Anisotropy of phase transition gravitational wave

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We study the anisotropy of phase transition gravitational wave (PTGW) and find it could be a new approach to probe the primordial density perturbation, especially at small scale. Generated long before recombination and free from Silk damping, PTGW could directly provide the information of density perturbation seeded from inflation or alternatives. The power spectrum of anisotropic PTGW is even enhanced at high-\(l\) due to the early integrated Sachs–Wolfe effect, making it a good probe into small scale density perturbation. We also find the amplitude of anisotropy of PTGW around nano Hertz is within the sensitive region of pulsar timing array projects like the Square Kilometre Array.

Introduction.— Unraveling the nature of the primordial seeds of our Universe is one of the most important scientific goals in fundamental physics. The density perturbation from inflation or other alternative theories provides the primordial seeds for evolution and structure formation of the Universe. The anisotropy of Cosmic Microwave Background (CMB) contains the information from the primordial density perturbation. Planck satellite has put strong constraints on the power spectrum \(C^{TT}_\ell\) of CMB temperature anisotropy in \(\ell\) range up to 2500 [1], which mainly comes from the primordial density perturbation at scales \(k < 0.2\) Mpc\(^{-1}\). However, due to the Silk damping, there is obvious suppression of the power spectrum at \(\ell > 1000\). Also, Sunyaev–Zel’dovich effect, cosmic infrared background and radio source dominate at \(\ell > 3000\) [2]. New approaches are needed to advance our understanding on the properties of density perturbation from very large scale to very small scale.

The discovery of gravitational wave (GW) at LIGO [3] opens a new window to understand the fundamental physics. Among different GW sources, phase transition gravitational wave (PTGW) [4–7] is produced long before CMB photons being last scattered. Motivated by the theoretical connection to the early universe, PTGW attracts a lot of attention which might be tested in various GW experiments including LIGO, TianQin [8, 9], LISA [10], Taiji [11], Square Kilometre Array (SKA) [12], FAST [13], etc. Recent studies from NanoGrav [14], PPTA [15], and EPTA [16] also discuss possible GW hints that might be explained by PTGW from a dark QCD phase transition. Like in the CMB case, PTGW can be also separated into an isotropic part that is relatively well studied, and an anisotropic part which lacks well understanding. The anisotropy behavior of PTGW could provide a novel approach to explore the primordial density perturbation, the dynamics of cosmic evolution, the production of dark matter, the origin of baryon-antibaryon asymmetry, the dynamics of electroweak phase transition, dark QCD phase transition, the formation of primordial black holes (PBH), etc. Especially, with the renaissance of PBH, exploring small scale information becomes attractive, since large density perturbation at small scale is needed to form PBH in most models.

Refs. [17, 18] study the effects of small-scale density perturbation on the isotropic PTGW energy spectra. Ref. [19] estimates the anisotropy in the GW background from cosmological phase transition and Ref. [20] studies large scale anisotropy of the stochastic GW background from cosmic domain wall. Here we count in the Sachs-Wolfe (SW) and integrated Sachs-Wolfe (ISW) effects, and study how the primordial density perturbation is imprinted on anisotropic PTGW analytically and numerically. We investigate the anisotropy of GW from dark QCD phase transition motivated by various dark matter models [4, 21]. Compared to the CMB temperature anisotropy, we find a stronger power spectrum in our case. Moreover, in contrast to the damping behavior of CMB, we point out that the power spectrum of PTGW anisotropy even get enhanced in high-\(\ell\) range because of the ISW effect. Therefore the anisotropic PTGW may serve as a good probe into the primordial density perturbation, especially at small scale.

