Modeling the impact of the vaccine on the COVID-19 epidemic transmission via fractional derivative

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Abstract  To achieve the goal of ceasing the spread of COVID-19 entirely it is essential to understand the dynamical behavior of the proliferation of the virus at an intense level. Studying this disease simply based on experimental analysis is very time consuming and expensive. Mathematical modeling might play a worthy role in this regard. By incorporating the mathematical frameworks with the available disease data it will be beneficial and economical to understand the key factors involved in the spread of COVID-19. As there are many vaccines available globally at present, henceforth, by including the effect of vaccination into the model will also support to understand the visible influence of the vaccine on the spread of COVID-19 virus. There are several ways to mathematically formulate the effect of disease on the population like deterministic modeling, stochastic modeling or fractional order modeling etc. Fractional order derivative modeling is one of the fundamental methods to understand real-world problems and evaluate accurate situations. In this article, a fractional order epidemic model $Sp Ep Ip Er p Rp Dp Q p Vp$ on the spread of COVID-19 is presented. $Sp Ep Ip Er p Rp Dp Q p Vp$ consists of eight compartments of population namely susceptible, exposed, infective, recovered, the quarantine population, recovered-exposed, and dead population. The fractional order derivative is considered in the Caputo sense. For the prophecy and tenacity of the epidemic, we compute the reproduction number $R_0$. Using fixed point theory, the existence and uniqueness of the solutions of fractional order derivative have been studied. Furthermore, we are using the generalized Adams–Bashforth–Moulton method, to obtain the approximate solution of the fractional-order COVID-19 model. Finally, numerical results and illustrative graphic simulation are given. Our results suggest that to reduce the number of cases of COVID-19 we should reduce the contact rate of the people if the population is not fully vaccinated. However, to tackle the issue of reducing the social distancing and lock down, which have very negative impact on the economy as well as on the mental health of the people, it is much better to increase the vaccine rate and get the whole nation to be fully vaccinated.

1 Introduction

COVID-19, a syndrome which is not very uncommon at present, was declared as a pandemic on January 22, 2020 by WHO. The most significant aspect of this pandemic is its antagonistic way of infecting the people. The elderly people and the people having chronic issues like diabetes, blood pressure or heart problems are the most fatally infected folks by COVID-19. The whole world is suffering from several crises due to COVID-19 outbreak including collapse of medical systems, subsiding of human social interactions, downfall of economy, and most importantly the rise of acute depression in the societies. Many researchers are making great efforts to study and analyze this disease from different aspects to assist in controlling the spread of this outbreak [1–10].

COVID-19 is transmitted to a person through the small droplets of nose or mouth from the infected individual who is sneezing or coughing. Such transmission might have occurred through the direct interaction with the infected person or by touching the objects containing the virus. To control such diffusion of virus many safety measures have been implemented at global level, including social distancing, wearing masks, lockdowns, closing country borders, hygiene and frequent hand sanitizing/washing. It is also advised to infected people to quarantine themselves at least for 14 days after they get their test reports positive.

During this pandemic era, at the moment vaccination has become the only hope to end or lower the intensity of this outbreak of COVID-19 with special care of testing practice and prevention measures. Since the start of this pandemic many pharmaceutical companies took the responsibility to develop the preeminent vaccines against the COVID-19 and this is the result of their efforts that many approved vaccines are available now not only in developed countries but also the developing countries are getting supply of...
bulk of these vaccines from developed countries. Though many vaccines have been introduced till date but still after several months
the full control on the spread of COVID-19 has not been achieved. This scenario motivates us to identify the key factors involved
in the spread of this pandemic which might help to quantify possible control and amenable stages of the infection. To understand
the complete dynamics of the disease, data analysis and mathematical modeling can play an essential role in measuring the effects
of mitigation strategies [11]. Mathematical frameworks constructed on the basis of infectious disease transmission are one of the
best approaches in forecasting, measuring and controlling prevalent outburst [12]. In Tunisia, people are getting vaccinated on a
regular basis; the most common vaccine available in Tunisia is BioNTech, Pfizer vaccine. Although this vaccination campaign has
decreased the intensity of the virus, COVID-19 remains a concern. So, we find it very essential to suggest an epidemic model to
predict the change in the trend of COVID-19 prevalence before and after vaccination campaigns. The forecast through our suggested
model might be helpful to understand the current situation of COVID-19 and to increase the efforts for vaccination campaigns.

Typically, a system of differential equations is used to describe the dispersal of infectious diseases [13]. Susceptible (S), exposed
(E), infected (I) and recovered (R) are the most common compartments of the epidemic models. Such a model is also known as the
SEIR model. This model might be further extended by adding more compartments based on the data availability and the case under
study. In an SEIR model, the system of differential equations describes the rate of change of susceptible (S), exposed (E), infected
(I) and recovered (R) compartments over time [14–17]. Recently, Fanelli et al. introduced a more complex SIRD model with death
(D) class [18] while Ndairou et al. presented a model SEIPAHRF with additional compartments namely new super-spreaders (P),
asymptomatic (A), hospitalized (H), fatality (F) [19].

Annas et al. in [20] developed an SEIR model to portray the infected cases in Indonesia as well as to show vaccine efficacy against
COVID-19 in Indonesia. Medrek and Pastuszak in [21] established a cellular automata (CA) model based on artificial intelligence
techniques for simulating the COVID-19 dispersion. They extended the SEIR framework to assess the epidemic behavior in Poland,
France, and Spain. They estimated new parameters into their model to reveal the true dependencies on age-dependent mortalities and
interaction rates. Berkane et al. proposed a mathematical framework to predict the spread of COVID-19 outbreak under the influence
of vaccine efficacy [22]. They extended the SIR model by adding a class of vaccinated people (V) and named it as SIRV model.
They pointed out that the vaccine is quite safe deprived of any long or short term side-effects. They also suggested that by extending
the current efforts for vaccination would be beneficial to save many lives in CANADA which are at risk at the moment. Ali et al.
[23] considered an SEIR model and found the conditions and Ulam’s type stability for suggested classification by implementing
the tools of nonlinear analysis. They studied the local and global asymptotical stabilities of the epidemic model upon the disease
free, endemic equilibrium and reproductive number and formulated that a good control on virus can be attained by reducing the
transmission rate as well as by increasing the rate of treatment.

