Abstract. In this article, we prove that there do not exist stable Schottky-Jacobi forms for the universal Jacobian locus and also prove that there exist non-trivial stable Schottky-Jacobi forms for the universal hyperelliptic locus.

1. Introduction

For a positive integer \( g \), we let

\[
\mathbb{H}_g = \left\{ \tau \in \mathbb{C}^{(g,g)} \mid \tau = \tau^t, \ \text{Im} \tau > 0 \right\}
\]

be the Siegel upper half plane of degree \( g \) and let

\[
Sp(2g, \mathbb{R}) = \left\{ M \in \mathbb{R}^{(2g,2g)} \mid \tau M \tau = J_g \right\}
\]

be the symplectic group of degree \( g \), where \( F^{(k,l)} \) denotes the set of all \( k \times l \) matrices with entries in a commutative ring \( F \) for two positive integers \( k \) and \( l \), \( \tau M \) denotes the transposed matrix of a matrix \( M \) and

\[
J_g = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}.
\]

Then \( Sp(2g, \mathbb{R}) \) acts on \( \mathbb{H}_g \) transitively by

\[
M \cdot \tau = (A \tau + B)(C \tau + D)^{-1},
\]

where \( M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, \mathbb{R}) \) and \( \Omega \in \mathbb{H}_n \). Let

\[
\Gamma_g = Sp(2g, \mathbb{Z}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, \mathbb{R}) \mid A, B, C, D \text{ integral} \right\}
\]

be the Siegel modular group of degree \( g \). This group acts on \( \mathbb{H}_g \) properly discontinuously.

Let \( \mathcal{A}_g := \Gamma_g \backslash \mathbb{H}_g \) be the Siegel modular variety of degree \( g \), that is, the moduli space of \( g \)-dimensional principally polarized abelian varieties, and let \( \mathcal{M}_g \) be the the moduli space of projective curves of genus \( g \). Then according to Torelli’s theorem, the Jacobi mapping

\[
T_g : \mathcal{M}_g \longrightarrow \mathcal{A}_g
\]

defined by

\[
C \longmapsto J(C) := \text{the Jacobian of } C
\]

is injective. The Jacobian locus \( J_g := T_g(\mathcal{M}_g) \) is a \((3g - 3)\)-dimensional subvariety of \( \mathcal{A}_g \).
The Schottky problem is to characterize the Jacobian locus or its closure \( J_g \) in \( A_g \).

At first this problem had been investigated from the analytical point of view: to find explicit equations of \( J_g \) (or \( J_g \)) in \( A_g \) defined by Siegel modular forms on \( \mathbb{H}_g \), for example, polynomials in the theta constant \( \theta \). Later on, the Schottky problem was due to Friedrich Schottky \([22]\) who gave the simple and beautiful equation satisfied by the theta constants of Jacobians of dimension 4. Much later the fact that this equation characterizes the Jacobian locus \( J_g \) was proved by J. Igusa \([14]\) (see also \([9, 11]\) and \([13]\)). Past decades there has been some progress on the characterization of Jacobians by some mathematicians.

For two positive integers \( g \) and \( h \), we consider the Heisenberg group

\[
H^{(g, h)}_\mathbb{R} = \{ (\lambda, \mu; \kappa) \mid \lambda, \mu \in \mathbb{R}^{(h, g)}, \kappa \in \mathbb{R}^{(h, h)}, \kappa + \mu^t \lambda \text{ symmetric} \}
\]

endowed with the following multiplication law

\[
(\lambda, \mu; \kappa) \circ (\lambda', \mu'; \kappa') = (\lambda + \lambda', \mu + \mu'; \kappa + \kappa' + \lambda^t \mu' - \mu^t \lambda')
\]

with \((\lambda, \mu; \kappa), (\lambda', \mu'; \kappa') \in H^{(g, h)}_\mathbb{R}\). We define the Jacobi group \( J^g \) of degree \( g \) and index \( h \) that is the semidirect product of \( Sp(2g, \mathbb{R}) \) and \( H^{(g, h)}_\mathbb{R} \)

\[
J^g = Sp(2g, \mathbb{R}) \ltimes H^{(g, h)}_\mathbb{R}
\]

endowed with the following multiplication law

\[
(M, (\lambda, \mu; \kappa)) \cdot (M', (\lambda', \mu'; \kappa')) = (MM', (\lambda + \lambda', \mu + \mu'; \kappa + \kappa' + \lambda^t \mu' - \mu^t \lambda'))
\]

with \(M, M' \in Sp(2g, \mathbb{R}), (\lambda, \mu; \kappa), (\lambda', \mu'; \kappa') \in H^{(g, h)}_\mathbb{R}\) and \((\tilde{\lambda}, \tilde{\mu}) = (\lambda, \mu)M'\). Then \( J^g \) acts on \( \mathbb{H}_g \times \mathbb{C}^{(h, g)} \) transitively by

\[
(M, (\lambda, \mu; \kappa)) \cdot (\Omega, Z) = \left(M \cdot \Omega, (Z + \lambda \Omega + \mu)(C \Omega + D)^{-1}\right),
\]

where \(M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, \mathbb{R}), (\lambda, \mu; \kappa) \in H^{(g, h)}_\mathbb{R}\) and \((\Omega, Z) \in \mathbb{H}_g \times \mathbb{C}^{(h, g)}\). We note that the Jacobi group \( J^g \) is not a reductive Lie group and the homogeneous space \( \mathbb{H}_g \times \mathbb{C}^{(h, g)} \) is not a symmetric space. From now on, for brevity we write \( \mathbb{H}_{g, h} = \mathbb{H}_g \times \mathbb{C}^{(h, g)}\). The homogeneous space \( \mathbb{H}_{g, h} \) is called the Siegel-Jacobi space of degree \( g \) and index \( h \).

