Further inequalities for the $A$-numerical radius of certain $2 \times 2$ operator matrices

Kais Feki$^{a,b}$ and Satyajit Sahoo$^{2}$

Abstract. Let $A = \begin{pmatrix} A & O \\ O & A \end{pmatrix}$ be a $2 \times 2$ diagonal operator matrix whose each diagonal entry is a bounded positive (semidefinite) linear operator $A$ acting on a complex Hilbert space $\mathcal{H}$. In this paper, we derive several $A$-numerical radius inequalities for $2 \times 2$ operator matrices whose entries are bounded with respect to the seminorm induced by the positive operator $A$ on $\mathcal{H}$. Some applications of our inequalities are also given.

1. Introduction and Preliminaries

Throughout this article, $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ stands for a complex Hilbert space with associated norm $\| \cdot \|$. If $\mathcal{M}$ is a given linear subspace of $\mathcal{H}$, then $\overline{\mathcal{M}}$ denotes its closure in the norm topology of $\mathcal{H}$. Further, the orthogonal projection onto a closed subspace $S$ of $\mathcal{H}$ will be denoted by $P_S$. Let $\mathcal{B}(\mathcal{H})$ denote the $C^*$-algebra of all bounded linear operators acting on $\mathcal{H}$ with the identity operator $I_\mathcal{H}$ (or simply $I$ if no confusion arises). If $T \in \mathcal{B}(\mathcal{H})$, then $\mathcal{N}(T)$, $\mathcal{R}(T)$ and $T^*$ are denoted by the kernel, the range and the adjoint of $T$, respectively. An operator $T \in \mathcal{B}(\mathcal{H})$ is called positive (semi-definite) $\langle Ax, x \rangle \geq 0$, for every $x \in \mathcal{H}$. For the rest of this paper, by an operator we mean a bounded linear operator. Further, we suppose that $A \in \mathcal{B}(\mathcal{H})$ is a nonzero positive operator which induces the following semi-inner product

$$\langle \cdot, \cdot \rangle_A : \mathcal{H} \times \mathcal{H} \to \mathbb{C}, \quad (x, y) \mapsto \langle x, y \rangle_A := \langle Ax, y \rangle = \langle A^{1/2}x, A^{1/2}y \rangle.$$ 

Here $A^{1/2}$ denotes the square root of $A$. Notice that the seminorm induced by $\langle \cdot, \cdot \rangle_A$ is given by $\|x\|_A = \|A^{1/2}x\|$, for all $x \in \mathcal{H}$. It can checked that $\| \cdot \|_A$ is a norm on $\mathcal{H}$ if and only if $A$ is injective, and that the seminormed space $(\mathcal{H}, \| \cdot \|_A)$ is complete if and only if $\mathcal{R}(A)$ is a closed subspace of $\mathcal{H}$.

Let $T \in \mathcal{B}(\mathcal{H})$. An operator $S \in \mathcal{B}(\mathcal{H})$ is called an $A$-adjoint of $T$ if $\langle Tx, y \rangle_A = \langle x, Sy \rangle_A$ for all $x, y \in \mathcal{H}$ (see [2]). Thus, the existence of an $A$-adjoint of $T$ is equivalent to the existence of a solution of the equation $AX = T^*A$. Notice that this kind of equations can be investigated by using a theorem due to Douglas [15] which establishes the equivalence between the following statements:

(i) The operator equation $TX = S$ has a bounded linear solution $X$.

*Date: April 4, 2022.

2020 Mathematics Subject Classification. Primary 47A12, 46C05; Secondary 47B65, 47A05.

Key words and phrases. Positive operator, $A$-adjoint operator, numerical radius, operator matrix, inequality.
(ii) \( R(S) \subseteq R(T) \).

(iii) There exists a positive number \( \lambda \) such that \( \| S^*x \| \leq \lambda \| T^*x \| \) for all \( x \in \mathcal{H} \).

Moreover, among many solutions of \( AX = S \), it has only one, say \( Q \), which satisfies \( R(Q) \subseteq R(T^*) \). Such \( Q \) is said the reduced solution of the equation \( TX = S \). If we denote by \( \mathcal{B}_A(\mathcal{H}) \), the subspace of all operators admitting \( A \)-adjoins, then by Douglas theorem, we have

\[
\mathcal{B}_A(\mathcal{H}) = \{ T \in \mathcal{B}(\mathcal{H}) ; R(T^*A) \subseteq R(A) \}.
\]

If \( T \in \mathcal{B}_A(\mathcal{H}) \), the reduced solution of the equation \( AX = T^*A \) will be denoted by \( T^{\sharp A} \). We mention here that, \( T^{\sharp A} = A^1T^*A \) in which \( A^1 \) is the Moore-Penrose inverse of \( A \) (see [3]). In addition, if \( T \in \mathcal{B}_A(\mathcal{H}) \), then \( T^{\sharp A} \in \mathcal{B}_A(\mathcal{H}) \), \( (T^{\sharp A})^{\sharp A} = P_{R(A)}TP_{R(A)} \) and \( ((T^{\sharp A})^{\sharp A})^{\sharp A} = T^{\sharp A} \). Moreover, if \( S \in \mathcal{B}_A(\mathcal{H}) \), then \( TS \in \mathcal{B}_A(\mathcal{H}) \) and \( (TS)^{\sharp A} = S^{\sharp A}T^{\sharp A} \). Furthermore, for every \( T \in \mathcal{B}_A(\mathcal{H}) \), we have

\[
\|T^{\sharp A}T\|_A = \|TT^{\sharp A}\|_A = \|T\|_A^2 = \|T^{\sharp A}\|_A^2.
\] (1.1)

An operator \( U \in \mathcal{B}_A(\mathcal{H}) \) is called \( A \)-unitary if \( \|UX\|_A = \|U^{\sharp A}X\|_A = \|x\|_A \), for all \( x \in \mathcal{H} \). It is worth mentioning that, an operator \( U \in \mathcal{B}_A(\mathcal{H}) \) is \( A \)-unitary if and only if \( U^{\sharp A}U = (U^{\sharp A})^{\sharp A}U^{\sharp A} = P_{R(A)} \) (see [2]). For an account of the results, we invite the reader to consult [2, 3].

