Magnetic phase diagram of a spin $S = 1/2$ antiferromagnetic two-leg ladder in the presence of modulated along legs Dzyaloshinskii-Moriya interaction

N. Avalishvili$^{1,3}$, B. Beradze$^2$, and G.I. Japaridze$^{1,3}$

$^1$Ilia State University, Faculty of Natural Sciences and Medicine, 0162, Tbilisi, Georgia
$^2$IVane Javakhishvili Tbilisi State University, Faculty of Exact and Natural Sciences, Chavchavadze Avenue 3, 0112, Tbilisi, Georgia and
$^3$Andronikashvili Institute of Physics, 0177, Tbilisi, Georgia

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We study the ground-state magnetic phase diagram of a spin $S = 1/2$ antiferromagnetic two-leg ladder in the presence of period two lattice units modulated, Dzyaloshinskii-Moriya (DM) interaction along the legs. We consider the case of collinear DM vectors and strong rung exchange and magnetic field. In this limit we map the initial ladder model onto the effective spin $\sigma = 1/2$ XXZ chain and study the latter using the continuum-limit bosonization approach. We identified four quantum phase transitions and corresponding critical magnetic fields, which mark transitions from the spin gapped regimes into the gapless quantum spin-liquid regimes. In the gapped phases the magnetization curve of the system shows plateaus at magnetisation $M = 0$ and to its saturation value per rung $M = M_{\text{sat}} = 1$. We have shown that the very presence of alternating DM interaction leads to opening of a gap in the excitation spectrum at magnetization $M = 0$. The width of the magnetization plateau at $M = 0.5M_{\text{sat}}$, is determined by the associated with the dynamical generation of a gap in the spectrum is calculated and is shown that its length scales as $(D_0D_1/J^2)^\alpha$ where $D_0, D_1$ are uniform and staggered components of the DM term, $J$ is the intraleg exchange and $\alpha \leq 3/4$ and weakly depends on the DM couplings.

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I. THE MODEL

Low-dimensional quantum magnetism has been the subject of intense research activity since the pioneering paper by Bethe [1]. Interest in study of these systems is determined by their remarkably rich and unconventional low-energy properties (see for review Refs. [2, 3]. An increased current activity in this field is connected with the large number of qualitatively new and dominated by the quantum effects phenomena discovered in these systems [4–7] as well as with the opened wide perspectives for use low-dimensional magnetic materials in modern nanoscale technologies.

A significant fraction of current research in the field of low-dimensional magnetism is focused on studies of helical structures and chiral order in the frustrated quantum magnetic systems [8–20]. The key couplings, responsible for stabilization of non-collinear magnetic configurations in these systems, is the Dzyaloshinskii-Moriya (DM) interaction [21]

$$\mathcal{H}_{\text{DM}} = \sum_n \mathbf{D}(n) \cdot [\mathbf{S}_n \times \mathbf{S}_{n+1}],$$

(1)

where $\mathbf{D}(n)$ is an axial DM vector. Although generally the vectors $\mathbf{D}(n)$ may spatially vary both in direction and magnitude, the symmetry restrictions based on the properties of real solid state materials usually rule out most of the possibilities and confine the majority of theoretical discussion to two principal cases – uniform DM interaction, $\mathbf{D}$ vector remains unchanged over the system [9, 12, 13, 16] and the case of staggered DM interaction, with antiparallel orientation of $\mathbf{D}$ on adjacent bonds [10, 11]. In both these cases the DM term can be eliminated by the gauge transformation and absorbed in a boundary conditions not affecting the structure of the basic Hamiltonian [8, 13].

The spin $S=1/2$ two-leg ladders represent the another subclass of low-dimensional quantum magnets which has been also the subject of perpetual intensive studies during last three decades [22–46]. Because the gapped character of
the excitation spectrum of a two-leg antiferromagnetic ladder for arbitrary ratio of the intraleg and interleg couplings\(^2\), in the presence of a magnetic field these systems show dominated by quantum effects complex behavior, which is manifestly displayed in the magnetic field driven quantum phase transitions in ladder systems\(^{28, 35, 38, 41}\). During the last few years interest in \(S = 1/2\) ladder materials has been enhanced by the discovery of various systems characterized by the presence of the uniform Dzyaloshinskii-Moriya interaction along the ladder legs and characterized by unconventional low-temperature magnetic properties\(^{43, 46}\). Among them are as materials characterized by the strong-leg exchange such as \((\text{C}_{7}\text{H}_{10}\text{N})_{2}\text{CuBr}_4\)\(^{44, 45}\) as well as the strong rung compounds such as \(\text{Cu}(\text{C}_8\text{H}_6\text{N}_2)\text{Cl}_2\)\(^{46}\).

Recently it has been demonstrated that the DM interaction can be efficiently tailored with an substantial efficiency factor by structural modulations\(^{47}\) or by external electric field\(^{48, 50}\). This unveils the possibility to control DM interaction and magnetic anisotropy via the electric field and opens a wide area for theoretical consideration of the effects caused by the spatially modulated DM interaction on the properties of low-dimensional spin systems. In the recent publication it has been shown that the effect of modulated spatially DMI on the ground state properties of the spin \(S = 1/2\) Heisenberg chain is substantial, leading to the appearance of the additional quantum phase transition point in the ground state phase diagram, which marks opening of a gap in the Luttinger-liquid phase of the spin chain and to the formation of the new gapped phase in the ground state phase diagram characterized by the coexisting spin dimerization and alternating spin chirality pattern\(^{51}\).

