A nonperturbative light-front coupled-cluster method

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(Dated: August 31, 2012)

Abstract

The nonperturbative Hamiltonian eigenvalue problem for bound states of a quantum field theory is formulated in terms of Dirac’s light-front coordinates and then approximated by the exponential-operator technique of the many-body coupled-cluster method. This approximation eliminates any need for the usual approximation of Fock-space truncation. Instead, the exponentiated operator is truncated, and the terms retained are determined by a set of nonlinear integral equations. These equations are solved simultaneously with an effective eigenvalue problem in the valence sector, where the number of constituents is small. Matrix elements can be calculated, with extensions of techniques from standard coupled-cluster theory, to obtain form factors and other observables.

PACS numbers: 12.38.Lg, 11.15.Tk, 11.10.Ef

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a Presented at QCD@Work2012, the International Workshop on QCD Theory and Experiment, June 18-21, 2012, Lecce, Italy.
I. INTRODUCTION

We wish to construct a method to compute hadron structure in terms of wave functions by solving the light-front Hamiltonian eigenvalue problem for quantum chromodynamics. The use of light-front quantization [1] allows for a well-defined Fock expansion [2], as indicated schematically in Fig. 1 for a proton. The fundamental eigenvalue problem
\[(K.E. + V_{QCD})|p\rangle = E_p|p\rangle,\]  
(1.1)
with K.E. representing the kinetic energy, \(V_{QCD}\) the interactions of QCD and \(E_p = \sqrt{m_p^2 + p^2}\), is equivalent to a coupled set of integral equations, depicted in Fig. 2, when projected onto Fock sectors.

In light-front quantization, the mass eigenvalue problem is
\[P^-|P\rangle = M^2 + P_+^2|P\rangle, \quad P|P\rangle = P|P\rangle.\]  
(1.2)
Here we define the light-front energy as \(P^- = (E - P^z)\), conjugate to the light-front time \(x^+ = t + z\); and the light-front momentum \(P = (P^+ \equiv E + P^z, \vec{P}_\perp = (P^x, P^y))\), conjugate to the spatial coordinates \(\vec{x} = (x^- \equiv t - z, x, y)\). Because \(p^+\) is positive for all constituents, there are no spurious vacuum contributions to eigenstates; particles cannot be produced from the vacuum and still conserve \(p^+\). We also have a boost-invariant separation of internal and external momenta. Wave functions are then functions of longitudinal momentum fractions \(x_i \equiv p^+_i/P^+\) and relative transverse momenta \(\vec{k}_{i\perp} \equiv \vec{p}_{i\perp} - x_i\vec{P}_\perp\) of the constituents.

To have a finite set of equations for the wave functions, there must be a truncation of some sort. The standard truncation is one of Fock space, to limit the number of constituents included in the calculation. Unfortunately, this introduces various complications, particularly uncanceled divergences [3]. For example, the Ward identity of gauge theories is destroyed by the truncation because a truncation of the number of photons in flight can eliminate vertex loops and self-energy loops on one leg but not the other [4]. We avoid these difficulties by

![FIG. 1. Fock-state expansion for a proton in terms of quarks and gluons.](image1)

![FIG. 2. Coupled equations for the Fock-state wave functions of a proton.](image2)
making a different sort of truncation, within the context of the light-front coupled-cluster (LFCC) method \cite{5-8}. The LFCC method writes the eigenstate in the form $\psi = \sqrt{Z} e^T |\phi\rangle$, with $|\phi\rangle$ a valence state with the smallest number of constituents consistent with the desired quantum numbers, $T$ an operator that adds constituents, and $\sqrt{Z}$ a normalization factor. The operator $T$ conserves $J_z$, light-front momentum $P$, charge and any other quantum number that should be conserved. We then define an effective Hamiltonian $\mathcal{H} = e^{-T} \mathcal{P} - e^T$ and a projection $P_v$ onto the valence sector, and the original eigenvalue problem (1.2) is equivalent to

$$P_v \mathcal{P} - |\phi\rangle = M^2 + P^2_{\perp} |\phi\rangle$$

and

$$(1 - P_v) \mathcal{P} - |\phi\rangle = 0.$$

The second equation is an auxiliary equation for the operator $T$.

At this point, the formulation is still exact and still reduces to an infinite set of equations for the infinite set of terms possible for $T$. The LFCC approximation is a restriction of $T$ to one or a few terms and a corresponding restriction of $1 - P_v$ to generate only enough auxiliary equations to solve for the terms kept in $T$. Many of the mathematical manipulations needed to carry out this approach can be found in the original many-body coupled-cluster method \cite{9, 10}. There the valence state is one of many particles, not a few, and the $T$ operator introduces correlated excitations of the single-particle states without changing particle number. Despite the large difference in the physics, the mathematics is much the same.

II. SAMPLE APPLICATION

To illustrate the use of the LFCC method and to compare it with ordinary Fock-space truncation, we consider a light-front analog of the Greenberg–Schweber model \cite{11}. In the model, a heavy fermionic source emits and absorbs bosons without changing its spin. The full $T$ operator would then be an expansion in multiple emissions of bosons, as shown in Fig. 3. The action of the exponentiation of $T$ is given in Fig. 4. This is to be compared with a Fock-state expansion of the eigenstate, as presented in Fig. 5 where the wave function $\psi_0$ is equivalent to the LFCC valence-state amplitude $\phi$.

