On Quantization of Field Theories in Polymomentum Variables\textsuperscript{1}

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Abstract. Polymomentum canonical theories, which are manifestly covariant multi-parameter generalizations of the Hamiltonian formalism to field theory, are considered as a possible basis of quantization. We arrive at a multi-parameter hypercomplex generalization of quantum mechanics to field theory in which the algebra of complex numbers and a time parameter are replaced by the space-time Clifford algebra and space-time variables treated in a manifestly covariant fashion. The corresponding covariant generalization of the Schrödinger equation is shown to be consistent with several aspects of the correspondence principle such as a relation to the De Donder-Weyl Hamilton-Jacobi theory in the classical limit and the Ehrenfest theorem. A relation of the corresponding wave function (over a finite dimensional configuration space of field and space-time variables) with the Schrödinger wave functional in quantum field theory is examined in the ultra-local approximation.

INTRODUCTION

The canonical quantization is based on the Hamiltonian formalism. The conventional Hamiltonian formalism in field theory is an infinite dimensional version of the one in mechanics. As a result, the quantum field theory based on it is essentially the quantum mechanics of systems with an infinite number of degrees of freedom. Most of the difficulties and ambiguities of quantum field theory are due to this infinite dimensionality. However, should quantum fields always be understood in this way? Does this picture exhaust all aspects of quantum fields? Is there a “genuine quantum field theory” more general that just quantum mechanics applied to fields? It is clear that in perturbative regime, i.e. in the vicinity of a free field theory which can be represented as a continuum of harmonic oscillators, the above picture can work well, and it really does as the experimental triumph of pertubative

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quantum field theory demonstrates. However, applicability of this picture in non-pertubative domain and in curved space-time, where no natural particle concept exists in general, can be more limited.

A conceivable approach to the above posed questions can be based on the (not widely acknowledged yet) fact that the conventional version of the Hamiltonian formalism in field theory is not the only one possible. In fact, there exist different alternative extensions of the Hamiltonian formulation to field theory which all reduce to the Hamilton formalism in mechanics if the number of space-time dimensions equals to one. These extensions originate from the calculus of variations of multiple integrals [1–4]. Unlike the conventional Hamiltonian formalism, all these formulations are constructed in a manifestly covariant way not requiring any singling out of a time dimension. They can be applied even if the signature of the space-time is not Minkowskian. This is achieved by assigning the canonical momentum like variables, which we called polymomenta [5], to the whole set of space-time derivatives of a field: \( \partial_\mu y^a \to p^\mu_a \). An analogue of the phase space is then a finite dimensional phase space of variables \((y^a, p^\mu_a, x^\nu)\) which we call the polymomentum phase space. Corresponding generalizations of the canonical formalism will be referred to as polymomentum canonical theories. In the geometric (Cartan’s) approach to the calculus of variations these theories (a version of which is also known as the multisymplectic formalism [7]) appear as a result of a certain choice of the so-called Lepagean equivalents of a field-theoretic (multidimensional) analogue of the Poincaré-Cartan form [4,6–9]. Unfortunately, applications of these theories in physics have been so far rather rare (see for references [4,5,7]).

The simplest example of a polymomentum canonical theory is the so-called De Donder-Weyl (DW) theory [1,2,4,7]. Given a Lagrangian density \( L = L(y^a, \partial_\mu y^a, x^\nu) \), the polymomenta are introduced by the formula \( p^\mu_a := \partial L/\partial (\partial_\mu y^a) \). An analogue of the Hamilton canonical function defined as \( H := \partial_\mu y^a p^\mu_a - L \) is referred to as the DW Hamiltonian function in what follows. Note, that \( H \) is a function of variables \((y^a, p^\mu_a, x^\mu) =: z^M \). In these variables the Euler-Lagrange field equations can be rewritten in the form of DW Hamiltonian field equations

\[
\partial_\mu y^a = \partial H/\partial p^\mu_a, \quad \partial_\mu p^\mu_a = -\partial H/\partial y^a.
\]

(1)

Clearly, this formulation reproduces the standard Hamiltonian formulation in mechanics at \( n = 1 \). At \( n > 1 \) it provides us with a kind of multi-parameter, or "multi-time", manifestly covariant generalization of the Hamiltonian formalism. In doing so fields are treated not as infinite dimensional mechanical systems evolving with time, but rather as systems varying in space-time, with the DW Hamiltonian function controlling such a variation (similarly to the usual Hamiltonian controlling the time evolution).

