Contrasting interactions between dipole oscillators in classical and quantum theories: illustrations of unretarded van der Waals forces

Timothy H Boyer

Department of Physics, City College of the City University of New York, New York, NY 10031, United States of America

E-mail: boyer@sci.ccny.cuny.edu

Received 20 October 2017, revised 25 January 2018
Accepted for publication 8 February 2018
Published 17 April 2018

Abstract

Students encounter harmonic-oscillator models in many aspects of basic physics, within widely varying theoretical contexts. Here we highlight the interconnections and varying points of view. We start with the classical mechanics of masses coupled by springs and trace how the same essential systems are reanalysed in the unretarded van der Waals interactions between dipole oscillators within classical and quantum theories. We note how classical-mechanical ideas from kinetic theory lead to energy equipartition which determines the high-temperature van der Waals forces of atoms and molecules modelled as dipole oscillators. In this case, colliding heat-bath particles can be regarded as providing local hidden variables for the statistical mechanical behaviour of the oscillators. Next we note how relativistic classical electrodynamical ideas conflict with the assumptions of nonrelativistic classical statistical mechanics. Classical electrodynamics which includes classical zero-point radiation leads to van der Waals forces between dipole oscillators, and these classical forces agree at all temperatures with the forces derived from quantum theory. However, the classical theory providing this agreement is not a local theory, but rather a non-local hidden-variables theory. The classical theory can be regarded as involving hidden variables in the random phases of the plane waves spreading throughout space which provide the source-free random radiation.

Keywords: harmonic oscillators, unretarded van der Waals forces, classical zero-point radiation, thermal equilibrium
1. Introduction

Students see harmonic-oscillator systems in many of their physics classes. Individual oscillators and coupled oscillators appear in classical mechanics, in electrodynamics, in statistical mechanics, in quantum mechanics, and in modern physics. The oscillator Hamiltonians involved in these areas may be very similar or even identical, but the theoretical contexts behind the oscillator systems can be strikingly different. Charged harmonic oscillators are often used as simple models of atoms and molecules, and the classical and quantum interpretations vary sharply. Here we illustrate some of the contrasting ideas of classical and quantum physics, with the unretarded van der Waals forces between dipole oscillator systems at finite temperature providing the unifying element. We believe that both students and instructors will be interested in the contrasts in the theoretical contexts which often go unmentioned. We start by recalling aspects of harmonic oscillators in classical mechanics which every advanced undergraduate student has seen. We then turn to oscillators in thermal equilibrium within three different theoretical contexts: classical statistical mechanics, classical electrodynamics, and quantum theory.

2. Interacting dipole oscillators in classical mechanics

2.1. Single oscillator

The description of our interacting dipole oscillators begins where students begin, namely with classical mechanics. The Hamiltonian for a single harmonic oscillator of mass $m$ and natural (angular) frequency $\omega_0$ with displacement $x$ and momentum $p$, is given by

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2x^2.$$  

(1)

The Hamiltonian equations of motion $\dot{x} = \partial H / \partial p = p/m$ and $\dot{p} = -\partial H / \partial x = -m\omega_0^2x$ yield the harmonic oscillator solution

$$x(t) = \sqrt{\frac{2E}{m\omega_0}} \cos(\omega_0 t + \phi),$$  

(2)

where the constant $E$ corresponds to the energy of the oscillator

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega_0^2x^2$$  

(3)

and the constant $\phi$ is an arbitrary phase. The time average of the kinetic energy equals the time average of the potential energy

$$\langle \frac{1}{2}m\dot{x}^2 \rangle = \frac{1}{2}m\omega_0^2 \frac{2E}{m\omega_0^2} \langle \sin^2(\omega_0 t + \phi) \rangle = \frac{1}{2}E,$$

$$\langle \frac{1}{2}m\omega_0^2x^2 \rangle = \frac{1}{2}m\omega_0^2 \frac{2E}{m\omega_0^2} \langle \cos^2(\omega_0 t + \phi) \rangle = \frac{1}{2}E,$$

(4)

since the average of the squares of the sine and cosine functions are both $1/2$.

2.2. Two interacting dipole oscillators

2.2.1. Interacting Hamiltonian. Next we consider two interacting dipole oscillators, since this provides the unifying model for our discussion. Interacting harmonic oscillators are often
used in the treatments of unretarded van der Waals forces between atoms and molecules. We introduce an electric dipole interaction \[^{1}\] \(-2e^2x_1x_2/R^3\) between two identical charged harmonic oscillators \(A\) and \(B\) separated by a distance \(R\) along the \(x\)-axis which is parallel to the oscillation direction for each oscillator, corresponding to the situation in a standard quantum mechanics text \[^{2}\]. For this situation, the Hamiltonian takes the form \[^{2}\] (see footnote 1):

\[
H = \frac{p_1^2}{2m} - \frac{1}{2}m\omega_1^2x_1^2 + \frac{p_2^2}{2m} - \frac{1}{2}m\omega_2^2x_2^2 - \frac{2e^2x_1x_2}{R^3}.
\]

Since we are still in the classical mechanics section of our discussion, we note that this same Hamiltonian can be interpreted in terms of masses interacting through springs. In the classical mechanics texts, this Hamiltonian corresponds to two particles of equal mass \(m\), one of which is attached to a wall to the left and the other is attached to a wall to the right by springs of spring constant \(\kappa\), while the masses are coupled together by a spring between the masses of spring constant \(\kappa'\), where \(\kappa = m\omega_0^2 = 2e^2/R^3\) and \(\kappa' = 2e^2/R^3\) \[^{3}\]. Proceeding with the usual classical-mechanical treatment, the Hamiltonian can be rewritten in terms of the normal coordinates \(x_+\) and \(x_-\) given by

\[
x_{\pm} = \frac{1}{\sqrt{2}}(x_A \pm x_B) \text{ and } p_{\pm} = \frac{1}{\sqrt{2}}(p_A \pm p_B),
\]

so that the Hamiltonian now takes the form \[^{2}\] (see footnote 1)

\[
H = \left[ \frac{p_+^2}{2m} + \frac{1}{2}m\omega_+^2x_+^2 \right] + \left[ \frac{p_-^2}{2m} + \frac{1}{2}m\omega_-^2x_-^2 \right],
\]

corresponding to two uncoupled oscillators with associated (angular) frequencies of oscillation given by

\[
\omega_{\pm}^2 = \omega_0^2 \mp \frac{2e^2}{mR^3}.
\]

Note the arrangement of the \(\pm\) signs. In the symmetric normal mode labelled by \(x_+\), the oscillators move in-phase. Thus in the classical-mechanical spring model, the central coupling spring is not compressed in this mode; in the electrostatic force model, the inter-oscillator electric dipole field tends to oppose the spring restoring force on each oscillator. Hence the interaction frequency \(\omega_+\) is the lower frequency. The situation is reversed for the antisymmetric \(x_-\) normal mode, and the oscillation frequency \(\omega_-\) has the higher frequency.

The electric dipole oscillators in the Hamiltonian of equation (5) are treated in the electrostatic approximation which includes the electrostatic interaction between the dipole oscillators but which neglects any radiation emission which would carry energy away from the interacting dipole oscillators.

\[\text{2.2.2. Normal modes of oscillation.}\] The normal mode solutions can be written in the form given above in equation (2) for a single oscillator

\[^{1}\] In the printing of 2005, there is an error of a factor of 2 in going from equation (6.98) to equation (6.99). Between Griffiths’ text and the present article, there is a small shift in notation. Also, the present article uses Gaussian units because of the interest in comparing with a full classical electrodynamic calculation. Gaussian units are natural to classical electrodynamics.
where $E_+$ and $E_-$ are the energies associated with the normal modes of oscillation of the interacting system.

Then writing $x_A = (x_+ + x_-)/\sqrt{2}$, $x_B = (x_+ - x_-)/\sqrt{2}$ and using the expressions in equation (9), we have the solutions for the motions of the two oscillators $A$ and $B$. The two oscillators exchange energy [3] through the interaction term in equation (5). If we start with all the energy in one oscillator and none in the second, the displacement of the first oscillator provides a force on the second, so that, after a suitable time interval, we will find that all the energy has been transferred to the second oscillator and there is none in the first [3]. The two identical oscillators exchange energy, with the (angular) frequency of exchange given by the beat frequency $\omega_{\text{exchange}} = \omega_+ - \omega_-$, which, from equation (8), depends on the strength of the electric dipole interaction $-(2e^2/R^3)x_A x_B$, corresponding to the strength of the spring constant $k'$ in the classical-mechanical texts [3]. As the interaction between the dipole oscillators becomes weaker, the energy exchange frequency $\omega_{\text{exchange}}$ decreases, vanishing in the uncoupled limit. However, for any finite coupling, no matter how small, the energy is always exchanged between the two oscillators $A$ and $B$.

2.2. Correlation between the oscillators. We will be interested in the correlations between displacements of the spatially-separated oscillators. Thus for our two interacting oscillators, we can calculate the time average of the product of the oscillator displacements using equation (9). We find

$$\langle x_A x_B \rangle = \langle (x_+ + x_-)(x_+ - x_-)/2 \rangle = \langle x_+^2 - x_-^2 \rangle/2 = \frac{E_+}{2m\omega_+^2} - \frac{E_-}{2m\omega_-^2}. \quad (10)$$

Evidently, the spatial correlation function for the two oscillator displacements depends upon both the energies $E_+$, $E_-$ in the normal modes and also the frequencies $\omega_+$, $\omega_-$ of the normal modes. For example, if all the energy is in the symmetric mode with frequency $\omega_+$ and none in the antisymmetric mode $\omega_-$, then the two oscillators move together in phase, and, in this case, $\langle x_A x_B \rangle = E_+/(2m\omega_+^2)$ is positive. The magnitude of the correlation depends upon the amplitude of the $x_+$-motion given in equation (9) which involves $[E_+/(m\omega_+^2)]^{1/2}$. The sign of the correlation is reversed for the antisymmetric mode. Thus the correlation in equation (10) has an immediate natural interpretation in terms of the normal modes of oscillation of the system.

3. Interacting dipole oscillators in classical statistical mechanics

Having reviewed the classical-mechanical behaviour of harmonic oscillators, we now re-examine the oscillators’ behaviour in the theoretical context of classical statistical mechanics.