A truncation of the $T$ operator to only the first term, single-boson emission, leaves only the $t_1$ terms of Fig. 4. The valence eigenvalue problem for $\phi$ and the auxiliary equation for $t_1$ are then the coupled system of nonlinear equations depicted in Fig. 6. These are to be compared with the equation for the Fock wave function $\psi_1$ in Fig. 7 obtained by truncating the Fock-state expansion at two bosons and then integrating out the zero-boson and two-boson wave functions. A key distinction between the two approaches is in the self-energy.
FIG. 4. The action of $e^T$ on the valence state $|\phi\rangle$. Factorials and other combinatoric factors are not shown.

FIG. 5. Fock-state expansion for the eigenstate $|\psi\rangle$. The circles are labeled by the relevant wave function $\psi_n$.

loop for the fermion; in the LFCC approach, it is the same everywhere it appears in Fig. 6.

For the Fock-space truncation approach, it is different in each Fock sector, and the form that appears in Fig. 7 is dependent on the momentum of the spectator boson.

In [5] we show that the simplest truncation of $T$ leads to an analytically solvable auxiliary equation and provides for this simple model the exact answer! The solution can be used to compute matrix elements, in particular the Dirac form factor for the fermion.$^1$

FIG. 6. Valence-sector eigenvalue problem and auxiliary equation in the LFCC method. The crosses represent kinetic energy contributions. The inhomogeneous term on the right of the auxiliary equation is the bare emission vertex from the original model Hamiltonian.

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$^1$ The limitation of the model to a heavy fermion that cannot flip its spin eliminates consideration of the Pauli form factor; this is not a limitation of the LFCC method.
FIG. 7. Equation for the one-boson/one-fermion wave function $\psi_1$ obtained in a Fock-space truncation to no more than two bosons.

III. SUMMARY

The LFCC method provides a nonperturbative approach to the solution of Hamiltonian eigenvalue problems in quantum field theories. It assumes that the Hamiltonian is suitably regulated, so that the integrals involved are rendered finite. The advantages of the method are that it does not require a Fock-space truncation and that it does not generate sector-dependent or spectator-dependent self-energies. Also, as an approximation, it is systematically improvable through the addition of terms to the $T$ operator, each constructed to add more particles to the valence state.

An application to a simple model has been discussed here and more fully in [5]. An application to the dressed-electron state in QED is investigated in [8], and additional work there is anticipated [6, 7]. The method should also be applicable to QCD, with light-front holographic QCD [12] as a natural starting point; there light-front holography may provide a useful approximation to the LFCC valence eigenvalue problem for QCD.

ACKNOWLEDGMENTS

This work was done in collaboration with S.S. Chabysheva and supported in part by the US Department of Energy.

[1] P.A.M. Dirac, Rev. Mod. Phys. 21, 392–399 (1949).
[2] For reviews of light-cone quantization, see M. Burkardt, Adv. Nucl. Phys. 23, 1–74 (2002); S.J. Brodsky, H.-C. Pauli, and S.S. Pinsky, Phys. Rep. 301, 299–486 (1998).
[3] S.S. Chabysheva and J.R. Hiller, Ann. Phys. 325, 2435–2450 (2010).
[4] D. Mustaki, S. Pinsky, J. Shigemitsu, and K. Wilson, Phys. Rev. D 43, 3411–3427 (1991).
[5] S.S. Chabysheva and J.R. Hiller, Phys. Lett. B 711, 417–422 (2012).
[6] J.R. Hiller and S.S. Chabysheva, Few Body Syst. 52, 315–321 (2012); S.S. Chabysheva and J.R. Hiller, Few Body Syst. 52, 323–329 (2012).
[7] J.R. Hiller, PoS(QNP2012), 113:1–6 (2012); S. Chabysheva, PoS(QNP2012), 123:1–6 (2012).
[8] S.S. Chabysheva and J.R. Hiller, “An application of the light-front coupled-cluster method to the nonperturbative solution of QED,” arXiv:1203.0250 [hep-ph].
[9] F. Coester, Nucl. Phys. 7, 421–424 (1958); F. Coester and H. Kümmel, Nucl. Phys. 17, 477–485 (1960).
For reviews of the many-body coupled-cluster method, see R.J. Bartlett and M. Musial, *Rev. Mod. Phys.* 79, 291–352 (2007); T.D. Crawford, and H.F. Schaefer, *Rev. Comp. Chem.* 14, 33–136 (2000); R. Bishop, A.S. Kendall, L.Y. Wong, and Y. Xian, *Phys. Rev. D* 48, 887–901 (1993); H. Kümmel, K.H. Lührmann, and J.G. Zabolitzky, *Phys. Rep.* 36, 1–63 (1978).

S.J. Brodsky, J.R. Hiller, and G. McCartor, *Phys. Rev. D* 58, 025005:1–16 (1998); O. Greenberg, and S.S. Schweber, *Nuovo Cimento* 8, 378–406 (1958).

G.F. de Teramond, and S.J. Brodsky, *Phys. Rev. Lett.* 102, 081601:1–4 (2009).