Linear Dimension Reduction Approximately Preserving a Function of the 1-Norm

MICHAEL P. CASEY

Duke Mathematics, 120 Science Drive, 117 Physics Building, Campus Box 90320, Durham, NC 27708-0320 E-mail: mpcasey@math.duke.edu

For any finite point set in $D$-dimensional space equipped with the 1-norm, we present random linear embeddings to $k$-dimensional space, with a new metric, having the following properties. For any pair of points from the point set that are not too close, the distance between their images is a strictly concave increasing function of their original distance, up to multiplicative error. The target dimension $k$ need only be quadratic in the logarithm of the size of the point set to ensure the result holds with high probability. The linear embeddings are random matrices composed of standard Cauchy random variables, and the proofs rely on Chernoff bounds for sums of iid random variables. The new metric is translation invariant, but is not induced by a norm.

MSC 2010 subject classifications: Primary 60; secondary 46B09, 46B85, 60E07, 60G50.

Keywords: dimension reduction, embeddings of finite metric spaces, random projection, metric preserving function, Cauchy random variables, Cauchy projections, stable distributions, concentration of measure.

1. Introduction

The Johnson-Lindenstrauss lemma \[9\] states that given a finite set of points $P \subset \mathbb{R}^D$ and $0 < \epsilon < 1$, there are random linear maps $F : \mathbb{R}^D \to \mathbb{R}^k$ satisfying, for any $x, y \in P$,

$$(1 - \epsilon) \|x - y\|_2 \leq \|F(x) - F(y)\|_2 \leq (1 + \epsilon) \|x - y\|_2$$

with high probability, provided $k = \Theta(\epsilon^{-2} \ln |P|)$. It is sufficient to draw the entries of $F$ i.i.d. sub-Gaussian \[14\]. These random linear projections have provided improved worst case performance bounds for many problems in theoretical computer science, machine learning, and numerical linear algebra. Ailon and Chazelle \[1\] show how $F$ may be computed quickly and apply it to the approximate nearest-neighbor problem, working on the projected points $F(P)$. Vempala \[21\] gives a review of problems that may be reduced to analyzing a set of points $P \subset \mathbb{R}^D$, so that after the random projection $F : \mathbb{R}^D \to \mathbb{R}^k$ is applied, the recovery of approximate solutions is possible with time and space bounds depending on $k$, the target dimension, instead of $D$, the ambient dimension.

In numerical linear algebra, Drineas et al. \[6\] use the lemma to approximate the leverage scores of a given matrix $A$; such scores are used to inform subsampling schemes for $A$, resulting in sketches $\tilde{A}$ of smaller dimensions that preserve desired properties of $A$. 

1
Drineas and Mahoney [7] give a further review of using randomness in numerical linear algebra.

The Johnson-Lindenstrauss lemma is a metric embedding result; the map $F$ sends the finite metric space $P \subset \mathbb{R}^D$ equipped with the 2-norm to a corresponding metric space $F(P) \subset \mathbb{R}^k$, also equipped with the 2-norm, such that distances are preserved well. Ailon and Chazelle [1] also showed that equipping the target space $\mathbb{R}^k$ with the 1-norm is also possible; the target dimension is still proportional to $\ln |P|$, but the dependence on $\epsilon$ may be a bit worse. However, analogous results using the 1-norm on the domain do not hold. For example, in [3] and [11], specific $N$-point subsets of $\mathbb{R}^D$ equipped with the 1-norm are shown to embed only in $\mathbb{R}^k$ with $k = N^{1/\epsilon^2}$ if one requires

$$\|x - y\|_1 \leq \|F(x) - F(y)\|_1 \leq c\|x - y\|_1.$$ 

In particular, Brinkman and Charikar [3] show the target dimension $k$ must be at least $N^{1/2 - O(\epsilon \ln(1/\epsilon))}$ if one wants $c = 1 + \epsilon$.

In light of these negative results, people have tried estimating $\|x - y\|_1$ from the coordinates of $F(x) - F(y)$. When the entries of $F$ are i.i.d. standard Cauchy random variables, the coordinates are distributed i.i.d. like $\|x - y\|_1 X$ with $X \sim$ Cauchy $(1)$. The median of $\|x - y\|_1 |X|$ is $\|x - y\|_1$, so estimating the median from the coordinates of $F(x) - F(y)$ would estimate the distance this way. Indyk [8] considers the sample median as an estimator, while Li, Hastie, and Church [13] consider 1-homogeneous functions of these coordinates for estimators. None of the estimators considered are metrics on $\mathbb{R}^k$. For $k$-nearest neighbor methods, we should like to have a metric on the target space $\mathbb{R}^k$ and prefer a low number of coordinates for each point.

Relaxing the problem as follows, we wish to find linear maps $F : \mathbb{R}^D \rightarrow \mathbb{R}^k$ satisfying, for any $x, y \in P$,

$$(1 - \epsilon)\mu(\|x - y\|_1) \leq \rho(F(x), F(y)) \leq (1 + \epsilon)\mu(\|x - y\|_1)$$

with high probability. We have changed the metric on $\mathbb{R}^k$ to $\rho$ instead of the one induced by the 1-norm, and we have introduced a nonlinear function $\mu$ in place of the identity function. We want $k = \Theta(\epsilon^{-c} \ln |P|)$, with $c < 4$ or better.

Here, $\mu : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a concave increasing function with $\mu(0) = 0$. Such $\mu$ are called “metric preserving” by Corazza [5], for the following reason:

$$\mu(\|x - y\|_1) \leq \mu(\|x - z\|_1) + \mu(\|z - y\|_1)$$

for any $x, y, z \in \mathbb{R}^D$, that is, they admit a new metric on the space that is “compatible” with the old one. In particular, spheres for the new metric about a particular point $y \in \mathbb{R}^D$, that is, the level sets $\{x \in \mathbb{R}^D \mid \|x - y\|_1 = t\}$, look like scaled versions of spheres for the 1-norm about that point; the scaling however is nonlinear. The 1-norm is used here as an example, but any other input metric will still satisfy the triangle inequality under such $\mu$. Not all metric preserving functions are concave increasing, but such a choice ensures the new metric generates the same topology as the old one.
1.1. Main Theorem

Throughout, $\ln^a(x) := (\ln(x))^a$, and for $p \geq 1$, $\ell_p^k$ denotes $\mathbb{R}^k$ with metric induced by the $p$-norm.

**Theorem 1.1.1.** Let $P \subset \mathbb{R}^D$ be a set of $N$ points. For $a \geq 0$, set

$$\xi(a) := \ln(1 + \sqrt{a}) + \frac{1}{2}\ln(1 + a) \quad \text{and} \quad \mu(a) := \text{atanh} \left( \frac{\sqrt{2a}}{1 + a} \right) + \frac{1}{2}\ln(1 + a^2).$$

Equip $\mathbb{R}^k$ with the metric

$$\rho(x, y) := \frac{1}{k} \sum_{i=1}^{k} \xi(|x_i - y_i|).$$

For $1 \leq j \leq D$ and $1 \leq i \leq k$, let $F_{ij}^{\text{i.i.d.} \text{Cauchy}(1)}$ be the entries of $F : \mathbb{R}^D \to \mathbb{R}^k$. If $c \geq 3$, $1/4 \geq \epsilon \geq N^{-c}$, and $k \geq \frac{C}{\epsilon^2(1-\epsilon)^2}(\ln^2(N^c))$,

then if $\|x - y\|_1 \geq \sqrt{1 + \epsilon},$

$$\mu\left( \frac{\|x - y\|_1}{1 + \epsilon} \right) \leq \rho(F(x), F(y)) \leq \mu((1 + \epsilon)\|x - y\|_1)$$

while if $\sqrt{1 + \epsilon} \geq \|x - y\|_1 \geq 8\epsilon^2,$

$$(1 - \epsilon)\mu(\|x - y\|_1) \leq \rho(F(x), F(y)) \leq (1 + \epsilon)\mu(\|x - y\|_1)$$

and finally if $\|x - y\|_1 \leq 8\epsilon^2,$

$$(1 - \epsilon)(1 - 4\epsilon^2)\mu(\|x - y\|_1) \leq \rho(F(x), F(y))$$

for all $x, y \in P$ with probability at least $1 - N^{-c-2}$.

**Remark 1.1.2.** We have not been able to establish an upper bound result

$$\rho(F(x), F(y)) \leq (1 + \epsilon)\mu(\|x - y\|_1)$$

with high probability when $\|x - y\| < 8\epsilon^2$. Our proofs break down or require a much higher estimate for the target dimension $k$. We conjecture that $k = O(\ln^2(N^c)/\epsilon^2)$ still suffices. In either case, $C$ is a constant independent of $\epsilon$ and $N$, but the estimates found for it here are not expected to be tight.

Just like the median estimator approaches, the main idea for the proof is to use the Cauchy random variables $F_{ij}$ to encode $\|x - y\|_1$ in the coordinates of $F(x - y)$. These
coordinates are still random, but applying the $\rho$ metric yields a sum of i.i.d. random variables that concentrate about their mean, which necessarily depends on $\|x - y\|_1$. We are thus able to recover a function of $\|x - y\|_1$ this way. We had to choose the function $\xi$ to grow logarithmically because Cauchy random variables only have fractional moments: concentration phenomena usually require moments of all (or very high) orders. We say more about this particular choice for $\xi$ in section A.

Independent of its interest as an analog of the Johnson-Lindenstrauss lemma, theorem 1.1.1 also contributes to the study of $p$-stable projections. In fact, we make the following conjecture for $1 < p < 2$:

**Conjecture 1.1.3.** For $1 \leq j \leq D$ and $1 \leq i \leq k$, draw the entries of the matrix $F : \mathbb{R}^D \to \mathbb{R}^k$, $F_{ij}$, as i.i.d. copies of a standard $p$-stable random variable. Then with $\rho$, $\epsilon$, and $k$ as in theorem 1.1.1, and with $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ defined as

$$\mu(\lambda) := E\xi(\lambda F_{11}),$$

the following bounds hold: if $\|x - y\|_p = O(\epsilon^2)$

$$(1 - \epsilon)\mu(\|x - y\|_p) \leq \rho(F(x), F(y)) \leq (1 + \epsilon)\mu(\|x - y\|_p)$$

and if $\|x - y\|_p = \Omega(\epsilon^2)$

$$\mu\left(\frac{\|x - y\|_p}{1 + \epsilon}\right) \leq \rho(F(x), F(y)) \leq \mu((1 + \epsilon)\|x - y\|_p)$$

for all $x, y \in P$ with probability at least $1 - N^{-c-2}$.

The setup for the proof would be the same as for theorem 1.1.1; however, because the density for a $p$-stable random variable is only implicitly defined, the needed 1st and 2nd moment estimates are not so straightforward, but could be empirically found on the computer using methods such as [4] to draw the $p$-stable random variables. This approach, in which we directly project the points from $\mathbb{R}^D$, may be contrasted to embedding $\ell_p^D \hookrightarrow \ell_1^k$ and applying theorem 1.1.1 there. Pisier [19] (see also [16, chapter 8] and [10, chapter 9]) shows that such embeddings exist with distortion $(1 + \epsilon)$, with $n$ proportional to $D$ and depending on $p$ and $\epsilon$.

1.2. Outline

The remainder of the paper is as follows. Section 2 shows how we reduce the problem to providing estimates of an appropriate moment generating function and necessary auxiliary lemmas. Section 3 gives the desired estimates and how they inform the choice of target dimension $k$. The final bound for $k$ is proved in corollary 3.4.7 there. The proofs here depend crucially on estimates for the 1st moment, 2nd moment, and the variance; these are made in appendix A. The background in complex analysis and related special functions used throughout appendix A is provided in appendix B.
2. Setup for Proving Concentration Behavior

2.1. Reduction to Studying a Single Coordinate

Let $F : \mathbb{R}^D \to \mathbb{R}^k$ be a matrix $(F_{ij})$. If $v \in \mathbb{R}^D$, then the $i$th coordinate of $F(v)$ is $(F(v))_i = \sum_{j=1}^D F_{ij}v_j$. We show below that if the $F_{ij}$ are independent identically distributed $p$-stable random variables, then by remark 2.1.4, $(F(v))_1 = \|v\|_p F_{1D}$. Consequently, $\rho(F(v), 0) = \frac{1}{p} \sum_{i=1}^k \xi(\|v\|_p F_{iD})$, and our goal is to show that this sum concentrates about its mean when $k$ is large enough.

**Definition 2.1.1 (p-Norms).** For $1 \leq p < \infty$ and $v \in \mathbb{R}^D$, the $p$-norm of $v$ is

$$
\|v\|_p := (\sum_{j=1}^D |v_j|^p)^{1/p}.
$$

The associated metric induced by the $p$-norm on $\mathbb{R}^D$ is

$$
\rho_p(x, y) := \|x - y\|_p = (\sum_{j=1}^D |x_j - y_j|^p)^{1/p}.
$$

These norms are convenient in part because they are “positively” 1-homogeneous,

$$
\|Cv\|_p = C\|v\|_p \quad \text{for} \quad C \geq 0.
$$

In particular, when $\|v\|_p > 0$, $v/\|v\|_p$ has $p$-norm 1. Given the nonexistence results in [3] and [11], the metric $\rho$ that we choose will not have this scaling property. It will still be translation invariant though.

The following definition is modified from [16, chapter 8].

**Definition 2.1.2 (Standard Symmetric $p$-Stable Random Variables).** For $0 < p \leq 2$, a random variable $W$ is drawn from the standard symmetric $p$-stable distribution if $\mathbb{E}\exp(itW) = \exp(\frac{-|t|^p}{p})$.

Such random variables have the following useful property.

**Lemma 2.1.3.** For $1 \leq j \leq D$, let $W_j \overset{i.i.d.}{\sim} W$ with $W$ standard symmetric and $p$-stable with $2 \geq p \geq 1$. Then if $v \in \mathbb{R}^D$, $\mathbb{E}\exp(it\sum_{j=1}^D W_jv_j) = \exp(\frac{-|t|^p\|v\|^p}{p})$.

**Remark 2.1.4.** So if $\|v\|_p = 1$, we have a new standard symmetric $p$-stable random variable $\sum W_jv_j$, and if $x \neq 0 \in \mathbb{R}^D$,

$$
\sum_{j=1}^D W_jx_j = \|x\|_p \sum_{j=1}^D W_j \frac{x_j}{\|x\|_p} \sim \|x\|_p W \left\| \frac{x_j}{\|x\|_p} \right\|_p = \|x\|_p W.
$$

That is, the distribution of the sum carries the $p$-norm information of $x$. We shall show in lemma 2.1.8 that Cauchy random variables are 1-stable.

**Proof.** By independence,

$$
\mathbb{E}e^{it\sum_{j=1}^D W_jv_j} = \prod_{j=1}^D \mathbb{E}e^{itv_jW_j} = \prod_{j=1}^D e^{-|tv_j|^p/p} = e^{-\frac{|t|^p}{p} \sum_{j=1}^D |v_j|^p} = e^{-|t|^p\|v\|^p/p}.
$$

\[\square\]
2.1.1. Cauchy Distribution

**Definition 2.1.5.** The symmetric Cauchy distribution with parameter \( \lambda > 0 \), denoted \( \text{Cauchy}(\lambda) \), has probability density function \( f_\lambda(x) := \frac{\lambda}{\pi(\lambda^2 + x^2)} \).

**Lemma 2.1.6.** Let \( X \sim \text{Cauchy}(1) \). Then the distribution function of \(|X|\) is

\[
\mathbb{P}\{|X| \leq t\} = \frac{2}{\pi} \arctan(t).
\]

**Remark 2.1.7.** So, by the inversion formula for arctan B.3.1,

\[
\mathbb{P}\{|X| > t\} = 1 - \frac{2}{\pi} \arctan(t) = \frac{2}{\pi} \arctan\left(\frac{1}{t}\right).
\]

**Proof.** For \( t \geq 0 \), compute

\[
\mathbb{P}\{|X| \leq t\} = \mathbb{P}\{-t \leq X \leq t\} = \frac{1}{\pi} \int_{-t}^{t} \frac{1}{1 + x^2} \, dx = \frac{2}{\pi} \int_{0}^{t} \frac{1}{1 + x^2} \, dx
\]

with \( x = \tan(v) \)

\[
= \frac{2}{\pi} \int_{0}^{\arctan(t)} (1 + \tan^2(v))^{-1} \sec^2(v) \, dv = \frac{2}{\pi} (\arctan(t) - \arctan(0)) = \frac{2}{\pi} \arctan(t).
\]

We verify that \( X \sim \text{Cauchy}(1) \) is 1-stable.

**Lemma 2.1.8.** Let \( X \sim \text{Cauchy}(1) \), then for \( t \in \mathbb{R} \),

\[
\mathbb{E}\exp(itX) = \exp(-|t|).
\]

**Proof.** We consider the contour integral \( \pi^{-1} \int_{C} \frac{e^{itz}}{(1+z^2)} \, dz \) with the contour \( C \) the half-circle of radius \( R \) in the upper half plane, together with the interval \([-R, R]\) for \( R > 0 \). The contour is oriented counter clockwise. We intend to take \( R \to \infty \), and as soon as \( R > 1 \), the contour encloses \( z = i \). Because \( 1/(1+z^2) = 1/((z-i)(z+i)) \), the integrand contains an isolated simple pole at \( z = i \), so by the residue formula [20, page 75-76, chapter 3],

\[
\frac{1}{\pi} \int_{C} \frac{\exp(itz)}{1+z^2} \, dz = \frac{1}{\pi} \frac{\exp(itz)}{\text{res}_{z=i} \frac{1}{1+z^2}} = 2i \lim_{z \to i} \frac{\exp(itz)}{1+z^2} = 2i \lim_{z \to i} \exp(itz) \frac{1}{z+i} = \exp(-t).
\]
For $z = re^{i\theta}$ with $0 < \theta < \pi$,
\[
\exp(itz) = \exp(itR(\cos \theta + i \sin \theta)) = \exp(itR \cos \theta) \exp(-tR \sin \theta)
\]
which goes to 0 as $R \to \infty$ because $t > 0$. We may also assume $R > \sqrt{2}$ in order to use lemma 2.1.9:
\[
\left| \frac{1}{1 + z^2} \right| \leq \frac{1}{R^2 R^{-4} + (1 - 2R^{-2})}.
\]
Consequently, with $C_+$ the semicircle of radius $R$,
\[
\left| \int_{C_+} \frac{\exp(itz)}{1 + z^2} \, dz \right| \leq \int_{C_+} \left| \frac{\exp(itz)}{1 + z^2} \right| \, dz \leq \frac{\exp(-tR \sin \theta)}{R^2} \frac{\pi R}{R^{-4} + (1 - 2R^{-2})}
\]
which goes to 0 for $t > 0$ when $R \to \infty$. We can conclude
\[
\exp(t) = \lim_{R \to \infty} \frac{1}{\pi} \int_{-R}^R \frac{\exp(itx)}{1 + x^2} \, dx + \lim_{R \to \infty} \frac{1}{\pi} \int_{C_+} \frac{\exp(itz)}{1 + z^2} \, dz = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\exp(itx)}{1 + x^2} \, dx.
\]
When $t < 0$, we have to use the opposite semicircle, with the closed contour now oriented clockwise. The same bounds now hold, as $-\pi < \theta < 0$ makes
\[
\exp(itz) = \exp(itR \cos \theta) \exp(-tR \sin \theta)
\]
have magnitude at most 1, while the residue is now taken at $z = -i$:
\[
\frac{1}{\pi} \int_{C} \frac{\exp(itz)}{1 + z^2} \, dz = -\frac{2i\pi}{\pi \text{ res}_{z=-i}} \exp(itz) \frac{1}{1 + z^2} = -2i \lim_{z \to -i} (z - (-i)) \frac{\exp(itz)}{1 + z^2} = -2i \lim_{z \to -i} \exp(itz) \frac{1}{z - (-i)} = \exp(t)
\]
with the initial minus sign because the contour is clockwise. \hfill \Box

**Lemma 2.1.9.** Let $z = re^{i\theta} \in \mathbb{C}$ with $r > 0$. Then if $r > \sqrt{2}$,
\[
\left| \frac{1}{1 + z^2} \right| \leq \frac{1}{r^2 \sqrt{r^{-4} + (1 - 2r^{-2})}}, \text{ while if } |\theta| \leq \pi/4, \left| \frac{1}{1 + z^2} \right| \leq \frac{1}{\sqrt{1 + r^4}}.
\]

