Derivative-free SMR conjugate gradient method for constraint nonlinear equations

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Abstract

Based on the SMR conjugate gradient method for unconstrained optimization proposed by Mohamed et al. [N. S. Mohamed, M. Mamat, M. Rivaie, S. M. Shaharuddin, Indones. J. Electr. Eng. Comput. Sci., \textbf{11} (2018), 1188–1193] and the Solodov and Svaiter projection technique, we propose a derivative-free SMR method for solving nonlinear equations with convex constraints. The proposed method can be viewed as an extension of the SMR method for solving unconstrained optimization. The proposed method can be used to solve large-scale nonlinear equations with convex constraints because of derivative-free and low storage. Under the assumption that the underlying mapping is Lipschitz continuous and satisfies a weaker monotonicity assumption, we prove its global convergence. Preliminary numerical results show that the proposed method is promising.

Keywords: Nonlinear equations, conjugate gradient method, projection method, global convergence.

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1. Introduction

Mathematically, nonlinear systems of equations with convex constraint, can be express as

\[ j(c) = 0, \quad c \in \Theta, \]  \hspace{1cm} (1.1)

where \( j : \Theta \to \mathbb{R} \) is a continuous nonlinear mapping, and \( \Theta \subseteq \mathbb{R}^n \) is a closed convex set. Nonlinear equations of the form (1.1) commonly appears in various applications such as financial forecasting problems [9], nonlinear compressed sensing [6], non-negative matrix factorisation [4, 25], economic equilibrium

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problems [13] and many others. Consequently, a number of different iterative methods have been developed to solve (1.1). For instance, see [10, 11, 28, 32]. La Cruz [7] recently presented a spectral method which uses the residual vector as a search direction to solve large-scale systems of nonlinear equations involving monotone mapping. Solodov and Svaiter [30] suggested a method in which projection, proximal point and Newton method were combined. Motivated by the work of Solodov [30], Zhang and Zhou [34] proposed a spectral gradient projection method for solving nonlinear equation involving monotone mapping. Our interest in this paper, is on the conjugate gradient method. The conjugate gradient method is widely used for solving the unconstrained optimization problem. In the last years, various conjugate gradient methods for large-scale unconstrained optimization have been extended to solve (1.1). For instance, using the projection technique of Solodov [30], Ibrahim et al. [20] extended the hybrid LS-FR conjugate gradient method proposed by Djordjević [14] to solve nonlinear equation with convex constraint. At each iteration, the proposed method does not store any matrix. Also, inspired by the RMILL method [29] for unconstrained optimization, Fang [16] developed three derivative-free conjugate gradient procedures. For more articles on derivative-free algorithms, interested readers may refer to the recent papers [1–3, 18, 19, 21–24].

Quite recently, Mohammed et al. [27] designed a SMR conjugate gradient method for solving unconstrained optimization problem. Based on the efficiency performance of the SMR method and the projection technique of Solodov and Svaiter [30] we describe a class of new derivative-free conjugate gradient method for solving (1.1). The global convergence of the method is proved under the assumption that the underlying mapping is Lipschitz continuous and satisfies a weaker monotonicity assumption.

The remaining part of this paper is organized as follows. In the next section, we propose a derivative-free SMR method for solving the constraint nonlinear equation (1.1). Under mild assumption, the global convergence is established in Section 3. In Section 4, preliminary numerical results are presented to show that our method are efficient and promising. Finally, we have the conclusion. Throughout this manuscript, \( \| \cdot \| \) denotes the Euclidean norm.

2. The method

In this section, we present our method for solving the nonlinear equation with convex constraints (1.1). Our method is based on the SMR conjugate gradient method which we will review below. The SMR conjugate gradient method proposed by Mohamed et al. [27] for solving the following unconstrained optimization problem

\[
\min g(c), \ c \in \mathbb{R}^n,
\]

where \( g : \mathbb{R}^n \to \mathbb{R} \) is a continuously differentiable function. Let \( \nabla g(c_t) \) denote the gradient of \( g \) at \( c_t \). The Mohammed et al. [27] method generates a sequence of iterates \( \{c_t\} \) by the following recursive formula

\[
c_{t+1} = c_t + \alpha_t p_t, \quad t \geq 0
\]

(2.1)

where \( c_t \) is the current iterative point and \( c_0 \in \mathbb{R}^n \) is set to be a starting point of the sequence. From (2.1), \( \alpha_t > 0 \) is known as a step size and \( p_t \) is the search direction defined by the rule:

\[
p_t := \begin{cases} 
-\nabla g(c_t), & \text{if } t = 0, \\
-\nabla g(c_t) + \beta_{t}^{\text{SMR}} p_{t-1}, & \text{if } t > 0,
\end{cases}
\]

where the conjugate gradient parameter \( \beta_{t}^{\text{SMR}} \) is defined as

\[
\beta_{t}^{\text{SMR}} := \max \left\{ 0, \frac{\| \nabla g(c_t) \|^2 - \| \nabla g(c_t)^T \nabla g(c_{t-1}) \|}{\| p_{t-1} \|^2} \right\}.
\]

Based on the SMR method, we introduce a derivative-free projection method for solving (1.1). Our proposed method first generate a trial point \( d_t \) by the following relation:

\[
d_t = c_t + \alpha_t p_t
\]
and the search direction \( p_t \) is computed by

\[
p_t := \begin{cases} 
-j_t, & \text{if } t = 0, \\
-j_t + \beta_t^{\text{ESMR}} p_{t-1}, & \text{if } t > 0,
\end{cases}
\]  

(2.2)

where \( j_t = j(c_t) \) and \( \beta_t^{\text{ESMR}} \) is defined as

\[
\beta_t^{\text{ESMR}} := \max \left\{ 0, \frac{\| j_t \|^2 - |j_t^T j_{t-1}|}{\| p_{t-1} \|^2} \right\}.
\]  

(2.3)

It can be observed that the search direction \( p_t \) defined by (2.2) is likely not a descent direction for all \( t \). To ensure the decency, we choose a vector from a subspace \( \varphi_t = \{ q \mid j_t^T q = 0 \} \) to replace the second term \( \beta_t^{\text{ESMR}} p_{t-1} \) of the direction (2.2). We obtain

\[
p_t = -j_t + q, \quad q \in \varphi_t.
\]

For instance, if we choose \( q = 0 \in \varphi_t \), the steepest descent direction is obtained. If we choose

\[
q = \beta_t^{\text{ESMR}} \left( p_{t-1} - \frac{j_t^T p_{t-1}}{\| j_t \|^2} j_t \right) \in \varphi_t,
\]

which is obviously motivated by the Gram-Schmidt (MGS) process, we get the direction used in [17, 31].

**Definition 2.1.** Let \( \Theta \subseteq \mathbb{R}^n \) be a nonempty closed convex set. Then for any \( y \in \mathbb{R}^n \), its projection onto \( \Theta \), denoted by \( P_{\Theta}[y] \), is defined by

\[
P_{\Theta}[y] := \arg \min \{ \| y - x \| : x \in \Theta \}.
\]

The projection operator \( P_{\Theta} \) has a well-known property, that is, for any \( y, x \in \mathbb{R}^n \) the following nonexpansive property hold

\[
\| P_{\Theta}[y] - P_{\Theta}[x] \| \leq \| y - x \|.
\]  

(2.4)

In what follows, we state the iterative procedures/steps of our method.

**Algorithm 1**

**Input.** Set an initial point \( c_0 \in \Theta \), the positive constants: \( \text{Tol} > 0, \tau \in (0, 1), \alpha \in (0, 2), \lambda > 0, \mu > 0 \). Set \( t = 0 \).

**Step 0.** Compute \( j_t \). If \( \| j_t \| \leq \text{Tol} \), stop. Otherwise, generate the search direction \( p_t \) using the following

\[
p_t := \begin{cases} 
-j_t, & \text{if } t = 0, \\
-j_t + \beta_t^{\text{ESMR}} \left( p_{t-1} - \frac{j_t^T p_{t-1}}{\| j_t \|^2} j_t \right), & \text{if } t > 0,
\end{cases}
\]  

(2.5)

where \( \beta_t^{\text{ESMR}} \) is computed by (2.3).