PTGW from QCD-like phase transition.— We consider the PTGW from QCD-like phase transition motivated by various dark matter models [4, 21]. The first-order phase transition happened from 1 to 100 MeV can produce PTGW with peak frequency in the vicinity of nano Hertz, which is within the sensitive frequency range of pulsar timing array [22] experiments such as SKA. Three mechanisms contribute to PTGW, namely, bubble collision [6, 23], sound wave [7, 24, 25], and magnetohydrodynamic turbulence [6, 26]. The isotropic PTGW
spectrum from bubble collision is given by [23, 27]

\[
h^2 \Omega_{\text{GW}}(f) \approx 1.67 \times 10^{-5} \left( \frac{R_\phi H_*}{(8\pi)^{\frac{3}{2}}} \right)^2 \left( \frac{\kappa_\phi \alpha}{1 + \alpha} \right)^2 \left( \frac{100}{g_*} \right)^{\frac{1}{2}} \frac{0.11 v_b}{0.42 + v_b^2} \frac{3.8 (f/f_{co})^{2.8}}{1 + 2.8 (f/f_{co})^{3.8}}
\]

(1)

with the peak frequency

\[
f_{co} \approx 1.65 \times 10^{-5} \text{ Hz} \left( \frac{(8\pi)^{\frac{3}{2}}}{H_* R_*} \right) \left( \frac{0.62 v_b}{1.8 - 0.1 v_b + v_b^2} \right)
\]

(2)

where \( \alpha \) is the strength parameter of phase transition and \( g_* \) is the total degrees of freedom. \( H_* R_* \) describes the mean bubble separation scaled by Hubble rate at temperature \( T_* \) that GW generated and \( v_b \) denotes the bubble wall velocity. The efficiency parameter \( \kappa_\phi \) is the fraction of the energy released by the phase transition that converted into gradient energy of the order-parameter field. Recent studies [28–34] on bubble collision show some improvements compared to Eq. (1). These differences give some quantitative corrections to the isotropic calculation, but do not affect our anisotropy calculations.

The energy spectrum is proportional to the energy density since \( H_*^2 = \rho/3M_p^2 \). We study here with the assumption that the density perturbation would produce the anisotropy of PTGW. We choose the following benchmark parameters for instance

Benchmark 1: \( T_* = 1 \text{ MeV}, \) \( H_* R_* = 0.15 \)

Benchmark 2: \( T_* = 5 \text{ MeV}, \) \( H_* R_* = 0.2 \)

(3)

with \( \alpha = 0.9, v_b = 0.8, g_* = 10, \) and \( \kappa_\phi = 0.1 \). The efficiency parameter \( \kappa_\phi \) could be relatively large if the underlying theory can give a sufficiently strong first-order phase transition [35].

**Anisotropy of PTGW.**— After having the isotropic PTGW, we calculate its anisotropy. We employ a line-of-sight integration to describe the free-stream of gravitons and calculate the power spectrum of the PTGW anisotropy. Refs. [36, 37] develop similar method to calculate power spectrum of stochastic GW background, but we use differently in our case. We expand the distribution function of GW as

\[
f(\eta, x, \hat{p}) = \bar{f}(\eta, p) - p \frac{\partial \bar{f}(\eta, p)}{\partial p} \mathcal{G}(\eta, \mathbf{x}, \hat{p})
\]

(4)

The dimensionless quantity \( \mathcal{G}(\eta, \mathbf{x}, \hat{p}) \) characterizes the perturbation of the distribution function \( f \) and \( \mathbf{p} \) is the momentum vector of GW. Although PTGW is produced after inflation, the anisotropy is sourced from primordial density perturbation [19]. Also the anisotropy we considered here is in super-horizon scale at the time of phase transition. Thus we assume a frequency independent anisotropy in Eq. (4). We use the conformal-Newtonian gauge

\[
ds^2 = -(1 + 2\Phi)dt^2 + a^2(1 - 2\Phi)\delta_{ij}dx^idx^j
\]

(5)

The equation that rules the free-stream of GW can be derived from the Boltzmann equation

\[
\mathcal{G}' + ik\mu \mathcal{G} = \Phi' - ik\mu \Psi
\]

(6)

