Quantum mechanical bound for efficiency of quantum Otto heat engine

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The second law of thermodynamics constrains that the efficiency of heat engines, classical or quantum, cannot be greater than the universal Carnot efficiency. We discover another bound for the efficiency of a quantum Otto heat engine consisting of a harmonic oscillator. Dynamics of the engine is governed by the Lindblad equation for the density matrix, which is mapped to the Fokker-Planck equation for the quasi-probability distribution. Applying stochastic thermodynamics to the Fokker-Planck equation system, we obtain the $h$-dependent quantum mechanical bound for the efficiency. It turns out that the bound is tighter than the Carnot efficiency. The engine achieves the bound in the low temperature limit where quantum effects dominate. Our work demonstrates that quantum nature could suppress the performance of heat engines in terms of efficiency bound, work and power output.

I. INTRODUCTION

A heat engine is a device harvesting work making use of a heat flow between multiple thermal reservoirs. One of the main concerns for the heat engine is efficiency. When the heat engine is in contact with two thermal reservoirs at temperatures $T_1$ and $T_2 (< T_1)$, the second law of thermodynamics constrains that the efficiency cannot be greater than the Carnot efficiency $\eta_C = 1 - T_2/T_1$. The upper bound is universal and independent of specific properties of heat engines.

We address the question of whether the Carnot efficiency is the unique fundamental bound for a quantum heat engine, a heat engine whose working substance is governed by quantum mechanics. Suppose that the temperature is so low that the thermal energy is comparable to or even less than the relevant energy scale. Then, quantum mechanical effects may show up and be reflected in the efficiency and its bound. Various quantum heat engine models have been studied to find the traces of quantum effects. On the one hand, some quantum heat engines behave similarly to classical engines as far as they are in contact with thermal reservoirs: the efficiency is bounded by the Carnot efficiency from above and the efficiency at the maximum power is given by the Curzon-Ahlborn efficiency. On the other hand, coherence and entanglement effects have been observed in quantum engines in contact with nonequilibrium reservoirs or with non-commutative operations.

In this paper, we investigate the quantum mechanical bound for the efficiency of the quantum Otto heat engine which uses a simple harmonic oscillator as a working substance. The quantum Otto heat engine has gathered more attention as it became realizable experimentally. The quantum mechanical state of the engine is described by the density matrix. We find that the quasi-probability distribution representation of the density matrix is useful. The equation of motion for the density matrix can be mapped to a classical Fokker-Planck equation for the quasi-probability distribution. By applying stochastic thermodynamics to the effective Fokker-Planck equation, we obtain the $h$-dependent quantum mechanical bound for the efficiency. Interestingly, the bound is tighter than the Carnot efficiency. Our work elucidates that the quantum mechanical effects could suppress the performance of heat engines in terms of efficiency bound, work and power output.

This paper is organized as follows. We introduce the quantum Otto heat engine model in Sec. III. The engine cycle consists of the adiabatic and isochoric processes. Dynamics of the density operator during the processes is described. In Sec. IV, we introduce the quasi-probability distribution and derive the equation of motion for it. The quasi-probability distribution satisfies the Fokker-Planck equation, to which one can apply the classical thermodynamics. In Sec. V, we derive the quantum mechanical bound for the engine efficiency by analyzing the Fokker-Planck equation system. Quantum mechanical effects on the heat engine are discussed in Sec. VI. We conclude the paper with summary and discussions in Sec. VII.

II. QUANTUM OTTO HEAT ENGINE

We consider a quantum Otto heat engine model with a simple harmonic oscillator as a working substance. The system Hamiltonian $\hat{H}(t)$ is given by

$$\hat{H}(t) = \hbar \omega(t) \hat{a}^\dagger \hat{a},$$

where $\hat{a}^\dagger$ and $\hat{a}$ are the creation and the annihilation operators satisfying $[\hat{a}, \hat{a}^\dagger] = 1$. The frequency parameter $\omega(t)$ is varied cyclically in time in a prescribed manner. The quantum mechanical state of the system is described by the density operator $\hat{\rho}(t)$. The same Hamiltonian was studied to find the optimal condition for the quantum heat engine operating in the Carnot cycle.
Isochoric processes are denoted as $I^1$. The Lindblad master equation is adopted to describe the thermal reservoir of temperature $T_\beta$.

