Entanglement spectrum and entangled modes of random XX spin chains

Mohammad Pouranvar and Kun Yang
National High Magnetic Field Laboratory and Department of Physics, Florida State University, Tallahassee, Florida 32306, USA
(Dated: May 11, 2014)

We study in this work the ground state entanglement properties of finite XX spin-1/2 chains with random couplings, using Jordan-Wigner transformation. We divide the system into two parts and study reduced density matrices (RDMs) of its subsystems. Due to the free-fermion nature of the problem, the RDMs take the form of that of a free fermion thermal ensemble. Finding spectrum of the corresponding entanglement Hamiltonian and corresponding eigenvectors, and comparing them with real space renormalization group (RSRG) treatment, we establish the validity of the RSRG approach for entanglement in the limit of strong disorder, but also find its limitations when disorder is weak. In the latter case our work provides a way to visualize the “effective spins” that form long distance singlet pairs.

I. INTRODUCTION

Presence of entanglement is perhaps the most fundamental difference between quantum and classical physics. Recently entanglement has been widely used to characterize phases and phase transitions in condensed matter physics. Among various ways to quantify entanglement, the most frequently used is the entanglement entropy be-

\[ S_{A} = -tr(\rho_A \ln \rho_A) \]

ment entropy is

\[ S_{A} = -tr(\rho_A \ln \rho_A) \]

While this number indeed describes how much subsystem \( A \) and subsystem \( B \) (most often in real space, which is the focus of this work), we can obtain the reduced density matrix (RDM) of each subsystem by tracing over degrees of freedom of the other subsystem: \( \rho^{A/B} = tr_{B/A}(\rho) \). The entangle-

\[ \rho^{A/B} = tr_{B/A}(\rho) \]

ments. Refael and Moore calculated the entanglement entropy of the random singlet phase using the RSRG method. Within this approximation scheme, entanglement between \( A \) and \( B \) comes from singlet pairs formed by a spin in \( A \) and another spin in \( B \). If it were exactly correct, then in the fermion language above there should be a zero mode associated with each of such pair, and the mode wave function is (ultra)localized at the sites of these entangled spins. In our exact diagonalization study of the random XX model we find, of course, this is not the case unless the randomness is very strong (which is the regime RSRG works well); in general the (entangle-

\[ \rho^{A/B} \]

It seems natural to ask if even more useful information is available in the eigenstates of \( \rho^{A/B} \). The answer is obviously yes, as they are the basis of numerical im-

\[ \rho^{A} = \frac{1}{Z} e^{-H^A} \]

is characterized by a free-fermion entanglement Hamiltonian \( H^A \). It is easy to obtain (at least numerically) the single particle eigenmodes (which make up, but are different from the multi-particle eigenstates of \( \rho^A \)), and their counterparts in part \( B \); we call such pairs entangled (single-particle) modes. For the present model these entangled modes represent singlet pairs formed by effective spins, often over long distance (see below).

This model is a special case of random XXZ model, which could also be studied using the real space renor-

\[ \rho^{A/B} \]

mation group (RSRG, also referred to as strong dis-

\[ \rho^{A/B} \]

order RG in literature)\(^3\) Within this approximation, it was found that the long-distance, low-energy behavior of this entire class of models is dominated by the random singlet fixed point, in which singlet bonds form between spins in opposite sub-lattices over arbitrarily large distances. Refael and Moore\(^3\) calculated the entanglement entropy of the random singlet phase using the RSRG method. Within this approximation scheme, entanglement between \( A \) and \( B \) comes from singlet pairs formed by a spin in \( A \) and another spin in \( B \). If it were exactly correct, then in the fermion language above there should be a zero mode associated with each of such pair, and the mode wave function is (ultra)localized at the sites of these entangled spins. In our exact diagonalization study of the random XX model we find, of course, this is not the case unless the randomness is very strong (which is the regime RSRG works well); in general the (entanglement) “energies” of these modes are not zero, and the mode wave functions do have finite extent. We do find that for modest disorder strength, the entanglement “energies” of the low-lying entangled modes (corresponding to long bonds that cross the subsystem boundary) indeed approaches zero with increasing system size and bond length, supporting the asymptotically free nature of RSRG (including in entanglement calculation). On the other hand the spatial extent of these modes do not decrease with such increase in bond length. These mode wave functions are understood as real space images of the effective spins that form singlets over long distance; the profiles of such effective spins cannot be obtained within RSRG. Thus entanglement provides a new way to probe such effective spins.
In a recent work, Fagotti, et al. also used the free fermion mapping to calculate disorder-averaged moments of entanglement spectrum in this model. A number of other papers studied entanglement entropy of random Heisenberg and other closely related models using various numerical methods. Our work is complementary to Ref. 13 and the other earlier works as our emphasis is on entangled modes of specific realizations of certain random distributions of couplings, not just disorder-averaged quantities.