**Step 1.** Determine the step-size \( \alpha_t = \max \{ \alpha r^m : m \geq 0 \} \) such that

\[
j(c_t + \alpha_t p_t)^T p_t \geq \mu \alpha_t \| p_t \|^2.
\]  

(2.6)

**Step 2.** Compute \( d_t = c_t + \alpha_t p_t \), where \( d_t \) is a trial point.

**Step 3.** If \( d_t \in \Theta \) and \( \| j(d_t) \| = 0 \), stop. Otherwise, compute the next iterate by

\[
c_{t+1} = P_{\Theta} \left[ c_t - \frac{j(d_t)^T (c_t - d_t)}{\| j(d_t) \|^2} j(d_t) \right],
\]

**Step 4.** Finally we set \( t = t + 1 \) and return to step 1.
3. Theoretical analysis

In this section, we obtain the global convergence property of Algorithm 1. We also make the following assumptions on the mapping $j$.

**Assumption 3.1.**

(i) The solution set of the constrained nonlinear (1.1), denoted by $\Theta^*$, is nonempty.
(ii) The mapping $j$ is Lipschitz continuous on $\mathbb{R}^n$. That is, there exists a constant $L > 0$ such that

$$||j(\alpha) - j(\beta)|| \leq L||\alpha - \beta||, \ \forall \alpha, \beta \in \mathbb{R}^n. \ (3.1)$$

(iii) For any $\beta \in \Theta^*$ and $\alpha \in \mathbb{R}^n$, it holds that

$$j(\alpha)^T(\alpha - \beta) \geq 0.$$

**Lemma 3.2.** Let $p_t$ be the search direction generated by Algorithm 1, then $p_t$ is a sufficient descent direction. That is for all $t \geq 0$,

$$j_t^Tp_t = -c||j_t||^2, \ c > 0. \ (3.2)$$

**Proof.** The proof follows. $\square$

**Lemma 3.3.** Let $\{p_t\}$ and $\{c_t\}$ be two sequences generated by Algorithm 1. Then, there exists a step size $\alpha_t$ satisfying the line search (2.6) for all $t \geq 0$.

**Proof.** For any $m \geq 0$, suppose (2.6) does not hold for the iterate $t_0$-th, then we have

$$-j(c_{t_0} + ar^m p_{t_0})^T p_{t_0} < \mu ar^m||p_{t_0}||^2.$$

Thus, by the continuity of $j$ and with $0 < r < 1$, it follows that by letting $m \to \infty$, we have

$$-j(c_{t_0})^T p_{t_0} \leq 0,$$

which contradicts (3.2). $\square$

**Lemma 3.4.** Let the sequences $\{c_t\}$ and $\{d_t\}$ be generated by the Algorithm 1 method under Assumption 3.1, then

$$\alpha_t \geq \max \left\{ \alpha, \frac{rc||j_t||^2}{(L + \mu)||p_t||^2} \right\}. \ (3.3)$$

**Proof.** Let $\bar{\alpha}_t = \alpha_t r^{-1}$. Assume $\alpha_t \neq \alpha$, from (2.6), $\bar{\alpha}_t$ does not satisfy (2.6). That is,

$$-j(c_t + \bar{\alpha}_tp_t)^T p_t < \mu \bar{\alpha}_t||p_t||^2.$$

From (3.1) and (3.2), it can be obviously seen that

$$c||j_t||^2 \leq -j_t^Tp_t = (j(c_t + \bar{\alpha}_tp_t) - j_t)^T p_t - j(c_t + \bar{\alpha}_tp_t)^T p_t \leq L\bar{\alpha}_t||p_t||^2 + \mu \bar{\alpha}_t||p_t||^2 \leq \bar{\alpha}_t(L + \mu)||p_t||^2.$$

This gives the desired inequality (3.3). $\square$
Lemma 3.5. Suppose that Assumption 3.1 holds. Let \( \{c_t\} \) and \( \{d_t\} \) be sequences generated by the Algorithm 1, then for any solution \( c^* \) contained in the solution set \( \Theta^* \) the inequality
\[
\|c_{t+1} - c^*\|^2 \leq \|c_t - c^*\|^2 - \mu^2\|c_t - d_t\|^4
\]
holds. In addition, \( \{c_t\} \) is bounded and
\[
\sum_{t=0}^{\infty} \|c_t - d_t\|^4 < +\infty. \tag{3.4}
\]

Proof. First, we begin by using the weakly monotonicity assumption (Assumption 3.1 (iii)) on the mapping \( j \). Thus, for any solution \( c^* \in \Theta^* \),
\[
j(d_t)^T(c_t - c^*) \geq j(d_t)^T(c_t - d_t).
\]
The above inequality together with (2.6) gives
\[
j(c_t + \alpha_t p_t)^T(c_t - d_t) \geq \mu \alpha_t^2 \|p_t\|^2 > 0. \tag{3.5}
\]
From (2.4) and (3.5), we have the following
\[
\|c_{t+1} - c^*\|^2 = \left\|p_t \left[ c_t - x \frac{j(d_t)^T(c_t - d_t)}{\|j(d_t)\|^2} j(d_t) \right] - c^* \right\|^2
\]
\[
\leq \left\|[c_t - x \frac{j(d_t)^T(c_t - d_t)}{\|j(d_t)\|^2} j(d_t)] - c^* \right\|^2
\]
\[
= \|c_t - c^*\|^2 - 2x \left( \frac{j(d_t)^T(c_t - d_t)}{\|j(d_t)\|^2} \right) j(d_t)^T(c_t - c^*) + x^2 \left( \frac{j(d_t)^T(c_t - d_t)}{\|j(d_t)\|^2} \right)^2
\]
\[
= \|c_t - c^*\|^2 - 2x \left( \frac{j(d_t)^T(c_t - d_t)}{\|j(d_t)\|^2} \right) j(d_t)^T(c_t - d_t) + x^2 \left( \frac{j(d_t)^T(c_t - d_t)}{\|j(d_t)\|^2} \right)^2
\]
\[
= \|c_t - c^*\|^2 - x(2 - x) \left( \frac{j(d_t)^T(c_t - d_t)}{\|j(d_t)\|^2} \right)^2
\]
\[
\leq \|c_t - c^*\|^2.
\]
Thus, the sequence \( \{\|c_t - c^*\|\} \) has a nonincreasing and convergent property. Therefore, this makes \( \{c_t\} \) to be bounded and therefore the following holds
\[
\sigma^2 \sum_{t=0}^{\infty} \|c_t - d_t\|^4 < \|c_0 - c^*\|^2 < +\infty.
\]
\[\square\]

Remark 3.6. Taking into account of the definition of \( d_t \) and also by (3.4), it can be deduced that
\[
\lim_{t \to \infty} \alpha_t \|p_t\| = 0. \tag{3.6}
\]

Theorem 3.7. Suppose Assumption 3.1 holds. Let \( \{c_t\} \) and \( \{d_t\} \) be sequences generated by Algorithm 1, then
\[
\liminf_{t \to \infty} \|j_t\| = 0. \tag{3.7}
\]
Proof. Suppose (3.7) is not valid, that is, there exist a constant say \( s > 0 \) such that \( s \leq \| \text{j}_t \|, \ t \geq 0 \). Then this along with (3.2) implies that

\[ \| \text{p}_t \| \geq cs, \ \forall t \geq 0. \]

It can be obviously seen from Lemma 3.5 and Remark 3.6, that the sequences \( \{ c_t \} \) and \( \{ d_k \} \) are bounded. In addition with the continuity of \( j \), it further implies that \( \| j_t \| \) is bounded by a constant say \( u \). From (2.5), it follows that for all \( t \geq 1, \)

\[
\| \text{p}_t \| = \| -j_t + \beta^{FSMR}_t (p_{t-1} - \frac{j_T p_{t-1}^T}{\|j_t\|^2} j_t) \| \\
= \left\| -j_t + \frac{\|j_t\|^2 - \|j_T j_{t-1}\|}{\|p_{t-1}\|^2} (p_{t-1} - \frac{j_T p_{t-1}^T}{\|j_t\|^2} j_t) \right\| \leq \|j_t\| + 2\frac{\|j_t\|^2 + \|j_t\|\|j_{t-1}\|}{\|p_{t-1}\|} \leq u + \frac{4u^2}{cs} \triangleq \gamma.
\]

It is worth mentioning since \( \beta^{FSMR}_t = 0 \), it is easy to see that \( p_{t-1} \) is still bounded. From (3.3), we have

\[
\alpha_t \| \text{p}_t \| \geq \max \left\{ \alpha, \frac{r \| \text{j}_t \|^2}{(L + \mu \| \text{p}_t \|^2)} \right\} \| \text{p}_t \| \geq \max \left\{ \frac{acs}{(L + \mu \gamma)} \right\} > 0,
\]

which contradicts (3.6). Hence (3.7) is valid. \( \square \)

4. Numerical experiments

This section evaluates the numerical efficiency of the proposed algorithm using the Dolan and More performance profile [15]. The metrics taking into consideration using the Dolan and More performance profile includes; the number of iterations, the number of function evaluations and the CPU running time. The performance of Algorithm 1 is compared with the derivative-free iterative method for nonlinear monotone equations with convex constraints proposed in [26]. In what follows, we refer to the algorithm proposed in [26] as Algorithm 2. All codes were coded and implemented in Matlab environment.