The system is in contact with the thermal reservoirs during the isochoric processes. The wavy arrows indicate the direction of work and heat flows.

Our engine system operates in the Otto cycle consisting of adiabatic and isochoric processes as illustrated in Fig. 1. During the adiabatic process, the system is isolated from the heat reservoir and the frequency parameter is kept constant at $\omega_1$. During the isochoric process, the system is connected to the thermal reservoir of temperature $T_i$ while the frequency parameter is varied between $\omega_1$ and $\omega_2$. The density matrix is governed by the von Neumann equation (2)

$$\frac{d}{dt} \hat{\rho}(t) = \frac{-i}{\hbar} \left[ \hat{H}, \hat{\rho}(t) \right] + D_i(\hat{\rho}(t))$$

with the dissipator $D_i$ defined by

$$D_i(\hat{\rho}) = \gamma (\bar{n}_i + 1) \left[ \hat{a} \hat{\rho} \hat{a}^\dagger - \frac{1}{2} (\hat{a}^\dagger \hat{a} \hat{\rho} + \hat{\rho} \hat{a}^\dagger \hat{a}) \right] + \gamma \bar{n}_i \left[ \hat{a}^\dagger \hat{\rho} \hat{a} - \frac{1}{2} (\hat{a} \hat{\rho} \hat{a}^\dagger + \hat{a}^\dagger \hat{\rho} \hat{a}) \right].$$

Here, $\gamma$ is a damping rate and

$$\bar{n}_i = (e^{\beta_i \hbar \omega_1} - 1)^{-1}$$

is the Planck distribution at inverse temperature $\beta_i = 1/(k_B T_i)$. The Boltzmann constant $k_B$ will be set to unity.

The Lindblad equation has the thermal equilibrium state

$$\dot{\hat{\rho}}_{th} = (1 - e^{-\beta_i \hbar \omega}) e^{-\beta_i \hat{H}_i}$$

as its steady state solution.

It takes $t_m$ for each process $m = A_1, A_2, I_1$, and $I_2$ so that the total engine cycle time is $\tau = t_{A_1} + t_{A_2} + t_{I_1} + t_{I_2}$. Repeating the cycles, the system will reach the cyclic steady state. We find that the density matrix in the cyclic steady state is of the form

$$\dot{\hat{\rho}}(t) = \left(1 - e^{-c(t)}\right) e^{-c(t)\hat{a}^\dagger \hat{a}}$$

with a periodic function $c(t) = c(t + \tau)$. In this state, the expectation value of the number operator $\hat{N} = \hat{a}^\dagger \hat{a}$ is given by $N(t) \equiv \text{tr}\left\{\hat{N} \hat{\rho}(t)\right\} = (e^{c(t)} - 1)^{-1}$. Thus, the cyclic steady state is fully characterized by $N(t) = N(t + \tau)$.

The von Neumann equation (2) yields that $N(t) = N_i$ is a time-independent constant during the adiabatic process $A_i$. On the other hand, during the isochoric process $I_1$, the Lindblad equation (3) yields that

$$\frac{d}{dt} N(t) = -\gamma (N(t) - \bar{n}_i).$$

The solution $N(t) = \bar{n}_i + (N(t_0) - \bar{n}_i) e^{-\gamma (t - t_0)}$ provides a self-consistent equation for $N_1$ and $N_2$, which leads to

$$\Delta N = N_1 - N_2 = (\bar{n}_1 - \bar{n}_2) \frac{(1 - e^{-\gamma t_1})(1 - e^{-\gamma t_2})}{1 - e^{-\gamma (t_1 + t_2)}}.$$ 

Time-dependence of $N(t)$ over the engine cycle is plotted in Fig. 2.

