On Segal–Sugawara vectors and Casimir elements for classical Lie algebras

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Abstract

We consider the centers of the affine vertex algebras at the critical level associated with simple Lie algebras. We derive new formulas for generators of the centers in the classical types. We also give a new formula for the Capelli-type determinant for the symplectic Lie algebras and calculate the Harish-Chandra images of the Casimir elements arising from the characteristic polynomial of the matrix of generators of each classical Lie algebra.

1 Introduction

Let \( \mathfrak{g} \) be a simple Lie algebra over \( \mathbb{C} \) equipped with a standard symmetric invariant bilinear form. The affine Kac–Moody algebra \( \hat{\mathfrak{g}} \) is defined as the central extension

\[
\hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C} K
\]

of the Lie algebra of Laurent polynomials in \( t \). The vacuum module \( V_{\text{cri}}(\mathfrak{g}) \) at the critical level over \( \hat{\mathfrak{g}} \) is the quotient of the universal enveloping algebra \( U(\hat{\mathfrak{g}}) \) by the left ideal generated by \( \mathfrak{g}[t] \) and \( K + h^\vee \), where \( h^\vee \) is the dual Coxeter number for \( \mathfrak{g} \). The vacuum module has a vertex algebra structure and is known as the \((universal) affine vertex algebra\); see e.g. [4] and [8] for definitions. The center of the vertex algebra \( V_{\text{cri}}(\mathfrak{g}) \) is defined by

\[
\mathfrak{z}(\hat{\mathfrak{g}}) = \{ S \in V_{\text{cri}}(\mathfrak{g}) \mid \mathfrak{g}[t] S = 0 \}.
\]

Any element of \( \mathfrak{z}(\hat{\mathfrak{g}}) \) is called a Segal–Sugawara vector. The vertex algebra axioms imply that the center is a commutative associative algebra which can be regarded as a subalgebra of \( U(t^{-1}\mathfrak{g}[t^{-1}]) \). The algebra \( \mathfrak{z}(\hat{\mathfrak{g}}) \) is equipped with the derivation \( T = -d/dt \) arising from the vertex algebra structure. By a theorem of Feigin and Frenkel [3], the differential algebra \( \mathfrak{z}(\hat{\mathfrak{g}}) \) possesses generators \( S_1, \ldots, S_n \) so that \( \mathfrak{z}(\hat{\mathfrak{g}}) \) is the algebra of polynomials

\[
\mathfrak{z}(\hat{\mathfrak{g}}) = \mathbb{C}[T^r S_l \mid l = 1, \ldots, n, \ r \geq 0],
\]

where \( n = \text{rank} \mathfrak{g} \); see also [4]. The algebra \( \mathfrak{z}(\hat{\mathfrak{g}}) \) is known as the Feigin–Frenkel center, and we call \( S_1, \ldots, S_n \) a complete set of Segal–Sugawara vectors. According to [3] (see also [4]),
the center can be identified with the classical $\mathcal{W}$-algebra associated with the Langlands dual Lie algebra $Lg$ via an affine version of the Harish-Chandra isomorphism
\[ \mathfrak{z} (\hat{g}) \cong \mathcal{W} (Lg). \]

Explicit formulas for complete sets of Segal–Sugawara vectors were given in [1] and [2] for the Lie algebras $g$ of type $A$, and in [11] for types $B$, $C$ and $D$ with the use of the Brauer algebra. The images of the vectors with respect to the Harish-Chandra isomorphism (1.2) are easy to find in type $A$ [1], whereas the calculations in the remaining classical types in [13] rely on the $q$-character formulas for certain Yangian representations. A detailed exposition of these results together with applications to commutative subalgebras in enveloping algebras and to higher order Hamiltonians in the Gaudin models can be found in [12]. A complete set of Segal–Sugawara vectors for the Lie algebra of type $G_2$ was produced in [14] by using computer-assisted calculations. A different method to construct generators of $\mathfrak{z} (\hat{g})$ was developed in [17] which lead to new explicit formulas in the case of the Lie algebras of types $B, C, D$ and $G_2$.

In this paper we derive new uniform expressions for the Segal–Sugawara vectors in all classical types. In types $B, C$ and $D$ we transform the formulas produced in [11] by eliminating the dependence on the Brauer diagrams with horizontal edges. In particular, the vectors are given more explicitly in the symplectic case thus resolving the ‘analytic continuation’ procedure used in [11]; see also [12, Ch. 8]. We also show that both in the orthogonal and symplectic case the Segal–Sugawara vectors of [11] coincide with those in [17].

In all classical types we also consider the Casimir elements obtained by the application of the symmetrization map to basic $g$-invariants in the symmetric algebra $S(g)$, arising from the characteristic polynomial of the matrix of generators. We calculate their Harish-Chandra images in terms of shifted invariant polynomials.

Certain analogues of the Capelli determinant for the orthogonal and symplectic Lie algebras were obtained in [10] with the use of the Sklyanin determinants; see also [5] and [16] for different approaches. Here we give a new symmetrized determinant expression for the Capelli-type determinant in the symplectic case.

## 2 Segal–Sugawara vectors

In all classical types, the new formulas for Segal–Sugawara vectors will take the form of linear combinations of certain symmetrized $\lambda$-minors or $\lambda$-permanents associated with partitions $\lambda$. We consider the Lie algebras of type $A$ first, where the formulas are derived from the results of [1] and [2]; see also [12, Ch. 7].

### 2.1 Generators of $\mathfrak{z} (\hat{g}[N])$

The affine Kac–Moody algebra $\hat{g}[N] = g[N][t, t^{-1}] \oplus \mathbb{C} K$ has the commutation relations
\[ [E_{ij}[r], E_{kl}[s]] = \delta_{kj} E_{il}[r + s] - \delta_{il} E_{kj}[r + s] + r \delta_{r,-s} K \left( \delta_{k,l} - \frac{\delta_{ij} \delta_{kl}}{N} \right), \]
and the element $K$ is central. Here and below we write $X[r]$ for the element $Xt^r$ of the Lie algebra of Laurent polynomials $\mathfrak{g}[t, t^{-1}]$ with $X \in \mathfrak{g}$ and $r \in \mathbb{Z}$. The elements $E_{ij}$ form a standard basis of $\mathfrak{gl}_N$. The critical level $K = -N$ coincides with the negative of the dual Coxeter number for $\mathfrak{sl}_N$.

Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be a partition of $m$ of length $\ell = \ell(\lambda)$, so that $\lambda_1 \geq \cdots \geq \lambda_\ell > 0$ and $\lambda_1 + \cdots + \lambda_\ell = m$. We will denote by $c_\lambda$ the number of permutations in the symmetric group $\mathfrak{S}_m$ of cycle type $\lambda$; see (2.12) below for an explicit expression. The symmetrized $\lambda$-minors $D(\lambda)$ and symmetrized $\lambda$-permanents $P(\lambda)$ are elements of $V_{crs}(\mathfrak{gl}_N) \cong U(t^{-1} \mathfrak{gl}_N[t^{-1}])$ defined by

$$D(\lambda) = \frac{1}{\ell!} \sum_{i_1, \ldots, i_\ell = 1}^N \sum_{\sigma \in \mathfrak{S}_\ell} \text{sgn} \sigma \cdot E_{i_{\sigma(1)}i_1}[-\lambda_1] \cdots E_{i_{\sigma(\ell)}i_\ell}[-\lambda_\ell]$$

and

$$P(\lambda) = \frac{1}{\ell!} \sum_{i_1, \ldots, i_\ell = 1}^N \sum_{\sigma \in \mathfrak{S}_\ell} E_{i_{\sigma(1)}i_1}[-\lambda_1] \cdots E_{i_{\sigma(\ell)}i_\ell}[-\lambda_\ell].$$

**Theorem 2.1.** All elements

$$\phi_m = \sum_{\lambda \vdash m} \binom{N}{\ell}^{-1} c_\lambda D(\lambda) \quad \text{and} \quad \psi_m = \sum_{\lambda \vdash m} \binom{N + \ell - 1}{\ell}^{-1} c_\lambda P(\lambda)$$

belong to the Feigin–Frenkel center $z(\hat{\mathfrak{gl}}_N)$. Moreover, each family $\phi_1, \ldots, \phi_N$ and $\psi_1, \ldots, \psi_N$ is a complete set of Segal–Sugawara vectors for $\mathfrak{gl}_N$.

