SOME ESTIMATES OF FUNDAMENTAL SOLUTION ON NONCOMPACT MANIFOLDS WITH TIME-DEPENDENT METRICS

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Abstract. In this article, we obtain some further estimates of fundamental solutions comparing to Chau-Tam-Yu [1] and give some applications of the estimates on asymptotic behaviors of fundamental solutions.

1. Introduction

Let \( \{g(t)\} \) be a smooth family of complete Riemannian metrics on noncompact manifold \( M^n \) of dimension \( n \) such that \( g(t) \) satisfies:

\[
\frac{\partial}{\partial t} g_{ij}(x,t) = 2h_{ij}(x,t)
\]

on \( M \times [0,T] \), where \( h_{ij}(x,t) \) is a smooth family of symmetric tensors.

Without further confusions and indications, \( \langle \cdot, \cdot \rangle, \| \cdot \|, \Delta, \nabla, \ldots \) etc. mean the time-dependent inner product, norm, Laplacian operator and covariant derivative, etc.

Consider the equation:

\[
\frac{\partial u}{\partial t} - \Delta u + qu = 0.
\]

Let us make the following assumptions on the family \( g(t) \) and equation (1.2).

(A1) \( \| h \|, \| \nabla h \| \) are uniformly bound on space-time, where the norm is taken with respect to \( t \).

(A2) The sectional curvatures of the metrics \( g(t) \) are uniformly bounded on space-time.

(A3) \( |q|, \| \nabla q \|, |\Delta q| \) are uniformly bounded on space-time.
Let $H(t)$ be the trace of $h_{ij}(t)$ with respect to $g(t)$.

In [1], Chau-Tam-Yu, using the same trick as in Grigor’yan [4], obtained some weighted local $L^2$-estimate of $u$, and using this weighted local $L^2$-estimate, they obtained a weighted $L^2$-estimate for the fundamental solution of equation (1.2). In this article, we first, using the same technic as in Grigor’yan [5], obtained some local weighted $L^2$-estimates of $\nabla u$ and $\Delta u$. Then, by the local weighted $L^2$-estimates, we get some $L^p$-estimates ($p \in (0, 2]$) of the gradient and Laplacian of the fundamental solution of (1.2). Finally, as an application of the integral estimates of the fundamental solution, we derive some asymptotic behaviors of the fundamental solution which are the foundation of the proof of non-positivity of Perelman’s new Li-Yau-Hamilton type expression in Chau-Tam-Yu [1].

2. SOME LOCAL INTEGRAL ESTIMATES

In [1], using the same trick as in Grigor’yan [4], Chau-Tam-Yu get the following local weighted $L^2$-estimate.

**Lemma 2.1.** Let $\Omega$ be a relative compact domain of $M$ with smooth boundary and let $K$ be a compact set with $K \subset \subset \Omega$. Let $u$ be any solution to the problem:

$$
\begin{align*}
&u_t - \Delta u + qu = 0, \text{ in } \Omega \times [0, T] \\
&u|_{\partial \Omega \times [0, T]} = 0 \\
&\text{supp } u(\cdot, 0) \subset K.
\end{align*}
$$

(2.1)

Let $f$ be a regular function with the constants $\gamma$ and $A$. Suppose

$$
\int_{\Omega} u^2 dV_t \leq \frac{1}{f(t)}
$$

for any $t > 0$. Then there is a positive constant $C$ depending only on $\gamma$, the uniform upper bound of $|q|$ and $|H|$, and a positive constant $D$ depending only on $T$, $\gamma$ and the uniformly upper bound of $\|h\|$, such that

$$
\int_{\Omega} u^2(x, t)e^{-r_0^2(x, K)}dV_t \leq \frac{4A}{f(t/\gamma)}e^{Ct}
$$

for any $t > 0$, where $r_0(x, K)$ denotes the distance between $x$ and $K$ with respect to the initial metric $g(0)$. 

In this section, using basically the same trick as in Grigor’yan [5], we get some local weighted $L^2$ estimates of $\nabla u$ and $\Delta u$.

**Lemma 2.2.** Let $u$ be the same as in Lemma 2.1. Then, there is a positive constant $D$ depending only on $\gamma$, $T$ and the uniformly upper bound of $\|h\|$, and a positive constant $C$ depending only on the uniformly upper bounds of $|H|$, $|q|$, such that

$$
\int_\Omega (u^2 + \|\nabla u\|^2)e^{\frac{\|\xi\|}{Dt}}dV_t \leq \frac{4Ae^{Ct}}{\int_0^t f(s/\gamma)ds}
$$

for any $t > 0$.

**Proof.** Let $D$ be larger than the $D$ in the statement of Lemma 2.1 such that the function $\xi = \frac{ru(x,K)}{Dt}$ satisfies

$$
\xi_t + 8\|\nabla \xi\|^2 \leq 0
$$

on $M \times (0,T]$. By the boundary conditions, $u_t = 0$ on $\partial \Omega \times [0,T]$. By integration by parts,

$$
\frac{d}{dt} \int_\Omega \|\nabla u\|^2 e^\xi dV_t = 2 \int_\Omega \langle \nabla u_t, \nabla u \rangle e^\xi dV_t - 2 \int_\Omega h(\nabla u, \nabla u) e^\xi dV_t + \int_\Omega \|\nabla u\|^2 \xi_t e^\xi dV_t + \int_\Omega H\|\nabla u\|^2 e^\xi dV_t
$$

$$
= -2 \int_\Omega u_t \Delta u e^\xi dV_t - 2 \int_\Omega u_t \langle \nabla u, \nabla \xi \rangle e^\xi dV_t - 2 \int_\Omega h(\nabla u, \nabla u) e^\xi dV_t + \int_\Omega \|\nabla u\|^2 \xi_t e^\xi dV_t
$$

$$
+ \int_\Omega H\|\nabla u\|^2 e^\xi dV_t
$$

$$
\leq -2 \int_\Omega |\Delta u|^2 e^\xi - 2 \int_\Omega \Delta u \langle \nabla u, \nabla \xi \rangle e^\xi dV_t - 8 \int_\Omega \|\nabla u\|^2 \|\nabla \xi\|^2 e^\xi dV_t
$$

$$
+ 2 \int_\Omega qu \Delta u e^\xi dV_t + 2 \int_\Omega qu \langle \nabla u, \nabla \xi \rangle dV_t - 2 \int_\Omega h(\nabla u, \nabla u) e^\xi dV_t + \int_\Omega H\|\nabla u\|^2 e^\xi dV_t
$$

$$
\leq C_1 \int_\Omega (\|\nabla u\|^2 + u^2) e^\xi dV_t - \int_\Omega \|\nabla u\|^2 \|\nabla \xi\|^2 e^\xi dV_t - \int_\Omega |\Delta u|^2 e^\xi dV_t
$$

where $C_1$ is a positive constant depending only the upper bounds of $|q|$, $\|X\|$ and $\|h\|$.
Similar computations give us that
\[
\frac{d}{dt} \int_{\Omega} u^2 e^\xi dV_t \leq C_2 \int_{\Omega} u^2 e^\xi dV_t - \int_{\Omega} \|\nabla u\|^2 e^\xi dV_t - \int_{\Omega} u^2 \|\nabla \xi\|^2 e^\xi dV_t.
\]
with \(C_2 > 0\) depending only on the upper bounds of \(|q|, \|h\|\) and \(\|X\|\).

