Recursive percolation

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We introduce a simple fractal process in which percolation is constructed on top of critical percolation clusters, and show that this construction can be repeated recursively any number of times. We carry out Monte Carlo simulation in two dimensions up to the $(n = 5)$th generation, and determine percolation thresholds and several sets of critical exponents. It is found that critical percolation clusters enjoy distinct critical exponents for any $n$, and that they become more and more compact as $n$ increases. Further, it is observed that recursive percolation clusters violate the domain Markov property except for the original percolation ($n = 0$). A similar study is also performed in three dimensions.

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The study of fractal, scale invariant structures in random geometries has become paradigmatic in theoretical physics and probability theory, in particular in two dimensions where exact results can be obtained by powerful analytical tools such as conformal field theory (CFT) and stochastic Loewner evolution (SLE). Applications include domain walls in magnetic systems, polymers, entanglement in quantum information theory, quantum Hall wave functions, and turbulence, to name but a few examples.

It was realized early on that the fine structure of fractal geometries can be further elucidated by defining other scale invariant processes on top of the fractals themselves, or said more simply, studying fractals on fractals. One of the best studied models is undoubtedly that of self-avoiding walks (SAW) defined on percolation clusters. Because of the self-avoidance, the SAW is confined to the backbone of the cluster. The resulting critical properties have been studied by a variety of techniques. One key result is that while the SAW enjoys its usual critical properties above the percolation threshold, $p > p_c$, it defines a new universality class exactly at $p = p_c$ with interesting multifractal properties. From the point of view of the renormalization group (RG) this means that $p_c$ is an unstable fixed point from which the system may flow to either the usual SAW fixed point at $p = 1$, or to a trivial fixed point at $p = 0$. Very recently, loop-erased random walks (LERW) were defined on critical percolation clusters, and their scaling behavior was found to be compatible with SLE, but with a diffusion coefficient outside the usual range.

In this Letter we shall focus on a fractal process that can be defined recursively on top of itself: to be precise, we shall study percolation on percolation clusters. For simplicity our study takes place in two and three dimensions, but we expect similar conclusions in higher dimensions. We shall show that if the original clusters are placed at criticality, $p = p_c$, there exists a non-trivial critical threshold, $p_c > p_c^0$, for the new percolation process which acts, once again, as an unstable fixed point with critical properties different from those of the original percolation clusters. This means in particular that the construction can be repeated recursively: on top of the new critical clusters, one may again study a percolation process and search for its threshold. We shall see that the same scenario takes place, so that the construction may in fact be repeated any number of times. The $n$th generation of percolation clusters thus generated enjoys, at the proper threshold $p = p_c^n$, distinct critical exponents for any $n$. We shall in particular characterize the limiting process when $n \to \infty$.

Model. We study recursive percolation defined on periodic $L \times L$ square lattices. The starting point is standard bond percolation, corresponding to generation $n = 0$ of the recursive process, with the known threshold $p_c^n = 1/2$. From a given set of percolation clusters $C_0$, henceforth called standard clusters for clarity, we define a set of dense clusters $C_n$ by filling in all bonds between neighboring sites in the same cluster.

Suppose, as an induction hypothesis, that the thresholds $p_c^1, \ldots, p_c^{n-1}$ have already been determined. A configuration of clusters $C_n$ at generation $n \geq 1$, with a given occupation probability $p^n$, is then defined as follows: For each $i = 1, \ldots, n$ in turn, produce $C_i$ by performing bond percolation on $C_{i-1}$ with probability $p = p_c^i$ if $i < n$, and $p = p^n$ if $i = n$.

Simulations. Each repetition of the above prescription produces an independent realization of $C_n$. We have studied the statistical properties of the clusters $C_n$ by performing extensive simulations on lattices of linear size $L = 2^p$, with $p = 4, 5, \ldots, 12$. For each generation $n$, we ran preliminary simulations within a relatively wide range of $p^n$ and for relatively small values of $L$, and obtained a rough estimate $p_c^n$ by studying the finite-size scaling of dimensionless observables. Simulations were then performed at and near the value of $p_c^n$ estimated in
the initial runs. This procedure was iterated a number of times, and finally we simulated at the best estimate \( p_c^n \) to determine several sets of critical exponents. The number of samples (in units of \( 10^6 \)) is about 6.0, 6.0, 5.0, 3.0, and 3.0 for \( n = 1, 2, \ldots, 5 \), respectively.

