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Quantum electrodynamics of accelerated atoms in free space and in confined cavities

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Intriguing properties of vacuum as viewed by accelerated observers have been the subject of intense investigation for almost three decades. One of the most remarkable results is the so-called Fulling-Unruh effect predicted and analyzed by Davies, Fulling, Unruh and DeWitt [1], and others [2]. In essence, it was shown that ground state atoms, accelerated through vacuum, are promoted to an excited state just as if they were in contact with a blackbody thermal field. These studies predict that a (two-level) ground state atom, having transition frequency \( \omega \), is not of experimental interest.

To begin with, we note that the physical picture of the Unruh effect in any setting is quite straightforward. In particular, it is the counter-rotating terms in the interaction Hamiltonian that describe the process of an excitation of an atom with simultaneous emission of a photon into a field mode. In the case of a slow switching on of the interaction, the ratio of emission and absorption probabilities is exponentially small and is described by the Unruh factor. In the opposite case of sharp cavity boundaries the above ratio is much greater and radiation is produced with an intensity which can exceed the intensity of Unruh acceleration radiation in free space by many orders of magnitude. In both cases real photons are produced, contrary to the opinion that a uniformly accelerated atom does not radiate. The cavity field at steady state is described by a thermal density matrix. However, under some conditions laser gain is possible. We present a detailed discussion of how the acceleration of atoms affects the generated cavity field in different situations, progressing from a simple physical picture of Unruh radiation to more complicated situations.

I. INTRODUCTION

We consider a gedanken experiment with a beam of atoms in their ground state that are accelerated through a single-mode microwave cavity. We show that taking into account of the "counter-rotating" terms in the interaction Hamiltonian leads to the excitation of an atom with simultaneous emission of a photon into a field mode. In the case of a slow switching on of the interaction, the ratio of emission and absorption probabilities is exponentially small and is described by the Unruh factor. In the opposite case of sharp cavity boundaries the above ratio is much greater and radiation is produced with an intensity which can exceed the intensity of Unruh acceleration radiation in free space by many orders of magnitude. In both cases real photons are produced, contrary to the opinion that a uniformly accelerated atom does not radiate. The cavity field at steady state is described by a thermal density matrix. However, under some conditions laser gain is possible. We present a detailed discussion of how the acceleration of atoms affects the generated cavity field in different situations, progressing from a simple physical picture of Unruh radiation to more complicated situations.
FIG. 1: (a) Resonant absorption or emission: an atom is excited (deexcited) as it simultaneously absorbs (emits) a photon. (b) Counter-resonant absorption or emission processes that are usually neglected in the “rotating wave approximation”: an atom is excited (deexcited) as it simultaneously emits (absorbs) a photon. (c) The energy for counter-resonant emission is drawn from work done by a force accelerating an atom. (d) Atoms or ions in their ground state $|b\rangle$ are accelerated through a single-mode microwave or optical cavity.

Other processes where counter-rotating terms play a crucial role include, e.g., parametric resonance and the anomalous Doppler effect. The similarity between the Unruh effect and the anomalous Doppler effect, in which an atom moving faster than the velocity of light in a medium is also excited to an upper state while simultaneously emitting a photon, has been emphasized in [6]. In both cases the energy for the excitation of an atom and emission of a photon is taken from the work done by the force supporting the motion of an atom along the given trajectory. However, for the Doppler effect there are no time-changing parameters: the excitation occurs due to the existence of the Cerenkov resonance.

In section II we formulate the model and write down the Hamiltonian. In section III the transition probabilities for emission are shown to yield a simple physical picture of the Unruh radiation. Next, the master equation is derived and the steady-state solution for the photon density matrix is derived and analyzed. In sections IV-VI we analyze the mechanism of emission and absorption by the accelerated atoms and its relation to the standard Unruh effect. The emission and absorption probabilities are calculated. The resulting integrals are evaluated by the method of stationary phase in all physically interesting asymptotic limits. The possibility of amplification and laser action is discussed. The interpretation of the results and discussion are presented in section VII.

II. THE MODEL

We start from writing Hamiltonian for the system consisting of a two-level atom interacting with the electromagnetic field:

$$\hat{H} = \hat{H}_a + \hat{H}_f + \hat{V}. \quad (1)$$
Here $\hat{H}_a = \hbar \omega \hat{\sigma}_z$ is the Hamiltonian for an atom with ground and excited states $|b\rangle$ and $|a\rangle$, respectively, separated by energy difference $E_a - E_b = \hbar \omega$, $\hat{\sigma}_z = 1/2(|a\rangle \langle b| - |b\rangle \langle a|)$ the Pauli matrix, and $\hat{H}_f = \sum_k \hbar \nu_k \hat{a}_k^+ \hat{a}_k$ is the field Hamiltonian, where $\hat{a}_k^+$ and $\hat{a}_k$ are photon creation and annihilation operators and the k-summation is taken over the electromagnetic modes of a cavity or a free space, depending on the formulation of the problem.