In the present work we study the effect of the alternating Dzyaloshinskii-Moriya interaction on the ground state phase diagram of the spin-1/2 strong-rung Heisenberg ladder in the presence of magnetic field. The Hamiltonian of the model under consideration is given by

\[
\hat{H} = H_{\text{leg}}^1 + H_{\text{leg}}^2 + H_{\text{rung}},
\]

where the Hamiltonian for leg \(\alpha\) is

\[
H_{\text{leg}}^\alpha = \sum_{n=1}^{N} \left\{ J_{\alpha} \mathbf{S}_n \cdot \mathbf{S}_{n+1} + D_{\alpha}(n) \cdot [\mathbf{S}_n \times \mathbf{S}_{n+1}] \right\},
\]

and the term corresponding to the interleg interaction and coupling with magnetic field is given by

\[
H_{\text{rung}} = \sum_{n=1}^{N} \left\{ J_{\perp} \mathbf{S}_{n,1} \cdot \mathbf{S}_{n,2} - H \left( S_{n,1}^z + S_{n,2}^z \right) \right\}.
\]

Here \(\mathbf{S}_{n,\alpha}\) is the spin \(S = 1/2\) operators at the \(n\)-th rung, \(D_{\alpha}(n)\) is a DM vector and the index \(\alpha = 1, 2\) denotes the ladder legs. Both the intraleg and interleg exchange constant are antiferromagnetic \((J, J_{\perp} > 0)\). In what follows we restric our consideration by the case where the vectors \(D_1(n)\) and \(D_2(n)\) are \textit{collinear} and the applied magnetic field is parallel to \(D_{\alpha}(n)\). We take \(D_{\alpha}(n) = (0, 0, D_{\alpha}(n))\), where

\[
D_{\alpha}(n) = D_{\alpha}^0 + (-1)^n D_{\alpha}^1.
\]

It is obvious, that at \(D_{\alpha}(n) \neq 0\) the spin-rotation symmetry of the Hamiltonian is \(U(1)\) and reflects invariance of the system with respect to the rotation around the vector \(\mathbf{D} (\hat{z} \text{ axis})\). In addition, at \(D_{\alpha}^1\) the translational symmetry of the Hamiltonian is restored only after the shift on two lattice units.

The outline of the paper is as follows: In the forthcoming section we derive transformation which allows to gauge away the DM coupling. Starting from the section III we restrict our consideration by the limit of strong rung exchange and magnetic field \((J_{\perp}, H \gg J_{\alpha})\) and derive the effective spin-chain Hamiltonian. In the Section IV we use the continuum-limit bosonization approach to study the ground state properties of the effective model. Finally, we conclude and summarize our results in section V.
II. GAUGING AWAY THE DM INTERACTION

Because the alternating DM term breaks the translation symmetry, it is convenient to rewrite the leg Hamiltonian in a form, which explicitly incorporates doubling of the unit cell of the model. We define the dimensionless parameters $d_\pm = (d_0^\alpha \pm d_1^\alpha) / J$, where $d_0^\alpha = D_0^\alpha / J$ and $d_1^\alpha = D_1^\alpha / J$ and rewrite the $\alpha$-leg Hamiltonian in the following way

$$H_{\text{leg}}^\alpha = \frac{J}{2} \sum_{m=1}^{N/2} \left[ \left( S_{2m-1,\alpha}^+ S_{2m,\alpha}^- + S_{2m-1,\alpha}^- S_{2m,\alpha}^+ \right) + \left( S_{2m,\alpha}^+ S_{2m+1,\alpha}^- + S_{2m,\alpha}^- S_{2m+1,\alpha}^+ \right) \right]$$

$$+ i d_0^\alpha \left( S_{2m-1,\alpha}^+ S_{2m,\alpha}^- - S_{2m-1,\alpha}^- S_{2m,\alpha}^+ \right) + i d_1^\alpha \left( S_{2m,\alpha}^+ S_{2m+1,\alpha}^- - S_{2m,\alpha}^- S_{2m+1,\alpha}^+ \right)$$

$$+ 2S_{2m,\alpha}^z \left( S_{2m-1,\alpha}^z + S_{2m+1,\alpha}^z \right) ,$$

where $S_{n,\alpha}^+ = S_{n,\alpha}^x \pm i S_{n,\alpha}^y$. Thus, at $d_0^\alpha \neq d_1^\alpha$, the translation symmetry of a $\alpha$-leg Hamiltonian is broken by the presence of nonequal amplitudes of the spin current operator

$$J^{0}_{\alpha} \sim i \left( S_{n,\alpha}^+ S_{n+1,\alpha}^- - S_{n,\alpha}^- S_{n+1,\alpha}^+ \right) ,$$

on odd and even links of a leg.

The next useful step is to rewrite the Hamiltonian [51] in a physically more suggestive manner by rotating spins and gauging away the DM interaction term in each leg. Here we follow the route, developed in the Ref. [51], in the case of a single chain with alternating DM interaction. By performing a site-dependent rotation around the $z$ axis of spins along the leg $\alpha$ with relative angle $\theta_\alpha^z$ for spins at consecutive odd-even sites $(2m-1, 2m)$ and $\theta_\alpha^z$ for spins at consecutive even-odd sites $(2m, 2m+1)$, we introduce new spin variables $\tau_{2m,\alpha}$ and $\tau_{2m+1,\alpha}$ by

$$\tau_{2m-1,\alpha}^z = e^{-i(m-1)(\theta_\alpha^z + \theta_\alpha^z)} S_{2m-1,\alpha}^z ,$$

$$\tau_{2m,\alpha}^z = e^{-i m \theta_\alpha^z - i (m-1) \theta_\alpha^z} S_{2m,\alpha}^z ,$$

$$\tau_{2m,\alpha}^{\pm} = S_{2m,\alpha}^{\pm} , \quad \tau_{2m+1,\alpha}^{\pm} = S_{2m+1,\alpha}^{\pm} .$$

