Decomposition of some Reshetikhin-Turaev representations into irreducible factors

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Abstract

We give the decomposition into irreducible factors of the $SU(2)$ Reshetikhin-Turaev representations of the mapping class group of surfaces when the level is $p = 4r$ or $p = 2r^2$ with $r$ an odd prime or when $p = 2r_1r_2$ with $r_1, r_2$ two distinct odd primes, under certain technical assumptions.

Keywords: Reshetikhin-Turaev representations, mapping class group, quantum representations, Topological Quantum Field Theory.

1 Introduction

Witten gave in [Wit89] convincing arguments for the existence of Topological Field Theories, as defined in [Ati88, Wit88], giving a three dimensional interpretation of the Jones polynomial when the gauge group is $SU(2)$. Each of these TQFTs gives a family of projective finite dimensional representations of the mapping class group $\text{Mod}(\Sigma_g)$ of a genus $g$ closed oriented surface $\Sigma_g$. Reshetikhin and Turaev gave a rigorous construction of these TQFTs [RT91] using representations of quantum groups. In this paper we will follow the skein theoretical construction of Lic91, BHMV95 to define these representations.

We can lift these projective representations to linear representations of some central extension $\tilde{\text{Mod}}(\Sigma_g)$ of $\text{Mod}(\Sigma_g)$ noted:

$$\rho_{p,g} : \tilde{\text{Mod}}(\Sigma_g) \to \text{GL}(V_{p,g}).$$

Here $p = 2(k + 2) \geq 3$ is an even integer indexing the representations and $V_{p,g}$ is a finite dimensional complex vector space. These representations are equipped with an invariant scalar product $\langle \cdot, \cdot \rangle_{p,g}$ with respect to which they are unitary.

The goal of this paper is to decompose some of these representations into irreducible factors. Only few results are known concerning their decomposition. In BHMV95, an explicit proper submodule of $V_{p,g}$ is given whenever 4 divides $p$. In [Rob01] it is shown that $V_{p,g}$ is irreducible when $p$ is an odd prime. Robert’s proof extends word-by-word to show that the modules $V_{18g}$ are also irreducible. In [AF10] the authors showed that for $p = 24, 36, 60$ then...
V_{p,g} contains at least three invariant submodules. Finally we gave in [Kor13] an explicit decomposition into irreducible factors of the modules V_{p,1} for arbitrary level p ≥ 3.

The main results of this paper are summarized in the two following Theorems:

**Theorem 1.1.**

1. If r is an odd prime, then V_{4r,2} is the sum of two irreducible subrepresentations.
2. If r is an odd prime, then V_{2r,2} is irreducible.
3. If r_1, r_2 are two distinct odd primes, then V_{2r_1r_2,2} is irreducible.

Given a level p ≥ 3, there exists a set of complex numbers called 6j-symbols at level p which will be defined in the next section. We refer to [MV94] for explicit formulas.

**Definition 1.2.** We say that a level p ≥ 3 is generic if every 6j-symbol at level p is not null.

**Theorem 1.3.**

1. The modules V_{18,g} are irreducible for arbitrary g ≥ 2.
2. If 50 is generic then the module V_{50,3} is irreducible.
3. If r is an odd prime, p = 4r is generic and g < r − 2, then V_{4r,g} is sum of two irreducible subrepresentations.
4. If r_1, r_2 are two distinct odd primes, p = 2r_1r_2 is generic and 2g < min(r_1, r_2), then V_{2r_1r_2,g} is irreducible.

Unfortunately, the author did not found any better way to test genericity than using a computer. However we obtained:

**Corollary 1.4.** (Computer assisted proof):

1. The modules V_{50,3} and V_{77,3} are irreducible.
2. The modules V_{28,g} for 3 ≤ g ≤ 4, V_{44,g} for 3 ≤ g ≤ 8 and V_{58,g} for 3 ≤ g ≤ 10 decompose into two irreducible submodules.

**Remark.** In [BHMV95] some representations ρ_{p,g} are also defined when p is odd. They verify ρ_{2p,g} ≅ ρ_{p,g} ⊗ ρ'_{2,g} where ρ'_{2,g} is the Weil representation at level 2. In particular if an odd level r is such that V_{2r,g} is irreducible, then so is V_{r,g}. This extends the two previous theorems to the SO(3) cases as well.

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2 Skein construction of the Reshetikhin-Turaev representations

Following [BHMV95], we will briefly define the representations $\rho_{p,g}$ and fix some notations.

2.1 The spaces $V_{p,g}$

Given an even integer $p \geq 6$, we denote by $A \in \mathbb{C}$ an arbitrary primitive $2p$-th roots of unity. Using the Kauffman skein relation of Figure 1, we associate to any framed link $L \subset S^3$ an invariant $\langle L \rangle_p \in \mathbb{C}$.

![Figure 1: Skein relations defining the framed link invariants.](image)

Choose $g \geq 1$ and denote by $C_g$ the set of isotopy classes of framed links (including the empty link) in an oriented genus $g$ handlebody $H_g$. We fix a genus $g$ Heegaard splitting of the sphere, i.e. an element $S \in \text{Mod}(\Sigma_g)$ and two handlebodies so that:

$$H^1_g \cup_{S \cdot m_1^g - \lambda m_1^g} H^2_g \cong S^3$$

Take $L_1, L_2 \in C_g$ and embed $L_1$ in $H^1_g$ and $L_2$ in $H^2_g$. The above gluing defines a link $L_1 \cup_{S} L_2 \subset S^3$. We call Hopf pairing the bilinear form:

$$(\cdot, \cdot)^H_{p,g} : \mathbb{C}[C_g] \times \mathbb{C}[C_g] \to \mathbb{C}$$

defined by

$$(L_1, L_2)^H_{p,g} := \langle L_1 \cup_{S} L_2 \rangle_p$$

Eventually we define the spaces $V_{p,g}$ as the quotients:

$$V_{p,g} := \mathbb{C}[C_g] / \ker((\cdot, \cdot)^H_{p,g})$$

The vector spaces $V_{p,g}$ are finite dimensional ([BHMV95]) and we can find explicit basis as follows. Let $g \geq 2$, choose a trivalent graph $\Gamma \subset H_g$ so that $H_g$ retracts on $\Gamma$ by deformation. If $g = 1$, $\Gamma$ represents the circle $S^1 \times \{0\} \subset S^1 \times D^2 \cong H_1$. We denote by $E(\Gamma)$ the set of its edges.

A triple $(i, j, k) \in \{0, \ldots, \frac{p-1}{2}\}$ is said $p$-admissible if:
1. \(|i - j| \leq k \leq i + j|

2. \(i + j + k\) is even and is smaller or equal to \(p - 4\).

A map \(\sigma : E(\Gamma) \to \{0, \ldots, \frac{p-1}{2}\}\) is a \(p\)-admissible coloring of \(\Gamma\) if for every three edges \(e_1, e_2, e_3 \in E(\Gamma)\) adjacent to a vertex, the triple \((\sigma(e_1), \sigma(e_2), \sigma(e_3))\) is \(p\)-admissible.

In [Jon83, Wen87] the authors defined some idempotents \(\{f_0, \ldots, f_{\frac{p-4}{2}}\}\) of the Temperley-Lieb algebra with coefficient in \(\mathbb{Q}(A)\) called Jones-Wenzl idempotents. To \(\sigma\) a \(p\)-admissible coloring of \(\Gamma\) we associate a vector \(u_\sigma \in V_{p,g}\) as follows. We replace each edge \(e \in E(\Gamma)\) by the Jones-Wenzl idempotent \(f_\sigma(e)\). If \((e_1, e_2, e_3)\) are three edges adjacent to a vertex of \(\Gamma\), we connect the idempotents using the link \(T_{\sigma(e_1), \sigma(e_2), \sigma(e_3)}\) defined in Figure 2.