Let \( \Gamma^g = \Gamma_g \ltimes H^{(g, h)}_\mathbb{Z} \) be the Jacobi modular group. Let

\[
A_{g, h} := \Gamma^g \setminus \mathbb{H}_{g, h}
\]

be the universal abelian variety. Consider the natural projection map

\[
\pi_{g, h} : A_{g, h} \rightarrow A_g.
\]

Let

\[
J_{g, h} := \pi_{g, h}^{-1}(J_g)
\]

be the universal Jacobian locus and let

\[
Hyp_{g, h} := \pi_{g, h}^{-1}(Hyp_g)
\]

be the universal hyperelliptic locus, where \( Hyp_g \) is the hyperelliptic locus in \( A_g \).
Let $2\mathcal{M}$ be a positive definite, even unimodular integral symmetric matrix of degree $h$. According to Theorem 3.6 in [27], if $g + h > 2k + 1$ with a nonnegative integer $k$, the Siegel-Jacobi operator
\[ \Psi_{g,\mathcal{M}} : J_{k,\mathcal{M}}(\Gamma_g) \rightarrow J_{k,\mathcal{M}}(\Gamma_{g-1}) \]
is an isomorphism (see also Theorem 2.2). Using this fact, we define the notion of stable Jacobi forms of weight $k$ and index $\mathcal{M}$. A Jacobi form $F \in J_{k,\mathcal{M}}(\Gamma_g)$ is said to be a Schottky-Jacobi form for $J_{g,h}$ (resp. $Hyp_{g,h}$) if it vanishes along $J_{g,h}$ (resp. $Hyp_{g,h}$). In a natural way, we can define the notion of stable Schottky-Jacobi forms for $J_{g,h}$ and $Hyp_{g,h}$. For precise definitions, we refer to Definition 2.3 and Definition 4.2.

The aim of this paper is to prove the non-existence of stable Schottky-Jacobi forms for the universal Jacobian locus and also to prove that there exist non-trivial stable Schottky-Jacobi forms for the universal hyperelliptic locus.

This article is organized as follows. In Section 2, we review some properties of the Siegel-Jacobi operator and the notion of stable Jacobi forms introduced by J.-H. Yang [30]. In Section 3, we review the notion of stable Schottky-Siegel forms and the works that were done recently by G. Codogni and N. I. Shepherd-Barron [3, 4]. In Section 4, we introduce the notion of stable Schottky-Jacobi forms and prove the following two theorems.

**Theorem 1.1.** Let $2\mathcal{M}$ be a positive definite, even unimodular integral symmetric matrix of degree $h$. Then there do not exist stable Schottky-Jacobi forms of index $\mathcal{M}$ for the universal Jacobian locus.

**Theorem 1.2.** Let $2\mathcal{M}$ be a positive definite, even unimodular integral symmetric matrix of degree $h$. Then there exist non-trivial stable Schottky-Jacobi forms of index $\mathcal{M}$ for the universal hyperelliptic locus.

In the final section, we make some comments and present several questions.

**Notations:** We denote by $\mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ the field of rational numbers, the field of real numbers and the field of complex numbers respectively. We denote by $\mathbb{Z}$ and $\mathbb{Z}^+$ the ring of integers and the set of all positive integers respectively. $\mathbb{R}^+$ denotes the set of all positive real numbers. $\mathbb{Z}_+$ and $\mathbb{R}_+$ denote the set of all nonnegative integers and the set of all nonnegative real numbers respectively. The symbol “:=” means that the expression on the right is the definition of that on the left. For two positive integers $k$ and $l$, $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring $F$. For a square matrix $A \in F^{(k,k)}$ of degree $k$, $\sigma(A)$ denotes the trace of $A$. For any $M \in F^{(k,l)}$, $^tM$ denotes the transpose of a matrix $M$. $I_n$ denotes the identity matrix of degree $n$. We put $i = \sqrt{-1}$.

2. Stable Jacobi Forms

For a non-negative integer $k$, we denote by $[\Gamma_g, k]$ the vector space of all Siegel modular forms of weight $k$. The Siegel $\Phi$-operator
\[ \Phi_g : [\Gamma_g, k] \rightarrow [\Gamma_{g-1}, k] \]
is an important linear map defined by
\[(\Phi_g f)(\tau) := \lim_{t \to \infty} f \left( \frac{\tau}{0} \right), \quad f \in [\Gamma_g, k], \ \tau \in \mathbb{H}g-1.\]

H. Maass [18] proved that if \( k \) is even and \( k > 2g \), then \( \Phi_g \) is surjective. E. Freitag [7] proved that if \( g > 2k \), then \( \Phi_g \) is injective. Using the theory of singular modular forms developed by Freitag [8, 10], he showed the following:

(SO1) \([\Gamma_g, k] = 0 \) for \( g > 2k, \ k \neq 0 \) (mod 4).

(SO2) \( \Phi_g \) is an isomorphism if \( g > 2k + 1 \).

**Definition 2.1.** A collection \((f_g)_{g \geq 0}\) is called a stable modular form of weight \( k \) if it satisfies the following conditions (SM1) and (SM2):

(SM1) \( f_g \in [\Gamma_g, k] \) for all \( g \geq 0 \).

(SM2) \( \Phi_g f_g = f_{g-1} \) for all \( g > 0 \).

