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ON THE NUMBER OF ISOLATED ZEROS OF PSEUDO-ABELIAN INTEGRALS: DEGENERACIES OF THE CUSPIDAL TYPE

Aymen Braghtha

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Abstract

We consider a multivalued function of the form

\[ H_\varepsilon = P_0^\varepsilon \prod_{i=1}^k P_i^{\alpha_i}, \]

where \( P_i \in \mathbb{R}[x,y], \alpha_i \in \mathbb{R}_+^*, \) which is a Darboux first integral of polynomial one-form \( \omega = M \frac{dH}{dH_\varepsilon} = 0, M_\varepsilon = P_\varepsilon \prod_{i=1}^k P_i. \) We assume, for \( \varepsilon = 0, \) that the polycycle \( \{H_0 = H = 0\} \) has only cuspidal singularity which we assume at the origin and other singularities are saddles.

We consider families of Darboux first integrals unfolding \( H_\varepsilon \) (and its cuspidal point) and pseudo-Abelian integrals associated to these unfolding. Under some conditions we show the existence of uniform local bound for the number of zeros of these pseudo-Abelian integrals.

Keywords. integrable systems, blowing-up, singular foliations, singularities, abelian functions

1 Formulation of main results

In this paper, we study a non generic case. Other non generic cases have been studied in [1,3,4,5]. Pseudo-Abelian integrals appear as the linear principal part of the displacement function in polynomial perturbation of Darboux integrable case.

More precisely consider Darboux integrable system \( \omega \) given by

\[ \omega = Md\log H, \]

where

\[ M = \prod_{i=0}^k P_i, \quad H = \prod_{i=0}^k P_i^{\alpha_i}, \quad \alpha_i > 0, \quad P_i \in \mathbb{R}[x,y]. \]

Now we consider an unfolding \( \omega_\varepsilon \) of Darboux integrable system \( \omega, \) where \( \omega_\varepsilon \) are one-forms with first integral

\[ H_\varepsilon = P_\varepsilon^\alpha \prod_{i=1}^k P_i^{\alpha_i}, \quad \omega_\varepsilon = M_\varepsilon d\log H_\varepsilon, \quad M_\varepsilon = P_\varepsilon \prod_{i=1}^k P_i. \]

where the polynomial \( P_0 \) has a cuspidal singularity at \( p_0 = (0,0), \) i.e. \( P_0(x,y) = y^2 - x^3 + \mathcal{O}((x,y)^4). \)

For non zero \( \varepsilon, \) the polynomial \( P_\varepsilon = y^2 - x^3 - \varepsilon x^2 + \mathcal{O}((x,y,\varepsilon)^4). \)

Choose a limit periodic set i.e. bounded component of \( \mathbb{R}^2 \setminus \{ \prod_{i=0}^k P_i = 0 \} \) filled cycles \( \gamma(h) \subset \{H = h\}, h \in (0,a). \) Denote by \( D \subset H^{-1}(0) \) the polycycle which is in the boundary of this limit periodic set.

Consider the unfolding \( \omega_\varepsilon = M_\varepsilon d\log H_\varepsilon \) of the form \( \omega. \) The foliation \( \omega_\varepsilon \) has a maximal nest of cycles \( \gamma(\varepsilon,h) \subset \{H_\varepsilon = h\}, h \in (0,a(\varepsilon)), \) filling a connected component of \( \mathbb{R}^2 \setminus \{H_\varepsilon = 0\} \) whose boundary is a polycycle \( D_\varepsilon \) close to \( D. \) Assume moreover that the foliation \( \omega_\varepsilon = 0 \) has no singularities on \( \text{Int}D_\varepsilon. \)

Consider pseudo-Abelian integrals of the form

\[ I(\varepsilon, h) := \int_{\gamma(\varepsilon,h)} \eta_2, \quad \eta_2 = \frac{\eta_1}{M_\varepsilon} \]

(4)
where $\eta_1$ is a polynomial one-form of degree at most $n$.

This integral appears as the linear term with respect to $\beta$ of the displacement function of a polynomial perturbation

$$\omega_{\varepsilon, \beta} = \omega_{\varepsilon} + \beta \eta_1 = 0.$$  

(5)

We assume the following genericity assumptions

1. The level curves $P_i = 0, i = 1, \ldots, k$ are smooth and $P_i(0,0) \neq 0$.

2. The level curves $P_\varepsilon = 0, P_i = 0, i = 1, \ldots, k$, intersect transversally two by two.

**Theorem 1.** Under the genericity assumptions there exists a bound for the number of isolated zeros of the pseudo-Abelian integrals $I(\varepsilon, h) = \int_{\gamma(\varepsilon,h)} \eta_2$ in $(0,a(\varepsilon))$. The bound is locally uniform with respect to all parameters in particular in $\varepsilon$.

Let $\mathcal{F}_1 : \{\omega_{\varepsilon} = 0\}, \mathcal{F}_2 : \{d\varepsilon = 0\}$ are the foliations of dimension two in complex space of dimension three with coordinates $(x,y,\varepsilon)$.

