ON THE K-THEORY OF LINEAR GROUPS

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Abstract. We prove that for a finitely generated linear group over a field of positive characteristic the family of quotients by finite subgroups has finite asymptotic dimension. We use this to show that the $K$-theoretic assembly map for the family of finite subgroups is split injective for every finitely generated linear group $G$ over a commutative ring with unit under the assumption that $G$ admits a finite-dimensional model for the classifying space for the family of finite subgroups. Furthermore, we prove that this is the case if and only if an upper bound on the rank of the solvable subgroups of $G$ exists.

1. Introduction

For every group $G$ and every ring $A$ there is a functor $\mathbb{K}_A : OrG \to \text{Spectra}$ from the orbit category of $G$ to the category of spectra sending $G/H$ to (a spectrum weakly equivalent to) the $K$-theory spectrum $\mathbb{K}(A[H])$ for every subgroup $H \leq G$. For any such functor $F : OrG \to \text{Spectra}$, a $G$-homology theory $F$ can be constructed via

$$F(X) := \text{Map}_G(\_ \times X) \wedge_{OrG} F,$$

see Davis and Lück [DL98]. We will denote its homotopy groups by $H^n_G(X; F) := \pi_nF(X)$. The assembly map for the family of finite subgroups is the map

$$H^n_G(EG; \mathbb{K}_A) \to H^n_G(pt; \mathbb{K}_A) \cong K_n(A[G])$$

induced by the map $EG \to pt$. Here $EG$ denotes the classifying space for the family of finite subgroups, see Lück [Luc00]. The assembly map is a helpful tool to relate the $K$-theory of the group ring $A[G]$ to the $K$-theory of the group rings over the finite subgroups $H \leq G$. It can more generally be defined for any additive $G$-category instead of $A$, see Bartels and Reich [BR07]. Note that additive categories will always be small and that $K$-theory will always mean non-connective $K$-theory constructed by Pedersen and Weibel [PW85].

Theorem 1.1. Let $R$ be a commutative ring with unit and let $G \leq GL_n(R)$ be finitely generated. If $G$ admits a finite-dimensional model for the classifying space $EG$, then the assembly map

$$H^n_G(EG; \mathbb{K}_A) \to K_n(A[G])$$

is split injective for every additive $G$-category $A$.
If $A$ is an additive $G$-category with involution such that for every virtually nilpotent subgroup $A \leq G$ there exists $i_0 \in \mathbb{N}$ such that for $i \geq i_0$ we have $K_{-i}(A[A]) = 0$, then the $L$-theoretic assembly map

$$H_n^G(EG; \mathbb{L}_A^{(-\infty)}) \to L_n^{(-\infty)}(\mathcal{A}[G])$$

is split injective.

Theorem 1.1 implies the (generalized integral) Novikov conjecture for these groups by [Kas, Section 6], since virtually nilpotent groups satisfy the Farrell–Jones conjecture by Wegner [Weg15]. The (rational) Novikov conjecture for these groups is already known by Guentner, Higson and Weinberger [GHW05], where it is shown that the Baum-Connes assembly map is split injective for linear groups.

We will use inheritance properties to reduce the proof of the theorem to the case where the ring $R$ has trivial nilradical and show that in this case the family $\{F \setminus G\}_{F \in \mathcal{F}_{in}}$ has finite decomposition complexity, where $\mathcal{F}_{in}$ denotes the family of finite subgroups of $G$. Then the theorem follows from the main theorem of [Kas14]. For convenience, the necessary results of [Kas14] are recalled in the appendix.

By a result of Alperin and Shalen [AS82] a finitely generated subgroup $G$ of $GL_n(F)$, where $F$ is a field of characteristic zero, has finite virtual cohomological dimension if and only if there is a bound on the Hirsch rank of the unipotent subgroups of $G$. This in particular implies that it has a finite-dimensional model for the classifying space $EG$. In positive characteristic, a finitely generated subgroup $G \leq GL_n(F)$ always admits a finite-dimensional model for $EG$ by Degrijse and Petrosyan [DP, Corollary 5]. We generalize this in the following way.

**Proposition 1.2.** Let $R$ be a commutative ring with unit and let $G \leq GL_n(R)$ be finitely generated. Then $G$ admits a finite-dimensional model for $EG$ if and only if there exists $N \in \mathbb{N}$ such that $l(A) \leq N$ for every solvable subgroup $A \leq G$, where $l(A)$ denotes the Hirsch rank of $A$.

Let $G$ be a solvable group and $1 = G_0 \leq G_1 \leq \ldots G_{n-1} \leq G_n = G$ a normal series with abelian factors. The *Hirsch rank* (or *Hirsch length*) $l(G)$ of $G$ is

$$l(G) = \sum_{i=1}^{n} \dim_{\mathbb{Q}} \mathbb{Q} \otimes_{\mathbb{Z}} (A_i/A_{i-1}).$$

We will prove the proposition in Section 5.

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## 2. Finite decomposition complexity

Let $X$ be a metric space. A decomposition $X = \bigcup_{i \in I} U_i$ is called *$r$-disjoint*, if $d(U_i, U_j) > r$ for all $i \neq j \in I$. We then denote the decomposition by

$$X = \bigcup_{r\text{-disj.}} U_i.$$

A *metric family* is a set of metric spaces. A metric family $\{X_i\}_{i \in I}$ has *finite asymptotic dimension uniformly* if there exists an $n \in \mathbb{N}$ such that for every $r > 0$, **
$i \in I$ there exists decompositions $X_i = \bigcup_{k=0}^{n_i} U_i^k$ and

$$U_i^k = \bigcup_{j \in J_{i,k}} U_{i,j},$$

such that $\sup_{i,j,k} U_{i,j} < \infty$.

