CONSERVATION LAWS OF COUPLED SEMILINEAR
WAVE EQUATIONS

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Abstract. A complete classification of all low-order conservation laws is carried out for
a system of coupled semilinear wave equations which is a natural two-component general-
ization of the nonlinear Klein-Gordon equation. The conserved quantities defined by these
conservation laws are derived and their physical meaning is discussed.

1. Introduction

We study a system of coupled semilinear wave equations

\begin{align}
\tag{1a}
    u_{tt} - a^2 u_{xx} + f(u) + p(v) &= 0,
\end{align}

\begin{align}
\tag{1b}
    v_{tt} - b^2 v_{xx} + g(v) + q(u) &= 0,
\end{align}

where \( a > 0 \) and \( b > 0 \) are constant wave speeds, \( f(u) \) and \( g(v) \) are self-interaction terms,
\( p(v) \) and \( q(u) \) are nonlinear coupling terms, such that \( p''(v) \neq 0, q''(u) \neq 0, \) and \( f(0) = p(0) = g(0) = q(0) = 0. \) With these conditions, the system is nonlinear (i.e. both equations
contain a nonlinear term), coupled (i.e. neither equation is decoupled), and homogeneous
(i.e. \( u = v = 0 \) is a solution).

In general this nonlinear system does not have a Lagrangian formulation. When \( a = b \)
and \( f' = g' = \text{const.}, \) however, there is a Lagrangian given by

\begin{align}
\tag{2}
    L &= -u_t v_t + c^2 u_x v_x \pm m^2 uv + \int p(v) \, dv + \int q(u) \, du
\end{align}

with \( c = \text{const.} \) and \( m = \text{const.}, \) which yields

\begin{align}
\tag{3}
    0 &= \frac{\delta L}{\delta u} = v_{tt} - c^2 v_{xx} \pm m^2 v + q(u), \quad 0 = \frac{\delta L}{\delta v} = u_{tt} - c^2 u_{xx} \pm m^2 u + p(v).
\end{align}

Note that these Euler-Lagrange equations are twisted in the sense that the variational der-
ivative of \( L \) with respect to \( u,v \) yields the wave equations for \( v,u \), respectively. In both
equations, \( c > 0 \) is the constant wave speed and \( m \geq 0 \) is the mass coefficient.

The semilinear system (1) is interesting as it is a natural two-component generalization
of the nonlinear Klein-Gordon equation, where each wave component has a different speed.
Systems of this type arise in many physical applications (see e.g. Ref. [1]), and analytical
aspects of these systems have been studied recently [1, 2]. Conservation laws have been found in the case of cubic and other power nonlinearities [3].

The aim in the present paper is to derive a complete classification of all low-order conservation laws for the semilinear system (1) without special assumptions on the nonlinearities. Conservation laws are important for many reasons: they yield conserved quantities and constants of motion; they can detect integrability; they provide potentials and nonlocally-related systems; they can be used to check accuracy of numerical solution methods; and they provide a way to develop good discretizations.

In section 2 we apply the direct method of multipliers [4, 5, 6, 7] to set up the standard determining equations for finding the conservation laws admitted by the semilinear system (1).

In section 3, we first present the classification of all low-order conservation laws and their multipliers. Then we examine the conserved quantities defined by these conservation laws, and we discuss their physical meaning as well as their connection to variational symmetries in the case when the semilinear system (1) has a Lagrangian formulation (3).

Several interesting results are obtained. In the case when the wave speeds are equal, \(a = b\), we find that the semilinear system (1) has conserved quantities representing energy, momentum, and boost momentum. This case also has a conserved energy-momentum quantity that depends on an arbitrary function of light-cone derivatives of \(u, v\) when the nonlinearities are related by \(f(u) = (1/\alpha)q(u), g(v) = \alpha p(v)\), with \(\alpha = \text{const.}\). In the Lagrangian case, we find that for nonlinearities given by powers of \(u, v\), the semilinear system (3) has two additional conserved quantities representing a dilational energy-momentum and a \(SO(2)\) rotation charge. Somewhat surprisingly, without any conditions relating the wave speeds \(a\) and \(b\), the semilinear system (1) also admits elementary conserved quantities, which are linear in derivatives of \(u, v\).

In section 4, we make some concluding remarks.

2. Conservation laws

A conservation law of the semilinear system (1) is a space-time divergence such that

\[
D_t T(t, x, u, v, u_t, v_t, u_x, v_x, \ldots) + D_x X(t, x, u, v, u_t, v_t, u_x, v_x, \ldots) = 0
\]

holds for all solutions \((u(t, x), v(t, x))\) of the system (1). The spatial integral of the conserved density \(T\) formally satisfies

\[
\frac{d}{dt} \int_{-\infty}^{\infty} T dx = -X\bigg|_{-\infty}^{\infty}
\]

and so if the spatial flux \(X\) vanishes at spatial infinity, then

\[
C[u, v] = \int_{-\infty}^{\infty} T dx = \text{const.}
\]

formally yields a conserved quantity for the semilinear system (1). Conversely, any such conserved quantity arises from a conservation law (1). Two conservation laws are equivalent if their conserved densities \(T(t, x, u, v, u_t, v_t, u_x, v_x, \ldots)\) differ by a total \(x\)-derivative \(D_x \Theta(t, x, u, v, u_t, v_t, u_x, v_x, \ldots)\) on all solutions \((u(t, x), v(t, x))\), thereby giving the same conserved quantity \(C[u, v]\) up to boundary terms. Correspondingly, the fluxes
X(t, x, u, v, u_t, v_t, u_x, v_x, ...) of two equivalent conservation laws differ by a total time derivative \(-D_t \Theta(t, x, u, v, u_t, v_t, u_x, v_x, \ldots)\) on all solutions \((u(t, x), v(t, x))\). A conservation law is called \textit{locally trivial} if

\begin{align}
T = \Phi + D_x \Theta, \quad X = \Psi - D_t \Theta
\end{align}

such that \(\Phi = \Psi = 0\) holds on all solutions \((u(t, x), v(t, x))\). Thus, equivalent conservation laws differ by a locally trivial conservation law.