The expectation value of the internal energy $E(t) = \text{tr}\left\{\hat{H}(t) \hat{\rho}(t)\right\}$ varies in time at the rate

$$\dot{\mathcal{E}} = \text{tr}\left\{ \left( \frac{d}{dt} \hat{H}(t) \right) \hat{\rho}(t) \right\} + \text{tr}\left\{ \hat{H}(t) \left( \frac{d}{dt} \hat{\rho}(t) \right) \right\},$$

where the former (latter) is designated as the rate for work (heat). During the isochoric processes, the Hamiltonian is time-independent and the system absorbs or dissipates the heat performing no work. During the adiabatic processes, the system performs the work without heat exchange. We will use the sign convention for the work and heat as specified in Eq. (10).

Since the internal energy is given by $\mathcal{E}(t) = \hbar \omega(t) N(t)$, the heat and the net work $W = W_1 + W_2$ for the single engine cycle are written in terms of $\Delta N$ as

$$Q_1 = \hbar \omega_1 \Delta N,$$

$$Q_2 = \hbar \omega_2 \Delta N,$$

$$W = \hbar (\omega_1 - \omega_2) \Delta N.$$ 

Note that $W = Q_1 - Q_2$, which corresponds to the first law of thermodynamics. The system acts as a heat engine.
When $Q_1 \geq 0$ and $W \geq 0$ or $(\Delta N) \geq 0$ and $w_1 \geq w_2$. Then, the efficiency is given by

$$\eta = \frac{W}{Q_1} = 1 - \frac{Q_2}{Q_1} = 1 - \frac{\omega_2}{\omega_1}. \quad (12)$$

The condition $\Delta N \geq 0$ requires that $\omega_1/T_1 \leq \omega_2/T_2$ (see [8]). Consequently, the engine efficiency cannot be larger than the Carnot efficiency

$$\eta_C = 1 - \frac{T_2}{T_1}. \quad (13)$$

The Carnot efficiency is also derived from the thermodynamic principle. Consider the von Neumann entropy

$$S_{\text{VN}}(t) = -\text{tr} \{ \hat{\rho}(t) \ln \hat{\rho}(t) \}. \quad (14)$$

Over the isochoric processes governed by the Lindblad equation [8], the system should satisfy the second law of thermodynamics [25]

$$(\Delta S_{\text{VN}})_{i_1} - \frac{Q_1}{T_1} \geq 0,$$

$$(\Delta S_{\text{VN}})_{i_2} + \frac{Q_2}{T_2} \geq 0. \quad (15)$$

On the other hand, the entropy is invariant $(\Delta S_{\text{VN}})_{A_1} = (\Delta S_{\text{VN}})_{A_2} = 0$ during the adiabatic processes [20]. The von Neumann entropy changes over the entire engine cycle add up to be zero. Consequently, (15) leads to the inequality

$$- \frac{Q_1}{T_1} + \frac{Q_2}{T_2} \geq 0 \quad (16)$$

and the Carnot bound.

We remark that the Carnot efficiency is the universal bound irrespective of system-dependent details. The same thermodynamic bound was also found in the previous studies of the quantum Otto heat engine [3, 8, 9]. It may suggest that the quantum mechanical nature does not impose an additional constraint on the efficiency. In the following section, however, we will discover another bound for the efficiency that is tighter than the Carnot efficiency.

III. QUASI-PROBABILITY DISTRIBUTION

The quasi-probability distribution allows a semiclassical description of a quantum mechanical system [24]. Recently, the quasi-probability distribution proved to be useful for the study of thermodynamics of open quantum systems [19, 26, 27]. We investigate the quantum Otto heat engine using the quasi-probability distribution.

