Almost Designs and Their Links with Balanced Incomplete Block Designs

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Abstract
Almost designs (t-adesigns) were proposed and discussed by Ding as a certain generalization of combinatorial designs related to almost difference sets. Unlike t-designs, t-adesigns need not also be (t−1)-designs. In this correspondence we discuss a special class of 3-adesigns, i.e., 3-adesigns which are also balanced incomplete block designs. We construct several classes of these, and discuss some of the restrictions on the sets of feasible parameters of such a class, as well as construct several new classes of 2-adesigns, and discuss some of their properties as well.

Key words and phrases: Combinatorial design, difference set, almost difference set, t-adesign.

Mathematics subject classifications: 05B10

1 Introduction
Combinatorial designs are an interesting subject of combinatorics closely related to finite geometry [2], [8], [20], [27], with applications in experiment design [4], [18], coding theory [1], [10], [22] and cryptography [6], [30], [32].

1.1 Finite incidence structures
A (finite) incidence structure is a triple (V, B, I) such that V is a finite set of elements called points, B is a finite set of elements called blocks, and I (⊆ V × B) is a symmetric binary relation between V and B. Since, in the sequel, all incidence structures (V, B, I) are such that B is a collection (i.e., a multiset) of nonempty subsets of V, and I is given by membership (i.e., a point p ∈ V and a block B ∈ B are incident if and only if p ∈ B), we will denote the incidence structure (V, B, I) simply by (V, B). An incidence structure that has no repeated blocks is called simple. All of the incidence structures

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discussed in the sequel are also assumed to be simple. A \( t-(v,k,\lambda) \) design (or \( t \)-design, for short) (with \( 0 < t < k < v \)) is an incidence structure \((V,\mathcal{B})\) where \( V \) is a set of \( v \) points and \( \mathcal{B} \) is a collection of \( k \)-subsets of \( V \) such that any \( t \)-subset of \( V \) is contained in exactly \( \lambda \) blocks \([2]\). In the literature, \( t \)-designs with \( t = 1 \) are sometimes referred to as tactical configurations, and those with \( t = 2 \) are sometimes referred to as balanced incomplete block designs. We will denote the number of blocks of an incidence structure by \( b \), and the number of blocks containing a given subset \( A \subseteq V \) of points by \( r^\mathcal{B}_A \) (when \( A \) is a singleton, and \( (V,\mathcal{B}) \) is a tactical configuration, simply by \( r^\mathcal{B} \)). Then the identities

\[
bk = vr^\mathcal{B},
\]

and

\[
r^\mathcal{B}(k-1) = (v-1)\lambda
\]

restrict the possible sets of parameters of \( t \)-designs. A \( t \)-design in which \( b = v \) and \( r^\mathcal{B} = k \) is called symmetric. The dual \((V,\mathcal{B})^\perp\) of the incidence structure \((V,\mathcal{B})\) is the incidence structure \((\mathcal{B},V)\) with the roles of points and blocks interchanged. A symmetric incidence structure has the same parameters as its dual.

### 1.2 Difference sets, partial difference sets and almost difference sets

One important way of obtaining symmetric balanced incomplete block designs is by constructing difference sets, another extensively studied combinatorial object \([2],[32]\). Let \( G \) be a group of order \( v \), and let \( k \) and \( \lambda \) be integers satisfying \( 2 \leq k < v \). A \((v,k,\lambda)\) difference set in \( G \) is a \( k \)-subset \( D \subseteq G \) such that the multiset \(* xy^{-1} | x, y \in D, x \neq y *\) contains every nonidentity member of \( G \) exactly \( \lambda \) times.

Skew Hadamard difference sets are especially relevant to this correspondence. A difference set \( D \) in a group \( G \) is called skew Hadamard if \( G \) is the disjoint union of \( D, D^{-1} \) and \( \{0\} \), where \( D^{-1} = \{d^{-1} | d \in D\} \). Skew Hadamard difference sets must (up to complementation) have parameters \((4l-1,2l-1,l-1)\) for some nonegative integer \( l \). Constructions of skew Hadamard difference sets can be found in \([12],[13],[15],[16],[17],[26],[28],[35]\). For a good survey on skew Hadamard difference sets the reader is referred to \([17]\).

Almost difference sets are a generalization of difference sets. In the literature there are two different definitions of almost difference sets \([7],[9]\). The following unification was given in \([11]\). A \((v,k,\lambda,s)\) almost difference set in \( G \) is a \( k \)-subset \( D \subseteq G \) such that the multiset \(* xy^{-1} | x, y \in D, x \neq y *\) contains \( s \) nonidentity members of \( G \) that appear \( \lambda \) times, and \( v-1-s \) nonidentity members that appear \( \lambda + 1 \) times.

A difference set can be viewed as an almost difference set with \( s = 0 \) or \( s = v-1 \). The complement \( G \setminus D \) of a \((v,k,\lambda,s)\) almost difference set is a \((v, v-k, v-2k+\lambda, s)\) almost difference set. A simple restriction which can be applied to the parameters of almost difference sets is that \((v-1)(\lambda+1)-s = k(k-1)\) must hold for any \((v,k,\lambda,s)\) almost difference set.

Difference sets and almost difference sets also have extensive applications in various fields such as communications, sequence design, error correcting codes, and CDMA and cryptography \([6],[10],[11],[19],[31]\). For a good survey on almost difference sets, the reader is referred to \([10]\) or \([29]\).
Another related combinatorial object, called partial difference sets, is also relevant to this correspondence. A $k$-subset $D$ of a group $G$ (with identity $Id$) is called a $(v, k, \lambda, \mu)$ partial difference set if each nonidentity member of $D$ occurs in the multiset $\{xy^{-1} | x, y \in D, x \neq y \}$ with multiplicity $\lambda$, and each nonidentity member of $G \setminus D$ occurs with multiplicity $\mu$. Thus, any partial difference set with $|\mu - \lambda| = 1$ is an almost difference set. Moreover, we say that $D$ is regular if, in addition, $Id \notin D$ and $D = D^{-1}$ (i.e., $D$ is reversible). Partial difference sets have been extensively studied and are related to other combinatorial objects such as Schur Rings, two-weight codes, strongly regular graphs and partial geometries. For further reading on partial difference sets, the reader is referred to [5] and [24].

1.3 Adesigns

Recent interest in almost difference sets and their codes is the main motivation for studying adesigns (almost designs). Let $V$ be a $v$-set and $B$ a collection of subsets of $V$, called blocks, each having cardinality $k$. If there is a positive integer $\lambda$ such that every $t$-subset of $V$ is incident with either $\lambda$ blocks or with $\lambda + 1$ blocks, and $(V, B)$ is not a $t$-design, then $(V, B)$ is called a $t$-$(v, k, \lambda)$ adesign (or $t$-adesign for short). We have the following lemma concerning the relation between almost difference sets and adesigns. The relation is analogous to that between difference sets and 2-designs. The proof is omitted as it is a simple counting exercise.

Lemma 1.1. Let $D$ be a $(v, k, \lambda, t)$ almost difference set in an abelian group $G$. Then $(G, \text{Dev}(D))$ is a $2$-$(v, k, \lambda)$ adesign. Moreover,

$$r_{\{x, y\}}^{\text{Dev}(D)} = \begin{cases} 
\lambda, & \text{if } d_D(x - y) = \lambda, \\
\lambda + 1, & \text{otherwise},
\end{cases}$$

for all distinct $x, y \in G$.

Adesigns were first coined by Ding [10], and several constructions of adesigns and their applications were further investigated in [25] and, indirectly in [14] and [34], as it was shown in [25] that almost difference families give 2-adesigns. In this correspondence we will study a special class of 3-adesigns, i.e., 3-adesigns which are also balanced incomplete block designs. We give several constructions of such 3-adesigns and we discuss some restrictions on their parameters as well as their links to some other combinatorial objects such as $\lambda$-coverings. Moreover, we construct several new families of 2-adesigns and discuss some of the restrictions on their parameters.

