Explicit formulae for Chern–Simons invariants of the hyperbolic orbifolds of the knot with Conway’s notation \( C(2n, 3) \)

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Abstract We calculate the Chern–Simons invariants of the hyperbolic orbifolds of the knot with Conway’s notation \( C(2n, 3) \) using the Schlafli formula for the generalized Chern–Simons function on the family of \( C(2n, 3) \) cone-manifold structures. We present the concrete and explicit formula of them. We apply the general instructions of Hilden, Lozano, and Montesinos-Amilibia and extend the Ham and Lee’s methods. As an application, we calculate the Chern–Simons invariants of cyclic coverings of the hyperbolic \( C(2n, 3) \) orbifolds.

Keywords Chern–Simons invariant · \( C(2n, 3) \) · Orbifold · Riley–Mednykh polynomial · Orbifold covering

Mathematics Subject Classification 57N10 · 57R19 · 57M99 · 57M25 · 57M27 · 57M50

1 Introduction

Chern–Simons invariant [1,21] was defined to be a geometric invariant and became a topological invariant after the Mostow Rigidity Theorem [23]. Various methods of finding Chern–Simons invariant using ideal triangulations have been introduced [2–
4, 24, 25, 32] and implemented [6, 9]. But, for orbifolds, to our knowledge, there does not exist a single convenient program which computes Chern–Simons invariant.

Instead of working on complicated combinatorics of 3-dimensional ideal tetrahedra to find the Chern–Simons invariants of the hyperbolic orbifolds of the knot with Conway’s notation $C(2n, 3)$, we deal with simple one-dimensional singular loci. Similar methods for volumes can be found in [11, 13]. We use the Schläfli formula for the generalized Chern–Simons function on the family of $C(2n, 3)$ cone-manifold structures [15]. In [16] a method of calculating the Chern–Simons invariants of two-bridge knot orbifolds were introduced but without explicit formulae. In [10], the Chern–Simons invariants of the twist knot orbifolds are computed. Similar approaches for $SU(2)$-connections can be found in [19] and for $SL(2, C)$-connections in [18]. For explanations of cone-manifolds, you can refer to [5, 13, 14, 20, 26, 27, 30].

The main purpose of the paper was to find the explicit and efficient formulae for Chern–Simons invariants of the hyperbolic orbifolds of the knot with Conway’s notation $C(2n, 3)$. Using the instruction in [15, Theorem 3.9] one can compute Chern–Simons invariants of the hyperbolic orbifolds of any knot or using the instruction in [16, Theorem 1.4], one can compute Chern–Simons invariants of the hyperbolic orbifolds of any two bridge knot one by one, theoretically. But the actual computations require some caution. For $C(2n, 3)$, to get a nice expression for the longitude, we used different coordinates for the representation of the knot group from those used in [13]. After this paper, we could present Chern–Simons invariants of the hyperbolic orbifolds of the knot with Conway’s notation $C(2n, 4)$ [12]. We had to elevate the precision to higher degree than the normal in Mathematica to find the right variety as $n$ gets larger. We expect we can at least handle a few more families.

For a two-bridge hyperbolic link, there exists an angle $\alpha_0 \in [\frac{2\pi}{3}, \pi)$ for each link $K$ such that the cone-manifold $K(\alpha)$ is hyperbolic for $\alpha \in (0, \alpha_0)$, Euclidean for $\alpha = \alpha_0$, and spherical for $\alpha \in (\alpha_0, \pi)$ [14, 20, 26, 27]. We will use the Chern–Simons invariant of the lens space $L(6n + 1, 4n + 1)$ calculated in [16]. Let us denote $C(2n, 3)$ by $T_{2n}$. The following theorem gives the formulae for $T_{2n}$. Note that if $2n$ of $T_{2n}$ is replaced by an odd integer, then $T_{2n}$ becomes a link with two components. Also, note that the Chern–Simons invariant of hyperbolic cone-manifolds of the knot with Conway’s notation $C(-2n, -3)$ is the same as that of the knot with Conway’s notation $C(2n, 3)$ up to sign. For the Chern–Simons invariant formula, since the knot $T_{2n}$ has to be hyperbolic, we exclude the case when $n = 0$.