**Proof.** The theorem will follow from the relations

$$\phi_{mm} = \left( \begin{array}{c} N \\ m \end{array} \right)^{-1} \phi_m \quad \text{and} \quad \psi_{mm} = \left( \begin{array}{c} N + m - 1 \\ m \end{array} \right)^{-1} \psi_m \tag{2.2}$$

for the Segal–Sugawara vectors $\phi_{mm}$ and $\psi_{mm}$ used in [12, Ch. 7]. To make the connection, for any $r \in \mathbb{Z}$ combine the elements $E_{ij}[r]$ into the matrix $E[r]$ so that

$$E[r] = \sum_{i,j=1}^N e_{ij} \otimes E_{ij}[r] \in \text{End} \mathbb{C}^N \otimes U(\hat{\mathfrak{gl}}_N),$$

where the $e_{ij}$ denote the standard matrix units. For each $a \in \{1, \ldots, m\}$ introduce the element $E[r]_a$ of the algebra

$$\underbrace{\text{End} \mathbb{C}^N \otimes \cdots \otimes \text{End} \mathbb{C}^N}_m \otimes U(\hat{\mathfrak{gl}}_N) \tag{2.3}$$

by

$$E[r]_a = \sum_{i,j=1}^N 1^{\otimes (a-1)} \otimes e_{ij} \otimes 1^{\otimes (m-a)} \otimes E_{ij}[r].$$

3
The symmetric group $\mathfrak{S}_m$ acts on the space
\[
(C^N)^\otimes_m = C^N \otimes C^N \otimes \cdots \otimes C^N
\] (2.4)
by permuting the tensor factors. Denote by $H^{(m)}$ and $A^{(m)}$ the elements of the algebra (2.3) (with the identity components in $U(\hat{gl}_N)$) which are the respective images of the symmetrizer $h^{(m)}$ and anti-symmetrizer $a^{(m)}$ defined by
\[
h^{(m)} = \frac{1}{m!} \sum_{s \in \mathfrak{S}_m} s \quad \text{and} \quad a^{(m)} = \frac{1}{m!} \sum_{s \in \mathfrak{S}_m} \text{sgn } s \cdot s,
\] (2.5)
under the action of $\mathfrak{S}_m$. By [12, Thms 7.1.3 & 7.1.4], the corresponding claims of Theorem 2.1 hold for the elements
\[
\phi_{mm} = \text{tr}_{1, \ldots, m} A^{(m)} \left( T + E[-1]_1 \right) \cdots \left( T + E[-1]_m \right) 1
\] (2.6)
and
\[
\psi_{mm} = \text{tr}_{1, \ldots, m} H^{(m)} \left( T + E[-1]_1 \right) \cdots \left( T + E[-1]_m \right) 1,
\] (2.7)
where the vacuum vector of $V_{\text{cri}}(\hat{gl}_N)$ is identified with the element $1 \in U \left( t^{-1} \hat{gl}_N [t^{-1}] \right)$ which is annihilated by the derivation $T = -d/dt$. The trace is taken with respect to all $m$ copies of the endomorphism algebra $\text{End } C^N$ in (2.3). Expand the product in (2.6) by using the relations $[T, E[-r]_a] = r E[-r-1]_a$ so that $\phi_{mm}$ will take the form of a linear combination of the traces
\[
\text{tr}_{1, \ldots, m} A^{(m)} E[-r_1]_{a_1} \cdots E[-r_s]_{a_s}
\] (2.8)
with $a_1 < \cdots < a_s$ and $r_i \geq 1$. The defining relations (2.1) imply that for $a < b$ we have
\[
E[-r]_a E[-s]_b - E[-s]_b E[-r]_a = E[-r-s]_a P_{ab} - P_{ab} E[-r-s]_a,
\]
where $P_{ab}$ is the permutation operator
\[
P_{ab} = \sum_{i,j=1}^N 1^{-\otimes(a-1)} \otimes e_{ij} \otimes 1^{-\otimes(b-a-1)} \otimes e_{ji} \otimes 1^{-\otimes(m-b)}.
\] (2.9)
Hence, by the cyclic property of trace, any permutation of the factors $E[-r_i]_{a_i}$ in the expression (2.8) does not change its value. Therefore, applying conjugations by suitable permutations of the index set $1, \ldots, m$ and using the cyclic property of trace, we can write
\[
\text{tr}_{1, \ldots, m} A^{(m)} \left( T + E[-1]_1 \right) \cdots \left( T + E[-1]_m \right) 1 = \text{tr}_{1, \ldots, m} A^{(m)} \sum_{\lambda + m} c_\lambda E[-\lambda],
\] (2.10)
for certain nonnegative integers $c_\lambda$, where we set
\[
E[-\lambda] = E[-\lambda_1]_1 \cdots E[-\lambda_\ell]_\ell
\]
for any partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ with $\ell = \ell(\lambda)$. Identifying partitions with their Young diagrams, we can write the expression on the left hand side of (2.10) as

$$\text{tr}_1,\ldots,m A^{(m)} T \sum_{\mu^{\ell+1} = 1} c_\mu E[-\mu] + \text{tr}_1,\ldots,m A^{(m)} \sum_{\mu^{\ell+1} = 1} c_\mu E[-\mu^+]$$

where $\mu^+$ is the diagram obtained from $\mu$ by adding one box to the first column. Hence the coefficients $c_\lambda$ satisfy the recurrence relation

$$c_\lambda = \sum_{\mu} \gamma(\mu, \lambda) c_\mu, \quad (2.11)$$

summed over the diagrams $\mu$ which are obtained from $\lambda$ by removing one box, where

$$\gamma(\mu, \lambda) = \begin{cases} \mu_i \cdot \text{mult}(\mu_i) & \text{if } \mu_i \geq 1, \\ 1 & \text{if } \mu_i = 0, \end{cases}$$

assuming the box is removed in row $i$, and $\text{mult}(\mu_i)$ denotes the multiplicity of $\mu_i$ as a part of $\mu$. Writing the partition $\lambda$ in the multiplicity form $\lambda = (1^{\alpha_1} 2^{\alpha_2} \ldots m^{\alpha_m})$, we derive from (2.11) by induction on $m$ that

$$c_\lambda = \frac{m!}{1^{\alpha_1} \alpha_1! 2^{\alpha_2} \alpha_2! \ldots m^{\alpha_m} \alpha_m!} \quad (2.12)$$

which is the number of permutations in $S_m$ of cycle type $\lambda$.

Note that the partial traces of the anti-symmetrizer are found by

$$\text{tr}_{\ell+1,\ldots,m} A^{(m)} = \left( \begin{array}{c} N \\ m \end{array} \right) \left( \begin{array}{c} N \\ \ell \end{array} \right)^{-1} A^{(\ell)}, \quad (2.13)$$

and so (2.10) implies

$$\phi_{mm} = \left( \begin{array}{c} N \\ m \end{array} \right) \sum_{\lambda^{\ell+1} = 1} \left( \begin{array}{c} N \\ \ell \end{array} \right)^{-1} c_\lambda \text{tr}_{1,\ldots,\ell} A^{(\ell)} E[-\lambda],$$

which proves the first relation in (2.2) because

$$D(\lambda) = \text{tr}_{1,\ldots,\ell} A^{(\ell)} E[-\lambda]. \quad (2.14)$$

The second relation in (2.2) is verified by the same argument, where the partial traces of the symmetrizer are evaluated by

$$\text{tr}_{\ell+1,\ldots,m} H^{(m)} = \left( \begin{array}{c} N + m - 1 \\ m \end{array} \right) \left( \begin{array}{c} N + \ell - 1 \\ \ell \end{array} \right)^{-1} H^{(\ell)}, \quad (2.15)$$

and the identity

$$P(\lambda) = \text{tr}_{1,\ldots,\ell} H^{(\ell)} E[-\lambda],$$

is used in place of (2.14).
As was pointed out in [17, Sec. 2], since the Feigin–Frenkel center \( \check{\mathfrak{g}}(\mathfrak{g}_N) \) is invariant under the automorphism \( \theta \) of \( U(t^{-1}\mathfrak{g}_N(t^{-1})) \) taking \( E_{ij}[r] \) to \( -E_{ji}[r] \), the Segal–Sugawara vectors of Theorem 2.1 can be modified to become eigenvectors of \( \theta \). Note that both \( A^{(\ell)} \) and \( H^{(\ell)} \) are stable under the simultaneous transpositions with respect to all \( \ell \) copies of \( \text{End} \mathbb{C}^N \) and so

\[
\theta : D(\lambda) \mapsto (-1)^\ell D(\lambda) \quad \text{and} \quad \theta : P(\lambda) \mapsto (-1)^\ell P(\lambda).
\]

The same symmetry properties hold for the anti-automorphism of \( U(t^{-1}\mathfrak{g}_N(t^{-1})) \) taking \( E_{ij}[r] \) to \( -E_{ij}[r] \). As in [12, Ch. 7], this leads to the following.

**Corollary 2.2.** All elements

\[
\phi_m^0 = \sum_{\lambda+m, \; m-\ell \text{ even}} \binom{N}{\ell}^{-1} c_\lambda D(\lambda) \quad \text{and} \quad \psi_m^0 = \sum_{\lambda+m, \; m-\ell \text{ even}} \binom{N+\ell-1}{\ell}^{-1} c_\lambda P(\lambda)
\]

belong to the Feigin–Frenkel center \( \check{\mathfrak{g}}(\mathfrak{g}_N) \). Moreover, each family \( \phi_1^0, \ldots, \phi_N^0 \) and \( \psi_1^0, \ldots, \psi_N^0 \) is a complete set of Segal–Sugawara vectors for \( \mathfrak{g}_N \).