In \cite{GTY13} Guentner, Tessera and Yu introduced the following generalization of finite asymptotic dimension.

**Definition 2.1.** Let $r > 0$. A metric family $\mathcal{X} = \{X_\alpha\}_{\alpha \in A}$ $r$-decomposes over a class of metric families $\mathcal{D}$ if for every $\alpha \in A$ there exists a decomposition $X_\alpha = U_\alpha^r \cup V_\alpha^r$ and $r$-disjoint decompositions

$$U_\alpha^r = \bigcup_{i \in I(\alpha,r)} U_{\alpha,i}^r, \quad V_\alpha^r = \bigcup_{j \in J(\alpha,r)} V_{\alpha,j}^r$$

such that the families $\{U_{\alpha,i}^r\}_{\alpha \in A, i \in I(\alpha,r)}$ and $\{V_{\alpha,j}^r\}_{\alpha \in A, j \in J(\alpha,r)}$ lie in $\mathcal{D}$. A metric family $\mathcal{X}$ decomposes over $\mathcal{D}$ if it $r$-decomposes over $\mathcal{D}$ for all $r > 0$.

Let $\mathcal{B}$ denote the class of bounded families, i.e. $\mathcal{X} \in \mathcal{B}$ if there exists $R > 0$ such that $\text{diam} X < R$ for all $X \in \mathcal{X}$. We set $\mathcal{D}_0 = \mathcal{B}$. For a successor ordinal $\gamma + 1$ we define $\mathcal{D}_{\gamma+1}$ to be the class of all metric families which decompose over $\mathcal{D}_{\gamma}$. For a limit ordinal $\lambda$ we define

$$\mathcal{D}_\lambda = \bigcup_{\gamma < \lambda} \mathcal{D}_{\gamma}.$$

A metric family $\mathcal{X}$ has finite decomposition complexity (FDC) if $\mathcal{X} \in \mathcal{D}_\gamma$ for some ordinal $\gamma$.

A metric space $X$ has FDC if the family $\{X\}$ consisting only of $X$ has FDC. A group $G$ has FDC if it has FDC with any (and thus every) proper left-invariant metric.

A subfamily $Z$ of a metric family $\mathcal{Y}$ is a metric family $Z$ such that for each $Z \in \mathcal{Z}$ there exists an $Y \in \mathcal{Z}$ with $Y \subseteq X$.

A map $F: \mathcal{X} \rightarrow \mathcal{Y}$ is between metric families $\mathcal{X}, \mathcal{Y}$ is a set of maps from elements of $\mathcal{X}$ to elements of $\mathcal{Y}$ such that every $X \in \mathcal{X}$ is the domain of at least one $f \in F$. The inverse image $F^{-1}(Z)$ of a subfamily $Z$ of $\mathcal{Y}$ is the metric family $\{f^{-1}(Z) \mid Z \in \mathcal{Z}, f \in F\}$. A map $F: \mathcal{X} \rightarrow \mathcal{Y}$ is called uniformly expansive, if there exists a non-decreasing function $\rho: [0, \infty) \rightarrow [0, \infty)$ such that for every $f: X \rightarrow Y$ in $F$ and every $x, y \in X$ we have

$$d(f(x), f(y)) \leq \rho(d(x, y)).$$

We will use the following three results about FDC.

**Theorem 2.2 (\cite{GT13}, Fibering Theorem 3.1.4).** Let $\mathcal{X}$ and $\mathcal{Y}$ be metric families and let $F: \mathcal{X} \rightarrow \mathcal{Y}$ be uniformly expansive. Assume $\mathcal{Y}$ has FDC and that for every bounded subfamily $Z$ of $\mathcal{Y}$ the inverse image $F^{-1}(Z)$ has FDC. Then $\mathcal{X}$ also has FDC.

**Theorem 2.3 (\cite{GT13}, Theorem 4.1).** A metric space $X$ with finite asymptotic dimension has FDC.

While the above theorem is stated only for metric spaces it also holds for metric families which have finite asymptotic dimension uniformly.
Theorem 2.4 ([GTY13, 3.1.7]). Let \( X \) be a metric space, expressed as a union of finitely many subspaces \( X = \bigcup_{i=0}^{n} X_i \). If the metric family \( \{X_i\}_{i=0,...,n} \) has FDC, so does \( X \).

This theorem again holds for metric families instead of metric spaces, i.e. a metric family \( \{\bigcup_{i=0}^{n} X_{ij}\}_{j \in J} \) has FDC if and only if the family \( \{X_{ij}\}_{j \in J, i=0,...,n} \) has FDC.

We will also need the following two results about finite asymptotic dimension.

Lemma 2.5. Let \( X \rightarrow Y \) be a family of maps such that there is \( k > 0 \) and each \( p \in P \) is \( k \)-Lipschitz. Suppose that \( Y \) has finite asymptotic dimension uniformly and that for each \( R > 0 \) the family \( \{p^{-1}(B_R(y)) \mid X \in \mathcal{X}, Y \in \mathcal{Y}, y \in Y, (p: X \rightarrow Y) \in P\} \) has finite asymptotic dimension uniformly. Then \( \mathcal{X} \) has finite asymptotic dimension uniformly.

This is [Roe03, Lemma 9.16] for metric families instead of metric spaces. The proof is the same.