Therefore,
\[
\frac{d}{dt} \int_{\Omega} (\|\nabla u\|^2 + u^2) e^\xi dV_t
\]
\[
\leq C_3 \int_{\Omega} (\|\nabla u\|^2 + u^2) e^\xi dV_t - \int_{\Omega} \|\nabla u\|^2 \|\nabla \xi\|^2 e^\xi dV_t - \int_M |\Delta u|^2 e^\xi dV_t
\]
where \(C_3 = C_1 + C_2\), and

(2.2)
\[
\frac{d}{dt} \int_{\Omega} e^{-C_4 t}(\|\nabla u\|^2 + u^2) e^\xi dV_t
\]
\[
\leq - \int_{\Omega} e^{-C_4 t}(\|\nabla u\|^2 + u^2) e^\xi dV_t - \int_{\Omega} e^{-C_4 t} |\Delta u|^2 e^\xi dV_t - \int_{\Omega} e^{-C_4 t} \|\nabla u\|^2 \|\nabla \xi\|^2 e^\xi dV_t
\]
where \(C_4 = C_3 + 1\). On the other hand,

(2.3)
\[
\int_{\Omega} (\|\nabla u\|^2 + u^2) e^\xi dV_t
\]
\[
= - \int_{\Omega} u \Delta u e^\xi dV_t - \int_{\Omega} u \langle \nabla u, \nabla \xi \rangle e^\xi dV_t + \int_{\Omega} u^2 e^\xi dV_t
\]
\[
\leq \left( \int_{\Omega} |u|^2 e^\xi dV_t \right)^{1/2} \left( \left( \int_{\Omega} |\Delta u|^2 e^\xi dV_t \right)^{1/2} + \left( \int_{\Omega} \|\nabla u\|^2 \|\nabla \xi\|^2 e^\xi dV_t \right)^{1/2} + \left( \int_{\Omega} |u|^2 e^\xi dV_t \right)^{1/2} \right].
\]
Let
\[
Q = \int_{\Omega} e^{-C_4 t}(\|\nabla u\|^2 + u^2) e^\xi dV_t,
\]
\[
E = \int_{\Omega} e^{-C_4 t} u^2 e^\xi dV_t, \quad F = \int_{\Omega} e^{-C_4 t} \|\nabla u\|^2 \|\nabla \xi\|^2 e^\xi dV_t, \quad \text{and} \quad G = \int_{\Omega} e^{-C_4 t} |\Delta u|^2 e^\xi dV_t.
\]
Then, by equation (2.2) and equation (2.3),

\[
Q' \leq -(E + F + G) \leq -(E^{1/2} + F^{1/2} + G^{1/2})^2 \leq -\frac{Q^2}{E}.
\]

This implies that

(2.4)
\[
\left( \frac{1}{Q} \right)' \geq \frac{1}{E}.
\]
We can assume that $C_4$ is greater than the constant $C$ in the statement of Lemma 2.1. By Lemma 2.1,

$$E \leq \frac{4A}{\bar{f}(t/\gamma)}.$$ 

By equation (2.4),

$$Q(t) \leq \frac{1}{\int_0^t \frac{f(s/\gamma)}{4A} ds}.$$ 

So,

$$\int_\Omega \|\nabla u\|^2 e^{\frac{r_0^2(x,K)}{Dt}} dV_t \leq \frac{4A e^{C_4 t}}{\int_0^t f(s/\gamma) ds}.$$ 

□

**Lemma 2.3.** Let $u$ be the same as in Lemma 2.1. Then, there is a positive constant $D$ depending only on $\gamma, T$ and the uniformly upper bound of $\|h\|$, and a positive constant $C$ depending only on the uniformly upper bounds of $\|h\|, \|\nabla q\|$, and $|q|$, such that

$$\int_\Omega (u^2 + \|\nabla u\|^2 + \|\Delta u\|^2) e^{\frac{r_0^2(x,K)}{Dt}} dV_t \leq \frac{24 A e^{C t}}{\int_0^t \int_0^s f(s/\gamma) ds d\sigma}$$

for any $t > 0$.

**Proof.** Let $D$ be larger than the $D$ in the statement of Lemma 2.2 such that the function $\xi = \frac{r_0(x,K)}{Dt}$ satisfies

$$\xi_t + 8\|\nabla \xi\|^2 \leq 0.$$
on $M \times (0, T]$. By the boundary condition, $\Delta u = 0$ on $\partial \Omega \times [0, \delta]$. So, integration by parts is valid in the following computations.

\[
\frac{d}{dt} \int_{\Omega} |\Delta u|^2 e^t dV_t
\]

\[
= 2 \int_{\Omega} \Delta u \Delta u_t e^t dV_t - 4 \int_{\Omega} \Delta u (h_{ij} u_{ij} + h_{ik} u_k) e^t dV_t
\]

\[
+ 2 \int_{\Omega} \Delta u (\nabla H, \nabla u) e^t dV_t + \int_{\Omega} |\Delta u|^2 e^t dV_t + \int_{\Omega} H |\Delta u|^2 e^t dV_t
\]

\[
\leq -2 \int_{\Omega} (\nabla \Delta u, \nabla u_t) e^t dV_t - 2 \int_{\Omega} \Delta u (\nabla u_t, \nabla \xi) e^t dV_t + 4 \int_{\Omega} h(\nabla \Delta u, \nabla u) e^t dV_t
\]

\[
+ 4 \int_{\Omega} (\Delta u) h(\nabla u, \nabla \xi) e^t dV_t + 2 \int_{\Omega} \Delta u (\nabla H, \nabla u) e^t dV_t - 8 \int_{\Omega} |\Delta u|^2 |\nabla \xi|^2 e^t dV_t
\]

