Generic appearance of objective results in quantum measurements

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Measurement is of central interest in quantum mechanics as it provides the link between the quantum world and the world of everyday experience. One of the features of the latter is its robust, objective character, contrasting the delicate nature of quantum systems. Here we analyze in a completely model-independent way the celebrated von Neumann measurement process, using recent techniques of information flow, studied in open quantum systems. We show the generic appearance of objective results in quantum measurements, provided we macroscopically coarse-grain the measuring apparatus and wait long enough. To study genericity, we employ the widely-used Gaussian Unitary Ensemble of random matrices and the Hoeffding inequality. We derive generic objectivization timescales, given solely by the interaction strength and the systems’ dimensions. Our results are manifestly universal and are a generic property of von Neumann measurements.

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Understanding quantum measurements has been one of the central problems of quantum theory since its beginning [1,2]. It not only provides the crucial link between the theory and experiment, the macro- and macro-worlds, but is at the heart of the modern quantum technologies (see e.g. [3]). The fundamental measurement theory dates back to von Neumann [4] and since then has been further developed in various directions, e.g. the decoherence theory [5,6]. To be readable, measurement results must inevitably be encoded into macroscopic degrees of freedom and one of the crucial features expected from a good measurement process is an objective character of the results: They can be read out by arbitrary many observers and without causing any disturbance by the mere read-out. This has been realized as early as in 1929 by Mott [7]. Achieved in well engineered measurements by a proper coupling to macroscopic degrees of freedom, it is not at all obvious if such a situation is a generic feature of a quantum measurement process with a macroscopic recording.

In a broader context of open quantum systems [5,6], this may be seen as a question about how information flows from the system to its environment. Pioneering research along this direction has been undertaken under the quantum Darwinism idea [8], arguing that in some situations (see e.g. [9,10]) perfect information about the system can be redundantly stored in the environment and becomes effectively classical [11] and objective. The generic character of some of the quantum Darwinism features was shown in [12] and the universality of decoherence was shown on short time-scales in [13–16]. A further step was recently made in [17,18] by formulating information flow and objectivity in the fundamental language of quantum states with the introduction of the, so called, Spectrum Broadcast Structures (SBS’s). The latter has been proven to be a useful tool allowing to obtain novel results in some of the emblematic models of decoherence [17,19,21]. Finally, questions of genericity have traditionally been the domain of statistical mechanics and thermodynamics (see e.g. [22,24]). Phrased in this language, we may ask to what form a generic state equilibrates during a von Neumann measurement.

In this communication we study information flow during a von Neumann measurement process with a macroscopic (in a sense of a number of degrees of freedom) measuring apparatus. Applying random matrix theory techniques [25,26], we show that generically the post-measurement state approaches, after a coarse-graining, a form carrying almost perfect, multiple records of the measurement result, thus making the latter objective. To study genericity, we use a properly structured Gaussian Unitary Ensemble (GUE) [25,26]. Since the seminal works of Wigner and Dyson on statistics of various experimentally observed spectra, it has been the basic choice for random Hamiltonians due to its universality and agreement with the experiment [25,26]. The apparatus is assumed to be noisy, with the initial state distributed according to some physically motivated measures of mixed states [27]. For large-dimensional measured systems, we provide estimates on the time-scale of the objectivization process. Since the only assumptions we make concern the genericity measures, our results are manifestly universal and apply to the whole class of von Neumann measurements, thus showing a generic and robust character of the emergence of objectivity. It is a bit of a surprise that this property of von Neumann measurements was so
where \( \hat{A} \) is the measured observable (assumed non-degenerate) and the \( \hat{B}_k \) are some general measuring observables. This leads to the evolution (setting \( \hbar = 1 \))

\[
\hat{U} \equiv e^{-i\hat{H}_{\text{int}} t} = \sum_a |a\rangle\langle a| \otimes \bigotimes_{k=1}^N e^{-i\hat{a}_k t}, \quad \text{where} \quad \hat{A} = \sum_{a=1}^{d^2} a|a\rangle\langle a|.
\]

Our main object of study is a partially testable and decoupled state \( \rho_{\text{SE obs}} \), with a fraction \( E_{\text{uno}} \) of size \( N_{\text{uno}} \) of unobserved subsystems traced out. This represents an inevitable loss of information during a measurement. Assuming \( \rho_{\text{SE}}(0) = \rho_{\text{obs}} \otimes \bigotimes_{k=1}^N \rho_{0k} \) we obtain:

\[
\rho_{\text{SE obs}}(t) = \sum_a p_a |a\rangle\langle a| \otimes \bigotimes_{k=1}^N \rho_{ak}(t) + \sum_a \sum_{a' \neq a} c_{aa'}
\]

\[
\times \left\{ \prod_{k=1}^{N_{\text{obs}}} \text{Tr}[e^{-i(a-a')\hat{B}_k t} \rho_{0k}] \right\} |a\rangle\langle a'| \otimes \bigotimes_{k=1}^N e^{-i\hat{a}_k t} \rho_{0k} e^{i\hat{a}_k t},
\]

where \( p_a \equiv \langle a|\rho_{\text{obs}}|a\rangle \), \( c_{aa'} \equiv \langle a|\rho_{\text{obs}}|a'\rangle \), \( \rho_{ak}(t) \equiv e^{-i\hat{a}_k t} \rho_{0k} e^{i\hat{a}_k t} \), \( N_{\text{uno}} + N_{\text{obs}} = N \). We define the decoherence factor for the unobserved fraction \( E_{\text{uno}} \):

\[
\Gamma_{\text{uno}}^{\text{obs}}(t) = \prod_{k=1}^{N_{\text{uno}}} \left| \text{Tr}[e^{-i(a-a')\hat{B}_k t} \rho_{0k}] \right|^2.
\]

If for all \( a \neq a' \): i) \( \Gamma_{\text{uno}}^{\text{obs}}(t) = 0 \), i.e. decoherence takes place, and ii) \( \rho_{ak}(t) \perp \rho_{a'k}(t) \), i.e. \( \rho_{ak}(t) \) are perfectly distinguishable, then we say that \( \rho_{\text{SE obs}}(t) \) is of a Spectrum Broadcast Structure (SBS) [17, 18] with respect to (w.r.t.) the basis \(|a\rangle\) (this context-dependence is of a fundamental importance, see e.g. [29]), defined as [30].

\[
\rho_{\text{SBS}} = \sum_a p_a |a\rangle\langle a| \otimes \rho_a \otimes \cdots \otimes \rho_a, \quad \rho_a \perp \rho_{a' \neq a}.
\]

The basis \(|a\rangle\) becomes then the, so-called, pointer basis in which the system has decohered and the result of the measurement, \( a \), appearing with the probability \( p_a \), is stored in the measuring setup in many, perfect copies. Crucially, their readouts, through projections on the supports of \( \rho_{ak}(t) \), will not disturb (on average) the joint state \( \rho_{\text{SE obs}}(t) \). This leads to a form of objectivity of the measurement result: It can be read out by multiple observers without disturbing neither the (decohered) system nor themselves [8, 17, 18]. In quantum-information terms, this objectivization process is a weaker form of quantum state broadcasting [31, 32]. We can thus reformulate the original question as: Are SBS’s generic for the interactions [1]? To address it, we introduce an ensemble of random Hamiltonians of the form (1) and random initial conditions \( \rho_{0k} \). We then estimate the average trace distance between the actual state (2) and an ideal SBS in the following steps: i) calculate the averages over \( \hat{B}_k \) of the decoherence factor (3) and the, so-called, super-fidelity bound for the states \( \rho_{ak}(t) \); ii) average them over \( \rho_{0k} \); iii) coarse-grain the apparatus; iv) further average over \( \hat{A} \); v) use the central result of (21) to bound the average distance and show that it vanishes in the macroscopic limit. We then use the concentration inequality of Hoeffding [33], following from the classical Chernoff bound, to show genericity.