**Proof.** With $z = re^{i\theta}$,
\[
1 + z^2 = 1 + r^2 e^{2i\theta} = 1 + r^2 (\cos(2\theta) + i \sin(2\theta)) = 1 + r^2 \cos(2\theta) + ir^2 \sin(2\theta)
\]
so
\[
|1 + z^2| = (1 + r^4 \cos^2(2\theta) + 2r^2 \cos(2\theta) + r^4 \sin^2(2\theta))^{1/2} = (1 + r^4 + 2r^2 \cos(2\theta))^{1/2} \\
\geq (1 + r^4 (1 - 2/r^2))^{1/2} = r^2 (1/r^4 + (1 - 2/r^2))^{1/2}
\]

```impar-bj ver. 2014/10/16 file: ms.tex date: June 11, 2019```
If \( r = |z| > \sqrt{2} \), all terms in the lower bound are positive. Hence,
\[
\left| \frac{1}{1 + z^2} \right| \leq \frac{1}{r^2} \frac{1}{\sqrt{r^2 - 4} + (1 - 2r^{-2})}.
\]

On the other hand, if \( \theta \leq \pi/4 \),
\[
\left| 1 + z^2 \right| = (1 + r^4 + 2r^2 \cos(2\theta))^{1/2} \geq (1 + r^4)^{1/2} = r^2(1 + 1/r^4)^{1/2}
\]
so that
\[
\left| \frac{1}{1 + z^2} \right| \leq \frac{1}{\sqrt{1 + r^4}}.
\]

### 2.1.2. Concentration and the Moment Generating Function

From our initial discussion of \( p \)-stable random variables and remark 2.1.4, taking each entry \( F_{ij} \) of the matrix \( F : \mathbb{R}^D \rightarrow \mathbb{R}^k \) as a standard symmetric \( p \)-stable random variable \( W_i \sim W \) makes each of the \( k \) coordinates \( F(v) \) have a distribution like \( \|v\|_p W \). These \( k \) coordinates are still random though, so more work has to be done to recover information related to \( \|v\|_p \). If \( \xi : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is strictly increasing, and hence invertible, one would hope that the empirical average \( \frac{1}{k} \sum_{i=1}^k \xi(\|v\|_p |W_i|) \) deviates little from its mean \( \mathbb{E}\xi(\|v\|_p |W|) \).

When the empirical average behaves this way, we say it **concentrates** about its mean.

The following lemma, which bounds the probabilities that the empirical average can be far from the mean, is a standard first step in showing concentration. The lemma will allow us to transition from considering sums of independent random variables to just the behavior of a single random variable.

**Lemma 2.1.10.** For \( 1 \leq i \leq k \), let \( W_i \overset{i.i.d.}{\sim} W \). Let \( \xi : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) and
\[
\mu(\lambda) := \mathbb{E}\xi(\lambda |W|).
\]

Then for \( s > 0 \), and \( \lambda_+ > \lambda > \lambda_- \),
\[
\mathbb{P}\left\{ \frac{1}{k} \sum_{i=1}^k \xi(\lambda |W_i|) > \mu(\lambda_+) \right\} \leq \left( \exp(-s\mu(\lambda_+))\mathbb{E}\exp(s\xi(\lambda |W|)) \right)^k,
\]
and
\[
\mathbb{P}\left\{ \frac{1}{k} \sum_{i=1}^k \xi(\lambda |W_i|) < \mu(\lambda_-) \right\} \leq \left( \exp(s\mu(\lambda_-))\mathbb{E}\exp(-s\xi(\lambda |W|)) \right)^k.
\]

**Remark 2.1.11.** Alternatively with
\[
\Delta_+ := \mu(\lambda_+) - \mu(\lambda) \quad \text{and} \quad \Delta_- = \mu(\lambda) - \mu(\lambda_-),
\]
the linearity of the expectation allows us to rewrite the above bounds as
\[
\Pr \left\{ \frac{1}{k} \sum_{i=1}^{k} \xi(\lambda | W_i |) > \mu(\lambda_+) \right\} \leq \left( \exp(-s\Delta) \mathbb{E} \exp(s(\xi(\lambda | W |) - \mu(\lambda))) \right)^k,
\]
and
\[
\Pr \left\{ \frac{1}{k} \sum_{i=1}^{k} \xi(\lambda | W_i |) < \mu(\lambda_-) \right\} \leq \left( \exp(-s\Delta) \mathbb{E} \exp(-s(\xi(\lambda | W |) - \mu(\lambda))) \right)^k.
\]
This formulation allows knowledge of the variance to come into play, but makes the lower tail proof less straightforward.

**Proof.** We use Markov’s inequality for nonnegative random variables. With \( s > 0 \),
\[
\Pr \left\{ \frac{1}{k} \sum_{i=1}^{k} \xi(\lambda | W_i |) > \mu(\lambda_+) \right\} = \Pr \left\{ s \sum_{i=1}^{k} \xi(\lambda | W_i |) > sk\mu(\lambda_+) \right\} = \Pr \left\{ \exp \left( s \sum_{i=1}^{k} \xi(\lambda | W_i |) \right) > e^{sk\mu(\lambda_+)} \right\} \leq e^{-sk\mu(\lambda_+)} \mathbb{E} \exp \left( s \sum_{i=1}^{k} \xi(\lambda | W_i |) \right)
\]
using independence of the \( W_i \) and then that \( W_i \sim W \) in the last line. Similarly,
\[
\Pr \left\{ \frac{1}{k} \sum_{i=1}^{k} \xi(\lambda | W_i |) < \mu(\lambda_-) \right\} = \Pr \left\{ -s \sum_{i=1}^{k} \xi(\lambda | W_i |) > -sk\mu(\lambda_-) \right\} = \Pr \left\{ \exp \left( -s \sum_{i=1}^{k} \xi(\lambda | W_i |) \right) > e^{-sk\mu(\lambda_-)} \right\} \leq e^{sk\mu(\lambda_-)} \mathbb{E} \exp \left( -s \sum_{i=1}^{k} \xi(\lambda | W_i |) \right)
\]
using independence of the \( W_i \) and then that \( W_i \sim W \) in the last line.

The plan is then to minimize the right hand sides over \( s \), which usually requires finding good upper bounds for the moment generating function
\[
\mathbb{E} \exp(\pm sY) \quad \text{with} \quad Y = \xi(\lambda | W |) \quad \text{or} \quad Y = \xi(\lambda | W |) - \mu(\lambda)
\]
as a function of \( s \). Even in cases where the moment generating function \( \mathbb{E} \exp(s\xi(\lambda | W |)) \) is explicitly known, such minimization might not be easy to do, sometimes because the
derivatives in $s$ are functions which are difficult to bound well. Often however, having a good upper bound on the moment generating function for which $s$ can be optimized is sufficient, as will be the case here. In the next chapter, we shall derive the actual estimates for the Cauchy case, and show how they dictate the choice of the target dimension $k$. The following lemmas will be used there.

2.1.3. Common Lemmas for Estimating the MGF

**Lemma 2.1.12.** Let $Y$ be a random variable with distribution function $F$ and density $f$ continuous on $[a,b]$. Let $g : \mathbb{R} \to \mathbb{R}$ a continuously differentiable function. Then, if $\mathbb{E}g(Y) < \infty$, and $a \leq b \in \mathbb{R}$,

$$
\mathbb{E}g(Y)\mathbb{I}\{a \leq Y \leq b\} = \int_a^b g'(t)\mathbb{P}\{Y > t\} \, dt + g(a)(1 - F(a)) - g(b)(1 - F(b)).
$$

**Proof.** The proof is via integration by parts. If $F'(t) := \mathbb{P}\{Y \leq t\}$ is the distribution function for $Y$, then $1 - F(t)$ goes to 0 as $t \to \infty$. If $F' = f$ with $f$ continuous on $[a,b]$,

$$
\int_a^b g(y) \, dF(y) = \int_a^b g(y) f(y) \, dy = \int_a^b g(y) \frac{d}{dy}(-(1 - F(y))) \, dy
$$

$$
= g(y)(-1 - F(y))\big|_a^b - \int_a^b g'(y)(-1 - F(y)) \, dy
$$

$$
= -g(y)\mathbb{P}\{Y > y\}\big|_a^b + \int_a^b g'(y)\mathbb{P}\{Y > y\} \, dy
$$

$$
= g(a)\mathbb{P}\{Y > a\} - g(b)\mathbb{P}\{Y > b\} + \int_a^b g'(y)\mathbb{P}\{Y > y\} \, dy.
$$

\[\square\]

**Lemma 2.1.13.** For $0 < s$ and $0 < u$,

$$
\exp(-s/u) \leq \left(\frac{2}{es}\right)^2 u^2
$$

The statement is only useful for small $u$, say $0 < u \leq 1$.

**Proof.** We want to compare $\exp(-s/u)$ to $c^2u^2$ with $c$ depending on $s$. Taking logs,

$$
-s \frac{u}{u} \leq 2 \ln(c) + 2 \ln(u) \quad \text{that is} \quad -s \leq 2u \ln(c) + 2u \ln(u)
$$

Minimize the right-hand side in $u$

$$
0 = 2 \ln(c) + 2 \ln(u) + 2 \Rightarrow -\ln(c) - 1 = \ln(u) \Rightarrow \frac{1}{ce} = u
$$
Preserving a Function of the 1-Norm

and at this value of \( u \),

\[
2u^* \ln(c) + 2u^* \ln(u^*) = \frac{2 \ln(c)}{ce} - \frac{2}{ce} \ln(ce) = \frac{2}{ce} (\ln(c) - \ln(c) - \ln(e)) = \frac{-2}{ce}
\]

So we require \( c \) to be

\[-s \leq -2/(ce) \Rightarrow -c \leq -2/(es) \Rightarrow c \geq 2/(es)\]

We take equality.

Lemma 2.1.14. For \( 0 \leq t \leq 1 \),

\[\exp(t) \leq 1 + t + \frac{e - 1}{2} t^2.\]

For \( t \leq 0 \),

\[\exp(t) \leq 1 + t + \frac{t^2}{2}.\]

In particular, for all \( t \leq 1 \),

\[\exp(t) \leq 1 + t + \frac{e - 1}{2} t^2 \leq 1 + t + t^2.\]

Proof. Because \( \exp(u) \) is convex, if \( 0 \leq u \leq 1 \) we may write \( \exp(u) \) as

\[\exp(u \cdot 1 + (1 - u) \cdot 0) \leq u \exp(1) + (1 - u) \exp(0) = 1 + (e - 1)u \leq 1 + 2u.\]

Consequently,

\[\exp(t) - 1 = \int_0^t \exp(u) \, du \leq \int_0^t (1 + (e - 1)u) \, du = t + \frac{e - 1}{2} t^2\]

that is,

\[\exp(t) \leq 1 + t + \frac{e - 1}{2} t^2 \leq 1 + t + t^2\]

For the \( t \leq 0 \) case, Taylor's theorem with Lagrange remainder (about \( t = 0 \)) gives

\[\exp(t) = 1 + t + \frac{t^2}{2} \exp(\xi)\]

for some \( \xi \leq 0 \). Because \( \exp(\xi) \) is monotone increasing, we have

\[\exp(t) \leq 1 + t + \frac{t^2}{2} \exp(0) = 1 + t + \frac{t^2}{2} \leq 1 + t + t^2.\]

Note Taylor's theorem with Lagrange remainder about \( t = 0 \) also shows \( \exp(t) \geq 1 + t \) for all \( t \in \mathbb{R} \) as the remainder term is always nonnegative. \( \square \)
3. Proving Concentration

In this section, we shall prove bounds of the form

\[
\exp(s\mu(\lambda_-))\mathbb{E}\exp(-s\xi(\lambda|W|)) \leq \exp\left(-\frac{\Delta^2}{4(V^2 + A_-)}\right)
\]

and

\[
\exp(-s\mu(\lambda_+))\mathbb{E}\exp(s\xi(\lambda|W|)) \leq \exp\left(-\frac{\Delta^2}{4(V^2 + A_+)}\right)
\]

for special choices of \(s\), with \(A_\pm\) functions of \(\lambda\) and \(V^2\) an upper bound on either the second moment or the variance for \(\xi(\lambda|W|)\). We provide estimates for the reciprocals of the exponential rates in order to estimate the target dimension \(k\). By lemma 2.1.10, taking \(k = \ln(2/\delta) \max\left(\frac{4(V^2 + A_+)}{\Delta^2}, \frac{4(V^2 + A_-)}{\Delta^2}\right)\) ensures \(\mu(\lambda_-) \leq \frac{1}{k} \sum_{j=1}^{k} \xi(\lambda|W_i|) \leq \mu(\lambda_+)\) with probability at least \(1 - \delta\). Taking \(\delta < N^{-c}\) with \(c \geq 3\) ensures that the above bound holds for all \(\binom{N}{2} < N^2\) pairs of points, with total probability at least

\[
1 - \delta N^2 > 1 - \frac{N^2}{N^c} \geq 1 - \frac{1}{N}.
\]

3.1. Estimating the Moment Generating Function

We modify an argument from [14], which will allow us to focus on estimating \(\mathbb{P}\{Y > t\}\) for \(Y\) the desired random variable. The next lemma is the crux of that argument.

**Lemma 3.1.1.** Let \(0 < u < 1\) and \(Y\) a random variable with continuous distribution function \(F\) and continuous density \(f\) on \((0, \infty)\). Then if \(\mathbb{E}\exp(uY) < \infty\),

\[
\mathbb{E}\exp(uY) \mathbb{P}\{Y \leq 1/u\} \leq 1 + u\mathbb{E}Y + u^2\mathbb{E}Y^2
\]

and

\[
\mathbb{E}\exp(uY) \mathbb{P}\{Y > 1/u\} = e\mathbb{P}\{Y > 1/u\} + \int_1^{\infty} \exp(t)\mathbb{P}\{Y > t/u\} \, dt.
\]

**Proof.** For the first integral, let \(F\) be the distribution function for \(Y\), that is, \(F(t) := \mathbb{P}\{Y \leq t\}\). By lemma 2.1.14, as \(uy \leq 1\),

\[
\mathbb{E}\exp(uY) \mathbb{P}\{Y \leq 1/u\} = \int_{-\infty}^{1/u} e^{uy} dF(y) \leq \int_{-\infty}^{1/u} (1 + uy + u^2y^2) \, dF(y)
\]

\[
\leq \int_{-\infty}^{\infty} (1 + uy + u^2y^2) \, dF(y) = 1 + u\mathbb{E}Y + u^2\mathbb{E}Y^2
\]
Preserving a Function of the 1-Norm

For the second integral, use lemma 2.1.12.

\[
E e^{uY} \mathbb{I} \{Y > 1/u\} = \int_{1/u}^{\infty} e^{uy} dF(y) = eP \{Y > 1/u\} + \int_{1/u}^{\infty} u e^{uy} P \{Y > y\} \, dy
\]

in which we have assumed the survival function \( P \{Y > y\} \) decays faster than \( e^{-y} \) in order to address the boundary term. Having assumed this function is also continuous, the usual change of variables \( t = uy \) yields

\[
E \exp(uY) \mathbb{I} \{Y > 1/u\} = eP \{Y > 1/u\} + \int_{1}^{\infty} \exp(t)P \{Y > t/u\} \, dt
\]

To estimate the survival functions, we first establish what they are for us.

**Lemma 3.1.2.** For \( 0 < \alpha \leq 1 \), \( 0 < \lambda, t \), and \( X \sim \text{Cauchy}(1) \),

\[
P \{\ln(1 + \lambda^\alpha |X|^\alpha) > t\} = \frac{2}{\pi} \arctan \left( \frac{\lambda}{(\exp(t) - 1)^{1/\alpha}} \right).
\]

and is differentiable for \( t > 0 \).

**Proof.** We have by the arctan inversion formula B.3.1

\[
P \{\ln(1 + \lambda^\alpha |X|^\alpha) > t\} = P \{\lambda^\alpha |X|^\alpha > \exp(t) - 1\} = \frac{2}{\pi} \arctan \left( \frac{\lambda}{(\exp(t) - 1)^{1/\alpha}} \right) = 1 - \frac{2}{\pi} \arctan \left( \frac{\exp(t) - 1^{1/\alpha}}{\lambda} \right).
\]

As a composition of differentiable functions, the survival function above is differentiable for \( t > 0 \). Because \( 0 < \alpha \leq 1 \), the derivative is continuous too with a finite limit as \( t \) goes to 0.

We specialize lemma 3.1.2 to \( \alpha = 1/2 \) for more workable estimates.

**Lemma 3.1.3.** Let \( X \sim \text{Cauchy}(1) \) and \( \lambda > 0 \). Then if \( t \geq 2 \),

\[
P \{\xi(\lambda |X|) > t\} \leq C_1(\lambda) \exp(-t) \quad \text{with} \quad C_1(\lambda) := \frac{2}{\pi} \frac{\lambda}{(1 - 1/e)^2}
\]

While if \( 2 \ln(1 + \sqrt{\lambda}) \leq t \),

\[
P \{\xi(\lambda |X|) > t\} \leq C_2(\lambda) \exp(-t/2) \quad \text{with} \quad C_2(\lambda) := \frac{2}{\pi}(1 + \sqrt{\lambda}).
\]
Proof. First, because $\xi(a) \leq 2\ln(1 + \sqrt{a})$ for all $a \geq 0$,

$$\{\xi(\lambda |X|) > t\} \subseteq \left\{2\ln(1 + \sqrt{\lambda |X|}) > t\right\}$$

so that, using lemma 3.1.2 with $\alpha = 1/2$,

$$\mathbb{P}\{\xi(\lambda |X|) > t\} \leq \mathbb{P}\left\{2\ln(1 + \sqrt{\lambda |X|}) > t\right\} = \mathbb{P}\left\{\ln(1 + \sqrt{\lambda |X|}) > t/2\right\}$$

$$= \frac{2}{\pi} \arctan \left(\frac{\lambda}{\exp(t/2) - 1}\right) \leq \frac{2}{\pi} \frac{\lambda}{\exp(t/2) - 1}$$

Now, for all $\lambda > 0$,

$$\frac{2}{\pi} \arctan \left(\frac{\lambda}{\exp(t/2) - 1}\right) \leq \frac{2}{\pi} \frac{\lambda}{\exp(t/2) - 1} = \frac{2\lambda}{\pi} \frac{\exp(-t)}{1 - \exp(-t/2)^2} \leq \frac{2\lambda}{\pi(1 - \exp(-t_0/2))^2} \exp(-t)$$

for all $t \geq t_0 > 0$ because

$$\frac{d}{dt}(1 - \exp(-t/2)) = \frac{1}{2} \exp(-t) > 0.$$ 

On the other hand, if $t \geq 2\ln(1 + \sqrt{\lambda})$, we then have

$$\frac{2}{\pi} \arctan \left(\frac{\lambda}{\exp(t/2) - 1}\right) \leq \frac{2}{\pi} \frac{\lambda}{\exp(t/2) - 1} \leq \frac{2}{\pi} \exp(-t/2) \sqrt{\lambda(1 - 1/(1 + \sqrt{\lambda}))} = \frac{2}{\pi} \exp(-t/2) \sqrt{\lambda(1 + \sqrt{\lambda})}$$

which is bounded above by $2/\pi$ for the $t$ in question. \hfill \Box

3.2. Large Scales

Lemma 3.2.1 (Upper Tail, Large Scales). For $1/2 > u > 0$, $X \sim \text{Cauchy}(1)$, $Y = \xi(\lambda |X|) - \mu(\lambda)$, and $V^2 \geq EY^2$,

$$\exp(-u\Delta_+) \mathbb{E}\exp(uY)$$

can be minimized to

$$\exp \left( -\frac{\Delta_+^2}{4(V^2 + A_+)} \right) \quad \text{at} \quad u = \frac{\Delta_+}{2(V^2 + A_+)}.$$
Preserving a Function of the 1-Norm

with $A$ a bounded nonnegative function of $\lambda \geq 1$.

In particular, for $\epsilon \leq 1/4$, $\lambda > 1/\sqrt{1+\epsilon}$, and $\Delta_+ = \mu((1+\epsilon)\lambda) - \mu(\lambda)$, we have the bound

$$\frac{4(V^2 + A_+)}{\Delta_+^2} \leq \frac{64}{\epsilon^2(1-\epsilon)^2} \left( \frac{\pi^2}{2} + \frac{64\pi}{\epsilon(\pi^2-1/2)} \right).$$

**Remark 3.2.2.** This bound is not tight; I believe there are better ways to estimate the $A_+$ term, possibly by iterating the argument at the end of the proof.

**Proof.** We break up $\mathbb{E}\exp(uY)$ into two integrals using lemma 3.1.1. With $V^2 \geq \mathbb{E}Y^2$, the first integral is

$$\mathbb{E}\exp(uY)\mathbb{I}\{uY \leq 1\} \leq \mathbb{E}(1 + uY + u^2Y^2)\mathbb{I}\{uY \leq 1\} \leq \mathbb{E}(1 + uY + u^2Y^2) \leq 1 + u^2V^2$$

The second integral is

$$\mathbb{E}\exp(uY)\mathbb{I}\{uY > 1\} \leq e\mathbb{P}\{Y > 1/u\} + \int_1^\infty \exp(t)\mathbb{P}\{Y > t/u\} \, dt$$

We thus need an upper bound on $\mathbb{P}\{Y > t/u\} = \mathbb{P}\{\xi(\lambda|X)| > \mu(\lambda) + t/u\}$ for $t \geq 1$.

We want to use lemma 3.1.3 to estimate these tail probabilities, so we compare

$$\mu(\lambda) + t/u \geq 2\ln(1 + \sqrt{\lambda})$$

Using the exact formula for $\mu(\lambda)$ from lemma A.1.1 and noting the atanh contribution is nonnegative by lemma A.1.2,

$$\mu(\lambda) + t/u \geq \frac{1}{2}\ln(1 + \lambda^2) + t/u \geq \frac{1}{2}\ln(1 + \lambda^2) + 2$$

because $1/2 > u > 0$. For $\lambda \leq 1$, we certainly have

$$2\ln(1 + \sqrt{\lambda}) \leq 2\sqrt{\lambda} \leq 2 \leq \frac{1}{2}\ln(1 + \lambda^2) + 2.$$ 

For $\lambda \geq 1$, we have

$$2\ln(1 + \sqrt{\lambda}) = 2\ln(\sqrt{\lambda}) + 2\ln(1 + 1/\sqrt{\lambda}) = \ln(\lambda) + 2\ln(1 + 1/\sqrt{\lambda}) \leq \ln(\lambda) + 2\ln(2)$$

while

$$\frac{1}{2}\ln(1 + \lambda^2) + 2 = 2 + \ln(\lambda) + \frac{1}{2}\ln(1 + 1/\lambda^2) > 2 + \ln(\lambda).$$

Because $\ln(2) < 1$, we are ok here too.

With $C_2(\lambda)$ the function in lemma 3.1.3,

$$e\mathbb{P}\{Y > 1/u\} = e\mathbb{P}\{\xi(\lambda|X)| > \mu + 1/u\} \leq e C_2 \exp\left(-\frac{\mu + 1/u}{2}\right)$$

**Remark 3.2.2.** This bound is not tight; I believe there are better ways to estimate the $A_+$ term, possibly by iterating the argument at the end of the proof.

**Proof.** We break up $\mathbb{E}\exp(uY)$ into two integrals using lemma 3.1.1. With $V^2 \geq \mathbb{E}Y^2$, the first integral is

$$\mathbb{E}\exp(uY)\mathbb{I}\{uY \leq 1\} \leq \mathbb{E}(1 + uY + u^2Y^2)\mathbb{I}\{uY \leq 1\} \leq \mathbb{E}(1 + uY + u^2Y^2) \leq 1 + u^2V^2$$

The second integral is

$$\mathbb{E}\exp(uY)\mathbb{I}\{uY > 1\} \leq e\mathbb{P}\{Y > 1/u\} + \int_1^\infty \exp(t)\mathbb{P}\{Y > t/u\} \, dt$$

We thus need an upper bound on $\mathbb{P}\{Y > t/u\} = \mathbb{P}\{\xi(\lambda|X)| > \mu(\lambda) + t/u\}$ for $t \geq 1$.

We want to use lemma 3.1.3 to estimate these tail probabilities, so we compare

