Multi-valued vortex solutions to the Schrödinger equation and radiation

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Abstract

This paper addresses the single-valued requirement for quantum wave functions when they are analytically continued in the spatial coordinates. This is particularly relevant for de Broglie-Bohm, hydrodynamic, or stochastic models of quantum mechanics where the physical basis for single-valuedness has been questioned. It first constructs a large class of multi-valued wave functions based on knotted vortex filaments familiar in fluid mechanics, and then it argues that for free particles, these systems will likely radiate electromagnetic radiation if they are charged or have multipolar moments, and only if they are single-valued will they definitely be radiationless. Thus, it is proposed that electromagnetic radiation is possibly the mechanism that causes quantum wave functions to relax to states of single-valuedness, and that multi-valued states might possibly exist in nature for transient periods of time. If true, this would be a modification to the standard quantum mechanical formalism. A prediction is made that electrons in vector Aharonov-Bohm experiments should radiate energy at a rate dependent on the solenoid’s magnetic flux.

1 Introduction

The question whether the Schrödinger equation must always be single-valued occupied a number of the founders of early quantum theory, starting with Schrödinger himself [60], then notably Pauli [52], and subsequently a number of others [57, 47, 79]. The question received renewed interest after the Aharonov-Bohm effect was discovered [1]. The general consensus reached was that even in the Aharonov-Bohm systems the wave function must still be single-valued [53, 79, 47], but there were some dissenters [64]. The subject received a new burst of interest from the insights of Wallstrom regarding the non-obvious motivation for the single-valued constraint in de Broglie-Bohm, hydrodynamic, and stochastic formulations of quantum mechanics [69, 70, 71, 73, 72]. Goldstein also independently commented on multi-valued wave functions in stochastic mechanics [23]. Derakhshani has recently suggested an interesting explanation for the

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single-valued constraint based on zitterbewegung [17, 18]. Smolin has also con-
sidered this issue [65]. Here I take a different approach to this question which
is based on electromagnetic radiation.

The prototypical example of a multi-valued wave function is one of the form
\[ \psi(x) = f(x)e^{il_z\varphi}, \]
where \( f(x) \) is a single-valued function and where \( \varphi \) is the
azimuthal angle, and \( l_z \) is a real constant. The wave function is single-valued
only if \( l_z = \pm n, n \in \mathbb{Z} \). The single-valued constraint leads to quantization
effects. In this simple case, when \( l_z \) is not an integer, we can consider the
wavefunction to have multiple Riemann sheets that differ from one another by
constant phase factors of the form \( e^{il_z2\pi m} \) for \( m \) integer. The fact that the
different sheets differ by a constant phase factor which is independent of \( x \), is
important, and consequently I shall consider only multi-valued wave functions
which have this property here.

Firstly I introduce a broad class of suitable multi-valued wave functions. I do
this by borrowing results from the theory of vortex filaments in fluid mechanics.
The fluid version of the Biot-Savart law allows one to construct a local potential
function which is generated by a vortex filament taking an arbitrary stringlike
shape, either closed or open. This includes arbitrary knots and links, and this
function is generally multi-valued when analytically continued in \( x \). From these
known solutions, I construct multi-valued solutions of the Schrödinger equation.
For the usual azimuthal example, the string would be the whole \( z \) axis, and be
infinitely long. The analysis presented applies to such cases as well as to knots
and links. The subject of quantum vortices applied to the Schrödinger equation
is not new [4, 5, 44, 7]. Here we generalize these treatments to multi-valued wave
functions.

Then I show, by using a nonlinear identity of the Schrödinger equation,
that the multi-valued linear Schrödinger equation can be replaced by an equiv-
alent single-valued but nonlinear equation. I then argue from this that if the
particle is charged, then the nonlinear term will likely produce spontaneous
bremsstrahlung even for a free particle, so that the multi-valued solution may
not be stable to radiative decay. Consequently, I argue that the equilibrium
state of the system that is approached must be single-valued, and since this
could account for the observed fact of single-valuedness, at least for charged par-
ticles, that perhaps the single-valued condition of quantum mechanics might
not be universally true, and in particular it might be violated for short time
periods where, after a collapse of the wave function for example, a transient
multi-valued wave function is produced which then subsequently decays into a
single-valued one. Although this radiative relaxation to single-valuedness is eas-
est to understand for charged particles, it might be expected to hold for neutral
particles too if they have some multipolar electromagnetic moments, since these
too would radiate when accelerated. Neutrons or neutral atoms are examples.
So we might expect that multi-valued wave functions if ever created would typi-
cally be short-lived for them as well. The exceptions might be neutrinos, or dark
matter particles (if they exist), where the lack of any electromagnetic interaction
would allow multi-valued wave functions to persist for longer periods of time.
This transient multi-valuedness might yield experimentally detectable effects.
We discuss one such test below in the vector Aharonov-Bohm discussion. Also, these results might also prove to be useful in the mathematical theory of knots.

When restricted by the single-valued constraint, the vortex solutions here are similar to the quantum vortex solutions in superconductors, and superfluids. Here they are considered as solutions to the single particle Schrödinger equation with or without a potential. I’m not aware of the general knot solutions found here having been considered previously in the physics literature, although a number of special cases were elegantly analyzed in [5].

