Richardson extrapolation for the discrete iterated modified projection solution

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Abstract

Approximate solutions of Urysohn integral equations using projection methods involve integrals which need to be evaluated using a numerical quadrature formula. It gives rise to the discrete versions of the projection methods. For $r \geq 1$, a space of piecewise polynomials of degree $\leq r - 1$ with respect to an uniform partition is chosen to be the approximating space and the projection is chosen to be the interpolatory projection at $r$ Gauss points. Asymptotic expansion for the iterated modified projection solution is available in literature. In this paper, we obtain an asymptotic expansion for the discrete iterated modified projection solution and use Richardson extrapolation to improve the order of convergence. Our results indicate a choice of a numerical quadrature which preserves the order of convergence in the continuous case. Numerical results are given for a specific example.

Keywords  Urysohn integral operator · Interpolatory projection · Gauss points · Nyström approximation · Richardson extrapolation

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1 Introduction

Let $\mathcal{X} = L^\infty[0, 1]$ and consider the following Urysohn integral operator

$$\mathcal{K}(x)(s) = \int_0^1 \kappa(s, t, x(t))dt, \quad s \in [0, 1], \; x \in \mathcal{X},$$

(1.1)
where $\kappa(s, t, u)$ is a continuous real valued function defined on $\Omega = [0, 1] \times [0, 1] \times \mathbb{R}$. Then $\mathcal{K}$ is a compact operator from $L^\infty[0, 1]$ to $C[0, 1]$. Assume that for $f \in C[0, 1],\quad x - \mathcal{K}(x) = f$ \hspace{1cm} (1.2) has a unique solution $\varphi$. We further assume that $(I - \mathcal{K}'(\varphi))^{-1}$ is a bounded linear operator from $C[0, 1]$ to itself, which is true for most applications. Then $\varphi$ has a nonzero Schauder-Leray index as a fixed point of the non-linear operator $\mathcal{K}(x) + f$.

We are interested in computable approximate solutions of the above equation. We consider some projection methods associated with a sequence of interpolatory projections converging to the Identity operator pointwise. For $r \geq 1$, let $X_n$ denote the space of piecewise polynomials of degree $\leq r - 1$ with respect to a uniform partition of $[0, 1]$ with $n$ subintervals. Let $h = \frac{1}{n}$ denote the mesh of the partition and let $Q_n : C[0, 1] \to X_n$ be the interpolation operator at $r$ Gauss points. In Grammont et al. [4], the following modified projection method is investigated:

$$x_n - \mathcal{K}_n^M(x_n) = f, \quad (1.3)$$

where

$$\mathcal{K}_n^M(x) = Q_n \mathcal{K}(x) + \mathcal{K}(Q_n x) - Q_n \mathcal{K}(Q_n x). \quad (1.4)$$

Note that for $y \in C([0, 1])$, $Q_n y \to y$ as $n \to \infty$. The existence of a unique solution $\varphi_n^M$ of (1.3) in a neighbourhood of $\varphi$ is proved in Theorem 1 of Grammont [3].

The iterated modified projection solution is defined as follows:

$$\tilde{\varphi}_n^M = \mathcal{K}(\varphi_n^M) + f.$$

If $\frac{\partial \kappa}{\partial u} \in C^{3r}(\Omega)$ and $f \in C^{2r}[0, 1]$, then the following orders of convergence are proved in Theorem 2.6 and Theorem 3.5 of Grammont et al. [4]:

$$\|\varphi - \varphi_n^M\|_\infty = O(h^{3r}), \quad \|\varphi - \tilde{\varphi}_n^M\|_\infty = O(h^{4r}).$$

Under the assumption of $\frac{\partial \kappa}{\partial u} \in C^{3r+3}(\Omega)$ and $f \in C^{2r+2}[0, 1]$, the following asymptotic series expansion is obtained in Theorem 3.3 of Kulkarni-Nidhin [5]:

$$\varphi_n^M = \varphi + \zeta h^{4r} + O(h^{4r+2}). \quad (1.5)$$

where the function $\zeta$ is independent of $h$.

In practice, it is necessary to replace all the integrals by a numerical quadrature formula, giving rise to the discrete versions of the above methods. We choose a composite numerical quadrature formula with a degree of precision $d - 1$ and with respect to a uniform partition of $[0, 1]$ with $m = np$, $p \in \mathbb{N}$, subintervals. Let $\tilde{h} = \frac{1}{m}$ denote the mesh of this fine partition. Then the Nyström operator $\mathcal{K}_m$ is obtained by replacing the integral in the definition (1.1) of $\mathcal{K}$ and the discrete version of the operator $\mathcal{K}_n^M$ defined in (1.4) is obtained on replacing $\mathcal{K}$ by $\mathcal{K}_m$ and is denoted by $\mathcal{K}_n^M$. The Nyström solution is denoted by $\varphi_m$ and it satisfies the following:

$$\varphi_m - \mathcal{K}_m(\varphi_m) = f.$$
The discrete modified projection and the discrete iterated modified projection solutions are denoted respectively by $z^M_n$ and $\tilde{z}^M_n$ and are defined as following:

\[ z^M_n - \mathcal{H}_n(z^M_n) = f, \quad \tilde{z}^M_n = \mathcal{H}_m(z^M_n) + f. \]

The following estimates are proved in Theorem 3.7 and Theorem 4.8 of Kulkarni-Rakshit [6]:

If $d \geq 2r$, $\kappa \in C^d(\Omega)$, $\frac{\partial \kappa}{\partial u} \in C^{2r}(\Omega)$ and $f \in C^d([0, 1])$, then

\[ \|z^M_n - \varphi\|_\infty = O\left(\max\left\{\tilde{h}^d, h^{3r}\right\}\right). \quad (1.6) \]

If $d \geq 2r$, $\frac{\partial \kappa}{\partial u} \in C^{\max\{d, 3r\}}(\Omega)$ and $f \in C^d([0, 1])$, then

\[ \|\tilde{z}^M_n - \varphi\|_\infty = O\left(\max\left\{\tilde{h}^d, h^{4r}\right\}\right). \quad (1.7) \]

In this paper, we show that if $d \geq 2r$, $\frac{\partial \kappa}{\partial u} \in C^{\max\{d, 3r+3\}}(\Omega)$, and $f \in C^{\max\{d, 2r+2\}}[0, 1]$, then

\[ \tilde{z}^M_n = \varphi_m + \alpha h^{4r} + O\left(h^{2r} \max\left\{\tilde{h}^d, h^{2r+2}\right\}\right), \quad (1.8) \]

where the term $\alpha$ is independent of $h$. If we choose $\tilde{h}$ and $d$ such that $\tilde{h}^d \leq h^{2r+2}$, then using the Richardson extrapolation, an estimate of $\varphi_m$ of the order of $h^{4r+2}$ could be obtained.

The proof of (1.5) is based on an asymptotic expansion for the term $\mathcal{H}'(\varphi)(Q_nv - v)$, where $\mathcal{H}'(\varphi)$ denotes the Fréchet derivative of $\mathcal{H}$ at $\varphi$ and $v$ is a smooth function. This is obtained by using the Euler-MacLaurin formula. The main difficulty in proving (1.8) is that a discrete version of the Euler-MacLaurin formula is not available which is needed for obtaining an asymptotic expansion for $\mathcal{H}'_m(\varphi_m)(Q_nv - v)$. We prove (1.8) using a different approach.

The paper is arranged as follows. In Section 2 we define discrete versions of the modified projection method and of its iterated version. Section 3 is devoted to the asymptotic series expansions for $\mathcal{H}'_m(\varphi_m)(Q_nv - v)$ and for terms involving higher order Fréchet derivatives of the operator $\mathcal{H}_m$ at $\varphi_m$. In Section 4 we prove our main result, that is, the expansion (1.8). Numerical results are given in Section 5.

## 2 Discrete projection methods

In the discrete projection methods, all the integrals are replaced by a numerical quadrature formula. We first define a numerical quadrature formula and the Nyström operator $\mathcal{H}_m$ which approximates $\mathcal{H}$. Let $m \in \mathbb{N}$ and consider the following uniform partition of $[0, 1]$: $\Delta_m : 0 < \frac{1}{m} < \ldots < \frac{m-1}{m} < 1.$ (2.1)
Let \( \tilde{h} = \frac{1}{m} \) and \( s_i = \frac{i}{m}, \ i = 0, \ldots, m \).