**Theorem 1.1** Let $X_{2n}(\alpha), 0 \leq \alpha < \alpha_0$ be the hyperbolic cone-manifold with underlying space $S^3$ and with singular set $T_{2n}$ of cone-angle $\alpha$. Let $k$ be a positive integer such that $k$-fold cyclic covering of $X_{2n}(\frac{2\pi}{k})$ is hyperbolic. Then the Chern–Simons invariant of $X_{2n}(\frac{2\pi}{k})$ (mod $\frac{1}{k}$ if $k$ is even or mod $\frac{1}{2k}$ if $k$ is odd) is given by the following formula:

$$
\text{cs} \left( X_{2n} \left( \frac{2\pi}{k} \right) \right) \equiv \frac{1}{2} \text{cs} \left( L(6n + 1, 4n + 1) \right) + \frac{1}{4\pi^2} \int_{\frac{2\pi}{k}}^{\alpha_0} \text{Im} \left( 2 \log \left( -M^{-4n-2}M^{-2} + x \right) \right) \, d\alpha
$$

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$$+ \frac{1}{4\pi^2} \int_{\alpha_0}^{\pi} I m \left( \log \left( -M^{-4n-2} \frac{M^{-2} + x_1}{M^2 + x_1} \right) \right) d\alpha,$$

where for $M = e^{i\frac{\alpha}{2}}$, $x$ ($Im(x) \leq 0$), $x_1$, and $x_2$ are zeroes of Riley–Mednykh polynomial $P_{2n} = P_{2n}(x, M)$ which is given recursively by

$$P_{2n} = \begin{cases} 
Q P_{2(n-1)} - M^8 P_{2(n-2)} & \text{if } n > 1, \\
Q P_{2(n+1)} - M^8 P_{2(n+2)} & \text{if } n < -1,
\end{cases}$$

with initial conditions

$$P_{-2} = M^2 x^2 + \left( M^4 - M^2 + 1 \right) x + M^2,$$

$$P_0 = M^{-2} \text{ for } n < 0 \text{ and } P_0 = 1 \text{ for } n > 0,$$

$$P_2 = -M^4 x^3 + \left( -2M^6 + M^4 - 2M^2 \right) x^2$$

$$+ \left( -M^8 + M^6 - 2M^4 + M^2 - 1 \right) x + M^4, \text{ and } M = e^{i\alpha} \text{ and }$$

$$Q = -M^4 x^3 + \left( -2M^6 + 2M^4 - 2M^2 \right) x^2$$

$$+ \left( -M^8 + 2M^6 - 3M^4 + 2M^2 - 1 \right) x + 2M^4,$$

where $x_1$ and $x_2$ approach common $x$ as $\alpha$ decreases to $\alpha_0$ and they come from the components of $x$ and $\bar{x}$.

2 Two bridge knots with Conway’s notation $C(2n, 3)$

A knot $K$ is a two bridge knot with Conway’s notation $C(2n, 3)$ if $K$ has a regular two-dimensional projection of the form in Fig. 1. For example, Fig. 2 is knot $C(4, 3)$. $K$ has 3 left-handed horizontal crossings and $2n$ right-handed vertical crossings. Recall that we denote it by $T_{2n}$. One can easily check that the slope of $T_{2n}$ is $3/(6n + 1)$ which is equivalent to the knot with slope $(4n + 1)/(6n + 1)$ [29]. For example, Fig. 2 shows the regular projections of knot 73 with slope 3/13 which is equivalent to the knot with slope 9/13.

We will use the following fundamental group of the knot with Conway’s notation $C(2n, 3)$ [11,17,28]. The following proposition can also be obtained by reading off the fundamental group from the Schubert normal form of $T_{2n}$ with slope $\frac{4n+1}{6n+1}$ [28,29].

Proposition 2.1

$$\pi_1(X_{2n}) = \left\langle s, t \mid swt^{-1}w^{-1} = 1 \right\rangle,$$

where $w = (ts^{-1}tst^{-1}s)^n$. 

