The Large-$N$ Limit of the Two-Hermitian-matrix model by the hidden BRST method

J. Alfaro*
Theory Division, CERN
CH-1211, Geneva 23

March 28, 2022

Abstract

This paper discusses the large $N$ limit of the two-Hermitian-matrix model in zero dimensions, using the hidden BRST method. A system of integral equations previously found is solved, showing that it contained the exact solution of the model in leading order of large $N$.

*Permanent address: Fac. de Fisica, Universidad Católica de Chile, Casilla 306, Santiago 22, Chile.
1 INTRODUCTION

Recently there has been a considerable amount of progress in understanding
the physics of two-dimensional quantum gravity coupled to \( c \leq 1 \) matter.
This was brought by the discovery of the double scaling limit in matrix models
\[1\]: The perturbative expansion of the matrix model provides a discretization
of the two-dimensional surface, classified according to its genus, which in the
matrix model is a given order in \( 1/N \) (\( N \) being the range of the matrix). By
tuning the couplings in the matrix model in such a way that the perturbative
series diverges, one can approach the continuum limit of the two-dimensional
surface. In this way it is possible to get a sum over all topologies in the first
quantized string, using as a discretization of the two-dimensional world sheet
a dense sum of Feynman graphs of an associated matrix model. In certain
cases, this sum can be calculated exactly, thus providing us with a definition
of the non-perturbative string.

Critical exponents have been computed using this method. Furthermore
it has been shown that a Virasoro algebra naturally arises in these matrix
models \[2\]. This algebra is a realization of the Schwinger-Dyson equations in
the \( U(N) \) invariant set of operators of the matrix model. Using this algebra,
the equivalence of zero-dimensional matrix models to topological 2\textit{d} gravity
has been proved \[3\]. Very recently, an application of these ideas to compute
intersection indices on moduli spaces of Riemann surfaces has been made \[4\].

It is clear that further progress in understanding the large-\( N \) limit will
provide new grounds to explore the physics of two-dimensional quantum
gravity and non-critical strings.

Three years ago we proposed an alternative approach to the large-\( N \) limit
based upon a hidden BRST symmetry which is present in any quantum field
theory \[5\]. In particular we were able to prove a theorem that permits the
solution of the model in leading order of large-\( N \) in a very simple way \[6\]. An
important application of these ideas was made to the two-Hermitian-matrix
model \[7\], which corresponds in the string picture to the spherical topology
of the colored string with two colors \[8\]. In such a case we were able to find
a set of integral equations that determine completely all correlations of the
traces of powers of the two matrices, in contrast with previous results where
only the partition function was found \[9\]. Very recently, a \( W_3 \) algebra has
emerged from this model \[10\].

In this paper we study the solutions of these equations in a systematic
way. We show that the system of integral equations can be reduced to a set of algebraic equations depending on some arbitrary constants. The arbitrary constants are fixed by the requirement of having either one or more cuts of an associated analytic function.

Since our main objective is to test the hidden BRST method, we discuss in detail several simple examples, the simplest non-trivial case corresponding to a cubic potential. We find that the solution of the problem is given by the same cubic equation as in ref. [11]. To show explicitly how the arbitrary constants are determined, we discuss the cubic potential with one free matrix. As a final example we show how to solve the two-matrix model with a quartic potential. In this case the system of integral equations reduces to a quartic algebraic equation.

This paper is organized as follows: In section 2 we review the derivation of the Schwinger-Dyson equations using the hidden BRST method. As an illustration, in section 3 we present a construction of the BRST charge encoding the Virasoro constraints of the zero-dimensional, one-hermitian-matrix model. In section 4 we derive the integral equations that describe the leading order in large-$N$ of the Two-Hermitian-matrix model. Section 5 show how to reduce the integral equations to an algebraic system. Section 7 contains the examples, and in section 8 we draw some conclusions.

2 The Hidden BRST method

In this section we review the BRST-invariant derivation of Schwinger-Dyson equations of ref.[6]. Usually we study separately the invariances of the classical action and the path integral measure. In the hidden BRST method we prefer to write a BRST-invariant action that reflects the symmetries of both the classical action and the functional measure. To illustrate the point let us consider the following functional integral describing the quantum properties of a bosonic field (A more general case is considered in [8]):

$$Z = \int [d\phi] \exp[-S(\phi)]$$  \hspace{1cm} (1)

Here $\phi(x)$ is a scalar field defined in the space-time point $x$. The functional measure $[d\phi]$ is invariant under field translations:
\[
\phi(x) \rightarrow \phi(x) + \epsilon(x) \quad (2)
\]

We want to build an action that is invariant under (2). In order to do this we introduce a collective field \( B(x) \) and write \( Z \) (up to a multiplicative constant) as follows:

\[
Z = \int [d\phi][dB] \exp[-S(\phi - B)] \quad (3)
\]

\( S(\phi - B) \) is invariant under (2) provided we transform \( B \) at the same time by:

\[
B(x) \rightarrow B(x) + \epsilon(x) \quad (4)
\]

The gauge invariant functions are \( F[\phi - B] \) for any \( F \). In the gauge \( B(x) = 0 \) we get back the original path integral (1).

