FACTORIZATION OF COMPLETELY BOUNDED
BILINEAR OPERATORS AND INJECTIVITY

Allan M. Sinclair(∗)
Department of Mathematics
University of Edinburgh
Edinburgh EH9 3JZ
SCOTLAND
allan@mathematics.edinburgh.ac.uk

Roger R. Smith(∗)(†)
Department of Mathematics
Texas A&M University
College Station, TX 77843
U.S.A.
rsmith@math.tamu.edu

ABSTRACT

A completely bounded bilinear operator $\phi: \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ on a von Neumann algebra $\mathcal{M}$ is said to have a factorization in $\mathcal{M}$ if there exist completely bounded linear operators $\psi_j, \theta_j: \mathcal{M} \to \mathcal{M}$ such that

$$\phi(x, y) = \sum_{j \in \Lambda} \psi_j(x)\theta_j(y), \quad x, y \in \mathcal{M},$$

where convergence of the sum is made precise below. The main result of the paper is that all completely bounded bilinear operators $\phi: \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ have factorizations in $\mathcal{M}$ if and only if $\mathcal{M}$ is injective.

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§1. Introduction

There are several conditions on a von Neumann algebra $\mathcal{N}$ that are known to be equivalent to the injectivity of $\mathcal{N}$. The outstanding, and fundamental, result is Connes’ proof [10] that injective factors on a separable Hilbert space are hyperfinite (see also [32]). Subsequently Haagerup [19] and Popa [26] gave simpler treatments of this result which avoided the technical theory of automorphism groups of von Neumann algebras in [10].

One result in the development of the subject prior to [10] plays a role below. Effros and Lance [15, Corollary 4.6] showed that a von Neumann factor $\mathcal{N}$ is semidiscrete (equivalently injective) if and only if the $C^*$-algebra $C^*(\mathcal{N},\mathcal{N}')$ is isomorphic to $\mathcal{N} \otimes_{\text{min}} \mathcal{N}'$; this is used in proving Theorem 4.4 below. The operators between von Neumann algebras which appear in [15] are all completely positive, but there are characterizations of injectivity of a von Neumann algebra $\mathcal{N}$ based on properties of completely bounded linear operators associated with $\mathcal{N}$. For example, Haagerup [20] has shown that $\mathcal{N}$ is injective if and only if each completely bounded linear operator from $\ell^\infty$ into $\mathcal{N}$ is a linear combination of completely positive linear operators from $\ell^\infty$ into $\mathcal{N}$; this is used in proving Theorem 5.3 below.

Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras with $\mathcal{N}$ acting on a Hilbert space $H$ and $\mathcal{M}$ infinite dimensional. The representation theorem of a completely bounded bilinear operator from $\mathcal{M} \times \mathcal{M}$ into $B(H)$ provides a factorization of such an operator into $\mathcal{N}$. Strengthening the hypotheses on this factorization for all completely bounded bilinear operators $\phi: \mathcal{M} \times \mathcal{M} \to \mathcal{N}$ provides another characterization of injectivity of $\mathcal{N}$ as we shall explain. If $\phi: \mathcal{M} \times \mathcal{M} \to \mathcal{N} \subseteq B(H)$ is a completely bounded bilinear operator then there is a representation $\pi: \mathcal{M} \to B(K)$ and continuous linear operators $W: H \to K$, $T: K \to K$, and $V: K \to H$ such that

$$\phi(m_1, m_2) = V\pi(m_1)T\pi(m_2)W, \quad m_1, m_2 \in \mathcal{M}$$

and $\|\phi\|_{cb} = \|V\| \|T\| \|W\|$. (See [5, 6, 23, 27]). However there is little control over $V, T,$ and $W$ other than the norm estimate. If $\psi, \theta: \mathcal{M} \to \mathcal{N}$ are completely bounded then $\phi: \mathcal{M} \times \mathcal{M} \to \mathcal{N}$ defined by

$$\phi(m_1, m_2) = \psi(m_1)\theta(m_2), \quad m_1, m_2 \in \mathcal{M}$$

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is a completely bounded bilinear operator, and (1.2) represents a factorization of the bilinear operator $\phi$ as a product of linear operators. More generally, suitable weakly convergent sums $\sum_{j \in \Lambda} \psi_j(m_1)\theta_j(m_2)$ of such products may define a completely bounded bilinear operator $\phi: \mathcal{M} \times \mathcal{M} \to \mathcal{N}$, and we refer to such a sum as a factorization of $\phi$. The main result of the paper is that injectivity of $\mathcal{N}$ is equivalent to all completely bounded operators $\phi: \mathcal{M} \times \mathcal{M} \to \mathcal{N}$ having such factorizations. A consequence of our work is that factorizations with $\psi_j, \theta_j$ mapping into $B(H)$ are always possible; the crucial point is to require $\psi_j, \theta_j$ to map into $\mathcal{N}$. Indeed it suffices to take $\mathcal{M} = \mathcal{N}$, which gives a characterization of injectivity in terms of completely bounded bilinear operators which is internal to $\mathcal{N}$.

We now give a brief description of the contents of the paper. Section 2 contains the basic notation and definitions, and also a short account of the $w^*$-Haagerup tensor product of $CB(\mathcal{X}, \mathcal{N}) \otimes_{w^*h} CB(\mathcal{Y}, \mathcal{N})$, where $\mathcal{X}, \mathcal{Y}$ are operator spaces, $\mathcal{N}$ is a von Neumann algebra, and $CB(\mathcal{X}, \mathcal{N})$ is the space of completely bounded linear operators from $\mathcal{X}$ into $\mathcal{N}$. This tensor product provides a convenient language for the formulation of our results. However we have delayed its appearance until the last section to reduce the technicalities for readers who are unfamiliar with it.

Section 3 contains a theorem on module map extensions in the bilinear case, extending a result of Wittstock [33] for one variable. This is used to obtain Proposition 3.3, a technical result on the representation of modular bilinear operators, which is important subsequently. The fourth and fifth sections are the heart of the paper, each devoted to one implication in the equivalence of injectivity and the factorization of completely bounded bilinear operators. Injectivity implies factorization is Theorem 4.4, while the reverse implication is Theorem 5.4. The final section is a brief summary of results and includes some other equivalences formulated in terms of the $w^*$-Haagerup tensor product.

We refer the reader to [22] for an account of the theory of completed bounded linear operators. The subsequent development of the multilinear case may be found in the survey article [6] or the book [27]. We also refer to [3, 7, 8, 19, 21, 24, 25, 26, 32] for related results on injectivity and multilinear operators.
§2. Notations and definitions

Throughout $\mathcal{M}$ and $\mathcal{N}$ will denote von Neumann algebras acting on a Hilbert space $H$ with commutants $\mathcal{M}'$ and $\mathcal{N}'$. $C^*$-algebras are denoted by $\mathcal{A}$ and operator spaces by $\mathcal{E}, \mathcal{F}, \mathcal{X}$ or $\mathcal{Y}$. Recall that an operator space $\mathcal{X}$ is a norm closed subspace of $B(H)$, the algebra of bounded linear operators on $H$, together with the norms and structure of $M_n(\mathcal{X})$ in $M_n(B(H)) = B(H^n)$, where $M_n$ denotes the $n \times n$ matrices. We refer to [22] and [6, 27] respectively for the theories of completely bounded linear operators and completely bounded multilinear operators. Recall that a completely bounded linear operator $\phi: \mathcal{X} \to B(H)$, where $\mathcal{X}$ is an operator space in a $C^*$-algebra $\mathcal{A}$, has a representation of the form

$$\phi(x) = U \pi(x) V, \quad x \in \mathcal{X}. \quad (2.1)$$

Here $\pi$ is a representation of $\mathcal{A}$ on a Hilbert space $K$ and $V: H \to K$, $U: K \to H$ are continuous linear operators satisfying

$$\|U\| = \|V\| = \|\phi\|_{cb}^{1/2}. \quad (2.2)$$

The corresponding result for completely bounded bilinear operators is the following. Given operator spaces $\mathcal{X}$ and $\mathcal{Y}$ in a $C^*$-algebra $\mathcal{A}$ and a completely bounded bilinear operator $\phi: \mathcal{X} \times \mathcal{Y} \to B(H)$, there exist a representation $\pi: \mathcal{A} \to B(K)$ and continuous linear operators $W: H \to K$, $T: K \to K$, and $U: K \to H$ such that

$$\phi(x, y) = U \pi(x) T \pi(y) W, \quad x \in \mathcal{X}, y \in \mathcal{Y}. \quad (2.3)$$

Moreover $U, T$ and $W$ may be chosen to satisfy the optimal condition

$$\|U\| = \|T\| = \|W\| = \|\phi\|_{cb}^{1/3}. \quad (2.4)$$

The usual formulation of (2.3) is

$$\phi(x, y) = U \pi(x) T \rho(y) W, \quad x \in \mathcal{X}, y \in \mathcal{Y} \quad (2.5)$$

where $\pi$ and $\rho$ are possibly distinct representations of $\mathcal{A}$, but this may be reduced to the form of (2.3) by writing

$$\phi(x, y) = (U, 0) \left( \begin{array}{cc} \pi(x) & 0 \\ 0 & \rho(x) \end{array} \right) \left( \begin{array}{cc} 0 & T \\ 0 & 0 \end{array} \right) \left( \begin{array}{cc} \pi(y) & 0 \\ 0 & \rho(y) \end{array} \right) \left( \begin{array}{c} 0 \\ W \end{array} \right). \quad (2.6)$$
If $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are operator spaces, $CB(\mathcal{X}, \mathcal{Y})$ denotes the space of completely bounded linear operators of $\mathcal{X}$ into $\mathcal{Y}$, while $CB^2(\mathcal{X} \times \mathcal{Y}, \mathcal{Z})$ denotes the space of completely bounded bilinear operators of $\mathcal{X} \times \mathcal{Y}$ into $\mathcal{Z}$. When $\mathcal{X} = \mathcal{Y}$, we abbreviate this latter space to $CB^2(\mathcal{X}, \mathcal{Z})$.