Consider a basic quadrature rule as follows:

\[
\int_0^1 f(t) dt \approx \sum_{i=1}^\rho \omega_i f(\mu_i)
\]

and let

\[
\zeta_j^i = s_{j-1} + \mu_i \tilde{h}, \ i = 1, \ldots, \rho, \ j = 1, \ldots, m.
\]

A composite integration rule with respect to the partition (2.1) is then defined as follows:

\[
\int_0^1 f(t) dt = m \sum_{j=1}^m \int_{s_{j-1}}^{s_j} f(t) dt \approx \tilde{h} \sum_{j=1}^m \sum_{i=1}^\rho \omega_i f(\zeta_j^i). \tag{2.2}
\]

For \( k \geq 1 \), let

\[
C^k_{\Delta_m}([0, 1]) = \{ g \in L^\infty[0, 1] : g|_{[s_{j-1}, s_j]} \in C^k([s_{j-1}, s_j]), \ j = 1, \ldots, m \}.
\]

The error in the numerical quadrature is assumed to be of the following form: There is a positive integer \( d \) such that if \( f \in C^d_{\Delta_m}([0, 1]) \), then

\[
\left| \int_0^1 f(t) dt - \tilde{h} \sum_{j=1}^m \sum_{i=1}^\rho \omega_i f(\zeta_j^i) \right| \leq C_1 \left\| f^{(d)} \right\|_{\infty} \tilde{h}^d, \tag{2.3}
\]

where \( f^{(d)} \) denotes \( d \)th derivative of \( f \) and \( C_1 \) is a constant independent of \( \tilde{h} \).

Assume that \( \kappa \in C^d(\Omega) \) and that \( f \in C^d([0, 1]) \). Then it follows that \( \varphi \in C^d([0, 1]) \). We also assume throughout that \((I - \mathcal{K}'(\varphi))^{-1}\) is a bounded linear operator from \( C[0, 1] \) to itself.

We replace the integral in (1.1) by the numerical quadrature formula (2.2) and define the Nyström operator as follows:

\[
\mathcal{K}_m(x)(s) = \tilde{h} \sum_{j=1}^m \sum_{i=1}^\rho \omega_i \kappa(s, \zeta_j^i, x(\zeta_j^i)), \ s \in [0, 1], \ x \in C([0, 1]).
\]

Then, since \( \kappa \in C^d(\Omega) \) and \( \varphi \in C^d([0, 1]) \), it follows

\[
\| \mathcal{K}(\varphi) - \mathcal{K}_m(\varphi) \|_{\infty} = O\left(\tilde{h}^d\right). \tag{2.4}
\]

In the Nyström method, (1.2) is approximated by the following:

\[
x_m - \mathcal{K}_m(x_m) = f. \tag{2.5}
\]

Fix \( \delta_0 > 0 \) and define the following:

\[
B(\varphi, \delta_0) = \{ x \in \mathcal{Y} : \| x - \varphi \|_{\infty} \leq \delta_0 \}.
\]
Then for \( m \) sufficiently large, (2.5) has a unique solution \( \varphi_m \) in \( B(\varphi, \delta_0) \). See Theorem 4 of Atkinson [1]. Also,

\[
\| \varphi - \varphi_m \| \leq \gamma \| \mathcal{K}(\varphi) - \mathcal{K}_m(\varphi) \| = O\left( \tilde{h}^d \right).
\]

Let

\[
\ell(s, t) = \frac{\partial K}{\partial u}(s, t, \varphi(t)), \quad s, t \in [0, 1].
\]

Note that the Fréchet derivatives of \( \mathcal{K} \) and \( \mathcal{K}_m \) are given by the following:

\[
\mathcal{K}'(\varphi)v(s) = \int_0^1 \ell(s, t)v(t)dt
\]

\[
\mathcal{K}_m'(\varphi)v(s) = \tilde{h} \sum_{j=1}^m \sum_{i=1}^\rho \omega_i \ell \left( s, \zeta^j_i \right) v \left( \zeta^j_i \right), \quad s \in [0, 1], \quad v \in C([0, 1]).
\]

If \( \frac{\partial K}{\partial u} \in C^d(\Omega) \) and \( v \in C_{\Delta_m}^d([0, 1]) \), then from (2.3),

\[
\| \mathcal{K}'(\varphi)v - \mathcal{K}_m'(\varphi)v \| \leq C_2 \| v \|_{d, \infty} \tilde{h}^d,
\]

where \( \| v \|_{d, \infty} = \max_{0 \leq j \leq d} \| v(j) \|_\infty \) and \( C_2 \) is a constant independent of \( \tilde{h} \).

We now define the interpolatory projection at \( r \) Gauss points. Let \( n \in \mathbb{N} \) and consider the following uniform partition of \([0, 1]\):

\[
\Delta_n : 0 < \frac{1}{n} < \cdots < \frac{n-1}{n} < 1.
\]

Define the following:

\[
t_j = \frac{j}{n}, \quad j = 0, \ldots, n \quad \text{and} \quad h = t_j - t_{j-1} = \frac{1}{n}.
\]

For a positive integer \( r \), let

\[
\mathcal{X}_n = \left\{ g \in L^\infty[0, 1] : g|_{[t_{j-1}, t_j]} \text{ is a polynomial of degree } \leq r-1, \quad j = 1, \ldots, n \right\}.
\]

Let \( \{q_1, \ldots, q_r\} \) denote the Gauss-Legendre zeros of order \( r \) in \([0, 1]\). The \( nr \) collocation nodes are chosen as follows:

\[
t_i^k = t_{k-1} + q_i h, \quad i = 1, \ldots, r, \quad k = 1, \ldots, n.
\]

Define the interpolation operator \( Q_n : C[0, 1] \to \mathcal{X}_n \) as follows:

\[
(Q_nx)(t_i^k) = x(t_i^k), \quad i = 1, \ldots, r, \quad k = 1, \ldots, n.
\]

Using the Hahn-Banach extension theorem as in Atkinson et al. [2], \( Q_n \) can be extended to \( L^\infty[0, 1] \). Note that for \( x \in C([0, 1]) \), \( Q_nx \to x \) as \( n \to \infty \). As a consequence, there exists a constant \( C_3 \) such as the following:

\[
\sup_n \| Q_n|_{C[0, 1]} \| \leq C_3.
\]

Also, if \( u \in C^r([0, 1]) \), then

\[
\| u - Q_nu \| \leq C_4 \left\| u^{(r)} \right\|_\infty h^r.
\]
where \( C_4 \) is a constant independent of \( n \). Throughout this paper, we choose \( d \geq 2r \) and \( m = pn \) for some \( p \in \mathbb{N} \). Then \( \tilde{h} = \frac{h}{p} \leq h \). It follows from (2.6), (2.8) and (2.9) that
\[
\| \varphi_m - Q_n \varphi_m \|_{\infty} \leq \| \varphi - Q_n \varphi \|_{\infty} + \| (I - Q_n)(\varphi_m - \varphi) \|_{\infty} = O \left( \max \left\{ h^r, \tilde{h}^d \right\} \right).
\]
Hence, it follows that
\[
\| \varphi_m - Q_n \varphi_m \|_{\infty} = O (h^r). \tag{2.10}
\]

The discrete modified projection operator is defined by replacing \( \mathcal{K} \) with \( \mathcal{K}_m \) in (1.4) as follows:
\[
\tilde{K}_M (x) = Q_n K_m(x) + K_m(Q_n x) - Q_n K_m(Q_n x). \tag{2.11}
\]

Discrete Modified Projection Method: Consider
\[
x_n - \tilde{K}_M (x_n) = f. \tag{2.12}
\]

In Remark 3.5 of Kulkarni-Rakshit [6], it is shown that there exists \( 0 < \delta \leq \delta_0 \) such that for \( n \) and \( m \) big enough, the above equation has a unique solution \( z^M_n \) in \( B(\varphi, \delta) \).