2 A multi-valued initial state based on a knotted vortex filament solution using the Biot-Savart law

It is well known that the single particle Schrödinger equation can be cast in the form of inviscid fluid mechanical equations by the Madelung transformation [45, 46]. The Euler equations take the following form in a conservative force field \( V \) with pressure \( p \) and density \( \rho \)

\[
\left( \frac{\partial}{\partial t} + u \cdot \nabla \right) u = -\frac{1}{\rho} \nabla p - \nabla V
\]  

and with the Madelung ansatz for the quantum force, this becomes

\[
\left( \frac{\partial}{\partial t} + u \cdot \nabla \right) u = -\nabla (Q + V)
\]

where

\[
Q = -\frac{\hbar^2}{2m} \frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}}
\]

and where \( \rho \) is a conserved density. Of course, these equations are identical to those in Bohmian mechanics for the single particle case [8]. For multiple particle states, especially when entangled, the Bohmian theory describes the system more easily than a hydrodynamic picture as it does away with a universal guiding fluid for all the particles, but if the presence of the particles in the fluid can affect the fluid currents that the other particles see, as in a theory in which the particles are actually solitons for example, then it’s still conceivable to construct a many-particle hydrodynamic model of quantum mechanics for this case too. These equations can also be considered as diffusion equations for a Brownian motion process as in stochastic mechanics [51]. In fact Wallstrom’s original interest in this subject arose from the equations of stochastic mechanics [69].

We first consider an incompressible fluid which satisfies \( \nabla \cdot u = 0 \) and is described by the knotted vortex filament solutions from 3D fluid mechanics [58, 39, 38]. Let \( u(x) \) denote such a solution, which is written in terms of the
“Biot-Savart” law applied to an inviscid incompressible fluid with a vorticity filament forming a knot, as in figure 1. The vortex filament is a thin tube inside of which the vorticity is non-zero, and pointing along the tangent to the tube. Outside of the filament the vorticity is zero.

A 3-space curve for the shape of a filament knot is given by a vector function \( \mathbf{R}_f(\sigma) \) where \( \sigma \) is arc-length along the curve which is assumed to be continuous and smooth, and the tangent vector \( d\mathbf{R}_f/d\sigma \) is in the direction given by the right hand rule applied to the circulation about the filament. For example, the simple trefoil knot illustrated in figure 1 is topologically equivalent to the following space curve, where \( \beta \) varies from 0 to \( 2\pi \):

\[
\begin{align*}
x &= \sin(\beta/2\pi) + 2 \sin(2 \star \beta/2\pi) \\
y &= \cos(\beta/2\pi) - 2 \cos(2 \star \beta/2\pi) \\
z &= -\sin(3 \star \beta/2\pi)
\end{align*}
\]

In this simple example formula, \( \beta \) is not equal to the arclength, but one could reparametrize it in principle to express the curve in terms of arclength \( \sigma \) if needed. It is straightforward to also include cases where the filament is not closed, but goes off to infinity in both directions, or where there are more than one knot superimposed with the same value of the vortex circulation constant forming a link. The velocity field of the fluid at an arbitrary point \( \mathbf{x} \) produced by a single closed filament is given by the “Biot-Savart” law with \( \Gamma \) being the circulation constant of the vortex as in section 2.3 of [58]:

\[
\mathbf{u}_f(\mathbf{x}) = \frac{\Gamma}{4\pi} \oint d\hat{\sigma} \times \frac{\mathbf{x} - \mathbf{R}_f(\sigma)}{|\mathbf{x} - \mathbf{R}_f(\sigma)|^3} = \frac{\Gamma}{4\pi} \nabla \times \oint \frac{d\hat{\sigma}}{|\mathbf{x} - \mathbf{R}_f(\sigma)|} \tag{5}
\]

If we have more than one filament, we just add their \( \mathbf{u}_f \) together to get the resultant. As \( \mathbf{u}_f(\mathbf{x}) \) is irrotational, it follows that in a simply connected domain \( \Omega_x \) which does not intersect the knot curve we have

\[
\mathbf{u}_f(\mathbf{x}) = \nabla \phi_f(x), \quad \mathbf{x} \in \Omega_x \tag{6}
\]

for some scalar function \( \phi_f(x) \). This domain can be expanded to include the whole space, but as it must exclude the filament, the resulting domain then
becomes multiply connected, and consequently \( \phi_f(x) \) will generally be multi-valued as \( x \) is analytically continued around the filament as shown in figure 1. Since \( \nabla \cdot u_f = 0 \) is required for an incompressible fluid, it follows that

\[
\Delta \phi_f(x) = 0
\]  

(7)

and it also follows from Stoke’s theorem that

\[
\oint_C u(x) \cdot dx = \Gamma W(C)
\]  

(8)

where \( W(C) \) is the winding or circulation number of the integration loop around the filament. It follows that although \( \phi_f(x) \) is multi-valued, the different values at the same \( x \) differ by additive constants which do not depend on \( x \). Consequently \( u(x) \), being a gradient of \( \phi_f \), is not changed by these constants, and is single-valued in this case. Now if we add to \( u \) a second solenoidal velocity field \( w(x) \) which is derived from a single-valued potential, \( w(x) = \nabla \phi_w(x) \) which is valid for all \( x \), then since \( \oint_C w(x(s)) \cdot ds = 0 \) for any closed curve \( C \), we must then have that

\[
\oint_C (u_f(x(s)) + w(x(s))) \cdot dx = \Gamma W(C)
\]  

(9)

The combined velocity in this case no longer divergence free, and it can describe a compressible fluid as is required for the initial velocity field of a Schrödinger equation in the Madelung construction. The vorticity filament will move advectively with the fluid as time progresses, as described by the Kelvin circulation theorem.