The coarse-graining is one of the crucial steps. As we will show, on the microscopic level of the individual apparatuses, the residual noise is too strong to allow a SBS formation even asymptotically. This can be overcome if we group the \( N_{\text{obs}} \) observed apparatuses into fractions scaling with \( N \) (called macrofractions) and pass to the thermodynamic limit \( N \to \infty \) [17]. The number \( \mathcal{M} \) of such groups (assumed for simplicity equal) is irrelevant, provided their sizes \( N_{\text{mac}} \equiv N_{\text{obs}}/\mathcal{M} \) satisfy \( N_{\text{mac}} \sim N \). These macrofractions may be understood as reflecting some detection threshold, e.g. a minimum bunch of photons the eye can detect.

Randomizing measurement Hamiltonians.— We introduce an ensemble of random Hamiltonians [1] using the widely-used Gaussian Unitary Ensemble [25, 26] in the following way (cf. [34, 35]): i) \( \hat{B}_k \) are independently, identically distributed (i.i.d.) according to a GUE with a scale factor \( \eta \); ii) \( \hat{A} \) is distributed according to its own GUE with a scale factor \( \eta_{\text{SBS}} \). We recall that the GUE measure is defined as:

\[
d\mu_{\text{GUE}}(\hat{H}) = \frac{1}{Z} e^{-\frac{1}{2} \sum_{i<j} \lambda_i \lambda_j} \prod_{i} \mu(\lambda_i) d\lambda d\hat{U},
\]

with \( Z \) the normalization, \( \lambda_i \) the eigenvalues, \( \eta \) a scale factor, and \( d\hat{U} \) the Haar measure on the unitary group.

The simultaneous vanishing of the decoherence factor (4) and of the generalized overlaps (31) [36] \( F_{\text{ad}'} \equiv F(\rho_{\text{ad}'}, \rho_{\text{ad}}') \equiv \text{Tr} \sqrt{\rho_{\text{ad}'} \rho_{\text{ad}}'} \leq \text{G}(\rho, \sigma) \equiv \text{Tr} (\rho \sigma) + (1 - \text{Tr} \rho^2)(1 - \text{Tr} \sigma^2) \) (although we note that it is not tight if both states are mixed, as e.g. for \( \rho \perp \sigma, \text{G}(\rho, \sigma) = 0 \)), which here reads:

\[
\text{G}(\rho_{ak}(t), \rho_{a'k}(t)) = \text{Tr} (\rho_{ak}(t)\rho_{a'k}(t)) + S_{\text{lin}} (\rho_{ak}) \equiv G_{\text{ad}}(t),
\]
where $S_{\text{lin}}(\rho_0) \equiv 1 - \text{Tr} \rho_0^2$ is the linear entropy of the initial state of an individual apparatus.

We now average (36) over the interaction and the initial conditions. We first average over $\{\hat{B}_k\}$, fixing the levels $a, a'$ of $\hat{A}$. We have:

$$\langle \Gamma_{ad}^{\text{uno}}(t) \rangle_{\hat{B}_k} = \frac{N_{\text{uno}}}{\sum_{k=1}^{N_{\text{uno}}} \langle | \text{Tr}e^{-i(a-a')\hat{B}_k}\rho_{0k}|^2 \rangle_{\hat{B}_k}},$$  

(7)

since $\hat{B}_k$ are i.i.d. Modulo $\rho_{0k}$, all the factors are identical and we calculate the average over a single $\hat{B}_k$, dropping the index $k$ for simplicity. Performing the Haar integration first ([38], Section I) and then the eigenvalue one ([38], Section IIB), we obtain [39]:

**Result 1.** The GUE averages of the single environment decoherence and super-fidelity factors read:

$$\langle \Gamma_{ad}(t) \rangle = \frac{1 + \text{Tr} \rho_0^2}{d+1} + (f_1(a, \lambda)) \frac{2(d - \text{Tr} \rho_0^2)}{d(d^2 - 1)},$$  

(8)

$$\langle G_{ad}(t) \rangle = S_{\text{lin}}(\rho_0) + \frac{1 + \text{Tr} \rho_0^2}{d+1} + (f_1(a, \lambda)) \frac{2(d \text{Tr} \rho_0^2 - 1)}{d(d^2 - 1)},$$  

(9)

with $f_1(a, \lambda) \equiv \sum m \sum_{m>n} \cos [(a - a')(\lambda_n - \lambda_m)t]$ and:

$$\langle f_1(a, \lambda) \rangle = p(d, \Delta_1) e^{-\Delta_1^2},$$  

(10)

$$p(d, \Delta_1) \equiv$$

(11)

$$\sum_n \sum_{m>n} L_0(n, m) \sum L_0(n, m) \left[ \frac{\Delta_2^2}{m!} \frac{\Delta_1^2}{n!} \frac{\Delta_1^2}{(m-n)!} \right]$$

where $\Delta_1 \equiv (a - a')/\sqrt{\eta_1}$ and $L_{ij}^{(m)}$ are the associated Laguerre polynomials.

The above results are exact. Although the average $\langle f_1(a, \lambda) \rangle$ with the GUE eigenvalue distribution $P_{\text{gue}}(\lambda)$ involves only the two-point correlation function [25]:

$$R_2(\lambda_1, \lambda_2) \equiv \frac{d!}{(d-2)!} \int \cdots \int d\lambda_3 \cdots d\lambda_d P_{\text{gue}}(\lambda_1, \ldots, \lambda_d)$$

(due to the symmetry), and the large-$d$ asymptotics of $R_2(\lambda_1, \lambda_2)$ are well known [25], they are of no use here. One can show that [25]:

$$R_2(\lambda_1, \lambda_2) \equiv K_d(\lambda_1, \lambda_1) K_d(\lambda_2, \lambda_2) - K_d(\lambda_1, \lambda_2)^2, \quad K_d(\lambda_1, \lambda_2) \equiv \sum_{l=0}^{d-1} \phi_l(\lambda_1) \phi_l(\lambda_2),$$

with $\phi_l(\lambda)$ the oscillator wave-functions, and while the first term approaches the Wigner semicircle distribution, integrable with $f_1(a, \lambda)$, the second term approaches a function of $|\lambda_1 - \lambda_2|$ only [25] and makes the integral divergent. That is the integration with $f_1(a, \lambda)$ and the large-$d$ limit are not interchangeable here.

Both [8,9] depend on $\rho_0$ only through its purity $\text{Tr} \rho_0^2$ and we can use the known results of generic state purity to effectively get rid of the initial state dependence. Although there is no canonical choice of a measure over mixed states, there are several popular ones e.g. the Hilbert-Schmidt and the Bures measures [27] giving:

$$\langle \text{Tr} \rho_0^2 \rangle_{\text{HS}} = \frac{2d}{d^2+1}, \quad \langle \text{Tr} \rho_0^2 \rangle_{\text{BU}} = \frac{5d^2 + 1}{2d(d^2+2)}.$$  

(12)

Especially the Bures measure is physically important as it: i) is directly connected to quantum metrology [40]; ii) reproduces the correct measure for pure states. In what follows we will assume that $\rho_{0k}$ are i.i.d. with one of the above measures and are averaged over.