An operator \( T \) is called \( A \)-bounded if there exists \( \lambda > 0 \) such that \( \|Tx\|_A \leq \lambda \|x\|_A \), \( \forall x \in \mathcal{H} \). By applying Douglas theorem, one can easily see that the subspace of all operators admitting \( A^{1/2} \)-adjoins, denoted by \( \mathcal{B}_{A^{1/2}}(\mathcal{H}) \), is equal the collection of all \( A \)-bounded operators, i.e.,

\[
\mathcal{B}_{A^{1/2}}(\mathcal{H}) = \{ T \in \mathcal{B}(\mathcal{H}) ; \exists \lambda > 0 ; \|Tx\|_A \leq \lambda \|x\|_A, \forall x \in \mathcal{H} \}.
\]

Notice that \( \mathcal{B}_A(\mathcal{H}) \) and \( \mathcal{B}_{A^{1/2}}(\mathcal{H}) \) are two subalgebras of \( \mathcal{B}(\mathcal{H}) \) which are, in general, neither closed nor dense in \( \mathcal{B}(\mathcal{H}) \). Moreover, we have \( \mathcal{B}_A(\mathcal{H}) \subset \mathcal{B}_{A^{1/2}}(\mathcal{H}) \) (see [2, 4]). Clearly, \( \langle \cdot, \cdot \rangle_A \) induces a seminorm on \( \mathcal{B}_{A^{1/2}}(\mathcal{H}) \). Indeed, if \( T \in \mathcal{B}_{A^{1/2}}(\mathcal{H}) \), then it holds that

\[
\|T\|_A := \sup_{x \in R(A), x \neq 0} \frac{\|Tx\|_A}{\|x\|_A} = \sup \{ \|Tx\|_A ; x \in \mathcal{H}, \|x\|_A = 1 \} < \infty.
\] (1.2)

Saddi [30] in 2012 defined the \( A \)-numerical radius of an operator \( T \in \mathcal{B}(\mathcal{H}) \) by

\[
\omega_A(T) := \sup \{ \|Tx\|_A ; x \in \mathcal{H}, \|x\|_A = 1 \}.
\]

Faghih-Ahmadi and Gorjizadeh [23] in 2016 showed that for \( T \in \mathcal{B}_{A^{1/2}}(\mathcal{H}) \), we have

\[
\|T\|_A = \sup \{ \|Tx, y\|_A ; x, y \in \mathcal{H}, \|x\|_A = \|y\|_A = 1 \}.
\] (1.3)

We notice here that it may happen that \( \|T\|_A \) and \( \omega_A(T) \) are equal to +\( \infty \) for some \( T \in \mathcal{B}(\mathcal{H}) \) (see [16]). However, \( \| \cdot \|_A \) and \( \omega_A(\cdot) \) are equivalent seminorms on \( \mathcal{B}_{A^{1/2}}(\mathcal{H}) \). More precisely, In 2018, Baklouti et al. [6] showed that for every \( T \in \mathcal{B}_{A^{1/2}}(\mathcal{H}) \), we have

\[
\frac{1}{2}\|T\|_A \leq \omega_A(T) \leq \|T\|_A.
\] (1.4)
For the sequel, for any arbitrary operator $T \in \mathcal{B}_A(H)$, we write

$$
\Re_A(T) := \frac{T + T^A}{2} \quad \text{and} \quad \Im_A(T) := \frac{T - T^A}{2i}.
$$

Recently, in 2019 Zamani [31, Theorem 2.5] showed that if $T \in \mathcal{B}_A(H)$, then

$$
\omega_A(T) = \sup_{\theta \in \mathbb{R}} \|\Re_A(e^{i\theta}T)\|_A = \sup_{\theta \in \mathbb{R}} \|\Im_A(e^{i\theta}T)\|_A. \tag{1.5}
$$

Notice that (1.5) is also proved in a general context in [8]. In 2020, the concept of the $A$-spectral radius of $A$-bounded operators was introduced by the first author in [16] as follows:

$$
r_A(T) := \inf_{n \geq 1} \|T^n\|_A^{\frac{1}{n}} = \lim_{n \to \infty} \|T^n\|_A^{\frac{1}{n}}. \tag{1.6}
$$

Here we want to mention that the proof of the second equality in (1.6) can also be found in [16, Theorem 1]. Like the classical spectral radius of Hilbert space operators, it was shown in [16] that $r_A(\cdot)$ satisfies the commutativity property, i.e.

$$
r_A(TS) = r_A(ST), \tag{1.7}
$$

for all $T, S \in \mathcal{B}_{A^{1/2}}(H)$. For the sequel, if $A = I$, then $\|T\|$, $r(T)$ and $\omega(T)$ denote respectively the classical operator norm, the spectral radius and the numerical radius of an operator $T$.

An operator $T \in \mathcal{B}(H)$ is called $A$-selfadjoint if $AT$ is selfadjoint, i.e., $AT = T^*A$ and it is called $A$-positive if $AT \geq 0$. If $T$ is $A$-positive, we will write $T \geq_A 0$.

In recent years, several results covering some classes of operators on a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$ were extended to $(H, \langle \cdot, \cdot \rangle_A)$. Of course, the extension is not trivial since many difficulties arise. For instance, as mentioned above, it may happen that $\|T\|_A = \infty$ for some $T \in \mathcal{B}(H)$. Moreover, no operator admits an adjoint operator for the semi-inner product $\langle \cdot, \cdot \rangle_A$. In addition, for $T \in \mathcal{B}_A(H)$, we have $(T^A)^\sharp_A = \frac{P_{\Re(A)}T}{P_{\Re(A)}} \neq T$. For further details about $A$-numerical radius, interested readers can follow [5, 6, 7, 9, 14, 10, 18, 19, 31, 28] and the references therein.

In this paper, we consider the $2 \times 2$ operator diagonal matrix $A = \begin{pmatrix} A & O \\ O & A \end{pmatrix}$. Clearly, $A \in \mathcal{B}(H \oplus H)^+$. So, $A$ induces the following semi-inner product

$$
\langle x, y \rangle_A = \langle Ax, y \rangle = \langle x_1, y_1 \rangle_A + \langle x_2, y_2 \rangle_A,
$$

for all $x = (x_1, x_2) \in H \oplus H$ and $y = (y_1, y_2) \in H \oplus H$.

Recently, several inequalities for the $A$-numerical radius of $2 \times 2$ operator matrices have been established by Bhunia et al. when $A$ is a positive injective operators (see [12]). Moreover, different upper and lower bounds of $A$-numerical radius when $A$ is a positive semidefinite operator has been recently investigated by the first author in [20], by Rout et al. in [29] and by Kittaneh et al. in [27]. In this article, we will continue working in this direction and we will prove several new $A$-numerical radius inequalities of certain $2 \times 2$ operator matrices. The inspiration for our investigation comes from [1, 25, 26].
2. Results

In this section, we present our results. Throughout this section $A$ is denoted to be the $2 \times 2$ operator diagonal matrix whose each diagonal entry is the positive operator $A$. To prove our first result, the following lemmas are required.

**Lemma 2.1.** ([16]) Let $T \in \mathcal{B}(\mathcal{H})$ is an $A$-self-adjoint operator. Then,

$$\|T\|_A = \omega_A(T) = r_A(T).$$

**Lemma 2.2.** ([17]) Let $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$ be such that $T_{ij} \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$ for all $i, j \in \{1, 2\}$. Then, $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H} \oplus \mathcal{H})$ and

$$r_A(T) \leq r \left[ \left( \frac{\|T_{11}\|_A}{\|T_{21}\|_A} \right) \left( \frac{\|T_{12}\|_A}{\|T_{22}\|_A} \right) \right].$$

**Lemma 2.3.** ([10, 20]) Let $P, Q, R, S \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$. Then, the following assertions hold

(i) $\omega_A \left[ \begin{pmatrix} P & O \\ O & S \end{pmatrix} \right] = \max \{\omega_A(P), \omega_A(S)\}$.

