Classically Simulating Quantum Circuits with Local Depolarizing Noise

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Abstract

We study the effect of noise on the classical simulatability of quantum circuits defined by computationally tractable (CT) states and efficiently computable sparse (ECS) operations. Examples of such circuits, which we call CT-ECS circuits, are IQP, Clifford Magic, and conjugated Clifford circuits. This means that there exist various CT-ECS circuits such that their output probability distributions are anti-concentrated and not classically simulatable in the noise-free setting (under plausible assumptions). First, we consider a noise model where a depolarizing channel with an arbitrarily small constant rate is applied to each qubit at the end of computation. We show that, under this noise model, if an approximate value of the noise rate is known, any CT-ECS circuit with an anti-concentrated output probability distribution is classically simulatable. This indicates that the presence of small noise drastically affects the classical simulatability of CT-ECS circuits. Then, we consider an extension of the noise model where the noise rate can vary with each qubit, and provide a similar sufficient condition for classically simulating CT-ECS circuits with anti-concentrated output probability distributions.

1 Introduction

1.1 Background and Main Results

A key step toward realizing a large-scale universal quantum computer is to demonstrate quantum computational supremacy [11], i.e., to perform computational tasks that are classically hard. As such a task, many researchers have focused on simulating quantum circuits, or more concretely, sampling the output probability distributions of quantum circuits. They have shown that, under plausible complexity-theoretic assumptions, this task is classically hard for various quantum circuits that seem easier to implement than universal ones. However, these classical hardness results have been obtained in severely restricted settings, such as a noise-free setting with additive approximation [5, 18, 3, 21] and a noise setting with multiplicative approximation [9]: the former requires us to sample the output probability distribution of a quantum circuit with additive error and the latter to sample the output probability distribution of a quantum circuit under a noise model with multiplicative error. Thus, there is great interest in considering the above task in a more reasonable setting.

We study the classical simulatability of quantum circuits in a noise setting with additive approximation, which requires us to sample the output probability distribution of a quantum circuit under a noise model with additive error. This setting is more reasonable than the noise-free setting with additive approximation since the presence of noise is unavoidable in realistic situations. Moreover, our setting is more reasonable than a noise setting with multiplicative approximation in the sense that we adopt a more realistic notion of approximation [1, 5], although noise in this paper is more restrictive than that in [9]. We consider a noise model where a depolarizing channel with an arbitrarily small constant rate \(0 < \varepsilon < 1\), which is denoted as \(D_\varepsilon\), is applied to each qubit at the end of computation. This channel leaves a qubit unaffected with probability \(1 - \varepsilon\) and replaces its state with the completely mixed one with probability \(\varepsilon\). We call this model noise model \(A\). We also consider its extension where the noise rate can vary with each qubit. More concretely, when a quantum circuit has \(n\) qubits, \(D_{\varepsilon_j}\) is applied to the \(j\)-th qubit at the end of computation for any \(1 \leq j \leq n\). We call this model noise...
model B. These noise models are simple, but analyzing them is a meaningful step toward studying more general models [10], such as one where noise exists before and after each gate in a quantum circuit. This is because, for example, this general noise model is equivalent to noise model A when we focus on instantaneous quantum polynomial-time (IQP) circuits, which are described below, with a particular type of intermediate noise [6].

A representative example of a quantum circuit that is not classically simulatable (in the noise-free setting) is an IQP circuit, which consists of Z-diagonal gates sandwiched by two Hadamard layers. In fact, there exists an IQP circuit such that its output probability distribution is anti-concentrated and not classically samplable in polynomial time with certain constant accuracy in $l_1$ norm (under plausible assumptions) [5]. On the other hand, Bremner et al. [6] studied the classical simulatability of IQP circuits under noise model A. They showed that, if the exact value of the noise rate is known, any IQP circuit with an anti-concentrated output probability distribution is classically simulatable in the sense that the resulting probability distribution is classically samplable in polynomial time with arbitrary constant accuracy in $l_1$ norm. This indicates that, under noise model A, if the exact value of the noise rate is known, the presence of small noise drastically affects the classical simulatability of IQP circuits.

In this paper, first, under a weaker assumption on the knowledge of the noise rate, we extend Bremner et al.’s result to quantum circuits that are defined by two concepts: computationally tractable (CT) states and efficiently computable sparse (ECS) operations [20]. Examples of such circuits, which we call CT-ECS circuits, are IQP circuits, Clifford Magic circuits [21], and conjugated Clifford circuits [3]. This means that there exist various CT-ECS circuits such that their output probability distributions are anti-concentrated and not classically simulatable in the sense described above for IQP circuits (under plausible assumptions). Constant-depth quantum circuits [19, 4, 2] are also CT-ECS circuits and not classically simulatable, although we do not know whether their output probability distributions are anti-concentrated. We postpone the explanation of CT states and ECS operations until Section 2, but, as depicted in Fig. 1(a), a CT-ECS circuit on $n$ qubits is a polynomial-size quantum circuit $C = VU$ such that $U|0^n\rangle$ is CT and $V^\dagger Z_j V$ is ECS for any $1 \leq j \leq n$, where $Z_j$ is a Pauli-Z operation on the $j$-th qubit. After performing $C$, we perform $Z$-basis measurements on all qubits. The CT-ECS circuit $C$ under noise model B is depicted in Fig. 1(b).

Our first result assumes noise model A, which corresponds to the case where $\varepsilon_j = \varepsilon$ for any $1 \leq j \leq n$ in Fig. 1(b). We show that, if an approximate value of the noise rate is known, any CT-ECS circuit with an anti-concentrated output probability distribution is classically simulatable:

**Theorem 1** (informal). Let $C$ be an arbitrary CT-ECS circuit on $n$ qubits such that its output probability distribution $p$ is anti-concentrated, i.e., $\sum_{x \in \{0,1\}^n} p(x)^2 \leq \alpha/2^n$ for some known constant $\alpha \geq 1$. We assume that a depolarizing channel with (possibly unknown) constant rate $\delta < \varepsilon < 1$ is applied to each qubit after performing $C$, which yields the probability distribution $\tilde{p}_A$. Moreover, we assume that it is possible to choose a constant $\lambda$ such that

$$1 \leq \frac{\varepsilon}{\lambda} \leq 1 + c,$$

where $c$ is a certain constant depending on $\alpha$. Then, $\tilde{p}_A$ is classically samplable in polynomial time with constant accuracy in $l_1$ norm.

Throughout the paper, the base of the logarithm is 2. If $\varepsilon$ is known, we can choose $\lambda = \varepsilon$ in Theorem 1. This case with IQP circuits precisely corresponds to Bremner et al.’s result [6]. As described above, there exist various CT-ECS circuits such that their output probability distributions are anti-concentrated and not classically simulatable in the noise-free setting (under plausible assumptions). Thus, Theorem 1 indicates that, under noise model A, if an approximate value of the noise rate is known, the presence of small noise drastically affects the classical simulatability of CT-ECS circuits.

Theorem 1 assumes noise model A where noise exists only at the end of computation, but, in some cases, it can be applied to an input-noise model. For example, Theorem 1 holds for IQP circuits when

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1Bremner et al. also dealt with quantum circuits for Simon’s algorithm. Our results can be directly extended to such circuits with access to an oracle, although we omit the details for simplicity.
Figure 1: (a): CT-ECS circuit $C = VU$, where $U|0^n\rangle$ is CT and $V^\dagger Z_j V$ is ECS for any $1 \leq j \leq n$. After performing $C$, we perform $Z$-basis measurements on all qubits. (b): CT-ECS circuit $C = VU$ under noise model $B$, where $\varepsilon_j$ represents the depolarizing channel $D_{\varepsilon_j}$ for any $1 \leq j \leq n$.

noise model $A$ is replaced with a noise model where $D_{\varepsilon}$ is applied to each qubit only at the start of computation, although Bu et al.’s main result implies a similar property of IQP circuits \[7\]. Moreover, Theorem 2, which is described below and assumes noise model $B$, also holds for IQP circuits when noise model $B$ is replaced with an input-noise model where the noise rate can vary with each qubit. Our main result is similar in spirit to Bu et al.’s, which provides classical algorithms for simulating Clifford circuits with nonstabilizer product input states (corresponding to input-noise models). However, we note that, in general, it is difficult to relate the output probability distributions of CT-ECS circuits under output-noise models to those of Clifford circuits under input-noise models.

Our main focus is on a noise setting, but, from a purely theoretical point of view, it is valuable to analyze the classical simulatability of quantum circuits in the noise-free setting. The proof method of Theorem 1 is based on computing the Fourier coefficients of an output probability distribution, and is useful in the noise-free setting. In fact, it implies that, when only $O(\log n)$ qubits are measured, any quantum circuit in a class of CT-ECS circuits on $n$ qubits is classically simulatable. More precisely, its output probability distribution is classically samplable in polynomial time with polynomial accuracy in $l_1$ norm. This class of CT-ECS circuits is defined by a restricted version of ECS operations, and includes IQP, Clifford Magic, conjugated Clifford, and constant-depth quantum circuits. It is known that the above property or a similar one holds for these quantum circuits (although the notions of approximation vary), but the proofs provided have depended on each circuit class \[19, 4, 12, 3\]. Our analysis unifies the previous ones and clarifies a class of quantum circuits for which the above property holds.

Our second result assumes noise model $B$, which is depicted in Fig. 1(b). For classically simulating CT-ECS circuits with anti-concentrated output probability distributions, we provide a sufficient condition, which is similar to Theorem 1.

**Theorem 2 (informal).** Let $C$ be an arbitrary CT-ECS circuit on $n$ qubits such that its output probability distribution $p$ satisfies $\sum_{x \in \{0,1\}^n} p(x)^2 \leq \alpha/2^n$ for some known constant $\alpha \geq 1$. We assume that a depolarizing channel with (possibly unknown) constant rate $0 < \varepsilon_j < 1$ is applied to the $j$-th qubit after performing $C$ for any $1 \leq j \leq n$, which yields the probability distribution $\tilde{p}_B$. Moreover, we assume that it is possible to choose a constant $\lambda_{\min}$ such that

$$1 \leq \frac{\varepsilon_{\min}}{\lambda_{\min}} \leq 1 + c,$$

where $\varepsilon_{\min} = \min\{\varepsilon_j | 1 \leq j \leq n\}$ and $c$ is a certain constant depending on $\alpha$, and we assume that it is possible to choose a constant $\lambda_j$ such that

$$0 \leq \varepsilon_j - \lambda_j \leq c\lambda_{\min}$$

for any $1 \leq j \leq n$ with $\varepsilon_j \neq \varepsilon_{\min}$. Then, $\tilde{p}_B$ is classically samplable in polynomial time with constant accuracy in $l_1$ norm.
To the best of our knowledge, this is the first analysis of the classical simulatability of quantum circuits under noise model $B$. Theorem 2 indicates that, under this noise model, if approximate values of the minimum noise rate and the other noise rates are known, the presence of small noise drastically affects the classical simulatability of CT-ECS circuits.