A prime denotes derivative with respect to the conformal time \( \eta \). Here \( \mu = \hat{k} \cdot \hat{p} \) is the cosine of angle between \( \mathbf{p} \) and the Fourier mode \( \hat{k} \). Integrating over \( \eta \) from \( \eta_0 \) to \( \eta \), we get the current anisotropy as

\[
\mathcal{G}(\eta, k, \mu) = \mathcal{G}(\eta_{pt}, k, \mu) e^{ik\mu(\eta_{pt} - \eta_0)}
\]

\[
+ \int_{\eta_{pt}}^{\eta_0} d\eta \left[ \phi'(\eta, k) - ik\mu \dot{\Phi}(\eta, k) \right] e^{ik\mu(\eta - \eta_0)}
\]

\[
= \left[ \mathcal{G}(\eta_{pt}, k) + \Phi(\eta_{pt}, k) \right] e^{ik\mu(\eta_{pt} - \eta_0)}
\]

\[
+ \int_{\eta_{pt}}^{\eta_0} d\eta \left[ \phi'(\eta, k) + \Phi'(\eta, k) \right] e^{ik\mu(\eta - \eta_0)}
\]

(7)

\( \eta_{pt} \) is the conformal time when PTGW produced. \( \mathcal{G}(\eta_{pt}) \) has no dependence on GW direction in our assumption for simplicity. We integrate the \( ik\mu \Phi \) term by parts and drop the \( \mu \) independent term \( \Phi(\eta_0, k) \) which is an undetectable alteration to the monopole of PTGW. The same procedure can be also found in CMB case, but a surface term is removed because the plasma is extremely opaque to photons, which is not the case for PTGW. The initial term for CMB vanishes for the same reason. However, in our case, the initial term \( [\mathcal{G}(\eta_{pt}, k) + \Phi(\eta_{pt}, k)] e^{ik\mu(\eta_{pt} - \eta_0)} \) is what we need and is given by the GW relic from phase transition in the early stage. This term shows exactly the SW effect for PTGW and the second term gives rise to the ISW effect which comes from the time evolution of gravitational potential. These two effects dominate the anisotropy of PTGW. Similar to the last scattering surface of CMB, here we have an emitting surface of PTGW at time \( \eta_{pt} \ll \eta_\Phi \). Free from scattering, PTGW offers a great probe into the very early stage closely after inflation.

Now we calculate the anisotropy of \( \Omega_{\text{GW}} \) from fluctuation of the distribution function. The GW energy density can be expressed as

\[
\rho_{\text{GW}}(\eta, \mathbf{x}) \equiv \int d^3p f(\eta, \mathbf{x}, \mathbf{p}) = \int d\mathbf{p} dp^3 f(\eta, \mathbf{x}, \mathbf{p})
\]

(8)
We also have
\[ \rho_{GW}(\eta, x) = \rho_c \int d\ln p \Omega_{GW}(\eta, x, p) \]  
(9)
\[ \rho_c \] is the critical density of the Universe. Thus
\[ \Omega_{GW}(\eta, x, p) = \int \frac{dp}{4\pi} \Omega_{GW}(\eta, p)[1 + \delta_{GW}(\eta, x, p, \tilde{p})] \]  
(10)
Then we separate the GW energy spectrum into an isotropic part plus the fluctuation
\[ \Omega_{GW}(\eta, x, p) = \int \frac{dp}{4\pi} \Omega_{GW}(\eta, p)[1 + \delta_{GW}(\eta, x, p, \tilde{p})] \]  
(11)
with the energy density contrast of GW as
\[ \delta_{GW}(\eta, x, p, \tilde{p}) = \left[ 4 - \frac{\partial \ln \Omega_{GW}(\eta, p)}{\partial \ln p} \right] G(\eta, x, \tilde{p}) \]  
(12)
We define the frequency dependence in the square bracket as \( g(p) \) and write in Fourier space
\[ \delta_{GW}(\eta, k, p, \mu) = g(p) \tilde{G}(\eta, k, \mu) \]  
(13)
Then the power spectrum of GW energy spectrum anisotropy writes
\[ C_{SW}^{\delta} (p) = g^2(p)C_{SW}^{\delta} \]  
(14)
For PTGW from bubble collision in radiation era, \( \Omega_{GW} \) is proportional to the energy density of radiation. Thus
\[ (\mathcal{G} + \Psi)(\eta_{pt}, k) = \frac{1}{3} R(k) \]  
(15)
where \( R \) is the primordial curvature perturbation and is conserved in super-horizon scale. All scales we are interested in today were outside the horizon at such an early time. The SW part of \( C_{SW}^{\delta} \) can be integrated simply using the spherical Bessel function \( j_\ell \) as
\[ C_{SW}^{\delta} = \frac{2}{9\pi} \int_0^\infty dk k^2 P_R(k) j_\ell^2[k(\eta_0 - \eta_{pt})] \]  
(16)
In the following, we use conventional power law parameterization \( P_R(k) \) at a pivot scale \( k_p \)
\[ P_R(k) = \frac{2\pi^2}{k^3} A_s \left( \frac{k}{k_p} \right)^{n_s - 1} \]  
(17)
Then, we have
\[ C_{SW}^{\delta} = 2^{n_s - 2} \frac{2\pi^2}{9} A_s [k_p(\eta_0 - \eta_{pt})]^{1 - n_s} \times \frac{\Gamma(l + \frac{n_s}{2} - \frac{1}{2}) \Gamma(3 - n_s)}{\Gamma(l + \frac{n_s}{2} - \frac{1}{2}) \Gamma^2(2 - \frac{n_s}{2})} \]  
(18)
In a scale-invariant case, \( \ell(\ell + 1)C_{SW}^{\delta} \) is a constant.

