Scale Invariant Monte Carlo under Linear Function Approximation with Curvature based Step-size

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Abstract
We study the feature-scaled version of the Monte Carlo algorithm with linear function approximation. This algorithm converges to a scale-invariant solution, which is not unduly affected by states having feature vectors with large norms. The usual versions of the MCMC algorithm, obtained by minimizing the least-squares criterion, do not produce solutions that give equal importance to all states irrespective of feature-vector norm – a requirement that may be critical in many reinforcement learning contexts. To speed up convergence in our algorithm, we introduce an adaptive step-size based on the curvature of the iterate convergence path – a novelty that may be useful in more general optimization contexts as well. A key contribution of this paper is to prove convergence, in the presence of adaptive curvature based step-size and heavy-ball momentum. We provide rigorous theoretical guarantees and use simulations to demonstrate the efficacy of our ideas.

1 INTRODUCTION
Feature scaling and data normalization is a common practice in machine learning and has been shown to be effective in a variety of areas such as deep learning (Bishop, 1995; Sola and Sevilla, 1997), nearest neighbour classifiers (Li et al., 2016; Singh and Singh, 2020), SVMs (Stolcke et al., 2008), PCA (Casella and Berger, 2001) and data mining (Han et al., 2011). Their main utility is when the norm of the input vector is not a true reflection of its importance Bishop (1995). Normalization is also known to often help increase the speed of learning Ba et al. (2016b) as well as reduce the dependence on outliers (Ben-Gal, 2005; Botchkarev, 2019).

Finding the optimal policy in Markov Decision Processes (MDPs) remains the central goal of reinforcement learning. In the context of optimal control, the value-iteration (Bellman, 1958) and policy iteration algorithms (Howard, 1960) have remained the cornerstones of dynamic programming (DP) methods to solve this problem. When one doesn’t know the model (transition probabilities) in the MDP explicitly, algorithms like Monte Carlo, TD(0) Learning, TD(\(\lambda\)) Learning and Q-Learning, or their variations are often used (Sutton and Barto, 1998).

For a large class of problems, the state space becomes large enough that explicitly maintaining the values associated with each state becomes infeasible Szepesvari (2010). In such cases, one uses approximation techniques to model the values associated with states. Two such approximation techniques that are often used are linear function approximation and neural network approximation.
To rigorously prove that the above methods work as expected, one needs to provide theoretical guarantees of their convergence. In the tabular setting (with no function approximation), several theoretical results provide such guarantees (for example, see Dayan (1992); Dayan and Sejnowski (1994); Tsitsiklis (2002)). In the context of linear function approximation, stochastic approximation techniques and ODE methods such as those listed in Ljung (1978); Borkar and Meyn (2000); Borkar (2008); Kushner and Yin (1997) are often used to provide such guarantees. Convergence guarantees under the linear function approximation regime have been explored in works by Tsitsiklis and Van Roy (1996); Korda and La (2015); Bertsekas et al. (2004); Konda and Tsitsiklis (1999); Perkins and Precup (2002); Bertsekas (2011).

In the linear function approximation setup, the value assigned to any state is approximated by a linear function of the feature vector associated with the state. For instance, in an \( m \)-state MDP, if the feature vector associated with state \( i, i \in \{1, \ldots, m\} \) is \( \phi_i \), then for some weight vector \( w \), the value \( V_i \) associated with the state \( i \) is approximated by \( \phi_i^\top w \). If \( \Phi \) is the matrix with rows as the feature vectors, and \( V \) is the vector of values associated with the states, i.e. \( \Phi = \{\phi_i\}_{i \in \{1, \ldots, m\}} \), and \( V = \{V_i\}_{i \in \{1, \ldots, m\}} \), then we are approximating \( V \) by \( \Phi^\top w \). Using the least-squares criterion to find \( w \) leads us to \( w = \arg \min_w \|\Phi^\top w - V\|_2^2 \). More generally, if we assign weights \( d_i \) to each state \( i \) such that \( \sum_{i \in \{1, \ldots, m\}} d_i = 1 \), then, the least squares (LS) criterion gives a weight \( w = \arg \min_w \sum_{i \in \{1, \ldots, m\}} d_i (\phi_i^\top w - V_i)^2 \). The major reinforcement learning algorithms using linear function approximation (listed previously) obtain weight vectors that conform to this criterion.

**Illustrative example for issues with LS:** The least squares method provides solutions which are more skewed towards feature vectors with larger norm. We illustrate this with a toy-example as follows. Consider a two-state system with features \( \phi_1, \phi_2 \in \mathbb{R} \). Say the values associated with these two states are \( V_1, V_2 \). Let \( \phi_1 = 1, \phi_2 = 2 \) and \( V_1 = 2, V_2 = 1 \). Then we want some \( w \in \mathbb{R} \) such that \( w \simeq 2 \) and \( 2w \simeq 1 \). One may expect the answer to be the mean of \( 2 \) and \( \frac{1}{2} \), i.e. \( w = \frac{5}{4} \), but the least squares solution for this system is \( w = \frac{1}{2} \). The least squares solution is dominated by the second feature vector, viz \( \phi_2 = 2 \), thus gives a solution that approximates the second linear equation better. The issue highlighted by this example is exacerbated when states that have features that are outliers.

To address this issue, in this paper, we propose a solution calculated as per the alternative criterion: \( w = \arg \min_w \sum_{i \in \{1, \ldots, m\}} d_i (\phi_i^\top w - V_i)^2 / ||\phi_i||_2^2 \) which is the minimizer of the weighted sum of squares of distances from \( w \) to the hyperplanes \( \phi_i^\top w = V_i, i \in \{1, \ldots, m\} \). This criterion has the following two advantages over least squares. Firstly, the solution under our criterion is scale invariant, i.e., irrespective of the norm (scale) of the feature vectors, the solution gives importance to states proportional to the chosen \( d_i \) values. Secondly, the solution is more robust to outlier rows. Unlike in the least-squares solution, large \( \phi_i \) values which may be outliers will not unduly affect the solution. Further, the solution remains unchanged even if the individual equations are re-scaled.

While scaling of feature vectors is often used in practice in a variety of machine learning as well as reinforcement learning contexts (Ioffe and Szegedy, 2015; Santurkar et al., 2018a; Huang et al., 2020; Bhatt et al., 2019), the current work contributes to the theory relating to feature-scaling in the context of RL algorithms. We provide convergence guarantees in the presence of momentum as well as an adaptive step size method.

We now present related work that the current paper builds upon. These broadly touch upon three aspects – adaptive step size, convergence under momentum and feature-normalization and scaling.

### 1.1 Additional Related Work

Adaptive step sizes have been explored classically by Schumer and Steiglitz (1968); Ang and Farhang-Boroujeny (2001); Kushner and Yang (1994) amongst others. In the context of reinforcement learning, adaptive step sizes have been explored in the context of policy gradient (Pirotta et al., 2013), and temporal difference learning (Dabney and Barto, 2012).

In optimization literature, several stochastic gradient descent (sgd) based algorithms use some form of adaptive step size (Ruder, 2016). Many like Adagrad (Duchi et al., 2011) and Adadelta (Zeiler, 2012) modify the step size. Others like Adam (Kingma and Ba, 2015) also add additional momentum terms to speed up convergence. A recent work also adapts the Polyak step-sizes to be stochastically updated (Loizou et al., 2021). Convergence of some of these methods in the presence of momentum

\(*\) We use \([m]\) to indicate \(\{1, \ldots, m\}\)
We now outline the basic (linear) framework under which our Algorithms operate. We formulate and study the convergence of scale-invariant RL algorithms with linear function approximation of the input features does not matter to the output solution. In RL systems, every state might be equally important irrespective of the feature vector norms. Our algorithms converge to a solution that satisfies this property of not being unduly influenced by outliers, which have been studied recently in works by Reddi et al. (2018); Défossez et al. (2020); Mai and Johansson (2020); Chen et al. (2019); Yang et al. (2016)

In other threads of work, normalization and feature scaling have been studied to good effect in the non-convex landscape of neural networks. For instance, layer normalization (Ba et al., 2016a) and batch normalization (Ioffe and Szegedy, 2015) have been used to "normalize" activations in intermediate layers of neural networks. Group normalization (Wu and He, 2018), self-normalization (Klambauer et al., 2017), weight normalization (Salimans and Kingma, 2016) and other variants have also been considered. Some works propose that these techniques make the optimization landscape smoother (Santurkar et al., 2018b), and other works propose that they help reduce covariate shift (Ioffe and Szegedy, 2015). We note that these normalization techniques rescale the inputs based on statistics per set of inputs, rather than a re-scaling of each input to have norm 1.

Normalization and feature scaling are less often used in linear settings – possibly because the drawbacks of using inputs that are not feature-scaled are not apparent. As highlighted in our toy example, using features without scaling in methods like minimization of least squares, can lead to solutions that are more skewed towards data where the feature-norms are higher.

Our work focuses on this problem of feature scaling in linear settings where we provide convergence guarantees in the presence of adaptive step size and momentum.

1.2 Our Contributions

We formulate and study the convergence of scale-invariant RL algorithms with linear function approximation in the presence of momentum and adaptive step size. Our algorithm uses a variant of the stochastic Kaczmarz method (Strohmer and Vershynin, 2009b) to seek a scale-invariant solution. Note that the original method solves overdetermined \( \Phi w = V \) systems that are consistent, and requires access to exact value \( (V) \) estimates. We provide a convergence guarantee even with only noisy samples of the value function. This is crucial in RL applications, where we get access to some noisy estimate of the value either by a one step temporal-difference (TD) or by summing rewards (Monte Carlo).

In RL systems, every state might be equally important irrespective of the feature vector norms. Our algorithms converge to a solution that satisfies this property of not being unduly influenced by outliers, or states with high feature-vector norms. Hence, we call our algorithm scale-invariant — as the scale of the input features does not matter to the output solution.

We now outline the basic (linear) framework under which our Algorithms operate.

1.2.1 The Update Rule

Consider any overdetermined linear system \( \Phi w = V \), consisting of \( m \) rows of the form \( \phi_i^\top w = V_i, \phi_i, w \in \mathbb{R}^n \). Let \( D \) be a diagonal weight matrix with entries \( d_1, \ldots, d_m \). If we wish to solve

\[
  w^* = \min_w \sum_{i=1}^{m} d_i (\phi_i w - V_i)^2 / ||\phi_i||^2
\]  

Then the stochastic update (with say \( \tau \) samples) takes the form

\[
  w_{k+1} = w_k - \alpha_k \frac{1}{\tau} \sum_{i=1}^{\tau} \frac{\phi_i^\top w_k - V_i}{||\phi_i||^2} \phi_i
\]

where the rows are sampled with probability \( d_i \) and \( \alpha_k \) is some step-size sequence. Since each expression of the form \( (\phi_i^\top w_k - V_i)\phi_i / ||\phi_i||^2 \) is a projection from \( w_k \) onto the hyperplane \( \phi_i^\top w = V_i \), we call the update map from \( w_k \) to \( w_{k+1} \) for all iterations \( k \) as Total Projections (TP) map. In general, such a map changes per iteration as a different rows of the form \( \phi_i^\top w = V_i \) are chosen. Depending on such a choice, the map at step \( k \) may be called \( TP_k : \mathbb{R}^n \rightarrow \mathbb{R}^n \). In other words, \( w_{k+1} = TP_k(w_k) \). This is a Kaczmazr based algorithm (Kaczmarz, 1937) which converges to a consistent solution in the presence of no noise.

For our full update rule, we need to add a momentum and our choice of step size. For the momentum part we use heavyball momentum with constant \( \beta \) (reasons in Section 5). For our step size, we use an osculating circle based step choice, which we call the curvature step (details in Section 4.1). We provide evidence that the step size works in the section 1.2.2. With these in place, we now describe our full update rule.
Let $TP_k(w) = \frac{1}{2} \sum_{i=1}^{\tau} (\phi_i w - V_i) \phi_i / ||\phi_i||^2$, and $\Delta TP_k(w_k) = TP_k((w_k - TP_k(w_k))) - TP_k(w_k)$.

Here the stochastic gradient update on $w_k$ with respect to our error term is given by $TP_k(w_k)$ and $\Delta TP_k(w_k)$ indicates the change in gradient. Our update rule is then given by:

$$w_{k+1} = w_k - \eta_k \frac{||TP_k(w_k)||}{||\Delta TP_k(w_k)||} TP_k(w_k) + \beta (w_k - w_{k-1})$$

(3)

where $\beta \in (0, 1)$ and $\eta_k = 1/k^{\beta}$; $p \in (0.5, 1)$. We now provide an intuition for each of the terms.

The second term in Equation 3 indicates the gradient update. Note that $\eta_k$ is a decreasing step-size sequence. Typically, one might use some sequence $\{\eta_k\}$ such that $\sum_{k=1}^{\infty} \eta_k = \infty$ and $\sum_{k=1}^{\infty} \eta_k^2 < \infty$. Such a requirement is satisfied by $\eta_k = 1/k^{p}$ where $p \in (0.5, 1)$ (Robbins and Monro, 1951; Blum, 1954). Recall that $TP_k$ is a map that gives the projections over the sampled hyperplanes $\phi_i w = V_i$. Therefore $TP_k(w_k)$ is an update in the direction of the required (total) projection from $w_k$ onto the hyperplanes chosen, i.e. a gradient descent update from $w_k$ towards $w^*$ in our chosen error metric.

The updates to $w_k$ at discrete time steps may be assumed to be a noisy discretization to a continuous curve $\omega(t)$ at some time $t$ such that $\omega(t) = w_k$. Then $\omega'(t)$ – the tangent to the curve $\omega(t)$ – may be approximated by the update $TP_k(w_k)$. The unit tangent to the curve $\omega(t)$ is given by $TP_k(w_k)/||TP_k(w_k)||$. Further, $\omega''(t)$ is approximated by the update $\Delta TP_k(w_k)$. But the radius of curvature $\kappa = ||dT/d\omega||$ where $T(t)$ is the unit tangent at time $t$, and $\omega(t)$ is the parameterized curve (Kuhnel, 2015).

Then by the Chain Rule, $\kappa = ||dT/dt||/||d\omega/dt|| = ||dT/dt||/||\omega'(t)|| = ||\omega''(t)||/||\omega'(t)||^2$ (Tapp, 2016). Then we find the approximation $\kappa \approx ||\Delta TP_k(w_k)||/||TP_k(w_k)||^2$ for the discrete setting and the radius of curvature $R = 1/\kappa = ||TP_k(w_k)||^2/||\Delta TP_k(w_k)||$. Then the update rule becomes $R$-Unit gradient vector $= R \cdot TP_k(w_k)/||TP_k(w_k)|| / ||\Delta TP_k(w_k)|| = ||TP_k(w_k)|| / ||\Delta TP_k(w_k)|| \cdot TP_k(w_k)$.

The third term in Equation 3 is a heavy-ball momentum term, where we add some constant ($\beta$) times the previous updates. This momentum term, is less useful in the context where we have no noise, but can be useful in the case of noisy updates (Gitman et al., 2019; Sutskever et al., 2013; Polyak, 1964).

In light of the multiple expressions in the update rule given by Equation 3, showing convergence is not straightforward. We use the theory of stochastic approximation to establish almost sure (a.s.) convergence for the algorithms we propose. This is a key technical contribution of this work.

1.2.2 Evidence for Adaptive Step Size

As outlined in the previous section, the adaptive step size that we choose is derived from the radius of curvature of the continuous curve that approximates our discrete updates in $w_k$. Such a step size sequence, performs quite well in simulations as outlined below. Note that in this simulation, the updates are not noisy. Even allowing for this, the exponential convergence was surprising.
In figure 1a, we plot the errors (as measured by distance from the error minimizer \( w^* \)) for the modified error function as given in equation 1) with number of iterations for total projections with curvature step algorithm. The number of states \( m = 100 \) and number of features \( n = 30 \). We note the exponential convergence and that the error decreases monotonically on a log-scale. This shows that with the increased curvature-step size, we still have a contraction on the error function.

1.2.3 The RL Context

Using Equation 3, we propose an algorithm Scale Invariant Monte-carlo (SIM-Algorithm) with curvature step. In the SIM-Algorithm, the role of \( \Phi \) is played by the feature vectors for the states. The value vector for the states \( V_i \) is estimated by the First-visit monte carlo where we sum the rewards from state \( i \) until termination. Thus, \( V_i = \sum_{t=1}^{n} \gamma^{t-1} R_t \) where the state of the Markov Chain at time \( t = 0 \), \( s(0) = i \). We note that the sampling of states in the Markov Chain happens as per the stationary distribution of the transition matrix (asymptotically). Thus \( \mathbb{P}(s(t) = i : \forall t > T_0) = d_i \) for some large \( T_0 \).

2 NOTATION AND PRELIMINARIES

Let us consider an RL setting with state space \( S \), where \( |S| = m \). Let the states be labeled \( \{1 \ldots m\} \). Consider an Markov Decision Process (MDP) given by \( M = (S, \mathbb{A}, P, R) \) Szepesvari (2010) and a discount factor \( \gamma \). Consider a deterministic stationary policy \( \mu : S \rightarrow \mathbb{A} \). This induces a transition matrix \( P \in \mathbb{R}^{m \times m} \). \( P \) gives a probability distribution over next states for each given state. The probability of transition from states \( s \) to \( s' \) \( (s, s' \in S) \) is given by \( P_{ss'} \). Given \( s \), the vector of transition probabilities over all \( s' \in S \) is given by \( P_s \). We will assume full mixing and ergodicity. Then let \( \pi \in \mathbb{R}^m \) be the stationary distribution associated with \( P \), and \( D \in \mathbb{R}^{m \times m} \) be the diagonal matrix associated with vector \( \pi \).