### 1.2 Overview of Techniques

To prove Theorem 1, we generalize Bremner et al.’s proof for IQP circuits [6]. There are two key points. The first one is to provide a general method for approximating the Fourier coefficients of the output probability distribution $p$. It is known that the probability distribution $\hat{p}_A$, which we want to approximate, can be simply represented by the noise rate $\varepsilon$ and the Fourier coefficients $\hat{p}(s)$ of $p$ for all $s \in \{0,1\}^n$ [15]. We show that, for any CT-ECS circuit on $n$ qubits, there exists a polynomial-time classical algorithm for approximating the low-degree Fourier coefficients of $p$, i.e., $\hat{p}(s)$ for any $s = s_1 \cdots s_n \in \{0,1\}^n$ with $\sum_{j=1}^n s_j = O(1)$. Bremner et al. [6] showed that such an algorithm exists for IQP circuits through a direct calculation of the Fourier coefficients for them. In contrast, we first provide a general relation between a quantum circuit and the Fourier coefficients of its output probability distribution. We then approximate each of the low-degree Fourier coefficients by combining this general relation with Nest’s classical algorithm for approximating the inner product value of a particular form defined by a CT state and an ECS operation [20]. This general relation seems to be a new tool to investigate the output probability distribution of a quantum circuit and thus may be of independent interest.

The second key point is to approximate $\hat{p}_A$ using an approximate value of $\varepsilon$. We define a function $q$ that seems to be close to $\hat{p}_A$ on the basis of its representation with $\varepsilon$ and the Fourier coefficients $\hat{p}(s)$ for all $s \in \{0,1\}^n$. Unfortunately, in contrast to Bremner et al.’s setting, we do not know $\varepsilon$. Thus, using an approximate value $\lambda$ of $\varepsilon$, we first choose an appropriate (polynomial) number of the low-degree Fourier coefficients used in $q$, and define $q$ based on the above representation of $\hat{p}_A$. More precisely, this number depends on $\lambda$, the constant $\alpha$ associated with the anti-concentration assumption, and the desired approximation accuracy. We then evaluate the approximation accuracy of $q$. Here, we need to care about the error caused by the difference between $\lambda$ and $\varepsilon$, and we upper-bound this error using the anti-concentration assumption.

To prove Theorem 2, we represent the effect of noise under noise model $B$ as the effect of noise under noise model $A$ with rate $\varepsilon_{\text{min}}$ and the remaining effects. We do this by transforming the representation of the probability distribution $\hat{p}_B$, which we want to approximate, with several basic properties of noise operators on real-valued functions over $\{0,1\}^n$ [15]. The obtained representation means that, to sample $\hat{p}_B$, it suffices to sample the probability distribution $\hat{p}_A$ (resulting from $p$) under noise model $A$ with rate $\varepsilon_{\text{min}}$ and then to classically simulate noise corresponding to the remaining effects. By Theorem 1, $\hat{p}_A$ is classically samplable in polynomial time with arbitrary constant accuracy in the $l_1$ norm. Moreover, we can simulate noise corresponding to the remaining effects using the approximate values of $\varepsilon_{\text{min}}$ and $\varepsilon_j$’s, which are not equal to $\varepsilon_{\text{min}}$.

### 2 Preliminaries

#### 2.1 Quantum Circuits and Their Output Probability Distributions

Pauli matrices $X$, $Y$, $Z$, and $I$ are

$$
X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
$$

The Hadamard operation $H$ is defined as $H = (X + Z)/\sqrt{2}$. For any real number $\theta$, the rotation operations $R_x(\theta)$ and $R_z(\theta)$ are defined as

$$
R_x(\theta) = \cos \frac{\theta}{2} I - i \sin \frac{\theta}{2} X, \quad R_z(\theta) = \cos \frac{\theta}{2} I - i \sin \frac{\theta}{2} Z.
$$

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### Footnotes

1. To the best of our knowledge, this is the first analysis of the classical simulatability of quantum circuits under noise model $B$.
2. Theorem 2 indicates that, under this noise model, if approximate values of the minimum noise rate and the other noise rates are known, the presence of small noise drastically affects the classical simulatability of CT-ECS circuits.

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### References

1. Bremner et al. [6] showed that such an algorithm exists for IQP circuits through a direct calculation of the Fourier coefficients for them.
2. Nest’s classical algorithm for approximating the inner product value of a particular form defined by a CT state and an ECS operation [20].
In particular, \( R_z(\pi/4) \) and \( R_z(\pi/2) \) are denoted as \( T \) and \( S \), respectively. It is easy to verify that the inverse of \( R_z(\theta) \) is \( R_z(-\theta) \) and, similarly, the inverse of \( R_z(\theta) \) is \( R_z(-\theta) \). The controlled-\( Z \) operation \( CZ \) and the controlled-controlled-\( Z \) operation \( CCZ \) are defined as

\[
CZ = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes Z, \quad CCZ = |00\rangle\langle 00| \otimes I + |01\rangle\langle 01| \otimes I + |10\rangle\langle 10| \otimes I + |11\rangle\langle 11| \otimes Z,
\]

where the states \( |0\rangle \) and \( |1\rangle \) are ±1-eigenstates of \( Z \), respectively. The operations \( H \), \( S \), and \( CZ \) are called Clifford operations.

A quantum circuit consists of elementary gates, each of which is in the gate set \( \mathcal{G} \). Here, \( \mathcal{G} = \{ R_x(\theta), R_z(\theta), CZ, \theta = \pm 2\pi/2^t \} \) with integer \( t \geq 1 \). Each \( R_x(\theta) \) has its inverse in \( \mathcal{G} \) and so does \( R_z(\theta) \). Moreover, each of the one-qubit operations \( X, Y, Z, I, \) and \( H \) can be decomposed into a constant number of \( R_x(\theta) \)'s and \( R_z(\theta) \)'s (up to an unimportant global phase). Similarly, \( CCZ \) can be decomposed into a constant number of \( R_x(\theta) \)'s, \( R_z(\theta) \)'s, and \( CZ \)'s. Since the gate set \( \{ H, T, CZ \} \) is approximately universal for quantum computation [13], so is \( \mathcal{G} \). The complexity measures of a quantum circuit are its size and depth. The size of a quantum circuit is the number of elementary gates in the circuit. To define the depth, we regard the circuit as a set of layers \( 1, \ldots, d \) consisting of elementary gates, where gates in the same layer act on pairwise disjoint sets of qubits and any gate in layer \( j \) is applied before any gate in layer \( j + 1 \). The depth is defined as the smallest possible value of \( d \) [5].

We deal with a (polynomial-time) uniform family of polynomial-size quantum circuits \( \{ C_n \}_{n \geq 1} \). Each \( C_n \) has \( n \) input qubits initialized to \( |0^n\rangle \). After performing \( C_n \), we perform \( Z \)-basis measurements on all qubits. The output probability distribution \( p \) of \( C_n \) over \( \{0, 1\}^n \) is defined as \( p(x) = |\langle x|C_n|0^n\rangle|^2 \). A symbol denoting a quantum circuit also denotes its matrix representation in the \( Z \) basis. The (polynomial-time) uniformity means that the function \( 1^n \mapsto \mathcal{C}_n \) is computable by a polynomial-time classical Turing machine, where \( \mathcal{C}_n \) is the classical description of \( C_n \) [14].

### 2.2 Fourier Expansions and Effects of Noise

Let \( f : \{0, 1\}^n \to \mathbb{R} \) be an arbitrary (real-valued) function. Then, \( f \) can be uniquely represented as an \( \mathbb{R} \)-linear combination of \( 2^n \) basis functions

\[
f(x) = \sum_{s \in \{0, 1\}^n} \widehat{f}(s)(-1)^{s \cdot x},
\]

which is called the Fourier expansion of \( f \) [15]. Here, \( \widehat{f}(s) \) is called the Fourier coefficient of \( f \) and the symbol \( \langle \cdot, \cdot \rangle \) represents the inner product of two \( n \)-bit strings, i.e., \( s \cdot x = \sum_{j=1}^n s_jx_j \) for any \( s = s_1 \cdots s_n \), \( x = x_1 \cdots x_n \in \{0, 1\}^n \). It holds that

\[
\widehat{f}(s) = \frac{1}{2^n} \sum_{x \in \{0, 1\}^n} f(x)(-1)^{s \cdot x}
\]

for any \( s \in \{0, 1\}^n \). The \( l_k \) norm of \( f \) is defined as

\[
||f||_k = \left( \sum_{x \in \{0, 1\}^n} |f(x)|^k \right)^{1/k},
\]

where \( k = 1, 2 \). It is known that \( ||f||_1^2 \leq 2^n ||f||_2^2 = 2^{2n} ||\hat{f}||_2^2 \) [15].

Let \( C \) be an arbitrary quantum circuit on \( n \) qubits and \( p \) be its output probability distribution over \( \{0, 1\}^n \). We consider \( C \) under noise model A where a depolarizing channel \( D_\varepsilon \) with rate \( 0 < \varepsilon < 1 \) is applied to each qubit after performing \( C \). Here, \( D_\varepsilon(\rho) = (1 - \varepsilon)\rho + \varepsilon I/2 \) for any density operator \( \rho \) of a qubit. We perform \( Z \)-basis measurements on all qubits and let \( \tilde{p}_A \) be the resulting probability distribution over \( \{0, 1\}^n \). As shown in [5], we can sample \( \tilde{p}_A \) by sampling an \( n \)-bit string according to \( p \) and then flipping each bit of the string with probability \( \varepsilon/2 \). This implies the following Fourier expansion of \( \tilde{p}_A \) [15]:

\[
\tilde{p}_A(x) = \sum_{s \in \{0, 1\}^n} (1 - \varepsilon)^{|s|} \tilde{p}(s)(-1)^{s \cdot x},
\]
constant-depth quantum circuits on $n$ ECS for any $1 \leq j \leq n$. We perform Z-basis measurements on all qubits and let $\tilde{p}_B$ be the resulting probability distribution over $\{0,1\}^n$. As with $\tilde{p}_A$, we can sample $\tilde{p}_B$ by sampling an $n$-bit string according to $p$ and then flipping its $j$-th bit with probability $\varepsilon_j/2$ for any $1 \leq j \leq n$. The Fourier expansion of $\tilde{p}_B$ is as follows [13]:

$$\tilde{p}_B(x) = \sum_{s \in \{0,1\}^n} \prod_{j=1}^{n} (1 - \varepsilon_j)^{s_j} \hat{p}(s)(-1)^{s \cdot x}.$$  

2.3 CT States and ECS Operations

We introduce CT states and (a restricted version of) ECS operations [20]. Let $|\phi\rangle$ be an arbitrary (pure) quantum state on $n$ qubits and $p$ be the probability distribution over $\{0,1\}^n$ defined as $p(x) = |\langle x|\phi\rangle|^2$. Then, $|\phi\rangle$ is CT if $p$ is classically samplable in polynomial time and, for any $x \in \{0,1\}^n$, $\langle x|\phi\rangle$ is classically computable in polynomial time. An example of a CT state is a product state.