3.1. Harmonic oscillators as models for van der Waals forces

The harmonic oscillator systems, which involve masses coupled by springs in the classical-mechanical texts, reappear in modern physics texts [4] in connection with van der Waals forces between atoms and molecules which are modelled as fluctuating electric dipole oscillators. The oscillations of the dipole oscillator models are taken as those associated with thermal equilibrium. The forces between the dipole oscillators are regarded as modelling the forces between neutral atoms and molecules.
3.2. Single oscillator in a heat bath

The basic classical-mechanical ideas for dipole oscillators, which we considered above, reappear in classical statistical mechanics. We emphasise that the ‘mechanics’ both of the oscillators and of classical statistical mechanics are part of nonrelativistic theory, and that classical statistical mechanics might well be termed ‘nonrelativistic classical statistical mechanics’.

3.2.1. Understanding energy equipartition. Classical statistical mechanics was developed during the 19th century by the application of statistical ideas to classical mechanics. The theory is an outgrowth of classical kinetic theory, based upon the idea that thermal equilibrium is achieved by the energy exchange of point masses upon collisions. A single harmonic oscillator can be regarded as in thermal equilibrium with a heat bath when we imagine the oscillator exchanging energy with point particles. The point particles are free particles which exchange energy upon collision with the other particles of the bath, and then exchange energy upon collision with the mass of the dipole oscillator. In equilibrium, the kinetic energy of the oscillator particle will match (on average) the kinetic energy of the free particles providing the heat bath. Thus the average energy of the oscillator in the heat bath provided by particle collisions is directly related to the average kinetic energy of the heat-bath particles and has nothing to do with the natural frequency \( \omega_0 \) of the oscillator. The average kinetic energy (in one spatial dimension) of the heat-bath particles is denoted by \((1/2)k_BT\). Thus the average kinetic energy of the (one-dimensional) oscillator which is in thermal equilibrium with the heat bath is also \((1/2)k_BT\). Now according to classical-mechanical theory, the kinetic energy of a harmonic oscillator is shared equally with the potential energy of the oscillator motion (as noted above in equation (4)); in a heat bath the mechanical motion is merely interrupted and is changed by the collisions with the heat-bath particles. Since the average kinetic energy is equal to \((1/2)k_BT\), the average potential energy must also be \((1/2)k_BT\). Thus the average total energy of the oscillator must be \((1/2)k_BT + (1/2)k_BT = k_BT\). This is the basic physics underlying the usual energy equipartition theorem of classical statistical mechanics [5].

3.2.2. Probability distribution in classical statistical mechanics. According to classical statistical mechanics, the probability distribution on phase space for the oscillator displacement \(x\) and momentum \(p\) is given by \(P(x, p)\) where [6]

\[
P(x, p) = \left( \frac{\omega_0^2}{2\pi \xi_s} \right) \exp \left( - \frac{p^2}{2m} + \frac{m\omega_0^2x^2}{2\xi_s} \right)
\]

and where \(\xi_s = k_BT\) is the average energy of a single dipole oscillator in classical statistical mechanics. The subscript \(s\) on \(\xi_s\) refers to ‘statistical mechanics’. The probability distribution in equation (11) can be written as a product of two distributions

\[
P(x, p) = P_x(x, \xi_s)P_p(p, \xi_s)
\]

with

\[
P_x(x, \xi_s) = \left( \frac{m\omega_0^2}{2\pi \xi_s} \right)^{1/2} \exp \left( - \frac{m\omega_0^2x^2}{2\xi_s} \right)
\]

and

\[
P_p(p, \xi_s) = \exp \left( - \frac{p^2}{2m} \right)
\]
and

\[ P_x(p, E_x) = \left( \frac{1}{2 \pi m E_x} \right)^{1/2} \exp \left( -\frac{p^2}{2m} \right). \tag{14} \]

The random variables \( x \) and \( p \) have independent probability distributions. The average value of powers of \( x \) and \( p \) is given by

\[ \langle x^{2l} p^{2l} \rangle = \frac{(2k)! (2l)!}{k! l! 2^{k+l}} \left( \frac{E_x}{m \omega_0^2} \right)^{k} \left[ m E_x \right]^{l} \tag{15} \]

for all even powers, with vanishing average value for any odd powers of \( x \) or of \( p \).

#### 3.3. Two interacting dipole oscillators in a heat bath

When there are two interacting dipole oscillators described by the Hamiltonian in equation (5), then there are correlations between the oscillator motions, and also there are van der Waals forces tending to pull the two oscillators together.

**3.3.1. Correlation function \( \langle x_A x_B \rangle \)**. When the two electric dipole oscillators interact electrostatically through their dipole moments, they become correlated just as calculated in equation (10) above in the section on classical mechanics of coupled oscillators. Here we have the same equipartition energy for the oscillators \( E_+ = E_0 = \frac{1}{2} k_B T \), but the frequencies of oscillation are different, so that corresponding to equation (10)

\[ \langle x_A x_B \rangle = \frac{E_+}{2m \omega_0^2} - \frac{E_0}{2m \omega_0^2} \]

\[ = \frac{E_0}{2m} \left( \frac{1}{\omega_0^2} - 2e^2/(mR^3) - \frac{1}{\omega_0^2 + 2e^2/(mR^3)} \right) \]

\[ \approx \frac{E_0}{2m \omega_0^2} \frac{4e^2}{m \omega_0^2 R^3} = \frac{2 \alpha^2 E_0}{e^2 R^3} = \frac{2 \alpha^2 k_B T}{e^2 R^3}, \tag{16} \]

where we have expanded in the approximation \( 2e^2/R^3 \ll m \omega_0^2 \), and the last line introduces the static electric polarizability \([7]\) of the oscillators where

\[ \alpha = e^2/(m \omega_0^2). \tag{17} \]

**3.3.2. van der Waals attraction between the oscillators calculated from electrostatic forces.** In electromagnetism classes, students are asked to calculate the electrostatic attraction between dipole oscillators. Therefore as our first calculation of unretarded van der Waals forces, we will use the electromagnetic context with only the final correlation given by classical statistical mechanics. The unretarded van der Waals force \( F_{BA} \) on oscillator \( B \) due to oscillator \( A \) corresponds to the electrostatic force of one dipole upon the other \([8]\) \( F_{BA} = (\hat{e} x_B \cdot \nabla) E_A(r_B) \) where the electric field \( E_A \) corresponds to the electric dipole field of oscillator \( A \). In the present case, all the forces are in the \( x \)-direction, corresponding to the direction of orientation of the oscillators and to the direction of the separation between the oscillators. Thus the average force on the electric dipole \( \hat{e} x_B \) at \( r_B \) due to the electric dipole \( \hat{e} x_A \) at \( r_A \) is given by the time average of the electrostatic force \( \langle F_{BA} \rangle = \langle F_{BA} \rangle \)

\[\]
The needed correlation function was obtained in equation (16) giving

\[ F_{\text{zrln}} = -6 \frac{e^2}{R^4} (x_A x_B), \]  

where \( x_A x_B \) corresponds to the electric polarizability of the oscillator and where \( \mathcal{E}_s = k_B T \) is the average energy of an isolated single oscillator. The force can be regarded as arising from a potential function \( U_s(R, T) \) as \( F_{\text{zrln}} = -\partial U_s(R, T)/\partial R \), where \( U_s(R, T) \) is given by

\[ U_s(R, T) \approx -2\alpha^2 \frac{\mathcal{E}_s}{R^6} = -2\alpha^2 \frac{k_B T}{R^6}. \]  

### 3.3.3. van der Waals force from Helmholtz free energy

Although the electromagnetic point of view above is direct, one obtains further insight from the fully statistical mechanic point of view which teaches students to work with the Hamiltonian function. The potential function \( U_s(R, T) \) in equation (20) which provides the van der Waals force between the oscillators does not correspond to the internal energy \( U \) of the interacting dipole oscillator system. The thermal energy of the interacting dipole oscillator system is just \( 2\mathcal{E}_s = 2k_B T \) which is the sum of the thermal energies of the two individual oscillators and makes no reference to the distance \( R \) separating the two oscillators. On the other hand, the van der Waals force in equation (19) clearly depends strongly on the separation between the oscillators. We recall that in nonrelativistic statistical mechanics, the kinetic energy of the oscillator masses is determined by the collisions of the point masses providing the thermal bath, and there is no dependence of this energy upon the oscillator frequencies. When the separation between the oscillators changes at constant temperature, the thermal internal energy \( U \) of the system does not change, but rather the entropy \( S \) of the situation changes, corresponding to \( dQ = T dS = dU + dW \) where \( dQ \) is the heat added, and \( dW \) is the work done by the system. According to thermodynamics, the force between the oscillators depends upon the change in Helmholtz free energy \( \mathcal{F} \).

The partition function for the two interacting oscillators in equation (5) is given by [9]

\[ Z_s = \int_{-\infty}^{\infty} dp_A \int_{-\infty}^{\infty} dp_B \int_{-\infty}^{\infty} dx_A \int_{-\infty}^{\infty} dx_B \exp ( -H/\mathcal{E}_s ). \]  

If we introduce the normal coordinates \( x_\pm \) and \( p_\pm \) with frequencies \( \omega_\pm = \omega_0^2 \pm 2e^2/(mR^3) \), then the partition function becomes

\[ Z_s(\omega_0, R, T) = \int_{-\infty}^{\infty} dp_\pm \int_{-\infty}^{\infty} dp_\pm \int_{-\infty}^{\infty} dx_+ \int_{-\infty}^{\infty} dx_- \exp ( -H/\mathcal{E}_s ) \]

\[ = (2\pi m k_B T)^{1/2} \left( \frac{2\pi k_B T}{m \omega_+^2} \right)^{1/2} \left( \frac{2\pi k_B T}{m \omega_-^2} \right)^{1/2} = (2\pi m k_B T)^{3/2} \frac{1}{\omega_+ \omega_-}. \]
Next we evaluate the Helmholtz free energy as

$$F_s(\omega_0, R, T) = -k_B T \ln Z_s = k_B T \left( \frac{1}{2} \ln \omega_1^2 + \frac{1}{2} \ln \omega_2^2 \right) - k_B T \ln [(2\pi k_B T)^2].$$

(23)

Now we omit the terms which are independent of the oscillator separation $R$, and we expand $\omega_1^2$ and $\omega_2^2$ appearing in equation (8) about $\omega_0^2$ using $\ln(1 + x) = x - x^2/2 + x^3/3 - \ldots$. Then we have

$$\ln \omega_1^2 = \ln \left( \omega_0^2 + \frac{2e^2}{mR^3} \right) = \ln \omega_0^2 + \ln \left( 1 + \frac{\pm 2e^2}{m\omega_0^2 R^3} \right)$$

$$= \ln \omega_0^2 + \left( \frac{\pm 2e^2}{m\omega_0^2 R^3} \right) - \frac{1}{2} \left( \frac{\pm 2e^2}{m\omega_0^2 R^3} \right)^2 + \ldots.$$  

(24)

The interparticle potential $U_s(R, T) = F_s(\omega_0, R, T) - \text{const}$ for the force between the two oscillators from the Helmholtz free energy $F_s(\omega_0, R, T)$ then becomes

$$U_s(R, T) = -\frac{2\alpha^2 k_B T}{R^6} = -\frac{2\alpha^2 \mathcal{E}_s}{R^6},$$

(25)

where $\alpha = e^2/(m\omega_0^2)$ and $\mathcal{E}_s = k_B T$. This result is exactly the same as that found from the electrostatic force calculation in equation (20). Note that $\mathcal{E}_s = k_B T$ here is the average energy for one oscillator.

