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SET-THEORETIC GEOLOGY

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Abstract. A ground of the universe $V$ is a transitive proper class $W \subseteq V$, such that $W \models \text{ZFC}$ and $V$ is obtained by set forcing over $W$, so that $V = W[G]$ for some $W$-generic filter $G \subseteq P \in W$. The model $V$ satisfies the ground axiom GA if there are no such $W$ properly contained in $V$. The model $W$ is a bedrock of $V$ if $W$ is a ground of $V$ and satisfies the ground axiom. The mantle of $V$ is the intersection of all grounds of $V$. The generic mantle of $V$ is the intersection of all grounds of all set-forcing extensions of $V$. The generic HOD, written gHOD, is the intersection of all HODs of all set-forcing extensions. The generic HOD is always a model of ZFC, and the generic mantle is always a model of ZF. Every model of ZFC is the mantle and generic mantle of another model of ZFC. We prove this theorem while also controlling the HOD of the final model, as well as the generic HOD. Iteratively taking the mantle penetrates down through the inner mantles to what we call the outer core, what remains when all outer layers of forcing have been stripped away. Many fundamental questions remain open.

The technique of forcing in set theory is customarily thought of as a method for constructing outer as opposed to inner models of set theory. A set theorist typically has a model of set theory $V$ and constructs a larger model $V[G]$, the forcing extension, by adjoining a $V$-generic filter $G$ over some partial order $P \in V$. A switch in perspective, however, allows us to view forcing as a method of describing inner models as well. The idea is simply to search inwardly for how the model $V$ itself might have arisen by forcing. Given a set theoretic universe $V$, we consider the classes $W$ over which $V$ can be realized as a forcing extension $V = W[G]$ by some $W$-generic filter $G \subseteq P \in W$. This change in viewpoint is the basis for a collection of questions leading to the topic we refer to as set-theoretic geology. In this article, we present some of the most interesting initial results in the topic, along with an abundance of open questions, many of which concern fundamental issues.

1. The mantle

We assume that the reader is familiar with the technique of forcing in set theory. Working in ZFC set theory and sometimes in GBC set theory, we suppose $V$ is the universe of all sets. A class $W$ is a ground of $V$, if $W$ is a transitive class model of ZFC and $V$ is obtained by set forcing over $W$, that is, if there is some forcing notion $P \in W$ and a $W$-generic filter $G \subseteq P$ such that $V = W[G]$. Laver [Lav07] (and later, independently, Woodin [Woo04b] [Woo04a]) proved in this case that $W$ is a

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definable class in $V$, using parameters in $W$. Building on Laver’s ideas, Hamkins and Reitz [Ham05, Rei06, Rei07] introduced the following axiom.

**Definition 1.** The ground axiom GA is the assertion that the universe $V$ is not obtained by set forcing over any strictly smaller subclass.

Because of the quantification over classes, the ground axiom assertion appears at first to be fundamentally second-order in nature, but Reitz [Rei07, Rei06] proved that it is actually first-order expressible (an equivalent claim is implicit, independently, in [Woo04a]).

**Definition 2.** A class $W$ is a bedrock for $V$ if it is a ground of $V$ and minimal with respect to the forcing extension relation.

Since a ground of a ground is a ground, we may equivalently define that $W$ is a bedrock of $V$ if it is a ground of $V$ and satisfies the ground axiom. Also, since by fact 7 any inner model $U$ of ZFC with $W \subseteq U \subseteq V$ for some ground $W$ of $V$ is both a forcing extension of $W$ and a ground of $V$, we may equivalently define that $W$ is a bedrock for $V$ if it is a ground of $V$ that is minimal with respect to inclusion among all grounds of $V$. It remains an open question whether there can be a model $V$ having more than one bedrock model.

In this article, we attempt to carry the investigation deeper underground, bringing to light the structure of the grounds of the set-theoretic universe $V$ and how they relate to the grounds of the forcing extensions of $V$. Continuing the geological metaphor, the principal new concept is:

**Definition 3.** The mantle $M$ of a model of set theory is the intersection of all of its grounds.

The ground axiom can be reformulated as the assertion $V = M$, that is, as the assertion that $V$ is its own mantle. The mantle was briefly mentioned, unnamed, at the conclusion of [Rei07], where the question was raised whether it necessarily models ZFC. Our main theorem in this article is a converse of sorts:

**Main Theorem 4.** Every model of ZFC is the mantle of another model of ZFC.

This theorem is a consequence of the more specific claims of theorems 62 and 63, in which we are able not only to control the mantle of the target model, but also what we call the generic mantle, as well as the HOD and generic HOD. We begin by proving that the mantle, although initially defined with second-order reference to the meta-class of all ground models, actually admits a first-order definition. This is a consequence of the following fact, formulated in second order set theory.

**Theorem 5** (Laver, independently Woodin). Every ground model $W$ of $V$ is definable in $V$ using parameters from $W$. That is, there is a formula $\phi(y,x)$ such that if $V = W[G]$ is a forcing extension of a ground $W$ by $W$-generic filter $G \subseteq P \in W$, then there is $r \in W$ such that

$$W = \{ x \mid \phi(r, x) \}.$$ 

The uniformity of the definition was observed and highlighted by Reitz in his dissertation [Rei06] and in [Rei07] and used to formulate the ground axiom and prove that it is first-order expressible. The exact nature of this definition is not important at this point—an explicit definition is written out in [Rei06, Rei07]—but let us mention that in Laver’s argument, the parameter $r$ is $P(\delta)^W$, where
$W \subseteq V$ exhibits the $\delta$-approximation and cover properties of [Ham03], and this is true for $\delta = \text{ro}(P)^+$, as well as for all larger cardinals $\delta$. (And this fact allows one to prove the theorem for class forcing as well, provided that the extension exhibits the $\delta$-approximation and cover properties for some $\delta$, which many of the usual class forcing iterations do.) For the subsequent theorems, we will make use of the following folklore facts. An inner model is a transitive class containing all the ordinals and satisfying ZF. We shall also frequently speak of inner models of ZFC as well.

**Fact 6.** If $W = \{ \langle r, x \rangle \mid \phi(r, x) \}$ is a definable class, then the class of all $r$ for which the section $W_r = \{ x \mid \langle r, x \rangle \in W \}$ is an inner model of ZFC is definable.

**Proof.** This amounts to a standard result in Jech [Jec03, Theorem 13.9], asserting that the question of whether a transitive proper class containing all the ordinals satisfies full ZFC is expressible by a single first-order formula. What needs to be said is that $W_r$ is transitive, contains all the ordinals, is closed under the Gödel operations, is almost universal, meaning that every $A \subseteq W_r$ is covered by some $B \in W_r$, and satisfies the axiom of choice. \qed

**Fact 7 ([Jec03, Corollary 15.43]).** Suppose that $V \subseteq V[G]$ is the forcing extension arising from a $V$-generic filter $G \subseteq P \in V$. If $U$ is a transitive model of ZFC with $V \subseteq U \subseteq V[G]$, then $U$ is a forcing extension of $V$ and a ground of $V[G]$. In fact, $U$ is a forcing extension of $V$ by a filter $G_0$ on a complete subalgebra $\mathcal{B}_0$ of the complete Boolean algebra $\mathcal{B} = \text{RO}(P)$.

**Proof.** We first show a general fact about forcing extensions: If $A \subseteq \kappa$ is a set of ordinals in $V[G]$, then there is a complete subalgebra $\mathcal{C} \subseteq \mathcal{B}$ such that $V[A] = V[\mathcal{C} \cap G]$. To see this, fix a name $\dot{A}$ for $A$ and consider the set $X_A$ of boolean values $[\alpha \in \dot{A}]$ for $\alpha < \kappa$ determining membership in $A$. Taking $\mathcal{C} \subseteq \mathcal{B}$ to be the complete subalgebra generated by $X_A$ in $V$, it is straightforward to verify that $V[A] = V[\mathcal{C} \cap G]$ (here we use the fact that $\mathcal{C} \cap G$ is determined by $X_A \cap G$, which follows from the fact that $X_A$ generates $\mathcal{C}$).

Now consider our models $V \subseteq U \subseteq V[G]$. Let $A \subseteq \kappa$ be a set of ordinals in $U$ coding $P(\mathcal{B})^U$. Let $\mathcal{B}_0$ be the complete subalgebra obtained by applying the general fact given above to the set $A$, and let $G_0 = \mathcal{B}_0 \cap G$ be the corresponding generic filter. It remains to argue that $V[G_0] = U$. For the forward inclusion, note that $A \in U$ and so $V[G_0] = V[A]$ is contained in $U$. For the reverse inclusion, it suffices to show that every set of ordinals in $U$ is in $V[G_0]$. But this also follows from the general fact above: each such set of ordinals $B$ is determined (over $V$) by $\mathcal{C}_B \cap G$ for an appropriate subalgebra $\mathcal{C}_B$. However, $\mathcal{C}_B \cap G$ is in $P(\mathcal{B})^U$, and therefore appears in $A$. It follows that $B \in V[A]$, and so $V[G_0] = U$. The remaining claim, that $U$ is a ground of $V$, relies on a general fact about complete embeddings of partial orders: If $\mathcal{B}_0 \subseteq \mathcal{B}$ is a complete subalgebra, then $\mathcal{B}$ is forcing equivalent to the iteration $\mathcal{B}_0 * \mathcal{B}/G_0$, where the second factor is (a name for) the quotient of $\mathcal{B}$ by the generic filter $G_0$ on $\mathcal{B}_0$. The details of this relationship are explored in [Kun80, pp243–244, Ex. D3–D4]. \qed

The following theorem summarizes several senses in which we have first-order definable access to the family of ground models of the universe.
Theorem 8. There is a parameterized family \( \{ W_r \mid r \in V \} \) of classes such that

1. Every \( W_r \) is a ground of \( V \), and \( r \in W_r \).
2. Every ground of \( V \) is \( W_r \) for some \( r \).
3. The classes \( W_r \) are uniformly definable in the sense that \( \{ \langle r, x \rangle \mid x \in W_r \} \) is first-order definable without parameters.
4. The relation \( V = W_r[G] \), where \( G \subseteq P \in W_r \) is \( W_r \)-generic” is first-order expressible in \( V \) in the arguments \( (r, G, P) \).
5. The definition relativizes down from \( V \) in the sense that if \( W_r \subseteq U \subseteq V \) for an inner model \( U \models \text{ZFC} \), then \( W_r^U = W_r \).
6. The definition relativizes up from \( V \) in the sense that for any \( r \) and any forcing extension \( V[G] \), there is \( s \) with \( W_r = W_s = W_s^{V[G]} \).

Proof. By theorem 5, there is a formula \( \phi(r, x) \) such that every ground \( W \) of \( V \) has the form \( W_r = \{ x \mid \phi(r, x) \} \) for some \( r \in W \). Furthermore, by fact 6, the question of whether for a given \( r \) the class \( U_r = \{ x \mid \phi(r, x) \} \) defines a transitive inner model of \( \text{ZFC} \) is first-order expressible. Thus, by quantifying over the possible partial orders \( P \) in this class and the possible \( U_r \)-generic filters \( G \subseteq P \), we can express in a first-order manner whether \( U_r \) is in fact a ground of \( V \) or not. Thus, we define \( W_r = \{ x \mid \phi(r, x) \} \), if this is a ground of \( V \), and otherwise \( W_r = V \). It follows that \( W_r \) is defined for all \( r \), that \( r \in W_r \), that the ground models of \( V \) are exactly the classes \( W_r \) and that the relation “\( x \in W_r \)” is first-order definable. So (1), (2) and (3) hold. Using this, we conclude that the relation of (4) is also first-order definable, since we merely express that \( P \) is a partial order in \( W_r \), that \( G \) is a filter meeting every dense subset of \( P \) in \( W_r \) and that every set is the interpretation of a \( P \)-name in \( W_r \) by \( G \). For (5), suppose that \( W_r \subseteq U \subseteq V \) and \( U \models \text{ZFC} \). Since \( W_r \) is a ground of \( V \), we may exhibit it as \( V = W_r[G] \), where \( G \subseteq P \) is \( W_r \)-generic. Furthermore, by the details of the definition of \( \phi \) and fact 6, we may assume that \( r = P(\delta)^{W_r} \), where \( \delta > |\text{ro}(P)| \). Since \( W_r \subseteq U \subseteq W_r[G] = V \), it follows by fact 7 that \( U = W_r[G_0] \) is a forcing extension of \( W_r \) by a complete subalgebra of \( \text{ro}(P) \). Since that complete subalgebra also has size less than \( \delta \), we may use the same parameter \( r = P(\delta)^{W_r} \) to define \( W_r \) inside \( U \). Thus, \( W_r^V = W_r^U \), as desired for (5). For (6), because \( W_r \) is a ground of \( V \), it is also a ground of \( V[G] \), and so \( W_r = W_s^{V[G]} \) for some \( s \). By (5), since \( V \) is intermediate between \( W_s^{V[G]} \) and \( V[G] \), we have \( W_r^V = W_s^{V[G]} \), establishing (6). \( \square \)

We shall fix the notation \( W_r \) of theorem 8 for the rest of this article, so that \( W_r \) now refers to the ground of \( V \) defined by parameter \( r \), and this definition is completely uniform, so that the classes \( W_r^U \) index the grounds of \( U \) for any model \( U \) of \( \text{ZFC} \), as \( r \) ranges over \( U \).

Theorem 8 shows that we are essentially able to treat the meta-class collection of all ground models of \( V \) in an entirely first-order manner. Quantifying over the ground models becomes simply quantifying over the parameters \( r \) used to define them as \( W_r \), and the basic questions about the existence and relations among various ground models become first-order properties of these parameters. For example, the ground axiom asserts \( \forall r V = W_r \), and a ground model \( W_r \) is a bedrock model if \( \forall s (W_r \subseteq W_r \rightarrow W_s = W_r) \). Alternatively, one could also have defined that \( W_r \) is a bedrock if \( (\text{GA})^{W_r} \), that is, if the ground axiom holds when relativized to \( W_r \).

Corollary 9. The mantle of any model of set theory is a parameter-free uniformly first-order-definable transitive class in that model, containing all ordinals.
Proof. We’ve already done the work above. If M is the mantle of V, then
\[ M = \bigcap_r W_r, \]
or in other words, \( x \in M \iff \forall r x \in W_r \), which is a uniform parameter-free definition. And being the intersection of grounds, the mantle is clearly transitive and contains all ordinals. \( \square \)

**Definition 10.** The ground models of the universe V are **downward directed** under inclusion if the intersection of any two contains a third, that is, if for every r and s, there is t with \( W_t \subseteq W_r \cap W_s \). The ground models are **locally** downward directed if for every r and s and every set B, there is t with \( W_t \cap B \subseteq W_r \cap W_s \). And similarly there are the upward directed notions.

Numerous fundamental open questions arise once one is situated in the context of the collection of all ground models.

**Question 11** ([Rei06, Rei07]). Is there a unique bedrock model when one exists?

Reitz [Rei06, Rei07] shows that there can be models of ZFC having no bedrock models at all. A more general question is:

**Question 12** ([Rei06]). Are the ground models of the universe downward directed?

If the ground models are downward directed, then of course there cannot be distinct bedrock models, since the bedrock models are exactly the minimal ground models with respect to inclusion. So an affirmative answer to question 12 implies an affirmative answer to question 11. Allowing larger intersections, let us say that the ground models of the universe are downward **set-directed** under inclusion if the intersection of any set-indexed collection of ground models contains a ground model, that is, if for every set A there is t such that \( W_t \subseteq \bigcap_{r \in A} W_r \). Relaxing this somewhat, we say that the ground models are **locally** downward set-directed if for any sets A and B there is t such that \( W_t \cap B \subseteq \bigcap_{r \in A} W_r \).

**Question 13.** Are the ground models of the universe downward set-directed? Are they locally downward set-directed? Are they locally downward directed?

Affirmative answers to all of the questions we have asked so far are consistent with ZFC. The answers are trivially affirmative under the ground axiom, for example, since in this case the only ground model is the universe itself. More generally, the questions all have affirmative answers if there is a least member of the collection of grounds under inclusion, or equivalently, in other words, if the mantle itself is a ground. This last hypothesis is true in any set-forcing extension of L.

**Definition 14.** An inner model W is a **solid bedrock** for V if it is a ground of V and contained in all other grounds of V. The **solid bedrock axiom** is the assertion that there is a solid bedrock.

The bedrock axiom of [Rei07], in contrast, asserts merely that there is a bedrock, but not necessarily that this bedrock is solid. In other words, the bedrock axiom asserts that there is a minimal ground, while the solid bedrock axiom asserts that there is a smallest ground.
Definition 15. The downward directed grounds hypothesis DDG is the assertion that the ground models of the universe are downward directed. The strong DDG is the assertion that the ground models of the universe are downward set-directed. The generic DDG is the assertion that the DDG holds in all forcing extensions. The generic strong DDG is the assertion that the strong DDG holds in all forcing extensions.

If $W$ is a solid bedrock, it follows of course that $W$ is precisely the mantle. Thus, the solid bedrock axiom is equivalent to the assertion that the mantle is a ground, and it immediately implies the strong DDG and all of the directedness properties that we have considered. Reitz observed in [Rei06] that under the continuum coding axiom CCA, the assertion that every set is coded into the GCH pattern, it follows that the universe is contained in every ground of every forcing extension. Thus, CCA implies that the universe is a solid bedrock in all its forcing extensions, and so the strong DDG holds in $V$ and all forcing extensions. That is, the CCA implies the generic strong DDG hypothesis. An affirmative answer to (the first part of) question 13 implies an affirmative answer to question 12, which we have said implies an affirmative answer to question 11.

Let us now prove that we can obtain affirmative answers to all of the above questions in the case where the universe is constructible from a set. Define that a generic ground of $V$ is a ground of a forcing extension of $V$.

Theorem 16. If $V = L[a]$ for a set $a$, then the generic strong DDG holds. Indeed, all the models of the form

$$H^\alpha = \text{HOD}^{V_{\text{Coll}(\omega, \alpha)}}$$

are grounds of $V$ and they are downward set-directed and dense in the grounds and indeed in the generic grounds, in the sense that every ground $W$ of $V$ or of any forcing extension $V[G]$ has $H^\alpha \subseteq W$ for all sufficiently large $\alpha$. These grounds therefore exhibit the desired downward set-directedness.

Proof. For any ordinal $\alpha$, we let $H^\alpha$ denote the class $\text{HOD}^{V_{\text{Coll}(\omega, \alpha)}}$ consisting of all $x$ forced by $1$ to be in the HOD of the extension. The second part of the theorem can be formalized as the statement that for any poset $P$ and any $P$-name $\dot{x}$ we have that $P$ forces $H^\alpha \subseteq W_{\dot{x}}^{V[G]}$ for all sufficiently large $\alpha$.

To begin the proof, note that if $G$ is $\text{Coll}(\omega, \alpha)$-generic over $V$ for some ordinal $\alpha$, then since the forcing is almost homogeneous and ordinal-definable, it follows that $\text{HOD}^{V[G]} \subseteq \text{HOD}^V$ and moreover $x \in \text{HOD}^{V[G]}$ if and only if $1 \vdash \dot{x} \in \text{HOD}$, which implies that $\text{HOD}^{V[G]}$ does not depend on $G$ and is equal to $H^\alpha$. In particular, $H^\alpha$ is an inner model of $V$ that satisfies ZFC. Similar homogeneity reasoning shows that $H^\beta \subseteq H^\alpha$ whenever $\alpha < \beta$, and so the collection of $H^\alpha$ is downward set-directed.