(ii) $\left\| \begin{pmatrix} P & O \\ O & S \end{pmatrix} \right\|_A = \left\| \begin{pmatrix} O & P \\ S & O \end{pmatrix} \right\|_A = \max \{\|P\|_A, \|S\|_A\}$.

(iii) If $P, Q, R, S \in \mathcal{B}_A(\mathcal{H})$, then $\begin{pmatrix} P & Q \\ R & S \end{pmatrix}^{2A} = \begin{pmatrix} P^{2A} & R^{2A} \\ Q^{2A} & S^{2A} \end{pmatrix}$.

Now, we are ready to prove our first result which generalizes [1, Theorem 2.7].

**Theorem 2.1.** Let $P, Q, R, S \in \mathcal{B}_A(\mathcal{H})$. Then, for $\lambda \in [0, 1]$, we have

$$\omega_A \left[ \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \right] \leq \frac{1}{2} \left( \|P\|_A + 2\omega_A(S) + \sqrt{\|\lambda^2 PP^{\sharp_A} + QQ^{\sharp_A}\|_A + \sqrt{(1 - \lambda)^2 PP^{\sharp_A} + RR^{\sharp_A}}} \right) + \sqrt{(1 - \lambda)^2 PP^{\sharp_A} + RR^{\sharp_A}} \right).$$

**Proof.** Let $T = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$. It is not difficult to see that $\Re_A(e^{i\theta}T)$ is an $A$-selfadjoint operator. So, by Lemma 2.1 we have

$$\|\Re_A(e^{i\theta}T)\|_A = \omega_A \left( \Re_A(e^{i\theta}T) \right).$$
So, by applying Lemma 2.3 (i), we see that

\[
\|R_A(e^{i\theta}T)\|_A
= \frac{1}{2}\omega_A \left[ \begin{pmatrix}
\lambda(e^{i\theta}P + e^{-i\theta}P^*_A) & e^{i\theta}Q \\
e^{-i\theta}Q^*_A & 0
\end{pmatrix}
\right]
\leq \frac{1}{2}\omega_A \left[ \begin{pmatrix}
(1 - \lambda)(e^{i\theta}P + e^{-i\theta}P^*_A) & e^{-i\theta}R^*_A \\
e^{-i\theta}R & 0
\end{pmatrix}
\right]
+ \frac{1}{2}\omega_A \left[ \begin{pmatrix}
0 & 0 \\
e^{i\theta}S + e^{-i\theta}S^*_A & 0
\end{pmatrix}
\right]
+ \omega_A(S),
\]

where the last inequality follows by Lemma 2.3 (i) together with the triangle inequality. Moreover, it can be observed that

\[
A \begin{pmatrix}
\lambda(e^{i\theta}P + e^{-i\theta}P^*_A) & e^{i\theta}Q \\
e^{-i\theta}Q^*_A & 0
\end{pmatrix} = \begin{pmatrix}
\lambda(e^{i\theta}AP + e^{-i\theta}AP^*_A) & e^{i\theta}AQ \\
e^{-i\theta}AQ^*_A & 0
\end{pmatrix}
= \begin{pmatrix}
\lambda(P^*_A)^*A + e^{-i\theta}P^*A & e^{i\theta}(Q^*_A)^*A \\
e^{-i\theta}Q^*A & 0
\end{pmatrix}
= \begin{pmatrix}
\lambda(e^{-i\theta}P^*_A + e^{i\theta}P) & e^{i\theta}Q \\
e^{-i\theta}Q^*_A & 0
\end{pmatrix}^* A.
\]

Hence \(\begin{pmatrix}
\lambda(e^{i\theta}P + e^{-i\theta}P^*_A) & e^{i\theta}Q \\
e^{-i\theta}Q^*_A & 0
\end{pmatrix}\) is \(A\)-selfadjoint operator. Similarly one can show that \(\begin{pmatrix}
(1 - \lambda)(e^{i\theta}P + e^{-i\theta}P^*_A) & e^{-i\theta}R^*_A \\
e^{-i\theta}R & 0
\end{pmatrix}\) is \(A\)-selfadjoint operator. So by applying Lemma 2.1 we see that

\[
\|R_A(e^{i\theta}T)\|_A \leq \frac{1}{2}r^*_A \left[ \begin{pmatrix}
\lambda(e^{i\theta}P + e^{-i\theta}P^*_A) & e^{i\theta}Q \\
e^{-i\theta}Q^*_A & O
\end{pmatrix}
\right]
+ \frac{1}{2}r^*_A \left[ \begin{pmatrix}
(1 - \lambda)(e^{i\theta}P + e^{-i\theta}P^*_A) & e^{-i\theta}R^*_A \\
e^{-i\theta}R & O
\end{pmatrix}
\right] + \omega_A(S).
\]

So, by using (1.7) we infer that

\[
\|R_A(e^{i\theta}T)\|_A \leq \frac{1}{2}r^*_A \left[ \begin{pmatrix}
e^{-i\theta}I & O \\
\lambda P & Q \\
\lambda P^*_A & \lambda e^{i\theta}I
\end{pmatrix}
\right]
+ \frac{1}{2}r^*_A \left[ \begin{pmatrix}
(1 - \lambda)e^{-i\theta}P^*_A \\
e^{-i\theta}I \\
(1 - \lambda)^2P^*_A + \lambda P^*_A
\end{pmatrix}
\right] + \omega_A(S).
\]
So, by using Lemma 2.2 we get
\[
\| R_A(e^{i\theta} T) \|_A \\
\leq \frac{1}{2} r \left[ \left( \lambda \| P \|_A + \frac{1}{\lambda} \| P \|_A \right) \left( \lambda \| P \|_A + \frac{1}{\lambda} \| P \|_A \right) \right] + \frac{1}{2} r \left[ \left( \| (1 - \lambda)^2 P P T_A + R T_A \|_A \right) \left( 1 - \lambda \| P \|_A \right) + \omega_A(S) \right] + \frac{1}{2} \left[ \| P \|_A + 2 \omega_A(S) + \sqrt{\| \lambda^2 P P T_A + Q Q T_A \|_A} + \sqrt{\| (1 - \lambda)^2 P P T_A + R T_A \|_A} \right].
\]

So, by taking the supremum over all \( \theta \in \mathbb{R} \) in the last inequality and then using 1.5 we get desired result. \( \Box \)

The following Lemma is useful in the sequel. Notice that its assertion generalize recent results done by Rout et al. in [29] for operators in \( B_A(H) \).