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Fig. 1 A two bridge knot with Conway’s notation $C(2n, 3)$ for $n > 0$ (left) and for $n < 0$ (right)

Fig. 2 The knot 7_3

3 The Riley–Mednykh polynomial

Given a set of generators, $\{s, t\}$, of the fundamental group for $\pi_1(X_{2n})$, we define a representation $\rho : \pi_1(X_{2n}) \to \text{SL}(2, \mathbb{C})$ by

$$
\rho(s) = \begin{bmatrix} M & 1 \\ 0 & M^{-1} \end{bmatrix}, \quad \rho(t) = \begin{bmatrix} M & 0 \\ 2 - M^2 - M^{-2} - x & M^{-1} \end{bmatrix}.
$$

Then $\rho$ can be identified with the point $(M, x) \in \mathbb{C}^2$. Using [11], when $M$ varies we have an algebraic set whose defining equation is the following Riley–Mednykh polynomial:
Theorem 3.1  \( \rho \) is a representation of \( \pi_1(X_{2n}) \) if and only if \( x \) is a root of the following Riley–Mednykh polynomial \( P_{2n} = P_{2n}(x, M) \) which is given recursively by

\[
P_{2n} = \begin{cases} 
Q P_{2(n-1)} - M^8 P_{2(n-2)} & \text{if } n > 1, \\
Q P_{2(n+1)} - M^8 P_{2(n+2)} & \text{if } n < -1,
\end{cases}
\]

with initial conditions

\[
P_{-2} = M^2 x^2 + (M^4 - M^2 + 1) x + M^2, \\
P_0 = M^{-2} \quad \text{for } n < 0 \quad \text{and} \quad P_0 = 1 \quad \text{for } n > 0,
\]

\[
P_2 = -M^4 x^3 + \left(-2M^6 + M^4 - 2M^2\right) x^2 \\
+ \left(-M^8 + 2M^6 - 2M^4 + 2M^2 - 1\right) x + M^4,
\]

and

\[
Q = -M^4 x^3 + \left(-2M^6 + 2M^4 - 2M^2\right) x^2 \\
+ \left(-M^8 + 2M^6 - 3M^4 + 2M^2 - 1\right) x + 2M^4.
\]

3.1 Longitude

Let \( l = w w^* s^{-4n} \), where \( w^* \) is the word obtained by reversing \( w \). Let \( L = \rho(l)_{11} \). Then \( l \) is the longitude which is null-homologous in \( X_{2n} \). And we have

Theorem 3.2 \([11]\)

\[
L = -M^{-4n-2} \frac{M^{-2} + x}{M^2 + x}.
\]

4 Schläfli formula for the generalized Chern–Simons function

The general references for this section are \([10,15,16,22,31]\). We introduce the generalized Chern–Simons function on the family of \( C(2n, 3) \) cone-manifold structures. For the oriented knot \( T_{2n} \), we orient a chosen meridian \( s \) such that the orientation of \( s \) follows by orientation of \( T_{2n} \) coincides with orientation of \( S^3 \). Hence, we use the definition of Lens space in \([16]\) so that we can have the right orientation when the definition of Lens space is combined with the following frame field. On the Riemannian manifold \( S^3 - T_{2n} - s \) we choose a special frame field \( \Gamma \). A special frame field \( \Gamma = (e_1, e_2, e_3) \) is an orthonormal frame field such that for each point \( x \) near \( T_{2n} \), \( e_1(x) \) has the knot direction, \( e_2(x) \) has the tangent direction of a meridian curve, and \( e_3(x) \) has the knot to point direction. A special frame field always exists by Proposition 3.1 of \([15]\). From \( \Gamma \) we obtain an orthonormal frame field \( \Gamma_\alpha \) on \( X_{2n}(\alpha) - s \) by the Schmidt orthonormalization process with respect to the geometric structure.
of the cone manifold $X_{2n}(\alpha)$. Moreover it can be made special by deforming it in a neighborhood of the singular set and $s$ if necessary. $\Gamma'$ is an extension of $\Gamma$ to $S^3 - T_{2n}$. For each cone-manifold $X_{2n}(\alpha)$, we assign the real number:

$$I(X_{2n}(\alpha)) = \frac{1}{2} \int_{\Gamma'} Q - \frac{1}{4\pi} \tau(s, \Gamma') - \frac{1}{4\pi} \left( \frac{\beta \alpha}{2\pi} \right),$$

where $-2\pi \leq \beta \leq 2\pi$, $Q$ is the Chern–Simons form:

$$Q = \frac{1}{4\pi^2} (\theta_{12} \wedge \theta_{13} \wedge \theta_{23} + \theta_{12} \wedge \Omega_{12} + \theta_{13} \wedge \Omega_{13} + \theta_{23} \wedge \Omega_{23}),$$

