A novel type of intermittency in a non-linear dynamo in a compressible flow

Erico L. Rempel,1,2★ Michael R. E. Proctor1★ and Abraham C.-L. Chian2,3★

1Department of Applied Mathematics and Theoretical Physics (DAMTP), University of Cambridge, Cambridge CB3 0WA
2Institute of Aeronautical Technology (IEFM/ITA) and World Institute for Space Environment Research (WISER), São José dos Campos – SP 12228–900, Brazil
3National Institute for Space Research (INPE) and World Institute for Space Environment Research (WISER), PO Box 515, São José dos Campos – SP 12227–010, Brazil

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ABSTRACT
The transition to an intermittent mean-field dynamo is studied using numerical simulations of magnetohydrodynamic turbulence driven by a helical forcing. The low-Prandtl number regime is investigated by keeping the kinematic viscosity fixed while the magnetic diffusivity is varied. Just below the critical parameter for the onset of dynamo action, a transient mean field with low magnetic energy is observed. After the transition to a sustained dynamo, the system is shown to evolve through different types of intermittency until a large-scale coherent field with small-scale turbulent fluctuations is formed. Prior to this coherent field stage, a new type of intermittency is detected, where the magnetic field randomly alternates between phases of coherent and incoherent large-scale spatial structures. The relevance of these findings to the understanding of the physics of mean-field dynamo and the physical mechanisms behind intermittent behaviour observed in stellar magnetic field variability are discussed.

Key words: magnetic fields – MHD – turbulence.

1 INTRODUCTION
The observation of strong magnetic fields in astrophysical bodies (planets, stars and galaxies) suggests the existence of a dynamo process, whereby a weak (seed) magnetic field is amplified due to the conversion of kinetic energy into magnetic energy. Dynamos can be classified as large-scale (or mean-field) dynamos or small-scale (or fluctuation) dynamos, according to whether the magnetic fields grow on spatial scales larger or smaller than the energy carrying scale of the fluid motion. A typical manifestation of a large-scale dynamo is the solar cycle, where the distribution of sunspots in space and time display large-scale spatial coherence and long-term temporal correlation, as seen in the butterfly diagram (Solanki 2003; Solanki, Inhester & Schüssler 2006; Thomas & Weiss 2008). Although the maxima and minima of solar activity form a recurrent 11-yr cycle, long periods of very low solar activity, grand minima such as the Maunder minimum (Beer, Tobias & Weiss 1998), have led several authors to look for a description of the solar cycle as an intermittent event due to the chaotic nature of the dynamo, driving the system to random alternations between phases of ‘regular magnetic activity’ and grand minima.

Many works employ low-dimensional models based on ordinary differential equations to investigate these modulations (e.g. Covas, Ashwin & Tavakol 1997; Wilmot-Smith et al. 2005). Others have focused on high-dimensional mean-field models based on partial differential equations. Covas & Tavakol (1999) reported the existence of crisis-induced and Pomeau–Manneville type-I intermittencies (Manneville & Pomeau 1979; Chian, Borotto & Gonzalez 1998; Chian et al. 2006, 2007) in an axisymmetric mean-field dynamo model, suggesting the multiple-intermittency hypothesis, since more than one type of intermittency may be responsible for the minima observed in solar cycle data. Ossendrijver (2000) showed that a 2D mean-field dynamo model that features an effect based on the buoyancy instability of magnetic flux tubes could produce grand minima. The grand minima observed by this model were later described by Ossendrijver & Covas (2003) as a manifestation of crisis-induced intermittency (Grebogi & Ott 1983; Rempel et al. 2004c; Rempel & Chian 2005, 2007; Rempel, Chian & Miranda 2007). Intermittency in a 2D mean-field model was also reported by Moss & Brooke (2000), where the reaction of the Lorentz force on the rotation is used as the non-linear effect that limits the magnetic field at finite amplitude. Charbonneau (2004) showed that an axisymmetric 2D solar cycle model based on the Babcock–Leighton mechanism of poloidal field regeneration can exhibit intermittency in the presence of low-amplitude noise. Recently, Brandenburg & Spiegel (2008) observed on–off intermittency in a mean-field dynamo.
dynamo model after imposing stochastic fluctuations in the α effect or using a fluctuating electromotive force. A review on the use of intermittent chaotic models to capture the main qualitative aspects of the temporal and spatial variability of the solar cycle is provided by Spiegel (2009).

