Fuzzy $\alpha$-translation AB-ideal of AB-algebras

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Abstract. In this paper, we present the idea of fuzzy (normalized) fuzzy translation and enlarged mysterious extensions of semi-foggy AB and fuzzy AB-ideal on AB-algebra and examine their characteristics and related matters.

Keywords: AB-algebra, fuzzy AB-subalgebra, fuzzy AB-ideal, fuzzy translation, fuzzy extension.

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1. Introduction

Many authors ([8, 10]) made algebra of BCK. The concept of fuzzy set, presented by L.A. Zadeh [12]. Hameed et al., ([5, 13, and 15]) made a KUSideal in KUSalgebra and presented vague KUSsubalgebras concepts, and like KUSideal of KUSalgebras and studied their interconnections. In [11], they applied the concept of the fuzzy set to BCK / BCI algebra and gave some of its characteristics. At. Hameed et al ([1, 2, 7, 9, and 14]) fuzzy discussed translation, (normalized) occult S-occult extension (KUS / CI / QS) sub algebras (KUS / CI / QS) algebra. They discussed fuzzy translation and fuzzy extension of fuzzy (KUS / CI / QS) ideals in (KUS / CI / QS) algebra. Dr. Areej Tawfiq Hameed et al. ([3, 4, 6]) provided an ideal model for AB-algebras and introduced the concepts of fuzzy AB-subalgebras, and like AB-fuzzy of AB-algebras, and achieved interdependence between them. Now, we define a fuzzy translation AB-subalgebras and a fuzzy translation AB-ideals of AB-algebras and look for some of their characteristics accurately by using the concepts of fuzzy AB-subalgebra and fuzzy AB-ideal. We're looking at whether $\mu$ and $\delta$ are fuzzy AB-ideals of AB-algebras X, then $\mu^M \times \delta^M$ is a fuzzy AB-ideal of X×X.
2. Preliminaries
We review some definitions and characteristics that will be useful in our results

2.1. Definition
Let Ψ be a set with a binary operation "\*" and a constant 0. Then (Ψ, \*, 0) is named an AB-algebra (AB – g) if the come axioms satisfied: for all π, ε, ω ∈ Ψ:
(i) \(((π \* ε) \* (ω \* ε)) \* (π \* ω) = 0,\)
0 *π = 0,
π *0 = π,
Note that: Define a binary relation (≤) on Ψ by π * ε = 0 ⇔ π ≤ ε.

2.2. Proposition
In any (Ψ, \*, 0), for all π, ε, ω ∈ Ψ, the come characteristics hold:
(1) (π \* ε) \* π = 0.
(2) π ≤ ω implies π \* ω ≤ ε \* ω.
(3) π ≤ ω implies ω \* ε ≤ ω \* π.

2.3. Notice
An AB-algebra (Ψ, \*, 0) is satisfies for all π, ε, π ∈ Ψ
(1) (π \* ε) \* ω = (π \* ω) \* ε,
(2) (π \* π) \* ε = 0.

2.4. Definition
Let (Ψ, \*, 0) be an (Ψ, \*, 0), and Ɨ ⊆ Ψ. Ɨ is named an AB-ideal of Ψ if it satisfies the come conditions:
0 ∈ Ɨ,
π \* (ε \* ω) ∈ Ɨ and ε ∈ Ɨ imply π \* ω ∈ Ɨ.

2.5. Definition
A set μ is named a fuzzy relation on any set S, if μ is a fuzzy subset μ : Ψ × Ψ → [0,1].

2.6. Definition
A fuzzy subset μ of (Ψ, \*, 0) is named a fuzzy AB-subalgebra of X if
μ(π \* ε) ≥ min{ μ(π), μ(ε)}, for all π, ε, ω ∈ Ψ.

2.7. Definition
A fuzzy subset μ of AB-algebra (Ψ, \*, 0) is named a fuzzy AB-ideal of X if it satisfies:
FAB1) μ(0) ≥ μ(π);
FAB2) μ(π \* ω) ≥ min{ μ(π \* ε), μ(ε)}, for all π, ε, ω ∈ Ψ.

2.8. Proposition
1- Every (Ψ, \*, 0) is an (Ψ, \*, 0).
2- Every (Ψ, \*, 0) is a (Ψ, \*, 0).

2.9. Definition
Let f : (Ψ, \*, 0) → (Ψ, \*, 0) be a mapping from an AB – g Ψ into an AB – g Y. If μ is a fuzzy subset of Ψ, then
f(μ)(ε) = \{sup_{π∈f^{-1}(ε)} μ(π), f^{-1}(ε) = \{π ∈ Ψ | f(π) = ε \} \neq \emptyset \}
0 otherwise
is said to be the image of μ under f and is denoted by f(μ).
2.10. Definition
Let \( f : (\Psi; \ast, 0) \rightarrow (Y; \ast', 0') \) be a mapping from an \( AB - g \), \( \Psi \) into an \( AB - g \), \( Y \). If \( \beta \) is a fuzzy subset of \( AB - g \), \( Y \), \( \Rightarrow \) the fuzzy subset \( \mu = \beta \circ f \) of \( \Psi \) (i.e., the fuzzy subset defined by \( \mu (\pi) = \beta (f(\pi)) \), for all \( \pi \in \Psi \)) is named the pre-image of \( \beta \) under \( f \).

2.11. Proposition
Let \( f : (\Psi; \ast, 0) \rightarrow (Y; \ast', 0') \) be a homo. from \( \Psi \) into \( Y \) and \( A \) be a fuzzy subset of \( \Psi \), the image \( f(A) \) is a fuzzy subset of \( Y \).

2.12. Proposition
Let \( f : (\Psi; \ast, 0) \rightarrow (Y; \ast', 0') \) be a homo. from \( \Psi \) into \( Y \) and \( B \) be a fuzzy subset of \( Y \), the inverse image \( f^{-1}(B) \) is a fuzzy subset of \( \Psi \).

2.13. Definition
Let \( (\Psi; \ast, 0) \) be a nonempty set and \( \mu \) be a fuzzy subset of \( \Psi \) and let \( \alpha \in [0, T] \). A mapping \( \mu^T : \Psi \rightarrow [0, 1] \) is named a translation fuzzy subset of \( \mu \) if it satisfies: \( \mu^T(\pi) = \mu(\pi) + \alpha \), for all \( \pi \in \Psi \), where \( T = 1 - \sup \{ \mu(\pi) : \pi \in \Psi \} \).

3. Fuzzy \( \alpha \)-translations \( AB \)-subalgebra of \( AB \)-algebra
In this part, we discuss translation on \( AB \)-algebra and we get some of relations, theorems, propositions and give Examples "of translation " of fuzzy \( AB \)-subalgebra. We show the notion of translation fuzzy \( AB \)-subalgebra of \( AB \)-algebra and investigate some of their characteristics.

In what follows, let \( (\Psi; \ast, 0) \) denote an \( AB - g \), and for any fuzzy subset \( \mu \) of \( \Psi \), we denote \( T = 1 - \sup \{ \mu(\pi) : \pi \in \Psi \} \).

3.1. Definition
Let \( \mu \) be a fuzzy subset of an \( AB - g \) \( (\Psi; \ast, 0) \) and let \( \alpha \in [0, T] \). A mapping \( \mu^T_\alpha : \Psi \rightarrow [0, 1] \) is named a translation fuzzy subset of \( \mu \) if it satisfies: \( \mu^T_\alpha(\pi) = \mu(\pi) + \alpha \), for all \( \pi \in \Psi \).

3.2. Definition
Let \( (\Psi; \ast, 0) \) be an \( AB \)-algebra, a fuzzy subset \( \mu \) in \( \Psi \) is named a translation fuzzy \( AB \)-subalgebra of \( \Psi \) if \( \forall \pi, \varepsilon \in \Psi, \mu^T_\alpha(\pi \ast \varepsilon) \geq \min \{ \mu^T_\alpha(\pi), \mu^T_\alpha(\varepsilon) \} \).  

3.3. Example
Let \( \Psi = \{0,1,2,3\} \) in which \( \ast \) be defined by the come table:

| \ast | 0 | 1 | 2 | 3 |
|-----|---|---|---|---|
| 0   | 0 | 0 | 0 | 0 |
| 1   | 1 | 0 | 0 | 0 |
| 2   | 2 | 0 | 0 | 0 |
| 3   | 3 | 3 | 3 | 0 |

\( TABLE (1) \)

(\( \Psi; \ast, 0 \)) is an \( AB - g \). It is easy to show that \( S_1 = \{0,1\}, S_2 = \{0,2\}, S_3 = \{0,3\} \) and \( S_4 = \{0,1,2,3\} \) are \( AB - sg \) of \( \Psi \).

Define a fuzzy subset \( \mu : \Psi \rightarrow [0,1] \not\exists \mu(0) = t_1, \mu(1) = \mu(2) = \mu(3) = t_2 \), where \( t_1, t_2 \in [0, 1] \) and \( t_1 > t_2 \).
Gives account that $\mu$ is the $FAB - sg$ for $\Psi$.

3.4. Theorem

Let $\mu$ be a $FAB - sg$ of $AB - g (\Psi, *, 0)$ and $\alpha \in [0,T] \Rightarrow \mu^T_{\alpha}$ is a $FAB - sg$ of $\Psi$.

