Rényi relative entropies and noncommutative $L_p$-spaces

Anna Jenčová *

Mathematical Institute, Slovak Academy of Sciences
Štefánikova 49, 814 73 Bratislava, Slovakia

Abstract

We propose an extension of the sandwiched Rényi relative $\alpha$-entropy for states on arbitrary von Neumann algebra, for the values $\alpha > 1$. For this, we use Kosaki’s definition of noncommutative $L_p$-spaces with respect to a state. Some properties of these extensions are proved, in particular the data processing inequality with respect to positive trace preserving maps. It is also shown that equality in data processing inequality characterizes sufficiency of quantum channels.

1 Introduction

The classical Rényi relative entropies were introduced by an axiomatic approach in [29], as the unique family of divergences satisfying certain natural properties. As it turned out, these quantities play a central role in many information-theoretic tasks, see e.g. [6] for an overview. A straightforward quantum generalization is given by standard quantum Rényi relative $\alpha$-entropies, defined as

$$D_{\alpha}(\rho\|\sigma) = \begin{cases} \frac{1}{\alpha-1} \log \left( \text{Tr} \rho^{\alpha} \sigma^{1-\alpha} \right) & \text{if } \alpha \in (0, 1) \text{ or } \text{supp}(\rho) \subseteq \text{supp}(\sigma) \\ \infty & \text{otherwise,} \end{cases}$$

where supp$(\rho)$ denotes the support of $\rho$ and $\alpha > 0$, $\alpha \neq 1$. These quantities share the useful properties of the classical Rényi relative

*jenca@mat.savba.sk
entropy, but not for all values of the parameter $\alpha$. In particular, for a quantum channel $\Phi$ and a pair of states $\rho$ and $\sigma$, the data processing inequality (DPI)

$$D_\alpha(\Phi(\rho)\|\Phi(\sigma)) \leq D_\alpha(\rho\|\sigma)$$ (1)

holds for $\alpha$ in the range $(0, 2]$, [25][11]. Moreover, for $\alpha \in (0, 1)$ the standard Rényi relative entropies appear as error exponents and cutoff rates in hypothesis testing [1][10][18].

Another quantum version of Rényi relative entropy was introduced in [37][22]. It is the sandwiched Rényi relative $\alpha$-entropy, defined as

$$\tilde{D}_\alpha(\rho\|\sigma) = \begin{cases} \frac{1}{\alpha-1} \log \text{Tr} \left[ (\sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}})^\alpha \right] & \text{if supp}(\rho) \subseteq \text{supp}(\sigma) \\ \infty & \text{otherwise} \end{cases}$$

for $\alpha > 0$, $\alpha \neq 1$. The sandwiched entropies satisfy DPI for $\alpha \in [1/2, 1) \cup (1, \infty)$, [22][37][2][8]. For $\alpha > 1$, $\tilde{D}_\alpha$ have an operational meaning as strong converse exponents in quantum hypothesis testing and channel coding, [20][19]. Moreover, both $D_\alpha$ and $\tilde{D}_\alpha$ yield the Umegaki relative entropy

$$D_1(\rho\|\sigma) = \begin{cases} \text{Tr} \rho (\log(\rho) - \log(\sigma)) & \text{if supp}(\rho) \subseteq \text{supp}(\sigma) \\ \infty & \text{otherwise} \end{cases}$$

in the limit as $\alpha \to 1$.

Let $(\Phi, \rho, \sigma)$ be a triple consisting of a quantum channel $\Phi$ and a pair of states $\rho$, $\sigma$ on the input space of $\Phi$. A channel $\Psi$ satisfying $\Psi \circ \Phi(\rho) = \rho$ and $\Psi \circ \Phi(\sigma) = \sigma$ is called a recovery map for $(\Phi, \rho, \sigma)$. If a recovery map exists, we say that the channel $\Phi$ is sufficient (or reversible) with respect to $\{\rho, \sigma\}$. This terminology was introduced in [27][28], by analogy with the classical notion of a sufficient statistic. Clearly, if $\Phi$ is sufficient with respect to $\{\rho, \sigma\}$, equality must be attained in DPI. It is much less obvious that the opposite implication holds in some cases. This was first observed in [27][28] for $D_1$ and $D_{1/2}$ and later extended to a large class of quantum divergences, including $D_\alpha$ with $\alpha \in (0, 2]$, [11]. The same property for $\tilde{D}_\alpha$ with $\alpha > 1$ was proved in [12].

Quantum versions of relative entropies are usually studied in the finite dimensional setting. Nevertheless, the standard version $D_\alpha$ is derived from the quasi-entropies [25], which were defined in [26] also
in the more general context of von Neumann algebras. A definition of sandwiched Rényi entropies for states on von Neumann algebras was recently proposed in [4]. These entropies are called the Araki-Masuda divergences and are based on the Araki-Masuda definition of non-commutative $L_p$ spaces with respect to a state.

The aim of the present work is to propose a von Neumann algebraic extension of $\tilde{D}_\alpha$ for $\alpha > 1$ using the interpolating family of Kosaki’s $L_p$-spaces, [34, 15]. This was inspired by the work by Beigi [2], where a similar family of norms (in finite dimensions) was used to prove DPI for $\tilde{D}_\alpha$ with $\alpha > 1$. It was later observed [21] that this method works even for positive trace-preserving maps. This framework was also used in [12] to show that equality in DPI implies sufficiency of the channel.

We will prove the following properties of the proposed quantities $\tilde{D}_\alpha$, $\alpha > 1$. We assume that $\psi, \varphi$ are normal states on a von Neumann algebra $\mathcal{M}$.

(i) Positivity: $\tilde{D}_\alpha(\psi\|\varphi) \geq 0$, with equality if and only if $\psi = \varphi$.

(ii) Monotonicity: if $\psi \neq \varphi$, the function $\alpha \mapsto \tilde{D}_\alpha(\psi\|\varphi)$ is increasing and strictly increasing whenever $\tilde{D}_\alpha(\psi\|\varphi) < \infty$.

(iii) Order relations: $\tilde{D}_\alpha$ can be extended to all positive normal functionals on $\mathcal{M}$. With this extension, $\psi_0 \leq \psi$ and $\varphi_0 \leq \varphi$ imply

$$\tilde{D}_\alpha(\psi_0\|\varphi) \leq \tilde{D}_\alpha(\psi\|\varphi), \quad \tilde{D}_\alpha(\psi\|\varphi_0) \geq \tilde{D}_\alpha(\psi\|\varphi).$$

(iv) Lower semicontinuity: the map $(\psi, \varphi) \mapsto \tilde{D}_\alpha(\psi\|\varphi)$ is jointly lower semicontinuous (on the positive part of the predual of $\mathcal{M}$).

(v) Generalized mean: let $\psi = \psi_1 \oplus \psi_2$, $\varphi = \varphi_1 \oplus \varphi_2$. Then

$$\exp\{(\alpha-1)\tilde{D}_\alpha(\psi\|\varphi)\} = \exp\{(\alpha-1)\tilde{D}_\alpha(\psi_1\|\varphi_1)\} + \exp\{(\alpha-1)\tilde{D}_\alpha(\psi_1\|\varphi_1)\}.$$

(vi) Data processing inequality: $\tilde{D}_\alpha(\Phi(\psi)\|\Phi(\varphi)) \leq \tilde{D}_\alpha(\psi\|\varphi)$ holds for any $\alpha > 1$ and any positive trace preserving maps. We also give some lower and upper bounds on the value of $\tilde{D}_\alpha(\psi\|\varphi) - \tilde{D}_\alpha(\Phi(\psi)\|\Phi(\varphi))$.

(vii) Characterization of sufficiency: if $\tilde{D}_\alpha(\psi\|\varphi)$ is finite, then equality in DPI for a 2-positive trace preserving map $\Phi$ implies that $\Phi$ is sufficient with respect to $\{\psi, \varphi\}$.

For $D_\alpha$, similar properties follow from the properties of the corresponding quasi-entropy [26]. In particular, DPI holds for $\alpha \in (0, 2]$ if the map satisfies a stronger positivity, more precisely, if $\Phi$ is the
pre-adjoint of a normal unital Schwarz map, \[23\]. The statements (i), (ii) and (vi) were proved also for the extension defined in \[4\], but note that complete positivity is required for (vi). Our proof of DPI is close to that of \[2\] and only positivity is assumed. On the other hand, the Araki-Masuda divergences are defined for \(\alpha \in \left[\frac{1}{2}, 1\right] \cup (1, \infty)\) and the limit values for \(\alpha \to 1, \infty\) are proved to be equal to the Umegaki relative entropy and max-relative entropy, respectively. A discussion of the limiting cases for the present setting, as well as the relation to the Araki-Masuda divergences, is left for future work.

The outline of the paper is as follows. In Section 2 we introduce the Kosaki’s \(L_p\)-spaces and give an overview of their properties, together with some technical results needed later. In Section 3 we give the definition of \(\tilde{D}_\alpha\) and prove the properties (i)–(vi). The last section deals with sufficiency of maps. A brief review on Haagerup’s \(L_p\)-spaces and complex interpolation method is given in the Appendices.

2 Non-commutative \(L_p\) spaces with respect to a state

Let \(\mathcal{M}\) be a (\(\sigma\)-finite) von Neumann algebra acting on a Hilbert space \(\mathcal{H}\). We denote the predual by \(\mathcal{M}^*\) and set of normal states by \(\mathcal{S}_*(\mathcal{M})\). For \(\psi \in \mathcal{M}^+\), we will denote by \(s(\psi)\) the projection onto the support of \(\psi\). For \(1 \leq p \leq \infty\), let \(L_p(\mathcal{M})\) be the Haagerup’s \(L_p\)-space over \(\mathcal{M}\). Below, we will use the identification of \(\mathcal{M}^*\) with \(L_1(\mathcal{M})\), so that \(\mathcal{S}_*(\mathcal{M})\) is identified with the subset of elements in \(L_1(\mathcal{M})^+\) with unit trace. See Appendix A for details.

In this section, we describe the noncommutative \(L_p\)-spaces with respect to a faithful normal state \(\varphi\) obtained by complex interpolation. These spaces were defined in \([15, 35, 38]\) and also in \([34]\), where \(\varphi\) is allowed to be a weight. We will follow the construction by Kosaki, details can be found in \([15]\).

2.1 The space \(L_\infty(\mathcal{M}, \varphi)\)

Fix a faithful normal state \(\varphi\) on \(\mathcal{M}\). To apply the complex interpolation method, we first show that \(\mathcal{M}\) can be continuously embedded into \(L_1(\mathcal{M}) \simeq \mathcal{M}_*\). For \(x \in \mathcal{M}\), we put

\[
h_x := h_{\varphi}^{1/2} x h_{\varphi}^{1/2}.
\]
By Hölder’s inequality (A.2), we have $h_x \in L_1(\mathcal{M})$ and $\|h_x\|_1 \leq \|x\|$. Moreover, $x$ is given uniquely and $h_x \in L_1(\mathcal{M})^+$ if and only if $x$ is positive. Note also that for $y \in \mathcal{M}$,

$$\text{Tr} \, h_x y = \text{Tr} \, h_{\varphi}^{1/2} x h_{\varphi}^{1/2} y = \text{Tr} \, h_{\varphi}^{1/2} y h_{\varphi}^{1/2} x = \text{Tr} \, h_y x$$

The map $x \mapsto h_x$ is obviously linear and defines a continuous positive embedding of $\mathcal{M}$ into $L_1(\mathcal{M})$. The image of $\mathcal{M}$ is the dense linear subspace

$L_\infty(\mathcal{M}, \varphi) := \{h_x, \ x \in \mathcal{M}\} \subseteq L_1(\mathcal{M})$.

The norm in $L_\infty(\mathcal{M}, \varphi)$ is introduced as

$$\|h_x\|_{\infty, \varphi} = \|x\|.$$ 

The next lemma shows that positive elements in $L_\infty(\mathcal{M}, \varphi)$ can be easily characterized.

