Shocks in nonlocal media

We investigate the formation of collisionless shocks along the spatial profile of a gaussian laser beam propagating in nonlocal nonlinear media. For defocusing nonlinearity the shock survives the smoothing effect of the nonlocal response, though its dynamics is qualitatively affected by the latter, whereas for focusing nonlinearity it dominates over filamentation. The patterns observed in a thermal defocusing medium are interpreted in the framework of our theory.

Shocks waves are a general phenomenon thoroughly investigated in disparate area of physics (fluids and water waves, plasma physics, gas dynamics, sound propagation, physics of explosions, etc.), entailing the propagation of discontinuous solutions typical of hyperbolic PDE models \[1,2\]. They are also expected in (non-hyperbolic) universal models for dispersive nonlinear media, such as the Korteweg-De Vries (KdV) and nonlinear Schrödinger (NLS, or analogous Gross-Pitaevskii) equations, since hydrodynamical approximations of such models hold true in certain regimes (typically, in the weakly dispersive or strongly nonlinear case) \[3,4,5\]. However, in the latter cases, no true discontinuous solutions are permitted. The general scenario, first investigated by Gurevich and strongly nonlinear case) \[3,4,5\].

In this Letter we investigate how nonlocality of the nonlinearity allows the shock to form also in the focusing regime where, contrary to the local case, it can prevail over filamentation or modulational instability (MI).

Theory. We start from the paraxial wave equation obeyed by the envelope \( A \) of a monochromatic field \( E = (\frac{2}{c_0^2})^{1/2} A \exp(\imath k z - \imath \omega t) \) (\(|A|^2 \) is the intensity)

\[
\frac{\partial A}{\partial Z} + \frac{1}{2k} \left( \frac{\partial^2 A}{\partial X^2} + \frac{\partial^2 A}{\partial Y^2} \right) + k_0 \Delta n A = -i \frac{\alpha_0}{2} A. \tag{1}
\]

where \( k = k_0 n = \frac{\omega}{c_0} \) is the wave-number, and \( \alpha_0 \) the intensity loss rate. A sufficiently general nonlocal model can be obtained by coupling Eq. (1) to an equation that rules the refractive index change \( \Delta n \) of nonlocal origin. Introducing the scaled coordinates \( x, y, z = X/w_0, Y/w_0, Z/L \), and complex variables \( \psi = A/\sqrt{T_0} \) and \( \theta = k_0 L n \Delta n \), where \( L n = (k_0 n_2 |L|)^{-1} \) is the non-linear length scale associated with peak intensity \( L_0 \) and a local Kerr coefficient \( n_2 \) (\( \Delta n = n_2 |A|^2 \)), \( L_d = k w_0^2 \) is the characteristic diffraction length associated with the input spot-size \( w_0 \), and \( L \equiv \sqrt{L_0 L_d} \), such model can be conveniently written as follows \[12\]

\[
\begin{align}
\imath \varepsilon \frac{\partial \psi}{\partial z} + \frac{\varepsilon^2}{2} \nabla^2 \psi + \chi \theta \psi &= -i \frac{\alpha}{2} \varepsilon \psi, \tag{2} \\
-\sigma^2 \nabla^2 \theta + \theta &= |\psi|^2, \tag{3}
\end{align}
\]

where \( \alpha = \alpha_0 L, \nabla^2 = \nabla_x^2 + \nabla_y^2, \chi = n_2/n_2 = \pm 1 \) is the sign of the nonlinearity, and \( \sigma^2 \) is a free parameter that measures the degree of nonlocality. The peculiar dimensionless form of Eqs. \[2,3\] where \( \varepsilon = L_{nl}/L = \sqrt{L_0/L_d} \) is a small quantity, highlights the fact that we will deal with the weakly defocusing (or strongly non-linear) regime, such that the local \( \sigma = 0 \) and lossless \( \alpha = 0 \) limit yields a semiclassical Schrödinger equation with cubic potential \( (\varepsilon \psi \text{ replace Planck constant}) \).
and time, respectively). We study Eqs. (2-3) subject to the axi-symmetric gaussian input \( \psi_0(r) = \exp(-r^2) \), \( r \equiv \sqrt{x^2 + y^2} \), describing a fundamental laser mode at its waist. For \( \varepsilon \ll 1 \), its evolution can be studied in the framework of the WKB trasformation \( \psi(r, z) = \sqrt{\rho(r, z)} \exp[i\phi(r, z)/\varepsilon] \) [4]. Substituting in Eqs. (2-3) and retaining only leading orders in \( \varepsilon \), we obtain