Lemma 2.6. Let \( \mathcal{X} = \{U_\alpha \cup V_\alpha\}_{\alpha \in A} \) be a metric family. Then
\[
\text{asdim } \mathcal{X} = \max\{\text{asdim} \{U_\alpha\}_\alpha, \text{asdim} \{V_\alpha\}_\alpha\}.
\]

This is [Roe03, Proposition 9.13] for metric families instead of metric spaces. The proof is the same.

In the next section it will be more convenient to work with pseudometrics instead of metrics, i.e. allowing \( d(x, y) = 0 \) for \( x \neq y \). Finite asymptotic dimension and FDC are defined in the same way for pseudometrics. If \( d \) is a pseudometric on \( X \), then we can define a metric \( d' \) on \( X \) by setting \( d'(x, y) := \max\{1, d(x, y)\} \) for \( x \neq y \). The metric \( d' \) is proper resp. left-invariant if and only if \( d \) is. It has finite asymptotic dimension resp. FDC if and only if \( d \) does. Therefore, to show that a group has finite asymptotic dimension or FDC, it suffices to show this for \( G \) equipped with a left-invariant proper pseudometric.

Notation 2.7. If \( G \) is a group, then by \( \{F \backslash G\}_{F \in \mathcal{F}_{\text{fin}}} \) we will always mean the family of quotients by all finite subgroups of \( G \), i.e. \( \mathcal{F}_{\text{fin}} \) will always refer to the family of finite subgroups of the group of which we take the quotients.

3. Linear groups over fields of positive characteristic

In this section \( K \) will always denote a field of positive characteristic. Every finitely generated subgroup \( G \) of \( GL_n(K) \) has finite asymptotic dimension by [GTY12, Theorem 3.1]. Here we want to show that even the family \( \{F \backslash G\}_{F \in \mathcal{F}_{\text{fin}}} \) has finite asymptotic dimension uniformly. We begin by recalling the argument from [GTY12].

A length function on a group \( G \) is function \( l: G \rightarrow [0, \infty) \) such that for all \( g, h \in G \)
\[
(1) \ l(e) = 0,
(2) \ l(g) = l(g^{-1}), \text{ and }
(3) \ l(gh) \leq l(g) + l(h).
\]

We do not require that \( l \) is proper, nor that \( l(g) = 0 \) if and only if \( g = e \). By setting \( d(g, h) := l(g^{-1}h) \) we obtain a pseudometric.

A discrete norm on a field \( K \) is a map \( \gamma: K \rightarrow [0, \infty) \) satisfying that for all \( x, y \in K \) we have
We want to apply Proof.\[660\times 950\]

spect to the associated pseudometric.

finite asymptotic dimension with respect to the pseudometric $d$.

normal. Considering the extension $1 \to G \to T \to H$.

normal.

Lemma 3.1.\[660\times 950\] For every discrete norm $\gamma$ on $K$ a length function $l_{\gamma}$ on $GL_n(K)$ by

\[l_{\gamma}(g) = \log \max_{i,j} \{ \gamma(g_{ij}), \gamma(g^{ij}) \},\]

where $g_{ij}$ and $g^{ij}$ are the matrix coefficients of $g$ and $g^{-1}$, respectively. By [GTY13, Proposition 5.2.4] the group $GL_n(K)$ equipped with the pseudometric $d(g,h) = l_{\gamma}(g^{-1}h)$ has finite asymptotic dimension for every discrete norm $\gamma$. Let us review the proof.

The subset $\mathcal{O} := \{ x \in K \mid \gamma(x) \leq 1 \}$ is a subring of $K$ called the ring of integers and $\mathfrak{m} := \{ x \in K \mid \gamma(x) < 1 \}$ is a principal ideal in $\mathcal{O}$. Let $\pi$ be a fixed generator of $\mathfrak{m}$ and let $D$ denote the subgroup of diagonal matrices with powers of $\pi$ on the diagonal. Let $U$ denote the unipotent upper triangular matrices. By [GTY13, Lemma 5.2.5] the group $U$ has asymptotic dimension zero. We have $D \cong \mathbb{Z}^n$ and the restriction of $l_{\gamma}$ to $D$ is given by

\[l_{\gamma}(a) := \max_i |k_i| \log \gamma(\pi^{-1})\]

where $a$ is the diagonal matrix with entries $\pi^{k_i}$ on the diagonal. The group $D$ thereby is quasi-isometric to $\mathbb{Z}^n$ with the standard metric and has asymptotic dimension $n$. Setting $T := DU$, $T$ is again a subgroup of $GL_n(K)$ for which the entries of $g$ and $g^{-1}$ are in $\mathcal{O}$. Then $GL_n(K) = TH$ by [GHW05, Lemma 5]. For $h \in H$ let $h_{ij}$ and $h^{ij}$ denote the matrix coefficients of $h$ and $h^{-1}$ respectively. By definition

\[0 \leq l_{\gamma}(h) = \log \max_{i,j} \{ \gamma(h_{ij}), \gamma(h^{ij}) \} \leq 0.\]

This implies that the inclusion $T \to GL_n(K)$ is isometric and metrically onto, i.e. for every $g \in GL_n(K)$ there exists a $t \in T$ with $d(g,t) = 0$. Hence, $GL_n(K)$ has finite asymptotic dimension with respect to the pseudometric $d$.

**Lemma 3.1.** For every discrete norm the family $\{ F \backslash GL_n(K) \}_F$, where $F$ ranges over all finite subgroups of $U$, has finite asymptotic dimension uniformly with respect to the associated pseudometric.

