On Solution of Singular Integral Equations by Operator Method
Najem A. Mohammad* and Mohammad Shami Hasso

1,2Department of Mathematics, Faculty of Science and Health, Koya University Daniel Mitterrand Boulevard, Koya KOY45 AB64, Kurdistan Region – Iraq
najem.mohammad@koyauniversity.org

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Abstract. In this paper, we study the exact solution of singular integral equations using two methods, including Adomian decomposition method and Elzaki transform method. We propose an analytical method for solving singular integral equations and system of singular integral equations, and have some goals in our paper related to suggested technique for solving singular integral equations. The primary goal is for giving analytical solutions of such equations with simple steps, another goal is to compare the suggested method with other methods used in this study.

1. Introduction

A singular integral equation has enormous applications in applied problems including fluid mechanics, bio-mechanics, and electromagnetic theory. An integral equation is called a singular integral equation if one or both limits of integration become infinite, or if the kernel \( K(x, t) \) of the equation becomes infinite at one or more points in the interval of integration. To be specific, the integral equation of the first kind is as follows:

\[
f(x) = \int_{p(x)}^{q(x)} K(x, t) u(t) \, dt
\]

or the integral equation of the second kind is as follow:

\[
u(x) = f(x) + \int_{p(x)}^{q(x)} K(x, t) u(t) \, dt
\]

is called singular if \( p(x) \), \( q(x) \) or both limits of integration are infinite [7]. Moreover, the equation (1) or (2) is called a singular equation if the kernel \( K(x, t) \) becomes infinite at one or more points in the domain of integration. Example of the first type of singular integral equations is given below:

\[
u(x) = f(x) + \int_{p(x)}^{\infty} K(x, t) u(t) \, dt,
\]

and example of the second type of singular integral equations is given by the following:

\[
u(x) = f(x) + \int_{p(x)}^{x} \frac{1}{(x-t)^{\alpha}} u(t) \, dt, \quad 0 < \alpha < 1.
\]

The weakly-singular integral equations of the second kind, given by:

\[
u(x) = f(x) + \int_{p(x)}^{x} \frac{1}{\sqrt{x-t}} u(t) \, dt,
\]
where \( p(x) \) is a constant function appearing frequently in many mathematical physics and chemistry applications such as heat conduction, crystal growth and electrochemistry. It is assumed that the function \( f(x) \) is sufficiently smooth so that a unique solution to equation (5) is guaranteed. The kernel \( k(x, t) = \frac{1}{\sqrt{x - t}} \) is weakly-singular kernel [3, 5,6].

2. Methods for Solving Singular Integral Equations

There are many different methods to solve singular integral equations analytical and numerical. We solve these singular integral equations by analytical methods to find the exact solution. In this paper we used the following methods.

2.1. The Adomian decomposition method [1,7]

The Adomian decomposition method consists of decomposing the unknown function \( u(x) \) of any equation into a sum of an infinite number of components defined by the decomposition series

\[
 u(x) = \sum_{n=0}^{\infty} u_n(x) \quad (6)
\]

Suppose we have a singular integral equation of the second kind:

\[
 u(x) = f(x) + \int_{0}^{x} \frac{1}{\sqrt{x - t}} u(t) \, dt, \quad (7)
\]

we assume that \( f(x) \) is continuous in \([0, a]\) and the kernel \( K(x, t) \) is continuous for \( 0 \leq x \leq a, \) and \( 0 \leq t \leq x. \)

To determine the solution \( u(x) \) of equation (7) we substitute the decomposition series (6) into both sides of equation (7) to obtain

\[
 \sum_{n=0}^{\infty} u_n(x) = f(x) + \int_{0}^{x} \frac{1}{\sqrt{x - t}} \sum_{n=0}^{\infty} u_n(t) \, dt.
\]

The components \( u_0(x), u_1(x), u_2(x), \ldots \) are usually determined by using the recurrence relation

\[
 u_0(x) = f(x),
\]

\[
 u_n(x) = f(x) + \int_{0}^{x} \frac{1}{\sqrt{x - t}} u_{n-1}(t) \, dt,
\]

and

\[
 u(x) = \sum_{n=0}^{\infty} u_n(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots + u_n(x) + \cdots,
\]

which is the solution of the singular integral equation (7).
For example [7] consider the singular integral equation:

\[ u(x) = \sqrt{x} - 2\pi x + 4\int_{0}^{x} \frac{1}{\sqrt{x-t}} u(t) dt \]

We solve the above singular integral equation by using Adomian decomposition method.

