BILLIARD FLOW AND EIGENFUNCTION CONCENTRATION
ON POLYHEDRONS

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ABSTRACT. We study dynamical properties of the billiard flow on 3 dimensional convex polyhedrons away from “pockets” and establish a finite tube condition for rational polyhedrons that extends well-known results in dimension 2. Furthermore, we establish a new quantitative estimate for lengths of periodic tubes in irrational polyhedrons. We then apply these dynamical results to prove a quantitative Laplace eigenfunction mass concentration near the pockets of convex polyhedral billiards. As a technical tool for proving our concentration results, we establish a control-theoretic estimate on a product space with an almost-periodic boundary condition that extends previously known estimates for periodic boundary conditions, which we believe should be of independent interest.

1. Introduction

In this paper our interests are twofold: first, we investigate several dynamical properties of the billiard flow on polyhedrons in $\mathbb{R}^n$. Historically, there has been a lot of interest in billiards flow on polygons, for example, see [13], [9], [4], [7], [5], [17] etc. For a more comprehensive overview, one might also consult the voluminous [18]. As is well known, even in the case of convex polygons, the study of even apparently simple looking questions can run into deep waters pretty quick, as is evidenced by Galperin’s example of non-periodic and not everywhere dense billiard trajectories in convex polygons (see [12]). It should not come as a surprise then that there is very little literature (to our knowledge) on the billiard flow on polyhedrons, convex or otherwise; for the meagre literature available, see [1], [2], [3] and references therein. In particular, [2] gives a generalization of Katok’s result regarding the entropy of the symbolic dynamics of the billiard flow on polygons to higher dimensions. Here, our main interest is in finer properties of the billiard flow on polyhedrons away from pockets, and our main results in this direction basically say that for rational polyhedrons, the “nonsingular part” of the phase space away from the pockets essentially “decomposes” into a collection of finitely many immersed solid tori and for irrational polyhedrons, the lengths of periodic tubes satisfy a certain asymptotic estimate. For a formal statement of our results, see Section 2 below.

Somewhat curiously, our interest in the billiard flow problem ultimately stems from a well-known spectral problem motivated from quantum physics: namely, the problem of high energy eigenfunction concentration. Let $\Delta$ denote the Dirichlet (or Neumann, as the case might be) Laplacian on a polyhedron $P \subset \mathbb{R}^n$. In the paper [15] concentration of eigenfunctions on certain flat polygonal surfaces has been investigated. The result basically states that given a polygon $P$ whose vertex
set is denoted by $V$ and an open neighbourhood $U$ of $V$, then there exists a constant $c(U) > 0$ such that for any Dirichlet eigenfunction $u$ the following concentration estimate holds:

$$(1.1) \quad \int_U |u|^2 \geq c \int_P |u|^2.$$  

Clearly, this can be thought of as an ergodicity phenomenon. We study this phenomenon in further settings, mainly higher dimensional polytopes. Since the Laplace eigenfunctions $\varphi_\lambda$ are eigenfunctions for the wave propagator $U(t) := e^{it\sqrt{-\Delta}}$, it is expected (at least heuristically) that the high-frequency limits $\lambda \to \infty$ should reflect the dynamics of the classical geodesic flow. In particular, when the geodesic flow is ergodic, one should expect the eigenfunctions to diffuse in phase space, which gives an intuitive feeling behind why estimates like (1.1) should be expected to hold.

As our overall plan of action, we aim to closely follow the scheme laid down in [15] who work in dimension $n = 2$. Their idea is to use the following two special properties of the billiard flow on polygonal domains: every billiard trajectory which avoids the neighbourhood $U$ is periodic (see [13]). Clearly, such periodic trajectories come in 1-parameter families, which form “cylinders” in $P \backslash U$. Furthermore, there are only finitely many such cylinders (see [16], [9]). These, together with propagation results for eigenfunctions, imply that if a sequence of eigenfunctions concentrate away from $U$, the corresponding semiclassical measure on the sphere bundle must concentrate along the families of periodic geodesics that sweep out such cylinders. This is ruled out using an argument from [21], which in turn is based on an argument by Burq and Zworski (see [5], [6]).

As we will demonstrate, these ideas will also generalize to dimensions $n \geq 3$.

The main innovation of this paper in the dynamical part is proving the finiteness of the family of cylinders or tubes for rational polyhedrons and the aforementioned estimate for lengths of periodic tubes in irrational polyhedrons, which turn out to be somewhat involved. In the analytical part, the main contribution is proving that a version of a control result due to Burq and Zworski (see Proposition 14 in [15]) also holds in higher dimensions for “almost periodic boundary conditions”, which corresponds to the case of “irrational polyhedrons”. Also, in the irrational case, one of the main technical problems is the lack of a finite tube condition, and the argument that works for rational polyhedrons requires appropriate modifications.

There are a few interesting questions left to consider. As mentioned before, a major question on the dynamical side is whether the finite tube condition (c.f. Definition 2.11) can be extended to irrational polyhedrons as well. Also interesting would be to speculate if any of the dynamical results stated here has any analogues for polyhedrons which are not necessarily convex. From the analytical point of view, an exciting question is to determine the dependence of the constant $c = c(\varepsilon)$ in Theorem 2.13 in the case when $U$ is an $\varepsilon$-neighbourhood of the singular set, i.e. to quantify the estimates in [15] and this paper, as least asymptotically as $\varepsilon \searrow 0$.

1.1. Structure of the paper. In Section 2 we state the preliminary definitions and state our main results. As advertised before, our paper splits naturally into a dynamical and an analytical part. Accordingly, Section 2 will have two subsections, devoted respectively to the dynamical and analytical parts. In Section 3 we will prove the main dynamical results, and finally in Section 4 we show the application
of the dynamical part to the eigenfunction concentration result, that is, give the proofs of the main results of the analytical part.

2. Statements and main results

2.1. Billiard Dynamics. First, let $P \subset \mathbb{R}^n$ be a polyhedron and let us define the singular set $S$, that is the union of all $(n - 2)$-dimensional “edges” of the polyhedron (in other words, the $(n - 2)$-skeleton), representing the higher dimension analogues of what are “corners” in polygonal billiards. Let $S \subset U \subset P$ be an open neighbourhood of the singular set. The billiard flow on $P$ is the usual one, a particle (or a point mass) travels to $P \setminus S$ with unit velocity, and then on striking one of the faces of the boundary, instantaneously changes direction according to the law of light reflection and continues along the reflected line. Trajectories which strike a singular point stop right there, and such trajectories are also called singular. If the above does not happen, then the motion is determined for all time. Now, we introduce some notations and definitions, where we mainly follow the influential paper [13].

Let us denote $\Gamma := \partial P$, and let $TT$ be the set of all unit tangent vectors with base points in $\Gamma$ and which are directed inside $P$. Denote by $f$ the first return (Poincaré) map of the Billiard flow to the set $TT$. $f$ and its iterates are defined and smooth everywhere except for the vectors whose billiard orbits hit $S$, which is a set of measure zero. Let the polyhedron $P$ have $k$ faces, and $\mathcal{P} = \{F_1, F_2, ..., F_k\}$ be the partition of the non-singular points (definition below) of $TT$, where all the vectors with base point on one face of $P$ form a member of the partition. Let $\Sigma^+_k := \{1, 2, ..., k\}^\mathbb{N}$ represent the forward string for the symbolic dynamics of the billiard flow, and let $\sigma$ be the usual forward shift operator. Now, define the “regular” part of $TT$ as $TT_1 := \{x \in TT : \text{the forward orbit of } x \text{ never hits } S\}$. It will become clear (see the proof of Theorem 2.5 below) that the basepoint of any regular vector cannot belong to $S$. In other words, a “generalized diagonal” of $P$, as defined in [17], must end at a singular point of $P$. Also, given a trajectory starting from $x \in TT_1$, the symbolic string for the forward orbit is given by $w(x)$, defined by $w(x)_i = j$ if $f^i(x) \in F_j$. This gives us the symbolic encoding $\Sigma^+_k := \{w \in \Sigma^+_k : \exists x \in TT_1 \text{ such that } w = w(x)\}$. For each such possible string $w \in \Sigma^+_k$, we define $X(w) := \{x \in TT_1 : w(x) = w\}$. In other words, $X(w)$ represents all tangent vectors whose billiard trajectories have the same symbolic representation $w$. Finally, a subset $Q$ of $TT_1$ is called a tube if all $x \in Q$ are parallel vectors whose base points form a connected set on one of the faces of the polyhedron. In dimension $n = 3$ in particular, the tube is polygonal (elliptical) if there is an open polygon (ellipse) $F$ such that $F \subset X(w) \subset \overline{F}$. An arbitrary element of $X(w)$ is denoted by $x(w)$.

We also take the space to make the following important distinction between rational and irrational polyhedrons. As previously remarked, most of the literature focusses on dimension $n = 2$, where the definition of rational (irrational) polyhedrons can be simply given in terms of the rationality (irrationality, respectively) of the angles at the vertices, namely, the polygon is called rational if all its angles are rational, otherwise it is called irrational. For reasons which will become clear in the course of this paper, many statements which we will concern ourselves with are easier to prove for rational polygons than irrational ones. As pointed out by several
authors, in dimensions $n \geq 3$, there is no unified definition of rational/irrational polyhedrons.