The objective of the present contribution is to discuss an approach to quantization of fields based on polymomentum canonical theories. Although we confine

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2) Throughout the paper \( y^a \) denote field variables, \( x^\mu \) are space-time variables \((\mu = 1, ..., n)\), \( \partial_\mu y^a \) are space-time derivatives (or first jets) of field variables, \( p^\mu_a \) denote polymomenta.
ourselves exclusively to the approach based on the DW theory, we believe that basic ideas presented in what follows can be extended to more general polymomentum theories.

**GRADED POISSON BRACKET AND QUANTIZATION**

The canonical quantization in mechanics is essentially based on the algebraic structure given by the Poisson bracket. One of the reasons why polymomentum canonical theories have not been used as a basis of quantization was the lack of an appropriate generalization of the Poisson bracket. In [5] we proposed such a generalization within the DW theory. The bracket is defined on horizontal differential forms $F = \frac{1}{p!} F_{\mu_1...\mu_p} (\varepsilon^M) dx^{\mu_1} \wedge ... \wedge dx^{\mu_p}$ of various degrees $p$ ($0 \leq p \leq n$), which play the role of dynamical variables (instead of functions in mechanics or functionals in the conventional Hamiltonian formalism in field theory). It leads to graded analogues of the Poisson algebra structure [5,10]. More specifically, the bracket on differential forms in DW theory leads to generalizations of the so-called Gerstenhaber algebra [11] (a graded analogue of the Poisson algebra with the grade of an element of the algebra with respect to the bracket differing by one from its grade with respect to the multiplication). For the purposes of the present paper it suffices to know a small subalgebra of the canonical brackets and a representation of the field equations in terms of the bracket operation.

Using the notation $\omega_\mu := (-1)^{(\mu-1)} dx^1 \wedge ... \wedge \hat{dx}^\mu \wedge ... \wedge dx^n$ the canonical brackets in the (Lie) subalgebra of forms of degree 0 and $(n-1)$ read [5]

$$\{p^a_\mu \omega_\mu, y^b \} = \delta^b_a, \quad \{p^\mu_\mu \omega_\mu, y^b \omega_\nu \} = \delta^b_\nu \omega_\mu, \quad \{p^a_\mu, y^b \omega_\nu \} = \delta^b_\nu \delta^a_\mu, \quad (2, b, c)$$

with other brackets vanishing. All brackets in (2) reduce to the canonical bracket in mechanics when $n = 1$; in this sense they are canonical and can be viewed as a starting point of quantization.

Let us adopt the Dirac correspondence rule that Poisson brackets go over to commutators divided by $i \hbar$ and apply it to the canonical brackets (2). Note that this is just an assumption: while this principle proved to work well for the usual Poisson bracket its precise form and applicability to graded Poisson bracket in DW theory has to be confirmed. By quantizing (2a) we immediately conclude that

$$\hat{p}_a \omega_\mu = i \hbar \partial_a,$$

where $\partial_a$ is a partial derivative with respect to the field variables. The commutator corresponding to (2c) leads to a realization of $\hat{\omega}_\mu$ and $\hat{p}_a^\mu$ in terms of Clifford imaginary units, or Dirac matrices, under the assumption that the law of composition of operators is the symmetrized Clifford (=matrix) product [12,13]

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3) For the reason of a limited space we avoid discussing properties of graded Poisson bracket in DW theory in details. In what follows we simply chose facts which we need and refer the interested reader for more details to [5,10,13].
\[ \hat{p}_a^\nu = -i\kappa \gamma^\nu \partial_a, \quad \hat{\omega}_\nu = -\kappa^{-1} \gamma_\nu. \]  

(3)

The quantity \( \kappa \) of the dimension \([\text{length}]^{n-1}\) appears here on dimensional grounds. Due to the infinitesimal nature of the volume element \( \omega_\mu \) we expect the absolute value of \( \kappa \) to be “very large”. Hence its relation to the ultra-violet cutoff scale [12] can be anticipated (see also the last section before Conclusion).