In order to quantize the gauge invariant theory that we have just introduced we use the BRST method[5]. The BRST and anti-BRST transformations are:

\[
\begin{align*}
\delta \phi &= \psi & \bar{\delta} \phi &= \bar{\psi} \\
\delta B &= \psi & \bar{\delta} B &= \bar{\psi} \\
\delta \psi &= 0 & \bar{\delta} \psi &= -ib \\
\delta \bar{\psi} &= ib & \bar{\delta} \bar{\psi} &= 0 \\
\delta b &= 0 & \bar{\delta} b &= 0
\end{align*}
\]

(5)

To reach the gauge \( B(x) = 0 \), we add the following term to the action:

\[
\delta \bar{\delta} [B^2] = \delta [B \bar{\psi}] = ibB - \bar{\psi} \psi \quad (6)
\]

If we integrate over the fields \( b \) and \( B \) the action reduces to:

\[
S[\phi] + \int [dx] \bar{\psi} \psi \quad (7)
\]
Using the equation of motion for $b$ and $B$ the BRST and anti-BRST symmetry (5) reduces to:

$$
\delta \psi(x) = \epsilon \psi(x) + \bar{\epsilon} \bar{\psi}(x)
$$

$$
\delta \psi(x) = \frac{\delta S}{\delta \phi(x)} \epsilon
$$

$$
\bar{\delta} \bar{\psi}(x) = -\frac{\delta S}{\delta \phi(x)} \bar{\epsilon}
$$

$$
\epsilon^2 = \bar{\epsilon}^2 = \epsilon \bar{\epsilon} + \epsilon \bar{\epsilon} = 0
$$

We can easily check that (8) is indeed a symmetry of (7). Associated to this symmetry is a set of Ward identities, for instance, for any function $F$:

$$
< \delta[F(\phi)\psi(y)] > = 0
$$

i.e.

$$
< \int dx \frac{\delta F}{\delta \phi(x)} [\epsilon \psi(x) + \bar{\epsilon} \bar{\psi}(x)] \psi(y) + F \frac{\delta S}{\delta \psi(y)} \bar{\epsilon} > = 0
$$

Computing the (trivial) average with respect to the fermionic variables we get:

$$
< \frac{\delta F}{\delta \phi(x)} - F \frac{\delta S}{\delta \phi(x)} > = 0
$$

The last identity encodes the most general Schwinger-Dyson equation of the model.

Notice that the symmetry (8) commutes with any symmetry of the classical action of the model. In particular, if we study a theory which is invariant under a large $N$ group, the invariance (8) will be respected order by order in the $1/N$ expansion. We will use this important fact to derive a BRST formulation of the Virasoro constraints satisfied by the zero dimensional matrix model in the next section.
3 Q CHARGE OF VIRASORO CONSTRAINTS

By now it is well known [2] that the partition function of the zero dimensional hermitian matrix model is annihilated by a set of operators that satisfy a Virasoro algebra. The existence of this algebra is fundamental to the proof of equivalence of two dimensional topological gravity and matrix models[3] and to the connection between matrix models and intersection indices in modular spaces of Riemann surfaces[4].

The Virasoro algebra we are discussing is a direct consequence of the Schwinger-Dyson equations of the matrix model. So we expect that the BRST-invariant derivation of these equations we presented in the previous chapter will produce a BRST extension of the Virasoro algebra. This is indeed the case as we are going to explain below. See also ([12] and[13]):

Let us consider the generating function of $U(N)$ invariants correlations functions of the zero-dimensional-hermitian matrix model:

$$Z[j] = \int [dM] \exp[-\sum_{n=0}^{\infty} j_n Tr M^n]$$

(12)

Particular critical potentials are obtained by expanding around different points in $j$-space.

Using the procedure of the last section we form the BRST-invariant extension of $Z[j]$:

$$Z[j, \eta, \bar{\eta}] = \int [dM][d\psi][d\bar{\psi}] \exp[-\sum_{n=0}^{\infty} (j_n Tr M^n + \eta_n Tr \bar{\psi} M^n + \bar{\eta}_n Tr \psi M^n)] \exp[-Tr \bar{\psi} \psi]$$

(13)

$\eta_n$ and $\bar{\eta}_n$ are anticommuting sources. The action $Tr \bar{\psi} \psi$ and the integration measure are invariant under a nilpotent BRST transformation:

$$\delta M = \psi \quad \delta \psi = 0 \quad \delta \bar{\psi} = 0$$

(14)

The Ward identity that $Z[j, \eta, \bar{\eta}]$ satisfies is:

$$<\sum_n j_n n Tr (M^{n-1}\psi) - \sum_n \eta_n \sum_m Tr (M^m \psi M^{n-1-m}\psi)> = 0$$

(15)

Replacing

$$\psi \rightarrow -\frac{\partial \exp[-Tr \bar{\psi} \psi]}{\partial \bar{\psi}}$$

(16)
and integrating by parts over $\bar{\psi}$ gives:

$$QZ[j, \eta, \bar{\eta}] = 0$$  \hspace{1cm} (17)

with the BRST charge $Q$ given by:

$$Q = \sum_{n=-1}^{\infty} \eta_n L_n - \sum_{n,m=-1}^{\infty} \frac{n-m}{2} \eta_n \eta_m \frac{\delta}{\delta \eta_{n+m}}$$  \hspace{1cm} (18)

