FINITE GROUP ACTIONS ON SYMPLECTIC CALABI-YAU
4-MANIFOLDS WITH $b_1 > 0$

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Abstract. This is the first of a series of papers devoted to the topology of symplectic Calabi-Yau 4-manifolds endowed with certain symplectic finite group actions. We completely determined the fixed-point set structure of a finite cyclic action on a symplectic Calabi-Yau 4-manifold with $b_1 > 0$. As an outcome of this fixed-point set analysis, the 4-manifold was shown to be a $T^2$-bundle over $T^2$ in some circumstances, e.g., in the case where the group action is an involution which fixes a 2-dimensional surface in the 4-manifold. Our project on symplectic Calabi-Yau 4-manifolds is based on an analysis of existence and classification of disjoint embeddings of certain configurations of symplectic surfaces in a rational 4-manifold. This paper laid the groundwork for such an analysis at the homological level. Some other results which are of independent interest, concerning the maximal number of disjointly embedded symplectic $(-2)$-spheres in a rational 4-manifold, were also obtained.

1. Introduction and the main results

In this paper, we study symplectic finite group actions on symplectic Calabi-Yau 4-manifolds with $b_1 > 0$. (Recall that a symplectic 4-manifold $M$ is called Calabi-Yau if $K_M$ is trivial.) Our starting point is the recent construction in [7], where to each symplectic 4-manifold $M$ equipped with a finite symplectic $G$-action, we associate a symplectic 4-manifold, denoted by $M_G$, and an embedding $D \to M_G$ of a disjoint union of configurations of symplectic surfaces. Roughly speaking, the 4-manifold $M_G$ is constructed by first de-singularizing the symplectic structure of the quotient orbifold $M/G$ along the 2-dimensional singular strata, making the underlying space $|M/G|$ into a symplectic 4-orbifold with only isolated singularities. Then $M_G$ is taken to be the minimal symplectic resolution of the symplectic 4-orbifold $|M/G|$, and $D$ is simply the pre-image of the singular set of the original orbifold $M/G$ in $M_G$. See [7] for more details. The idea is to recover the $G$-action on $M$, particularly the 4-manifold $M$, by analyzing the embedding $D \to M_G$. With this understood, it was shown (cf. [7], Theorem 1.9) that if $M$ is Calabi-Yau, then $M_G$ is either a symplectic 4-manifold with torsion canonical class, or a rational 4-manifold, or an irrational ruled 4-manifold over $T^2$. Then our basic observation is that, when $M_G$ is rational or ruled, it is possible to effectively recover the original 4-manifold $M$ by analyzing the embedding $D \to M_G$. 

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Moreover, as it turns out, one can also derive new constraints on the fixed-point set structure of the $G$-action from non-existence results for the embedding $D \to M_G$.

As an initial step toward understanding the topology of symplectic Calabi-Yau 4-manifolds endowed with a symplectic finite group action, we consider first the case where the 4-manifold $M$ has $b_1 > 0$, and determine the fixed-point set structure of a finite cyclic action on $M$. As a result of our analysis, we obtain the following

**Theorem 1.1.** Suppose $M$ is a symplectic Calabi-Yau 4-manifold with $b_1 > 0$ which is endowed with a finite symplectic $G$-action. If the resolution $M_G$ is irrational ruled, or $M_G$ is rational and $G = \mathbb{Z}_2$, then $M$ must be diffeomorphic to a $T^2$-bundle over $T^2$ with homologically essential fibers.

We remark that in Theorem 1.1, $M$ is in fact diffeomorphic to a hyperelliptic surface in the case of $G = \mathbb{Z}_2$ and $M_G$ is rational. On the other hand, we note that in the case of $G = \mathbb{Z}_2$, $M_G$ is rational or ruled if and only if the fixed-point set $M^G$ contains a 2-dimensional component. We state this special case in the following

**Corollary:** Let $M$ be a symplectic Calabi-Yau 4-manifold with $b_1 > 0$, which is equipped with a symplectic $\mathbb{Z}_2$-action whose fixed-point set contains a 2-dimensional component. Then $M$ must be diffeomorphic to a $T^2$-bundle over $T^2$ with homologically essential fibers.

R. Inanc Baykur [2] informed us that he has examples of symplectic Calabi-Yau 4-manifolds with $b_1 = 2$ and 4, which are constructed using symplectic Lefschetz pencils, and which come with a natural symplectic $\mathbb{Z}_2$-action whose fixed-point set contains a 2-dimensional component. Our theorem shows that these symplectic Calabi-Yau 4-manifolds all have the standard smooth structure.

To put Theorem 1.1 in a perspective, recall that symplectic 4-manifolds can be classified into four classes according to their symplectic Kodaira dimension $\kappa^s$, which is a smooth invariant and takes values in $\{-\infty, 0, 1, 2\}$. (The classification is analogous to the classification in complex surface theory, but the relevant definitions are given in completely different ways. For Kähler surfaces, the two classifications coincide. See [26].) Furthermore, as a culmination of the seminal works of Gromov, McDuff, and Taubes [22, 31, 40], the case of $\kappa^s = -\infty$ is completely determined: these symplectic 4-manifolds are precisely the rational or ruled surfaces.

Much effort has also been devoted to the next case, i.e., $\kappa^s = 0$. First, based on Taubes’ theory [40], T.-J. Li (cf. [23]) showed that a minimal symplectic 4-manifold $M$ with $\kappa^s = 0$ is either Calabi-Yau (i.e., $K_M$ is trivial), or a double cover of $M$ is Calabi-Yau. Note that a symplectic Calabi-Yau 4-manifold is spin. Using the Bauer-Furuta theory of spin 4-manifolds, together with Taubes’ theorem [10] and the classical Rochlin Theorem, the following homological constraints were obtained, see [32, 26, 1, 27]:

- A symplectic Calabi-Yau 4-manifold $M$ either has the integral homology and intersection form of a $K3$ surface, or has the rational homology and intersection form of a $T^2$-bundle over $T^2$; in particular, $0 \leq b_1(M) \leq 4$, and if $b_1(M) > 0$, $M$ has zero Euler number and signature. (If $M$ is non-Calabi-Yau but a double cover of $M$ is Calabi-Yau, then $M$ is an integral homology Enriques surface.)
• In addition, for the case of $b_1(M) = 4$, the cohomology ring $H^*(M, \mathbb{Q})$ is isomorphic to $H^*(T^4, \mathbb{Q})$ (cf. [37]).

The above homological constraints are in sharp contrast to the flexibility known in higher dimensional symplectic Calabi-Yau manifolds, see e.g., [13]. Using a covering trick, one can also obtain interesting constraints on the fundamental group (as well as homotopy type in the case of $b_1 > 0$) of a symplectic Calabi-Yau 4-manifold (cf. [15]), e.g., in the case of $b_1 = 0$, the fundamental group has no subgroup of finite index.

As for examples, besides $K3$ surface, all orientable $T^2$-bundles over $T^2$ are symplectic Calabi-Yau 4-manifolds (cf. [20, 26]). (A topological classification of $T^2$-bundles over $T^2$ is given in [38].) We remark that not all $T^2$-bundles over $T^2$ admit a complex structure, and not all $T^2$-bundles over $T^2$ have homologically essential fibers (cf. [20]). If a complex surface is a symplectic Calabi-Yau 4-manifold, then it is either a $K3$ surface, a complex torus, a primary Kodaira surface, or a hyperelliptic surface. With this understood, the following has been an open question (cf. [26, 12]):

Does there exist a symplectic Calabi-Yau 4-manifold other than the known examples, i.e., $T^2$-bundles over $T^2$ or $K3$ surface?

We remark that the basic smooth invariants in 4-manifold theory (e.g., Seiberg-Witten invariants) are ineffective in distinguishing homeomorphic symplectic Calabi-Yau 4-manifolds, so one hopes to construct new examples which have different topological invariants such as fundamental group. On the other hand, concerning characterizing diffeomorphism type of a symplectic Calabi-Yau 4-manifold, Theorem 1.1 is the first result of such kind (under a finite symmetry condition). Finally, for connections of this question with hypsymplectic structures and Donaldson’s conjecture, we refer the readers to the recent article [14].

With the preceding understood, the idea of our project is to specialize in symplectic Calabi-Yau 4-manifolds $M$ which admits a $G$-action such that $M_G$ is rational or ruled, and through $D \to M_G$, to gain insight about the topology of $M$. Note that with Theorem 1.1, the case where $M_G$ is irrational ruled is closed.

Now we state the results on the fixed-point set structure of a finite cyclic action on symplectic Calabi-Yau 4-manifolds with $b_1 > 0$. The case of free actions is trivial and therefore omitted. For simplicity, assume the fixed-point set $M^G$ is nonempty. We shall separate the prime order and non-prime order cases.

**Theorem 1.2.** Let $M$ be a symplectic Calabi-Yau 4-manifold with $b_1 > 0$, which is equipped with a symplectic $G$-action of prime order. Then the fixed-point set structure of the $G$-action and the symplectic resolution $M_G$ must be one of the following cases:

1. Suppose $M_G$ has torsion canonical class. Then either $G = \mathbb{Z}_2$ or $G = \mathbb{Z}_3$. In the former case, $G$ either has 8 isolated fixed points, with $b_1(M) < 4$ and $M_G$ being an integral homology Enriques surface, or has 16 isolated fixed points, with $b_1(M) = 4$ and $M_G$ being an integral homology $K3$ surface. In the latter case where $G = \mathbb{Z}_3$, the fixed point set consists of 9 isolated points of type $(1, 2)$, with $b_1(M) = 4$ and $M_G$ being an integral homology $K3$ surface.

2. Suppose $M_G$ is irrational ruled. Then $G = \mathbb{Z}_2$ or $\mathbb{Z}_3$, the fixed point set consists of only tori with self-intersection zero, and $M_G$ is a $S^2$-bundle over $T^2$. 
Theorem 1.3. Let $G$ be a symplectic Calabi-Yau 4-manifold with $b_1 > 0$, equipped with a symplectic $G$-action where $G$ is cyclic of non-prime order. Suppose $M_G$ is rational or ruled, but for any prime order subgroup $H$, $M_H$ has torsion canonical class. Then $G = \mathbb{Z}_4$ or $\mathbb{Z}_8$. Moreover,

(i) If $G = \mathbb{Z}_4$, there are two possibilities:
   (a) the $G$-action has 4 isolated fixed points, where exactly 2 of the fixed points are of type $(1,1)$, and $M_G = \mathbb{CP}^2 \# 11\mathbb{CP}^2$; in this case, $b_1(M) = 2$,
   (b) the $G$-action has 4 isolated fixed points, all of type $(1,1)$, and 12 isolated points of isotropy of order 2, with $M_G = \mathbb{CP}^2 \# 13\mathbb{CP}^2$; in this case, $b_1(M) = 4$.

(ii) If $G = \mathbb{Z}_8$, there are two possibilities, where in both cases, $b_1(M) = 4$:
   (a) the $G$-action has 2 isolated fixed points, all of type $(1,3)$, and 2 isolated points of isotropy of order 4 of type $(1,3)$, and 12 isolated points of isotropy of order 2, with $M_G = \mathbb{CP}^2 \# 11\mathbb{CP}^2$;
   (b) the $G$-action has 2 isolated fixed points, all of type $(1,5)$, and 2 isolated points of isotropy of order 4 of type $(1,1)$, and 12 isolated points of isotropy of order 2, with $M_G = \mathbb{CP}^2 \# 11\mathbb{CP}^2$.

We remark that in Theorems 1.2 and 1.3, the cases where $M_G$ is rational has either $b_1(M) = 2$ or 4. One can easily write down examples of holomorphic actions on a hyperelliptic surface or a complex torus realizing the fixed-point set structures in these cases, except for the ones in Theorem 1.2(3)(iii) and Theorem 1.3(ii), where the group $G$ has order 5 or 8 and $b_1(M) = 4$. Holomorphic actions on a complex torus which realize these fixed-point set structures can be found in Fujiki [17], Table 6.

For a large part, the proofs of Theorems 1.2 and 1.3 employ the standard techniques in group actions, i.e., the Lefschetz fixed point theorem and the $G$-signature theorem, coupled with the standard results in symplectic topology of rational and ruled surfaces and the topological constraints of minimal symplectic 4-manifolds with $\kappa^s = 0$ through
the use of $M_G$. Some of the cases also require the use of $G$-index theorem for Dirac operators and Seiberg-Witten theory. These traditional methods are quite efficient in determining the fixed-point set structure for the isolated fixed points, however, for the 2-dimensional fixed components, these methods have their natural limitations. The reason is that the 2-dimensional fixed components (particularly the tori of self-intersection zero) often do not make any contribution in the various $G$-index theorem calculations, hence cannot be detected by these methods. (See [10], Section 3, for a summary of these traditional methods.)

With this understood, in order to obtain further constraints on the 2-dimensional fixed components, we analyze the embedding of $D$ in $M_G$. (Note that the 2-dimensional fixed components form part of $D$.) In particular, we shall examine the homology classes of the components of $D$, which are symplectic surfaces forming a disjoint union of configurations. It turns out that the main difficulty occurs when $M_G$ is rational.

To explain this aspect of the story, which is the main technical contribution in this paper, we let $(X, \omega)$ be a symplectic rational 4-manifold, where $X = \mathbb{CP}^2 \# N \overline{\mathbb{CP}}^2$. We shall denote the canonical line bundle of $(X, \omega)$ by $K_\omega$ to indicate the dependence on $\omega$. We also use $K_X$ when the dependence on $\omega$ needs not to be emphasized.

Let $\mathcal{E}_X$ be the set of classes in $H^2(X)$ which can be represented by a smooth ($-1$)-sphere, and let $\mathcal{E}_\omega := \{ E \in \mathcal{E}_X | c_1(K_\omega) \cdot E = -1 \}$. Then each class in $\mathcal{E}_\omega$ can be represented by a symplectic $(-1)$-sphere (cf. [28]); in particular, $\omega(E) > 0$ for any $E \in \mathcal{E}_\omega$. With this understood, a basis $H, E_1, \cdots, E_N$ of $H^2(X)$ is called a reduced basis of $(X, \omega)$ if the following are true:

- it has a standard intersection form, i.e., $H^2 = 1$, $E_i^2 = -1$ and $H \cdot E_i = 0$ for any $i$, and $E_i \cdot E_j = 0$ for any $i \neq j$;
- $E_i \in \mathcal{E}_\omega$ for each $i$, and moreover, the following area conditions are satisfied:
  - $\omega(E_N) = \min_{E \in \mathcal{E}_\omega} \omega(E)$, and for any $i < N$, $\omega(E_i) = \min_{E \in \mathcal{E}_i} \omega(E)$ where $\mathcal{E}_i := \{ E \in \mathcal{E}_\omega | E \cdot E_j = 0 \ \forall j > i \}$;
- $c_1(K_\omega) = -3H + E_1 + \cdots + E_N$.

We mention the following constraints on the symplectic areas:

- $\omega(H) > 0$, and $\omega(E_i) \geq \omega(E_j)$ for any $i < j$;
- for any $i \neq j$, $H - E_i - E_j \in \mathcal{E}_\omega$, so that $\omega(H - E_i - E_j) > 0$; and
- $\omega(H - E_i - E_j - E_k) > 0$ for any distinct $i, j, k$.

The readers are referred to [30] for more details. We remark that a reduced basis is not necessarily unique, however, the symplectic areas of its classes

$$(\omega(H), \omega(E_1), \cdots, \omega(E_N))$$

uniquely determine the symplectic structure $\omega$ up to symplectomorphisms, cf. [24]. Finally, we mention the following useful result from [24]:

**Suppose $N \geq 2$. Then for any $\omega$-compatible almost complex structure $J$, any class $E \in \mathcal{E}_\omega$ which has the minimal symplectic area can be represented by an embedded $J$-holomorphic sphere. In particular, the class $E_N$ in a reduced basis $H, E_1, \cdots, E_N$ can be represented by a $J$-holomorphic $(-1)$-sphere for any given $J$.**

With the preceding understood, we fix a reduced basis $H, E_1, E_2, \cdots, E_N$ of $(X, \omega)$. Then for any symplectic surface in $X$, its homology class $A$ can be expressed in terms
of the reduced basis $H, E_1, E_2, \cdots, E_N$:

$$A = aH - \sum_{i=1}^{N} b_i E_i, \text{ where } a \in \mathbb{Z}, b_i \in \mathbb{Z}. $$

The numbers $a, b_i$ are called the $a$-coefficient and $b$-coefficients of $A$. By the adjunction formula, the numbers $a$ and $b_i$ are bound by a set of equations involving the self-intersection number $A^2$ and the genus of the surface. It follows easily from these equations that for each fixed value of the $a$-coefficient, there are only finitely many possible values for the $b_i$-coefficients. However, for each given symplectic surface, there is no apriori upper bound for the $a$-coefficient of its class $A$, although there is a natural lower bound of the $a$-coefficient (cf. Lemmas 3.1 and 3.2).

Now suppose $D$ is a disjoint union of configurations of symplectic surfaces embedded in $X$, where its components are denoted by $F_k$. The first step in approaching the problem of existence and classification of $D \to X$ is to look at the classes of the components $F_k$ in a given reduced basis. This process often involves a case-by-case examination, hence it is important that for each component $F_k$, there are only finitely many possible homological expressions. Such a finiteness can be achieved by bounding the values of the $a$-coefficient of each $F_k$, as the self-intersection number $F_k^2$ and the genus of $F_k$ are all pre-determined by $D \to X$.

In the present situation, $c_1(K_\omega)$ is supported in $D$. More precisely,

$$c_1(K_\omega) = \sum_k c_k F_k, \text{ where } c_k \in \mathbb{Q} \text{ and } c_k \leq 0. $$

Since the $a$-coefficient of $c_1(K_\omega)$ equals $-3$, for those components $F_k$ with $c_k \neq 0$, the $a$-coefficient can not be arbitrarily large. However, if $F_k$ is a $(-2)$-sphere, which is either disjoint from other components, or appears in a configuration of only $(-2)$-spheres, then $c_k = 0$, and there is no bearing on the $a$-coefficient of $F_k$ from $c_1(K_\omega)$.

It turns out that we can remedy this issue by imposing an auxiliary area condition. More concretely, let $A$ be the class of a symplectic $(-\alpha)$-sphere where $\alpha = 2$ or $3$. If the area condition $\omega(A) < -c_1(K_\omega) \cdot [\omega]$ is satisfied, then $A$ must take the following expression in a given reduced basis:

$$A = aH - (a - 1)E_{j_1} - E_{j_2} - \cdots - E_{j_{2a+\alpha}}. $$

In particular, the $a$-coefficient of $A$ has an upper bound in terms of $N$:

$$a \leq \frac{1}{2}(N - \alpha) $$

(See Lemma 3.4.) On the other hand, for the topological problem of classifying embeddings of $D$ in $X$, one can always freely impose such an area condition by working with a different symplectic structure (cf. Lemma 4.1). Thus in principle, at least for the problem we have at hand, we have developed the necessary tools in this paper to classify the possible embeddings $D \to M_G$ at the homological level. In forthcoming papers (cf. [8]), we shall further develop techniques in order to understand the possible embeddings $D \to M_G$ beyond the homological level.

In the course of the proof of Theorem 1.1, we also discovered the following result which is of independent interest.
**Theorem 1.4.** Let $X = \mathbb{CP}^2 \# N\overline{\mathbb{CP}^2}$ where $N = 7, 8$ or $9$. There exist no $N$ disjointly embedded symplectic $(−2)$-spheres in $X$.

We remark that by a theorem of Ruberman [35], there exist $N$ disjointly embedded smooth $(−2)$-spheres in $X = \mathbb{CP}^2 \# N\overline{\mathbb{CP}^2}$ for any $N ≥ 2$. On the other hand, for $N = 7$ and $8$, there exist $N$ homology classes $F_1, F_2, \ldots, F_N \in H_2(X)$, where $F_i \cdot F_j = 0$ for any $i \neq j$, and each individual $F_i$ can be represented by a symplectic $(−2)$-sphere (cf. Lemma 5.1). The above theorem says that these homology classes can not be represented simultaneously by disjoint symplectic $(−2)$-spheres. For $N = 9$, the corresponding homology classes do not exist (cf. Lemma 5.1).

The proof of Theorem 1.4 relies on a recent theorem of Ruberman and Starkston, which asserts that the combinatorial line arrangement coming from the Fano plane has no topological $\mathbb{C}$-realization (cf. [36]). Our result and method raises naturally the following interesting

**Question:** For each $N ≥ 2$, what is the maximal number of disjointly embedded symplectic $(−2)$-spheres in the rational 4-manifold $\mathbb{CP}^2 \# N\overline{\mathbb{CP}^2}$?

We point out that for any $N ≥ 3$ and odd, there always exist $N − 1$ disjointly embedded symplectic $(−2)$-spheres in $\mathbb{CP}^2 \# N\overline{\mathbb{CP}^2}$. So for $N = 7$ and $9$, the maximal number is $6$ and $8$ respectively.

As for the proof of Theorem 1.1, the case where $G = \mathbb{Z}_2$ and $M_G$ is rational is the most delicate one. Here the key technical result, stated as Lemma 5.1, is a classification of all possible homological expressions (in a reduced basis) of the classes of any given set of $8$ disjointly embedded symplectic $(−2)$-spheres in the rational elliptic surface $\mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$, where the symplectic structure on $\mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$ is chosen to obey a certain set of delicate area constraints on the $(−2)$-spheres (such a symplectic structure always exists by Lemma 4.1). The proof of Theorem 1.4 also relies on this technical result.

The organization of the paper is as follows. In Section 2, we give an examination of the fixed-point set structure using the traditional methods in group actions, which is coupled with some standard results and techniques in symplectic 4-manifolds and Seiberg-Witten theory. Section 3 is occupied by a study of symplectic surfaces in rational 4-manifolds. We begin by deriving some basic constraints on the $a, b_i$-coefficients of a class $A$ which is represented by a connected, embedded symplectic surface. The later part of the section focuses on the classes of symplectic spheres; in particular, it contains Lemma 3.4, which gives an upper bound on the $a$-coefficient of a symplectic $(−2)$-sphere or $(−3)$-sphere under an area condition. In Section 4, we begin by proving a lemma (i.e., Lemma 4.1) which allows us to freely impose certain auxiliary area conditions. This lemma, especially when combined with Lemma 3.4, proves to be very critical in our analysis of the embedding $D \rightarrow M_G$. We then prove several non-existence results concerning certain symplectic configurations in rational 4-manifolds. These results are used to further remove some ambiguities concerning 2-dimensional fixed components in Section 2. In Section 5, we give proofs of the main theorems.

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2. THE FIXED-POINT SET: A PRELIMINARY EXAMINATION

In this section, we give a preliminary analysis of the fixed-point set structure, using mainly the traditional methods, i.e., the Lefschetz fixed point theorem and the \( G \)-signature theorem. Some of the cases also require additional tools such as the \( G \)-index theorem for Dirac operators and Seiberg-Witten theory. Throughout this section, \( M \) is a symplectic Calabi-Yau 4-manifold with \( b_1 > 0 \), equipped with a symplectic \( G \)-action where \( G \) is finite cyclic and the fixed-point set \( M^G \neq \emptyset \). Note that for any subgroup \( H \), the fixed-point set \( M^H \neq \emptyset \) as well. Throughout this section, we let \( \Sigma_i \) be the \( 2 \)-dimensional components in \( M^G \) and denote by \( g_i \) the genus of \( \Sigma_i \). Note that by the adjunction formula and the fact that \( c_1(K_M) = 0 \), one has \( \Sigma^2_i = 2g_i - 2 \) for each \( i \).

We begin with an analysis for the case where \( b_1(M) = 2 \) or \( 3 \), and the resolution of the \( G \)-action \( M_G \) has torsion canonical class.

**Lemma 2.1.** Suppose \( b_1(M) = 2 \) or \( 3 \), and \( G \) is of prime order \( p \) such that \( M_G \) has torsion canonical class. Then \( p = 2 \) and \( M^G \) consists of 8 isolated points. Furthermore, \( b_1(M/G) = 0 \) and \( b_2^+(M/G) = 1 \).

*Proof.* According to [7], Lemma 4.1, \( M_G \) is rational or ruled if and only if the orbifold \( M/G \) contains a \( 2 \)-dimensional singular component, or an isolated singular point of non-Du Val type. It follows immediately that \( M/G \) has only isolated Du Val singularities. Now recall the following version of the Lefschetz fixed point theorem,

\[
 p \cdot \chi(M/G) = \chi(M) + (p - 1) \cdot \#M^G.
\]

With \( \chi(M) = 0 \), and observing that the resolution of each singular point of \( M/G \) is a chain of \( p - 1 \) spheres, we obtain the following expression

\[
 \chi(M_G) = \chi(M/G) + (p - 1) \cdot \#M^G = (p - 1)(\frac{1}{p} + 1) \cdot \#M^G.
\]

On the other hand, \( M_G \) has torsion canonical class, so that \( \chi(M_G) = 0, 12, \) or \( 24 \). It is clear that \( \chi(M_G) > 0 \), so that \( \chi(M_G) = 12 \) or \( 24 \). We also note that \( b_1(M_G) = 0 \) in these two cases. Moreover, since \( b_1(M) = 2 \) or \( 3 \), we have \( b_2^+(M/G) \leq b_2^+(M) \leq 2 \), so that \( \chi(M_G) = 12 \) must be true. The equation \( (p - 1)(\frac{1}{p} + 1) \cdot \#M^G = 12 \) has only one solution: \( p = 2 \) and \( \#M^G = 8 \). Finally, note that \( b_1(M/G) = b_1(M_G) = 0 \), and \( b_2^+(M/G) = b_2^+(M_G) = 1 \). This finishes off the proof. \( \square \)

2.1. The case where \( b_1 = 2 \). We first assume \( G \) is of prime order \( p > 1 \) and let \( g \in G \) be a generator of \( G \). We begin with the easier case where \( M_G \) is irrational ruled. Note that this happens exactly when \( b_1(M/G) = 2 = b_1(M) \), which means that the action of \( G \) on \( H^1(M; \mathbb{R}) \) is trivial.

**Lemma 2.2.** Suppose \( G \) is of prime order and \( M_G \) is irrational ruled. Then the fixed-point set \( M^G \) consists of a disjoint union of tori of self-intersection zero.