The set of all conservation laws (up to equivalence) admitted by the semilinear system (11) forms a vector space on which there is a natural action \([8, 9, 10]\) by the group of all Lie symmetries of the system (11).

Each conservation law (11) has an equivalent \textit{characteristic form} in which \(u_{tt}, v_{tt}\), and all derivatives of \(u_{tt}\) and \(v_{tt}\) are eliminated from \(T\) and \(X\) through use of the system (11) and its differential consequences. There are two steps to obtaining the characteristic form. Let \(\mathcal{E}\) denote the space of all solutions \((u(t, x), v(t, x))\) of the system (11). First, we eliminate \(u_{tt}, v_{tt}, u_{txx}, v_{txx}, \ldots\) to get

\begin{align}
\hat{T} = T \big|_\mathcal{E} = T - \Phi, \quad \hat{X} = X \big|_\mathcal{E} = X - \Psi
\end{align}

where \(\hat{T}\) and \(\hat{X}\) are functions only of \(t, x, u, v, u_t, v_t, u_x, v_x, u_{tx}, u_{xx}, v_{xx}, u_{txx}, v_{txx}, \ldots\), so that, on all solutions of the semilinear system (11),

\begin{align}
(D_t \hat{T} + D_x \hat{X}) \big|_\mathcal{E} = 0
\end{align}

holds with

\begin{align}
D_t \big|_\mathcal{E} &= \partial_t + u_t \partial_u + v_t \partial_v + u_{tx} \partial_{u_x} + v_{tx} \partial_{v_x} + \cdots + \left( a^2 u_{xx} - f(u) - p(v) \right) \partial_{u_{tt}} \\
&+ \left( b^2 v_{xx} - g(v) - q(u) \right) \partial_{v_{tt}} + D_x \left( a^2 u_{xx} - f(u) - p(v) \right) \partial_{u_{tx}} \\
&+ D_x \left( b^2 v_{xx} - g(v) - q(u) \right) \partial_{v_{tx}} + \cdots
\end{align}

\begin{align}
D_x \big|_\mathcal{E} = D_x.
\end{align}

Next, moving off of solutions, we use the identity

\begin{align}
D_t = D_t \big|_\mathcal{E} + \left( u_{tt} - a^2 u_{xx} + f(u) + p(v) \right) \partial_{u_{tt}} + \left( v_{tt} - b^2 v_{xx} + g(v) + q(u) \right) \partial_{v_{tt}} \\
+ D_x \left( u_{tt} - a^2 u_{xx} + f(u) + p(v) \right) \partial_{u_{tx}} + D_x \left( v_{tt} - b^2 v_{xx} + g(v) + q(u) \right) \partial_{v_{tx}} + \cdots
\end{align}

This yields the characteristic form of the conservation law (11)

\begin{align}
D_t \hat{T} + D_x (\hat{X} + \hat{\Psi}) = \left( u_{tt} - a^2 u_{xx} + f(u) + p(v) \right) Q^u + \left( v_{tt} - b^2 v_{xx} + g(v) + q(u) \right) Q^v
\end{align}

holding identically, where

\begin{align}
\hat{\Psi} &= E_{u_{tx}} (\hat{T}) \left( u_{tt} - a^2 u_{xx} + f(u) + p(v) \right) + E_{v_{tx}} (\hat{T}) \left( v_{tt} - b^2 v_{xx} + g(v) + q(u) \right) \\
&+ E_{u_{txx}} (\hat{T}) D_x \left( u_{tt} - a^2 u_{xx} + f(u) + p(v) \right) + E_{v_{txx}} (\hat{T}) D_x \left( v_{tt} - b^2 v_{xx} + g(v) + q(u) \right) + \cdots
\end{align}

is a trivial flux, and where the pair of functions

\begin{align}
Q^u = E_{u_t} (\hat{T}), \quad Q^v = E_{v_t} (\hat{T})
\end{align}

is called a \textit{multiplier} (or a \textit{characteristic}). Here \(E_w = \partial_w - D_x \partial_{w_x} + D_x^2 \partial_{w_{xx}} - \cdots\) denotes the (spatial) Euler operator \([8, 7]\) with respect to a variable \(w\). This operator has the important
property \[8, 7\] that a function \(h(t, x, w, w_x, w_{xx}, \ldots)\) is annihilated by \(E_w\) iff the function is a total (spatial) derivative, \(h = D_x \theta(t, x, w, w_x, w_{xx}, \ldots)\).

The relation (15) between \(\hat{T}\) and \((Q^u, Q^v)\) shows that a multiplier will be uniquely determined as a function of \(t, x, u, v, u_t, v_t, u_x, v_x, u_{tx}, v_{tx}, u_{xx}, v_{xx}, u_{txx}, v_{txx}, \ldots\) if \(\hat{T}\) has no dependence on \(u_t, v_t\), and their derivatives. In particular, the multiplier for a locally trivial conserved density \(\hat{T} = D_x \Theta\) vanishes, since \(E_{u_t}(D_x \Theta) = E_{v_t}(D_x \Theta) = 0\). Conversely, for a trivial multiplier \((Q^u, Q^v)\) = 0, the relation (15) implies that \(\hat{T} = D_x \Theta + \hat{T}_0\) holds for some function \(\hat{T}_0(t, x, u, v, u_x, v_x, u_{xx}, v_{xx}, \ldots)\) with no dependence on \(u_t, v_t\), and their derivatives. Then the characteristic equation (13) yields the relation \(D_t \hat{T}_0 + D_x \hat{X}_0 = 0\), which holds identically, where \(\hat{X}_0 = D_t \Theta - \hat{X}\). This determines \(\hat{T}_0 = D_x \Theta_0\) and \(\hat{X}_0 = -D_t \Theta_0\), showing that \(\hat{T} = D_x (\Theta + \Theta_0)\) is locally trivial.