The remainder of this paper is organized as follows. In Section 2 we make an initial investigation into when a $(t+1)$-adesign is a $t$-design or a $t$-adesign. In Section 3 we give three generic constructions of 3-adesigns which are balanced incomplete block designs and, furthermore, we discuss the natural question of when a $(t+1)$-adesign is a $t$-design or $t$-adesign. In Section 4 we give some new constructions of 2-adesigns and we discuss some of the restrictions on the parameters of 2-adesigns. Section 5 concludes the paper.
2 A note on the parameter sets of \((t+1)\)-adesigns which are either \(t\)-designs or \(t\)-adesigns

It is well-known that \((t+1)\)-designs are always \(t\)-designs (see [2]); however, a \((t+1)\)-adesign needs not always be a \(t\)-design nor a \(t\)-adesign. In this section we make an investigation into when a \((t+1)\)-adesign is a \(t\)-design, or a \(t\)-adesign, by eliminating some of the possible parameter sets.

Suppose \((V, B)\) is a \((t+1)-(v, k, \lambda')\) adesign with \(b\) blocks. Let \(r_Y\) denote the number of blocks containing the \(t\)-set \(Y \subseteq V\), and define

\[
I_Y = \{(z, B) \mid z \in V \setminus Y \text{ and } Y \cup \{z\} \subseteq B \in B\}.
\]

We will count \(|I_Y|\) in two ways. There are \(v-t\) ways to choose \(z\), and since \((V, B)\) is a \((t+1)\)-adesign, neither \(|I_Y| = \lambda'(v-t)\) nor \(|I_Y| = (\lambda'+1)(v-t)\) can hold for all \(t\)-subsets \(Y \subseteq V\), otherwise \((V, B)\) would be a \((t+1)\)-design. Thus \(\lambda'(v-t) \leq |I_Y| \leq (\lambda'+1)(v-t)\). We also have \(r_Y\) ways to choose a block \(B\) containing \(Y\), and for each choice of \(B\), there are \(k-t\) ways to choose \(z\). This gives us \(\lambda'(v-t) \leq r_Y(k-t) \leq (\lambda'+1)(v-t)\) for all possible \(t\)-subsets \(Y \subseteq V\), whence

\[
\lambda' \leq r_Y \frac{k-t}{v-t} \leq \lambda'+1. \quad (1)
\]

Notice that if \(\frac{v-t}{v-k} < 2\), then

\[
\left[ \lambda' \frac{v-t}{k-t} \right] \leq r_Y \leq \left[ \lambda' \frac{v-t}{k-t} + \frac{v-t}{k-t} \right] < \left[ \lambda' \frac{v-t}{k-t} \right] + 2,
\]

so that \((V, B)\) is either a \(t-(v, k, \lambda)\) adesign with \(\lambda = r_Y\) or \(r_Y - 1\), or a \(t-(v, k, \lambda)\) design with \(\lambda = r_Y\). Also notice that if \((V, B)\) is in fact a \(t\)-design, then by [11] we must have \(\lambda' \frac{k-t}{v-t} - 1 < \lambda' < \lambda' \frac{k-t}{v-t}\) so that \(\lambda' = \lfloor \lambda' \frac{k-t}{v-t} \rfloor\) and, moreover, we have \(\lambda' \frac{v}{t+1}/\binom{v}{t+1} < \lambda' \frac{k}{t+1}/\binom{k}{t+1} < (\lambda' + 1) \frac{v}{t+1}/\binom{v}{t+1} < (\lambda' + 1) \frac{k}{t+1}/\binom{k}{t+1}\) from which it follows that

\[
\lambda' \binom{v}{t+1}/\binom{k}{t+1} < b < (\lambda' + 1) \binom{v}{t+1}/\binom{k}{t+1}.
\]

We have thus shown the following.

**Lemma 2.1.** Let \((V, B)\) be a \((t+1)-(v, k, \lambda')\) adesign with \(b\) blocks. Then for any \(t\)-subset \(Y \subseteq V\) we have \(\lambda' \leq r_Y \frac{k-t}{v-t} \leq \lambda' + 1\). If \(\frac{k-t}{v-t} > \frac{1}{2}\) then \((V, B)\) is either a \(t-(v, k, \lambda)\) adesign with \(\lambda = r_Y - 1\) or \(r_Y\), or a \(t-(v, k, \lambda)\) design with \(\lambda = r_Y\). Moreover, if \((V, B)\) is a \(t-(v, k, \lambda)\) design, then \(\lambda' = \lfloor \lambda' \frac{k-t}{v-t} \rfloor\) and \(\lambda' \binom{v}{t+1}/\binom{k}{t+1} < b < (\lambda' + 1) \binom{v}{t+1}/\binom{k}{t+1}\).

We will now give some constructions of 3-adesigns which are also balanced incomplete block designs.

3 Constructions of 3-adesigns which are balanced incomplete block designs

All of the constructions in this section follow the same basic method. We take two distinct difference sets (or almost difference sets), \(D_1\) and \(D_2\), over the same group \(G\), and consider the union of their
developments \(\text{Dev}(D_0) \cup \text{Dev}(D_1)\). We begin with a construction based on the twin prime almost difference sets which were discussed in [38].

**Theorem 3.1.** Let \(q\) be a prime power. Let \(G = (\mathbb{F}_q, +) \times (\mathbb{F}_q, +)\), and define \(D = \{(a, b) \in G \mid a \text{ and } b \text{ are both squares or both nonsquares}\}\), and \(\tilde{D} = \{(a, b) \in G \mid \text{one of } a, b \text{ is square and the other is nonsquare}\}\).

Denote \(\text{Dev}(D \cup \mathbb{F}_q \times \{0\})\) by \(B_0\) and \(\text{Dev}(\tilde{D} \cup \{0\} \times \mathbb{F}_q)\) by \(B_1\). Then \((G, B_0 \cup B_1)\) is a \(2-(q^2, \frac{q^2+1}{2}, \frac{q^2-1}{2})\) design and a \(3-(q^2, \frac{q^2+1}{2}, \frac{q^2-1}{2})\) adesign.

**Proof.** Denote \(\frac{q^2+1}{2}\) by \(k\). It was shown in [38] (also apply Lemma 1.1) that, for two distinct elements \(x, y \in G\),

\[
\begin{align*}
    r_{B_0}^{\mathbb{F}_q, \{x, y\}}(x, y) &= \begin{cases} 
        \frac{q^2+3}{4}, & \text{if } x - y \in D \cup \mathbb{F}_q \times \{0\} \\
        \frac{q^2-1}{4}, & \text{if } x - y \in \tilde{D} \cup \{0\} \times \mathbb{F}_q
    \end{cases}, \\
    r_{B_1}^{\mathbb{F}_q, \{x, y\}}(x, y) &= \begin{cases} 
        \frac{q^2+1}{4}, & \text{if } x - y \in D \cup \mathbb{F}_q \times \{0\} \\
        \frac{q^2+3}{4}, & \text{if } x - y \in \tilde{D} \cup \{0\} \times \mathbb{F}_q
    \end{cases}.
\end{align*}
\]

(2)

To show that \((G, B_0 \cup B_1)\) is a 2-design, we need to compute \(r_{B_0}^{\mathbb{F}_q, \{x, y\}}(x, y)\), which we have done in the appendix since it requires arguments very similar to those used in [38]. By Lemma 5.1 (see appendix), for two distinct elements \(x, y \in G\),

\[
\begin{align*}
    r_{B_0}^{\mathbb{F}_q, \{x, y\}}(x, y) &= \begin{cases} 
        \frac{q^2+3}{4}, & \text{if } x - y \in D \cup \mathbb{F}_q \times \{0\} \\
        \frac{q^2-1}{4}, & \text{if } x - y \in \tilde{D} \cup \{0\} \times \mathbb{F}_q
    \end{cases}, \\
    r_{B_1}^{\mathbb{F}_q, \{x, y\}}(x, y) &= \begin{cases} 
        \frac{q^2+1}{4}, & \text{if } x - y \in D \cup \mathbb{F}_q \times \{0\} \\
        \frac{q^2+3}{4}, & \text{if } x - y \in \tilde{D} \cup \{0\} \times \mathbb{F}_q
    \end{cases}.
\end{align*}
\]