For $\phi \in CB^2(\mathcal{A}, B(H))$, recall that the $n$-fold amplification $\phi_n \in CB^2(M_n(\mathcal{A}), M_n(B(H)))$ is defined by

$$\phi_n((x_{ij}), (y_{ij})) = \left( \sum_{k=1}^{n} \phi(x_{ik}, y_{kj}) \right)$$

for $(x_{ij}), (y_{ij}) \in M_n(\mathcal{A})$. Then $\phi$ is said to be completely positive if

$$\phi_n((x_{ij}), (x_{ij})^*) \geq 0, \quad (x_{ij}) \in M_n(\mathcal{A}), \quad n \in \mathbb{N}. \quad (2.8)$$

In contrast to the linear case, completely positive bilinear operators need not be completely bounded. This is well known [5], but we include an elementary example. Let $\psi: B(\ell_2) \to B(\ell_2)$ be the transpose on infinite matrices and define $\phi: B(H) \times B(H) \to B(H)$ by

$$\phi(x, y) = \psi(x)\psi(y^*)^*, \quad x, y \in B(H). \quad (2.8)$$

It is easy to check that

$$\phi_n((x_{ij}), (x_{ij})^*) = \psi_n(x_{ij})\psi_n(x_{ij})^* \geq 0, \quad (2.9)$$

and so $\phi$ is completely positive. However $\psi(x) = \phi(x, 1)$, and so $\phi$ cannot be completely bounded, since $\psi$ is not. Thus decomposing bilinear completely bounded operators as a linear combination of completely positive bilinear operators is (seemingly) less restrictive than similar decompositions in the linear case. We note in passing that the representation (2.3) may be extended to the multilinear case [5, 23], but will not be needed here.

Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be bimodules over a $C^*$-algebra $\mathcal{A}$ and let $\phi$: $\mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ be bilinear. Then $\phi$ is said to be $\mathcal{A}$-modular if, for $x \in \mathcal{X}$, $y \in \mathcal{Y}$, $a \in \mathcal{A}$, the following relations hold:

$$\phi(ax, y) = a\phi(x, y), \quad (2.10)$$
$$\phi(xa, y) = \phi(x, ay), \quad (2.11)$$
$$\phi(x, ya) = \phi(x, y)a. \quad (2.12)$$
Such operators have played a major role in the theory of completely bounded operators and their applications for several years (see [4, 8, 13, 27, 28]), and will also be useful in subsequent sections of this paper. Recall that a von Neumann algebra $\mathcal{N}$ is injective if for any containing von Neumann algebra $\mathcal{M}$ there is a conditional expectation $E: \mathcal{M} \to \mathcal{N}$. By this we mean a completely positive projection of $\mathcal{M}$ onto $\mathcal{N}$, and such projections are automatically $\mathcal{N}$-modular [31].

The $C^*$-algebra generated by two $C^*$-subalgebras $\mathcal{A}$ and $\mathcal{B}$ of $B(H)$ is denoted by $C^*(\mathcal{A}, \mathcal{B})$. If $\Lambda$ is a (non-empty) index set and $H$ is a Hilbert space, let $\ell_2(\Lambda, H) = \ell_2(\Lambda) \otimes_2 H$ denote the Hilbert space of “sequences” in $H$ indexed by $\Lambda$. For a minimal projection $e$ onto a standard basis vector of $\ell_2(\Lambda)$, we let

$$R(\Lambda) = B(\ell_2(\Lambda))e,$$

$$C(\Lambda) = eB(\ell_2(\Lambda)),$$

be the $\Lambda$-row and -column operator spaces respectively. Although we require general index sets for the proper formulation of our results, the reader will not be misled by thinking of $\Lambda$ as $\mathbb{N}$.

We now review several tensor products which will be needed subsequently. The minimal (also called injective or spatial) tensor product of $C^*$-algebras $\mathcal{A}$ and $\mathcal{B}$ is denoted by $\mathcal{A} \otimes_{\min} \mathcal{B}$ [30], while $\mathcal{M} \otimes \mathcal{N}$ denotes the von Neumann algebra tensor product of von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$ [30]. The Haagerup tensor product $\mathcal{A} \otimes_h \mathcal{B}$ [13, 18] is the completion of the algebraic tensor product $\mathcal{A} \otimes \mathcal{B}$ in the norm

$$\|u\|_h = \inf \left\{ \left( \sum_{j=1}^{n} a_j a_j^* \right)^{1/2} \left( \sum_{j=1}^{n} b_j^* b_j \right)^{1/2} \right\}$$

(2.13)

taken over all representations $u = \sum_{j=1}^{n} a_j \otimes b_j \in \mathcal{A} \otimes \mathcal{B}$. There are several weak versions of this tensor product and we will require the $w^*$-Haagerup tensor product $\otimes_{w^* h}$, introduced for pairs of dual operator spaces in [2]. Our interest will focus on $CB(\mathcal{X}, \mathcal{N}) \otimes_{w^* h} CB(\mathcal{Y}, \mathcal{N})$ where $\mathcal{X}$ and $\mathcal{Y}$ are operator spaces and $\mathcal{N} \subseteq B(H)$ is a von Neumann algebra, and we give a straightforward definition in this case which is equivalent to the original formulation. Note that $CB(\mathcal{X}, \mathcal{N})$ is a dual operator space which can be identified with the dual of the
operator space projective tensor product $X \hat{\otimes} N_*$ [1, 16]. We omit further discussion of $\hat{\otimes}$ since it will not be needed subsequently.

Consider the vector space $V$ of all formal sums $\sum_{j \in \Lambda} \psi_j \otimes \theta_j$ where $\psi_j \in CB(X, N)$, $\theta_j \in CB(Y, N)$ and, for all $x \in X$, $y \in Y$, and finite subsets $F$ of $\Lambda$, there exists a constant $K$ such that

$$\left\| \sum_{j \in F} \psi_j(x)^* \psi_j(x) \right\| \leq K \|x\|^2,$$

(2.14)

$$\left\| \sum_{j \in F} \theta_j(y)^* \theta_j(y) \right\| \leq K \|y\|^2.$$  

(2.15)

For vectors $\xi, \eta \in H$,

$$\sum_{j \in F} \|\psi_j(x)^* \eta\|^2 = \sum_{j \in F} \langle \psi_j(x)^* \psi_j(x), \eta \eta \rangle \leq K \|x\|^2 \|\eta\|^2,$$

(2.16)

and similarly

$$\sum_{j \in F} \|\theta_j(y) \xi\|^2 \leq K \|y\|^2 \|\xi\|^2.$$  

(2.17)

These two inequalities then remain valid for the sum over all $j \in \Lambda$ (as a consequence of which only countably many terms in the sums are non-zero), and the Cauchy-Schwarz inequality then shows that $\sum_{j \in \Lambda} \langle \psi_j(x)^* \theta_j(y), \xi, \eta \rangle$ is an absolutely convergent series, bounded in absolute value by $K \|x\| \|y\| \|\xi\| \|\eta\|$. Thus $\sum_{j \in \Lambda} \psi_j(x)^* \theta_j(y)$ is weakly convergent, so defines an element of $N$. Essentially the same argument shows that

$$\sum_{j \in \Lambda} \langle \psi_j(x)^* t \theta_j(y), \xi, \eta \rangle, \quad x \in X, y \in Y, t \in B(H), \xi, \eta \in H$$

(2.18)

is always an absolutely convergent series and so $\sum_{j \in \Lambda} \psi_j(x)^* t \theta_j(y)$ converges weakly to an element of $B(H)$ for all $t \in B(H)$. We now declare two sums $\sum_{j \in \Lambda} \psi_j \otimes \theta_j, \sum_{j \in \Lambda} \tilde{\psi}_j \otimes \tilde{\theta}_j$ to be equal in $V$ if

$$\sum_{j \in \Lambda} \psi_j(x)^* t \theta_j(y) = \sum_{j \in \Lambda} \tilde{\psi}_j(x)^* t \tilde{\theta}_j(y).$$  

(2.19)
for all \( t \in B(H) \). By taking \( t = I \) it is then clear that \( \sum_{j \in \Lambda} \psi_j(x)\theta_j(y) \) is independent of the particular representation chosen.

An element \( v = \sum_{j \in \Lambda} \psi_j \otimes \theta_j \) leads to bounded bilinear maps \( \Psi: \mathcal{X} \times \mathcal{X}^* \to \mathcal{N} \) and \( \Theta: \mathcal{Y}^* \times \mathcal{Y} \to \mathcal{N} \) defined by

\[
\Psi(x_1, x_2^*) = \sum_{j \in \Lambda} \psi_j(x_1)\psi_j(x_2)^*, \tag{2.20}
\]

\[
\Theta(y_1^*, y_2) = \sum_{j \in \Lambda} \theta_j(y_1)^*\theta_j(y_2), \tag{2.21}
\]

with weak convergence in the sums. For each \( v = \sum_{j \in \Lambda} \psi_j \otimes \theta_j \in \mathcal{V} \) we define

\[
|||v||| = \inf \{ \|\Psi\|^{1/2}_{cb} \|\Theta\|^{1/2}_{cb} \}
\]

(2.22)

where the infimum is taken over all possible representations of \( v \). If no representation of \( v \) has associated operators which are completely bounded, then we set \( |||v||| = \infty \). It is not immediately clear that \( |||\cdot||| \) is a norm on the set of elements for which \( |||v||| \) is finite. This will follow from the next proposition, whose purpose is to give an alternative description of the \( w^*\)-Haagerup norm on \( CB(\mathcal{X}, \mathcal{N}) \otimes_{w^* h} CB(\mathcal{Y}, \mathcal{N}) \).