$$\mu(\lambda) + t/u \geq 2\ln(1 + \sqrt{\lambda})$$

Using the exact formula for $\mu(\lambda)$ from lemma A.1.1 and noting the atanh contribution is nonnegative by lemma A.1.2,

$$\mu(\lambda) + t/u \geq \frac{1}{2}\ln(1 + \lambda^2) + t/u \geq \frac{1}{2}\ln(1 + \lambda^2) + 2$$

because $1/2 > u > 0$. For $\lambda \leq 1$, we certainly have

$$2\ln(1 + \sqrt{\lambda}) \leq 2\sqrt{\lambda} \leq 2 \leq \frac{1}{2}\ln(1 + \lambda^2) + 2.$$ 

For $\lambda \geq 1$, we have

$$2\ln(1 + \sqrt{\lambda}) = 2\ln(\sqrt{\lambda}) + 2\ln(1 + 1/\sqrt{\lambda}) = \ln(\lambda) + 2\ln(1 + 1/\sqrt{\lambda}) \leq \ln(\lambda) + 2\ln(2)$$

while

$$\frac{1}{2}\ln(1 + \lambda^2) + 2 = 2 + \ln(\lambda) + \frac{1}{2}\ln(1 + 1/\lambda^2) > 2 + \ln(\lambda).$$

Because $\ln(2) < 1$, we are ok here too.

With $C_2(\lambda)$ the function in lemma 3.1.3,

$$e\mathbb{P}\{Y > 1/u\} = e\mathbb{P}\{\xi(\lambda|X)| > \mu + 1/u\} \leq e C_2 \exp\left(-\frac{\mu + 1/u}{2}\right)$$
We also have from that lemma
\[ \int_1^\infty \exp(t) \mathbb{P}\{Y > t/u\} \, dt = \int_1^\infty \exp(t) \mathbb{P}\{\xi(\lambda|X) > \mu + t/u\} \, dt \]
\[ \leq C_2 \int_1^\infty \exp(t) \exp\left(-\frac{\mu + t/u}{2}\right) = C_2 \exp(-\mu/2) \int_1^\infty \exp(t(1 - 1/(2u))) \, dt \]

The integral makes sense only for \(1 - 1/(2u) < 0\), that is \(2u < 1\).
\[ = \frac{C_2 \exp(-\mu/2)}{1 - 1/(2u)} \int_1^\infty (\exp(t(1 - 1/(2u))))' \, dt \]
\[ = \frac{C_2 \exp(-\mu/2)}{1 - 1/(2u)} \exp(t(1 - 1/(2u)))|_1^\infty = (-1) \frac{eC_2 \exp(-\mu/2) \exp(-1/(2u))}{1 - 1/(2u)} \]
\[ = eC_2 \exp(-\mu/2) \exp(-1/(2u)) \frac{2u}{1 - 2u} \]

By lemma 2.1.13,
\[ \exp(-1/(2u)) \leq \left(\frac{2}{e/2}\right)^2 u^2 = \frac{16}{e^2} u^2, \]
so we can estimate everything together as
\[ \mathbb{E}\exp(uY) \mathbb{I}\{uY > 1\} \leq e\mathbb{P}\{Y > 1/u\} + \int_1^\infty \exp(t) \mathbb{P}\{Y > t/u\} \, dt \]
\[ \leq eC_2 \exp(-\mu/2) \exp(-1/(2u)) \left(1 + \frac{2u}{1 - 2u}\right) \]
\[ = eC_2 \exp(-\mu/2) \exp(-1/(2u)) \frac{1}{1 - 2u} \]
\[ \leq C_2 \exp(-\mu/2) \frac{16}{e} \frac{u^2}{1 - 2u} \]

Note that
\[ C_2(\lambda) \exp(-\mu/2) \leq \frac{2}{\pi} \frac{(1 + \sqrt{\lambda})}{(1 + \lambda^2)^{1/4}}. \]

By subadditivity,
\[ \frac{2}{\pi} \frac{(1 + \sqrt{\lambda})}{(1 + \lambda^2)^{1/4}} \geq \frac{2}{\pi} \frac{(1 + \sqrt{\lambda})}{1^{1/4} + (\lambda^2)^{1/4}} = \frac{2}{\pi}. \]

For an upper bound, if \(\lambda \leq 1\),
\[ \frac{2}{\pi} \frac{(1 + \sqrt{\lambda})}{(1 + \lambda^2)^{1/4}} \leq \frac{2}{\pi} (1 + \sqrt{\lambda}) \leq \frac{4}{\pi}. \]
Preserving a Function of the 1-Norm

On the other hand, if $\lambda \geq 1$,

$$\frac{2}{\pi} \frac{(1 + \sqrt{\lambda})}{(1 + \lambda^2)^{1/4}} \leq \frac{4}{\pi} \frac{\sqrt{\lambda}}{(1 + \lambda^2)^{1/4}} < \frac{4 \sqrt{\lambda}}{\pi \lambda^{3/4}} = \frac{4}{\pi}.$$  

If we choose an upper bound on $u \leq u_0 < 1/2$, we then have

$$\mathbb{E} \exp(uY) I \{uY > 1\} \leq C_2 \exp(-\mu/2) \frac{16}{e} \frac{u^2}{1 - 2u} \leq \frac{4}{\pi} \frac{16}{e} \frac{u^2}{1 - 2u} := A_+ u^2.$$  

We then want to optimize $u$ for

$$\exp(-u\Delta_+) \mathbb{E} \exp(uY) \leq \exp(-u\Delta_+) (1 + V^2u^2 + A_+ u^2) \leq \exp(-u\Delta_+ + u^2(V^2 + A_+))$$

If

$$k(u) := -u\Delta_+ + u^2(V^2 + A_+)$$

which is convex, then

$$0 = k'(u^*) = -\Delta_+ + 2(u^*)(V^2 + A_+) \Rightarrow \frac{\Delta_+}{2(V^2 + A_+)} = u^*$$

and at $u^*$,

$$k(u^*) = \frac{\Delta_+}{2(V^2 + A_+)}\Delta_+ + \left(\frac{\Delta_+}{2(V^2 + A_+)}\right)^2 (V^2 + A_+)$$

$$= \frac{\Delta_+^2}{2(V^2 + A_+)} + \frac{\Delta_+^2}{4(V^2 + A_+)} = \frac{\Delta_+^2}{4(V^2 + A_+)}.$$  

We need to make sure $u^* < 1/2$. We have a lower bound on $A_+$ of

$$\frac{4}{\pi} \frac{16}{e} \frac{1}{1 - 2u_0} \geq \frac{8}{\pi} \frac{16}{e}$$

if we choose $u_0 = 1/4$. We have to verify then that $u^* \leq 1/4$. In this case,

$$u^* \leq \frac{\Delta_+}{2A_+} \leq \frac{\Delta_+}{(16/\pi)(16/e)} < 0.034\Delta_+.$$  

If we choose $\Delta_+ = \mu((1 + \epsilon)\lambda) - \mu(\lambda)$, then $\Delta_+ < \epsilon$ for $\lambda \geq 1/\sqrt{1 + \epsilon}$ by lemma A.2.5.

We can now estimate $A_+$ and $u^*$ a bit better. For $\lambda \geq 1/\sqrt{1 + \epsilon}$, we take $V^2 = \pi^2/2$ as our upper bound for the variance by remark A.4.2. Consequently, $u^* < \Delta_+/\pi^2 < 0.102\epsilon$ as $A_+$ is positive and $\Delta < \epsilon$. We can now estimate $A_+$ as

$$A_+ = \frac{4}{\pi} \frac{16}{e} \frac{1}{1 - 2u} \leq \frac{4}{\pi} \frac{16}{e} \frac{1}{1 - 2\epsilon/\pi^2} = \frac{64\pi}{e(\pi^2 - 2\epsilon)} \leq \frac{64\pi}{e(\pi^2 - 1/2)}.$$
if $\epsilon \leq 1/4$. We then have, using lemma A.2.5 again
\[ \frac{4(V^2 + A_+)}{\Delta_+^2} \leq \frac{16}{\epsilon^2(1 - \epsilon)^2} 4(V^2 + A_+) \leq \frac{64}{\epsilon^2(1 - \epsilon)^2} \left( \frac{\pi^2}{2} + \frac{64\pi}{e(\pi^2 - 1/2)} \right). \]

Lemma 3.2.3 (Lower Tail, Large Scales). Let $1 > u > 0$ and $X \sim \text{Cauchy}(1)$. If
\[ Y = \xi(\lambda | X) - \mu(\lambda) \quad \text{and} \quad V^2 \geq EY^2, \]
then
\[ \exp(-u\Delta_-)E\exp(-uY) \]
can be minimized to
\[ \exp \left( -\frac{\Delta_-^2}{4(V^2 + A_-)} \right) \] at \[ u = \frac{\Delta_-}{2(V^2 + A_-)}, \]
with $A_-$ a bounded nonnegative function of $\lambda$ for $\lambda \geq 1$.
In particular, for $\epsilon \leq 1/4$, $\lambda > \sqrt{1 + \epsilon}$, and $\Delta_- = \mu(\lambda) - \mu((1 + \epsilon)^{-1}\lambda)$, we have the bound
\[ \frac{4(V^2 + A_-)}{\Delta_-^2} \leq \frac{64}{\epsilon^2(1 - \epsilon)^2} \left( \frac{\pi^2}{2} + \frac{8\pi^2}{e(\pi^2 - 1/4)} \sqrt{2} \right). \]

Remark 3.2.4. Again, the bound is not sharp, as there should be better ways to estimate $A_-$, possibly by iterating the argument found in the proof.

Remark 3.2.5. Again, by the discussion at the beginning of this section, the target dimension $k$ may be taken to be linear in $\ln(N)$ for these scales.

Proof. Note that $Y$ does not have a sign, so we try breaking up the corresponding integral again.
\[ E\exp(-uY) = E\exp(-uY)1\{ -uY \leq 1 \} + E\exp(-uY)1\{ -uY > 1 \}. \]
We can still use lemma 3.1.1 applied to $-Y$. Just as in the upper tail computations,
\[ E\exp(-uY)1\{ -uY \leq 1 \} \leq E(1 - uY + (-uY)^2) = 1 + V^2u^2. \]
and
\[ E\exp(-uY)1\{ -uY > 1 \} \leq eP\{ -Y > 1/u \} + \int_1^\infty \exp(t)P\{ -Y > t/u \} \ dt. \]
Now,
\[ P\{ -Y > t/u \} = P\{ \mu - \xi(\lambda | X) > t/u \} = P\{ \xi(\lambda | X) - \mu < -t/u \} = P\{ \xi(\lambda | X) < \mu - t/u \} \]
By subadditivity of \(\sqrt{a}\),
\[
\xi(a) = \ln(1 + \sqrt{a}) + \frac{1}{2} \ln(1 + a) \geq \ln(\sqrt{1 + a}) + \frac{1}{2} \ln(1 + a) = \ln(1 + a).
\]

So
\[
P\{Y > t/u\} = P\{\xi(\lambda |X|) < \mu - t/u\} \leq P\{\ln(1 + \lambda |X|) < \mu - t/u\}
= P\{1 + \lambda |X| < \exp(\mu - t/u)\} = P\left\{|X| < \frac{\exp(\mu - t/u) - 1}{\lambda}\right\}
= \frac{2}{\pi} \arctan\left(\frac{\exp(\mu - t/u) - 1}{\lambda}\right) \leq \frac{2}{\pi} \arctan\left(\frac{\exp(\mu - t/u) - \exp(-t/u)}{\lambda}\right)
= \frac{2}{\pi} \arctan\left(\frac{\exp(-t/u)}{\lambda}\right)
\]

Finally, using the basic upper bound for \(\arctan\),
\[
P\{Y > t/u\} \leq \frac{2}{\pi} \exp\left(\frac{1}{2} \ln(1 + \lambda^2)\right) = \frac{1}{\lambda} \sqrt{1 + \lambda^2} = \frac{1}{\lambda^2 + 1}
\]
which is bounded for \(\lambda \geq \lambda_0 > 0\). We thus have \textit{provided} \(1 - 1/u < 0\) that is, \(u < 1\),
\[
\int_1^\infty \exp(t)P\{Y > t/u\} dt < C_1 \int_1^\infty \exp(t)\exp(t(1 - 1/u)) dt
\]
\[
= \frac{C_1}{1 - 1/u} \int_1^\infty (\exp(t(1 - 1/u)))' dt = \frac{C_1}{1 - 1/u} \exp(t(1 - 1/u))|_1^\infty
\]
\[
= -\frac{C_1 e}{1 - 1/u} \exp(-1/u) = C_1 \frac{e}{1 - u} \exp(-1/u).
\]

Putting things together
\[
\mathbb{E} \exp(-uY)I\{-uY > 1\} \leq eP\{-Y > 1/u\} + \int_1^\infty \exp(t)P\{-Y > t/u\} dt
< eC_1 \exp(-1/u) + \frac{C_1 e}{1 - u} \exp(-1/u) = \frac{eC_1}{1 - u} \exp(-1/u) \leq \frac{eC_1}{1 - u} 4 u^2
= \frac{4u^2}{e(1 - u)} \frac{2}{\sqrt{1 + \frac{1}{\lambda^2}}} \leq \frac{8u^2}{e\pi(1 - u)} \sqrt{1 + \frac{1}{1 + \epsilon}}
\leq A_\epsilon u^2
\]
if we have a bound $1 > u_0 \geq u$ and assuming $\lambda \geq \sqrt{1 + \epsilon}$.

So

$$\exp(-u \Delta_-) \mathbb{E} \exp(-uY) < \exp(-u \Delta_-)(1 + (V^2 + A_-)u^2) \leq \exp(-u \Delta_- + u^2(V^2 + A_-)).$$

Because

$$k(u) := -u \Delta_- + u^2(V^2 + A_-)$$

is convex, we can find the global minimizer $u^*$ at

$$0 = k'(u^*) = -\Delta_- + 2u^*(V^2 + A_-) \Rightarrow \frac{\Delta_-}{2(V^2 + A_-)} = u^*$$

so that

$$k(u^*) = -\frac{\Delta_-^2}{2(V^2 + A_-)} + \frac{\Delta_-^2}{4(V^2 + A_-)^2}(V^2 + A_-) = -\frac{\Delta_-^2}{4(V^2 + A_-)}$$

We need to ensure $u^* < 1$. By remark A.4.2, we take $V^2 = \pi^2/2$ as our upper bound for the variance. Consequently, $u^* < \Delta_-/\pi^2$ as $A_-$ is positive. By lemma A.2.5 and the discussion following, $\Delta_- < \epsilon$, when $\lambda \geq \sqrt{1 + \epsilon}$ making $u^* < \epsilon/9 < 1/2$ for $\epsilon < 1$ certainly.

We can now estimate $A_-$ as

$$A_- \leq \frac{8}{e\pi(1 - \epsilon/\pi^2)} \sqrt{1 + \frac{1}{1 + \epsilon}} \leq \frac{8\pi^2}{e\pi(\pi^2 - \epsilon)} \sqrt{1 + \frac{1}{1 + \epsilon}} \leq \frac{8\pi^2 \sqrt{2}}{e\pi(\pi^2 - 1/4)}$$

for $\epsilon \leq 1/4$. Finally, using lemma A.2.5 again

$$\frac{4(V^2 + A_-)}{\Delta_-^2} \leq \frac{16}{\epsilon^2(1 - \epsilon)^2} 4(V^2 + A_-) \leq \frac{64}{\epsilon^2(1 - \epsilon)^2} \left( \frac{\pi^2}{2} + \frac{8\pi^2 \sqrt{2}}{e\pi(\pi^2 - 1/4)} \right).$$

\[\square\]

3.3. Small Scales

In the last section dividing by $\Delta_\pm$ was ok as these quantities were bounded around $\epsilon$ and away from 0. In this section, we shall have $\Delta_\pm = \pm \epsilon \mu(\lambda) \to 0$ when $\lambda \to 0$, which will need slightly different arguments. Even here, we shall not take $\lambda$ too small, as the target dimension $k$ will then grow accordingly.

**Lemma 3.3.1 (Upper Tail, Small Scales).** For $1/2 > u > 0$, $X \sim Cauchy(1)$, $1 \geq \lambda > 8\epsilon^2$, and $Y = \xi(\lambda | X|)$, if $V^2 \geq EY^2$,

$$\exp(-u(1 + \epsilon)\mu(\lambda))\mathbb{E} \exp(uY)$$
can be minimized to

\[
\exp \left( - \frac{\epsilon^2 \mu^2(\lambda)}{4(V^2 + A_+)} \right) \quad \text{at} \quad u = \frac{\epsilon \mu(\lambda)}{2(V^2 + A_+)}.
\]

with \( A_+ \) a bounded nonnegative function of \( \lambda \leq 1 \).

In particular, for \( \epsilon \leq 1/4 \),

\[
\frac{4(V^2 + A_+)}{\epsilon^2 \mu^2(\lambda)} \leq \frac{8}{\epsilon^2} \left( 3.126 + 1 + \frac{4}{\pi} - \frac{4}{\pi} \ln(\lambda) + 8 + 2\sqrt{2} + \frac{1}{4} \right)
\]

**Remark 3.3.2.** Note how the logarithmic term blows up when \( \lambda \) becomes small. We shall discuss this in section 3.4. The restriction that \( \lambda \geq 8\epsilon^2 \) prevents us from using this lemma to show concentration at moderately small scales. There may be a different proof technique that could do so, possibly using a particular moment instead of the full moment generating function; compare \([18]\).

**Proof.** We break up \( \mathbb{E} \exp(uY) \) into two integrals using lemma 3.1.1. With \( V^2 \geq \mathbb{E}Y^2 \), the first integral is

\[
\mathbb{E} \exp(uY) \mathbb{P}\{uY \leq 1\} \leq \mathbb{E}(1 + uY + u^2Y^2) \mathbb{P}\{uY \leq 1\}
\]

\[
\leq \mathbb{E}(1 + uY + u^2Y^2) \leq 1 + u\mu + u^2V^2.
\]

The second integral is

\[
\mathbb{E} \exp(uY) \mathbb{P}\{uY > 1\} \leq e^\mathbb{P}\{Y > 1/u\} + \int_1^\infty \exp(t)\mathbb{P}\{Y > t/u\} \, dt
\]

We thus need an upper bound on \( \mathbb{P}\{Y > t/u\} = \mathbb{P}\{\xi(\lambda|X|) > t/u\} \) for \( t \geq 1 \).

We want to use lemma 3.1.3 with \( C_1(\lambda) \) to estimate these tail probabilities as we are assuming \( \lambda \) is bounded here. If we assume \( u < 1/2 \), then \( t/u > 1 \). In this case, the lemma says

\[
e^\mathbb{P}\{Y > 1/u\} \leq eC_1(\lambda) \exp(-1/u)
\]

while (noting that \( 1 - 1/u < 0 \) for us)

\[
\int_1^\infty \exp(t)\mathbb{P}\{Y > t/u\} \, dt \leq C_1(\lambda) \int_1^\infty \exp(t - t/u) \, dt
\]

\[
= C_1(\lambda) \int_1^\infty (\exp(t(1 - 1/u)))' \, dt = C_1(\lambda) \frac{1}{1 - 1/u} \exp(t(1 - 1/u)) |_{t=1}^{t=\infty}
\]

\[
= (-1) \frac{C_1(\lambda)}{1 - 1/u} \exp(1 - 1/u) = \frac{C_1(\lambda)e^u \exp(-1/u)}{1 - u}.
\]

By lemma 2.1.13,

\[
\exp(-1/u) \leq \left( \frac{2}{e} \right)^2 u^2 = \frac{4}{e^2} u^2,
\]
so we can estimate everything together as

\[ E \exp(Y) \mathbb{1} \{ Y > 1/u \} \leq e \mathbb{P} \{ Y > 1/u \} + \int_1^\infty \exp(t) \mathbb{P} \{ Y > t/u \} \, dt \]

\[ \leq eC_1(\lambda) \exp(-1/u) + \frac{C_1(\lambda) e u \exp(-1/u)}{1 - u} \]

\[ = \frac{C_1(\lambda)}{1 - u} e \exp(-1/u) \leq \frac{C_1(\lambda)}{1 - u} e \left( \frac{4}{e} u^2 \right) \leq \frac{C_1(\lambda)}{1 - u} e \left( \frac{4}{e} u^2 \right) \]

\[ = \frac{2}{\pi} \frac{\lambda}{(1 - 1/e)^2} e \frac{1}{1 - u_0} u^2 = \frac{8e}{\pi (e - 1)^2 (1 - u_0)} \lambda u^2 \]

assuming an upper bound on \( u < u_0 \leq 1/2 \). Choosing \( u_0 = 1/2 \), we set

\[ A_+(\lambda) := \frac{8e}{\pi (e - 1)^2 (1 - u_0)} \lambda = \frac{16e}{\pi (e - 1)^2} \lambda. \]

Consequently,

\[ \exp(-u(1 + \epsilon)\mu(\lambda)) E \exp(uY) \]

\[ \leq \exp(-u(1 + \epsilon)\mu(\lambda)) \left( 1 + u \mu(\lambda) + u^2 (V^2 + A_+(\lambda)) \right) \]

\[ \leq \exp \left( -u(1 + \epsilon)\mu(\lambda) + u^2 (V^2 + A_+(\lambda)) \right) \]

\[ = \exp \left( -u \epsilon \mu(\lambda) + u^2 (V^2 + A_+(\lambda)) \right) \]

and we want to minimize this last quantity in \( u \). Let

\[ k(u) := -u \epsilon \mu(\lambda) + u^2 (V^2 + A_+(\lambda)) \]

Then, setting the derivative to 0 yields

\[ 0 = -\epsilon \mu(\lambda) + 2u^* (V^2 + A_+(\lambda)) \Rightarrow u^* = \frac{\epsilon \mu(\lambda)}{2(V^2 + A_+(\lambda))}. \]

Because \( k(u) \) is a convex function, \( u^* \) is a global minimizer at which

\[ k(u^*) = -\epsilon \mu(\lambda) \frac{\epsilon \mu(\lambda)}{2(V^2 + A_+(\lambda))} + \frac{\epsilon^2 \mu^2(\lambda)}{4(V^2 + A_+(\lambda))^2} (V^2 + A_+(\lambda)) \]

\[ = -\frac{\epsilon^2 \mu^2(\lambda)}{4(V^2 + A_+(\lambda))}. \]

We need to check \( u^* < 1/2 \). Because \( V^2 \geq \mathbb{E}Y^2 \geq (\mathbb{E}Y)^2 = \mu^2(\lambda) \) by Jensen’s inequality, we always have for \( 0 < \lambda \leq 1 \)

\[ u^* = \frac{\epsilon \mu(\lambda)}{2(V^2 + A_+(\lambda))} < \frac{\epsilon \mu(\lambda)}{2V^2} \leq \frac{\epsilon (1 + \lambda)}{2 \sqrt{2N}} \leq \frac{\epsilon}{\sqrt{2N}} < 1/4 < 1/2 \]
provided $4\varepsilon < \sqrt{2}\lambda$, that is, $8\varepsilon^2 < \lambda$. In this case, we can estimate $A_+(\lambda)$ as

$$A_+(\lambda) \leq \frac{C_1(\lambda)}{1 - 1/4\varepsilon} = \frac{162\lambda}{3e\pi(1 - 1/e)^2} = \lambda\frac{32e}{3\pi(e - 1)^2} < 3.126\lambda.$$  

Consequently using lemma A.4.3 for the bound $V^2$, we can give the bound

$$\frac{4(V^2 + A_+)}{\varepsilon^2\mu^2(\lambda)} \leq \frac{(1 + \lambda)^2}{2\varepsilon^2\lambda} \leq 4\frac{V^2 + A_+}{\varepsilon^2} \leq \frac{2(1 + \lambda)^2}{\varepsilon^2} \left(\frac{V^2}{\lambda} + 3.126\right)$$

$$= \frac{2(1 + \lambda)^2}{\varepsilon^2} \left(\frac{V^2}{\lambda} + 3.126\right)$$

When $\lambda \leq 1$, this is

$$\frac{2(1 + \lambda)^2}{\varepsilon^2} \left(3.126 + 1 + \frac{4}{\pi} - \frac{4}{\pi} \ln(\lambda) + \frac{8}{(1 + \lambda)^2} + 2\lambda\sqrt{2}\lambda + \frac{\lambda^3}{4}\right)$$

$$\leq \frac{8}{\varepsilon^2} \left(3.126 + 1 + \frac{4}{\pi} - \frac{4}{\pi} \ln(\lambda) + 8 + 2\sqrt{2} + \frac{1}{4}\right).$$

\[\square\]

**Lemma 3.3.3** (Lower Tail, Small Scales). Let $t < \mu(\lambda)$, $X \sim \text{Cauchy}(1)$, and $Y = \xi(\lambda|X)$ with $EY^2 \leq V^2$. Then at $u = (\mu(\lambda) - t)/V^2$,

$$\exp(tu) \mathbb{E}\exp(-uY) \leq \exp \left(\frac{-(t - \mu)^2}{2V^2}\right)$$

In particular, for $t = (1 - \varepsilon)\mu(\lambda)$, $\varepsilon \leq 1/4$, and $0 \leq \lambda \leq 1$

$$\frac{2V^2}{(t - \mu)^2} \leq \frac{4}{\varepsilon^2} \left(1 + \frac{4}{\pi} - \frac{4}{\pi} \ln(\lambda) + 8 + 2\sqrt{2} + \frac{1}{4}\right)$$

and for $1 \leq \lambda \leq 2$,

$$\frac{2V^2}{(t - \mu)^2} \leq \frac{9}{\varepsilon^2} \left(\frac{\pi^2}{2} + 4 + 2\sqrt{2}\right).$$

**Remark 3.3.4.** I do not think the bound is tight. Again, note how the bound deteriorates when $\lambda$ becomes small. We shall discuss this in section 3.4.

**Proof.** By lemma 2.1.14,

$$\exp(tu) \mathbb{E}\exp(-uY) \leq \exp(tu) \mathbb{E} \left(1 - uY + \frac{u^2}{2}Y^2\right)$$

$$\leq \exp(tu) \left(1 - u\mu + \frac{u^2}{2}V^2\right) \leq \exp \left(u(t - \mu) + \frac{u^2}{2}V^2\right).$$
We want to minimize 

\[ k(u) := u(t - \mu) + \frac{u^2}{2}V^2. \]

Setting the derivative to 0 yields

\[ 0 = (t - \mu) + uV^2 \Rightarrow \frac{\mu - t}{V^2} = u^* > 0. \]

The minimizer \( u^* \) is a global minimizer because \( k(u) \) is convex. At \( u^* \),

\[ k(u^*) = -\left(\frac{(\mu - t)^2}{V^2} + \frac{V^2}{2}\left(\frac{\mu - t}{V^2}\right)^2\right) = -\frac{(\mu - t)^2}{2V^2} \]

so that

\[ \exp(tu^*)\mathbb{E}\exp(-u^*Y) \leq \exp\left(-\frac{(t - \mu)^2}{2V^2}\right). \]

Using lemma A.4.3 for the bound \( V^2 \), we can give the bound for \( \lambda \leq 1 \)