**Residual noise and coarse-graining.** As $p(d, \Delta_1)$ is an even polynomial of degree $2(2d-3)$, [10] implies that the time dependent part in [8,9] decays for any fixed $d$ and a gap $|a - a'| \neq 0$ with a characteristic time $\tau_{ad} \equiv |a - a'|^{-1} \sqrt{\eta_1}/(d+1)$ ([38], Section II). The remaining constant terms: A common one of the order $O(1/d)$ (cf. (12)), called "white noise", and additionally $\langle S_{\text{lin}}(\rho_0) \rangle$ in (9). The latter, arising from the non-tight bound (6), is intuitively understood—the noisier the apparatus is initially, the lesser information, measured by the state distinguishability, it can accumulate. These factors, reflecting residual background fluctuations in the ensemble, pertain to a single apparatus and prevent a SBS formation. However, coming back to (3), using (7) and (12), we actually obtain an exponential decay with $N_{\text{uno}}$ of the collective decoherence factor:

$$0 \leq \langle \Gamma_{ad}^{\text{uno}}(t) \rangle \leq \langle \Gamma_{ad}(t) \rangle^{N_{\text{uno}}} \xrightarrow{t \gg \tau_{ad}} O \left( d^{-N_{\text{uno}}} \right),$$  

(13)

showing that for a large local dimension $d$ and/or large unobserved fraction $N_{\text{uno}}$, measurement dynamics (1) generically leads to decoherence (cf. [13]). The same step can be preformed on the observed fraction too [17]: We group the $N_{\text{obs}}$ observed apparatuses into $M$ groups of $N_{\text{mac}}$ each, described by states $\rho_{0k}^{\text{mac}}(t) \equiv \otimes_{k \in \text{mac}} R_{\text{obs}}(t)$. Due to the factorization of fidelity w.r.t. the tensor product and the i.i.d. property, the resulting super-fidelity bound (6) for the group also decays (cf. [12]):

$$0 \leq \langle \Gamma_{mac}(t) \rangle \leq \langle G_{mac}(t) \rangle^{N_{\text{mac}}} \xrightarrow{t \gg \tau_{mac}} O \left( e^{-N_{\text{mac}}/d} \right).$$  

(14)

If both $N_{\text{uno}}, N_{\text{mac}}$ scale with $N$, [13,14] can be made small in the macroscopic/thermodynamic limit $N \to \infty$. Crucially, increasing $d$ alone is not enough—it damps the white noise, but $\langle S_{\text{lin}}(\rho_0) \rangle \sim 1 - O(1/d)$ by [12].

**Generic post measurement state and objectivity.**—Results [8,9] still depend on the $\hat{A}$’s level differences $|a - a'|$. To study a completely general behavior, a further averaging of $\langle \Gamma_{ad}^{\text{uno}}(t) \rangle$, $\langle G_{ad}(t) \rangle$ over the levels $a, a'$ should be performed with the corresponding two-point correlation function $R_2(a, a')$ (the average is independent of the labels $a, a'$ due to the symmetry). The resulting integrals are intractable, but from (8-A52) they will eventually reach the noise-floor (see Fig. 1). Lower bounds on the relevant timescales can be obtained from
state on the coarse-grained level of macrofractions:

\[
\epsilon_{SBS}(t) \equiv \frac{1}{2} \lVert \rho_{SBS}^a(t) - \rho_{SBS}^b \rVert_{Tr}
\]  

(18)

\[
\leq \sum_{a} \sum_{a' \neq a} |c_{aa}^a| \sqrt{F_{aa}^a(t)} + \sqrt{F_{aa}^a(t)} \sum_{mac} \sqrt{F_{aa}^{mac}(t)}
\]

Using \(p_{aa'}, |c_{aa'}| \leq 1, \langle \sqrt{T} \rangle \leq \langle \sqrt{T} \rangle \) for \( f \geq 0 \), the super-fidelity bound, and the Result 2, estimation (18) gives:

**Result 3.** Averaged over all the von Neumann measurements (1) and the initial conditions, the optimal distance of the actual state to an ideal SBS state satisfies:

\[
\langle \langle \epsilon_{SBS}(t) \rangle \rangle \xrightarrow{t \gg \tau_{SBS}} O \left[ d_S^2 \left( 2^{-N_{uno} \log d} + \mathcal{M} e^{-N_{mac} \log d} \right) \right]
\]  

(19)

where \( \tau_{SBS} \) is the largest of (15) and \( \mathcal{M} \) is the number of macrofractions into which the observed degrees of freedom of the apparatus are coarse-grained.

Finally, since \( 0 \leq \epsilon_{SBS}(t) \leq 1 \) is a bounded random variable for any \( t \), it follows from the Hoeffding inequality that:

\[
|\langle \langle \epsilon_{SBS}(t) \rangle \rangle - \langle \epsilon_{SBS}(t) \rangle| \leq 2 e^{-2d^2} \text{ for any } d > 0.
\]

This, together with Result 3, shows the genericity of the SBS formation for large enough apparatuses and long enough times.

**Conclusions.**—A measurement is an inevitable part of any quantum experiment and the results must inevitably be encoded into macroscopic degrees of freedom and become effectively classical for us to read. This in particular entails becoming objective. We studied this process using the general von Neumann measurement scheme (1) with a macroscopic measuring apparatus. A huge amount of degrees of freedom \((N \sim 10^{23})\) makes it in practice impossible to observe them all and to control each individual coupling. A way to model this physical situation is to introduce some randomness and ask questions about genericity. We did it in two steps: First we randomized the measurement device side (the observables and the generically noisy initial states) and showed that after including the inevitable losses and macroscopic coarse-graining, a post measurement state approaches the so called SBS form asymptotically for almost any initial conditions and couplings. The timescales of this process depended on the spectral gap of the measured observable on the system side. Afterwards, to get rid of this dependence, we went beyond a single experiment scenario, randomizing the measured observable too. An interesting aspect of that second randomization is that this may be viewed as a quite natural assumption of any quantum system interacting with many objects. Indeed it is natural to assume that it interacts with each of the objects with some fixed, yet different than with the others, way. Since there are many objects, then the averaging effect comes from that variety of the interactions and can be viewed as a self-averaging of the system plus environment com-

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**FIG. 1:** (Color online). Time dependence of the exact full averages of the decoherence factor (a),(c) and the super-fidelity respectively:

- FIG. 1: (a) \( N_{uno} = 1 \)
- FIG. 1: (b) \( N_{mac} = 1 \)
- FIG. 1: (c) \( N_{uno} = 20 \)
- FIG. 1: (d) \( N_{mac} = 20 \)

The interaction and initial state averages satisfy:

\[
\langle \langle G^{uno}(t) \rangle \rangle \xrightarrow{t \gg \tau_{dec}} O \left( e^{-N_{uno} \log d} \right),
\]

(16)

\[
\langle \langle G^{mac}(t) \rangle \rangle \xrightarrow{t \gg \tau_{fid}} O \left( e^{-N_{mac} \log d} \right).
\]

(17)

Next crucial step is to use the result of (21) estimating an optimal trace distance between (2) and an ideal SBS...
plex. This led to our central result: Almost any quantum measurement produces objective outcomes on the macroscopic level on the timescale given by the larger of \(t\). This is a universal, model-independent result.

