T. A. Sedrakyan
The Abdus Salam International Centre for Theoretical Physics, Strada Costiera 11, 34100 Trieste, Italy
email: tsedraky@ictp.it

Abstract

We construct a replica field theory for a random matrix model with logarithmic confinement [K. A. Muttalib et.al., Phys. Rev. Lett. 71, 471 (1993)]. The corresponding replica partition function is calculated exactly for any size of matrix $N$. We make a color-flavor transformation of the original model and find corresponding Toda lattice equations for the replica partition function in both formulations. The replica partition function in the flavor space is defined by generalized Itzikson-Zuber (IZ) integral over homogeneous factor space of pseudo-unitary supergroups $SU(n | M, M)/SU(n | M - N, M)$ (Stiefel manifold) with $M \to \infty$, which is evaluated and represented in a compact form.
Recently a considerable interest appeared towards analyzing of integrable structure behind Gaussian random matrix theories (RMT). An important development in this context was made by Kanzieper in \( \text{Ref.}^3 \), where the link between nonlinear replica-\( \sigma \)-models and Painlevé hierarchy of exactly solvable Toda lattice equations was established. More precisely it has been shown, that the replica partition function of the Gaussian unitary ensemble (GUE), which in its original and dual (in the replica space) \( \sigma \)-model representation satisfies the Toda lattice equation (TLE), can be reduced to Painlevé transcendent \( \varphi_{11} \). From here exact nonperturbative results were obtained demonstrating the ability of the method to overcome the critique of replica trick. In the chiral unitary ensemble (chUE), the replica trick on the basis of TLE was analyzed in \( \text{Ref.}^2,9 \).

In this paper we show how the above picture may arise and work in a unitary ensemble defined by probability distribution function \( P(H) = \exp[-\text{tr}V(H)] \) with the confining potential

\[
V(x) = \sum_{n=0}^{\infty} \ln[1 + 2q^{n+1} \cosh \beta \chi + q^{2n+2}],
\]

that behaves as \( V(x) \propto \ln^2(|x|) \) at \( |x| \to \infty \). Here \( x = \sinh \beta \chi /2 \) and the parameter \( q = e^{-\beta} \), with \( \beta > 0 \).

Defined and solved using the orthogonal polynomials method in \( \text{Ref.}^1 \), this unitary random matrix model has attracted considerable attention during the last decade. One reason for that is that it leads to the eigenvalue distribution which for not very large \( \beta < \pi^2 \) is intermediate between the Wigner-Dyson and the Poisson distribution and has the features typical of the critical level statistics in\( ^{17,18} \).

Another reason, perhaps more important, is that the potential Eq.(1) fits very well the function \( a[\ln(1 + bx^2)]^2 \) in a wide region of the variable \( x \). This behavior was obtained numerically in the tight binding Anderson model with random site energies as a distribution potential of variables \( x_i, i = 1, \ldots, N \), connected with the conductance as \( g = \sum_{i=1}^{N}(1 + x_i^2)^{-1} \), reflecting a log-normal behavior of the conductance \( g \) in a product of random matrices. Therefore, by treating matrices \( H \) as \( H^2 = (TT^* + T^*T - 2I)/4 \), where \( T \) is the transfer matrix characterizing the disordered conductor, the model with the potential Eq.(1) can be considered as a qualitatively correct phenomenological model for the Anderson insulator with the parameter \( \beta \) playing a role of the ratio of the system size to the localization length \( L/\xi \) and \( \chi/\xi \) playing a role of the Lyapunov exponent (\( I \) is an identity matrix). At large \( \beta \) the probability density \( \rho(\chi) \) is strongly peaked near positive integers exhibiting the phenomenon which is known in theory of quasi-1d Anderson localization as transmission eigenvalue crystallization\( ^{13,14,22,31} \).

Up to now the model Eq.(1) was investigated only within the approach of orthogonal polynomials which was successful because the corresponding set of orthogonal polynomials (q-deformed Hermite polynomials) is explicitly known. Though this method gives an exact solution to the problem it does not tell us anything about the connection of this problem to exactly integrable lattice models. On the other hand, such a connection is expected by analogy with the Wigner-Dyson RMT which is solvable by the ordinary Hermite polynomials. Thus the main question we ask is as to whether or not the RMT Eq.(1) is related with exactly integrable lattice models and what precisely these models are.

As a first step we compute the replica partition function exactly using the pole structure of the distribution function \( P(H) \). Indeed, the form of the potential allows us to integrate over poles in complex plane and obtain the new exact expression for the partition function as an infinite sum of residues. Such a result of nonperturbative nature turns out to be exactly deducible from the dual representation of the partition function in a form of the Itzikson-Zuber (IZ) type matrix integral over homogeneous factor space of pseudounitary supergroups \( \text{SU}(n|M, M) \) (Stiefel manifold) with \( M \to \infty \). Thus one of the results of the paper is a calculation of this specific IZ integral by use of Duistermaat-Heckman theorem. To the best of our knowledge this integral was not calculated earlier in the literature.

One of the main results of the present paper is the establishment of the link of RMT Eq.(1) with integrable lattice models. For the original replica partition functions we infer the same Toda lattice equations as in the case of the Gaussian RMT. However the dual partition functions \( Z_{n,N} \) obtained by a super-symmetric mapping onto a variant of the replica \( \sigma \) model (color-flavor transformation)\( ^{10} \) has a more delicate relationship with the Toda lattice models. Namely, semi-infinite hierarchies of fermionic and bosonic flavors involved in this mapping manifest themselves as an extension of graded TLE first found in\( ^{21} \) in the context of chiral unitary ensemble relevant for QCD.

Finally, we analyze the new representation for the two-level correlation function \( R_2(x) \) and the probability density \( \rho(\chi) \) and discuss the phenomenon of \( \chi^\text{c}-\text{crystallization} \).

The paper is organized as follows. In the second section we make a color-flavor transformation in the model under consideration and give its sigma-model formulation defined on the homogeneous factor space of pseudo-unitary supergroups. We calculate the replica partition function in both, color and flavor spaces respectively. In the third section we infer Toda lattice equations in both spaces. In the fourth section, by use of the method of pole integration,
we calculate exactly the density of states and the two point correlation function at large $N$. Comparison with known results follows. Section 5 contains our conclusions. The calculation of the Itzikson-Zuber matrix integral over Stiefel manifold and the derivation of TLE in flavor space are presented in two Appendices.

II. THE REPLICa PARTITION FUNCTION.

The replica partition function of the model Eq. (1), defined by $Z_{n,N}(\epsilon) = \langle \det^n (\epsilon - H) \rangle_H$, can be represented as an integral

$$
(C_M)^N \int dH \det^n (\epsilon - H) \prod_{k=1}^M \frac{1}{\det \[(\mu_k + iH)(\mu_k - iH)\]},
$$

(2)

where $\mu_k = \cosh kN/2$, $C_M = (\prod_{n=1}^M 4^n)^{-1}$ and the limit $M \to \infty$ is understood. The main idea of the replica trick is the possibility of analytic continuation of $Z_{n,N}$ to non-integer values of $n$, which was shown to be exact in GUE.