\[ \frac{2V^2}{(\mu - t)^2} = \frac{2}{\epsilon^2 \mu^2(\lambda)} V^2 \leq \frac{(1 + \lambda)^2 V^2}{\epsilon^2} \lambda \]

\[ \leq \frac{4}{\epsilon^2} \left( 1 + \frac{4}{\pi} - \frac{4}{\pi} \ln(\lambda) + \frac{8}{(1 + \lambda)^2} + 2\lambda \frac{\sqrt{2}\lambda}{1 + \lambda} + \frac{\lambda^3}{4} \right) \]

\[ \leq \frac{4}{\epsilon^2} \left( 1 + \frac{4}{\pi} - \frac{4}{\pi} \ln(8\epsilon^2) + 8 + 2\sqrt{2} + \frac{1}{4} \right) \]

and for \( 1 \leq \lambda \leq 2 \),

\[ \frac{2V^2}{(\mu - t)^2} \leq \frac{(1 + \lambda)^2 V^2}{\epsilon^2} \lambda \leq \frac{9}{\epsilon^2} \left( \frac{\pi^2}{2} + 2 + 2\sqrt{2} + 2 \right) = \frac{9}{\epsilon^2} \left( \frac{\pi^2}{2} + 4 + 2\sqrt{2} \right) \].

\[ \Box \]

### 3.4. Really Small Scales

In the last section 3.3, we saw the reciprocals of the concentration rates blow up like \( \ln(1/\lambda) \) as \( \lambda \to 0 \). In this section, we show that we can stop that blow up at a particular \( \lambda_0 = \Theta(\delta) \) with \( \delta > 0 \) the failure probability.

We shall show in lemma 3.4.3 that \( \xi(a) \approx \sqrt{a} \) for small \( a \) which will play well with \( \mu(\lambda) = \Theta(\sqrt{\lambda}) \) for \( \lambda \leq 1 \), as seen in remark A.1.3. The following lemma 3.4.1 shows how this approximate homogeneity could be used. Specifically, let \( X_i \overset{i.i.d.}{\sim} \text{Cauchy}(1) \) for \( 1 \leq i \leq k \). We show in section 3.4.1 that with high probability, \( \max_i |X_i| \leq C_k \) for some increasing function \( C_k \) of \( k \). The hope would be to invoke the approximate homogeneity above to conclude that if the concentration results hold for \( \lambda \approx \epsilon/C_k \), it holds for all \( \lambda \leq \epsilon/C_k \) too.
Unfortunately, at least for Cauchy random variables, $C_k$ grows quickly with $k$, so that one already needs a concentration result for moderately small $\lambda$. We were able to give a lower tail concentration result in lemma 3.3.3 with no restriction on how small $\lambda$ can be, but the upper tail concentration result in lemma 3.3.1 required $\lambda \geq 8\epsilon^2$.

**Lemma 3.4.1.** For $1 \leq i \leq k$, let $X_i \overset{i.i.d.}{\sim} \text{Cauchy}(1)$. For $0 < \epsilon < 1/4$ and $0 < \lambda_0 \leq 1$, suppose

$$(1 - \epsilon)\mu(\lambda_0) \leq \frac{1}{k} \sum_{i=1}^{k} \xi(\lambda_0 | X_i) \leq (1 + \epsilon)\mu(\lambda_0)$$

and $\lambda_0 \max_i |X_i| \leq c_0 \leq 1/6$.

Then if $0 < \eta < 1$,

$$(1 - \epsilon')\mu(\eta \lambda_0) \leq \frac{1}{k} \sum_{i=1}^{k} \xi(\eta \lambda_0 | X_i) \leq (1 + \epsilon')\mu(\eta \lambda_0)$$

with $\epsilon'$ depending on $\epsilon$, $c_0$, and $\lambda_0$. If $\lambda_0 = O(\epsilon^2)$, then $\epsilon'$ may be made to be $O(\epsilon)$ for $\epsilon$ small enough.

**Remark 3.4.2.** We shall see in section 3.4.1 that $\lambda_0$ must be taken very small in order for $\lambda_0 \max_i |X_i| \leq c_0$ with high probability.

**Proof.** Because $\max_i |X_i| \leq c_0 \leq .16$ and $0 < \eta < 1$, we can invoke lemma 3.4.3 once to say

$$\sqrt{\eta \lambda_0 | X_i|} \leq \xi(\eta \lambda_0 | X_i|) \leq \sqrt{\eta \lambda_0 | X_i|} \left(1 + \frac{\eta \lambda_0 | X_i|}{2}\right)$$

and then again, writing $\sqrt{\eta \lambda_0 | X_i|} = \sqrt{\eta} \sqrt{\lambda_0 | X_i|}$

$$\frac{\sqrt{\eta}}{1 + \lambda_0 | X_i|/2} \xi(\lambda_0 | X_i|) \leq \xi(\eta \lambda_0 | X_i|) \leq \sqrt{\eta} \xi(\lambda_0 | X_i|) \left(1 + \frac{\eta \lambda_0 | X_i|}{2}\right)$$

leaving a bound of

$$\frac{\sqrt{\eta}}{1 + \lambda_0 c_0/2} \xi(\lambda_0 | X_i|) \leq \xi(\eta \lambda_0 | X_i|) \leq \sqrt{\eta} \xi(\lambda_0 | X_i|) \left(1 + \frac{\eta \lambda_0 c_0}{2}\right)$$

By assumption, summing over $i$ and dividing by $k$ yields

$$(1 - \epsilon)\frac{\sqrt{\eta}}{1 + \lambda_0 c_0/2} \mu(\lambda_0) \leq \frac{1}{k} \sum_{i=1}^{k} \xi(\eta \lambda_0 | X_i|) \leq (1 + \epsilon)\frac{\sqrt{\eta} \mu(\lambda_0)}{1 + \frac{\eta \lambda_0 c_0}{2}}.$$
We shall now use remark A.1.3 (twice) to “absorb” \( \sqrt{\eta} \) into \( \mu \), recalling
\[
\frac{\sqrt{2\lambda}}{1 + \lambda} \leq \mu(\lambda) \leq \frac{\sqrt{2\lambda}}{1 + \lambda} \left(1 + \frac{2\lambda}{1 + \lambda} + \frac{\lambda^2}{2}\right) = \sqrt{2\lambda} \frac{1 + \lambda + \lambda^2}{1 + \lambda^2}.
\]

It will help to rewrite the last bound as
\[
\sqrt{2\lambda} \frac{1 + \lambda + \lambda^2}{1 + \lambda^2} = \sqrt{2\lambda} \left(\frac{1 + \lambda}{1 + \lambda^2} + (\lambda/2)^{3/2}\right).
\]

We look at the bounds individually.

For the lower bound,
\[
\frac{1}{k} \sum_{i=1}^{k} \xi(\eta \lambda_0 | X_i |) \geq (1 - \epsilon) \frac{\sqrt{\eta}}{1 + \lambda_0 c_0/2} \mu(\lambda_0)
\geq (1 - \epsilon) \sqrt{2\lambda_0} \left(\frac{1 + \lambda_0}{1 + \lambda_0^2} + (\lambda_0 / 2)^{3/2}\right)^{-1} \frac{1}{(1 + \lambda_0 c_0 / 2)(1 + \lambda_0)}
\geq (1 - \epsilon) \mu(\eta \lambda_0) \left(\frac{1 + \eta \lambda_0}{1 + \eta \lambda_0^2} + (\eta \lambda_0 / 2)^{3/2}\right)^{-1} \frac{1}{(1 + \lambda_0 c_0 / 2)(1 + \lambda_0)}
\]

We need to control the multiplier above. Its inverse is
\[
\left(\frac{1 + \eta \lambda_0}{1 + \eta \lambda_0^2} + (\eta \lambda_0 / 2)^{3/2}\right) (1 + \lambda_0 c_0 / 2)(1 + \lambda_0)
\]

If \( \lambda_0 = O(\epsilon^2) \), then this multiplier is \( 1 + O(\epsilon^2) \) for \( \epsilon \) small enough.

For the upper bound,
\[
\frac{1}{k} \sum_{i=1}^{k} \xi(\eta \lambda_0 | X_i |) \leq (1 + \epsilon) \sqrt{2\eta} \mu(\lambda_0) \left(1 + \frac{\eta \lambda_0 c_0}{2}\right)
\leq (1 + \epsilon) \sqrt{2\eta} \lambda_0 \left(\frac{1 + \lambda_0}{1 + \lambda_0^2} + (\lambda_0 / 2)^{3/2}\right) \left(1 + \frac{\eta \lambda_0 c_0}{2}\right)
\leq (1 + \epsilon) \mu(\eta \lambda_0) (1 + \eta \lambda_0) \left(\frac{1 + \lambda_0}{1 + \lambda_0^2} + (\lambda_0 / 2)^{3/2}\right) \left(1 + \frac{\eta \lambda_0 c_0}{2}\right)
\]

The multiplier
\[
(1 + \eta \lambda_0) \left(\frac{1 + \lambda_0}{1 + \lambda_0^2} + (\lambda_0 / 2)^{3/2}\right) \left(1 + \frac{\eta \lambda_0 c_0}{2}\right)
\]
is \( 1 + O(\epsilon^2) \) as soon as \( \lambda_0 = O(\epsilon^2) \) when \( \epsilon \) is small enough.

We now show \( \xi(a) \approx \sqrt{a} \) for small \( a \) which will play well with \( \mu(\lambda) = \Theta(\sqrt{\lambda}) \) for \( \lambda \leq 1 \), as seen in remark A.1.3.
Lemma 3.4.3. For $0 < a < 1/6 \approx .16$,
\[
\sqrt{a} \leq \xi(a) = \ln(1 + \sqrt{a}) + \frac{1}{2} \ln(1 + a) \leq \sqrt{a} \left(1 + \frac{a}{2}\right).
\]

Remark 3.4.4. Note the upper bound is better than the upper bound of
\[
\sqrt{a} + \frac{a}{2} = \sqrt{a} \left(1 + \frac{\sqrt{a}}{2}\right).
\]

Proof. We focus on
\[
f(x) = \ln(1 + x) + \frac{1}{2} \ln(1 + x^2),
\]
using Taylor’s theorem with Lagrange remainder about $x = 0$.
We have
\[
f'(x) = \frac{1}{1 + x} + \frac{x}{1 + x^2}
\]
which is strictly positive for all $x > 0$.
We also have
\[
f''(x) = -\frac{1}{(1 + x)^2} + \frac{1}{(1 + x^2)^2} (1 + x^2 - 2x^2)
\]
\[
= -\frac{1}{(1 + x)^2} + \frac{1}{(1 + x^2)^2} (1 - x^2)
\]
\[
= \frac{1}{(1 + x)^2(1 + x^2)^2} (-1 - 2x^2 - x^4 + 1 + 2x + x^2 - x^2(1 + 2x + x^2))
\]
\[
= \frac{1}{(1 + x)^2(1 + x^2)^2} (-2x^2 - x^4 + 2x + x^2 - x^2 - 2x^3 - x^4)
\]
\[
= \frac{1}{(1 + x)^2(1 + x^2)^2} (-2x^2 - 2x^4 + 2x - 2x^3)
\]
\[
= \frac{2x}{(1 + x)^2(1 + x^2)^2} (1 - x - x^2 - x^3)
\]
For $0 \leq x \leq 1/2$, the 2nd derivative is positive, so for some $z \in (0, 1/2)$,
\[
f(x) = f(0) + f'(0)x + f''(z)x^2 \geq f'(0)x = x\]
We then have (from the first simplification of \( f''(x) \) above)

\[
f'''(x) = \left( -\frac{1}{(1+x)^2} + \frac{1}{(1+x^2)^2} \right)'
\]

\[
= \left( -\frac{1}{(1+x)^2} + \frac{1}{(1+x^2)^2} - \frac{x^2}{(1+x^2)^2} \right)'
\]

\[
= 2 \left( 1 + x^2 \right)^{-3} - \frac{2(2x)}{(1 + x^2)^3} = \frac{((1+x)^2(2x) - x^2(1+x^2)2x)}{(1+x^2)^4}
\]

\[
= \frac{2}{(1 + x)^3} - \frac{4x}{(1 + x^2)^3} - 2\frac{1}{(1 + x^2)^2}(2x(1 + x^2) - 4x^3)
\]

\[
= \frac{2}{(1 + x)^3} - 2\frac{6x}{(1 + x^2)^3} + \frac{2x^3}{(1 + x^2)^3}
\]

\[
= \frac{2}{(1 + x)^3} - \frac{6x}{(1 + x^2)^3}
\]

We finally have

\[
f^{(4)}(x) = -\frac{6}{(1+x)^4} - \frac{((1+x)^2(6-6x^2)-(6x-2x^3)(1+x^2)^22x)}{(1+x^2)^6}
\]

\[
= -\frac{6}{(1+x)^4} - \frac{6}{(1+x)^4}(1+x^2)(1-x^2) - x(6x-2x^3)
\]

\[
= -\frac{6}{(1+x)^4} - \frac{6}{(1+x^2)^4}(1-x^4-6x^2+2x^4)
\]

\[
= -\frac{6}{(1+x)^4} - \frac{6}{(1+x^2)^4}(1-6x^2+x^4)
\]

Certainly for \(|x| < 1/\sqrt{6}\), all terms are negative, so for some \(z \in (0, 1/\sqrt{6})\),

\[
f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \frac{x^4}{4!}f^{(4)}(z)
\]

\[
= x + \frac{x^3}{3} + \frac{x^4}{3!}f^{(4)}(z) \leq x + \frac{x^3}{3}.
\]

Putting both bounds together, as \(1/2 > 1/\sqrt{6}\), we have for all \(0 \leq x < 1/\sqrt{6}\),

\[
x \leq f(x) \leq x \left( 1 + \frac{x^2}{2} \right)
\]

Setting \(x = \sqrt{a}\), we have our result. \(\square\)

**Lemma 3.4.5.** For \(0 < \epsilon\)

\[
1 + \frac{\epsilon}{2} \left( 1 - \frac{\epsilon}{4} \right) \leq \sqrt{1+\epsilon} \leq 1 + \frac{\epsilon}{2}.
\]
Preserving a Function of the 1-Norm

**Proof.** For the upper bound,
$$\sqrt{1 + \epsilon} \leq \sqrt{1 + \epsilon + \epsilon^2/4} = 1 + \epsilon/2.$$ Let $f(x) = (1 + x)^{1/2}$. Then
$$f'(x) = \frac{1}{2(1 + x)^{1/2}}, \quad f''(x) = \frac{3}{8(1 + x)^{5/2}}, \quad \text{and} \quad f^{(3)}(x) = \frac{3}{8(1 + x)^{5/2}} > 0.$$ So for some $x \in (0, \epsilon)$ and all $\epsilon \geq 0$,
$$f(\epsilon) = f(0) + f'(0)\epsilon + f''(0)\frac{\epsilon^2}{2} + f^{(3)}(x)\frac{\epsilon^3}{3!} = 1 + \frac{\epsilon}{2} - \frac{\epsilon^2}{8} + f^{(3)}(x)\frac{\epsilon^3}{3!} \geq 1 + \frac{\epsilon}{2} - \frac{\epsilon^2}{8}.$$ Thus,
$$1 + \frac{\epsilon}{2} \left(1 - \frac{\epsilon}{4}\right) \leq \sqrt{1 + \epsilon} \leq 1 + \frac{\epsilon}{2}.$$  

**3.4.1. Bounds on Maxima**

**Lemma 3.4.6.** Let $X_i$ for $1 \leq i \leq k$ be independent identically distributed random variables. Let $Z$ be the largest of $|X_i|$. Then provided $\alpha = 1/(kp_t) > 1$, with $p_t = \Pr \{|X_i| > t\}$, there is the bound
$$\Pr \{Z > t\} \leq \exp(-H(\alpha)kp_t) \quad \text{with} \quad H(x) := x \ln(x) + 1 - x.$$ **Proof.** Let $Y_i(t)$ be the indicator function $\mathbb{I}(|X_i| > t)$ which is a Bern $(p_t)$ random variable with $p_t = \Pr \{|X_i| > t\}$. If $Z > t$, then at least one of the $X_i$ is greater than $t$:
$$\Pr \{Z_j > t\} = \Pr \left\{ \sum_{i=1}^{k} Y_i(t) > 1 \right\}, \quad \text{while} \quad \mathbb{E} \sum_{i=1}^{k} Y_i(t) = kp_t.$$ If $\alpha = 1/(kp_t) > 1$, the Chernoff-Hoeffding bounds for the binomial distribution apply, (See [2, page 255-56].)
$$\Pr \{Z > t\} = \Pr \left\{ \sum_{i=1}^{k} Y_i(t) > \alpha kp_t \right\} \leq \exp(-H(\alpha)kp_t)$$ with $H(\alpha) := \alpha \ln(\alpha) + 1 - \alpha$.  

For the above to be useful, we link $\alpha$ to $k$ as follows.
Let $p_t = 1/(kC_k)$ with $C_k > 1$ possibly depending on $k$ so that $\alpha = C_k$ and
$$H(\alpha)kp_t = H(C_k)\frac{1}{C_k} = (C_k \ln(C_k) + 1 - C_k)\frac{1}{C_k} = \ln(C_k) + \frac{1}{C_k} - 1.$$
which is nonnegative and increasing in $C_k$ for $C_k \geq 1$ because
\[
\frac{d}{dc}(\ln(c) + \frac{1}{c} - 1) = \frac{1}{c} - \frac{1}{c^2} = \frac{c - 1}{c^2} > 0 \quad \text{for } c > 1.
\]

If the desired failure probability is at most $\delta \in (0, 1)$, taking $C_k = e/\delta$ makes
\[
\exp(-H(C_k)k\mu_t) = \exp(-\ln(e/\delta) - (\delta/e) + 1) = e^{-\delta/e} < \delta.
\]

Note that none of the above calculations use the actual behavior of $p_t$ with respect to $t$.

We now specialize to Cauchy random variables. If $X_i \sim \text{Cauchy}(1)$,
\[
p_t = \mathbb{P}\{|\lambda X_i| > t\} = \frac{2}{\pi} \arctan(\lambda/t) \leq \frac{2\lambda}{\pi t}.
\]
Consequently,
\[
t \leq \frac{2\lambda}{\pi p_t} = \frac{2\lambda k e}{\pi \delta}.
\]

Typically, we want $\delta = N^{-c}$ with $c \geq 3$ say in order for the dimension reduction guarantee to hold for all pairs of points. Picking a larger value for the failure probability $\delta$ would make $t$ smaller though. The alternative is to take $\lambda$ small. We can now use lemma 3.4.1.

**Corollary 3.4.7 (Lower Tail).** For all $0 < \eta < 1$, $N^{-c} \leq \epsilon < 1/4$, $X_j \overset{i.i.d.}{\sim} \text{Cauchy}(1)$, and $\lambda_0 = e^2\pi/(8keN^c)$, the following bound holds, with failure probability at most $2/N^c$,
\[
(1 - \epsilon)(1 - 4\epsilon^2)\mu(\eta\lambda_0) \leq \frac{1}{k} \sum_{j=1}^{k} \xi(\eta\lambda_0 | X_j|)(1 + \epsilon)(1 + 4\epsilon^2)\mu(\eta\lambda_0).
\]

**Proof.** By lemma 3.4.6,
\[
\max_i \{|\lambda_0 | X_i|\} \leq \frac{\epsilon^2}{4} < 1/6 \quad \text{with failure probability at most } 1/N^c.
\]
On the other hand, we want to use lemma 3.3.3 to say
\[
(1 - \epsilon)\mu(\lambda_0) \leq \frac{1}{k} \sum_{i=1}^{k} \xi(\lambda_0 | X_i|) \leq (1 + \epsilon)\mu(\lambda_0)
\]
with failure probability at most $1/N^c$, noting there is no restriction on the size of $\lambda_0$.

Following the discussion at the beginning of this section, we have to choose $k$ as
\[
k \geq \ln(N^c) \frac{4}{e^2} \left(1 + \frac{4}{\pi} - \frac{4}{\pi} \ln(\lambda_0) + 8 + 2\sqrt{2} + \frac{1}{4}\right) = \frac{C}{\epsilon^2} \ln^2(N^c)
\]
for some constant $C$, having used $\ln(N^c) > \ln(1/\epsilon)$ and our choice of $\lambda_0$. We are now free to use lemma 3.4.1 to conclude, with failure probability at most $2/N^c$,
\[
(1 - \epsilon)(1 - 4\epsilon^2)\mu(\eta\lambda_0) \leq \frac{1}{k} \sum_{j=1}^{k} \xi(\eta\lambda_0 | X_j|) \leq (1 + \epsilon)(1 + 4\epsilon^2)\mu(\eta\lambda_0).
\]
Appendix A: The First and Second Moments

In the definition of metric $\rho$ on $\mathbb{R}^k$, we could have used $\ln(1 + \lambda |X|)$ or $\ln(1 + \sqrt{\lambda |X|})$, as the function applied to each coordinate, and we know the exact moments of such functions in terms of polylogarithms, using lemmas B.0.1 and B.0.13. However, it turns out using the linear combination

$$\xi(\lambda |X|) := \ln(1 + \sqrt{\lambda |X|}) + \frac{1}{2} \ln(1 + \lambda |X|)$$

greatly simplifies the first moment and estimates of the second moment in terms of known functions. This first moment is also approximately $1/2$-homogeneous at small scales (that is, for small $\lambda$), which will allow us to recover concentration properties there too. This homogeneity is lost if we use either of the logarithms individually, as a $-\lambda \ln(\lambda)$ term appears in those cases, as can already be seen in computing $\mathbb{E} \ln(1 + \lambda |X|)$ in lemma A.3.1. That term instead will appear in our estimates for the second moment and will become important when proving concentration at small scales.

For both moments, the contour integral setup below will greatly facilitate computations; in particular, it will allow us to avoid estimating $\mathbb{E} \ln^2(1 + \sqrt{\lambda |X|})$ and $\mathbb{E} \ln^2(1 + \lambda |X|)$ individually, which while possible, is not necessary for our results.

**Lemma A.0.1** (Contour Integral Setup). For $\lambda > 0$, $b > 0$, and $X \sim \text{Cauchy}(1)$,

$$\mathbb{E} \ln^b(1 + \sqrt{\lambda |X|}) = \ln^b(1 + i\lambda) + \ln^b(1 + -i\lambda)$$

$$- \frac{1}{2} \mathbb{E} \ln^b(1 + i\sqrt{\lambda |X|}) - \frac{1}{2} \mathbb{E} \ln^b(1 - i\sqrt{\lambda |X|}).$$

**Remark A.0.2.** The task is then to simplify the complex logarithms on the right hand side when particular values of $b$ are chosen. We shall do so in the next sections.

**Proof.** We want to compute

$$I(\lambda) := \mathbb{E} \ln^b(1 + \sqrt{\lambda |X|}) = \frac{2}{\pi} \int_0^\infty \ln^b(1 + \sqrt{\lambda x}) \frac{dx}{1 + x^2}$$

via contour integration. Extending to $z \in \mathbb{C} - (-\infty, 0]$, let

$$f(z) := \frac{2 \ln^b(1 + \sqrt{\lambda z})}{\pi} \frac{1}{1 + z^2}$$

which has simple poles at $z = \pm i$. 

We shall compute $I(\lambda)$ be using two different contours given by

\[ C^+ := \partial \{ re^{i\theta} \mid 0 \leq r \leq R \text{ and } 0 \leq \theta \leq \pi - \epsilon \} \text{ oriented counterclockwise} \]

and

\[ C^- := \partial \{ re^{i\theta} \mid 0 \leq r \leq R \text{ and } -\pi + \epsilon \leq \theta \leq 0 \} \text{ oriented clockwise.} \]

Setting

\[ C^\pm(R) := \{ Re^{\pm i\theta} \mid 0 \leq \theta \leq \pi - \epsilon \} \]

and

\[ C^\pm_\epsilon(R) := \{ re^{\pm i(\pi - \epsilon)} \mid 0 \leq r \leq R \} \]

we have

\[ C^+ := [0, R] \cup C^+(R) \cup C^+_\epsilon(R) \text{ and } C^- := [0, R] \cup C^-(R) \cup C^-_\epsilon(R) \]

We shall show that

\[ \lim_{R \to \infty} \int_{C^\pm_\epsilon(R)} f(z) \, dz = 0 \]

and

\[ \lim_{R \to \infty} \lim_{\epsilon \to 0} \left( \int_{C^+_\epsilon(R)} f(z) \, dz + \int_{C^-_\epsilon(R)} f(z) \, dz \right) = E \ln^b(1 + i \sqrt{\lambda |X|}) + E \ln^b(1 - i \sqrt{\lambda |X|}). \]

On the other hand, keeping in mind the orientations of the contours, the residue theorem dictates for $R > 1$,

\[ \int_{C^+} f(z) \, dz = 2\pi i \Res_{z=i} f(z) = 2\pi i \lim_{z \to i} (z-i) \frac{2 \ln^b(1 + \sqrt{\lambda z})}{(z-i)(z+i)} = 2 \ln^b(1 + \sqrt{i \lambda}) \]