Thus, there is a one-to-one relation between multipliers and conserved densities (up to equivalence). All multipliers

\[
(Q^u(t, x, u, v, u_t, v_t, u_x, v_x, u_{tx}, v_{tx}, u_{xx}, v_{xx}, u_{txx}, v_{txx}, \ldots),
Q^v(t, x, u, v, u_t, v_t, u_x, v_x, u_{tx}, v_{tx}, u_{xx}, v_{xx}, u_{txx}, v_{txx}, \ldots))
\]  

(16)

(with a finite differential order) are determined by the condition that their summed product with the equations in the semilinear system (1) is a total space-time divergence. Such divergences have the characterization that their variational derivative with respect to \(u\) and \(v\) vanishes identically \[8, 7\]. This condition

\[
\frac{\delta}{\delta u} \left( (u_{tt} - a^2 u_{xx} + f(u) + p(v))Q^v + (u_{tt} - b^2 v_{xx} + g(v) + q(u))Q^v \right) = 0 \tag{17a}
\]

\[
\frac{\delta}{\delta v} \left( (u_{tt} - a^2 u_{xx} + f(u) + p(v))Q^u + (u_{tt} - b^2 v_{xx} + g(v) + q(u))Q^v \right) = 0 \tag{17b}
\]

can be split in an explicit form with respect to \(u_t, v_t, u_{ttt}, v_{ttt}, u_{ttxx}, v_{ttxx}, \ldots\), which yields an equivalent set of equations \[4, 5, 6\]

\[
D^2_t Q^u - a^2 D^2_x Q^u + f'(u)Q^u + q'(u)Q^v = 0, \quad D^2_t Q^v - b^2 D^2_x Q^v + g'(v)Q^v + p'(v)Q^u = 0, \quad \text{(18)}
\]

and

\[
\partial_u Q^u = E_{u_t}(Q^u), \quad \partial_u Q^v = E_{v_t}(Q^v), \quad \partial_v Q^u = E_{u_t}(Q^v), \quad \partial_v Q^v = E_{v_t}(Q^v),
\]

\[
\partial_{u_{xx}} Q^u = -E_{u_t}^{(1)}(Q^u), \quad \partial_{u_{xx}} Q^v = -E_{v_t}^{(1)}(Q^v), \quad \partial_{v_{xx}} Q^u = -E_{u_t}^{(1)}(Q^v), \quad \partial_{v_{xx}} Q^v = -E_{v_t}^{(1)}(Q^v),
\]

\[
\partial_{u_{txx}} Q^u = -E_{u_t}^{(2)}(Q^u), \quad \partial_{u_{txx}} Q^v = -E_{v_t}^{(2)}(Q^v), \quad \partial_{v_{txx}} Q^u = -E_{u_t}^{(2)}(Q^v), \quad \partial_{v_{txx}} Q^v = -E_{v_t}^{(2)}(Q^v),
\]

etc.

(19)

holding for all solutions \((u(t, x), v(t, x))\) of the semilinear system (1). Here \(E_{u_t}^{(1)} = \partial_{u_x} - 2D_x \partial_{u_{xx}} + 3D_x^2 \partial_{u_{xxx}} - \cdots, E_{u_t}^{(2)} = \partial_{u_x} - 3D_x \partial_{u_{xx}} + 6D_x^2 \partial_{u_{xxx}} - \cdots, \) etc. denote higher (spatial) Euler operators \[8\]. These equations (18)–(19) constitute the standard determining system for multipliers \[5, 6, 7\].

In the case when the semilinear system (1) is a Lagrangian system \[3\], the first equation (18) is simply the determining equation for symmetries \(X = Q^u \partial_u + Q^v \partial_v\) in evolutionary form \[8, 11, 17\]. The second equation (19) is then equivalent to the condition that the Lagrangian \(\theta\) is invariant under \(X = Q^u \partial_u + Q^v \partial_v\) to within a total space-time divergence.
Hence, in the Lagrangian case, the determining equations (18) and (19) for multipliers are equivalent to the condition \[12\] that \(X = Q^u \partial_u + Q^v \partial_v\) is a variational symmetry of the semilinear system.

In the general case, the first equation (18) instead is the adjoint of the symmetry determining equation, and its solutions \((Q^u, Q^v)\) are called **adjoint-symmetries**. The second equation (19) then comprises the Helmholtz conditions which are necessary and sufficient for \((Q^u, Q^v)\) to be an Euler-Lagrange expression (15). Consequently, in this case \[13, 8, 5, 6\], multipliers are simply adjoint-symmetries that have a variational form, and the determination of conservation laws via multipliers is a kind of adjoint problem \[4\] of the determination of symmetries.

For any solution of the multiplier determining equations (18) and (19), a conserved density and a flux can be recovered either by \[14, 7\] directly integrating the relation (15) between \((Q^u, Q^v)\) and \(\hat{T}\), or by \[8, 5, 6\] using a homotopy integral formula which expresses \(\hat{T}\) in terms of \((Q^u, Q^v)\) (see also Ref.\[15, 16\]).

A conservation law of the semilinear system (1) is said to be of **low order** \[17\] if the only derivatives of \(u, v\) that appear in its multiplier are related to the leading derivatives \(u_{tt}, v_{tt}, u_{xx}, v_{xx}\) in the system by differentiation with respect to \(t, x\). This means that the multiplier for a low-order conservation law is at most first-order,

\[
Q = (Q^u(t, x, u, v, u_t, v_t, u_x, v_x), Q^v(t, x, u, v, u_t, v_t, u_x, v_x)).
\] (20)

In general, for wave equations, conservation laws of physical importance, such as energy and momentum, are of low order, while conservation laws connected with integrability are typically of higher order.