\[
+ \int_{\Omega} H |\Delta u|^2 e^t dV_t
\]

\[
= -2 \int_{\Omega} \|\nabla \Delta u\|^2 e^t dV_t - 2 \int_{\Omega} \Delta u (\nabla \Delta u, \nabla \xi) e^t dV_t - 8 \int_{\Omega} |\Delta u|^2 |\nabla \xi|^2 e^t dV_t
\]

\[
+ 2 \int_{\Omega} (\nabla \Delta u, \nabla (qu)) e^t dV_t + 2 \int_{\Omega} \Delta u (\nabla (qu), \nabla \xi) e^t dV_t + 4 \int_{\Omega} h(\nabla \Delta u, \nabla u) e^t dV_t
\]

\[
+ 4 \int_{\Omega} (\Delta u) h(\nabla u, \nabla \xi) e^t dV_t + 2 \int_{\Omega} \Delta u (\nabla H, \nabla u) e^t dV_t + \int_{\Omega} H |\Delta u|^2 e^t dV_t
\]

\[
\leq C_1 \int_{\Omega} (u^2 + \|\nabla u\|^2 + |\Delta u|^2) e^t dV_t - \int_{\Omega} \|\nabla \Delta u\|^2 e^t dV_t - \int_{\Omega} \|\Delta u\|^2 |\nabla \xi|^2 e^t dV_t
\]

where $C_1 > 0$ depends on the upper bounds of $|q|$, $\|\nabla q\|$ and $\|h\|$.

Hence, combining the computations in the proof of Lemma 2.2 we have

\[
\frac{d}{dt} \int_{\Omega} (u^2 + \|\nabla u\|^2 + |\Delta u|^2) e^t dV_t
\]

\[
\leq C_2 \int_{\Omega} (u^2 + \|\nabla u\|^2 + |\Delta u|^2) e^t dV_t - \int_{\Omega} \|\nabla \Delta u\|^2 e^t dV_t - \int_{\Omega} \|\Delta u\|^2 e^t dV_t
\]

\[
- \int_{\Omega} \|\nabla u\|^2 e^t dV_t - \int_{\Omega} |\Delta u|^2 |\nabla \xi|^2 e^t dV_t - \int_{\Omega} \|\nabla u\|^2 |\nabla \xi|^2 e^t dV_t - \int_{\Omega} u^2 |\nabla \xi|^2 e^t dV_t
\]

where $C_2 > 0$ depends on the upper bounds of $|q|$, $\|\nabla q\|$ and $\|h\|$.
Then, 
\[
\frac{d}{dt} Q := \frac{d}{dt} \int_{\Omega} e^{-C_3 t} (u^2 + \|\nabla u\|^2 + |\Delta u|^2) e^{\xi} dV_t \\
\leq - \int_{\Omega} e^{-C_3 t} \|\nabla u\|^2 e^{\xi} dV_t - \int_{\Omega} e^{-C_3 t} |\Delta u|^2 e^{\xi} dV_t - \int_{\Omega} e^{-C_3 t} \|\nabla u\|^2 e^{\xi} dV_t - \int_{\Omega} e^{-C_3 t} |\Delta u|^2 e^{\xi} dV_t \\
- \int_{\Omega} e^{-C_3 t} |\Delta u|^2 \|\nabla \xi\|^2 e^{\xi} dV_t - \int_{\Omega} e^{-C_3 t} \|\nabla u\|^2 |\Delta \xi| e^{\xi} dV_t - \int_{\Omega} e^{-C_3 t} |\Delta u|^2 \|\nabla \xi\|^2 e^{\xi} dV_t \\
:= - E_3 - E_2 - E_1 - E_0 - \tilde{E}_2 - \tilde{E}_1 - \tilde{E}_0,
\]
where \(C_3 = C_2 + 1\). On the other hand, 
\[
\int_{\Omega} (\Delta u)^2 e^{\xi} dV_t \\
= - \int_{\Omega} \langle \nabla u, \nabla \Delta u \rangle e^{\xi} dV_t - \int_{\Omega} \Delta u \langle \nabla u, \nabla \xi \rangle e^{\xi} dV_t \\
\leq \left( \int_{\Omega} \|\nabla u\|^2 e^{\xi} dV_t \right)^{\frac{1}{2}} \left( \int_{\Omega} |\Delta u|^2 e^{\xi} dV_t \right)^{\frac{1}{2}} + \left( \int_{\Omega} \|\nabla u\|^2 e^{\xi} dV_t \right)^{\frac{1}{2}} \left( \int_{\Omega} |\Delta u|^2 |\nabla \xi|^2 e^{\xi} dV_t \right)^{\frac{1}{2}}.
\]
Hence, 
\[
E_2 \leq E_3^{1/2} E_1^{1/2} + E_1^{1/2} \tilde{E}_2^{1/2}.
\]
Then, 
\[
(E_0 + E_1 + E_2 + E_3 + \tilde{E}_0 + \tilde{E}_1 + \tilde{E}_2)(E_0 + E_1) \\
\geq E_0^2 + E_1^2 + E_1 E_3 + E_1 \tilde{E}_2 \\
\geq E_0^2 + E_1^2 + \frac{(E_1^{1/2} E_3^{1/2} + E_1^{1/2} \tilde{E}_2^{1/2})^2}{2} \\
\geq \frac{E_0^2 + E_1^2 + E_2^2}{2} \\
\geq \frac{Q^2}{6}.
\]
Therefore, 
\[
\frac{dQ}{dt} \leq - \frac{Q^2}{6(E_0 + E_1)},
\]
and 
\[
(2.5) \quad \left( \frac{1}{Q} \right)' \geq \frac{1}{6(E_0 + E_1)}.
\]
We can assume that $C_3$ is bigger than the constant $C$ in the statement of Lemma 2.2. Then
\[ E_0 + E_1 \leq \frac{4A}{\int_0^t f(s/\gamma)ds}, \]
and
\[ Q \leq \frac{24A}{\int_0^t \int_0^\sigma f(s/\gamma)dsd\sigma}. \]
This completes the proof. \qed

3. SOME INTEGRAL ESTIMATES OF FUNDAMENTAL SOLUTIONS

Let $Z(x, t; y, s)$ and $Z_k(x, t; y, s)$ be the same as in Chau-Tam-Yu [1]. In [1], Chau-Tam-Yu get the following weighted $L^2$-estimates of $Z$.

