Density of states for Random Band Matrix

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Abstract

By applying the supersymmetric approach we rigorously prove smoothness of the averaged density of states for a three dimensional random band matrix ensemble, in the limit of infinite volume and fixed band width. We also prove that the resulting expression for the density of states coincides with the Wigner semicircle with a precision $1/W^2$, for $W$ large but finite.

1 Introduction

Random Matrix Theory (RMT) has proved to be relevant in the study of several physical models. It was initially applied to the study of resonance spectra of complex nuclei and later to the study of the quantum properties of weakly disordered conductors, and the spectral properties of quantum systems which are chaotic in their classical limit \( [1][2] \). RMT also appears in other fields, such as statistics, number theory and random permutations. See for example \([3][4][5]\) for recent developments.

In this article we study the density of states for a class of Hermitian random matrices \( H_{ij} \) whose elements are Gaussian with mean zero and covariance

\[ \langle H_{ij} H_{kl} \rangle = \delta_{jk} \delta_{il} J_{ij}. \]  

In the classical case of GUE, Gaussian unitary ensembles, the indices \( i \) and \( j \) range from 1 to \( N \) and \( J_{ij} = 1/N \). For this case the density of states (DOS) is given by Wigner’s famous semicircle law

\[ \rho_{SC}(E) = \begin{cases} \frac{1}{\pi} \sqrt{1 - \frac{E^2}{4}} & |E| \leq 2 \\ 0 & |E| > 2 \end{cases} \]  

in the limit \( N \uparrow \infty \). Our analysis will focus on band random matrices for which the indices \( i, j \) range over a box \( \Lambda \cap \mathbb{Z}^d \) and \( J_{ij} \) is small when \( |i - j| \) is larger then some fixed band width \( W \). As we let \( \Lambda \uparrow \mathbb{Z}^d \) the spectral properties of such matrices should be quite similar to that of a random Schrödinger operator on a lattice \( \mathbb{Z}^d \) given by

\[ H = -\Delta + \lambda V(j) \]  

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where \( V(j) \) are independent random variables and \( \Delta \) is the discrete Laplacian. For example in one dimension the spectra of the random Schrödinger and the random band matrix are pure point with exponentially decaying eigenfunctions \([6] [7]\). This is called localization. In two or more dimensions, localization also holds for energies outside some interval depending on \( d, \lambda, W \). Thus band random matrices are a way of interpolating between classical random matrix ensembles (GUE or GOE) and random Schrödinger.

The goal of this article is to obtain detailed information about the density of states for a special class of random band matrices in 3 dimensions. We shall consider energies at which extended rather than localized states are expected. More precisely let \( i, j \in \Lambda \cap \mathbb{Z}^3 \), \( \Lambda \) a set of cubes of side \( W \), and define

\[
J_{ij} := \left( \frac{1}{-W^2 \Delta + 1} \right)_{ij} \sim \frac{1}{4\pi W^2} \left( \frac{1}{1 + |i-j|} \right) e^{-\frac{|i-j|}{W}} \tag{1.4}
\]

where \( \Delta \) is the Laplacian with periodic boundary conditions in the volume \( \Lambda \) and \( W \) is large but fixed. Our estimates are valid uniformly in the size of \( \Lambda \subseteq \mathbb{Z}^3 \).

The average density of states is given by

\[
\langle \rho_{\Lambda}(E) \rangle = -\frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \text{Im} \left\langle \left( \frac{1}{E + i\varepsilon - H} \right)_{00} \right\rangle . \tag{1.5}
\]

Note that as \( \Lambda \uparrow \mathbb{Z}^3 \), the density of states \( \rho(E) \) does not depend on the configuration with probability one. The derivative of \( \rho_{\Lambda}(E) \) is

\[
\frac{d}{dE} \langle \rho_{\Lambda}(E) \rangle = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \sum_{x \in \Lambda} R(E + i\varepsilon; 0, x) \tag{1.6}
\]

where

\[
R(E + i\varepsilon; 0, x) = \left\langle \left( \frac{1}{E + i\varepsilon - H} \right)_{0x} \left( \frac{1}{E + i\varepsilon - H} \right)_{x0} \right\rangle \tag{1.7}
\]

Note that for \( x \neq 0 \)

\[
\left\langle \left( \frac{1}{E + i\varepsilon - H} \right)_{0x} \right\rangle = 0 \tag{1.8}
\]

because of the symmetry \( H_{ij} \rightarrow -H_{ij} \).

Our main result is that for large \( W \) and \( E \) inside the interval \([-2, 2]\), \( \langle \rho_{\Lambda}(E) \rangle \) equals the Wigner semicircle distribution \([12]\) plus corrections of order \( W^{-2} \). Moreover \( R(x) \) decays exponentially fast and \( \rho(E) \) is smooth in \( E \). These results hold for fixed \( W \) and are uniform in \( \varepsilon \) as \( \varepsilon \downarrow 0 \) and in the volume as \( \Lambda \uparrow \mathbb{Z}^3 \). See Theorem 2.1 for a precise statement. When \( |E| > 2 + O(W^{-1}) \) we expect that \( \rho(E) \) is smaller than any power of \( W^{-1} \) and that localization holds.

For random Schrödinger operators given by \([13]\) we have the classic bound by Wegner, \( \rho(E) \leq \text{const} \lambda^{-1} \) for small \( \lambda \). This estimate is far from optimal since for small \( \lambda \) we expect the density of states to approach that of the Laplacian.
Unfortunately there are no uniform bounds on $\rho(E)$ as $\lambda \to 0$ or estimates on the smoothness of $\rho(E)$ unless either the distribution of $V$ is Cauchy (in which case the density of states can be explicitly computed) or $E$ lies in an interval for which localized states are proved to exist.

Note that for both the random Schrödinger and the random band matrix ensembles it is conjectured that for $d = 3$

$$\pi \sum_x \left| \left( \frac{1}{E + i\varepsilon - H} \right) \left|_{0x} \right|^2 \right| e^{ixp} \approx \frac{\rho(E)}{Dp^2 + \varepsilon}$$

where $D$ is the diffusion constant. Here $E$ must be inside $[0, 4d]$ for the case of random Schrödinger or inside $[-2, 2]$ for our band matrix ensemble, and both $W^{-1}$ and $\lambda$ are small. This paper does not address this important conjecture. Instead we are using the phase oscillations of the Green’s functions to obtain exponential decay for $R(x)$.

To establish our results on Green’s functions we use the supersymmetric formalism of K. Efetov [11] [12] which has its roots in earlier work by Wegner [3] [4]. We recommend the survey article of Mirlin [1] and also the paper of Mirlin and Fyodorov [10] which studies random band matrices in 1 dimension. In the mathematics literature, A. Klein studied the density of states using supersymmetric methods [13] but only at energies where localization holds.

The supersymmetric method enables one to explicitly average the Green’s function over the randomness. This technique involves the use of both real and anticommuting variables. However when we perform our estimates all anticommuting variables are integrated out so that the resulting integrals is just over real variables. As a result of this averaging, the problem is converted into a problem in statistical mechanics whose action has approximately the form

$$\mathcal{A}(\phi) = \sum_{j \in \Lambda} W^2 (\nabla \phi)^2 (j) + U(\phi)(j)$$

where the potential $U$ is a function of the field $\phi(j)$ and has two saddle points. In some respects this problem looks like a double well $\phi^4$ interaction. A more careful analysis of the integral over $\phi$ shows that one saddle dominates and it yields the Wigner semicircle distribution. The second saddle is suppressed by a determinant as we shall explain later. The large parameter $W$ ensures that the integral is governed by the saddle and its Gaussian fluctuations. There are similar integrals which appear for random Schrödinger operators, however the path integral is much more oscillatory and we can not yet control them unless there are long range correlations in the $V(j)$.

The average of $\left| \left( E + i\varepsilon - H \right)_{0x}^{-1} \right|^2$ can also be calculated with the supersymmetric formalism but the statistical mechanics is now more complicated. Instead of two saddle points there is a non compact saddle manifold and fluctuations have massless modes which are responsible for the power law (1.4).

The remainder of this article is organized as follows. In Sec. 2 we give a precise statement of our results. In Sec. 3 we use the supersymmetric formalism
to convert averages of the Green’s function to a model in statistical mechanics. The advantage of this representation is that for large $W$ we see that the integral is dominated by two saddle points. These saddle points and their Hessians are discussed in Sec. 4. The following section is devoted to obtaining our results in a box $\Lambda \subseteq \mathbb{Z}^3$ of side $W$. In the last section we show that the analysis in the box can be extended to $\mathbb{Z}^3$ using a variant of the cluster expansion.

**Notation** As in the paper we will need to insert many constants in the different bounds we will denote by $K$ any large positive constant, independent from $W$ and $\Lambda$, and by $c$ any small positive constant independent from $W$ and $\Lambda$. These constants need not be the same in different estimates. Also we will sometimes use the symbol $\lesssim$ to indicate that there is a constant factor $K$ on the right side of the inequality ($\lesssim$ stands for $\leq K$) without writing $K$ explicitly.

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## 2 Model

As we said in the introduction, we consider the set $\mathcal{H}$ of Hermitian matrices $H$ with entries $i, j \in \Lambda \subset \mathbb{Z}^d, d > 0$. From (1.1) we see that the probability density is

$$P(H) = \prod_{\substack{i \in \Lambda \atop i < j}} \frac{dH_{ij} dH^*_{ij}}{2\pi J_{ij}} e^{-\frac{|J_{ij}|^2}{4}} \prod_{i \in \Lambda} \frac{dH_{ii}}{\sqrt{2\pi J_{ii}}} e^{-\frac{H_{ii}^2}{2J_{ii}}}$$  \hspace{1cm} (2.1)

where $<$ is an order relation on $\Lambda$ and $J$ is defined in (1.4). With these definitions $\mathcal{H}$ is a set of hermitian random band matrices with band width $W$. Note that in $d = 1$ for $\Lambda = [1, N]$ we have $|\Lambda| = N$ and $J_{ij} = N^{-1} \exp[-|i-j|/N]$ which is very close to GUE. For any function of $H$ $F(H)$ we define the average $\langle F(H) \rangle$ as

$$\langle F(H) \rangle = \int dH P(H) F(H)$$  \hspace{1cm} (2.2)

We study the averaged density of states $\bar{\rho}_\Lambda(E)$:

$$\bar{\rho}_\Lambda(E) =: -\frac{1}{\pi} \lim_{\varepsilon \to 0^+} \text{Im} \left\langle \frac{\text{Tr} \frac{1}{|\Lambda|} \frac{1}{E+i\varepsilon - H}}{E+i\varepsilon - H} \right\rangle = -\frac{1}{\pi} \lim_{\varepsilon \to 0^+} \text{Im} \langle G^+_0 \rangle$$  \hspace{1cm} (2.3)

where “Im” indicates the imaginary part and $G^+$ is the retarded Green’s function:

$$G^+ := \frac{1}{E + i\varepsilon - H}$$  \hspace{1cm} (2.4)
In the following we restrict to $d = 3$ and we consider energies well inside the spectrum. For technical reasons we also avoid the energy $E = 0$. Therefore we consider only energies in the interval

$$\mathcal{I} = \{ E : |E| \leq 1.8 \text{ and } |E| > \eta \}$$

with $\eta > 0$. We assume that our region $\Lambda$ is a union of cubes in $\mathbb{Z}^3$ of side $W$.

The paper is devoted to the proof of the following theorem

**Theorem 1** For $d = 3$, there exists a value $W_0$ such that for all $W \geq W_0$ the averaged density of states $\bar{\rho}_\Lambda(E)$ is smooth in $E$, in the interval $\mathcal{I}$, uniformly in $W$ and $\Lambda$ (hence also in the limit $\Lambda \uparrow \mathbb{Z}^3$):

$$|\partial^n_E \bar{\rho}_\Lambda(E)| \lesssim C_n \ \forall \ n < n_0(W)$$

where $\lim_{W \to \infty} n_0(W) = \infty$. Moreover $\bar{\rho}_\Lambda(E)$ is the semicircle law with a precision $1/W^2$.

$$|\bar{\rho}_\Lambda(E) - \rho_{SC}(E)| \lesssim \frac{1}{W^2}$$

where $\rho_{SC}(E)$ is the semicircle law $\rho_{SC}$ defined in (1.2). Note that the first equation (2.6) means in particular that for $x \neq 0$

$$R(x) = |\langle G^+_0 G^+_x \rangle| \lesssim \frac{1}{W^3} e^{-c|x|}$$

Outline of the paper

In Sec. 5 we establish Theorem 1 on a cube $\Lambda$ of side $W$. We use the supersymmetric formalism to write $\langle G^+_0 \rangle$ as a functional integral where a saddle point analysis can be performed. Actually there are two saddles. For $d = 3$ one saddle is suppressed by a factor $e^{-W}$ (note that this is not true for $d \leq 2$). The fluctuations around the saddle are controlled using small probability arguments while the integral near the dominant saddle is estimated by a Brascamp-Lieb inequality [14][15].

Sect. 6 is devoted to the cluster expansion which enables us to analyze the limit $\Lambda \uparrow \mathbb{Z}^3$. The cluster expansion expresses $\langle G^+_0 \rangle$ as a sum over finite volume contributions $Y \subset \mathbb{Z}^3$ which are again unions of cubes of side $W$. We show that large $Y$ terms give small contributions to $\langle G^+_0 \rangle$. This expansion also enables us to prove the bound on $R(x)$.

## 3 Supersymmetric approach

In this section we shall use the algebraic formalism of supersymmetry to express our average Green’s function in terms of a functional integral, which, apart form a determinant, is local. Let $J$ be given by [14].
Lemma 1 The averaged Green’s function can be written as

\[ \langle G^{00}_+ \rangle = \int da \, db \, \exp \left[ -\frac{1}{2} (a^T J^{-1} a + b^T J^{-1} b) \right] \prod_{i \in \Lambda} \frac{(E_\varepsilon - i b_i)}{(E_\varepsilon - a_i)} \times \frac{1}{(E_\varepsilon - a_0)} \det[J^{-1} - F(a, b) - F'(a_0, b_0)] \] (3.1)

where \( a \) and \( b \) are vectors whose components \( a_i, b_i \) \((i = 1, ..., |\Lambda|)\) are real variables and \( a^T \) and \( b^T \) are the corresponding transposed vectors. We defined \( E_\varepsilon = E + i \varepsilon \) and \( F(a, b) \) and \( F'(a_0, b_0) \) are matrices with elements

\[ F(a, b)_{ij} := \frac{1}{(E_\varepsilon - a_i)(E_\varepsilon - i b_i)} \] (3.2)

\[ F'(a_0, b_0)_{ij} := \frac{1}{(E_\varepsilon - a_0)(E_\varepsilon - i b_0)} \] (3.3)

Note that each \( a_i \) has a pole at \( a_i = E + i \varepsilon \) while \( b_i \) has no singularity (as it appears only in the numerator). This expression is then well defined only for \( \varepsilon > 0 \).

By the same technique we obtain a similar formulas for \( \langle G^{0j}_+ G^{j0}_+ \rangle \) and in general for \( \langle G^{0j}_+ G^{j1}_+ ... G^{jn}_+ \rangle \).