However, a very important point of difference exists, which will dictate a convenient definition in higher dimensions. To explain this, let us first recall the well-known tool or method of unfolding a trajectory. Let $\gamma$ be a billiard trajectory of $x \in T\Gamma_1$. Starting from an initial point, we follow $\gamma$ till it strikes a face $P_1$ (say) of $P$. Then we reflect $P$ about $P_1$, and keep following it inside the reflected polyhedron till it strikes another face, whence we reflect the polyhedron again. Continuing this process indefinitely gives a sequence $P, P_1, P_2, ..., P_m, ...$ of polyhedra which are skewed on the forward ray determined by $\gamma$. We call this object an infinite corridor along the ray $\gamma$. Obviously, refolding the corridor the line folds back (immerses) to $\gamma$.

We are soon going to describe the difference in behaviour displayed by different polyhedra with respect to the above mentioned unfolding process and motivate a definition of rational polyhedra contained in [2]. Before that, we record a few preliminary results, whose proofs we relegate to Section 3.

**Lemma 2.1** ([13], Lemma 1). In any polyhedron $P \subset \mathbb{R}^n$, if $w(x) = w(y)$, then $x$ and $y$ are parallel vectors.

Next, we just record the fact that any trajectory in a polygon whose forward closure does not contain any singular point must be periodic. In other words, no trajectory can continue indefinitely inside the polyhedron without coming arbitrarily close to the pockets. The proof appears in [13] (see Theorem 2, Corollary 2) for polygons.

**Theorem 2.2.** For an arbitrary polygon $P$, and for any $x \in TP$, if the closure of the set $\pi_1\{f^i x : i \geq 0\}$ does not contain any singular point, then the orbit of $x$ is periodic, where $\pi : TP \to P$ is the usual projection map.

Now, let us consider a polyhedron $P$. In higher dimensions, depending on the type of the polyhedron, it turns out that an orbit generated by $x \in T\Gamma_1$ with a periodic symbol $w(x)$ might or might not be periodic. Now, observe that if the forward closure of the orbit of $x$, called $\gamma$ henceforth, does not intersect $S$, then $\gamma$ can be “thickened” to form a tubular neighbourhood $T$ around $\gamma$ such that each trajectory in $T$ parallel to $\gamma$ also has the symbolic representation $w(x)$. It turns out that such tubes $T$ can themselves be periodic (see below). This brings us to the following definition, which is taken from [2].

**Definition 2.3.** Let $P \subset \mathbb{R}^n$ be a polyhedron and let $\rho_i$ represent the reflection map of $P$ over the $i$th-face of $P$. Then $P$ is called rational if the group $G$ generated by the $\rho_i$ is finite, otherwise the polyhedron is called irrational.

With that in place, the following result, which is Theorem 5 of [13], should not come as a total surprise:

**Theorem 2.4.** Let $P \subset \mathbb{R}^3$ be an arbitrary convex polyhedron and $w \in \Sigma^+_p$ is a periodic sequence with period $k$. The following hold:

1. There exists $x(w)$ so that $x(w)$ is periodic with period $k$.
2. In addition, one of the following two cases holds:
   a. There exists $q \geq 1$ such that all $y(w) \in X(w) \setminus x(w)$ are periodic with period $qk$ and the set $X(w)$ is an open polygonal tube.
(b) The set $X(w)$ is an elliptical tube and there exists a unique point $x(w)$ which is periodic and is at the center of mass of $X(w)$.

(3) If $k$ is odd, then only the first case (2)(a) above can happen and $q = 2$.

(4) If $P$ is rational then only the first case above can happen.

As an example of when 2(a) above might occur, consider a right prism whose horizontal cross-section is an equilateral triangle. Consider a trajectory which lies on a plane perpendicular to the height of the prism, and strikes an equilateral triangular cross-section exactly at the mid-point of the three sides. This is periodic with period 3, whereas any $y(w) \in X(w) \setminus x(w)$ is periodic with period 6.

For an example when 2(b) might occur, it is enough to consider a regular tetrahedron and the closed orbit corresponding to the word $w = (abcd)$, where $a, b, c$ and $d$ encode the faces of the tetrahedron. Then it is possible to show that there is a unique closed orbit $x(w)$, that $X(w)$ is an elliptical tube and the nearby parallel trajectories “come back” rotated by an irrational angle. See [1, Section 8] for explicit computations.

Theorem 2.4 shows that the result of Theorem 2.2 does not generalize to higher dimensions in a completely straightforward way, that is, it gives the existence of trajectories whose closure does not contain any singular point, but which are not themselves periodic (for irrational polyhedra), but are contained in periodic tubes. However, the following result says that Theorem 2.4 contains all such possible obstructions:

**Theorem 2.5 ([13]).** Let $P \in \mathbb{R}^3$ be a convex polyhedron and $w \in \Sigma^+_P$ which is not periodic. Let $x \in X(w)$. Then, any tubular neighbourhood containing an open tube around the trajectory of $x$ intersects the singular set $S$.

The above result basically says that $X(w)$ is at best a codimension 1 “strip” when $P$ is convex, which includes the case that $X(w)$ consists of a single point. As a consequence we obtain the following:

**Corollary 2.6.** For any billiard trajectory $\gamma$ in a convex polyhedron $P$, either $\gamma$ is contained in a periodic tube, or the closure of $\gamma$, $C_\gamma$, meets $S$.

We relegate the proofs of Theorems 2.4 and 2.5 to Section 8.

Now, we are ready to state our main results regarding the billiard dynamics of $P$. We introduce the notation $D := (P \sqcup \sigma P)/\sim$ to denote the double of $P$ – an open Euclidean 3-manifold; the singular set is now of codimension 2. Here $\sigma$ denotes a reflection in a plane in $\mathbb{R}^3$. This space plays the role of Euclidean surfaces with conical singularities (ESCS) from [15]. Before proceeding, note that an eigenfunction $u_n$ on $P$ induces a $C^\infty$ eigenfunction on $D$, that is equal to $u_n$ on $P$ and to $-u_n \circ \sigma$ in $\sigma P$. A lower bound on the mass near the singular set in $D$ translates to a similar bound in $P$.

**Lemma 2.7.** Let $\varepsilon > 0$. Define $U_\varepsilon$ to be an $\varepsilon$-neighbourhood of the singular set of the doubled polyhedron $D$. Order the lengths of closed geodesics not hitting $U_\varepsilon$ by

$L_1 \leq L_2 \leq \cdots$

Then we have the following asymptotic formula:

$$\sum_{i=1}^{\infty} \frac{1}{L_i} = O(\varepsilon^{-4})$$
as \( \varepsilon \to 0 \). In general, if \( \dim P = n \) we have:

\[
\sum_{i=1}^{\infty} \frac{1}{i^\alpha} = O(\varepsilon^{2(n-1)})
\]

**Remark 2.8.** In the case of polygons, i.e. \( n = 2 \), this lemma together with Corollary 2.6 shows the (finite) cylinder condition of [15]. Note that for \( n \geq 3 \), we do not immediately get the analogous claim. For this, we devised a different argument that works in the case of rational polyhedrons, see the proof of Theorem 2.12.

By Theorems 2.4, 2.5 and Corollary 2.6 any geodesic not hitting \( U_\varepsilon \) is contained in a tube with disc or polygonal cross-section. Inside \( D \setminus U_\varepsilon \), consider any periodic tube \( T_i \) with cross-section being a disc of radius \( \frac{\varepsilon}{10} \), and \( \gamma_i \) representing the central periodic trajectory inside \( T_i \). Also, note that these tubes are given by immersing honest solid cylinders \( p_i : \mathbb{R} \times D_{\varepsilon/10} \to D \).

Assume two such tubes \( T_i \) and \( T_j \) intersect. We will prove a lower bound on the angle of their intersection. Observe that there is no loss of generality involved in assuming that the tubes have disc cross-section. More precisely, assume furthermore that a geodesic \( \gamma'_i \) lying in \( T_i \) intersects a geodesic \( \gamma'_j \) lying in \( T_j \) at a point \( m \in D \), at an angle \( \alpha \). Then we have:

**Lemma 2.9.** The following lower bounds holds:

\[
\frac{1}{\sin \alpha} \leq \min \left( \frac{L_i}{L_j} \right)
\]

**Remark 2.10.** Consider the equilateral triangle and two periodic geodesics: the central one and the “bouncing ball” type, which hits the middle of an edge by an angle of \( \frac{\pi}{6} \). Then the corresponding immersed cylinders are contained in one another, but we cannot draw (in the pre-image) a picture with one strip contained in the other one, since the bouncing ball exits one side of the strip of the central geodesic and re-enters.

Now, following [15], we give the following

**Definition 2.11.** Let \( D \) be the double of \( P \) as defined above. A region \( U \subset D \) is said to satisfy the “tube condition” if there exists a finite collection of immersed tubes \( T_i \) such that any orbit that avoids \( U \) belongs to some \( T_i \).