**Lemma 1.** Let $k \in L_1(\mathcal{M})^+$. Then $k \in L_\infty(\mathcal{M}, \varphi)$ if and only if $k \leq \lambda h_{\varphi}$ for some $\lambda > 0$. In this case,

$$\|k\|_{\infty, \varphi} = \|x\| = \inf \{\lambda > 0, k \leq \lambda h_{\varphi}\}.$$

**Proof.** Let $x \in \mathcal{M}^+$, then for all $a \in \mathcal{M}^+$,

$$\text{Tr} \, h_x a \leq \|h_x a\|_1 = \|x h_{\varphi}^{1/2} a h_{\varphi}^{1/2}\|_1 \leq \|x\| \text{Tr} \, h_{\varphi} a,$$

by Hölder’s inequality, so that $h_x \leq \|x\| h_{\varphi}$. Conversely, let $0 \leq k \leq \lambda h_{\varphi}$ and let $k = h_\psi$, $\psi \in \mathcal{M}_\varphi^+$. Consider the standard form $\langle l(\mathcal{M}), L_2(\mathcal{M}), J, L_2(\mathcal{M})^+ \rangle$, where $l(\mathcal{M})$ is the left action of $\mathcal{M}$ on $L_2(\mathcal{M})$, see Appendix A. Then for $y \in \mathcal{M}$, $\varphi(y) = (h_{\varphi}^{1/2}, y h_{\varphi}^{1/2})$ and by the commutant Radon-Nikodym theorem [30, Section 5.19], there is some $x \in \mathcal{M}$ such that $0 \leq x \leq \lambda$ and

$$\psi(y) = (J x h_{\varphi}^{1/2}, y h_{\varphi}^{1/2}) = (h_{\varphi}^{1/2} x^+, y h_{\varphi}^{1/2}) = \text{Tr} \, x h_{\varphi}^{1/2} y h_{\varphi}^{1/2} = \text{Tr} \, h_{\varphi}^{1/2} x h_{\varphi}^{1/2} y.$$

It follows that $k = h_\psi = h_x$. The last assertion follows from the fact that for positive $x \in \mathcal{M}$, $\|x\| = \inf \{\lambda > 0, x \leq \lambda\}$. 

To characterize arbitrary elements in $L_\infty(\mathcal{M}, \varphi)$, let $\mathcal{M}_2 := M_2(\mathcal{M}) \simeq \mathcal{M} \otimes \mathcal{M}_2(\mathbb{C})$ be the algebra of $2 \times 2$ matrices over $\mathcal{M}$. The predual of $\mathcal{M}_2$ can be identified with $M_2(\mathcal{M}_\varphi)$, where for $\psi \in (\mathcal{M}_2)_\varphi$, we put $\psi_{ij}(a) = \psi(a \otimes |i\rangle \langle j|)$. This means that we also identify $L_1(\mathcal{M}_2)$ with $M_2(L_1(\mathcal{M}))$. 

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Lemma 2. Let \( k \in L_1(\mathcal{M}) \). Let \( h_2, k_2 \in L_1(\mathcal{M}_2) \) be defined as
\[
h_2 := \begin{pmatrix} h_\varphi & 0 \\ 0 & h_\varphi \end{pmatrix}, \quad k_2 := \begin{pmatrix} 0 & k \\ k^* & 0 \end{pmatrix}
\]
Then \( k \in L_\infty(\mathcal{M}, \varphi) \) if and only if \( k_2 \leq \lambda h_2 \) for some \( \lambda > 0 \). In this case,
\[
\|k\|_{\infty, \varphi} = \inf\{\lambda > 0, k_2 \leq \lambda h_2 \}.
\]

Proof. Let \( k = h_x \) and let \( \lambda \in \mathbb{R} \). Note that \( \lambda h_2 - k_2 = h_2^{1/2} x h_2^{1/2} \), where
\[
x_\lambda := \begin{pmatrix} \lambda & -x \\ -x^* & \lambda \end{pmatrix},
\]
and that \( \|x\| = \|-x\| = \inf\{\lambda > 0, \ x_\lambda \geq 0 \} \). Hence \( k_2 \leq \lambda h_2 \) for any \( \lambda \geq \|x\| \). It is also clear that \( \|x\| \) is the smallest \( \lambda \) such that this inequality holds. Conversely, assume that \( k_2 \leq \lambda h_2 \) for some \( \lambda > 0 \). Let \( a = \begin{pmatrix} a_{11} & a_{12} \\ a_{12}^* & a_{22} \end{pmatrix} \in \mathcal{M}_2^+ \), then also
\[
a_- := \begin{pmatrix} a_{11} & -a_{12} \\ -a_{12}^* & a_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathcal{M}_2^+
\]
and note that \( \text{Tr} k_2 a_- = -\text{Tr} k_2 a, \text{Tr} h_2 a_- = \text{Tr} h_2 a \). It follows that we have \( \pm k_2 \leq \lambda h_2 \), so that \( 0 \leq k_2 + \lambda h_2 \leq 2\lambda h_2 \). Since \( h_2 \) defines a faithful positive normal linear functional on \( \mathcal{M}_2 \), Lemma 1 applies, so that there is some \( y = \begin{pmatrix} y_{11} & x \\ x^* & y_{22} \end{pmatrix} \in \mathcal{M}_2^+ \) such that
\[
\begin{pmatrix} \lambda h_\varphi & k \\ k^* & \lambda h_\varphi \end{pmatrix} = k_2 + \lambda h_2 = h_2^{1/2} y h_2^{1/2} = \begin{pmatrix} h_2^{1/2} y_{11} h_\varphi^{1/2} & h_2^{1/2} x h_\varphi^{1/2} \\ h_\varphi^{1/2} x^* h_\varphi^{1/2} & h_\varphi^{1/2} y_{22} h_\varphi^{1/2} \end{pmatrix}
\]
It follows that \( y_{11} = y_{22} = \lambda \) and \( k = h_x \). Moreover, since \( y = \begin{pmatrix} \lambda & x \\ x^* & \lambda \end{pmatrix} \) is positive, \( \|x\| \leq \lambda \).

\[\square\]

2.2 The interpolation spaces \( L_p(\mathcal{M}, \varphi) \)

We now define the \( L_p \)-space over \( \mathcal{M} \) with respect to \( \varphi \) as
\[
L_p(\mathcal{M}, \varphi) := C_{1/p}(L_\infty(\mathcal{M}, \varphi), L_1(\mathcal{M})).
\]
For definition of the space $C_\eta$, see Appendix B. The norm in $L_p(\mathcal{M}, \varphi)$ will be denoted by $\| \cdot \|_{p,\varphi}$. For $1 \leq p \leq \infty$ and $1/q + 1/p = 1$, put

$$i_p : L_p(\mathcal{M}) \to L_1(\mathcal{M}), \quad k \mapsto h_\varphi^{1/2q}kh_\varphi^{1/2q}.$$ 

**Theorem 1.** ([7, Theorem 9.1]) The map $i_p$ is an isometric isomorphism of $L_p(\mathcal{M})$ onto $L_p(\mathcal{M}, \varphi)$.

Using the polar decomposition in $L_p(\mathcal{M})$, we obtain that elements in $L_p(\mathcal{M}, \varphi)$ have the form $h_\varphi^{1/2q}uh_\varphi^{1/p}h_\varphi^{1/2q}$, where $h \in L_1(\mathcal{M})^+$ and $u \in \mathcal{M}$ is a partial unitary such that $u^*u = s(h)$, with norm

$$\|h_\varphi^{1/2q}uh_\varphi^{1/p}h_\varphi^{1/2q}\|_{p,\varphi} = (\text{Tr } h)^{1/p}.$$ 

We now list some important properties of the spaces $L_p(\mathcal{M}, \varphi)$. Let $1 \leq p \leq p' \leq \infty$. Then $L_{p'}(\mathcal{M}, \varphi) \subseteq L_p(\mathcal{M}, \varphi)$ and

$$\|k\|_{p,\varphi} \leq \|k\|_{p',\varphi}, \quad \forall k \in L_{p'}(\mathcal{M}, \varphi). \quad (2)$$

This follows easily by Theorem 1 and Hölder's inequality, but it is also a consequence of the abstract theory of complex interpolation, see [3, Theorem 4.2.1]. The space $L_\infty(\mathcal{M}, \varphi)$ is dense in $L_1(\mathcal{M})$ and therefore also in $L_p(\mathcal{M}, \varphi)$ for each $p > 1$ by [3, Theorem 4.2.2]. It follows that $L_{p'}(\mathcal{M}, \varphi)$ and $L_p(\mathcal{M}, \varphi)$ are compatible Banach spaces. By the reiteration theorem ([3, Theorem 4.6.1]), we have

$$C_\eta(L_{p'}(\mathcal{M}, \varphi), L_p(\mathcal{M}, \varphi)) = L_{p_\eta}(\mathcal{M}, \varphi), \quad 0 \leq \eta \leq 1, \quad (3)$$

where $1/p_\eta = \eta/p + (1-\eta)/p'$.

Let now $1 \leq p \leq \infty$, $1/p + 1/q = 1$. The duality

$$\langle k, h_x \rangle := \text{Tr } kx, \quad x \in \mathcal{M}, \quad k \in L_1(\mathcal{M})$$

extends to a duality between $L_p(\mathcal{M}, \varphi)$ and $L_q(\mathcal{M}, \varphi)$, given by

$$\langle h_\varphi^{1/2q}k_1h_\varphi^{1/2q}, h_\varphi^{1/2p}k_2h_\varphi^{1/2p} \rangle = \text{Tr } k_1k_2, \quad (4)$$

where $k_1 \in L_p(\mathcal{M})$, $k_2 \in L_q(\mathcal{M})$. For $1 \leq p < \infty$, $L_q(\mathcal{M}, \varphi)$ is isometrically isomorphic to the Banach space dual of $L_p(\mathcal{M}, \varphi)$. This follows immediately from Theorem 1.

For each $1 \leq p \leq \infty$, we have the following Clarkson’s and McCarthy’s inequalities:
Lemma 3. Let \( f \) be in \( L_p(\mathcal{M}, \varphi) \). For \( 2 \leq p \leq \infty \), we have
\[
(\|h+k\|_{p,\varphi}^p + \|h-k\|_{p,\varphi}^p)^{1/p} \leq \sqrt{2}(\|h\|_{p,\varphi}^2 + \|k\|_{p,\varphi}^2)^{1/2} \leq 2^{1-1/p}(\|h\|_{p,\varphi}^p + \|k\|_{p,\varphi}^p)^{1/p}.
\]
For \( 1 \leq p \leq 2 \) the inequalities reverse.

Theorem 3. Let \( h, k \in L_p(\mathcal{M}, \varphi) \), \( 1 \leq p \leq 2 \) and \( 1/p + 1/q = 1 \). Then
\[
(\|h+k\|_{p,\varphi}^q + \|h-k\|_{p,\varphi}^q)^{1/q} \leq 2^{1/q}(\|h\|_{p,\varphi}^p + \|k\|_{p,\varphi}^p)^{1/p}.
\]
For \( 2 \leq p \leq \infty \) the inequality reverses.

This implies that for \( 1 < p < \infty \) the space \( L_p(\mathcal{M}, \varphi) \) is uniformly convex and uniformly smooth. Then \( L_p(\mathcal{M}, \varphi) \) is strictly convex, hence for each \( 0 \neq h \in L_p(\mathcal{M}, \varphi) \), there is a unique element \( T_{q,\varphi}(h) \) in the unit ball of \( L_q(\mathcal{M}, \varphi) \) such that
\[
\langle T_{q}(h), h \rangle = \|h\|_{p,\varphi}.
\]
Let \( h = h_{\varphi}^{-1}u^k^{1/p}h_{\varphi}^{-1/2q} \) for some \( k \in L_1(\mathcal{M})^+ \) and a partial isometry \( u \in \mathcal{M} \) such that \( u^*u = s(k) \). Then we clearly have
\[
T_{q,\varphi}(h) := \|h\|_{p,\varphi}^{-p}h_{\varphi}^{1/2p}k^{1/q}u^*h_{\varphi}^{-1/2p}.
\]
Restricted to the unit ball of \( L_p(\mathcal{M}, \varphi) \), the map \( T_{q,\varphi} \) is a uniformly continuous bijection onto the unit ball of \( L_q(\mathcal{M}, \varphi) \) and we have \( T_{q,\varphi}^{-1} = T_{p,\varphi} \) for this restriction.