The task of finding the potential function \( \phi_f(x) \) is identical to two different problems in electromagnetism. The first and most common is the magnetic field \( B \) generated by a charge current \( I \) flowing along a knot, and expressed as the gradient of a magnetic scalar potential. The second is the problem of calculating the vector potential \( A \) which is generated by a magnetic flux filament pointing along a knot curve. The magnetic scalar potential is discussed in many sources, for example [36, 22, 54, 55, 28, 40]. A classical result for a vortex loop which is unknotted, ie. topologically equivalent to a circle, is (equation 2.5 in [58])

\[
\phi_f(x) = -\frac{\Gamma}{4\pi} \Omega(x)
\]  

(10)

where \( \Omega(x) \) is the solid angle subtended by the vortex loop when viewed from the point \( x \). In integral form this is

\[
\Omega(x) = \int_S \frac{(x - R) \cdot dS}{|x - R|^3}
\]  

(11)

where \( R \) is a point on the surface, and the surface \( S \) is bounded by the loop. This result, originally due to Maxwell [6], can be applied to knotted vortex loops too by using a Seifert surface for the knot [61, 75]. These are compact,
connected, and oriented surfaces with the knot as its boundary. A beautiful tool for visualizing them is a software program called SeifertView, created by Professor Jarke van Wick, Technische Universiteit Eindhoven. An example of a Seifert surface is shown in figure 2.

The Seifert surface acts as a cut analogous to the Riemann sheet cut familiar from complex analysis. If we analytically continue \( \phi_f(x) \) along some curve, but avoid ever passing through the Seifert surface, it will remain single-valued, but the difference between its value on one side versus the opposite side of the surface will be non-zero, and independent of where on the surface the discontinuity is calculated. When dealing with links we can superimpose the potential functions for the individual knots that make up the link because of the linearity of Laplace’s equation. The vorticity constant for the knots making up a link need not be the same in general, but for our purposes here, we shall assume that they are the same so that the discontinuity of \( \phi_f(x) \) across the surface will always be the same, up to a sign. Links have Seifert surfaces too, and although they are not unique, I presume that one always exists which can act as a suitable cut surface.

3 The Schrödinger equation in the de Broglie-Bohm-Madelung pilot wave formalism

We consider the single-particle Schrödinger equation

\[
-\frac{\hbar^2}{2m} \Delta + V \psi = i\hbar \frac{\partial \psi}{\partial t} \tag{12}
\]

and we write

\[
\psi(x, t) = R(x, t)e^{iS(x,t)/\hbar} \tag{13}
\]

where both \( R(x) \) and \( S(x) \) are real functions. The guidance equation is given by

\[
\frac{dX(t)}{dt} = \frac{1}{m} \nabla S(X(t)) \tag{14}
\]
so we can equate the fluid velocity in a hydrodynamic picture with this. We wish to incorporate the potential from the vortex solution into the initial state of a wave function. We assume at an initial time $t = 0$ that we have

$$S(x, 0) = \phi_f(x) + \phi_w(x)$$

(15)

where $\phi_f$ is the knot potential calculated above, and $\exp(i m \phi_w(x)/\hbar)$ is single-valued, but otherwise arbitrary. Then the initial value for the wave function is

$$\psi(x, 0) = R(x, 0) e^{i m (\phi_f(x) + \phi_w(x))/\hbar}$$

(16)

where here $R(x, 0)$ is an arbitrary positive and single-valued function of $x$. Let us define a mapping by continuing $x$ around the filament with a winding number of $N_w$ as

$$\psi(x, 0, N_w) = \psi(x, 0) e^{i m \Gamma N_w/\hbar}$$

(17)

and we see that this is in general multi-valued due to factor $e^{i m \Gamma N_w/\hbar}$. Because the Schrödinger equation is linear, this factorization will be preserved in time, so that

$$\psi(x, t, N_w) = \psi(x, t) e^{i m \Gamma N_w/\hbar}$$

(18)

where it is assumed here that the analytic continuation curve is advectively adjusted to account for motion of the filament in time so that the winding number doesn’t change. And so if the wave function is multi-valued initially, it will remain so for all time. The vortex filament will change its shape and undulate in time as we integrate the equations forward or backward in time, but because the equations are equivalent to an Eulerian fluid, the Kelvin circulation theorem will remain in effect, and the topological knot and link structure of the filament or filaments will remain invariant. In order for the wave function to be single-valued we must require

$$e^{i m \Gamma/\hbar} = 1$$

(19)

or equivalently

$$m \Gamma/\hbar = 2 \pi N, N \in \mathbb{Z}$$

(20)

The question is then why should this quantization condition be true?