- Control parameters: For Algorithm 1, we select \( a = 1, \ \gamma = 0.8, \ \mu = 10^{-4}, \ \alpha = 2, \ \text{Tol} = 10^{-6} \). As for Algorithm 2, we select all parameters as in [26].
- Dimensions: 1000, 5000, 10, 000, 50, 000, 100, 000.
- Initial points: \( c_1 = (0.1,0.1,\ldots,0.1)^T, \ c_2 = (0.2,0.2,\ldots,0.2)^T, \ c_3 = (0.5,0.5,\ldots,0.5)^T, \ c_4 = (1.2,1.2,\ldots,1.2)^T, \ c_5 = (1.5,1.5,\ldots,1.5)^T, \ c_6 = (2,2,\ldots,2)^T, \ c_7 = \text{rand}(0,1). \)

The test problems with \( j = (j_1, j_2, \ldots, j_n) \) are given below.

**Problem 4.1** ([8]). Exponential function:

\[ j_1(c) = e^{c_1} - 1, \ j_2(c) = e^{c_1} + c_1 - 1, \ \text{for} \ i = 2,3,\ldots,n, \ \text{and} \ \Theta = \mathbb{R}_+^n. \]

**Problem 4.2** ([8]). Modified logarithmic function:

\[
j_i(c) = \ln(c_i + 1) - \frac{c_i}{n}, \ \text{for} \ i = 1,2,3,\ldots,n,
\]

\[
\Theta = \left\{ c \in \mathbb{R}^n : \sum_{i=1}^n c_i \leq n, c_i > -1, i = 1,2,\ldots,n \right\}.
\]

**Problem 4.3** ([7]).

\[
j_i(c) = \min \{ \min(|c_i|, c_i^2), \max(|c_i|, c_i^2) \}, \ \text{for} \ i = 2,3,\ldots,n, \ \text{and} \ \Theta = \mathbb{R}_+^n.
\]
Problem 4.4 ([8]). Strictly convex function I:
\[ j_i(c) = e^{c_i} - 1, \quad \text{for } i = 1, 2, \ldots, n, \quad \text{and } \quad \Theta = \mathbb{R}_+^n. \]

Problem 4.5 ([8]). Strictly convex function II:
\[ j_i(c) = \frac{i}{n}e^{c_i} - 1, \quad \text{for } i = 1, 2, \ldots, n, \quad \text{and } \quad \Theta = \mathbb{R}_+^n. \]

Problem 4.6 ([5]). Tridiagonal exponential function:
\[
\begin{align*}
{j_1(c)} &= c_1 - e^{\cos(h(c_1 + c_2))}, \\
{j_i(c)} &= c_i - e^{\cos(h(c_{i-1} + c_i + c_{i+1}))}, \quad \text{for } i = 2, \ldots, n - 1, \\
{j_n(z)} &= c_n - e^{\cos(h(c_{n-1} + c_n))}, \\
{h} &= \frac{1}{n + 1}.
\end{align*}
\]

Problem 4.7 ([33]). Nonsmooth function:
\[ j_i(c) = c_i - \sin|c_i - 1|, \quad i = 1, 2, 3, \ldots, n, \]
\[ \Theta = \left\{ c \in \mathbb{R}^n : \sum_{i=1}^{n} c_i \leq n, c_i \geq -1, i = 1, 2, \ldots, n \right\}. \]

Problem 4.8 ([8]). The Trig exp function
\[
\begin{align*}
{j_1(c)} &= 3c_1^3 + 2c_2 - 5 + \sin(c_1 - c_2)\sin(c_1 + c_2), \\
{j_i(c)} &= 3c_i^3 + 2c_{i+1} - 5 + \sin(c_i - c_{i-1})\sin(c_i + c_{i+1}) + 4c_i - c_{i-1}e^{c_{i-1} - c_i} - 3 \quad \text{for } i = 2, 3, \ldots, n - 1, \\
{j_n(z)} &= c_{n-1}e^{c_{n-1} - c_n} - 4c_n - 3, \quad \text{where } \quad h = \frac{1}{m + 1} \quad \text{and } \quad \Theta = \mathbb{R}_+^n.
\end{align*}
\]

Problem 4.9 ([12]).
\[ t_i = \sum_{i=1}^{n} c_i^2, \quad d = 10^{-5}, \quad j_i(c) = 2d(c_i - 1) + 4(t_i - 0.25)c_i, \quad i = 1, 2, 3, \ldots, n, \quad \text{and } \quad \Theta = \mathbb{R}_+^n. \]

Figure 1: Performance profiles based on number of iterations.  
Figure 2: Performance profiles based on number of function evaluations.
The assessment results are given in Tables 1-9 of the appendix section. In Tables 1-9, “dm” denotes the dimension, “inp” denotes the initial point, “it” denotes the number of iteration, “nf” denotes the number of function evaluation, and “tm” denotes the CPU running time. Figure 1 displays the performance profile of the number of iterations. It can be seen that Algorithm 1 is the top curve. Algorithm 1 outperforms Algorithm 2, with Algorithms 1 been able to solve 72% of the test problems with less number of iterations while, Algorithm 2 was able to solve around 32%. Similarly, in Figure 2, Algorithm 1 achieved less number of function evaluations compared to Algorithm 2. Figure 3 is the performance profile measured by the CPU time. The top curve is the Algorithm 1 that solved the most problems in a time that was within a factor of the best time. In particular, the Algorithm 1 solves about 62% of the test problems with the least CPU time, while Algorithm 2 solves around 39%. Based on the above comparisons, it indicates that the Algorithm 1 outperforms Algorithm 2 well-known methods for all metric, that is the number of iteration, the total number of function evaluations and the CPU time.

5. Conclusion

We have proposed a derivative-free SMR conjugate gradient method for constraint nonlinear equation. The search direction of the proposed method satisfies the sufficient descent condition. The global convergence is proved under the assumption that the underlying operator is Lipschitz continuous and satisfies a weaker monotonicity condition. Numerical experiment shows that the proposed method is efficient.

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Table 1: Numerical result for Problem 4.1.