Let \( \rho \) be a rational representation of \( GL(g, \mathbb{C}) \) on a finite dimensional complex vector space \( V_{\rho} \). Let \( \mathcal{M} \in \mathbb{R}^{(h,h)} \) be a symmetric half-integral semi-positive definite matrix of degree \( h \). The canonical automorphic factor
\[ J_{\rho, \mathcal{M}} : G^J \times \mathbb{H}_{g,h} \longrightarrow GL(V_{\rho}) \]
for \( G^J \) on \( \mathbb{H}_{g,h} \) is given as follows:
\[ J_{\rho, \mathcal{M}}((M, (\lambda, \mu; \kappa)), (\tau, z)) = e^{2\pi i \sigma(M(z+\lambda \tau+\mu)(C\tau+D)^{-1}C^t(z+\lambda \tau+\mu))} \times e^{-2\pi i \sigma(M(\lambda \tau^2+2\lambda \tau z+\kappa+\mu \lambda))} \rho(C\tau+D), \]
where \( M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, \mathbb{R}), \ (\lambda, \mu; \kappa) \in \mathbb{H}^{(g,h)}_{\mathbb{R}} \) and \((\tau, z) \in \mathbb{H}_{g,h} \). We refer to [29] for a geometrical construction of \( J_{\rho, \mathcal{M}} \).

Let \( C^\infty(\mathbb{H}_{g,h}, V_\rho) \) be the algebra of all \( C^\infty \) functions on \( \mathbb{H}_{g,h} \) with values in \( V_\rho \). For \( f \in C^\infty(\mathbb{H}_{g,h}, V_\rho) \), we define
\[ (f|_{\rho, \mathcal{M}})((M, (\lambda, \mu; \kappa)), (\tau, z)) \]
\[ = J_{\rho, \mathcal{M}}((M, (\lambda, \mu; \kappa)), (\tau, z))^{-1} \]
\[ f((A\tau + B)(C\tau + D)^{-1}, (z + \lambda \tau + \mu)(C\tau + D)^{-1}), \]
where \( M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, \mathbb{R}), \ (\lambda, \mu; \kappa) \in \mathbb{H}^{(g,h)}_{\mathbb{R}} \) and \((\tau, z) \in \mathbb{H}_{g,h} \).

**Definition 2.2.** Let \( \rho \) and \( \mathcal{M} \) be as above. Let
\[ H^{(g,h)}_{\mathbb{Z}} := \{ (\lambda, \mu; \kappa) \in \mathbb{H}^{(g,h)}_{\mathbb{R}} \mid \lambda, \mu, \kappa \text{ integral} \} \]
be the discrete subgroup of \( \mathbb{H}^{(g,h)}_{\mathbb{R}} \). A Jacobi form of index \( \mathcal{M} \) with respect to \( \rho \) on a subgroup \( \Gamma \) of \( \Gamma_g \) of finite index is a holomorphic function \( f \in C^\infty(\mathbb{H}_{g,h}, V_\rho) \) satisfying the following conditions (A) and (B):

(A) \( f|_{\rho, \mathcal{M}}[\tilde{\gamma}] = f \) for all \( \tilde{\gamma} \in \tilde{\Gamma} := \Gamma \rtimes H^{(g,h)}_{\mathbb{Z}} \).

(B) For each $M \in \Gamma_g$, $f|_{\rho,\mathcal{M}}[M]$ has a Fourier expansion of the following form:

$$(f|_{\rho,\mathcal{M}}[M])(\tau, z) = \sum_{T = tT \geq 0, \text{half-integral}} \sum_{R \in \mathbb{Z}^{(g, h)}} c(T, R) \cdot e^{\frac{2\pi i}{\lambda_1}(T \tau \cdot \frac{1}{2} R)} \cdot e^{2\pi i \sigma(Rz)}$$

with $\lambda_\Gamma(\not= 0) \in \mathbb{Z}$ and $c(T, R) \not= 0$ only if $\left(\frac{tT}{\lambda_1 R} \frac{1}{2} R \not= 0\right)$.

If $g \geq 2$, the condition (B) is superfluous by Köcher principle (cf. [33] Lemma 1.6). We denote by $J_{\rho,\mathcal{M}}(\Gamma)$ the vector space of all Jacobi forms of index $\mathcal{M}$ with respect to $\rho$ on $\Gamma$. Ziegler (cf. [33] Theorem 1.8 or [6] Theorem 1.1) proved that the vector space $J_{\rho,\mathcal{M}}(\Gamma)$ is finite dimensional. In the special case $\rho(A) = (\det(A))^k$ with $A \in GL(g, \mathbb{C})$ and a fixed $k \in \mathbb{Z}$, we write $J_{k,\mathcal{M}}(\Gamma)$ instead of $J_{\rho,\mathcal{M}}(\Gamma)$ and call $k$ the weight of the corresponding Jacobi forms. For more results about Jacobi forms with $g > 1$ and $h > 1$, we refer to [26, 27, 28, 29, 30, 31, 32] and [33]. Jacobi forms play an important role in lifting elliptic cusp forms to Siegel cusp forms of degree 2 (cf. [16, 17]).

Now we consider the special case $\rho = \det^k$ with $k \in \mathbb{Z}_+$. We define the Siegel-Jacobi operator

$$\Psi_{g,\mathcal{M}} : J_{k,\mathcal{M}}(\Gamma_g) \rightarrow J_{k,\mathcal{M}}(\Gamma_{g-1})$$

by

$$(\Psi_{g,\mathcal{M}}F)(\tau, z) := \lim_{t \rightarrow \infty} F\left(\begin{pmatrix} \tau & 0 \\ 0 & it \end{pmatrix}, (z, 0) \right),$$

where $F \in J_{k,\mathcal{M}}(\Gamma_g)$, $\tau \in \mathbb{H}_{g-1}$ and $z \in \mathbb{C}^{(h-g-1)}$. We observe that the above limit exists and $\Psi_{g,\mathcal{M}}$ is a well-defined linear map (cf. [33]).

J.-H. Yang [27] proved the following theorems.