and

$$\tau(s, \Gamma') = -\int_{\Gamma}(s) \theta_{23},$$

where $(\theta_{ij})$ is the connection 1-form, $(\Omega_{ij})$ is the curvature 2-form of the Riemannian connection on $X_{2n}(\alpha)$ and the integral is over the orthonormalizations of the same frame field. When $\alpha = \frac{2\pi}{k}$ for some positive integer, $I(X_{2n}(\frac{2\pi}{k})) \mod \frac{1}{k}$ if $k$ is even or mod $\frac{1}{2k}$ if $k$ is odd) is independent of the frame field $\Gamma$ and of the representative in the equivalence class $\beta$ and hence an invariant of the orbifold $X_{2n}(\frac{2\pi}{k})$. $I(X_{2n}(\frac{2\pi}{k})) \mod \frac{1}{k}$ if $k$ is even or mod $\frac{1}{2k}$ if $k$ is odd) is called the Chern–Simons invariant of the orbifold and is denoted by $\text{cs}(X_{2n}(\frac{2\pi}{k}))$.

On the generalized Chern–Simons function on the family of $C(2n, 3)$ cone-manifold structures we have the following Schläfli formula:

**Theorem 4.1** (Theorem 1.2 of [16]) For a family of geometric cone-manifold structures, $X_{2n}(\alpha)$, and differentiable functions $\alpha(t)$ and $\beta(t)$ of $t$ we have

$$dI(X_{2n}(\alpha)) = -\frac{1}{4\pi^2} \beta d\alpha.$$ 

**5 Proof of the Theorem 1.1**

For $n \geq 1$ and $M = e^{i \frac{x}{2}}$, $P_{2n}(x, M)$ have $3n$ component zeros, and for $n \leq -1$, $- (3n + 1)$ component zeros. The component which gives the maximal volume is the geometric component [7, 8, 11] and in [11] it is identified. For each $T_{2n}$, there exists an angle $\alpha_0 \in [\frac{2\pi}{3}, \pi]$ such that $T_{2n}$ is hyperbolic for $\alpha \in (0, \alpha_0)$, Euclidean for $\alpha = \alpha_0$, and spherical for $\alpha \in (\alpha_0, \pi]$ [14, 20, 26, 27]. Denote by $D(X_{2n}(\alpha))$ be the set of zeros of the discriminant of $P_{2n}(x, e^{i \frac{x}{2}})$ over $x$. Then $\alpha_0$ will be one of $D(X_{2n}(\alpha))$.

On the geometric component we can calculate the Chern–Simons invariant of an orbifold $X_{2n}(\frac{2\pi}{k})$ (mod $\frac{1}{k}$ if $k$ is even or mod $\frac{1}{2k}$ if $k$ is odd), where $k$ is a positive integer such that $k$-fold cyclic covering of $X_{2n}(\frac{2\pi}{k})$ is hyperbolic:
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Table 1  Chern–Simons invariant of \( X_{2n} \) for \( n \) between 1 and 9 and for \( n \) between −9 and −1

| \( 2n \) | \( \alpha_0 \) | \( \text{cs} (X_{2n}) \) | \( 2n \) | \( \alpha_0 \) | \( \text{cs} (X_{2n}) \) |
|---|---|---|---|---|---|
| 2  | 2.40717 | 0.346796 | 2  | 2.09440 | 0 |
| 4  | 2.75511 | 0.187220 | 4  | 2.68404 | 0.202492 |
| 6  | 2.87826 | 0.116482 | 6  | 2.84713 | 0.287081 |
| 8  | 2.94175 | 0.0787607 | 8  | 2.92433 | 0.300333 |
| 10 | 2.98054 | 0.0554891 | 10 | 2.96942 | 0.356274 |
| 12 | 3.00671 | 0.030565 | 12 | 2.99899 | 0.373511 |
| 14 | 3.02556 | 0.0283589 | 14 | 3.01989 | 0.385781 |
| 16 | 3.03978 | 0.0200137 | 16 | 3.03545 | 0.394957 |
| 18 | 3.05090 | 0.0117308 | 18 | 3.04747 | 0.402076 |