This paper employs 3D magnetohydrodynamics (MHD) simulations to investigate the onset of intermittent mean-field dynamo as a function of the magnetic Prandtl number $Pr$, defined as the ratio between the kinematic viscosity $\nu$ and the magnetic diffusivity $\eta$. Although the model geometry is highly idealized, the full induction and momentum equations are solved, so that there is no need to appeal to an averaging process. Early numerical works in similar models focused on $Pr \geq 1$ regimes due to difficulties in exciting a non-linear dynamo for low values of Pr (see, e.g. Nordlund et al. 1992). Nevertheless, the low-$Pr$ ($Pr \leq 1$) regime is crucial for understanding astrophysical plasmas such as stellar dynamo, where $Pr$ is usually much less than one (e.g. $Pr \sim 10^{-7}-10^{-4}$ in the solar convective zone, depending on the depth (Schekochihin et al. 2007)). Furthermore, the role of compressibility of the velocity field in astrophysical plasmas should not be ignored, even in low-Mach-number regimes (Rogachevskii & Kleeruin 1997; Haugen, Brandenburg & Mee 2004). Thus, we adopt a compressible MHD code and explore low-$Pr$ regimes. Two conditions have been shown to be important for a fluid flow to act as a large-scale dynamo: chaotic stream lines and kinetic helicity (Childress & Gilbert 1995; Biskamp 2000). Depending on the characteristics of the velocity field and other parameters, such as $\eta$, the resulting magnetic field lines may display regular or irregular motion or a decay to a purely hydrodynamic state. In order to obtain a chaotic and helical velocity field, the flow is driven by an ABC (Arnold–Beltrami–Childress) forcing, which is a superposition of three helical waves (Arnold & Hebd 1965; Galloway & Frisch 1984) with a characteristic wave number $k_f$, which sets the energy carrying scale of the flow. The emergence of a mean field is studied by forcing the flow at scale $k_f = 5$ and observing the energy transfer towards larger scales.

This paper is divided as follows. Section 2 contains a description of the dynamo model. The main results are described in Section 3. The kinetic viscosity is set at a value such that the velocity field is weakly turbulent. Then, a seed magnetic field is applied and the magnetic diffusivity $\eta$ is progressively reduced until the onset of dynamo is observed. Three types of dynamo regimes are observed as a function of $\eta$. In the first regime, right after the onset of dynamo, there is an intermittent switching between bursty phases of high-amplitude magnetic activity and quiescent phases of low magnetic energy, similar to the on–off spatiotemporal intermittency reported in other works (Sweet et al. 2001a,b; Rempel & Chian 2007; Rempel et al. 2007). The second regime reveals an intermittent switching between coherent and incoherent large-scale structures and, up to our knowledge, has not been reported in previous dynamo studies. This intermittent process persists up to a certain threshold value of the magnetic diffusivity, where the third regime is observed, with the system self-organizing into a spatially coherent mean-field that exhibits complex temporal dynamics. An analysis of the spatiotemporal complexity of the patterns observed in each regime is also described. The conclusions are given in Section 4.

## 2 THE MODEL

We consider a compressible isothermal gas (the ratio of specific heats $\gamma = c_p/c_v = 1$, where $c_p$ is the heat capacity at constant pressure and $c_v$ is the heat capacity at constant volume) with constant sound speed $c_s$, constant dynamical viscosity $\mu$, constant magnetic diffusivity $\eta$ and constant magnetic permeability $\mu_0$. The continuity equation is solved in terms of the logarithm of the density $ln(\rho)$, since this quantity varies spatially much less than density.

$$\frac{\partial\ln\rho}{\partial t} + \mathbf{u} \cdot \nabla \ln\rho + \nabla \cdot \mathbf{u} = 0,$$

where $\mathbf{u}$ is the fluid velocity. The momentum equation is given by

$$\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{\nabla p}{\rho} + \frac{\mathbf{J} \times \mathbf{B}}{\rho} + \frac{\mu}{\rho} \left( \nabla^2 \mathbf{u} + \frac{1}{3} \nabla (\nabla \cdot \mathbf{u}) \right) + \mathbf{f},$$

where $\mathbf{J} = \nabla \times \mathbf{B}/\mu_0$ is the current density, $p$ is the pressure and $\mathbf{f}$ is an external forcing. The pressure gradient is obtained from the entropy equation for an ideal gas $s = c_v \ln(p\rho^{-\gamma})$. Thus, $\nabla p/\rho = c_v^2 \nabla \ln\rho$, where $c_v = \gamma p/\rho$ is assumed to be constant. The induction equation is written in terms of the magnetic vector potential $\mathbf{A}$, so that $\nabla \cdot \mathbf{B} = 0$, since $\mathbf{B} = \nabla \times \mathbf{A}.