*Proof.* Let \( \{q_j\} \) be the set of isolated fixed points and let \( z := \#\{q_j\} \). We shall first compute the Lefschetz number \( L(g, M) = \sum_{k=0}^4 (-1)^k\text{tr}(g|_{H^k(M;\mathbb{R})}) \) and the Sign number \( \text{Sign}(g, M) = \text{tr}(g|_{H^2^+ (M;\mathbb{R})}) - \text{tr}(g|_{H^2^- (M;\mathbb{R})}) \). To this end, we first observe
that $b_2^s(M/G) = 1$. To see this, note that $b_2^s(M) = 1$, so that either $b_2^s(M/G) = 0$ or $b_2^s(M/G) = 1$. Suppose to the contrary that $b_2^s(M/G) = 0$. Then $G = \mathbb{Z}_2$ must be true. With this understood, the Lefschetz fixed point theorem gives

$$
\sum_i (2 - 2g_i) + z = L(g, M) = 2 - 2 \times 2 + 1 - 1 = -2,
$$

where on the other hand, the $G$-signature theorem gives

$$
\sum_i \Sigma_i^2 = \text{Sign}(g, M) = \text{tr}(g|_{H^2_{+}}) - \text{tr}(g|_{H^2_{-}}) = 1 - (-1) = 2.
$$

With $\Sigma_i^2 = 2g_i - 2$ for each $i$, it follows easily that $z = 0$, i.e., there are no isolated fixed points. As a consequence, we note that the underlying space of $M/G$ is smooth, and it is simply the resolution $M_G$, which is an irrational ruled 4-manifold by the assumption. But this implies that $b_2^s(M/G) = b_2^s(M_G) \geq 1$, contradicting the assumption $b_2^s(M/G) = 0$. Hence we must have $b_2^s(M/G) = 1$. With $b_2^s(M/G) = 1$, it follows easily that $L(g, M) = 0$ and $\text{Sign}(g, M) = 0$.

The equation $L(g, M) = 0$ implies $z = \sum_i (2g_i - 2) = \sum_i \Sigma_i^2$. Suppose to the contrary that $z > 0$. Then there is a component $\Sigma_i$ such that $\Sigma_i^2 > 0$. Since $b_2^s(M/G) = 1$, it follows easily that there is only one such component. As a consequence, if we denote by $c_i$ the normal weight along each $\Sigma_i$, then by replacing $g$ by a suitable power of it, we may assume that the normal weight of the action of $g$ along the component $\Sigma_i$ with $\Sigma_i^2 > 0$ equals 1. Let $(a_j, b_j)$ be the weights of the action of $g$ at the isolated fixed point $q_j$. Then with $z = \sum_i \Sigma_i^2$, and observing that $\cot\left(\frac{a_i\pi}{p}\right) \cdot \cot\left(\frac{b_i\pi}{p}\right) < \csc^2\left(\frac{\pi}{p}\right)$ for each $j$, it follows from the $G$-signature theorem that

$$
\text{Sign}(g, M) = - \sum_j \cot\left(\frac{a_j\pi}{p}\right) \cdot \cot\left(\frac{b_j\pi}{p}\right) + \sum_i \csc^2\left(\frac{c_i\pi}{p}\right) \Sigma_i^2
$$

$$
> \sum_i \left(\csc^2\left(\frac{c_i\pi}{p}\right) - \csc^2\left(\frac{\pi}{p}\right)\right) \cdot \Sigma_i^2.
$$

Since $c_i = 1$ when $\Sigma_i^2 > 0$, it follows easily that $\sum_i \left(\csc^2\left(\frac{c_i\pi}{p}\right) - \csc^2\left(\frac{\pi}{p}\right)\right) \cdot \Sigma_i^2 \geq 0$. This leads to a contradiction that $\text{Sign}(g, M) > 0$, hence $z = 0$ must be true.

With $z = 0$, $M_G$ is simply the underlying manifold of $M/G$, which must be a $S^2$-bundle over $T^2$. This is because it is a ruled surface over $T^2$ (cf. [7]), and because $b_2^s(M/G) = 1$. It remains to show that each $\Sigma_i$ is a torus. This follows easily by observing that $\sum_i (2g_i - 2) = z = 0$, and that $g_i > 0$ for each $i$. The latter is true because if $\Sigma_i$ is a sphere, then $\Sigma_i^2 = -2$, so that $\Sigma_i$ descends to a $(-2p)$-sphere in $M_G$. But $M_G$ is a $S^2$-bundle over $T^2$, it does not contain any $(-2p)$-sphere. This finishes off the proof.

Next we consider the case where $M_G$ is rational; note that this happens exactly when $b_1(M/G) = 0$. First observe that with $b_1(M) = 2$, the action of $G$ on $H^1(M; \mathbb{R})$ is given by rotations. We choose a generator $g \in G$ such that the action of $g$ on $H^1(M; \mathbb{R})$ is given by a rotation of angle $\frac{2\pi}{p}$. By Poincaré duality, the action of $g$ on $H^3(M; \mathbb{R})$ is also given by a rotation of angle $\frac{2\pi}{p}$. This implies that the Lefschetz
numbers \( L(g^k, M) = 2 - 4 \cos(\frac{2\pi}{p}) + 1 + tr(g^k|_{H^2_{-}}) \). Since \( b_2^- (M) = 1 \), we see that 
\[ \text{tr}(g^k|_{H^2_{-}}) = \pm 1, \] where \( \text{tr}(g^k|_{H^2_{-}}) = -1 \) only if \( G = \mathbb{Z}_2 \). On the other hand, by the
Lefschetz fixed point theorem, \( L(g^k, M) \) is independent of \( k \). This implies easily that 
\( G \) is either \( \mathbb{Z}_2 \) or \( \mathbb{Z}_3 \).

**Lemma 2.3.** Suppose \( M_G \) is rational and \( G = \mathbb{Z}_2 \). Then \( G \) has 8 isolated fixed points. 
Furthermore, \( \{ \Sigma_i \} \neq \emptyset \) and \( \sum_i \Sigma_i^2 = 2(1 - b_2^- (M/G)) \).

**Proof.** For \( G = \mathbb{Z}_2 \), we first observe that the \( G \)-Signature theorem gives
\[ 1 - \text{tr}(g|_{H^2_{-}}) = \text{Sign}(g, M) = \sum_i \Sigma_i^2 = \sum_i (2g_i - 2). \]

On the other hand, let \( z \) be the number of isolated fixed points of \( G \). Then the
Lefschetz fixed point theorem implies that
\[ z + \sum_i (2 - 2g_i) = L(g, M) = 2 - 4 \times (-1) + 1 + \text{tr}(g|_{H^2_{-}}) = 8 - \sum_i (2g_i - 2). \]
It follows that \( z = 8 \). Finally, \( \sum_i \Sigma_i^2 = 1 - \text{tr}(g|_{H^2_{-}}) = 2(1 - b_2^- (M/G)) \) because 
\( b_2^- (M) = 1 \). Note that \( \{ \Sigma_i \} \neq \emptyset \) because \( M_G \) is rational. This finishes the proof.

**□**

**Lemma 2.4.** Suppose \( M_G \) is rational and \( G = \mathbb{Z}_3 \). Then \( G \) has 6 isolated fixed points, 
either three of which are of type (1, 1). Furthermore, \( \sum_i \Sigma_i^2 = 0, \) and at most one of 
the components in \( \{ \Sigma_i \} \) is a sphere.

**Proof.** First of all, observe that \( b_2^- (M/G) = 1 \) as \( G = \mathbb{Z}_3 \), and consequently,
\[ L(g, M) = 2 - 4 \times (-\frac{1}{2}) + 1 + 1 = 6, \ \text{Sign}(g, M) = 1 - 1 = 0. \]
Hence if we let \( x, y \) be the number of isolated fixed points of \( G \) which are of type (1, 1) 
and (1, 2) respectively. Then the Lefschetz fixed point theorem and the \( G \)-Signature theorem imply, respectively, that
\[ x + y + \sum_i (2 - 2g_i) = 6, \]
and
\[ -\cot\left(\frac{\pi}{3}\right) \cdot \cot\left(\frac{\pi}{3}\right) \cdot x - \cot\left(\frac{\pi}{3}\right) \cdot \cot\left(\frac{2\pi}{3}\right) \cdot y + \sum_i \csc^2\left(\frac{c_i \pi}{3}\right) \Sigma_i^2 = 0, \]
where \( c_i \) is the normal weight of the action of \( g \) along \( \Sigma_i \). With \( \Sigma_i^2 = 2g_i - 2 \) and 
\( \csc^2\left(\frac{c_i \pi}{3}\right) = \frac{4}{3} \) for any \( i \), we obtain easily that
\[ x + y = 6 + \sum_i \Sigma_i^2, \] and 
\[ -\frac{1}{3}(x - y) + \frac{4}{3} \sum_i \Sigma_i^2 = 0. \]
To proceed further, we eliminate the variable \( x \) and obtain \( 2y + 3 \sum_i \Sigma_i^2 = 6 \). On the 
other hand, observe that \( b_2^- (M/G) = 1 \) implies that there is at most one component 
\( \Sigma_i \) such that \( \Sigma_i^2 < 0 \) (note that these are precisely the spherical components in \( \{ \Sigma_i \} \)).
Consequently, it is easily seen that \( \sum_i \Sigma_i^2 \geq -2 \), and with this, it follows easily that 
\( y = 0 \) or \( 3 \) are the only possibilities, where \( x = 8 \) or \( 3 \) and \( \sum_i \Sigma_i^2 = 2 \) or \( 0 \) respectively.
It remains to eliminate the possibility that \( x = 8, y = 0 \) and \( \sum_i \Sigma_i^2 = 2 \). To this end, we observe that the \( G \)-action is spin because the order of \( G \) is an odd prime (cf. \cite{[10]}). With this understood, the index of the Dirac operator of the spin orbifold \( M/G \) must be zero because \( b_2^-(M/G) = b_2^-(M/G) = 1 \) (cf. Fukumoto-Furuta \cite{[18]}, Corollary 1). We shall compute the index via the Spin number \( \text{Spin}(g, M) := \text{tr}(g|_{\text{Ker} D}) - \text{tr}(g|_{\text{Coker} D}) \), where \( D \) is the Dirac operator on \( M \). If we write \( \text{Ker} D = \oplus_{k=0}^2 V_k^+, \text{Coker} D = \oplus_{k=0}^2 V_k^− \), where \( V_k^+, V_k^− \) are the eigenspaces of \( g \) with eigenvalue \( \mu^k_3 := \exp \frac{2k\pi i}{3} \), then \( \text{Spin}(g, M) = \sum_{k=0}^2 d_k \mu^k_3 \), where \( d_k \equiv \dim \mathbb{C} V_k^+ - \dim \mathbb{C} V_k^- \). Since both \( \text{Ker} D \) and \( \text{Coker} D \) are quaternion vector spaces, and the quaternions \( i \) and \( j \) are anti-commutative, it follows that \( V_k^± \) are quaternion vector spaces, and that multiplication by \( j \) maps \( V_k^± \) isomorphically to \( V_{2-k}^± \) for \( k = 1, 2 \). This implies that \( d_0 \) is even and \( d_1 = d_2 \). With this understood, \( \text{Spin}(g, M) = d_0 + d_1 \mu_3 + d_2 \mu_3^2 = d_0 - d_1 \). On the other hand, \( \text{Index} D = d_0 + d_1 + d_2 = -\frac{1}{2} \text{Sign}(M) = 0 \). Combining the two equations, we obtain \( \text{Spin}(g, M) = \frac{3}{2} d_0 \). Note that \( d_0 \) equals the index of the Dirac operator on the spin orbifold \( M/G \).

With the preceding understood, we shall compute \( \text{Spin}(g, M) \) using the \( G \)-index theorem for Dirac operators given in the formula in Lemma 3.8 of \cite{[10]}, because the \( G \)-action is symplectic. We first determine the contribution to \( \text{Spin}(g, M) \) from an isolated fixed point \( m \). Let \( (a_m, b_m) \) be the weights of the action of \( g \) at \( m \). In order to compute the contribution, we need to first determine the number \( k(g, m) \) in the formula in Lemma 3.8 of \cite{[10]}. The number \( k(g, m) \) is given by an equation \( k(g, m) \cdot p = 2r_m + a_m + b_m \), where \( 0 \leq r_m < p = 3 \). It follows easily that if \( (a_m, b_m) = (1, 1) \), then \( r_m = 2 \), so that \( k(g, m) = 2 \). In this case, the contribution is

\[
-(-1)^{k(g, m)} \frac{1}{4} \csc \frac{a_m \pi}{p} \csc \frac{b_m \pi}{p} = -\frac{1}{3}.
\]

A similar calculation shows that if \( (a_m, b_m) = (2, 2) \), the contribution is \(-\frac{1}{3} \) as well (in this case, \( r_m = 1 \) and \( k(g, m) = 2 \)), and if \( (a_m, b_m) = (1, 2) \), the contribution equals \( \frac{1}{3} \) (in this case, \( r_m = 0 \) and \( k(g, m) = 1 \)).

The contribution from a fixed component \( \Sigma_i \) determined similarly. The corresponding number \( k(g, \Sigma_i) \) is given by the equation \( k(g, \Sigma_i) \cdot p = 2r_i + c_i \) for some \( 0 < r_i < p \), where \( c_i \) is the normal weight of the action of \( g \) along \( \Sigma_i \). If \( c_i = 1 \), then \( r_i = 1 \), so that \( k(g, \Sigma_i) = 1 \). The contribution in this case equals

\[
-(-1)^{k(g, \Sigma_i)} \frac{1}{4} \csc \frac{c_i \pi}{p} \cot \frac{c_i \pi}{p} = -\frac{1}{6} \Sigma_i^2.
\]

If \( c_i = 2 \), we have \( r_i = 2 \) and \( k(g, \Sigma_i) = 2 \). However, the contribution still equals \(-\frac{1}{6} \Sigma_i^2 \), which turns out to be independent of \( c_i \).

With these understood, the Spin number

\[
\text{Spin}(g, M) = -\frac{1}{3} x + \frac{1}{3} y + \sum_i (-\frac{1}{6} \Sigma_i^2) = -\frac{1}{3} \cdot 8 + \frac{1}{3} \cdot 0 - \frac{1}{6} \cdot 2 = -3.
\]

Consequently, \( d_0 = \frac{2}{3} \text{Spin}(g, M) = -2 \), which is non-zero. This finishes the proof. \( \square \)
It remains to consider the case where $G$ is of non-prime order $n$, $M_G$ is rational or ruled, but for any subgroup $H$ of prime order, $M_H$ has torsion canonical class.

First, by Lemma 2.1, the order $n$ of $G$ must be a power of 2; more precisely, $n = 2^k > 2$. Furthermore, note that $b_1(M/G) = 0$ by Lemma 2.1. With this understood, we claim that $n = 4$ must be true. Suppose to the contrary that $n \geq 8$. Then there must be an element $g_0 \in G$ of order 8. The action of $g_0$ on $H^1(M;\mathbb{R}) = \mathbb{R}^2$ is given by a rotation of an angle $\theta = \frac{2q\pi}{n}$ for some odd number $q$. We note that $\cos \theta = \pm \frac{\sqrt{n}}{2}$. This is a contradiction to the Lefschetz fixed point theorem because the Lefschetz number $L(g_0, M) = 2 - 4 \cos \theta + 1 \pm 1$ is not an integer. Hence the claim $n = 4$. Note that since $b_1(M/G) = 0$, $M_G$ must be rational.

With the preceding understood, we fix a generator $g$ of $G$, and let $H$ be the subgroup of order 2 generated by $h := g^2$. Then by our assumption, $M_H$ has torsion canonical class, so that by Lemma 2.1, $H$ has exactly 8 isolated fixed points in $M$ and has no 2-dimensional fixed components. Since $M^G$ is contained in $M^H$, the action of $G$ has no 2-dimensional fixed components as well.

To proceed further, note that there are two possibilities: $b_1^-(M/G) = 0$ or 1. Consider first the case where $b_1^-(M/G) = 0$. In this case, $L(g, M) = 2 - 4 \times 0 + 1 - 1 = 2$, so the $G$-action has 2 isolated fixed points. Examining the induced action of $G$ on $M^H$, the remaining 6 fixed points of $H$ are of isotropy of order 2, and consequently, the orbifold $M/G$ has 5 singular points – two of order 4 and three of order 2. Let $x, y$ be the number of fixed points of $G$ of type $(1,1)$ and $(1,3)$ respectively. Note that the resolution of a type $(1,1)$ fixed point in $M_G$ is a $(-4)$-sphere and the resolution of a type $(1,3)$ fixed point is a linear chain of three $(-2)$-spheres. A point of isotropy of order 2 gives rise to a $(-2)$-sphere in $M_G$. As a result, we have

$$b_1^-(M_G) = b_1^-(M/G) + x + 3y + 3 = x + 3y + 3.$$ 

On the other hand, $c_1(K_{M_G}) = \sum_i -\frac{1}{4}E_i$, where $E_i$ is the $(-4)$-sphere in $M_G$ coming from the resolution of a type $(1,1)$ fixed point of $G$ (cf. [7], Proposition 3.2). Thus $c_1(K_{M_G})^2 = \sum_i \frac{1}{4}E_i^2 = -x$. Since $M_G$ is rational, we have $c_1(K_{M_G})^2 = 9 - b_1^-(M_G)$, which is $-x = 9 - (x + 3y + 3)$. It follows that $y = 2$, and $x = 2 - y = 0$. But this is a contradiction as it implies that $c_1(K_{M_G}) = 0$. Hence the case where $b_1^-(M/G) = 0$ is eliminated.

For the case where $b_2^-(M/G) = 1$, it is easy to see that $L(g, M) = 4$, so the $G$-action has 4 isolated fixed points. A similar calculation results

$$b_2^-(M_G) = b_2^-(M/G) + x + 3y + 2 = x + 3y + 3$$

and $c_1(K_{M_G})^2 = -x$. The equation $c_1(K_{M_G})^2 = 9 - b_2^-(M_G)$ implies $y = 2$ as well. However, this time we have $x = 4 - y = 2$, i.e., $G$ has 2 isolated fixed points of type $(1,1)$. We summarize our discussions in the following

**Lemma 2.5.** Suppose $M_G$ is rational or ruled, but for any subgroup $H$ of prime order, $M_H$ has torsion canonical class. Then $M_G$ must be rational, and $G$ is of order 4. Furthermore, the fixed-point set $M^G$ consists of 4 isolated points, exactly two of which are of type $(1,1)$, and there are 4 isolated points of isotropy of order 2 in $M$. 

2.2. The case where \( b_1 = 3 \). Assume \( M_G \) is rational or ruled. We first claim that in this case, \( G \) must be \( \mathbb{Z}_2 \). To see this, note that \( b_2^+(M) = 2 \) and \( b_2^+(M/G) = 1 \), so there must be a subgroup \( G' \) of index 2 such that \( b_2^+(M/G') = 2 \). Let \( H \) be a subgroup of \( G' \) which has prime order. Then note that \( M^H \) is nonempty because \( M^G \) is nonempty, and \( b_2^+(M/H) = 2 \) implies that \( M_H \) can not be rational or ruled. But this contradicts Lemma 2.1 as \( b_2^+(M/H) = 2 \). Hence \( G' \) must be trivial, and \( G = \mathbb{Z}_2 \) as claimed.

**Lemma 2.6.** Suppose \( M_G \) is rational or ruled. Then \( G \) must be of order 2. Moreover,

(i) if \( M_G \) is irrational ruled, then the fixed-point set \( M^G \) consists of a disjoint union of tori of self-intersection zero;

(ii) if \( M_G \) is rational, then the fixed-point set \( M^G \) contains 8 isolated points, with \( \{ \Sigma_i \} \neq \emptyset \) and \( \sum_i \Sigma_i^2 = 2(1 - b_2^+(M/G)) \).

**Proof.** Since \( M_G \) is rational or ruled and \( G = \mathbb{Z}_2 \), \( \{ \Sigma_i \} \neq \emptyset \) must be true. We denote by \( z \) the number of isolated fixed points and let \( 1 \neq g \in G \). Then by the \( G \)-Signature theorem,

\[
\sum_i \Sigma_i^2 = \text{Sign}(g, M) = (1 - 1) - tr(g|_{H^2_-}) = -tr(g|_{H^2_-}).
\]

First, consider case (i) where \( M_G \) is irrational ruled. In this case, \( b_1(M/G) = 2 \), so the Lefschetz fixed point theorem implies that

\[
z + \sum_i (2 - 2g_i) = L(g, M) = 2 - 2 \times (1 + 1 - 1) + (1 - 1) + tr(g|_{H^2_-}) = tr(g|_{H^2_-}).
\]

With \( \Sigma_i^2 = 2g_i - 2 \) for each \( i \), it follows immediately that \( z = 0 \). As a consequence, \( M_G \) is simply the underlying manifold of \( M/G \). This immediately ruled out the possibility that \( b_2^+(M/G) = 0 \), because as an irrational ruled 4-manifold, \( M_G \) has non-zero \( b_2^+ \).

Next, we assume \( b_2^+(M/G) = 1 \). Then \( \sum_i \Sigma_i^2 = -tr(g|_{H^2_-}) = -(1 - 1) = 0 \). As \( \Sigma_i^2 = 2g_i - 2 \) for each \( i \), the claim that each \( \Sigma_i \) is a torus of self-intersection zero follows immediately if we show that none of \( \Sigma_i \) is a sphere. This can be seen as follows. Note that \( b_2^+(M/G) = 1 \) implies that \( M_G \) must be a \( S^2 \)-bundle over \( T^2 \). If a \( \Sigma_i \) is a sphere, then its descendant in \( M_G \) is a \((-4)\)-sphere, which does not exist in \( S^2 \)-bundle over \( T^2 \). Hence the claim.

Finally, we rule out the possibility that \( b_2^+(M/G) = 2 \). In this case, \( \sum_i \Sigma_i^2 = -tr(g|_{H^2_-}) = -(1 + 1) = -2 \), so that there must be a \( \Sigma_i \) which is a \((-2)\)-sphere. On the other hand, \( b_2^+(M/G) = 2 \) implies that \( M_G \) is a \( S^2 \)-bundle over \( T^2 \) blown up at one point. The descendant of \( \Sigma_i \) is a symplectic \((-4)\)-sphere in \( M_G \). However, there is also no symplectic \((-4)\)-sphere in a \( S^2 \)-bundle over \( T^2 \) blown up at one point. To see this, suppose there is a symplectic \((-4)\)-sphere in \( M_G \), denoted by \( C \), and let \( F \) and \( E \) be the fiber class and the exceptional \((-1)\)-class respectively. Note that \( c_1(K_{M_G}) \cdot F = -2 \) and \( c_1(K_{M_G}) \cdot E = -1 \). With this understood, since \( \pi_2(M_G) \) is generated by \( F \) and \( E \), we write \( C = aF + bE \). Then \(-4 = C^2 = -b^2 \) and \( 2 = c_1(K_{M_G}) \cdot C = -2a - b \), giving either \( C = -2F + 2E \) or \( C = -2E \). But this is a contradiction because in both cases \( C \) has a negative symplectic area. Hence the possibility \( b_2^+(M/G) = 2 \) is ruled out.

For case (ii) where \( M_G \) is rational, \( b_1(M/G) = 0 \). In this case, the Lefschetz number \( L(g, M) = 2 - 2 \times (-1 - 1 - 1) + (1 - 1) + tr(g|_{H^2_-}) = 8 + tr(g|_{H^2_-}) \), which implies...
$z = 8$. The assertion $\sum_j \Sigma^2 = 2(1 - b_2(M/G))$ follows easily from the fact that $tr(g|_{H^2(-)}) = 2(b_2(M/G) - 1)$. This finishes the proof.

2.3. The case where $b_1 = 4$. The fact that the cohomology ring $H^*(M, \mathbb{R})$ is isomorphic to that of $T^4$ (cf. [37]) played a crucial role in the analysis of the fixed-point set structure in this case. In particular, this fact has the following two corollaries: (1) it allows us to express the action of $G$ on the entire cohomology $H^*(M, \mathbb{R})$ in terms of its action on $H^1(M, \mathbb{R})$, and (2) since the Hurwitz map $\pi_2(M) \to H_2(M)$ has trivial image, the fixed-point set $M^G$ does not have any spherical components. With the help of the adjunction formula, this is equivalent to the statement that all the 2-dimensional fixed components have nonnegative self-intersection.

For the first point above, to be more concrete, let $g \in G$ be any nontrivial element. Since the action of $g$ on $M$ is orientation-preserving, the representation of $g$ on $H^1(M, \mathbb{R})$ splits into a sum of two complex 1-dimensional representations. This said, there is a basis $\{\alpha_i\}$, $i = 1, 2, 3, 4$, of $H^1(M, \mathbb{R})$ such that $\alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \alpha_4 \in H^4(M, \mathbb{R})$ is positive according to the natural orientation of $M$. Furthermore, we assume that the span of $\alpha_1, \alpha_2$ and the span of $\alpha_3, \alpha_4$ are invariant under the action of $g$, and with respect to the orientation given by the above order, the action of $g$ is given by a rotation of angle $\theta_1, \theta_2$ respectively.

**Lemma 2.7.** With $g, \theta_1, \theta_2$ as given above, the following hold true:

1. $2(\cos \theta_1 + \cos \theta_2) \in \mathbb{Z}$.
2. The Lefschetz number $L(g, M) = 4(1 - \cos \theta_1)(1 - \cos \theta_2)$.
3. The representation of $g$ on $H^{2,+}(M, \mathbb{R})$ (resp. $H^{2,-}(M, \mathbb{R})$) splits into a trivial 1-dimensional representation and a 2-dimensional one on which $g$ acts as a rotation of angle $\theta_1 + \theta_2$ (resp. $\theta_1 - \theta_2$). Consequently,

\[ \text{Sign}(g, M) = 2(\cos(\theta_1 + \theta_2) - \cos(\theta_1 - \theta_2)). \]

**Proof.** Let $\gamma_1 := \alpha_1 \cup \alpha_3$, $\gamma_2 := \alpha_1 \cup \alpha_4$, $\gamma_3 := \alpha_2 \cup \alpha_3$, and $\gamma_4 := \alpha_2 \cup \alpha_4$. Then a straightforward calculation gives

\[ g \cdot (\alpha_1 \cup \alpha_2) = \alpha_1 \cup \alpha_2, \quad g \cdot (\alpha_3 \cup \alpha_4) = \alpha_3 \cup \alpha_4, \]

\[ g \cdot \gamma_1 = \cos \theta_1 \cos \theta_2 \gamma_1 + \cos \theta_1 \sin \theta_2 \gamma_2 + \sin \theta_1 \cos \theta_2 \gamma_3 + \sin \theta_1 \sin \theta_2 \gamma_4, \]

\[ g \cdot \gamma_2 = -\cos \theta_1 \sin \theta_2 \gamma_1 + \cos \theta_1 \cos \theta_2 \gamma_2 - \sin \theta_1 \sin \theta_2 \gamma_3 + \sin \theta_1 \cos \theta_2 \gamma_4, \]

\[ g \cdot \gamma_3 = -\sin \theta_1 \cos \theta_2 \gamma_1 - \sin \theta_1 \sin \theta_2 \gamma_2 + \cos \theta_1 \cos \theta_2 \gamma_3 + \cos \theta_1 \sin \theta_2 \gamma_4, \]

and

\[ g \cdot \gamma_4 = \sin \theta_1 \sin \theta_2 \gamma_1 - \sin \theta_1 \cos \theta_2 \gamma_2 - \cos \theta_1 \sin \theta_2 \gamma_3 + \cos \theta_1 \cos \theta_2 \gamma_4. \]

The action on $H^3(M, \mathbb{R})$ can be similarly determined. From these calculations we deduce easily that

\[ L(g, M) = 2 - 4(\cos \theta_1 + \cos \theta_2) + (2 + 4 \cos \theta_1 \cos \theta_2) = 4(1 - \cos \theta_1)(1 - \cos \theta_2). \]

In order to understand the action of $g$ on $H^{2,+}(M, \mathbb{R})$ and $H^{2,-}(M, \mathbb{R})$, and to compute $\text{Sign}(g, M)$, we note that $H^{2,+}(M, \mathbb{R})$ is spanned by $\beta_i$, $i = 1, 2, 3$, where

\[ \beta_1 = \alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \alpha_4, \quad \beta_2 = \alpha_1 \cup \alpha_3 - \alpha_2 \cup \alpha_4, \quad \beta_3 = \alpha_1 \cup \alpha_4 + \alpha_2 \cup \alpha_3. \]
Likewise, \( H^{2,-}(M, \mathbb{R}) \) is spanned by \( \beta'_i, i = 1, 2, 3 \), where
\[
\beta'_1 = \alpha_1 \cup \alpha_2 - \alpha_3 \cup \alpha_4, \quad \beta'_2 = \alpha_1 \cup \alpha_3 + \alpha_2 \cup \alpha_4, \quad \beta'_3 = \alpha_1 \cup \alpha_4 - \alpha_2 \cup \alpha_3.
\]
With this understood, the action of \( g \) on \( H^{2,+}(M, \mathbb{R}) \) and \( H^{2,-}(M, \mathbb{R}) \) is as follows: both \( \beta_1 \) and \( \beta'_1 \) are fixed by \( g \), and \( g \) acts on the span of \( \beta_2, \beta_3 \) and the span of \( \beta'_2, \beta'_3 \) as a rotation of angle \( \theta_1 + \theta_2, \theta_1 - \theta_2 \) respectively. It follows in particular that
\[
\text{Sign}(g, M) := \text{tr}(g|_{H^{2,+}} - \text{tr}(g)|_{H^{2,-}}) \text{ is given by}
\]
\[
\text{Sign}(g, M) = 2(\cos(\theta_1 + \theta_2) - \cos(\theta_1 - \theta_2)).
\]
Finally, note that \( \text{tr}(g|_{H^1(M, \mathbb{R})}) = 2(\cos\theta_1 + \cos\theta_2) \). On the other hand, the action of \( g \) on \( H^1(M; \mathbb{Z})/\text{Tor} \) is given by a \( 4 \times 4 \) matrix with integer entries. Hence the assertion \( 2(\cos\theta_1 + \cos\theta_2) \in \mathbb{Z} \). This completes the proof of the lemma.

