A PARTIAL CAYLEY TRANSFORM OF SIEGEL-JACOBI DISK

JAE-HYUN YANG

Abstract. Let $H_g$ and $D_g$ be the Siegel upper half plane and the generalized unit disk of degree $g$ respectively. Let $\mathbb{C}^{(h,g)}$ be the Euclidean space of all $h \times g$ complex matrices. We present a partial Cayley transform of the Siegel-Jacobi disk $D_g \times \mathbb{C}^{(h,g)}$ onto the Siegel-Jacobi space $H_g \times \mathbb{C}^{(h,g)}$ which gives a partial bounded realization of $H_g \times \mathbb{C}^{(h,g)}$ by $D_g \times \mathbb{C}^{(h,g)}$. We prove that the natural actions of the Jacobi group on $D_g \times \mathbb{C}^{(h,g)}$ and $H_g \times \mathbb{C}^{(h,g)}$ are compatible via a partial Cayley transform. A partial Cayley transform plays an important role in computing differential operators on the Siegel-Jacobi disk $D_g \times \mathbb{C}^{(h,g)}$ invariant under the natural action of the Jacobi group on $D_g \times \mathbb{C}^{(h,g)}$ explicitly.

1. Introduction

For a given fixed positive integer $g$, we let $H_g = \{ \Omega \in \mathbb{C}^{(g,g)} \mid \Omega = t\Omega, \quad \text{Im} \Omega > 0 \}$ be the Siegel upper half plane of degree $g$ and let $Sp(g, \mathbb{R}) = \{ M \in \mathbb{R}^{(2g,2g)} \mid tM J_g M = J_g \}$ be the symplectic group of degree $g$, where $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring $F$ for two positive integers $k$ and $l$, $tM$ denotes the transpose matrix of a matrix $M$ and

$$J_g = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}.$$ We see that $Sp(g, \mathbb{R})$ acts on $H_g$ transitively by

$$M \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1},$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g, \mathbb{R})$ and $\Omega \in H_g$.

Let $D_g = \{ W \in \mathbb{C}^{(g,g)} \mid W = tW, \quad I_g - W \bar{W} > 0 \}$ be the generalized unit disk of degree $g$. The Cayley transform $\Phi: D_g \rightarrow H_g$ defined by

$$\Phi(W) = i(I_g + W)(I_g - W)^{-1}, \quad W \in D_g$$

is a biholomorphic mapping of $D_g$ onto $H_g$ which gives the bounded realization of $H_g$ by $D_g$ (cf. [8 pp. 281-283]). And the action (2.8) of the symplectic group on $D_g$ is compatible with the action (1.1) via the Cayley transform $\Phi$.

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1
For two positive integers $g$ and $h$, we consider the Heisenberg group
\[
H^{(g,h)}_\mathbb{R} = \left\{ (\lambda, \mu; \kappa) \mid \lambda, \mu \in \mathbb{R}^{(h,g)}, \kappa \in \mathbb{R}^{(h,h)}, \kappa + \mu^t \lambda \text{ symmetric} \right\}
\]
endowed with the following multiplication law
\[
(\lambda, \mu; \kappa) \circ (\lambda', \mu'; \kappa') = (\lambda + \lambda', \mu + \mu'; \kappa + \kappa' + \lambda^t \mu' - \mu^t \lambda').
\]
The Jacobi group $G_J$ is defined as the semidirect product of $Sp(g, \mathbb{R})$ and $H^{(g,h)}_\mathbb{R}$
\[
G_J = Sp(g, \mathbb{R}) \rtimes H^{(g,h)}_\mathbb{R}
\]
endowed with the following multiplication law
\[
\left( M, (\lambda, \mu; \kappa) \right) \cdot \left( M', (\lambda', \mu'; \kappa') \right) = \left( MM', (\tilde{\lambda} + \lambda', \tilde{\mu} + \mu'; \kappa + \kappa' + \tilde{\lambda}^t \mu' - \tilde{\mu}^t \lambda') \right)
\]
with $M, M' \in Sp(g, \mathbb{R}), (\lambda, \mu; \kappa), (\lambda', \mu'; \kappa') \in H^{(g,h)}_\mathbb{R}$ and $(\tilde{\lambda}, \tilde{\mu}) = (\lambda, \mu)M'$. Then $G_J$ acts on $\mathbb{H}_g \times \mathbb{C}^{(h,g)}$ transitively by
\[
(M, (\lambda, \mu; \kappa)) \cdot (\Omega, Z) = \left( M \cdot \Omega, (Z + \lambda \Omega + \mu)(C \Omega + D)^{-1} \right),
\]
where $M = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \in Sp(g, \mathbb{R}), (\lambda, \mu; \kappa) \in H^{(g,h)}_\mathbb{R}$ and $(\Omega, Z) \in \mathbb{H}_g \times \mathbb{C}^{(h,g)}$. In [9, p. 1331], the author presented the natural construction of the action (1.3).