The ISW part of \( C_{SW}^{\delta} \) has no simple analytic expression and we estimate it at large-\( \ell \) as follows. The multiple moment is
\[ G_{SW}^{ISW}(\eta_0, k) = \int_{\eta_{pt}}^{\eta_0} d\eta (\mathcal{G} + \Psi)(\eta, k) j_\ell[k(\eta_0 - \eta)] \]  
(19)
A perturbation with wavenumber \( k \) contributes most to multiples \( \ell \sim k(\eta_0 - \eta_{pt}) \). For large scale \( k(\eta_0 - \eta_{pt}) \ll \ell \), the peak of \( j_\ell \) is always far outside the integration range. Thus the transfer function will be zero approximately. We only consider wavenumber range \( k(\eta_0 - \eta_{pt}) \gtrsim \ell \). For small scale \( k(\eta_0 - \eta_{pt}) \ll 1 \), perturbations decay rapidly after entering horizon in the very early stage. Thus we can approximate
\[ (\mathcal{G} + \Psi)(\eta, k) \approx -(\Phi + \Psi)(\eta_{pt}, k) \delta(\eta - \eta_{k}) \]  
(20)
where \( \eta_k \) is the conformal time when \( k \) mode decays. Phase transition happened long before matter-radiation equality, so we set \( \eta_{pt} \) to zero for our integration. The above approximation is valid because the peak width of \( j_\ell \) is about \( k\Delta \eta \sim 1/3 \), which is much larger compared to the peak width of \( (\Phi + \Psi) \). Thus we get the final expression for multiples at large-\( \ell \)
\[ G_{SW}(\eta_0, k) \approx [(\mathcal{G} + \Psi) - (\Phi + \Psi)(\eta_{pt}, k)] j_\ell[k(\eta_0 - \eta_{pt})] \]  
(21)
Then the initial condition is
\[ [(\mathcal{G} + \Psi) - (\Phi + \Psi)](\eta_{pt}, k) = R(k) \]  
(22)
Thus the overall power spectrum at large-\( \ell \) is 9 times larger than \( C_{SW}^{\delta} \) only. This is why PTGW has advantage over CMB to probe small scale density perturbation.

With the power spectrum, we can compute the variance of fluctuation with
\[ \text{Var}^{\delta} = \frac{1}{4\pi} \sum_{\ell = 0}^{\infty} (2\ell + 1) C_{\ell}^{\delta} \]  
(23)
For \( \delta_{GW}(\eta_0, x_0, p, \mu) \), we have \( \text{Var}^{\delta_{GW}}(p) = g^2(p)\text{Var}^{\delta} \). We employ the frequency dependent standard deviation
\[ \sigma_{GW}(p) = h^2 \Omega_{GW}(p) \sqrt{\text{Var}^{\delta_{GW}}(p)} \]  
(24)
to quantify the amplitude of PTGW anisotropy. It is worth noticing that one should use the normalized variance \( \text{Var}^{\delta} \) for evaluation of density perturbation.