Let \( R_{ss'} \) indicates the reward on transition between state \( s \) and \( s' \) \( (s, s' \in S) \). Let \( \phi_s \in \mathbb{R}^n \) be the set of features associated with each state and \( \Phi \) be the corresponding matrix of all features. In the value estimation problem, we want to find the value \( V \in \mathbb{R}^m \), under a policy \( \mu \), for each state. Then, for each state \( s \) we have Szepesvari (2010) that \( V_s = E[\sum_{t=0}^\infty \gamma^t R_{t+1}|S_0 = s] \). Under the linear function approximation, we estimate \( V \) as \( \Phi_w \), where \( w \in \mathbb{R}^n \) denotes the feature weights. We denote the error function for the iterate in the SIM Algorithm as \( G(\cdot) \).

Let the weight to which the regular Monte Carlo algorithm converges be called \( \tilde{w} \) and the best approximation to the value vector \( V = \Phi \tilde{w} = \tilde{V} \). Note that \( \tilde{w} = \min_w \sum_{i=1}^m d_i ||\phi_i w - V_i||^2 \). Similarly, let the weight vector to which we want SIM Algorithm to converge be \( w^* \). Then \( w^* = \min_w \sum_{i=1}^m d_i ||\phi_i w - V_i||^2 / ||\phi_i||^2 \). Let \( V^* = \Phi w^* \) be our approximation of the value vector \( V \).

We denote the length of episode in Monte Carlo as \( T \) with number of unique states seen as \( \tau \).

3 MAIN ALGORITHM AND ITS ANALYSIS

We outline our Total Projections (TP) method as a general method to find the scale invariant solution to an overdetermined system, through repeated projections. Our main method is given in algorithm 1, where we run through a trajectory sampled from the stationary distribution. This method calls as a subroutine algorithm 2, for a one step stochastic weight update. This method is inspired by Randomized Kaczmarz (our main modifications are highlighted in appendix B). We speed up the algorithm through a novel step size method (section 4.1) and momentum (section 5).

The algorithm follows the same design of the regular Monte Carlo Algorithm for reinforcement learning in the outer loop (Sutton and Barto, 1998). This is indicated in Algorithm 1. Here we run a trajectory as per an \( \epsilon \)-greedy policy with respect to the calculated weight vector \( w_k \). We set the \( \epsilon \) to be some sequence that decays to 0. Asymptotically, this algorithm is greedy with respect to the approximated Value vectors \( V_k = \Phi w_k \). In other words, at every state, it chooses the action that maximizes the one step reward plus the value at the next state.

The above Algorithm runs the improved Algorithm 2, TP subroutine, which incorporate our major ideas. As noted in the discussion in Section 1.2, we use heavy-ball momentum and also use curvature-step with a decreasing multiplier \( \eta = 1/k \).
We envisage that this sub-routine can be utilized in other reinforcement learning algorithms under Algorithm 1.

Algorithm 1 SIM Algorithm for First-visit MC with curvature-step
1: Input: \( \Phi \), max Iterations
2: Output: \( w^* \) - estimated ideal output weights
3: Initialize weight vector \( w_0 \).
4: while \( ||w_k - w_{k-1}|| > \varepsilon \) do
5: Value Function Estimate \( V_k = \Phi w_k \)
6: Let policy \( \mu_k \) be \( \varepsilon \)-greedy with respect to \( V_k \)
7: GetTrajectory as per policy \( \mu_k \)
8: \( w_{k+1} \leftarrow \text{TP}(w_k, \Phi) \)
9: \( k \leftarrow k + 1 \)
10: \( w^* \leftarrow w_k \), \( V^* \leftarrow \Phi w^* \)
11: \( \mu \leftarrow \text{greedy policy with respect to } V^* \)
12: return \( w^*, V^*, \mu^* \)

Remark 1. Our main improvements are in the inner subroutine, Algorithm 2, of the SIM-Algorithm. We envisage that this sub-routine can be utilized in other reinforcement learning algorithms under linear function approximation. The requirement is an ability to approximate the value function at each state, which in the case of Monte Carlo is the discounted sum of rewards from any state to the terminal state in the trajectory.

3.1 Analysis of Convergence

Theorem 1. The stochastic approximation algorithm

\[
w_{k+1} = w_k - \eta_k \frac{||TP_k(w_k)||}{||\Delta TP_k(w_k)||} TP_k(w_k) + \beta (w_k - w_{k-1})
\]

converges a.s. to

\[
w^* = [(\Phi^T N D N \Phi)^{-1} \Phi^T N D N] V
\]

where \( N \) is diagonal with \( N_{i,i} = \frac{1}{||\phi_i||^2} \), \( \beta \in (0, 1) \), \( \eta_k = 1/k^p \); \( p \in (0.5, 1] \)

To prove the above theorem, we first propose a simpler Theorem 2, which does not involve the momentum term. We state and prove this below.

Theorem 2. \( w_{k+1} = w_k - \alpha_k \cdot TP_k(w_k) \) without momentum converges to \( w^* \) (a.s)

To prove convergence, we need to show that four conditions are satisfied.

The major claims that we use in this proof are the following:

Fact: \( V_t \)'s are bounded. In other words, if \( R_{\text{max}} = \max_{s,s'} [R_{ss'}] \), then \( V_i \leq R_{\text{max}}/(1 - \gamma) \) \( \forall i \)

Fact: \( \tau \) is bounded as it is the number of unique states

Now let the filtration be \( \mathcal{F}_k = \{w_0, \ldots, w_k\} \). For the stochastic update equation in theorem 2, let the expected update be \( h_{k+1}(w_k) = \mathbb{E}[TP_k(w_k)|\mathcal{F}_k] \). Then, the update rule in standard form is \( w_{k+1} = w_k - \alpha h_{k+1}(w_k) + M_{k+1} \)

Proposition 3.1. \( h_{k+1}(w_k) \) is Lipschitz

Proof. Proof in Appendix C.4 and appendix D.3.1

Proposition 3.2. The step size sequence \( \{\alpha_i\} \) satisfy \( \sum_{i=0}^{\infty} \alpha_i = \infty \) and \( \sum_{i=0}^{\infty} \alpha_i^2 < \infty \)

Proof Sketch. This proceeds from our construction of the step size sequence in section 4.1. See appendix D.3.2 for full proof.
Proposition 3.3. \( \{M_k\} \) is a zero-mean martingale difference noise sequence

Proof. We show this in appendix D.3.3.

Proposition 3.4. The iterates remain bounded almost surely. In other words, \( \sup_k w_k < \infty \) (a.s).

Proof. First note that \( V_i \)'s are upper-bounded. Thus the estimates for the hyperplanes are upper-bounded. Now, in a fully determined system, there is at least one, and at most \( m \) intersection points in \( \mathbb{R}^n \) of the \( m \) hyperplanes. Since each iteration brings us closer to at least one of these intersection points (by the Pythagoras theorem, as we are doing projections), and the intersection points are all bounded, the iterates are almost surely bounded.

Proposition 3.5. Let \( h(\cdot) \) be the function which our update equation tracks asymptotically, then the unique globally asymptotically stable equilibrium point for the limiting o.d.e given by \( \dot{w}(t) = h(w(t)) \) is given as \( w^* = \left[ ( \Phi^\top NDN \Phi )^{-1} \Phi^\top NDN \right] V \)

Proof. We show this in appendix D.

Proof of Theorem 2. From propositions 3.1, 3.2, 3.3, 3.4, we satisfy the assumptions A1-A4 required to show convergence of a stochastic approximation equation Borkar (2008). Based on proposition 3.5 we converge to the unique globally asymptotically stable equilibrium point given by \( w^* = \left[ ( \Phi^\top NDN \Phi )^{-1} \Phi^\top NDN \right] V \)

3.2 Convergence using Momentum

Momentum methods have been shown to converge by Défossez et al. (2020); Reddi et al. (2018). Convergence under heavy-ball momentum has been shown by Ghadimi et al. (2015). Avrachenkov et al. (2020) have used two-time scale methods to show convergence under momentum terms. We consider one such adaptation of these general techniques here.

Proof of Theorem 1. We cover the full proof in appendix F. Here we provide two propositions (from appendix F) that show that the final iterate is the same as the iterate without momentum, added with perturbation terms and a zero-mean martingale noise sequence. Given that the martingale noise and perturbations have zero expectation and are multiplied with a decaying scalar (that is square summable, but not summable), the convergence properties are the same as for the case without momentum.

Proposition 3.6. The stochastic approximation equation with momentum can be rewritten as

\[
\begin{align*}
  w_{k+1} - w_k &= \alpha_k z_k \\
  z_i &= z_{i-1} + \zeta_{i,k} \left[ h(w_k) + \varepsilon_{i,k} + M_{i,k} \right] \quad \forall i \in [1, k] \\
  z_0 &= [h(w_k) + M_{0,k}]
\end{align*}
\]

where \( M_{i,k} \) are martingale difference noise, coefficients \( \zeta_{i,k} = \beta^i \frac{\alpha_k}{\alpha_n} \) provide exponential decay, expected update \( h(\cdot) \) converges to \( w^* \) and \( \varepsilon_{i,k} \) are perturbation terms.

Proposition 3.7. The above set of equations collapse into the stochastic equation \( w_{k+1} - w_k = \alpha_k [h(w_k) + \hat{\varepsilon}_k + M_k] \) where \( h(w_k) \) converges to \( w^* \), \( \{\hat{\varepsilon}_k\} \) are perturbation terms and \( \{M_k\} \) are martingale difference noise terms.

Note that the perturbation terms don’t affect convergence and Martingales difference random variables have expectation 0. Therefore convergence mainly depends on the first term. But the first term is the same as in Theorem 2. Therefore the iterates converge to the same point as in Theorem 2, even in the presence of momentum.

4 DISCUSSION ON STEP SIZE

In this section we cover in detail our curvature step size and choice of momentum method.
4.1 Adaptive step sizes for Total Projections Algorithm

Choice of step size is extremely important for ML practitioners. We propose a novel variation for a step size sequence.

To achieve convergence for a stochastic approximation algorithm, we need the step size sequences \( \{ \alpha_k \} \) to be such that \( \sum_{k=1}^{\infty} \alpha_k = \infty \) and \( \sum_{k=1}^{\infty} \alpha_k^2 < \infty \) (Borkar, 2008). To achieve this, our step size sequence takes the form \( \eta_k \cdot \theta_k ||TP_k(\cdot)|| \), where \( \eta_k = 1/k^p; \ p \in (0.5, 1] \). The second term \( \theta_k \) is the term of interest currently, and the third term makes the existing update term \( TP_k(\cdot) \) unit norm.

4.1.1 Idea for Curvature Step

Figure 2: Illustration for how the curvature-step works

Figure 2 illustrates the working of our curvature step on the step size based on the radius of the osculating circle. \( w_k \) is the iterate, and C is the center of the circle formed by the osculating circle. We calculate the radius based on intermediate points \( w'_k \) and \( w''_k \), to finally get to point \( w_{k+1} \).

4.1.2 Estimating Radius of Osculating circle

Let \( w_k(t) \) be some stochastic gradient curve we are descending, with some subset of hyperplanes fixed. Then the curvature is given by \( \kappa = ||w''(\cdot)|| \), where \( w \) is parameterized to some unit vector in the space, and radius \( R = 1/\kappa \).

Note that our updates, \( TP_k(\cdot) \) are tangents to \( w(t) \). Since our estimates are not unit parameterized, we need an appropriate change of scale (re-parametrization). In other words, we divide our estimate for tangent by \( ||TP_k(w_k)|| \), to get the unit tangent. Similar re-scaling of our estimate for curvature yields \( ||TP_k(w_k)||^2 / ||\Delta TP_k(w_k)|| \) in the denominator Chappers (2017).

Let \( \Delta TP_k(w_k) = TP_k(w_k - TP_k(w_k)) - TP_k(w_k) \). Then, our guess for the second derivative is \( ||\Delta TP_k(w_k)|| \), which after re-parametrization gives \( ||\Delta TP_k(w_k)|| / ||TP_k(w_k)||^2 \). Then we have \( R = 1/\kappa = ||TP_k(w_k)||^2 / ||\Delta TP_k(w_k)|| \). Thus:

\[
\theta_k = \frac{||TP_k(w_k)||^2}{||\Delta TP_k(w_k)||}
\]  \hspace{1cm} (6)

Thus our update equation (without momentum) becomes:

\[
w_{k+1} = w_k - \eta_{\theta_k} \frac{TP_k(w_k)}{||TP_k(w_k)||}
= w_k - \eta_{\theta_k} \frac{||TP_k(w_k)||}{||\Delta TP_k(w_k)||} (TP_k(w_k))
\]  \hspace{1cm} (7)

We call the step size sequence \( \alpha_k \) as curvature-step sequence. We now provide a visual illustration and rationale for the curvature-step, for consideration alongside Figure 1a.
5 EXPERIMENTS

We carried out simulations for systems with 25 states and 10 features \((m=25,n=10)\) in the presence of noise to see efficacy of our proposed algorithm. We carry out two experiments. The first is to determine the momentum method to be used with our curvature step size method. The second experiment is to compare the efficacy of using a normal step size, using curvature step size with no momentum, and using curvature step size with (Polyak’s) heavyball momentum. We outline these experiments below.

Momentum Method Used: Of the various momentum optimization methods used in gradient descent algorithms Ruder (2017), our comparisons (figure 3a) showed Heavy Ball momentum with \(\beta = 0.5\) works best (reasons in appendix E). We use this for our step size sequence. We notice that the decrease in error using some of the momentum methods is not monotonic, meaning that there could be bad updates that are amplified by the momentum method used. In this sense, the heavy-ball momentum is conservative, and ensures convergence so long as the original iterates converge, even in the presence of noise. We next look at whether using the heavy-ball momentum so chosen, we get better convergence rates than without using momentum, in the noisy setting.

Advantage of using curvature step and momentum with noisy updates: In Figure 3b, we compare convergence using (1) No Curvature Step size (2) Curvature Step and (3) Momentum. This is for the setting with \(m = 25, n = 10\). We notice that the setting with curvature-step and heavy-ball momentum (with \(\beta = 0.5\)) works best.

Remark 2. Our experiments show that the setting without curvature step has very poor convergence rates. This is in-line with the convergence rate of the Randomized Kaczmarz algorithm (Strohmer and Vershynin, 2009b) which is inversely proportional to the square of the condition number of the linear system. Our experiments show that the curvature step reduces the dependence of convergence rate on the condition number (see figure 1a). We further note this reduced dependence continues in the noisy setting as well (see figure 3b).

6 CONCLUSIONS AND DISCUSSION

In this work, we presented a scale-invariant version of the popular Monte Carlo algorithm for reinforcement learning. We gave a rationale for why Least Squares criterion fails in many instances, and the feature-scaled version should be used in the linear-function approximation setting. We then proposed a novel adaptive step size sequence based on the curvature of the path of convergence of the iterate \(w_k\). We provided a convergence proof for this algorithm in the presence of momentum. Finally we experimentally validated our algorithm through simulations and showed that in the presence of noise we have significant speedups over the regular algorithm. Without noise, our algorithm in fact has exponentially faster convergence than the usual stochastic gradient update rule.

A possible extension of our work would be to use the proposed step size in the context of non-linear, non-convex settings. Further, we believe there is merit in applying our scale-invariant algorithm (rather than some least-squares variant) in various other linear settings where we wish to give equal importance to all data points – irrespective of norm.
REFERENCES

Ang, W.-P. and Farhang-Boroujeny, B. (2001). A new class of gradient adaptive step-size lms algorithms. *IEEE transactions on signal processing*, 49(4):805–810.

Avrachenkov, K., Patil, K., and Thoppe, G. (2020). Online algorithms for estimating change rates of web pages.

Axler, S. J. (1997). *Linear Algebra Done Right*. Undergraduate Texts in Mathematics. Springer, New York.

Ba, J. L., Kiros, J. R., and Hinton, G. E. (2016a). Layer normalization. *arXiv preprint arXiv:1607.06450*.

Ba, L. J., Kiros, J. R., and Hinton, G. E. (2016b). Layer normalization. *CoRR*, abs/1607.06450.

Bellman, R. (1958). Dynamic programming and stochastic control processes. *Information and Control*, 1(3):228–239.

Ben-Gal, I. (2005). *Outlier Detection*, pages 131–146. Springer US, Boston, MA.

Bertsekas, D. P. (2011). Approximate policy iteration: A survey and some new methods. *Journal of Control Theory and Applications*, 9(3):310–335.

Bertsekas, D. P., Borkar, V. S., and Nedic, A. (2004). Improved temporal difference methods with linear function approximation. *Learning and Approximate Dynamic Programming*, pages 231–255.

Bhatt, A., Argus, M., Amiranashvili, A., and Brox, T. (2019). Crossnorm: Normalization for off-policy td reinforcement learning. *arXiv preprint arXiv:1902.05605*.

Bishop, C. M. (1995). *Neural Networks for Pattern Recognition*. Oxford University Press, Inc., USA.

Blum, J. R. (1954). Approximation Methods which Converge with Probability one. *The Annals of Mathematical Statistics*, 25(2):382 – 386.

Borkar, V. S. (2008). *Stochastic Approximations, A Dynamical Systems Viewpoint*. Cambridge University Press.

Borkar, V. S. and Meyn, S. P. (2000). The ode method for convergence of stochastic approximation and reinforcement learning. *SIAM Journal on Control and Optimization*, 38(2):447–469.

Botchkarev, A. (2019). A new typology design of performance metrics to measure errors in machine learning regression algorithms. *Interdisciplinary Journal of Information, Knowledge, and Management*, 14:045–076.

Boyd, S. and Vandenberghe, L. (2004). *Convex Optimization*. Cambridge University Press, USA.