4 Requirement that the filament must be a nodal curve for the wave function

It is typically assumed that if a Schrödinger wave function has a vortex filament, that it must be a nodal filament, so that the wave function vanishes along it. The reason for this is continuity of the wave function. In a neighborhood of a point on the vortex filament, the wave function’s phase takes on multiple values. If it does not vanish on the filament, then the different phases in a
neighborhood would lead to a discontinuity. Since we typically assume that the quantum mechanical wave function must be continuous, this then requires that the complex wave function vanish along the vortex. See for example [32]. If we want to create such a state, then we must require that the initial value for $\psi$ must vanish on the filament curve so that it will be continuous there

$$\psi(R_f(\sigma), 0) = 0, \ \sigma \in [0, l]$$

(21)

where $l$ is the length of the filament knot. This requires that

$$R(x, 0)|_{x=R_f(\sigma)} = 0$$

(22)

so we must find a single-valued function which vanishes on the filament curve. Such a function is easy to create from positive definite integrals of the following type

$$I_n^f(x) = \int_0^l \frac{d\sigma}{|x - R_f(\sigma)|^n}, \ n \ a \ positive \ integer$$

(23)

then, letting $D_f$ denote the set of points on the filament, we have

$$\frac{1}{I_n^f(x)|_{x \in D_f}} = 0, \ if \ n \geq 1$$

(24)

then we can create a wave function by writing

$$\psi(x, 0) = \Phi(x) \frac{e^{im\phi_f(x)/\hbar}}{I_n^f(x)}$$

(25)

where $\Phi(x)$ is an arbitrary smooth single-valued complex function which falls off at infinity fast enough to make $\psi(x, 0)$ normalizable. Let us also assume that $\Phi(x)$ has no nodes so that even if raised to a non-integer power, it remains single-valued. In order that $\psi(x, 0)$ be nodal along the filament curve, we must require the $n = 2$ or greater. With these conditions met, we have achieved a wave function which has the desired knotted vorticity, along with the correct nodal property, and solving the Schrödinger equation forward in time should preserve these properties. Actually doing this time evolution requires numerical techniques in general. If we want to study analytic solutions for vortex loops, the elegant methods presented in [5] can be considered. These analytic methods are not as general as the methods presented here based on the Biot-Savart law though, and they have only been studied for the single-valued case.

5 Comparison with Dirac strings

When the quantization condition that ensures single-valuedness (19) is satisfied, the wave function with a vortex instantaneously looks similar to the wave function for a charged particle in the presence of a Dirac String [20, 19, 74, 29], except that there is no monopole at the end of our vortex string as it is either
a closed curve, or one that goes off to infinity in both directions. A reasonable question to ask in this case is whether our vortex curves are essentially Dirac strings tied in a knot. To answer this question, we must consider some complications in the analysis of the Dirac string.

The vector potential outside of Dirac string can be viewed as due to a magnetic flux inside the string, or equivalently as a chain of magnetic dipoles. But the magnetic flux inside the string is then mathematically subtracted out [19, 74] by Dirac. He developed an action principle for this situation which required for consistency that a charged particle never pass through the string. In the case of a wave function in the vicinity of such a string, this would require that the wave function vanish along the string. This is called the “Dirac veto”, and is precisely the nodal behavior that we have given to our multi-valued vortex solutions in order to make the wave function be continuous. Therefore I believe that the vortex solutions here are essentially the same as Dirac strings when they satisfy the Dirac veto and when the single-valued quantization condition is satisfied by our vortex.

The theory of magnetic monopoles was modified by Wu and Yang [77] in such a way as to eliminate the Dirac string altogether, and thus avoid the Dirac veto which was considered a problem. The physical effect and description of a monopole should depend only on the position of the monopole, and not on the shape of the Dirac string attached to it. A modified Dirac string theory based on the Wu-Yang paper, which avoided the Dirac veto was also introduced by Brandt and Primack [9]. The Wu-Yang approaches provide a better model of the magnetic monopole than Dirac’s original version. These theories are clearly physically different from Dirac’s original theory because of the Dirac veto. Although the Wu-Yang theory is a better description of a monopole, the original Dirac string theory, with the Dirac veto, is very similar if not the same as our vortex model.

In this paper we go beyond the Dirac string theory, because we consider multi-valued wave functions which were not considered by Dirac. Moreover, although our vortex knots look just like Dirac strings when they are single-valued, they can exist without any monopoles attached to them, and they complement the seminal analysis of [4, 5, 44, 7] of such systems.

6 An identity for the Schrödinger equation

The following two equations are equivalent at points of analyticity for the real functions $R$ and $S$

\[
\left[-\frac{\hbar^2}{2m}\Delta + V\right]Re^{iS/\hbar} = i\hbar \frac{\partial}{\partial t} \left(Re^{iS/\hbar}\right)
\]