If $W$ is a ground of $V$, so that $V = W[g]$ for some $W$-generic $g \subseteq P \in W$, then let $\theta$ be any cardinal at least as large as $\text{card}(P)$ and suppose that $G$ is $V$-generic for $\text{Coll}(\omega, \theta)$. We may absorb the $P$ forcing into the collapse, since $P \times \text{Coll}(\omega, \theta)$ is forcing equivalent to $\text{Coll}(\omega, \theta)$ by [Jec03, Lemma 26.7], and so

$$V[G] = W[g][G] = W[G']$$

for some $W$-generic filter $G' \subseteq \text{Coll}(\omega, \theta)$. Furthermore, we may observe that

$$\text{HOD}^{V[G]} = \text{HOD}^{W[G']} \subseteq W,$$
where the inclusion holds because \( \text{Coll}(\omega, \theta) \) is almost homogeneous and ordinal definable in \( W \). Finally, using our hypothesis that \( V = L[a] \), we argue that \( \text{HOD}^{|G|} \) is a ground of \( V \). This is a consequence of Vopěnka’s theorem \cite[Theorem 15.46]{Jec03}, which asserts that every set is generic over \( \text{HOD} \). In our case, since \( V[G] = L[a, G] \), we have that \( a \) and \( G \) are generic over \( \text{HOD}^{|G|} \), and so \( \text{HOD}^{|G|} \subseteq V \subseteq \text{HOD}^{|G|}[a, G] = V[G] \), trapping \( V \) between \( \text{HOD}^{|G|} \) and its forcing extension \( V[G] \). It therefore follows by fact 7 that \( \text{HOD}^{|G|} \) is a ground of \( V \), as desired. The downward set-directedness of the grounds is now an obvious consequence.

Finally, to see that the models \( H^\alpha \) are dense in the generic grounds, let \( W \) be a ground of \( V[h] \), so that \( W[g] = V[h] \) for some \( W \)-generic filter \( g \) and \( V \)-generic \( h \subseteq Q \in V \). By applying what we have proved so far, but in \( V[h] \), we know that \( \text{HOD}^{|h|}[\text{Coll}(\omega, \alpha)] \subseteq W \) for sufficiently large \( \alpha \). But by insisting that \( \alpha \) is also larger than \( Q \), the forcing \( Q \) is absorbed into the collapse via \( Q * \text{Coll}(\omega, \alpha) \equiv \text{Coll}(\omega, \alpha) \), and from this it follows that \( \text{HOD}^{|h|}[\text{Coll}(\omega, \alpha)] = \text{HOD}^{|h|}[\text{Coll}(\omega, \alpha)] = H^\alpha \), and so \( H^\alpha \subseteq W \) as desired. Lastly, again it is easy to see as a consequence that the generic grounds are downward set-directed, and so the generic DDG holds in \( V \).

To what extent generally can we expect the grounds to be directed, locally set directed or even fully set-directed? A model of ZFC having two bedrock models, providing a negative answer to question 11, would be very interesting. Aside from the naturalness of these hypotheses, their merit is that if they hold, the mantle is well behaved. Before examining their effect on the mantle, let’s step back a little and look at a more general setup. We say that \( \mathcal{W} \) is parameterized family of inner models if each \( U_p \) is an inner model and there a formula \( \varphi \) such that \( x \in U_p \) if and only if \( \varphi(x, p) \) holds.

**Lemma 17.** If \( \mathcal{W} = \{U_p \mid p \in I\} \) is a parameterized family of inner models, then \( \bigcap \mathcal{W} \) is an inner model if and only if \( V_\alpha \cap (\bigcap \mathcal{W}) \in \bigcap \mathcal{W} \) for every ordinal \( \alpha \). Furthermore, this is indeed the case if \( \bigcap \mathcal{W} \) is a definable class in every \( U \in \mathcal{W} \).

**Proof.** For the forward direction of the claimed equivalence, assume that \( \bigcap \mathcal{W} \) is an inner model, that is, a transitive class model of ZF containing all ordinals. Given an ordinal \( \alpha \), it follows that \( (V_\alpha \cap \bigcap \mathcal{W}) \in \bigcap \mathcal{W} \). By absoluteness, \( (V_\alpha \cap \bigcap \mathcal{W}) = V_\alpha \cap \bigcap \mathcal{W} \), which proves the forward direction. For the reverse direction, note that \( \bigcap \mathcal{W} \) is transitive, contains all the ordinals, and is closed under the eight Gödel operations, since this is true of every member of \( \mathcal{W} \), as each member of \( \mathcal{W} \) is an inner model. So by fact 6, in order to prove that \( \bigcap \mathcal{W} \) is an inner model, all that’s left to show is that it is almost universal. So let \( A \) be a subset of \( \bigcap \mathcal{W} \) and let \( \alpha \) be the rank of \( A \). Then \( A \subseteq V_\alpha \cap (\bigcap \mathcal{W}) \subseteq \bigcap \mathcal{W} \) by assumption, showing almost universality, and hence proving the equivalence.

Now assume that \( \bigcap \mathcal{W} \) is a definable subclass of every \( U \in \mathcal{W} \). For any ordinal \( \alpha \) and any \( U \in \mathcal{W} \), we have \( V_\alpha \cap (\bigcap \mathcal{W}) = V_\alpha \cap U \cap (\bigcap \mathcal{W}) = V_\alpha^U \cap (\bigcap \mathcal{W}) \in U \), by the Separation axiom in \( U \). Since \( U \) was arbitrary, it follows that \( V_\alpha \cap (\bigcap \mathcal{W}) \in \bigcap \mathcal{W} \), as desired. \( \square \)

Let’s now return to the mantle and explore the effects our concepts of directedness have on it.
Theorem 18.

1. If the DDG holds, that is, if the grounds are downward directed, then the mantle is constant across these ground models.
2. If the mantle is constant across the grounds, then it is a model of ZF.
3. If the strong DDG holds, that is, if the grounds are downward set-directed, then the mantle is a model of ZFC.
4. Indeed, this latter conclusion can be made if the grounds are merely downward directed and locally downward set-directed.

Proof. (1) Suppose that the ground models of $V$ are downward directed. Fix a ground $W$. Since any ground of $W$ is also a ground of $V$, it follows that the mantle of $V$ is contained in the mantle of $W$. Conversely, if $a$ is not in the mantle of $V$, then it is not in some ground $W'$, and so it is not in $W \cap W'$. By directedness, however, there is a ground $W \subseteq W \cap W'$. Clearly, $W$ is a ground of $W$, so that the mantle of $W$ is contained in $W$. But $a \notin W$, so $a$ is not in the mantle of $W$. So the mantle is constant among the ground models of $V$.

(2) If the mantle is constant across the grounds, then in particular, it is a definable subclass of every ground. Since the mantle is the intersection of all the grounds, it is an inner model by lemma 17, which we proved precisely for this purpose.

For (3) and (4), suppose that the ground models are downward directed and locally downward set-directed. By (1) and (2), the mantle satisfies ZF. For ZFC, consider any set $y$ in the mantle $M$. Every ground model $W$, being a model of ZFC, has various well orderings of $y$; what we must show is that there is a well ordering of $y$ in common to all the grounds. For each well ordering $z$ of $y$ that is not in the mantle, there is a ground model $W_{r_z}$ with $z \notin W_{r_z}$. By local set-directedness, the family $\{ W_{r_z} \mid z \text{ is a well order of } y \text{ with } z \notin M \}$ has a ground model $W$ with $W \cap B \subseteq W_{r_z}$ for all such relations $z$, where $B$ is the set in $V$ of all relations on $y$. That is, any relation $z$ excluded from the mantle is excluded by reason of not being in a particular ground model $W_{r_z}$, and consequently it is also excluded from $W$. Any well ordering of $y$ in $W$, therefore, is in the mantle, and since $W \models ZF$, there are many such well orderings. So $M = ZFC$, as desired for (3) and (4).

The argument of theorem 18 can be viewed as an instance of the following general phenomenon.

Definition 19. A parameterized family $W = \{ N_i \mid i \in I \}$ of transitive sets or classes, where $I$ is a class, is locally downward set-directed, if for any set $B$ and any set $J \subseteq I$, there is an $i_0 \in I$ such that $B \cap N_{i_0} \subseteq B \cap (\bigcap_{i \in J} N_i)$.

Clearly, it is equivalent to require merely that $B \cap N_{i_0} \subseteq \bigcap_{i \in J} N_i$.

Corollary 20. If $W$ is a locally downward set-directed parameterized family of inner models of ZFC and $V_\alpha \cap (\bigcap W) \in \bigcap W$ for every ordinal $\alpha$, then $\bigcap W$ satisfies ZFC. If $W = \{ W_\alpha \mid \alpha \in ORD \}$ is any definable sequence of inner models that is descending in the sense that $\alpha < \beta$ implies $W_\beta \subseteq W_\alpha$, then $\bigcap W$ is an inner model, and if every $W_\alpha$ satisfies ZFC, then so does $\bigcap W$.

Proof. The first claim follows by lemma 17 and the argument of theorem 18. For the second claim, suppose that $W$ is a descending sequence of inner models. To see that $\bigcap W$ is an inner model, it suffices to show by lemma 17 that $V_\alpha \cap (\bigcap W) \in \bigcap W$ for every ordinal $\alpha$. Fix any ordinal $\alpha$. Since $V_\alpha$ is a set, we may simply wait for all the
elements of $V_\alpha$ that will eventually fall out of the $V_\alpha^{W_\beta}$ to have fallen out—simply apply the Replacement axiom to the falling-out times—and thereby find an ordinal $\beta$ such that $V_\alpha^{W_\beta'} = V_\alpha^{W_\beta}$ for all $\beta' \geq \beta$. Then $V_\alpha^{W_\beta} = V_\alpha \cap W_\beta = V_\alpha \cap (\bigcap \mathcal{W})$, so all that needs to be shown is that $V_\alpha^{W_\beta} \in W_\gamma$, for every $\gamma$. But this is clear for $\gamma < \beta$, as $V_\alpha^{W_\beta} \subseteq W_\beta \subseteq W_\gamma$ in that case. And for $\gamma \geq \beta$, we have that $V_\alpha^{W_\beta} = V_\alpha^{W_\gamma} \in W_\gamma$. So $\bigcap \mathcal{W}$ is an inner model. To see that this model satisfies choice if every member of $\mathcal{W}$ does, it suffices to observe that $\mathcal{W}$ is downward set-directed. □

Let us remark that having a descending sequence of set-sized length does not suffice for the conclusion, even when the intersection of all of its members is definable in each of the models. In [McA74], for example, a model of ZFC is produced in which the sequence $\langle HOD_n \mid n < \omega \rangle$ is definable, where $HOD^{n+1} = HOD^{HOD^n}$, and where the intersection $HOD^\omega = \bigcap_{\alpha < \omega} HOD^n$ does not satisfy the axiom of choice. Since the model produced there is constructible from a set, it follows from [Gri75] that the sequence of iterated HODs is definable in every iterated HOD, and so $HOD^\omega$ is a class in every HOD there. So by lemma 17, we know that $HOD^\omega$ is an inner model in that case, but still not a model of ZFC. Note that by corollary 20, if the sequence $\langle HOD^n \mid \alpha \in \text{ORD} \rangle$ is definable, then the intersection $HOD^\infty = \bigcap_{\alpha < \infty} HOD^n$ is a model of ZFC, as every $HOD^{n+1}$ is a model of ZFC and $HOD^\infty = \bigcap_{\alpha < \infty} HOD^{\alpha+1}$. Another natural example is given by [Deh78], where it is shown that if $\kappa$ is a measurable cardinal and $\mu$ is a normal measure on $\kappa$, then the intersection of, for example, the first $\omega^2$ many iterated ultrapowers of $V$ by $\mu$ fails to satisfy the axiom of choice. Clearly, though, this intersection is definable in each of the earlier iterates, and so by lemma 17 it is an inner model. There is also an interesting example, given by a 1974 result due to Harrington appearing in [Zad83, section 7], where a model is constructed in which the intersection $\bigcap_{n<\omega} HOD^n$ is not a model of ZF, and actually, neither the sequence $\langle HOD^n \mid n < \omega \rangle$ nor $\bigcap_{n<\omega} HOD^n$ is a class in that model.

It follows by fact 7 that if the mantle $M$ is a ground of $V$, then there are only a set number of possible intermediate models $M \subseteq W \subseteq V$, since each is determined by its filter on a certain subalgebra of the forcing from $M$ to $V$. Thus, under the solid bedrock axiom, there are only set many grounds, in the sense that there is a set $I$ such that every ground of $V$ is $W_r$ for some $r \in I$. We don’t know if the converse holds.

**Question 21.** Is the solid bedrock axiom equivalent to the assertion that there are only set many grounds?

A strong counterexample to this would be a model $V$ having only set many grounds, but no minimal ground. Any such model would of course also be a counterexample to downward set-directedness and the generic strong DDG hypothesis.

**Theorem 22.** If the universe is constructible from a set, then question 21 has a positive answer. Indeed, if the universe is constructible from a set, then the following are equivalent:

1. There are only set many grounds.
2. The bedrock axiom.
3. The solid bedrock axiom.
Proof. Suppose $V$ is constructible from a set. For every ordinal $\alpha$, let $H^\alpha = \text{HOD}^{V_{\text{Coll}(\omega, \alpha)}}$, as in the proof of theorem 16.

(1 $\implies$ 2) Assume that $V$ has only set many grounds $\{W_r \mid r \in I\}$, for some set $I$. By theorem 16, for each $r \in I$ there is an ordinal $\alpha_r$ for which $H^{\alpha_r} \subseteq W_r$. Thus, if $\alpha = \sup_r \alpha_r$, we have the ground $H^\alpha$ contained in all grounds $W_r$. In other words, $H^\alpha$ is a smallest ground, verifying both the bedrock axiom, as well as the solid bedrock axiom.

(2 $\implies$ 3) Suppose that $W$ is a minimal ground. By theorem 16, it follows that $W$ must have the form $H^\alpha$ for some $\alpha$, and moreover that $W = H^\beta$ for all larger $\beta$, since the $H^\gamma$ sequence is non-increasing with respect to inclusion. Thus, $W$ is contained in every $H^\gamma$, and since these models are dense in the grounds, it follows that $W$ is contained in every ground. So $W$ is a solid bedrock, as desired.

(3 $\implies$ 1) This implication was proved in the remarks before the statement of question 21.  

\begin{theorem}
Every model $V$ of ZFC has a class forcing extension $V[G]$ in which the grounds are downward set-directed, but there is no bedrock.
\end{theorem}

\begin{proof}
This is the essence of [Rei06, Theorem 24], where Reitz showed that every model has a class forcing extension having no bedrock. Beginning with any model $V$, we first move to a class forcing extension $\V$ exhibiting the continuum coding axiom (CCA), which asserts that every set of ordinals is coded into the GCH pattern of the continuum function, by iteratively coding sets into this pattern. Then, in $\V$, let $P$ be the Easton support class product of forcing adding a Cohen subset to every regular cardinal $\lambda$ for which $2^{\omega \cdot \lambda} = \lambda$, and suppose that $\V[G]$ is the corresponding class forcing extension. This forcing preserves all cardinals, cofinalities and the GCH pattern over $\V$. Consequently, the sets of $\V$ remain coded unboundedly often into the GCH pattern of $\V[G]$. If $W$ is any ground of $\V[G]$, therefore, it follows as well that the sets of $\V$ are coded into the GCH pattern of $W$, and so $\V \subseteq W$. Furthermore, if $P^\alpha$ is the tail portion of the forcing, at coordinates at and above $\alpha$, then it follows that all initial segments of any coordinate of $G^\alpha = G \upharpoonright P^\alpha$, construed as a set of ordinals, are in $\V$ and hence in $W$. If $\alpha$ is above the size of the forcing between $W$ and $\V[G]$, then it follows by the chain condition of the forcing over $W$ that $G^\alpha$ is amenable to $W$, for otherwise there would be a large antichain. Thus, the ground model $W$ is intermediate in the interval $\V[G^\alpha] \subseteq W \subseteq \V[G]$. Since $\V[G]$ arises by set forcing over $\V[G^\alpha]$ with the initial factor $P_\alpha$, at coordinates below $\alpha$, we have a strictly descending sequence of grounds $\V[G^\beta]$, of order type ORD, that are dense below the grounds of $\V[G]$. It follows that the grounds are set-directed and have no minimal member, establishing the theorem. 

The class forcing extension $V[G]$ of the previous theorem has the nice property that for every ordinal $\alpha$, we have $V_\alpha^{V[G]} = V_\alpha^{V[G_{\omega_1}]}$ for some $G_\alpha$ that is set-generic over $V$. So in a sense, $V[G]$ is “close to $V$”. Since the assumption that the universe is constructible from a set considerably simplifies the geology, the following is an interesting strengthening of the previous theorem. The class forcing extension constructed here will not be close to $V$ in the previous sense, though.

\begin{theorem}
Every model of set theory has a class forcing extension of the form $L[r]$, where $r \subseteq \omega$, in which there is no bedrock, but the grounds are downward set-directed.
\end{theorem}
Proof. First of all, the grounds of any model of the form $L[r]$ are downward set-directed by theorem 16, so the final claim of the theorem will come for free.

The desired model $L[r]$ is produced by Jensen coding, and the content of our proof is the observation that this model has no bedrock. In the Jensen construction (see [BJW82] for a standard reference), one passes first to a model $(W, A)$, with $A \subseteq \text{ORD}$ specifically arranged so that $H^W_\alpha = L_\alpha[A]$ whenever $\alpha$ is a cardinal of $W$. One next extends $W$ by forcing with Jensen’s class partial order $\mathbb{P}$. Suppose that $G \subseteq \mathbb{P}$ is $W$-generic, and let $W' = W[G]$, which has the form $L[r]$ for some real $r$ in $W'$. The forcing $G$ adds a class $D \subseteq \text{ORD}$ such that $W[G] = L[D]$. If $\tau$ is a cardinal of $W$, then $\mathbb{P}$ splits as $\mathbb{P}_\tau \ast \mathbb{P}^{D_\tau}$. The forcing $\mathbb{P}_\tau$ is a class forcing coding $V$ as a set $D_\tau = D \cap \tau^+$, in the sense that $W[G | \mathbb{P}_\tau] = L[D_\tau]$, and $\mathbb{P}^{D_\tau}$ is a set forcing, essentially an iteration of almost disjoint coding, which brings the coding from $\tau^+$ down to $\omega$ in the sense that it codes $D_\tau$ as a subset $D_0$ of $\omega$. In particular, $L[D_\tau]$ is a ground of $W'$. Furthermore, if $\tau' > \tau$ is a larger cardinal of $W$, then $L[D_{\tau'}]$ is a nontrivial set-forcing extension of $L[D_\tau]$, and so $W'$ has class-many grounds. Since $W' = L[r]$ is constructible from a set, it follows from this and theorem 22 that the bedrock axiom fails there. \hfill \Box

Our most fundamental lack of knowledge about the mantle is that, without any additional hypotheses, we do not yet know whether or not it is necessarily a model of ZFC or even of ZF.

Question 25. Does the mantle necessarily satisfy ZF? Or ZFC?

