Error estimate of a decoupled numerical scheme for the Cahn–Hilliard–Stokes–Darcy system

Wenbin Chen, Shufen Wang, Yichao Zhang
School of Mathematical Sciences, Fudan University, Shanghai 200433, China

Daozhi Han*
Department of Mathematics and Statistics, Missouri University of Science and Technology, Rolla, MO 65409, USA
*Corresponding author: handaoz@mst.edu

Cheng Wang
Department of Mathematics, University of Massachusetts Dartmouth, North Dartmouth, MA 02747, USA

and

Xiaoming Wang
Department of Mathematics, SUSTech International Center for Mathematics, National Center for Applied Mathematics Shenzhen, Guangdong Provincial Key Laboratory of Computational Sicence and Material Design, Southern University of Science and Technology, Shenzhen 518055, China

[Received on 14 June 2020; revised on 30 January 2021]

We analyze a fully discrete finite element numerical scheme for the Cahn–Hilliard–Stokes–Darcy system that models two-phase flows in coupled free flow and porous media. To avoid a well-known difficulty associated with the coupling between the Cahn–Hilliard equation and the fluid motion, we make use of the operator-splitting in the numerical scheme, so that these two solvers are decoupled, which in turn would greatly improve the computational efficiency. The unique solvability and the energy stability have been proved in Chen et al. (2017, Uniquely solvable and energy stable decoupled numerical schemes for the Cahn–Hilliard–Stokes–Darcy system for two-phase flows in karstic geometry. Numer. Math., 137, 229–255). In this work, we carry out a detailed convergence analysis and error estimate for the fully discrete finite element scheme, so that the optimal rate convergence order is established in the energy norm, i.e., in the $ℓ^∞(0, T; H^1) ∩ ℓ^2(0, T; H^2)$ norm for the phase variables, as well as in the $ℓ^∞(0, T; H^1) ∩ ℓ^2(0, T; H^2)$ norm for the velocity variable. Such an energy norm error estimate leads to a cancelation of a nonlinear error term associated with the convection part, which turns out to be a key step to pass through the analysis. In addition, a discrete $ℓ^2(0; T; H^3)$ bound of the numerical solution for the phase variables plays an important role in the error estimate, which is accomplished via a discrete version of Gagliardo–Niremberg inequality in the finite element setting.

Keywords: phase field model; two-phase flow; error analysis; unconditional stability.

1. Introduction

In many applications such as contaminant transport in karst aquifer, oil recovery in karst oil reservoir, proton exchange membrane fuel cell technology and cardiovascular modeling, multiphase flows in conduit and in porous media interact with each other, and therefore have to be considered together.
Geometric configurations that consist of both conduit and porous media are termed as karstic geometry. In this article we aim to analyze a decoupled numerical algorithm for solving the Cahn–Hilliard–Stokes–Darcy model (CHSD) for two-phase flows in karst geometry—a domain configuration with conduit interfacing porous media. We first recall the CHSD system derived in Han et al. (2014b). Let $\Omega_c$ denote the conduit region and $\Omega_m$ denote the porous media. The interface between the two parts (i.e., $\partial\Omega_c \cap \partial\Omega_m$) is denoted by $\Gamma_{cm}$, on which $n_{cm}$ is the unit normal to $\Gamma_{cm}$ pointing from $\Omega_c$ to $\Omega_m$. Then, we define $\Gamma_c = \partial\Omega_c \setminus \Gamma_{cm}$ and $\Gamma_m = \partial\Omega_m \setminus \Gamma_{cm}$, with $n_c, n_m$ being the unit outer normals to $\Gamma_c$ and $\Gamma_m$. On the interface $\Gamma_{cm}$, we denote by $\{\tau_i\}$ ($i = 1, ..., d-1$) a local orthonormal basis for the tangent plane to $\Gamma_{cm}$. A two-dimensional geometry is illustrated in Fig. 1.

In turn, the CHSD system takes the following form:

$$\rho_0 \partial_t u_c = \nabla \cdot T(u_c, P_c) - \varphi_c \nabla \mu_c, \quad \text{in } \Omega_c,$$

$$\nabla \cdot u_c = 0, \quad \text{in } \Omega_c,$$

$$\partial_t \varphi_c + \nabla \cdot (u_c \varphi_c) = \text{div}(M(\varphi_c) \nabla \mu_c), \quad \text{in } \Omega_c,$$

$$\frac{\rho_0}{\chi} \partial_t u_m + \nu(\varphi_m) \Pi^{-1} u_m = - (\nabla P_m + \varphi_m \nabla \mu_m), \quad \text{in } \Omega_m,$$

$$\nabla \cdot u_m = 0, \quad \text{in } \Omega_m,$$

$$\partial_t \varphi_m + \nabla \cdot (u_m \varphi_m) = \text{div}(M(\varphi_m) \nabla \mu_m), \quad \text{in } \Omega_m.$$

The chemical potentials $\mu_c, \mu_m$ turn out to be

$$\mu_j = \gamma \left[ \frac{1}{\epsilon} (\varphi_j^3 - \varphi_j) - \epsilon \Delta \varphi_j \right], \quad j \in \{c, m\},$$

and the Cauchy stress tensor $T$ is given by

$$T(u_c, P_c) = 2 \nu(\varphi_c) \mathbb{D}(u_c) - P_c \mathbb{I},$$

in which $\mathbb{D}(u_c) = \frac{1}{2}(\nabla u_c + \nabla u_c^T)$ and $\mathbb{I}$ is the $d \times d$ identity matrix. Here, $\rho_0$ is the density of the fluid, $M$ is the mobility satisfying $0 < M_0 \leq M \leq M_1$, $\chi$ is the porosity and $\nu$ is the viscosity satisfying $0 < \nu_0 \leq \nu \leq \nu_1$. In addition, we assume that both the mobility $M$ and the viscosity $\nu$ are Lipschitz continuous. $\Pi$ is the permeability matrix of size $d \times d$ that is assumed to be bounded, symmetric and uniformly positive definite. The parameter $\gamma$ in (1.7) is a positive constant related to the surface tension.

The CHSD system is subject to the following boundary and interface conditions.

**Boundary conditions on $\Gamma_c$ and $\Gamma_m$:**

$$u_c = 0, \quad \frac{\partial \varphi_c}{\partial n_c} = \frac{\partial \mu_c}{\partial n_c} = 0, \quad \text{on } \Gamma_c,$$

$$u_m \cdot n_m = 0, \quad \frac{\partial \varphi_m}{\partial n_m} = \frac{\partial \mu_m}{\partial n_m} = 0, \quad \text{on } \Gamma_m.$$
Interface conditions on $\Gamma_{cm}$:

\begin{align*}
\varphi_m &= \varphi_c, \quad \frac{\partial \varphi_m}{\partial n_{cm}} = \frac{\partial \varphi_c}{\partial n_{cm}}, \quad \text{on } \Gamma_{cm}, \quad (1.11) \\
\mu_m &= \mu_c, \quad M(\varphi_m) \frac{\partial \mu_m}{\partial n_{cm}} = M(\varphi_c) \frac{\partial \mu_c}{\partial n_{cm}}, \quad \text{on } \Gamma_{cm}, \quad (1.12) \\
u_m \cdot n_{cm} &= u_c \cdot n_{cm}, \quad \text{on } \Gamma_{cm}, \quad (1.13) \\
-2\nu(\varphi_c) n_{cm} \cdot \mathbb{D}(u_c) n_{cm} + P_c &= P_m, \quad \text{on } \Gamma_{cm}, \quad (1.14) \\
-\nu(\varphi_c) \tau_i \cdot \mathbb{D}(u_c) n_{cm} &= \alpha_{BJSJ} \frac{\nu(\varphi_m)}{2\sqrt{\text{tr}(\Pi)}} \tau_i \cdot u_c, \quad i = 1, \ldots, d - 1, \text{on } \Gamma_{cm}, \quad (1.15)
\end{align*}

where $\alpha_{BJSJ}$ is an empirical parameter in the Beavers–Joseph–Saffman–Jones (BJSJ) condition and $\text{tr}(\Pi)$ is the trace of $\Pi$.

Define the total energy of the system as follows:

\begin{align*}
\mathcal{E}(t) := \int_{\Omega_c} \frac{\rho_0}{2} |u_c|^2 \, dx + \int_{\Omega_m} \frac{\rho_0}{2\chi} |u_m|^2 \, dx + \gamma \int_{\Omega} \left[ \frac{\epsilon}{2} |\nabla \varphi|^2 + \frac{1}{\epsilon} F(\varphi) \right] \, dx, \quad (1.16)
\end{align*}

where $F(\varphi) = \frac{1}{4}(\varphi^2 - 1)^2$. The CHSD system (1.1)–(1.15) obeys a dissipative energy law (Chen et al., 2017):

\begin{align*}
\frac{d}{dt} \mathcal{E}(t) = -\mathcal{Q}(t) \leq 0, \quad \forall t \geq 0, \quad (1.17)
\end{align*}
where the rate of energy dissipation $\mathcal{D}$ is given by

$$
\mathcal{D}(t) = \int_{\Omega_m} v(\varphi_m) \Pi^{-1} |\mathbf{u}_m|^2 \, dx + \int_{\Omega_c} 2v(\varphi_c) |\mathbf{D}(\mathbf{u}_c)|^2 \, dx \\
+ \int_{\Omega} M(\varphi) |\nabla \mu(\varphi)|^2 \, dx + \int_{r_{cm}} \alpha_{BJS} \frac{v(\varphi)}{\sqrt{\text{trace}(\Pi)}} \sum_{i=1}^{d-1} |\mathbf{u}_c \cdot \tau_i|^2 \, dS \geq 0.
$$

(1.18)

The CHSD system (1.1)–(1.15) is systematically derived via Onsager’s extremum principle in Han et al. (2014b). The well-posedness of a variant of the CHSD model is studied in Han et al. (2014a). A decoupled unconditionally stable numerical algorithm for solving the CHSD system is proposed in Chen et al. (2017). Here, we focus on the error analysis of a similar decoupled numerical scheme (cf. Section 2) in which the computation of Stokes equations and Darcy equations are nevertheless coupled. The decoupling between the Cahn–Hilliard equation and fluid equations is accomplished by a special technique of operator splitting in which an intermediate velocity for advection in the Cahn–Hilliard equation is defined in terms of the capillarity from fluid equations. The application of this specific fractional step method for solving phase field models is first reported in Minjeaud (2013) and later in Shen & Yang (2015). To the best of our knowledge, error analysis of the decoupled scheme via the aforementioned operator splitting has not been reported elsewhere for any phase field model coupled with fluid motion.

There have been some convergence analysis works for either the Cahn–Hilliard–Navier–Stokes (Stokes) (CHNS, CHS) or the Cahn–Hilliard–Darcy (Hele–Shaw) system (CHD, CHHS) in recent years. The convergence of certain finite element numerical solutions to weak solutions of the CHNS equations was proved in Feng (2006), and a similar analysis is perform for the CHHS system in Feng & Wise (2012). Diegel et al. (2015) have established optimal convergence rates for a mixed finite element method for solving the CHS system, with first-order temporal accuracy. More recently, an optimal rate error estimate is presented for a second-order accurate numerical scheme for solving the CHNS equations in Diegel et al. (2017). A similar error estimate was also reported in Cai & Shen (2018), based on a finite element discretization of a linear, weakly coupled energy stable scheme for the CHNS system. As for the CHHS system, in which the kinematic diffusion term is replaced by a damping one, an optimal error analysis has been presented in Chen et al. (2016) and Liu et al. (2017), in the framework of finite difference and finite element spatial approximations, respectively.

The CHSD system consist of the CHS and the CHD equations, coupled together via a set of domain interface boundary conditions. Hence, the advection in the Cahn–Hilliard flow is involved with both the Stokes and the Darcy velocity fields. While the Stokes velocity has a regularity of $L^2(0, T; H^1)$, the Darcy velocity is only of $L^\infty(0, T; L^2)$. With the $L^2(0, T; H^1)$ bound of the velocity field, a uniform maximum norm estimate of the phase has been derived, which significantly simplifies the error analysis for the CHNS system (Diegel et al., 2017) and the CHS equations (Diegel et al., 2015). On the other hand, for the CHD system, only an $L^p(0, T; L^\infty)$ bound (with a finite value of $p$) could be established for the phase variable, as analyzed in Liu et al. (2017). The lack of uniform bound of the phase variable has dramatically complicated the error analysis of the nonlinear advection associated with the Cahn–Hilliard equation. A similar difficulty is encountered here for the error analysis of the CHSD system. To overcome this subtle difficulty, we perform an $L^2(0, T; H^3)$ bound estimate of the phase variable in the numerical solution, which is accomplished by the usage of a discrete Gagliardo–Nirenberg inequality in the finite element setting. This bound will play an important role to
pass through the error estimate. Such a technique has been applied in the analysis for the CHHS system in the existing literature, as reported in Chen et al. (2016, 2019) and Liu et al. (2017). Moreover, the CHSD system contains a coupling between the CHS and CHD equations, the corresponding estimates are expected to be even more challenging than the ones for the CHHS model.

The rest of the article is organized as follows. In Section 2, we introduce the weak formulation of the CHSD system and present the decoupled numerical scheme. Some preliminary analysis including the stability estimates are gathered in Section 3. The detailed error analysis of the numerical scheme is carried out in Section 4. Finally, some concluding remarks are provided in Section 5.

2. The numerical scheme

2.1 The weak formulation

For the CHSD problem, we introduce the following spaces:

\[ \mathbf{H}(\text{div}; \Omega) := \{ \mathbf{w} \in L^2(\Omega_j) \mid \nabla \cdot \mathbf{w} \in L^2(\Omega_j) \}, \quad j \in \{c, m\}, \]

\[ \mathbf{H}_{c,0} := \{ \mathbf{w} \in H^1(\Omega_c) \mid \mathbf{w} = 0 \text{ on } \Gamma_c \}, \]

\[ \mathbf{H}_{c,\text{div}} := \{ \mathbf{w} \in \mathbf{H}_{c,0} \mid \nabla \cdot \mathbf{w} = 0 \}, \]

\[ \mathbf{H}_{m,0} := \{ \mathbf{w} \in \mathbf{H}(\text{div}; \Omega_m) \mid \mathbf{w} \cdot \mathbf{n}_m = 0 \text{ on } \Gamma_m \}, \]

\[ \mathbf{H}_{m,\text{div}} := \{ \mathbf{w} \in \mathbf{H}_{m,0} \mid \nabla \cdot \mathbf{w} = 0 \}, \]

\[ X_m := H^1(\Omega_m) \cap L^2(\Omega_m). \]

Here, \( L^2_0(\Omega_m) \) is a subspace of \( L^2 \) whose elements are of mean zero. We also use the notation \( L^2_0(\Omega) \), which is defined similarly and will be used later. We denote \((\cdot, \cdot)_c, (\cdot, \cdot)_m\) the inner products on the spaces \( L^2(\Omega_c), L^2(\Omega_m) \), respectively (also for the corresponding vector spaces). The inner product on \( L^2(\Omega) \) is simply denoted by \((\cdot, \cdot)\). In turn, it is clear that

\[ (u, v) = (u_m, v_m)_m + (u_c, v_c)_c, \quad \| u \|^2_{L^2(\Omega)} = \| u_m \|^2_{L^2(\Omega_m)} + \| u_c \|^2_{L^2(\Omega_c)}, \]

where \( u_m := u|_{\partial \Omega_m} \) and \( u_c := u|_{\partial \Omega_c}. \) We will suppress the dependence on the domain in the \( L^2 \) norm if there is no ambiguity. And also, \( H' \) stands for the dual space of \( H \) with the duality induced by the \( L^2 \) inner product. For simplicity, we denote \( \| \cdot \| := \| \cdot \|_{L^2} \) and \( \| \cdot \|_p := \| \cdot \|_{L^p} \) for \( 1 \leq p \leq \infty, p \neq 2. \) In addition, the notation \( \| \cdot \|_{cm} \) is introduced as the \( L^2 \) norm on the interface \( \Gamma_{cm}. \) For all the functions \( f, \bar{f} \) represents the mean value of \( f \) on its domain.

The definition of the weak formulation of the three-dimensional CHSD system is given below. The two-dimensional case could be similarly defined with slight changes in time integrability of the functions.

**Definition 2.1** Suppose that \( d = 3 \) and \( T > 0 \) is arbitrary. We consider the initial data \( \varphi_0 \in H^1(\Omega), u_c(0) \in H_{c,\text{div}}, u_m(0) \in H_{m,\text{div}}. \) The functions \((u_c, P_c, u_m, P_m, \varphi, \mu)\) with the following properties

\[ u_c \in L^\infty(0, T; L^2(\Omega_c)) \cap L^2(0, T; H_{c,0}'), \quad \frac{d u_c}{dt} \in L^2(0, T; (H_{c,0}')'), \]

are unique solutions of the CHSD system (2.1).
\(u_m \in L^\infty(0, T; L^2(\Omega_m)) \cap L^2(0, T; H_{m,0}), \frac{\partial u_m}{\partial t} \in L^4(0, T; (H_{m,0})'), \)
\(P_c \in L^4(0, T; L^2(\Omega_c)), \quad P_m \in L^4(0, T; X_m), \)
\(\varphi \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)), \varphi_i \in L^2(0, T; (H^1(\Omega))'), \)
\(\mu \in L^2(0, T; H^1(\Omega)), \)

is called a finite energy weak solution of the CHSD system (1.1)–(1.15), if the following conditions are satisfied.

