Proper spaces are spectral

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Abstract. Since Hochster’s work, spectral spaces have attracted increasing interest. Through this note we intend to show that the set of proper ideals of a ring endowed with coarse lower topology is a spectral space.

1. Introduction. Hochster, in [7] has characterised spectral spaces (i.e., quasi-compact, sober, admits a basis of quasi-compact open subspaces that is closed under finite intersections) as spectra of prime ideals of rings endowed with Zariski topology. Since then spectrality has been studied in different contexts. Here are some examples. Apart from Zariski topology, several other topologies (like Lawson, Scott, lower, upper, ultra-filter, patch, inverse etc.) have been considered on prime ideals (see [2] for a detailed account). Among other classes of ideals, the set of all ideals with lower topology is a spectral space has been proved in [8], whereas [3] has shown that the spaces of all ideals, proper ideals, and radical ideals of a ring endowed with a hull-kernel-type topology are spectral. In [4], the space of the semistar operations (of finite type), endowed with a Zariski-like topology has been shown to be spectral. In [9], it has been shown that the collection of continuous valuations on a topological monoid with topology determined by any finitely generated ideal is a spectral space. In [1], the authors have considered the prime spectra of modules with different topologies and introduced some conditions under which these are spectral spaces.

Our aim is to prove that the set of proper ideals of a ring endowed with a coarse lower topology is spectral. A coarse lower topology on an arbitrary class of ideals of a ring has been studied in [5], and they are called ideal spaces. Although some of the results appearing here have been discussed in [5], but to make the present work self-contained, we have provided proofs of those results as well. To prove the spectrality, we have adapted

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the technique that avoids the checking of the existence of a basis of quasi-compact open subspaces that is closed under finite intersections. These properties rather follow from proper ideals to be an open subspace of the spectral space of all ideals (see the discussion before Lemma 2.2).

2. Proper spaces and spectrality. Suppose $A$ be a commutative ring with identity and $\text{Idl}(A)$ denotes the set of all ideals of $A$. For $\mathcal{I} \in \text{Idl}(A)$, we define $v(\mathcal{I}) = \{ \mathcal{G} \in \text{Idl}(A) \mid \mathcal{I} \subseteq \mathcal{G} \}$. The collection $\mathcal{F}_v = \{ v(\mathcal{I}) \mid \mathcal{I} \in \text{Idl}(A) \}$ of subsets of $\text{Idl}(A)$, as a closed subbase, indices a unique (coarse lower) topology on $\text{Idl}(A)$. The set $\text{Idl}(A)$ endowed with coarse lower topology is called an ideal space. Let $\text{Prp}(A)$ be the set of all proper ideals (i.e., $A \not\in \text{Prp}(A)$) of $A$. The set $\text{Prp}(A)$ with the subspace topology of the ideal space $\text{Idl}(A)$ is called a proper space.

It is well known (see [8, Theorem 4.2]) that

**Theorem 2.1.** Every ideal space $\text{Idl}(A)$ is spectral.

The proof of Theorem 2.1 is based on the fact that $\text{Idl}(A)$ is an algebraic lattice, and that is no longer true for $\text{Prp}(A)$. The proof of spectrality of $\text{Prp}(A)$ relies on the following key Lemma 2.2, which allows us to use the spectrality of $\text{Idl}(A)$. Furthermore, we can avoid proving the existence of a compact open base and closedness of open compact sets under finite intersections. Thanks to Lemma 2.2, these two properties follow from the spectrality of $\text{Idl}(A)$.

**Lemma 2.2.** A quasi-compact, sober, open subspace of a spectral space is spectral.

**Proof.** Let $S$ be a quasi-compact, sober, open subspace of a spectral space $X$. Since $S$ is quasi-compact and sober, all we need to prove is that the set $\mathcal{B}_S$ of compact open subsets of $S$ form a basis of a topology that is closed under finite intersections. The following two facts are obvious:

- A subset of $S$ is open in $S$ if and only if it is open in $X$.
- Therefore, a subset of $S$ belongs to $\mathcal{B}_S$ if and only if it belongs to $\mathcal{B}_X$.

Using these facts, we argue as follows:

Let $U$ be an open subset of $S$. Since $U$ is also open in $X$, we have $U = \bigcup \mathcal{U}$, for some subset $\mathcal{U}$ of $\mathcal{B}_X$. But each element of $\mathcal{U}$ being a subset of $U$ is a subset of $S$, and it belongs to $\mathcal{B}_S$. Therefore every open subset of $S$ can be presented as a union of compact open subsets of $S$. Now it remains to prove that $\mathcal{B}_S$ is closed under finite intersections, but this immediately follows from the fact that $\mathcal{B}_X$ is closed under finite intersections. Q.E.D.

To have the claim of a proper space $\text{Prp}(A)$ to be spectral, now it is sufficient to show that:
Then, in particular, we obtain with subbasic closed sets, we also require Alexander subbase theorem here.

which we need in order to apply finite intersection property. Since the argument goes with subbasic closed sets, we also require Alexander subbase theorem here.