Casella, G. and Berger, R. (2001). *Statistical Inference*. Duxbury Resource Center.

Chappers (2017). Curvature derivation for arbitrary parameterization. Mathematics Stack Exchange. Author: Chappers, https://math.stackexchange.com/users/221811/chappers, URL:https://math.stackexchange.com/q/2153902 (version: 2017-02-21).

Chen, X., Liu, S., Sun, R., and Hong, M. (2019). On the convergence of a class of adam-type algorithms for non-convex optimization. In *International Conference on Learning Representations*.

Cinlar, E. (2011). *Martingales and Stochastics*, pages 172–242. Springer New York, New York, NY.

Dabney, W. and Barto, A. G. (2012). Adaptive step-size for online temporal difference learning. In *Twenty-Sixth AAAI Conference on Artificial Intelligence*.

Dayan, P. (1992). The convergence of td (λ) for general λ. *Machine learning*, 8(3):341–362.

Dayan, P. and Sejnowski, T. J. (1994). Td(lambda) converges with probability 1. *Mach. Learn.*, 14(3):295–301.
Défossez, A., Bottou, L., Bach, F., and Usunier, N. (2020). On the convergence of adam and adagrad. *CoRR*, abs/2003.02395.

Duchi, J., Hazan, E., and Singer, Y. (2011). Adaptive subgradient methods for online learning and stochastic optimization. *Journal of machine learning research*, 12(7).

Ghadimi, E., Feyzmahdavian, H. R., and Johansson, M. (2015). Global convergence of the heavy-ball method for convex optimization. In *2015 European control conference (ECC)*, pages 310–315. IEEE.

Gitman, I., Lang, H., Zhang, P., and Xiao, L. (2019). Understanding the role of momentum in stochastic gradient methods. *Advances in Neural Information Processing Systems*, 32.

Han, J., Kamber, M., and Pei, J. (2011). *Data Mining: Concepts and Techniques*. Morgan Kaufmann Publishers Inc., San Francisco, CA, USA, 3rd edition.

Howard, R. A. (1960). *Dynamic programming and markov processes*. John Wiley.

Huang, L., Qin, J., Zhou, Y., Zhu, F., Liu, L., and Shao, L. (2020). Normalization techniques in training dnns: Methodology, analysis and application. *arXiv preprint arXiv:2009.12836*.

Ioffe, S. and Szegedy, C. (2015). Batch normalization: Accelerating deep network training by reducing internal covariate shift. In *International conference on machine learning*, pages 448–456. PMLR.

Kaczmarz, S. (1937). Angenäherte auflösung von systemen linearer gleichungen. *Bulletin International de l’Académie Polonaise des Sciences et des Lettres. Classe des Sciences Mathématiques et Naturelles. Série A, Sciences Mathématiques*, 35:355–357.

Kingma, D. P. and Ba, J. (2015). Adam: A method for stochastic optimization. In Bengio, Y. and LeCun, Y., editors, *3rd International Conference on Learning Representations, ICLR 2015, San Diego, CA, USA, May 7-9, 2015, Conference Track Proceedings*.

Klambauer, G., Unterthiner, T., Mayr, A., and Hochreiter, S. (2017). Self-normalizing neural networks. *Advances in neural information processing systems*, 30.

Konda, V. and Tsitsiklis, J. (1999). Actor-critic algorithms. *Advances in neural information processing systems*, 12.

Korda, N. and La, P. (2015). On td (0) with function approximation: Concentration bounds and a centered variant with exponential convergence. In *International conference on machine learning*, pages 626–634. PMLR.

Kuhnel, W. (2015). *Differential Geometry*. Student Mathematical Library. American Mathematical Society.

Kushner, H. J. and Yang, J. (1994). Analysis of adaptive step size sa algorithms for parameter tracking. In *Proceedings of 1994 33rd IEEE Conference on Decision and Control*, volume 1, pages 730–737. IEEE.

Kushner, H. J. and Yin, G. G. (1997). *Stochastic Approximation Algorithms and Applications*. Springer New York, New York, NY.

Lakshminarayanan, C. and Bhatnagar, S. (2017). A stability criterion for two timescale stochastic approximation schemes. *Automatica*, 79:108 – 114.

Li, D., Zhang, B., and Li, C. (2016). A feature-scaling-based k-nearest neighbor algorithm for indoor positioning systems. *IEEE Internet of Things Journal*, 3(4):590–597.

Ljung, L. (1978). Strong convergence of a stochastic approximation algorithm. *The Annals of Statistics*, 6(3):680–696.

Loizou, N., Vaswani, S., Laradji, I. H., and Lacoste-Julien, S. (2021). Stochastic polyak step-size for sgd: An adaptive learning rate for fast convergence. In *International Conference on Artificial Intelligence and Statistics*, pages 1306–1314. PMLR.
Mai, V. and Johansson, M. (2020). Convergence of a stochastic gradient method with momentum for non-smooth non-convex optimization. In International Conference on Machine Learning, pages 6630–6639. PMLR.

Murthy, K. R. (2021). Show \( \sum_{i=0}^{k-1} r_i^k \) for \( r \in (0, 1) \) goes to 0 as \( k \to \infty \). Mathematics Stack Exchange. URL:https://math.stackexchange.com/q/4005591 (version: 2021-01-30).

Perkins, T. and Precup, D. (2002). A convergent form of approximate policy iteration. Advances in neural information processing systems, 15.

Pirotta, M., Restelli, M., and Bascetta, L. (2013). Adaptive step-size for policy gradient methods. Advances in Neural Information Processing Systems, 26.

Polyak, B. T. (1964). Some methods of speeding up the convergence of iteration methods. Ussr computational mathematics and mathematical physics, 4(5):1–17.

Reddi, S. J., Kale, S., and Kumar, S. (2018). On the convergence of adam and beyond. In International Conference on Learning Representations.

Robbins, H. and Monro, S. (1951). A stochastic approximation method. The annals of mathematical statistics, pages 400–407.

Ruder, S. (2016). An overview of gradient descent optimization algorithms. arXiv preprint arXiv:1609.04747.

Ruder, S. (2017). An overview of gradient descent optimization algorithms.

Salimans, T. and Kingma, D. P. (2016). Weight normalization: A simple reparameterization to accelerate training of deep neural networks. Advances in neural information processing systems, 29.

Santurkar, S., Tsipras, D., Ilyas, A., and Madry, A. (2018a). How does batch normalization help optimization? Advances in neural information processing systems, 31.

Santurkar, S., Tsipras, D., Ilyas, A., and Madry, A. (2018b). How does batch normalization help optimization? In Bengio, S., Wallach, H., Larochelle, H., Grauman, K., Cesa-Bianchi, N., and Garnett, R., editors, Advances in Neural Information Processing Systems, volume 31. Curran Associates, Inc.

Schumer, M. and Steiglitz, K. (1968). Adaptive step size random search. IEEE Transactions on Automatic Control, 13(3):270–276.

Singh, D. and Singh, B. (2020). Investigating the impact of data normalization on classification performance. Applied Soft Computing, 97:105524.

Sola, J. and Sevilla, J. (1997). Importance of input data normalization for the application of neural networks to complex industrial problems. IEEE Transactions on Nuclear Science, 44(3):1464–1468.

Stolcke, A., Kajarekar, S., and Ferrer, L. (2008). Nonparametric feature normalization for svm-based speaker verification. In 2008 IEEE International Conference on Acoustics, Speech and Signal Processing, pages 1577–1580.

Strohmer, T. and Vershynin, R. (2009a). Comments on the randomized kaczmarz method. Journal of Fourier Analysis and Applications, 15(4):437–440.

Strohmer, T. and Vershynin, R. (2009b). Randomized kaczmarz for sampling distribution. Journal of Fourier Analysis and Applications, 15(262).

Sutskever, I., Martens, J., Dahl, G., and Hinton, G. (2013). On the importance of initialization and momentum in deep learning. In International conference on machine learning, pages 1139–1147. PMLR.
Sutton, R. S. and Barto, A. G. (1998). *Introduction to Reinforcement Learning*. MIT Press, Cambridge, MA, USA, 1st edition.

Szepesvari, C. (2010). *Algorithms for Reinforcement Learning*. Morgan and Claypool Publishers.

Tapp, K. (2016). *Curves*, pages 1–60. Springer International Publishing, Cham.

Tsitsiklis, J. and Van Roy, B. (1996). Analysis of temporal-difference learning with function approximation. *Advances in neural information processing systems*, 9.

Tsitsiklis, J. N. (2002). On the convergence of optimistic policy iteration. *Journal of Machine Learning Research*, 3(Jul):59–72.

Wu, Y. and He, K. (2018). Group normalization. In *Proceedings of the European conference on computer vision (ECCV)*, pages 3–19.

Yang, T., Lin, Q., and Li, Z. (2016). Unified convergence analysis of stochastic momentum methods for convex and non-convex optimization. *arXiv preprint arXiv:1604.03257*.

Zeiler, M. D. (2012). Adadelta: an adaptive learning rate method. *arXiv preprint arXiv:1212.5701*. 
Appendices

A CONVERGENCE POINT OF THE MONTE CARLO - LEAST SQUARES SOLUTION

We now calculate the convergence point of the Monte Carlo algorithm. The first visit Monte Carlo is an unbiased estimator for the value corresponding to states. Further, the updates under the Monte Carlo algorithm with linear function approximation correspond to a stochastic gradient descent on the least squares error function Szepesvari (2010); Sutton and Barto (1998). We will show here that the convergence point of the algorithm is given by

\[ w^{**} = (\Phi^\top \Phi)^{-1}(\Phi^\top V) \]

Proposition A.1. \[ w^{**} = (\Phi^\top \Phi)^{-1}(\Phi^\top V) \] and \[ V^{**} = \Phi (\Phi^\top \Phi)^{-1}(\Phi^\top V) \]

Proof.

\[ w^{**} = \arg \min_{w \in \mathbb{R}^n} \sum_{s \in S} (\phi_s^\top w - V_s)^2 \]  \hspace{1cm} (8)

Taking the derivative and setting it to 0 for the arg-min:

\[ \frac{d}{dw} \left( \sum_{s \in S} (\phi_s^\top w^{**} - V_s)^2 \right) = 0 \]  \hspace{1cm} (9)

taking the derivative:

\[ \sum_{s \in S} 2\phi_s (\phi_s^\top w^{**} - V_s) = 0 \]

\[ \left( \sum_{s \in S} \phi_s \phi_s^\top \right) w^{**} = \sum_{s \in S} V_s \phi_s \]  \hspace{1cm} (10)

\[ \left( \sum_{s \in S} \phi_s \phi_s^\top \right) = \Phi \cdot \Phi^\top \text{ and } \left( \sum_{s \in S} \phi_s V_s \right) = \Phi^\top \cdot V. \text{ Thus:} \]

\[ \Phi^\top \Phi w^{**} = \Phi^\top V \]

Then we have:

\[ w^{**} = (\Phi^\top \Phi)^{-1}(\Phi^\top V) \]
\[ V^{**} = \Phi w^{**} = \Phi (\Phi^\top \Phi)^{-1}(\Phi^\top V) \]  \hspace{1cm} (11)

Thus in the case of Least Squares we have the solution given by \[ V^{**} = \Phi w^{**} = \Phi (\Phi^\top \Phi)^{-1}(\Phi^\top V) \]

In figure 4, we illustrate the \( \mathbb{R}^m \) perspective of the least squares solution. The least squares solution is a projection onto the column space of \( \Phi \). In other words, the solution is the point on the column space of \( \Phi \), which is at least distance from \( V \). Our claim is that such a solution may be unduly affected by rows which have large feature-norm.

For comparison, this solution can be compared with figure 5, where we illustrate in \( \mathbb{R}^n \) why distances to hyperplanes might be a scale invariant solution, which is unaffected by the feature norms.
B DIFFERENCES BETWEEN OUR ALGORITHM AND THE KACZMARZ ALGORITHM

Our algorithm is a variation on the Randomized Kaczmarz algorithm described in Strohmer and Vershynin (2009b). We note the major differences below

1. The Randomized Kaczmarz algorithm samples the hyperplanes with a probability proportional to the square of the feature-norm, viz \( ||\phi_s||^2 \) Strohmer and Vershynin (2009b). This approach has been criticized in literature Strohmer and Vershynin (2009a). (In our own simulations, this sampling did not provide any benefits). To sample proportional to the feature-norm square of the states, one needs to know the features-norms of all states, which may not be possible

2. In the RL context, obtaining all possible features ab-initio is difficult, and so is sampling as per feature-norm square. Natural sampling would be as per the stationary distribution of the ergodic Markov Chain and we allow for this.

3. The original Randomized Kaczmarz method was meant for a fully determined \( Ax = b \) system. Therefore, in the original setup, the iterates lie on hyperplanes onto which one projects. On the other hand, our iterates don’t lie on any hyperplane. This makes it easier to identify the sequence of iterates with a gradient field (of our error function).

4. We obtain major speedups (up to a few orders of magnitude) over the regular Kaczmarz method due to our usage of momentum and step size based on radius of osculatory-circle.

5. Our formulation makes the algorithm directly a gradient descent on the error function

\[
\sum_{s \in S} d_s \left( \frac{\phi_s^T w - V_s}{||\phi_s||} \right)^2
\]

where \( d_s \) are some positive weights corresponding the hyperplanes \( \mathcal{H}_s \equiv \phi_s^T w - V_s \). For example, \( d_s = \frac{1}{|S|} \forall s \in S \) may correspond to a uniform sampling. Another example is where \( d_s = \pi_s \) where \( \pi \) is the stationary distribution corresponding to the Transition Matrix of a Markov Chain.
C PROPERTIES OF THE TOTAL PROJECTION (TP) OPERATOR

In this section, we will consider properties of the Total Projections operation $TP(w) = \frac{1}{|S|} \sum_{s \in S} d_s \left[ \frac{\phi_s^\top w - V_s}{2||\phi_s||_2^2} \right] \phi_s$ and the error function $G(w) = \frac{1}{|S|} \sum_{s \in S} d_s \left[ \frac{\phi_s^\top w - V_s}{2||\phi_s||_2^2} \right]^2$ such that $\sum_{s \in S} d_s \leq |S|$ where $d_s$ are some positive weights attached to hyperplanes $H_s \equiv \phi_s^\top w - V_s$.

The properties shown below hold in general for positive weights $\{d_s\}_{s \in S}$ as long as $\sum_{s \in S} d_s \leq |S|$. But it may be worthwhile to consider what these positive weights may be. One example set of weights is $d_s = 1 \forall s \in S$, which may be considered as uniform weights. Another weight set is $d_s = \pi_s |S|$ where $\pi_s$ is the probability of occurrence of state $s$ in the stationary distribution, which will be of interest to us in our algorithms.

We will now show the following properties in the section numbers given:

C.1 $TP(\cdot) = \nabla_w G(\cdot)$

C.2 $G(\cdot)$ is convex

C.3 $G(\cdot)$ is strongly convex

C.4 $\nabla G(\cdot)$ is a Lipschitz function

C.5 Hess $(G(\cdot))$ is bounded above

C.6 The batch version of the Total Projections algorithm converges

C.7 Conditions on the step size of the total projection algorithm

C.8 Convergence Rate of the Total Projections Algorithm

C.1 Total Projection is a gradient descent on the error function

Proposition C.1.1. Let

$$TP(w) = \frac{1}{|S|} \sum_{s \in S} d_s \left[ \frac{\phi_s^\top w - V_s}{2||\phi_s||_2^2} \right] \phi_s$$

Then $TP(w) = \nabla_w G(w)$

Proof. We obtain this by just differentiating $G(\cdot)$ with respect to $w$.

Figure 5 is an illustration of the convergence point of the Total Projections Algorithm. We have three hyperplanes in $\mathbb{R}^2$ and we attempting to find a $w$ such that $w$ is the point that minimizes the total sum of squares of distances to these hyperplanes. Note that hyperplanes are scale invariant in the sense, $\phi_s^\top w = V_s$ is the same hyperplane as $c \cdot \phi_s^\top w = c \cdot V_s$ for any arbitrary $c \in \mathbb{R}$. Thus our solution remains invariant under a multiplication of any row by a constant $c \in \mathbb{R}$.