By considering the values \(r_{B_0}^{\mathbb{F}_q, \{x, y\}}(x, y) + r_{B_1}^{\mathbb{F}_q, \{x, y\}}(x, y)\), it is now clear that \((G, B_0 \cup B_1)\) is a \(2-(q^2, \frac{q^2+1}{2}, \frac{q^2-1}{2})\) design. It remains to show that \((G, B_0 \cup B_1)\) is a \(3-(q^2, \frac{q^2+1}{2}, \frac{q^2-1}{2})\) adesign. Now let \(x = (x_1, x_2)\), \(y = (y_1, y_2)\) and \(z = (z_1, z_2)\) be three distinct members of \(B_0\). Suppose that \(x, y, z\) appear together in \(\mu\) blocks of \(B_0\). Notice that if \((a, b) \in D\) then \(-a\) and \(-b\) are either both square or both nonsquare so that \(-a, b) \in D\), i.e., \(D\) is a reversible subset of \(G\). The same can be said for \(\tilde{D}\). We will repeatedly use the fact that \(D \cup \mathbb{F}_q \times \{0\}\) and \(\tilde{D} \cup \{0\} \times \mathbb{F}_q\) are both reversible subsets of \(G\) without making reference to it.

We want to count the number of blocks of \(B_0 \cup B_1\) containing \(x, y, z\) by imposing restrictions on the three points \(x - y, x - z\) and \(y - z\). For the convenience of the reader, we have listed the 20 representative possibilities for the distribution of the three points \(x - y, x - z\) and \(y - z\) (comprising the cases which need to be checked) in Table 3 along with the number of blocks (either \(\frac{q^2-1}{4}\) or \(\frac{q^2+3}{4}\)) in \(B_0 \cup B_1\) containing \(x, y, z\); the cases listed with an asterisk (*) are readily seen to be impossible, and so are omitted. In fact, all of the cases are very similar, and so we will only show a select few of them, and the rest are omitted.

**Case 1** Assume \(x - y, x - z, y - z \in D\). By (2), and the principle of inclusion and exclusion, we have that \(x, y, z\) appear together in \(q^2 - 3k + 3\frac{q^2+3}{4} - \mu\) blocks in \(\text{Dev}(D \cup \mathbb{F}_q \times \{0\}) = \text{Dev}(\tilde{D} \cup \{0\} \times \mathbb{F}_q)\). Suppose that \(x, y, z \in D \cup \mathbb{F}_q \times \{0\}\). Then \(x - y, z - y \in D \cup \mathbb{F}_q \times \{0\}\). But since \(y_1 - x_1 \neq 0 \neq z_1 - x_1\), this gives us that \(y - x, z - y \in \tilde{D}\), a contradiction. By a similar argument,
Table 1: Possible distributions for the points \(x, y, z\) and \(y - z\).

| case no. | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8* | 9* | 10* | 11 | 12* | 13* | 15* | 16* | 17 | 18 | 19 | 20 |
|----------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| \(D\)   | \(\alpha\) | \(\beta\) | \(\gamma\) | \(\alpha\) | \(\beta\) | \(\gamma\) | \(\alpha\) | \(\beta\) | \(\gamma\) | \(\alpha\) | \(\beta\) | \(\gamma\) | \(\alpha\) | \(\beta\) | \(\gamma\) | \(\alpha\) | \(\beta\) | \(\gamma\) | \(\alpha\) | \(\beta\) | \(\gamma\) |
| \(F_q^* \times \{0\}\) | \(\alpha\) | \(\beta\) | \(\gamma\) | \(\alpha\) | \(\beta\) | \(\gamma\) | \(\alpha\) | \(\beta\) | \(\gamma\) | \(\alpha\) | \(\beta\) | \(\gamma\) | \(\alpha\) | \(\beta\) | \(\gamma\) | \(\alpha\) | \(\beta\) | \(\gamma\) | \(\alpha\) | \(\beta\) | \(\gamma\) |
| \(\hat{D}\) | \(\alpha\) | \(\beta\) | \(\gamma\) | \(\alpha\) | \(\beta\) | \(\gamma\) | \(\alpha\) | \(\beta\) | \(\gamma\) | \(\alpha\) | \(\beta\) | \(\gamma\) | \(\alpha\) | \(\beta\) | \(\gamma\) | \(\alpha\) | \(\beta\) | \(\gamma\) | \(\alpha\) | \(\beta\) | \(\gamma\) |
| \(\{0\} \times F_q^*\) | \(\alpha\) | \(\beta\) | \(\gamma\) | \(\alpha\) | \(\beta\) | \(\gamma\) | \(\alpha\) | \(\beta\) | \(\gamma\) | \(\alpha\) | \(\beta\) | \(\gamma\) | \(\alpha\) | \(\beta\) | \(\gamma\) | \(\alpha\) | \(\beta\) | \(\gamma\) | \(\alpha\) | \(\beta\) | \(\gamma\) |
| \(r_{B_0 \cup B_1}^{B_0 \cup B_1}(x,y,z)\) | \(\lambda_2\) | \(\lambda_2\) | \(\lambda_2\) | \(\lambda_2\) | \(\lambda_1\) | \(\lambda_1\) | \(\lambda_1\) | \(\lambda_1\) | \(\lambda_2\) | \(\lambda_2\) | \(\lambda_2\) | \(\lambda_2\) | \(\lambda_1\) | \(\lambda_1\) | \(\lambda_1\) | \(\lambda_1\) | \(\lambda_1\) | \(\lambda_1\) | \(\lambda_1\) | \(\lambda_1\) |

Note: We denote \(x - y, z - z\) and \(y - z\), respectively, by \(\alpha, \beta\) and \(\gamma\); cases which are clearly impossible are given with an asterisk (*), and we let \(\lambda_1\) resp. \(\lambda_2\) denote \(\frac{2^2 + 1}{4}\) resp. \(\frac{2^2 + 3}{4}\).

**Case 5** Assume \(x - y, x - z \in D\) and \(y - z \in F_q^* \times \{0\}\). By (2), and the principle of inclusion and exclusion, we have that \(x, y\) and \(z\) appear together in \(q^2 - 3k + 3\frac{q^2 + 3}{4} - \mu\) blocks in \(\text{Dev}(D \cup F_q \times \{0\})\).

If any one of \(D \cup F_q \times \{0\} \cup \{0\} + x, D \cup F_q \times \{0\} \cup \{0\} + y\) or \(D \cup F_q \times \{0\} \cup \{0\} + z\) contains all three of \(x, y\) and \(z\) then we will have two of \(x - y, x - z\) and \(y - z\) contained in \(D \cup \{0\} \times F_q^*\) which is a contradiction. Thus the number of blocks in \(B_0 \cup B_1\) containing \(x, y\) and \(z\) is \(q^2 - 3k + 3\frac{q^2 + 3}{4} = \frac{q^2 + 3}{4}\).

**Case 6** Assume \(x - y, x - z \in \hat{D}\) and \(y - z \in \hat{D}\). By (2), and the principle of inclusion and exclusion, we have that \(x, y\) and \(z\) appear together in \(q^2 - 3k + 2\frac{q^2 + 3}{4} + \frac{q^2 - 1}{4} - \mu\) blocks in \(\text{Dev}(\hat{D} \cup F_q \times \{0\})\).

If any one of \(D \cup F_q \times \{0\} \cup \{0\} + x, D \cup F_q \times \{0\} \cup \{0\} + y\) or \(D \cup F_q \times \{0\} \cup \{0\} + z\) contains all three of \(x, y\) and \(z\) then we will have two of \(x - y, x - z\) and \(y - z\) contained in \(\hat{D}\) which is a contradiction. Thus the number of blocks in \(B_0 \cup B_1\) containing \(x, y\) and \(z\) is \(q^2 - 3k + 2\frac{q^2 + 3}{4} + \frac{q^2 - 1}{4} = \frac{q^2 - 1}{4}\).