**Lemma 2.4.** Let \( T, S \in B_{1/2}(H) \). Then,
\[
(i) \quad \omega_A \left[ \begin{pmatrix} T & S \\ S & T \end{pmatrix} \right] = \max \{ \omega_A(T + S), \omega_A(T - S) \},
\]
\[
(ii) \quad \omega_A \left[ \begin{pmatrix} T & -S \\ S & T \end{pmatrix} \right] = \max \{ \omega_A(T + iS), \omega_A(T - iS) \}.
\]

In order to prove Lemma 2.4 we need the following result.

**Lemma A.** ([10]) Let \( T \in B_{1/2}(H) \). Then,
\[
\omega_A(U T_A U) = \omega_A(T),
\]
for any \( A \)-unitary operator \( U \in B_A(H) \).

Now, we state the proof of Lemma 2.4.

**Proof of Lemma 2.4.** (i) Let \( U = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ -I & I \end{pmatrix} \). By using Lemma 2.3 (iii), we see that
\[
U T_A = \frac{1}{\sqrt{2}} \begin{pmatrix} P_{R(A)} & -P_{R(A)} \\ P_{R(A)} & P_{R(A)} \end{pmatrix}.
\]

So, by using the fact that \( A P_{R(A)} = A \), we can verify that \( \| U x \|_A = \| U T_A x \|_A = \| x \|_A \) for all \( x \in H \oplus H \). Hence \( U \) is \( A \)-unitary. So, by Lemma A we see that
\[
\omega_A \left[ \begin{pmatrix} T & S \\ S & T \end{pmatrix} \right] = \omega_A \left[ U T_A \begin{pmatrix} T & S \\ S & T \end{pmatrix} \right] = \omega_A \left[ \begin{pmatrix} P_{R(A)} & O \\ O & P_{R(A)} \end{pmatrix} \begin{pmatrix} T - S & O \\ O & T + S \end{pmatrix} \right] = \omega_A \left[ \begin{pmatrix} T - S & O \\ O & T + S \end{pmatrix} \right] = \max \{ \omega_A(T + S), \omega_A(T - S) \},
\]
where the last equality follows from Lemma 2.3 (i).
(ii) By considering the $A$-unitary operator $U = \frac{1}{\sqrt{2}} \begin{pmatrix} I & iI \\ iI & I \end{pmatrix}$ and proceeding as above we get the desired result. □

As an application of Theorem 2.1 together with Lemma 2.4, we state the following corollary.

**Corollary 2.1.** Let $P, Q \in \mathcal{B}_A(\mathcal{H})$. Then for all $\lambda \in [0, 1]$, we have

$$\omega_A(P \pm Q) \leq \min\{\mu, \nu\},$$

where

$$\mu = \omega_A(P) + \frac{1}{2} \left( \|P\|_A + \sqrt{\|\lambda^2 PP^A + QQ^A\|_A} + \sqrt{(1 - \lambda)^2 PP^A + QQ^A} \right),$$

and

$$\nu = \frac{3}{2} \omega_A(P) + \frac{1}{2} \left[ \sqrt{\lambda^2 \omega_A^2(P) + \|P\|^2_A} + \sqrt{(1 - \lambda)^2 \omega_A^2(P) + \|P\|^2_A} \right].$$

**Proof.** Recall first that it has been recently proved in [11] that

$$\omega_A \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \leq \frac{1}{2} \left[ \omega_A(P) + 2\omega_A(S) + \sqrt{\lambda^2 \omega_A^2(P) + \|P\|^2_A} + \sqrt{(1 - \lambda)^2 \omega_A^2(P) + \|R\|^2_A} \right].$$

So, we get the desired result by applying Theorem 2.1 together with (2.1) and Lemma 2.4 □

Now we state the following theorem which generalizes a recent result proved by Rout et al. in [29] since $\mathcal{B}_A(\mathcal{H})$ is in general a proper subspace of $\mathcal{B}_{A^{1/2}}(\mathcal{H})$.

**Theorem 2.2.** Let $T = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$ be such that $P, Q, R, S \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$. Then,

$$\omega_A \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \leq \max\{\omega_A(P), \omega_A(S)\} + \frac{\omega_A(Q + R) + \omega_A(Q - R)}{2}.$$  \hspace{1cm} (2.2)

**Proof.** Using triangle inequality we have

$$\omega_A \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \leq \omega_A \begin{pmatrix} P & Q \\ O & S \end{pmatrix} + \omega_A \begin{pmatrix} O & Q \\ R & O \end{pmatrix} = \max\{\omega_A(P), \omega_A(S)\} + \omega_A \begin{pmatrix} O & Q \\ R & O \end{pmatrix},$$

where

$$\omega_A \begin{pmatrix} O & Q \\ R & O \end{pmatrix} \leq \omega_A \begin{pmatrix} O & Q \\ R & O \end{pmatrix}.$$  \hspace{1cm} (2.3)
where the last equality follows by Lemma 2.3 (i). Let $\mathbb{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} I & -I \\ I & I \end{pmatrix}$. By proceeding as above we prove that $\mathbb{U}$ is $A$-unitary. So, by Lemma A we see that

$$\omega_A \left[ \begin{pmatrix} O & Q \\ R & O \end{pmatrix} \right] = \omega_A \left[ \mathbb{U}^{-1} \begin{pmatrix} O & Q \\ R & O \end{pmatrix} \mathbb{U} \right]$$

$$= \frac{1}{2} \omega_A \left[ \begin{pmatrix} P_{R(A)} & O \\ O & P_{R(A)} \end{pmatrix} \begin{pmatrix} R + Q & -R + Q \\ R - Q & -R - Q \end{pmatrix} \right]$$

$$= \frac{1}{2} \omega_A \left[ \begin{pmatrix} R + Q & -R + Q \\ R - Q & -R - Q \end{pmatrix} \right]$$

$$\leq \frac{1}{2} \omega_A \left[ \begin{pmatrix} R + Q & O \\ O & -R - Q \end{pmatrix} \right] + \frac{1}{2} \omega_A \left[ \begin{pmatrix} O & -R + Q \\ R - Q & O \end{pmatrix} \right]$$

$$= \frac{1}{2} \left( \omega_A(Q + R) + \omega_A(Q - R) \right),$$

where the last equality follows from Lemma 2.4. So,

$$\omega_A \left[ \begin{pmatrix} O & Q \\ R & O \end{pmatrix} \right] \leq \frac{1}{2} \left( \omega_A(Q + R) + \omega_A(Q - R) \right). \quad (2.4)$$

Combining (2.4) together with (2.3) yields to the desired result. \hfill \square

Our next objective is to present an improvement of the inequality (2.2). To do this, we need the following lemma.