| dm  | inp | it | nf   | tm     | nm | it | nf   | tm     | nm  |
|-----|-----|----|------|--------|----|----|------|--------|-----|
| 1000|     |    |      |        |    |    |      |        |     |
|     | c1  | 3  | 11   | 0.22387| 0  | 16 | 64   | 0.041872| 3.45E-07|
|     | c2  | 2  | 7    | 0.02453| 0  | 16 | 64   | 0.021963| 7.03E-07|
|     | c3  | 3  | 11   | 0.019411| 0  | 17 | 68   | 0.020096| 6.22E-07|
|     | c4  | 2  | 7    | 0.008967| 0  | 18 | 72   | 0.026791| 4.54E-07|
|     | c5  | 2  | 7    | 0.019794| 0  | 18 | 72   | 0.022601| 3.65E-07|
|     | c6  | 2  | 7    | 0.010403| 0  | 18 | 72   | 0.020692| 3.80E-07|
|     | c7  | 19 | 75   | 0.087602| 0  | 17 | 68   | 0.017729| 7.12E-07|
| 5000|     |    |      |        |    |    |      |        |     |
|     | c1  | 2  | 7    | 0.035455| 0  | 16 | 64   | 0.073457| 7.61E-07|
|     | c2  | 2  | 7    | 0.021848| 0  | 17 | 68   | 0.062864| 5.15E-07|
|     | c3  | 2  | 7    | 0.020868| 0  | 18 | 72   | 0.065321| 4.63E-07|
|     | c4  | 2  | 7    | 0.020796| 0  | 19 | 76   | 0.072119| 3.38E-07|
|     | c5  | 2  | 7    | 0.089639| 0  | 18 | 72   | 0.064029| 8.12E-07|
|     | c6  | 2  | 7    | 0.025163| 0  | 18 | 72   | 0.10743 | 8.10E-07|
|     | c7  | 72 | 287  | 11.9443| 0  | 18 | 72   | 0.063575| 5.37E-07|
| 10000|    |    |      |        |    |    |      |        |     |
|     | c1  | 2  | 7    | 0.065315| 0  | 17 | 68   | 0.11957 | 3.55E-07|
|     | c2  | 2  | 7    | 0.02504 | 0  | 17 | 68   | 0.096546| 7.27E-07|
|     | c3  | 2  | 7    | 0.022788| 0  | 18 | 72   | 0.10444 | 6.55E-07|
|     | c4  | 2  | 7    | 0.11322 | 0  | 19 | 76   | 0.10849 | 4.77E-07|
|     | c5  | 2  | 7    | 0.041054| 0  | 20 | 80   | 0.12003 | 4.52E-07|
|     | c6  | 2  | 7    | 0.030381| 0  | 19 | 76   | 0.14566 | 5.51E-07|
|     | c7  | 244| 975  | 11.9443| 0  | 18 | 72   | 0.099343| 7.51E-07|
| 50000|    |    |      |        |    |    |      |        |     |
|     | c1  | 2  | 7    | 0.10549 | 0  | 17 | 68   | 0.41409 | 7.93E-07|
|     | c2  | 2  | 7    | 0.13639 | 0  | 18 | 72   | 0.42281 | 5.44E-07|
|     | c3  | 2  | 7    | 0.13746 | 0  | 19 | 76   | 0.39061 | 4.86E-07|
|     | c4  | 2  | 7    | 0.1435  | 0  | 20 | 80   | 0.43511 | 9.70E-07|
|     | c5  | 2  | 7    | 0.11929 | 0  | 22 | 88   | 0.5831  | 8.63E-07|
|     | c6  | 2  | 7    | 0.18758 | 0  | 23 | 92   | 0.64319 | 8.62E-07|
|     | c7  | 77 | 307  | 7.5127  | 0  | 19 | 76   | 0.43164 | 5.61E-07|
| 100000|   |    |      |        |    |    |      |        |     |
|     | c1  | 2  | 7    | 0.21193 | 0  | 18 | 72   | 0.75865 | 3.76E-07|
|     | c2  | 2  | 7    | 0.18253 | 0  | 18 | 72   | 0.85139 | 7.69E-07|
|     | c3  | 2  | 7    | 0.17591 | 0  | 19 | 76   | 0.8479  | 6.88E-07|
|     | c4  | 2  | 7    | 0.20545 | 0  | 23 | 92   | 1.0896  | 3.63E-07|
|     | c5  | 2  | 7    | 0.381   | 0  | 23 | 92   | 1.043   | 9.61E-07|
|     | c6  | 2  | 7    | 0.20449 | 0  | 26 | 104  | 1.3727  | 3.39E-07|
|     | c7  | 141| 563  | 32.7692 | 0  | 20 | 80   | 0.8401  | 7.79E-07|
Table 2: Numerical result for Problem 4.2.

| Algorithm 1 | Algorithm 2 |
|-------------|-------------|
| **dm** | **inp** | **it** | **nf** | **tm** | **nm** | **it** | **nf** | **tm** | **nm** |
| 1000 | | | | | | | | | |
| c₁ | 7 | 22 | 0.057927 | 1.58E-09 | 13 | 51 | 0.010627 | 7.68E-07 |
| c₂ | 7 | 22 | 0.012172 | 2.12E-09 | 15 | 59 | 0.013658 | 3.49E-07 |
| c₃ | 6 | 19 | 0.005371 | 7.52E-09 | 16 | 63 | 0.018126 | 6.98E-07 |
| c₄ | 8 | 25 | 0.025657 | 1.95E-09 | 18 | 71 | 0.048427 | 3.52E-07 |
| c₅ | 6 | 19 | 0.009033 | 8.43E-09 | 18 | 71 | 0.019308 | 5.13E-07 |
| c₆ | 9 | 28 | 0.008674 | 1.04E-09 | 18 | 71 | 0.016487 | 8.59E-07 |
| c₇ | 11 | 35 | 0.010357 | 6.79E-07 | 17 | 67 | 0.014273 | 4.49E-07 |
| 5000 | | | | | | | | | |
| c₁ | 6 | 20 | 0.017838 | 2.97E-07 | 14 | 55 | 0.052256 | 5.44E-07 |
| c₂ | 6 | 20 | 0.051993 | 4.05E-07 | 15 | 59 | 0.057197 | 7.63E-07 |
| c₃ | 6 | 19 | 0.016847 | 9.12E-10 | 17 | 67 | 0.052789 | 5.12E-07 |
| c₄ | 7 | 23 | 0.025412 | 3.74E-07 | 18 | 71 | 0.074039 | 7.37E-07 |
| c₅ | 6 | 19 | 0.017764 | 1.42E-09 | 19 | 75 | 0.061007 | 3.75E-07 |
| c₆ | 7 | 22 | 0.014747 | 7.12E-09 | 19 | 75 | 0.061742 | 6.27E-07 |
| c₇ | 12 | 38 | 0.033328 | 6.79E-07 | 17 | 67 | 0.072130 | 9.83E-07 |
| 10000 | | | | | | | | | |
| c₁ | 5 | 16 | 0.024083 | 9.23E-07 | 14 | 55 | 0.117640 | 7.66E-07 |
| c₂ | 6 | 20 | 0.075326 | 3.06E-07 | 16 | 63 | 0.081869 | 3.55E-07 |
| c₃ | 6 | 19 | 0.032398 | 4.32E-10 | 17 | 67 | 0.103390 | 7.23E-07 |
| c₄ | 7 | 24 | 0.029118 | 2.82E-07 | 19 | 75 | 0.096522 | 3.63E-07 |
| c₅ | 6 | 19 | 0.017764 | 1.42E-09 | 19 | 75 | 0.117560 | 5.29E-07 |
| c₆ | 7 | 22 | 0.137500 | 4.21E-09 | 19 | 75 | 0.106360 | 9.51E-07 |
| c₇ | 11 | 38 | 0.077944 | 1.97E-07 | 18 | 71 | 0.104950 | 4.63E-07 |
| 50000 | | | | | | | | | |
| c₁ | 7 | 26 | 0.136900 | 1.84E-07 | 15 | 59 | 0.342390 | 5.78E-07 |
| c₂ | 9 | 34 | 0.126310 | 3.87E-07 | 16 | 63 | 0.373650 | 7.92E-07 |
| c₃ | 6 | 21 | 0.081129 | 5.88E-07 | 18 | 71 | 0.406850 | 5.36E-07 |
| c₄ | 10 | 37 | 0.155930 | 3.60E-07 | 21 | 84 | 0.456620 | 3.43E-07 |
| c₅ | 7 | 25 | 0.093439 | 1.16E-07 | 21 | 84 | 0.458420 | 4.72E-07 |
| c₆ | 8 | 28 | 0.107060 | 7.93E-07 | 21 | 84 | 0.505290 | 4.77E-07 |
| c₇ | 10 | 37 | 0.163890 | 9.96E-07 | 19 | 75 | 0.445200 | 3.45E-07 |
| 100000 | | | | | | | | | |
| c₁ | 7 | 26 | 0.594930 | 2.56E-07 | 15 | 59 | 0.621580 | 8.17E-07 |
| c₂ | 9 | 34 | 0.292540 | 5.47E-07 | 17 | 67 | 0.694700 | 3.76E-07 |
| c₃ | 6 | 21 | 0.313950 | 7.65E-07 | 18 | 72 | 0.770960 | 9.65E-07 |
| c₄ | 10 | 37 | 0.390740 | 5.09E-07 | 22 | 88 | 1.010100 | 8.28E-07 |
| c₅ | 7 | 25 | 0.340980 | 1.55E-07 | 22 | 88 | 1.009800 | 8.18E-07 |
| c₆ | 9 | 32 | 0.244090 | 1.09E-07 | 22 | 88 | 0.992440 | 7.87E-07 |
| c₇ | 11 | 41 | 0.370310 | 1.15E-07 | 20 | 80 | 0.929080 | 5.48E-07 |
Table 3: Numerical result for Problem 4.3.