Proof:

Let's say $\mu$ is a $FAB - sg$ of $\Psi$, and $\alpha \in [0,T]$. Let $\pi, \varepsilon \in \Psi \Rightarrow \mu(\pi \ast \varepsilon) \geq \min\{\mu(\pi), \mu(\varepsilon)\}$. Thus

$$\mu(\pi \ast \varepsilon) + \alpha \geq \min\{\mu(\pi) + \alpha, \mu(\varepsilon) + \alpha\},$$

and so

$$\mu(\pi \ast \varepsilon) + \alpha = \mu^T_{\alpha}(\pi) \ast \varepsilon \geq \min\{\mu^T_{\alpha}(\pi), \mu^T_{\alpha}(\varepsilon)\}.$$

Subsequently, $\mu^T_{\alpha}$ is a $FAB - sg$ of $\Psi$. $\square$

3.5. Theorem

Let $\mu$ be a fuzzy subset of $AB - g (\Psi, *, 0)$ such that $\mu^T_{\alpha}$ of $\mu$ is a $FAB - sg$ of $\Psi$, for some $\alpha \in [0,T]$. Let $\pi, \varepsilon \in \Psi$, $\mu^T_{\alpha}(\pi) \ast \varepsilon = \min\{\mu^T_{\alpha}(\pi), \mu^T_{\alpha}(\varepsilon)\} = \min\{\mu(\pi) + \alpha, \mu(\varepsilon) + \alpha\}$ and so $\mu(\pi \ast \varepsilon) \geq \min\{\mu(\pi), \mu(\varepsilon)\}$.

Subsequently, $\mu$ is a $FAB - sg$ of $\Psi$. $\square$

3.6. Definition

For a fuzzy subset $\mu$ of an $AB - g (\Psi, *, 0)$, $\alpha \in [0,T]$ and $t \in \text{Im}(\mu)$ with $t \geq \alpha$, let $U_{\alpha}(\mu; t) := \{\pi \in \Psi | \mu(\pi) \geq t - \alpha \}$. 

3.7. Notice

If $\mu$ is a $FAB - sg$ of $X$, for all $t \in \text{Im}(\mu)$ with $t \geq \alpha$, $U_{\alpha}(\mu; t) \Rightarrow \mu(\pi \ast \varepsilon) \geq \min\{\mu(\pi), \mu(\varepsilon)\} \geq t - \alpha$, that is $\mu(\pi \ast \varepsilon) \geq \min\{\mu(\pi), \mu(\varepsilon)\} \geq t - \alpha$, therefor $\pi \ast \varepsilon \in \text{Im}(\mu)$. However, if we do not give a condition that $\mu$ is a $FAB - sg$ of $\Psi$, $\Rightarrow U_{\alpha}(\mu; t)$ is not an $AB - sg$ of $\Psi$ as seen in the come Example.

3.8. Example

Consider $\Psi = \{0, 1, 2, 3\}$ is an $AB - g$ which is given in Example 2.3.

Define a fuzzy subset $\lambda$ of $\Psi$:

| $\Psi$ | 0 | 1 | 2 | 3 |
|--------|---|---|---|---|
| $\lambda$ | 0.7 | 0.6 | 0.4 | 0.3 |

$\lambda$ is not a $FAB - sg$ of $\Psi$.

Ago, $\lambda(1 \ast 2) = 0.3 < 0.4 = \min\{\lambda(1), \lambda(2)\}$. For $\alpha = 0.1$, $\beta = 0.9$ and $t = 0.5$, we obtain $U_{\alpha}(\lambda; t) = \{0, 1, 2\}$ which is not an $AB - sg$ of $\Psi$. Ago, $1 \ast 2 = 3 \notin U_{\alpha}(\lambda; t)$.

3.9. Proposition

Let $\mu$ be a fuzzy subset of an $AB - g \Psi$ and $\alpha \in [0,T]$. $\mu^T_{\alpha}$ is a $FAB - sg$ of $\Psi$ $\Rightarrow U_{\alpha}(\mu; t)$ is an $AB - sg$ of $\Psi$, for all $t \in \text{Im}(\mu)$ with $t \geq \alpha$.

Proof:
Necessity is clear (let's say, $\mu^T_\alpha$ is a FAB $- sg$ by Theorem (2.5), $\Rightarrow \mu$ is a FAB $- sg$, by Notice (2.7), $\Rightarrow U_\alpha (\mu; t)$ is a FAB $- sg$).

To prove the conversely, Let's say $x, y \in U_\alpha (\mu; t)$ and $\mu^T_\alpha$ of $\mu$ is not a FAB $- sg$ of $X$, therefore $\mu^T_\alpha (\pi * \varepsilon) < t \leq \min \{\mu^T_\alpha (\pi), \mu^T_\alpha (\varepsilon)\}$.

$\Rightarrow \mu(\pi) \geq t - \alpha$ and $\mu(\varepsilon) \geq t - \alpha$. However, $\mu(\pi * \varepsilon) < t - \alpha$. This shows that $\pi * \varepsilon \notin U_\alpha (\mu; t)$. This is a contradiction, and so $\mu^T_\alpha (\pi * \varepsilon) \geq \min \{\mu^T_\alpha (\pi), \mu^T_\alpha (\varepsilon)\}$, for all $\pi, \varepsilon \in X$. Subsequently, $\mu^T_\alpha$ is a FAB $- sg$ of $\Psi$. □

3.10. Theorem

Let $f : (\Psi; \ast, 0) \rightarrow (Y; \ast', 0')$ be an onto homo. between $AB - g \Psi$ and $AB - g Y$.

For every $T \text{FAB} - sg \mu^T_\alpha$ of $\Psi$, $f (\mu^T_\alpha)$ is a $T \text{FAB} - sg$ of $Y$.

Proof:

By definition $\lambda^T_\alpha (\varepsilon') = f (\mu^T_\alpha (\varepsilon')) = \sup_{\mu \in f^{-1}(\varepsilon')} \mu(\pi) + \alpha$, for all $\varepsilon' \in Y$ (sup $\emptyset = 0$).

By Proposition (1.10). Subsequently, $f (\mu^T_\alpha)$ is a FAB $- sg$ of $Y$. □

3.11. Theorem

An homomorphic pre-image of a $T \text{FAB} - sg$ of $AB - g$ is also a $T \text{FAB} - sg$ of $AB - g$.

Proof:

Let $f : (\Psi; \ast, 0) \rightarrow (Y; \ast', 0')$ be a homo. of $AB$-algebra, $\lambda$ the $T \text{FAB} - sg$ of $Y$ and $\mu$ the pre-image of $\lambda$ under $f \Rightarrow \mu^T_\alpha (\pi) = \lambda^T_\alpha (f (\pi))$, $\forall \pi \in \Psi$.

By Proposition (1.11). Subsequently, $\mu^T_\alpha$ is an $AB - sg$ of $\Psi$. □

3.12. Definition

Let $(\Psi; \ast, 0)$ be a $AB - g$, $\mu_1$ and $\mu_2$ be fuzzy subsets of $\Psi$. $\Rightarrow \mu_2$ is named a fuzzy S-extension of $\mu_1$ if the come assertions are valid:

(S1) $\mu_2$ is a fuzzy extension of $\mu_1$.

(S2) If $\mu_1$ is a FAB $- sg$ of $\Psi$, $\Rightarrow \mu_2$ is a FAB $- sg$ of $\Psi$.

3.13. Proposition

Let $\mu$ be a FAB $- sg$ of an $AB - g (\Psi, \ast, 0)$ and $\alpha \in [0,T]$ $\Rightarrow$ the translation fuzzy subset $\mu^T_\alpha$ of $\mu$ is a fuzzy S-extension of $\mu$.

Proof:

$\Lambda_\alpha \mu (\pi) = \mu (\pi) + \alpha \geq \mu (\pi)$, $\Rightarrow \mu^T_\alpha (\pi)$ is a fuzzy extension of $\mu (\pi)$, for all $\pi \in \Psi$ and $\Lambda_\alpha \mu$ is a FAB $- sg$ of $\Psi$ $\Rightarrow \mu^T_\alpha$ of $\mu$ is a FAB $- sg$ (by Theorem (2.4)). □

In general, the converse of Proposition (2.13) is not true as seen in the come Example .

3.14. Example

Let $\Psi = \{0, 1, 2, 3\}$ be an $AB - g$ which is given in Example (2.2).

Define a FAB $- sg$ $\mu$ of $\Psi$ by:

| $\Psi$ | 0 | 1 | 2 | 3 |
|-------|---|---|---|---|
| $\mu$ | 0.8 | 0.5 | 0.7 | 0.5 |

TABLE (3)
μ is a $FAB - sg$ of Ψ. Let ν be fuzzy subsets of Ψ where $\alpha=0.1$ given by:

| $\Psi$ | 0 | 1 | 2 | 3 |
|--------|---|---|---|---|
| $\mu^T_T$ | 0.9 | 0.6 | 0.8 | 0.6 |

TABLE (4)

$\mu^T_T$ is a fuzzy S-extension of $\mu$. However, the $\mu$ is not a fuzzy S-extension of $\mu^T_T$.

**3.15. Proposition**

The interpart of fuzzy S-extension of a fuzzy subset $\mu$ of $(\Psi, *, 0)$ is a fuzzy S-extension of $\mu$.

