in the metric (5), (23) where the matter occupies a sphere with the size less or an order of the Schwarzschild radius (see Fig. 5), (a), (b). As could be expected, the potential \( \alpha \), which was finite in the case of a point source, remains finite. The potential \( \lambda \), which was infinite in the point where the point-like source was located, becomes finite inside a uniformly mass distribution within a ball.

The internal structure of a compact object in the metric (27) is defined by the internal \( R_i \) and external \( R_f \) radii, while there is only single external radius \( r_f \) in

FIG. 5. (a) A compact object of uncompressible fluid \((\rho_0 = 0.43 \, M_\text{p}^2 r_g^{-2})\) with the radius \( r_f = 2r_g \) in the conformally-unimodular metric (23) looks as a shell (b) with the boundaries \( R_i = R(0) = 1.34 \, r_g \) and \( R_f = R(r_f) = 1.52 \, r_g \) in the Schwarzschild type metric (27).

(c), (d) Low density object \( \rho_0 = 5.0117 \times 10^{-10} \, M_\text{p}^2 r_g^{-2} \) looks as a solid ball \( R_f \approx r_f = 1000r_g \) in both metrics if parameter \( r_0 \) in (16) equals \( r_0 = -96.75 \).
The pressure obtained by solving the TOV equation (points) and the equations (6), (7), (8), (9) (solid curve). The values of parameters correspond to Fig. 5 (a), (b).

Thus, the meaning of this additional parameter $r_0$ becomes clear. Namely, it defines the internal structure of an object. It is not surprising that partial information about the pseudo-BH structure is contained in the Schwarzschild external solution in the form of parameter $r_0$ because no real BH in the conformally-unimodular metric exists.

Certainly, the pressure $P(R)$, $R \in \{R_i, R_f\}$ obtained by the solution of the TOV equation matches the pressure recovered from Eqs. (8), (9), (12) in the parametric form $p(r), R(r), r \in \{0, r_f\}$ as it is shown in Fig. 6.

2. Supermassive object

Recently, the existence of supermassive compact objects in galaxy nuclei was confirmed, and their masses were estimated as $m = 6.5 \times 10^9 m_\odot$. Assuming the existence of some maximal density in nature $\rho_{\text{max}} \simeq M_p^4$, after conversion to the units $M_p^2 r_g^{-2}$, results in $\rho_0 = \rho_{\text{max}} = 3.4 \times 10^{95} M_p^2 r_g^{-2}$. For the conformally-unimodular metric, the size of this object turns out to be very small and, as calculations show, the potentials $\lambda$ and $\alpha$ inside a ball can be estimated by taking expressions (14), (16) for empty space (i.e., the boundary conditions affect $\alpha(r)$ stronger than the
structure" of an object). Moreover, one has at a small \( r/r_g \)
\[
\alpha(r) \approx \ln \left( a_0(r_0) + \frac{r^3 - r_0^3}{6r_g^3k(r_0)} \right),
\]
(33)
because the value of \( a \) tends to some constant \( a_0 \) at \( r \to 0 \). The parameters \( a_0(r_0) \) and \( k(r_0) \) are the functions of \( r_0 \). The expression (33) has been derived by the expansion of the function \( f(a) \) into Taylor’s series at the point \( a_0 \) up to the first order in \( a - a_0 \). After this expansion finding of the inverse function \( f^{-1} \) becomes elementary. The value of \( a_0 \) is a root of the equation \( f(a) - \frac{r_0^3}{6r_g k} = 0 \) and \( k = f'(a_0) \). Further, as an example, \( r_0 = 0 \) will be considered, when \( a_0 \approx 0.54 \), and \( k = 25.2 \).

The calculation of mass using (32) yields
\[
m \approx \frac{4\pi}{3} \rho_0 \frac{a_0}{k(1 - a_0^2/4)^{3/2}} r_f^3,
\]
giving the estimation for \( r_f \approx 2.6 \times 10^{-32} r_g = 1.5 \times 10^{16} M^{-1}_p \). The radius of a boundary surface in the Schwarzschild metric can be approximated from (24), (33)
\[
R(r) \approx \frac{r_g}{1 - \left( a_0 + \frac{r^3}{6r_g k} \right)^2},
\]
(35)
which gives \( R_i = R(0) \approx 1.4 r_g \).