Let $U$ be an arbitrary quantum operation on $n$ qubits that is both unitary and Hermitian. The operation $U$ is sparse if there exists a polynomial $s$ in $n$ such that, for any $x \in \{0,1\}^n$, $U(x)$ is a linear combination of at most $s(n)$ computational basis states. When $U$ is a sparse operation (associated with $s$), for any $1 \leq j \leq s(n)$, we define two functions, $\beta_j : \{0,1\}^n \to \mathbb{C}$ and $\gamma_j : \{0,1\}^n \to \{0,1\}^n$, as follows: for any $x \in \{0,1\}^n$, if the $j$-th non-zero entry exists in the column indexed by $x$ when traversing this column from top to bottom, $\beta_j(x)$ is this entry and $\gamma_j(x)$ is the row index associated with $\beta_j(x)$. If the $j$-th non-zero entry does not exist in this column, $\beta_j(x) = 0$ and $\gamma_j(x) = 0^n$. The sparse operation $U$ is ECS if, for any $x \in \{0,1\}^n$ and $1 \leq j \leq s(n)$, $\beta_j(x)$ and $\gamma_j(x)$ are classically computable in polynomial time. In particular, an ECS operation with $s(n) = O(1)$ is called ECS$_1$. Moreover, an ECS operation with $s(n) = 1$ is called efficiently computable basis-preserving.

An efficiently computable basis-preserving operation preserves the class of CT states [20]:

**Theorem 3** ([20]). Let $|\phi\rangle$ be an arbitrary CT state on $n$ qubits and $U$ be an arbitrarily efficiently computable basis-preserving operation on $n$ qubits. Then, $U|\phi\rangle$ is CT.

The following theorem is a rephrased version of the one in [20]:

**Theorem 4** ([20]). Let $U$ be an arbitrary quantum operation on $n$ qubits such that $U|0^n\rangle$ is CT, and $O$ be an arbitrary observable with $||O|| \leq 1$, where $||O||$ is the absolute value of the largest eigenvalue of $O$. Let $V$ be an arbitrary quantum operation on $n$ qubits such that $V^\dagger OV$ is ECS, and $f$ be an arbitrary polynomial in $n$. Then, there exists a polynomial-time randomized algorithm which outputs a real number $r$ such that

$$\Pr \left[ \left| \langle 0^n|U^\dagger V^\dagger O V U |0^n\rangle - r \right| \leq \frac{1}{f(n)} \right] \geq 1 - \frac{1}{\exp(n)}.$$  

3 Target Quantum Circuits and Associated Fourier Coefficients

3.1 CT-ECS Circuits

We focus on a new class of quantum circuits defined by CT states and ECS operations:

**Definition 1.** A quantum circuit $C$ on $n$ qubits initialized to $|0^n\rangle$ is CT-ECS if $C$ consists of two blocks $C = VU$ such that $U$ and $V$ are polynomial-size quantum circuits, $U|0^n\rangle$ is CT, and $V^\dagger Z_j V$ is ECS for any $1 \leq j \leq n$, where $Z_j$ is a Pauli-Z operation on the $j$-th qubit.

To provide examples of CT-ECS circuits, we define IQP, Clifford Magic, conjugated Clifford, and constant-depth quantum circuits on $n$ qubits as follows:

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\[\text{For simplicity, we require perfect accuracy in sampling probability distributions and computing values, although irrational numbers may be involved. Precisely speaking, it suffices to require exponential accuracy. This is also applied to similar situations in this paper, such as the definition of ECS operations.}\]
• An IQP circuit is of the form $H^\otimes nDH^\otimes n$, where $D$ is a polynomial-size quantum circuit consisting of $Z$, $CZ$, and $CCZ$ gates [5].

• A Clifford Magic circuit is of the form $ET^\otimes nH^\otimes n$, where $E$ is a polynomial-size Clifford circuit, which consists of $H$, $S$, and $CZ$ gates [21].

• A conjugated Clifford circuit is of the form $R_x(-\theta)^\otimes nR_z(-\phi)^\otimes nER_z(\phi)^\otimes nR_x(\theta)^\otimes n$ for arbitrary real numbers $\phi, \theta$, where $E$ is a polynomial-size Clifford circuit [3].

• A constant-depth quantum circuit is a polynomial-size quantum circuit $F$ whose depth is constant [19, 4, 2].

In the common definition of an IQP circuit, $D$ consists of more general $Z$-diagonal gates. However, for simplicity, we adopt the above definition. The resulting class includes a quantum circuit of the form $H^\otimes nDH^\otimes n$ such that its output probability distribution is anti-concentrated and not classically simulatable (under plausible assumptions).

We show that the above circuits are CT-ECS:

**Lemma 1.** Let $C$ be one of the following quantum circuits on $n$ qubits: an IQP, a Clifford Magic, a conjugated Clifford, or a constant-depth quantum circuit. Then, $C$ is CT-ECS.

**Proof.** When $C$ is an IQP circuit, $C = H^\otimes nDH^\otimes n$, where $D$ is a polynomial-size quantum circuit consisting of $Z$, $CZ$, and $CCZ$ gates. We consider $U = DH^\otimes n$ and $V = H^\otimes n$. Since $H^\otimes n|0^n\rangle$ is a product state and $D$ is an efficiently computable basis-preserving operation, by Theorem [3] $U|0^n\rangle = DH^\otimes n|0^n\rangle$ is CT. Moreover, $V^\dagger Z_j V = X_j$, which is obviously ECS (in fact, efficiently computable basis-preserving), where $X_j$ is a Pauli-$X$ operation on the $j$-th qubit. Thus, $C$ is CT-ECS.

When $C$ is a Clifford Magic circuit, $C = ET^\otimes nH^\otimes n$, where $E$ is a polynomial-size Clifford circuit. We consider $U = T^\otimes nH^\otimes n$ and $V = E$. Since $U|0^n\rangle = T^\otimes nH^\otimes n|0^n\rangle$ is a product state, it is CT. Moreover, since $E$ is a Clifford circuit, $V^\dagger Z_j V = E^\dagger Z_j E$ is a Pauli operation on $n$ qubits, which is obviously ECS (in fact, efficiently computable basis-preserving). Thus, $C$ is CT-ECS.

When $C$ is a conjugated Clifford circuit, $C = R_x(-\theta)^\otimes nR_z(-\phi)^\otimes nER_z(\phi)^\otimes nR_x(\theta)^\otimes n$ for arbitrary real numbers $\phi, \theta$, where $E$ is a polynomial-size Clifford circuit. We consider $U = R_x(\phi)^\otimes nR_x(\theta)^\otimes n$ and $V = R_x(-\theta)^\otimes nR_z(-\phi)^\otimes n E$. Since $U|0^n\rangle = R_x(\phi)^\otimes nR_x(\theta)^\otimes n|0^n\rangle$ is a product state, it is CT. Moreover,

$$V^\dagger Z_j V = E^\dagger R_x(\phi)_jR_x(\theta)_jZ_jR_x(-\theta)_jR_z(-\phi)_j E.$$ 

The coefficients $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{C}$ satisfying the following relation are classically computable in constant time:

$$\alpha_1 I + \alpha_2 Z + \alpha_3 X + \alpha_4 Y = R_x(\phi)_jR_x(\theta)_jZ_jR_x(-\theta)_jR_z(-\phi)_j.$$

This implies that $V^\dagger Z_j V = \alpha_1 I + \alpha_2 E^\dagger Z E + \alpha_3 E^\dagger X E + \alpha_4 E^\dagger Y E$. Since $E$ is a Clifford circuit, the operations $E^\dagger Z E$, $E^\dagger X E$, and $E^\dagger Y E$ are Pauli operations on $n$ qubits, which implies that $V^\dagger Z_j V$ is ECS. Thus, $C$ is CT-ECS.

When $C$ is a constant-depth quantum circuit, we consider $U = I$ and $V = C$. Of course, $U|0^n\rangle = |0^n\rangle$ is CT. Moreover, since each elementary gate in this paper acts only on a constant number of qubits, $V^\dagger Z_j V$ also acts only on a constant number of qubits, which implies that $V^\dagger Z_j V$ is ECS. Thus, $C$ is CT-ECS.

Lemma [1] implies that there exist various CT-ECS circuits such that their output probability distributions are anti-concentrated and not classically simulatable (under plausible assumptions), although we do not know whether the output probability distributions of constant-depth quantum circuits are anti-concentrated.

The above proof implies that, for any quantum circuit $C$ in Lemma 1, the associated ECS operation $V^\dagger Z_j V$ satisfies the condition where, for any $x \in \{0,1\}^n$, $V^\dagger Z_j V|x\rangle$ can be represented as a linear combination of at most $O(1)$ computational basis states. In other words, $V^\dagger Z_j V$ is ECS$_1$. This defines a class of CT-ECS circuits, whose elements we call CT-ECS$_1$ circuits. In Section 4.3, we consider the classical simulatability of CT-ECS$_1$ circuits on $n$ qubits in the noise-free setting when only $O(\log n)$ qubits are measured.
3.2 Approximating the Associated Fourier Coefficients

We provide a general relation between a quantum circuit and the Fourier coefficients of its output probability distribution:

**Lemma 2.** Let $C$ be an arbitrary quantum circuit on $n$ qubits initialized to $|0^n\rangle$ and $p$ be its output probability distribution over $\{0, 1\}^n$. Then,

$$\hat{p}(s) = \frac{1}{2^n} \langle 0^n|C^\dagger Z^s C|0^n\rangle$$

for any $s = s_1 \cdots s_n \in \{0, 1\}^n$, where $Z^s = \bigotimes_{j=1}^n Z^{s_j}$, i.e., the tensor product of a Pauli-Z operation on the $j$-th qubit with $s_j = 1$ for any $1 \leq j \leq n$.