### 2.2 Generators of \( \check{\mathfrak{g}}(\mathfrak{g}) \) for types \( B, C \) and \( D \)

We will regard the orthogonal Lie algebras \( \mathfrak{o}_N \) with \( N = 2n+1 \) and \( N = 2n \) and symplectic Lie algebra \( \mathfrak{sp}_N \) with \( N = 2n \) as subalgebras of \( \mathfrak{g}_N \) spanned by the elements \( F_{ij} \),

\[
F_{ij} = E_{ij} - E_{ji}, \quad \text{and} \quad F_{ij} = E_{ij} - \varepsilon_i \varepsilon_j E_{ji},
\]

respectively, for \( \mathfrak{o}_N \) and \( \mathfrak{sp}_N \), where \( i' = N - i + 1 \). In the symplectic case we set \( \varepsilon_i = 1 \) for \( i = 1, \ldots, n \) and \( \varepsilon_i = -1 \) for \( i = n+1, \ldots, 2n \). As before, we will write \( F_{ij}[r] = F_{ij}t^r \) with \( r \in \mathbb{Z} \) for elements of the Kac–Moody algebra \( \check{\mathfrak{g}} \) for \( \mathfrak{g} = \mathfrak{o}_N \) or \( \mathfrak{sp}_N \), as defined in (1.1).

Let \( \lambda = (\lambda_1, \ldots, \lambda_\ell) \) be a partition of \( m \) of length \( \ell = \ell(\lambda) \). In the case \( \mathfrak{g} = \mathfrak{sp}_{2n} \) we introduce the corresponding *symmetrized \( \lambda \)-minor* by

\[
D(\lambda) = \frac{1}{\ell!} \sum_{i_1, \ldots, i_\ell=1}^{2n} \sum_{\sigma \in \Sigma_\ell} \text{sgn} \sigma \cdot F_{i_{\sigma(1)}i_1}[-\lambda_1] \cdots F_{i_{\sigma(\ell)}i_\ell}[-\lambda_\ell].
\]

In the case \( \mathfrak{g} = \mathfrak{o}_N \) the *symmetrized \( \lambda \)-permanent* is defined by

\[
P(\lambda) = \frac{1}{\ell!} \sum_{i_1, \ldots, i_\ell=1}^{N} \sum_{\sigma \in \Sigma_\ell} F_{i_{\sigma(1)}i_1}[-\lambda_1] \cdots F_{i_{\sigma(\ell)}i_\ell}[-\lambda_\ell].
\]

Both \( D(\lambda) \) and \( P(\lambda) \) are zero unless \( \ell(\lambda) \) is even; see also (2.30) below.

We will keep using the numbers \( c_\lambda \) given by (2.12) which count the permutations in \( \Sigma_m \) of cycle type \( \lambda \). Recall a distinguished Segal–Sugawara vector \( \text{Pf } F[-1] \) for \( \mathfrak{g} = \mathfrak{o}_{2n} \), which is the (noncommutative) *Pfaffian* of the matrix \( F[-1] = [F_{ij}[-1]] \); see [11], [12, Sec. 8.1].
Theorem 2.3. All elements
\[ \phi_m = \sum_{\lambda \vdash m, \ell(\lambda) \text{ even}} \left( \frac{2n + 1}{m} \right)^{\ell(\lambda)} \lambda D(\lambda) \quad \text{and} \quad \phi_m = \sum_{\lambda \vdash m, \ell(\lambda) \text{ even}} \left( \frac{N + \ell - 2}{m} \right)^{\ell(\lambda)} \lambda P(\lambda) \]
belong to the Feigin–Frenkel center \( \mathfrak{z}(\hat{\mathfrak{g}}) \) in the symplectic and orthogonal case, respectively. Moreover, the family \( \phi_2, \phi_4, \ldots, \phi_{2n} \) is a complete set of Segal–Sugawara vectors for \( \mathfrak{g} = \mathfrak{sp}_{2n} \) and \( \mathfrak{g} = \mathfrak{o}_{2n+1} \), while \( \phi_2, \phi_4, \ldots, \phi_{2n-2}, \text{Pf } F[-1] \) is complete set of Segal–Sugawara vectors for \( \mathfrak{g} = \mathfrak{o}_{2n} \).

Proof. We will derive the theorem from the relations
\[ \phi_{mm} = \left( \frac{2n + 1}{m} \right) \phi_m \quad \text{and} \quad \phi_{mm} = \left( \frac{N + m - 2}{m} \right) \phi_m \tag{2.16} \]
in the symplectic and orthogonal case, respectively, for the Segal–Sugawara vectors \( \phi_{mm} \) considered in [12, Ch. 8]. We will regard the \( N \times N \) matrix \( F[r] = [F_{ij}[r]] \) as the element
\[ F[r] = \sum_{i,j=1}^{N} e_{ij} \otimes F_{ij}[r] \in \text{End } \mathbb{C}^N \otimes \text{U}(\hat{\mathfrak{g}}). \]
It has the skew-symmetry property \( F[r] + F[r]^t = 0 \) with respect to the transposition defined by
\[ t : e_{ij} \mapsto \begin{cases} e_{j'i'} & \text{in the orthogonal case}, \\ \varepsilon_i \varepsilon_j e_{j'i'} & \text{in the symplectic case}. \end{cases} \tag{2.17} \]
For each \( a \in \{1, \ldots, m\} \) introduce the element \( F[r]_a \) of the algebra
\[ \text{End } \mathbb{C}^N \otimes \ldots \otimes \text{End } \mathbb{C}^N \otimes \text{U}(\hat{\mathfrak{g}}) \tag{2.18} \]
by
\[ F[r]_a = \sum_{i,j=1}^{N} 1^{\otimes (a-1)} \otimes e_{ij} \otimes 1^{\otimes (m-a)} \otimes F_{ij}[r]. \]
The \( a \)-th partial transposition \( t_a \) on the algebra (2.18) acts as the map (2.17) on the \( a \)-th copy of \( \text{End } \mathbb{C}^N \) and as the identity map on all other tensor factors.

The Segal–Sugawara vectors \( \phi_{mm} \) are constructed with the use of the Brauer algebra \( \mathcal{B}_m(\omega) \) whose definition we will now recall. Consider an \( m \)-diagram \( d \) which is a collection of \( 2m \) dots arranged into two rows with \( m \) dots in each row labelled by \( 1, \ldots, m \); the dots are connected by \( m \) edges in such a way that any dot belongs to only one edge. An edge is called horizontal if it connects two dots in the same row. The product \( dd' \) of two diagrams \( d \) and \( d' \) is determined by placing \( d \) under \( d' \) and identifying the vertices of the bottom row of \( d' \) with the corresponding vertices in the top row of \( d \). Let \( s \) be the number of closed loops obtained in this placement. The product \( dd' \) is given by \( \omega^s \) times the resulting diagram without loops. The algebra \( \mathcal{B}_m(\omega) \) is defined as the \( \mathbb{C}(\omega) \)-linear span of the \( m \)-diagrams with this multiplication.

For \( 1 \leq a < b \leq m \) denote by \( s_{ab} \) and \( \epsilon_{ab} \) the respective diagrams of the form
They generate the algebra $\mathcal{B}_m(\omega)$. Its subalgebra spanned over $\mathbb{C}$ by the diagrams without horizontal edges will be identified with the group algebra of the symmetric group $\mathbb{C}[\mathcal{S}_m]$ so that $s_{ab}$ is identified with the transposition $(ab)$.

We will use a special element $s^{(m)} \in \mathcal{B}_m(\omega)$, known as the symmetrizer. Several explicit expressions for $s^{(m)}$ are collected in [12, Ch. 1]; we will recall one of them, as appeared in [6],

$$
    s^{(m)} = \frac{1}{m!} \sum_{r=0}^{[m/2]} (-1)^r \left( \frac{\omega}{2} + m - 2 \right)^{-1} \sum_{d \in D^{(r)}} d,
$$

where $D^{(r)} \subset \mathcal{B}_m(\omega)$ denotes the set of diagrams which have exactly $r$ horizontal edges in the top (and hence in the bottom) row. Since $D^{(0)} = \mathcal{S}_m$, the element

$$
    h^{(m)} = \frac{1}{m!} \sum_{d \in D^{(0)}} d
$$

coincides with the symmetrizer in $\mathbb{C}[\mathcal{S}_m]$ in (2.5).

For every $a \in \{1, \ldots, m\}$ introduce the transposition $t_a$ as the linear map

$$
    t_a : \mathcal{B}_m(\omega) \to \mathcal{B}_m(\omega), \quad d \mapsto d^{t_a},
$$

where the diagram $d^{t_a}$ is obtained from $d$ by swapping the $a$-th dots in the top and bottom rows, as the ends of edges. In particular, $s_a^{t_a} = \epsilon_a$ and $\epsilon_a^{t_a} = s_{ab}$. Denote by $J_m$ the subspace of $\mathcal{B}_m(\omega)$ spanned by all sums $d + d^{t_a}$ with $d \in \mathcal{B}_m(\omega)$ and $a = 1, \ldots, m$. Note that if $\tau = t_{a_1} \circ \cdots \circ t_{a_k}$ is the composition of an odd number of distinct transpositions, then the sum $d + d^{\tau}$ belongs to $J_m$.