**Proposition 2.1.** Let \( v \in \mathcal{V} \). Then \( v \in CB(\mathcal{X}, \mathcal{N}) \otimes_{w^* h} CB(\mathcal{Y}, \mathcal{N}) \) if and only if \( |||v||| < \infty \), and in this case

\[
|||v||| = \|v\|_{w^* h}. \tag{2.23}
\]

**Proof.** Suppose that \( |||v||| < \infty \). Since it is clear that \( |||\lambda v||| = |\lambda| \|v||| \), we may assume that \( |||v||| = 1 \). Thus, given \( \epsilon > 0 \), there exists a representation \( v = \sum_{j \in \Lambda} \psi_j \otimes \theta_j \) and \( \|\Psi\|_{cb}, \|\Theta\|_{cb} < (1 + \epsilon)^{1/2} \). For any finite subset \( \mathcal{F} = \{1, \ldots, n\} \subseteq \Lambda \) (after renumbering), the norm of \( (\psi_1, \ldots, \psi_n) \) as an element of \( \mathbf{M}_n(CB(\mathcal{X}, \mathcal{N})) \) is its cb-norm as an element of \( CB(\mathcal{X}, \mathbf{M}_n(\mathcal{N})) \), so, for \( X \in \mathbf{M}_k(\mathcal{X}), k \geq 1, \)

\[
\|(\psi_1 \otimes I_k(X), \ldots, \psi_n \otimes I_k(X))\|^2 = \left\| \sum_{i=1}^n (\psi_i \otimes I_k(X))(\psi_i \otimes I_k(X))^* \right\| \leq \|\Psi_k(X, X^*)\| \leq \|\Psi\|_{cb} \|X\|^2, \tag{2.24}
\]
and a similar estimate holds for columns with $\psi_i$'s replaced by $\theta_i$'s. It follows from (2.24) that
\[
\left\| (\psi_1, \ldots, \psi_n) \right\| \leq \left\| \Psi \right\|_{cb}^{1/2}
\]  
and
\[
\left\| (\theta_1, \ldots, \theta_n)^T \right\| \leq \left\| \Theta \right\|_{cb}^{1/2}
\]  
for any finite subset $F$ of $\Lambda$. By [2, Theorem 3.1], there exists a representation $v = \sum_{j=1}^{n} \psi_j \otimes \theta_j$, where the norms of the row of $\psi_j$'s and column of $\theta_j$'s are both 1. For any finite subset $F = \{1, 2, \ldots, n\}$ of $\Lambda$ (after renumbering), let
\[
\Psi_F(x_1, x_2^*) = \sum_{j=1}^{n} \psi_j(x_1)\psi_j(x_2)^*
\]  
and
\[
\Theta_F(y_1^*, y_2) = \sum_{j=1}^{n} \theta_j(y_1)^*\theta_j(y_2).
\]  
It is then immediate from the definition of the norms in $M_k(CB(\mathcal{X}, \mathcal{N}))$ and $M_k(CB(\mathcal{Y}, \mathcal{N}))$ that $\left\| \Psi_F \right\|_{cb}, \left\| \Theta_F \right\|_{cb} \leq 1$, from which it follows that $\left\| \Psi \right\|_{cb}, \left\| \Theta \right\|_{cb} \leq 1$. Thus $v \in \mathcal{V}$ and $\left\| v \right\| \leq \left\| v \right\|_{w^* h}$, proving the reverse inequality.

There is a natural map $\nu$: $CB(\mathcal{X}, \mathcal{N}) \otimes_{w^* h} CB(\mathcal{Y}, \mathcal{N}) \to CB^2(\mathcal{X} \times \mathcal{Y}, \mathcal{N})$ defined by
\[
\nu \left( \sum_{j \in \Lambda} \psi_j \otimes \theta_j \right)(x, y) = \sum_{j \in \Lambda} \psi_j(x)\theta_j(y).
\]  
The sum on the right hand side of (2.23) may be viewed as the product of elements from $R(\Lambda)\overline{\otimes} \mathcal{N}$ and $C(\Lambda)\overline{\otimes} \mathcal{N}$, from which the estimate
\[
\left\| \sum_{j \in \Lambda} \psi_j(x)\theta_j(y) \right\| \leq \left\| \sum_{j \in \Lambda} \psi_j(x)\psi_j(x)^* \right\|^{1/2} \left\| \sum_{j \in \Lambda} \theta_j(y)^*\theta_j(y) \right\|^{1/2}
\]  
\[
= \left\| \Psi(x, x)^* \right\|^{1/2} \left\| \Theta(y^*, y) \right\|^{1/2}
\]  
\[
\leq \left\| \Psi \right\|_{cb}^{1/2} \left\| \Theta \right\|_{cb}^{1/2} \left\| x \right\| \left\| y \right\|
\]  
(2.31)
is immediate. This inequality lifts to the $n$-fold amplification, showing that $\nu$ is a contraction. Subsequently we show that $\nu$ is a complete quotient map when $\mathcal{N}$ is injective.


§3. $\mathcal{A}$-modular bilinear operators

In this section we generalize Wittstock’s one variable completely bounded modular extension theorem [33] to two variables. The proof involves standard techniques of modifying a completely bounded operator to one that has a representation with good computational properties. Our approach would apply to any number of variables, and in the case of one variable is perhaps simpler than Wittstock’s original method.

**Theorem 3.1.** Let $\mathcal{A}$ be a $C^*$-subalgebra of $B(H)$, let $\mathcal{E}$ and $\mathcal{F}$ be norm closed subspaces of $B(H)$ which are also $\mathcal{A}$-modules, and let $\phi: \mathcal{E} \times \mathcal{F} \to B(H)$ be a completely bounded $\mathcal{A}$-modular bilinear operator. Then $\phi$ extends to a completely bounded $\mathcal{A}$-modular bilinear operator $\psi: B(H) \times B(H) \to B(H)$ with preservation of norm. Moreover, $\psi$ has a representation

$$
\psi(x, y) = V \pi(x) T \pi(y) W, \quad x, y \in B(H),
$$

where $\pi: B(H) \to B(K)$ is a representation, and $V, T, W$ are continuous linear operators $H \xrightarrow{W} K \xrightarrow{T} K \xrightarrow{V} H$ (3.2) satisfying $\|\psi\|_{cb} = \|V\| \|T\| \|W\|$, and

$$
aV = V \pi(a), \quad \pi(a)T = T \pi(a), \quad \pi(a)W = Wa.
$$

**Proof.** A completely bounded bilinear operator can be extended with preservation of completely bounded norm, so let $\theta: B(H) \times B(H) \to B(H)$ be any such extension of $\phi$. By (2.3), $\theta$ has a representation

$$
\theta(x, y) = V_1 \pi(x) T_1 \pi(y) W_1, \quad x, y \in B(H)
$$

with $\|\theta\|_{cb} = \|V_1\| \|T_1\| \|W_1\|$. These operators are successively replaced by inserting suitable projections from $B(K)$ into (3.4).

By $\mathcal{A}$-modularity,

$$
(aV_1 - V_1 \pi(a)) \pi(e) T_1 \pi(f) W_1 \xi = 0
$$

for all $a \in \mathcal{A}, e \in \mathcal{E}, f \in \mathcal{F}$, and $\xi \in H$. Let

$$
K_1 = \text{span}\{\pi(e) T_1 \pi(f) W_1 \xi: e \in \mathcal{E}, f \in \mathcal{F}, \xi \in H\}.
$$
Then $K_1$ is a closed $\pi(A)$-invariant subspace of $K$, since $\pi(a)e = \pi(ae) \in \pi(E)$. Thus the projection $P_1$ of $K$ onto $K_1$ is in $\pi(A)'$. Let $V = V_1P_1$ and define $\theta_1 \in CB^2(B(H), B(H))$ by

$$\theta_1(x, y) = V\pi(x)T_1\pi(y)W_1, \quad x, y \in B(H).$$

(3.6)

Clearly $\|\theta_1\|_{cb} \leq \|\theta\|_{cb} = \|\phi\|_{cb}$ and we now verify that $\theta_1$ is an extension of $\phi$. For all $\xi, \eta \in H$, $e \in E$, and $f \in F$,

$$\langle \theta_1(e, f)\xi, \eta \rangle = \langle V_1P_1\pi(e)T_1\pi(f)W_1\xi, \eta \rangle$$

$$= \langle V_1\pi(e)T_1\pi(f)W_1\xi, \eta \rangle$$

$$= \langle \theta(e, f)\xi, \eta \rangle$$

$$= \langle \phi(e, f)\xi, \eta \rangle,$$

(3.7)

and so $\theta_1$ extends $\phi$. The second equality in (3.7) is immediate from the definition of $K_1$. From (3.5), $aV_1 - V_1\pi(a)$ annihilates $K_1$ for all $a \in A$, so $(aV_1 - V_1\pi(a))P_1 = 0$. Since $P_1$ commutes with $\pi(A)$, we obtain

$$aV - V\pi(a) = aV_1P_1 - V_1P_1\pi(a) = (aV_1 - V_1\pi(a))P_1 = 0$$

(3.8)

for all $a \in A$. We now make a second modification to $\theta$.