1. For any \(v, \phi \in H^1(\Omega), \)
\[
(\partial_t \varphi, v) + (\rho(\varphi) \nabla \mu(\varphi), \nabla v) - (u \varphi, \nabla v) = 0,
\]
\[
\gamma \left[ \frac{1}{\varepsilon} (f(\varphi), \phi) + \epsilon (\nabla \varphi, \nabla \phi) \right] - (\mu(\varphi), \phi) = 0, \quad f(\varphi) := \varphi^3 - \varphi.
\]

2. For any \(v_c \in H_{c,0} \) and \(q_c \in L^2(\Omega_c), \)
\[
\rho_0 (\partial_t u_c, v_c) + a_c (u_c, v_c) + b_c (v_c, P_c) + \int_{\Gamma_{cm}} P_m (v_c \cdot n_{cm}) \, ds - b_c (u_c, q_c) + (\varphi_c \nabla \mu(\varphi_c), v_c)_c = 0,
\]
where
\[
a_c (u_c, v_c) = 2 \left( v(\varphi_c) \nabla (u_c), \nabla (v_c) \right)_c + \sum_{I=1}^{d-1} \int_{\Gamma_{cm}} \alpha_{BJ} \frac{v(\varphi)}{\sqrt{\text{tr}([T])}} (u_c \cdot \tau_i) (v_c \cdot \tau_i) \, ds,
\]
\[
b_c (v_c, q_c) = - (\nabla \cdot v_c, q_c)_c.
\]

3. For any \(v_m \in H_{m,0} \) and \(d_m \in H^1(\Omega_m), \)
\[
\rho_0 (\partial_t u_m, v_m) + a_m (u_m, v_m) + b_m (v_m, P_m) - b_m (u_m, q_m) + (\varphi_m \nabla \mu(\varphi_m), v_m)_m - \int_{\Gamma_{cm}} u_c \cdot n_{cm} q_m \, ds = 0,
\]
where
\[
a_m (u_m, v_m) = (v(\varphi_m) [T]^{-1} u_m, v_m)_m,
\]
\[
b_m (v_m, q_m) = (v_m, \nabla q_m)_m.
\]

4. \(\varphi|_{t=0} = \varphi_0(x), u_c|_{t=0} = u_c(0), u_m|_{t=0} = u_m(0). \)
(5) The finite energy solution satisfies the energy inequality
\[ E(t) + \int_s^t \mathcal{D}(\tau) \, d\tau \leq E(s), \]  
for all \( t \in [s, T) \) and almost all \( s \in [0, T) \) (including \( s = 0 \)), where the total energy \( E \) is given by (1.16).

### 2.2 The numerical scheme

Let \( \tau > 0 \) be the time step size, \( K = [T/\tau] \), and set \( t^k = k\tau \) for \( 0 \leq k \leq K \). Similarly, we denote \( \mathbf{u}^k \) as a numerical approximation to \( \mathbf{u}(t^k) = \mathbf{u}(k\tau) \), with a notation \( \mathbf{u}(i) := \mathbf{u}(i, t) \) for simplicity. Let \( \mathcal{T}_h \) and \( \mathcal{T}_m \) be a quasi-uniform triangulation of the domain \( \Omega_c \) and \( \Omega_m \) with mesh size \( h \). Then, \( \mathcal{T}^h := \mathcal{T}_c \cup \mathcal{T}_m \) forms a triangulation of the whole domain \( \Omega \). \( \mathcal{T}_c^h \) and \( \mathcal{T}_m^h \) coincide on the interface \( \Gamma_{cm} \). Let \( Y_h \) denote the finite element approximation of \( H^1(\Omega) \), such as
\[ Y_h = \{ \varphi_h \in C(\bar{\Omega}) | \varphi_h |_{K} \in P_r(K), \forall K \in \mathcal{T}_h \}. \]

Additionally, we introduce \( \hat{Y}_h := Y_h \cap L^2(\Omega) \). Let \( X_h, Y_h, X_m, Y_m \) be the finite element approximation of \( H_{c,0}, L^2(\Omega_c), H_{m,0}, X_m \), respectively, while the approximation polynomials have adequate degrees. We assume that \( X_h \) and \( Y_h \) are stable approximation spaces for Stokes velocity and pressure in the sense that
\[ \sup_{\varphi_h \in X_h} \frac{\langle \nabla \cdot \varphi_h, q_h \rangle}{||\varphi_h||_{H^1}} \geq c ||q_h||, \forall q_h \in Y_h. \]  
(2.11)

The validity of such an inf-sup condition for some standard finite element spaces can be found in Layton et al. (2002). The classical P2-P0, Taylor–Hood finite element spaces and the Mini finite element spaces are commonly adopted in practice for \( X_h \) and \( Y_h \); cf. Girault & Raviart (1986) and Layton et al. (2002). The spaces \( X_m \) and \( Y_m \) are assumed to be stable in the sense that
\[ \sup_{\varphi_h \in X_m} \frac{\langle \nabla \varphi_h, q_h \rangle}{||\varphi_h||} \geq c ||q_h||, \forall q_h \in Y_m. \]  
(2.12)

In particular, we notice that the Taylor–Hood finite element spaces satisfy the above condition.

We will focus on the error analysis of the following unconditionally energy stable scheme that decouples the computation of the Cahn–Hilliard flow from that of fluid equations, i.e., for a totally decoupled scheme; see the related descriptions in Chen et al. (2017). Given \( 0 \leq k \leq K - 1 \), find \( (\varphi^{k+1}_h, \mu^{k+1}_h, \mathbf{u}^{k+1}_{c,h}, \mathbf{u}^{k+1}_{m,h}, \mathbf{q}^{k+1}_{c,h}, \mathbf{q}^{k+1}_{m,h}) \in \bar{Y}_h \times Y_h \times X_h \times X_h \times X_h \times X_h \) such that for all \( (\psi, \varphi, \psi, \varphi) \) \( \in \bar{Y}_h \times Y_h \times X_h \times X_h \times X_h \times X_h \), there holds
\[ (\delta \varphi^{k+1}_h, \nu) + (M(\varphi^{k}_h) \nabla \mu^{k+1}_h, \nabla \nu) - (\mathbf{u}^{k+1}_h \varphi^{k}_h, \nabla \nu) = 0, \]  
(2.13a)
\[ \gamma \left[ \frac{1}{\varepsilon} (f(\varphi^{k+1}_h, \varphi^{k}_h), \phi) + \varepsilon (\nabla \varphi^{k+1}_h, \nabla \phi) \right] - (\mu^{k+1}_h, \phi) = 0, \]  
(2.13b)
\[\rho_0(\delta u_{c,h}^{k+1}, v_c) + a_c^k(u_{c,h}^{k+1}, v_c) + b_c(v_c, p_{c,h}^{k+1}) + \int_{\Gamma_{cm}} p_{m,h}^{k+1}(v_c \cdot n_{cm}) \, dS - b_c(u_{c,h}^{k+1}, q_c) + (\varphi_{c,h}^k \nabla \mu_{c,h}^{k+1}, v_c) = 0,\]  
\[\rho_0(\delta u_{m,h}^{k+1}, v_m) + a_m^k(u_{m,h}^{k+1}, v_m) + b_m(v_m, p_{m,h}^{k+1}) + (\varphi_{m,h}^k \nabla \mu_{m,h}^{k+1}, v_m) = 0,\]  
(2.13c)

where

\[f(\varphi_{h}^{k+1}, \varphi_{h}^k) := (\varphi_{h}^{k+1})^3 - \varphi_{h}^k, \quad \delta_{h} \varphi_{h}^{k+1} := \frac{\varphi_{h}^{k+1} - \varphi_{h}^k}{\tau},\]  
(2.14)

\[\tilde{u}_{h}^{k+1} = \begin{cases} 
\tilde{u}_{m,h}^{k+1}, & x \in \Omega_m, \\
\tilde{u}_{c,h}^{k+1}, & x \in \Omega_c,
\end{cases} \begin{cases} 
\rho_0 \tilde{u}_{m,h}^{k+1} - u_{m,h}^{k+1} + \varphi_{m,h}^k \nabla \mu_{m,h}^{k+1} = 0, \\
\rho_0 \tilde{u}_{c,h}^{k+1} - u_{c,h}^{k+1} + \varphi_{c,h}^k \nabla \mu_{c,h}^{k+1} = 0,
\end{cases}\]  
(2.15)

\[a_c^k(u_{c,h}^{k+1}, v_c) = 2(\nu(\varphi_{c,h}) \nabla(u_{c,h}^{k+1}), \nabla(v_c)) \|$ of the weak solution to CHSD system (1.1)–(1.15), we define
\[\tilde{\nabla} = \tilde{\nabla}_{J,S}, \quad \tilde{\nabla}_{J,S}(\nu(\varphi_{c,h}) \nabla u_{c,h}^{k+1}) \|$ of the weak solution to CHSD system (1.1)–(1.15), we define
\[\frac{1}{\sqrt{\text{tr}(I)}} \]  
(2.16)

\[b_c(v_c, q_c) = - (\nabla \cdot v_c, q_c)_c,\]  
(2.17)

\[a_m^k(u_{m,h}^{k+1}, v_m) = (\nu(\varphi_{m,h}) I^{-1} u_{m,h}^{k+1}, v_m) \]$ of the weak solution to CHSD system (1.1)–(1.15), we define
\[\frac{1}{\sqrt{\text{tr}(I)}} \]  
(2.18)

\[b_m(v_m, q_m) = (v_m, \nabla q_m)_m.\]  
(2.19)

The initial values are taken as follows:

\[\varphi_{0}^{h} = \mathcal{P} \varphi^{0}, \quad u_{j,h}^{0} = \mathcal{P}_{j,u} u_{j}^{0}, \quad j \in \{c, m\}.\]  
(2.20)

The unique solvability of the proposed scheme (2.13a)–(2.19) has been proved via a convexity analysis, and the energy stability is ensured by a careful estimate; the details could be found in Chen et al. (2017). In this article, we focus on the optimal rate convergence analysis and error estimate.

3. Some preliminary estimates

Some projections are needed in the later analysis. Ritz projection \(\mathcal{P} : H^1(\Omega) \rightarrow Y_h\),

\[(\nabla (\mathcal{P} \varphi - \varphi), \nabla v) = 0, \quad \forall \, v \in Y_h, \quad (\mathcal{P} \varphi - \varphi, 1) = 0,\]  
(3.1)

and for \(\phi = \varphi(t), \forall \, t \in [0, T]\), where \(\varphi\) is of the weak solution to CHSD system (1.1)–(1.15), we define the modified Ritz projection \(\mathcal{P}^{\phi} : H^1(\Omega) \rightarrow Y_h\),

\[(M(\phi) \nabla (\mathcal{P}^{\phi} \mu - \mu), \nabla v) = 0, \quad \forall \, v \in Y_h, \quad (\mathcal{P}^{\phi} \mu - \mu, 1) = 0.\]  
(3.2)
Stokes–Darcy projection \( \left( \mathcal{P}_{c,u}^{\phi}, \mathcal{P}_{c,p}^{\phi}, \mathcal{P}_{m,u}^{\phi}, \mathcal{P}_{m,p}^{\phi} \right) : (H_{c,0}^1(\Omega_c), H_{m,0}^1(\Omega_m)) \to (X_c^h, X_m^h, X_c^h, X_m^h) \),

which, for all \( v_c \in X_c^h, q_c \in X_m^h, v_m \in X_m^h, q_m \in M_m^h \), satisfies the following equalities:

\[
2 \left( \langle \phi \rangle_c, \mathcal{D}(\mathcal{P}_{c,u}^{\phi} u_c), \mathcal{D}(v_c) \right) + \sum_{i=1}^{d-1} \int_{\Gamma_c} \alpha_{BJSJ} \frac{\langle \phi \rangle_c}{\sqrt{\alpha(\Omega)}} \left( (\mathcal{P}_{c,u}^{\phi} u_c) \cdot \tau_i \right) (v_c \cdot \tau_i) \, dS
\]

\[
- \left( \mathcal{P}_{c,p}^{\phi} P_c, \nabla \cdot v_c \right)_c + \int_{\Gamma_c} (\mathcal{P}_{m,p}^{\phi} P_m) (v_c \cdot n_{cm}) \, dS + \left( \nabla \cdot (\mathcal{P}_{c,u}^{\phi} u_c), q_c \right)_c
\]

\[
= 2 \left( \langle \phi \rangle_c, \mathcal{D}(u_c), \mathcal{D}(v_c) \right) + \sum_{i=1}^{d-1} \int_{\Gamma_c} \alpha_{BJSJ} \frac{\langle \phi \rangle_c}{\sqrt{\alpha(\Omega)}} \left( u_c \cdot \tau_i \right) (v_c \cdot \tau_i) \, dS
\]

\[
- (P_c, \nabla \cdot v_c)_c + \int_{\Gamma_c} P_m (v_c \cdot n_{cm}) \, dS + (\nabla \cdot u_c, q_c)_c, \tag{3.3}
\]

\[
(\langle \phi \rangle_m P_i - (\mathcal{P}_{m,u}^{\phi} u_m), v_m)_m + (\nabla (\mathcal{P}_{m,p}^{\phi} P_m), v_m)_m - (\mathcal{P}_{m,u}^{\phi} u_m, \nabla q_m)_m - \int_{\Gamma_m} (\mathcal{P}_{m,u}^{\phi} u_c) \cdot n_{cm} q_m \, dS
\]

\[
= (\langle \phi \rangle_m P_i - (\mathcal{P}_{m,u}^{\phi} u_m), v_m)_m + (\nabla P_m, v_m)_m - (u_m, \nabla q_m)_m - \int_{\Gamma_m} u_c \cdot n_{cm} q_m \, dS. \tag{3.4}
\]

Especially, for \( 0 \leq k \leq K \), we rewrite the notation of the projections above as follows:

\[
\mathcal{P}^k := \mathcal{P}(\psi^k), \tag{3.5}
\]

\[
(\mathcal{P}^k_{c,u}, \mathcal{P}^k_{c,p}, \mathcal{P}^k_{m,u}, \mathcal{P}^k_{m,p}) := \mathcal{P}(\psi^k, \psi^k, \psi^k, \psi^k). \tag{3.6}
\]

What follows is a standard result of Ritz projection \( \text{Brenner & Scott (2008)} \). There exists a constant \( C > 0 \) depending on \( M_0, M_1 \), such that the Ritz projections \( \mathcal{P} \) and \( \mathcal{P}^k \) satisfies

\[
\| \mathcal{P} \varphi - \varphi \|_p + h \| \nabla (\mathcal{P} \varphi - \varphi) \|_p \leq Ch^{q+1} \| \varphi \|_{W^{q+1}_p}, \tag{3.7}
\]

\[
\| \mathcal{P}^k \varphi - \varphi \| + h \| \nabla (\mathcal{P}^k \varphi - \varphi) \| \leq Ch^{q+1} \| \varphi \|_{H^{q+1}}, \tag{3.8}
\]

for all \( \varphi \in H^{q+1}(\Omega) \), \( q \geq 0 \), \( p \in [2, \infty) \) and all \( 0 \leq k \leq K \) with \( Y_h \) consisting of polynomials of order \( q \) or higher.

For the Stokes–Darcy projection, the following error estimates have been established in Rivière & Yotov (2005), Mu & Zhu (2010) and Chen et al. (2013):

\[
\| u_c - \mathcal{P}^k_{c,u} u_c \|_{H^1(\Omega_c)} + \| u_m - \mathcal{P}^k_{m,u} u_m \| \leq h^q \left( \| u_c \|_{H^{q+1}(\Omega_c)} + \| u_m \|_{H^{q+1}(\Omega_m)} \right). \tag{3.9}
\]
Here, we introduce the linear operator $T_h : \hat{Y}_h \rightarrow \hat{Y}_h$, which is defined via the variational problem: given $\zeta \in \hat{Y}_h$, find $T_h(\zeta) \in \hat{Y}_h$ such that

$$
(\nabla T_h(\zeta), \nabla \xi) = (\zeta, \xi), \quad \forall \xi \in \hat{Y}_h. \tag{3.10}
$$

With this operator, we are able to define the following $\|\cdot\|_{-1,h}$ norm:

$$
\|\zeta\|_{-1,h} := \left\| \nabla T_h(\zeta) \right\| = \sqrt{(\nabla T_h(\zeta), \nabla T_h(\zeta))} = \sqrt{(\zeta, T_h(\zeta))}, \quad \forall \zeta \in \hat{Y}_h. \tag{3.11}
$$

We also define the discrete Laplacian, $\Delta_h : \hat{Y}_h \rightarrow \hat{Y}_h$ as follows: for any $v_h \in \hat{Y}_h$, $\Delta_h v_h \in \hat{Y}_h$ denotes the unique solution to the problem

$$
(\Delta_h v_h, \xi) = -(\nabla v_h, \nabla \xi), \quad \forall \xi \in \hat{Y}_h. \tag{3.12}
$$

We recall the following discrete Gagliardo–Nirenberg inequality from Heywood & Rannacher (1982) and Liu et al. (2017), which is needed for the uniform estimate of the order parameter $\phi_{h}^{k+1}$.

**Lemma 3.1** Suppose that $\Omega$ is a convex and polyhedral domain. Then, for any $\phi_h \in Y_h$,

$$
\|\phi_h\|_{L^\infty} \leq C \left\| \Delta_h \phi_h \right\|^{\frac{d}{2(6-d)}} \| \phi_h \|_{L^6}^{\frac{3(4-d)}{2(6-d)}} + C \| \phi_h \|_{L^6}, \quad \forall \phi_h \in Y_h, \quad d = 2, 3, \tag{3.13}
$$

and consequently,

$$
\|\phi_h - \overline{\phi_h}\|_{L^\infty} \leq C \left\| \nabla \Delta_h \phi_h \right\|^{\frac{d}{2(6-d)}} \left\| \nabla \phi_h \right\|_{L^6}^{\frac{2(1-5d)}{2(6-d)}} + C \left\| \nabla \phi_h \right\|, \quad d = 2, 3, \tag{3.14}
$$

where $\overline{\phi_h}$ is the mean value of $\phi_h$.

The following technical lemma has been proven in Diegel et al. (2017).

**Lemma 3.2** Suppose $g \in H^1(\Omega)$ and $v \in \hat{Y}_h$. Then,

$$
| (g, v) | \leq C \| \nabla g \| \| v \|_{-1,h} \tag{3.15}
$$

holds for some $C > 0$ that is independent of $h$. 

We also recall the inverse inequality
\[
\|\varphi_h\|_{W^q} \leq Ch^{d/q - d/p} h^{l-m} \|\varphi_h\|_{W^l_p}, \quad \forall \varphi_h \in Y_h, \tag{3.16}
\]
for all \(1 \leq p \leq q \leq \infty, \ 0 \leq l \leq m \leq 1\).

The following trace theorem is necessary for the estimate of certain interface boundary terms.

