Stability analysis of de Sitter solutions in models with the Gauss-Bonnet term

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\textbf{Abstract}

We investigate the scalar field dynamics of models with non-minimally coupled scalar fields in the presence of the Gauss-Bonnet term and derive the structure of effective potential and conditions for stable de Sitter solutions in general. Specializing to specific couplings, we explore the possibility of realizing the stable de Sitter configurations which may have implications for both the early Universe and late time evolution.

\section{Introduction}

Among the curvature corrections to the Einstein-Hilbert action, the Gauss-Bonnet term is of a special relevance. The Gauss-Bonnet term is a topological invariant quantity in four dimensions but may have dynamical relevance if coupled to an evolving scalar field. The Gauss-Bonnet term coupled to a scalar field arises naturally in the string theory \cite{1} and uses in string inspired cosmological models \cite{2, 3, 4, 5, 6, 7}, because have interesting cosmological implications both for early Universe \cite{8, 9, 10, 11, 12, 13, 14, 15} and for the late time dynamics \cite{16, 17, 18, 19, 20, 21}.

In case the scalar field has to account for late time cosmic evolution, it is more than desirable to ensure that it does not disturb the thermal history and that the late time

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\textsuperscript{1}Non-local cosmological models with the Gauss-Bonnet term are actively developed as well \cite{22, 23, 24, 25, 26, 27, 28, 29, 30}. 
dynamics is free from initial conditions of the field. In this case, it is mandatory that dynamics exhibits scaling behaviour where scalar field, in a sense, spends most of its time. In that case, thermal history remains intact. However, scaling solution which is an attractor of dynamics mimics background matter and one therefore needs to device a an exit mechanism from this regime to acceleration. In case of the Einstein–Hilbert action, the steep exponential potential with exponential form of coupling function, gives rise to exit from scaling regime to acceleration \cite{6,7}. Indeed, the Gauss-Bonnet coupling induces a minimum in the run away potential where the field can settle such that the de Sitter is a late time attractor of the dynamics \cite{6,7}. Let us note that there exist other mechanism capable of executing the said transition. For instance, coupling with massive neutrino matter to scalar field proportional to trace of its energy-momentum tensor of neutrino matter can facilitate exit from scaling regime. In this case, coupling dynamically builds up only at late stages resulting into the minimum of the effective potential \cite{27}. In general, addition of the Gauss-Bonnet term modifies the field potential and may induce interesting features in it \cite{6,7}.

In order to find out the de Sitter solutions in a model with a minimally coupled scalar field with a potential $V$ it is enough to find zeros of the first derivative of $V$. The sign of the second derivative of the potential $V$ at a de Sitter point determines the stability of the solution. Similar analysis of de Sitter solutions is possible in the case of non-minimal coupling if one introduces an effective potential such that zeroes of its first derivative correspond to de Sitter points whereas the sign of its second derivative determines their stability \cite{28}.

In this paper, we analyze models with the Gauss-Bonnet term in case of a non-minimally coupled scalar field; we look for the general structure of the corresponding effective potential and investigate the specific cases. Using the framework of effective potential, our goal is to find out de Sitter solutions in these models and compare them with their counter part in the corresponding models without the Gauss-Bonnet term.

The paper is organized as follows. In Section 2, we remind the Friedmann equations for models with a non-minimally coupled scalar field and the Gauss-Bonnet term. In Section 3, we propose the effective potential as a useful tool for the search and stability analysis of de Sitter solutions. In Section 4, we present specific examples of models with de Sitter solutions. Section 5 is devoted to our conclusions.