4. Interacting dipole oscillators in classical electrodynamics

4.1. Dipole oscillators in classical electrodynamics

Electric dipole oscillators are a staple of classical electrodynamics in connection with dispersion, radiation emission, and radiation scattering. The mechanical oscillator behaviour is given by the same Hamiltonians as appear in equations (1) and (5). However, now the dipole oscillator systems include not only electrostatic dipole–dipole interactions, but also connections to radiation emission and absorption.

4.2. Contrasts between classical mechanics and classical electrodynamics

4.2.1. Mismatch between classical mechanics and classical electrodynamics

Classical electrodynamics was developed in the 19th century in the same time period as the development of classical statistical mechanics. However, it was noted that the two theories had contrasting and conflicting aspects. Electrodynamics incorporated a fundamental velocity $c$ (derivable from the constants of electrostatics and magnetostatics), whereas classical mechanics (which is the basis for classical statistical mechanics) had no such fundamental velocity. In complete contrast with classical mechanics, there is no such thing as energy transfer due to a point collision within classical electrodynamics. Sudden collisions of charged particles would involve large energy losses to radiation. Indeed, classical electrodynamics involves long-range Coulomb forces which do not fit into classical statistical mechanics. Energy transfer in electromagnetic systems involves forces from electromagnetic fields associated with charged particles or electromagnetic waves. In complete contrast to the situation in classical statistical mechanics, electric dipole oscillators
of different frequencies interacting through electromagnetic fields are not forced to have the same average energy in situations of steady-state behaviour.

4.2.2. Classical electromagnetic radiation equilibrium. At the end of the 19th century and beginning of the 20th century, the mismatch between classical mechanics and classical electrodynamics led to attempts to pin down the inertial frame of the ‘luminiferous ether’ and also to attempts to understand the spectrum of classical radiation equilibrium (the blackbody problem). By the early years of the 20th century, it was realised that classical electrodynamics was a relativistic theory satisfying Lorentz transformations whereas nonrelativistic mechanics satisfied Galilean transformations. Furthermore, nonrelativistic classical statistical mechanics did not fit with the blackbody radiation spectrum except at low frequencies.

It was Planck who recognised that, in equilibrium, the average energy of a charged harmonic oscillator matched that of the random radiation at the oscillator frequency \([10]\). This possible dependence of the oscillator kinetic energy on the frequency of the oscillator motion is totally different from the kinetic-theory idea that the equilibrium kinetic energy must be the same for all systems. However, around 1900, the importance of relativity was not recognised and the possibility of classical zero-point radiation was not considered. Indeed, the influence of the idea of kinetic energy equipartition from kinetic theory was so strong that there was little thought that Nature might contain random classical electromagnetic radiation which did not vanish at the absolute zero of temperature. The physicists in the early 20th century applied the ideas of nonrelativistic classical statistical mechanics to the modes of electromagnetic radiation and claimed that classical physics led inevitably to the Rayleigh–Jeans spectrum for classical electromagnetic radiation in thermal equilibrium. Today we are aware that this claim is wrong \([11]\), despite the fact that it is still repeated in the textbooks of modern physics \([12]\).

Like classical statistical mechanics, quantum mechanics which developed during the early part of the 20th century is a particle theory which takes no account of random radiation at zero temperature. The quantum theory of thermal radiation is a particle theory involving ‘photons’ leading to a Planck spectrum which does not include any zero-point radiation \([13]\).

Today we are much more aware of the importance of special relativity and of the presence of random classical radiation even at zero temperature. Classical zero-point radiation (which is a recognised possibility in classical electrodynamics) is required in order to account for Casimir forces within classical electromagnetism \([14, 15]\). Today we know that relativistic classical electrodynamics which includes classical electromagnetic zero-point radiation leads to classical thermal equilibrium at the Planck spectrum including zero-point radiation \([11, 16, 17]\), corresponding to an energy per normal mode given by

\[
E_{p}(\omega, T) = \frac{1}{2} \hbar \omega \coth \left( \frac{\hbar \omega}{2k_B T} \right) = \frac{1}{2} \hbar \omega + \frac{\hbar \omega}{\exp[\hbar \omega/k_B T] - 1}. \tag{26}
\]

It is this spectrum of random radiation which we will use when discussing thermal equilibrium for classical dipole oscillators within classical electrodynamics.

4.3. Single dipole oscillator in thermal radiation

4.3.1. Radiation energy balance. At thermal equilibrium, a dipole oscillator in classical electrodynamics must be in equilibrium with the random radiation which surrounds it. Thus a dipole oscillator must radiate away its energy, as calculated in classes in electromagnetism, but it must also absorb energy from the random radiation field which surrounds it. Indeed, it was Planck who first performed this calculation involving thermal equilibrium for a dipole oscillator in random classical radiation. Although Planck’s calculation was important in the
history of physics, the behaviour of a charged harmonic oscillator in random classical radiation is unlikely to be familiar to advanced undergraduate physics students today. Today our courses often do not introduce quantum theory from a historical perspective.

4.3.2. Planck’s calculation. Here we sketch the basic aspects of Planck’s calculation [10]. Random classical radiation can be treated as source-free plane waves taken as having periodic boundary conditions in a large cubic box of side $a$,

$$\mathbf{E}_\mathcal{R}(\mathbf{r}, t) = \text{Re} \sum_k \sum_{\lambda=1}^2 \tilde{e}(\mathbf{k}, \lambda) \left( \frac{8\pi \mathcal{E}_\mathcal{R}(\omega)}{a^3} \right)^{1/2} \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t + \theta(\mathbf{k}, \lambda))], \quad (27)$$

$$\mathbf{B}_\mathcal{R}(\mathbf{r}, t) = \text{Re} \sum_k \sum_{\lambda=1}^2 \tilde{k} \times \tilde{e}(\mathbf{k}, \lambda) \left( \frac{8\pi \mathcal{E}_\mathcal{R}(\omega)}{a^3} \right)^{1/2} \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t + \theta(\mathbf{k}, \lambda))], \quad (28)$$

where the wave vector $\mathbf{k}$ takes the values $\mathbf{k} = \hat{x}l2\pi/a + \hat{y}m2\pi/a + \hat{z}n2\pi/a$ for $l, m, n$ running over all positive and negative integers, the constant $a$ refers to the length of a side of the box for periodic boundary conditions with volume $V$, and the random phases $\theta(\mathbf{k}, \lambda)$ are distributed uniformly and independently over the interval $(0, 2\pi)$ [18]. The function $\mathcal{E}_\mathcal{R}(\omega)$ is the electromagnetic energy in the radiation normal mode of frequency $\omega = ck$, and the subscript $\mathcal{R}$ refers to ‘random classical radiation’. A small charged harmonic oscillator of mass $m$, natural (angular) frequency $\omega_0$, and charge $e$ located at position $\mathbf{r}$ in this radiation has a classical equation of motion following Newton’s second law [19]

$$m\ddot{x} = -m\omega_0^2 x + m\tau \dddot{x} + e\mathbf{E}_\mathcal{R}(\mathbf{r}, t). \quad (29)$$

Here the forces involve the spring restoring force $-m\omega_0^2 x$ of the oscillator appearing in the Hamiltonian in equation (1), a radiation damping force $m\tau \dddot{x}$ where

$$\tau = 2e^2/(3mc^3) \quad (30)$$

and the driving force $e\mathbf{E}_\mathcal{R}(\mathbf{r}, t)$ of the random electric field in equation (27), where $\mathbf{r}$ gives the location of the (small) oscillator. Since we are dealing with linear systems, we can consider separately the effects of one of the (complex) plane waves $\mathbf{E}_{k\lambda}(\mathbf{r}, t)$ in the sum appearing in equation (27)

$$\mathbf{E}_{k\lambda}(\mathbf{r}, t) = \tilde{e}(\mathbf{k}, \lambda) \left( \frac{8\pi \mathcal{E}_\mathcal{R}(\omega)}{a^3} \right)^{1/2} \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t + \theta(\mathbf{k}, \lambda))]. \quad (31)$$

With this (complex) plane wave (31) as the source electric field, the oscillator equation (29) has the (complex) steady-state solution

$$x(t) = \frac{e\mathcal{E}_k(\mathbf{k}, \lambda)}{m} \left( \frac{8\pi \mathcal{E}_\mathcal{R}(\omega)}{a^3} \right)^{1/2} \frac{\exp[i(\mathbf{k} \cdot \mathbf{r} - i\omega t + i\theta(\mathbf{k}, \lambda))]}{-\omega^2 + \omega_0^2 + i\tau\omega^3}. \quad (32)$$

Summing over all the plane waves of the random radiation in equation (27) and taking the real part, we have the (real) displacement of the dipole oscillator in random radiation given by

$$x(t) = \sum_{k=1}^2 \sum_{\lambda=1}^2 \frac{e\mathcal{E}_k(\mathbf{k}, \lambda)}{m} \left( \frac{2\pi \mathcal{E}_\mathcal{R}(\omega)}{a^3} \right)^{1/2} \left\{ \frac{\exp[i(\mathbf{k} \cdot \mathbf{r} - i\omega t + i\theta(\mathbf{k}, \lambda))]}{-\omega^2 + \omega_0^2 + i\tau\omega^3} + \text{cc} \right\}, \quad (33)$$

where ‘cc’ stands for ‘complex conjugate.’ The average values for $x(t)$, $\dot{x}(t)$, etc are found by averaging over the random phases of the radiation modes with
\begin{align}
\langle \exp[-i\omega t + i\theta(k, \lambda)] \exp[-i\omega't + i\theta(k', \lambda')] \rangle &= 0 \\
\langle \exp[-i\omega t + i\theta(k, \lambda)] \exp[i\omega't - i\theta(k', \lambda')] \rangle &= \delta_{\omega,\omega'} \delta_{k,k'}.
\end{align}