The quasi-probability distributions can be defined by the Fourier transform of a joint moment generating function of $\hat{a}^\dagger$ and $\hat{a}$ [24]. Unlike the probability distributions for classical observables, the quasi-probability distributions do not have a unique representation due to the nonvanishing commutator of the operators. Most commonly studied are the P-representation $P(\alpha, \alpha^*)$, the Husimi Q-distribution $Q(\alpha, \alpha^*)$, and the Wigner function $W(\alpha, \alpha^*)$ [20]. In this paper, we present the results mainly from the Husimi Q-distribution or the Q-function. The results from the other distributions will be mentioned briefly.

Let $|\alpha\rangle$ and $\langle \alpha|$ be the coherent states satisfying $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ and $\langle \alpha|\hat{a}^\dagger = \alpha^*\langle \alpha|$ with a complex number $\alpha$ and its complex conjugate $\alpha^*$. The Q-function is defined as [20, 28]

$$Q(\alpha, \alpha^*, t) = \frac{1}{\pi} \langle \alpha|\hat{\rho}(t)|\alpha\rangle. \quad (17)$$

It allows one to evaluate moments of $\hat{a}$ and $\hat{a}^\dagger$ in the antinormal order conveniently as [20]

$$\langle \hat{a}^m\hat{a}^\dagger^n\rangle = \text{tr} \{ \hat{\rho}(t)\hat{a}^m\hat{a}^\dagger^n \} = \int d^2\alpha \: \alpha^m\alpha^{*n}Q(\alpha, \alpha^*, t).$$

All the expressions involving the density operator can be rewritten in terms of the Q-function. Mathematical tools for that purpose are found in the literature. Thus, we present the following relations without derivation. We refer readers to Ref. [20] for details. First of all, the von Neumann equation [2] becomes

$$\frac{\partial}{\partial t}Q(\alpha, \alpha^*, t) = i\omega(t) \frac{\partial}{\partial \alpha}Q - i\omega(t) \frac{\partial}{\partial \alpha^*}\alpha^*Q. \quad (18)$$

It has the solution

$$Q(\alpha, \alpha^*, t) = Q_0 \left( \alpha e^{i\Omega(t)}, \alpha^* e^{-i\Omega(t)} \right), \quad (19)$$

where $Q_0(\alpha, \alpha^*)$ is the initial distribution at time $t_0$ and $\Omega(t) \equiv \int_{t_0}^t \omega(t')dt'$. It can be easily derived from the definition $Q(t) = \frac{1}{\pi} \langle \alpha|\hat{\rho}(t)|\alpha\rangle$. Note that $\hat{\rho}(t) = \sum_{\alpha} |\alpha\rangle$$\langle \alpha| \hat{\rho}(t) |\alpha\rangle\langle \alpha|$.
\( \hat{U}(t) \hat{\rho}(t_0) \hat{U}^\dagger(t) \) with the unitary time evolution operator \( \hat{U}(t) = e^{-i \Omega(t) \hat{N}} \). The identity \( e^{-i \Omega \hat{N}} e^{i \Omega \hat{N}} = e^{i \hat{\alpha} \hat{\alpha}^\dagger} \) leads to \( \hat{U}^\dagger(t)|\alpha\rangle \propto |\alpha e^{i \Omega(t)}\rangle \). Thus, during the adiabatic process, the Q-function rotates in the complex \( \alpha \) plane by the angle \( \Omega(t) \) maintaining its shape.

The Lindblad equation [3] for the isochoric process is rewritten as

\[
\frac{\partial}{\partial t} Q(\alpha, \alpha^*, t) = -\left( \frac{\partial}{\partial \alpha} J + \frac{\partial}{\partial \alpha^*} J^* \right),
\]

where \( J \), which will be called the probability current, is given by

\[
J = \left( i \omega + \frac{\gamma}{2} \right) \alpha Q - D \frac{\partial}{\partial \alpha^*} Q
\]

(21)

and \( D = \gamma (\bar{n} + 1)/2 \), called the diffusion constant. Note that \( \omega = \omega_i + \bar{n} \) and \( \bar{n} = n_i \) for the adiabatic process \( t_i \) \((i = 1, 2)\). We remark that [20] also covers the adiabatic process when one sets \( \gamma = 0 \) and replaces \( \omega \) with the time-dependent \( \omega(t) \).