2 Evolution equations for models with the Gauss-Bonnet term

In this paper, we investigate the model with the Gauss-Bonnet term in a general background described by the following action,

$$S = \int d^4x\sqrt{-g}\left(U(\phi)R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) - F(\phi)G\right),$$

where the functions $U$, $V$, and $F$ are differentiable ones, and $G$ is the Gauss-Bonnet term,

$$G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}.$$

In what follows, we specialize to spatially flat Friedmann–Lemaître–Robertson–Walker (FLRW) universe,

$$ds^2 = -dt^2 + a^2(t)\left(dx_1^2 + dx_2^2 + dx_3^2\right).$$
Variations of action (1) with respect to $g_{\mu\nu}$ and $\phi$ lead to the following evolution equations [8]:

$$6H^2U + 6HU'\dot{\phi} = \frac{1}{2}\dot{\phi}^2 + V + 24H^3F'\dot{\phi},$$

$$4\left(U - 4H\dot{F}\right)\dot{H} = -\dot{\phi}^2 - 2U + 2HU + 8H^2\left(\ddot{F} - H\dot{F}\right),$$

$$\ddot{\phi} + 3H\dot{\phi} - 6\left(\dot{H} + 2H^2\right)U' + V' + 24H^2F'\left(\dot{H} + H^2\right) = 0,$$

where $H = \dot{a}/a$ is the Hubble parameter, dots and primes denote the derivatives with respect to cosmic time and $\phi$ respectively.

It would be convenient to cast Eqs. (5) and (4) as a dynamical system,

$$\dot{\phi} = \psi,$$

$$\dot{\psi} = \frac{1}{2(B - 4F'H\psi)} \left\{ 2H \left[ 3B + 4F'V' - 6U' - 6U \right] \psi - 2V^2 \psi X \right. + 12 \left( (2U' + 3)F' + 2U'F'' \right) \psi^2 \},$$

$$\dot{H} = \frac{1}{4(B - 4F'H\psi)} \left\{ 8 \left( U' - 4F'H^2 \right) H\psi - 2V^2 \left( 4F'H^2 - U' \right) X \right. + \left( 8F''H^2 - 2U'' - 1 \right) \psi^2 \},$$

where

$$B = 3 \left( 4H^2F' - U' \right)^2 + U, \quad X = \frac{U^2}{V^2} \left[ 24F'H^4 - 12U'H^2 + V' \right].$$

The dynamical system (6) is suitable for the search of de Sitter solutions which we shall undertake in the following section.

3 The effective potential and de Sitter solutions

3.1 The search of de Sitter solutions

Our first goal is to find de Sitter solutions in this model described by (1) and to compare them with de Sitter solutions in the corresponding model without the Gauss-Bonnet term. In principle, solutions with a constant Hubble parameter $H$ and the field $\phi$ depending on time can exist, but these solutions correspond to varying effective gravitational constant proportional to $1/U(\phi)$ and have properties different from the standard de Sitter solutions. For this reason, we restrict ourselves to de Sitter solutions with a constant $\phi$. Putting $\phi = \phi_{dS}$ and $H = H_{dS}$, we have

$$6H_{dS}^2U_{dS} = V_{dS},$$

$$12H_{dS}^2U_{dS}' = V_{dS}' + 24H_{dS}^4F_{dS},$$

where $V_{dS} = V(\phi_{dS})$, $U_{dS} = U(\phi_{dS})$, and $F_{dS} = F(\phi_{dS})$. We see that the value of the Hubble parameter at the de Sitter point is the same as in the corresponding model without the Gauss-Bonnet term,

$$H_{dS}^2 = \frac{V_{dS}}{6U_{dS}}.$$
The value of $F'(\phi dS)$ is fixed by (9):

$$F'_dS = \frac{3U_dS(2U'_dSV_dS - V'_dSU_dS)}{2V^2_dS}.$$  

(11)

Therefore, we come to conclusion that for arbitrary functions $U$ and $V$ with $VU > 0$, we can choose $F(\phi)$ such that the corresponding point becomes a de Sitter solution, with the Hubble parameter defined by (10).