The atom-field interaction Hamiltonian in the atomic rest frame can be written in the dipole approximation as

$$\hat{V} = \hbar \sum_k g_k (\hat{a}_k^+ + \hat{a}_k) u_k(z(\tau)) (\hat{\sigma} + \hat{\sigma}^+) .$$

(2)

Here $u_k(r)$ form a set of orthogonal, normalized functions, $\hat{\sigma}^+ = |a\rangle \langle b|$ and $\hat{\sigma} = |b\rangle \langle a|$ are the atomic raising and lowering operators, and $g_k = \mu_E k / \hbar$ is the atom-field coupling frequency, which depends on the atomic dipole moment $\mu$ and the electric field amplitude $E_k$ in the rest frame of an atom, evaluated on the trajectory $z(\tau)$ of an atom as a function of proper time $\tau$.

In the interaction representation, the master equation for the density operator $\hat{\rho}$ can be written as

$$i\hbar \frac{d\hat{\rho}}{d\tau} = \left[ \hat{V}(\tau) \hat{\rho} \right] ,$$

(3)

where the interaction Hamiltonian in this representation can be obtained by replacing $\hat{a}_k \rightarrow \hat{a}_k \exp(-i \nu_k t(\tau))$, $\hat{a}_k^+ \rightarrow \hat{a}_k^+ \exp(i \nu_k t(\tau))$, $\hat{\sigma} \rightarrow \hat{\sigma} \exp(-i \omega t)$, and $\hat{\sigma}^+ \rightarrow \hat{\sigma}^+ \exp(i \omega t)$, where $t$ is the time in the inertial laboratory frame. Note that we do not intend to use rotating wave approximation since the excitation of an atom from the ground state with simultaneous emission of a photon is described by counter-rotating terms of the type $\hat{a}^+ \hat{\sigma}^+$.

Our main goal will be to solve master equation Eq. (3) and analyze its solution in various limits. However, we can get some insight into the physics of an accelerated atom-field interaction by solving first a simpler problem, namely, calculating photon emission and absorption probabilities within the standard first-order perturbation theory.

### III. TRANSITION PROBABILITIES, MASTER EQUATION, AND THE PHOTON DENSITY MATRIX

#### A. Probabilities of photon emission and absorption

Consider an atom entering the cavity at a proper time $\tau_i$, at which moment the interaction with a cavity mode is assumed to be turned on. If the interaction is weak enough, the state vector of the system atom+field at any subsequent time $\tau$ can be found using the first-order perturbation theory:

$$|\psi(\tau)\rangle = |\psi(\tau_i)\rangle - \frac{i}{\hbar} \int_{\tau_i}^{\tau} \hat{V}(\tau') d\tau' |\psi(\tau_i)\rangle .$$

(4)

The probability of transition $|\psi(\tau_i)\rangle \rightarrow |\psi(\tau)\rangle$ is therefore given by

$$P = \frac{1}{\hbar^2} \left| \int_{\tau_i}^{\tau} \langle \psi(\tau') | \hat{V}(\tau') | \psi(\tau_i) \rangle d\tau' \right|^2 .$$

(5)

In particular, if an atom was initially in its ground state $|b\rangle$, the probability of excitation of an atom with simultaneous photon emission into the $k$th mode can be calculated as

$$P(1_k, a) = \frac{1}{\hbar^2} \left| \int_{\tau_i}^{\tau} g_k(\tau') \langle 1_k | \hat{a}_k^+ | 0_k \rangle \langle a | \hat{\sigma}^+ | b \rangle d\tau' \right|^2 .$$

(6)

The probability of photon absorption from the $k$th mode by a ground-state atom, when there is only photon in this mode, is given by

$$P(0_k, a) = \frac{1}{\hbar^2} \left| \int_{\tau_i}^{\tau} g_k(\tau') \langle 0_k | \hat{a}_k | 1_k \rangle \langle a | \hat{\sigma}^+ | b \rangle d\tau' \right|^2 .$$

(7)

Now consider a uniformly accelerated atom moving along the trajectory $t(\tau) = t_0 + \frac{1}{\alpha} \sinh(\alpha \tau), \quad z(\tau) = \frac{c}{\alpha} \left[ \cosh(\alpha \tau) - 1 \right]$.
where $t_0 = t(\tau = 0)$ is the moment of time in the laboratory (cavity) frame when the atom starts its acceleration. Expanding the electromagnetic field in terms of running waves with wave vectors $k, k_z = k \cdot v/v$, the atom-field interaction Hamiltonian in the atomic frame is given by

$$\dot{\hat{V}}(\tau) = \sum_k \hbar g_k(\tau) \left[ \hat{a}_k e^{-i\omega(\tau + i\alpha \tau/\alpha)} + \text{h.c.} \right] \left[ \hat{\sigma} e^{-i\omega \tau} + \text{h.c.} \right].$$

For the dipole moment $\mu$ oriented in $x$-direction, the corresponding electric field amplitude in the atomic frame $E'_x$ is related to the $x$-component of the electric field amplitude in the lab frame as $E'_x = (c - v)/(c + v)E_x$. Since $v = c \tanh(\alpha \tau)$ for a uniformly accelerated particle, we have $E'_x = e^{-\alpha \tau}E_x$ and $g_k(\tau) = g_k e^{-\alpha \tau}$.

