Voronoi’s conjecture for extensions of Voronoi parallelohedra

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1. Definitions and notation. A parallelohedron (see [1]) is a convex polytope $P$ such that its translates form a face-to-face tiling $T(P)$ of the affine space.

Let $\Lambda$ be a $d$-lattice in $\mathbb{R}^d$ with $0 \in \Lambda$, and let $\Omega$ be a positive-definite quadratic form of $d$ variables. Then the polyhedron

$$P_V = P_V(\Lambda, \Omega) = \left\{ y \in \mathbb{R}^d : y^T \Omega y = \min_{x' \in \Lambda} (y - x')^T \Omega (y - x') \right\}$$

is called the Voronoi parallelohedron (Voronoi cell) for the lattice $\Lambda$ with respect to $\Omega$.

Voronoi [2] stated the conjecture that all parallelohedra are Voronoi cells. A complete proof or disproof of it is still an open problem. Nevertheless, Voronoi’s conjecture holds in many important special classes of parallelohedra (see [2]–[5]).

We look at the special class of parallelohedra of the form $P + I$, where $P$ is some parallelohedron, $I$ is a line segment, and $+$ denotes the Minkowski sum. We call a parallelohedron of the form $P + I$ an extension of $P$. (Note that for an arbitrary pair of a parallelohedron $P$ and a line interval $I$ the Minkowski sum $P + I$ is not necessarily a parallelohedron.)

Grishukhin [6] proved Voronoi’s conjecture for extensions of $(d - 2)$-primitive parallelohedra. (In view of Zitomirskij’s result in [3], the latter form a subclass of the Voronoi cells.) In [6], he also stated the problem of proving Voronoi’s conjecture for all parallelohedra of the form $P + I$, where $P$ is a Voronoi cell.

Here we give a sketch of the proof of Voronoi’s conjecture for extensions of Voronoi cells and thereby solve Grishukhin’s problem and improve Erdahl’s result [4] that Voronoi’s conjecture holds for zonotopes which are parallelohedra.

Now we give the requisite definitions.

If $P$ is a parallelohedron and $I$ a line interval such that $P + I$ is also a parallelohedron, then we say that the direction of $I$ is free for $P$ (see, for instance, [6]).

Following [5], we say that a parallelohedron $P$ is reducible if it can be represented as a direct sum $P = P_1 \oplus P_2$ of lower-dimensional parallelohedra (that is, as a Minkowski sum in which the linear spans of the terms $P_1$ and $P_2$ are complementary subspaces).

For a polytope $Q$ with symmetry centre let $c(Q)$ denote this centre. By definition, the facet vector of a facet $F$ of a parallelohedron $P$ is equal to $2c(F) - 2c(P)$ (that both $P$ and $F$ are centrally symmetric follows from Minkowski’s paper [7]).

We say that a $d$-dimensional parallelohedron $P$ has a cross enveloping the facet vectors if there exist hyperplanes $\alpha_1$ and $\alpha_2$ in $\mathbb{R}^d$ such that each facet vector of $P$ is parallel to one of these hyperplanes (or both).

2. Main results. The main result of this note is as follows.

Theorem 1 [8]. Let $P$ and $P + I$ be parallelohedra, where $P$ is a Voronoi cell. Then $P + I$ is also a Voronoi cell.

The following results are crucial for the proof of Theorem 1.

Theorem 2. Let $P$ be a Voronoi cell with infinitely many free directions. Then $P$ is reducible.
**Theorem 3.** Let $P$ be a Voronoi cell with a cross enveloping the facet vectors. Then $P$ is reducible.

We sketch the proof of Theorem 1.

First we show that Theorems 1 and 2 are equivalent. Note that it was stated (although without proof) in Theorem 3.18 of [9] that Theorem 1 follows from Theorem 2.

Next we prove Theorem 2 using induction on the dimension of $P$, and at the same time we establish Theorem 3.

We denote the proposition “Theorem 2 holds for all parallelohedra $P$ of dimension at most $d$” by $A(d)$ and the analogous proposition for Theorem 3 by $B(d)$.

Induction base: the proposition $A(d)$ holds for $d \leq 4$, and the proposition $B(d)$ holds for $d \leq 2$. This is an immediate consequence of the well-known classification of parallelohedra up through dimension 4 (see [10]).

The induction proper proceeds in accordance with the following scheme:

$$\cdots \implies B(n) \implies A(n+2) \implies B(n+1) \implies A(n+3) \implies \cdots$$

In this way $A(d)$ and $B(d)$ will be proved for any positive integer $d$, giving us Theorems 2 and 3.

Since Theorem 2 holds, so does Theorem 1.

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