Let us note that at de Sitter point, system of equations (6) has the following form,

$$\dot{\phi}_{dS} = 0,$$  

(12)

$$\dot{\psi}_{dS} = -\frac{V^2_dS}{B(\phi_{dS})U_{dS}}X(\phi_{dS}),$$  

(13)

$$\dot{H}_{dS} = -\frac{(4F'_{dS}H^2_dS - U'_{dS})V^2_dS}{2B(\phi_{dS})U^2_{dS}}X(\phi_{dS}).$$  

(14)

It would be convenient, if all the necessary information on the existence and stability of de Sitter solutions is obtained from single combination of functions $U$, $V$, and $F$ dubbed effective potential $V_{eff}$. The de Sitter solutions would correspond to zeros of the first derivative of $V_{eff}$ and stability of the solutions would correspond to its second derivative being positive. In the present paper, we show that while the general situation is more complicated, we achieve the above-mentioned goal if we restrict ourselves to the case of $U > 0$.

De Sitter solutions in the model with nonminimal coupling and without the Gauss-Bonnet term have been considered in [28], where the effective potential has been introduce. In the model with the Gauss-Bonnet term, we can also introduce such $V_{eff}(\phi)$ that

$$V'_{eff}(\phi_dS) = X(\phi_{dS}) = 0.$$  

(15)

Indeed, let

$$V_{eff} = -\frac{U^2}{V} + \frac{2}{3}F.$$  

(16)

Using Eq. (10), we get

$$X(\phi_{dS}) = \frac{2}{3}F'_{dS} - \frac{2U'_{dSV_dS}}{V_dS} + \frac{V'_{dSU^2_dS}}{V^2_dS} = V'_{eff}(\phi_{dS}),$$  

(17)

and de Sitter solutions correspond to extremum points of the effective potential $V_{eff}$. It should be noted that the first term in (16) can be expressed through the effective potential without the Gauss-Bonnet term $V_{eff}$ introduced in [28] as $-\bar{V}_{eff}$. A disadvantage of the form (16) for the effective potential is that it diverges at $V = 0$. However, in such a case we have a Minkowski solution instead of de Sitter one, so this feature does not spoil the analysis of de Sitter solutions.
3.2 The stability analysis

To investigate the Lyapunov stability of a de Sitter solution we use the following expansions,

\[ H = H_{dS} + \delta H_1(t), \quad \phi(t) = \phi_{dS} + \delta \phi_1(t), \quad \psi(t) = \delta \psi_1(t), \quad (18) \]

where \( \delta \) is a small parameter. From (3), in the first order in \( \delta \) we get the following linear system

\[ \dot{\phi}_1 = A_{11} \phi_1 + A_{12} \psi_1 + A_{13} H_1, \quad \text{(19)} \]
\[ \dot{\psi}_1 = A_{21} \phi_1 + A_{22} \psi_1 + A_{23} H_1, \quad \text{(20)} \]
\[ \dot{H}_1 = A_{31} \phi_1 + A_{32} \psi_1 + A_{33} H_1, \quad \text{(21)} \]

where

\[ A = \begin{bmatrix} \quad 0, & 1, & 0 \\ -\frac{V^2}{U B} X'_\phi, & H_{dS} \left(1 - 4\frac{U}{B}\right), & -\frac{V^2}{U B} X'_H \\ \frac{V X'_\phi}{2U B^2} (V'U - U'V), & \frac{2H_{dS}}{B V} (V'U - U'V), & \frac{V X'_H}{2U B^2} (V'U - U'V) \end{bmatrix} \]

and all functions are taken at \( \phi = \phi_{dS} \).