**Lemma 3.3** Suppose \(v \in H^1(\Omega)\). Then,
\[
\|v\|_{L^2(\partial \Omega)} \leq C\|v\|_{H^1(\Omega)}. \tag{3.17}
\]
In particular, for \(u_h \in H_{c,0}\), there holds
\[
\|u_h\|_{L^2(F_{\gamma,0})} \leq C\|\mathbb{D}(u_h)\|_{L^2(\gamma,0)}. \tag{3.18}
\]

Now, we derive some stability estimate of the scheme (2.13a)–(2.20). The following estimates are direct consequence of the discrete energy law established in Chen et al. (2017).

**Lemma 3.4** Let \((\varphi_h^{k+1}, \mu_h^{k+1}, u_{c,h}^{k+1}, P_{c,h}^{k+1}, u_{m,h}^{k+1}, P_{m,h}^{k+1}) \in Y_h \times Y_h \times \mathbb{X}_c^h \times M_c^h \times \mathbb{X}_m^h \times M_m^h\) be the unique solution of (2.13a)–(2.20) for \(0 \leq k \leq K - 1\). Then, there exists a constant \(C > 0\) dependent on the initial data such that
\[
\max_{0 \leq k \leq K} \left[ \|u_h^k\|^2 + \|u_{m,h}^k\|^2 + \|\mu_h^k\|^2 + \|\varphi_h^k - 1\|^2 + \|\nabla \varphi_h^k\|^2 \right] \leq C, \tag{3.19}
\]
\[
\max_{0 \leq k \leq K} \|\varphi_h^k\|_{H_1} \leq C, \tag{3.20}
\]
\[
\sum_{k=0}^{K-1} \left[ \tau \|\nabla \mu_h^{k+1}\|^2 + \tau \alpha_k(u_{c,h}^{k+1}, u_{c,h}^k) + \tau \|u_{m,h}^{k+1}\|^2 + \|u_{m,h}^{k+1} - u_{m,h}^k\|^2 + \|u_{c,h}^{k+1} - u_{c,h}^k\|^2 + \|\nabla (\varphi_h^{k+1} - \varphi_h^k)\|^2 \right] \leq C \tag{3.21}
\]
hold for every \(0 \leq k \leq K - 1, d = 2, 3\).

For the error analysis, we also need the uniform bound of the order parameter and the chemical potential for which we derive the following stability estimates; see also (Diegel et al., 2015, Lemma 2.13).

**Lemma 3.5** Let \((\varphi_h^{k+1}, \mu_h^{k+1}, u_{c,h}^{k+1}, P_{c,h}^{k+1}, u_{m,h}^{k+1}, P_{m,h}^{k+1}) \in Y_h \times Y_h \times \mathbb{X}_c^h \times M_c^h \times \mathbb{X}_m^h \times M_m^h\) be the unique solution of (2.13a)–(2.20) for \(0 \leq k \leq K - 1\). Then, there exists some constant \(C > 0\) dependent on \(\gamma\) and \(\epsilon\) such that
\[
\|\Delta_h \varphi_h^{k+1}\|^2 \leq C\|\mu_h^{k+1}\|^2 + C, \tag{3.22}
\]
\[
\|\mu_h^{k+1}\|^2 \leq \|\nabla \mu_h^{k+1}\|^2 + C. \tag{3.23}
\]

Downloaded from https://academic.oup.com/imajna/article/42/3/2621/6307434 by guest on 02 September 2022
\[
\tau \sum_{k=0}^{K-1} \left[ \| \Delta_h \varphi^{k+1}_h \|^2 + \| \mu^{k+1}_h \|_{H^1}^2 \right] \leq C(T+1),
\]

(3.24)

\[
\tau \sum_{k=0}^{K-1} \| \varphi^{k+1}_h \|_{\infty}^{\frac{4(d-6)}{d}} \leq C(T+1),
\]

(3.25)

\[
\tau \sum_{k=0}^{K-1} \left[ \| \nabla \Delta_h \varphi^{k+1}_h \|^2 + \| \varphi^{k+1}_h \|_{\infty}^{\frac{8(d-6)}{d}} \right] \leq C(T+1)
\]

(3.26)

hold for every \(0 \leq k \leq K-1, d = 2, 3\).

**Proof.** Setting \(\phi_h = \Delta_h \varphi^{k+1}_h\) in (2.13b), by the uniform bound of \(\| \varphi^{k+1}_h \|_{H^1}\) and \(\| \varphi^k_h \|\) in Lemma 3.4, we have

\[
\| \Delta_h \varphi^{k+1}_h \|^2 = -(\nabla \varphi^{k+1}_h, \nabla \Delta_h \varphi^{k+1}_h)
\]

\[
= \frac{1}{\epsilon^2} \left( f(\varphi^{k+1}_h, \varphi^k_h), \Delta_h \varphi^{k+1}_h \right) - \frac{1}{\gamma \epsilon} (\mu^{k+1}_h, \Delta_h \varphi^{k+1}_h)
\]

\[
\leq \frac{1}{\epsilon^2} \left( \| f(\varphi^{k+1}_h, \varphi^k_h) \| \| \Delta_h \varphi^{k+1}_h \| + \frac{1}{\gamma \epsilon} \| \mu^{k+1}_h \| \| \Delta_h \varphi^{k+1}_h \| \right)
\]

\[
\leq \frac{1}{\epsilon^2} \left( \| \varphi^{k+1}_h \|_{H^1}^3 + \| \varphi^k_h \| \right) \| \Delta_h \varphi^{k+1}_h \| + \frac{1}{\gamma \epsilon} \| \mu^{k+1}_h \| \| \Delta_h \varphi^{k+1}_h \| \right)
\]

\[
\leq \frac{1}{\epsilon^2} \left( C \| \varphi^{k+1}_h \|_{H^1}^3 + \| \varphi^k_h \| \right) \| \Delta_h \varphi^{k+1}_h \| + \frac{1}{\gamma \epsilon^2} \| \mu^{k+1}_h \|^2 + \frac{1}{4} \| \Delta_h \varphi^{k+1}_h \|^2
\]

\[
\leq \frac{C}{\epsilon^4} + \frac{1}{\gamma \epsilon^2} \| \mu^{k+1}_h \|^2 + \frac{1}{2} \| \Delta_h \varphi^{k+1}_h \|^2.
\]

(3.27)

Therefore, we get

\[
\| \Delta_h \varphi^{k+1}_h \|^2 \leq \frac{2}{\gamma \epsilon^2} \| \mu^{k+1}_h \|^2 + \frac{2C}{\epsilon^4},
\]

(3.28)

which in turn proves (3.22). Likewise, by taking \(\phi = \mu^{k+1}_h\) in (2.13b), one derives

\[
\| \mu^{k+1}_h \|^2 = \frac{\gamma}{\epsilon} \left( f(\varphi^{k+1}_h, \varphi^k_h), \mu^{k+1}_h \right) + \gamma \epsilon \left( \nabla \varphi^{k+1}_h, \nabla \mu^{k+1}_h \right)
\]

\[
\leq \frac{\gamma}{\epsilon} \| f(\varphi^{k+1}_h, \varphi^k_h) \| \| \mu^{k+1}_h \| + \gamma \epsilon \| \nabla \varphi^{k+1}_h \| \| \nabla \mu^{k+1}_h \|
\]

\[
\leq \frac{\gamma^2}{2 \epsilon^2} \| f(\varphi^{k+1}_h, \varphi^k_h) \|^2 + \frac{1}{2} \| \mu^{k+1}_h \|^2 + \frac{\gamma^2 \epsilon^2}{2} \| \nabla \varphi^{k+1}_h \|^2 + \frac{1}{2} \| \nabla \mu^{k+1}_h \|^2
\]

\[
\leq \frac{\gamma^2}{2 \epsilon^2} \left( \| \varphi^{k+1}_h \|_{L^6}^3 + \| \varphi^k_h \| \right)^2 + \frac{1}{2} \| \mu^{k+1}_h \|^2 + \frac{\gamma^2 \epsilon^2}{2} \| \nabla \varphi^{k+1}_h \|^2 + \frac{1}{2} \| \nabla \mu^{k+1}_h \|^2
\]

\[
\leq \frac{1}{2} \| \mu^{k+1}_h \|^2 + \frac{1}{2} \| \nabla \mu^{k+1}_h \|^2 + \frac{C \gamma^2}{2 \epsilon^2} + \frac{C \gamma^2 \epsilon^2}{2}.
\]

(3.29)
As a result, inequality (3.23) holds, i.e.,

\[ \left\| \mu_h^{k+1} \right\|^2 \leq \left\| \nabla \mu_h^{k+1} \right\|^2 + \frac{C \gamma^2}{\varepsilon^2} + C \gamma^2 \varepsilon^2. \]  \hspace{1cm} (3.30)

Moreover, the inequality (3.24) follows from (3.22), (3.23) and (3.21). By Lemma 3.1, one has

\[ \left\| \varphi_{h}^{k+1} \right\|_{\infty} \leq C \left\| \Delta_h \varphi_{h}^{k+1} \right\|_{2^{(d-6)/d}} \left\| \varphi_{h}^{k+1} \right\|_{L^6}^{\frac{3(6-d)}{6-d}} + C \left\| \varphi_{h}^{k+1} \right\|_{L^6} \]
\[ \leq C \left\| \Delta_h \varphi_{h}^{k+1} \right\|_{2^{(d-6)/d}} + C. \]  \hspace{1cm} (3.31)

Thus, an application of Young’s inequality gives

\[ \left\| \varphi_{h}^{k+1} \right\|_{\infty}^{\frac{4(6-d)}{d}} \leq \left( C \left\| \Delta_h \varphi_{h}^{k+1} \right\|_{2^{(d-6)/d}} + C \right)^{\frac{4(6-d)}{d}} \leq \left( C \left\| \Delta_h \varphi_{h}^{k+1} \right\|^{2} + C \right) . \]  \hspace{1cm} (3.32)

Subsequently, a combination of (3.24), (3.28) and (3.32) yields (3.25).

For the inequality (3.26), we observe the following identity for any \( v_h \in Y_h, \Delta_h v_h, \Delta_h^2 v_h \in \bar{Y}_h :\)

\[ \left( \nabla v_h, \nabla \Delta_h^2 v_h \right) = \left\| \nabla \Delta_h v_h \right\|^2 = \left\| \Delta_h^2 v_h \right\|_{-1, h} \]  \hspace{1cm} (3.33)

and that

\[ \left\| \left( \varphi_{h}^{k+1} \right)^3 - \varphi_{h}^{k} \right\|_{H^1}^2 = \left\| \left( \varphi_{h}^{k+1} \right)^3 - \varphi_{h}^{k} \right\|^2 + \left\| \nabla \left( \left( \varphi_{h}^{k+1} \right)^3 - \varphi_{h}^{k} \right) \right\|^2 \]
\[ \leq 2 \left\| \left( \varphi_{h}^{k+1} \right)^3 \right\|^2 + 2 \left\| \varphi_{h}^{k} \right\|^2 + 2 \left\| \nabla \left( \varphi_{h}^{k+1} \right)^3 \right\|^2 + 2 \left\| \nabla \varphi_{h}^{k} \right\|^2 \]
\[ = 2 \left\| \left( \varphi_{h}^{k+1} \right)^6 \right\|_{L^6} + 2 \left\| \varphi_{h}^{k} \right\|_{H^1}^2 + 2 \left\| \nabla \left( \varphi_{h}^{k+1} \right)^3 \right\|^2 \]
\[ \leq C \left\| \left( \varphi_{h}^{k+1} \right)^6 \right\|_{H^1} + 2 \left\| \varphi_{h}^{k} \right\|_{H^1}^{2} + 2 \left\| \nabla \varphi_{h}^{k+1} \right\|_{H^1} \]
\[ \leq C \left\| \left( \varphi_{h}^{k+1} \right)^6 \right\|_{H^1} + 2 \left\| \varphi_{h}^{k} \right\|_{H^1} + 6 \left\| \varphi_{h}^{k+1} \right\|_{H^1} \left( \frac{d}{6-d} \left\| \varphi_{h}^{k+1} \right\|_{\infty}^{\frac{4(6-d)}{d}} + \frac{6 - 2d}{6 - d} \right) \]
\[ \leq C \left\| \varphi_{h}^{k+1} \right\|_{\infty}^{\frac{4(6-d)}{d}} + C. \]  \hspace{1cm} (3.34)
Then, by taking $\phi_h = \Delta_h^2 \psi^{k+1}_h$ in (2.13b), one obtains
\[
\left\| \nabla \Delta_h \psi^{k+1}_h \right\|^2 \leq \frac{1}{\gamma^2 \epsilon^2} \left\| \nabla \mu^{k+1}_h \right\|^2 + \frac{1}{\epsilon^4} \left\| \psi^{k+1}_h \right\|_{\infty} \left\| \psi^{k+1}_h \right\|_{\infty}^{4(6-d)} + \frac{C}{\epsilon^4} + \frac{1}{2} \left\| \nabla \Delta_h \psi^{k+1}_h \right\|^2,
\]
which yields that
\[
\left\| \nabla \Delta_h \psi^{k+1}_h \right\|^2 \leq \frac{2}{\gamma^2 \epsilon^2} \left\| \nabla \mu^{k+1}_h \right\|^2 + \frac{1}{\epsilon^4} \left\| \psi^{k+1}_h \right\|_{\infty} \left\| \psi^{k+1}_h \right\|_{\infty}^{4(6-d)} + \frac{C}{\epsilon^4}.
\]
(3.36)

Also, notice that $(\psi^k_h, 1) \equiv (\psi^0_h, 1) = C$, $\forall 0 \leq k \leq K$, by taking $v_h = 1$ in (2.13a). By Lemma 3.1, we derive
\[
\left\| \psi^{k+1}_h \right\|_{\infty} \leq \left\| \psi^{k+1}_h - \psi^{k+1}_h \right\|_{\infty} + \left\| \psi^{k+1}_h \right\|_{\infty} \leq C \left\| \nabla \Delta_h \psi^{k+1}_h \right\|_{\infty} \left\| \nabla \psi^{k+1}_h \right\|_{\infty} + C \left\| \nabla \psi^{k+1}_h \right\|_{\infty} + \left| \phi^{k+1}_0 \right|,
\]
so that
\[
\left\| \psi^{k+1}_h \right\|_{\infty} \leq C \left\| \nabla \Delta_h \psi^{k+1}_h \right\|^2 + C.
\]
(3.38)

Combining (3.36), (3.38), (3.21) and (3.25), one readily derives (3.26). This completes the proof.

4. The optimal rate error analysis

In this section, we provide a convergence analysis and error estimate for the numerical scheme (2.13a)–(2.20). Further regularity assumptions for the weak solution are needed in the analysis.

Assumption 1 We assume that weak solutions to the CHSD system (2.6)–(2.9) have the following additional regularities:
\[
\psi \in L^\infty \left( 0, T; W^{1,6}(\Omega) \right) \cap L^4 \left( 0, T; H^1(\Omega) \right) \cap H^2 \left( 0, T; L^2(\Omega) \right) \cap L^\infty \left( 0, T; H^{9+1}(\Omega) \right).
\]
(4.1)
\( \mu \in L^\infty \left( 0, T; H^{q+1}(\Omega) \right), \quad (4.2) \)
\[
\begin{align*}
 \mathbf{u}_c &\in L^\infty \left( 0, T; \left[ H^{q+1}(\Omega_c) \right]^d \right) \cap W^{1,4} \left( 0, T; \left[ L^2(\Omega_c) \right]^d \right) \cap H^2 \left( 0, T; \left[ L^2(\Omega_c) \right]^d \right), \\
 \mathbf{u}_m &\in L^\infty \left( 0, T; \left[ H^{q+1}(\Omega_m) \right]^d \right) \cap W^{1,4} \left( 0, T; \left[ L^2(\Omega_m) \right]^d \right) \cap H^2 \left( 0, T; \left[ L^2(\Omega_m) \right]^d \right),
\end{align*}
\]  
\( (4.3), (4.4) \)

where \( q \geq 1 \) is the spatial approximation order.