With Lemma 2.7 at hand, we shall first examine the fixed-point set structure when \( G \) is of prime order.

**Lemma 2.8.** Suppose \( G \) is of prime order \( p > 1 \). Then the following hold true.

1. Either \( b^+_2(M/G) = 1 \) or \( b^+_2(M/G) = 3 \). Moreover, \( M_G \) has torsion canonical class if and only if \( b^+_2(M/G) = 3 \) and \( b_1(M/G) = 0 \).
2. If \( M_G \) has torsion canonical class, then \( p = 2 \) or \( p = 3 \), where in the former case, the fixed-point set \( M^G \) consists of 16 isolated points, and in the latter case, \( M^G \) consists of 9 isolated points of type \((1, 2)\).
3. If \( M_G \) is irrational ruled, then \( M^G \) consists of a disjoint union of tori of self-intersection zero.
4. If \( M_G \) is rational, then \( p \neq 2 \) and \( p \leq 5 \).

**Proof.** For (1), note that by Lemma 2.7, \( b^+_2(M/G) = 3 \) if and only if \( \theta_1 + \theta_2 = 2\pi \) for a generator \( g \) of \( G \). If \( \theta_1 + \theta_2 \neq 2\pi \), then \( b^+_2(M/G) = 1 \). Hence either \( b^+_2(M/G) = 1 \) or \( b^+_2(M/G) = 3 \) as claimed. It remains to show that if \( M_G \) has torsion canonical class, then \( b^+_2(M/G) \neq 1 \) but \( b_1(M/G) = 0 \). To see this, suppose \( M_G \) has torsion canonical class. Then the same argument as in Lemma 2.1 shows that \( \chi(M_G) = 12 \) or 24, and \( b_1(M/G) = 0 \). If \( b^+_2(M/G) = 1 \), then \( \chi(M_G) = 12 \), and as in Lemma 2.1, \( p = 2 \) must be true. With \( p = 2 \) and \( b_1(M/G) = 0 \), the angles \( \theta_1, \theta_2 \) in Lemma 2.7 must be both equal to \( \pi \). But this implies that \( b^+_2(M/G) = 3 \), contradicting the assumption of \( b^+_2(M/G) = 1 \). Hence part (1) is proved.

Part (2) follows readily from the same argument as in Lemma 2.1. Note that when \( \chi(M_G) = 24, p = 2, 3 \) or 5. The case of \( p = 5 \) can be further eliminated by the (weak version) \( G \)-signature theorem.

For part (3), if \( M_G \) is irrational ruled, then \( b_1(M/G) = 2 \). This means that in Lemma 2.7, one of the angles \( \theta_1, \theta_2 \) must be 0. As a corollary, \( L(g, M) = \text{Sign}(g, M) = 0 \) for any nontrivial element \( g \in G \), and \( b_T^+(M/G) = 1 \). With this understood, part (3) follows by the same argument as in Lemma 2.2.

Finally, for part (4) we assume \( M_G \) is rational. Then \( b^+_2(M/G) = 1 \) and \( b_1(M/G) = 0 \), so that by Lemma 2.7, \( p \neq 2 \). On the other hand, assume \( p \geq 5 \). We fix a generator \( g \in G \) such that in Lemma 2.7, the angles \( \theta_1 = \frac{2\pi}{p} \) and \( \theta_2 = \frac{2q\pi}{p} \) for some \( 0 < q < p - 1 \).
Then it follows easily from \( p \geq 5 \) that \( L(g, M) = 4(1 - \cos \theta_1)(1 - \cos \theta_2) \) satisfies the bound \( L(g, M) \leq 7 \). With this understood, we appeal to the following version of Lefschetz fixed point theorem

\[
p \cdot \chi(M/G) = \chi(M) + (p - 1) \cdot L(g, M),
\]
where \( \chi(M) = 0 \) and \( L(g, M) \in \mathbb{Z} \). It follows easily that \( L(g, M) \) is divisible by \( p \), and with \( p \geq 5 \) and \( L(g, M) \leq 7 \), we have \( L(g, M) = p \), which equals either 5 or 7, and \( \chi(M/G) = p - 1 \), which equals either 4 or 6. Suppose \( p = 5 \). Then \( \chi(M/G) = 4 \), which implies that \( b_2^1(M/G) = 1 \), so that \( \theta_2 = \frac{2\pi}{p} \) for \( q = 2 \) or 3 by Lemma 2.7. For these values, one can easily check that \( L(g, M) = 5 \). Suppose \( p = 7 \). Then \( \chi(M/G) = 6 \), which implies that \( b_2^1(M/H) = 3 \), so that \( \theta_2 = \theta_1 = \frac{2\pi}{p} \) by Lemma 2.7. In this case \( L(g, M) \neq 7 \). Hence \( p \leq 5 \) must be true. (We remark that when \( p = 5 \), \( 2(\cos \theta_1 + \cos \theta_2) = -1 \in \mathbb{Z} \), so Lemma 2.7(1) is not violated.) This finishes the proof of the lemma.

In the next two lemmas, we shall determine the fixed-point set structure where \( M_G \) is rational and \( G = \mathbb{Z}_3 \) or \( \mathbb{Z}_5 \). Let \( g \in G \) be a generator.

**Lemma 2.9.** Assume \( M_G \) is rational and \( G = \mathbb{Z}_3 \). Then the fixed-point set \( M^G \) consists of 9 isolated points of type \((1, 1)\), plus possible \( 2 \)-dimensional components \( \{ \Sigma_i \} \) which are tori of self-intersection zero.

**Proof.** We shall first apply the Lefschetz fixed point theorem and the \( G \)-signature theorem. To this end, observe that since \( M_G \) is rational, \( b_1(M/G) = 0 \), which implies that the angles \( \theta_1, \theta_2 \) in Lemma 2.7 are both nonzero, and furthermore, \( b_2^1(M/G) = 1 \), which implies that \( \theta_1 = \theta_2 \) must be true. As a consequence, \( L(g, M) = 9, b_2^1(M/G) = 3, \) and \( \text{Sign}(M/G) = 1 - 3 = -2 \).

With this understood, we let \( x, y \) be the number of isolated fixed points of type \((1, 1)\) and \((1, 2)\) respectively. Then by the Lefschetz fixed point theorem,

\[
x + y - \sum_i \Sigma_i^2 = L(g, M) = 9.
\]

In order to apply the \( G \)-signature theorem (the weak version), we note that the signature defect for a type \((1, 1)\) and \((1, 2)\) isolated fixed point is \(-\frac{1}{3}(p-1)(p-2) = -\frac{2}{3}\), \(\frac{1}{3}(p-1)(p-2) = \frac{2}{3}\) respectively (cf. [I]). Hence

\[
-\frac{2}{3}x + \frac{2}{3}y + \sum_i \frac{3^2 - 1}{3} \Sigma_i^2 = 3 \cdot \text{Sign}(M/G) - \text{Sign}(M) = -6.
\]

Combining the two equations, we get \( x + \frac{5}{3}y = 9 \). It is easy to see that the solutions are \( x = 9, y = 0 \) or \( x = 4, y = 3 \). In the former case, \( \sum_i \Sigma_i^2 = 0 \), while in the latter case, \( \sum_i \Sigma_i^2 = -2 \). The latter case is not possible since \( \Sigma_i^2 \geq 0 \) for all \( i \). For the same reason, we must have \( \Sigma_i^2 = 0 \) for all \( i \) in the former case. By the adjunction formula, each \( \Sigma_i \) is a torus. This finishes the proof.

□
Lemma 2.10. Assume $M_G$ is rational and $G = \mathbb{Z}_5$. Then the fixed-point set $M^G$ consists of 5 isolated points of type $(1,2)$, plus possible 2-dimensional components \( \{\Sigma_i\} \) which are tori of self-intersection zero.

Proof. We shall first apply the Lefschetz fixed point theorem and the weak version of the $G$-signature theorem. To this end, note that the signature defect for an isolated fixed point of type $(1,1)$, $(1,2)$ (the same as $(1,3)$) and $(1,4)$ is $-4$, $0$, $4$ respectively (cf. [9]). Thus if we let $x$, $y$, $z$ be the number of fixed points of type $(1,1)$, $(1,4)$ and $(1,2)$ respectively, then

$$x + y + z - \sum_i \Sigma_i^2 = L(g,M) = 5,$$

and

$$-4x + 4y + \sum_i \frac{5^2 - 1}{3} \Sigma_i^2 = 0.$$

Combining the two equations, we have $x + 3y + 2z = 10$. Note that $x + y + z$ must be odd, because $\sum_i \Sigma_i^2 = \sum_i (2g_i - 2)$ is even. It follows that $z$ must be odd. The solutions of $x, y, z$ and $\sum_i \Sigma_i^2$ are listed below:

1. $x = 8$, $y = 0$, $z = 1$, and $\sum_i \Sigma_i^2 = 4$,
2. $x = 5$, $y = 1$, $z = 1$, and $\sum_i \Sigma_i^2 = 2$,
3. $x = 2$, $y = 2$, $z = 1$, and $\sum_i \Sigma_i^2 = 0$,
4. $x = 4$, $y = 0$, $z = 3$, and $\sum_i \Sigma_i^2 = 2$,
5. $x = 1$, $y = 1$, $z = 3$, and $\sum_i \Sigma_i^2 = 0$,
6. $x = 0$, $y = 0$, $z = 5$, and $\sum_i \Sigma_i^2 = 0$.

Next we shall first eliminate cases (1), (2), and (4) where $\sum_i \Sigma_i^2 \neq 0$ by computing with the $G$-index theorem for Dirac operators, using the formula in Lemma 3.8 of [10] (note that the $G$-action is spin). To this end, let $(a_m, b_m)$ be the weights of the action of $g$ at an isolated fixed point $m$, and let $c_i$ be the weight of the action in the normal direction along a fixed component $\Sigma_i$. Then observe that the integers $k(g,m)$ and $k(g,\Sigma_i)$ in the formula in Lemma 3.8 of [10] have the same parity with $a_m + b_m$ and $c_i$ respectively. With this understood, we can divide the isolated fixed points of each type and fixed components into two groups, I and II, according to the following rule: for type $(1,1)$, group I consists of fixed points with $(a_m, b_m) = (1, 1)$ or $(4, 4)$ (and the rest are group II), for type $(1,4)$, a fixed point $m$ belongs to group I if $(a_m, b_m) = (1, 4)$, and group II if $(a_m, b_m) = (2, 3)$, and for type $(1,2)$, group I consists of fixed points with $(a_m, b_m) = (1, 2)$ or $(3, 4)$, and group II consists of fixed points with $(a_m, b_m) = (2, 4)$ or $(1, 3)$, and finally, for a fixed component $\Sigma_i$, it belongs to group I if and only if $c_i = 1$ or $4$. With this understood, the contribution to the Spin number $Spin(g,M)$ from an isolated fixed point $m$, which is $-(-1)^{k(g,m)} \frac{1}{4} \csc \frac{a_m \pi}{p} \csc \frac{b_m \pi}{p}$, takes values as follows:

- $-\frac{1}{4} \csc^2 \frac{a_m \pi}{p}$ if $m$ is in group I and of type $(1,1)$,
- $-\frac{1}{4} \csc^2 \frac{b_m \pi}{p}$ if $m$ is in group II and of type $(1,1)$,
- $\frac{1}{8} \csc^2 \frac{a_m \pi}{p}$ if $m$ is in group I and of type $(1,4)$,
- $\frac{1}{8} \csc^2 \frac{b_m \pi}{p}$ if $m$ is in group II and of type $(1,4)$,
\[ \frac{1}{4} \text{csc} \frac{\pi}{5} \text{csc} \frac{2\pi}{5} \text{ if } m \text{ is in group I and of type } (1, 2), \]
\[ -\frac{1}{4} \text{csc} \frac{\pi}{5} \text{csc} \frac{2\pi}{5} \text{ if } m \text{ is in group II and of type } (1, 2), \]
and the contribution from a fixed component \( \Sigma_i \), \((-1)^k(g, \Sigma_i) \frac{1}{4} \Sigma_i^2 \text{csc} \frac{\pi}{p} \cot \frac{\pi}{p} \), takes values as follows:
\[ -\frac{1}{4} \Sigma_i^2 \text{csc} \frac{\pi}{5} \cot \frac{\pi}{5} \text{ if } \Sigma_i \text{ is in group I}, \]
\[ \frac{1}{4} \Sigma_i^2 \text{csc} \frac{\pi}{5} \cot \frac{\pi}{5} \text{ if } \Sigma_i \text{ is in group II}. \]

If we denote by \( x_k, y_k, z_k \), for \( k = 1, 2 \), the number of fixed points \( m \) belonging to group I, II, of type \((1, 1), (1, 4), \) and \((1, 2)\) respectively, and we denote by \( w_1, w_2 \) the sum of \( \Sigma_i^2 \) for \( \Sigma_i \) belonging to group I, II respectively, then the Spin number is
\[ \text{Spin}(g, M) = \frac{1}{4} \sum_{k=1}^{2} (y_k - x_k) \text{csc}^2 \frac{k\pi}{5} + (-1)^k w_k \text{csc} \frac{k\pi}{5} \cot \frac{k\pi}{5} + (z_1 - z_2) \text{csc} \frac{\pi}{5} \cot \frac{\pi}{5}. \]

Now the key observation is that for \( g^2 \), the contributions to the Spin number for group I and group II switch values. It follows easily then, with the identities \( \sum_{k=1}^{2} \text{csc}^2 \frac{k\pi}{5} = 4 \) and \( \sum_{k=1}^{2} (-1)^k \text{csc} \frac{k\pi}{5} \cot \frac{k\pi}{5} = -2 \), that
\[ \text{Spin}(g, M) + \text{Spin}(g^2, M) = \sum_{k=1}^{2} (y_k - x_k - \frac{1}{2} w_k) = y - x - \frac{1}{2} \sum_{i} \Sigma_i^2 = -\frac{5}{2} \sum_{i} \Sigma_i^2. \]

On the other hand, in the \( G \)-index for Dirac operators, the quaternion structure implies that the representation of weight \( s \) must be complex linearly isomorphic to the representation of weight \( 5 - s \). Consequently, in the Spin number
\[ \text{Spin}(g, M) = d_0 + d_1 \mu + d_2 \mu^2 + d_3 \mu^3 + d_4 \mu^4, \text{ where } \mu = \exp(2\pi i/5), \]
one has \( d_1 = d_4, d_2 = d_3. \) Note that \( \text{Spin}(g^2, M) = d_0 + d_1 \mu^2 + d_2 \mu^4 + d_3 \mu + d_4 \mu^3. \) It follows easily that
\[ \text{Spin}(g, M) + \text{Spin}(g^2, M) = 2d_0 - d_1 - d_2 = -\frac{5}{2} \sum_{i} \Sigma_i^2. \]

Finally, \( d_0 + d_1 + d_2 + d_3 + d_4 = -\text{Sign}(M)/8 = 0. \) It follows immediately that \( d_0 = -\sum_{i} \Sigma_i^2 \). The integer \( d_0 \) is the index of Dirac operator for the spin orbifold \( M/G \), which equals 0 because \( b^2_2(M/G) = b^2_2(M/G) = 1 \) (see Fukumoto-Furuta [18], Corollary 1). This rules out the cases (1),(2),(4), where \( d_0 = -\sum_{i} \Sigma_i^2 \neq 0 \).

The above calculation also shows that in the remaining cases, \( d_0 = d_1 + d_2 = 0. \) Moreover, note that each \( \Sigma_i \) is a torus with \( \Sigma_i^2 = 0. \) In particular, \( w_1 = w_2 = 0. \)

To deal with the remaining possibilities, we shall use the Mod \( p \) vanishing theorem of Seiberg-Witten invariants of Nakamura (cf. [23]). We shall first compute with the \( G \)-signature theorem (not the weak version). First of all, without loss of generality, assume the angles \( \theta_1 = \frac{2\pi}{5} \) and \( \theta_2 = \frac{4\pi}{5} \) in Lemma 2.7. With this we have
\[ \text{Sign}(g, M) = 2(\cos \frac{6\pi}{5} - \cos \frac{-2\pi}{5}) = -2(\cos \frac{\pi}{5} + \cos \frac{2\pi}{5}). \]

On the other hand, we observe that the same division of fixed points or components into group I or group II works here too, and the values of the contributions to \( \text{Sign}(g, M) \) are listed below: for an isolated fixed point \( m, \)
• $-\cot^2 \frac{\pi}{5}$ if $m$ is in group I and of type $(1, 1)$,
• $-\cot^2 \frac{2\pi}{5}$ if $m$ is in group II and of type $(1, 1)$,
• $\cot^2 \frac{\pi}{5}$ if $m$ is in group I and of type $(1, 4)$,
• $\cot^2 \frac{2\pi}{5}$ if $m$ is in group II and of type $(1, 4)$,
• $-\cot \frac{\pi}{5} \cot \frac{2\pi}{5}$ if $m$ is in group I and of type $(1, 2)$,
• $\cot \frac{\pi}{5} \cot \frac{2\pi}{5}$ if $m$ is in group II and of type $(1, 2)$,

and for a fixed component $\Sigma_i$,

• $\Sigma_i^2 \cdot \csc^2 \frac{\pi}{5}$ if $\Sigma_i$ is in group I,
• $\Sigma_i^2 \cdot \csc^2 \frac{2\pi}{5}$ if $\Sigma_i$ is in group II.

Consequently, with $w_1 = w_2 = 0$,

$$\text{Sign}(g, M) = \sum_{k=1}^{2} (y_k - x_k) \cot \frac{k\pi}{5} + (z_2 - z_1) \cot \frac{\pi}{5} \cot \frac{2\pi}{5}.$$  

Next we observe that $\text{Sign}(g^2, M) = 2(\cos \frac{12\pi}{5} - \cos \frac{4\pi}{5}) = -\text{Sign}(g, M)$, and moreover, for $g^2$ the contributions to the Sign number for group I and group II switch values. Taking the difference $\text{Sign}(g, M) - \text{Sign}(g^2, M)$, and using the identities (see Lemma 6.4 in [10])

$$\cot^2 \frac{\pi}{5} - \cot^2 \frac{2\pi}{5} = \csc^2 \frac{\pi}{5} - \csc^2 \frac{2\pi}{5} = 4 \cot \frac{\pi}{5} \cot \frac{2\pi}{5},$$

we obtain

$$\text{Sign}(g, M) = (2(y_1 - y_2 + x_2 - x_1) + (z_2 - z_1)) \cdot \cot \frac{\pi}{5} \cot \frac{2\pi}{5}.$$  

Now finally, observing the identity $5 \cot \frac{\pi}{5} \cot \frac{2\pi}{5} = 2(\cos \frac{\pi}{5} + \cos \frac{2\pi}{5}) = -\text{Sign}(g, M)$, we obtain the following constraint

$$2(y_1 - y_2 + x_2 - x_1) + z_2 - z_1 = -5.$$  

With these preparations, we examine the remaining cases (3), (5) in more detail. First consider case (3), where $x = y = 2, z = 1$. Observe that $y_1 - y_2 + x_2 - x_1$ is always even. It follows easily that $z_2 - z_1 = -1$ and $y_1 - x_1 = -(y_2 - x_2) = -1$ in this case. For case (5) where $x = y = 1, z = 3$, note that $y_1 - y_2 + x_2 - x_1 = \pm 2$. It follows that $z_2 - z_1 = -1$ and $y_1 - x_1 = -(y_2 - x_2) = -1$ as well.

Next we check this against the formula for the Spin number $\text{Spin}(g, M)$. To this end, we will use the following identities:

$$\csc \frac{\pi}{5} \csc \frac{2\pi}{5} = 4 \cot \frac{\pi}{5} \cot \frac{2\pi}{5}, \quad \csc \frac{\pi}{5} \cot \frac{\pi}{5} + \csc \frac{2\pi}{5} \cot \frac{2\pi}{5} = 6 \cot \frac{\pi}{5} \cot \frac{2\pi}{5},$$

which can be easily verified by direct calculation. Now with this understood, note that on the one hand,

$$\text{Spin}(g, M) = \sum_{k=0}^{4} d_k \mu^k = 2d_1 (\cos \frac{\pi}{5} + \cos \frac{2\pi}{5}) = 5d_1 \cot \frac{\pi}{5} \cot \frac{2\pi}{5},$$
and on the other hand, we have
\[ \text{Spin}(g, M) = \frac{1}{4}(-\csc^2 \frac{\pi}{5} + \csc^2 \frac{2\pi}{5} + \csc \frac{\pi}{5} \csc \frac{2\pi}{5}) = 0. \]

It follows immediately that in cases (3), (5), we have \( d_1 = 0 \), and as a result, \( d_k = 0 \) for all \( k = 0, 1, \cdots, 4 \).

With the preceding understood, recall that the condition in the Mod \( p \) vanishing theorem of Seiberg-Witten invariants (cf. [33]) is
\[ 2d_k < 1 - b_1^G + b_2^G \]
for any \( k = 0, 1, \cdots, 4 \), where \( b_1^G = b_1(M/G) = 0 \) and \( b_2^G = b_2^+(M/G) = 1 \) (note that since \( b_1(M/G) = 0 \), the fixed-point set \( J^G \) in Nakaruma’s theorem consists of a single point, i.e., \( [0] \), so the integers \( \{k_j\} \) in Nakaruma’s theorem are given by \( \{d_k\} \) for any \( l \), and the integer \( d(c) = 0 \)). With \( d_k = 0 \) for all \( k \), the condition in the Mod \( p \) vanishing theorem of Seiberg-Witten invariants is satisfied, so the Seiberg-Witten invariant for the canonical \( \text{Spin}^c \) structure (which is induced by a spin structure on \( M \)) vanishes (mod \( 5 \)). But by Taubes’ theorem, the Seiberg-Witten invariant equals 1, which is a contradiction. Hence, cases (3), (5) are ruled out.

Finally, case (6) where \( x = y = 0, z = 5 \) remains. This finishes the proof.

\[ \square \]

It remains to consider the case where \( G \) is of non-prime order, \( M_G \) is rational or ruled, but for any prime order subgroup \( H \), \( M_H \) has torsion canonical class. Let \( n \) be the order of \( G \). Then by Lemma 2.8, \( n = 2^k 3^l \) where \( k > 1 \) or \( l > 1 \), and moreover, for any nontrivial element \( g \in G \), the angles \( \theta_1, \theta_2 \) in Lemma 2.7 are both nonzero. In particular, \( b_1(M/G) = 0 \), and \( M_G \) must be rational. First of all, we have

**Lemma 2.11.** Suppose \( G = \mathbb{Z}_4 \) and for the order 2 subgroup \( H \), \( M_H \) has torsion canonical class. Then there are two possibilities:

(i) \( b_2^+(M/G) = 3 \), and the \( G \)-action has 4 isolated fixed points, all of type (1,3), and 12 isolated points of isotropy of order 2.

(ii) \( M_G \) is rational, and the \( G \)-action has 4 isolated fixed points, all of type (1,1), and 12 isolated points of isotropy of order 2.

**Proof.** Fix a generator \( g \in G \). Then it is easy to see that in Lemma 2.7, either \( \theta_1 = 2\pi - \theta_2 \) or \( \theta_1 = \theta_2 \). So either \( b_2^+(M/G) = 3 \), \( b_2^-(M/G) = 1 \), or \( b_2^+(M/G) = 1 \), \( b_2^-(M/G) = 3 \). In any case, we have \( \chi(M/G) = 6 \). Finally, observe that \( L(g, M) = 4 \) in both cases.

On the other hand, by examining the action of \( G \) on \( M^H \), which consists of 16 isolated points, and with \( L(g, M) = 4 \), it follows easily that \( M/G \) has 10 isolated singularities. With \( \chi(M/G) = 6 \), it follows that \( \chi(M_G) > 12 \), so that if \( M_G \) has torsion canonical class, then \( b_2^+(M/G) = 3 \) must be true. Case (i) follows immediately.

Suppose \( M_G \) is rational, and let \( x, y \) be the number of fixed points of type (1,1) and (1,3) respectively. Then note that each type (1,1) fixed point contributes a \((-4)\)-sphere in \( M_G \), which in turn contributes \(-1\) to \( c_1(K_{M_G})^2 \). The other singular points of \( M/G \) contribute zero, hence \( c_1(K_{M_G})^2 = -x \). On the other hand, note that \( \chi(M_G) = \chi(M/G) + 3y + 6 = 12 + x + 3y \). As \( M_G \) is rational, \( c_1(K_{M_G})^2 = 12 - \chi(M_G) \), which implies \( y = 0 \). Hence \( x = 4 \), and case (ii) follows. This finishes the proof.

\[ \square \]
Now finally, we have

**Lemma 2.12.** Suppose $M_G$ is rational, but for any prime order subgroup $H$, $M_H$ has torsion canonical class. Then the order $n$ of $G$ must be a power of 2. Moreover, if $n \geq 8$, then $n = 8$, and the $G$-action falls into one of the following two cases:

(i) the $G$-action has 2 isolated fixed points, all of type $(1,3)$, 2 isolated points of isotropy of order 4 of type $(1,3)$, and 12 isolated points of isotropy of order 2;

(ii) the $G$-action has 2 isolated fixed points, all of type $(1,5)$, 2 isolated points of isotropy of order 4 of type $(1,1)$, and 12 isolated points of isotropy of order 2.