**Proof:**

Let $\{\mu_i|i\in\Lambda\}$ be a family of fuzzy S-extension of $\mu$ of $AB - g$ $(\Psi, *, 0) \Rightarrow$ for any $\pi, \xi \in \Psi, i \in \Lambda,

$$(\bigwedge_{i \in \Lambda} \mu_i) (\pi \ast \xi) = \inf (\mu_i (\pi \ast \xi)) \geq \inf (\min \{\mu_i(\pi), \mu_i(\xi)\})
$$

$ = \min \{\inf (\mu_i(\pi)), \inf (\mu_i(\xi))\} = \min \{(\bigwedge_{i \in \Lambda} \mu_i)(\pi), (\bigwedge_{i \in \Lambda} \mu_i)(\xi)\}. \square$

**3.16. Notice**

The union of fuzzy S-extension of a fuzzy subset $\mu$ of $X$, is not a fuzzy S-extension of $\mu$ as seen in the come Example.

**3.17. Example**

Let $\Psi = \{0, 1, 2, 3\}$ be a $FAB - g$ which is given in Example (2.2). Define a fuzzy subset $\mu$ of $\Psi$ by:

| $\Psi$ | 0 | 1 | 2 | 3 |
|--------|---|---|---|---|
| $\mu$  | 0.8 | 0.5 | 0.7 | 0.5 |

TABLE (5)

$\mu$ is a $FAB - sg$ of $\Psi$. Let $\nu$ and $\delta$ be fuzzy subsets of $X$ given by:

| $\Psi$ | 0 | 1 | 2 | 3 |
|--------|---|---|---|---|
| $\nu$  | 0.8 | 0.6 | 0.7 | 0.7 |
| $\delta$ | 0.9 | 0.5 | 0.8 | 0.5 |
| $\nu \cup \delta$ | 0.9 | 0.6 | 0.8 | 0.7 |

TABLE (6)

$\nu$ and $\delta$ are fuzzy S-extinsion of $\mu$. However, the union $\nu \cup \delta$ is not a fuzzy S-extinsion of $\mu$ Ago $(\nu \cup \delta)(3 \ast 2) = 0.6 < 0.7 = \min\{(\nu \cup \delta)(3), (\nu \cup \delta)(2)\}$.

**3.18. Definition**

For a fuzzy subset $\mu$ of an $AB - g$ $(\Psi, *, 0)$, $\alpha \in [0, T]$ and $t \in [0, 1]$ with $t \geq \alpha$, let $U_\alpha(\mu; t) = \{ \pi \in \Psi | \mu(\pi) \geq t - \alpha \}$.

If $\mu$ is a $FAB - sg$ of $X$, $\Rightarrow U_\alpha(\mu; t)$ is an $AB - sg$ of $\Psi$, $\forall t \in \Im(\mu)$ with $t \geq \alpha$.

However, if we do not give a condition that $\mu$ is a $FAB - sg$ of $\Psi$, $\Rightarrow U_\alpha(\mu; t)$ is not an $AB - sg$ of $\Psi$ as seen in the come Example.
3.19. Example

Let $\Psi = \{0, 1, 2, 3\}$ be an $AB - g$ which is given in Example (2.2). Define a fuzzy subset $\lambda$ of $\Psi$ by:

| $\Psi$ | 0 | 1 | 2 | 3 |
|-------|---|---|---|---|
| $\lambda$ | 0.7 | 0.6 | 0.4 | 0.3 |

TABLE (7)

$\lambda$ is not a $FAB - sg$ of $\Psi$. Ago $\lambda(1 \ast 2) = \lambda(3) = 0.3 < 0.4 = \min\{\lambda(1), \lambda(2)\}$. For $\alpha = 0.1$ and $t = 0.5$, we obtain $U_\alpha(\lambda; t) = \{0, 1, 2\}$ which is not an $AB - sg$ of $\Psi$. Ago $1 \ast 2 = 3 \in U_\alpha(\lambda; t)$.

3.20. Proposition

Let $\mu$ be a $FAB - sg$ of $AB - g$ ($\Psi$, $\ast$, 0) and $\alpha$, $\lambda \in [0, T]$. If $\alpha \geq \lambda$, the $TFAB - sg \mu_\alpha^T$ of $\mu$ is a fuzzy S-extension of the $TFAB - sg \mu_\lambda^T$ of $\mu$.

Proof:

For every $\pi \in \Psi$ and $\alpha, \lambda \in [0, T]$ and $\alpha \geq \lambda$, we have

$\mu_\alpha^T(\pi) = \mu(\pi) + \alpha \geq \mu(\pi) + \lambda = \mu_\lambda^T(\pi)$, therefore $\mu_\alpha^T(\pi)$ is a fuzzy extension of $\mu_\lambda^T(\pi)$. Ago $\mu$ is a $FAB - sg$ of $\Psi$, $\Rightarrow \mu_\lambda^T$ of $\mu$ is a $FAB - sg$ (by Theorem (2.4)).

Subsequently, $\mu_\alpha^T$ of $\mu$ is a fuzzy S-extension of the $FAB - sg \mu_\lambda^T$ of $\mu$. □

3.21. Proposition

Let $\mu$ be a $FAB - sg$ of $AB - g$ $\Psi$ and $\lambda \in [0, T]$. For every fuzzy S-extension $\nu$ of the $TFAB - sg \mu_\lambda^T$ of $\mu$, there exists $\alpha \in [0, T]$ such that $\alpha \geq \lambda$ and $\nu$ is a fuzzy S-extension of the $TFAB - sg \mu_\alpha^T$ of $\mu$.

Proof:

Ago $\mu$ is a $FAB - sg$ of an $AB - g$ ($\Psi$, $\ast$, 0) and $\lambda \in [0, T]$, the fuzzy translation subset $\mu_\lambda^T$ of $\mu$ is a $FAB - sg$ of $\Psi$. If $\nu$ is a fuzzy S-extension of $\mu_\lambda^T$, $\Rightarrow \exists \alpha \in [0, T] \exists \alpha \geq \lambda$ and $\nu(\pi) \geq \mu_\alpha^T(\pi), \forall \pi \in \Psi$, Subsequently, $\nu$ is a fuzzy S-extension of the $TFAB - sg \mu_\alpha^T$ of $\mu$. □

The come Example illustrates Proposition (2.21).

3.22. Example

Let $\Psi = \{0, 1, 2, 3\}$ be an $AB - g$ which is given in Example (1.1.16). Define a fuzzy subset $\mu$ of $\Psi$ by:

| $\Psi$ | 0 | 1 | 2 | 3 |
|-------|---|---|---|---|
| $\mu$ | 0.7 | 0.5 | 0.4 | 0.4 |

TABLE (8)

$\mu$ is a $FAB - sg$ of $\Psi$ and $T=0.3$. If we take $\lambda = 0.2$, $\Rightarrow$ the $TFAB - sg \mu_\lambda^T$ of $\mu$ is given by:

| $\Psi$ | 0 | 1 | 2 | 3 |
|-------|---|---|---|---|
| $\mu_\lambda^T$ | 0.9 | 0.7 | 0.6 | 0.6 |

TABLE (9)
Let \( \nu \) be a fuzzy subset of \( \Psi \) defined by:

\[
\begin{array}{c|cccc}
\Psi & 0 & 1 & 2 & 3 \\
\nu & 0.94 & 0.76 & 0.64 & 0.64 \\
\end{array}
\]

\[\text{TABLE (10)}\]

\( \nu \) is clearly a \( FAB - sg \) of \( \Psi \) which is fuzzy extinsion of \( \mu^T \) and subsequently, \( \nu \) is a fuzzy S-extinsion. Translation fuzzy subset \( \mu^T \) of \( \mu \). Take \( \alpha = 0.23 \), \( \lambda = 0.23 > 0.2 = \lambda \), and the \( TFAB - sg \mu^T \) of \( \mu \) is given as follows: Note that \( \nu(\pi) \geq \mu^T(\pi) \), for all \( \pi \in \Psi \), and subsequently, \( \nu \) is a fuzzy S-extinsion of the \( TFAB - sg \mu^T \) of \( \mu \).

3.23. Definition

A fuzzy S-extinsion \( \nu \) of a \( FAB - sg \mu \) in an \( AB - g \) (\( \Psi, \ast, 0 \)) is said to be normalized if there exists \( \pi \in \Psi \) such that \( \nu(\pi) = 1 \).

Let \( \mu \) be a \( FAB - sg \) of \( \Psi \). A fuzzy subset \( \nu \) of \( \Psi \) is named a maximal fuzzy S-extension of \( \mu \) if it satisfies:

1. \( (M_1) \) \( \nu \) is a fuzzy S-extension of \( \mu \),
2. \( (M_2) \) there is not exists another \( FAB - sg \) of an \( AB - g \) \( \Psi \), which is a fuzzy extinsion of \( \nu \).

3.24. Example

Let \( \Psi = \{0, a, b, c\} \) be an \( AB - g \) which is given in Example (2.22).

Let \( \mu \) and \( \nu \) be fuzzy subsets of \( \Psi \) which are defined by \( \mu(\pi) = \frac{1}{5} \) and \( \nu(\pi) = 1 \), for all \( \pi \in \Psi \).

Clearly \( \mu \) and \( \nu \) are \( FAB - sg \) of \( \Psi \). It is easy to verify that \( \nu \) is a maximal fuzzy S-extension of \( \mu \).