Finally, we have:

**Theorem 2.12.** Let \( P \) be a rational convex polyhedron. Then, any neighbourhood \( U \) of the singular set \( S \) satisfies the tube condition.

Assume more generally \( P \) is any convex polyhedron, and \( U \) and \( S \) as before. Then the directions determined by rational periodic tubes in \( SP \setminus SU \) are isolated in the set of all directions determined by periodic tubes.

2.2. Eigenfunctions and semiclassical defect measures. We first start by stating our main analytical result.

**Theorem 2.13.** Given a neighbourhood \( U \) of \( S \) inside the polyhedron \( P \), there exists a positive constant \( c = c(U) \) such that for any \( L^2 \)-normalized eigenfunction \( u_k \), we have that

\[
\int_U |u_k|^2 \geq c.
\]
Notation. Before we proceed further, we first state a notational convention. Throughout the rest of the paper, we will slightly abuse notation, and treat \( U \) as a neighbourhood of the singular points in the double \( D \) rather than in \( P \), as we have done before. It will be clear from the proofs that the main eigenfunction concentration result will also be established for the double \( D \), but it will readily imply Theorem 2.13 (see our discussion immediately before Lemma 2.7). Also, to clarify, one can define the Laplacian on \( D \) by taking the Friedrichs extension of the operator with domain \( C_C^\infty(D_0) \), where \( D \) can be written as the disjoint union \( D := S' \cup D_0 \), where \( S' \) is the singular set in \( D \). It is known that this is self-adjoint with compact resolvent, so has discrete spectrum \( \lambda_j \) going to infinity, and a complete orthonormal \( L^2 \)-basis \( u_j \) of Laplace eigenfunctions.

Observe that we are claiming that Theorem 2.13 above holds for all eigenfunctions, as opposed to a density 1 subsequence of \( \lambda_j \). We follow the analytic argument inspired by [15]. To the contrary assume there is no concentration in a neighbourhood \( U \) of the surrounding hypersurfaces of \( P \), that is, there exists a subsequence \( u_n \) satisfying

\[
\lim_{n \to \infty} \int_U |u_n|^2 = 0.
\]

Let \( \mu \) be an arbitrary semiclassical measure associated to the sequence \( u_n \) in the standard way. That is, for any \( a \in C_C^\infty(S^*D_0) \) (unit cotangent bundle on \( D_0 = D \setminus S \)) and any zeroth-order pseudodifferential operator \( A \) on \( D_0 \) with principal symbol \( a \) we have

\[
\lim_{n \to \infty} \langle Au_n, u_n \rangle = \int_{S^*D_0} ad\mu.
\]

We will look at the interaction of this semiclassical measure with the geodesic flow on \( D \). To that end, we have the following result. Let \( U_0 := U \setminus S \).

**Theorem 2.14.** The support of \( \mu \) is disjoint from \( \pi^{-1}(U_0) \) and \( \mu \) is a probability measure which is invariant under the geodesic flow.

In proving Theorem 2.13 our main technical result is a version of a control theory estimate in [6], which also works for periodic boundary conditions, as observed in [15]. We prove an analogous version, but in dimension 3, but more importantly, for “almost periodic boundary conditions” (see Theorem 4.5 below). Among other consequences of such a control result would be a stronger and more general versions of Theorem 2 of [21], which has been stated in the particular context of partially rectangular billiards.

**Theorem 2.15.** For any periodic tube \( T \) immersed in \( P \), and a neighbourhood \( U \) of \( \partial T \) inside \( P \), there exists a constant \( C > 0 \) such that

\[
-\Delta u = \lambda u \implies \int_U |u(x)|^2dx \geq C \int_T |u(x)|^2dx,
\]

that is, no eigenfunction can concentrate in \( T \) and away from \( \partial T \).

The proofs of Theorem 2.15 is absolutely similar in spirit to the proof in [21], except that in three dimensions, we need to use our version of the control result, given by Theorem 4.5. We will skip the details.
3. PROOFS OF THE DYNAMICAL RESULTS

Now we provide proofs for the claims made in Subsection 2.1. First, Lemma 2.1

Proof. Unfold the polyhedron $P$ along trajectories determined by $x$ and $y$. Unless they are parallel, the distance between their trajectories grows linearly, and cannot be contained in the same corridor. The first time they move into different corridors, their symbolic representations $w(x)$ and $w(y)$ must also differ. □

Next, we give a proof Theorem 2.4 for completeness, see Theorem 5 of [13].

Proof. Suppose $w$ has period $k$. Suppose $k$ is even, and if not, consider $2k$. Since $k$ is even, the polyhedra $P$ and $P^k$ have the same orientation. Observe that $X(w)$ is a convex simply-connected set. Extend $f$ (recall from page 3 that $f$ is the Poincaré return map) to the boundary of $X(w)$ by continuity from the inside. Since $k$ is finite, such an extension is possible when $X(w)$ is convex, and we have that $f^k(X(w)) = X(w)$. By the Brouwer fixed point theorem we can locate an $x(w) \in X(w)$ such that $f^k(x(w)) = x(w)$. This implies that $P^k$ can be gotten from $P$ by a translation along the straightened trajectory of $x(w)$ (possibly) composed with a rotation in $\mathbb{R}^3$ by an angle $\alpha$, and if $\alpha \neq 0$, then $x(w)$ must be at the center of mass of $X(w)$ since $X(w)$ is invariant under a rotation around $x(w)$.

If $\alpha = 2\pi p/q$ (where $p, q$ are integers), then $P^q(w)$ can be gotten from $P$ by translation alone, and all $y(w) \in X(w)$ are periodic. Now, take the basepoints of the straightened tube $X(w)$ on each of the faces $F_i$ that it strikes. Project each of them on a plane perpendicular to the direction of $x$, and taking the intersection over all the $k$ projections, we have that the cross-section of the tube is a convex polygon.

The periodicity of each $y(w)$ implies that its orbit stays a bounded distance away from the singular points and hence the tube is open.

If $\alpha$ is irrational, then $P^m$ is never a translation alone of $P$. Thus no point in $X(w) \setminus x(w)$ can be periodic. The cross-section of the tube lying on a plane perpendicular to the axis $\gamma$ is exactly a disc, and its footprint (or the set of its basepoints) on each face of $P$ will be an ellipse. □

Remark 3.1. Observe that the above proof will not work when $P$ is not convex, as one can easily come up with examples of non-convex polyhedrons where the set $X(w)$ is not simply-connected, which disallows the application of the Brouwer fixed point theorem.

Now we prove Theorem 2.5. Following [13, Theorem 6], we write a proof for dimension $n = 3$ for ease of presentation, but as remarked in [13], it will be clear that both Theorem 2.4 and Theorem 2.5 will extend with obvious modifications to all dimensions.

Proof. Suppose that the result is not true, and $X(w)$ contains a set $X'$ of parallel vectors generating a tubular neighbourhood $T$. Fix a point $x \in T$ such that $x$ is in the interior of $X'$. Let $\gamma$ be the billiard trajectory generated by $x$. We find and fix a $\delta > 0$ such that dist($\gamma, S$) $> \delta$. Let $\tilde{\gamma}$ represent the lift of the geodesic $\gamma$ to the unit tangent bundle $S(P)$. Consider the set $Y$ of forward limit points of $\tilde{\gamma}$. Since $Y$ is compact, we can apply Furstenberg’s uniform recurrence theorem (see [11, Theorem 1.16]) to find a point $x^* \in S(P)$ that is uniformly recurrent in $Y$. If $G$ represents the billiard trajectory generated by $x^*$ through $\pi(x^*)$, and $\tilde{G}$
represents the lift of $G$ to $S(P)$, recall that uniform recurrence means that for any neighbourhood $\tilde{W}$ of $x^*$ that intersects $\tilde{G}$, there exists $L \in \mathbb{R}$ such that
\[
\forall t \in [t, t + L] \text{ such that } (G(s), \tilde{G}(s)) \in \tilde{W}.
\]
If the basepoint of $x^*$ lies on a face $P_j$ of $P$, observe that $x^*$ is not tangent to $P_j$, as since then the forward orbit of $x$ would come arbitrarily close to one of the edges of $P$. Now, suppose that the sequence $x_i := f^n x \to x^*$. Define $S_i(x) := f^n(S)$, $S_i(x)$ to be the maximal tube around $x_i$, and $\hat{S}(x^*)$ to be the maximal tube around $x^*$. Unfold $S(x^*)$ along a corridor to get a tube $S^\infty(x^*)$, and fix $\varepsilon > 0$ much smaller than the minimal diameter of $S$, in particular, $\varepsilon \ll \delta$. Then, let $N_\varepsilon(S^\infty(x^*)) := T_\varepsilon(S^\infty(x^*)) \setminus \text{Int} (S^\infty(x^*))$ denote the outer $\varepsilon$-tubular neighbourhood of $(S^\infty(x^*))$.