and similarly

\[ \int_{C^-} f(z) \, dz = -2\pi i \lim_{z \to -i} (z-i) \frac{2 \ln^b(1 + \sqrt{\lambda z})}{(z-i)(z+i)} = 2 \ln^b(1 + \sqrt{-i \lambda}). \]

For the $C^\pm(R)$ integrals, using lemmas A.0.4 and 2.1.9, we have when $|z| = R > \sqrt{2}$

\[ |f(z)| \leq \frac{(\ln(1 + \sqrt{R}) + (\pi + \ln(2))/2)^b}{R^2 \sqrt{R^{-4} + (1 - 2R^{-2})}} \]
Consequently, with $z = Re^{i\theta}$ so that $dz = Rei\theta d\theta$,

$$\left| \int_{C^+(R)} f(z) \, dz \right| \leq \int_0^{\pi - \epsilon} \left| f(Re^{i\theta}) Re^{i\theta} \right| \, d\theta \leq \frac{(\ln(1 + \sqrt{\lambda R}) + (\pi + \ln(2))/2)^b}{R \sqrt{R^{-4} + (1 - 2R^{-2})}} (\pi - \epsilon) \to 0$$

when $R \to \infty$ using lemma A.0.3. Similar reasoning applies to the $C^- (R)$ integral.

For the $C^\pm (R)$ integrals, note that $\sqrt{r e^{\pm i(\pi - \epsilon)}} = \sqrt{r e^{\mp i\epsilon/2}} = \pm i \sqrt{r e^{\mp i\epsilon/2}}$, which approaches $\pm i r$ when $\epsilon \to 0$. Consequently, when $z = re^{i(\pi - \epsilon)} = -re^{-i\epsilon}$,

$$\lim_{\epsilon \to 0} \int_{C^+(R)} f(z) \, dz = \lim_{\epsilon \to 0} \int_0^R f(-re^{-i\epsilon}) (-e^{-i\epsilon}) \, dr = \lim_{\epsilon \to 0} \int_0^R \frac{e^{-i\epsilon} \ln b(1 + i \sqrt{\lambda r} e^{-i\epsilon/2})}{1 + r^2 e^{-2i\epsilon}} \, dr$$

We want to use the dominated convergence theorem to take the limit inside the integral. Using lemmas A.0.4 and 2.1.9 again, now assuming $\epsilon < \pi/8$,

$$\left| \frac{e^{-i\epsilon} \ln b(1 + i \sqrt{\lambda r} e^{-i\epsilon/2})}{1 + r^2 e^{-2i\epsilon}} \right| \leq \frac{(\ln(1 + \sqrt{\lambda r}) + (\pi + \ln(2))/2)^b}{\sqrt{1 + r^2}}$$

which is not only bounded for $0 \leq r \leq R$, but also stays integrable when $r \to \infty$, again by lemma A.0.3. The dominated convergence theorem now can say

$$\lim_{\epsilon \to 0} \int_{C^+(R)} f(z) \, dz = \int_0^R \lim_{\epsilon \to 0} \frac{e^{-i\epsilon} \ln b(1 + i \sqrt{\lambda r} e^{-i\epsilon/2})}{1 + r^2 e^{-2i\epsilon}} \, dr = \int_0^R \frac{\ln b(1 + i \sqrt{\lambda r})}{1 + r^2} \, dr.$$
Putting everything together, we have
\[ 2 \ln^b(1 + \sqrt{i\lambda}) + 2 \ln^b(1 + \sqrt{-i\lambda}) = 2I(\lambda) + \mathbb{E} \ln^b(1 + i\sqrt{\lambda |X|}) + \mathbb{E} \ln^b(1 - i\sqrt{\lambda |X|}) \]
that is
\[ \ln^b(1 + \sqrt{i\lambda}) + \ln^b(1 + \sqrt{-i\lambda}) = I(\lambda) + \frac{1}{2} \mathbb{E} \ln^b(1 + i\sqrt{\lambda |X|}) + \frac{1}{2} \mathbb{E} \ln^b(1 - i\sqrt{\lambda |X|}) \]
as claimed.

Lemma A.0.3. For \( b > 0, \lambda > 0, \) and \( c > 0, \)
\[ \frac{(\ln(1 + \sqrt{\lambda r}) + c)^b}{r} \to 0 \quad \text{as} \quad r \to \infty. \]

Proof. For \( b < 2, \) we can just use
\[ \frac{(\ln(1 + \sqrt{\lambda r}) + c)^b}{r} \leq \frac{(\sqrt{\lambda r} + c)^b}{r} = \frac{(\lambda r)^{b/2}}{r} (1 + c/\sqrt{\lambda r})^b \to 0 \]
when \( r \to \infty \) as \( r^{1-\epsilon}/r^2 \to 0 \) in that case.
For the larger \( b, \) the proof is by induction. Because we are sending \( r \to \infty, \) we may assume \( r > 1/\lambda \) so that \( \sqrt{\lambda r} < \lambda r. \) In this case, similar reasoning to the above shows
\[ \lim_{r \to \infty} \frac{(\ln(1 + \sqrt{\lambda r}) + c)^b}{r} = 0 \]
for \( b < 1, \) which we take as the base case.
For the induction step, L’Hospital’s rule dictates
\[ 0 \leq \lim_{r \to \infty} \frac{(\ln(1 + \lambda r) + c)^b}{r} = \lim_{r \to \infty} \frac{b(\ln(1 + \lambda r) + c)^{b-1}}{1 + \lambda r} \]
\[ < \lim_{r \to \infty} \frac{b(\ln(1 + \lambda r) + c)^{b-1}}{1} \frac{\lambda}{\lambda r} = \lim_{r \to \infty} \frac{b(\ln(1 + \lambda r) + c)^{b-1}}{r} \]
\[ = b \lim_{r \to \infty} \frac{(\ln(1 + \lambda r) + c)^{b-1}}{r} \]
so if the limit is 0 for \( b' \leq b - 1, \) it is 0 for \( b' \leq b \) as well.

Lemma A.0.4. Let \( z \in \mathbb{C} - (-\infty, 0] \) and \( \alpha \leq 1/2. \) Then, with \( r = |z|, \)
\[ |\ln(1 + z^\alpha)| \leq \ln(1 + r^\alpha) + \frac{\pi + \ln(2)}{2}. \]
Proof. Let \( z = re^{i\theta} \) with \( |\theta| < \pi \). Then
\[
z^\alpha = \exp(\alpha \ln(z)) = \exp(\alpha(\ln(r) + i\theta)) = r^\alpha e^{i\alpha \theta}.
\]
Consequently,
\[
\ln(1 + z^\alpha) = \ln(1 + r^\alpha \cos(\alpha \theta) + ir^\alpha \sin(\alpha \theta))
\]
\[
= \frac{1}{2} \ln((1 + r^\alpha \cos(\alpha \theta))^2 + r^{2\alpha} \sin^2(\alpha \theta)) + i \arctan \left( \frac{r^\alpha \sin(\alpha \theta)}{1 + r^\alpha \cos(\alpha \theta)} \right).
\]
The cosine term is nonnegative as \( |\alpha \theta| < \alpha \pi \leq \pi/2 \) by our assumption on \( \alpha \).

By the AM-GM inequality,
\[
|\ln(1 + z^\alpha)|^2 \leq \frac{1}{4} \ln^2(1 + 2r^\alpha \cos(\alpha \theta) + r^{2\alpha}) + \arctan^2 \left( \frac{r^\alpha \sin(\alpha \theta)}{1 + r^\alpha \cos(\alpha \theta)} \right)
\]
\[
\leq \frac{1}{4} \ln^2(2 + 2r^{2\alpha}) + \frac{\pi^2}{4}
\]
as \( |\arctan(x)| \leq \pi/2 \) for \( x \in \mathbb{R} \).

Because the square root function is subadditive, we finally have
\[
|\ln(1 + z^\alpha)| \leq \left( \frac{1}{4} \ln^2(2 + 2r^{2\alpha}) + \frac{\pi^2}{4} \right)^{1/2}
\]
\[
\leq \frac{1}{2} \ln(2 + r^{2\alpha}) + \frac{\pi}{2}
\]
\[
= \frac{\ln(2)}{2} + \frac{\ln(1 + r^{2\alpha}) + \pi}{2}
\]
\[
\leq \frac{\ln(2)}{2} + \ln(1 + r^{\alpha}) + \frac{\pi}{2}.
\]

In the following sections, we specialize to the case \( b = 1 \) and \( b = 2 \) in order to compute the 1st and 2nd moments respectively. The complex integrals and residues then simplify to more identifiable functions.

### A.1. 1st Moment

**Lemma A.1.1.** If \( \lambda > 0 \) and \( X \sim \text{Cauchy}(1) \), then
\[
\mathbb{E} \ln(1 + \sqrt{\lambda |X|}) = \text{atanh} \left( \frac{\sqrt{2\lambda}}{1 + \lambda} \right) + \frac{1}{2} \ln(1 + \lambda^2) - \frac{1}{2} \mathbb{E} \ln(1 + \lambda |X|)
\]
that is,

\[ \mu(\lambda) := \mathbb{E}(\xi(\lambda | X)) = \text{atanh} \left( \frac{\sqrt{2\lambda}}{1 + \lambda} \right) + \frac{1}{2} \ln(1 + \lambda^2). \]

**Proof.** Starting from lemma A.0.1 with \( b = 1, \)

\[ \mathbb{E}(\ln(1 + \sqrt{\lambda | X|})) \]

\[ = \ln(1 + \sqrt{\lambda}) + \ln(1 + \sqrt{-\lambda}) - \frac{1}{2} \mathbb{E}(\ln(1 + i \sqrt{\lambda | X|}) - \frac{1}{2} \mathbb{E}(\ln(1 - i \sqrt{\lambda | X|})). \]

By lemma B.1.9 and the atanh addition formula B.1.11,

\[ \ln(1 + \sqrt{\lambda}) + \ln(1 + \sqrt{-\lambda}) \]

\[ = \text{atanh}(\sqrt{\lambda}) + \text{atanh}(\sqrt{-\lambda}) + \frac{1}{2} \ln(1 - (\sqrt{\lambda})^2) + \frac{1}{2} \ln(1 - (\sqrt{-\lambda})^2) \]

\[ = \text{atanh} \left( \frac{\sqrt{\lambda} + \sqrt{-\lambda}}{1 + \sqrt{\lambda}\sqrt{-\lambda}} \right) + \frac{1}{2} \ln(1 - i\lambda) + \frac{1}{2} \ln(1 - (-i\lambda)) \]

\[ = \text{atanh} \left( \frac{\sqrt{2\lambda}}{1 + \lambda} \right) + \frac{1}{2} \ln(1 + \lambda^2). \]

By lemma B.0.12,

\[ \ln(1 + i \sqrt{\lambda | X|}) + \ln(1 - i \sqrt{\lambda | X|}) = \ln(1 - (i \sqrt{\lambda | X|})^2) = \ln(1 + \lambda | X|). \]

Consequently,

\[ \mathbb{E}(\ln(1 + \sqrt{\lambda | X|}) + \frac{1}{2} \mathbb{E}(\ln(1 + \lambda | X|)) = \text{atanh} \left( \frac{\sqrt{2\lambda}}{1 + \lambda} \right) + \frac{1}{2} \ln(1 + \lambda^2) \]

as claimed. \( \square \)

We shall be using the following lemma to show that \( \mu(\lambda) = \Theta(\sqrt{\lambda}) \) as well when \( \lambda \) is small.

**Lemma A.1.2.** For \( \lambda > 0, \)

\[ \frac{\sqrt{2\lambda}}{1 + \lambda} < \text{atanh} \left( \frac{\sqrt{2\lambda}}{1 + \lambda} \right) < \frac{\sqrt{2\lambda}}{1 + \lambda} \left( 1 + 2\lambda \frac{1}{1 + \lambda^2} \right) \leq \sqrt{2} \]

and approaches \( \theta \) as \( \lambda \to \infty. \) Further, for any \( \lambda \leq \lambda_0 \leq 1, \)

\[ \frac{\sqrt{2\lambda}}{1 + \lambda} < \text{atanh} \left( \frac{\sqrt{2\lambda}}{1 + \lambda} \right) < \frac{\sqrt{2\lambda}}{1 + \lambda} \left( 1 + \frac{2\lambda_0}{1 + \lambda_0^2} \right) \]
Remark A.1.3. Numerically, the upper bound overestimates by a factor of $\sqrt{2}$ when $\lambda = 1$, but the estimate gets much better for larger $\lambda$.

By lemma A.1.1, we now also have the bound

$$\frac{\sqrt{2\lambda}}{1 + \lambda} \leq \mu(\lambda) \leq \frac{\sqrt{2\lambda}}{1 + \lambda} \left(1 + \frac{2\lambda_0}{1 + \lambda_0^2}\right) + \frac{\lambda^2}{2}$$

using

$$0 < \frac{1}{2} \ln(1 + \lambda^2) \leq \frac{\lambda^2}{2}$$

for $\lambda > 0$.

**Proof.** The limit for large $\lambda$ is immediate. We first show that the input has a unique maximum at $\lambda = 1$. It will be easier to view it as a function of $\nu = \sqrt{\lambda}$ as $\nu$ then has a positive derivative with respect to $\lambda$.

$$\frac{d}{d\nu} \frac{\nu \sqrt{2}}{1 + \nu^2} = \frac{\sqrt{2}}{(1 + \nu^2)^2} \left((1 + \nu^2) - \nu(2\nu)\right) = \frac{\sqrt{2}}{(1 + \nu^2)^2} (1 - \nu^2)$$

which is positive for $\sqrt{\lambda} = \nu < 1$ and negative for $\sqrt{\lambda} = \nu > 1$. Because $\text{atanh}$ is monotone increasing on $\mathbb{R}_+$, we have a unique maximum at $\nu = 1 = \lambda$, at which point the input is $1/\sqrt{2}$.

For the lower bound, note from the power series for $\text{atanh}$,

$$\text{atanh}(x) = \sum_{j=0}^{\infty} \frac{x^{2j+1}}{2j + 1}$$

all terms are nonnegative when $x > 0$, so $\text{atanh}(x) > x$ in this case.

For the upper bound, use lemma B.1.10:

$$\text{atanh}(u) \leq \frac{u}{1 - u^2}$$

for $|u| < 1$. Consequently,

$$\text{atanh} \left(\frac{\sqrt{2\lambda}}{1 + \lambda}\right) \leq \frac{\sqrt{2\lambda}}{1 + \lambda} \left(1 - \frac{2\lambda}{1 + 2\lambda + \lambda^2}\right)^{-1} = \frac{\sqrt{2\lambda}}{1 + \lambda} \left(1 + \frac{1 + \lambda^2}{1 + 2\lambda + \lambda^2}\right)^{-1}$$

$$= \frac{\sqrt{2\lambda}}{1 + \lambda} \left(\frac{1 + 2\lambda + \lambda^2}{1 + \lambda^2}\right) = \frac{\sqrt{2\lambda}}{1 + \lambda} \left(1 + \frac{2\lambda}{1 + \lambda^2}\right).$$

At the beginning of the proof, we showed $\lambda/(1 + \lambda^2)$ is strictly increasing for $\lambda \leq 1$, so for any $\lambda \leq \lambda_0 \leq 1$,

$$\text{atanh} \left(\frac{\sqrt{2\lambda}}{1 + \lambda}\right) < \frac{\sqrt{2\lambda}}{1 + \lambda} \left(1 + \frac{2\lambda_0}{1 + \lambda_0^2}\right)$$

\[\square\]
A.2. Estimating Deviations of the Mean

We derive the estimates used in the large scale concentration proofs given above.

\[ \mu((1 + \epsilon)\lambda) - \mu(\lambda) \quad \text{and} \quad \mu(\lambda) - \mu((1 + \epsilon)^{-1}\lambda) \]

when \( \lambda \) is not too small. Because

\[ \mu(\lambda) = \text{atanh} \left( \frac{\sqrt{2\lambda}}{1 + \lambda} \right) + \frac{1}{2} \ln(1 + \lambda^2) \]

both deviations will be sums of two terms, an atanh term and a ln term. The first evidence that this deviations are bounded in \( \lambda \) is the following.

**Lemma A.2.1.** For \( \lambda > 0 \) and \( a > 1 \),

\[ \frac{1}{2} \ln(1 + (a\lambda)^2) - \frac{1}{2} \ln(1 + \lambda^2) = \frac{1}{2} \ln \left( 1 + \frac{(a^2 - 1)\lambda^2}{1 + \lambda^2} \right). \]

**Remark A.2.2.** Note this is bounded above by

\[ \frac{1}{2} \ln \left( 1 + (a^2 - 1) \right) = \ln(a) \]

for all \( \lambda > 0 \) and by

\[ \frac{1}{2} \ln \left( 1 + \frac{(a^2 - 1)}{2} \right) \]

for all \( 1 \geq \lambda \geq 0 \).

**Proof.** We have

\[ \frac{1}{2} \ln(1 + (a\lambda)^2) - \frac{1}{2} \ln(1 + \lambda^2) = \frac{1}{2} \ln \left( 1 + \frac{a^2\lambda^2}{1 + \lambda^2} \right) \]

\[ = \frac{1}{2} \ln \left( 1 + \frac{\lambda^2 + (a^2 - 1)\lambda^2}{1 + \lambda^2} \right) \]

\[ = \frac{1}{2} \ln \left( 1 + \frac{(a^2 - 1)\lambda^2}{1 + \lambda^2} \right) \]

**Lemma A.2.3.** For \( \lambda > 0 \) and \( a > 1 \),

\[ \text{atanh} \left( \frac{\sqrt{2a\lambda}}{1 + a\lambda} \right) - \text{atanh} \left( \frac{\sqrt{2\lambda}}{1 + \lambda} \right) = (\sqrt{a} - 1)\sqrt{2\lambda} \frac{1 - \lambda\sqrt{a}}{(1 - \lambda\sqrt{a})^2 + \lambda(1 + a)}. \]
Preserving a Function of the 1-Norm

Remark A.2.4. Note the change in sign when $\lambda$ crosses $1/\sqrt{a}$. We shall need it in some of the later bounds.

Proof. By the atanh addition formula B.1.11,

$$\text{atanh}(u) - \text{atanh}(v) = \text{atanh}(u) + \text{atanh}(-v) = \text{atanh} \left( \frac{u + (-v)}{1 + u(-v)} \right)$$

for $u, v \in (-1, 1)$, which is the case for us here. With

$$u = \frac{\sqrt{2a\lambda}}{1 + a\lambda} \quad \text{and} \quad v = \frac{\sqrt{2\lambda}}{1 + \lambda}$$

$$1 - uv = 1 - \frac{2\lambda\sqrt{a}}{(1 + \lambda)(1 + a\lambda)}$$

while

$$u - v = \frac{\sqrt{2a\lambda}}{1 + a\lambda} - \frac{\sqrt{2\lambda}}{1 + \lambda} = \sqrt{2\lambda} \left( \frac{\sqrt{a}}{1 + a\lambda} - \frac{1}{1 + \lambda} \right)$$

$$= \frac{\sqrt{2\lambda}}{(1 + a\lambda)(1 + \lambda)} ((1 + \lambda)\sqrt{a} - 1 - a\lambda)$$

so that

$$\frac{u - v}{1 - uv} = \frac{\sqrt{2\lambda}((1 + \lambda)\sqrt{a} - 1 - a\lambda)}{(1 + a\lambda)(1 + \lambda)} \left( \frac{(1 + \lambda)(1 + a\lambda) - 2\lambda\sqrt{a}}{(1 + a\lambda)(1 + \lambda)} \right)^{-1}$$

$$= \frac{\sqrt{2\lambda}((1 + \lambda)\sqrt{a} - 1 - a\lambda)}{(1 + a\lambda)(1 + \lambda)} \frac{1}{(1 + a\lambda)(1 + \lambda)}$$

$$= \sqrt{2\lambda} \frac{\sqrt{a} - 1 + \lambda\sqrt{a} - a\lambda}{1 + a\lambda^2 + \lambda + a\lambda - 2\lambda\sqrt{a}}$$

$$= \sqrt{2\lambda} \frac{\sqrt{a} - 1 + \lambda\sqrt{a}(1 - \sqrt{a})}{1 + a\lambda^2 + \lambda(1 + a) - 2\lambda\sqrt{a}}$$

$$= (\sqrt{a} - 1)\sqrt{2\lambda} \frac{1 - \lambda\sqrt{a}}{(1 - \lambda\sqrt{a})^2 + \lambda(1 + a)}$$

Because atanh is an odd function, taking atanh of the above will give negative numbers when $\lambda \sqrt{a} > 1$.

Lemma A.2.5. For $1 \leq a$ and $1/\sqrt{a} \leq \lambda$,

$$a - 1 > \mu(a\lambda) - \mu(\lambda) \geq \frac{a-1}{4}(1 - (a - 1))$$
Proof. The atanh contribution is negative for $\lambda \geq 1/\sqrt{a}$ with input

$$\frac{(\sqrt{a} - 1)\sqrt{2\lambda}}{(1 - \lambda\sqrt{a})^2 + \lambda(1 + a)}$$

Let

$$v := \frac{(\sqrt{a} - 1)\sqrt{2\lambda}}{(\lambda\sqrt{a} - 1)^2 + \lambda(1 + a)}$$

We estimate

$$v = \frac{(\sqrt{a} - 1)\sqrt{2\lambda}(\lambda\sqrt{a} - 1)}{(\lambda\sqrt{a} - 1)^2 + \lambda(1 + a)} \leq \frac{2}{(\lambda\sqrt{a} - 1)^2 + \lambda(1 + a)}$$

$$= \frac{\sqrt{a} - 1}{\sqrt{2\lambda}} \leq \frac{\sqrt{a}}{\sqrt{2\lambda}} < \frac{1}{\sqrt{2}}$$

So

$$w := \frac{\sqrt{a} - 1}{\sqrt{2\lambda}} < 1$$

We may write

$$\text{atanh}(v) \leq \text{atanh}(w) \leq \frac{w}{1 - w^2}$$

$$= \frac{\sqrt{a} - 1}{\sqrt{2\lambda}} \left(1 - \frac{(\sqrt{a} - 1)^2}{2(1 + a)}\right)^{-1} = \frac{2}{(\lambda\sqrt{a} - 1)^2 + \lambda(1 + a)}$$

$$= \frac{(\sqrt{a} - 1)\sqrt{2\lambda}}{2 + 2a - (a + 1 - 2\sqrt{a})} = \frac{(\sqrt{a} - 1)\sqrt{2\lambda}}{1 + a + 2\sqrt{a}} = \frac{(\sqrt{a} - 1)\sqrt{2\lambda}}{(1 + \sqrt{a})^2}$$

$$\leq \frac{\sqrt{a} - 1}{2}$$
as the remaining factor is decreasing in $a$:

\[
\frac{d}{da} \frac{(1 + a)^{1/2}}{(1 + \sqrt{a})^2}
= \frac{1}{1 + \sqrt{a}} \left( (1 + \sqrt{a})^2 \frac{1}{2(1 + a)^{1/2}} - 2(1 + a)^{1/2}(1 + \sqrt{a}) \frac{1}{2\sqrt{a}} \right)
= \frac{1}{2(1 + \sqrt{a})^2(1 + a)^{1/2}} \left( 1 - 2 \frac{(1 + a)}{\sqrt{a} + a} \right)
= \frac{1}{2(1 + \sqrt{a})^{3/2}(1 + a)^{1/2} \sqrt{a}} (a + \sqrt{a} - 2 - 2a)
\]

\[
< 0 \quad \text{certainly for } a \geq 1.
\]

On the other hand, because $\lambda \geq 1/\sqrt{a}$, the ln contribution is now

\[
\frac{1}{2} \ln \left( 1 + \frac{(a^2 - 1)\lambda^2}{1 + \lambda^2} \right) \geq \frac{1}{2} \ln \left( 1 + \frac{(a^2 - 1)(1/a)}{1 + 1/a} \right)
= \frac{1}{2} \ln \left( 1 + \frac{(a^2 - 1)}{a + 1} \right)
\]

using lemma A.2.6 to lower bound the logarithm.

Hence, for $\lambda \geq 1/\sqrt{a}$, and using lemma 3.4.5 to approximate the square root in the 2nd line,

\[
\mu(a\lambda) - \mu(\lambda) \geq \frac{(a - 1)}{2} \left( 1 - \frac{a - 1}{2} \right) - \frac{\sqrt{a} - 1}{2}
\geq \frac{(a - 1)}{2} \left( 1 - \frac{a - 1}{2} \right) - \frac{a - 1}{4} = \frac{(a - 1)}{2} - \frac{a - 1}{4} - \frac{(a - 1)^2}{4}
= \frac{a - 1}{4} - \frac{(a - 1)^2}{4} = \frac{a - 1}{4} (1 - (a - 1)).
\]

If we drop the negative atanh contribution, we can find an upper bound

\[
\mu(a\lambda) - \mu(\lambda) \leq \frac{1}{2} \ln \left( 1 + \frac{(a^2 - 1)\lambda^2}{1 + \lambda^2} \right) < \ln(a) \leq a - 1.
\]

Note this need not be an upper bound if the atanh contribution were positive.

\[\square\]

**Lemma A.2.6.** For $-1 < t$,

\[
\ln(1 + t) \geq t(1 - t/2).
\]
Remark A.2.7. One may also use the alternating series test if $t < 1$.

Proof. Let $f(t) = \ln(1 + t)$. Then

$$f'(t) = \frac{1}{1+t}, \quad f''(t) = -\frac{1}{(1+t)^2}, \quad \text{and} \quad f^{(3)}(t) = \frac{2}{(1+t)^3} > 0.$$ 

Because $t > -1$, all these terms have a sign. By Taylor’s theorem with Lagrange remainder, with $x \in (0, t)$,

$$f(t) = f(0) + f'(0)t + f''(0)\frac{t^2}{2} + f^{(3)}(x)\frac{t^3}{3!} \geq t - \frac{t^2}{2} = t(1 - t/2)$$

□

A.3. An Auxiliary Mean

The following mean will be useful in some of the later estimates for the second moment.

Lemma A.3.1. For $\lambda \geq 0$ and $X \sim \text{Cauchy}(1)$,