![Figure 2: The link \(T_{i,j,k}\) used to connect three idempotents \(f_i, f_j\) and \(f_k\). The numbers above each three arcs denotes the number of parallel copies of the arc used to define the link.](image)

The Theorem 4.11 of [BHLMV95] asserts that the elements \(u_\sigma\) for \(\sigma\) a \(p\)-admissible coloring of \(\Gamma\), form a basis of \(V_{p,g}\). Moreover there exists a non-degenerate bilinear form \((\cdot, \cdot)_{p,g}\) on \(V_{p,g}\) invariant under the action of \(\widetilde{\text{Mod}}(\Sigma_g)\), for which the vectors \(u_\sigma\) are pairwise orthogonal.

The basis \(u_\sigma\) depends on the choice of the trivalent graph. We can transform a trivalent graph into one another by a sequence of Whitehead moves. Suppose that \(\Gamma_1\) and \(\Gamma_2\) are two trivalent graphs of genus \(g \geq 2\), which only differ by a single Whitehead move, inside a ball \(B^3\), as drawn in Figure 3.

Fix a \(p\)-admissible coloring of the graphs outside \(B^3\) and denote by \(\sigma(i)\) (resp. \(\sigma(j)\)) the vector associated to the coloration of \(\Gamma_1\) (resp of \(\Gamma_2\)) with the edge \(i\) colored by \(\sigma(i)\) (resp with the edge \(j\) colored by \(\sigma(j)\)).

Then the vectors \(\sigma(i)\) belong to the subspace spanned by the vectors \(\sigma(j)\)
and decompose using the so-called ‘fusion rules’ formula:

\[
\begin{array}{c}
\tau_i = \sum_j \left\{ \begin{array}{cccc}
 a & b & j \\
 c & d & i
\end{array} \right\}
\end{array}
\]

(1)

where the sum runs through \( p \)-admissible colorings and the coefficient \( \left\{ \begin{array}{cccc}
 a & b & j \\
 c & d & i
\end{array} \right\} \) only depends on the colors of the edges \( a, b, c, d, i \) and \( j \) and is called recoupling coefficient or 6j-symbol in literature. We refer to [MV94] for a proof and an explicit computation of these coefficients. Numerical experiments suggest that they are never null.

2.2 The Reshetikhin-Turaev representations

We fix an orientation preserving homeomorphism

\[ \alpha : \Sigma_g \to \partial H_g \]

Choose a class \( \phi \in \text{Mod}(\Sigma_g) \) associated to a homeomorphism which extends to \( H_g \) through \( \alpha \). Then \( \phi \) acts on \( C_g \) and preserves the kernel of the Hopf pairing so acts on \( V_{\rho, g} \) by passing to the quotient. Denote by \( \tilde{\rho}_{\rho,g}(\phi) \in \text{GL}(V_{\rho, g}) \) the resulting operator.

Now choose \( \phi \in \text{Mod}(\Sigma_g) \) so that the corresponding homeomorphisms extend to \( H_g \) through \( \alpha \circ S \). This extension also defines, by quotient, an operator on \( V_{\rho, g} \). We denote by \( \tilde{\rho}_{\rho,g}(\phi) \) the dual of this operator for the Hopf pairing.

The elements of \( \text{Mod}(\Sigma_g) \) which extend to \( H_g \) either through \( \alpha \) or through \( \alpha \circ S \), generate the whole group \( \text{Mod}(\Sigma_g) \). It is a non trivial fact that the associated operators \( \tilde{\rho}_{\rho,g}(\phi) \) generate a projective representation:

\[ \tilde{\rho}_{\rho,g} : \text{Mod}(\Sigma_g) \to \text{PGL}(V_{\rho, g}) \]

We consider a central extension \( \tilde{\text{Mod}}(\Sigma_g) \) of \( \text{Mod}(\Sigma_g) \) that lifts the above projective representations to linear ones (see [MR95, GM09]):

\[ \rho_{\rho,g} : \tilde{\text{Mod}}(\Sigma_g) \to \text{GL}(V_{\rho, g}) \]
These are the so-called Reshetikhin-Turaev representations.

Now to each edge $e \in E(\Gamma)$, choose a disc $D_e$ properly embedded in $H_\Sigma$, that intersects $\Gamma$ transversely once in $e$. Note that the set of boundary curves $\gamma_e := \partial D_e \subset \partial H_\Sigma \rightarrow \Sigma_\Sigma$ forms a pants decomposition of $\Sigma_\Sigma$.

A classical property of the Jones-Wenzl idempotents asserts that, if $T_e \in \text{Mod}(\Sigma_\Gamma)$ denotes the Dehn twist along $\gamma_e$, then:

$$\tilde{\rho}_{p,\Sigma}(T_e) \cdot u_\sigma = \mu_\sigma(e) u_\sigma$$

where $\mu_i := (-1)^{i(i+2)}$.

We fix the lift of $T_e$ in $\tilde{\text{Mod}}(\Sigma_\Sigma)$, still denoted $T_e$, so that $\rho_{p,\Sigma}(T_e) \cdot u_\sigma = \mu_\sigma(e) u_\sigma$.

We also fix the lift $S \in \tilde{\text{Mod}}(\Sigma_\Sigma)$ so that the matrix of $\rho_{p,\Sigma}(S)$ is the matrix of the Hopf pairing $(\cdot|\cdot)_{p,\Sigma}$ multiplied by an element $\eta \in \mathbb{C}$ which verifies $|\eta| = |\frac{\sqrt{p}}{A_{-i}^{-2}}|$. We refer to [BHMV95], where $\eta$ represents the quantum invariant of $S^3$, for a detailed discussion on $\eta$.

Since $S$ and the $\{T_e\}_{e \in E(\Gamma)}$ generate $\tilde{\text{Mod}}(\Sigma_\Sigma)$ for some trivalent graphs, we have an explicit description of $\rho_{p,\Sigma}$.

## 3 Cyclicality of the vacuum vector

Denote by $\mathcal{A}_{p,\Sigma}$ the subalgebra of $\text{End}(V_{p,\Sigma})$ generated by the operators $\rho_{p,\Sigma}(\phi)$ for $\phi \in \tilde{\text{Mod}}(\Sigma_\Sigma)$. The key ingredient to prove Theorem 1.1 is to show that the vacuum vector $v_0 \in V_{p,\Sigma}$, associated to the class of the empty link, is cyclic, i.e. that $\mathcal{A}_{p,\Sigma} \cdot v_0 = V_{p,\Sigma}$.

### 3.1 The genus one case

In [Kor13] we gave an explicit decomposition of the Weil representations into irreducible factors. An easy generalization of the arguments of the proof of Lemma 3 of [FK06] leads to an explicit isomorphism of $SL_2(\mathbb{Z})$-modules between $V_{p,1}$ and the odd submodule of the Weil representation at level $p$. Proving that $v_0 \in V_{p,1}$ is cyclic reduces to show that its projection on each irreducible submodule of $V_{p,1}$ is not null.

Denote by $\{u_0, \ldots, u_{p-1}\}$ the basis of $V_{p,1}$ where $u_i$ is the class of the closure of the $i$-th Jones-Wenzl idempotent along a longitude in $H_1$. Also denote by $\{e_i, i \in \mathbb{Z}/p\mathbb{Z}\}$ the basis of the Weil $SL_2(\mathbb{Z})$-module $U_p$ at level $p$ as described in [Kor13].

In this basis, the Weil projective representations in genus one are defined by the matrices:

$$\pi_p(S) = \frac{1}{\sqrt{p}}(A^{-i}j)_{i,j \in \mathbb{Z}/p\mathbb{Z}}$$

$$\pi_p(T) = (A^2 \delta_i)_{i,j \in \mathbb{Z}/p\mathbb{Z}}$$
Here the level is an integer \( p \geq 2 \) not necessary even. When \( p \) is even, we take \( A \) to be a primitive \( 2p - th \) roots of unity. When \( p \) is odd, \( A \) is a primitive \( p - th \) roots of unity.

The vectors \( \{ e_i^r := e_i - e_{-i}, i \in \{1, \ldots, \frac{p-2}{2}\} \} \) span a submodule \( U_p^- \subset U_p \).

**Lemma 3.1.** Let \( p = 2r \geq 6 \) be an even integer. Then the following map:

\[
\Psi : \begin{cases}
U_p^- & \rightarrow \ V_{p,1} \\
e_i^r & \mapsto u_{i+r-1}
\end{cases}
\]

is an isomorphism of \( SL_2(\mathbb{Z}) \)-projective modules.