The Virasoro generators $L_n$ are given by:

$$L_m = \sum_{n=0}^{m} \frac{\partial}{\partial j_{n+m}} \frac{\partial}{\partial j_n} + \sum_{n=0}^{\infty} n j_n \frac{\partial}{\partial j_{n+m}}$$  \hspace{1cm} (19)

Notice that due to the identity $Tr1 = N$ we must replace

$$\frac{\partial}{\partial j_{0}} \rightarrow -N$$

These are the same $L_n$ found in \cite{14}. The nilpotency of $Q$ follows because the $L_n$ satisfy a Virasoro algebra:

$$[L_n, L_m] = (n-m)L_{n+m} \quad n, m = -1, 0, 1, \ldots$$  \hspace{1cm} (20)

The same procedure can be used to get a nilpotent BRST charge for the Virasoro constraints of the large $N$ vector models discussed in \cite{15}.

4 THE INTEGRAL EQUATIONS

In this section we review the derivation of the system of integral equations that describe the large-$N$ limit of the Two-Hermitian-matrix model in zero dimension \cite{9}.

The action of the model is:

$$S = Tr[V_1(M_1) + V_2(M_2) - cM_1M_2],$$  \hspace{1cm} (21)

where $M_a$ are $N \times N$ Hermitian matrices and $V_i(M_i)$ are arbitrary potentials.
We know that $S$ is invariant under $M_a \to UM_a U^\dagger U eU(N)$. Following section 2 [6] we form the BRST-invariant extension of $S$:

$$\bar{S} = S + \text{Tr}[\bar{\psi}_1 \psi_1 + \bar{\psi}_2 \psi_2],$$  
(22)

where $\psi_a$ and $\bar{\psi}_a$ are Grassman valued $N \times N$ matrices; $\bar{S}$ is invariant under the following BRST transformation:

$$\delta M_a = \epsilon \psi_a + \bar{\epsilon} \bar{\psi}_a$$  
(23)

$$\delta \psi_a = \bar{\epsilon} \delta S/\delta M_a$$  
(24)

$$\delta \bar{\psi}_a = -\epsilon \delta S/\delta M_a$$  
(25)

$$\epsilon^2 = \bar{\epsilon}^2 = \epsilon \bar{\epsilon} + \bar{\epsilon} \epsilon = 0.$$  
(26)

As explained in section 2, the Ward identities corresponding to this BRST symmetry are the Schwinger-Dyson equations of the model. Moreover we can prove the following important proposition, [7, 9]:

**The $U(N)$ invariants of the quantum field are BRST invariants in leading order of large-$N$.**

The proof of this proposition relies heavily on the existence of the exact symmetry of both the measure and the action provided by (26).

In what follows, we are going to show how to use this proposition to solve the Two-matrix model in leading order of large $N$.

Let us write:

$$(M_a)_{ij} = \sum_\alpha m_\alpha^a (T^\alpha_a)_{ij},$$  
(27)

where $m_\alpha^a$ are the eigenvalues of the matrix $M_a$ and $T^\alpha_a$ the corresponding projectors satisfying:

$$T^\alpha_a T^\beta_a = \delta_{\alpha\beta} T^\alpha_a \quad \sum_\alpha T^\alpha_a = 1 \quad Tr T^\alpha_a = 1$$  
(28)

then, since $m_\alpha^a$ are BRST invariants ,according to the proposition stated above, we get:

$$\delta (M_a)_{ij}^k = \sum_\alpha (m_\alpha^a)^k \delta (T^\alpha_a)_{ij} \text{ with } k=1,2...$$  
(29)

It follows that the BRST variation of the projectors is:

$$\delta T^\alpha_a = \sum_{\gamma \neq \alpha} T^\alpha_a \delta M_a T^\gamma_a + T^\alpha_a \delta M_a T^\alpha_a \frac{m_\gamma^a - m_\alpha^a}{m_\alpha^a - m_\gamma^a}.$$  
(30)
From the proposition, it follows that, in leading order of $1/N$,
\[
\delta T_r[T^\alpha_1 T^\beta_1 \cdots T^\alpha_n T^\beta_n \psi_a] = 0 \tag{31}
\]

After computing the fermionic average, we obtain:
\[
(\Delta(x_{\alpha_1}) - cy_{\beta_n} - \sum_{\gamma \neq \alpha} \frac{1}{x_{\alpha_1} - x_\gamma})T_{\alpha_1 \beta_1 \cdots \alpha_n \beta_n} = 
\sum_{k=2}^{n} \sum_{\gamma \neq \alpha_1} T_{\alpha_1 \beta_1 \cdots \alpha_{k-1} \beta_{k-1} \alpha_k \beta_k+1 \cdots \alpha_n \beta_n} \frac{1 - \delta_{\gamma \alpha_k}}{x_{\alpha_1} - x_\gamma} + \sum_{\gamma \neq \alpha_1} T_{\gamma \beta_1 \alpha_2 \beta_2 \alpha_n \beta_n} \tag{32}
\]

We have introduced the definitions:
\[
\Delta_1(x) = \frac{\partial V_1(x)}{\partial x} \tag{33}
\]
\[
\Delta_2(y) = \frac{\partial V_2(y)}{\partial y}
\]
\[
T_{\alpha_1 \beta_1 \cdots \alpha_n \beta_n} = TrT^\alpha_1 T^\beta_2 \cdots T^\alpha_n T^\beta_n \tag{34}
\]

These relations provide a whole set of identities that determine completely the $U(N)$ invariants correlations of the model. Notice that the exact BRST symmetry of the model was very helpful in the derivation of the identities. Although a complete set of identities follows from the previous equation, in this paper we will discuss in detail the case $n = 1$.