**Percolation threshold.** Let \( R_p^2 \) be the probability that one cluster in \( C_n \) wraps both periodic lattice directions. The wrapping probabilities are known to be universal at criticality. Plots of \( R_p^2 \) against \( p^n \) present very clear crossings for \( n = 1, 2, 3, 4 \) (see Fig. 1). but starting from \( n = 5 \) the scaling becomes more problematic in the sense that larger sizes \( L \) are needed to see a clearer crossing. Near the threshold \( p_c^n \), it should scale like

\[
R_p^2 = (R^2_p)_n + a_1 (p^n - p_c^n) L y_1^n + a_2 (p^n - p_c^n)^2 L^2 y_2^n + b_1 L y_3^n + c (p^n - p_c^n) L^3 y_4^n + b_2 L^2 + \cdots, \tag{1}
\]

defining the thermal exponent \( y_1 \) and the first correction to scaling exponent \( y_2 \). The coefficients \( a_1, a_2, b_1, b_2 \), and \( c \) are non-universal constants. The finite-size scaling clearly shows that \( p_c^n \) acts as an unstable fixed point for \( n \)th generation clusters, sustaining flows to the trivial fixed points \( p^n \approx 1 \) and \( p^n = 0 \) respectively; one illustration is given in Fig. 1 of the Suplementary Material.

**TABLE I. Thresholds \( p_c^n \) for \( n \)th generation percolation.**

| \( n \) | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| \( p_c^n \) | 0.50 | 0.654902(1) | 0.73954(4) | 0.79450(6) | 0.8342(8) | 0.861(2) |
| \( p_n + 1 \) | 0.50 | 0.666667 | 0.75 | 0.8 | 0.8333 | 0.857 |

We have determined the thresholds \( p_c^n \) for \( n \leq 5 \) (see Table I), but the various critical exponents only for \( n \leq 4 \) (see below). The leading correction exponent \( y_1^n \) is found around \(-1\). The error bars in Table I include the uncertainties in estimating \( y_1 \), including some terms in Eq. (1) and varying the minimum size for a lower cutoff \( L \geq L_{\text{min}} \). Notice that the values of \( p_c^n \) are close to the simple fraction \((n+1)/(n+2)\), especially for larger \( n \). This gives compelling evidence that \( p_c^n \to 1 \) for \( n \to 1 \), meaning that recursive percolation can be defined for any number of generations. Below we shall characterize the limiting process \( n \to \infty \) in various ways.

**Bond density.** The density of occupied bonds, \( \rho^n \) and \( \overline{\rho}^n \), is given in Table II. Here and elsewhere quantities with (resp. without) an overline refer to the dense clusters \( C_n \) (resp. standard clusters \( C_n \)). The results are close, albeit not perfectly equal, to \( \rho^n = \frac{1}{2} \) independently of \( n \), respectively \( \overline{\rho}^n = \frac{n+3}{2(n+2)} \), with the common limit \( \rho^n, \overline{\rho}^n \to \frac{1}{2} \) as \( n \to \infty \). Below we shall present more substantial evidence that the difference between standard and dense clusters disappears when \( n \to \infty \).

**TABLE II. Bond density \( \rho^n \) and \( \overline{\rho}^n \) for standard and dense clusters.**

| \( n \) | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| \( \rho^n \) | 0.50 | 0.491(1) | 0.489(1) | 0.491(1) | 0.494(1) |
| \( \overline{\rho}^n \) | 0.75 | 0.661(1) | 0.618(1) | 0.592(1) | 0.577(1) |

**TABLE III. Wrapping probabilities \( R_p^n \) and \( \overline{R}_p^n \).** For \( n = 0, R_p^0 \) and \( \overline{R}_p^0 \) are exactly known as 0.521 058 290 and 0.351 642 855, respectively [14, 15].

| \( n \) | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| \( R_p^n \) | 0.521(1) | 0.636(1) | 0.675(1) | 0.693(2) | 0.705(2) |
| \( \overline{R}_p^n \) | 0.902(1) | 0.820(1) | 0.779(1) | 0.759(2) | 0.749(2) |
| \( R_p^2 \) | 0.352(1) | 0.495(1) | 0.547(1) | 0.571(2) | 0.586(2) |
| \( \overline{R}_p^2 \) | 0.853(1) | 0.735(1) | 0.682(1) | 0.657(2) | 0.644(2) |