Introduce a rational function in $\omega$ by

$$
    \gamma_m(\omega) = \frac{\omega + m - 2}{\omega + 2m - 2}.
$$

**Lemma 2.4.** In the algebra $\mathcal{B}_m(\omega)$ we have

$$
    \gamma_m(\omega) s^{(m)} \equiv h^{(m)} \mod J_m.
$$

**Proof.** If $m$ is odd, then the claim is trivial because both $s^{(m)}$ and $h^{(m)}$ belong to the subspace $J_m$. Now suppose $m = 2k$ is even. We will start with the formula (2.19) for $s^{(2k)}$ and use an inductive procedure to apply a sequence of reductions modulo $J_{2k}$ to eliminate all diagrams containing horizontal edges from the sum. Any diagram $d$ containing an edge of the form $(a, a)$ belongs to $J_{2k}$, so that such diagrams can be ignored in the procedure. As a first step, for each $r = 0, 1, \ldots, k$ split the set of diagrams $D^{(r)}$ into three subsets,

$$
    D^{(r)} = D^{(r,-)} \cup D^{(r,0)} \cup D^{(r,+)},
$$

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where $d \in D^{(r,-)}$ if and only if the vertices 1 in the top and bottom rows are the ends of horizontal edges; $d \in D^{(r,+)}$ if and only if the vertices 1 are the ends of different non-horizontal edges, and the remaining diagrams belong to $D^{(r,0)}$. In particular, $D^{(k)} = D^{(k,-)}$. It is clear by the application of the transposition $t_1$ that for $r \geq 0$

$$\sum_{d \in D^{(r,0)}} d \equiv 0 \mod J_{2k} \quad \text{and} \quad \sum_{d \in D^{(r+1,-)}} d \equiv - \sum_{d \in D^{(r+)}} d \mod J_{2k}.$$ 

Taking into account the relation

$$\left(\frac{\omega/2 + 2k - 2}{r}\right)^{-1} + \left(\frac{\omega/2 + 2k - 2}{r + 1}\right)^{-1} = \frac{\omega + 4k - 2}{\omega + 4k - 4} \left(\frac{\omega/2 + 2k - 3}{r}\right)^{-1},$$

we can conclude from (2.19) that the reduction modulo $J_{2k}$ yields the equivalence

$$\gamma_{2k}(\omega) s^{(2k)} \equiv \gamma_{2k-2}(\omega) + \sum_{r=0}^{k-1} (-1)^r \left(\frac{\omega/2 + 2k - 3}{r}\right)^{-1} \sum_{d \in D^{(r+)}} d. \quad (2.22)$$

Note that the inverse binomial coefficients in this expression coincide with those in (2.19) for $m = 2k - 2$ with the parameter $\omega$ replaced with $\omega + 2$.

For the second step of the reduction, represent each set $D^{(r,+)}$ as the union

$$D^{(r,+)} = \bigcup_{a,b=2}^{k} D^{(r,+)}_{a,b},$$

where the subset $D^{(r,+)}_{a,b}$ consists of the diagrams $d$ containing the (non-horizontal) edges $(1, a)$ with the dot 1 in the top row, and $(1, b)$ with the dot 1 in the bottom row. Re-arrange expression (2.22) to include an extra sum by writing

$$\sum_{d \in D^{(r+)}} d = \sum_{a,b=2}^{k} \sum_{d \in D^{(r+)}} d$$

and changing the order of summation to take the external sum over $a$ and $b$. If $a = b$, then we proceed by applying the same reduction modulo $J_{2k}$ as in the first step, by ignoring the vertices 1 and $a$ in the top and bottom rows. If $a \neq b$, then split the union of sets $D^{(r,+)}_{a,b} \cup D^{(r,+)}_{b,a}$ as in (2.21). 

$$D^{(r,+)}_{a,b} \cup D^{(r,+)}_{b,a} = D^{(r,+)}_{\{a,b\}} \cup D^{(r,+)}_{\{a\}} \cup D^{(r,+)}_{\{b\}},$$

where $d \in D^{(r,+)}_{\{a\}}$ if and only if the remaining vertices $a$ and $b$ are the ends of horizontal edges; $d \in D^{(r,+)}_{\{a\}}$ if and only if the remaining vertices $a$ and $b$ are the ends of different non-horizontal edges, and the remaining diagrams belong to $D^{(r,0)}_{\{a,b\}}$. Similar to the first reduction step, the application of the composition of transpositions $t_1 \circ t_a \circ t_b$ shows that for $r \geq 0$

$$\sum_{d \in D^{(r,0)}_{\{a,b\}}} d \equiv 0 \mod J_{2k} \quad \text{and} \quad \sum_{d \in D^{(r+,+)}_{\{a,b\}}} d \equiv - \sum_{d \in D^{(r+,+)}_{\{a,b\}}} d \mod J_{2k}.$$
This leads to the second step reduction formula analogous to (2.22), and the argument continues in the same way, where compositions of $2r - 1$ distinct transpositions are used at the $r$-th step. As a result of the $k$-th step of the reduction procedure, we get the sum of diagrams without horizontal edges with the overall coefficient $1/(2k)!$, as required.

The Brauer algebra $B_m(\omega)$ with the special values $\omega = N$ and $\omega = -2n$ acts on the tensor space (2.4) so that the action centralizers the respective diagonal actions of the orthogonal and symplectic groups. In the orthogonal case, the generators of $B_m(N)$ act by the rule

$$s_{ab} \mapsto P_{ab}, \quad \epsilon_{ab} \mapsto Q_{ab}, \quad 1 \leq a < b \leq m,$$

(2.23)

where $P_{ab}$ is defined by (2.9), while

$$Q_{ab} = \sum_{i,j=1}^{N} 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(b-a-1)} \otimes e_{ij'} \otimes 1^{\otimes(m-b)}.$$

In the symplectic case, the action of $B_m(-N)$ with $N = 2n$ in the space (2.4) is defined by

$$s_{ab} \mapsto -P_{ab}, \quad \epsilon_{ab} \mapsto -Q_{ab}, \quad 1 \leq a < b \leq m,$$

(2.24)

where

$$Q_{ab} = \sum_{i,j=1}^{2n} \varepsilon_i \varepsilon_j 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(b-a-1)} \otimes e_{ij'} \otimes 1^{\otimes(m-b)}.$$

We will denote by $S^{(m)}$ the image of the symmetrizer $s^{(m)} \in B_m(\omega)$ under the respective actions (2.23) and (2.24), assuming $m \leq n$ in the symplectic case. The elements $\phi_{mm}$ of the vacuum module $V_{\text{cri}}(\mathfrak{g}) \cong U\left(t^{-1} \mathfrak{g}[t^{-1}]\right)$ are defined by

$$\phi_{mm} = \gamma_m(\omega) \operatorname{tr}_{1,...,m} S^{(m)}(T + F[-1]_1) \cdots (T + F[-1]_m) 1$$

(2.25)

where $\omega = N$ and $\omega = -2n$, respectively, for the orthogonal and symplectic case. In the symplectic case the values of $m$ are restricted to $1 \leq m \leq 2n + 1$ with an additional justification of formula (2.25) for the values $n + 1 \leq m \leq 2n + 1$ via an ‘analytic continuation’ argument; see [12, Sec. 8.3]. As proved in [11] (see also [12, Ch. 8]), all elements $\phi_{mm}$ belong to the Feigin–Frenkel center $\mathcal{Z}(\mathfrak{g})$. Moreover, the elements $\phi_{22}, \phi_{44}, \ldots, \phi_{2n2n}$, form a complete set of Segal–Sugawara vectors for $\mathfrak{g} = \mathfrak{o}_{2n+1}$ and $\mathfrak{sp}_{2n}$, whereas $\phi_{22}, \phi_{44}, \ldots, \phi_{2n-2n-2}, \text{Pf} F[-1]$ form a complete set of Segal–Sugawara vectors for $\mathfrak{g} = \mathfrak{o}_{2n}$.