**Proof.** Let $F$ be a finite subgroup of $U$. Then we can consider the map

\[F \backslash T \xrightarrow{\rho_F} D.\]

We want to apply Lemma 2.5 to the family $\{ \rho_F : F \backslash T \xrightarrow{\rho} D \}_{F \subseteq U \ fin.}$. For this we have to show that for every $R > 0$ the family $\{ \rho_F^{-1}(B_R(d)) \}_{d \in D, F \subseteq U \ fin.}$ has finite asymptotic dimension uniformly. The preimage $\rho_F^{-1}(d) = \{ Fu \mid u \in U \}$ of a point $d \in D$ is isometric to $(F)^d \backslash U$ by mapping $F u d$ to $d^{-1} F u d^{-1} u d$, where $(F)^d := \{ d^{-1} f d \mid f \in F \}$. Therefore, the preimage of $B_R(d)$ for any $R > 0$ is a finite union of spaces isometric to spaces of the form $(F)^d \backslash U$ with $d' \in D$. The
Lemma 2.6

Let $a$ be a proper, left-invariant metric on $G$. Then the family $\{F\setminus U\}_{F \subseteq U \text{ fin.}}$ has asymptotic dimension zero uniformly. This also holds for the subfamily $\{F\setminus U\}_{F \subseteq U \text{ fin.}}$ of $\{F\setminus U\}_{F \subseteq U \text{ fin.}}$.

Proof. By the main theorem of Alperin [Alp87] there exists a normal subgroup $G' \leq G$ with index $[G : G'] =: N < \infty$ such that every finite subgroup of $G'$ is unipotent. Therefore, every finite subgroup $F \subseteq U$ is conjugate in $GL_n(K)$ to a finite subgroup $F' \subseteq U$. Let $g = th$ with $t \in T, h \in H$ be such that $g^{-1}F'g = F$. Since $U$ is normal in $T$, we have that $t^{-1}F't \subseteq U$ and we can assume $g \in H$ and in particular $t_r(g) = 0$. This implies that conjugation by $g$ is an isometry and induces an isometry between $F'\setminus GL_n(K)$ and $F\setminus GL_n(K)$. By Lemma 3.1 the family $\{F'\setminus GL_n(K)\}_{F' \subseteq U \text{ fin.}}$ has finite asymptotic dimension uniformly and by the above isometry therefore the family $\{F\setminus GL_n(K)\}_{F \subseteq U \text{ fin.}}$ also has finite asymptotic dimension uniformly. This also holds for the subfamily $\{F\setminus G\}_{F \subseteq G' \text{ fin.}}$.

Since $[G : G'] = N$ every finite subgroup $\tilde{F}$ of $G$ has a normal subgroup $F$ of index at most $N$ lying in $G'$. The quotient group $F\setminus \tilde{F}$ acts isometrically on $F\setminus G$. Thus, projecting the covers that give finite asymptotic dimension for $\{F\setminus G\}_{F \subseteq G' \text{ fin.}}$ down to the quotient $\{F\setminus G\}_{F \subseteq G' \text{ fin.}}$ shows that this family still has finite asymptotic dimension uniformly. \qed

Proposition 3.2. Let $G \leq GL_n(K)$ be a finitely generated subgroup. Then for every discrete norm $\gamma$ the family $\{F\setminus G\}_{F \subseteq \mathcal{F}_{\text{fin.}}}$ has finite asymptotic dimension uniformly with respect to the associated pseudometric.

Proof. Let $R > 0$ be given and let $S$ denote the partition of $U$ into $R$-connected components, i.e. two elements $u, u' \in U$ lie inside the same $S \in S$ if and only if there exists a sequence $u_0, \ldots, u_n$ with $u = u_0, u' = u_n$ and $d(u_{i-1}, u_i) \leq R$ for all $i = 1, \ldots, n$. Since $U$ has asymptotic dimension zero uniformly we have that $r := \sup_{S \in S} \text{diam } S < \infty$. Since the left action of $F$ on $U$ is isometric, if $fu = u'$ for some $f \in F, u, u' \in U$, then $f$ maps the $R$-connected component of $u$ bijectively onto the $R$-connected component of $u'$. This implies that every $R$-connected component of $F\setminus U$ is a quotient of an $R$-connected component of $U$ and in particular has diameter at most $r$. Therefore, the family $\{F\setminus U\}_{F \subseteq U \text{ fin.}}$ has asymptotic dimension zero uniformly as claimed. \qed

Theorem 3.3. Let $G \leq GL_n(K)$ be a finitely generated subgroup. There exists a proper, left-invariant metric on $G$ such that the family $\{F\setminus G\}_{F \subseteq \mathcal{F}_{\text{fin.}}}$ has finite asymptotic dimension uniformly.

Proof. The subring of $K$ generated by the matrix entries of a finite generating set for $G$ is a finitely generated domain $A$ with $G \leq GL_n(A)$ and we may replace $K$ by the (finitely generated) fraction field of $A$, thus we can assume that $K$ is a finitely generated field of positive characteristic. By [GY12, Proposition 3.4] for every finitely generated subring $A$ of $K$ there exists a finite set $N_A$ of discrete norms such that for every $k \in \mathbb{N}$ the set

$$B_A(k) = \{a \in A \mid \forall \gamma \in N_A : \gamma(a) \leq e^k\}$$
Proposition 3.2

