Variational Constraints on Masses and Radii of $B_c$ Mesons along with Properties of $B_c$ Mesons

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Abstract

Within the non-relativistic quark model framework, the spectrum, radii, radial wave function at origin and decay constants for $B_c$ mesons are derived through finding the numerical solution of the Schrodinger equation by shooting method. Momentum width is determined for $B_c$ meson states with angular momentum quantum numbers and radii. Masses of $B_c$ meson states are calculated through constraints on mass and $r_{rms}$ derived with uncertainty and variational principles. Results are compared with the experimentally observed and theoretically available results. These results have implications for scalar form factors and leptonic decays of $B_c$ mesons.

I. Introduction

$B_c$ meson, with beauty(b) and charm(c) quarks, is discovered in 1998 at Fermilab in Collider Detector [1]. Its mass lies in between charmonium and bottomonium mesons. It is the most important meson for understanding of Quantum Chromodynamics (QCD) due to its different flavoured heavy quark-antiquark pair. Experimentally, only two $B_c$ meson states ($B_c(1S),B_c(2S)$) are discovered with mass $6.2749\pm0.0008$ GeV and $6.842\pm0.004\pm0.005$ GeV respectively. Theoretically, $B_c$ mesons have been studied through quark potential model [2, 3, 4, 5, 6, 7, 8, 9], QCD sum rule [10, 11, 12, 13], the heavy quark effective theory [14] and lattice QCD [15, 16, 17].

A variety of numerical techniques are available in the literature to solve the Schrodinger equation. In this paper, Schrodinger equation with non-relativistic potential model is solved numerically for $B_c$ meson using Born Openheimer formalism and adiabatic approximation for ground, radially and orbitally excited states of $B_c$ meson. This numerical solution is used to find mass, radial wave function at origin, decay constant, root mean square radii and momentum width.

In Ref.[18], variational principle is combined with uncertainty principle to derive the constraints to the mass of $c\bar{c}$ meson with same flavour of quark and antiquark. In this paper, work is extended for $B_c$ mesons with different flavour of quark and antiquark.

Potential model used for conventional mesons is discussed in the section II of this paper which was further used to calculate radial wave functions for the ground and radially excited state $B_c$ mesons by solving the Schrödinger equation numerically. The expressions used to find masses, radial wave function at origin, decay constant, root mean square radii and momentum width of $B_c$ mesons are also written in section II. In section III, a more simpler technique, developed by combining uncertainty principle and variational principle, is used to calculate the mass of ground state of $B_c$ meson while the results are discussed in section IV.

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II. Conventional $B_c$ mesons

0.1 Schrödinger Equation

Properties of mesons can be derived by solving the Schrödinger equation:

$$H \Psi = E \Psi \tag{1}$$

where $H$ is the energy operator known as Hamiltonian and $E$ is the total energy of the system. Hamiltonian can be defined as:

$$H = H_T + H_V \tag{2}$$

$H_T$ is the kinetic energy part of the Hamiltonian and is defined as:

$$H_T = \frac{P^2}{2\mu} \tag{3}$$

with $P$ as the momentum and $\mu$ as the reduced mass of the quark-antiquark system. Potential energy part of the Hamiltonian for mesons is modelled as \[19\] :

$$H_V = V(r) = H^{\text{conf}} + H^{\text{cont}} + H^{\text{tens}} + H^{s.o}, \tag{4}$$

Here

$$H^{\text{conf}} = \frac{-4\alpha_s}{3r} + br, \tag{5}$$

$$H^{\text{cont}} = \frac{32\pi\alpha_s}{9m_qm_\sigma} (\sigma \sqrt{\pi})^3 e^{-\sigma r^2} S_qS_\bar{q}, \tag{6}$$

$$H^{\text{tens}} = \frac{4\alpha_s}{m_qm_\sigma^3} S_T, \tag{7}$$

$$H^{s.o} = \left(\frac{S_q + S_\bar{q}}{4m_q^2} \right) L \left(\frac{4\alpha_s}{3r^3} - \frac{b}{r} \right) + \frac{S_q + S_\bar{q}}{2m_qm_\sigma} L \frac{4\alpha_s}{3r^3}. \tag{8}$$