Let $\mathcal{F}$ be the foliation of dimension one on the complex space of dimension three with coordinates $(x,y,\varepsilon)$ which is given by the intersection of leaves of $\mathcal{F}_1$ and $\mathcal{F}_2$ (i.e. given by the 2-form $\Omega = \omega_{\varepsilon} \wedge d\varepsilon$). This foliation has a cuspidal singularity at the origin (a cusp).

We want to study the analytical properties of the foliation $\mathcal{F}$ in a neighborhood of the cusp. For this reason we make a global blowing-up of the cusp of the product space $(x,y,\varepsilon)$ of phase and parameter spaces. We want our blow-up to separate the two branches of the cusp. This requirement leads to the quasi-homogeneous blowing-up of weight $(2,3,2)$.

**Remark 1.** In term of first integrals, the foliation $\mathcal{F}$ is given by two first integrals

$$H(x,y,\varepsilon) = h, \quad \varepsilon = s.$$  

## 2 Quasi-homogeneous blowing-up of $\mathcal{F}$

Recall the construction of the quasi-homogenous blowing-up. We define the weighted projective space $\mathbb{P}^2_{2,3,2}$ as factor space of $\mathbb{C}^3$ by the $\mathbb{C}^*$ action $(x,y,\varepsilon) \mapsto (t^2 x, t^3 y, t^2 \varepsilon)$. The quasi-homogenous blowing-up of $\mathbb{C}^3$ at the origin is defined as the incidence three dimensional manifold $W = \{ (p,q) \in \mathbb{P}^2_{2,3,2} \times \mathbb{C}^3 : \exists t \in \mathbb{C} : (q_1, q_2, q_3) = (t^2 p_1, t^3 p_2, t^2 p_3) \}$, where $(q_1, q_2, q_3) \in \mathbb{C}$ and $[(p_1, p_2, p_3)] \in \mathbb{P}^2_{2,3,2}$.

The quasi-homogeneous blowing-up $\sigma : W \to \mathbb{C}^3$ is just the restriction to $W$ of the projection $\mathbb{P}^2_{2,3,2} \times \mathbb{C}^3$.

We will need explicit formula for the blow-up in the standard affine charts of $W$. The projective space $\mathbb{P}^2_{2,3,2}$ is covered by three affine charts: $U_1 = \{ x \neq 0 \}$ with coordinates $(y_1, z_1)$, $U_2 = \{ y \neq 0 \}$ with coordinates $(x_2, z_2)$ and $U_3 = \{ \varepsilon \neq 0 \}$ with coordinates $(x_3, y_3)$.

The transition formula follow from the requirement that the points $(1,y_1,1), (x_2,1,z_2)$ and $(x_3, y_3,1)$ lie on the same orbit of the action:

$$F_2 : (y_1, z_1) \mapsto \left( x_2 = 1/y_1^{2/3}, z_2 = z_1/y_1 \sqrt{y_1} \right)$$

$$F_3 : (y_1, z_1) \mapsto \left( x_3 = 1/z_1, y_3 = y_1/z_1 \sqrt{z_1} \right).$$

These affine charts define affine charts on $W$, with coordinates $(y_1, z_1, t_1)$, $(x_2, z_2, t_2)$ and $(x_3, y_3, t_3)$. The blow-up $\sigma$ is written as

$$\sigma_1 : x = t_1^2, \quad y = y_1 t_1^3, \quad \varepsilon = t_1^3 z_1$$

(6)

$$\sigma_2 : x = t_2^2 x_2, \quad y = t_2^3, \quad \varepsilon = t_2^3 z_2$$

(7)

$$\sigma_3 : x = t_3^2 x_3, \quad y = t_3^3 y_3, \quad \varepsilon = t_3^3.$$  

(8)
We apply this blow-up \( \sigma \) to the one-dimensional foliation \( \mathcal{F} \). Let \( \sigma^{-1}\mathcal{F} \) the lifting of the foliation \( \mathcal{F} \) to the complement. This foliation has a cuspidal singularity at the origin. The pull-back foliation \( \sigma^*\mathcal{F} \) will be called the strict transform of the foliation \( \mathcal{F} \) is defined by the pull-back \( \sigma^*\Omega = \sigma^*(\omega_x \wedge d\varepsilon) \) divided by a suitable power of the function defining the exceptional divisor. In this charts \( U_j, j = 1, 2, 3 \) we have

\[
\sigma_1^*\Omega = x^3\Omega_1, \quad \sigma_2^*\Omega = y^3\Omega_2, \quad \sigma_3^*\Omega = \varepsilon^2\Omega_3,
\]

where

\[
\begin{align*}
\Omega_1 &= (6y^2 - 6 - 4z_1)dx \wedge dz_1 + 4y_1z_1dy_1 \wedge dx + 2xy_1dy_1 \wedge dz_1, \\
\Omega_2 &= (6 - 6x_2^2 - 4x_2^2z_2)dy \wedge dz_2 + (-6z_2x_2^2 - 4x_2^2z_2^2)dx_2 \wedge dy \\
+ (-3yx_2^2 - 2yx_2z_2)dx_2 \wedge dz_2, \\
\Omega_3 &= (-6x_3^2 - 4x_3)dx_3 \wedge dz + 4y_3dy_3 \wedge d\varepsilon.
\end{align*}
\]

