Zero vorticity condition in calculation of ground state energy of Josephson junction lattices

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We employ the charge neutrality condition in the form of zero vorticity for a Josephson junction lattice in calculating its ground state. We consider a Fibonacci ladder and the Penrose lattice to test our method. We also compare the results with those of the model that treats the plaquettes independently. We show that the zero vorticity condition improves on the results of the independent plaquette model.

PACS numbers: 74.81.Fa, 61.44.-n

I. INTRODUCTION

A Josephson junction is characterized by its current-phase relation $I(\phi)$, where $\phi$ is the phase difference across the junction. $I(\phi)$ must have two general properties: it must be $2\pi$-periodic and it must be an odd function. Therefore $I(\phi)$ can be any periodic, odd function. In here we focus on sinusoidal Josephson junctions. In this case, the Hamiltonian of system is a cosine function of the gauge invariant phase difference. A Josephson junction lattice is a lattice with nodes that are superconducting islands and bonds that represent Josephson junctions. This lattice of Josephson junctions is a model that can be used for the description of other important systems like stacked Josephson junctions, high-temperature superconductors, and a rotating BEC.

If we ignore screening effects, we can write the Hamiltonian of the system as a sum of Hamiltonians of Josephson junctions. In here we focus on minimization of this Hamiltonian as a function of superconducting phase in presence of external magnetic field. By this minimization we obtain the ground state energy as a function of the external magnetic field, or frustration factor, which is defined as the ratio of flux through a plaquette with unit area to the quantum of flux. Behavior of ground state energy $E$ as a function of frustration factor $f$ is interesting. When $f$ is zero all junctions can attain their absolute minimum, but when $f$ becomes nonzero, the bonds of the lattice cannot generally obtain this absolute minimum simultaneously and produce some superconducting current — this is the origin of the term frustration.

We approximate the behavior of the lattice in ground state by the Independent Plaquette Model (IPM), which considers the plaquettes independently. Also we suppose that total vorticity of lattice is zero. This condition can be understood once we look at the relation of this model with Charged Coulomb Gas (CCG) model. The Hamiltonian of the Josephson junction lattice can be mapped to CCG on dual lattice, where the charge of each point in dual lattice is proportional to vorticity of corresponding plaquette in original lattice. This shows that the zero vorticity condition in CCG model means that the system is neutral.

This condition was applied before for the description of numerical results on the square lattice. In here we apply this condition for lattices with two types of plaquettes, corresponding to CCG models with four types of charge. Our result is in agreement with those without this explicit condition and is an improvement on them.

Behavior of $E(f)$ is in close relation with behavior of critical temperature $T_C(f)$ of lattice. In fact, when the ground state energy is increased, we expect that the critical temperature is lowered and vice versa. Our result is in agreement with the mean field approach for $T_C(f)$.

This model can be mapped onto the frustrated XY model, which is the XY model with unitary coupling which is dependent on a parameter equivalent to the frustration factor. This model is in close relation to the Frenkel-Kontorova model which describes the behavior of the ground state for a system of particles under the effect of the lattice potential.

The structure of the paper is as follows: in section II, we introduce our model with the zero vorticity condition; in section III we compare the results of the model with the numerical study, comparing energies and corresponding Fourier spectra; the final section is devoted to Conclusions.

II. THE MODEL

For a Josephson junction lattice, the Hamiltonian is,

$$H = - \sum_{<i,j>} \cos(\gamma_{ij})$$

where $<i,j>$ means that $i$th node and $j$th node are connected through a bond of lattice. $\gamma_{ij}$ is the gauge invariant phase difference. It is equal to

$$\gamma_{i,j} = \theta_i - \theta_j - A_{i,j}$$

where $\theta$ is the superconducting phase and $A_{i,j}$ is the integral of the vector potential from $i$th node to $j$th node. We define the frustration factor as the ratio of magnetic flux through a plaquette with unit area to the flux quantum.
The gauge invariant phase difference satisfies the fluxoid quantization:

$$V_k = \sum \gamma_{ij} = 2\pi(n_k - a_k f)$$ (2.3)

where \(n_k\) is an integer and \(a_k\) is the area of \(k\)th plaquette. Summation is over the edges of \(k\)th plaquette. \(V_k\) is vorticity of plaquette, and in transformation to CCG, is proportional to charges on \(k\)th site of the dual lattice point. \(n_k\) can have two values: \([a_k f]\) or \([a_k f] + 1\) where \([x]\) means greatest integer less than \(x\). It means that if we have \(l\) types of plaquettes, then we have \(2l\) different values for \(V_k\) and therefore \(2l\) types of charges in the equivalent CCG model. We want to minimize this Hamiltonian with respect to the superconducting phases. We use the Monte-Carlo (MC) method for the numerical minimization. We denote this minimum by \(E\) which is normalized by the number of junctions.