**Proof.** We compute the matrix elements:

\[
\langle \psi(e_i^r), \rho_{p,1}(S)\psi(e_i^r) \rangle = \frac{\eta \cdot (-1)^{i+j}}{A^2 - A^{-2}} (A^{2(i+j)} - A^{-2(i+j)})
\]

\[
= \frac{\eta \cdot (-1)^r \sqrt{p}}{A^2 - A^{-2}} \langle e_i^r, \pi_p(S)e_i^r \rangle
\]

where the scalar \( \frac{\eta \cdot (-1)^r \sqrt{p}}{A^2 - A^{-2}} \) has norm one.

\[
\langle \psi(e_i^r), \rho_{p,1}(T)\psi(e_i^r) \rangle = (-1)^{r-1} A^{-1} \langle e_i^r, \pi_p(T)e_i^r \rangle
\]

\[
= (-1)^{r-1} A^{-1} \langle e_i^r, \pi_p(T)e_i^r \rangle
\]

\[\square\]

The decomposition into irreducible submodules of \( U_p \) is described by the following:

**Proposition 3.2 ([Kor13]).** We have the following decompositions where \( \cong \) denotes an isomorphism of \( SL_2(\mathbb{Z}) \)-modules:

1. If \( a \) and \( b \) are coprime, then \( U_{ab} \cong U_a \otimes U_b \).
2. If \( r \) is prime and \( n \geq 1 \), then \( U_{r^n} \cong U_r \otimes W_{r^n} \) where \( W_{r^n} \) denotes another module.
3. If \( r \) is an odd prime, then \( U_r \cong 1 \otimes W_r \) where \( 1 \) is the trivial representation.
4. The modules \( U_p \) for \( r > 2 \) and \( W_r \) split into two submodules: \( U_p \cong U_p^- \oplus U_p^+ \), \( W_r \cong W_r^- \oplus W_r^+ \).
5. The modules \( B_1 \otimes \ldots \otimes B_k \), where the \( B_i \) have the form \( U_r^*, U_r^-, U_2, U_4^*, U_4^-, W_r^+ \) or \( W_r^- \) and have pairwise coprime levels, are irreducible.

We can now prove:

**Proposition 3.3.** Let \( p \geq 6 \) be an even integer. Then the vacuum vector \( v_0 \in V_{p,1} \) is cyclic if and only if one of the following three cases holds:

- \( p = 2r_1 \ldots r_k \) with \( r_i \) distinct odd primes.
- \( p = 2r^2 \) with \( r \) prime.
• $p = 4r$ with $r$ prime.

Proof. We will use Proposition 3.2 and the explicit isomorphisms given in the main Theorem of [Kor13] to study whether the vector $v := \psi^{-1}(v_0) = e_{r-1} - e_{r} \in U_p^-$ has non trivial projection on each submodule of $U_p^-$ or not.

Given two integers $x$ and $n$, we will denote by $[x]_n \in \mathbb{Z}/n\mathbb{Z}$ the class of $x$ modulo $n$. We write

$$v = e_{x} - e_{x-1} \in U_p^-$$

with $x = \frac{p-2}{2}$.

First, when $p = 2r^2$, with $r$ prime, the module $U_p^-$ is irreducible so the vector is cyclic.

When $p = 4r$, with $r$ an odd prime, the module decomposes into two irreducible submodules:

$$U_p^- \cong U_4^- \oplus U_r^+ \oplus U_4^+ \oplus U_r^-$$

The vector $v$ decomposes as follows:

$$v = e_{x} \oplus e_{x-1} \oplus e_{x-1} \oplus e_{x}$$

$$= \left(\frac{1}{2}(e_{x} - e_{x-1}) \otimes (e_{x} + e_{x-1})\right)$$

$$+ \left(\frac{1}{2}(e_{x} + e_{x-1}) \otimes (e_{x} - e_{x-1})\right)$$

Where the first term lies in $U_4^- \otimes U_r^+$ and the second in $U_r^+ \otimes U_r^-$. Since $x = 2r - 1$, neither 4 nor $r$ divide $x$, so $[x]_4 \neq [x]_4$ and $[x]_r \neq [x]_r$, and the two projections are not null.

When $p = 2r_1 \ldots r_s$, with $r_i$ distinct odd primes, we have the following decomposition:

$$U_p^- \cong \bigoplus_{e = (e_1, e_2 + 1)^s} X_e$$

where

$$X_e := U_2 \otimes U_{r_1}^{e_1} \otimes \ldots \otimes U_{r_s}^{e_s}$$

Let us fix $e$ and denote:

$$e_e := e_{[x]} \otimes e_{[x]}^{e_1} \otimes \ldots \otimes e_{[x]}^{e_s} \in X_e$$

where we used the notation $e_{[x]}^\pm := e_{x} \pm e_{-x}$. By using the facts that $\langle e_i, e_i \rangle = 1$ and $\langle e_{-i}, e_i \rangle = (-1)^{\frac{1+d}{2}}$, we compute:

$$\langle v, e_e \rangle = \left(e_{x} \otimes e_{x} \otimes \ldots \otimes e_{x} \otimes e_{x}^{e_1} \otimes \ldots \otimes e_{x}^{e_s}\right)$$

$$- \left(e_{x} \otimes e_{x-1} \otimes \ldots \otimes e_{x-1} \otimes e_{x}^{e_1} \otimes \ldots \otimes e_{x}^{e_s}\right)$$

$$= 1 - (-1)^{\frac{s+1}{2}} = 2 \neq 0$$
So the projection of $v$ on each irreducible submodule $X_\epsilon$ is not null.

Now suppose that $p = 2r_1^{n_1} \ldots r_k^{n_k}$ with $k \geq 2$, $r_i$ distinct primes and $n_1 \geq 2$. Since $r_1$ does not divide $x$, the vector $v$ has a null projection on the submodule:

$$
\begin{cases}
U_{r_1^{n_1}}^* \otimes U_{2^{n_2} \ldots r_k^{n_k}}^- & \text{if } r_1 \neq 2, \\
U_{2r_2}^* \otimes U_{r_1^{n_1} \ldots r_k^{n_k}}^- & \text{if } r_1 = 2
\end{cases}
$$

Next if $p = 2^n$, with $r$ an odd prime and $n \geq 2$, the projection of $v$ on $U_2 \otimes U_{n-1}$ is null.

Finally if $p = 2^n$, with $n \geq 3$, the projection of $v$ on $U_{2^{n-2}}$ is null. □

3.2 Cyclicity in higher genus

The goal of this subsection is to prove the following:

**Proposition 3.4.** When $g \geq 2$, the vacuum vector $v_0 \in V_{p,g}$ is cyclic in the following cases:

1. When $p = 4r$ with $r$ an odd prime and if $g = 2$ or if $p$ is generic and $g < r - 2$.
2. When $p = 2r^2$ with $r$ an odd prime and $g = 2$ or if $p = 50$ and $g = 3$.
3. When $p = 2r_1 r_2$ with $r_1, r_2$ distinct odd primes and $g = 2$ or if $p$ is generic and $2g < \min(r_1, r_2)$.

Fix a trivalent graph $\Gamma \subset H_g$ as in section 2. Two $p$-admissible colorings $\sigma_1, \sigma_2$ of $\Gamma$ will be said equivalent if:

$$(-1)^{r(e)} A^{\sigma_1(e)\sigma_2(e)+2} = (-1)^{\sigma_1(e) A^{\sigma_2(e)\sigma_2(e)+2}}$$

for all $e \in E(\Gamma)$

We denote by $\text{col}_p(\Gamma)$ the set of equivalence classes of colorings for this relation.