Remarks Note that if we omit the observable, that is we omit \( (E_\varepsilon - a_0)^{-1} \) and \( F' \) in (3.1), we are actually computing \( \langle 1 \rangle = 1 \), thus

\[ 1 = \int da \, db \, \exp \left[ -\frac{1}{2} (a^T J^{-1} a + b^T J^{-1} b) \right] \prod_{i \in \Lambda} \frac{(E_\varepsilon - i b_i)}{(E_\varepsilon - a_i)} \det[J^{-1} - F(a, b)] . \] (3.4)

Proof Note that the Green’s function can be written as a functional integral:

\[ G_{kl}^+ = -i \int dS^* dS \ exp \left[ i S^+(E_\varepsilon - H) S \right] \det[-i(E_\varepsilon - H)] S_k S_l^* \] (3.5)

where the determinant is the normalization factor and we defined

\[ S = \begin{pmatrix} S_1 \\ \vdots \\ S_{|\Lambda|} \end{pmatrix} \quad S^+ = \begin{pmatrix} S_1^* \\ \vdots \\ S_{|\Lambda|}^* \end{pmatrix} \] (3.6)

and \( S_1, ..., S_{|\Lambda|} \) are complex bosonic fields. In order to insert all the \( H \) dependence in the argument of the exponential we introduce integrals over fermionic fields:

\[ \det[-i(E_\varepsilon - H)] = \int d\chi^* d\chi \ exp \left[ i \chi^+(E_\varepsilon - H) \chi \right] \] (3.7)
\[ \chi = \begin{pmatrix} \chi_1 \\ \vdots \\ \chi_{|\Lambda|} \end{pmatrix} \quad \chi^+ = \begin{pmatrix} \chi_1^* \\ \cdots \\ \chi_{|\Lambda|}^* \end{pmatrix} \] (3.8)

and \( \chi_1, \ldots, \chi_{|\Lambda|} \) are fermionic complex fields. Therefore we can write

\[ G_{kl} = -i \int d\Phi^* d\Phi \ \exp \left[ i\Phi^+ (E_{\varepsilon} - H) \Phi \right] S_k S_l^* \] (3.9)

where we have introduced the superfields \( \Phi_1, \ldots, \Phi_{|\Lambda|} \) \((i = 1, \ldots, |\Lambda|)\)

\[ \Phi_i = \begin{pmatrix} S_i \\ \chi_i \end{pmatrix} \quad \Phi_i^+ = (S_i^*, \chi_i^*) \] (3.10)

These superfields can be seen as components of a supervector \( \Phi \)

\[ \Phi = \begin{pmatrix} \Phi_1 \\ \vdots \\ \Phi_{|\Lambda|} \end{pmatrix} \quad \Phi^+ = \begin{pmatrix} \Phi_1^* \\ \cdots \\ \Phi_{|\Lambda|}^* \end{pmatrix} \] (3.11)

We adopted the conventions in the review by Mirlin [1] We summarize supersymmetric formalism and notation in App. A.

Now we can perform the average over \( H \):

\[ \langle \exp \left[ -i\Phi^+ H \Phi \right] \rangle = \exp \left[ -\frac{1}{2} \sum_{ij} J_{ij} (\Phi_i^+ \Phi_j)(\Phi_j^+ \Phi_i) \right] \] (3.12)

To convert this quartic interaction into a quadratic one we perform a Hubbard-Stratonovich transformation:

\[ \sum_{ij} J_{ij} (\Phi_i^+ \Phi_j)(\Phi_j^+ \Phi_i) = \sum_{ij} [A_i J_{ij} A_j - B_i J_{ij} B_j - 2 P_i^* J_{ij} P_j] \] (3.13)

where

\[ A_i = S_i^* S_i, \quad B_i = \chi_i^* \chi_i, \quad P_i = S_i^* \chi_i, \quad P_i^* = S_i \chi_i^* \] (3.14)

and there is no sum over \( i \). Note that \( A_i \) and \( B_i \) are commuting variables while \( P_i \) and \( P_i^* \) are anticommuting ones. Now

\[ \exp \left[ -\frac{1}{2} A^T J A \right] = (2\pi)^{-|\Lambda|} \int \prod_{i \in \Lambda} \frac{da_i}{\sqrt{\det J}} \ e^{-\frac{1}{2} a^T J a - i a^T A} \] (3.15)

\[ \exp \left[ +\frac{1}{2} B^T J B \right] = (2\pi)^{-|\Lambda|} \int \prod_{i \in \Lambda} \frac{db_i}{\sqrt{\det J}} \ e^{-\frac{1}{2} b^T J b + b^T B} \]

\[ \exp \left[ -P^+ J P \right] = (2\pi)^{|\Lambda|} \int \prod_{i \in \Lambda} \frac{d\rho_i}{\det J^{-1}} \ e^{-\rho^+ J^{-1} \rho - i \rho^+ P - i P^+ \rho} \]
where \(a_i, b_i\) are real bosonic fields and \(\rho_i\) is a complex fermionic field for any \(i = 1, \ldots, |\Lambda|\). Therefore

\[
\langle \exp \left[ -i \Phi^+ H \Phi \right] \rangle = \int da \, db \, d\rho^* \, d\rho \, e^{-\frac{i}{2} (a^T J^{-1} a + b^T J^{-1} b + 2 \rho^* J^{-1} \rho) - i \Phi^+ R \Phi} \tag{3.16}
\]

where

\[
\Phi^+ R \Phi := \sum_{i=1}^{|\Lambda|} \Phi_i^+ R_i \Phi_i, \quad R_i := \begin{pmatrix} a_i & \rho_i^* \\ \rho_i & i b_i \end{pmatrix} \tag{3.17}
\]

\(R_i\) is actually a supermatrix, containing both bosonic and fermionic variables. Therefore the integration over the fermionic fields can be performed exactly. Using this formalism we have

\[
\left\langle \int d\Phi^* d\Phi \, e^{i \Phi^+ (E_s - H) \Phi} S_k S_l \right\rangle = \frac{\delta_{kl}}{i} \left( \frac{1}{E_s - R_k} \right)^{11} \prod_i \frac{1}{\text{Sdet}(E_s - R_i)} \tag{3.18}
\]

where

\[
\text{Sdet}(E_s - R_i) = \frac{(E_s - a_i)}{(E_s - i b_i)} \left[ 1 - \rho_i^* \rho_i (E_s - a_i)(E_s - i b_i) \right] \tag{3.19}
\]

\[
\left( \frac{1}{E_s - R_k} \right)^{11} = \frac{1}{(E_s - a_k)} \left[ 1 - \rho_k^* \rho_k (E_s - a_k)(E_s - i b_k) \right]^{-1} \tag{3.20}
\]

Therefore

\[
\langle G_{00}^+ \rangle = \int da \, db \, d\rho^* \, d\rho \, e^{-\frac{i}{2} (a^T J^{-1} a + b^T J^{-1} b + 2 \rho^* J^{-1} \rho)} \prod_{i \in \Lambda} \frac{(E_s - i b_i)}{(E_s - a_i)} \prod_i \left[ 1 - \frac{\rho_i^* \rho_i (E_s - a_i)(E_s - i b_i)}{(E_s - a_i)(E_s - i b_i)} \right]^{-1} \tag{3.21}
\]

The integration over the fermionic fields can be performed exactly. Using the property: \(\rho_i^2 = (\rho_i^*)^2 = 0 \forall i\) we observe that

\[
\left[ 1 - \frac{\rho_i^* \rho_i (E_s - a_i)(E_s - i b_i)}{(E_s - a_i)(E_s - i b_i)} \right]^{-1} = \exp \left[ \frac{\rho_i^* \rho_i}{(E_s - a_i)(E_s - i b_i)} \right] \tag{3.22}
\]

therefore the integration over \(\rho\) and \(\rho^*\) reduces to the following expression

\[
\int d\rho^* \, d\rho \, e^{-\rho^* [J^{-1} + F(a, b) + F'(a_0, b_0)] \rho} = \det [J^{-1} - F(a, b) - F'(a_0, b_0)] \tag{3.23}
\]

where \(F(a, b)\) and \(F'(a_0, b_0)\) are defined in (3.2,3.3). We obtain then the expression (3.1).
4 Saddle point analysis

In this section we shall deform the integral (3.1) over \(a_j\) and \(b_j\) so that they pass through certain complex saddle points. If we ignore the determinant in (4.9) and the kinetic term, we show that the resulting integrand has a double well structure, with the two wells of the same height. In the Sec. 5.1.2 (Lemma 6) we will see that the determinant actually suppresses one of the two saddles by a factor \(e^{-W}\).

Saddle points Observing the integrand in (3.1) we remark that the factor 
\[-W^2\Delta\] in \(a^TJ^{-1}a + b^TJ^{-1}b\) forces the fields \(a\) and \(b\) to be approximately constant. Therefore if we ignore the determinant, the leading contribution to the integrand (hence also to the saddle structure) is then

\[
e^{-\frac{W}{2}(a^2+b^2)} \left[ \frac{(E_\nu - ib)}{(E_\nu - a)} \right]^{\kappa |\Lambda|} = \left[ e^{-[f_1(a)+f_2(b)]} \right]^{\kappa |\Lambda|} \tag{4.1}
\]

where the fields \(a\) and \(b\) are constant \((a_i = a, b_i = b\) for all \(i\)) and we defined

\[
f_1(a) = \frac{a^2}{2} + \ln(E - a)
\]
\[
f_2(b) = \frac{b^2}{2} - \ln(E - ib). \tag{4.2}
\]

Note that in this approximation the saddle points for GUE and Random Band Matrix are the same. The critical points of \(f_1\) and \(f_2\) are given by

\[
a_s = E_r \pm iE_i
\]
\[
b_s = -iE_r \pm E_i \tag{4.3}
\]

where we defined

\[
\mathcal{E} := E_r - iE_i := \frac{E}{2} - i\sqrt{1 - \frac{E^2}{4}}. \tag{4.4}
\]

Note that \(\mathcal{E}\) satisfies

\[
E - \mathcal{E} = \mathcal{E}^*, \quad \mathcal{E} \mathcal{E}^* = 1 \quad \forall |E| < 2.
\tag{4.5}
\]

Spectrum Note that, if \(|E| < 2\) the saddle \(a_s, b_s\) have non zero imaginary parts even as \(\varepsilon \downarrow 0\). For \(|E| \geq 2 + O(W^{-1})\) we expect that the density of states is smaller than any power of \(W^{-1}\), for \(W\) large.

Contour deformation We deform the integration contour in order to pass through a saddle point. To avoid crossing the pole \(a_i = E_\nu\) we have to pass through the saddle \(a_s = \mathcal{E}\). On the other hand the choice for \(b_s\) is arbitrary,
as there is no pole in \( b \), but it turns out (see Sec. 5.1) that \( b_s = -i\mathcal{E} \) is the dominant contribution. Note that

\[
f''(\mathcal{E}) = f''(-i\mathcal{E}) = (1 - \mathcal{E}^2) .
\]

Hence the Hessian at this saddle point is

\[
B^{-1} = -W^2 \Delta + (1 - \mathcal{E}^2)
\]

**Lemma 2** We perform inside (3.1) the translation

\[
a_j \rightarrow a_j + \mathcal{E} \\
b_j \rightarrow b_j - i\mathcal{E} \quad \forall j \in \Lambda
\]

and take the limit \( \varepsilon \downarrow 0 \). The integral can then be written as

\[
\langle G_{60}^+ \rangle_{\varepsilon=0} = \int d\mu_B(a, b) \det[1 + (D + D'_0)B] e^{V'_0 + \sum_{j \in \Lambda} V_j}
\]

where the measure \( d\mu_B(a, b) \) has covariance \( B \) given by (4.7). The factor \( \exp[\sum_j V_j] \) is what remains in the exponential after the Hessian has been extracted, \( \det[1 + DB] \) corresponds to \( \det[J^{-1} + F] \) after the normalization factor \( \det B^{-1} \) has been extracted. Finally \( \exp V'_0 \) and \( D'_0 \) are the contributions from the observable.

More precisely we define the measure as

\[
d\mu_B(a, b) = d\mu_B(a) d\mu_B(b)
\]

and \( B \) is the Hessian around the saddle, defined in (4.7). The normalization factor for the measure has been extracted from the determinant. The interactions are given by \( V_j = V_j(a_j) + V_j(b_j) \) and \( D_{ij} = \delta_{ij} D_i \), where

\[
V_j(a_j) = \int_0^1 dt (1-t)^2 \frac{a_j^3}{(\mathcal{E}^* - ta_j)^3}
\]

\[
V_j(b_j) = -\int_0^1 dt (1-t)^2 \frac{(ib_j)^3}{(\mathcal{E}^* - ib_j)^3}
\]

\[
D_i = \left[ \mathcal{E}^2 - F(a + \mathcal{E}, b - i\mathcal{E})_{ii} \right] = \mathcal{E}^2 - \frac{1}{(\mathcal{E}^* - a_i)(\mathcal{E}^* - ib_i)}
\]

\[
= -\int_0^1 dt \left[ \frac{a_i}{(\mathcal{E}^* - ta_i)^2(\mathcal{E}^* - ib_i t)^2} + \frac{ib_i}{(\mathcal{E}^* - ta_i)(\mathcal{E}^* - ib_i t)^2} \right] .
\]

Finally the contributions from the observable are given by \( V'_0 \) and \( D'_0 \) where

\[
V'_0 = -\ln(\mathcal{E}^* - a_0)
\]

\[
(D'_0)_{ij} = -F'(a_0 + \mathcal{E}, b_0 - i\mathcal{E})_{ij} = -\delta_{i0} \delta_{j0} \frac{1}{(\mathcal{E}^* - a_0)(\mathcal{E}^* - ib_0)} .
\]
The proof is a straightforward change of variables and a reorganization of the resulting expression. Note that for any $|E| < 2$ there is no pole in $a$ as the factor $E - E - a_i$ is always at a distance at least $E_i$ from zero.

For the special value $E = 0$, a singularity in $b_i = 1$ seems to appear from the factor $1/i(1 - b_i)$ in the argument of the determinant. This is not a real singularity as there is the same factor in the numerator outside the determinant. Nevertheless to avoid additional technical problems we avoid $E = 0$ in the following. This is the reason why we chose $\eta > 0$ in $I$ in Theorem 1.