**Case 19** Assume \(x - y \in D, x - z \in \hat{D}\) and \(y - z \in \{0\} \times F_q^*\). By (2), and the principle of inclusion and exclusion, we have that \(x, y\) and \(z\) appear together in \(q^2 - 3k + 2\frac{q^2 - 1}{4} + 2\frac{q^2 + 3}{4} - \mu\) blocks in \(\text{Dev}(D \cup F_q \times \{0\}) = \text{Dev}(\hat{D} \cup \{0\} \times F_q^*)\). The only one of \(D \cup F_q \times \{0\} \cup \{0\} + x, D \cup F_q \times \{0\} \cup \{0\} + y\) and \(D \cup F_q \times \{0\} \cup \{0\} + z\) containing the three points \(x, y\) and \(z\) is \(D \cup F_q \times \{0\} \cup \{0\} + z\), since \(x - z, y - z \in D \cup F_q \times \{0\}\). Thus, the number of blocks in \(B_0 \cup B_1\) containing \(x, y\) and \(z\) is \(q^2 - 3k + 2\frac{q^2 - 1}{4} + 2\frac{q^2 + 3}{4} = \frac{q^2 + 3}{4}\).

The remaining cases are similar in the sense that they all involve an application of (2) and the principle of inclusion and exclusion, followed by an argument about which of \(D \cup F_q \times \{0\} \cup \{0\} + x, D \cup F_q \times \{0\} \cup \{0\} + y\) and \(D \cup F_q \times \{0\} \cup \{0\} + z\) contain all three of \(x, y\) and \(z\).
Example 3.2. Let \( q = 5 \), and let \( D \) and \( \tilde{D} \) be defined as in Theorem 3.1. Then

\[
D = \{(2, 3), (1, 4), (3, 2), (4, 1), (3, 3), (1, 1), (4, 4), (2, 2)\}
\]

and

\[
\tilde{D} = \{(4, 3), (1, 2), (1, 3), (3, 1), (2, 1), (2, 4), (3, 4), (4, 2)\},
\]

and the incidence structure \((\mathbb{F}_5 \times \mathbb{F}_5, \text{Dev}(D \cup \mathbb{F}_5 \times \{0\}) \cup \text{Dev}(\tilde{D} \cup \{0\} \times \mathbb{F}_5))\) is a 2-(25, 13, 12) design with 50 blocks and the property that each 3-subset of \( \mathbb{F}_5 \times \mathbb{F}_5 \) appears in either six or seven blocks.

Our next construction involves partial difference sets. The following is a well-known characterization of regular partial difference sets.

Lemma 3.3. \[24\] Let \( G \) be an abelian group of order \( v \) and let \( D \) be a nontrivial regular \((v, k, \lambda, \lambda + 1)\)-partial difference set. Then, up to complementation, one of the following occurs:

1. \( (v, k\lambda, \lambda + 1) = (v, \frac{v-1}{2}, \frac{v^5-1}{4}, \frac{v^2-1}{4}) \) where \( v \equiv 1 \pmod{4} \), or
2. \( (v, k, \lambda, \lambda + 1) = (243, 22, 1, 2) \).

We have the following construction.

Theorem 3.4. Let \( D \subset G \setminus \{0\} \) be a regular \((v, \frac{v-1}{2}, \frac{v^5-1}{4}, \frac{v^2-1}{4})\) partial difference set in a group \((G, +)\). Then \((G, \text{Dev}(D) \cup \text{Dev}(\overline{D} \setminus \{0\}))\) is a 2-\((v, \frac{v-1}{2}, \frac{v^3-1}{2})\) design and a 3-\((v, \frac{v-1}{2}, \frac{v^2-1}{4})\) adesign.

Proof. Denote \( \frac{v-1}{2} \) by \( k \) and \( \frac{v^5-1}{4} \) by \( \lambda \). Let \( x, y \) and \( z \) be distinct members of \( G \). By Lemma 1.1 we have that

\[
r_{\{x,y\}}^{\text{Dev}(D')} = \begin{cases} 
\lambda, & \text{if } x - y \in D', \\
\lambda + 1, & \text{otherwise}.
\end{cases}
\]

(3)

for all distinct \( x, y \in G \), where \( D' = D \) or \( \overline{D} \setminus \{0\} \), whence \((G, \text{Dev}(D) \cup \text{Dev}(\overline{D} \setminus \{0\}))\) is a 2-\((v, \frac{v-1}{2}, \frac{v^3-1}{2})\) design. We will now show that \((G, \text{Dev}(D) \cup \text{Dev}(\overline{D} \setminus \{0\}))\) is a 3-\((v, \frac{v-1}{2}, \frac{v^2-1}{4})\) adesign. To count the number of blocks of \( \text{Dev}(D) \cup \text{Dev}(\overline{D} \setminus \{0\}) \) in which \( x, y, z \) appear together, we first count the number of blocks of \( \text{Dev}(D) \cup \text{Dev}(\overline{D}) \) in which \( x, y, z \) appear together. Suppose the points \( x, y \) and \( z \) appear together in \( \mu \) blocks of \( \text{Dev}(D) \). We will consider cases based on the distribution of the points \( x - y, x - z \) and \( y - z \) in \( G \). Again, we will repeatedly use the fact that \( D \) and \( \overline{D} \setminus \{0\} \) are reversible without making reference to it.

Case 1: Assume \( x - y, x - z, y - z \in D \). By (3), and the principle of inclusion and exclusion, there are \( v - 3k + 3\lambda - \mu \) blocks in \( \text{Dev}(\overline{D}) \) containing \( x, y \) and \( z \). Thus, there are \( v - 3k + 3\lambda \) blocks in \( \text{Dev}(D) \cup \text{Dev}(\overline{D}) \) containing \( x, y \) and \( z \). We want to know how many of these are also contained in \( \{\overline{D} + x, \overline{D} + y, \overline{D} + z\} \). If we suppose that any one of \( \overline{D} + x, \overline{D} + y \) or \( \overline{D} + z \) contains all of \( x, y \) and \( z \), then we will have two of \( x - y, x - z \) and \( y - z \) contained in \( \overline{D} \), a contradiction. Thus, in this case, \( x, y \) and \( z \) appear together in exactly \( v - 3k + 3\lambda = \lambda - 1 \) blocks of \( \text{Dev}(D) \cup \text{Dev}(\overline{D} \setminus \{0\}) \).

Case 2: Assume \( x - y, x - z \in D \) and \( y - z \in \overline{D} \setminus \{0\} \). By (3), and the principle of inclusion and exclusion, there are \( v - 3k + 3\lambda + 1 - \mu \) blocks in \( \text{Dev}(\overline{D}) \) containing \( x, y \) and \( z \). As in the last case, if
we suppose that any one of $D + x, D + y$ or $D + z$ contains all of $x, y$ and $z$, then we will have two of $x - y, x - z$ and $y - z$ contained in $D$, a contradiction. Thus, in this case, $x, y$ and $z$ appear together in exactly $v - 3k + 3\lambda + 1 = \lambda$ blocks of $Dev(D) \cup Dev(D \setminus \{0\})$. 

**Case 3:** Assume $x - y \in D$ and $x - z, y - z \in D \setminus \{0\}$. By (3), and the principle of inclusion and exclusion, there are $v - 3k + 3\lambda + 2 - \mu$ blocks in $Dev(D)$ containing $x, y$ and $z$. The only one of $D + x, D + y$ or $D + z$ containing all of $x, y$ and $z$ is $D + z$ since $x - z, y - z \in D \setminus \{0\}$. Thus, in this case, $x, y$ and $z$ appear together in exactly $v - 3k + 3\lambda + 1 = \lambda$ blocks of $Dev(D) \cup Dev(D \setminus \{0\})$.

The remaining case, where $x - y, x - z, y - z \in D \setminus \{0\}$, is similar to those just shown, and is omitted. 

There is also the complementary construction. Its proof is similar and so it is omitted.

