Existence of positive equilibria for quasilinear models of structured population

Stefano Bertoni∗

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Abstract

In this paper I prove the existence of a positive stationary solution for a generic quasilinear model of structured population. The existence is proved using Schauder’s fixed point theorem. The theorem is applied to a hierarchically size–structured population model.

Keywords: structured population model, stationary solution, net reproduction function, compactness, Schauder’s fixed point theorem.

1 Introduction

The size–structured population model IBVP (Initial Boundary Value Problem, see [3]), in the autonomous case, has the following general form:

\[
\begin{aligned}
    u_t + (g(x, u(t, \cdot)) u_x + \mu(x, u(t, \cdot)) u &= 0 \\
    g(0, u(t, \cdot)) u(t, 0) &= \int_J \beta(x, u(t, \cdot)) u(t, x) \, dx,
\end{aligned}
\]

(1)

where \( x \in J = [0, \infty) \) represents age or size, \( t \geq 0 \) is time, \( u \) is the population density, \( u(t, \cdot) \in L^1(J) \) for each \( t \geq 0 \).

The model equations involve the following vital rates: \( \mu = \mu(x, u) \) — mortality, \( \beta = \beta(x, u) \) — fertility and \( g = g(x, u) \) — growth rate. These coefficients depend on the size \( x \) and on the total population behaviour through \( u \) in a general (also nonlinear) way.

The total population at the instant \( t \) is given by \( P(t) = \int_J u(t, x) \, dx \), the flow of the newborns is \( B(t) = \int_0^\infty \beta(x, u(t, \cdot)) u(t, x) \, dx \). In this paper we obtain for Pbm. (1) a theorem of existence of a positive equilibrium.

∗Dipartimento di Matematica, Università di Trento (Italy). E-mail: bertoni@science.unitn.it
In general, however, the well-posedness of this class of PDE models is still an open question ([4], Introduction).

The first nonlinear population model was introduced and analysed in the seminal paper [8] of Gurtin and MacCamy in 1974, with nonlinearities depending only on $P(t)$. It was followed in the eighties by several other papers with generic nonlinearities in $u$ for the case $g = 1$ e.g. by J. Prüss that gave some sufficient conditions for the existence of a positive equilibrium [9] [10] [11].

In 2003 Diekmann et al. [5] managed the case of nonconstant $g$ and $n$ scalar biomasses $S_1, S_2, \ldots S_n$ depending on $u$, using a very different mathematical formulation; they proved the existence of nonzero equilibria and gave bifurcation conditions.

In 2006 Farkas e Hagen [7] studied the stability of stationary solutions of the IBVP, in the case of nonlinear dependence on the total population $P$, via linearization and semigroup and spectral methods. They give stability criteria in terms of a modified net reproduction rate.

In this paper I establish Thm. 5, that gives sufficient conditions for the existence of a positive equilibrium for Pbm. (1), under generic dependence on $u$. I use a compactness hypothesis. I set also preliminarily some positivity and boundedness hypotheses on the coefficients $\mu$ and $g$.

The problem is transformed in a fixed point problem and the existence of a solution is obtained through Schauder’s fixed point theorem.

However there is no uniqueness in general. I give a made–up counterexample. I give also a condition for the non–existence of positive equilibria using suitable assumptions of monotonicity on the coefficients $\mu$, $g$ and $\beta$.

At the end of Sec. 3 I show as application the existence of a positive stationary solution for a nonlinear model of structured population of Ackleh and Ito [2].

In the Appendix, I resume some propositions on compactness.

2 Preliminaries

2.1 Notations

$J = [0, \infty)$ is the interval of definition of $x$.

$<g, f> = \int_J g(x) f(x) \, dx$ for $f \in L^1(J)$ and $g \in L^\infty(J)$.

$L^1_+(J) = \{ \phi \in L^1(J) \mid \phi(x) \geq 0 \mathrm{\ a.e.\ } x \in J \}$ is the positive cone of $L^1(J)$.

Given two functions $u_1, u_2: [0, \infty) \to [0, \infty)$, we will write $u_1 < u_2$ if $0 \leq u_1(x) \leq u_2(x)$ and $u_1 \neq u_2 \mathrm{\ a.e.\ } x \in J$. The relation $<$ is a partial order on the cone $L^1_+(J)$.