**Lemma 2.5. ([24])** Let $T = [t_{ij}] \in M_n(\mathbb{C})$ be such that $t_{ij} \geq 0$ for all $i, j = 1, 2, \ldots, n$. Then

$$\omega(T) = \frac{1}{2} r([t_{ij} + t_{ji}]).$$

**Theorem 2.3.** Let $\mathbb{T} = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$ be such that $P, Q, R, S \in \mathcal{B}_A(\mathcal{H})$. Then

$$\omega_A \left[ \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \right]$$

$$\leq \frac{1}{2} \left( \omega_A(P) + \omega_A(S) + \sqrt{(\omega_A(P) - \omega_A(S))^2 + (\omega_A(Q + R) + \omega_A(Q - R))^2} \right).$$

Moreover, the inequality is sharper than the inequality (2.2).
Proof. It follows from [20, Theorem 2.3] that

\[ \omega_A(T) \leq \omega \begin{bmatrix} \omega_A(P) & \omega_A(Q) \\ O & R \end{bmatrix} = r \begin{bmatrix} \omega_A(P) & \omega_A(Q) \\ O & R \end{bmatrix} = \frac{1}{2} \left[ \omega_A(P) + \omega_A(S) + \sqrt{(\omega_A(P) - \omega_A(S))^2 + 4\omega_A^2(OQRO)} \right], \]

where the last equality follows from Lemma 2.5. So, by applying (2.4) we get

\[ \omega_A(\begin{bmatrix} P & Q \\ R & S \end{bmatrix}) \leq \frac{1}{2} \left[ \omega_A(P) + \omega_A(S) + \sqrt{(\omega_A(P) - \omega_A(S))^2 + (\omega_A(Q + R) + \omega_A(Q - R))^2} \right]. \]

This proves the desired inequality. Moreover, it can be observed that

\[
\max\{\omega_A(P), \omega_A(S)\} + \frac{\omega_A(Q + R) + \omega_A(Q - R)}{2} \\
= \frac{\omega_A(P) + \omega_A(S) + |\omega_A(P) - \omega_A(S)|}{2} + \frac{\omega_A(Q + R) + \omega_A(Q - R)}{2} \\
\geq \frac{1}{2} \left( \omega_A(P) + \omega_A(S) + \sqrt{(\omega_A(P) - \omega_A(S))^2 + (\omega_A(Q + R) + \omega_A(Q - R))^2} \right).
\]

Hence, the proof is complete. \qed

As a consequence of Theorem 2.3 we state the following corollary.

**Corollary 2.2.** Let \( P, Q, R, S \in B_A(H) \). Then,

\[
r_A(PQ + RS) \leq \frac{1}{2} \left[ \omega_A(QP) + \omega_A(SR) \right] + \frac{1}{2} \sqrt{[\omega_A(QP - \omega_A(SR))^2 + [\omega_A(QR + SP) + \omega_A(QR - SP)]^2}. 
\]

In order to prove Corollary 2.2, we need the following lemma.

**Lemma 2.6.** ([16]) If \( T \in B_{A^{1/2}}(H) \), then

\[
r_A(T) \leq \omega_A(T). 
\]

Now we are in a position to establish Corollary 2.2.
Proof of Corollary 2.2. It can be observed that
\[
A(PQ + RS) = r_A \begin{pmatrix} PQ + RS & O \\ O & O \end{pmatrix} \\
= r_A \begin{pmatrix} P & R \\ O & O \end{pmatrix} \begin{pmatrix} Q & O \\ S & O \end{pmatrix} \\
= r_A \begin{pmatrix} Q & O \\ S & O \end{pmatrix} \begin{pmatrix} P & R \\ O & O \end{pmatrix} \quad \text{(by (1.7))} \\
= r_A \begin{pmatrix} QP & QR \\ SP & SR \end{pmatrix} \\
\leq \omega A \begin{pmatrix} QP & QR \\ SP & SR \end{pmatrix} \quad \text{(by Lemma 2.6).}
\]

So, by applying Theorem 2.3 we reach the required result. □

Remark 2.1. Recently the first author proved in [17] that for \( P, Q, R, S \in \mathcal{B}_A(\mathcal{H}) \) it holds
\[
r_A(PQ + RS) \leq \frac{1}{2} [\omega_A(QP) + \omega_A(SR)] \\
+ \frac{1}{2} \sqrt{[\omega_A(QP) - \omega_A(SR)]^2 + 4 \|QR\|_A \|SP\|_A^2}. \tag{2.5}
\]

If \( QR = SP \), then it can be seen that the inequality in Corollary 2.2 is sharper than (2.5).

Remark 2.2. Notice that by letting \( Q = S = I \) in Corollary 2.2 we get
\[
r_A(P + R) \leq \frac{1}{2} [\omega_A(P) + \omega_A(R)] \\
+ \frac{1}{2} \sqrt{(\omega_A(P) - \omega_A(R))^2 + (\omega_A(P + R) + \omega_A(P - R))^2}.
\]

To establish further upper bounds for the \( A \)-numerical radius of the operator matrix \( \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \), we need the following lemmas.

Lemma 2.7. ([17]) Let \( \mathbb{T} = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \) be such that \( P, Q, R, S \in \mathcal{B}_{A/2}(\mathcal{H}) \). Then, \( \mathbb{T} \in \mathcal{B}_{A/2}(\mathcal{H} \oplus \mathcal{H}) \) and
\[
\|\mathbb{T}\|_A \leq \left\| \begin{pmatrix} \|P\|_A & \|Q\|_A \\ \|R\|_A & \|S\|_A \end{pmatrix} \right\|.
\]

Lemma 2.8. Let \( T, S \in \mathcal{B}_A(\mathcal{H}) \). Then,
\[
\|T^{2A}S\|_A = \|S^{2A}T\|_A.
\]
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Proof. By using the fact that $P_{\mathcal{R}(\mathcal{A})} A = A P_{\mathcal{R}(\mathcal{A})} = A$ together with (1.3), we see that

$$\|T^{\sharp A} S\|_A = \|S^{\sharp A} P_{\mathcal{R}(\mathcal{A})} T P_{\mathcal{R}(\mathcal{A})}\|_A$$

$$= \sup \left\{ |\langle A P_{\mathcal{R}(\mathcal{A})} x, (S^{\sharp A} P_{\mathcal{R}(\mathcal{A})} T)^{\sharp A} y \rangle| \mid x, y \in \mathcal{H}, \|x\|_A = \|y\|_A = 1 \right\}$$

$$= \sup \left\{ |\langle S^{\sharp A} P_{\mathcal{R}(\mathcal{A})} T x, y \rangle_A| \mid x, y \in \mathcal{H}, \|x\|_A = \|y\|_A = 1 \right\}$$

$$= \sup \left\{ |\langle A P_{\mathcal{R}(\mathcal{A})} T x, S y \rangle| \mid x, y \in \mathcal{H}, \|x\|_A = \|y\|_A = 1 \right\}$$

$$= \sup \left\{ |\langle S^{\sharp A} T x, y \rangle_A| \mid x, y \in \mathcal{H}, \|x\|_A = \|y\|_A = 1 \right\}$$

$$= \|S^{\sharp A} T\|_A.$$ 

This proves the desired result. \qed

Now, we are in a position to state the following theorem.