**Theorem 2.1.** Let $2\mathcal{M}$ be a positive even unimodular symmetric integral matrix of degree $h$ and let $k$ be an even nonnegative integer. If $g + h > 2k$, then the Siegel-Jacobi operator $\Psi_{g,\mathcal{M}}$ is injective.

*Proof. See Theorem 3.5 in [27].* □

**Theorem 2.2.** Let $2\mathcal{M}$ be as above in Theorem 2.1 and let $k$ be an even nonnegative integer. If $g + h > 2k + 1$, then the Siegel-Jacobi operator $\Psi_{g,\mathcal{M}}$ is an isomorphism.

*Proof. See Theorem 3.6 in [27].* □

**Remark 2.1.** A Jacobi form in $J_{k,\mathcal{M}}(\Gamma_g)$ is said to be singular if it admits a Fourier expansion such that a Fourier coefficient $c(T, R)$ vanishes unless

$$\det\left(\frac{T}{\frac{1}{2} t R} \frac{1}{2} R \not= 0\right) = 0.$$ 

Let $2\mathcal{M}$ be as above in Theorem 2.1. Yang proved that if $k$ is an even nonnegative integer and $g + \text{rank}(\mathcal{M}) > 2k$, then any non-zero Jacobi form in $J_{k,\mathcal{M}}(\Gamma_g)$ is singular (cf. [28] Theorem 4.5]).

**Theorem 2.3.** Let $2\mathcal{M}$ be as above in Theorem 2.1 and let $k$ be an even nonnegative integer. If $2k > 4g + h$, then the Siegel-Jacobi operator $\Psi_{g,\mathcal{M}}$ is surjective.
Proof. See Theorem 3.7 in [27]. □

Remark 2.2. Yang [27, Theorem 4.2] proved that the action of the Hecke operators on Jacobi forms is compatible with that of the Siegel-Jacobi operator.

Definition 2.3. A collection \((F_g)_{g \geq 0}\) is called a stable Jacobi form of weight \(k\) and index \(\mathcal{M}\) if it satisfies the following conditions (SJ1) and (SJ2):

(SJ1) \( F_g \in J_{k, \mathcal{M}}(\Gamma_g) \) for all \( g \geq 0 \).

(SJ2) \( \Psi_{g, \mathcal{M}} F_g = F_{g-1} \) for all \( g \geq 1 \).

Remark 2.3. The concept of a stable Jacobi forms was introduced by Yang [30].

Example. Let \( S \) be a positive even unimodular symmetric integral matrix of degree \( 2k \) and let \( c \in \mathbb{Z}^{(2k, h)} \) be an integral matrix. We define the theta series \( \vartheta_{S,c}^{(g)}(\tau, z) \) by

\[
\vartheta_{S,c}^{(g)}(\tau, z) := \sum_{\lambda \in \mathbb{Z}^{(2k, g)}} e^{\pi i \left\{ \sigma(\lambda \tau t^\lambda) + 2\sigma(t^\lambda c \lambda^t z) \right\}}, \quad (\tau, z) \in \mathbb{H}_{g,h}.
\]

It is easily seen that \( \vartheta_{S,c}^{(g)} \in J_{k, \mathcal{M}}(\Gamma_g) \) with \( \mathcal{M} := \frac{1}{2} t^g c S c \) for all \( g \geq 0 \) and \( \Psi_{g, \mathcal{M}} \vartheta_{S,c}^{(g)} = \vartheta_{S,c}^{(g-1)} \) for all \( g \geq 1 \). Thus the collection \( \Theta_{S,c} := \left( \vartheta_{S,c}^{(g)} \right)_{g \geq 0} \) is a stable Jacobi form of weight \( k \) and index \( \mathcal{M} \).

3. Stable Schottky-Siegel Forms

Let \( A^{\text{Sat}}_g \) be the Satake compactification of the Siegel modular variety \( A_g \) (cf. [21]).

\[
A^{\text{Sat}}_g = A_g \cup A_{g-1} \cup \cdots A_1 \cup A_0.
\]

W. Baily [1] proved that \( A^{\text{Sat}}_g \) is a normal projective variety in which \( A_g \) is Zariski open. In particular, we have a closed embedding

\[
t_g : A^{\text{Sat}}_{g-1} \hookrightarrow A^{\text{Sat}}_g.
\]

The collection \( (A^{\text{Sat}}_g)_{g \geq 0} \) and the above embeddings \( (t_g)_{g \geq 0} \) define the projective limit

\[
A^{\text{Sat}}_\infty := \bigcup_{g \geq 0} A^{\text{Sat}}_g = \lim_{\leftarrow} A^{\text{Sat}}_g
\]

which is called the stable Satake compactification. Let \( L_g \) be the determinant line bundle of the Hodge bundle over \( A_g \). The we have the isomorphism

\[
H^0(A_g, L_g^{\otimes k}) \cong [\Gamma_g, k].
\]

Let \( J^{\text{Sat}}_g \) (resp. \( \text{Hyp}^{\text{Sat}}_g \)) be the closure of \( J_g \) (resp. \( \text{Hyp}_g \)) inside \( A^{\text{Sat}}_g \). We define

\[
J_\infty := \bigcup_{g \geq 0} J^{\text{Sat}}_g \quad \text{and} \quad \text{Hyp}_\infty := \bigcup_{g \geq 0} \text{Hyp}^{\text{Sat}}_g.
\]
Definition 3.1. A pair \((\Lambda, Q)\) is called a quadratic form if \(\Lambda\) is a lattice and \(Q\) is an integer-valued bilinear symmetric form on \(\Lambda\). The rank of \((\Lambda, Q)\) is defined to be the rank of \(\Lambda\). For \(v \in \Lambda\), the integer \(Q(v,v)\) is called the norm of \(v\). A quadratic form \((\Lambda, Q)\) is said to be even if \(Q(v,v)\) is even for all \(v \in \Lambda\). A quadratic form \((\Lambda, Q)\) is said to be unimodular if \(\det(Q) = 1\).