**Proposition 2.3.** Every proper space $\text{Prp}(A)$ is quasi-compact.

**Proof.** Let $\{C_\alpha\}_{\alpha \in \Lambda}$ be a family of subbasic closed sets of an ideal space $\text{Prp}(A)$ such that $\bigcap_{\alpha \in \Lambda} C_\alpha = \emptyset$. Let $\{\mathcal{J}_\alpha\}_{\alpha \in \Lambda}$ be a family of ideals of $\text{Idl}(A)$ such that $\forall \alpha \in \Lambda, C_\alpha = v(\mathcal{J}_\alpha)$. Since $\bigcap_{\alpha \in \Lambda} v(\mathcal{J}_\alpha) = v(\bigcup_{\alpha \in \Lambda} \mathcal{J}_\alpha)$, we get $v(\bigcup_{\alpha \in \Lambda} \mathcal{J}_\alpha) = \emptyset$, and that $(\ast)$ gives $\bigcup_{\alpha \in \Lambda} \mathcal{J}_\alpha = A$. Then, in particular, we obtain $1 = \sum_{\alpha \in \Lambda} x_\alpha$, where $x_\alpha \in \mathcal{J}_\alpha$ and $x_\alpha \neq 0$ for $i = 1, \ldots, n$. This implies $A = \sum_{i=1}^n \mathcal{J}_a_i$. Therefore, $\bigcap_{i=1}^n C_{a_i} = \emptyset$, and hence by Alexander subbase theorem, $\text{Prp}(A)$ is quasi-compact.

Q.E.D.

(II). To show $\text{Prp}(A)$ is sober, we show that $\text{Prp}(A)$ is $T_0$ and every non-empty irreducible closed subset of $\text{Prp}(A)$ has a generic point. Since for a topological space to be $T_0$, it is necessary and sufficient that its specialisation order is a partial order (see [8]), a proper space $\text{Prp}(A)$ has the same property. In other words,

**Proposition 2.4.** Every proper space $\text{Prp}(A)$ is $T_0$.

For the proof of existence of generic points of non-empty irreducible closed sets, we need the following result.

**Lemma 2.5.** Every non-empty subbasic closed subset of $\text{Prp}(A)$ is irreducible.

**Proof.** We show that $v(\mathcal{J}) = c1(\mathcal{J})$ for every non-empty subbasic closed set $v(\mathcal{J})$ of $\text{Prp}(A)$, and that will prove the lemma. Since $c1(\mathcal{J})$ is the smallest closed set containing $\mathcal{J}$, and since $v(\mathcal{J})$ is a closed set containing $\mathcal{J}$, we immediately have $c1(\mathcal{J}) \subseteq v(\mathcal{J})$. For the reverse inclusion, we first consider the case that $c1(\mathcal{J}) = \text{Prp}(A)$. Then the above containment forces $v(\mathcal{J}) = c1(\mathcal{J})$. Now suppose that $c1(\mathcal{J}) \neq \text{Prp}(A)$. Being a closed set, $c1(\mathcal{J})$ is expressible as $c1(\mathcal{J}) = \bigcap_{\alpha \in \Omega} (\bigcup_{i=1}^{n\alpha} v(\mathcal{J}_{\alpha}))$, for some index set $\Omega$, such that, for each $\alpha \in \Omega$, there is a positive integer $n_\alpha$ and ideals $\mathcal{J}_{\alpha_1}, \ldots, \mathcal{J}_{\alpha_{n\alpha}}$ of $\text{Idl}(A)$. Since
If $C$ is an irreducible closed subset of a topological space $X$ and $\mathcal{S}$ is a closed subbase of $X$, then it is known (see [6, Section 7.2, p. 308]) that $C$ is the intersection of members of $\mathcal{S}$. For a proper space we get more in the sense that the converse of Lemma 2.5 is also true.

**Lemma 2.6.** If $C$ is a non-empty irreducible closed subset of a proper space $Prp(A)$, then $C = v(\mathcal{I})$ for some $\mathcal{I} \in Prp(A)$.

We are now ready to prove the sobriety of $Prp(A)$.

**Proposition 2.7.** Every proper space $Prp(A)$ is sober.

**Proof.** Since by Lemma 2.6, every non-empty irreducible closed subset of $Prp(A)$ is of the form $v(\mathcal{I})$, where $\mathcal{I} \in Prp(A)$. Suppose $v(\mathcal{J})$ be a non-empty irreducible closed subset of $Prp(A)$. From the fact $\mathcal{I} \in v(\mathcal{J})$, we immediately have $c1(\mathcal{J}) \subseteq v(\mathcal{J})$. Therefore, to show $v(\mathcal{J})$ has a generic point, it is sufficient to prove that $c1(\mathcal{J}) \supseteq v(\mathcal{J})$. Since $\mathcal{S}$ is a closed subbase of $Prp(A)$, the required containment now follows from Lemma 2.5. Moreover, by Proposition 2.4, an ideal space $Prp(A)$ is $T_0$. Therefore, we obtain the uniqueness of a generic point. Q.E.D.

(III). What remains is to show that

**Proposition 2.8.** A proper space $Prp(A)$ is open in $Idl(A)$.

**Proof.** Since $A \in Idl(A)$, by Lemma 2.5, $A = v(A) = c1(A)$, and hence $Idl(A) \setminus Prp(A)$ is closed, and that implies $Prp(A)$ is open. Q.E.D.

Since $Prp(A)$ is quasi-compact (Proposition 2.3), sober (Proposition 2.7), and an open subspace of $Idl(A)$ (Proposition 2.8), by Lemma 2.2, we have

**Theorem 2.9.** Every proper space is spectral.

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