and using Lemma 2.5 there exists it suffices to show that the Theorem 3.3

isometric to $F$ and similarly for $(p_1(F)\times p_2(F))^{(h,h')}$, where $(F)^{(h,h')}\times p_2(F))^{(h,h')}$ and similarly for $(p_1(F)\times p_2(F))^{(h,h')}$. By Theorem 2.4 it suffices to show that the
family \( \{ F \setminus F' \}_{F \leq F'} \) has FDC where \( F \leq F' \) ranges over all pairs of finite subgroups of \( H_1 \times H_2 \). Let \( S_R \) denote the family of finite subgroups of \( H_1 \times H_2 \) generated by elements from \( B_R(e) \) and let \( s_R := \sup_{S \in S_R} \text{diam} \, S \). Let \( F \leq H_1 \times H_2 \) be finite. Then for every \( R > 0 \) the group \( F \) is the \( R \)-disjoint union of the cosets of \( (F \cap B_R(e)) \) and each of these has diameter at most \( s_R \). We see that the family of finite subgroups of \( H_1 \times H_2 \) has asymptotic dimension zero uniformly. This implies that the above family \( \{ F \setminus F' \}_{F \leq F'} \) also has asymptotic dimension zero uniformly since every \( R \)-connected component of \( F \setminus F' \) is a quotient of an \( R \)-connected component of \( F' \) and thus has uniformly bounded diameter. \( \square \)

**Lemma 4.2** ([GTY13, Lemma 5.2.3]). Let \( R \) be a finitely generated commutative ring with unit and let \( n \) be the nilpotent radical of \( R \),

\[
    n = \{ r \in R \mid \exists n : r^n = 0 \}.
\]

The quotient ring \( S = R/n \) contains a finite number of prime ideals \( p_1, \ldots, p_k \) such that the diagonal map

\[
    S \to S/p_1 \oplus \cdots \oplus S/p_k
\]

embeds \( S \) into a finite direct sum of domains.

**Theorem 4.3.** Let \( R \) be a commutative ring with unit and trivial nilradical and let \( G \) be a finitely generated subgroup of \( GL(n, R) \). Then \( \{ F \setminus G \}_{F \leq \mathcal{F}_{\text{fin}}} \) has FDC.

**Proof.** Because \( G \) is finitely generated we can assume that \( R \) is finitely generated as well. Since the nilradical of \( R \) is trivial, we have \( R = S \) in the notation of the previous lemma and there is an embedding

\[
    GL_n(S) \hookrightarrow GL_n(S/p_1) \times \cdots \times GL_n(S/p_k) \hookrightarrow GL_n(Q(S/p_1)) \times \cdots \times GL_n(Q(S/p_k))
\]

where \( Q(S/p_i) \) is the quotient field of \( S/p_i \). Let \( G_i \) be the image of the group \( G \) in \( GL_n(Q(S/p_i)) \). If \( S/p_i \) has positive characteristic, the family \( \{ F \setminus G_i \}_{F \leq \mathcal{F}_{\text{fin}}} \) has FDC by **Theorem 3.3**. If \( S/p_i \) has characteristic zero, then \( G_i \) is virtually torsion-free by Selberg’s Lemma and thus \( \{ F \setminus G_i \}_{F \leq \mathcal{F}_{\text{fin}}} \) has FDC by [Kas15, Theorem 4.10]. Now **Lemma 4.1** implies that the family \( \{ F \setminus G \}_{F \leq \mathcal{F}_{\text{fin}}} \) also has FDC. \( \square \)

**Proof of Theorem 1.1:** This follows directly from **Theorem 4.3** and [Kas14, Theorems 3.2.2 and 3.3.1] if \( R \) has trivial nilradical. Note that [Kas14, Theorems 3.2.2 and 3.3.1] are stronger than the similar [Kas15, Theorem A and Theorem 9.1], where an upper bound on the order of the finite subgroups is needed. For convenience we show in the appendix how the results from [Kas15] can be used to prove the theorems from [Kas14].

If the nilradical \( n \) of \( R \) is non-trivial, we have an exact sequence

\[
    1 \to (1 + M_n(n)) \cap G \to G \to H \to 1
\]

where \( H \) denotes the image of \( G \) in \( GL_n(R/n) \). Now the \( K \)-theoretic assembly map for \( H \) is split injective and \( (1 + M_n(n)) \cap G \) is nilpotent. Therefore, the preimage of every virtually cyclic subgroup of \( H \) is virtually solvable and satisfies the Farrell–Jones conjecture by [Weg15]. By [Kas, Proposition 4.1] this implies that the \( K \)-theoretic assembly map for \( G \) is split injective as well. The \( L \)-theory version of the theorem follows in the same way from the results in [Kas, Section 6]. \( \square \)
5. Dimension of the classifying space

In this section we want to prove Proposition 1.2. We will need the following result about classifying spaces. The proof is the same as the proof of Lück [Lüc00, Theorem 3.1].

**Theorem 5.1.** Let $1 \to K \to G \xrightarrow{\pi} Q \to 1$ be an exact sequence of groups. Assume that $Q$ admits a finite-dimensional model for $EQ$ and that there exists an $N \in \mathbb{N}$ such that for every finite subgroup $F \subseteq Q$ the preimage admits a model for $E_{\pi^{-1}}(F)$ of dimension at most $N$. Then there exists a finite-dimensional model for $EG$.

**Proof of Proposition 1.2.** For a group $G$ let $cdG$ be the shortest length of a projective resolution of $\mathbb{Z}$ as a $\mathbb{Z}[G]$-module and let $hdG$ be the shortest length of a flat resolution of $\mathbb{Z}$ as a $\mathbb{Z}[G]$-module. Let $gdG$ denote the minimal dimension of a model for $EG$. For a countable group $G$ by Nucinkis [Nuc04, Theorem 4.1] we have

$$hdG \leq cdG \leq hdG + 1.$$ 

Furthermore,

$$cdG \leq gdG \leq \max\{cdG, 3\},$$

where the first inequality follows from taking the cellular chain complex of $EG$ as a resolution and the second inequality follows from Lück [Lüc89, Theorem 13.19].