Taking the continuum version, $N \to \infty$, of the case $n = 1$, we get:
\[
\Delta_1(x) - 2P \int dx' u(x')/(x - x') = c \int dyv(y)yF(x, y) \tag{35}
\]
\[
\Delta_2(y) - 2P \int dy' v(y')/(y - y') = c \int dxu(x)xF(x, y) \tag{36}
\]
\[
P \int dx' u(x')F(x', y)/(x - x') = [\Delta_1(x) - cy - P \int dx' u(x')/(x - x')]F(x, y) \tag{37}
\]
\[
P \int dy' v(y')F(x, y')/(y - y') = [\Delta_2(y) - cx - P \int dy' v(y')/(y - y')]F(x, y) \tag{38}
\]
\[
\int dy' v(y')\Delta_2(y')F(x, y') = cx \tag{39}
\]
\[ \int dx' u(x') \Delta_1(x') F(x', y) = cy \] 
\[ \int dx u(x) = \int dx u(x) F(x, y) = 1 \] 
\[ \int dy v(y) = \int dy v(y) F(x, y) = 1. \]

plus the normalization conditions:

and \( P \) stands for the principal value of the integral. \( u \) (\( v \)) is the density of eigenvalues of \( M_1(M_2) \) and \( F(x, y) \) is the continuum limit of \( Tr T_1^a T_2^b \). Actually (37) and (38) plus the normalization conditions (42), and (43) imply the other equations, as can be easily seen by multiplying by \( v(y) \) and integrating over \( y \) in (37) to get (38).

5 SOLUTION OF THE SYSTEM OF INTEGRAL EQUATIONS

In this section we will show how to solve the integral system of last section. Our strategy will follow closely the standard procedure[16]. We will explain how the system of integral equations reduces to an algebraic equation satisfied by an associated analytic function. The different phases of the model are determined by the different number of cuts the analytic function may have.

Let us introduce the following functions:

\[ U(x, y) = \int_{d_2}^{d_1} dx' u(x') F(x', y)/(x - x') \] 
\[ H(x) = \int_{d_2}^{d_1} dx' u(x')/(x - x'). \]

These functions are analytic everywhere in the complex \( x \) plane cut along the real axis between \( d_1 \) and \( d_2 \). Across the cut we have:

\[ (U_+ - U_-)/2 = -i \pi u(x) F(x, y) \]
\[
(U_+ + U_-)/2 = P \int dx' u(x') F(x', y)/(x - x') \quad (47)
\]
\[
(H_+ - H_-)/2 = -i\pi u(x) \quad (48)
\]
\[
(H_+ + H_-)/2 = P \int dx' u(x')/(x - x') \quad (49)
\]

By the index +(-) we indicate the value of the function in the upper(lower) part of the cut. Moreover, for large \(x\) we have:

\[
U(x, y) \sim 1/x \quad (50)
\]
\[
H(x) \sim 1/x. \quad (51)
\]

According to eq. (37) we have:

\[
U_+ = \Delta_1(x) - cy - \frac{P \int dx' u(x')/(x - x') + i\pi u(x)}{\Delta_1(x) - cy - \frac{P \int dx' u(x')/(x - x') - i\pi u(x)}}. \quad (52)
\]

But \(U(x, y)\) is analytic in the same region of the \(x\) complex plane than \(H(x)\) is; therefore we must have:

\[
U(x, y) = \frac{\lambda(x, y)}{\Delta_1(x) - cy - H(x)} \quad (53)
\]

with \(\lambda(x, y)\) an integral function of \(x\) which is a polynomial in \(x\) if \(\Delta_1\) is a polynomial also; \(\lambda\) shall be chosen to get the right asymptotic behavior for large \(x\). That is:

\[
U(x, y) \sim 1/x, \text{ for large } x \quad (54)
\]

Plugging (53) in (49), we obtain

\[
F(x, y) = \frac{\lambda(x, y)}{[(\Delta_1 - cy - H_+)(\Delta_1 - cy - H_-)]}. \quad (55)
\]

In a similar way we can introduce analytic functions in the variable \(y\):

\[
V(x, y) = \int_{e_2}^{e_1} dy' v(y') F(x, y')/(y - y') \quad (56)
\]
\[
I(y) = \int_{e_2}^{e_1} dy' v(y')/(y - y'), \quad (57)
\]
These functions are analytic everywhere in the complex $y$ plane cut along the real axis between $e_1$ and $e_2$. Across the cut we have:

\[
\begin{align*}
(V_+ - V_-)/2 &= -i\pi v(y)F(x,y) \\
(V_+ + V_-)/2 &= P \int dy' v(y')F(x,y')/(y - y') \\
(I_+ - I_-)/2 &= -i\pi v(y) \\
(I_+ + I_-)/2 &= P \int dy' v(y')/(y - y')
\end{align*}
\]  