(26)

is equivalent to
\[
\left[ \frac{(\nu \hbar)^2}{2m} \triangle + \left( V + \frac{\hbar^2}{2m} (\nu^2 - 1) \frac{\triangle R}{R} \right) \right] \text{Re}^{iS/(\nu \hbar)} = i (\nu \hbar) \frac{\partial}{\partial t} \left( \text{Re}^{iS/(\nu \hbar)} \right)
\]

where \(\nu\) is an arbitrary complex-valued constant. An elementary proof of this is given in [13] where it was used as the basis for generalizing stochastic mechanics to arbitrary values of the diffusion constant. In another paper it was used as a symmetry for Brownian motion [12]. This identity is true in any number of dimensions, and therefore it can be used with multiparticle wave functions too, with suitable rescaling of the particle coordinates to compensate for different masses in the equation. The generalized model allows one to consider stochastic mechanics, Bohmian mechanics, and Heisenberg operator quantum mechanics as all part of a covering diffusion theory [62, 11]. In this paper we find a new application for this identity. Suppose that \(S\) is of the form of the vortex filament solution. Let us define

\[
\psi_{\nu}(x, t) = R(x, t) e^{iS(x, t)/(\nu \hbar)}
\]

and for an analytically continued version of this function

\[
\psi_{\nu}(x, t, N_w) = R(x, t) e^{iS(x, t)/(\nu \hbar)} e^{i m \Gamma N_w/(\nu \hbar)}
\]

we see that for certain special values of \(\nu\), \(\psi_{\nu}(x, t, N_w)\) becomes single-valued. The condition for this to happen is

\[
e^{im \Gamma N_w/(\nu \hbar)} = 1, \forall N_w
\]

and so, if \(\Gamma \neq 0\), then either \(\nu = \infty\), or

\[
m \Gamma / (\nu \hbar) = 2\pi M, \ M \in \mathbb{Z}
\]

\[
\nu = \frac{m \Gamma}{2\pi M \hbar}
\]

In order to obtain a single-valued and linear equation, we must have \(\nu = 1\) which requires \(\Gamma = 2\pi M \hbar / m\). So single-valued knotted solutions of this type can exist in standard quantum mechanics, in the absence of any external electromagnetic fields, provided this condition is satisfied. Note that if the vorticity constant \(\Gamma\) is zero, then \(\psi_{\nu}(x, t)\) is single valued for all values of \(\nu\).

When these knotted filaments are such as to produce linear as well as single-valued wave function, then the filaments are called quantum vortices in the physics literature. In this case, if the wave function is continuous at the vortex, then the filament is a nodal curve for the Schrödinger wave function as required [32]. In the multi-valued case, I assume that this nodal property is still true.
For a free particle, the nonlinear term may cause radiation and relaxation to a linear and single-valued state. But, when there is an attractive potential present, the situation is more complicated. There might be non-linear bound states which don’t radiate.

Notice that $\nu$ is not uniquely defined by (32) since $M$ can be any integer. Without loss of generality we can restrict consideration to positive values of $M$. The value $M = 1$ gives the largest value of $\nu$. As we shall see below, the electromagnetic radiation from this particle grows with increasing $\nu^2$. Therefore the case $M = 1$ also gives the maximum rate of radiation.

7 Bremsstrahlung for charged particles

For a classical charged particle undergoing acceleration, the instantaneous radiated power is given by Larmor’s formula

$$P_{\text{Rad–Cl}} = \frac{2}{3} \frac{q^2 c^3 a^2}{\alpha}$$  \hspace{1cm} (33)

For a quantum particle, with a wave function $\psi$ which passes through a force field producing radiation, there are a number of calculations of Larmor’s formula for scalar particles [30, 31, 78, 35]. These calculations are based on rigorous second-quantized scalar electrodynamics. Since we are considering a non-plane wave wave packet here, I find it more suitable to consider a result that I obtained in [15] which is very simple. It applies to the situation where a wave-packet moves through a localized force field due to a potential $U$ that causes radiation. This result was the lowest order approximation in $\alpha$. The result is, for instantaneous power radiated by a non-relativistic wave packet simply, and to a first approximation, given by the following quantum-Larmor formula:

$$P_{\text{Rad–QED}}(t) = \frac{2}{3} \frac{q^2 c^3}{\alpha} \left| \langle \psi(t) | \hat{a}^2 | \psi(t) \rangle \right|$$

$$= \frac{2}{3} \frac{q^2 c^3}{\alpha} \int d^3 x \rho(x, t) (\nabla U/m)^2$$  \hspace{1cm} (34)

My derivation of this formula in [15] leaned heavily on the radiation treatment by Schiff [59], which gives the most detailed description of the effects of the wave function’s form on radiation phenomenon, and not just plane wave analysis. It is plausible then that this expression can give a reasonable approximation to the radiation from the nonlinear equation we found in achieving single-valuedness [27]. I think is worthwhile to see if it can be reconciled with the methods of [30, 31, 78, 35], but I will not attempt that here. Higher order corrections to this formula will not change the basic feature that is important here, and that is that the nonlinear system will not be radiation free, but will lose energy by radiating it away. I also found that a charged Bose-Einstein condensate beam radiates as a combination of two simple terms [14].
\[ P_{\text{Bose–Einstein}}(t) = \]
\[ N^2 \frac{2}{3} \frac{q^2}{c^3} (\psi(t) | \hat{a} | \psi(t))^2 + N^2 \frac{2}{3} \frac{q^2}{c^3} (\psi(t) | \hat{a}^2 | \psi(t)) \]

where \( N \) here is the mean number of particles in the condensate. The first term is a kind of coherent Larmor radiation term which calculates the radiation due to a classical charge current which is proportional to the Schrödinger probability current. In the present situation, due to (40) below, it follows that \( \langle \psi(t) | \hat{a} | \psi(t) \rangle = 0 \) for the acceleration caused by the nonlinear force term in (27). This is consistent with the fact that in general for a single-valued free-particle Schrödinger equation, the radiation that would be generated from the current density treated as a classical source is always zero [16]. It’s not clear though, that for the multi-valued case this still holds for all higher multipole moment terms in the radiation expansion. In any event a single particle state radiates approximately according to (34).