We mention that studying the Siegel-Jacobi space or the Siegel-Jacobi disk associated with the Jacobi group is useful to the study of the universal family of polarized abelian varieties (cf. [10], [12]). The aim of this paper is to present a partial Cayley transform of the Siegel-Jacobi disk $\mathbb{D}_g \times \mathbb{C}^{(h,g)}$ onto the Siegel-Jacobi space $\mathbb{H}_g \times \mathbb{C}^{(h,g)}$ which gives a partially bounded realization of $\mathbb{H}_g \times \mathbb{C}^{(h,g)}$ by $\mathbb{D}_g \times \mathbb{C}^{(h,g)}$ and to prove that the natural actions of the Jacobi group on $\mathbb{D}_g \times \mathbb{C}^{(h,g)}$ and $\mathbb{H}_g \times \mathbb{C}^{(h,g)}$ are compatible via a partial Cayley transform. The main reason that we study a partial Cayley transform is that this transform is usefully applied to computing differential operators on the Siegel-Jacobi disk $\mathbb{D}_g \times \mathbb{C}^{(h,g)}$ invariant under the action (3.5) of the Jacobi group $G_J^t$ (cf. (3.2)) explicitly.

This paper is organized as follows. In Section 2, we review the Cayley transform of the generalized unit disk $\mathbb{D}_g$ onto the Siegel upper half plane $\mathbb{H}_g$ which gives a bounded realization of $\mathbb{H}_g$ by $\mathbb{D}_g$. In Section 3, we construct a partial Cayley transform of the Siegel-Jacobi disk $\mathbb{D}_g \times \mathbb{C}^{(h,g)}$ onto the Siegel-Jacobi space $\mathbb{H}_g \times \mathbb{C}^{(h,g)}$ which gives a partially bounded realization of $\mathbb{H}_g \times \mathbb{C}^{(h,g)}$ by $\mathbb{D}_g \times \mathbb{C}^{(h,g)}$ (cf. (3.6)). We prove that the action (1.3) of the Jacobi group $G_J$ is compatible with the action (3.5) of the Jacobi group $G_J^t$ through a partial Cayley transform (cf. Theorem 3.1). In the final section, we present the canonical automorphic factors of the Jacobi group $G_J^t$.

**Notations:** We denote by $\mathbb{R}$ and $\mathbb{C}$ the field of real numbers, and the field of complex numbers respectively. For a square matrix $A \in F^{(k,k)}$ of degree $k$, $\sigma(A)$ denotes the trace of $A$. For $\Omega \in \mathbb{H}_g$, $\text{Re} \Omega$ (resp. $\text{Im} \Omega$) denotes the real (resp. imaginary) part of $\Omega$. For a matrix $A \in F^{(k,k)}$ and $B \in F^{(k,l)}$, we write $A|B| = ^t BAB$. $I_n$ denotes the identity matrix of degree $n$. 
2. The Cayley Transform

Let
\[ T = \frac{1}{\sqrt{2}} \begin{pmatrix} I_g & I_g \\ iI_g & -iI_g \end{pmatrix} \]
be the $2g \times 2g$ matrix represented by $\Phi$. Then
\[ T^{-1}Sp(g, \mathbb{R}) T = \left\{ \begin{pmatrix} P & Q \\ Q & P \end{pmatrix} \mid \begin{array}{l} \quad \top PQ = I_g, \quad \top PQ = \top P \end{array} \right\}. \]

Indeed, if $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g, \mathbb{R})$, then
\[ T^{-1}MT = \begin{pmatrix} P & Q \\ Q & P \end{pmatrix}, \]
where
\[ P = \frac{1}{2} \left\{ (A + D) + i(B - C) \right\} \]
and
\[ Q = \frac{1}{2} \left\{ (A - D) - i(B + C) \right\}. \]

For brevity, we set
\[ G^* = T^{-1}Sp(g, \mathbb{R}) T. \]

Then $G^*$ is a subgroup of $SU(g, g)$, where
\[ SU(g, g) = \left\{ h \in \mathbb{C}^{(g,g)} \mid \top h I_{g,g} h = I_{g,g} \right\}, \quad I_{g,g} = \begin{pmatrix} I_g & 0 \\ 0 & -I_g \end{pmatrix}. \]

In the case $g = 1$, we observe that
\[ T^{-1}Sp(1, \mathbb{R}) T = T^{-1}SL_2(\mathbb{R}) T = SU(1, 1). \]

If $g > 1$, then $G^*$ is a proper subgroup of $SU(g, g)$. In fact, since $\top T J_g T = -i J_g$, we get
\[ G^* = \left\{ h \in SU(g, g) \mid \top h J_g h = J_g \right\} = SU(g, g) \cap Sp(g, \mathbb{C}), \]
where
\[ Sp(g, \mathbb{C}) = \left\{ \alpha \in \mathbb{C}^{(2g,2g)} \mid \top \alpha J_g \alpha = J_g \right\}. \]

Let
\[ P^+ = \left\{ \begin{pmatrix} I_g & Z \\ 0 & I_g \end{pmatrix} \mid Z = \top Z \in \mathbb{C}^{(g,g)} \right\} \]
be the $P^+$-part of the complexification of $G^* \subset SU(g, g)$. We note that the Harish-Chandra decomposition of an element $\begin{pmatrix} P & Q \\ P & \bar{Q} \end{pmatrix}$ in $G^*$ is
\[ \begin{pmatrix} P & Q \\ Q & P \end{pmatrix} = \begin{pmatrix} I_g & \bar{Q}P^{-1} \end{pmatrix} \begin{pmatrix} P & 0 \\ Q & P \end{pmatrix} \left( \begin{pmatrix} I_g & 0 \\ 0 & I_g \end{pmatrix} \right). \]