### 2.3 Hadamard three lines theorem

We first note that the infimum in the definition of the interpolation norm \( \| \cdot \|_{p,\varphi} \) is attained. Let \( h \in L_p(\mathcal{M}, \varphi) \) be of the form \( h = h_{\varphi}^{-1/2q}k_{\varphi}^{-1/2q} \) for some \( k \in L_p(\mathcal{M}) \) and let \( k = u^l_{1/p} \) be the polar decomposition of \( k \). Put
\[
f_{h,p}(z) := \|l\|_{1/p-z}h_{\varphi}^{(1-z)/2}u^z h_{\varphi}^{(1-z)/2}, \quad z \in S.
\]
Then \( f_{h,p}F := F(L_{\infty}(\mathcal{M}, \varphi), L_1(\mathcal{M})) \), \( f_{h,p}(1/p) = h \) and we have \( \|h\|_{p,\varphi} = \|f_{h,p}\|_F \).

Lemma 3. Let \( f \in F \) and assume that \( \|f(\theta)\|_{1/\theta,\varphi} = ||f||_F \) for some \( \theta \in (0,1) \). Then
\[
\|f(x+it)\|_{1/x,\varphi} = ||f||_F, \quad \forall x \in [0,1], \ t \in \mathbb{R}.
\]
Proof. Let \( p = 1/\theta, q = 1/(1-\theta). \) Put \( h := f(\theta) \) and let \( g := fT_{q,\varphi}(h),q. \) Let
\[
K(z) := \langle g(1-z), f(z) \rangle, \quad z \in S.
\]
Note that for \( z = x + it, f(z) \in L_{1/\varphi}(\varphi), g(1-z) \in L_{1/(1-x)}(\varphi) \) and \( \|f(z)\|_{1/\varphi} \leq \|f\|_\varphi, \|g(1-z)\|_{1/(1-x)} \leq \|g\|_\varphi = 1. \) It follows that
\[
|K(x+it)| \leq \|g(1-x-it)\|_{1/(1-x)} \|f(x+it)\|_{1/x} \leq \|f\|_\varphi.
\]
Moreover, \( K(\theta) = \|f(\theta)\|_{p,\varphi} = \|f\|_\varphi. \) By the maximum modulus principle, \( K \) must be a constant, so that \( K(z) = \|f\|_\varphi \) for all \( z \in S. \) It follows that we must have \( \|f(x+it)\|_{1/x,\varphi} = \|f\|_\varphi \) for all \( x \) and \( t. \)

The next lemma shows that the infimum in the definition of the interpolation norm is attained also for the reiterated spaces.

**Lemma 4.** Let \( 1 \leq p \leq p' \leq \infty \) and let \( \eta \in (0,1), \) \( p_\eta = \eta/p + (1-\eta)/p'. \) Let \( h \in L_{p_\eta}(\mathcal{M}, \varphi) \) and put \( g(\zeta) = fh_{p_\eta}(\zeta/p + (1-\zeta)/p'), \) \( \zeta \in S. \) Then \( g \in \mathcal{F}_{p',p} : \mathcal{F}(L_{p'}(\mathcal{M}, \varphi), L_p(\mathcal{M}, \varphi)), g(\eta) = h \) and \( \|h\|_{p_\eta,\varphi} = \|g\|_{\mathcal{F}_{p',p}}. \)

**Proof.** By [5][32.3], for any \( f \in \mathcal{F} \) the function \( q(z) = f(z/p + (1-z)/p') \) belongs to \( \mathcal{F}_{p',p} \) and
\[
\||g\|_{\mathcal{F}_{p',p}} = \max_t \{\sup_t \|q(it)\|_{p',\varphi}, \sup_t \|q(1+it)\|_{p,\varphi}\} \leq \|f\|_\varphi.
\]
By reiteration [3],
\[
\|h\|_{p_\eta,\varphi} \leq \||g\|_{\mathcal{F}_{p',p}} \leq \|fh_{p_\eta}\|_{\varphi} = \|h\|_{p_\eta,\varphi}.
\]
The statement follows also by Lemma [3] by noticing that for any \( x \in [0,1], \) \( t \in \mathbb{R}, \)
\[
\|g(x+it)\|_{p,x,\varphi} = \|fh_{p_\eta}\|_{\varphi} = \|h\|_{p_\eta,\varphi}.
\]

Assume that \( h \in L_p(\mathcal{M}, \varphi), \|h\|_{p,\varphi} = 1. \) Note that by Lemma [3] the values of the function \( fh_{p} \) run through the unit balls of all the spaces \( L_{p'}(\mathcal{M}, \varphi). \) The next lemma shows that by applying the map \( T_{q,\varphi} \) we again obtain an element of \( \mathcal{F}. \)
Lemma 5. Let $1 < p < \infty$, and let $h \in L_p(\mathcal{M}, \varphi)$, with $\|h\|_{p, \varphi} = 1$. Then for all $z = x + it$, $x \in (0, 1)$,

$$T_{1/(1-x),\varphi}(f_{h,p}(z)) = f_{T_{\eta,\varphi}(h),q}(1-z).$$

Proof. Since $\|h\|_{p, \varphi} = 1$, we have $h = h^{1/2}u^*_\psi h^{1/2}$ for some $\psi \in \mathcal{G}_*(\mathcal{M})$ and $u^*u = s(\psi)$. By Lemma 3 $\|f_{h,p}(x+it)\|_{1/x, \varphi} = \|f_{h,p}\|_{\mathcal{F}} = 1$ and $\|f_{T_{\eta,\varphi}(h),q}(1-x-it)\|_{1/(1-x), \varphi} = \|f_{T_{\eta,\varphi}(h),q}\|_{\mathcal{F}} = 1$ for all $x \in [0, 1]$ and $t \in \mathbb{R}$. By (4), we have

$$\langle f_{h,p}(z), f_{T_{\eta,\varphi}(h),q}(1-z) \rangle = \text{Tr} \left( h^{i-it/2}u^*_\psi h^{i-it/2} \left( h^{i-it/2}h^{1-z}u^*h^{i-it} \right) \right) = \text{Tr} h^{i} = 1.$$

By uniqueness, we must have $T_{1/(1-x),\varphi}(f_{h,p}(z)) = f_{T_{\eta,\varphi}(h),q}(1-z)$ for all $x \in (0, 1)$, $t \in \mathbb{R}$. $lacksquare$

The inequality part of the following result is a version of Hadamard’s three lines theorem. For convenience of the reader, we add a proof.

Theorem 4. Let $1 \leq p \leq p' \leq \infty$ and let $0 < \eta < 1$. Then for $f \in \mathcal{F}(L_p' (\varphi), L_{p'} (\varphi))$,

$$\|f(\eta)\|_{p_\eta, \varphi} \leq \left( \sup_{t \in \mathbb{R}} \|f(it)\|_{p, \varphi} \right)^{1-\eta} \left( \sup_{t \in \mathbb{R}} \|f(1+it)\|_{p, \varphi} \right)\eta,$$

where $1/p_\eta = \eta/p + (1-\eta)/p'$. Moreover, equality is attained if and only if

$$f(z) = f_{h,p_\eta}(z/p + (1-z)/p')M^{-\eta},$$

for $h = f(\eta)$ and $M > 0$.

Proof. Let $q$, $q'$ and $q_\eta$ be the duals of $p$, $p'$ and $p_\eta$, so that $1/q_\eta = \eta/q + (1-\eta)/q'$ and $L_{q_\eta}(\varphi) = C_{1-\eta}(L_q(M, \varphi), L_{q'}(M, \varphi))$. Put $h = f(\eta)$ and let

$$g(z) = f_{T_{q_\eta,\varphi}(h),q_\eta}(z/q' + (1-z)/q).$$

By Lemma 3 $g \in \mathcal{F}_{q,q'}$ and

$$g(1-\eta) = T_{q_\eta,\varphi}(h), \quad \|g\|_{\mathcal{F}_{q,q'}} = 1 = \|T_{q_\eta,\varphi}(h)\|_{q_\eta, \varphi}.$$ 

As in the proof of lemma 3, $K(z) := \langle g(1-z), f(z) \rangle$ defines a bounded continuous function on $S$, analytic in the interior of $S$. By the usual
Hadamard’s three lines theorem,
\[
\|f(\eta)\|_{p,\psi} = |K(\eta)| \leq (\sup_{t \in \mathbb{R}} |K(it)|)^{1-n}(\sup_{t \in \mathbb{R}} |K(1+it)|)^{n} \\
\leq (\sup_{t \in \mathbb{R}} \|f(it)\|_{p',\psi})^{1-n}(\sup_{t \in \mathbb{R}} \|f(1+it)\|_{p,\psi})^{n}.
\]

Now assume that equality is attained. Let \( M_0 = \sup_{t \in \mathbb{R}} \|f(it)\|_{p',\psi} \), \( M_1 = \sup_{t \in \mathbb{R}} \|f(1+it)\|_{p,\psi} \) and let
\[
F(z) := K(z)M_0^{z-1}M_1^{-z}, \quad z \in S.
\]

Then \( |F(z)| \leq 1 \) for all \( z \in S \) and \( F(\eta) = 1 \). By the maximum modulus principle, \( F(z) = F(\eta) = 1 \) for all \( z \), that is,
\[
(g(1-z), f(z)M_0^{z-1}M_1^{-z}) = 1, \quad z \in S. \tag{7}
\]

Suppose first that \( \|h\|_{p,\psi} = 1 \). Note that by Lemma 5
\[
g(1-z) = f_{T_{q,\psi}(h),q_0}(u) = T_{1/Re(u),\phi}(f_{h,p}(1-u)),
\]
where \( u = z/q + (1-z)/q' \). Hence
\[
g(1-z) = T_{q,\psi}(f_{h,p}(z/p + (1-z)/p')).
\]

Since \( \|f(x+it)M_0^{x+it}M_1^{-x-it}\|_{p,\psi} \leq 1 \) by the first part of the proof, (7) implies that we must have
\[
f(z)M_0^{z-1}M_1^{-z} = f_{h,p}(z/p + (1-z)/p'),
\]
by definition and properties of \( T_{q,\psi} \). If \( \|h\|_{p,\psi} = a \neq 1 \), then we may replace \( f \) by \( a^{-1}f \). Note that the equality still holds, with \( M_0 \) and \( M_1 \) replaced by \( a^{-1}M_0 \) and \( a^{-1}M_1 \). We obtain
\[
f(z) = a^{-1}f(z) = af_{a^{-1}h,p}(z/p + (1-z)/p')(a^{-1}M_0)^{1-z}(a^{-1}M_1)^z
\]
\[
= f_{a^{-1}h,p}(z/p + (1-z)/p')M_0^{-z}M_1^z
\]
\[
= a^{-1/p}f_{h,p}(z/p + (1-z)/p')M_0^{-z}M_1^z
\]
\[
= f_{h,p}(z/p + (1-z)/p')AM^z,
\]
where \( A > 0 \) and \( M = M_1/M_0 \). Since \( f(\eta) = f_{h,p}(1/p_n) = h \), we must have \( AM^\eta = 1 \). It follows that
\[
f(z) = f_{h,p}(z/p + (1-z)/p')M^{z-\eta}.
\]

For the converse, note that using Lemma 8 we obtain \( M_0 = \sup_t \|f(it)\|_{p',\psi} = \|h\|_{p,\psi}M^{-\eta} \) and \( M_1 = \sup_t \|f(1+it)\|_{p,\psi} = \|h\|_{p,\psi}M^{1-\eta} \). It follows that \( M_1/M_0 = M \) and \( M_0^{1-\eta}M_1^\eta = \|h\|_{p,\psi} \). 
\( \square \)
### 2.4 The positive cone in $L_p(M, \varphi)$.