Properties of the Hessian. The Hessian $B^{-1}$, which is the covariance of the Gaussian measure after the translation, has now a complex mass term:

\[(1 - \mathcal{E}^2) = 2 \left(1 - \frac{E^2}{4}\right) + iE \sqrt{1 - \frac{E^2}{4}} =: m_r^2 + im_i^2. \tag{4.15}\]

Note that for $|E| < 2$ the real part $m_r^2$ is positive and this ensures the convergence of the integral. In the following, as we will need to treat in a different way the real and imaginary part of $B^{-1}$, we introduce the real covariance $C$:

\[C := \frac{1}{-W^2\Delta + m_r^2}, \tag{4.16}\]

therefore

\[B^{-1} = C^{-1} + im_i^2. \tag{4.17}\]

Note that $C$ is positive as a quadratic form and pointwise. In momentum space $C$ is written as

\[\hat{C}(\vec{k}) = \frac{1}{W^2} \sum_{i=1}^{d} \frac{1}{(1 - \cos k_i) + (m_r/W)^2} \quad k_i = 2\pi \frac{n_i}{|\Lambda|^{1/d}}, \quad n_i = 0, \ldots, |\Lambda|^{1/d} - 1. \tag{4.18}\]

When $|\Lambda| \uparrow \infty$, $k_i$ becomes a continuum variable $k_i \in [0, 2\pi]$. The spatial decay depends on the dimension. In the particular case of $d = 3$

\[0 < C_{ij} \lesssim \frac{1}{W^2(1 + |i - j|)} e^{-|i-j|m_r W}. \tag{4.19}\]

The covariance $B$ has the same expression as $C$, but with an imaginary term in the mass. It is easy to prove that $B$ decays in the same way as $C$.

\[|B_{ij}| \lesssim \frac{1}{W^2(1 + |i - j|)} e^{-|i-j|m_r W}. \tag{4.20}\]
Properties of the interaction  After the translation the functions $f_1$, $f_2$ introduced in (4.2) become

$$f_1(a) = -\frac{1}{2} (1 - \varepsilon^2) a_j^2 + V_j(a_j)$$

$$f_2(b) =: -\frac{1}{2} (1 - \varepsilon^2) b_j^2 + V_j(b_j) \quad (4.21)$$

Note that, after the translation there also constant factors arising from $f_1$ and $f_2$ which cancel. In the following we will insert absolute values in the integral, in order to obtain our estimates. We then have to study the behavior of

$$F_1(a) =: \left| e^{-f_1(a)} \right| \quad F_2(b) =: \left| e^{-f_2(a)} \right| \quad (4.22)$$

It is easy to prove that for $|E| \leq 1.8$ $F_1(a)$ has only one maximum, in $a = 0$, of height 1 (see Fig. 1). Note that when $1.8 < |E| \leq 2$ zero is no longer the maximum of $F_1(a)$ and this is why we restrict $E$ to $I$ given by (2.5). Nevertheless, there is still a single saddle point so we expect that by suitable deformation of the contour we should be able to extend our result to the interval $|E| \leq 2 - O(W^{-1})$.

On the other hand, for any value of $|E| < 2$, $F_2(b)$ has two maxima, which do correspond to the two saddles, one in $b = 0$ and one in $b = 2\varepsilon_i$. Both maxima have height 1 (Fig. 1). We will see in the next section that the second maximum is suppressed by a factor $e^{-W}$ from the determinant appearing in (4.9).

5 Finite volume estimate

We prove now Theorem 1 in a fixed cube $\Lambda$ of side $W$, with $0 \in \Lambda$. We prove the boundness of $\rho_\Lambda(E)$ in Theorem 2, then in Theorem 3 we prove the bounds on
the derivatives and on \( |\bar{\rho}_\Lambda(E) - \rho_{SC}| \): these bounds follow by the same technique used for the bounds on \( \bar{\rho}_\Lambda(E) \), with some slight modifications.

**Theorem 2** For \( \Lambda \) as above, there exists a value \( W_0 \) such that for all \( W \geq W_0 \) and for all \( E \in I \), where \( I \) is defined in (2.3), the averaged density of states \( \bar{\rho}_\Lambda(E) \) is bounded uniformly in \( W \) and \( \Lambda \)

\[
|\bar{\rho}_\Lambda(E)| \leq K
\]  \( (5.1) \)

**Theorem 3** For \( \Lambda \) as above, there exists a value \( W_0 \) such that for all \( W \geq W_0 \) and for all \( E \in I \), where \( I \) is defined in (2.3), we have

\[
|\partial_E \bar{\rho}_\Lambda(E)| \leq C_n \quad \forall \ n < n_0(W)
\]  \( (5.2) \)

\[
|\bar{\rho}_\Lambda(E) - \rho_{SC}(E)| \leq \frac{K}{W^2}
\]  \( (5.3) \)

uniformly in \( \Lambda \) and \( W \).

5.1 Proof of Theorem 2

Inserting the absolute values in the expression (4.9) we have

\[
|\langle G_{00}^+ \rangle_{\epsilon=0}| \leq \int |d\mu_B(a,b)||\text{det}[1 + (D + D_0^*)B]| \left| e^{V_0 + \sum_{j \in \Lambda} V_j} \right| (5.4)
\]

The absolute values of \( d\mu_B \) and \( \text{det}[1 + (D + D_0^*)B] \) are bounded through Lemma 3 and 4 respectively.

**Lemma 3** The total variation of the complex measure is bounded by

\[
|d\mu_B(a,b)| \lesssim e^{O_{\frac{\text{det}}{W^2}}} \ d\mu_C(a,b)
\]  \( (5.5) \)

**Proof** The measure \( d\mu_B(a,b) \) can be written as

\[
|d\mu_B(a,b)| = \left| \frac{\text{det} B^{-1}}{\text{det} C^{-1}} \right| \ d\mu_C(a,b)
\]  \( (5.6) \)

where the determinants are the normalization factors for the two measures and can be written as

\[
\left| \frac{\text{det} B^{-1}}{\text{det} C^{-1}} \right| = \left| \text{det} 1 + i\lambda^2 C \right|
\]  \( (5.7) \)

Note that, for any normal matrix \( A \), with \( \text{Tr} A^+ A < \infty \), the following inequality is true

\[
|\text{det}(1 + A)| \leq \left| e^{\text{Tr}^A} \right| e^{\frac{1}{2} \text{Tr} A^+ A}
\]  \( (5.8) \)
In our case $A = i\delta C$, therefore $\text{Tr} A$ is imaginary, and the norm of the first exponential is one. The second exponent gives

$$\text{Tr} A^+ A = m_4^4 \text{Tr} C^2 = m_4^4 \sum_{i,j \in \Lambda} \frac{1}{W^4 |i-j|^2} e^{-\frac{m_4 |i-j|}{W}} \leq \frac{m_4^4}{m_r^2} \sum_{i \in \Lambda} \frac{1}{W^3} \quad (5.9)$$

The bound in (5.3) then follows.

**Lemma 4** The determinant of $1 + (D + D_0') B$ is bounded by

$$|\det[1 + (D + D_0') B]| \leq K^{\frac{|\Lambda|}{W}} |e^{\text{Tr} (D + D_0') B}| \quad (5.10)$$

**Proof** The proof is obtained by applying (5.8) and repeating the same arguments as in the Lemma above. Note that we applied $\sup_{a,b} |D(a, b) + D_0'(a, b)| = K$ for some constant $K$ independent from $W$.

Applying the Lemmas above we have

$$|\langle G_0^+ \rangle_{\varepsilon=0}| \leq K^{\frac{|\Lambda|}{W}} \int d\mu_C(a, b) \left| e^{\text{Tr} (D + D_0') B} \right| \left| e^{|\sum_{j \in \Lambda} V_j} \right| \quad (5.11)$$

where we bounded $|\exp(V')| = |E^* - a_0|^{-1} \leq K$, and $V'$ in defined in (4.13).

**Partitioning the domain of integration** In order to distinguish small field and large field regions we partition the integration domain by inserting

$$1 = \sum_{k=1}^5 \chi[I^k] \quad (5.12)$$

as follows

$$\int d\mu_C(a, b) \left| e^{\text{Tr} (D + D_0') B} \right| \left| e^{|\sum_{j \in \Lambda} V_j} \right| = \sum_{k=1}^5 T_k$$

$$T_k =: \int d\mu_C(a, b) \left| e^{\text{Tr} (D + D_0') B} \right| \left| e^{|\sum_{j \in \Lambda} V_j} \right| \chi[I^k] \quad (5.13)$$

where $\chi[I^k]$ is the characteristic function of the set $I^k$ and

$$I^1 = \left\{ a, b : |a_j|, |b_j - b_{j'}| \leq \frac{1}{W^*} \forall j, j' \in \Lambda \text{ and } |b_0| \leq \frac{2}{W^*} \right\}$$

$$I^2 = \left\{ a, b : |a_j|, |b_j - b_{j'}| \leq \frac{1}{W^*} \forall j, j' \in \Lambda \text{ and } |b_0 - 2E_i| \leq \frac{2}{W^*} \right\} \quad (5.14)$$
\[ I^3 = \left\{ a, b : b_j \in \mathbb{R} \forall j \text{ and } \exists j \in \Lambda \text{ s.t. } |a_j| > \frac{1}{W^*} \right\} \] (5.15)

\[ I^4 = \left\{ a, b : |a_j| \leq \frac{1}{W^*} \forall j \in \Lambda \text{ and } \exists j, j' \in \Lambda \text{ s.t. } |b_j - b_{j'}| > \frac{1}{W^*} \right\} \]

\[ I^5 = \left\{ a, b : |a_j|, |b_j - b_{j'}| \leq \frac{1}{W^*} \forall j, j' \in \Lambda \text{ and } |b_0|, |b_0 - 2E_i| > \frac{2}{W} \right\} \]

**Small field region**  The first two intervals correspond to the *small field region*. \( T_1 \) is the leading contribution and corresponds to the case when all \( a \) fields and all \( b \) fields are near zero. In this case the interacting terms of the measure do not destroy the log convexity of the Gaussian \( d\mu_C \), therefore we can apply a Brascamp-Lieb inequality \([14][15]\) which states

**Brascamp-Lieb Inequality:**  Let

\[ d\mu_H(x) =: dx_1...dx_N \frac{1}{Z(H)} e^{-\frac{1}{2}H(x)} \] (5.16)

where \( x = (x_1, ..., x_N) \in \mathbb{R}^N \), \( H(x) \) is a positive function symmetric under \( x \rightarrow -x \), and the partition function is

\[ Z(H) =: \int dx_1...dx_N e^{-\frac{1}{2}H(x)} \] (5.17)

Then if \( H'' \geq C^{-1} > 0 \) the following inequalities hold

\[ \int d\mu_H(x) |x_i|^n \leq \int d\mu_C(x) |x_i|^n \quad n > 0 \] (5.18)

\[ \int d\mu_H(x) e^{(f,x)} \leq \int d\mu_C(x) e^{(f,x)} \] (5.19)

where \( d\mu_C(x) \) is the free measure with covariance \( C \), \( f \) is any vector in \( \mathbb{R}^N \), and \( (f, x) = \sum_i f_i x_i \).

The second term corresponds to the case when all the \( a \) fields are near zero and all the \( b \) fields are near the second saddle \( 2E_i \) (see Fig. [1]). In this case we bound the interaction (trace and \( V_j \) factors) by sup norm. The large contributions are now suppressed by a small \( \exp[-W] \) factor, from the trace bound.

**Large field region**  The last three intervals correspond to the *large field region*. In all these cases we bound the interaction terms (the trace and \( V_j \)) by sup norm in terms of quadratic and linear expressions in \( a \) and \( b \). The large contributions from this bound are then compensated by the small probability factor (as the large field region is very unlikely). Note that the \( b \) field bounds are more delicate because of the double well structure (see Fig. [1]).

Below we analyze the integration restricted to each interval.
5.1.1 Small field region: leading contribution $T_1$

We consider the leading contribution $T_1$. In the region $I^1$ all the $a$ fields and the $b$ fields are near 0. We apply

\[
\text{Re} V(a_j) \leq K |a_j|^3 \quad \text{if } |a_j| << 1
\]
\[
\text{Re} V(b_j) \leq K |b_j|^3 \quad \text{if } |b_j| << 1
\]

and we bounded the $\text{Tr}(D + D'_0)$ by applying

\[
|D_j| \lesssim |a_j| + |b_j|
\]
\[
|D'_0| \lesssim 1
\]

Therefore we can write

\[
T_1 \leq \int d\mu C(a, b) e^{K \sum_j (|a_j|^3 + |b_j|^3)} e^{K \sum_j (|a_j| + |b_j|) W^{-2}} \chi[I^1]
\]

Now we insert the cubic and linear contributions in the measure by this definition

\[
H(a) =: a^T C^{-1} a - \left[ K \sum_j \left( |a_j|^3 + \frac{|a_j|}{W^2} \right) \right]
\]
\[
Z(H) =: \int da e^{-\frac{1}{2} H(a)} \chi[I^1]
\]
\[
d\mu_H(a) =: \frac{1}{Z(H)} da e^{-\frac{1}{2} H(a)} \chi[I^1]
\]

The same definitions hold for the $b$ fields. Now we have

\[
\int d\mu_C(a, b) e^{K \sum_j (|a_j|^3 + |b_j|^3)} e^{K \sum_j (|a_j| + |b_j|) W^{-2}} \chi[I^1] = \left( \frac{Z(H)}{Z_0} \right)^2
\]

where the free partition function is

\[
Z_0 = \int da e^{-\frac{1}{2} a^T C^{-1} a}.
\]

Therefore we actually have to estimate the normalization factor of the interacting measure This is done through Lemma 5 below.

**Lemma 5** With $Z(H)$ and $Z_0$ defined by (5.23) and (5.26) we have

\[
Z(H) \leq e^{O(\frac{W}{W^4})} Z_0
\]
Proof  Let $H(t)$ defined as

$$H(t)(a) := a^T C^{-1} a - \left[ K \sum_j t \left( |a_j|^3 + \frac{|a_j|}{W^2} \right) \right] \tag{5.28}$$

interpolate between $H$ and $C^{-1}$. Note that on $I_1$

$$\ln \left[ \frac{Z(H(1))}{Z(H(0))} \right] = \int_0^1 dt \frac{d}{dt} \ln Z(H(t))$$

$$= \sum_j \int_0^1 dt \int d\mu_{H(t)}(a) \left[ |a_j|^3 + \frac{|a_j|}{W^2} \right]$$

$$\leq \sum_j \int_0^1 dt \int d\mu_{C_j}(a) \left[ |a_j|^3 + \frac{|a_j|}{W^2} \right] \leq K \frac{|A|}{W^3} \tag{5.29}$$

where we defined $d\mu_{H(t)}(a)$ as in (5.24). In the last line we used Brascamp-Lieb (5.18) together with

$$H'' \geq C^{-1} - f m_i^2 = C_f^{-1} > 0 \tag{5.30}$$

which is valid on $I_1$ and for $f = O \left( W^{-\frac{1}{3}} \right)$. In general we will use this definition of $C_f$ for $f$ a constant $0 < f < 1$ or, when $\Lambda$ is a set of cubes, $f$ a diagonal matrix constant on each cube. Now

$$Z(H(0)) = \left[ \int d\mu_C \chi[I^1] \right] \leq Z_0 \tag{5.31}$$

This ends the proof.

Applying Lemma 5 we have

$$T_1 \leq e^{O\left(\frac{|A|}{W^2}\right)} \leq K \tag{5.32}$$

5.1.2 Small field region: contribution from the second saddle

In this section we show that $T_2 \leq e^{-cW}$. This means that the fields have actually the same behavior as in a large field region. Note that this property holds only in three dimensions.

In the interval $I^2$ all the $a$ fields are near $0$ and the $b$ fields are near the second saddle $2\mathcal{E}_i$. Recall that $\mathcal{E}_i = \sqrt{1 - E^2/4}$. Note that for all $|a_j| \leq W^{-\frac{1}{6}}$ and $|b_j - 2\mathcal{E}_i| \leq W^{-\frac{1}{6}}$ we have

$$\text{Re} \, V(a_j) \leq \frac{m^2}{2} f_a a_j^2 \tag{5.33}$$

$$\text{Re} \, V(b_j) \leq \frac{m^2}{2} f_b b_j^2 + (1 - f_b) m_i^2 2\mathcal{E}_i (b_j - \mathcal{E}_i) \tag{5.34}$$
with \( f_a = f_b = O(W^{-\frac{1}{2}}) \). Note that for \( b \) there is a linear contribution coming from the translation to the second saddle. Moreover can be bounded \( \text{Tr}(D + D'_0)B \) applying Lemma 6 below.