The following assumptions are also made, on the parameters of the problem
\[
M_0 \leq M(\varphi) \leq M_1, \quad |M'| \leq C, \quad v_0 \leq v(\varphi) \leq v_1, \quad |v'| \leq C. \quad (4.5)
\]

For the weak solution \((\mathbf{u}_c, P_c, \mathbf{u}_m, P_m, \varphi, \mu)\) to the CHSD system \((2.6)-(2.9)\), we set
\[
\rho^\varphi(x, t) := \varphi(x, t) - \mathcal{P}\varphi(x, t), \quad \rho^\mu(x, t) := \mu(x, t) - \mathcal{P}^\mu(x, t),
\]
\( (4.6) \)
\[
\rho^\mathbf{u}(x, t) \bigg|_{\Omega_j} = \rho^\mathbf{u}_j(x, t) := \mathbf{u}_j(x, t) - \mathcal{P}^\mathbf{u}_j(x, t), \quad j \in \{c, m\};
\]
\( (4.7) \)

specially, for \( 0 \leq k \leq K \), \( j \in \{c, m\}, \)
\[
\rho^\varphi_k \bigg|_{\Omega_j} = \rho^\varphi_{k,j} := (\varphi^k - \mathcal{P}\varphi^k) \bigg|_{\Omega_j}, \quad \rho^\mu_k \bigg|_{\Omega_j} = \rho^\mu_{k,j} := (\mu^k - \mathcal{P}\mu^k) \bigg|_{\Omega_j},
\]
\( (4.8) \)
\[
\rho^\mathbf{u}_k \bigg|_{\Omega_j} = \rho^\mathbf{u}_{k,j} := \mathbf{u}_j^k - \mathcal{P}^\mathbf{u}_j^k, \quad \rho^\mathbf{p}_k \bigg|_{\Omega_j} = \rho^\mathbf{p}_{k,j} := \mathbf{p}_j^k - \mathcal{P}^\mathbf{p}_j^k,
\]
\( (4.9) \)

and for \( 0 \leq k \leq K - 1 \), \( j \in \{c, m\}, \)
\[
R^\varphi_{k+1} \bigg|_{\Omega_j} = R^\varphi_{k+1,j} := (\delta_j \mathcal{P}\varphi^{k+1} - \delta_j \varphi^{k+1}) \bigg|_{\Omega_j}, \quad R^\mathbf{u}_{k+1} \bigg|_{\Omega_j} = R^\mathbf{u}_{k+1,j} := \delta_j \mathcal{P}^\mathbf{u}_{j,k} - \delta_j \mathbf{u}_{j,k}.
\]
\( (4.10) \)

The error functions are defined as follows, for \( j \in \{c, m\} \) and \( 0 \leq k \leq K \):
\[
\begin{align*}
\sigma^\varphi_k \bigg|_{\Omega_j} &= \sigma^\varphi_{k,j} := (\mathcal{P}\varphi^k - \varphi^k_h) \bigg|_{\Omega_j}, \quad \epsilon^\varphi_k \bigg|_{\Omega_j} = \epsilon^\varphi_{k,j} := (\varphi^k - \varphi^k_h) \bigg|_{\Omega_j}, \\
\sigma^\mu_k \bigg|_{\Omega_j} &= \sigma^\mu_{k,j} := (\mathcal{P}\mu^k - \mu^k_h) \bigg|_{\Omega_j}, \quad \epsilon^\mu_k \bigg|_{\Omega_j} = \epsilon^\mu_{k,j} := (\mu^k - \mu^k_h) \bigg|_{\Omega_j}, \\
\sigma^\mathbf{u}_k \bigg|_{\Omega_j} &= \sigma^\mathbf{u}_{k,j} := \mathcal{P}^\mathbf{u}_{j,k} - \mathbf{u}_{j,k} - \mathcal{P}^\mathbf{u}_{j,h} - \mathbf{u}_{j,h}, \quad \epsilon^\mathbf{u}_k \bigg|_{\Omega_j} = \epsilon^\mathbf{u}_{k,j} := \mathbf{u}_j^k - \mathbf{u}_{j,h}, \\
\sigma^\mathbf{p}_k \bigg|_{\Omega_j} &= \sigma^\mathbf{p}_{k,j} := \mathcal{P}^\mathbf{p}_{j,k} - \mathbf{p}_{j,k} - \mathcal{P}^\mathbf{p}_{j,h} - \mathbf{p}_{j,h}, \quad \epsilon^\mathbf{p}_k \bigg|_{\Omega_j} = \epsilon^\mathbf{p}_{k,j} := \mathbf{p}_j^k - \mathbf{p}_{j,h}.
\end{align*}
\]  
\( (4.11)-(4.14) \)

Note that the numerical solution \( \varphi_h^k \) satisfies mass conservation by choosing \( v_h = 1 \) in \((2.13a)\), same as the weak solution \( \varphi \). Recall also that \( \varphi_0^h = \mathcal{P}\varphi^0 \). Then, by the definition of Ritz projection, we see that \( (\varphi^k, 1) = (\mathcal{P}\varphi^k, 1) = (\varphi_h^k, 1) \equiv C_0 \) for all \( 0 \leq k \leq K \). This enables one to apply Poincaré inequality to \( \rho^\varphi_k, \sigma^\varphi_k, \epsilon^\varphi_k, \delta_j \sigma^\varphi_{k+1} \) for \( 0 \leq k \leq K \). We shall also make use of the fact that \( \sigma^\varphi_k, \delta_j \sigma^\varphi_{k+1} \in \mathcal{Y}_h \).
Given any $t \in [0, T]$, the solution to the CHSD system satisfies
\begin{equation}
(\delta_t \mathcal{P} \phi^{k+1}, v) + (M(\phi^{k+1}) \nabla \phi^{k+1} \mu^{k+1}, \nabla v) - (u^{k+1} \phi^{k+1}, \nabla v) = (R^{u,k+1}, v), \tag{4.15a}
\end{equation}

\begin{equation}
\gamma \epsilon (\nabla \mathcal{P} \phi^{k+1}, \nabla \phi) - (\mathcal{P} \phi^{k+1}, \mu^{k+1}, \phi) + \frac{\gamma}{\epsilon} f(\phi^{k+1}), \phi) = (\rho^{\mu,k+1}, \phi), \tag{4.15b}
\end{equation}

\begin{equation}
\rho_0(\delta_t c_{,u} u^{k+1}_c, v_c)_c + a_c(\mathcal{P} \phi^{k+1} u^{k+1}_c, v_c) + b_c(v_c, \mathcal{P} \phi^{k+1} p^{k+1} + \int_{\Gamma_{cm}} \mathcal{P} m v^{k+1} (v_c \cdot n_m) dS - b_c(\mathcal{P} \phi^{k+1} u^{k+1}_c, q_c) + (\phi^{k+1}_c \nabla \mu^{k+1}_c, v_c)_c = \rho_0(R^{u,k+1}, v_c)_c, \tag{4.15c}
\end{equation}

\begin{equation}
\frac{\rho_0}{\chi} (\delta_t \mathcal{P} m u^{k+1}_m, v_m)_m + a_m(\mathcal{P} \phi^{k+1} u^{k+1}_m, v_m)_m + b_m(v_m, \mathcal{P} m p^{k+1} m) - b_m(\mathcal{P} \phi^{k+1} u^{k+1}_m, q_m) - \int_{\Gamma_{cm}} \mathcal{P} c_{,u} u^{k+1}_c \cdot n_m q_m dS + (\phi^{k+1}_m \nabla \mu^{k+1}_m, v_m)_m \right) = \frac{\rho_0}{\chi} (R^{u,k+1}, v_m)_m, \tag{4.15d}
\end{equation}

for all $v, \phi \in Y_h, v_j \in X^h, q_j \in M^h, j \in \{c, m\}$ and $0 \leq k \leq K - 1$.

Subtracting (2.13a)–(2.13d) from (4.15a)–(4.15d), we obtain
\begin{equation}
(\delta_t \sigma^{\phi,k+1}, v) + (M(\phi^{k+1}) \nabla \sigma^{\mu,k+1}, \nabla v) = -\left((M(\phi^{k+1}) - M(\phi^{k})) \nabla \mathcal{P} \phi^{k+1} \mu^{k+1}, \nabla v) + (u^{k+1} \phi^{k+1} - \mathcal{P} m \phi^{k}, \nabla v) + (R^{\phi,k+1}, v), \right) \tag{4.16a}
\end{equation}

\begin{equation}
\gamma \epsilon (\nabla \sigma^{\phi,k+1}, \nabla \phi) - (\sigma^{\mu,k+1}, \phi) = (\rho^{\mu,k+1}, \phi) - \frac{\gamma}{\epsilon} f(\phi^{k+1}) - f(\phi^{k+1}, \phi) \tag{4.16b}
\end{equation}

\begin{equation}
\rho_0(\delta_t \sigma^{u,k+1}, v)_c + \int_{\Gamma_{cm}} \sigma^{p,k+1}_m (v_c \cdot n_m) dS + a^k_c(\sigma^{u,k+1}, v_c) + b^k_c(v_c, \sigma^{p,k+1}) - b^k_c(\sigma^{u,k+1}, q_c) = \rho_0(R^{u,k+1}, v)_c - \left(\phi^{k+1}_c \nabla \mu^{k+1}_c - \phi^{k,c,h}_c \nabla \mu^{k+1}_c, v_c\right) - 2 \left((v(\phi^{k+1}_c) - v(\phi^{k}_c))DF(\mathcal{P} \phi^{k+1} u^{k+1}_c, \nabla (\mathcal{P} \phi^{k+1} u^{k+1}_c)) \right) \tag{4.16c}
\end{equation}

\begin{equation}
+ \sum_{i=1}^{d-1} \int_{\Gamma_{cm}} o_B(s) \nabla (\mathcal{P} \phi^{k+1} u^{k+1}_c \cdot \tau_i)(v_c \cdot \tau_i) dS, \tag{4.16c}
\end{equation}

\begin{equation}
\frac{\rho_0}{\chi} (\delta_t \sigma^{u,k+1}, v_m)_m - \int_{\Gamma_{cm}} \sigma^{u,k+1}_m \cdot n_m q_m dS + a^k_m(\sigma^{u,k+1}, v_m)_m + b_m(v_m, \sigma^{p,k+1}) - b_m(\sigma^{u,k+1}, q_m) = \frac{\rho_0}{\chi} (R^{u,k+1}, v)_m - \left(\phi^{k+1}_m \nabla \mu^{k+1}_m - \phi^{k,m}_m \nabla \mu^{k+1}_m, v_m\right) + \left((v(\phi^{k+1}) - v(\phi^{k}_m))N^{-1}(\mathcal{P} \phi^{k+1} u^{k+1}_c, \nabla (\mathcal{P} \phi^{k+1} u^{k+1}_c)) \right)_m, \tag{4.16d}
\end{equation}

for all $0 \leq k \leq K - 1, v, \phi \in Y_h, v_j \in X^h, q_j \in M^h, j \in \{c, m\}$.
Setting $v = \sigma^{\mu,k+1}$ in (4.16a), $\phi = \delta_{i}\sigma^{\phi,k+1}$ in (4.16b), $v_c = \sigma^{u,k+1}_c$, $q_c = \sigma^{p,k+1}_c$ in (4.16c), $v_m = \sigma^{u,k+1}_m$, $q_m = \sigma^{p,k+1}_m$ in (4.16d), adding the resulting equations and noticing that for $d = 2, 3$,

$$
M_0 \leq M(\varphi) \leq M_1, \quad v_0 \leq v(\varphi) \leq v_1, \quad \lambda_{\max}(\Pi) \leq \lambda, \quad \text{tr}(\Pi) \leq d\lambda,
$$

$$
\|u\|^2 = \|\Pi^{1/2}\Pi^{-1/2}u\|^2 \leq \|\Pi^{1/2}\Pi^{-1/2}u\|^2 = \lambda_{\max}(\Pi) \|\Pi^{-1/2}u\|^2 \leq \lambda \|\Pi^{-1/2}u\|^2, \quad (4.17)
$$

we derive the following error equation for the numerical scheme:

$$
M_0 \left\| \nabla \sigma^{\mu,k+1} \right\|^2 + \frac{\gamma\epsilon}{2\tau} \left( \|\nabla \sigma^{\varphi,k+1}\|^2 - \|\nabla \sigma^{\varphi,k}\|^2 + \|\nabla(\sigma^{\varphi,k+1} - \sigma^{\varphi,k})\|^2 \right) \\
+ \frac{\rho_0}{2\tau} \left( \|\sigma^{u,k+1}_c\|^2 - \|\sigma^{u,k}_c\|^2 + \|\sigma^{u,k+1}_c - \sigma^{u,k}_c\|^2 \right) + \alpha_{BJS} \frac{v_0}{\sqrt{d\lambda}} \sum_{i=1}^{d-1} \|\sigma^{u,k+1}_c \cdot \tau_i\|_{cm}^2 \\
+ 2v_0 \|D(\sigma^{u,k+1}_c)\|^2 \\
= - \left( (M(\varphi^{k+1}) - M(\varphi^k)) \nabla \theta^{k+1} \mu^{k+1}, \nabla \sigma^{\mu,k+1} \right) \\
- 2 \left( (v(\varphi^{k+1}) - v(\varphi^k)) D(\sigma^{u,k+1}_c), D(\sigma^{u,k+1}_c) \right)_c \\
- \sum_{i=1}^{d-1} \int_{S_{cm}} \frac{\alpha_{BJS}}{\sqrt{\text{tr}(\Pi)}} \frac{\nu(\varphi^{k+1}) - \nu(\varphi^k)}{\nu(\varphi_m^{k+1}) - \nu(\varphi_m^k) + \frac{\rho_0}{\chi} \left( R^{u,k+1}_m, \sigma^{u,k+1}_m \right)_m} \left( \theta^{k+1} \mu^{k+1}_c \cdot \tau_i \right) \left( \sigma^{u,k+1}_c \cdot \tau_i \right) dS \\
- \left( (v(\varphi_m^{k+1}) - v(\varphi_m^k)) \Pi^{-1} \sigma^{u,k+1}_m, \sigma^{u,k+1}_m \right)_m \\
+ \frac{\rho_0}{\chi} \left( R^{u,k+1}_c, \sigma^{u,k+1}_c \right)_c + \left( R^{u,k+1}_c, \sigma^{u,k+1}_c \right)_c \\
+ \left( \rho^{\mu,k+1}, \delta_{i}\sigma^{\phi,k+1} \right) + \left( u^{k+1} - u^{k+1}_h, \sigma^{u,k+1} \right) \\
- \frac{\gamma}{\epsilon} \left( f(\varphi^{k+1}, \varphi^{k+1}) - f(\varphi^k, \varphi^k), \delta_{i}\sigma^{\phi,k+1} \right) - \left( \varphi^{k+1} \nabla \mu^{k+1} - \varphi^k \nabla \mu^{k+1}_h, \sigma^{u,k+1} \right) \\
:= \sum_{j=1}^{11} I_j, \quad (4.18)
$$

where we have designated the eleven terms on the right-hand side of (4.18) by $I_j, j = 1, 2 \cdots 11$. Now, we estimate the $I_j$s in a series of lemmas.

**Lemma 4.1** (Estimate of the first term $I_1$). Suppose $(\varphi, \mu, u_c, u_m, P_c, P_m)$ is a weak solution to (4.15a)–(4.15d) with the additional regularities described in Assumption 1, $d = 2, 3$. Set $M_0$ as the
lower bound of the mobility $M(\phi)$. Then, the first term $I_1$ of RHS of (4.18) satisfies
\[
- \left( (M(\varphi^{k+1}) - M(\varphi_h^k)) \nabla \tilde{\theta}^{k+1} \mu^{k+1}, \nabla \sigma^{\mu,k+1} \right) \leq C \left( R^{k+1} + \| \nabla \varphi^k \|_2^2 \right) + \frac{M_0}{12} \| \nabla \sigma^{\mu,k+1} \|_2^2,
\]
for a constant $C$ independent of $\tau$ and $h$.

**Proof.** We split the term into two parts as follows:
\[
- \left( (M(\varphi^{k+1}) - M(\varphi_h^k)) \nabla \tilde{\theta}^{k+1} \mu^{k+1}, \nabla \sigma^{\mu,k+1} \right) = \left( (M(\varphi^{k+1}) - M(\varphi_h^k)) \nabla \rho^{\mu,k+1}, \nabla \sigma^{\mu,k+1} \right) \]
\[
- \left( (M(\varphi^{k+1}) - M(\varphi_h^k)) \nabla \mu^{k+1}, \nabla \sigma^{\mu,k+1} \right).
\]

By the inverse inequality, there exists a constant $\theta_1 > 0$ such that for all $0 \leq k \leq K - 1$, we have
\[
\left\| \left( (M(\varphi^{k+1}) - M(\varphi_h^k)) \nabla \rho^{\mu,k+1}, \nabla \sigma^{\mu,k+1} \right) \right\| \\
\leq C \left\| M(\varphi^{k+1}) - M(\varphi_h^k) \right\|_6 \| \nabla \rho^{\mu,k+1} \| \| \nabla \sigma^{\mu,k+1} \|_3 \\
\leq C \left\| \varphi^{k+1} - \varphi_h^k \right\|_6 h \| \mu^{k+1} \|_{H^2} h^{d/3 - d/2} \| \nabla \sigma^{\mu,k+1} \| \\
\leq Ch^{1-d/6} \left\| \varphi^{k+1} - \varphi_h^k \right\|_{H^1} \| \nabla \sigma^{\mu,k+1} \| \\
\leq \frac{C}{\theta_1} \left\| \varphi^{k+1} - \varphi_h^k \right\|_{H^1}^2 + \frac{\theta_1}{2} \| \nabla \sigma^{\mu,k+1} \|^2 \\
\leq \frac{C}{\theta_1} \left( R^{k+1} + \| \nabla \varphi^k \|_2^2 \right) + \frac{\theta_1}{2} \| \nabla \sigma^{\mu,k+1} \|^2,
\]
and similarly,
\[
\left\| \left( (M(\varphi^{k+1}) - M(\varphi_h^k)) \nabla \mu^{k+1}, \nabla \sigma^{\mu,k+1} \right) \right\| \\
\leq C \left\| M(\varphi^{k+1}) - M(\varphi_h^k) \right\|_6 \| \nabla \mu^{k+1} \|_3 \| \nabla \sigma^{\mu,k+1} \| \\
\leq C \left\| \varphi^{k+1} - \varphi_h^k \right\|_6 \| \nabla \sigma^{\mu,k+1} \| \\
\leq C \left\| \varphi^{k+1} - \varphi_h^k \right\|_{H^1} \| \nabla \sigma^{\mu,k+1} \| \\
\leq \frac{C}{\theta_1} \left( R^{k+1} + \| \nabla \varphi^k \|_2^2 \right) + \frac{\theta_1}{2} \| \nabla \sigma^{\mu,k+1} \|^2.
\]
Combining (4.21) and (4.22) and choosing $\theta_1 = \frac{M_0}{12}$, one obtains (4.19). This completes the proof. \qed
The estimates of $I_2, I_3, I_4$ in (4.18) are summarized in the following lemma.

**Lemma 4.2** (Estimates of $I_2, I_3, I_4$). The assumptions are the same as in Lemma 4.1. Then, $I_2, I_3, I_4$ of RHS in (4.18) satisfy

\[
-2 \left( \left( v(\phi_c^{k+1}) - v(\phi_{c,h}^k) \right) \mathbb{D}(A_{c,u}^{k+1} \mathbf{u}_c^{k+1}), \mathbb{D}(\sigma_c^{u,k+1}) \right)_{c} \\
\leq C \left( R^{k+1} \right. + \left. \| \nabla \phi^{k+1} \|^2 + \frac{v_0}{2} \| \mathbb{D}(\sigma_c^{u,k+1}) \|^2 \right),
\]

(4.23)

\[
- \sum_{i=1}^{d-1} \int_{I_{cm}} \mathbf{BJSI} \frac{v(\phi_c^{k+1}) - v(\phi_{c,h}^k)}{\sqrt{\text{tr}(\mathbf{I})}} \left( r_{c,u}^{k+1} \mathbf{u}_c^{k+1} \cdot \mathbf{r}_i \right) \left( r_{c}^{u,k+1} \cdot \mathbf{r}_i \right) dS \\
\leq C \left( R^{k+1} \right. + \left. \| \nabla \phi^{k+1} \|^2 + \frac{\alpha_{BJSI}}{2} \sqrt{d} \sum_{i=1}^{d-1} \| r_{c}^{u,k+1} \cdot \mathbf{r}_i \|_{cm}^2 \right),
\]

(4.24)

\[
- \left( \left( v(\phi_m^{k+1}) - v(\phi_{m,h}^k) \right) \Pi^{-1} A_{m,u}^{k+1} \mathbf{u}_m^{k+1}, \sigma_m^{u,k+1} \right)_m \\
\leq C \left( R^{k+1} \right. + \left. \| \nabla \phi^{k+1} \|^2 + \frac{v_0}{4d} \| \sigma_m^{u,k+1} \|^2 \right),
\]

(4.25)

where $C$s are constants independent of $\tau$ and $h$.