**Observables and scaling.** The critical clusters \( C_n \) and \( \overline{C}_n \) are more precisely characterized by their critical exponents. We shall be interested in five sets of exponents, \( d_X^n \), \( \delta_X^n \), \( \alpha_X^n \), \( \beta_X^n \), \( \gamma_X^n \), and \( \rho_X^n \), where the letter \( X = F, R, H, B, S \) stands for respectively the cluster’s fractal dimension, the dimension of red bonds (or, more precisely, pseudo-bridges; see below), the hull dimension, the dimension of the backbone, and the shortest path dimension. The \( X = F, R \) codimensions are also known in the SLE literature as the one and two-arm exponents, respectively.
Considerable insight into the geometric structure of percolation clusters has been gained by decomposing incipient clusters into backbones (i.e., two-connected components) and bridges (i.e., occupied bonds whose removal makes the cluster fall apart). It is useful to represent the clusters via the loops on the medial lattice that separate clusters and their duals [16]. The loop surrounding a non-wrapping cluster is then its hull. The loop representation also facilitates the identification of bridges and non-bridges [17]. By definition, a pseudo-bridge is an occupied bond that is not a bridge, but both of whose sides are adjacent to the same loop. These concepts are illustrated in Fig. 2.

The exponents $d^p_X$ (and similarly $d^n_X$) can be obtained from the finite-size scaling of suitable observables at criticality $p^n_c$. The largest-cluster size scales like $L^{d^n_B}$. The number of pseudo-bridges goes like $L^{d^p_B}$. The largest-loop length has the behavior $L^{d^n_R}$, of the largest backbone grows like $L^{d^n_B}$. And finally, the shortest path (as measured using breadth-first growth [17]) has the scaling $L^{d^n_S}$.

Critical exponents. The Coulomb gas (CG) approach to CFT [1,2], and more recently SLE [3,4], have provided a plenitude of information about ordinary percolation clusters (i.e., $n = 0$). In particular we have the exact results [18,19]

$$d^0_F = 2 - (6 - g)(g - 2)/(8g) = 91/48,$$

$$d^0_H = 1 + 2/g = 7/4,$$

$$d^0_R = (4 - g)(4 + 3g)/(8g) = 3/4,$$

where $g = \frac{\beta}{\Delta}$ for percolation, in which case the above results are even rigorous [3,4].

We note that the CG duality transformation $g \rightarrow 16/g$ relates $d^0_H \rightarrow d^n_H$. Moreover, our definitions imply that $d^0_F = d^0_R = d^n_B$. The second equality is based on the assumption that removing the branches in the dense clusters does not affect the fractal dimension of the resulting clusters, and is supported by our Monte Carlo data. Our numerical estimates for the remaining eight exponents are shown in Table IV. The effect of uncertainties of $p^i_c$ $(i = 1, \ldots, n)$ is not taken into account in the error bars in Table IV, which is expected to be smaller or (at most) comparable to those in Table IV.

As for $n = 0$, the red-bond dimension $d^n_R$ is found to be identical to the thermal exponent $y^n_i$ in Eq. (1), which is numerically determined as 0.433(1), 0.273(4), 0.182(3), 0.116(6), 0.09(1) for $n = 1, \ldots, 5$, respectively. The equality $y^n_i = d^n_H$ holds true in any spatial dimension [23,24].

The dependence on $n$ of all sets of critical exponents can be convincingly fitted to ratios of low-degree polynomials, with a common limit for the standard and dense exponents (see Supplementary Material for details). The plots corresponding to all exponents of Table IV are shown in Fig. 3. The common limiting values for $n \rightarrow \infty$ are estimated as:

$$d^n_H = d^n_S = d^n_C \simeq 1.42(3),$$

$$d^n_B = d^n_R = d^n_F = d^n_S \simeq 1.826(6),$$

$$d^n_R = d^n_S \simeq 1.045(5).$$

It is noteworthy that the numerical values of Table IV are compatible with $d^n_R + d^n_B = 0$ for any $n$. In particular the common $n \rightarrow \infty$ limit must be $d^n_C = d^n_R = 0$.

The limiting clusters. The above results can be used to characterize the limiting clusters $C_\infty = C_\infty$ in various ways. The fact that $d^n_C = 0$ means that the number of red bonds in the limiting clusters does not grow with $L$. This is compatible with the observation (see Fig. 3(b)) that the difference between the clusters and their backbones vanishes in the limit. In other words, the limiting clusters are dense objects, with only few leaves or dangling ends. Moreover, they are devoid of deep fjords, since

| $n$ | 0 | 1 | 2 | 3 | 4 |
|-----|---|---|---|---|---|
| $d^0_H$ | 1.75 | 1.0683(1) | 1.5358(1) | 1.4967(1) | 1.4723(2) |
| $d^0_H$ | 1.3333 | 1.3739(1) | 1.3929(1) | 1.4026(2) | 1.4075(2) |
| $d^0_R$ | 1.6433(1) | 1.7596(1) | 1.7942(1) | 1.8078(2) | 1.8148(2) |
| $d^0_R$ | 1.8958(1) | 1.8573(1) | 1.8424(1) | 1.8357(2) | 1.8323(2) |
| $d^0_S$ | 1.1308(2) | 1.0813(2) | 1.0645(10) | 1.0576(10) | 1.053(1) |
| $d^0_S$ | 1.1433(1) | 1.0813(1) | 1.0645(10) | 1.0576(10) | 1.053(1) |