3.25. Proposition

If a fuzzy subset \( \nu \) of an \( AB - g \) \( \Psi \) is a normalized fuzzy S-extension of a \( FAB - sg \mu \) of \( \Psi \), \( \nu(0) = 1 \).

Proof:

It is clear because \( \nu(0) \geq \nu(\pi) \), for all \( \pi \in \Psi \). □

3.26. Notice

Let \( \mu \) be a \( FAB - sg \) of an \( AB - g \) (\( \Psi, \ast, 0 \)) \( \Rightarrow \) every maximal fuzzy S-extension of \( \mu \) is normalized.

4. translation of Fuzzy AB-ideals

In this part, we shall define the notion of translation of fuzzy AB-ideals, and we study some of the relations, theorems, propositions and Example s of translation of fuzzy AB-ideals of \( AB - g \).

4.1. Definition

Let \( (\Psi, \ast, 0) \) be an \( AB - g \) and \( I \) be a nonempty subset of \( \Psi \). \( \Rightarrow \) \( I \) is named an AB-ideal of \( \Psi \) (\( TFAB - i \)), if it satisfies:

\( 0 \in I \),

\( ((\pi \ast \varepsilon) \ast \omega) \in I \) and \( \varepsilon \in I \) imply \( (\pi \ast \omega) \in I \), for all \( \pi, \varepsilon, \omega \in \Psi \).
4.2. Example

Let $\Psi = \{0, 1, 2, 3\}$ in which $(\ast)$ be defined by the come table:

\[
\begin{array}{cccc}
0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
2 & 2 & 2 & 0 & 0 \\
3 & 3 & 3 & 3 & 0 \\
\end{array}
\]

\textit{TABLE (11)}

($\Psi; \ast, 0$) is an $AB - g$. It is easy to show that $I_1 = \{0, 1\}$ and $I_2 = \{0, 1, 2, 3\}$ are $AB$-ideals of $\Psi$.

4.3. Definition

Let $(\Psi, \ast, 0)$ be an $AB - g$, a translation fuzzy subset $\mu$ of $\Psi$ is named a translation fuzzy $AB$-ideal of $\Psi$, if it satisfies the come conditions: $\forall \pi, \epsilon, \omega \in \Psi$,

- (FAB1) $\mu^T(0) \geq \mu^T(\pi)$,
- (FAB2) $\mu^T(\pi \ast \omega) \geq \min \{ \mu^T((\pi \ast \epsilon) \ast \omega), \mu^T(\epsilon) \}$.

4.4. Example

Let $\Psi = \{0, 1, 2, 3\}$ be a $AB - g$ which is given in Example (2.2).

Define a fuzzy subset $\mu : \Psi \rightarrow [0, 1]$ by $\mu(\pi) = \begin{cases} 0.7 & \text{if } \pi \in \{0,1\} \\ 0.3 & \text{otherwise} \end{cases}$

$I = \{0, 1\}$ is an $AB - \ast$ of $\Psi$. Routine calculation gives that $\mu$ is a $FAB - \ast$ of $\Psi$.

Consider $\Psi = \{0, 1, 2, 3\}$ is an $AB - g$ which is given in Example (3.2).

Define a fuzzy subset $\mu : \Psi \rightarrow [0, 1]$ such that $\mu(0) = t_1$, $\mu(1) = \mu(2) = \mu(3) = t_2$, where $t_1, t_2 \in [0, 1]$ and $t_1 > t_2$. Routine calculation gives that $\mu$ is a $FAB - \ast$ of $\Psi$.

4.5. Theorem

Let $\mu$ is a $FAB - \ast$ of an $AB - g$, $(\Psi, \ast, 0)$ $\Rightarrow \mu^T_\alpha$ is a $FAB - \ast$ of $\Psi$, $\forall \alpha \in [0, T]$.

Proof : Assume that $\mu$ is a $FAB - \ast$ of $\Psi$ and let $\alpha \in [0, T] \Rightarrow \forall \pi, \epsilon, \omega \in \Psi$,

- $\mu^T_\alpha(0) = \mu(0) + \alpha \geq \mu(\pi) + \alpha = \mu^T_\alpha(\pi)$.
- $\mu^T_\alpha(\pi \ast \omega) = \mu(\pi \ast \omega) + \alpha \geq \min\{\mu(\pi \ast (\epsilon \ast \omega)), \mu(\epsilon)\} + \alpha$
- $= \min\{\mu((\pi \ast \epsilon) \ast \omega) + \alpha, \mu(\epsilon) + \alpha\} = \min\{\mu^T_\alpha((\pi \ast \epsilon) \ast \omega), \mu^T_\alpha(\epsilon)\}$.

Subsequently, $\mu^T_\alpha$ is a $FAB - \ast$ of $\Psi$. □

4.6. Theorem

Let $\mu$ be a fuzzy subset of $AB - g$, $(\Psi, \ast, 0) \ni \mu^T_\alpha$ is a $FAB - \ast$ of $\Psi$, for some $\alpha \in [0, T] \Rightarrow \mu$ is a $FAB - \ast$ of $\Psi$.

Proof :
Let’s say $\mu_a^T$ is a TFAB $- i$ of $\Psi$, for some $\alpha \in [0, T]$. Let $\pi, \varepsilon, \omega \in \Psi$, we have $\mu(0) + \alpha = \mu_a^T(0) \geq \mu_a^T(\pi) = \mu(\pi) + \alpha$ and so $\mu(0) \geq \mu(\pi)$.

$$\mu(\pi * \omega) + \alpha = \mu_a^T(\pi * \omega) \geq \min\{\mu_a^T((\pi * \varepsilon) * \omega), \mu_a^T(\varepsilon)\},$$

where $0$ and $0'$ are the zero elements of $\Psi$ and $\Psi$, respectively.

Conversely, suppose that $U_a(\mu; t)$ is an $AB - i$ of $\Psi$, for every $t \in \operatorname{Im}(\mu)$ with $t > \alpha$. If $\exists \pi \in \Psi$ such that $\mu_a^T(0) < t \leq \mu_a^T(\pi), \Rightarrow \mu(\pi) > t = \alpha$.

In such a case, we have $\mu(\pi; t)$, for all $\pi \in U_a(\mu; t)$, hence $U_a(\mu; t)$ is a $AB - i$ of $\Psi$.

Therefore, as $\mu_a^T(\pi * \omega) \geq \min\{\mu_a^T((\pi * \varepsilon) * \omega), \mu_a^T(\varepsilon)\}$, $\forall \pi, \varepsilon, \omega \in \Psi$.

Subsequently, $\mu_a^T$ is a $FAB - i$ of $\Psi$. $\square$

$4.7. \textbf{Theorem}$

For $\alpha \in [0, T]$, let $\mu_a^T$ be the translation fuzzy subset $\mu$ of $AB - g (\Psi, *, 0)$. The come are equivalent:

1. $\mu_a^T$ is a $FAB - i$ of $\Psi$.
2. $\forall t \in \operatorname{Im}(\mu), t > \alpha \Rightarrow U_a(\mu; t)$ is an $AB - i$ of $\Psi$.

Proof:

Let’s say $\mu_a^T$ is a TFAB $- i$ of $\Psi$ and let $t \in \operatorname{Im}(\mu)$ be $\exists t > \alpha$. Ago, $\mu_a^T(0) \geq \mu_a^T(\pi), \forall \pi \in \Psi$, we have $\mu(0) + \alpha = \mu_a^T(0) \geq \mu_a^T(\pi) = \mu(\pi) + \alpha \geq t, \forall \pi \in U_a(\mu; t)$.

Subsequently, $0 \in U_a(\mu; t)$. Let $\pi, \varepsilon, \omega \in \Psi, \beta \neq 0$ such that $(\pi * (\varepsilon * \omega)) \in U_a(\mu; t)$ and $\exists \in U_a(\mu; t)$, $\mu((\pi * (\varepsilon * \omega)) \geq t = \alpha$.

i.e., $\mu_a^T((\pi * \varepsilon) * \omega) = \mu((\pi * \varepsilon) * \omega) + \alpha \geq t$ and $\mu_a^T(\varepsilon) = \mu(\varepsilon) + \alpha \geq t$.

Ago $\mu_a^T$ is TFAB $- i$ of $\Psi$, it follows that $\mu(\pi * \varepsilon) + \alpha = \mu_a^T(\pi * \varepsilon) \geq \min\{\mu_a^T((\pi * \varepsilon) * \omega), \mu_a^T(\varepsilon)\} \geq t$, that is $\mu(\pi * \varepsilon) \geq t = \alpha$, so that $(\pi * \varepsilon) \in U_a(\mu; t)$, therefore $U_a(\mu; t)$ is a $AB - i$ of $\Psi$.

4.8. $\textbf{Theorem}$

Let $f : (\Psi, *, 0) \rightarrow (Y, *, 0')$ be an onto homomorphism between $AB - g \Psi$ and $AB - g Y$. For every $\lambda_a^T \in \Psi$, $f(\mu)$ is a TFAB $- i$ of $Y$.

Proof:

Let $f : (\Psi, *, 0) \rightarrow (Y, *, 0')$ be an onto homomorphism of $AB - g, \mu$ is a TFAB $- i$ of $\Psi$ and $\lambda_a^T$ the image of $\mu_a^T$ under $f$. Ago is a TFAB $- i$ of $\Psi$, we have $\mu_a^T(0) \geq \mu_a^T(\pi), \forall \pi \in \Psi$.