Let \( u_0(x) = f(x) = \sqrt{x} - 2\pi x \), then

\[
\begin{align*}
    u_1(x) &= 4\int_{0}^{x} \frac{1}{\sqrt{x-t}} u_0(t) dt \\
    &= 4\int_{0}^{x} \frac{\sqrt{t} - 2\pi t}{\sqrt{x-t}} dt \\
    \text{Hence } u_1(x) &= 2\pi x - \frac{32}{3} \pi \sqrt{x^3}
\end{align*}
\]

\[
\begin{align*}
    u_2(x) &= 4\int_{0}^{x} \frac{1}{\sqrt{x-t}} u_1(t) dt \\
    &= 4\int_{0}^{x} \frac{2\pi t - \frac{32}{3} \pi \sqrt{t^3}}{\sqrt{x-t}} dt \\
    \text{we get } u_2(x) &= \frac{32}{3} \sqrt{x^3} - \frac{128}{8} \pi x^2
\end{align*}
\]

and so on. Using \( u_n(x) = 4\int_{0}^{x} \frac{1}{\sqrt{x-t}} u_{n-1}(t) dt \) gives the series solution

\[
    u(x) = \sum_{n=0}^{\infty} u_n(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots + u_n(x) + \cdots
\]

\[
    = (\sqrt{x} - 2\pi x) + \left(2\pi x - \frac{32}{3} \pi \sqrt{x^3}\right) + \left(\frac{32}{3} \pi \sqrt{x^3} - \frac{128}{8} \pi x^2\right) + \cdots
\]

\[
    = \sqrt{x} - 2\pi x + 2\pi x - \frac{32}{3} \pi \sqrt{x^3} + \frac{32}{3} \pi \sqrt{x^3} - \frac{128}{8} \pi x^2 + \cdots. \quad (8)
\]

We can easily observe the appearance of the noise terms, i.e. identical terms with opposite signs. Cancelling these noise terms in the equation (8) gives the exact solution \( u(x) = \sqrt{x} \).

2.2. The Elzaki Transform Method [4]

The Elzaki Transform defined for function of exponential order, we consider functions in the set \( A \) defined by:

\[
A = \left\{ f(t) : \exists M, k_1, k_2 > 0, |f(t)| < Me^{k_1t}, \text{ if } t \in (-1)^j \times [0, \infty) \right\}.
\]
For a given function in the set \( A \), the constant \( M \) must be finite number and \( k_1, k_2 \) may be finite or infinite. The Elzaki Transform is denoted by the operator \( E(.) \) and defined by the integral equations

\[
E[f(t)] = T(v) = v \int_0^{\infty} e^{-\frac{t}{v}} f(t) dt, \quad t \geq 0, k_1 \leq v \leq k_2.
\]

The following table is Elzaki Transform for some functions

| \( f(x) \) | 1 | \( x \) | \( x^n \) | \( e^{ax} \) | \( \sin x \) | \( \cos x \) | \( e^{ax}\sin bx \) | \( e^{ax}\cos bx \) |
|---|---|---|---|---|---|---|---|---|
| \( E(f(x)) \) | \( v^2 \) | \( v^3 \) | if \( n \) fractional \( \Gamma(n+1)v^{n+2} \) if \( n \) positive integer \( \frac{n!}{v^{n+2}} \) | \( v^2 \) | \( av^3 \) | \( \frac{v^2}{1 + a^2 v^2} \) | \( \frac{bv^3}{(1 - av)^2 + b^2 v^2} \) | \( \frac{(1 - av)v^2}{(1 - av)^2 + b^2 v^2} \) |

**Theorem 2.1** [2]

Let \( f(t) \) and \( g(t) \) be two functions in \( A \) and \((f * g)(t)\) is a convolution of \( f(t) \) and \( g(t) \), then

\[
E(f \ast g) = \frac{1}{v}E(f)E(g),
\]

where \( E(f) \) is the Elzaki Transform of \( f \).