**Lemma 6** If \(|a_j| \leq W^{-\frac{1}{2}}\) and \(|b_j - 2\xi_i| \leq W^{-\frac{1}{2}}\), then the real part of \([(D + D'_0)B]_{jj}\) is bounded by

\[
\text{Re} \left( D + D'_0 \right)_{jj} B_{jj} \leq -c W^{-2}
\]

where \( c > 0 \) is some constant independent from \( W \).

**Proof** Note that

\[
\text{Re} \left( DB \right)_{jj} = \left[ \text{Re} D_j \text{ Re} B_{jj} - \text{Im} D_j \text{ Im} B_{jj} \right]
\]

The key point is that, for \( a_j \) near zero and \( b_j \) near to the second saddle we have

\[
\text{Re} D_j = -m_r^2 + O \left( \frac{1}{W^{\frac{2}{3}}} \right)
\]

\[
\text{Im} D_j = -m_i^2 + O \left( \frac{1}{W^{\frac{3}{3}}} \right)
\]

Note that this estimates are not true in other regions. If both \( a_j \) and \( b_j \) are near zero \( D_j \approx 0 \) while for \( a_j \) or \( b_j \) far from the saddle we can only say that \(|D_j| \leq \text{const.}\). For \( D'_0 \) we only need to know that for any \( a_0, b_0 \in \mathbb{R} \)

\[
D'_0 = O(1)
\]

Now, by simple Fourier space analysis we see that

\[
\text{Re} B_{jj} \geq c \frac{1}{W^2}
\]

\[
\text{Im} B_{jj} = O \left( \frac{1}{W^3} \right)
\]

Inserting these estimates in (5.36) the proof follows. \( \blacksquare \)

Inserting all this results in \( T_2 \) we have

\[
T_2 \leq e^{-c \frac{1}{W^2}} \int d\mu_C(a, b) e^{\frac{a^2}{2} \sum_j f_a \sum_j a_j^2 + f_0 \sum_j b_j^2} e^{(1-f_0)m_r^2 2\xi_i \sum_j (b_j - \xi_i)}
\]

We insert the quadratic terms in the measure. The normalization ratio are bounded using Lemma 7 below

**Lemma 7** For any \( 0 < f < 1 \) we have

\[
\det \left( \frac{C^{-1}}{C_f^{-1}} \right) \lesssim \frac{1}{1 - f} e^{O \left( \frac{1}{W^3} \right)}
\]

where \( C_f \) is defined in (5.37).
**Proof** Diagonalizing the matrices we can write

\[
\det \left( \frac{C^{-1}}{C_f} \right) = \prod_k \frac{2 \sum_{i=1}^3 (1 - \cos k_i) W^2 + m_i^2}{2 \sum_{i=1}^3 (1 - \cos k_i) W^2 + m_i^2 (1 - f)} 
\leq \frac{1}{1 - f} \left| e^{-f m_0^2 T_C C_f^0} \right| \leq \frac{1}{1 - f} e^{O \left( \frac{\Lambda}{W^2} \right)}
\]

where we defined \( C_f^0 \) as the covariance \( C_f \) where the zero mode has been extracted. This ends the proof.

Therefore we can write

\[
T_2 \leq \frac{1}{(1 - f_a) (1 - f_b)} e^{O \left( \frac{\Lambda(W - \frac{1}{2})}{W^2} \right)} e^{-c \frac{\Lambda}{W^2}} \int d\mu_{C_f} (b) e^{(1 - f_b) m_0^2 2 \varepsilon, \Sigma_j (b_j - \varepsilon_i)} 
\leq e^{O(W W^{-\frac{1}{2}})} e^{-c W} \lesssim e^{-c W}
\]

where we inserted \( |\Lambda| = W^{3} \) and \( f_a = f_b = O(W^{-\frac{1}{4}}) \) and we applied

\[
\int d\mu_{C_f} (b) e^{(1 - f_a) m_0^2 2 \varepsilon, \Sigma_j (b_j - \varepsilon_i)} = 1.
\]

**5.1.3 Large field region**

This is the region selected by the intervals \( I^3 \) (one \( a \) fields large) \( I^4 \) (one pair of \( b \) fields with \( |b_j - b_{j'}| \) large) and \( I^5 \) (all \( b \) fields far from both saddles). We apply the following inequalities

\[
\text{Re} (D + D_0')_{ii} \leq \sup_{a, b} |(D + D_0')_{ii}| |B_{ii}| \lesssim O \left( \frac{1}{W^2} \right)
\]

\[
\text{Re} V (a_j) \leq \frac{m_j^2}{2} f_a a_j^2
\]

\[
\text{Re} V (b_j) \leq \frac{m_j^2}{2} f_b b_j^2 + O (1 - f_b)
\]

with \( 1/2 < f_a < 1, f_b = 1 - W^{-3} \). These estimates are true for any value of \( a_j \) and \( b_j \in \mathbb{R} \). On the other hand, when we are in the interval \( I^5 \), all \( b \) fields must be far from both saddles the interaction in exponentially small, therefore we gain an additional small factor:

\[
\text{Re} V (b_j)_{I^5} \leq \frac{m_j^2}{2} f_b b_j^2 + O (1 - f_b) - c \left( \frac{1}{W^{\frac{3}{2}}} \right)^2
\]

Note that the factor \( O (1 - f_b) \) comes from the contribution of the second saddle (see Fig. [3]). Therefore we can write

\[
T_3 + T_4 + T_5 \leq e^{O \left( \frac{\Lambda}{W^2} + (1 - f_b) |\Lambda| \right)} \cdot \int d\mu_C (a, b) e^{\frac{m_j^2}{2} [f_a \Sigma_j a_j^2 + f_b \Sigma_j b_j^2]} \left\{ \chi [I^3] + \chi [I^4] + e^{-c W^{3} W^{-\frac{1}{2}}} \right\}
\]

19
We insert the quadratic terms in the measure:
\[ T_3 + T_4 + T_5 \leq \frac{1}{(1 - f_a)(1 - f_b)} e^{O\left( \frac{(1 + f_a)(1 + f_b)}{W^2} \right)} e^{O\left( \frac{1 - f_a}{W^3} + (1 - f_b)|A| \right)} , \]
\[ \cdot \int d\mu_{C_{f_a}}(a) d\mu_{C_{f_b}}(b) \left\{ \chi[I^3] + \chi[I^4] + e^{-cW^3W^{-\frac{3}{4}}} \right\} \]  
(5.51)
where we defined \( C_{f_a} \) and \( C_{f_b} \) as in (5.30) and we applied Lemma 7.

To bound the contributions from \( I^3 \) and \( I^4 \) we apply the following Lemma.

**Lemma 8** The probability of having one \( |a_j| > W^{-\frac{3}{4}} \) or one pair \( |b_j - b_{j'}| > W^{-\frac{3}{4}} \) is exponentially small
\[ \int d\mu_{C_{f_a}}(a) \chi[I^3] \lesssim W^3 e^{-cW^2W^{-\frac{3}{4}}} \]  
(5.52)
\[ \int d\mu_{C_{f_b}}(b) \chi[I^4] \lesssim W^6 e^{-cW^2W^{-\frac{3}{4}}} \]  
(5.53)

**Proof** We consider first the integral for \( a \)
\[ \int d\mu_{C_{f_a}}(a) \chi[I^3] \leq \sum_j \int d\mu_{C_{f_a}}(a) \chi \left( |a_j| > W^{-\frac{3}{4}} \right) \]
\[ \leq \sum_j \int d\mu_{C_{f_a}}(a) \frac{(e^{-xa_j} + e^{xa_j})}{1 + e^{-2xW^{-\frac{3}{4}}}} e^{-xW^{-\frac{3}{4}}} \]
\[ \leq \sum_j 2e^{\frac{1}{2}x^2(C_{f_a})_{jj}} e^{-xW^{-\frac{3}{4}}} \lesssim W^3 e^{-cW^2W^{-\frac{3}{4}}} \]  
(5.54)
where we applied \((C_{f_a})_{jj} = O(1/W^2)\) and we set \( x = O(W^{-\frac{3}{4}}W^2)\). The same proof holds for the \( b \) field. In this case the presence of a difference \( b_j - b_{j'} \) is crucial to ensure that the factor \( [(C_{f_b})_{jj} + (C_{f_b})_{jj'} - 2(C_{f_b})_{jj'}] \) is of order \( W^{-2} \) and does not depend on the mass (which could be very tiny for \( C_{f_b} \)). The factor \( W^6 \) comes from the sum over \( j \) and \( j' \).

Putting together all the factors we have
\[ T_3 + T_4 + T_5 \leq \frac{1}{(1 - f_a)(1 - f_b)} e^{O\left( \frac{(1 + f_a)(1 + f_b)}{W^2} \right)} e^{O\left( \frac{1 - f_a}{W^3} + (1 - f_b)|A| \right)} , \]
\[ \cdot \left\{ (W^3 + W^6)e^{-cW^2W^{-\frac{3}{4}}} + e^{-cW^3W^{-\frac{3}{4}}} \right\} \lesssim e^{-cW^3W^{-\frac{3}{4}}} \]  
(5.55)
where we have inserted \(|A| = O(W^3)\), \( f_a = 3/4 \) and \( f_b = 1 - (1/W^3) \). Note that there is an additional factor \( W^3 \) from the zero mode \((1 - f_b)^{-1}\) of the determinant.
5.1.4 Sum over the different regions

Summing the bounds on different intervals we have finally

\[ |\langle G_{00}^+ \rangle_{\varepsilon=0}| \lesssim \left[ 1 + e^{-\varepsilon W} + e^{-e W^{3/4}} \right] \lesssim K \]  \hspace{1cm} (5.56)

This completes the proof of (5.1).

5.1.5 Large volume

It is straightforward to extend the above estimates to the case when \( \Lambda \) is a union of cubes.

**Corollary 1** The density of states in a union of cubes \( \Lambda \) is bounded by

\[ \| \bar{\rho}_\Lambda(E) \| \lesssim e^{O\left( \frac{\Lambda}{W^2} \right)} \]  \hspace{1cm} (5.57)

**Proof** In each cube we apply the bounds above. The result is written as a quadratic form \( \exp[v^T C_f v] \) where \( v \) is a vector which depends on the bounds on each particular cube and \( f \) is now a diagonal matrix which is constant on each cube. The key point is that

\[ C_f \leq \frac{1}{-W^2 \Delta_N + (1 - f)m^2_r} \]  \hspace{1cm} (5.58)

where \( \Delta_N \) is a Laplacian with Neumann boundary conditions on the cubes, and decouples the cubes automatically. Now we can perform the estimates in each cube separately. This completes the proof of (5.57).

5.2 Proof of Theorem 3

To prove this result we integrate by parts to generate perturbative terms. To control the remainder we apply the bounds of Theorem 2.

5.2.1 Semicircle law

We prove that \( \bar{\rho}(E) = \rho_{SC} \) with a precision of order \( W^{-2} \):

\[ \bar{\rho}_\Lambda(E) = \rho_{SC} + O\left( \frac{1}{W^2} \right). \]  \hspace{1cm} (5.59)

Note that \( \bar{\rho}_\Lambda(E) = -\frac{1}{\pi} \text{Im}\langle G_{00}^+ \rangle \) therefore we have to study

\[ \langle G_{00}^+ \rangle = \int d\mu_B(a,b) \frac{1}{(E^* - a_0)} e^{V \det[1 + (D + D_0^*)B]} \]  \hspace{1cm} (5.60)
We have to perform a few steps of perturbative expansion on the observable $(E^* - a_0)^{-1}$ and $D'_0$. These are more clear if we write the determinant as a fermionic integral.

\[
\det[1 + (D + D'_0)B] = \int d\mu_B(\rho^*, \rho) \ e^{-\rho^* \rho D} (1 - D'_0\rho_0^*\rho_0) \tag{5.61}
\]

where we defined

\[
d\mu_B(\rho^*, \rho) = \det B \ e^{-\rho^* B^{-1} \rho} \tag{5.62}
\]

\[
\rho^* \rho D = \sum_j \rho^*_j \rho_j D_j \tag{5.63}
\]

and $D_j$ and $D'_0$ are introduced in (4.12) and (4.14). The density of states is then written as

\[
\langle \mathcal{O}_{00} \rangle = \int d\mu_B(a, b, \rho^*, \rho) \ e^{V - \rho^* \rho D} \mathcal{O}_0 \tag{5.64}
\]

where we defined $d\mu_B(a, b, \rho^*, \rho) = d\mu_B(a, b)d\mu_B(\rho^*, \rho)$ and the observable $\mathcal{O}_0$ is

\[
\mathcal{O}_0 = \frac{1}{(E^* - a_0)} (1 - D'_0\rho_0^*\rho_0) \tag{5.65}
\]

\[
= E + a_0 \int_0^1 dt \frac{1}{(E^* - ta_0)^2} - \frac{1}{(E^* - a_0)} D'_0\rho_0^*\rho_0
\]

The first term is a constant and gives the semicircle law $-\frac{1}{\pi} \text{Im} E = \rho_{SC}$. Note that we apply

\[
\int d\mu_B(a, b, \rho^*, \rho) \ e^{V - \rho^* \rho D} = 1 \tag{5.66}
\]

The remaining two terms give the corrections

\[
\delta \rho_1 = \int d\mu_B(a, b, \rho^*, \rho) \ e^{V - \rho^* \rho D} \left[ a_0 \int_0^1 dt \frac{1}{(E^* - ta_0)^2} \right] \tag{5.67}
\]

\[
\delta \rho_2 = \int d\mu_B(a, b, \rho^*, \rho) \ e^{V - \rho^* \rho D} \left[ -D'_0\rho_0^*\rho_0 \right] \tag{5.68}
\]

**Estimate of $\delta \rho_2$** We first consider the estimate on the second integral, as it is the easiest one. We partition the integral domain inserting (5.12) as in Sec. 5.1 and we perform the fermionic integral in a different way depending on the region.

Near the first saddle (interval $I^1$) we apply

\[
\int d\mu_B(\rho^*, \rho) \ e^{-\rho^* \rho D} \rho_0^*\rho_0 = \left( \frac{1}{B^{-1} + D} \right)_{00} \det[1 + DB] \tag{5.69}
\]
It is easy to see that for \( a_j \) and \( b_j \) near zero \( D_j \approx 0 \) and \( |(B^{-1} + D)^{-1}_{00}| = O(W^{-2}) \). Therefore we have

\[
|\delta \rho_2(1^1)| \lesssim \frac{1}{W^2} \int d\mu_B(a, b) e^V \det(1 + DB) \left[ \frac{-D'_0}{(\xi^* - a_0)} \right] \lesssim \frac{1}{W^2}
\]  

(5.70)

where we applied the same bounds as in Sec. 5.1.1.