By $A$-modularity,

$$\langle (V\pi(e)(\pi(a)T_1 - T_1\pi(a))\pi(f)W_1\xi, \eta \rangle = \langle (\theta_1(ea, f) - \theta_1(e, af))\xi, \eta \rangle$$

$$= \langle (\phi(ea, f) - \phi(e, af))\xi, \eta \rangle$$

$$= 0$$

(3.9)

for all $a \in A$, $e \in E$, $f \in F$ and $\xi, \eta \in H$. Thus

$$\langle (\pi(a)T_1 - T_1\pi(a))\pi(f)W_1\xi, \pi(e)^*V^*\eta \rangle = 0.$$  

(3.10)

Let

$$K_2 = \text{span}\{\pi(f)W_1\xi: \ f \in F, \xi \in H\},$$

(3.11)

$$K_3 = \text{span}\{\pi(e)^*V^*\eta: \ e \in E, \eta \in H\},$$

(3.12)
and let $P_2$ and $P_3$ respectively be the projections onto these subspaces of $K$. The relations

$$\pi(a)\pi(f) = \pi(af) \in \pi(F), \quad \pi(a)\pi(e)^* = \pi( ea^* )^* \in \pi(E)^*$$

(3.13)

show that $K_2$ and $K_3$ are invariant subspaces for $\pi(A)$, and so $P_2, P_3 \in \pi(A)'$. Let $T = P_3T_1P_2$, and define $\theta_2 \in CB^2(B(H), B(H))$ by

$$\theta_2(x, y) = V\pi(x)T\pi(y)W_1, \quad x, y \in B(H).$$

(3.14)

Then, for $e \in E$, $f \in F$, $\xi, \eta \in H$,

$$\langle \theta_2(e, f)\xi, \eta \rangle = \langle T_1P_2\pi(f)W_1\xi, P_3\pi(e)^*V^*\eta \rangle$$

$$= \langle T_1\pi(f)W_1\xi, \pi(e)^*V^*\eta \rangle$$

$$= \langle \theta_1(e, f)\xi, \eta \rangle$$

$$= \langle \phi(e, f)\xi, \eta \rangle,$$

(3.15)

verifying that $\theta_2$ extends $\phi$, and norm preservation is clear. Moreover

$$\langle (\pi(a)T_1 - T_1\pi(a))\pi(f)W_1\xi, \pi(e)^*V^*\eta \rangle$$

$$= \langle (\theta_1(ea, f) - \theta_1(e, af))\xi, \eta \rangle$$

$$= \langle (\phi(ea, f) - \phi(e, af))\xi, \eta \rangle$$

$$= 0$$

(3.16)

and so each operator $\pi(a)T_1 - T_1\pi(a)$, $(a \in A)$, maps $K_2$ into $K_3^\perp$, by (3.11) and (3.12). Thus

$$\pi(a)T - T\pi(a) = \pi(a)P_3T_1P_2 - P_3T_1P_2\pi(a)$$

$$= P_3(\pi(a)T_1 - T_1\pi(a))P_2$$

$$= 0$$

(3.17)

for $a \in A$ since $P_2, P_2 \in \pi(A)'$. Thus (3.8) and (3.17) show that, for $a \in A$, $x, y \in B(H)$,

$$\theta_2(ax, y) = a\theta_2(x, y), \quad \theta_2(xa, y) = \theta_2(x, ay).$$

(3.18)
The final modification is to define $P_4$ to be the projection of $K$ onto

$$K_4 = \overline{\text{span}}\{\pi(f)^*T^*\pi(e)^*V^*\eta; \; e \in \mathcal{E}, f \in \mathcal{F}, \eta \in H\}, \quad (3.19)$$

and to let $\psi \in CB^2(B(H), B(H))$ be the bilinear operator obtained by replacing $W_1$ by $W = P_4W_1$ in (3.14):

$$\psi(x,y) = V\pi(x)T\pi(y)W, \quad x,y \in B(H). \quad (3.20)$$

As before, one can verify that $\psi$ extends $\phi$ with preservation of norm, and that the relations (3.2) hold. The details are so similar that we omit them.

**Remark 3.2.** Note that if $F = E^*$ and $\phi$ is in addition completely positive, then $\psi$ can be chosen to be completely positive with $W = V^*$ and $T \geq 0$.

We end this section with a technical result which will be needed subsequently; since it deals with modular bilinear operators we include it here. Below, the important point is the normality condition on the representation.

**Proposition 3.3.** Let $\mathcal{R}, \mathcal{S} \subseteq B(H)$ be a commuting pair of von Neumann algebras and let $\phi \in CB^2(C^*(\mathcal{R}, \mathcal{S}), B(H))$ be $\mathcal{S}$-modular. Then there exist a representation $\pi: C^*(\mathcal{R}, \mathcal{S}) \to B(L)$, whose restriction to $\mathcal{S}$ is normal, and continuous linear operators $W: H \to L$, $T: L \to L$ and $V: L \to H$ such that

$$\phi(x,y) = V\pi(x)T\pi(y)W, \quad x,y \in C^*(\mathcal{R}, \mathcal{S}), \quad (3.21)$$

$$\|\phi\|_{cb} = \|V\| \|T\| \|W\|, \quad \text{and}$$

$$sV = V\pi(s), \quad \pi(s)T = T\pi(s), \quad \pi(s)W = Ws, \quad s \in \mathcal{S}. \quad (3.22)$$

**Proof.** By Theorem 3.1, $\phi$ has a representation

$$\phi(x,y) = V_1\rho(x)T_1\rho(y)W_1, \quad x,y \in C^*(\mathcal{R}, \mathcal{S}) \quad (3.23)$$

where $\rho: C^*(\mathcal{R}, \mathcal{S}) \to B(K)$ is a representation and $W_1: H \to K$, $T_1: K \to K$, and $V_1: K \to H$ satisfy $\|\phi\|_{cb} = \|V_1\| \|T_1\| \|W_1\|$. Moreover

$$sV_1 = V_1\rho(s), \quad \rho(s)T_1 = T_1\rho(s), \quad \rho(s)W_1 = W_1s. \quad (3.24)$$
By the decomposition of a representation into its normal and singular parts [30, Theorem III.2.14] there is a central projection $p \in \rho(S)''$ so that

$$s \rightarrow \langle \rho(s)p\xi, \eta \rangle, \quad s \rightarrow \langle \rho(s)(1-p)\xi, \eta \rangle \quad (3.25)$$

are respectively normal and singular linear functionals on $S$ for all $\xi, \eta \in K$. Now

$$s \rightarrow \langle \phi(rs, y)\xi, \eta \rangle = \langle \phi(sr, y)\xi, \eta \rangle = \langle \phi(r, y)p\xi, \eta \rangle = \langle \rho(s)T_1\rho(y)W_1\xi, \rho(r)^*V_1^*\eta \rangle \quad (3.26)$$

is normal on $S$ for all $r \in \mathcal{R}, y \in C^*(\mathcal{R}, S)$, and $\xi, \eta \in H$ by the second equality in (3.26). Thus

$$\langle \phi(rs, y)\xi, \eta \rangle = \langle V_1\rho(rs)pT_1\rho(y)W_1\xi, \eta \rangle \quad (3.27)$$

for $r \in \mathcal{R}, s \in S, y \in C^*(\mathcal{R}, S), \xi, \eta \in H$.

Now $p$ is central in $\rho(S)''$ and so commutes with $\rho(S)$. Moreover, by hypothesis, $\rho(\mathcal{R}) \subseteq \rho(S)' = (\rho(S)')'$, so $p \in (\rho(S)')'$ commutes with $\rho(r)$ for $r \in \mathcal{R}$. It follows that $p \in C^*(\mathcal{R}, S)'$. By (3.24), $T_1 \in \rho(S)'$. Thus $T_1$ commutes with all operators in $\rho(S)''$, and in particular with $p$. It follows from (3.27) that

$$\langle \phi(rs, y)\xi, \eta \rangle = \langle V_1p\rho(rs)pT_1p\rho(y)pW_1\xi, \eta \rangle \quad (3.28)$$

and hence, for $x, y \in C^*(\mathcal{R}, S)$,

$$\phi(x, y) = V_1pp(x)pT_1p\rho(y)pW_1. \quad (3.29)$$

Let $L \subseteq K$ be the range of $p$, and observe that this is invariant for $C^*(\mathcal{R}, S)$ since $p \in C^*(\mathcal{R}, S)'$. Thus we may define $\pi = \rho|_L$, $W = pW_1$: $H \rightarrow L$, $T = pT_1|_L$: $L \rightarrow L$, and $V = V_1|_L$: $L \rightarrow H$, to obtain the representation

$$\phi(x, y) = V\pi(x)T\pi(y)W; \quad x, y \in C^*(\mathcal{R}, S) \quad (3.30)$$

from (3.29). For $\xi, \eta \in L$ and $s \in S$,

$$\langle \pi(s)\xi, \eta \rangle = \langle \pi(s)p\xi, \eta \rangle = \langle \rho(s)p\xi, \eta \rangle, \quad (3.31)$$

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which, by (3.25), defines a normal functional on $S$. Thus the restriction of $\pi$ to $S$ is a normal representation. The required relations (3.22) follow easily from (3.24). For $\xi \in L$,

$$(sV - V\pi(s))\xi = (sV_1 - V_1\rho(s))\xi = 0$$  \hspace{1cm} (3.32)

since $\rho(s)\xi \in L$, and so $sV = V\pi(s)$. The other two equalities in (3.22) are verified similarly. Finally, $\|\phi\|_{cb} = \|V\| \|T\| \|W\|$ is clear from the construction.
§4. Factorization for injective ranges

The first two propositions are steps on the way to Theorem 4.4, which is the main result of this section. Throughout, the sum $\sum_{j \in \Lambda}$ of operators over the index set $\Lambda$ is weakly convergent. Note that $\Lambda$ may be uncountable even for von Neumann algebras with separable predual if $\phi$ is not normal, for example. The conclusions can be strengthened slightly, which is discussed in Remark 4.5. In this section, all operators denoted by upper case greek letters $\Psi, \Theta, \Gamma, \Delta$, are defined as in (2.20) and (2.21).