Using (7-9) we map the initial $\alpha$-leg Hamiltonian onto

$$H_{\text{leg}}^\alpha = \frac{J}{2} \sum_{m=1}^{N/2} \left[ \left( \cos \theta_\alpha^z + d_0^\alpha \sin \theta_\alpha^z \right) \left( \tau_{2m-1,\alpha}^+ \tau_{2m,\alpha}^- + \tau_{2m,\alpha}^- \tau_{2m-1,\alpha}^+ \right) \right]$$

$$+ \left( \cos \theta_\alpha^z + d_1^\alpha \sin \theta_\alpha^z \right) \left( \tau_{2m,\alpha}^+ \tau_{2m+1,\alpha}^- + \tau_{2m,\alpha}^- \tau_{2m+1,\alpha}^+ \right)$$

$$- i \left( \sin \theta_\alpha^z - d_0^\alpha \cos \theta_\alpha^z \right) \left( \tau_{2m-1,\alpha}^+ \tau_{2m,\alpha}^- - \tau_{2m,\alpha}^- \tau_{2m-1,\alpha}^+ \right)$$

$$- i \left( \sin \theta_\alpha^z - d_1^\alpha \cos \theta_\alpha^z \right) \left( \tau_{2m,\alpha}^+ \tau_{2m+1,\alpha}^- - \tau_{2m,\alpha}^- \tau_{2m+1,\alpha}^+ \right)$$

$$+ 2 \tau_{2m} \left( \tau_{2m-1} + \tau_{2m+1} \right) .$$

Choosing $\tan \theta_\alpha^z = d_\alpha^z$, gives

$$J(\sin \theta_\alpha^z - d_0^\alpha \cos \theta_\alpha^z) = 0 ,$$

$$J(\cos \theta_\alpha^z + d_0^\alpha \sin \theta_\alpha^z) = J \sqrt{1 + (d_\alpha^z)^2} \equiv J^{(\pm)}_\alpha ,$$

and thus we eliminate the modulated DM interaction and obtain the Hamiltonian of a each leg characterized by the alternating transverse exchange interaction

$$H_{\text{leg}}^\alpha = \sum_{m=1}^{N/2} \left[ \left( J^{(\pm)}_\alpha \right) \left( \tau_{2m-1,\alpha}^+ \tau_{2m,\alpha}^- + \tau_{2m-1,\alpha}^- \tau_{2m,\alpha}^+ \right) + \left( J^{(\pm)}_\alpha \right) \left( \tau_{2m,\alpha}^+ \tau_{2m+1,\alpha}^- + \tau_{2m,\alpha}^- \tau_{2m+1,\alpha}^+ \right) \right]$$

$$+ J \tau_{2m,\alpha} \left( \tau_{2m-1,\alpha} + \tau_{2m+1,\alpha} \right) .$$
It is instructive to rewrite the Hamiltonian \( H^\text{leg} \) in the following, more common, form

\[
H^\text{leg} = J_\alpha \sum_n \left[ \frac{1}{2} (1 + (-1)^n \delta_\alpha) \left( \tau_{n,\alpha}^+ \tau_{n+1,\alpha}^- + \tau_{n,\alpha}^- \tau_{n+1,\alpha}^+ \right) + \gamma_\alpha \tau_{n,\alpha}^z \tau_{n+1,\alpha}^z \right],
\]

(13)

where, at \( d_i^\alpha \ll 1 \) (\( i = \pm \)),

\[
J_\alpha = \frac{J_\alpha^{(+)} + J_\alpha^{(-)}}{2} \approx J / \gamma_\alpha + \mathcal{O} ((d_i^\alpha)^4),
\]

(14)

\[
\delta_\alpha = \frac{J_\alpha^{(+)} - J_\alpha^{(-)}}{J_\alpha^{(+)} + J_\alpha^{(-)}} \approx d_0^\alpha d_1^\alpha \gamma_\alpha + \mathcal{O} ((d_i^\alpha)^4)
\]

(15)

and

\[
\gamma_\alpha = J / J_\alpha = \frac{1}{\sqrt{1 + (d_0^\alpha)^2 + (d_1^\alpha)^2}}.
\]

(16)

Thus at \( J_\alpha^{(+)} \neq J_\alpha^{(-)} \) the Hamiltonian \( H^\text{leg} \) is recognized as a Hamiltonian of the XXZ chain with easy-plane anisotropy \( (\gamma_\alpha^2 < 1) \) and alternating transverse exchange. Note that the alternation of the transverse exchange \( \delta_\alpha \neq 0 \) only for finite \( D_1^\alpha \neq 0 \) and \( D_0^\alpha \neq 0 \). In the following we will discard \( \mathcal{O} (d_i^\alpha) \) corrections.

Gauging away the DM interaction does not affect coupling with the magnetic field and, as it can easily check by inspection, results only via appearance of the site dependents phase factor in the transverse part of the on-rung exchange. Inserting (7) - (9) in (4) we obtain

\[
H_{\text{rung}} = \sum_{n=1}^N \left[ \frac{J_\alpha}{2} \left( e^{-i \Phi_n^{\prime}} \tau_{n,1}^+ \tau_{n,2}^- + h.c. \right) + J_{\perp} \tau_{n,1}^z \tau_{n,2}^- - H (\tau_{n,1}^z + \tau_{n,2}^z) \right],
\]

(17)

where

\[
\Phi_{2m-1} = (m-1)(\vartheta_1^+ + \vartheta_1^- - \vartheta_2^+ - \vartheta_2^-),
\]

(18)

\[
\Phi_{2m} = m(\vartheta_1^+ + \vartheta_1^- - \vartheta_2^+ - \vartheta_2^-) - (\vartheta_1^+ - \vartheta_2^-).
\]

(19)

In the particular case of a ladder with identical legs and in-phase modulation of the DM interaction \( (D_1(n) = D_2(n)) \), \( \Phi_{2m-1} = 0 \) and \( \Phi_{2m} = \vartheta_1^+ - \vartheta_2^- \).