| dm  | inp | it | nf | tm    | nm  | it | nf | tm    | nm  |
|-----|-----|----|----|-------|-----|----|----|-------|-----|
| 1000|     |    |    |       |     |    |    |       |     |
|     | c1  | 2  | 6  | 0.047463 | 0   | 2  | 6  | 0.004935 | 0   |
|     | c2  | 2  | 6  | 0.004985 | 0   | 2  | 6  | 0.005237 | 0   |
|     | c3  | 2  | 6  | 0.007876 | 0   | 2  | 6  | 0.004172 | 0   |
|     | c4  | 3  | 11 | 0.007798 | 0   | 2  | 6  | 0.003795 | 0   |
|     | c5  | 3  | 11 | 0.007161 | 0   | 2  | 6  | 0.00432  | 0   |
|     | c6  | 3  | 11 | 0.008865 | 0   | 2  | 6  | 0.004887 | 0   |
|     | c7  | 3  | 10 | 0.008971 | 0   | 2  | 6  | 0.005036 | 0   |
| 5000|     |    |    |       |     |    |    |       |     |
|     | c1  | 2  | 6  | 0.012999 | 0   | 2  | 6  | 0.010986 | 0   |
|     | c2  | 2  | 6  | 0.015079 | 0   | 2  | 6  | 0.011065 | 0   |
|     | c3  | 2  | 6  | 0.01514  | 0   | 2  | 6  | 0.01249  | 0   |
|     | c4  | 3  | 11 | 0.051959 | 0   | 2  | 6  | 0.010016 | 0   |
|     | c5  | 3  | 11 | 0.023692 | 0   | 2  | 6  | 0.012051 | 0   |
|     | c6  | 3  | 11 | 0.04786  | 0   | 2  | 6  | 0.009662 | 0   |
|     | c7  | 3  | 10 | 0.029942 | 0   | 2  | 6  | 0.01144  | 0   |
| 10000|    |    |    |       |     |    |    |       |     |
|     | c1  | 2  | 6  | 0.023004 | 0   | 2  | 6  | 0.021632 | 0   |
|     | c2  | 2  | 6  | 0.023611 | 0   | 2  | 6  | 0.019348 | 0   |
|     | c3  | 2  | 6  | 0.020701 | 0   | 2  | 6  | 0.032078 | 0   |
|     | c4  | 3  | 11 | 0.046703 | 0   | 2  | 6  | 0.01331  | 0   |
|     | c5  | 3  | 11 | 0.029767 | 0   | 2  | 6  | 0.045565 | 0   |
|     | c6  | 3  | 11 | 0.030729 | 0   | 2  | 6  | 0.015755 | 0   |
|     | c7  | 3  | 10 | 0.024323 | 0   | 2  | 6  | 0.028824 | 0   |
| 50000|    |    |    |       |     |    |    |       |     |
|     | c1  | 2  | 6  | 0.075796 | 0   | 2  | 6  | 0.15535  | 0   |
|     | c2  | 2  | 6  | 0.078935 | 0   | 2  | 6  | 0.097748 | 0   |
|     | c3  | 2  | 6  | 0.18919  | 0   | 2  | 6  | 0.14607  | 0   |
|     | c4  | 3  | 11 | 0.23517  | 0   | 2  | 6  | 0.084355 | 0   |
|     | c5  | 3  | 11 | 0.14586  | 0   | 2  | 6  | 0.08018  | 0   |
|     | c6  | 3  | 11 | 0.23072  | 0   | 2  | 6  | 0.21474  | 0   |
|     | c7  | 3  | 10 | 0.10979  | 0   | 2  | 7  | 0.13907  | 0   |
| 100000|   |    |    |       |     |    |    |       |     |
|     | c1  | 2  | 6  | 0.15384  | 0   | 2  | 6  | 0.21944  | 0   |
|     | c2  | 2  | 6  | 0.16301  | 0   | 2  | 6  | 0.15044  | 0   |
|     | c3  | 2  | 6  | 0.15059  | 0   | 2  | 6  | 0.13129  | 0   |
|     | c4  | 3  | 11 | 0.31484  | 0   | 2  | 6  | 0.10428  | 0   |
|     | c5  | 3  | 11 | 0.53464  | 0   | 2  | 6  | 0.094099 | 0   |
|     | c6  | 3  | 11 | 0.28063  | 0   | 2  | 6  | 0.13453  | 0   |
|     | c7  | 3  | 10 | 0.21087  | 0   | 2  | 7  | 0.22669  | 0   |
Table 4: Numerical result for Problem 4.4.

| dm  | inp | it | nf | tm    | nm | it | nf | tm    | nm    |
|-----|-----|----|----|-------|----|----|----|-------|-------|
| 1000| c_1 | 2  | 7  | 0.021528 | 0  | 15 | 60 | 0.011562 | 5.13E-07 |
|     | c_2 | 2  | 7  | 0.004349 | 0  | 16 | 64 | 0.011605 | 3.59E-07 |
|     | c_3 | 2  | 7  | 0.005195 | 0  | 16 | 64 | 0.009487 | 9.42E-07 |
|     | c_4 | 2  | 7  | 0.006398 | 0  | 15 | 60 | 0.012216 | 6.44E-07 |
|     | c_5 | 2  | 7  | 0.00616  | 0  | 17 | 68 | 0.017389 | 3.91E-07 |
|     | c_6 | 2  | 7  | 0.006822 | 0  | 17 | 68 | 0.017897 | 7.89E-07 |
|     | c_7 | 2  | 7  | 0.00622  | 0  | 17 | 68 | 0.015525 | 5.18E-07 |

| c_1 | 2  | 7  | 0.014377 | 0  | 16 | 64 | 0.050295 | 3.86E-07 |
| c_2 | 2  | 7  | 0.019962 | 0  | 16 | 64 | 0.12969  | 8.02E-07 |
| c_3 | 2  | 7  | 0.02006  | 0  | 17 | 68 | 0.04733  | 7.00E-07 |
| c_4 | 2  | 7  | 0.020821 | 0  | 16 | 64 | 0.16294  | 4.74E-07 |
| c_5 | 2  | 7  | 0.053722 | 0  | 17 | 68 | 0.075782 | 8.74E-07 |
| c_6 | 2  | 7  | 0.023505 | 0  | 17 | 68 | 0.28728  | 5.11E-07 |
| c_7 | 2  | 7  | 0.3848   | 0  | 18 | 72 | 0.077147 | 3.67E-07 |

| 5000| c_1 | 2  | 7  | 0.018114 | 0  | 16 | 64 | 0.15223  | 5.46E-07 |
| c_2 | 2  | 7  | 0.015551 | 0  | 17 | 68 | 0.11014  | 3.76E-07 |
| c_3 | 2  | 7  | 0.019111 | 0  | 17 | 68 | 0.10562  | 9.90E-07 |
| c_4 | 2  | 7  | 0.050521 | 0  | 19 | 76 | 0.091896 | 3.70E-07 |
| c_5 | 2  | 7  | 0.11758  | 0  | 18 | 72 | 0.10508  | 4.15E-07 |
| c_6 | 2  | 7  | 0.049299 | 0  | 19 | 76 | 0.25366  | 7.22E-07 |
| c_7 | 2  | 7  | 0.35227  | 0  | 18 | 72 | 0.073714 | 5.08E-07 |

| 10000| c_1 | 2  | 7  | 0.068523 | 0  | 17 | 68 | 0.34355  | 4.04E-07 |
| c_2 | 2  | 7  | 0.11952  | 0  | 17 | 68 | 0.41732  | 8.40E-07 |
| c_3 | 2  | 7  | 0.06705  | 0  | 18 | 72 | 0.50331  | 7.39E-07 |
| c_4 | 2  | 7  | 0.076185 | 0  | 20 | 80 | 0.46207  | 6.25E-07 |
| c_5 | 2  | 7  | 0.13758  | 0  | 20 | 80 | 0.41103  | 8.13E-07 |
| c_6 | 2  | 7  | 0.090147 | 0  | 22 | 88 | 0.45026  | 9.65E-07 |
| c_7 | 155 | 619 | 3.5227  | 0  | 18 | 72 | 0.073714 | 5.08E-07 |

| 50000| c_1 | 2  | 7  | 0.1258  | 0  | 17 | 68 | 0.9023   | 5.71E-07 |
| c_2 | 2  | 7  | 0.12298 | 0  | 18 | 72 | 0.62902  | 3.98E-07 |
| c_3 | 2  | 7  | 0.14553 | 0  | 19 | 76 | 0.55966  | 9.57E-07 |
| c_4 | 2  | 7  | 0.17899 | 0  | 22 | 88 | 0.79304  | 3.99E-07 |
| c_5 | 2  | 7  | 0.15521 | 0  | 24 | 96 | 0.87754  | 3.66E-07 |
| c_6 | 2  | 7  | 0.17655 | 0  | 26 | 104 | 0.86724  | 3.55E-07 |
| c_7 | 261 | 1043 | 19.5919 | 0  | 19 | 76 | 0.61574  | 6.74E-07 |

| 100000| c_1 | 2  | 7  | 0.1258  | 0  | 17 | 68 | 0.9023   | 5.71E-07 |
| c_2 | 2  | 7  | 0.12298 | 0  | 18 | 72 | 0.62902  | 3.98E-07 |
| c_3 | 2  | 7  | 0.14553 | 0  | 19 | 76 | 0.55966  | 9.57E-07 |
| c_4 | 2  | 7  | 0.17899 | 0  | 22 | 88 | 0.79304  | 3.99E-07 |
| c_5 | 2  | 7  | 0.15521 | 0  | 24 | 96 | 0.87754  | 3.66E-07 |
| c_6 | 2  | 7  | 0.17655 | 0  | 26 | 104 | 0.86724  | 3.55E-07 |
| c_7 | 516 | 2063 | 86.4794 | 0  | 19 | 76 | 0.5515   | 9.54E-07 |
Table 5: Numerical result for Problem 4.5.