Consider a cross-section $C$ to $N_\varepsilon(S^\infty(x^*))$ which contains the orbit of $x^*$, which is a union of strips. Due to the uniform recurrence of $x^*$, any such cross-section $C$ intersects edges of $P$ with (uniformly) bounded gaps between the heights of their occurrences. But since $x_i \to x^*$, one can find such cross-sections $C$ such that there is an $\varepsilon$-wide rectangle in $C \cap S^\infty(x^*) \cap S^\infty(x_i)$ whose height goes to $\infty$ as $i \to \infty$. Thus such a cross-section must eventually intersect an edge, contradicting the maximality of the tube and the fact that the trajectory of $x^*$ does not intersect any edge. This proves our claim. □

Now we prove Lemma 2.9.

Proof. Enlarge $T_i$ to a solid cylinder $T_i'$ of radius $\frac{2\varepsilon}{5}$, not intersecting $U_{\varepsilon/5}$, such that $T_i'$ intersects $\gamma_j$. This is possible since the distance between $\gamma_i$ and $\gamma_j$ is at most $\frac{\varepsilon}{5}$. Similarly, extend $T_j$ to a solid cylinder $T_j'$ intersecting $\gamma_i$. By a slight abuse of notation, we will sometimes identify the immersed tubes $T_j'$ and their preimages under $p_j$ (think of an unfolding).

Consider the point $p$ on $\gamma_i$ with minimal distance $d < \varepsilon/5$ to $\gamma_j$ ($\gamma_i$ and $\gamma_j$ are not parallel since the tubes $T_i$ and $T_j$ are distinct maximal tubes, so $p$ is unique). Consider the strip $S_i$ contained in $T_j'$ determined by $\gamma_i$ and the direction of the tube $T_j'$ and let it have width $h_j$. By triangle inequality
\[
\frac{1}{2} h_j > \frac{3\varepsilon}{5} - \frac{\varepsilon}{5} = \frac{2\varepsilon}{5}
\]
We introduce the parameter $l_i := \frac{h_j}{\sin \alpha}$, i.e. the length of the portion of $\gamma_i$ lying in the strip $S_j$. Assume that $l_i > L_i$ for the sake of contradiction.

This means that $\gamma_i$ remains in the tube $T_j'$ for all time and has the same symbolic description as $\gamma_j$. By Lemma 2.1, we see that $\gamma_i$ will hit the boundary of the solid cylinder and will have a different symbolic description from $T_j$, unless parallel to $\gamma_j$ by Lemma 2.1. This implies that $T_i$ and $T_j$, up to orientation, generate the same symbolic sequence and must be part of the same maximal tube, which contradicts the choice of the tubes.

Therefore, we must have $l_i \leq L_i$ and so similarly, $l_j := \frac{h_j}{\sin \alpha} \leq L_j$. We combine these two inequalities together with (3.1) to get the (2.1).

We now discuss the idea of partitioning the phase space alluded to before. For this purpose, we introduce the sets
\[
V_i := \{(x, \theta) \in SD_0 \mid x \in T_i, \ |\gamma_i - \theta| < \frac{4\varepsilon/5}{2L_i}\}
\]
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where $\dot{\gamma}_i$ is the unit speed of the geodesic parametrised by arc-length. We then have $V_i$ is locally isometric to a tube of radius $\varepsilon/10$ times a spherical cap of radius $\frac{4\varepsilon/5}{2L_i}$. Thus, we have

$$vol(V_i) = L_i \times \left(\frac{\varepsilon}{10}\right)^2 \pi \times \left(1 - \cos\left(\frac{4\varepsilon/5}{2L_i}\right)\right)$$

(3.3)

$$\geq \varepsilon^2 \times L_i \times \sin^2\left(\frac{\varepsilon}{5L_i}\right)$$

$$= \varepsilon^4 \times O\left(\frac{1}{L_i}\right)$$

as $L_i \to \infty$. Note also for this, we may take $\varepsilon$ small enough (the distances between faces in $D \setminus U_\varepsilon$ are at least $\varepsilon$), so that $L_i$ are bounded from below by $O(\varepsilon)$. The sets $V_i$ are disjoint by Lemma 2.9 and hence by summing the volume estimates above we obtain Lemma 2.7.

Finally, we prove the finite tube condition, stated in Theorem 2.12.

Proof. For prove the first claim, let us assume to the contrary. That is, let us assume that there are infinitely many periodic tubes $T_i$ in $D \setminus U_\varepsilon$, assumed to have disc cross-section without any loss of generality. The central trajectory of $T_i$ is $\gamma_i$, generated by $x_i$, and also assume that the periods of said tubes are given by $L_i$. By our previous discussions, we know that the number of tubes with period length $\leq L$ is finite (more precisely, the number of such tubes is $o(L)$). This implies that $L_i \to \infty$. Due to the compactness of the phase space $T\Gamma$, we have that a subsequence, still called $x_i$ by minor abuse of notation, converges to $x$. By a straightforward continuity argument, the trajectory $\gamma$ determined by $x$ also stays in $D \setminus U_\varepsilon$. By Theorems 2.4 and 2.3, such a geodesic $\gamma$ is contained in a rational periodic tube $T$. This tube $T$ has a central geodesic $\gamma'$, say, with period $L$. We now apply our angle estimate Lemma 2.9 to tubes of radius $\varepsilon/10$ centred at the basepoint of $x_i$ and directions of $\gamma_i$ and $\gamma$ for large $i$. If $\alpha_i = \angle(\gamma_i, \gamma') = \angle(\gamma_i, \gamma)$, then $\sin \alpha \geq\frac{\varepsilon}{\min(L, L_i)} = \frac{\varepsilon}{L}$ for large enough $i$. But since $x_i \to x$, we must have $\alpha_i \to 0$, which gives a contradiction.

To obtain the second claim, we simply repeat the previous argument with the assumption that the limiting tube $T$ is of rational type. The finishes the proof. \qed

Remark 3.2. The reason why we consider the case of rational polyhedrons only in Theorem 2.12 is the following. Assume $P$ any convex polyhedron. We may arrive to the point where $x_i \to x$ as in the proof above. If the limiting tube $T$ is irrational, we run into trouble: the tubes $T_i$ might not contain a periodic geodesic in the tube $T$ (note that the unique such geodesic is $\gamma'$). Thus we cannot apply the angle estimate in Lemma 2.9. So our “enemy” is the existence of such thin, long tubes.

4. PROOFS OF THE ANALYTICAL RESULTS

As a general reference to much of the material in this section, the reader is referred to [24]. First, we start by proving Theorem 2.14. The method of the proof is standard, and we closely follow the proof in [15].

Proof. Suppose that there is a point $q \in \text{supp} \mu$ with $\pi(q) \in U_0$. Let $\phi \in C^\infty(D_0)$ satisfy $\text{supp} \phi \subset U_0$, with $\phi \equiv 1$ in a small neighbourhood $G$ of $\pi(q)$. Since
\( \phi \geq 0 \), and \( \mu \) is a positive measure, \( \langle \mu, \phi \rangle \geq 0 \). If \( \langle \mu, \phi \rangle = 0 \), then \( \langle \mu, \chi \rangle = 0 \) for every \( \chi \in C^\infty(S^*D_0) \) supported in \( \pi^{-1}(G) \), since we have \( a = \phi \chi \), and \( |\langle \mu, \phi \chi \rangle| \leq \langle \mu, \phi \rangle \| \chi \|_{L^\infty} \). This would imply that \( \pi^{-1}(G) \cap \text{supp} \mu = \emptyset \), which is a contradiction. But \( \langle \mu, \phi \rangle > 0 \) means that \( \lim_{n \to \infty} \int_D |u_n|^2 \phi > 0 \), which is a contradiction again.

Now, consider a cutoff function \( \rho \) that is identically 1 near \( S \) and identically equal to 0 outside \( U \). Our main task is to prove that \( \int_{S^*D_0} d\mu = 1 \), and the invariance of \( \mu \) under the geodesic flow is well-known. One can check that

\[
\int_{S^*D_0} d\mu = \int_{S^*D_0} (1 - \rho) d\mu.
\]

But we also have, \( \int_{S^*D_0} \rho |u_n|^2 \to 0 \), which proves our contention.

Before starting with the proof of Theorem 2.12, we establish some preliminaries. For the sake of completeness, we begin by proving a result in unpiblished lecture notes of N. Burq on the control and observability for an elliptic PDE, as stated in the footnote of [6], pp. 17. Our main aim is to ultimately discuss a non-periodic version of the statement for \( M_x \times [0, L] \) for some \( L > 0 \).

To this end, let \( (M_x, g_x) \) and \( (M_y, g_y) \) be two compact Riemannian manifolds with smooth boundary. We will consider the manifold \( (M_x \times M_y, g) \), where \( g = g_x \oplus g_y \) is the product metric and denote the product by \( M = M_x \times M_y \). Denote by \( -\Delta_g \) the positive-definite Laplace-Beltrami operator.

We say a subset \( A \subset M_x \) satisfies the geometric control condition or just (GCC) if every geodesic \( \gamma \) in \( M_x \) hits \( A \) in finite time.