To $[\sigma] \in \text{col}_p(\Gamma)$, we associate the subspace:

$$W_{[\sigma]} := \text{Span}\{u_{\sigma'}, \sigma' \in [\sigma]\} \subset V_{p,g}$$

**Lemma 3.5.** If $X \subset V_{p,g}$ is a Mod($\Sigma_g$)-submodule, then:

$$X = \bigoplus_{[\sigma] \in \text{col}_p(\Gamma)} X \cap W_{[\sigma]}$$

**Proof.** The matrices $p_{p,g}(T_e)$, for $e \in E(\Gamma)$, generate a commutative subalgebra of $A_{p,g}$. The set $\text{col}_p(\Gamma)$ indexes its characters and the spaces $W_{[\sigma]}$ are the associated common eigenspaces of the $p_{p,g}(T_e)$. The orthogonal projector on $X$ must commute with the $p_{p,g}(T_e)$ and thus preserves the subspaces $W_{[\sigma]}$. □
The strategy to prove Proposition 3.4 is to apply Lemma 3.5 to

\[ X := (\mathcal{A}_{p,g} \cdot v_0)^\perp \]

the orthogonal (for the invariant form) of the cyclic space generated by the vacuum vector.

**Definition 3.6.**

1. We call \( \Gamma_g \) a *fly eyes graph* of genus \( g \) if it is a trivalent graph obtained by the following inductive method:
   - \( \Gamma_2 \) is the Theta graph \( \quad \)
   - A graph \( \Gamma_{g+1} \) is obtained from a \( \Gamma_g \) by choosing arbitrary a vertex and inserting a triangle as drawn on the left-hand side of Figure 4.

The right-hand side gives an example of a genus 8 fly eyes graph.

![Figure 4](image)

Figure 4: On the left: the operation transforming a fly eyes graph of genus \( g \) into a one of genus \( g + 1 \). On the right: an example of genus 8 fly eye graph.

2. The genus 3 fly eyes graph is unique and is called the tetrahedron graph. We say that a level \( p \geq 3 \) is generic if for any coloring \( \sigma \) of \( \Gamma_3 \), we have:

\[ (u_\sigma, v_0)^H_{p,3} \neq 0 \]

The complex numbers \( (u_\sigma, v_0)^H_{p,3} \) are called tetrahedron coefficients in literature and are related to the \( 6j \)-symbols defined in the previous section. In particular it is equivalent to say that the \( 6j \)-symbols or the tetrahedron coefficients are not null for a level \( p \). It follows from fusion-rules (equation (1)) that if \( p \) is generic, then for any \( g \geq 3 \), for any fly eyes graph \( \Gamma_g \) and for any \( p \)-admissible coloring \( \sigma \) of \( \Gamma_g \), we have:

\[ (u_\sigma, v_0)^H_{p,3} \neq 0 \]

Fix \( g \geq 2 \) and embed a fly eyes graph \( \Gamma_g \) in \( S^3 \). Denote by \( H_g \) the embedded handlebody

\[ H_g := S^3 \setminus V(\Gamma_g) \]
where \( V(\Gamma_g) \) denotes a tubular neighborhood of \( \Gamma_g \). For each edge \( e \in E(\Gamma_g) \), fix a curve \( \gamma_e \subset H_g \) which bounds a disc intersecting \( \Gamma_g \) only once along \( e \).

We construct a map:

\[
w : N\!E(\Gamma_g) \to V_{p,g}
\]
as follows. To \( f : E(\Gamma_g) \to \mathbb{N} \) we associate the class in \( V_{p,g} \) of the link made of \( f(e) \) parallel copies of \( \gamma_e \) for each edge \( e \in E(\Gamma_g) \).

When \( g = 2 \), we will note \( w_{a,b,c} \in V_{p,2} \) the class of the link made of \( a \) parallel copies of \( \gamma_1 \), \( b \) copies of \( \gamma_2 \) and \( c \) of \( \gamma_3 \).

The Figure 5 shows the curves \( \gamma_e \) when \( g = 2 \) and \( g = 3 \).

Lemma 3.7. If \( p = 4r \), with \( r \) an odd prime, or if \( p = 2r_1r_2 \), with \( r_1, r_2 \) two distinct odd primes, then:

\[
w_{a,b,c} \in \mathcal{A}_{p,2} \cdot v_0, \text{ for any } a, b, c \in \{0, 1\}
\]

Proof. Choose a longitude \( L \) and a meridian \( M \) of \( \Sigma_1 \). The space \( \mathcal{A}_{p,1} \cdot v_0 = V_{p,1} \) is generated by juxtaposition of properly embedded parallel copies of \( L \) and \( M \) in \( H_1 \), colored by the element \( \omega \in V_{p,1} \) as defined in [BHMV92].

By embedding the skein elements \( L(\omega) \) and \( M(\omega) \) in \( H_2 \cong H_1 \# H_1 \) in both handles, we see that the vectors \( w_{i,j,i} \) belong to \( \mathcal{A}_{p,2} \cdot v_0 \) for arbitrary \( i \) and \( j \). So do the vectors \( w_{i,j,0} \) by action of the mapping class group.

It remains to show that \( w_{1,1,1} \in \mathcal{A}_{p,2} \cdot v_0 \). It follows from the definition of Jones-Wenzl idempotents that:

\[
w_{1,1,1} = 2 \left( \begin{array}{c} 2 \\ 2 \end{array} \right) + w_{2,0,0} + w_{0,2,0} + (A^2 + A^{-2})v_0
\]

Thus we just have to show that \( 2 \left( \begin{array}{c} 2 \\ 2 \end{array} \right) \in \mathcal{A}_{p,2} \cdot v_0 \).
Using fusion rules (see [MV94]), we have that:

\[
\begin{array}{c}
2 \quad 0 \quad 2 \\
\end{array}
= \sum_{k=0,2,4} \left\{ \begin{array}{ccc} 2 & 2 & k \\ 2 & 2 & 0 \end{array} \right\}
\begin{array}{c}
2 \quad k \quad 2 \\
\end{array}
\]

where \( T_e \) is (a lift of) the Dehn twist around the middle edge of the Theta graph (labeled 0).

Since both vectors belong to \( V_{p,2} \cdot v_0 \) and since the recoupling coefficients \( \left\{ \begin{array}{ccc} 2 & 2 & k \\ 2 & 2 & 0 \end{array} \right\} \) and \( \left\{ \begin{array}{ccc} 2 & 2 & k \\ 2 & 2 & 2 \end{array} \right\} \) are not zero and \( \mu_2 \neq 1 \), we know that the 3-dimensional space generated by

\[
\begin{array}{c}
2 \quad 0 \quad 2 \\
\end{array}
\begin{array}{c}
2 \quad 2 \quad 2 \\
\end{array}
\begin{array}{c}
2 \quad 4 \quad 2 \\
\end{array}
\]

is included in \( A_{p,2} \cdot v_0 \). So does the vector \( 2 \begin{array}{c} 2 \end{array} \).

\[\square\]

**Lemma 3.8.** When \( p = 2r^2 \), with \( r \) an odd prime, then

\[
\begin{array}{c}
2 \quad 0 \quad 2 \\
\end{array}
\begin{array}{c}
2 \quad 2 \quad 2 \\
\end{array}
\begin{array}{c}
2 \quad 4 \quad 2 \\
\end{array}
\end{array}
\in A_{p,2} \cdot v_0,
\]

for all \( 0 \leq a, b, c \leq \frac{r-3}{2} \).

Moreover, if \( \sigma \) is a \( p \)-admissible coloring of \( \Gamma = \begin{array}{c} \end{array} \) such that:

\[
\sigma(e) \equiv -1 \pmod{r}, \text{ for all } e \in E(\Gamma)
\]

then \( u_\sigma \in A_{p,2} \cdot v_0 \).

**Proof.** Note first that \( i, j \in \left\{ 0, \ldots, \frac{p-1}{2} \right\} \) are such that:

- \( \mu_i = \mu_j \),
- \( i \neq j \),

if and only if \( i \equiv j \equiv -1 \pmod{r} \) and \( i \) and \( j \) have same parity (and are distinct).

Thus when \( \sigma \) satisfies the condition of the Lemma, the subspace \( W_{[a]} \) is one-dimensional. The Lemma 3.8 implies that this subspace is either in \( A_{p,2} \cdot v_0 \), or in its orthogonal. Now note that the Hopf pairing \( (u_\sigma, v_0)_H \) is not zero for it is equal to a 3j-symbol. This prove the second part of the Lemma.