If $e_1, e_2 \in L^1(J)$ and $e_1 < e_2$, then write

$[e_1, e_2] = \{ \phi \in L^1(J) \mid e_1(x) \leq \phi(x) \leq e_2(x) \mathrm{\ a.e.\ } x \in J \}$.

Functions $f(u(\cdot))$ defined for $u \in L^1_+(J)$ will be usually briefly denoted as $f(u)$. 

2
2.2 Hypotheses and definitions

Hypothesis (A)

a) The functions \( x \mapsto g(x,u), \mu(x,u) \) are \( L^\infty(J) \) for each \( u \in L_1^+(J) \) and there exist constants \( \underline{g}, \bar{g}, \underline{\mu}, \bar{\mu} \):

\[
0 < \underline{g} \leq g(x,u) \leq \bar{g}, \quad 0 < \underline{\mu} \leq \mu(x,u) \leq \bar{\mu}
\]

for each \( u \in L_1^+(J) \), a. e. \( x \in J \).

b) \( \beta(x,u) \geq 0 \) for a.e. \( x \in J \), for each \( u \geq 0 \) and there exists a constant \( \bar{\beta} > 0 \):

\[
\beta(x,u) \leq \bar{\beta} \text{ for each } u \geq 0, \text{ a.e. } x \in J.
\]

c) \( u \mapsto g(x,u), \mu(x,u), \beta(x,u) \) are continuously depending on \( u \in L_1^+(J) \) for a.e. \( x \in J \).

Auxiliary functions. For \( x \in J, u \in L_1^+(J) \), we set:

\[
\Pi(x,u) := \frac{1}{g(x,u)} e^{-\int_0^x \frac{\mu(y,u)}{g(y,u)} \, dy}.
\]

Under the boundedness assumptions of Hyp. (A), we define the auxiliary functions \( e_1, e_2 \):

\[
e_1(x) := e^{-(\bar{\mu}/\bar{g}) x} \frac{x}{\bar{g}}, \quad e_2(x) := e^{-(\underline{\mu}/\underline{g}) x} \frac{x}{\underline{g}}.
\]

Lemma 1 (Properties of \( \Pi \)) Using the assumptions on the lower and upper bounds of \( \mu \) and \( g \), given in Hyp. (A), we obtain for each \( x, u \):

\[
e_1(x) \leq \Pi(x,u) \leq e_2(x).
\]

Moreover \( \Pi(\cdot, u) \in L^1(J) \cap L^\infty(J) \) for each \( u \in L_1^+(J) \).

The interval \([e_1, e_2]\) is a closed convex subset of \( L_1^+(J) \).

"Onion" set. Set \( U := \bigcup_{\lambda > 0} [\lambda e_1, \lambda e_2] \).

It is simple to prove that the sets \( U \) and \( \overline{U} = U \cup \{0\} \) are convex.

Hypothesis (C) (Uniformly bounded variation).

\[
\forall T > 0 : \lim_{h \to 0} \sup_{u \in U} \int_0^T |g(x+h,u) - g(x,u)| \, dx = 0.
\]

We mean that \( g \) is extended as 0 for \( x < 0 \).

Remark 1 Condition (C) means that \( \sup \) has to be considered on functions of the form \( u = \lambda v \), with \( v \in [e_1, e_2] \). Since \( U \neq L_1^+(J) \) (e.g. \( x^{-1/2} e^{-x} \notin U \)) this is an effective reduction of the requests.

Under Hyp. (A), Condition (C) is satisfied also for \( u = 0 \) (therefore it holds for \( u \in \overline{U} \)) because \( g(x,u) \) is continuous in \( u \).
Hypothesis (D)

∀T > 0 : ∃kT > 0 : |g(x, u)| ≤ kT, for each u ∈ U and a.e. x ∈ [0, T].

Hyp. (D) implies Hyp. (C).