The Discrete Iterated Modified Projection solution is defined as follows:
\[
\tilde{z}^M_n = K_m(z^M_n) + f. \tag{2.13}
\]

### 3 Asymptotic series expansions

By assumption, \( I - \mathcal{K}'(\varphi) \) is invertible. Let
\[
M = \left( I - \mathcal{K}'(\varphi) \right)^{-1} \mathcal{K}'(\varphi), \quad \Psi(t) = (t - q_1) \cdots (t - q_r), \quad t \in [0, 1].
\]

We quote the following asymptotic series expansions from (2.33) and (2.34) of Kulkarni-Nidhin [5].

If \( \frac{\partial \kappa}{\partial u} \in C^{2r+2}(\Omega) \) and \( \psi \in C^{2r+2}([0, 1]) \), then
\[
\mathcal{K}'(\varphi)(Q_n \psi - \psi) = T(\psi)h^{2r} + O(h^{2r+2}), \tag{3.1}
\]
\[
M(Q_n \psi - \psi) = U(\psi)h^{2r} + O(h^{2r+2}), \tag{3.2}
\]
where for \( s \in [0, 1] \),
\[
T(\psi)(s) = d_{2r,2r} \mathcal{K}'(\varphi)\psi^{(2r)}(s) + \sum_{i=r}^{2r-1} d_{2r,i} \left[ \left( \frac{\partial}{\partial t} \right)^{2r-1-i} \ell(s,t)\psi^{(i)}(t) \right]_{t=0}^{1},
\]
\[
U(\psi)(s) = d_{2r,2r} M\psi^{(2r)}(s) + \sum_{i=r}^{2r-1} d_{2r,i} \left[ \left( \frac{\partial}{\partial t} \right)^{2r-1-i} m(s,t)\psi^{(i)}(t) \right]_{t=0}^{1},
\]
\[
d_{2r,i} = -\int_0^1 \Phi_i(\tau) B_{2r-i}(\tau) \frac{B_{2r-i}}{(2r-i)!} \Psi(\tau) d\tau, \quad i = r, \ldots, 2r.
\]
with

Note that the kernel of $M$ is denoted by $m(s, t)$.

$$
\Phi_i(\tau) = \int_0^1 \frac{(\sigma - \tau)^i - r}{(i - r)!} \frac{[q_1, q_2, \ldots, q_r, \tau](-\sigma)^{r-1}}{(r - 1)!} d\sigma.
$$

Note that the coefficients $d_{2r,i}$ are independent of $h$. Also,

$$
\mathcal{K}''(\varphi)(Q_n \psi - \psi)^2 = V_1(\psi)h^{2r} + O(h^{2r+2}),
\mathcal{K}^{(3)}(\varphi)(Q_n \psi - \psi)^3 = V_2(\psi)h^{3r} + O(h^{3r+1}),
$$

where

$$
V_1(\psi) = \left( \int_0^1 \psi(\tau)^2 \Phi_r(\tau)^2 d\tau \right) \mathcal{K}''(\varphi) \left( \psi(\tau)^2 \right)
$$

and

$$
V_2(\psi) = \left( \int_0^1 \psi(\tau)^3 \Phi_r(\tau)^3 d\tau \right) \mathcal{K}^{(3)}(\varphi) \left( \psi(\tau)^3 \right).
$$

Let $\frac{\partial^4 \kappa}{\partial u^4} \in C^{2r}(\Omega)$ and $v_k \in C([0, 1])$, $1 \leq k \leq 4$. The Fréchet derivatives of $\mathcal{K}_m$ are as follows:

$$
\mathcal{K}_m^{(k)}(x)(v_1, \ldots, v_k)(s) = \tilde{h} \sum_{j=1}^{m} \sum_{i=1}^{\rho} \omega_i \frac{\partial^k \kappa}{\partial u^k} \left( s, \zeta_i^j, x \left( \xi_i^j \right) \right) v_1 \left( \xi_i^j \right) \cdots v_k \left( \xi_i^j \right),
$$

$s \in [0, 1]$.

We introduce the following notations:

$$
D^{(i,j,k)}(s, t, u) = \frac{\partial^{i+j+k} \kappa}{\partial s^i \partial t^j \partial u^k}(s, t, u), \quad C_5 = \max_{0 \leq i+j \leq 2r} \max_{s,t \in [0,1]} \left| D^{(i,j,k)}(s, t, u) \right|.
$$

Using the mean value theorem, it can be seen as follows:

$$
\left\| \mathcal{K}_m'(\varphi) - \mathcal{K}_m'(\varphi_m) \right\| \leq \left( \sum_{i=1}^{\rho} |\omega_i| \right) C_5 \tilde{h}^d. \quad (3.3)
$$

We now prove asymptotic series expansions for the first three Fréchet derivatives of the Nyström operator $\mathcal{K}_m$ at $\varphi_m$.

**Proposition 3.1** If $\frac{\partial \kappa}{\partial u} \in C^{\max\{d, 2r+2\}}(\Omega)$ and $f \in C^{\max\{d, 2r+2\}}([0, 1])$, then

$$
\mathcal{K}_m'(\varphi_m)(Q_n \varphi_m - \varphi_m) = T(\varphi)h^{2r} + O \left( \max \left\{ \tilde{h}^d, h^{2r+2} \right\} \right), \quad (3.4)
$$

$$
\mathcal{K}_m''(\varphi_m)(Q_n \varphi_m - \varphi_m)^2 = V_1(\varphi)h^{2r} + O \left( \max \left\{ \tilde{h}^d, h^{2r+2} \right\} \right), \quad (3.5)
$$

$$
\mathcal{K}_m^{(3)}(\varphi_m)(Q_n \varphi_m - \varphi_m)^3 = V_2(\varphi)h^{3r} + O \left( \max \left\{ \tilde{h}^d, h^{3r+1} \right\} \right). \quad (3.6)
$$
Proof We write the following:

\[ \mathcal{H}_m'(\phi)(Q_n\phi - \phi) = \mathcal{H}'(\phi)(Q_n\phi - \phi) + (\mathcal{H}_m'(\phi) - \mathcal{H}'(\phi))(Q_n\phi - \phi). \]

Since \( d \geq 2r \) and \( \tilde{h} \leq h \), from the estimate (2.7), we obtain the following:

\[ \| (\mathcal{H}_m'(\phi) - \mathcal{H}'(\phi))(Q_n\phi - \phi) \|_\infty \leq C_2 \| Q_n\phi - \phi \|_{d, \infty} \tilde{h}^d = C_2 \| \phi \|_{d, \infty} \tilde{h}^d. \]

Since \( \phi \in C^{2r+2}[0, 1] \), it follows from (3.1) that

\[ \mathcal{H}_m'(\phi)(Q_n\phi - \phi) = T(\phi) h^{2r} + O(h^{2r+2}). \]

Since by (2.6), \( \| \phi - \phi_m \|_\infty = O(\tilde{h}^d) \), we obtain the following:

\[ \mathcal{H}_m'(\phi)(Q_n\phi_m - \phi_m) = \mathcal{H}_m'(\phi)(Q_n\phi - \phi) + \mathcal{H}_m'(\phi)(Q_n - I)(\phi_m - \phi) = T(\phi) h^{2r} + O \left( \max \left\{ \tilde{h}^d, h^{2r+2} \right\} \right). \]

Using the estimate (3.3), we thus obtain the following:

\[ \mathcal{H}_m'(\phi_m)(Q_n\phi_m - \phi_m) = \mathcal{H}_m'(\phi)(Q_n\phi_m - \phi_m) + [\mathcal{H}_m'(\phi_m) - \mathcal{H}_m'(\phi)](Q_n\phi_m - \phi_m) = T(\phi) h^{2r} + O \left( \max \left\{ \tilde{h}^d, h^{2r+2} \right\} \right). \]

This completes the proof of (3.4). The proofs of (3.5) and (3.6) are similar.

4 Discrete iterated modified projection method

In this section, we prove our main result about the asymptotic series expansion for the discrete iterated modified projection solution \( \tilde{z}_n^M \).