One may of course restrict the concept of grounds to certain classes of forcing notions. Thus, let us say that $W$ is a $\sigma$-closed ground if $W$ is a ground such that the universe is obtainable from $W$ by a $\sigma$-closed forcing $\mathbb{P}$ (note that $\mathbb{P}$ is $\sigma$-closed in $W$ if and only if it is so in $V$). As a reminder, let us mention that by definition, a ground must be a model of ZFC. The $\sigma$-closed mantle is simply the intersection of all $\sigma$-closed grounds. More generally, we define the $\Gamma$-Mantle, for any definable class $\Gamma$ of forcing notions, to be the intersection of all grounds $W$ for which $V$ is a forcing extension of $W$ by a forcing notion in $\Gamma^W$. Theorem 27 shows that the $\sigma$-closed mantle need not be a model of ZFC, even if the universe is constructible from a set. We will make use of the following folklore result, emphasizing that its proof makes no use of the axiom of choice.

Lemma 26 ([Sol70, Lemma 2.5]). Assuming ZF, if $G \times H$ is $V$-generic for product forcing $\mathbb{P} \times \mathbb{Q}$, then $V[G] \cap V[H] = V$.

Proof. Certainly $V \subseteq V[G] \cap V[H]$. Conversely, suppose that $x \in V[G] \cap V[H]$. We may assume by $\epsilon$-induction that $x \subseteq V$. By assumption, there is a $\mathbb{P}$-name $\tau$ and a $\mathbb{Q}$-name $\sigma$ such that $x = \tau_G = \sigma_H$. So there must be a condition $(p, q) \in G \times H$ forcing that $\tau_G = \sigma_H$. We claim that $p$ already decides the elements of $\tau$. If not, we could find $V[H]$-generic filters $G'$ and $G''$, both containing $p$, such that $\tau_{G'} \neq \tau_{G''}$. But both must be equal to $\sigma_H$, since $G' \times H$ and $G'' \times H$ are both $V$-generic and contain $(p, q)$, a contradiction. Thus, $p \Vdash \tau \in V$, and so $x \in V$. \hfill \Box

Theorem 27. If ZFC is consistent, then so is the theory consisting of the following axioms:

1. ZFC,
2. $V = L[A]$, for a set $A \subseteq \omega_1$. 

(3) the $\sigma$-closed mantle is the same as $L(\mathbb{R})$ and fails to satisfy the axiom of choice.

Proof. Starting with a model of ZFC, we can pass to a forcing extension $M$ whose $L(\mathbb{R})$ fails to satisfy the axiom of choice. For example, adding $\mathbb{R}_1$ Cohen reals will do this; see [Kun80, P. 245, ex. (E3) and (E4)]. The model $L(\mathbb{R})^M$ will satisfy ZF $+$ DC, though, since this always holds in the $L(\mathbb{R})$ of a model of ZFC. Now let $a, b$ be mutually generic Cohen subsets of $\omega_1$ over $L(\mathbb{R})^M$. Since this forcing adds a well-ordering of $\mathbb{R}$, it follows that $L(\mathbb{R})[a]$ is a model of ZFC, and in fact $L(\mathbb{R})[a] = L[a]$; and the same is true for $L(\mathbb{R})[b] = L[b]$. Note that since DC holds in $L(\mathbb{R})$, adding $a$ or $b$ adds no new $\omega$-sequences, and hence doesn’t change $L(\mathbb{R})$.  

So we may unambiguously write $L(\mathbb{R})$ for $L(\mathbb{R})^M = L(\mathbb{R})L(\mathbb{R})[a] = L(\mathbb{R})^M[b]$. Consider now the model $N = L(\mathbb{R})[a, b] = L[a, b]$. Both $L[a]$ and $L[b]$ are $\sigma$-closed grounds of $N$, and so the $\sigma$-closed mantle of $N$ is contained in the intersection of $L(\mathbb{R})[a]$ and $L(\mathbb{R})[b]$. But since $a$ and $b$ are mutually generic, however, this intersection is precisely $L(\mathbb{R})$ by lemma 26. Conversely, $L(\mathbb{R})$ is contained in every $\sigma$-closed ground of $N$, since every such ground must contain all $\omega$-sequences of ordinals which are in $N$, and hence all the reals of $N$. So the $\sigma$-closed mantle of $N$ is precisely $L(\mathbb{R})^N$, which is the $L(\mathbb{R})$ we specifically arranged at the beginning to violate the axiom of choice.

The proof actually shows that every model $V$ of ZFC has a forcing extension $\mathcal{V}$ in which there is a set $A \subseteq \omega_1$, such that $L[A] \models$ ZFC, but the $\sigma$-closed mantle of $L[A]$ is $L(\mathbb{R})^\mathcal{V}$ and fails to satisfy the axiom of choice. Note also that statement (2) of the previous theorem is optimal, in the sense that if $V = L[r]$, where $r \subseteq \omega$, then trivially $V$ is its own $\sigma$-closed mantle.

What we said about $\sigma$-closed geology remains true when restricting to the $\sigma$-distributive notions. The underlying concept in this context is that a ground $W$ is a $\sigma$-distributive ground if there is a notion of forcing $\mathbb{P} \in W$ such that $W$ thinks that $\mathbb{P}$ is $\sigma$-distributive, and $V$ is a forcing extension of $W$ by $\mathbb{P}$. Note that $\mathbb{P}$ might fail to be $\sigma$-distributive in $V$ in that case.

**Question 28.** Under what circumstances is the mantle also a ground model of the universe? That is, when does the solid bedrock axiom hold?

**Question 29.** When does the universe remain a solid bedrock in all its forcing extensions?

We regard this latter property, the forcing invariance of being a solid bedrock, as an inner-model-like structural property, indicating when it holds that the universe is highly regular or close to highly regular in some sense. For example, the property holds in $L$, $L[\theta^+]$, $L[\mu]$ and many other canonical models, and it is also an easy consequence of CCA. Question 29 has an affirmative answer in exactly the models that are their own generic mantle, a concept defined in section 3.

Let us conclude this section with a brief combinatorial analysis of the structure of the grounds of the universe.

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1. It is an amusing observation of the first author that over ZF, the principle of dependent choices is in fact equivalent to the statement that $\sigma$-closed forcing adds no new countable sequences of ordinals.
Theorem 30.

(1) The collection of grounds between a fixed ground $W$ and the universe $V$ is an upper semi-lattice.

(2) If the grounds of $V$ are downward directed, then the grounds of $V$ are an upper semi-lattice.

(3) In this case, however, the upper semi-lattice of grounds of $V$ need not be a complete upper semi-lattice, even if the universe is constructible from a set.

(4) The grounds of $V$ need not be a lattice, even when the grounds are downward set-directed, and even if the universe is constructible from a set. There can be two grounds $W_0$ and $W_1$ whose intersection contains no largest model of ZFC.

Proof. (1) By fact 7, any ZFC model between $W$ and $V$ is a forcing extension $W[G]$ of $W$, and a ground of $V$. Suppose that $W[G]$ and $W[H]$ are two such extensions, with $G \subseteq P \in W$ and $H \subseteq P \in W$ both $W$-generic and $W[G], W[H] \subseteq V$. Irrespective of mutual genericity concerns, in $V$ we may form the object $G \times H$, a subset of $P \times Q \in W$. Since $P \times Q$ is well-orderable in $W$, it follows that the model $W[G \times H]$, consisting of all sets constructible from elements of $W$ and $G \times H$, is the smallest model of ZFC containing $W$ and $G \times H$. This is a model of ZFC between $W$ and $V$; it is a ground of $V$; and clearly it contains both $W[G]$ and $W[H]$. Also, any other model of ZFC containing both $W[G]$ and $W[H]$ will contain $W$ and $G$ and $H$ and therefore also $G \times H$, and so $W[G \times H]$ is the least upper bound of $W[G]$ and $W[H]$, as desired.

(2) Suppose that the grounds of $V$ are directed. If $W_r$ and $W_s$ are two grounds, then there is some deeper ground $W_t \subseteq W_r \cap W_s$. By applying (1) to the grounds between $W_t$ and $V$, it follows that $W_r$ and $W_s$ have a least upper bound.

For (3) and (4), we force over $L$ to construct a model having the desired features for its grounds. Using Easton forcing, let $P$ be the forcing over $L$ to force $2^\aleph_n = \aleph_{n+2}$ for all $n < \omega$. More specifically, $P$ is the full support product of the forcing $Q_n = \text{Add}(\aleph_n, \aleph_{n+2})$, as defined in $L$. Suppose that $G \subseteq P$ is $L$-generic for this forcing. Next, we add a Cohen real $c$ over $L[G]$. In the model $V = L[G][c]$, we construct two grounds, so as to witness (4). The first ground is simply $L[G]$. The second is $L[G \upharpoonright c]$, where $G \upharpoonright c$ is the filter $G$ restricted to the coordinates $n$ with $n \in c$, viewing $c$ as a subset of $\omega$. We claim that $L[G]$ and $L[G \upharpoonright c]$ have no greatest lower bound among the grounds. Suppose towards contradiction that $W$, a ground of $V$, is the greatest lower bound of $L[G]$ and $L[G \upharpoonright c]$. As all cardinals are preserved between $L$ and $L[G][c]$, it follows that the models $W$, $L[G]$ and $L[G \upharpoonright c]$ have all the same $\aleph_n$’s. Note that if $n \in c$, then $L[G]$ and $L[G \upharpoonright c]$ both contain $L[G(\aleph_n)]$, where $G(\aleph_n)$ is the part of the generic filter $G$ on coordinate $n$, to pump up the GCH at $\aleph_n$. Since $L[G(\aleph_n)]$ is a ground of both $L[G]$ and $L[G \upharpoonright c]$ for such $n$, it follows by our assumption on $W$ being a greatest lower bound that $L[G(\aleph_n)] \subseteq W$ for any $n \in c$. This implies that the GCH fails in $W$ at every $\aleph_n$ for which $n \in c$. If $n \notin c$, however, then the GCH must hold in $W$ at $\aleph_n$, since it holds in $L[G \upharpoonright c]$ at these $\aleph_n$ and cardinals are preserved. Thus, the GCH pattern in $W$ on the $\aleph_n$’s exactly conforms with the real $c$. But $c$ is not in $W$, because it is not in $L[G]$, contradicting our assumption that $W$ satisfied ZFC. Thus, we have proved that the grounds $L[G]$ and $L[G \upharpoonright c]$ can have no greatest lower bound, establishing (4). Statement (3) follows on general lattice theoretic grounds, since if a downward directed upper semi-lattice is complete with respect to joins, then for any pair of nodes, the join
of all lower bounds of these nodes will be a greatest lower bound, and so it would be a lattice. By (4), it need not be a lattice, and so we conclude that (3), it need not be complete as an upper semi-lattice.

In particular, theorem 30 statement (4) provides an example of a model $L[G][c]$ having two grounds $L[G]$ and $L[G \upharpoonright c]$ whose intersection is not a model of ZF. Namely, the separation axiom fails in $L[G] \cap L[G \upharpoonright c]$, since in this model one can define the set of natural numbers $n$ for which the power set of $\aleph_n$ is not bijective with $\aleph_{n+1}$, but this set corresponds exactly to $c$, which does not exist in $L[G] \cap L[G \upharpoonright c]$.

2. A brief upward glance

Set-theoretic geology naturally has a downward-oriented focus, towards deeper grounds and mantles, but let us cast a brief upward glance in this section by considering several upwards closure issues. The second author heard the following observation from Woodin in the early 1990s.

**Observation 31.** If $W$ is a countable model of ZFC set theory, then there are forcing extensions $W[c]$ and $W[d]$, both obtained by adding a Cohen real, which are non-amalgamable in the sense that there can be no model of ZFC with the same ordinals as $W$ containing both $W[c]$ and $W[d]$. Thus, the family of forcing extensions of $W$ is not upward directed.

**Proof.** Since $W$ is countable, let $z$ be a real coding the entirety of $W$. Enumerate the dense subsets $\langle D_n \mid n < \omega \rangle$ of the Cohen forcing $\text{Add}(\omega, 1)$ in $W$. We construct $c$ and $d$ in stages. We begin by letting $c_0$ be any element of $D_0$. Let $d_0$ consist of exactly as many 0s as $|c_0|$, followed by a 1, followed by $z(0)$, and then extended to an element of $D_0$. Continuing, $c_{n+1}$ extends $c_n$ by adding 0s until the length of $d_n$, and then a 1, and then extending into $D_{n+1}$; and $d_{n+1}$ extends $d_n$ by adding 0s to the length of $c_{n+1}$, then a 1, then $z(n)$, then extending into $D_{n+1}$. Let $c = \cup c_n$ and $d = \cup d_n$. Since we met all the dense sets in $W$, we know that $c$ and $d$ are $W$-generic Cohen reals, and so we may form the forcing extensions $W[c]$ and $W[d]$. But if $W \subseteq U \models \text{ZFC}$ and both $c$ and $d$ are in $U$, then in $U$ we may reconstruct the map $n \mapsto (c_n, d_n)$, by giving attention to the blocks of 0s in $c$ and $d$. From this map, we may reconstruct $z$ in $U$, which reveals all the ordinals of $W$ to be countable, a contradiction if $U$ and $W$ have the same ordinals. □

Let us remark that many of the results in this section concern forcing extensions of an arbitrary countable model of set theory, which of course includes the case of ill-founded models. Although there is no problem with forcing extensions of ill-founded models, when properly carried out, the reader may prefer to focus on the case of countable transitive models for the results in this section, and such a perspective will lose very little of the point of our observations.

The method of observation 31 is easily generalized to produce three $W$-generic Cohen reals $c_0$, $c_1$ and $c_2$, such that any two of them can be amalgamated, but the three of them cannot. More generally, for any finite $n$, there are forcing extensions $W[c_i]$ for $i < n$, such that any proper subset of them is amalgamable, and in fact any subsequence $\vec{c}$ omitting at least one $c_i$ is mutually $W$-generic for adding several Cohen reals, but such that there is no forcing extension $W[G]$ simultaneously extending all $W[c_i]$ for all $i < n$. In particular, the full finite sequence $\langle c_i \mid i < n \rangle$ cannot be added by forcing over $W$. 
Let us turn now to infinite linearly ordered sequences of forcing extensions. We show first in theorem 32 and observation 33 that one mustn’t ask for too much; but nevertheless, in theorem 34 we prove the surprising positive result, that any increasing sequence of forcing extensions over a countable model $W$, with forcing of uniformly bounded size, is bounded above by a single forcing extension $W[G]$.

**Theorem 32.** If $W$ is a countable model of ZFC, then there is an increasing sequence of set-forcing extensions of $W$ having no upper bound in the generic multiverse of $W$.

$$W[G_0] \subseteq W[G_1] \subseteq \cdots \subseteq W[G_n] \subseteq \cdots$$

**Proof.** Since $W$ is countable, there is an increasing sequence $\langle \gamma_n \mid n < \omega \rangle$ of ordinals that is cofinal in the ordinals of $W$. Let $G_n$ be $W$-generic for the collapse forcing $\text{Coll}(\omega, \gamma_n)$, as defined in $W$. (By absorbing the smaller forcing, we may arrange that $W[G_n]$ contains $G_m$ for $m < n$.) Since every ordinal of $W$ is eventually collapsed, there can be no set-forcing extension of $W$, and indeed, no model with the same ordinals as $W$, that contains every $W[G_n]$. □

But that was cheating, of course, since the sequence of forcing notions is not even definable in $W$, as the class $\{ \gamma_n \mid n < \omega \}$ is not a class of $W$. A more intriguing question would be whether this phenomenon can occur with forcing notions that constitute a set in $W$, or (equivalently, actually) whether it can occur using always the same poset in $W$. For example, if $W[c_0] \subseteq W[c_0][c_1] \subseteq W[c_0][c_1][c_2] \subseteq \cdots$ is an increasing sequence of generic extensions of $W$ by adding Cohen reals, then does it follow that there is a set-forcing extension $W[G]$ of $W$ with $W[c_0] \cdots [c_n] \subseteq W[G]$ for every $n$? For this, we begin by showing that one mustn’t ask for too much:

**Observation 33.** If $W$ is a countable model of ZFC, then there is a sequence of forcing extensions $W \subseteq W[c_0] \subseteq W[c_0][c_1] \subseteq W[c_0][c_1][c_2] \subseteq \cdots$, adding a Cohen real at each step, such that there is no forcing extension of $W$ containing the sequence $\langle c_n \mid n < \omega \rangle$.

**Proof.** Let $\langle d_n \mid n < \omega \rangle$ be any $W$-generic sequence for the forcing to add $\omega$ many Cohen reals over $W$. Let $z$ be any real coding the ordinals of $W$. Let us view these reals as infinite binary sequences. Define the real $c_n$ to agree with $d_n$ on all digits except the initial digit, and set $c_n(0) = z(n)$. That is, we make a single-bit change to each $d_n$, so as to code one additional bit of $z$. Since we have made only finitely many changes to each $d_n$, it follows that $c_n$ is an $W$-generic Cohen real, and also $W[c_0] \cdots [c_n] = W[d_0] \cdots [d_n]$. Thus, we have $W \subseteq W[c_0] \subseteq W[c_0][c_1] \subseteq W[c_0][c_1][c_2] \subseteq \cdots$, adding a generic Cohen real at each step. But there can be no forcing extension of $W$ containing $\langle c_n \mid n < \omega \rangle$, since any such extension would have the real $z$, revealing all the ordinals of $W$ to be countable. □

We can modify the construction to allow $z$ to be $W$-generic, but collapsing some cardinals of $W$. For example, for any cardinal $\delta$ of $W$, we could let $z$ be $W$-generic for the collapse of $\delta$. Then, if we construct the sequence $\langle c_n \mid n < \omega \rangle$ as above, but inside $W[z]$, we get a sequence of Cohen real extensions $W \subseteq W[c_0] \subseteq W[c_0][c_1] \subseteq W[c_0][c_1][c_2] \subseteq \cdots$ such that $W[\langle c_n \mid n < \omega \rangle] = W[z]$, which collapses $\delta$.

But of course, the question of whether the models $W[c_0][c_1] \cdots [c_n]$ have an upper bound is not the same question as whether one can add the sequence $\langle c_n \mid n < \omega \rangle$, since an upper bound may not have this sequence. And in fact, this is exactly what occurs, and we have a surprising positive result:
Theorem 34. Suppose that $W$ is a countable model of ZFC, and
\[ W[G_0] \subseteq W[G_1] \subseteq \cdots \subseteq W[G_n] \subseteq \cdots \]
is an increasing sequence of forcing extensions of $W$, with $G_n \subseteq Q_n \in W$ being $W$-generic. If the cardinalities of the $Q_n$'s in $W$ are bounded in $W$, then there is a set-forcing extension $W[G]$ with $W[G_n] \subseteq W[G]$ for all $n < \omega$.

Proof. Let us first make the argument in the special case that we have
\[ W \subseteq W[g_0] \subseteq W[g_0][g_1] \subseteq \cdots \subseteq W[g_0][g_1] \cdots [g_n] \subseteq \cdots , \]
where each $g_n$ is generic over the prior model for forcing $Q_n \in W$. That is, each extension $W[g_0][g_1] \cdots [g_n]$ is obtained by product forcing $Q_0 \times \cdots \times Q_n$ over $W$, and the $g_n$ are mutually $W$-generic. Let $\delta$ be a regular cardinal with each $Q_n$ having size at most $\delta$, built with underlying set a subset of $\delta$. In $W$, let $\theta = 2^{\delta}$, let $\langle R_\alpha | \alpha < \theta \rangle$ enumerate all posets of size at most $\delta$, with unbounded repetition, and let $P = \prod_{\alpha < \theta} R_\alpha$ be the finite support product of these posets. Since each factor is $\delta^+$-c.c., it follows that the product is $\delta^+$-c.c. Since $W$ is countable, we may build a filter $H \subseteq P$ that is $W$-generic. In fact, we may find such a filter $H \subseteq P$ that meets every dense set in $\bigcup_{n<\omega} W[g_0][g_1] \cdots [g_n]$, since this union is also countable. In particular, $H$ and $g_0 \times \cdots \times g_n$ are mutually $W$-generic for every $n < \omega$. The filter $H$ is determined by the filters $H_\alpha \subseteq R_\alpha$ that it adds at each coordinate.