Let us denote $L_p(M, \varphi)^+ := L_p(M, \varphi) \cap L_1(M)^+$. Then it is clear that

$$L_p(M, \varphi)^+ = \{ h_{\varphi}^{1/2q} h_1^{1/p} h_{\varphi}^{1/2q}, h \in L_1(M)^+ \}.$$ 

It follows by the properties of $L_p(M)^+$ ([33]) that $L_p(M, \varphi)^+$ is a closed convex cone which is pointed and generates all $L_p(M, \varphi)$. Note also that $L_\infty(M, \varphi)^+$ is dense in $L_p(M, \varphi)^+$, for any $1 \leq p$.

Let $k \in L_p(M, \varphi)$, $k = h_{\varphi}^{1/2q} u h_1^{1/p} h_{\varphi}^{1/2q}$, $h \in L_1(M)^+$. Then $k$ has a polar decomposition of the form

$$k = h_{\varphi}^{1/2q} u h_1^{1/p} h_{\varphi}^{1/2q} |k|_{p, \varphi} = \sigma_{-1/2q}(u)|k|_{p, \varphi},$$

where $|k|_{p, \varphi} = h_{\varphi}^{1/2q} h_1^{1/p} h_{\varphi}^{1/2q} \in L_p(M, \varphi)^+$ and $\sigma^p$ denotes the modular group of $\varphi$. We next look at self-adjoint elements in $L_p(M, \varphi)$.

**Lemma 6.** Let $k = k^* \in L_p(M, \varphi)$. Then there is a decomposition

$$k = k_{p, \varphi,+} - k_{p, \varphi,-},$$

where $k_{p, \varphi, \pm} = h_{\varphi}^{1/2q} h_\pm^{1/p} h_{\varphi}^{1/2q} \in L_p(M, \varphi)^+$, $h_+, h_- \in L_1(M)i^+$, $h_+ h_- = 0$ and we have

$$\|k\|_{p, \varphi} = (\|k_{p, \varphi,+}\|_{p, \varphi}^p + \|k_{p, \varphi,-}\|_{p, \varphi}^p)^{1/p}.$$ 

**Proof.** If $k = k^*$, then $k = h_{\varphi}^{1/2q} u h_1^{1/p} h_{\varphi}^{1/2q}$, where $l = l^* \in L_p(M)$. It follows that $l = u h_1^{1/p}$, where $h = h^* \in L_1(M)$, $h = h_+ - h_-$, $h_+, h_- \in L_1(M)^+$, $h_+ h_- = 0$. Moreover, $u = e_+ - e_-$, where $e_\pm := s(h_\pm)$ and $|h_+|^{1/p} = (h_+ + h_-)^{1/p} = h_+^{1/p} + h_-^{1/p}$. It follows that $k$ has the above form. As for the norm, we have

$$\|k\|_{p, \varphi} = \text{Tr} |h| = \text{Tr} h_+ + \text{Tr} h_- = \|k_{p, \varphi,+}\|_{p, \varphi}^p + \|k_{p, \varphi,-}\|_{p, \varphi}^p.$$

**Corollary 1.** Let $h \in L_p(M, \varphi)^+$ and let $h_1 \in L_1(M)^+$ be such that $h_1 \leq h$. Then $h_1 \in L_p(M, \varphi)^+$ and $\|h_1\|_{p, \varphi} \leq \|h\|_{p, \varphi}$.

**Proof.** Let $x \in M^+$, then

$$0 \leq \langle h_1 x, h_1 \rangle = \text{Tr} h_1 x \leq \|h_1\|_{q, \varphi} \|h\|_{p, \varphi}.$$ 

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Since $L_\infty(\mathcal{M}, \varphi)^+$ is dense in $L_q(\mathcal{M}, \varphi)^+$, it follows that $\langle k, h_1 \rangle \leq \|k\|_{q, \varphi} \|h\|_{p, \varphi}$ for all $k \in L_q(\mathcal{M}, \varphi)^+$. Let now $k = k^* \in L_q(\mathcal{M}, \varphi)$, with decomposition $k = k_{q, \varphi}^+ - k_{q, \varphi}^-$ as in Lemma 6. Then

$$\langle k, h_1 \rangle \leq \|k\|_{q, \varphi} \|h\|_{p, \varphi} \|k_{q, \varphi}^+\|_{q, \varphi} + \|k_{q, \varphi}^-\|_{q, \varphi}$$

the last inequality follows by classical Hölder’s inequality. For $k \in L_p(\mathcal{M}, \varphi)$, we have $k = \text{Re}(k) + i\text{Im}(k)$, with the usual definition of the self-adjoint elements $\text{Re}(k)$ and $\text{Im}(k)$ in $L_p(\mathcal{M}, \varphi)$. Then

$$\langle k, h_1 \rangle \leq \|\text{Re}(k)\|_{q, \varphi} \|h\|_{p, \varphi} \|k_{q, \varphi}^+\|_{q, \varphi} + \|\text{Im}(k)\|_{q, \varphi} \|k_{q, \varphi}^-\|_{q, \varphi}$$

Hence $h_1$ defines a bounded positive linear functional on $L_q(\mathcal{M}, \varphi)$ and therefore $h_1 \in L_p(\mathcal{M}, \varphi)^+$. To prove the last statement, note that by (5), $T_{q, \varphi}(h_1)$ is a positive element in the unit ball of $L_q(\mathcal{M}, \varphi)$, so that

$$\|h_1\|_{p, \varphi} = \langle T_{q, \varphi}(h_1), h_1 \rangle \leq \langle T_{q, \varphi}(h_1), h \rangle \leq \|h\|_{p, \varphi}.$$  

3 The Rényi relative entropy

We will need to extend the definition of $L_p(\mathcal{M}, \varphi)$ to all (not necessarily faithful) normal states. So let $\varphi \in \mathcal{S}_*(\mathcal{M})$ and let $s(\varphi) = \varepsilon$. Then $\varphi$ restricts to a faithful normal state on $\varepsilon \mathcal{M} \varepsilon$ and we may identify the predual $(\varepsilon \mathcal{M} \varepsilon)^*$ with the set of all $\psi \in \mathcal{M}_*$ such that $\varepsilon \psi \varepsilon = \psi$, where $\varepsilon \psi \varepsilon(x) = \psi(e \varepsilon x \varepsilon)$, $x \in \mathcal{M}$. By [33], Theorem 7], $h_\psi = h_{\varepsilon \psi \varepsilon} = e h_\psi e$ for all such $\psi$. Hence we may identify $L_1(\varepsilon \mathcal{M} \varepsilon) \cong \varepsilon L_1(\mathcal{M}) \varepsilon$ and using the polar decomposition, $L_p(\varepsilon \mathcal{M} \varepsilon) \cong \varepsilon L_p(\mathcal{M}) \varepsilon$ for all $p \geq 1$. The space $L_p(\mathcal{M}, \varphi)$ is then defined as

$$L_p(\mathcal{M}, \varphi) = \{h \in L_1(\mathcal{M}), \ h = e h e \in L_p(\varepsilon \mathcal{M} \varepsilon, \varphi|_{\varepsilon \mathcal{M} \varepsilon})\},$$

with the corresponding norm.

3.1 Definition and basic properties

Let $1 < \alpha < \infty$ and let $\varphi, \psi \in \mathcal{S}_* \mathcal{M}$. We define
\[
\tilde{D}_\alpha(\psi\|\varphi) = \begin{cases} 
\frac{1}{\alpha-1} \log(\|h_\psi\|_{\alpha,\varphi}) & \text{if } h_\psi \in L_\alpha(M,\varphi) \\
\infty & \text{otherwise}
\end{cases}
\] (8)

We first show that this definition is an extension of the sandwiched Rényi relative \(\alpha\)-entropy. Assume that \(\dim(M) < \infty\) and let \(\tau_0\) be a faithful normal trace on \(M\). Any state \(\varphi \in \mathcal{G}_\ast(M)\) is given by a density operator \(\rho_\varphi \in M^+\), such that \(\varphi(x) = \tau_0(\rho_\varphi x), x \in M\).

**Proposition 1.** Let \(\dim(H) < \infty\) and let \(\psi, \varphi \in \mathcal{G}_\ast(M)\), with density operators \(\rho_\varphi = \sigma, \rho_\psi = \rho\). Then

\[
\tilde{D}_\alpha(\psi\|\varphi) = \begin{cases} 
\frac{1}{\alpha-1} \log \tau_0[(\sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}})^\alpha], & \text{if } \text{supp}(\rho) \subseteq \text{supp}(\sigma) \\
\infty & \text{otherwise}
\end{cases}
\]

**Proof.** We may assume that \(\text{supp}(\rho) \subseteq \text{supp}(\sigma)\), otherwise \(\psi \notin L_p(M,\varphi)\) and both quantities are infinite. Then \(\sigma = e\sigma e\), similarly for \(\rho\). Put

\[
S := \sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}},
\]

where the powers are defined on the support of \(\sigma\). Then we have

\[
\rho = \sigma^{1/2\beta} S^{1/2\beta},
\]

where \(1/\alpha + 1/\beta = 1\). By [A.3], we have \(S \otimes \exp(\cdot/\alpha) \in L_\alpha(eMe)\) and

\[
h_\psi = \rho \otimes \exp(\cdot) = \sigma^{1/2\beta} S^{1/2\beta} \otimes \exp(\cdot) = h_\varphi^{1/2\beta} (S \otimes \exp(\cdot/\alpha)) h_\varphi^{1/2\beta}.
\]

It follows that \(h_\psi \in L_\alpha(M,\varphi)\) and

\[
\|h_\psi\|_{\alpha,\varphi}^\alpha = \|S \otimes \exp(\cdot/\alpha)\|_{\alpha}^\alpha = \tau_0(|S|^\alpha).
\]

We next prove the properties (i)–(v) of \(\tilde{D}_\alpha\).

**Proposition 2.** Let \(\psi, \varphi \in \mathcal{G}_\ast(M)\).

(i) For any \(1 < \alpha < \infty\), \(\tilde{D}_\alpha(\psi\|\varphi) \geq 0\), with equality if and only if \(\psi = \varphi\).

(ii) If \(\psi \neq \varphi\), the function \(1 < \alpha \mapsto D_\alpha(\psi\|\varphi)\) is strictly increasing.
Proof. By (2), \( \|h_\psi\|_{\alpha,\varphi} \geq \|h_\psi\|_1 = 1 \), hence \( \tilde{D}_\alpha(\psi\|\varphi) \geq 0 \). Since 
\[ h_\varphi = h_\varphi^{1/2}h_\varphi^{1/2} \in L_\infty(\mathcal{M}, \varphi) \]
and
\[ 1 = \|h_\varphi\|_1 \leq \|h_\varphi\|_{\alpha,\varphi} \leq \|h_\varphi\|_{\infty,\varphi} = \|1\|_1 = 1, \]
we have \( \tilde{D}_\alpha(\varphi\|\varphi) = 0 \), for all \( \alpha \). Assume now that \( \tilde{D}_\alpha(\psi\|\varphi) = 0 \).
Choose any \( 1 < p < \alpha \), then \( 1 \leq \|h_\psi\|_{p,\varphi} \leq \|h_\psi\|_{\alpha,\varphi} = 1 \), so that 
\( \|h_\psi\|_{p,\varphi} = 1 \). Let \( f \) be the constant function \( f(z) = h_\psi \) for all \( z \in S \),
then clearly \( f \in \mathcal{F}_{\alpha,1} \). Let \( \eta \in (0,1) \) be such that \( 1/p = \eta + (1-\eta)/\alpha \),
then \( f \) satisfies equality in the Hadamard three lines theorem (Theorem 4) at \( \eta \). It follows that \( h_\psi \equiv f_{h_\psi,p}(z + (1-z)/\alpha), z \in S \)
(note that in this case \( M = M_1/M_0 = 1 \)). Hence \( f_{h_\psi,p}(z) \equiv h_\psi \) for all 
\( z \in S \). Putting \( z = 0 \), we get \( h_\psi = ch_\varphi \), where \( c = \|h_\psi\|_{\alpha,\varphi} \). As \( \psi \) is a state,
we must have \( c = 1 \) and \( \psi = \varphi \). This finishes the proof of (i).