Note that the realization of operators in terms of Clifford imaginary units implies a certain generalization of the formalism of quantum mechanics. Namely, whereas the conventional quantum mechanics is built up on complex numbers which are essentially the Clifford numbers corresponding to the one-dimensional space-time (= the time dimension in mechanics), the present approach to quantization of fields viewed as multi-parameter Hamiltonian systems (of the De Donder-Weyl type) makes use of the hypercomplex (Clifford) algebra of the underlying space-time manifold [14,15].

In order to guess the form of quantum equations of motions within the present approach it is important to know how the field equations are represented in terms of the bracket operation and what is the meaning of the bracket with the DW Hamiltonian function. In fact, the bracket with \( H \) exists only for forms of degree higher than \((n-1)\) [5]. Using \((n-1)\)-form canonical variables appearing in (2) DW Hamiltonian equations (1) can be written in Poisson bracket formulation as follows [5] (cf. [13])

\[ d(y^a \omega _\mu ) = *\{[H, y^a \omega _\mu]\} = *\partial H/\partial p^\mu a, \quad d(p^\mu _a \omega _\mu ) = *\{[H, p^\mu _a \omega _\mu]\} = -* \partial H/\partial y^a, \]  

(4)

where * is the Hodge duality operator acting on horizontal forms, and \( d \) is the total exterior differential \( dF := \frac{1}{p!}\partial_M F_{\mu_1}...\mu_p \partial_{\mu} z^M dx^\mu \wedge dx^{\mu_1} \wedge ... \wedge dx^{\mu_p} \), with \( z^M \) denoting the set of variables \((y^a, p^\mu_a, x^\mu)\). For more general dynamical variables represented by \( p \)-forms \( F \) we need a notion of the bracket with an \( n \)-form \( H_\omega \), where \( \omega := dx^1 \wedge ... \wedge dx^n \), which allows us to write the equations of motion in the symbolic form [5]

\[ dF = \{[H_\omega, F]\} + d^h F, \]

where \( d^h \) is the exterior differential with respect to the space-time (=horizontal) variables. Hence, we conclude that the DW Hamiltonian “generates” infinitesimal space-time variations of dynamical variables corresponding to the total exterior differentiation, much like the Hamilton function in mechanics generates the infinitesimal evolution along the time dimension.

Now, an analogue of the Schrödinger equation can be expected to have a form \( i\hbar \partial_\Psi \sim \hat{H} \Psi \), where \( i \) and \( \partial \) denote appropriate analogues of the imaginary unit and the exterior differentiation respectively. Keeping in mind the above remark on a hypercomplex generalization of quantum mechanics appearing here, an analogy between the exterior differential and the Dirac operator (in fact, the latter is \( d - *^{-1} d* \) [14]), and natural requirements imposed by the correspondence principle, the following generalization of the Schrödinger equation can be formulated [12,13,18]

\[ i\hbar \kappa \gamma^\mu \partial_\mu \Psi = \hat{H} \Psi, \]  

(5)
where \( \hat{H} \) is the operator corresponding to the DW Hamiltonian function, the constant \( \kappa \) of dimension \([\text{length}]^{-(n-1)}\) appears again on dimensional grounds, and \( \Psi = \Psi(y^a, x^\mu) \) is a wave function over the configuration space of field and space-time variables. In the following section we demonstrate that this equation fulfills several aspects of the correspondence principle. Note also that it reproduces the quantum mechanical Schrödinger equation at \( n = 1 \).

Let us construct the DW Hamiltonian operator for the system of interacting scalar fields \( y^a \) in flat space-time given by the Lagrangian density

\[
L = \frac{1}{2} \partial_\mu y^a \partial^\mu y_a - V(y).
\]

(6)

Then the polymomenta and the DW Hamiltonian function are given by

\[
p^a_\mu = \partial_\mu y^a, \quad H = \frac{1}{2} p^a_\mu p_\mu^a + V(y).
\]

(7)

DW Hamiltonian field equations take the form

\[
\partial_\mu y^a = p^a_\mu, \quad \partial_\mu p_\mu^a = -\partial V/\partial y^a,
\]

(8)

which is essentially a first order form of a system of coupled Klein-Gordon equations.