Thus for our example in equation (33), we have the averages
\begin{equation}
\langle x(t) \rangle = 0, \quad \langle \dot{x}(t) \rangle = 0,
\end{equation}

\begin{equation}
\langle x^2(t) \rangle = \sum_k \sum_{\lambda=1}^2 \epsilon_k^2(k, \lambda) \left( \frac{2\pi E_R(\omega)}{a^3} \right) \left( \frac{2e^2}{m^2[(-\omega^2 + \omega_0^2)^2 + (\tau\omega^3)^2]} \right),
\end{equation}

\begin{equation}
\langle \dot{x}^2(t) \rangle = \sum_k \sum_{\lambda=1}^2 \epsilon_k^2(k, \lambda) \left( \frac{2\pi E_R(\omega)}{a^3} \right) \left( \frac{2e^2\omega^2}{m^2[(-\omega^2 + \omega_0^2)^2 + (\tau\omega^3)^2]} \right)
\end{equation}

and the average oscillator energy
\begin{equation}
E_x(\omega_0) = m \langle \dot{x}^2(t) \rangle / 2 + m\omega_0^2 \langle x^2(t) \rangle / 2,
\end{equation}
becomes
\begin{equation}
E_x(\omega_0) = \sum_k \sum_{\lambda=1}^2 \epsilon_k^2(k, \lambda) \left( \frac{2\pi E_R(\omega)}{a^3} \right) \left( \frac{e^2(\omega_0^2 + \omega^2)}{m[(-\omega^2 + \omega_0^2)^2 + (\tau\omega^3)^2]} \right).
\end{equation}

If the box for the periodic boundary conditions is taken as large, then the normal modes are closely spaced, and the sums over \( k \) can be replaced by integrals in \( d^3k \), \( \sum_k \to (a/2\pi)^3 \int d^3k \), so that
\begin{equation}
\langle x^2(t) \rangle = \langle \dot{x}^2(t) \rangle = \frac{a}{2\pi} \int d^3k \sum_{\lambda=1}^2 \epsilon_k^2(k, \lambda) \left( \frac{2\pi E_R(\omega)}{a^3} \right) \left( \frac{2e^2}{m^2[(-\omega^2 + \omega_0^2)^2 + (\tau\omega^3)^2]} \right),
\end{equation}

with analogous expressions for \( \langle \dot{x}^2(t) \rangle \) and \( E_x(\omega_0) \). Summing over polarisations \( \lambda \), we have
\begin{equation}
\sum_{\lambda=1}^2 \epsilon_k^2(k, \lambda) = 1 - \frac{k^2}{k^2}.
\end{equation}

Then the angular integrations in \( k \) give
\begin{equation}
\int_0^{2\pi} d\phi \int_0^\pi d\theta \sin(\theta)(1 - \cos^2\theta) = \frac{8\pi}{3}.
\end{equation}

In order to evaluate the final integrals over the magnitude of \( k \), we assume that the charge \( e \) is small so that the damping parameter \( \tau\omega_0^2 \ll \omega_0 \). Then the integrands in equations (37)–(40) are sharply peaked. We replace every appearance of the frequency \( \omega = \omega_0 \) by \( \omega_0 \) except for the combination \( \omega - \omega_0 \), we take the limits of the integral as running from \(-\infty\) to \(+\infty\), and we use
\begin{equation}
\int_{-\infty}^{\infty} \frac{du}{a^2u^2 + b^2} = \frac{\pi}{ab}.
\end{equation}

Then we find
\begin{equation}
\langle x^2(t) \rangle = \frac{E_R(\omega_0)}{m\omega_0^2}, \quad \langle \dot{x}^2(t) \rangle = \frac{E_R(\omega_0)}{m}.
\end{equation}
and from equation (39) we have
\[ \mathcal{E}_r(\omega_0) = \mathcal{E}_R(\omega_0). \]
We find that the average energy \( \mathcal{E}_r(\omega_0) \) of the oscillator of natural frequency \( \omega_0 \) in random radiation is the same as the average energy \( \mathcal{E}_R(\omega_0) \) of the radiation normal mode at the frequency \( \omega_0 \). This is Planck’s historic calculation of the average energy of an oscillator immersed in random classical radiation [10]. In a similar fashion, we can start with the expression (33) and evaluate averages for general products of positions and momenta. We find [20] that the average values of the positions and momenta correspond to
\[
\langle x^{2k}p^{2l} \rangle = \frac{(2k)!(2l)!}{k!l!2^{k+l}} \left( \frac{\mathcal{E}_r(\omega_0)}{m\omega_0^2} \right)^{k-l} \langle m\mathcal{E}_r(\omega_0) \rangle^l
\]
with all average values involving odd powers vanishing. This is the same form as equation (15) found from nonrelativistic classical statistical mechanics; however, in the electromagnetic case here, the average energy \( \mathcal{E}_r(\omega_0) \) may depend on the frequency \( \omega_0 \) of the oscillator, whereas in the classical statistical mechanical case, the average energy \( \mathcal{E}_r = k_bT \) is independent of \( \omega_0 \).

4.3.3. Surprising aspects of the radiation balance. The classical electrodynamics description may be regarded as surprising since the charge \( e \) which coupled the oscillator to the random classical radiation does not appear in the final results equations (45) and (46). It turns out that any electromagnetic coupling for the oscillator (for example a quadrupole coupling involving two positive charges at the end of a spring) will give the same zero-coupling limit [21]. Even in the zero-coupling limit, the influence of the random radiation is still evident in the non-vanishing random behaviour of the classical oscillator. The independence from the details of the electromagnetic interaction arises since both the energy loss and the energy pick-up from the random radiation are increased or decreased in the same fashion as the electromagnetic interaction is altered.

It is clear from the results in equations (15) and (46), that both classical descriptions of the single harmonic oscillator involve a Gaussian probability distribution
\[
P(x, p) = \left( \frac{\omega_0}{2\pi\mathcal{E}} \right) \exp \left( -\frac{p^2/(2m) + m\omega_0^2x^2/2}{\mathcal{E}} \right) = P_d(x, \mathcal{E})P_p(p, \mathcal{E})
\]
with different assumptions regarding the average oscillator energy \( \mathcal{E} \). In kinetic theory and in classical statistical mechanics, the average kinetic energy for each particle is the same in equilibrium; equilibrium kinetic energy is completely independent from any oscillation frequency of the particle. In contrast, electromagnetic systems in random radiation do not show this kinetic-energy equipartition; rather, the particle kinetic energy (of the oscillator) depends upon the random energy \( \mathcal{E}_R(\omega_0) \) in the radiation at the natural frequency \( \omega_0 \) of the oscillator, and can be quite different for oscillators of different frequencies.

4.3.4. Limitations of dipole oscillator systems. There are two aspects of a dipole oscillator which we wish to emphasise. The first is that a point dipole oscillator can be regarded as the low-velocity limit of a relativistic oscillator system. This allows us to consider point dipole oscillators within relativistic classical electrodynamics. This small-oscillator low-velocity limit has been discussed at length [17]. Second, a dipole oscillator does not determine the frequency spectrum of thermal radiation. The oscillator energy will match the energy of the random radiation but does not determine the spectrum. A dipole oscillator does not act as a
'black particle' in rescattering radiation into different frequencies. Rather a harmonic
oscillator is a linear system which will scatter radiation into different directions, tending to
make the radiation pattern more nearly isotropic; [21] however, the scattered radiation is of
exactly the same frequency as the incident radiation. Thus a point dipole oscillator does not
push radiation toward the equilibrium frequency spectrum of thermal radiation. It was
precisely because of this failure of a point oscillator to act as a black particle that Planck
turned to statistical mechanics applied to the oscillator in trying to determine the radiation
spectrum of thermal equilibrium [10].

However, if one applies nonrelativistic classical statistical mechanics to the oscillator (as
in the modern physics texts [12]) or uses a nonrelativistic nonlinear classical scatterer (as has
been done in the research literature [22]), then one indeed finds the Rayleigh–Jeans spectrum.
Only by going to relativistic considerations including Lorentz-invariant classical zero-point
radiation, does one find the Planck spectrum with zero-point radiation $T_{Pzp}$ of

\[\text{equation (26)}\]

as the equilibrium spectrum of classical physics [11, 16, 17]. The Planck spectrum for thermal radiation depends crucially upon relativity within classical physics.

4.4. Two interacting dipole oscillators

4.4.1. Oscillator motion in random classical radiation. When two electric dipole harmonic
oscillators are located in random classical electromagnetic radiation $E_N(r, t)$, they are driven
into random oscillation. The oscillator equations of motion take the form

\[\begin{align*}
    m\ddot{x}_A &= -m\omega_A^2 x_A + m\tau\dot{x}_A + eE_{Br}(r_A, t) + eE_{Rr}(r_A, t), \\
    m\ddot{x}_B &= -m\omega_B^2 x_B + m\tau\dot{x}_B + eE_{Ar}(r_B, t) + eE_{Rr}(r_B, t),
\end{align*}\]

(48)

where $E_{Br}(r_A, t)$ is the $x$-component of the electric dipole field of oscillator $B$ at the location of
oscillator $A$, and $E_{Ar}(r_B, t)$ is the analogous dipole field due to oscillator $A$. Again for point
dipole oscillators, this is a system of linear equations, and accordingly we can treat separately
the contribution (31) from each plane wave $E_{k\lambda}$ appearing in the random radiation sum of
equation (27). Introducing $E_{k\lambda}(r, t)$ from equation (31) into the equations (48), the (complex)
steady-state equations become

\[\begin{align*}
    -\omega_A^2 m\ddot{x}_A &= -m\omega_A^2 x_A + i\omega_A^3 m\tau x_A - m\eta(k, R)x_A + eE_{k\lambda}(r_A, t), \\
    -\omega_B^2 m\ddot{x}_B &= -m\omega_B^2 x_B + i\omega_B^3 m\tau x_B - m\eta(k, R)x_B + eE_{k\lambda}(r_B, t),
\end{align*}\]

(49)

where

\[\eta(k, R) = -\frac{2e^2}{mR^3} (1 - ikR) \exp[ikR].\]

(50)

The terms involving $\eta$ give the full electric fields of the oscillating dipoles (and not just the
electrostatic field) at the position of the other dipole.