We begin with a technical lemma which will be needed subsequently. The same result for different pairs of tensor products may be found in [14, 17]. On the algebraic tensor product $A \otimes B \otimes C \otimes D$ of $C^*$-algebras, the shuffle map $S$ is defined by

$$S(a \otimes b \otimes c \otimes d) = a \otimes c \otimes b \otimes d.$$  \hspace{1cm} (4.1)

Lemma 4.1. Let $A, B, C$ and $D$ be $C^*$-algebras. The shuffle map induces a complete contraction

$$S: (A \otimes_{\min} B) \otimes_h (C \otimes_{\min} D) \to (A \otimes_h C) \otimes_{\min} (B \otimes_h D).$$ \hspace{1cm} (4.2)

Proof. We will use the fact, [23], that there is an isometric identification between $CB^2(A \times B, B(H))$ and $CB(A \otimes_h B, B(H))$ for $C^*$-algebras $A$ and $B$. Let $\phi_1: A \otimes_h C \to B(K_1)$ and $\phi_2: B \otimes_h D \to B(K_2)$ be completely isometric embeddings. Then $\phi_1 \otimes \phi_2: (A \otimes_h C) \otimes_{\min} (B \otimes_h D) \to B(K_1) \otimes_{\min} B(K_2)$ is a completely isometric embedding [1]. From (2.3) and (2.4), $\phi_1$ and $\phi_2$ may be expressed by

$$\phi_1(a \otimes c) = V_1 \pi_1(a) T_1 \rho_1(c) W_1,$$ \hspace{1cm} (4.3)

$$\phi_2(b \otimes d) = V_2 \pi_2(b) T_2 \rho_2(d) W_2,$$ \hspace{1cm} (4.4)

where $\pi_i, \rho_i$ are $*$-representations and $V_i, T_i$ and $W_i$ are contractive operators between appropriate Hilbert spaces, for $i = 1, 2$. By the definition of the minimal tensor product [30], $\pi_1 \otimes \pi_2$ and $\rho_1 \otimes \rho_2$ define $*$-representations of $A \otimes_{\min} B$ and $C \otimes_{\min} D$ respectively. Define $\psi \in CB^2((A \otimes_{\min} B) \times (C \otimes_{\min} D), B(K_1 \otimes_2 K_2))$ by

$$\psi(a \otimes b, c \otimes d) = (V_1 \otimes V_2) (\pi_1(a) \otimes \pi_2(b)) (T_1 \otimes T_2) (\rho_1(c) \otimes \rho_2(d)) (W_1 \otimes W_2).$$ \hspace{1cm} (4.5)
Then $\psi$ is a completely contractive bilinear operator, and

$$
\psi(a \otimes b, c \otimes d) = \phi_1(a \otimes c) \otimes \phi_2(b \otimes d).
$$

(4.6)

Letting $\tilde{\psi}$ be the associated completely contractive linear operator on

$$(A \otimes_{\min} B) \otimes_h (C \otimes_{\min} D),$$

the complete contractivity of $S$ follows from the relation

$$
S = (\phi_1 \otimes \phi_2)^{-1} \tilde{\psi}, \quad (4.7)
$$

where $(\phi_1 \otimes \phi_2)^{-1}$ is defined on the range of $\phi_1 \otimes \phi_2$.

**Proposition 4.2.** Let $\mathcal{M} \subseteq \mathcal{N} \subseteq B(H)$ be an inclusion of injective von Neumann algebras with $\mathcal{M}$ a factor. If $\phi \in CB^2(\mathcal{M}, \mathcal{N})$ then there exist $\psi_j, \theta_j \in CB(\mathcal{M}, \mathcal{N})$ satisfying

$$
\phi(m_1, m_2) = \sum_{j \in \Lambda} \psi_j(m_1) \theta_j(m_2), \quad m_1, m_2 \in \mathcal{M} \quad (4.8)
$$

and

$$
\|\phi\|_{cb} = \|\Psi\|_{cb}^{1/2} \|\Theta\|_{cb}^{1/2}. \quad (4.9)
$$

**Proof.** We will use the results of Effros and Lance [15, Proposition 4.5 and Corollary 4.6] that for an injective von Neumann algebra $\mathcal{M}$ the map $\eta_{\mathcal{M}}$: $m \otimes m' \to mm'$ from $\mathcal{M} \otimes_{\min} \mathcal{M}'$ into $C^*(\mathcal{M}, \mathcal{M}')$ is a bounded surjective $*$-homomorphism, and is additionally a $*$-isomorphism if $\mathcal{M}$ is a factor. These were proved originally for semidiscrete von Neumann algebras, but semidiscreteness is equivalent to injectivity.

Define the operator $\tilde{\phi}$: $C^*(\mathcal{M}, \mathcal{N}') \times C^*(\mathcal{M}, \mathcal{N}') \to C^*(\mathcal{N}, \mathcal{N}')$ by

$$
\tilde{\phi}(m_1 m_1', m_2 m_2') = \phi(m_1, m_2) m_1' m_2'. \quad (4.10)
$$

for $m_1, m_2 \in \mathcal{M}$, $n_1', n_2' \in \mathcal{N}'$. To see that $\tilde{\phi}$ is a well defined completely bounded bilinear operator observe that it is the composition of the following completely bounded maps.

1. The inclusion

$$
I: C^*(\mathcal{M}, \mathcal{N}') \to C^*(\mathcal{M}, \mathcal{M}'). \quad (4.11)
$$

Here $\mathcal{N}' \subseteq \mathcal{M}'$ since $\mathcal{M} \subseteq \mathcal{N}$.

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(2) The inverse of the Effros-Lance isomorphism

\[ \eta_{M}^{-1} : C^*(M, M') \to M \otimes_{\min} M' \]  

which exists since \( \mathcal{M} \) is an injective factor [15]. Note that the range of \( \eta_{M}^{-1} I \) is \( M \otimes_{\min} \mathcal{N}' \).

(3) The completely bounded bilinear operator \( \phi \otimes \lambda \) from \( (M \otimes_{\min} M') \times (M \otimes_{\min} M') \) to \( \mathcal{N} \otimes_{\min} M' \) given by

\[ (m_1 \otimes m'_1) \times (m_2 \otimes m'_2) \to \phi(m_1, m_2) \otimes m'_1 m'_2, \]  

where \( \lambda : M' \times M' \to M' \) is the completely contractive bilinear multiplication map \( \lambda(m'_1, m'_2) = m'_1 m'_2 \). There are several ways to show that \( \phi \otimes \lambda \) is completely bounded. One such is to observe that the shuffle map \( S : (A \otimes_{\min} B) \otimes_{h} (C \otimes_{\min} D) \to (A \otimes_{h} C) \otimes_{\min} (B \otimes_{h} D) \) for \( C^* \)-algebras \( A, B, C \) and \( D \), defined in (4.1), is completely contractive, by Lemma 4.1. Then recall from [23] that a completely bounded bilinear operator \( \psi : A \times B \to C \) induces a completely bounded linear operator \( \phi : A \otimes B \to C \) of the same norm by \( a \otimes b \to \phi(a, b) \). Combining these results, we obtain a completely bounded linear operator on \( (M \otimes_{\min} M') \otimes_{h} (M \otimes_{\min} M') \) by

\[ (m_1 \otimes m'_1) \otimes (m_2 \otimes m'_2) \to (m_1 \otimes m_2) \otimes (m'_1 \otimes m'_2) \to \phi(m_1, m_2) \otimes m'_1 m'_2 \]  

and this is the linearization of \( \phi \otimes \lambda \). It is easy to check that \( \| \phi \otimes \lambda \|_{cb} = \| \phi \|_{cb} \). Note that \( (\phi \otimes \lambda) \eta_{M}^{-1} I \) has range in \( M \otimes_{\min} \mathcal{N}' \).

(4) The Effros-Lance homomorphism \( \eta_{\mathcal{N}} : \mathcal{N} \otimes_{\min} \mathcal{N}' \to C^*(\mathcal{N}, \mathcal{N}') \), which is continuous since \( \mathcal{N} \) is injective [15].

We now see that

\[ \tilde{\phi} = \eta_{\mathcal{N}}(\phi \otimes \lambda)(\eta_{M}^{-1} \otimes \eta_{M}^{-1})(I \otimes I) \]  

and so \( \tilde{\phi} \) is well defined and completely bounded, as asserted, and it is easily checked that \( \| \tilde{\phi} \|_{cb} = \| \phi \|_{cb} \). Moreover the construction of \( \tilde{\phi} \) shows that this operator is \( \mathcal{N}' \)-modular.