Diagonalizing Hermitian $N \times N$ matrices $H = V \text{diag}(E_i) V^\dagger$ and taking into account the Vandermonde determinant one can calculate Eq. (2) by closing the contour of integration at infinity (which is correct for $n < 2M - N$) and using the Cauchy formula of pole integration:

$$
Z_{n,N}(\epsilon) = (2\pi)^N (C_M)^N \sum_{\{k_i\}} \prod_{i=1}^N \frac{\prod_{j=1}^{2M-N}(\epsilon - i\mu_k)}{\prod_{j=1}^M (\mu_k - \mu_p)},
$$

(3)

where the sum is taken over all possible subsets $\{k_1, \ldots, k_N\}$ of the set of indices $k \in \{1, \ldots, M\}$ of the poles $\mu_k + iE_i$ on the upper half plane and $\{p_1, \ldots, p_{2M-N}\} = \{1, \ldots, M; M+1, \ldots, 2M\} \setminus \{k_1, \ldots, k_N\}$ includes the numeration of poles $\mu_k - iE_i$ with the condition $\mu_{M+j} = -\mu_j$, $j = 1, \ldots, M$. This formula will be a starting point of the further analyzes.

A. Color–flavor transformation

Now we define another, dual representation of the partition function in which the replica space and the space of matrix indices interchange. Usually (see e.g. [19]) such a representation is obtained from the fermionic replica sigma-model derived using the Hubbard-Stratonovich transformation.

In our case the potential [11] is not Gaussian and the Hubbard-Stratonovich transformation is not useful. But nevertheless it appears that one can make a transformation of the partition function given by an integral over original $N \times N$ Hermitian matrices (following [19]) let us call the space where they are acting a color space) and pass to the integral over matrices, which are acting in replica space (let us call it flavor space). Similar type of transformations effective in supersymmetric $\sigma$-models are called color–flavor transformations.

In order to make the color–flavor transformation let us transform the determinants in the integrand of Eq. (2) into the Gaussian integral over superfields $\Phi^a_i = \{\Phi^{(1),k}_{i}; \{\phi^{(2),p}_{i}\}\}$ with $a = \{\alpha; k; p\} = \{1; \ldots; n; 1; \ldots; M; 1; \ldots; M\}$, where $n$ is the number of fermionic components $\Phi^a_i$ while $M$ is the number of bosonic ones $\phi^{(1),k}_{i}; \phi^{(2),p}_{i}$. The supersymmetric structure of the fields $\Phi^a_i$ which can be regarded as $i = 1, \ldots, N$ vectors in $(n+2M)$ dimensional flavor superspace, is dictated by the fact that the partition function Eq. (2) is a product of a certain number of determinants and inverse determinants. Using these superfields we obtain

$$
Z_{n,N}(\epsilon) = \frac{(i^{-n}C_M)^N}{(2\pi)^{2MN}} \int dH \prod_{i,a} d\Phi^{(1),a}_i d\Phi^{(2),a}_i \exp(-\Phi^{(1),a}_i \tilde{V}^{ab}_i \Phi^{(2),a}_i). \quad (4)
$$

The indices $i, j = 1 \ldots N$ correspond to the original color space while indices $a$ and $b$ are forming a basis in the flavor space. The matrix $\tilde{V}^{ab}_i = \text{diag}\{i[\delta_{ij} - iH_{ij}]\delta_{ab} + [\mu_k\delta_{ij} + iH_{ij}]\delta_{kp} - [\mu_k\delta_{ij} - iH_{ij}]\delta_{kp}\}$ is diagonal with $\alpha, \beta = 1 \ldots n$; $k, p = 1 \ldots M$ and Kronecker $\delta$-s. As one can see the integral over Hermitian random matrices $H$ in the expression [4] simply produces a Dirac delta-function $\langle \text{diag}\{\delta_{ij}, \delta_{kp}, -\delta_{kp}\}\rangle$.

For further treatment it is convenient to make use of the following trick: we introduce a set of real numbers $\omega_1, \ldots, \omega_N$ and instead of the delta-function $\prod_{i,j} \delta[\Phi^{(1),a}_i \Phi^{(2),a}_j]$ we substitute $\lim_{\{\omega_i\} \to 0} \prod_{i,j} \delta[\Phi^{(1),a}_i \Phi^{(2),a}_j - \omega_i \omega_j \delta_{ij}]$. 


Then after the rescaling of the superfields $\Phi_i^a \to \omega_i^a \Phi_i^a$ one obtains
\[
Z_{n,N}(\epsilon) = \frac{(i^{-n}C_M)^N}{(2\pi)^{2MN-N(N+1)/2}} \lim_{\omega_i \to 0} \prod_{i,j,k=1}^{N} \prod_{a} \omega_i^{2(2M-n-N)} \int d\Phi_{k}^{+a} d\Phi_{k}^{a} \\
\cdot \prod_{i,j} \delta(\Phi_i^{+a}g^{ab}\Phi_j^b - \delta_{ij}) \exp(-\omega_i^2 \Phi_i^{+a}g^{ab}V_{ij} V_{ij}^b \Phi_j^c),
\] (5)
with
\[
V_{ij} = \left( \begin{array}{ccc} i\epsilon \delta_{ij} \delta_{\alpha\beta} & 0 & 0 \\
0 & \mu_k \delta_{ij} \delta_{kp} & 0 \\
0 & 0 & -\mu_k \delta_{ij} \delta_{kp} \end{array} \right)
\] (6)
and the summation is understood over repeating indices $b$. The argument of the new $\delta$- function in Eq. (5) imposes that all possible field configurations contributing to the integral are $N$ normalized vectors in the $(n+2M)$ dimensional complex superspace with metric $g^{ab}$. By definition this space can be called complex Stiefel manifold and is equivalent to the homogeneous factor space $S = SU(n|M,M)$ of pseudounitary supergroups. Therefore the formula (5) is nothing but a replica $\sigma$ model formulation of the model (2) in the form of graded IZ type integral over Stiefel manifold in the $(n+2M)$ dimensional flavor superspace:
\[
Z_{n,N}(\epsilon) = \frac{(i^{-n}C_M)^N}{(2\pi)^{2MN-N(N+1)/2}} \lim_{\omega_i \to 0} \prod_{i=1}^{N} \omega_i^{2(2M-n-N)} \int_{U \in S} dU \exp\{-Str(\Omega U^{-1}VU)\},
\] (7)
where $U^{-1} = g U^* g$, the matrices $V = \text{diag}\{\epsilon_1, \ldots, \epsilon_n, \mu_1, \ldots, \mu_M, -\mu_1, \ldots, -\mu_M\}$; $\Omega = \text{diag}\{0, \ldots, 0, \omega_1^2, \ldots, \omega_n^2, 0, \ldots, 0\}$ and the limit $\epsilon_1, \ldots, \epsilon_n \to \epsilon$ is supposed.