The classical calculus of an integer-order is generalized by fractional calculus. In the eighteenth century Liouville, Riemann,
Fourier and Euler struggled to produce substantial results in classical calculus, simultaneously, significant progress was made in the
field of fractional calculus as well. This was due to the numerous applications of fractional calculus in the field of mathematical
modeling, where calculus was not able to clearly describe several inherited materials and memory processes [24]. The accumulated
benefit of fractional differential equation (FDE) to model complex problems in real life, such benefit is due to its different properties
that were not found in integer-order differential equation (IDEs). Unlike IDEs, which are local by default, FDEs are composed of the
memory effects that optimize them. Consequently, in various situations, the future state of the model depends not only on the
present state, but also on earlier history. The dynamics of certain infectious diseases, such as measles dynamics [25], chickenpox
[26], dengue fever [27], HIV [28], rubella disease [29], tuberculosis [30], have recently been analyzed by several researchers as the
fractional model.

Fractional-order derivatives provides more accurate analysis for modeling COVID-19 outbreak using the combination of memory
and transmissible properties, as fractional operators add full degrees of freedom in the system, while infinite memory of fractional
operators also provide multiple advantages as compared to restricted memory of integer models [31]. In [32], authors developed
an SIRD model based on fractional order differential equations to integrate the long and short term memory effect in pandemic
development. The authors extended the model SEIPAHRF introduced in [19] using fractional-order derivative to demonstrate the
significance of assuming the Caputo fractional different, in view of the fact that the fractional order derivative plays an important
role to fit the number of cases reported in the regions of Spain, Portugal and Galicia [33] accurately. A fractional order SIR model
was developed in [34] to study the spread of COVID-19 outbreak using a general incidence rate function and a nonlinear recovery
rate. Khan et al. [35] proposed a novel mathematical model to investigate the impact of various useful public safety measures against
COVID-19 dynamics in the Pakistan. Gao et al. [36] investigated the COVID-19 mathematical model in the Caputo sense. Asamoah
et al. [37] used an optimal control technique to create a mathematical model using the environmental viral load and recommended
some efficient intervention strategies.

Based on the previous analysis on COVID-19 using mathematical frameworks, we are motivated to use the Caputo fractional-
order operators to model and analyze COVID-19 epidemics for disease transmission in this study. One main objective in this
context is to include the vaccine effect in the model proposed by Fredj and Cherif [38]. The advantage of the Caputo derivative
over other fractional derivatives is that it gives natural modeling i.e., the Caputo derivative of a constant is zero. Thus, Caputo
fractional differential equation (FDO) allows for the inclusion of local initial conditions in the model derivation [39]. A fractional
order COVID-19 model using different techniques and analysis was presented in [40] for the proper identification of the individuals
in community of Saudi Arabia to reduce the intensity of the virus. In [41], author proposed a fractional order model to examine
infective compartments: solution of the proposed fractional model. The Adams-Bashforth-Moulton method is applied in Sect. 5 to obtain the general solution of the model. The impact of the vaccine based on the proposed model is given in Sect. 6. We also illustrate our findings using the model having constant population.

2 Generalized mathematical COVID-19 model

The following definitions and notations of fractional differential calculus will be used in the remaining paper.

**Definition 2.1** [42]: The Riemann–Liouville of the continuous fractional integral function $\psi : (0, \infty) \rightarrow \mathbb{R}$ of the order $\nu > 0$ with $t$ is described as

$$D_0^\nu \psi(t) = \frac{1}{\Gamma(n - \nu)} \int_0^t (t - u)^{n-1-\nu} \psi(u)du, \quad t > 0.$$

**Definition 2.2** [43]: The Caputo derivative of fractional order $\nu > 0$ of a function $\psi : (0, \infty) \rightarrow \mathbb{R}$ is expressed as

$$^c D_0^\nu \psi(t) = \frac{1}{\Gamma(n - \nu)} \int_0^t (t - u)^{n-1-\nu} \psi^{(n)}(u)du, \quad n - 1 < \nu < n.$$

**Definition 2.3** [44]: If $\nu > 0$, then gamma function is defined as

$$\Gamma(\nu) = \int_0^\infty x^{\nu-1} e^{-x}dx.$$

For more information on the Caputo derivative and its utility we refer to [45, 46]; other recent general references to fractional differential equations and fractional calculus can be found in [47–49].

The classical model SIR (susceptible-infected-recovered) in epidemiology makes it possible to explain the critical condition of the development of the disease inside the population, independently of the total size of the population for a short period. The simplest ODE system of SIR considered as is following:

$$\frac{dS_p}{dt} = -\frac{\mu}{M}S_pI_p,$$

$$\frac{dI_p}{dt} = -\frac{\mu}{M}S_pI_p - \beta I_p,$$

$$\frac{dR_p}{dt} = \beta I_p.$$

The term $\frac{\mu}{M}S_pI_p$ describes the rate of transmission of disease suitable for interaction between susceptible and infected persons, and $\beta I_p$ indicates the rate of heal of infected person. The recovered individuals are now considered not to enter the susceptible class; i.e., the recovered individuals are resistant to the disease; they can not be re-infected, and susceptible individuals can not be infected.