The remainder of the paper is organized as the following. In Sec. II we introduce our model and notions like entanglement energy and entangled modes. Sec. III is a brief review of RSRG from the viewpoint of entanglement. Our numerical results are presented and analyzed in Sec. IV. Sec. V offers a summary and some concluding remarks.

II. MODEL AND ENTANGLED MODES

The model we work with is a one-dimensional (1D) spin-1/2 antiferromagnetic XX model with \( N \) sites and with random nearest neighbour couplings. The Hamiltonian of the system is

\[
H = \sum_{n=1}^{N-1} J_n (s^x_n s^x_{n+1} + s^y_n s^y_{n+1}).
\]

Note that we use open instead of periodic boundary condition here. By mapping spin operators to fermion operators via Jordan-Wigner transformation:

\[
c_n = e^{(i\pi \sum_{j<n} s^+_j s^-_j)} s^-_n,
\]

where \( c \) is a fermionic operator, the Hamiltonian becomes

\[
H = \frac{1}{2} \sum_{n=1}^{N-1} J_n (c^+_n c_{n+1} + c^+_{n+1} c_n).
\]

The \((N-1)\) \( J \)'s in Eq. (2) will be generated based on random distribution functions to be specified later. It is a special case of the most general free Fermi Hamiltonian:

\[
H = \sum_{i=1}^{N} \sum_{j=1}^{N} h_{ij} c^+_i c_j,
\]

with

\[
h_{ij} = \begin{cases} (1/2) J_i & \text{if } j = i + 1 \\ (1/2) J_{i-1} & \text{if } j = i - 1 \\ 0 & \text{otherwise} \end{cases}
\]

and \( h_{ij} = 0 \) otherwise.

The number of fermions is related to magnetization:

\[
N_F = \frac{1}{2} N + \sum_{n=1}^{N} s^x_n;
\]

for the ground state we expect \( N_F = N/2 \) (we always work with even \( N \)).

Now, we divide the system into two subsystems \( A \) and \( B \), often (but not necessarily) with equal number of sites. Subsystem \( A \) is from site 1 to \( N_A \), and subsystem \( B \) is from site \( N_A + 1 \) to \( N \). We know that we can write the RDM \( \rho^A \) (since it is a positive definite operator) as the exponential of a Hermitian operator, \( H^A \), as in Eq. (1). For the special case of free fermion system here, the entanglement Hamiltonian \( H^A \) is also a free fermion Hamiltonian:

\[
H^A = \sum_{i,j=1}^{N_A} h^A_{ij} c^+_i c_j.
\]

To determine eigenmodes and eigenvalues of \( h^A \) we follow Ref. 12 by defining correlation function

\[
C^A_{ij} = \langle c^+_i c_j \rangle,
\]

which is an \( N_A \times N_A \) symmetric matrix. Its \( N_A \) eigenmodes are the same as those of \( h^A \), while the corresponding eigenvalues are related to each other:

\[
n^A_k = \frac{1}{1 + e^{\epsilon^A_k}},
\]

where \( \epsilon^A_k \) is an eigenvalue of \( h^A \) referred to as entanglement energy (not to be confused with eigenvalues of the original Hamiltonian \( h \)), and \( n_k \) is the probability of finding a fermion in the corresponding mode.