By quantizing the bracket

\[
\{[p^b_\mu p^a_\mu, y^b \omega]\} = 2p^b\nu
\]

(9)

we obtain \([13]\)

\[
\tilde{p}^a_\mu \tilde{p}_\mu^a = -\hbar^2 \kappa^2 \Delta,
\]

where \( \Delta := \partial_\alpha \partial^\alpha \) is the Laplacian operator in the space of field variables. Thus the DW Hamiltonian operator of the system of interacting scalar fields takes the form

\[
\tilde{H} = -\frac{1}{2} \hbar^2 \kappa^2 \Delta + V(y).
\]

(10)

Note that for a free scalar field \( V(y) = (1/2\hbar^2)m^2y^2 \), so that the DW Hamiltonian operator becomes similar to the Hamiltonian operator of the harmonic oscillator in the space of field variables. Its eigenvalues divided by \( \kappa \) read \( m_N = m(N + \frac{1}{2}) \).

Separating variables \( \Psi(y, x^\mu) = \Phi(x)f(y) \) from (5) we obtain

\[
\tilde{H}f_N = \kappa m_N f_N, \quad \hbar \gamma^\mu \partial_\mu \Phi = m_N \Phi.
\]

Then for a free scalar field any solution of (5) is a linear combination of

\[
\Psi_{N,k,r}(y, x, t) = u_{N,r}(k) f_N(y)e^{i\epsilon_r(\omega_{N,k} t - ik \cdot x)},
\]

(11)

where \( \omega_{N,k} := \sqrt{k^2 + m_N^2/\hbar^2} \), \( u_{N,r}(k) \) is a properly normalized constant spinor, \( \epsilon_r = +1(-1) \) for positive (negative) energy solutions, and \( f_N \) are eigenfunctions of
the harmonic oscillator in $y$-space. As a consequence, any Green function of (5) is given by [18]

$$ K(y', x'; y, x, t) = \sum_{N=0}^{\infty} \bar{f}_N(y') f_N(y) D_N(x' - x, t' - t), \quad (12) $$

where $D_N$ denotes a Green function of the spinor field of mass $m_N$. In doing so the type of the Green function $D$ should coincide with the type of the Green function $K$. Note that at large space-time separations $|x' - x| \gg \hbar/m$ the contribution of the term with $N = 0$ dominates, so that the asymptotic space-time behavior of corresponding Green functions is that of a spinor particle with mass $\frac{1}{2}m$. We hope to present a more detailed analysis elsewhere.

**THE CORRESPONDENCE PRINCIPLE**

In this section we discuss three properties of Eq. (5) which make it a proper candidate to the Schrödinger equation within the polymomentum quantization. All three are in fact different aspects of the correspondence principle.

Let us recall first that the DW canonical theory leads to its own field theoretic generalization of the Hamilton-Jacobi theory [2,4]. The corresponding Hamilton-Jacobi equation is a partial differential equation on $n$ functions $S^\mu = S^\mu(y^a, x^\nu)$

$$ \partial_\mu S^\mu + H(x^\mu, y^a, p_\mu^a = \partial S^\mu / \partial y^a) = 0. $$

In a simple example of scalar fields (6) the DW Hamilton-Jacobi equation reads

$$ \partial_\mu S^\mu = - \frac{1}{2} \partial_\alpha S^\mu \partial_\alpha S_\mu - \frac{1}{2} \frac{m^2}{\hbar^2} y^2. \quad (13) $$

Now, if we substitute (a hypercomplex analogue of) the quasiclassical ansatz

$$ \Psi = R \exp(i S^\mu \gamma_\mu / \hbar \kappa) \eta, \quad (14) $$

where $\eta$ is a constant reference spinor, to (5) and (10) we obtain a set of equations which can be transformed to the form [13]

$$ \partial_\mu S^\mu = - \frac{1}{2} \partial_\alpha S^\mu \partial_\alpha S_\mu - \frac{1}{2} \frac{m^2}{\hbar^2} y^2 + \frac{1}{2} \frac{\hbar^2 \kappa^2}{\hbar} \Delta R / R, \quad (15) $$

$$ \partial_\alpha S^\mu \partial_\alpha S_\mu = \partial_\alpha |S| \partial_\alpha |S|, \quad \partial_\mu S^\mu = \frac{S^\mu}{|S|} \partial_\mu |S|. \quad (16a, b) $$

In the first of these we recognize the DW Hamilton-Jacobi equation (13) with an additional term $\frac{1}{2} \frac{\hbar^2 \kappa^2}{\hbar} \Delta R / R$ which is similar to the so-called quantum potential known in quantum mechanics [16] and vanishes in the classical limit $\hbar \to 0$. Last two equations are supplementary conditions which appear most likely due to the
fact that the quasiclassical ansatz (14) does not represent a most general spinor, thus imposing certain restrictions on dynamics of the wave function. Note that in the case of quantum mechanics, \( n = 1 \), conditions (16a,b) reduce to trivial identities.