**Theorem 4.1.** Let \( z \in \mathbb{R} \) and assume \( f \in L^2(M) \). Assume \( u \in H^1(M) \) satisfies

\[
(\Delta - z)u = f, \quad u|_{\partial M} = 0.
\]

Now let \( \omega \subset M_x \) be an open, non-empty set, satisfying (GCC). Then there exists a constant \( C = C(M, g) > 0 \) such that the following observability estimate holds:

\[
\|u\|_{L^2(\omega \times M_y)} \leq C(\|f\|_{L^2(\omega \times M_y)} + \|u\|_{H^1(\omega \times M_y)})
\]

**Proof.** Consider the basis of orthonormal Dirichlet eigenfunctions \( \{e_k\}_{k=1}^\infty \subset L^2(M_y) \), such that \( -\Delta_g e_k = \lambda_k e_k \), where \( 0 < \lambda_1 \leq \lambda_2 \leq \ldots \). We consider \( u \) and \( f \) satisfying (4.1), and consider the Fourier expansion into \( e_k \):

\[
u(x, y) = \sum_{k=1}^\infty u_k(x) e_k(y), \quad f(x, y) = \sum_{k=1}^\infty f_k(x) e_k(y)
\]

The sums converge in \( L^2(M) \) and we have the formula for all \( k \in \mathbb{N} \)

\[
u_k(x) = \int_{M_y} e_k(y) u_k(x, y) dvol_y, \quad f_k(x) = \int_{M_y} e_k(y) f(x, y) dvol_y.
\]

Here \( dvol_y \) denotes the volume density on \( M_y \). Therefore, we see that \( u_k|_{\partial M_y} = f_k|_{\partial M_y} = 0 \). Then we have, by using (4.1) and the eigenfunction condition:

\[
(-\Delta_g - z) \sum_{k=1}^\infty e_k u_k = \sum_{k=1}^\infty e_k (-\Delta_g - z + \lambda_k) u_k = \sum_{k=1}^\infty e_k f_k
\]

Therefore, we conclude for \( k \in \mathbb{N} \)

\[
(-\Delta_g - (z - \lambda_k)) u_k(x) = f_k(x)
\]
The remaining ingredient is an observability estimate on \((M_x, g_x)\).

**Lemma 4.2.** There exists a \(C = C(g_x, \omega) > 0\), such that for any \(s \in \mathbb{R}\) and any
\[(4.4) \quad (-\Delta_{g_x} - s)v = g, \quad v|_{\partial M_x} = 0\]
with \(v \in H^1_0(M_x)\) and \(g \in H^{-1}(M_x)\), we have the apriori estimate
\[
(4.5) \quad \|v\|_{L^2(M_x)} \leq C(\|g\|_{H^{-1}(M_x)} + \|v\|_{L^2(\omega)})
\]

We first prove the main claim given the above lemma. This follows by applying the lemma to equation (4.3) for each \(k \geq 1\)
\[
(4.6) \quad \|u\|_{L^2(M_x)}^2 = \sum_{k=1}^{\infty} \|u_k\|_{L^2(M_x)}^2 \leq C \left( \sum_{k=1}^{\infty} \|f_k\|_{H^{-1}(M_x)}^2 + \sum_{k=1}^{\infty} \|u_k\|_{L^2(\omega)}^2 \right) = C(\|f\|_{H^{-1}(M_x)}^2 + \|u\|_{L^2(\omega)}^2).
\]

Now we prove the lemma in question.

**Proof of Lemma 4.2.** Assume without loss of generality \(g, v\) are real-valued. We split the proof to cases according to the value of \(s \in \mathbb{R}\). We will use semiclassical defect measures in the case of large \(s\).

**Case 0: fixed \(s\).** Assume inequality (4.5) does not hold for this fixed \(s\), so there is a sequence \(v_k \in H^1_0(M_x)\) with \(\|v_k\|_{L^2(M_x)} = 1\) and \(g_k \in H^{-1}(M_x)\) with
\[
(4.7) \quad \|g_k\|_{H^{-1}(M_x)} + \|v_k\|_{L^2(\omega)} < \frac{1}{k};
\]

Then clearly \(\|g_k\|_{H^{-1}(M_x)} \to 0\) and \(\|v_k\|_{L^2(\omega)} \to 0\) as \(k \to \infty\). Elliptic estimates give us that \(\|v_k\|_{H^1(M_x)}\) is bounded in \(k\)\(^2\) so by Rellich’s theorem we may assume \(v_k \to v\) in \(L^2(M_x)\). By the assumptions, we have in the sense of distributions
\[
(4.8) \quad (-\Delta_{g_x} - s)v = 0, \quad v|_{\omega} = 0
\]
Elliptic regularity gives \(v\) is \(C^\infty\) in the interior of \(M_x\) and therefore by the unique continuation principle \(v = 0\), which contradicts \(\|v\|_{L^2(M_x)} = 1\).

**Case 1: bounded \(s\).** Here we show that \(C(s)\) is locally bounded. The proof is by contradiction and is very similar to the previous case.

**Case 2: \(s < -\varepsilon < 0\).** Here \(\varepsilon > 0\) fixed. Integrating by parts, we see
\[
(4.9) \quad \|dv\|_{L^2(M_x)}^2 - s\|v\|_{L^2(M_x)}^2 = (g, v)_{H^{-1}(M_x) \times H^1_0(M_x)}
\]
Estimating right hand side using the boundedness of the \(H^{-1} \times H^1_0\) pairing and using \(s < -\varepsilon\), we obtain \(\|v\|_{L^2(M_x)}^2 \leq \frac{1}{\varepsilon}\|g\|_{H^{-1}(M_x)}^2\), proving (4.5) for \(s < -\varepsilon\).

**Case 3: \(s \to \infty\).** Now it suffices to argue by contradiction and assume \(s_k \to \infty\) with \(C(s_k) \to \infty\). Then, there is a sequence \(v_k \in H^1_0(M_x)\), with \(\|v_k\|_{L^2(M_x)} = 1\) such that
\[
(4.10) \quad (-\Delta_{g_x} - s_k)v_k = g_k
\]
1By elliptic estimates, we have \(\|v\|_{H^1(M_x)} \leq C'(\|g\|_{H^{-1}(M_x)} + \|v\|_{L^2(M_x)})\) for some \(C' > 0\), for \(\varepsilon\) and \(g\) as in the statement of the Lemma.
Assume also \( \|g_k\|_{H^{-1}(M_x)} \to 0 \) and \( \|v_k\|_{L^2(\omega)} \to 0 \) as \( k \to \infty \). We introduce a small semiclassical parameter \( \hbar_k > 0 \) by \( \hbar_k^2 := \frac{1}{\omega} \). Then we have\(^2\)

\[
(4.10) \quad (-\hbar_k^2 \Delta_{x} - 1)v_k = o_{H^{-1}}(\hbar_k)
\]

So by \(^{10}\) Theorems E.42., E.43., E.44., we know there is a semiclassical (Radon) measure \( \mu \) on \( S^*M_x \) associated to a subsequence \( v_k \) (we re-label things), such that it is invariant by the geodesic flow\(^3\) of \( g_x \) and

\[
(4.11) \quad \langle Op_{h_k}(a)v_k, v_k \rangle \to \int_{S^*M_x} ad\mu, \quad \text{as } k \to \infty
\]

Here \( a \in C_0^{\infty}(T^*M_x) \), compactly supported in the interior of \( M_x \) and \( Op_{h_k}(a) \) is a semiclassical quantisation procedure on (the interior of) \( M_x \).

Now, since \( \|v_k\|_{L^2(\omega)} \to 0 \) as \( k \to \infty \), we have by \((4.11)\) that \( \mu = 0 \) on \( \pi^{-1}(\omega) \), where \( \pi : S^*M_x \to M_x \) is the projection. Since \( \mu \) is invariant by the geodesic flow and \( \omega \) satisfies (GCC), we thus have \( \mu \equiv 0 \), contradicting the assumption that \( \|v_k\|_{L^2(M_x)} = 1 \). \( \square \)

**Remark 4.3.** The case of \( M_x = [0, a] \) and \( M_y = [0, b] \) in Theorem 4.4 and so Lemma 4.2 for \( M_x = [0, a] \) was considered in \(^{12}\), by using elementary means to prove the inequality directly. Note that any open subset of \([0, a]\) satisfies the (GCC). Note also that if \( \omega \subset M_x \) is a neighbourhood of the boundary, then it satisfies the (GCC).

We can actually handle periodic conditions similarly, so we have

**Theorem 4.4.** Assume the same setup as in the previous theorem, just with \( M_y = I = [0, 1] \). This time we take periodic boundary conditions in \((4.11)\), i.e.

\[
u(x, 0) = u(x, 1) \quad \text{for } x \in M_x, \quad u|_{\partial M_x \times [0, 1]} = 0
\]

Then the estimate \((4.14)\) holds.