FIG. 2. A configuration of percolation clusters, of which one contributes to $R^2$. Bridges are denoted by the green lines, and non-bridges are shown as blue and red bonds with the latter for pseudo-bridges. The loops separating clusters from their duals are shown as dotted lines on the medial (tilted square) lattice.
their hulls and external perimeter scale in the same way (see Fig. 3a).

Another set of clusters having similar characteristics are the Fortuin-Kasteleyn (FK) clusters of the $q = 4$ state Potts model, whose hulls behave as the level lines of a free Gaussian field with central charge $c = 1$. These Potts clusters can be described by the CG construction with the selfdual choice of the coupling, $g = 4$ [3, 4].

Despite of this resemblance, the $C_\infty$ clusters are most definitely different from the $q = 4$ Potts clusters. The latter have indeed $d_H = 0$ from Eq. (2), $d_H = d_H$ by the selfduality, and it has been shown numerically that $d_H = d_\infty$ [20]. However, the exact values $d_\infty = \frac{15}{3} = 5$ coming again from Eq. (2), are in sharp contrast with the estimates [3]. Even if we were to eschew the $n \to \infty$ extrapolation and invoke only the monotonicity properties of Table IV the resulting upper bounds, $d_H \leq 1.4723(2)$ and $d_\infty \leq 1.8323(2)$ coming from the $n = 4$ data, rule out the $q = 4$ Potts universality class by a very substantial numerical confidence. Actually, the fractal dimensions $d_\infty$ for $n \geq 1$ fail to be described by the exact $d_\infty$ formula in Eq. (4) for any real value $g$, since it has a minimum $d_{\infty, \min} = (2 + \sqrt{3})/2 \approx 1.866$ at $g = 2\sqrt{3}$.

One key difference between $C_\infty$ and the Potts clusters is that the SLE$_{c=4}$ process that is known to produce the hulls of the latter satisfies the domain Markov property by construction [3, 4]. But apart from the $n = 0$ generation, the recursive percolation clusters explicitly violate the domain Markov property. This is seen most convincingly from the data for $d_\infty$ and $d_\infty$, which are no longer related by interchanging $g \to 16/g$ in Eq. (2) when $n \geq 1$.

**Scaling function.** To further substantiate the RG scenario, in which $p_c$ acts as an unstable fixed point sustaining flows to $p^* = 0$ and $p^* = 1$, we now study the universal scaling functions corresponding to the crossover. For simplicity we focus on the case $n = 1$.

In Fig. 4 we show the estimates for the effective hull dimension $d_H^0$ against the scaling variable $u = (p^1 - p_c) L^{y_1}$, cf. Eq. (11), defined as $d_H^0(L) = \log_2(H^0_L)/(2L/H^0_L(L))$. With the choice $y_1 = 0.433(1)$ the data for all sizes $L$ collapse perfectly to reveal the universal scaling function. For $u = 0$ one has the $C_n$ universality class, here with $d_H = 1.6083(1)$, while for $u > 0$ there is a flow to the $C_{n-1}$ universality class, as expected, with now $d_H^0 = 4/3$. The flow for $u < 0$ is to the trivial fixed point with $d_H = 0$. Similar results have been obtained for the other exponents $d_\infty$, and for general values of $n \geq 1$.

**Three dimensions.** Simulations on the simple-cubic lattice yield that $p_c = 0.56472(7)$, $y_1 = 0.432(8)$, and $y_1 = 2.417(2)$. As in two dimensions, the value of $y_1$ is significantly smaller than the reported fractal dimensions for the q-state FK clusters $d_{p} = 2.5835(5)$, $2.52295(15)$, $2.4816(1)$, and $2.492(4)$ for $q = 0, 1, 2, 2.2$ respectively [27]. It is also interesting that $y_1 = 0.432(8)$ agrees with the two-dimensional value.