**Remark 2.** In term of first integrals, the foliation \( \sigma^*\mathcal{F} \) is given by two first integrals

\[
\sigma^*H(x, y, \varepsilon) = h, \quad \sigma^*\varepsilon = s,
\]

In particular in a neighborhood of the exceptional divisor the restrictions of the foliation \( \sigma^*\mathcal{F} \) to the charts \( U_1 \) and \( U_3 \) are given respectively, by

\[
\begin{align*}
\psi_1 &= H(t_1^2, t_1^3y_1, t_1^3z_1) = x^3(y_1^2 - 1) = h, \quad \varphi_1 = xz_1 = s, \\
\psi_3 &= H(t_3^2x_3, t_3^3y_3, t_3^3z_3) = \varepsilon^3(y_3^2 - x_3^3) = h, \quad \varphi_3 = \varepsilon = s,
\end{align*}
\]

where \( \{x = 0\} \) and \( \{\varepsilon = 0\} \) are local equations of the exceptional divisor respectively.

### 3 Singular locus of the foliation \( \sigma^*\mathcal{F} \)

In this section, we compute the singular locus of the pull-back \( \sigma^*\Omega \) in a neighborhood of the exceptional divisor \( \mathbb{CP}^2_{2;3;2} \). We check it in each chart separately.

In the chart \( U_1 \), the zeros locus of the form \( \Omega_1 \) in a neighborhood of the exceptional divisor \( \{x = 0\} \) consists of germs of two curves \( \{y_1 = \pm 1, z_1 = 0\} \) and a two singular points \( p_1 = (0, 1, 0), p_2 = (0, -1, 0) \) generated by the quasi-homogeneous blowing-up.

In the chart \( U_3 \), the zeros locus of the form \( \Omega_3 \) in a neighborhood of the exceptional divisor \( \{\varepsilon = 0\} \) consists of \( p_3 = (0, 0, 0) \) (Morse point) and \( p_4 = (-\frac{2}{3}, 0, 0) \) (center). The singularities of this foliation are the line of Morse points \( x_3 = 0, y_3 = 0 \), the lines of centers \( x_3 = -\frac{2}{3}, y_3 = 0 \) and the transform strict of \( \{y^2 - x^3 - x^2\varepsilon = 0\} \).

**Proposition 1.** The singularities of \( \sigma^*\mathcal{F} \) are located at the points \( p_1, p_2, p_3 \) and \( p_4 \). The points \( p_1, p_2 \) and \( p_3 \) are linearisable saddles and the point \( p_4 \) is a center.

**Proof.** Since \( \sigma : \mathbb{C} \to \mathbb{C}^3 \) is a biholomorphism outside the exceptional divisor \( \mathbb{CP}^2_{2;3;2} \), all singularities of \( \sigma^*\mathcal{F} \) on \( \mathbb{C}^3 \setminus \{x = 0\} \) correspond to singularities of \( \mathcal{F} \). Thus, it suffices to compute the singularities of \( \sigma^*\mathcal{F} \) on the exceptional divisor \( \{x = 0\} \). More precisely, we consider the foliation on neighborhood of \( \mathbb{CP}^2_{2;3;2} \) (the exceptional divisor) generated by the blown-up one-form \( \sigma^*\Omega \). Let \( \psi_1, \psi_3 \) are the functions given in (13) and (14).

(1) In the chart \( U_1 \), near the divisor exceptional and for \( |z_1| \leq \varepsilon \) for \( \varepsilon \) sufficiently small, the foliation \( \sigma^*\mathcal{F} \) is given by two first integrals

\[
G_1 = \varphi_1^3\psi_1^{-1} = z_1^3(y_1^2 - (1 + z_1))^{-1}V^{-1} = s^3h^{-1}, \quad \varphi_1 = xz_1 = s.
\]
where $V$ is analytic function such that $V(0,0,0) \neq 0$. In particular on the exceptional divisor \{x = 0\} the foliation $\sigma^*F$ is given by the levels $G_1 = s^3 h^{-1} = t$.

Now we calculate the eigenvalues at $p_1$ and $p_2$. The vector field $V_1$ generating the foliation $\sigma^* F$ is given by

$$V_1(x,y_1,z_1) = \beta_1 x \frac{\partial}{\partial x} + \beta_2 y_1 \frac{\partial}{\partial y_1} + \beta_3 z_1 \frac{\partial}{\partial z_1},$$

where the vector $(\beta_1, \beta_2, \beta_3)$ satisfies the following equations

$$< (\beta_1, \beta_2, \beta_3), (3,1,0) > = 0, < (\beta_1, \beta_2, \beta_3), (1,0,1) > = 0$$

here $<,>$ be the usual scalar product on $\mathbb{C}^3$. By simple computation, we obtain $\beta_1 = 1, \beta_2 = -3$ and $\beta_3 = -1$.