We may have three different types of \( \epsilon \). First, it can be a very big positive number, which means the probability of finding a fermion at mode \( k \) is almost 1; thus this fermion is almost exclusively localized in \( A \). Second, it can be a very big positive number, which means the probability of finding a fermion at mode \( k \) is almost zero; this means that there is a fermion that is almost localized completely in the complementary mode (to be discussed later) localized in subsystem \( B \). These modes contribute very little to entanglement. Third, it can be neither of the above (and possibly close to zero), which means there is a substantial probability \( n^A \) less than 1 that this particle is in subsystem \( A \). This necessarily implies that there is a corresponding mode in \( B \) where the same particle resides with probability \( n^B = 1 - n^A \). These pairs of modes, referred to as entangled modes, dominate entanglement. To find the relation between eigenvalues of these modes, we consider the case of a single fermion occupying a mode that is split into two parts. Since the total probability is

\[
n = n^A + n^B = \frac{1}{1 + e^{\epsilon^A}} + \frac{1}{1 + e^{\epsilon^B}} = 1,
\]

we have \( \epsilon^A = -\epsilon^B \). It is straightforward to establish the corresponding relation \( C^A_{ij} = C^A_{ji} \) in the multi particle case: it follows from the Schmidt decomposition that \( \rho^A \) and \( \rho^B \),
and thus $H^A$ and $H^B$ must have the same spectra, while the corresponding eigenstates must have their particle numbers add up to $N$; as a result the spectra of $h^A$ and $h^B$ are related by particle-hole transformation. In the following we will use these relations to identify such pairs of entangled modes.

In fact the entangled modes can be glued together to form a set of $N_F$ orthonormal modes occupied by the fermions in the ground state, and the weight of these modes in subsystems $A$ and $B$ correspond to $n^A$ and $n^B$ respectively. One way to obtain these modes directly (without using the RDMs) is using Klich’s method as outlined below.

We divide the Hilbert space of one particle states into two parts corresponding to our subsystem $A$ and $B$, and introduce corresponding projection operators $P_A$ and $P_B$ as:

$$P_A = \sum_{i=1}^{N_A} |i\rangle \langle i|; \quad P_B = \sum_{i=N_A+1}^{N} |i\rangle \langle i|.$$  \hspace{1cm} (12)

Introduce a Hermitian matrix

$$M_{kk'} = \langle P_A k' | P_A k \rangle,$$  \hspace{1cm} (13)

where $|k\rangle$’s are the lowest $N_F$ eigenvectors of matrix $h$ in Eq. (5) occupied by fermions in the ground state. Then diagonalize this $N_F \times N_F$ matrix by a unitary matrix $V$ as $M = V d V^\dagger$, we obtain these new modes:

$$|l\rangle^A = \frac{1}{\sqrt{d_l}} \sum_k V_{ik}^\dagger P_A |k\rangle = \frac{1}{\sqrt{d_l}} \sum_{k=1}^{N_F} \sum_{i=1}^{N_A} V_{ik}^\dagger U_{ik} |i\rangle,$$  \hspace{1cm} (14)

$$|l\rangle^B = \frac{1}{\sqrt{1-d_l}} \sum_k V_{ik}^\dagger P_B |k\rangle = \frac{1}{\sqrt{1-d_l}} \sum_{k=1}^{N_F} \sum_{i=N_A+1}^{N} V_{ik}^\dagger U_{ik} |i\rangle,$$  \hspace{1cm} (15)

where $U$ is the unitary matrix that diagonalize matrix $h$ in Eq. (5) and $l$ goes from 1 to $N_F$. $|l\rangle^A$ corresponds to eigenvalue $d_l$ (which is the same as $n_l^A$ when $N_F = N_A$) and it is in subsystem $A$. $|l\rangle^B$ corresponds to eigenvalue $1-d_l$ and it is in subsystem $B$. Now, by sticking these modes together we obtain a mode in whole system (Klich eigenmodes):

$$|l\rangle = \sqrt{d_l} |l\rangle^A + \sqrt{1-d_l} |l\rangle^B = \sum_{k=1}^{N_F} V_{ik} |k\rangle.$$  \hspace{1cm} (16)

As shown by Klich, the single Slater determinant formed by these modes is identical to the original ground state, while projecting them to subsystems $A$ and $B$ gives rise to the entangled modes with proper weight. These Klich eigenmodes are thus particularly useful in studies of entanglement.