Definition 3.2. Let \((\Lambda, Q)\) be an even unimodular positive definite quadratic form of rank \(m\). For a positive integer \(g\), the theta series \(\theta_{Q,g}\) associated to \((\Lambda, Q)\) is defined to be

\[
\theta_{Q,g}(\tau) := \sum_{x_1, \ldots, x_g \in \Lambda} \exp \left( \pi i \sum_{p,q=1}^g Q(x_p, x_q) \tau_{pq} \right), \quad \tau = (\tau_{pq}) \in \mathbb{H}_g.
\]

It is well known that \(\theta_{Q,g}(\tau)\) is a Siegel modular form on \(\mathbb{H}_g\) of weight \(\frac{m}{2}\). We easily see that

\[
\Phi_{g+1}(\theta_{Q,g+1}) = \theta_{Q,g} \quad \text{for all } g \geq 0.
\]

Therefore the collection of all theta series associated to \((\Lambda, Q)\)

\[
(3.1) \quad \Theta_Q := (\theta_{Q,g})_{g \geq 0}
\]

is a stable modular form of weight \(\frac{m}{2}\).

Freitag \[8\] proved the following theorem.

Theorem 3.1. The ring of stable modular forms is a polynomial ring in countably many theta series \(\Theta_Q = (\theta_{Q,g})_{g \geq 0}\) associated to irreducible positive even unimodular quadratic forms.

Proof. See Theorem 2.5 in \[8\].

Definition 3.3. A modular form \(f \in [\Gamma_g, k]\) is called a Schottky-Siegel form of weight \(k\) for \(J_g\) (resp. \(\text{Hyp}_g\)) if it vanishes along \(J_g\) (resp. \(\text{Hyp}_g\)). A collection \((f_g)_{g \geq 0}\) is called a stable Schottky-Siegel form of weight \(k\) for the Jacobian locus (resp. the hyperelliptic locus) if \((f_g)_{g \geq 0}\) is a stable modular form of weight \(k\) and \(f_g\) vanishes along \(J_g\) (resp. \(\text{Hyp}_g\)) for every \(g \geq 0\).

G. Codogni and N. I. Shepherd-Barron \[4\] proved the following.

Theorem 3.2. There do not exist stable Schottky-Siegel form for the Jacobian locus.

Proof. See Theorem 1.3 and Corollary 1.4 in \[4\].

Remark 3.1. Let

\[
(3.2) \quad \varphi_g(\tau) := \theta_{E_8 \oplus E_8, g}(\tau) - \theta_{D_{16}^+, g}(\tau), \quad \tau \in \mathbb{H}_g
\]

be the Igusa modular form, that is, the difference of the theta series in genus \(g\) associated to the two distinct positive even unimodular quadratic forms \(E_8 \oplus E_8\) and \(D_{16}^+\) of rank 16. We see that \(\varphi_g(\tau)\) is a Siegel modular form on \(\mathbb{H}_g\) of weight 8. Since \(\Phi_g \varphi_g = \varphi_{g-1}\) for all \(g \geq 1\), a collection \((\varphi_g)_{g \geq 0}\) is a stable modular form of weight 8. Igusa \[14, 15\] showed that the Schottky-Siegel form discovered by Schottky \[22\] is an explicit rational multiple of \(\varphi_4\). In \[14\], he also showed that the Jacobian locus \(J_4\) is reduced and irreducible, and so cuts out exactly \(J_4\) in \(A_4\). Indeed, \(\varphi_4(\tau)\) is a degree 16 polynomial in the Thetanullwerte of genus 4. On the other hand, Grushevsky and Salvati Manni \[12\] showed that the Igusa modular
form \( \varphi_5 \) of genus 5 cuts out exactly the trigonal locus in \( J_5 \) and so does not vanish along \( J_5 \). Thus \( (\varphi_g)_{g \geq 0} \) is not a stable Schottky-Siegel form.

G. Codogni [3] proved the following.

**Theorem 3.3.** There exist non-trivial stable Schottky-Siegel form for the hyperelliptic locus. Precisely the ideal of stable Schottky-Siegel forms for the hyperelliptic locus is generated by differences of theta series

\[ \Theta_P - \Theta_Q, \]

where \( P \) and \( Q \) are positive definite even unimodular quadratic forms of the same rank.

**Proof.** See Theorem 1.2 in [3]. \( \square \)

**Remark 3.2.** Let \( P \) and \( Q \) be two positive even unimodular quadratic forms of the same rank. We let

\[ \Theta_P := (\theta_{P,g})_{g \geq 0} \quad \text{and} \quad \Theta_Q := (\theta_{Q,g})_{g \geq 0} \]

be two stable modular forms. Codogni [3, Theorem 1.4] showed that the difference of theta series

\[ \Theta_P - \Theta_Q \]

is a stable Schottky-Siegel form for the hyperelliptic locus when one of the following conditions (1)–(3):

1. \( \text{rank}(P) = \text{rank}(Q) = 24 \) and the two quadratic forms have the same number of vectors of norm 2;
2. \( \text{rank}(P) = \text{rank}(Q) = 32 \) and the two quadratic forms do not have any vectors of norm 2;
3. \( \text{rank}(P) = \text{rank}(Q) = 48 \) and the two quadratic forms do not have any vectors of norm 2 and 4.

4. **Stable Schottky-Jacobi Forms and Proofs of Main Theorems**

In this section, we introduce the notion of stable Schottky-Jacobi forms and prove the main theorems.