The evaluation of standard (super) IZ integral over the group manifold is based on the analyze of heat kernel equation in the Hermitian matrix space and on calculation of induced metric on the subspace of diagonal matrices. In our case, since the tangent to the Stiefel manifold vector defined by $dH = UdU^{-1} \setminus GdG^{-1}$, with $U \in SU(n | M, M)$, $G \in SU(n | M - N, M)$, the corresponding induced metric does not coincide with the Vandermonde determinant, but with some modification.

Another elegant calculation of IZ integral over unitary supergroups has been made in33 by use of Duistermaat-Heckman theorem. In33 it has been shown, that the theorem can be applied to pseudounitary supergroups as well. In Appendix A we apply this theorem to our IZ integral over manifold $SU(n | M, M)/SU(n | M - N, M)$. Leaving the mathematical details of calculations aside we present here only the answer. Let us introduce a family of maps $\sigma : \{1,2,\ldots,N\} \to \{1,\ldots,M\}$ of the indices of the original color space into the subset of indices of the positive root part of the flavor superspace. Correspondingly $\tilde{\sigma}$ will denote the complement part of the $\sigma$: $\tilde{\sigma} = \{1,\ldots,2M\} \setminus \{\sigma(1),\ldots,\sigma(N)\}$. Then the answer is (see Appendix A for details)
\[
\int_{U \in S} dU \exp\{-Str(\Omega U^{-1}VU)\} = (2\pi)^{2MN-N(N-1)/2} \sum_{\{\sigma\}} \det[\exp(-\omega_{\sigma(p)}^2\mu_{\sigma(p)})] \Delta_{\sigma}(V) \Delta_{\tilde{\sigma}}(V)
\] (8)
where $\alpha, \beta = 1, \ldots, N$, $k, p = 1, \ldots, N$ are fermionic and bosonic components correspondingly, the sum is taken over all possible maps $\sigma$ and $\Delta_{\sigma}(\omega)$, $\Delta_{\tilde{\sigma}}(V)$ are defined by the induced metric as
\[
\Delta_{\sigma}(\nu) = \frac{\prod_{i=1}^{N} \prod_{p=1}^{2M-N} (\nu_{\sigma(i)} - \nu_{\sigma(p)})(\nu_{\sigma(i)} - \nu_{\sigma(j)}))}{\prod_{\alpha=1}^{n} \prod_{i=1}^{N} (\mu_{\alpha} - \nu_{\sigma(i)}).}
\] (9)
The formulae (8) and (9) present the extension of IZ matrix integral for the case of integration over Stiefel manifold $S$. It is clear, that the expressions (8), (9) were obtained for integer values of $n$. Now taking the limit $\omega_i \to 0$ in (8) we will reproduce the same formula (3) for replica partition function as it was obtained in original color formulation. This demonstrates the equivalence of the color and flavor formulations of the model under consideration.

III. TODA LATTICE EQUATIONS

In the present section we derive Toda lattice equations in both, color and flavor spaces respectively. The first part is devoted to the so called $\tau$ function, defined for a unitary ensemble with any given distribution function, and TLE for it in color space with initial conditions corresponding to the RMT Eq. (11). In the second part we show how the fermionic and bosonic flavors involved in the replica partition functions $Z_{n,N}$, connect them with integrable hierarchies.
A. TLE in color space

It is easy to see, that the replica partition function (2) is the $T \to 0$ limit of so called $\tau$-function $\tau_N(T, \epsilon) = \int dH \exp TR[H + \tilde{V}(H)]$ of the model with the potential $V(H) = n \ln(\epsilon - H) + \tilde{V}(H)$, which can be regarded also as a matrix Fourier transform. $N \times N$ matrix $T$, being conjugate variable to energy matrix $H$, can be regarded as an effective time. The $\tau$-function can be calculated exactly as 27

$$\tau_N(T; \epsilon) = \frac{1}{\Delta(T)} \det[W_\delta(t_j)]_{\delta,j=0,\ldots,N-1},$$

(10)

where $\Delta(T)$ is the Vandermonde determinant corresponding to the matrix $T$ and

$$W_\delta(t) = \int d\lambda \delta^\tau \exp{\lambda t + \tilde{V}(\lambda)}$$

(11)

with $\delta = 0, \ldots, N - 1$. In the $t_i \to 0$ limit 10 gives us the replicated partition function $Z_{N,N}(\epsilon) = \tau_N(0, \epsilon)$ and $\tau_N(t, \epsilon) = \tau_N(T, \epsilon) |_{(t_i) = t}$ can be expressed via Hankel determinant

$$\tau_N(t, \epsilon) = \frac{1}{\Delta(t)} \det[\partial_i^j W_\delta(t)].$$

(12)

Then it is clear 27 28, that $\tau_N(t, \epsilon)$ will satisfy the TLE

$$\tau_N(t, \epsilon) \partial^2 \tau_N(t, \epsilon) - (\partial_i \tau_N(t, \epsilon))^2 = \tau_{N+1}(t, \epsilon) \tau_{N-1}(t, \epsilon)$$

with the initial condition

$$\tau_1(0, \epsilon) = 2\pi C(M) \sum_{k=1}^{2M} (\epsilon - ik\epsilon)^1 / \prod_{p=1; p \neq k}^{2M} (\mu_k - \mu_p).$$

(13)

B. TLE in flavor (supersymmetric) space

After some algebraic manipulations the replicated partition function 3 can be rewritten as

$$Z_{n,N}(\epsilon) = \left(\frac{C_M}{V_{2M}(\mu_1 \ldots \mu_{2M})}\right)^N \lim_{\epsilon_n \to \epsilon} \frac{1}{V_n(\epsilon_1 \ldots \epsilon_n)} \sum_{\{\sigma\}} (-1)^{\pi(\sigma)} V_{N+n}(\epsilon_1 \ldots \epsilon_n, \mu_{\sigma(1)} \ldots \mu_{\sigma(N)}) V_{2M-N}(\mu_{\bar{\sigma}(1)} \ldots \mu_{\bar{\sigma}(2M-N)})$$

(14)

where $V(\{\ldots\})$ is the Vandermonde determinant defined by the set of arguments $\{\ldots\}$ and $\pi(\sigma)$ is the parity of permutation of $1, \ldots, N, N + 1, \ldots, M$ to $\sigma(1), \ldots, \sigma(N), \bar{\sigma}(1), \ldots, \bar{\sigma}(M - N)$. This form of $Z_{n,N}(\epsilon)$ can be understood as a formula for a determinant calculated by minors

$$Z_{n,N}(\epsilon) = \frac{(i^{-n} C_M)^N}{V_{2M}(\mu_1 \ldots \mu_{2M})} \lim_{\epsilon_n \to \epsilon} I^{-1} \det[A]$$