Note that $0 \in f(0')$, where $0$ and $0'$ are the zero elements of $\Psi$ and $Y$, respectively.

Thus $\lambda_a^T(0') = f(\mu_a^T(0)) = \sup_{\lambda \in f^{-1}(0')} \mu(\lambda) + \alpha = \mu_a^T(0) \geq \mu_a^T(\pi)$, for all $\pi \in \Psi$, which implies that $\lambda_a^T(0') \geq \sup_{\lambda \in f^{-1}(\alpha)} \mu(\lambda) + \alpha = \lambda_a^T(\pi')$.

For any $\pi', \varepsilon', \omega' \in Y$, let $0 \in f^{-1}(0')$, $E0 \in f^{-1}(\varepsilon')$, $\omega_0 \in f^{-1}(\omega)$ be such that $f(\mu_a^T(\pi' * \varepsilon') * \omega') = \sup_{\lambda \in f^{-1}(\pi') + \alpha = \lambda_a^T(\pi')}$.
\[ f(\mu_i^T) \left( (\pi' \ast' e') \ast' \omega' \right) = \lambda_{\alpha}^T \left( (\pi' \ast' e') \ast' \omega' \right) \\
= \sup_{t \in f^{-1}((\pi', e') \ast (\omega))} \mu \left( (\pi_0 \ast e_0) \ast \omega_0 \right) + \alpha = \sup_{t \in f^{-1}((\pi', e') \ast (\omega))} \mu \left( (\pi_0 \ast e_0) \ast \omega_0 \right) + \alpha \\
\Rightarrow \lambda_{\alpha}^T \left( (\pi' \ast' e') \ast' \omega' \right) = \sup_{t \in f^{-1}((\pi', e') \ast (\omega))} \mu \left( (\pi_0 \ast e_0) \ast \omega_0 \right) + \alpha = \mu_i^T \left( (\pi_0 \ast e_0) \ast \omega_0 \right) \\
\geq \min \left\{ \mu_i^T \left( (\pi_0 \ast e_0) \ast \omega_0 \right), \mu_i^T \left( (\pi_0 \ast e_0) \ast \omega_0 \right) \right\} \\
= \min \left\{ \sup_{t \in f^{-1}((\pi', e') \ast (\omega))} \mu \left( (\pi_0 \ast e_0) \ast \omega_0 \right), \sup_{t \in f^{-1}((\pi', e') \ast (\omega))} \mu \left( (\pi_0 \ast e_0) \ast \omega_0 \right) \right\} \\
\Rightarrow \lambda_{\alpha}^T \left( (\pi' \ast' e') \ast' \omega' \right) = \lambda_{\alpha}^T \left( (\pi' \ast' e') \ast' \omega' \right) \\
\geq \{ \mu_i^T \left( (\pi_0 \ast e_0) \ast \omega_0 \right), \mu_i^T \left( (\pi_0 \ast e_0) \ast \omega_0 \right) \} \\
= \{ \lambda_{\alpha}^T \left( (\pi' \ast' e') \ast' \omega' \right), \lambda_{\alpha}^T \left( (\pi' \ast' e') \ast' \omega' \right) \} \\
\Rightarrow \lambda_{\alpha}^T \left( (\pi' \ast' e') \ast' \omega' \right) = \lambda_{\alpha}^T \left( (\pi' \ast' e') \ast' \omega' \right)
\]

Subsequently, \( \lambda_{\alpha}^T = f(\mu_i^T) \) is a \( FAB - i \) of \( Y \).

4.9. Theorem
An homomorphic pre-image of a \( TFAB - i \) of \( AB - g \) \((\Psi, \ast, 0)\) is also a \( TFAB - i \).

Proof:
Let \( f: (\Psi; \ast', 0) \rightarrow (Y; \ast', 0') \) be a homomorph of \( AB - g \), \( \lambda \) the \( TFAB - i \) of \( Y \) and \( \mu \) the pre-image of \( \lambda \) under \( f \). By Theorem (3.8), we have that \( \mu_i^T \) is a \( FAB - i \) of \( \Psi \).

4.10. Definition
Let \( \mu_1 \) and \( \mu_2 \) be fuzzy subsets of an \( AB - g \) \((\Psi, \ast, 0)\). \( \mu_2 \) is named a fuzzy extension \( AB - i \) of \( \mu_1 \) if the come assertions are valid:

\( (I_i) \) \( \mu_i \) is a fuzzy extension of \( \mu_i \).
\( (I_ii) \) If \( \mu_i \) is a \( FAB - i \) of \( \Psi \) \( \Rightarrow \) \( \mu_i \) is a \( FAB - i \) of \( \Psi \).

4.11. Proposition
Let \( \mu \) be a \( FAB - i \) of \( (\Psi, \ast, 0) \) and let \( \alpha, \gamma \in [0,T] \). If \( \alpha \geq \gamma \), then the fuzzy subset translation \( T_{\alpha}^\mu \) of \( \mu \) is a fuzzy extension \( AB - i \) of \( \mu_i^T \) of \( \mu \).

For every \( FAB - i \) \( \mu \) of \( \Psi \) and \( \gamma \in [0,T] \), the fuzzy subset translation \( \mu_{\gamma}^\mu \) of \( \mu \) is a \( TFAB - i \) of \( \Psi \). If \( \nu \) is a fuzzy extension \( AB - i \) of \( \mu \), \( \exists \alpha \in [0,T] \) such that \( \nu(\pi) \geq T_{\alpha}^\mu(\pi) \), \( \forall \pi \in \Psi \).

4.12. Proposition
Let \( \mu \) be a \( FAB - i \) of an \( AB - g \) \((\Psi, \ast, 0) \) and \( \gamma \in [0,T] \). For every fuzzy extension \( AB - i \), \( \nu \) of the \( TFAB - i \) \( \mu_{\gamma}^\mu \) of \( \mu \), \( \exists \alpha \in [0,T] \) such that \( \alpha \geq \gamma \) and \( \nu(\pi) \geq T_{\alpha}^\mu(\pi) \).

The come Example illustrates Proposition (3.12).

4.13. Example
Let \( \Psi = \{0, 1, 2\} \) in which \((\ast)\) be given by:

| * | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 2 |
| 2 | 2 | 2 | 0 |

\[ TABLE (12) \]
$(\Psi, *, 0)$ is an $AB - g$. Define a fuzzy subset $\mu$ of $\Psi$ by:

$$
\begin{array}{c|ccc}
\mu & 0 & 1 & 2 \\
\hline
0.8 & 0.7 & 0.6 \\
\end{array}
$$

*TABLE (13)*

$\mu$ is a $FAB - i$ of $X$ and $T = 0.2$. If we take $\gamma = 0.12$, $\Rightarrow$ the $TFAB - i \mu^T_\gamma$ of $\mu$ is given by:

$$
\begin{array}{c|cc}
\Psi & 0 & 1 \\
\hline
\mu^T_\gamma & 0.92 & 0.82 \\
\end{array}
$$

*TABLE (14)*

Let $\nu$ be a fuzzy subset of $\Psi$ defined by:

$$
\begin{array}{c|ccc}
\nu & 0 & 1 & 2 \\
\hline
0.98 & 0.89 & 0.81 \\
\end{array}
$$

*TABLE (15)*

$\nu$ is clearly a fuzzy extension $AB - i$ of the $TFAB - i \mu^T_\gamma$ of $\mu$. However, $\nu$ is not a $TFAB - i \mu^T_\alpha$ of $\mu$ for all $\alpha \in [0, T]$. Take $\alpha = 0.17$, $\Rightarrow \alpha = 0.17 > 0.12 = \gamma$, and the $TFAB - i \mu^T_\alpha$ of $\mu$ is given as follows:

$$
\begin{array}{c|ccc}
\Psi & 0 & 1 & 2 \\
\hline
\mu^T_\alpha & 0.97 & 0.87 & 0.77 \\
\end{array}
$$

*TABLE (16)*

Note that $\nu(\pi) \geq \mu^T_\alpha(\pi)$, for all $\pi \in \Psi$, and Subsequently, $\nu$ is a fuzzy extension $AB - i$ of the $TFAB - i \mu^T_\alpha$ of $\mu$.

### 4.14. Proposition

Let $\mu$ be a $FAB - i$ of an $AB - g$ $(\Psi, *, 0)$ and $\alpha \in [0, T] \Rightarrow$ the fuzzy subset translation $\mu^T_\alpha$ of $\mu$ is a fuzzy extension $AB - i$ of $\mu$.

A fuzzy extension $AB - i$ of a $FAB - i \mu$ may not be represented as a $TFAB - i \mu^T_\alpha$ of $\mu$, that is, the converse of proposition (3.14) is not true in general, as shown by the come Example.