(58) \hspace{1cm} (59) \hspace{1cm} (60) \hspace{1cm} (61)

By the index $+(-)$ we indicate the value of the function in the upper(lower) part of the cut. Moreover, for large $y$ we have:

\[
\begin{align*}
V(x,y) &\sim 1/y \\
I(y) &\sim 1/y.
\end{align*}
\]  

(62) \hspace{1cm} (63)

In order to satisfy (58) we must have:

\[
V_-/V_+ = \frac{\Delta_2(y) - cx - P \int dy' v(y')/(y - y') + i\pi v(y)}{\Delta_2(y) - cx - P \int dy' v(y')/(y - y') - i\pi v(y)}.
\]  

(64)

Therefore we also have:

\[
V(x,y) = \frac{\mu(x,y)}{[\Delta_2(y) - cx - I(y)]},
\]  

(65)

with $\mu(x,y)$ being an integral function in $y$ chosen to match the asymptotic behavior of $V$ for large $y$. We get

\[
F(x,y) = \frac{\mu(x,y)}{[\Delta_2(y) - cx - I_+(\Delta_2 - cx - I_-)]}.
\]  

(66)

Of course, (58) and (66) must coincide for $x, y$ in the support of $u(x)$ and $v(y)$. This is a strong requirement, which permits us to find $H$ and $I$, as will be clear after we work out some examples. The main conclusion of this section is that the complicated integral system of the last section has been reduced to a simpler equality between two functional forms for $F(x,y)$. As far as I know this procedure to decouple an integral system was not known before.
In this section we want to discuss some simple models to illustrate the method of section 5. We will see that for given $\Delta_1$ and $\Delta_2$, the consistency requirement for $F(x, y)$ will imply a set of equations for the values on the cut of the analytic functions we have introduced in the last chapter depending on some arbitrary constants. For polynomial $\Delta$’s, this set of equations can be reduced to algebraic equations by using the Cauchy theorem of residues. The arbitrary constants are the order parameters of the different phases of the model. They are determined by the number of disconnected cuts we allow in the analytic functions.

The simplest situation we want to discuss is the ”free” case. Namely:

$$
\Delta_1 = x \quad (67)
$$

$$
\Delta_2 = y. \quad (68)
$$

Actually, we can compute $H$ and $I$ directly by replacing (15) and (16) into (63). By the usual method we find:

$$
u(x) = (1 - c^2)\sqrt{d^2 - x^2}/2\pi \quad (69)
$$

$$
d = 2/\sqrt{1-c^2} \quad (70)
$$

$$
u(y) = (1 - c^2)\sqrt{d^2 - y^2}/2\pi. \quad (71)
$$

Therefore:

$$
U(x, y) = \frac{1}{[x(1 + c^2)/2 - cy + (1 - c^2)\sqrt{x^2 - d^2}/2\pi]} \quad (72)
$$

$$
V(x, y) = \frac{1}{[y(1 + c^2)/2 - cx + (1 - c^2)\sqrt{y^2 - d^2}/2\pi]}. \quad (73)
$$

The square root takes its principal value with a cut along the real axis between $-d$ and $d$. It follows that:

$$
F(x, y) = \frac{1}{[c^2(x^2 + y^2) + 1 - c^2 - c(1 + c^2)xy]}. \quad (74)
$$
It is easy to check that (74) gives the right perturbative expansion in $c$.

Let us examine a more general case:

$$\Delta_1(x) = x + g_1 x^2$$
$$\Delta_2(y) = y + g_2 y^2$$

(75)

(76)

According to (53) and (65) we must have:

$$U(x,y) = \frac{g_1 x + \lambda_1(y)}{x + g_1 x^2 - cy - H(x)}$$
$$V(x,y) = \frac{g_2 y + \mu_1(x)}{y + g_2 y^2 - cx - I(y)}$$

(77)

(78)

The comparison of the $F(x,y)$ coming from $U(x,y)$ and from $V(x,y)$ gives:

$$\mu_1 c^2 - cg_2(2\Delta_1 - H_+ - H_-) = a_2 x^2 + a_1 x + a_0$$
$$g_2(\Delta_1 - H_+)(\Delta_1 - H_-) - c\mu_1(2\Delta_1 - H_+ - H_-) = b_2 x^2 + b_1 x + b_0$$
$$\mu_1(\Delta_1 - H_+)(\Delta_1 - H_-) = g_1 c^2 x^3 + h_2 x^2 + h_1 x$$

(79)

(80)

(81)

and

$$\lambda_1 c^2 - cg_1(2\Delta_2 - I_+ - I_-) = a_2 y^2 + b_2 y + h_2$$
$$g_2(\Delta_2 - I_+)(\Delta_2 - I_-) - c\lambda_1(2\Delta_2 - I_+ - I_-) = a_1 y^2 + b_1 y + h_1$$
$$\lambda_1(\Delta_2 - I_+)(\Delta_2 - I_-) = g_2 c^2 y^3 + a_0 y^2 + b_0 y$$

(82)

(83)

(84)

for certain constants $a_i, b_i, h_i$.