By Proposition 3.3 (with \( R = M, S = \mathcal{N}' \)), \( \tilde{\phi} \) has a representation

\[ \tilde{\phi}(x, y) = V \pi(x) T \pi(y) W, \quad x, y \in C^*(M, \mathcal{N}') \]  

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where \( \pi: C^*(\mathcal{M}, \mathcal{N}') \to B(L) \) is a representation whose restriction to \( \mathcal{N}' \) is normal,

\[
n'V = V\pi(n'), \quad \pi(n')T = T\pi(n'), \quad \pi(n')W = Wn', \quad n' \in \mathcal{N}', \tag{4.17}
\]

and \( \|\phi\|_{cb} = \|V\| \|T\| \|W\| \). By the structure theory of normal representations [11] we may assume that

\[
L \subseteq \ell_2(\Lambda, H) = \ell_2(\Lambda) \otimes_2 H, \tag{4.18}
\]

where \( \Lambda \) is a sufficiently large index set, \( L \) is invariant for the von Neumann subalgebra \( I \otimes \mathcal{N}' \) of \( B(\ell_2(\Lambda)) \otimes B(H) \), and \( \pi(n') = (I \otimes n')q \) where \( q \) is the projection in \( (I \otimes \mathcal{N}')' = B(\ell_2(\Lambda)) \overline{\otimes} \mathcal{N} \) onto \( L \). Writing matrices relative to the decomposition \( \ell_2(\Lambda) \otimes_2 H = L \oplus L^\perp \), we have

\[
\tilde{\phi}(x, y) = (V, 0) \begin{pmatrix} \pi(x) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \pi(y) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} W \\ 0 \end{pmatrix} \tag{4.19}
\]

for all \( x, y \in C^*(\mathcal{M}, \mathcal{N}') \). Since \( T \) commutes with \( \pi(\mathcal{N}') \) by (4.17), \( \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix} \) commutes with \( \begin{pmatrix} \pi(n') & 0 \\ 0 & z \end{pmatrix} \) for all \( z \in B(L^\perp) \), and in particular \( T \) commutes with \( I \otimes n' \), taking \( z = (1 - q)(I \otimes n')(1 - q) \). Hence \( \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix} \in B(\ell_2(\Lambda)) \overline{\otimes} \mathcal{N} \). Similarly

\[
n'(V, 0) = (n'V, 0) \\
= (V\pi(n'), 0) \\
= (V, 0) \begin{pmatrix} \pi(n') & 0 \\ 0 & z \end{pmatrix} \tag{4.20}
\]

for all \( z \in B(L^\perp) \) so the same choice for \( z \) as before shows that \( n'(V, 0) = (V, 0)(I \otimes n') \) for all \( n' \in \mathcal{N}' \). Thus \((V, 0) \in R(\Lambda) \overline{\otimes} \mathcal{N} \) where \( R(\Lambda) \) is the row space of \( B(\ell_2(\Lambda)) \). A similar calculation shows that \( \begin{pmatrix} W \\ 0 \end{pmatrix} \in C(\Lambda) \overline{\otimes} \mathcal{N} \) where \( C(\Lambda) \) is the column space of \( B(\ell_2(\Lambda)) \). Finally \( \begin{pmatrix} \pi(m) & 0 \\ 0 & 0 \end{pmatrix} \) commutes with \( \begin{pmatrix} \pi(n') & 0 \\ 0 & z \end{pmatrix} \) for all \( m \in \mathcal{M}, n' \in \mathcal{N}', z \in B(L^\perp) \), so the same choice of \( z \) shows that \( \begin{pmatrix} \pi(m) & 0 \\ 0 & 0 \end{pmatrix} \) commutes with \( I \otimes \mathcal{N}' \) and thus lies in \( B(\ell_2(\Lambda)) \overline{\otimes} \mathcal{N} \).

Now for each \( j \in \Lambda \), let \( e_j \) be the orthogonal projection onto the basis vector of \( \ell_2(\Lambda) \) with a 1 in the \( j^{th} \) place and 0 in every other position. Then define \( \psi_j, \theta_j: \mathcal{M} \to \mathcal{N} \) for
each $j \in \Lambda$ by letting $\psi_j(m)$ be the $j^{th}$ component of $(V, 0) \left( \begin{array}{cc} \pi(m) & 0 \\ 0 & 0 \end{array} \right)$ in $R(\Lambda) \otimes \mathcal{N}$, and by letting $\theta_j(m)$ be the $j^{th}$ component of $\left( T \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right) \left( \begin{array}{cc} \pi(m) & 0 \\ 0 & 0 \end{array} \right) \left( W \begin{array}{cc} 0 & 0 \end{array} \right)$ in $C(\Lambda) \otimes \mathcal{N}$. Thus

$$\psi_j(m) = (V, 0) \left( \begin{array}{cc} \pi(m) & 0 \\ 0 & 0 \end{array} \right) (e_j \otimes I) \quad (4.21)$$

and

$$\theta_j(m) = (e_j \otimes I) \left( T \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right) \left( \begin{array}{cc} \pi(m) & 0 \\ 0 & 0 \end{array} \right) \left( W \begin{array}{cc} 0 & 0 \end{array} \right) \quad (4.22)$$

for all $m \in \mathcal{M}$. From (4.21), (4.22) and the restriction of (4.19) to $\mathcal{M} \times \mathcal{M}$, it follows that

$$\phi(m_1, m_2) = \sum_{j \in \Lambda} \psi_j(m_1) \theta_j(m_2) \quad (4.23)$$

for all $m_1, m_2 \in \mathcal{M}$.

Now

$$\Psi(m_1, m_2) = \sum_{j \in \Lambda} \psi_j(m_1) \psi_j(m_2)^*$$

$$= \sum_{j \in \Lambda} (V, 0) \left( \begin{array}{cc} \pi(m_1) & 0 \\ 0 & 0 \end{array} \right) (e_j \otimes I) \left( \begin{array}{cc} \pi(m_2) & 0 \\ 0 & 0 \end{array} \right) \left( V^* \right)$$

$$= V \pi(m_1) I \pi(m_2) V^* \quad (4.24)$$

which shows that $\Psi$ is completely bounded and that $\|\Psi\|_{cb} = \|V\|^2$. Similarly

$$\Theta(m_1, m_2) = \sum_{j \in \Lambda} \theta_j(m_1)^* \theta_j(m_2)$$

$$= \sum_{j \in \Lambda} (W^*, 0) \left( \begin{array}{cc} \pi(m_1) & 0 \\ 0 & 0 \end{array} \right) \left( T^* \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right) (e_j \otimes I) \left( \begin{array}{cc} \pi(m_2) & 0 \\ 0 & 0 \end{array} \right) \left( W \begin{array}{cc} 0 & 0 \end{array} \right)$$

$$= W^* \pi(m_1) T^* T \pi(m_2) W \quad (4.25)$$

which shows that $\Theta$ is completely bounded and that $\|\Theta\|_{cb} \leq \|W\|^2 \|T\|^2$. Then (4.9) is an immediate consequence of $\|\phi\|_{cb} \geq \|\Psi\|_{cb}^{1/2} \|\Theta\|_{cb}^{1/2}$ and (2.22).

The next step is to remove the hypothesis $\mathcal{M} \subseteq \mathcal{N}$ from Proposition 4.2.
Proposition 4.3. Let $\mathcal{M} \subseteq B(H)$ be an injective factor and let $\mathcal{N}$ be an injective von Neumann algebra. If $\phi \in \text{CB}^2(\mathcal{M}, \mathcal{N})$ then there exist $\psi_j, \theta_j \in \text{CB}(\mathcal{M}, \mathcal{N})$ satisfying
\begin{equation}
\phi(m_1, m_2) = \sum_{j \in \Lambda} \psi_j(m_1)\theta_j(m_2), \quad m_1, m_2 \in \mathcal{M}
\end{equation}
and
\begin{equation}
\|\phi\|_{\text{cb}} = \|\Psi\|_{\text{cb}}^{1/2} \|\Theta\|_{\text{cb}}^{1/2}.
\end{equation}

Proof. Identify $\mathcal{M}$ with the subalgebra $I \otimes \mathcal{M}$ of $\mathcal{N} \otimes B(H)$, and let $e$ be a rank one projection in $B(H)$. Then define $\tilde{\phi}: \mathcal{M} \times \mathcal{M} \to \mathcal{N} \otimes B(H)$ by $\tilde{\phi}(m_1, m_2) = \phi(m_1, m_2) \otimes e$, and note that $\|\tilde{\phi}\|_{\text{cb}} = \|\phi\|_{\text{cb}}$. The injective factor $\mathcal{M}$ is now a subalgebra of the injective von Neumann algebra $\mathcal{N} \otimes B(H)$ so Proposition 4.2 applies to $\tilde{\phi}$. Hence
\begin{equation}
\tilde{\phi}(m_1, m_2) = \sum_{j \in \Lambda} \gamma_j(m_1)\delta_j(m_2), \quad m_1, m_2 \in \mathcal{M}
\end{equation}
where $\gamma_j, \delta_j \in \text{CB}(\mathcal{M}, \mathcal{N} \otimes B(H))$ and $\|\tilde{\phi}\|_{\text{cb}} = \|\Gamma\|_{\text{cb}}^{1/2} \|\Delta\|_{\text{cb}}^{1/2}$. Thus
\begin{equation}
\phi(m_1, m_2) \otimes e = \sum_{j \in \Lambda} (1 \otimes e)\gamma_j(m_1)\delta_j(m_2)(1 \otimes e), \quad m_1, m_2 \in \mathcal{M}.
\end{equation}

Let $\{\xi_k: k \in \Omega\}$ be an orthonormal basis for $H$, let $\{f_k: k \in \Omega\}$ be the associated rank one projections and choose partial isometries $\{v_k: k \in \Omega\}$ such that $v_k^*v_k = f_k$ and $v_kv_k^* = e$. Then define $\psi_{jk}, \theta_{jk} \in \text{CB}(\mathcal{M}, \mathcal{N})$ by
\begin{equation}
\psi_{jk}(m) \otimes v_k = (1 \otimes e)\gamma_j(m)(1 \otimes f_j)
\end{equation}
and
\begin{equation}
\theta_{jk}(m) \otimes v_k^* = (1 \otimes f_k)\delta_j(m)(1 \otimes e).
\end{equation}

It follows from (4.29) that
\begin{equation}
\phi(m_1, m_2) = \sum_{j \in \Lambda} \psi_{jk}(m_1)\theta_{jk}(m_2),
\end{equation}
giving the required factorization. Moreover
\begin{align}
\Psi(m_1, m_2) \otimes e &= \sum_{j \in \Lambda} (1 \otimes e)\gamma_j(m_1)(1 \otimes f_k)\gamma_j(m_2^*)(1 \otimes e) \\
&= (1 \otimes e)\Gamma(m_1, m_2)(1 \otimes e)
\end{align}
(4.33)
so $\|\Psi\|_{cb} \leq \|\Gamma\|_{cb}$. Similarly $\|\Theta\|_{cb} \leq \|\Delta\|_{cb}$, and (4.27) follows from the corresponding result for $\Gamma$ and $\Delta$.