In particular, we just proved that:

\[
\text{Span} \left\{ u \begin{array}{c} v \end{array} w \mid 0 \leq u, v, w \leq r-2 \right\} \subset A_{p,2} \cdot v_0
\]

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We finish the proof by noticing that the vector $w_{a,b,c}$ belongs to this space whenever we have:
\[
\begin{align*}
  a + c & \leq r - 2 \\
  b + c & \leq r - 2 \\
  a + b & \leq r - 2
\end{align*}
\]
□

Lemma 3.9. The vector $w_f$ belongs to $A_{p,g} \cdot v_0$ for $f \in \{0,1\}^{E(\Gamma)}$ when $p$ is generic and:

- $p = 4r$ with $r$ an odd prime such that $g \leq r - 2$.
- $p = 2r_1r_2$ with $r_1, r_2$ distinct odd primes and $2g \leq \min(r_1, r_2)$.
- $p = 50$ and $g = 3$.

Proof. We proceed like in the proof of Lemma 3.8: first we note that if $f \in \{0,1\}^{E(\Gamma)}$, then:
\[
w_f \in \text{Span}(u_\sigma, 0 \leq \sigma(e) \leq g \text{ for all } e \in E(\Gamma_g))
\]
Then we note that if $\sigma$ is such that $0 \leq \sigma(e) \leq g$ for all $e \in \Gamma_g$, then $W_{[\sigma]}$ is one-dimensional so is included in $A_{p,g} \cdot v_0$ for $(u_\sigma, v_0)^H_{p,g} \neq 0$ by assumption. The fact that these $W_{[\sigma]}$ are one-dimensional is deduced from the following two facts:

1. When $p = 4r$, and $i, j \in \{0, \ldots, \frac{p^2}{2}\}$, then $\mu_i = \mu_j$ if and only if:
   - $i = j$,
   - or $i = \frac{p^2}{2} - j$ and $i$ is even.

2. When $p = 2r_1r_2$, and $i, j \in \{0, \ldots, \frac{p^2}{2}\}$, then $\mu_i = \mu_j$ if and only if:
   - $i = j$,
   - or $j$ is the only element satisfying
     \[
     \begin{align*}
     i & \equiv j \pmod{2r_1} \\
     i & \equiv -j - 2 \pmod{r_1} \\
     i & \equiv j \pmod{2r_2} \\
     i & \equiv -j - 2 \pmod{r_2}
     \end{align*}
     \]
□

Proof of Proposition 3.4. Fix a fly eyes graph $\Gamma$, a class $[\sigma] \in \text{col}(\Gamma)$, and choose a vector
\[
v = \sum_{\sigma' \in [\sigma]} \alpha_{\sigma'} u_{\sigma'} \in W_{[\sigma]} \bigcap (A_{p,g} \cdot v_0)^\perp
\]
By Lemma 3.5, we must show that $v = 0$ to conclude. We will find dim $(W_{[\sigma]})$ independent equations verified by the coefficients $\alpha_{\sigma'}$.

Note $F \subset \text{N}^{E(\Gamma)}$ the set of functions $f$ so that:

- $f(e) \in \{0,1\}, \forall e \in E(\Gamma)$, if $p = 4r$ or $p = 2r_1r_2$,
- $f(e) \in \{0, \ldots, \frac{p^2}{2}\}, \forall e \in E(\Gamma)$, if $p = 2r_2$. 

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Using Lemmas 3.7, 3.8 and 3.9, we know that

\[ w_f \in \mathcal{A}_{p,g}, \quad \forall f \in F \]

By definition of \( v \), we have that:

\[
\left( w_f, v_0 \right)_{p,g}^H = 0, \quad \text{for all } f \in F \quad (2)
\]

\[
\Leftrightarrow \sum_{f \in \sigma} \left( \prod_{e \in E} \lambda^{f(e)}_{\sigma'(e)} \right) a_{\sigma'} (u_{\sigma'}, v_0)_{p,g} = 0, \quad \text{for all } f \in F 
\]

where \( \lambda_i = -\left(A^{2i+1} + A^{-2i+1}\right) \).

Since the complex numbers \( (u_{\sigma'}, v_0)_{p,g} \) are non null when \( p \) is generic, it is enough to show that the matrix:

\[
M := \left( \prod_{e \in E} \lambda^{f(e)}_{\sigma'(e)} \right)_{f \in F}
\]

has independent lines to conclude the proof.

We now define an invertible square matrix \( \tilde{M} \) such that \( M \) is obtained from \( \tilde{M} \) by removing some lines.

When \( i \in \{0, \ldots, \frac{p-4}{2}\} \) we define the set:

\[
\omega(i) := \left\{ j \in \{0, \ldots, \frac{p-4}{2}\}, \text{ so that } \mu_i = \mu_j \right\}
\]

And the Vandermonde matrix:

\[
N[i] := \left( \lambda^j_{\omega(i)} \right)_{0 \leq j \leq \#\omega(i) - 1}
\]

Since \( \lambda_i \neq \lambda_j \) when \( i \neq j \in \omega(i) \), the matrix \( N[i] \) is invertible.

Now note \( E(\Gamma) = \{ e_1, \ldots, e_{3g-3} \} \) and choose \( \sigma \in [\sigma] \) arbitrary. The matrix

\[
\tilde{M} := N(e_1) \otimes \ldots \otimes N(e_{3g-3})
\]

is clearly invertible and \( M \) is obtained from \( \tilde{M} \) by removing the lines corresponding to non \( p \)-admissible colorings of \( \Gamma \).

4 Decomposition into irreducible factors

In this section, we will prove the Theorems 1.1 and 1.3. Denote by \((\mathcal{A}_{p,g})'\) the commutant of the algebra \( \mathcal{A}_{p,g} \), i.e. the subspace of \( \text{End}(V_{p,g}) \) of operators commuting with all the \( p_{p,g}(\phi) \) for \( \phi \in \text{Mod}(\Sigma_g) \).

The dimension of \((\mathcal{A}_{p,g})'\) is equal to the number of irreducible submodules of \( V_{p,g} \). We thus have to show that \( \text{dim}(\mathcal{A}_{p,g})' \) is one if \( p = 2r^2 \) and \( p = 2r_1r_2 \) and is two when \( p = 4r \) with the additional assumptions of the two Theorems.
Consider the following linear map:

\[ f : \left\{ \frac{\mathcal{A}_{p,g}}{\theta} \right\} \rightarrow V_{p,g} \]

The cyclicity of \( v_0 \) (Proposition 3.4) implies that \( f \) is injective. Moreover if \( \phi \in \text{Mod}(\Sigma_g) \) is the lift of a homeomorphism of \( \Sigma_g \) that extends to \( H_g \) through \( \alpha : \Sigma_g \rightarrow \partial H_g \), then:

\[ \rho_{p,g}(\phi) \cdot v_0 = v_0 \]

Denote by \( \widetilde{\text{Mod}}(H_g) \subset \text{Mod}(\Sigma_g) \) the subgroup generated by these \( \phi \). By definition, we have:

\[ \text{Range}(f) \subset \left\{ v \in V_{p,g} \text{ so that } \rho_{p,g}(\phi) \cdot v = v, \text{ for all } \phi \in \widetilde{\text{Mod}}(H_g) \right\} \]

In particular, for any trivalent graph \( \Gamma \), we have \( \text{Range}(f) \subset W_{[0]}(\Gamma) \) where \( [0] \) is the class of the coloring sending every edges of \( \Gamma \) to 0. As an immediate consequence, we get the:

Proof of Theorems 1.1 and 1.3 when \( p = 2r^2 \). When \( p = 2r^2 \), with \( r \) an odd prime, then \( W_{[0]} \) is one-dimensional, generated by \( v_0 \). Thus \( \text{Range}(f) = \{v_0\} \) and \( (\mathcal{A}_{2r^2, g})' = \{1\} \). The Schur Lemma implies that the module \( V_{2r^2, g} \) is irreducible. □

Remark. When \( p = 50 \), we remark that the numbers \( \mu_0, \mu_1, \ldots, \mu_{23} \) are pairwise distinct. The proof of Roberts [Rob01] applies word-by-word in this case to show that \( V_{50,1} \) is irreducible. Indeed the fact that the \( \mu_i \) are distinct implies that the null vector \( v_0 \in V_{50,1} \) is cyclic for the action of the group generated by the Dehn twist along the longitude of \( H_1 \). This easily implies that \( v_0 \in V_{50,1} \) is cyclic for the action of \( \widetilde{\text{Mod}}(\Sigma_g) \) for arbitrary \( g \geq 1 \) and we conclude as above by noticing that \( W_{[0]} \) is one dimension generated by \( v_0 \).