Thus, it is argued that in the classical limit equation (5) leads to the DW Hamilton-Jacobi equation (with two supplementary conditions which are specific to field theory and probably are due to restrictions imposed by the chosen in (14) analogue of the quasiclassical ansatz).

Another aspect of the correspondence principle we are to consider is the Ehrenfest theorem. Let us assume that expectation values of operators are given by

\[
\langle \hat{O} \rangle := \int dy \overline{\Psi} \hat{O} \Psi,
\]

where \( \overline{\Psi} \) is the Dirac conjugate of \( \Psi \). These expectation values depend on space-time points as the averaging is performed only over the field space. Using generalized Schrödinger equation (5) with the DW Hamiltonian (10) we can show that [13]

\[
\partial_\mu \langle \hat{p}_a^\mu \rangle = -\langle \partial_\mu \hat{H} \rangle, \quad \partial_\mu \langle \hat{y}_a^\mu \omega^\mu \rangle = \langle \hat{p}_a^\mu \omega^\mu \rangle.
\]

By comparing (18) with DW Hamiltonian field equations (8) we conclude that the latter are fulfilled "in average" as a consequence of the representation of operators (3), generalized Schrödinger equation (5), and the definition of expectation values (17). However, it should be noted that this property is fulfilled only for specially chosen operators (try e.g. to evaluate \( \partial_\mu \langle \hat{y}_a^\mu \rangle \) to see that this will not yield the desired result \( \langle \hat{p}_a^\mu \rangle \) for scalar fields). Moreover, the scalar product \( \int dy \overline{\Psi} \Psi \) implied by definition (17) in general is not positive definite and depends on points of the space-time. Therefore, it can not be used for a probabilistic interpretation. These drawbacks urge us to look for a more appropriate version of the Ehrenfest theorem.

An alternative is suggested by the fact that generalized Schrödinger equation (5) possesses a positive definite and time independent scalar product

\[
\int dx \int dy \overline{\Psi} \beta \Psi,
\]

where we introduced the notation \( \gamma^\mu =: (\gamma^i, \beta) \) \((i, j = 1, ..., n - 1)\), thus explicitly singling out the time variable \( t := x^n \) and the time component of \( \gamma \)-matrices: \( \beta := \gamma^t \) \((\beta^2 = 1)\). The existence of the satisfactory scalar product of this kind implies that the probabilistic interpretation of the wave function which fulfills generalized Schrödinger equation (5) is possible only if a time dimension is singled out. The wave function \( \Psi(y^a, x, t) \) is interpreted then as a probability amplitude of obtaining the field value \( y \) in the space point \( x \) in the moment of time \( t \). As a result, the theory becomes very much similar to usual quantum mechanics of a fictitious (spinor) particle in the space of variables \((y^a, x)\).
Now, new (global) expectation values of operators can be defined by

$$\langle \hat{O} \rangle := \int dy \int dx \overline{\Psi} \hat{O} \Psi.$$ \hspace{1cm} (20)

These expectation values depend only on time. Using definition (20) and generalized Schrödinger equation (5) written in the form

$$i\hbar \frac{\partial}{\partial t} \Psi = -i\hbar \alpha^i \frac{\partial}{\partial t} \Psi + \frac{1}{\kappa} \beta \hat{H} \Psi,$$ \hspace{1cm} (21)

where $\alpha^i := \beta \gamma^i$, we obtain

$$\frac{\partial}{\partial t} \langle y^a \rangle = \langle \hat{p}^a \rangle, \quad \frac{\partial}{\partial t} \langle \hat{p}^a \rangle = -\langle \hat{\partial}_a \hat{p}^a \rangle - \langle \partial_a \hat{H} \rangle.$$ \hspace{1cm} (22)

Note that in (22) we identified $\hat{\partial}_a \hat{p}^a$ with $-2i\hbar \kappa \gamma^i \partial_a \partial_i$. This identification is consistent with yet another aspect of the correspondence principle a discussion of which follows.