For more detail, we refer to [2, p.155]. Thus the $P^+$-component of the following element
\[ \begin{pmatrix} P & Q \\ Q & P \end{pmatrix} \cdot \begin{pmatrix} I_g & W \\ 0 & I_g \end{pmatrix}, \quad W \in \mathbb{D}_g \]
of the complexification of $G^*$ is given by

$$ (2.7) \quad \left( \begin{pmatrix} I_g & (PW + Q)(QW + P)^{-1} \\ 0 & I_g \end{pmatrix} \right). $$

We note that $Q \overline{P}^{-1} \in \mathbb{D}_g$. We get the Harish-Chandra embedding of $\mathbb{D}_g$ into $P^+$ (cf. [2, p. 155] or [5, pp. 58-59]). Therefore we see that $G^*$ acts on $\mathbb{D}_g$ transitively by

$$ (2.8) \quad \left( \begin{pmatrix} P & Q \\ Q & P \end{pmatrix} \right) \cdot W = (PW + Q)(QW + P)^{-1}, \quad \left( \begin{pmatrix} P & Q \\ Q & P \end{pmatrix} \right) \in G^*, \; W \in \mathbb{D}_g. $$

The isotropy subgroup at the origin $o$ is given by

$$ K = \left\{ \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix} \right| P \in U(g) \right\}. $$

Thus $G^*/K$ is biholomorphic to $\mathbb{D}_g$. It is known that the action (1.1) is compatible with the action (2.8) via the Cayley transform $\Phi$ (cf. (1.2)). In other words, if $M \in Sp(g, \mathbb{R})$ and $W \in \mathbb{D}_g$, then

$$ (2.9) \quad M \cdot \Phi(W) = \Phi(M^* \cdot W), $$

where $M^* = T^{-1}MT \in G_*$. For a proof of Formula (2.9), we refer to the proof of Theorem 3.1.

For $\Omega = (\omega_{ij}) \in \mathbb{H}_g$, we write $\Omega = X + iY$ with $X = (x_{ij}), \; Y = (y_{ij})$ real and $d\Omega = (d\omega_{ij})$. We also put

$$ \frac{\partial}{\partial \Omega} = \left( \begin{array}{cc} 1 + \delta_{ij} & \frac{\partial}{\partial \omega_{ij}} \\ 2 & \frac{\partial}{\partial \omega_{ij}} \end{array} \right) \quad \text{and} \quad \frac{\partial}{\partial \overline{\Omega}} = \left( \begin{array}{cc} 1 + \delta_{ij} & \frac{\partial}{\partial \overline{\omega}_{ij}} \\ 2 & \frac{\partial}{\partial \overline{\omega}_{ij}} \end{array} \right). $$

Then

$$ (2.10) \quad ds^2 = \sigma \left( Y^{-1}d\Omega Y^{-1}d\overline{\Omega} \right) $$

is a $Sp(g, \mathbb{R})$-invariant metric on $\mathbb{H}_g$ (cf. [3]) and H. Maass [3] proved that its Laplacian is given by

$$ (2.11) \quad \Delta = 4\sigma \left( \begin{array}{c} Y \end{array} \right)^t \left( \begin{array}{c} \frac{\partial}{\partial \Omega} \end{array} \right) \frac{\partial}{\partial \Omega}). $$

For $W = (w_{ij}) \in \mathbb{D}_g$, we write $dW = (dw_{ij})$ and $d\overline{W} = (d\overline{w}_{ij})$. We put

$$ \frac{\partial}{\partial W} = \left( \begin{array}{cc} 1 + \delta_{ij} & \frac{\partial}{\partial w_{ij}} \\ 2 & \frac{\partial}{\partial w_{ij}} \end{array} \right) \quad \text{and} \quad \frac{\partial}{\partial \overline{W}} = \left( \begin{array}{cc} 1 + \delta_{ij} & \frac{\partial}{\partial \overline{w}_{ij}} \\ 2 & \frac{\partial}{\partial \overline{w}_{ij}} \end{array} \right). $$

Using the Cayley transform $\Phi : \mathbb{D}_g \rightarrow \mathbb{H}_g$, H. Maass proved (cf. [3]) that

$$ (2.12) \quad ds_*^2 = 4\sigma \left( (I_g - W\overline{W})^{-1}dW(I_g - \overline{W}W)^{-1}d\overline{W} \right) $$

is a $G_*$-invariant Riemannian metric on $\mathbb{D}_g$ and its Laplacian is given by

$$ (2.13) \quad \Delta_* = \sigma \left( (I_g - W\overline{W})^t \left( (I_g - W\overline{W}) \frac{\partial}{\partial W} \right) \frac{\partial}{\partial W} \right). $$
3. A Partial Cayley Transform

In this section, we present a partial Cayley transform of $D_g \times \mathbb{C}^{(h,g)}$ onto $H_g \times \mathbb{C}^{(h,g)}$ and prove that the action (1.3) of $G^J$ on $H_g \times \mathbb{C}^{(h,g)}$ is compatible with the natural action (cf. (3.5)) of the Jacobi group $G^*_J$ on $D_g \times \mathbb{C}^{(h,g)}$ via a partial Cayley transform.

From now on, for brevity we write $H_{g,h} = H_g \times \mathbb{C}^{(h,g)}$. We can identify an element $g = (M, (\lambda, \mu; \kappa))$ of $G^J$, $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g, \mathbb{R})$ with the element

$$
\begin{pmatrix}
A & 0 & B & A^t \mu - B^t \lambda \\
\lambda & I_h & \mu & \kappa \\
C & 0 & D & C^t \mu - D^t \lambda \\
0 & 0 & 0 & I_h
\end{pmatrix}
$$

of $Sp(g + h, \mathbb{R})$. This subgroup plays an important role in investigating the Fourier-Jacobi expansion of a Siegel modular form for $Sp(g + h, \mathbb{R})$ (cf. [4]) and studying the theory of Jacobi forms (cf. [1], [7-9], [15]).