Table 2  Chern–Simons invariant of the hyperbolic orbifold, \( \text{cs}(X_{2n}(\frac{2\pi}{k})) \) for \( n \) between 1 and 9 and for \( k \) between 3 and 10, and of its cyclic covering, \( \text{cs}(M_k(X_{2n})) \)

| \( k \) | \( \text{cs} (X_2(\frac{2\pi}{k})) \) | \( \text{cs} (M_k(X_2)) \) | \( \text{cs} (X_4(\frac{2\pi}{k})) \) | \( \text{cs} (M_k(X_4)) \) |
|---|---|---|---|---|
| 3  | 0.0200137 | 0.0600411 | 0.163905 | 0.491714 |
| 4  | 0.186810 | 0.747239 | 0.207480 | 0.829920 |
| 5  | 0.00166425 | 0.00832123 | 0.0602662 | 0.301331 |
| 6  | 0.0504594 | 0.302756 | 0.140577 | 0.843464 |
| 7  | 0.0163411 | 0.114387 | 0.0610011 | 0.427008 |
| 8  | 0.116987 | 0.935894 | 0.00457501 | 0.036600 |
| 9  | 0.0292866 | 0.263580 | 0.0181733 | 0.163560 |
| 10 | 0.0595395 | 0.595395 | 0.0302655 | 0.302655 |

| \( k \) | \( \text{cs} (X_6(\frac{2\pi}{k})) \) | \( \text{cs} (M_k(X_6)) \) | \( \text{cs} (X_8(\frac{2\pi}{k})) \) | \( \text{cs} (M_k(X_8)) \) |
|---|---|---|---|---|
| 3  | 0.0117308 | 0.0351925 | 0.0392668 | 0.117800 |
| 4  | 0.0254160 | 0.101664 | 0.115898 | 0.463593 |
| 5  | 0.0770172 | 0.385086 | 0.0209964 | 0.104982 |
| 6  | 0.130155 | 0.780930 | 0.149082 | 0.894495 |
| 7  | 0.0343996 | 0.240797 | 0.0382671 | 0.267870 |
| 8  | 0.0925471 | 0.740377 | 0.0866540 | 0.693232 |
| 9  | 0.0295838 | 0.266254 | 0.0170042 | 0.153038 |
| 10 | 0.0810442 | 0.810442 | 0.0636841 | 0.636841 |

\[
\text{cs} \left( \frac{2\pi}{k} \right) \equiv I \left( X_{2n} \left( \frac{2\pi}{k} \right) \right) \left( \text{mod} \frac{1}{k} \right)
\equiv I \left( X_{2n}(\pi) \right) + \frac{1}{4\pi^2} \int_{\frac{\pi}{k}}^{\frac{\pi}{k}} \beta \, d\alpha \left( \text{mod} \frac{1}{k} \right)
\]
Table 2 continued

| $k$ | $cs\left(X_{10}\left(\frac{2\pi}{X}\right)\right)$ | $cs\left(M_k(X_{10})\right)$ | $cs\left(X_{12}\left(\frac{2\pi}{X}\right)\right)$ | $cs\left(M_k(X_{12})\right)$ |
|-----|------------------------------------------------|--------------------------------|--------------------------------|--------------------------------|
| 3   | 0.0749335                                      | 0.224800                       | 0.116132                       | 0.348396                       |
| 4   | 0.218720                                       | 0.874878                       | 0.078447                       | 0.214260                       |
| 5   | 0.0783315                                      | 0.391658                       | 0.042852                       | 0.214260                       |
| 6   | 0.0150995                                      | 0.0905970                      | 0.055083                       | 0.330499                       |
| 7   | 0.0560983                                      | 0.392688                       | 0.0098623                       | 0.0690364                      |
| 8   | 0.0948488                                      | 0.758790                       | 0.110442                       | 0.883540                       |
| 9   | 0.0185935                                      | 0.167341                       | 0.110442                       | 0.883540                       |
| 10  | 0.0605490                                      | 0.605490                       | 0.0648550                      | 0.648550                       |

| $k$ | $cs\left(X_{14}\left(\frac{2\pi}{X}\right)\right)$ | $cs\left(M_k(X_{14})\right)$ | $cs\left(X_{16}\left(\frac{2\pi}{X}\right)\right)$ | $cs\left(M_k(X_{16})\right)$ |
|-----|------------------------------------------------|--------------------------------|--------------------------------|--------------------------------|
| 3   | 0.161005                                       | 0.483014                       | 0.0416866                       | 0.125060                       |
| 4   | 0.192332                                       | 0.769328                       | 0.058936                       | 0.235574                       |
| 5   | 0.0116320                                      | 0.0581602                      | 0.0831339                       | 0.415670                       |
| 6   | 0.0993703                                      | 0.596222                       | 0.146399                       | 0.878396                       |
| 7   | 0.0393825                                      | 0.275677                       | 0.000227239                    | 0.00159067                     |
| 8   | 0.00537856                                     | 0.0430285                      | 0.0280750                      | 0.224600                       |
| 9   | 0.0409719                                      | 0.368747                       | 0.00154689                     | 0.0139220                      |
| 10  | 0.0735205                                      | 0.735205                       | 0.0849545                      | 0.849545                       |