Hypothesis (Lβ) (Limit of β). For each x ≥ 0,

\[
\lim_{\|u\|_1 \to +\infty, u > 0} \beta(x, u) = 0.
\]

Definition 2 (Net reproduction function) (Cmp. ([10], p. 330) For u ∈ L^1(J)

\[
R(u) := \int_J \beta(x, u) \Pi(x, u) \, dx.
\]

Under Hyp. (A), R(u) is well-defined and if also (Lβ) holds, then

\[
\lim_{\|u\|_1 \to \infty, u \geq 0} R(u) = 0.
\]

2.3 Compactness

The (closed, convex) interval [e_1, e_2] ⊆ [0, e_2] ⊆ L^1(J) is invariant with respect to Π, i.e. Π(·, [e_1, e_2]) ⊆ Π(·, [0, e_2]) ⊆ [e_1, e_2].

Lemma 3 (Compactness) Under Hyp. (A) and (C), the function u ↦ Π(·, u), defined on U and U in L^1(J), is compact.

The lemma of compactness is proved in Appendix, Sec. [B].

3 Existence of equilibria

In this section we prove the existence of a positive stationary solution u^* for Pbm. ([1]) as fixed point of a suitable transformation of L^1(J).

3.1 Stationary solutions

The equilibria are the time-independent solutions u = u^*(x) of Problem ([1]). These are determined from

\[
\left\{ \begin{array}{l}
\frac{\partial}{\partial x} \left( g(x, u^*(\cdot)) u^*(x) \right) + \mu(x, u^*(\cdot)) u^*(x) = 0 \\
g(0, u^*(\cdot)) u^*(0) = \int_0^\infty \beta(x, u^*(\cdot)) u^*(x) \, dx
\end{array} \right.
\]

and (see [10], Eq. (8)) they corresponds to the solutions of the functional equation

\[
u(x) = \int_0^\infty \beta(x', u) u(x') \, dx' \frac{\mu(x, u)}{\mu(x, u)} e^{-\int_0^x \frac{\mu(y, u)}{\mu(y, u)} \, dy} \quad \text{for } x \in J,
\]

the only premises being g > 0, \( \frac{\mu(x, u)}{\mu(y, u)} \in L^1_{\text{loc}}(J) \).

This equation is translated immediately in a fixed point problem.
Proposition 4 Under Hyp. (A) the stationary solutions of Pbm. (1) are the fixed points of the functional $T: L^1(J) \to L^1(J)$ defined as

$$
(T \phi)(x) = \frac{G(\phi)}{g(x, \phi)} e^{-\int_0^x \frac{g(x, \phi(y))}{\phi(y)} dy}
$$

and vice versa, where $G: L^1(J) \to \mathbb{R}$ is given by

$$
G(\phi) = \int J \beta(x, \phi(\cdot)) \phi(x) dx,
$$

for $\phi \in L^1(J)$. The functional equation $u = Tu$ can be written in a more compact form as

$$
u(x) = G(u(\cdot)) \Pi(x, u(\cdot)),
$$

that we discuss.

Theorem 5 (Existence of equilibria) Assume Hyp. (A) and (C). Suppose there is a constant $\rho_0 > 0$ such that for $u \in L^1(J)$, $\|u\|_1 \geq \rho_0$ implies $R(u) \leq 1$. If $R(0) > 1$ then Problem (1) admits at least a positive stationary solution. The solution satisfies the functional equation

$$
u^*(x) = \lambda^* \Pi(x, u^*(\cdot)),
$$

where $\lambda^* > 0$ is a suitable number and the corresponding population is constant and given by $P^* = \lambda^* \|\Pi(\cdot, u^*)\|_1$.

From (6) we have the following statement:

Corollary 6 Under Hyp. (A), (C) and (L$_\beta$), if $R(0) > 1$ then Problem (1) admits a positive stationary solution.

3.2 Proof of Thm. 5

Prop. 4 reduces the search for equilibria of Pbm. (1) to Eq. (10).

$G(0) = 0$ gives the trivial equilibrium $u = 0$ so we exclude this case. The proof is divided into two steps.