### 4.15. Example

Let $\Psi = \{0, 1, 2, 3\}$ be an $AB - g$ with the come table:

$$
\begin{array}{c|cccc}
* & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 \\
2 & 2 & 2 & 0 & 0 \\
3 & 3 & 3 & 3 & 0 \\
\end{array}
$$

*TABLE (17)*
Define a fuzzy subset \( \mu \) of \( \Psi \) by:

\[
\begin{array}{cccc}
\Psi & 0 & 1 & 2 \\
\mu^T & 0.93 & 0.73 & 0.63 \\
\end{array}
\]

\textit{TABLE (18)}

\( \mu \) is a FAB \( - \) i of \( \Psi \).

\[
\begin{array}{cccc}
\Psi & 0 & 1 & 2 \\
\mu & 0.9 & 0.6 & 0.8 \\
\end{array}
\]

\textit{TABLE (19)}

Let \( \nu \) be a fuzzy subset of \( \Psi \) defined by:

\[
\begin{array}{cccc}
\Psi & 0 & 1 & 2 \\
\nu & 0.82 & 0.46 & 0.59 \\
\end{array}
\]

\textit{TABLE (20)}

\( \nu \) is a fuzzy extension \( \Psi \). However, \( \nu \) is not the TFA \( \Psi \) of \( \mu \) for all \( \alpha \in [0,T] \).

\[4.16. \text{ Proposition}\]

The interpart of any set of TFA \( - \) i of \( \Psi \) is also TFA \( - \) i of \( \Psi \).

Proof:

Let \( \{\mu_i | i \in \Lambda\} \) be a family of TFA \( - \) i of \( AB \) \( - g \) \( \Psi \), \( \Rightarrow \) for any \( \pi, \varepsilon, \omega \in \Psi \), \( i \in \Lambda \),

\[
\begin{align*}
(\bigcap_{i \in \Lambda} (\mu^-_i)_{i}(\pi)) &= (\bigcap_{i \in \Lambda} (\mu^-_i)_{i}(\pi)) + \alpha \\
&\geq \inf (\mu_i(\pi)+\alpha) = \inf (\mu_i(\pi)) = (\bigcap_{i \in \Lambda} (\mu^-_i)_{i}(\pi)), \quad \text{and}
\end{align*}
\]

\[
(\bigcap_{i \in \Lambda} (\mu^-_i)_{i}(\pi * \omega)) = \inf ((\mu^-_i)_{i}(\pi * \omega)) = \inf (\mu_i(\pi * \omega))^+\alpha
\]

\[
\begin{align*}
&\geq \inf (\min \{\mu_i ((\pi * \varepsilon) * \omega), \mu_i(\varepsilon)\}) + \alpha \\
&= \inf (\min \{\mu_i ((\pi * \varepsilon) * \omega) + \alpha, \mu_i(\varepsilon) + \alpha\}) \\
&= \min \{(\inf (\mu_i ((\pi * \varepsilon) * \omega) + \alpha), \inf (\mu_i(\varepsilon) + \alpha)\} \\
&= \min \{(\bigcap_{i \in \Lambda} (\mu^-_i)_{i}(\pi * \varepsilon) * \omega), (\bigcap_{i \in \Lambda} (\mu^-_i)_{i}(\varepsilon)) \}. \quad \square
\end{align*}
\]

Clearly, the union of fuzzy extension s of a fuzzy subset translation of \( AB \) \( - g \) \( \Psi \) is not a fuzzy extension of \( \mu \) as seen in the come Example.

\[4.17. \text{ Example}\]

Let \( \Psi = \{0, 1, 2, 3\} \) be an \( AB \) \( - g \) which is given in Example \( (3.15) \).
Define a fuzzy subset $\mu$ of $\Psi$ by:

| $\Psi$ | 0 | 1 | 2 | 3 |
|--------|---|---|---|---|
| $\mu$  | 0.8 | 0.5 | 0.6 | 0.5 |

*TABLE (21)*

Let $\alpha = 0$, $\mu$ is a $TFAB - i$ of $\Psi$. Let $\nu$ and $\delta$ be fuzzy subsets translation of $\Psi$ given by:

| $\Psi$ | 0 | 1 | 2 | 3 |
|--------|---|---|---|---|
| $\nu$  | 0.9 | 0.6 | 0.7 | 0.6 |
| $\delta$ | 0.9 | 0.6 | 0.6 | 0.7 |

*TABLE (22)*

$\nu$ and $\delta$ are fuzzy extensions of $\mu$. However, the union $\nu \cup \delta$ is not a fuzzy extension of $\mu$ Ago $(\nu \cup \delta)(3*2) = 0.6 < 0.7 = \min\{(\nu \cup \delta)(3), (\nu \cup \delta)(2)\}$.

4.18. Theorem

Let $\alpha \in [0,T]$, $\mu^T_\alpha$ be the translation fuzzy subset of $\mu$. The come are equivalent:

1. $\mu^T_\alpha$ is a $TFAB - i$ of $\Psi$.
2. $\forall t \in \text{Im}(\mu), t > \alpha \Rightarrow U_\alpha(\mu; t)$ is $AB - i$ of $\Psi$.

Proof:

Let's say $\mu^T_\alpha$ is a $TFAB - i$ of $\Psi$ and let $t \in \text{Im}(\mu)$ be such that $t > \alpha$. Ago $\mu^T_\alpha(0) \geq \mu^T_\alpha(\pi), \forall \pi \in \Psi$, we have $\mu(0)+\alpha = \mu^T_\alpha(0) \geq \mu^T_\alpha(\pi) = \mu(\pi) + \alpha$ that mean $\mu(0) \geq \mu(\pi), \forall \pi \in \Psi$.

Let $\pi \in U_\alpha(\mu; t)$, $\Rightarrow \mu(\pi) > t-\alpha$ and $\mu(0) \geq \mu(\pi)$ imply $\mu(0) \geq \mu(\pi) \geq t-\alpha$.

Subsequently, $0 \in U_\alpha(\mu; t)$.

Let $\pi, \epsilon, \omega \in \Psi$ be $\exists ((\pi \ast \epsilon) \ast \omega) \in U_\alpha(\mu; t)$ and $\epsilon \in U_\alpha(\mu; t)$ $\Rightarrow$ $\mu((\pi \ast \epsilon) \ast \omega) \geq t - \alpha$ and $\mu(\epsilon) \geq t - \alpha$,

i.e., $\mu^T_\alpha((\pi \ast \epsilon) \ast \omega) = \mu((\pi \ast \epsilon) \ast \omega) + \alpha \geq t$ and $\mu^T_\alpha(\epsilon) = \mu(\epsilon) + \alpha \geq t$. Ago $\mu^T_\alpha$ is a $TFAB - i$ of $\Psi$, it follows that $\mu((\pi \ast \epsilon) \ast \omega) + \alpha = \mu^T_\alpha((\pi \ast \epsilon) \ast \omega) \geq \min\{\mu^T_\alpha((\pi \ast \epsilon) \ast \omega), \mu^T_\alpha(\epsilon)\} \geq t$, that is, $\mu((\pi \ast \epsilon) \ast \omega) \geq t - \alpha$ so that $(\pi \ast \epsilon) \in U_\alpha(\mu; t)$. Therefore $U_\alpha(\mu; t)$ is $AB - i$ of $\Psi$.

Conversely, suppose that $U_\alpha(\mu; t)$ is an $AB - i$ of $\Psi$ for every $t \in \text{Im}(\mu)$ with $t > \alpha$. If there exists $\pi \in \Psi \exists \mu^T_\alpha(0) < \lambda \leq \mu^T_\alpha(\pi) \Rightarrow \mu(\pi) \geq \lambda - \alpha$. However, $\mu(0) < \lambda - \alpha$. This shows that $\pi \in U_\alpha(\mu; t)$ and $0 \in U_\alpha(\mu; t)$. This is a contradiction, and so $\mu^T_\alpha(0) \geq \mu^T_\alpha(\pi)$ for all $\pi \in \Psi$.

Now Let's say $\exists \pi, \omega \in \Psi \exists \mu^T_\alpha((\pi \ast \epsilon) \ast \omega)$ $\mu^T_\alpha((\pi \ast \epsilon) \ast \omega) \leq \gamma \leq \min\{\mu^T_\alpha((\pi \ast \epsilon) \ast \omega), \mu^T_\alpha(\epsilon)\} \Rightarrow \mu((\pi \ast \epsilon) \ast \omega) \geq \gamma - \alpha$ and $\mu(\epsilon) \geq \gamma - \alpha$.

However, $\mu((\pi \ast \epsilon) \ast \omega) < \gamma - \alpha$. Subsequently, $((\pi \ast \epsilon) \ast \omega) \in U_\alpha(\mu; \gamma)$ and $\epsilon \in U_\alpha(\mu; \gamma)$. However, $(\pi \ast \epsilon) \in U_\alpha(\mu; \gamma)$. This is a contradiction.

Therefore $\mu^T_\alpha((\pi \ast \omega) \geq \min\{\mu^T_\alpha((\pi \ast \epsilon) \ast \omega), \mu^T_\alpha(\epsilon)\}$, for all $\pi, \epsilon, \omega \in \Psi$. Subsequently, $\mu^T_\alpha$ is a $TFAB - i$ of $\Psi$. ◇
In Theorem (3.17(2)), if \( t \leq \alpha \), \( U_\alpha (\mu; t) = \Psi \).

4.19. Proposition

Let \( \mu \) be a \( FAB - i \) of an \( AB - g (\Psi, *, 0) \) and let \( \alpha \in [0,T] \), \( \Rightarrow \) the fuzzy subset translation \( \mu^T_\alpha \) of \( \mu \) is a \( FAB - sg \) of \( \Psi \).