In $H^{\text{conf}}$, first term describes coulomb like interaction while the second one is due to linear confinement. $H^{\text{cont}}$, $H^{\text{tens}}$, and $H^{s.o}$ describe the colour contact, colour tensor, and spin orbit interactions respectively. $\alpha_s$, $b$ and $S_T$ are the strong coupling constants, string tension and the tensor operator respectively. $S_T$ is defined as:

$$S_T = S_q\hat{r}S_\bar{q}\hat{r} - \frac{1}{3} S_qS_\bar{q}, \tag{9}$$

such that

$$\langle L, J | S_T | L, J \rangle = \begin{cases} -\frac{1}{6(2L+3)}, & J = L + 1 \\ \frac{1}{6}, & J = L \\ -\frac{1}{6(2L-1)}, & J = L - 1. \end{cases} \tag{10}$$

$$\overrightarrow{L} \cdot \overrightarrow{S} = \frac{J(J+1) - L(L+1) - S(S+1)}{2}, \tag{11}$$

Here, $L$ is the relative orbital angular momentum of the quark-antiquark and $S$ is the total spin angular momentum. $H^{s.o}$ and $H^{tens}$ are equal to zero \[20\] for $L = 0$, where in the eq.(6)

$$S_q, S_\bar{q} = \frac{S(S+1)}{2} - \frac{3}{4}, \quad \text{and } m_q \text{ is the constituent mass of the quarks.}$$
To find the mass of $B_c$ mesons, numerical solution of the radial Schrödinger equation:

$$U''(r) + 2\mu(E - H_V - \frac{L(L+1)}{2\mu r^2})U(r) = 0.$$  \hspace{1cm} (12)

is found by using the shooting method. Here $U(r) = rR(r)$, product of interquark distance $r$ and the radial wave function $R(r)$. The parameters $m_b = 4.825$ GeV, $m_c = 1.4794$ GeV are taken from [21, 22] while parameters ($\alpha_s, b, \sigma$) are found by fitting the meson’s mass with experimentally known mass, we got the following values. $\alpha_s = 0.4791$, $\sigma = 1.0999$ GeV, $b = 0.1371$ GeV$^2$. Mass of a $b\bar{c}$ state is obtained after the addition of constituent quark masses in the energy $E$. Radial wave function of $^1S_0$, $^3S_1$, $^3P_2$ and $^3P_0$ are shown in Fig. (1-4) for first three radially excited states.

### 0.3 Mixed States

Mesons having same mass of quark and antiquark satisfy the parity and charge conservation laws. But mesons with different mass of quark and antiquark do not satisfy the charge conservation law. $B_c$ mesons with different flavoured quark-antiquark are not eigenstates of the charge conjugation. So the states with same $J$ and $P$, but with different $S$ can mix. $B_c(^1P_1)$ and $B_c(^3P_1)$ are the states with same $J$ and $P$.

\begin{align*}
|P\rangle &= \cos \phi_M |^1P_1\rangle + \sin \phi_M |^3P_1\rangle, \\
|P'\rangle &= -\sin \phi_M |^1P_1\rangle + \cos \phi_M |^3P_1\rangle,
\end{align*}

(13) \hspace{1cm} (14)

where $\theta_M$ is the mixing angle. In heavy quark limit $\phi_M^0 = \tan^{-1}\left(\sqrt{\frac{L}{L+1}}\right)$. For $L = 1$, $\phi_M^0 = 35.3^\circ$. For $L = 2$, $\phi_M^0 = 39.2^\circ$. Mass for mixed states $P$, $P'$ and $D, D'$ are reported in Table 1.