Next comes the key step. Externally to $W$, we may find an increasing sequence $\langle \theta_n | n < \omega \rangle$ of ordinals cofinal in $\theta$, such that $R_{\theta_n} = Q_n$. This is possible because the posets are repeated unboundedly, and $\theta$ is countable in $V$. Let us modify the filter $H$ by surgery to produce a new filter $H^*$, by changing $H$ at the coordinates $\theta_n$ to use $g_n$ rather than $H_{\theta_n}$. That is, let $H^*_n = g_n$ and otherwise $H^*_n = H_{\theta_n}$, for $\alpha \notin \{ \theta_n | n < \omega \}$. It is clear that $H^*$ is still a filter on $P$. We claim that $H^*$ is $W$-generic. To see this, suppose that $A \subseteq P$ is any maximal antichain in $W$. By the $\delta^+$-chain condition and the fact that $\text{cof}(\theta)^W > \delta$, it follows that the conditions in $A$ have support bounded by some $\gamma < \theta$. Since the $\theta_n$ are increasing and cofinal in $\theta$, only finitely many of them lay below $\gamma$, and we may suppose that there is some largest $\theta_m$ below $\gamma$. Let $H**$ be the filter derived from $H$ by performing the surgical modifications only on the coordinates $\theta_0, \ldots, \theta_m$. Thus, $H^*$ and $H**$ agree on all coordinates below $\gamma$. By construction, we had ensured that $H$ and $g_0 \times \cdots \times g_m$ are mutually generic over $W$ for the forcing $P \times Q_0 \times \cdots \times Q_m$. This poset has an automorphism swapping the latter copies of $Q_i$ with their copy at $\theta_i$ in $P$, and this automorphism takes the $W$-generic filter $H \times g_0 \times \cdots \times g_m$ exactly to $H** \times H_{\theta_0} \times \cdots \times H_{\theta_m}$. In particular, $H**$ is $W$-generic for $P$, and so $H**$ meets the maximal antichain $A$. Since $H^*$ and $H**$ agree at coordinates below $\gamma$, it follows that $H^*$ also meets $A$. In summary, we have proved that $H^*$ is $W$-generic for $P$, and so $W[H^*]$ is a set-forcing extension of $W$. By design, each $g_n$ appears at coordinate $\theta_n$ in $H^*$, and so $W[g_0][g_1] \cdots [g_n] \subseteq W[H^*]$ for every $n < \omega$, as desired.

Finally, we reduce the general case to this special case. Suppose that $W[G_0] \subseteq W[G_1] \subseteq \cdots \subseteq W[G_n] \subseteq \cdots$ is an increasing sequence of forcing extensions of $W$, with $G_n \subseteq Q_n \in W$ being $W$-generic and each $Q_n$ of size at most $\kappa$ in $W$. By the standard facts surrounding finite iterated forcing, we may view each model as a forcing extension of the previous model
\[ W[G_{n+1}] = W[G_n][H_n], \]
where $H_n$ is $W[G_n]$-generic for the corresponding quotient forcing $Q_n/G_n$ in $W[G_n]$. Let $g \in \text{Coll}(\omega, \kappa)$ be $\bigcup_n W[G_n]$-generic for the collapse of $\kappa$, so that it is mutually generic with every $G_n$. Thus, we have the increasing sequence of extensions $W[g][G_0] \subseteq W[g][G_1] \subseteq \cdots$, where we have added $g$ to each model. Since each $Q_n$ is countable in $W[g]$, it is forcing equivalent there to the forcing to add a Cohen real. Furthermore, the quotient forcing $Q_n/G_n$ is also forcing equivalent in $W[g][G_n]$ to adding a Cohen real. Thus, $W[g][G_{n+1}] = W[g][G_n][H_n] = W[g][G_n][h_n]$, for some $W[g][G_n]$-generic Cohen real $h_n$. Unwrapping this recursion, we have $W[g][G_{n+1}] = W[g][G_0][h_1] \cdots [h_n]$, and consequently

$$W[g] \subseteq W[g][G_0] \subseteq W[g][G_0][h_1] \subseteq W[g][G_0][h_1][h_2] \subseteq \cdots,$$

which places us into the first case of the proof, since this is now product forcing rather than iterated forcing. □

**Definition 35.** A collection $\{W[G_n] \mid n < \omega\}$ of forcing extensions of $W$ is **finitely amalgamable** over $W$ if for every $n < \omega$ there is a forcing extension $W[H]$ with $W[G_m] \subseteq W[H]$, for all $m \leq n$. It is **amalgamable** over $W$ if there is $W[H]$ such that $W[G_n] \subseteq W[H]$, for all $n < \omega$.

The next corollary shows that we cannot improve the non-amalgamability result of Woodin to the case of infinitely many Cohen reals, with all finite subsets amalgamable.

**Corollary 36.** If $W$ is a countable model of ZFC and $\{W[G_n] \mid n < \omega\}$ is a finitely amalgamable collection of forcing extensions of $W$, using forcing of bounded size in $W$, then this collection is fully amalgamable. That is, there is a forcing extension $W[H]$ with $W[G_n] \subseteq W[H]$, for all $n < \omega$.

**Proof.** Since the collection is finitely amalgamable, for each $n < \omega$ there is some $W$-generic $K$ such that $W[G_m] \subseteq W[K]$, for all $m \leq n$. Thus, we may form the minimal model $W[G_0][G_1] \cdots [G_n]$ between $W$ and $W[K]$, and thus $W[G_0][G_1] \cdots [G_n]$ is a forcing extension of $W$. We are thus in the situation of theorem 34, with an increasing chain of forcing extensions.

$$W \subseteq W[G_0] \subseteq W[G_0][G_1] \subseteq \cdots \subseteq W[G_0][G_1] \cdots [G_n] \subseteq \cdots$$

Therefore, by theorem 34, there is a model $W[H]$ containing all these extensions, and in particular, $W[G_n] \subseteq W[H]$, as desired. □

3. The generic mantle

Let us turn now to a somewhat enlarged context, where we have found an intriguing and perhaps more robust version of the mantle. The concept is inspired by [Fuc08], where the generic HOD is introduced, a topic we shall explore further in section 4. The idea is to consider not only the grounds of $V$, but also the grounds of the forcing extensions of $V$. A **generic ground** of $V$ is a ground of some set-forcing extension of $V$, that is, a model $W$ having $V[G] = W[H]$ for some $V$-generic filter $G \subseteq P \in V$ and $W$-generic filter $H \subseteq Q \in W$. This is equivalent to saying that $V$ and $W$ have a common forcing extension, and this is therefore a symmetric relation. The **generic mantle** of $V$ is the intersection of all generic grounds of $V$. Since this includes $V$ itself, as well as every ground of $V$, it follows that the generic mantle of $V$ will be included in $V$ and indeed, in the mantle of $V$. 
Definition 37. The generic mantle, denoted \( gM \), is the class of all \( x \) such that every forcing notion \( P \) forces that \( \dot{x} \) belongs to every ground of \( V^P \). That is, \[
gM = \{ x \mid \forall P \ 1 \models_P \forall r \ \dot{x} \in W_r \},\]
where \( W_r \) is the ground of the forcing extension defined by parameter \( r \) using the uniform definition of theorem 8 applied in the forcing extension. Equivalently, \[
gM = \{ x \mid \forall P \ 1 \models \dot{x} \in M \},\]
where \( M \) defines the mantle as interpreted in the forcing extension.

In particular, the generic mantle is uniformly definable without parameters. If \( P \) is a poset, we shall write \( M^P \) for the class \( \{ x \mid 1 \models P \dot{x} \in M \} \), so that \( gM = \bigcap_P M^P \).

Observation 38. \( gM = \bigcap_\alpha M^{\text{Coll}(\omega, \alpha)} \).

Proof. The inclusion from left to right is trivial, as the intersection on the right ranges only over a subclass of all forcing notions. For the reverse inclusion, suppose that \( x \in \bigcap_\alpha M^{\text{Coll}(\omega, \alpha)} \), and let \( P \) be an arbitrary poset, of some cardinality \( \alpha \). It follows that \( P \times \text{Coll}(\omega, \alpha) \) is forcing equivalent to \( \text{Coll}(\omega, \alpha) \), and so \( x \in M^{\text{Coll}(\omega, \alpha)} = M^{\text{Col}(\omega, \alpha)} \). But \( M^{\text{Col}(\omega, \alpha)} \subseteq M^P \), as is easily verified, so we are done. \( \square \)

In many models of set theory, the generic mantle and the mantle coincide, and although we have introduced them as distinct concepts, we do not actually know them to be different, for we have no model yet in which we know them to differ (see question 55). We introduce the generic mantle in part because the evidence indicates that it may be a more robust notion. For example, we are able to prove that the generic mantle satisfies the ZF axioms of set theory without needing the directedness hypotheses of theorem 18.

Theorem 39. The generic mantle of \( V \) is a definable transitive class in \( V \), containing all ordinals, invariant under set forcing, and a model of the ZF axioms of set theory.

Proof. The generic mantle is a definable transitive class containing all ordinals and closed under the G"odel operations, because it is the intersection of the generic grounds, each of which has those closure properties in the corresponding extension of \( V \). So by fact 6, in order to show that \( gM \) is an inner model, it remains to show merely that it is almost universal, which we do below.

Let’s first prove that \( gM \) is invariant under set forcing, meaning that if \( V[G] \) is a forcing extension of \( V \) by set forcing \( P \), then \( gM^V = gM^{V[G]} \). Observe first that every ground model of a forcing extension of \( V[G] \) is also a ground model of a forcing extension of \( V \). Thus, the generic mantle of \( V[G] \) is the intersection of a sub-collection of the models used to form the generic mantle of \( V \), and so \( gM^V \subseteq gM^{V[G]} \). For the reverse inclusion, suppose that \( x \notin gM^V \). Thus, there is a forcing extension \( V[H] \) via some forcing notion \( Q \) for which there is a ground \( W \subseteq V[H] \) that does not contain \( x \). There is a condition \( q \in H \) forcing that \( W \) is like this. By forcing below \( q \) with \( Q \) over \( V[G] \), we may assume that \( H \) is not merely \( V \)-generic, but also \( V[G] \)-generic, and consequently that \( G \times H \) is \( V \)-generic for the product forcing. Thus, \( W \) is also a ground of \( V[H]|G[H] \), which is the same as \( V[G]|H \), and so we have found a forcing extension of \( V[G] \) that has a ground \( W \) that omits \( x \). So \( x \notin gM^{V[G]} \) and so \( gM^V = gM^{V[G]} \), establishing that the generic mantle is invariant by set forcing.
To see that the generic mantle is a model of ZF, we claim first that $V_\alpha \cap gM \in gM$, for every $\alpha$. It is clear for any ground $W$ that $V_\alpha \cap gM = V_\alpha^W \cap gM^W \in W$, by the forcing invariance of $gM$. Since this is true for every ground, it follows that $V_\alpha \cap gM \in M$. And so we have shown in ZFC that for every $\alpha$, we have $V_\alpha \cap gM \in M$. So this is true in every forcing extension, which again by the forcing absoluteness of $gM$ (and of $V_\alpha \cap gM$) means that $V_\alpha \cap gM \in M^{\text{Coll}(\omega, \gamma)}$, for every $\gamma$ and every $\alpha$. So $V_\alpha \cap gM \in gM$, for every $\alpha$, by observation 38. It now follows that if $A$ is a subset of $gM$ of rank $\alpha$, then $A \subseteq V_\alpha \cap gM \in gM$, and so $gM$ is almost universal and hence is an inner model, as desired. □

The generic mantle of a model of set theory seems most naturally considered in a broad context that includes all of the forcing extensions of that model, their ground models, the forcing extensions of those models, and so on. Woodin [Woo04a] introduced the concept of the generic multiverse of a model $U$ of set theory, the smallest collection of models containing $U$ and closed under forcing extensions and grounds, and this seems to be an ideal context in which to undertake our project of set-theoretic geology. The generic multiverse naturally partitions the larger multiverse of models of set theory into equivalence (meta)classes, consisting of models reachable from one another by passing to forcing extensions and ground models.

Because the generic multiverse concept is clearly second-order or higher-order, however, there are certain difficulties of formalization and meta-mathematical issues that need to be addressed. This is particularly true when one wants to consider the generic multiverse of the full set-theoretic universe $V$, rather than merely the generic multiverse of a toy countable model. The standard approaches to second-order set theory, after all, such as Gödel-Bernays set theory or Kelly-Morse set theory, do not seem to provide a direct account of the generic multiverse of $V$, whose forcing extensions are of course not directly available, even as GBC or KM classes.

Nevertheless, it turns out that many of the natural questions one might want to ask about the generic multiverse of $V$ can in fact be formalized in first order ZFC set theory, and since we are indeed principally interested in the features of the generic multiverse of the full set-theoretic universe $V$, we shall prefer to formalize our concepts this way whenever this is possible, so that they would be available for that purpose. For example, even though a naive treatment of the generic mantle would seem to present difficulties arising from the fact that we do not have full access inside $V$ to the forcing extensions of $V$ and to their respective grounds, we have nevertheless proved above that the generic mantle is a definable class in $V$, and that it is invariant by forcing over any model and to any ground. Thus, it is entirely a first-order project to state that the generic mantle satisfies a given statement or has such-and-such relation to other definable classes, and such questions can therefore be investigated without any meta-mathematical difficulties.

The standard treatments of forcing over $V$, such as the method of working under the forcing relation, the method of Boolean-valued models or the naturalist account of forcing (see [HS]), provide a first-order means of treating truth in the forcing extensions of $V$. Thus, it is a first-order statement of set theory to state that a given assertion $\varphi$ holds in some forcing extension of $V$ (and thus the modal assertions in the modal logic of forcing as in [HL08] are all first order expressible). Iterating this method, one can similarly state in an entirely first order manner whether $\varphi$ holds in every forcing extension of every ground of every forcing extension of $V$, or
whether a given set $x$ is an element of all such models, and so on. In this way, we see that many natural questions about the nature of the generic multiverse of $V$ are actually first-order questions about $V$.

Nevertheless, it is sometimes illuminating to consider the generic multiverse of a model in a context where we can legitimately grasp the generic multiverse as a whole, rather than only from the filtered perspective of one of the models in it. In these circumstances, we temporarily adopt what we call the toy model perspective. In this method, which is analogous to the countable-transitive-model approach to forcing as opposed to the various forcing-over-$V$ approaches, one has a countable model $W$ of ZFC, and one considers the collection of all forcing extensions of $W$, as constructed in the current set-theoretic background universe $V$, as well as the grounds of these universes and their forcing extensions and so on, forming ultimately the smallest collection of models closed under the forcing extensions and grounds that are available. The particular generic multiverse of $W$ therefore depends on the set-theoretic background in which it is constructed; if we were to force over $V$ to add a Cohen real $c$, for example, then for a given countable transitive model $W$, the generic multiverse of $W$ as computed in $V[c]$ will include $W[c]$, but in $V$ this model is of course not available. In any case, the generic multiverse of a countable model $W$ will be a family of continuum many models, essentially a set of reals under a suitable coding scheme. With the generic multiverse thus becoming concrete for a given toy model $W$, one may now imagine living inside $W$, treating it as the full universe and using that constructed generic multiverse as the generic multiverse of it.

It is essentially the toy model approach that one uses, for example, when undertaking forcing only with countable transitive models. Although this way of doing forcing was formerly quite common, it has now largely been supplanted with the various formalizations of forcing in ZFC, which enable one to make sense of forcing over an arbitrary model of ZFC. Since as we have mentioned we are principally interested in the questions of set-theoretic geology and the generic multiverse in the context of the full set-theoretic universe $V$, we intend to fall back on the toy approach only as a last resort, when it may not be clear how a given concept might be formalized. For example, we adopted the toy model approach in observation 31, since it is unclear how to formalize that assertion as a claim about the generic multiverse of the full universe $V$. Similarly, the inner model hypothesis IMH of [Fri06] is formalized via the toy model approach, because it is unclear how to formalize it as a claim about the full universe $V$.

The models of the generic multiverse can be specified by the corresponding finite zigzag sequence of forcing extensions and ground models on the path leading to it from the original model; let us say that $\langle U_0, U_1, \ldots, U_k \rangle$ is a multiverse path leading from $U_0$ to $U_k$ if each $U_{i+1}$ is either a forcing extension or a ground model of $U_i$. Woodin has argued that any statement true in a model of the generic multiverse is true already in a model reachable by a multiverse paths of length at most three from any fixed original model, specifically, in a forcing extension of a ground model of a forcing extension of the original model.\footnote{In [Woo04a], he appears to make the stronger claim that the models themselves are reachable in three steps, but in personal conversation with the second author, he clarified this to the claim we have just stated here. Meanwhile, under the hypothesis of theorem 45, every model in the generic multiverse of $V$ is reachable in only two steps, as a ground extension of $V$.}
finite bound on the number of steps) allows us to express in a first-order manner the
generic-multiverse “possibility” modality ♦ ϕ, asserting that ϕ is generic-multiverse
possible, that is, that ϕ is true in some model of the generic multiverse. This is
because by the three-step fact, ♦ ϕ is equivalent to the assertion that there is a
forcing notion P forcing that there is a ground having a forcing notion forcing ϕ
over that ground. These forcing modalities are further explored in [HL08], [HLa],
[HLb], with the latter article specifically treating the interaction of the upward
and downward forcing possibility modalities and offering improvements and further
results on this particular issue.

Woodin introduced the generic multiverse in [Woo04a] in part to make a philo-
sophical argument against a certain view of mathematical truth, namely, truth as
true-in-the-generic-multiverse. Although we investigate the generic multiverse, we
do not hold this view of truth. Rather, we are attempting to understand the funda-
mental features of the generic multiverse, a task we place at the foundation of
any deep understanding of forcing, and we consider the generic multiverse to be
the natural and illuminating background context for our project of set theoretic
geology. Apart from any grand Platonist-multiverse view of truth, we find that
the ordinary Tarskian view of truth, considered individually within each model of
the multiverse, is sufficient to provide a global understanding of truth across the
multiverse, for the definability of the forcing relation gives every model access to
the truth concepts of its forcing extensions, and conversely, Laver’s theorem on
the definability of the ground model gives every forcing extension direct access to
its grounds. Our view is that the generic multiverse is a grand new set-theoretic
context for us to explore: what are the features of the generic multiverse? What is
or can it be like? This is what set-theoretic geology is about.3

**Corollary 40.** The generic mantle is constant across the generic multiverse. In-
deed, the generic mantle is the intersection of the generic multiverse.