In [38], authors formulated a mathematical model of COVID-19 based on the compartments susceptible ($S_p$), exposed ($E_p$), infective ($I_p$), recovered-exposed ($Er_p$), recovered ($R_p$), dead population ($D_p$) and the quarantine population ($Q_p$). We have proposed the following generalized fractional-order COVID-19 model by adding the population of the vaccinated people in their model having constant population $M = S_p + E_p + I_p + Er_p + R_p + D_p + Q_p + V_p$, which is divided into eight epidemiological compartments:

\[
\begin{align*}
^c D_0^\nu S_p &= -\frac{\mu_1}{M}S_pE_p - \frac{\mu_2}{M}S_pI_p - (\alpha_1 + \mu_3)S_p - \beta_2 S_p, \\
^c D_0^\nu E_p &= \frac{\mu_1}{M}S_pE_p + \frac{\mu_2}{M}S_pI_p + \alpha_1 V_p I_p - (\alpha_1 + \alpha_2)E_p - \beta_2 E_p, \\
^c D_0^\nu I_p &= \alpha_1 E_p - (\beta_1 + \beta_2)I_p, \\
^c D_0^\nu Er_p &= \alpha_2 E_p - \beta_2 Er_p, \\
^c D_0^\nu R_p &= \beta_1 I_p - \beta_2 R_p, \\
^c D_0^\nu D_p &= \beta_2 I_p, \\
^c D_0^\nu Q_p &= \mu_3 S_p - \beta_2 Q_p, \\
^c D_0^\nu V_p &= \alpha_3 S_p - \sigma \mu_1 V_p I_p - \beta_2 V_p.
\end{align*}
\]
Suppose that $Z_p C$ with $0 \leq 3$

### Stability analysis

The parametric values along with their description, of the model presented in (1), are given in Table 1 as follows:

| Parameters | Description                                      | Values               |
|------------|--------------------------------------------------|----------------------|
| $\mu_1^c$  | Contact rate with $S_p$ and $E_p$                 | $0.8^{(day^{-1})}$   |
| $\mu_2^c$  | Contact rate with $S_p$ and $I_p$                | $0.02^{(day^{-1})}$  |
| $\mu_3^c$  | Home quarantine rate of $S_p$                    | $0.166^{(day^{-1})}$ |
| $M$        | Total population                                 | $11 \times 10^6$ person |
| $\alpha_1^c$ | Incubation rate                             | $0.0109^{(day^{-1})}$ |
| $\alpha_2^c$ | Recover rate of $E_p$                         | $0.1^{(day^{-1})}$  |
| $\beta_1^c$ | Recover rate of $I_p$                          | $0.003^{(day^{-1})}$ |
| $\beta_2^c$ | Fatality rate                                  | $0.0037^{(day^{-1})}$ |
| $\alpha^v$ | Vaccine rate                                    | $0.00000035^{(day^{-1})}$ |
| $\sigma^v$ | Vaccine inefficacy                              | $0.005^{(day^{-1})}$ |

First, we transform the system (1) in the following matrix form:

$$\frac{D^v}{D^v} Z_p(t) = F(t, Z_p(t)), \text{ provided } F = (F_1, \ldots, F_8),$$

where $Z_p = (S_p, E_p, I_p, E_{rp}, R_p, D_p, Q_p, V_p)$ and $F_i, i = 1, \ldots, 8$ are defined as follows

\[
\begin{align*}
F_1(t, Z_p(t)) &= -\mu_1^c S_p E_p - \frac{\mu_2^c}{M} S_p I_p - (\alpha^v + \mu_3^c)S_p - \beta_2^c S_p, \\
F_2(t, Z_p(t)) &= \frac{\mu_1^c}{M} S_p E_p + \frac{\mu_2^c}{M} S_p I_p + \alpha^v \mu_1^c V_p I_p - (\alpha_1^c + \alpha_2^c)E_p - \beta_2^c E_p, \\
F_3(t, Z_p(t)) &= \alpha_1^c E_p - (\beta_1^c + \beta_2^c)I_p, \\
F_4(t, Z_p(t)) &= \alpha_2^c E_p - \beta_3^c E_r, \\
F_5(t, Z_p(t)) &= \beta_1^c I_p - \beta_3^c R_p, \\
F_6(t, Z_p(t)) &= \beta_2^c I_p, \\
F_7(t, Z_p(t)) &= \mu_3^c S_p - \beta_2^c Q_p, \\
F_8(t, Z_p(t)) &= \alpha^v S_p - \sigma^v \mu_1^c V_p I_p - \beta_2^c V_p,
\end{align*}
\]

with non-negative initial conditions

\[
S_p(0) = S_{p0} \geq 0, \quad E_p(0) = E_{p0} \geq 0, \quad I_p(0) = I_{p0} \geq 0, \quad E_{rp}(0) = E_{rp0} \geq 0, \\
R_p(0) = R_{p0} \geq 0, \quad D_p(0) = D_{p0} \geq 0, \quad Q_p(0) = Q_{p0} \geq 0, \quad V_p(0) = V_{p0} \geq 0.
\]

The parametric values along with their description, of the model presented in (1), are given in Table 1 as follows:

### 3 Stability analysis

In this section, we will establish the positivity of the proposed fractional model (1). Similarly, the basic reproduction number $R_0$ will be computed using the next generation matrix method [50] and the disease free equilibrium points will be derived.

#### 3.1 The positivity and boundedness of the solution

To establish the positivity of the epidemiological model, we recall the following lemma to prove the main theorem on the positivity of the solution of fractional model (1).

**Lemma 3.2 (Generalized mean value theorem).** Let $\psi(t) \in \mathbb{C}[a, b]$ and Caputo operator for fractional derivative $\frac{D^v}{D^v} \psi(t) \in \mathbb{C}(a, b)$ for $0 < v \leq 1$. Then

$$\psi(t) = \psi(s) + \frac{1}{\Gamma(v)} \frac{D^v}{D^v} \psi(u)(t - s)^v,$$

with $0 \leq u \leq t, \forall t \in (a, b]$.