| $k$ | $cs\left(X_{18}\left(\frac{2\pi}{X}\right)\right)$ | $cs\left(M_k(X_{18})\right)$ |
|-----|------------------------------------------------|--------------------------------|
| 3   | 0.0907588                                      | 0.272277                       |
| 4   | 0.177274                                       | 0.709096                       |
| 5   | 0.0564774                                      | 0.282387                       |
| 6   | 0.0286139                                      | 0.171683                       |
| 7   | 0.0343586                                      | 0.240510                       |
| 8   | 0.0526332                                      | 0.421066                       |
| 9   | 0.0195418                                      | 0.175876                       |
| 10  | 0.0982547                                      | 0.982547                       |

\[
\equiv \frac{1}{2} \left( L(6n + 1, 4n + 1) + \frac{1}{4\pi^2} \int_{\alpha_0}^{\alpha_0} \text{Im} \left( 2 * \log \left( -M^{-4n-2} \frac{M^{-2} + x}{M^2 + x} \right) \right) d\alpha \right) \\
+ \frac{1}{4\pi^2} \int_{\alpha_0}^{\pi} \text{Im} \left( \log \left( -M^{-4n-2} \frac{M^{-2} + x_1}{M^2 + x_1} \right) + \log \left( -M^{-4n-2} \frac{M^{-2} + x_2}{M^2 + x_2} \right) \right) d\alpha \\
\left( \mod \frac{1}{k} \text{ if } k \text{ is even or } \mod \frac{1}{2k} \text{ if } k \text{ is odd} \right)
\]

where the second equivalence comes from Theorem 4.1 and the third equivalence comes from the fact that \( I (X_{2n}(\pi)) \equiv \frac{1}{2} cs\left(L(6n + 1, 4n + 1)\right) \left( \mod \frac{1}{2} \right), \)
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Table 3  Chern–Simons invariant of the hyperbolic orbifold, \( cs(X_{2n}(\frac{2\pi n}{k})) \) for \( n \) between \(-9\) and \(-2\) and for \( k \) between \(3\) and \(10\), and of its cyclic covering, \( cs(M_k(X_{2n})) \)

| \( k \) | \( cs(X_{-4}(\frac{2\pi}{k})) \) | \( cs(M_k(X_{-4})) \) | \( cs(X_{-6}(\frac{2\pi}{k})) \) | \( cs(M_k(X_{-6})) \) |
|-------|-------------------------------|-----------------|-------------------------------|-----------------|
| 3     | 0.0578105                     | 0.173431        | 0.0502767                     | 0.150830        |
| 4     | 0.0141698                     | 0.0566791       | 0.206063                      | 0.824252        |
| 5     | 0.0771122                     | 0.385561        | 0.0724185                     | 0.362092        |
| 6     | 0.113440                      | 0.680638        | 0.136957                      | 0.821740        |
| 7     | 0.0647357                     | 0.453150        | 0.0334583                     | 0.234208        |
| 8     | 0.0262590                     | 0.210072        | 0.0770408                     | 0.616327        |
| 9     | 0.0506565                     | 0.455908        | 0.0530941                     | 0.477846        |
| 10    | 0.0693643                     | 0.693643        | 0.0324771                     | 0.324771        |

| \( k \) | \( cs(X_{-8}(\frac{2\pi}{k})) \) | \( cs(M_k(X_{-8})) \) | \( cs(X_{-10}(\frac{2\pi}{k})) \) | \( cs(M_k(X_{-10})) \) |
|-------|-------------------------------|-----------------|-------------------------------|-----------------|
| 3     | 0.0260938                     | 0.0782813       | 0.159369                      | 0.478108        |
| 4     | 0.121024                      | 0.484097        | 0.0211627                     | 0.804659        |
| 5     | 0.0343014                     | 0.171507        | 0.0799373                     | 0.396986        |
| 6     | 0.123924                      | 0.743545        | 0.0941609                     | 0.564965        |
| 7     | 0.0354455                     | 0.248118        | 0.0204861                     | 0.143403        |
| 8     | 0.0887397                     | 0.709918        | 0.0833782                     | 0.667026        |
| 9     | 0.0158804                     | 0.142923        | 0.0170947                     | 0.153852        |
| 10    | 0.0555635                     | 0.555635        | 0.0614793                     | 0.614793        |