$$\frac{\partial\mathbf{A}}{\partial t} = \mathbf{u} \times \mathbf{B} - \eta \mathbf{u} \mathbf{A}.$$}

We adopt non-dimensional units, such that $c_s = \rho_0 = \mu_0 = 1$, where $\rho_0 = \langle \rho \rangle$ is the spatial average of $\rho$. Equations (1)–(3) are solved in a box with sides $L = 2\pi$ and periodic boundary conditions. The PENCIL code is a compressible MHD code that employs high-order finite differences and has been extensively used in astrophysical simulations (Brandenburg & Subramanian 2005 and references therein). The initial conditions are $\rho = u = 0$, and $\mathbf{A}$ is a set of normally distributed, uncorrelated random numbers with zero mean and standard deviation equal to $10^{-3}$. For the forcing function $\mathbf{f}$, we adopt the form of ABC flow used in Sun, Subramanian & Brandenburg (2007),

$$f(x) = \frac{A_f}{\sqrt{3}} \left( \sin k_f z + \cos k_f y \right) \left( \sin k_f x + \cos k_f z \right),$$

where $A_f$ is the amplitude and $k_f$ the wavenumber of the forcing function, which is isotropic with respect to the three coordinate directions. The ABC forcing is an interesting choice for dynamo studies, since it is a Beltrami flow, $\nabla \times \mathbf{u} \propto \mathbf{u}$, and, therefore, is maximally helical, that is $(H_1)^2 = (\mathbf{u} \cdot \nabla \times \mathbf{u})^2 = \langle u_i^2 \rangle \langle (\nabla \times \mathbf{u})^2 \rangle$, where the angle brackets denote spatial integration over the periodic domain.

In all the following sections, we use $A_f = 0.1$, which yields Mach numbers below 0.5. A numerical resolution of $64^3$ mesh points is chosen. As in Brandenburg (2001), we set $k_f = 5$ in order to be able to see the emergence of a large-scale magnetic field, with spatial scales larger than the energy injection scale. Spatial averages are denoted by $\langle \cdot \rangle$ and time averages by $\langle \cdot \rangle_t$. Unless otherwise stated, references to kinetic ($Re$) and magnetic ($Rm$) Reynolds numbers are based on the forcing scale,

$$Re = \frac{\lambda_f U}{v}, \quad Rm = \frac{\lambda_f U}{\eta},$$

where $v = \mu/\rho_0$ is the average kinematic viscosity, $\lambda_f = 2\pi/k_f$ is the forcing spatial scale, and $U = \langle u_i^2 \rangle^{1/2}$ is the mean velocity at a time when the magnetic field is saturated.
3 RESULTS

3.1 Hydrodynamic simulations

We start with hydrodynamic simulations in the absence of magnetic fields. We set $k_f = 5$, $A_f = 0.1$, and vary the average kinematic viscosity $\nu$. Fig. 1(a) displays the velocity components of an asymptotically stable solution of equations (1) and (2) at $\nu = 0.02$, which corresponds to $Re \sim 12.38$. The initial velocity field $u = 0$ evolves with time until it takes the shape of the ABC flow. After converging to this attractor, the amplitudes of the velocity components do not change with time, so this is a steady state of the system. As the kinematic viscosity is reduced, the ABC flow becomes unstable and a sequence of symmetry breaking bifurcations takes place. At $\nu = 0.005$ ($Re \sim 100$ based on the forcing scale or $Re \sim 500$ based on the box scale), the system is in a weakly turbulent regime, as illustrated in Fig. 1(b). The flow is weakly compressible, as can be seen in Fig. 2, which shows that the density fluctuations are within 5 per cent of the mean value $\rho_0 = 1$.