Using the generalised Taylor series expansion, we obtain the following:

\[ \tilde{z}_n^M - \phi_m = \mathcal{H}_m(z_n^M) - \mathcal{H}_m(\phi_m) = \mathcal{H}_m'(\phi_m)(z_n^M - \phi_m) + R(z_n^M - \phi_m), \]

where,

\[ R(z_n^M - \phi_m)(s) = \int_0^1 (1 - \theta) \mathcal{H}_m''(\phi_m + \theta(z_n^M - \phi_m))(z_n^M - \phi_m)^2(s) d\theta. \]

It follows that

\[ \| R(z_n^M - \phi_m) \|_\infty \leq \frac{C_5}{2} \left( \sum_{i=1}^p |\omega_i| \right) \| z_n^M - \phi_m \|_\infty^2. \]

If \( d \geq 2r \), \( \kappa \in C^d(\Omega) \), \( \frac{\partial \kappa}{\partial u} \in C^{2r}(\Omega) \) and \( f \in C^d([0, 1]) \), then using (1.6), we deduce the following:

\[ \tilde{z}_n^M - \phi_m = \mathcal{H}_m'(\phi_m)(z_n^M - \phi_m) + O \left( \max \left\{ \tilde{h}^d, h^{3r} \right\} \right). \]  

(4.1)

Our aim is to obtain an asymptotic series expansion for the first term on the right-hand side of the above equation.
We first prove the following preliminary results which are needed later on.

**Lemma 4.1** If $\frac{\partial \kappa}{\partial u} \in C^{3r+3}(\Omega)$, then for $x \in B(\varphi, \delta)$ and for $1 \leq k \leq 4$,

$$\left\| (I - Q_n) \mathcal{K}_m^{(k)}(x) \right\| = O\left(h^r\right),$$

(4.2)

$$\left\| \mathcal{K}_m'(\varphi_m)(I - Q_n) \mathcal{K}_m^{(k)}(x) \right\| = O\left(h^{2r}\right)$$

(4.3)

and

$$\left\| \mathcal{K}_m'(\varphi_m)(I - Q_n) \mathcal{K}_m'(\varphi_m)(I - Q_n) \mathcal{K}_m^{(k)}(x) \right\| = O\left(h^{4r}\right).$$

(4.4)

**Proof** Note that for $1 \leq k \leq 4$ and for $v_1, \ldots, v_k \in C([0, 1])$,

$$\mathcal{K}_m^{(k)}(x)(v_1, \ldots, v_k)(s) = \tilde{h} \sum_{j=1}^{m} \sum_{i=1}^{\rho} \omega_i \frac{\partial^k \kappa}{\partial u^k} (s, \zeta_j^i, x(\zeta_j^i)) v_1(\zeta_j^i) \cdots v_k(\zeta_j^i).$$

and for $p = 1, \ldots, 2r$,

$$\left( \mathcal{K}_m^{(k)}(x)(v_1, \ldots, v_k) \right)^{(p)}(s) = \tilde{h} \sum_{j=1}^{m} \sum_{i=1}^{\rho} \omega_i \frac{\partial^{p+k} \kappa}{\partial s^p \partial u^k} (s, \zeta_j^i, x(\zeta_j^i)) v_1(\zeta_j^i) \cdots v_k(\zeta_j^i).$$

Hence, for $x \in B(\varphi, \delta)$,

$$\left\| \left( \mathcal{K}_m^{(k)}(x)(v_1, \ldots, v_k) \right)^{(p)} \right\|_{\infty} \leq C_5 \left( \sum_{i=1}^{\rho} |\omega_i| \right) \|v_1\|_{\infty} \cdots \|v_k\|_{\infty}.$$

From the estimate (2.9),

$$\left\| (I - Q_n) \mathcal{K}_m^{(k)}(x)(v_1, \ldots, v_k) \right\|_{\infty} \leq C_4 \left\| \left( \mathcal{K}_m^{(k)}(x)(v_1, \ldots, v_k) \right)^{(p)} \right\|_{\infty} \tilde{h}^r$$

$$\leq C_4 C_5 \left( \sum_{i=1}^{\rho} |\omega_i| \right) \|v_1\|_{\infty} \cdots \|v_k\|_{\infty} \tilde{h}^r.$$
and $C_6 = \frac{1}{r!} 2^r \|\Psi\|_\infty \left( \sum_{i=1}^\rho |\omega_i| \right)$ is a constant independent of $n$. Since $\|\ell_m\|_{r,\infty} \leq C_5$, it follows that

$$\|\mathcal{K}_m'(\varphi_m)(I - Q_n)\mathcal{K}_m^{(k)}(x)(v_1, \ldots, v_k)\|_\infty \leq (C_5)^2 C_6 \left( \sum_{i=1}^\rho |\omega_i| \right) \|v_1\|_\infty \ldots \|v_k\|_\infty h^{2r}.$$  

By taking the supremum over the set $\{(v_1, \ldots, v_k) : \|v_1\|_\infty \leq 1, \ldots, \|v_k\|_\infty \leq 1\}$, we obtain (4.3).

In order to prove (4.4), we recall the following estimate from Proposition 4.4 of Kulkarni-Rakshit [6]: If $u \in C^2[0, 1]$, then

$$\|\mathcal{K}_m'(\varphi_m)(I - Q_n)\mathcal{K}_m^{(k)}(x)v\|_\infty \leq (C_6)^2 \|\ell_m\|_{r,\infty} \|\ell_m\|_{3r,\infty} \|u\|_{2r,\infty} h^{4r}. \quad (4.5)$$

Then for $v \in C[0, 1]$,

$$\|\mathcal{K}_m'(\varphi_m)(I - Q_n)\mathcal{K}_m^{(k)}(\varphi_m)(I - Q_n)v\|_\infty \leq (C_6)^2 \|\ell_m\|_{r,\infty} \|\ell_m\|_{3r,\infty} \|\mathcal{K}_m'(x)v\|_{2r,\infty} h^{4r} \leq (C_5)^2 (C_6)^2 \left( \sum_{i=1}^\rho |\omega_i| \right) \|\ell_m\|_{3r,\infty} \|v\|_\infty h^{4r}.$$  

The estimate (4.4) then follows by taking the supremum over the set $\{v \in C[0, 1] : \|v\|_\infty \leq 1\}$.  

From Proposition 4.2 of Kulkarni-Rakshit [6] we recall that for all $m$ big enough, $I - \mathcal{K}_m'(\varphi_m)$ is invertible and that $\| (I - \mathcal{K}_m'(\varphi_m))^{-1} \| \leq C_7$, where $C_7$ is a constant independent of $m$. It can be easily checked as follows:

$$z_n^M - \varphi_m = -\left[ I - \mathcal{K}_m'(\varphi_m) \right]^{-1} \left\{ \mathcal{K}_m(\varphi_m) - \mathcal{K}_m'(\varphi_m)\varphi_m - \tilde{\mathcal{K}}_n^M(z_n^M) + \mathcal{K}_m'(\varphi_m)z_n^M \right\}. \quad (4.6)$$

Let $L_m = (I - \mathcal{K}_m'(\varphi_m))^{-1}$. We write as follows:

$$\mathcal{K}_m'(\varphi_m)(z_n^M - \varphi_m)$$

$$= -L_m \mathcal{K}_m'(\varphi_m) \left\{ \mathcal{K}_m(\varphi_m) - \tilde{\mathcal{K}}_n^M(\varphi_m) \right\} + L_m \mathcal{K}_m'(\varphi_m) \left\{ \tilde{\mathcal{K}}_n^M(z_n^M) - \tilde{\mathcal{K}}_n^M(\varphi_m) - \mathcal{K}_m'(\varphi_m)(z_n^M - \varphi_m) \right\} + L_m \mathcal{K}_m'(\varphi_m) \left\{ \mathcal{K}_m'(\varphi_m)(z_n^M - \varphi_m) \right\}. \quad (4.6)$$

We obtain an asymptotic expansion for the first term on the right hand side of (4.6) and show that the second and the third terms are of the order of $h^{2r} \max\left\{ \tilde{h}^d, h^{2r+2} \right\}$. The following two lemmas are needed to obtain the results for the first term.
From now onwards, we assume the following:
\[ d \geq 2r, \quad \frac{\partial \kappa}{\partial u} \in C^{\max\{d, 3r+3\}}(\Omega) \quad \text{and} \quad f \in C^{\max\{d, 2r+2\}}[0, 1]. \]