### 0.4 Radial wave function at origin

For normalized wave function

$$U'(0) = R(0) = \sqrt{4\pi}\psi(0).$$  \hspace{1cm} (15)

$U'(0)$ is calculated to find the radial wave function at origin whose magnitudes are reported in Table 2. For the states with $L > 0$, wave function becomes zero at the origin.
Table 1: Masses of ground, radially, and orbitally excited state $B_c$ mesons. Calculated masses are rounded to 0.0001 GeV.

| Meson State | $J^P$ | My Calculated Mass | Theo. Mass | Theor. Mass | Exp. Mass |
|-------------|------|-------------------|------------|-------------|-----------|
|             |      | GeV               | [2]        | [27] Model  | [29]      |
| $(1^3S_1)$  | 1−   | 6.3154            | 6.314      | 6.326       | 6.27 ± 0.0008 |
| $(1^1S_0)$  | 0−   | 6.2749            | 6.274      | 6.271       |           |
| $(2^3S_1)$  | 1−   | 6.8558            | 6.855      | 6.890       |           |
| $(2^1S_0)$  | 0−   | 6.8424            | 6.841      | 6.871       | 6.842 ± 0.004 ± 0.005 |
| $(3^3S_1)$  | 1−   | 7.2069            | 7.206      | 7.252       |           |
| $(3^1S_0)$  | 0−   | 7.1979            | 7.197      | 7.239       |           |
| $(4^3S_1)$  | 1−   | 7.4962            | 7.495      | 7.550       |           |
| $(4^1S_0)$  | 0−   | 7.4893            | 7.488      | 7.540       |           |
| $(1^3P_2)$  | 2+   | 6.7508            | 6.753      | 6.787       |           |
| $(1P_1^1)$  | 1+   | 6.7436            | 6.744      | 6.776       |           |
| $(1P_1)$    | 1+   | 6.7263            | 6.7271     | 6.757       |           |
| $(1^3P_0)$  | 0+   | 6.6926            | 6.701      | 6.714       |           |
| $(2^3P_2)$  | 2+   | 7.1132            | 7.111      | 7.160       |           |
| $(2P_1^1)$  | 1+   | 7.0648            | 7.098      | 7.150       |           |
| $(2P_1)$    | 1+   | 7.1057            | 7.0947     | 7.134       |           |
| $(2^3P_0)$  | 0+   | 7.0936            | 7.086      | 7.107       |           |
| $(3^3P_2)$  | 2+   | 7.4098            | 7.406      | 7.464       |           |
| $(3P_1^1)$  | 1+   | 7.3656            | 7.393      | 7.458       |           |
| $(3P_1)$    | 1+   | 7.4030            | 7.4039     | 7.441       |           |
| $(3^3P_0)$  | 0+   | 7.3915            | 7.398      | 7.420       |           |
| $(1^3D_3)$  | 3−   | 6.9901            | 6.998      | 7.030       |           |
| $(1D_2^1)$  | 2−   | 6.9873            | 6.984      | 7.032       |           |
| $(1D_2)$    | 2−   | 6.9803            | 6.986      | 7.024       |           |
| $(1^3D_1)$  | 1−   | 6.987             | 6.964      | 7.020       |           |

Table 2: Radial wave function at origin and decay constant of $B_c$ mesons.

| Meson State | $J^P$ | $|\psi(o)|^2$ [GeV$^3$] | My Calculated $f_p$ [GeV] | $f_p$ [25], $f_p$ [24], $f_p$ [28] |
|-------------|------|------------------------|--------------------------|----------------------------------|
| $(1^3S_1)$  | 1−   | 0.2358                 | 0.6694                   | 0.411, 0.604, 0.435              |
| $(1^1S_0)$  | 0−   | 0.247                  | 0.6877                   | 0.412, 0.607, 0.432              |
| $(2^3S_1)$  | 1−   | 0.1222                 | 0.4625                   | 0.356                            |
| $(2^1S_0)$  | 0−   | 0.1264                 | 0.4707                   | 0.355                            |
| $(3^3S_1)$  | 1−   | 0.0830                 | 0.3719                   | 0.326                            |
| $(3^1S_0)$  | 0−   | 0.0851                 | 0.3767                   | 0.325                            |
| $(4^3S_1)$  | 1−   | 0.0633                 | 0.3184                   | 0.308                            |
| $(4^1S_0)$  | 0−   | 0.0646                 | 0.3218                   | 0.307                            |
Figure 2: $^3S_1$ state for $n=1,2,3$. Line curve is for $n = 1$, the curve with line plus dots is for $n = 2$ and dashed curve is for $n = 3$