Thus, we can use the equation of motion [20] to describe both the adiabatic and isochoric processes. The other quasi distributions have the same equations of motion with their own diffusion constants. The P-representation has \( D = \gamma \bar{n}/2 \) and the Wigner function has \( D = \gamma (\bar{n} + 1)/2 \).

The thermal equilibrium state [4] is rewritten as

\[
Q_{\text{th}}(\alpha, \alpha^*) = \frac{1}{\pi(\bar{n} + 1)} e^{-\alpha \alpha^*/(\bar{n} + 1)},
\]

(22)

while the cyclic steady state solution [7] becomes

\[
Q(\alpha, \alpha^*, t) = \frac{1}{\pi(N(t) + 1)} e^{-\alpha \alpha^*/(N(t) + 1)}.
\]

(23)

They are obtained by using the identity \( e^{-\lambda \hat{a}^\dagger} = : e^{-1 - e^{-\lambda}} \hat{a}^\dagger : \), where \( : \hat{O} : \) represents the normal ordered form of an operator \( \hat{O} \).

The expectation value of the number operator is also rewritten in terms of the Q-function:

\[
N(t) = \text{tr} \{ \hat{a}^\dagger \hat{a} \hat{\rho} \} = \text{tr} \{ (\hat{a} \hat{a}^\dagger - 1) \hat{\rho} \} = \int d^2 \alpha \ \left[ (\alpha \alpha^* - 1) Q \right].
\]

(24)

The internal energy and the heat absorption rate are written similarly as

\[
E = \hbar \omega \int d^2 \alpha \ \left[ (\alpha \alpha^* - 1) Q \right]
\]

(25)

and

\[
\dot{Q} = \hbar \omega \int d^2 \alpha \ \left[ (\alpha \alpha^* - 1) \frac{\partial Q}{\partial t} \right]
\]

\[
= \gamma \hbar \omega \int d^2 \alpha \ \left[ (\bar{n} + 1) - \alpha \alpha^* \right] Q.
\]

(26)

The last equality is obtained by using [20].

\section{Fokker-Planck Equation and Thermodynamics}

The quasi-probability distribution \( Q(\alpha, \alpha^*) \) is a real-valued nonnegative and normalized function. Furthermore, for the harmonic oscillator system, the second-order partial differential equation for \( Q \) as shown in [20] has the same structure as the Fokker-Planck equation for a classical Markov system. We exploit the correspondence to map the quantum Otto heat engine to a classical thermodynamic system.

Consider first the isochoric process. We introduce a position-like variable \( x = (\alpha + \alpha^*)/2 \) and a momentum-like variable \( p = (\alpha - \alpha^*)/(2i) \). Then, the Lindblad equation [20] is rewritten as

\[
\frac{\partial}{\partial t} Q(x, p, t) = -\frac{1}{\gamma} \sum_{k=x, p} \left( \partial_k A_k - T \partial_k \phi \right) Q
\]

(27)

where \( \partial_k \) denotes the partial differentiation with respect to \( k = (x, p) \), the drift force \( A_k(x, p) \) is given by

\[
A_x = \left( \frac{-2 \hbar \omega}{-\omega \gamma - 2i \hbar \omega} \right) \left( x \right)
\]

(28)

and the parameters are given by

\[
\alpha = \frac{4 \hbar \omega}{\gamma}
\]

\[
T = (\bar{n} + 1) \hbar \omega = \frac{\hbar \omega}{1 - e^{-\beta \hbar \omega}}.
\]

(29)

This is equivalent to the Fokker-Planck equation for a Brownian particle in the two-dimensional phase space \((x, p)\) under the drift force \( A_k(x, p) \). The particle is immersed in the thermal reservoir characterized by the effective damping coefficient \( \alpha \) and the effective temperature \( T \). The drift force \( A_k \) are linear in \( x \) and \( p \). Such a linear system is called the Ornstein-Uhlenbeck process, whose properties are well documented in the literature [30, 32].