**Proposition 3.1.** There are some positive constants $C$ and $D$ with $C$ depending only on $T, n$, the lower bound of the Ricci curvature of the initial metric and the upper bounds of $|q|$ and $|h|$, and $D$ depending only on $T$ and the upper bound of $|h|$, such that for $0 \leq s < t \leq T$,
\[
\int_M Z^2(x, t; y, s) e^{\frac{r_0^2(x, y)}{D(t-s)}} dV_t(x) \leq \frac{C}{V^0_y(\sqrt{t-s})} \quad \text{and} \quad \int_M Z^2(x, t; y, s) e^{\frac{r_0^2(x, y)}{D(t-s)}} dV_s(y) \leq \frac{C}{V^0_x(\sqrt{t-s})}.
\]

In this section, we get some integral estimates of $\nabla Z$ and $\Delta Z$.

**Corollary 3.1.** For any $p \in (0, 2]$, there is a positive constants $C$ depending only on $T, n, p$ the lower bound of the Ricci curvature of the initial metric and the upper bounds of $|q|$ and $||h||$, and a positive constant $D$ depending only on $p, T$ and the upper bound of $||h||$, such that
\[
\int_M Z^p(x, t; y, s) e^{\frac{r_0^2(x, y)}{D(t-s)}} dV_t(x) \leq \frac{C}{[V^0_y(\sqrt{t-s})]^{p-1}} \quad \text{and} \quad \int_M Z^p(x, t; y, s) e^{\frac{r_0^2(x, y)}{D(t-s)}} dV_s(y) \leq \frac{C}{[V^0_x(\sqrt{t-s})]^{p-1}}.
\]
Proof. Let $C_1, D_1 > 0$ be such that
\[ \int_M Z^2(x, t, y; s)e^{\frac{r^2(x, y)}{2(t - s)}}dV_0(x) \leq \frac{C_1}{V_y^0(\sqrt{t - s})}. \]

Let $D = \frac{4D_1}{p}$ and $R = \sqrt{t - s}$. Then
\[ \int_{B^0_y(R)} Z^p(x, t; y, s)e^{\frac{r^2(x, y)}{2(t - s)}}dV_0(x) \]
\[ \leq C_2 \int_{B^0_y(2R) \setminus B^0_y(2^k - 1)} Z^p(x, t; y, s)e^{\frac{r^2(x, y)}{2(t - s)}}dV_0(x) \]
\[ \leq C_2 (V^0_y(2^k R) - V^0_y(2^{k - 1} R))^{1 - \frac{p}{2}} \left( \int_{B^0_y(2^k R) \setminus B^0_y(2^{k - 1} R)} Z^2(x, t; y, s)e^{\frac{r^2(x, y)}{2(t - s)}}dV_0(x) \right)^{\frac{p}{2}} \]
\[ = C_2 (V^0_y(2^k R) - V^0_y(2^{k - 1} R))^{1 - \frac{p}{2}} \left( \int_{B^0_y(2^k R) \setminus B^0_y(2^{k - 1} R)} Z^2(x, t; y, s)e^{\frac{r^2(x, y)}{2(t - s)}}dV_0(x) \right)^{\frac{p}{2}} \]
\[ \leq \frac{C_3}{[V^0_y(\sqrt{t - s})]^{\frac{p}{2}}} \times (V^0_y(2^k R) - V^0_y(2^{k - 1} R))^{1 - \frac{p}{2}} e^{-\frac{4k - 2pR^2}{2p(t - s)}} \]
\[ \leq \frac{C_3}{[V^0_y(\sqrt{t - s})]^{\frac{p}{2}}} \times e^{-\frac{4k - 2pR^2}{2p(t - s)}} \]
\[ = \frac{C_3}{[V^0_y(\sqrt{t - s})]^{p - 1}} \times e^{C_2k - \frac{4kR^2}{D_2}}, \]

where $C_2$ depends on the equivalent constant of the family $g(t)$, $C_3$ depends on $C_1, p$ and $C_2$, $C_4$ depends on the lower bound of the $Rc_0$ and $n$, $C_5$ depends on $C_4, T$ and $n$, and $D_2$ depends on $p$ and $D_1$.

The same argument using Hölder inequality give us
\[ \int_{B^0_y(R)} Z^p(x, t; y, s)e^{\frac{r^2(x, y)}{2(t - s)}}dV_0(x) \leq \frac{C_6}{[V^0_y(\sqrt{t - s})]^{p - 1}} \]

where $C_6$ depends on $C_1$ and $p$.

Summing the above inequalities together, we get the first inequality. The proof of the second one is similar. \( \square \)

**Proposition 3.2.** There is a positive constant $C$ depending only on $T$, $n$, the lower bound of the Ricci curvature of the initial metric and the upper bounds of $|q|$ and $\|h\|$, and a positive constant $D$ depending only on $T$ and the upper bound
of $\|h\|$, such that

$$
\int_M \|\nabla_x Z(x, t; y, s)\|^2 e^{\frac{r(y)}{2(t-s)}} dV_t(x) \leq \frac{C}{(t-s)V_y^0(\sqrt{t-s})}
$$

and

$$
\int_M \|\nabla_y Z(x, t; y, s)\|^2 e^{\frac{r(y)}{2(t-s)}} dV_s(y) \leq \frac{C}{(t-s)V_x^0(\sqrt{t-s})}
$$

for any $0 \leq s < t \leq T$.

**Proof.** We only prove the first inequality, the proof of the second one is similar.

Note that

$$
\int_0^t V_y^0(\sqrt{s}) ds = V_y^0(\sqrt{t}) \int_0^t \frac{V_y^0(\sqrt{s})}{V_y^0(\sqrt{t})} ds
\geq V_y^0(\sqrt{t}) \int_0^t \frac{s^n}{t^n} e^{-C_1 \sqrt{t}} ds
= e^{-C_1 T} V_y^0(\sqrt{t}) \int_0^1 x^n dx
= c_2 t V_y^0(\sqrt{t}).
$$

By Lemma 2.1 and Lemma 2.2 there is some $D_1 > 0$ and $C_3 > 0$, such that

$$
\int_{\Omega_k} \|\nabla_x Z_k(x, t; y, s)\|^2 e^{\frac{r(y)}{2(t-s)}} dV_t(x) \leq \frac{C_3}{(t-s)V_y^0(\sqrt{t-s})}
$$

for any $k$ and $0 \leq s < t \leq T$.