This aspect is a relation between the classical equations of motion and the Heisenberg equations of motion of operators. From (21) it follows that the time evolution is given by the operator

$$\hat{\mathcal{E}} := -i\hbar \alpha^i \frac{\partial}{\partial t} + \frac{1}{\kappa} \beta \hat{H}.$$ \hspace{1cm} (23)

Then, proceeding according to the standard quantum mechanics we obtain

$$\frac{\partial}{\partial t} y^a = \frac{i}{\hbar} [\hat{\mathcal{E}}, y^a] = \hat{p}^a, \quad \frac{\partial}{\partial t} \hat{p}^a = \frac{i}{\hbar} [\hat{\mathcal{E}}, \hat{p}^a] = -\hat{\partial}_a \hat{p}^a - \partial_a \hat{H},$$ \hspace{1cm} (24)

where we assumed as before

$$\hat{\partial}_a \hat{p}^a = -2i\hbar \kappa \gamma^i \partial_a \partial_i.$$ \hspace{1cm} (25)

Hence, as a consequence of generalized Schrödinger equation (5) and the representation of operators (3), the Heisenberg equations of motion have the same form as classical DW Hamiltonian equations (1) written in the form with a singled out time dimension.

**RELATION TO THE SCHRÖDINGER WAVE FUNCTIONAL**

In this section a possible relationship between the Schrödinger wave functional in quantum field theory [17] and our wave function is examined\(^4\). We confine ourselves

\(^4\) The presentation here essentially follows an unpublished preprint by the author [18].
to the simplest example of a free real scalar field. For the sake of simplicity we henceforth put \( n = 3 + 1 \) and \( \hbar = 1 \).

The idea is as follows. On the one hand, the Schrödinger wave functional \( \Psi(y(x), t) \) is known to be a probability amplitude of the field configuration \( y = y(x) \) to be observed in the moment of time \( t \). On the other hand, our wave function \( \Psi(y, x, t) \) can be interpreted as a probability amplitude of finding the value \( y \) of the field in the point \( x \) in the moment of time \( t \). Hence, the wave functional could in principle be related to a certain composition of single amplitudes given by our wave function.

Let us consider the Schrödinger functional corresponding to the vacuum state of a free scalar field [17]

\[
\psi_0(y(x), t) = \eta \exp \left( iE_0 t - \frac{1}{2} \int \frac{dk}{(2\pi)^3} \omega_k \bar{y}(k) \bar{y}(-k) \right),
\]

(26)

where the Fourier expansion \( y(x) = \int \frac{dk}{(2\pi)^3} y(k) e^{ikx} \) is used, \( \eta \) is a normalization factor, \( \omega_k := \sqrt{m^2 + k^2} \), and \( E_0 \) is the vacuum state energy

\[
E_0 = \lim_{Q \to \infty} \frac{1}{2} \int_V dx \int_Q \frac{dk}{(2\pi)^3} \omega_k
\]

(27)

which is divergent if either the ultraviolet cutoff \( Q \) of the volume of integration in \( k \)-space or the infrared cutoff \( V \) of the volume of integration over \( x \)-space go to infinity. The symbol \( \lim \) has a formal meaning throughout.

By replacing the Fourier integral by the Fourier series according to the rule \( \int \frac{dk}{(2\pi)^3} \to \lim_{V \to \infty} \frac{1}{V} \sum_{|k|} \), \( |k| \in \mathbb{Z}^3 \), the Schrödinger vacuum state functional can be written in the form of an infinite product of the harmonic oscillator ground state wave functions over all cells in \( k \)-space

\[
\psi_0(y(x), t) = \eta \lim_{V \to \infty} \prod_{|k|} \exp \left( \frac{1}{2} \left( i\omega_k t - \frac{1}{V} \omega_k y^2(|k|) \right) \right).
\]

(28)

Now, let us consider the ground state \((N = 0)\) wave functions (cf. Eq. (11)) of generalized Schrödinger equation (5) for a free scalar field

\[
\psi_{N=0,k}(y_k, x, t) = u_{N=0}(k)e^{i\omega_{0,k}t - ik \cdot x} e^{-\frac{m}{2\pi} y^2},
\]