**Theorem 2.4.** Let $P, Q, R, S \in \mathcal{B}_A(\mathcal{H})$. Then,

$$\omega_{\mathcal{A}} \left[ \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \right] \leq \min\{\mu, \nu\},$$

where

$$\mu = \frac{\sqrt{2}}{2} \sqrt{\|P\|^2_A + \|Q\|^2_A + \sqrt{\|P\|^2_A - \|Q\|^2_A}^2 + 4\|P^{\sharp A} Q\|^2_A}$$

$$+ \frac{\sqrt{2}}{2} \sqrt{\|R\|^2_A + \|S\|^2_A + \sqrt{\|R\|^2_A - \|S\|^2_A}^2 + 4\|S^{\sharp A} R\|^2_A},$$

and

$$\nu = \frac{\sqrt{2}}{2} \sqrt{\|P\|^2_A + \|R\|^2_A + \sqrt{\|P\|^2_A - \|R\|^2_A}^2 + 4\|P R^{\sharp A}\|^2_A}$$

$$+ \frac{\sqrt{2}}{2} \sqrt{\|Q\|^2_A + \|S\|^2_A + \sqrt{\|Q\|^2_A - \|S\|^2_A}^2 + 4\|S Q^{\sharp A}\|^2_A}.$$ 

Proof. We first prove that

$$\omega_{\mathcal{A}} \left[ \begin{pmatrix} P & Q \\ O & O \end{pmatrix} \right] \leq \frac{\sqrt{2}}{2} \sqrt{\|P\|^2_A + \|Q\|^2_A + \sqrt{\|P\|^2_A - \|Q\|^2_A}^2 + 4\|P^{\sharp A} Q\|^2_A}. \quad (2.6)$$
By using (1.4) together with (1.1) we see that
\[
\omega_A \left[ \begin{pmatrix} P & Q \\ O & O \end{pmatrix} \right] \leq \left\| \begin{pmatrix} P & Q \\ O & O \end{pmatrix} \right\|_A
\]
\[
= \left\| \begin{pmatrix} P & Q \\ O & O \end{pmatrix} \right\|_A^{\frac{1}{2}}
\]
\[
= \left\| \begin{pmatrix} P^{\sharp A} & Q^{\sharp A} \\ O & O \end{pmatrix} \begin{pmatrix} P & Q \\ O & O \end{pmatrix} \right\|_A^{\frac{1}{2}}
\]
\[
= \left\| \begin{pmatrix} P^{\sharp A} & P^{\sharp A}Q \\ Q^{\sharp A}P & Q^{\sharp A}Q \end{pmatrix} \right\|_A
\]

So, by using Lemma 2.7 together with Lemma 2.8 we see that
\[
\omega_A^2 \left[ \begin{pmatrix} P & Q \\ O & O \end{pmatrix} \right] \leq \left\| \left( \begin{array}{cc} P^{\sharp A} & P \\ Q^{\sharp A}P & P^{\sharp A}Q \end{array} \right) \right\|_A
\]
\[
= \left( \frac{1}{2} \right) \left( \left\| P \right\|_A^2 + \left\| Q \right\|_A^2 + \sqrt{\left( \left\| P \right\|_A^2 - \left\| Q \right\|_A^2 \right)^2 + 4 \left\| P^{\sharp A}Q \right\|_A^2} \right)
\]

This proves (2.6). Let \( U = \begin{pmatrix} O & I \\ P & Q \end{pmatrix} \). In view of Lemma 2.3 (iii) we have \( U \in B_A(\mathcal{H} \oplus \mathcal{H}) \) and \( U^{\sharp A} = \begin{pmatrix} O & P_R & R \\ P_R^\perp & O & S \end{pmatrix} \). Further, it can be seen that \( U \) is \( \mathcal{A} \)-unitary operator. So, by using Lemma A together with (2.6) we get
\[
\omega_A \left[ \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \right] \leq \omega_A \left[ \begin{pmatrix} P & Q \\ O & O \end{pmatrix} \right] + \omega_A \left[ \begin{pmatrix} O & R \\ O & S \end{pmatrix} \right]
\]
\[
= \omega_A \left[ \begin{pmatrix} P & Q \\ O & O \end{pmatrix} \right] + \omega_A \left[ \begin{pmatrix} O & R \\ P_R & S \end{pmatrix} \right]
\]
\[
= \omega_A \left[ \begin{pmatrix} P & Q \\ O & O \end{pmatrix} \right] + \omega_A \left[ \begin{pmatrix} P_R & O \\ O & S \end{pmatrix} \right]
\]
\[
= \omega_A \left[ \begin{pmatrix} P & Q \\ O & O \end{pmatrix} \right] + \omega_A \left[ \begin{pmatrix} S & R \\ O & O \end{pmatrix} \right]
\]
\[
\leq \frac{\sqrt{2}}{2} \sqrt{\left\| P \right\|_A^2 + \left\| Q \right\|_A^2 + \sqrt{\left( \left\| P \right\|_A^2 - \left\| Q \right\|_A^2 \right)^2 + 4 \left\| P^{\sharp A}Q \right\|_A^2}}
\]
\[
+ \frac{\sqrt{2}}{2} \sqrt{\left\| R \right\|_A^2 + \left\| S \right\|_A^2 + \sqrt{\left( \left\| R \right\|_A^2 - \left\| S \right\|_A^2 \right)^2 + 4 \left\| S^{\sharp A}R \right\|_A^2}}.
\]
Now, since $\omega_A \left[ \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \right] = \omega_A \left[ \begin{pmatrix} P R^*_A & R S^*_A \\ Q R^*_A & S S^*_A \end{pmatrix} \right]$, then by using similar arguments as above we obtain

$$\omega_A \left[ \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \right] \leq \frac{\sqrt{2}}{2} \sqrt{\|P\|_A^2 + \|R\|_A^2 + \sqrt{(||P\|_A^2 - ||R\|_A^2)^2 + 4\|PR^*_A\|_A^2}}$$

$$+ \frac{\sqrt{2}}{2} \sqrt{\|Q\|_A^2 + \|S\|_A^2 + \sqrt{(||Q\|_A^2 - ||S\|_A^2)^2 + 4\|(S^*A)Q^*_A\|_A^2}}$$

Hence, the proof is complete. □

Our next result provides a lower bound for the $A$-numerical radius of a $2 \times 2$ operator matrix such that its second row consists of zero operators. We mention here that our result improves a recent result proved by Rout et al. in [29] since $\mathcal{B}_A(\mathcal{H}) \subseteq \mathcal{B}_{A^{1/2}}(\mathcal{H})$.