We set

$$
T_* = \frac{1}{\sqrt{2}} \begin{pmatrix} I_{g+h} & I_{g+h} \\ iI_{g+h} & -iI_{g+h} \end{pmatrix}.
$$

We now consider the group $G^J_*$ defined by

$$
G^*_J = T_*^{-1} G^J T_*.
$$

If $g = (M, (\lambda, \mu; \kappa)) \in G^J$ with $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g, \mathbb{R})$, then $T_*^{-1} g T_*$ is given by

$$
(3.1) \quad T_*^{-1} g T_* = \left( \frac{P_*}{Q_*} \frac{Q_*}{P_*} \right),
$$

where

$$
P_* = \left( P, \begin{pmatrix} \frac{i}{2}(\lambda + i\mu) & \frac{i}{2} \left( P^t(\lambda + i\mu) - P^t(\lambda - i\mu) \right) \end{pmatrix}, \frac{1}{2} \left( I_h + i\frac{\kappa}{2} \right) \right),
$$

$$
Q_* = \left( Q, \begin{pmatrix} \frac{i}{2}(\lambda - i\mu) & \frac{i}{2} \left( Q^t(\lambda - i\mu) - Q^t(\lambda + i\mu) \right) \end{pmatrix}, -i\frac{\kappa}{2} \right),
$$

and $P, Q$ are given by Formulas (2.4) and (2.5). From now on, we write

$$
\left( \begin{pmatrix} P \\ Q \end{pmatrix}, \frac{1}{2}(\lambda + i\mu), \frac{1}{2}(\lambda - i\mu); -i\frac{\kappa}{2} \right) = \left( \frac{P_*}{Q_*} \frac{Q_*}{P_*} \right).
$$

In other words, we have the relation

$$
T_*^{-1} \left( \begin{pmatrix} A \\ C \end{pmatrix}, (\lambda, \mu; \kappa) \right) T_* = \left( \begin{pmatrix} P \\ Q \end{pmatrix}, \frac{1}{2}(\lambda + i\mu), \frac{1}{2}(\lambda - i\mu); -i\frac{\kappa}{2} \right).
$$

Let

$$
H_{C}^{(g,h)} = \left\{ (\xi, \eta; \zeta) \mid \xi, \eta \in \mathbb{C}^{(h,g)}, \zeta \in \mathbb{C}^{(h,h)}, \zeta + \eta^t \xi \text{ symmetric} \right\}
$$

be the Heisenberg group endowed with the following multiplication

$$
(\xi, \eta; \zeta) \circ (\xi', \eta'; \zeta') = (\xi + \xi', \eta + \eta'; \zeta + \zeta' + \xi^t \eta' - \eta^t \xi').
$$
We define the semidirect product
\[
SL(2g, \mathbb{C}) \ltimes H_{\mathbb{C}}^{(g,h)}
\]
endowed with the following multiplication
\[
\begin{pmatrix}
P & Q \\
R & S
\end{pmatrix}
\begin{pmatrix}
P' & Q' \\
R' & S'
\end{pmatrix}
= \begin{pmatrix}
P & Q \\
R & S
\end{pmatrix}
\begin{pmatrix}
P' & Q' \\
R' & S'
\end{pmatrix},
\]
where \(\tilde{\xi} = \xi P' + \eta R'\) and \(\tilde{\eta} = \xi Q' + \eta S'\).

If we identify \(H_{\mathbb{C}}^{(g,h)}\) with the subgroup
\[
\left\{ (\xi, \bar{\xi}; i\kappa) \mid \xi \in \mathbb{C}^{(h,g)}, \kappa \in \mathbb{R}^{(h,h)} \right\}
\]
of \(H_{\mathbb{C}}^{(g,h)}\), we have the following inclusion
\[
G^J \subset SU(g, g) \ltimes H_{\mathbb{R}}^{(g,h)} \subset SL(2g, \mathbb{C}) \ltimes H_{\mathbb{C}}^{(g,h)}.
\]
More precisely, if we recall \(G_* = SU(g, g) \cap Sp(g, \mathbb{C})\) (cf. (2.6)), we see that the Jacobi group \(G^J_*\) is given by
\[
(3.2) \quad G^J_* = \left\{ \left( \begin{pmatrix} P & Q \\ R & S \end{pmatrix}, (\lambda, \mu; \kappa) \right) \mid \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \in G_*, \xi \in \mathbb{C}^{(m,m)}, \kappa \in \mathbb{R}^{(m,m)} \right\}.
\]

We define the mapping \(\Theta : G^J \rightarrow G^J_*\) by
\[
(3.3) \quad \Theta \left( \begin{pmatrix} A & B \\ C & D \end{pmatrix}, (\lambda, \mu; \kappa) \right) = \left( \begin{pmatrix} P & Q \\ R & S \end{pmatrix}, \frac{1}{2}(\lambda + i\mu), \frac{1}{2}(\lambda - i\mu); -i\frac{\kappa}{2} \right),
\]
where \(P\) and \(Q\) are given by Formulas (2.4) and (2.5). We can see that if \(g_1, g_2 \in G^J\), then
\[
\Theta(g_1g_2) = \Theta(g_1)\Theta(g_2).
\]