### 3. Classification results

We will now classify all low-order conservation laws (9) admitted by the semilinear system (1). This means finding all multipliers (15) with a general first-order form (20). The corresponding conserved densities and fluxes will then have the following general form.

From relation (15) and properties of the (spatial) Euler operator, a first-order multiplier (20) determines that a conserved density has the form \(\hat{T} = T(t, x, u, v, u_t, v_t, u_x, v_x) + T_0 + D_x \Theta\) where, without loss of generality, the function \(T_0\) contains no purely first-order terms and no \(t\)-derivatives of \(u, v\). The characteristic equation \[13\] then yields \(D_t|_{\varepsilon} T + D_t T_0 + D_x \hat{X} = 0\). This relation splits with respect to all derivatives of \(u, v\) with order greater than two. Starting at the highest order terms, the splitting leads to \(T_0 = D_x \Theta + \Phi\) and \(\hat{X} = - D_t \Theta + X + \Psi\) by a standard descent argument in derivatives of \(u, v, \) where \(\Phi|_{\varepsilon} = 0\) and \(\Psi|_{\varepsilon} = 0\) are trivial terms, and where \(\hat{X}(t, x, u, v, u_t, v_t, u_x, v_x)\) is first-order. Hence, the non-trivial part of \(\hat{T}\) and \(\hat{X}\) consists of only the first-order terms \(T\) and \(X\). Conversely, a first-order conserved density \(\hat{T}(t, x, u, v, u_t, v_t, u_x, v_x)\) and a first-order flux \(\hat{X}(t, x, u, v, u_t, v_t, u_x, v_x)\) determine a first-order multiplier (20) directly by the relation (15).

Thus, we have established the following characterization of low-order conservation laws.

**Lemma 3.1.** For the semilinear system (1), a non-trivial conservation law (9) will have a multiplier of first-order (20) iff the conserved density and the flux, up to equivalence, are of first-order

\[
\hat{T} = T(t, x, u, v, u_t, v_t, u_x, v_x), \quad \hat{X} = X(t, x, u, v, u_t, v_t, u_x, v_x).
\] (21)
Therefore, the class of low-order conservation laws consists of all admitted non-trivial conserved densities and fluxes with the form (22).

3.1. Multipliers, conserved densities and fluxes. The classification of low-order conservation laws will be carried out modulo the equivalence transformations

\[
\begin{align*}
  u &\rightarrow v, \quad f(u) \rightarrow q(u), \quad p(v) \rightarrow g(v), \\
  v &\rightarrow u, \quad q(u) \rightarrow f(u), \quad g(v) \rightarrow p(v)
\end{align*}
\]

and

\[
\begin{align*}
  u &\rightarrow \alpha u, \quad f(u) \rightarrow \alpha f(\alpha u), \quad p(v) \rightarrow \alpha p(\beta v), \quad \alpha \neq 0 \\
  v &\rightarrow \beta v, \quad q(u) \rightarrow \beta q(\alpha u), \quad g(v) \rightarrow \beta p(\beta v), \quad \beta \neq 0
\end{align*}
\]

under which the general form of the semilinear system (11) is preserved.

Based on the form of the equations (1a)–(1b) in the system, it is computationally convenient to divide the classification into three main cases: (I) under which the general form of the semilinear system (1) is preserved.

Proposition 3.1. For the semilinear system of wave equations (11) in the case \( f(u) = g(v) = 0 \), the admitted conservation law multipliers consist of:

\[
\begin{align*}
  (a) \quad & a = b, \quad \text{arbitrary } p(v), q(u) \\
  & Q = (b^2 tv_x + xv, b^2 tu_x + xu_t) \\
  & Q = (v_x, u_x) \\
  & Q = (v_t, u_t) \\
  (b) \quad & a = b, \quad p(v) = \beta u^{k-1}, \quad q(u) = \alpha u^{k-1} \\
  & Q = (ktv_t + kxv_x + 2v, ktu_t + kxu_x - 2u) \\
  (c) \quad & a = b, \quad p(v) = \beta e^{q_v}, \quad q(u) = \alpha e^{q_u} \\
  & Q = ((v_t \pm bv_x)A(t, x) \pm 2(b/\delta)A_x(t, x), (u_t \pm bu_x)A(t, x) \pm 2(b/\gamma)A_x(t, x))
\end{align*}
\]

where \( A_t \mp bA_x = 0 \)

Proposition 3.2. For the semilinear system of wave equations (11) in the case \( f(u) \neq 0 \) and \( g(v) = 0 \), the admitted conservation law multipliers consist of:

\[
\begin{align*}
  (a) \quad & a = b, \quad f(u) = \kappa q(u), \quad \text{arbitrary } p(v), q(u) \\
  & Q = (b^2 tv_x + xv, b^2 tu_x - \kappa v_x + x(u_t - \kappa v_t)) \\
  & Q = (v_x, u_x - \kappa v_x) \\
  & Q = (v_t, u_t - \kappa v_t) \\
  (b) \quad & a = b, \quad f(u) = \kappa q(u), \quad p(v) = \beta e^{q_v}, \quad q(u) = \alpha e^{q_u} \\
  & Q = (v_t \pm bv_x)A(t, x) + 2(b/\delta)A_x(t, x), \\
  & \quad (u_t - \kappa v_t \pm b(u_x - \kappa v_x))A(t, x) + 2(1/\gamma - \kappa/\delta)A_x(t, x)
\end{align*}
\]

where \( A_t \mp bA_x = 0 \)
Proposition 3.3. For the semilinear system of wave equations \((1)\) in the case \(f(u) \neq 0\) and \(g(v) \neq 0\), the admitted conservation law multipliers consist of:

(a) \(a \neq b, \ f(u) = (1/\alpha)q(u) + \delta u, \ g(v) = \alpha p(v) + \gamma v, \ \text{arbitrary } p(v), q(u)\)

\[
Q = (-\alpha A(t)B(x), A(t)B(x))
\tag{33}
\]

where \((a^2 - b^2)A'' + (a^2\gamma - b^2\delta)A = 0, \ (a^2 - b^2)B'' + (\gamma - \delta)B = 0\)

(b) \(a = b, \ f(u) = \beta q(u) + \gamma u, \ g(v) = \alpha p(v) + \gamma v, \ \text{arbitrary } p(v), q(u)\)

\[
Q = (b^2t(\alpha u_x - v_x) + x(\alpha u_t - v_t), b^2t(\beta v_x - u_x) + x(\beta v_t - u_t))
\tag{34}
\]

\[
Q = (\alpha u_x - v_x, \beta v_x - u_x)
\tag{35}
\]

\[
Q = (\alpha u_t - v_t, \beta v_t - u_t)
\tag{36}
\]

(c) \(a = b, \ f(u) = (1/\alpha)q(u) + \gamma u, \ g(v) = \alpha p(v) + \gamma v, \ \text{arbitrary } p(v), q(u)\)

\[
Q = (-\alpha A(t, x), A(t, x))
\tag{37}
\]

where \(A_{tt} - b^2 A_{xx} + \gamma A = 0\)

(d) \(a = b, \ f(u) = (1/\alpha)q(u), \ g(v) = \alpha p(v), \ \text{arbitrary } p(v), q(u)\)

\[
Q = (-\alpha A(\zeta), A(\zeta))
\tag{38}
\]

where \(\zeta = \alpha u_t - v_t \pm b(\alpha u_x - v_x)\)

(e) \(a = b, \ f(u) = \beta q(u), \ g(v) = \alpha p(v), \ p(v) = \kappa e^{\mu v}, \ q(u) = \lambda e^{\delta u}\)

\[
Q = ((\alpha u_t - v_t \pm b(\alpha u_x - v_x))A(t, x) + 2(\alpha/\delta - 1/\mu)A_t(t, x),
\]

\[
(\beta v_t - u_t \pm b(\beta v_x - u_x))A(t, x) + 2(\beta/\mu - 1/\delta)A_t(t, x))
\tag{39}
\]

where \(A_t \pm bA_x = 0\)

(f) \(a = b, \ f(u) = \gamma u, \ g(v) = \gamma v, \ p(v) = \alpha/v, \ q(u) = \beta/u\)

\[
Q = (-v, u)
\tag{40}
\]

Each multiplier \(Q\) determines a conserved density \(\hat{T}\) and a flux \(\hat{X}\), up to equivalence. The simplest way to obtain explicit expressions for them is by first splitting the characteristic equation \((13)\) with respect to \(u_{xx}, v_{xx}, u_{tx}, v_{tx}\), and next integrating the resulting linear system

\[
\hat{T}_{ut} - Q^u = 0, \quad \hat{T}_{vt} - Q^v = 0,
\tag{41}
\]

\[
\hat{X}_{ut} + \hat{T}_{ux} = 0, \quad \hat{X}_{vt} + \hat{T}_{vx} = 0,
\tag{42}
\]

\[
\hat{X}_{ux} + a^2\hat{T}_{ux} = 0, \quad \hat{X}_{vx} + b^2\hat{T}_{vx} = 0,
\tag{43}
\]

\[
\hat{T}_t + u_t\hat{T}_u + v_t\hat{T}_v + \hat{X}_x + u_x\hat{X}_u + v_x\hat{X}_v - (f(u) + p(v))\hat{T}_{ut} - (g(v) + q(u))\hat{T}_{vt} = 0.
\tag{44}
\]

This leads to the following main classification result, after the various overlapping cases in Propositions 3.1, 3.2, and 3.3 have been merged.

Theorem 3.1. All low-order conservation laws \((21)\) admitted by the semilinear system of wave equations \((1)\) are given by, up to equivalence, the conserved densities and the fluxes:

(a) \(a \neq b, \ f(u) = (1/\alpha)q(u) + \delta u, \ g(v) = \alpha p(v) + \gamma v, \ \text{arbitrary } p(v), q(u)\)
\[ T_1 = ((v_t - \alpha u_t)A(t) + (\alpha u - v)A'(t))B(x) \]
\[ X_1 = ((\alpha a^2 u_x - b^2 v_x)B(x) + (b^2 v - \alpha a^2 u)B'(x))A(t) \]

where \((a^2 - b^2)A'' + (a^2 \gamma - b^2 \delta)A = 0, \quad (a^2 - b^2)B'' + (\gamma - \delta)B = 0\)

(b) \(a = b, \quad f(u) = \beta q(u) + \gamma u, \quad g(v) = \alpha p(v) + \gamma v, \quad \text{arbitrary } p(v), q(u)\)

\[ T_2 = (\alpha \beta - 1) \left( \int f(v) dv + \int g(u) du \right) - (u_t v_t + b^2 u_x v_x + \gamma uv) \]
\[ + \frac{1}{2} \alpha (v_t^2 + b^2 v_x^2 + \gamma v^2) + \frac{1}{2} \beta (u_t^2 + b^2 u_x^2 + \gamma u^2) \]
\[ X_2 = b^2 (u_t (v_x - \beta u_x) + v_t (u_x - \alpha v_x)) \]
\[ T_3 = (\beta u_x - v_x) u_t + (\alpha v_x - u_x) v_t \]
\[ X_3 = (\alpha \beta - 1) \left( \int f(v) dv + \int g(u) du \right) + u_t v_t + b^2 u_x v_x - \gamma uv \]
\[ - \frac{1}{2} \alpha (v_t^2 + b^2 v_x^2 - \gamma v^2) - \frac{1}{2} \beta (u_t^2 + b^2 u_x^2 - \gamma u^2) \]
\[ T_4 = (\alpha \beta - 1) x \left( \int f(v) dv + \int g(u) du \right) - x (u_t v_t + b^2 u_x v_x + \gamma uv) \]
\[ + \frac{1}{2} \alpha x (u_t^2 + b^2 u_x^2 + \gamma u^2) + \frac{1}{2} \beta x (v_t^2 + b^2 v_x^2 + \gamma v^2) \]
\[ + b^2 t (\alpha u_t u_x + \beta v_t v_x - u_t v_x - u_x v_t) \]
\[ X_4 = b^2 (\alpha \beta - 1) t \left( \int f(v) dv + \int g(u) du \right) + b^2 t (u_t v_t + b^2 u_x v_x - \gamma uv) \]
\[ - \frac{1}{2} \alpha b^2 t (u_t^2 + b^2 u_x^2 - \gamma u^2) - \frac{1}{2} \beta b^2 t (v_t^2 + b^2 v_x^2 - \gamma v^2) \]
\[ + b^2 x (u_t v_x + u_x v_t - \alpha u_t u_x - \beta v_t v_x) \]