**Proof.** The proof is completely analogous to the proof of the previous theorem, by just noting that we may look at \( u \) as a function on \( M_x \times S^1 \). Then we expand \( u \) into Fourier basis on \( S^1 \) and apply Lemma 4.2 \( \square \)

We can handle even non-periodic admissible conditions of simple form. We call an isometry \( \varphi : M_x \to M_x \), restricting to an isometry on the boundary \( \varphi|_{\partial M_x} : \partial M_x \to \partial M_x \), admissible, if for every \( \varepsilon > 0 \), the set

\[
(4.12) \quad S(\varphi, \varepsilon) = \{ k \in \mathbb{N} \mid \text{dist}(\varphi^k, Id) < \varepsilon \}
\]

is relatively dense in \( \mathbb{N} \). A set \( A \subset \mathbb{N} \) is relatively dense if there exists an \( N \) such that every consecutive \( N \) positive integers contain an element of \( A \); i.e. every set of the form \( \{ k, k+1, \ldots, k+N-1 \} \) contains an element of \( A \). Here \( \text{dist}(\cdot, \cdot) \) denotes distance between mappings in \( C^{\infty}(M_x, M_x) \).

\(^2\)Recall that \( \|u\|_{H^{-1}} = \|h\xi\|^{-1}\tilde{u}(\xi)\|_{L^2} \) and \( \|u\|^2_{H^{-1}} = \|\xi\|^2\tilde{u}(\xi)\|_{L^2} \). So if \( g_k = o_{H^{-1}}(1) \), then \( \|h^2g_k\|_{H^{-1}} = h\|\xi\|^2\tilde{u}(\xi)\|_{L^2} = o(h) \) by using above formulas.

\(^3\)For the invariance of the semiclassical measure under the billiard or the broken geodesic flow, the reader is referred to \(^{23}\).
Theorem 4.5. Let \( \varphi : M_x \to M_x \) be an admissible isometry. Assume this time that \( u \) is a function in \( H^1(M_x \times \mathbb{R}) \cap C(M_x, H^1(M_x)) \), such that \( u(x, y + L) = u(\varphi(x), y) \) for all \((x, y) \in M_x \times \mathbb{R} \), where \( L > 0 \). Define \( M_x = M_x \times [0, L] / \{(x, 1) \sim (\varphi(x), 0) \} \) to be the mapping cylinder determined by \( \varphi \), with the inherited Riemannian metric from \( M_x \times \mathbb{R} \). Assume \( u \) satisfies
\[
(-\Delta_{g_x} - \partial_y^2 - z)u = f \quad \text{on} \quad M_x \times \mathbb{R}, \quad u|_{\partial M_x \times \mathbb{R}} = 0
\]
Let \( \omega \subset M_x \) be an open, non-empty set, satisfying (GCC) and assume \( \omega \) invariant under \( \varphi \). Denote the mapping cylinder over \( \omega \) by \( \omega_x \). Then there exists a constant \( C = C(M_x, g_x, \omega) > 0 \), such that the following observability estimate holds:
\[
\|u\|_{L^2(M_x)} \leq C(\|f\|_{H^{-1}_x L^2(M_x)} + \|u\|_{\omega_x} L^2(\omega_x))
\]
Proof. We use the theory of almost periodic functions outlined below in Appendix A. In particular, by the admissibility condition the map
\[
u : \mathbb{R} \ni y \mapsto u(\cdot, y) \in L^2(M_x)
\]
is almost periodic. Therefore, there exists a countable set \( \{\lambda_n\}_{n=1}^\infty \) such that
\[
u(y) \sim \sum_n u_n(\lambda_n; u)e^{i\lambda_n y}
\]
Here we recall the Bohr transformation
\[
u_n(\lambda_n; u) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^T u(y)e^{-i\lambda_n y}dy =: M_T\{f(y)e^{-i\lambda_n y}\}
\]
Therefore, we have \( u_n(\lambda_n; u) = u_n(x) \in H^1_0(M_x) \).
Now, since \( \varphi \) an isometry, we have \( f : \mathbb{R} \ni y \mapsto f(\cdot, y) \in L^2(M_x) \) also almost periodic. One can check (we use uniform convergence of the Bohr transformation here, to interchange limits when differentiating) that
\[
(-\Delta_{g_x} - \partial_y^2 - z)\nu(y) \sim \sum_{n=1}^\infty (-\Delta_{g_x} - (z - \lambda_n^2))u_n(x) \cdot e^{i\lambda_n y}
\]
Similarly, one has \( f(y) \sim \sum_n f_n(x) \cdot e^{i\lambda_n y} \), and by uniqueness of Fourier expansions of almost periodic functions, we have for all \( n \in \mathbb{N} \)
\[
f_n \equiv (-\Delta_{g_x} - (z - \lambda_n^2))u_n
\]
Now estimates in Lemma 4.2 give
\[
\|u_n\|_{L^2(M_x)} \leq C(\|f_n\|_{H^{-1} L^2(M_x)} + \|u_n\|_{\omega_x} L^2(\omega))
\]
Now Parseval’s identity (A.5) applied twice to \( f \) and \( u \) gives
\[
M\{\|u(y)\|^2\} = \|u\|_{L^2(M_x)}^2 = \sum_n \|u_n\|_{L^2(M_x)}^2 \leq C(\sum_n \|f_n\|_{H^{-1} L^2(M_x)}^2 + \|u_n\|_{\omega_x}^2 L^2(\omega)) = C(\|f\|_{H^{-1} L^2(M_x)}^2 + \|u\|_{L^2(M_x)}^2)
\]
This finishes the proof. \( \square \)
Remark 4.6. It might be possible to prove above by using a more direct approach of approximating \( u \) by periodic functions on \( M \times [0, kL] \) for \( k \) large, applying Theorem 4.4 and then taking a limiting procedure.

Now we begin the proof of Theorem 2.13. Before tackling the general case, we first give a proof that works for rational polyhedrons, and contains the essential ideas. This proof is essentially a higher dimensional analogue of [15].

Proof. Let \( \mu \) be a limit semiclassical measure associated to the eigenfunction sequence \( u_n \) as above, and let \((z, \xi) \in S^*D_0\) be in the support of \( \mu \). According to Theorem 2.14, the finite tube condition implies that \( z \) belongs to a tube periodic in the direction of \( \xi \). Also, the support of \( \mu \) is included in the union of the maximal tubes \( T_i \).

Choose such a particular tube \( T \). By definition, \( T \) is locally isometric to \((0, l) \times \Omega\), where \( \Omega \) denotes a disc or a convex polygon, as the case may be. Using this local isometry, we pull back the eigenfunction \( u_n \) to \( T \). We now apply the black box concentration techniques of [5], [6] and [15] to this setting. We choose an “inner shell” of thickness \( \varepsilon \) (that is, the complement of a tube periodic of thickness \( \varepsilon \)), denoted by \( \Omega_\varepsilon \subset \Omega \), such that \([0, l] \times \Omega_\varepsilon\) does not intersect \( U_\varepsilon \). Now we consider a smooth cut-off function \( \chi \) such that \( \chi \equiv 1 \) inside \( \Omega_\varepsilon \) and \( \chi = 0 \) outside \( \Omega_{\varepsilon/2} \). Then, \( \chi u_n \) vanishes near the walls of the tube \( T \). Now, consider a semiclassical measure \( \nu \) associated with the sequence \( v_n := \chi u_n \) on \( T \). By compactness, the \( v_n \) are bounded in \( L^2 \), so there exists at least one semiclassical measure associated with \( v_n \) on \( T \). Since \( \mu \) is supported on a finite number of tubes, there are only a finite number of directions in the support of \( \nu \). So we can find a constant-coefficient pseudodifferential operator \( \Phi \) on \( T \) that is microlocally 1 in a neighbourhood of directions parallel to \( dz \) (the direction of unwrapped geodesic paths), but vanishes microlocally in a neighbourhood of every other direction in the support of \( \nu \).

Now, consider the sequence of functions \( \Phi v_n \) on \( T \). The semiclassical measures \( \nu' \) associated to this sequence are related to those for the sequence \( v_n \) by \( \nu' = |\sigma(\Phi)|^2 \nu \), where \( \sigma(\Phi) \) is the principal symbol of the operator \( \Phi \). Thus, the support of \( \nu' \) is restricted to directions parallel to \( dz \) and to the geodesics parametrized by \( x, y \) such that \( \chi(x, y) = 1 \).