We believe one can go beyond the genericity notion used here (Hoeffding inequality) and show the concentration of measure phenomenon, e.g. by combining the results for the Wigner-type matrices \([36]\) with the methods of \([22, 42]\). Another possible future direction is to consider non-trivial dynamics of the system and the measuring device. A candidate tool for such an analysis already exists in the form of dynamical SBS \([19]\).

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Appendix A: Ensemble average over the apparatus

1. Average over the Haar distributed unitary transformations $U$

In this Section we average the decoherence and the super-fidelity factors over the Haar measure. Due to the assumed independent identical distribution (i.i.d.) of the apparatus observables $B_k$, $k = 1, \ldots, N$, it is enough to calculate the averages over a single observable only. This is what we shall calculate, neglecting for brevity the index $k$. We start with the decoherence factor and prove that:

**Theorem 4.** The decoherence factor for the single copy of the environment average over the Haar distributed unitary transformations $U$ is equal to:

$$
\langle \Gamma_{aa'}(t) \rangle_U = | \text{Tr} D |^2 \frac{d - \text{Tr}[\rho_0^2]}{d(d^2 - 1)} + \frac{d \text{Tr}[\rho_0^2] - 1}{d^2 - 1}
$$

where $d$ is the local dimension of the environment.

We first write the decoherence factor as:

$$
\Gamma_{aa'}(t) = \left| \text{Tr}[e^{-i(a-a')Bt} \rho_0] \right|^2 = \text{Tr}[e^{-i(a-a')Bt} \rho_0] \text{Tr}[e^{-i(a-a')Bt} \rho_0]^\dagger = \text{Tr}[e^{-i(a-a')Bt} \rho_0] \text{Tr}[(e^{-i(a-a')Bt})^\dagger \rho_0]
$$

where we diagonalized the observable $B$ as $B = U \text{diag}[\lambda_1, \ldots, \lambda_d] U^\dagger$ and defined:

$$
D \equiv \text{diag}[e^{-i\Delta t \lambda_1}, \ldots, e^{-i\Delta t \lambda_d}], \quad \Delta t \equiv (a-a')t.
$$

We also used $\text{Tr} A \text{Tr} B = \text{Tr}(A \otimes B)$ in the second line and the following fact in the first step:

**Fact 1.** For any operator $X$ the following is true

$$
| \text{Tr} X |^2 = \text{Tr} X \text{Tr} X^\dagger,
$$

where $^\dagger$ stands for hermitian conjugation.

**Proof.**

$$
| \text{Tr} X |^2 = \text{Tr} X \overline{X} = \text{Tr} X X^T = \text{Tr} X X^\dagger,
$$

where $\overline{X}$ stands for the complex conjugation of $X$ and we used $\text{Tr} X = X^T$, and $X^\dagger = X^T$.

We will also need two more well known facts:

**Fact 2.** For any operators $A$, $B$ and the SWAP operator $\mathbb{V}$, we have that:

$$
\text{Tr}[\mathbb{V} A \otimes B] = \text{Tr}(AB).
$$

**Proof.** Let us write the SWAP operator as:

$$
\mathbb{V} = \sum_{ij} |ij \rangle \langle ji|
$$

Inserting (A7) into (A6) we have that:

$$
\text{Tr}[\mathbb{V} A \otimes B] = \text{Tr} \left( \sum_{ij} |ij \rangle \langle ji| A \otimes B \right) = \text{Tr} \left( \sum_{ij} |ij \rangle \langle ij| A \otimes B \right) = \sum_{ijkl} \langle ji| A |kl \rangle \langle kl| B |ij \rangle = \text{Tr}(AB).
$$
**Fact 3.** For any hermitian operator $X$ from $\mathbb{C}^d$ to $\mathbb{C}^d$, it holds:

\[
\int d\mathcal{U}I \otimes UX \otimes XU^\dagger \otimes U^\dagger = \frac{2}{d(d + 1)} \text{Tr}[\Pi_{\text{sym}}X \otimes X]\Pi_{\text{sym}} + \frac{2}{d(d - 1)} \text{Tr}[\Pi_{\text{asym}}X \otimes X]\Pi_{\text{asym}},
\]

(A9)

where $\Pi_{\text{sym}}$ and $\Pi_{\text{asym}}$ are the orthogonal projectors onto the symmetric and antisymmetric subspaces, respectively, equal to

\[
\Pi_{\text{sym}} \equiv \frac{I + V}{2}, \quad \Pi_{\text{asym}} \equiv \frac{I - V}{2},
\]

(A10)

where $V$ is the SWAP operator.

We now integrate Eq. (A2) over $U \otimes U$. Using linearity of the trace we pull the integral inside the trace:

\[
\langle \Gamma_{ad}(t) \rangle_{\mathcal{U}} = \int d\mathcal{U} \text{Tr}(U^d \otimes U^d \rho_0 \otimes \rho_0 U \otimes UD \otimes D^\dagger) = \text{Tr}\left[\left(\int d\mathcal{U}U \rho_0 \otimes \rho_0 U^\dagger \otimes U^\dagger\right)d \otimes D^\dagger\right]
\]

(A11)

We then use Fact 3 with $X \equiv \rho_0$. We can easily calculate $\text{Tr}[\Pi_{\text{sym}}\rho_0 \otimes \rho_0]$ and $\text{Tr}[\Pi_{\text{asym}}\rho_0 \otimes \rho_0]$ using Facts 1 and 2 and obtain:

\[
\text{Tr}[\Pi_{\text{sym}}\rho_0 \otimes \rho_0]\Pi_{\text{sym}} = \frac{1}{2} \text{Tr}[\{I + V\}(\rho_0 \otimes \rho_0)] \frac{I + V}{2} = \frac{1 + \text{Tr}[\rho_0^2]}{2} \frac{I + V}{2},
\]

(A12)

and for the antisymmetric projector

\[
\text{Tr}[\Pi_{\text{asym}}\rho_0 \otimes \rho_0]\Pi_{\text{asym}} = \frac{1}{2} \text{Tr}[\{I - V\}(\rho_0 \otimes \rho_0)] \frac{I - V}{2} = \frac{1 - \text{Tr}[\rho_0^2]}{2} \frac{I - V}{2}.
\]

(A13)

We then again use Facts 1 and 2 to calculate the remaining traces $\text{Tr}\left[(I \pm V)D \otimes D^\dagger\right]$, keeping in mind that $D$ is hermitian and that $\text{Tr}D^2 = d$. This finally gives:

\[
\langle \Gamma_{ad}(t) \rangle_{\mathcal{U}} = |\text{Tr}D|^2 \frac{d - \text{Tr}[\rho_0^2]}{d(d^2 - 1)} + \frac{d\text{Tr}[\rho_0^2] - 1}{d^2 - 1},
\]

(A14)

proving our Theorem.