(c) \(a = b, \quad f(u) = (1/\alpha) q(u) + \gamma u, \quad g(v) = \alpha p(v) + \gamma v, \quad \text{arbitrary } p(v), q(u)\)

\[ T_5 = (\alpha u_t - v_t) A(t, x) + (v - \alpha u) A_x(t, x) \]
\[ X_5 = b^2 ((v_x - \alpha x u) A(t, x) + (\alpha u - v) A_x(t, x)) \]

where \(A_{tt} - b^2 A_{xx} + \gamma A = 0\)

(d) \(a = b, \quad f(u) = (1/\alpha) q(u), \quad g(v) = \alpha p(v), \quad \text{arbitrary } p(v), q(u)\)

\[ T_6 = B(\zeta) \]
\[ X_6 = \pm b B(\zeta) \]

where \(\zeta = \alpha u_t - v_t \pm b (\alpha u_x - v_x)\)

(e) \(a = b, \quad f(u) = \beta q(u), \quad g(v) = \alpha p(v), \quad p(v) = \kappa e^\mu v, \quad q(u) = \lambda e^{\delta u}\)
$$T_7 = ((\alpha \beta - 1)(\lambda \gamma \exp(\delta u) + \delta \kappa \exp(\gamma v)) + \frac{1}{2}\delta \gamma (\alpha (u_t \pm bu_x)^2 + \beta (v_t \pm bv_x)^2)$$
$$- \delta \gamma (u_t \pm bu_x)(v_t \pm bv_x)A(t,x) + 2(\delta \beta - \gamma)v_t + (\gamma \alpha - \delta)u_t)A_t(t,x)$$
$$+ 2((\delta - \gamma \alpha)u + (\gamma - \delta \beta)v)A_{tt}(t,x)$$
$$X_7 = \pm b\left( ((\alpha \beta - 1)(\lambda \gamma \exp(\delta u) + \delta \kappa \exp(\gamma v)) - \frac{1}{2}\delta \gamma (\alpha (u_t \pm bu_x)^2 + \beta (v_t \pm bv_x)^2)$$
$$+ \delta \gamma (u_t \pm bu_x)(v_t \pm bv_x)A(t,x) + 2(\delta - \gamma \alpha)u_x + (\gamma - \delta \beta)v_x)A_x(t,x)$$
$$+ 2b^2((\delta \beta - \gamma)v + (\gamma \alpha - \delta)u)A_{xx}(t,x) \right)$$

where $A_t \mp bA_x = 0$

(f) $a = b, \quad f(u) = \gamma u, \quad g(v) = \gamma v, \quad p(v) = \alpha / v, \quad q(u) = \beta / u, \quad \gamma \neq 0$

$$T_g = uv_t - vv_t$$

$$X_g = b^2(vu_x - uv_x) + (\beta - \alpha)x$$

(g) $a = b, \quad f = g = 0, \quad p(v) = \alpha v^{k-1}, \quad q(u) = \beta u^{-k-1}$

$$T_h = kt(v_xu_t + b^2v_xu_x + kx(v_xu_t + u_xv_t) + 2(vu_t - uv_t) + t(\alpha v^k - \beta u^{-k})$$

$$X_h = -b^2kt(v_xu_x + u_xv_x) - kx(v_xu_t + b^2v_xu_x) + x(\alpha v^k - \beta u^{-k})$$

3.2. Conserved quantities. We now discuss the conserved quantities (6) that arise from the conservation laws in Theorem 3.1 for the semilinear system (1).

The case of unequal wave speeds $a \neq b$ will be considered first.

The conservation law (45) in part (a) is a linear expression in $u, v, u_t, u_x, v_t, v_x$. This conservation law is admitted because the wave equations (1a) and (1b) in the case $f(u) = (1/\alpha)q(u) + \delta u, g(v) = \alpha p(v) + \gamma v, a \neq b$ can be combined to yield a linear equation

$$(\alpha u - v)_{tt} - (\alpha a^2 u - b^2 v)_{xx} + \delta \alpha u - \gamma v = 0.$$ (54)

Any linear PDE admits elementary conserved quantities which are given by the projection of solutions of the PDE onto modes that satisfy the adjoint linear system. The modes for the linear equation (51) consist of the multiplier expression (53) which has the form $Q = (-\alpha F, F)$ where $F(t, x) = A(t)B(x)$ is a mode function satisfying the adjoint of equation (51),

$$(a^2 - b^2)F_{xx} + (\gamma - \delta)F = 0, \quad (a^2 - b^2)F_{tt} + (\gamma a^2 - \delta b^2)F = 0.$$ (55)

From this adjoint system, in the general case when $\gamma \neq \delta$ and $a^2 \gamma \neq b^2 \delta$, the mode function is a harmonic travelling wave $F = \exp(i(kx + wt))$ where

$$k = \sqrt{(\gamma - \delta)/(a^2 - b^2)}$$ (56)

is the reciprocal wave length, and

$$w = \sqrt{(a^2 \gamma - b^2 \delta)/(a^2 - b^2)}$$ (57)

is the frequency, given in terms of the wave speeds $a, b$ and the mass coefficients $\delta, \gamma$ in the wave equations (1a) and (1b). The resulting conserved quantities have the form

$$C[u, v] = \int_{-\infty}^{\infty} ((v_t - \alpha u_t) + iw(\alpha u - v)) \exp(i(kx + wt)) \, dx.$$ (58)
and the first two do not contain $t, x$; in all other cases, the mode function degenerates into a form where its dependence on $x$ or $t$ is a linear polynomial.