Now, let \( w_n := \Phi(\chi u_n) \). Then, on calculation we can see that

\[
(-\Delta_{\Omega} - \partial_z^2 - \lambda)w_n = -\Phi((\Delta \chi)u_n) - 2\Phi(\partial_x \chi \partial_x u_n) - 2\Phi(\partial_y \chi \partial_y u_n).
\]

We are now in a position to apply Theorem 4.4 for \( u = w_n \), \( z = \lambda \), \( f = -\Phi((\Delta \chi)u_n) - 2\Phi(\partial_x \chi \partial_x u_n) - 2\Phi(\partial_y \chi \partial_y u_n) \) and \( \omega \) contained in the set \( \{ \chi \equiv 0 \} \). By our choice of \( \omega \), we have \( \|w_n\|_{L^2(\omega)}^2 \to 0 \). Also, it is clear from the choice of our cut-off that since \( \text{supp}(\nabla \chi) \) is disjoint from \( \text{supp} \nu' \), \( \|f\|_{H^{-1}_x L^1_y(M_f)} \to 0 \) as \( n \to \infty \). This implies that \( \nu' \), and hence \( \nu \) does not have any mass in the direction of the tube \( T_i \). Since the number of tubes is finite this implies that \( \mu = 0 \), which contradicts the latter conclusion of Theorem 2.14 and finally proves the claim. \( \square \)

Now we begin with the proof in the general case, that is, the case of irrational polyhedrons. Observe that in this setting there are several issues that render the above proof invalid. Firstly, the boundary condition, as used in Theorem 4.4, is not periodic, but almost periodic in the the irrational case. Even more problematic is the fact that we do not have the finite tube condition, as pointed out in Remark 3.2 above.
Proof of Theorem 2.12. We start as before. Let $\mu$ be the semiclassical measure associated to the eigenfunction sequence $u_n$ as above, and let $(z, \zeta) \in S^*D_\alpha$ be in the support of $\mu$. According to Theorem 2.4 and Corollary 2.9, this direction belongs to a maximal tube $T$ periodic in the direction of $\zeta$, of length say $L$. By definition, there is a local isometry $p : \Omega \times \mathbb{R} \to T$, where $\Omega$ is a polygon or a disc, as the case might be. Using this local isometry, we identify $\Omega \times \mathbb{R}$ with $T$ and $p^*u_n$ with $u_n$ by a slight abuse of notation; we also pull back the measure $\mu$. Write $(x, y)$ for the coordinates on $\Omega$ and $z$ for the $\mathbb{R}$-coordinate. Note also that $p \circ \varphi_\alpha = p$, where $\varphi_\alpha(x, y, z) = (R_\alpha(x, y), z + L)$ is the isometry of $\mathbb{R}^3$; $R_\alpha$ is the rotation by an angle $\alpha$ which is associated to the tube. Thus $\mu$ is invariant under $\varphi_\alpha$. Denote by $\varphi_{\alpha,0}$ the rotation $\varphi_{\alpha,0}(x, y, z) = (R_\alpha(x, y), z)$.

Denote by $\Omega_\varepsilon \subset \Omega$ the inner shell of thickness $\varepsilon$ inside the tube $T$, i.e. the complement of the $\varepsilon$-neighbourhood of $\partial \Omega$. Consider now $(p_0, z_0) \in \partial \Omega_{\varepsilon/2} \times \mathbb{R}$. Then we claim that there exists $\delta > 0$ and $l > 0$ independent of such $p_0$ and $z_0$, such that there exists a point $(p_0, z)$ with $|z - z_0| \leq l$, and the $\delta$-neighbourhood of $(p_0, z)$ is contained in the set where $\mu = 0$.

Assume firstly $\alpha$ is irrational; then $\Omega \subset \mathbb{R}^2$ is a disc. Now $T$ contains in its boundary at least one singular point $(q, r)$ — a singular point closest to the axis of the tube (c.f. proof of Theorem 2.4). We know $\mu = 0$ in an $\varepsilon$-neighbourhood of $(q, r)$ by Theorem 2.12, therefore $\mu = 0$ in an $\varepsilon/2$-neighbourhood of $(q', r')$, denoted by $B$, where $q'$ is the closest point to $q$ lying on $\partial \Omega_{\varepsilon/2}$. By the invariance property of $\mu$ under $\varphi_\alpha$, we have $\mu = 0$ on $\varphi_\alpha^k(B)$ for any $k \in \mathbb{Z}$. By unique ergodicity of the irrational rotation on $S^1$ (c.f. Appendix A) and by a compactness argument, we know there is a finite integer $K > 0$ and $\varepsilon/2 > \delta > 0$, such that

$$N_\delta(\partial \Omega_{\varepsilon/2} \times \{r\}) \subset \bigcup_{i=0}^K (\varphi_\alpha^i(B))$$

Here $N_\delta$ denotes the $\delta$-neighbourhood. This proves the claim with $l = KL + C$, where $C > 0$ is a constant, since the independence of $\delta$ and $l$ of $p_0$ and $z_0$ follows from the symmetry of the problem.

For $\alpha$ rational, we argue similarly (with the difference that no ergodicity statement gets used).

Furthermore, by compactness we may take a uniform $\delta$ and $l$ such the claim holds in an $\eta$-neighbourhood of $\partial \Omega \times \mathbb{R}$, for some $\eta > 0$.

We return to the analytical argument. Let $\chi \in C_0^\infty(\Omega)$ be such that

$$\chi(x, y) = \begin{cases} 1, & (x, y) \in \Omega_{\varepsilon/2-\eta} \\ 0, & (x, y) \in \Omega \setminus \Omega_{\varepsilon/2} \end{cases}$$

Denote $v_n := \chi u_n$ and by $T_\alpha$ the mapping cylinder determined by $\varphi_\alpha$ and $T$. By compactness and since the length of the tube is finite, we have $\|v_n\|_{L^2(T_\alpha)}$ bounded in $n$, so there exists a semiclassical measure $\nu$ on $T_\alpha$ associated to this sequence (after relabeling we take it to be $v_n$).

Take now $\Phi$ a semiclassical PDO in $\mathbb{R}^3$ with symbol independent of the space variable and in $C_0^\infty(T^*\mathbb{R}^3)$, such that it microlocally cuts off near $\zeta$ (recall that $\zeta$ points in the $z$-direction). We may take the symbol $\sigma(\Phi) := a(\xi)$ supported in a small cone $\Gamma \subset \mathbb{R}^3 \setminus 0$ around $\zeta$, such that all lines in $\Gamma$ with $(x, y)$ basepoint at $\Omega_{\varepsilon/2} \setminus \Omega_{\varepsilon/2-\eta}$ hit the set where $\mu = 0$ in finite time. Moreover, we may take
a(ξ) equal to 1 near ζ and invariant under rotations around ζ. In particular, by
definition a ◦ ϕ _{α,0} = a.

Let w _{n} := Φ _{n}(v _{n}). Observe that ϕ _{α} ◦ Φ(ϕ _{α}) _{n} = Φ (see Proposition 4.7). Therefore
w _{n} descends to a function on T _{α} and the semiclassical measure ν' associated to w _{n}
satisfies ν' = |α|^2ν. By our choice of Φ and invariance of μ under the geodesic flow,
we have ν' = 0 on (Ω _{ε/2} \ \Ω _{ε/2−η}) × R. By a computation and since Φ commutes
with constant coefficient differential operators, we have

\begin{equation}
(4.20) \quad \|w _{n}\| _{L^2(T_α)} + \|\|\Phi((\Delta_Ω)u _{n}) + 2\Phi(\nabla x,y \cdot \nabla x,y u _{n})\|_{H^{-1}(Ω)}
\end{equation}

Choose ω ⊂ Ω to be a small enough neighbourhood of ∂Ω, for example ω = Ω \ \Ω _{ε/2+η} would work. We are now in a position to apply Theorem 4.10 to u = w _{n},
z = λ, f = −Φ((Δ_Ω)u _{n}) − 2Φ(∂x∂y∂yw _{n}) − 2Φ(∂y∂yw _{n}) and ω as above to get

\begin{equation}
(4.21) \quad \|\Phi((\Delta_Ω)u _{n})\| _{H^{-1}(Ω)} + \|\Phi(u _{n} \nabla x,y)\| _{H^{-1}(Ω)}
\end{equation}

This is further bounded by

\begin{equation}
(4.22) \quad \|\Phi((\Delta_Ω)u _{n})\| _{L^2(Ω)} + \|\Phi(u _{n} \nabla x,y)\| _{L^2(Ω)}
\end{equation}

The L^2 norm of (4.21) on T _{α} goes to zero as n → ∞, since the semiclassical measure
ν' vanishes on (Ω _{ε/2} \ \Ω _{ε/2−η}) × R and supp ∇χ ⊂ Ω _{ε/2} \ \Ω _{ε/2−η}. This shows that the
right hand side of (4.20) goes to zero as n → ∞, which implies ν' = 0, and
further implies μ ≡ 0 since the choice of (z, ζ) was arbitrary. But μ is a probability,
which is a contradiction to the initial hypothesis. □

**Proposition 4.7.** The following properties hold for Φ a semiclassical PDO in R^3
with symbol as in the proof of Theorem 4.11 above.

1. ϕ _{α} ◦ Φ = Φ _{α} ◦ ϕ _{α}
2. ΦP = ΦP for P a constant coefficient differential operator.

**Proof.** For the item 1. above, we have by definition and the change of coordinates
y' = ϕ _{α}(y)

\[ F_h(u ◦ ϕ _{α})(ξ) = \int e^{-i\frac{ξ}{\hbar} x} u ◦ ϕ _{α}(y) dy = \int e^{-i\frac{ξ}{\hbar} x} \frac{ϕ _{α}^{-1}(y')} {ϕ _{α}(y')} u(y') dy' = e^{iξ \frac{ξ}{\hbar}} \int e^{-i\frac{ξ}{\hbar} x} \frac{ϕ _{α}^{-1}(y')}{ϕ _{α}(y')} u(y') dy' = e^{iξ \frac{ξ}{\hbar}} F_h(u)(ϕ _{α}(ξ)), \]

where \( F_h \) denotes the semiclassical Fourier transform. This further implies, after a
change of coordinate ξ' = ϕ _{α}(ξ) and using a ◦ ϕ _{α,0} = a,

\[ (2\pi \hbar)^3 \Phi (ϕ _{α} u)(x) = \int e^{i\frac{ξ}{\hbar} x} F_h(u ◦ ϕ _{α})(ξ) a(ξ) dξ = \int e^{i\frac{ξ}{\hbar} x} e^{iξ' (ϕ _{α})^{-1}(ξ')} F_h(u)(ξ') a(ξ') dξ' = \int e^{iξ' (ϕ _{α})^{-1}(ξ')} F_h(u)(ξ') a(ξ') dξ', \]

One should consult [24] Chapter 5.3.] for a similar theory on the n-torus.
which is interpreted as $(2\pi h)^n \varphi^*_{\alpha}(\Phi u)(x)$.