**Proof.** We transform the representation of the Fourier coefficient described in Section 2.2 as follows:

$$\hat{p}(s) = \frac{1}{2^n} \sum_{x \in \{0, 1\}^n} p(x)(-1)^{s \cdot x} = \frac{1}{2^n} \sum_{x \in \{0, 1\}^n} \langle 0^n|C^\dagger|x\rangle \langle x|C|0^n\rangle (-1)^{s \cdot x}$$

$$= \frac{1}{2^n} \sum_{x \in \{0, 1\}^n} \langle 0^n|C^\dagger X^x|0^n\rangle \langle 0^n|X^x C|0^n\rangle (-1)^{s \cdot x}, \tag{1}$$

where $X^x$ denotes $H^\otimes n Z^x H^\otimes n$ for any $x \in \{0, 1\}^n$. Since it holds that

$$\langle 0^n|0^n\rangle = \frac{1}{2^n} \sum_{t \in \{0, 1\}^n} Z^t,$$

the above representation (1) of $\hat{p}(s)$ implies that

$$\hat{p}(s) = \frac{1}{2^{2n}} \sum_{x \in \{0, 1\}^n} \sum_{t \in \{0, 1\}^n} \langle 0^n|C^\dagger X^x Z^t X^x C|0^n\rangle (-1)^{s \cdot x}$$

$$= \frac{1}{2^{2n}} \sum_{t \in \{0, 1\}^n} \langle 0^n|C^\dagger Z^t C|0^n\rangle \sum_{x \in \{0, 1\}^n} (-1)^{(s+t) \cdot x} = \frac{1}{2^n} \langle 0^n|C^\dagger Z^s C|0^n\rangle,$$

where $s + t$ is the bit-wise addition of $s$ and $t$ modulo 2. This is the desired representation. \qed

Using Theorem 3 and Lemma 2, we show that the low-degree Fourier coefficients of the output probability distribution of a CT-ECS circuit can be approximated classically in polynomial time:

**Lemma 3.** Let $C$ be an arbitrary CT-ECS circuit on $n$ qubits and $p$ be its output probability distribution over $\{0, 1\}^n$. Let $f$ be an arbitrary polynomial in $n$ and $s$ be an arbitrary element of $\{0, 1\}^n$ with $|s| = O(1)$. Then, there exists a polynomial-time randomized algorithm which outputs a real number $\tilde{p}(s)$ such that

$$\Pr \left[ |\hat{p}(s) - \tilde{p}(s)| \leq \frac{1}{2^n f(n)} \right] \geq 1 - \frac{1}{\exp(n)}.$$

**Proof.** Since $C$ is CT-ECS, it can be represented as $C = VU$ such that $U|0^n\rangle$ is CT and $V^\dagger Z_j V$ is ECS for any $1 \leq j \leq n$. Let $s$ be an arbitrary element of $\{0, 1\}^n$ with $|s| = O(1)$, and we assume that $s_{j_1} = \cdots = s_{j_{|s|}} = 1$. In this case,

$$V^\dagger Z^s V = \prod_{k=1}^{|s|} (V^\dagger Z_{j_k} V).$$

Since $V^\dagger Z_{j_k} V$ is ECS, $V^\dagger Z^s V$ is the product of a constant number of ECS operations. A simple calculation shows that such a product is also ECS. Thus, $V^\dagger Z^s V$ is ECS.

Since $U|0^n\rangle$ is CT and $||Z^s|| \leq 1$, by Theorem 3 there exists a polynomial-time randomized algorithm which outputs a real number $r(s)$ such that

$$\Pr \left[ |\langle 0^n|U^\dagger V^\dagger Z^s V U|0^n\rangle - r(s)| \leq \frac{1}{f(n)} \right] \geq 1 - \frac{1}{\exp(n)}.$$
By Lemma 2 with $C = VU$,
\[
\hat{p}(s) = \frac{1}{2^n}(0^n|U^\dagger V^\dagger Z^n V U|0^n)
\]
and thus the desired relation holds by defining $\hat{p}'(s) = r(s)/2^n$. \hfill \qed

For any probability distribution $p$ over $\{0, 1\}^n$, it holds that
\[
\hat{p}(0^n) = \frac{1}{2^n} \sum_{x \in \{0, 1\}^n} p(x) = \frac{1}{2^n}.
\]

Thus, when we consider a classical algorithm for approximating $\hat{p}(s)$, we only consider an algorithm that outputs $1/2^n$ when $s = 0^n$. This slightly simplifies the analysis of the classical simulatability of CT-ECS circuits in the following sections.

4 CT-ECS Circuits under Noise Model A

In this section, we prove Theorem 1. Its precise statement is as follows:

**Theorem 1.** Let $C$ be an arbitrary CT-ECS circuit on $n$ qubits such that its output probability distribution $p$ over $\{0, 1\}^n$ is anti-concentrated, i.e., $\sum_{x \in \{0, 1\}^n} p(x)^2 \leq \alpha/2^n$ for some known constant $\alpha \geq 1$. Let $0 < \delta < 1$ be an arbitrary constant. We assume that

- a depolarizing channel with (possibly unknown) constant rate $0 < \varepsilon < 1$ is applied to each qubit after performing $C$, which yields the probability distribution $\tilde{p}_A$ over $\{0, 1\}^n$, and
- it is possible to choose a constant $\lambda$ such that
  \[
  1 \leq \varepsilon \lambda \leq 1 + \frac{1}{10\sqrt{\alpha} \log \frac{10\sqrt{\alpha}}{\delta}}.
  \]

Then, there exists a polynomial-time randomized algorithm which outputs (a classical description of) a probability distribution $\tilde{q}_A$ over $\{0, 1\}^n$ such that

\[
\Pr [||\tilde{p}_A - \tilde{q}_A||_1 \leq \delta] \geq 1 - \frac{1}{\exp(n)}
\]

and $\tilde{q}_A$ is classically samplable in polynomial time.

First, we define a function over $\{0, 1\}^n$ that is close to $\tilde{p}_A$, which we want to approximate. This is done by using the approximate value $\lambda$ of the noise rate $\varepsilon$ and the approximate values $\hat{p}'(s)$ of the low-degree Fourier coefficients of $p$ obtained by Lemma 3. Then, we sample a probability distribution close to the function.

4.1 Function Close to the Target Probability Distribution

We show that the probability distribution $\tilde{p}_A$ can be approximated by a function whose Fourier coefficients can be obtained classically in polynomial time:

**Lemma 4.** Let $C$ be an arbitrary CT-ECS circuit on $n$ qubits such that its output probability distribution $p$ satisfies $\sum_{x \in \{0, 1\}^n} p(x)^2 \leq \alpha/2^n$ for some known constant $\alpha \geq 1$. Let $0 < \delta < 1$ be an arbitrary constant. We assume that

- a depolarizing channel with constant rate $0 < \varepsilon < 1$ is applied to each qubit after performing $C$, which yields the probability distribution $\tilde{p}_A$, and
- it is possible to choose a constant $\lambda$ such that
  \[
  1 \leq \varepsilon \lambda \leq 1 + \frac{1}{10\sqrt{\alpha} \log \frac{10\sqrt{\alpha}}{\delta}}.
  \]
Then, there exists a polynomial-time randomized algorithm which outputs the Fourier coefficients of a function \( q \) over \( \{0,1\}^n \) such that

\[
\Pr \left[ \| \widehat{p}_\Lambda - q \|_1 \leq \frac{\delta}{3} \right] \geq 1 - \frac{1}{\exp(n)}.
\]

**Proof.** As described in Section 2.2, the probability distribution \( \widehat{p}_\Lambda \) is represented as

\[
\widehat{p}_\Lambda(x) = \sum_{s \in \{0,1\}^n} (1 - \varepsilon)^{|s|} \widehat{p}(s)(-1)^{s \cdot x}.
\]

Using the known constants \( \alpha, \delta, \) and \( \lambda \), we fix an integer constant

\[
c = \left\lfloor \frac{1}{\lambda} \log \frac{10\sqrt{\alpha}}{\delta} \right\rfloor.
\]

Since \( 0 < \lambda < 1 \) and \( 10\sqrt{\alpha}/\delta > 10, c > 3 \). The definition of \( c \) implies that

\[
\frac{1}{\lambda} \log \frac{10\sqrt{\alpha}}{\delta} \leq c \leq \frac{1}{\lambda} \log \frac{10\sqrt{\alpha}}{\delta} + 1 \leq \frac{2}{\lambda} \log \frac{10\sqrt{\alpha}}{\delta}.
\]

Thus, \( \alpha/2^{3c} \leq \delta^2/100 \). Moreover, \( 0 \leq c(\varepsilon - \lambda) \leq \frac{\delta}{5\sqrt{\alpha}} \) since

\[
\lambda \leq \varepsilon \leq \lambda + \frac{\lambda}{10\sqrt{\alpha} \log 10\sqrt{\alpha}} \leq \lambda + \frac{\lambda}{10\sqrt{\alpha} \cdot \frac{c}{2}} \leq \lambda + \frac{\delta}{5\sqrt{\alpha}}.
\]

By Lemma 3 with \( f(n) = 10(n^c + 1)/\delta \) and an arbitrary \( s \in \{0,1\}^n \) with \( |s| \leq c \), there exists a polynomial-time randomized algorithm which outputs \( \widehat{p}(s) \) such that

\[
\Pr \left[ \| \widehat{p}(s) - \widehat{p}(s) \|_1 \leq \frac{1}{2^n f(n)} \right] \geq 1 - \frac{1}{\exp(n)}.
\]

We compute \( \widehat{p}(s) \) for all \( s \in \{0,1\}^n \setminus \{0^n\} \) with \( |s| \leq c \), and define \( \widehat{p}(0^n) = 1/2^n \) (as described at the end of Section 3.2) and \( \widehat{p}(s) = 0 \) for all \( s \in \{0,1\}^n \) with \( |s| > c \). Since \( \{s \in \{0,1\}^n \ | \ |s| \leq c \} \leq n^c + 1 \), it takes polynomial time to compute all these values. Moreover,

\[
\Pr \left[ \forall s \in \{0,1\}^n \ | \ |s| \leq c, \ |\widehat{p}(s) - \widehat{p}(s)| \leq \frac{1}{2^n f(n)} \right] \geq 1 - \frac{1}{\exp(n)}.
\]

This can be simply shown by a direct application of the inequality \( (1 - a)^r \geq 1 - ra \) for an arbitrary real number \( 0 \leq a \leq 1 \) and integer \( r \geq 1 \).