We note in general the Klich eigenmodes are not the same as the occupied eigemodes of Hamiltonian matrix $h$ in Eq. (5). We illustrate this by calculating the $N_F \times N_F$ overlap matrix between these modes, in which each row represents an eigenmode of matrix $h$ and each column represents one particle Klich eigenmode. Table I is overlap matrix of a sample with $N = 30, N_F = 15, \alpha = 0.1$ (see below for definition of $\alpha$ and its physical meaning). In this case of strong disorder, each Klich eigenmode has large overlap(s) with one (or at most two) eigenmode(s) of matrix $h$, and one can (almost) identify a one-to-one correspondence between them. Table II is overlap matrix of a sample with $N = 30, N_F = 15, \alpha = 0.9$. In this case of weaker disorder, such a correspondence is not possible.

### III. REAL SPACE RENORMALIZATION GROUP METHOD

We can obtain an approximate ground state of the Hamiltonian of Eq. (2) (with $J$’s distributed with a distribution function) by real space renormalization group (RSRG) method. As described by Dasgupta and Ma and Fisher, we first pick up the maximum $J = J_{\text{max}}$. Two spins that are coupled by this $J_{\text{max}}$ will form a singlet in the ground state of the Hamiltonian if we ignore their couplings to other spins. Then we remove the two spins that are coupled by this coupling constant. The two other couplings $J_L, J_R$ that couple this singlet pair with the rest of chain are treated perturbatively, resulting in a new coupling constant $J \approx J_{\text{max}}/J_{\text{max}}$ between the further neighbor spins (see Fig. 1). We note that this new coupling is smaller than $J_L$ and $J_R$, resulting in a reduction of overall energy scale. As we repeat this procedure many times we have singlet bonds that form between spins that are very far apart (thus forming a random singlet state). In the meantime the distribution of $J$’s broadens (with probability of finding smaller effective bonds increases), and approaches a power-law distribution function:

$$P(J) = \left\{ \begin{array}{ll} \alpha J^{-1+\alpha}, & \text{if } 0 \leq J \leq 1 \\ 0, & \text{otherwise} \end{array} \right. \hspace{1cm} (17)$$

with $\alpha$ slowly decreasing with energy scale. We thus use this type of distributions of $J$’s for our Hamiltonian in Eqs. (2) or Eq. (4), with smaller $\alpha$ corresponding to broader distribution (on logarithmic scale) and stronger disorder. Fig. 2 shows an example of the random singlet state generated by the RSRG procedure.
Such \textit{approximate} ground state is made of a direct product of singlet pairs, and only those singlets that cross
the boundary contribute to entanglement. In the fermion language introduced earlier, there is one fermion living on
the two sites that form a singlet, with equal probability. Thus the entanglement energy of the eigenmodes of $h^A$
for subsystems $A$ would only take 3 possible values for such a state: $+\infty$, $-\infty$ and 0; the last corresponds to
singlets formed by spins on opposite sides of the boundary, with entangled modes having equal weight on the
two sites of these spins, and zero weight elsewhere. We will show that this is \textit{not} exactly the case for the \textit{exact}
ground state.

\newpage

\section{RESULTS}

As mentioned before, two important predictions of RSRG method are as follows. First, entangled modes
are ultra localized at entangled spin sites; second, entanglement energy of these modes are \textit{exactly} zero, and
that corresponds to $n^A = n^B = 0.5$. In this section we show that these predictions are not accurate; on the
other hand the entanglement energy does approach zero asymptotically in the limit of large separation between
the entangled modes, reflect the asymptotic exactness of

\begin{table}
\centering
\begin{tabular}{cccccccccccc}
\hline
& 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 1.00 \\
1.00 & 0.00 & 1.00 & 0.81 & 0.19 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.17 & 0.00 & 0.80 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.96 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\

\hline
\end{tabular}
\caption{Overlaps between Klich eigenmodes and filled eigenmodes of original Hamiltonian ($N_F \times N_F$ matrix) when $N = 30, N_F = 15, \alpha = 0.1$. Each row represents an eigenmode of matrix $h$ and each column represents one particle Klich eigenmode. One can find a one-to-one correspondence between them in most cases.}
\end{table}

\begin{table}
\centering
\begin{tabular}{cccccccccccc}
\hline
& 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.27 & 0.35 & 0.18 & 0.01 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\