We come now to the main result of the section.

**Theorem 4.4.** Let $\mathcal{X}$ and $\mathcal{Y}$ be operator spaces and let $\mathcal{N} \subseteq B(K)$ be an injective von Neumann algebra. If $\phi \in CB^2(\mathcal{X} \times \mathcal{Y}, \mathcal{N})$ then there exist $\psi_j \in CB(\mathcal{X}, \mathcal{N})$ and $\theta_j \in CB(\mathcal{Y}, \mathcal{N})$ such that

$$\phi(x, y) = \sum_{j \in \Lambda} \psi_j(x) \theta_j(y), \quad x \in \mathcal{X}, y \in \mathcal{Y},$$

and

$$\|\phi\|_{cb} = \|\Psi\|^{1/2}_{cb} \|\Theta\|^{1/2}_{cb}. \quad (4.35)$$

**Proof.** Let $H$ be a Hilbert space such that $\mathcal{X}, \mathcal{Y} \subseteq B(H)$ as operator spaces, and extend $\phi$ to $\phi_1: B(H) \times B(H) \to B(K)$ with preservation of norm [23]. If $E$ is the conditional expectation from $B(K)$ to $\mathcal{N}$, let $\phi_2 = E\phi_1$. Then $\phi_2$ is a norm preserving extension of $\phi$ and $\phi_2 \in CB^2(B(H), \mathcal{N})$. Proposition 4.3 now gives the required completely bounded linear operators by restricting to $\mathcal{X}$ and $\mathcal{Y}$ those defined on $B(H)$.

**Remark 4.5.** (i) We have stated Theorem 4.4 in the full generality of operator spaces so that generalizations to multilinear operators are immediate. For example, if $\phi: \mathcal{A} \times \mathcal{B} \times \mathcal{C} \to \mathcal{N}$ were a completely bounded trilinear operator, a factorization could be obtained by identifying $\phi$ with a bilinear operator $\psi \in CB^2((\mathcal{A} \otimes_h \mathcal{B}) \times \mathcal{C}, \mathcal{N})$ and applying Theorem 4.4. Even if $\mathcal{A}$ and $\mathcal{B}$ were $C^*$-algebras, $\mathcal{A} \otimes_h \mathcal{B}$ is only an operator space in general. This technique of employing the Haagerup tensor product comes from [23].

(ii) An examination of the proofs of the first two propositions shows that if $\phi$ is separately normal in each variable, then the resulting operators $\psi_j, \theta_j$ may be chosen to be normal. However this is not necessarily true in Theorem 4.4 for dual operator spaces since the conditional expectation may destroy normality.

(iii) If $\mathcal{Y} = \mathcal{X}^*$ in Theorem 4.4 then it makes sense to consider the extra hypothesis of complete positivity for $\phi$. Again an examination of the proofs, being careful to choose completely positive extensions at each stage, shows that we may take $\theta_j$ to be $\psi_j^*$, where $\psi_j^*(x^*)$ is defined to be $\psi_j(x)^*$. 
(iv) For the simplest injective von Neumann algebra $C$, Theorem 4.4 recaptures, in different language, the result of [2] that the dual of $\mathcal{X} \otimes_h \mathcal{Y}$ is $\mathcal{X}^* \otimes_{w^*h} \mathcal{Y}^*$ for operator spaces $\mathcal{X}$ and $\mathcal{Y}$. Thus Theorem 4.4 may be viewed as a generalization of [2, Theorem 3.2].
§5. Completely bounded factorization implies injectivity

In the previous section we considered factorizations \( \phi(x, y) = \sum_{j \in \Lambda} \psi_j(x) \theta_j(y) \) in \( CB^2(X \times Y, N) \) of completely bounded bilinear operators, where the associated bilinear operators \( \Psi \) and \( \Theta \) were completely bounded. We now wish to broaden the set of admissible factorizations by considering ones for which there exists a constant \( K \) such that

\[
\left\| \sum_{j \in \Lambda} \psi_j(x) \psi_j(x)^* \right\| \leq K \|x\|^2, \quad (5.1)
\]

\[
\left\| \sum_{j \in \Lambda} \theta_j(y)^* \theta_j(y) \right\| \leq K \|y\|^2. \quad (5.2)
\]

Then the associated bilinear operators \( \Psi \) and \( \Theta \) are still completely positive and bounded, but perhaps not completely bounded. We will distinguish these two factorizations by calling them type CB (for completely bounded) and type B (for bounded) respectively. We emphasize that the operators \( \psi_j, \theta_j \) are completely bounded in both cases, and the names reflect the nature of \( \Psi \) and \( \Theta \).

The following lemma records a standard decomposition of certain completely bounded bilinear operators as a linear combination of continuous completely positive bilinear operators in exactly the correct form for subsequent use. The technique is well known in the theory of quadratic forms.

**Lemma 5.1.** Let \( \mathcal{M} \) and \( \mathcal{N} \) be von Neumann algebras and suppose that \( \phi \in CB^2(\mathcal{M}, \mathcal{N}) \) has a factorization of type B. Then there exist continuous completely positive bilinear maps \( \phi_k: \mathcal{M} \times \mathcal{M} \to \mathcal{N} \), \( 1 \leq k \leq 4 \), such that

\[
\phi = (\phi_1 - \phi_2) + i(\phi_3 - \phi_4). \quad (5.3)
\]

**Proof.** Suppose that

\[
\phi(m_1, m_2) = \sum_{j \in \Lambda} \psi_j(m_1) \theta_j(m_2), \quad m_1, m_2 \in \mathcal{M} \quad (5.4)
\]

is a factorization with \( \psi_j, \theta_j \in CB(\mathcal{M}, \mathcal{N}) \) satisfying (5.1) and (5.2). The algebraic identity

\[
\psi_j(m_1) \theta_j(m_2) = \frac{1}{4} \sum_{k=1}^{4} i^k (\psi_j(m_1) + i^k \theta_j^*(m_1))(\psi_j^*(m_2) + i^{-k} \theta_j(m_2)) \quad (5.5)
\]
expresses each bilinear operator \( \psi_j(m_1)\theta_j(m_2) \) as a linear combination of continuous completely positive bilinear operators. Then (5.3) follows by summing (5.5) over \( j \in \Lambda \), provided that the resulting sums on the right hand side define continuous bilinear operators. We examine \( \sum_{j \in \Lambda} (\psi_j(m_1) + \theta_j^*(m_1))(\psi_j^*(m_2) + \theta_j(m_2)) \), which is typical. If \( \xi, \eta \) are arbitrary vectors, then

\[
\left| \sum_{j \in \Lambda} \langle (\psi_j(m_1) + \theta_j^*(m_1))(\psi_j^*(m_2) + \theta_j(m_2))\xi, \eta \rangle \right| \\
\leq \sum_{j \in \Lambda} |\langle \psi_j(m_2)^*\xi, \psi_j(m_1)^*\eta \rangle| + \sum_{j \in \Lambda} |\langle \theta_j(m_2)\xi, \theta_j(m_1)^*\eta \rangle| \\
+ \sum_{j \in \Lambda} |\langle \theta_j(m_2)\xi, \psi_j(m_1)^*\eta \rangle| + \sum_{j \in \Lambda} |\langle \psi_j(m_2)^*\xi, \theta_j(m_1)^*\eta \rangle|. \tag{5.6}
\]