| dm   | Algorithm 1       | Algorithm 2       |
|------|-------------------|-------------------|
|      | dm    | inp | it  | nf   | tm     | nm     | it  | nf   | tm   | nm   |
|      | c_1   | 21  | 80  | 0.24362 | 4.32E-07 | 19  | 75  | 0.011016 | 6.70E-07 |
|      | c_2   | 30  | 113 | 0.03131 | 7.77E-07 | 19  | 75  | 0.020435 | 6.02E-07 |
|      | c_3   | 61  | 240 | 0.11984 | 1.94E-07 | 20  | 79  | 0.013558 | 8.17E-07 |
| 1000 | c_4   | 78  | 310 | 0.37303 | 2.03E-07 | 20  | 80  | 0.013375 | 4.14E-07 |
|      | c_5   | 107 | 425 | 0.25091 | 3.70E-07 | 20  | 80  | 0.013224 | 3.51E-07 |
|      | c_6   | 199 | 792 | 0.74997 | 5.00E-07 | 21  | 84  | 0.016755 | 3.89E-07 |
|      | c_7   | 115 | 459 | 0.24034 | 9.24E-07 | 21  | 84  | 0.024595 | 9.45E-07 |
|      | c_1   | 55  | 212 | 0.24991 | 1.67E-07 | 20  | 79  | 0.046749 | 6.26E-07 |
|      | c_2   | 34  | 131 | 0.17639 | 8.67E-07 | 20  | 79  | 0.067916 | 5.64E-07 |
|      | c_3   | 117 | 466 | 1.0295  | 2.48E-07 | 21  | 83  | 0.044109 | 7.12E-07 |
| 5000 | c_4   | 149 | 594 | 1.6382  | 1.85E-07 | 21  | 84  | 0.056726 | 3.38E-07 |
|      | c_5   | 239 | 956 | 3.1091  | 9.65E-08 | 21  | 84  | 0.050799 | 4.47E-07 |
|      | c_6   | 319 | 1272 | 4.0241 | 3.52E-07 | 21  | 84  | 0.05015  | 6.59E-07 |
|      | c_7   | 212 | 844 | 2.144   | 9.57E-07 | 29  | 115 | 0.076333 | 3.95E-07 |
|      | c_1   | 55  | 212 | 0.63431 | 2.05E-07 | 20  | 79  | 0.097986 | 9.79E-07 |
|      | c_2   | 46  | 179 | 0.56127 | 4.30E-07 | 20  | 79  | 0.10212  | 8.67E-07 |
|      | c_3   | 183 | 729 | 3.3313  | 4.24E-07 | 22  | 87  | 0.11089  | 4.07E-07 |
| 10000| c_4   | 205 | 819 | 4.1869  | 1.98E-07 | 23  | 92  | 0.094223 | 4.76E-07 |
|      | c_5   | 388 | 1550 | 10.1934 | 1.07E-07 | 21  | 84  | 0.096989 | 7.05E-07 |
|      | c_6   | 439 | 1755 | 10.8848 | 9.10E-07 | 21  | 84  | 0.08221  | 5.31E-07 |
|      | c_7   | 294 | 1174 | 6.1579  | 2.53E-07 | 23  | 92  | 0.096241 | 9.31E-07 |
|      | c_1   | 86  | 336 | 4.2367  | 3.00E-07 | 23  | 92  | 0.37093  | 4.69E-07 |
|      | c_2   | 77  | 304 | 4.0044  | 9.55E-08 | 23  | 92  | 0.3743   | 4.37E-07 |
|      | c_3   | 444 | 1773 | 51.5065 | 4.96E-07 | 22  | 88  | 0.34276  | 8.93E-07 |
| 50000| c_4   | 483 | 1932 | 73.1707 | 1.11E-07 | 24  | 96  | 0.40374  | 5.83E-07 |
|      | c_5   | NaN | NaN | NaN     | NaN     | 24  | 96  | 0.44806  | 5.87E-07 |
|      | c_6   | NaN | NaN | NaN     | NaN     | 23  | 92  | 0.42548  | 8.28E-07 |
|      | c_7   | NaN | NaN | NaN     | NaN     | 26  | 104 | 0.44434  | 4.70E-07 |
|      | c_1   | 118 | 467 | 18.4469 | 4.85E-07 | 24  | 96  | 0.79669  | 8.11E-07 |
|      | c_2   | 153 | 607 | 27.5528 | 1.46E-07 | 24  | 96  | 0.77883  | 7.59E-07 |
|      | c_3   | 364 | 1449 | 72.9234 | 7.58E-07 | 23  | 92  | 0.70613  | 4.30E-07 |
| 10000| c_4   | 564 | 2254 | 127.6077 | 2.47E-07 | 25  | 100 | 0.80085  | 3.79E-07 |
|      | c_5   | NaN | NaN | NaN     | NaN     | 25  | 100 | 0.85008  | 5.83E-07 |
|      | c_6   | NaN | NaN | NaN     | NaN     | 26  | 104 | 0.8879   | 3.96E-07 |
|      | c_7   | NaN | NaN | NaN     | NaN     | 24  | 96  | 0.7836   | 9.28E-07 |
Table 6: Numerical result for Problem 4.6.

| Algorithm 1 | Algorithm 2 |
|-------------|-------------|
| dm | inp | it | nf | tm | nm | it | nf | tm | nm |
| 1000 | | | | | | | | | |
| c₁ | 9 | 36 | 0.040786 | 8.24E-07 | 18 | 72 | 0.038391 | 4.82E-07 |
| c₂ | 9 | 36 | 0.009086 | 7.93E-07 | 18 | 72 | 0.01727 | 4.64E-07 |
| c₃ | 9 | 36 | 0.010292 | 6.98E-07 | 18 | 72 | 0.017788 | 4.08E-07 |
| c₄ | 9 | 36 | 0.009416 | 4.78E-07 | 17 | 68 | 0.023552 | 8.34E-07 |
| c₅ | 9 | 36 | 0.015915 | 3.83E-07 | 17 | 68 | 0.015462 | 6.69E-07 |
| c₆ | 9 | 36 | 0.009282 | 2.26E-07 | 17 | 68 | 0.016267 | 3.94E-07 |
| c₇ | 9 | 36 | 0.00898 | 7.01E-07 | 18 | 72 | 0.01953 | 4.12E-07 |
| 5000 | | | | | | | | | |
| c₁ | 10 | 40 | 0.030792 | 1.85E-07 | 19 | 76 | 0.074066 | 3.58E-07 |
| c₂ | 10 | 40 | 0.032701 | 1.78E-07 | 19 | 76 | 0.073348 | 3.44E-07 |
| c₃ | 10 | 40 | 0.030782 | 1.57E-07 | 18 | 72 | 0.071951 | 9.14E-07 |
| c₄ | 10 | 40 | 0.041305 | 1.07E-07 | 18 | 72 | 0.071951 | 9.14E-07 |
| c₅ | 9 | 36 | 0.032659 | 8.61E-07 | 18 | 72 | 0.079743 | 5.02E-07 |
| c₆ | 9 | 36 | 0.029737 | 5.08E-07 | 17 | 68 | 0.075313 | 8.83E-07 |
| c₇ | 10 | 40 | 0.033865 | 1.58E-07 | 18 | 72 | 0.081145 | 9.22E-07 |
| 10000 | | | | | | | | | |
| c₁ | 10 | 40 | 0.058927 | 2.62E-07 | 21 | 84 | 0.19184 | 4.00E-07 |
| c₂ | 10 | 40 | 0.053023 | 2.52E-07 | 21 | 84 | 0.17222 | 3.85E-07 |
| c₃ | 10 | 40 | 0.056913 | 2.22E-07 | 20 | 80 | 0.17532 | 5.83E-07 |
| c₄ | 10 | 40 | 0.063291 | 1.52E-07 | 18 | 72 | 0.14316 | 8.85E-07 |
| c₅ | 10 | 40 | 0.052705 | 1.22E-07 | 18 | 72 | 0.13431 | 7.10E-07 |
| c₆ | 9 | 36 | 0.044005 | 7.18E-07 | 18 | 72 | 0.13244 | 4.19E-07 |
| c₇ | 10 | 40 | 0.067819 | 2.23E-07 | 20 | 80 | 0.17845 | 5.88E-07 |
| 50000 | | | | | | | | | |
| c₁ | 10 | 40 | 0.20709 | 5.85E-07 | 24 | 96 | 0.82383 | 7.08E-07 |
| c₂ | 10 | 40 | 0.18537 | 5.63E-07 | 24 | 96 | 0.77604 | 6.81E-07 |
| c₃ | 10 | 40 | 0.22992 | 4.96E-07 | 23 | 92 | 0.74561 | 7.26E-07 |
| c₄ | 10 | 40 | 0.21739 | 3.40E-07 | 21 | 84 | 0.64599 | 5.18E-07 |
| c₅ | 10 | 40 | 0.2151 | 2.72E-07 | 21 | 84 | 0.66372 | 4.16E-07 |
| c₆ | 10 | 40 | 0.18323 | 1.61E-07 | 18 | 72 | 0.59112 | 9.36E-07 |
| c₇ | 10 | 40 | 0.2104 | 5.00E-07 | 23 | 92 | 0.73695 | 7.32E-07 |
| 100000 | | | | | | | | | |
| c₁ | 10 | 40 | 0.42899 | 8.28E-07 | 29 | 116 | 2.308 | 5.93E-07 |
| c₂ | 10 | 40 | 0.39985 | 7.96E-07 | 28 | 112 | 2.191 | 6.09E-07 |
| c₃ | 10 | 40 | 0.40936 | 7.01E-07 | 26 | 104 | 1.9182 | 6.39E-07 |
| c₄ | 10 | 40 | 0.43014 | 4.80E-07 | 23 | 92 | 1.604 | 7.03E-07 |
| c₅ | 10 | 40 | 0.45867 | 3.85E-07 | 22 | 88 | 1.4573 | 3.66E-07 |
| c₆ | 10 | 40 | 0.39766 | 2.27E-07 | 20 | 80 | 1.2933 | 5.97E-07 |
| c₇ | 10 | 40 | 0.41225 | 7.08E-07 | 26 | 104 | 1.9593 | 6.44E-07 |
## Table 7: Numerical result for Problem 4.7.