Now let’s apply the quantum-Larmor formula (34) to our knotted Schrödinger solutions. Let us rewrite (27) as

\[
\left[ -\frac{\hbar^2}{2(m/\nu)} \nabla + \frac{1}{\nu} \left( V + \frac{\hbar^2}{2m} (\nu^2 - 1) \left( \frac{\Delta R}{R} \right) \right) \right] \text{Re}^{iS/\nu\hbar} = i\hbar \frac{\partial}{\partial t} \left( \text{Re}^{iS/\nu\hbar} \right)
\]

We can define an effective mass

\[ m_{\text{eff}}(\nu) = m/\nu \]

Consider a free particle case, so we set the potential \( V \) to zero in (27). We really don’t know how to calculate the radiation from a multi-valued wave function. The best we can hope for is that the usual radiation formula’s can be applied only when the wave function is single-valued. This is plausible, but definitely not rigorously derivable from any complete theory. If experimental evidence supports this approach, then we can test the idea further and zero in on a better theory if needed. Recall the Bohmian quantum mechanical potential

\[ Q_B = -\frac{\hbar^2}{2m} \frac{\Delta R}{R} \]

In terms of this, the nonlinear equation is

\[
\left[ -\frac{\hbar^2}{2m_{\text{eff}}(\nu)} \nabla - \frac{1}{\nu} (\nu^2 - 1) Q_B \right] \text{Re}^{iS/\nu\hbar} = i\hbar \frac{\partial}{\partial t} \left( \text{Re}^{iS/\nu\hbar} \right)
\]
Incidentally, it is easy to show that

$$\int \rho \nabla Q_B d^3x = -\hbar^2/2m \int R^2 \nabla \left( \frac{\Delta R}{R} \right) d^3x = 0 \tag{40}$$

If we assume that the quantum Larmor formula (34) is still valid, considering this a plausible but not rigorously derivable hypothesis, the power radiated would then be

$$P(t) = \frac{2}{3} \frac{q^2}{c^3} \left( \nu - \frac{1}{\nu} \right)^2 \int \left( \frac{\nabla Q_B(x,t)}{m_{eff}(\nu)} \right)^2 R(x,t)^2 d^3x \tag{41}$$

or

$$P(t) = \frac{2}{3} \frac{q^2}{c^3} (\nu^2 - 1)^2 \int \left( \frac{\nabla Q_B(x,t)}{m} \right)^2 \rho(x,t) d^3x \tag{42}$$

This is zero if $\nu^2 = 1$, but otherwise it’s positive. Now we can see a problem since $\nu$ is not unique owing to (32). So the question is, what determines which value of $\nu$ to use, and what would happen to the wave function of the particle as it lost energy due to radiation? It’s difficult to say without a full theory for radiative effects in this situation. If, in such a complete theory, the vorticity $\Gamma$ or possibly the mass $m$ could change, then it could change in such a way that $\nu$ approaches 1 asymptotically. Considering (32) this implies (assuming $M = 1$ in (32))

$$\nu \Rightarrow 1 \text{ implies that } \Gamma m \Rightarrow 2\pi \hbar \tag{43}$$

So, in this case the multi-valued solution would transiently radiate and might approach an equilibrium state which is both single-valued and linear. This could give a radiative explanation for the question posed by Wallstrom [69] at least for the free particle state. Alternatively, it’s possible that the state could just keep radiating energy away until it was so spread out in position that the radiation rate slowly approached zero as the wave function became more and more spread out.

Let me make a mathematical argument based on continuity in favor of the choice $M = 1$ in (32). Suppose that $\Delta \Gamma/2\pi \hbar \approx 1$. So the multi-valued wave function is only slightly multi-valued. It’s reasonable then to expect, in this case, that $\nu$ should be also close to 1, and this implies that $M = 1$ in (32). Granted this is not a compelling physical argument, but accepting it allows us to continue and explore if there is any experimental evidence for this phenomenon, and if there is, then a more satisfactory resolution of this non-uniqueness problem might be found.

For the case where the particle is not free, but subject to a binding potential $V$, then it seems plausible that some new unexpected nonlinear solutions to (27) might exist that do not radiate, and so these could be stable, and perhaps a new form of quantum state. In this case, different values of $M$ might be interesting.
to consider. One might also imagine that the vortex knots could have some relevance for string theory.

So to sum it up, we have presented a plausibility argument that multi-valued wave functions for free particles might well radiate, and these could either reach a single-valued equilibrium, or else just get more and more spread out over time, faster than free particle quantum theory would predict, due to radiated energy loss. In the next section we consider a system that might yield experimental confirmation of this effect, and if so, then it would open up a new field of research into a more complete understanding of this phenomenon.