(i) Splitting variables. Consider Eq. (10): assume that $u$ is a solution of

$$
\begin{align*}
\lambda &= G(u) (\neq 0) \\
v &= \frac{1}{\lambda} u.
\end{align*}
$$

By substitution we obtain: $\lambda v = \lambda \Pi(x, \lambda v)$ and $\lambda = \int_0^\infty \beta(x, \lambda v) \lambda \Pi(x, \lambda v) dx$ so that $1 = \int_0^\infty \beta(x, \lambda v) \Pi(x, \lambda v) dx$. Hence $(v, \lambda) \in [e_1, e_2] \times (0, \infty)$ is a solution of the system:

$$
\begin{align*}
v(x) &= \Pi(x, \lambda v(\cdot)), \\
R(\lambda v) &= 1.
\end{align*}
$$

5
Vice versa, if \((v, \lambda)\) is a solution of (12), then \(u = \lambda v\) is a solution of the equation 
\[ u = G(u) \Pi(\cdot, u). \]

The condition \(R(0) > 1\) implies that \(\lambda^* \neq 0\). For each solution \((v, \lambda)\) of Pbm. (12) we have \((v, \lambda) \in [e_1, e_2] \times (0, \infty)\).

(ii) **Fixed point.** In this step we apply Schauder’s fixed point theorem — see [6], [12]. We write Pbm. (12) in the form
\[
\begin{align*}
\begin{bmatrix} v(-) = \Pi(\cdot, \lambda v), & v \in [e_1, e_2], \\
\lambda = \max\{\lambda + R(\lambda v) - 1; 0\}, & \lambda \geq 0
\end{bmatrix}
\end{align*}
\]
that is \((v, \lambda) = A((v, \lambda))\), with \((v, \lambda) \in [e_1, e_2] \times (0, \infty)\) and \(A\) defined by the second members of (13).

The map \(u \mapsto \Pi(\cdot, u)\) is continuous and compact on \(U\); the function \(R(u)\) is continuous and bounded from \(L^1_u(J)\) to \((0, \infty)\), since \(0 < R(u) \leq \beta \|e_2\|_1\); therefore \(A: [e_1, e_2] \times (0, \infty) \to L^1(J) \times (0, \infty)\) is continuous and compact.

\(A_1(v, \lambda) := \Pi(\cdot, \lambda v)\) has image in \([e_1, e_2]\).

Now prove that for a fixed \(M > \frac{\rho_0}{\|e_1\|_1}, \frac{\rho_0}{\|e_1\|_1} + \beta \|e_2\|_1 - 1\), then
\(A_2(v, \lambda) := \max\{\lambda + R(\lambda v) - 1; 0\}\) maps \([e_1, e_2] \times [0, M]\) on \([0, M]\).

If \(\frac{\rho_0}{\|e_1\|_1} \leq \lambda \leq M\), then \(\lambda \geq \frac{\rho_0}{\|v\|_1}\) and \(R(\lambda v) \leq 1\), so that
\[ \lambda + R(\lambda v) - 1 \leq 1 + \lambda - 1 = \lambda \leq M. \]

If \(0 \leq \lambda < \frac{\rho_0}{\|e_1\|_1}\), then \(\lambda + R(\lambda v) - 1 \leq \frac{\rho_0}{\|e_1\|_1} + \beta \|e_2\|_1 - 1 \leq M\).

So \(A\) maps \([e_1, e_2] \times [0, M]\), a closed convex subset of \(L^1(J) \times (0, \infty)\), in itself.

Since \(A\) is compact, by Schauder’s fixed point theorem, Eq. (13) has at least a fixed point \((v^*, \lambda^*) \in [e_1 e_2] \times [0, M]\) and it is different from 0 for the initial remark; \((v^*, \lambda^*)\) is a fixed point also for Eq. (12).

Finally, Eq. (10) is satisfied by \(u^* = \lambda^* v^*\) and the corresponding stationary population is
\[ P^* = \int_J u(x) \, dx = \lambda^* \int_J v^*(x) \, dx. \]

**Remark 2** \(R(0) > 1\) implies \(\beta \|e_2\|_1 > 1\), therefore in the proof it is possible to assume \(M = \frac{\rho_0}{\|e_1\|_1} + \beta \|e_2\|_1 - 1\) and to have the estimate \(P^* \leq M \|e_2\|_1\).