**Proof.** The inequality (4.23) is derived the same way as (4.19), that is,

\[
-2 \left( \left( v(\phi_c^{k+1}) - v(\phi_{c,h}^k) \right) \mathbb{D}(A_{c,u}^{k+1} \mathbf{u}_c^{k+1}), \mathbb{D}(\sigma_c^{u,k+1}) \right)_{c} \\
\leq 2 \left( \left( v(\phi_c^{k+1}) - v(\phi_{c,h}^k) \right) \mathbb{D}(\sigma_c^{u,k+1}), \mathbb{D}(\sigma_c^{u,k+1}) \right)_{c} + 2 \left( \left( v(\phi_c^{k+1}) - v(\phi_{c,h}^k) \right) \mathbb{D}(\mathbf{u}_c^{k+1}), \mathbb{D}(\sigma_c^{u,k+1}) \right)_{c} \\
\leq 2 \left( \| v(\phi_c^{k+1}) - v(\phi_{c,h}^k) \|_6 \| \mathbb{D}(\sigma_c^{u,k+1}) \| + \| \mathbb{D}(\sigma_c^{u,k+1}) \| \| \mathbb{D}(\mathbf{u}_c^{k+1}) \| + 2 \| v(\phi_c^{k+1}) - v(\phi_{c,h}^k) \|_6 \| \mathbb{D}(\mathbf{u}_c^{k+1}) \| \| \mathbb{D}(\sigma_c^{u,k+1}) \| \\
\leq C h^{1-d/6} \| \phi_c^{k+1} - \phi_{c,h}^k \|_6 \| \mathbf{u}_c^{k+1} \|_{H^2} \| \mathbb{D}(\sigma_c^{u,k+1}) \| + C \| \phi_c^{k+1} - \phi_{c,h}^k \|_6 \| \mathbb{D}(\sigma_c^{u,k+1}) \| \\
\leq C \| \phi_c^{k+1} - \phi_{c,h}^k \|_{H^1} \| \mathbb{D}(\sigma_c^{u,k+1}) \| \\
\leq \frac{C}{\theta} \left( R^{k+1} + \| \nabla \phi^{k+1} \|^2 + \theta \| \mathbb{D}(\sigma_c^{u,k+1}) \|^2 \right).
\]

(4.26)
With an application of Lemma 3.3, one has
\[
- \sum_{i=1}^{d-1} \int_{\Gamma_{cm}} \frac{\nu(\phi_c^{k+1}) - \nu(\phi_{c,h}^k)}{\sqrt{\nu(I)}} \left( \phi_{c,u}^{k+1} \mathbf{u}_c^{k+1} \cdot \mathbf{\tau}_i \right) \left( \sigma_c^{u,k+1} \cdot \mathbf{\tau}_i \right) \, dS
\]
\[
\leq C \sum_{i=1}^{d-1} \left\| \phi_{c}^{k+1} - \phi_{c,h}^k \right\|_{L^2(\Gamma_{cm})} \left( \left\| \rho_c u_{k+1} \right\|_{L^2(\Gamma_{cm})} + \left\| \mathbf{u}_c^{k+1} \right\|_{L^2(\Gamma_{cm})} \right) \left\| \sigma_c^{u,k+1} \cdot \mathbf{\tau}_i \right\|_{cm}
\]
\[
\leq C \sum_{i=1}^{d-1} \left\| \phi_{c}^{k+1} - \phi_{c,h}^k \right\|_{H^1(\Omega_c)} \left( \left\| \mathbf{u}_c^{k+1} \right\|_{H^2(\Omega_c)} + \left\| \mathbf{u}_c^{k+1} \right\|_{H^2(\Omega_c)} \right) \left\| \sigma_c^{u,k+1} \cdot \mathbf{\tau}_i \right\|_{cm}
\]
\[
\leq \sum_{i=1}^{d-1} \left\| \phi_{c}^{k+1} - \phi_{c,h}^k \right\|_{H^1(\Omega_c)} \left( \left\| \mathbf{u}_c^{k+1} \right\|_{H^2(\Omega_c)} + \left\| \mathbf{u}_c^{k+1} \right\|_{H^2(\Omega_c)} \right) \left\| \sigma_c^{u,k+1} \cdot \mathbf{\tau}_i \right\|_{cm}
\]
\[
\leq \frac{C}{\theta_3} \left( R^{k+1} + \left\| \nabla \phi^{k+1} \right\| \right) + \theta_3 \sum_{i=1}^{d-1} \left\| \sigma_c^{u,k+1} \cdot \mathbf{\tau}_i \right\|_{cm}^2. \tag{4.27}
\]

Likewise,
\[
- \left( \left( \nu(\phi_m^{k+1}) - \nu(\phi_{m,h}^k) \right) \Pi^{-1} \phi_{m,u}^{k+1} \mathbf{u}_{m,h}^{k+1} \sigma_{m,k+1} \right)_m \leq C \left\| \phi_{m}^{k+1} - \phi_{m,h}^k \right\|_6 \left( \left\| \rho_m u_{k+1} \right\|_{3} + \left\| \mathbf{u}_{m,h}^{k+1} \right\|_{3} + \left\| \sigma_{m,k+1} \right\| \right)
\]
\[
\leq C \left\| \phi_{m}^{k+1} - \phi_{m,h}^k \right\|_6 \left( \left\| \mathbf{u}_{m,h}^{k+1} \right\|_{H^1} + \left\| \mathbf{u}_{m,h}^{k+1} \right\|_{H^1} + \left\| \sigma_{m,k+1} \right\| \right)
\]
\[
\leq \frac{C}{\theta_3} \left( R^{k+1} + \left\| \nabla \phi^{k+1} \right\| \right) + \theta_3 \left\| \sigma_{m,k+1} \right\|_m^2. \tag{4.28}
\]

By choosing \( \theta_2 = \frac{\nu_0}{2}, \theta_3 = \alpha_{BJS} \frac{\nu_0}{\sqrt{\nu(I)}}, \theta_4 = \frac{\nu_0}{4\lambda} \), we complete the proof of the lemma. \(\square\)

The next lemma contains the estimates of \( I_5, I_6, I_7, I_8 \). The assumptions are the same as in Lemma 4.1. One has the following estimates on the terms \( I_5, I_6, I_7, I_8 \) of RHS in (4.18):
\[
\left| \frac{\rho_{0} \left( R_{m}^{u,k+1}, \sigma_{m}^{u,k+1} \right)_m }{\chi} \right| \leq C \left\| R_{m}^{u,k+1} \right\|_m^2 + \frac{\nu_0}{4\lambda} \left\| \sigma_{m,k+1} \right\|_m^2, \tag{4.29}
\]
\[
\left| \rho_{0} \left( R_{c}^{u,k+1}, \sigma_{c}^{u,k+1} \right)_c \right| \leq C \left\| R_{c}^{u,k+1} \right\|_c^2 + \frac{\nu_0}{2} \left\| \nabla (\sigma_{c}^{u,k+1}) \right\|_c^2, \tag{4.30}
\]
\[
\left| \left( R_{\phi}^{\mu,k+1}, \sigma_{\mu,k+1} \right) \right| \leq C \left\| R_{\phi}^{\mu,k+1} \right\|_c^2 + \frac{M_0}{12} \left\| \nabla \sigma_{\mu,k+1} \right\|_c^2, \tag{4.31}
\]
\[
\left| \left( \rho_{\mu,k+1}, \delta_{\phi} \sigma_{\phi,k+1} \right) \right| \leq \frac{C}{\theta_8} \left\| \nabla \rho_{\mu,k+1} \right\|_{-1,h}^2 + \theta_8 \left\| \delta_{\phi} \sigma_{\phi,k+1} \right\|_{-1,h}^2. \tag{4.32}
\]
Proof. In fact, (4.29) is a direct result of the Cauchy–Schwarz inequality. Thanks to the Poincaré inequality and Korn’s inequality (Brenner & Scott, 2008), for any \( \theta_6 > 0 \), we have

\[
\left| \rho_0 \left( R_{c}^{u,k+1}, \sigma_c^{u,k+1} \right) \right|_c \leq \left\| \rho_0 R_{c}^{u,k+1} \right\| \left\| \sigma_c^{u,k+1} \right\| \leq C \left\| \rho_0 R_{c}^{u,k+1} \right\| \mathbb{D}(\sigma_c^{u,k+1}) \leq \frac{C}{\theta_6} \left( R_{c}^{u,k+1} \right)^2 + \theta_6 \mathbb{D}(\sigma_c^{u,k+1})^2, \quad \forall \theta_6 > 0. \tag{4.33}
\]

We notice that \( (R_{c}^{u,k+1}, 1) = 0 \) holds for all \( 0 \leq k \leq K - 1 \) by choosing the test function \( v = 1 \) in (2.6) and using the mass conservation of Ritz projection. Let \( \overline{\sigma_{\mu,k+1}} \) be the mean value of \( \sigma_{\mu,k+1} \) on \( \Omega \), it follows that

\[
\left| (R_{c}^{u,k+1}, \sigma_{\mu,k+1}) \right| = \left| (R_{c}^{u,k+1}, \sigma_{\mu,k+1} - \overline{\sigma_{\mu,k+1}}) \right| \leq \left\| R_{c}^{u,k+1} \right\| \left\| \sigma_{\mu,k+1} - \overline{\sigma_{\mu,k+1}} \right\| \leq C \left\| R_{c}^{u,k+1} \right\| \left\| \nabla \sigma_{\mu,k+1} \right\| \leq \frac{C}{\theta_7} \left( R_{c}^{u,k+1} \right)^2 + \theta_7 \left( R_{c}^{u,k+1} \right)^2, \quad \forall \theta_7 > 0. \tag{4.34}
\]

For the eighth term of the RHS of (4.18), we apply Lemma 3.2 and recall \( \delta_j \sigma_{\varphi,k+1} \in \mathcal{V}_h \) for all \( 0 \leq k \leq K - 1 \). Thus, for any \( \theta_8 > 0 \), one gets

\[
\left| \left( \rho_{\mu,k+1}, \delta_j \sigma_{\varphi,k+1} \right) \right| \leq C \left\| \nabla \rho_{\mu,k+1} \right\| \left\| \delta_j \sigma_{\varphi,k+1} \right\|_{-1,h} \leq \frac{C}{\theta_8} \left( \nabla \rho_{\mu,k+1} \right)^2 + \theta_8 \left( \delta_j \sigma_{\varphi,k+1} \right)^2. \tag{4.35}
\]

The proof is complete upon setting \( \theta_6 = \frac{\nu_0}{\tau}, \theta_7 = \frac{M_0}{12} \). \( \square \)

The following lemma gives an estimate of the ninth term \( I_9 \) on the RHS of (4.18).

LEMMA 4.4 (Estimate of \( I_9 \)). The assumptions are the same as in Lemma 4.1. Then, for any \( 0 \leq k \leq K - 1 \), the following inequality holds for a constant \( C \) that is independent of \( \tau \) and \( h \):

\[
\left( u^{k+1} \varphi^{k+1} - u_h^{k+1} \varphi_h^{k}, \nabla \sigma_{\mu,k+1} \right) \leq \frac{\tau}{\rho_0} \left( \varphi_h^{k+1} \nabla \sigma_{\mu,k+1} \right)^2 - \frac{\tau X}{\rho_0} \left( \varphi_h^{k+1} \nabla \sigma_{\mu,k+1} \right)^2 + \frac{M_0}{12} \left( \nabla \sigma_{\mu,k+1} \right)^2 + C \left( 1 + \left( \nabla \rho_{\mu,k+1} \right)^2 \right) + R^{k+1} + \left( \nabla e_{\varphi,k} \right)^2 + \left( \varphi_h^{k+1} \right)^2 \left( R^{k+1} + \left( e_{u,k} \right)^2 \right). \tag{4.36}
\]
Proof. We first split the term on $\Omega_m$ as follows:

$$u^{k+1}_m \varphi^{k+1}_m - \overline{u}^{k+1}_{m,h} \varphi^{k}_{m,h}$$

$$= u^{k+1}_m \varphi^{k+1}_m - \left( u^{k+1}_m - \frac{\tau X}{\rho_0} \varphi^{k+1}_{m,h} \nabla \mu^{k+1}_{m,h} \right) \varphi^{k}_{m,h} = u^{k+1}_m \varphi^{k+1}_m - \overline{u}^{k+1}_{m,h} \varphi^{k}_{m,h} + \frac{\tau X}{\rho_0} (\varphi^{k}_{m,h})^2 \nabla \mu^{k+1}_{m,h}$$

$$= u^{k+1}_m \left( \varphi^{k+1}_m - \varphi^{k}_m + e^{\varphi,k}_m \right) + \varphi^{k}_{m,h} \left( u^{k+1}_m - \overline{u}^{k+1}_{m,h} + e^{u,k}_m \right) - \frac{\tau X}{\rho_0} (\varphi^{k}_{m,h})^2 \left( \nabla \rho^\mu_{m,k+1} + \nabla \sigma_{m,k+1} - \nabla \mu_{m}^{k+1} \right)$$

$$= I_m - \frac{\tau X}{\rho_0} (\varphi^{k}_{m,h})^2 \nabla \sigma_{m,k+1},$$

(4.37)

where

$$I_m \triangleq u^{k+1}_m \left( \varphi^{k+1}_m - \varphi^{k}_m + e^{\varphi,k}_m \right) + \varphi^{k}_{m,h} \left( u^{k+1}_m - \overline{u}^{k+1}_{m,h} + e^{u,k}_m \right) - \frac{\tau X}{\rho_0} (\varphi^{k}_{m,h})^2 \left( \nabla \rho^\mu_{m,k+1} + \nabla \mu_{m}^{k+1} \right).$$

(4.38)

In light of $\| \varphi^k_h \|_{H^1} \leq C$ and $\| e^{\varphi,k} \|_{H^1} \leq C \| \nabla e^{\varphi,k} \|$, one has

$$\| I_m \| \leq \| u^{k+1}_m \left( \varphi^{k+1}_m - \varphi^{k}_m + e^{\varphi,k}_m \right) \| + \| \varphi^{k}_{m,h} \left( u^{k+1}_m - \overline{u}^{k+1}_{m,h} + e^{u,k}_m \right) \| + \frac{\tau X}{\rho_0} (\varphi^{k}_{m,h})^2 \left( \nabla \rho^\mu_{m,k+1} + \nabla \mu_{m}^{k+1} \right)$$

$$\leq \| u^{k+1}_m \|_4 \| \varphi^{k+1}_m - \varphi^{k}_m + e^{\varphi,k}_m \|_4 + \| \varphi^{k}_{m,h} \|_\infty \| u^{k+1}_m - \overline{u}^{k+1}_{m,h} + e^{u,k}_m \| + C \| \varphi^{k}_{m,h} \|_6^2 \| \nabla \rho^\mu_{m,k+1} + \nabla \mu_{m}^{k+1} \|_6$$

$$\leq C \| \varphi^{k+1}_m - \varphi^{k}_m + e^{\varphi,k}_m \|_{H^1} + \| \varphi^{k}_{m,h} \|_\infty \left( \| u^{k+1}_m - \overline{u}^{k+1}_{m,h} + e^{u,k}_m \| \right) + C \| \varphi^{k}_{m,h} \|_{H^1}^2 \left( \| \nabla \rho^\mu_{m,k+1} \|_6 + C \right)$$

$$\leq C \| \varphi^{k+1}_m - \varphi^{k}_m \|_{H^1} + C \| \nabla e^{\varphi,k} \| + \| \varphi^{k}_{m,h} \|_\infty \left( \| u^{k+1}_m - \overline{u}^{k+1}_{m,h} + e^{u,k}_m \| \right) + C \left( 1 + \| \nabla \rho^\mu_{m,k+1} \|_6 \right).$$

(4.39)

By Young’s inequality, we obtain

$$\| I_m \|^2 \leq C \tau^2 \left( 1 + \| \nabla \rho^\mu_{m,k+1} \|_6^2 \right) + C \| \varphi^{k+1}_m - \varphi^{k}_m \|_{H^1}^2 + C \| \nabla e^{\varphi,k} \|^2$$

$$+ C \| \varphi^{k}_{m,h} \|_\infty^2 \left( \| u^{k+1}_m - \overline{u}^{k+1}_{m,h} \|^2 + \| e^{u,k}_m \|^2 \right)$$

$$\leq C \tau^2 \left( 1 + \| \nabla \rho^\mu_{m,k+1} \|_6^2 \right) + CR^{k+1} + C \| \nabla e^{\varphi,k} \|^2 + C \| \varphi^{k}_{m,h} \|_\infty^2 \left( K^{k+1} + \| e^{u,k}_m \|^2 \right).$$

(4.40)

Similarly, with the following definition:

$$I_c \triangleq u^{k+1}_c \left( \varphi^{k+1}_c - \varphi^{k}_c + e^{\varphi,c}_k \right) + \varphi^{k}_{c,h} \left( u^{k+1}_c - \overline{u}^{k+1}_{c,h} + e^{u,c}_c \right) - \frac{\tau}{\rho_0} (\varphi^{k}_{c,h})^2 \left( \nabla \rho^\mu_{c,k+1} + \nabla \mu_{c}^{k+1} \right),$$

(4.41)
one gets

$$\| I_c \|^2 \leq C \tau^2 \left( 1 + \| \nabla \rho^{\mu,k+1} \|_6^2 \right) + CR^{k+1} + C \| \nabla e^{\psi,k} \|^2 + C \| \varphi_h \|_\infty^2 \left( R^{k+1} + \| e^{u,k} \|^2 \right).$$