C.2 $G(\cdot)$ is convex

In this subsection, we will show:

(a) $G(w) = \frac{1}{|S|} \sum_{s \in S} d_s \left[ \frac{\phi_s^\top w - V_s}{2||\phi_s||_2^2} \right]^2$ is convex

using:

(b) $\phi \phi^\top$ is a positive semi definite matrix for all $\phi \in \mathbb{R}^n$

Proposition C.2.1. $G(\cdot)$ is convex in $\mathbb{R}^n$

Proof. We have already seen in Section C.1.1 that $\nabla_w G(w) = \frac{1}{|S|} \sum_{s \in S} d_s \left[ \frac{\phi_s^\top w - V_s}{2||\phi_s||_2^2} \right] \phi_s$. Now we have

1. $\mathbb{R}^n$ is a convex set
2. $G(\cdot)$ is twice differentiable
Thus it is sufficient to show that $Hess(G(\cdot))$ is positive semi-definite.

$$\nabla_w G(w) = \frac{1}{|S|} \sum_{s \in S} d_s \left[ \frac{\phi_s^T w - V_s}{2 ||\phi_s||_2} \right] \phi_s$$

Then,

$$Hess(G(\cdot)) = \nabla_w (\nabla_w G(w))$$

$$= \nabla_w \left( \frac{1}{|S|} \sum_{s \in S} d_s \left[ \frac{\phi_s^T w - V_s}{2 ||\phi_s||_2} \right] \phi_s \right)$$

$$= \frac{1}{|S|} \sum_{s \in S} d_s \nabla_w \left( \frac{(\phi_s^T w - V_s)\phi_s}{||\phi_s||_2^2} \right)$$

$$= \frac{1}{|S|} \sum_{s \in S} d_s \left( \frac{\nabla_w (\phi_s^T w)\phi_s}{||\phi_s||_2^2} \right)$$

$$= \frac{1}{|S|} \sum_{s \in S} d_s \left( \frac{\phi_s \nabla_w (\phi_s^T w)}{||\phi_s||_2^2} \right)$$

$$= \frac{1}{|S|} \sum_{s \in S} d_s \left( \phi_s \frac{\phi_s^T w}{||\phi_s||_2} \right)$$

By proposition C.2.2, $Hess(G(\cdot))$ is the sum of $|S|$ positive definite matrices, weighted by some positive coefficients $\frac{\pi_s}{|S| \cdot ||\phi_s||_2^2}$. Thus $Hess(G(\cdot))$ is positive semi definite. Thus $G(\cdot)$ is a convex function.
An alternate method to show $G(\cdot)$ is convex, would be to show that $w^T [\text{Hess}(G(w))] w \geq 0$. We showed earlier that $\text{Hess}(G(\cdot)) = \frac{1}{|S|} \sum_{s \in S} d_s \left( \frac{\phi_s \phi_s^T}{||\phi_s||^2_2} \right)$. Let $w \in \mathbb{R}^n$. Then we have

$$w^T [\text{Hess}(G(w))] w = w^T \frac{1}{|S|} \sum_{s \in S} d_s \left( \frac{\phi_s \phi_s^T}{||\phi_s||^2_2} \right) w$$

where $\phi_s \neq \mathbf{0}$ ∀i

$$= \frac{1}{|S|} \sum_{s \in S} d_s \left( \frac{w^T \phi_s \phi_s^T w}{||\phi_s||^2_2} \right)$$

$$= \frac{1}{|S|} \sum_{s \in S} d_s \left( \frac{(\phi_s^T w)^T (\phi_s^T w)}{||\phi_s||^2_2} \right)$$

$$= \frac{1}{|S|} \sum_{s \in S} d_s \left( \frac{(y_i)^T (y_i)}{||\phi_s||^2_2} \right)$$

where $y_i = \phi_s^T w$

$$= \frac{1}{|S|} \sum_{s \in S} d_s \left( \frac{||y_i||^2_2}{||\phi_s||^2_2} \right)$$

$$\geq 0 \quad \forall w \in \mathbb{R}^n$$

This shows that $G(\cdot)$ is convex.

\[ \square \]

**Proposition C.2.2.** $\phi \phi^T$ is a positive semi definite matrix for all $\phi \in \mathbb{R}^n$

**Proof.** Let $\Phi = \phi \phi^T$ (for this proposition). Then to prove $\Phi$ is psd, it is sufficient to show $w^T \Phi w \geq 0 \quad \forall w \in \mathbb{R}^n$

$$w^T \Phi w = w^T \phi \phi^T w$$

$$= (\phi^T w)^T (\phi^T w)$$

$$= y^T y \quad \text{where } y = \phi^T w$$

$$= ||y||^2_2 \quad \text{2-norm squared of } y$$

$$\geq 0$$

Thus $\Phi = \phi \phi^T$ is positive semi-definite for all $a \in \mathbb{R}^n$. \[ \square \]

### C.3 $G(\cdot)$ is strongly convex and thereby strictly convex

In this subsection, we will show:

(a) $G(\cdot)$ is strongly convex when $\text{rank}(\Phi) = n$

(b) $G(\cdot)$ is strictly convex when $\text{rank}(\Phi) = n$

**Proposition C.3.1.** $G(\cdot)$ is strongly convex if $\Phi \in \mathbb{R}^{m \times n}$ has rank $n$

**Proof.** If $\Phi$ has rank $n$, then the vectors $\{\phi_s\}_{s \in S}$ span $\mathbb{R}^n$. Then we have to show that if $\lambda_{min}$ is the least eigenvalue of $\text{Hess}(G(\cdot)) = \frac{1}{|S|} \sum_{s \in S} d_s \left( \frac{\phi_s \phi_s^T}{||\phi_s||^2_2} \right)$, then $\lambda_{min} > 0$. We show this as follows:
$\phi_s\phi_s^T$ is a rank 1 symmetric matrix. Symmetric matrices have real eigenvalues. Further,

\[
(\phi_s^T w)^2 \geq 0 \quad \forall \phi_s, w \in \mathbb{R}^n \\
(\phi_s^T w)(\phi_s^T w) \geq 0 \\
(w^T \phi_s)(\phi_s^T w) \geq 0 \\
w^T (\phi_s \phi_s^T) w \geq 0 \\
w^T \left( \frac{\phi_s \phi_s^T}{||\phi_s||_2^2} \right) w \geq 0 \\
\frac{1}{|S|} \sum_{s \in S} d_s w^T \left( \frac{\phi_s \phi_s^T}{||\phi_s||_2^2} \right) w \geq 0 \\
w^T \left( \frac{1}{|S|} \sum_{s \in S} d_s \left( \frac{\phi_s \phi_s^T}{||\phi_s||_2^2} \right) \right) w \geq 0
\]

This means the eigenvalues of $\frac{1}{|S|} \sum_{s \in S} d_s \left( \frac{\phi_s \phi_s^T}{||\phi_s||_2^2} \right)$ are non-negative. It remains to be shown that no eigenvalue is equal to 0. This is true as if some eigenvalue is equal to 0, then for the corresponding eigenvector, say $w$,

\[
w^T \left( \frac{1}{|S|} \sum_{s \in S} d_s \left( \frac{\phi_s \phi_s^T}{||\phi_s||_2^2} \right) \right) w = 0 \\
\frac{1}{|S|} \sum_{s \in S} d_s w^T \left( \frac{\phi_s \phi_s^T}{||\phi_s||_2^2} \right) w = 0 \\
w^T \left( \frac{\phi_s \phi_s^T}{||\phi_s||_2^2} \right) w = 0 \quad \forall \phi_s \in A \\
w^T \phi_s \phi_s^T w = 0 \quad \forall \phi_s \in A \\
(\phi_s^T w)^2 = 0 \quad \forall \phi_s \in A \\
\phi_s^T w = 0 \quad \forall \phi_s \in A
\]

But this is a contradiction as $\{\phi_s\}_{i=1}^n$ is a spanning set for $\mathbb{R}^n$. Thus minimum eigenvalue of $Hess(G(\cdot)) = \frac{1}{|S|} \sum_{s \in S} d_s \left( \frac{\phi_s \phi_s^T}{||\phi_s||_2^2} \right)$ is greater than 0. Thus by definition of strong convexity we have that $G(\cdot)$ is strongly convex when $\Phi \in \mathbb{R}^{m \times n}$ has rank $n$.

\[\square\]

**Proposition C.3.2.** Let $\lambda_{\text{min}}$ be the least eigen value of $\frac{1}{|S|} \sum_{s \in S} d_s \left( \frac{\phi_s \phi_s^T}{||\phi_s||_2^2} \right)$. Then, $G(\cdot)$ is $\mu$–strongly convex where $\mu = \lambda_{\text{min}}$

**Proof.** Note that we can show $\mu$–strongly convex when we show the following. Consider

\[
w^T (Hess(G(\cdot)) - \mu I) w = w^T \left( \frac{1}{|S|} \sum_{s \in S} d_s \left( \frac{\phi_s \phi_s^T}{||\phi_s||_2^2} \right) - \mu I \right) w \quad \text{where} \quad \phi_s \neq 0 \quad \forall i
\]

\[
= w^T \left( \frac{1}{|S|} \sum_{s \in S} d_s \phi_s \phi_s^T \right) w - \mu w^T w
\]
If $\lambda_{\text{min}}$ is the least eigenvalue of \( \frac{1}{|S|} \sum_{s \in S} \phi_s \phi_s^\top \), $\lambda_{\text{min}} > 0$ as we have already shown
\[
w^\top (\text{Hess}(G(\cdot)) - \mu I)w \geq \lambda_{\text{min}} w^\top w - \mu w^\top w \\
\geq (\lambda_{\text{min}} - \mu) ||w||^2
\]
Thus $\exists \mu \in [0, \lambda_{\text{min}})$ such that
\[
w^\top (\text{Hess}(G(\cdot)) - \mu I)w > 0
\]
Thus we see that $G(\cdot)$ is $\mu$ strongly convex where $\mu = \lambda_{\text{min}}$.

**Proposition C.3.3.** $G(\cdot)$ is strictly convex.

**Proof.** Strict convexity is a subset of strong convexity. Thus $G$ is strictly convex.

**C.4 $\nabla G(\cdot)$ is a Lipschitz function**

In this subsection, we will show:

(a) $\nabla_w G(\cdot)$ is Lipschitz continuous with Lipschitz constant equal to 1 using $s$.

(b) $\frac{\phi_s \phi_s^\top}{||\phi_s||_2^2}$ has one eigenvalue 1 and rest $n - 1$ eigenvalues 0.

**Proposition C.4.1.** Let $\phi_s \neq 0$. Then, $V_s = \frac{\phi_s \phi_s^\top}{||\phi_s||_2^2}$ has one eigenvalue 1 and rest $n - 1$ eigenvalues 0.

**Proof.** Let $\phi_s \neq 0$. Then, $V_s = \frac{\phi_s \phi_s^\top}{||\phi_s||_2^2}$. Then, $V_s$ is symmetric thus has real eigen values (Axler (1997)). The second part follows from proposition C.2.2.

**Claim 1:** $V_s$ is a rank 1 matrix: We note that the rank of $vv^\top = 1$ for any $v \neq 0, v \in \mathbb{R}^n$. This is because the rank is the dimension of the column space of the matrix. Since the columns of $vv^\top$ are all scalar multiples of $v$, rank is 1.

**Claim 2:** $V_s$ is an eigenvector of $V_s$ with eigenvalue 1.

Let $\nu_s = \frac{\phi_s}{||\phi_s||_2}$. Then $V_s \nu_s = \nu_s V_s$. Then let $\nu_s \nu_s^\top = V_s$. Then we have to show $\nu_s$ is an eigenvector of $V_s = \nu_s \nu_s^\top$. But this is easy to see.

$V_s \nu_s = \nu_s V_s \nu_s = \nu_s \frac{\phi_s \phi_s^\top}{||\phi_s||_2^2} = \nu_s$. Thus $\nu_s$ is an eigenvector of $V_s$ with eigenvalue 1.

**Claim 3:** $\phi_s$ is an eigenvector of $\frac{\phi_s \phi_s^\top}{||\phi_s||_2^2}$ with eigenvalue 1.

Let $\nu_s = \frac{\phi_s}{||\phi_s||_2}$. Then $\nu_s \nu_s^\top = V_s$. Then let $\nu_s \nu_s^\top = V_s$. Then we have to show $\nu_s$ is an eigenvector of $V_s = \nu_s \nu_s^\top$. But this is easy to see.

$V_s \nu_s = \nu_s V_s \nu_s = \nu_s \frac{\phi_s \phi_s^\top}{||\phi_s||_2^2} = \nu_s$. Thus $\nu_s$ is an eigenvector of $V_s$ with eigenvalue 1.

**Claim 4:** The other eigenvectors are orthogonal to eigenvector with eigenvalue 1.

First we note that $\nu_s$ is the only eigenvector of $V_s$ with eigenvalue 1. Then we show in general that in a real symmetric matrix, eigenvectors with distinct eigenvalues are orthogonal.

Let $\nu_1$ and $\nu_2$ be two eigenvectors of $V_s$ with distinct eigenvalues $\mu_1$ and $\mu_2$. Then $V_s \nu_1 = \mu_1 \nu_1$ and $V_s \nu_2 = \mu_2 \nu_2$. Consider $\mu_1 \nu_1 \nu_2^\top \nu_1$. This is equal to $\nu_1^\top (\mu_1 \nu_1) = \nu_2^\top \nu_3 \nu_1 = \nu_2^\top \nu_3 \nu_1 = (V_s \nu_2)^\top \nu_1 = (\mu_2 \nu_2)^\top \nu_1 = \mu_2 \nu_2 \nu_1$. Thus $\mu_1 \nu_1 \nu_2^\top \nu_1 = \mu_2 \nu_2 \nu_1$ for distinct $\mu_1, \mu_2$, implying that $\nu_2^\top \nu_1 = 0$, or in other words $\nu_1$ and $\nu_2$ are orthogonal.

**Claim 5:** eigenvalue 0 has a multiplicity of $n - 1$.

First we note that there are $n$ eigenvectors for $V_s = \nu_s \nu_s^\top$ in $\mathbb{R}^n$. We have found one eigenvector $\nu_s$ with eigenvalue 1. We have also shown that all other eigenvectors are orthogonal to $\nu_s$. Consider any eigenvector $\nu$ orthogonal to $\nu_s$. Then $\nu_s \nu^\top = 0$. Now consider $V_s \nu = (\nu_s \nu_s^\top) \nu = \nu_s (\nu_s^\top \nu) = \nu_s \nu^\top = 0 = 0$. Thus for all $n - 1$ eigenvectors orthogonal to $\nu_s$, eigenvalue is 0.

Thus $\frac{\phi_s \phi_s^\top}{||\phi_s||_2^2}$ has eigenvalue 1 with multiplicity 1, and eigenvalue 0 with multiplicity $n - 1$.

\[\square\]
Now we are ready to show the Lipschitz property of $\nabla G(\cdot)$

**Proposition C.4.2.** $\nabla G(\cdot)$ is Lipschitz continuous

**Proof.** We already showed that:

$$
\nabla G(w) = \frac{1}{|S|} \sum_{s \in S} d_s \frac{(\phi_s^T w - V_s)\phi_s}{||\phi_s||^2_2}
$$

Then we have

$$
\nabla G(w) - \nabla G(y) = \left( \frac{1}{|S|} \sum_{s \in S} d_s \frac{(\phi_s^T w - V_s)\phi_s}{||\phi_s||^2_2} \right) - \left( \frac{1}{|S|} \sum_{s \in S} d_s \frac{(\phi_s^T y - V_s)\phi_s}{||\phi_s||^2_2} \right)
$$

$$
= \left( \frac{1}{|S|} \sum_{s \in S} d_s \frac{(\phi_s^T (w - y))\phi_s}{||\phi_s||^2_2} \right)
$$

$$
= \left( \frac{1}{|S|} \sum_{s \in S} d_s \frac{\phi_s (\phi_s^T (w - y))}{||\phi_s||^2_2} \right)
$$

$$
= \left( \frac{1}{|S|} \sum_{s \in S} d_s \frac{\phi_s \phi_s^T}{||\phi_s||^2_2} \right) (w - y)
$$

Since max eigen value of $\frac{\phi_s \phi_s^T}{||\phi_s||^2_2}$ is 1

$$
||\nabla G(w) - \nabla G(y)|| \leq \frac{1}{|S|} \sum_{s \in S} d_s ||w - y||
$$

$$
||\nabla G(w) - \nabla G(y)|| \leq \frac{|S|}{|S|} ||w - y|| = ||w - y||
$$

Thus the function $G(\cdot)$ is Lipschitz where the Lipschitz constant, $L \leq 1$

C.5 The Hessian of $G(\cdot)$ is bounded above

In this subsection, we will show:

(a) The Hessian of $G(\cdot)$ is bounded above, or

$\text{Hess}G(\cdot) \leq I$ where $I$ is the identity matrix

**Proposition C.5.1.** $\text{Hess}G(\cdot) \leq I$ where $I$ is the identity matrix

**Proof.** The proposition is equivalent to showing $w^T \text{Hess}(G(w))w \leq ||w||^2_2$

Using $\text{Hess}(G(\cdot)) = \frac{1}{|S|} \sum_{s \in S} d_s \left[ \frac{\phi_s \phi_s^T}{||\phi_s||^2_2} \right]$, for some $w \in \mathbb{R}^n$, we have

$$
w^T \text{Hess}(G(w))w = w^T \frac{1}{|S|} \sum_{s \in S} d_s \left[ \frac{\phi_s \phi_s^T}{||\phi_s||^2_2} \right] w \quad \text{where } \phi_s \neq \mathbf{0} \quad \forall i
$$

$$
= \frac{1}{|S|} \sum_{s \in S} d_s w^T \left[ \frac{\phi_s \phi_s^T}{||\phi_s||^2_2} \right] w
$$
Since the maximum eigenvalue of \( \frac{\phi_s \phi_s^\top}{\|\phi_s\|_2^2} = 1 \) by proposition C.4.1,

\[
w^\top \text{Hess}(G(w))w = \frac{1}{|S|} \sum_{s \in S} d_s w^\top \left( \frac{\phi_s \phi_s^\top}{\|\phi_s\|_2^2} w \right)
\leq \frac{1}{|S|} \sum_{s \in S} d_s w^\top w = \frac{1}{|S|} \sum_{s \in S} d_s \|w\|_2^2 \leq \|w\|_2^2
\]

\( \Box \)

### C.6 The batch version of the Total Projections algorithm converges

Now we proceed to prove convergence of the batch version (non stochastic version) of the TP algorithm. We have already shown \( G(\cdot) \) is convex. Thus, we know that it has a unique optimum point. Thus if our algorithm converges to some optimum, it is guaranteed that we will converge to the unique optimum.

**Proposition C.6.1.** Let \( w^* \) be the minimizer of \( G(w) = 1 \frac{1}{|S|} \sum_{s \in S} d_s \left[ \frac{|\phi_s^\top w - V_s|^2}{2\|\phi_s\|_2^2} \right] \). Then if the sequence \( \{w_1, w_2, w_3, \ldots\} \) is obtained by successive total projection operations, starting from some arbitrary point \( w_0 \in \mathbb{R}^n \), then \( w_\infty = w^* \).