From equations (11) and (12), we get:

$$\mu_1 = g_2[\Delta_1 - (H_+ - H_-)]/c + 1,$$
$$\lambda_1 = g_1[\Delta_2 - (I_+ - I_-)]/c + 1,$$

(85)

(86)
from which we can fix several of the arbitrary constants:

\[ a_2 = -g_1 g_2 c \]  \hspace{1cm} (87)
\[ a_1 = -g_2 c \]  \hspace{1cm} (88)
\[ a_0 = c^2 \]  \hspace{1cm} (89)
\[ b_2 = -g_1 c \]  \hspace{1cm} (90)
\[ h_2 = c^2 \]  \hspace{1cm} (91)

The remaining system of equations is:

\[
H^2_+ + H_+ H_- + H^2_- - (H_+ + H_-)(2\Delta_1 + c/g_2) + \\
\Delta_1^2 + 2\Delta_1 c/g_2 + B = 0 \tag{92}
\]

\[
(g_2(\Delta_1 - (H_+ + H_-))/c + 1)[\Delta_1 - H_+][\Delta_1 - H_-] = g_1 c^2 x^3 + c^2 x^2 + h_1 x \tag{93}
\]

where we have defined

\[
\bar{B} = (-g_1 c^2 x^2 + b_1 x + b_0)/g_2 \tag{94}
\]

To solve this system, multiply eq. (92) by \((H_+ - H_-)\). A remarkable property of the resulting expression is that it depends solely on the discontinuities of powers of \(H(x)\) along the cut. Using this property, we can reduce the system (92) and (93) to a cubic equation for \(H(x)\). In fact, consider the integral:

\[
\frac{1}{2\pi i} \int dx \frac{H(x)^3}{x - y}
\]

along a curve that encloses the cut of \(H(x)\). In the region of the complex \(x\) plane that does not contain the cut, the integral equals \(H(y)^3\). On the cut, we can use (92) to express \(H_+(x)^3 - H_-(x)^3\) as a function of \(H_+(x)^2 - H_-(x)^2\) and \(H_+(x) - H_-(x)\). Then the remaining integrals can be computed in the region that do not contain the cut to get the following equation for \(H\):

\[
H(y)^3 - (2\Delta_1 + c/g_2)H(y)^2 + F(y)H(y) - F'(y) - F''(y)(< x > -y)/2 - \\
F'''(y)(< x^2 > -2 < x > y + y^2)/6 - F^{(4)}(y)(< x^3 > -3 < x^2 > y + \\
3 < x > y^2 - y^3)/24 + 2g_1 = 0 \tag{95}
\]
for certain additional constants $< x^i >$. We have introduced the notation:

$$F(y) = \Delta_1^2 + 2c\Delta_1/g_2 + \bar{B}$$  \hfill (96)

The condition (51) has been explicitly used to derive (95), which coincides with equation (4) of ref. [11].

Let us review what we have obtained up to here. The first step in the solution of the set of integral equations of section 4 was to express $U(x,y)$ ($V(x,y)$) in terms of $H(x)(I(y)$ plus certain arbitrary functions of $x(y)$. The next step was to impose the equality

$$\frac{U_+(x,y) - U_-(x,y)}{H_+(x) - H_-(x)} = \frac{V_+(x,y) - V_-(x,y)}{I_+(x) - I_-(x)} = F(x,y)$$  \hfill (97)

In this way, we found conditions that relate these arbitrary functions. Remarkable enough, these conditions can be solved completely, reducing the arbitrariness to a set of constants. This is the main result of this paper. It is clear that the same procedure will work for an arbitrary potential.

In what follows, we will sketch that solution of the cubic equation which is relevant for the phase of the Two-matrix model containing the perturbative solution. That is, we will explore the conditions under which $H(x)$ has only one cut on the real axis. This requirement will fix all the arbitrary constants. In principle, we could find all the phases of the model allowing more cuts, but we will not do this here.

### 6.1 Analysis of the cubic equation

The cubic equation $H^3 + a_2H^2 + a_1H + a_0 = 0$ has the following solutions:

- $H_n = w_n s_1 + w_n^2 s_2 - a_2/3$  \hfill (98)
- $s_1 = [r + (r^2 + q^2)^{1/2}]^{1/3}$  \hfill (99)
- $s_1s_2 = -q$  \hfill (100)
- $q = a_1/3 - (a_2/3)^2$  \hfill (101)
- $r = (a_1a_2 - 3a_0)/6 - (a_2/3)^3$  \hfill (102)

where $w_n$ is the n-th root of unity.
The equations (92) and (93) are satisfied if:

\[ q = \frac{B}{3} + \frac{(\Delta_1 - c/g_2)^2}{9} \]  

\[ r = c(g_1 c^2 x^3 + c^2 x^2 + h_1 x + h_0)/(2g_2) + q(\Delta_1 - c/g_2)/2 + \frac{(\Delta_1 - c/g_2)^3}{54} \]  

\[ b_1 = g_1 g_2 - c(c^2 + 1) \]

The arbitrary constants will be determined by demanding that the analytic continuation of the solution of the cubic that behaves as \(1/x\) for large \(x\) must have only two real branch points.