Thus, after gauging away the DM interaction the Hamiltonian under consideration takes the following form

\[
\mathcal{H} = \sum_{n,\alpha} \left[ \frac{J_\alpha}{2} (1 + (-1)^n \delta_\alpha) \left( \tau_{n,\alpha}^+ \tau_{n+1,\alpha}^- + \tau_{n,\alpha}^- \tau_{n+1,\alpha}^+ \right) + \tau_{n,\alpha}^z \tau_{n+1,\alpha}^z - H (\tau_{n,1}^z + \tau_{n,2}^z) \right]
\]

\[
+ J_\perp \sum_{n=1}^N \left[ \frac{1}{2} \left( e^{-i \Phi_n^{\prime}} \tau_{n,1}^+ \tau_{n,2}^- + h.c. \right) + \tau_{n,1}^z \tau_{n,2}^- \right].
\]

(20)

## III. EFFECTIVE HAMILTONIAN IN THE CASE OF STRONG RUNG EXCHANGE AND MAGNETIC FIELD

In what follows we restrict ourselves by consideration of the limit of strong rung exchange and magnetic field \( (J_\perp, H \gg J_\alpha) \) and follow the route developed already decades ago to study magnetic phase diagram of a two-leg ladder in this limit \cite{28, 29}.

We start from the case \( J_\alpha = 0 \), where the system decouples into a set of noninteracting rungs in a magnetic field

\[
\mathcal{H} = \sum_{n=1}^N \left[ \frac{J_{\perp}}{2} \left( e^{-i \Phi_n} \tau_{n,1}^+ \tau_{n,2}^- + h.c. \right) + J_{\perp} \tau_{n,1}^z \tau_{n,2}^- - H (\tau_{n,1}^z + \tau_{n,2}^z) \right]
\]

(21)
and all eigenstates of the Hamiltonian (21) can be written as a product of rung states. The phase factor $\Phi_n$ does not change the eigenvalues of a pair of coupled spins on a rung – at each rung, two spins form either a singlet state $|s\rangle$ with energy $E_s = -0.75J_\perp$ or in one of the triplet states $|t^+\rangle$, $|t^0\rangle$ and $|t^-\rangle$ with energies $E_{t^+} = 0.25J_\perp - H$, $E_{t^0} = 0.25J_\perp$ and $E_{t^-}(n) = 0.25J_\perp + H$, respectively. When $H$ is small, the ground state consists of a product of rung singlets. As the field $H$ increases, the energy of the triplet state $|t^+\rangle$ decreases and at $H \simeq J_\perp$ forms, together with the singlet state, a doublet of almost degenerate low energy state, split from the remaining high energy two triplet states. To project the Hamiltonian (20) on the corresponding low-energy sector we introduce the effective spin operator $\sigma$ which act on these states as

$$
\begin{align*}
\sigma_n^+ |s >_n & = -\frac{1}{2} |s >_n, \\
\sigma_n^- |t^+ >_n & = \frac{1}{2} |t^+ >_n, \\
\sigma_n^- |t^> >_n & = 0,
\end{align*}
$$

(22)

The relation between the initial ladder spin operator $\tau_n$ and the pseudo-spin operator $\sigma_n$ in this restricted subspace can be easily derived by inspection,

$$
\tau_{n,\alpha}^\pm = (-1)^\alpha \frac{1}{\sqrt{2}} \sigma_n^\pm, \quad \tau_{n,\alpha}^z = \frac{1}{2} \left( \frac{1}{2} + \sigma_n^z \right).
$$

(23)

Using (23), to the first order and up to a constant, we easily obtain the following effective Hamiltonian

$$
\mathcal{H}_{\text{eff}} = J^* \sum_{n=1}^N \left[ \frac{1}{2} (1 + (-1)^n \delta^*) (\sigma_n^+ \sigma_{n+1}^- + \sigma_n^- \sigma_{n+1}^+) + \gamma^* \sigma_n^z \sigma_{n+1}^- - h \sigma_n^z \right]
$$

(24)

which describes a single spin 1/2 chain with a fixed XY anisotropy of $\gamma^*$ and alternating transverse exchange in an effective uniform magnetic field $H_{\text{eff}}$. Here

$$
J^* = \frac{J_1 + J_2}{2},
$$

(25)

$$
\delta^* = \frac{\delta_1 J_1 + \delta_2 J_2}{J_1 + J_2},
$$

(26)

$$
\gamma^* = \frac{1}{2} \frac{\gamma_1 J_1 + \gamma_2 J_2}{J_1 + J_2}
$$

(27)

and

$$
H_{\text{eff}} = J^* h = H - J_\perp - \gamma^* J^*.
$$

(28)

It is worth to notice that a similar problem has been studied intensively in past years [30, 31, 52, 53].

In absence of the DM interaction, ($\delta_\alpha = 0$ and $\gamma_\alpha = 1$) the Hamiltonian (21) is the Hamiltonian of XXZ chain with a fixed XY anisotropy of 1/2 and coincides with the corresponding effective Hamiltonian for a standard ladder derived in the strong-rung and limit [28]. As it follows from (26) in the strong on-rung coupling limit the effect of modulated DM interaction, displayed via the alternation of the transverse exchange, is most pronounced in the case of in-phase modulation of the DM coupling in both legs, where $\delta_1$ and $\delta_2$ are of the same sign and is the weakest in the case of anti-phase modulation where $\delta_1$ and $\delta_2$ are of different signs.