**Remark 3.3** Suppose that $\psi(t) \in \mathbb{C}[0, b]$ and the Caputo derivative of $\frac{D^v}{D^v} \psi \in (0, b)$ for $0 < v \leq 1$. Lemma 3.2 shows that if $\frac{D^v}{D^v} \psi(t) \geq 0, \forall t \in (0, b]$, then $\psi(t)$ is non-decreasing if $\frac{D^v}{D^v} \psi(t) \leq 0, \forall t \in (0, b]$, then $\psi(t)$ is non-increasing for all $t \in (0, b]$. 
Theorem 3.4 The solution to the proposed fractional order model (1) together with initial conditions (2) is unique and limited in $R^8_+$. 

Proof The existence and uniqueness of the model solution (1) can be achieved by Wei [51]. We observe from the model (1)

\[
\begin{align*}
\dot{c}_0D^\nu_t S_p |_{S_p=0} & = 0, \\
\dot{g}_0D^\nu_t E_p |_{E_p=0} & = \frac{\mu_v^1}{M} S_p I_p \geq 0, \\
\dot{g}_0D^\nu_t I_p |_{I_p=0} & = \alpha_v^1 E_p \geq 0, \\
\dot{g}_0D^\nu_t E_r |_{E_r=0} & = \alpha_v^2 E_p \geq 0, \\
\dot{g}_0D^\nu_t R_p |_{R_p=0} & = \beta_v^1 I_p \geq 0, \\
\dot{g}_0D^\nu_t D_p |_{D_p=0} & = \beta_v^2 I_p \geq 0, \\
\dot{g}_0D^\nu_t Q_p |_{Q_p=0} & = \mu_v^3 \geq 0, \\
\dot{g}_0D^\nu_t V_p |_{V_p=0} & = \alpha_v^v S_p \geq 0.
\end{align*}
\]

From Remark 3.3 and system (3), $Z_p(0) \in R^8_+$. Also in each line bounding the non-negative octant, the vector field points will remain in $R^8_+$. Therefore, the fractional model (1) with solution $Z_p(t)$ is non negative if the initial condition is set positively invariant. $\square$

3.2 Disease free equilibrium

In the disease-free equilibrium state, we have a disappearance of infection. Therefore, all infected groups are null and only susceptible individuals can make up the entire population which implies that

\[E_p = I_p = R_p = D_p = Q_p = E_r = V_p = 0.\]

Thus, the disease-free equilibrium points of our model is

\[Z_p^0 = (M, 0, 0, 0, 0, 0, 0).\]

3.3 Basic reproduction number

In proportion to the spread of the disease in a population, the basic reproduction number indicates an important role in the nature and mechanism of the current epidemic. The average number of circumstances in a population that would not have been infected during their infectious making a prediction by an infected person may be accepted [52]. Physically, if $R_0 < 1$, the infection will vanish, but if $R_0 > 1$, the infection is confirmed and the disease persists. To calculate the basic reproduction number $R_0$, we employ new generation matrix method by assuming the matrices of generations $F$ and $V$.

Let $X = (E_p, I_p)^T$. System (1) can be written as

\[X' = (F + V)X,\]

where

\[
F = \begin{bmatrix}
\frac{\mu_v^1}{M} S_p E_p + \frac{\mu_v^2}{M} S_p I_p + \sigma_v^v \mu_v^1 V_p I_p \\
0
\end{bmatrix},
\]

\[
V = \begin{bmatrix}
(\alpha_v^1 + \alpha_v^2) E_p + \beta_v^2 E_p \\
-\alpha_v^1 E_p + (\beta_v^1 + \beta_v^2) I_p
\end{bmatrix}.
\]

Then, the transmission matrix $F$ and transition matrix $V$ are given by

\[
F = \begin{bmatrix}
\frac{\mu_v^1}{M} S_p & \frac{\mu_v^2}{M} S_p + \alpha_v^v \mu_v^1 V_p I_p \\
0 & 0
\end{bmatrix},
\]

\[
V = \begin{bmatrix}
(\alpha_v^1 + \alpha_v^2) E_p + \beta_v^2 E_p & 0 \\
-\alpha_v^1 E_p + (\beta_v^1 + \beta_v^2) I_p & (\beta_v^1 + \beta_v^2)
\end{bmatrix}.
\]

The basic reproduction number is equal to the largest eigenvalue of $FV^{-1}$, thus

\[R_0 = \frac{\mu_v^1(\beta_v^1 + \beta_v^2) + \mu_v^2 \alpha_v^v}{(\alpha_v^1 + \alpha_v^2)(\beta_v^1 + \beta_v^2)}.\]
4 Existence and uniqueness of the solution

In this section it is shown that the system (1) has a unique solution. Taking both sides of the following matrix form as an integral form, we get

$$Z_p(t) - Z_p(0) = \frac{1}{\Gamma(v)} \int_0^t (t - \kappa)^{v-1} F(\kappa, Z_p) d\kappa,$$

provided $F = (F_1, \ldots, F_8)$.

Assume that $Z_p$ is nonnegative and bounded, then we have

\begin{align*}
\|S_p(t)\| &\leq z_1; \|E_p(t)\| \leq z_2; \|I_p(t)\| \leq z_3; \|E r_p(t)\| \leq z_4; \\
\|K_p(t)\| &\leq z_5; \|D_p(t)\| \leq z_6; \|Q_p(t)\| \leq z_7; \|V_p(t)\| \leq z_8,
\end{align*}

where $z_1, \ldots, z_8$ are positive constants. We show that $F = (F_1, \ldots, F_8)$ follows the condition of Lipschitz’s and contraction.