**Corollary 3.5.** Let $D \subseteq G \setminus \{0\}$ be a regular $(v, \frac{v-1}{2}, \frac{v-5}{4}, \frac{v-1}{4})$ partial difference set in a group $(G, +)$. Then $(G, Dev(D \cup \{0\}) \cup Dev(D))$ is a $2-(v, \frac{v+1}{2}, \frac{v-3}{2})$ design and a $3-(v, \frac{v+1}{2}, \frac{v-1}{4})$ adesign.

**Example 3.6.** Let $q \equiv 1 \pmod{4}$ be a prime power. It is well-known [24] that the quadratic residues modulo $q$, $D_0^{(2,q)}$, form a regular partial difference set. Then by Theorem 3.4 $(F_q, Dev(D_0^{(2,q)}) \cup Dev(D_1^{(2,q)}))$ is a $2-(v, \frac{v-1}{2}, \frac{v-3}{2})$ design with $2q$ blocks and the property that at each 3-subset of $F_q$ are contained in either $\frac{q^2-9}{4}$ or $\frac{q^2-5}{4}$ blocks.

**Example 3.7.** Let $q$ be an odd prime power, and let $I \subseteq \{0, 1, ..., q\}$ with $|I| = \frac{q-1}{2}$ and define $D = \cup_{i \in I} D_i^{(q+1, q^2)}$. It was shown in [5] that $D$ is a regular partial difference set. Then by Theorem 3.4 $(F_q, Dev(D) \cup Dev(D \setminus \{0\}))$ is a $2-(q^2, \frac{q^2-1}{2}, \frac{q^2-3}{2})$ design with $2q^2$ blocks and the property that each 3-subset of $F_{q^2}$ is contained in either $\frac{q^2-9}{4}$ or $\frac{q^2-5}{4}$ blocks.

Next we give a construction using skew Hadamard difference sets, which is actually a generalization of Theorem 6.1 in [25].

**Theorem 3.8.** Let $D$ be a $(v, k, \lambda)$ skew Hadamard difference set in a group $(G, +)$. Then $(G, Dev(D) \cup Dev(D \setminus \{0\}))$ is a $2-(v, k, 2\lambda)$ design and a $3-(v, k, \lambda - 1)$ adesign.

**Proof.** Let $l$ be the integer such that $v = 4l - 1, k = 2l - 1$ and $\lambda = l - 1$. The fact that $r_{(x,y)}^{Dev(D')} is the constant $\lambda$ for all $x, y \in G$ distinct, where $D' = D$ or $D \setminus \{0\}$, implies that $(G, Dev(D) \cup Dev(D \setminus \{0\}))$ is a $2-(v, k, 2\lambda)$ design. We need to show that $(G, Dev(D) \cup Dev(D \setminus \{0\}))$ is a $3-(v, k, \lambda - 1)$ adesign. Let $x, y, z \in G$ be distinct. Like in the previous two constructions, we will assume that $x, y$ and $z$ appear together in $\mu$ blocks of $Dev(D)$, and we will first count the number of blocks of $Dev(D) \cup Dev(D)$ in which $x, y$ and $z$ appear together. By the principle of inclusion and exclusion, there are $v - 3k + 3\lambda - \mu$ blocks in $Dev(D) \cup Dev(D)$ containing $x, y$ and $z$. Then there must be $v - 3k + 3\lambda$ blocks in $Dev(D)$ containing $x, y$ and $z$. We want to know how many of these blocks are also in the set $\{-D \cup \{0\} + x, -D \cup \{0\} + y, -D \cup \{0\} + z\}$. Without loss of generality, suppose that both $-D \cup \{0\} + x$ and $-D \cup \{0\} + y$ contain the points $x, y$ and $z$. Then we must have $y - x, z - x \in (-D) \cap D$, which is a contradiction, as $(-D) \cap D = \emptyset$. Thus, no more than one of the blocks $-D \cup \{0\} + x, -D \cup \{0\} + y$ and $-D \cup \{0\} + z$ contain the points $x, y$ and $z$. We need only show that $(G, Dev(D) \cup Dev(D))$ is...
not a 3-design, and we will be done. If \((G, \Dev(D) \cup \Dev(T \setminus \{0\}))\) were a 3-design, then, by the above arguments, the only choices for the constant \(r_{\{x,y,z\}}^{\Dev(D) \cup \Dev(T \setminus \{0\})}\) \(= \lambda'\) would be \(\lambda\) or \(\lambda - 1\). The number of blocks in \(\Dev(D) \cup \Dev(T \setminus \{0\})\) is given by \(\lambda'(v) / \binom{k}{3}\), whence the equation

\[
2v = \lambda' \binom{v}{3}/\binom{k}{3}
\]

must hold. If \(\lambda' = \lambda\) then \(\text{I}\) becomes \(v = v + 3\), a contradiction, and if \(\lambda' = \lambda - 1\) then \(\text{I}\) becomes \((k - 1)(k - 2) = (\lambda - 1)(v - 2)\), another contradiction.

The following is the complementary construction. The proof is omitted.

**Corollary 3.9.** Let \(D\) be a \((v, k, \lambda)\) skew Hadamard difference set in a group \((G, +)\). Then \((G, \Dev(D \cup \{0\}) \cup \Dev(T))\) is a 2-(\(v, k, 2\lambda + 1\)) design and a 3-(\(v, k, \lambda\)) adesign.

**Example 3.10.** Let \((A, +)\) and \((B, +)\) be two abelian groups of the same order \(n\). Let \(f\) be a 2-to-1 planar function from \(A\) to \(B\), and let \(D = \Im(f) \setminus \{0\}\). If \(n \equiv 3 \pmod{4}\) then \(D\) is a skew Hadamard difference set in \((B, +)\) \([36]\). Then by Theorem 3.8, \((\mathbb{F}_q, \Dev(D) \cup \Dev(T \setminus \{0\}))\) is a 2-(\(n, \frac{n - 1}{2}, \frac{n - 3}{2}\)) design with the property that each 3-subset of \(B\) appears in either exactly \(\frac{n - 3}{4}\) or exactly \(\frac{n - 3}{4}\) blocks.

**Example 3.11.** Let \(p_1\) be a prime, \(N \equiv 7 \pmod{8}\) a positive integer, and let \(p \equiv 3 \pmod{4}\) be a prime such that \(f := \ord_N(p) = \phi(N)/2\). Let \(s\) be any odd integer and \(I\) any subset of \(\mathbb{Z}_N\) such that \(\{i \pmod{p_1^n} \mid i \in I\} = \mathbb{Z}_{p_1^n}\) and let \(D = \cup_{i \in I} \Dev(N, q)\) where \(q = p^s\). Then \(D\) is a skew Hadamard difference set in \((\mathbb{F}_q, +)\) \([17]\). Then by Theorem 3.8, \((\mathbb{F}_q, \Dev(D) \cup \Dev(T \setminus \{0\}))\) is a 2-(\(q, \frac{q - 1}{2}, \frac{q - 3}{2}\)) design with the property that each 3-subset of \(\mathbb{F}_q\) appears in either exactly \(\frac{q - 7}{4}\) or exactly \(\frac{q - 7}{4}\) blocks.

**Remark 3.1.** It is clear from their proofs that the balanced incomplete block designs resulting from the 3-adesigns constructed in this section can also be realized as difference families, each consisting of two distinct difference sets (or almost difference sets). To the best of our knowledge, the balanced incomplete block designs resulting from the 3-adesigns constructed in Theorem 3.7 have not been reported on previously. The balanced incomplete block designs resulting from the 3-adesigns constructed in Theorems 3.4 and 3.8 were both discussed by Wilson in \([37]\) as difference families, and also by Liu and Ding in \([23]\) as balanced incomplete block designs.

**Remark 3.2.** Assuming that \((V, \mathcal{B})\) is a 2-(\(v, k, \lambda\)) design, the condition \(\lambda' = \lfloor \lambda k - 2 \rfloor\) stated in Lemma 2.1 is necessary for \((V, \mathcal{B})\) to be a 3-(\(v, k, \lambda'\)) adesign, but not sufficient, as the following example, whose construction method was discussed in \([25]\), illustrates. We have yet to find an example of a 3-adesign which is also a 2-adesign.