Figure 3: $^3P_0$ state for $n=1,2,3$. Line curve is for $n = 1$, the curve with line plus dots is for $n = 2$ and dashed curve is for $n = 3$

Figure 4: $^3P_2$ state for $n=1,2,3$. Line curve is for $n = 1$, the curve with line plus dots is for $n = 2$ and dashed curve is for $n = 3$
Table 3: Radii, Momentum Widths, and Mass of $B_c$ mesons. Calculated masses are rounded to 0.0001 GeV.

| Meson State | $J^P$ | Our Calculated Radii | $\beta$ with NR potential Model | Mass by Varitional | % error |
|-------------|-------|----------------------|---------------------------------|-------------------|---------|
|             |       | GeV                  | GeV                             | GeV               |         |
| $(1^3S_1)$  | $1^-$ | 0.3344               | 0.7224                          | 6.3919            | 1.21    |
| $(1^1S_0)$  | $0^-$ | 0.3186               | 0.7582                          | 6.3785            | 1.65    |
| $(2^3S_1)$  | $1^-$ | 0.7324               | 0.3298                          | 6.6898            | 2.42    |
| $(2^1S_0)$  | $0^-$ | 0.7229               | 0.3341                          | 6.682             | 2.34    |
| $(3^3S_1)$  | $1^-$ | 1.0576               | 0.2284                          | 6.9434            | 3.66    |
| $(3^1S_0)$  | $0^-$ | 1.0509               | 0.2298                          | 6.9383            | 2.93    |
| $(4^3S_1)$  | $1^-$ | 1.315                | 0.1837                          | 7.1375            | 4.78    |
| $(4^1S_0)$  | $0^-$ | 1.3113               | 0.1842                          | 7.1348            | 4.73    |
| $(1^3P_2)$  | $2^+$ | 0.5976               | 0.4042                          | 6.6773            | 3.58    |
| $(1^1P_0)$  | $0^+$ | 0.5329               | 0.4533                          | 6.6501            | 0.65    |
| $(2^3P_2)$  | $2^+$ | 0.9457               | 0.2554                          | 6.8958            | 3.07    |
| $(2^3P_0)$  | $0^+$ | 0.8887               | 0.2718                          | 6.8565            | 2.96    |
| $(3^3P_2)$  | $2^+$ | 1.2306               | 0.1963                          | 7.0971            | 4.23    |
| $(3^3P_0)$  | $0^+$ | 1.1818               | 0.2044                          | 7.0623            | 4.13    |

0.5 Decay Constants

$|\psi(0)|^2$ is used to find the decay constants ($f_p$) of pseudo scalar and pseudo vector mesons. Following Van-Royen-Weisskopf formula [23] is used to find decay constants.

$$f_p = R(0) = \sqrt{\frac{12|\psi(0)|^2}{M_p}}.$$ (16)

where $M_p$ is the mass of corresponding meson. I used the numerically calculated masses (given in Table 1) for Pseudo scalar and vector meson.