**Theorem 4.1** Suppose that condition (4) is satisfied and the following inequality

$$0 \leq r = \max\{r_1, r_2, \ldots, r_8\} < 1$$

holds, where $r_i, i = 1, \ldots, 8$ are Lipschitz’s constants, then the functions $F_i$ for $i = 1, \ldots, 8$ fulfill the condition of Lipschitz’s and are contraction mapping.

**Proof** Consider the difference and using (4), we have

$$\begin{align*}
\|F_i(t, Z_p) - F_i(t, Z_{p_k})\| &\leq \left\| \frac{\mu^1}{M} E_p(t)(S_p(t) - S_{p_1}(t)) \right\| + \left\| \frac{\mu^2}{M} I_p(t)(S_p(t) - S_{p_1}(t)) \right\| \\
&\quad + \left\| (\alpha^v + \mu^2)(S_p(t) - S_{p_1}(t)) \right\| + \left\| \beta^2(S_p(t) - S_{p_1}(t)) \right\| \\
&\leq \left[ \frac{\mu^1}{M} \|E_p(t)\| + \frac{\mu^2}{M} \|I_p(t)\| + \|\alpha^v + \mu^2\| \right] \|S_p(t) - S_{p_1}(t)\| \\
&\leq \left[ \frac{\mu^1}{M} \|E_p(t)\| + \frac{\mu^2}{M} \|I_p(t)\| + \alpha^v + \mu^2 \right] \|S_p(t) - S_{p_1}(t)\| \\
&\leq \left[ \frac{\mu^1}{M} \|E_p(t)\| + \frac{\mu^2}{M} \|I_p(t)\| + \alpha^v + \mu^2 \right] \|S_p(t) - S_{p_1}(t)\| \\
&= r_1 \|S_p(t) - S_{p_1}(t)\|,
\end{align*}$$

where $r_1 = \frac{\mu^1}{M} z_2 + \frac{\mu^2}{M} z_3 + (\alpha^v + \mu^2) \|S_p(t) - S_{p_1}(t)\|$ is a contraction mapping. Similarly, we can prove that $F_i$, for $i = 2, \ldots, 8$ fulfill the Lipschitz’s condition and we can write

$$\|F_i(t, Z_p) - F_i(t, Z_{p_k})\| \leq r \|S_p(t) - S_{p_1}(t)\|,$$

where $r = \max\{r_1, r_2, \ldots, r_8\}$ and $0 \leq r_1 < 1, i = 1, \ldots, 8$ using condition (5). Hence, $F$ is a contraction. □

Depending on the system (1), consider the following recursive forms:

$$Z_{p_k}(t) = \frac{1}{\Gamma(v)} \int_0^t (t - \kappa)^{v-1} F(\kappa, Z_{p_{k-1}}) d\kappa.$$

The difference between the two terms can be expressed as follows:

$$Z_{p_k}(t) - Z_{p_{k-1}}(t) = \frac{1}{\Gamma(v)} \int_0^t (t - \kappa)^{v-1} F(\kappa, Z_{p_{k-1}}) d\kappa,$$

using the initial condition $Z_{p_0}(0) = Z_p(0)$, where $\psi_n = (\psi_{1n}, \ldots, \psi_{8n})$. Consider

$$\|\psi_{1n}(t)\| = \|S_{p_n}(t) - S_{p_{n-1}}(t)\|$$

$$\leq \frac{1}{\Gamma(v)} \int_0^t (t - \kappa)^{v-1} F(\kappa, Z_{p_{k-1}}) d\kappa$$

$$\leq \frac{r_1}{\Gamma(v)} \int_0^t (t - \kappa)^{v-1} \|S_{p_n}(t) - S_{p_{n-1}}(t)\| d\kappa$$

$$\leq \frac{r_1}{\Gamma(v)} \int_0^t (t - \kappa)^{v-1} \|\psi_{1(n-1)}(t)\| d\kappa.$$
and we have
\[ \| \varphi_{1n}(t) \| \leq \frac{r_1}{\Gamma(v)} \int_0^t (t - \kappa)^{v-1} \| \varphi_{1(n-1)}(\kappa) \| d\kappa. \] (6)

Similarly, we get
\[ \| \varphi_{nn}(t) \| \leq \frac{r_1}{\Gamma(v)} \int_0^t (t - \kappa)^{v-1} \| \varphi_{1(n-1)}(\kappa) \| d\kappa \]
for some \( r_i \), where \( i = 2, \ldots, 8 \). Thus, we can write
\[ \| \varphi_n(t) \| \leq \frac{r}{\Gamma(v)} \int_0^t (t - \kappa)^{v-1} \| \varphi_{n-1}(\kappa) \| d\kappa, \]
where \( r = \max(r_1, \ldots, r_8) \).

Further, we can write \( Z_{pn}(t) = \sum_{i=1}^n \psi(t) \), where \( \psi(t) = (\varphi_{1n}, \ldots, \varphi_{8n}) \) such that
\[
\begin{align*}
S_{pn}(t) &= \sum_{i=1}^n \varphi_{1i}(t), \quad E_{pn}(t) = \sum_{i=1}^n \varphi_{2i}(t), \\
I_{pn}(t) &= \sum_{i=1}^n \varphi_{3i}(t), \quad E_{pn}(t) = \sum_{i=1}^n \varphi_{4i}(t), \\
R_{pn}(t) &= \sum_{i=1}^n \varphi_{5i}(t), \quad D_{pn}(t) = \sum_{i=1}^n \varphi_{6i}(t), \\
Q_{pn}(t) &= \sum_{i=1}^n \varphi_{7i}(t), \quad \psi(t) = \sum_{i=1}^n \varphi_{8i}(t).
\end{align*}
\] (7)

In the following theorem, we prove the existence of the solution.