These conserved quantities $\mathcal{L}_k$ are the only ones admitted by the semilinear system (1) when the wave equations (1a) and (1b) in the case $a \neq b$. Next, the case of equal wave speeds, but with no Lagrangian being admitted, will be considered.

The conservation law (49) in part (c) is a linear expression in $u, v, u_t, u_x, v_t, v_x$. It is admitted because the wave equations (1a) and (1b) in the case $f(u) = (1/\alpha)q(u) + \gamma u$, $g(v) = \alpha p(v) + \gamma v$, $a = b$ can be combined into a linear wave equation for $\alpha u - v$,

$$(\alpha u - v)_{tt} - b^2(\alpha u - v)_{xx} + \gamma(\alpha u - v) = 0. \quad (59)$$

Similarly to part (a), there are elementary conserved quantities given by the projection of solutions of this linear equation onto modes that satisfy the adjoint linear system. Since the equation (59) is self-adjoint, the modes have the form given by the multiplier expression (37) where the mode function $A(t, x)$ is the general solution of the wave equation

$$A_{tt} - b^2 A_{xx} + \gamma A = 0. \quad (60)$$

Note that $b > 0$ is wave speed and $\gamma$ is the squared-mass coefficient. Solutions of this wave equation can be expressed as Fourier modes $A = \exp(i(kx \pm wt))$ where $w^2 = b^2k^2 + \gamma$ is the standard dispersion relation, with $k$ being arbitrary. In the case $\gamma = 0$, the general solution is simply a linear combination of left-moving and right-moving travelling waves, $A = A_\pm(x \pm bt)$. In the case $\gamma \neq 0$, the general solution depends on two arbitrary functions which can be identified with the amplitudes for the two Fourier modes, giving $A = \int_{-\infty}^{\infty} \exp(i(kx \pm wt)) \hat{A}_\pm(k) dk$. In both cases, the resulting conserved quantities are given by the expression (58), where $k$ is now an arbitrary parameter.

The three conservation laws (46), (47), (48) in part (b) depend quadratically on $u_t, u_x, v_t, v_x$ and the first two do not contain $t, x$. They respectively yield a conserved energy and momentum

$$\mathcal{C}[u, v]_{\text{ener.}} = \int_{-\infty}^{\infty} \left((\alpha - 1) \left( \int f(v) dv + \int g(u) du \right) + \frac{1}{2}(\alpha - 1)(v_t^2 + b^2v_x^2 + \gamma v^2) \right. \right.$$

$$+ \frac{1}{2}(\beta - 1)(u_t^2 + b^2u_x^2 + \gamma u^2) + \frac{1}{2}((u_t - v_t)^2 + b^2(u_x - v_x)^2 + \gamma(u - v)^2) \left. \right) dx, \quad (61)$$

$$\mathcal{C}[u, v]_{\text{mom.}} = \int_{-\infty}^{\infty} \left((\alpha - 1)v_t u_x + (\beta - 1)u_t u_x + (u_t - v_t)(u_x - v_x) \right) dx, \quad (62)$$

and a conserved boost-momentum

$$\mathcal{C}[u, v]_{\text{boost mom.}} = \int_{-\infty}^{\infty} \left((\alpha - 1)x \left( \int f(v) dv + \int g(u) du \right) + \frac{1}{2}x((\alpha - 1)(v_t^2 + b^2v_x^2 + \gamma v^2) \right.$$ \n
$$+ (\beta - 1)(u_t^2 + b^2u_x^2 + \gamma u^2) + (u_t - v_t)^2 + b^2(u_x - v_x)^2 + \gamma(u - v)^2) \right.$$ \n
$$+ b^2t((\alpha - 1)v_t u_x + (\beta - 1)u_t u_x + (u_t - v_t)(v_x - u_x)) \right) dx. \quad (63)$$
The conserved energy will be positive when $\alpha > 1$, $\beta > 1$, and $\gamma > 0$. In contrast, the sign of the conserved momentum depends on the relative signs of $u_t, u_x$ as well as $v_t, v_x$. When these signs are positive, corresponding to right-moving wave motion, the momentum is positive if $\alpha > 1$ and $\beta > 1$, whereas when the signs are negative, corresponding to left-moving wave motion, the momentum is negative if $\alpha > 1$ and $\beta > 1$.

The conservation law (51) in part (e) also depends quadratically on $u_t, u_x, v_t, v_x$. It yields a conserved hyperbolic-energy

$$\mathcal{C}[u, v; A(\xi)]_{\text{hyper. ener.}} = \int_{-\infty}^{\infty} \left( \left((\alpha \beta - 1)(\lambda \gamma \exp(\delta u) + \delta \kappa \exp(\gamma v)) + \frac{1}{2} \delta \gamma ((\alpha - 1)(u_t \pm bu_x)^2 + (\beta - 1)(v_t \pm bv_x)^2 + (u_t - v_t \pm b(u_x - v_x))^2) \right) A(x \pm bt) 
+ 2b((\delta \beta - \gamma)v_t + (\gamma \alpha - \delta)u_t)A'(x \pm bt) + 2b^2((\delta - \gamma \alpha)u 
+ (\gamma - \delta \beta)v)A''(x \pm bt) \right) dx$$