\hline
\end{tabular}
\caption{Overlaps between Klich eigenmodes and filled eigenmodes of original Hamiltonian ($N_F \times N_F$ matrix) when $N = 30, N_F = 15, \alpha = 0.9$. Each row represents an eigenmode of matrix $h$ and each column represents one particle Klich eigenmode. The overlaps spread among quite a few eigenmodes, and a one-to-one correspondence between them is not possible.}
\end{table}
modes, the Klich eigenmode with \( \alpha \) exactly. Panel (b) shows the zero (entanglement) energy eigenmodes and eigenvalues of entanglement Hamiltonian entanglement. On the other hand, we can also calculate singlet pair that crosses the boundary, and contributes to according to RSRG procedure. We can see there is just one disorder). Panel (a) shows the singlet pair formation ac-

\textbf{A. Entangled modes}

Particle-hole symmetry of Hamiltonian Eq. (2) forces the entanglement energies to be in \( \pm \) pairs. If we choose number of spin sites as: \( N = 2 \times \) odd integer, then \( N_A = N/2 \) is odd, we will have at least one pair of entangled modes whose entanglement energy is exactly zero. In this subsection we focus on these modes.

Fig. 2 shows a sample with \( N = 102 \), and we choose \( N_P = N_A = N_B = N/2 = 51 \). J’s are generated using the distribution function of Eq. (17) with \( \alpha = 0.1 \) (strong disorder). Panel (a) shows the singlet pair formation according to RSRG procedure. We can see there is just one singlet pair that crosses the boundary, and contributes to entanglement. On the other hand, we can also calculate eigenmodes and eigenvalues of entanglement Hamiltonian exactly. Panel (b) shows the zero (entanglement) energy mode, the Klich eigenmode with \( n^A = n^B = 0.5 \). We see that this Klich eigenmode is indeed strongly localized around the two spins that are entangled (consistent with RSRG).

For weaker disorder RSRG method may not be as reliable. Fig. 3 shows a similar sample of Fig. 2 but with \( \alpha = 0.9 \). The RSRG method generates almost the same singlet pair configuration [see panel (a)]. However in panel (b) we see the zero (entanglement) energy Klich eigenmode is much more spread out compared to the previous case, even though it still peaks at the location of the two spins RSRG predicts to form the entangled pair. In such cases (and including the earlier strong disorder example) we should view the entanglement as coming from two effective spins that actually spread over some finite spatial extent, whose profile are described by these entangled modes. This is appropriate as the profiles of these entangled modes are not sensitive to the position of the boundaries of these systems (see panel c of Figs. 3 and 4), indicating that they are intrinsic properties of the degrees of freedom that are entangled over long distance. We note that such information about the effective spin(s) (that form singlets) cannot be obtained from RSRG, but it may also be accessed by looking at spin-spin correlation functions in an exact numerical calculation.

To illustrate the point above quantitatively, we consider the same samples of \( N = 102 \), but choose different locations of the boundary: \( N_A = 19, 39, 51, 69, 79, 89, \) and calculate the overlaps of zero energy Klich eigenmodes among them, forming a \( 6 \times 6 \) overlap matrix. The results are presented in Table III for \( \alpha = 0.1 \) and in Table IV for \( \alpha = 0.9 \). RSRG predicts the same entangled spin pair for them, except for the first choice of \( N_A = 19 \). As a result the overlaps are all close to 1 (closer to 1 for the stronger disorder case \( \alpha = 0.1 \)), except for those involving \( N_A = 19 \).

To quantify the spatial extent of such effective spins, we use inverse participation ratio (IPR) of the corresponding Klich eigenmode:

\[
\text{IPR} = \frac{1}{\sum_i |\psi_i|^4}. \tag{18}
\]

For example, if a wave function is localized over just one site, \( \psi_i = \delta_{ij} \), then IPR = 1, and if a wave function
TABLE III: Overlaps between zero energy Klich eigenmodes for a sample with $\alpha = 0.1$, $N = 102$, $N_F = 51$ corresponding to different choices of boundary, $N_A = 19, 39, 51, 69, 79, 89$. The first one, $N_A = 19$, corresponds to a different entanglement configuration.