**Lemma 4.2** If \( d \geq 2r, \quad \frac{\partial \kappa}{\partial u} \in C^{\max\{d, 3r+3\}}(\Omega) \quad \text{and} \quad f \in C^{\max\{d, 2r+2\}}[0, 1], \) then

\[
L_m \mathcal{K}_m^{\prime}(\varphi_m)(I - Q_n) \mathcal{K}_m^{\prime}(\varphi_m)(Q_n - I) \varphi_m
= U(T(\varphi))h^{4r} + O\left(h^{2r} \max\{\tilde{h}^d, h^{2r+2}\}\right) \tag{4.7}
\]

and

\[
L_m \mathcal{K}_m^{\prime}(\varphi_m)\mathcal{K}_m^{\prime}(I - Q_n)\mathcal{K}_m^{\prime}(\varphi_m)((Q_n - I)\varphi_m)^2
= U(V_1(\varphi))h^{4r} + O\left(h^{2r} \max\{\tilde{h}^d, h^{2r+2}\}\right). \tag{4.8}
\]

**Proof** Using the resolvent identity, we write the following:

\[
L_m = \left[I - \mathcal{K}^{\prime}(\varphi)\right]^{-1} + \left[I - \mathcal{K}^{\prime}(\varphi)\right]^{-1} \left[\mathcal{K}_m^{\prime}(\varphi_m) - \mathcal{K}_m^{\prime}(\varphi)\right] L_m
+ \left[I - \mathcal{K}^{\prime}(\varphi)\right]^{-1} \left[\mathcal{K}_m^{\prime}(\varphi) - \mathcal{K}_m^{\prime}(\varphi)\right] L_m.
\]

Let

\[
y_n = \mathcal{K}_m^{\prime}(\varphi_m)(I - Q_n)\mathcal{K}_m^{\prime}(\varphi_m)(Q_n - I)\varphi_m
= \mathcal{K}_m^{\prime}(\varphi_m)(I - Q_n)\mathcal{K}_m^{\prime}(\varphi_m)(Q_n - I)\varphi
+ \mathcal{K}_m^{\prime}(\varphi_m)(I - Q_n)\mathcal{K}_m^{\prime}(\varphi_m)(Q_n - I)(\varphi_m - \varphi).
\]

From (2.6), (4.3) and (4.5), it follows that

\[
\|y_n\|_{\infty} = O\left(h^{2r} \max\{\tilde{h}^d, h^{2r}\}\right) = O\left(h^{4r}\right). \tag{4.9}
\]

Note that

\[
L_m y_n = \left[I - \mathcal{K}^{\prime}(\varphi)\right]^{-1} y_n
+ \left[I - \mathcal{K}^{\prime}(\varphi)\right]^{-1} \left[\mathcal{K}_m^{\prime}(\varphi_m) - \mathcal{K}_m^{\prime}(\varphi)\right] L_m y_n
+ \left[I - \mathcal{K}^{\prime}(\varphi)\right]^{-1} \left[\mathcal{K}_m^{\prime}(\varphi) - \mathcal{K}_m^{\prime}(\varphi)\right] L_m y_n. \tag{4.10}
\]

Consider the first term as follows:

\[
\left[I - \mathcal{K}^{\prime}(\varphi)\right]^{-1} y_n = \left[I - \mathcal{K}^{\prime}(\varphi)\right]^{-1} \mathcal{K}^{\prime}(\varphi)(I - Q_n)\mathcal{K}_m^{\prime}(\varphi_m)(Q_n \varphi_m - \varphi_m)
+ \left[I - \mathcal{K}^{\prime}(\varphi)\right]^{-1} (\mathcal{K}_m^{\prime}(\varphi_m) - \mathcal{K}^{\prime}(\varphi)) (I - Q_n)\mathcal{K}_m^{\prime}(\varphi_m)
\times (Q_n \varphi_m - \varphi_m).
\]

Note that from (3.2) and (3.4),

\[
\left[I - \mathcal{K}^{\prime}(\varphi)\right]^{-1} \mathcal{K}^{\prime}(\varphi)(I - Q_n)\mathcal{K}_m^{\prime}(\varphi_m)(Q_n \varphi_m - \varphi_m)
= U(T(\varphi))h^{4r} + O\left(h^{2r} \max\{\tilde{h}^d, h^{2r+2}\}\right).
\]
On the other hand,
\[(I - Q_n)\mathcal{K}_m'(\varphi_m)(Q_n\varphi_m - \varphi_m) = (I - Q_n)\mathcal{K}_m'(\varphi)(Q_n\varphi - \varphi)\]
\[+ (I - Q_n)\mathcal{K}_m'(\varphi)(Q_n - I)(\varphi_m - \varphi)\]
\[+ (I - Q_n)(\mathcal{K}_m'(\varphi_m) - \mathcal{K}_m'(\varphi))(Q_n\varphi_m - \varphi_m)\].

From Kulkarni-Rakshit [6, Proposition 2.3], we have the following:
\[\|(I - Q_n)\mathcal{K}_m'(\varphi)(Q_n\varphi - \varphi)\|_\infty = O(\tilde{h}^3r)\].

Combining the above estimate with the estimates (2.6), (2.10), (3.3) and (4.2), we obtain the following:
\[
\left\| \left[ \begin{array}{c} I - \mathcal{K}'(\varphi) \\ \mathcal{K}_m'(\varphi_m) - \mathcal{K}_m'(\varphi) \end{array} \right] (I - Q_n)\mathcal{K}_m'(\varphi_m)(Q_n\varphi_m - \varphi_m) \right\|_\infty
= O\left(\tilde{h}^d h^{2r}\right).
\]

Thus,
\[\left[ I - \mathcal{K}'(\varphi) \right]^{-1} y_n = U(T(\varphi))h^{4r} + O\left(h^{2r} \max\left\{\tilde{h}^d, h^{2r+2}\right\}\right).\]

Using the estimate (4.9), it then follows that the second term of (4.10) is of the order of \(\tilde{h}^d h^{4r}\). We write the third term of (4.10) as follows:
\[
\left[ I - \mathcal{K}'(\varphi) \right]^{-1} \left[ \mathcal{K}_m'(\varphi) - \mathcal{K}_m'(\varphi) \right] L_m(I - Q_n)\mathcal{K}_m'(\varphi_m)(Q_n - I)(\varphi_m - \varphi + \varphi).
\]

From (2.7), for \(v \in C[0, 1]\),
\[
\left\| \left( \mathcal{K}_m'(\varphi) - \mathcal{K}_m'(\varphi) \right) \mathcal{K}_m'(\varphi_m)v \right\|_\infty \leq C_2 \left\| \mathcal{K}_m'(\varphi_m)v \right\|_{d, \infty} \tilde{h}^d \leq C_2 C_5 \left( \sum_{i=1}^{\rho} |\omega_i| \right) \|v\|_\infty \tilde{h}^d.
\]

It follows that
\[\left\| \left( \mathcal{K}_m'(\varphi) - \mathcal{K}_m'(\varphi) \right) \mathcal{K}_m'(\varphi_m) \right\| = O\left(\tilde{h}^d\right)\],
\[\text{(4.11)}\]
whereas as in Lemma 4.1, it can be proved that
\[\|(I - Q_n)\mathcal{K}_m'(\varphi_m)(I - Q_n)\varphi\|_\infty = O\left(h^{3r}\right).
\]

Thus, the third term of (4.10) is of the order of \(\tilde{h}^d h^{3r}\).