**Theorem 2.5.** Let $P, Q \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$. Then

$$\omega_A \left[ \begin{pmatrix} P & Q \\ O & O \end{pmatrix} \right] \geq \frac{1}{2} \max\{\alpha, \beta\},$$

(2.7)

where $\alpha = \omega_A(P + Q) + \omega_A(P - Q)$ and $\beta = \omega_A(P + iQ) + \omega_A(P - iQ)$.

**Proof.** Let $U = \begin{pmatrix} O & I \\ I & O \end{pmatrix}$. By using Lemma 2.3 (iii), we see that

$$U^{*A} = \begin{pmatrix} P_{R(A)} & O \\ O & P_{R(A)} \end{pmatrix} = \begin{pmatrix} O & P_{R(A)} \\ P_{R(A)} & O \end{pmatrix}. $$

So, by using the fact that $AP_{R(A)} = A$, we can verify that $\|Ux\|_A = \|U^{*A}x\|_A = \|x\|_A$ for all $x \in \mathcal{H} \oplus \mathcal{H}$. Hence $U$ is $A$-unitary. So, using Lemma 2.4 (i) we
observes that
\[
\max \{\omega_A(P + Q), \omega_A(P - Q)\} = \omega_A \left[ \begin{pmatrix} P & Q \\ Q & P \end{pmatrix} \right] \\
\leq \omega_A \left[ \begin{pmatrix} P & Q \\ O & O \end{pmatrix} \right] + \omega_A \left[ \begin{pmatrix} O & O \\ Q & P \end{pmatrix} \right] \\
= \omega_A \left[ \begin{pmatrix} P & Q \\ O & O \end{pmatrix} \right] + \omega_A \left[ \begin{pmatrix} P_{R(A)} & O \\ O & P_{R(A)} \end{pmatrix} \right] \left[ \begin{pmatrix} P & Q \\ O & O \end{pmatrix} \right] \\
= \omega_A \left[ \begin{pmatrix} P & Q \\ O & O \end{pmatrix} \right] + \omega_A \left[ \begin{pmatrix} P & Q \\ O & O \end{pmatrix} \right] \\
= 2\omega_A \left[ \begin{pmatrix} P & Q \\ O & O \end{pmatrix} \right].
\]

On the other hand, by considering the $A$-unitary operator $V = \begin{pmatrix} I & 0 \\ O & -I \end{pmatrix}$ and proceeding as above we see that
\[
\max \{\omega_A(P + iQ), \omega_A(P - iQ)\} = \omega_A \left[ \begin{pmatrix} P & -Q \\ Q & P \end{pmatrix} \right] \text{ (by Lemma 2.4 (ii))} \\
\leq 2\omega_A \left[ \begin{pmatrix} P & Q \\ O & O \end{pmatrix} \right].
\]

Hence we get our desired result. \qed

Now, in order to prove a lower bound for $\omega_A \left[ \begin{pmatrix} P & Q \\ O & O \end{pmatrix} \right]$, we need the following lemma.

**Lemma 2.9.** ([21]) Let $T, S \in B_A(H)$. Then,
\[
\omega_A(TS \pm ST^{\sharp_A}) \leq 2\|T\|_A \omega_A(S). \tag{2.8}
\]

In the next result, Lemma 2.4 enables us to present another application of the inequality (2.8).

**Theorem 2.6.** Let $P, Q \in B_A(H)$. Then
\[
\omega_A \left[ \begin{pmatrix} P & Q \\ O & O \end{pmatrix} \right] \geq \frac{1}{2} \max \{\omega_A(P + iQ), \omega_A(P - iQ)\}. \tag{2.9}
\]

*Proof.* Let $T = \begin{pmatrix} P & Q \\ O & O \end{pmatrix}$ and $U = \begin{pmatrix} O & I \\ I & -I \end{pmatrix}$. It is not difficult to verify that $U$ is $A$-unitary. Moreover, since $P_{R(A)}^{\sharp_A}X^{\sharp_A} = X^{\sharp_A}P_{R(A)}^{\sharp_A} = X^{\sharp_A}$ for all $X \in B_A(H)$ (see [22]), then it can be verified that
\[
U^{\sharp_A}T^{\sharp_A} + T^{\sharp_A}(U^{\sharp_A})^{\sharp_A} = \begin{pmatrix} Q^{\sharp_A} & P^{\sharp_A} \\ -P^{\sharp_A} & Q^{\sharp_A} \end{pmatrix} = \begin{pmatrix} Q & -P \end{pmatrix}^{\sharp_A}.
\]
By using Lemma 2.9 we see that
\[ \omega_A(U^\dagger A T^\dagger A + T^\dagger A U^\dagger A) \leq 2 \omega_A(T^\dagger A), \]
which, in turn, implies that
\[
\omega_A \left[ \begin{pmatrix} P & Q \\ O & O \end{pmatrix} \right] \geq \frac{1}{2} \omega_A \left[ \begin{pmatrix} Q & -P \\ P & Q \end{pmatrix} \right] = \frac{1}{2} \max \{ \omega_A(P + iQ), \omega_A(P - iQ) \},
\]
where the last equality follows by applying Lemma 2.4(ii).

As an application of the above theorem, we can derive the following \(A\)-numerical radius inequality.

**Theorem 2.7.** Let \(T \in \mathcal{B}_A(H)\). Then
\[
\omega_A(T) \leq 2 \min \left\{ \omega_A \left[ \begin{pmatrix} \Re A(T) & O \\ \Im A(T) & O \end{pmatrix} \right], \omega_A \left[ \begin{pmatrix} O & -i \Im A(T) \\ \Re A(T) & O \end{pmatrix} \right] \right\}.
\]

To prove Theorem 2.7, we need the following lemma.

**Lemma 2.10.** Let \(T, S \in \mathcal{B}_A(H)\). Then
\[
\omega_A(T \pm iS) \leq 2 \omega_A \left[ \begin{pmatrix} O & T \\ iS & O \end{pmatrix} \right]. \tag{2.10}
\]

**Proof.** Let \(X = \begin{pmatrix} I & I \\ O & O \end{pmatrix}\) and \(Y = \begin{pmatrix} O & T \\ S & O \end{pmatrix}\). It can be observed that
\[
XYX^\dagger A = \begin{pmatrix} TP_{\Re A} + SP_{\Re A} & O \\ O & O \end{pmatrix},
\]
and
\[
\|X\|_A^2 = \|XX^\dagger A\|_A = \left\| \begin{pmatrix} 2P_{\Re A} & O \\ O & O \end{pmatrix} \right\|_A = 2\|P_{\Re A}\|_A = 2.
\]

So, by using the fact that \(AP_{\Re A} = A\) we see that
\[
\omega_A(T + S) = \omega_A(TP_{\Re A} + SP_{\Re A})
\]
\[
= \omega_A \left[ \begin{pmatrix} TP_{\Re A} + SP_{\Re A} & O \\ O & O \end{pmatrix} \right]
\]
\[
= \omega_A(XYX^\dagger A) \quad \text{(by Lemma 2.3)}
\]
\[
\leq \|X\|_A^2 \omega_A(Y) \quad \text{(by [31, Lemma 4.4])}
\]
\[
= 2 \omega_A(Y).
\]
This gives
\[
\omega_A(T + S) \leq 2 \omega_A\left[ \begin{pmatrix} O & T \\ S & O \end{pmatrix} \right].
\] (2.11)

By replacing \( S \) by \(-S\) in (2.11) and then using the fact that \( \omega_A\left[ \begin{pmatrix} O & T \\ S & O \end{pmatrix} \right] = \omega_A\left[ \begin{pmatrix} O & T \\ -S & O \end{pmatrix} \right] \) we get
\[
\omega_A(T \pm S) \leq 2 \omega_A\left[ \begin{pmatrix} O & T \\ S & O \end{pmatrix} \right].
\] (2.12)

Finally, by replacing \( S \) by \( iS \) in (2.12), we reach the desired results. \( \square \)

Now we are ready to prove Theorem 2.7.