### 3.3 A counterexample

Thm. [5] is a sufficient condition but not a necessary one. We can have also \(R(0) < 1\) if there exists \(u_0 \in L^1_u(J)\) such that \(R(u_0) > 1\). In this case it is possible to need other conditions on \(u_0\) to prove a statement of existence. The idea is to construct explicitly an example with a positive equilibrium but \(R(0) < 1\).

Set \(\mu(x, u) = g(x, u) = g\) so that \(\Pi(x, u) = \frac{1}{g} e^{-x}\), independent of \(u\).
Define $e_0(x) := e^{-x}$. Take $F : L^+_1(J) \to \mathbb{R}$, $u \mapsto F(u)$, such that $F(0) < 1$, $F(e_0) = 1$ and $\lim_{\|u\|_1 \to \infty, u > 0} F(u) = 0$, $F$ continuous but obviously nonmonotonic.

Now set $\beta(x, u) = 2g(1 - e^{-x})F(u)$ so that $R(u) = F(u)$.

Then $R(e_0) = 1$ and $u = e_0$ is a solution of the fixed point equation and a positive equilibrium.

As example of function $F$ we can take $F(u) := f(\|u\|_1)$, where

$$f(a) := \begin{cases} 
\frac{1}{2} + 3a & \text{for } 0 \leq a \leq \frac{1}{2}, \\
\frac{3}{2} - 2a & \text{for } \frac{1}{2} < a \leq \frac{5}{4}, \\
\frac{e^{-a}}{2} & \text{for } a > \frac{5}{4}.
\end{cases}$$

(14)

In this case we have two positive equilibria, $u(x) = e^{-x}$ and $u(x) = \frac{1}{6} e^{-x}$, corresponding to the two solutions of $f(a) = 1$, i.e. $a = 1$, $a = 1/6$.

### 3.4 A nonexistence result and a sufficient and necessary condition

Under suitable monotonicity hypotheses, $R(0) > 1$ becomes a necessary and sufficient condition.

A function $f$, defined on ordered spaces, is increasing if $u_1 < u_2$ implies $f(u_1) < f(u_2)$. The other monotonicity definitions are extended in the same ways.

Now assume $u \in L^+_1(J)$ in the following statements.

**Assumption (M) (Monotonicity)**

- $u \mapsto \mu(x, u)/g(x, u)$ is nondecreasing (or increasing) for each $x \geq 0$ (mortality–growth ratio),
- $u \mapsto \beta(x, u)/\mu(x, u)$ is decreasing (or nonincreasing) for each $x \geq 0$ (fertility–mortality ratio),
- $x \mapsto \beta(x, u)/\mu(x, u)$ is nondecreasing (or increasing) for each $u$.

The hypotheses between parentheses are in alternative: $u \mapsto \beta/\mu$ must be strictly decreasing and the other two functions are only nondecreasing, or, vice versa, $u \mapsto \beta/\mu$ nonincreasing and the others have to be two strictly increasing.

To prove the nonexistence condition we need the following statement:

**Lemma 7 (Monotonicity)** Assume Hypotheses (A), (C), $(L_{\beta})$ and Assumption (M).

Then the functional $R : L^+_1(J) \to (0, \infty)$ is continuous, decreasing and

$$\lim_{\|u\|_1 \to +\infty, u > 0} R(u) = 0.$$
I do not give the details of the proof of this lemma, but the main idea is to write \( R(u) \) as \( \int_J dx \frac{R(x,u)}{\mu(x,u)} \int_J dy \) and to study the properties of monotonicity of the integral \( \int_J dx \gamma(x) e^{-\int_{x_0}^x \mu(y)g(y) dy} \) with respect to suitable \( \gamma \) and \( h \).

For a detailed proof, see Bertoni [1].

**Proposition 8 (Non existence of positive stationary solutions)**

Under Hypotheses of Lemma 7, if \( R(0) \leq 1 \) then Pbm. [1] has no positive stationary solutions.

**Proof.** If \( R(0) \leq 1 \) then \( R(u) = 1 \) does not have positive solutions by monotonicity.

Since existence of positive equilibria is equivalent to positive solutions of \( u = G(u) \Pi(\cdot, u) \) and so of Eq. (12), the conclusion follows.