Hence,
\[
L_m y_n = L_m \mathcal{K}_m'(\varphi_m)(I - Q_n)\mathcal{K}_m'(\varphi_m)(Q_n\varphi_m - \varphi_m)
= U(T(\varphi))h^{4r} + O\left(h^{2r} \max\left\{\tilde{h}^d, h^{2r+2}\right\}\right),
\]

which proves (4.7). The proof of (4.8) is similar.
Lemma 4.3 If $d \geq 2r$, $\frac{\partial u}{\partial x} \in C^{\max\{d, 3r+3\}}(\Omega)$ and $f \in C^{\max\{d, 2r+2\}}[0,1]$, then
\[
\left\| L_m \mathcal{K}_m'(\varphi_m)(I - Q_n)\mathcal{K}_m^{(3)}(\varphi_m)(Q_n\varphi_m - \varphi_m)^3 \right\|_{\infty} = O\left(h^{4r+2}\right).
\]

Proof Let
\[
w_n = \mathcal{K}_m'(\varphi_m)(I - Q_n)\mathcal{K}_m^{(3)}(\varphi_m)(Q_n\varphi_m - \varphi_m)^3.
\]
Then, using (2.10) and (4.3), we obtain the following:
\[
\left\| w_n \right\|_{\infty} = O\left(h^{5r}\right). \quad (4.12)
\]
As in Lemma 4.2, we write the following:
\[
L_m w_n = \left[ I - \mathcal{K}'(\varphi) \right]^{-1} w_n
+ \left[ I - \mathcal{K}'(\varphi) \right]^{-1} \left[ \mathcal{K}_m'(\varphi_m) - \mathcal{K}_m'(\varphi) \right] L_m w_n
+ \left[ I - \mathcal{K}'(\varphi) \right]^{-1} \left[ \mathcal{K}_m'(\varphi) - \mathcal{K}'(\varphi) \right] L_m w_n. \quad (4.13)
\]
Note that
\[
\left[ I - \mathcal{K}'(\varphi) \right]^{-1} w_n = \left[ I - \mathcal{K}'(\varphi) \right]^{-1} \mathcal{K}'(\varphi)(I - Q_n)\mathcal{K}_m^{(3)}(\varphi_m)(Q_n\varphi_m - \varphi_m)^3
+ \left[ I - \mathcal{K}'(\varphi) \right]^{-1} \left( \mathcal{K}_m'(\varphi_m) - \mathcal{K}_m'(\varphi) \right)(I - Q_n)\mathcal{K}_m^{(3)}(\varphi_m)
\times (Q_n\varphi_m - \varphi_m)^3.
\]
Using the asymptotic expansions (3.2) and (3.6), we obtain the following:
\[
\left[ I - \mathcal{K}'(\varphi) \right]^{-1} \mathcal{K}'(\varphi)(I - Q_n)\mathcal{K}_m^{(3)}(\varphi_m)(Q_n\varphi_m - \varphi_m)^3
= U(V_2(\varphi))h^{5r} + O\left(h^{2r} \max\left\{ \tilde{h}^d, h^{3r+1} \right\}\right).
\]
If $r = 1$, then $V_2(\varphi) = 0$ and if $r \geq 2$, then $5r \geq 4r + 2$. Hence, it follows that
\[
\left\| \left[ I - \mathcal{K}'(\varphi) \right]^{-1} \mathcal{K}'(\varphi)(I - Q_n)\mathcal{K}_m^{(3)}(\varphi_m)(Q_n\varphi_m - \varphi_m)^3 \right\|_{\infty} = O\left(h^{4r+2}\right).
\]
On the other hand, using (2.7), (2.10), (3.3) and (4.2), we obtain the following:
\[
\left\| \left[ I - \mathcal{K}'(\varphi) \right]^{-1} \left( \mathcal{K}_m'(\varphi_m) - \mathcal{K}_m'(\varphi) \right)(I - Q_n)\mathcal{K}_m^{(3)}(\varphi_m)(Q_n\varphi_m - \varphi_m)^3 \right\|_{\infty}
= O(\tilde{h}^d h^{3r}).
\]
Sine $d \geq 2r$, it follows that
\[
\left\| \left[ I - \mathcal{K}'(\varphi) \right]^{-1} w_n \right\|_{\infty} = O\left(h^{4r+2}\right). \quad (4.14)
\]
Using the estimates (3.3) and (4.12), we see the following:
\[
\left\| \left[ I - \mathcal{K}'(\varphi) \right]^{-1} \left[ \mathcal{K}_m'(\varphi_m) - \mathcal{K}_m'(\varphi) \right] L_m w_n \right\|_{\infty} = O\left(\tilde{h}^d h^{5r}\right). \quad (4.15)
\]
We write the third term of (4.13) as follows:
\[
\left[ I - \mathcal{K}'(\varphi) \right]^{-1} \left[ \mathcal{K}_m'(\varphi) - \mathcal{K}'(\varphi) \right] \mathcal{K}_m'(\varphi_m) L_m(I - Q_n)\mathcal{K}_m^{(3)}(\varphi_m)(Q_n\varphi_m - \varphi_m)^3.
\]
From (2.10), (4.2) and (4.11), it follows that the above term is of the order of $\tilde{h}^d h^{4r}$. Thus,
\[
\left\| I - \mathcal{K}'(\varphi) \right\|^{-1} \left[ \mathcal{K}_m(\varphi) - \mathcal{K}'(\varphi) \right] L_m w_n \right\|_{\infty} = O \left( \tilde{h}^d h^{4r} \right).
\]
(4.16)
Since $d \geq 2r$, it follows from (4.13)–(4.16) that
\[
\|

\begin{align*}
L_m w_n \right\|_{\infty} = \left\| L_m \mathcal{K}'(\varphi_m)(I - Q_n) \mathcal{K}_m(\varphi_m)(Q_n \varphi_m - \varphi_m) \right\|_{\infty} = O \left( h^{4r+2} \right).
\end{align*}
\]
This completes the proof.

We now obtain the asymptotic expansion for the first term of (4.6).

**Proposition 4.4** If $d \geq 2r$, $\frac{\partial^k}{\partial u} \in C^{\max\{d, 3r+3\}}(\Omega)$ and $f \in C^{\max\{d, 2r+2\}}[0, 1]$, then
\[
L_m \mathcal{K}_m(\varphi_m) \left\{ \mathcal{K}_m(\varphi_m) - \mathcal{K}_m(\varphi_m) \right\} = -U \left( T(\varphi) + \frac{V_1(\varphi)}{2} \right) h^{4r} + O \left( h^{2r} \max \left\{ \tilde{h}^d, h^{2r+2} \right\} \right).
\]

**Proof** Note that
\[
\mathcal{K}_m(\varphi_m) - \mathcal{K}_m(\varphi_m) = (I - Q_n) \left( \mathcal{K}_m(\varphi_m) - \mathcal{K}_m(Q_n \varphi_m) \right).
\]
By Taylor’s theorem,
\[
\mathcal{K}_m(Q_n \varphi_m) - \mathcal{K}_m(\varphi_m) = \mathcal{K}_m(\varphi_m)(Q_n \varphi_m - \varphi_m) + \frac{\mathcal{K}_m(\varphi_m)}{2} (Q_n \varphi_m - \varphi_m)^2
\]
\[
+ \frac{\mathcal{K}_m(\varphi_m)}{6} (Q_n \varphi_m - \varphi_m)^3 + \frac{\mathcal{K}_m(\varphi_m)}{24} (Q_n \varphi_m - \varphi_m)^4,
\]
where $\xi_m \in B(\varphi, \delta)$.

Hence,
\[
L_m \mathcal{K}_m(\varphi_m) \left\{ \mathcal{K}_m(\varphi_m) - \mathcal{K}_m(\varphi_m) \right\} = -L_m \mathcal{K}_m(\varphi_m)(I - Q_n) \mathcal{K}_m(\varphi_m)(Q_n \varphi_m - \varphi_m)
\]
\[
- \frac{1}{2} L_m \mathcal{K}_m(\varphi_m)(I - Q_n) \mathcal{K}_m(\varphi_m)(Q_n \varphi_m - \varphi_m)^2
\]
\[
- \frac{1}{6} L_m \mathcal{K}_m(\varphi_m)(I - Q_n) \mathcal{K}_m(\varphi_m)(Q_n \varphi_m - \varphi_m)^3
\]
\[
- \frac{1}{24} L_m \mathcal{K}_m(\varphi_m)(I - Q_n) \mathcal{K}_m(\varphi_m)(Q_n \varphi_m - \varphi_m)^4.
\]

Using (2.10) and (4.3), we deduce the following:
\[
\left\| L_m \mathcal{K}_m(\varphi_m)(I - Q_n) \mathcal{K}_m(\varphi_m)(Q_n \varphi_m - \varphi_m)^4 \right\|_{\infty} = O \left( h^{6r} \right).
\]
Using the above estimate, Lemma 4.2 and Lemma 4.3, we obtain the following:
\[
L_m \mathcal{K}_m(\varphi_m) \left\{ \mathcal{K}_m(\varphi_m) - \mathcal{K}_m(\varphi_m) \right\} = -U \left( T(\varphi) + \frac{V_1(\varphi)}{2} \right) h^{4r} + O \left( h^{4r+2} \right),
\]
which completes the proof.
We quote the following result from Kulkarni-Rakshit [6, Proposition 4.6]:
\[
\| \tilde{X}_n^M (z_n^M) - X_n^M (\varphi_m) - (X_n^M)' (\varphi_m)(z_n^M - \varphi_m) \|_{\infty} = O \left( \max \left\{ \bar{h}^d, h^{3r} \right\}^2 \right).
\]
(4.17)
It follows that the second term on the right hand side of (4.6) is of the order of
\[
\max \left\{ \tilde{h}^d, h^{3r} \right\}.
\]
We now want obtain the order of convergence of the third term on the right hand side of (4.6). For this purpose we first prove the following result.