If we divide through by $m$, and then add and subtract the two equations in (49) while
dividing by $\sqrt{2}$, we can deal with the normal modes $x_+$ and $x_-$ introduced in equations (6) so
that we have

\[\begin{align*}
    -\omega^2 x_+ &= -\omega_0^2 x_+ + i\omega_0^3 \tau x_+ - \eta(k, R)x_+ + (e/m)[E_{k\lambda}(r_A, t) + E_{k\lambda}(r_B, t)]/\sqrt{2}, \\
    -\omega^2 x_- &= -\omega_0^2 x_- + i\omega_0^3 \tau x_- + \eta(k, R)x_- + (e/m)[E_{k\lambda}(r_A, t) - E_{k\lambda}(r_B, t)]/\sqrt{2}
\end{align*}\]

(51)
and
\[ x_\pm = \frac{(e/m) [E_{k\lambda}(r_A, t) \pm E_{k\lambda}(r_B, t)] / \sqrt{2}}{-\omega^2 + \omega_0^2 - i\omega^3 \pm \eta(k, R)}. \] (52)

Reintroducing the full sum over all plane waves in the random radiation (27), we find the (real) normal modes
\[ x_\pm(t) = \sum_{k, \lambda=1}^{2} \frac{e_c(k, \lambda)}{m} \left( \frac{2\pi \mathcal{E}_R(\omega)}{a^3} \right)^{1/2} \times \left\{ \begin{array}{c} \{ \exp[ik \cdot r_A] \pm \exp[ik \cdot r_B] \} \exp[-i\omega t + i\theta(k, \lambda)] / \sqrt{2} \\ -\omega^2 + \omega_0^2 - i\omega^3 \pm \eta(k, R) \end{array} \right\} \sum_{l=1}^{2} \exp \left[ ik \cdot (r_A - r_B) \right] + \text{cc}. \] (53)

Now squaring, and averaging over the random phases \( \theta(k, \lambda) \) as in equations (34) and (35), we find
\[ \langle x_\pm^2(t) \rangle = \sum_{k, \lambda=1}^{2} \frac{e_c^2(k, \lambda)}{m^2} \left( \frac{2\pi \mathcal{E}_R(\omega)}{a^3} \right)^{1/2} \times \left\{ \begin{array}{c} \{ \exp[ik \cdot r_A] \pm \exp[ik \cdot r_B] \} \\ -\omega^2 + \omega_0^2 - i\omega^3 \pm \eta(k, R) \end{array} \right\}^2 \]
\[ = \sum_{k, \lambda=1}^{2} \frac{e_c^2}{m^2} \left( \frac{2\pi \mathcal{E}_R(\omega)}{a^3} \right)^{1/2} \times \left\{ \begin{array}{c} \{ 1 \pm \cos(k \cdot (r_A - r_B)) \} \\ -\omega^2 + \omega_0^2 - i\omega^3 \pm \eta(k, R) \end{array} \right\} \] (54)

Again assuming that the box for the periodic boundary conditions is taken as large so that the normal modes are closely spaced, then the sums over \( k \) can be replaced by integrals in \( d^3k \), \( \sum_k \to (a/2\pi)^3 \int d^3k \). The sum over polarisation is the same as in equation (42). Next we carry out the angular integrations in \( k \) as
\[ \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta (1 - \cos^2 \theta)(1 \pm \cos kR \cos \theta) \]
\[ = 2\pi \int_1^1 dx (1 - x^2)(1 \pm \cos kx) \]
\[ = 8\pi \left[ \frac{1}{3} \pm \frac{\cos(kR)}{(kR)^2} \right] \frac{\sin(kR)}{(kR)^3} \]
\[ = \frac{4\pi}{(e^2/m)k^3} \text{Im}(i\omega^3 \mp \eta). \] (55)

Then the expression (54) for the displacement squared collapses to
\[ \langle x_\pm^2(t) \rangle = \int_{k=0}^{k=\infty} dk \left( \frac{2\mathcal{E}_R(\omega)}{\pi mk} \right) \left\{ \frac{\text{Im}(i\omega^3 \mp \eta)}{[-\omega^2 + \omega_0^2 + \text{Re} \eta]^2 + [\text{Im}(i\omega^3 \mp \eta)]^2} \right\}. \] (56)

We are assuming that the interaction between the oscillators is small so that the frequency shift associated with \( \text{Re} \eta(\omega_0/c, R) \) is small compared with the natural oscillator frequency \( \omega_0 \),
\[ \text{Re} \eta(\omega_0/c, R) = -\frac{2e^2}{mR^2} [\cos(kR) + kR \sin(kR)] \ll \omega_0^2. \] (57)
Then the integrand is strongly peaked at the resonant angular frequency \( \omega_\pm \), with

\[
\omega_\pm = \omega_0^2 + \text{Re} \, \eta(\omega_0/c, R),
\]

\[
\omega_\pm \approx \omega_0 + \frac{1}{2\omega_0} \text{Re} \, \eta(\omega_0/c, R).
\] (58)

Inserting \( \omega_\pm \) for every frequency which does not involve the difference \( \omega - \omega_\pm \), and extending the integrations from \(-\infty\) to \(+\infty\), we have

\[
\langle x_\pm^2(t) \rangle = \int_{-\infty}^{\infty} \frac{2\mathcal{E}_R(\omega_\pm)}{\pi \omega_\pm} \left\{ \frac{\text{Im}[\text{Re} \omega_\pm + \eta(\omega_0/c, R)]}{4\omega_\pm^2 (\omega - \omega_\pm)^2 + \{\text{Im}[\text{Re} \omega_\pm + \eta(\omega_\pm/c, R)]\}^2} \right\}. \] (59)

Now using the definite integral in equation (44), our expression becomes

\[
\langle x_\pm^2(t) \rangle = \left\{ \frac{\pi}{2\omega_\pm} \right\} \left\{ \frac{\mathcal{E}_R(\omega_\pm)}{m \omega_\pm^2} \right\}. \] (60)

Proceeding in a similar fashion and setting \( \mathcal{E}_r(\omega_\pm) = m \langle \dot{x}_\pm^2 \rangle / 2 + m \omega_\pm^3 \langle x_\pm^2 \rangle / 2 \), we obtain

\[
\langle x_\pm^2(t) \rangle = \frac{\mathcal{E}_R(\omega_\pm)}{m \omega_\pm^2}, \quad \langle \dot{x}_\pm^2(t) \rangle = \frac{\mathcal{E}_R(\omega_\pm)}{m}, \quad \text{and} \quad \mathcal{E}_r(\omega_\pm) = \mathcal{E}_R(\omega_\pm). \] (61)

These expressions take the same form as in equation (45) found for a single dipole oscillator in random classical electromagnetic radiation. In the unretarded approximation, \( kR \ll 1 \), the resonant frequencies become

\[
\omega_\pm^2 = \omega_0^2 + \text{Re} \, \eta(\omega_0/c, R) \approx \omega_0^2 + \frac{2e^2}{mR^3}
\] (62)

and

\[
\omega_\pm \approx \left( \omega_0^2 + \frac{2e^2}{mR^3} \right)^{1/2} \approx \omega_0 \left[ 1 + \frac{e^2}{m\omega_0^2 R^3} - \frac{1}{2} \left( \frac{e^2}{m\omega_0^2 R^3} \right)^2 \right]
\] (63)

with

\[
\omega_+ - \omega_- \approx -\frac{2e^2}{m\omega_0 R^3}. \] (64)

4.4.2 Correlation function \( \langle x_A x_B \rangle \). The correlation function \( \langle x_A x_B \rangle \) takes a form similar to that found above in equation (16), here becoming

\[
\langle x_A x_B \rangle = \langle (x_+ + x_-)(x_+ - x_-)/2 \rangle = \langle x_+^2 - x_-^2 \rangle/2 = \frac{\mathcal{E}_r(\omega_+)}{2m\omega_+^2} - \frac{\mathcal{E}_r(\omega_-)}{2m\omega_-^2}
\] (65)

where now (in contrast to the situation in equation (16)) the energy of the normal mode can depend upon the frequency of the normal mode. If we introduce the Planck spectrum with zero-point radiation given in equation (26) and note equation (64), then we find
This correlation function for the two oscillators immersed in random radiation is quite different from the correlation found in equation (16) for two oscillators in a classical statistical mechanical bath precisely because energy equipartition does not hold for thermodynamic equilibrium within classical electrodynamics.

4.4.3. van der Waals forces. The average force on dipole $B$ is given by

$$F_{\text{vdW}} = \langle (\mathbf{e}_i \chi_B) \cdot \nabla_B \mathbf{E}(\mathbf{r}_B, t) \rangle$$

For oscillators oriented along the $x$-axis and separated along the $x$-axis, this gives a force along the $x$-axis $F = ie_i \phi_t (\partial E_i / \partial x)$. The electric field $E_i$ at dipole $B$ is the sum of the electric field $E_{A_i}$ arising from the oscillating electric dipole $A$ and the source-independent random radiation $E_{R_i}$. Thus the average force $F_{\text{vdW}} = \hat{F} F_{\text{vdW}}$ on oscillator $B$ is given by

$$F_{\text{vdW}} = \langle (\mathbf{e}_i \chi_B) \cdot \nabla_B [E_{R_i}(\mathbf{r}_B, t) + E_{A_i}(\mathbf{r}_B, t)] \rangle$$

$$= \hat{F} \left( e_B \frac{\partial}{\partial R} E_{R_i}(\mathbf{r}_B, t) \right) + \hat{F} \left( e_B \frac{\partial}{\partial R} E_{A_i}(\mathbf{r}_B, t) \right).$$

In the unretarded limit considered here, the contribution from the first term vanishes; the first term gives a non-zero contribution only for the retarded forces [23]. The calculation of unretarded van der Waals forces proceeds exactly as discussed above where we obtain the average electrostatic force of one dipole upon the other using equation (18). Thus here we have

$$F_{\text{vdW}} = \left( e_B \frac{\partial}{\partial R} \left( \frac{2eB}{R^3} \right) \right) = -\frac{6e^2}{R^4} \langle \chi_A \chi_B \rangle$$

$$= \frac{6e^2}{R^4} \frac{e^2}{m^2 \omega_B R^3} \frac{\partial}{\partial \omega} \left( \frac{h \coth [h \omega / (2k_B T)]}{2 \omega} \right)_{\omega = \omega_0}. \quad (68)$$

The potential function $U_c(R)$ associated with the van der Waals force is

$$U_c(R, T) = \frac{e^4}{m^2 \omega_0 R^6} \frac{\partial}{\partial \omega} \left( \frac{h \coth [h \omega / (2k_B T)]}{2 \omega} \right)_{\omega = \omega_0}. \quad (69)$$

There are two special cases which are worth considering separately.