We define a function \( q \) over \( \{0,1\}^n \) as

\[
q(x) = \sum_{s \in \{0,1\}^n, |s| \leq c} (1 - \lambda)^{|s|} \widehat{p}(s)(-1)^{s \cdot x}.
\]

In the following, we show that \( \| \widehat{p}_\Lambda - q \|_1 \leq \delta/3 \) under the assumption that, for all \( s \in \{0,1\}^n \) with \( |s| \leq c \), \( |\widehat{p}(s) - \widehat{p}(s)| \leq \frac{1}{2^n f(n)} \). A direct calculation with the relations described in Section 2.2 shows that

\[
\| \widehat{p}_\Lambda - q \|_1^2 \leq 2^n \| \widehat{p}_\Lambda - q \|_2^2 = 2^{2n} \| \widehat{p}_\Lambda - \widehat{q} \|_2^2
\]

\[
= 2^{2n} \sum_{s:|s|>c} (1 - \varepsilon)^{2|s|} \widehat{p}(s)^2 + 2^{2n} \sum_{s:|s| \leq c} (1 - \varepsilon)^{|s|} \widehat{p}(s) - (1 - \lambda)^{|s|} \widehat{p}(s)^2.
\]

Using the bounds \( \sum_{x \in \{0,1\}^n} p(x)^2 \leq \alpha/2^n \) and \( \alpha/2^{3c} \leq \delta^2/100 \), we upper-bound the first term of (2) as follows:

\[
2^{2n} \sum_{s:|s|>c} (1 - \varepsilon)^{2|s|} \widehat{p}(s)^2 \leq 2^{2n}(1 - \lambda)^{2c} \sum_{s:|s|>c} \widehat{p}(s)^2 \leq 2^{2n}(1 - \lambda)^{2c} \sum_{s \in \{0,1\}^n} \widehat{p}(s)^2
\]

\[
= 2^n (1 - \lambda)^{2c} \sum_{x \in \{0,1\}^n} p(x)^2 \leq \frac{2^n}{e^{2c}} \cdot \frac{\alpha}{2n} \leq \frac{\alpha}{2^{2c}} \leq \frac{\delta^2}{100}.
\]
Then, we upper-bound the second term of $\square$ as follows:

$$2^{2n} \sum_{s : |s| \leq c} \left[ (1 - \varepsilon)^{|s|} \tilde{p}(s) - (1 - \lambda)^{|s|} \tilde{p}'(s) \right]^2$$

$$= 2^{2n} \sum_{s : |s| \leq c} \left\{ \left[ (1 - \varepsilon)^{|s|} - (1 - \lambda)^{|s|} \right] \tilde{p}(s) + (1 - \lambda)^{|s|} (\tilde{p}(s) - \tilde{p}'(s)) \right\}^2$$

$$\leq 2^{2n} \sum_{s : |s| \leq c} \left\{ \left[ (1 - \lambda)^{|s|} - (1 - \varepsilon)^{|s|} \right] \tilde{p}(s) + (1 - \lambda)^{|s|} (\tilde{p}(s) - \tilde{p}'(s)) \right\}^2$$

$$\leq 2^{2n} \sum_{s : |s| \leq c} \left( c(\varepsilon - \lambda) |\tilde{p}(s)| + \frac{1}{2^n f(n)} \right)^2,$$

where the last inequality is due to the fact that $(1 - \lambda)^{|s|} - (1 - \varepsilon)^{|s|} \leq |s| (\varepsilon - \lambda) \leq c(\varepsilon - \lambda)$ and $|\tilde{p}(s) - \tilde{p}'(s)| \leq \frac{1}{2^n f(n)}$, where $f(n) = 10(n^c + 1)/\delta$.

Using the bounds $0 \leq c(\varepsilon - \lambda) \leq \frac{\delta}{5\sqrt{\alpha}}$, $|\tilde{p}(s)| \leq 1/2^n$, $|\{ s \in \{0,1\}^n | |s| \leq c \}| \leq n^{c+1}$, and $\sum_{x \in \{0,1\}^n} p(x)^2 \leq \alpha/2^n$, we upper-bound the value $\square$ as follows:

$$2^{2n} \sum_{s : |s| \leq c} \left( c(\varepsilon - \lambda) |\tilde{p}(s)| + \frac{1}{2^n f(n)} \right)^2 \leq 2^{2n} \sum_{s : |s| \leq c} \left( \frac{\delta |\tilde{p}(s)|}{5\sqrt{\alpha}} + \frac{1}{2^n f(n)} \right)^2$$

$$= 2^{2n} \sum_{s : |s| \leq c} \left( \frac{\delta^2 |\tilde{p}(s)|^2}{25\alpha} + \frac{2\delta |\tilde{p}(s)|}{5\sqrt{\alpha} f(n)} + \frac{1}{2^n f(n)^2} \right)$$

$$\leq \frac{2n \delta^2}{25\alpha} \sum_{x \in \{0,1\}^n} p(x)^2 + \frac{2\delta (n^{c+1})}{5\sqrt{\alpha} f(n)} + \frac{n^{c+1}}{f(n)^2} \leq \frac{\delta^2}{25} + \frac{\delta^2}{25} + \frac{\delta^2}{100} = \frac{9\delta^2}{100}.$$

Combining the upper bounds of the above two terms implies that

$$||\tilde{p}_A - q||_1^2 \leq \frac{\delta^2}{100} + \frac{9\delta^2}{100} \leq \frac{\delta^2}{9},$$

which is the desired bound. \qed

### 4.2 Sampling a Probability Distribution Close to the Function

The remaining problem is to sample a probability distribution close to the function $q$ defined in the proof of Lemma 4. To do this, we use a classical sampling algorithm proposed by Bremner et al. [6], although the following analysis is slightly simpler than the previous one. We represent $q$ as

$$q(x) = \sum_{s \in \{0,1\}^n, |s| \leq c} \hat{q}(s)(-1)^{s \cdot x},$$

where $\hat{q}(s) = (1 - \lambda)^{|s|} \tilde{p}(s)$. It holds that $\sum_{x \in \{0,1\}^n} q(x) = 2^n \hat{q}(0^n) = 2^n \tilde{p}'(0^n) = 1$.

We define $S_\epsilon = 1$ and

$$S_y = \sum_{x \in \{0,1\}^n, x_1 \cdots x_k=y} q(x)$$

for any $1 \leq k \leq n$ and $y \in \{0,1\}^k$, where $\epsilon$ denotes the empty string. The sampling algorithm is described as follows:

1. Set $y \leftarrow \epsilon$.

2. Perform the following procedure $n$ times:

   (a) If $S_{yz} < 0$ for some $z \in \{0,1\}$, set $y \leftarrow y \bar{z}$, where $\bar{z} = 1 - z$.  

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(b) Otherwise, set $y \leftarrow y0$ with probability $S_{y0}/S_y$ and $y \leftarrow y1$ with probability $1 - S_{y0}/S_y$.

3. Output $y \in \{0,1\}^n$.

A direct calculation shows that

$$S_y = 2^{n-k} \sum_{s \in \{0,1\}^n, |s| \leq c, s_{k+1} \ldots s_n = 0^{n-k}} \hat{q}(s)(-1)^{s_1 \ldots s_k \cdot y}$$

for any $1 \leq k \leq n$ and $y \in \{0,1\}^k$. Since $S_y$ is classically computable in polynomial time (when we have $\hat{q}(s)$ for all $s \in \{0,1\}^n$ with $|s| \leq c$), the runtime of the sampling algorithm is also polynomial. This sampling algorithm defines a real-valued function, denoted as $\text{Alg}(q)$, that maps $y \in \{0,1\}^n$ to the probability that the sampling algorithm outputs $y$.

To make the analysis of $\text{Alg}(q)$ easier, Bremner et al. [6] defined another function, $\text{Fix}(q) = \left(\sum_{x \in \{0,1\}^n} q(x)\right) \text{Alg}(q)$, and showed that $\text{Fix}(q)/\sum_{x \in \{0,1\}^n} q(x)$, which is $\text{Alg}(q)$, is a probability distribution and

$$||q - \text{Fix}(q)||_1 = 2 \sum_{x \in \{0,1\}^n, q(x) < 0} |q(x)|.$$

As described above, we can assume that $\sum_{x \in \{0,1\}^n} q(x) = 1$, and thus can deal with the situation where $\text{Fix}(q) = \text{Alg}(q)$. Therefore, Bremner et al.’s analysis of $\text{Fix}(q)$ directly works for $\text{Alg}(q)$. Moreover, to prove Theorem [1], it suffices to show the following lemma, which corresponds to a special case of the property of $\text{Alg}(q)$ shown by Bremner et al. [6]:

**Lemma 5.** We assume that there exists a constant $0 < \delta < 1$ such that $||\tilde{p}_A - q||_1 \leq \delta/3$. Then, $||\tilde{p}_A - \text{Alg}(q)||_1 \leq \delta$.

**Proof.** We upper-bound the value $||\tilde{p}_A - \text{Alg}(q)||_1$ as follows:

$$||\tilde{p}_A - \text{Alg}(q)||_1 \leq ||\tilde{p}_A - q||_1 + ||q - \text{Alg}(q)||_1 \leq \frac{\delta}{3} + 2 \sum_{x \in \{0,1\}^n, q(x) < 0} |q(x)|$$

$$\leq \frac{\delta}{3} + 2 \sum_{x \in \{0,1\}^n, q(x) < 0} |\tilde{p}_A(x) - q(x)| \leq \frac{\delta}{3} + 2||\tilde{p}_A - q||_1 \leq \delta,$$

which is the desired upper bound. $\Box$

Combining the above lemmas immediately implies Theorem [1].