In the other regions \( I^k, k \neq 0 \) we cannot apply (5.69) as \((B^{-1} + D)^{-1}\) is not well defined (\(D\) is big and may cancel \(B^{-1}\)). Therefore we apply

\[
\int d\mu_B(\rho^*, \rho) e^{-\rho^* \varphi D} \rho_0^0 = \det M
\]

(5.71)

where \(M\) is the matrix \(1 + DB\) with the row 0 substituted with \(B\):

\[
\begin{align*}
M_{ij} &= (1 + DB)_{ij} \quad i, j \neq 0 \\
M_{i0} &= D_i B_{00} \quad i \neq 0 \\
M_{0j} &= B_{0j} \quad \forall j
\end{align*}
\]

(5.72)

If we apply (5.8) we obtain the same bounds as in Sec. 5.1. Therefore performing the same bounds as in Sec. 5.1.2-5.1.3, we have

\[
\sum_{k \neq 1} |\delta \rho_2(I^k)| \lesssim e^{-c W}
\]

(5.73)

Hence

\[
|\delta \rho_2| \lesssim \frac{1}{W^2} + e^{-c W} \lesssim \frac{1}{W^2}
\]

(5.74)

**Estimate of \( \delta \rho_1 \)**

Now we consider the first error term. Before inserting the partition (5.12) and integrating over the fermionic integrals we have to perform one step of integration by parts

\[
\delta \rho_1 = \sum_k B_{0k} \int d\mu_B(a, b, \rho^*, \rho) e^{V - \rho^* \varphi D} \]

\[
\times \left[ \frac{\delta}{\delta a_k} (V_k - \rho_k^* \rho_k D_k) + \delta_{0k} \frac{\delta}{\delta a_0} \int_0^1 dt \frac{1}{(\xi^* - ta_0)^2} \right]
\]

(5.75)

Note that \(\frac{\delta}{\delta a_k} V_k \lesssim |a_k|^2 + |a_k|^3\). In the region around the first saddle \((I^1)\), applying the Brascamp-Lieb inequality (5.18), these fields give a factor \(W^{-2}\). In the other regions they are bounded by the exponential mass decay. The contribution from \(\rho_k^* \rho_k\) is estimated as in \(\delta \rho_2\) above. Therefore

\[
|\delta \rho_1| \lesssim \left( \frac{1}{W^2} \sum_k |B_{0k}| \right) + |B_{00}| \lesssim \frac{1}{W^2}
\]

(5.76)

where we applied \(\sum_k |B_{0k}| \leq const\) and \(|B_{00}| = O(W^{-2})\). This ends the proof of (5.59).
5.2.2 Smoothness

Now we consider the derivatives. Note that it is easier to compute the derivatives on the starting expression \( \langle G_{(00)}^+ \rangle \) than directly on the functional integral \([5.60]\).

The derivative at order \( n \) is given by

\[
\partial^n \overline{\rho}_A(E) = -n!(1)^n \frac{1}{\pi} \text{Im} \langle (G^n)_{00} \rangle \propto \text{Im} \sum_{j_1, \ldots, j_n} \langle G_{0j_1} \cdots G_{j_n0} \rangle \tag{5.77}
\]

Applying the supersymmetric approach and the saddle point analysis as in Sec. 3-4, we can write for instance \( R(x) \) as

\[
R(x) = \langle G_{0x} G_{x0} \rangle = \langle O_0 O_x \rangle_{\text{SUSY}} - \langle O_0 \rangle_{\text{SUSY}} \langle O_x \rangle_{\text{SUSY}} \tag{5.78}\]

where we defined

\[
\langle F(a, b, \rho^*, \rho) \rangle_{\text{SUSY}} = \int d\mu \chi[I] \prod_{k=1}^p \rho_{i_k}^* \rho_{j_k} \tag{5.79}\]

and the observables are \([5.65]\) and

\[
O_x = \frac{1}{(E_0^* - a_x)} (1 - D_x^* \rho_x \rho_x) . \tag{5.80}\]

A similar formula holds for the general case.

We perform now integration by parts starting from \( O_0 \) until we have a path of connected vertices that connects 0 to \( j \) or we have enough vertices to extract a factor \( W^{-3} \) for each observable \( O_j \). This factor ensures that we can sum over the position of \( j \) inside the cube \( \Lambda \).

Note that, as in general we will have to estimate products of fields, both fermionic and bosonic, we will need the two Lemmas below.

**Lemma 9** Let consider the average of the product of \( p \) fermionic fields

\[
\int d\mu B(\rho^*, \rho) e^{-\rho^* \rho D} \prod_{k=1}^p \rho_{i_k}^* \rho_{j_k} \tag{5.81}\]

Note that \( i_k \) and \( j_k \) are not necessarily equal. This integral gives different estimates depending on the region we are considering. If \( a \) and \( b \) are near zero we have

\[
\left| \int d\mu B(\rho^*, \rho) e^{-\rho^* \rho D} \chi[I^1] \prod_{k=1}^p \rho_{i_k}^* \rho_{j_k} \right| \leq \frac{\rho^2}{W_{2p}} |\det(1 + DB)| \chi[I^1] \tag{5.82}\]

On the other hand, in the other regions \( I^s, s \neq 1 \) we have

\[
\int d\mu B(\rho^*, \rho) e^{-\rho^* \rho D} \chi[I^s] \prod_{k=1}^p \rho_{i_k}^* \rho_{j_k} = \sigma [\det J_1 M] \chi[I^s] \tag{5.83}\]
where $M$ is the matrix $1 + DB$ where the rows $i_1,...i_p$ are substituted by the corresponding rows of $B$, and the columns $j_1,...j_p$ are substituted by the corresponding columns of $DB$:

$$M_{ij} = (1 + DB)_{ij} \quad i \neq i_1,...i_p, j \neq j_1,...j_p$$

$$M_{ijk} = D_i B_{ijk} \quad i \neq i_1,...i_p$$

$$M_{ikj} = B_{ikj} \quad \forall j$$

(5.84)

Finally $I$ are the set of indices $I = \{i_1, ..., i_p\}$ $J = \{j_1, ..., j_p\}$ and $\det_{IJ} M$ is the determinant of the matrix $M$ without the rows $j_1, ..., j_p$ and the columns $i_1, ..., i_p$, and $\sigma$ is a sign. This new determinant can be bounded as usual.

$$|\det_{IJ} M| \lesssim e^{O(\max_i |\Lambda| W^3)}$$

(5.85)

Proof To obtain the first bound (5.82) we apply (5.69) for a product of fermionic fields. The result is the determinant of a $p \times p$ matrix whose elements are $(B^{-1} + D)_{ij}^{-1}$ with $i = i_1, ..., i_p$ and $j = j_1, ..., j_p$. This determinant is easily bounded by $p! \sup |(B^{-1} + D)_{ij}^{-1}|$. Applying $|(B^{-1} + D)_{ij}^{-1}| = O(W^{-2})$ we obtain the result.

The second expression (5.83) is easily obtained using the anticommuting properties of the fermionic fields. Finally (5.85) holds because the only error terms come from the absence of a term 1 in $p$ diagonal elements.

Lemma 10 We consider the integral

$$I_a(n_1, ..., n_p) =: \int d\mu(a, b) \ |e^V| \prod_{k=1}^p |a_{jk}|^{n_k} e^{\text{Re} Tr DB}$$

(5.86)

where $p > 0$, $n_k > 0$ for all $k$ and $n = \sum_k n_k$. Then

$$I_a(n_1, ..., n_p)[I^1] \lesssim n! \left( \frac{K}{W} \right)^n$$

(5.87)

$$I_a(n_1, ..., n_p)[I^2] \lesssim K^n e^{-cW}$$

(5.88)

$$I_a(n_1, ..., n_p)[I^q] \lesssim K^n \prod_k \sqrt{n_k!} e^{-cW^q} \quad q > 2$$

(5.89)

If instead of $a$ fields we have $b$ fields the result is the same, but in the large field region we pay a larger factor, because we have a very small mass remaining in the covariance

$$I_b(n_1, ..., n_p)[I^q] \lesssim K^n W^q n \prod_k \sqrt{n_k!} e^{-cW^q} \quad q > 2$$

(5.90)
Proof As in Sec. 5 we partition the integration domain \(1 = \sum_{q=1}^{6} \chi[I^q]\).
When we are near the first saddle \((I_1)\) we write
\[
|a_{j_1}|^{n_1} \ldots |a_{j_p}|^{n_p} \leq \frac{1}{n} \sum_{k=1}^{p} n_k |a_{j_k}|^{n_k}
\]
Now we can apply Brascamp-Lieb inequality as stated in (5.18).
In the region near the second saddle \((I_2)\) we can bound the field \(a\) by a constant.
In the large field region we bound the fields \(a\) using a fraction of the exponential decay of the mass term
\[
|a_j|^n \leq \left( \frac{K}{\sqrt{\delta}} \right)^n \sqrt{n!} e^{\frac{1}{2}\delta m^2 a_j^2}
\]
where \(\delta > 0\) is a small constant \(\delta < 1\) which must be smaller than the mass of \(C_{f_a}\). Note that, for the \(b\) fields in the region \(I_3\) or \(I_4\) \(\delta\) must be of order \(\delta = O(W^{-3})\) as this is the mass of \(C_{f_b}\). This completes the proof.

6 Infinite volume limit

In this section we shall establish bounds on \(\langle G_{00}^+ \rangle\) and the exponential decay of \(R(x)\) uniformly as \(\Lambda \uparrow \mathbb{Z}^3\). This is done by a standard method (see [16] or [17], ch.III.1) in statistical mechanics called the cluster expansion. These expansions are possible when there is a single dominant saddle point (in our case \(a = b = 0\)) whose fluctuations are close to that of a massive Gaussian i.e. a Gaussian whose covariance \(B\) has exponential decay. We are going to use a standard expansion with a few modifications. By supersymmetry some terms of the expansion are one (see Lemma 11) thus simplifying the expression. On the other hand for technical reasons the treatment of the covariance in the measure is slightly different form the usual one.

We prove the following theorem

**Theorem 4** There exists \(W_0\) such that for all \(W > W_0\) \(\lim_{\Lambda \uparrow \mathbb{Z}^3} \tilde{\rho}_\Lambda(E)\) is bounded in \(\mathcal{I}\) uniformly in \(W\)

\[
\left| \lim_{\Lambda \uparrow \mathbb{Z}^3} \tilde{\rho}_\Lambda(E) \right| \leq K
\]
for some constant independent from \(W\). Moreover

\[
\left| \lim_{\Lambda \uparrow \mathbb{Z}^3} \partial_E \tilde{\rho}_\Lambda(E) \right| \leq C_n \quad \forall n < n_0(W)
\]

\[
\lim_{\Lambda \uparrow \mathbb{Z}^3} \tilde{\rho}_\Lambda(E) = \rho_{SC}(E) + O(W^{-2})
\]
uniformly in \(\Lambda\) and \(W\). In particular, for \(x \neq 0\)

\[
\left| \lim_{\Lambda \uparrow \mathbb{Z}^3} R(x) \right| \leq \frac{K}{W^3} e^{-c \frac{|x|}{W}}
\]

26
Outline of the proof: Note that the exponential decay (4.20) of $B$ means that regions at a distance higher than $W$ are approximately decoupled. As the observable depends only on the fields at zero or at zero and $x$ we expect that all interactions take place in a volume of order $W^3$ around $i = 0$. To exploit this fact we partition $\Lambda$ in cubes of side $W$ forming the lattice $L$. For that we introduce the function

$$\chi_\Delta(i) = \begin{cases} 1 & \text{if } i \in \Delta \\ 0 & \text{otherwise} \end{cases} \tag{6.5}$$

satisfying $\sum_{\Delta \in L} \chi_\Delta(i) = \chi_\Lambda(i)$. In the following we call root cube the cube containing $i = 0$ and we denote it by $\Delta_0$.

The cluster expansion expresses $\langle G_{00}^+ \rangle_{\epsilon=0}$ and $R(x)$ as a sum of finite volume contributions. Let $Y$ be a union of cubes $\Delta$ containing zero (zero and $x$ when we estimate $R(x)$). Then the cluster expansion gives

$$\langle G_{00}^+ \rangle_{\epsilon=0} = \sum_{Y \ni \Delta_0} c_Y Z_Y' \tag{6.6}$$

where $Z_Y'$ is a functional integral over fields $a_j$, $b_j$, $\rho^*_j$, $\rho_j$ present. We shall show, using the ideas of Sec.5.1.5, that $Z_Y' \leq (KW^\frac{3}{2})|Y|$. Lemma 11 If we restrict to the set of cubes $Y^c = \Lambda \setminus Y$ and there is no observable contribution we have

$$Z_{Y^c} = \int d\mu_{B_{Y^c}}(a,b,\rho^*,\rho) e^{\sum_{j \in \Lambda'} (V_j - \rho^*_j \rho_j D_j)} = 1 \tag{6.7}$$

where $d\mu_{B_{Y^c}}(a,b,\rho^*,\rho)$ is defined after (5.64) and $B_{Y^c}$ is the covariance $B$ restricted to the volume $Y^c$.

Proof We perform the translation $a_j \to a_j - \mathcal{E}$, $b_j \to b_j + i \mathcal{E}$ for all $j \in \Lambda'$. Note that, for a general $\Lambda' \subset \Lambda$, $B_{\Lambda'}^{-1} \neq -W^2 \Delta + (1 - \mathcal{E}^2)$. Therefore the translation gives some linear and constant terms. The constant terms are cancelled when we add the contributions from the $a$ and $b$ fields. By performing the inverse Hubbard-Stratonovich transformation we obtain

$$R_{\Lambda'} = \left\langle d\Phi^* d\Phi^e e^{i\Phi^e (E_e + A - H) \Phi} \right\rangle_1 = 1 \tag{6.8}$$

27
where the average \( \langle \cdot \rangle_1 \) is computed with the probability distribution (2.1) with covariance \( \bar{J} \) instead of \( J \), with \( \bar{J}_{ij} = B^{-1} + E^2 \). The matrix \( A \) is a diagonal matrix

\[
\Phi^+ A \Phi = \sum_i \Phi^+_i A_i, \quad A_i = E \sum_k (\bar{J}_{ik} - \delta_{ik}) \tag{6.9}
\]

This completes the proof. 

6.1 Cluster expansion

We derive the cluster expansion formula. The result is stated in Lemma 12 below. We construct the expansion by an inductive argument.