**IV. MAGNETIC PROPERTIES**

**A. The first critical field $H_{on}$ and the saturation field $H_{sat}$**

The performed mapping allows to determine critical fields $H_{on}$ corresponding to the onset of magnetization in the system and the saturation field $H_{sat}$ [28]. The easiest way to express $H_{on}$ and $H_{sat}$ in terms of ladder parameters is
to perform the Jordan-Wigner transformation which maps the problem onto a system of interacting spinless fermions:

\[
\sigma_n^- = e^{i\pi \sum_{m<n} c_m^\dagger c_n}, \\
\sigma_n^+ = c_n^\dagger e^{-i\pi \sum_{m<n} c_m^\dagger c_n}, \\
\sigma_n^z = c_n^\dagger c_n - 1/2 \equiv \rho_n,
\]

the model could be rewritten in the following form:

\[
H_{sf} = t \sum_n (1 + (-1)^n)\delta^\ast(a_n^\dagger a_{n+1} + \text{h.c.}) + V \sum_n \rho_n \rho_{n+1} - \mu \sum_n \rho_n,
\]

where \( t = V = J^\ast / 2 \) and \( \mu = H - J_\perp \). The lowest critical field \( H_{on} \) corresponds to that value of the chemical potential \( \mu_c \) for which the band of spinless fermions starts to fill up. In this limit we can neglect the interaction term in Eq. (30) and obtain the model of free massive particles with spectrum

\[
E^\pm(k) = -\mu \pm J^\ast (\cos^2(k) + \delta^\ast \sin^2(k))^{1/2}.
\]

This gives

\[
H_{on} = J_\perp - J^\ast.
\]

Thus at \( H < H_{on} \) the effective spin-chain model is fully polarized with magnetization (per site) \( m = -1/2 \). This corresponds to the state, where on all rungs pairs of spins form singlets and respectively, as it follows from (23) the net magnetization (per site) of the of the initial ladder system

\[
M = \frac{1}{N} \sum_{n,\alpha} \tau^z_{n,\alpha} = 0.
\]

A similar argument can be used to determine \( H_{sat} \). In the limit of almost saturated magnetization, it is useful to make a particle-hole transformation and estimate \( H_{sat} \) from the condition where the transformed hole band starts to fill, what gives

\[
H_{sat} = J_\perp + J^\ast.
\]

Therefore for \( H > H_{sat} \) the effective spin-chain model is in the fully polarized state with \( m = 1/2 \), all on-rung pairs of spins form triplets and respectively, as it follows from (23) the net magnetization of the of the initial ladder system

\[
M = \frac{1}{N} \sum_{n,\alpha} \tau^z_{n,\alpha} = 1.
\]

### B. Magnetization plateau at \( M = 0.5 \)

To characterize excitation spectrum and magnetic properties of the system at intermediate values of the magnetic fields \( H_{on} < H < H_{sat} \) we use the continuum-limit bosonization treatment of the model (24). Following the usual procedure in the low energy limit, we bosonize the spin degrees of freedom at fixed magnetization \( m \) and the interaction term becomes

\[
\sigma^z_n = m + \sqrt{\frac{K}{\pi}} \partial_x \phi(x) + \frac{A_1}{\pi} \sin \left( \sqrt{4\pi K} \phi(x) + (2m + 1)\pi n \right), \quad \sigma^+_n = \frac{B_1}{\pi} e^{-i \sqrt{\pi K} \phi(x)} \sin \left( \sqrt{4\pi K} \phi(x) + (2m + 1)\pi n \right),
\]

where \( t = V = J^\ast / 2 \) and \( \mu = H - J_\perp \). The lowest critical field \( H_{on} \) corresponds to that value of the chemical potential \( \mu_c \) for which the band of spinless fermions starts to fill up. In this limit we can neglect the interaction term in Eq. (30) and obtain the model of free massive particles with spectrum

\[
E^\pm(k) = -\mu \pm J^\ast (\cos^2(k) + \delta^\ast \sin^2(k))^{1/2}.
\]

This gives

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H_{on} = J_\perp - J^\ast.
\]

Thus at \( H < H_{on} \) the effective spin-chain model is fully polarized with magnetization (per site) \( m = -1/2 \). This corresponds to the state, where on all rungs pairs of spins form singlets and respectively, as it follows from (23) the net magnetization (per site) of the of the initial ladder system

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A similar argument can be used to determine \( H_{sat} \). In the limit of almost saturated magnetization, it is useful to make a particle-hole transformation and estimate \( H_{sat} \) from the condition where the transformed hole band starts to fill, what gives

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H_{sat} = J_\perp + J^\ast.
\]

Therefore for \( H > H_{sat} \) the effective spin-chain model is in the fully polarized state with \( m = 1/2 \), all on-rung pairs of spins form triplets and respectively, as it follows from (23) the net magnetization of the of the initial ladder system

\[
M = \frac{1}{N} \sum_{n,\alpha} \tau^z_{n,\alpha} = 1.
\]
where $A_1$ and $B_1$ are non-universal real constants of the order of unity and $m$ is the magnetization (per site) of the chain. Here $\phi(x)$ and $\theta(x)$ are dual bosonic fields, $\partial_x \phi = v_x \partial_x \theta$, and satisfy the following commutation relations

$$[\phi(x), \theta(y)] = i \Theta(y-x),$$

$$[\phi(x), \theta(x)] = i/2,$$  \hspace{1cm} (35)

and $K(\gamma^*, m)$ is the spin-stiffness parameter for a chain with anisotropy $\gamma^*$ and magnetization $m$. At zero magnetization

$$K(\gamma^*, 0) = \frac{\pi}{2(\pi - \arccos \gamma^*)}. \hspace{1cm} (36)$$

Using (33) and (34) we get the following bosonized Hamiltonian

$$\mathcal{H} = u \int dx \left[ \frac{1}{2} (\partial_x \phi)^2 + \frac{1}{2} (\partial_x \theta)^2 - h \sqrt{\frac{K}{\pi}} \partial_x \phi + \frac{\Delta^*}{2 \pi^2 \alpha^2} \cos \left( \sqrt{4\pi K} \phi + 2\pi mx \right) \right], \hspace{1cm} (37)$$

where $u \simeq J^*/K$. In deriving (37) the strongly irrelevant at $\gamma^* < 1/2$ (i.e. for $K > 3/4$) term $\sim \cos \left( \sqrt{16\pi K} \phi + 2\pi mx \right)$ has been omitted.