Now we proceed in the same way as in the proof of Theorem 2.1 by expanding the product in (2.25) with the use of [12, Lemmas 8.1.5 & 8.3.1] to get

$$\phi_{mm} = \gamma_m(\omega) \operatorname{tr}_{1,...,m} S^{(m)} \sum_{\lambda \vdash m} c_{\lambda} F[-\lambda],$$

(2.26)

where we set

$$F[-\lambda] = F[-\lambda_1] \cdots F[-\lambda_\ell]$$
for \( \lambda = (\lambda_1, \ldots, \lambda_\ell) \) with \( \ell = \ell(\lambda) \). Note that the summands in (2.26) with odd values of \( \ell \) are equal to zero because the matrices \( F[-\lambda] \) are skew-symmetric with respect to the transposition \( t \), while \( S^{(m)} \) is stable under the simultaneous transpositions with respect to all \( m \) copies of \( \text{End} \ \mathbb{C}^N \). Now use [12, Eqs (5.32) & (5.43)] to calculate partial traces to get

\[
\gamma_m(\omega) \text{tr}_{1,\ldots,\ell} S^{(m)} F[-\lambda] = \text{tr}_{1,\ldots,\ell} H^{(\ell)} F[-\lambda],
\]

where \( H^{(\ell)} \) denotes the image of the element \( h^{(\ell)} \) defined in (2.20), under the respective actions (2.23) and (2.24) of the Brauer algebra. Relation (2.29) follows from Lemma 2.4 because the transposition \( t_a \) on the Brauer algebra is consistent with the partial transposition \( t_a \) on the tensor product (2.18). That is, if an element \( s \in B_m(\omega) \) has the form \( s = d + d t_a \), then for the image \( S \) of \( s \) with respect to the actions (2.23) and (2.24) we have \( \text{tr}_{1,\ldots,\ell} S F[-\lambda] = 0 \). Indeed, one verifies easily that \( S \) is stable under the transposition \( t_a \) on the space (2.18), so that the relation holds since \( F[-\lambda_a] + F[-\lambda_a]^t = 0 \). It remains to note that

\[
D(\lambda) = \text{tr}_{1,\ldots,\ell} H^{(\ell)} F[-\lambda] \quad \text{and} \quad P(\lambda) = \text{tr}_{1,\ldots,\ell} H^{(\ell)} F[-\lambda]
\]

in the symplectic and orthogonal case, respectively.

\[\square\]

Remark 2.5. All coefficients of the polynomial in \( T \) given by

\[
\gamma_m(\omega) \text{tr}_{1,\ldots,\ell} S^{(m)} \left( T + F[-1] \right) \cdots \left( T + F[-1] \right) = \phi_{m_0} T^{m_0} + \phi_{m_1} T^{m_1-1} + \cdots + \phi_{m_m}
\]

belong to \( z(\hat{g}) \); see [12, Ch. 8]. By replacing \( T \mapsto T + u \) for a variable \( u \), one can show that

\[
\phi_{m_k} = \binom{N + m - 2}{m - k} \phi_{kk} \quad \text{and} \quad \phi_{m_k} = \binom{2n - k + 1}{m - k} \phi_{kk}
\]

in the orthogonal and symplectic case, respectively. \[\square\]

2.3 Symmetrization map

In her recent work [17], Yakimova gave new formulas for Segal–Sugawara vectors in types \( B, C, D \) and \( G_2 \) by using the canonical symmetrization map. We will show that these vectors in the classical types coincide with those found in [11].
Recall that for a Lie algebra \( \mathfrak{a} \) the symmetrization map \( \varpi : S(\mathfrak{a}) \to U(\mathfrak{a}) \) is defined by
\[
\varpi : x_1 \ldots x_n \mapsto \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} x_{\sigma(1)} \ldots x_{\sigma(n)}, \quad x_i \in \mathfrak{a}.
\] (2.31)

Regarding \( T = -d/dt \) as a derivation of the Lie algebra \( t^{-1}\mathfrak{g}[t^{-1}] \), we will apply the map \( \varpi \) for the Lie algebra \( \mathbb{C}T \oplus t^{-1}\mathfrak{g}[t^{-1}] \).

For any element \( S \in S(\mathfrak{g}) \) we will denote by \( S[\cdot] \) the image of \( S \) under the embedding \( S(\mathfrak{g}) \hookrightarrow S(t^{-1}\mathfrak{g}[t^{-1}]) \) defined by \( X \mapsto X[\cdot] \) for \( X \in \mathfrak{g} \).

**Type A.** Take \( \mathfrak{g} = \mathfrak{gl}_N \) and introduce elements \( \Delta_k \) and \( \Phi_k \) of the symmetric algebra \( S(\mathfrak{gl}_N) \) by the expansions
\[
\det(u + E) = u^N + \Delta_1 u^{N-1} + \cdots + \Delta_N
\]
and
\[
\det(1 - qE)^{-1} = 1 + \sum_{k=1}^{\infty} \Phi_k q^k,
\]
for the matrix \( E = [E_{ij}] \). Then
\[
S(\mathfrak{gl}_N^{\mathfrak{gl}_N}) = \mathbb{C}[\Delta_1, \ldots, \Delta_N] = \mathbb{C}[\Phi_1, \ldots, \Phi_N].
\] (2.32)

The first formula in the next proposition (along with its closely related versions) was pointed out in [17].

**Proposition 2.6.** The Segal–Sugawara vectors (2.6) and (2.7) can be written in the form
\[
\phi_{mm} = \sum_{k=1}^{m} \binom{N - k}{m - k} \varpi \left( T^{m-k} \Delta_k [-1] \right) 1
\] (2.33)
and
\[
\psi_{mm} = \sum_{k=1}^{m} \binom{N + m - 1}{m - k} \varpi \left( T^{m-k} \Phi_k [-1] \right) 1.
\] (2.34)

**Proof.** Expand the product in (2.6) as
\[
\text{tr}_{1, \ldots, m} A^{(m)} \sum_{k=1}^{m} \sum_{1 \leq i_1 < \cdots < i_{m-k} \leq m} E[-1]_{i_1} \ldots E[-1]_{i_{m-k}} T E[-1]_{i_{m-k}+1} \ldots E[-1]_{m} 1,
\]
so that the factors \( T \) occur in the places \( i_1, \ldots, i_{m-k} \). Now apply conjugations by suitable elements of \( \mathfrak{S}_m \) and use the cyclic property of trace to bring this expression to the form where the labels of the factors \( E[-1] \) take consecutive values:
\[
\text{tr}_{1, \ldots, m} A^{(m)} \sum_{k=1}^{m} \sum_{1 \leq j_1 < \cdots < j_{m-k} \leq k} E[-1]_{j_1} \ldots E[-1]_{j_{m-k}-1} T E[-1]_{j_{m-k}} \ldots E[-1]_{k} 1
\]
with the factors $T$ occurring just before $E[-1]_{ja}$ for $a = 1, \ldots, m-k$. Now apply formula (2.13) for the partial traces of $A^{(m)}$ with $\ell$ replaced by $k$ to come to the expression

$$
\sum_{k=1}^{m} \frac{(N-k)!}{(N-m)!} m! \operatorname{tr}_{1, \ldots, k} A^{(k)} \sum_{1 \leq j_1 \leq \cdots \leq j_{m-k} \leq k} E[-1]_1 \cdots E[-1]_{j_1-1} T E[-1]_{j_1} \cdots E[-1]_k 1.
$$

For a fixed value of $k$ we have the relation

$$
\operatorname{tr}_{1, \ldots, k} A^{(k)} \sum_{1 \leq j_1 \leq \cdots \leq j_{m-k} \leq k} E[-1]_1 \cdots E[-1]_{j_1-1} T E[-1]_{j_1} \cdots E[-1]_k 1 = \left( \frac{m}{k} \right) \varpi \left( T^{m-k} \Delta_k [-1] \right) 1. \quad (2.35)
$$

This proves (2.33). The proof of (2.34) is the same, with the use of (2.15).

**Type C.** Write the elements $F_{ij}$ of the symplectic Lie algebra $\mathfrak{sp}_{2n}$ into the matrix $F = [F_{ij}]$. Introduce elements $\Delta_{2l}$ of the symmetric algebra $S(\mathfrak{sp}_{2n})$ by

$$
\det(u + F) = u^{2n} + \Delta_2 u^{2n-2} + \cdots + \Delta_{2n}.
$$

We have

$$
S(\mathfrak{sp}_{2n})^{\mathfrak{sp}_{2n}} = \mathbb{C} [\Delta_2, \Delta_4, \ldots, \Delta_{2n}].
$$

**Proposition 2.7.** The Segal–Sugawara vectors (2.25) can be written in the form

$$
\phi_{mm} = \sum_{\ell=1}^{[m/2]} \left( \frac{2n - 2\ell + 1}{m - 2\ell} \right) \varpi \left( T^{m-2\ell} \Delta_{2\ell} [-1] \right) 1 \quad (2.36)
$$

for $m = 1, \ldots, 2n+1$.