(4.42)

Consequently, the following inequality is valid:

$$\| I_c \|^2 + \| I_m \|^2 \leq C \tau^2 \left( 1 + \| \nabla \rho^{\mu,k+1} \|_6^2 \right) + CR^{k+1} + C \| \nabla e^{\psi,k} \|^2 + C \| \varphi_h \|_\infty^2 \left( R^{k+1} + \| e^{u,k} \|^2 \right).$$

(4.43)

Thus, for constant $\theta_9 > 0$, there holds

$$\left( u^{k+1} \varphi^{k+1} - \bar{u}_h^{k+1} \varphi_h^k, \nabla \sigma^{\mu,k+1} \right)$$

$$= \left( u_c^{k+1} \varphi_c^{k+1} - \bar{u}_{c,h}^{k+1} \varphi_{c,h}^k, \nabla \sigma_c^{\mu,k+1} \right) + \left( u_m^{k+1} \varphi_m^{k+1} - \bar{u}_{m,h}^{k+1} \varphi_{m,h}^k, \nabla \sigma_m^{\mu,k+1} \right)$$

$$= \left( I_c - \frac{\tau}{\rho_0} \left( \varphi_c^k \right)^2 \nabla \sigma_c^{\mu,k+1}, \nabla \sigma_c^{\mu,k+1} \right) + \left( I_m - \frac{\tau X}{\rho_0} \left( \varphi_m^k \right)^2 \nabla \sigma_m^{\mu,k+1}, \nabla \sigma_m^{\mu,k+1} \right)$$

$$\leq \frac{-\tau}{\rho_0} \left\| \varphi_{c,h}^k \nabla \sigma_c^{\mu,k+1} \right\|^2 - \frac{\tau X}{\rho_0} \left\| \varphi_{m,h}^k \nabla \sigma_m^{\mu,k+1} \right\|^2 + \frac{1}{4\theta_9} \left( \| I_c \|^2 + \| I_m \|^2 \right) + \theta_9 \left\| \nabla \sigma^{\mu,k+1} \right\|^2$$

$$\leq \frac{-\tau}{\rho_0} \left\| \varphi_{c,h}^k \nabla \sigma_c^{\mu,k+1} \right\|^2 - \frac{\tau X}{\rho_0} \left\| \varphi_{m,h}^k \nabla \sigma_m^{\mu,k+1} \right\|^2 + \theta_9 \left\| \nabla \sigma^{\mu,k+1} \right\|^2 + C \frac{\tau^2}{\theta_9} \left( 1 + \| \nabla \rho^{\mu,k+1} \|_6^2 \right)$$

$$+ R^{k+1} + \| \nabla e^{\psi,k} \|^2 + \| \varphi_h^k \|_\infty^2 \left( R^{k+1} + \| e^{u,k} \|^2 \right).$$

(4.44)

This proves the lemma by choosing $\theta_9 = \frac{M_0}{4\tau}$.  

The term $I_{10}$ is estimated in the following lemma.

**Lemma 4.5** (Estimate of the term $I_{10}$). The assumptions are the same as in Lemma 4.1. Then, the tenth term of RHS in (4.18) satisfies

$$\left\| \frac{\gamma'}{e} \left( f(\varphi^{k+1} - \varphi^{k+1}) - f(\varphi_h^{k+1}, \varphi_h^k, \delta_i \sigma^{\psi,k+1}) \right) \right\|$$

$$\leq \theta_{10} \left\| \delta_i \sigma^{\psi,k+1} \right\|^2_{-1,h} + \frac{C}{\theta_{10}} \left( R^{k+1} + \left( 1 + \| \varphi_h^{k+1} \|_\infty^4 \right) \left\| \nabla e^{\psi,k+1} \right\|^2 + \left\| \nabla e^{\psi,k} \right\|^2 \right).$$

(4.45)

for a constant $C$ independent of $\tau$ and $h$. 
Proof. First, we need to estimate \(|∇(f(ϕ^{k+1}, ϕ^{k+1}) - f(ϕ_h^{k+1}, ϕ_h^{k}))|\). Recall that \(f(a, b) = a^3 - b\). Hence,

\[
\|∇(f(ϕ^{k+1}, ϕ^{k+1}) - f(ϕ_h^{k+1}, ϕ_h^{k}))\| \\
≤ \|∇(ϕ^{k+1})^3 - ∇(ϕ_h^{k+1})^3\| + ∇(ϕ^{k+1} - ϕ_h^{k})\| \\
= 3(ϕ^{k+1})^2∇ϕ^{k+1} - 3(ϕ_h^{k+1})^2∇ϕ_h^{k+1} + ∇(ϕ^{k+1} - ϕ_h^{k})\| + ∇(ϕ^{k+1} - ϕ_h^{k})\|
\]

\[
≤ 3\left[(ϕ^{k+1})^2 - (ϕ_h^{k+1})^2\right]∇ϕ^{k+1} + (ϕ_h^{k+1})^2∇(ϕ^{k+1} - ϕ_h^{k})\| + ∇(ϕ^{k+1} - ϕ_h^{k})\| + ∇(ϕ^{k+1} - ϕ_h^{k})\|
\]

\[
≤ 3\left[|ϕ^{k+1} + ϕ_h^{k+1}|_6 + |e_6| + |∇ϕ^{k+1}|_6 + 3|ϕ_h^{k+1}|^2|∇ϕ_h^{k+1}| + |ϕ^{k+1} - ϕ_h^{k}| + |∇(ϕ^{k+1} - ϕ_h^{k})|\right]
\]

\[
≤ C\left(|ϕ^{k+1} + ϕ_h^{k+1}|_6 + C|ϕ_h^{k+1}|_6 + 3|ϕ_h^{k+1}|^2|∇ϕ_h^{k+1}| + |ϕ^{k+1} - ϕ_h^{k}| + |∇(ϕ^{k+1} - ϕ_h^{k})|\right)
\]

\[
≤ |∇(ϕ^{k+1} - ϕ_h^{k})| + C\left(1 + |ϕ_h^{k+1}|^2\right)|∇ϕ_h^{k+1}| + |∇(ϕ^{k+1} - ϕ_h^{k})|,
\]

which in turn yields

\[
|∇(f(ϕ^{k+1}, ϕ^{k+1}) - f(ϕ_h^{k+1}, ϕ_h^{k}))| \leq C\left[|∇(ϕ^{k+1} - ϕ_h^{k})|^2 + C\left(1 + |ϕ_h^{k+1}|^2\right)|∇ϕ_h^{k+1}|^2 + C|∇(ϕ^{k+1} - ϕ_h^{k})|^2\right] .
\]

(4.47)

Thus, by Lemma 3.2, we derive the following estimate for any \(θ_{10} > 0\):

\[
\left|\frac{γ}{ε}(f(ϕ^{k+1}, ϕ^{k+1}) - f(ϕ_h^{k+1}, ϕ_h^{k}), δ_tσ_ϕ^{k+1})\right| \\
≤ C\left[|∇(f(ϕ^{k+1}, ϕ^{k+1}) - f(ϕ_h^{k+1}, ϕ_h^{k}))| δ_tσ_ϕ^{k+1}\right]|_{-1,h} \\
≤ θ_{10}\left[δ_tσ_ϕ^{k+1}\right]^2_{-1,h} + \frac{C}{θ_{10}}\left|∇(f(ϕ^{k+1}, ϕ^{k+1}) - f(ϕ_h^{k+1}, ϕ_h^{k}))|\right|^2 \\
≤ θ_{10}\left[δ_tσ_ϕ^{k+1}\right]^2_{-1,h} + \frac{C}{θ_{10}}\left[κ^{k+1} + \left(1 + |ϕ_h^{k+1}|^4\right)|∇ϕ_h^{k+1}|^2 + |∇(ϕ^{k+1} - ϕ_h^{k})|^2\right].
\]

(4.48)

This completes the proof. □

Finally, we estimate the last term \(I_{11}\) in the following lemma.
Lemma 4.6 (Estimate of the $I_{11}$). The assumptions are the same as in Lemma 4.1. Then, for the last term $I_{11}$ of RHS in (4.18), the following inequality holds for a constant $C$ independent of $\tau$ and $h$:

$$\begin{align*}
\left| - \left( \varphi^{k+1} \nabla \mu^{k+1} - \varphi^{k}_h \nabla \mu^{k+1}_h, \sigma \mathbf{u}^{k+1} \right) \right| & \leq C \left( R^{k+1} + \left\| \nabla e_{\varphi, k} \right\|_2^2 + \frac{M_0}{12} \left\| \nabla e_{\mu, k+1} \right\|^2 + \left( 1 + C \left\| \varphi^{k}_h \right\|_{\infty} \right) \left\| \sigma \mathbf{u}^{k+1} \right\|^2. \tag{4.49} \right.
\end{align*}$$

Proof. We make use of the following decomposition:

$$\begin{align*}
\left\| \varphi^{k+1} \nabla \mu^{k+1} - \varphi^{k}_h \nabla \mu^{k+1}_h \right\| & = \left\| \left( \varphi^{k+1} - \varphi^{k}_h \right) \nabla \mu^{k+1} + \varphi^{k}_h \nabla (\mu^{k+1} - \mu^{k+1}_h) \right\|
\leq \left\| \varphi^{k+1} - \varphi^{k}_h \right\| + \left\| \varphi^{k}_h \right\| \left\| \nabla (\mu^{k+1} - \mu^{k+1}_h) \right\|.
\end{align*}$$

Then, for any $\theta_{11} > 0$, there holds

$$\begin{align*}
\left| \left( \varphi^{k+1} \nabla \mu^{k+1} - \varphi^{k}_h \nabla \mu^{k+1}_h, \sigma \mathbf{u}^{k+1} \right) \right| & \leq \left\| \varphi^{k+1} \nabla \mu^{k+1} - \varphi^{k}_h \nabla \mu^{k+1}_h \right\| \left\| \sigma \mathbf{u}^{k+1} \right\|
\leq \left[ C \left( \left\| \varphi^{k+1} \right\|_{H^1} + \left\| e_{\varphi, k} \right\|_{H^1} \right) + \left\| \varphi^{k}_h \right\|_{\infty} \left\| \nabla e_{\mu, k+1} \right\| \right] \left\| \sigma \mathbf{u}^{k+1} \right\|
\leq C \left( \left\| \varphi^{k+1} \right\|_{H^1} + \left\| e_{\varphi, k} \right\|_{H^1} \right) \left\| \sigma \mathbf{u}^{k+1} \right\| + \left\| \nabla e_{\mu, k+1} \right\| \left\| \varphi^{k}_h \right\|_{\infty} \left\| \sigma \mathbf{u}^{k+1} \right\|
\leq C \left( \left\| \varphi^{k+1} \right\|_{H^1}^2 + \left\| e_{\varphi, k} \right\|_{H^1}^2 \right) + \left\| \sigma \mathbf{u}^{k+1} \right\|^2 + \theta_{11} \left\| \nabla e_{\mu, k+1} \right\|^2 + \frac{C}{\theta_{11}} \left\| \varphi^{k}_h \right\|_{\infty}^2 \left\| \sigma \mathbf{u}^{k+1} \right\|^2
\leq C \left( R^{k+1} + \left\| \nabla e_{\varphi, k} \right\|_2^2 \right) + \theta_{11} \left\| \nabla e_{\mu, k+1} \right\|^2 + \left( 1 + \frac{C}{\theta_{11}} \left\| \varphi^{k}_h \right\|_{\infty}^2 \right) \left\| \sigma \mathbf{u}^{k+1} \right\|^2. \tag{4.51} \right.
\end{align*}$$

The proof is complete by choosing $\theta_{11} = \frac{M_0}{12}$. \hfill \Box

The next lemma gives an estimate of $\left\| \delta_t \sigma_{\varphi, k+1} \right\|_{-1, h}$.

Lemma 4.7 The assumptions are the same as in Lemma 4.1. There exists a constant $C > 0$ independent of $\tau$ and $h$ such that

$$\begin{align*}
\left\| \delta_t \sigma_{\varphi, k+1} \right\|_{-1, h}^2 & \leq C \tau^2 + C \tau^2 \left\| \nabla \rho \mu_{k+1} \right\|_6^2 + \left( \frac{25M^2_1}{4} + C_1 \tau + 1 \right) \left\| \nabla \mu_{k+1} \right\|^2 + C \left\| R_{\varphi, k+1} \right\|^2
+ \left( 1 + \left\| \varphi^{k}_h \right\|_{\infty} \right) R^{k+1} + C \left\| \nabla e_{\varphi, k} \right\|^2 + C \left\| \varphi^{k}_h \right\|_{\infty} \left\| \mathbf{u}_k \right\|^2. \tag{4.52} \right.
\end{align*}$$
Proof. Recall that \( \| \zeta \|_{L^1}^2 = \| \nabla T_h(\zeta) \|^2 = (\nabla T_h(\zeta), \nabla T_h(\zeta)) = (\zeta, T_h(\zeta)) \) for all \( \zeta \in \tilde{Y}_h \). Noticing that \( \delta \sigma_{\phi,k+1} \in \tilde{Y}_h \), setting \( v = T_h(\delta \sigma_{\phi,k+1}) \) in (4.16a), using (4.19), (4.37) and (4.43), we derive

\[
\| \delta \sigma_{\phi,k+1} \|^2_{-1,h} = (\delta \sigma_{\phi,k+1}, T_h(\delta \sigma_{\phi,k+1}))
\]

\[
= - \left( (M(\phi_{k+1}^h) - M(\phi_h)) \nabla \phi_{k+1}^h, \nabla T_h(\delta \sigma_{\phi,k+1}) \right) + (R_{\phi,k+1}, T_h(\delta \sigma_{\phi,k+1}))
\]

\[
+ \left( \nabla u_{k+1}^h \psi_{k+1}^h - \nabla u_{h}^k \psi_{h}^k, \nabla T_h(\delta \sigma_{\phi,k+1}) \right)
\]

\[
\leq C \left( R_{\phi,k+1}^2 + \| \nabla \psi_{\phi,k} \|^2 \right) + \frac{1}{5} \| \nabla T_h(\delta \sigma_{\phi,k+1}) \|^2 + \| R_{\phi,k+1} \| \| T_h(\delta \sigma_{\phi,k+1}) \|
\]

\[
+ \| M(\phi_h^k) \nabla \sigma_{\mu,k+1} \| \| \nabla T_h(\delta \sigma_{\phi,k+1}) \| + \| u_{k+1}^h \psi_{k+1}^h - u_{h}^k \psi_{h}^k \| \| \nabla T_h(\delta \sigma_{\phi,k+1}) \|
\]

\[
\leq CR_{\phi,k+1}^2 + C \| \nabla \psi_{\phi,k} \|^2 + \frac{1}{5} \| \nabla T_h(\delta \sigma_{\phi,k+1}) \|^2 + C \| R_{\phi,k+1} \| \| \nabla T_h(\delta \sigma_{\phi,k+1}) \|
\]

\[
+ \frac{5}{4} \| M(\phi_h^k) \nabla \sigma_{\mu,k+1} \|^2 + \frac{1}{5} \| \nabla T_h(\delta \sigma_{\phi,k+1}) \|^2 + \frac{5}{4} \| u_{k+1}^h \psi_{k+1}^h - u_{h}^k \psi_{h}^k \|^2 + \frac{5}{4} \| \nabla T_h(\delta \sigma_{\phi,k+1}) \|^2
\]

\[
\leq CR_{\phi,k+1}^2 + C \| \nabla \psi_{\phi,k} \|^2 + \frac{5M_{\phi,k}^2}{4} \| \nabla \sigma_{\mu,k+1} \|^2 + C \| R_{\phi,k+1} \|^2 + \frac{4}{5} \| \nabla T_h(\delta \sigma_{\phi,k+1}) \|^2
\]

\[
+ \frac{5}{4} \left( \frac{1}{2} \| L_c \|^2 + \| L_m \|^2 \right) + \frac{5}{4} \left( \frac{1}{2} \| \sigma_{\phi,k} \|^2 \| \sigma_{\mu,k+1} \|^2 + \frac{5}{4} \| \nabla \sigma_{\phi,k,h} \|^2 \right)
\]

\[
\leq CR_{\phi,k+1}^2 + C \| \nabla \psi_{\phi,k} \|^2 + \left( \frac{5M_{\phi,k}^2}{4} + C \tau \| \phi_h^k \|^4 \| \nabla \sigma_{\mu,k+1} \|^2 + C \| R_{\phi,k+1} \|^2 + \frac{4}{5} \| \delta \sigma_{\phi,k+1} \|^2 \right)
\]

\[
+ C \| \nabla \rho_{\mu,k+1} \|^2_{6} + C \left( 1 + \| \phi_h^k \|^2 \right) R_{\phi,k+1}^2 + C \| \nabla \psi_{\phi,k} \|^2 + C \| \psi_{\phi,k} \|^2 \| e_{u,k} \|^2
\]

\[
\leq C \tau^2 + \left( \frac{5M_{\phi,k}^2}{4} + C \tau \| \phi_h^k \|^4 \| \nabla \sigma_{\mu,k+1} \|^2 + C \| R_{\phi,k+1} \|^2 + \frac{4}{5} \| \delta \sigma_{\phi,k+1} \|^2 \right)
\]

\[
+ C \left( 1 + \| \phi_h^k \|^2 \right) R_{\phi,k+1}^2 + C \| \nabla \psi_{\phi,k} \|^2 + C \| \nabla \rho_{\mu,k+1} \|^2_{6} + C \| \phi_h^k \|^2 \| e_{u,k} \|^2.
\]
4.8 Suppose result. (4.18), choosing independent of \(\phi_k\), \(\phi_1\), \(\|\phi_k\|_\infty \leq C\|\phi_1\|_\infty \leq 1\), to the left-hand side of the inequality. \(\Box\)

With all these estimates of the RHS terms in place, the error equation (4.18) leads to the following result.