$$\mathbb{E}\ln(1 + \lambda |X|) = -\frac{2}{\pi} \ln(\lambda) \arctan(\lambda) + \frac{1}{2} \ln(1 + \lambda^2) + \frac{2}{\pi} \text{Ti}_2(\lambda).$$

Remark A.3.2. It is clear that the expectation is nonnegative when $0 < \lambda \leq 1$. When $\lambda > 1$, see lemma A.3.3.

Proof. From lemma B.0.1

$$\mathbb{E}\ln(1 + \lambda |X|) = \frac{1}{i\pi} (\text{Li}_2(1 + i\lambda) - \text{Li}_2(1 - i\lambda))$$

$$= \frac{2}{\pi} \frac{1}{2i} (\text{Li}_2(1 + i\lambda) - \text{Li}_2(1 - i\lambda))$$

We use the reflection formula B.2.1 to expand the dilogarithm terms.

Recall from lemma B.2.1, for $z \in (\mathbb{C} - \mathbb{R}) \cup (0, 1)$,

$$\text{Li}_2(z) + \text{Li}_2(1 - z) - \text{Li}_2(1) = -\ln(z)\ln(1 - z).$$

So we have

$$\text{Li}_2(1 - i\lambda) = -\ln(i\lambda)\ln(1 - i\lambda) - \text{Li}_2(i\lambda) + \text{Li}_2(1)$$

and

$$\text{Li}_2(1 + i\lambda) = -\ln(-i\lambda)\ln(1 + i\lambda) - \text{Li}_2(-i\lambda) + \text{Li}_2(1).$$
Consequently,

\[
\frac{1}{2i}(\text{Li}_2(1 + i\lambda) - \text{Li}_2(1 - i\lambda))
\]

\[
= \frac{1}{2i} \left( -\ln(-i\lambda) \ln(1 + i\lambda) - \text{Li}_2(-i\lambda) + \text{Li}_2(1) \right)
\]

\[
- \frac{1}{2i} \left( -\ln(i\lambda) \ln(1 - i\lambda) - \text{Li}_2(i\lambda) + \text{Li}_2(1) \right)
\]

\[
= \frac{1}{2i} \left( -\ln(-i\lambda) \ln(1 + i\lambda) + \ln(i\lambda) \ln(1 - i\lambda) + \text{Li}_2(i\lambda) - \text{Li}_2(-i\lambda) \right)
\]

\[
= \frac{1}{2i} \left( -\left(\ln(\lambda) - \frac{i\pi}{2}\right) \ln(1 + i\lambda) + \left(\ln(\lambda) + \frac{i\pi}{2}\right) \ln(1 - i\lambda) \right) + \text{Ti}_2(\lambda)
\]

\[
= \frac{1}{2i} \ln(\lambda)(\ln(1 - i\lambda) - \ln(1 + i\lambda)) + \frac{i\pi}{4}(\ln(1 - i\lambda) + \ln(1 + i\lambda)) + \text{Ti}_2(\lambda)
\]

By lemma B.0.11 (really the remark there) and the definition of \text{arctan},

\[
\frac{1}{2i}(\text{Li}_2(1 + i\lambda) - \text{Li}_2(1 - i\lambda)) = -\ln(\lambda) \arctan(\lambda) + \frac{\pi i}{2} \ln(1 + \lambda^2) + \text{Ti}_2(\lambda).
\]

Thus,

\[
\mathbb{E} \ln(1 + \lambda |X|) = \frac{2}{\pi} \frac{1}{2i}(\text{Li}_2(1 + i\lambda) - \text{Li}_2(1 - i\lambda))
\]

\[
= -\frac{2}{\pi} \ln(\lambda) \arctan(\lambda) + \frac{1}{2} \ln(1 + \lambda^2) + \frac{2}{\pi} \text{Ti}_2(\lambda).
\]

We take some time to better understand how \(\mathbb{E} \ln(1 + \lambda |X|)\) behaves as a function of \(\lambda\). We know it is increasing from its definition as an expectation of increasing functions of \(\lambda\), but depending on the size of \(\lambda\), certain terms contribute much more than others.

**Lemma A.3.3.** For \(\lambda > 0\), let

\[
f(\lambda) := \text{Ti}_2(\lambda) - \ln(\lambda) \arctan(\lambda).
\]

Then

\[
0 < f(\lambda) = f \left(\frac{1}{\lambda}\right) \leq f(1) = \text{Ti}_2(1) < 1.
\]

and goes to 0 as \(\lambda \to \infty\) or \(\lambda \to 0\).

**Proof.** Take the derivative

\[
\frac{d}{d\lambda} \left(\text{Ti}_2(\lambda) - \ln(\lambda) \arctan(\lambda)\right)
\]

\[
= \frac{\arctan(\lambda)}{\lambda} - \frac{\arctan(\lambda)}{1 + \lambda^2} - \frac{\ln(\lambda)}{1 + \lambda^2}
\]

\[
= \frac{\ln(\lambda)}{1 + \lambda^2}.
\]
which is positive for $\lambda < 1$ and negative for $\lambda > 1$ and hence $\lambda = 1$ is the unique maximizer.

For showing the equality, consider the inversion formula B.3.3 for $T_{2}(\lambda)$,

$$T_{2}(\lambda) = T_{2}(1/\lambda) + \frac{\pi}{2} \ln(\lambda)$$

so that, using the inversion formula B.3.1 for $\arctan(\lambda)$,

$$f(\lambda) = T_{2}(\lambda) - \ln(\lambda) \arctan(\lambda) = T_{2}(1/\lambda) + \ln(\lambda) \left( \frac{\pi}{2} - \arctan(\lambda) \right)$$

$$= T_{2}(1/\lambda) + \ln(\lambda) \arctan(1/\lambda) = T_{2}(1/\lambda) - \ln(1/\lambda) \arctan(1/\lambda)$$

$$= f(1/\lambda)$$

For the lower bound, all terms of $f(\lambda)$ are nonnegative for $0 < \lambda < 1$, and hence $f(1/\lambda)$ must be nonnegative too. Alternatively, the 3rd line above has all terms nonnegative for $\lambda \geq 1$: the power series for $T_{2}(1/\lambda)$ is alternating with terms of strictly decreasing magnitude:

$$T_{2}(1/\lambda) = \sum_{j=0}^{\infty} (-1)^{j} \frac{\lambda^{-(2j+1)}}{(2j+1)^{2}}.$$  

This series also shows

$$\frac{1}{\lambda} > T_{2}(1/\lambda) > \frac{1}{\lambda} - \frac{1}{9\lambda^{2}} > 0.$$  

The same reasoning yields $1 > T_{2}(1) > 1 - 1/9$.  

A.4. 2nd Moment

To estimate the 2nd moment $\mathbb{E}\xi^{2}(\lambda | X)$, note that for any $a, b > 0$,

$$(a + b)^{2} = a^{2} + b^{2} + 2ab = a^{2} + b^{2} + 2\sqrt{a^{2}b^{2}}$$

$$\leq a^{2} + b^{2} + 2 \left( \frac{a^{2}}{2} + \frac{b^{2}}{2} \right) = 2a^{2} + 2b^{2}$$

by the AM-GM inequality. Consequently,

$$\mathbb{E}\xi^{2}(\lambda | X) = \mathbb{E}\left( \ln(1 + \sqrt{\lambda | X|}) + \frac{1}{2} \ln(1 + \lambda | X|) \right)^{2}$$

$$\leq \mathbb{E}\left( 2\ln^{2}(1 + \sqrt{\lambda | X|}) + 2 \frac{1}{4} \ln^{2}(1 + \lambda | X|) \right)$$

$$= \mathbb{E}\left( 2\ln^{2}(1 + \sqrt{\lambda | X|}) + \frac{1}{2} \ln^{2}(1 + \lambda | X|) \right).$$

It turns out this last expression also arises from a contour integral.
Lemma A.4.1. If $\lambda > 0$ and $X \sim \text{Cauchy}(1)$, then

$$\mathbb{E}\left(2 \ln^2(1 + \sqrt{\lambda |X|}) + \frac{1}{2} \ln^2(1 + \lambda |X|)\right) = 2\mathbb{E}\arctan^2(\sqrt{\lambda |X|}) + \mu^2(\lambda) - \arctan^2(\lambda) + 2 \arctan(\lambda) h(\sqrt{\lambda}) - h^2(\sqrt{\lambda})$$

with

$$h(\sqrt{\lambda}) = \frac{\pi}{2} + \arctan\left(\frac{\sqrt{\lambda}}{\sqrt{2}} - \frac{1}{\sqrt{2} \lambda}\right).$$

Proof. The computations will be a bit more involved than those for the first moment. Starting from lemma A.0.1 with $b = 2$,

$$\mathbb{E} \ln^2(1 + \sqrt{\lambda |X|})
= \ln^2(1 + \sqrt{i\lambda}) + \ln^2(1 + \sqrt{-i\lambda})
- \frac{1}{2} \mathbb{E} \ln^2(1 + i\sqrt{\lambda |X|}) - \frac{1}{2} \mathbb{E} \ln^2(1 - i\sqrt{\lambda |X|}),$$

that is,

$$\mathbb{E}2 \ln^2(1 + \sqrt{\lambda |X|}) + \mathbb{E} \ln^2(1 + i\sqrt{\lambda |X|}) + \mathbb{E} \ln^2(1 - i\sqrt{\lambda |X|})
= 2 \ln^2(1 + \sqrt{i\lambda}) + 2 \ln^2(1 + \sqrt{-i\lambda}).$$

By lemma A.4.4,

$$\mathbb{E} \ln^2(1 + i\sqrt{\lambda |X|}) + \mathbb{E} \ln^2(1 - i\sqrt{\lambda |X|})
= \mathbb{E} \left(\frac{1}{2} \ln^2(1 + \lambda |X|)^2 - 2\mathbb{E} \arctan^2(\sqrt{\lambda |X|})\right)
= \mathbb{E} \frac{1}{2} \ln^2(1 + \lambda |X|) - 2\mathbb{E} \arctan^2(\sqrt{\lambda |X|}).$$

For the residue terms, we use lemma A.4.7:

$$2 \ln^2(1 + \sqrt{i\lambda}) + 2 \ln^2(1 + \sqrt{-i\lambda})
= \frac{1}{4} \ln^2(1 + \lambda^2) - \arctan^2(\lambda)
+ \ln(1 + \lambda^2) g(\sqrt{\lambda}) + 2 \arctan(\lambda) h(\sqrt{\lambda}) + g^2(\sqrt{\lambda}) - h^2(\sqrt{\lambda})$$

with

$$g(\sqrt{\lambda}) = \text{atanh} \left(\frac{\sqrt{2\lambda}}{1 + \lambda}\right) \quad \text{and} \quad h(\sqrt{\lambda}) = \frac{\pi}{2} + \arctan \left(\frac{\sqrt{\lambda}}{\sqrt{2}} - \frac{1}{\sqrt{2} \lambda}\right).$$
Recalling our computation of $\mu(\lambda)$ in lemma A.1.1, we can further simplify:

$$
2 \ln^2(1 + \sqrt{i\lambda}) + 2 \ln^2(1 - \sqrt{i\lambda}) \\
= \left( \frac{1}{4} \ln^2(1 + \lambda^2) + \ln(1 + \lambda^2) g(\sqrt{\lambda}) + g^2(\sqrt{\lambda}) \right) \\
- \arctan^2(\lambda) + 2 \arctan(\lambda) h(\sqrt{\lambda}) - h^2(\sqrt{\lambda}) \\
= \mu^2(\lambda) - \arctan^2(\lambda) + 2 \arctan(\lambda) h(\sqrt{\lambda}) - h^2(\sqrt{\lambda})
$$

Putting everything together we may conclude

$$
\mathbb{E} \left( 2 \ln^2(1 + \sqrt{\lambda |X|}) + \frac{1}{2} \ln^2(1 + \lambda |X|) \right) \\
= 2 \mathbb{E} \arctan^2(\sqrt{\lambda |X|}) + \mu^2(\lambda) - \arctan^2(\lambda) + 2 \arctan(\lambda) h(\sqrt{\lambda}) - h^2(\sqrt{\lambda}).
$$

\[ \blacksquare \]

**Remark A.4.2.** Using lemma A.4.5, we thus have the upper bound

$$
\mathbb{E} \xi^2(\lambda |X|) \\
\leq \min \left\{ 2 \mathbb{E} \ln(1 + \lambda |X|), \frac{\pi^2}{2} \right\} + \mu^2(\lambda) \\
- \arctan^2(\lambda) + 2 \arctan(\lambda) h(\sqrt{\lambda}) - h^2(\sqrt{\lambda}) \\
= \min \left\{ 2 \mathbb{E} \ln(1 + \lambda |X|), \frac{\pi^2}{2} \right\} + \mu^2(\lambda) - \left( \arctan(\lambda) - h(\sqrt{\lambda}) \right)^2 \\
\leq \min \left\{ 2 \mathbb{E} \ln(1 + \lambda |X|), \frac{\pi^2}{2} \right\} + \mu^2(\lambda).
$$

In particular, the variance is bounded from above by $\pi^2/2$ for all $\lambda > 0$.

**Lemma A.4.3.** For $0 < \lambda \leq 1$

$$
\frac{\mathbb{E} \xi^2(\lambda |X|)}{\lambda} \leq \lambda + \frac{4}{\pi} \ln(\lambda) + \frac{8}{(1 + \lambda)^2} + 2\lambda \sqrt{2\lambda} + \frac{\lambda^3}{4}
$$

while for $\lambda \geq 1$,

$$
\frac{\mathbb{E} \xi^2(\lambda |X|)}{\lambda} \leq \frac{\pi^2}{2} + 2 + \lambda \sqrt{2} + \frac{\lambda^3}{4}.
$$

**Proof.** By remark A.4.2 and lemma A.3.1, we have

$$
\mathbb{E} \xi^2(\lambda |X|) \leq \ln(1 + \lambda^2) + \frac{4}{\pi} T_{12}(\lambda) - \frac{4}{\pi} \ln(\lambda) \arctan(\lambda) + \mu^2(\lambda) \\
\leq \lambda^2 + \frac{4}{\pi} \lambda - \frac{4}{\pi} \lambda \ln(\lambda) + \mu^2(\lambda)
$$
because $T_2(\lambda)$ is an alternating series with terms of decreasing magnitude for $\lambda < 2$ and that for $\lambda \leq 1$, $\ln(\lambda)$ is nonnegative. By remark A.1.3,

$$\mu^2(\lambda) \leq \left( 2 \frac{\sqrt{2\lambda}}{1+\lambda} + \frac{\lambda^2}{2} \right)^2 = \frac{8\lambda}{(1+\lambda)^2} + 2\lambda^2 \frac{\sqrt{2\lambda}}{1+\lambda} + \frac{\lambda^4}{4}.$$  

Consequently, for $\lambda \leq 1$,

$$\frac{E \xi^2(\lambda | X)}{\lambda} \leq \lambda + \frac{4}{\pi} - \frac{4}{\pi} \ln(\lambda) + \frac{8\lambda}{(1+\lambda)^2} + 2\lambda \frac{\sqrt{2\lambda}}{1+\lambda} + \frac{\lambda^3}{4}.$$  

When $\lambda \geq 1$, we can instead use the other upper bound for the atanh term, using lemma A.1.2.

$$\frac{E \xi^2(\lambda | X)}{\lambda} \leq \frac{\pi^2}{2\lambda} \frac{1}{\lambda} \left( \sqrt{2} + \frac{\lambda^2}{2} \right)^2 \leq \frac{\pi^2}{2} + 2 + \lambda \sqrt{2} + \frac{\lambda^3}{4}.$$  

\[\square\]

**Lemma A.4.4.** For $r > 0$,

$$\ln^2(1 + ir) + \ln^2(1 - ir) = \frac{1}{2} \ln^2(1 + r^2) - 2 \arctan^2(r).$$  

**Proof.** We are adding complex conjugates, so

$$\ln^2(1 + ir) + \ln^2(1 - ir) = 2\Re \ln^2(1 + ir)$$

$$= 2\Re \left( \frac{1}{2} \ln(1 + r^2) + i \arctan(r) \right)^2$$

$$= 2 \left( \frac{1}{4} \ln^2(1 + r^2) - \arctan^2(r) \right) = \frac{1}{2} \ln^2(1 + r^2) - 2 \arctan^2(r).$$  

\[\square\]

**Lemma A.4.5.** For $\nu > 0$,

$$\arctan^2(\sqrt{\nu}) \leq \min \left\{ \ln(1 + \nu), \frac{\pi^2}{4} \right\}$$

is strictly concave. Consequently, for any $\lambda > 0$ and $X \sim \text{Cauchy}(1)$,

$$\mathbb{E} \arctan^2(\sqrt{\lambda | X|}) \leq \min \left\{ \mathbb{E} \ln(1 + \lambda | X|), \frac{\pi^2}{4} \right\}.$$  

**Remark A.4.6.** The function is also strictly increasing and 0 when $\nu = 0$, so the function is subadditive too. This bound contains a $-\lambda \ln(\lambda)$ term for small $\lambda$.  

imsart-bj ver. 2014/10/16 file: ms.tex date: June 11, 2019
Proof. Upon taking the derivative,
\[
\frac{d}{d\nu} \arctan^2(\sqrt{\nu}) = 2 \arctan(\sqrt{\nu}) \frac{1}{1+(\sqrt{\nu})^2} - \frac{1}{\sqrt{\nu}} \arctan(\sqrt{\nu}) \frac{1}{1+\nu}
\]
which is positive for \( \nu > 0 \). We shall show the derivative is decreasing as well, as a product of decreasing functions. We focus on the \( \arctan \) fraction, as \( 1/(1+\nu) \) is decreasing.
\[
\frac{d}{d\nu} \frac{\arctan(\sqrt{\nu})}{\sqrt{\nu}} = \frac{1}{\nu} \left( \frac{\sqrt{\nu}}{1+\nu} - \frac{1}{2} \frac{1}{2\sqrt{\nu}} - \frac{1}{1+\nu} \right)
\]
so we just need to show the bracketed term is nonpositive. It is 0 when \( \nu = 0 \), and we show it is decreasing:
\[
\frac{d}{d\nu} \left( \frac{\sqrt{\nu}}{1+\nu} - \arctan(\sqrt{\nu}) \right) = \frac{1}{2\nu^{3/2}} \left( \frac{\sqrt{\nu}}{1+\nu} - \arctan(\sqrt{\nu}) \right) < 0
\]
as desired.

For the upper bounds, the constant follows from \( \arctan(x) \leq \pi/2 \) for all \( x \in \mathbb{R} \), while the \( \ln(1+\nu) \) bound follows from comparing derivatives, noting that both functions are 0 when \( \nu = 0 \).
\[
\frac{d}{d\nu} \arctan^2(\sqrt{\nu}) = \frac{\arctan(\sqrt{\nu})}{\sqrt{\nu}} \frac{1}{1+\nu} \leq \frac{\sqrt{\nu}}{\sqrt{\nu} \frac{1}{1+\nu}} = \frac{1}{1+\nu} = \frac{d}{d\nu} \ln(1+\nu).
\]

\[\square\]

Lemma A.4.7. For \( \nu > 0 \),
\[
\ln^2(1+\nu\sqrt{i}) + \ln^2(1+\nu\sqrt{-i}) = \frac{1}{8} \ln^2(1+\nu^4) - \frac{1}{2} \arctan^2(\nu^2) + \frac{1}{2} \ln(1+\nu^4) g(\nu) + \arctan(\nu^2) h(\nu) + \frac{1}{2} (g^2(\nu) - h^2(\nu))
\]
with
\[
g(\nu) = \operatorname{atanh} \left( \frac{\nu\sqrt{2}}{1+\nu^2} \right) \quad \text{and} \quad h(\nu) = \frac{\pi}{2} + \arctan \left( \frac{\nu}{\sqrt{2}} - \frac{1}{\nu\sqrt{2}} \right).
\]

Proof. Using lemma B.1.9,
\[
\ln^2(1+\nu\sqrt{i}) = \left( \operatorname{atanh} (\nu\sqrt{i}) + \frac{1}{2} \ln(1-i\nu^2) \right)^2 = \operatorname{atanh}^2(\nu\sqrt{i}) + \operatorname{atanh}(\nu\sqrt{i}) \ln(1-i\nu^2) + \frac{1}{4} \ln^2(1-i\nu^2)
\]
and similarly
\[ \ln^2(1 + \nu \sqrt{-i}) = \left( \text{atanh}(\nu \sqrt{-i}) + \frac{1}{2} \ln(1 + i\nu^2) \right)^2 \]
= atanh^2(\nu \sqrt{-i}) + \text{atanh}(\nu \sqrt{-i}) \ln(1 + i\nu^2) + \frac{1}{4} \ln^2(1 + i\nu^2)

Adding yields several terms:

1. \[ (\nu) := \frac{1}{4} \ln^2(1 + i\nu^2) + \frac{1}{4} \ln^2(1 - i\nu^2) \]
2. \[ (\nu) := \text{atanh}(\nu \sqrt{i}) \ln(1 - i\nu^2) + \text{atanh}(\nu \sqrt{-i}) \ln(1 + i\nu^2) \]
3. \[ (\nu) := \text{atanh}^2(\nu \sqrt{i}) + \text{atanh}^2(\nu \sqrt{-i}) \]

From lemma A.4.4,

\[ (\nu) = \frac{1}{4} \left( \frac{1}{2} \ln^2(1 + \nu^4) - 2 \arctan^2(\nu^2) \right) = \frac{1}{8} \ln^2(1 + \nu^4) - \frac{1}{2} \arctan^2(\nu^2). \]

We also have

\[ (\nu) = \text{atanh}(\nu \sqrt{i}) \left( \frac{1}{2} \ln(1 + \nu^4) - i \arctan(\nu^2) \right) \]
+ \text{atanh}(\nu \sqrt{-i}) \left( \frac{1}{2} \ln(1 + \nu^4) + i \arctan(\nu^2) \right)
= \frac{1}{2} \ln(1 + \nu^4) \left( \text{atanh}(\nu \sqrt{i}) + \text{atanh}(\nu \sqrt{-i}) \right)
- i \arctan(\nu^2) \left( \text{atanh}(\nu \sqrt{i}) - \text{atanh}(\nu \sqrt{-i}) \right)
= \frac{1}{2} \ln(1 + \nu^4) g(\nu) + \arctan(\nu^2) h(\nu).

Let
\[ g(\nu) := \text{atanh}(\nu \sqrt{i}) + \text{atanh}(\nu \sqrt{-i}) \]
= atanh \left( \frac{\nu \sqrt{1 + \sqrt{-i}^2}}{1 + \nu^2 \sqrt{-i}^2} \right) = atanh \left( \frac{\nu \sqrt{2}}{1 + \nu^2} \right) 
by the atanh addition formula B.1.11, as \sqrt{\pm i} = (1 \pm i)/\sqrt{2} are conjugates of each other.

Let
\[ h(\nu) := -i \left( \text{atanh}(\nu \sqrt{i}) - \text{atanh}(\nu \sqrt{-i}) \right). \]

Then
\[ g^2(\nu) - h^2(\nu) \]
= atanh^2(\nu \sqrt{i}) + atanh^2(\nu \sqrt{-i}) + 2 \text{atanh}(\nu \sqrt{i}) \text{atanh}(\nu \sqrt{-i})
+ (\text{atanh}(\nu \sqrt{i}) - \text{atanh}(\nu \sqrt{-i}))^2
= 2(\text{atanh}^2(\nu \sqrt{i}) + \text{atanh}^2(\nu \sqrt{-i}) = 2(\nu).
So we are left to understand \( h(\nu) \). By lemma A.4.8, it is

\[
h(\nu) = \frac{\pi}{2} + \arctan \left( \frac{\nu}{\sqrt{2}} - \frac{1}{\nu \sqrt{2}} \right).
\]

\[
\text{Lemma A.4.8.} \quad \text{For } \nu > 0,
\]

\[
h(\nu) := -i \left( \tanh(\nu \sqrt{i}) - \tanh(\nu \sqrt{-i}) \right) = \frac{\pi}{2} + \arctan \left( \frac{\nu}{\sqrt{2}} - \frac{1}{\nu \sqrt{2}} \right).
\]

\[
\text{Remark A.4.9.} \quad \text{For } \nu < 1, \text{ we can rewrite the above as}
\]

\[
\frac{\pi}{2} - \arctan \left( \frac{1 - \nu^2}{\nu \sqrt{2}} \right) = \arctan \left( \frac{\nu \sqrt{2}}{1 - \nu^2} \right).
\]

\[
\text{Proof.} \quad \text{We cannot directly use the \( \tanh \) addition formula because there is a singularity when } \nu \text{ crosses 1. However, by definition of \( \tanh \) B.1.6, we can convert } h(\nu) \text{ as follows}
\]

\[
h(\nu) := -i \left( \tanh(\nu \sqrt{i}) - \tanh(\nu \sqrt{-i}) \right)
\]

\[
= -i \left( -i \arctan(i\nu \sqrt{i}) - (-i) \arctan(i\nu \sqrt{-i}) \right)
\]

\[
= - \arctan(i\nu \sqrt{i}) + \arctan(i\nu \sqrt{-i})
\]

\[
= - \arctan(i\nu \sqrt{i}) + \arctan(i\nu (-i \sqrt{i}))
\]

\[
= - \arctan(i\nu \sqrt{i}) + \arctan(\nu \sqrt{i})
\]

using \( \sqrt{-i} = -i \sqrt{i} \). We now use the inversion formula B.3.1 for \( \arctan \).

\[
h(\nu) = - \arctan(i\nu \sqrt{i}) + \frac{\pi}{2} - \arctan(1/(\nu \sqrt{i}))
\]

\[
= \frac{\pi}{2} - ( \arctan(i\nu \sqrt{i}) + \arctan(-i \sqrt{i}/\nu) )
\]

We claim

\[
( \arctan(i\nu \sqrt{i}) + \arctan(-i \sqrt{i}/\nu) ) = - \arctan \left( \frac{\nu}{\sqrt{2}} - \frac{1}{\nu \sqrt{2}} \right).
\]

Both expressions are 0 at \( \nu = 1 \), so we just need to show the derivatives match for \( \nu > 0 \).
On one hand,

\[
\frac{d}{d\nu} (-1) \arctan \left( \frac{\nu}{\sqrt{2}} - \frac{1}{\nu \sqrt{2}} \right) \\
= - \left( 1 + \left( \frac{\nu}{\sqrt{2}} - \frac{1}{\nu \sqrt{2}} \right)^2 \right)^{-1} \frac{d}{d\nu} \left( \frac{\nu}{\sqrt{2}} - \frac{1}{\nu \sqrt{2}} \right) \\
= - \left( 1 + \frac{1}{2} \left( \frac{\nu^2 - 1}{\nu} \right)^2 \right)^{-1} \left( \frac{1}{\sqrt{2}} + \frac{1}{\nu^2 \sqrt{2}} \right) \\
= - \frac{1}{\sqrt{2}} \left( \frac{2\nu^2 + \nu^4 - 2\nu^2 + 1}{2\nu^2} \right)^{-1} \left( \frac{\nu^2 + 1}{\nu^2} \right) \\