**Theorem 4.2** The functions defined in (7) exist and smooth. If there exists \( t_0 > 1 \) such that \( \frac{r_1^v}{\Gamma(v+1)} \leq 1 \) for \( i = 1, \ldots, 8 \), then there exist at least one solution of system given by the fractional-order COVID-19 model \( S_{pn}, E_{pn}, E_{pn}, R_{pn}, D_{pn}, Q_{pn}, \psi(t) \).

**Proof** Suppose there exists \( t_1 \) such that \( \frac{r_1^v}{\Gamma(v+1)} < 1 \). From the recursive scheme and from (6), we have
\[ \| \varphi_{1n}(t) \| \leq \frac{r_1}{\Gamma(v)} \int_0^t (t - \kappa)^{v-1} \| \varphi_{1(n-1)}(\kappa) \| d\kappa \leq \frac{r_1^v}{\Gamma(v+1)} \| \varphi_{1n}(t) \|. \] (8)

Replacing \( n \) by \( n - 1 \) in (6), we get
\[ \| \varphi_{1(n-1)}(t) \| \leq \frac{r_1}{\Gamma(v)} \int_0^t (t - \kappa)^{v-1} \| \varphi_{1(n-2)}(\kappa) \| d\kappa \leq \frac{r_1^v}{\Gamma(v+1)} \| \varphi_{1(n-1)}(t) \|. \] (9)

Again replacing \( n \) by \( n - 2 \) in (6), we have
\[ \| \varphi_{1(n-2)}(t) \| \leq \frac{r_1}{\Gamma(v)} \int_0^t (t - \kappa)^{v-1} \| \varphi_{1(n-3)}(\kappa) \| d\kappa \leq \frac{r_1^v}{\Gamma(v+1)} \| \varphi_{1(n-2)}(t) \|. \] (10)

Using (8), (9) and (10), we have
\[ \| \varphi_{1n}(t) \| \leq \left( \frac{r_1^v}{\Gamma(v+1)} \right)^2 \| \varphi_{1n}(t) \| \leq \left( \frac{r_1^v}{\Gamma(v+1)} \right)^3 \| \varphi_{1n}(t) \|. \]

If we keep substituting in this way and use the initial condition, we obtain
\[ \| \varphi_{1n}(t) \| \leq \| S_{pn}(0) \| \left[ \frac{r_1}{\Gamma(v+1)} \right]^n t^n \]
and hence we can write
\[ \| \varphi_{n}(t) \| \leq \| Z_{pn}(0) \| \left[ \frac{r}{\Gamma(v+1)} \right]^n t^n. \]

where \( \phi_{n}(t) = (\varphi_{1n}, \varphi_{2n}, \ldots, \varphi_{8n}) \) and \( r = \max(r_1, \ldots, r_8) \). This yields existence and smoothness of the function given in (7).

Next, we will show that the sequence \( \{ S_{pn}, E_{pn}, I_{pn}, E_{pn}, R_{pn}, D_{pn}, Q_{pn}, \psi(t) \} \) converges to the solutions of the system (1). For this, we define \( \Psi_n = (\Psi_{1n}, \Psi_{2n}, \ldots, \Psi_{8n}) \), as remainder terms after \( n \)-iterations as follows
\[ Z_{pn}(t) - Z_{pn}(0) = Z_{pn}(t) - \Psi_n(t). \]
Employing the triangular inequality and Lipschitz condition, we have
\[ \|\Psi_{1n}(t)\| \leq \left[ \frac{r_1}{\Gamma(v+1)} t^v \right]^{n+1} z_1. \]
At \( t_0 \), we get
\[ \|\Psi_{11}(t)\| \leq \left[ \frac{r_1}{\Gamma(v+1)} t^v \right]^{n+1} z_1. \]
This gives
\[ \lim_{n \to \infty} \|\Psi_{1n}(t)\| \leq \lim_{n \to \infty} \left[ \frac{r_1}{\Gamma(v+1)} t^v \right]^{n+1} z_1. \]
By hypothesis \( \frac{r_1}{\Gamma(v+1)} t^v \leq 1 \), we have
\[ \lim_{n \to \infty} \|\Psi_{1n}(t)\| = 0. \]
Likewise, we attain
\[ \lim_{n \to \infty} \|\Psi_{in}(t)\| = 0, \quad i = 2, \ldots, 8. \]
Hence \( \|\Psi_n\| \to 0 \), which implies the existence of the solution of the model (7). \( \Box \)

**Theorem 4.3** If the condition \( 1 - \frac{r_1}{\Gamma(v+1)} t^v > 0 \) for \( i = 1, \ldots, 8 \) holds, then the \( S_p E_p I_p R_p D_p Q_p V_p \) fractional-order COVID-19 model has a unique solution.

**Proof** We assume that another solution is possible for the system to highlight the uniqueness of the solution such as \( Z_{p1}(t) = (S_{p1}(t), E_{p1}(t), I_{p1}(t), R_{p1}(t), D_{p1}(t), Q_{p1}(t), V_{p1}(t)). \) Then we have
\[ S_p(t) - S_{p1}(t) = \frac{1}{\Gamma(v)} \int_0^t (t - \kappa)^{v-1} (F_1(\kappa, S_p) - F_1(\kappa, S_{p1})) d\kappa. \]
This yields
\[ \|S_p(t) - S_{p1}(t)\| = \left\| \frac{1}{\Gamma(v)} \int_0^t (t - \kappa)^{v-1} (F_1(\kappa, S_p) - F_1(\kappa, S_{p1})) d\kappa \right\| \\
\leq \frac{1}{\Gamma(v)} \int_0^t (t - \kappa)^{v-1} \|F_1(\kappa, S_p) - F_1(\kappa, S_{p1})\| d\kappa. \]
From the Lipschitz condition (6), it follows that
\[ \|S_p(t) - S_{p1}(t)\| \leq \frac{r_1}{\Gamma(v+1)} t^v \|S_p(t) - S_{p1}(t)\| \]
and consequently
\[ \|S_p(t) - S_{p1}(t)\| - \frac{r_1}{\Gamma(v+1)} t^v \|S_p(t) - S_{p1}(t)\| \leq 0 \]
implies that
\[ \|S_p(t) - S_{p1}(t)\| \left( 1 - \frac{r_1}{\Gamma(v+1)} t^v \right) \leq 0. \]
As \( 1 - \frac{r_1}{\Gamma(v+1)} t^v > 0 \), inequality (11) implies that
\[ \|S_p(t) - S_{p1}(t)\| = 0. \]
This means that \( S_p(t) = S_{p1}(t) \). Apply similar technique to all solution for \( i = 2, \ldots, 8 \) and we can write...
\[ \| Z_p(t) - Z_{p_1}(t) \| = 0. \]