We are at liberty to assume that the momentum-like variable \( p \) is odd under the time reversal while \( x \) is even. Following Ref. [30], one can show that the dynamics satisfies the detailed balance. Thus, the Fokker-Planck equation describes an equilibrium system. The distribution function in [22] corresponds to the equilibrium Boltzmann distribution \( Q_{\text{th}}(x, p) = \frac{1}{Z} e^{-\beta V(x, p)} \), where

\[
V(x, p) = \hbar \omega (x^2 + p^2 - 1)
\]

(30)

is the energy function, \( \beta = 1/T \) is the effective inverse temperature, and \( Z \) is the partition function. Due to the choice \( T = (\bar{n} + 1) \hbar \omega \), we have the equivalence

\[
E(t) = \int dx dp V(x, p) Q(x, p, t)
\]

(31)

between the energy expectation value of the quantum system and the ensemble average of the energy function \( V(x, p) \) of the effective classical system.
The same Fokker-Planck equation with $\gamma = 0$ and $\omega = \omega(t)$ covers the adiabatic process. The system is detached from the heat reservoir and driven out of equilibrium with the time-dependent $\omega(t)$.

We are now ready to apply classical thermodynamics to the Fokker-Planck system. The second law of thermodynamics for the Fokker-Planck system states that [21]

$$\Delta S_{\text{tot}} = \Delta S_Q + \frac{-\Delta Q}{T} \geq 0,$$

(32)

where $\Delta S_Q$ is the change in the Shannon entropy

$$S_Q(t) = -\int d^2\alpha Q(\alpha, \alpha^*, t) \ln Q(\alpha, \alpha^*, t)$$

$$= -\int dx dp Q(x, p, t) \ln Q(x, p, t)$$

(33)

of the system and $-(\Delta Q)/T$ is the Clausius entropy change of the heat reservoir of temperature $T$ losing the heat $(\Delta Q)$. The Shannon entropy for the quasi-probability distribution is called the Wehrl entropy [33]. Due to the equivalence [31], the heat dissipations in the quantum and the classical systems are the same. On the other hand, the Wehrl entropy, in general, is different from the von Neumann entropy

$$S_{vN} = -\text{tr} \{ \hat{\rho} \ln \hat{\rho} \} = -\frac{1}{2} \int d^2\alpha \langle \alpha | \hat{\rho} \ln \hat{\rho} | \alpha \rangle$$

which involves $| \alpha \rangle | \alpha' \rangle$ with $\alpha' \neq \alpha$. Thus, the inequality in (32) for the effective system may provide an additional information that is unavaiable from the second law (16) for the quantum system.

Applying the second law of thermodynamics to the effective system, one obtains the following relations:

$$\Delta S_{\text{tot}} \Delta S_Q = Q_1 \geq 0,$$

(34)

$$\Delta S_{\text{tot}} \Delta S_Q = Q_2 \geq 0,$$

(35)

$$\Delta S_{\text{tot}} \Delta S_Q = 0 = 0.$$

During the adiabatic processes, the total entropy does not change since the shape of $Q(x, p)$ is invariant (see [19]) and there are no heat dissipations. Since the Wehrl entropy is a state function, the sum of the Wehrl entropy changes over the complete engine cycle adds up to zero. Therefore, we obtain

$$-\frac{Q_1}{T_1} + \frac{Q_2}{T_2} \geq 0.$$  

(36)

FIG. 3. The quantum mechanical bounds $\eta_q$ are plotted as a function of $q$ for dimensionless parameters $(r_T, r_\omega) = (0.1, 0.9)$ (red), $(0.2, 0.8)$ (blue), and $(0.3, 0.7)$ (cyan). The dashed line and the dashed-dotted line indicate the Carnot efficiency $\eta_C$ and the efficiency $\eta$, respectively.