\[
\rho_z + \left[\frac{(D-1)}{r}\rho u + (\rho u)_r\right] = -\alpha \rho; \quad u_z + uu_r - \chi \theta_r = 0,
\]

\[
-\sigma^2 \left( \theta_{rr} + \frac{D-1}{r} \theta_r \right) + \theta = \rho.
\]

where \( u \equiv \phi_r \) is the phase chirp, and \( D = 2 \) is the transverse dimensionality. The 1D case described by Eqs. (1) with \( D = 1 \) and \( r \rightarrow x (\partial_y = 0) \) illustrates the basic physics with least complexity. In the defocusing case \( (\chi = -1) \) for an ideal medium \( (\sigma = \alpha = 0, \theta = \rho) \), Eqs. (4) are a well known hyperbolic system of conservation laws (Euler and continuity equations) with real celerities (or eigenspeeds, i.e. velocities of Riemann invariants) \( \pm \chi = u \pm \sqrt{-\chi \rho} \), which rules gas dynamics \( (u \text{ and } \rho \text{ are velocity and mass density of a gas with pressure } \propto \rho^2) \). A gaussian input is known to develop two symmetric shocks at finite \( z \) [4]. Importantly the diffraction, which is initially of order \( \varepsilon^2 \), starts to play a major role in the proximity of the overtaking point, and regularize the wave-breaking through the appearance of fast (wavelength \( \sim z \)) oscillations which connect the high and low sides of the front and expand outwards (far from the beam center) [3]. Such oscillations, characteristic of a collisionless shock, appear simultaneously in intensity and phase chirp \( (u) \) as clearly shown in Fig. 1(a),c).

In the nonlocal case, the index change \( \theta(x) \) can be wider than the gaussian mode (for large \( \sigma \) and the shock dynamics is essentially driven by the chirp \( u \) with \( \rho \) adiabatically following. This can be seen by means of the following approximate solution of Eqs. (4): considering that the equation for \( \rho \) is of lesser order \( [O(\varepsilon)] \), with respect to those for \( \theta \) and \( u \) \( [O(1)] \), we assume \( \rho = \exp(-2x^2) \) unchanged in \( z \) and solve exactly the third of Eq. (4) for \( \theta(x) \) (though derived easily, its full expression is quite cumbersome). Then, applying the theory of characteristics \( \hat{1} \), the second of Eqs. (4) is reduced to the following ODEs, where dot stands for \( df/dz \)

\[
\dot{x} = u; \quad \dot{u} = \chi \theta_x,
\]

equivalent to the motion of a unit mass in the potential \( \psi(x) = -\chi \theta \) with conserved energy \( E = u(x)^2 + \psi(x) \).

The solution of Eqs. (4) with initial condition \( x(0) = s, u(0) = 0 \) yields \( x(s, z), u(s, z) \) in parametric form, from which overtaking is found whenever \( u(x, z) \) (obtained by eliminating \( s \)) becomes a multivalued function of \( x \) at finite \( z = z_s \). The shock point corresponding to \( |du/dx| \rightarrow \infty \) is found from the solution \( u(x, z) \) displayed in Fig. 2(a) [2(b)], at positions \( x = \pm x_s \neq 0 \) (defocusing case) or \( x_s = 0 \) (focusing case). The shock distance \( z_s \) increases with \( \sigma \) in both cases, as shown in Fig. 2(c).