(29)

where \( \omega_{0,k} = \sqrt{(\frac{m}{2})^2 + k^2} \). To simplify a subsequent analysis, which is in any case of preliminary character, we ignore in what follows the spinor nature of the wave function encoded in \( u_{N=0}(k) \). Taking into consideration the probabilistic interpretation of solutions (29) and assuming that there are no correlations between the field values in space-like separated points, the amplitude of finding in the vacuum state the whole configuration \( y = y(x) \) can be represented as an infinite product of single amplitudes given by the ground state solutions (29) with \( y = y(x) \) over
all points \( \mathbf{x} \) of the space. In order to ensure the spatial isotropy and homogeneity which are expected for the vacuum state we also have to take a product over all possible values of wave numbers because each separate mode with a wave number \( \mathbf{k} \) violates these properties. This also agrees with an idea of the vacuum state in which all possible \( \mathbf{k} \)-states are filled. Hence, the following symbolic formula for the approximate composed vacuum amplitude can be written (up to a normalization)

\[
\prod_{\mathbf{k}} \prod_{\mathbf{x}} e^{i\omega_0, \mathbf{k} \cdot \mathbf{x}} e^{-\frac{m}{2V} y(\mathbf{x})^2}.
\] (30)

This expression can be assigned a meaning if a certain discretization in both \( \mathbf{x} \)- and \( \mathbf{k} \)-spaces is assumed. This discretization can be related to finite values of cutoff parameters \( V \) and \( Q \) which imply minimal volume elements in \( \mathbf{k} \)-space and in \( \mathbf{x} \)-space to be, respectively, \( (2\pi)^3/V =: \xi^3 \) and \( (2\pi)^3/Q =: \lambda^3 \). Then coordinates in \( \mathbf{x} \)- and \( \mathbf{k} \)-space are integers \( \lfloor \mathbf{x} \rfloor \in \mathbb{Z}^3 \) and \( \lfloor \mathbf{k} \rfloor \in \mathbb{Z}^3 \) such that \( \mathbf{x} = \lfloor \mathbf{x} \rfloor \lambda, \mathbf{k} = \lfloor \mathbf{k} \rfloor \xi \).

The continuum limit formally corresponds to \( V \rightarrow \infty \) and \( Q \rightarrow \infty \), however, an analysis of its existence in mathematical sense is beyond the scope of the present consideration. Using this discretization, the obvious identity \( \prod_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}} = 1 \), and the Fourier series expansion \( y(\mathbf{x}) = \frac{1}{V} \sum_{|\mathbf{k}|} y_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}} \), we obtain

\[
\prod_{\mathbf{k}} \prod_{\mathbf{x}} e^{i\omega_0, \mathbf{k} \cdot \mathbf{x}} e^{-\frac{m}{2V} y(\mathbf{x})^2}
= \lim_{V \rightarrow \infty} \lim_{Q \rightarrow \infty} \prod_{\lfloor \mathbf{k} \rfloor} \prod_{\lfloor \mathbf{x} \rfloor} \exp \left( -\frac{m}{2\kappa} \frac{1}{V^2} \sum_{|\mathbf{q}'|} \sum_{|\mathbf{q}''|} y_{\mathbf{q}'} y_{\mathbf{q}''} e^{i(\mathbf{q}' + \mathbf{q}'') \cdot \mathbf{x}} \right)
= \lim_{Q \rightarrow \infty} \lim_{V \rightarrow \infty} \prod_{\lfloor \mathbf{k} \rfloor} \prod_{\lfloor \mathbf{x} \rfloor} \exp \left( -\frac{m}{2\kappa} \frac{1}{V^2} \sum_{|\mathbf{q}'|} \sum_{|\mathbf{q}''|} y_{\mathbf{q}'} y_{\mathbf{q}''} e^{i(\mathbf{q}' + \mathbf{q}'') \cdot \mathbf{x}} \right)
= \lim_{Q \rightarrow \infty} \lim_{V \rightarrow \infty} \prod_{\lfloor \mathbf{k} \rfloor} \prod_{\lfloor \mathbf{x} \rfloor} \exp \left( -\frac{m}{2\kappa} \frac{QV}{(2\pi)^3 V^2} \sum_{|\mathbf{q}|} y_{\mathbf{q}} y_{-\mathbf{q}} \right)
= \lim_{Q \rightarrow \infty} \lim_{V \rightarrow \infty} \prod_{\lfloor \mathbf{k} \rfloor} \exp \left( i\omega_0, \mathbf{k} - \frac{m}{2V} \frac{Q}{V} y_{\mathbf{k}} y_{-\mathbf{k}} \right),
\] (31)

where in passing to the fourth line we have taken into account that the number of cells both in \( \mathbf{x} \)- and \( \mathbf{k} \)-space is equal to \( QV/(2\pi)^3 \).