(2) In the chart $U_3$, near the exceptional divisor \{\varepsilon = 0\}, the foliation $\sigma^* F$ is given by

$$G_3 = \varphi_3^3 \psi_3^{-1} = (y_3^2 - x_3^2(1 + x_3))^{-1} = s^3 h^{-1}, \quad \varphi_3 = \varepsilon = s.$$

In particular the restriction of this foliation to the exceptional divisor \{\varepsilon = 0\}, by Morse lemma we can put the function $1/G_3$ to the normal form $y_3^2 - x_3^2$ in a neighborhood of $p_3$ (we put the variable change $z_3 = \pm x_3(1 + x_3)^{1/2}$). On other hand the Hessian matrix of $1/G_3$ at the point $p_4$ has two positive eigenvalues.

4 The different scaled variations of $\delta(s,t)$

In this section, we compute the scaled variations with respect to different variables $s$ and $t$ of the integrals of the blown-up one form $\sigma^*_1 \eta_2$ along the different relatives cycles using the same technics of [5].

Proposition 2. The computation of the different scaled variations of the cycle $\delta(s,t)$ us gives

1. For $t \in [0,2N]$, the cycle $\delta(s,t)$ satisfies a iterated scaled variations with respect to $t$ of the form

$$\text{Var}_{(t,3)} \circ \text{Var}_{(t,-1)} \circ \text{Var}_{(t,-\alpha_1)} \circ \ldots \circ \text{Var}_{(t,-\alpha_k)} \delta(s,t) = 0. \quad (15)$$

2. For $t \in [N, +\infty)$, the cycle $\delta(s,t)$ satisfies a iterated scaled variations with respect to $1/t$ of the form

$$\text{Var}_{(1/t,-3)} \circ \text{Var}_{(1/t,1)} \circ \text{Var}_{(1/t,\alpha_1)} \circ \ldots \circ \text{Var}_{(1/t,\alpha_k)} \delta(s,1/t) = 0. \quad (16)$$

3. Near $s = 0$, we have

$$\text{Var}_{(s,1)} \circ \text{Var}_{(s,1)} \delta(s,t) = \text{Var}_{(s,1)}(\hat{\delta}(s,t)) = 0, \quad (17)$$

where $\text{Var}_{(s,1)} \delta(s,t) = \hat{\delta}(s,t)$ is a figure eight cycle.

Proof. As in [5], there exist a some local chart with coordinates $(u,v,w)$ defined in a some neighborhood of each separatrix of polycycle such that the foliation $\sigma^*F$ is defined by two first integrals. Precisely:

1. for $t \in [0,2N]$, there exist a local chart $(V_{div},(u,v,w))$ defined in neighborhood of the separatrix $\delta_{div}$ such the foliation $\sigma^*F$ by two first integrals

$$F_1 = w^3(v-1)^{-1}(v+1)^{-1} = t, \quad F_2 = uw = s,$$
2. for $t \in [N, +\infty]$, there exists a local chart $(V_{div}^+, (u, v, w))$ defined in neighborhood of the separatix $\delta_{div}^+$ such that the foliation $\sigma_1^+ F$ is defined by two first integrals

$$F_1 = u^3(v + 2)^{-1}v^{-1} = t, \quad F_2 = uw = s.$$ 

3. for $t \in [N, +\infty]$, there exists a local chart $(V_{div}^-, (u, v, w))$ defined in a neighborhood of the separatix $\delta_{div}^-$ such that the foliation $\sigma_1^- F$ is defined by two first integrals

$$F_1 = u^3(v - 2)^{-1}v^{-1} = t, \quad F_2 = uw = s.$$ 

In second step we prove that each relative cycle can be chosen as a lift of a path contained in the separatix associated to this relative cycle. Precisely:

1. on the chart $(V_{div}, (u, v, w))$, the linear projection $\pi(u, v, w) = v$ is everywhere transverse to the levels of the foliation $\sigma^* F$ which corresponds simply to the graphs of the multivalued functions

$$v \mapsto (u, w) = \left( st^{-\frac{1}{2}}v^{-\frac{1}{2}}(v - 1)^{-\frac{1}{2}}v + 1)^{\frac{1}{2}}, t^{1/2}(v - 1)^{-1/2}(v + 1)^{1/2} \right),$$

2. on the chart $(V_{div}^+, (u, v, w))$, the linear projection $\pi(u, v, w) = v$ is everywhere transverse to the levels of the foliation $\sigma^+ F$ which corresponds simply to the graphs of the multivalued functions

$$v \mapsto (u, w) = \left( st^{-\frac{1}{2}}v^{-\frac{1}{2}}(v + 2)^{-\frac{1}{2}}, t^{1/2}(v + 2)^{1/2} \right),$$