**Proof.** We first show that the order of $G$ is a power of 2, i.e., $n = 2^k$ for some $k > 1$. This is done in two steps. First, we claim that there is no element of order 9 in $G$. Suppose to the contrary, $g \in G$ has order 9. Without loss of generality, assume $\theta_1 = \frac{2\pi}{9}$, $\theta_2 = \frac{2\pi a}{9}$ for some $0 < q < 9$ in Lemma 2.7. Then since $g^2$ has order 3, for the subgroup $H$ generated by $g^2$, $M_H$ has torsion canonical class. In particular, by Lemma 2.8(1), $b_2^+(M/H) = 3$, so that $3(\theta_1 + \theta_2)$ must be a multiple of $2\pi$. This means that $q = 2, 5$ or 8. But for $q = 2, 5$ or 8, Lemma 2.7(1) is violated as $2(\cos \theta_1 + \cos \theta_2)$ is not an integer. Hence there is no element of order 9 in $G$.

Secondly, if the order of $G$ is divisible by 3 but $G$ contains no elements of order 9, there must be an element of order 12 in $G$ (i.e., $k > 1$ in $n = 2^k 3^l$). Let $g \in G$ be of order 12, and write $\theta_1 = \frac{2\pi}{12}$, $\theta_2 = \frac{2\pi a}{12}$ for some $0 < q < 12$ in Lemma 2.7. Then both $6(\theta_1 + \theta_2)$ and $4(\theta_1 + \theta_2)$ must be a multiple of $2\pi$, which means $q = 5$ or 11. But for both $q = 5$ and 11, $2(\cos \theta_1 + \cos \theta_2)$ is not an integer, so Lemma 2.7(1) is violated. This shows that the order of $G$ is not divisible by 3, so it must be a power of 2.

With the preceding understood, if $n \geq 8$, there must be an element $g \in G$ of order 8. Let $G'$ be the subgroup generated by $g$. Without loss of generality, let $\theta_1 = \frac{2\pi}{8}$, $\theta_2 = \frac{2\pi a}{8}$ for some $0 < q < 8$ in Lemma 2.7. We first note that if $q = 1, 7$, $2(\cos \theta_1 + \cos \theta_2)$ is not an integer so that Lemma 2.7(1) is violated. Hence $q \neq 1$ or 7.

On the other hand, let $H$ be the subgroup of order 4 generated by $g^2$. Then by Lemma 2.11, there are two cases, (i) and (ii), as listed therein.

Suppose we are in case (i) of Lemma 2.11 where $M_H$ has torsion canonical class. In this case, $b_2^+(M/H) = 3$, which easily implies that $q = 3$ in $\theta_2$. As a corollary, $L(g, M) = 2$ and $b_2^+(M/G') = 1$. Examining the action of $g$ on $M^H$, with $L(g, M) = 2$, it follows easily that $M/G'$ has 6 isolated singular points, where two of them have isotropy of order 8, one of isotropy of order 4, and three of isotropy of order 2. Now we determine the action of $g$ at the two fixed points. First, since the fixed points of $g^2$ are all of type $(1,3)$, it follows easily that the fixed points of $g$ are of either type $(1,3)$ or $(1,7)$. With this understood, we observe that with $q = 3$ in $\theta_2$, $b_2^+(M/G') = b_2^+(M/G'') = 1$, so that $\chi(M/G') = 4$. It follows easily that $\chi(M/G') > 12$. With $b_2^+(M/G') = 1$, this implies that $M_G$ is rational. Continuing with the analysis of the action of $g$ at the two fixed points, we note that the minimal resolution of a singular point of order 8 of type $(1,3)$ in $M_{G'}$ is a pair of $(-3)$-spheres intersecting transversely and positively at one point. Its contribution to $c_1(K_{M_{G'}})^2$ is easily seen to be $-1$. All other types of singular points of $M/G$ are Du Val singularities, so make zero contribution. The minimal resolution of a singular point of order 8 of type $(1,7)$ in $M_{G'}$ is a linear chain of seven $(-2)$-spheres, so its contribution to $\chi(M_{G'})$ is 7.
It follows easily that the equation \( c_1(K_{M/G'})^2 = 12 - \chi(M_{G'}) \), which is satisfied by rational 4-manifolds, implies that there can not be any fixed point of \( g \) which is of type \((1,7)\). This finishes the discussion on case (i) of the lemma.

The analysis for case (ii) of Lemma 2.11, where \( M_H \) is rational, is similar. In this case we have \( q = 5 \) in \( \theta_2 \). We continue to have \( L(g,M) = 2 \) and \( b^+_2(M/G') = b^+_2(M/G') = 1 \). Furthermore, \( M_{G'} \) is automatically rational because \( M_H \) is rational. By Lemma 2.11, the singular point of isotropy of order 4 in \( M/G' \) is of type \((1,1)\), and the two singular points of isotropy of order 8 are of either type \((1,1)\) or \((1,5)\). Using the equation \( c_1(K_{M/G'})^2 = 12 - \chi(M_{G'}) \), one can eliminate the possibility that a singular point of isotropy of order 8 in \( M/G' \) is of type \((1,1)\). This finishes the discussion on case (ii) of the lemma.

(One can check that neither (i) nor (ii) violates the G-signature theorem, which was not used in the analysis above.)

Finally, suppose \( n > 8 \). Then there must be an element \( h \in G \) of order 16. Without loss of generality, for the element \( h \), we assume \( \theta_1 = \frac{2q}{16} \), \( \theta_2 = \frac{2q}{16} \) for some \( 0 < q < 16 \) in Lemma 2.7. Then \( g = h^2 \) is of order 8, and from the analysis above, it is easy to see that \( q = 3, 5, 11 \) or \( 13 \). However, one can check that for any of the choices of \( q \) above, \( 2(\cos \theta_1 + \cos \theta_2) \) is not an integer so that Lemma 2.7(1) is violated. The contradiction eliminates the possibility of \( n > 8 \). This finishes the proof of the lemma.

\[ \square \]

### 3. Symplectic surfaces in a rational 4-manifold

Let \((X, \omega)\) be a symplectic rational 4-manifold where \( X = \mathbb{CP}^2 \# N \mathbb{CP}^2 \). Recall that \( \mathcal{E}_X \) is the set of classes in \( H^2(X) \) which are represented by smooth \((-1)\)-spheres, and

\[ \mathcal{E}_\omega = \{ E \in \mathcal{E}_X | c_1(K_\omega) \cdot E = -1 \}, \]

where \( K_\omega \) is the canonical line bundle.

We fix a reduced basis \( H, E_1, \cdots, E_N \) of \((X, \omega)\). For any \( A \in H^2(X) \), we shall write

\[ A = aH - \sum_{i=1}^{N} b_iE_i, \quad \text{where} \ a, b_i \in \mathbb{Z}. \]

We begin by deriving some general constraints on the coefficients \( a \) and \( b_i \) when \( A \) is represented by a connected, embedded symplectic surface, particularly, when \( A \) is the class of a symplectic \((-\alpha)\)-sphere for \( \alpha > 1 \). These constraints are consequences of the fundamental work of Li-Liu [29] and Li-Li [25] on symplectic rational 4-manifolds.

We first collect a few useful facts. For a generic \( \omega \)-compatible almost complex structure \( J \), the class \( H \) and any class \( E \in \mathcal{E}_\omega \) can be represented by a \( J \)-holomorphic sphere (cf. [28]). In particular, this implies that for any \( E \in \mathcal{E}_\omega \), where \( E \neq E_i \), \( 1 \leq i \leq N \), the coefficients in \( E = aH - \sum_{i=1}^{N} b_iE_i \) satisfy \( a > 0 \), \( b_i \geq 0 \) for all \( i \) by the positivity of intersection of \( J \)-holomorphic curves. Similarly, if \( A = aH - \sum_{i=1}^{N} b_iE_i \) is the class of a connected, embedded symplectic surface \( \Sigma \) with \( A^2 \geq 0 \), then by choosing an \( \omega \)-compatible almost complex structure \( J \) such that \( \Sigma \) is \( J \)-holomorphic, we see easily that \( a > 0 \) and \( b_i \geq 0 \) for all \( i \).

The situation is more subtle when \( A^2 < 0 \) and \( A \) is not a class in \( \mathcal{E}_\omega \). We begin with the following lemma.
Lemma 3.1. Suppose \( A = aH - \sum_{i=1}^{N} b_i E_i \) is the class of a connected, embedded symplectic surface of genus \( g \).

1. If \( a > 0 \), then \( b_i \geq 0 \) for all \( i \).
2. The \( a \)-coefficient of \( A \) satisfies the following inequality: \( (a-1)(a-2) \geq 2g \), with “=” if and only if \( b_i = 0 \) or \( 1 \) for all \( i \).

Proof. For part (1), we begin by noting that the genus \( g \) of the symplectic surface representing \( A \) is given by the adjunction formula

\[
g = \frac{1}{2}(A^2 + c_1(K_\omega) \cdot A) + 1.
\]

Suppose to the contrary that \( a > 0 \) but \( b_k < 0 \) for some \( k \). Then we consider the reflection \( R(E_k) \) on \( H^2(X) \) defined by the class \( E_k \), where

\[
R(E_k)\beta = \beta + 2(\beta \cdot E_k)E_k, \quad \forall \beta \in H^2(X).
\]

If we let \( \tilde{A} \) be the image of \( A \) under \( R(E_k) \) and write \( \tilde{A} = \tilde{a}H - \sum_{i=1}^{N} \tilde{b}_i E_i \), then \( \tilde{a} = a, \tilde{b}_k = -b_k > 0 \), and \( \tilde{b}_i = b_i \) for all \( i \neq k \). It follows easily that \( \tilde{A}^2 = A^2 \) and \( c_1(K_\omega) \cdot \tilde{A} - c_1(K_\omega) \cdot A = 2b_k > 0 \). Finally, since \( R(E_k) \) is induced by an orientation-preserving diffeomorphism of \( X \) (cf. [23]), the class \( \tilde{A} \) is represented by a smoothly embedded, connected surface of genus \( g \).

Now the condition \( a > 0 \) enters the argument. Pick a sufficiently small \( \varepsilon > 0 \), and let \( e := H - \sum_{i=1}^{N} \epsilon E_i \in H^2(X, \mathbb{R}) \). Then \( a > 0 \) implies that \( e \cdot \tilde{A} = a - \sum_{i=1}^{N} \tilde{b}_i > 0 \) for sufficiently small \( \varepsilon > 0 \). On the other hand, we claim that \( e \) lies in the symplectic cone associated to the symplectic canonical class \( c_1(K_\omega) \). To see this, we only need to verify that (i) \( e^2 = 1 - Ne^2 > 0 \), which is obviously true when \( \epsilon > 0 \) is sufficiently small, and (ii) \( e \cdot E > 0 \) for any class \( E \in E_\omega \) (cf. [21]). To see (ii) is true, we write \( E = uH - \sum_{i=1}^{N} v_i E_i \). Then \( u^2 = \sum_i v_i^2 \geq 1 \) and \( u \geq 0 \), and \( e \cdot E = u - \epsilon \sum_i v_i \). If \( E = E_l \) for some \( l \), then \( e \cdot E = \epsilon > 0 \). If \( u > 0 \), then \( e \cdot E = \sqrt{\sum_i v_i^2} - 1 - \epsilon \sum_i v_i > 0 \) when \( \epsilon > 0 \) is sufficiently small. Hence the claim that \( e \) lies in the symplectic cone associated to the symplectic canonical class \( c_1(K_\omega) \).

Now the fact that \( e \cdot \tilde{A} > 0 \) together with the fact that \( e \) lies in the symplectic cone associated to the symplectic canonical class \( c_1(K_\omega) \) imply the following inequality on the symplectic genus \( \eta(\tilde{A}) \) of \( \tilde{A} \) (cf. [25], Definition 3.1, p. 130):

\[
\eta(\tilde{A}) \geq \frac{1}{2}(\tilde{A}^2 + c_1(K_\omega) \cdot \tilde{A}) + 1.
\]

On the other hand, the minimal genus is bounded from below by the symplectic genus (cf. [25], Lemma 3.2). Thus \( g \geq \eta(\tilde{A}) \), which implies that \( c_1(K_\omega) \cdot A \geq c_1(K_\omega) \cdot \tilde{A} \). But this is a contradiction because \( c_1(K_\omega) \cdot \tilde{A} - c_1(K_\omega) \cdot A = 2b_k > 0 \). This finishes off part (1) of the lemma.

For part (2), the adjunction formula \( A^2 + c_1(K_\omega) \cdot A + 2 = 2g \) gives

\[
a^2 - \sum_{i=1}^{N} b_i^2 - 3a + \sum_{i=1}^{N} b_i + 2 = 2g.
\]
Proof. Let \( a \neq 0 \). Furthermore, as the composition of the reflections \( b \) with \( \omega = \) if and only if \( a \), we obtain easily \( (a - 1)(a - 2) \geq 2g \), with \( = \) if and only if \( b_i = 0 \) or \( 1 \) for all \( i \). This finishes off part (2), and the proof of the lemma is complete. \( \square \)

The following lemma deals with the case where the \( a \)-coefficient of \( A \) is negative.

**Lemma 3.2.** Let \( A = aH - \sum_{i=1}^{N} b_i E_i \) be the class of a connected, embedded symplectic surface of genus \( g \) such that \( a < 0 \). Then

1. the symplectic surface representing \( A \) must be a symplectic \( (-\alpha) \)-sphere where \( \alpha > 2 \), i.e., \( g = 0 \) and \( A^2 < -2 \), and
2. the expression \( A = aH - \sum_{i=1}^{N} b_i E_i \) must be in the following form:
   \[
   A = aH + (|a| + 1)E_{j_1} - E_{j_2} - \cdots - E_{j_s},
   \]
   in particular, \( 2|a| < \alpha \). Moreover, \( E_{j_1} = E_1 \) and \( \omega(E_1) > \omega(E_i) \) for any \( i > 1 \).

**Proof.** Let \( b_i^- = \max(0, -b_i) \) and \( b_i^+ = \max(0, b_i) \), and consider the class

\[
\tilde{A} = |a|H - \sum_{i=1}^{N} (b_i^- + b_i^+) E_i.
\]

Since \( b_i^- = |b_i| \) when \( b_i < 0 \) and equals \( 0 \) otherwise, and \( b_i^+ = b_i \) when \( b_i > 0 \) and equals \( 0 \) otherwise, it follows easily that \( \tilde{A} \) is the image of \( -A \) under the action of the composition of the reflections \( R(E_k) \), where \( k \) is running over the set of indices such that \( b_k > 0 \). In particular, \( \tilde{A} \) is represented by a smoothly embedded surface of genus \( g \) because each \( R(E_k) \) can be realized by an orientation-preserving diffeomorphism.

As in the proof of the previous lemma, \( e := H - \sum_{i=1}^{N} \epsilon E_i \) lies in the symplectic cone associated to the symplectic canonical class \( c_1(K_\omega) \) when \( \epsilon > 0 \) is sufficiently small. Furthermore, as \( a \neq 0 \), we have \( e \cdot \tilde{A} > 0 \), so that

\[
g \geq \eta(\tilde{A}) \geq \frac{1}{2}(\tilde{A}^2 + c_1(K_\omega) \cdot \tilde{A}) + 1,
\]

where \( \eta(\tilde{A}) \) denotes the symplectic genus of \( \tilde{A} \) (cf. \([23]\)). The above inequality is equivalent to

\[
-3|a| + \sum_{i=1}^{N} (b_i^- + b_i^+) \leq -A^2 + 2g - 2.
\]

On the other hand, the adjunction formula for \( A \) gives the equation \(-3a + \sum_{i=1}^{N} b_i = -A^2 + 2g - 2 \), which implies easily, when combined with the above inequality, that

\[
\sum_{i=1}^{N} b_i^+ \leq -A^2 + 2g - 2.
\]

It follows that \( \sum_{i=1}^{N} b_i^- \leq 3|a| \).

Note that the adjunction formula \( A^2 + c_1(K_\omega) \cdot A + 2 = 2g \) also implies easily that

\[
2g + \sum_{i=1}^{N} b_i(b_i - 1) = a^2 - 3a + 2 = (a - 1)(a - 2) = (|a| + 1)(|a| + 2).
\]

(The last equality is due to the assumption that \( a < 0 \).) It follows that \( b_i^- \leq |a| + 1 \) for each \( i \), and moreover, if \( b_i^- = |a| + 1 \) for some \( i \), then \( g = 0 \), and for any \( j \neq i \),
Suppose to the contrary that \( b_i^\leq |a| \) for all \( i \). Then we will write \( A \) as follows:

\[
A = -(|a|H - \sum_{i=1}^{N} b_i^- E_i) - \sum_{i=1}^{N} b_i^+ E_i.
\]

Since \( b_i^\leq |a| \) for all \( i \) and \( \sum_{i=1}^{N} b_i^- \leq 3|a| \), the class \( |a|H - \sum_{i=1}^{N} b_i^- E_i \) can be written as a sum of classes of the form \( H, H - E_i, H - E_i - E_j, \) or \( H - E_i - E_j - E_k \), where distinct indices stand for distinct classes. Since all these classes have non-negative symplectic areas, it follows that \( \omega(A) \leq 0 \), which is a contradiction. Hence

\[
A = aH + (|a| + 1)E_{j_1} - E_{j_2} - \cdots - E_{j_s}, \quad \text{where } s = -A^2 - 2|a|.
\]

In particular, \( A^2 = -2|a| - s < -2 \), and moreover, \( 2|a| < -A^2 \). Finally, if there is a class \( E_i \) such that \( \omega(E_{j_1}) \leq \omega(E_i) \), then

\[
\omega(aH + (|a| + 1)E_{j_1}) \leq -(|a| - 1)\omega(H - E_{j_1}) - \omega(H - E_{j_1} - E_i) < 0,
\]

which implies that \( \omega(A) < 0 \). It follows easily that \( E_{j_1} = E_1 \), and \( \omega(E_1) > \omega(E_i) \) for any \( i > 1 \). This finishes off the proof.

In the rest of this section, we shall be focusing on the possible homological expressions of a symplectic \((-\alpha)\)-sphere, in particular, for \( \alpha = 2 \) and \( 3 \). The constraints in Lemmas 3.1 and 3.2 allow us to easily determine all the possible expressions of the class \( A \) of a symplectic \((-\alpha)\)-sphere in terms of the reduced basis \( H, E_1, \cdots, E_N \) when the \( a \)-coefficient of \( A \) is relatively small, say \( a \leq 3 \).

To this end, write \( A = aH - \sum_{i=1}^{N} b_i E_i \), and observe that in the following equation

\[
\sum_{i=1}^{N} b_i(b_i - 1) = a^2 - 3a + 2 = (a - 1)(a - 2)
\]

which is satisfied by the coefficients \( a, b_i \) of \( A \), the left-hand side is always a non-negative, even integer. In particular, when \( a = 1 \) or \( 2 \), \( b_i \) must be either 0 or 1. For \( a = 0 \), the area condition \( \omega(A) > 0 \) implies that exactly one of the \( b_i \)'s equals \(-1\) and the rest are either 0 or 1. So in this case,

\[
A = E_i - E_{j_1} - E_{j_2} - \cdots - E_{j_{\alpha - 1}}, \quad \text{where } i, j_1, j_2, \cdots, j_{\alpha - 1} \text{ are distinct.}
\]

Furthermore, with \( \omega(A) > 0 \) it is necessary that \( i > j_1, \cdots, j_{\alpha - 1} \).

With the preceding understood, we examine the class \( A \) more closely for the case where \( \alpha = 2 \) or \( 3 \). First, assume \( A \) is the class of a symplectic \((-2)\)-sphere. Then for \( a = 0, 1, 2 \), the class \( A \) takes the following forms respectively:

\[
A = E_i - E_j, \quad \text{where } i > j; \quad A = H - E_i - E_j - E_k, \quad \text{where } i, j, k \text{ are distinct, and}
\]

\[
A = 2H - E_j - E_{j_2} - \cdots - E_{j_6}, \quad \text{where } j_1, j_2, \cdots, j_6 \text{ are distinct.}
\]

For \( a = 3 \), exactly one of the \( b_i \)'s equals either 2 or \(-1\), so that

\[
A = 3H - 2E_j - E_{j_2} - \cdots - E_{j_8}, \quad \text{where } j_1, j_2, \cdots, j_8 \text{ are distinct,}
\]
or
\[ A = 3H - E_{j_1} - E_{j_2} - \cdots - E_{j_{11}} + E_{j_{12}}, \]
where \( j_1, j_2, \ldots, j_{11} \) are distinct.

However, the latter case can be ruled out by Lemma 3.1 because \( A \) is represented by a connected, embedded symplectic surface. By Lemma 3.2, the case \( a < 0 \) cannot occur.

Next we look at the case where \( A \) is the class of a symplectic \((-3)\)-sphere. For \( a = 0, 1, 2 \), \( A \) takes the following forms respectively:
\[ A = E_i - E_j - E_k, \quad \text{where } i, j, k \text{ are distinct, and } i > j, k, \]
\[ A = H - E_i - E_j - E_k - E_l, \quad \text{where } i, j, k, l \text{ are distinct,} \]
and
\[ A = 2H - E_{j_1} - E_{j_2} - \cdots - E_{j_7}, \quad \text{where } j_1, j_2, \ldots, j_7 \text{ are distinct.} \]

For \( a = 3 \), exactly one of the \( b_i \)'s equals either 2 or \(-1\), so that
\[ A = 3H - 2E_{j_1} - E_{j_2} - \cdots - E_{j_9}, \quad \text{where } j_1, j_2, \ldots, j_9 \text{ are distinct,} \]
or
\[ A = 3H - E_{j_1} - E_{j_2} - \cdots - E_{j_{11}} + E_{j_{12}}, \quad \text{where } j_1, j_2, \ldots, j_{12} \text{ are distinct.} \]

Again, the latter case is ruled out by Lemma 3.1. For \( a < 0 \), the only possible expression is \( A = -H + 2E_1 \) by Lemma 3.2.

We have listed all the possibilities for the class \( A = aH - \sum_{i=1}^{N} b_i E_i \) when \( a \leq 3 \). If the coefficient \( a \) lies outside this range, the expression for \( A \) can be much more complicated. With this understood, the following lemma plays a key role in determining the expression of \( A \) when the \( a \)-coefficient is large.

**Lemma 3.3.** Let \( A = aH - \sum_{i=1}^{N} b_i E_i \) be any class which satisfies
\[ A^2 = -\alpha, \quad c_1(K_\omega) \cdot A = \alpha - 2, \quad \text{where } \alpha = 2, 3. \]

If \( a > 3 \), then there are at least 9 terms in the expression of \( A \) with non-zero \( b_i \)-coefficient when \( \alpha = 2 \), and there are at least 10 terms with non-zero \( b_i \)-coefficient when \( \alpha = 3 \).

**Proof.** We begin by recalling a reduction procedure useful in this kind of problems. For any distinct indices \( i, j, k \), we set \( H_{ijk} := H - E_i - E_j - E_k \). Then \( H_{ijk} \) satisfies the following conditions:
\[ H_{ijk}^2 = -2, \quad c_1(K_\omega) \cdot H_{ijk} = 0, \quad \text{and } \omega(H_{ijk}) \geq 0. \]

Furthermore, there is a reflection \( R_{ijk} \) on \( H^2(M) \) associated to \( H_{ijk} \), which is defined by the following formula:
\[ R_{ijk}(A) := A + (A \cdot H_{ijk}) H_{ijk}, \quad \forall A \in H^2(X). \]

To ease the notation, let \( \tilde{A} := R_{ijk}(A) \). Then it is easy to see that
\[ \tilde{A}^2 = A^2, \quad c_1(K_\omega) \cdot \tilde{A} = c_1(K_\omega) \cdot A, \quad \text{and } \tilde{A} \cdot H_{ijk} = -A \cdot H_{ijk}. \]

The last equality implies that
\[ A = R_{ijk}(\tilde{A}) = \tilde{A} + (\tilde{A} \cdot H_{ijk}) H_{ijk}. \]
Finally, note that the operation $R_{ijk}$ will decrease (resp. increase) the $a$-coefficient in the expression of $A$ if and only if $A \cdot H_{ijk} < 0$ (resp. $A \cdot H_{ijk} > 0$), where $A \cdot H_{ijk} = a - (b_i + b_j + b_k)$. See [30] or [3] for further discussions on this reduction procedure.

With the preceding understood, let $A = aH - \sum_{i=1}^{N} b_iE_i$ be any class satisfying the conditions in the lemma, i.e., $A^2 = -\alpha$, $c_1(K_{\omega}) \cdot A = \alpha - 2$, where $\alpha = 2, 3$, and assume $a > 3$. Suppose to the contrary that $A$ has no more than 8 terms in the expression with non-zero $b_i$-coefficient when $\alpha = 2$, and has no more than 9 terms with non-zero $b_i$-coefficient when $\alpha = 3$.

**Claim:** There are distinct indices $i, j, k$ such that (i) $b_i, b_j, b_k$ are positive, and (ii) $A \cdot H_{ijk} = a - (b_i + b_j + b_k) < 0$.

**Proof of Claim:** We shall prove the claim by contradiction. But first, we observe that there are at least 3 terms in $A$ with the $b_i$-coefficient positive. To see this, note that the conditions $A^2 = -\alpha$, $c_1(K_{\omega}) \cdot A = \alpha - 2$ are equivalent to $a^2 - \sum_{i=1}^{N} b_i^2 = -\alpha$, $-3a + \sum_{i=1}^{N} b_i = \alpha - 2$, which imply that

$$\sum_{i=1}^{N} b_i(b_i - 1) = (a - 1)(a - 2).$$

Since $a > 3$, it follows that for any $i$, if $b_i > 0$, then $b_i \leq a - 1$ must be true. Therefore, if there were at most 2 terms in $A$ with the $b_i$-coefficient positive, then $\sum_{i=1}^{N} b_i \leq 2(a - 1)$, which contradicts $-3a + \sum_{i=1}^{N} b_i = \alpha - 2$, where $a > 3$ and $\alpha = 2, 3$.

With the preceding understood, suppose the claim is not true. Then it follows that $b_i + b_j + b_k \leq a$ holds true for any distinct indices $i, j, k$, where $b_i, b_j, b_k$ are not necessarily positive or non-zero. Consider first the case where $\alpha = 2$. Pick a $b_i$-coefficient, say $b_s$, such that $b_s > 0$. Then we have

$$\sum_{i=1}^{N} b_i = \sum_{i=1}^{N} b_i + b_s - b_s \leq 3a - b_s \leq 3a - 1.$$

(Note that $\sum_{i=1}^{N} b_i + b_s \leq 3a$, because by the assumption, there are no more than 8 terms in the expression of $A$ with non-zero $b_i$-coefficient, and $b_i + b_j + b_k \leq a$ for any distinct indices $i, j, k$. But the above inequality contradicts $-3a + \sum_{i=1}^{N} b_i = \alpha - 2 = 0$, which proves the claim for the case of $\alpha = 2$. A similar argument also confirms the claim for $\alpha = 3$. This finishes off the proof of the claim.