We let

\[ A_{g,h} := \Gamma_{g,h} \backslash \mathbb{H}_{g,h} \]

be the universal abelian variety and let

\[ A_{g,h}^{\text{Sat}} := A_{g,h} \cup A_{g-1,h} \cup \cdots \cup A_{1,h} \cup A_{0,h} \]

be the Satake compactification of \( A_{g,h} \). We consider the natural projection map

\[ \pi_{g,h} : A_{g,h} \to A_g \]

of \( A_{g,h} \) onto \( A_g \). Let

\[ J_{g,h} := \pi_{g,h}^{-1}(J_g) \]

be the universal Jacobian locus and let

\[ H_{yp_{g,h}} := \pi_{g,h}^{-1}(H_{yp_g}) \]
be the universal hyperelliptic locus. Let $J^S_{g,h}$ (resp. $Hyp^S_{g,h}$) be the closure of $J_{g,h}$ (resp. $Hyp_{g,h}$) in $A^g_{Sat}$. We put
$$A^\infty_{g,h} := \bigcup_{g \geq 0} A^g_{Sat},$$
$$J^\infty_{g,h} := \bigcup_{g \geq 0} J^S_{g,h},$$
and
$$Hyp^\infty_{g,h} := \bigcup_{g \geq 0} Hyp^S_{g,h}.$$

**Definition 4.1.** Let $M$ be a half-integral semi-positive symmetric matrix of degree $h$ and $k \in \mathbb{Z}_+$. A Jacobi form $F \in J^k_{\Gamma_g,M}$ is called a Schottky-Jacobi form of weight $k$ and index $M$ for the universal Jacobian (resp. hyperelliptic) locus if it vanishes along $J_{g,h}$ (resp. $Hyp_{g,h}$).

**Definition 4.2.** Let $M$ be a half-integral semi-positive symmetric matrix of degree $h$ and $k \in \mathbb{Z}_+$. A collection $(F_g)_{g \geq 0}$ is called a stable Schottky-Jacobi form of weight $k$ and index $M$ if it satisfies the following conditions (SSJ1) and (SSJ2):

(SSJ1) $F_g \in J^k_{\Gamma_g,M}$ is a Schottky-Jacobi form of weight $k$ and index $M$ for all $g \geq 0$.

(SSJ2) $\Psi_{g,M} F_g = F_{g-1}$ for all $g \geq 1$.

**Theorem 4.1.** Let $2M$ be a positive even unimodular symmetric integral matrix of degree $h$. Then there do not exist stable Schottky-Jacobi forms of index $M$ for the universal Jacobian locus.

**Proof.** We first observe that $h \equiv 0 \pmod{8}$ (cf. [2]). Assume that there exists a non-trivial stable Schottky-Jacobi form $(F_g)_{g \geq 0}$ of weight $k$ and index $M$ for the universal Jacobian locus.

**Case 1:** $k$ is even.

Using the Shimura isomorphism (cf. [24, 25]), we obtain the following

$$J^k_{\Gamma_g,M} = [\Gamma_g, k_s] \cdot \vartheta^{[g]}_{2M}(\tau, z),$$

where $k_s := k - \frac{h}{2}$ and

$$\vartheta^{[g]}_{2M}(\tau, z) := \sum_{\lambda \in \mathbb{Z}^{(h,g)}} e^{2\pi i \sigma(M(\lambda \tau^t \lambda + 2\lambda^t z))}.$$  

We refer to [23, Theorem 3.3] for the proof of the formula (4.1). We see from (SO1) in Section 2 that $[\Gamma_g, k_s] = 0$ if $g + h > 2k$ and $k \equiv 0 \pmod{4}$. So $k \equiv 0 \pmod{4}$. We observe that the Siegel-Jacobi operator $\Psi_{g,M} : J^k_{\Gamma_g} \rightarrow J^k_{\Gamma_{g-1}}$ is an isomorphism if $g + h > 2k + 1$ (see Theorem 2.2 in Section 2). It is easy to see that

$$\Psi_{g,M} \vartheta^{[g]}_{2M} = \vartheta^{[g-1]}_{2M} \quad \text{for all } g \geq 1.$$  

According to the formula (4.1), we may write

$$F_g(\tau, z) = f_g(\tau) \cdot \vartheta^{[g]}_{2M}(\tau, z), \quad f \in [\Gamma, k_s].$$
Now we have, for \((\tau, z) \in \mathbb{H}_{g-1, h}\),
\[
(\Psi_{g,\mathcal{M}}F_{g})(\tau, z) = \lim_{t \to \infty} F_{g}\left(\begin{pmatrix} \tau & 0 \\ 0 & it \end{pmatrix}, (z, 0) \right)
= \lim_{t \to \infty} f_{g}\left(\begin{pmatrix} \tau & 0 \\ 0 & it \end{pmatrix}, (z, 0) \right)
= (\Phi_{g}f_{g})(\tau) \cdot \vartheta^{[g-1]}_{2M}(\tau, z).
\]
Here \(\Phi_{g}\) is the Siegel \(\Phi\)-operator defined by (2.1).

On the other hand, by the assumption that \((F_{g})_{g \geq 0}\) is a stable Schottky-Jacobi form, we have
\[
\Psi_{g,\mathcal{M}}F_{g} = F_{g-1} = f_{g-1} \cdot \vartheta^{[g-1]}_{2M} \quad \text{for some } f_{g-1} \in [\Gamma_{g-1}, k]
\]
for all \(g \geq 1\). Therefore
\[
\Phi_{g}f_{g} = f_{g-1} \quad \text{for all } g \geq 1.
\]
Obviously \(f_{g}\) vanishes along \(J_{g}\) for all \(g \geq 0\). Thus \((f_{g})_{g \geq 0}\) is a non-trivial stable Schottky-Siegel form of weight \(k\). This contradicts the non-existence of a non-trivial stable Schottky-Siegel form for the Jacobian locus (see Theorem 3.2).