By Fatou’s lemma,

$$
\int_M \|\nabla_x Z(x, t; y, s)\|^2 e^{\frac{r(y)}{2(t-s)}} dV_t(x)
\leq \liminf_{k \to \infty} \int_{\Omega_k} \|\nabla_x Z_k(x, t; y, s)\|^2 e^{\frac{r(y)}{2(t-s)}} dV_t(x)
\leq \frac{C_3}{(t-s)V_y^0(\sqrt{t-s})}.
$$

□

By the same arguments as in the proof of Corollary 3.1, we have the following corollary.
Corollary 3.2. For any $p \in (0, 2]$, there is a positive constant $C$ depending only on $T, n, p$ the lower bound of the Ricci curvature of the initial metric and the upper bounds of $|q|$ and $\|h\|$, and a positive constant $D$ depending only on $p, T$ and the upper bound of $\|h\|$, such that

$$
\int_M \|\nabla^t_x Z(x, t; y, s)\|^{p e^{\frac{r_0^2(x,y)}{d(t-s)}}} dV_t(x) \leq \frac{C}{(t-s)^{\frac{p}{2}} [V^0_y(\sqrt{t-s})]^{p-1}} \quad \text{and} \\
\int_M \|\nabla^s_y Z(x, t; y, s)\|^{p e^{\frac{r_0^2(x,y)}{d(t-s)}}} dV_s(y) \leq \frac{C}{(t-s)^{\frac{p}{2}} [V^0_x(\sqrt{t-s})]^{p-1}}
$$

for any $0 \leq s < t \leq T$.

By the same arguments as in the proof of Proposition 3.1 and Corollary 3.1 using Lemma 2.3. We have the following integral estimate of $\Delta Z$.

Proposition 3.3. For any $p \in (0, 2]$, there is a positive constant $C$ depending only on $T, n, p$ the lower bound of the Ricci curvature of the initial metric and the upper bounds of $|q|, \|\nabla q\|$ and $\|h\|$, and a positive constant $D$ depending only on $p, T$ and the upper bound of $\|h\|$, such that

$$
\int_M \|\Delta^t_x Z(x, t; y, s)\|^{p e^{\frac{r_0^2(x,y)}{d(t-s)}}} dV_t(x) \leq \frac{C}{(t-s)^p [V^0_y(\sqrt{t-s})]^{p-1}} \quad \text{and} \\
\int_M \|\Delta^s_y Z(x, t; y, s)\|^{p e^{\frac{r_0^2(x,y)}{d(t-s)}}} dV_s(y) \leq \frac{C}{(t-s)^p [V^0_x(\sqrt{t-s})]^{p-1}}
$$

for any $0 \leq s < t \leq T$.

4. Gaussian upper bound of the gradient of fundamental solution

Proposition 4.1. For any $\delta \in [0, 1)$ there is a positive constant $C$ depending only on $n, T$ and the upper bounds of $|q|, \|h\|$ and $\|Rc\|$, such that

$$
\frac{\|\nabla^t_x Z(x, t; y, s)\|^2}{Z^{1+\delta}(x, t; y, s)} \leq \frac{C}{(1-\delta)(t-s)[V^0_y(\sqrt{t-s})]^{1-\delta}} \quad \text{and} \\
\frac{\|\nabla^s_y Z(x, t; y, s)\|^2}{Z^{1+\delta}(x, t; y, s)} \leq \frac{C}{(1-\delta)(t-s)[V^0_x(\sqrt{t-s})]^{1-\delta}}.
$$
Proof. Fixed $0 \leq s < T$ and $\sigma \in (0, T - s - \sigma)$. Let $u(x, \tau) = Z(x, \tau + \sigma + s; y, s)$. The domain of $\tau$ is $(0, T - s]$. By Lemma 5.2 in Chau-Tam-Yau [1],

$$u(x, \tau) = \mathcal{Z}(x, \tau + \sigma + s; y, s) \leq \frac{C_1}{\sqrt[\gamma]{\tau + \sigma}} \leq \frac{C_1}{\sqrt[\gamma]{\sqrt{\sigma}}} := N(\sigma),$$

for any $\tau \in [0, T - s - \sigma]$. By Lemma 6.3 in Chau-Tam-Yau [1] (It was proved in [1] only for Ricci flow, but we can prove it by the same argument without any difficulty within our setting.) ,

$$\left\| \nabla^{\tau+\sigma+s}u \right\|^2_{u^{1+\delta}} \leq \frac{C_2u^{1-\delta}(\log \frac{N}{u} + 1)}{\tau} \leq \frac{C_2u^{1-\delta}(1-\delta)\log \frac{N}{u} + 1}{(1-\delta)\tau} \leq \frac{C_2N^{1-\delta}}{(1-\delta)\tau} \leq \frac{C_2}{(1-\delta)\tau[V_y^0(\sqrt{\sigma})]^{1-\delta}},$$

where we have used the inequality $\log(1 + x) \leq x$.

Hence, for any $s < t$, by letting $\tau = \sigma = \frac{t-s}{2}$,

$$\left\| \nabla^t_{x} Z(x, t; y, s) \right\|^2_{Z^{1+\delta}(x, t; y, s)} \leq \frac{C_3}{(1-\delta)(t-s)[V_y^0(\sqrt{t-s})]^{1-\delta}}.$$

This completes the proof of the first inequality. The proof of the second inequality is just the same. \hfill \square

**Corollary 4.1.** There is a positive constants $C$ depending only on $n, T$ and the upper bounds of $|q|, \|h\|$ and $\|Rc\|$, and a positive constant $D$ depending only on $T$ and the upper bound of $\|h\|$, such that

$$\left\| \nabla^t_{x} Z(x, t; y, s) \right\| \leq \frac{Ce^{-\frac{\phi^2(x,y)}{D(t-s)}}}{[(t-s)V_y^0(\sqrt{t-s})V_y^0(\sqrt{t-s})]^{\frac{1}{2}}},$$

$$\left\| \nabla^s_{y} Z(x, t; y, s) \right\| \leq \frac{Ce^{-\frac{\phi^2(x,y)}{D(t-s)}}}{[(t-s)V_y^0(\sqrt{t-s})V_y^0(\sqrt{t-s})]^{\frac{1}{2}}},$$

$$\left\| \nabla^t_{x} Z(x, t; y, s) \right\| \leq \frac{Ce^{-\frac{\phi^2(x,y)}{D(t-s)}}}{\sqrt{t-s}V_y^0(\sqrt{t-s})} \text{ and}$$

$$\left\| \nabla^s_{y} Z(x, t; y, s) \right\| \leq \frac{Ce^{-\frac{\phi^2(x,y)}{D(t-s)}}}{\sqrt{t-s}V_x^0(\sqrt{t-s})}. $$
Proof. Straight forward from Corollary 5.2 in Chau-Tam-Yu [1] and Proposition 4.1 with $\delta = 0$. □