Proof of Theorem 2.7. Clearly \( T \) can written as \( T = \Re_A(T) + i\Im_A(T) \) where
\[
\Re_A(T) := \frac{T + T^{z_A}}{2} \quad \text{and} \quad \Im_A(T) := \frac{T - T^{z_A}}{2i}.
\]
So, \( T^{z_A} = [\Re_A(T)]^{z_A} - i[\Im_A(T)]^{z_A} \). Moreover, a short calculation reveals that
\( (T^{z_A})^{z_A} = [\Re_A(T)]^{z_A} + i[\Im_A(T)]^{z_A} \). So, by applying Theorem 2.6 we see that
\[
\omega_A\left[ \begin{pmatrix} \Re_A(T) & O \\ \Im_A(T) & O \end{pmatrix} \right] \\
= \omega_A\left[ \begin{pmatrix} [\Re_A(T)]^{z_A} & [\Im_A(T)]^{z_A} \\ O & O \end{pmatrix} \right] \\
\geq \frac{1}{2} \max \left\{ \omega_A\left( [\Re_A(T)]^{z_A} + i[\Im_A(T)]^{z_A} \right), \omega_A\left( [\Re_A(T)]^{z_A} - i[\Im_A(T)]^{z_A} \right) \right\} \\
= \frac{1}{2} \max \{ \omega_A(T^{z_A}), \omega_A((T^{z_A})^{z_A}) \} \\
= \frac{1}{2} \omega_A(T).
\]

On the other hand, by applying Lemma 2.10 we get
\[
\omega_A\left[ \begin{pmatrix} O & -i\Im_A(T) \\ \Re_A(T) & O \end{pmatrix} \right] \\
= \omega_A\left[ \begin{pmatrix} O & [\Re_A(T)]^{z_A} \\ i[\Im_A(T)]^{z_A} & O \end{pmatrix} \right] \\
\geq \frac{1}{2} \max \left\{ \omega_A\left( [\Re_A(T)]^{z_A} + i[\Im_A(T)]^{z_A} \right), \omega_A\left( [\Re_A(T)]^{z_A} - i[\Im_A(T)]^{z_A} \right) \right\} \\
= \frac{1}{2} \omega_A(T).
\]

Based on Lemma 2.4, the forthcoming theorem which provides a lower bound for \( \overline{A} \)-numerical radius of a \( 2 \times 2 \) operator matrix is also an application of the inequality (2.8).
Further inequalities for the $A$-numerical radius of certain $2 \times 2$ operator matrices

**Theorem 2.8.** Let $P, Q, R, S \in \mathcal{B}_A(\mathcal{H})$. Then

$$\omega_A \left[ \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \right] \geq \frac{1}{2} \max\{\mu, \nu\},$$

(2.13)

where

$$\mu = \max\{\omega_A(Q + R + S + P), \omega_A(Q + R - S - P)\},$$

and

$$\nu = \max\{\omega_A(Q - R + i(S + P)), \omega_A(Q - R - i(S + P))\}.$$ 

**Proof.** Let $U = \begin{pmatrix} O & I \\ I & O \end{pmatrix}$. By using Lemma 2.3 (iii), we see that

$$U^* A U = \begin{pmatrix} P_{R(A)} & O \\ O & P_{R(A)} \end{pmatrix} = \begin{pmatrix} O & P_{R(A)} \\ P_{R(A)} & O \end{pmatrix}. $$

So, by using the fact that $AP_{R(A)} = A$, we can verify that $\| U x \|_A = \| U^* A U x \|_A = \| x \|_A$ for all $x \in \mathcal{H} \oplus \mathcal{H}$. Hence $U$ is $A$-unitary. Moreover, clearly $(U^* A U)^* = U^* A U$.

Now, let $T = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$. Since, $P_{R(A)} X^* A = X^* A P_{R(A)} = X^* A$ for all $X \in \mathcal{B}_A(\mathcal{H})$ (see [22]), then a short calculation shows that

$$U^* A T U^* A + T U^* A U = \begin{pmatrix} Q^* A + R^* A & S^* A + P^* A \\ S^* A + P^* A & Q^* A + R^* A \end{pmatrix} = \begin{pmatrix} Q + R & S + P \\ S + P & Q + R \end{pmatrix}^* A.$$ 

By using Lemma 2.9, we see that

$$\omega_A(U^* A T U^* A + T U^* A U) \leq 2 \omega_A(T^* A),$$

which, in turn, implies that

$$\omega_A \left[ \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \right] \geq \frac{1}{2} \omega_A \left[ \begin{pmatrix} Q + R & S + P \\ S + P & Q + R \end{pmatrix} \right] \geq \frac{1}{2} \max\{\omega_A(Q + R + S + P), \omega_A(Q + R - S - P)\} := \frac{1}{2} \mu,$$

where the last equality follows by applying Lemma 2.4 (i). On the other hand, by choosing $U = \begin{pmatrix} O & I \\ -I & O \end{pmatrix}$ and proceeding as above we obtain

$$\omega_A \left[ \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \right] \geq \frac{1}{2} \omega_A \left[ \begin{pmatrix} Q - R & -S + P \\ S + P & Q - R \end{pmatrix} \right] \geq \frac{1}{2} \max\{\omega_A(Q - R + i(S + P)), \omega_A(Q - R - i(S + P))\} := \frac{1}{2} \nu,$$

where the last equality follows by applying Lemma 2.4 (ii). This completes the proof of the theorem. \qed


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[1a] University of Monastir, Faculty of Economic Sciences and Management of Mahdia, Mahdia, Tunisia

[1b] Laboratory Physics-Mathematics and Applications (LR/13/ES-22), Faculty of Sciences of Sfax, University of Sfax, Sfax, Tunisia

Email address: kais.feki@hotmail.com; kais.feki@fsegma.u-monastir.tn

[2] P.G. Department of Mathematics, Utkal University, Vanivihar, Bhubaneswar 751004, India.

Email address: satyajitsahoo2010@gmail.com