We have tested these predictions by integrating numerically Eqs. (2-3). Simulations with \( \chi = -1 \) [see Fig. 2(b,d)] show indeed steepening and post-shock oscillations in the spatial chirp \( u \), which are accompanied by a steep front in \( \rho \) moving outward. The shock location in \( x \) and \( z \) is in good agreement with the results of our approximate analysis summarized in Fig. 2.

Numerical simulations of Eqs. (2-3) validates also the focusing scenario. The field evolution displayed in Fig. 3(a) exhibits shock formation at the focus point \( (x_0 = 0, z_s \simeq 8, \sigma = 5) \) driven the phase whose chirp is shown in Fig. 3(b). This is remarkable because, in the local limit \( \sigma = 0 \), the celerities become imaginary (the equivalent gas would have pressure decreasing with increasing density \( \rho \)), and no shock could be claimed to exist. In this limit, the reduced problem [4] is elliptic and the initial value problem is ill-posed [12], an ultimate consequence of the onset of MI: modes with transverse (nor-
FIG. 3: (Color online) Level plot of the intensity in the focusing case ($\chi = 1, \varepsilon = 0.01$): (a) nonlocal case ($\sigma^2 = 25$); (b) chirp profile for various $z$ for (a); (c) quasi-local case ($\sigma^2 = 10^{-5}$).

FIG. 4: 2D evolution according to Eqs. (2.3) with filamentation.

$\Delta T = (\partial^2_x + \partial^2_y)\Delta T - C\Delta T = -|A|^2$ (6)

where the source term accounts for absorption proportional to intensity through the coefficient $\gamma = \alpha_0/(\rho_0c_0D_T)$, where $\rho_0$ is the material density, $c_0$ the specific heat at constant pressure, and $D_T$ is the thermal diffusivity (see e.g. [16]). Eq. (6) has been already employed to model a refractive index of thermal origin in Ref. [10], and in Ref. [11] in the limit $C = 0$ which is equivalent to consider the range of nonlocality (measured by $1/C$, see below) to be infinite. Starting from the 3D heat equation $\nabla^2 \Delta T = -|A|^2$, the latter regime of strong absorption, we need to account for longitudinal temperature profiles that are known from solutions of the 3D heat equations to be peaked at characteristic distance $Z$ in the middle of sample and decay to room temperature on the facets [14]. Since highly nonlinear phenomena occurs in the neighborhood of $Z$ where the index change is maximum, we can use a (longitudinal) parabolic approximation with characteristic width $L_{\text{eff}}(\sim L)$ of the 3D temperature field $\Delta T(X,Y,Z) = \left[1-(Z-Z_0)^2/2L_{\text{eff}}^2\right] \Delta T_{\perp}(X,Y)$ and consequently approximate $\nabla^2 \Delta T \approx (\partial^2_x + \partial^2_y)\Delta T - L_{\text{eff}}^2/\Delta T_{\perp}$, so that the 3D heat equation reduces to Eq. (6) with $C = 1/L_{\text{eff}}^2$. Following this approach, Eq. (6) coupled to Eq. (11) can be cast in the form of Eqs. (2.3) by posing $\theta = k_0L_{\text{eff}}|dn/dT|\Delta T_{\perp}$ and $\sigma^2 = 1/(C\omega_0^2) = L_{\text{eff}}^2/w_0^2$. The model reproduces the infinite range nonlocality for negligible losses ($L_{\text{eff}} \to \infty$); while for thin samples [$\left|\left(\partial^2_x + \partial^2_y\right)\Delta T_{\perp}\right| << \left|\left(\partial^2_x + \partial^2_y\right)\Delta T\right|$], $L_{\text{eff}}$ can be related to the Kerr coefficient $n_2$ as

$L_{\text{eff}} = \sqrt{n_2/\gamma|dn/dT|} = \frac{D_T\rho_0c_0n_2}{\alpha_0|dn/dT|}$ (7)

which establishes a link between the degree of nonlocality and the strength of the nonlinear response (similarly to other nonlocal materials [12]).