**Discussion.** We conclude by suggesting that the recursive construction presented here, via the study of percolation on percolation clusters, may carry over more generally to the $q$-state Potts model. For instance, it is well known that $q$-state FK clusters arise by considering percolation with $p_c = \sqrt{q}/(1 + \sqrt{q})$ on top of $q$-state Potts spin clusters [24]. Both types of clusters are well defined for arbitrary real $0 \leq q \leq 4$ [27, 28]. It is thus tempting to speculate that on top of $q$-state FK clusters one may define new $q_1$-state FK clusters, and that the latter will be critical for a suitable non-trivial choice of the temperature variable, with distinct critical exponents. Future
work will show whether this construction is possible and can be repeated recursively.

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[1] P. Di Francesco, P. Mathieu and D. Sénéchal, Conformal Field Theory (Springer, New York, 1997).
[2] J.L. Jacobsen, Conformal field theory applied to loop models, in A.J. Guttmann (ed.), Polygons, polyominoes and polycubes, Lecture Notes in Physics 775, 347–424 (Springer, 2009).
[3] O. Schramm, Israel J. Math. 118, 221 (2000); S. Rohde and O. Schramm, Ann. Math. 161, 883 (2005); G.F. Lawler, Conformally Invariant Processes in the Plane (American Mathematical Society, Providence, 2005).
[4] W. Kager and B. Nienhuis, J. Stat. Phys. 115, 1149 (2004); J. Cardy, Ann. Phys. 318, 81 (2005).
[5] R. Rammal and G. Toulouse, J. Physique Lett. 44, 13–22 (1983).
[6] R. Rammal, G. Toulouse and J. Vannimenus, J. Phys. France 45, 389–394 (1984).
[7] M Sahimi, J. Phys. A: Math. Gen. 17, L379 (1984).
[8] C. Vanderzande and A. Komoda, Phys. Rev. A 45, 5335(R) (1992).
[9] C. von Ferber, V. Blavatska, R. Folk and Yu. Holovatch, Phys. Rev. E 70, 035104(R) (2004).
[10] V. Blavatska and V. Janke, Phys. Rev. Lett. 101, 125701 (2008).
[11] E. Daryaei and S. Rouhani, Phys. Rev. E 89, 062101 (2014); E. Daryaei, Phys. Rev. E 90, 022129 (2014).
[12] D. Stauffer and A. Aharony, Introduction to Percolation Theory (Taylor & Francis, London, 2003), 2nd ed.
[13] G. Grimmett, Percolation (Springer, Berlin, 1999), 2nd ed.
[14] H. T. Pinson, J. Stat. Phys. 75, 1167 (1994).
[15] R. M. Ziff, C. D. Lorenz, and P. Kleban, Physica A 266, 17 (1999).
[16] R.J. Baxter, S.B. Kelland and F.Y. Wu, J. Phys. A: Math. Gen. 9, 397 (1976).
[17] X. Xu, J. Wang, Z. Zhou, T. M. Garoni, and Y. Deng, Phys. Rev. E 89, 012102 (2014).
[18] B. Nienhuis, E.K. Riedel and M. Schick, J. Phys. A: Math. Gen. 13, L189 (1980).
[19] H. Saleur and B. Duplantier, Phys. Rev. Lett. 58, 2325–2328 (1987).
[20] Y. Deng, H.W.J. Blöte and B. Nienhuis, Phys. Rev. E 69, 026114 (2004).
[21] J.L. Jacobsen and P. Zinn-Justin, J. Phys. A: Math. Gen. 35, 2131 (2002).
[22] Z. Zhou, J. Yang, Y. Deng, and R. M. Ziff, Phys. Rev. E 86, 061101 (2012).
[23] A. Coniglio, J. Phys. A: Math. Gen. 15, 3829 (1982).
[24] R. Vasseur, J.L. Jacobsen and H. Saleur, J. Stat. Mech. L07001 (2012).
[25] A. Pelissetto and E. Vicari, Phys. Rep. 368, 549 (2002); Y. Deng, T. M. Garoni, and A. D. Sokal, Phys. Rev. Lett. 98, 030602 (2007); ibid, 99, 055701 (2007).
[26] L. Chayes and J. Machta, Physica A 254, 477 (1998).
[27] J. Dubail, J.L. Jacobsen and H. Saleur, J. Phys. A: Math. Theor. 43, 482002 (2010); J. Stat. Mech. (2010) P12026.
[28] R. Vasseur and J.L. Jacobsen, J. Phys. A: Math. Theor. 45, 165001 (2012).
Supplementary Material for: Recursive Percolation

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This supplementary material provides details on simulations and fitting procedures in both two and three dimensions.