*Proof.* This is an immediate consequence of theorem 39. Specifically, since the
generic mantle is invariant by set forcing, it is preserved as one moves either from a
ground to a forcing extension or from a forcing extension to a ground. Since these
are the operations that generate the multiverse, all the models in the multiverse
have the same generic mantle. This means, in particular, that the generic mantle is
contained within the intersection of the multiverse. Conversely, the generic mantle
is defined to be the intersection of some of the models of the multiverse, and so the
intersection of the multiverse is contained within the generic mantle. Thus, they
are equal.

**Corollary 41.** The generic mantle is the largest forcing-invariant definable class.

*Proof.* We have argued above that the generic mantle is a forcing-invariant definable
class in any model of set theory. Any other class that is definable and invariant
by forcing over any model of set theory will be preserved to any forcing extension
V[G] and then to any subsequent ground W of V[G], and therefore be contained
within the generic mantle. So the generic mantle is the largest such forcing-invariant
class.

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3Haim Gaifman has pointed out (with humor) that the term ‘set-theoretic geology’ suggests
a Platonist view of forcing as a natural process or force of nature, and that for those who view
forcing as an intentional human activity, the alternative term should be ‘set-theoretic archæology.’
To our way of thinking, the previous corollary identifies the generic mantle as a highly canonical object, the largest forcing-invariant definable class. Because of this, we would expect it to become a focus of attention for those set-theorists interested in forcing and models of set theory.

We would like now to consider the downward directed grounds hypothesis as it arises in connection not only with the grounds of $V$, but with the grounds of the forcing extensions of $V$. Recall that the generic grounds of a model are the grounds of its forcing extensions. We say that the grounds are dense below the generic grounds, if every generic ground contains a ground. This concept is first-order expressible in set theory as the assertion that for every poset $P$ and every $P$-name $\dot{r}$, every condition in $P$ forces “there is an index $s \in \dot{V}$ such that $(W_s)^{\dot{V}} \subseteq W_{\dot{r}}$". A ground extension of $V$ is a forcing extension of a ground of $V$, or in other words, a model having a common ground with $V$: all such models have the form $W_r[G]$ for some $W_r$-generic filter $G \subseteq P \in W_r$, but we may not assume here that $G$ is $V$-generic.

**Lemma 42.** Any generic ground of $V$ that is contained in $V$ is a ground of $V$.

*Proof.* If $W$ is a ground of a forcing extension $V[G]$ and $W \subseteq V \subseteq V[G]$, then by lemma 7 it follows that $W$ is a ground of $V$. □

What we don’t know, more generally, is the following analogue of lemma 42 for models in the generic multiverse.

**Question 43.** If $W$ is in the generic multiverse of $V$ and $W \subseteq V$, must $W$ be a ground of $V$? In other words, for models within the same generic multiverse, is the inclusion relation the same as the “is a ground model of” relation?

The answer is yes if the grounds are dense in the multiverse, since then every model in the multiverse contained in $V$ is trapped between a ground of $V$ and $V$, and hence is itself a ground by lemma 42.

**Theorem 44.** The following are equivalent.

1. The DDG holds. That is, the grounds of $V$ are downward directed.
2. The DDG holds in some forcing extension of $V$.
3. The DDG holds in every ground of $V$.

*Proof.* Since $V$ is a forcing extension of itself, the implications $1 \implies 2$ and $3 \implies 1$ are immediate. For $2 \implies 3$, suppose that the grounds of a forcing extension $V[G]$ are downward directed and that $W$ and $W'$ are grounds of $U$, which is a ground of $V$. Thus, $W$ and $W'$ are also grounds of $V[G]$, and so by 2 there is some ground $\dot{W}$ of $V[G]$ with $\dot{W} \subseteq \dot{W} \cap W'$. Since $\dot{W} \subseteq U \subseteq V[G]$, it follows by lemma 42 that $W$ is a ground of $U$, and so 3 holds. □

Consider now in contrast the generic DDG, which asserts that the DDG holds in all forcing extensions.

**Theorem 45.** The following are equivalent.

1. The generic DDG holds. That is, in every forcing extension of $V$, the grounds are downward directed.
2. The grounds of $V$ are downward directed and dense below the generic grounds.
(3) The grounds of $V$ are downward directed and dense below the grounds of every ground extension.

Proof. (1 $\iff$ 3) Suppose that the grounds of every forcing extension of $V$ are downward directed. In particular, since $V$ is a forcing extension of itself, the grounds of $V$ are downward directed. Suppose that $W$ is a ground of a ground extension $W_r[G]$, where $G \subseteq P \in W_r$ is $W_r$-generic. Thus, $W$ has the form $W = W_t^{W_r[G]}$ for some parameter $t \in W_r[G]$, having a suitable $P$-name with $t = \dot{t}_G$.

If we temporarily imagine that $G$ is actually $V$-generic for the forcing $P$, then we may also form $V[G]$ and notice that $W_r[G]$ is a ground of $V[G]$. In this case, $V$, $W$ and $W_r$ are all grounds of $V[G]$, and so by our assumption there is a ground $W_s^{V[G]}$ contained in each of them. By lemma 42, it follows that $W_s^{V[G]}$ is a ground of $V$, which we may therefore denote simply by $W_s$, as well as a ground of $W_r$.

Thus, in this case where $G$ is $V$-generic, we have found the desired ground $W_s$ of $V$ contained in $W$. Since $W_r[G]$ can see that $W_s \subseteq W_r^{W_r[G]}$, there must be a condition $p \in P$ forcing this property about $s$ over $W_r$. Furthermore, by arguing the same below any given condition, $W_r$ can see that the set of such $p$ having such an $s$ is dense in $P$.

If we now drop the assumption that $G$ is $V$-generic, we may nevertheless assume by genericity that there is a condition $p \in G$ with $s$ as above, ensuring that $W_s^{W_r[G]} \subseteq W$. But $W_s^{W_r}$ is a ground of $W_r$ and therefore a ground of $V$. So we have found a ground of $V$ in $W_r$, as desired for 3.

(3 $\implies$ 2) This is immediate, since any ground of a forcing extension of $V$ is a ground of a ground extension, since $V$ is a ground of itself.

(2 $\implies$ 1) Suppose that the grounds of $V$ are downward directed and dense below the generic grounds, and suppose that $V[G]$ is a forcing extension of $V$, with grounds $W_r^{V[G]}$ and $W_s^{V[G]}$. Since the grounds are dense below the generic grounds, we may find grounds $W_r'$ and $W_s'$ with $W_r' \subseteq W_r^{V[G]}$ and $W_s' \subseteq W_s^{V[G]}$. Since the grounds are downward directed, there is a ground $W_t$ with $W_t \subseteq W_r' \cap W_s'$, and this is contained in $W_r^{V[G]} \cap W_s^{V[G]}$. Since $W_t$ is also a ground of $V[G]$, we conclude that the grounds of $V[G]$ are downward directed, as desired. □

Corollary 46. If the generic DDG holds, then

(1) the mantle is the same as the generic mantle;

(2) the class of ground extensions is closed under forcing extensions and grounds;

(3) and consequently, the generic multiverse of $V$ consists of the ground extensions of $V$.

Proof. Suppose that the generic DDG holds. By theorem 45, the grounds are dense below the generic grounds, and so the mantle and generic mantle coincide, establishing statement (1). By the same theorem, any ground $W$ of a ground extension $W_r[G]$ contains some ground $W_s$ of both $V$ and $W_r$, and so $W_s \subseteq W \subseteq W_r[G]$. Since $W_s$ is a ground of $W_r[G]$, it follows by lemma 42 that $W$ is a forcing extension of $W_s$, and so $W$ is a ground extension of $V$. Thus, the collection of ground extensions of $V$ is closed under grounds and also (clearly) under forcing extensions, establishing statement (2). Statement (3) now follows easily. □

Theorem 47. If there is a model of ZFC, then there is a model of ZFC having a ground extension that is not a generic ground. In particular, the generic grounds of such a model do not exhaust its generic multiverse.
Proof. Suppose that \( W \) is a countable model of ZFC and consider the non-amalgamable extensions \( W[c] \) and \( W[d] \) arising in observation 31. Focus on the model \( W[c] \), and observe that the model \( W[d] \) is a forcing extension of a ground of \( W[c] \), but there can be no extension \( W[c][G] \) of which \( W[d] \) is a ground, since any such extension would amalgamate \( W[c] \) and \( W[d] \).\( \Box \)

On the positive side, it is also relatively consistent with ZFC that the generic grounds do exhaust the generic multiverse. This is true in \( L \), for example, since the generic multiverse of \( L \) consists precisely of the forcing extensions of \( L \). We say that a generic ground \( W \) is a generic bedrock if it is minimal among the generic grounds (meaning that there is no generic ground which is properly contained in it) and it is a solid generic bedrock if it is least among the generic grounds (meaning that it is contained in every generic ground). Since a generic bedrock is contained in the generic ground \( V \), it follows by lemma 42 that it is a ground, and it is straightforward to formalize the statement that a ground \( W_s \) is a bedrock or a solid bedrock. Let the solid generic bedrock axiom assert that there is a solid generic bedrock. By lemma 42 again, the solid generic bedrock axiom is equivalent to the assertion that the generic mantle is a ground (and one may take this as the formal definition of this axiom). In the presence of the solid generic bedrock axiom, of course, the generic grounds will be downward set-directed, and furthermore, the generic multiverse will consist precisely of the forcing extensions of the solid generic bedrock.

Let us consider now the set-directed analogue of theorem 45.

**Theorem 48.** The following are equivalent

1. The generic strong DDG holds. That is, in every forcing extension of \( V \), the grounds are downward set-directed.
2. The grounds of \( V \) are downward set-directed and dense below the generic grounds.
3. The grounds of \( V \) are downward set-directed and dense below the grounds of every ground extension.

Proof. (1 \( \implies \) 3) If in every forcing extension, the grounds are downward set-directed, then in particular in \( V \) they are downward set directed. And they are dense below the grounds of any ground extension by theorem 45, since the generic strong DDG implies the generic DDG, so statement 3 holds.

(3 \( \implies \) 2) This is immediate, since every generic ground is trivially a ground of a ground extension, since \( V \) is a ground of \( V \).

(2 \( \implies \) 1) Suppose that the grounds are downward set-directed and every generic ground contains a ground. Consider any forcing extension \( V[G] \), where \( G \subseteq \mathbb{P} \) is \( V \)-generic, and any set \( I \) of indices in \( V[G] \), giving rise the generic grounds \( W^V_r[G] \) for \( r \in I \). By our assumption, for every such \( r \in I \) there is \( t \in V \) with \( W_t \subseteq W^V_r[G] \). If \( \dot{r} \) is a name for \( r \), then there is a condition \( p \in G \) forcing that \( W^V_{\dot{r}} \subseteq W^V_r[G] \). Fix a name \( \dot{I} \) for \( I \) that is full, in the sense that whenever \( [\dot{r}_0] \neq 0 \), then there is \( \dot{r} \in \text{dom}(\dot{I}) \) such that \( [\dot{r}_0 = \dot{r}] = 1 \). By collecting the witnesses \( t \) as above for each \( \dot{r} \in \text{dom}(\dot{I}) \) and condition \( p \in \mathbb{P} \), we may find a set \( J \in V \) such that whenever \( p \models \dot{r} \in I \), then there is \( q \leq p \) and \( t \in J \) such that \( q \models W^V_{\dot{r}} \subseteq W^V_r[G] \). A simple density argument now shows that for every \( r \in I \), there is \( t \in J \) with \( W_t \subseteq W^V_r[G] \).

Next, by the downward directedness of the grounds in \( V \), there is a single ground...
$W_s$ contained in $W_t$ for all $t \in J$ and hence also in $W_r^{V[G]}$ for every $r \in I$. Since $W_t$ is a ground of $V$ and hence also of $V[G]$, we have therefore established that the grounds of $V[G]$ are downward directed, as desired. 

Consider now the local version of downward directedness. Extending our previous terminology, let us define the local DDG hypothesis as the assertion that the grounds are locally downward directed, meaning that for every $r$ and $s$ and every set $B$ there is $t$ with $W_t \cap B \subseteq W_r \cap W_s$; the strong local DDG asserts that the grounds are locally downward set-directed, meaning that for every set $I$ and set $B$ there is $t$ with $W_t \cap B \subseteq \bigcap_{r \in I} W_r$; and the generic strong local DDG asserts that this holds in every forcing extension.

**Theorem 49.** If the generic strong local DDG holds, then the generic mantle is a model of ZFC.

**Proof.** We already know by theorem 39 that the generic mantle $gM$ satisfies ZF. Unfortunately, we cannot apply corollary 20 directly here in order to prove the axiom of choice holds in the generic mantle, because the generic grounds are not classes in $V$. But we can adopt the argument of theorem 18. Assume toward contradiction that $y$ is a member of $gM$ which has no well-order in $gM$. Let $R$ be the set of well-orders of $y$. Since none of these are in $gM$, we may find for each $r \in R$ a poset $P_r$ which forces “$\hat{r} \notin M^{V[G_r]}$,” where $G_r$ is the canonical $P_r$-name of the generic filter. Let $P = \prod_r P_r$ be the full-support product of these forcing notions and suppose $G \subseteq P$ is $V$-generic. In $V[G]$, we have the extension $V[G_r]$ arising from coordinate $r$, and by the choice of $P_r$ it follows that $r$ is not in the mantle of $V[G_r]$, and hence also not in the mantle of $V[G]$. Thus, in $V[G]$, for each $r \in R$ we may find a ground $W_r^{V[G]}$ omitting $r$. Since we have assumed that the grounds of $V[G]$ are locally downward set directed, there is a single ground $W \subseteq V[G]$ omitting every $r \in R$. Since $y \in gM^V$, it follows that $y \in W$, but we have proved that $y$ has no well-ordering in $W$, contrary to $W \models ZFC$. 

The generic strong local DDG follows of course from the generic strong DDG, which we have mentioned holds trivially in $L$, for example, simply because $L$ is a ground below any ground of any forcing extension of $L$. In other words, $L$ is a solid generic bedrock. There are many other models having this feature, but we do not fully know the extent of the phenomenon.

**Question 50.** When does the universe have a solid generic bedrock? In other words, under what circumstances is the generic mantle also a ground model of the universe?

This phenomenon is not universal, in light of the following.

**Corollary 51.** Every model $V$ of ZFC has a class forcing extension $V[G]$ that satisfies the generic strong local DDG, but which has no bedrock and no generic bedrock.

**Proof.** This is a corollary to theorem 23. We first move to $\bigvee$ satisfying the CCA. Then, we perform an Easton support product $P$ adding a Cohen subset over $\bigvee$ to every regular cardinal $\lambda$ for which $2^\lambda = \lambda$. Suppose that $\bigvee[G]$ is the resulting extension, and that $\bigvee[G][g]$ is obtained by further set forcing over $\bigvee[G]$. If $W$ is a ground of $\bigvee[G][g]$, then since the sets in $\bigvee$ remain coded unboundedly in the GCH
pattern of $\mathcal{V}[G][g]$, it follows that $\mathcal{V} \subseteq W$. Since $\mathcal{V}[G][g]$ is obtained by set forcing over $W$, it follows that for $\lambda$ above the size of this forcing, since the generic sets added by $\mathcal{P}$ at stage $\lambda$ have all their initial segments in $\mathcal{V}$ and hence in $W$, that these generic objects must be already in $W$. It follows that $\mathcal{V}[G^\alpha] \subseteq W \subseteq \mathcal{V}[G][g]$ for some sufficiently large $\alpha$. Thus, once again the models $\mathcal{V}[G^\alpha]$ form a strictly descending sequence of grounds, which are dense below the generic grounds of $\mathcal{V}[G]$. It follows that the generic grounds are downward set-directed, but there is no minimal ground or generic ground, as desired. \hfill \qed

As before, this can be strengthened by the following corollary to theorem 24. Again, the model obtained there does not have the property that every set in it is set-generic over $V$, in contradistinction to the one constructed above.

**Corollary 52.** Every model of set theory has a class forcing extension of the form $L[r]$, where $r \subseteq \omega$, which satisfies the generic strong local DDG, but which has no bedrock or generic bedrock.

*Proof.* This is a consequence of theorem 24. Starting with an arbitrary model of set theory, by that theorem, there is a proper class forcing extension of the form $L[r]$ in which the bedrock axiom fails. Since this model is constructible from a set, we know by theorem 16 that the generic grounds are downward set-directed.

Now the original argument showing that $L[r]$ has no bedrock also shows that it has no generic bedrock. The point is that any forcing extension $L[r][g]$ of $L[r]$ has class many grounds, just like $L[r]$, as any ground of $L[r]$ is a ground of $L[r][g]$. But $L[r][g]$ is still constructible from a set, so by theorem 22, it follows that the bedrock axiom fails in $L[r][g]$. So there is no generic bedrock, since a generic bedrock would be a bedrock in some set-forcing extension. \hfill \qed

The argument of the previous proof can be expanded to show the following.

**Theorem 53.** If the universe is constructible from a set, then this is true throughout the generic multiverse. If in addition the bedrock axiom fails, then it fails throughout the generic multiverse.

*Proof.* We have to show that the properties in question are preserved when passing to set-forcing extensions and to grounds. If the universe is constructible from a set, then this is obviously true in forcing extensions. To see that it is true in grounds as well, suppose that $V = L[x]$ and $W$ is a ground of $V$. Let $\mathcal{P} \in W$ be a partial order for which $G$ is generic, such that $L[x] = W[G]$. Let $\tau$ be a name such that $x = \tau\mathcal{P}$. Let $\alpha \in W$ be a set of ordinals which codes the transitive closure $t = \text{TC}(\{\mathcal{P}, \tau\})$, and consider the ZFC model $L[\alpha]$. Notice that $L[\alpha] \subseteq W$ since $\alpha \in W$ and $L[\alpha][G] = L[x]$, since $x = \tau\mathcal{P} \in L[\alpha][G]$. Moreover, $G$ is $L[\alpha]$-generic, and so $L[\alpha] \subseteq W \subseteq L[\alpha][G]$ traps the ZFC model $W$ between $L[\alpha]$ and its forcing extension $L[\alpha][G] = L[x]$, and so it follows by fact 7 that $W$ is a set-forcing extension of $L[\alpha]$, and hence constructible from a set.

Let’s now assume that the universe is constructible from a set and the bedrock axiom fails. Clearly, the failure of the bedrock axiom is preserved when passing to grounds, since a bedrock of a ground would be a bedrock, by fact 7. This holds in general, without assuming that the universe is constructible from a set. The argument establishing corollary 52 shows that the failure of the bedrock axiom persists from models which are constructible from a set to their set-forcing extension. \hfill \qed
Let us say that a model having no bedrock model is *bottomless* and a model having no generic bedrock is *generically bottomless*.

We are of course interested more generally in the question of in what sense the generic multiverse itself may be downward directed, downward set-directed or locally downward set-directed. These questions are easy enough to formalize directly in the toy model approach, but in the full context of forcing over \( V \), the questions can present meta-mathematical difficulties for the general case, although they are also consequences of other axioms, such as the solid generic bedrock axiom, which are formalizable. At least part of the interest in such questions, of course, has to do with their relation to the following question.

**Question 54.** Does the generic mantle necessarily satisfy ZFC?

Although we have introduced the mantle and the generic mantle as distinct notions, we do not actually know that they are different. We have as yet no models in which the mantle differs from the generic mantle.

**Question 55.** Is it consistent with ZFC that the mantle is different from the generic mantle?

At the end of the next section, we shall see that all the previous questions have simple answers if the universe is constructible from a set.