4.1 The case where \( p = 4r \)

Let \( p \geq 3 \) be such that \( p \equiv 4 \pmod{8} \). Consider a link \( L \subset \Sigma_g \times [0, 1] \) inside the cylinder \( \Sigma_g \times [0, 1] \) and color \( L \) by \( p \) parallel copies of \( \omega \) or, equivalently, by the \( (\frac{\pi}{4}) - \text{th} \) Jones-Wenzl idempotent. The gluing of the above cobordism on \( H_g \) induces an operator acting on \( V_{p,g} \). In [BHMV95] it is shown that this operator only depends the homology class of \( L \) in \( H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z}) \) and we get this way an injective morphism of algebras:

\[ i : \mathbb{C}[H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z})] \hookrightarrow \mathcal{A}_{p,g} \]

Its action on \( v_0 \) gives the space \( W_{[0]} \equiv \mathbb{C}[H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z})] \).

We denote by \( P \) the projector of \( V_{p,g} \) on the subspace of vectors fixed by the operators of \( i(\mathbb{C}[H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z})]) \). Clearly \( P \in (\mathcal{A}_{p,g})' \).
Note \( x_i, y_i \in H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z}) \) the meridian and longitude around the \( i \)-th hole and note:

\[
\Theta_i := \frac{1}{\sqrt{2}}(-1 + x_i + y_i + x_i y_i) \in \mathbb{C}[H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z})]
\]

The \( \Theta_i \)'s are symmetries which pairwise commute and

\[
P = \frac{1}{2^g} \left( \sqrt{2}(\Theta_1 + \ldots + \Theta_g) + g + 1 \right)
\]

The symmetric group \( \sigma_g \) acts by permutation on the generators of \( \mathbb{C}[\Theta_1, \ldots, \Theta_g] \). We note \( \mathcal{W}_g \subset i\left( \mathbb{C}[H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z})] \right) \) the subalgebra of \( \mathbb{C}[\Theta_1, \ldots, \Theta_g] \) of elements fixed by \( \sigma_g \).

Finally we denote by \( I \subset i\left( \mathbb{C}[H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z})] \right) \) the ideal generated by the elements \((x_i - 1)\) for \( 1 \leq i \leq g \). We have:

\[
\mathbb{C}[H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z})]/I \cong \mathbb{C}[H_g(\Sigma_g, \mathbb{Z}/2\mathbb{Z})] \equiv \mathcal{W}_{[0]}
\]

**Lemma 4.1.** Consider the action of \( Sp(2^g, \mathbb{Z}/2\mathbb{Z}) \) on \( i\left( \mathbb{C}[H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z})] \right) \). Then:

1. The vectors fixed by this action are the ones of \( \text{Span}(I, P) \).
2. For every \( w \in \mathcal{W}_g \) and \( \phi \in Sp(2^g, \mathbb{Z}/2\mathbb{Z}) \) we have:

\[
\phi \cdot w - w \in I
\]

**Proof.** The first point follows from the fact that the action of \( Sp(2^g, \mathbb{Z}/2\mathbb{Z}) \) on \( H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z}) \) has two orbits: the singleton containing the neutral element and the set containing the other elements. Indeed by taking an appropriate \( \mathbb{Z}/2\mathbb{Z} \)-basis of \( H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z}) \), this action is described by the usual Birman generators of \( Sp(2^g, \mathbb{Z}) \) ([Bir71]) passed to the quotient in \( Sp(2^g, \mathbb{Z}/2\mathbb{Z}) \), that is the \( 2^g \times 2^g \) matrices:

\[
\left( \begin{array}{cc} A & 0_g \\ 0_g & A^* \end{array} \right), \left( \begin{array}{cc} 0_g & B \\ 1_g & 0_g \end{array} \right) \text{ and } \left( \begin{array}{cc} 0_g & 1_g \\ 1_g & 0_g \end{array} \right)
\]

where \( A \in \text{GL}(g, \mathbb{Z}/2\mathbb{Z}) \) and \( B \) is symmetric. We just have to remark that the commutant of the algebra generated by these matrices consists of the scalar matrices to conclude.

To prove the second point, denote by \( X_i, Y_i, Z_{i,j} \) for \( 1 \leq i, j \leq g \) the class in \( H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z}) \) of the Dehn twists of Figure 3 generating \( H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z}) \). First note that the operators \( \Theta_i \) are invariant under the action of the \( X_i \) and \( Y_i \) and that the element of the algebra \( \mathcal{W}_g \) are invariant under permutation of the handles. We are reduced to show that for \( w \in \mathcal{W}_g \) we have \( Z_{i,j} \cdot w - w \in I \).
First note that \( Z_{1,2} \cdot \Theta_i = \Theta_i \) when \( i \not\in \{1, 2\} \). Then we compute:

\[
\begin{align*}
Z_{1,2} \cdot \Theta_1 - \Theta_1 &= \frac{1}{\sqrt{2}}(y_1 + x_1y_1)(x_2 - 1) \\
Z_{1,2} \cdot \Theta_2 - \Theta_1 &= \frac{1}{\sqrt{2}}(y_2 + x_2y_2)(x_1 - 1) \\
Z_{1,2} \cdot (\Theta_1 \Theta_2) - (\Theta_1 \Theta_2) &= \frac{1}{2}((x_1x_2 - 1)(x_1y_1 + y_1)(x_2y_2 + y_2) \\
&\quad + (x_2 - 1)(y_1x_1 + y_1)(-1 + x_2) \\
&\quad + (x_1 - 1)(x_2y_2 + y_2)(-1 + x_1)) \in I
\end{align*}
\]

□

The case \( p = 4r \) of Theorems 1.1 and 1.3 are easily deduced from the:

**Proposition 4.2.** If \( p \equiv 4 \pmod{8} \) and \( v_0 \in V_{p, g} \) is cyclic, then

\[
f^{-1}(\mathcal{C}[H_1(H_{p,g}, \mathbb{Z}/2\mathbb{Z})]) = \text{Span}(1, P)
\]

**Proof.** Let \( \Theta \in (\mathcal{A}_{p,g})' \). Since \( \Theta \cdot v_0 \) lies in \( W_{0|0} \) and is invariant under permutation of the handles, there exists an element \( w \in W_g \) such that \( w \cdot v_0 = \Theta \cdot v_0 \). Now if \( \phi \in \text{Mod}(\Sigma_g) \), then:

\[
\Theta \circ \rho_{p,g}(\phi) \cdot v_0 = \rho_{p,g}(\phi) \circ \Theta \cdot v_0 = \rho_{p,g}(\phi) \circ w \cdot v_0 = w \circ \rho_{p,g}(\phi) \cdot v_0
\]

where we used the second point of Lemma 4.1 in the last equality. Using the cyclicity of \( v_0 \) we get that \( \Theta = w \in W_g \). We conclude using the first point of Lemma 4.1.

□

4.2 The case where \( p = 2r_1r_2 \)

In this subsection, we suppose that \( p = 2r_1r_2 \) with \( r_1, r_2 \) distinct odd primes.

In this case, there exists an unique integer \( x \in \{1, \ldots, r_1r_2 - 2\} \) such that \( \mu_x = 1 \).