| dm   | inp | it  | nf  | tm     | nm   | it  | nf  | tm     | nm   |
|------|-----|-----|-----|--------|------|-----|-----|--------|------|
| 1000 | c₁  | 5   | 20  | 0.02835 | 3.24E-07 | 17  | 68  | 0.015365 | 6.92E-07 |
|      | c₂  | 5   | 0.00849 | 1.43E-07 | 17  | 68  | 0.013959 | 4.34E-07 |
|      | c₃  | 5   | 0.02459 | 1.68E-08 | 5   | 20  | 0.007568 | 4.50E-08 |
|      | c₄  | 6   | 24  | 0.009991 | 9.16E-09 | 18  | 72  | 0.016767 | 8.82E-07 |
|      | c₅  | 6   | 0.008829 | 1.23E-08 | 19  | 76  | 0.015507 | 8.09E-07 |
|      | c₆  | 6   | 0.010687 | 1.04E-07 | 18  | 71  | 0.018398 | 5.23E-07 |
|      | c₇  | 14  | 56  | 0.019819 | 8.59E-08 | 19  | 76  | 0.016317 | 4.01E-07 |
| 5000 | c₁  | 6   | 24  | 0.020684 | 7.25E-07 | 18  | 72  | 0.05213 | 5.59E-07 |
|      | c₂  | 5   | 0.019194 | 3.20E-07 | 17  | 68  | 0.0552 | 9.70E-07 |
|      | c₃  | 5   | 0.021205 | 3.75E-08 | 5   | 20  | 0.018794 | 1.01E-07 |
|      | c₄  | 6   | 24  | 0.023976 | 2.05E-08 | 19  | 76  | 0.057299 | 7.14E-07 |
|      | c₅  | 6   | 0.020556 | 2.75E-08 | 20  | 80  | 0.068109 | 6.56E-07 |
|      | c₆  | 6   | 0.021564 | 2.32E-07 | 19  | 75  | 0.05434 | 4.22E-07 |
|      | c₇  | 12  | 48  | 0.045511 | 1.54E-07 | 19  | 76  | 0.071732 | 9.23E-07 |
| 10000| c₁  | 6   | 24  | 0.03415 | 5.12E-09 | 18  | 72  | 0.092741 | 7.90E-07 |
|      | c₂  | 5   | 0.036905 | 4.52E-07 | 18  | 72  | 0.096713 | 4.95E-07 |
|      | c₃  | 5   | 0.038781 | 5.31E-08 | 5   | 20  | 0.022365 | 1.42E-07 |
|      | c₄  | 6   | 24  | 0.033419 | 2.90E-08 | 20  | 80  | 0.14743 | 3.66E-07 |
|      | c₅  | 6   | 0.031117 | 3.89E-08 | 20  | 80  | 0.099993 | 9.28E-07 |
|      | c₆  | 6   | 0.029565 | 3.28E-07 | 21  | 84  | 0.12772 | 4.36E-07 |
|      | c₇  | 19  | 76  | 0.10999 | 1.97E-07 | 20  | 80  | 0.11279 | 4.60E-07 |
| 50000| c₁  | 6   | 24  | 0.1595 | 1.15E-08 | 19  | 76  | 0.37052 | 6.42E-07 |
|      | c₂  | 6   | 0.15396 | 5.06E-09 | 19  | 76  | 0.36384 | 4.02E-07 |
|      | c₃  | 5   | 0.099997 | 1.19E-07 | 5   | 20  | 0.092183 | 3.18E-07 |
|      | c₄  | 6   | 24  | 0.11421 | 6.48E-08 | 21  | 84  | 0.45906 | 8.23E-07 |
|      | c₅  | 6   | 0.13876 | 8.70E-08 | 21  | 84  | 0.43542 | 7.14E-07 |
|      | c₆  | 6   | 0.11134 | 7.35E-07 | 21  | 84  | 0.46337 | 9.75E-07 |
|      | c₇  | 15  | 60  | 0.35451 | 2.17E-07 | 21  | 84  | 0.45523 | 3.79E-07 |
| 100000| c₁ | 6   | 24  | 0.22565 | 1.62E-08 | 20  | 80  | 0.77259 | 7.45E-07 |
|      | c₂  | 6   | 0.22915 | 7.15E-09 | 19  | 76  | 0.68732 | 5.69E-07 |
|      | c₃  | 5   | 0.19636 | 1.68E-07 | 5   | 20  | 0.22767 | 4.50E-07 |
|      | c₄  | 6   | 24  | 0.23499 | 9.16E-08 | 22  | 88  | 1.4285 | 4.22E-07 |
|      | c₅  | 6   | 0.27449 | 1.23E-07 | 22  | 88  | 0.90341 | 7.50E-07 |
|      | c₆  | 7   | 27  | 0.26052 | 5.19E-09 | 22  | 88  | 0.87859 | 5.00E-07 |
|      | c₇  | 12  | 48  | 0.53364 | 1.96E-08 | 20  | 80  | 0.85813 | 6.66E-07 |
Table 8: Numerical result for Problem 4.8.