Since \( \det(A) = 0 \), functions \( H_1(t), \phi_1(t) \) and \( \psi_1(t) \) are not independent. From Eq. (3), we obtain

\[ H_1 = \frac{V'_{dS} U_{dS} - U'_{dS} V_{dS}}{2U_{dS} V_{dS}} (H_{dS} \phi_1 - \psi_1). \quad \text{(22)} \]

Substituting (22) into (19) and (20), we get:

\[ \dot{\phi}_1 = \bar{A}_{11} \phi_1 + \bar{A}_{12} \psi_1, \quad \text{(23)} \]
\[ \dot{\psi}_1 = \bar{A}_{21} \phi_1 + \bar{A}_{22} \psi, \quad \text{(24)} \]

where

\[ \bar{A} = \begin{bmatrix} \quad 0, & \frac{V X'_\phi}{2U B} - \frac{V X'_H (V'U - U'V) H_{dS}}{2U^2 B}, & H_{dS} \left(1 - 4\frac{U}{B}\right) + \frac{V X'_H (V'U - U'V)}{2U^2 B} \end{bmatrix} \]

The condition on the determinant of the characteristic matrix

\[ \det(\bar{A} - \lambda \cdot I) = 0 \quad \text{(25)} \]

gives the following expressions for \( \lambda \):

\[ \lambda_\pm = \frac{Z \pm \sqrt{Z^2 + Y}}{4U^2 B}, \quad \text{(26)} \]

where

\[ Z = -\frac{3U^2}{V^2} \sqrt{6 V} \left[ \frac{7}{9} U V^2 + (V'U - U'V)^2 \right], \]
and

\[ Y = 8VB \left( X'_H H \dot{d}S U^2 V' - X'_H H \ddot{d}S U^3 V' - 2U^3 V X''_\phi \right) = -16U^3 V^2 B V''_{\text{eff}}. \]

A de Sitter solution is stable only if both \( Z/B < 0, \) and \( Y < 0. \) Situation considerably simplifies in the case of a positive \( U \) and, therefore, a positive \( V. \) Indeed, if at the de Sitter point both \( U \) and \( V \) are positive then \( Z < 0. \) Further from Eq. (7), it follows that \( B > 0 \) for any \( U > 0. \) This means that the condition \( Z/B < 0 \) at any de Sitter point is satisfied automatically.

Thus, we finally reach a conclusion that for any \( U(\phi_{dS}) > 0, \) a de Sitter solution is stable if \( V''_{\text{eff}}(\phi_{dS}) > 0 \) and unstable if \( V''_{\text{eff}}(\phi_{dS}) < 0. \) In the next section, we consider several examples of models and explore the existence and stability of de Sitter solutions.

4 Examples

4.1 Models with exponential potential

The string theory inspired cosmological model with

\[ U = U_0, \quad V = ce^{-\lambda \phi}, \quad F = \frac{\alpha}{\mu} e^{\mu \phi}, \tag{27} \]

where \( U_0, \alpha, c, \lambda, \) and \( \mu \) are positive constants, has been considered in \( [6]. \) In this model, the effective potential is

\[ V_{\text{eff}} = -\frac{U_0^2}{c} e^{\lambda \phi} + \frac{2\alpha}{3\mu} e^{\mu \phi}. \tag{28} \]

The condition \( V_{\text{eff}}'(\phi_{dS}) = 0 \) gives

\[ \phi_{dS} = \frac{1}{\lambda - \mu} \ln \left( \frac{2ac}{3U_0^2 \lambda} \right). \tag{29} \]

There exists a de Sitter solution for all \( \mu \neq \lambda. \) It is easy to see that \( V''_{\text{eff}} = 0 \) at

\[ \phi_2 = \frac{1}{\lambda - \mu} \ln \left( \frac{2ac\mu}{3U_0^2 \lambda^2} \right) = \phi_{dS} - \frac{\ln(\lambda) - \ln(\mu)}{\lambda - \mu}, \tag{30} \]

and \( \phi_{dS} > \phi_2 \) for any \( \lambda \neq \mu. \)

If \( \mu > \lambda, \) then \( V''_{\text{eff}} \) is positive at large \( \phi, \) so the second derivative is positive at the de Sitter point and this point is stable. In the opposite case, \( \mu < \lambda, \) \( V''_{\text{eff}} < 0 \) at large \( \phi \) and the de Sitter solution is unstable. This result coincides with the result obtained in \( [6] \) by another method.