| $N_A = 19$ | $N_A = 39$ | $N_A = 51$ | $N_A = 69$ | $N_A = 79$ | $N_A = 89$ |
|-----------|-----------|-----------|-----------|-----------|-----------|
| 1.0       | 0.0       | 0.0       | 0.0       | 0.0       | 0.0       |
| $N_A = 39$ | 0.0       | 1.0       | 0.99      | 0.99      | 0.99      |
| $N_A = 51$ | 0.0       | 0.99      | 1.0       | 0.99      | 0.99      |
| $N_A = 69$ | 0.0       | 0.99      | 1.0       | 0.99      | 0.99      |
| $N_A = 79$ | 0.0       | 0.99      | 0.99      | 1.0       | 0.99      |
| $N_A = 89$ | 0.0       | 0.99      | 0.99      | 0.99      | 1.0       |

TABLE IV: Overlaps between zero energy Klich eigenmodes for a sample with $\alpha = 0.9$, $N = 102$, $N_F = 51$ corresponding to different choices of boundary, $N_A = 19, 39, 51, 69, 79, 89$. The first one, $N_A = 19$, corresponds to a different entanglement configuration.

| $N_A = 19$ | $N_A = 39$ | $N_A = 51$ | $N_A = 69$ | $N_A = 79$ | $N_A = 89$ |
|-----------|-----------|-----------|-----------|-----------|-----------|
| 1.0       | $3.1 \times 10^{-6}$ | $3.2 \times 10^{-6}$ | $3.2 \times 10^{-6}$ | $3.1 \times 10^{-6}$ | $3.0 \times 10^{-6}$ |
| $N_A = 39$ | 0.841     | 0.843     | 0.834     | 0.834     | 0.812     |
| $N_A = 51$ | 0.841     | 0.933     | 0.982     | 0.982     | 0.956     |
| $N_A = 69$ | 0.843     | 1.0       | 0.988     | 0.988     | 0.962     |
| $N_A = 79$ | 0.834     | 0.982     | 1.0       | 0.968     | 0.968     |
| $N_A = 89$ | 0.812     | 0.956     | 0.962     | 0.968     | 1.0       |

FIG. 4: [Color online] (a) Singlet bond configuration generated by RSRG method for a sample with $\alpha = 0.9$ (weaker disorder). Panels (b) and (c) are zero energy Klich eigenstates corresponding to two different boundaries. Panel (b): Zero (entanglement) energy Klich eigenmode which is peaked at the entangled spins predicted by RSRG but much more spread out, $N_A = 51$ (red line) and panel (c) corresponds to $N_A = 69$ (green line).

spreads equally over all sites, $\psi_i = 1/\sqrt{N}$, then IPR= $N$; thus IPR is a measure of how much the wave function spreads. In particular, if RSRG were exact, it would predict IPR=2 for the Klich modes that correspond to the singlet pair contributing to entanglement. It is thus reasonable to identify $\frac{IPR}{N}$ as the spatial extent of an effective spin that contribute to entanglement, through formation of a singlet with another effective spin in the other subsystem.

To see how IPR depends on $\alpha$, we fix the chain length to be $N = 2 \times 41 = 82$, $N_A = N_B = N_F = 41$, and vary $\alpha$ between 0.1 and 1. We only keep samples in which we have only one singlet bond crossing the boundary (and contributing to entanglement) according to RSRG. Result is shown in Fig. 5. We see IPR is very close to 2 for strong disorder, indicating the entangled modes are strongly localized on individual spins (as RSRG predicts). On the other hand it is also clear that IPR increases with increasing $\alpha$ (or decreasing disorder strength), indicating the entangled modes are spreading out over multiple spins, a piece of physics RSRG is unable to capture.

We find that the spatial spreading of the entangled modes is determined by the disorder strength, and insensitive to system size or length of the entangled bond (which typically grows with system size). To demonstrate this point we calculate the average IPR of the zero (entanglement) energy Klich mode versus number of spins, $N$, for fixed $\alpha$. Again we only keep samples with only one singlet bond crossing the boundary according to RSRG. Result is shown in Fig. 6. For both strong and weak disorder, we find little dependence on size $N$.