Now going back to the proof of the lemma, we pick the indices $i, j, k$ given by the claim above, and perform the operation $R_{ijk}$ to reduce $A$ to $\tilde{A} := R_{ijk}(A)$, which continues to obey the conditions on $A$, i.e.,

$$\tilde{A}^2 = -\alpha$$

and $c_1(K_{\omega}) \cdot \tilde{A} = \alpha - 2$.

Set $c := b_i + b_j + b_k - a > 0$. We need to derive an upper bound on $c$. To this end, note that

$$b_i(b_i - 1) + b_j(b_j - 1) + b_k(b_k - 1) \leq (a - 1)(a - 2).$$
Using the inequality $3(b_i^2 + b_j^2 + b_k^2) \geq (b_i + b_j + b_k)^2$, we obtain

$$\frac{b_i + b_j + b_k}{3} (\frac{b_i + b_j + b_k}{3} - 1) \leq \frac{b_i^2 + b_j^2 + b_k^2}{3} - \frac{b_i + b_j + b_k}{3} \leq \frac{1}{3}(a - 1)(a - 2).$$

Since $a > 3$, this gives $\frac{b_i + b_j + b_k}{3} - 1 \leq \frac{1}{\sqrt{3}}(a - 1)$, and consequently, $c \leq \sqrt{3}(a - 1) + 3 - a$. This estimate shows that the $a$-coefficient of $A$, denoted by $\tilde{a}$, will be non-negative, because as long as $a > 3$,

$$\tilde{a} = a - c = (2 - \sqrt{3})a + \sqrt{3} - 3 \geq (2 - \sqrt{3}) \cdot 4 + \sqrt{3} - 3 = 5 - 3\sqrt{3} > -1.$$ 

Finally, because $b_i, b_j, b_k$ are non-zero, this operation does not introduce any new terms with non-zero $b_i$-coefficient, so $\tilde{A}$ continues to have no more than 8 terms with non-zero $b_i$-coefficient when $a = 2$, and no more than 9 terms when $a = 3$.

After finitely many steps, we will arrive at a class, continuing to be denoted by $\tilde{A}$, whose $a$-coefficient lies in the range $0 \leq \tilde{a} \leq 3$. Without loss of generality, we may assume $\tilde{A}$ is the first class whose $a$-coefficient lies in this range; in particular, the $a$-coefficient of $A$ obeys $a > 3$. We shall examine $\tilde{A}$ according to the value of $\tilde{a}$ below.

To this end, we denote by $\tilde{b}_i$ the $b_i$-coefficients of $\tilde{A}$. Then it is helpful to observe that $\tilde{b}_i + \tilde{b}_j + \tilde{b}_k - \tilde{a} = -c < 0$, because of the relation $\tilde{A} \cdot H_{ijk} = -A \cdot H_{ijk}$.

Suppose $\tilde{a} = 0$. Then $\tilde{A} = E_i - E_{j_1} - \cdots - E_{j_{a-1}}$. It is easy to see that with $\tilde{b}_i + \tilde{b}_j + \tilde{b}_k - \tilde{a} = -c < 0$, where $\tilde{a} = 0$, the only possibility is $\tilde{b}_i = -1$ and $\tilde{b}_j = \tilde{b}_k = 0$. It follows that $c = 1$. But this implies that $a = \tilde{a} + c = 0 + 1 = 1$, which contradicts $a > 3$. Hence $\tilde{a} \neq 0$.

Suppose $\tilde{a} = 1$. Then in this case, $\tilde{A} = H - E_{j_1} - \cdots - E_{j_{a-1}}$, and it follows quickly that $\tilde{b}_i = \tilde{b}_j = \tilde{b}_k = 0$ and $c = \tilde{a} = 1$. This gives $a = \tilde{a} + c = 2$, which also contradicts $a > 3$. This shows that $\tilde{a} \neq 1$.

Suppose $\tilde{a} = 2$. Then $\tilde{A} = 2H - E_{j_1} - \cdots - E_{j_{a-4}}$. The condition $a > 3$ requires that in this case we must have $c \geq 2$, and consequently, $\tilde{b}_i + \tilde{b}_j + \tilde{b}_k - \tilde{a} = -c \leq -2$. Since the $b_i$-coefficients of $\tilde{A}$ are non-negative and $\tilde{a} = 2$, it follows that $\tilde{b}_i = \tilde{b}_j = \tilde{b}_k = 0$ and $c = \tilde{a} = 2$ must be true. In particular, the indices $i, j, k$ are not appearing in the expression of $\tilde{A}$, and it follows that $\tilde{A}$ takes the form

$$A = 4H - 2E_i - 2E_j - 2E_k - E_{j_1} - \cdots - E_{j_{a-4}}$$

where $j_i \neq i, j, k$.

But this contradicts the assumption that $A$ has no more than 8 terms in the expression with non-zero $b_i$-coefficient when $a = 2$, and has no more than 9 terms with non-zero $b_i$-coefficient when $a = 3$. Hence $\tilde{a} \neq 2$.

Finally, suppose $\tilde{a} = 3$. The expressions for $\tilde{A}$ are

$$\tilde{A} = 3H - 2E_{j_1} - \cdots - E_{j_{a+6}}$$

or

$$\tilde{A} = 3H - E_{j_1} - \cdots - E_{j_{a+8}} + E_{j_{a+9}}.$$ 

Since $c \geq 1$, with $\tilde{a} = 3$, it follows that $\tilde{b}_i + \tilde{b}_j + \tilde{b}_k \leq 3 - 1 = 2$. In the former case, the following are the possibilities for $\tilde{b}_i, \tilde{b}_j, \tilde{b}_k$:

$$(\tilde{b}_i, \tilde{b}_j, \tilde{b}_k) = (2, 0, 0), (1, 1, 0), (1, 0, 0), (0, 0, 0).$$
With this understood, note that $b_l = \tilde{b}_l + c$ for $l = i, j, k$. Since at least one of $\tilde{b}_i, \tilde{b}_j, \tilde{b}_k$ is zero, it follows that the number of terms in the expression of $A$ with non-zero $b_l$-coefficient is at least 1 more than the number of terms with non-zero $b_l$-coefficient in $\tilde{A}$. Now $\tilde{A}$ has $\alpha + 6$ many terms of non-zero $b_l$-coefficient, so $A$ must have at least $\alpha + 7$ many terms, which is easily seen a contradiction to the assumption that $A$ has no more than 8 terms in the expression with non-zero $b_l$-coefficient when $\alpha = 2$, and has no more than 9 terms with non-zero $b_l$-coefficient when $\alpha = 3$. In the latter case, $\tilde{A}$ has $\alpha + 9 \geq 11$ many terms with non-zero $b_l$-coefficient, which is also a contradiction. Hence $\tilde{a} \neq 3$, which completes the proof of the lemma.

With the preceding understood, we now state a lemma which is of fundamental importance for our project on the symplectic Calabi-Yau 4-manifolds. The key observation is that, when combined with Lemma 3.3, the area condition $\omega(A) < -c_1(K_\omega) \cdot [\omega]$ will give severe constraints on the $a, b_l$-coefficients of $A$; in particular, it implies an upper bound on the $a$-coefficient of $A$ in terms of $N$ for the case of $\alpha = 2$ or 3.

**Lemma 3.4.** Let $A = ah\ - \sum_{i=1}^{N} b_i E_i$ be the class of a symplectic $(-\alpha)$-sphere where $\alpha = 2$ or 3, such that $\omega(A) < -c_1(K_\omega) \cdot [\omega]$. Then $A$ must be of the following form

$$A = ah - (a - 1)E_{j_1} - E_{j_2} - \cdots - E_{j_{2a+\alpha}}.$$ 

In particular, $a \leq \frac{1}{2}(N - \alpha)$.

**Proof.** For $a \leq 3$, we have seen that $A$ takes the claimed expression. So we may assume $a > 3$ for the sake of the proof of the lemma.

With this understood, first note from Lemma 3.1 that $b_l \geq 0$, and from Lemma 3.3, that there are at least $\alpha + 7$ terms in the expression of $A$ with non-zero $b_l$-coefficients. With this understood, we let $b_l^+ = \max(1, b_l)$. Then $b_l^+ = b_l$ when $b_l > 0$ and $b_l^+ = 1$ when $b_l = 0$. It is easy to see that we can write

$$A = -c_1(K_\omega) + (a - 3)H - \sum_{i=1}^{N} (b_l^+ - 1)E_i + \sum_{i=1}^{N} \max(0, 1 - b_l)E_i.$$ 

Next we observe that $-3a + \sum_{i=1}^{N} b_i = \alpha - 2$. Let $n$ be the number of $b_l$’s which are non-zero. Then because $n \geq \alpha + 7$, we have

$$\sum_{i=1}^{N} (b_l^+ - 1) = \sum_{i=1}^{N} b_i - n = 3a + \alpha - 2 - n \leq 3(a - 3).$$

On the other hand, we claim that there must be a $b_i$ such that $b_i = a - 1$. Suppose to the contrary that this is not true. Then for each $i$, $b_i^+ - 1 \leq a - 3$ must be true. With this understood, note that the class $(a - 3)H - \sum_{i=1}^{N} (b_l^+ - 1)E_i$ can be written as a sum of classes of the form $H, H - E_i, H - E_i - E_j, \text{ or } H - E_i - E_j - E_k$, where distinct indices stand for distinct classes, because $\sum_{i=1}^{N} (b_l^+ - 1) \leq 3(a - 3)$, and for each $i$, $b_l^+ - 1 \leq a - 3$. But this easily gives us the inequality $\omega(A) \geq -c_1(K_\omega) \cdot [\omega]$, which contradicts the area assumption in the lemma. Hence the claim.

Now we observe that in the equation $\sum_{i=1}^{N} b_i(b_l - 1) = (a - 1)(a - 2)$ which is satisfied by the $a, b_l$-coefficients of $A$, if $b_l = a - 1$ for some $i$, then the rest of the $b_l$’s are all
equal to either 0 or 1. With this understood, the equation 
\[ -3a + \sum_{i=1}^{N} b_i = \alpha - 2 \]
implies that the number of \( b_i \)'s equaling 1 must be \( 2a + \alpha - 1 \). It follows immediately that \( A \) takes the expression
\[ A = aH - (a - 1)E_{j_1} - E_{j_2} - \cdots - E_{j_{2a+\alpha}}, \]
as claimed in the lemma.

We remark that if \( A \) is the class of a symplectic \((-4)\)-sphere whose \( a \)-coefficient satisfies \( a > 3 \) and there are at least 11 terms in the expression of \( A \) having non-zero \( b_i \)-coefficients, then the same proof shows that the condition \( \omega(A) < -c_1(K_\omega) \cdot [\omega] \) would imply that \( A \) also takes the special expression in Lemma 3.4. However, in general it is not true that there are always at least 11 terms having non-zero \( b_i \)-coefficients in the expression of a symplectic \((-4)\)-sphere. For example, the following class can be represented by a symplectic \((-4)\)-sphere, which has only 10 terms with non-zero \( b_i \)-coefficients:
\[ A = 6H - 2E_{j_1} - 2E_{j_2} - \cdots - 2E_{j_{10}}. \]

4. Non-existence of certain symplectic configurations

In this section, we give several results concerning nonexistence of certain configurations of symplectic surfaces in rational 4-manifolds. To prove these results, we examine the possible homological expressions of the components in a certain reduced basis, using the constraints established in Section 3, and show that the configurations can not exist even at the homology level. These nonexistence results will then be used in Section 5 to eliminate several possibilities of the fixed-point set structure obtained in Section 2 concerning the 2-dimensional fixed components, which have resisted all the known obstructions available so far.

First, we prove a lemma which allows us to impose certain auxiliary area conditions.

**Lemma 4.1.** Let \( (X, \omega) \) be a symplectic 4-manifold, and let \( D = \sqcup D_i \subset X \), where each \( D_i = \sqcup C_{ij} \) is a configuration of symplectic surfaces intersecting transversely and positively according to a negative definite plumbing graph \( \Gamma_i \). Then for any given collection of positive real numbers \( \{a_{ij}\} \), there exists a \( \delta_0 > 0 \), such that for any choice of \( \{\delta_i\} \) where \( 0 < \delta_i < \delta_0 \), there is a symplectic 4-manifold \( (\tilde{X}, \tilde{\omega}) \) with \( D \subset \tilde{X} \), which has the following significance:

- \( D = \sqcup D_i \) is a set of symplectic configurations in \( (\tilde{X}, \tilde{\omega}) \), and there is a diffeomorphism \( \psi : \tilde{X} \to X \) which is identity on \( D \), such that \( \psi^*c_1(K_\omega) = c_1(K_{\tilde{\omega}}) \),
- the \( \tilde{\omega} \)-symplectic area of each surface \( C_{ij} \) equals \( \delta_i a_{ij} \), i.e., \( \tilde{\omega}(C_{ij}) = \delta_i a_{ij} \).

**Proof.** First of all, we may assume without loss of generality that the intersections of \( C_{ij} \) are \( \omega \)-orthogonal, because we can always slightly perturb the symplectic surfaces to achieve this (cf. [21]). With this understood, since the plumbing graph \( \Gamma_i \) is negative definite, each configuration \( D_i \) has a regular neighborhood \( U_i \) such that \( L_i := \partial U_i \) is a convex contact boundary (in the strong sense), cf. [19]. Furthermore, by a theorem of Park and Stipsicz [34], the contact structure on \( L_i \) is the Milnor fillable contact
structure (cf. [3]). We denote by \( \alpha_i \), the contact form on \( L_i \), where \( \omega = d\alpha_i \) on \( L_i \). It is clear that we can arrange so that \( \{U_i\} \) are disjoint in \( X \).

Now for any given collection of real positive numbers \( \{a_{ij}\} \), let \( (U'_i, \omega'_i) \) be a convex regular neighborhood of \( D_i = \cup_j C_{ij} \) constructed in [19] such that \( \omega'_i(C_{ij}) = a_{ij} \). Fixing an identification \( \partial U'_i = L_i \), we let \( \alpha'_i \) denote the contact form on \( L_i \) such that \( \omega'_i = d\alpha'_i \) on \( L_i \). Then by [33], \( \alpha'_i = e^{f_i} \alpha_i \) for some smooth function \( f_i \) on \( L_i \). With this understood, we set \( \delta_0 > 0 \) by the condition \( \delta_0^{-1} := \max_i \{sup_{x \in L_i} e^{f_i(x)} \} \).

Given any \( \{\delta_i\} \) where \( 0 < \delta_i < \delta_0 \), we set \( C_i := \log \delta_i \). Then it is easy to see that \( C_i + f_i(x) < 0 \) for any \( x \in L_i \). With this understood, we let

\[
W_i := \{(x, t) \in L_i \times \mathbb{R} | C_i + f_i(x) \leq t \leq 0\},
\]

given with the symplectic structure \( d(e^{t \alpha_i}) \). We define \( (\tilde{U}_i, \tilde{\omega}_i) \) to be the symplectic 4-manifold obtained by gluing \( (U'_i, \delta_i \omega'_i) \) to \( W_i \) via the contactomorphism sending \( x \in L_i = \partial U'_i \) to \( (x, C_i + f_i(x)) \in W_i \). Note that each \( (\tilde{U}_i, \tilde{\omega}_i) \) has a convex contact boundary \( \partial \tilde{U}_i = L_i \) where \( \tilde{\omega}_i = d\alpha_i \) on \( L_i \). This understood, we define \( (\tilde{X}, \tilde{\omega}) \) to be the symplectic 4-manifold obtained by removing \( \cup_i U_i \) from \( X \) and then gluing back \( \cup_i \tilde{U}_i \) along \( \cup_i L_i \). It is easy to see that there is a diffeomorphism \( \psi : \tilde{X} \to X \) which is identity on \( D \), such that \( \psi^* c_1(K_{\omega}) = c_1(K_{\tilde{\omega}}) \), and the \( \tilde{\omega} \)-symplectic area of each surface \( C_{ij} \) equals \( \delta_i a_{ij} \). This finishes the proof of the lemma.

The second lemma contains two useful observations. In particular, the first observation implies that in a configuration of symplectic surfaces there is at most one symplectic sphere with negative \( a \)-coefficient.

**Lemma 4.2.** (1) Let \( A_1, A_2 \) be the classes of two symplectic spheres whose \( a \)-coefficients are negative. Then \( A_1 \cdot A_2 < 0 \).

(2) Let \( B = aH - \sum_{i=1}^{N} b_i E_i \) be a nonzero class satisfying \( B^2 = c_1(K_{\omega}) \cdot B = 0 \). If \( a \geq 0 \), then \( a \geq 3 \). Moreover, for \( a = 3 \), the following are the only possible expressions for \( B \):

\[
B = 3H - E_{j_1} - \cdots - E_{j_{10}}.
\]

**Proof.** For (1), let \( a_1, a_2 \) be the \( a \)-coefficients of \( A_1, A_2 \) respectively, which are negative by assumption. Then it follows easily from the expression in Lemma 3.2 that

\[
A_1 \cdot A_2 \leq a_1 a_2 - (|a_1| + 1)(|a_2| + 1) = -(|a_1| + |a_2| + 1) < 0.
\]

For (2), we first note that \( B \neq 0 \) and \( B^2 = 0 \) imply easily that \( a \neq 0 \) in \( B \). With this understood, we note that the conditions \( B^2 = c_1(K_{\omega}) \cdot B = 0 \) are equivalent to

\[
a^2 - \sum_{i=1}^{N} b_i^2 = -3a + \sum_{i=1}^{N} b_i = 0.
\]

It follows easily that \( a(a - 3) = \sum_{i=1}^{N} b_i(b_i - 1) \geq 0 \). With the assumption that \( a \geq 0 \), it follows immediately that \( a \geq 3 \). Moreover, if \( a = 3 \), each \( b_i \) must be either 0 or 1, from which the expression of \( B \) follows easily. This finishes the proof of the lemma.

With these preparations, we now prove the aforementioned nonexistence results.
Proposition 4.3. Let \( \{B_i\} \) be a nonempty set of disjoint symplectic surfaces in \( X = \mathbb{CP}^2\#10\overline{\mathbb{CP}}^2 \), where there is at most one spherical component, and \( F_1, F_2, F_3 \) be a disjoint union of symplectic \((-3)\)-spheres in the complement of \( B_i \), such that

\[
c_1(K_X) = -\frac{2}{3} \sum_i B_i - \frac{1}{3}(F_1 + F_2 + F_3).
\]

Suppose \( F_{4,1}, F_{4,2} \) are a pair of symplectic \((-2)\)-spheres in the complement of \( B_i \) and \( F_1, F_2, F_3 \), such that \( F_{4,1}, F_{4,2} \) intersect transversely and positively at one point. Then \( \{B_i\} \) must consist of one component which is a torus.

Proof. First of all, since \( c_1(K_X) \) is represented by \( F_1, F_2, F_3 \) and \( B_i \), which are disjoint from the two \((-2)\)-spheres \( F_{4,1}, F_{4,2} \), it is clear that, by Lemma 4.1, we may assume without loss of generality that the following area condition holds:

\[
\omega(F_{4,1}) = c_1(K_X) < -c_1(K_X) \cdot [\omega].
\]

Then by Lemma 3.4, the \( a \)-coefficients of \( F_{4,1}, F_{4,2} \) lie in the range \( 0 \leq a \leq 4 \), and moreover, their classes take the special form in Lemma 3.4. Furthermore, again by Lemma 4.1, we can also arrange so that \( F_1, F_2, F_3 \) have the same area, which is sufficiently small, so that \( \omega(F_k) < \omega(B_i) \) for each \( i,k \).

With this understood, we next derive some basic information about \( B_i \). First, \( c_1(K_X) = -\frac{2}{3} \sum_i B_i - \frac{1}{3}(F_1 + F_2 + F_3) \) implies that \( c_1(K_X)^2 = \frac{1}{9} \sum_i B_i^2 - 1 \), and with \( X = \mathbb{CP}^2\#10\overline{\mathbb{CP}}^2 \), it follows easily that \( \sum_i B_i^2 = 0 \). On the other hand, if we denote by \( g_i \) the genus of \( B_i \), then the adjunction formula applied to each \( B_i \) gives us \(-\frac{2}{3} B_i^2 + B_i = 2g_i - 2 \), which is equivalent to \( B_i^2 = 6(g_i - 1) \) for each \( i \). In particular, \( B_i^2 < 0 \) if and only if \( B_i \) is spherical, hence by our assumption, there is at most one component \( B_i \) with \( B_i^2 < 0 \), and such a component must be a \((-6)\)-sphere.

With the preceding understood, we observe that the proposition follows readily if there is no \( B_i \) such that \( B_i^2 < 0 \). This is because, with \( \sum_i B_i^2 = 0 \), \( B_i^2 = 0 \) must be true for each \( i \), and each \( B_i \) is a torus. To see that there is only one component in \( \{B_i\} \), we note that by Lemma 4.2(2), the \( a \)-coefficient of each \( B_i \) is at least 3. On the other hand, each \( B_i \) contributes at least \( \frac{4}{3} \cdot 3 = 2 \) to the \( a \)-coefficient of \(-c_1(K_X) \), which equals 3, while the total contribution from \( F_1, F_2, F_3 \) to the \( a \)-coefficient of \(-c_1(K_X) \) is at least \( \frac{1}{3} \cdot (-1) = -\frac{1}{3} \) by Lemmas 3.2(2) and 4.2(1). Hence the claim. Therefore, it boils down to show that there is no \( B_i \) such that \( B_i^2 < 0 \).

Suppose to the contrary that there is a component, call it \( B_1 \), such that \( B_1^2 < 0 \). Then as we have seen, \( B_1 \) must be a symplectic \((-6)\)-sphere. Now since \( \sum_i B_i^2 = 0 \), there must be exactly one \( B_i \), call it \( B_2 \), such that \( B_2^2 > 0 \) (because by assumption there is at most one \( B_i \) which is spherical). It is clear that \( B_2 \) is a genus-2 surface with \( B_2^2 = 6 \). The rest of \( B_i \) must be torus if there is any. By Lemma 3.2(2), the \( a \)-coefficient of \( B_1 \) is at least \(-2 \), and by Lemma 3.1(2), the \( a \)-coefficient of \( B_2 \) is at least 4. If we apply Lemma 4.2(2) to \( B := B_1 + B_2 \), we see that the sum of the \( a \)-coefficients of \( B_1, B_2 \) must be at least 3. Their contribution to the \( a \)-coefficient of \(-c_1(K_X) \) is then at least \( \frac{4}{3} \cdot 3 = 2 \). It follows again from Lemma 4.2(2) that there are no torus components in \( \{B_i\} \), i.e., \( B_1, B_2 \) are the only components in \( \{B_i\} \). Finally, note that the sum of the \( a \)-coefficients of \( F_1, F_2, F_3 \) is at most 3.
Case (1): Suppose \( a = -2 \) in \( B_1 \). Then by Lemma 3.2, we can write \( B_1 = -2H + 3E_1 - E_p \) for some \( E_p \). We consider the possibilities for the classes of \( F_1, F_2, F_3 \). Note that by Lemma 4.2(1), \( a \geq 0 \) in \( F_1, F_2, F_3 \). Consequently, \( a \leq 3 \) in \( F_1, F_2, F_3 \). Suppose \( a = 3 \) in one of them, say \( F_1 \). Then \( B_1 \cdot F_1 = 0 \) easily implies that

\[
F_1 = 3H - 2E_1 - E_{i_1} - \cdots - E_{i_8},
\]

where \( E_p \) does not show up in \( F_1 \). But this is a contradiction as

\[
\omega(F_1 - B_1) = \omega(5H - 5E_1 - E_{i_1} - \cdots - E_{i_8}) + \omega(E_p) > 0,
\]

as the class \( 5H - 5E_1 - E_{i_1} - \cdots - E_{i_8} \) can be written as a sum of classes of the form \( H - E_i - E_j \) and \( H - E_i - E_j - E_k \), which all have nonnegative areas. If \( a = 2 \) in \( F_1 \), then one can check easily that \( F_1 \cdot B_1 < 0 \) is always true. If \( a = 1 \) in \( F_1 \), then \( F_1 \) must take the form \( F_1 = H - E_i - E_p - E_q - E_r \) for some \( E_q, E_r \). In particular, since \( F_2, F_3 \) are disjoint from \( F_1 \), we must have \( a \neq 1 \) in \( F_2, F_3 \). It follows that both \( F_2, F_3 \) should have \( a = 0 \). Since the sum of the \( a \)-coefficients of \( F_1, F_2, F_3 \) is always an odd number, it follows that the sum must equal 1. Consequently, we must have

\[
F_1 = H - E_1 - E_p - E_q - E_r,
\]

and both \( F_2, F_3 \) have zero \( a \)-coefficients. It follows that the sum of the \( a \)-coefficients of \( B_1, B_2 \) equals 4, so that \( a = 6 \) in \( B_2 \).

To proceed further, we write \( B_2 = 6H - \sum_{i=1}^{10} b_i E_i \). Note that \( B_2 \) has genus 2, so that \( c_1(K_X) \cdot B_2 + D_2^2 = 2 \times 2 - 2 = 2 \). With \( D_2^2 = 6 \), this implies that

\[
-18 + \sum_{i=1}^{10} b_i = 2, \quad 36 - \sum_{i=1}^{10} b_i^2 = 6.
\]

Consequently, \( \sum_{i=1}^{10} b_i(b_i - 1) = 16 \), and as a result, note that \( b_i \leq 4 \) for each \( i \). On the other hand, \( B_2 \cdot B_1 = 0 \), which gives \(-12 + 3b_1 - b_p = 0 \). Since \( b_1 \leq 4 \), we must have \( b_p = 0 \) and \( b_1 = 4 \). Then \( \sum_{i=2}^{10} b_i(b_i - 1) = 16 - 4 \times 3 = 4 \) implies that in \( b_2, \ldots, b_{10} \), there are exactly two of them equaling 2; the rest are either 1 or 0. With \( B_1 \cdot B_2 = 0 \), it follows easily that

\[
B_2 = 6H - 4E_1 - E_q - E_r - 2E_{i_1} - 2E_{i_2} - E_{i_3} - \cdots - E_{i_6}.
\]

With this understood, we note that

\[
2(B_1 + B_2) + F_1 = 9H - 3E_1 - 3E_p - 3E_q - 3E_r - 4E_{i_1} - 4E_{i_2} - 2E_{i_3} - \cdots - 2E_{i_6}.
\]

This implies that without loss of generality,

\[
F_2 = E_{i_1} - E_{i_3} - E_{i_4}, \quad F_3 = E_{i_2} - E_{i_5} - E_{i_6}.
\]

With the preceding understood, let \( A \) be the class of any of the \((-2)\)-spheres \( F_{4,1}, F_{4,2} \). Then recall that because of the area condition we imposed at the beginning, the \( a \)-coefficient of \( A \) lies in the range \( 0 \leq a \leq 4 \), and its expression must be of the form specified in Lemma 3.4. With this understood, if \( a = 4 \) in \( A \), then

\[
A = 4H - 3E_{j_1} - E_{j_2} - \cdots - E_{j_{10}},
\]

containing all 10 \( E_\ell \)-classes. It is easy to see that \( A \cdot F_2 \neq 0 \), which rules out this possibility. If \( a = 3 \) in \( A \), then we can write \( A = 3H - 2E_{j_1} - E_{j_2} - \cdots - E_{j_8} \). Then
This implies that $b_i = E_1$ must be true, and $E_p$ is not contained in $A$. With this understood, $A \cdot F_2 = A \cdot F_3 = 0$ implies that one of the $E_r$-classes in each pair $(E_{i3}, E_{i4})$, $(E_{i5}, E_{i6})$ can not appear in $A$. Together with $E_p$, there are 3 $E_r$-classes not contained in $A$, which is a contradiction as there are only 10 $E_r$-classes in total. If $a = 2$ in $A$, then it is easy to see that $A \cdot B_1 < 0$. Hence we must have either $a = 1$ or $a = 0$ in $A$. If $a = 1$ in $A$, then $A \cdot B_1 = 0$ implies that $A$ contains both $E_1$ and $E_p$. But this leads to $A \cdot F_1 < 0$, which is a contradiction. This shows that $A = E_s - E_t$ for some $E_s, E_t$. It is easy to check that there are only 3 possibilities: $E_q - E_r, E_{i3} - E_{i4}$, and $E_{i5} - E_{i6}$. We just showed that the classes of $F_{4,1}, F_{4,2}$ must be from the three classes above. But they mutually intersect trivially with each other, contradicting the fact that $F_{4,1} \cdot F_{4,2} = 1$. Hence Case (1) is ruled out.