**Example 3.12.** Let \(n\) be an odd integer divisible by 3. Consider, for fixed \(a \in \mathbb{Z}_n\), all pairs \(\{a - i \pmod{n}, a + i \pmod{n}\}\), for \(i = 1, \ldots, n - 1\). The union of any two distinct pairs gives a block consisting of four points. Denote, for fixed \(a \in \mathbb{Z}_n\), the set of all blocks obtained in this way by \(\mathcal{B}_a\). Then \((\mathbb{Z}_n, \cup_{a \in \mathbb{Z}_n} \mathcal{B}_a)\) is a 2-(\(n, 4, n\)) design. Also notice that \(|n^{2 - 2}| = 2\) for all \(n \geq 9\), and the number of blocks is \(b = n(\frac{n - 1}{2}) \) so that \(2\binom{n}{3} / \binom{4}{3} < b < 3\binom{n}{3} / \binom{4}{3}\) is satisfied; however, since \(n\) is divisible by 3, we can find 3-subsets of \(\mathbb{Z}_n\) not contained in any block (choose three points \(x, y\) and \(z\) so that \(|x - y| = |x - z| = |y - z|\)).
4 Constructions of 2-adesigns

We begin this section with a discussion on the possible number of blocks of 2-adesigns.

4.1 Possible number of blocks of 2-adesigns

Let \((V, B)\) be a 2-\((v, k, \lambda)\) adesign with \(b\) blocks. According to Lemma 3.3, if \((V, B)\) is a tactical configuration, then \(\lambda = \lfloor r \frac{B}{v-1} \rfloor\) and

\[
\lambda\left(\frac{v}{2}\right) / \left(\frac{k}{2}\right) < b < (\lambda + 1)\left(\frac{v}{2}\right) / \left(\frac{k}{2}\right).
\]

In [21], Horsely generalized Fisher’s inequality to coverings and packings. Let \(v, k\) and \(\lambda\) be positive integers and let \((V, B)\) be an incidence structure with \(|V| = v\) and \(|B| = k\) for all \(B \in \mathcal{B}\). If each pair of points occurs in at least \(\lambda\) blocks, then \((V, B)\) is a \((v, k, \lambda)\)-covering. If each pair of points occurs in at most \(\lambda\) blocks then \((V, B)\) is a \((v, k, \lambda)\)-packing.

Denote the incidence matrix of a \((v, k, \lambda)\)-covering resp. -packing by \(M_c\) resp. \(M_p\). Also note that, if \(b\) is the number of blocks in \(B\) then

\[
b \geq \text{rank}(M) \geq \text{rank}(MM^T)
\]

where \(M\) is either \(M_c\) or \(M_p\).

**Lemma 4.1.** [21] Let \(v, k\) and \(\lambda\) be positive integers such that \(3 \leq k < v\), and let \(r\) and \(d\) be the integers such that \(\lambda(v-1) = r(k-1) - d\) and \(0 \leq d < k - 1\). If \(d < r - \lambda\), then

\[
\text{rank}(M_c M_c^T) \geq \left\lceil \frac{v(r + 1)}{k + 1} \right\rceil.
\]

**Lemma 4.2.** [21] Let \(v, k\) and \(\lambda\) be positive integers such that \(3 \leq k < v\), and let \(r\) and \(d\) be the integers such that \(\lambda(v-1) = r(k-1) + d\) and \(0 \leq d < k - 1\). If \(d < r - \lambda\), then

\[
b \leq \left\lfloor \frac{v(r - 1)}{k - 1} \right\rfloor.
\]

Again let \((V, B)\) be a 2-(\(v, k, \lambda\)) adesign with \(b\) blocks. Clearly \((V, B)\) is a \((v, k, \lambda)\)-covering and a \((v, k, \lambda + 1)\)-packing. Let \(r_1, d_1\) and \(\lambda\) be defined, respectively, as \(r, d\) and \(\lambda\) were defined in Lemma 4.1 and let \(r_2, d_2\) and \(\lambda + 1\) be as defined, respectively, as \(r, d\) and \(\lambda\) were defined in Lemma 4.2. Then, combining Lemmas 4.1 and 4.2 gives us

\[
\left\lfloor \frac{v(r_1 + 1)}{k + 1} \right\rfloor \leq b \leq \left\lfloor \frac{v(r_2 - 1)}{k - 1} \right\rfloor.
\]  

(5)
4.2 Constructions of 2-adesigns

In this section we will need some facts about cyclotomic classes and cyclotomic numbers. Let \( q = ef + 1 \) be a prime power, and \( \gamma \) a primitive element of the finite field \( \mathbb{F}_q \) with \( q \) elements. The cyclotomic classes of order \( e \) are given by \( D_i^{(e,q)} = \langle \gamma^i \rangle \) for \( i = 0, 1, \ldots, e - 1 \). The cyclotomic numbers of order \( e \) are given by \( (i, j)_e = |D_i^{(e,q)} \cap D_j^{(e,q)} + 1| \). It is obvious that there are at most \( e^2 \) different cyclotomic numbers of order \( e \). When it is clear from the context, we will simply denote \( (i, j)_e \) by \( (i, j) \).

We will need to use the cyclotomic numbers of order 2.

**Lemma 4.3.** [33] For a prime power \( q \), if \( q \equiv 1 \pmod{4} \), then the cyclotomic numbers of order two are given by

\[
(0,0) = \frac{q - 5}{4}, \\
(0,1) = (1,0) = (1,1) = \frac{q - 1}{4}.
\]

If \( q \equiv 3 \pmod{4} \) then the cyclotomic numbers of order two are given by

\[
(0,1) = \frac{q + 1}{4}, \\
(0,0) = (1,0) = (1,1) = \frac{q - 3}{4}.
\]

Here we again begin with a construction based on the twin prime almost difference sets discussed in [38].

**Theorem 4.4.** Let \( q \) be a prime power. Let \( G = (\mathbb{F}_q, +) \times (\mathbb{F}_q, +) \), and define

\[
D = \{(a, b) \in G \mid a \text{ and } b \text{ are both squares or both nonsquares}\},
\]

and

\[
\tilde{D} = \{(a, b) \in G \mid \text{one of } a, b \text{ is square and the other is nonsquare}\}.
\]

Then \((G, \text{Dev}(D) \cup \text{Dev}(\tilde{D}))\) is a 2-(\( q^2, \frac{q^2-2q+1}{2}, \frac{q^2-4q+3}{2} \)) adesign.

**Proof.** It was shown in [38] that, for \( x, y \in G \) distinct,

\[
r_{\{x,y\}}^\text{Dev(D)} = \begin{cases} 
\frac{q^2-4q+7}{4}, & \text{if } x - y \in D, \\
\frac{q^2-4q+3}{4}, & \text{otherwise}.
\end{cases}
\]

(6)

To show that \((G, \text{Dev}(D) \cup \text{Dev}(\tilde{D}))\) is a 2-(\( q^2, \frac{q^2-2q+1}{2}, \frac{q^2-4q+3}{2} \)) adesign, we will compute \( r^\text{Dev(\tilde{D})}_{\{x,y\}} \). We want to count the number of solutions to the equation

\[
(a, b) = (a_1, b_1) - (a_2, b_2)
\]

(7)
where \((a_1, b_1), (a_2, b_2) \in \tilde{D}\). We use a method similar to that used in [38]. Assume that \(a\) and \(b\) are both square. If \(a_1\) and \(a_2\) are square and \(b_1\) and \(b_2\) are nonsquare, then, using Lemma 4.8, the number of solutions to (7) is \((0, 0)_{2}(1, 1)_{2}\). There are three other cases depending on which of \(a_1, a_2, b_1\) and \(b_2\) are square and which are nonsquare, and the number of solutions to (7), as we run over these other possibilities, is one of \((0, 1)_{2}(1, 0)_{2}, (1, 0)_{2}(0, 1)_{2}\) or \((1, 1)_{2}(0, 0)_{2}\). Summing over all four possibilities, the total number of solutions to (7) when \(a\) and \(b\) are both square is \(\frac{q^2-4q+3}{4}\) (regardless of the residue of \(q\) modulo 4).