(15)

where we have introduced the $(n + 2M) \times (n + 2M)$ matrix $A$ as

$$A = \begin{pmatrix}
I_1 & \delta_1 I_1 & \ldots & \delta_1^{n+N-1} I_1 & 0 & \ldots & 0 \\
I_2 & \delta_2 I_2 & \ldots & \delta_2^{n+N-1} I_2 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
I_n & \delta_n I_n & \ldots & \delta_n^{n+N-1} I_n & 0 & \ldots & 0 \\
I_1 & \delta_1 I_1 & \ldots & \delta_1^{n+N-1} I_1 & I_1 & \ldots & \delta_1^{2M-N-1} I_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
I_M & \delta_M I_M & \ldots & \delta_M^{n+N-1} I_M & I_M & \ldots & \delta_M^{2M-N-1} I_M \\
0 & 0 & \ldots & 0 & I_{M+1} & \ldots & \delta_{M+1}^{2M-N-1} I_{M+1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & I_{2M} & \ldots & \delta_{2M}^{2M-N-1} I_{2M}
\end{pmatrix},$$

(16)
involving the functions $I_\alpha = \exp(\text{i} \alpha_\gamma)$, $\alpha = 1, \ldots, n$; $\tilde{I}_r = \exp \mu_r$, $r = 1, \ldots, 2M$; $I = \prod_{\alpha,r=1}^{n,2M} I_\alpha \tilde{I}_r$ and the notations $\delta_\alpha = \epsilon_\alpha \frac{\partial}{\partial \mu_\alpha}$; $\tilde{\delta}_r = \mu_r \frac{\partial}{\partial \mu_r}$. Define $\delta^{1+2M} = \epsilon \frac{\partial}{\partial \epsilon} + \sum_{r=1}^{2M} (\tilde{\delta}_r)$.

Now, following the technique developed in [28], the modified graded Toda lattice equation for replicated partition function $Z_{n,N}(\epsilon)$ will follow directly from its expression [15] as a determinant and two sets of Sylvester identity [28] for the matrix $A$ (see Appendix B for a details). As a result, the following equation is established

$$n - i \partial_\epsilon \delta^{1+2M} \ln[Z_{n,N}(\epsilon)] = \frac{n}{Z_{n+1,N}(\epsilon) Z_{n-1,N}(\epsilon)} + \frac{n}{Z_{n+1,N-1}(\epsilon) Z_{n-1,N+1}(\epsilon)} Z_{n,N}(\epsilon),$$

where $n < 2M - N$, when $Z_{n,N}$ is finite. It has a graded form since contains derivatives over both, fermionic and bosonic parameters. One can see here, that in the replica limit $n \to 0$ the right hand side (RHS) of this equation involves a factorized product of fermionic and bosonic partition functions $Z_{n=1}$ and $Z_{n=-1}$ correspondingly. Moreover, the second term in RHS shows that indeed fermionic, bosonic and supersymmetric partition functions for any $N$ belongs to the same integrable hierarchy.

In the limit $N \to \infty$ (but $N < M$), when $Z_{n,N-1} = Z_{n,N+1} = Z_{n,\infty}$, the equation (17) simplifies to

$$n - i \partial_\epsilon \delta^{1+\infty} \ln[Z_{n,\infty}(\epsilon)] = 2n Z_{n+1,\infty}(\epsilon) Z_{n-1,\infty}(\epsilon),$$

where $\delta^{1+\infty} = \epsilon \frac{\partial}{\partial \epsilon} + \sum_{r=1}^{\infty} (\tilde{\delta}_r)$.

Similar to (15) type of equations, but for a finite amount of mass parameters $\mu$ were considered earlier in [20, 21, 29, 30]. The factorization properties of Toda lattice equations in the replica limit have been analyzed in [20].

As we see the equation (17) differs essentially from the known equation (18), which defines integrable Toda lattice model [3]. Since the equation (17) appears for the exactly calculable partition function (3) of the unitary matrix model with potential $\mathcal{V}$, one can conjecture the existence of corresponding integrable lattice chain model. It is definitely interesting to investigate the interplay between color and flavor degrees of freedom in the equation (17) further and analyze its consequences.

IV. DENSITY OF STATES, MEAN CONDUCTANCE AND TWO POINT CORRELATION FUNCTION

In the original articles [10, 16] where the model with the potential $\mathcal{V}$ was initiated, the authors, by use of $q$-Hermite orthogonal polynomials, have analytically calculated the $N \to \infty$ asymptotic of the two point correlation function. For the appropriately renormalized two point kernel in the bulk of the spectrum exact expression approximately becomes (except very close to the origin)

$$K(\xi, \eta) \approx \frac{\beta}{2 \pi} \sin[\pi(\xi - \eta)] \sinh[(\xi - \eta)\beta/2],$$

which is translational invariant. In its $\beta \to 0$ limit $K(\xi, \eta)$ reproduces the universal behavior of Wigner-Dyson RMT, while for the large $\beta$, as authors of [29] have argued, the model approaches to the uncorrelated systems with Poisson distribution.

In our approach the calculation of the density of states $\rho(x)$ will be reduced to Cauchy integration over poles of the distribution function in the

$$\rho(x) = \frac{1}{Z_{0,N}(\epsilon)} \int_{-\infty}^{\infty} dH \text{tr}(x - H) P(H) = \frac{N(C_M)^N}{Z_{0,N}(\epsilon)} \prod_{i=2}^{N} dx_i \prod_{i,j=1}^{N} (x_i - x_j) \prod_{k,j=1}^{M,N} \frac{1}{(\mu_k + i x_j)(\mu_k - i x_j)},$$

where the normalization factor $Z_{0,N}(\epsilon)$ is defined by the expression (3), for the case $n = 0$. In order to analyze the large $N$ limit we will consider $N = M \to \infty$. In this case the calculation of the Cauchy integrals over poles in our problem essentially simplifies. From (3) one can deduce

$$Z_{0,M}(\epsilon) = (2\pi)^M (C_M)^M M! \prod_{k,p=1}^{M} \frac{1}{(\mu_k + \mu_p)}. \tag{21}$$

Now, in order to take the Cauchy integral over poles in (20) we close the integration contours at infinity of upper half plane and notice, that because of presence of Vandermonde determinant, the integration variables $x_i$, $i = 2, \ldots, M$
FIG. 1: $\chi$ eigenvalue densities at $\beta = 6$ (dashed line) and $\beta = 20$ (solid line). The connection with transmission eigenvalues of $T$ is given by $T = \cosh[\beta \chi/2]^{-2}$.

will utilize $M - 1$ poles, all different. Only one pole will be left untouched and we will have the summation over it in the result:

$$
\rho(x) = \frac{1}{2\pi} \sum_{k=1}^{M} \frac{2\mu_k}{x^2 + \mu_k^2} \prod_{q \neq k}^{M} \frac{x - i\mu_q}{x + i\mu_q} (\mu_k + \mu_q) = \frac{1}{2\pi} \sum_{k=1}^{M} \frac{\mu_k}{x^2 + \mu_k^2}.
$$

Constituents of this formula have the following meaning. The factor $(x^2 + \mu_k^2) \prod_{q \neq k}^{M} (x + i\mu_q)$ in the denominator comes from the distribution function of the remnant eigenvalue $x$, the part of which, namely $\prod_{q \neq k}^{M} (x - i\mu_q)$, was cancelled with the similar bracket in the Vandermonde determinant. The factor $\prod_{q \neq k}^{M} (x - i\mu_q)$ in the nominator is the remnant from the Vandermonde determinant. The factor $\prod_{q \neq k}^{M} (\mu_k + \mu_q)$ is left after mutual cancellations between the residues of integrated poles, the Vandermonde determinant and the normalization factor $Z_{0,N}(x)$ defined by formula (21). The second part of the formula (22) is an identity.