= - \frac{1}{\sqrt{2}} \frac{2\nu^2 + \nu^4 - 1}{\nu^2} = - \frac{\sqrt{2} (1 + \nu^2)}{1 + \nu^4}.
\]

On the other hand,

\[
\frac{d}{d\nu} \left( \arctan(\nu \sqrt{i}) + \arctan(-i \sqrt{i}/\nu) \right) \\
= \frac{i \sqrt{7}}{1 - i \nu^2} + \frac{-i \sqrt{7}}{1 - i/\nu^2} \frac{(-1)}{\nu^2} = \frac{i \sqrt{7}}{1 - i \nu^2} + \frac{i \sqrt{7}}{1 - i/\nu^2} + \frac{-\sqrt{7}}{1 + \nu^2 + 1} \\
= \frac{\sqrt{7}}{1 + \nu^4} (i(1 + i\nu^2) - (1 - i\nu^2)) = \frac{\sqrt{7}}{1 + \nu^4} (i - \nu^2 - 1 + i\nu^2) \\
= \frac{\sqrt{7}}{1 + \nu^4} (i - 1)(1 + \nu^2) = \frac{-\sqrt{2} \sqrt{1 + i\nu^2}}{1 + \nu^4} (1 + \nu^2) \\
= - \frac{\sqrt{2} (1 + \nu^2)}{1 + \nu^4}.
\]

Because the derivatives match, we are done. \(\square\)

**Appendix B: Polylogarithms and Their Friends**

The polylogarithms \(\text{Li}_b(z)\) arise when we compute or estimate the first and second moments of the coordinate projections; they will help us give quantitative bounds which are needed in some of the proofs. References for polylogarithms are [12] and [15].

As initial motivation for studying such functions, we have the following lemma.

**Lemma B.0.1.** Let \(X \sim \text{Cauchy}(1)\) and \(\nu > 0\). Then for \(b > -1\),

\[
\mathbb{E} \ln^b (1 + \nu |X|) = \frac{\Gamma(b + 1)}{i \pi} (\text{Li}_{b+1}(1 + iv) - \text{Li}_{b+1}(1 - iv)).
\]
Proof. We have

\[
I_b(\nu) := E \ln^b(1 + \nu |X|) = \frac{2}{\pi} \int_0^\infty \frac{\ln^b(1 + \nu x)}{1 + x^2} \, dx
\]

With \( u = 1 + \nu x \),

\[
\frac{2}{\pi} \int_0^\infty \frac{\ln^b(1 + \nu x)}{1 + x^2} \, dx = \frac{2 \nu}{\pi} \int_1^\infty \frac{\ln^b(u)}{1 + \nu^2(u - 1)^2} \, du
\]

and with \( t = \ln(u) \) so that \( dt = du/u \Rightarrow e^t \, dt = du \),

\[
= \frac{2 \nu}{\pi} \int_0^\infty \frac{t^b e^t}{\nu^2 + (e^t - 1)^2} \, dt
\]

Because

\[
\frac{1 + i\nu}{e^t - (1 + i\nu)} - \frac{1 - i\nu}{e^t - (1 - i\nu)} = \frac{2i\nu e^t}{(e^t - (1 + i\nu))(e^t - (1 - i\nu))}
\]

we may write

\[
I_b(\nu) = \frac{1}{i\pi} \int_0^\infty \frac{t^b}{(e^t - (1 + i\nu))(e^t - (1 - i\nu))} \frac{2i\nu e^t}{(e^t - (1 + i\nu))(e^t - (1 - i\nu))} \, dt
\]

\[
= \frac{1}{i\pi} \int_0^\infty \frac{t^b(1 + i\nu)}{e^t - (1 + i\nu)} - \frac{t^b(1 - i\nu)}{e^t - (1 - i\nu)} \, dt
\]

\[
= \frac{\Gamma(b + 1)}{i\pi} (\text{Li}_{b+1}(1 + i\nu) - \text{Li}_{b+1}(1 - i\nu)).
\]

The polylogarithms are defined because \( \nu > 0 \), and if \( b > 0 \), the value at \( \nu = 0 \) is also defined.

General references for complex analysis are [20] for proofs and [17] for intuition. If \( z = x + iy \in \mathbb{C} \) with \( x, y \in \mathbb{R} \), then \( \Re(z) := x \) and \( \Im(z) := y \). If \( z = re^{i\theta} = x + iy \in \mathbb{C} \),
denote $z^* = re^{-i\theta} = x - iy$ for the complex conjugate. Further $|z|^2 = zz^* = x^2 + y^2$. Thus, if $w = se^{i\phi}$, we have

$$(zw)^* = (rse^{i(\theta + \phi)})^* = rse^{-i(\theta + \phi)} = z^* w^*.$$  

Further, if $w \neq 0$,

$$\frac{|z|^2}{|w|^2} = \frac{zz^*}{ww^*} = \frac{r^2}{s^2} = \frac{|z|^2}{|w|^2}.$$  

For us, analytic functions are synonymous with holomorphic ones. We shall be using two theorems from complex analysis repeatedly. Cf. [20, page 52,96].

**Theorem B.0.2** (Analytic Continuation). *Let f and g be analytic functions in a connected open subset $\Omega$ of $\mathbb{C}$. If $f(z) = g(z)$ for all $z$ in a non-empty open subset of $\Omega$, then $f(z) = g(z)$ throughout $\Omega$.*

**Theorem B.0.3** (Primitives). *Let $f$ be an analytic function in a simply connected subset $\Omega$ of $\mathbb{C}$. Then for $z_0, z \in \Omega$, the function

$$F(z) := \int_{z_0}^z f(w) \, dw = \int_{\gamma} f(w) \, dw$$

is analytic too, with $\gamma$ any path from $z_0$ to $z$ lying in $\Omega$.*

**Definition B.0.4** (The Logarithm). *For $z = re^{i\theta} \in \mathbb{C} - (-\infty, 0]$, define (the principle branch of) the logarithm of $z$, $\ln(z)$ as

$$\ln(z) := \ln(r) + i\theta = \int_1^z \frac{dw}{w}$$

for any path from 1 to $z$ in $\mathbb{C} - (-\infty, 0]$.*

**Remark B.0.5.** Note that

$$\ln(z^*) = \ln(r) - i\theta = \ln(z)^*.$$  

The map $w \mapsto 1/w$ takes $\mathbb{C} - (-\infty, 0]$ to itself; for if $w = se^{i\phi}$, with $|\phi| < \pi$, then $1/w = (1/s)e^{-i\phi}$ which also lives in $\mathbb{C} - (-\infty, 0]$. With this choice of principle branch, the logarithm still satisfies

$$-\ln(1/w) = \ln(w)$$

via

$$-\ln(1/w) = -(\ln(1/s) + i(-\phi)) = \ln(s) + i\phi = \ln(w).$$  

Similarly, note that if $\Re(z), \Re(w) > 0$, then

$$zw = rse^{i(\theta + \phi)} \quad \text{with} \quad |\theta + \phi| < \pi$$
so \( \arg(zw) = \theta + \phi \) and
\[
\ln(zw) = \ln(rs) + i(\theta + \phi) = \ln(z) + \ln(w)
\]
in this case. However, the general identity \( \ln(z_1 z_2) = \ln(z_1) + \ln(z_2) \) does not hold.

**Definition B.0.6** (The Polylogarithm of Order 1). Define the polylogarithm of order 1, \( \text{Li}_1(z) \) as
\[
\text{Li}_1(z) := \sum_{j=1}^{\infty} \frac{z^j}{j} \quad \text{for} \quad |z| < 1
\]
and
\[
\text{Li}_1(z) := -\ln(1-z) = \ln \left( \frac{1}{1-z} \right) \quad \text{for} \quad z \in \mathbb{C} - [1, \infty).
\]

For general \( z \), the domain makes sense, as \( 1 - z = -(z - 1) \in \mathbb{C} - (-\infty, 0] \) for the \( z \) in question. Recall when \( |z| < 1 \),
\[
-\ln(1-z) = \sum_{j=1}^{\infty} \frac{z^j}{j},
\]
noting that both sides agree when \( z = 0 \), and upon differentiating,
\[
\frac{d}{dz} \sum_{j=1}^{\infty} \frac{z^j}{j} = \sum_{j=0}^{\infty} z^j = \frac{1}{1-z} = \frac{d}{dz}(-\ln(1-z))
\]
which means \(-\ln(1-z)\) and the sum differ by a constant, namely 0.

Unlike \( \text{Li}_1(z) \), the higher order polylogarithms extend to the unit circle.

**Definition B.0.7** (Higher Integral Order Polylogarithms). For \( 2 \leq n \in \mathbb{N} \), the polylogarithm of order \( n \), \( \text{Li}_n(z) \) is defined as
\[
\text{Li}_n(z) := \sum_{j=1}^{\infty} \frac{z^j}{j^n} \quad \text{for} \quad |z| \leq 1
\]
and by
\[
\text{Li}_n(z) := \int_0^z \frac{\text{Li}_{n-1}(w)}{w} \, dw \quad \text{for} \quad z \in \mathbb{C} - (1, \infty).
\]

The order of the polylogarithms may be extended; the general integral form below will be useful for some of the computations later.

**Definition B.0.8**. For \( b > 0 \), define the polylogarithm of order \( b \) as
\[
\text{Li}_b(z) := \sum_{j=1}^{\infty} \frac{z^j}{j^b} \quad \text{for} \quad |z| < 1
\]
Preserving a Function of the 1-Norm

and

\[ \text{Li}_b(z) := \frac{1}{\Gamma(b)} \int_0^\infty \frac{zt^{b-1}}{e^t - z} \, dt = \frac{1}{\Gamma(b)} \int_0^\infty \frac{zt^{b-1} e^{-t}}{1 - e^{-t} z} \, dt. \]

for \( z \in \mathbb{C} - [1, \infty) \).

To check that the definitions are consistent, note that if \( |z| < 1 \), then \( |e^{-t} z| < 1 \) too, and we may use the geometric series to rewrite:

\[ \text{Li}_b(z) = \frac{1}{\Gamma(b)} \int_0^\infty \frac{z t^{b-1} e^{-t}}{1 - e^{-t} z} \, dt \]

and if we can exchange the integral and the sum,

\[ = \sum_{j=0}^\infty \frac{z^{j+1}}{\Gamma(b)} \int_0^\infty t^{b-1} e^{-t} \sum_{j=0}^\infty e^{-tj} z^j \, dt \]

Now, with \( s/j = t \),

\[ \frac{1}{\Gamma(b)} \int_0^\infty t^{b-1} e^{-tj} \, dt = \frac{1}{\Gamma(b)} \int_0^\infty (s/j)^{b-1} e^{-s} \frac{ds}{j} = \frac{1}{j^b} \frac{1}{\Gamma(b)} \int_0^\infty (s)^{b-1} e^{-s} \, ds = \frac{1}{j^b} \]

so we recover, when \( |z| < 1 \),

\[ \text{Li}_b(z) = \sum_{j=1}^\infty \frac{z^j}{j^b} \]

The nonintegral order polylogarithms also extend to the unit circle when the order is greater than 1.

**Lemma B.0.9.** For \( b > 1 \) and \( z \in \mathbb{C} \) with \( |z| = 1 \),

\[ \text{Li}_b(z) < b. \]

**Proof.** By definition,

\[ \text{Li}_b(z) = \sum_{j=1}^\infty \frac{z^j}{j^b} \]

so that when \( |z| = 1 \), \( |\text{Li}_b(z)| \leq \sum_{j=1}^\infty \frac{|z|^j}{j^b} = \sum_{j=1}^\infty \frac{1}{j^b} \)

The series is finite because \( b > 1 \); concretely, by the integral test (because \( 1/x^b \) is convex),

\[ \sum_{j=1}^\infty \frac{1}{j^b} = 1 + \sum_{j=2}^\infty \frac{1}{j^b} \leq 1 + \int_1^\infty \frac{1}{x^b} \, dx = 1 + (b - 1) \left( \frac{1}{x^{b-1}} \right) |x| = b < \infty. \]
For all $b > 1$, we also have

$$\text{Li}_b(z) = \int_0^z \frac{\text{Li}_{b-1}(w)}{w} \, dw$$

just note from the power series

$$\frac{\text{Li}_{b-1}(z)}{z} = z^{j-1} = \sum_{j=1}^\infty \frac{z^j}{j^b} = \frac{d}{dz} \text{Li}_b(z).$$

Because $1/z$ and $\text{Li}_{b-1}(z)$ analytic on $\mathbb{C} - [1, \infty)$ away from 0, $\text{Li}_{b-1}(z)/z$ is too. Still from the power series, all terms are degree 0 or higher, so $\text{Li}_{b-1}(z)/z$ is also analytic for $|z| < 1$ and hence on all of $\mathbb{C} - [1, \infty)$. Consequently,

$$\int_0^z \frac{\text{Li}_{b-1}(w)}{w} \, dw$$

is analytic there too. Because this integral agrees with $\text{Li}_b(z)$ for $|z| < 1$, analytic continuation dictates it agrees with $\text{Li}_b(z)$ on the full domain $\mathbb{C} - [1, \infty)$.

**Lemma B.0.10.** For $b > 1$, and $0 < x < 1$,

$$\text{Li}_b(x) \leq x \text{Li}_b(1).$$

**Proof.** From the power series,

$$\text{Li}_b(x) = \sum_{j=1}^\infty \frac{x^j}{j^b} = x \sum_{j=1}^{\infty} \frac{x^{j-1}}{j^b} \leq x \sum_{j=1}^{\infty} \frac{1}{j^b} = x \text{Li}_b(1)$$

having used $x^k \leq 1$ for $0 < x < 1$ and $k \geq 0$. □

**Lemma B.0.11.** For $z \in (\mathbb{C} - \mathbb{R}) \cup (-1, 1)$ and $b > 0$,

$$\text{Li}_b(z) + \text{Li}_b(-z) = \frac{1}{2^{b-1}} \text{Li}_b(z^2).$$

If $b > 1$, the equality also holds when $z = \pm 1$.

**Remark B.0.12.** When $b = 1$, recover

$$\ln(1 - z) + \ln(1 + z) = -\left(\text{Li}_1(z) + \text{Li}_1(-z)\right) = -\text{Li}_1(z^2) = \ln(1 - z^2).$$

**Proof.** First assume $|z| < 1$. From the power series,

$$\text{Li}_b(z) + \text{Li}_b(-z) = \sum_{j=1}^{\infty} \frac{z^j + (-z)^j}{j^b} = \sum_{j=1}^{\infty} \frac{z^j + (-1)^j}{j^b} = 2 \sum_{j=1}^{\infty} \frac{z^{2j}}{(2j)^b} = \frac{1}{2^{b-1}} \sum_{j=1}^{\infty} \frac{(z^2)^j}{j^b} = \frac{1}{2^{b-1}} \text{Li}_b(z^2).$$
Both sides are analytic functions on \((\mathbb{C} - \mathbb{R}) \cup (-1, 1)\), so by analytic continuation, the identity continues to hold there. If \(b > 1\), the power series are also defined at \(z = \pm 1\).

A useful property of the polylogarithms and the logarithm that we shall use repeatedly in computations is that they are all symmetric about the real axis, that is, \(\text{Li}_b(z^*) = \text{Li}_b(z)\) or concretely

\[
\Re \text{Li}_b(z^*) = \Re \text{Li}_b(z) \quad \text{and} \quad \Im \text{Li}_b(z^*) = -\Im \text{Li}_b(z).
\]

Powers and polynomials of such functions also have this property. Intuitively this symmetry follows from the real coefficients in their power series expansions, so that \(\text{Li}(x) \in \mathbb{R}\) when \(x < 1\). Rigorously, we use the Schwarz reflection principle; because \(\text{Li}_b(z)\) is analytic in \(\mathbb{C} - [1, \infty)\) when \(0 \leq \arg(z) < \pi\) and real valued on \((-\infty, 1)\), \(\text{Li}_b(z)\) may be extended to the rest of \(\mathbb{C} - [1, \infty)\) in an analytic fashion. Analytic continuation then dictates that this extension coincides with the original definition of \(\text{Li}_b(z)\). See [20] pages 57-59 for the Schwarz reflection principle, page 56 for showing the integral definitions of \(\text{Li}_b(z)\) are analytic, and page 52 for the principle of analytic continuation.

**Lemma B.0.13.** Let \(\lambda > 0\), \(X \sim \text{Cauchy}(1)\) and \(b > -1\). Then

\[
\mathbb{E} \ln^b(1 + \sqrt{\lambda |X|}) = \frac{\Gamma(b+1)}{i\pi} \left( \text{Li}_{b+1}(1 + \sqrt{i\lambda}) + \text{Li}_{b+1}(1 - \sqrt{i\lambda}) \right) - \frac{\Gamma(b+1)}{i\pi} \left( \text{Li}_{b+1}(1 + i\sqrt{\lambda}) + \text{Li}_{b+1}(1 - i\sqrt{\lambda}) \right).
\]

**Proof.** With \(\lambda y = x\), we have

\[
I_b(\lambda) := \mathbb{E} \ln^b(1 + \sqrt{\lambda X}) := \frac{2}{\pi} \int_0^\infty \frac{\ln^b(1 + \sqrt{\lambda y})}{1 + y^2} \, dy = \frac{2\lambda}{\pi} \int_0^\infty \frac{\ln^b(1 + \sqrt{x})}{1 + \lambda^{-2} x^2} \, dx
\]

If \(u^2 = x\) so that \(2u \, du = dx\),

\[
I_b(\lambda) = \frac{2\lambda}{\pi} \int_0^\infty \frac{\ln^b(1 + u)}{\lambda^2 + u^4} \, 2u \, du = \frac{2\lambda}{\pi} \int_0^\infty \frac{\ln^b(1 + u)}{(u^2 + i\lambda)(u^2 - i\lambda)} \, du
\]

Now

\[
\left( \frac{1}{u - i\sqrt{i\lambda}} - \frac{1}{u + i\sqrt{i\lambda}} \right) = \frac{2i\sqrt{i\lambda}}{u^2 + i\lambda}
\]
So we can write
\[ I_b(\lambda) = \frac{\sqrt{\lambda}}{i\pi \sqrt{i}} \int_0^\infty \ln^b(1 + u) \left( \frac{1}{u - i\sqrt{i\lambda}} - \frac{1}{u + i\sqrt{i\lambda}} \right) \left( \frac{1}{(u - \sqrt{i\lambda}) + (u + \sqrt{i\lambda})} \right) \, du, \]
and with \( v = u + 1, \)
\[ I_b(\lambda) = \frac{\sqrt{\lambda}}{i\pi \sqrt{i}} \int_1^\infty \ln^b(v) \left( \frac{1}{v - 1 - i\sqrt{i\lambda}} - \frac{1}{v - 1 + i\sqrt{i\lambda}} \right) \left( \frac{1}{(v - 1 - \sqrt{i\lambda}) + (v - 1 + \sqrt{i\lambda})} \right) \, dv \]
upon setting \( a_\pm := 1 \pm i\sqrt{i\lambda} \) and \( c_\pm := 1 \pm \sqrt{i\lambda}. \)
So
\[ I_b(\lambda) = E \ln^b(1 + \sqrt{\lambda}X) = [a_+, c_+] - [a_-, c_] + [a_-, c_+] - [a_+, c_-] \]
with
\[ \frac{z_1, z_2}{\ln^b(v) (v - z_1)(v - z_2)} = \frac{\sqrt{\lambda} \Gamma(b + 1)}{(z_1 - z_2)i\pi \sqrt{i}} (\text{Li}_{b+1}(z_1) - \text{Li}_{b+1}(z_2)). \]
by lemma B.0.14.
Compute,
\[ \frac{[a_+, c_+] - [a_-, c_+]}{[a_-, c_+] - [a_+, c_-]} = \frac{\Gamma(b + 1) \sqrt{\lambda}}{i\pi \sqrt{i}} \left( \frac{\text{Li}_{b+1}(a_+) - \text{Li}_{b+1}(c_+)}{(i - 1)\sqrt{i\lambda}} - \frac{\text{Li}_{b+1}(a_-) - \text{Li}_{b+1}(c_-)}{-(i + 1)\sqrt{i\lambda}} \right) \]
\[ = \frac{\Gamma(b + 1)}{i^2 \pi} \left( \frac{\text{Li}_{b+1}(a_+) - \text{Li}_{b+1}(c_+)}{(i - 1)} - \frac{\text{Li}_{b+1}(a_-) - \text{Li}_{b+1}(c_-)}{-(i + 1)} \right) \]
\[ = \frac{\Gamma(b + 1)}{i^2 \pi} \left( \frac{\text{Li}_{b+1}(a_+) - \text{Li}_{b+1}(c_+)}{(i - 1)} + \frac{\text{Li}_{b+1}(a_-) - \text{Li}_{b+1}(c_-)}{(i + 1)} \right) \]
\[ = \frac{\Gamma(b + 1)}{i^2 \pi (-2)} ((i + 1) \text{Li}_{b+1}(a_+) + (i - 1) \text{Li}_{b+1}(a_-) - 2i \text{Li}_{b+1}(c_+)) \]
and similarly
\[
\begin{align*}
&= \frac{\Gamma(b + 1)\sqrt{\lambda}}{i\pi\sqrt{i}} \left( \frac{\text{Li}_{b+1}(a_+) - \text{Li}_{b+1}(c_-)}{(i + 1)\sqrt{i\lambda}} - \frac{\text{Li}_{b+1}(a_-) - \text{Li}_{b+1}(c_-)}{-(i - 1)\sqrt{i\lambda}} \right) \\
&= \frac{\Gamma(b + 1)}{i\pi} \left( \frac{\text{Li}_{b+1}(a_+) - \text{Li}_{b+1}(c_-)}{(i + 1)} + \frac{\text{Li}_{b+1}(a_-) - \text{Li}_{b+1}(c_-)}{(i - 1)} \right) \\
&= \frac{\Gamma(b + 1)}{i^2\pi(-2)} \left( (i - 1) \text{Li}_{b+1}(a_+) + (i + 1) \text{Li}_{b+1}(a_-) - 2i \text{Li}_{b+1}(c_-) \right).
\end{align*}
\]

Thus,
\[
\mathbb{E} \ln^b(1 + \sqrt{\lambda X}) = \frac{\Gamma(b + 1)(2i)}{i^2\pi(-2)} \left( \text{Li}_{b+1}(a_+) + \text{Li}_{b+1}(a_-) - \text{Li}_{b+1}(c_-) \right)
\]
\[
= \frac{\Gamma(b + 1)}{i\pi} \left( \text{Li}_{b+1}(c_+) + \text{Li}_{b+1}(c_-) - \text{Li}_{b+1}(a_+) - \text{Li}_{b+1}(a_-) \right)
\]
as desired. \qed

**Lemma B.0.14.** Let \( z_1, z_2 \in \mathbb{C} - [1, \infty) \) with \( z_1 \neq z_2 \), and \( b > -1 \). Then
\[
\int_1^{\infty} \frac{\ln^b(v)}{(v - z_1)(v - z_2)} \, dv = \frac{\Gamma(b + 1)}{z_1 - z_2} (\text{Li}_{b+1}(z_1) - \text{Li}_{b+1}(z_2)).
\]

**Proof.** Because
\[
\frac{z_1}{v - z_1} - \frac{z_2}{v - z_2} = \frac{z_1 v - z_1 z_2 - z_2 v + z_1 z_2}{(v - z_1)(v - z_2)} = \frac{(z_1 - z_2)v}{(v - z_1)(v - z_2)},
\]
we can write, with \( w = \ln(v) \) so that \( dw = dv/v \),
\[
\int_1^{\infty} \frac{\ln^b(v)}{(v - z_1)(v - z_2)} \, dv = \frac{1}{z_1 - z_2} \int_1^{\infty} \frac{\ln^b(v)}{v - z_1} \left( \frac{z_1}{v - z_1} - \frac{z_2}{v - z_2} \right) \, dv
\]
\[
= \frac{1}{z_1 - z_2} \int_0^{\infty} \left( \frac{z_1 u^b}{e^w - z_1} - \frac{z_2 u^b}{e^w - z_2} \right) \, dw
\]
\[
= \frac{\Gamma(b + 1)}{z_1 - z_2} (\text{Li}_{b+1}(z_1) - \text{Li}_{b+1}(z_2)).
\]
\qed
B.1. Arctan and the Inverse Tangent Integrals

From lemma 2.1.6, we have seen \( \arctan(t) \) is proportional to the distribution function for the standard Cauchy distribution. It is then perhaps not surprising that \( \arctan \) and its relatives arise in working with functions of Cauchy random variables. We outline the properties we shall be using here.

The following definition is opaque but most useful to us.

**Definition B.1.1.** Define \( \arctan(z) \) as

\[
\arctan(z) := \sum_{j=0}^{\infty} (-1)^j \frac{z^{2j+1}}{2j+1} \text{ for } |z| < 1,
\]

and

\[
\arctan(z) = \int_{0}^{z} \frac{1}{1 + w^2} \, dw \text{ for } z \in (C - i\mathbb{R}) \cup (-i, i).
\]

Equivalently,

\[
\arctan(z) := \frac{1}{2i}(\ln(1 + iz) - \ln(1 - iz)) = \frac{1}{2i}(\text{Li}_1(iz) - \text{Li}_1(-iz)).
\]

**Remark B.1.2.** The function \( \arctan \) is related to the usual tangent function as follows. On \((-\pi/2, \pi/2)\), recall \( \tan(\theta) \) is strictly monotone increasing, so its inverse function is well-defined:

\[
\frac{d}{d\theta} \tan(\theta) = \frac{d}{d\theta} \frac{\sin(\theta)}{\cos(\theta)} = \frac{1}{\cos^2(\theta)}(\cos^2(\theta) - (-\sin^2(\theta))) = \sec^2(\theta) = 1 + \tan^2(\theta) \geq 1.