This integer is even and verifies either

\[
\begin{align*}
x \equiv -2 \pmod{r_1} \\
x \equiv 0 \pmod{r_2}
\end{align*}
\]

We begin by stating a technical Lemma which proof will be the subject of the next subsection:
Lemma 4.3. If \((x, x, x)\) is \(p\)-admissible, then we have the following:

\[
\begin{pmatrix}
  x & x & 2 \\
  x & x & 0 \\
  x & x & x
\end{pmatrix}
\neq
\begin{pmatrix}
  x & x & 4 \\
  x & x & 0 \\
  x & x & x
\end{pmatrix}
\]

Lemma 4.4. Let \(p \geq 3\) be such that \((x, x, x)\) is \(p\)-admissible. Let \(\Gamma_1, \Gamma_2\) be two trivalent graphs which only differ by a single Whitehead move inside a ball \(B^3\) as drawn in Figure 3. Then:

\[
W_{[0]}(\Gamma_1) \cap W_{[0]}(\Gamma_2) \subset \text{Span} \left( \nu_0^{\Gamma_1}, \text{such that } \sigma(a)\sigma(b)\sigma(c)\sigma(d) = 0 \right)
\]

Proof. Let \(\sigma_1, \sigma_2\) be two \(p\)-admissible colorings of \(\Gamma_1\), with colors 0 or \(x\), such that:

\[
\sigma_1(e) = \sigma_2(e), \forall e \in E(\Gamma_1) - \{i\}
\]

and with \(\sigma_i(a) = \sigma_i(b) = \sigma_i(c) = \sigma_i(d) = x \) and \(\sigma_1(i) = 0, \sigma_2(i) = x\).

Suppose there exists \((\alpha, \beta) \in \mathbb{C}^2\) so that:

\[
v := \alpha \begin{pmatrix}
  x & x & 2 \\
  x & x & 0 \\
  x & x & x
\end{pmatrix} + \beta \begin{pmatrix}
  x & x & 4 \\
  x & x & 0 \\
  x & x & x
\end{pmatrix}
\]

\[
\neq
\begin{pmatrix}
  x & x & 2 \\
  x & x & 0 \\
  x & x & 4
\end{pmatrix}
\]

We must show that \(\alpha = \beta = 0\) to conclude. Using the fusion rule equation (1) of section 2.1, we get:

\[
v = \left( \alpha \begin{pmatrix}
  x & x & 2 \\
  x & x & 0 \\
  x & x & x
\end{pmatrix} \right) + \left( \alpha \begin{pmatrix}
  x & x & 4 \\
  x & x & 0 \\
  x & x & x
\end{pmatrix} \right) + \left( \beta \begin{pmatrix}
  x & x & 2 \\
  x & x & 0 \\
  x & x & x
\end{pmatrix} \right) + \left( \beta \begin{pmatrix}
  x & x & 4 \\
  x & x & 0 \\
  x & x & x
\end{pmatrix} \right)
\]

where \(2\) and \(4\) represent the vectors associated to colorations of \(\Gamma_2\) by the same colors that \(\sigma_1, \sigma_2\) outside the ball \(B^3\) and with the edge \(j\) colored respectively by 2 and 4.

The vector \(v'\) is orthogonal to the two previous ones.

Now since \(v \in W_{[0]}(\Gamma_2)\), we have the following system:

\[
\begin{pmatrix}
  x & x & 2 \\
  x & x & 0 \\
  x & x & 4
\end{pmatrix}
\neq
\begin{pmatrix}
  x & x & 2 \\
  x & x & 0 \\
  x & x & x
\end{pmatrix}
\]

We conclude using Lemma 4.3. \(\square\)

If \(i \in \{1, \ldots, g\}\), we note \(b_i \in V_p, g\) the vector representing a single ribbon colored by \(x\) around the \(i - th\) hole.

Lemma 4.5. If \(G^g\) represents the set of all trivalent graph of genus \(g\), then:

\[
\bigcap_{\Gamma \in G^g} W_{[0]}(\Gamma) = \text{Span} \left( \nu_0, b_i, 1 \leq i \leq g \right)
\]
**Proof.** Let \( \sigma \) be a coloring of the graph of Figure 7 such that:

1. \( \sigma(e) \in \{0, x\} \), for all \( e \in E(\Gamma) \),
2. There exists \( i < j \) with \( \sigma(a_i) = \sigma(b_i) = \sigma(a_j) = \sigma(b_j) = x \).

We can suppose that for every \( i < k < j \), then \( \sigma(a_k) = \sigma(b_k) = 0 \).

Using Lemma 4.4 with \( a = a_i, b = a_j, c = b_i \) and \( d = b_j \), we have that the projection of \( u_\sigma \) on \( \bigcap_{\Gamma \in \mathcal{G}} W_{[0]}(\Gamma) \) is null.

We conclude by noticing that if \( \sigma \) is a coloring of \( \Gamma \), with colors in \( \{0, x\} \), that does not satisfies 2 then \( u_\sigma = b_i \) for some \( i \in \{1, \ldots, g\} \) or \( u_\sigma = v_0 \). \( \square \)

**Lemma 4.6.** There exists an element \( a \in \mathcal{A}_{2r_1 r_2, 1} \) so that:

\[
\begin{align*}
\{ a \cdot u_0 &= u_x \\
\psi \cdot u_0 &= u_0 \\
\psi \cdot u_x &= u_0 \\
\psi \cdot u_x &= u_0
\end{align*}
\]

**Proof.** It is enough to show that there exists a symmetry \( \psi \in (\mathcal{A}_{2r_1 r_2, 1})' \) so that:

\[
\psi \cdot u_0 = u_x \text{ and } \psi \cdot u_x = u_0
\]

Indeed, the cyclicity of \( u_0 \) (Proposition 3.3) implies the existence of \( a \in \mathcal{A}_{2r_1 r_2, 1} \) so that

\[
a \cdot u_0 = u_x
\]

If such a \( \psi \) does exist, we then have:

\[
a \cdot u_x = a \cdot u_0 \cdot \psi = \psi \cdot u_0 \cdot a = u_0
\]

The symmetry \( \psi \) is defined as follows: choose \( i \in \{0, \ldots, r_1 r_2 - 2\} \), then only one of the following two cases occurs:

- Either there exists \( j \in \{0, \ldots, r_1 r_2 - 2\} \) so that

\[
\begin{align*}
j &\equiv i \pmod{2r_1} \\
j &\equiv -i - 2 \pmod{r_2}
\end{align*}
\]

and we put \( \psi(u_i) := +u_j \).
• Or there exists \( j \in \{0, \ldots, r_1 r_2 - 2\} \) so that
\[
\begin{align*}
\begin{cases}
    j \equiv i \pmod{2r_2} \\
    j \equiv -i - 2 \pmod{r_1}
\end{cases}
\end{align*}
\]

and we put \( \psi(u_i) := -u_j \).

A straightforward computation shows that \( \psi \) commutes with \( \rho_{p,1}(T) \) and \( \rho_{p,1}(S) \) and either \( \psi \) or \( -\psi \) sends \( u_0 \) to \( u_x \). □

The proof of the Theorems 1.1 and 1.3 when \( p = 2r_1 r_2 \) follows from the following:

**Proposition 4.7.** Let \( r_1, r_2 \) be two distinct odd primes, \( p = 2r_1 r_2 \) and \( g \geq 2 \) be such that \( v_0 \in V_{p,g} \) is cyclic. Then \( V_{p,g} \) is irreducible.

**Proof.** Using Lemma 4.5 and the fact that the vectors of \( \text{Range}(f) \) must be invariant under permutation of the handles, we have that:
\[
\text{Range}(f) \subset \text{Span} \left( v_0, b_1 + \ldots + b_g \right)
\]

By contradiction, suppose there exists \( \Theta \in (\mathcal{A}_{p,g})' \) so that:
\[
\Theta \cdot v_0 = b_1 + \ldots + b_g = (a \otimes 1 \otimes \ldots \otimes 1 + \ldots + 1 \otimes \ldots \otimes 1 \otimes a) \cdot v_0
\]
where \( a \otimes 1 \otimes \ldots \otimes 1 \) denotes the embedding of the element \( a \in \mathcal{A}_{p,1} \), seen as a linear combination of \( \omega \)-colored link in \( \Sigma_1 \times [0, 1] \), in the first handle of \( \Sigma_g \times [0, 1] \).