| \( k \) | \( cs(X_{-12}(\frac{2\pi}{k})) \) | \( cs(M_k(X_{-12})) \) | \( cs(X_{-14}(\frac{2\pi}{k})) \) | \( cs(M_k(X_{-14})) \) |
|-------|-------------------------------|-----------------|-------------------------------|-----------------|
| 3     | 0.119699                       | 0.359097        | 0.0758416                     | 0.227525        |
| 4     | 0.163139                       | 0.652556        | 0.0503095                     | 0.201238        |
| 5     | 0.0170874                      | 0.0854371       | 0.0493320                     | 0.246660        |
| 6     | 0.0558073                      | 0.334844        | 0.0125167                     | 0.0751005       |
| 7     | 0.0683200                      | 0.478240        | 0.0397753                     | 0.278427        |
| 8     | 0.0693583                      | 0.554866        | 0.0503822                     | 0.403058        |
| 9     | 0.00963738                     | 0.0867365       | 0.0527761                     | 0.474985        |
| 10    | 0.0587154                      | 0.587154        | 0.0509898                     | 0.509989        |

| \( k \) | \( cs(X_{-12}(\frac{2\pi}{k})) \) | \( cs(M_k(X_{-12})) \) | \( cs(X_{-14}(\frac{2\pi}{k})) \) | \( cs(M_k(X_{-14})) \) |
|-------|-------------------------------|-----------------|-------------------------------|-----------------|
| 3     | 0.0291847                      | 0.0875541       | 0.147267                      | 0.441800        |
| 4     | 0.184443                       | 0.737773        | 0.0665438                     | 0.266175        |
| 5     | 0.0785018                      | 0.392509        | 0.00562151                    | 0.0281075       |
| 6     | 0.132806                       | 0.796838        | 0.0843746                     | 0.506248        |
| 7     | 0.00813976                     | 0.0569783       | 0.0458762                     | 0.321134        |
| 8     | 0.0283132                      | 0.226505        | 0.00418700                    | 0.0334960       |
| 9     | 0.0372653                      | 0.335388        | 0.0196972                     | 0.177275        |
| 10    | 0.0401699                      | 0.401699        | 0.0272925                     | 0.272925        |
Theorem 3.2, and geometric interpretations of hyperbolic and spherical holonomy representations. The following theorem gives the Chern–Simons invariant of the Lens space $L(6n + 1, 4n + 1)$:

**Theorem 5.1** (Theorem 1.3 of [16])

$$
\text{cs}(L(6n + 1, 4n + 1)) \equiv \frac{4n + 4}{12n + 2} \pmod{1}.
$$

6 Chern–Simons invariants of the hyperbolic orbifolds of the knot with Conway’s notation $C(2n, 3)$ and of its cyclic coverings

The Table 1 gives the approximate Chern–Simons invariant of $T_{2n}$ for each $n$ between $-9$ and 9 except the unknot and the amphicheiral knot. We used Simpson’s rule for the approximation with $10^4$ ($5 \times 10^3$ in Simpson’s rule) intervals from 0 to $\alpha_0$ and $10^4$ ($5 \times 10^3$ in Simpson’s rule) intervals from $\alpha_0$ to $\pi$. The Table 2 (resp. the Table 3) gives the approximate Chern–Simons invariant of the hyperbolic orbifold, $\text{cs}(X_{2n}(\frac{2\pi}{k}))$ for $n$ between 1 and 9 (resp. for $n$ between $-9$ and $-2$) and for $k$ between 3 and 10, and of its cyclic covering, $\text{cs}(M_k(X_{2n}))$. We again used Simpson’s rule for the approximation with $10^4$ ($5 \times 10^3$ in Simpson’s rule) intervals from $\frac{2\pi}{k}$ to $\alpha_0$ and $10^4$ ($5 \times 10^3$ in Simpson’s rule) intervals from $\alpha_0$ to $\pi$.

We used Mathematica for the calculations. We record here that our data in Table 1 and those obtained from SnapPy match up up to six decimal points.

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