(64)

which involves an arbitrary travelling wave function $A(\xi)$ and the light-cone derivatives $u_t \pm bu_x$ and $v_t \pm bv_x$ which represent the characteristic directions for the semilinear system (1) when $a = b$. This energy quantity is positive when $\alpha > 1$, $\beta > 1$, $\lambda \gamma > 0$, $\delta \kappa > 0$, $\delta \gamma > 0$, and $A = \text{const.} > 0$. Unlike parts (a) and (c), the appearance of an arbitrary function in the energy is not due to any linearity in the system (1) but instead comes from the exponential form of the nonlinear terms $p(v) = \kappa e^{\mu v}$, $q(u) = \lambda e^{\delta u}$, $g(v) = \alpha p(v)$, $f(u) = \beta q(u)$. With this type of nonlinearity, the system (1) is analogous to the Liouville wave equation $w_{tt} - b^2 w_{xx} + \gamma \exp(w) = 0$ for which it is known that all solutions $w(t, x)$ are functions only of the moving coordinate $x \pm bt$. Consequently, an arbitrary function of $x \pm bt$ is compatible with conservation of energy in such systems.

The conservation law (50) in part (d) involves an arbitrary function of a linear combination of the light-cone derivatives $u_t \pm bu_x$ and $v_t \pm bv_x$ as given by $\zeta = \alpha u_t - v_t \pm b(\alpha u_x - v_x)$. Similarly to part (c), the wave equations (13) and (14) in the case $f(u) = (1/\alpha)q(u)$, $g(v) = \alpha p(v)$, $a = b$ can be combined to yield the ordinary linear wave equation

$$(\alpha u - v)_{tt} - b^2(\alpha u - v)_{xx} = 0.$$  

(65)

This wave equation is well-known to admit any function of $\zeta$ as a conserved density. As a result, the corresponding conserved quantity for the semilinear system (1) is given by

$$\mathcal{C}[u, v; B(\zeta)]_{\text{ener./moment.}} = \int_{-\infty}^{\infty} B(\alpha u_t - v_t \pm b(\alpha u_x - v_x)) \, dx$$

(66)

where $B$ is an arbitrary function of $\zeta$. When $B$ is linear in $\zeta$, this yields an elementary conserved quantity. When $B$ is nonlinear in $\zeta$, the conserved quantity is an energy-type quantity if $B$ is an even function or a momentum-type quantity if $B$ is an odd function.

Lastly, the Lagrangian case will be considered.

Among the preceding conserved quantities, the ones that are admitted in the case of a Lagrangian formulation (3) consist of the energy (61), momentum (62), boost momentum (63), and hyperbolic energy (64). Their multipliers thereby correspond to variational symmetries $X = Q^u \partial_u + Q^v \partial_v$, which are respectively given by a time-translation (36), a space-translation
a Lorentz boost (34), and a generalized space-time translation combined with a general-
ized shift on \( u, v \) (39).

There are two additional conservation laws (52) and (53), in part (f) and (g) respectively, holding in the massless Lagrangian case \( f = g = 0, p(v) = \alpha v^{k-1}, q(u) = \beta u^{-k-1}, a = b \) for which the nonlinearities are powers of \( u \) and \( v \). In this case, the wave equations (1a) and (1b) are Euler-Lagrange equations with respect to \( v \) and \( u \), where the Lagrangian is given by

\[
L = -u_t v_t + b^2 u_x v_x + \alpha \int v^{k-1} dv + \beta \int u^{-k-1} du
\]

with

\[
\int w^{n-1} dw = \begin{cases} 
(1/n) w^n & n \neq 0 \\
\ln(w) & n = 0 
\end{cases}.
\]

The multipliers (40) and (27) for these two conservation laws correspond to variational symmetries \( X = Q^u \partial_u + Q^v \partial_v \) of the Lagrangian system. The first multiplier (40) represents an infinitesimal rotation on \((u, v)\), so the resulting conserved quantity

\[
\mathcal{C}[u, v]_{\text{rot. charge}} = \int_{-\infty}^{\infty} (uv_t - vu_t) \, dx
\]

is an \( SO(2) \) rotation charge in the space \((u, v)\). The second multiplier (27) represents an infinitesimal scaling on \((t, x, u, v)\), and hence the resulting conserved quantity is a dilational energy-momentum

\[
\mathcal{C}[u, v]_{\text{ener.-mom.}} = \int_{-\infty}^{\infty} (kt(v_t u_t + b^2 v_x u_x) + kx(v_x u_t + u_x v_t) + 2(vu_t - vu_t) + t(\alpha v^k - \beta u^{-k})) \, dx.
\]

4. Concluding remarks

The system of nonlinearly coupled wave equations (11) has a fairly rich structure of low-
order conserved quantities, particularly when the wave speeds are equal, \( a = b \). These quantities include energy, momentum, and boost momentum, plus a dilational energy-momentum and an \( SO(2) \) rotation charge in the Lagrangian case (when the only nonlinearities are due to the coupling between \( u \) and \( v \)). In two non-Lagrangian cases (when the nonlinearities also comprise a self-coupling for both \( u \) and \( v \)), additional energy-type conserved quantities arise, one involving arbitrary functions of the travelling wave coordinates \( x \pm bt \) and the other involving arbitrary functions of a linear combination of the light-cone derivatives \( u_t \pm bu_x \) and \( v_t \pm bv_x \).

All of these conserved quantities will be useful in the analysis of solutions of this system. We will leave this analysis for elsewhere.

An interesting problem for future work will be to look for higher-order conservation laws and higher-order symmetries, as this will detect if the system (11) is integrable for any special types of nonlinearities. In particular, this system is a natural two-component generalization of the Klein-Gordon equation which includes as special cases the Liouville equation, the sine-Gordon/sinh-Gordon equation, and the Tzetzecia equation, each of which are integrable in the sense of possessing an infinite hierarchy of conservation laws and symmetries (see e.g. Ref. [18, 5]).
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