Hence, we have obtained a simple formula:

$$
\rho(x) = \frac{1}{\pi} \sum_{k=1}^{M} \text{Im} \left( \frac{1}{x - i\mu_k} \right).
$$

One will arrive at the same formula (23) taking a replica limit for the one point Green’s function $G(x) = \lim_{n \to 0} n^{-1} \partial Z_{n,M}(x)/\partial x$ and calculating the DOS.

The model under consideration with potential (1) first was considered in [10], in order to describe the disordered conductors connecting matrices $H$ with the ensemble of random transfer matrices $T$ as

$$
H^2 = \frac{1}{4} (TT^* + T^*T - 2I).
$$

Therefore substituting $x = \sinh[\beta \chi/2]$ ($e^{\beta \chi/2}$ is the eigenvalue of $T$) into the $\rho(x)$ one will obtain an exact expression for the density of $\chi$:

$$
\rho(\chi) = \frac{dx}{d\chi} \rho(x).
$$

which exhibits crystallization at the integer values $n$, as it is presented on Fig.1 for two values of $\beta$. This phenomenon for different ensembles was observed earlier in the articles [15,22,31].

The eigenvalues $x_i$ of $H$ and $\chi_i$ are connected with the conductance $g = tr(t^+ t)/2$ ($t$ is the transmission matrix) as

$$
g = \sum_{i=1}^{N} \frac{1}{1 + x_i^2} = \sum_{i=1}^{N} \frac{1}{\cosh[\beta \chi_i/2]^2}
$$

and by the same type of Cauchy integration over poles as in $\rho(x)$ one can find the average conductance

$$
\langle g \rangle = \sum_{n=1}^{M} \frac{1}{1 + \mu_n^2} = \sum_{n=1}^{M} \frac{1}{1 + \cosh[n\beta/2]^2}.
$$
FIG. 2: The two point correlation function $R_2[\xi - \eta]$ calculated exactly by Cauchy integration over poles and after unfolding with $\rho(x)$ (dots). The solid line represents $R_2[\xi - \eta]$ obtained from the formula (19), both at $\beta = 0.3$.

In the same way one can calculate the two point correlation function $R_2[x, y]$, again in the limit $N = M \to \infty$

$$R_2(x, y) = \frac{1}{Z_{0,N}(\epsilon)} \int_{-\infty}^{\infty} dH \text{tr} \delta(x - H) \text{tr} \delta(y - H) P(H). \quad (28)$$

By Cauchy integration over poles, in this case, we should now leave untouched two poles of the distribution function $P(H)$ (see formula (20)). Therefore, in the result we will have a sum over all possible positions $m$ and $n$ of that poles. The answer reads:

$$R_2(x, y) = \frac{1}{\pi^2} \sum_{m,n} \frac{\mu_{m} \mu_{n}}{m^{2} + \mu_{n}^{2}} \frac{\mu_{m} \mu_{n}}{m^{2} + \mu_{m}^{2}} \prod_{k \neq m, n} \frac{(x - i\mu_{k})(\mu_{m} + i\mu_{k})(y - i\mu_{k})(\mu_{n} + i\mu_{k})}{(x + i\mu_{k})(\mu_{m} - i\mu_{k})(y + i\mu_{k})(\mu_{n} - i\mu_{k})}. \quad (29)$$

Details of the calculation of $R_2(x, y)$ repeat the details of the one for the density of states $\rho(x)$, presented above. Since here we hold two eigenvalues $x$ and $y$ untouched, the formula (29) is similar (but not equal) to the product of two structures (22) of $\rho(x)$ with corresponding adjustment of the summation procedure. Unlike to the case of $\rho(x)$, I was not able to find a simplification of the final formula (29), as it was done in the second part of Eq.(22).

In spite of explicit presence of the imaginary unite $i$ in the expression (29) the resultant $R_2(x, y)$ is real. After unfolding, namely after the re-scaling $x \to \xi$, defined by $d\xi/dx = \rho(x)$, $R_2(\xi, \eta)$ is pictured in Fig.2 by dots for $\beta = 0.3$. Moreover, the numerical analyze of the expression (29) after unfolding shows that $R_2(\xi, \eta)$ in fact is a function of difference $\xi - \eta$.

Numerical analyses show, that consideration of other potentials, say by taking the product in formula (2) from $k = 0$ or $k = 2$, after appropriate unfolding gives us the same behavior as approximately defined by formula (19).

Conclusions

We give a sigma-model formulation of the RMT with confining potential (11). The replica partition function of the model has been calculated exactly in both, original color space and in flavor space (after a color-flavor transformation) with the same result. In both spaces we have established connections between the RMT Eq.(11) and exactly solvable lattice equations. A generalization of graded TLE is defined and solved.

We have exactly calculated the probability density and the two point correlation function of the model at $N \to \infty$ and confirmed the known results obtained earlier. The effect of crystallization of transmission eigenvalues has been observed.

Acknowledgments

I am greatly thankful to V. E. Kravtsov for numerous stimulating discussions and critical reading of the paper. I
Appendix A. Evaluation of Supersymmetric Itzikson-Zuber Integral over \( SU(n \mid M, M)/SU(n \mid M - N, M) \)

In this appendix we present a simple derivation of the supersymmetric extension of the Itzikson-Zuber formula over Stiefel manifold \( SU(n \mid M, M)/SU(n \mid M - N, M) \).

\[
\mathcal{I}(X, Y; r) = \int_{U \in SU(n \mid M, M)/SU(n \mid M - N, M)} dU \exp\{r \text{Str}(XU^{-1}YU)\} \tag{30}
\]

In this calculation one may follow the standard technique developed in \(^{25, 26}\) for unitary supergroups based on the solution of the heat kernel equation. In \(^{25}\) it was argued, that the heat kernel equation technique can be applied to pseudounitary supergroups. However we will present here the calculations based on the application of Duistermaat-Heckman theorem to integrals over supermanifolds, which states that under certain conditions the quasi-classical saddle-point approximation is exact. This theorem was successfully applied to IZ integral over unitary supergroups \( U(n \mid m) \) in \(^{25}\) and pseudounitary supergroups \( U(n \mid m, k) \) in \(^{33}\). In what follows we will assume, that this criteria are met by our integral \(^{25}\) as well and evaluate the integral in the saddle-point approximation. The fact, that final result reproduces the same answer \(^{34}\) for \( Z_{n,N}(\epsilon) \), as we got by simple calculation of the integral \(^{33}\) by the Cauchy formula confirms, that this assumption is correct. The same answer one will obtain following the standard technique \(^{25, 26}\).