Proof:

Ago \( \mu \) be a \( FAB - i \) of an \( AB - g \Psi \), \( \Rightarrow \) by proposition (1.8(2)), \( \mu \) be a fuzzy \( AB - sg \) of an \( AB - g \Psi \) and let \( \alpha \in [0,T] \), \( \Rightarrow \) by proposition (2.2), the fuzzy subset translation \( \mu^T_\alpha \) of \( \mu \) is a \( TFAB - sg \) of \( \Psi \).

In general, the converse of the proposition (3.19) is not true.

4.20. Example

Consider an AB-algebra \( \Psi = \{0, 1, 2\} \) with the Example (3.13). Define a fuzzy subset \( \mu \) of \( \Psi \) by:

| \( \Psi \) | 0 | 1 | 2 |
|----------|---|---|---|
| \( \mu \) | 0.7 | 0.5 | 0.6 |

Table (23)

\( \mu \) is not \( FAB - i \) of \( \Psi \). Ago \( \mu (1 \ast 0) = \mu (1) = 0.5 < 0.6 = \min \{ \mu (1 \ast (2 \ast 0)), \mu (2) \} \)

= \( \min \{ \mu (2), \mu (2) \} \), and \( T = 0.3 \).

However, if we take \( \alpha = 0.2 \) the fuzzy translation \( \mu^T_\alpha \) of \( \mu \) is given as follows:

| \( \Psi \) | 0 | 1 | 2 |
|----------|---|---|---|
| \( \mu^T_\alpha \) | 0.9 | 0.7 | 0.8 |

Table (24)

\( \mu^T_\alpha \) is a \( FAB - sg \) of \( \Psi \). \( \lambda^M_\Psi (\pi' \ast ' \omega') = \min \{ \lambda^M_\Psi ((\pi' \ast ' \epsilon') \ast ' \omega'), \lambda^M_\Psi (\epsilon') \} \).

Subsequently, \( \lambda^M_\Psi = f(\mu^M_\Psi) \) is a \( FAB - i \) of \( \Psi \). □

5. Cartesian product on fuzzy AB-ideal of AB-algebra

In this part, we discuss the Cartesian product of fuzzy translation of AB-algebra and establish some of its characteristics in detail on the basis of fuzzy AB-ideal as [9].

5.1. Definition

Let \( \mu^T_\alpha \) and \( \delta^T_\alpha \) be fuzzy translation \( s \) of an \( AB - g \Psi \). The Cartesian product \( \mu^T_\alpha \times \delta^T_\alpha : \Psi \times \Psi \rightarrow [0,1] \) is defined by \( (\mu^T_\alpha \times \delta^T_\alpha)(\pi, \epsilon) = \min \{ \mu^T_\alpha (\pi), \delta^T_\alpha (\epsilon) \} \), for all \( \pi, \epsilon \in \Psi \).

5.2. Theorem

Let \( \mu \) and \( \nu \) be two \( FAB - i \) of an \( AB - g (\Psi, *, 0) \). Let \( T = \min \{ T_\mu, T_\nu \} \) where \( T_\mu = 1 - \sup \{ \mu (\pi) : \pi \in \Psi \} \) and \( T_\nu = 1 - \sup \{ \nu (\pi) : \pi \in \Psi \} \) and \( \alpha \in [0,T] \). \( \Rightarrow \) the fuzzy translation of Cartesian product \( \mu \times \nu \) is a \( FAB - i \) of \( \Psi \times \Psi \).

Proof:
Let μ and ν be two FAB - i of an AB - g Ψ. let α∈[0,T].

Now, by Theorem (2.2) $\mu^T_a, \nu^T_a$ are FAB - i of Ψ and by Theorem (6.9) of [4], $\mu^T_a \times \nu^T_a$ is a FAB - i of Ψ×Ψ. Also ,$\forall\pi, \forall\varepsilon \in \Psi \times \Psi$.

\[
(\mu \times \nu)^T_a(\pi, \varepsilon) = (\mu \times \nu)(\pi, \varepsilon) + \alpha = min\{\mu(x), \nu(y)} + \alpha
\]

Subsequently, $(\mu \times \nu)^T_a(\pi, \varepsilon) = (\mu^T_a \times \nu^T_a)(\pi, \varepsilon)$.

5.3. Theorem

Let μ and δ be fuzzy subsets in an AB - g Ψ. We have μT_a × δT_a is a FAB - i of Ψ×Ψ. Then :

(i) Either $\mu^T_a(0) \geq \mu^T_a(\pi)$ or $\delta^T_a(0) \geq \delta^T_a(\pi)$ , for all $\pi \in \Psi$.

(ii) If $\mu^T_a(0) \geq \mu^T_a(\pi)$ and $\delta^T_a(0) < \delta^T_a(\pi)$ , for some $\pi, \epsilon \in \Psi \Rightarrow (\mu^T_a \times \delta^T_a)(\pi, \epsilon) = min\{\mu^T_a(\pi), \delta^T_a(\epsilon)\}

\[
= \min\{u(\pi) + \alpha, v(\epsilon) + \alpha\} = \min\{\mu^T_a(\pi), \nu^T_a(\epsilon)\} = (\mu^T_a \times \nu^T_a)(\pi, \epsilon)
\]

Subsequently, $(\mu \times \nu)^T_a$ is a TFAB - i of Ψ×Ψ . □

5.4. Theorem

Let μ and δ be fuzzy subsets of an AB - g X such that $\mu^T_a \times \delta^T_a$ is a FAB - i of Ψ×Ψ.

(iii) If contradiction . Subsequently, if $\mu^T_a(0)$ ≥ $\mu^T_a(\pi)$ and $\delta^T_a(0)$ ≥ $\delta^T_a(\pi)$ .

\[
\begin{align*}
\text{If we take } &\text{min}\{\mu^T_a(0), \mu^T_a(\pi)\} = (\mu^T_a \times \delta^T_a)(0,0) = (\mu^T_a \times \nu^T_a)(0,0)
\end{align*}
\]

\[
\mu^T(0) \geq \mu^T(\pi) \Rightarrow \delta^T_a(0) \geq \delta^T_a(\pi)
\]

\[
\text{or } \delta^T_a(0) = \delta^T_a(\pi)
\]

\[
\text{or } \delta^T_a(0) < \delta^T_a(\pi)
\]

\[
\text{which is a contradiction .}
\]

\[
\text{Subsequently, either } \delta^T_a(0) \geq \mu^T_a(\pi) \text{ or } \delta^T_a(0) \geq \delta^T_a(\pi)
\]

\[
\text{(iii)} \Rightarrow (\text{ii}) \Rightarrow (\text{i}) \Rightarrow (\text{iii}) \Rightarrow □
\]
At least, we will prove that \( \mu \) is a \( FAB - i \) of \( \Psi \). Let \( \mu^{T_\pi}(0) \geq \mu^{T_\pi}(\pi) \Rightarrow \mu(0) \geq \mu(\pi) \) Ago by Theorem (6.2(ii)) either \( \delta^{T_\pi}(0) \geq \mu^{T_\pi}(\pi) \) or \( \delta^{T_\pi}(0) \geq \delta^{T_\pi}(\pi) \) If \( \delta^{T_\pi}(0) \geq \mu^{T_\pi}(\pi) \) for any \( \pi \in \Psi \). Subsequently, \( (\mu^{T_\pi} \times \delta^{T_\pi})(\pi,0) = \min\{\mu^{T_\pi}(\pi),\delta^{T_\pi}(0)\} \)

\[ = \mu^{T_\pi}(\pi), \text{taking } \mu^{T_\pi}(\pi) \]

\[ = (\mu^{T_\pi} \times \delta^{T_\pi})(\pi_1,0) \]

If we take \( \pi_2 = \pi_3 = \omega = 0 \) in \( (A) \Rightarrow \mu(\pi_1 \times \omega_1) + \alpha = \mu^{T_\pi}(\pi_1 * \omega_1) = (\mu^{T_\pi} \times \delta^{T_\pi})(\pi_1 + \omega_1,0) \geq \min \{\mu^{T_\pi} \times \delta^{T_\pi}\} \left( (\pi_1 \times \omega_1),0 \right), (\mu^{T_\pi} \times \delta^{T_\pi}(\epsilon,0) \right) \}

\[ = \min\{\min\{\mu^{T_\pi}(\pi_1 \times \omega_1),\delta^{T_\pi}(\epsilon,0)\}, \min\{\mu^{T_\pi}(\epsilon),\delta^{T_\pi}(0)\}\} \]

\[ = \min\{\mu^{T_\pi}(\pi_1 \times \omega_1),\delta^{T_\pi}(\epsilon,0)\} = \min\{(\pi_1 \times \omega_1), \epsilon \} \]

\[ = \min\{(\pi_1 \times \omega_1), \epsilon \} + \alpha \]

\[ = \min\{\mu(\pi_1 \times \omega_1), \epsilon \}, \alpha \}

which proves that \( \mu \) is a \( FAB - i \) of \( \Psi \).

Subsequently, \( \epsilon \) or \( \delta \) is a \( FAB - i \) of \( \Psi \). □

5.5. Theorem

\( \mu \) and \( \delta \) are \( FAB - i \) s in an \( AB - g(\Psi, *, 0) \), \( \Rightarrow \mu^{M} \times \delta^{M} \) is a \( FAB - i \) in \( \Psi \times \Psi \).