**Proof.** Consider the algorithm \( w_{k+1} = w_k - \alpha_k (TP(w_k)) \) where \( TP(w_k) = \nabla_w G(w_k) = 1 \frac{1}{|S|} \sum_{s \in S} d_s \left[ \frac{\phi_s^\top w - V_s}{\|\phi_s\|_2^2} \phi_s \right] \) and \( \alpha_k \) is some step size sequence. This is a gradient descent algorithm on \( G(\cdot) \). It has been proved in literature Boyd and Vandenberghe (2004) that a (batch) gradient descent algorithm converges to the local minimizer. Since we have shown that \( G(\cdot) \) is a convex function over a convex set, it has a single local minimizer, which is also the global optimum.

We start with the second order Taylor series expansion of \( G(\cdot) \) at some point \( y \in \mathbb{R}^n \) in the neighborhood of \( w \in \mathbb{R}^n \), and some \( z \) between \( w \) and \( y \), we have

\[
G(y) = G(w) + \nabla G(w)^\top (y - w) + \frac{1}{2} (y - w)^\top \text{Hess}(G(z))(y - w)
\]

By proposition C.5.1, \( H(G(\cdot)) \) is bounded above by 1

\[
G(y) \leq G(w) + \nabla G(w)^\top (y - w) + \frac{1}{2} \|y - w\|_2^2
\]

In gradient descent, we proceed in the opposite direction of the gradient. \( \therefore y - w = -\alpha \nabla G(w) \)

\[
G(y) \leq G(w) - \alpha \nabla G(w)^\top \nabla G(w) + \frac{1}{2} \|y - w\|_2^2
\]

Then, \( \|y - w\|_2 = \alpha \|\nabla G(w)\|_2 \)

\[
G(y) \leq G(w) - \alpha \|\nabla G(w)\|_2^2 + \frac{1}{2} (\alpha \|\nabla G(w)\|_2)^2 \leq G(w) - (\alpha - \frac{\alpha^2}{2}) \|\nabla G(w)\|_2^2
\]
We want $\alpha - \frac{\alpha^2}{2} = \alpha \left(1 - \frac{\alpha}{2}\right) > 0$. Setting $\alpha - \frac{\alpha^2}{2} = c > 0$ we get $\alpha \in (0, 2)$:

$$G(y) \leq G(w) - c||\nabla G(w)||^2_2$$

(for some constant $c > 0$)

Since $TP(w) = \nabla G(w)$, we have $y = w - \alpha TP(w)$ where $\alpha \in (0, 2)$

$$G(w - \alpha TP(w)) < G(w)$$

If we label the successive iterates as $w_k$ and $w_{k+1}$, and the step size for the k'th step as $\alpha_k$:

$$G(w_{k+1}) < G(w_k)$$

(for $\alpha_k \in (0, 2)$)

Let $w^* = \arg \min_{w \in \mathbb{R}^n} G(w)$. Then:

$$G(w_{k+1}) - G(w^*) < G(w_k) - G(w^*)$$

Then for some constant $\gamma_k < 1$, $\gamma_k \in \mathbb{R}$:

$$G(w_{k+1}) - G(w^*) = \gamma_k G(w_k) - G(w^*)$$

Similarly, for some constant $\gamma_k < 1$, $\gamma_k \in \mathbb{R}$:

$$G(w_{k+1}) - G(w^*) = \gamma_k G(w_{k-1}) - G(w^*)$$

$$= \ldots$$

$$G(w_{k+1}) - G(w^*) = \left(\prod_{i=0}^{k} \gamma_i\right)(G(w_0) - G(w^*))$$

(where $\gamma_i < 1 \ \forall i \in \{0, \ldots, k\}$)

Now we take the limit as $k \to \infty$

$$\lim_{k \to \infty} (G(w_{k+1}) - G(w^*)) = \lim_{k \to \infty} \left(\prod_{i=0}^{k} \gamma_i\right)(G(w_0) - G(w^*))$$

(where $\gamma_i < 1 \ \forall i \in \lim_{k \to \infty} \{0, \ldots, k\}$)

$$G(w_\infty) - G(w^*) = \left(\lim_{k \to \infty} \prod_{i=1}^{k} \gamma_i\right)(G(w_1) - G(w^*))$$

(where $\gamma_i < 1 \ \forall i \in \lim_{k \to \infty} \{1, \ldots, k\}$)

Since the product of infinite numbers less than 1 is 0, we have:

$$G(w_\infty) - G(w^*) = 0$$

$$G(w_\infty) = G(w^*)$$

Since $G(\cdot)$ is convex over $\mathbb{R}^n$, there $w^*$ is the unique minimizer

$$w_\infty = w^*$$

Thus we show convergence. To get rate of convergence, we need to make some assumptions about $\alpha$.

### C.7 Conditions on the step size of the total projection algorithm

We showed in proposition C.6.1 that the batch version of Total Projections converges to the global optimum for $\alpha_k \in (0, 2)$. Now we will study what is the ideal step size to take in this above range as part of the TP algorithm.
Proposition C.7.1. The optimal step-size $\alpha_{OPT} = 1$

Proof. We have already seen in proposition C.6.1 that for some $w_{k+1}$ in the neighborhood of $w_k$, we have

$$G(w_{k+1}) \leq G(w_k) - (\alpha - \frac{\alpha^2}{2}) ||\nabla G(w_k)||^2_2$$

which is quadratic in $\alpha$. If we want to minimize the LHS, with respect to $\alpha$, we set the derivative of the RHS with respect to $\alpha$ to 0. Thus for an optimal alpha, viz. $\alpha_{OPT}$ we have:

$$\nabla_\alpha G(w_{k+1}) = 0$$

$$\nabla_\alpha \left( G(w_k) - \left( \alpha_{OPT} - \frac{\alpha^2_{OPT}}{2} \right) ||\nabla G(w_k)||^2_2 \right) = 0$$

Since $\nabla_\alpha G(w_k) = 0$ and $\nabla G(w_k)||^2_2$ is independent of $\alpha$

$$\nabla_\alpha \left[ \alpha_{OPT} - \frac{\alpha^2_{OPT}}{2} \right] ||\nabla G(w_k)||^2_2 = 0$$

which leads to:

$$1 - \alpha_{OPT} = 0$$

Thus

$$\alpha_{OPT} = 1$$

In light of this, the stochastic update equation for the batch version of the TP algorithm is

$$w_{k+1} = w_k - \left( \frac{1}{|S|} \sum_{s \in S} d_s \left[ \phi_s^T w - V_s \right] \frac{1}{||\phi_s||^2} \phi_s \right)$$

C.8 Convergence Rate of the Total Projections Algorithm

Now we are ready to show the exponential convergence rate for the Total Projections algorithm. We will now show the rate of convergence of the TP algorithm is exponential when $\Phi$ has full column rank using:

(a) $G(w_{k+1}) - G(w^*) \leq G(w_k) - G(w^*) - \frac{1}{2} ||\nabla G(w_k)||^2_2$

(b) $||\nabla G(w_k)||^2_2 \geq \frac{2(G(w_k) - G(w^*))}{2 - \lambda_{min}}$

Proposition C.8.1. Rate of convergence of the TP algorithm is exponential when $\Phi$ has full column rank

Proof. Firstly, from proposition C.8.2, we have:

$$G(w_{k+1}) - G(w^*) \leq G(w_k) - G(w^*) - \frac{1}{2} ||\nabla G(w_k)||^2_2$$

Then from proposition C.8.3 we have:

$$||\nabla G(w_k)||^2_2 \geq \frac{2(G(w_k) - G(w^*))}{2 - \lambda_{min}}$$

(Note: $\lambda_{min} \leq \lambda_{max} \leq 1$)
Combining, we get:

\[ G(w_{k+1}) - G(w^*) \leq G(w_k) - G(w^*) - \frac{1}{2} \frac{2(G(w_k) - G(w^*))}{2 - \lambda_{min}} \]

\[ = [G(w_k) - G(w^*)] \left[ 1 - \frac{1}{2 - \lambda_{min}} \right] \]

\[ = [G(w_k) - G(w^*)] \left[ 1 - \frac{\lambda_{min}}{2 - \lambda_{min}} \right] \]

We now can create a telescoping product. For successive iterates \{w_1, \ldots, w_k\}:

\[ G(w_{k+1}) - G(w^*) \leq [G(w_k) - G(w^*)] \left[ 1 - \frac{\lambda_{min}}{2 - \lambda_{min}} \right]^n \]

Thus we have a Q-linear rate of convergence, also known as exponential rate of convergence.

**Proposition C.8.2.** \( G(w_{k+1}) - G(w^*) \leq G(w_k) - G(w^*) - \frac{1}{2} \|\nabla G(w_k)\|^2_2 \)

**Proof.** From equation (12) in proposition C.7.1, we can see \( G(w_{k+1}) \leq G(w_k) - (\alpha - \frac{\alpha^2}{2})\|\nabla G(w_k)\|^2_2 \). Substituting \( \alpha = 1 \) from proposition C.7.1, we get

\[ G(w_{k+1}) \leq G(w_k) - \frac{1}{2} \|\nabla G(w_k)\|^2_2 \]

Then subtracting \( G(w^*) \) from both sides:

\[ G(w_{k+1}) - G(w^*) \leq G(w_k) - G(w^*) - \frac{1}{2} \|\nabla G(w_k)\|^2_2 \]

**Proposition C.8.3.** \( \|\nabla G(w_k)\|^2_2 \geq \frac{2}{2 - \lambda_{min}} (G(w_k) - G(w^*)) \)

**Proof.** From proposition C.3.2 we note that when \( \Phi \) has full column rank, then \( G(\cdot) \) is \( \mu \)-strongly convex, with \( \mu = \lambda_{min} \), where \( \lambda_{min} \) is the least eigenvalue of \( Hess(G(\cdot)) \).

Let \( w_{k+1} \in \mathbb{R}^n \) be some point in the neighborhood of \( w_k \in \mathbb{R}^n \), and \( z \) be a point in the interval \([w_k, w_{k+1}]\). Then by second order Taylor series expansion,

\[ G(w_{k+1}) = G(w_k) + \nabla w G(w_k)^\top (w_{k+1} - w) + \frac{1}{2} (w_{k+1} - w_k)^\top Hess(G(z))(w_{k+1} - w_k) \]

Since the Hessian is bounded below:

\[ G(w_{k+1}) \geq G(w_k) + \nabla w G(w_k)^\top (w_{k+1} - w_k) + \frac{1}{2} (w_{k+1} - w_k)^\top \lambda_{min}(w_{k+1} - w_k) \]

\[ = G(w_k) + \nabla w G(w_k)^\top (w_{k+1} - w_k) + \frac{\lambda_{min}}{2} \|w_{k+1} - w_k\|^2_2 \]
But if \( w_{k+1} = w_k - \nabla G(w_k) \) or \( w_{k+1} - w_k = -\nabla G(w_k) \). Thus,

\[
G(w_{k+1}) \geq G(w_k) + \nabla w G(w_k) \top (-\nabla G(w_k)) + \frac{\lambda_{min}}{2} ||-\nabla G(w_k)||^2 \]

\[
= G(w_k) - ||\nabla w G(w_k)||^2 + \frac{\lambda_{min}}{2} ||\nabla G(w_k)||^2 \\
= G(w_k) - \left[ 2 - \frac{\lambda_{min}}{2} \right] ||\nabla w G(w_k)||^2 \\
\geq G(w_k) - (w_{k+1})
\]

But \( G(w_k) - G(w_{k+1}) > G(w_k) - G(w^*) \). Thus

\[
\left[ 2 - \frac{\lambda_{min}}{2} \right] ||\nabla G(w_k)||^2 \geq G(w_k) - G(w^*)
\]

\[
||\nabla G(w_k)||^2 \geq \left[ \frac{2}{2 - \lambda_{min}} \right] (G(w_k) - G(w^*))
\]

(Note: \( \lambda_{min} \leq \lambda_{max} \leq 1 \)

□

D CONVERGENCE OF SCALE INVARIANT MONTE CARLO WITHOUT MOMENTUM

In this section, we will show the convergence point of the Scale Invariant Monte Carlo.

D.1 Notation and Problem Setup

Firstly note that under the linear function approximation regime, we are solving the overdetermined system \( \Phi w = \hat{V} \) with \( |S| = m \) hyperplanes of the form \( \mathcal{H}_s \equiv \phi_i^\top w - \tilde{V}_s = 0 \). We know that the first visit Monte Carlo, is an unbiased estimator of the value function \( V \) for each state. Thus \( \hat{V} \) is an unbiased estimator of \( V \).

The sampling of the hyperplanes is as per the stationary distribution of the transition matrix \( P \). The stationary distribution is given by \( \pi \) with the probability of \( \mathcal{H}_s \) given by \( \pi_s \). We denote the diagonal matrix associated with \( \pi \) as \( D \). Finally, we define a normalization matrix \( N \) where \( N \) is a diagonal matrix with \( N_{(i,i)} = \frac{1}{||\phi_i||^2} \).

Let us define \( TP_k(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n \) to be the function that takes a point \( w_k \) and gives us the shift in \( w_k \) for the \( k \)th iteration.

Thus \( TP_k(w_k) = \frac{1}{\tau} \sum_{\phi_i} \phi_i^\top w_k - \tilde{V}_s \phi_i \). Further, let us define \( TP(\cdot) \) (ref. section C) as \( TP(w_k) = \sum_{s \in S} \pi_s \phi_i^\top w_k - \tilde{V}_s \phi_i \).

\( TP_k(w_k) \) then depends on the trajectory for the Monte Carlo. In other words, \( \tau \) depends on the set of hyperplanes sampled (which is random), where the number of hyperplanes sampled is also random. We will assume that the stopping time \( \tau \) is obtained by some independent random process \( \text{Cinlar} \) (2011). In other words, \( \tau \) is an independent Random Variable. We make this assumption as if \( \tau \) is dependent explicitly on landing at certain states in the Markov Chain, then we lose the stationarity of the distribution as all states will eventually reach the absorbing states.

The limiting ODE that the stochastic update equation \( w_{k+1} = w_k - \alpha_k TP_k(w_k) \) tracks is given by \( \dot{w}(t) = h_{k+1}(w(t)) \) where \( h_{k+1}(w) = \mathbb{E} [TP_k(w_k) | F_k] \) for the filtration \( F_k = \{ w_0, \ldots, w_k \} \). Note that \( \dot{w}(t) = h_{k+1}(w(t)) \) is a well studied o.d.e which converges to the point where \( \dot{w}(t) = 0 \) \( \text{Borkar} \) (2008). Let us denote this point as \( w^* \). Then the problem in this section is to find the point of convergence, \( w^* \).
D.2 Putting the update equation in standard form:

Consider the update equation $w_{k+1} = w_k - \alpha_k TP_k(w_k)$. Given the filtration $\mathcal{F}_k = \{w_0, \ldots, w_k\}$, we wish to find $h_{k+1}(w)$. Let $\{1 \ldots \tau\}$ be the set of unique hyperplanes sampled on the k’th run of trajectory. Then:

$$h_{k+1}(w) = \mathbb{E}[TP_k(w_k)|\mathcal{F}_k] = \mathbb{E}\left[\frac{1}{\tau} \sum_{i=1}^{\tau} \frac{\phi_i^T w_k - \bar{V}_i}{||\phi_i||^2} \phi_i | \mathcal{F}_k\right]$$

We note that $\tau$, the set of hyperplanes $\{1 \ldots \tau\}$ sampled, as well as $\bar{V}$ are all random variables. To simplify from the three random variables, first we write the above expression as an expectation over the conditional expectation given $\tau$. Then:

$$h_{k+1}(w) = \mathbb{E}_{\tau}\left[\mathbb{E}\left[\frac{1}{\tau} \sum_{i=1}^{\tau} \frac{\phi_i^T w_k - \bar{V}_i}{||\phi_i||^2} \phi_i | \mathcal{F}_k, \tau\right]\right]$$

By linearity of expectation, we can take the expectation inside the brackets:

$$= \mathbb{E}_{\tau}\left[\sum_{i=1}^{\tau} \mathbb{E}\left[\frac{\phi_i^T w_k - \bar{V}_i}{||\phi_i||^2} \phi_i | \mathcal{F}_k, \tau\right]\right]$$

But any hyperplane $\mathcal{H}_i \equiv \phi_i^T w - V_i$ is chosen with probability equal to $\pi_i$ where $\pi$ is the stationary distribution. Thus the weights $d_s$ that we used in Section C now take the form $d_s = \pi_s |S|$. Thus $\forall i \in \{1, \ldots, \tau\}$:

$$\mathbb{E}_{(i,\bar{V})}\left[\frac{\phi_i^T w_k - \bar{V}_i}{||\phi_i||^2} \phi_i | \mathcal{F}_k, \tau\right] = \frac{1}{|S|} \sum_{s \in S} \pi_s |S| \cdot \mathbb{E}_{\bar{V}}\left[\frac{\phi_s^T w_k - \bar{V}_s}{||\phi_s||^2} \phi_s\right]$$

Substituting this back in (15), we get:

$$h_{k+1}(w) = \mathbb{E}_{\tau}\left[\frac{1}{\tau} \sum_{i=1}^{\tau} \left(\sum_{s \in S} \pi_s |S| \cdot \mathbb{E}_{\bar{V}}\left[\frac{\phi_s^T w_k - \bar{V}_s}{||\phi_s||^2} \phi_s\right]\right)\right]$$

Since each of the terms in the sum is the same:

$$= \mathbb{E}_{\tau}\left[\sum_{s \in S} \pi_s |S| \cdot \mathbb{E}_{\bar{V}}\left[\frac{\phi_s^T w_k - \bar{V}_s}{||\phi_s||^2} \phi_s\right]\right]$$

Since each term inside is independent of $\tau$, the expectation stays the same. Thus:

$$h_{k+1}(w) = \sum_{s \in S} \pi_s |S| \cdot \mathbb{E}_{\bar{V}}\left[\frac{\phi_s^T w_k - \bar{V}_s}{||\phi_s||^2} \phi_s\right]$$

But the RHS is simply $TP(w_k)$ for the Monte Carlo. Thus:

$$h_{k+1}(w) = TP(w_k)$$

(17)

Since the function $h_k(\cdot)$ is constant for all $k$, i.e. $h_k(\cdot) = TP(\cdot)$, we can simply refer to this as $h(\cdot) = TP(\cdot)$.