3. on the chart $(V_{div}^-, (u, v, w))$, the linear projection $\pi(u, v, w) = v$ is everywhere transverse to the levels of the foliation $\sigma^- F$ which corresponds simply to the graphs of the multivalued functions

$$v \mapsto (u, w) = \left( st^{-\frac{1}{2}}v^{-\frac{1}{2}}(v - 2)^{-\frac{1}{2}}, t^{1/2}(v - 2)^{1/2} \right).$$

In third step, we compute the different scaled variations of relatives cycles using the local expression of two first integrals $F_1$ and $F_2$ above near the singular points $p_1, p_2$ and $p_3$. Recall that the scaled variation of a relative cycle $\delta(s)$ is given by

$$\text{Var}_{(s, \beta)} \delta(s) = \delta(se^{i\pi \beta}) - \delta(se^{-i\pi \beta}).$$

In the local chart $(V_{div}^+, (u, v, w))$, the restriction of the blown-up foliation $\sigma_1^+ F$ to the transversals sections $\Sigma_{div}^- = \{ w = 1 \}$ (near the point $p_3$) and $\Omega_+ = \{ u = 1 \}$ (near the point $p_1$) is given respectively by

$$F_1|_{\Sigma_{div}^-} = \frac{1}{v} = t, \quad F_2|_{\Sigma_{div}^-} = u = s,$$

$$F_1|_{\Omega_+} = \frac{w^3}{v} = t, \quad F_2|_{\Omega_+} = w = s.$$ 

Let us fix $t \in [N, +\infty]$. We observe that the restriction of the foliation $\sigma_1^+ F$ to the transversal section $\Sigma_{div}^- = \{ w = 1 \}$ is analytic with respect to $s$. Then, after taking an scaled variation with respect to $s$, the relative cycle $\delta_{div}^+ (s, t)$ is replaced by a loop $\theta_1$, modulo homotopy, which consists of line segment $\ell_{31} = [p_3, p_1]$ connecting the Morse point $p_3$ with the point $p_1$ encircling the latter along a small counterclockwise circular arc $\alpha_1$ and then returning along the segment $\ell_{13} = [p_1, p_3]$. The loop $\theta_1$ can be moved along the complex curve $\{ u = 0 \}$. Then, we have

$$\text{Var}_{(s, \beta)} \delta_1^+ (s, t) = \theta_1 = \ell_{31} \alpha_1 \ell_{13}.$$ 

The same computation of the scaled variation with respect to $s$ for the relative cycle $\delta_{div}^-(s, t)$ gives us a loop $\theta_3$, modulo homotopy, which can be moved along the complex plane $\{ u = 0 \}$. The loop $\theta_3$
consists of line segment $\ell_{32} = [p_3, p_2]$ connecting the point $p_3$ with the point $p_2$ encircling the latter along a small counterclockwise circular arc $\alpha_3$ and then returning along the segment $\ell_{23} = [p_2, p_3]$. Then, we have

$$\text{Var}_{(s, t)} \delta_{\text{div}}(s, t) = \theta_3 = \ell_{32} \alpha_3 \ell_{23}.$$ 

In the local chart $(V_{\text{div}}, (u, v, w))$, we define the transversal section $\Omega_+ = \{ u = 1 \}$ (resp $\Omega_+ = \{ u = 1 \}$) near $p_1$ (resp near $p_2$). The restriction of the foliation $\sigma_1^* F$ to the transversal section $\Omega_+$ is given by

$$F_1|_{\Omega_+} = \frac{w^3}{v} = t, \quad F_2|_{\Omega_+} = w = s.$$ 

On the second step let us fix $t \in [0, 2N]$. After taking an scaled variation with respect to $s$, the relative cycle $\delta_{\text{div}}(s, t)$ is replaced by a figure eight cycle which can be moved along the complex line $C'_\text{div} = \{ x = 0, G_1 = t \}$ of the foliation $\sigma_1^* F$. This case is similar to the classical situation which is studied by Bobiński and Mardešić in [2].

Now using the analyzability of the lifting $\sigma^{-1} F$ with respect to $s$, the scaled variation of the cycle of integration $\delta(s, t)$ with respect to $s$ is equal to the scaled variation with respect to $s$ of the following difference $\delta_{\text{div}}^+(s, t) - \delta_{\text{div}}^-(s, t)$ which is equal, modulo homotopy, to the cycle $\theta_1 \theta_3^{-1}$, where $\theta_3^{-1}$ is the inverse of the loop $\theta_3$. Schematically, the loop $\theta_1 \theta_3^{-1}$ is a figure eight cycle.

**Remark 3.**

- In the local chart $(V_{\text{div}}^+, (u, v, w))$ (resp $(V_{\text{div}}^-, (u, v, w))$, the loop $\theta_1$ (resp $\theta_3$) generating the fundamental group of the complex plane $\{ u = w = 0 \} \setminus \{ p_1 \}$ (resp $\{ u = w = 0 \} \setminus \{ p_2 \}$) with base point $p_3$.