We need to find the extremum of the function

\[
\mathcal{A}(U) = \text{Str}(XU^{-1}YU),
\]

where \( U \in SU(n \mid M, M)/SU(n \mid M - N, M) \) and \( X \) and \( Y \) are diagonal matrices from the expression \(^{9}\). It is easy to find the saddle-point equation

\[
[X, U^{-1}YU] = 0 \tag{32}
\]

from the condition \( \delta \mathcal{A}(U) = \text{Str}\{XU^{-1}YU - U^{-1}YUXU^{-1}\delta U\} = 0 \).

Since we have \( X = \text{diag}\{0 \ldots 0, x_1 \ldots x_N, 0 \ldots 0\} \) and \( U \in SU(n \mid M, M)/SU(n \mid M - N, M) \) the only solutions \( U \) of \(^{32}\) (up to irrelevant \( U \in SU(n \mid M - N, M) \)) are those matrices, which permute the diagonal elements of the matrix \( Y \) in three-fermionic, positive and negative metric parts of the bosonic sectors separately. In other words the set of solutions of \(^{32}\) is

\[
U^{(0)}_{\alpha \beta} = \begin{pmatrix} 0 & 0 \\ 0 & \prod_{k,p} \prod_{k',p'} & 0 \end{pmatrix}, \quad \begin{array}{ll} \alpha, \beta = 1, \ldots, n \\ k, p = 1, \ldots, M \\ k', p' = 1, \ldots, M \end{array} \tag{33}
\]

where \( \prod_{\alpha \beta} \) and \( \prod_{k,p}(\prod_{k',p'}) \) are the elements of the permutation groups \( S_n \) and \( S_M \) of the fermionic and bosonic sectors respectively. Notice, that the set of permutation elements \( \prod_{ij}, i, j = 1, \ldots, N \) coincides with the set of maps \( \sigma : \{1, \ldots, N\} \rightarrow \{1, \ldots, M\} \) defined earlier.

We now set \( U = U^{(0)}e^{\Lambda L} \) in \(^{34}\), with an infinitesimal Hermitian supermatrix \( L = U^{-1}\delta U \in su(n \mid M, M) \setminus su(n \mid M - N, M) \) and expand the function \( \mathcal{A}(U) \) up to the quadratic order in \( L \). In this factor space \( L \) has only following nonzero matrix elements \( L_{ij}, i, j = 1, \ldots, N; L_{ik}, i = 1, \ldots, N; k = 1, \ldots, 2M - N \) and \( L_{\alpha \beta}, \alpha = 1, \ldots, n \).

In the second order over \( L, \mathcal{A}(U) \) is

\[
\mathcal{A}(U) = \text{Str}XU^{(0)}YU^{(0)} + \text{Str}XLU^{(0)}YU^{(0)}L - \text{Str}\frac{1}{2}[X, U^{(0)}YU^{(0)}]L^2
= \text{Str}XU^{(0)}YU^{(0)} + \frac{1}{2}\text{Str}[X, L][U^{(0)}YU^{(0)}, L]. \tag{34}
\]

Due to Duistermaat-Heckman theorem in evaluation of IZ integral \(^{30}\) we should take the sum over all extrema, therefore substituting \(^{33}\) to \(^{30}\) we obtain

\[
\text{Str}XU^{(0)}YU^{(0)} + \frac{1}{2}\text{Str}[X, L][U^{(0)}YU^{(0)}, L].
\]
$$I(X,Y; r) = \sum_{\sigma \in S_{2M}} \exp(r \sum_{i=1}^{N} x_i \gamma_{\sigma(i)}) \int \prod_{i,j=1}^{N} dL_{ij} \prod_{i,k=1}^{N} dL_{ik} \prod_{i,\alpha=1}^{N,n} dL_{i\alpha}$$

$$\times \exp r \left[ \frac{1}{2} \sum_{i,j=1}^{N} |L_{ij}|^2 (x_i - x_j)(y_{\sigma(i)} - y_{\sigma(j)}) \right] + \frac{1}{2} \sum_{i,k=1}^{N,2M-N} |L_{ik}|^2 x_i(y_{\sigma(i)} - y_{\sigma(k)})$$

$$+ \sum_{\iota,\alpha=1}^{N,n} |L_{i\alpha}|^2 x_i(y_{\sigma(i)} - y_{\alpha}) \right],$$

(35)

where \(\bar{y}_\alpha\) are fermionic components of the diagonal matrix \(Y\) and \(\bar{\sigma}\) is the complement to \(\sigma\) permutation, defined after the formula (7). Here we essentially have used \(\text{diag}(0, \ldots, 0, x_1 \ldots x_N, 0 \ldots 0)\) form of the matrix \(X\).

Evaluating now the Gaussian integrals in (35) over complex bosonic \(L_{ij}, i \neq j\); \(L_{i,k}, i \neq k\) and the complex Grassmann variables \(L_{i\alpha}\), we arrive at

$$I(X,Y; r) = (2\pi)^{2MN-N(N-1)/2} (-r)^{Nn-2MN+(N-1)N/2}$$

$$\times \sum_{\{\sigma\}} \det[\exp r x_i y_{\sigma(j)}] \Delta_{\sigma}(X) \Delta_{\sigma}(Y)$$

(36)

with

$$\Delta_{\sigma}(\nu) = \frac{\prod_{i=1}^{N} \prod_{p=1}^{2M-N} (\nu_{\sigma(i)} - \nu_{\sigma(p)}) \prod_{\alpha=1}^{n} \prod_{i=1}^{n} (\nu_{\alpha} - \nu_{\sigma(i)})}{\prod_{i=1}^{n} \prod_{i,j=1}^{2M-N} (\nu_{\sigma(i)} - \nu_{\sigma(j)})} \nu = X, Y.$$  

(37)

**Appendix B. Graded Toda Lattice Equation**

In this Appendix we show how the graded Toda lattice equation for the replica partition function \(Z_{n,N}(\epsilon)\) appears. First we would like to demonstrate, that \(Z_{n,N}(\epsilon)\) is a ratio of certain determinants. The expression (3) can be regarded as \(\epsilon_\alpha \to \epsilon; \alpha = 1, \ldots n\) limit of \(n\) different fermionic eigenvalues. For the convenience and compactness of formulas we will drop from further transformations the constant factor \((2\pi)^N(C_M)^{N^M}N^{nN}\) and introduce the notation \(\epsilon_\alpha = -i\epsilon_\alpha\) for a while. Also, instead of fixed \(\mu_1 \ldots \mu_{2M}, -\mu_1, \ldots -\mu_{2M}\) set of parameters of the model, we consider here general situation with different \(\mu_k, k = 1, \ldots 2M\). Condition \(\mu_{k+M} = -\mu_k\) can be easily imposed at the end. We have

$$Z_{n,N}(\epsilon) = \lim_{\{\epsilon_\alpha\} \to \epsilon} \sum_{\{\sigma\}} \prod_{\alpha=1}^{n} \prod_{i=1}^{2M-N} (\epsilon_{\alpha} - \mu_{\sigma(i)} - \mu_{\sigma(j)})$$

$$\times \prod_{\iota < \iota = 1}^{N} (\mu_{\sigma(i)} - \mu_{\sigma(j)}) \prod_{i,k=p=1}^{2M-N} (\mu_{\sigma(k)} - \mu_{\sigma(p)})$$