Now, we taking the Elzaki transform for the given example [7] to get:

\[
E(u(x)) = E(\sqrt{x}) - 2\pi E(x) + 4E \left( \int_0^{\frac{x}{\sqrt{x-t}}} \frac{1}{\sqrt{x-t}} u(t) dt \right)
\]

\[
E(u(x)) = \left( \Gamma \left( \frac{3}{2} \right) \sqrt{\pi} v^{\frac{1}{2}} \right) - 2\pi v^3 + 4 \left( E(x)^{-\frac{1}{2}} u(v) \right), \quad \text{by Theorem 2.1, we get that}
\]

\[
T(v) = \left( \frac{1}{2} \sqrt{\pi} v^2 \right) - 2\pi v^3 + 4 \left( \frac{1}{\sqrt{\pi} v^2} u(v) \right), \quad \text{where} \ E(u(x)) = T(v)
\]

\[
= \left( \frac{1}{2} \sqrt{\pi} v^2 \right) - 2\pi v^3 + 4 \left( \frac{1}{\sqrt{\pi} v^2} u(v) \right)
\]

\[
= \left( \frac{1}{2} \sqrt{\pi} v^2 \right) - 2\pi v^3 + 4 \left( \sqrt{\pi} v^2 u(v) \right)
\]

\[
= \left( \frac{1}{2} \sqrt{\pi} v^2 \right) - 2\pi v^3 + 4 \left( \sqrt{\pi} v^2 u(v) \right)
\]

\[
T(v) - 4\sqrt{\pi} v \frac{1}{v^2} T(v) = \left( \frac{1}{2} \sqrt{\pi} v^2 \right) - 2\pi v^3
\]

\[
T(v)(1 - 4\sqrt{\pi} v^2) = \left( \frac{1}{2} \sqrt{\pi} v^2 \right) - 2\pi v^3
\]
this yields \( T(\nu) = \frac{\frac{1}{4\sqrt{\pi \nu^2}} - 2\pi v^3}{1 - 4\sqrt{\pi \nu^2}} \).  

By using the long division, we get the first term of equation (9) follows:

\[
\frac{1}{2} \sqrt{\pi \nu^2} \frac{\frac{5}{1}}{1 - 4\sqrt{\pi \nu^2}} = -\frac{\frac{1}{8} v^2}{1 - 4\sqrt{\pi \nu^2}} + \frac{\frac{1}{8} v^2}{1 - 4\sqrt{\pi \nu^2}}.
\]

and the second term of Equation (9) satisfies

\[
\frac{2\pi v^3}{1 - 4\sqrt{\pi \nu^2}} = -\frac{\frac{1}{2} \sqrt{\pi \nu^2}}{1 - 4\sqrt{\pi \nu^2}} - \frac{\frac{1}{8} v^2}{1 - 4\sqrt{\pi \nu^2}} + \frac{\frac{1}{8} v^2}{1 - 4\sqrt{\pi \nu^2}}.
\]

Substitution equation (10) and equation (11) in equation (9), we get that

\[
T(\nu) = -\frac{\frac{1}{8} v^2}{1 - 4\sqrt{\pi \nu^2}} + \frac{\frac{1}{8} v^2}{1 - 4\sqrt{\pi \nu^2}} + \frac{\frac{1}{2} \sqrt{\pi \nu^2}}{1 - 4\sqrt{\pi \nu^2}} + \frac{\frac{1}{8} v^2}{1 - 4\sqrt{\pi \nu^2}} - \frac{\frac{1}{8} v^2}{1 - 4\sqrt{\pi \nu^2}}.
\]

Therefore,

\[
T(\nu) = \frac{1}{2} \sqrt{\pi \nu^2}, \text{ by Elzaki inverse, we obtain}
\]

\[
E^{-1}(T(\nu)) = E^{-1}\left(\frac{1}{2} \sqrt{\pi \nu^2}\right) = E^{-1}\left(E(u(x)) = E^{-1}\left(\Gamma\left(\frac{3}{2}\right) \nu^2\right)\right),
\]

where \( E(\sqrt{x}) = \Gamma\left(\frac{3}{2}\right) \nu^2 \).

So \((E(u(x))) = E^{-1}(E(\sqrt{x})) \Rightarrow u(x) = \sqrt{x} \).

Therefore the solution of the singular integral equation in the given example is \( u(x) = \sqrt{x} \).

3. Results and Discussions

In this paper we propose analytical technique for solving singular integral equations. The suggested method is named by operator method. The method depends on the Carson transformation.