As was already mentioned, it is possible to “approach” closer to the Schwarzschild radius if to take another value of parameter \( r_0 \). The thickness of surface \( \Delta R = R_f - R_i \approx |\frac{dR(r)}{dr}|_{r \to r_f} \approx 7.5 \times 10^{-97} r_g \approx 4.3 \times 10^{-49} M^{-1}_p \). So small thickness \( \Delta R \) of surface results from the hugeness of its area. The second equation of (12), using (33) and setting boundary condition \( p(r_f) = 0 \) allow estimating the pressure
\[
p(r) \approx \frac{r_f^3 - r^3}{r^3 + 6a_0 k r_g^3} \rho_0.
\]
(36)
The maximum of the pressure is \( p \approx 0.07 M^2_p r_g^{-2} \), i.e., it is much lower than the density \( \rho_0 \), due to low potential gradient \( \alpha(r) \) inside an “eicheon” given by Eq. (33), or from the extremely small surface thickness \( \Delta R \) in the terms of the TOV approach.
D. Low density objects

Low density objects (recall, for example, that the sun radius is $R_f \approx 236000r_g$) illustrated in Figs. 5 (c), (d), which represents a solid ball in the Schwarzschild metric (27). Although they are not related to the compact objects but could be considered for the completeness of the picture. It turns out to be that, in this case, the value of parameter $r_0$ in external metric (16) is fixed by the requirement $r = 0$ when $R = 0$. As is shown in Fig. 7, the condition of $R = 0$ at $r = 0$ meets only if $r_0 = -96.75$ for a non-compact object of the radius of $r_f = 1000r_g$. Then the value of $R(r)$ becomes almost the same with the $r$-growth, as it is shown in Fig. 7. Thus, one may conclude that the “friable” objects can also be described consistently in the conformally-unimodular metric.

![Graph: Dependence of $R(r)$ for a ball of $R_f \approx r_f = 1000r_g$ filled with the “friable” matter at different values of the parameter $r_0$ in the external metric: $r_0 = -120$ (dashed line), $-96.75$ (solid), $-70$ (dash-dotted). Density of matter is of $\rho_0 \approx 5 \times 10^{-10} M_p^2 r_g^{-2}$.]

In this context, it is interesting to imagine a low density object, but with an empty core surrounded by a firm “artificial surface” composed from an incompressible liquid, for instance. Such an object can also be described in the conformally-unimodular metric, with the $r$-coordinate running from 0 to $r_f$ and $r_0$ should have a small negative value, that corresponds to an internal cavity in the Schwarzschild
coordinates.

The difference between the “friable” and compact dense objects is, that for the first one, internal cavity in the Schwarzschild metric could eliminate an by taking a larger value of \( r_0 \) in the metric \( (23) \). In contrast, for a dense object, the cavity in the Schwarzschild metric cannot be eliminated in any way.

V. OBJECTS MADE OF DUST

Let’s consider the motion of a sample dust particle in the metric \( (23) \) in the neighborhood of \( r = 0 \), where

\[
\alpha \approx \text{const}, \quad \lambda = \alpha + \ln \left( \left(1 - e^{2\alpha}\right) \frac{r}{r_g} \right) \approx \text{const} + \ln r. \tag{37}
\]

The radial geodesics satisfy the equation

\[
\ddot{\eta} = 0, \quad \ddot{r} + \frac{2\dot{r}^2}{r} = 0, \tag{38}
\]

where a dot denotes a derivative over the proper time \( s \). The solution of Eq. \( (38) \)

\[
r(\eta) = r_{in}^{2/3} \left( r_{in} - 3v(\eta - \eta_{in}) \right)^{1/3} \tag{39}
\]

shows that the sample particle, placed initially at the point \( r_{in}, \eta = \eta_{in} \) and having the speed \( v \) directed towards center, reaches the point \( r = 0 \) for the finite time.

Qualitatively, the formation of objects with the equation of state of the dust type, i.e., having very low pressure, can be imagined as the radial falling of the dust particles in the “eicheon” field. As a result, dust particles are accumulated in the vicinity of \( r < r_g \), where the gradient of potential \( \alpha \) is negligible, i.e., the gravitational field is absent. In this conformally-unimodular metric, “eicheon” is similar to a trap because a particle needs to overcome the region of large potential to escape from such a trap. This picture is quite similar to those discussed in Refs. \[10, 42\].

On the other hand, in the Schwarzschild metric, a layer, where the dust particles are accumulated, is very thin. The thickness is determined by the residual pressure
if to consider that some small pressure is still present. This picture resembles a surface discussed in [11, 17, 18, 42], where it originates from the non-zero mass of graviton.

VI. CONCLUSION

We considered the conformally-unimodular gauge, which was chosen for the sake of avoiding the problem of vacuum energy. A requirement that the bulk vacuum energy $\rho_{\text{vac}} \sim M_p^4$ does not influence the curvature of space-time leads to the gauge invariance violation and restricts the class of the possible the metrics. That results in the absence of BH and the appearance of “eiheons” instead. All the compact real astrophysical objects in this class of the metrics look like solid balls of different sizes without any singular surfaces (“horizons”). If such the compact objects $r_f \leq r_g$ are considered in the Schwarzschild metric, they look like a matter layer distributed over the impenetrable spherical shell with a radius greater than the Schwarzschild one.

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