### 4. The generic HOD

Let us now introduce the generic HOD, a concept generalizing the classical HOD in the same way that the generic mantle generalizes the mantle. We assume that the reader is familiar with the basic theory of HOD, the class of hereditarily ordinal definable sets (consult [Jec03] for a review).

**Definition 56.** The *generic* HOD of \( V \), denoted \( gHOD \), is the intersection of all the HODs of all set-forcing extensions of \( V \). That is, \( x \in gHOD \) if and only if \( x \in HOD^{V[G]} \) for all set-forcing extensions \( V[G] \). So
\[
gHOD = \{ x | \forall P \forces x \in HOD \}.
\]

The generic HOD has been referred to as the limit HOD, or \( \text{lim}_\omega \text{HOD} \) when it was introduced in [Fuc08, p. 298] as a special case of \( \text{lim}_\alpha \text{HOD} \). The following characterization was used there.

**Lemma 57.** \( gHOD = \{ x | \forall \alpha \forces_{\text{Coll}(\omega, \alpha)} x \in HOD \} = \bigcap_{\alpha < \infty} HOD^{\text{Coll}(\omega, \alpha)} \).

**Proof.** Only the first identity needs a proof. The inclusion from left to right is trivial. For the opposite direction, let \( x \in HOD^{\text{Coll}(\omega, \alpha)} \) for every \( \alpha \). Let \( P \) be an arbitrary poset, and let \( \alpha \) be at least the cardinality of \( P \). Then \( P \times \text{Coll}(\omega, \alpha) \) embeds densely into \( \text{Coll}(\omega, \alpha) \). Let \( G \times H \) be \( P \times \text{Coll}(\omega, \alpha) \)-generic over \( V \), and let \( H' \) be \( \text{Coll}(\omega, \alpha) \)-generic over \( V \) such that \( V[G][H] = V[H'] \). Then it follows from the homogeneity of \( \text{Coll}(\omega, \alpha) \), and from the fact that \( \text{Coll}(\omega, \alpha) \) is ordinal definable, that
\[
x \in \text{HOD}^{V[H']} = \text{HOD}^{V[G][H]} \subseteq \text{HOD}^{V[G]}.
\]
So \( P \) forces that \( x \in \text{HOD} \), and as \( P \) was arbitrary, it follows that \( x \in gHOD \), as desired. \( \square \)
The proof of the previous lemma also showed the following easy fact.

**Remark 58.** If $\alpha \leq \beta$, then $\text{HOD}^{V_{\text{Coll}(\omega, \beta)}} \subseteq \text{HOD}^{V_{\text{Coll}(\omega, \alpha)}}$.

**Theorem 59 ([Fuc08]).** In any model of set theory, $\text{gHOD}$ is a parameter-free uniformly first-order definable class, containing all ordinals, invariant by set forcing, and a model of ZFC.

**Proof.** The standard treatments of HOD show that the relation $x \in \text{HOD}$ is parameter-free uniformly first-order definable in any model of set theory. So definition 56 is a definition of $\text{gHOD}$ which has the desired form. It is easy to see that $\text{gHOD}$ is transitive and that it contains all ordinals.

To see that it is invariant by forcing, consider any set-forcing extension $V \subseteq V[G]$ by a poset $P$. If $\alpha$ is at least the cardinality of $P$ and $H$ is Coll($\omega, \alpha$)-generic over $V[G]$, then there is an $H'$ which is Coll($\omega, \alpha$)-generic over $V$ such that $V[H'] = V[G][H]$. So $(\text{HOD}^{V_{\text{Coll}(\omega, \alpha)}})^{V[G]} = \text{HOD}^{V[G][H]} = \text{HOD}^{V[H']} = \text{HOD}^{V_{\text{Coll}(\omega, \alpha)}}$. So by remark 58, it follows that

$$\text{gHOD}^{V[G]} = \left(\bigcap_{|P| \leq \beta} \text{HOD}^{V_{\text{Coll}(\omega, \beta)}}\right)^{V[G]} = \bigcap_{|P| \leq \beta} \text{HOD}^{V_{\text{Coll}(\omega, \beta)}}.$$

It follows immediately from corollary 20 that $\text{gHOD} \models \text{ZFC}$, as it is the intersection of the definable decreasing sequence $\langle \text{HOD}^{V_{\text{Coll}(\omega, \alpha)}} \mid \alpha < \infty \rangle$ of ZFC-models.

**Corollary 60.** The generic HOD is constant across the generic multiverse and is contained in the generic mantle. The classes exhibit the following inclusions:

$$\text{HOD} \quad \bigcup \quad \text{gHOD} \subseteq \text{gM} \subseteq M$$

**Proof.** Because theorem 59 shows that $\text{gHOD}$ is invariant by forcing, every model in the multiverse has the same $\text{gHOD}$. Thus, $\text{gHOD}$ is contained within the intersection of the multiverse, which is the generic mantle, and so $\text{gHOD} \subseteq \text{gM}$. The other inclusions $\text{gHOD} \subseteq \text{HOD}$ and $\text{gM} \subseteq M$ are immediate.

These basic inclusion relations will be separated by the theorems of the next section, except that the exact nature of the generic mantle remains somewhat unsettled, since we have been unable to separate it either from the generic HOD or from the mantle, although we do separate these latter two. Before moving on to those results, however, let us first prove the following theorem, which explains in part why proper class forcing will loom so large in our subsequent arguments.

**Theorem 61.** If the universe is constructible from a set, $V = L[a]$, then

$$\text{gHOD} = \text{gM} = M.$$

**Proof.** We know already that $\text{gHOD} \subseteq \text{gM} \subseteq M$, by corollary 60, so the only thing left to prove is that $M \subseteq \text{gHOD}$. But this follows immediately from theorem 16, which states among other things that the inner models of the form $\text{HOD}^{V_{\text{Coll}(\omega, \alpha)}}$ are dense in the grounds, together with lemma 57, which says that the generic HOD is the intersection of these models. □
5. Controlling the mantle and the generic mantle

We now prove our main theorems, which control the mantle and generic mantle of the target models, and also the HOD and generic HOD.

**Theorem 62.** Every model $V$ of ZFC has a class forcing extension $V[G]$ in which $V$ is the mantle, the generic HOD and the HOD.

$$V = M^V[G] = gM^V[G] = gHOD^V[G] = HOD^V[G]$$

**Proof.** Our strategy will be to perform class forcing $V \subseteq V[G]$ in such a way that the various forces acting on the mantles and HODs in $V[G]$ are perfectly balanced, in each case giving $V$ as the result. Pushing upward, expanding these classes up to $V$, we will force in such a way that every set in $V$ is coded explicitly into the continuum function of $V[G]$, thereby ensuring that each such set is in the mantle, the generic mantle, the HOD and the generic HOD. Pressing downward, holding these classes down to $V$, we will maintain certain factor and homogeneity properties on the forcing that ensure that no additional sets are added to the mantles and HODs.

For each ordinal $\alpha$, let $\delta_\alpha$ be the $\alpha^{th}$ cardinal of the form $\lambda^+$, where $\lambda$ is a strong limit cardinal, but not a limit of strong limit cardinals. More precisely, $\delta_\alpha = \beth^+(\alpha+1)$, the successor of $\lambda = \beth^+(\alpha+1)$. The $\delta_\alpha$ will be the cardinals at which we code information, one bit each time, by forcing either the GCH or its failure at $\delta_\alpha$. Every $\delta_\alpha$ is a regular uncountable cardinal, and the sequence of $\delta_\alpha$ is increasing, conveniently spaced apart in order to avoid interference between the various levels of coding. We could easily modify the argument to allow the $\delta_\alpha$ to be spaced more closely together—and if the GCH holds in $V$, we could actually code at every successor cardinal—but spacing the cardinals more distantly as we have seems to produce the most transparent general argument. Since $\lambda$ is not a limit of strong limit cardinals, there is a largest strong limit cardinal $\gamma$ below $\lambda$, namely $\gamma = \beth^\omega\cdot\alpha$, and furthermore, $\delta_\beta \leq \gamma^+$ for all $\beta < \alpha$. Note also that $\alpha \leq \beth^\omega\cdot\alpha = \gamma$, and so $\alpha < \gamma^+ < \lambda < \delta_\alpha$. Let $Q_\alpha$ be the forcing that generically chooses whether to force the GCH or its negation at $\delta_\alpha$. To force the GCH at $\delta_\alpha$, we use the forcing $\text{Add}(\delta_\alpha^+,1)$ to add a Cohen subset to $\delta_\alpha^+$, which is equivalent to the canonical forcing to collapse $2^{\delta_\alpha}$ to $\delta_\alpha^+$. Forcing the failure of the GCH at $\delta_\alpha$, on the other hand, requires a little care in the case that the GCH fails below $\delta_\alpha$ in $V$, for adding generic subsets to $\delta_\alpha$ may collapse cardinals above $\delta_\alpha$, and even adding $(\delta_\alpha^{++})^V$ many subsets to $\delta_\alpha$ may not in general suffice to ensure that the GCH fails at $\delta_\alpha$ in the extension. Since the standard $\Delta$-system argument (see [Kun80, Lemma 6.10]) establishes that $\text{Add}(\delta_\alpha,\theta)$ is $(2^{\left<\delta_\alpha\right>^{++}})^\omega\cdot\alpha$-c.c. for any ordinal $\theta$, however, it does in general suffice to add $(2^{\left<\delta_\alpha\right>^{++}})^V$ many subsets to $\delta_\alpha$. Thus, we take $Q_\alpha$ as the side-by-side forcing of these two alternatives, or in the terminology of [Ham00], the poset $Q_\alpha$ is the lottery sum $\text{Add}(\delta_\alpha^+,1) \oplus \text{Add}(\delta_\alpha,\left<\delta_\alpha\right>^{++})^V$.\footnote{More generally, for any family $A$ of forcing notions, the lottery sum $\oplus A$ is defined to be $\{ (P,p) \mid P \in A \} \cup \{1\}$, ordered with $1$ above everything and otherwise $(P,p) \leq (Q,q)$ if $P = Q$ and $p \leq q$. The generic filter must in effect choose a single $P \in A$ and force with it. For two posets, we use infix notation: $P \oplus Q = \oplus \{ P, Q \}.$} Conditions opting for the first poset force the GCH at $\delta_\alpha$ and those opting for the second force its failure. Note that $Q_\alpha$ is $\delta_\alpha^{++}$-closed and has size $(2^{\left<\delta_\alpha\right>^{++}})^V$, which is strictly less than the next strong limit above $\delta_\alpha$, which is itself less than $\delta_{\alpha+1}$.

Let $\mathbb{P}$ be the class forcing product $\prod_\alpha Q_\alpha$, with set support. That is, conditions in $\mathbb{P}$ are set functions $p$, with $\text{dom}(p) \subseteq \text{ORD}$ and $p(\alpha) \in Q_\alpha$, ordered
by extension of the domain and strengthening in each coordinate. (In particular, we do not use Easton support, which would not work here, because it would create new unwanted definable subsets of the inaccessible cardinals, if any.) Suppose that $G \subseteq P$ is $V$-generic and consider the model $V[G]$. The class forcing $P$ factors at every ordinal $\alpha$ as $P_\alpha \times P_\alpha^\alpha$, where $P_\alpha = \prod_{\alpha' < \alpha} Q_{\alpha'}$ and $P_\alpha^\alpha = \prod_{\alpha' \geq \alpha} Q_{\alpha'}$, where again the products have set support, which for $P_\alpha$ means full support. We claim that $|P_\alpha| < \delta_\alpha$ for every $\alpha$. To see this, suppose that $\delta_\alpha = \lambda^+$, where $\lambda$ is a strong limit cardinal, but not a limit of strong limit cardinals. We have argued that $\delta_\beta \leq \gamma^+$ for every $\beta < \alpha$, where $\gamma$ is the largest strong limit cardinal below $\lambda$. It follows that $|Q_\beta| \leq (2^{\gamma^+})^{++}$, and so $P_\alpha$ is the product of $\alpha$ many posets of at most this size. Since $\alpha \leq \gamma^+$, this implies that $P_\alpha$ has size at most $((2^{\gamma^+})^{++})^{++}$, and since $\lambda$ is a strong limit cardinal, this is less than $\lambda$ and hence less than $\delta_\alpha$, as desired. Combining this with the fact that the tail forcing $P_\alpha^\alpha$ is $<\delta_\alpha$-closed, it follows that every set in $V[G]$ is added by some large enough initial factor $P_\alpha$. And using this, the standard arguments show that $V[G]$ satisfies ZFC. (For example, in the terminology of [Rei06], this is a progressively closed product, and these always preserve ZFC.)

Let us verify that indeed the various levels of GCH coding in our forcing do not interfere with each other. For any ordinal $\alpha$, factor $P$ as $P_\alpha \times Q_\alpha \times P_\alpha^{++}$. The tail forcing $P^{++}$ is $<\delta_{\alpha+1}$-closed and therefore does not affect the GCH at $\delta_\alpha$. The initial factor $P_\alpha$ has size less than $\delta_\alpha$, and therefore does not affect the GCH at $\delta_\alpha$. So the question whether the GCH holds at $\delta_\alpha$ in $V[G]$ is determined by what $G$ does on $Q_\alpha$. In other words, the overall GCH pattern in $V[G]$ on the cardinals $\delta_\alpha$ is determined in accordance with the choices that $G$ makes in the individual lotteries at each coordinate. A similar argument shows that the forcing $P$ preserves all strong limit cardinals and creates no new strong limit cardinals. Thus, the class \{ $\delta_\alpha \mid \alpha \in \text{ORD}$ \} remains definable in $V[G]$. Let us now make the key observations about $V[G]$. First, pushing upward, we claim that every set of ordinals in $V$ is coded into the GCH pattern of $V[G]$. Suppose that $x$ is a set of ordinals in $V$ and $p$ is any condition in $P$. Choose ordinals $\beta$ and $\xi$ with $x \subseteq \beta$ and $\text{dom}(p) \subseteq \xi$. Since $x$ is a set and $P$ uses set support, we may extend $p$ to a stronger condition $q \leq p$ with $\text{dom}(q) = \text{dom}(p) \cup \{ \xi, \xi + \beta \}$, where $q$ opts on the interval $[\xi, \xi + \beta]$ to force the GCH or its negation according to the pattern determined by $x$. That is, we build $q$ so that for every $\alpha < \beta$, if $\alpha \in x$, then $q(\xi + \alpha)$ opts to force the GCH at $\delta_{\xi+\alpha}$, and if $\alpha \notin x$, then it opts for its failure. Since we have argued that there is no interference between the levels of coding, the condition $q$ forces that the GCH pattern in $V[G]$ for those values of $\delta_{\xi+\alpha}$ is exactly the same as the pattern of $x$ on $\beta$. Thus, it is dense that $x$ is coded in this way, and so generically every set in $V$ will be coded into the GCH pattern of $V[G]$, since every set in $V$ is coded by a set of ordinals in $V$. Since we have mentioned that the class \{ $\delta_\alpha \mid \alpha \in \text{ORD}$ \} is definable in $V[G]$, we may conclude immediately that every set in $V$ is ordinal definable in $V[G]$. For the generic HOD, we consider the set-forcing extensions of $V[G]$. Suppose that $V[G][h]$ is obtained by further forcing $h \subseteq Q \in V[G]$. Since the continuum function of $V[G][h]$ and $V[G]$ agree above $|Q|$, it follows that $V[G]$ and $V[G][h]$ have the same strong limit cardinals and the same GCH patterns above $|Q|$. This implies that a tail segment of \{ $\delta_\alpha \mid \alpha \in \text{ORD}$ \} remains definable in $V[G][h]$, and the GCH pattern on this segment is the same in $V[G][h]$ as in $V[G]$. Since every set of ordinals $x$ in $V$
was coded into the the GCH pattern of $V[G]$ on the cardinals \{ $\delta_\alpha \mid \alpha \in \text{ORD}$ \}, a simple padding argument shows that this implies that every set in $V$ is in fact coded unboundedly often into the GCH pattern of $V[G]$ on these cardinals. So we conclude that every set in $V$ remains ordinal definable in the extension $V[G][h]$. Since the forcing $h$ was arbitrary, we conclude $V \subseteq g\text{HOD}^V[G]$, and consequently also $V \subseteq g\text{HOD} \subseteq gM \subseteq M$.

Conversely, we now argue that the mantle $M$ of $V[G]$ is contained in $V$. For any ordinal $\alpha$, factor the forcing at $\alpha$ as $\mathbb{P} = \mathbb{P}_\alpha \times \mathbb{P}^\alpha$. The generic filter $G$ similarly factors as $G_\alpha \times G^\alpha$. Since $\mathbb{P}_\alpha$ is set forcing in $V$, it follows that the tail extension $V[G^\alpha]$ is a ground of $V[G]$, and so the mantle of $V[G]$ is contained within every $V[G^\alpha]$. Since $\mathbb{P}^\alpha$ is $<\delta_\alpha$-closed, it follows in particular that $V_\alpha^{V[G^\alpha]} = V_\alpha$, and so $\bigcap V[G^\alpha] = V$. Altogether, we have established $V \subseteq g\text{HOD} \subseteq gM \subseteq M \subseteq V$, and so all these are equal, as we claimed. Finally, let us consider $\text{HOD}^V[G]$. We have argued that $V \subseteq \text{HOD}^V[G]$, and it remains for us to prove the converse inclusion. For this, in order to control $\text{HOD}^V[G]$, it would be expected to appeal to homogeneity properties of the forcing $\mathbb{P}$. Unfortunately, the forcing $\mathbb{P}$ is not almost homogeneous, because different conditions can make fundamentally different choices in the lotteries about how the forcing will proceed. Nevertheless, we claim that there is sufficient latent homogeneity in the forcing for an argument to succeed. In order to show $\text{HOD}^V[G] \subseteq V$, it suffices to show that $\text{HOD}^V[G]$ is contained in every tail extension $V[G^\alpha]$, as the intersection over these is $V$, as we argued. Consider $V[G]$ as a forcing extension of the tail extension $V[G^\alpha]$ by the initial forcing $G_\alpha \subseteq \mathbb{P}_\alpha$. Although $\mathbb{P}_\alpha$ is not almost homogenous, it is densely almost homogeneous, meaning that there is a dense set of conditions $q$ such that the lower cone $\mathbb{P}_\alpha \upharpoonright q$ is almost homogeneous. The point is simply that because we have used full support, rather than Easton support, we may extend any condition in $\mathbb{P}_\alpha$ to a condition with support $\alpha$. Furthermore, we may extend to a condition that makes a definite selection in each of the lotteries before stage $\alpha$ as to which of the two posets should be used in that coordinate. Since each of these individual posets is almost homogenous, the lower cone $\mathbb{P}_\alpha \upharpoonright q$ is the full product of almost homogeneous forcing, and consequently is itself almost homogeneous. Thus, by genericity, there is some $q \in G_\alpha$ such that $\mathbb{P}_\alpha \upharpoonright q$ is almost homogenous. It follows that every ordinal definable set of ordinals added by this forcing is definable in the ground model $V[G^\alpha]$ from ordinal parameters and the poset $\mathbb{P}_\alpha \upharpoonright q$, used as an additional parameter. So we have proved that $\text{HOD}^V[G]$ is included in every tail extension $V[G^\alpha]$, and thus in $V$. So we have proved all the desired equalities $V = M^V[G] = gM^V[G] = g\text{HOD}^V[G] = \text{HOD}^V[G]$. \hfill \square

**Theorem 63.** Every model $V$ of ZFC has a class forcing extension $V[G]$ in which $V$ is the mantle, the generic mantle and the generic HOD, but $V[G]$ is the HOD.

$$V = M^V[G] = gM^V[G] = g\text{HOD}^V[G], \quad \text{but} \quad \text{HOD}^V[G] = V[G].$$

**Proof.** For this theorem, we must balance the various forces on the classes differently, to keep the mantles and the generic HOD low, while allowing $\text{HOD}^V[G]$ to expand. Pushing the classes up at least to $V$, our strategy will be once again to force that every set of ordinals in $V$ is coded unboundedly into the continuum function of $V[G]$. In order to push $\text{HOD}^V[G]$ fully up to $V[G]$, however, we will force that every new set of ordinals in $V[G]$ is coded into the continuum function, but
these new sets will be coded each time only boundedly often. This makes these sets ordinal definable in \( V[G] \), while allowing the factor argument of theorem 62 to hold down the mantle to \( V \) and consequently also the generic mantle and generic HOD. The subtle effect is that the new sets become ordinal definable, but only temporarily so, for further forcing can erase the bounded coding and make them drop out of HOD.