3.1. Operator method for solving singular Integral Equations

Let \( D = \frac{d}{dx}, \quad D^{-1} = \int_0^x f(t) dt \) – operators of differentiation and integration, respectively. Used Carson transformation [8, 9]

\[
F(D) = D^\circ \left[\int_0^x e^{-Dt} f(t) dt\right],
\]

where the function \( F(D) \) is called the operator representation of \( f(t) \).
If \( f(x) = 1 \) applying equation (12) we get

\[
F(D) = D \int_0^\infty e^{-Dt} \, dt = 1,
\]

if \( f(x) = x \) then

\[
F(D) = D \int_0^\infty e^{-Dt} \, dt = \frac{1}{D},
\]

if \( f(x) = \sqrt{x} \) then

\[
F(D) = D \int_0^\infty e^{-Dt} \sqrt{t} \, dt = \frac{\sqrt{\pi}}{2\sqrt{D}},
\]

if \( f(x) = \frac{1}{\sqrt{x}} \) then

\[
F(D) = D \int_0^\infty e^{-Dt} \frac{1}{\sqrt{t}} \, dt = \sqrt{\pi D}.
\]

The following table is Carson Transformation for some functions

| \( f(x) \) | \( 1 \) | \( x \) | \( \sqrt{x} \) | \( 1/\sqrt{x} \) | \( e^{\lambda x} \) | \( \sin \lambda x \) | \( \cos \lambda x \) | \( e^{\lambda x} \sin \alpha x \) | \( e^{\lambda x} \cos \alpha x \) |
|------------|------|------|-------|-----------|-----------|---------|---------|----------------|----------------|
| \( F(D) \)  | \( 1 \) | \( \frac{1}{D} \) | \( \frac{\sqrt{\pi}}{2\sqrt{D}} \) | \( \sqrt{\pi D} \) | \( \frac{D}{D-\lambda} \) | \( \frac{\lambda D}{D^2+\lambda^2} \) | \( \frac{D^2}{D^2+\lambda^2} \) | \( \frac{aD}{(D-\lambda)^2+a^2} \) | \( \frac{D(D-\lambda)}{(D-\lambda)^2+a^2} \) |

Consider the singular integral equation

\[
u(x) = f(x) + \lambda \int_0^x \frac{1}{\sqrt{x-t}} \, u(t) \, dt,
\]

Taking the Carson transformation of the both sides of equation (13) leads to

\[C(u(x)) = C(f(x)) + C(K(x)) \cdot C(u(x)),\]

where \( C \) is Carson transformation or equivalently

\[U(D) = F(D) + K(D) \cdot U(D), \text{ this gives } U(D) = \frac{F(D)}{1 - \lambda \sqrt{\pi D}},\]

where \( U(D) = C(u(x)) \) and \( F(D) = C(f(x)) \).

Applying Carson inverse to both sides of Equation (14) gives the formula

\[u(x) = C^{-1} \left( \frac{F(D)}{1 - \lambda \sqrt{\pi D}} \right).\]

That will be used for determination of the solution \( u(x) \).

Now we solve the singular integral equation in the given example by operator method.

First let \( f(x) = \sqrt{x - 2\pi}, \)
This yields $F(D) = C(f(x)) = C(\sqrt{x} - 2\pi) = C(\sqrt{x}) - 2\pi C(x)$

$$= \frac{\sqrt{\pi}}{2\sqrt{D}} - 2\pi \frac{1}{D}.$$ 

$$u(x) = C^{-1} \left( \frac{F(D)}{1 - \lambda \sqrt{\pi D}} \right), \quad \lambda = 4,$$

$$F(D) = \frac{\sqrt{\pi}}{2\sqrt{D}} - 2\pi \frac{1}{D}.$$ 

We have

$$u(x) = C^{-1} \left( \frac{\sqrt{\pi}}{2\sqrt{D}} - \frac{2\pi}{1 - 4\sqrt{\pi D}} \right),$$

this implies that

$$C^{-1} \left( \frac{\sqrt{\pi}}{1 - 4\sqrt{\pi D}} \right)$$

Then

Hence $u(x) = C^{-1} \left( \frac{\sqrt{\pi}}{1 - 4\sqrt{\pi D}} \right)$

we get $u(x) = \sqrt{x}$

Hence, the solution of the singular integral equation is $u(x) = \sqrt{x}$.

4. Conclusion

There are many difficulties in solving singular integral equations analytically. These difficulties generally depend on the mixture of the kernel functions of such equations. In this paper, an analytical method is proposed for solving singular integral equations because it has less steps compared to the Adomian decomposition method and Elzaki transform method. The suggested method can also be applied to any integral equations, and system of singular integral equations.

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