TWO DIMENSIONS

Simulations and Sampled Quantities. We investigate the recursive percolation on periodic $L \times L$ square lattices. To generate a bond configuration for a given $i$th generation, we independently visit each edge on the lattice, and randomly place an occupied bond with probability $p^i$ if the visiting edge connects two sites in the same cluster for the $(i-1)$th generation. The percolation clusters are then constructed via a breadth-first growing procedure similar to the Leath-Alexandrowicz algorithm [1]. For the $n$th generation of recursive percolation, this procedure is performed recursively for $i = 0, 1, \ldots, n$, with the bond probability $p^i$ being set at the previously determined critical value for $i < n$, and at any desired value $p$ for $i = n$.

To characterize the geometric properties of the $n$th generation of percolation clusters, the occupied bonds are classified into bridges and non-bridges: a bond is a bridge if its deletion leads to the disconnection of a cluster. A bond configuration is mapped onto the corresponding Baxter-Kelland-Wu loop configuration [2], defined on the medial graph of the square lattice (see Fig. 2 in the manuscript). A pseudo-bridge is defined as a non-bridge but both of whose sides are adjacent to the same loop. One efficient algorithm has been introduced in Ref. [3] for the classification of occupied bonds into these three classes.

From the simulations of the $n$th generation of recursive percolation, we sampled the following quantities:

- Wrapping probability $R^a_n = \langle r_x + r_y \rangle / 2$, with $\langle \cdots \rangle$ for ensemble average. If a configuration connects to itself along the $\alpha$-direction ($\alpha = x, y$) on the lattice, we set $r_\alpha = 1$; otherwise, $r_\alpha = 0$.
- Wrapping probability along both directions $R_n^2 = \langle r_x \cdot r_y \rangle$.
- The mean density of occupied bonds $\rho^n$.
- The mean number of pseudo-bridges $B^n_R$.
- The mean size of the largest cluster $C^n_B$.
- The mean size of the largest backbone cluster $C^n_B$.
- The mean size of the largest non-backbone cluster $C^n_{B1}$.
- The mean length of the largest loop $H^n_1$.
- The mean number of pseudo-bridges $B^n_P$.
- The mean maximum time step $S^n_\eta$ for constructing percolation clusters. When growing a percolation cluster via the breadth-first method from a seed site, the time step is recorded and measures the shortest-path length between the seed site and any activated site at the current Monte Carlo step [4]. The time step $S$ for a percolation cluster is then the maximum shortest-path length between the seed site and any site in the cluster. Given a bond configuration, $S^n_\eta$ is defined as the maximum time step among all the clusters.

The dense percolation clusters are obtained by filling all the edges that connect two neighboring sites in the same cluster. In other words, the dense clusters are the $(n + 1)$th generation of percolation clusters with probability $p^{n+1} = 1$. For the dense clusters, the above quantities are measured and denoted by $R^n_R, R^n_2, \ldots$, with an overline.

By definition, one has $C^n_1 = C^n_B$.

According to finite-size scaling and our previous studies [2], one expects that at criticality,

$$B^n_R \propto L^{d^n_R}, \quad C^n_1 \propto L^{d^n_F}, \quad C^n_{B1} \propto L^{d^n_B}, \quad H^n_1 \propto L^{d^n_H}, \quad S^n_\eta \propto L^{d^n_\eta},$$

where $d^n_R$ is the dimension of red bonds, $d^n_F$ is the cluster’s fractal dimension, $d^n_B$ is the fractal dimension of backbone clusters, $d^n_H$ is the hull dimension, and $d^n_\eta$ is the shortest-path dimension.

Wrapping probability. The wrapping probabilities are known to be powerful in locating a phase transition. Indeed, the non-trivial percolation threshold in the $(n = 1)$th recursive percolation is clearly demonstrated by Fig. 4 which plots the $R^n_2$ data versus bond probability $p^n$. The approximately common intersection for different sizes yields an approximate threshold $p^n_c \approx 0.65$. For $p^n < p^n_c$, the wrapping probability $R^n_2$ quickly drops to zero, illustrating the non-percolating phase. For $p^n > p^n_c$, as $L$ increases, $R^n_2$ converges to a non-trivial value $R^n_{2,d} \approx 0.85$. This implies that for $p^n > p^n_c$, the system is in a quasi-long-range ordered phase within the same universality class—namely, the two-point correlation function decays algebraically as a function of distance with a $p^n$-independent exponent. In the terminology of the
renormalization group, one has an unstable fixed point at \( p^1 = p^1_c \), a trivial fixed point \( p^1 = 0 \), and a non-trivial fixed point at \( p^1 \approx 1 \) (the second intersection at about \( p^1 \approx 0.95 \) for small sizes may move to the right for larger systems.) It is interesting to observe that using \( u = (p^1 - p^1_c) \) the fit results are shown in Table I. There is an unphysical artifact: the backbone dimension \( d^B \) may represent, respectively, the number of standard and the dense percolation clusters decreasing, namely \( (d^B, d^D) \to (d^B, d^D) \to (d^B, d^D) \to \infty \) for \( n \to \infty \), the hull dimension \( d^H \) and the maximum construction time step \( S^1_n \).