The other three cases where neither \(a\) nor \(b\) are zero can be argued similarly. When \(a\) and \(b\) are both nonsquare, the total number of solutions to (7) is \(\frac{q^2-4q+3}{4}\), and when one of \(a\) and \(b\) is square and the other is nonsquare, the total number of solutions is \(\frac{q^2-4q+7}{4}\). If \(a \neq 0\) and \(b = 0\) then (7) becomes \((a_2a^{-1}, b_2) + (1, 0) = (a_1a^{-1}, b_1)\) which, using Lemma 4.8 again, has \(((0, 0)_{2} + (1, 1)_{2})\frac{q-1}{2} = \frac{q^2-4q+3}{4}\) solutions. A similar argument shows that when \(a = 0\) and \(b \neq 0\) the number of solutions to (7) is again \(\frac{q^2-4q+3}{4}\). Thus, if \(x, y \in G\) are distinct,

\[
D_{\text{Dev}}(\tilde{D})_{\text{Dev}} = \begin{cases} \frac{q^2-4q+7}{4}, & \text{if } x - y \in \tilde{D}, \\ \frac{q^2-4q+3}{4}, & \text{otherwise.} \end{cases}
\]

Combining (6) and (8) gives us, for \(x, y \in G\) distinct,

\[
D_{\text{Dev}(D) \cup \text{Dev}(\tilde{D})} = \begin{cases} \frac{q^2-4q+7}{4} + \frac{q^2-4q+3}{4} = \frac{q^2-4q+5}{2}, & \text{if } x - y \in D \cup \tilde{D}, \\ \frac{q^2-4q+3}{4} + \frac{q^2-4q+3}{4} = \frac{q^2-4q+3}{2}, & \text{otherwise.} \end{cases}
\]

This completes the proof.

**Example 4.5.** Let \(q = 5\), and let \(D\) and \(\tilde{D}\) be defined as in Example 3.2, i.e.,

\[D = \{(2, 3), (1, 4), (3, 2), (4, 1), (3, 3), (1, 1), (4, 4), (2, 2)\}\]

and

\[\tilde{D} = \{(4, 3), (1, 2), (1, 3), (3, 1), (2, 1), (2, 4), (3, 4), (4, 2)\}.

Then the incidence structure \((F_5 \times F_5, D_{\text{Dev}(D) \cup \text{Dev}(\tilde{D})})\) is a 2-(25, 8, 4) adesign with 50 blocks.

Our next construction is a modification of Bose’s Steiner triple systems [3].

**Theorem 4.6.** Let \(n > 3\) be an odd integer, let \(G = \mathbb{Z}_n \times \mathbb{Z}_3\), and let “\(>\)” be any total ordering on \(\mathbb{Z}_n\) \((\text{e.g. } 0 < 1 < \cdots < n - 1)\). Define

\[
\mathcal{B} = \left\{ \{(a, i), (b, i), \left(\frac{n+1}{2}(a+b), i+1\right) \mid a, b \in \mathbb{Z}_n, a < b \} \right. \\
\cup \left\{ \{(a, i), (b-1, i), \left(\frac{n+1}{2}(a+b), i+1\right) \mid a, b \in \mathbb{Z}_n, a \neq b, a \neq b - 1 \} \right. \\
\cup \left\{ \{(a, 0), (a, 1), (a, 2) \mid a \in \mathbb{Z}_n \} \right. .
\]

Then \((G, \mathcal{B})\) is a 2-(3n, 3, 1) adesign \((\text{with } 3n^2 - 2n \text{ blocks})\).
Proof. Let \((\alpha, j), (\beta, k) \in G\). It is clear that each block is incident with three points. If \(\alpha = \beta\) then the pair occurs in the block \(\{ (\alpha, 0), (\alpha, 1), (\alpha, 2) \}\) or in the block \(\{ (\alpha, j), (\alpha - 1, j), (\alpha, j + 1) \}\) \((= \{ (\alpha, j), (\beta - 1, j), ((n + 1)/2 \cdot (\alpha + \beta), j + 1) \})\) and in no other block.

Now assume \(\alpha \neq \beta\). Without loss of generality, we can assume that \(\alpha < \beta\). There are three cases depending on the residues \(k\) and \(j\) modulo 3:

1) If \(k = j\), the pair \((\alpha, j), (\beta, k)\) occurs in the block \(\{ (\alpha, k), (\beta, k), ((n + 1)/2 \cdot (\alpha + \beta), k + 1) \}\) and in no other block.

2) If \(k = j + 1\) (mod 3), the equation \(\frac{n+1}{2}(x+\alpha) = \beta\) has a unique solution \(x = \gamma\). Since \(\frac{n+1}{2} = a\) for all \(a \in \mathbb{Z}_n\), i.e., the binary operation \(f(a, b) := \frac{n+1}{2}\) is idempotent, and \(\alpha \neq \beta\), we have \(\gamma \neq \alpha\). If \(\gamma < \alpha\), the pair \((\alpha, j), (\beta, k)\) occurs in the block \(\{ (\alpha, j), (\gamma, j), (\beta, k) \}\) \((= \{ (\alpha, j), (\gamma, j), ((n+1)/2 \cdot (\alpha + \gamma), j + 1) \})\), as well as in the block \(\{ (\alpha, j), (\gamma - 1, j), (\beta, k) \}\) \((= \{ (\alpha, j), (\gamma - 1, j), ((n+1)/2 \cdot (\alpha + \gamma), j + 1) \})\), and in no other block. If \(\alpha < \gamma\) the pair occurs in in the block \(\{ (\alpha, j), (\gamma, j), (\beta, k) \}\) and in no other block.

3) If \(j = k + 1\) (mod 3) then the equation \(\frac{n+1}{2}(x + \beta) = \alpha\) has a unique solution \(x = \gamma\). Since the binary operation \(f(a, b) := \frac{n+1}{2}\) is idempotent, and \(\alpha \neq \beta\), we have \(\gamma \neq \beta\). If \(\gamma < \beta\) then the pair \((\alpha, j), (\beta, k)\) occurs in the block \(\{ (\gamma, k), (\beta, k), (\alpha, j) \}\) \((= \{ (\gamma, k), (\beta, k), ((n+1)/2 \cdot (\gamma + \beta), k + 1) \})\), as well as in the block \(\{ (\beta, k), (\gamma - 1, k), (\alpha, j) \}\) \((= \{ (\beta, k), (\gamma - 1, k), ((n+1)/2 \cdot (\beta + \gamma), k + 1) \})\), and in no other block. If \(\gamma > \beta\) then the pair occurs in in the block \(\{ (\beta, k), (\gamma, k), (\alpha, j) \}\) and in no other block.

This completes the proof.

Let \((V, \mathcal{B})\) be an incidence structure. Let \(p \in V\) and define \(\mathcal{B}_p = \{ B \setminus \{ p \} \mid p \in B, B \in \mathcal{B} \}\). The incidence structure \((V \setminus \{ p \}, \mathcal{B}_p)\) is called the contraction of \((V, \mathcal{B})\) at the point \(p\). We can obtain new symmetric 2-adesigns by contracting on a point of any one of the 3-adesigns constructed in Section 3. It is easy to see that contracting at points of a 3-adesign will result in a 2-adesign as long as not all three subsets of points occur in the same number of blocks of the contraction.

Remark 4.1. Let \((V, \mathcal{B})\) be any one of the 3-(v, k, \lambda) adesigns constructed in Section 3 and let \(p\) be any point of \(V\). It is clear from their proofs that contracting at \(p\) gives a 2-(v, k, \lambda) adesign since we can always find a pair \(x, y \in V \setminus \{ p \}\) such that \(x, y\) and \(p\) appear together in \(\lambda\) blocks of \(\mathcal{B}\), and we can also find another pair \(x', y' \in V \setminus \{ p \}\) such that \(x', y'\) and \(p\) appear together in \(\lambda + 1\) blocks of \(\mathcal{B}\). Moreover, contracting at a point \(p\) of \((V, \mathcal{B})\) gives a \(\lambda\)-covering which meets the bound given in Lemma 4.1, i.e., it is a minimal \(\lambda\)-covering.