**Proposition 4.2.** Let $f$ be a bounded smooth function on $M$ and $F(x,t)$ be a bounded smooth function on $M \times (0,T]$. Then, the Cauchy problem

$$\begin{cases}
u_t - \Delta u + qu = F \\ u(x,0) = f(x)
\end{cases}$$

has a unique bounded solution

$$u(x,t) = \int_M Z(x,t;y,0)f(y)dV_0(y) + \int_0^t \int_M Z(x,t;y,s)F(y,s)dV_s(y)ds.$$ 

Proof. By estimates of $Z(x,t;y,s)$, $u(x,t)$ is well defined and bounded. Moreover, by the estimate of the gradient of $Z(x,t;y,0)$ and Lebesgue’s dominant convergence theorem,

$$\nabla u = \int_M \nabla_x^t Z(x,t;y,0)f(y)dV_0(y) + \int_0^t \int_M \nabla_x^t Z(x,t;y,s)F(y,s)dV_s(y)ds.$$ 

So, a direct computation shows that $u$ is a weak solution of the Cauchy problem. By regularity theory, it is actually a classical solution. Uniqueness comes directly from the maximum principle. □

**Remark 4.1.** This representation formula was obtained by Guenther ([6]) on compact manifolds.

5. Some Asymptotic Behavior of the fundamental solution

The following asymptotic behavior of the fundamental solution is basically the same as in Garofalo-Lanconelli [2]. Since our setting is different with the setting in Garofalo-Lanconelli [2], we give a detailed proof here.

**Proposition 5.1.** Let $Z(x,y,t) = Z(x,t;y,0)$. Then, for any relative compact domain $\Omega$, there is positive constant $\delta$, and a positive function $u_0 \in C^\infty(M \times M \times [0,\delta])$ with $u_0(x,x,t) = 1$ for any $x \in \Omega$, such that

$$\left| Z(x,y,t) - \frac{1}{(4\pi t)^{n/2}} e^{-\frac{r^2(x,y)}{4t}} u_0(x,y,t) \right| \leq Ct^{1-\frac{n}{2}}$$
on $\Omega \times \Omega \times (0, \delta]$, for some positive constant $C$.

Proof. We enlarge $\Omega$ to a compact domain $\Omega'$ such that $\Omega' \supset \Omega$. Let $\delta > 0$ be such that
\[ \frac{1}{4} g(0) \leq g(t) \leq 4g(0), \]
for any $t \in [0, \delta]$. Let $\epsilon > 0$ be such that $B_{t_y}(2\epsilon)$ is a convex geodesic ball of $(M, g(t))$ for any $t \in [0, \delta]$ and $y \in \Omega'$.

For each $t \in [0, \delta]$ and $y \in \Omega'$, let $(r^t_y, \theta^t_y)$ be the polar coordinate at $y$ of $(B_{t_y}(2\epsilon), g(t))$. Then
\[ \Delta = \frac{\partial^2}{\partial r^2} + \left( \frac{n-1}{r} + \frac{\partial \log \sqrt{\det g(r, \theta)}}{\partial r} \right) \frac{\partial}{\partial r} + \Delta_{s_r}. \]

We divide the proof into the following steps.

Step 1. Construction of $u_0$.

Let
\[ U = \{(x, y, t) \in M \times \Omega' \times [0, \delta] \mid r_t(x, y) \leq \epsilon \}. \]

Let
\[ G(x, y, t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{r^2(x, y)}{4t}}. \]

Then, $G(x, y, t)$ is smooth on $\overline{U}$, and
\[ \frac{\partial}{\partial t} G(x, y, t) - \Delta^t_{x} G(x, y, t) \]
\[ = \left( - \frac{r_t \partial r_t}{2t} \frac{\partial}{\partial t} + \frac{r_t \partial \log \sqrt{\det g(t)(r, \theta)}}{\partial r} \right) G. \]

Let $u_0(x, y, t)$ be a function to be determined. Let $\Box$ be the operator
\[ \Box = \frac{\partial}{\partial t} - \Delta + q. \]

Then
\[ \Box_{x} (Gu_0) = G \Box_{x} u_0 + \left( - \frac{r_t \partial r_t}{2t} \frac{\partial}{\partial t} + \frac{r_t \partial \log \sqrt{\det g(t)(r, \theta)}}{\partial r} \right) Gu_0 - 2(\nabla^t_x G, \nabla^t_x u_0) \]
\[ = G \Box_x u_0 + \left( \frac{r_t \partial u_0}{t} \frac{\partial}{\partial r} + \left( - \frac{r_t \partial r_t}{2t} \frac{\partial}{\partial t} + \frac{r_t \partial \log \sqrt{\det g(t)(r, \theta)}}{\partial r} \right) u_0 \right) G. \]
We require \( u_0 \) to be such that the coefficient of \( G \) is vanished and \( u(y, y, 0) = 1 \). That is to solve the ODE:

\[
\frac{\partial u_0}{\partial r} = \frac{1}{2} \left( \frac{\partial r_t}{\partial t} - \frac{\partial \log \sqrt{\det g(r, \theta)}}{\partial r} \right) u_0
\]

with initial data 1. \( u_0(\rho_t^x, \theta_t^y, y, t) = \exp \left( \frac{1}{2} \int_0^t \left[ \frac{\partial r_t}{\partial t} - \frac{\partial \log \sqrt{\det g(t)}(r, \theta)}{\partial r} \right] dr \right) \) is the solution of the ODE with the initial data. So \( u_0 \) is positive and smooth function on \( \Omega \). For this \( u_0 \), we have

\[
\square \chi(Gu_0) = G\square \chi u_0.
\]

Step 2. Let \( \zeta \) be a smooth function \( M \) such that \( \zeta \equiv 1 \) on \( \Omega \) and \( \zeta \equiv 0 \) on \( M \setminus \Omega'' \), where \( \Omega \subset \subset \Omega'' \subset \subset \Omega' \). Let \( \eta \) be a smooth function on \( \mathbb{R} \), such that \( \eta \equiv 1 \) on \( [0, 1/3] \) and \( \eta \equiv 0 \) on \( [2/3, \infty) \). Let

\[
\chi(x, y) = \eta(r_0(x, y)/\epsilon)\zeta(y).
\]

Then \( \chi \in C_0^\infty(M \times M) \) and \( \chi \equiv 1 \) on \( V \) where

\[
V = \{(x, y) \in M \times \Omega | r_0(x, y) \leq \epsilon/3\}.
\]

It is clear that \( V \times [0, \delta] \subset U \) and \( \text{supp} \chi \subset \subset U \). So \( \chi u_0 \in C_0^\infty(M \times M) \).