The essential component, for any regular cardinal \( \kappa \), is the \textit{self-encoding forcing} at \( \kappa \), which we now describe. This is the forcing iteration \( Q \) of length \( \omega \) that begins by adding a Cohen subset of \( \kappa \), and then proceeds in each subsequent stage to code the generic filter from the prior stage into the GCH pattern at the next block of cardinals. All coding will take place in the interval \( I = [\kappa, \lambda] \), where \( \lambda = \beth_\kappa \) is the least beth fixed point above \( \kappa \) (and the forcing will preserve all beth fixed points).

The end result in the corresponding extension \( V[G] \) is that the initial Cohen subset of \( \kappa \) and the entire generic filter \( G \) is coded into the GCH pattern at cardinals in \( I \). To be more precise, the forcing begins at stage 0 with \( Q_0 = \text{Add}(\kappa, 1) \), adding a Cohen subset \( g_0 \subseteq \kappa_0 = \kappa \). The stage 1 forcing \( Q_1 \) will code \( g_0 \) into the GCH pattern at the next \( \kappa \) many cardinals. For this, we first force if necessary to ensure that the GCH holds at the next \( \kappa \) many cardinals (this may collapse cardinals, but there is no need to exceed or even reach the beth fixed point \( \lambda = \beth_\kappa \)), and then perform suitable Easton forcing at these cardinals, so that for the next \( \kappa \) many cardinals \( \nu \) above \( \kappa \) in the corresponding extension \( V[g_0 * g_1] \), we have either \( 2^\nu = \nu^+ \) or \( 2^\nu = \nu^{++} \), according to whether the corresponding ordinal is in \( g_0 \). In order to continue the iteration, we use a canonical pairing function on ordinals in order to view \( g_1 \) as a subset of \( \kappa_1 \), the supremum of the next \( \kappa \) many surviving cardinals above \( \kappa \). In general, the generic filter for the stage \( n \) forcing \( Q_n \) is determined by a subset \( g_n \subseteq \kappa_n \), and the stage \( n + 1 \) forcing \( Q_{n+1} \) first forces if necessary the GCH to hold at the next \( \kappa_n \) many cardinals, and then uses Easton forcing to code \( g_n \) into the GCH pattern on those cardinals. There is sufficient room to carry out each stage of forcing below the next beth fixed point \( \lambda = \beth_\kappa \), and it is not difficult to see that \( \lambda = \sup_n \kappa_n \). The entire iteration \( Q \) consequently has size \( \lambda^\omega \). The end result is that if \( G \subseteq Q \) is V-generic, then \( G \) is coded explicitly into the GCH pattern of \( V[G] \) on the interval \( I \). Note that \( Q \) is \( < \kappa \)-closed and does not affect the continuum function on cardinals outside the interval \( I \) provided the GCH holds on \( I \) in \( V \). In the event that the GCH fails in \( V \) the situation is not much worse, as \( Q \) does not affect the continuum function outside the interval \( (\gamma, \lambda^\omega) \) where \( 2^\gamma \leq \kappa \).

We now assemble these components into the overall class forcing. For every ordinal \( \alpha \), let \( \kappa_\alpha \) be the \( \alpha \)th cardinal of the form \( (2^\lambda)^+ \), where \( \lambda = \beth_\alpha \) is a beth fixed point. Let \( Q_\alpha \) be the self-encoding forcing at \( \kappa_\alpha \), which adds a generic filter that encodes itself into the GCH pattern on the interval \( I_\alpha = [\kappa_\alpha, \lambda_\alpha] \), where \( \lambda_\alpha \) is the next beth fixed point above \( \kappa_\alpha \). Note that these intervals are disjoint. Let \( P = \prod_\alpha Q_\alpha \) be the set support class product of these posets. As in theorem 62, we may for any ordinal \( \alpha \) factor this forcing as \( P_\alpha \times P^\alpha \), where \( P_\alpha = \prod_{\beta < \alpha} Q_\beta \) and \( P^\alpha = \prod_{\beta \geq \alpha} Q_\beta \), using again set support in these products, which for \( P_\alpha \) means full support. The initial factor \( P_\alpha \) is the product of posets \( Q_\beta \) of size \( \lambda_\beta^\gamma \), for \( \beta < \alpha \), which therefore has size at most \( \lambda^\omega \leq 2^\lambda \), where \( \lambda = \sup_{\beta < \alpha} \lambda_\beta \). This is strictly less than \( \kappa_\alpha \). In summary, we have established the convenient factor properties that \( |P_\alpha| < \kappa_\alpha \) and \( P^\alpha \) is \( \kappa_\alpha \)-closed. The usual arguments now show that if \( G \subseteq P \) is \( V \)-generic, then \( V[G] \) satisfies ZFC and every set in \( V[G] \) is added by some stage.
Also, for any ordinal $\alpha$, we may factor the forcing as $P_\alpha \times Q_\alpha \times P_\alpha^{+1}$. Because the final factor $P^{+1}$ is $<\kappa_\omega+1$-closed and the initial factor $P_\alpha$ has size less than $\kappa_\alpha$, neither of these affects the GCH pattern on the interval $I_\alpha = [\kappa_\alpha, \lambda_\alpha)$. Thus, the GCH pattern on $I_\alpha$ in $V[G]$ is determined by what the generic filter $G$ does on the $\alpha$th coordinate $Q_\alpha$.

We now observe that $V[G]$ exhibits the desired coding features. If $x$ is any set of ordinals in $V$, then in any coordinate stage $\alpha$ of forcing above $\text{sup}(x)$, it is dense for $x$ to appear as an interval in the generic object added at the very first stage of forcing in $Q_\alpha$. The subsequent stages of forcing in $Q_\alpha$ will therefore have the effect of coding $x$ into the GCH pattern in $I_\alpha$. Thus, the set $x$ is coded into the GCH pattern of $V[G]$, and is consequently ordinal definable there. Since $x$ is coded unboundedly often in this way, $x$ will remain ordinal definable in any set-forcing extension of $V[G]$, because any such extension $V[G][h]$ has the same GCH pattern as $V[G]$ above the size of the forcing $h$. Thus, $x$ is ordinal definable in any such $V[G][h]$, and so $V \subseteq g\text{HOD}^{V[G]}$. We know in general that $g\text{HOD} \subseteq gM \subseteq M$. In order to complete the cycle, we now argue $M \subseteq V$. Observe that the tail forcing extension $V[G^{\alpha}]$ is a ground of $V[G]$, because the initial factor $P_\alpha$ is set forcing. Thus, the mantle of $V[G]$ is contained within the intersection of all $V[G^{\alpha}]$. But as before, the intersection of all of the tail extensions $V[G^{\alpha}]$ is simply $V$, by the increasing closedness of $P^{\alpha}$, and so the mantle of $V[G]$ is contained in $V$. This establishes that $V \subseteq g\text{HOD}^{V[G]} \subseteq gM^{V[G]} \subseteq M^{V[G]} \subseteq V$, and hence all are equal, as desired.

It remains to compute $\text{HOD}^{V[G]}$. We have observed that every set in $V[G]$ is added by some initial factor forcing $P_\alpha$ for some ordinal $\alpha$, and therefore every object in $V[G]$ has the form $\tau_{G_\alpha}$, for some ordinal $\alpha$ and some $P_\alpha$-name $\tau$ in $V$. Since we have already established that $V \subseteq \text{HOD}^{V[G]}$, it follows that the name $\tau$ is ordinal definable in $V[G]$. Furthermore, for every $\beta < \alpha$, the generic filter $G(\beta) \subseteq Q_\beta$ added at stage $\beta$ is coded into the continuum function on the interval $I_\beta$, and in $V[G]$ we may definably assemble these filters into the generic filter $G_\alpha$ on the product $P_\alpha = \prod_{\beta < \alpha} Q_\beta$. Thus, $G_\alpha$ is also ordinal definable in $V[G]$. So $\tau_{G_\alpha}$ is ordinal definable in $V[G]$ and so $\text{HOD}^{V[G]} = V[G]$, as desired. \hfill \Box

In the previous theorem, we have kept the mantles low, while pushing up the HOD. Next, in contrast, we keep the HODs low, while pushing up the mantle.

**Theorem 64.** The constructible universe $L$ has a class-forcing extension $L[G]$ in which $L$ is the HOD and generic HOD, but $L[G]$ is the mantle.

$$L = \text{HOD}^{L[G]} = g\text{HOD}^{L[G]}, \quad \text{but} \quad M^{L[G]} = L[G].$$

**Proof.** This theorem is a consequence of the main result of [HRW08]. In that article, Hamkins, Reitz and Woodin showed that if one performs the Easton support Silver iteration over $L$, successively adding a Cohen subset to each regular cardinal, then the resulting model $L[G]$ satisfies the ground axiom. In other words, $L[G]$ has no nontrivial grounds, and therefore is its own mantle. Since the forcing is almost homogeneous and ordinal-definable, it creates no new ordinal-definable sets, and so $\text{HOD}^{L[G]} = L$. Since $g\text{HOD} \subseteq \text{HOD}$, it follows that $g\text{HOD}^{L[G]} = L$ as well. \hfill \Box

Unfortunately, we have not yet managed to determine the generic mantle of $L[G]$. 
Question 65. What is the generic mantle of the Hamkins-Reitz-Woodin model $L[G]$?

In order to prove a generalized version of theorem 64, we shall need the following concepts and results.

Definition 66 ([Ham03]). Suppose that $W \subseteq V$ are both transitive models of (a suitable small fragment of) ZFC and $\delta$ is a cardinal in $V$.

1. The extension $W \subseteq V$ exhibits the $\delta$ cover property if for every $A \in V$ with $A \subseteq W$ and $|A|^V < \delta$, there is $B \in W$ such that $A \subseteq B$ and $|B|^W < \delta$.
2. The extension $W \subseteq V$ exhibits the $\delta$ approximation property if whenever $A \in V$ with $A \subseteq W$ and $A \cap B \in W$ for all $B \in W$ with $|B|^W < \delta$, then $A \in W$.

Lemma 67 ([Ham03]). If $V \subseteq V[G]$ is a forcing extension by forcing of the form $\mathbb{Q}_1 \ast \mathbb{Q}_2$, where $\mathbb{Q}_1$ is nontrivial and \#\mathbb{Q}_1 “strategically closed”, then $V \subseteq V[G]$ satisfies the $\delta$ cover and $\delta$ approximation properties for $\delta = |\mathbb{Q}_1|^+$.

Lemma 68 (Hamkins, Laver [Lav07]). Suppose that $W$, $W'$ and $V$ are transitive models of ZFC and $\delta$ is a regular cardinal of $V$. Suppose further that $W \subseteq V$ and $W' \subseteq V$, that these extensions both exhibit the $\delta$ cover and approximation properties, and that $\mathcal{P}(\delta)^W = \mathcal{P}(\delta)^W$ and $(\delta^+)^W = (\delta^+)W' = (\delta^+)^V$. Then $W = W'$.

An alternative proof of lemma 67 is provided by [Mit06] (see also [HJ10, Lemma 12]). In lemma 67, the second factor $\mathbb{Q}_2$ may be trivial. Laver’s original version of lemma 68 applied only to small forcing, i.e., assumed that $W$ and $W'$ both are grounds of $V$ by a forcing of size less than $\delta$, which Hamkins improved to extensions with the $\delta$ cover and approximation properties. An analogous version of the lemma for small forcing was proved independently by Woodin in [Woo04a, Lemma 21].

Theorem 69. Every model $V$ of ZFC has a class forcing extension $V[G]$ in which $V$ is the HOD and generic HOD, but $V[G]$ is the mantle.

$$V = \text{HOD}^{V[G]} = \text{gHOD}^{V[G]}, \quad \text{but} \quad M^{V[G]} = V[G].$$

Proof. For this theorem, we adapt the main argument and result of [HRW08]. The forcing takes place in two steps. First, we perform the forcing of theorem 62, coding into the GCH pattern on the cardinals $\mathbb{Q}_a = \prod_{\alpha \in \omega^+} Q_\alpha$, where $Q_\alpha$ generically chooses whether to force the GCH at $\alpha$ or its failure. If $V[H]$ is the resulting forcing extension, then the argument of theorem 62 shows that $V = \text{HOD}^{V[H]} = \text{gHOD}^{V[H]} = M^{V[H]} = gM^{V[H]}$.

The second step is to perform an Easton-support Silver iteration $\mathbb{P}$ over $V[H]$, a class length forcing iteration that at stage $\alpha$ adds a Cohen subset to $(2^{\omega_0})^+$. These cardinals lay conveniently within the interval $(\delta_\alpha, \delta_{\alpha+1})$ between the successive coding points of the first step. Suppose that $K \subseteq \mathbb{P}$ is $V[H]$-generic, and we consider the final extension $V[H][K]$. Because the Silver iteration does not affect the GCH coding at the various $\delta_\alpha$ and preserves all strong limit cardinals and therefore the definability of the class $\{\delta_\alpha | \alpha \in \text{ORD}\}$, it follows that every set in $V$ is coded arbitrarily highly in the continuum function of $V[H * K]$. This coding survives into set generic extensions of $V[H * K]$ and their HODs, and so $V \subseteq g\text{HOD}^{V[H * K]}$. Conversely, the Silver iteration $\mathbb{P}$ is almost homogeneous and ordinal definable, and so $\text{HOD}^{V[H * K]} \subseteq \text{HOD}^{V[H]} = V$. We conclude $V = \text{HOD}^{V[H * K]} = \text{gHOD}^{V[H * K]}$. 
We claim now $V[H*K]$ has no nontrivial grounds, and is therefore its own mantle. In other words, $V[H*K]$ satisfies the ground axiom. This part of the argument follows the main method and result of [HRW08]. Let us suppose towards a contradiction that $W$ is a nontrivial ground, so that $V[H*K] = W[h]$ for some nontrivial forcing $h \subseteq Q \in W$ over $W$. We may assume that conditions in $Q$ are ordinals less than $|Q|^W$. Since this is set forcing, the model $W$ has the same continuum function as $W[h]$ on cardinals above $|Q|^W$. In particular, these models eventually have the same strong limit cardinals, and so a final segment of the class \{ $\delta_\alpha \mid \alpha \in \ORD$ \} is definable in $W$. Because the sets of ordinals in $V$ are all coded unboundedly often into the GCH pattern of $V[H*K] = W[h]$ on these cardinals, it follows that they are also coded in $W$ and so $V \subseteq W$. Let $\lambda$ be the least strong limit above $|Q|^W$, so that $\lambda^+ = \delta_\alpha$ for some ordinal $\alpha$. Factor both forcing notions at $\alpha$, resulting in $V[H*K] = V[H_1][H_2][K_1][K_2]$, where $H_1$ is the part of $H$ at coordinates $\beta < \alpha$ and $H_2$ is the part of $H$ at coordinates $\beta \geq \alpha$, and $K_1 \subseteq \mathbb{P}_\alpha$ is the part of $K$ below stage $\alpha$ and $K_2 \subseteq \mathbb{P}_{\alpha,\alpha}$ is the tail part of the $K$ iteration from stage $\alpha$ onwards. Thus, we have $V[H_1][H_2][K_1][K_2] = W[h]$. Note that the forcing adding $H_1$ and $K_1$ has size strictly less than $\lambda$. Let $\delta$ be a regular cardinal above the size of these initial factors and also $|Q|^W$, but less than $\lambda$. Since the forcing that adds $H_2$ is $<\delta$-closed in $V$, every $\delta$-small subset of $H_2$ in $V[H][K]$ is covered by a condition in $V$. To see this, let $x \subseteq H_2$ have size less than $\delta$ in $V[H][K]$. By the closedness of the last factor, $x \in V[H][K_1] = V[H_2][H_3][K_1]$. Since $H_1*K_1$ is generic for forcing of size less than $\delta$ in $V[H_2]$, we can conclude by the $\delta$-cover property (see lemma 67) that there is $x' \in V[H_2]$ such that $x \subseteq x'$ and $x'$ has size less than $\delta$ in $V[H_2]$. Since $H_2$ is a class in $V[H_2]$, we may pick $x'$ in such a way that $x' \subseteq H_2$. Now it follows by the $<\delta$-closedness of $\mathbb{P}_2$ in $V$ that $x' \in V$. And by genericity of $H_2$, there is a common strengthening of all the conditions in $x'$ in $V$. Let’s call it a master condition for $x$. Since $V \subseteq W$, such conditions are also in $W$. Thus, every $\delta$-small approximation $H_2 \cap B$ to $H_2$, with $B \in W$ of size less than $\delta$ in $W$, will be the set of conditions in $B$ which are weaker than a master condition for $H_2 \cap B$, and will therefore itself be in $W$. Since the forcing adding $h$ over $W$ is small relative to $\delta$ and therefore exhibits the $\delta$-approximation property by lemma 67, it follows that $H_2$ is amenable to $W$, and hence that $V[H_2] \subseteq W$. In $W$, let $A \subseteq \lambda$ code $(2^{<\lambda})^W$, as well as $\mathbb{Q}$-names $H_1, K_1 \in W$ such that $H_1 = (\dot{H}_1)_h$ and $K_1 = (\dot{K}_1)_h$; for example, we could simply ensure that $H_\mathbb{Q}^W \in L[A]$. Since $A \in W$, it follows that $V[H_2][A] \subseteq W$. Since $A$ codes all bounded subsets of $\lambda$ in $W$, we have $\dot{Q} \in V[H_2][A]$. The filter $h$ is $V[H_2][A]$-generic for $Q$, and we may consider the forcing extension $V[H_2][A][h]$. By the choice of $A$, this model has the names $\dot{H}_1$ and $\dot{K}_1$, allowing it to build $H_1$ and $K_1$ using $h$, and so $V[H][K_1] \subseteq V[H_2][A][h]$. Conversely, the objects $A$ and $h$ could not have been added by the tail forcing $K_2$ over $V[H][K_1]$, since this forcing is $\leq \lambda$-closed over $V[H][K_1]$, and so $V[H_2][A][h] \subseteq V[H][K_1]$. Consequently, $V[H_2][A][h] = V[H][K_1]$. We may therefore view the model $V[H*K]$ as having arisen by forcing over $V[H_2][A]$, adding $h*K_2$, since $V[H_2][A][h*K_2] = V[H_2][A][h][K_2] = V[H][K_1][K_2] = V[H*K]$. Alternatively, we may view $V[H*K] = W[h]$ as having arisen by forcing over $W$, adding $h$. Each of these extensions exhibits the $\delta$-approximation and cover properties. Furthermore, since $P(\delta)^W = P(\delta)^{V[H_2][A]}$ and the cardinal successor of $\delta$ is the same in each of the models at hand, it follows by lemma 68 that $W = V[H_2][A]$. Thus, $V[h] = V[H_2][A][h]$, which we already established was $V[H][K_1]$, ...
contrary to our assumption that $W[h] = V[H][K]$. So there can be no such ground $W$. □

**Theorem 70.** Every model $V$ of ZFC has a class forcing extension $V[G]$ in which $V[G]$ is its own mantle, generic mantle, generic HOD and HOD.

$$V[G] = M^V[G] = gM^V[G] = g\text{HOD}^V[G] = \text{HOD}^V[G]$$

*Proof.* The main result of [Rei06], using ideas of [McA71], shows that every model $V$ of ZFC has a class forcing extension $V[G]$ in which every set of ordinals is coded unboundedly often into the continuum function. (This assertion was called the continuum coding axiom.) In this case, it is easy to see that $V[G] = M^V[G] = gM^V[G] = g\text{HOD}^V[G] = \text{HOD}^V[G]$, since these sets remains coded in this way in all set-forcing extensions and grounds of such extensions by set forcing. □

**Remark 71.** We have chosen to use GCH coding in the arguments above. However, the arguments are easily modified to accommodate other methods of coding, which would be consistent with GCH in the target model. For example, coding via $\Diamond^\kappa$, in the style of [BT, BT09], seems perfectly acceptable. The result would be that one could add $V[G] \models \text{GCH}$ to each of theorems 62, 63, 69.