From symmetry of a single plaquette, we can say that all four gauge invariant phase differences are equal, and we denote it by \(\gamma\). Then from Eq. (2.3) we can find \(\gamma = \pi/2(n - a f)\), for two choices of \(n\), and giving the ground state energy of the plaquette as:

$$E_i = -\cos(\pi/2(\lfloor a f \rfloor + i - a f))$$ (2.4)

for \(i = 0, 1\). Now, we approximate the Hamiltonian as follows: suppose that \(k\)th plaquette has vorticity \([a_k f] + i_k\), then it has energy \(E_{1k}^k\) in IPM approximation, and we can write Hamiltonian as sum of these energies in IPM approximation:

$$H_{IPM} = \sum_k E_{1k}^k$$ (2.5)

where \(i_k\) determines vorticity of \(k\)th plaquette. Now we minimize above Hamiltonian with respect to \(i_k\). In original IPM, we choose \(i_k\) such that \([a_k f] + i_k\) becomes nearest integer to \(a_k f\). Because of interaction between plaquettes this is not exact, and some of the plaquettes give the furthest integer as their vorticity. We suppose that this deviation from IPM is such that total vorticity of lattice becomes as small as possible. This is the zero vorticity condition, with the total vorticity given as,

$$V_{total} = \sum_k V_k = 2\pi(n_{total} - a_{lattice, f})$$ (2.6)

where \(n_{total} = \sum_k n_k\) and \(a_{lattice, f}\) is the area of the whole lattice. The condition for zero total vorticity reads,

$$V_{total}/N = \sum_k V_k/N \approx 0$$ (2.7)

where \(N\) is total number of plaquettes.

This condition can be understood if we look at the relation between Josephson junction lattice and CCG. It can be shown that Josephson junction hamiltonian can be mapped to CCG on the dual lattice, and in this transformation \(V_k\) is proportional to charge on the equivalent point in the dual lattice. Hence, the zero vorticity condition is equivalent to the charge neutrality of CCG system.

Suppose that we have \(k\) types of plaquettes in a lattice, then each type can take on two energies, either \(E_0\) or \(E_1\), bringing in \(2k\) variables. But the total number of each type of plaquette is determined and therefore we have \(k\) constraints on these variables and there are \(k\) independent variables. Using the above equation lets us remove one of these variables and finally we have \(k - 1\) variables. If we apply the IPM approximation, the Hamiltonian becomes linear in terms of these variables and easily can be minimized. We denote this model as IPM0.

We use the power spectrum of \(E(f)\) to characterize it,

$$S(\omega) = \hat{E}\hat{E}^*$$ (2.8)

where \(\hat{E}^*\) is the complex conjugate of \(\hat{E}\), the Fourier transform of \(E(f)\),

$$\hat{E}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E(f)exp(2\pi i \omega f)df$$ (2.9)

The power spectrum was calculated in MATLAB by the FFT method.

Now we focus on lattices with two types of plaquettes. We choose two lattices: the Fibonacci ladder and the Penrose lattice. These two cases are in close relation with each other. Ratio of areas of plaquettes in both are the same and both make use of the Fibonacci order. Therefore we can compare their energies to give some insight about the difference between them. From the above discussion, we can find that these Josephson junction lattices are equivalent to the CCG model with four types of charge.

III. LATTICE WITH TWO TYPES OF PLAQUETTES

In here we consider the above formulation for lattices with two types of plaquettes which we denote them by \(L\) and \(S\). \(S\) plaquette has area 1 and \(L\) plaquette has area \(\alpha\). Density of \(S\) plaquettes is \(N_S\) and that of the \(L\) plaquettes is \(N_L\), therefore \(N_L + N_S = 1\). We suppose that the density of \(S\) (\(L\)) plaquettes with energy \(E_0\) is \(n_S(n_L)\), therefore from Eq. (2.7) we can find that

$$n_S + n_L = 1 + N_L(\lfloor|\alpha f| - \alpha f\rfloor) + N_S(|f| - f)$$ (3.1)