The results for the various critical exponents, as well as the corresponding “dense” ones (if applicable), are summarized in Table IV and shown in Fig. 3 in the manuscript. As \( n \) increases, it can be seen that the difference in each pair of exponents decreases, namely \( (d^B, d^D), (d^F, d^D), (d^S, d^D), \) and \( (d^R, d^D) \). This suggests, once again, that as \( n \) increases, the distinction between the standard and the dense percolation clusters disappears. Further, the decreasing of \( d^B \) implies that the percolation clusters become more and more compact.

To see whether the “standard” and “dense” critical exponents have a common limiting value for \( n \to \infty \), we simply apply a low-degree polynomial ansatz

\[
d^B_\infty = d^B_\infty + a_1 \frac{1}{n} + a_2 \frac{1}{n^2}.
\]

The fitting results are summarized in Table II. There is an unphysical artifact: the backbone dimension \( d^B_\infty \) of percolation clusters is bigger than the corresponding fractal dimension \( d^F_\infty \), and the hull dimension \( d^H_\infty \) is smaller than the dense one \( d^D_\infty \). Actually, the estimate \( d^H_\infty \) is already smaller than \( d^H_\infty \) for \( n = 2, 3, 4 \) (see Table IV in the

![Figure 1](image1.png)

FIG. 1. Wrapping probability \( R^1_2 \) versus bond probability \( p^1 \) for different sizes on square lattice. The inset shows the \( R^1_2 \) data as a function of \( u = (p^1 - p^1_c) \) \( L^{p^1_c} \), where \( p^1_c = 0.654902 \) and \( y^1_c = 0.433 \) are taken from the fit.

![Figure 2](image2.png)

FIG. 2. The wrapping probabilities \( R^1_c \) (\( R^1_c \)) and \( R^2_2 \) (\( R^2_2 \)) are shown in frames (a) and (b) respectively as functions of \( n \) at the percolation threshold \( p^c \).

The critical thresholds \( p^c \) are determined by least-squares fitting of the \( R_2 \) (and \( R_1 \)) data near \( p^c \) by Eq. (2) in the manuscript. The results for \( p^c \) are given in Table I in the manuscript. The critical values \( R^1_c \) and \( R^2_c \) are shown in Fig. 2, where the “dense” wrapping probabilities \( \overline{R^1} \) and \( \overline{R^2} \) are obtained from the ansatz

\[
\mathcal{O}(L) = \mathcal{O}_c + b_1 L^{-1} + b_2 L^{-2},
\]

with fitting parameters \( b_1 \) and \( b_2 \).

Figure 2 implies that as \( n \) increases, the difference between the “standard” and the “dense” wrapping probabilities becomes smaller and smaller and probably vanishes for \( n \to \infty \).

| \( d^O \) | \( n_{min} \) | \( a_1 \) | \( a_2 \) | \( d^S \) | \( \chi^2/DF \) |
|---|---|---|---|---|---|
| \( d^H \) | 2 | 0.43(3) | -0.23(1) | 1.38(2) | 0/0 |
| \( d^H \) | 2 | -0.058(3) | 1.422(4) | 0/1 |
| \( d^H \) | 1 | -0.080(2) | 0.028(1) | 1.428(5) | 0/1 |
| \( d^H \) | 2 | 0.040(3) | -0.014(2) | 1.822(4) | 0/1 |
| \( d^H \) | 1 | 0.051(2) | -0.014(2) | 1.820(5) | 0/1 |
| \( d^D \) | 2 | -0.0822(8) | -0.13(3) | 1.833(4) | 0/1 |
| \( d^D \) | 1 | -0.086(2) | 0.018(2) | 1.837(5) | 0/1 |
| \( d^R \) | 1 | 0.87(3) | -0.37(2) | -0.070(8) | 3/1 |
| \( d^R \) | 2 | 1.2(2) | -0.8(2) | -0.13(3) | 0/0 |
| \( d^D \) | 2 | -0.049(3) | -0.03(3) | 0/1 |
| \( d^D \) | 1 | -0.070(6) | 0.26(4) | -0.01(2) | 0/1 |
| \( d^S \) | 2 | 0.045(6) | -0.017(9) | 0.040(3) | 0/1 |
| \( d^S \) | 1 | 0.05(1) | -0.017(9) | 0.040(3) | 0/1 |