We generalize this result assuming that the constants can be negative. For

\[ V_{\text{eff}} = c_1 e^{-N_1 \phi} + c_2 e^{-N_2 \phi}, \tag{31} \]

the de Sitter point \( \phi_{dS} = \frac{1}{N_1 - N_2} \ln \left( -\frac{c_1 N_1}{c_2 N_2} \right) \) exists only if \( c_1 N_1/c_2 N_2 < 0 \) and \( N_2 \neq N_1. \) If we assume functions \( U \) and \( V \) in the form given by (27), then \( c_1 < 0. \) At the same time one and the same the effective potential corresponds to different choice of functions \( F, V, \) and
If two of these functions are given, then we can get the third function using the given form of the effective potential. It is a way of constructing models with de Sitter solutions. For example, the model with a non-minimally coupled scalar field defined by functions

\[
U = U_0 \left( \xi \phi^2 + 1 \right) e^{m \phi}, \quad \text{and} \quad V = V_0 \phi^4 e^{n \phi},
\]

has the effective potential given by (31) if

\[
F = \frac{3}{2} \left( \frac{4U_0^2 e^{2m \phi - n \phi}}{V_0} \left( \xi + \frac{1}{\phi^2} \right)^2 + c_1 e^{-N_1 \phi} + c_2 e^{-N_2 \phi} \right).
\]

In this model, \( c_i \) and \( N_i \) are arbitrary constants. The analysis of the second derivative of \( V_{eff} \) gives the following stability conditions:

- if \( c_1 > 0 \) and \( c_2 > 0 \), then the de Sitter solution is stable;
- if \( c_1 < 0 \) and \( c_2 < 0 \), then the de Sitter solution is unstable;
- if \( c_1 > 0 \) and \( c_2 < 0 \), then the de Sitter solution is stable at \( |N_1| > |N_2| \) and unstable at \( |N_1| < |N_2| \);
- if \( c_1 < 0 \) and \( c_2 > 0 \), then the de Sitter solution is stable at \( |N_1| < |N_2| \) and unstable at \( |N_1| > |N_2| \).

The effective potential can be used not only to simplify the analysis of the stability of de Sitter solutions in a given model, but also to construct a new model with de Sitter solutions.

### 4.2 Models with \( V = CU^2 \)

Let us consider the case \( V = CU^2 \), where \( C \) is a positive constant. In this case, a model without the Gauss-Bonnet term transforms to a model with a constant potential in the Einstein frame. If the Gauss-Bonnet term is presented, then the function \( F(\phi) \) plays a role of the effective potential, fully determining the position and stability of the de Sitter solutions, because

\[
V_{eff} = -\frac{1}{C} + \frac{2}{3} F.
\]

So, values of \( \phi_{dS} \) satisfy the condition \( F'(\phi_{dS}) = 0 \). From Eq. (26), it follows

\[
\lambda_\pm = -\frac{\sqrt{6CU}}{4} \pm \frac{\sqrt{6CU[9(3U^2 + U) - 16CU^2 F'']} + 12 \sqrt{3U^2 + U}}{12 \sqrt{3U^2 + U}}.
\]

For \( U(\phi_{dS}) > 0 \), a de Sitter solution is unstable at \( F'' < 0 \) and stable at \( F'' > 0 \).

Note that the only difference between minimal and non-minimal coupling cases is that values of the Hubble parameter at de Sitter points

\[
H_{dS}^2 = \frac{C}{6} U(\phi_{dS}),
\]

can be different if \( U \) is not a constant.
In what follows, we shall demonstrate the working of de Sitter search algorithm through concrete examples.