Now we are in a position to put our update equation in standard form. Let $M_{k+1} = TP_k(w_k) - \mathbb{E}[TP_k(w_k)|\mathcal{F}_k]$, then we can write the update rule as:
where $\alpha_k$ is as defined in section 4.1. Further, $h(w_k) = TP(w_k)$ and $M_{k+1} = TP_k(w_k) - TP(w_k)$

In the next section we will show that the four conditions required for convergence (Borkar (2008)) are satisfied. In the section after that we will show the point it converges to.

### D.3 Showing satisfaction of assumptions A1-A4 required for convergence

We need to show the following assumptions are satisfied:

1. The map $h(\cdot)$ is Lipschitz
2. Step sizes $\{\alpha_k\}$ are positive scalars satisfying $\sum_{k=0}^{\infty} \alpha_k = \infty$ and $\sum_{k=0}^{\infty} \alpha_k^2 < \infty$
3. $\{M_k\}$ is a martingale difference sequence with respect to the filtrations $\mathcal{F}_k$.
   Further $\{M_k\}$ are square integrable with $\mathbb{E}\left[||M_{k+1}||^2 | F_k \right] \leq K(1 + ||w_k||^2)$ a.s. for some positive constant $K$
4. The iterates $\{w_k\}$ remain bounded almost surely

We will show these in order.

#### D.3.1 The map $h(\cdot)$ is Lipschitz

In Appendix section C.4, we showed that the TP update is Lipschitz for general weights $d_s$ as long as $\sum_{s \in S} d_s \leq |S|$. Now we are considering the specific case where $d_s = \pi_s |S|$. Since $\pi$ is a probability distribution (and therefore sums to 1), we satisfy $\sum_{s \in S} d_s \leq |S|$. Thus $h(\cdot)$ is Lipschitz.

#### D.3.2 The sequence $\{\alpha_k\}$ is square summable but not summable

**Proposition D.3.1.** The step size sequence $\{\alpha_k\}_{k=1}^{\infty}$ satisfies $\sum_{k=0}^{\infty} \alpha_k = \infty$ and $\sum_{k=0}^{\infty} \alpha_k^2 < \infty$

**Proof.** We provide the full proof for proposition 3.2 as follows.

$$\sum_{k=0}^{\infty} \alpha_k = \sum_{k=0}^{\infty} \frac{\theta_k \eta_k}{||TP_k(w_k)||}$$

Expanding $\theta_k = \frac{||TP_k(w_k)||^2}{||\Delta TP_k(w_k)||}$, we get:

$$\sum_{k=0}^{\infty} \frac{||TP_k(w_k)||}{||\Delta TP_k(w_k)||}$$

let $\vartheta_k = \frac{||TP_k(w_k)||}{||\Delta TP_k(w_k)||}$. Then:

$$\sum_{k=0}^{\infty} \frac{\vartheta_k}{k^p}$$

We first show the almost sure lower bounds on $TP_k(w_k)$ and $\Delta TP_k(w_k)$. Note that $TP_k(w_k) = 1/\tau \sum_{i=1}^{\tau} \left[ \frac{\phi_i^T w_k - \tilde{V}_k}{||\phi_i||^2} \right] \phi_i = 1/\tau \sum_{i=1}^{\tau} \left[ \frac{\phi_i \phi_i^T}{||\phi_i||^2} \right] w_k - 1/\tau \sum_{i=1}^{\tau} \left[ \frac{\tilde{V}_k \phi_i}{||\phi_i||^2} \right]$ is almost surely not equal to 0 for random
$w_k \in \mathbb{R}^n$. For $\Delta TP_k(w_k)$, we write:

$$\Delta TP_k(w_k) = TP_k(w_k - TP_k(w_k)) - TP_k(w_k)$$

$$= \frac{1}{\tau} \sum_{i=1}^{\tau} \left[ \frac{\phi_i^T (w_k - TP_k(w_k)) - \bar{V}_i}{\|\phi_i\|^2} \right] \phi_i - \frac{1}{\tau} \sum_{i=1}^{\tau} \left[ \frac{\phi_i^T w_k - \bar{V}_i}{\|\phi_i\|^2} \right] \phi_i$$

$$= -\frac{1}{\tau} \sum_{i=1}^{\tau} \frac{\phi_i \phi_i^T}{\|\phi_i\|^2} TP_k(w_k)$$

We firstly note that $TP_k(w_k) \neq 0$ almost surely as the iterate doesn’t lie on the hyperplanes that $TP_k(\cdot)$ uses. (WLOG, if we do lie on the intersection of the hyperplanes, then we may choose other hyperplanes). Further, for any given vector $TP_k(w_k)$, the chance of $TP_k(w_k)$ being perpendicular to all the vectors $\{\phi_i\}_{i=1}^{\tau}$ is almost surely 0. Thus $\Delta TP_k(w_k) \neq 0 \ a.s.$.

For the upper bounds, we first note that the iterates $\{w_k\}$ are bounded a.s. as per proposition 3.4 and Appendix F.7. Then we further have that the estimates $\bar{V}$ are bounded by $R_{max}/1 - \gamma$ where $R_{max}$ is the maximum reward on transitions and $\gamma$ is the discounting factor. Since, the iterates $\{w_k\}$ are bounded, $TP_k(w_k)$ and $\Delta TP_k(w_k)$ are upper bounded.

Now by these statements, $\vartheta_k = \frac{\|TP_k(w_k)\|}{\|\Delta TP_k(w_k)\|}$ is upper and lower bounded almost surely. \(^2\)

Then, let $\bar{\vartheta} = \sup_k \vartheta_k$ and $\underline{\vartheta} = \inf_k \vartheta_k$. Then

$$\sum_{k=0}^{\infty} \alpha_k = \sum_{i=0}^{\infty} \frac{\vartheta_k}{k^p}$$

$$\geq \sum_{k=0}^{\infty} \frac{\underline{\vartheta}}{k^p}$$

$$= \underline{\vartheta} \sum_{k=0}^{\infty} \frac{1}{k^p}$$

$$= \underline{\vartheta} \times \infty$$

$$= \infty$$

Similarly,

$$\sum_{k=0}^{\infty} \alpha_k^2 = \sum_{k=0}^{\infty} \vartheta^2_k \eta_k^2$$

$$\leq \bar{\vartheta} \sum_{k=0}^{\infty} \eta_k^2$$

$$= \bar{\vartheta} \sum_{k=0}^{\infty} \frac{1}{k^{2p}}$$

Now since $\sum_{k=0}^{\infty} \frac{1}{k^{2p}}$ is finite, and $\bar{\vartheta}$ is finite. Thus:

$$\sum_{k=0}^{\infty} \alpha_k^2 < \infty$$

---

\(^2\)Note: In our simulations, such points where $\|\Delta TP_k(w_k)\| \sim 0$ were never reached and iterates were stable even very close to the solution (see Figure 1a). But to ensure algorithmic stability (given limited floating point precision), we can physically set the updates to not occur when $\Delta TP_k(w_k)$ is below a certain $\varepsilon$ (say $10^{-6}$) threshold.
D.3.3 \( \{M_k\} \) is a martingale difference sequence that is square integrable:

We need to show that \( E[M_{k+1} | \mathcal{F}_k] = 0 \) a.s. and \( E[|M_{k+1}|^2 | \mathcal{F}_k] \leq K(1 + |w_k|^2) \) where \( M_{k+1} = TP_k(w_k) - TP(w_k) \).

For the first part, we have that

\[
E[M_{k+1} | \mathcal{F}_k] = E[TP_k(w_k) - TP(w_k) | \mathcal{F}_k] = E[TP_k(w_k) | \mathcal{F}_k] - E[TP(w_k) | \mathcal{F}_k]
\]

Then since \( M_{k+1} \) is a martingale difference sequence that is square integrable:

\[
E[|M_{k+1}|^2 | \mathcal{F}_k] \leq K(1 + |w_k|^2)
\]

But we already computed in appendix D.2 that \( E[TP_k(w_k) | \mathcal{F}_k] = TP(w_k) \).

For the second part, we write

\[
M_{k+1} = TP_k(w_k) - TP(w_k)
\]

\[
= 1 \sum_{i=1}^{\tau} \frac{\phi_i w_k - \overline{V}_i}{||\phi_i||^2} \phi_i - \sum_{s \in S} \pi_s \left[ \frac{\phi_s w_k - V_s}{||\phi_s||^2} \right] \phi_s
\]

\[
= \left[ \frac{1}{\tau} \sum_{i=1}^{\tau} \frac{\phi_i \phi_i^T}{||\phi_i||^2} - \sum_{s \in S} \pi_s \frac{\phi_s \phi_s^T}{||\phi_s||^2} \right] w_k - \left[ \frac{1}{\tau} \sum_{i=1}^{\tau} \overline{V}_i \phi_i \phi_i^T - \sum_{s \in S} \pi_s \frac{V_s \phi_s \phi_s^T}{||\phi_s||^2} \right]
\]

We can call \( \left[ \frac{1}{\tau} \sum_{i=1}^{\tau} \frac{\phi_i \phi_i^T}{||\phi_i||^2} - \sum_{s \in S} \pi_s \frac{\phi_s \phi_s^T}{||\phi_s||^2} \right] \) as \( A \) and \( \left[ \frac{1}{\tau} \sum_{i=1}^{\tau} \overline{V}_i \phi_i \phi_i^T - \sum_{s \in S} \pi_s \frac{V_s \phi_s \phi_s^T}{||\phi_s||^2} \right] \) as \( b \). Then:

\[
M_{k+1} = Aw_k - b
\]

Note that the eigenvalues of \( A \) are bounded as each term \( \frac{\phi_i \phi_i^T}{||\phi_i||^2} \) has a maximum eigenvalue of 1 as per proposition C.4.1. Similarly \( b \) is bounded as \( \overline{V} \) is bounded by \( \frac{R_{max}}{1 - \gamma} \) where \( R_{max} \) is the maximum reward on transitions between states and \( \gamma \) is the discounting factor.

Now we see that \( M_{k+1} = Aw_k - b \) is linear in \( w_k \) with bounded coefficients. Thus \( E[||M_{k+1}||^2 | \mathcal{F}_k] \) is quadratic in \( w_k \). Now it’s straightforward to see that there exists some constant \( K \) such that \( E[||M_{k+1}||^2 | \mathcal{F}_k] \leq K(1 + |w_k|^2) \).

D.3.4 The iterates remain bounded almost surely

We have already shown this in proposition 3.4. We also provide a proof based on stability criterion from Lakshminarayanan and Bhatnagar (2017) in appendix section F.7.

Now that we satisfy conditions A1-A4 for iterate convergence Borkar (2008) in sections D.3.1 to D.3.4, we know that the iterates will converge. It remains to be seen where it convergess to, which we will cover in the next section.

D.4 Convergence point of the Scale Invariant Monte Carlo

In this section we will show that:

(a) If \( w^* \) is the point of convergence of the Scale Invariant Monte Carlo Algorithm, then

\[
w^* = \left( \Phi^T ND N \Phi \right)^{-1} \Phi^T ND V
\]

using
\[ (\sum_{s \in S} \pi_s \left[ \frac{\phi_s \phi_s^\top}{||\phi_s||^2} \right]) w^* = \sum_{s \in S} \pi_s \left[ \frac{\phi_s V_s}{||\phi_s||^2} \right] \]

\[ \sum_{s \in S} \pi_s \left[ \frac{\phi_s \phi_s^\top}{||\phi_s||^2} \right] = \Phi^\top NDN \Phi \]

\[ \sum_{s \in S} \pi_s \left[ \frac{\phi_s V_s}{||\phi_s||^2} \right] = \Phi^\top NDNV \]

Proposition D.4.1. The convergence point \( w^* \) of our algorithm, which is the stable point of the o.d.e that our stochastic update equation tracks, satisfies the condition

\[ w^* = \left[ (\Phi^\top NDN \Phi)^{-1} \Phi^\top NDN \right] V \]

Proof. Since we are looking for the point \( w^* \) where \( E TP(w^*) = 0 \), from proposition D.4.2 we have:

\[ \left( \sum_{s \in S} \pi_s \left[ \frac{\phi_s \phi_s^\top}{||\phi_s||^2} \right] \right) w^* = \sum_{s \in S} \pi_s \left[ \frac{\phi_s V_s}{||\phi_s||^2} \right] \]

From proposition D.4.4, we have that \( \sum_{s \in S} \pi_s \left[ \frac{\phi_s V_s}{||\phi_s||^2} \right] = \Phi^\top NDNV \). Thus:

\[ \left( \sum_{s \in S} \pi_s \left[ \frac{\phi_s \phi_s^\top}{||\phi_s||^2} \right] \right) w^* = \Phi^\top NDNV \]

From proposition D.4.3, we have that \( \sum_{s \in S} \pi_s \left[ \frac{\phi_s \phi_s^\top}{||\phi_s||^2} \right] = \Phi^\top NDN \Phi \). Thus:

\[ \Phi^\top NDN \Phi w^* = \Phi^\top NDNV \]

Multiplying by \( (\Phi^\top NDN \Phi)^{-1} \) on both sides:

\[ (\Phi^\top NDN \Phi)^{-1} \Phi^\top NDN \Phi w^* = (\Phi^\top NDN \Phi)^{-1} \Phi^\top NDNV \]

To finally get:

\[ w^* = \left[ (\Phi^\top NDN \Phi)^{-1} \Phi^\top NDN \right] V \quad (19) \]

Proposition D.4.2. The convergence point \( w^* \) of our algorithm, which is the stable point of the o.d.e that our stochastic update equation tracks, satisfies the condition

\[ \left( \sum_{s \in S} \pi_s \left[ \frac{\phi_s \phi_s^\top}{||\phi_s||^2} \right] \right) w^* = \sum_{s \in S} \pi_s \left[ \frac{\phi_s V_s}{||\phi_s||^2} \right] \]

Proof. We are looking for the point where \( h(w(t)) = 0 \). In other words, we are looking for a point \( w^* \) where \( E TP(w^*) = 0 \). Then we have:

\[ \sum_{s \in S} \pi_s \left[ \frac{\phi_s \phi_s^\top}{||\phi_s||^2} \right] w^* - V_s = 0 \]

Which we can directly rewrite to:

\[ \left( \sum_{s \in S} \pi_s \left[ \frac{\phi_s \phi_s^\top}{||\phi_s||^2} \right] \right) w^* = \sum_{s \in S} \pi_s \left[ \frac{\phi_s V_s}{||\phi_s||^2} \right] \]
Proposition D.4.3. \[ \sum_{s \in S} \pi_s \left[ \frac{\phi_s \phi_s^\top}{||\phi_s||_2^2} \right] = \Phi^\top NDN \Phi \]

**Proof.** Note that the LHS and RHS are both matrices of size \( n \times n \). We will show the equality explicitly for each \((i,j)\)'th entry of this matrix.

For the LHS, the entry at position \((i,j)\) is given by \( \sum_{s \in S} \pi_s \left[ \frac{\phi_s(i)\phi_s(j)}{||\phi_s||_2^2} \right] \).

For the RHS, first note that \( NDN \) is a diagonal matrix of size \(|S| \times |S|\). The diagonal entries are given by \( [NDN]_{(s,s)} = \pi_s ||\phi_s||_2^2 \). Then \( NDN\Phi \) has \(|S|\) rows of the form \( \frac{\pi_s}{||\phi_s||_2^2} \phi_s \). Finally, the entry at the \((i,j)\)'th location of \( \Phi^\top NDN\Phi \), which is a \( n \times n \) matrix is given by \( \sum_{s \in S} \phi_s(i) \frac{\pi_s}{||\phi_s||_2^2} \phi_s(j) \).

Note that this can be rewritten as \( \sum_{s \in S} \pi_s \left[ \frac{\phi_s(i)\phi_s(j)}{||\phi_s||_2^2} \right] \), which is the same as the LHS. \( \square \)

Proposition D.4.4. \[ \sum_{s \in S} \pi_s \left[ \frac{\phi_s V_s}{||\phi_s||_2^2} \right] = \Phi^\top NDNV \]

**Proof.** In this case we are dealing with a vector in \( \mathbb{R}^n \) for both the LHS and the RHS. We will show equality by showing the \( i \)'th entry of this vector on both LHS and RHS are the same.

For the LHS, we have a sum of \(|S|\) vectors of the form \( \pi_s V_s \frac{\phi_s(i)}{||\phi_s||_2^2} \). Then the entry at \( i \)'th location is given by \( \sum_{s \in S} \frac{\pi_s V_s}{||\phi_s||_2^2} \phi_s(i) \).

For the RHS, note that \( [NDN]_{(s,s)} = \frac{\pi_s}{||\phi_s||_2^2} \) as in proposition D.4.3. Then \( NDNV \) is a vector of size \( |S| \) where the entry for state \( s \) is given as \( [NDNV]_s = \frac{\pi_s V_s}{||\phi_s||_2^2} \). Finally, we have that the entry at the \( i \)'th row \( (i \in \{1, \ldots, n\}) \) in \( \Phi^\top NDNV \) is given by \( \frac{\pi_s V_s}{||\phi_s||_2^2} \phi_s(i) \), which is the same as the LHS \( \square \)

E  CHOICE OF MOMENTUM MULTIPLIER FOR HEAVYBALL MOMENTUM

We plot the mean error with iterations for different \( \beta \) values to do a comparison between the various constant values in Figures 6a and 6b. This will enable us to see reasons for our choice of \( \beta = 0.5 \).