- By the univalness of the blown-up one form $\sigma_1^* \eta_2$, we have

$$\text{Var}_{(t, \alpha)} \int_{\delta(s, t)} \sigma_1^* \eta_2 = \int_{\text{Var}_{(t, \alpha)} \delta(s, t)} \sigma_1^* \eta_2.$$ 

## 5 Proof of the Theorem

In this section we first take benefit from the blowing-up in the family to prove our principal theorem. The proof is analogous of the following:

**Theorem 2.** There exists a bound of the number of zeros of the function $t \mapsto J(s, t)$, for $t \in [0, +\infty]$ and $s > 0$ sufficiently small. This bound is locally with respect to all parameters uniform, in particular with respect to $s$.

Let $\beta = (\beta_1, \ldots, \beta_{k+2})$ where $\beta_1 = 3, \beta_2 = -1, \beta_3 = -\alpha_1, \ldots, \beta_{k+2} = -\alpha_k$. Let $D_1$ is slit annulus in the complex plane $\mathbb{C}_1^*$ with boundary $\partial D_1$. This boundary is decomposed as follows $\partial D_1 = C_{R_1} \cup C_r \cup C_{\pm}$, where $C_{R_1} = \{ |t| = R_1, |\arg t| \leq \alpha \pi \}, C_r = \{ r_1 < |t| < R_1, |\arg t| = \alpha \pi \}$ and $C_{\pm} = \{ |t| = r_1, |\arg t| \leq \alpha \pi \}$.

Petrov’s method gives us that the number of zeros $\# Z(J(s, t))$ of the function $J(s, t)$ in slit annulus $D_1$ is bounded by the increment of the argument of $J(s, t)$ along $\partial D_1$ divided by $2\pi$ i.e.

$$\# Z(J(s, t)|_{D_1}) \leq \frac{1}{2\pi} \Delta \arg(J(s, t)|_{\partial D_1}) = \frac{1}{2\pi} \Delta \arg(J(s, t)|_{C_{R_1}})$$

$$+ \frac{1}{2\pi} \Delta \arg(J(s, t)|_{C_{\pm}}) + \frac{1}{2\pi} \Delta \arg(J(s, t)|_{C_r})$$

**A** The increment of argument $\Delta \arg(J(s, t)|_{C_{R_1}})$ is uniformly bounded by Gabrielov’s theorem [6].
(B) We use the Schwartz’s principle

\[ \text{Im}(J(s,t))|_{C^\pm} = \mp 2i\text{Var}_{(t,\alpha)} J(s,t). \]

Thus, the increments of argument along segments \( C^\pm \) are bounded by zeros of the variation \( \text{Var}_{(t,\alpha)} J(s,t) \) on segment \((r, R)\). By identity (18), the function \( \text{Var}_{(t,\beta)} J(s,t) \) can be written as follows

\[
\text{Var}_{(t,\beta)} J(s,t) = K(t^{\frac{d_1}{\pi}}, \ldots, t^{\frac{d_k}{\pi}}, s; \log s)
= K(e^{\frac{d_1}{2} \log t}, \ldots, e^{\frac{d_k}{2} \log t}, e^{\log s}, \log s)
\]

where \( K \) is a meromorphic function. The function \( \text{Var}_{(t,\beta)} J(s,t) \) is logarithmico-analytic function of type 1 in the variable \( s \) (see [9]). Then, there exist a finite recover of \( \mathbb{R}^{k+\mu+1} \times \mathbb{R} \) by a logarithmico-exponential cylinders, using Rolin-Lion’s theorem [9], such that on each cylinder of this family we have

\[
\text{Var}_{(t,\beta)} J(s,t) = y_0^\mu y_1^\nu A(t)U(t, y_0, y_1),
\]

with \( y_0 = s - \theta_0(t), y_1 = \log y_0 - \theta_1(t) \), where \( \theta_0, \theta_1, A \) are logarithmico-exponential functions and \( U \) is a logarithmico-exponential unity function. As the number of zeros of a logarithmico-exponential function is bounded, the number of zeros of \( \text{Var}_{(t,\beta)} J(s,t) \) is bounded.

(C) Finally, we estimate the increment of argument of \( J \) along the small arc \( C_{r_1} \). Then, it is necessarily to study the increment of argument of the leading term of the function \( J \) at \( t = 0 \).

**Lemma 1.** The increment of the argument of \( J(s,t) \) along the small circular arc \( C_{r_1} \), can be estimated by the increment of the argument of a some meromorphic function \( F(s,t) \).