For \( F = A_4 \phi^4 + A_2 \phi^2 \), de Sitter points defined by the condition \( F' = 0 \) are

\[
\phi_{\text{dS}} = \pm \sqrt{\frac{A_2}{2A_4}}, \quad \phi_{\text{dS}0} = 0.
\]  

(34)

It is evident that \( \phi_{\text{dS}} \) are real only if \( A_2 \) and \( A_4 \) have different signs. The values of the second derivative of \( F \) at the de Sitter points are

\[
F''|_{\phi_{\text{dS}} = \pm} = -4A_2, \quad F''|_{\phi_{\text{dS}0}} = 2A_2.
\]

Thus, the de Sitter solution in points \( \phi_{\text{dS}} \) is unstable for any \( A_2 > 0 \) and \( A_4 < 0 \) and is stable for any \( A_2 < 0 \) and \( A_4 > 0 \). At the point \( \phi_{\text{dS}0} \), the de Sitter solution is stable for any \( A_2 > 0 \) and unstable at \( A_2 < 0 \). At \( A_2 = 0 \), the only de Sitter point is \( \phi_{\text{dS}} = 0 \) and we get \( \lambda_+ = 0 \) and \( \lambda_- = -\sqrt{6CU}/2 \). The left picture in Fig. 1 illustrates different possibilities.

A more complicated example is

\[
\tilde{F} = A_4 \phi^4 + A_2 \phi^2 + C \sin(\omega \phi),
\]

(35)

where \( C \) and \( \omega \) are constants. The central and right pictures in Fig. 1 demonstrate that in dependence of values of the parameters \( C \) and \( \omega \) the number of de Sitter solution changes. In the central picture, the blue and black lines correspond to models with one stable de Sitter solution, whereas the green line corresponds to a model with 3 stable and 2 unstable de Sitter solutions. In the right picture, the red line corresponds to a model with 2 stable and 1 unstable de Sitter solutions, the black line correspond to models with one stable de Sitter solution, and the green line corresponds to a model with 4 stable and 3 unstable de Sitter solutions. Therefore, using graphical representation of the function \( \tilde{F} \), one can get the structure and stability properties of de Sitter solutions.

Thus, for an arbitrary positive \( U(\phi) \), we obtain de Sitter solutions given by an arbitrary function \( V_{\text{eff}} \), if we choose \( V = CU^2 \) and \( F = V_{\text{eff}} \).

### 4.3 Models with a massive scalar field

In the previous subsection, the de Sitter solutions appear due to properties of the coupling function \( F \) solely. It is more interesting to get de Sitter solutions via an interplay between the scalar field potential \( V \) and the coupling function \( F \) so as these two functions, being simple monomials, give rise to de Sitter solution. Using the conception of the effective potential, it is easy to create such models. For example, let the potential be of the simplest massive form

\[
V = m^2 \phi^2,
\]

(36)

with the coupling function

\[
U = \xi \phi^2
\]

(37)

for the coupling function with curvature. In this situation the effective potential is

\[
V_{\text{eff}} = -\frac{\xi^2}{m^2} \phi^2 + \frac{2}{3} F.
\]

(38)
Without the Gauss-Bonnet contribution the effective potential is a monotonic function, so there are no de Sitter solutions. However, it is clear that addition of any monomial $F = F_0 \phi^n$ with $n > 2$ and $F_0 > 0$ gives us a stable de Sitter solution. Straightforward calculation shows that

$$\phi_{dS}^{n-2} = \frac{3\xi^2}{nF_0m^2}$$

and consequently the de Sitter solution exists if $n \neq 2$. Using the second derivative of the effective potential we easily obtain

$$V''(\phi) = \frac{4\xi^2}{m^2}(n-2),$$  \hspace{1cm} (39)$$

which implies that the de Sitter solution is unstable for $n < 2$. Looking at the plot of the effective potential in Fig. 2 (the right picture) one can clearly distinguish between cases corresponding to $n > 2 \& n < 2$.