As we observe, for arbitrary $m \neq 0$ the cosine term in (37) contains oscillating factor and therefore has to be ignored in the continuum limit. Therefore in this case the effective spin chain model is critical and its long-wavelength excitations are described by the standard Gaussian theory and the magnetic field term can be easily absorbed by a shift $\partial_x \phi \rightarrow \partial_x \phi + K/\alpha h$ resulting to linear in $h$ magnetization of a gapless critical phase.

At $m = 0$ i.e. $M = 0.5$ the oscillating factor in the cosine term is absent and the continuum-limit bosonized version of the effective spin-chain model is given by the Hamiltonian

$$\mathcal{H} = u \int dx \left[ \frac{1}{2} (\partial_x \phi)^2 + \frac{1}{2} (\partial_x \theta)^2 - h \sqrt{\frac{K}{\pi}} \partial_x \phi + \frac{\Delta^*}{2 \pi^2 \alpha^2} \cos \sqrt{4\pi K} \phi \right]. \hspace{1cm} (38)$$

The Hamiltonian (38) is easily recognized as the standard Hamiltonian for the commensurate-incommensurate phase transition which has been intensively studied in the past using bosonization approach and the Bethe ansatz technique. Below we use the results obtained in [61, 62] to describe magnetization plateau and the transitions from a gapped (plateau) to gapless paramagnetic phases.

Let us first consider $h = 0$. In this case, the continuum theory of the initial spin-chain model in the magnetic field $H = J_{\perp} + \gamma^* J^*$ is given by the quantum sine-Gordon (SG) model with a massive term $\sim \delta^* \cos(\sqrt{4\pi K} \phi)$ with $3/4 \leq K < 1$. From the exact solution of the SG model, it is known that at $3/4 \leq K < 1$ the excitation spectrum of the model consists of solitons and antisolitons with masses $m \geq \sqrt{\frac{\pi}{K}}$

$$m = J^* C(K) \left( \delta^* \right)^{1/(2-K)}, \hspace{1cm} (39)$$

where $C(K)$ is a constant of the order for all $K$ from the considered sector $0.75 \leq K < 1$.

$$C(K) = \frac{2}{\sqrt{\pi}} \sqrt{\Gamma\left(\frac{K}{4\pi} \right)} \left[ \frac{\Gamma(1-K/4)}{2 \Gamma(K/4)} \right]^{2/(4-K)} \Gamma\left(\frac{K}{4\pi} \right),$$

and $\Gamma$ is the Gamma function.

In the gapped phase, the ground state properties of the system are determined by the dominant potential energy term $\sim \cos(\sqrt{4\pi K} \phi)$ and therefore in the gapped phases, the field $\phi$ is pinned in one of the vacua

$$\langle 0 | \sqrt{4\pi K} \phi | 0 \rangle = (2n + 1)\pi, \hspace{1cm} (40)$$

to ensure the minimum of the energy. From (33) and (34) it is clearly seen in the ordered phase local spin degrees of freedom are fully suppressed and only the on link dimer order

$$D_n = (-1)^n \left( \sigma^+_n \sigma^+_{n+1} + \sigma^-_n \sigma^+_{n+1} \right) \sim \cos(\sqrt{4\pi K} \phi)$$
show a long range order. In terms of initial ladder system it corresponds to the formation by singlet and triplet located on neighboring rungs of a plaquette of the coherent entangled pair.

At $h \neq 0$ (i.e. $H \neq J_\perp + \gamma^* J^*$), the presence of the gradient term in the Hamiltonian (38) makes it necessary to consider the ground state of the sine-Gordon model in sectors with nonzero topological charge. The effective chemical potential

$$\sim h^0_{\text{eff}} \sqrt{\frac{K}{\pi}} \partial_x \phi$$

tends to change the number of particles in the ground state i.e. to create finite and uniform density of solitons. It is clear that the gradient term in (38) can be again eliminated by a gauge transformation

$$\phi \to \phi + h^0_{\text{eff}} \sqrt{\frac{K}{\pi}} x,$$

however this immediately implies that the vacuum distribution of the field $\phi$ will be shifted with respect of the minima (40). This competition between contributions of the smooth and modulated components of the magnetic field is resolved as a continuous phase transition from a gapped state at $|h| < \Delta$ to a gapless (paramagnetic) phase at $|h| > \Delta$, where $\Delta$ is the soliton mass [59]. This condition gives two critical values of the magnetic field for each plateau

$$H^\pm_c = J_\perp + \gamma^* J^* \pm J^* C(K) (\delta^*)^{1/(2-K)}$$

and respectively determines the width of the magnetization plateau by

$$H^+ - H^- \simeq 2 J^* (\delta^*)^{1/(2-K)}.$$  \hspace{1cm} (42)