We want to show that

\[
Z(x, y, t) - \chi Gu_0(x, y, t) = -\int_0^t \int_M Z(x, t; z, s)\square \chi(\chi Gu_0)(z, y, s)dV_0(z)ds
\]

for any \( (x, y, t) \in M \times \Omega \times (0, \delta] \).

For any \( \varphi \in C_0^\infty(\Omega) \), consider the function

\[
\psi(x, t) = \int_M (Z(x, y, t) - (\chi Gu_0)(x, y, t))\varphi(y)dV_0(y).
\]
We have

\[ \square \psi = - \int_M \square_x (\chi Gu_0)(x, y, t) \varphi(y) dV_0(y) \]

\[ = - \int_M (\Delta^t \chi) Gu_0(x, y, t) \varphi(y) dV_0(y) - \int_M (\chi \square_x u_0)(x, y, t) \varphi(y) dV_0(y) \]

\[ + 2 \int_M \langle \nabla_x \chi, \nabla^t_x (Gu_0) \rangle(x, y, t) \varphi(y) dV_0(y) \]

\[ := f_1(x, t) + f_2(x, t) + f_3(x, t) \]

\[ := f(x, t), \]

It is clear that \( f_1, f_2 \) is bounded on \( M \times (0, \delta] \). For \( f_3 \), note that \( \nabla^t \chi = 0 \) when \( y \) near \( x \). This implies that \( f_3 \) is also bounded. So, \( f \) is bounded. Moreover, \( \psi \) it also clearly bounded.

We are interested in the initial value of \( \psi \). It is clear that

\[ \lim_{t \to 0^+} \int_M Z(x, y, t) \varphi(y) dV(y) = \varphi(x). \]

By direct computation,

\[ \lim_{t \to 0^+} \int_M \chi(x, y)G(x, y, t)u_0(x, y, t) \varphi(y) dV_0(y) \]

\[ = \lim_{t \to 0^+} \int_M G(x, y, t)[\chi(x, y)u_0(x, y, t) \varphi(y)] dV_t(y) \]

\[ = \chi(x, x)u_0(x, x, 0) \varphi(x) = \varphi(x), \]

since \( \chi(x, x) = 1, u_0(x, x, 0) = 1 \) when \( x \in \text{supp} \varphi \subset \Omega \). Therefore,

\[ (5.3) \lim_{t \to 0^+} \psi(x, t) = 0. \]

By Proposition 4.2,

\[ \psi(x, t) = \int_0^t \int_M Z(x, t; z, s) f(z, s) dV_s(z) ds \]

\[ = - \int_0^t \int_M Z(x, t; z, s) \int_M \square_z (\chi Gu_0)(z, y, s) \varphi(y) dV_0(y) dV_s(z) ds \]

\[ = - \int_M \left( \int_0^t \int_M Z(x, t; z, s) \square_z (\chi Gu_0)(z, y, s) dV_s(z) ds \right) \varphi(y) dV_0(y). \]
Note that $\varphi \in C_0^\infty(\Omega)$ is arbitrary. We get the identity (5.2).

Step 3. Verification of the asymptotic behavior.

By the identity (5.2), we have

$$
\mathcal{Z}(x, y, t) - \chi Gu_0(x, y, t)
= - \int_0^t \int_M \mathcal{Z}(x, t; z, s) \Delta_z (\chi Gu_0)(z, y, s) dV_s(z) ds
= - \int_0^t \int_M \mathcal{Z}(x, t; z, s) [(\Delta^s_z \chi)Gu_0](z, y, s) dV_s(z) ds
- \int_0^t \int_M \mathcal{Z}(x, t; z, s) (\chi G \square_z u_0)(z, y, s) dV_s(z) ds
+ 2 \int_0^t \int_M \mathcal{Z}(x, t; z, s) (\nabla^s_z \chi, \nabla^s_z (Gu_0))(z, y, s) dV_s(z) ds
=: I_1 + I_2 + I_3.
$$

We first estimate $I_1$,

$$
|I_1| \leq C_1 \int_0^t \int_{M \setminus B^\delta_y(\epsilon/6)} \mathcal{Z}(x, t; z, s) G(z, y, s) dV_s(z) ds
\leq C_2 \int_0^t s^{-\frac{n}{2}} e^{-\frac{a^2}{4s}} \int_M \mathcal{Z}(x, t; z, s) dV_s(z) ds
\leq C_3 \int_0^t s^{-\frac{n}{2}} e^{-\frac{a^2}{4s}} ds
\leq C_4 t
$$

for any $t \in (0, \delta]$, where $a$ is some positive constant independent of $t$.

Similarly,

$$
|I_3| \leq C_5 \int_0^t s^{-\frac{n}{2} - 1} e^{-\frac{a^2}{4s}} ds \leq C_6 t.
$$
We come to estimate $I_2$. By Corollary 5.2 in Chau-Tam-Yu [1],

$$|I_2| \leq C_7 \int_0^t \int_M \mathcal{Z}(x, t; z, s)G(z, y, s)dV(z)ds$$

$$\leq C_8 \int_0^t (t-s)^{-\frac{n}{2}} s^{-\frac{n}{2}} \int_M \exp \left(-\frac{b^2r_0^2(x, z)}{4(t-s)} - \frac{b^2r_0^2(z, y)}{4s} \right)dV_0(z)ds$$

$$\leq C_8 \int_0^t t^{-\frac{n}{2}} \int_M \frac{1}{(t-s)^{\frac{n}{2}}} e^{-\frac{b^2 \min(r_0^2(x, z), r_0^2(x, y))}{4(t-s)}} ds$$

$$\leq C_8 t^{-\frac{n}{2}} \int_0^t \int_{\{z \in M | r_0(x, z) \leq r_0(y, z)\}} \tau^{-\frac{n}{2}} e^{-\frac{b^2r_0^2(x, x)}{4t}} dV_0(z)ds$$

$$+ C_8 t^{-\frac{n}{2}} \int_0^t \int_{\{z \in M | r_0(y, z) \leq r_0(x, z)\}} \tau^{-\frac{n}{2}} e^{-\frac{b^2r_0^2(x, y)}{4t}} dV_0(z)ds$$

$$\leq C_9 t^{-\frac{n}{2}} \int_0^t \int_M \tau^{-\frac{n}{2}} e^{-\frac{b^2r_0^2(x, x)}{4t}} dV_0(z)ds + C_8 t^{-\frac{n}{2}} \int_0^t \int_M \tau^{-\frac{n}{2}} e^{-\frac{b^2r_0^2(x, y)}{4t}} dV_0(z)ds$$

$$\leq C_9 t^{1-\frac{n}{2}}$$

where $b$ is some positive constant independent of $t$ and $\tau = \frac{(t-s)s}{t}$.