(38)

where the nominator and denominator have been multiplied by the same expressions. Introduce now a bosonic Vandermonde determinant:

$$V_s(\{x_1, \ldots x_s\}) = \prod_{i<j=1}^{s} (x_i - x_j) = \text{Det} \begin{bmatrix} 1 & x_1 & \ldots & x_{s-1} \\ 1 & x_2 & \ldots & x_{s-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_s & \ldots & x_{s-1} \end{bmatrix}$$

(39)
The expression \( \frac{\partial}{\partial e} \) for the partition function can be rewritten as
\[
Z_{n,N}(e) = \lim_{\{e_{\alpha}\} \to \{e_{\alpha}\}} \sum_{\{\sigma\}} (-1)^P \frac{\mathcal{V}_{n+N}(\{\theta_1, \ldots, \theta_{n+N}\}) \mathcal{V}_{2M-N}(\{\mu_{\sigma(1)}, \ldots, \mu_{\sigma(2M-N)}\})}{\mathcal{V}_n(\{e_1, \ldots, e_n\}) \mathcal{V}_{2M}(\{\mu_1, \ldots, \mu_{2M}\})},
\]
where \( \{\theta_1, \ldots, \theta_{n+N}\} = \{e_1, \ldots, e_n, \mu_{\sigma(1)}, \ldots, \mu_{\sigma(N)}\} \) and \((-1)^P\) is the parity of permutation
\[
P : \{e_1, \ldots, e_n, \mu_1, \ldots, \mu_{2M}\} \to \{e_1, \ldots, e_n, \mu_{\sigma(1)}, \ldots, \mu_{\sigma(N)}, \mu_{\sigma(1)}, \ldots, \mu_{\sigma(2M-N)}\}.
\]
The denominator in the expression \((40)\) does not depend on \(\sigma\) and can be taken out of summation, while \(\sum\) of nominators is nothing but the determinant of the matrix \(A_0:\)
\[
A_0 = \begin{pmatrix}
1 & e_1 & e_1^2 & \ldots & e_1^{n+N-1} & 0 & 0 & \ldots & 0 \\
1 & e_2 & e_2^2 & \ldots & e_2^{n+N-1} & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & e_n & e_n^2 & \ldots & e_n^{n+N-1} & 0 & 0 & \ldots & 0 \\
1 & \mu_1 & \mu_1^2 & \ldots & \mu_1^{n+N-1} & 1 & \mu_1 & \ldots & \mu_1^{2M-N-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \mu_M & \mu_M^2 & \ldots & \mu_M^{n+N-1} & 1 & \mu_M & \ldots & \mu_M^{2M-N-1} \\
0 & 0 & 0 & \ldots & 0 & 1 & \mu_{M+1} & \ldots & \mu_{M+1}^{2M-N-1} \\
0 & 0 & 0 & \ldots & 0 & 1 & \mu_{2M} & \ldots & \mu_{2M}^{2M-N-1}
\end{pmatrix},
\]
calculated by \(N + n\) and \(2M - N\) dimensional minors and by use of the formula \(\text{(39)}\).

Now let us consider the operator \(e^{-x}(\delta_e)^P e^x = x^P + \mathcal{P}^{(P-1)}(x)\)
\[
e^{-x}(\delta_e)^p e^x = x^p + \mathcal{P}^{(P-1)}(x)
\]
where \(\mathcal{P}^{(P-1)}(x)\) is the polynomial of \(p - 1\) order of \(x\). From this it follows, that if we replace the monomials \(x^p\) in the matrix \(A_0\) with \(e^{-x}(\delta_e)^p e^x\), we will obtain the matrix \(A\) defined in \(\text{(16)}\). Their determinants relate as
\[
det[A_0] = det[I^{-1}A]
\]
and define the nominators in the expressions \(\text{(15)}\) and \(\text{(40)}\) for the partition function \(Z_{n,N}(e)\). The notation \(I = \exp(\sum_{\alpha=1}^{n} e_{\alpha} + \sum_{k=1}^{2M} \mu_k)\) is defined just after formula \(\text{(15)}\).

The next step is the consideration of the degenerate limit \(e_{\alpha} \to e; \alpha = 1, \ldots, n\). We put \(e_1 = e\) and consider Taylor expansions \(f(e_{\alpha}) = f(e) + (e_{\alpha} - e) \partial_e f(e) + \ldots + \frac{(e_{\alpha} - e)^{n-1}}{(n-1)!} \partial_e^{n-1} f(e) + \ldots \) around \(e\) of all entities in the upper-left \(n \times n\) corner of the matrix \(A\) in \(\text{(15)}\). It is easy to see that in the limit \(e_\alpha \to e; \alpha = 1, \ldots, n\), higher than \(n - 1\) orders of the expansions will not contribute into the matrix \(A\), while the coefficients \((e_{\alpha} - e)^{k}\) will form a Vandermonde determinant \(\text{(39)}\), which will cancel the same \(\mathcal{V}_n(\{e_1, \ldots, e_n\})\) in the denominator of \(Z_{n,N}(e)\) in the formula \(\text{(40)}\).