By Flores and Nucinkis [FN07, Theorem 1] for a solvable group with finite Hirsch length $l(G)$ it holds that $l(G) = hdG$. Note that Flores and Nucinkis use Hillman’s definition of the Hirsch rank for elementary amenable groups. It can be shown by a simple transfinite induction that for solvable groups this agrees with the definition given in the introduction. Furthermore, every solvable group with infinite Hirsch length has a subgroup with arbitrary large Hirsch length. In particular, the existence of a finite-dimensional model $X$ for $EG$ directly implies that the Hirsch rank of the solvable subgroups of $G$ is bounded by $\dim X$. It remains to prove the other direction.

Let $R$ be a fixed commutative ring with unit and let $G \leq GL_n(R)$ be finitely generated with $N \in \mathbb{N}$ an upper bound on the Hirsch rank of the solvable subgroups of $G$. Since $G$ is finitely generated, we can assume that $R$ is also finitely generated and let $n, S$ and $p_1, \ldots, p_k$ be as in Lemma 4.2. Furthermore, let $H$ denote the image of $G$ in $GL_n(S)$ and $\rho: GL_n(R) \to GL_n(S)$ the projection. Let $A$ be a finitely generated abelian subgroup of $H$. Then $\rho^{-1}(A)$ is solvable. This implies that the rank of the finitely generated abelian subgroups of $H$ is also bounded by $N$.

First let us show that $H$ admits a finite-dimensional model for $EH$. By Lemma 4.2 $H$ embeds into $GL_n(S/p_1) \times \cdots \times GL_n(S/p_k)$ and since $H$ is finitely generated, we can assume that all the domains $S/p_i$ are as well. Order them in such a way that $S/p_1, \ldots, S/p_q$ are of positive characteristic and $S/p_{q+1}, \ldots, S/p_k$ are of characteristic zero. Then $GL_n(S/p_{q+1}) \times \cdots \times GL_n(S/p_k)$ embeds into $GL_{n(k-q)}(\mathbb{C})$. Let $\pi$ denote the projection of $H$ to $GL_n(S/p_1) \times \cdots \times GL_n(S/p_q)$ and let $\pi_i$ denote the projection of $H$ to $GL_n(S/p_i)$ for $i = 1, \ldots, q$, then $\pi_i(H)$ admits a finite-dimensional model $E_i$ for $\mathbb{C}^{\pi_i(H)}$ by [DP, Corollary 5] and thus $E_1 \times \cdots \times E_q$ is a finite-dimensional model for $\mathbb{C}^{\pi(H)}$. By Theorem 5.1 it remains to show that for every finite subgroup $F \subseteq \pi(H)$ the preimage $\pi^{-1}(F)$ admits a finite-dimensional model with dimension bounded uniformly in $F$. Let $\rho$ denote the projection from $H$ to $GL_{n(k-q)}(\mathbb{C})$. Then $\rho(H)$ is virtually torsion free by Selberg’s Lemma [Sel60].
The group $\rho(\ker \pi)$ is isomorphic to $\ker \pi$ and thus $N$ is a bound on the rank of its finitely generated abelian subgroups. Furthermore, $\rho(\ker \pi)$ has finite index in $\rho(\pi^{-1}(F))$ for every finite subgroup $F \leq \pi(H)$. Thus, the rank of the finitely generated abelian subgroups of $\rho(\pi^{-1}(F))$ is also bounded by $N$. By [Kas, Proposition 3.1] this implies that the rank of the finitely generated unipotent subgroups of $\rho(\pi^{-1}(F))$ is bounded by $\frac{N(N+1)}{2}$. This implies that $\rho(\pi^{-1}(F))$ has finite virtual cohomological dimension bounded uniformly in $F$, see [AS82, Remark after Theorem 3.3]. The order of the finite subgroups in $\rho(\pi^{-1}(F))$ is bounded uniformly in $F$ since they are all contained inside the virtually torsion-free group $\rho(H)$. By [Lüć00, Theorem 1.10] this implies that there exist finite-dimensional models for $\rho(\pi^{-1}(F))$ with dimension bounded uniformly in $F$ and since $\rho: \pi^{-1}(F) \to \rho(\pi^{-1}(F))$ has finite kernel they are also models for $\rho(\pi^{-1}(F))$. This completes the proof that $H$ admits a finite-dimensional model for $E$. For every finite subgroup $F \leq H$, its preimage $A$ in $G$ is virtually nilpotent and thus elementary amenable, and the Hirsch rank of $A$ is bounded by $N$. By the inequalities from the beginning of the proof this implies that there is a model for $E$ of dimension at most $N + 2$. Using again Theorem 5.1 we conclude that there exists a finite-dimensional model for $E$.