\]

Note \( |\tan(\theta)| \to \infty \) as \( |\theta| \to \pi/2 \). We can thus define \( \arctan : \mathbb{R} \to (-\pi/2, \pi/2) \) by

\[
\arctan(\tan(\theta)) = \theta.
\]

Take the derivative to find

\[
1 = \arctan'(\tan(\theta))(1 + \tan^2(\theta))
\]

\[
\frac{1}{1 + \tan^2(\theta)} = \arctan'(\tan(\theta))
\]

so that with \( x = \tan(\theta) \),

\[
\frac{1}{1 + x^2} = \arctan'(x) \text{ or } \int_{0}^{r} \frac{1}{1 + x^2} = \arctan(r)
\]

by the fundamental theorem of calculus, noting that \( \arctan(0) = 0 \). The definition B.1.1 here, thus agrees with what one would expect for \( \arctan \) on \([0, \infty)\). It also shows \( \arctan \)
is an analytic function on \((\mathbb{C} - i\mathbb{R}) \cup (-i, i)\) as this domain is simply connected and \((1 + w^2)^{-1}\) is analytic there. The power series for \(\arctan(z)\) then follows by considering any path from 0 to \(z\) contained in the interior of the unit disk. The integrand \((1 + w^2)^{-1}\) may then be expressed as a geometric series:

\[
\arctan(z) = \int_{0}^{z} \frac{1}{1 + w^2} \, dw = \int_{0}^{z} \sum_{j=0}^{\infty} (-w^2)^j \, dw = \sum_{j=0}^{\infty} (-1)^j \int_{0}^{z} w^{2j} \, dw = \sum_{j=0}^{\infty} (-1)^j \frac{z^{2j+1}}{2j+1}.
\]

From the integral formulation, we also immediately have, with \(v = -w\),

\[
\arctan(-z) = \int_{0}^{-z} \frac{1}{1 + w^2} \, dw = -\int_{0}^{z} \frac{1}{1 + v^2} \, dv = -\arctan(z).
\]

The last definition for \(\arctan(z)\) follows from

\[
\frac{d}{dz} \frac{1}{2i} \left( \ln(1 + iz) - \ln(1 - iz) \right) = \frac{1}{2i} \left( \frac{i}{1 + iz} - \frac{(-i)}{1 - iz} \right) = \frac{1}{2} \left( \frac{1}{1 + iz} + \frac{1}{1 - iz} \right) = \frac{1}{1 + z^2} = \frac{d}{dz} \arctan(z)
\]

and that \(\arctan(0) = 0\).

We can generalize.

**Definition B.1.3.** For \(z \in \mathbb{C} - i\mathbb{R} \cup (-i, i)\) and \(b > 0\), define the inverse tangent integral of order \(b\) as

\[
T_{ib}(z) := \sum_{j=0}^{\infty} (-1)^j \frac{z^{2j+1}}{(2j+1)^b} \quad \text{for} \quad |z| < 1
\]

and

\[
T_{ib}(z) = \frac{L_{ib}(iz) - L_{ib}(-iz)}{2i} \quad \text{for} \quad z \in \mathbb{C} - i\mathbb{R} \cup (-i, i).
\]

**Remark B.1.4.** Note if \(|y| < 1\), we find

\[
L_{ib}(iy) - L_{ib}(-iy) = \sum_{j=1}^{\infty} \frac{(iy)^j - (-iy)^j}{j^b} = \sum_{j=1}^{\infty} \frac{y^j}{j^b} (1 - (-1)^j) = 2 \sum_{j=0}^{\infty} \frac{y^{2j+1}}{(2j+1)^b} = 2i \sum_{j=0}^{\infty} \frac{(-1)^j y^{2j+1}}{(2j+1)^b} =: 2i T_{ib}(y) \in i\mathbb{R}.
\]
Hence,

\[ Ti_b(y) = \frac{L_i(by) - L_i(-iy)}{2i} \]

when \(|y| < 1\) and \(b > 0\). The right hand side continues to make sense for \(y \in (\mathbb{C} - i\mathbb{R}) \cup (-i, i)\), so we may define

\[ Ti_b(z) := \frac{L_i(iz) - L_i(-iz)}{2i} \]

as an analytic function on \(z \in (\mathbb{C} - i\mathbb{R}) \cup (-i, i)\) that agrees with the power series on the interior of the unit circle.

**Remark B.1.5.** In particular, we have

\[ Ti_1(z) = \arctan(z). \]

and for \(b > 1\),

\[ Ti_b(z) := \int_0^z \frac{Ti_{b-1}(w)}{w} \, dw. \]

To see the latter, note that \(Ti_b(0) = 0\) from the power series, while differentiating yields

\[
\frac{d}{dz} Ti_b(z) = \frac{d}{dz} \left( \frac{L_i(iz) - L_i(-iz)}{2i} \right) = \frac{1}{2i} \left( \frac{L_i(iz)}{z} - \frac{L_i(-iz)}{z} \right) = \frac{1}{z} Ti_{b-1}(z).
\]

These formulas also follow from lemma B.1.8, noting that

\[ Ti_b(z) = \frac{1}{i} \chi_b(iz) \]

which we introduce further below.

To focus on the behavior of \(\arctan\) on \((-i, i)\) which was not addressed in the inversion formula B.3.1, we change points of view through a rotation of the complex plane.

**Definition B.1.6.** Define the function \(\text{atanh}\) as

\[ \text{atanh}(x) = \sum_{j=0}^{\infty} \frac{x^{2j+1}}{2j+1} \quad \text{for} \quad |x| < 1, \]

and as

\[ \text{atanh}(z) = \int_0^z \frac{1}{1-w^2} \, dw = -i \arctan(iz) \quad \text{for} \quad z \in (\mathbb{C} - \mathbb{R}) \cup (-1, 1). \]

or equivalently as

\[ \text{atanh}(z) = \frac{1}{2} (\ln(1+z) - \ln(1-z)) = \frac{1}{2} (L_1(z) - L_1(-z)). \]
To see that the definitions are consistent, note first from the power series, \( \text{atanh}(0) = 0 = \arctan(0) \), while on the other hand,

\[
\frac{d}{dz}(-i) \arctan(iz) = \frac{(-i)}{1 + (iz)^2} = \frac{1}{1 - z^2} = \frac{d}{dz} \text{atanh}(z).
\]

Of course, we can generalize,

**Definition B.1.7.** Define \( \chi_b, \) the Legendre \( \chi \) function of order \( b > 0 \), as

\[
\chi_b(z) = \sum_{j=1}^{\infty} \frac{z^j}{j^b} \quad \text{for} \quad |z| < 1
\]

and

\[
\chi_b(z) = \frac{1}{2} (\text{Li}_b(z) - \text{Li}_b(-z)) \quad \text{for} \quad z \in (\mathbb{C} - \mathbb{R}) \cup (-1, 1)
\]

In particular, \( \chi_1(z) = \text{atanh}(z) \).

**Lemma B.1.8.** For \( b > 1 \) and \( z \in (\mathbb{C} - \mathbb{R}) \cup (-1, 1) \),

\[
\chi_b(z) = \int_0^z \frac{\chi_{b-1}(w)}{w} \, dw.
\]

**Proof.** By definition,

\[
\int_0^z \frac{\chi_{b-1}(w)}{w} \, dw = \frac{1}{2} \int_0^z \frac{\text{Li}_{b-1}(w) - \text{Li}_{b-1}(-w)}{w} \, dw
\]

\[
= \frac{1}{2} \int_0^z \frac{\text{Li}_{b-1}(w)}{w} \, dw - \frac{1}{2} \int_0^z \frac{\text{Li}_{b-1}(-w)}{w} \, dw
\]

and with \( v = -w \),

\[
= \frac{1}{2} \text{Li}_b(z) - \frac{1}{2} \int_0^{-z} \frac{\text{Li}_{b-1}(v)}{-v} (-dv) = \frac{1}{2} \text{Li}_b(z) - \frac{1}{2} \int_0^{-z} \frac{\text{Li}_{b-1}(v)}{v} \, dv
\]

\[
= \frac{1}{2} \text{Li}_b(z) - \frac{1}{2} \text{Li}_b(-z) = \chi_b(z).
\]

**Lemma B.1.9.** Let \( z \in (\mathbb{C} - \mathbb{R}) \cup (-1, 1) \) then

\[
\ln(1 + z) = \text{atanh}(z) + \frac{1}{2} \ln(1 - z^2)
\]

and for \( b > 0 \),

\[
\text{Li}_b(z) = \chi_b(z) + \frac{1}{2^b} \text{Li}_b(z^2)
\]
**Proof.** Just split into even and odd degree terms.

\[
\text{Li}_b(z) = \sum_{j=1}^{\infty} \frac{z^j}{j^b} = \sum_{j=0}^{\infty} \frac{z^{2j+1}}{(2j+1)^b} + \sum_{j=1}^{\infty} \frac{z^{2j}}{(2j)^b} = \chi_b(z) + \frac{1}{2b} \sum_{j=1}^{\infty} \frac{(z^2)^j}{j^b} = \chi_b(z) + \frac{1}{2b} \text{Li}_b(z^2).
\]

The equality extends to \((\mathbb{C} - \mathbb{R}) \cup (-1, 1)\) as both sides are analytic there.

When \(b = 1\), we have

\[
\ln(1 + z) = -\text{Li}_b(-z) = -\text{atanh}(-z) + \frac{1}{2} \ln(1 - (-z)^2) = \text{atanh}(z) + \frac{1}{2} \ln(1 + z^2)
\]
as desired. \(\square\)

A useful property of \text{atanh} is

**Lemma B.1.10.** For \(0 \leq u < 1\),

\[
\text{atanh}(u) \leq \frac{u}{1 - u^2}
\]

**Proof.** From the power series,

\[
\text{atanh}(u) = \sum_{j=0}^{\infty} \frac{u^{2j+1}}{2j+1} = u \sum_{j=0}^{\infty} \frac{(u^2)^j}{2j+1} \leq u \sum_{j=0}^{\infty} (u^2)^j = \frac{u}{1 - u^2}.
\]

\(\square\)

Here is the addition formula.

**Lemma B.1.11 (Atanh Addition Formula).** If \(-1 < x, y < 1\),

\[
\text{atanh}(x) + \text{atanh}(y) = \text{atanh} \left( \frac{x + y}{1 + xy} \right).
\]

If \(z \in \mathbb{C} - \mathbb{R}\),

\[
\text{atanh}(z) + \text{atanh}(z^*) = \text{atanh} \left( \frac{2 \Re(z)}{1 + |z|^2} \right)
\]

**Proof.** Because \text{atanh} is odd, the addition formula also covers subtraction via

\[
-\text{atanh}(y) = \text{atanh}(-y).
\]

Recall

\[
\frac{d}{dz} \text{atanh}(z) = \frac{1}{1 - z^2}.
\]
Preserving a Function of the 1-Norm

while

\[
\frac{d}{dz} \text{atanh} \left( \frac{z + w}{1 + zw} \right) = \left( 1 - \left( \frac{z + w}{1 + zw} \right)^2 \right)^{-1} \left( \frac{(1 + zw) - (z + w)w}{(1 + zw)^2} \right)
= ((1 + zw)^2 - (z + w)^2)^{-1} (1 - w^2)
= (1 + 2zw + (zw)^2 - z^2 - w^2 - 2zw)^{-1} (1 - w^2)
= \frac{(1 - w^2)}{(1 - z^2)(1 - w^2)} = \frac{1}{1 - z^2}
\]

so

\[
\text{atanh} \left( \frac{z + w}{1 + zw} \right) = \text{atanh}(z) + c
\]

with \(c\) a constant. Taking \(z = 0\) forces \(c = \text{atanh}(w)\) as desired.

For \(z, w \in (\mathbb{C} - \mathbb{R}) \cup (-1, 1)\), let

\[
f(z, w) := \frac{z + w}{1 + zw}.
\]

We want to know when \(f(z, w)\) also lies in the domain of \text{atanh}. When \(w = z^*\),

\[
\frac{z + z^*}{1 + zz^*} = \frac{2\Re(z)}{1 + |z|^2} = \frac{2\Re(z)/|z|}{1/|z| + |z|}
\]

so that

\[
\left| \frac{z + z^*}{1 + zz^*} \right| \leq \frac{2 |\Re(z)/|z||}{1/|z| + |z|} \leq \frac{2}{|z| + |z|} \leq \frac{1}{\sqrt{|z|/|z|}} = 1.
\]

by the AM-GM inequality. The equality case occurs just if \(|z| = 1\), but in that case,

\[
\left| \frac{\Re(z)}{|z|} \right| < 1
\]

as \(z = \pm 1\) is not allowed for \text{atanh}. We are thus ok for all \(z \in (\mathbb{C} - \mathbb{R}) \cup (-1, 1)\) in this \(w = z^*\) case.

When \(x, y \in (-1, 1)\), we may consider

\[
\partial_x f(x, y) = \frac{1}{1 + xy} - \frac{(x + y)(1 + y^2)}{(1 + xy)^2(1 + xy - xy - y^2)} = \frac{1 - y^2}{(1 + xy)^2} > 0
\]

and by symmetry,

\[
\partial_y f(x, y) = \frac{1 - x^2}{(1 + xy)^2} > 0.
\]
So $f$ is increasing in each of the individual coordinates. In particular, when $-1 < x < y < 1$,

$$\frac{2x}{1 + x^2} = f(x, x) < f(x, y) < f(y, y) = \frac{2y}{1 + y^2}.$$ 

If $0 \leq x < y$, then we have

$$0 \leq f(x, y) < \frac{2y}{1 + y^2} < \frac{y}{y} = 1$$

and if $x < y \leq 0$, we have

$$0 \leq |f(x, y)| < \frac{2|x|}{1 + x^2} < \frac{|x|}{|x|} = 1$$

and finally if $x < 0 < y$, with $t = \max\{|x|, |y|\}$,

$$|f(x, y)| < \frac{2t}{1 + t^2} < 1$$

all by the AM-GM inequality, with strict inequality because $|x|, |y| < 1$.

\hfill \Box

### B.2. Dilogarithm Properties

The dilogarithm is the polylogarithm of order 2.

**Lemma B.2.1** (Reflection Formula). For $z \in (\mathbb{C} - \mathbb{R}) \cup (0, 1)$,

$$\text{Li}_2(z) + \text{Li}_2(1 - z) - \text{Li}_2(1) = -\ln(z)\ln(1 - z).$$

**Proof.** (Compare to [12, page 5].) Consider

$$\frac{d}{dz}(\text{Li}_2(z) + \text{Li}_2(1 - z)) = \frac{\text{Li}_1(z)}{z} + \frac{\text{Li}_1(1 - z)}{1 - z}(-1) = \frac{-\ln(1 - z)}{z} + \frac{\ln(z)}{1 - z}. $$

On the other hand,

$$\frac{d}{dz}(-\ln(z)\ln(1 - z)) = \frac{-\ln(1 - z)}{z} + \frac{\ln(z)}{1 - z}.$$  

Because the domain $(\mathbb{C} - \mathbb{R}) \cup (0, 1)$ is simply connected and the derivative above is analytic there, we have

$$-\ln(z)\ln(1 - z) + \ln(z_0)\ln(1 - z_0) = \text{Li}_2(z) + \text{Li}_2(1 - z) - (\text{Li}_2(z_0) + \text{Li}_2(1 - z_0))$$

for some $z_0$ which we may take to lie on $(0, 1)$. Taking the limit as $z_0 \to 0$ is safe, as the Taylor series for $\ln(1 - z_0)$ ensures $\ln(z_0)\ln(1 - z_0) \to 0$, while the dilogarithm is continuous on $(-\infty, 1]$. Hence,

$$-\ln(z)\ln(1 - z) = \text{Li}_2(z) + \text{Li}_2(1 - z) - \text{Li}_2(1)$$

as desired. Note that proving the identity via integration by parts has to make this same limiting argument.  

\hfill \Box
B.3. Inversion Formulas

The following lemma allows us to describe the survival function of $|X|$ with $X \sim \text{Cauchy} (1)$ in a convenient way. Note that the survival function for $|X|$ will only consider $z = x > 0$.

**Lemma B.3.1.** For $z \in \mathbb{C} - i\mathbb{R}$,

$$\arctan(z) + \arctan\left(\frac{1}{z}\right) = \begin{cases} \pi/2 & \text{if } \Re(z) > 0 \\ -\pi/2 & \text{if } \Re(z) < 0 \end{cases}.$$

**Remark B.3.2.** On the imaginary axis, $\arctan(ir) = i \text{atanh}(r)$ and atanh is only defined for $r \in (-1, 1)$ so $1/r$ does not make sense there. Consequently the domain in question has two connected components, so different constants should not be unexpected.

**Proof.** First note that the left hand side is a constant

$$\frac{d}{dz} \left( \arctan(z) + \arctan\left(\frac{1}{z}\right) \right) = \frac{1}{1 + z^2} + \frac{1}{1 + \frac{1}{z^2}} = 0.$$

The constant is determined by representative points in the right and left hand planes. From the case $z = 1$,

$$\arctan(1) + \arctan\left(\frac{1}{1}\right) = 2 \arctan(1) = 2 \frac{\pi}{4} = \frac{\pi}{2},$$

and similarly from the case $z = -1$,

$$\arctan(-1) + \arctan\left(\frac{1}{-1}\right) = -2 \arctan(1) = -2 \frac{\pi}{4} = -\frac{\pi}{2}.$$

\[ \square \]

**Lemma B.3.3.** Let $z \in \mathbb{C} - i\mathbb{R}$. If $\Re(z) > 0$,\n
$$\text{T}_2(z) = \text{T}_2\left(\frac{1}{z}\right) + \frac{\pi}{2} \ln(z)$$

and if $\Re(z) < 0$,

$$\text{T}_2(z) = \text{T}_2\left(\frac{1}{z}\right) - \frac{\pi}{2} \ln(-z).$$

**Proof.** By definition,

$$\frac{d}{dz} \text{T}_2(z) = \frac{\arctan(z)}{z}.$$
On the other hand,
\[\frac{d}{dz} \left( T_{i2} \left( \frac{1}{z} \right) \pm \frac{\pi}{2} \ln(\pm z) \right) = \frac{\arctan(z) \left( \frac{-1}{z^2} \right)^{\pm 1} \pm \frac{1}{2} \pm \frac{1}{z} \left( \pm \frac{\pi}{2} - \arctan \left( \frac{1}{z} \right) \right)}{\left( \frac{1}{z} \right)^{\pm 1}}.\]

Now use lemma B.3.1. If \(\Re(z) > 0\),
\[\frac{d}{dz} \left( T_{i2} \left( \frac{1}{z} \right) + \frac{\pi}{2} \ln(z) \right) = \frac{1}{z} \left( \frac{\pi}{2} - \arctan \left( \frac{1}{z} \right) \right) = \frac{1}{z} \arctan(z)\]
and if \(\Re(z) < 0\),
\[\frac{d}{dz} \left( T_{i2} \left( \frac{1}{z} \right) - \frac{\pi}{2} \ln(z) \right) = \frac{1}{z} \left( -\frac{\pi}{2} - \arctan \left( \frac{1}{z} \right) \right) = \frac{1}{z} \arctan(z).\]

So in both cases,
\[T_{i2}(z) = c_\pm + T_{i2} \left( \frac{1}{z} \right) \pm \frac{\pi}{2} \ln(\pm z)\]

Taking \(z = 1\) in the "+" case and \(z = -1\) in the "-" case shows that \(c_\pm = 0\).

\[\square\]

Acknowledgments

This work was supported in part by Duke University while completing my Ph.D. thesis. I should like to thank my advisor Professor Sayan Mukherjee for encouraging me in completing this work. I should like to thank Mom, Dad, Katie, and everyone who has been praying for me throughout my time at Duke. I should finally like to thank the Blessed Virgin Mary, Saint Joseph, and the Holy Trinity for helping me be patient throughout this work.

References

[1] Ailon, N. and Chazelle, B. (2009). The Fast Johnson-Lindenstrauss Transform and Approximate Nearest Neighbors*. SIAM Journal on Computing 39 302–322.
[2] Biau, G. and Devroye, L. (2015). Lectures on the Nearest Neighbor Method. Springer Series in the Data Sciences. Springer International Publishing, Cham. DOI: 10.1007/978-3-319-25388-6.
[3] Brinkman, B. and Charikar, M. (2005). On the Impossibility of Dimension Reduction in \(L_1\). J. ACM 52 766–788.
[4] Chambers, J. M., Mallows, C. L. and Stuck, B. W. (1976). A Method for Simulating Stable Random Variables. Journal of the American Statistical Association 71 340–344.
[5] Corazza, P. (1999). Introduction to Metric-Preserving Functions. The American Mathematical Monthly 106 309–323.
Preserving a Function of the 1-Norm

[6] Drineas, P., Magdon-Ismail, M., Mahoney, M. W. and Woodruff, D. P. (2012). Fast Approximation of Matrix Coherence and Statistical Leverage. Journal of Machine Learning Research 13 32. arXiv: 1109.3843.

[7] Drineas, P. and Mahoney, M. W. (2016). RandNLA: Randomized Numerical Linear Algebra. Communications of the ACM 59 80–90.

[8] Indyk, P. (2006). Stable Distributions, Pseudorandom Generators, Embeddings, and Data Stream Computation. J. ACM 53 307–323.

[9] Johnson, W. B. and Lindenstrauss, J. (1984). Extensions of Lipschitz Mappings into a Hilbert Space. Contemporary Mathematics 26 189–206.

[10] Ledoux, M. and Talagrand, M. (1991). Probability in Banach Spaces. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)] 23. Springer-Verlag, Berlin. MR1102015

[11] Lee, J. R. and Naor, A. (2004). Embedding the Diamond Graph in $L_p$ and Dimension Reduction in $L_1$. Geometric & Functional Analysis GAFA 14 745–747. arXiv: math/0407520.

[12] Lewin, L. (1981). Polylogarithms and Associated Functions. North-Holland Publishing Co., New York-Amsterdam. MR618278

[13] Li, P., Hastie, T. J. and Church, K. W. (2007). Nonlinear Estimators and Tail Bounds for Dimension Reduction in $l_1$ Using Cauchy Random Projections. Journal of Machine Learning Research (JMLR) 8 2497–2532. arXiv: cs/0610155. MR2353840

[14] Matoušek, J. (2008). On Variants of the Johnson–Lindenstrauss Lemma. Random Structures and Algorithms 33 142–156.

[15] Maximon, L. C. (2003). The Dilogarithm Function for Complex Argument. Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences 459 2807–2819.

[16] Milman, V. D. and Schechtman, G. (1986). Asymptotic Theory of Finite-Dimensional Normed Spaces. Lecture Notes in Mathematics 1200. Springer-Verlag, Berlin With an appendix by M. Gromov. MR856576

[17] Needham, T. (1997). Visual Complex Analysis. The Clarendon Press, Oxford University Press, New York. MR1446490

[18] Philips, T. K. and Nelson, R. (1995). The Moment Bound is Tighter than Chernoff’s Bound for Positive Tail Probabilities. The American Statistician 49 175–178. MR1347726

[19] Pisier, G. (1983). On the Dimension of the $l^n_p$-Subspaces of Banach Spaces, for $1 \leq p < 2$. Transactions of the American Mathematical Society 276 201–211. MR684503

[20] Stein, E. M. and Shakarchi, R. (2003). Complex analysis. Princeton Lectures in Analysis 2. Princeton University Press, Princeton, NJ. MR1976398

[21] Vempala, S. S. (2004). The Random Projection Method. DIMACS Series in Discrete Mathematics and Theoretical Computer Science 65. American Mathematical Society, Providence, RI. MR2073630