3.2 Onset of dynamo action

We now fix $\nu = 0.005$, and change the magnetic diffusivity $\eta$ as we look for the onset of non-linear dynamo action. Since we are interested in the low-Prandtl number limit, $\eta$ is made larger than $\nu$, as $Pr$ is varied from 0.0625 to 0.5. Fig. 3 shows the bifurcation diagrams for the time-averaged magnetic ($\langle E_m \rangle_t$, red triangles) and kinetic ($\langle E_k \rangle_t$, black circles) energies in linear [Fig. 3(a)] and linear-log [Fig. 3(b)] scales as a function of $\eta$ (lower axes) or $Rm$ (upper axes). For each value of $\eta$, the initial variations of $\langle B^2 \rangle/2$ and $\langle u^2 \rangle/2$ are discarded, and time averages are computed for the saturated energy values. For large values of $\eta$, the seed magnetic field decays rapidly and there is no dynamo. At the onset of dynamo action at $\eta \sim 0.053$ ($Rm \sim 9.5$ in the forcing scale or $Rm \sim 47.5$ in the box scale), the magnetic energy starts to grow at the expense of kinetic energy. Two different dynamo behaviours can be identified in Fig. 3. In the first range, between $\eta \sim 0.053$ and $\eta \sim 0.04$, there is a sharp increase in the saturated magnetic energy. Between $\eta \sim 0.04$ and $\eta \sim 0.01$, the magnetic energy increases more slowly as an exponential function of $\eta$ until it is comparable to the kinetic energy at $\eta = 0.01$.

We turn to the dynamics near the transition to non-linear dynamo. Fig. 4 plots the time series of magnetic energy for different initial conditions at $\eta = 0.055$, just before the onset of dynamo action. For a small seed magnetic field there is an initial growth of the magnetic energy [Fig. 4(a)]. Then the field saturates and starts to decay. Even during the growth phase the magnetic field has a much lower magnitude than the velocity field. Consequently, the impact of the Lorentz force on the velocity field is negligible and the same kind of behaviour was observed in kinematic simulations, where the Lorentz force term $J \times B/\rho$ was removed from equation (2).

Figure 1. (Colour online) Contour plots of the velocity components for hydrodynamic simulations at (a) $\nu = 0.02$ and (b) $\nu = 0.005$.

Figure 2. (Colour online) Contour plot of $\ln(\rho)$ for the hydrodynamic simulation at $\nu = 0.005$.
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Figure 4. Time series of magnetic energy at \( \eta = 0.055 \) exhibiting transient dynamics for different initial conditions. The rms velocity, \( u_{\text{rms}} \), is approximately 0.4, thus the turnover time is \( \tau = \lambda_f / u_{\text{rms}} \sim \pi \) time units, where \( \lambda_f = 2\pi k_f \) is the forcing scale.

Figs 4(b)–(d) show other instances of a ‘transient dynamo’ for different types of initial conditions. Thus, the dynamo state is not an attractor of the system and we believe this transient and apparently chaotic dynamics displayed by the magnetic field is a signature of non-attracting chaotic sets known as chaotic saddles (Hsu, Ott & Grebogi 1988; Rempel et al. 2004a,b; Chian et al. 2007), which have attracted wide attention recently due to their role in transition to hydrodynamic turbulence (Hof et al. 2006; Peixinho & Mullin 2006; Willis & Kerswell 2007; Hof et al. 2008).

3.3 Intermittent dynamo

At \( \eta = 0.053 \), there is a transition to sustained dynamo action. The time series of magnetic energy exhibit random switching between phases of bursty and quiescent activity, as seen in Fig. 5(a), including long periods of minima with almost zero magnetic activity such as the one between \( t \sim 17,500 \) and \( t \sim 20,000 \). This is the same type of behaviour reported by Sweet et al. (2001a,b) for low Reynolds number (\( Re = 6.3 \)) dynamo simulations with Prandtl number close to one and ABC forcing at scale \( k_f = 1 \). A blowout bifurcation, whereby the hydrodynamic state looses transversal stability, was characterized as responsible for the intermittency, which in this case is called on–off intermittency, since the solutions go arbitrarily close to a manifold defined by the purely hydrodynamic state \( B = 0 \) (‘on’ phases) and suddenly depart from the manifold during the strong magnetic bursts (‘off’ phases).