In our case this requirement is satisfied if \(r^2 + q^3\), which is a polynomial in \(x\) of degree 9, can be written as follows:

\[ r^2 + q^3 = -A(x^3 + p_2 x^2 + p_1 x + p_0) \]  

for some constants \(p_i, s_i\) and \(A\).

Actually we can have different solutions of the problem by permitting a larger number of branch points [17], but we will not consider them here. In the next subsection we study a simpler limiting case of the last equation.

### 6.2 One of the matrices is free: \(g_1 = 0\)

In this section we consider the limit \(g_1 = 0\). The equation for \(I(y)\) reduces to a quadratic equation, which can be solved easily. The one cut solution implies the following values for the arbitrary constants:

\[ h_1 = 0 \]  

\[ bb_0 = 1 \]

and \(h_0\) satisfies the following equation:

\[ 64g_2 h_0^3 + h_0^2(96c^2 g_2^2 - 96g_2^2 - (c^2 - 1)^4) + h_0(30c^4 g_2^2 - 60c^2 g_2^2 + 30g_2^2 - (c^2 - 1)^5) + 27g_2^4 - c^6 g_2^2 + 3c^4 g_2^2 + 3c^2 g_2^2 + g_2^2 = 0. \]

Now we want to check the consistency of the whole procedure by finding \(h_1, bb_0, h_0\) using (106). For \(g_1 = 0\), \(r^2 + q^3\) is a polynomial of degree 5; in order to get just two branch points, we must thus have:
\[ r^2 + q^3 = -A(x + p_0)^2(x^3 + s_2x^2 + s_1x + s_0). \]

We find the following equations to determine \( p_0 \) and \( h_0 \):

\[
h_0 = -(c^3g_2 + c^5g_2 - 5cg_2^3 - 6c^3g_2^3 + c^4p_0 - 2c^4p_0 + c^8p_0 - 8c^2g_2^2p_0 - 2c^4g_2^2p_0 - 12c^6g_2^2p_0 + g_2^4p_0 - 3c^3g_2p_0^2 - 3c^5g_2p_0^2 + 12c^7g_2p_0^2 - 6c^9g_2p_0^2 + 3c^3g_2p_0^2 - 2c^2g_2^2p_0^2 + 16c^4g_2^2p_0^2 - 16c^6g_2^2p_0^2 - 10c^3g_2^3p_0^4) /
(3c^3g_2 + 9c^5g_2 + 9c^3g_2^3 + 6c^2g_2^2p_0 - 36c^4g_2^2p_0 - 6cg_2^3p_0^2),
\] (110)

\[
0 = c^2g_2^2 + 4c^4g_2 + 3c^6g_2 + g_2^3 + (c^3 + c^5 - 5c^7 + 3c^9 - 12c^3g_2^2) p_0 + (-c^2g_2 - 10c^4g_2 + 11c^6g_2)p_0^2 + 8c^3g_2^2p_0^3.
\] (111)

The last two equations for \( p_0 \) have a common root if \( h_0 \) satisfies (109).

As usual the asymptotic expansion of the functions \( H(x) \) and \( I(y) \) around \( \infty \) gives Green’s functions of the eigenvalues of the corresponding matrices. For instance, we get:

\[
<x> = \frac{c(-1 + c^2 + h_0)}{g_2},
\]

\[
<x^2> = \frac{c^2 - 2c^4 + c^6 + g_2^2 - c^2h_0 + c^4h_0}{g_2^2}.
\] (112)

When \( g_2 \neq 0 \), the system of equations to determine the constants in (106) turns out to be too complicated to be solved analytically, so we will not discuss it here.

Instead we will mention how these results generalize to a quartic potential.

### 6.3 Quartic Potential

For the sake of completeness, we will sketch the solution of the two-matrix model with a quartic potential, which is defined by:

\[
\Delta_1 = ax + g_1x^3,
\]

\[
\Delta_2 = by + g_2y^3,
\] (113) (114)
So, we must have:

\[
U(x, y) = \frac{g_1x^2 + \lambda_1x + \lambda_0}{g_1x^3 + ax - cy - H(x)} \quad (115)
\]

\[
V(x, y) = \frac{g_2y^2 + \mu_1y + \mu_0}{g_2y^3 + by - cx - I(y)} \quad (116)
\]

Again, imposing the equality of \( F(x, y) \) coming from the two previous functions, we get a system of algebraic equations to determine the arbitrary functions \( \lambda_0, \lambda_1, \mu_0, \mu_1 \). One of these equations is:

\[
g_2(H_+^4 - H_-^4) - (3g_2\Delta_1 + c)(H_+^3 - H_-^3) + (3g_2\Delta_1^2 + B + 3c\Delta_1)(H_+^2 - H_-^2) - \Delta_1(g_2\Delta_1^2 + 2B + 3c\Delta_1)(H_+ - H_-) = Dc(H_+ - H_-) \quad (117)
\]

with

\[ B = b_2x^2 + b_1x + b_0 \]

and

\[ D = -g_1cbx^3 + d_2x^2 + d_1x + d_0 \]

This system can be solved using the Cauchy theorem of residues as was used to derive (95). We get a quartic equation for \( H(x) \) and a similar one for \( I(y) \).