For (ii), let \( 1 < \alpha < \alpha' < \infty \). We may assume that \( \tilde{D}_{\alpha'}(\psi\|\varphi) < \infty \),
otherwise there is nothing to prove. Hence \( h_\psi \in L_{\alpha'}(\mathcal{M}, \varphi) \subseteq L_p(\mathcal{M}, \varphi) \)
for all \( 1 \leq p \leq \alpha' \). Let \( \eta \) be such that \( \eta + (1-\eta)/\alpha' = 1/\alpha \).
Then \( 1 - \eta = \beta'/\beta \), where \( 1/\beta + 1/\alpha = 1 \) and \( 1/\alpha' + 1/\beta' = 1 \). We
consider again the constant function \( f(z) \equiv h_\psi \), which this time is an element of \( \mathcal{F}_{\alpha',1} \). By Theorem 4
with \( p = 1 \) and \( p' = \alpha' \), we obtain
\[ \|h_\psi\|_{\alpha,\varphi} \leq \|h_\psi\|_{1-\eta,\varphi} = \|h_\psi\|^{\beta'/\beta}_{\alpha',\varphi}. \]
Taking the logarithm proves \( \tilde{D}_\alpha(\psi\|\varphi) \leq \tilde{D}_{\alpha'}(\psi\|\varphi) \). Assume now that equality holds, then it follows that 
\( h_\psi \equiv f_{h_\psi,\alpha}(z + (1-z)/\alpha')M^{z-\eta} \),
similarly as in the proof of (i). Putting \( u := z + (1-z)/\alpha' \), we obtain
\( h_\psi \equiv f_{h_\psi,\alpha}(u)Ma^u+b \) for all \( u \) with \( Re(u) \) between \( 1/\alpha' \) and \( 1 \), here \( a, b \in \mathbb{R} \). This equality clearly extends to all \( u \in S \). Again, putting 
\( u = 0 \), we obtain \( h_\psi = ch_\varphi \), which implies \( \psi = \varphi \).

It is clear from Theorem 11 and the remarks at the beginning of Section 3 that the spaces \( L_p(\mathcal{M}, \varphi) \)
can be defined for \( 0 \neq \varphi \in \mathcal{M}_*^+ \) and that for \( \lambda > 0 \), 
\( \|h\|_{p,\lambda \varphi} = \lambda^{1/q}\|h\|_{p,\varphi} \) for any \( h \in L_p(\mathcal{M}, \varphi) = L_p(\mathcal{M}, \lambda \varphi) \).
The definition of \( \tilde{D}_\alpha \) can thus be extended to positive normal functionals. It is easy to see that for \( \mu, \lambda > 0 \) and \( 0 \neq \psi, \varphi \in \mathcal{M}_*^+ \), we have
\[ \tilde{D}_\alpha(\mu \psi\|\lambda \varphi) = \tilde{D}_\alpha(\psi\|\varphi) + \frac{\alpha}{\alpha - 1} \log \mu - \log \lambda. \quad (9) \]
With this extension, we have the following order relations.
It was proved by Kosaki \cite[Theorem 4.2]{Kosaki} that the map \( h \mapsto D_\alpha(h) \) is jointly lower semicontinuous.

**Proposition 4.** \( D_\alpha : \mathcal{M}_*^+ \times \mathcal{M}_*^+ \to [0, \infty] \) is jointly lower semicontinuous.

**Proof.** It suffices to prove that the set \( \{(\psi, \varphi) \in \mathcal{M}_*^+ \times \mathcal{M}_*^+, \|h\|_{\alpha, \varphi} \leq a\} \) is closed in \( \mathcal{M}_* \times \mathcal{M}_* \) for each \( a \geq 0 \). So let \( \psi_n \) and \( \varphi_n \) be sequences of positive normal functionals, converging in \( \mathcal{M}_* \) to \( \psi \) and \( \varphi \), respectively, and such that \( \|h_{\psi_n}\|_{\alpha, \varphi_n} \leq a \). By Theorem 3 we have

\[
h_{\psi_n} = h_{\varphi_n}^{1/2\beta} k_n h_{\varphi_n}^{1/2\beta}, \quad k_n \in L_\alpha(\mathcal{M})^+, \|k_n\|_\alpha = \|h_{\psi_n}\|_{\alpha, \varphi_n} \leq a.
\]

Since the space \( L_\alpha(\mathcal{M}) \) is reflexive and \( \{k_n\} \) is bounded, we may assume that \( k_n \) converges to some \( k \) weakly in \( L_\alpha(\mathcal{M}) \), and then \( \|k\|_\alpha \leq a \).

Let \( h := h_{\varphi_n}^{1/2\beta} k h_{\varphi_n}^{1/2\beta} \), so that \( h \in L_\alpha(\mathcal{M}, \varphi) \) with \( \|h\|_{\alpha, \varphi} = \|k\|_\alpha \leq a \). We will show that \( h_{\psi_n} \) converges to \( h \) weakly in \( L_1(\mathcal{M}) \) and hence we must have \( h_{\psi_n} = h \). So let \( x \in \mathcal{M} \). Then by Hölder’s inequality,

\[
|\text{Tr} (h_{\psi_n} - h) x| = |\text{Tr} (h_{\varphi_n}^{1/2\beta} k_n h_{\varphi_n}^{1/2\beta} - h_{\varphi_n}^{1/2\beta} k h_{\varphi_n}^{1/2\beta}) x| \leq |\text{Tr} (h_{\varphi_n}^{1/2\beta} - h_{\varphi_n}^{1/2\beta}) k_n h_{\varphi_n}^{1/2\beta} x| + |\text{Tr} h_{\varphi_n}^{1/2\beta} k_n (h_{\varphi_n}^{1/2\beta} - h_{\varphi_n}^{1/2\beta}) x| + |\text{Tr} h_{\varphi_n}^{1/2\beta} (k_n - k) h_{\varphi_n}^{1/2\beta} x| \leq \|h_{\varphi_n}^{1/2\beta} - h_{\varphi_n}^{1/2\beta}\|_\alpha \|k\|_\alpha (\varphi(1^{1/2\beta}) + \varphi(1^{1/2\beta})) \|x\| \leq \|k\|_\alpha \|h_{\varphi_n} - h\|_{\alpha, \varphi_n} \|x\|.
\]

It was proved by Kosaki \cite[Theorem 4.2]{Kosaki} that the map \( L_1(\mathcal{M})^+ \ni h \mapsto h_{1/\beta}^{1/p} \in L_\beta(\mathcal{M})^+ \) is norm continuous. Hence the first part of the last expression converges to 0. Since \( h_{\varphi_n}^{1/2\beta} x h_{\varphi_n}^{1/2\beta} \in L_\beta(\mathcal{M}) \) for any \( x \in \mathcal{M} \), the second part goes to 0 as well.

\[\square\]
Let now \( N = M \oplus M \) and \( \varphi_1, \varphi_2 \in \mathcal{G}_s(M) \) be faithful, \( \varphi = \varphi_1 \oplus \varphi_2 \). By \([33]\), \( L_p(N) = L_p(M) \times L_p(M) \) and \( \|(k_1, k_2)\|_p = (\|k_1\|_p^p + \|k_2\|_p^p)^{1/p} \), \( 1 \leq p \leq \infty \). By this and Theorem \([1]\) we obtain that \( L_p(N, \varphi) = L_p(M, \varphi_1) \times L_p(M, \varphi_2) \) and for \( h = (h_1, h_2) \in L_p(N, \varphi) \),

\[
\|(h_1, h_2)\|_{p, \varphi} = (\|h_1\|_{p, \varphi_1}^p + \|h_2\|_{p, \varphi_2}^p)^{1/p}.
\] (10)

**Proposition 5.** Let \( \psi_1, \psi_2, \varphi_1, \varphi_2 \in \mathcal{G}_s(M) \) and let \( \psi = \psi_1 + \psi_2 \), \( \varphi = \varphi_1 + \varphi_2 \). Then

\[
\exp\{(\alpha - 1)\tilde{D}_\alpha(\psi, \varphi)\} = \exp\{(\alpha - 1)\tilde{D}_\alpha(\psi_1, \varphi_1)\} + \exp\{(\alpha - 1)\tilde{D}_\alpha(\psi_1, \varphi_1)\}.
\]

**Proof.** Follows immediately from (10) and the definition of \( \tilde{D}_\alpha \).

**\( \square \)**

### 3.2 The Petz dual and the data processing inequality

Let \( N \) be another von Neumann algebra and let \( \Phi : L_1(M) \to L_1(N) \) be a positive linear trace-preserving map. Then \( \Phi \) defines a positive linear map \( M_+ \to N_+ \), also denoted by \( \Phi \), mapping states to states. The adjoint \( \Phi^* : N \to M \) is normal, positive and unital. The map \( \Phi \) will be fixed throughout this section, together with some \( \varphi \in \mathcal{G}_s(M) \).

We put \( e := s(\varphi) \) and \( e' := s(\Phi(\varphi)) \).

We first show that \( \Phi \) maps \( L_1(M, \varphi) \) into \( L_1(N, \Phi(\varphi)) \), see the remarks at the beginning of Section \([3]\). From \( \varphi(\Phi^*(1 - e')) = \Phi(\varphi)(1 - e) = 0 \), it follows that \( e\Phi^*(1 - e')e = 0 \) and hence \( e\Phi^*(e') = e \), so that \( e \leq \Phi^*(e') \). Let now \( h = che \in L_1(M)^+ \), then

\[
\text{Tr} \, h = \text{Tr} \, he \leq \text{Tr} \, h\Phi^*(e') = \text{Tr} \, \Phi(h)e' \leq \text{Tr} \, \Phi(h) = \text{Tr} \, h,
\]

hence \( e' \Phi(h)e' = \Phi(h) \) and \( \Phi(h) \in L_1(N, \Phi(\varphi)) \). Since \( L_1(M, \varphi) \) is generated by positive elements, this implies that \( \Phi \) maps \( L_1(M, \varphi) \) into \( L_1(N, \Phi(\varphi)) \).

Assume next that \( h = h_x \) for some \( x \in eM^+ e \). Then \( h_x \leq \|x\|_\varphi h_\varphi \) and since \( \Phi \) is positive, we also have \( \Phi(h_x) \leq \|x\|_\varphi \Phi(h_\varphi) \). By Lemma \([1]\) there is some \( x' \in e'N^+ e' \) such that

\[
\Phi(h_x) = \Phi(h_\varphi^{1/2}xh_\varphi^{1/2}) = \Phi(h_\varphi^{1/2})\Phi(\varphi)^{1/2}x'\Phi(\varphi)^{1/2} = \Phi(h_\varphi)x' \in L_\infty(N, \Phi(\varphi))^+.
\]

Since \( M^+ \) generates \( M \), it follows that \( \Phi \) maps \( L_\infty(M, \varphi) \) into \( L_\infty(N, \Phi(\varphi)) \). By linearity, the map \( x \mapsto x' \) extends to a linear map \( \Phi^*_\varphi : eMe \to e'Ne' \), which is obviously positive, unital and normal.
**Proposition 6.** For any $1 \leq p \leq \infty$, $\Phi$ restricts to a contraction $L_p(\mathcal{M}, \varphi) \to L_p(\mathcal{N}, \Phi(\varphi))$.