| \( d^O \) | \( n_{min} \) | \( a_1 \) | \( a_2 \) | \( d^S \) | \( \chi^2/DF \) |
|---|---|---|---|---|---|
| \( d^H \) | 2 | 0.43(3) | -0.23(1) | 1.38(2) | 0/0 |
| \( d^H \) | 2 | -0.058(3) | 1.422(4) | 0/1 |
| \( d^H \) | 1 | -0.080(2) | 0.028(1) | 1.428(5) | 0/1 |
| \( d^H \) | 2 | 0.040(3) | -0.014(2) | 1.822(4) | 0/1 |
| \( d^H \) | 1 | 0.051(2) | -0.014(2) | 1.820(5) | 0/1 |
| \( d^D \) | 2 | -0.0822(8) | -0.13(3) | 1.833(4) | 0/1 |
| \( d^D \) | 1 | -0.086(2) | 0.018(2) | 1.837(5) | 0/1 |
| \( d^R \) | 1 | 0.87(3) | -0.37(2) | -0.070(8) | 3/1 |
| \( d^R \) | 2 | 1.2(2) | -0.8(2) | -0.13(3) | 0/0 |
| \( d^D \) | 2 | -0.049(3) | -0.03(3) | 0/1 |
| \( d^D \) | 1 | -0.070(6) | 0.26(4) | -0.01(2) | 0/1 |
| \( d^S \) | 2 | 0.045(6) | -0.017(9) | 0.040(3) | 0/1 |
| \( d^S \) | 1 | 0.05(1) | -0.017(9) | 0.040(3) | 0/1 |

Critical exponents. To determine the critical exponents in Eq. (1), we carry out simulations at the estimated thresholds \( p^c \) for \( n = 1, 2, 3, 4 \). The Monte Carlo data for the quantities in Eq. (1) are analyzed by

\[
\mathcal{O}(L) = L^{d^c} (b_0 + b_1 L^{-1} + b_2 L^{-2}),
\]
This is due to the fact that we have only a limited number of data points and the ansatz is probably oversimplified. On the other hand, this supports that for $n \to \infty$, the distinction between the standard and the dense clusters vanishes, and the convergence is faster than $1/n$. On the basis of these fitting results, we estimate the limiting values of the critical exponents as in Eq. (3) of the main manuscript.

**THREE DIMENSIONS**

We have also simulated the recursive percolation model on the simple-cubic lattice for generation $n = 1$. The system size is taken as $L = 8, 16, \ldots, 256$, and periodic boundary condition are applied. The total number of samples is about $10^9$. Figure shows the wrapping probability $R^3_1$ (the probability that there exists at least one cluster wrapping all three periodic lattice directions) as a function of $p^1$ for different sizes. As for the two-dimensional case, it is clearly demonstrated that there exists a percolation threshold $p^1_c$ separating the non-percolating phase for $p^1 < p^1_c$ and the critical phase for $p^1 > p^1_c$. The crossing for small sizes suggests that the non-trivial fixed point be at $p^1 \approx 0.95$. Nevertheless, we cannot exclude that this is due to some combined effect of various correction terms, and the actual fixed point may go to $p^1 = 1$ if larger sizes were considered. The clarification of this point would request accurate numerical data, since the curves for $p^1 > p^1_c$ become close to horizontal for large $L$ — a generic property of stable fixed points. The inset of Fig. is for the wrapping probability $R^1_e$ along either of the three directions, and yields a rough estimate of the percolation threshold $p^1_c = 0.5645$. The least-squares fit of the $R^1_e$ data by Eq. (1) in the manuscript gives $p^1_c = 0.5647(2)$ and a thermal exponent $y^1_t = 0.432(8)$. It is interesting to observe that, within the error bars, the critical exponents $d^R_1 \equiv y^1_t$ in two and three dimensions are identical to each other.

![FIG. 3. Wrapping probability $R^3_1$ as a function of $p^1$ for different sizes on the simple cubic lattice. The inset is for the wrapping probability $R^1_e$ along either of the three directions.](image)

[1] P. L. Leath, Phys. Rev. B 14 5046 (1976); Z. Alexandrowicz, Phys. Lett. A 80, 284 (1980).
[2] R.J. Baxter, S.B. Kelland and F.Y. Wu, J. Phys. A: Math. Gen. 9, 397 (1976).
[3] X. Xu, J. Wang, Z. Zhou, T. M. Garoni, and Y. Deng, Phys. Rev. E 89 012120 (2014).
[4] Z. Zhou, J. Yang, Y. Deng, and R. M. Ziff, Phys. Rev. E 86, 061101 (2012).