**Proof of Theorem [1].** By Lemma [4] there exists a polynomial-time randomized algorithm which outputs the Fourier coefficients of a function $q$ over $\{0,1\}^n$ such that

$$\Pr\left[||\tilde{p}_A - q||_1 \leq \frac{\delta}{3}\right] \geq 1 - \frac{1}{\exp(n)}.$$ 

We define $\tilde{q}_A = \text{Alg}(q)$. By Lemma [5] when $||\tilde{p}_A - q||_1 \leq \delta/3$, it holds that

$$||\tilde{p}_A - \tilde{q}_A||_1 = ||\tilde{p}_A - \text{Alg}(q)||_1 \leq \delta.$$ 

By the sampling algorithm described above, $\tilde{q}_A$ is classically samplable in polynomial time. $\Box$

### 4.3 Applications of Theorem [1]

We first deal with an input-noise model where $D_\epsilon$ is applied to each qubit (initialized to $|0\rangle$) only at the start of computation, and consider IQP circuits under this input-noise model as depicted in Fig. 2(a). We show that, when noise model $A$ is replaced with this input-noise model, Theorem [1] holds for IQP circuits. Let $C = H^\otimes n DH^\otimes n$ be an arbitrary IQP circuit on $n$ qubits, where $D$ is a polynomial-size
quantum circuit consisting of $Z$, $CZ$, and $CCZ$ gates. Let $\tilde{p}_m$ be the resulting probability distribution over $\{0,1\}^n$. The input state $|0^n\rangle$ affected by noise is represented as

$$
\left[\left(1 - \frac{\varepsilon}{2}\right)|0\rangle\langle 0| + \frac{\varepsilon}{2}|1\rangle\langle 1|\right]^{\otimes n} = \sum_{y \in \{0,1\}^n} \left(1 - \frac{\varepsilon}{2}\right)^{n-|y|} \left(\frac{\varepsilon}{2}\right)^{|y|} X^y|0^n\rangle\langle 0^n|X^y.
$$

A direct calculation shows that $\tilde{p}_m = p_A$, where $p_A$ is the output probability distribution of $C$ under noise model $A$ (with rate $\varepsilon$). This is because $H^{\otimes n}DH^{\otimes n}X^y = X^yH^{\otimes n}DH^{\otimes n}$ for any $y \in \{0,1\}^n$. Thus, when the output probability distribution of $C$ is anti-concentrated, $p_A$ is classically simulatable by Theorem 1 and so is $\tilde{p}_m$. Similarly, when noise model $B$ is replaced with the input-noise model, Theorem 2 holds for IQP circuits. To show this, it suffices to represent the input state $|0^n\rangle$ affected by noise as

$$
\bigotimes_{j=1}^{n} \left[\left(1 - \frac{\varepsilon_j}{2}\right)|0\rangle\langle 0| + \frac{\varepsilon_j}{2}|1\rangle\langle 1|\right] = \sum_{y \in \{0,1\}^n} \prod_{j=1}^{n} \left(1 - \frac{\varepsilon_j}{2}\right)^{1-y_j} \left(\frac{\varepsilon_j}{2}\right)^{y_j} X^y|0^n\rangle\langle 0^n|X^y.
$$

and to apply the relation $H^{\otimes n}DH^{\otimes n}X^y = X^yH^{\otimes n}DH^{\otimes n}$ as above.

We then consider CT-ECS circuits in the noise-free setting as depicted in Fig. 2(b). Using a proof method similar to the one of Theorem 1, we show that, when only $O(\log n)$ qubits are measured, any quantum circuit in a class of CT-ECS circuits on $n$ qubits is classically simulatable in the noise-free setting. Let $C = Vu$ be an arbitrary CT-ECS circuit on $n$ qubits such that only $m = O(\log n)$ qubits are measured, and let $p$ be its output probability distribution over $\{0,1\}^m$. We can represent $p$ as

$$
p(x) = \sum_{s \in \{0,1\}^m} \hat{p}(s)(-1)^{sx}.
$$

Since $m = O(\log n)$, roughly speaking, $p(x)$ can be computed as the sum of a polynomial number of the Fourier coefficients. A direct calculation similar to the proof of Lemma 2 implies that

$$
\hat{p}(s) = \frac{1}{2^m}(0^n|U^\dagger V^\dagger (Z^s \otimes I_n-m)VU|0^n)
$$

for any $s = s_1 \cdots s_m \in \{0,1\}^m$, where $I_{n-m}$ is the identity on the qubits that are not measured. By an argument similar to the proof of Lemma 3, we want to approximate $\hat{p}(s)$ for all $s \in \{0,1\}^m$. However, in the present case, $V^\dagger (Z^s \otimes I_{n-m})V$ is the product of at most $m$ ECS operations, which is not always ECS.

We assume that $V^\dagger (Z^s \otimes I_{n-m})V$ is ECS for all $s \in \{0,1\}^m$. In this case, all the Fourier coefficients can be approximated classically in polynomial time with polynomial accuracy. Then, we define a function $q$ over $\{0,1\}^m$ as

$$
q(x) = \sum_{s \in \{0,1\}^m} \hat{p}'(s)(-1)^{sx},
$$

where $\hat{p}'(s)$ is the quantum Fourier transform of $\hat{p}(s)$.
where \( \hat{p}'(s) \) is the approximate value of \( \hat{p}(s) \) for all \( s \in \{0,1\}^m \) and \( \hat{p}'(0^m) = 1/2^m \). Since \( \|p - q\|_1 \) is upper-bounded by some inverse polynomial in \( n \), the classical sampling algorithm described in Section 4.2 implies that \( p \) is classically simulatable. More precisely, it can be approximated by a classically samplable probability distribution, which is \( \text{Alg}(q) \), with polynomial accuracy in \( l_1 \) norm.

For example, we consider a CT-ECS \( \epsilon \) circuit, which is a CT-ECS circuit such that the associated ECS operation \( V^\dagger Z_j V \) is ECS. As described at the end of Section 3.1, this means that, for any \( x \in \{0,1\}^n \), \( V^\dagger Z_j V|x\rangle \) can be represented as a linear combination of at most \( O(1) \) computational basis states. In this case, a simple calculation shows that \( V^\dagger(Z^* \otimes I_{n-m})V \) is ECS for all \( s \in \{0,1\}^m \), since it is the product of at most \( m \) ECS \( \epsilon \) (not ECS) operations. Thus, when only \( O(\log n) \) qubits are measured, any CT-ECS \( \epsilon \) circuit on \( n \) qubits is classically simulatable in the noise-free setting.

5 CT-ECS Circuits under Noise Model B

In this section, we prove Theorem 2. Its precise statement is as follows:

**Theorem 2.** Let \( C \) be an arbitrary CT-ECS circuit on \( n \) qubits such that its output probability distribution \( p \) over \( \{0,1\}^n \) satisfies \( \sum_{x \in \{0,1\}^n} p(x)^2 \leq \alpha/2^n \) for some known constant \( \alpha \geq 1 \). Let \( 0 < \delta < 1 \) be an arbitrary constant. We assume that

- a depolarizing channel with (possibly unknown) constant rate \( 0 < \epsilon_j < 1 \) is applied to the \( j \)-th qubit after performing \( C \) for any \( 1 \leq j \leq n \), which yields the probability distribution \( \tilde{p}_B \) over \( \{0,1\}^n \),
- it is possible to choose a constant \( \lambda_{\min} \) such that
  \[
  1 \leq \frac{\epsilon_{\min}}{\lambda_{\min}} \leq 1 + \frac{1}{10\sqrt{\alpha} \log \frac{10\sqrt{\alpha}}{\delta}} \]
  where \( \epsilon_{\min} = \min\{\epsilon_j | 1 \leq j \leq n\} \), and
- it is possible to choose a constant \( \lambda_j \) such that
  \[
  0 \leq \epsilon_j - \lambda_j \leq \frac{\lambda_{\min}}{10\sqrt{\alpha} \log \frac{10\sqrt{\alpha}}{\delta}}
  \]
  for any \( 1 \leq j \leq n \) with \( \epsilon_j \neq \epsilon_{\min} \), where all numbers \( j \) with \( \epsilon_j \neq \epsilon_{\min} \) are known.

Then, there exists a polynomial-time randomized algorithm which outputs (a classical description of) a probability distribution \( \tilde{q}_B \) over \( \{0,1\}^n \) such that

\[
\Pr\left[ \|\tilde{p}_B - \tilde{q}_B\|_1 \leq \left(1 + \frac{1}{1 - \lambda_{\min}}\right) \frac{1}{\delta} \right] \geq 1 - \frac{1}{\exp(n)}
\]

and \( \tilde{q}_B \) is classically samplable in polynomial time.

Although the approximate accuracy of \( \tilde{q}_B \) depends on \( \lambda_{\min} \), a typical situation might be when \( \epsilon_{\min} \) is not so large, such as \( \epsilon_{\min} \leq 1/2 \), and thus \( \lambda_{\min} \leq 1/2 \). In this case, \( \|\tilde{p}_B - \tilde{q}_B\|_1 \leq 3\delta \).

We represent the effect of noise under noise model \( B \) as the effect under noise model \( A \) with rate \( \epsilon_{\min} \) and the remaining effects. To do this, for any \( 1 \leq j \leq n \) and \( 0 \leq \delta_j \leq 1 \), we consider the noise operator \( T_{\delta_j} \) on real-valued functions over \( \{0,1\}^n \) defined as

\[
T_{\delta_j} f(x) = \mathbb{E}_{y_j \sim \mathcal{N}_{\delta_j}(x_j)}[f(x_1, \ldots, x_{j-1}, y_j, x_{j+1}, \ldots, x_n)],
\]

where \( y_j \sim \mathcal{N}_{\delta_j}(x_j) \) means that the random string \( y_j \) is drawn as

\[
y_j = \begin{cases} 
  x_j & \text{with probability } 1 - \delta_j/2, \\
  1 - x_j & \text{otherwise}
\end{cases}
\]
for any \( x_j \in \{0, 1\} \). Let \( p \) be an arbitrary probability distribution over \( \{0, 1\}^n \). It is easy to verify that \( T_{\delta_1}^1 \cdots T_{\delta_n}^n p(x) \) is equal to the probability of obtaining \( x \) by sampling an \( n \)-bit string according to \( p \) and then flipping its \( j \)-th bit with probability \( \delta_j/2 \) for any \( 1 \leq j \leq n \). Thus, the representation of \( \tilde{\rho}_B \), which is described in Section \( 2.2 \), is obtained as \( T_{\delta_1}^1 \cdots T_{\delta_n}^n p(x) \). In particular, the representation of \( \tilde{\rho}_A \) is obtained as \( T_{\varepsilon_1}^1 \cdots T_{\varepsilon_n}^n p(x) \). A key property of the operator \( T_{\delta_1}^1 \cdots T_{\delta_n}^n \) is that it is a contraction operator, i.e., \( \|T_{\delta_1}^1 \cdots T_{\delta_n}^n f\|_1 \leq \|f\|_1 \) for any real-valued function \( f \) over \( \{0, 1\}^n \). This can be simply shown by applying the triangle inequality for real numbers.