**Proof.** As we have seen, $\Phi$ maps $L_1(\mathcal{M}, \varphi)$ into $L_1(\mathcal{N}, \Phi(\varphi))$ and $L_\infty(\mathcal{M}, \varphi)$ into $L_\infty(\mathcal{N}, \Phi(\varphi))$. For any $h \in L_1(\mathcal{M}, \varphi)$,

$$\|\Phi(h)\|_1 = \sup_{x_0 \in \mathcal{N}, \|x_0\| \leq 1} \text{Tr} \Phi(h)x_0 = \sup_{x_0 \in \mathcal{N}, \|x_0\| \leq 1} \text{Tr} h\Phi^*(x_0) \leq \|h\|_1,$$

the last inequality follows from the fact that $\Phi^*$ is a unital positive map, hence a contraction by the Russo-Dye theorem. Let us again use the standard form (11).

$$\langle h_0, \Phi(h) \rangle = \langle h_0, \Phi(h_\varphi) \rangle = \text{Tr} h_0 \Phi^*(x) = \langle \Phi(\varphi)(h_0), h_x \rangle.$$  \hfill (11)

By the uniqueness part in [3, Theorem 1], this extends to

$$\langle h_0, \Phi(h) \rangle = \langle \Phi(\varphi)(h_0), h \rangle, \quad h \in L_p(\mathcal{M}, \varphi), \quad h_0 \in L_q(\mathcal{N}, \Phi(\varphi)).$$   \hfill (12)

Moreover, for $x \in e\mathcal{M}e$,

$$\text{Tr} \Phi(\varphi)(\Phi(h_\varphi))x = \text{Tr} \Phi(\varphi)\Phi^*(x) = \text{Tr} \Phi(h_\varphi)^{1/2}\Phi^*(x)\Phi(h_\varphi)^{1/2} = \text{Tr} \Phi(h_x) = \text{Tr} h_x = \text{Tr} h_\varphi x,$$

so that $\Phi(\varphi)(\Phi(h_\varphi)) = h_\varphi$. By Proposition 5, $\Phi(\varphi)$ defines a positive contraction $L_p(\mathcal{N}, \Phi(\varphi)) \to L_p(\mathcal{M}, \varphi)$, for $1 \leq p \leq \infty$.

**Remark 1.** Let us again use the standard form $(l(\mathcal{M}), L_2(\mathcal{M}), \mathcal{J}, L_2(\mathcal{M})^+)$.

As in the proof of Lemma 1 we have

$$\text{Tr} h_x y = (Jxh_\varphi^{1/2}, yh_\varphi^{1/2}), \quad y \in e\mathcal{M}e$$

for $x \in e\mathcal{M}^+ e$ and by linearity, this holds for all $x \in e\mathcal{M}e$. It follows that $\Phi^*_\varphi$ is determined by

$$\langle Jxh_\varphi^{1/2}, \Phi^*_\varphi(y_0)h_\varphi^{1/2} \rangle = \text{Tr} h_x \Phi^*_\varphi(y_0) = \text{Tr} \Phi(h_x)y_0 = \text{Tr} \Phi(h_\varphi)^{1/2}x' \Phi(h_\varphi)^{1/2}y_0 = (J_0 \Phi^*_\varphi(x)\Phi(h_\varphi)^{1/2}, y_0\Phi(h_\varphi)^{1/2})$$

for $x \in e\mathcal{M}^+ e$.
for all \( y_0 \in e'N e' \) and \( x \in e' M e \), here \( J_0 \) is the adjoint operation on \( L_2(e'N e') \). In this form, the map \( \Phi^* \) was defined by Petz in [28] and is therefore called the Petz dual. Moreover, it was proved that for any \( n \), \( \Phi^* \) is \( n \)-positive if and only if \( \Phi \) is.

We are now ready to prove the data processing inequality for \( \tilde{D}_\alpha \), together with some bounds.

**Theorem 5.** Let \( 1 < \alpha < \infty \), \( 1/\alpha + 1/\beta = 1 \). Let \( \psi, \varphi \in \mathcal{S}_*(M) \) and assume that \( h_\psi \in L_\alpha(M, \varphi) \). Let us denote \( h := T_{\beta,\varphi}(h_\psi) \), \( h_0 = T_{\beta,\Phi(\varphi)}(\Phi(h_\psi)) \). Then for \( 1 < \alpha \leq 2 \),

\[
\tilde{D}_\alpha(\psi\|\varphi) - \tilde{D}_\alpha(\Phi(\psi)\|\Phi(\varphi)) \geq -\beta \log \left( \left( 2^\beta - \|h - \Phi_\varphi(h_0)\|_\beta^{\beta} \right)^{1/\beta} - 1 \right) \geq 0
\]

and for \( 2 \leq \alpha < \infty \),

\[
\tilde{D}_\alpha(\psi\|\varphi) - \tilde{D}_\alpha(\Phi(\psi)\|\Phi(\varphi)) \geq -\beta \log \left( \left( 2^\alpha - \|h - \Phi_\varphi(h_0)\|_\beta^{\alpha} \right)^{1/\alpha} - 1 \right) \geq 0
\]

If \( 1 < \alpha < \infty \) and \( \|h - \Phi_\varphi(h_0)\|_{\beta,\varphi} < 1 \), we also have an upper bound

\[
\tilde{D}_\alpha(\psi\|\varphi) - \tilde{D}_\alpha(\Phi(\psi)\|\Phi(\varphi)) \leq -\beta \log \left( 1 - \|h - \Phi_\varphi(h_0)\|_{\beta,\varphi} \right).
\]

**Proof.** By (12), we obtain

\[
\frac{\|\Phi(h_\psi)\|_{\alpha,\Phi(\varphi)}}{\|h_\psi\|_{\alpha,\varphi}} = \frac{\langle h_0, \Phi(h_\psi) \rangle}{\|h_\psi\|_{\alpha,\varphi}} = \langle \Phi_\varphi(h_0), \|h_\psi\|^{-1}_{\alpha,\varphi} h_\psi \rangle = \langle \Phi_\varphi(h_0) + h, \|h_\psi\|^{-1}_{\alpha,\varphi} h_\psi \rangle - \langle h, \|h_\psi\|^{-1}_{\alpha,\varphi} h_\psi \rangle \leq \|\Phi_\varphi(h_0) + h\|_{\beta,\varphi} - 1.
\]

For \( 1 < \alpha \leq 2 \), Clarkson’s inequality (Theorem 2) implies

\[
\|\Phi_\varphi(h_0) + h\|_{\beta,\varphi} \leq \left( 2^\beta - \|h - \Phi_\varphi(h_0)\|_{\beta,\varphi}^{\beta} \right)^{1/\beta} \leq 2.
\]

For \( 2 \leq \alpha < \infty \), we apply McCarthy’s inequality (Theorem 3) and obtain

\[
\|\Phi_\varphi(h_0) + h\|_{\beta,\varphi} \leq \left( 2^\alpha - \|h - \Phi_\varphi(h_0)\|_{\beta,\varphi}^{\alpha} \right)^{1/\alpha} \leq 2.
\]

The inequalities in (i) and (ii) now follow by taking the logarithms. On the other hand, we have a lower bound

\[
\frac{\|\Phi(h_\psi)\|_{\alpha,\Phi(\varphi)}}{\|h_\psi\|_{\alpha,\varphi}} = \langle \Phi_\varphi(h_0), \|h_\psi\|^{-1}_{\alpha,\varphi} h_\psi \rangle = \langle h - (h - \Phi_\varphi(h_0)), \|\psi\|^{-1}_{\alpha,\varphi} h_\psi \rangle \geq 1 - \|h - \Phi_\varphi(h_0)\|_{\beta,\varphi}.
\]

If \( 1 - \|h - \Phi_\varphi(h_0)\|_{\beta,\varphi} > 0 \), this implies (iii).
Corollary 2. For $1 < \alpha < \infty$, the map $(\psi, \varphi) \mapsto \exp\{ (\alpha - 1)\tilde{D}_\alpha \}$ is jointly convex.

Proof. The following arguments are quite standard. Let $\psi_1, \psi_2, \varphi_1, \varphi_2 \in \mathbb{S}_*(M)$. Let $\psi, \varphi \in \mathbb{S}_*(M \oplus M)$ be given by $\psi = \lambda \psi_1 \oplus (1 - \lambda) \psi_2$ and $\varphi = \lambda \varphi_1 \oplus (1 - \lambda) \varphi_2$. By Proposition 5 and (9), we obtain

$$\exp\{ (\alpha - 1)\tilde{D}_\alpha (\psi \| \varphi) \} = \exp\{ (\alpha - 1)\tilde{D}_\alpha (\lambda \psi_1 \| \lambda \varphi_1) \} + \exp\{ (\alpha - 1)\tilde{D}_\alpha ((1 - \lambda) \psi_2 \| (1 - \lambda) \varphi_2) \} = \lambda \exp\{ (\alpha - 1)\tilde{D}_\alpha (\psi_1 \| \varphi_1) \} + (1 - \lambda) \exp\{ (\alpha - 1)\tilde{D}_\alpha (\psi_2 \| \varphi_2) \}.$$

Let $\Phi : L_1(M \oplus M) \to L_1(M)$ be given by $(h_1, h_2) \mapsto h_1 + h_2$, then $\Phi$ is obviously positive and trace preserving and $\Phi(\varphi) = \lambda \varphi_1 + (1 - \lambda) \varphi_2, \quad \Phi(\psi) = \lambda \psi_1 + (1 - \lambda) \psi_2.$

The statement now follows by Theorem 5.

We also obtain a characterization of equality, which will be useful in the next section.

Corollary 3. Let $\psi, \varphi \in \mathbb{S}_*(M)$ and assume that $\psi \in L_\alpha(M, \varphi)$. Then $D_\alpha(\psi \| \varphi) = D_\alpha(\Phi(\psi) \| \Phi(\varphi))$ if and only if

$$\Phi(\varphi) \circ T_{\beta, \Phi(\varphi)} \circ \Phi(h_\psi) = T_{\beta, \varphi}(h_\psi).$$

If $\alpha = 2$, this is equivalent to $\Phi(\varphi) \circ \Phi(\psi) = \psi$.

Proof. The first statement is immediate from Theorem 5. Let now $\alpha = 2$, then

$$\|\Phi(h_\psi)\|_{2, \Phi(\varphi)}^2 = \langle \Phi(h_\psi), \Phi(h_\psi) \rangle = \langle h_\psi, \Phi(\psi) \circ \Phi(h_\psi) \rangle \leq \|h_\psi\|_{2, \varphi} \|\Phi(\psi) \circ \Phi(h_\psi)\|_{2, \varphi} \leq \|h_\psi\|_{2, \varphi}^2.$$

The statement now follows by equality condition in the Schwarz inequality.

\[\square\]
4 Sufficiency of channels

In this section, we study the case of equality in DPI for $\tilde{D}_n$. The aim is to show that this equality implies existence of a recovery map for $(\Phi, \psi, \varphi)$. For this, we need that the map $\Phi$ is 2-positive, which will be assumed in the rest of the paper.

Let $\psi, \varphi \in \mathcal{S}_p(M)$ and let $\Phi : L_1(M) \to L_1(N)$ be a 2-positive trace preserving map. We say that $\Phi$ is sufficient with respect to $\{\psi, \varphi\}$ if there exists a 2-positive trace preserving map $\Psi : L_1(N) \to L_1(M)$, such that $\Psi \circ \Phi(h_\psi) = h_\psi$ and $\Phi \circ \Psi(h_\varphi) = h_\varphi$.

Remark 2. In the above definition, we may also assume that both $\Phi$ and $\Psi$ are completely positive and trace preserving maps, such maps are usually called quantum channels. This definition seems stronger, but in fact it is fully equivalent, in the sense that if $\Phi$ is 2-positive and trace preserving, and sufficient in the 2-sense, then there are quantum channels $\tilde{\Phi}$ and $\tilde{\Psi}$ that coincide with $\Phi$ and $\Psi$ when restricted to $\{\psi, \varphi\}$ and $\{\Phi(\psi), \Phi(\varphi)\}$, respectively.

The following theorem is one of the crucial results of [28]. Note that it implies that $\Phi_\varphi$ is a universal recovery map.

Theorem 6. [28, 13] Let $\Phi : L_1(M) \to L_1(N)$ be a trace preserving 2-positive map. Let $\varphi \in \mathcal{S}_p(M)$ be faithful and assume that $\Phi(\varphi)$ is faithful as well. Then $\Phi$ is sufficient with respect to $\{\psi, \varphi\}$ if and only if $\Phi_\varphi \circ \Phi(h_\psi) = h_\psi$.