According to \cite[p. 250]{11}, \(G^J_*\) is of the Harish-Chandra type (cf. \cite[p. 118]{5}). Let
\[
g_* = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}, (\lambda, \mu; \kappa)
\]
be an element of \(G^J_*\). Since the Harish-Chandra decomposition of an element \(\begin{pmatrix} P & Q \\ R & S \end{pmatrix}\) in \(SU(g, g)\) is given by
\[
\begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} I_g & QS^{-1} \\ 0 & I_g \end{pmatrix} \begin{pmatrix} P - QS^{-1}R & 0 \\ 0 & I_g \end{pmatrix} \begin{pmatrix} I_g & 0 \\ S^{-1}R & I_g \end{pmatrix},
\]
the \(P^+_*\)-component of the following element
\[
g_*, \begin{pmatrix} I_g & W \\ 0 & I_g \end{pmatrix}, (0, \eta; 0), \quad W \in \mathbb{D}_g
\]
of \(SL(2g, \mathbb{C}) \ltimes H_{\mathbb{C}}^{(g,h)}\) is given by
\[
(3.4) \quad \begin{pmatrix} I_g & (PW + Q)(\overline{Q}W + \overline{P})^{-1} \\ 0 & I_g \end{pmatrix}, (0, (\eta + \lambda W + \mu)(\overline{Q}W + \overline{P})^{-1}; 0).
\]
We can identify \( \mathbb{D}_g \times \mathbb{C}^{(h,g)} \) with the subset
\[
\left\{ \left( \begin{array}{cc} I_g & W \\ 0 & I_g \end{array} \right), (0, \eta; 0) \mid W \in \mathbb{D}_g, \ \eta \in \mathbb{C}^{(h,g)} \right\}
\]
of the complexification of \( G^J_* \). Indeed, \( \mathbb{D}_g \times \mathbb{C}^{(h,g)} \) is embedded into \( P_+^* \) given by
\[
P_+^* = \left\{ \left( \begin{array}{cc} I_g & W \\ 0 & I_g \end{array} \right), (0, \eta; 0) \mid W = tW \in \mathbb{C}^{(g,g)}, \ \eta \in \mathbb{C}^{(h,g)} \right\}.
\]
This is a generalization of the Harish-Chandra embedding (cf. [5, p. 119]). Hence \( G^J_* \) acts on \( \mathbb{D}_g \times \mathbb{C}^{(h,g)} \) transitively by
\[
(3.5) \quad \left( \begin{array}{cc} P & Q \\ Q & P \end{array} \right), (\lambda, \mu; \kappa) \cdot (W, \eta) = \left( (PW + Q)(\overline{QW + P})^{-1}, (\eta + \lambda W + \mu)(\overline{QW + P})^{-1} \right).
\]

From now on, for brevity we write \( \mathbb{D}_{g,h} = \mathbb{D}_g \times \mathbb{C}^{(h,g)} \). We define the map \( \Phi_* \) of \( \mathbb{D}_{g,h} \) into \( \mathbb{H}_{g,h} \) by
\[
(3.6) \quad \Phi_*(W, \eta) = \left( i(I_g + W)(I_g - W)^{-1}, 2i \eta (I_g - W)^{-1} \right), \quad (W, \eta) \in \mathbb{D}_{g,h}.
\]

We can show that \( \Phi_* \) is a biholomorphic map of \( \mathbb{D}_{g,h} \) onto \( \mathbb{H}_{g,h} \) which gives a partial bounded realization of \( \mathbb{H}_{g,h} \) by the Siegel-Jacobi disk \( \mathbb{D}_{g,h} \). We call this map \( \Phi_* \) the partial Cayley transform of the Siegel-Jacobi disk \( \mathbb{D}_{g,h} \).

**Theorem 3.1.** The action (1.3) of \( G^J \) on \( \mathbb{H}_{g,h} \) is compatible with the action (3.5) of \( G^J_* \) on \( \mathbb{D}_{g,h} \) through the partial Cayley transform \( \Phi_* \). In other words, if \( g_0 \in G^J \) and \( (W, \eta) \in \mathbb{D}_{g,h} \),
\[
(3.7) \quad g_0 \cdot \Phi_*(W, \eta) = \Phi_*(g_* \cdot (W, \eta)),
\]
where \( g_* = T^{-1}_* g_0 T_* \). We observe that Formula (3.7) generalizes Formula (2.9). The inverse of \( \Phi_* \) is
\[
(3.8) \quad \Phi_*^{-1}(\Omega, Z) = \left( (\Omega - iI_g)(\Omega + iI_g)^{-1}, Z(\Omega + iI_g)^{-1} \right).
\]

**Proof.** Let
\[
g_0 = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right), (\lambda, \mu; \kappa)
\]
be an element of \( G^J \) and let \( g_* = T^{-1}_* g_0 T_* \). Then
\[
g_* = \left( \begin{array}{cc} P & Q \\ Q & P \end{array} \right), \left( \frac{1}{2}(\lambda + i\mu), \frac{1}{2}(\lambda - i\mu), -i\kappa \right),
\]
where \( P \) and \( Q \) are given by Formulas (2.4) and (2.5).