4.4.3.1. High-temperature limit. The first special case is the high-temperature limit, $\hbar \omega_0 / 2 \ll k_B T$, of the Planck spectrum with zero-point radiation (26) where the energy goes over fully to the Rayleigh–Jeans form and so the energy per normal mode is independent of the frequency, $E(\omega_r) = E(\omega_c) = k_B T$. In this case, we need the expansion for coth $x$ for small $x$, $\coth x = 1/x + x/3 - x^3/45 + ...$, so that

$$F_{\text{vdW}} \approx 6 \frac{e^2}{R^4} \frac{e^2}{m^2 \omega_0 R^3} \frac{\partial}{\partial \omega} \left( \frac{h \coth [h \omega / (2k_B T)]}{2 \omega} \right)_{\omega = \omega_0} = -12 \alpha^2 k_B T R^3 \omega_0$$

for $k_B T \gg \hbar \omega_0$. 

$$\approx 6 \frac{e^2}{R^4} \frac{e^2}{m^2 \omega_0 R^3} \frac{\partial}{\partial \omega} \left( \frac{h \coth [h \omega / (2k_B T)]}{2 \omega} \right)_{\omega = \omega_0} = -12 \alpha^2 k_B T R^3 \omega_0.$$ 

(70)
with \( \alpha = e^2 / (m \omega_d^2) \) corresponding to the electric polarizability of the oscillator. The associated potential function \( U \), for the force is

\[
U(R, T) = -2\alpha^2 k_B T / R^6, \tag{71}
\]

just as in equation (20) for the case of nonrelativistic classical statistical mechanics.

### 4.4.3.2. Low-temperature limit

The second special case is that of low temperature, \( k_B T \ll \hbar \omega_0 \), where the Planck spectrum including classical zero-point radiation (26) goes over to the zero-point spectrum, \( E_{zp}(\omega) = (1/2)\hbar \omega \). \( E_{zp}(\omega) = (1/2)\hbar \omega \). In this case, we need the limit of \( \cosh x \) for large \( x \), \( \cosh x \to 1 \). Then the force between the two oscillators is

\[
F_{\text{conh}} \approx 6 \frac{e^2}{R^4} \frac{\alpha}{m^2 \omega_0 R^3} \frac{\partial}{\partial \omega} \left( \frac{\hbar}{2\omega} \right)_{\omega=\omega_0} = -6\frac{\alpha^2 \hbar \omega_0}{R^6} \quad \text{for} \quad \hbar \omega_0 \gg k_B T, \tag{72}
\]

where \( \alpha = e^2 / (m \omega_d^2) \) corresponds to the electric polarizability of the oscillator. The force can be regarded as arising from a potential function \( U(R, 0) \) as \( F_{\text{conh}} = -\partial U(R, 0) / \partial R \), where \( U(R, 0) \) is given by

\[
U(R, 0) \approx -\frac{\alpha^2 }{R^6} E_{zp}(\omega_0) = -\frac{\alpha^2 \hbar \omega_0}{R^6} / 2. \tag{73}
\]

### 4.5. Zero-point energy in relativistic classical electrodynamics

We have seen here that classical electrodynamics allows both a high-temperature and low-temperature limit for van der Waals forces. This situation is in sharp contrast with nonrelativistic classical statistical mechanics which has only one form for the van der Waals forces at all temperatures. The nonrelativistic classical statistical mechanical result in equation (25) vanishes at zero temperature where \( \varepsilon_\xi = k_B T \to 0 \). However, the general classical electromagnetic spectrum \( E_{zp}(\omega, T) \) in equation (26) contains temperature-independent random classical zero-point radiation which persists even at zero-temperature. It is the random classical zero-point radiation \( E_{zp}(\omega) \) which accounts for the van der Waals forces between the dipole oscillators in the low-temperature limit of equation (73).

Classical electromagnetic zero-point radiation corresponds to an energy \( E_{zp}(\omega) = \hbar \omega / 2 \) per normal mode. Up to an over-all multiplicative constant, this is the unique spectrum of random radiation which is invariant under Lorentz transformation [24]. The zero-point radiation spectrum takes the same form in every inertial frame and so has no preferred frame of reference. The spectrum also corresponds to a divergent energy density, as must hold true for a Lorentz-invariant spectrum. The zero-point radiation spectrum is quite different from the thermal radiation spectrum at non-zero temperature which has a preferred reference frame, namely the reference frame in which the enclosing container is at rest. Furthermore, the thermal radiation \( E_{zp}(\omega, T) \) above the zero-point radiation \( E_{zp}(\omega) \) must have a finite energy density which is involved in thermodynamic relations [25].

A nonrelativistic classical-mechanical theory such as classical statistical mechanics cannot support the idea of a zero-point energy because all kinetic energy is shared in collisions between point masses. On the other hand, a point dipole oscillator immersed in classical zero-point radiation will acquire the average energy of random radiation at the oscillator’s natural frequency; and this average energy includes zero-point energy. A point classical oscillator can support the idea of a zero-point energy if it shares its energy by interactions through classical electromagnetic forces.
We should note that the zero-point energy of an oscillator \( \mathcal{E}_r(\omega_0, 0) = \frac{\hbar \omega_0}{2} \) is adiabatic invariant under a change in the natural frequency \( \omega_0 \) of the oscillator, just as the spectrum of classical electromagnetic zero-point energy is invariant under an adiabatic compression [25].

5. Interacting dipole oscillators in quantum electrodynamics

There are strong connections between quantum electrodynamics and classical electrodynamics with classical electromagnetic zero-point radiation. If one considers the interaction of point electric dipole oscillators with the quantum electromagnetic field, the equations of motion in the Heisenberg picture take the same form as the classical electrodynamic equations. Since the equations are linear, one finds the same average values, now taken as vacuum expectation values, for \( \langle \hat{x}_r^2 \rangle, \langle \hat{x}_d \rangle, \langle \hat{x}_d \hat{x}_d \rangle \), and indeed for all the quadratic expressions. Since quadratic expressions are involved, the van der Waals force predicted by quantum electrodynamics and by classical electrodynamics with classical zero-point radiation are the same [26]. Indeed, there is general agreement at all temperatures between classical electrodynamics with zero-point radiation and quantum electrodynamics for free fields and for harmonic oscillator systems provided all products of quantum operators are completely symmetrized [20].

6. Interacting dipole oscillators in quantum mechanics

6.1. Connection of quantum electrodynamics and quantum mechanics

Quantum mechanics is often assumed to be a suitable limit of quantum electrodynamics [20]. Because of the agreement between classical electrodynamics and quantum electrodynamics for linear systems, we expect that the quantum mechanics of harmonic oscillator systems should correspond to the small-charge limit \( e \to 0 \) of charged harmonic oscillators in classical electrodynamics, provided all quantum operator products are completely symmetrized.

In the small-\( e \) limit, a classical dipole oscillator in random classical radiation takes the same distribution function as the random radiation at the natural frequency of the oscillator. Thus in equations (46) and (47) for a single oscillator in random radiation, there is no dependence upon the charge \( e \) associated with the dipole moment.

Classical electrodynamics with classical electromagnetic zero-point radiation is sometimes termed ‘stochastic electrodynamics’. The small-\( e \) limit for the dipole oscillators coupled to random zero-point radiation is sometimes termed ‘stochastic mechanics’ in analogy with the small-\( e \) limit of quantum electrodynamics becoming quantum mechanics. For harmonic oscillator systems, the results of stochastic mechanics agree with the results of quantum mechanics provided all the quantum operator products are completely symmetrized [20].

6.2. Single oscillator

6.2.1. Harmonic oscillator in quantum mechanics. The quantum theory of the harmonic oscillator [2] (see footnote 1) is familiar to every advanced undergraduate physics student. The quantum Hamiltonian is the same as that for the classical-mechanical oscillator given in equation (1), but the position \( x \) and momentum \( p \) now become quantum operators \( \hat{x} \) and \( \hat{p} \). The energy eigenstates \( |n\rangle \) of the oscillator are labelled by the integer index \( n = 0, 1, 2, \ldots \) and correspond to energies \( \mathcal{E}_q(n) = (n + 1/2) \hbar \omega_0 \). Here the subscript \( q \) on \( \mathcal{E}_q(n) \) denotes ‘quantum’. The ground state corresponds to \( n = 0 \).
6.2.2. Contrasts between classical and quantum descriptions. For the quantum oscillator, the vacuum expectation values of position-squared \( \langle 0 | \hat{x}^2 | 0 \rangle \) and momentum-squared \( \langle 0 | \hat{p}^2 | 0 \rangle \) correspond exactly to equation (45), provided \( \mathcal{E}_R \) is replaced by \( \mathcal{E}_q \). Indeed the expectation values for powers of \( \hat{x} \) alone or of powers of \( \hat{p} \) alone give

\[
\langle 0 | \hat{x}^{2k} | 0 \rangle = \left( \frac{2k!}{k!2^k} \right) \mathcal{E}_q \quad \text{and} \quad \langle 0 | \hat{p}^{2l} | 0 \rangle = \left( \frac{2l!}{l!2^l} \right) [m\mathcal{E}_q]^l,
\]

vanish for all odd powers, and agree with the results of classical electrodynamics with classical zero-point radiation given in equation (46). However, in contrast to classical theories leading to independent probability distributions for \( x \) and \( p \) as in equations (15) and (46), the energy of the quantum operator corresponds to an eigenvalue, so that average values involving \( \hat{x} \) and \( \hat{p} \) cannot be regarded as involving independent random variables. The quantum variables cannot be regarded as described by a probability distribution such as given in equation (47) for the classical random oscillators. The quantum description of the harmonic oscillator involves a ground-state energy which takes a unique value \( \mathcal{E}_q(0) = (1/2)\hbar\omega_0 \) with no dispersion. The energy uncertainty in a state \( |n\rangle \) involves \( \langle \Delta \mathcal{E}_q(n) \rangle^2 = \langle n|\hat{H}^2|n\rangle - \langle n|\hat{H}|n\rangle^2 \) where \( \langle n|\hat{H}|n\rangle = \langle n|\hat{p}^2/(2m) + m\omega_0\hat{x}^2/2|n\rangle \), and so involves a very specific and not-completely-symmetrized operator order (such as \( \hat{p}\hat{p}^2 + \hat{x}\hat{x}^2 \)) for the operators \( \hat{x} \) and \( \hat{p} \).