**Proof.** As in the proof of Theorem 2.3, it will be sufficient to assume that $m \leq n$. The arguments used in [12, Sec. 8.3] will then allow one to extend the result to all remaining values of $m$. Expand the product in (2.25) and apply conjugations by suitable permutations to get

$$
\phi_{mm} = \gamma_m (-2n) \operatorname{tr}_{1, \ldots, m} S^{(m)} \sum_{\beta} d_{\beta} F[-\beta], \quad (2.37)
$$

summed over compositions $\beta = (\beta_1, \ldots, \beta_\ell)$ of $m$ with $\beta_i \geq 1$, where we set

$$
F[-\beta] = F[-\beta_1]_1 \cdots F[-\beta_\ell]_\ell,
$$

while $d_{\beta}$ are certain integer coefficients. As with the expansion (2.26), the summands with odd values of $\ell$ are equal to zero. Now calculate partial traces by using (2.27) and apply relation (2.29), which holds in the same form for $F[-\lambda]$ replaced by $F[-\beta]$, to get

$$
\phi_{mm} = \sum_{\beta} d_{\beta} \left( \frac{2n - 2\ell + 1}{m - 2\ell} \right) \left( \frac{m}{2\ell} \right)^{-1} \operatorname{tr}_{1, \ldots, 2\ell} H^{(2\ell)} F[-\beta],
$$

for $m = 1, \ldots, 2n+1$.
summed over the compositions $\beta = (\beta_1, \ldots, \beta_2)$. As with relation (2.35), for a fixed value of $l$ we have
\[
\sum_\beta d_\beta \text{tr}_{1,\ldots,2l} H^{(2l)} F[-\beta] = \left(\frac{m}{2l}\right) \varpi \left(T^{m-2l} \Delta_2[-1]\right) 1,
\]
thus completing the proof.

Proposition 2.7 shows that $\phi_{m,m}$ with $m = 2k$ coincides with the Segal–Sugawara vector produced in [17, Theorem 4.4].

**Types B and D.** Introduce elements $\Phi_{2l}$ of the symmetric algebra $S(\mathfrak{o}_N)$ by
\[
\det(1 - qF)^{-1} = 1 + \sum_{k=1}^{\infty} \Phi_{2l} q^{2l}
\]
for the matrix $F = [F_{ij}]$. Then
\[
S(\mathfrak{o}_{2n+1})^{\mathfrak{o}_{2n+1}} = \mathbb{C}[\Phi_2, \ldots, \Phi_{2n}] \quad \text{and} \quad S(\mathfrak{o}_{2n})^{\mathfrak{o}_{2n}} = \mathbb{C}[\Phi_2, \ldots, \Phi_{2n-2}, \text{Pf} F[-1]].
\]

**Proposition 2.8.** The Segal–Sugawara vectors (2.25) can be written in the form
\[
\phi_{m,m} = \sum_{l=1}^{[m/2]} \left(N + m - 2 \frac{m}{2l}\right) \varpi \left(T^{m-2l} \Phi_{2l}[-1]\right) 1 \quad (2.38)
\]
for $m \geq 1$.

**Proof.** The argument is the same as for Proposition 2.7, where we use the partial trace formula (2.28) instead of (2.27), and the corresponding version of relation (2.29) for compositions.

Proposition 2.8 implies that $\phi_{m,m}$ with $m = 2k$ coincides with the Segal–Sugawara vector given by [17, Theorem 7.6], because the binomial coefficient in (2.38) coincides with the expression $R(k, k - 1)$ used therein.

Comparing the binomial coefficients in (2.36) and (2.38) with those occurring in (2.13) and (2.15), one can write an equivalent formula for the Segal–Sugawara vector $\phi_{m,m}$ in the symbolic form
\[
\phi_{m,m} = \text{tr}_{1,\ldots,m} \overline{H}^{(m)} \left(T + F[-1]_1\right) \ldots \left(T + F[-1]_m\right) 1, \quad (2.39)
\]
where the symbol $\overline{H}^{(m)}$ is interpreted as a modified operator $H^{(m)}$ via an expansion similar to that of (2.25), with the partial traces calculated by the rules
\[
\text{tr}_{m} \overline{H}^{(m)} = \frac{N + m - 2}{m} \overline{H}^{(m-1)} \quad \text{and} \quad \text{tr}_{m} \overline{H}^{(m)} = \frac{2n - m + 2}{m} \overline{H}^{(m-1)}
\]
in the orthogonal and symplectic case, respectively.
To derive a formal expression in the symplectic case, introduce the \((2n + 1) \times (2n + 1)\) matrix \(F^o\) by inserting the zero row and column in the middle of the matrix \(F = [F_{ij}]\), as illustrated:

\[
F^o = \begin{bmatrix}
F_{11} & \ldots & F_{1n} & 0 & F_{1n'} & \ldots & F_{11'} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
F_{n1} & \ldots & F_{nn} & 0 & F_{nn'} & \ldots & F_{n1'} \\
0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
F_{n'1} & \ldots & F_{nn'} & 0 & F_{nn''} & \ldots & F_{n'1'} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
F_{1'1} & \ldots & F_{1'\ell} & 0 & F_{1'\ell'} & \ldots & F_{1'1'}
\end{bmatrix}.
\]  

(2.40)

Define also the corresponding matrices \(F^o[r] = [F^o_{ij}[r]]\) and label their rows and columns by the symbols \(1, \ldots, n, 0, n', \ldots, l'\) which will also label the canonical basis elements of the space \(\mathbb{C}^{2n+1}\). Consider the tensor space (2.18) for \(\mathfrak{g} = \mathfrak{sp}_{2n}\) with \(N = 2n + 1\) and let \(A^{(m)}\) be the anti-symmetrization operator in this space, as defined in Sec. 2.1. Now introduce elements \(\phi_{ma} \in V_{cri}(\mathfrak{sp}_{2n}) \cong U\left(t^{-1}\mathfrak{sp}_{2n}[t^{-1}]\right)\) by the expansion

\[
\text{tr}_{1,\ldots,m} A^{(m)} \left( T + F^o[-1]_1 \right) \ldots \left( T + F^o[-1]_m \right) = \phi_{m0} T^m + \phi_{m1} T^{m-1} + \cdots + \phi_{mm}. \tag{2.41}
\]

**Corollary 2.9.** All elements \(\phi_{ma}\) belong to the Feigin–Frenkel center \(\mathfrak{z}(\mathfrak{sp}_{2n})\). Moreover, the family \(\phi_{22}, \phi_{44}, \ldots, \phi_{2n\cdot 2n}\) is a complete set of Segal–Sugawara vectors.

**Proof.** Similar to (2.37), expand the product on the left hand side of (2.41) to bring it to the form

\[
\text{tr}_{1,\ldots,m} A^{(m)} \sum_{k=0}^{m} \left( \sum_{\beta} d^{(m)}_{\beta} F^o[-\beta] \right) T^{m-k}, \tag{2.42}
\]

with the second sum over compositions \(\beta = (\beta_1, \ldots, \beta_\ell)\) of \(k\) with \(\beta_i \geq 1\), where

\[
F^o[-\beta] = F^o[-\beta_1] \ldots F^o[-\beta_\ell],
\]

and \(d^{(m)}_{\beta}\) are certain integer coefficients. Now use (2.13) to write (2.42) as

\[
\sum_{k=0}^{m} \left( \sum_{\beta} d^{(m)}_{\beta} \binom{2n+1}{m} \binom{2n+1}{\ell}^{-1} \text{tr}_{1,\ldots,\ell} A^{(\ell)} F^o[-\beta] \right) T^{m-k}. \tag{2.43}
\]

Now observe that \(\text{tr}_{1,\ldots,\ell} A^{(\ell)} F^o[-\beta]\) coincides with the trace \(\text{tr}_{1,\ldots,\ell} A^{(\ell)} F[-\beta]\) calculated over the tensor space (2.18) for \(\mathfrak{g} = \mathfrak{sp}_{2n}\) with \(N = 2n\).

On the other hand, the arguments used in the proof of Theorem 2.3 show that the polynomial in \(T\) defined by

\[
\gamma_m(-2n) \text{tr}_{1,\ldots,m} S^{(m)} \left( T + F[-1]_1 \right) \ldots \left( T + F[-1]_m \right) \tag{2.44}
\]

is given by the same expression (2.43). Hence, the polynomials (2.41) and (2.44) coincide, and the corollary follows from the results of [12, Sec. 8.3].  

\[
\square
\]
3 Casimir elements and Harish-Chandra images

3.1 Symmetrized basic invariants

Type $A$. Applying the symmetrization map (2.31) for the Lie algebra $\mathfrak{gl}_N$, and using (2.32), we get algebraically independent generators of the center $Z(\mathfrak{gl}_N)$ of the universal enveloping algebra $U(\mathfrak{gl}_N)$,

$$Z(\mathfrak{gl}_N) = \mathbb{C}[\varpi(\Delta_1), \ldots, \varpi(\Delta_N)] = \mathbb{C}[\varpi(\Phi_1), \ldots, \varpi(\Phi_N)].$$

Given an $N$-tuple of complex numbers $\lambda = (\lambda_1, \ldots, \lambda_N)$, the corresponding irreducible highest weight representation $L(\lambda)$ of $\mathfrak{gl}_N$ is generated by a nonzero vector $\xi \in L(\lambda)$ such that

$$E_{ij} \xi = 0 \quad \text{for} \quad 1 \leq i < j \leq N, \quad \text{and} \quad E_{ii} \xi = \lambda_i \xi \quad \text{for} \quad 1 \leq i \leq N.$$

Any element $z \in Z(\mathfrak{gl}_N)$ acts in $L(\lambda)$ by multiplying each vector by a scalar $\chi(z)$. As a function of the highest weight, $\chi(z)$ is a shifted symmetric polynomial in the variables $\lambda_1, \ldots, \lambda_N$ which can be regarded as the image of $z$ under the Harish-Chandra isomorphism $\chi$. This polynomial is symmetric in the shifted variables $\lambda_1, \lambda_2 - 1, \ldots, \lambda_N - N + 1$.