Hence fractional-order COVID-19 model has a unique solution. \( \square \)

5 Numerical scheme for the solution

For our proposed fractional order epidemic model \( S_p E_p I_p E_r R_p D_p Q_p V_p \), this section provides the numerical solution. We use the generalized Adams-Bashforth-Moulton method to construct the numerical solution of (1). This method has been introduced and briefly discussed in [53] and some more information is given in [54]. A number of additional results for a specific initial value problem are contained in [55], a detailed mathematical analysis is provided in [56], and additional practical remarks can be found in [57]. Numerical experiments and comparisons with other methods are reported in [58, 59]. Assume the following fractional differential nonlinear equation to construct the numerical solution using this approach.

\[
\frac{\varepsilon}{\nu} D_t^\nu \psi(t) = g(t, \psi(t)), \quad 0 \leq t \leq T
\]

by the following initial condition

\[
\psi^{(j)} = \psi_0^j, \quad j = 0, 1, \ldots, [\nu] - 1.
\]

From (12), we get the solution \( \psi(t) \) by solving the following equation:

\[
\psi(t) = \sum_{j=0}^{[\nu]-1} \frac{\psi_0^j}{j!} t^j + \frac{h^\nu}{\Gamma(\nu+2)} \int_0^t (t-u)^{\nu-1} f(u, \psi(u)) du.
\]

The above Eq. (13) is known as the Volterra integral equation. For the integration of (13) [57, 60, 61] we apply the predictor-corrector formula based on the Adams-Bashforth-Moulton integration algorithm. Suppose \( \frac{T}{nh}, \ t_n = nh \) and \( n = 0, 1, \ldots, N \in \mathbb{Z}^+ \), Eq. (13) can be discretized as follows:

\[
\psi(t_{n+1}) = \sum_{j=0}^{[\nu]-1} \frac{\psi_0^j}{j!} t_{n+1}^j + \frac{h^\nu}{\Gamma(\nu+2)} f(t_{n+1}, \psi(t_{n+1})) + \frac{h^\nu}{\Gamma(\nu+2)} \sum_{i=0}^n a_{i,n+1} f(t_i, \psi(t_i)),
\]

where

\[
a_{i,n+1} = \begin{cases} \frac{n+1}{\nu} - (n-\nu)(n+1)^\nu, & \text{if } i = 0, \\ (n-i+2)^{\nu+1} + (n-i)^{\nu+1} - 2(n-i+1)^{\nu+1}, & \text{if } 0 < i \leq n, \\ 1, & \text{if } i = n+1. \end{cases}
\]

Then \( \psi^{pr}(t_{n+1}) \) is calculated as

\[
\psi^{pr}(t_{n+1}) = \sum_{j=0}^{[\nu]-1} \frac{\psi_0^j}{j!} t_{n+1}^j + \frac{1}{\Gamma(\nu)} \sum_{i=0}^n b_{i,n+1} f(t_i, \psi(t_i))
\]

with

\[
b_{i,n+1} = \frac{h^\nu}{\nu} ((n+1-i)^\nu - (n-i)^\nu).
\]

5.1 Numerical scheme for COVID-19 Caputo fractional model

In this section, the nonlinear fractional model \( S_p E_p I_p E_r R_p D_p Q_p V_p \) is numerically solved. To employ the generalized Adams-Bashforth-Moulton method, write the model in the following matrix form

\[
X = \begin{bmatrix} S_p \\ E_p \\ I_p \\ E_r \\ R_p \\ D_p \\ Q_p \\ V_p \end{bmatrix} \quad \text{and} \quad \phi X = \begin{bmatrix} \phi S_p \\ \phi E_p \\ \phi I_p \\ \phi E_r \\ \phi R_p \\ \phi D_p \\ \phi Q_p \\ \phi V_p \end{bmatrix}
\]

in scheme (14).
\[ Z_p(t_{n+1}) = \sum_{j=0}^{[v-1]} \frac{\psi_0^j}{j!} t_{n+1}^j + \sum_{i=0}^{n} a_i n+1 \phi Z_p(t_i, Z_p(t_i)) + \frac{h^v}{\Gamma(v+2)} \phi Z_p(t_{n+1}, Z^{pr}_p(t_{n+1})), \]

where

\[ \phi Z_p = (\phi_{Sp}, \phi_{Ep}, \phi_{Ip}, \phi_{Erp}, \phi_{Rp}, \phi_{Qp}, \phi_{Vp}) \]

and

\[ Z^{pr}_p(t_{n+1}) = \sum_{j=0}^{[v-1]} \frac{\psi_0^j}{j!} t_{n+1}^j + \frac{1}{\Gamma(v)} \sum_{i=0}^{n} b_i n+1 \phi Z_p(t_i, Z_p(t_i)). \]