**Case 2:** \(k\) is odd.

Using the Shimura isomorphism, we may write
\[
F_{g}(\tau, z) = \psi_{g}(\tau) \cdot \vartheta^{[g]}_{2M}(\tau, z) \quad \text{for all } g \geq 1,
\]
where \(\vartheta^{[g]}_{2M}(\tau, z)\) is the theta series defined by Formula (4.2) and \(f_{g}(\tau)\) satisfies the following behaviours (4.3) and (4.4):
\[
\begin{align*}
(4.3) \quad & \psi_{g}(\tau + S) = \psi_{g}(\tau) \quad \text{for all } S = ^tS \in \mathbb{Z}^{(g, g)}; \\
(4.4) \quad & \psi_{g}(-\tau^{-1}) = \det(-\tau)^k \det\left(\frac{\tau}{i}\right)^{-\frac{h}{2}} \psi_{g}(\tau), \quad \tau \in \mathbb{H}_{g}.
\end{align*}
\]
We put
\[
\xi_{g}(\tau) := \left\{\psi_{g}(\tau)\right\}^2 \quad \text{for all } g \geq 1.
\]
Then we see easily that a collection \(\xi = (\xi_{g})_{g \geq 0}\) is a non-trivial stable Schottky-Siegel form of weight \(2k - h\) for the Jacobian locus. This contradicts the non-existence of a non-trivial stable Schottky-Siegel form for the Jacobian locus. Hence we complete the proof of the above theorem (Theorem 1.1). \(\Box\)

**Theorem 4.2.** Let \(2\mathcal{M}\) be a positive even unimodular symmetric integral matrix of degree \(h\). Then there exist non-trivial stable Schottky-Jacobi forms of \(\mathcal{M}\) for the universal hyperelliptic locus.

**Proof.** According to Theorem 3.3, there exists a non-trivial stable Schottky-Siegel form \((f_{g})_{g \geq 0}\) of weight \(k\) for the hyperelliptic locus. We see from (SO1) in Section 2 that \(k \equiv 0 \pmod{4}\) and \(k \in \mathbb{Z}^+\). We put \(\ell := k + \frac{h}{2}\). Then using the Shimura isomorphism, we have
\[
J_{\ell,\mathcal{M}}(\Gamma_{g}) = [\Gamma_{g}, k] \cdot \vartheta^{[g]}_{2M}(\tau, z),
\]
where \( \vartheta_{2M}^{[g]}(\tau, z) \) is the theta series defined by Formula (4.2). We define the Jacobi forms
\[
F_g(\tau, z) := f_g(\tau) \cdot \vartheta_{2M}^{[g]}(\tau, z), \quad g \geq 0.
\]
Since \( f_g \in [\Gamma_g, k] \) is a Jacobi form of weight \( k \) and index 0, we get \( F_g \in J_{\ell,M}(\Gamma_g) \) for all \( g \geq 0 \). For \( [(\tau, z)] \in Hyp_{g,h} \),
\[
F_g(\tau, z) = 0, \quad g \geq 0.
\]
By a simple calculation, we obtain
\[
\Psi_{g,M} F_g = F_{g-1} \quad \text{for all } g \geq 1.
\]
Thus \((F_g)_{g \geq 0}\) is a non-trivial stable Schottky-Jacobi form of weight \( \ell \) and index \( M \) for the universal hyperelliptic locus \( Hyp_{\infty,h} \). This completes the proof of the above theorem (Theorem 1.2). \( \square \)

We define the invariant \( \mu(Q) \) of a quadratic form \((Q, \Lambda)\) by
\[
\mu(Q) := \min \{ Q(v, v) \mid v \in \Lambda, v \neq 0 \}.
\]

**Theorem 4.3.** Let \( 2M \) be a positive even unimodular symmetric integral matrix of degree \( h \). Let \((Q, \Lambda)\) and \((P, \Gamma)\) be two positive even unimodular quadratic forms of rank \( m \). Assume that
\[
\frac{m}{\mu} \leq 8, \quad \text{where } \mu := \min \{ \mu(Q), \mu(P) \}.
\]
We put
\[
F_g(\tau, z) := \{ \theta_{Q,g}(\tau) - \theta_{P,g}(\tau) \} \cdot \vartheta_{2M}^{[g]}(\tau, z), \quad g \geq 0.
\]
Then \((F_g)_{g \geq 0}\) is a stable Schottky-Jacobi form of weight \( \frac{1}{2}(m + h) \) and index \( M \) for the universal hyperelliptic locus.

**Proof.** It is easily seen that
\[
\theta_{Q,g}, \theta_{P,g} \in J_{\frac{m}{\mu}0}(\Gamma_g) \quad \text{and} \quad \theta_{Q,g} \cdot \vartheta_{2M}^{[g]}, \theta_{P,g} \cdot \vartheta_{2M}^{[g]} \in J_{\frac{1}{2}(m+h),M}(\Gamma_g)
\]
for all \( g \geq 0 \). The proof follows immediately from Theorem 5.5 in [3] and the above facts (4.5). \( \square \)

5. Final Remarks

In the final section, we make some remarks and present several open questions.