The following is a standard result of ergodic theory.

Lemma 7. Let $\Omega : L_1(M) \to L_1(M)$ be 2-positive and trace preserving, admitting a faithful normal invariant state. Then there is a faithful normal conditional expectation $E$ on $M$ such that any $\psi \in \mathcal{S}_p(M)$ is invariant under $\Omega$ if and only if $\psi \circ E = \psi$.

Proof. Let $\mathcal{S}$ be the set of all normal invariant states of $\Omega$ and let $\mathcal{I}$ be the set of all 2-positive unital normal maps $T : M \to M$, such that $\psi \circ T = \psi$ for all $\psi \in \mathcal{S}$. Then $\mathcal{I}$ is a semigroup (i.e. closed under composition), convex and closed with respect to the pointwise weak*-topology. By the mean ergodic theorem [17], $\mathcal{I}$ contains a conditional expectation $E$, such that

$$T \circ E = E \circ T = E, \quad \forall T \in \mathcal{I}.$$
Since \( E \in \mathcal{T} \), \( \psi \circ E = \psi \) for all \( \psi \in \mathcal{S} \). On the other hand, let \( \psi \in \mathcal{S}_* (\mathcal{M}) \) be such that \( \psi \circ E = \psi \), then

\[
\psi \circ \Omega^* = \psi \circ E \circ \Omega^* = \psi \circ E = \psi,
\]

because \( \Omega^* \in \mathcal{T} \).

**Lemma 8.** Let \( \varphi \in \mathcal{S}_* (\mathcal{M}) \) be faithful. Let \( 1 < p < \infty \) and let \( \psi \in \mathcal{S}_* (\mathcal{M}) \) be such that

\[
h_\psi = c h_\varphi^{1/2q} h_\omega^{1/p} h_\varphi^{1/2q}
\]

for some \( c > 0 \) and \( \omega \in \mathcal{S}_* (\mathcal{M}) \). Let \( \Phi : L_1 (\mathcal{M}) \to L_1 (\mathcal{N}) \) be a 2-positive trace preserving map. Then \( \Phi \) is sufficient with respect to \( \{ \psi, \varphi \} \) if and only if it is sufficient with respect to \( \{ \omega, \varphi \} \).

**Proof.** Let \( \Omega = \Phi \varphi \circ \Phi \), then \( \varphi \) is a faithful invariant state for \( \Omega \). By Lemma 7 and Theorem 6 there is a faithful normal conditional expectation \( E \) such that \( \Phi \) is sufficient with respect to \( \{ \psi, \varphi \} \) if and only if \( \psi \circ E = \psi \). Let us denote the range of \( E \) by \( \mathcal{M}_0 \).

Since the space \( L_p (\mathcal{M}) \) is essentially independent from the choice of the f.n.s. weight \( \phi \) (see Appendix), we may assume that \( \phi = \varphi \), so that the results of Appendix A.2 can be applied.

Let \( \psi \circ E = \psi \), that is, \( E_1 (h_\psi) = h_\psi \). By (A.5) and (A.6),

\[
h_\psi = E_1 (h_\psi) = c E_1 (h_\varphi^{1/2q} h_\omega^{1/p} h_\varphi^{1/2q}) = c h_\varphi^{1/2q} E_p (h_\omega^{1/p}) h_\varphi^{1/2q}.
\]

Since \( i_p \) is an isomorphism (see Theorem 11), we see that we must have \( h_\omega^{1/p} = E_p (h_\omega^{1/p}) \in L_p (\mathcal{M}_0) \). But then also \( h_\omega \in L_1 (\mathcal{M}_0) \), so that \( \omega \circ E = \omega \) and \( \Phi \) is sufficient with respect to \( \{ \omega, \varphi \} \). Conversely, if \( \omega \circ E = \omega \), then \( h_\omega^{1/p} \in L_p (\mathcal{M}_0) \), so that \( h_\psi \in L_1 (\mathcal{M}_0) \) and \( \psi \circ E = \psi \).

**Lemma 9.** Let \( \Phi : L_1 (\mathcal{M}) \to L_1 (\mathcal{N}) \) be a positive trace preserving map and let \( h \in L_p (\mathcal{M}, \varphi) \) be such that \( \| \Phi (h) \|_{p, \Phi (\varphi)} = \| h \|_{p, \varphi} \). Then

\[
\| \Phi (f_{p, h} (\theta)) \|_{1/\theta, \Phi (\varphi)} = \| f_{p, h} (\theta) \|_{1/\theta, \varphi}, \quad \forall \theta \in (0, 1).
\]

**Proof.** By Proposition 6 \( \Phi \circ f_{p, h} \in F (L_\infty (\mathcal{N}, \Phi (\varphi)), L_1 (\mathcal{N})) =: \mathcal{F}_0 \) and \( \| \Phi \circ f_{p, h} \|_{\mathcal{F}_0} \leq \| f_{p, h} \|_{\mathcal{F}} \). Since \( \Phi \circ f_{p, h} (1/p) = \Phi (h) \), we have

\[
\| \Phi (h) \|_{p, \Phi (\varphi)} \leq \| \Phi \circ f_{p, h} \|_{\mathcal{F}_0} \leq \| f_{p, h} \|_{\mathcal{F}} = \| h \|_{p, \varphi} = \| \Phi (h) \|_{p, \Phi (\varphi)},
\]

hence \( \| \Phi \circ f_{p, h} (1/p) \|_{p, \Phi (\varphi)} = \| \Phi (f_{p, h}) \|_{\mathcal{F}_0} = \| f_{p, h} \|_{\mathcal{F}} \). The result now follows by Lemma 8.
We are now prepared to prove the main result of this section.

**Theorem 7.** Let $\Phi : L_1(\mathcal{M}) \rightarrow L_1(\mathcal{N})$ be a 2-positive trace preserving map and let $1 < \alpha < \infty$. Let $\varphi, \psi \in \mathcal{G}_s(\mathcal{M})$ be such that $h_\psi \in L_\alpha(\mathcal{M}, \varphi)$. Then $\Phi$ is sufficient with respect to $\{\psi, \varphi\}$ if and only if $\tilde{D}_\alpha(\psi \parallel \varphi) = \tilde{D}_\alpha(\Phi(\psi) \parallel \Phi(\varphi))$.

**Proof.** Note that by the assumptions, $s(\psi) \leq s(\varphi)$ and we may suppose that both $\varphi$ and $\Phi(\varphi)$ are faithful, by restriction to the corresponding compressed algebras. Further, we have $h_\psi = h_\varphi^{1/2} h_\omega^{1/\alpha} h_\varphi^{1/2\beta}$ for some $\omega \in \mathcal{M}_+^+$, here $1/\alpha + 1/\beta = 1$. Suppose that $\tilde{D}_\alpha(\psi \parallel \varphi) = \tilde{D}_\alpha(\Phi(\psi) \parallel \Phi(\varphi))$. Then $\|\Phi(h_\psi)\|_{\alpha, \Phi(\varphi)} = \|h_\psi\|_{\alpha, \varphi}$ and by Lemma 9

$$\|\Phi(f_{\alpha, h_\psi}(1/2))\|_{2, \Phi(\varphi)} = \|f_{\alpha, h_\psi}(1/2)\|_{2, \varphi}. $$

Note that

$$f_{\alpha, h_\psi}(1/2) = ch_\varphi^{1/4} h_\omega^{1/2} h_\varphi^{1/4} \in L_1(\mathcal{M})^+$$

for some constant $c > 0$ and there is some $\psi_1 \in \mathcal{G}_s(\mathcal{M})$, such that $f_{\alpha, h_\psi}(1/2) = dh_{\psi_1}$, where $d > 0$ is obtained by normalization. It follows that $h_{\psi_1} \in L_2(\mathcal{M}, \varphi)$ and we have

$$\|\Phi(h_{\psi_1})\|_{2, \Phi(\varphi)} = \|h_{\psi_1}\|_{2, \varphi}. $$

By Corollary 8, this implies that $\Phi$ is sufficient with respect to $\{\psi_1, \varphi\}$ and by Lemma 8 $\Phi$ is sufficient with respect to $\{\omega_1, \varphi\}$, where $\omega_1 = \omega(1)^{-1}\omega$. Using Lemma 8 again, we obtain that $\Phi$ is sufficient with respect to $\{\psi, \varphi\}$.

The converse statement follows immediately from DPI (Theorem 5). \hfill \square

## 5 Concluding remarks

In this paper, an extension of the sandwiched Rényi relative $\alpha$-entropies to the setting of von Neumann algebras is defined for $\alpha > 1$, using an interpolating family of non-commutative $L_p$-spaces with respect to a state. For this extension, we proved some of the basic properties, in particular the data processing inequality with respect to positive trace-preserving maps. In contrast, for the recently proposed Araki-Masuda divergences of [4], the proof of DPI requires the maps to be completely positive, but the divergences are defined also for $\alpha \in [1/2, 1)$ and the
limit values as $\alpha \to 1, \infty$ are obtained. Note that convergence of $\tilde{D}_\alpha$ to the Umegaki relative entropy as $\alpha \to 1$, would imply that the latter quantity satisfies DPI with respect to positive trace preserving maps, and not only adjoints of unital Schwarz maps, as is presently known \[36\]. In finite dimensions, this fact was recently observed in \[21\].

Another main result of the paper is the fact that preservation of the extended sandwiched entropies characterizes sufficiency of 2-positive trace preserving maps. Note that for most of the proofs 2-positivity was not needed, indeed, Lemma \[7\] is the only place where more than positivity is necessary. It would be interesting to see whether similar results can be proved assuming only positivity, since the results known so far on sufficiency of maps need stronger positivity conditions.

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**Appendices**

A. Haagerup’s $L_p$-spaces

We review the definition and basic properties of Haagerup’s $L_p$-spaces over $\mathcal{M}$ \[9\], see \[33\] for details.

Let $\phi$ be a faithful normal semifinite (f.n.s) weight on $\mathcal{M}$ and let $\sigma^\phi$ be the modular group with respect to $\phi$. Let us recall that the crossed product $\mathcal{R} = \mathcal{M} \times_{\sigma^\phi} \mathbb{R}$ is a von Neumann algebra acting on $L_2(\mathbb{R}, \mathcal{H}) \equiv L_2(\mathbb{R}) \otimes \mathcal{H}$, generated by the operators

$$\pi(x)\xi(t) = \sigma^\phi_{-t}(x)\xi(t), \quad \xi \in L_2(\mathbb{R}, \mathcal{H}), \quad t \in \mathbb{R}, \quad x \in \mathcal{M}$$

$$\lambda(s)\xi(t) = \xi(t-s), \quad \xi \in L_2(\mathbb{R}, \mathcal{H}), \quad t \in \mathbb{R}, \quad s \in \mathbb{R}.$$ 

The map $x \mapsto \pi(x)$ is a normal faithful representation of $\mathcal{M}$ and we will identify $\mathcal{M}$ with its image $\pi(\mathcal{M})$.

The dual action $\mathbb{R} \ni s \mapsto \theta_s$ is determined by

$$\theta_s(\pi(x)) = \pi(x), \quad \theta_s(\lambda(t)) = e^{-ist}\lambda(t), \quad x \in \mathcal{M}, \quad t, s \in \mathbb{R}.$$ 

The algebra $\mathcal{R}$ is equipped with a canonical f.n.s. trace $\tau$, satisfying $\tau \circ \theta_s = e^{-it}\tau$, $s \in \mathbb{R}$. The space $L_p(\mathcal{M})$, $1 \leq p \leq \infty$, is defined
as the space of \( \tau \)-measurable operators \( h \) affiliated with \( \mathcal{R} \), satisfying 
\[
\theta_s(h) = e^{-s/p}h, \text{ for all } s \in \mathbb{R}.
\]
Since the set of \( \tau \)-measurable operators is a *-algebra with respect to strong sum and strong product, \( L_p(\mathcal{M}) \) is a vector space if the sum is defined in the strong sense. Note that the weight \( \phi \) does not appear in the notation, since the resulting spaces are essentially independent on the choice of \( \phi \).