Having retrieved the model Eqs. (2.3), let us show next that the scenario illustrated previously applies substantially unchanged in bulk (2D case) even on account for the optical power loss ($\alpha \neq 0$). An example of the general dynamics is shown in Fig. 3 where we report a simulation of the full model (2.3) with $\sigma^2 = 1$ and relatively large loss $\alpha = 1$. In analogy to the 1D case, Fig. 4(a) clearly shows that the radial chirp $u = \phi_r$ steepens and then develop characteristic oscillations after the shock point ($z \approx 6$, where $|\partial_u u| \to \infty$). Correspondingly the intensity exhibits also an external front which is connected to a flat central region with a characteristic overshoot [see Fig. 4(b)] corresponding to a brighter
ring [inset in Fig. 4(c)]. For larger distances this structure moves outward following the motion of the shock. In the experiment such motion can be observed, at fixed physical length, by increasing the power, which amounts to decrease $\varepsilon$ while scaling $z$ and $\alpha$ accordingly ($z \propto 1/\varepsilon$, $\alpha \propto \varepsilon$), as displayed in Fig. 4(c) for $\sigma = 0.3$.

As a sample of a strongly absorbing medium we choose a 1 mm long cell filled with an aqueous solution of Rhodamine B (0.6 mM concentration). Our measurements of the linear and nonlinear properties of the sample performed by means of the Z-scan technique gives data consistent with the literature [16], and allows us to extrapolate at the operating vacuum wavelength of 532 nm, a linear index $n = 1.3$, a defocusing nonlinear index $n_2 = 7 \times 10^{-7}$ cm$^2$/W$^{-1}$, and $\alpha_0 = 62$ cm$^{-1}$. For our sample $D_T = 1.5 \times 10^{-7}$ m$^2$s$^{-1}$, $\rho_0 = 10^3$ kg m$^{-3}$, $c_p = 4 \times 10^2$JK$^{-1}K^{-1}$ and $|dn/dT| = 10^{-4}$ K$^{-1}$ ($\gamma \approx 4^4 K$ W$^{-1}$), and exploiting Eq. 7 we estimate $L_{\text{eff}} \approx 10 \mu$m ($L_{\text{eff}} << L$ because of the strong absorption that causes strong heating of our sample near the input facet), and correspondingly the degree of nonlocality $\sigma \approx 0.3$. We operate with an input gaussian beam with fixed intensity waist $w_{0I} = w_0/\sqrt{2} = 20 \mu$m ($L_d \approx 12$ mm) focused onto the input face of the cell. With these numbers, an input power $P = \pi w_{0I}^2 I_0 = 200$ mW yields a nonlinear length $L_{\text{NL}} \approx 8 \mu$m ($L \approx 0.3$ mm), which allows us to work in the semiclassical regime with $\varepsilon \approx 0.025$. The radial intensity profiles together with the 2D patterns imaged by means of a 40 $\times$ microscope objective and a recording CCD camera are reported in Fig. 5. As shown the beam exhibits the formation of the bright ring whose external front moves outward with increasing power, consistently with the reported simulations. We point out that, at higher powers, we observe (both experimentally and numerically) that the moving intensity front leaves behind damped oscillations that correspond to inner rings of lesser brightness, as reported in literature [16]. This, however, occurs well beyond the shock point that we have characterized so far.

In summary, the evolution of a gaussian beam in the strong nonlinear regime is characterized by occurrence of collisionless (i.e., regularized by diffraction) shocks that survive the smoothing effect of (even strong) nonlocality. While experimental results support the theoretical scenario in the defocusing case, the remarkable result that the nonlocality favours shock dynamics over filamentation requires future investigation.

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[18] Paraxial diffraction in defocusing media is well known to be isomorphic in 1D to propagation in a normally dispersive focusing medium as considered in Ref. [7].