Hence, $\chi u_0$ satisfies our requirements. \hfill \Box

Remark 5.1. We have frequently used the following fact in proof.

$$\int_M t^{-\frac{n}{2}} e^{-\frac{b^2r_0^2(x, y)}{4t}} dV(y) \leq C$$

for some positive $C$ constant depending only on $\delta$ on the lower bound of the Ricci curvature.

Another fact used in the proof is that

$$\lim_{t \to 0^+} \int_M \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{b^2r_0^2(x, y)}{4t}} \phi(y) dV(y) = \phi(y).$$

The proof is just the same as in the case of Euclidean space.

We come to derive some integral asymptotic behavior of the fundamental solution. Let $p$ be a fixed point and $u(x, t) = \mathcal{Z}(x, t, p, 0)$. Let $f$ be such that

$$u = \frac{e^{-f}}{(4\pi t)^{\frac{n}{2}}}.$$

Then

$$(2\Delta f - \|\nabla f\|^2)u = -2\Delta u + \frac{\|\nabla u\|^2}{u}.$$
Let $\epsilon$ be any small positive number. By Proposition 3.3 with $p = 1$,
\[ \int_M t|\Delta u|dV_t \leq C \]
for any $t \in (0, T]$. By Proposition 4.1 with $\delta = \frac{1}{2}$ and Corollary 3.1 with $p = \frac{1}{2}$,
\[ \int_M t\|\nabla u\|^2 u dV_t \leq C \]
for any $t \in (0, T]$. So,
\begin{equation}
\int_M t \left| 2\Delta f - \|\nabla f\|^2 \right| u dV_t \leq C
\end{equation}
for any $t > 0$.

By exactly the same computation as in the proof of Lemma 7.6 in Chau-Tam-Yu [1] using Proposition 5.1, we have the following integral asymptotic behavior.

**Proposition 5.2.** For any bounded nonnegative smooth function $h$ on $M \times [0, T]$,
\[ \lim_{t \to 0^+} \int_M fuh dV_t = \frac{n}{2} h(p, 0). \]

**Lemma 5.1.** For any nonnegative smooth function $h$ on $M \times [0, T]$ such that $\text{supp} h(t) \subset K$ for any $t$ and for some compact subset $K$ in $M$,
\[ \limsup_{t \to 0^+} \int_M t(2\Delta f - \|\nabla f\|^2)uh dV_t \leq \frac{n}{2} h(p, 0). \]

**Proof.** By Lemma 4.1 in Chau-Tam-Yu [1], for any $\alpha > 1$, $\epsilon > 0$,
\[ \frac{\|\nabla u\|^2}{u^2} - \alpha \frac{\Delta u}{u} \leq C_1(\alpha, \epsilon) + \frac{(n + \epsilon)\alpha^2}{2t}. \]

Hence
\[ \int_M t(2\Delta f - \|\nabla f\|^2)uh dV_t \]
\[ = \int_M t \left( \frac{\|\nabla u\|^2}{u} - 2\Delta u \right) h dV_t \]
\[ \leq C_1 t \int_M u\varphi dV_t + \frac{(n + \epsilon)\alpha^2}{2} \int_M uhdV_t + (\alpha - 2)t \int_M h\Delta u dV_t \]
\[ = C_1 t \int_M uhdV_t + \frac{(n + \epsilon)\alpha^2}{2} \int_M uhdV_t + (\alpha - 2)t \int_M u\Delta hdV_t \]
\[ \leq C_2 t + \frac{(n + \epsilon)\alpha^2}{2} \int_M uhdV_t. \]
Then,
\[
\limsup_{t \to 0^+} \int_M t(2\Delta f - \|\nabla f\|^2) u h dV_t \leq \frac{(n + \epsilon) \alpha^2}{2} h(p, 0).
\]

Letting \(\alpha \to 1^+\) and \(\epsilon \to 0^+\), we get the result. \(\square\)

**Proposition 5.3.** For any bounded nonnegative smooth function \(h\) on \(M \times [0, T]\),
\[
\limsup_{t \to 0} \int_M t(2\Delta f - \|\nabla f\|^2) u h dV_t \leq \frac{n}{2} h(p, 0).
\]

**Proof.** Let \(\{\rho_i\}\) be a partition of unit on \(M\). Then, by Lebesgue’s dominant convergence theorem, inequality (5.5), and that \(h\) is bounded,
\[
\int_M t(2\Delta f - \|\nabla f\|^2) u h dV_t = \sum_{i=1}^{\infty} \int_M t(2\Delta f - \|\nabla f\|^2) u \rho_i h dV_t.
\]
Furthermore, by Fatou’s lemma, inequality (5.5), the last lemma and that \(h\) is bounded,
\[
\limsup_{t \to 0} \int_M t(2\Delta f - \|\nabla f\|^2) u h dV_t \leq \sum_{i=1}^{\infty} \limsup_{t \to 0} \int_M t(2\Delta f - \|\nabla f\|^2) u \rho_i h dV_t
\]
\[
\leq \sum_{i=1}^{\infty} \frac{n}{2} \rho_i(p) h(p, 0) \]
\[
= \frac{n}{2} h(p, 0).
\]
\(\square\)

**Remark 5.2.** The proposition means that
\[
(5.6) \quad \limsup_{t \to 0^+} t(2\Delta f - \|\nabla f\|^2) u \leq \frac{n}{2} \delta_p
\]
in the sense of distribution.

**Remark 5.3.** By the last two propositions, we know the non-positivity of Perelman’s new Li-Yau-Hamilton expression \(v\) (Ref. Perelman [7] ) in the sense of distributions. By maximum principle, we know that \(v\) is non-positive.
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