First we want to test if there is any connection between the root cube \( \Delta_0 \) and some other \( \Delta \in \Lambda \). For that purpose we introduce an interpolated covariance \( B(s_1) \) with \( 0 \leq s_1 \leq 1 \), which satisfies \( B(1) = B \) while \( B(0) \) decouples the root cube \( \Delta_0 \) from the rest of the volume. The easiest choice for \( B(s_1) \) is

\[
B(s_1)_{ij} = s B_{ij} \quad \text{for } i \in \Delta_0 \text{ and } j \notin \Lambda \setminus \Delta_0, \quad \text{or vice versa, and } B(s_1)_{ij} = B_{ij} \text{ otherwise.}
\]

For technical reasons we choose the following (less natural) interpolation rule

\[
B(s_1)^{-1} = C(s_1)^{-1} + i m_i^2 \tag{6.10}
\]

where

\[
C(s_1)_{ij} = \begin{cases} 
    s_1 C_{ij} & \text{if } i \in \Delta_0, j \in \Delta \neq \Delta_0, \text{ or vice versa} \\
    C_{ij} & \text{otherwise} 
\end{cases} \tag{6.11}
\]

The reason we use this definition of \( B(s_1) \) is that we do not want to mix the real and imaginary part in \( B^{-1} \) in order to apply later the same estimates of Sec. 5. Note that (6.11) is equivalent to the definition

\[
C(s_1) = s_1 C + (1 - s_1) \left[ C_{\Delta_0 \Delta_0} + C_{\Delta_0 \Delta_0^c} \right] \tag{6.12}
\]

\[
(C_{\Delta \Delta'})_{ij} = \frac{1}{2} \left[ \chi_\Delta(i) C_{ij} \chi_\Delta'(j) + \chi_\Delta'(i) C_{ij} \chi_\Delta(j) \right] \tag{6.13}
\]

where \( \Delta_0^c = \Lambda \setminus \Delta_0 \). Therefore \( C(s) \) is still a positive operator, as it is a convex combination of the positive operators \( C \) and \( C_{\Delta \Delta'} \). This fact is essential to ensure the convergence of the integrals. With the interpolated covariance we define

\[
F_{A}[s_1] = \int d\mu_{B(s_1)}(a, b, \rho, \rho^* \rho) e^{\sum_{j \in V} (V_j - \rho_j \rho_j D_j)} \mathcal{O}_0. \tag{6.14}
\]

Note that for \( s_1 = 1 \) \( F_{A}[s_1]_{s_1=1} = \langle G^{+00}_{00} \rangle_{\varepsilon=0} \). Now we apply a first order Taylor formula to \( F_{A}[s_1] \)

\[
F_{A}[s_1]_{s_1=1} = F_{A}[s_1]_{s_1=0} + \int_0^1 ds_1 \partial_{s_1} F_{A}[s_1]. \tag{6.15}
\]
The first term \( F_\Lambda[s_1, s_1 = 0] = F_{\Delta_0} \) corresponds to decoupling \( \Delta_0 \) from the rest of the volume. The derivative in the second term of (6.15) gives

\[
\partial_s F_\Lambda[s_1] = \int \partial_s d\mu_{B(s_1)}(a, b, \rho^*, \rho) e^{\sum_j (V_j - \rho^*_j \rho_j D_j)} \mathcal{O}_0 .
\]  

(6.16)

Using integration by parts we have

\[
\int \partial_s d\mu_{B(s_1)}(a, b, \rho^*, \rho) = \int d\mu_{B(s_1)}(a, b, \rho^*, \rho) \sum_{ij} \partial_s B(s_1)_{ij}
\]

\[
\times \left[ \frac{\delta}{\delta a_i} \frac{\delta}{\delta a_j} + \frac{\delta}{\delta b_i} \frac{\delta}{\delta b_j} + \frac{\delta}{\delta \rho^*_j} \frac{\delta}{\delta \rho^*_i} \right].
\]

(6.17)

(6.18)

The derivative \( \delta/\delta a_i \) may apply to \( \exp[V_i - \rho^*_i \rho_i D_i] \) or to the observable \( \mathcal{O}_0 \) (this only for \( i = 0 \)):

\[
\frac{\delta}{\delta a_i} e^{\sum_j \rho^*_j \rho_j D_j} = \frac{\delta}{\delta a_i} \left( -\rho^*_i \rho_i \right) e^{\sum_j \rho^*_j \rho_j D_j}
\]

\[
\mathcal{O}_0 = (-\rho^*_0 \rho_0) \frac{\delta}{\delta a_0} D'_0.
\]

(6.19)

(6.20)

(6.21)

The same definitions hold for \( \delta/\delta b_i \). The fermionic derivative \( \delta/\delta \rho_i \) may apply to \( \exp[-\rho^*_i \rho_i D_i] \) or to the observable \( \mathcal{O}_0 \) (this only for \( i = 0 \)):

\[
\frac{\delta}{\delta \rho_i} e^{\sum_j \rho^*_j \rho_j D_j} = D_i \rho^*_i e^{\sum_j \rho^*_j \rho_j D_j}
\]

\[
\frac{\delta}{\delta \rho_0} \mathcal{O}_0 = \rho^*_0 D'_0.
\]

(6.22)

(6.23)

Similar formulas hold for \( \delta/\delta \rho^*_i \). Therefore

\[
\partial_s F_\Lambda[s_1] = \sum_{i_1, j_1} \left[ \partial_s B(s_1)_{i_1, j_1} \right] F_\Lambda[s_1](i_1, j_1))
\]

\[
(6.24)
\]

where

\[
F_\Lambda[s_1](i_1, j_1)) = \int d\mu_{B(s_1)}(a, b, \rho^*, \rho) \left[ \frac{\delta}{\delta a_{i_1}} \frac{\delta}{\delta a_{j_1}} + \frac{\delta}{\delta b_{i_1}} \frac{\delta}{\delta b_{j_1}} + \frac{\delta}{\delta \rho^*_{j_1}} \frac{\delta}{\delta \rho^*_{i_1}} \right]
\]

\[
\times \left[ e^{\sum_j (V_j - \rho^*_j \rho_j D_j)} \mathcal{O}_0 \right].
\]

(6.25)

Let us consider the propagator \( \partial_s B(s_1)_{i_1, j_1} \) extracted by the Taylor formula. If we choose the easiest interpolating rule, that is \( B(s)_{ij} = s B_{ij} \) when \( i \) and \( j \) are in different cubes and \( i \) or \( j \) \( \notin \Delta_0 \), the derivative is not zero only for \( i_1 \in \Delta_0 \) and \( j_1 \notin \Delta_0 \), or vice versa. Hence \( \partial_s B(s_1)_{i_1, j_1} \) connects explicitly \( \Delta_0 \) to a different
cube. With the definition (6.10) the derivation is different and instead of one line we extract three

\[ \partial_{s_1} B(s_1)_{i_1 j_1} = \partial_{s_1} \left( \frac{1}{C(s_1)^{-1} + i m^2} \right)_{i_1 j_1} \]  

(6.26)
\[ = \sum_{k_1, k_1'} \left( \frac{1}{1 + i m^2 C(s_1)} \right) [\partial_{s_1} C(s_1)]_{k_1 k_1'} \left( \frac{1}{1 + i m^2 C(s_1)} \right)_{k_1' j_1} \]
\[ = \sum_{\Delta \notin \Delta_0} \sum_{k_1, k_1'} G(s_1)_{i_1 k_1} C_{k_1 k_1'} G(s_1)_{k_1' j_1} \]

where we used (6.11) and \( G(s) \) is
\[ G(s_1) =: \frac{1}{1 + i m^2 C(s_1)} \]  

(6.27)

For each term \((k_1, k_1')\) with \(k_1 \in \Delta_0\) and \(k_1' \in \Delta_1\) we say there is a \textit{strong connection} between \(\Delta_0\) and \(\Delta_1\). We denote this by drawing a line from \(\Delta_0\) to \(\Delta_1\). Note that these points do not correspond to any derivative inside the functional integral, as the only derivatives occur on \(i_1\) and \(j_1\). If \(i_1\) and \(j_1\) belong to some cube \(\Delta \notin \Delta_0 \cup \Delta_1\) they give some additional strong connections.

Therefore the first step of the induction extracts a link \(l_1\), associated to four points \(i_1, j_1, k_1, k_1'\), connecting \(\Delta_0\) to a set of one, two or three new cubes depending on the positions of \(i_1\) and \(j_1\). We call this set the \textit{generalized cube} \(\tilde{\Delta}_1\). The different possible links inside \(\tilde{\Delta}_1\) are shown in Fig. 3. Now

\[ \partial_{s_1} F_A[s_1] = \sum_{(i_1, j_1)} G(s_1)_{i_1 k_1} C_{k_1 k_1'} G(s_1)_{k_1' j_1} F_A[s_1](i_1 j_1) \]  

(6.28)

Note that the functional integral after \(\partial_{s_1} B_{s_1}\) has been extracted is function only of \((i_1, j_1)\) and not of \((k_1, k_1')\), as only the first two indices correspond to a functional derivative inside the integral.

Now we fix the points \((i_1, j_1), (k_1, k_1')\) corresponding to a strong connection between \(\Delta_0\) and \(\Delta_1\). We want to test if there is any connection between the set \(\Delta_{0,1} = \Delta_0 \cup \Delta_1\) and any other cube \(\Delta' \in \Lambda \setminus \Delta_{0,1}\). We introduce then a new parameter \(s_2\) in \(C(s_1)\) in the functional integral:

\[ C(s_1, s_2)_{ij} = \begin{cases} 
  s_2 C(s_1)_{ij} & \text{if} & i \in \Delta_{0,1}, j \notin \Delta_{0,1}, \text{ or vice versa} \\
  C(s_1)_{ij} & \text{otherwise}.
\end{cases} \]  

(6.29)

As for \(C(s_1)\) we can write \(C(s_1, s_2)\) as a convex combination of positive operators

\[ C(s_1, s_2) = s_2 C(s_1) + (1 - s_2) \left[ C_{\Delta_{0,1} \Delta_{0,1}}(s_1) + C_{\Delta_{0,1} \Delta_{0,1}}(s_1) \right] \]  

(6.30)

where \(C(s_1)\) and \(C_{\Delta \Delta}(s_1)\) are positive. Therefore \(C(s_1, s_2)\) is still positive. Then \(F_A[s_1](i_1 j_1) = F_A[s_1, s_2](i_1 j_1)_{s_2=1, 2} \). We apply again the first order Taylor
formula

$$F_\Lambda[s_1, s_2]((i_1 j_1))_{s_2=1} = F_\Lambda[s_1, s_2]((i_1 j_1))_{s_2=0} + \int_0^1 ds_2 \partial_{s_2} F_\Lambda[s_1, s_2]((i_1 j_1)) .$$  

(6.31)

As before $F_\Lambda[s_1, s_2]((i_1 j_1))_{s_2=0}$ corresponds to a functional integral restricted to $\Delta_{0,1}$: $F_\Lambda[s_1, s_2]((i_1 j_1))_{s_2=0} = F_{\Delta_{0,1}}[s_1]((i_1 j_1))$. The other term gives

$$\partial_{s_2} F_\Lambda[s_1, s_2]((i_1 j_1)) = \sum_{(i_1 j_1),(i_2 j_2),(k_2,k_2')} \left[ G(s_1, s_2)_{i_1 k_2} C(s_1)_{k_2 k_2'} G(s_1, s_2)_{k_2' j_2} \right] \times F_\Lambda[s_1, s_2]((i_1 j_1),(i_2 j_2)) .$$  

(6.32)

We repeat this argument until we construct all the possible connected components containing the root cube. This is a finite sum, for $\Lambda$ fixed.

Definitions We give now some more precise statements. We define a generalized cube $\tilde{\Delta}$ as a set of one, two or three disjoint cubes in $\Lambda$. A generalized polymer $\tilde{Y}$ is then a disjoint set of generalized cubes $\tilde{\Delta}$. A tree $T$ on $\tilde{Y}$ is a set of links $l_1, \ldots, l_n$ connecting the generalized cubes in $\tilde{Y}$ and forming no loops (see Fig. 3). We call the set of all these cubes the polymer $Y$ contained in $\tilde{Y}$. Each link $l_r$ corresponds to four connected vertices $i_r, j_r, k_r, k_r'$ in $\tilde{Y}$. The corresponding propagators are $G_{i_r k_r}, C_{k_r k_r'}$ and $G_{k_r' j_r}$ defined in (5.34) and the corresponding links are shown in (see Fig. 3). Note that the same $\tilde{\Delta}$ and tree $T$ may correspond to several different polymers $Y$ (see Fig. 3).

We apply then the following formula.

Lemma 12 $\langle \tilde{G}_{\tilde{\Delta}} \rangle_{\varepsilon=0}$ can be written as

$$\langle \tilde{G}_{\tilde{\Delta}} \rangle_{\varepsilon=0} = \sum_{\tilde{Y} \supset \tilde{\Delta}_0} \sum_{T \text{ on } \tilde{Y}} \prod_{r=1}^{\lfloor \tilde{Y} \rfloor - 1} \int_0^1 ds_r \sum_{(i_r,j_r),(k_r,k_r')} M_T(s) \left[ \prod_{r=1}^{\lfloor \tilde{Y} \rfloor - 1} G(s)_{i_r k_r} C_{k_r k_r'} G(s)_{k_r' j_r} \right] F_{\tilde{Y}}[\{i_r, j_r\}]$$

(6.33)

where we sum over the generalized polymers $\tilde{Y}$ and over the ordered trees $T$ on $\tilde{Y}$ with root $\tilde{\Delta}_0$. Note that the ordering on a tree is the ordering on its links $l_r$, $r = 1, \ldots, |\tilde{Y}| - 1$. The product over $r$ is then the product over the links in the tree. The points $(i_r, j_r)$ $(k_r, k_r')$ fix the number of cubes inside each $\tilde{\Delta} \in \tilde{Y}$ and the links connecting them. Each tree link $l_r$ is associated to the parameter $0 \leq s_r \leq 1$. The product $G_{i_r k_r} C_{k_r k_r'} G_{k_r' j_r}$ corresponds to $\partial_{s_r} B(s_r)$ for each link $l_r \in T$ and the factor $M_T(s)$ is the product of $s$ factors extracted by the derivatives $\partial_r B(s_r)$. Finally $F_{\tilde{Y}}[\{i_r, j_r\}]$ is the functional integral remaining after the propagators $\partial_{s_r} B(s_r)$ have been extracted.
Figure 2: The two polymers and tree structures in b. and c. both correspond to the same generalized polymer $\tilde{Y}$ and tree $T$ in a.

More precisely, for each $r$ the points $(i_r, j_r)$ must satisfy the constraint $k_r \in \tilde{\Delta}_r'$ for some $r' < r$, $k'_r \in \tilde{\Delta}_r$, and $i_r, j_r$ may belong to any $\tilde{\Delta}_r'$ with $r' \leq r$. The propagators $C(s)$ $G(s)$ are defined as

$$G(s) := \frac{1}{1 + im_i^2 C(s)}.$$  
$$C(s)_{ij} := s_{ij} C_{ij}$$  \hspace{1cm} (6.34)

where

$$s_{ii} = 1$$
$$s_{ij} = 1 \quad \text{if} \quad \exists \ r \quad \text{s.t.} \quad i, j \in \tilde{\Delta}_r$$
$$s_{ij} = \prod_{k=r'}^{r} s_k \quad \text{if} \quad \exists r' < r \quad \text{s.t.} \quad i \in \tilde{\Delta}_r, \ j \in \tilde{\Delta}_{r'}$$
$$s_{ij} = 0 \quad \text{if} \quad i \in Y, \ j \notin Y.$$  \hspace{1cm} (6.35)

The remaining functional integral is

$$F_Y[s]\{(i_r, j_r)\} = \int d\mu_{B(s)}(a, b, \rho^*, \rho)$$
$$\prod_{r=1}^{n} \left[ \frac{\delta}{\delta a_{i_r}} \frac{\delta}{\delta a_{j_r}} + \frac{\delta}{\delta b_{i_r}} \frac{\delta}{\delta b_{j_r}} + \frac{\delta}{\delta \rho^*_{j_r}} \frac{\delta}{\delta \rho_{i_r}} \right] \left[ e^{\sum_{j \in Y} (V_j - \rho^*_t \rho_j) D_j} \mathcal{O}_0 \right]$$  \hspace{1cm} (6.36)

where $B(s)$ is

$$B(s)^{-1} := C(s)^{-1} + im_i^2.$$  \hspace{1cm} (6.37)

Note that, as we constructed the tree, the order on the tree lines ensures that, for each $1 \leq r \leq n$, the tree line $l_r$ connects $\tilde{\Delta}_r$ to some $\tilde{\Delta}_{r'}$ with $r' < r$.  

32
Figure 3: some examples of links of type 1, 2 and 3; the two point vertex is a filled dot, the three point one is an empty dot.