As consequence, Condition \( R(0) > 1 \) becomes a necessary and sufficient condition of existence of positive equilibria for Pbm. [1] under Hyp. (A), (C), (L\( \beta \)) and (M).

### 3.5 Applications

Ackleh e Ito [2] consider a hierarchically size-structured population model that can be reported to Eq. (1). They proved existence of measure-valued solutions for the Cauchy problem. We give a condition of existence of a stationary positive solution for a simple case of this model, by taking

\[
 g(x, u(\cdot)) = g + (\overline{g} - g) e^{-\int_{x_0}^x u(y) dy}.
\]

Hyp. (D) is equivalent to

\[
 \forall T > 0 : \sup_{u \in U} \sup_{0 \leq x \leq T} |g_x(x, u)| < \infty
\]

that is, for (15):

\[
 \forall T > 0 : \sup_{u \in U} \sup_{0 \leq x \leq T} |e^{-\int_{x_0}^x u(y) dy} \cdot u(x)| < \infty.
\]

For \( u = \lambda v \) with \( v \in [e_1, e_2] \) we use the inequality \( \sup_{\lambda > 0} \lambda e^{-\alpha \lambda} = \frac{1}{\alpha e} \): therefore

\[
 e^{-\int_{x_0}^x u(y) dy} \cdot u(x) = \lambda v(x) e^{-\lambda \int_{x_0}^x v(y) dy} \leq \frac{v(x)}{e \int_{x_0}^x v(y) dy} \leq \frac{e_2(x)}{e \int_{x_0}^x e_1(y) dy} < \infty.
\]

Assume \( \mu \) and \( \beta \) to satisfy Hyp. (A) and (L\( \beta \)). The other conditions on \( g \) of Cor. [6] are trivially satisfied, so we obtain the existence of at least one positive stationary solution if

\[
 \int_J dx \beta(x, 0) e^{-\int_{x_0}^x \mu(y, 0) dy} > \overline{g}.
\]
Appendix

A  Compactness conditions

As well known, the conditions for the relative compactness of a set \(W\) in \(L^1(0, \infty)\) are given by the Riesz–Kolmogorov Theorem. We use the following version:

i) \(W\) is bounded;

ii) \(\lim_{T \to \infty} \sup_{w \in W} \int_{x > T} |w(x)| \, dx = 0.\)

iii) \(\lim_{h \to 0} \sup_{u \in W} \int_0^T |w(x + h) - w(x)| \, dx = 0\) for each \(T > 0.\)

Sets of continuous, uniformly bounded variation functions in \(L^1(0, \infty)\) are (relatively) compact.

B  Compactness of \(\Pi\) (Proof of Lemma 3)

For each \(u \in L^1_+(J)\), the function \(\Pi(x, u)\) defined by (2) has the following properties:

1. \(\Pi(\cdot, u) \in [e_1, e_2]\), that implies (i) and (ii) of the Riesz–Kolmogorov Theorem;

2. \(x \mapsto \Pi(x, u)\) is continuous.

Now we prove (iii) for \(u \in \mathcal{U}\). Let be \(T > 0, h > 0:\)

\[
\int_0^T |\Pi(x+h, u) - \Pi(x, u)| \, dx \leq \\
\leq \int_0^T dx \left( e^{\int_0^x \frac{e^{y+h} \mu(y,u)}{g(y,u) g(x+y,u)} dy} - 1 \right) + \\
+ \int_0^T dx \left( e^{\int_0^x \frac{e^{y+h} \mu(y,u)}{g(y,u) g(x+y,u)} dy} - 1 \right) \\
\leq \frac{T \|T\|}{g^2} h + \frac{T \|T\|}{g^2} \int_0^T dx |g(x+h, u) - g(x, u)|,
\]

therefore, using Hyp. (C) for \(u \in \mathcal{U}\), this completes the proof. The case \(h < 0\) is managed analogously.

We obtain the \(\Pi\) sends \(U\) in a relatively compact subset of \([e_1, e_2]\) in the norm of \(L^1(J)\) i.e. the set \(\Pi(\cdot, U)\) is relatively compact.

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