0.6 Radii

The normalized wave functions are then used to calculate root mean square radii using the following relation:

$$\sqrt{\langle r^2 \rangle} = \int U^* r^2 U dr.$$ (17)

0.7 Momentum Width

Momentum width ($\beta$) for a system of quark-antiquark bound state is defined as [26]

$$\beta = \sqrt{\frac{3}{2}} \frac{1}{r_{rms}}.$$ (18)

Using the root mean square radii, $\beta$ is calculated.
III. Spectrum of Mesons by Uncertainty and Variational Principles

Heisenberg’s uncertainty principle can be written as

$$\beta \bar{x} \geq \frac{1}{2}$$  \hspace{1cm} (19)

with $\Delta p_x = \beta$ (momentum width of the wave function) and $\Delta x = \bar{x}$ (size of meson corresponding to wavefunction). As Hamiltonian is the sum of kinetic and potential energy, so can be written as

$$H = \frac{P^2}{2\mu} + H_V(r),$$ \hspace{1cm} (20)

or

$$H = \frac{p_x^2}{2\mu} + \frac{p_y^2}{2\mu} + \frac{p_z^2}{2\mu} + H_V(r).$$ \hspace{1cm} (21)

From uncertainty principle, $\beta \geq \frac{1}{2\bar{x}}$. Assuming $\beta = \frac{1}{2\bar{x}}$, Hamiltonian can be written as

$$H = \frac{1}{2\mu x} + \frac{1}{2\mu y} + \frac{1}{2\mu z} + H_V(\sqrt{x^2 + y^2 + z^2}).$$ \hspace{1cm} (22)

Assuming $\bar{x} = \bar{y} = \bar{z}$, minimize the Hamiltonian with $\bar{x}$. For $B_c(1S_0^1)$, $\bar{x} = 0.0964$ fm and for $B_c(1S_0^3)$, $\bar{x} = 0.1022$ fm. Using $r_{min} = \sqrt{3}x_{min}$, the ground state radii of $B_c(1S_0^1)$ and $B_c(1S_0^3)$ are 0.167 fm, 0.177 fm. The mass $B_c$ meson is calculated with following expression.

$$M = m_Q + m_{\bar{Q}} + \frac{1}{8\mu r_{min}^2} + H_V(r_{min}),$$ \hspace{1cm} (23)

This gives mass of $B_c(1^1S_0) = 6.3379$ GeV with 1.08% error with the experimental mass. Mass of $B_c(1^3S_0) = 6.2776$ GeV that have 8.25% error with experimental mass. It means this method is successful for ground state and not applicable on excited states. To apply this simpler technique to higher states, eq.(23) can be modified by replacing $r_{min}$ with $r_{rms}$. The modified form of Eq.(23) can be written as

$$M = m_Q + m_{\bar{Q}} + \frac{\beta^2}{2\mu} + V(r_{rms}) + \frac{L(L+1)}{2\mu r_{rms}^2},$$ \hspace{1cm} (24)

The mass obtained with $\beta$ and $r_{rms}$ are reported in Table 3.

IV. Discussion and conclusion

In Fig. (1-4), radial wave functions are plotted against quark-antiquark distance. Figures illustrate that the peaks are shifted away from the origin with the orbital excitations. It is observed that the number of nodes increases by going toward higher radially excited state. In Table 1, calculated masses are reported for the ground as well as radially and orbitally excited states of $B_c$ mesons in non relativistic framework along with the experimental and theoretical predictions of the other’s works. Calculated masses are in complete agreement with the already calculated theoretical masses as well as the experimental values. The results reported in Table 2 illustrate that wave function at origin and decay constants are decreasing toward higher radial excitations. Pseudo scalar $B_c$ mesons have higher values of $|\psi(0)|^2$ and $f_p$ as compare to vector mesons. In Table 3, radii, momentum widths are reported in 3rd and 4th column. Mass calculated by using the radii and momentum widths are reported in 5th column of Table 3. It is noted that radii of $BC$ mesons increase with radial and angular excitations. The % error between
this mass and numerically calculated mass (reported in Table 1) shows a good agreement. It is observed that the different states of \( B_c \) mesons described with momentum width depends on \( L \) and \( r_{\text{rms}} \). It is concluded that the constraints derived between mass and radius by combining the uncertainty and variational principles give accurate results for ground state as well as radially and orbitally excited states.

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