\section*{Appendix}

In this appendix we want to prove the following

\begin{theorem}[
([Kas14, Theorem 3.2.2])].\] Let $G$ be a discrete group such that \{\(H \backslash G\)\}_{H \in \mathcal{F}_{\text{in}}} has FDC and let \(A\) be a small additive $G$-category. Assume that there is a finite-dimensional $G$-CW-model for the classifying space for proper $G$-actions $EG$. Then the assembly map in algebraic $K$-theory $H^{\mathcal{G}}_*(E; \mathbb{K}_A) \to K_*(A[G])$ is a split injection.\end{theorem}

The analogous result in $L$-theory [Kas14, Theorem 3.3.1] follows in the same way from the results of [Kas15]. We will use the notation introduced in [Kas15]. Note that in the appendix metrics are allowed to take on the value $\infty$. We will need the following equivariant version of FDC.

\begin{definition} An equivariant metric family is a family $\{ (X_\alpha, G_\alpha) \}_{\alpha \in A}$ where $G_\alpha$ is a group and $X_\alpha$ is a metric $G_\alpha$-space.\end{definition}

\begin{definition} An equivariant metric family $\mathcal{X} = \{ (X_\alpha, G_\alpha) \}_{\alpha \in A}$ decomposes over a class of equivariant metric families $\mathcal{D}$ if for every $r > 0$ and every $\alpha \in A$ there exists a decomposition $X_\alpha = U^r_\alpha \cup V^r_\alpha$ into $G_\alpha$-invariant subspaces and $r$-disjoint decompositions

\begin{align*}
U^r_\alpha &= \bigcup_{i \in I(r, \alpha)} U^r_{\alpha, i} \quad \text{and} \quad V^r_\alpha = \bigcup_{j \in J(r, \alpha)} V^r_{\alpha, j},
\end{align*}

such that $G_\alpha$ acts on $I(r, \alpha)$ and $J(r, \alpha)$ and for every $g \in G_\alpha$ we have $g U^r_{\alpha, i} = U^r_{\alpha, g i}$ and $g V^r_{\alpha, j} = V^r_{\alpha, g j}$. Furthermore, the families

$$\left\{ \left( \bigprod_{i \in I(r, \alpha)} U^r_{\alpha, i}, G_\alpha \right) \right\}_{\alpha \in A} \quad \text{and} \quad \left\{ \left( \bigprod_{j \in J(r, \alpha)} V^r_{\alpha, j}, G_\alpha \right) \right\}_{\alpha \in A}$$

have to lie in $\mathcal{D}$.\end{definition}
Notice that the underlying sets of $U^r_\alpha$ and $\bigoplus_{i \in I(r,\alpha)} U^r_{\alpha,i}$ are canonically isomorphic and in this sense the $G_\alpha$-action on $\bigoplus_{i \in I(r,\alpha)} U^r_{\alpha,i}$ is the same as the action on $U^r_\alpha$, only the metric has changed.

**Definition A.4.** An equivariant metric family $\mathcal{X}$ is called *semi-bounded*, if there exists $R > 0$ such that for all $(X, G) \in \mathcal{X}$ and $x, y \in X$ we have $d(x, y) < R$ or $d(x, y) = \infty$.

Let $eB$ denote the class of semi-bounded equivariant families. We set $eD_0 = eB$ and given a successor ordinal $\gamma + 1$ we define $eD_{\gamma + 1}$ to be the class of all equivariant metric families which decompose over $eD_\gamma$. For a limit ordinal $\lambda$ we define

$$eD_\lambda = \bigcup_{\gamma < \lambda} eD_\gamma.$$  

An equivariant metric family $\mathcal{X}$ has *finite decomposition complexity (FDC)* if $\mathcal{X}$ lies in $eD_\gamma$ for some ordinal $\gamma$.

Note that the equivariant metric family $\{(X_\alpha, \{e\})\}_{\alpha \in A}$ has FDC if and only if the metric family $\{X_\alpha\}_{\alpha \in A}$ has FDC.

A metric family $\{X_\alpha\}_{\alpha \in A}$ has uniformly bounded geometry if for every $R > 0$ there exists $N \in \mathbb{N}$ such that for every $\alpha \in A, U \subseteq X_\alpha$ with $\text{diam}(U) \leq R$ the set $U$ contains at most $N$ elements.

The following is a generalization of Ramras, Tessera and Yu [RTY14, Theorem 6.4]. The proof is analogous to the proof of [RTY14, Theorem 6.4] and can be found in [Kas14]. The additive $G$-category $A_G(X)$ is defined in [Kas15, Definition 5.1] and $A_G^c(X)$ denotes the fixed-point category. For a definition of the bounded product see [Kas15, Definition 5.11].

**Theorem A.5.** Let $\mathcal{X} = \{(X_\alpha, G_\alpha)\}_{\alpha \in A}$ be an equivariant family with FDC, and let the family $\{X_\alpha\}_{\alpha \in A}$ have bounded geometry uniformly. Then

$$\colim s K_n \left( \prod_{\alpha \in A} A_{G_\alpha}^G (P_s X_\alpha) \right) = 0$$

for all $n \in \mathbb{Z}$, where the colimit is taken over the maps induced by the inclusion of the respective Rips complexes $P_s X_\alpha$.

Furthermore, recall the following

**Theorem A.6** ([Kas15, Theorem 7.6]). Let $G$ be a discrete group admitting a finite-dimensional model for $BG$ and let $X$ be a simplicial $G$-CW complex with a proper $G$-invariant metric. Assume that for every $G$-set $J$ with finite stabilizers

$$\colim K_n \left( \prod_{j \in J} A_G(GK) \right)^G = 0,$$

where the colimit is taken over all finite subcomplexes $K \subseteq X$. Then the assembly map

$$H_*^G(X; \mathbb{K}_A) \to K_*(A[G])$$

is a split injection.

Before we can prove Theorem A.1 we need the following.
**Proposition A.7 ([Kas14, Proposition 3.2.1])**. Let $G$ be a group such that the metric family $\{H \setminus G\}_{H \in \Fin}$ has FDC. Then the equivariant metric family $\{(G, H)\}_{H \in \Fin}$ has FDC as well.