Consider the \textit{elementary shifted symmetric polynomials}

$$e^*_m(\lambda_1, \ldots, \lambda_N) = \sum_{1 \leq i_1 < \cdots < i_m} \lambda_{i_1}(\lambda_{i_2} - 1) \cdots (\lambda_{i_m} - m + 1)$$

and the \textit{complete shifted symmetric polynomials}

$$h^*_m(\lambda_1, \ldots, \lambda_N) = \sum_{1 \leq i_1 \leq \cdots \leq i_m} \lambda_{i_1}(\lambda_{i_2} + 1) \cdots (\lambda_{i_m} + m - 1).$$

They are particular cases of the \textit{factorial or shifted Schur polynomials}; see [9] and [15].

Recall that the \textit{Stirling number of the second kind} \( \left\{ \binom{m}{k} \right\} \) counts the number of partitions of the set \( \{1, \ldots, m\} \) into \( k \) nonempty subsets.

\textbf{Theorem 3.1.} For the Harish-Chandra images we have

$$\chi : \varpi(\Delta_m) \mapsto \sum_{k=1}^{m} \binom{m}{k} \\left( \begin{array}{c} N \\ m \end{array} \right) \left( \begin{array}{c} N \\ k \end{array} \right)^{-1} e^*_k(\lambda_1, \ldots, \lambda_N)$$

and

$$\chi : \varpi(\Phi_m) \mapsto \sum_{k=1}^{m} \binom{m}{k} \left( \begin{array}{c} -N \\ m \end{array} \right) \left( \begin{array}{c} -N \\ k \end{array} \right)^{-1} h^*_k(\lambda_1, \ldots, \lambda_N).$$

\textbf{Proof.} We will use the matrix notation of Sec. 2.1 applied to the algebra $U(\mathfrak{gl}_N)$ in place of $U(\hat{\mathfrak{gl}}_N)$. Regarding the matrix $E = [E_{ij}]$ as the element

$$E = \sum_{i,j=1}^{N} e_{ij} \otimes E_{ij} \in \text{End} \mathbb{C}^n \otimes U(\mathfrak{gl}_N)$$
we get the following counterparts of (2.6) and (2.7):
\[
\varpi(\Delta_m) = \text{tr}_{1,\ldots,m} A^{(m)} E_1 \ldots E_m \quad \text{and} \quad \varpi(\Phi_m) = \text{tr}_{1,\ldots,m} H^{(m)} E_1 \ldots E_m.
\]

On the other hand, the Harish-Chandra images
\[
\chi : \text{tr}_{1,\ldots,m} A^{(m)} E_1 (E_2 - 1) \ldots (E_m - m + 1) \mapsto e_m^*(\lambda_1, \ldots, \lambda_N)
\]
and
\[
\chi : \text{tr}_{1,\ldots,m} H^{(m)} E_1 (E_2 + 1) \ldots (E_m + m - 1) \mapsto h_m^*(\lambda_1, \ldots, \lambda_N)
\]
are well-known; see e.g. [12, Secs 4.6 & 4.7] for proofs. By the same argument as used in the proof of Theorem 2.1, the identity
\[
x^m = \sum_{k=1}^{m} \binom{m}{k} x(x-1) \ldots (x-k+1) \quad \text{(3.1)}
\]
implies that
\[
\text{tr}_{1,\ldots,m} A^{(m)} E_1 \ldots E_m = \text{tr}_{1,\ldots,m} A^{(m)} \sum_{k=1}^{m} \binom{m}{k} E_1 (E_2 - 1) \ldots (E_k - k + 1)
\]
and
\[
\text{tr}_{1,\ldots,m} H^{(m)} E_1 \ldots E_m = \text{tr}_{1,\ldots,m} H^{(m)} \sum_{k=1}^{m} (-1)^{m-k} \binom{m}{k} E_1 (E_2 + 1) \ldots (E_k + k - 1).
\]
The required formulas now follow by calculating the partial traces over the spaces \(\text{End} \mathbb{C}^N\) labelled by \(k+1, \ldots, m\), with the use of (2.13) and (2.15).

**Types B, C and D.** Now use the notation of Secs 2.2 & 2.3 and let \(\mathfrak{g}\) be the orthogonal Lie algebra \(\mathfrak{o}_N\) with \(N = 2n + 1\) and \(N = 2n\) or symplectic Lie algebra \(\mathfrak{sp}_N\) with \(N = 2n\).

Given any \(n\)-tuple of complex numbers \(\lambda = (\lambda_1, \ldots, \lambda_n)\), the corresponding irreducible highest weight representation \(L(\lambda)\) of the Lie algebra \(\mathfrak{g}\) is generated by a nonzero vector \(\xi \in L(\lambda)\) such that
\[
F_{ij} \xi = 0 \quad \text{for} \quad 1 \leq i < j \leq N, \quad \text{and} \quad F_{ii} \xi = \lambda_i \xi \quad \text{for} \quad 1 \leq i \leq n.
\]

Any element \(z\) of the center of \(U(\mathfrak{g})\) acts in \(L(\lambda)\) by multiplying each vector by a scalar \(\chi(z)\). As a function of the highest weight, \(\chi(z)\) is a shifted invariant polynomial in the variables \(\lambda_1, \ldots, \lambda_n\) with respect to the action of the corresponding Weyl group. The polynomial \(\chi(z)\) can be regarded as the Harish-Chandra image of \(z\).

**Theorem 3.2.**

1. If \(\mathfrak{g} = \mathfrak{sp}_{2n}\), then for \(m = 2, 4, \ldots, 2n\) the Harish-Chandra images are
\[
\chi : \varpi(\Delta_m) \mapsto \sum_{k=1}^{m} \binom{m}{k} \binom{2n+1}{m} \binom{2n+1}{k}^{-1} e_k^*(\lambda_1, \ldots, \lambda_n, 0, -\lambda_n, \ldots, -\lambda_1).
\]
2. If \( g = \mathfrak{o}_{2n+1} \), then for even \( m \geq 2 \) the Harish-Chandra images are
\[
\chi : \varpi(\Phi_m) \mapsto \sum_{k=1}^{m} \frac{m}{k} \binom{m}{k} \left( \begin{array}{c} -2n \end{array} \right) \left( \begin{array}{c} -2n \end{array} \right)^{-1} h_k^\ast(\lambda_1, \ldots, \lambda_n, -\lambda_n, \ldots, -\lambda_1).
\]

3. If \( g = \mathfrak{o}_{2n} \), then for even \( m \geq 2 \) the Harish-Chandra images are
\[
\chi : \varpi(\Phi_m) \mapsto \sum_{k=1}^{m} \frac{m}{k} \binom{m}{k} \left( \begin{array}{c} -2n+1 \end{array} \right) \left( \begin{array}{c} -2n+1 \end{array} \right)^{-1}
\times \left( \frac{1}{2} h_k^\ast(\lambda_1, \ldots, \lambda_{n-1}, -\lambda_n, \ldots, -\lambda_1) + \frac{1}{2} h_k^\ast(\lambda_1, \ldots, \lambda_n, -\lambda_{n-1}, \ldots, -\lambda_1) \right).
\]