The estimation of each of these four terms is identical; we take the first as typical. Then, applying the Cauchy-Schwarz inequality,

\[
\sum_{j \in \Lambda} |\langle \psi_j(m_2)^*\xi, \psi_j(m_1)^*\eta \rangle| \\
\leq \sum_{j \in \Lambda} \|\psi_j(m_2)^*\xi\| \|\psi_j(m_1)^*\eta\| \\
\leq \left( \sum_{j \in \Lambda} \|\psi_j(m_2)^*\xi\|^2 \right)^{1/2} \left( \sum_{j \in \Lambda} \|\psi_j(m_1)^*\eta\|^2 \right)^{1/2} \\
= \left( \sum_{j \in \Lambda} \langle \psi_j(m_2)^*\psi_j(m_2)^*\xi, \xi \rangle \right)^{1/2} \left( \sum_{j \in \Lambda} \langle \psi_j(m_1)^*\psi_j(m_1)^*\eta, \eta \rangle \right)^{1/2} \\
\leq \left| \sum_{j \in \Lambda} \psi_j(m_2)^*\psi_j(m_2)^* \right|^{1/2} \left| \sum_{j \in \Lambda} \psi_j(m_1)^*\psi_j(m_1)^* \right|^{1/2} \|\xi\| \|\eta\| \\
\leq K \|m_1\| \|m_2\| \|\xi\| \|\eta\| \tag{5.7}
\]

by (5.1) and (5.2). Returning to (5.6), we obtain

\[
\left\| \frac{1}{4} \sum_{j \in \Lambda} (\psi_j(m_1) + \theta_j^*(m_1))(\psi_j^*(m_2) + \theta_j(m_2)) \right\| \leq K \|m_1\| \|m_2\|, \tag{5.8}
\]

and so each sum over \( j \in \Lambda \) on the right hand side of (5.5) is a continuous bilinear operator of norm at most \( K \).
Every infinite dimensional von Neumann algebra $\mathcal{M}$ contains a copy of $\ell^\infty$. Let us fix such a copy, and denote by $\mathcal{U}$ its abelian (and hence amenable) unitary group. Then let $\beta$ be a fixed normalized invariant mean on the space of complex valued bounded functions on $\mathcal{U}$. Letting $B^2(\mathcal{M}, \mathcal{N})$ denote the space of bounded bilinear maps on $\mathcal{M} \times \mathcal{M}$ into a von Neumann algebra $\mathcal{N}$, there is an induced map $\gamma: B^2(\mathcal{M}, \mathcal{N}) \to B^2(\mathcal{M}, \mathcal{N})$ defined as follows. For $x, y \in \mathcal{M}$ and $\omega \in \mathcal{N}^*$ the function $f_{x,y,\omega}(u) = \omega(\phi(xu,u^*y))$ is bounded by $\|\omega\| \|\phi\| \|x\| \|y\|$. We define $\gamma \phi(x,y) \in (\mathcal{N}^*)^* = \mathcal{N}$ by

$$
\gamma \phi(x,y)(\omega) = \beta(f_{x,y,\omega}).
$$

The technique of averaging between the variables in the next lemma comes from [12].

**Lemma 5.2.** The map $\gamma: B^2(\mathcal{M}, \mathcal{N}) \to B^2(\mathcal{M}, \mathcal{N})$ is a linear contraction and $\gamma \phi$ satisfies

$$
\gamma \phi(xa,y) = \gamma \phi(x,ay) \quad x, y \in \mathcal{M}, \quad a \in \ell^\infty
$$

(5.10)

for $\phi \in B^2(\mathcal{M}, \mathcal{N})$. Moreover $\gamma$ preserves both complete boundedness and complete positivity.

**Proof.** Equation (5.10) is a consequence of the invariance of $\beta$ and the standard fact that every unital $C^*$-algebra is the span of its unitary group. The other parts of the lemma are routine deductions from the definition of $\gamma$. For example, if $\mathcal{N}$ is represented on $H$, $\xi_1, \ldots, \xi_n \in H$, $(x_{ij}) \in \mathcal{M}_n(\mathcal{M})$, and $\phi$ is completely positive, then

$$
\left\langle (\gamma \phi)_n((x_{ij}),(x_{ij})^*), \left(\begin{array}{c} \xi_1 \\ \vdots \\ \xi_n \end{array}\right), \left(\begin{array}{c} \xi_1 \\ \vdots \\ \xi_n \end{array}\right) \right\rangle \geq 0
$$

(5.11)

because this inner product is obtained by applying $\beta$ to the non-negative function

$$
\langle \phi_n((x_{ij}u),(x_{ij}u)^*), \left(\begin{array}{c} \xi_1 \\ \vdots \\ \xi_n \end{array}\right), \left(\begin{array}{c} \xi_1 \\ \vdots \\ \xi_n \end{array}\right) \rangle.
$$

This shows that $\gamma \phi$ is completely positive, and we omit further details.
Theorem 5.3. Let $\mathcal{M}$ be an infinite dimensional von Neumann algebra and let $\mathcal{N}$ be a von Neumann algebra. If every $\phi \in CB^2(\mathcal{M}, \mathcal{N})$ is a linear combination of continuous completely positive bilinear operators, then $\mathcal{N}$ is injective.

Proof. This result is deduced from a theorem of Haagerup [20, Theorem 2.1] on decomposable completely bounded linear operators from $\ell^\infty$ into a von Neumann algebra $\mathcal{N}$. Fix a copy of $\ell^\infty$ with unitary group $\mathcal{U}$ in $\mathcal{M}$ and an invariant mean $\beta$ as in the previous lemma. Let $\mathcal{E}$ be the conditional expectation from $\mathcal{M}$ onto $\ell^\infty$. If $\phi \in CB(\ell^\infty, \mathcal{N})$, define $\theta \in CB^2(\mathcal{M}, \mathcal{N})$ by

$$
\theta(m_1, m_2) = \phi(\mathcal{E}(m_1 m_2)), \quad m_1, m_2 \in \mathcal{M}. \tag{5.12}
$$

Since $m_1 \times m_2 \to m_1 m_2$ is a completely bounded bilinear operator on $\mathcal{M} \times \mathcal{M}$ and $\mathcal{E}$ is completely positive, it is routine to check that $\theta$ is completely bounded. By hypothesis, there are continuous completely positive bilinear operators $\theta_j: \mathcal{M} \times \mathcal{M} \to \mathcal{N}$ ($j = 1, \ldots, 4$) such that

$$
\theta = \theta_1 - \theta_2 + i\theta_3 - i\theta_4. \tag{5.13}
$$

Now $\theta(m_1u, u^* m_2) = \theta(m_1, m_2)$ for all $u \in \mathcal{U}$, by (5.12), and so $\gamma \theta = \theta$. Applying $\gamma$ to (5.13) and noting that $\phi(x) = \theta(x, 1)$ for $x \in \ell^\infty$, we obtain

$$
\phi(x) = \gamma\theta_1(x, 1) - \gamma\theta_2(x, 1) + i\gamma\theta_3(x, 1) - i\gamma\theta_4(x, 1), \quad x \in \ell^\infty. \tag{5.14}
$$

Now define $\phi_j: \ell^\infty \to \mathcal{N}$ ($j = 1, \ldots, 4$) by $\phi_j(x) = \gamma\theta_j(x, 1)$. For $x \in \ell^\infty$,

$$
\phi_j(xx^*) = \gamma\theta_j(xx^*, 1) = \gamma\theta_j(x, x^*) \geq 0, \tag{5.15}
$$

by Lemma 5.2 and complete positivity of $\theta_j$. Positivity of $\phi_j$ is immediate from (5.15), and since $\ell^\infty$ is an abelian $C^*$-algebra, complete positivity of $\phi_j$ follows from [29]. Thus (5.14) expresses $\phi$ as a linear combination of completely positive maps, and the result follows from [20, Theorem 2.1].

Combining Lemma 5.1 and Theorem 5.3, we immediately have the following result.

Theorem 5.4. Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras with $\mathcal{M}$ infinite dimensional. If every $\phi \in CB^2(\mathcal{M}, \mathcal{N})$ has a factorization of type $B$, then $\mathcal{N}$ is injective.
§6. Summary of results

In this section we collect together the results of previous sections. Recall that a factorization $\phi(x, y) = \sum_{j \in \Lambda} \psi_j(x)\theta_j(y)$ is of type CB (respectively type B) if the associated bilinear maps $\Psi$ and $\Theta$ are completely bounded (respectively bounded). Also recall the product map $\nu: CB(X, \mathcal{M}) \otimes_{w^*h} CB(Y, \mathcal{M}) \to CB^2(X \times Y, \mathcal{M})$ from Section 2.

**Theorem 6.1.** Let $\mathcal{M}$ be a von Neumann algebra. Then the following are equivalent:

(i) $\mathcal{M}$ is injective,

(ii) each $\phi \in CB^2(\mathcal{M}, \mathcal{M})$ has a type CB factorization

$$\phi(m_1, m_2) = \sum_{j \in \Lambda} \psi_j(m_1)\theta_j(m_2), \quad \psi_j, \theta_j \in CB(\mathcal{M}, \mathcal{M}),$$

(iii) each $\phi \in CB^2(\mathcal{M}, \mathcal{M})$ has a type B factorization with $\psi_j, \theta_j \in CB(\mathcal{M}, \mathcal{M}),$

(iv) $\nu: CB(\mathcal{M}, \mathcal{M}) \otimes_{w^*h} CB(\mathcal{M}, \mathcal{M}) \to CB^2(\mathcal{M}, \mathcal{M})$ is surjective and a complete quotient map.

**Proof.** (i) $\Rightarrow$ (ii). This is Theorem 4.4 in the case $X = Y = N = M$.

(ii) $\Rightarrow$ (iii). This is obvious.

(iii) $\Rightarrow$ (i). This is Theorem 5.4. Of course there is nothing to prove if $\mathcal{M}$ is finite dimensional.

(iv) $\Rightarrow$ (ii). This is immediate from the definition of the $w^*$-Haagerup tensor product (see Section 2).

(i) $\Rightarrow$ (iv). The surjectivity of $\nu$ is Theorem 4.4. The fact that $\nu$ is a complete quotient map is just a restatement of (4.35) of that theorem.

**Remark 6.2.** In Theorem 6.1 we have only given the main equivalences that are internal to $\mathcal{M}$, but there are many others which could be extracted from the previous sections. For example (ii) and (iii) could be recast for general operator spaces $X$ and $Y$, while dropping the complete quotient map hypothesis from (iv) gives a statement which is clearly equivalent to (ii).
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