Using the same technique, one can also calculate the average of the super-fidelity factor. The only non-trivial part is the Hilbert-Schmidt product between the apparatus states $\rho_a(t)$ and $\rho_{a'}(t)$. Using the same notation as in Eq. (A2) we obtain:

\[
\text{Tr}(\rho_a(t)\rho_{a'}(t)) = \text{Tr}[\rho_0 e^{i(a-a')B} \rho_0 e^{-i(a-a')B}] = \text{Tr}[U^d \rho_0 UD^\dagger U^\dagger \rho_0 UD] = \text{Tr}[V(U^d \otimes U^d)(\rho_0 \otimes \rho_0)(U \otimes U)(D^\dagger \otimes D)],
\]

(A15)

where in the last step we used Fact 3. We note that the only difference between the above Hilbert-Schmidt factor and the decoherence factor (A2) is the presence of the SWAP operator $V$. Repeating the same steps as above gives:

\[
\langle G_{ad}(t) \rangle_{\mathcal{U}} = S_{\text{lin}}(\rho_0) + \frac{d - \text{Tr}[\rho_0^2]}{d^2 - 1} + |\text{Tr}D|^2 \frac{d\text{Tr}[\rho_0^2] - 1}{d(d^2 - 1)}.
\]

(A16)

Finally, we evaluate $|\text{Tr}D|^2$ from its definition in Eq. (A3):

\[
|\text{Tr}D|^2 \equiv d + 2f_\ell(a, \lambda), \quad f_\ell(a, \lambda) \equiv \sum_{m \geq n} \cos[[\lambda_n - \lambda_m]] = \sum_{m \geq n} \sum_{n > m} \cos[(a - a')(\lambda_n - \lambda_m)t],
\]

(A17)

which is a function of the eigenvalues $a$ of the observable $A$ and the eigenvalues $\lambda$ of $B$. 

2. Averaging over the eigenvalues

After averaging over the unitary group in Sec. [A.1] we perform the average over the GUE eigenvalue distribution:

\[ P_{\text{gue}}(\lambda) = \frac{1}{Z} e^{-\frac{1}{2} \eta_E \sum \lambda^2_i} \prod_{i<j} |\lambda_j - \lambda_i|^2. \]  

(A18)

Here \( \eta_E \) is the eigenvalue scale of the observable \( B \) and \( Z \) is a normalization constant (the GUE partition function). The task then is to find the following average:

\[ \langle fi(a, \lambda) \rangle = \langle \sum_{i<j} \cos(\Delta_t(\lambda_i - \lambda_j)) \rangle = \sum_i \sum_j \int d\lambda \, P_{\text{gue}}(\lambda) \cos(\Delta_t(\lambda_i - \lambda_j)) \]  

(A19)

Quite surprisingly, this average can be performed explicitly using the standard methods of dealing with GUE [25]. We first introduce the harmonic oscillator wave functions:

\[ \phi_n(x) \equiv \frac{1}{\sqrt{\sqrt{2\pi n!}}} e^{-\frac{x^2}{2}} He_n(x). \]  

(A20)

Notice that we define the wave functions using the so-called "probabilist" Hermite polynomials:

\[ He_n(x) \equiv (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}} \]

That is, they are orthogonal with respect to the weight function \( \exp[-x^2/2] \), and are related to the physicist’s polynomials \( H_n(x) \) via \( He_n(x) = 2^{-n/2} H_n(x/\sqrt{2}) \). Of course we still have \( \int \phi_n(x)\phi_m(x) = \delta_{nm} \). Then the GUE eigenvalue distribution takes on a very compact and elegant form, after rescaling \( \lambda_k \equiv \frac{\xi_k}{\sqrt{\eta_E}} \) (6.2.4):

\[ P_{\text{gue}}(\lambda)d\lambda = \frac{1}{d!} \det[\phi_{j-1}(\xi_i)]^2 d\xi, \]  

(A21)

where \( i, j = 1, \ldots, d \).

a. Exploiting the symmetry

A crucial step is the realization that this average has an index permutation symmetry. Let \( \sigma \in S_d \), be a permutation, then:

\[ P_{\text{gue}}(\lambda_{c(1)}, \ldots, \lambda_{c(d)}) = P_{\text{gue}}(\lambda_1, \ldots, \lambda_d). \]  

(A22)

Analogously we have (keeping the eigenvalues \( a \) fixed):

\[ fi(a, \lambda_{c(1)}, \ldots, \lambda_{c(d)}) = \sum_{i<j} \cos[\Delta_t(\lambda_{c(i)} - \lambda_{c(j)})] = \sum_{i<j} \cos[\Delta_t(\lambda_i - \lambda_j)] = fi(a, \lambda). \]  

(A23)

This is because in both expressions the eigenvalue functions are symmetric and all pairs of indices are taken (i.e. the product or sum is over all \( i < j \)). Equivalently, we can recall that \( \text{Tr} D = \sum_i e^{-i\Delta_t \lambda_i} \) and from Eq. (A17) \( fi(a, \lambda) = 1/2(|\text{Tr} D|^2 - d) \), which is clearly symmetric under the permutations. Hence, the calculation of the average \( fi(a, \lambda) \) reduces to a single term:

\[ \langle fi(a, \lambda) \rangle = \frac{d(d-1)}{2} \int d\lambda \, P_{\text{gue}}(\lambda) \cos(\Delta_t(\lambda_1 - \lambda_2)) \]  

(A24)
As the integrand only depends on two variables, we can take the marginal distribution, defined as \[25\] (6.1.2):

\[
R_2(\lambda_1, \lambda_2) \equiv \frac{d!}{(d-2)!} \int d\lambda_3 \cdots d\lambda_d \ P_{\text{gue}}(\lambda_1, \ldots, \lambda_d).
\]

Hence, we have reduced the problem to the integral:

\[
\langle f_1(a, \lambda) \rangle = \frac{1}{2} \int d\lambda_1 d\lambda_2 \ R_2(\lambda_1, \lambda_2) \cos[\Delta(\lambda_1 - \lambda_2)].
\]

**b. The crucial integral**

To calculate (A26) we will need the following integral:

\[
J_{n,m}(a) \equiv \int dx \ \phi_n(x)\phi_m(x)e^{iax}.
\]

(A27)

In fact (A27) may be interpreted as a special case of a matrix element of the displacement operator \(D(\beta) = \exp[\beta \alpha^\dagger - \beta^* \alpha]\) in the Fock basis \(\{|n\rangle\}_{n \in \mathbb{N}}\). We recover our integral setting \(\beta \equiv ia, \ a \in \mathbb{R}\). This turns out to be a well known quantity in quantum optics (see e.g. [44]), but for completeness we present its calculation below. Without a loss of generality, we will assume \(m \geq n\). First, we express the wavefunctions through the (probabilist) Hermite polynomials as in Eq. (A20) and use the generating function for \(He_n(x)\) with parameters \(r, s\) to perform the integral:

\[
J_{n,m}(a) = \frac{1}{\sqrt{2\pi n! m!}} \int dx \ H_n(x)H_m(x)e^{-\frac{1}{2}x^2+iax}
\]

(A28)

\[
= \frac{1}{\sqrt{2\pi n! m!}} \frac{\partial^n}{\partial r^n} \bigg|_{r=0} \frac{\partial^m}{\partial s^m} \bigg|_{s=0} \int dx \ e^{(ia+r)s-x/2(r+s)+x^2}e^{iax+ias+rs}
\]

(A29)

\[
= \frac{e^{-\frac{1}{2}a^2}}{\sqrt{n! m!}} \frac{\partial^n}{\partial r^n} \bigg|_{r=0} \frac{\partial^m}{\partial s^m} \bigg|_{s=0} (ia+r)^me^{iax}
\]

(A30)

\[
= \frac{e^{-\frac{1}{2}a^2}}{\sqrt{n! m!}} \frac{\partial^n}{\partial r^n} \bigg|_{r=0} \sum_k \binom{n}{k} \binom{m}{k} k!(ia)^{n+m-2k}
\]