Now we have:
\[
\Theta^2 \cdot v_0 = (a \otimes 1 \otimes \ldots \otimes 1 + \ldots + 1 \otimes \ldots \otimes 1 \otimes a)^2 \cdot v_0 = g v_0 + (a \otimes a \otimes 1 \otimes \ldots \otimes 1) \cdot v_0 + \ldots + (1 \otimes \ldots \otimes 1 \otimes a \otimes a) \cdot v_0
\]
We see that \( \Theta^2 \cdot v_0 \) does not belong to \( \bigcap_{\Gamma \in G} W_{[0]}(\Gamma) \) which contradicts the fact that \( \Theta^2 \in (\mathcal{A}_{p,g})' \). □

4.3 **Proof of Lemma 4.3**

In this subsection we put \( p = 2r_1 r_2 \), with \( r_1, r_2 \) two distinct odd primes. We suppose there exists \( x \in \{1, \ldots, r_1 r_2 - 2\} \) so that \( (x, x, x) \) is \( p \)-admissible and so that
\[
\begin{align*}
\begin{cases}
    x \equiv 0 \pmod{2r_1} \\
    x \equiv -2 \pmod{r_2}
\end{cases}
\end{align*}
\]

We also choose \( A_1 \) and \( A_2 \) some primitive \( r_1 - th \) and \( r_2 - th \) roots of unity, so that \( A_1^2 = A_2 \). In particular we have \( A_2^2 = A_1^{-2} \).

The goal of this subsection is to show that:
\[
D := \left\{ \begin{array}{ccc}
    x & x & 2 \\
    x & x & 0
\end{array} \right\} - \left\{ \begin{array}{ccc}
    x & x & 4 \\
    x & x & x
\end{array} \right\} - \left\{ \begin{array}{ccc}
    x & x & 4 \\
    x & x & 0
\end{array} \right\} - \left\{ \begin{array}{ccc}
    x & x & 2 \\
    x & x & x
\end{array} \right\} \neq 0
\]
A straightforward computation, using the formula of the recoupling coefficients ([MV94]), gives:

\[
D = (-1)^{x+y} \frac{[3][5][x][\frac{2}{2}x + 1] \Gamma(\frac{3}{x})^3}{[2][x+3][((x+2)!)[x+3] \left( \frac{x}{2} - 1 \right)^2 \left( \frac{x}{2} + 1 \right)^2 + 1} \] 

\[
-2[x-1][x+3] \left( \frac{x}{2} + 1 \right)^2 + [x-1][x+3] \left( \frac{x}{2} + 2 \right)^2 + 3 
\]

\[
+ [x-1][x+3] \left( \frac{x}{2} + 1 \right)^2 - [x-1][x+3] \left( \frac{x}{2} + 2 \right)[x+3] 
\]

\[
= (-1)^{x+y} \frac{[3][5][x][\frac{2}{2}x + 1] \Gamma(\frac{3}{x})^3}{[2][x+3][((x+2)!)[x+3] \left( A^2 - A^{-2} \right)^2 A^{10} A^{-10} P(A_1, A_2)} 
\]

where we put:

\[
P(x, y) := x^{20} y^{16} - x^{17} y^{17} - x^{16} y^{18} + x^{19} y^{13} - 4 x^{18} y^{14} + 3 x^{17} y^{15} + 2 x^{15} y^{17} - x^{19} y^{11} 
\]

\[
- 5 x^{17} y^{13} - 4 x^{15} y^{15} + 4 x^{14} y^{16} - 2 x^{13} y^{17} - x^{18} y^{10} + 2 x^{17} y^{11} + 6 x^{16} y^{12} + 2 x^{15} y^{13} 
\]

\[
- x^{14} y^{14} + 2 x^{13} y^{15} + x^{11} y^{17} + 2 x^{15} y^{11} + x^{14} y^{12} + x^{13} y^{13} - 6 x^{12} y^{14} + x^{10} y^{16} 
\]

\[
+ x^{10} y^{12} + 4 x^{9} y^{13} - x^{15} y^{12} - 2 x^{12} y^{2} - 6 x^{14} y^{6} - 4 x^{13} y^{2} + 2 x^{12} y^{2} - 8 x^{11} y^{8} - 6 x^{10} y^{10} 
\]

\[
- 6 x^{10} y^{11} - x^{11} y^{13} + x^{13} y^{5} + 6 x^{11} y^{7} + 6 x^{10} y^{8} + 8 x^{9} y^{9} - 2 x^{8} y^{10} + 4 x^{7} y^{11} + 6 x^{6} y^{12} + 2 x^{5} y^{13} 
\]

\[
+ x^{13} y^{3} + 4 x^{12} y^{4} + 4 x^{11} y^{6} - x^{9} y^{7} + 4 x^{6} y^{8} - x^{7} y^{9} - x^{11} y^{11} + 4 x^{6} y^{6} + 4 x^{8} y^{7} + x^{5} y^{8} - 4 x^{10} y^{10} - x^{7} y^{11} 
\]

\[
- x^{10} y^{2} + 6 x^{9} y^{4} - x^{7} y^{5} - x^{6} y^{6} - 2 x^{5} y^{7} - x y^{7} - 2 x^{5} y^{6} - 6 x^{8} y^{6} - 3 x^{9} y^{8} + 2 x^{7} y^{11} 
\]

\[
- 4 x^{6} y^{2} + 4 x^{5} y^{6} + 5 x^{3} y^{7} + x^{2} y^{7} - 2 x^{5} y^{3} - 3 x^{3} y^{9} + 4 x^{4} y^{8} - x y^{5} + x^{4} + x^{2} y^{7} - y^{2} 
\]

Note that \(P(x, y)\) does not depend on \(r_1, r_2\) or \(x\). The proof reduces to show that \(P(A_1, A_2) \neq 0\) for \(A_1, A_2\) any primitive \(r_1 - th\) and \(r_2 - th\) roots of unity.

Consider the following algebraic curves in \(\mathbb{C}^2:\)

\[
C := \{ (z_1, z_2) \in \mathbb{C}^2 \text{ so that } P(z_1, z_2) = 0 \} 
\]

\[
T := \{ (z_1, z_2) \in \mathbb{C}^2 \text{ so that } |z_1|^2 = |z_2|^2 = 1 \} 
\]

Note that \(T\) is a torus, has degree 3 and that these two curves share no irreducible components in common. The Bézout Theorem (see [Har77] Chap I Corollary 7.8) implies that:

\[#(C \cap T) \leq \text{deg}(C) \cdot \text{deg}(T) = 108\]

Now suppose there exist \(A_1\) and \(A_2\) some primitive \(r_1\) and \(r_2\) roots of unity so that \(P(A_1, A_2) = 0\). Since \(P(x, y) \in \mathbb{Z}[x, y]\), the equality \(P(A_1, A_2) = 0\) must hold for every \(r_1\) and \(r_2\) roots of unity. Thus we have:

\[r_1 \cdot r_2 \leq #(C \cap T) \leq \text{deg}(C) \cdot \text{deg}(T) = 108\]
So we just have the following possible cases:

\[ \{r_1, r_2\} \in \{[3, 5], [5, 7], [3, 11], [5, 11], [7, 11], [3, 13], [5, 13], [7, 13]\} \]

First if \( \{r_1, r_2\} = \{3, 5\}, \{5, 7\}, \{3, 11\} \) or \( \{5, 13\} \), then \( x = 10, 28, 22 \) and 50 respectively and we see that \( (x, x, x) \) is not 2\( r_1 r_2 \)-admissible.

We handle the four reminding cases by checking that \( P(e^{2\pi i \frac{3}{5}}, e^{2\pi i \frac{11}{13}}) \neq 0 \), \( P(e^{2\pi i \frac{5}{7}}, e^{2\pi i \frac{11}{13}}) \neq 0 \), \( P(e^{2\pi i \frac{3}{7}}, e^{2\pi i \frac{13}{11}}) \neq 0 \) and \( P(e^{2\pi i \frac{5}{7}}, e^{2\pi i \frac{13}{11}}) \neq 0 \).

This concludes the proof of Lemma 4.

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