We have \( L_\infty(\mathcal{M}) \simeq \mathcal{M} \) and for \( p = 1 \), there is a linear bijection 
\[
\psi \mapsto h_\psi \tag{A.1}
\]
between the predual \( \mathcal{M}_* \) and \( L_1(\mathcal{M}) \), such that the cone \( \mathcal{M}_*^+ \) of positive normal functionals is mapped onto the cone \( L_1(\mathcal{M})^+ \) of positive operators in \( L_1(\mathcal{M}) \). If \( \psi = u|\psi| \) is the polar decomposition of \( \psi \), then we have a polar decomposition \( h_\psi = u|h_\psi| \), where \( |h_\psi| = h_{|\psi|} \).

Moreover, a trace can be defined in \( L_1(\mathcal{M}) \) by 
\[
\text{Tr} h_\psi = \psi(1), \quad \psi \in \mathcal{M}_*.
\]
Equipped with the norm \( \|h\|_1 = \text{Tr}|h| \), \( L_1(\mathcal{M}) \) is a Banach space and the bijection \( \text{A.1} \) is an isometry.

For \( 1 < p < \infty \), the cone \( L_p(\mathcal{M})^+ \) of positive operators in \( L_p(\mathcal{M}) \) is precisely the set \( \{ h_\psi^{1/p}, \psi \in \mathcal{M}_*^+ \} \) and any element \( k \in L_p(\mathcal{M}) \) has a polar decomposition \( k = u|k| \), where \( u \in \mathcal{M} \) is a partial isometry and \( |k| \in L_p(\mathcal{M})^+ \). With the norm
\[
\|k\|_p = (\text{Tr}|k|^p)^{1/p},
\]
\( L_p(\mathcal{M}) \) is a Banach space. If \( 1 \leq p, q, r \leq \infty \) with \( 1/p + 1/q = 1/r \) and \( h \in L_p(\mathcal{M}), k \in L_q(\mathcal{M}) \), the (strong) product \( hk \) is in \( L_r(\mathcal{M}) \) and Hölder’s inequality holds
\[
\|hk\|_r \leq \|h\|_p \|k\|_q. \tag{A.2}
\]
For \( r = 1 \), we obtain
\[
|\text{Tr} hk| \leq \text{Tr} |hk| = \|hk\|_1 \leq \|h\|_p \|k\|_q.
\]
This defines a duality between \( L_p(\mathcal{M}) \) and \( L_q(\mathcal{M}) \) with \( 1/p + 1/q = 1 \), and for \( 1 \leq p < \infty \), the dual space \( L_p(\mathcal{M})^* \) is isometrically isomorphic to \( L_q(\mathcal{M}) \).

An important special case is \( p = 2 \). The space \( L_2(\mathcal{M}) \) is a Hilbert space, with inner product
\[
(h, k) = \text{Tr} h^*k, \quad h, k \in L_2(\mathcal{M}).
\]
The left action of $M$ on $L_2(M)$, defined by

$$l(x): h \mapsto xh, \quad h \in L_2(M), x \in M$$

is a faithful normal representation of $M$ and $(l(M), L_2(M), J, L_2(M)^+)$ is a standard form for $M$, [32, 30], where the conjugation $J$ is defined by $Jh = h^*$, $h \in L_2(M)$. Any element $\psi \in M_+^*$ has a unique implementing vector $h_{\varphi}^{1/2} \in L_2(M)^+$, such that

$$\varphi(x) = (h_{\varphi}^{1/2}, xh_{\varphi}^{1/2}), \quad x \in M$$

If $\varphi$ is faithful, that is, $\varphi(x) = 0$ for $x \geq 0$ implies that $x = 0$, then its implementing vector is cyclic and separating, which means that both $Mh_{\varphi}^{1/2}$ and $h_{\varphi}^{1/2}M$ are dense in $L_2(M)$.

### A.1 The semifinite case

Assume that $M$ is semifinite and let $\tau_0$ be a faithful normal semifinite trace on $M$. For $p \geq 1$, the spaces $L_p(M, \tau_0)$ are defined as the sets of $\tau_0$-measurable operators affiliated with $M$ such that $\tau_0(|h|^p) < \infty$, equipped with the norm $\|h\|_{p, \tau_0} = \tau_0(|h|^p)^{1/p}$. By [33] p. 62, we have $M \times_{\sigma_0} \mathbb{R} \simeq M \otimes L_\infty(\mathbb{R})$ and for $1 \leq p < \infty$

$$L_p(M) \simeq L_p(M, \tau_0) \otimes \exp(c/p), \quad (A.3)$$

where for $h \in L_1(M, \tau_0)$, we have $\text{Tr}(h \otimes \exp(\cdot)) = \tau_0(h)$.

### A.2 Extensions of conditional expectations

A conditional expectation $E$ on a von Neumann algebra $M$ is a positive contractive normal projection onto a von Neumann subalgebra $M_0 \subseteq M$. A conditional expectation is necessarily completely positive and satisfies the condition

$$E(xy) = xE(a)y, \quad x, y \in M_0, \ a \in M. \quad (A.4)$$

If a faithful normal state $\phi$ and a von Neumann subalgebra $M_0 \subseteq M$ are given, then there exists a (unique) conditional expectation $E$ such that $\phi \circ E = \phi$ with range $M_0$ if and only if $\sigma_t^\phi(M_0) \subseteq M_0$, for all $t \in \mathbb{R}$, [31]. In this case, the modular group of the restricted state $\phi_0 = \phi|_{M_0}$ coincides with the restriction of $\sigma_t^\phi$ to $M_0$. Moreover, we have $\sigma_t^\phi \circ E = E \circ \sigma_t^\phi$ for all $t \in \mathbb{R}$.
Note that the crossed product $\mathcal{R}_0 = \mathcal{M}_0 \times_{\sigma_{\phi_0}} \mathbb{R}$ is a subalgebra in $\mathcal{R}$ and for $1 \leq p \leq \infty$, the space $L_p(\mathcal{M}_0)$ for $1 \leq p \leq \infty$ can be identified with a subspace in $L_p(\mathcal{M})$. It was proved in [14] that $E$ can be extended to a projection $E_p$ of $L_p(\mathcal{M})$ onto $L_p(\mathcal{M}_0)$, $1 \leq p \leq \infty$.

The extension is defined as follows. Let $\mathcal{M}_\phi \subseteq \mathcal{M}$ be the subalgebra of analytic elements with respect to $\phi$, that is, of elements $x \in \mathcal{M}$ such that $t \mapsto \sigma^\phi_t(x)$ extends to an entire analytic function $\mathbb{C} \to \mathcal{M}$. Then $E$ maps $\mathcal{M}_\phi$ onto $(\mathcal{M}_0)_{\phi_0}$. For $x \in \mathcal{M}_\phi$, the map

$$h_{\phi}^{1/2p}xh_{\phi}^{1/2p} \mapsto h_{\phi}^{1/2p}E(x)h_{\phi}^{1/2p}$$

is a contractive projection and since the set $\mathcal{E}$ is $\phi$-bra of analytic elements with respect to $C$, $f$ is continuously embedded in $X$ here we used the property (A.4) of conditional expectations and the facts that $\phi$ is $0$-dense in $L_0(\mathcal{M})$, and similarly for $L_p(\mathcal{M})$, it extends to a contractive projection $E_p$ of $L_p(\mathcal{M})$ onto $L_p(\mathcal{M}_0)$. Moreover, $E_p$ is positive and satisfies

$$E_s(hk) = E_r(l)k, \quad h \in L_p(\mathcal{M}_0), k \in L_q(\mathcal{M}_0), l \in L_r(\mathcal{M}), \quad (A.5)$$

whenever $1 \leq p, q, r \leq \infty$ are such that $1/p + 1/q + 1/r = 1/s \leq 1$.

Note that for $p = 1$, we have

$$E_1 : h_\psi \mapsto h_\psi \circ E, \quad \psi \in \mathcal{M}_*.$$  \hfill (A.6)

Indeed, let $a, x \in \mathcal{M}_\phi$ and let $\phi_x$ be such that $h_{\phi_x} = h_{\phi}^{1/2}xh_{\phi}^{1/2}$. Then

$$\phi_x \circ E(a) = \text{Tr} h_{\phi}^{1/2}xh_{\phi}^{1/2}E(a) = \text{Tr} h_{\phi}^{1/2}x\sigma_{-i/2}^\phi(E(a))h_{\phi}^{1/2}$$

$$= \phi(x\sigma_{-i/2}^\phi(E(a))) = \phi(xE(\sigma_{-i/2}^\phi(a))) = \phi(E(x)\sigma_{-i/2}^\phi(a))$$

$$= \text{Tr} h_{\phi}^{1/2}E(x)h_{\phi}^{1/2}a = \text{Tr} E_1(h_{\phi_x})a.$$  

here we used the property (A.4) of conditional expectations and the facts that $\phi \circ E = \psi$, $\sigma^\phi \circ E = E \circ \sigma^\psi$ and $\sigma^\psi_0(a) = h_{\phi}^zh_{\phi}^{-z}$. Since $\mathcal{M}_\phi$ is $w^*$-dense in $\mathcal{M}$, $E_1(h_{\phi_x}) = h_{\phi_{\circ E}}$. Since the set $\{h_{\phi_x}, x \in \mathcal{M}_\phi\}$ is dense in $L_1(\mathcal{M})$, the statement follows.

**B  The complex interpolation method**

In this paragraph, we briefly describe the complex interpolation method, following [3], see also [15].

Let $(X_0, X_1)$ be a compatible pair of Banach spaces, with norms $\| \cdot \|_0$ and $\| \cdot \|_1$. For our purposes, it is enough to assume that $X_0$ is continuously embedded in $X_1$. Let $S \subset \mathbb{C}$ be the strip $S = \{z \in \mathbb{C}, \ 0 \leq Re(z) \leq 1\}$ and let $\mathcal{F} = \mathcal{F}(X_0, X_1)$ be the set of functions $f : S \to X_1$ such that
(a) $f$ is bounded, continuous on $S$ and analytic in the interior of $S$

(b) For $t \in \mathbb{R}$, $f(it) \in X_0$ and the map $t \in \mathbb{R} \mapsto f(it) \in X_0$ is
continuous and bounded.

For $f \in \mathcal{F}$, let
\[
\|f\|_{\mathcal{F}} = \max\{\sup_t \|f(it)\|_0, \sup_t \|f(1+it)\|_1\}
\]

Then $(\mathcal{F}, \|\cdot\|_{\mathcal{F}})$ is a Banach space. For $0 < \theta < 1$, the interpolation
space is defined as the set
\[
C_\theta(X_0, X_1) = \{f(\theta), f \in \mathcal{F}\}
\]
endowed with the norm
\[
\|x\|_\theta = \inf\{\|f\|_{\mathcal{F}}, f(\theta) = x, f \in \mathcal{F}\}. \quad (B.7)
\]

Since $C_\theta(X_0, X_1)$ is the quotient space $\mathcal{F}/K_\theta$ with respect to the
closed subspace $K_\theta = \{f \in \mathcal{F}, f(\theta) = 0\}$, we see that $C_\theta(X_0, X_1)$ is a
Banach space. Moreover, we have the continuous embeddings
\[
X_0 \subseteq C_\theta(X_0, X_1) \subseteq X_1
\]
and $C_\theta$ defines an exact interpolation functor of exponent $\theta$, which
means that the following abstract version of the Riesz-Thorin inter-
polation theorem holds.

**Theorem B.1.** Let $(X_0, X_1)$ and $(Y_0, Y_1)$ be pairs of compatible Ba-
nach spaces and let $T : X_1 \to Y_1$ be a bounded linear operator such
that $T(X_0) \subseteq Y_0$. If $\|Tx\|_{Y_1} \leq M_1\|x\|_{X_1}$, $x \in X_1$ and $\|Tx_0\|_{X_0} \leq M_0\|x_0\|_{X_0}$ for $x_0 \in X_0$, then
\[
\|Tx\|_\theta \leq M_0^{1-\theta} M_1^{\theta} \|x\|_\theta.
\]

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