Indeed, it has been shown [20] that if one completely symmetrizes the order of all factors in quantum operator products, then for free fields and harmonic oscillator systems, at any temperature, quantum electrodynamics and classic electrodynamics with classical zero-point radiation give the same average values. There is disagreement with such unsymmetrized quantum operator expressions as \( \langle 0 | \hat{x}^2 \hat{p}^2 | 0 \rangle \) or \( \langle 0 | \hat{x}^2 \hat{p}^2 + \hat{p}^2 \hat{x}^2 | 0 \rangle /2 \), whereas the classical theory agrees with only the completely symmetrized expression \( \langle 0 | \hat{x}^2 \hat{p}^2 + \hat{x} \hat{p}^2 \hat{x} + \hat{p} \hat{x} \hat{p} \hat{x} + \hat{x} \hat{p} \hat{x} \hat{p} + \hat{p} \hat{x} \hat{p} \hat{x} | 0 \rangle /6 \). For example, the van der Waals forces agree between the classical and the quantum theories, since these expressions are bilinear, but the theories disagree regarding the dispersion. Thus the Hamiltonian \( \hat{H} = \hat{p}^2/(2m) + m\omega_0\hat{x}^2/2 \) involves completely symmetrized quantum operator products, but the expression for the Hamiltonian squared \( \hat{H}^2 \) does not. The classical theory does not agree with the energy-eigenstate aspect of the quantum theory.

6.2.3. Quantum oscillator at non-zero temperature. At non-zero temperature \( T > 0 \), the expectation value of a quantum operator \( \hat{O} \) is given as an incoherent sum over excited states, so that the quantum operator \( \hat{O} \) takes the average value

\[
\langle \hat{O} \rangle_T = \sum_{n=0}^{\infty} \langle n|\hat{O}|n\rangle \frac{1}{Z_q} \exp \left( -\frac{\mathcal{E}_q(n)}{k_BT} \right),
\]

where \( Z_q \) is the quantum partition function for the system. The thermodynamics of a quantum oscillator can be described by the partition function [27]

\[
Z_q(\omega_0, T) = \sum_{n=0}^{\infty} \exp \left( -\frac{\mathcal{E}_q(n)}{k_BT} \right) = \sum_{n=0}^{\infty} \exp \left( -\frac{(n + 1/2)\hbar\omega_0}{k_BT} \right) = \frac{1}{2} \cosh \left( \frac{\hbar\omega_0}{2k_BT} \right)
\]

The average quantum oscillator energy \( \langle |\hat{H}| \rangle_T \) at non-zero temperature \( T \) is given by

\[
\mathcal{E}_q(T) = -\frac{\partial \ln Z_q(\omega_0, T)}{\partial (1/k_BT)} = \frac{1}{2} \hbar\omega_0 \coth \left( \frac{\hbar\omega_0}{2k_BT} \right) = \frac{1}{2} \hbar\omega_0 + \frac{\hbar\omega_0}{\exp (\hbar\omega_0/k_BT) - 1},
\]
which is the same as the expression for a classical dipole oscillator in thermal equilibrium in classical electrodynamics including zero-point radiation as in equation (26).

6.2.4. High- and low-temperature limits. It is interesting that the quantum theory of the harmonic oscillator includes the zero-point energy in equation (77), whereas the photon description giving the Planck spectrum \( \mathcal{E}_p(\omega, T) \) does not include any zero-point energy for the radiation [13]. In the limit of high temperature for the Planck spectrum, we have

\[
\mathcal{E}_p(\omega, T) = \frac{\hbar \omega_0}{\exp(\hbar \omega_0/k_B T) - 1} = \hbar \omega_0 \left[ 1 + \frac{\hbar \omega_0}{k_B T} + \frac{1}{2} \left( \frac{\hbar \omega_0}{k_B T} \right)^2 + \ldots - 1 \right]^{-1}
\]

\[
= k_B T - \frac{1}{2} \hbar \omega_0 + O \left( \frac{\hbar \omega_0}{k_B T} \right), \quad k_B T \gg \hbar \omega_0, \quad (78)
\]

which retains a finite correction \(-\hbar \omega_0/2\) at high temperature \(T\). Thus the average energy \(\mathcal{E}_q(T)\) of the quantum oscillator in equation (77) goes over fully to the classical statistical mechanical result \(k_B T\) (without any nonvanishing correction at high \(T\)) only if the zero-point energy is included

\[
\mathcal{E}_q(T) = \frac{1}{2} \hbar \omega_0 + \frac{\hbar \omega_0}{\exp(\hbar \omega_0/k_B T) - 1}
\]

\[
= \frac{1}{2} \hbar \omega_0 + \left[ k_B T - \frac{1}{2} \hbar \omega_0 + O \left( \frac{\hbar \omega_0}{k_B T} \right) \right]
\]

\[
= k_B T + O \left( \frac{\hbar \omega_0}{k_B T} \right), \quad k_B T \gg \hbar \omega_0. \quad (79)
\]

Sometimes the need for the zero-point energy in order to achieve the classical limit at high temperature is treated incorrectly in textbook accounts [27].

The difference between the average values \(\langle |\hat{O}| \rangle_T\) for unsymmetrized products of quantum operators and the corresponding average values obtained in classical electrodynamics with classical zero-point radiation persists at non-zero temperatures; however, the fractional discrepancy becomes ever smaller as the temperature becomes larger [20].

6.3. Two interacting dipole oscillators

6.3.1. van der Waals forces at zero temperature. The van der Waals forces between two dipole oscillators in quantum theory are treated in a problem in a standard quantum mechanics textbook [2] (see footnote 1). The unperturbed Hamiltonian corresponds to two harmonic oscillators, and the interaction is that of two electric dipoles \(i e x_A\) and \(i e x_B\) separated by a distance \(R\) along the \(x\)-axis, giving the interacting Hamiltonian in equation (5). When the normal-mode coordinates \(x_+\) and \(x_-\) are introduced, the Hamiltonian can be rewritten as in equation (7). Thus the Hamiltonian can be viewed as that of two uncoupled harmonic oscillators of frequencies \(\omega_\pm = [\omega_0^2 \mp 2e^2/(mR^3)]^{1/2}\) as in equation (8), giving the energy eigenvalues \(\mathcal{E}_q(k, l)\) for the two coupled oscillators as

\[
\mathcal{E}_q(k, l) = (k + 1/2) \hbar \omega_+ + (l + 1/2) \hbar \omega_- \quad \text{for} \quad k, l = 0, 1, 2, \ldots \quad (80)
\]

If the interaction between the oscillators is small \(e^2/R^3 \ll \hbar \omega_0^2\), we may use the binomial expansion through second order in the correction to the unperturbed ground-state energy. The system ground-state energy is given by
\[ \mathcal{E}_q(0, 0) = \frac{1}{2} \hbar \left[ \left( \frac{\omega_0^2}{mR^2} + 2e^2 \right)^{1/2} + \left( \frac{\omega_0^2}{mR^2} - 2e^2 \right)^{1/2} \right] \]
\[ \approx \hbar \omega_0 - \frac{1}{2} \left( \frac{\alpha^2}{m \omega_0^2} \right)^2 \frac{\hbar \omega_0}{R^6} = 2\mathcal{E}_q(\omega_0) - \frac{\alpha^2 \mathcal{E}_q(\omega_0)}{R^6}. \] (81)

where \( \mathcal{E}_q(\omega_0) = \hbar \omega_0 / 2 \) is the quantum ground-state energy for a single harmonic oscillator.

The interaction energy \( \mathcal{E}_q(0, 0) - \hbar \omega_0 \) (viewed as a function of the inter-oscillator separation \( R \)) provides a potential function \( \mathcal{U}_q(R, 0) = -\alpha^2 \mathcal{E}_q / R^6 \) for the force between the two dipole oscillators at zero temperature. We see that the quantum mechanical ground-state energy in equation (81) provides exactly the same potential energy as the potential function in equation (73) found at zero-temperature from classical electrodynamics with classical zero-point radiation.

### 6.3.2. van der Waals forces at non-zero temperature

At non-zero temperature \( T \), the Helmholtz free energy will provide the potential function for the van der Waals forces between the oscillators. The Helmholtz free energy \( \mathcal{F}_q(\omega_0, R, T) \) for the coupled quantum system is analogous to that found for a single oscillator following the partition function equation (76), but now involving

\[ Z_q(\omega_0, R, T) = \sum_{k,l=0}^{\infty} \exp \left( -\frac{\mathcal{E}_q(k, l)}{k_B T} \right) = \sum_{k,l=0}^{\infty} \exp \left( -\frac{(k + 1/2) \hbar \omega_+ + (l + 1/2) \hbar \omega_-}{k_B T} \right) \]
\[ = \frac{1}{2} \coth \left( \frac{\hbar \omega_+}{2k_B T} \right) \frac{1}{2} \coth \left( \frac{\hbar \omega_-}{2k_B T} \right), \] (82)

so that

\[ \mathcal{F}_q(\omega_0, R, T) = -k_B T \ln Z_q(\omega_0, R, T) \]
\[ = -k_B T \left\{ \ln \left[ \coth \left( \frac{\hbar \omega_+}{2k_B T} \right) \right] + \ln \left[ \coth \left( \frac{\hbar \omega_-}{2k_B T} \right) \right] - \ln 4 \right\}. \] (83)