**Lemma 4.5** If \( d \geq 2r \), \( \frac{\partial k}{\partial u} \in C^{\max\{d,3r+3\}}(\Omega) \) and \( f \in C^{\max\{d,2r+2\}}[0,1] \), then
\[
\| X_m'(\varphi_m)(I - Q_n) X_m'(\varphi_m)(I - Q_n) \left[ \tilde{X}_n^M (\varphi_m) - X_n^M (\varphi_m) \right] \|_{\infty} = O \left( h^{6r} \right),
\]
(4.18)
\[
\| X_m'(\varphi_m)(I - Q_n) X_m'(\varphi_m)(I - Q_n) (X_n^M)' (\varphi_m) \|_{\infty} = O \left( h^{3r} \right).
\]
(4.19)

**Proof** Note that
\[
\tilde{X}_n^M (\varphi_m) - X_n^M (\varphi_m) = (I - Q_n) (X_m(Q_n \varphi_m) - X_m(\varphi_m)) = (I - Q_n) X_m'(\xi_m)(Q_n \varphi_m - \varphi_m),
\]
where \( \xi_m \in B(\varphi, \delta) \). Hence
\[
\| X_m'(\varphi_m)(I - Q_n) X_m'(\varphi_m)(I - Q_n) X_m'(\xi_m)(Q_n \varphi_m - \varphi_m) \|_{\infty} \\
\leq \| X_m'(\varphi_m)(I - Q_n) X_m'(\varphi_m)(I - Q_n) X_m'(\xi_m)(Q_n \varphi - \varphi) \|_{\infty} \\
+ \| X_m'(\varphi_m)(I - Q_n) X_m'(\varphi_m)(I - Q_n) X_m'(\xi_m)(Q_n - I)(\varphi_m - \varphi) \|_{\infty}.
\]
(4.20)
Recall from (4.5) that
\[
\| X_m'(\varphi_m)(I - Q_n) X_m'(\varphi_m)(I - Q_n) X_m'(\xi_m)(Q_n \varphi - \varphi) \|_{\infty} \leq (C_6)^2 \| \ell_m \|_{r, \infty} \| \ell_m \|_{3r, \infty} \| X_m'(\xi_m)(Q_n \varphi - \varphi) \|_{2r, \infty} h^{4r}
\]
and it can be checked that
\[
\| X_m'(\xi_m)(Q_n \varphi - \varphi) \|_{2r, \infty} \leq C_5 C_6 \| \varphi \|_{2r, \infty} h^{2r}.
\]
Thus,
\[
\| X_m'(\varphi_m)(I - Q_n) X_m'(\varphi_m)(I - Q_n) X_m'(\xi_m)(Q_n \varphi - \varphi) \|_{\infty} = O \left( h^{6r} \right). 
\]
(4.21)
On the other hand, since \( d \geq 2r \) and \( \tilde{h} \leq h \), from (2.6) and (4.4),
\[
\| X_m'(\varphi_m)(I - Q_n) X_m'(\varphi_m)(I - Q_n) X_m'(\xi_m)(Q_n - I)(\varphi_m - \varphi) \|_{\infty} \\
\leq \| X_m'(\varphi_m)(I - Q_n) X_m'(\varphi_m)(I - Q_n) X_m'(\xi_m)(Q_n - I)(\varphi_m - \varphi) \|_{\infty} \\
= O(h^{4r} \tilde{h}^d) = O \left( h^{6r} \right).
\]
(4.22)
The estimate \((4.18)\) then follows from \((4.20), (4.21)\) and \((4.22)\).

In order to prove \((4.19)\), recall that
\[
\left(\tilde{K}_M^M\right)'(\varphi_m) = Q_nK'(\varphi_m) + (I - Q_n)K'(Q_n\varphi_m)Q_n.
\]

Hence
\[
K'(\varphi_m)(I - Q_n)K'(\varphi_m)(I - Q_n)\left(\tilde{K}_M^M\right)'(\varphi_m)
= K'(\varphi_m)(I - Q_n)K'(\varphi_m)(I - Q_n)\left(K'(Q_n\varphi_m) - K'(\varphi_m)\right)Q_n
+ K'(\varphi_m)(I - Q_n)K'(\varphi_m)(I - Q_n)K'(\varphi_m)Q_n.
\]

The result \((4.19)\) follows from \((2.10), (3.3), (4.3)\) and \((4.4)\).

We now obtain the order of convergence of the third term in \((4.6)\).

**Proposition 4.6** If \(d \geq 2r, \frac{\partial \kappa}{\partial u} \in C^{\max\{d, 3r + 3\}}(\Omega)\) and \(f \in C^{\max\{d, 2r + 2\}}[0, 1]\), then
\[
\left\|K'(\varphi_m)\left\{\left(\tilde{K}_M^M\right)'(\varphi_m) - K'(\varphi_m)\right\}(z_n^M - \varphi_m)\right\|_\infty = O\left(h^{3r} \max\left\{\tilde{h}^d, h^{3r}\right\}\right).
\]

**Proof** Note that
\[
\left(\tilde{K}_M^M\right)'(\varphi_m) - K'(\varphi_m) = (I - Q_n)(K'(Q_n\varphi_m) - K'(\varphi_m))Q_n
- (I - Q_n)K'(\varphi_m)(I - Q_n).
\]

By Taylor's theorem,
\[
(K'(Q_n\varphi_m) - K'(\varphi_m))Q_n(z_n^M - \varphi_m) = K''(\xi_m)\left(Q_n\varphi_m - \varphi_m, Q_n(z_n^M - \varphi_m)\right)
\]
where \(\xi_m \in B(\varphi, \delta)\). Hence, from \((1.6), (2.10)\) and \((4.3)\), we obtain the following:
\[
\left\|K'(\varphi_m)(I - Q_n)K''(\xi_m)\left(Q_n\varphi_m - \varphi_m, Q_n(z_n^M - \varphi_m)\right)\right\|_\infty
\leq \left\|K_m'(\varphi_m)(I - Q_n)K''(\xi_m)\right\|_\infty \left\|Q_n\varphi_m - \varphi_m\right\| \left\|Q_n\right\| \left\|z_n^M - \varphi_m\right\|_\infty
= O\left(h^{3r} \max\left\{\tilde{h}^d, h^{3r}\right\}\right). \tag{4.23}
\]

Note that
\[
K'(\varphi_m)(I - Q_n)K'(\varphi_m)(I - Q_n)(z_n^M - \varphi_m)
= K'(\varphi_m)(I - Q_n)K'(\varphi_m)(I - Q_n)
\]
\[
\times \left[\tilde{K}_n^M(z_n^M) - K_n^M(\varphi_m) - \left(\tilde{K}_n^M\right)'(\varphi_m)(z_n^M - \varphi_m)\right]
+ K'(\varphi_m)(I - Q_n)K'(\varphi_m)(I - Q_n)\left[\tilde{K}_n^M(\varphi_m) - K_n(\varphi_m)\right]
+ K'(\varphi_m)(I - Q_n)K'(\varphi_m)(I - Q_n)\left(\tilde{K}_n^M\right)'(\varphi_m)(z_n^M - \varphi_m).
\]
By (4.3) and (4.17), the first term on the right-hand side of the above equation is of the order of $h^{2r} \max \left\{ \tilde{h}^d, h^{3r} \right\}^2$. By (4.18) of Lemma 4.5, the second term is of the order of $h^{6r}$. By (4.19) of Lemma 4.5 and by (1.6), the third term is of the order of $h^{3r} \max \left\{ \tilde{h}^d, h^{3r} \right\}^2$. It follows that

$$\| \mathcal{K}_m'(\varphi_m)(I - Q_n) \mathcal{K}_m'(\varphi_m)(I - Q_n)(z_n^M - \varphi_m) \|_\infty = O \left( h^{3r} \max \left\{ \tilde{h}^d, h^{3r} \right\}^2 \right).$$

The required result follows from (4.23) and the above estimate.