For \( \eta = 0.05 \), the dynamo is strongly intermittent, but no grand minima are found in the time series of \( \langle B^2 \rangle / 2 \), shown in Fig. 5(b). There are strong bursts interspersed by lower peaks, but the energy is seldom close to zero. A look at the spatial structures of the magnetic field is helpful to distinguish the two types of intermittency exemplified by Figs 5(a) and (b). Fig. 6 shows the contour plots of \( B_i \) for the on–off intermittency (\( \eta = 0.053 \)) at four different times. The snapshots in the left-hand column (predominantly red) are taken at quiescent phases, and the ones in the right-hand column, at bursty phases. The magnetic field displays complex spatial structures in both phases, although they are more clearly seen in the bursty phases. The dynamics at \( \eta = 0.05 \) is very different. The contour plots of \( B_x \), shown in Fig. 7, reveal that a large-scale sinusoidal modulation of the magnetic field is evolving, but there is an intermittent switching in the preferred direction selected by the magnetic field. In the first snapshot (\( t = 5000 \)), the preferred direction for \( B_x \) is \( z \); in the second snapshot (\( t = 6000 \)), the field is restructuring and there is no preferred direction; in the third snapshot (\( t = 9000 \)) the preferred direction is \( y \) and there is no preferred direction at \( t = 10,000 \). This coherent-incoherent intermittency has been observed for all tested values of \( \eta \) in the interval from \( \eta = 0.05 \) to 0.02.

For \( \eta = 0.01 \), the magnetic field has settled to a large scale and approximately sinusoidal mean field, with complex small scale spatial structures and irregular oscillations, as shown in Figs 8 and 9. In Fig. 8(a), the magnetic energy \( \langle B^2 \rangle / 2 \) is strong enough to cause a reduction of the mean kinetic energy from about \( \langle u^2 \rangle / 2 = 0.4 \) in the previous examples, to \( \langle u^2 \rangle / 2 = 0.3 \). There is also a definite

Figure 5. Intermittent time series of the magnetic energy at (a) \( \eta = 0.053 \) and (b) \( \eta = 0.05 \). The turnover time is \( \tau \sim \pi \) in (a) and \( \tau \sim 3.3 \) in (b).

Figure 6. (Colour online) Contour plots of \( B_i \) for \( \eta = 0.053 \) at four different values of \( t \). The magnetic energy increases and decreases randomly in time, as in Fig. 5(a), resulting in on–off intermittency.
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3.4 Inverse cascade

The appearance of a mean-field dynamo is due to the transfer of magnetic energy from the energy injection scale to larger scales. This transfer is expected in helical flows, and in mean-field dynamo theory is attributed to the $\alpha$ effect (Moffatt 1978; Krause & Rädler 1980), whereby interactions between low-scale fluctuations of $B$ and $u$ produce a large-scale $B$ (other mechanisms can be responsible for the rise of a mean-field in the absence of a mean $\alpha$ effect, such as the interaction of a fluctuating $\alpha$ effect and large-scale shear flows (Vishniac & Brandenburg 1997; Proctor 2007; Hughes & Proctor 2009)). The kinetic and magnetic energy power spectra for the coherent mean-field state at $\eta = 0.01$ are shown in Fig. 10 for $t = 5000$. The 1D spectra are obtained from the 3D spectra by computing the integrals of the spectral energy along spherical shells with rays defined by the modulus of $k$. For the magnetic spectrum (solid red line), although the forcing scale is $k_f = 5$,
there is a backward transfer of magnetic energy, making $k = 1$ the predominant scale, resulting in a $B$ field with a typical scale of the size of the box. As for the kinetic energy (dashed black line), $k = 5$ is still the predominant scale. For this value of $Rm$, the spectra are particularly similar in the ‘inertial range’ (not properly defined yet, since they are not sufficiently extended). The $k^{-3}$ line is a guide to the eye, and is the power-law observed for the magnetic spectrum of dynamos with helical flows at low $Rm$ (following a $k^{-3}$ range) (Müller, Stieglitz & Horanyi 2004; Mininni 2007).

Fig. 11 plots the peak height for the Fourier modes $k = 1$ (dashed line) and $k = 5$ (solid line) of the time-averaged magnetic energy spectra as a function of $\eta$. It shows that the energy difference between these scales increases as the magnetic diffusivity decreases, giving rise to a progressively more coherent mean field, as discussed in the next section. The two distinct ranges previously identified in Fig. 3 can be seen in Fig. 11 as well. The first one with a steepest energy increase between $\eta = 0.053$ and 0.04, and the second one between $\eta = 0.04$ and 0.01, where energy seems to increase exponentially, as seen in the linear-log plot of Fig. 11(b).