**Remark 5.1.** Let \( 2M \) be a positive even unimodular symmetric integral matrix of degree \( h \). Assume that
\[
g + \text{rank}(M) > 2k + 1 \quad \text{and} \quad k \in \mathbb{Z}_+ \text{ is even.}
\]
We denote by \( C_{k,M} \) be the vector space of stable Jacobi forms of weight \( k \) and index \( M \).
According to Theorem 2.2, the Siegel-Jacobi operator \( \Psi_{g,M} : J_{k,M}(\Gamma_g) \rightarrow J_{k,M}(\Gamma_{g-1}) \) is an isomorphism, and hence we obtain
\[
\dim C_{k,M} = \dim J_{k,M}(\Gamma_g).
\]
From Formula (4.1), we see that
\[
J_{k,M}(\Gamma_g) = [\Gamma_g, k_\ast] \cdot \vartheta_{2M}^{[g]}(\tau, z), \quad \text{where } k_\ast := k - \frac{h}{2}.
\]
Therefore from (SO1) in Section 2, we get the vanishing result:
\[ J_{k,M}(\Gamma_g) = 0 \quad \text{if} \quad 2k \not\equiv h \pmod{8}. \]
Thus \( k \equiv 0 \pmod{4} \) if \( J_{k,M}(\Gamma_g) \neq 0 \). According to Yang [28], any Jacobi form in \( J_{k,M}(\Gamma_g) \) is singular. We note that any element in \( \Gamma_g, k - h \frac{1}{2} \) is a singular modular form (see [8, 10]). Hence we conclude that \( C_{k,M} \) is spanned by stable Jacobi forms of the form
\[ \left( \theta_{P,g}(\tau) \vartheta_{2M}(\tau, z) \right)_{g \geq 0}, \]
where \( P \) runs over the set of positive even unimodular quadratic forms of rank \( 2k - h \).

Remark 5.2. Let \( \varphi_g(\tau) \) be the Igusa modular form defined by the formula (3.2). We denote by \( [\Gamma_g, k]_0 \) be the space of all Siegel cuspidal Hecke eigenforms on \( \mathbb{H}_g \) of weight \( k \). It is known that \( [\Gamma_4, 8]_0 = \mathbb{C} \cdot \varphi_4 \) (for a nice proof of this, we refer to [5]). Poor [20] showed that \( \varphi_g(\tau) \) vanishes on the hyperelliptic locus \( \text{Hyp}_g \) for all \( g \geq 1 \), and the divisor of \( \varphi_g(\tau) \) in \( \mathbb{A}_g \) is proper and irreducible for all \( g \geq 4 \). And Ikeda [16, 17] proved that if \( g \equiv k \pmod{2} \), there exists a canonical lifting
\[ I_{g,k} : [\Gamma_1, 2k]_0 \rightarrow [\Gamma_{2g}, g + k]_0. \]
Considering the special cases of the Ikeda lift maps \( I_{2,6} \) and \( I_{6,6} \), Breulman and Kuss [2] showed that
\[ I_{2,6}(\Delta) = c \varphi_4, \quad c(\neq 0) \in \mathbb{C}, \]
and constructed a nonzero Siegel cusp form of degree 12 and weight 12 which is the image of \( \Delta(\tau) \), where
\[ \Delta(\tau) = (2\pi i)^{12} q \prod_{n=1}^{\infty} (1 - q^n)^{24}, \quad q := e^{2\pi i \tau}, \quad \tau \in \mathbb{H}_1 \]
is a cusp form of weight 12.

Remark 5.3. We consider a half-integral semi-positive symmetric integral matrix \( M \) such that \( 2M \) is not even or which is not unimodular. The natural questions arise:

Question 1. Are there non-trivial stable Schottky-Jacobi forms of index \( M \) for the universal Jacobian locus ?

Question 2. Are there non-trivial stable Schottky-Jacobi forms of index \( M \) for the universal hyperelliptic locus ?

Remark 5.4. Let \( 2M \) be a positive even unimodular symmetric integral matrix of degree \( h \). For two nonnegative integers \( k \) and \( \ell \), we let \( A_{k,\ell,M} \) be the vector space of stable Jacobi forms of weight \( k \) and index \( \ell M \). We put
\[ A(M) := \bigoplus_{\ell=0}^{\infty} \bigoplus_{k=0}^{\infty} A_{k,\ell,M}. \]
Then we see easily that
\[ A_{k,\ell,M} \ast A_{p,q,M} \subset A_{k+p, (\ell+q)M} \]
with respect to the natural multiplication \( \ast \). Thus \( A(M) \) is a bi-graded ring. Let \( I(M) \) be the space of all stable Schottky-Jacobi forms for the universal hyperelliptic locus contained in \( A(M) \). Then \( I(M) \) is an ideal of \( A(M) \).
According to Theorem 3.1, the subset

$$A(M)_0 := \bigoplus_{k=0}^{\infty} A_{k,0}$$

of $A(M)$ has a polynomial ring structure.

Let

$$A^{[4]}(M)_1 := \bigoplus_{k \equiv 0 \pmod{4}} A_{k,M}$$

and let $B^{[4]}(M)_1$ be the subspace of all stable Schottky-Jacobi forms for the universal hyperelliptic locus contained in $A^{[4]}(M)_1$. Using Theorem 4.2 in [3], we can show that

$$(\Theta_P - \Theta_Q) \Theta_{2M}$$

is a stable Schottky-Jacobi form for the universal hyperelliptic locus of weight $\frac{1}{2}(m + h)$ and index $M$, that is, $(\Theta_P - \Theta_Q) \Theta_{2M} \in B^{[4]}(M)_1$. Here $P$ and $Q$ are two positive even unimodular quadratic forms of the same rank $m$ ($m \in \mathbb{Z}^+$), and $\Theta_P$, $\Theta_Q$ are stable modular forms that are defined in Formula (3.1). The subspace $B^{[4]}(M)_1$ of $A^{[4]}(M)_1$ is spanned by all the stable Jacobi forms of type (5.1), where $m$ runs over the set of all positive integers $8n$ ($n \in \mathbb{Z}^+$).

**Question 3.** What kinds of structures does $A(M)$ have?

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Department of Mathematics, Inha University, Inchon 222212, Korea
E-mail address: jhyang@inha.ac.kr