Therefore we obtain
\[
Z_{n,N}(e) = \frac{(2\pi)^N (C_M)^N r_{n+N}}{\mathcal{V}_{2M}(\{\mu_1, \ldots, \mu_{2M}\})} I^{-1} \det[B_{n,N}],
\]
with \(I = \exp(ne + \sum_{\alpha} \mu_\alpha)\) and \(B_{n,N}\) equal to
\[
\begin{pmatrix}
I_1 & \delta_1 I_1 & \ldots & \delta_1^{n+N-1} I_1 & 0 & \ldots & 0 \\
\frac{1}{(n-1)!} \partial_e^{n-1} I_1 & \frac{1}{(n-1)!} \partial_e \delta_1 I_1 & \ldots & \frac{1}{(n-1)!} \partial_e \delta_1^{n+N-1} I_1 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{(n-1)!} \partial_e^{n-1} \delta_1 I_1 & \frac{1}{(n-1)!} \partial_e \delta_1 \delta_1 I_1 & \ldots & \frac{1}{(n-1)!} \partial_e \delta_1^{n+N-1} I_1 & 0 & \ldots & 0 \\
\frac{1}{(n-1)!} \partial_e^{n-1} \delta_1 \delta_1 I_1 & \frac{1}{(n-1)!} \partial_e \delta_1 \delta_1 \delta_1 I_1 & \ldots & \frac{1}{(n-1)!} \partial_e \delta_1 \delta_1^{n+N-1} I_1 & \frac{1}{(n-1)!} \partial_e \delta_1^{2M-N-1} I_1 & \ldots & \frac{1}{(n-1)!} \partial_e \delta_1 I_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{(n-1)!} \partial_e^{n-1} \delta_1 \delta_1 \delta_1 I_1 & \frac{1}{(n-1)!} \partial_e \delta_1 \delta_1 \delta_1 \delta_1 I_1 & \ldots & \frac{1}{(n-1)!} \partial_e \delta_1 \delta_1 \delta_1^{n+N-1} I_1 & \frac{1}{(n-1)!} \partial_e \delta_1 \delta_1^{2M-N-1} I_1 & \ldots & \frac{1}{(n-1)!} \partial_e \delta_1 \delta_1 I_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{(n-1)!} \partial_e^{n-1} \delta_1 \delta_1 \delta_1 \delta_1 I_1 & \frac{1}{(n-1)!} \partial_e \delta_1 \delta_1 \delta_1 \delta_1 \delta_1 I_1 & \ldots & \frac{1}{(n-1)!} \partial_e \delta_1 \delta_1 \delta_1 \delta_1^{n+N-1} I_1 & \frac{1}{(n-1)!} \partial_e \delta_1 \delta_1 \delta_1^{2M-N-1} I_1 & \ldots & \frac{1}{(n-1)!} \partial_e \delta_1 \delta_1 \delta_1 I_1 \\
0 & 0 & \ldots & 0 & 1 & \mu_{2M} & \ldots & \mu_{2M}^{2M-N-1} I_1 \\
0 & 0 & \ldots & 0 & 1 & \mu_{M+1} & \ldots & \mu_{M+1}^{2M-N-1} I_1 \\
0 & 0 & \ldots & 0 & 1 & \mu_{2M} & \ldots & \mu_{2M}^{2M-N-1} I_1 \\
\end{pmatrix}.
\]
In this expression $I_1 = \exp(e)$ and $\delta_1 = \epsilon D$. To derive the Toda lattice equation \cite{17}, we will follow the technique presented in the article \cite{21} in connection with QCD partition function. The method is based on use of the Sylvester identity \cite{28}. For the determinant of any given matrix $D$

$$C_{i,j}C_{p,q} - C_{i,q}C_{p,j} = \det[D]C_{i,j;p,q}, \quad (57)$$

where $C_{i,j}$ is the cofactor of the matrix element $ij$:

$$C_{i,j} = \frac{\partial \det[D]}{\partial D_{i,j}}, \quad (58)$$

and $C_{i,j;p,q}$ is the double cofactor of the matrix elements $ij$ and $pq$:

$$C_{i,j;p,q} = \frac{\partial^2 \det[D]}{\partial D_{i,j} \partial D_{p,q}}. \quad (59)$$

Let us consider now the $(2M+n+1) \times (2M+n+1)$ dimensional matrices $B_{n+1,N}$ and $B_{n+1,N-1}$ and write Sylvester identities \cite{17} for different set of indices $i, j; p, q$ for each of them.

For the $B_{n+1,N}$ we take $i = n + 1$, $j = n + N + 1$, $p = n$, $q = n + N$ and obtain

$$C_{n+1,n+N+1}C_{n,n+N} - C_{n,n+N+1}C_{n+1,n+N} = \det[B_{n+1,N}]C_{n+1,n+N+1;n,n+N}. \quad (60)$$

From analogous to \cite{10} structure of $B_{n+1,N}$ follows that

$$C_{n+1,n+N+1} = \det[B_{n,N}], \quad C_{n,n+N+1} = \frac{1}{n}\partial_e \det[B_{n,N}],$$

$$C_{n+1,n+N+1;n,n+N} = \det[B_{n-1,N}],$$

$$C_{n,n+N} = \frac{1}{n}\partial_e \det[B'_{n,N}], \quad C_{n+1,n+N} = \det[B'_{n,N}], \quad (61)$$

where the matrix $B'_{n,N}$ is defined by acting by the operator $\delta^{1+2M} = e\frac{\partial}{\partial e} + \sum_{r=1}^{2M} \mu_r \frac{\partial}{\partial \mu_r}$ on the $(n+N)$-th column of the matrix $B_{n,N}$.

Now consider the Sylvester identity for the matrix $B_{n+1,N-1}$. Setting $i = n + 1$, $j = n + 2M + 1$, $p = n$, $q = n + 2M$, we will have

$$C'_{n+1,n+2M+1}C'_{n,n+2M} - C'_{n,n+2M+1}C'_{n+1,n+2M} = \det[B_{n+1,N-1}]C'_{n+1,n+2M+1;n,n+2M}. \quad (62)$$

As in the previous case and from the structure of \cite{10} for $B_{n+1,N-1}$ it follows that

$$C'_{n+1,n+2M+1} = \det[B_{n,N}], \quad C'_{n,n+2M+1} = \frac{1}{n}\partial_e \det[B_{n,N}],$$

$$C'_{n+1,n+2M+1;n,n+N} = \det[B_{n-1,N+1}],$$

$$C'_{n,n+N} = \frac{1}{n}\partial_e \det[B''_{n,N}], \quad C'_{n+1,n+N} = \det[B''_{n,N}], \quad (63)$$

where the matrix $B''_{n,N}$ is defined by acting by the operator $\delta^{1+2M}$ on the $(n + 2M)$-th column of the matrix $B_{n,N}$.

It is easy to observe, that

$$\det[B'_{n,N}] + \det[B''_{n,N}] = \delta^{1+2M}\det[B_{n,N}]. \quad (64)$$

Inserting now expressions \cite{57} and \cite{58} into the Sylvester identities \cite{50} and \cite{52} respectively, taking their sum and using \cite{54} we get

$$\det[B_{n,N}]^2 \frac{\partial}{\partial e} \delta^{1+2M} \ln[\det[B_{n,N}]] = \quad (65)$$

$$= n \det[B_{n+1,N}]\det[B_{n-1,N}] + n \det[B_{n+1,N-1}]\det[B_{n-1,N+1}].$$
Finally, using the relation \[ \frac{\partial e}{\partial e} \delta_{1+2M} \ln[Z_{n,N}(\epsilon)] = nZ_{n+1,N}(\epsilon)Z_{n-1,N}(\epsilon) + nZ_{n+1,N-1}(\epsilon)Z_{n-1,N+1}(\epsilon), \]

which coincides with \[17\] after substitution $e = i\epsilon$. Notice that the Vandermonde determinants, $\nu_{2M}(\{\mu_1, \ldots, \mu_{2M}\})$, do not contribute to the left hand side of the Toda lattice equation since $\delta_{1+2M} \ln[\nu_{2M}(\{\mu_1, \ldots, \mu_{2M}\})]$ is a number, while the contribution of $I = \exp(ne + \sum_{r=1}^{2M} \mu_r)$ gives addition of $n$.

If we take $N = M$, then, since $B_{n-1,N+1} = 0$, the graded Toda equation \[56\] simplifies.