We now carry out a Taylor series expansion for \( f(\omega_0) \) about the argument \( \omega_0 \) as \( f(\omega_0) = f(\omega_0) + (\omega_0 - \omega_0) f'(\omega_0) + [(\omega_0 - \omega_0)^2 / 2] f''(\omega_0) + \ldots, \) and note the value for \( \omega_0 - \omega_0 \) through order \( 1/R^6 \) in equation (63). Then we find

\[ \mathcal{F}_q(\omega_0, R, T) \approx 2\mathcal{F}_q(\omega_0, 0) - k_B T \left\{ -\frac{\hbar}{2k_B T} \coth \left( \frac{\hbar \omega_0}{2k_B T} \right) \left[ (\omega_+ - \omega_0) + (\omega_- - \omega_0) \right] \right\} \]
\[ - k_B T \left\{ -\frac{\hbar}{2k_B T} \frac{\partial}{\partial \omega} \left( \coth \left( \frac{\hbar \omega_0}{2k_B T} \right) \right) \right\}_{\omega = \omega_0} \left[ (\omega_+ - \omega_0)^2 + (\omega_- - \omega_0)^2 \right]/2 \]
\[ = 2\mathcal{F}_q(\omega_0, 0) - k_B T \left\{ -\frac{\hbar}{2k_B T} \coth \left( \frac{\hbar \omega_0}{2k_B T} \right) \right\}_{\omega = \omega_0} \left[ 2 - \frac{\omega_0}{2} \left( \frac{e^2}{m \omega_0^2 R^4} \right)^2 \right] \]
\[ - k_B T \left\{ -\frac{\hbar}{2k_B T} \frac{\partial}{\partial \omega} \coth \left( \frac{\hbar \omega_0}{2k_B T} \right) \right\}_{\omega = \omega_0} \left[ \frac{1}{2} \left( \frac{e^2}{m \omega_0^2 R^4} \right)^2 \right]. \] (84)

The potential function \( \mathcal{U}_q(R, T) \) for the van der Waals force between the dipole oscillators can omit the terms in equation (84) which are independent of the spatial separation \( R \), and so takes the form
\[ \mathcal{U}_d(R, T) = \frac{e^4}{m^2 \omega_0^2 R^6} \frac{\partial}{\partial \omega} \left( \frac{\hbar \coth\left(\frac{\hbar \omega}{2k_B T}\right)}{2\omega} \right)_{\omega = \omega_0}, \] (85)

which is the same as that found in equation (69) from classical electrodynamics including zero-point radiation. The classical electrodynamic and quantum mechanical calculations for van der Waals forces agree at all temperatures.

7. Closing summary

In this article, we have discussed harmonic oscillators within various theoretical contexts of elementary physics. The classical mechanics of a mass at the end of a spring teaches us that a linear oscillator shares its energy equally between particle kinetic energy and spring potential energy. Furthermore, oscillators weakly coupled by springs share their energy between the oscillators, and can be described in terms of the normal modes of oscillation of the oscillator system. Because atoms and molecules are often described in terms of dipole oscillator systems, the physics of harmonic oscillators reappears in discussions of van der Waals forces within the contexts of classical statistical mechanics, classical electrodynamics, quantum electrodynamics, and quantum mechanics. At high temperatures, the van der Waals forces can be adequately described by nonrelativistic classical statistical mechanics which arose from a picture of energy transfer between the oscillators involving collisions between point masses which provide the thermal bath for the oscillators.

Classical electrodynamics has relativistic foundations which are very different from those of nonrelativistic classical mechanics. However, point dipole oscillators can be incorporated consistently into relativistic classical electrodynamics. Classical electrodynamics gives correct predictions [21] for Casimir forces, for van der Waals forces, for oscillator specific heats, for blackbody radiation, and for diamagnetism only if it incorporates Lorentz-invariant random classical zero-point radiation as the source-free solution of Maxwell’s equations. Because the textbooks [28] of classical electrodynamics do not include the source-free radiation in the general solution to Maxwell’s equations, many teachers of physics do not realise that the possibility of source-free radiation is an intrinsic part of classical electrodynamics, and that this source-free radiation forms a boundary condition on Maxwell’s differential equations [29]. Classical electromagnetic theory must choose this boundary condition so as to account for experimental observations. The choice of this boundary condition is ‘as much a part of the postulates of the theory as the form of the Lagrangian or the value of the electron charge’ [30]. Relativistic classical electrodynamics with classical zero-point radiation leads to Planck’s spectrum including zero-point radiation as the equilibrium spectrum of classical thermal radiation.

Any dipole oscillator in equilibrium with random radiation acquires the same random energy as is present in the radiation normal modes of the same frequency as the oscillator. Interacting dipole oscillators experience van der Waals forces between the oscillators on account of the random radiation. The van der Waals forces calculated from classical electrodynamics agree with the results of classical statistical mechanics at high temperatures, and agree with the results of quantum theory at all temperatures.

Nonrelativistic classical statistical mechanics can be regarded as providing a local hidden variables theory for the behaviour of interacting dipole oscillators, with the colliding heat-bath particles providing the hidden variables. This theory agrees with quantum theory only at high temperatures. Classical electrodynamics with classical electromagnetic zero-point radiation provides a classical theory which agrees with the results of quantum theory for free fields and harmonic oscillator systems. The classical theory can be regarded as providing a hidden variables theory for the behaviour of interacting dipole oscillators, with the random
phases of the source-free radiation modes providing the hidden variables. The classical
electrodynamic theory is not a local hidden variables theory, but rather is a nonlocal theory
involving correlations over finite distances. For free fields and harmonic oscillator systems,
the classical electrodynamic forces agree with the quantum results at all temperatures.

ORCID IDs

Timothy H Boyer @ https://orcid.org/0000-0003-2267-6290

References

[1] Griffiths D J 2013 Introduction to Electrodynamics 4th edn (Boston, MA: Pearson) p 172
[2] Griffiths D J 2005 Introduction to Quantum Mechanics 2nd edn (Upper Saddle River, NJ:
Pearson) pp 286–7
[3] See for example Fowles G R and Cassiday G L 1993 Analytical Mechanics 5th edn (Fort Worth,
TX: Saunders College Publishing) pp 382–99 Section 11.3
[4] See for example Eisberg R and Resnick R 1985 Quantum Physics of Atoms, Molecules, Solids,
Nuclei, and Particles 2nd edn (New York: Wiley) p 444
[5] See for example Morse P H 1969 Thermal Physics (Reading, MA: Benjamin-Cummins) pp 192, 222
[6] See for example Morse P H 1969 Thermal Physics (Reading, MA: Benjamin-Cummins) p 193
[7] See for example Griffiths D J 2013 Introduction to Electrodynamics 4th edn (Boston, MA:
Pearson) p 168
[8] See for example Griffiths D J 2013 Introduction to Electrodynamics 4th edn (Boston, MA:
Pearson) p 171 equation (4.5)
[9] See for example Morse P H 1969 Thermal Physics (Reading, MA : Benjamin-Cummins) p 270
[10] See for example the historical discussion in Lavenda B H 1991 Statistical Physics: A Probabilistic
Approach (New York: Wiley) pp 73–4
[11] Boyer T H Blackbody radiation in classical physics: a historical perspective (arXiv:1711.04179)
[12] See for example Eisberg R and Resnick R 1985 Quantum Physics of Atoms, Molecules, Solids,
Nuclei, and Particles 2nd edn (New York: Wiley) pp 6–13
[13] See for example Eisberg R and Resnick R 1985 Quantum Physics of Atoms, Molecules, Solids,
Nuclei, and Particles 2nd edn (New York: Wiley) p 16 Equations (1)–(26)
[14] Boyer T H 2011 Any classical description of nature requires classical electromagnetic zero-point
radiation Am. J. Phys. 79 1163–7
[15] Boyer T H 2016 Understanding zero-point energy in the context of classical electromagnetism
Eur. J. Phys. 37 055206
[16] Boyer T H 2010 Derivation of the Planck spectrum for relativistic classical scalar radiation from
thermal equilibrium in an accelerating frame Phys. Rev. D 81 105024
[17] Boyer T H 2016 Understanding the Planck blackbody spectrum and Landau diamagnetism within
classical electromagnetism Eur. J. Phys. 37 065102
[18] Rice S O 1954 Mathematical analysis of random noise Selected Papers on Noise and Stochastic
Processes ed N Wax (New York: Dover) pp 133–294
[19] See for example Griffiths D J 2013 Introduction to Electrodynamics 4th edn (Boston, MA:
Pearson) p 490
[20] Boyer T H 1975 General connection between random electrodynamics and quantum
electrodyamics for free electromagnetic fields and for dipole oscillator systems Phys. Rev.
D 11 809–30
[21] Boyer T H 1975 Random electrodynamics: the theory of classical electrodynamics with classical
electromagnetic zero-point radiation Phys. Rev. D 11 790–808
[22] van Vleck J H 1924 The absorption of radiation by multiply periodic orbits, and its relation to the
correspondence principle and the Rayleigh–Jeans law: II. Calculation of absorption by multiply
periodic orbits Phys. Rev. 24 347–65
Boyer T H 1976 Equilibrium of random classical electromagnetic radiation in the presence of a
nonrelativistic nonlinear electric dipole oscillator Phys. Rev. D 13 2832–45

23
Boyer T H 1978 Statistical equilibrium of nonrelativistic multiply periodic classical systems and random classical electromagnetic radiation Phys. Rev. A 18 1228–37
[23] Boyer T H 1973 Retarded van der Waals forces at all distances derived from classical electrodynamics with classical electromagnetic zero-point radiation Phys. Rev. A 7 1832–40
[24] Marshall T W 1965 Statistical electrodynamics Proc. Camb. Phil. Soc. 61 537–46
[25] Boyer T H 2003 Thermodynamics of the harmonic oscillator: Wienas displacement law and the Planck spectrum Am. J. Phys. 71 866–70
[26] Boyer T H 1975 Temperature dependence of Van der Waals forces in classical electrodynamics with classical electromagnetic zero-point radiation Phys. Rev. A 11 1650–63
[27] See for example Morse P H 1969 Thermal Physics (Reading, MA: Benjamin-Cummins) p 314 where the high-temperature expression for $U_{\text{vib}}$ is incorrect in equation (22-7)
[28] See for example Griffiths D J 2013 Introduction to Electrodynamics 4th edn (Boston, MA: Pearson) p 445
Jackson J D 1999 Classical Electrodynamics 3rd edn (New York: Wiley) p 246
Zangwill A 2013 Modern Electrodynamics (Cambridge: Cambridge University Press) p 725
Garg A 2012 Classical Electromagnetism in a Nutshell (Princeton, NJ: Princeton University Press) p 204
[29] Boyer T H 2017 The contrasting roles of Planck’s constant in classical and quantum theories Am. J. Phys. 86
[30] Quoted from Coleman S 1961 Classical Electron Theory from a Modern Standpoint Rand Corporation Report p 19 (Santa Monica, CA: RAND Corp)
Teplitz D (ed) Reprinted in 1982 Electromagnetism: Paths to Research (New York: Spring Science) ch 6 pp 183–210