**Case (2):** Suppose $a = -1$ in $B_1$. Then $B_1 = -H + 2E_1 - E_x - E_u - E_v$ for some $E_x, E_u, E_v$. Again by Lemma 4.2(1), $a \geq 0$ in $F_1, F_2, F_3$. If $a = 3$ in $F_1$, then it is easy to see from $F_1 \cdot B_1 = 0$, that $E_1$ must appear in $F_1$ with coefficient $-2$, and two of $E_x, E_u, E_v$ can not appear in $F_1$. But $F_1$ contains 9 $E_r$-classes and there are totally 10 $E_r$-classes, which is a contradiction. If $a = 2$ in $F_1$, then $F_1 \cdot B_1 = 0$ implies that $F_1 = 2H - E_1 - E_{i1} - \cdots - E_{i6}$. But this gives a contradiction

$$\omega(F_1 - B_1) = \omega(3H - 3E_1 - E_{i1} - \cdots - E_{i6}) + \omega(E_x + E_u + E_v) > 0,$$

as the class $3H - 3E_1 - E_{i1} - \cdots - E_{i6}$ can be written as a sum of classes of the form $H - E_{i1} - E_{i2} - E_{i3}$, which all have nonnegative areas. Consequently, $a = 1$ in $F_1$ and $a = 0$ in $F_2, F_3$, where

$$F_1 = H - E_1 - E_x - E_u - E_v$$

for some $E_u, E_v$. By the same argument as in Case (1), the sum of the $a$-coefficients of $B_1, B_2$ equals 4, so that $a = 5$ in $B_2$.

Let $B_2 = 5H - \sum_{i=1}^{10} b_i E_i$. Then $c_1(K_X) \cdot B_2 + B_2^2 = 2$ and $B_2^3 = 6$ imply that

$$-15 + \sum_{i=1}^{10} b_i + 6 = 2, \quad 25 - \sum_{i=1}^{10} b_i^2 = 6.$$ 

Consequently, $\sum_{i=1}^{10} b_i(b_i - 1) = 8$, and as a result, $b_i \leq 3$ for each $i$. With this understood, $B_2 \cdot B_1 = B_2 \cdot F_1 = 0$ implies that

$$-5 + 2b_1 - b_x - b_y - b_z = 0 \text{ and } 5 - b_1 - b_x - b_u - b_v = 0.$$ 

This implies that $b_1 = 3$, and either $b_x = 1$, $b_y = b_z = 0$, $b_u = 1$, $b_v = 0$, or $b_x = 0$, $b_u = b_v = 1$, $b_y = 1$, $b_z = 0$. With this understood, note that $\sum_{i=2}^{10} b_i(b_i - 1) = 8 - 6 = 2$ implies that exactly one of $b_2, \cdots, b_{10}$ equals 2, with the rest equaling 1 or 0. Furthermore, note that $\sum_{i=1}^{10} b_i = 11$, which implies that there are exactly six $b_i$'s equaling 1. It follows easily that the first case where $b_x = 1$, $b_y = b_z = 0$, $b_u = 1$, $b_v = 0$ is not possible, and we must have

$$B_2 = 5H - 3E_1 - E_y - E_u - E_v - 2E_{i1} - E_{i2} - E_{i3} - E_{i4}.$$ 

After computing $2(B_1 + B_2) + F_1$, we see that $E_y, E_{i1}$ must be the $E_r$-classes in $F_2, F_3$ which has a (+1)-coefficient. It follows then

$$F_2 = E_y - E_z - E_{i4}, \quad F_3 = E_{i1} - E_{i2} - E_{i3}.$$
without loss of generality.

With the preceding understood, let \( A \) be the class of any of the \((-2)\)-spheres \( F_{4,1}, F_{4,2} \). If \( a = 4 \) in \( A \), we have \( A \cdot F_2 \neq 0 \) which is not allowed as in Case (1). If \( a = 3 \) in \( A \), then we can write \( A = 3H - 2E_{j_1} - E_{j_2} - \cdots - E_{j_k} \). Then \( B_1 \cdot A = 0 \) implies that \( E_{j_1} = E_1 \) must be true, and exactly one of \( E_x, E_y, E_z \) appear in \( A \). With \( F_2 \cdot A = 0 \), we see that \( E_x \) is contained in \( A \). But this leads to \( A \cdot F_1 = -2 \), which is a contradiction. To proceed further, we rule out \( a = 2 \) in \( A \) by a similar argument as in Case (1). Now suppose \( a = 1 \) in \( A \). Then \( B_1 \cdot A = 0 \) implies that \( E_1 \) and exactly one of \( E_x, E_y, E_z \) appear in \( A \). Then \( A \cdot F_2 = 0 \) implies \( A \) must contain \( E_x \). But we then get \( A \cdot F_1 < 0 \) which is a contradiction. This leaves only two possibilities for \( A \): \( E_u - E_v, E_{i_2} - E_{i_3} \). But these two classes intersect trivially, contradicting \( F_{4,1} \cdot F_{4,2} = 1 \). Hence Case (2) is also eliminated.

**Case (3):** Suppose \( a = 0 \) in \( B_1 \). Then since \( a \geq 4 \) in \( B_2 \), we see immediately that the sum of the \( a \)-coefficients of \( F_1, F_2, F_3 \) is either 1 or \(-1 \). In the former case, \( a = 4 \) in \( B_2 \). If we write \( B_2 = 4H - \sum_{i=1}^{10} b_i E_i \), then \( c_1(K_X) \cdot B_2 + B_2^2 = 2 \) and \( B_2^2 = 6 \) imply that

\[
B_2 = 4H - 2E_{j_1} - E_{j_2} - \cdots - E_{j_7}.
\]

But \( B_1 \) takes the form of \( B_1 = E_{i_1} - E_{i_2} - \cdots - E_{i_6} \). The fact that there are totally 10 \( E_i \)-classes implies easily that \( B_1 \cdot B_2 < 0 \). In the latter case, \( a = 5 \) in \( B_2 \). But then by Lemma 3.2(2), exactly one of \( F_1, F_2, F_3 \) has \( a = -1 \). Suppose it is \( F_1 \). Then \( F_1 = -H + 2E_1 \). It is easy to see that \( F_1 \cdot B_2 \) is always odd because the \( a \)-coefficient of \( B_2 \) is 5. This rules out Case (3).

**Case (4):** Suppose \( a = 1 \) in \( B_1 \). Then \( a = 4 \) in \( B_2 \) and \( F_1 = -H + 2E_1 \). But note that \( B_1 \cdot F_1 \) is always odd, hence this is not possible. This rules out Case (4).

**Case (5):** Suppose \( a > 1 \) in \( B_1 \). Then with \( a \geq 4 \) in \( B_2 \), the total contribution of \( B_1, B_2 \) to the \( a \)-coefficient of \(-3c_1(K_X)\) is at least 12. But the \( a \)-coefficient of \(-3c_1(K_X)\) is 9, so \( F_1, F_2, F_3 \) must contribute \(-3 \) to \( a \)-coefficient of \(-3c_1(K_X)\). This is not possible by Lemmas 3.2(2) and 4.2(1). Hence Case (5) is eliminated.

The above discussions show that there is no component \( B_i \) with \( B_i^2 < 0 \). Hence the proposition is proved.

\[\Box\]

**Proposition 4.4.** Let \( F_1, F_2, \cdots, F_9 \) be a disjoint union of symplectic \((-3)\)-spheres in a rational 4-manifold \( X \), and let \( \{B_i\} \) be a set of disjoint symplectic surfaces, possibly empty, which lie in the complement of \( F_1, F_2, \cdots, F_9 \), such that

\[ c_1(K_X) = -\frac{1}{3}(F_1 + F_2 + \cdots + F_9) - \frac{2}{3} \sum_i B_i. \]

Then \( \{B_i\} \) must be empty if each \( B_i \) is a torus of self-intersection zero.

**Proof.** We shall prove by contradiction. Suppose \( \{B_i\} \neq \emptyset \), where each \( B_i \) is a torus with \( B_i^2 = 0 \). We first note that \( c_1(K_X)^2 = -3 + \frac{4}{9} \sum_i B_i^2 = -3 \), so that \( X = \mathbb{CP}^2 \# 12 \mathbb{CP}^2 \). Let \( a_i \) be the \( a \)-coefficient of \( B_i \). Then by Lemma 4.2(2), \( a_i \geq 3 \) for each \( i \). Note that the contribution of \( \{B_i\} \) to the \( a \)-coefficient of \(-c_1(K_X)\) is \( \frac{4}{9} \sum_i a_i \). Since the \( a \)-coefficient of \(-c_1(K_X)\) is 3, it follows easily that there is only one component in
\{B_i\}$, and moreover, the sum of the $a$-coefficients of $F_1, \cdots, F_9$ can be at most 3. We denote the single component of $\{B_i\}$ by $B$.

With the preceding understood, the following is the key observation:

The maximal number of disjoint symplectic $(-3)$-spheres in $\mathbb{CP}^2 \# 12\mathbb{CP}^2$ with $a$-coefficient equaling 0 is six, and moreover, such six $(-3)$-spheres must be of the form:

- $E_{i_1} - E_{i_2} - E_{i_3}$, $E_{i_2} - E_{i_3} - E_{i_4}$,
- $E_{j_1} - E_{j_2} - E_{j_3}$, $E_{j_2} - E_{j_3} - E_{j_4}$,
- $E_{k_1} - E_{k_2} - E_{k_3}$, $E_{k_2} - E_{k_3} - E_{k_4}$.

To see this, let $A = E_i - E_j - E_k$, $A' = E_r - E_s - E_t$ be two distinct symplectic $(-3)$-spheres such that $A \cdot A' = 0$. Then it is easy to see that if $E_i$ is not contained in $A$ and $E_i$ not in $A'$, the indices $i, j, k, r, s, t$ must be distinct. On the other hand, without loss of generality, assume that $E_r$ appears in $A$, say $r = j$, then $k = s$ or $t$ must be true. The above claim follows easily from the fact that we only have these two alternatives.

With the preceding understood, note that by Lemma 3.2(2), $a \geq -1$ in each $F_k$. Moreover, if $a = -1$, the class must be $-H + 2E_1$, and there is at most one such $(-3)$-sphere in $F_1, \cdots, F_9$ by Lemma 4.2(1).

We claim that the class $A = -H + 2E_1$ can not be represented by any of the $(-3)$-spheres $F_k$. To see this, note that if $A'$ is the class of one of $F_k$ which has positive $a$-coefficient, then $A \cdot A' \neq 0$ unless the $a$-coefficient of $A'$ is an even number. Now with the fact that the sum of the $a$-coefficients of $F_1, \cdots, F_9$ can be at most 3, it follows easily that at least six of the nine $(-3)$-spheres $F_1, \cdots, F_9$ have zero $a$-coefficient. But this is a contradiction because it is easy to see that $A = -H + 2E_1$ intersects nontrivially with one of the six $(3)$-spheres. Hence the claim that the class $A = -H + 2E_1$ can not occur. It follows easily that six of the nine $(-3)$-spheres $F_1, \cdots, F_9$ have zero $a$-coefficient, and three of them have $a$-coefficient equaling 1. Moreover, note that the $a$-coefficient of $B$ must be 3.

To proceed further, we note that since the surface $B$ is disjoint from the six $(-3)$-spheres with zero $a$-coefficient, it must be the following class:

$$B = 3H - E_{i_1} - E_{i_2} - E_{i_4} - E_{j_1} - E_{j_2} - E_{j_4} - E_{k_1} - E_{k_2} - E_{k_4}.$$ 

In other words, the three $E_i$-classes which are missing from $B$ are $E_{i_3}, E_{j_3}, E_{k_3}$. With this understood, let $A = H - E_{i_1} - E_{i_2} - E_{i_3} - E_{i_4}$ be any of the three $(3)$-spheres whose $a$-coefficient equals 1. Then $A \cdot B = 0$ implies that exactly three of the four $E_i$-classes $E_{i_1}, E_{i_2}, E_{i_3}, E_{i_4}$ must appear in $B$. Without loss of generality, let $E_{i_4}$ be the one not contained in $B$, and without loss of generality, assume $E_{i_4} = E_{i_3}$. Then since $A$ intersects trivially with the $(3)$-sphere $E_{i_2} - E_{i_3} - E_{i_4}$, it is easy to see that $A$ must also contain the class $E_{i_2}$. Now with both $E_{i_2}, E_{i_3}$ contained in $A$, the intersection of $A$ with the $(3)$-sphere $E_{i_1} - E_{i_2} - E_{i_3}$ must be negative. This is a contradiction, hence the proposition is proved.

\[\square\]

**Proposition 4.5.** Let $F_{j,1}, F_{j,2}$, where $1 \leq j \leq 5$, be a disjoint union of five pairs of symplectic $(-3)$-sphere and $(-2)$-sphere intersecting transversely and positively at one point in a rational 4-manifold $X$, and let $\{B_i\}$ be a set of disjoint symplectic surfaces,
possibly empty, lying in the complement of $F_{j,1}$, $F_{j,2}$, such that

$$c_1(K_X) = -5 \sum_{j=1}^{5} \left( \frac{2}{5} F_{j,1} + \frac{1}{5} F_{j,2} \right) - \frac{4}{5} \sum_i B_i.$$ 

Then $\{B_i\}$ must be empty if each $B_i$ is a torus of self-intersection zero.

**Proof.** We prove by contradiction. Suppose to the contrary that $\{B_i\}$ is nonempty, with each $B_i$ being a torus of self-intersection zero. Then again by Lemma 4.2(2), each $B_i$ has an $a$-coefficient greater than or equal to 3, so that there can be only one component in $\{B_i\}$. We call it $B$. Moreover, the $a$-coefficient of $B$ is either 4 or 3.

Before we proceed further, note that $c_1(K_X)^2 = -2$, so that $X = \mathbb{CP}^2 \# 11\mathbb{CP}^2$. In particular, there are only 11 $E_i$-classes in $X$.

**Case (1):** Suppose $a = 4$ in $B$. Then if we write $B = 4H - \sum_{i=1}^{11} b_i E_i$, the $b_i$'s satisfy the following equation: $4(4 - 3) = \sum_{i=1}^{11} b_i(b_i - 1)$ (see the proof of Lemma 4.2).

It follows easily that

$$B = 4H - 2E_{j_1} - 2E_{j_2} - E_{j_3} - \ldots - E_{j_{10}}.$$ 

With this understood, note that since the contribution of $B$ to the $a$-coefficient of $-5c_1(K_X)$ is 16 > 15, it follows easily that there must be a (−3)-sphere $F_{j,1}$ having $a = -1$, with the remaining four (−3)-spheres having $a = 0$. By Lemma 3.2(2), the class of the (−3)-sphere with $a = -1$ must be $-H + 2E_1$, and since its intersection with $B$ is zero, either $E_1 = E_{j_1}$ or $E_1 = E_{j_2}$ must be true. Without loss of generality, assume $E_{j_1} = E_1$. Then it is clear that none of the four (−3)-spheres with $a = 0$ can contain the class $E_1 = E_{j_1}$.

With the preceding understood, it is easy to see that the expressions of the four (−3)-spheres with $a = 0$ fall into the following two possibilities without loss of generality:

(!) $E_{i_1} - E_{i_2} - E_{i_3}$, $E_{i_2} - E_{i_3} - E_{i_4}$, $E_{i_5} - E_{i_6} - E_{i_7}$, $E_{i_6} - E_{i_7} - E_{i_8}$,

(!!) $E_{i_1} - E_{i_2} - E_{i_3}$, $E_{i_2} - E_{i_3} - E_{i_4}$, $E_{i_5} - E_{i_6} - E_{i_7}$, $E_{i_6} - E_{i_7} - E_{i_{10}}$.

Suppose we are in case (!). Consider the pair of (−3)-spheres $E_{i_1} - E_{i_2} - E_{i_3}$ and $E_{i_2} - E_{i_3} - E_{i_4}$. If the class $E_{i_1}$ is not contained in the expression of $B$, then it is easy to see that none of the four classes $E_{i_1}, E_{i_2}, E_{i_3}, E_{i_4}$ are contained in $B$. But this contradicts the fact that there are only 11 $E_i$-classes in total. Hence $E_{i_1}$ must be contained in $B$. We know that $E_{i_1} \neq E_{j_1}$. If $E_{i_1} = E_{j_2}$, then both $E_{i_2}, E_{i_3}$ are contained in $B$, and it follows that $E_{i_4}$ does not show up in the expression of $B$. On the other hand, if $E_{i_1} = E_{j_2}$ for some $s > 2$, then it is easy to see that $E_{i_3}$ can not show up in $B$. In any event, one of $E_{i_3}, E_{i_4}$ does not appear in the expression of $B$. With this understood, the same argument shows that one of $E_{i_7}, E_{i_{10}}$ also does not appear in the expression of $B$. But this clearly contradicts the fact that there are totally only 11 $E_i$-classes, hence case (!) is not possible. The argument for case (!!) is similar. First, note that one of $E_{i_3}, E_{i_4}$ does not appear in $B$ as we have argued in case (!). Secondly, consider the pair of (−3)-spheres $E_{i_5} - E_{i_6} - E_{i_7}$, and $E_{i_8} - E_{i_9} - E_{i_{10}}$. We observe that one of the classes $E_{i_5}, E_{i_6}$ is not equal to $E_{j_2}$. Without loss of generality, assume $E_{i_5} \neq E_{j_2}$. Then one of $E_{i_6}, E_{i_7}$ can not be contained in $B$. So totally there
are at least 2 $E_i$-classes not contained in $B$, which contradicts the fact that there are only 11 $E_i$-classes. Hence case (!!) is also not possible. This rules out Case (1).

**Case (2):** Suppose $a = 3$ in $B$. Then by Lemma 4.2(2),

$$B = 3H - E_{j_1} - \cdots - E_{j_{29}}.$$  

With this understood, we first observe that the class $-H + 2E_1$ intersects nontrivially with $B$, so none of the five $(-3)$-spheres can have $a < 0$. On the other hand, from the proof of Proposition 4.4, it is easy to see that the five $(-3)$-spheres cannot all have $a = 0$. Now observe that the contribution of $B$ to the $a$-coefficient of $-5c_1(K_X)$ is 12. It follows easily that exactly one of the five $(-3)$-spheres has $a = 1$, and the other four all have $a = 0$. The possible expressions of the four $(-3)$-spheres with $a = 0$ are given in either (!) or (!!!) listed in Case (1). In the second case (!!!), it is easy to see that there are three $E_i$-classes in the four $(-3)$-spheres with $a = 0$ which do not show up in $B$. This contradicts the fact that there are only 11 $E_i$-classes, hence (!!!) is not possible. In case (!), it is easy to see that $E_{i_3}$, $E_{i_7}$ are precisely the two $E_i$-classes that are not in the expression of $B$. To derive a contradiction, we consider the $(-3)$-sphere with $a = 1$. We write its class as $A = H - E_{i_1} - E_{i_2} - E_{i_3} - E_{i_4}$. Then we note that one of $E_{i_3}$ and $E_{i_7}$ say $E_{i_3}$, must appear in the above expression. It follows that $E_{i_1}, E_{i_2}, E_{i_4}$ all appear in $A$, but not $E_{i_3}$. Without loss of generality, assume $\{E_{i_1}, E_{i_2}, E_{i_4}\} = \{E_{i_1}, E_{i_2}, E_{i_3}\}$. Then $A \cdot B = 0$ implies easily that $E_{i_4}$ can not show up in $B$. It follows that $E_{i_4} = E_{i_2}$ must be true. But this implies that $A$ has nonzero intersection with the $(-3)$-sphere $E_{i_3} - E_{i_6} - E_{i_7}$, which is a contradiction. Hence (!) is also not possible. This rules out Case (2) as well, and the proof of the proposition is complete.

\[ \square \]

5. The proof of main theorems

We begin with the key technical lemma, which classifies the possible homological expressions of a disjoint union of 8 symplectic $(-2)$-spheres in $\mathbb{CP}^2 \# 9\mathbb{CP}^2$ under a very delicately chosen assumption on the symplectic structure.

**Lemma 5.1.** Let $F_1, F_2, \ldots, F_8$ be a disjoint union of 8 symplectic $(-2)$-spheres in $X = \mathbb{CP}^2 \# 9\mathbb{CP}^2$. Suppose the symplectic structure $\omega$ obeys the following constraints:

- one of $F_k$ has $\omega$-area $\delta_1$, the remaining seven have $\omega$-area $\delta_2$;
- $\delta_2 < \delta_1 < 2\delta_2$, and $7\delta_i < -c_1(K_X) \cdot [\omega]$ for $i = 1, 2$.

Then for any given reduced basis $H, E_1, E_2, \ldots, E_9$ of $(X, \omega)$, there are three possibilities for the classes of $F_1, F_2, \ldots, F_8$:

(a) $F_1 = 3H - 2E_{i_1} - E_{i_2} - \cdots - E_{i_{17}} - E_{i_8}$, and $F_2 = H - E_{i_2} - E_{i_3} - E_{i_4}$,
$F_3 = H - E_{i_2} - E_{i_5} - E_{i_6}$, $F_4 = H - E_{i_2} - E_{i_7} - E_{i_8}$, $F_5 = H - E_{i_3} - E_{i_5} - E_{i_7}$,
$F_6 = H - E_{i_3} - E_{i_6} - E_{i_8}$, $F_7 = H - E_{i_4} - E_{i_5} - E_{i_8}$, $F_8 = H - E_{i_4} - E_{i_6} - E_{i_7}$.

(b) $F_1 = H - E_{i_1} - E_{i_2} - E_{i_3} - F_2 = H - E_{i_1} - E_{i_4} - E_{i_5}$, $F_3 = H - E_{i_1} - E_{i_6} - E_{i_7}$,
$F_4 = H - E_{i_2} - E_{i_4} - E_{i_6}$, $F_5 = H - E_{i_3} - E_{i_5} - E_{i_7}$, $F_6 = H - E_{i_3} - E_{i_6} - E_{i_7}$,
$F_7 = H - E_{i_4} - E_{i_5} - E_{i_7}$, and $F_8 = E_{i_8} - E_{i_9}$.

\[ \square \]
Proof. By Lemma 3.4, $a \leq 3$ in each $F_k$.

**Case (1):** Suppose there is a $F_k$ whose $a$-coefficient equals 3. We may assume without loss of generality that it is $F_1$, and write

$$F_1 = 3H - 2E_{i_1} - E_{i_2} - \cdots - E_{i_7} - E_{i_8}.$$ 

Furthermore, we denote by $E_{i_9}$ the unique $E_i$-class that is missing in $F_1$. Let $A$ be the class of any of the remaining $(-2)$-spheres, i.e., $F_2, F_3, \cdots, F_8$. Our first observation is that $a \neq 3$ in $A$. To see this, we note that if the $a$-coefficient of $A$ equals 3, then $A \cdot F_1 = 0$ implies that $A$ must take the following form without loss of generality:

$$A = 3H - E_{i_1} - 2E_{i_2} - \cdots - E_{i_7} - E_{i_9}.$$ 

With this understood, we observe that

$$F_1 + A + c_1(K_X) = 3H - 2E_{i_1} - 2E_{i_2} - E_{i_3} - E_{i_4} - \cdots - E_{i_7},$$

which can be written as a sum of three terms of the form $H - E_i - E_j - E_k$. It follows that $\omega(A + F_1) \geq -c_1(K_X) \cdot [\omega]$, which is a contradiction. Hence the claim.

To proceed further, we first examine the classes $A$ whose $a$-coefficient equals 1. Note that if $A$ is a class with $a = 1$, then $A \cdot F_1 = 0$ implies that if $E_{i_1}$ appears in $A$, then so does $E_{i_9}$. This allows us to divide the classes $A$ with $a = 1$ into two types:

$$(\alpha) A = H - E_{i_1} - E_{i_9} - E_x, \quad (\beta) A = H - E_r - E_s - E_x,$$

where $E_x, E_r, E_s \in \{E_{i_2}, E_{i_3}, \cdots, E_{i_8}\}$.

**Claim:** There are no classes $A$ with $a = 2$.

**Proof of Claim:** We first observe that if $A$ is a class with $a = 2$, then $E_{i_1}$ is not contained in $A$. This is because if $E_{i_1}$ is contained in $A$, then $A \cdot F_1 = 0$ implies that $E_{i_9}$ must also be contained in $A$, and $A$ takes the following form

$$A = 2H - E_{i_1} - E_{i_9} - E_{k_1} - E_{k_2} - E_{k_3} - E_{k_4},$$

But this would lead to

$$\omega(F_1 + A) + c_1(K_X) \cdot [\omega] = \omega(2H - 2E_{i_1} - E_{k_1} - E_{k_2} - E_{k_3} - E_{k_4}) \geq 0,$$

which is a contradiction.

With the preceding understood, suppose to the contrary that there is a class $A$ with $a = 2$. Then without loss of generality, we may write it as

$$A_1 = 2H - E_{i_2} - E_{i_3} - E_{i_9} - E_{i_4} - E_{i_5} - E_{i_6} - E_{i_7}.$$ 

Moreover, if $A$ is another class of $F_2, F_3, \cdots, F_8$ with $a = 2$, then it is easy to check that $A_1 \cdot A < 0$. Hence $A_1$ is the only one with $a = 2$.