For brevity, we write
\[
(\Omega, Z) = \Phi_*(W, \eta) \quad \text{and} \quad (\Omega_*, Z_*) = g_0 \cdot (\Omega, Z).
\]
That is,
\[
\Omega = i(I_g + W)(I_g - W)^{-1} \quad \text{and} \quad Z = 2i \eta(I_g - W)^{-1}.
\]
Then we get
\[
\Omega_* = (A\Omega + B)(C\Omega + D)^{-1} \\
= \left\{ i A(I_g + W)(I_g - W)^{-1} + B \right\} \left\{ i C(I_g + W)(I_g - W)^{-1} + D \right\}^{-1} \\
= \left\{ i A(I_g + W) + B(I_g - W) \right\}(I_g - W)^{-1} \\
\quad \times \left\{ \left[ i C(I_g + W) + D(I_g - W) \right](I_g - W)^{-1} \right\}^{-1} \\
= \left\{ (i A - B)W + (i A + B) \right\} \left\{ (i C - D)W + (i C + D) \right\}^{-1}
\]
and
\[
Z_* = (Z + \lambda\Omega + \mu)(C\Omega + D)^{-1} \\
= \left\{ -i^2 (I_g - W)^{-1} + i \lambda(I_g + W)(I_g - W)^{-1} + \mu \right\} \left\{ i C(I_g + W)(I_g - W)^{-1} + D \right\}^{-1} \\
= \left\{ 2i \eta + i \lambda(I_g + W) + \mu(I_g - W) \right\}(I_g - W)^{-1} \\
\quad \times \left\{ \left[ i C(I_g + W) + D(I_g - W) \right](I_g - W)^{-1} \right\}^{-1} \\
= \left\{ 2i \eta + (\lambda i - \mu)W + \lambda i + \mu \right\} \left\{ (i C - D)W + (i C + D) \right\}^{-1}.
\]
On the other hand, we set
\[
(W_*, \eta_*) = g_* \cdot (W, \eta) \quad \text{and} \quad (\tilde{\Omega}, \tilde{Z}) = \Phi_*(W_*, \eta_*).
\]
Then
\[
W_* = (PW + Q)(\overline{Q}W + \overline{P})^{-1} \quad \text{and} \quad \eta_* = (\eta + \lambda_*W + \mu_*)(\overline{Q}W + \overline{P})^{-1},
\]
where \(\lambda_* = \frac{1}{2}(\lambda + i \mu)\) and \(\mu_* = \frac{1}{2}(\lambda - i \mu)\).

According to Formulas (2.4) and (2.5), we get
\[
\tilde{\Omega} = i (I_g + W_*)(I_g - W_*)^{-1} \\
= i \left\{ I_g + (PW + Q)(\overline{Q}W + \overline{P})^{-1} \right\} \left\{ I_g - (PW + Q)(\overline{Q}W + \overline{P})^{-1} \right\}^{-1} \\
= i \left( \overline{Q}W + \overline{P} + PW + Q \right)(\overline{Q}W + \overline{P})^{-1} \\
\quad \times \left\{ \left( \overline{Q}W + \overline{P} - PW - Q \right)(\overline{Q}W + \overline{P})^{-1} \right\}^{-1} \\
= i \left\{ (P + \overline{Q})W + \overline{P} + Q \right\} \left\{ (\overline{Q} - P)W + \overline{P} - Q \right\}^{-1} \\
= \left\{ (i A - B)W + (i A + B) \right\} \left\{ (i C - D)W + (i C + D) \right\}^{-1}
\]

Therefore \(\tilde{\Omega} = \Omega_*\). In fact, this result is the well-known fact (cf. Formula (2.9)) that the action (1.1) is compatible with the action (2.8) via the Cayley transform \(\Phi\). Here we gave
A PARTIAL CAYLEY TRANSFORM OF SIEGEL-JACOBI DISK

9

a proof of Formula (2.9) for convenience.

\[
\hat{Z} = 2i \eta (I_g - W^*)^{-1} = 2i (\eta + \lambda_\ast W + \mu_\ast) (\overline{Q}W + \overline{P})^{-1} \times \left\{ I_g - (PW + Q)(\overline{Q}W + \overline{P})^{-1} \right\}^{-1} = 2i (\eta + \lambda_\ast W + \mu_\ast) (\overline{Q}W + \overline{P})^{-1} \times \left\{ (\overline{Q}W + \overline{P} - PW - Q)(\overline{Q}W + \overline{P})^{-1} \right\}^{-1} = 2i (\eta + \lambda_\ast W + \mu_\ast) \left\{ (\overline{Q} - P)W + \overline{P} - Q \right\}^{-1}.
\]

Using Formulas (2.4) and (2.5), we obtain

\[
\hat{Z} = \left\{ 2i \eta + (\lambda i - \mu)W + \lambda i + \mu \right\} \left\{ (i C - D)W + i C + D \right\}^{-1}.
\]

Hence \( \hat{Z} = Z^* \). Consequently we get Formula (3.7). Formula (3.8) follows immediately from a direct computation. \( \square \)