6. **Inner mantles and the outer core**

The mantle $M$ of the universe $V$ arises by brushing away the outermost layers of forcing like so much accumulated dust and sand to reveal the underlying ancient structure, the mantle, on which these layers rest. If the mantle is itself a model of ZFC, then it makes sense to penetrate still deeper, computing the mantle of the mantle and so on, revealing still more ancient layers, and one naturally desires to iterate the process. We begin easily enough with $M^0 = V$ and then recursively define the successive inner mantles by $M^{n+1} = M^M^n$, that is, $M^{n+1}$ is the mantle of $M^n$, provided that this is a model of ZFC. Thus, $M^1$ is the mantle and $M^2$ is the mantle of the mantle, and so on. One would naturally expect to continue this recursion transfinitely with an intersection $M^\omega = \bigcap_{n<\omega} M^n$ at $\omega$, but here an interesting and subtle metamathematical obstacle rises up, preventing a simple success. The issue is that although we have provided definitions of the mantle $M^1$ and of the mantle-of-the-mantle $M^2$ and so on, these definitions become increasingly complex as the procedure is iterated, and an observant reader will notice that our recursive definition does not actually provide a definition of the $n^{\text{th}}$ mantle $M^n$ that is uniform in $n$. Instead, it is a recursion that takes place in the meta-theory rather than the object theory, and so on this definition we may legitimately refer to the $n^{\text{th}}$ mantle $M^n$ only for meta-theoretic natural numbers $n$. In particular, it does not provide a uniform definition of the inner mantles $M^n$, and consequently with it we seem unable to perform the intersection $\bigcap_n M^n$ in a definable manner.

Exactly the same issue arises when one attempts to iterate the class HOD, by considering the HOD-of-HOD and so on in the iterated HOD models. A 1974 result of Harrington appearing in [Zad83, section 7], with related work in [McA74], shows that it is relatively consistent with Gödel-Bernays set theory that $\text{HOD}^n$ exists for each $n < \omega$ but the intersection $\text{HOD}^\omega = \bigcap_n \text{HOD}^n$ is not a class. There simply is in general no uniform definition of the classes $\text{HOD}^n$. We expect an analogous result for the iterated mantles $M^n$, and this is a current focus of study for us.
Nevertheless, some models of set theory have a special structure that allows them to enjoy a uniform definition of these classes, and in these models we may continue the iteration transfinitely. Following the treatment of the iterated HOD\(^\alpha\) and gHOD\(^\alpha\) of [HKP], and working in Gödel-Bernays set theory, we define that a GB class \(\bar{\mathcal{M}}\) is a uniform presentation of the inner mantles \(\mathcal{M}^\alpha\) for \(\alpha < \eta\) if \(\bar{\mathcal{M}} \subseteq \{(x, \alpha) \mid \alpha < \eta\}\) and the slices \(\bar{\mathcal{M}}^\alpha = \{x \mid (x, \alpha) \in \bar{\mathcal{M}}\}\) for \(\alpha < \eta\) are all inner models of ZFC and obey the defining properties of the iterated mantle construction, namely, the base case \(\bar{\mathcal{M}}^0 = V\), the successor case \(\bar{\mathcal{M}}^{\alpha+1} = \mathcal{M}\bar{\mathcal{M}}^\alpha\) and the limit case \(\bar{\mathcal{M}}^\gamma = \bigcap_{\alpha < \gamma} \bar{\mathcal{M}}^\alpha\) for limit ordinals \(\gamma\). By induction, any two such classes \(\bar{\mathcal{M}}\) agree on their common coordinates, and when such a class \(\bar{\mathcal{M}}\) has been provided we may legitimately and unambiguously refer to the \(\alpha\)th mantle \(\mathcal{M}^\alpha\). We accordingly define the phrase “the \(\eta\)th inner mantle \(\mathcal{M}^\eta\) exists” to mean that \(\eta\) is an ordinal and there is a uniform presentation \(\bar{\mathcal{M}}\) of the inner mantles \(\mathcal{M}^\alpha\) for \(\alpha \leq \eta\). It is easy to see that the \(n\)th inner mantle \(\mathcal{M}^n\) exists for any (meta-theoretic) natural number \(n\), and if \(\mathcal{M}^\eta\) exists, so does \(\mathcal{M}^\alpha\) for any \(\alpha < \eta\). But as with the Harrington result in the case of HOD\(^\alpha\), we do not expect necessarily to be able to proceed through limit ordinals or even up to \(\omega\) uniformly. Note that even when the \(\eta\)th inner mantle \(\mathcal{M}^\eta\) exists, there seems little reason to expect that it is necessarily a definable class, even when \(\eta\) is definable or comparatively small, such as \(\eta = \omega\).

**Question 72.** Is there is a model of GBC in which there is no uniform presentation of the inner mantles \(\mathcal{M}^n\) for \(n < \omega\)? In particular, is there a model where the \(\omega\)th inner mantle \(\mathcal{M}^\omega\) does not exist? Is there a model where \(\mathcal{M}^\omega\) exists as a class but does not satisfy ZFC?

In analogy with the situation with the iterated HODs, we expect affirmative answers to these questions.

In the case that there is a uniform presentation of the inner mantles \(\mathcal{M}^\alpha\) for all ordinals \(\alpha\), that is, presented uniformly by a single GB class \(\bar{\mathcal{M}} \subseteq V \times \text{ORD}\), then it follows by corollary 20 that \(\mathcal{M}^{\text{ORD}} = \bigcap_\alpha \mathcal{M}^\alpha \models \text{ZFC}\) as well. That is, if one can iteratively and uniformly compute the inner mantles all the way through ORD, then the intersection of these inner mantles is a ZFC model, and we can imagine continuing past ORD by defining \(\mathcal{M}^{\text{ORD}+1} = \mathcal{M}\mathcal{M}^{\text{ORD}}\), and so on.

One way that this might happen, of course, is if the inner mantle process actually stabilizes before ORD, that is, if for some \(\alpha\) we have \(\mathcal{M}^\alpha = \mathcal{M}^\beta\) for all \(\beta > \alpha\); in this case, \(\mathcal{M}^\alpha\) is a model of ZFC, but not a nontrivial forcing extension of any deeper model (or equivalently, a model of ZFC plus the ground axiom). If \(\alpha\) is minimal with this property, then we say that the sequence of inner mantles stabilizes at \(\alpha\) and refer to the model \(\mathcal{M}^\alpha\) as the outer core of the universe in which it was computed. This is what remains when all outer layers of forcing have been successively stripped away. There seems to be no reason in general why the inner mantle process should necessarily stabilize at an early stage, or even after iterating through all the ordinals. For example, there seems no obvious reason why \(M^{\text{ORD}}\) cannot itself be a forcing extension of some other still-deeper model. Indeed, we make the following strong negative conjecture about this prospect.

**Conjecture 73.** Every model of ZFC is the \(\mathcal{M}^{\text{ORD}}\) of another model of ZFC in which the sequence of inner mantles does not stabilize. More generally, every model of ZFC is the \(\mathcal{M}^\alpha\) of another model of ZFC for any desired \(\alpha \leq \text{ORD}\), in which the sequence of inner mantles does not stabilize before \(\alpha\).
Let us explain the sense in which this conjecture, along with the main theorem of this article, can be viewed as philosophically negative. If one has adopted the philosophical view that beneath the set-theoretic universe lies a highly regular structure, some kind of canonical inner model, which may have become obscured over the eons by the accumulated layers of subsequent forcing constructions over that structure, then one would be led to expect that the mantle, or perhaps the inner mantles or the outer core, which sweep away all these subsequent layers of forcing, would exhibit highly regular structural features. But since our main theorem shows that every model of ZFC is the mantle of another model, one cannot prove in general that the mantle exhibits any extra structural features at all. And under conjecture 73, the same can be said of the inner mantles and the outer core. In particular, if the conjecture holds, then there are models whose outer core is realized first at stage $\alpha$ in $M^\alpha$, for any desired $\alpha \leq \text{ORD}$, since every model of ZFC + GA would be the $(M^\alpha)V$ of a suitable extension $V$. Thus, under the conjecture one should not expect to prove universal regularity features for the outer core even when it is realized as $M^{\text{ORD}}$, beyond what one can prove about arbitrary models of the ground axiom. And this is not much, since [HRW08] shows that models of GA need not even satisfy $V = \text{HOD}$.

Note that if the conjecture holds, then the inner mantle $M^{\text{ORD}}$ can be itself a forcing extension (since many models of ZFC are forcing extensions), and the process of stripping away the outer layers of forcing has not terminated even with ORD many iterations. Thus, the conjecture implies that there can be models of set theory having no outer core. It is also conceivable that the inner mantle calculation might break down at some ordinal stage $\alpha$ for the reason that $M^\alpha$ no longer satisfies ZFC, and in this case, the original model would have no outer core.

**Question 74. Under what circumstances does the outer core exist?**

We may similarly carry out the entire construction using generic mantles in place of mantles. Thus, we define that “the $\eta$th inner generic mantle $gM^\eta$ exists,” for $\eta \leq \text{ORD}$, if there is a class $\bar{M} \subseteq V \times \eta$, whose slices are inner models of ZFC respecting the iterative definition of the generic mantle, so that $\bar{M}^0 = V$ at the beginning, $\bar{M}^{\alpha+1} = gM^{\bar{M}^\alpha}$ as successor ordinals, and $\bar{M}^\lambda = \bigcap_{\alpha < \lambda} \bar{M}^\alpha$ at limits. If the process terminates by reaching a model of ZFC that is equal to its own generic mantle—thereby satisfying $V = gM^V$, the generic ground axiom—then we refer to this model as the generic outer core. One may similarly consider the iterated inner HOD construction and inner generic HODs, as in [HKP]. We extend conjecture 73 naturally to the following.

**Conjecture 75. If $V$ is any model of ZFC, then for any $\alpha \leq \text{ORD}$, there is another model $V$ of ZFC in which $V$ is (the $\alpha$th inner mantle, the $\alpha$th generic inner mantle, the $\alpha$th inner HOD and the $\alpha$th inner generic HOD).**

$$V = (M^\alpha)V = (gM^\alpha)V = (\text{HOD}^\alpha)V = (g\text{HOD}^\alpha)V$$

7. Large cardinal indestructibility across the multiverse

Laver [Lav78] famously proved that any supercompact cardinal $\kappa$ can be made indestructible by further $<\kappa$-directed closed forcing. In this section, we consider the possibility of large cardinal indestructibility not just in the upwards direction, but throughout the relevant portion of the generic multiverse.
For any class $\Gamma$ of forcing notions, we say that $W$ is a $\Gamma$-ground of $V$, or equivalently, that $V$ is a $\Gamma$-extension of $W$, if $V = W[G]$ for some $W$-generic $G \subseteq P \in W$, where $P$ has property $\Gamma$ in $W$. The $\Gamma$-generic multiverse is obtained by closing the universe under $\Gamma$-extensions and $\Gamma$-grounds.

**Theorem 76.** If $\kappa$ is supercompact, then there is a class forcing extension $V[G]$ in which $\kappa$ remains supercompact, becomes indestructible by further $\lt \kappa$-directed closed forcing, and the ground axiom holds. Thus, $\kappa$ remains supercompact in all $\lt \kappa$-directed closed extensions and (vacuously) in all $\lt \kappa$-directed closed grounds.

**Proof.** Suppose that $\kappa$ is supercompact. By Laver’s theorem [Lav78], we may find a set-forcing extension $V[g]$ in which $\kappa$ remains supercompact and becomes indestructible by all $\lt \kappa$-directed closed forcing. By coding the universe into the continuum function above $\kappa$, we may perform $\lt \kappa$-directed closed class forcing to find a model $V[g][G]$ in which every set is coded unboundedly often into the GCH pattern of $V[g][G]$. It follows that $V[g][G]$ satisfies the continuum coding axiom and therefore also the ground axiom, as in [Rei07, Rei06]. Because the GCH coding forcing is $\lt \kappa$-directed closed (and factors easily into set forcing followed by highly closed forcing), it follows that the supercompactness of $\kappa$ remains indestructible over $V[g][G]$ by further $\lt \kappa$-directed closed forcing. \hfill $\Box$

By combining the previous argument with the usual methods to attain global indestructibility, one may also accomplish this effect for many supercompact cardinals simultaneously. We show next that it is not possible to improve this to the entire $\lt \kappa$-directed closed generic multiverse.

**Theorem 77.** No supercompact cardinal $\kappa$ is indestructible throughout the $\lt \kappa$-directed closed generic multiverse. Indeed, for every cardinal $\kappa$, there is a $\lt \kappa$-directed closed generic ground in which $\kappa$ is not measurable.

**Proof.** Fix any cardinal $\kappa$. Let $P$ be the forcing to add a slim $\kappa$-Kurepa tree, a tree $T$ of height $\kappa$, whose $\alpha^{th}$ level (for infinite $\alpha$) has size at most $\alpha$, but which has $\kappa^+$ many $\kappa$-branches. The natural way to do this has the feature that $P$ is $\lt \kappa$-closed, but not $\lt \kappa$-directed closed. An easy reflection argument shows that no measurable cardinal can have a slim $\kappa$-Kurepa tree, and so $\kappa$ is not measurable in $V[T]$. But the forcing $P$ can be absorbed into the forcing $\text{Coll}(\kappa, 2^\kappa)$, since the combined forcing $P \ast \text{Coll}(\kappa, 2^\kappa)$ is $\lt \kappa$-closed, has size $2^\kappa$ and collapses $2^\kappa$ to $\kappa$, and $\text{Coll}(\kappa, 2^\kappa)$ is the unique forcing notion (up to isomorphism of the complete Boolean algebras) with this property. Thus, if $G \subseteq \text{Coll}(\kappa, 2^\kappa)$ is $V[T]$-generic for this collapse, we may rearrange the combined generic filter $T \ast G$ as a single $V$-generic filter $H \subseteq \text{Coll}(\kappa, 2^\kappa)$ such that $V[T][G] = V[H]$. Since the collapse forcing is $\lt \kappa$-directed closed, it follows that $V[T]$ is a $\lt \kappa$-directed closed generic ground of $V$, in which $\kappa$ is not measurable, as desired. \hfill $\Box$

The situation for indestructible weak compactness is somewhat more attractive.

**Theorem 78.** If $\kappa$ is supercompact, then there is a set-forcing extension $V[G]$ in which $\kappa$ is weakly compact and this is indestructible throughout the $\lt \kappa$-closed generic multiverse of $V[G]$.

**Proof.** By the Laver preparation [Lav78], there is a forcing extension $V[G]$ in which the supercompactness of $\kappa$ is indestructible by all $\lt \kappa$-directed closed forcing. By [HJ10, Theorem 22], this implies that the weak compactness of $\kappa$ is indestructible.
over $V[G]$ by all $<\kappa$-closed forcing, that is, generalizing $<\kappa$-directed closed to $<\kappa$-closed forcing.

**Lemma 78.1.** If $\kappa$ is weakly compact and this is indestructible by $<\kappa$-closed forcing, then $\kappa$ retains this property throughout the $<\kappa$-closed generic multiverse.

**Proof.** Suppose that $\kappa$ is weakly compact in a model $V$ and indestructible over $V$ by all $<\kappa$-closed forcing. Clearly, this property remains true in all $<\kappa$-closed extensions, since the two-step iteration of $<\kappa$-closed forcing is $<\kappa$-closed. We also argue conversely, however, that the property remains true in all $<\kappa$-closed grounds. Suppose that $W$ is a $<\kappa$-closed ground of $V$, so that $V = W[g]$ is $W$-generic for some $<\kappa$-closed forcing $g \subseteq P \in W$. First, we know that $\kappa$ is inaccessible in $W$ because inaccessibility is downward absolute. Second, in order to establish the tree property for $\kappa$ in $W$, suppose that $T$ is a $\kappa$-tree in $W$. By the tree property of $\kappa$ in $V$, there is a branch $b$ through $T$ in $V = W[g]$. Thus, $W$ has a $P$-name $b$ for this branch. In $W$, since the forcing $P$ is $<\kappa$-closed, we may build a pseudo-generic descending $\kappa$-sequence of conditions in $P$ that decide more and more of this name $b$. Thus, in $W$ we can build a branch through $T$, thereby establishing this instance of the tree property for $\kappa$ in $W$. It follows that $\kappa$ is weakly compact in $W$. Consider now any $<\kappa$-closed forcing $Q \in W$. Since $P$ is $<\kappa$-closed in $W$, it follows that $Q$ remains $<\kappa$-closed in $V = W[g]$. Thus, for any $W[g]$-generic filter $h \subseteq Q$, it follows by the indestructibility of $\kappa$ over $V$ that $\kappa$ is weakly compact in $V[h] = W[g|h]$. Since $P$ also remains $<\kappa$-closed in $W[h]$, it follows by the pseudo-forcing downward absoluteness argument that $\kappa$ is weakly compact in $W[h]$. Thus, there can be no condition in $Q$ forcing over $W$ that $\kappa$ is not weakly compact in the extension of $W$ by $Q$. Thus, we have proved that the weak compactness of $\kappa$ is indestructible over $W$ by $<\kappa$-closed forcing. Thus, the indestructibility property we desire is preserved as one moves through the $<\kappa$-closed generic multiverse. \qed

The theorem follows directly. \qed

In the case of small forcing, the well-known Lévy-Solovay theorem of [LS67] establishes that every measurable cardinal $\kappa$ is preserved by the move to any $\kappa$-small forcing extension as well as to any $\kappa$-small ground (that is, in each case, by forcing of size less than $\kappa$). It follows that any measurable cardinal $\kappa$ is measurable throughout the corresponding small-generic multiverse. Similarly, most of the other classical large cardinals are preserved to small forcing extensions and grounds, and therefore retain their large cardinal property throughout the small-generic multiverse. It would be natural to consider also the extent to which other set-theoretic axioms, such as the proper forcing axiom, could be indestructible throughout significant portions of the generic multiverse.

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