This condition for a lattice with one type of plaquette (\(a_L = a_S = 1, n_S = n_L = n\)) is reduced to

$$n = 1 + (\lfloor|f| - f\rfloor)$$ (3.2)

which is in agreement with the previous result. The Hamiltonian for a lattice with two types of plaquettes within the IPM is

$$H_{IPM} = (n_S E_0^S + (N_S - n_S) E_1^S + n_L E_0^L + (N_L - n_L) E_1^L)$$ (3.3)
Now we minimize Hamiltonian with respect to these two densities with condition Eq. (3.1). If we increase \( n_L \) by one (therefore decrease \( n_L \) by one), then change in Hamiltonian is \( \Delta E = E_0^L - E_1^L - E_0^S + E_1^S \), therefore if \( \Delta E > 0 \) then \( n_s = \min(N_s, q(f)) \) and if \( \Delta E < 0 \) then \( n_L = \min(N_L, q(f)) \) where \( \min(x, y) \) means the minimum between \( x \) and \( y \). We can write these results as

\[
\begin{align*}
    n_S &= N_S \theta(\Delta E) + (q(f) - N_L)(1 - \theta(\Delta E)) \quad (3.4) \\
    n_L &= N_L (1 - \theta(\Delta E)) + (q(f) - N_S) \theta(\Delta E) \quad (3.5)
\end{align*}
\]

where \( \theta(x) \) is the step function which is zero for negative \( x \) and one for positive \( x \), and \( q(f) \) represents the right hand side of Eq. (3.1).

Now we apply these results to a ladder with two types of plaquettes and the 2D Penrose lattice.

A. Fibonacci Ladder

For this lattice we have, \( \alpha = \tau = (1 + \sqrt{5})/2 \) and the plaquettes appear in the Fibonacci order, defined as follows. Denoting the structure in its \( m \)th step of construction by \( U_m \), the structure is constructed recursively by the rule \( U_{m+2} = U_{m+1} + U_m \), where summation means adjacent placement: to get the structure in step \( (m+2) \), put the structure in step \( (m+1) \), first and that of step \( m \), to its right. This generates the sequence

\[
S, L, LS, LSL, LSLLS, LSLLSLSL, ... \quad (3.6)
\]

In the limit of an infinite structure this defines a quasicrystalline lattice such that ratio of \( N_L \) to \( N_S \) is \( \tau \). The energy and Fourier spectra for MC and IPM0 are given in Figs. 1 and 2.

![FIG. 1: Numerical and IPM0 energy for the Fibonacci ladder](image)

As we see, the agreement between two results is very good. From Fourier spectrum comparison, we see that there are peaks on \( \omega = 1, \tau, 1 + \tau, \tau - 1 \), for both models. This agrees with the results of IPM without the zero vorticity condition.14

We also compare results of these two methods for other types of lattices with different ratio of areas and again find good agreement.

B. Two dimensional example: The Penrose lattice

Penrose lattice is an example of 2D quasicrystals, which has two types of rhombi with areas \( \sin(2\pi/5) \) and \( \sin(\pi/5) \). It is known that minimization of Josephson Junction Hamiltonian on this lattice, as well as on the Fibonacci ladder, does not result in a periodic \( E(f) \); similarly, \( T_C(f) \) is aperiodic24–27. Our numerical results agree with this observation.

We compare the ground state energy for this lattice with its one-dimensional counterpart, the Fibonacci ladder. This shown in Fig. 3. We see that the ground state

![FIG. 2: Numerical and IPM0 Fourier spectrum for the Fibonacci ladder](image)

energy for the Penrose lattice is greater than that of the Fibonacci ladder. This can be related to the constraints involved: in the two dimensional case there are more neighboring plaquettes than in the ladder geometry.

The ground state energy using MC and IPM0 and their corresponding Fourier spectra for the Penrose lattice are shown in Figs. 4 and 5.

![FIG. 3: Comparison between energy for Penrose lattice \( E_P \) and Fibonacci ladder \( E_F \)](image)

We can see that the results of IPM0 are in agreement generally with Monte-Carlo results. Also IPM0 predicts the main frequencies of \( E(f) \) correctly.
IV. CONCLUSION

We enter a condition in the calculation of the ground state energy of Josephson junction lattices, namely that the total vorticity of lattice be zero. This condition along with the Independent Plaquette Model (IPM) gives us the properties of the ground state energy with good approximation. It can describe the main peaks of the Fourier spectrum of $E(f)$ on areas of plaquettes and their sum and difference. In 2D the accuracy of the model is not as good as in the case of the ladder, because of greater effects of neighboring plaquettes: In 2D a plaquette has more neighbors than in the ladder, and therefore IPM is worse in 2D than in the effectively 1D ladder.

Zero vorticity condition can be understood when we look at the connection of Josephson junction Hamiltonian with CCG. In CCG model vorticity of plaquette plays the role of charge on its corresponding dual lattice site. Therefore zero vorticity condition in CCG means that system of charge is neutral, which naturally leads to a lower energy for the ground state.

V. ACKNOWLEDGMENTS

I thank the Institute for Advanced Studies in Basic Sciences for supporting this research. I also thank M. R. Kolahchi for useful comments on the manuscript.

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