8 Linear superposition

It we have two or more multi-valued solutions to Schrödinger equation, with different values of the vorticity constant, then the suitably normalized superposition of them is also a solution. However, the identity (27) we used for a single vorticity constant won’t work anymore to produce a single-valued wave function in this case. Let \( \psi_j \) be a solution to (12) with a vorticity constant \( \Gamma_j \). Let there be \( M \) such functions, and consider the normalized superposition of them

\[
\psi = \frac{1}{N} \sum_{j=1}^{N} \psi_j
\]

where \( N \) is a normalization factor. So \( \psi \) will also be a solution to (12). Now each \( \psi_j \) has a value of \( \nu_j = \frac{m \Gamma_j}{2 \pi \hbar} \) from our nonlinear identity which makes a single-valued transformed wave function. Writing \( \psi_j \) as \( \psi_j = R_j e^{iS_j/\hbar} \), we obtain a nonlinear equation for each term in the sum

\[
\left[ -\frac{h^2}{2 \langle m/\nu_j \rangle} \Delta + \frac{1}{\nu_j} \left( V + \frac{h^2}{2m} (\nu_j^2 - 1) \frac{\Delta R_j}{R_j} \right) \right] \frac{R_j e^{iS_j/\hbar}}{\nu_j} = i\hbar \frac{\partial}{\partial t} \left( \frac{R_j e^{iS_j/\hbar}}{\nu_j} \right)
\]

(45)

Here, each function \( R_j e^{iS_j/\hbar} \) is single-valued. So the linear Schrödinger equation for \( \psi \) is equivalent to a set of decoupled non-linear equations for \( \psi_j \) whose solutions are single-valued. If the various \( \psi_j \) are non-overlapping, then we could argue that the radiation emitted by them would be approximately just the quantum Larmor formula (34) for each one of them summed. But when the \( \psi_j \) have overlapping support, then it’s not obvious how to estimate the amount of radiation. However, we can see that for a free particle, the only situation which is clearly free of radiation is if \( \nu_j = 1 \) for all \( j \). Thus, even in the case of a superposition, it is quite plausible that radiation will occur and equilibrium will be reached only when the superposition \( \psi \) becomes single-valued. If experimental evidence can be found for this radiation, then it will provide clues on how to generalize the standard quantum theory to describe this radiation exactly without resorting to plausibility arguments.
9 Some comments on the Aharonov-Bohm effect, and predicted energy loss due to radiation

Consider an ideal cylindrical solenoid whose centerline is the z axis with a static magnetic flux inside it. Let it be infinitely long in both directions. Next consider a Schrödinger wave packet of electrons passing around this solenoid, and assume that the wave function vanishes at the solenoid’s surface. The space is no longer simply connected, and the phase function is therefore not automatically single-valued as we analytically continue the wave function around the solenoid. This is the vector Aharonov-Bohm system. We can still use the identity \(27\) for this case too, and so we can transform such a wave function into one that is single-valued but nonlinear. We again expect radiation from this system, as in \(44\). A toroidal solenoid can also produce the effect as in the very clean electron microscope experiments in \(68\). In these experiments, the toroidal solenoid has a superconducting cladding which requires that the magnetic flux be quantized in increments of \(2\pi \hbar c/2e\). This results in a phase shift of either 0 or \(\pi\), depending on the flux in the solenoid. When the phase shift is \(\pi\), the wave function analytically continued around the solenoid is double-valued. As the electron beam passes around the solenoid in this case, it should lose some energy due to radiation if the theory presented here is correct. In this case, we expect the kinetic energy of the particle to drop as a result of radiation. So the exiting electron energy for a \(\pi\) phaseshift should be slightly lower than for a zero phaseshift. I’m sure it would be difficult to measure this energy drop, but perhaps it’s not impossible. Probably a better way to look for the radiation would be to detect it directly with a lens focused on the toroidal solenoid, and imaging onto a sensitive electromagnetic radiation detector. The \(\pi\) phaseshift case should see radiation, but the zero phaseshift should not. The signal here would be proportional to the beam current, which would make it easier to detect at higher currents. This seems to me to be quite a doable experiment. There would undoubtedly be other radiation due to the focusing lenses of the electron optics in the electron microscope and also due to beam electrons entering the material of the solenoid, and so a difference between the two states of the solenoid having zero and \(\pi\) phaseshift would have to be measured. Hopefully there would be enough of a signal to show a difference.

The understanding of the time-dependent Aharonov-Bohm effect is incomplete at this time \(3, 10, 37, 63\). Introducing the possibility of transient multi-valued wave functions into the mix of possibilities surrounding this topic might enrich it, but it also would certainly complicate it. We might consider what would happen to the wave function if the magnetic field inside the solenoid varied with time. It seems that this might lead to transient multi-valued wave functions which could then radiate energy according to our formulas. Time-dependent cases require that a vector potential \(A\) be included. The generalized form of the nonlinear identity including a vector potential is presented in the Appendix.
10 Some comments on dissipation mechanisms for achieving single-valued equilibrium