Combining Lemma \[ \text{[15]} \] with these basic facts on the noise operator, we show the following lemma on the representation of \( \tilde{\rho}_B \) and its approximability:

**Lemma 6.** Let \( C \) be an arbitrary CT-ECS circuit on \( n \) qubits such that its output probability distribution \( p \) satisfies \( \sum_{x \in \{0, 1\}^n} p(x)^2 \leq \alpha/2^n \) for some known constant \( \alpha \geq 1 \). Let \( 0 < \delta < 1 \) be an arbitrary constant. We assume that

- a depolarizing channel with constant rate \( 0 < \varepsilon_j < 1 \) is applied to the \( j \)-th qubit after performing \( C \) for any \( 1 \leq j \leq n \), which yields the probability distribution \( \tilde{\rho}_B \), and
- it is possible to choose a constant \( \lambda_{\text{min}} \) such that
  \[
  1 \leq \frac{\varepsilon_{\text{min}}}{\lambda_{\text{min}}} \leq 1 + \frac{1}{\frac{10\sqrt{\alpha}}{\delta} \log \frac{10\sqrt{\alpha}}{\delta}},
  \]
  where \( \varepsilon_{\text{min}} = \min\{\varepsilon_j | 1 \leq j \leq n\} \).

Then, the following items hold:

(i) \( \tilde{\rho}_B(x) = T_{\delta_1}^1 \cdots T_{\delta_n}^n \tilde{\rho}_A(x) \), where
  \[
  \delta_j = \frac{\varepsilon_j - \varepsilon_{\text{min}}}{1 - \varepsilon_{\text{min}}} \quad \Rightarrow \quad \tilde{\rho}_A(x) = \sum_{s \in \{0, 1\}^n} (1 - \varepsilon_{\text{min}})^{|s|} \tilde{\rho}(s)(-1)^{s \cdot x}.
  \]

(ii) There exists a polynomial-time randomized algorithm which outputs the Fourier coefficients of a function \( q \) over \( \{0, 1\}^n \) such that
  \[
  \Pr \left[ \|\tilde{\rho}_A - q\|_1 \leq \frac{\delta}{3} \right] \geq 1 - \frac{1}{\exp(n)}.
  \]

(iii) Let \( 0 \leq \delta'_j \leq 1 \) be an arbitrary constant for any \( 1 \leq j \leq n \). We assume that there exists a constant \( \beta \geq 0 \) such that \( |\delta_j - \delta'_j| \leq \beta \) for any \( 1 \leq j \leq n \). Then,
  \[
  \Pr \left[ \|T_{\delta_1}^1 \cdots T_{\delta_n}^n q - T_{\delta'_1}^1 \cdots T_{\delta'_n}^n q\|_1 \leq c\beta\sqrt{\alpha + 1} \right] \geq 1 - \frac{1}{\exp(n)},
  \]
  where \( c = \left\lceil \frac{1}{\lambda_{\text{min}} \log \frac{10\sqrt{\alpha}}{\delta}} \right\rceil \).

**Proof.** (i): As described above, the probability distribution \( \tilde{\rho}_B \) is represented as
  \[
  \tilde{\rho}_B(x) = T_{\varepsilon_1}^1 \cdots T_{\varepsilon_n}^n p(x) = \sum_{s \in \{0, 1\}^n} \prod_{j=1}^n (1 - \varepsilon_j)^{s_j} \tilde{\rho}(s)(-1)^{s \cdot x}.
  \]

We can transform this representation of \( \tilde{\rho}_B \) as follows:
  \[
  \tilde{\rho}_B(x) = \sum_{s \in \{0, 1\}^n} \left[ \prod_{j=1}^n \frac{(1 - \varepsilon_j)^{s_j}}{(1 - \varepsilon_{\text{min}})^{s_j}} \right] (1 - \varepsilon_{\text{min}})^{|s|} \tilde{\rho}(s)(-1)^{s \cdot x}
  \]
  \[
  = \sum_{s \in \{0, 1\}^n} \left[ \prod_{j=1}^n \frac{(1 - \varepsilon_j - \varepsilon_{\text{min}})}{(1 - \varepsilon_{\text{min}})} \right] (1 - \varepsilon_{\text{min}})^{|s|} \tilde{\rho}(s)(-1)^{s \cdot x}
  \]
  \[
  = T_{\delta_1}^1 \cdots T_{\delta_n}^n \sum_{s \in \{0, 1\}^n} (1 - \varepsilon_{\text{min}})^{|s|} \tilde{\rho}(s)(-1)^{s \cdot x} = T_{\delta_1}^1 \cdots T_{\delta_n}^n \tilde{\rho}_A(x),
  \]
which is the desired representation.

(ii): The probability distribution \( \tilde{p}_A \) can be regarded as the output probability distribution of \( C \) under noise model \( A \) with rate \( \varepsilon_{\min} \). Since we have an approximate value \( \lambda_{\min} \) of \( \varepsilon_{\min} \), we can apply the proof of Lemma 4. More concretely, we define

\[
c = \left( \frac{1}{\lambda_{\min}} \log \frac{10\sqrt{\alpha}}{\delta} \right).
\]

Since \( 0 < \lambda_{\min} < 1 \) and \( 10\sqrt{\alpha}/\delta > 10 \), \( c > 3 \). We can obtain \( \tilde{p}'(s) \) for all \( s \in \{0, 1\}^n \setminus \{0^n\} \) with \( |s| \leq c \) such that

\[
\Pr \left[ |\tilde{p}(s) - \tilde{p}'(s)| \leq \frac{1}{2^n f(n)} \right] \geq 1 - \frac{1}{\exp(n)},
\]

where \( f(n) = 10(n^c + 1)/\delta \). We define \( \tilde{p}'(0^n) = 1/2^n \), which is equal to \( \tilde{p}(0^n) \), and \( \tilde{p}'(s) = 0 \) for all \( s \in \{0, 1\}^n \) with \( |s| > c \). It holds that

\[
\Pr \left[ \forall s \in \{0, 1\}^n \text{ with } |s| \leq c, \ |\tilde{p}(s) - \tilde{p}'(s)| \leq \frac{1}{2^n f(n)} \right] \geq 1 - \frac{1}{\exp(n)}.
\]

Moreover, we define a function \( q \) over \( \{0, 1\}^n \) as

\[
q(x) = \sum_{s:|s| \leq c} (1 - \lambda_{\min})^{|s|} \tilde{p}'(s)(-1)^{s \cdot x}
\]

and it holds that \( ||\tilde{p}_A - q||_1 \leq \delta/3 \) under the assumption that, for all \( s \in \{0, 1\}^n \) with \( |s| \leq c \), \( |\tilde{p}(s) - \tilde{p}'(s)| \leq \frac{1}{2^n f(n)} \).

(iii): The function \( T_{\delta_1} \cdots T_{\delta_n} q \) over \( \{0, 1\}^n \) can be represented as

\[
T_{\delta_1} \cdots T_{\delta_n} q(x) = \sum_{s:|s| \leq c} \left[ \prod_{j=1}^n (1 - \delta_j)^{s_j} \right] \tilde{q}(s)(-1)^{s \cdot x},
\]

where \( \tilde{q}(s) = (1 - \lambda_{\min})^{|s|} \tilde{p}'(s) \). Since \( T_{\delta_1} \cdots T_{\delta_n} q \) can be represented similarly,

\[
T_{\delta_1} \cdots T_{\delta_n} q(x) - T_{\delta_1} \cdots T_{\delta_n} q(x) = \sum_{s:|s| \leq c} \left[ \prod_{j=1}^n (1 - \delta_j)^{s_j} - \prod_{j=1}^n (1 - \delta_j')^{s_j} \right] \tilde{q}(s)(-1)^{s \cdot x}.
\]

This representation with the relation described in Section 2.2 implies that

\[
||T_{\delta_1} \cdots T_{\delta_n} q - T_{\delta_1} \cdots T_{\delta_n} q||_1^2 \leq 2^{2n} \sum_{s:|s| \leq c} \left[ \prod_{j=1}^n (1 - \delta_j)^{s_j} - \prod_{j=1}^n (1 - \delta_j')^{s_j} \right]^2 \tilde{q}(s)^2.
\]

A simple calculation shows that, for any \( s \in \{0, 1\}^n \) with \( s_{j_1} = \cdots = s_{j_k} = 1 \),

\[
\left| \prod_{j=1}^n (1 - \delta_j)^{s_j} - \prod_{j=1}^n (1 - \delta_j')^{s_j} \right| = \left| \prod_{k=1}^{|s|} (1 - \delta_{j_k}) - \prod_{k=1}^{|s|} (1 - \delta_{j_k}') \right| \leq \sum_{k=1}^{|s|} |\delta_{j_k} - \delta_{j_k}'|.
\]

Combining this with the inequality (4) and the assumption that \( |\delta_j - \delta_j'| \leq \beta \) for any \( 1 \leq j \leq n \), it holds that

\[
||T_{\delta_1} \cdots T_{\delta_n} q - T_{\delta_1} \cdots T_{\delta_n} q||_1^2 \leq 2^{2n} \sum_{s:|s| \leq c} (|s|\beta)^2 \tilde{q}(s)^2 \leq c^2 \beta^2 2^{2n} \sum_{s:|s| \leq c} \tilde{q}(s)^2.
\]

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The remaining problem is to show that \( 2^{2n} \sum_{s: |s| \leq c} \hat{q}(s)^2 \leq \alpha + 1 \) under the assumption that, for all \( s \in \{0, 1\}^n \) with \( |s| \leq c \), \( \left| \hat{p}(s) - \hat{p}'(s) \right| \leq \frac{1}{2^n f(n)} \). This can be done by using the bounds \( |\hat{p}(s)| \leq 1/2^n \), \( \{|s| \leq c\} \leq n^c + 1 \), and \( \sum_{x \in \{0, 1\}^n} p(x)^2 \leq \alpha/2^n \) as follows:

\[
2^{2n} \sum_{s: |s| \leq c} \hat{q}(s)^2 \leq 2^{2n} \sum_{s: |s| \leq c} \hat{p}'(s)^2 \leq 2^{2n} \sum_{s: |s| \leq c} (|\hat{p}(s)| + |\hat{p}(s) - \hat{p}'(s)|)^2
\]
\[
\leq 2^{2n} \sum_{s: |s| \leq c} \left( |\hat{p}(s)| + \frac{1}{2^n f(n)} \right)^2 = 2^{2n} \sum_{s: |s| \leq c} \left( \hat{p}(s)^2 + \frac{2 |\hat{p}(s)|}{2^n f(n)} + \frac{1}{2^{2n} f(n)^2} \right)
\]
\[
\leq 2^n \sum_{x \in \{0, 1\}^n} p(x)^2 + \frac{2(n^c + 1)}{f(n)^2} + \frac{n^c + 1}{f(n)^2} \leq \alpha + \frac{\delta}{5} + \frac{\delta^2}{100(n^c + 1)} \leq \alpha + 1,
\]

which is the desired bound.