Note that when we increase \( \beta \) beyond 0.5, we see non-smoothness in convergence of the stochastic case. Thus we do not go for \( \beta > 0.5 \) even though it sometimes leads to faster convergence.

We note that in the non-stochastic case, all values of \( \beta \in [0, 1) \) lead to convergence. Given enough iterations, we expect the same in the stochastic case as well.

F  SHOWING CONVERGENCE WITH MOMENTUM FOR THE SCALE INVARIANT MONTE-CARLO (SIM) ALGORITHM

F.1  Problem Setup

Our original stochastic approximation equation with momentum can be written as

\[ w_{k+1} = w_k - \alpha_k \sum_{i=1}^{n} \left[ \frac{\phi_i^\top (w - \bar{V}_i)}{||\phi_i||_2^2} \phi_i \right] + \beta (w_k - w_{k-1}) \]
Figure 6: Comparison of different Momentum

where the notations have the usual meaning explained in section 2 and further, \( \beta \in [0, 1) \). We want to show that this converges, where we have already shown that the update \( w_{k+1} = w_k - \alpha_k \sum_{i=1}^{\tau} \frac{\phi_i^T w - \tilde{V}_i}{||\phi_i||^2} \phi_i \) converges.

Approach used

Traditional algorithms may attempt such a momentum under the two timescale approximation scheme. These have been considered in Borkar (2008); Lakshminarayanan and Bhatnagar (2017). Two time scale approximation are also considered in Avrachenkov et al. (2020) in the context of web page change rate estimation. We take a different approach. First we convert the given stochastic approximation equation with momentum into a two timescale regime, with two iterates getting updated. Then we collapse the second iterate into a perturbation on the first iterate \( w_k \), and thus show convergence. We detail this in the following sections.

F.2 Adapting the stochastic-approximation equation with momentum into a two timescale structure:

**Proposition F.2.1.** The update equation \( w_{k+1} = w_k - \alpha_k \sum_{i=1}^{\tau} \frac{\phi_i^T w - \tilde{V}_i}{||\phi_i||^2} \phi_i \) + \( \beta(w_k - w_{k-1}) \) can also be written as the set of equations

\[
\begin{align*}
  w_{k+1} - w_k &= \alpha_k z_k \\
  z_0 &= TP_k(w_k) \\
  z_i &= z_{i-1} + \zeta_{(i,k)} TP_{k-i}(w_{k-i}) \quad \forall i \in [1, k]
\end{align*}
\]

where \( \zeta_{(i,k)} = \frac{\beta^i \alpha_{k-i}}{\alpha_k} \).

**Proof.** Consider:

\[
w_{k+1} = w_k - \alpha_k \sum_{i=1}^{\tau} \frac{\phi_i^T w - \tilde{V}_i}{||\phi_i||^2} \phi_i + \beta(w_k - w_{k-1}) \tag{20}
\]

Rewriting as a difference:

\[
w_{k+1} - w_k = \alpha_k \sum_{i=1}^{\tau} \frac{\phi_i^T w - \tilde{V}_i}{||\phi_i||^2} \phi_i + \beta(w_k - w_{k-1})
\]
We will call the term $\sum_{i=1}^T \left[ \frac{1}{T} w_k - \frac{1}{T} V_t \phi_i \right]$ as $TP_k(w_k)$

$$w_{k+1} - w_k = \alpha_k TP_k(w_k) + \beta(w_k - w_{k-1})$$

Expanding the momentum term

$$w_{k+1} - w_k = \alpha_k TP_k(w_k) + \beta(\alpha_{k-1} TP_{k-1}(w_{k-1}) + \beta(w_{k-1} - w_{k-2}))$$

$$= \alpha_k TP_k(w_k) + \beta \alpha_{k-1} TP_{k-1}(w_{k-1}) + \beta^2(w_{k-1} - w_{k-2})$$

$$= \alpha_k TP_k(w_k) + \beta \alpha_{k-1} TP_{k-1}(w_{k-1}) + \beta^2(\alpha_{k-2} TP_{k-2}(w_{k-2}) + \beta(w_{k-2} - w_{k-3}))$$

$$= \ldots$$

Thus we can write the whole thing as:

$$= \alpha_k TP_k(w_k) + \beta \alpha_{k-1} TP_{k-1}(w_{k-1}) + \beta^2 \alpha_{k-2} TP_{k-2}(w_{k-2}) + \cdots + \beta^k \alpha_0 TP_0(w_0) \quad (21)$$

We note that this is in the form of a discounted sum of vectors, which we have to bring into a form that is the sum of two iterates Kushner and Yin (1997).

We reverse the order of the second iterate set. We build $w_k$ bottom up as follows. Let:

$$z_0 = TP_k(w_k)$$

$$z_1 = z_0 + \beta \frac{\alpha_{k-1}}{\alpha_k} TP_{k-1}(w_{k-1})$$

$$\vdots \quad = \quad \vdots$$

$$z_k = z_{k-1} + \beta \frac{\alpha_0}{\alpha_k} TP_0(w_0)$$

Further, to simplify this set of equations, we let $\tilde{\zeta}_{(i,k)}$ be the step size corresponding to $z_i$ such that $\tilde{\zeta}_{(i,k)} = \beta^i \frac{\alpha_{k-i}}{\alpha_k}$. Then we have the set of equations as:

$$w_{k+1} - w_k = \alpha_k z_k$$

$$z_0 = TP_k(w_k)$$

$$z_1 = z_0 + \tilde{\zeta}_{(1,k)} TP_{k-1}(w_{k-1})$$

$$\vdots \quad = \quad \vdots$$

$$z_k = z_{k-1} + \tilde{\zeta}_{(k,k)} TP_0(w_0)$$

Or more generally if $\tilde{\zeta}_{(i,k)} = \beta^i \frac{\alpha_{k-i}}{\alpha_k}$,

$$w_{k+1} - w_k = \alpha_k z_k \quad (22)$$

$$z_0 = TP_k(w_k)$$

$$z_i = z_{i+1} + \tilde{\zeta}_{(i,i)} TP_i(w_{i-1}) \quad \forall i \in [1, k]$$

F.3 Collapsing the two iterate stochastic approximation equations into a single iterate form:

Now wish to express the above equation in terms of an expected update and a Martingale noise term (with respect to the filtration). For $z_0$, such an expression is straightforward: We add and subtract the expectation to change the equation from $z_0 = TP_k(w_k)$ to

$$z_0 = \mathbb{E} \left[ TP_k(w_k) | \mathcal{F}_k \right] + (TP_k(w_k) - \mathbb{E} \left[ TP_k(w_k) | \mathcal{F}_k \right]) \quad (23)$$

where the first term is the expected update term $H_{(0,k)} = \mathbb{E} \left[ TP_k(w_k) | \mathcal{F}_k \right]$ second term is a martingale difference noise term, $M_{(0,k)} = (TP_k(w_k) - \mathbb{E} \left[ TP_k(w_k) | \mathcal{F}_k \right])$. 

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Now let us focus on $z_i$ for $i \in [1, k]$ 

$$z_i = z_{i-1} + \zeta_{(i,k)}TP_{k-i}(w_{k-i})$$

can be rewritten as:

$$z_i = z_{i-1} + \zeta_{(i,k)}[TP_{k-i}(w_k) + (TP_{k-i}(w_{k-i}) - TP_{k-i}(w_k))]$$

Which can be further broken down as:

$$z_i = z_{i-1} + \zeta_{(i,k)} \left[ \mathbb{E}[TP_{k-i}(w_k)|\mathcal{F}_k] + (TP_{k-i}(w_k) - \mathbb{E}[TP_{k-i}(w_k)|\mathcal{F}_k]) + (TP_{k-i}(w_{k-i}) - TP_{k-i}(w_k)) \right]$$

Now we take an expectation of the first term over all possible $\mathcal{F}_k$. Thus the first term breaks into:

$$z_i = z_{i-1} + \zeta_{(i,k)} \left[ \mathbb{E}[TP_{k-i}(w_k)|\mathcal{F}_k] + \right. \left. (\mathbb{E}[TP_{k-i}(w_k)|\mathcal{F}_k] - \mathbb{E}[TP_{k-i}(w_k)|\mathcal{F}_k]) + (TP_{k-i}(w_k) - \mathbb{E}[TP_{k-i}(w_k)|\mathcal{F}_k]) + (TP_{k-i}(w_{k-i}) - TP_{k-i}(w_k)) \right]$$

Note that in the filtration, $\mathbb{E}[TP_{k-i}(w_k)|\mathcal{F}_k] - \mathbb{E}[TP_{k-i}(w_k)|\mathcal{F}_k]$ remains unaffected, and therefore, we can write $\mathbb{E}[TP_{k-i}(w_k)|\mathcal{F}_k] = \mathbb{E}[TP_{k-i}(w_k)|\mathcal{F}_k]$. For ease of notation, we simply write $\mathbb{E}[TP_{k-i}(w_k)|\mathcal{F}_k]$ as $\mathbb{E}[TP_{k-i}(w_k)|\mathcal{F}_k]$. Then we have:

$$z_i = z_{i-1} + \zeta_{(i,k)} \left[ \mathbb{E}[TP_{k-i}(w_k)|\mathcal{F}_k] + (TP_{k-i}(w_k) - \mathbb{E}[TP_{k-i}(w_k)|\mathcal{F}_k]) + (TP_{k-i}(w_{k-i}) - TP_{k-i}(w_k)) \right]$$

Notice that the third term above $TP_{k-i}(w_k) - \mathbb{E}[TP_{k-i}(w_k)|\mathcal{F}_k]$ is actually 0 as the filtration provides the exact hyperplanes as well as $w_k$. Thus the expression is deterministic. Therefore, $\mathbb{E}[TP_{k-i}(w_k)|\mathcal{F}_k] = TP_{k-i}(w_k)$. Thus we finally have

$$z_i = z_{i-1} + \zeta_{(i,k)} \left[ \mathbb{E}[TP_{k-i}(w_k)|\mathcal{F}_k] + (TP_{k-i}(w_k) - \mathbb{E}[TP_{k-i}(w_k)|\mathcal{F}_k]) \right]$$

Let

$$H_{(i,k)} = \mathbb{E}[TP_{k-i}(w_k)|\mathcal{F}_k]$$

$$M_{(i,k)} = \mathbb{E}[TP_{k-i}(w_k)|\mathcal{F}_k] - \mathbb{E}[TP_{k-i}(w_k)|\mathcal{F}_k]$$

$$\varepsilon_{(i,k)} = TP_{k-i}(w_{k-i}) - TP_{k-i}(w_k)$$

If $\zeta_{(i,k)} = \beta^i \frac{\partial}{\partial w_k}$. Further, let $h_{(i,k)}(\cdot)$ be some limiting o.d.e that asymptotically tracks $H_{(i,k)}(\cdot)$. Thus we have the set of equations:

$$w_{k+1} - w_k = \alpha_k z_k$$

$$z_0 = h_{(0,k)} + M_{(0,k)}$$

$$z_i = z_{i-1} + \zeta_{(i,k)}(h_{(i,k)} + M_{(i,k)} + \varepsilon_{(i,k)}) \quad \forall i \in [1, k]$$

We collapse these now into a single equation. Since $h_{(i,k)}(w_k) = TP(w_k)\forall i, k$ based on proposition F.7.1 We will label this simply as $h(w_k)$
Let \( \bar{h}(w_k) = h(w_k) \left( 1 + \sum_{i=1}^{k} \zeta_{(i,k)} \right) \), \( \bar{M}_k = M_{(0,k)} + \sum_{i=1}^{k} \zeta_{(i,k)} M_{(i,k)} \) and \( \bar{\varepsilon}_k = \sum_{i=1}^{k} \zeta_{(i,k)} \varepsilon_{(i,k)} \), then:

\[
w_{k+1} - w_k = \alpha_k \left[ \bar{h}(w_k) + \bar{\varepsilon}_k + \bar{M}_k \right]
\]

Now we have to show that this single equation follows the requirements for convergence. We will show each of the assumptions in order.

### F.4 Showing basic properties of required for convergence:

**Proposition F.4.1.** The step size sequence \( \{\alpha_i\}_{i=1}^{\infty} \) satisfies \( \sum_{i=1}^{\infty} \alpha_i = \infty \) and \( \sum_{i=0}^{\infty} \alpha_i^2 < \infty \)

**Proof.** The step size sequence remains the same as in proposition D.3.1. Thus the proof remains the same.

**Proposition F.4.2.** Let \( \bar{\zeta}_k := 1 + \sum_{i=1}^{k} \zeta_{(i,k)} \). Then \( \bar{\zeta}_k \) is bounded.

**Proof.** Consider \( \bar{\zeta}_k = \frac{1}{\alpha_k} \left[ \alpha_k + \beta \alpha_{k-1} + \beta^2 \alpha_{k-2} + \ldots \right] \)

Recall that \( \alpha_k = \eta_k \vartheta_k \) where \( \eta_k = \frac{1}{k^p} \); \( p \in (0.5, 1] \) and \( \vartheta_k = \frac{||TP_k(w_k)||}{||\Delta TP_k(w_k)||} \). Further, recall that \( \sup \vartheta_k = \vartheta \). Then:

\[
\bar{\zeta}_k \leq \frac{1}{\vartheta_k} \left[ 1 + \beta \left( \frac{k}{k-1} \right)^p + \beta^2 \left( \frac{k}{k-2} \right)^p + \ldots \right] \\
\leq \frac{1}{\vartheta_k} \left[ 1 + \beta \frac{k}{k-1} + \beta^2 \frac{k}{k-2} + \ldots \right] \\
= \frac{1}{\vartheta_k} \left[ 1 + \beta + \frac{\beta}{k-1} + \beta^2 + \frac{2 \beta^2}{k-2} + \beta^3 + \frac{3 \beta^3}{k-3} + \ldots \right] \\
= \frac{1}{\vartheta_k} \left[ (1 + \beta + \beta^2 + \ldots) + \left( \frac{\beta}{k-1} + \frac{2 \beta^2}{k-2} + \frac{3 \beta^3}{k-3} + \ldots \right) \right]
\]

As \( k \to \infty \), the first half of the above expression is \( \frac{1}{\vartheta_k \pi (1-p)} \). As \( k \to \infty \), the second half converges to 0 Murthy (2021). Thus the entire expression remains bounded.

**Proposition F.4.3.** The expected update for \( w, \bar{h} : \mathbb{R}^n \to \mathbb{R}^n \) is Lipschitz

**Proof.** We have already shown that \( h(\cdot) = TP(w) \) is Lipschitz (as can be seen from the fact that \( \sum_{s \in S} \pi_s \phi_s \phi_s^T \) where \( C = \sum_{s \in S} \pi_s \phi_s V_s \) is linear in \( w \)). Now we will show that \( \bar{h}(\cdot) = h(\cdot) (1 + \sum_{i=1}^{k} \zeta_i) \) is also Lipschitz. But we have shown that \( \bar{\zeta}_k = 1 + \sum_{i=1}^{k} \zeta_{(i,k)} \) is bounded in proposition F.4.2.

Thus we have that if \( h(\cdot) \) is Lipschitz, then \( \bar{\zeta}_k h(\cdot) = \bar{h}(\cdot) \) is also Lipschitz for some constant \( \bar{\zeta}_k \).
F.5 Showing that the noise term is a martingale difference sequence:

**Proposition F.5.1.** We specifically consider \(M_{(0,k)}\) first. \(E[M_{(0,k)}|\mathcal{F}_k] = 0\) and \(E[||M_{(0,k)}||^2|\mathcal{F}_k] \leq K[1 + ||w_k||^2]\)

**Proof.** We note that \(M_{(0,k)} = TP_k(w_k) - E[TP_k(w_k)|\mathcal{F}_k] = TP_k(w_k) - TP(w_k)\) as per appendix section D.2.

Now we have already shown in appendix section D.3.3 that \(E[TP_k(w_k) - TP(w_k)|\mathcal{F}_k] = 0\). Further we also showed \(\exists A_{(0,k)}, C_{(0,k)}\) such that \(M_{(0,k)} = A_{(0,k)}w_k - C_{(0,k)}\) whence \(E[||TP_k(w_k) - TP(w_k)||^2|\mathcal{F}_k] \leq K[1 + ||w_k||^2]\) for some \(K \in \mathbb{R}\)

\(\square\)

**Proposition F.5.2.** \(E[M_{(i,k)}|\mathcal{F}_k] = 0\) and \(E[||M_{(0,k)}||^2|\mathcal{F}_k] \leq K[1 + ||w_k||^2]\)

**Proof.** First we note that

\[
M_{(i,k)} = E[TP_{k-i}(w_k)|\mathcal{F}_k] - E_{\mathcal{F}_k}[E[TP_{k-i}(w_k)|\mathcal{F}_k]]
\]

Note that the second expectation remains unchanged given the filtration, thus we can rewrite this as:

\[
= E[TP_{k-i}(w_k)|\mathcal{F}_k] - E_{\mathcal{F}_k}[E[TP_{k-i}(w_k)|\mathcal{F}_k]|\mathcal{F}_k]
\]

Given such a definition, \(E[M_{(i,k)}|\mathcal{F}_k] = E_{\mathcal{F}_k}[E[TP_{k-i}(w_k)|\mathcal{F}_k]|\mathcal{F}_k] - E_{\mathcal{F}_k}[E[TP_{k-i}(w_k)|\mathcal{F}_k]|\mathcal{F}_k] = 0\)

For the second part, note that the filtration gives us the hyperplanes, say \(\{1, \ldots, r\}\) that have been sampled. Then:

\[
E[TP_{k-i}(w_k)|\mathcal{F}_k] = \left[\frac{1}{\tau} \sum_{i=1}^{\tau} \phi_i^T w_k - \tilde{V}_i \phi_i \right]
\]

We obtain from appendix proposition F.7.1 that \(E[TP_{k-i}(w_k)|\mathcal{F}_k]] = TP(w_k) = \sum_{s \in S} \pi_s \left[\phi_s^T w - V_s \phi_s \right]\). Therefore:

\[
M_{(i,k)} = \left[\frac{1}{\tau} \sum_{i=1}^{\tau} \phi_i^T w_k - \tilde{V}_i \phi_i \right] - \sum_{s \in S} \pi_s \left[\phi_s^T w - V_s \phi_s \right]
\]

\[
= \left[\frac{1}{\tau} \sum_{i=1}^{\tau} \phi_i^T w_k - \tilde{V}_i \phi_i \right] - \sum_{s \in S} \pi_s \phi_s^T \phi_s
\]

\[
= A_{(i,k)}w_k - C_{(i,k)}
\]

which is linear in \(w_k\) with bounded coefficients. Further note that \(A_{(i,k)}\) is bounded above as \(\frac{\phi_i \phi_i^T}{||\phi_i||^2}\) has maximum eigen value 1. Further, \(C_{(i,k)}\) is bounded as \(\tilde{V}\) is bounded above by \(\frac{R_{\text{max}}}{1 - \gamma}\) where \(R_{\text{max}}\) is the maximum reward and \(\gamma\) is the discounting factor.