Results and Discussion.— Planck [1] gives us currently the best constraint on the primordial power spectrum and we take this as the starting point: \( \ln(10^{10}A_s) = 3.044 \) and \( n_s = 0.966 \). The pivot scale is set to be \( 0.05 \text{ Mpc}^{-1} \). We modify CLASS [38] to count in the ISW effect. The power spectra are shown in Fig. 1. The blue line is \( \ell(\ell + 1)/2\pi C_{\ell}^{\delta} \) for PTGW anisotropy and the gray
Figure 1. Power Spectrum: The blue line is power spectrum $C^2_T$ for the anisotropy of phase transition gravitational waves and the gray dashed line is $C^2_{TT}$ for CMB temperature. Both of them are dimensionless, so one should multiply the TT power spectrum by the squared CMB temperature to get the conventional one in unit of $\mu K^2$.

Figure 2. Upper: The frequency dependence between the fluctuation $\delta_{GW}$ of the energy spectrum $\Omega_{GW}$ and the fluctuation $G$ of the distribution $f$. The dashed line is the frequency dependency factor $g(p)$ for benchmark 1 and the solid one is for benchmark 2. Lower: Realization of PTGW anisotropy power spectrum $C^2_{\ell GW}$.

Figure 3. The isotropic and anisotropic PTGW energy spectra: The black lines are the isotropic PTGW, dashed one for benchmark 1 and solid one for benchmark 2. The blue lines are the anisotropic PTGW, dashed one for benchmark 1 and solid one for benchmark 2.

The frequency dependence factor $g(p)$ between $\delta_{GW}$ and $G$ are shown in upper panel of Fig. 2. The dashed and solid lines are for benchmark 1 and 2 respectively. This factor equals 5 in the frequency range that is larger than the peak frequency $f_{\sigma}$. We absorb it into $C^2_{\ell GW}$ and give one realization map as an example in the lower panel. We remind readers that this is not the real distribution but only a realization of our PTGW power spectrum. The real one might be more connected to the CMB temperature map because they both have the same origin, which is the primordial density perturbation.

The amplitude of anisotropic PTGW is shown in Fig. 3. The black lines are the isotropic PTGW for benchmark 1 and 2. The blue lines show $\sigma_{GW}$ at different frequencies. The shape of $\sigma_{GW}$ differs from the isotropic PTGW $h^2\Omega_{GW}$ because of the factor $g(p)$ shown in upper panel in Fig. 2. Compared to the $4 \times 10^{-5}$ level CMB temperature anisotropy (can be computed from Planck map or result of the power spectrum), the anisotropic PTGW is at $1 \times 10^{-4}$ level for $G$ and $5 \times 10^{-4}$ for the frequency dependent $\delta_{GW}(p)$ in the range right to peak frequency. The variance is related to the resolution of observation, but the small-$\ell$ multiples contribute the most. So the value of $\sigma_{GW}$ does not change significantly if the resolu-
tion is not far from degree level. We can see anisotropy of PTGW is still within the sensitivity of SKA for both our benchmark parameter sets.

In conclusion, we have developed methodology for computing anisotropy of the phase transition gravitational wave from bubble collision. With this method, we calculated the power spectrum of the anisotropy both analytically and numerically. Sachs–Wolfe effect is considered and we find that a higher level of anisotropy show up than in CMB case and it even goes up in high-ℓ range due to the integrated Sachs–Wolfe effect. Thus we propose it to be an excellent probe of the density perturbation. The standard deviation of the fluctuation describes the anisotropy strength and we find it to be $10^{-4}$. For quantifying the detectability, we use the frequency dependent standard deviation and compute it to be as large as $5 \times 10^{-4}$ right to the peak frequency. We find that the anisotropy level is within the sensitivity region of SKA for phase transition gravitational wave around nano Hertz. We leave the detailed discussions on anisotropy of PTGW produced by sound wave and turbulence in a future work. We point out that, for these two mechanisms, ISW effect also contributes a large portion to the anisotropy.

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