Proof:

Let \( \mu \) and \( \delta \) be two \( FAB - i \) of an \( AB - g \Psi \). Let \( \gamma(0,1) \). Now by theorem (2.2) \( \mu^{M}, \delta^{M} \) are \( FAB - i \) of \( \Psi \) and by Theorem (6.9) of [4] .

\[ \mu^{M} \times \delta^{M} \] is a \( FAB - i \) of \( \Psi \times \Psi \). Also, \( \forall(\pi, \epsilon) \in \Psi \times \Psi \).

\( (\mu \times \delta)^{M}(\pi, \epsilon) = \gamma(\mu, \delta)(\pi, \epsilon) = \gamma(\min\{\mu(\pi), \delta(\epsilon)\}) \)

\[ = \min\{\gamma, \mu(\pi), \gamma(\delta(\epsilon))\} \]

\[ = \min\{\mu^{M}(\pi), \delta^{M}(\epsilon)\} = (\mu^{M} \times \delta^{M})(\pi, \epsilon) . \]

Subsequently, \( (\mu \times \delta)^{M} \) is a \( FAB - i \) of \( \Psi \times \Psi \).

5.6. Theorem

Let \( \mu \) and \( \delta \) be fuzzy subsets of an \( AB - g(\Psi, *, 0) \) \( \exists \mu^{M} \times \delta^{M} \) is a \( FAB - i \) of \( \Psi \times \Psi \). \( \Rightarrow \)

(i) Either \( \mu^{M}(0) \geq \mu^{M}(\pi) \) or \( \delta^{M}(0) \geq \delta^{M}(\pi), \forall \pi \in \Psi \).

(ii) If \( \mu^{M}(0) \geq \mu^{M}(\pi), \forall \pi \in \Psi \) \( \Rightarrow \) either \( \delta^{M}(0) \geq \delta^{M}(\pi) \) or \( \delta^{M}(0) \geq \delta^{M}(\pi) \).

(iii) If \( \delta^{M}(0) \geq \delta^{M}(\pi), \forall \pi \in \Psi \) \( \Rightarrow \) either \( \mu^{M}(0) \geq \mu^{M}(\pi) \) or \( \mu^{M}(0) \geq \delta^{M}(\pi) \).

Proof:

Let \( \mu^{M} \times \delta^{M} \) be a \( FAB - i \) of \( \Psi \times \Psi \).

(i) Suppose that \( \mu^{M}(0) < \mu^{M}(\pi) \) and \( \delta^{M}(0) < \delta^{M}(\pi) \) for some \( \pi, \epsilon \in \Psi \). \( \Rightarrow \) \( (\mu^{M} \times \delta^{M})(\pi, \epsilon) = \min\{\mu^{M}(\pi), \delta^{M}(\epsilon)\} \geq \min\{\mu^{M}(0), \delta^{M}(0)\} = (\mu^{M} \times \delta^{M})(0,0) \) which is a contradiction. Therefore \( \mu^{M}(0) \geq \mu^{M}(\pi) \) or \( \delta^{M}(0) \geq \delta^{M}(\pi), \forall \pi \in \Psi \).

(ii) Let's say \( \exists \pi, \epsilon \in \Psi \) such that \( \delta^{M}(0) < \mu^{M}(\pi) \) and \( \delta^{M}(0) < \delta^{M}(\pi) \) \( \Rightarrow \) \( (\mu^{M} \times \delta^{M})(0,0) = \min\{\mu^{M}(0), \delta^{M}(0)\} = \delta^{M}(0) \) and Subsequently, \( (\mu^{M} \times \delta^{M})(\pi, \epsilon) = \min\{\mu^{M}(\pi), \delta^{M}(\epsilon)\} \geq \delta^{M}(0) \) \( = (\mu^{M} \times \delta^{M})(0,0) \) which is a contradiction. Therefore, \( \delta^{M}(0) \geq \mu^{M}(\pi) \) and \( \delta^{M}(0) \geq \delta^{M}(\pi), \forall \pi \in \Psi \).

(iii) Subsequently, \( \mu^{M}(0) \geq \mu^{M}(\pi) \) or \( \delta^{M}(0) \geq \delta^{M}(\pi) \).

(iii) it is clear. □
5.7. Theorem

Let $\mu$ and $\delta$ be fuzzy subsets of an $AB - g$ $(\Psi, \ast, 0) \ni \mu^M \ast \delta^M$ is a $FAB - i$ of $\Psi \times \Psi$. \\
$\Rightarrow$ either $\mu$ or $\delta$ is a $FAB - i$ of $\Psi$.

Proof:

First we prove that $\delta$ is a $FAB - i$ of $\Psi$. By Theorem (4.6(i)) either \( \mu^M(0) \geq \mu^M(\pi) \) or \( \mu^M(0) \geq \delta^M(\pi) \), for all $\pi \in \Psi$. Let’s say $\delta^M(0) \geq \delta^M(\pi)$, for all $\pi \in \Psi \Rightarrow \gamma. \delta(0) \geq \gamma. \delta(\pi) \Rightarrow \delta(0) \geq \delta(\pi)$.

It follows from Theorem (6.6(iii)) that either $\mu^M(0) \geq \mu^M(\pi)$ or $\mu^M(0) \geq \delta^M(\pi)$.

If $\mu^M(0) \geq \delta^M(\pi)$, for any $\pi \in \Psi$ $\Rightarrow$ \( \mu^M(\pi) \times \delta^M(\pi) \) $\geq$ \( \mu^M(0) \), \( \mu^M(\pi) \) $\geq$ \( \mu^M(0) \times \delta^M(\pi) \).

Let $\pi_0, \pi_1, \pi_2$ be elements of $\Psi$. We have $\mu^M(\pi_0) \geq \mu^M(\pi_1) \ast \mu^M(\pi_2)$.

If we take $\pi_1 = \pi_2 = 0$ in (A), then $\delta(\pi_2 * \omega_2) + \alpha = \delta^M(\pi_2 * \omega_2) = \delta^M(\pi_2 * \omega_2)$.

If $\mu^M(0) \geq \mu^M(\pi) \Rightarrow \mu^M(0) \ast \delta^M(\pi) = \min \{ \mu^M(0), \delta^M(\pi) \}$.

At last, we will prove that $\mu$ is a $FAB - i$ of $\Psi$. Let $\mu^M(0) \geq \mu^M(\pi) \Rightarrow \mu^M(\pi) \ast \delta^M(\pi) \geq \mu^M(0)$.

5. References

[1] A.T. Hameed and A.K. Alkurdi, *Fuzzy Magnified Translation s of QS-algebras*, LAMBERT Academic Publishing , 2018.

[2] A.T. Hameed and A.K. Alkurdi, *Fuzzy Translation and Fuzzy multiplication of QS-algebras*, Journal University of Kerbala, vol.15, no. 4 (2017), pp:145-157.

[3] A.T. Hameed and B.N. Abbas, *Derivation of AB-ideal s and fuzzy AB-ideal s of AB-algebra*, LAMBERT Academic Publishing, 2018.
[4] A.T. Hameed and B.N. Abbas, \textit{AB-ideal s of AB-algebra}, Applied Mathematical Sciences, vol.11, no.35 (2017), pp:1715-1723.

[5] A.T. Hameed and B.N. Abbas, \textit{On Some Characteristics of AB-algebras}, Algebra Letters, vol.7 (2017), pp:1-12.

[6] A.T. Hameed and B.N. Abbas, \textit{Some characteristics of fuzzy AB-ideal of AB-algebras}, Journal of AL-Qadisiyah for computer science and mathematics, vol.10, no. 1(2018), pp:1-7.

[7] A.T. Hameed and N.Z. Mohammed, \textit{Fuzzy Translation and Fuzzy multiplication of CI-algebras}, Journal of Karbala University, vol.15, no.1(2017), pp:110-123.

[8] A.T. Hameed, \textit{Fuzzy ideal s of some algebras}, PH. Sc. Thesis, Ain Shams University, Faculty of Sciences, Egypt, 2015.

[9] A.T. Hameed, S.M. Mostafa and A.H. Abed, \textit{Big Generalized Fuzzy KUS-ideal s of KUS-algebra}, College of Education Journal in AL-Mustansiryah University 26-27 April 2017.

[10] J. Meng and Y. B. Jun, \textit{BCK-algebras}, Kyung Moon Sa Co., Korea, 1994.

[11] K. B. Lee, Y.B. Jun and M. I. Doh, \textit{Fuzzy Translation s and Fuzzy Multiplications of BCK/BCI-algebras}, Commum. Korean Math. Sco., vol.24 (2009), pp:353–360.

[12] L.A. Zadeh, \textit{Fuzzy Sets}, Inform. and Control, vol.8 (1965), pp:338-353.

[13] S.M. Mostafa, A.T. Hameed and A.H. Abed, \textit{Fuzzy KUS-ideal s of KUS-algebra}, Basra Journal of Science (A), vol.34, no.2 (2016), pp:73-84.

[14] S.M. Mostafa, A.T. Hameed and N.Z. Mohammed, \textit{Fuzzy \( \alpha \)-Translation of KUS-algebras}, Journal Al-Qadisyah for Computer Science and Mathematics, vol.8, no.2(2016), pp:8-16.

[15] S.M. Mostafa, M.A. Abdel Naby, F. Abdel-Halim and A.T. Hameed, \textit{On KUS-algebra}, International Journal of Algebra, vol.7, no.3(2013), pp:131-144.