**Proof.** The problem of the estimation of the increment of the argument of \( J(s,t) \) along the circular arc \( C_{r_1} \), consist that the principal part of the function \( J \) contains the term \( \log s \to -\infty \) as \( s \to 0 \). To resolve this problem we make a blowing-up at the origin in the total space with coordinates \((x, y, z)\) where

\[
x = J_1(s,t), \quad y = J_2(s,t), \quad z = (\log s)^{-1}.
\]

The function \( J(s,t) \) can be rewritten as follows

\[
J(s,t) = J_1(s,t) + J_2(s,t) \log s = ((\log s)^{-1} J_1(s,t) + J_2(s,t)) \log s = (x + y) z^{-1}.
\]

Thus, for \( z^{-1} \in \mathbb{R} \) be sufficiently small, we have

\[
\arg(J(s,t)) = \arg((x + y) z^{-1}) = \arg(x + y).
\]

To estimate the increment of argument of \( x + y \) uniformly with respect to \( s > 0 \) we make a quasi-homogeneous blowing-up \( \pi_1 \) with weight \((\frac{1}{2}, 1, \frac{1}{2})\) of the polynomial \( x + y \) at \( C_1 = \{x = y = z = 0\} \) (the centre of blowing-up). The explicit formula of the quasi-homogeneous blowing-up \( \pi_1 \) in the affine charts \( T_1 = \{x \neq 0\}, T_2 = \{y \neq 0\} \) and \( T_3 = \{z \neq 0\} \) is written respectively as

\[
\begin{align*}
\pi_{11} : x &= \sqrt{x_1}, \quad y = y_1 x_1, \quad z = z_1 \sqrt{x_1}, \\
\pi_{12} : x &= x_2 \sqrt{y_2}, \quad y = y_2, \quad z = z_2 \sqrt{y_2}, \\
\pi_{13} : x &= x_3 \sqrt{z_3}, \quad y = y_3 z_3, \quad z = \sqrt{z_3}.
\end{align*}
\]

The pull-back \( \pi_1^*(x + y) \) is given, in different charts, by

\[
\begin{align*}
\pi_{11}^*(x + y) &= x_1(z_1 + y_1) = d_1 P_1(x_1, y_1, z_1), \\
\pi_{12}^*(x + y) &= y_2(x_2 z_2 + 1) = d_2 P_2(x_2, y_2, z_2), \\
\pi_{13}^*(x + y) &= z_3(x_3 + y_3) = d_3 P_3(x_3, y_3, z_3).
\end{align*}
\]
where $d_i = 0$ and $P_i = 0$ are equations of exceptional divisor and the strict transform of $zx + y = 0$ respectively.

Observe that $P_i = 0, i = 1, 3$ has not a normal crossing with the exceptional divisor $d_i = 0, i = 1, 3$. To resolve this problem we make a second blowing-up $\pi_2$ with centre a subvariety $C_2$ which is given, in different charts, as following:

1. In the chart $T_1$, choose a local coordinate chart with coordinates $(x_1, y_1, z_1)$ in which $C_2 = \{y_1 = z_1 = 0\}$. Then $\pi_2^{-1}(C_2)$ is covered by two coordinates charts $U_{y_1}$ and $U_{z_1}$ with coordinate $(\tilde{x}_1, \tilde{y}_1, \tilde{z}_1)$ where in $y_1$-chart $U_{y_1}$ the blowing-up $\pi_2$ is given by $x_1 = \tilde{x}_1, y_1 = \tilde{y}_1, z_1 = \tilde{z}_1\tilde{y}_1$ and in $z_1$-chart $U_{z_1}$ the blowing-up $\pi_2$ is given by $x_1 = \tilde{x}_1, y_1 = \tilde{y}_1\tilde{z}_1, z_1 = \tilde{z}_1$.

2. In the chart $T_2$, the blowing-up $\pi_2$ is a biholomorphism ($\pi_2$ is a proper map).

3. In this chart $T_3$, choose a local coordinate chart with coordinates $(x_3, y_3, z_3)$ in which $C_2 = \{x_3 = y_3 = 0\}$. Then $\pi_2^{-1}(C_2)$ is covered by two coordinates charts $U_{x_3}$ and $U_{y_3}$ with coordinate $(\tilde{x}_3, \tilde{y}_3, \tilde{z}_3)$ where in $x_3$-chart $U_{x_3}$ the blowing-up $\pi_2$ is given by $x_3 = \tilde{x}_3, y_3 = \tilde{y}_3\tilde{x}_3, z_3 = \tilde{z}_3$ and in $y_3$-chart $U_{y_3}$ the blowing-up $\pi_2$ is given by $x_3 = \tilde{x}_3\tilde{y}_3, y_3 = \tilde{y}_3, z_3 = \tilde{z}_3$.

The pull-back $\pi_1^*(zx + y)$ is given, in different charts, by

- In the $y_1$-chart $U_{y_1}$, the transformation of the pull-back $\pi_1^*(zx + y)$ by the blowing-up $\pi_2$ is given by
  \[\pi_2 \circ \pi_1^*(zx + y) = \pi_2^*(d_1 P_1(x_1, y_1, z_1)) = \tilde{x}_1\tilde{y}_1(\tilde{z}_1 + 1) \equiv \tilde{x}_1\tilde{y}_1 = J_2(s, t) = F(s, t)\]

- In the $z_1$-chart $U_{z_1}$, the transformation of the pull-back $\pi_1^*(zx + y)$ by the blowing-up $\pi_2$ is given by
  \[\pi_2^* \circ \pi_1^*(zx + y) = \pi_2^*(d_1 P_1(x_1, y_1, z_1)) = \tilde{z}_1\tilde{x}_1(\tilde{y}_1 + 1) \equiv \tilde{x}_1(\log s)^{-1}J_1(s, t) = F(s, t)\]