Next we examine the possible classes of $A$ with $a = 1$, which intersects trivially with $F_1$ and $A_1$. It is easy to see that if $A$ is a class with $a = 1$ and $A \cdot A_1 = 0$, then $A$
can not be of type \((\alpha)\), and for a type \((\beta)\) class, \(A\) must contain \(E_{i_8}\). It is easy to see that maximally, there are three such type \((\beta)\) classes that are mutually disjoint, i.e.,

\[
A_2 = H - E_{i_2} - E_{i_3} - E_{i_8},\ A_3 = H - E_{i_4} - E_{i_5} - E_{i_8},\ A_4 = H - E_{i_6} - E_{i_7} - E_{i_8}
\]

without loss of generality. The remaining three classes of \(A\) must all have \(a\)-coefficient equaling 0, and it is easy to see that, without loss of generality, they are

\[
A_5 = E_{i_2} - E_{i_3},\ A_6 = E_{i_4} - E_{i_5},\ A_7 = E_{i_6} - E_{i_7}.
\]

To derive a contradiction, we appeal to the area constraints. First, we observe that the area of \(F_1\) must be greater than the area of any of \(A_5, A_6, A_7\). For example,

\[
\omega(F_1 - A_5) = \omega(3H - 2E_{i_1} - 2E_{i_2} - E_{i_4} - E_{i_5} - \cdots - E_{i_8}) \geq 0.
\]

Furthermore, note that if \(\omega(F_1 - A_5) = 0\), then \(\omega(H - E_{x} - E_{y} - E_{z}) = 0\) for any three classes \(E_{x}, E_{y}, E_{z}\) from the set \(\{E_{i_1}, E_{i_2}, E_{i_3}, E_{i_4}, \ldots, E_{i_8}\}\). In particular, \(E_{i_4}, E_{i_5}, E_{i_6}, E_{i_7}\) have the same area, contradicting \(\omega(A_6) > 0, \omega(A_7) > 0\). It follows that \(\omega(F_1) = \delta_1\) and the remaining classes have the same area equaling \(\delta_2 < \delta_1\). With this understood, we note that \(\omega(F_1 - A_5 - A_4) = \omega(2H - 2E_{i_1} - 2E_{i_2} - E_{i_3} - E_{i_6}) \geq 0\), contradicting the constraint \(\delta_1 < 2\delta_2\). This finishes off the proof of the Claim.

Now back to the discussion on Case (1), we claim that no type \((\alpha)\) classes can occur. Suppose to the contrary that there is a type \((\alpha)\) class, which, without loss of generality, is assumed to be \(A_1 = H - E_{i_1} - E_{i_6} - E_{i_8}\). It is easy to see that any other type \((\alpha)\) class has a negative intersection with \(A_1\), hence \(A_1\) is the only type \((\alpha)\) class. Now let \(A\) be any type \((\beta)\) class such that \(A \cdot A_1 = 0\). Then \(A\) must contain \(E_{i_8}\), and furthermore, it is easy to see that maximally, there are three such type \((\beta)\) classes which are mutually disjoint. Without loss of generality, they are

\[
A_2 = H - E_{i_2} - E_{i_3} - E_{i_8},\ A_3 = H - E_{i_4} - E_{i_5} - E_{i_8},\ A_4 = H - E_{i_6} - E_{i_7} - E_{i_8}
\]

The remaining three classes of \(A\) must all have \(a\)-coefficient equaling 0, and it is easy to see that, without loss of generality, they are

\[
A_5 = E_{i_2} - E_{i_3},\ A_6 = E_{i_4} - E_{i_5},\ A_7 = E_{i_6} - E_{i_7}.
\]

This possibility can be ruled out using the area constraints as we did in the proof of the Claim. Hence no type \((\alpha)\) classes can occur.

With the preceding understood, we further observe that no class \(A\) with \(a = 0\) can be realized by \(F_2, F_3, \ldots, F_8\). Suppose, without loss of generality, \(A_1 = E_{i_7} - E_{i_8}\) is realized. Let \(A\) be a type \((\beta)\) class which intersects trivially with \(A_1\). Then it is easy to see that either \(A\) contains both \(E_{i_7}, E_{i_8}\), or \(A\) contains neither \(E_{i_7}\) nor \(E_{i_8}\). It is clear that there can be at most one type \((\beta)\) class which contains both \(E_{i_7}, E_{i_8}\). Without loss of generality, we let it be \(A_2 = H - E_{i_7} - E_{i_8}\). Then any other type \((\beta)\) classes which intersect trivially with \(A_1, A_2\) must contain \(E_{i_2}\), and there are maximally two such classes:

\[
H - E_{i_2} - E_{i_3} - E_{i_4},\ H - E_{i_2} - E_{i_5} - E_{i_6}\]

With this understood, note that there are at most two other classes, both having \(a = 0\), that are allowed, i.e., \(E_{i_3} - E_{i_4}, E_{i_5} - E_{i_6}\), bringing total number of allowable classes for \(F_2, F_3, \ldots, F_8\) to 6. But apparently, there are not enough many classes, hence our claim.

The above discussions show that the classes of \(F_2, F_3, \ldots, F_8\) are all of type \((\beta)\). With this understood, we first rule out the possibility that no triple of \(F_2, F_3, \ldots, F_8\)
shares a common $E_i$-class. Suppose to the contrary that this is the case. Then without loss of generality, we write
\[
F_2 = H - E_{i_2} - E_{i_3} - E_{i_4}, \quad F_3 = H - E_{i_2} - E_{i_5} - E_{i_6},
\]
Note that by our assumption, $F_4$ cannot contain $E_{i_2}$. With this understood, $F_4 \cdot F_2 = F_4 \cdot F_3 = 0$ implies that we may write $F_4 = H - E_{i_3} - E_{i_5} - E_{i_7}$ without loss of generality. Now observe that $F_5$ cannot contain $E_{i_2}, E_{i_3}, E_{i_5}$. Hence $F_5 = H - E_{i_4} - E_{i_6} - E_{i_7}$ must be true. Now examining the class of $F_6$, by our assumption it cannot contain any of $E_{i_2}, E_{i_3}, \ldots, E_{i_7}$. This is clearly a contradiction. Hence the possibility that no triple of $F_2, F_3, \ldots, F_8$ shares a common $E_i$-class is ruled out.

With the preceding understood, we may write without loss of generality that
\[
F_2 = H - E_{i_2} - E_{i_3} - E_{i_4}, \quad F_3 = H - E_{i_2} - E_{i_5} - E_{i_6}, \quad F_4 = H - E_{i_2} - E_{i_7} - E_{i_8},
\]
With this given, it is easy to see that the other four $(-2)$-spheres must be
\[
F_5 = H - E_{i_3} - E_{i_5} - E_{i_7}, \quad F_6 = H - E_{i_3} - E_{i_6} - E_{i_8},
\]
and
\[
F_7 = H - E_{i_4} - E_{i_5} - E_{i_8}, \quad F_8 = H - E_{i_4} - E_{i_6} - E_{i_7}.
\]
This possibility of classes of $F_1, F_2, \ldots, F_8$ is listed as Case (a) of the lemma.

Case (2): Suppose $a \leq 2$ in all eight $(-2)$-spheres $F_1, F_2, \ldots, F_8$.

(i): Assume at least two of $F_1, F_2, \ldots, F_8$ have $a$-coefficient equaling 2. Without loss of generality, let $F_1, F_2$ be such two $(-2)$-spheres. It is easy to see from $F_1 \cdot F_2 = 0$ that $F_1, F_2$ must have exactly 4 $E_i$-classes in common. Hence without loss of generality, we may write them as
\[
F_1 = 2H - E_{j_1} - E_{j_2} - E_{j_3} - E_{j_4} - E_{j_6}, \quad F_2 = 2H - E_{j_1} - E_{j_2} - E_{j_3} - E_{j_4} - E_{j_7} - E_{j_8}.
\]
With this understood, we denote by $E_{j_9}$ the unique $E_i$-class that is missing in $F_1, F_2$. Moreover, we denote by $A$ the class of any of the remaining $(-2)$-spheres, i.e., $F_3, F_4, \ldots, F_8$.

Claim: There are no classes $A$ which contains $E_{j_9}$.

Proof of Claim: First, it is easy to see that if $A$ is a class with $a = 0$ which contains $E_{j_9}$, the intersection of $A$ with one of $F_1, F_2$ will be nonzero. Now suppose $A$ is a class with $a = 1$ which contains $E_{j_9}$. Then $A \cdot F_1 = A \cdot F_2 = 0$ implies that $A$ must be of the form $A = H - E_x - E_y - E_{j_9}$ for some $E_x, E_y \in \{E_{j_1}, \ldots, E_{j_4}\}$. With this understood, we note that
\[
F_1 + F_2 + A + c_1(K_X) = 2H - E_{j_1} - E_{j_2} - E_{j_3} - E_{j_4} - E_x - E_y,
\]
which leads to a contradiction in areas: $\omega(F_1 + F_2 + A) \geq -c_1(K_X) \cdot [\omega]$. Finally, suppose $A$ is a class with $a = 2$ which contains $E_{j_9}$. Then $A \cdot F_1 = A \cdot F_2 = 0$ implies that, without loss of generality, we may assume
\[
A = 2H - E_{j_1} - E_{j_2} - E_{j_3} - E_{j_5} - E_{j_7} - E_{j_9}.
\]
In this case, we have $F_1 + F_2 + A + c_1(K_X) = 3H - 2E_{j_1} - 2E_{j_2} - 2E_{j_3} - E_{j_4} - E_{j_5} - E_{j_7}$, which also leads to the contradiction in areas: $\omega(F_1 + F_2 + A) \geq -c_1(K_X) \cdot [\omega]$. Hence the Claim.
Now back to the discussion on Case (2), it is easy to see that there are two other classes $A$ with $a = 2$ and trivial mutual intersection, which intersect trivially with $F_1, F_2$; we denote them by $A_1, A_2$, where

$$A_1 = 2H - E_{j_1} - E_{j_2} - E_{j_5} - E_{j_6} - E_{j_7} - E_{j_8}, A_2 = 2H - E_{j_3} - E_{j_4} - E_{j_5} - E_{j_6} - E_{j_7} - E_{j_8}.$$ 

On the other hand, let $A$ be a class with $a = 1$ which intersects trivially with $F_1, F_2$. Then $A$ must be of the form $A = H - E_r - E_s - E_t$, where $E_r \in \{E_{j_1}, E_{j_2}, E_{j_3}, E_{j_4}\}$, $E_s \in \{E_{j_5}, E_{j_6}\}$, and $E_t \in \{E_{j_7}, E_{j_8}\}$.

With the preceding understood, if both of $A_1, A_2$ are realized by the $(-2)$-spheres, then it is easy to see that no classes $A$ with $a = 1$ can be realized. On the other hand, it is easy to see that there are maximally 4 classes $A$ with $a = 0$:

$$E_{j_1} - E_{j_2}, E_{j_3} - E_{j_4}, E_{j_5} - E_{j_6}, E_{j_7} - E_{j_8}.$$ 

Hence all of them must be realized. With this understood, it is easy to see that three of $F_1, F_2, A_1, A_2$ and all of the classes with $a = 0$ must have the smaller area $\delta_2$. As a consequence, we may assume without loss of generality that $\omega(F_1) = \omega(E_{j_1} - E_{j_2})$. Then

$$\omega(2H - 2E_{j_1} - E_{j_3} - E_{j_4} - E_{j_5} - E_{j_6}) = \omega(F_1) - \omega(E_{j_1} - E_{j_2}) = 0,$$

which implies that $E_{j_3}, E_{j_4}, E_{j_5}, E_{j_6}$ have the same area. But this contradicts the fact that the classes $E_{j_3} - E_{j_4}, E_{j_5} - E_{j_6}$ are realized by the symplectic $(-2)$-spheres. It follows that $A_1, A_2$ can not be both realized.

Suppose only one of $A_1, A_2$, say $A_1$, is realized. Then there are four classes $A$ with $a = 1$ that are possible, i.e.,

$$A_3 = H - E_{j_3} - E_{j_5} - E_{j_7}, A_4 = H - E_{j_3} - E_{j_6} - E_{j_8},$$

and

$$A_5 = H - E_{j_4} - E_{j_5} - E_{j_8}, A_6 = H - E_{j_4} - E_{j_6} - E_{j_7}.$$ 

If all of $A_3, A_4, A_5, A_6$ are realized, then the remaining $(-2)$-sphere must have $a$-coefficient equaling $0$, and it must be the class $A_7 = E_{j_1} - E_{j_2}$ without loss of generality. But this leads to a contradiction in areas as follows: note that

$$\omega(F_1 - A_7) = \omega(2H - 2E_{j_1} - E_{j_3} - E_{j_4} - E_{j_5} - E_{j_6}) \geq 0.$$ 

Furthermore, if $\omega(F_1 - A_7) = 0$, the four classes $E_{j_3}, E_{j_4}, E_{j_5}, E_{j_6}$ must have the same area. It follows easily that $\omega(A_7) = \delta_2 < \delta_1$. The same argument applies with $F_1$ being replaced by $F_2$ or $A_1$. Note that at least two of $F_1, F_2, A_1$ must have the smaller area $\delta_2$. It follows easily that the six classes $E_{j_3}, E_{j_4}, E_{j_5}, E_{j_6}, E_{j_7}, E_{j_8}$ must have the same area. But this would imply that all the eight $(-2)$-spheres have the same area, which is a contradiction. Finally, note that if any of $A_3, A_4, A_5, A_6$ is realized, $A_7$ is the only possible class with $a = 0$. If none of $A_3, A_4, A_5, A_6$ is realized, the allowable classes with $a = 0$ are $E_{j_3} - E_{j_4}, E_{j_5} - E_{j_6}, E_{j_7} - E_{j_8}$, in addition to $A_7$. It follows that neither $A_1$ nor $A_2$ can be realized.

The above discussion shows that $F_1, F_2$ are the only two $(-2)$-spheres with $a = 2$. From the discussion, it is also clear that the maximal number of mutually disjoint classes with $a = 1$ which intersect trivially with $F_1, F_2$ is 4, which, without loss of generality, are given by $A_3, A_4, A_5, A_6$. If any of them is realized, there is only one
possible class with \( a = 0 \), i.e., \( A_7 = E_{j_1} - E_{j_2} \). If none of the \( a = 1 \) classes are realized, then there are maximally 4 classes with \( a = 0 \) that are allowed. In any event, we do not have enough classes that can be realized. Thus (i) is eliminated.

(ii): Assume only one of \( F_1, F_2, \cdots, F_8 \) has \( a \)-coefficient equaling 2. Without loss of generality, assume it is \( F_1 \), and we write

\[
F_1 = 2H - E_{k_1} - E_{k_2} - E_{k_3} - E_{k_4} - E_{k_5} - E_{k_6}.
\]

We denote the remaining three \( E_i \)-classes by \( E_{k_7}, E_{k_8}, E_{k_9} \), and denote by \( A \) the class of any of the \((-2)\)-spheres \( F_2, F_3, \cdots, F_8 \).

Examining classes \( A \) with \( a = 1 \) which intersect trivially with \( F_1 \), we note that \( A \) must be of the form

\[
A = H - E_{r} - E_{s} - E_{t}, \quad \text{where} \quad E_{r}, E_{s} \in \{E_{k_1}, \cdots, E_{k_6}\}, \quad \text{and} \quad E_{t} \in \{E_{k_7}, E_{k_8}, E_{k_9}\}.
\]

Consider first the case where amongst the classes \( A \) with \( a = 1 \), the \( E_i \)-classes \( E_{k_1}, E_{k_2}, \cdots, E_{k_6} \) can only appear once. It is easy to see that in this case, all the \( a = 1 \) classes must have a common \( E_i \)-class which must be one of \( E_{k_7}, E_{k_8}, E_{k_9} \). It is clear that there are maximally three such classes with \( a = 1 \), i.e.,

\[
H - E_{k_1} - E_{k_2} - E_{k_7}, \quad H - E_{k_3} - E_{k_4} - E_{k_7}, \quad H - E_{k_5} - E_{k_6} - E_{k_7}
\]

without loss of generality. The remaining four \((-2)\)-spheres must have \( a \)-coefficient equaling 0, and they must be

\[
E_{k_1} - E_{k_2}, \quad E_{k_3} - E_{k_4}, \quad E_{k_5} - E_{k_6}, \quad E_{k_8} - E_{k_9}
\]

without loss of generality. With this understood, we note that the area of \( F_1 \) must be the larger \( \delta_1 \), with the remaining seven \((-2)\)-spheres having area \( \delta_2 \). However, this leads to

\[
\omega(F_1) - \omega(E_{k_1} - E_{k_2}) - \omega(E_{k_3} - E_{k_4}) = \omega(2H - 2E_{k_1} - 2E_{k_3} - E_{k_5} - E_{k_6}) \geq 0,
\]

which contradicts the constraint \( \delta_1 < 2\delta_2 \). Hence this first case is ruled out.

Next we assume that the \( E_i \)-classes \( E_{k_1}, E_{k_2}, \cdots, E_{k_6} \) can appear at most twice in the \( a = 1 \) classes, and at least one of them, say \( E_{k_1} \), appeared twice. Then without loss of generality, we may assume

\[
A_1 = H - E_{k_1} - E_{k_2} - E_{k_7}, \quad A_2 = H - E_{k_1} - E_{k_3} - E_{k_8}
\]

are realized by the \((-2)\)-spheres. Since there are at most 4 mutually disjoint classes with \( a = 0 \) that can possibly be realized by the \((-2)\)-spheres, we must have another \( a = 1 \) class, call it \( A_3 \). By our assumption, \( A_3 \) can not contain \( E_{k_1} \). The fact that \( A_3 \) intersects trivially with \( A_1, A_2 \) implies that either \( A_3 = H - E_{k_2} - E_{k_3} - E_{k_9} \), or without loss of generality, \( A_3 = H - E_{k_1} - E_{k_4} - E_{k_7} \). In the former case, none of \( E_{k_1}, E_{k_2}, E_{k_3} \) can appear anymore by our assumption, which implies easily that there can be no more \( a = 1 \) classes. On the other hand, there is only one possible \( a = 0 \) class, say \( E_{k_5} - E_{k_6} \). Hence the former case is not possible. In the latter case, \( E_{k_1}, E_{k_3}, E_{i_7} \) can no longer appear. We note that there is only one possible \( a = 0 \) class, i.e., \( E_{k_5} - E_{k_6} \), so there must be three more \( a = 1 \) classes. Call them \( A_4, A_5, A_6 \). Then observe that \( A_4, A_5, A_6 \) intersect trivially with \( A_2 \), so all of them must contain
which is listed as Case (b) of the lemma.

Finally, assume one of the $E_i$-classes $E_{k_1}, E_{k_2}, \cdots, E_{k_6}$, say $E_{k_4}$, appears in the $a = 1$ classes three times. Without loss of generality, we assume

$$A_1 = H - E_{k_1} - E_{k_2} - E_{k_7}, \quad A_2 = H - E_{k_1} - E_{k_3} - E_{k_8}, \quad A_3 = H - E_{k_1} - E_{k_4} - E_{k_9}$$

are realized by the $(-2)$-spheres. Again, there is only one possible $a = 0$ class, i.e., $E_{k_5} - E_{k_6}$, so there must be three more $a = 1$ classes, which are denoted by $A_4, A_5, A_6$.

It is easy to see that the following are the only possibility:

$$A_4 = H - E_{k_3} - E_{k_4} - E_{k_7}, \quad A_5 = H - E_{k_2} - E_{k_4} - E_{k_8}, \quad A_6 = H - E_{k_2} - E_{k_3} - E_{k_9}.$$ 

In order to rule out this last case, we observe that

$$F_1 + \sum_{i=1}^{6} A_i + c_1(K_X) = 5H - 3(E_{k_1} + \cdots + E_{k_4}) - E_{k_7} - E_{k_8} - E_{k_9}.$$ 

The right-hand side has non-negative area, which leads to a contradiction to the constraint $7\delta_i < -c_1(K_X) \cdot [\omega]$ for $i = 1, 2$. Hence (ii) is also eliminated.

(iii): It remains to consider the case where the $a$-coefficient of $F_1, F_2, \cdots, F_8$ equals either 1 or 0. We begin by noting that there are at least four $(-2)$-spheres with $a = 1$.

The first possibility is that each $E_i$-class appears amongst the $a = 1$ classes at most three times. To analyze this case, we take two of the $(-2)$-spheres with $a = 1$, say $F_1, F_2$, and we write them as

$$F_1 = H - E_{l_1} - E_{l_2} - E_{l_3}, \quad F_2 = H - E_{l_1} - E_{l_4} - E_{l_5}.$$ 

Assume $F_3$ also has $a$-coefficient equaling 1. Then there are two possibilities for $F_3$: either $F_3 = H - E_{l_1} - E_{l_6} - E_{l_7}$ or $F_3 = H - E_{l_2} - E_{l_4} - E_{l_6}$ without loss of generality. There is at least one more $(-2)$-sphere with $a = 1$, say $F_4$. Then if $F_3 = H - E_{l_1} - E_{l_6} - E_{l_7}$, we may assume without loss of generality that $F_4 = H - E_{l_2} - E_{l_4} - E_{l_6}$ because of our assumption that each $E_i$-class appears amongst the $a = 1$ classes at most three times. If $F_3 = H - E_{l_2} - E_{l_4} - E_{l_6}$ in the latter case, we may assume $F_4 = H - E_{l_3} - E_{l_5} - E_{l_6}$ (note that the other choice $F_4 = H - E_{l_1} - E_{l_6} - E_{l_7}$ is equivalent to the former case). In any event, with these choices for $F_1, F_2, F_3, F_4$, there can be at most one $(-2)$-sphere with $a = 0$. Consequently, there must be three more $(-2)$-spheres with $a = 1$. One can check easily that without loss of generality, in this case the eight $(-2)$-spheres are

$$F_1 = H - E_{l_1} - E_{l_2} - E_{l_3}, \quad F_2 = H - E_{l_1} - E_{l_4} - E_{l_5}, \quad F_3 = H - E_{l_1} - E_{l_6} - E_{l_7}, \quad F_4 = H - E_{l_2} - E_{l_4} - E_{l_6}, \quad F_5 = H - E_{l_3} - E_{l_5} - E_{l_6}, \quad F_6 = H - E_{l_2} - E_{l_5} - E_{l_7}, \quad F_7 = H - E_{l_3} - E_{l_4} - E_{l_7}, \quad F_8 = E_{l_8} - E_{l_9},$$

which is listed as Case (b) of the lemma.

The remaining possibility is that one of the $E_i$-classes appears in the $a = 1$ classes four times. In this case, it is easy to check that without loss of generality, the eight $(-2)$-spheres are

$$F_1 = H - E_{l_1} - E_{l_2} - E_{l_3}, \quad F_2 = H - E_{l_1} - E_{l_4} - E_{l_5}, \quad F_3 = H - E_{l_1} - E_{l_6} - E_{l_7},$$
\[ F_4 = H - E_{l_1} - E_{l_8} - E_{l_9}, F_5 = E_{l_2} - E_{l_3}, F_6 = E_{l_4} - E_{l_5}, F_7 = E_{l_6} - E_{l_7}, F_8 = E_{l_8} - E_{l_9}. \]

This is listed as Case (c) of the lemma. The proof of the lemma is complete.

In the following lemma, \( D \subseteq \mathbb{C} \) is an open disc centered at the origin, with radius unspecified. Let \( \Psi : D \times D \to \mathbb{C}^2 \) be a diffeomorphism onto a neighborhood of \( 0 \in \mathbb{C}^2 \), given by equations \( z_1 = \psi(z, w), z_2 = w \), where \( z_1, z_2 \) are the standard holomorphic coordinates on \( \mathbb{C}^2 \) and \( z, w \) are a local complex coordinate on the first and second factor in \( D \times D \). Furthermore, assume \( \Psi \) satisfies the following conditions: \( \psi(z, w) \) is holomorphic in \( w \in D \), and \( \psi(0, w) = 0 \) for all \( w \in D \). With this understood,

**Lemma 5.2.** Let \( C \subseteq \mathbb{C}^2 \) be an embedded holomorphic disc containing the origin, where \( C \) intersects the \( z_2 \)-axis with a tangency of order \( n > 1 \). Let \( F : D \to \mathbb{C}^2 \) be a holomorphic parametrization of \( C \) such that \( F(0) = 0 \). Then the map \( \pi_1 \circ \Psi^{-1} \circ F : D \to D \) is an \( n \)-fold branched covering in a neighborhood of \( 0 \in D \), ramified at 0, where \( \pi_1 : D \times D \to D \) is the projection onto the first factor.

**Proof.** Considering the parametrization \( \Psi^{-1} \circ F \) of \( C \) in the coordinates \( (z, w) \), it is clear that after a re-parametrization of the domain \( D \) if necessary, we may assume that \( \Psi^{-1} \circ F \) is given by \( z = f(\xi), w = \xi \), where \( \xi \) is a local holomorphic coordinate on the domain \( D \). We remark that \( \Psi^{-1} \circ F \) is \( J \)-holomorphic with respect to the almost complex structure \( J \) on \( D \times D \), where \( J \) is the pullback of the standard complex structure on \( \mathbb{C}^2 \) via \( \Psi \).

We shall compute \( \partial_w f \) for the function \( f \), where \( f \) is considered a function of \( w \) (as \( w = \xi \)). To this end, we set \( z_k = x_k + \sqrt{-1}y_k, k = 1, 2, \) and \( z = s + \sqrt{-1}t, w = u + \sqrt{-1}v \). Then with respect to the coordinates \((s, t, u, v)\) and \((x_1, y_1, x_2, y_2)\), the Jacobian of \( \Psi \) is given by the matrix

\[
D\Psi = \begin{pmatrix} A & B \\ 0 & I \end{pmatrix},
\]

where \( A = \begin{pmatrix} \frac{\partial x_1}{\partial s} & \frac{\partial x_1}{\partial t} \\ \frac{\partial y_1}{\partial s} & \frac{\partial y_1}{\partial t} \end{pmatrix} \), \( B = \begin{pmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial v} \\ \frac{\partial y_1}{\partial u} & \frac{\partial y_1}{\partial v} \end{pmatrix} \). Let \( J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) be the matrix representing the standard complex structure. Then the assumptions that \( \psi(z, w) \) is holomorphic in \( w \in D \) and \( \psi(0, w) = 0 \) for all \( w \in D \) imply that \( J_0B = BJ_0 \) and \( B = 0 \) along the disc \( z = 0 \).