**Proof** The proof follows directly from the inductive procedure we explained above.

Now we can bound $|\langle G_{00} \rangle|$ inserting the absolute value inside (6.33). Therefore we have to bound

$$
|\langle G_{00}^{+} \rangle_{\varepsilon=0} | \leq \sum_{\tilde{Y}} \sum_{T_{\text{on} \tilde{Y}}} \prod_{i=1}^{[\tilde{Y}] - 1} \left[ \int_{0}^{1} ds_{i} \sum_{(i_{r}, j_{r})} \sum_{(k_{r}, k_{r}')} |M_{T}(s)| \right]
$$

$$
\left[ \prod_{r=1}^{[\tilde{Y}] - 1} |G(s)_{i_{r}, k_{r}}||C_{k_{r}, k_{r}'}||G(s)_{k_{r}', j_{r}}| \right] |F_{Y}| \left[ s \{i_{r}, j_{r}\} \right].
$$

### 6.2 Bound on the functional integral

We bound the remaining functional integral $|F_{Y}[s](\{i_{r}, j_{r}\})|$ by a generalization of the arguments for the finite volume case (Sec.5).

**Definitions** We call $V_{d}$ the set of vertices derived by the cluster expansion appearing in (6.33):

$$
V_{d} = \{ j \in \tilde{Y} : \exists 1 \leq r \leq |\tilde{Y}| - 1 \text{ s.t. } j = i_{r} \text{ or } j = i_{r} \} \quad (6.39)
$$

For each $j \in V_{d}$ we call $d_{j}(a)$, $d_{j}(b)$, $d_{j}(\rho)$, $d_{j}(\rho^{*})$ the number of derivatives $\delta/\delta a_{j}$, $\delta/\delta b_{j}$, $\delta/\delta \rho_{j}$ or $\delta/\delta \rho_{j}^{*}$ respectively. As we see from (6.19-6.22) these derivatives apply to $V(a_{j})$, $V(b_{j})$ or $D_{j}\rho_{j}^{*}\rho_{j}$. We also have a contribution from the observable $O_{0}$, but only for $j = 0$.
Note that, by the properties of anticommuting variables, $d_j(\rho)$, $d_j(\rho^*)$ can be only 0 or 1. On the other hand there is no limit to $d_j(a)$, $d_j(b)$. We call $d_j = d_j(a) + d_j(b) + d_j(\rho) + d_j(\rho^*)$ the total number of derivatives in $j$. This number is actually fixed by the choice of $\{(i_r,j_r)\}_r$. For each $j \in \mathcal{V}_d$ we have to study

$$
\partial_{a_j}^{d_j(a)} \partial_{b_j}^{d_j(b)} \partial_{\rho_j}^{d_j(\rho)} \partial_{\rho_j^*}^{d_j(\rho^*)} \left( e^{V_j(a) + V_j(b)} (1 - \rho_j^* \rho_j D_j) \right)
$$

(6.40)

In the small field region we need to extract some structure. We can actually write the derivative as

$$
\sum_{r_j(a), r_j(b)} a_j^{r_j(a)} b_j^{r_j(b)} \rho_j^{r_j(\rho)} \rho_j^{r_j(\rho^*)} C(a_j)C(b_j)
$$

(6.41)

where $r_j(a)$, $r_j(b)$, $r_j(\rho)$, $r_j(\rho^*)$ are the number of fields remaining after the derivatives have been performed. Note that $r_j(\rho)$ and $r_j(\rho^*)$ can take only the values 0 and 1. On the other hand it is easy to see that $r_j(a) \leq 3d_j(b)$, and the same holds for $b$. Moreover the parameter $n_j = d_j + r_j \geq 3$, except for $j = 0$ (and $j = x$ if we are considering $R(x)$). The factors $C(a)$ and $C(b)$ no longer depend on the fermionic fields. By analytic tools we can show that in the small field region $I^1$

$$
|C(a)| \leq K_d^{d_j(a)} d_j(a)! \quad |C(b)| \leq K_d^{d_j(b)} d_j(b)!
$$

(6.42)

In the large field region (that means in $I^q$ for $q > 1$), we do not need to extract the whole structure, as the fields are large and the small factors come from the probability. We only need to extract explicitly the fermionic fields therefore we write the derivatives as

$$
\sum_{r_j(\rho), r_j(\rho^*)} \rho_j^{r_j(\rho)} \rho_j^{r_j(\rho^*)} C(a_j)C(b_j)
$$

(6.43)

Again by analytic tools we can show that in the

$$
|C(a)C(b)| \leq K_d^{d_j(a)} d_j(a)! d_j(b)! \left| e^{V_j(a) + V_j(b)} \right|
$$

(6.44)

In this region we have no factors $r_j$. Nevertheless we define $r_j = 0$ if $d_j \geq 3$ and $r_j = 3 - d_j$ otherwise. In this way we ensure $n_j \geq 3$ for all $j \in \mathcal{V}_d$.

With these definitions and results we prove the following Theorem.

**Theorem 5**

$$
|F_Y[s]((i_r,j_r))| \leq K_1^{Y_1} \sum_{\{r_j\}_{j \in \mathcal{V}_d}} \prod_{j \in \mathcal{V}_d} K_2^{n_\Delta} r_\Delta! \prod_{j \in \mathcal{V}_d \cap \Delta} d_j! \left( \frac{1}{W} \right)^{r_j}
$$

(6.45)

where $K_1$, $K_2$ are constants and $r_\Delta = \sum_{j \in \mathcal{V}_d \cap \Delta} r_j$, $n_\Delta = \sum_{j \in \mathcal{V}_d \cap \Delta} n_j$.  

34
The proof is a generalization of the arguments for Theorem 2 and 3 in Sec. 5. Note that Lemma 3 and 4 hold also after substituting $B$ and $C$ with $B(s)$ and $C(s)$. Then we can introduce the partition (5.13) in each cube separately

$$1 = \prod_{\Delta} \left[ \sum_{k_{\Delta}} \chi[k_{\Delta}] \right]$$

(6.46)

We perform the bounds in each cube as in Sec. 5. Note that, as the derivations bring several bosonic and fermionic fields out of the exponential we have to use some of the ideas of Theorem 3.

First we have to perform the integral over the fermionic variables extracting the correct factors $W - 1$. Now we have many different cubes, each in a different interval, therefore we cannot apply (5.82) as in Lemma 9, as $(B^{-1} + D)^{-1}$ is not be well defined. We apply then a generalization of (5.83) which allows to extract also the small factors.

We have to compute the integral

$$\int \mathcal{D}\mu_{B(s)}(\rho^*, \rho) \left[ e^{-\sum_{j \in Y \setminus \mathcal{V}_d} \rho_j^* \rho_j (D_j + D'_0)} \right]$$

$$\times \left[ \prod_{j \in \mathcal{U}} \rho_j \right] \left[ \prod_{j \in \mathcal{V}_d \cap \Delta} \rho_j^* \right] \left[ \prod_{j \in \mathcal{D}} \rho_j \right]$$

(6.47)

We partition now the set of $j \in Y$ as

$$\mathcal{U} = \{ j \in Y \mid j \not\in \mathcal{V}_d \}$$

$$\mathcal{D} = \{ j \in Y \mid j \in \mathcal{V}_d, \ r_j(\rho^*) = 1 \text{ or } r_j(\rho) = 1 \}$$

(6.48)

Note that the points $j \in \mathcal{V}_d$ with $r_j(\rho^*) = r_j(\rho) = 1$ do not give any contribution to the integral. For each cube we introduce $r_{\Delta}(\rho) = \sum_{j \in \mathcal{V}_d \cap \Delta} r_j(\rho)$ and $r_{\Delta}(\rho^*) = \sum_{j \in \mathcal{V}_d \cap \Delta} r_j(\rho^*)$. Note that the fields $\rho_j$ are columns in the resulting matrix and the fields $\rho_j^*$ are rows. With these definitions we have the following Lemma.

**Lemma 13** By multi-linearity of the determinant, the fermionic integral (6.47) above can be written as

$$\prod_{\Delta} \left[ \frac{r_{\Delta}(\rho)^{r_{\Delta}(\rho)} r_{\Delta}(\rho^*)^{r_{\Delta}(\rho^*)}}{W^{r_{\Delta}(\rho)^{r_{\Delta}(\rho)} + r_{\Delta}(\rho^*)}} \right] \sigma \det M$$

(6.49)

where $\sigma$ is a sign depending on how we order the rows and columns in $M$, and $M$ is the block matrix

$$M = \begin{pmatrix} M_{uu} & M_{ud} \\ M_{du} & M_{dd} \end{pmatrix}$$

(6.50)
$M_{uu}$ is a matrix corresponds to the elements $(M)_{ij}$ with both $i, j \in U$, $M_{dd}$ to the elements $(M)_{ij}$ with both $i, j \in D$ and $M_{ud}$ and $M_{du}$ are the mixed terms. Note that, if $j \in V_d$ and $r_j(\rho) = 1$ but $r_j(\rho^*) = 0$ $j$ appears only in a column of $M_{dd}$ and the element $M_{jj}$ is not present in the matrix. Therefore we order the lines and columns of $M_{dd}$:

$$(M_{dd})_{ij} = (M)_{l_i c_j}$$

where $\Delta(i)$ is the cube containing the vertex $i$.

**Proof**  The proof follows from the properties of the anticommuting variable $s$ and determinants.  

Now we can insert absolute values inside the bosonic integral. If we bound $\det M$ as in (5.8) we obtain the same bound as in Lemma 4, with an additional error term. The precise statement is given in Lemma (14) below.

**Lemma 14**  The determinant of the matrix $M$ defined as in (6.50-6.51) satisfies the bound

$$|\det M| \lesssim e^{Re\, Tr(M_{uu} - 1)} e^{O(|Y| + |V_d|)}$$

**Proof**  Using (5.8) we have

$$|\det M| \leq e^{Re\, Tr(M - 1)} e^{K \cdot Tr(M - 1)^* (M - 1)}$$

Now applying the definitions (6.50) we can write

$$Re \, Tr \, (M - 1) = Re \, Tr \, (M_{uu} - 1) + Re \, Tr \, (M_{dd} - 1)$$

$$Tr \, (M - 1)^* (M - 1) = Tr \, (M_{uu} - 1)^* (M_{uu} - 1) + Tr \, (M_{dd} - 1)^* (M_{dd} - 1) +$$

$$+ Tr \, [(M_{du} - 1)^* (M_{du} - 1) + (M_{ud} - 1)^* (M_{ud} - 1)]$$

By inserting the definitions (6.51) and using the decay of $B_{ij}$, it is easy to see that all terms are bounded by a constant per cube except for

$$Tr(M_{dd} - 1)^* (M_{dd} - 1) \lesssim \sum_{\Delta} r_{\Delta}(\rho) + r_{\Delta}(\rho^*)$$

This completes the proof.  

Now we perform the estimates as in Sec. 5. As in Sec.5.1.5, in order to decouple the estimates on different cubes we write all the bounds in terms of quadratic forms like $v^T C_f(s) v$ where $v$ is some vector and $C_f(s) = C(s)^{-1} -$
We perform now the sum over the clusters $Y$. We split the sum in several pieces. First fixing the cubes $\Delta$ we sum over the points $i_r$, $j_r$, $k_r$, $k_r'$ in the cubes. Note that after this operation is done, there is still a small factor associated to each cube. The factorials arising from the combinatorics are beaten by a piece from the decay of $GCG$. The remaining piece of the decay is used to sum over the cube positions, following the tree structure. Finally we sum over the tree choice $T$ using the fact that we have a small factor per cube.

To perform all these bounds we need now to study the spatial decay of the propagators $C$, $B$ and $G$. We know already the spatial decay of $C(s)$ (see (4.19). The decay of $G$ is given by the following Lemma.

Lemma 15 The propagator $G = (1 + im^2C(s))^{-1}$ decays as

$$|G_{ij}| \leq \delta_{ij} + \frac{m^2_i}{W^3} e^{-\frac{f}{m} |i-j|} + \frac{2}{W^3} e^{-f W |i-j|}$$

(6.57)

where $f = \text{inf}[1/2, g]$ and $g$ is some constant independent from $W$.

Proof By a Combes-Thomas argument we prove that

$$\left\| R^{-1} \frac{1}{1 + im^2 C(s)} R \right\| \leq 2$$

(6.58)

for $R$ a multiplication operator defined as $R|x> = \exp[\mu \vec{x}]|x>$ and $\mu$ any vector with $|\mu| < gm_r/W$. Now $G$ can be written as

$$\left( \frac{1}{1 + im^2 C(s)} \right)_{ij} = \delta_{ij} - im^2 C(s)_{ij} - \frac{m^4}{1 + im^2 C(s)} C(s)_{ij}$$

(6.59)

We need to study the decay of the last term. Actually we see that

$$\left| C(s) \frac{1}{1 + im^2 C(s)} C(s) \right|_{ij} e^{\mu|\vec{x}-\vec{y}|} = \left| R^{-1} C(s) \frac{1}{1 + im^2 C(s)} C(s) R \right|_{ij}$$

$$= \left| (R^{-1} C(s) R) \left( R^{-1} \frac{1}{1 + im^2 C(s)} R \right) (R^{-1} C(s) R) \right|_{ij}$$

$$= \left| (V, AW) \right| \leq \|V\| \|W\| \|A\| \leq \frac{2}{W^3}$$

(6.60)
where we defined $V_k = (R^{-1}C(s)R)_{ik}$, $W_j = (R^{-1}C(s)R)_{kj}$ and $A = R^{-1}(1 + im^2C(s))R$. We chose $\vec{\mu}$ such that $\mu|\vec{x} - \vec{y}| = \vec{\mu}(\vec{x} - \vec{y})$, and $\mu = |\vec{\mu}| \leq gm_r/W$, and $\mu \leq m_r/2W$. The last condition must ensure that the exponential decay of $C(s)$ controls the exponentials from $R^{-1}$ and $R$. This completes the proof.

Note that this Lemma gives also the decay of $B(s)$

$$|B(s)_{ij}| \leq \frac{1}{W^{n_j}} \prod_{r=1}^{n_j} e^{-\frac{1}{W}|i-j|} + \frac{2}{W^{n_j}} \prod_{r=1}^{n_j} e^{-\frac{1}{W}|i-j|}$$

as $B(s) = (G - 1)i/m_i^2$.

6.3.1 Extracting small factors

Before performing the estimates we extract some factors from the propagators $GCG$ for each tree line, to offset the factorials eventually arising in the estimates and to ensure we have a small factor per vertex.

Factorials. Constant powers of factorials such as $d^p_{\Delta}$, for $p$ fixed, can be beaten using a piece of the decay of $CGC$. Note that each tree line $l_r$ connects different cubes, therefore we have $d_{\Delta}$ disjoint cubes hooked to the cube $\Delta$ by the tree $T$. When $d_{\Delta}$ is large, since we are in a finite dimensional space, many of these cubes must be very far from $\Delta$. It is easy to see that, half of the $d_{\Delta}$ cubes must be at a distance from $\Delta$ of order $W d_{\Delta}^{3/4}$. Therefore we gain a factor $\exp[-c d_{\Delta}^{3/4}]$ which can beat any constant power of factorials.

$$\left[ \prod_{r=1}^{d_{\Delta}} e^{-\frac{1}{W}|i-r|} e^{-\frac{1}{W}|i-r'|} \right] \leq \left[ \prod_{\Delta} K_{n_{\Delta}}^{n_{\Delta}} \right]$$

(6.62)

Note that the constant $K$ depends on $p$.