V. QUANTUM MECHANICAL EFFECT

We discuss the implication of the quantum mechanical bound $\eta_q$. In order to quantify the quantum mechanical effect, we introduce a dimensionless parameter

$$q = \frac{\hbar \omega_1}{k_B T_1}.$$  

(37)

We also introduce positive dimensionless parameters $r_T = T_2/T_1$, $r_\omega = \omega_2/\omega_1$, and $r = r_\omega/r_T = \beta_2 \omega_2/(\beta_1 \omega_1)$. We only consider the region $r_T \leq 1$, $r_\omega \leq 1$, and $r \geq 1$ where the system acts as a heat engine. The quantum mechanical bound is then written as

$$\eta_q = 1 - r_\omega \left( \frac{1 - e^{-q}}{1 - e^{-r q}} \right).$$

(38)

The bound $\eta_q$ is a decreasing function of $q$ and equal to the Carnot efficiency at $q = 0$. Thus, we conclude that

$$\eta \leq \eta_q \leq \eta_C.$$  

(39)

The quantum mechanical bound is tighter than the Carnot efficiency. It reduces to the Carnot efficiency in the limiting case $q \to 0$ (classical limit) or $r \to 1$ (reversible limit). The $q$-dependence of $\eta_q$ is drawn in Fig. 3 for a couple of values of $(r_T, r_\omega)$.

The Carnot efficiency is realized ($\eta = \eta_C$) in the reversible limit $r \to 1$. On the other hand, the quantum mechanical bound is realized ($\eta = \eta_q$) in the $q \to \infty$ limit. Thus, the quantum mechanical bound $\eta_q$ is more useful than the Carnot efficiency $\eta_C$ as a fundamental bound for the efficiency.

It is also interesting to study a quantum mechanical effect on the power of the engine. From [9] and [11], the
The extracted work per engine cycle is given by
\[
W = W_{\text{max}} \frac{(1 - e^{-\gamma t_1})(1 - e^{-\gamma t_2})}{1 - e^{-\gamma (t_1 + t_2)}},
\]
(40)

where
\[
W_{\text{max}} = \hbar(\omega_1 - \omega_2)(\bar{n}_1 - \bar{n}_2)
= (1 - r_w)k_B T_1 \left( \frac{q}{e^q - 1} - \frac{q}{e^{q - 1} - 1} \right).
\]
(41)

As a function of the cycle times, it takes the maximum value \(W_{\text{max}}\) when \(t_1 = t_2 \to \infty\). After a little algebra, one can show that \(W_{\text{max}}\) is a decreasing function of \(q\) (see Fig. 4). It implies that the engine is most productive in the classical limit \(q \to 0\).

We also study the \(q\)-dependence of the power \(\mathcal{P} = W/\tau\) where \(\tau = t_1 + t_2 + t_{A_1} + t_{A_2}\) is the engine cycle time. The extracted work is independent of \(t_{A_1}\) and \(t_{A_2}\). Thus, for the optimal power, we will set \(t_{A_1} = t_{A_2} = 0\) and \(t_1 = t_2 = \tau/2\). Then, the extracted work per cycle and the average power are given by
\[
\mathcal{W} = W_{\text{max}} \frac{(1 - e^{-\gamma \tau/2})^2}{1 - e^{-\gamma \tau}},
\]
\[
\mathcal{P} = \mathcal{P}_{\text{max}} \frac{4(1 - e^{-\gamma \tau/2})^2}{\gamma \tau (1 - e^{-\gamma \tau})},
\]
(42)

with
\[
\mathcal{P}_{\text{max}} = \frac{\gamma W_{\text{max}}}{4}.
\]
(43)

The power decreases monotonically as \(\tau\) increases. It takes the maximum value \(\mathcal{P}_{\text{max}}\) in the \(\tau \to 0\) limit. Note that the maximum power is proportional to \(W_{\text{max}}\) Thus, the maximum power is a monotonically decreasing function of \(q\).