Let us compare the composed amplitude (31) with the standard vacuum functional in the form (28). Two additional parameters \( \kappa \) and \( Q \) appear in (31): \( Q \) is an (infinitely large) ultra-violet cutoff of the volume in \( \mathbf{k} \)-space, while \( \kappa \) is essentially the inverse of an infinitesimal (or very small) volume element in \( \mathbf{x} \)-space (cf. Eq. (3)), i.e a kind of fundamental length to the power 3. From the physical point of view it is quite natural to relate the inverse of the fundamental length to the ultraviolet cutoff. We thus identify \( \kappa = Q/(2\pi)^3 \) obtaining the composed amplitude

\[
\lim_{V \rightarrow \infty} \prod_{\lfloor \mathbf{k} \rfloor} \exp \left( i\omega_0, \mathbf{k} - \frac{m}{2V} \frac{Q}{V} y_{\mathbf{k}} y_{-\mathbf{k}} \right)
\] (32)
which is similar to (28) except that in (32) the proper mass $m$ appears instead of the frequency $\omega_k = \sqrt{m^2 + k^2}$ and $\omega_{0,k}$ replaces $\frac{1}{2}\omega_k$.

It is easy to see that the discrepancy between (28) and (32) disappears in the ultra-local limit $|k| \ll m$. In this limit the two-point Wightman function $\langle y(x_1) y(x_2) \rangle$ between space-like separated points $x_1$ and $x_2$ vanishes, so that there are no correlations between the field values in these points. This is, however, exactly the assumption which we made when writing the approximate composed amplitude in the form (30). Hence, in the ultra-local limit the composed amplitude constructed from the the ground state wave functions obeying generalized Schrödinger equation (5) is consistent with the Schrödinger wave functional of the vacuum state (28). Unfortunately, an attempt to extend this correspondence beyond the ultra-local limit leads to a difficulty of writing an expression for the composed amplitude similar to (30) which would account for all relevant correlations between the field values in space-like separated points.

Note, that another important byproduct of our analysis in this section is a conclusion that the constant $\kappa$ which appeared in (3) and (5) on purely dimensional grounds has to be identified with an ultraviolet cutoff scale quantity.

**CONCLUSION**

Field theories can be viewed as multi-parameter Hamiltonian-like systems in which space-time variables appear on equal footing as analogues of the time parameter in mechanics. A quantization of such a version of the Hamiltonian formalism leads to an extension of the formalism of quantum mechanics in which the Clifford algebra of underlying space-time manifold plays a key role similar to that of complex numbers in quantum mechanics. The latter thus appears as a special case of a theory with a single (time) parameter. In this formulation a description of quantized fields is achieved in terms of a (spinor) wave function on a finite dimensional analogue of the configuration space (the space of field and space-time variables). The wave function satisfies a multi-parameter covariant generalization of the Schrödinger equation, Eq. (5), which is a partial derivative equation similar to the Dirac equation with the mass term replaced by an operator corresponding to a multi-parameter (polymomentum) analogue of Hamilton’s canonical function. Note that despite the dynamics is formulated in a manifestly covariant manner the consideration of scalar products suggests that a proper probabilistic interpretation of the wave function still may require a time parameter to be singled out.

The description outlined above appears to be very different from that known in contemporary quantum field theory. A relation to the latter is a challenge to the theory presented here. In this paper we pointed out a relation to the Schrödinger wave functional which can thus far be followed only in ultra-local approximation. However, the latter is too rough for the real physics. Hence, further efforts are required to clarify possible connections with the standard quantum field theory.

Note in conclusion, that the present approach may have interesting applications
to the problem of quantization of gravity and field theories on non-Lorentzian space-times if the problems with the physical interpretation and the relationship to the standard quantum field theory are resolved. Further discussion can be found in [19] were a sketch of an approach to quantization of general relativity based on the present framework is presented.

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