It’s not clear how to develop a theory that would take into account the energy loss due to radiation, and the influence that this would have on the wave function. We might try and invoke a perturbation expansion along the lines of conventional theory. It’s not obvious that the vortex would be preserved. If it is, and if a single-valued limit is to be reached, then (43) must be satisfied. This means that either the mass or the vorticity of the wave function must change. Particle masses are usually taken as constants in non-relativistic quantum mechanics, although there are variable mass modifications which I find interesting to consider [24, 25, 26, 27, 21, 66, 67, 33, 34, 41]. This would represent another modification of standard quantum mechanics which would allow transient states of multi-valued wave functions to exist transiently off the mass shell, and they could stabilize to single-valued states over time due to radiation with the mass returning to its normal value. This phenomenon could perhaps be relevant to the interesting radiative decay to the mass shell value for a classical charged particle in a Stueckelberg type of off mass shell theory as described in [2]. A more prosaic possibility would be if the mass stayed constant and the expected value of the energy of the wave function decreased until the multi-valued vortex either dissipated away, or became single-valued by a change in the circulation constant $\Gamma$. Of course it’s also possible that both the mass and the vorticity constant could change.

11 Topological considerations

If we ignore radiative energy loss, then the free particle Shrödinger equation is equivalent to an inviscid compressible Eulerian fluid described by the Madelung theory. This would be the case if the vortex is resulting in a single-valued linear wave function, so that it definitely doesn’t radiate. Such fluids with vortex knots and links have been the subject of much research, and besides the circulation and the helicity, there are other topological invariants. For example, Liu and Ricca have studied the Jones polynomial as a dynamical invariant of an inviscid fluid [42, 43, 56], generalizing earlier results by Moffatt et al. [48, 49, 50]. The Jones polynomial has been a subject of much interest in quantum field theory due to the famous paper by Witten [76]. In the present circumstance we have another quantum system, namely the single particle Shrödinger equation, which can have knotted and linked vortex tubes whose topology can be associated with an invariant Jones polynomial. The connection with conformal quantum field theory was made made explicit in [42]. These various results are directly applicable to our present theory. This seems like a fertile area therefore for exploration. In the case of quantum vortices, it is known that circular ring vortex loops (or unknots) can shrink to zero and disappear [6] and the time reverse. I don’t know if this same phenomenon can occur with other knots, like the trefoil knot for example.
12 Discussion

A large class of multi-valued Schrödinger wave functions have been proposed and considered here, and it has been shown, using a nonlinear identity of Schrödinger’s equation, that they can be transformed into solutions of a single-valued but nonlinear differential equation. These were based on knotted vortex filament fluid methods. The Biot-Savart law commonly used in vortex analysis was supplemented by an additional factor along the filament to create wave functions which had the necessary nodal properties required by continuity in quantum mechanics along such a filamentary vortex curve. This satisfies the requirement that the wave function vanish along the vortex filament curve. Considering Larmor type formulas for bremsstrahlung suggests that even free charged particles with these wave functions would radiate due to the nonlinearity of the single-valued transformed equation in this case. Only when the wave function is single-valued, and also simultaneously satisfies a linear Schrödinger equation will it be clearly radiation free in the free particle case. Since radiative emission can be expected to change the wave function, it is conceivable that after a transient period the wave function will achieve an equilibrium state in which it is both single-valued and linear, so that the multi-valuedness may only be transient. This would be expected to apply also for neutral particles which had a magnetic moment, or other electromagnetic multipole moments, like say a neutron. This fact could be the physical origin of the single-valued constraint that is imposed in standard quantum mechanics. This effect would not be expected to operate for neutrinos however, and so perhaps their wave functions are not necessarily single-valued. If a set of non-interacting charged particles achieved a single-valued quantum state before coming together to interact through Coulomb potentials, then the single-valuedness would be preserved for all time because of the properties of the Schrödinger equation. The general problem of multi-valued solutions in an attractive potential is more complicated, and there may be stationary or oscillatory states, and it’s possible that some of these could perhaps be stable to radiative decay.

The theory presented here can also be applied to the vector Aharonov-Bohm system, and in particular for time dependent solenoids this might be interesting. The theory makes a simple prediction that electrons would radiate electromagnetic energy which depends on the magnetic flux in the solenoid in such systems.

These results may offer new methods of analysis for pure mathematical knot theory as well. The time-evolution of these Schrödinger knots could be studied in numerical simulation for example, with or without a potential function. Scattering of vortex knots could also be considered. The connection with the Jones polynomials is another subject for exploration.

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Appendix - A generalization of the nonlinear Schrödinger identity to include electromagnetic potentials

The following two equations are equivalent at points of analyticity for the real functions $R$ and $S$ provided that the vector potential $\mathbf{A}$ is expressed in the transverse gauge so that $\nabla \cdot \mathbf{A} = 0$

$$\left[ \frac{1}{2} (-i \nabla + q \mathbf{A})^2 + V \right] Re iS = \frac{i}{\partial t} (Re iS) \quad (46)$$

$$V - \frac{1}{2} \left( \left\| (-i \nabla + q \mathbf{A}) \right\|^2 - (-i \nabla + q \mathbf{A})^2 \right) Re iS/\nu = i\nu \frac{\partial}{\partial t} (Re iS/\nu) \quad (47)$$

The proof is elementary and straightforward but somewhat tedious. This is true in any number of dimensions, and therefore it can be applied to multi-particle wave equations if the coordinates of the particles are suitably scaled to account for mass differences. It allows the multi-valued analysis to be applied to particles in magnetic fields

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