### 3.5 Spatiotemporal complexity

Two operators computed in the Fourier space can be used to quantify the spatiotemporal complexity of a system. The amount of spatial disorder can be quantified by means of the spectral entropy (Powell & Percival 1979; Xi & Gunton 1995; Rempel et al. 2007),

$$S(t) = -\sum_{k=1}^{M} p_k(t) \ln[p_k(t)],$$

where $M$ is the number of spherical shells used in the magnetic or kinetic energy spectra and $p_k(t)$ is the relative weight of the shell-integrated energy $E_k$ at an instant $t$,

$$p_k(t) = \frac{E_k(t)}{\sum_{k=1}^{M} E_k(t)}.$$  

The spectral entropy is the maximum for a random system with uniform distribution, that is for all $k$, $p_k(t) = 1/M$, in which case $S(t) = \ln M$ (Badii & Politi 1997). For the present spatial resolution, $M = 32$ and the maximum entropy is $S_{MAX} \sim 3.47$.

Another quantity of interest is the average wave number or spectral average (Thyagaraja 1979; Lopes & Rizzato 1999; He & Chian 2003),

$$N(t) = \sqrt{\frac{\sum_{k=1}^{M} k^2 E_k(t)}{\sum_{k=1}^{M} E_k(t)}}.$$  

The spectral average is a measure of the number of active shells. It is the square root of the averaged $k^2$, where the average is weighted by the shell-integrated energy, $E_k$. Thus, it measures the energy spread in the $k$ spectrum, and should increase with time in systems with energy cascade until dissipative effects restrain its growth. In the absence of energy cascade, $N(t)$ is limited by the number of linearly unstable modes.

Fig. 12 shows the time-averaged spectral entropy $\langle S \rangle_t$ (top panel) and the time-averaged spectral average $\langle N \rangle_t$ (bottom panel) for the kinetic (black circles) and magnetic (red triangles) energy spectra as a function of $\eta$. It can be concluded from the top panel that the spatial structures of the velocity field become less complex with the onset of the on–off intermittent dynamo at $\eta \sim 0.053$, with a decrease in $\langle S \rangle_t$. This is due to the beginning of the action of the Lorentz force on the momentum equation. From $\eta = 0.055$ to $\eta = 0.04$, there is a strong decay not only in $\langle S \rangle_t$, but also in $\langle S_m \rangle_t$, reflecting the increase in the frequency of occurrence of large-scale coherent mean-field structures in the coherent–incoherent intermittent dynamo as $\eta$ is decreased. The spatial complexity of both the velocity and magnetic fields, then, starts to increase from $\eta \sim 0.04$ to $\eta \sim 0.01$. This tendency coincides with an increase in the magnetic spectral average $\langle N_m \rangle_t$ (bottom panel), which reflects a rise in magnetic energy in smaller scales. Note that the kinetic spectral average $\langle N_k \rangle_t$ is always close to the energy injection scale $k = 5$, and the magnetic spectral average is lower due to the high energy peak in $k = 1$ (see Figs 10 and 11). A comparison of the values of the spectral entropy shown in Fig. 12 with the maximum value $S_{MAX} \sim 3.47$ suggests an intermediate level of complexity in the spatial patterns observed. This is to be expected, since turbulence is not fully developed and most of the energy is concentrated in large-scale velocity and magnetic structures.

The spectral entropy $S_m(t)$ is able to distinguish the two types of intermittency displayed in Fig. 5 of Section 3.3. Fig. 13(a) shows the time series of the spectral entropy $S_m(t)$ and magnetic energy $\langle B^2 \rangle / 2$ for the on–off intermittency at $\eta = 0.053$. There is little correlation between the two series, since $S_m(t)$ oscillates erratically even during phases of grand minima in the magnetic energy time series. Fig. 13(b) plots the same time series as Fig. 13(a), but for the coherent–incoherent intermittency at $\eta = 0.05$. There is now strong...
correlation between $S_m(t)$ and $\langle B^2 \rangle /2$, as the entropy decreases whenever the magnetic energy increases abruptly during spatially coherent bursts, indicating a high variability not only in the energy amplitude, but also in the spatial complexity.