**Proof.** Let $\{\{X_i, G_i\}\}_{i \in I}$ be an equivariant metric family with $G_i \leq G$ a finite subgroup and assume $X_i \subseteq \bigsqcup A_i, G$ is a $G_i$-invariant subspace, where $A_i$ is a $G_i$-set. We prove by induction on the decomposition complexity that the family $\{\{X_i, G_i\}\}_{i \in I}$ lies in $\mathfrak{D}_\gamma$ if $\{G_i \setminus X_i\}_{i \in I} \in \mathfrak{D}_\gamma$. For the start of the induction let $\{G_i \setminus X_i\}_{i \in I}$ be in $\mathfrak{D}_0 = \mathfrak{B}$. Since $G_i \setminus X_i$ is bounded, there is $a_i \in A_i$ with $X_i \subseteq \bigsqcup_{G_i, a_i} G$. Then there exist $R > 0$ and $Y_i \subseteq G = \bigsqcup_{G_i, a_i} G \subseteq \bigsqcup A_i G$ with $\text{diam} Y_i < R$ for all $i \in I$ such that $X_i = G_i Y_i \subseteq \bigsqcup A_i G$. Let $G_i' \leq G_i$ be the stabilizer of $a_i$. Then this is equivalent

$$X_i \cong \bigsqcup_{[g] \in G_i/G_i'} gG_i' Y_i$$

with $G_i' Y_i \subseteq G$.

Let $r > 0$ be given and define $S_r := \{H \in \Fin \mid H = (S), S \subseteq B_{2R+r}(e)\}$ and $k := \max_{H \in S} |H|$. Let $g_i \in Y_i$ be a fixed base point. Let $H_i \leq G_i'$ be the subgroup generated by those $g_i \in G_i'$ with $d(Y_i, g_i Y_i) < r$. For these $g_i$ we have $d(e, g_i^{-1} g_i) < 2R + r$. Therefore, $g_i^{-1} H_i g_i \in S_r$ and $|H_i| \leq k$. We have the decomposition

$$X_i = \bigcup_{[g] \in G_i/G_i'} gH_i Y_i.$$ 

This decomposition is $r$-disjoint, since $d(g h y, g' h' y') < r$ with $g, g' \in G_i, h, h' \in H_i$ and $y, y' \in Y_i$ implies that $d(Y_i, h^{-1} g^{-1} g' h' Y_i) < r$ and so by definition the element $h^{-1} g^{-1} g' h'$ lies in $H_i$ which is equivalent to $g H_i = g' H_i$.

By definition of $H_i$ each $h \in H_i$ can be written as $h = g_1 \cdots g_l$, where $l \leq |H| \leq k$ and $g_j$ such that $d(Y_i, g_j Y_i) < r$. For every $y, y' \in Y_i$ by left-invariance and the triangle inequality we obtain

$$d(y, h y') \leq d(y, g_1 y') + d(g_1 y', g_1 g_2 y') + \cdots + d(g_1 \cdots g_l y', y') = d(y, g_1 y') + d(y, g_2 y') + \cdots + d(y, y') < lr.$$ 

Therefore $\text{diam} g H_i Y_i = \text{diam} H_i Y_i < kr$. Thus, $\{\{X_i, G_i\}\}_{i \in I}$ is $r$-decomposable over $\mathfrak{D}_0 = e \mathfrak{B}$ for every $r > 0$ and lies in $\mathfrak{D}_1$.

If $\{G_i \setminus X_i\}_{i \in I}$ lies in $\mathfrak{D}_\gamma$, then it decomposes over $\mathfrak{D}_\gamma$ and the preimages under the projection $X_i \to G_i \setminus X_i$ give a decomposition of $\{\{X_i, G_i\}\}$ over $\mathfrak{D}_{\gamma+1}$ by the induction hypothesis. Here $G_i$ acts trivially on the index set of the decomposition. The induction step for limit ordinals is trivial.

**Proof of Theorem A.1.** By [Kas15, Proposition 1.5] $G$ admits a finite dimensional model $X$ for $\underline{\mathbb{E}}G$ with a left-invariant proper metric. By Theorem A.6 we have to show that

$$\text{colim}_K K_n \left( \prod_{j \in J} A_G(GK) \right)^G = 0,$$

where the colimit is taken over all finite subcomplexes $K \subseteq X$. Since the category $\left( \prod_{j \in J} A_G(GK) \right)^G$ is equivalent to $\prod_{G_j \in G \setminus J} A_{G_j}^{G_j}(P_s G)$, where $G_j$ is the
stabilizer of $j \in J$, this is equivalent to showing that for every family of finite subgroups $\{G_i\}_{i \in I}$ over some index set $I$ the following holds
\[ \colim_{K} K_n \left( \prod_{i \in I} A_{G_i}^G (GK) \right) = 0. \]

By [Kas15, Lemma 1.8] and [Kas15, Proposition 6.3], for every finite subcomplex $K \subseteq X$ there exists $K' \subseteq X$ finite and $s > 0$ and maps $GK \to P_s G \to GK'$ such that the composition is metrically homotopic to the identity. In particular, the composition induces the identity in the $K$-theory of the associated controlled categories. Thus it remains to show
\[ \colim_{s} K_n \left( \prod_{i \in I} A_{G_i}^G (P_s G) \right) = 0. \]

Since $\{(G, H)\}_{H \in F}$ has FDC by Proposition A.7 and the category $A_{G_i}^G (P_s G)$ is equivalent to $A_{G_i}^{P_s G}$, this follows from Theorem A.5. □

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