As usual in the case of C-IC transition, the magnetic susceptibility of the system shows a square-root divergence at the transition points:

$$\chi(H) \sim \sqrt{(H/H^-)} - 1 \quad \text{for} \quad H \geq H^-$$

and

$$\chi(H) = \sqrt{1 - (H/H^+)} \quad \text{for} \quad H \leq H^+.$$

Thus we obtain the following magnetic phase diagram for a ladder with alternating DMI along the legs. For $H \leq H_{\text{on}}$, the system is in a rung-singlet phase with zero magnetization and vanishing magnetic susceptibility. For $H > H_{\text{on}}$ some of singlet rungs melt and the magnetization increase as $M \sim \sqrt{(H/H_{\text{on}}) - 1}$. With further increase of the magnetic field the system gradually crosses to a regime with linearly increasing magnetization. However, in the vicinity of the magnetization plateau at $M = 0.5$, for $H \leq H^-$ this linear dependence changes again into a square-root behavior $M = 0.5 - \sqrt{1 - (H/H^-)}$. For fields in the interval between $H^+ < H < H^+$ the magnetization is constant $M = 0.5$. At $H > H^+$ the magnetization increases as $M = 0.5 + \sqrt{(H/H^+)} - 1$, then passes again through a linear regime until, in the vicinity of the saturation field $H < H_{\text{sat}}$, it becomes $M = 1 - \sqrt{1 - (H/H_{\text{sat}})}$.

V. SUMMARY

We have studied the ground state magnetic phase diagram of a spin $S = 1/2$ antiferromagnetic two-leg ladder in the case where spins along the legs are affected by the modulated with period of two lattice units Dzyaloshinskii-Moriya interaction. The model is studied in the limit of strong rung exchange and magnetic field. It is shown, that the very presence of modulated DM term opens a gap in the excitation spectrum of the ladder at magnetization equal
to half of its saturated value $M_{sat}$. The value of a gap is determined simultaneously by the uniform and staggered components of the DMI and is absent if one of these components is zero. Respectively, at $M = 0.5M_{sat}$ the plateau at magnetization curve appears of a width which determined by the gap in the excitation spectrum. We have also shown, that in the gapped phase unconventional magnetic order, where triplet and singlet states localized on neighboring rungs form an entangled dimer states and this dimerized plaquettes form a long-range order phase with period two lattice sites wavelength.

To conclude we have to stress that the obtained phase diagram is not a particularity of the considered case of two lattice unit modulation of the DMI. The gapped phases, characterized by magnetization plateau and dimerized plaquette order, appear in the case of arbitrary, but commensurate with the lattice unit modulation of the DMI interaction $D_\alpha(n) = D_0^\alpha + \cos(qn)D_1^\alpha$ at corresponding values of the magnetization $M = p/qM_{sat}$, where $p$ and $q$ are natural numbers and $p < q$.