3.6 Numerical resolution

Since we are interested in the long-time behaviour of the solutions, due to restrictions in computational time we have opted for low-resolution simulations ($64^3$). However, simulations with a $128^3$ mesh indicate that the same behaviour is expected for higher resolutions. Fig. 14 shows the general agreement between the time series for the kinetic (a) and magnetic (b) energies obtained with different resolutions. The results for $64^3$ are shown in solid lines (red) and for $128^3$ in dashed lines (black). The computed values for spectral entropy and spectral average for both resolutions are given in Table 1 and reveal good agreement, as well.

4 CONCLUSIONS

We have described an intermittent route to mean-field dynamo as a function of the (low) magnetic Prandtl number $Pr$. Notably, the transition from the purely hydrodynamic regime to the dynamo regime suddenly gives rise to chaotic magnetic field oscillations, and no sequence of symmetry breaking bifurcations prior to the chaotic regime is observed. This is because the velocity field adopted is already in a strongly chaotic regime. A sequence of symmetry breaking bifurcations was observed in some simulations with lower kinetic Reynolds number and that is the subject of a future work.
The main contributions of this work are the quantification of spatial complexity of the magnetic field patterns from 3D compressible MHD dynamo simulations (Figs 12 and 13), and the identification of a new type of intermittency characterized by the switching between coherent and incoherent large-scale structures [Figs 5(b), 7 and 13(b)]. The mechanisms responsible for this intermittency have not yet been explored and, in future works, we expect that a better description can be achieved by means of the analysis of phase synchronization of Fourier modes (Koga et al. 2008; Chian & Miranda 2009).

This paper is also an addition to a series of previous works on the onset of non-linear dynamo action in ABC flows. Kinematic and non-linear dynamos were studied by Galanti, Sulem & Pouquet (1992) for several choices of the forcing scale $k_f$ and $Pr$ ranging from 1 to 13, with low kinetic Reynolds number $Re$ (up to 20); a bifurcation study with low $Re$, $Pr = 1$ and $k_f = 1$ was conducted by Seehafer, Feudel & Schmidtmann (1996); Sweet et al. (2001a,b) detected a blowout bifurcation responsible for on–off intermittency for $k_f = 1$, $Re = 6.3$ and $Pr$ close to one; Brummell, Cattaneo & Tobias (2001) studied kinematic and non-linear regimes with $Re$ between 50 and 100 and $k_f = 1$ in a time-dependent ABC flow previously investigated by Galloway & Proctor (1992); Mininni (2007) investigated the inverse energy cascade at small $Pr$ (down to 0.005) and $k_f = 3$, with $Re$ varying from 11 to 6200; Alexakis & Ponty (2008) studied the effect of the Lorentz force on on–off intermittency in ABC flows with $k_f = 1$ and varying both $Re$ and $Rm$. In contrast to all the aforementioned papers which performed incompressible MHD simulations, in this paper we opted for a compressible code. It is not yet clear whether the compressibility is a crucial feature of the dynamics observed, though certainly compressibility is a feature shared by the solar dynamo! Our simulations were performed with $k_f = 5$, $Re \sim 100$ (or $Re \sim 500$ for the box-scale $Re$) and $Pr \in [0.0625, 0.5]$. Despite the use of a model with simple geometry, reasonably small kinetic and magnetic Reynolds numbers, and without differential rotation, our results reveal some qualitative resemblance to certain aspects of the solar dynamo and we expect them to be useful in the analysis of more realistic models. Alternatively, one could be interested in a mean-field theory description of the problem by using an $\alpha^2$ dynamo model, but it is unclear whether a coherent–incoherent type of intermittency can be observed in that case or not. In fact, it is even possible to argue that the scale separation between the energy-carrying scale and the box scale employed in our model is not large enough to properly characterize a mean-field dynamo, so that is still an open question.

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**Table 1.** Spectral entropy and spectral average for different resolution runs.

| Res. | 64$^3$ | 128$^3$ |
|------|--------|--------|
| $(S_E)_t$ | 1.24 | 1.10 |
| $(N_E)_t$ | 4.77 | 4.96 |
| $(S_W)_t$ | 1.08 | 1.06 |
| $(N_W)_t$ | 2.85 | 2.93 |

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