**Lemma 4.8** Suppose \((\varphi, \mu, u_c, u_m, P_c, P_m)\) is a weak solution to (4.15a)–(4.15d) satisfying additional regularities prescribed in Assumption 1. Then, for any \(\tau, h > 0\), there exists a constant \(C > 0\), independent of \(h\) and \(\tau\), such that for any \(0 \leq k \leq K - 1\),

\[
\frac{M_0}{3} \left\| \nabla \sigma^{\mu,k+1} \right\|^2 + \frac{\gamma\epsilon}{2\tau} \left( \left\| \nabla \varphi^{k+1} \right\|^2 - \left\| \nabla \varphi^{k} \right\|^2 + \left\| \nabla (\varphi^{k+1} - \varphi^{k}) \right\|^2 \right)
+ \frac{\rho_0}{2\tau} \left( \left\| \sigma^{u,k+1} \right\|^2 - \left\| \sigma^{u,k} \right\|^2 + \left\| \sigma^{u,k+1} - \sigma^{u,k} \right\|^2 \right)
+ v_0 \left\| \mathcal{D}(\sigma^{u,k+1}) \right\|^2 + \alpha_{BJSJ} \frac{v_0}{\sqrt{d\lambda}} \sum_{i=1}^{d-1} \left\| \sigma^{u,k+1} \cdot \tau_i \right\|^2_{cm}
+ \frac{\rho_0}{2\tau} \left( \left\| \sigma^{u,k+1} \right\|^2 - \left\| \sigma^{u,k} \right\|^2 + \left\| \sigma^{u,k+1} - \sigma^{u,k} \right\|^2 \right)
+ \frac{\rho_0}{2\tau} \left\| \psi_k \right\|^2 + \frac{\tau}{\rho_0} \left\| \psi_{k+1} \right\|^2 + \left\| \sigma^{u,k+1} \right\|^2
\leq C \mathcal{B}^{k+1} + C \left( 1 + \left\| \varphi^{k+1} \right\|^4_\infty \right) \left\| \nabla \varphi^{k+1} \right\|^2 + \left( 1 + \left\| \varphi^{k+1} \right\|^2 \right) \left\| \sigma^{u,k+1} \right\|^2
+ C \left\| \varphi^{k+1} \right\|^2 \left\| \sigma^{u,k} \right\|^2 + C \left\| \nabla \varphi^{k} \right\|^2, \tag{4.54}
\]

where

\[
\mathcal{B}^{k+1} := \tau^2 + \left\| \rho^{u,k+1} \right\|^2 + \left\| \rho^{c,k+1} \right\|^2 + \left\| R^{\varphi,k+1} \right\|^2 + \left( 1 + \left\| \varphi^{k+1} \right\|^2_\infty \right) \left\| R^{k+1} \right\|^2
+ \left\| \varphi^{k+1}_\infty \right\|^2 + \left\| \nabla \varphi^{k} \right\|^2 + \left( 1 + \left\| \varphi^{k+1} \right\|^4_\infty \right) \left\| \nabla \varphi^{k+1} \right\|^2
+ (1 + \tau^2) \left\| \nabla \rho^{\mu,k+1} \right\|^2. \tag{4.55}
\]

**Proof.** Substituting the estimates in Lemmas 4.1–4.7 into the right-hand side of the error equation (4.18), choosing

\[
\theta_8 = \theta_{10} = \frac{M_0}{6 \left( \frac{25M_1^2}{4} + C_1 \tau (T + 1) \right)], \tag{4.56}
\]
with $C_1$ the positive constant defined in inequality (4.52), we get

$$\frac{M_0}{3} \left\| \nabla \sigma_{\mu,k+1} \right\|^2 + \frac{\nu \epsilon}{2 \tau} \left( \left\| \nabla \sigma_{\varphi,k+1} \right\|^2 - \left\| \nabla \sigma_{\varphi,k} \right\|^2 \right)$$

$$+ \frac{\rho_0}{2 \tau} \left( \left\| \sigma_{c,u,k+1} \right\|^2 - \left\| \sigma_{c,u,k} \right\|^2 + \left\| \sigma_{c,u,k+1} - \sigma_{c,u,k} \right\|^2 \right)$$

$$+ \nu_0 \left\| \mathbb{D}(\sigma_{c,u,k+1}) \right\|^2 + \alpha_{BJSJ} \frac{v_0}{2} \sum_{i=1}^{d-1} \left\| \sigma_{c,u,k+1} \cdot \tau_i \right\|^2$$

$$+ \frac{\rho_0}{2 \tau} \left( \left\| \sigma_{m,u,k+1} \right\|^2 - \left\| \sigma_{m,u,k} \right\|^2 + \left\| \sigma_{m,u,k+1} - \sigma_{m,u,k} \right\|^2 \right) + \nu_0 \left\| \sigma_{m,u,k+1} \right\|^2$$

$$+ \frac{\tau}{\rho_0} \left\| \psi_{ch} \right\|^2 + \left( \frac{\tau \chi}{\rho_0} \right) \left\| \sigma_{m,h} \right\|^2$$

$$\leq C \tau^2 + C \left\| R_{m,k+1} \right\|^2 + C \left\| R_{c,k+1} \right\|^2 + C \left\| R_{\varphi,k+1} \right\|^2 + C \left( 1 + \left\| \nabla \phi \right\|^2 \right) \left( 1 + \left\| \nabla \phi \right\|^2 \right) \left( 1 + \left\| \nabla \phi \right\|^2 \right)$$

$$+ C \left\| e_{u,k} \right\|^2 + \left( 1 + \left\| \phi_{k+1} \right\|^4 \right) \left\| \nabla e_{\varphi,k+1} \right\|^2$$

$$+ C \left\| \nabla \mu_{t,k+1} \right\|^2 + C \left\| \phi_{k+1} \right\|^2$$

$$+ \left( 1 + C \right) \left\| \phi_{k+1} \right\|^2 \left\| \sigma_{u,k+1} \right\|^2. \quad (4.57)$$

The proof is complete since $\left\| e_{u,k} \right\|^2 = \left\| \rho_{u,k} + \sigma_{u,k} \right\|^2 \leq 2 \left( \left\| \rho_{u,k} \right\|^2 + \left\| \sigma_{u,k} \right\|^2 \right)$ and $\left\| \nabla \mu_{t,k+1} \right\| = C \left\| \nabla \mu_{t,k+1} \right\|$. □

Regarding $\mathcal{R}_{k+1}$ in Equation (4.55), the following estimate could be derived.

**Lemma 4.9** Suppose $(\phi, \mu, u, u, P, P)$ is a weak solution to (4.15a)-(4.15d) satisfying additional regularities in Assumption 1. Then, for all $0 \leq l \leq K - 1$, there holds

$$\sum_{k=0}^{l} \mathcal{R}_{k+1} \leq C(T+1)\tau + \frac{2}{\tau} \int_0^T \left( \left\| \partial_t \rho_0 (\cdot, t) \right\|^2 + \left\| \partial_t \rho_1 (\cdot, t) \right\|^2 \right) dt$$

$$+ C\tau^{-1/2} \sqrt{T+1} \left( \sum_{k=0}^{l} \left\| \nabla \rho_{\varphi,k+1} \right\|^4 \right)^{1/2} + \sum_{k=0}^{l} \left( \left\| \nabla \rho_{\varphi,k} \right\|^2 + \left( 1 + \tau^2 \right) \left\| \nabla \rho_{\mu,k+1} \right\|^2 \right)^{1/2}$$

$$+ C\tau^{-1/2} \sqrt{T+1} \left( \sum_{k=0}^{l} \left\| \rho_{u,k} \right\|^4 \right)^{1/2}. \quad (4.58)$$
Proof. First, by Minkowski’s inequality and Hölder’s inequality, one obtains

\[ \left\| R^{\varphi,k+1} \right\|^2 = \left\| \delta_t \varphi^{k+1} - \partial_t \varphi^{k+1} \right\|^2 \]

\[ \leq 2 \left\| \delta_t \left( \varphi^{k+1} - \varphi^k \right) \right\|^2 + 2 \left\| \delta_t \varphi^{k+1} - \partial_t \varphi^{k+1} \right\|^2 \]

\[ = \frac{2}{\tau^2} \left\| \int_{t_k}^{t_{k+1}} \partial_t \rho \varphi(\cdot,t) \, dt \right\|^2 + \frac{2}{\tau^2} \left\| \int_{t_k}^{t_{k+1}} \left( t - t_k \right) \partial_t \varphi(\cdot,t) \, dt \right\|^2 \]

\[ \leq \frac{2}{\tau^2} \left( \int_{t_k}^{t_{k+1}} \left\| \partial_t \rho \varphi(\cdot,t) \right\|^2 \, dt \right)^2 + \frac{2}{\tau^2} \left( \int_{t_k}^{t_{k+1}} \frac{\left( t - t_k \right)}{3} \partial_t \varphi(\cdot,t) \right)^2 dt. \] (4.59)

Likewise, for \( j \in \{c,m\} \), one has

\[ \left\| R^{u,k+1}_j \right\|^2 \leq \frac{2}{\tau} \int_{t_k}^{t_{k+1}} \left\| \partial_t \rho^j(\cdot,t) \right\|^2 \, dt + \frac{2}{\tau^3} \int_{t_k}^{t_{k+1}} \left\| \partial_t \varphi(\cdot,t) \right\|^2 \, dt. \] (4.60)

Applying Minkowski’s inequality and Hölder’s inequality again gives, for \( j \in \{c,m\} \),

\[ \left\| \varphi^{k+1} - \varphi^k \right\|^4 = \int_{t_k}^{t_{k+1}} \left\| \partial_t \varphi(\cdot,t) \right\|^4 \, dt \leq \left( \int_{t_k}^{t_{k+1}} \left\| \partial_t \varphi(\cdot,t) \right\|^4 \, dt \right)^4 \]

\[ \leq \left( \int_{t_k}^{t_{k+1}} \left\| \partial_t \varphi(\cdot,t) \right\|^4 \, dt \right)^3 \left( \int_{t_k}^{t_{k+1}} \, dt \right) = \tau^3 \int_{t_k}^{t_{k+1}} \left\| \partial_t \varphi(\cdot,t) \right\|^4 \, dt, \] (4.61)

which in turn leads to

\[ \left( R^{k+1} \right)^2 = \left( \left\| \varphi^{k+1} - \varphi^k \right\|^2 + \left\| \nabla \left( \varphi^{k+1} - \varphi^k \right) \right\|^2 + \left\| u^{k+1} - u^k \right\|^2 \right)^2 \]

\[ \leq C \left( \left\| \varphi^{k+1} - \varphi^k \right\|^4 + \left\| \nabla \left( \varphi^{k+1} - \varphi^k \right) \right\|^4 + \left\| u^{k+1} - u^k \right\|^4 \right) \]

\[ \leq C \tau^3 \int_{t_k}^{t_{k+1}} \left( \left\| \partial_t \varphi(\cdot,t) \right\|^4 + \left\| \nabla \partial_t \varphi(\cdot,t) \right\|^4 + \left\| \partial_t u(\cdot,t) \right\|^4 \right) \, dt. \] (4.62)
Thus, we have
\[
\sum_{k=0}^{l} \left( 1 + \| \varphi_h^k \|_\infty^2 \right) R^{k+1} \leq \left( \sum_{k=0}^{l} C \left( 1 + \| \varphi_h^k \|_\infty^{2(d-6)/d} \right) \right)^{1/2} \left( \sum_{k=0}^{l} \left( R^{k+1} \right)^2 \right)^{1/2}
\]
\[
\leq \left( \sum_{k=0}^{l} C \left( 1 + \| \varphi_h^k \|_\infty^{4(6-d)/d} \right) \right)^{1/2} \left( C \tau^3 \int_0^{t_{n+1}} \left( \| \partial_t \varphi(\cdot, t) \|_4^4 + \| \nabla \varphi(\cdot, t) \|_4^4 + \| \partial_t \eta(\cdot, t) \|_4^4 \right) dt \right)^{1/2}
\]
\[
\leq C \tau \left( \sum_{k=0}^{l} \left( 1 + \| \varphi_h^k \|_\infty^{4(6-d)/d} \right) \right)^{1/2} \left( \int_0^{t_{n+1}} \left( \| \partial_t \varphi(\cdot, t) \|_4^4 + \| \nabla \varphi(\cdot, t) \|_4^4 + \| \partial_t \eta(\cdot, t) \|_4^4 \right) dt \right)^{1/2}
\]
\[
\leq C \tau \sqrt{T + 1} \left( \int_0^{t_{n+1}} \left( \| \partial_t \varphi(\cdot, t) \|_4^4 + \| \nabla \varphi(\cdot, t) \|_4^4 + \| \partial_t \eta(\cdot, t) \|_4^4 \right) dt \right)^{1/2} \leq C \tau \sqrt{T + 1}. \tag{4.63}
\]

Similarly, we have
\[
\sum_{k=0}^{l} \left( 1 + \| \varphi_h^{k+1} \|_\infty^4 \right) \| \nabla \varphi^{k+1} \|_2^2 \leq \tau^{-1/2} \left( C \tau \sum_{k=0}^{l} \left( 1 + \| \varphi_h^{k+1} \|_\infty^{4(6-d)/d} \right) \right)^{1/2} \left( \sum_{k=0}^{l} \left( \nabla \varphi^{k+1} \|_4^4 \right) \right)^{1/2}
\]
\[
\leq C \tau^{-1/2} \sqrt{T + 1} \left( \sum_{k=0}^{l} \left( \nabla \varphi^{k+1} \|_4^4 \right) \right)^{1/2}, \tag{4.64}
\]
\[
\sum_{k=0}^{l} \| \varphi_h^k \|_\infty^2 \left( \| \rho^{u,k} \|_2^2 \right) \leq \tau^{-1/2} \left( C \tau \sum_{k=0}^{l} \left( 1 + \| \varphi_h^k \|_\infty^{4(6-d)/d} \right) \right)^{1/2} \left( \sum_{k=0}^{l} \left( \rho^{u,k} \|_4^4 \right) \right)^{1/2}
\]
\[
\leq C \tau^{-1/2} \sqrt{T + 1} \left( \sum_{k=0}^{l} \left( \rho^{u,k} \|_4^4 \right) \right)^{1/2}. \tag{4.65}
\]

Henceforth, it follows that
\[
\sum_{k=0}^{l} \varphi^{k+1} = \sum_{k=0}^{l} \left[ \tau^2 + \| R^{\varphi,k+1} \|_2^2 + \| R^{u,k+1} \|_2^2 + \| R^{\eta,k+1} \|_2^2 + \left( 1 + \| \varphi_h^k \|_\infty^2 \right) R^{k+1} \right]
\]
\[
+ \left( 1 + \| \varphi_h^{k+1} \|_\infty^4 \right) \| \nabla \varphi^{k+1} \|_2^2 + \| \nabla \rho^{u,k} \|_6^2 + \| \rho^{k+1} \|_\infty^2 \| \rho^{u,k} \|_2^2 \right].
\]
≤ \left( T + C\sqrt{T + 1} \right) \tau + \frac{2\tau}{3} \int_0^{t_{i+1}} \left( \| \partial_t \varphi(\cdot, t) \|^2 + \| \partial_n u(\cdot, t) \|^2 \right) \, dt \\
+ \frac{2}{\tau} \int_0^{t_{i+1}} \left( \| \partial_t \rho(\cdot, t) \|^2 + \| \partial_t \rho(\cdot, t) \|^2 \right) \, dt + \sum_{k=0}^l \left( 1 + \tau^2 \right) \| \nabla \rho_{\mu,k+1} \|^2_6 + \| \nabla \rho_{\varphi,k} \|^2 \right) \\
+ C \tau^{-1/2} \sqrt{T + 1} \left( \sum_{k=0}^l \| \nabla \rho_{\varphi,k+1} \|^4 \right)^{1/2} + C \tau^{-1/2} \sqrt{T + 1} \left( \sum_{k=0}^l \| \rho_{u,k} \|^4 \right)^{1/2} \\
≤ \left( T + C\sqrt{T + 1} + \frac{2}{3} \right) \tau + \frac{2}{\tau} \int_0^{t_{i+1}} \left( \| \partial_t \rho(\cdot, t) \|^2 + \| \partial_t \rho(\cdot, t) \|^2 \right) \, dt \\
+ C \tau^{-1/2} \sqrt{T + 1} \left( \sum_{k=0}^l \| \nabla \rho_{\varphi,k+1} \|^4 \right)^{1/2} + \sum_{k=0}^l \left( 1 + \tau^2 \right) \| \nabla \rho_{\mu,k+1} \|^2_6 + \| \nabla \rho_{\varphi,k} \|^2 \right) \\
+ C \tau^{-1/2} \sqrt{T + 1} \left( \sum_{k=0}^l \| \rho_{u,k} \|^4 \right)^{1/2}. \\
(4.66)

This completes the proof. \qed

Now, we are ready to prove the main convergence theorem.