Lemma 6 implies Theorem 2 as follows:

**Proof of Theorem 2.** Using the function \( q \) obtained by Lemma 6, we define \( \tilde{q}_A = \text{Alg}(q) \) and \( \tilde{q}_B = T_{\delta_1}^1 \cdots T_{\delta_n}^n \tilde{q}_A \), where

\[
\delta_j = \frac{\lambda_j - \lambda_{\min}}{1 - \lambda_{\min}}
\]
for any \( 1 \leq j \leq n \) with \( \varepsilon_j \neq \varepsilon_{\min} \) and \( \delta_j = 0 \) for any other \( j \). In the following, we assume that, for all \( s \in \{0, 1\}^n \) with \( |s| \leq c \), \( |\hat{p}(s) - \hat{p}'(s)| \leq \frac{1}{2^n f(n)} \). In this case, \( ||\tilde{p}_A - q||_1 \leq \delta/3 \). Thus,

\[
||\tilde{p}_A - q||_1 + ||q - \tilde{q}_A||_1 \leq \delta
\]

by the proof of Lemma 5. This implies that

\[
||\tilde{p}_B - \tilde{q}_B||_1 = ||T_{\delta_1}^1 \cdots T_{\delta_n}^n \tilde{p}_A - T_{\delta_1}^1 \cdots T_{\delta_n}^n \tilde{q}_A||_1
\]
\[
\leq ||T_{\delta_1}^1 \cdots T_{\delta_n}^n \tilde{p}_A - T_{\delta_1}^1 \cdots T_{\delta_n}^n q - T_{\delta_1}^1 \cdots T_{\delta_n}^n q||_1
\]
\[
+ ||T_{\delta_1}^1 \cdots T_{\delta_n}^n q - T_{\delta_1}^1 \cdots T_{\delta_n}^n \tilde{q}_A||_1
\]
\[
\leq ||\tilde{p}_A - q||_1 + ||T_{\delta_1}^1 \cdots T_{\delta_n}^n q - T_{\delta_1}^1 \cdots T_{\delta_n}^n q||_1 + ||q - \tilde{q}_A||_1
\]
\[
= \delta + ||T_{\delta_1}^1 \cdots T_{\delta_n}^n q - T_{\delta_1}^1 \cdots T_{\delta_n}^n \tilde{q}_A||_1,
\]

(5)

where the second inequality is due to the fact that the noise operator is a contraction operator as described at the beginning of this section.

It is obvious that \( |\delta_j - \delta_j'| = 0 \) for any \( 1 \leq j \leq n \) with \( \varepsilon_j = \varepsilon_{\min} \). We show that

\[
|\delta_j - \delta_j'| \leq \frac{\delta}{1 - \lambda_{\min}} \cdot \frac{\lambda_{\min}}{10 \sqrt{\alpha} \log 10 \sqrt{\alpha}}
\]

for any \( 1 \leq j \leq n \) with \( \varepsilon_j \neq \varepsilon_{\min} \). In fact, \( \delta_j - \delta_j' \) is upper-bounded by using the bounds \( \lambda_{\min} \leq \varepsilon_{\min} < 1 \) and \( \varepsilon_j - \lambda_j \leq \frac{\lambda_{\min}}{10 \sqrt{\alpha} \log 10 \sqrt{\alpha}} \) as follows:

\[
\delta_j - \delta_j' = \left( 1 - \frac{1 - \varepsilon_j}{1 - \varepsilon_{\min}} \right) - \left( 1 - \frac{1 - \lambda_j}{1 - \lambda_{\min}} \right) = \frac{1 - \lambda_j}{1 - \lambda_{\min}} - \frac{1 - \varepsilon_j}{1 - \varepsilon_{\min}}
\]
\[
\leq \frac{1 - \varepsilon_j}{1 - \lambda_{\min}} \cdot \frac{1 - \lambda_j}{1 - \lambda_{\min}} \leq \frac{\delta}{10 \sqrt{\alpha} \log 10 \sqrt{\alpha}} \cdot \frac{\lambda_{\min}}{1 - \lambda_{\min}}.
\]

On the other hand, \( \delta_j - \delta_j' \) is lower-bounded by using the bounds \( \lambda_{\min} \leq \varepsilon_{\min} \leq \lambda_{\min} + \frac{\lambda_{\min}}{10 \sqrt{\alpha} \log 10 \sqrt{\alpha}} \) and \( \lambda_j \leq \varepsilon_j \) as follows:

\[
\delta_j - \delta_j' = \frac{\varepsilon_j - \varepsilon_{\min}}{1 - \varepsilon_{\min}} - \frac{\lambda_j - \lambda_{\min}}{1 - \lambda_{\min}} \geq \frac{\varepsilon_j - \varepsilon_{\min}}{1 - \lambda_{\min}} - \frac{\lambda_j - \lambda_{\min}}{1 - \lambda_{\min}} = \frac{\varepsilon_j - \lambda_j}{1 - \lambda_{\min}} - \frac{\varepsilon_{\min} - \lambda_{\min}}{1 - \lambda_{\min}}
\]
\[
\geq -\frac{\varepsilon_{\min} - \lambda_{\min}}{1 - \lambda_{\min}} \geq -\frac{\delta}{10 \sqrt{\alpha} \log 10 \sqrt{\alpha}} \cdot \frac{\lambda_{\min}}{1 - \lambda_{\min}}.
\]
By the definition of \( c \), it holds that \( 2 < c - 1 \leq \frac{1}{\lambda_{\min}} \log \frac{10\sqrt{\alpha}}{\delta} \). Thus, the above upper bound of \( |\delta_j - \delta'_j| \) implies that
\[
|\delta_j - \delta'_j| \leq \frac{\delta}{1 - \lambda_{\min}} \cdot \frac{\lambda_{\min}}{10\sqrt{\alpha} \log \frac{10\sqrt{\alpha}}{\delta}} \leq \frac{\delta}{1 - \lambda_{\min}} \cdot \frac{1}{10(c - 1)\sqrt{\alpha}}
\]
for any \( 1 \leq j \leq n \). By Lemma 6 with \( \beta = \frac{\delta}{1 - \lambda_{\min}} \cdot \frac{1}{10(c - 1)\sqrt{\alpha}} \),
\[
||T_{d_1} \cdots T_{d_n} q - T'_{d_1} \cdots T'_{d_n} q||_1 \leq \frac{\delta}{1 - \lambda_{\min}} \cdot \frac{c\sqrt{\alpha} + 1}{10(c - 1)\sqrt{\alpha}} \leq \frac{\delta}{1 - \lambda_{\min}},
\]
since \( c > 3 \) and \( \alpha \geq 1 \). This bound with the inequality (5) implies that
\[
||\tilde{q}_B - \tilde{q}_B||_1 \leq \delta + \frac{\delta}{1 - \lambda_{\min}} = \left( 1 + \frac{1}{1 - \lambda_{\min}} \right) \delta.
\]
The above analysis means that, to simulate \( \tilde{q}_B \), it suffices to sample \( \tilde{q}_B \), i.e., to sample \( \tilde{q}_A \) and flip its \( j \)-th bit using a biased coin with probability
\[
\frac{\delta_j'}{2} = \frac{\lambda_j - \lambda_{\min}}{2(1 - \lambda_{\min})} = \frac{1}{2} \left( 1 - \frac{1 - \lambda_j}{1 - \lambda_{\min}} \right)
\]
of heads for any \( 1 \leq j \leq n \) with \( \varepsilon_j \neq \varepsilon_{\min} \). By the sampling algorithm described in Section 4.2, \( \tilde{q}_A \) is classically samplable in polynomial time. Thus, \( \tilde{q}_B \) is also classically samplable in polynomial time. We note that, although it may not be possible to flip such a biased coin with perfect accuracy, it suffices to flip a biased coin whose probability of heads is exponentially close to that of the above coin by computing the value \( \frac{1 - \lambda_j}{1 - \lambda_{\min}} \) with exponential accuracy.

6 Conclusions and Future Work

We considered the effect of noise on the classical simulatability of CT-ECS circuits, such as IQP, Clifford Magic, conjugated Clifford, and constant-depth quantum circuits. We showed that, under noise model \( A \), if an approximate value of the noise rate is known, any CT-ECS circuit with an anti-concentrated output probability distribution is classically simulatable. This indicates that the presence of small noise drastically affects the classical simulatability of CT-ECS circuits. We also considered noise model \( B \) where the noise rate can vary with each qubit, and provided a similar sufficient condition for classically simulating CT-ECS circuits with anti-concentrated output probability distributions.

Interesting challenges would be to investigate the effect of noise on the classical simulatability of other quantum circuits that are not classically simulatable in the noise-free setting (under plausible assumptions), such as quantum circuits for Shor’s factoring and discrete logarithm algorithms [17]. At present, it seems difficult to apply our analysis directly to such quantum circuits based on the quantum Fourier transform. However, there exists a classical algorithm for simulating such circuits with sparse output probability distributions in the noise-free setting [10] and this algorithm might be useful in a noise setting. It would also be interesting to investigate fault-tolerant schemes to protect CT-ECS circuits from noise. As described in [6], a simple repetition code can be used to protect IQP circuits under noise model \( A \) with a known rate, and the circuits produced by using the error-correcting code are also IQP circuits. Similarly, the simple repetition code can be used for CT-ECS circuits under noise model \( A \) with a known rate. However, the produced circuits are in general completely different from the original ones. This would decrease the possibility of implementing such circuits, and thus it would be interesting to study how to avoid this problem.

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