Again, the complicated set of integral equations has been reduced to an algebraic equation, depending on a set of arbitrary constants. To completely determine the arbitrary constants we may demand that \( H(x) \) has only one cut in the \( x \) complex plane. Once this choice has been made all \( U(N) \)-invariant correlations of the two matrices which involve \( u(x), v(y) \) and \( F(x, y) \) can be computed.

7 CONCLUSIONS AND OPEN PROBLEMS

We have been able to solve the system of integral equations that describe the large-\( N \) limit of the two-matrix model (a particular subset of the Schwinger-Dyson equations) by reducing it to an algebraic equation satisfied by an associated analytic function which depends on some arbitrary constants. In particular, we have verified that the solution found in [11] is obtained. In certain simple cases, we have been able to find these constants explicitly by
choosing the analytic function with only one cut (the perturbative phase of the model). In more general situations, finding the constants for the one cut solution analytically is very complicated.

Our solution is appropriate to explore all the different phases of the model which are characterized by the set of arbitrary constants. The constants are no longer arbitrary if we fix the number of branch points of the associated analytic function. We see this result as a strong test of the hidden BRST method.

It is an interesting open problem to find the other solutions of (32), which will enable us to compute more general correlations functions of the form \( Tr M_1^{m_1} M_2^{m_2} M_1^{m_1} M_2^{m_2} \ldots \).

**ACKNOWLEDGEMENTS**

The author wants to thank Luis Alvarez-Gaumé for useful discussions and Spenta Wadia for having called his attention to ref. [11]. He also wants to express his gratitude to P.H. Damgaard and G. Shore for a critical reading of the manuscript. His work has been supported by a EC CERN Fellowship. He also acknowledges help from the Fundación Andes # C-11666/4.

**References**

[1] D. Gross and A. Migdal, Phys. Rev. Lett. 64(1990)717; M. Douglas and S. Shenker, Nucl. Phys. B335(1990)635; E. Brezin and V.A. Kazakov, Phys. Lett. B236(1990)144.

[2] R. Dijkgraaf, E. Verlinde and H. Verlinde, Nucl. Phys. B348(1991)435; M. Fukuma, H. Kawai and R. Nakayama, Int. J. Mod. Phys. A6(1991)1385.

[3] E. Witten, Nucl. Phys. B340(1990)281.

[4] M. Kontsevich, "Intersection theory on the moduli space of curves and the matrix Airy function", 30 Arbeitstagung Bonn, Max Planck Institute preprint MPI/91-47.

[5] M. Henneaux, Phys. Rep. C126(1985)1.
[6] J. Alfaro and P.H. Damgaard, Phys. Lett. 222B(1989)425, and Ann. Phys. 202(1990)398.

[7] J. Alfaro, Phys. Lett. 200B(1988)80.

[8] C. Itzykson and J.B. Zuber, J. Math. Phys. 21(1980)411; M.L. Mehta, Commun. Math. Phys. 79(1981)327.

[9] J. Alfaro and J.C. Retamal, Phys. Lett. B222(1989)429. J. Alfaro, ”Hidden BRST and large N”, Lecture at the Cargèse Workshop on Probabilistic Methods in Quantum Field Theory and Quantum Gravity, P.H. Damgaard, H. Huffel and A. Rosennblum eds., (Plenum Press, New York, 1990), p. 279.

[10] M.L. Mehta, Commun. Math. Phys. 79(1981)327; D.V. Boulatov and V.A. Kazakov, Phys. Lett. B186(1987)379; D.Gross and A. Migdal, Phys. Rev. Lett. 64(1990)717; E.Brézin, M. Douglas, V. Kazakov and S. Shenker, Phys. Lett. B237(1990)43.; C. Crnkovic, P. Ginsparg and G. Moore, Phys. Lett. B237(1990)196.

[11] E. Gava and K.S. Narain, Phys. Lett. B263(1991)213.

[12] J. Alfaro and A. Jevicki, ”BRST formulation of Virasoro conditions in the Non-perturbative string”, Proceedings of the ”VII Chilean Physics Symposium”, Santiago, December 18-21 1990.

[13] J. Alfaro, ”Collective Coordinates and Hidden BRST symmetry” in ”Quarks, Symmetries and Strings: A symposium in honour of B. Sakita’s 60 birthday”. Eds. M. Kaku et al., World Scientific 1991.

[14] L. Alvarez-Gaume, C. Gomez and J. Lacki, Phys. Lett. B253(1991)56; Gerasimov et al., matrix models of 2D gravity and Toda theory, Lebedev Institute preprint/ITEP(1990).

[15] Yoneya, T. and Nishigaki, S., Nucl. Phys. B348(1991)787.; P. Di Vecchia, M. Kato and N. Ohta, Nucl. Phys. B357(1991)495; P.H. Damgaard and K. Shigamotono, Phys. Lett. B262(1991)432.

[16] E. Brezin, C. Itzykson, G. Parisi and J.B. Zuber, Comm. Math. Phys. 59(1978)35
[17] K. Demeterfi, N. Deo, S. Jain and C.I. Tan, Phys. Rev. D42(1990)4105; F. David, Nucl. Phys. B348(1991)507.