These results suggest that the quantum effect suppresses the power of the heat engine. We note that a quantum coherence effect is absent in the quantum Otto engine model considered in this work. The Lindblad dynamics during the isochoric process and the simple form of the time-dependent Hamiltonian satisfying \([\hat{H}(t), \hat{H}(t')] = 0\) during the adiabatic process do not generate a quantum coherence [32]. Thus, the quantum effect comes into play only through the discreetness of the energy level of the engine system. In the classical limit with \(q \ll 1\), the energy gap is smaller than the thermal energy so that the heat flows freely between the system and the reservoir. However, in the quantum regime with \(q \gg 1\), the discreetness of the energy gap obstructs the heat flow, which makes the heat engine less efficient. Recently, there was a report that the quantum coherence can enhance the power of the heat engine [33]. It would be interesting to investigate the effects of the discreetness of the energy gap and the quantum coherence simultaneously, which is beyond the scope of the current work.

VI. SUMMARY AND DISCUSSIONS

We have investigated the thermodynamic properties of the quantum Otto heat engine consisting of a harmonic oscillator. The quantum system can be mapped to a classical thermodynamic system with the help of the quasi-probability distribution. Applying the second law of thermodynamics to the effective classical system, we have obtained the quantum mechanical bound for the efficiency. The Q-function leads to the inequality that \(\eta \leq \eta_Q\) with the h-dependent quantum mechanical bound \(\eta_Q\). The equality holds in the low temperature limit where \(k_B T_i \ll \hbar \omega_i\). Surprisingly, \(\eta_Q \leq \eta_C\) so that the quantum mechanical bound provides a tighter bound than the Carnot efficiency.

We also investigated the work and power of our engine model. The work per engine cycle takes the maximum value in the limit where the time intervals of the isothermal processes tend to infinity. The maximum value decreases as \(q\) increases. Thus, the engine produces the maximal work in the classical limit \(q \to 0\). In contrast to the work, the power takes the maximum in the small cycle time limit \(\tau \to 0\).

One can consider the other quasi-probability distributions such as the P-representation and the Wigner function instead of the Q-function. These choices only modify the effective temperature \(T\). That is, \(T = \hbar \omega \bar{n}\) for the P-representation and \(\hbar \omega (\bar{n} + 1/2)\) for the Wigner function, while \(T = \hbar \omega (\bar{n} + 1)\) for the Q-function as shown in (29). They yield the additional bounds
\[
\eta_P = 1 - \frac{\omega_2 \bar{n}_2}{\omega_1 \bar{n}_1},
\]
\[
\eta_W = 1 - \frac{\omega_2 (\bar{n}_2 + 1/2)}{\omega_1 (n_1 + 1/2)},
\]
(44)

FIG. 4. The maximum value of the work \(W_{\text{max}}\) is plotted as a function of \(q\) for dimensionless parameters \((r_T, r_w) = (0.01, 0.1)\) (red), \((0.04, 0.2)\) (blue), and \((0.09, 0.3)\) (cyan).
They are compared in Fig. 5. Note that $\eta_P$ is larger than the Carnot efficiency and does not provide useful information. On the other hand, $\eta_W$ is smaller than the Carnot efficiency, but larger than $\eta_Q$. The Q-function provides the most useful bound for the efficiency. It may be interesting to find another quasi-probability distribution leading to a tighter bound.

The exact mapping to the classical thermodynamic systems described by the Fokker-Planck equation is possible only for the harmonic oscillator system. Nevertheless, we expect that the similar quantum mechanical bound may exist for other quantum heat engines. For example, our system reduces to a two-level system in the low temperature limit. Since our formalism is still valid in that limit, we expect that the efficiency of the quantum Otto heat engine with the two-level system would be bounded by the quantum mechanical bound. We leave the extension to other quantum heat engines for future studies.

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