- In the chart $T_2$, we have
  \[\pi_2^* \circ \pi_1^*(zx + y) = \pi_2^*(d_2 P_2(x_2, y_2, z_2)) = d_2 P_2(x_2, y_2, z_2) = (\log s)^{-1}J_1(s, t) + J_2(s, t) = F(s, t)\]

- In the $x_3$-chart $U_{x_3}$, the transformation of the pull-back $\pi_1^*(zx + y)$ by the blowing-up $\pi_2$ is given by
  \[\pi_2^* \circ \pi_1^*(zx + y) = \pi_2^*(d_3 P_3(x_3, y_3, z_3)) = \tilde{z}_3\tilde{x}_3(\tilde{y}_3 + 1) \equiv \tilde{x}_3(\log s)^{-1}J_1(s, t) = F(s, t)\]

- In the $y_3$-chart $U_{y_3}$, the transformation of the pull-back $\pi_1^*(zx + y)$ by the blowing-up $\pi_2$ is given by
  \[\pi_2^* \circ \pi_1^*(zx + y) = \pi_2^*(d_3 P_3(x_3, y_3, z_3)) = \tilde{y}_3(\tilde{x}_3 + 1) \equiv \tilde{y}_3 = J_2(s, t) = F(s, t)\]

Finally, we distinguish three cases:

1. $\arg_{C_{r_1}} J(s, t) = \arg_{C_{r_1}} ((\log s)^{-1}J_1(s, t)) = \arg_{C_{r_1}} J_1(s, t), ((\log s)^{-1} \in \mathbb{R})$
2. $\arg_{C_{r_1}} J(s, t) = \arg_{C_{r_1}} J_2(s, t)$
3. In the chart $T_2$, the function $F(s, t) = ((\log s)^{-1}J_1(s, t)) + J_2(s, t)$ is meromorphic. 

\[\square\]
Now we define the functional space $P_\beta$ which are formed of coefficients of the polynomials $P_i$ of the Darboux first integral $H$, the coefficients of the polynomials $R, S$ of the perturbative one forme $\eta$, exponents $\alpha_i$ and degrees $n_i = \deg P_i, n = \max(\deg R, \deg S)$. Consider the following finite dimensional functional space $P_\beta$

$$P_\beta(m_\beta, M_\beta; \beta_1, \ldots, \beta_{k+2}) = \{ \sum_{j=1}^{k+2} \sum_{n, \ell} A_{j\ell n}(s) t^{\beta_j} n s^m \log^\ell(t) : A_{j\ell n}(s) \in \mathbb{C}, m_\beta < A_{j\ell n} < M_\beta, 0 \leq \ell \leq k+1 \}.$$  

For the first two cases, the function $J_i(s, t), i = 1, 2$ satisfies the following iterated variations equation with respect to $t$

$$\var{t, \beta_1} \circ \ldots \circ \var{t, \beta_{k+2}} J_i(s, t) = 0.$$  

Thus, by Lemma 4.8 from [2], there exists a non zero leading term $P_{i\beta} \in P_\beta$ of $J_i(s, t), i = 1, 2$ at $t = 0$ such that $|J_i(s, t) - P_{i\beta}(s, t)| = O(t^{\mu_1}), \mu_1 > 0$, uniformly in $s$. Moreover, the function $J_i(s, t), i = 1, 2$ satisfies the iterated variation equation

$$\var{(s, 1)} J_i(s, t) = 0.$$  

Thus, we have $J_i(s, t) = O(s^{\mu_2}), \mu_2 > 0$, uniformly in $t$.

For each element in the parameter space, we can choose the leading term of $P_{i\beta}$. The increment of argument of this leading term is bounded by a constant $C(M_\beta, k + 2, \beta_{k+2})$. Since the leading term of $P_{i\beta}$ is also the leading term of $J_i(s, t)$, the limit $\lim_{r_1 \to 0} \Delta \arg(J_i(s, t) | C_{r_1}) \leq C(M_\beta, k + 2, \beta_{k+2})$.

In the chart $T_2$, the function $F$ is meromorphic. Thus, this function can be rewritten as following

$$F(s, t) = (\log s)^{-1} J_1(s, t) + J_2(s, t) = G(t^{\beta_1}, \ldots, t^{\beta_s}, s, (\log s)^{-1})$$

where $G$ is meromorphic function. The number $\# Z(G)$ of zeros of the function $G$ is uniformly bounded. The latter claim is a direct application of fewnomials theory of Khovanskii [8]: since the functions $\epsilon_i(t) = t^{\beta_i}, \epsilon(s) = (\log s)^{-1}$ are Pfaffian functions (solutions of Pfaffian equations $tde_i - \beta_i \epsilon dt = 0$ and $sde + \epsilon^2 ds$, respectively), the upper bound for this number of zeros can be given, using Rolle-Khovanskii arguments of [7], in terms of the number of zeros of some polynomial and its derivatives. The latter are uniformly bounded by Gabrielov’s theorem [6].

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