(A31)

Then we use the binomial formula for the derivatives \(\frac{d^n}{dx^n} f(x)g(x) = \sum_{k=0}^n \binom{n}{k} \frac{d^{n-k}f(x)}{dx^{n-k}} \frac{d^k g(x)}{dx^k}\). Since \(m \geq n\) we don’t run into any unexpected problems and obtain:

\[
J_{n,m}(a) = \frac{e^{-\frac{1}{2}a^2}}{\sqrt{n! m!}} \sum_k \binom{n}{k} \binom{m}{k} k!(ia)^{n+m-2k}
\]

(A32)

This may be nicely expressed in terms of the associated Laguerre polynomials as (taking \(a \in \mathbb{R}\)):

\[
J_{n,m}(a) = e^{-\frac{1}{2}a^2} \sqrt{\frac{n!}{m!}} (ia)^{m-n} L_n^{(m-n)}(a^2),
\]

(A33)

where:

\[
L_n^{(m)}(x) \equiv \sum_k \binom{n+m}{n-k} \frac{(-x)^k}{k!}
\]

(A34)

(we adopt the common standardization for the Laguerre polynomials that the leading coefficient is equal to \((-1)^n/n!\)). Eq. (A27) can also be expressed more compactly in terms of the, so-called, 2D Laguerre functions introduced in [45]:

\[
\langle m|D(a)|n \rangle = (-1)^n \sqrt{n!} L_n^{(m-n)}(a, a^*)
\]

(A35)
for a general complex displacement $a$. The 2D Laguerre functions are defined as [43]:

$$l_{m,n}(z,z^*) \equiv \frac{1}{\sqrt{\pi}} e^{z^2} \frac{1}{\sqrt{m!n!}} \sum_{j=0}^{m} \binom{n}{j} \frac{j!(j-n)!}{j!(n-j)!} (-1)^{j-n} z^{m-j} z^{*n-j}.$$  \hspace{1cm} (A36)

**c. Putting the results together**

We return to calculating the integral (A26). We use Eq. (A21), rescale the variables, and introduce a more friendly notation $(x, y) \equiv (\zeta_1, \zeta_2)$:

$$\langle f_t(a, \lambda) \rangle = \frac{1}{2} \int dxdy \ R_2(x, y) \cos[\tilde{\Delta}_t(x - y)],$$  \hspace{1cm} (A37)

where:

$$\tilde{\Delta}_t \equiv \frac{(a - a') t}{\sqrt{T}}.$$  \hspace{1cm} (A38)

Now, Dyson’s Theorem will let us calculate the 2-point correlation function [25] Thm 5.14 , (6.2.6-7):

$$R_2(x, y) = K(x, x) K(y, y) - K(x, y)^2,$$  \hspace{1cm} (A39)

where the kernel is defined through the oscillator wave-functions (A20) as:

$$K(x, y) \equiv \sum_{j=0}^{d-1} \phi_j(x) \phi_j(y).$$  \hspace{1cm} (A40)

Hence, we can express our integral as the following sum:

$$\langle f_t(a, \lambda) \rangle = \sum_{n=0}^{d-1} \sum_{m=0}^{d-1} \int dxdy \ \left( \phi_n(x)^2 \phi_m(y)^2 - \phi_n(x) \phi_m(x) \phi_n(y) \phi_m(y) \right) \cos[\tilde{\Delta}_t(x - y)].$$  \hspace{1cm} (A41)

By expressing the cosine function in exponential form, the integrals become separable and we obtain:

$$\langle f_t(a, \lambda) \rangle = \frac{1}{4} \sum_{n=0}^{d-1} \sum_{m=0}^{d-1} \left( A_{n,m} + \tilde{A}_{n,m} - 2B_{n,m} \right),$$  \hspace{1cm} (A42)

where we introduced auxiliary functions:

$$A_{n,m} \equiv I_{n,m}(\tilde{\Delta}_t) I_{m,n}(\tilde{\Delta}_t),$$  \hspace{1cm} (A43)

$$\tilde{A}_{n,m} \equiv I_{n,m}(\tilde{\Delta}_t) I_{m,n}(\tilde{\Delta}_t),$$  \hspace{1cm} (A44)

$$B_{n,m} \equiv I_{n,m}(\tilde{\Delta}_t) I_{m,n}(\tilde{\Delta}_t).$$  \hspace{1cm} (A45)

Now, we are able to separate the $n,m$ summation into three parts $n < m$, $n = m$, and $n > m$. From the definition of the auxiliary functions, we easily see that the diagonal summation $n = m$ vanishes. The remaining sums $n < m$ and $n > m$ become the same, since $A_{n,m} = \tilde{A}_{n,m}$ and $B_{n,m} = B_{m,n}$. Hence, for convenience’s sake we will calculate the sum $m > n$ only. We use the explicit result (A32) for the $I_{n,m}$ and obtain:

$$A_{n,m} = e^{-\tilde{\Delta}_t^2} \sum_{k=0}^{n} \sum_{l=0}^{m} \binom{n}{k} \binom{m}{l} (-1)^{n+m-k} \tilde{\Delta}_t^{2(n+m-k-l)} \frac{(n-k)!(m-l)!}{(n-m)!}.$$  \hspace{1cm} (A46)
And by doing the same for $\tilde{A}$, we indeed realize that $A_{n,m} = \tilde{A}_{n,m}$. For $B_{n,m}$, in turn, we obtain:

$$B_{n,m} = e^{-\Delta_t^2} \sum_{k=0}^{n} \sum_{l=0}^{n} \binom{n}{k} \binom{m}{l} \frac{(-1)^{k+l} \Delta_t^{2(n+m-k-l)}}{(m-k)!(n-l)!}. \quad (A47)$$

Hence, putting it all together (with a factor of 2, since now we only sum $m > n$), we arrive at:

$$\langle f_t(a, \lambda) \rangle = e^{-\Delta_t^2} \sum_{n=0}^{d-2} \sum_{m=n+1}^{d-1} \left[ \sum_{k=0}^{n} \sum_{l=0}^{m} \binom{n}{k} \binom{m}{l} \frac{(-1)^{n+m-k-l} \Delta_t^{2(n+m-k-l)}}{(m-k)!(n-l)!} \right] - \sum_{k=0}^{n} \sum_{l=0}^{n} \binom{n}{k} \binom{m}{l} \frac{(-1)^{k+l} \Delta_t^{2(n+m-k-l)}}{(m-k)!(n-l)!}. \quad (A48)$$

This result may also be rewritten using the associated Laguerre polynomials and Eq. (A33):

$$\langle f_t(a, \lambda) \rangle = e^{-\Delta_t^2} \sum_{n=0}^{d-2} \sum_{m=n+1}^{d-1} \left[ L_n^{(0)}(\tilde{\Delta}_t^2)L_m^{(0)}(\tilde{\Delta}_t^2) - n! m! \Delta_t^{2(m-n)} [L_n^{(m-n)}(\tilde{\Delta}_t^2)]^2 \right] \quad (A49)$$

One of the sums can be performed using the following identity for the associated Laguerre polynomials:

$$\sum_{m=0}^{M} L_{d-1}^{(1)}(\tilde{\Delta}_t^2) = L_M^{(a+1)}(x), \quad (A50)$$

giving:

$$\langle f_t(a, \lambda) \rangle = e^{-\Delta_t^2} \left[ L_{d-1}^{(1)}(\tilde{\Delta}_t^2)L_{d-2}^{(1)}(\tilde{\Delta}_t^2) - \sum_{n=0}^{d-2} L_n^{(0)}(\tilde{\Delta}_t^2)L_m^{(1)}(\tilde{\Delta}_t^2) - \sum_{n=0}^{d-2} \sum_{m=n+1}^{d-1} \frac{n! m! \Delta_t^{2(m-n)} [L_n^{(m-n)}(\tilde{\Delta}_t^2)]^2} \right]. \quad (A51)$$

However, we keep (A49) in the main text since it is more compact.