Moreover \( \phi Z_p(t_i, Z_p(t_i)) \) are calculated from the following functions:

\[ \phi_{Sp}(t_i, Z_p(t_i)) = -\frac{\mu_1^v}{M} S_p E_p - \frac{\mu_2^v}{M} S_p I_p - (\alpha^v + \mu_3^v) S_p - \beta_1^v S_p, \quad (15) \]
\[ \phi_{Ep}(t_i, Z_p(t_i)) = \frac{\mu_1^v}{M} S_p E_p + \frac{\mu_2^v}{M} S_p I_p + \sigma^v \mu_1^v V_p I_p - (\alpha_1^v + \alpha_2^v) E - \beta_2^v E_p, \quad (16) \]
\[ \phi_{Ip}(t_i, Z_p(t_i)) = \alpha_1^v E_p - (\beta_1^v + \beta_2^v) I_p, \quad (17) \]
\[ \phi_{Erp}(t_i, Z_p(t_i)) = \alpha_2^v E_p - \beta_2^v E_p, \quad (18) \]
\[ \phi_{Rp}(t_i, Z_p(t_i)) = \beta_1^v I_p - \beta_2^v R_p, \quad (19) \]
\[ \phi_{Dp}(t_i, Z_p(t_i)) = \beta_2^v I_p - \beta_2^v D_p, \quad (20) \]
\[ \phi_{Qp}(t_i, Z_p(t_i)) = \mu_3 S_p - \beta_2 I_p, \quad (21) \]
\[ \phi_{Vp}(t_i, Z_p(t_i)) = \alpha^v S_p - \alpha^v \mu_1 V_p I_p - V_p \beta_2^v. \quad (22) \]

In accumulation, the quantities \( \phi Z_p(t_{n+1}, Z^{pr}_p(t_{n+1})) \) are calculated from Eqs. (15–22) correspondingly at the point \( t_{n+1}, n = 1, \ldots, k \).

### 6 Numerical simulations and impact of vaccination

Numerical simulations of the proposed COVID vaccine model are presented in Figs. 1, 2, 3 and 4 using the parametric values given in Table 1. Figure 1 compares the approximate of fractional COVID model with real data given in [38]. Figure 2 presents the approximate solution of the fractional model for different values of fractional-order \( 0 < \alpha \leq 1 \).

Tunisia received the first batch of Pfizer COVID-19 vaccinations via the COVAX Laboratory on March 2021 where the first 743 shots were injected to health care professional being the prioritized group. This first phase of vaccination was continued throughout the current year of 2021 and the goal of this phase was to cover the vaccination of 20% part of the nation by delivering almost 3.1
Fig. 2 COVID model governed by Caputo fractional operator and initial condition (10999782, 200, 18, 0, 0, 0, 0, 0, 0) million of doses. Although the rate of vaccination in Tunisia is comparatively high in Africa even then this is not enough to fully control the spread of the virus. In July 2021, Tunisia was hit by a third wave of COVID-19 and as a consequence the health system was collapsed badly and an economic crisis was occurred.
Fig. 3 Comparison of vaccinated and non vaccinated infected population for integer-order ($\nu = 1$) and fractional-order ($\nu = 0.85$) model with vaccine rate $\alpha_1 = 0.00000012$ and inefficacy rate $\sigma_1 = 0.0005$ with initial condition $(10999782, 200, 18, 0, 0, 0, 0, 0)$.

Figure 3 shows that infected vaccinated people reduce to 395, 305 and 240 for integer-order model $\nu = 1$ and for the fractional-order model $\nu = 0.85$ vaccinated people reduce to 105, 90 and 70 with contact rate of 0.8, 0.75 and 0.7, respectively. Figure 4 depicts that number of deaths of the vaccinated people reduce to 162, 125 and 101 for integer-order model $\nu = 1$ and for the fractional-order model $\nu = 0.85$ vaccinated people reduce to 58, 52 and 45 with contact rate of 0.8, 0.75 and 0.7, respectively. This shows that increasing the contact rate with unvaccinated people might reduce the effect of vaccination. As in Figs. 3 and 4, it is shown that in the current situation where the whole nation is not fully vaccinated the good control on the spread of COVID-19 can be achieved through reducing the contact rate which suggests that the complete vaccination of the whole population is essential to have a good control on the spread of COVID-19 outbreak. Thus, to overcome the stress of social distancing and quarantine and to bring the world
Fig. 4 Comparison of vaccinated and non-vaccinated dead population for integer-order ($\nu = 1$) and fractional-order ($\nu = 0.85$) model with vaccine rate $\alpha = 0.000000012$ and inefficacy rate $\sigma = 0.0005$ with initial condition $(10999782, 200, 18, 0, 0, 0, 0, 0)$

to the normal scenario, instead of reducing the contact rate, it is better to increase the number of vaccinated people. Governments should also increase the vaccine campaigns and give awareness to the common people to get vaccinated at their earliest.

7 Conclusion

Our study is based on the analysis of the $S_pE_pI_pE_r_pR_pD_pQ_pV_p$ epidemic fractional order model for the disease of COVID-19. The main objective of this research is to reveal the effect of vaccination on the population. Our numerical simulations are made using the real data from Tunisia. Our results show that our suggested model might be considered as the reference model to conclude the
vaccination effect on the population. We investigated the conditions for stability analysis and numerical solutions of the epidemic model $S_p E_p I_p E_r P D_p Q_p V_p$, using Caputo fractional derivative. The numerical technique used to approximate the solution of the model is the generalized Adams-Bashforth-Moulton method. Outcomes of our analysis indicate the decrease in number of deaths and infections of patients due to COVID-19 after increasing the rate of vaccination in the country. It also suggests that by advancing the campaign of vaccination would be very beneficial in controlling the spread of COVID-19 and lessening the severe effect of the disease.

**Data availability statement** The data used to support the findings of this study are cited at relevant places within the text as references.

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**Data availability statement** The data used to support the findings of this study are cited at relevant places within the text as references.
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