For a coordinate \((\Omega, Z) \in \mathbb{H}_{g,h}\) with \(\Omega = (\omega_{\mu\nu}) \in \mathbb{H}_g\) and \(Z = (z_{kl}) \in \mathbb{C}^{(h,g)}\), we put

\[
\begin{align*}
\Omega &= X + iY, & X &= (x_{\mu\nu}), & Y &= (y_{\mu\nu}) \text{ real}, \\
Z &= U + iV, & U &= (u_{kl}), & V &= (v_{kl}) \text{ real}, \\
d\Omega &= (d\omega_{\mu\nu}), & dX &= (dx_{\mu\nu}), & dY &= (dy_{\mu\nu}), \\
dZ &= (dz_{kl}), & dU &= (du_{kl}), & dV &= (dv_{kl}), \\
d\overline{\Omega} &= (d\overline{\omega}_{\mu\nu}), & d\overline{Z} &= (d\overline{z}_{kl}).
\end{align*}
\]

\[
\frac{\partial}{\partial \Omega} = \left( \frac{1 + \delta_{\mu\nu}}{2} \frac{\partial}{\partial \omega_{\mu\nu}} \right), \quad \frac{\partial}{\partial \overline{\Omega}} = \left( \frac{1 + \delta_{\mu\nu}}{2} \frac{\partial}{\partial \overline{\omega}_{\mu\nu}} \right),
\]

\[
\frac{\partial}{\partial Z} = \begin{pmatrix}
\frac{\partial}{\partial z_{11}} & \cdots & \frac{\partial}{\partial z_{1h}} \\
\vdots & \ddots & \vdots \\
\frac{\partial}{\partial z_{g1}} & \cdots & \frac{\partial}{\partial z_{gh}}
\end{pmatrix}, \quad \frac{\partial}{\partial \overline{Z}} = \begin{pmatrix}
\frac{\partial}{\partial \overline{z}_{11}} & \cdots & \frac{\partial}{\partial \overline{z}_{1h}} \\
\vdots & \ddots & \vdots \\
\frac{\partial}{\partial \overline{z}_{g1}} & \cdots & \frac{\partial}{\partial \overline{z}_{gh}}
\end{pmatrix}.
\]

Remark 3.1. The author proved in [13] that for any two positive real numbers \(A\) and \(B\), the following metric

\[
ds_{g,h;A,B}^2 = A \sigma \left( Y^{-1}d\Omega Y^{-1}d\overline{\Omega} \right) + B \left\{ \sigma \left( Y^{-1} V V Y^{-1} d\Omega Y^{-1} d\overline{\Omega} \right) + \sigma \left( Y^{-1} (dZ) d\overline{Z} \right) - \sigma \left( V Y^{-1} d\Omega Y^{-1} (dZ) \right) - \sigma \left( V Y^{-1} d\overline{\Omega} Y^{-1} (dZ) \right) \right\}
\]

(3.9)
is a Riemannian metric on $\mathbb{H}_{g,h}$ which is invariant under the action (1.3) of the Jacobi group $G^J$ and its Laplacian is given by

$$\Delta_{n,m;A,B} = \frac{4}{A} \left\{ \sigma \left( Y^t \left( Y \frac{\partial}{\partial \Omega} \right) \frac{\partial}{\partial \Omega} \right) + \sigma \left( V Y^{-1} V^t \left( Y \frac{\partial}{\partial Z} \right) \frac{\partial}{\partial Z} \right) \right. $$

$$+ \sigma \left( V^t \left( Y \frac{\partial}{\partial \Omega} \right) \frac{\partial}{\partial Z} \right) + \sigma \left( t V^t \left( Y \frac{\partial}{\partial Z} \right) \frac{\partial}{\partial \Omega} \right) \right\}$$

$$(3.10)$$

$$+ \frac{4}{B} \sigma \left( Y \frac{\partial}{\partial Z} \right) \left( \frac{\partial}{\partial Z} \right).$$

We observe that Formulas (3.9) and (3.10) generalize Formulas (2.10) and (2.11). The following differential form

$$dv_{g,h} = \left( \text{det} \ Y \right)^{-(g+h+1)} [dX] \wedge [dY] \wedge [dU] \wedge [dV]$$

is a $G^J$-invariant volume element on $\mathbb{H}_{g,h}$, where

$$[dX] = \wedge_{\mu \leq \nu} dx_{\mu \nu}, \quad [dY] = \wedge_{\mu \leq \nu} dy_{\mu \nu}, \quad [dU] = \wedge_{k,l} du_{kl} \quad \text{and} \quad [dV] = \wedge_{k,l} dv_{kl}.$$ 

Using the partial Cayley transform $\Phi_*$ and Theorem 3.1, we can find a $G^J_*$-invariant Riemannian metric on the Siegel-Jacobi disk $D_{g,h}$ and its Laplacian explicitly which generalize Formulas (2.12) and (2.13). For more detail, we refer to [14].

4. The Canonical Automorphic Factors

The isotropy subgroup $K^J_*$ at $(0,0)$ under the action (3.5) is

$$(4.1) \quad K^J_* = \left\{ \left( \begin{array}{cc} P & 0 \\ 0 & \mathcal{P} \end{array} \right), (0,0; \kappa) \right| P \in U(g), \kappa \in \mathbb{R}^{(h,h)} \right\}.$$ 

The complexification of $K^J_*$ is given by

$$(4.2) \quad K^J_{*,\mathbb{C}} = \left\{ \left( \begin{array}{cc} P & Q \\ 0 & -P^{-1} \end{array} \right), (0,0; \zeta) \right| P \in GL(g,\mathbb{C}), \zeta \in \mathbb{C}^{(h,h)} \right\}.$$ 