$W$ factors We need a factor $W^{-1}$ for each field hooked to a derived vertex $j \in \mathcal{V}_d$. We extract then a factor $W^{-1}$ from each $G$ propagator:

$$\left[ \prod_{r=1}^{d_{\Delta}} e^{-\frac{1}{W}|i-r|} e^{-\frac{1}{W}|i-r'|} \right] \leq \left[ \prod_{\Delta} K_{n_{\Delta}}^{n_{\Delta}} \right]$$

(6.63)

Note that $n_j \geq 3$ for all $j \in \mathcal{V}_d$ except for $j = 0$. As the each cube has volume $W^3$, this ensure that we can choose the position of each vertex without paying any factor $W$. 

38
Finally, after extracting a fraction \( \varepsilon \) of the exponential decay and the factors \( W^{-1} \) we separate in the remaining factors the polynomial and exponential decay

\[
\prod_{r=1}^{[\gamma]-1} \tilde{G}_{i_r k_r} \tilde{C}_{k_r k'_r} \tilde{G}_{k'_r j_r} \prod_{r=1}^{n} e^{-\frac{f'}{4W}d(\Delta(i_r), \Delta(k_r))} e^{-\frac{f'}{4W}d(\Delta(k_r), \Delta(k'_r))} e^{-\frac{f'}{4W}d(\Delta(k'_r), \Delta(j_r))} \\
(6.64)
\]

where

\[
\tilde{G}_{ij} =: \left[ \delta_{i_r k_r} W^2 + \frac{1}{1+|i-j|} + \frac{1}{W} \right] \quad \tilde{C}_{ij} =: \frac{W^{-4}}{1+|i-k|} \quad (6.65)
\]

where \( f' = fm_r - \varepsilon \) is the remaining mass and \( d(\Delta, \Delta') \) is the distance between the center of the cube \( \Delta \) and \( \Delta' \).

### 6.3.2 Sum over the vertex positions

Now using the decay \( 1/|i-j| \) in \( \tilde{G} \) and \( \tilde{C} \) we sum over the positions of all vertices inside their cube (the cube is fixed). Each line of the cluster expansion corresponds to four vertices \( i_r, j_r, k_r, k'_r \), where \( k_r \) and \( k'_r \) correspond to two point vertices and must belong to different cubes while \( i_r, j_r \) may belong to the same cube. For each \( j = i_r \) or \( j = j_r \) we distinguish two cases

- \( j \) contracts to \( j' \) and \( j' \) has never been extracted before in the cluster expansion. Then we say \( j' \) is new with respect to \( j \).
- \( j \) contracts to \( j' \) and \( j' \) has already been extracted before by the cluster expansion. Then we say \( j' \) is old with respect to \( j \).

We consider first the case \( k_r \neq i_r \) and \( k'_r \neq j_r \) so that the factors \( \delta_{i_r, k_r} \) and \( \delta_{j_r, k'_r} \) disappear. Note that we sum over the position of \( i_r \) (\( j_r \)) only when it is new. We consider the different cases.

**\( i_r \neq j_r \) and both \( i_r \) and \( j_r \) new** Then we sum over \( i_r \) and \( j_r \).

\[
\sum_{k_r \in \Delta(i_r)} \sum_{i_r \in \Delta(i_r)} \tilde{G}_{i_r k_r} \sum_{k'_r \in \Delta(k'_r)} \tilde{C}_{k_r k'_r} \sum_{j_r \in \Delta(j_r)} \tilde{G}_{k'_r j_r} \leq W^{5} = \left( W^2 \right)^{2} \left( W^2 \right)
\]

(6.66)

Therefore we pay a factor \( W^{5/2} \) for \( i_r \) and the same factor for \( j_r \).

**\( i_r \neq j_r \) and \( i_r \) new, \( j_r \) old** Then we sum only over \( i_r \).

\[
\sum_{k'_r \in \Delta(k'_r)} \tilde{G}_{k'_r j_r} \sum_{k_r \in \Delta(k_r)} \tilde{C}_{k_r k'_r} \sum_{i_r \in \Delta(i_r)} \tilde{G}_{i_r k_r} \leq W^{2}
\]

(6.67)

Therefore we pay a factor \( W^2 \) for \( i_r \) and no factor for \( j_r \).
\(i_r \neq j_r\) and \(i_r\) and \(j_r\) old

Then \(i_r\) and \(j_r\) are both fixed.

\[
\sum_{k_r \in \Delta(i_r)} \tilde{G}_{i_r, k_r} W^{-4} \sum_{k_r' \in \Delta(j_r)} \tilde{G}_{k_r', j_r} \leq O(1)
\] (6.68)

where we bounded \(\tilde{C}_{k_r, k_r'} \leq W^{-4}\).

\(i_r = j_r\) and \(i_r\) new

Then we sum over \(i_r\).

\[
\sum_{k_r \in \Delta(i_r)} \sum_{k_r' \in \Delta(j_r)} \tilde{C}_{k_r, k_r'} \left\{ \sum_{|i_r - k_r| \geq |i_r - k_r'|} \left[ \tilde{G}_{i_r, k_r'} \right]^2 + \sum_{|i_r - k_r| < |i_r - k_r'|} \left[ \tilde{G}_{i_r, k_r} \right]^2 \right\} \leq W^2
\]

(6.69)

Therefore we pay a factor \(W^2\) for \(i_r\).

\(i_r = j_r\) and \(i_r\) old

Then \(i_r\) is fixed.

\[
\sum_{k_r \in \Delta(i_r)} \tilde{G}_{i_r, k_r} W^{-4} \sum_{k_r' \in \Delta(i_r)} \tilde{G}_{k_r', j_r} \leq O(1)
\] (6.70)

where we bounded \(\tilde{C}_{k_r, k_r'} \leq W^{-4}\). When \(i_r = k_r\) or \(j_r = k_r'\) it is easy to see that the same estimations hold.

Note that for each \(j = i_r\) (\(j = j_r\)), with \(j \neq 0\), we pay a factor \(W^{5/2}\) when it is new and some constant \(K\) in any other case. Therefore, applying \(n_j \geq 3\), we have

\[
W^{\frac{5}{2}} K^{n_j - 1} \frac{1}{W^{n_j}} = \left( \frac{K}{W^{1-\frac{5}{2}}} \right)^{n_j} \leq \left( \frac{K}{W^{5/2}} \right)^{n_j}
\] (6.71)

Therefore we have a factor \(W^{-\frac{5}{2}}\) for each tree line hooked to \(j\). This means we have a factor \(W^{-\frac{5}{2}}\) for each generalized cube \(\Delta\) is \(\bar{Y}\). The case \(j = 0\) is special as \(n_j \geq 0\). Nevertheless the position of \(0\) is fixed so that we do not pay the factor \(W^{5/2}\). Therefore for \(j = 0\) we have

\[
K^{n_0} \frac{1}{W^{n_0}}
\] (6.72)

6.3.3 Combinatorial bounds

The combinatoric inside each cube costs a factor

\[
\sum_{r_{\Delta} \leq d_{\Delta}} K^{d_{\Delta}} (d_{\Delta})! (r_{\Delta})! \leq K^{d_{\Delta}} (d_{\Delta})! = K^{|\bar{Y}|} (d_{\Delta})^4
\] (6.73)

The factorials are beaten by a piece of the exponential decay of the tree lines (6.62), while the constant factor will be bounded later when we will sum over the tree structure.
6.3.4 Sum over the cube positions and the tree structure

We sum over the cube positions using the exponential decay and following the tree from the leaves towards the root. The result is $K|Y|$. The remaining sum is now

$$\sum_{\text{unordered}} \prod_{i=1}^{|Y|-1} \int_0^1 ds_i |M_T(s)| \prod_{\Delta \in \tilde{Y}} \left[ \frac{K}{W^\Delta} \right]$$

(6.74)

where we have split the sum over ordered rooted trees as the sum over unordered rooted trees, and the sum over orders. This last sum is performed by the integral over the interpolating factors (see [16] or [17], Lemma III.1.1) We give here a sketch of the proof.

**Lemma 16** The sum over all the orders on the tree $T$ is bounded using the interpolating factors $s_i$ as follows

$$\sum_{\text{orders}} \prod_{i=1}^{|Y|-1} \int_0^1 ds_i |M_T(s)| = 1$$

(6.75)

**Proof** We introduce the variables $\varepsilon_{ij}$ for all $\tilde{\Delta}_i \tilde{\Delta}_j \in \tilde{Y}$. Then we introduce the function $F(\varepsilon) = \prod_{(ij) \in T} (1 + \varepsilon_{ij})$ where $T$ is unordered. Now we perform the tree expansion as we did in Lemma 12 Sec. 6.1. We define $\varepsilon_{ij}(s_1) = s_1 \varepsilon_{ij}$ if $i$ or $j = 0$ and $\varepsilon_{ij}(s_1) = \varepsilon_{ij}$ otherwise. We apply the first order Taylor expansion and we go on until we extract all the $\varepsilon_{ij}$. The term proportional to all $\varepsilon_{ij}$ is then

$$\prod_{(ij) \in T} \varepsilon_{ij} \sum_{\text{orders}} \prod_{i=1}^{|Y|-1} \int_0^1 ds_i |M_T(s)|$$

(6.76)

If we expand $F(\varepsilon)$ in powers of $\varepsilon$ we see that the term $\prod_{(ij) \in T} \varepsilon_{ij}$ has coefficient 1. Therefore by comparing powers of $\varepsilon$ we obtain (6.75).

Finally the sum over the structure can be written as

$$\sum_{|Y|} \sum_{\text{unordered}} g^{|Y|} = \sum_{d_{\Delta_0} \geq 0} \prod_{i=1}^{d_{\Delta_0}} g \left[ \sum_{d_{\Delta_i} \geq 0} \prod_{i'=1}^{d_{\Delta_i}} g \sum \cdots \right]$$

(6.77)

where we defined $g = K/W^\Delta < 1$ and for each generalized cube we sum over the coordination number. We start summing from the leaves going towards the root. The leaves give

$$g \sum_{d \geq 0} g^d = g \frac{1}{1-g} \leq g \frac{1}{g} \quad \text{if} \quad g \frac{1}{1-g} < 1$$

(6.78)
The following step gives
\[ g \sum_{d \geq 0} g^\frac{d}{2} = g \frac{1}{1 - g^\frac{1}{2}} \leq g^{\frac{1}{2}}, \quad \text{if} \quad g \frac{1}{1 - g^\frac{1}{2}} < 1 \quad (6.79) \]

Then we repeat inductively. Finally
\[ \sum_{|\tilde{Y}|} \sum_{\text{unordered}} g^{S} \leq K + 1 \quad (6.80) \]

where the constant comes from the bound for \( n = 0 \). This completes the proof of the first part of Theorem 4, namely the boundness of the density of states (6.3).

### 6.4 Smoothness and exponential decay

To bound the derivatives of the density of states (6.2), and in particular the decay of \( L(x) \) (6.4), we perform the cluster expansion as in Sec. 6.1.

For \( L(x) \), note that contributions where \( Y \) does not contain both 0 and \( x \) are cancelled. When 0 and \( x \) both belong to \( Y \) we can extract the exponential decay directly from the tree lines \( GCG \). For derivatives \( \partial^n \hat{\rho}_L(E) \) the idea is the same. Only contributions from \( Y \) containing all the observables are not cancelled. The fine structure (the factor \( W^{-3} \) in \( L(x) \)) are then extracted by a few steps of perturbative expansion as in Sec. 5.2.2. In the same way we can prove the semicircle law behavior (6.3). This completes the proof of Theorem 4.

### Appendix A

**Supersymmetric formalism**

We summarize the conventions and notations we adopted in this work (they are based on the review by Mirlin [1]).

**Grassmann variables** A set of Grassmann variables and their complex conjugates \( \chi_1, \chi_1^*, \ldots \chi_N, \chi_N^* \) has the following properties:
\[ \chi_i \chi_j = -\chi_j \chi_i, \quad \chi_i^* \chi_j = -\chi_j^* \chi_i, \quad \chi_i^* \chi_j^* = -\chi_j \chi_i \quad (A.1) \]
\[ (\chi_i^*)^* = -\chi_i, \quad (\chi_i \chi_j)^* = \chi_i^* \chi_j^* \quad (A.2) \]
\[ \int d\chi_i \ 1 = \int d\chi_i^* \ 1 = 0, \quad \int d\chi_i \ \chi_i = \int d\chi_i^* \ \chi_i^* = \frac{1}{\sqrt{2\pi}} \quad (A.3) \]
With these definitions we introduce a vector and its adjoint as usual

$$\chi = \begin{pmatrix} \chi_1 \\ \vdots \\ \chi_N \end{pmatrix} \quad \chi^+ = (\chi_1^*, \cdots, \chi_N^*) \quad (A.4)$$

Now $\chi^+ \chi$ is a real commuting variable and

$$\int \prod_i d\chi_i^* d\chi_i \ e^{-\chi^+ M \chi} = \det \frac{M}{2\pi} \quad (A.5)$$

for any matrix $M$.

**Supervectors and supermatrices** A supervector is defined as

$$\Phi = \begin{pmatrix} S_1 \\ \vdots \\ S_N \\ \chi_1 \\ \vdots \\ \chi_N \end{pmatrix} \quad \Phi^+ = (S_1^*, \cdots, S_N^*, \chi_1^*, \cdots, \chi_N^*) \quad (A.6)$$

where $S_i$ are the commuting and $\chi_i$ are the anticommuting components. Similarly a supermatrix is a matrix with both commuting and anticommuting entries

$$M = \begin{pmatrix} a & \sigma \\ \rho & b \end{pmatrix} \quad (A.7)$$

where $a$ and $b$ are ordinary matrices while $\sigma$ and $\rho$ have anticommuting elements. We identify the element of a supermatrix by four indices $M_{\alpha\beta}^{ij}$ where $\alpha, \beta$ specify in which sector we are: $(0,0)$ corresponds to $a$ (boson-boson); $(1,1)$ corresponds to $b$ (fermion-fermion); $(0,1)$ corresponds to $\sigma$ (boson-fermion); $(1,0)$ corresponds to $\rho$ (fermion-boson). $(i,j)$ identify the matrix element inside each sector. For example $M_{00}^{00} = a_{ij}$.

The notions equivalent to trace and determinant are supertrace and superdeterminant

$$\text{Str} M = \text{Tra} - \text{Trb}, \quad \text{Sdet} M = \det(a - \sigma b^{-1}) \det b^{-1} \quad (A.8)$$

With these definitions we have

$$\text{Str} \ln M = \ln \text{Sdet} M \quad (A.9)$$

$$\int d\Phi^* d\Phi \ e^{-\Phi^+ M \Phi} = \text{Sdet} M^{-1} \quad (A.10)$$
\[ \int d\Phi^* d\Phi \Phi_{\alpha,k} \Phi_{\beta,l}^* e^{-\Phi^* M \Phi} = (M^{-1})_{\alpha\beta} \text{Sdet} M^{-1} \]  \hspace{1cm} (A.11)

Note that some properties are different from that of the usual matrices, in particular:

\[ \text{Sdet } z M = \text{Sdet } M \]  \hspace{1cm} (A.12)

for any complex number \( z \).

Finally from these formulas one can derive the inverse of the supermatrix \( M \) \hspace{1cm} (A.7):

\[
M^{-1} = \left( \begin{array}{cc}
(a - \sigma b^{-1} \rho)^{-1} & - (a - \sigma b^{-1} \rho)^{-1} \sigma b^{-1} \\
- b^{-1} \rho (a - \sigma b^{-1} \rho)^{-1} & b^{-1} \left[ 1 + \rho (a - \sigma b^{-1} \rho)^{-1} \sigma b^{-1} \right]
\end{array} \right) \]  \hspace{1cm} (A.13)

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