For the second point, simply recall that $\mathcal{F}_h(D^\alpha u) = \frac{e^{i\alpha}}{n!} \mathcal{F}_h(u)$, where $D = -i\partial$ and $\alpha$ is any multiindex. The proof then follows from a straightforward computation. □

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Appendix A. Almost periodic functions

We introduce some theory of almost periodic functions, as developed by H. Bohr in 1920s and later generalised by others. We will follow mostly the first two chapters of [19]. For this purpose, let $X$ be a Banach space with norm $\|\cdot\|$.

We will say a number $\tau > 0$ is an $\varepsilon$-almost period of $f : \mathbb{R} \to X$ if

$$\sup_{t \in \mathbb{R}} \|f(t + \tau) - f(t)\| \leq \varepsilon \quad (A.1)$$

We also say a subset $E \subset \mathbb{R}$ is relatively dense if there is an $l > 0$ such that for any interval $(\alpha, \alpha + l) \subset \mathbb{R}$ of length $l$ contains an element of $E$.

We start with a basic definition:

Definition A.1. A continuous function $f : \mathbb{R} \to X$ is called almost periodic if for every $\varepsilon > 0$, there is an $l = l(\varepsilon) > 0$ such that each interval $(\alpha, \alpha + l) \subset \mathbb{R}$ contains a number $\tau = \tau(\varepsilon)$ such that (A.1) holds.

Example A.2 (An almost periodic map in a mapping torus). Take a continuous function $u : \mathbb{R} \times D^2 \to \mathbb{C}$, where $D^2 \subset \mathbb{R}^2$ is the unit disc and there is a rotation $R_\alpha$ and $L > 0$ such that $u(R_\alpha(x, y), z) = u(x, y, z + L)$ for all $(x, y, z)$. Then the map

$$f : \mathbb{R} \ni z \mapsto u(\cdot, \cdot, z) \in L^2(D^2)$$

is almost periodic. To see this, consider two cases: $\alpha$ rational and irrational.

If $\alpha = \frac{p}{q}$ rational, then we clearly see that $f$ is periodic for period $T = qL$.

If $\alpha$ irrational (multiple of $\pi$), then by the unique ergodicity of the rotation map we know the set

$$\{k\alpha \mod 2\pi \mid k \in \mathbb{N}\} \subset S^1 \cong [0, 2\pi)$$

is dense. Moreover, from this one deduces that for any $A \subset [0, 2\pi)$ non-empty and open, the set $S(A) := \{k \in \mathbb{N} \mid k\alpha \mod 2\pi \in A\}$ is relatively dense, in the sense that there is an $l \in \mathbb{N}$, such that for any $k \in \mathbb{N}$ we have $\{k, k + 1, \ldots, k + l - 1\}$ containing an element of $S(A)$.
Therefore, we are left to observe that given any $\varepsilon > 0$, we may take $\delta = \delta(\varepsilon) > 0$ small enough and so $S(\delta) := S((-\delta, \delta))$, so that $S(\delta)L \subset \mathbb{R}$ is relatively dense and each $\tau \in S(\delta)L$ is an $\varepsilon$-almost period for $f$.

**Example A.3** (Admissible maps and almost periodicity). More generally, if $\varphi : M_x \to M_x$ is admissible (c.f. definition (1.12)) and $u : M_x \times \mathbb{R} \to \mathbb{C}$ continuous, satisfying that $u(x, y + L) = u(\varphi(x), y)$ for all $(x, y)$ and some $L > 0$ fixed. Then the map

$$f : \mathbb{R} \ni y \mapsto u(\cdot, y) \in L^2(M_x)$$

is almost periodic, by definition and using the end of the previous example.

By the previous example, the rotation map $R_\alpha : D^2 \to D^2$ is admissible.

**Remark A.4** (Higher dimensional rotations are admissible). More generally, one can show that the higher dimensional rotations $\varphi = R_{\alpha_1, \ldots, \alpha_n} : \left(\frac{\mathbb{R}}{\mathbb{Z}}\right)^n \to \left(\frac{\mathbb{R}}{\mathbb{Z}}\right)^n$, given by

$$(A.2) \quad R_{\alpha_1, \ldots, \alpha_n} : (x_1, \ldots, x_n) \mapsto (x_1 + \alpha_1, \ldots, x_n + \alpha_n) \mod \mathbb{Z}^n$$

are admissible. By ergodic theory of $\mathbb{T}^n$, we know that $\varphi$ is ergodic iff $1, \alpha_1, \ldots, \alpha_n$ are linearly independent (l.i.) over $\mathbb{Q}$ using Fourier expansions, i.e. iff $\alpha_1, \ldots, \alpha_n$ are l.i. over $\mathbb{Z}$ taken modulo $\mathbb{Z}$.

By a change of coordinates in $SL_n(\mathbb{Z})$, one can then reduce the rotation to the form

$$R = R_{\alpha'_1, \ldots, \alpha'_k} : (x_1, \ldots, x_n) \mapsto (x_1 + \alpha'_1, \ldots, x_k + \alpha'_k, x_{k+1}, \ldots, x_n) \mod \mathbb{Z}^n$$

where $k$ is the largest integer such that $k$ of $\alpha_1, \ldots, \alpha_n$ are l.i. in the above sense, and so are $\alpha'_1, \ldots, \alpha'_k$. Thus we may apply ergodic theory and deduce from this that for any $A \subset [0,1]^n$ non-empty and open, the set $S(A) := \{k \in \mathbb{N} \mid R^k(0) \mod \mathbb{Z}^n \in A\}$ is relatively dense. Then we finish as in Example A.2.

Now, for a general rotation $R \in SO(n)$, one may take a complex matrix $P$ so that $P^{-1}RP = Q$ is diagonal and has eigenvalues coming in pairs $(e^{i\alpha}, e^{-i\alpha})$ for some angles $\alpha \in \mathbb{R}$. By the discussion above and the since the action of $Q$ is conjugate to a rotation on a torus $A \subset [0,1]^n$, we get that for any $\varepsilon > 0$, the set of indices $k$ such that $Q^k$ is $\varepsilon$-close to $Id$ is relatively dense. This implies that $R : D^n \to D^n$ is admissible, where $D^n \subset \mathbb{R}^n$ is the closed unit disc.

We collect a few basic facts about almost periodic functions: $f$ periodic implies $f$ almost periodic, $f$ is uniformly continuous if almost periodic (Chapter 1). A simple example is given by $f(t) = \sin t + \sin(\sqrt{2}t)$. Another equivalent definition is due to S. Bochner and says that (c.f. [19] p. 4.)

**Definition A.5.** Let $f : \mathbb{R} \to X$ be continuous. Then $f$ is almost periodic if and only if the family of functions $\{f^h(t) = f(t + h) \mid -\infty < h < \infty\}$ is compact in $C(X)$ equipped with uniform norm.

Next, we discuss expansion into trigonometric polynomials, similarly to the case of periodic functions. The fundamental theorem in the area is the Approximation theorem which says that every almost periodic $f$ is a uniform limit of sums of trigonometric polynomials $e^{i\lambda_k t}a_k(\varepsilon)$, where $\{\lambda_k\}_{k=-\infty}^{\infty} \subset \mathbb{R}$ is a countable set of coefficients (c.f. [19] p. 17.)]. Clearly for periodic functions on $[0,1]$, we may take $\lambda_k = 2k\pi$. 


We introduce the notation of a mean value for an almost periodic function $f$

$$M\{f\} := \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(t) dt$$

Now we define the Bohr transformation $a(\lambda; f)$ for $\lambda \in \mathbb{R}$ as a “mean value Fourier transform”

$$a(\lambda; f) := \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(t) e^{-i\lambda t} dt = M_t\{f(t)e^{-i\lambda t}\} \quad (A.3)$$

The values $\lambda = \lambda_k$ are then exactly the values for which $a(\lambda; f) \neq 0$ and the set $\{\lambda_k \mid k \in \mathbb{N}\}$ is sometimes called the spectrum of $f$. We will write (formally, with no convergence implied)

$$f(t) \sim \sum_{k = -\infty}^{\infty} a_k(\lambda_k; f)e^{i\lambda_k t} \quad (A.4)$$

However, one can show that certain Bochner-Fejer sums converge uniformly to $f$.

Next, we discuss a Parseval-type identity when $X$ is a Hilbert space, which says that if $(A.4)$ holds, then

$$M_t\{\|f(t)\|^2\} = \sum_{k = -\infty}^{\infty} \|a_k\|^2 \quad (A.5)$$

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