We now prove our main result.

**Theorem 4.7** If $d \geq 2r$, $\frac{\partial \kappa}{\partial u} \in C^{\max\{d, 3r+3\}}(\Omega)$, and $f \in C^{\max\{d, 2r+2\}}[0, 1]$, then

$$z_n^M - \varphi_m = \left( U(T(\varphi)) + \frac{U(V_1(\varphi))}{2} \right) h^{4r} + O \left( h^{2r} \max \left\{ \tilde{h}^d, h^{2r+2} \right\} \right).$$

**Proof** Recall from (4.1) that

$$z_n^M - \varphi_m = \mathcal{K}_m'(\varphi_m)(z_n^M - \varphi_m) + O \left( \max \left\{ \tilde{h}^d, h^{3r} \right\}^2 \right).$$

Recall from (4.6) that

$$\mathcal{K}_m'(\varphi_m)(z_n^M - \varphi_m) = -L_m \mathcal{K}_m'(\varphi_m) \left\{ \mathcal{K}_m(\varphi_m) - \mathcal{K}_n^M(\varphi_m) \right\} + L_m \mathcal{K}_m'(\varphi_m) \left\{ \mathcal{K}_n^M(z_n^M - \varphi_m) - \left( \mathcal{K}_n^M \right)'(\varphi_m)(z_n^M - \varphi_m) \right\} + L_m \mathcal{K}_m'(\varphi_m) \left\{ \left( \mathcal{K}_n^M \right)'(\varphi_m) - \mathcal{K}_m'(\varphi_m) \right\} (z_n^M - \varphi_m).$$

Thus, by Proposition 4.4, estimate (4.17) and Proposition 4.6, we obtain the following:

$$\mathcal{K}_m'(\varphi_m)(z_n^M - \varphi_m) = \left( U(T(\varphi)) + \frac{U(V_1(\varphi))}{2} \right) h^{4r} + O \left( h^{2r} \max \left\{ \tilde{h}^d, h^{2r+2} \right\} \right)$$

and the proof is complete. \(\square\)

We can now apply one step of Richardson extrapolation and obtain approximations of $\varphi$ of higher order. Define the following:

$$z_n^{E} = \frac{2^{4r} z_n^M - z_{2n}^M}{2^{4r} - 1}.$$
Corollary 1 Under the assumptions of Theorem 4.7,
\[ \| z_n^E - \varphi_m \|_\infty = O \left( h^{2r} \max \{ \tilde{h}^d, h^{2r+2} \} \right). \] (4.24)
If we choose \( \tilde{h} \) and \( d \) such that \( \tilde{h}^d \leq h^{2r+2} \), then
\[ \| z_n^E - \varphi_m \|_\infty = O \left( h^{4r+2} \right). \] (4.25)

5 Numerical results

In this section, we quote the results from Kulkarni-Nidhin [5] to illustrate the improvement of orders of convergence by the Richardson extrapolation obtained in (4.25).

Consider the following:
\[ \varphi(s) - \int_0^1 \frac{ds}{s + t + \varphi(t)} = f(s), \quad 0 \leq s \leq 1, \] (5.1)
where \( f \) is so chosen that \( \varphi(t) = \frac{1}{t+1} \) is a solution of (5.1).

We consider the following uniform partition of \([0, 1]\):
\[ 0 < \frac{1}{n} < \frac{2}{n} < \cdots < \frac{n}{n} = 1. \] (5.2)

For \( r = 1, 2 \), let \( \mathcal{P}_n \) be the space of piecewise polynomials of degree \( \leq r - 1 \) with respect to the partition (5.2). The collocation points are chosen to be \( r \) Gauss points in each subinterval.

If \( \mathcal{P}_n \) is the space of piecewise constant functions, then we choose the composite 2 point Gaussian quadrature with respect to the uniform partition of \([0, 1]\) with 256 intervals. The computations are done for \( n = 2, 4, 8, 16 \) and 32. Thus,
\[ r = 1, \quad d = 4, \quad \tilde{h} = 2^{-8}, \quad h \geq 2^{-5} \quad \text{and hence} \quad \tilde{h}^d = 2^{-32} \leq 2^{-20} \leq h^{2r+2} \]
and the conditions of Theorem 4.7 are satisfied (Table 1).

Table 1 Interpolation at Midpoints: \( r = 1 \)

| \( n \) | \( \| \varphi_m - z_n^M \|_\infty \) | \( \delta^M \) | \( \| \varphi_m - z_n^M \|_\infty \) | \( \delta^M \) | \( \| \varphi_m - z_n^E \|_\infty \) | \( \delta^E \) |
|---|---|---|---|---|---|---|
| 2 | \( 6.92 \times 10^{-3} \) | 3.30 \times 10^{-4} | | | | |
| 4 | \( 1.02 \times 10^{-3} \) | 2.76 | 2.13 \times 10^{-5} | 3.95 | 7.31 \times 10^{-7} | |
| 8 | \( 1.40 \times 10^{-4} \) | 2.87 | 1.34 \times 10^{-6} | 3.99 | 6.81 \times 10^{-9} | 6.75 |
| 16 | \( 1.82 \times 10^{-5} \) | 2.94 | 8.37 \times 10^{-8} | 4.00 | 8.46 \times 10^{-11} | 6.33 |
| 32 | \( 2.27 \times 10^{-6} \) | 3.00 | 5.23 \times 10^{-9} | 4.00 | 1.01 \times 10^{-12} | 6.39 |
Table 2  Interpolation at Gauss 2 points: \( r = 2 \)

| \( n \) | \( \| \psi_m - \tilde{z}_n^M \|_\infty \) | \( \delta^M \) | \( \| \psi_m - \tilde{z}_n^{IM} \|_\infty \) | \( \delta^{IM} \) | \( \| \psi_m - \tilde{z}_n^E \|_\infty \) | \( \delta^E \) |
|-------|------------------|------|------------------|------|------------------|------|
| 2     | 5.06 \times 10^{-4} |      | 6.47 \times 10^{-5} |      |                  |      |
| 4     | 1.07 \times 10^{-5} | 5.56 | 2.09 \times 10^{-7} | 8.27 | 4.37 \times 10^{-8} |      |
| 8     | 1.85 \times 10^{-7} | 5.86 | 8.45 \times 10^{-10} | 7.95 | 2.67 \times 10^{-11} | 10.68 |
| 16    | 3.07 \times 10^{-9} | 5.90 | 3.35 \times 10^{-12} | 7.98 | 4.73 \times 10^{-14} | 9.14  |
| 32    | 4.74 \times 10^{-11} | 6.02 | 1.34 \times 10^{-14} | 7.96 | 2.11 \times 10^{-15} | 4.49  |

If \( \mathcal{X}_n \) is the space of piecewise linear functions, then we chose the composite 2 point Gaussian quadrature with respect to the uniform partition of \([0, 1]\) with \( n^2 \) intervals. Thus,

\[
r = 2, \quad \tilde{h} = h^2, \quad d = 4 \quad \text{and hence} \quad \tilde{h}^d = h^8 \leq h^6 = h^{2r+2}
\]

and the conditions of Theorem 4.7 are satisfied (Table 2).

The expected orders of convergence from (1.6), (1.7) and (4.24) are as follows:

- Discrete Modified Projection Solution: \( \delta^M = 3r \),
- Discrete Iterated Modified Projection Solution: \( \delta^{IM} = 4r \),
- Extrapolated Solution: \( \delta^E = 4r + 2 \).

It is clear from the above tables that the computed orders of convergence match well with the theoretical orders of convergence and that the extrapolated solution improves upon the discrete iterated modified projection solution.

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