4.4 Models with the Higgs potential

Let us consider model with

$$U = U_0 + \xi \phi^2, \quad V = V_0 \phi^4,$$  \hspace{1cm} (40)$$

where $U_0$, $\xi$ and $V_0$ are positive constants. The corresponding model without the Gauss-Bonnet term is the physically motivated inflationary model dubbed Higgs-driven inflation perfectly consistent with cosmological observations. Inflationary scenario proposed

In the point $\phi = 0$ the function $U = 0$, so this extremum of $V_{eff}$ does not corresponds to a de Sitter solution.
in Ref. [8] uses the functional form of $F$ given by $F = F_0/\phi^4$. In this case effective potential has the following form,

$$V_{\text{eff}} = -\frac{(U_0 + \xi \phi^2)^2}{V_0 \phi^4} + \frac{2F_0}{3\phi^4}. \quad (41)$$

The de Sitter solutions correspond to real values of $\phi$ only if $F_0 > 3U_0^2/(2V_0)$:

$$\phi_{dS} = \pm \sqrt{\frac{(2F_0V_0 - 3U_0^2)}{3\xi U_0}}. \quad (42)$$

At the de Sitter points, the second derivative of the effective potential is

$$V''_{\text{eff}}(\phi_{dS}) = \frac{72U_0^3\xi^3}{(2F_0V_0 - 3U_0^2)^2 V_0} > 0. \quad (43)$$

Thus, evidently, all de Sitter solutions of the considering model are stable. In the central plot of Fig. 2, the red curve corresponds to a model without de Sitter solution, whereas the blue curve corresponds to a model with de Sitter solutions.

In another interesting case of the function

$$F = \frac{3U_0^2}{2V_0\phi^4} + \frac{U_0\xi}{V_0\phi^2} + \xi_2\phi^2 + \xi_3\phi^4, \quad (44)$$

extrema of the effective potential are $\phi_{1,2} = \pm \sqrt{-\xi_2/(2\xi_3)}$ and $\phi_0 = 0$. The point $\phi_0 = 0$ is a singular point of the function $F$ and we do not consider this point. Evidently the de Sitter points $\phi_{1,2}$ are real only if the parameters $\xi_3$ and $\xi_2$ have different signs.

In the points $\phi_{1,2}$ the second derivative is $V''_{\text{eff}} = -8\xi_2/3$. So, the points $\phi_{1,2}$ are unstable at $\xi_2 > 0$ and stable at $\xi_2 < 0$.

![Figure 2](image)

Figure 2: The effective potential $V_{\text{eff}}(\phi)$ for $U = U_0 + \xi \phi^2$, $V = V_0 \phi^n$, $F = F_0 \phi^\alpha$. In the left picture, $V_{\text{eff}}(\phi)$ for $n = 2$, $U_0 = 0$, $F_0 = 1$, $\xi = 1$ and $V_0 = 1$ is presented: $\alpha = 4$ (red curve) and $\alpha = 1$ (blue curve). In the central picture, $V_{\text{eff}}(\phi)$ for $n = 4$, $U_0 = 1$, $\xi = 1$ and $F_0 = 1$ is presented: $V_0 = 1$ (red curve) and $V_0 = 2$ (blue curve). In the right picture, $V_{\text{eff}}(\phi)$ for $n = 6$, $U_0 = 1$ and $V_0 = 1$ is presented: $\alpha = -8$, $\xi = 0.72$, and $F_0 = 2$ (red curve), $\alpha = -2$, $\xi = 1$ and $F_0 = 15$ (blue curve), $\alpha = -2$, $\xi = 1$ and $F_0 = 0.1$ (cyan curve).
4.5 Six degree potential

In this subsection, we consider more general functions $U$ and $V$, than in the previous subsection, and power-law functions $F$:

$$U = U_0(1 + \xi \phi^N), \quad V = V_0 \phi^n, \quad F = F_0 \phi^\alpha, \quad (45)$$

where $U_0, V_0$ and $\xi$ are positive constants. The effective potential of the model has the following form:

$$V_{\text{eff}} = -\frac{U_0^2 (1 + \xi \phi^N)^2}{V_0 \phi^n} + \frac{2}{3} F_0 \phi^\alpha. \quad (46)$$

In the case $n > 2N$, at $F_0 = 0$ the function $V_{\text{eff}}$ is a monotonically increasing one, so there is no de Sitter solution for $U > 0$ that has been mentioned in [31].