Example 4.7. Let \(q\) be a prime power and let \(D\) and \(\hat{D}\) be defined as in Theorem 3.7. Denote \(\mathbb{F}_q \times \mathbb{F}_q\) by \(V\) and \(\text{Dev}(D \cup \mathbb{F}_q \times \{ 0 \}) \cup \text{Dev}(\hat{D} \cup \{ 0 \} \times \mathbb{F}_q)\) by \(\mathcal{B}\). By Theorem 3.7 the incidence structure \((V, \mathcal{B})\) is a 3-(\(q^2\), \(\frac{q^2+1}{2}\), \(\frac{q^2-1}{4}\)) adesign with \(2q\) blocks. If we contract at the point \((0, 0)\), then we get the incidence structure \((V \setminus \{(0, 0)\}, \mathcal{B}_{(0,0)})\), which, by Remark 4.1, is a 2-(\(q^2 - 1\), \(\frac{q^2-1}{2}\), \(\frac{q^2-1}{4}\)) adesign with \(b = q^2 + 1\) blocks. Now let \(r = \frac{q^2+1}{2}\) and \(d = \frac{q^2-1}{4}\). Then with \(v = q^2 - 1\), \(k = \frac{q^2+1}{2}\) and \(\lambda = \frac{q^2-1}{4}\), we
have that \( \lambda(v - 1) = r(k - 1) - d \) with \( 0 \leq d < r - \lambda \), and \( r \) and \( d \) satisfy the conditions of Lemma 4.1. Then, since \( (q^2 - 1)(q^2 + 3)/(q^2 + 1) > q^2 \), we have

\[
\frac{v(r + 1)}{(k + 1)} = \left\lfloor \frac{(q^2 - 1)(q^2 + 3)}{(q^2 + 1)} \right\rfloor = q^2 + 1 = b,
\]

i.e., \( b \) meets the bound given in Lemma 4.1 and \((V \setminus \{(0, 0)\}, B_{(0,0)})\) is a minimal \( \lambda \)-covering.

### 5 Concluding Remarks

In this correspondence we investigated \( t \)-adesigns and their links with other combinatorial objects such as balanced incomplete block designs, and coverings and packings. We first considered the question of when a \((t + 1)\)-adesign is a \( t \)-design or a \( t \)-adesign, and then we constructed several classes of \( 3 \)-adesigns which are, in fact, balanced incomplete block designs. We have also discussed some of the restrictions on the possible sets of feasible parameters for both \( 2 \)-adesigns and \( 3 \)-adesigns. The \( 2 \)-adesigns we constructed have new parameters, and some of them have the interesting property that they are minimal \( \lambda \)-coverings.

### Appendix

**Lemma 5.1.** Let \( q \) be a prime power. Let \( G = (\mathbb{F}_q, +) \times (\mathbb{F}_q, +) \), and define

\[
D = \{(a, b) \in G \mid a \text{ and } b \text{ are both squares or both nonsquares}\},
\]

and

\[
\tilde{D} = \{(a, b) \in G \mid \text{one of } a, b \text{ is square and the other is nonsquare}\}.
\]

Then

\[
r_{B_1}^G(x, y) = \begin{cases} 
\frac{q^2 - 1}{4}, & \text{if } x - y \in D \cup \mathbb{F}_q^* \times \{0\}, \\
\frac{q^2 + 3}{4}, & \text{if } x - y \in \tilde{D} \cup \{0\} \times \mathbb{F}_q^*, 
\end{cases}
\]

for all \( x, y \in G \) distinct.

**Proof.** Let \((a, b) \in G \setminus \{(0, 0)\}\). We want to count the number of solutions to the equation

\[
(a, b) = (a_1, b_1) - (a_2, b_2)
\]

where \((a_1, b_1), (a_2, b_2) \in \tilde{D} \cup \{(0, 0)\} \times \mathbb{F}_q^*\). Let \( M \) denote the multiset

\[
\{\ast (a_1, b_1) - (a_2, b_2) \mid (a_1, b_1), (a_2, b_2) \in \tilde{D} \cup \{(0, 0)\} \times \mathbb{F}_q^*\}.
\]

**Case where \( a \) and \( b \) are both square:** We have the following possibilities: if \((a_1, b_1), (a_2, b_2) \in \tilde{D}\)
then (9) has \(\frac{q^2 - 4q + 3}{4}\) solutions, and if one of \((a_1, b_1), (a_2, b_2)\) is \((0, 0)\) then (9) has two solutions. If one of \((a_1, b_1), (a_2, b_2)\) is a member of \(\{0\} \times \mathbb{F}_q\), i.e., we have

\[(a, b) = (0, b_1) - (-a, b_2) \text{ or } (a, b) = (a, b_1') - (0, b_2'),\]  

(10) then there are three possibilities: if \(q \equiv 3 \pmod{4}\) then \(-a\) is nonsquare and \(b_2\) is square, thus, by Lemma 4.3, (10) has \((0, 0)_2\) resp. \((0, 1)_2\) solutions when \(b_1\) is square resp. nonsquare; if \(q \equiv 1 \pmod{4}\) then \(-a\) is square, thus if \(a_1 = 0\) then (10) has \((1, 0)_2\) resp. \((1, 1)_2\) solutions when \(b_1\) is square resp. nonsquare, and if \(a_1 = a\) then (10) has \((0, 1)_2\) resp. \((1, 1)_2\) solutions when \(b_2\) is square resp. nonsquare. Thus, the total number of solutions of (10) is \(2(0, 1)_2 + (0, 0)_2 + (1, 1)_2 = q - 1\) (regardless of \(q\)'s residue modulo 4), whence the total number of solutions to (9) when \(a\) and \(b\) are both square is \(\frac{q^2 - 4q + 3}{4} + 2 + q - 1 = \frac{q^2 - 1}{4}\).

**Case where \(a\) and \(b\) are both nonsquare:** A similar argument as that of the previous case gives the total number of solutions to (9) when \(a\) and \(b\) are both nonsquare is \(\frac{q^2 - 1}{4}\).

**Case where one of \(a\) or \(b\) in zero and the other is nonzero:** If a member of \(\mathbb{F}_q^* \times \{0\}\) appears as a difference of two members of \(\tilde{D} \cup \{0\} \times \mathbb{F}_q\), \(\lambda\) times, then counting \(M\) in two ways gives us

\[(q - 1)\lambda = \frac{(q - 1)^2 - 2}{2} - 1 + (q - 1)^2\]

whence \(\lambda = \frac{q^2 - 1}{4}\). If a member of \(\{0\} \times \mathbb{F}_q\) appears as a difference of two members of \(\tilde{D} \cup \{0\} \times \mathbb{F}_q\), \(\lambda'\) times, then counting \(M\) in two ways gives us

\[(q - 1)\lambda' = q(q - 1) + \frac{(q - 1)^2 - 2}{2}\]

whence \(\lambda' = \frac{q^2 - 1}{4}\).

**Case where one of \(a\) or \(b\) is square and the other is nonsquare:** Now let \(\lambda''\) denote the number of times a member of \(\tilde{D}\) appears as a difference of two members of \(\tilde{D} \cup \{0\} \times \mathbb{F}_q\). Let \(\lambda\) and \(\lambda'\) be defined as in the previous case. Notice

\[
\frac{q^2 + 1}{2} \cdot \frac{q^2 - 1}{2} = |M| = |\tilde{D} \cup \{0\} \times \mathbb{F}_q|(|\tilde{D} \cup \{0\} \times \mathbb{F}_q| - 1)
\]

\[= \lambda''|\tilde{D}| + \frac{q^2 - 1}{4}|\tilde{D}| + (q - 1)(\lambda + \lambda')
\]

\[= (\lambda'' + \frac{q^2 - 1}{4})(\frac{(q - 1)^2}{2} + (q - 1)(\frac{q^2 + 1}{2}) - 1).
\]

Then \(\lambda'' = \frac{q^2 + 3}{4}\). The result follows immediately.

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