Please note that in the main text we do not directly use $\langle f_t(a, \lambda) \rangle$, but rather separate its Gaussian and polynomial parts, i.e.

$$\langle f_t(a, \lambda) \rangle \equiv e^{-\Delta_t^2} p(d, \tilde{\Delta}_t). \quad (A52)$$

### Appendix B: Short time analysis

In this Section we perform the final averaging over the system observable $A$. In the previous Section, we have used the i.i.d. property of the apparatus ensemble to reduce the big, compound averages to single copy ones. Here, however, we cannot do so, as ultimately we are interested in the macroscopic quantities $\Gamma^{uno}_{aad'}$, $G^{mac}_{aad'}$. Thus we need:

$$\langle \langle X^f(t) \rangle \rangle_{aad'} \equiv \int da P_{Gue}(a) \langle X_{aad'}(t) \rangle^{N_f}, \quad (B1)$$

where $X_{aad'} = \Gamma_{aad'}$ or $G_{aad'}$ and $f$ = uno or mac respectively. We note that both $\langle \Gamma_{aad'}(t) \rangle$, $\langle G_{aad'}(t) \rangle$ depend on $A$ only through the eigenvalue differences $|a - a'|$. Thus, the $A$-averaging reduces to averaging over the eigenvalues only with its own GUE eigenvalue distribution:

$$P_{Gue}(a) \equiv P_{Gue}(a_1, \ldots, a_d) \equiv e^{-\frac{1}{2} \eta_S \Sigma_i a_i^2} \prod_{i<j} |a_i - a_j|^2, \quad (B2)$$

where $d_S$ is the system dimension and $\eta_S$ is the eigenvalue scale of the system observable $A$. From the permutational symmetry of the GUE distribution (best seen through the Vandermonde determinant), the integral in Eq. (B1) reduces to the integration with the same 2-point correlation function $\langle X^f(t) \rangle$, but now defined for the distribution (B2) and thus all the averages for different pairs $aa'$ are the same and equal to:

$$\langle \langle X^f(t) \rangle \rangle = \left[ \frac{d_S!}{(d_S - 2)!} \right]^{-1} \int dada' R_2(a, a') \langle X_{aad'}(t) \rangle^{N_f}. \quad (B3)$$
The resulting integral is too complicated to be performed analytically and actually this is in fact not needed as we see from Eq. (A52) that \((f_i(a, \lambda))\) will eventually decay so that both factors will approach their noise-floor values (cf. Result 1 from the main text). What we are interested in are the relevant timescales. We can estimate lower bounds on those timescales from the decay times of \((A52)\). We will perform this analysis in the following steps: i) assume a short-time limit; ii) approximate the polynomial \(p(d, \hat{\Delta})\) of Eq. (A52); iii) approximate the \(N_f\) power; iv) using Eqs. (A39), (A40), and (A20) estimate the fastest decaying term in (B5). This will then give lower bounds on the desired times of the asymptotic approach. The latter are for sure greater than the initial decay times.

First, we assume \(\hat{\Delta} \ll 1\), or \(t \ll \sqrt{\eta E} / |a - a'|\). The maximum of \(|a - a'|\) is of the order of \(\sqrt{d_S / \eta S}\) from the Wigner semi-circle law, defining the short-time limit:

\[
t \ll \sqrt{\frac{\eta E \eta S}{d_S}} = \frac{1}{g \sqrt{d_S}},
\]

where \(g \equiv 1 / \sqrt{\eta E \eta S}\) is the effective interaction strength.

We now explicitly calculate the coefficients of the lowest order terms in \(p(d, \hat{\Delta})\) directly from Eq. (A48). One immediately sees that the polynomial is even so the lowest terms are the constant and the quadratic ones. First, we assume \(\hat{\Delta} \ll 1\), or \(t \ll \sqrt{\eta E} / |a - a'|\). The maximum of \(|a - a'|\) is of the order of \(\sqrt{d_S / \eta S}\) from the Wigner semi-circle law, defining the short-time limit:

\[
\sum_{n < m} [1] = \frac{d(d - 1)}{2}
\]

The quadratic term occurs when the indices fulfill the condition \(k + l + 1 = m + n\). On the first term, this can occur if \((k = n, l = m - 1)\) or if \((k = n - 1, l = m)\). On the second term this can occur only when the \(m\) index is \(m + 1\) and the inner indices are \(k = l = n\). Thus we obtain:

\[
\sum_{n < m} \left[ m(-1)\hat{\Delta}^2 \right] + \sum_{n < m} \left[ n(-1)\hat{\Delta}^2 \right] - \sum_n \left[ (n + 1)\hat{\Delta}^2 \right] = -\frac{d^2(d - 1)}{2} \hat{\Delta}^2
\]

Thus, for short times we have:

\[
p(d, \hat{\Delta}) = \frac{d(d - 1)}{2} \left(1 - d\hat{\Delta}^2\right) + O(\hat{\Delta}^4).
\]

To proceed further, we upper bound the above expression by the Gaussian function: \(1 - d\hat{\Delta}^2 \leq e^{-d\hat{\Delta}^2}\) resulting from Eq. (A52) in the short-time bound:

\[
\langle f_i(a, \lambda) \rangle \lesssim e^{-\frac{(d+1)\hat{\Delta}^2}{2}} \approx e^{-\left(t / \tau_{ad}\right)^2}, \quad \tau_{ad} \approx \sqrt{\eta E / d + 1 |a - a'|}
\]

We then use the following approximation of the power:

\[
(\alpha + \beta e^{-x})^{N_f} \approx (\alpha + \beta)^{N_f} e^{-N_f \frac{\beta}{\alpha + \beta} x}
\]

for \(x \ll 1\). We apply it to the single-copy averaged factors:

\[
\langle \Gamma_{aad}(t) \rangle = \frac{1 + \text{Tr} \rho_0^2}{d + 1} + \langle f_i(a, \lambda) \rangle \frac{2(d - \text{Tr} \rho_0^2)}{d(d^2 - 1)}
\]

\[
\langle G_{aad}(t) \rangle = S \text{lin}(\rho_0) + \frac{1 + \text{Tr} \rho_0^2}{d + 1} + \langle f_i(a, \lambda) \rangle \frac{2(d - \text{Tr} \rho_0^2 - 1)}{d(d^2 - 1)}
\]

identifying \(\beta\) with the constant terms and \(\alpha\) with the multiplicative one. After a simple algebra, this leads to the following short time approximations of the macroscopic factors:

\[
\langle \Gamma_{aad}^{\text{mac}}(t) \rangle = \langle \Gamma_{aad}(t) \rangle^{N_{aad}} \approx \exp \left[-N_{aad} \hat{\Delta}^2 \left(1 - \langle \text{Tr} \rho_0^2 \rangle\right)\right]
\]

\[
\langle G_{aad}^{\text{mac}}(t) \rangle = \langle G_{aad}(t) \rangle^{N_{aad}} \approx \exp \left[-N_{aad} \hat{\Delta}^2 \left(1 - \langle \text{Tr} \rho_0^2 \rangle - 1\right)\right]
\]
For \( \langle \text{Tr} \rho_0^2 \rangle \) we can use either the Hilbert-Schmidt or Bures measure from the main text. It is interesting to note that in any case \( \langle \text{Tr} \rho_0^2 \rangle \propto 1/d \), so that (B12) shows a dependence on both \( N_{\text{uno}} \) and \( d \) while (B13) shows a dependence mainly on \( N_{\text{mac}} \).