With the preceding understood, we note that the almost complex structure \( J \) is given by the matrix

\[
J = D\Psi \begin{pmatrix} J_0 & 0 \\ 0 & J_0 \end{pmatrix} (D\Psi)^{-1} = \begin{pmatrix} A_{J_0}A^{-1} & (-AJ_0A^{-1} + J_0)B \\ 0 & J_0 \end{pmatrix}. 
\]

Now the Jacobian of \( \Psi^{-1} \circ F \) is \( \begin{pmatrix} Df \\ I \end{pmatrix} \) where \( Df \) is the Jacobian of \( f \). It follows easily that the \( J \)-holomorphic equation satisfied by \( \Psi^{-1} \circ F \), i.e.,

\[
J \begin{pmatrix} Df \\ I \end{pmatrix} = \begin{pmatrix} Df \\ I \end{pmatrix} J_0
\]
is equivalent to the equation $Df + (AJ_0A^{-1}) \cdot Df \cdot J_0 = (AJ_0A^{-1}J_0 + I)B$. Intrinsically, this can be written as

$$\partial_w f = \frac{1}{2}(AJ_0A^{-1}J_0 + I)B.$$  

With the above understood, we note that since $B = 0$ along the disc $z = 0$, we have $|B| \leq C_1|z|$ near $z = 0$ for some constant $C_1 > 0$. It follows easily that the function $f$ obeys the inequality $|\partial_w f| \leq C_2|f|$ for some constant $C_2 > 0$. By the Carleman similarity principle (e.g. see Siébert-Tian [39], Lemma 2.9), there is a complex valued function $g$ of class $C^\alpha$ and a holomorphic function $\phi$, such that $f(w) = \phi(w)g(w)$, where $g(0) \neq 0$. Note that $\phi$ vanishes at $w = 0$ of order $n$ because by the assumption, the holomorphic disc $C$ intersects the $z_2$-axis with a tangency of order $n$. After a further change of coordinate, we may assume that $f(w) = w^n g(w)$ for a $C^\alpha$-class function $g$, where $w \in D$.

Our next goal is to show that for any $c \neq 0$, with $|c|$ sufficiently small, the equation

$$f(w) = c$$

has exactly $n$ distinct solutions lying in a small neighborhood of $0 \in D$. To see this, we take $h(w)$ to be an $n$-th root of the function $g(w)$, i.e., $h(w)^n = g(w)$, which is also of $C^\alpha$-class. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the $n$-th roots of $c$. For each $i = 1, 2, \ldots, n$, we consider the equation

$$wh(w) = \lambda_i.$$  

Set $P(w) := \frac{1}{h(0)}(\lambda_i - w(h(w) - h(0)))$. Then the above equation becomes $w = P(w)$.

With this understood, let $B(r) \subset D$ be the closed disc of radius $r$. Then for $r > 0$ sufficiently small, $P : B(r) \to B(r)$ is a well-defined continuous map, as long as $|\lambda_i| \leq \frac{1}{2}|h(0)| \cdot r$. Now we pick any $w_1 \in B(r)$ and define inductively $w_{k+1} = P(w_k)$ for $k \geq 1$. Since $B(r)$ is compact, the sequence $\{w_k\}$ has a convergent subsequence. The limit $w_0 \in B(r)$ satisfies the equation $w_0 = P(w_0)$.

It follows easily that when $c \neq 0$ lies in the disc of radius $(\frac{1}{2}|h(0)| \cdot r)^n$, the equation $f(w) = c$ has at least $n$ distinct solutions, all lying in the disc $B(r)$. On the other hand, the local intersection number of the holomorphic disc $C$ with each holomorphic disc $z = c$ equals $n$. This implies that the equation $f(w) = c$ has precisely $n$ distinct solutions in $B(r)$, and the intersections of $C$ with each holomorphic disc $z = c$, where $c \neq 0$ and $|c|$ is sufficiently small, are all transversal. It follows immediately that the map $\pi_1 \circ \Psi^{-1} \circ F : D \to D$ is an $n$-fold branched covering in a neighborhood of $0 \in D$, ramified at $0$. This finishes the proof of the lemma.

With these preparations, we now prove the main theorems.

**Proof of Theorem 1.1:**

We first consider the case where $M_G$ is irrational ruled. Recall that $M_G$ is rational or ruled if and only if the singular set of $M/G$ either contains a 2-dimensional component or a non-Du Val isolated singularity (cf. Lemma 4.1 in [7]). It follows easily that there is a subgroup $H$ of prime order $p$ such that $M_H$ is rational or ruled. On the other hand, $b_1(M/H) \geq b_1(M/G) = 2$ as $M_G$ is irrational ruled. It follows that $M_H$ is in fact also irrational ruled.
By Lemma 2.2 and Lemma 2.6(i), the fixed-point set of \( H \) consists of only tori of self-intersection zero. Moreover, from the proofs it is known that \( M_H \) is a \( S^2 \)-bundle over \( T^2 \), and \( M \) is simply a branched cover of \( M_H \) along the fixed-point set.

With this understood, we denote by \( \{B_i\} \) the image of the fixed-point set of \( H \) in \( M_H \), which is a disjoint union of symplectic tori of self-intersection zero. Let \( F \) be the fiber class of the \( S^2 \)-fibration on \( M_H \). Then we note that \( c_1(K_{M_H}) = \frac{1}{p} \sum_i B_i \) (cf. Proposition 3.2 in [7]), and \( c_1(K_{M_H}) \cdot F = -2 \). It follows easily that \( p = 2 \) or \( 3 \), and \( (\sum_i B_i) \cdot F = 4 \) or \( 3 \) accordingly.

To proceed further, we choose an \( \omega \)-compatible almost complex structure \( J \) on \( M_H \), where \( \omega \) denotes the symplectic structure on \( M_H \), such that \( J \) is integrable in a neighborhood of each \( B_i \). Note that this is possible because \( \omega \) admits a standard model near each \( B_i \). Now by Gromov’s theory, there exists a \( S^2 \)-bundle structure on \( M_H \), with base \( T^2 \) and each fiber \( J \)-holomorphic. We denote by \( \pi : M_H \to T^2 \) the corresponding projection onto the base. Then by Lemma 5.2, the restriction \( \pi|_{B_i} : B_i \to T^2 \) is a branched covering where the ramification occurs exactly at the non-transversal intersection points of \( B_i \) with the fibers. But each \( B_i \) is a torus, so that \( \pi|_{B_i} \) must be unramified, or equivalently, \( B_i \) intersects each fiber transversely. With this understood, it follows easily that the pre-image of each fiber of the \( S^2 \)-bundle in \( M \) is a symplectic torus (here we use the fact that \( (\sum_i B_i) \cdot F = 4 \) or \( 3 \) respectively according to whether \( p = 2 \) or \( 3 \)), giving \( M \) a structure of a \( T^2 \)-bundle over \( T^2 \) with symplectic fibers. This finishes the proof for the case where \( M_G \) is irrational ruled.

Next we assume \( M_G \) is rational and \( G = \mathbb{Z}_2 \). By Lemma 2.3 and Lemma 2.6(ii), the fixed-point set \( M_G \) consists of 8 isolated points and a disjoint union of 2-dimensional components \( \Sigma_i \), where \( \sum_i \Sigma_i^2 = 2(1 - b_2(M/G)) \), and \( b_2(M/G) \in \{0, 1, 2\} \). We denote by \( B_i \) the image of \( \Sigma_i \) in \( M_G \). Then \( c_1(K_{M_G}) = -\frac{1}{2} \sum_i B_i \) (cf. [7], Proposition 3.2) and \( B_i^2 = 2\Sigma_i^2 \) for each \( i \), so that

\[
c_1(K_{M_G})^2 = \frac{1}{4} \sum_i B_i^2 = 1 - b_2(M/G).
\]

It follows easily that \( M_G = \mathbb{CP}^2 \# N\mathbb{CP}^2 \) where \( N = 8, 9 \) or \( 10 \), if \( b_2(M/G) = 0, 1 \) or \( 2 \) respectively. Moreover, note that \( M_G \) contains 8 symplectic \((-2)\)-spheres coming from the resolution of the 8 isolated singular points of \( M/G \).

By Theorem 1.4, the case where \( M_G = \mathbb{CP}^2 \# 8\mathbb{CP}^2 \) is immediately ruled out. The case where \( M_G = \mathbb{CP}^2 \# 10\mathbb{CP}^2 \) is ruled out as follows. We consider the double branched cover \( Y \) of \( M_G \) with branch loci \( \{B_i\} \). Then \( Y \) is easily seen a symplectic Calabi-Yau 4-manifold with \( b_1 = 0 \), which is an integral homology \( K3 \) surface. Note that \( Y \) contains 16 embedded \((-2)\)-spheres in the complement of the branch set. Now observe that in the case of \( M_G = \mathbb{CP}^2 \# 10\mathbb{CP}^2 \), \( \sum_i \Sigma_i^2 = -2 \), so that there must be one \( \Sigma_i \) with \( \Sigma_i^2 < 0 \). This \( \Sigma_i \) gives rise to an embedded \((-2)\)-sphere in \( Y \), in addition to the 16 embedded \((-2)\)-spheres, so that \( Y \) contains 17 disjointly embedded \((-2)\)-spheres. But this contradicts a theorem of Ruberman in [35], which says that an integral homology \( K3 \) surface can contain at most 16 disjointly embedded \((-2)\)-spheres. Hence \( M_G = \mathbb{CP}^2 \# 10\mathbb{CP}^2 \) is ruled out. Finally, we note that the same
argument shows that in the case of $M_G = \mathbb{CP}^2 \# 9 \overline{\mathbb{CP}^2}$, the surfaces $B_i$ must be tori of self-intersection zero, because in this case, $\sum_i \Sigma_i^2 = 0$ where each $\Sigma_i^2 \geq 0$.

We continue by analyzing the case of $M_G = \mathbb{CP}^2 \# 9 \overline{\mathbb{CP}^2}$ in more detail. First, we note that there are at most two components in $\{B_i\}$. This is because $c_1(K_M) = -\frac{1}{2} \sum_i B_i$, and the $a$-coefficient of each $B_i$ with respect to a given reduced basis is at least 3 (cf. Lemma 4.2(2)). Next, we determine the homology classes of the 8 symplectic $(-2)$-spheres $F_1, F_2, \ldots, F_8$ in $M_G$. By Lemma 4.1, we can choose a symplectic structure on $M_G$ so that the area constraints in Lemma 5.1 are satisfied. (Note that this is possible because $-c_1(K_M) \cdot [\omega] = \frac{1}{2} \sum_i \omega(B_i) > 0$.) Then the classes of $F_1, F_2, \ldots, F_8$ are given in 3 cases as listed in Lemma 5.1. We claim that case (a) and case (b) cannot occur. To see this, suppose we are in case (a). It is easy to check, with the area constraints in Lemma 5.1, that the class $E_1$ has the smallest area among the $E_i$-classes in the reduced basis. With this understood, we choose an almost complex structure $J$ such that each symplectic $(-2)$-sphere $F_k$ is $J$-holomorphic. By [24], the class $E_{iy}$ can be represented by a $J$-holomorphic $(-1)$-sphere $C$. Symplectically blow down $M_G$ along $C$, noting that $C$ is disjoint from the $(-2)$-spheres $F_k$ as $C \cdot F_k = 0$, we obtain 8 disjointly embedded symplectic $(-2)$-spheres in $\mathbb{CP}^2 \# 8 \overline{\mathbb{CP}^2}$, contradicting Theorem 1.4. Case (b) is similarly eliminated. Consequently, the homology classes of $F_1, F_2, \ldots, F_8$ are given by case (c) of Lemma 5.1.

Our next step is to show that there is an embedded symplectic sphere with self-intersection zero, denoted by $F$, which lies in the complement of $F_1, F_2, \ldots, F_8$ and intersects transversely and positively with $B_i$. This can be seen as follows. It is easy to check that in case (c) of Lemma 5.1, the class $E_{i_1}$ has the largest area. By [24], we can choose $\omega$-compatible almost complex structures $J$ so that $B_i$ and $F_k$ are all $J$-holomorphic, and successively represent the classes $E_{i_s}$, $s \geq 2$, beginning with the one of the smallest area, by a $J$-holomorphic $(-1)$-sphere. By successively symplectically blowing down the classes $E_{i_s}$, $s \geq 2$, we reach $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$, with $E_{i_1}$ being the $(-1)$-class. Note that the $(-2)$-spheres $F_1, F_2, F_3, F_4$ descend to 4 disjointly embedded symplectic spheres of self-intersection zero (they all have class $H - E_{i_1}$); in fact there is a symplectic $S^2$-fibration of $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ containing them as fibers. With this understood, we can take a fiber $F$ in the complement which intersects transversely and positively with the descendant of $B_i$ in $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$. We then symplectically blow up $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ successively, reversing the symplectic blowing down procedure, in order to go back to $M_G$. In this way, we recover the 8 symplectic $(-2)$-spheres $F_1, F_2, \ldots, F_8$ and the tori $B_i$. Note that the symplectic structure on $M_G$ may be different since we don’t keep track of the sizes of the symplectic blowing up.

Now we symplectically blow down $F_1, F_2, \ldots, F_8$, which results a symplectic 4-orbifold $X$ with 8 isolated singular points, all of isotropy of order 2. In the complement of the singularities, there lies the embedded symplectic sphere $F$ with $F^2 = 0$, and the tori $B_i$. By [21], we can assume that $F$ and $B_i$ intersect symplectically orthogonally without loss of generality.

With the preceding understood, we consider the set $J$ of $\omega$-compatible almost complex structures on $X$ which satisfy the following conditions: fix a sufficiently small regular neighborhood $V$ of $\cup_i B_i$, not containing any singular points of $X$, and fix
Then $C_i$ contains $k > 0$ singular points of $X$. We can pick an orbifold Riemann sphere $\Sigma$ with $k$ orbifold points of order 2, which are denoted by $z_1, z_2, \cdots, z_k$, and find a $J$-holomorphic map $f: \Sigma \to X$ parametrizing $C_i$. In general, such a map $f$ near an orbifold point $z_j$, which has order $m_j$, is given by a pair $(\hat{f}_j, \rho_j)$, where $\hat{f}_j: D \to \mathbb{C}^2$ is a local lifting of $f$ near $z_j$ to the uniformizing system at $f(z_j) \in X$, and $\rho_j: \mathbb{Z}_{m_j} \to G_{f(z_j)}$ is an injective homomorphism to the isotropy group $G_{f(z_j)}$ at $f(z_j) \in X$, with respect to which $\hat{f}_j$ is equivariant. With this understood, we let $g \in \mathbb{Z}_{m_j}$ be the generator acting on $D$ by a rotation of angle $2\pi/m_j$, and let $(m_{j,1}, m_{j,2})$, $0 \leq m_{j,1}, m_{j,2} < m_j$, be the weights of the action of the element $\rho_j(g) \in G_{f(z_j)}$ on $\mathbb{C}^2$. Then the virtual dimension of the moduli space of $J$-holomorphic curves containing $C_i$ equals $2d$, where $d \in \mathbb{Z}$ and is given by

$$d = c_1(TX) \cdot C_i + 2 - \sum_{j=1}^k \frac{m_{j,1} + m_{j,2}}{m_j} - (3 - k).$$

See [11][6]. Note that in the present situation, $m_j = 2$ and $m_{j,1} = m_{j,2} = 1$ for each $j$. It follows easily that $d = c_1(TX) \cdot C_i - 1$; in particular, $c_1(TX) \cdot C_i \in \mathbb{Z}$. Moreover, since $J$ is generic, we have $d \geq 0$, which implies that $c_1(TX) \cdot C_i \geq 1$.

An immediate corollary is that $\{C_i\}$ either consists of two components, each with multiplicity 1, or a single component with multiplicity 2, and moreover, $c_1(TX) \cdot C_i = 1$ for each $i$. This is because $c_1(TX) \cdot F = 2$, and $F = \sum_i m_iC_i$. We can further rule out the possibility of two components as follows. Suppose to the contrary that there are two components $C_1, C_2$ in $\{C_i\}$. Then $C_1^2 + 2C_1 \cdot C_2 + C_2^2 = F^2 = 0$ implies that one of $C_1^2, C_2^2$ must be negative. Without loss of generality, assume $C_1^2 < 0$. Then $C_2^2 \geq 0$ because $b_2^+(X) = 1$. With this understood, we note that $C_1 \cdot C_2 \geq \frac{1}{2}$ by the orbifold intersection formula in [5] (see also [6]). This implies $C_1^2 \leq -1$. Now we
apply the orbifold adjunction inequality (cf. [5, 6]) to \( C_1 \), which gives

\[
C_1^2 - c_1(TX) \cdot C_1 + 2 \geq k \cdot (1 - \frac{1}{2}).
\]

With \( C_1^2 \leq -1 \) and \( c_1(TX) \cdot C_1 = 1 \), it follows that \( k = 0 \), which is a contradiction. Hence the claim that there is only one component in \( \{ C_i \} \).

Let \( C \) denote the single component which has multiplicity 2, and let \( f : \Sigma \to X \) be a \( J \)-holomorphic parametrization of \( C \). Then we note that \( C^2 = 0 \) and \( c_1(TX) \cdot C = 1 \).

Applying the orbifold adjunction formula to \( C \) (cf. [5, 6]), we get

\[
C^2 - c_1(TX) \cdot C + 2 = k(1 - \frac{1}{2}) + \sum k_{[z,z']} + \sum k_z,
\]

where \( k_{[z,z']}, k_z \geq 0 \). It follows easily that \( k \leq 2 \), and if \( k = 2 \), then all \( k_{[z,z']}, k_z = 0 \), which means that \( C \) is an embedded 2-dimensional suborbifold. To rule out the possibility that \( k = 1 \), we first observe that in this case, \( k_{[z,z']} \in \mathbb{Z} \). This is because as \( k = 1 \), we can not have a pair of points \( z, z' \in \Sigma \), where \( z \neq z' \), such that \( f(z) = f(z') \) is a singular point of \( X \). It follows easily that all \( k_{[z,z']} \) must be zero, and \( k_z = \frac{1}{2} \) at the unique singular point \( f(z) \) on \( C \). The number \( k_z \) is the local self-intersection number of \( C \) at the singular point, and \( k_z = \frac{1}{2} \) means that in the uniformizing system near the singular point, \( C \) is given by a \( J \)-holomorphic (singular) disc with a local self-intersection 1 at the origin. It follows that the singularity at the origin must be a cusp singularity and the \( J \)-holomorphic disc is parametrized by a pair of functions \( z_1 = t^2, z_2 = t^3 + \cdots \), where \( t \in D \). However, it is clear that a such defined \( J \)-holomorphic disc is not invariant under the \( \mathbb{Z}_2 \)-action \( (z_1, z_2) \mapsto (-z_1, -z_2) \), which is a contradiction. Hence \( k = 1 \) is ruled out. This finishes the proof of the lemma.

\[ \square \]

It follows easily that the compactified moduli space \( \overline{M}_J \) gives rise to a \( J \)-holomorphic \( S^2 \)-fibration on \( X \), which contains 4 multiple fibers, each with multiplicity 2. We denote by \( \tau : X \to B \) the \( S^2 \)-fibration. It is easy to see that the base \( B \) is an orbifold sphere, with 4 orbifold points of order 2. Furthermore, note that for each \( i \), \( \pi|_{B_i} : B_i \to B \) is a branched covering in the complement of the multiple fibers by Lemma 5.2.

To proceed further, we note that \( c_1(K_X) = -\frac{1}{2} \sum_i B_i \), so that \( \left( \sum_i B_i \right) \cdot F = 4 \). Let \( z_1, z_2, z_3, z_4 \) be the orbifold points of \( B \), and let \( w_1, \cdots, w_k \in B \) be the points parametrizing those regular fibers which does not intersect transversely with \( \cup_i B_i \). We denote by \( x_i \) the number of intersection points of \( \cup_i B_i \) with the multiple fiber at \( z_l \), \( l = 1, 2, 3, 4 \), and denote by \( y_j \) the number of intersection points of \( \cup_i B_i \) with the regular fiber at \( w_j \), where \( j = 1, 2, \cdots, k \). Then note that \( x_i \leq 2 \) and \( y_j < 4 \) for each \( l, j \). On the other hand, we observe the following relation in Euler numbers:

\[
\sum_i \chi(B_i) - \sum_{j=1}^k y_j - \sum_{l=1}^4 x_l = 4(\chi(|B|) - k - 4),
\]

where \( |B| = S^2 \) is the underlying space of \( B \). With \( x_i \leq 2 \) and \( y_j < 4 \), it follows easily that \( k \) must be zero, and \( x_l = 2 \) for each \( l \). This means that \( \cup_i B_i \) intersects each regular fiber transversely at 4 points and intersects each multiple fiber at 2 points.
Finally, we observe that $X = |M/G|$, i.e., $X$ is the symplectic 4-orbifold obtained by de-singularizing $M/G$ along the 2-dimensional singular components. With this understood, it is easy to see that under the projection $M \to X = |M/G|$, the pre-image of each regular fiber in the $\mathbb{S}^2$-fibration on $X$ is a symplectic $T^2$ in $M$, giving rise to a $T^2$-fibration over $B$ on $M$ (here we use the fact that $\cup_i B_i$ intersects each regular fiber transversely at 4 points and the projection $M \to X$ is a double cover branched over $\cup_i B_i$). Moreover, the pre-image of each multiple $S^2$-fiber is a multiple $T^2$-fiber of multiplicity 2 in the $T^2$-fibration on $M$. It is known that such a 4-manifold $M$ is diffeomorphic to a hyperelliptic surface or a secondary Kodaira surface, see \[16\]. Since $b_1(M) \neq 1$, $M$ can not be diffeomorphic to a secondary Kodaira surface. Hence $M$ must be diffeomorphic to a hyperelliptic surface, and as such, it is diffeomorphic to a $T^2$-bundle over $T^2$ with homologically essential fibers. This finishes the proof of Theorem 1.1.

**Proof of Theorems 1.2 and 1.3:**

Suppose $G$ is of prime order $p$. The case where $M_G$ has torsion canonical class is contained in Lemmas 2.1 and 2.8(2), and the case where $M_G$ is irrational ruled is in Lemmas 2.2 and 2.6(i), with $p = 2$ or 3 from the proof of Theorem 1.1.

Suppose $M_G$ is rational. Then by Lemmas 2.3, 2.4, 2.6 and 2.8, the order $p = 2, 3$ or 5. Concerning the fixed-point set structure, the case of $G = \mathbb{Z}_2$ follows readily from the proof of Theorem 1.1. For $G = \mathbb{Z}_3$, the fixed-point set structure for the isolated points is determined in Lemmas 2.4 and 2.9. Regarding the 2-dimensional fixed components, we need to explore the embedding $D \to M_G$. In order to determine $M_G$ in each case, we use the formula in Proposition 3.2 of \[17\] to determine $c_1(K_{M_G})$, based on the singular set structure of the quotient orbifold $M/G$, then we compute $c_1(K_{M_G})^2$. This allows us to determine the diffeomorphism type of $M_G$ as $M_G$ is a rational 4-manifold. In the case of $b_1(M) = 2$, it is easy to see that $M_G = \mathbb{CP}^2 \# 10\overline{\mathbb{CP}^2}$. If the set of 2-dimensional fixed components is nonempty, Proposition 4.3 implies that it must consist of a single torus. In the case of $b_1(M) = 4$, $M_G = \mathbb{CP}^2 \# 12\overline{\mathbb{CP}^2}$, and Proposition 4.4 implies that there are no 2-dimensional fixed components. For $G = \mathbb{Z}_5$ where $b_1(M) = 4$, the fixed-point set structure for the isolated points is determined in Lemma 2.10. The possible 2-dimensional fixed components are excluded by Proposition 4.5.

For the case where $G$ is of non-prime order, the order of $G$ and the fixed-point set structure are determined in Lemmas 2.11 and 2.12. This completes the discussion on Theorems 1.2 and 1.3.

**Proof of Theorem 1.4:**

First, consider the case of $N = 8$. We begin by showing that one can choose a symplectic structure on $X$ such that the area constraints in Lemma 5.1 are fulfilled. To see this, by Lemma 4.1 we can choose symplectic structures $\omega$ on $X$ such that one of the 8 symplectic $(-2)$-spheres has area $\delta_1$ and the remaining 7 symplectic $(-2)$-spheres have area $\delta_2$, where $\delta_2 < \delta_1 < 2\delta_2$, and $\delta_1, \delta_2$ can be arbitrarily small. It remains to show that one can arrange so that $7\delta_i < -c_1(K_X) \cdot [\omega]$, $i = 1, 2$, hold true. For this, we need to use the fact that for $X = \mathbb{CP}^2 \# N\overline{\mathbb{CP}^2}$, where $N \leq 8$, $-c_1(K_X)$ can be represented by pseudo-holomorphic curves, and moreover, one can require the
pseudo-holomorphic curves to pass through any given point in \( X \), see Taubes [40]. We pick a point \( x_0 \in X \) in the complement of the 8 symplectic \((-2)\)-spheres and require the pseudo-holomorphic curves representing \(-c_1(K_X)\) to pass through \( x_0 \). Then it is easy to see that no matter how small we choose the areas \( \delta_1, \delta_2 \), \(-c_1(K_X) \cdot [\omega] > \delta_0 \) for some \( \delta_0 \) independent of the choice of \( \delta_1, \delta_2 \). It follows that we can arrange so that \( 7\delta_i < -c_1(K_X) \cdot [\omega], i = 1, 2 \), hold true. Hence there is a symplectic structure on \( X \) such that the area constraints in Lemma 5.1 are fulfilled.

With the preceding understood, by the same argument as in Lemma 5.1, we can show that the homology classes of the 8 symplectic \((-2)\)-spheres must be given as in case (a) of Lemma 5.1. Then by successively symplectically blowing down the \( E_{i_s} \) classes for \( s \geq 2 \), we reach to the 4-manifold \( \mathbb{C}P^2 \# \mathbb{C}P^2 \), with \( E_{i_1} \) being the \((-1)\)-class. We notice that the 7 symplectic \((-2)\)-spheres \( F_2, F_3, \cdots, F_8 \) descend to a configuration of symplectic spheres of the class \( H \), which intersect transversely and positively according to the incidence relation of the Fano plane; that is, the 7 spheres intersect in 7 points, where each point is contained in 3 spheres. By a theorem of Ruberman and Starkston (cf. [36]), such a configuration can not exist in \( \mathbb{C}P^2 \). Thus to derive a contradiction, we need to represent the class \( E_{i_1} \) by a symplectic \((-1)\)-sphere in the complement of the configuration of 7 symplectic spheres, to further blow down \( \mathbb{C}P^2 \# \mathbb{C}P^2 \) to \( \mathbb{C}P^2 \).

To this end, we note that the configuration of 7 symplectic spheres in \( \mathbb{C}P^2 \# \mathbb{C}P^2 \) is \( J \)-holomorphic with respect to some compatible almost complex structure \( J \). On the other hand, the class \( E_{i_1} \) is represented by a finite set of \( J \)-holomorphic curves \( \sum_i m_i C_i \) by Taubes’ theorem (cf. [28]). Now the key observation is that if there are more than one components in \( \{ C_i \} \), then one of them must have a negative \( a \)-coefficient in the reduced basis \( H, E_{i_1} \). But such a component intersects negatively with the 7 \( J \)-holomorphic spheres in the configuration, whose class is \( H \). This is a contradiction, hence \( E_{i_1} \) must be represented by a single \( J \)-holomorphic curve, which is a \((-1)\)-sphere and lies in the complement of the configuration of 7 symplectic spheres. This finishes the proof for the case of \( N = 8 \).

The argument for the case of \( N = 7 \) is similar. For \( N = 9 \), it is easy to see from Lemma 5.1 that the homology class for the 9-th symplectic \((-2)\)-sphere does not exist. This completes the proof of Theorem 1.4.

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