B. Entanglement energy

Now, we focus on entanglement energy. If we choose the number of spin sites, $N$, to be $2 \times$ even integer, then $N_A = N/2$ is even, RSRG would predict we will have an even number of entangled pairs of spins (possibly zero). In the following we focus on samples with exactly two of such entangled pairs based on RSRG, which then predicts there should be two modes whose entanglement energies are exactly zero, while all other entanglement energies
should be $\pm \infty$. We find that while never literally true, the above is close to being the case for strong disorder, but not quite so for weaker ones. As example we consider a system of $N = 100$, $N_F = N_A = N_B = N/2 = 50$. Table VI lists the entanglement energies of subsystem $A$, $\epsilon_A^k$, and their corresponding probabilities, $n_A^k$, when we use $\alpha = 0.1$; while Table VII lists $\epsilon_A^k$ and $n_A^k$ when we use $\alpha = 0.9$. We can see from comparison that for strong disorder, there is a pair of entanglement energies very close to zero, while (most) others have big magnitudes, consistent with RSRG prediction. For weaker disorder, on the other hand, the lowest entanglement energy is not as close to zero, and there are more entanglement energies that are of order 1. Fig. 7 (for strong disorder) and Fig. 8 (for weak disorder) shows RSRG generated singlet pair configuration for these two samples, and the two Klich eigenmodes with entanglement energy closest to zero.

To better quantify this point, we calculate smallest (in magnitude) entanglement energy averaged over many
excitation energies have been performed before.\textsuperscript{17,18} This is also the case for entanglement energy (the test for real dynamical properties.\textsuperscript{2}) In the long distance and low-energy limit, the reliability of RSRG increases with decreasing bond length, and approaches zero as bond length \( \rightarrow \infty \). This is consistent with asymptotic exactness of RSRG. More quantitatively, we find that for two different disorder strengths, \( \alpha = 0.9 \) and \( \alpha = 0.6 \), \( \epsilon \) depends on logarithm of bond length linearly (beyond certain length scale), indicating

\[ \epsilon_{\text{typical}} \sim L^{-a}, \quad \begin{cases} a = 8.3 \pm 0.5, & \alpha = 0.9, \\ a = 8.5 \pm 0.6, & \alpha = 0.6, \end{cases} \tag{19} \]

where \( L \) is bond length, \( \epsilon_{\text{typical}} = e^{c \log \epsilon} \) is the typical value of entanglement energy (in contrast to average value, which is often dominated by rare fluctuations for broad distributions). Such power law behavior is consistent with the quantum critical nature of the random singlet phase. The corresponding exponent is the same for the two cases within error bars, indicating its universality. We note that within RSRG, the effective bond strength (and corresponding singlet to triplet excitation energy) scales with bond length in a (stretched) exponential fashion. Here we are studying entanglement energy, a different quantity, and find it scales with bond length in a power-law fashion. We would like to caution though that our study is limited to moderately large sizes, and cannot completely rule out the possibility that true asymptotic behavior could be different.

V. SUMMARY AND CONCLUDING REMARKS

In this paper we have studied the entanglement spectrum and in particular, the pairs of entangled modes of random spin-1/2 XX chains using a free fermion mapping, and compared them with predictions of real space renormalization group (RSRG) treatment. We find that RSRG results are qualitatively valid, but not quantitatively accurate, especially for modest disorder strength. On the other hand, in the large distance limit its prediction about entanglement energies becomes asymptotically exact. It would be interesting in the future to study if such asymptotic exactness still holds in the presence of relevant perturbations that drives the RSRG flow away from the random singlet fixed point, like dimerization.\textsuperscript{15} On the other hand, the spreading of the entangled modes
FIG. 10: [Color online] Average logarithm of smallest entanglement energy ($\epsilon$) versus logarithm of bond length. For $\alpha = 0.9$ we study samples ranging from size $N = 100$ to $N = 1000$ with interval of $\Delta N = 100$. For $\alpha = 0.6$, we study samples with $N = 100, 200$ and $N = 300$. For each $N$ we calculate smallest (in magnitude) $\epsilon$ and the corresponding bond length for 500 samples. Logarithm of bond length is divided into 40 segments, with $\log \epsilon$ within each segment averaged over. The blue line is best linear fit for $\alpha = 0.9$ data with slope of $-8.3$ with standard deviation 0.5. Green line is the best linear fit for the case of $\alpha = 0.6$ data with slope of $-8.5$ with standard deviation 0.6.

In a broader context, we argue that eigenstates of the reduced density matrices contain much useful information about entanglement, just like their spectra which are widely studied now. While the example presented here is a very simple one, we hope it serves as a starting point for future studies of entanglement taking advantage of eigenstates of the reduced density matrices.

Acknowledgments

This research is supported by the National Science Foundation through grant No. DMR-1004545.