| dm | Algorithm 1 | Algorithm 2 |
|----|-------------|-------------|
|    | dm | inp | it | nf | tm | nm | it | nf | tm | nm |
| 1000 | c_1 | 13 | 48 | 0.28499 | NaN | 36 | 144 | 0.19199 | 6.34E-07 |
|      | c_2 | 206 | 824 | 2.7554 | 3.27E-07 | 35 | 140 | 0.17837 | 9.13E-07 |
|      | c_3 | 71 | 284 | 1.2452 | 3.58E-07 | 35 | 140 | 0.1615 | 7.34E-07 |
|      | c_4 NaN | NaN | NaN | NaN | NaN | 33 | 132 | 0.14206 | 2.30E-07 |
|      | c_5 NaN | NaN | NaN | NaN | NaN | 31 | 124 | 0.14692 | 8.06E-07 |
|      | c_6 | 4 | 14 | 0.033612 | NaN | 24 | 96 | 0.13093 | 9.72E-07 |
|      | c_7 | 25 | 97 | 0.32507 | NaN | 29 | 116 | 0.13481 | 2.49E-07 |
| 5000 | c_1 | 199 | 796 | 11.1914 | 9.18E-07 | 34 | 136 | 0.72482 | 8.36E-07 |
|      | c_2 | 10 | 37 | 0.56969 | NaN | 34 | 136 | 0.7476 | 7.93E-07 |
|      | c_3 NaN | NaN | NaN | NaN | NaN | 34 | 136 | 0.73337 | 6.18E-07 |
|      | c_4 NaN | NaN | NaN | NaN | NaN | 31 | 124 | 0.67258 | 3.90E-07 |
|      | c_5 | 21 | 79 | 1.0588 | NaN | 30 | 120 | 0.69797 | 8.11E-07 |
|      | c_6 | 28 | 103 | 1.2647 | NaN | 29 | 116 | 0.52499 | 7.51E-07 |
|      | c_7 | 8 | 29 | 0.23404 | NaN | 24 | 96 | 0.54913 | 2.91E-07 |
| 10000 | c_1 NaN | NaN | NaN | NaN | NaN | 34 | 136 | 1.3986 | 6.78E-07 |
|      | c_2 NaN | NaN | NaN | NaN | NaN | 34 | 136 | 1.3769 | 6.42E-07 |
|      | c_3 | 61 | 244 | 5.0283 | 3.56E-07 | 33 | 132 | 1.3628 | 7.57E-07 |
|      | c_4 | 125 | 500 | 11.3843 | 8.81E-07 | 30 | 120 | 1.2608 | 3.94E-07 |
|      | c_5 | 17 | 63 | 1.3152 | NaN | 30 | 120 | 1.2407 | 5.57E-07 |
|      | c_6 | 275 | 1098 | 24.2426 | NaN | 27 | 108 | 1.1274 | 3.65E-07 |
|      | c_7 | 107 | 425 | 14.0895 | NaN | 26 | 104 | 10.1937 | 9.05E-07 |
| 50000 | c_1 | 41 | 159 | 18.506 | NaN | 34 | 136 | 6.0333 | 6.35E-07 |
|      | c_2 | 37 | 144 | 29.7738 | NaN | 33 | 132 | 5.9281 | 6.12E-07 |
|      | c_3 | 113 | 451 | 52.1981 | 4.09E-07 | 32 | 128 | 6.1533 | 7.22E-07 |
|      | c_4 | 11 | 41 | 4.0036 | NaN | 24 | 96 | 5.3594 | 3.36E-07 |
|      | c_5 | 22 | 84 | 13.935 | NaN | 29 | 116 | 6.3535 | 5.83E-07 |
|      | c_6 | 12 | 44 | 5.9726 | NaN | 31 | 124 | 8.4474 | 7.91E-07 |
|      | c_7 | 11 | 42 | 3.1947 | NaN | 26 | 104 | 10.049 | 3.36E-07 |
| 100000 | c_1 | 40 | 156 | 51.3509 | NaN | 33 | 132 | 16.1737 | 8.00E-07 |
|       | c_2 | 5 | 17 | 2.7955 | NaN | 33 | 132 | 14.174 | 7.49E-07 |
|       | c_3 | 126 | 503 | 115.4434 | 4.01E-07 | 40 | 160 | 17.8687 | 9.75E-07 |
|       | c_4 | 55 | 215 | 64.8415 | NaN | 30 | 120 | 12.3492 | 9.85E-07 |
|       | c_5 | 5 | 17 | 2.8864 | NaN | 28 | 112 | 10.6799 | 9.46E-07 |
|       | c_6 NaN | NaN | NaN | NaN | NaN | 26 | 104 | 10.1937 | 9.05E-07 |
|       | c_7 | 10 | 38 | 7.026 | NaN | 30 | 120 | 11.5463 | 2.57E-07 |
Table 9: Numerical result for Problem 4.9.

| dm   | inp | it | nf | tm    | nm    | tm | nm    |
|------|-----|----|----|-------|-------|----|-------|
|      |     |    |    |       |       |    |       |
| 1000 | c1  | 10 | 34 | 0.025676 | 1.06E-07 | 11 | 42 | 0.00812 | 2.67E-07 |
|      | c2  | 10 | 34 | 0.006497 | 1.06E-07 | 11 | 42 | 0.008082 | 2.67E-07 |
|      | c3  | 10 | 34 | 0.006769 | 1.06E-07 | 11 | 42 | 0.009859 | 2.67E-07 |
|      | c4  | 10 | 34 | 0.006506 | 1.06E-07 | 11 | 42 | 0.008471 | 2.67E-07 |
|      | c5  | 10 | 34 | 0.006659 | 1.06E-07 | 11 | 42 | 0.01137 | 2.67E-07 |
|      | c6  | 10 | 34 | 0.005753 | 1.06E-07 | 12 | 46 | 0.009152 | 2.67E-07 |
|      | c7  | 10 | 34 | 0.006338 | 1.06E-07 | 11 | 42 | 0.007492 | 2.67E-07 |
| 5000 | c1  | 7  | 25 | 0.017159 | 6.89E-08 | 8  | 31 | 0.032027 | 1.59E-07 |
|      | c2  | 7  | 25 | 0.016181 | 6.89E-08 | 8  | 31 | 0.030725 | 1.59E-07 |
|      | c3  | 7  | 25 | 0.019768 | 6.89E-08 | 8  | 31 | 0.025114 | 1.59E-07 |
|      | c4  | 7  | 25 | 0.015824 | 6.89E-08 | 9  | 35 | 0.036701 | 1.59E-07 |
|      | c5  | 7  | 25 | 0.018099 | 6.89E-08 | 9  | 35 | 0.032259 | 1.59E-07 |
|      | c6  | 7  | 25 | 0.019327 | 6.89E-08 | 9  | 35 | 0.047628 | 1.59E-07 |
|      | c7  | 7  | 25 | 0.019407 | 6.89E-08 | 8  | 31 | 0.035816 | 1.59E-07 |
| 10000| c1  | 6  | 22 | 0.036419 | 8.13E-08 | 11 | 43 | 0.074354 | 7.22E-07 |
|      | c2  | 6  | 22 | 0.041491 | 8.13E-08 | 11 | 43 | 0.077893 | 7.22E-07 |
|      | c3  | 6  | 22 | 0.037815 | 8.13E-08 | 11 | 43 | 0.08316 | 7.22E-07 |
|      | c4  | 6  | 22 | 0.038765 | 8.13E-08 | 12 | 47 | 0.10473 | 7.22E-07 |
|      | c5  | 6  | 22 | 0.039708 | 8.13E-08 | 13 | 51 | 0.1345 | 7.22E-07 |
|      | c6  | 6  | 22 | 0.033133 | 8.13E-08 | 13 | 51 | 0.13929 | 7.22E-07 |
|      | c7  | 6  | 22 | 0.038275 | 8.13E-08 | 11 | 43 | 0.10201 | 7.22E-07 |
| 5000 | c1  | 5  | 19 | 0.18677 | 1.41E-07 | 10 | 40 | 0.36425 | 7.59E-07 |
|      | c2  | 5  | 19 | 0.15923 | 1.41E-07 | 10 | 40 | 0.35055 | 7.59E-07 |
|      | c3  | 5  | 19 | 0.18752 | 1.41E-07 | 11 | 44 | 0.4686 | 7.59E-07 |
|      | c4  | 5  | 19 | 0.16679 | 1.41E-07 | 13 | 52 | 0.67096 | 7.59E-07 |
|      | c5  | 5  | 19 | 0.15913 | 1.41E-07 | 14 | 56 | 0.83025 | 7.59E-07 |
|      | c6  | 5  | 19 | 0.16621 | 1.41E-07 | 16 | 64 | 1.0331 | 7.59E-07 |
|      | c7  | 5  | 19 | 0.17155 | 1.41E-07 | 11 | 44 | 0.4487 | 7.59E-07 |
| 10000| c1  | 6  | 23 | 0.49529 | 2.10E-07 | 9  | 36 | 0.65953 | 2.19E-07 |
|      | c2  | 6  | 23 | 0.49149 | 2.10E-07 | 9  | 36 | 0.66909 | 2.19E-07 |
|      | c3  | 6  | 23 | 0.46091 | 2.10E-07 | 11 | 44 | 1.0605 | 2.19E-07 |
|      | c4  | 6  | 23 | 0.55906 | 2.10E-07 | 14 | 56 | 1.7111 | 2.19E-07 |
|      | c5  | 6  | 23 | 0.47421 | 2.10E-07 | 16 | 64 | 2.2975 | 2.19E-07 |
|      | c6  | 6  | 23 | 0.47836 | 2.10E-07 | 18 | 72 | 3.0334 | 2.19E-07 |
|      | c7  | 6  | 23 | 0.48422 | 2.10E-07 | 11 | 44 | 1.0618 | 2.19E-07 |