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[1] H. Bethe, Z. Phys. 71, 205 (1931).
[2] H.-J. Mikeska and A. K. Kolezhuk, in Quantum Magnetism, edited by U. Schollwöck, J. Richter, D.J.J. Farnell and R.F. Bishop, (Lect. Notes in Phys. 645, Springer, Berlin 2004.)
[3] A. Vasiliev, O. Volkova, E. Zvereva, and M. Markina, Milestones of low-D quantum magnetism, npj Quantum Materials 18 (2018).
[4] S. Sachdev, Nature Physics 4, 173 (2008).
[5] C. Broholm et al., Magnetized States of Quantum Spin Chains in High Magnetic Fields, Lecture Notes in Physics 595, 211-234, (2008).
[6] Masashi Takigawa and Frédérik Mila in (Introduction to Frustrated Magnetism Editors C. Lacroix, P Mendels and F. Mila, Springer 2011)
[7] Lucile Savary, Leon Balents, Rep. Prog. Phys. 80, 016502 (2017).
[8] J.H.H. Perk and H.W. Capel, Physics Letters A 58, 115 (1976); ibid, Physica A 92, 163 (1978).
[9] A.A. Zvyagin, Fiz. Niz. Temp. 15 977 (1989); A.A. Zvyagin, Zh. Eksp. Teor. Fiz. 98, 1396 (1990).
[10] M. Oshikawa and I. Affleck, Phys. Rev. Lett. 79, 2883 (1997). I. Affleck and M. Oshikawa, Phys. Rev. B 60 1038 (1999).
[11] J. Z. Zhao, X. Q. Wang, T. Xiang, Z. B. Su, and L. Yu, Phys. Rev. Lett. 90, 207204 (2003).
[12] D. N. Aristov and S. V. Maleyev, Phys. Rev. B 62, R751 (2000).
[13] M. Bocquet, F.H.L. Essler, A.M. Tsvelik, and A.O. Gogolin, Phys. Rev. B 64, 094425 (2001).
[14] S. A. Zvyagin, A. K. Kolezhuk, J. Krzystek, and R. Fisher, Phys. Rev. Lett. 83, 027201 (2004).
[15] S. A. Zvyagin, A. K. Kolezhuk, J. Krzystek, and R. Fisher, Phys. Rev. Lett. 85, 017207 (2005).
[16] S. Gangadharaiah, J. Sun, and O. A. Starykh, Phys. Rev. B 78, 054436 (2008).
[17] I. Garate and I. Affleck, Phys. Rev. B 81, 144419 (2010).
[18] Zhihao Hao, Yuan Wan, Ioannis Rousochatzakis, Julia Wildeboer, A. Seidel, F. Mila, and O. Tchernyshyov, Phys. Rev. B 84, 094452 (2011).
[19] S. Peotta, L. Mazza, E. Vicari, M. Polini, R. Fazio, and D. Rossini, J. Stat. Mech. P09005 (2014).
[20] Yang-Hao Chan, Wen Jin, Hong-Chen Jiang, and O. A. Starykh, Phys. Rev. B 96 214441 (2017).
[21] I. E. Dzyaloshinskii, Sov. Phys. JETP, 5, 1259 (1957); T. Moriya, Phys. Rev. Lett. 4, 288 (1960).
[22] H.J. Schulz, Phys. Rev. B 34, 6372, (1986).
[23] I. Affleck, J. Phys. Condens. Matter 1, 3047 (1991).
[24] D.G. Shelton, A.A. Nersesyan and A.M. Tsvelik, Phys. Rev. B 53, 8521 (1996).
[25] C.A. Hayward, D. Poilblanc, and L.P. Lévy, Phys. Rev. B 54, R12649 (1996).
[26] E. Dagotto and T. M. Rice, Science, 271, 618 (1996).
[27] E. Dagotto, Rep. Prog. Phys. 62, 1525 (1999).
[28] F. Mila, Eur. Phys. J. B 6, 201 (1998).
[29] K. Totsuka, Phys. Rev. B 57 3454 (1998).
[30] R. Chitra and T. Giamarchi, Phys. Rev. B 55, 5816 (1997).
[31] D. C. Cabra, A. Honecker, and P. Pujol, Phys. Rev. Lett. 79, 5126 (1997); *ibid* Phys. Rev. B 58, 6241 (1998).
[32] M. Usami and S. I. Suga, Phys. Rev. B 58, 14401 (1998).
[33] T. Giamarchi and A. M. Tsvelik, Phys. Rev. B 59, 11398 (1999).
[34] Y.-J. Wang, F. H. L. Essler, M. Fabrizio, and A. A. Nersesyan, Phys. Rev. B 66, 024412 (2002).
[35] Y.-J. Wang Phys. Rev. B 68, 214428 (2003).
[36] T. Vekua, G. I. Japaridze, and H.-J. Mikeska, Phys. Rev. B 67, 064419 (2003).
[37] K. Hida, M. Shino and W. Chen, J. Phys. Soc. Japan 73, 1587 (2004).
[38] T. Vekua, G. I. Japaridze, and H.-J. Mikeska, Phys. Rev. B 70, 014425 (2004).
[39] J.-B. Fouet, F. Mila, D. Clarke, H. Youk, O. Tchernyshyov, P. Fendley, R. M. Noack, Phys. Rev. B 73, 214405 (2006).
[40] G. I. Japaridze and E. Pogosyan, Jour. Phys. C: Cond. Matt. 18, 9297 (2006).
[41] G. I. Japaridze, A. Langari and S. Mahdavifar, Jour. Phys. C: Cond. Matter 19 076201 (2007).
[42] G. I. Japaridze and Saeed Mahdavifar, Eur. Phys. J. B 68, 59 (2009).
[43] K. Yu. Povarov, W. E. A. Lorenz, F. Xiao, C. P. Landee, Y. Krasnikova, and A. Zheludev, J. Magn. Magn. Mater. 370, 62 (2014).
[44] M. Ozerov, M. Maksymenko, J. Wsosnitza, A. Honecker, C. P. Landee, M. M. Turnbull, S. C. Furuya, T. Giamarchi, S. A. Zvyagin, Phys. Rev. B 92, 241113 (2015).
[45] V. N. Glazkov, M. Fayzullin, Yu. Krasnikova, G. Skoblin, D. Schmidiger, S. Mühlbauer, and A. Zheludev, Phys. Rev. B 92 184403 (2015).
[46] A. N. Ponomaryov, M. Ozerov, L. Zviagina, J. Wsosnitza, K. Yu. Povarov, F. Xiao, A. Zheludev, C. Landee, E. Čižmár, A. A. Zvyagin, and S. A. Zvyagin, Phys. Rev. B 93, 134416 (2016).
[47] O. M. Volkov, D. D. Shela, Y. Gaididei, V. P. Kravchuk, U.K. Rössler, J. Fassbender, and D. Makarov, Scientific Reports 8, Article number: 866 (2018).
[48] H. Yang, O. Boule, V. Cros, A. Fert, and M. Chshiev, Scientific Reports, 8, Article N: 12356 (2018).
[49] W. Zhang, et al., App. Phys. Lett. 113, 122406 (2018).
[50] T. Srivastava et al., Nano Lett. 18, 4871 (2018).
[51] N. Avalishvili G.I.Japaridze and G. Rossini, Phys. Rev. B 99, 205159 (2019).
[52] T. Verkholyak, O. Derzhko, T. Krokhmalskii, and J. Stolze, Phys. Rev. B 76, 144418 (2007).
[53] D.V. Dmitriev, V. Ya. Krivnov, Phys. Rev. B 86, 134407 (2012).
[54] P. Jordan and E. Wigner, Z. Physik 47, 631 (1928).
[55] A.O. Gogolin, A.A. Nersesyan and A.M. Tsvelik, *Bosonization and strongly correlated systems*, Cambridge University Press, Cambridge (1998).
[56] T. Hikihara and A. Furusaki, Phys. Rev. B 63, 134438 (2001).
[57] A. Luther and I. Peschel, Phys. Rev. B 12, 3908 (1975).
[58] For details see chapter IV in [54].
[59] G.I. Japaridze and A.A. Nersesyan, JETF Pis’ma 27, 356 (1978); [JETP Lett. 27, 334 (1978)].
[60] V.L. Pokrovsky and A.L. Talapov Phys. Rev. Lett. 42, 65 (1979).
[61] G.I. Japaridze and A.A. Nersesyan, J. Low Temp. Phys. 37, 95 (1979).
[62] G.I. Japaridze, A.A. Nersesyan, and P.B. Wiegmann, Nucl. Phys. B 230, 511 (1984).
[63] R.F. Dashen, B. Hasslacher and A. Neveu, Phys. Rev. D 10 1449 (1974); *ibid* Phys. Rev. D 11 3424 (1975).
[64] V.E. Korepin and L.D. Faddeev, Sov. Theor. Math. Phys., 25, 147 (1975).
[65] A. B. Zamolodchikov, Int. J. Mod. Phys. A10 1125 (1995).