Thus \(||M_{(i,k)}||^2\) is quadratic in \(w_k\). Now it is easy to see that there would exist some \(K\) such that \(E[||M_{(i,k)}||^2|\mathcal{F}_k] \leq K(1 + ||w_k||^2)\)

\(\square\)

**Proposition F.5.3.** Consider the filtration \(\mathcal{F}_k = \{w_0, \ldots, w_k\}\). Then the sequence \(\{\hat{N}_k\}\) is a zero-mean martingale difference noise sequence. Specifically, we have that:

1. \(E[\hat{N}_k|\mathcal{F}_k] = 0\)
2. \( \mathbb{E}[||\dot{\mathcal{M}}_k||^2 | \mathcal{F}_k] \leq K_i (1 + ||w_k||^2) \)

**Proof.** For the first part, we need to show \( \mathbb{E}[\dot{\mathcal{M}}_k | \mathcal{F}_k] = 0 \) where \( \dot{\mathcal{M}}_k = \mathcal{M}_{(0,k)} + \sum_{i=1}^{k} \zeta_{(i,k)} M_{(i,k)}. \)

We have:

\[
\mathbb{E}[\dot{\mathcal{M}}_k | \mathcal{F}_k] = \mathbb{E}[\mathcal{M}_{(0,k)} + \sum_{i=1}^{k} \zeta_{(i,k)} M_{(i,k)} | \mathcal{F}_k]
\]

By linearity of expectation:

\[
= \mathbb{E}[\mathcal{M}_{(0,k)} | \mathcal{F}_k] + \sum_{i=1}^{k} \zeta_{(i,k)} \mathbb{E}[M_{(i,k)} | \mathcal{F}_k]
\]

But we have from proposition F.5.2 that \( \mathbb{E}[M_{(i,k)} | \mathcal{F}_k] = 0 \) and from proposition F.5.1 that \( \mathbb{E}[\mathcal{M}_{(0,k)} | \mathcal{F}_k] = 0 \). Therefore

\[
\mathbb{E}[\dot{\mathcal{M}}_k | \mathcal{F}_k] = 0 + \sum_{i=1}^{k} \zeta_{(i,k)} 0
\]

For the second part, we see this by linearity.

\[
\dot{\mathcal{M}}_k = \mathcal{M}_{(0,k)} + \sum_{i=1}^{k} \zeta_{(i,k)} M_{(i,k)}
\]

From propositions F.5.1 and F.5.2, we can write the above as:

\[
= (A_{(0,k)} + \sum_{i=1}^{k} \zeta_{(i,k)} A_{(i,k)}) w_k - (C_{(0,k)} + \sum_{i=1}^{k} \zeta_{(i,k)} C_{(i,k)})
\]

Since \( A_{(i,k)}, C_{(i,k)} \) are bounded \( \forall i \in \{0, \ldots, k\} \), we can write the above as:

\[
\dot{\mathcal{M}}_k = \dot{A} w_k - \dot{C}
\]

where \( \dot{A} = A_{(0,k)} + \sum_{i=1}^{k} \zeta_{(i,k)} A_{(i,k)} \) is bounded and \( \dot{C} = C_{(0,k)} + \sum_{i=1}^{k} \zeta_{(i,k)} C_{(i,k)} \) is bounded. Now we see that \( \dot{\mathcal{M}}_k \) is linear in \( w_k \) with bounded coefficients.

Thus \( ||\dot{\mathcal{M}}_k||^2 \) is quadratic in \( w_k \), whence \( \exists \mathcal{K} \in \mathbb{R} \) such that \( \mathbb{E}[||\dot{\mathcal{M}}_k||^2 | \mathcal{F}_k] \leq \mathcal{K}(1 + ||w_k||^2) \)

\[\square\]

**F.6 Showing that the momentum terms sum to a perturbation:**

**Proposition F.6.1** (Helper proposition for F.6.2). \( \sum_{i=1}^{k} \zeta_i \cdot ||w_k - w_{k-i}|| \to 0 \) as \( k \to \infty \)

**Proof.**

\[
\sum_{i=1}^{\infty} \zeta_i \cdot ||w_k - w_{k-i}|| = \sum_{i=1}^{m} \zeta_i \cdot ||w_k - w_{k-i}|| + \sum_{i=m+1}^{\infty} \zeta_i \cdot ||w_k - w_{k-i}||
\]

Now given any \( \epsilon \), there \( \exists m \) such that \( \sum_{i=m+1}^{\infty} \zeta_i \cdot ||w_k - w_{k-i}|| < \epsilon \). This is because \( \zeta_i \sim \beta^i \) go to 0 and \( ||w_k - w_{k-i}|| \) are bounded (shown separately when we show stability of iterates).
Then given any finite $m$, at the asymptote as $k \to \infty$, we have $||w_k - w_{k-i}|| \to 0$ as $\alpha_k \to 0$. Thus
$$\sum_{i=1}^{k} \zeta_i \cdot ||w_k - w_{k-i}|| < \epsilon$$
as $k \to \infty$ for any arbitrary $\epsilon$.

Thus $\sum_{i=1}^{k} \zeta_i \cdot ||w_k - w_{k-i}|| \downarrow 0$ as $k \to \infty$.

**Proposition F.6.2.** $\hat{\epsilon}_k$ are perturbation terms that satisfy $||\hat{\epsilon}_k|| \leq d_k (1 + ||w_k||)$ where $d_k$ are a sequence of positive scalars such that $\lim_{k \to \infty} d_k = 0$.

*Proof.* First note that $\hat{\epsilon}_k = \sum_{i=1}^{k} \zeta_i \epsilon_{i,k}$ and $\epsilon_{i,k} = TP_{k-i}(w_{k-i}) - TP_{k-i}(w_k)$. Thus
$$\epsilon_i = \frac{1}{\tau} \sum_{j=1}^{\tau} \left[ \frac{\phi_j^T w_{k-i} - V_j}{||\phi_j||^2} \right] \phi_j - \frac{1}{\tau} \sum_{j=1}^{\tau} \left[ \frac{\phi_j^T w_k - V_j}{||\phi_j||^2} \right] \phi_j$$

Taking terms common:
$$\frac{1}{\tau} \left( \sum_{j=1}^{\tau} \left[ \frac{\phi_j^T w_{k-i} - V_j}{||\phi_j||^2} \right] \phi_j - \sum_{j=1}^{\tau} \left[ \frac{\phi_j^T w_k - V_j}{||\phi_j||^2} \right] \phi_j \right) (w_{k-i} - w_k)$$

$$||\epsilon_i|| \leq ||w_{k-i} - w_k||$$

(29)

Now we extend this by using $\hat{\epsilon}_k = \sum_{i=1}^{k} \zeta_i \epsilon_i$

$$\hat{\epsilon}_k = \sum_{i=1}^{k} \zeta_i \epsilon_i$$

Expanding $\epsilon_i$ using the inequality in (29)

$$||\hat{\epsilon}_k|| \leq \sum_{i=1}^{k} \zeta_i \cdot ||w_k - w_{k-i}||$$

Now we note that asymptotically as $k \to \infty$, we have for finite $i$, $||w_k - w_{k-i}|| \to 0$ and for large $i$, $\zeta_i \to 0$. Thus by proposition F.6.1 the above is bounded above by some arbitrary $\epsilon$.

$$\leq \epsilon$$

Thus asymptotically we see that this perturbation term is $o(1)$.

**F.7 Stability Criterion: Iterates remain bounded**

In Borkar (2008), we have to prove that the iterates of in the update equation remain bounded. Lakshminarayanan and Bhatnagar (2017) have provided a stability criterion to ensure that the iterates remain bounded. While we have already shown that the iterates on after the expected update remain bounded in proposition 3.4, here we will explicitly show the stability criterion is satisfied.

But first a basic proposition:

**Proposition F.7.1.** $E[E[TP_{k-i}(w_k)|F_k]] = TP(w_k)$

*Proof.* Note that we are considering the expectation over all possible filtrations. Note that the random variables under consideration are $\tau$ - the number of hyperplanes sampled, $i \in \{1, \ldots, \tau\}$ - the set of hyperplanes sampled, and $V$ - the value function. The filtration gives us $w_k$ and the set of hyperplanes chosen in a particular trajectory. Let’s label the unique hyperplanes in the trajectory by $\{1, \ldots, \tau\}$.

Then:

$$E_{F_k}[E[TP_{k-i}(w_k)|F_k]] = E_{F_k=(\tau,i,V)} \left( \frac{1}{\tau} \sum_{i=1}^{\tau} \left[ \frac{\phi_i^T w_k - V_i}{||\phi_i||^2} \phi_i \right] \right)$$

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By linearity we rewrite this as:

\[
E_\tau \left( \frac{1}{\tau} \sum_{i=1}^{\tau} E_{\iota_i, \tilde{V}} \left[ \frac{\phi_i^T w_k - \tilde{V}_i}{||\phi_i||^2} \phi_i \right] \right)
\]

Over all possible filtrations, we can write the expectation of the inner term as:

\[
E_{\iota_i, \tilde{V}} \left[ \phi_i^T w_k - \tilde{V}_i \right] = \sum_{s \in S} \pi_s \phi_s^T w_k - V_s \frac{||\phi_s||^2}{||\phi_s||^2} \phi_s
\]

Substituting this in the previous expression, we get:

\[
E_{\tau} \left( \frac{1}{\tau} \sum_{i=1}^{\tau} \left[ \sum_{s \in S} \pi_s \phi_s^T w_k - V_s \frac{||\phi_s||^2}{||\phi_s||^2} \phi_s \right] \right)
\]

But the inner expression is now independent of \( \tau \). Thus:

\[
E_{\tau} \left( \frac{1}{\tau} \sum_{i=1}^{\tau} \left[ \sum_{s \in S} \pi_s \phi_s^T w_k - V_s \frac{||\phi_s||^2}{||\phi_s||^2} \phi_s \right] \right)
\]

We note that the RHS is \( TP(w_k) \)

**Proposition F.7.2.** Let us define the sequence of functions \( \tilde{h}_c(w) : \mathbb{R}^n \mapsto \mathbb{R}^n \) such that \( \tilde{h}_c(w) = \frac{h(cw)}{c} \); \( c \geq 1 \). Then

1. \( \tilde{h}_c(\cdot) \mapsto \tilde{h}_\infty(\cdot) \) as \( c \to \infty \) uniformly on compact sets Further,

2. The limiting o.d.e, \( \dot{w}(t) = \tilde{h}_\infty(w(t)) \) has a unique globally asymptotically stable equilibrium at the origin.

**Proof.** First note that

\[
\tilde{h}(w) = h(w)(1 + \sum_{i=1}^{k} \zeta_i)
\]

But \( \zeta = (1 + \sum_{i=1}^{k} \zeta_i) \). Then:

\[
= \zeta h(w)
\]

Expanding \( h(w) \):

\[
= \zeta \sum_{s \in S} \pi_s \left[ \frac{\phi_s^T w - V_s}{||\phi_s||^2} \right] \phi_s
\]

Now we write \( \tilde{h}_c(w) \) from its definition:

\[
\tilde{h}_c(w) = \zeta \sum_{s \in S} \pi_s \left[ \frac{\phi_s^T cw - V_s}{c||\phi_s||^2} \right] \phi_s
\]

But the constants \( c \) can be cancelled for the \( w \) term:

\[
= \zeta \left( \sum_{s \in S} \pi_s \left[ \frac{\phi_s \phi_s^T}{||\phi_s||^2} \right] w - \frac{1}{c} \sum_{s \in S} \pi_s \left[ \frac{V_s \phi_s}{||\phi_s||^2} \right] \right)
\]

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We observe the uniform convergence of this set of functions \( h_c(w) \) to \( h_\infty(w) \) in the limit \( c \to \infty \) as the term \( c \) is only involved with a constant coefficient given by \( \sum_{s \in S} \pi_s \left[ \frac{V_s \phi_s}{||\phi_s||^2} \right] \). Thus the first part is proved.

For the second part, we note the following:

\[
\hat{h}_\infty(w) = \lim_{c \to \infty} \hat{c} \left( \sum_{s \in S} \pi_s \left[ \frac{\phi_s \phi_s^\top}{||\phi_s||^2} \right] w - \frac{1}{c} \sum_{s \in S} \pi_s \left[ \frac{V_s \phi_s}{||\phi_s||^2} \right] \right)
\]

Now we apply the limit only on the second term:

\[
\hat{h}_\infty(w) = \hat{c} \sum_{s \in S} \pi_s \left[ \frac{\phi_s \phi_s^\top}{||\phi_s||^2} \right] w - \left( \lim_{c \to \infty} \frac{\hat{c}}{c} \right) \sum_{s \in S} \pi_s \left[ \frac{V_s \phi_s}{||\phi_s||^2} \right]
\]

Evaluating the limit, we get 0 for the second term:

\[
\hat{h}_\infty(w) = \hat{c} \sum_{s \in S} \pi_s \left[ \frac{\phi_s \phi_s^\top}{||\phi_s||^2} \right] w
\]

Thus:

\[
\hat{h}_\infty(w) = \hat{c} \sum_{s \in S} \pi_s \left[ \frac{\phi_s \phi_s^\top}{||\phi_s||^2} \right] w
\]

Now consider the system \( \dot{w}(t) = \hat{h}_\infty(w(t)) = \hat{c} \sum_{s \in S} \pi_s \left[ \frac{\phi_s \phi_s^\top}{||\phi_s||^2} \right] w \). At the equilibrium point,

\[
\dot{w}(t) = 0
\]

\[
\hat{c} \sum_{s \in S} \pi_s \left[ \frac{\phi_s \phi_s^\top}{||\phi_s||^2} \right] w = 0
\]

But \( \Phi \) has full column rank (by assumption). Thus no eigen value of \( \sum_{s \in S} \pi_s \left[ \frac{\phi_s \phi_s^\top}{||\phi_s||^2} \right] \) is 0. Thus:

\[
w = 0
\]

Thus \( \dot{w}(t) = \hat{h}_\infty(w(t)) \) has a unique globally asymptotically stable equilibrium at the origin. \( \square \)

### F.8 The stochastic update equation with momentum converges:

In section F.4, we showed the assumptions A1 and A2 required for convergence. In section F.5 we showed that the noise term is a martingale difference sequence - assumption A3 (per Borkar (2008)). In section F.7, we showed assumption A4, which was the stability criterion required to show that the iterates remain bounded Lakshminarayanan and Bhatnagar (2017). Finally, in section F.6, we showed that the momentum terms added a perturbation term to the o.d.e that we are asymptotically tracking. All that is left to see is where we converge to.

**Proposition F.8.1.** The globally asymptotically stable equilibrium for the limiting o.d.e \( \dot{w}(t) = \hat{h}(w(t)) \) that our stochastic approximation equation tracks is given by \( w^* = \left[ (\Phi^\top NDN\Phi)^{-1} \Phi^\top NDN \right] V \)

**Proof.** The update equation, \( \dot{w}(t) = \hat{h}(w(t)) \) can be written as \( \dot{w}(t) = \hat{c} h(w(t)) = \hat{c} \sum_{s \in S} \pi_s \left[ \frac{\phi_s^\top w - V_s}{||\phi_s||^2} \right] \phi_s \). Considering that the states are sampled from the stationary distribution \( \pi \), we have:

\[
\dot{w}(t) = \hat{c} \sum_{s \in S} \pi_s \left[ \frac{\phi_s^\top w - V_s}{||\phi_s||^2} \right] \phi_s
\]
Then the equilibrium point is given by the point where:

\[ \dot{w}(t) = 0 \]

\[ \zeta \sum_{s \in S} \pi_s \left[ \frac{\phi_s^T w - V_s}{||\phi_s||^2} \right] \phi_s = 0 \]

But \( \zeta \) is just a constant. Therefore:

\[ \sum_{s \in S} \pi_s \left[ \frac{\phi_s^T w - V_s}{||\phi_s||^2} \right] \phi_s = 0 \tag{30} \]

But this is an equation that we have already solved in section D.4. The solution is given by:

\[ w^* = \left[ (\Phi^T N D N \Phi)^{-1} \Phi^T N D N \right] V \]

We’ve now satisfied all the criteria and also shown the point to which we converge. Thus we show the convergence for the full algorithm with momentum.