**Theorem 1** Suppose \((\varphi, \mu, u_c, u_m, P_c, P_m)\) is a weak solution to (4.15a)–(4.15d) with the additional regularities described in Assumption 1. Recall the definition of error functions \(\sigma\)s in Equations 4.11–4.14 and the \(\rho\), \(\rho_u\), \(\rho_\mu\) in Equations 4.6–4.9. Then, provided that \(0 < \tau < \tau_1\) for some sufficiently small \(\tau_1 > 0\),

\[
\max_{0 \leq k \leq K-1} \left( \| \nabla \rho_{\varphi,k+1} \|^2 + \| \sigma_{c,k+1} \|^2 + \| \sigma_{m,k+1} \|^2 \right) + \tau \sum_{k=0}^{K-1} \| \nabla \rho_{\mu,k+1} \|^2 \\
+ \sum_{k=0}^{K-1} \left( \| \nabla (\sigma_{c,k+1} - \sigma_{c,k}) \|^2 + \| \sigma_{c,k+1} - \sigma_{c,k} \|^2 + \| \sigma_{m,k+1} - \sigma_{m,k} \|^2 \right) \\
+ \tau \sum_{k=0}^{K-1} \left( \| \nabla (\sigma_{c,k+1}) \|^2 + \sum_{i=1}^{d-1} \| \sigma_{c,k+1} \cdot \tau_i \|^2_{cm} + \| \sigma_{m,k+1} \|^2 \right) + \tau^2 \sum_{k=0}^{K-1} \| \rho_{h,k} \| \nabla \rho_{\mu,k+1} \|^2 \\
≤ C(T) \left[ \tau^2 + \int_0^T \left( \| \partial_t \rho(\cdot, t) \|^2 + \| \partial_t \rho(\cdot, t) \|^2 \right) \, dt + \tau^{1/2} \left( \sum_{k=0}^K \| \nabla \rho_{\varphi,k+1} \|^4 \right)^{1/2} \\
+ \tau \sum_{k=0}^K \left( \| \nabla \rho_{\varphi,k} \|^2 + (1 + \tau^2) \| \nabla \rho_{\mu,k+1} \|^2_6 + \| \rho_{u,k} \|^4 \right)^{1/2} \right]. \\
(4.67)
\]
Proof. Applying $\tau \sum_{k=0}^{l}$ to (4.54), and observing that $\sigma^{\psi,k} = 0$ and $\sigma^{u,k}_j = 0$ for $k = 0$, $j \in \{c, m\}$, it follows that

$$\frac{\gamma e}{2} \|\nabla \sigma^{\psi,l+1}\|^2 + \frac{\rho_0}{2} \|\sigma^{\psi,l+1}_c\|^2 + \frac{\rho_0}{2\chi} \|\sigma^{\psi,l+1}_m\|^2 + \tau \sum_{k=0}^{l} \left( \frac{M_0}{3} \|\nabla \sigma^{\mu,k+1}\|^2 \right)$$

$$+ \sum_{k=0}^{l} \left( \frac{\gamma e}{2} \|\nabla (\sigma^{\psi,k+1} - \sigma^{\psi,k})\|^2 + \frac{\rho_0}{2} \|\sigma^{\psi,k+1}_c - \sigma^{\psi,k}_c\|^2 + \frac{\rho_0}{2\chi} \|\sigma^{\psi,k+1}_m - \sigma^{\psi,k}_m\|^2 \right)$$

$$+ \tau \sum_{k=0}^{l} \left[ v_0 \|\mathcal{D}(\sigma^{u,k+1}_c)\|^2 + \alpha_{BJSJ} \frac{v_0}{2\sqrt{d\lambda}} \sum_{i=1}^{d-1} \|\sigma^{u,k+1}_c \cdot \tau_i\|^2 + \frac{v_0}{2\lambda} \|\sigma^{u,k+1}_m\|^2 \right]$$

$$+ \frac{\tau^2}{\rho_0} \sum_{k=0}^{l} \left[ \|\phi^{k,h}_c \nabla \sigma^{\mu,k+1}_c\|^2 + \chi \|\phi^{k,h}_m \nabla \sigma^{\mu,k+1}_m\|^2 \right]$$

$$\leq C_T \sum_{k=0}^{l} \|\nabla \sigma^{\psi,k}\|^2 + C_T \sum_{k=0}^{l} \left( 1 + \|\phi^{k,l}_h\|_{\infty}^{4(6-d)} \right) \|\nabla \sigma^{\psi,k+1}\|^2 + \tau \sum_{k=0}^{l} \left( 1 + C \|\phi^{k}_h\|_{\infty}^{2(6-d)} \right) \|\sigma^{u,k+1}\|^2$$

$$+ C_T \sum_{k=1}^{l} \|\nabla \sigma^{\psi,k}\|^2 + C_T \sum_{k=1}^{l} \|\phi^{k}_h\|_{\infty}^{2(6-d)} \|\sigma^{u,k}\|^2$$

$$\leq C_T \sum_{k=0}^{l} \|\nabla \sigma^{\psi,k}\|^2 + C_T \left( 1 + \|\phi^{l+1}_h\|_{\infty}^{4(6-d)} \right) \|\nabla \sigma^{\psi,l+1}\|^2 + \tau \left( 1 + C \|\phi^{l}_h\|_{\infty}^{2(6-d)} \right) \|\sigma^{u,l+1}\|^2$$

$$+ C_T \sum_{k=1}^{l} \left( 1 + \|\phi^{k}_h\|_{\infty}^{4(6-d)} \right) \|\nabla \sigma^{\psi,k}\|^2 + C_T \sum_{k=1}^{l} \left( 1 + 2 \|\phi^{k}_h\|_{\infty}^{2(6-d)} \right) \|\sigma^{u,k}\|^2.$$  (4.68)

Moving all the terms indexed $(l + 1)$ to the left-hand side, one has

$$\left( \frac{\gamma e}{2} - C_T \left( 1 + \|\phi^{l+1}_h\|_{\infty}^{4(6-d)} \right) \right) \|\nabla \sigma^{\psi,l+1}\|^2 + \left( \frac{\rho_0}{2} - \tau (1 + C \|\phi^{l}_h\|_{\infty}^{2(6-d)}) \right) \|\sigma^{u,l+1}_c\|^2$$

$$+ \left( \frac{\rho_0}{2\chi} - \tau (1 + C \|\phi^{l}_h\|_{\infty}^{2(6-d)}) \right) \|\sigma^{u,l+1}_m\|^2 + \tau \sum_{k=0}^{l} \left( \frac{M_0}{3} \|\nabla \sigma^{\mu,k+1}\|^2 \right)$$

$$+ \sum_{k=0}^{l} \left( \frac{\gamma e}{2} \|\nabla (\sigma^{\psi,k+1} - \sigma^{\psi,k})\|^2 + \frac{\rho_0}{2} \|\sigma^{\psi,k+1}_c - \sigma^{\psi,k}_c\|^2 + \frac{\rho_0}{2\chi} \|\sigma^{\psi,k+1}_m - \sigma^{\psi,k}_m\|^2 \right)$$

$$+ \tau \sum_{k=0}^{l} \left[ v_0 \|\mathcal{D}(\sigma^{u,k+1}_c)\|^2 + \alpha_{BJSJ} \frac{v_0}{2\sqrt{d\lambda}} \sum_{i=1}^{d-1} \|\sigma^{u,k+1}_c \cdot \tau_i\|^2 + \frac{v_0}{2\lambda} \|\sigma^{u,k+1}_m\|^2 \right]$$
It follows from (4.69) that
\[
\nabla \rho \tau_k \leq C \tau \sum_{k=0}^{l} \rho_0 \sum_{k=0}^{l} \left\| \phi_{c,h}^{k} \nabla \sigma_{c}^{\mu,k+1} \right\|^2 + \chi \left\| \phi_{m,h}^{k} \nabla \sigma_{m}^{\mu,k+1} \right\|^2.
\]

By Lemma 3.5 we have, for all \(0 \leq l \leq K - 1\),
\[
\tau^{\frac{1}{2}} \left\| \phi_{h}^{l+1} \right\|_{\infty}^{\frac{4(6-d)}{d}} = \left( \tau \left\| \phi_{h}^{l+1} \right\|_{\infty}^{\frac{8(6-d)}{d}} \right)^{\frac{1}{2}} \leq \left( \tau \sum_{k=0}^{K-1} \left\| \phi_{h}^{k+1} \right\|_{\infty}^{\frac{8(6-d)}{d}} \right)^{\frac{1}{2}} \leq C \sqrt{T + 1}.
\] (4.70)

Hence, we can choose a sufficiently small \(\tau_1\) such that for all \(0 < \tau < \tau_1\) and \(0 \leq l \leq K - 1\)
\[
C \tau \left( 1 + \left\| \phi_{h}^{l+1} \right\|_{\infty}^{\frac{4(6-d)}{d}} \right) \leq C \tau + C \tau^{\frac{1}{2}} \left( C \sqrt{T + 1} \right) \leq \frac{\gamma \epsilon}{4},
\] (4.71)
\[
\frac{\gamma \epsilon}{2} - C \tau \left( 1 + \left\| \phi_{h}^{l+1} \right\|_{\infty}^{\frac{4(6-d)}{d}} \right) \geq \frac{\gamma \epsilon}{4},
\] (4.72)
\[
\frac{\rho_0}{2} - C \tau \left( 1 + \left\| \phi_{h}^{l+1} \right\|_{\infty}^{\frac{4(6-d)}{d}} \right) \geq \frac{\rho_0}{4},
\] (4.73)
\[
\frac{\rho_0}{2 \chi} - C \tau \left( 1 + \left\| \phi_{h}^{l+1} \right\|_{\infty}^{\frac{4(6-d)}{d}} \right) \geq \frac{\rho_0}{4 \chi}.
\] (4.74)

It follows from (4.69) that
\[
\frac{\gamma \epsilon}{4} \left\| \nabla \sigma_{c}^{\mu,l+1} \right\|^2 + \frac{\rho_0}{4} \left\| \sigma_{c}^{u,l+1} \right\|^2 + \frac{\rho_0}{4 \chi} \left\| \sigma_{m}^{u,l+1} \right\|^2 + \tau \sum_{k=0}^{l} \left( M_0 \left\| \nabla \sigma_{c}^{\mu,k+1} \right\|^2 \right)
\]
\[
+ \sum_{k=0}^{l} \left( \frac{\gamma \epsilon}{2} \left\| \nabla (\sigma_{c}^{\mu,k+1} - \sigma_{c}^{k}) \right\|^2 + \frac{\rho_0}{2} \left\| \sigma_{c}^{u,k+1} - \sigma_{c}^{k} \right\|^2 + \frac{\rho_0}{2 \chi} \left\| \sigma_{m}^{u,k+1} - \sigma_{m}^{k} \right\|^2 \right)
\]
\[
+ \tau \sum_{k=0}^{l} \left[ v_0 \left\| \nabla (\sigma_{c}^{u,k+1}) \right\|^2 + \alpha_{BJSJ} \frac{v_0}{2 \sqrt{d \lambda}} \sum_{i=1}^{d-1} \left\| \sigma_{c}^{u,k+1} \cdot \tau_{i} \right\|_{c_{m}}^2 + \frac{v_0}{2 \lambda} \left\| \sigma_{m}^{u,k+1} \right\|^2 \right]
\]
\[
+ \frac{\tau^2}{\rho_0} \sum_{k=0}^{l} \left( \left\| \phi_{c,h}^{k} \nabla \sigma_{c}^{\mu,k+1} \right\|^2 + \chi \left\| \phi_{m,h}^{k} \nabla \sigma_{m}^{\mu,k+1} \right\|^2 \right)
\]
\[
\leq C \tau \sum_{k=0}^{l} \left( 1 + \left\| \phi_{h}^{k} \right\|_{\infty}^{\frac{4(6-d)}{d}} \right) \left\| \nabla \sigma_{c}^{k} \right\|^2 + C \tau \sum_{k=1}^{l} \left( 1 + 2 \left\| \phi_{h}^{k} \right\|_{\infty}^{\frac{4(6-d)}{d}} \right) \left\| \sigma_{c}^{u,k} \right\|^2.
\]
Noticing that $\sum_{k=0}^{K} \| \varphi_h^k \|_1^{p(6-d) \over d} \leq C(T+1)$ for $p = 2, 4$, and in light of Lemma 4.9, we arrive at the error estimate (4.67) by setting $l = K - 1$ and applying discrete Gronwall’s inequality. This completes the proof.

**Corollary 4.1** Suppose $(\varphi, \mu, u_c, u_m, P_c, P_m)$ is a weak solution to (4.15a)–(4.15d) satisfying the regularities Assumption 1. Then, there exists $\tau_1 > 0$ such that for all $\tau < \tau_1$ the following optimal convergence rates hold:

$$\max_{0 \leq k \leq K-1} \left( \| \nabla \varphi_{k+1} \|_2^2 + \| e_{k+1} \|_2^2 + \| e_m^{k+1} \|_2^2 \right) + \tau \sum_{k=0}^{K-1} \| \nabla \mu_{k+1} \|_2^2 + \tau \sum_{k=0}^{K-1} \| D(\varepsilon_{k+1}) \|_2^2 \leq C(T)(\tau^2 + h^2q),$$

where $q \geq 1$ is the spatial approximation order.

For numerical evidence of the convergence results, we refer to Chen et al. (2017).

**Remark 4.1** In the discrete energy dissipation analysis established in Chen et al. (2017), for the numerical scheme, a cancelation of a nonlinear error term associated with the convection part has played a very important role. Meanwhile, in the optimal rate error estimate presented in this section, such a cancelation technique is not needed in the convergence proof, due to the subtle fact that, a growth constant for the velocity error term, namely $(1 + C\| \varphi_h \|_\infty^2)$ appearing in (4.49), would not lead to a theoretical difficulty in the derivation of discrete Gronwall inequality. This fact is associated with Navier–Stokes nature for the fluid velocity, in which the higher order kinematic diffusion and the temporal derivative of the velocity variable have greatly facilitated the analysis at both the analytic and numerical levels. In comparison, for the Cahn–Hilliard–Hele–Shaw system, in which the fluid velocity is statically determined by the phase field variables, such a cancelation technique is necessary to pass through the optimal rate convergence analysis because of lack of regularity for the velocity field; see the related works Chen et al. (2016); Diegel et al. (2017); Liu et al. (2017), etc.

5. Concluding remarks

In this article, we provide an optimal rate convergence analysis and error estimate of a fully discrete finite element numerical scheme for the CHSD system that models two-phase flows. An operator splitting is applied in the numerical scheme, so that a coupling between the Cahn–Hilliard and the fluid solvers is avoided. The unique solvability and the energy stability have already been proven in the existing literature. The optimal rate error estimate is established in the energy norm, $\ell^\infty(0, T; H^1) \cap \ell^2(0, T; H^2)$ norm for the phase variables and $\ell^\infty(0, T; H^1) \cap \ell^2(0, T; H^2)$ norm for the velocity variable. A discrete $\ell^2(0; T; H^3)$ bound of the numerical solution for the phase variables also plays an important role, which is accomplished via a discrete version of Gagliardo–Nirenberg inequality in the finite element space.

**Funding**

National Key R&D Program of China (2019YFA0709502 to W.C.); National Science Foundation of China (12071090 to W.C., 11871159 to X.W.); National Science Foundation (DMS-1912715 to D.H.,
DMS-2012669 to C.W.); Key Laboratory of Mathematics for Nonlinear Sciences, Fudan University (C.W.); Guangdong Provincial Key Laboratory of Computational Science and Material Design (2019B030301001 to X.W.).

REFERENCES

Brenner, S. C. & Scott, L. R. (2008) *The Mathematical Theory of Finite Element Methods*, vol. 15, 3rd edn. Texts in Applied Mathematics. New York: Springer, p. xviii+397.

Cai, Y. & Shen, J. (2018) Error estimates for a fully discretized scheme to a Cahn–Hilliard phase-field model for two-phase incompressible flows. *Math. Comp.*, 87, 2057–2090.

Chen, W., Feng, W., Liu, Y., Wang, C. & Wise, S. M. (2019) A second order energy stable scheme for the Cahn–Hilliard–Hele–Shaw equations. *Discrete Contin. Dyn. Syst. Ser. B*, 24, 149–182.

Chen, W., Gunzburger, M., Sun, D. & Wang, X. (2013) Efficient and long-time accurate second-order methods for the Stokes–Darcy system. *SIAM J. Numer. Anal.*, 51, 2563–2584.

Chen, W., Han, D. & Wang, X. (2017) Uniquely solvable and energy stable decoupled numerical schemes for the Cahn–Hilliard–Stokes–Darcy flows for two-phase flows in karstic geometry. *Numer. Math.*, 137, 229–255.

Chen, W., Liu, Y., Wang, C. & Wise, S. M. (2016) Convergence analysis of a fully discrete finite difference scheme for the Cahn–Hilliard–Hele–Shaw equation. *Math. Comp.*, 85, 2231–2257.

Diegel, A. E., Feng, X. H. & Wise, S. M. (2015) Analysis of a mixed finite element method for a Cahn–Hilliard–Darcy–Stokes system. *SIAM J. Numer. Anal.*, 53, 127–152.

Diegel, A. E., Wang, C., Wang, X. & Wise, S. M. (2017) Convergence analysis and error estimates for a second order accurate finite element method for the Cahn–Hilliard–Navier–Stokes system. *Numer. Math.*, 137, 495–534.

Feng, X. (2006) Fully discrete finite element approximations of the Navier–Stokes–Cahn–Hilliard diffuse interface model for two-phase fluid flows. *SIAM J. Numer. Anal.*, 44, 1049–1072 (electronic).

Feng, X. & Wise, S. (2012) Analysis of a Darcy–Cahn–Hilliard diffuse interface model for the Hele–Shaw flow and its fully discrete finite element approximation. *SIAM J. Numer. Anal.*, 50, 1320–1343.

Girault, V. & Raviart, P.-A. (1986) *Finite Element Methods for Navier–Stokes Equations: Theory and Algorithms*. Springer Series in Computational Mathematics, vol. 5. Berlin: Springer, p. x+374.

Han, D., Wang, X. & Wu, H. (2014a) Existence and uniqueness of global weak solutions to a Cahn–Hilliard–Stokes–Darcy system for two phase incompressible flows in karstic geometry. *J. Differential Equations*, 257, 3887–3933.

Han, D., Sun, D. & Wang, X. (2014b) Two-phase flows in karstic geometry. *Math. Methods Appl. Sci.*, 37, 3048–3063.

Heywood, J. G. & Rannacher, R. (1982) Finite element approximation of the nonstationary Navier–Stokes problem I. Regularity of solutions and second-order error estimates for spatial discretization. *SIAM J. Numer. Anal.*, 19, 275–311.

Layton, W. J., Schieweck, F. & Yotov, I. (2002) Coupling fluid flow with porous media flow. *SIAM J. Numer. Anal.*, 40, 2195–2218.

Liu, Y., Chen, W., Wang, C. & Wise, S. M. (2017) Error analysis of a mixed finite element method for a Cahn–Hilliard–Hele–Shaw system. *Numer. Math.*, 135, 679–709.

Minjeaud, S. (2013) An unconditionally stable uncoupled scheme for a triphasic Cahn–Hilliard/Navier–Stokes model. *Numer. Methods Partial Differential Equations*, 29, 584–618.

Mu, M. & Zhu, X. (2010) Decoupled schemes for a non-stationary mixed Stokes–Darcy model. *Math. Comp.*, 79, 707–731.

Riviè re, B. & Yotov, I. (2005) Locally conservative coupling of Stokes and Darcy flows. *SIAM J. Numer. Anal.*, 42, 1959–1977.

Shen, J. & Yang, X. (2015) Decoupled, energy stable schemes for phase-field models of two-phase incompressible flows. *SIAM J. Numer. Anal.*, 53, 279–296.