By a complicated computation, we can show that if

$$(4.3) \quad g_* = \left( \begin{array}{cc} P & Q \\ Q & \mathcal{P} \end{array} \right), (\lambda, \mu; \kappa)$$

is an element of $G^J_*$, then the $K^J_{*,\mathbb{C}}$-component of

$$g_* \cdot \left( \begin{array}{cc} I_g & W \\ 0 & I_g \end{array} \right), (0, \eta; 0)$$

is given by

$$(4.4) \quad \left( \begin{array}{cc} P & (PW+Q)(\mathcal{Q}W+\mathcal{P})^{-1}Q \\ 0 & \mathcal{Q}W+\mathcal{P} \end{array} \right), (0,0; \kappa_*),$$

where

$$\kappa_* = \kappa + \lambda^t \eta + (\eta + \lambda W + \mu)^t \eta - (\eta + \lambda W + \mu)^t (\mathcal{Q}W+\mathcal{P})^{-1} \lambda (\eta + \lambda W + \mu)$$

$$= \kappa + \lambda^t \eta + (\eta + \lambda W + \mu)^t \lambda - (\eta + \lambda W + \mu)(\mathcal{Q}W+\mathcal{P})^{-1} \mathcal{Q}^t (\eta + \lambda W + \mu).$$

Here we used the fact that $(\mathcal{Q}W+\mathcal{P})^{-1} \mathcal{Q}$ is symmetric.
For \( g_* \in G_*^J \) given by (4.3) with \( g_0 = \begin{pmatrix} P & Q \\ Q & P \end{pmatrix} \in G_* \) and \((W, \eta) \in D_{g,h}\), we write

\[
J(g_*(W, \eta)) = a(g_*(W, \eta)) b(g_0, W),
\]

where \( a(g_*(W, \eta)) = (I_{2g}, (0, 0; \kappa_*)) \), where \( \kappa_* \) is given in (4.4) and

\[
b(g_0, W) = \left( \begin{pmatrix} P - (PW + Q)(QW + P)^{-1}QW + P \\ 0 \end{pmatrix}, (0, 0; 0) \right).
\]

**Lemma 4.1.** Let \( \rho : GL(g, \mathbb{C}) \rightarrow GL(V_{\rho}) \) be a holomorphic representation of \( GL(g, \mathbb{C}) \) on a finite dimensional complex vector space \( V_{\rho} \) and \( \chi : \mathbb{C}^{(h,h)} \rightarrow \mathbb{C}^\times \) be a character of the additive group \( \mathbb{C}^{(h,h)} \). Then the mapping

\[
J_{\chi,\rho} : G_*^J \times D_{g,h} \rightarrow GL(V_{\rho})
\]

defined by

\[
J_{\chi,\rho}(g_*(W, \eta)) = \chi(a(g_*(W, \eta))) \rho(b(g_0, W))
\]

is an automorphic factor of \( G_*^J \) with respect to \( \chi \) and \( \rho \).

**Proof.** We observe that \( a(g_*(W, \eta)) \) is a summand of automorphic, i.e.,

\[
a(g_1 g_2, (W, \eta)) = a(g_1, (W, \eta)) + a(g_2, (W, \eta)),
\]

where \( g_1, g_2 \in G_*^J \) and \((W, \eta) \in D_{g,h}\). Together with this fact, the proof follows from the fact that the mapping

\[
J_{\rho} : G_* \times D_{g,h} \rightarrow GL(V_{\rho})
\]

defined by

\[
J_{\rho}(g_0, W) := \rho(b(g_0, W)), \quad g_0 \in G_*, \ W \in D_{g}
\]

is an automorphic factor of \( G_* \). \( \square \)

**Example 4.1.** Let \( \mathcal{M} \) be a symmetric half-integral semi-positive definite matrix of degree \( h \). Then the character

\[
\chi_{\mathcal{M}} : \mathbb{C}^{(h,h)} \rightarrow \mathbb{C}^\times
\]

defined by

\[
\chi_{\mathcal{M}}(c) = e^{-2\pi i \sigma(\mathcal{M}c)}, \quad c \in \mathbb{C}^{(h,h)}
\]

provides the automorphic factor

\[
J_{\mathcal{M},\rho} : G_*^J \times D_{g,h} \rightarrow GL(V_{\rho})
\]

defined by

\[
J_{\mathcal{M},\rho}(g_*(W, \eta)) = e^{-2\pi i \sigma(\mathcal{M}k_*)} \rho(QW + P),
\]

where \( g_* \) is an element in \( G_*^J \) given by (4.3) and \( \kappa_* \) is given in (4.4). Using \( J_{\mathcal{M},\rho} \), we can define the notion of Jacobi forms on \( D_{g,h} \) of index \( \mathcal{M} \) with respect to the Siegel modular group \( T^{-1}Sp(g, Z) T \) (cf. [7-9]).

**Remark 4.2.** The \( P_*^- \)-component of

\[
g_* \cdot \begin{pmatrix} I_g & W \\ 0 & I_g \end{pmatrix}, (0, \eta; 0)
\]
is given by

\[
(4.6) \quad \left( \begin{array}{c} I_g \\ \bar{Q}W + \bar{P} \end{array} \right)^{-1} \left( \begin{array}{c} I_g \\ 0 \end{array} \right), \quad \left( \lambda - (\eta + \lambda W + \mu) \left( \bar{Q}W + \bar{P} \right)^{-1} \bar{Q}, \ 0 ; 0 \right),
\]

where

\[
P_\ast^{-} = \left\{ \left( \begin{array}{c} I_g \\ W \end{array} \right), \ (\xi, 0 ; 0) \ \bigg| \ \ W = ^tW \in \mathbb{C}^{(g,g)}, \ \xi \in \mathbb{C}^{(h,g)} \right\}.
\]

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Department of Mathematics, Inha University, Incheon 402-751, Korea
E-mail address: jhyang@inha.ac.kr