We are now ready to estimate the lower bound on the decay time of the integral (B3). Substituting Eq. (B12) or (B13) and using Eqs. (A39), (A40), and (A20), we are left with a sum of integrals of the following structure:

\[
\int d\alpha d' \left| H_{\alpha}(a)H_{\alpha'}(a') \right|^2 e^{-\mu_1(a-a')^2 - \frac{1}{2}(a^2 + a'^2)} \sim \int d\alpha d' \left| a e^{-\mu_1(a-a')^2 - \frac{1}{2}(a^2 + a'^2)} \right|,
\]

where the powers satisfy \( 0 \leq r, s \leq 2(d_5 - 1) \) (cf. Eq. (A40)) and \( \mu_1 \equiv N_{\text{uno}}(gt)^2 \left( d - \langle \text{Tr} \rho_0^2 \rangle \right) \) for the decoherence factor and \( \mu_t \equiv N_{\text{mac}}(gt)^2 \left( d \langle \text{Tr} \rho_0^2 \rangle - 1 \right) \) for the super-fidelity (cf. Eqs. (B12, B13)). To separate the integrals, we diagonalize the quadratic form in the exponent. Its eigenvalues read (B12)

\[
\langle \langle X^f(t) \rangle \rangle \sim \frac{1}{\sqrt{1 + 4\mu_t}} + \frac{1}{1 + 4\mu_t} + \frac{1}{(1 + 4\mu_t)^{3/2}} + \cdots + \frac{1}{(1 + 4\mu_t)^{2(d_5 - 1) + \epsilon}}.
\]

The fastest decaying is the last term, which for \( \mu_t \ll 1 \) or \( t \ll O \left( \frac{1}{\sqrt{2d_5}} \right) \) can be approximated by an exponential:

\[
\frac{1}{(1 + 4\mu_t)^{2(d_5 - 1) + \epsilon}} \approx \exp \left[ -8d_5 c(d) N_f (gt)^2 \right],
\]

where we neglected the factor \(-2 + \epsilon = -1 \) or \(-3/2\) in the power as compared to \( 2d_5 \) and \( c(d) \equiv d - \langle \text{Tr} \rho_0^2 \rangle \) for the decoherence factor and \( c(d) \equiv d \langle \text{Tr} \rho_0^2 \rangle - 1 \) for the super-fidelity. This finally gives the following lower bounds on the decay times:

\[
\tau_{\text{dec}} \equiv \left[ 8g^2 N_{\text{uno}} d_5 \left( d - \langle \text{Tr} \rho_0^2 \rangle \right) \right]^{-1/2} \sim \left[ 8g^2 N_{\text{uno}} d_5 d \right]^{-1/2},
\]
\[
\tau_{\text{fid}} \equiv \left[ 8g^2 N_{\text{mac}} d_5 \left( d \langle \text{Tr} \rho_0^2 \rangle - 1 \right) \right]^{-1/2} \sim \left[ 8g^2 N_{\text{mac}} d_5 \right]^{-1/2},
\]

where the simplified estimates are in the limit of a large local dimension of the apparatus \( d \). The above times are clearly within the short-time approximation range (B4). It is interesting to note that the above separation of physical time-scales of decoherence and information accumulation is, on the mathematical level, a consequence of a simple symmetry difference in the initial formulas (A2) and (A15). The latter has an additional SWAP operator \( V \) under the trace.

As mentioned, the above timescales are only lower bounds for the actual times of the noise floor approach. Perhaps they can be tightened using different analytical tools, but for the purpose of this work we will use the above \( \tau_{\text{dec}}, \tau_{\text{fid}} \).

**Appendix C: Upperbounding the exact expression for \( \langle f_1(a, \lambda) \rangle \)**

This Section is complimentary to the main line of reasoning and is not necessary for understanding it. The exact expression (A45) is algebraically complicated and a simplified expression, reproducing short- and long-time behavior and preferably upper bounding it would be desirable. Here we attempt a construction of such an upper bound. For short times, we may neglect in the average \( \langle f_1(a, \lambda) \rangle = p(d, \lambda) e^{-\lambda^2} \) all terms except the first two giving
\begin{align}
\langle f_t(a, \lambda) \rangle & \lesssim [d(d-1)/2] e^{-(d+1)\Delta^2_t} \quad \text{(see Eq. (B8))}. \quad \text{On the other hand, for large times the highest order term } \Delta^{2(d-3)}_t \quad \text{dominates and it always comes with a negative coefficient as can be seen from Eq. (A48). These observations suggest the following Ansatz:} \\
\tilde{\Gamma}_{aa'}(t) & \equiv \frac{1+\langle \text{Tr}\rho_0^2 \rangle}{d+1} + \frac{d-\langle \text{Tr}\rho_0^2 \rangle}{d+1} e^{-(d+1)\Delta^2_t}, \quad \text{(C1)} \\
\tilde{G}_{aa'}(t) & \equiv \langle S_{\text{lin}}(\rho_0) \rangle + \frac{1+\langle \text{Tr}\rho_0^2 \rangle}{d+1} + \frac{d(\langle \text{Tr}\rho_0^2 \rangle - 1)}{d+1} e^{-(d+1)\Delta^2_t}, \quad \text{(C2)}
\end{align}

for the decoherence and the superfidelity factors respectively. By construction, the expressions (C1, C2) reproduce the correct short time behavior as \( \langle \Gamma_{aa'}(t) \rangle = \tilde{\Gamma}_{aa'}(t) + O(\Delta_t^4) \), and similarly \( \langle G_{aa'}(t) \rangle \), for \( t \ll \tau_{aa'} \), where \( \tau_{aa'} \) is the characteristic time of the Gaussian decay in (C1, C2):

\[ \tau_{aa'} \equiv \frac{\sqrt{\eta}E}{\sqrt{d+1}|a-a'|}. \quad \text{(C3)} \]

They are also upper bounds since \( 1-x \leq e^{-x} \) (cf. (B7)). Similarly, for long times \( t \gg \tau_{aa'} \), (C1, C2) reproduce the correct white noise factors and also upper bound the exact averages, since \( p(d, \Delta_t) \) is then negative. These two facts together suggest that (C1, C2) may be upper bounds for all times \( t \) and dimensions \( d \). Unfortunately we
were unable to prove it analytically, but numerical evidence for \( d \leq 20 \) suggests that it is indeed so. In Fig. B1 we present some sample plots of both the exact expressions and \((C_1,C_2)\) together with the errors. The price to pay for working with the simplified expressions \((C_1,C_2)\) is that one loses the important physical information on non-Markovianity, reflected by the non-monotonic behaviour of the exact averages for low ratios \( N/d \). Also the exact functions approach their asymptotic limits from below, signalizing recovery of coherences/loss of information in the environment, while \((C_1,C_2)\) approach them from above. However, if one is solely interested in an SBS formation, an upper bound decaying to the correct noise level is just enough.