**Proof.** The trace
\[
\text{tr}_{1,\ldots,m} H^{(m)} F_1 \ldots F_m
\]
coincides with \( \varpi(\Delta_m) \) and \( \varpi(\Phi_m) \), respectively, in the symplectic and orthogonal case. By applying Lemma 2.4 as in the proof of Theorem 2.3, we get
\[
\text{tr}_{1,\ldots,m} H^{(m)} F_1 \ldots F_m = \gamma_m(\omega) \text{tr}_{1,\ldots,m} S^{(m)} F_1 \ldots F_m,
\]
where \( \omega = N \) and \( \omega = -2n \), respectively, for the orthogonal and symplectic case. Using (3.1) again, and adjusting the arguments of the proof of Theorem 2.3 to the case of Lie algebra \( g \), we derive the relations
\[
\gamma_m(-2n) \text{tr}_{1,\ldots,m} S^{(m)} F_1 \ldots F_m = \gamma_m(-2n) \text{tr}_{1,\ldots,m} S^{(m)} \sum_{k=1}^{m} \frac{m}{k} F_1(F_2 - 1) \ldots (F_k - k + 1)
\]
in the symplectic case, and
\[
\gamma_m(N) \text{tr}_{1,\ldots,m} S^{(m)} F_1 \ldots F_m
\]
\[
= \gamma_m(N) \text{tr}_{1,\ldots,m} S^{(m)} \sum_{k=1}^{m} (-1)^{m-k} \frac{m}{k} F_1(F_2 + 1) \ldots (F_k + k - 1)
\]
in the orthogonal case. By the results of [13, Sec. 6] (see also [12, Sec. 13.4]), we have the Harish-Chandra images
\[
\chi : \gamma_k(-2n) \text{tr}_{1,\ldots,k} S^{(k)} F_1(F_2 - 1) \ldots (F_k - k + 1) \mapsto e_k^\ast(\lambda_1, \ldots, \lambda_n, 0, -\lambda_n, \ldots, -\lambda_1)
\]
for \( g = \mathfrak{sp}_{2n} \),
\[
\chi : \gamma_k(N) \text{tr}_{1,\ldots,k} S^{(k)} F_1(F_2 + 1) \ldots (F_k + k - 1) \mapsto h_k^\ast(\lambda_1, \ldots, \lambda_n, -\lambda_n, \ldots, -\lambda_1)
\]
for \( g = \mathfrak{o}_{2n+1} \), and
\[
\chi : \gamma_k(N) \text{tr}_{1,\ldots,k} S^{(k)} F_1(F_2 + 1) \ldots (F_k + k - 1)
\]
\[
\mapsto \left( \frac{1}{2} h_k^\ast(\lambda_1, \ldots, \lambda_{n-1}, -\lambda_n, \ldots, -\lambda_1) + \frac{1}{2} h_k^\ast(\lambda_1, \ldots, \lambda_n, -\lambda_{n-1}, \ldots, -\lambda_1) \right)
\]
for \( g = \mathfrak{o}_{2n} \). The proof is completed by calculating the partial traces of \( \gamma_m(\omega) S^{(m)} \) over the spaces \( \text{End} \mathbb{C}^N \) labelled by \( k + 1, \ldots, m \), with the use of (2.27) and (2.28). \( \square \)
Remark 3.3. As the proof of Theorem 3.2 shows, the formulas for the Harish-Chandra images extend to odd values of $m$, assuming that $\Delta_m = \Phi_m = 0$. This provides linear dependence relations for the elementary and complete shifted symmetric polynomials. 

3.2 Capelli-type determinant for $\mathfrak{sp}_{2n}$

The coefficients of the polynomial in $u$ defined by

$$D_m(u) = \gamma_m(\omega) \text{tr}_{1,\ldots,m} S^{(m)} \left( F_1 + u + \frac{m-1}{2} \right) \left( F_2 + u + \frac{m-3}{2} \right) \times \cdots \times \left( F_m + u - \frac{m-1}{2} \right),$$

where $\omega = N$ and $\omega = -2n$ in the cases $\mathfrak{g} = \mathfrak{o}_N$ and $\mathfrak{g} = \mathfrak{sp}_{2n}$, respectively, are Casimir elements for $\mathfrak{g}$. Their images under the Harish-Chandra isomorphism were found in [7]; see also [12, Ch. 5]. As with the Segal–Sugawara vectors, the arguments of Sec. 2.3 imply the interpretations for $D_m(u)$ in a symbolic form, analogous to (2.39):

$$D_m(u) = \text{tr}_{1,\ldots,m} H^{(m)} \left( F_1 + u + \frac{m-1}{2} \right) \left( F_2 + u + \frac{m-3}{2} \right) \cdots \left( F_m + u - \frac{m-1}{2} \right).$$

In the rest of this section we assume that $\mathfrak{g} = \mathfrak{sp}_{2n}$. Using the matrix (2.40) introduce the symmetrized determinant $C(u)$ of the matrix $u + F^\circ := uI + F^\circ$ by

$$u C(u) = \frac{1}{(2n+1)!} \sum_{\sigma,\tau} \text{sgn} \sigma \tau \cdot \left( u + n + F^\circ \right)_{\sigma(1)\tau(1)} \left( u + n - 1 + F^\circ \right)_{\sigma(2)\tau(2)} \times \cdots \times \left( u - n + F^\circ \right)_{\sigma(2n)\tau(2n)},$$

summed over all permutations $\sigma, \tau$ of the set $\{1, \ldots, n, 0, n', \ldots, 1'\}$.

Proposition 3.4. The symmetrized determinant is an even polynomial in $u$ of the form

$$C(u) = u^{2n} + C_1 u^{2n-2} + \cdots + C_n,$$

and the coefficients $C_1, \ldots, C_n$ are algebraically independent generators of the center of the algebra $\mathcal{U}(\mathfrak{sp}_{2n})$. Moreover, for the Harish-Chandra image we have

$$\chi : C(u) \mapsto (u^2 - l_1^2) \cdots (u^2 - l_n^2),$$

where $l_i = \lambda_i + n - i + 1$ for $i = 1, \ldots, n$.

Proof. Consider the tensor product algebra (2.18) for $N = 2n + 1$ with the last tensor factor replaced by $\mathcal{U}(\mathfrak{sp}_{2n})$. As with Corollary 2.9, we can write an equivalent expression for the polynomial $D_m(u)$ in the form

$$D_m(u) = \text{tr}_{1,\ldots,m} A^{(m)} \left( F_1^\circ + u + \frac{m-1}{2} \right) \left( F_2^\circ + u + \frac{m-3}{2} \right) \cdots \left( F_m^\circ + u - \frac{m-1}{2} \right).$$
Then clearly \( uC(u) = D_{2n+1}(u) \) so that all claims follow from [12, Cor. 5.5.5] and the identity
\[
(u^2 - l_1^2) \cdots (u^2 - l_n^2) = \sum_{k=0}^{n} (-1)^k e_k(l_1^2, \ldots, l_n^2 | a)(u^2 - 1^2) \cdots (u^2 - (n-k)^2)
\]
for the factorial elementary symmetric polynomials \( e_k(l_1^2, \ldots, l_n^2 | a) \) associated with the sequence \( a = (1^2, 2^2, \ldots) \); see [9, Eq. (6.5)].

Proposition 3.4 provides a new formula for the Capelli-type determinant introduced in [10], which also plays the role of a noncommutative characteristic polynomial of the matrix \( F \).

**References**

[1] A. V. Chervov and A. I. Molev, *On higher order Sugawara operators*, Int. Math. Res. Not. (2009), 1612–1635.

[2] A. Chervov and D. Talalaev, *Quantum spectral curves, quantum integrable systems and the geometric Langlands correspondence*, arXiv:hep-th/0604128.

[3] B. Feigin and E. Frenkel, *Affine Kac–Moody algebras at the critical level and Gelfand–Dikii algebras*, Int. J. Mod. Phys. A7, Suppl. 1A (1992), 197–215.

[4] E. Frenkel, *Langlands correspondence for loop groups*, Cambridge Studies in Advanced Mathematics, 103. Cambridge University Press, Cambridge, 2007.

[5] R. Howe and T. Umeda, *The Capelli identity, the double commutant theorem, and multiplicity-free actions*, Math. Ann. 290 (1991), 569–619.

[6] J. Hu and Z. Xiao, *On tensor spaces for Birman–Murakami–Wenzl algebras*, J. Algebra 324 (2010), 2893–2922.

[7] N. Iorgov, A. I. Molev and E. Ragoucy, *Casimir elements from the Brauer–Schur–Weyl duality*, J. Algebra 387 (2013), 144–159.

[8] V. Kac, *Vertex algebras for beginners*, University Lecture Series, 10. American Mathematical Society, Providence, RI, 1997.

[9] I. G. Macdonald, *Schur functions: theme and variations*, in: “Actes 28-e Séminaire Lotharingien”, pp. 5–39, Publ. I.R.M.A. Strasbourg, 1992, 498/S–27.

[10] A. I. Molev, *Sklyanin determinant, Laplace operators and characteristic identities for classical Lie algebras*, J. Math. Phys. 36 (1995), 923–943.

[11] A. I. Molev, *Feigin–Frenkel center in types B, C and D*, Invent. Math. 191 (2013), 1–34.

[12] A. Molev, *Sugawara operators for classical Lie algebras*. Mathematical Surveys and Monographs 229. AMS, Providence, RI, 2018.
[13] A. I. Molev and E. E. Mukhin, *Yangian characters and classical $\mathcal{W}$-algebras*, in “Conformal Field Theory, Automorphic Forms and Related Topics” (W. Kohnen, R. Weissauer, Eds), Springer, 2014, pp. 287–334.

[14] A. I. Molev, E. Ragoucy and N. Rozhkovskaya, *Segal–Sugawara vectors for the Lie algebra of type $G_2$*, J. Algebra 455 (2016), 386–401.

[15] A. Okounkov and G. Olshanski, *Shifted Schur functions*, St. Petersburg Math. J. 9 (1998), 239–300.

[16] A. Wachi, *Central elements in the universal enveloping algebras for the split realization of the orthogonal Lie algebras*, Lett. Math. Phys. 77 (2006), 155–168.

[17] O. Yakimova, *Symmetrisation and the Feigin–Frenkel centre*, arXiv:1910.10204v2.

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