Let us consider the case $N = 2$ and $n = 6$. If we add the function $F = F_0/\phi^2$, then unstable de Sitter can be obtained for some values of parameter (see the blue and cyan curves in the right picture of Fig. [2]). The analysis of $V'_{\text{eff}}$ shows that no more than two values of $\phi_{\text{dS}}$ are real either

$$\phi_{\text{dS}1,2} = \pm \sqrt{\frac{3}{2F_0 V_0 - 3U_0^2 \xi^2}} \left(2U_0 \xi \pm \sqrt{U_0^2 \xi^2 + 2F_0 V_0}\right) \frac{U_0}{U_0^2 \xi^2}, \quad (47)$$

or

$$\phi_{\text{dS}3,4} = \pm \sqrt{\frac{3}{2F_0 V_0 - 3U_0^2 \xi^2}} \left(2U_0 \xi - \sqrt{U_0^2 \xi^2 + 2F_0 V_0}\right) \frac{U_0}{U_0^2 \xi^2}. \quad (48)$$

In order to find out the stable de Sitter solutions, we choose $F = F_0/\phi^8$ and obtain $V_{\text{eff}}$ which has minima, see the red curve in the right picture of Fig. [2].

5 Conclusion

In the present paper, we have investigated a homogeneous cosmological dynamics of non-minimally coupled scalar field with both the curvature and the Gauss-Bonnet terms. We were mainly interested in looking for the fixed points of scalar field dynamics which correspond to de Sitter solutions. We have shown that, in the case of a positive coupling function $U(\phi)$, it is possible to introduce an effective potential $V_{\text{eff}}$ which can be expressed through curvature $U$, the scalar field potential $V$ and the coupling function with the Gauss-Bonnet term denoted by $F$. We show that it is convenient to investigate the structure of fixed points using the effective potential, indeed, the stable de Sitter solutions correspond to minima of the effective potential. The latter implies that the existence and stability of de Sitter solutions in the system under consideration can be studied with the help of function $V_{\text{eff}}$, since the stability of de Sitter solutions is analogous to the stability of a classical mechanical system moving in the potential field $V_{\text{eff}}$. In the model under consideration, the effective potential is a sum of two terms. The first one includes the contributions from $U$ and $V$.^

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3Similar model has been considered in [30]
the effective potential for models in absence of the Gauss-Bonnet term, proposed in [28]; the second term includes the function $F$ or the contribution from the Gauss-Bonnet term. Thus, the use of the effective potential is the simplest way to compare the results on existence and stability of de Sitter solutions in models with and without the Gauss-Bonnet term. We in general derive the structure of the effective potential and conditions for existence and stability de Sitter solutions. Using this approach, we have studied concrete models with the Gauss-Bonnet term and described a number of situations where de Sitter solutions exist due to the presence of the Gauss-Bonnet term and disappear otherwise.

Let us note that the effective potential for models without the Gauss-Bonnet term has a simple physical meaning: it is an invariant under the conformal rescaling of the metric that coincides with the potential in the Einstein frame [32]. It would be interesting to get similar interpretation for the effective potential proposed in this paper.

Let also emphasize that the unstable de Sitter solutions can be useful to describe inflation, whereas stable de solution is often used in models of late time acceleration of Universe. We should, however, stress that the effective potential formalism in Gauss-Bonnet cosmology works only for study of exact de Sitter solution. When we deviate from exact de Sitter to quasi-de Sitter, the effective potential as a single construction from three functions, entering the action of the theory, may not be enough. We leave the detailed study of quasi-de Sitter solutions in the considered models for our future investigations.

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