For simplicity, consider the case of a single-mode cavity and co-propagating atom and field: $k_z = |k| = \nu/c$. Substituting Eqs. (8), (11) into Eqs. (6), (7), we obtain

$$P(0_k, a) = g^2|I_a(\omega)|^2; \quad P(1_k, a) = g^2|I_e(\omega)|^2,$$

where the absorption and emission amplitudes are given, respectively, by

$$I_a(\omega) = \int^{\tau_e}_{\tau_i} \exp \left[ i\frac{\omega}{\alpha} (e^{-\alpha \tau} - 1) + i\omega \tau - \alpha \tau \right] d\tau,$$

$$I_e(\omega) = I_a(-\omega),$$

and it is assumed that an atom enters the cavity at $\tau = \tau_i$ and exits cavity at $\tau = \tau_e$. Hereafter we skip the index $k$ in the coupling constant $g_k$ for the single-mode cavity. For a counter-propagating wave one needs to replace $\alpha \to -\alpha$ in Eq. (11).

We also note that the amplitude of the process of photon emission by an excited atom is $I_3 = I_a^2$. Therefore, the probability of this process is $P(1_k, b) = P(0_k, a)$.

If we remove the cavity walls to infinity by letting $\tau_i \to -\infty$ and $\tau_e \to \infty$, the integrals in Eq. (11) are reduced to gamma-functions by making the substitution $x = \frac{\omega}{\alpha} e^{-\alpha \tau}$:

$$I_{a,e} = \frac{1}{\nu} e^{-\frac{i\omega}{\nu}} \left( \frac{\omega}{\nu} \right)^{\frac{1}{2}} \int_{0}^{\infty} e^{i\frac{\omega}{\nu} x} \frac{x^{\frac{1}{2}}}{\nu} dx = \frac{i}{\nu} e^{-\frac{i\omega}{\nu}} \left( \frac{\omega}{\nu} \right)^{\frac{1}{2}} e^{-\frac{\omega}{\nu} x} \Gamma \left( 1 - \frac{i\omega}{\alpha} \right),$$

where the upper and lower signs correspond to the absorption and emission amplitudes respectively. Using the equality

$$\Gamma \left( 1 - \frac{i\omega}{\alpha} \right) \Gamma \left( 1 + \frac{i\omega}{\alpha} \right) = \frac{\pi \omega}{\sinh \frac{\pi \omega}{\alpha}},$$

we arrive at

$$|I_a|^2 = \frac{1}{\nu^2} e^{-\frac{\omega}{\nu}} \frac{\pi \omega}{\sinh \left( \frac{\pi \omega}{\alpha} \right)},$$

and

$$|I_e|^2 = \frac{1}{\nu^2} e^{-\frac{\omega}{\nu}} \frac{\pi \omega}{\sinh \left( \frac{\pi \omega}{\alpha} \right)} = \frac{1}{\nu^2} e^{\frac{2\pi \omega}{\alpha}} - 1.$$

Note the familiar Planck factor $\left( e^{-\frac{\hbar \omega}{2\pi \nu \alpha}} - 1 \right)^{-1}$ in the emission probability with temperature equal to the Unruh temperature $T_u = \frac{\hbar \alpha}{2\pi \nu \alpha}$. The ratio of emission to absorption probabilities is also in agreement with Unruh result in free space:

$$\frac{P(1_k, a)}{P(0_k, a)} = e^{-2\pi \omega/\alpha}.$$
Consider again uniformly accelerated atoms moving along a trajectory [8] and interacting with a single-mode field in a cavity. Our goal is to find the solution to the master equation Eq. (6) within the perturbation theory following the approach described in [10, 11]. In particular, we will find the steady state number of photons in a cavity mode as a result of interaction with a beam of atoms.

Atom in free space.

First, we derive the results for a single atom in free space within the approach of the quantum theory of the laser [10, 11]. Equation (3) can be rewritten in the form convenient to apply perturbation theory expansion:

\[
\frac{d\rho}{d\tau} = -\frac{i}{\hbar} [\hat{V}, \rho] + \left( -\frac{i}{\hbar} \right)^2 \int_0^\tau \left[ \hat{V}(\tau), \left[ \hat{V}(\tau'), \rho(\tau') \right] \right] d\tau'.
\] (17)

In Markov approximation, assuming weak interaction with many field modes we decompose the density matrix as

\[
\rho_{\text{atom}} = \rho_{\text{atom}}(\tau) \otimes \rho_{\text{fld}}(0).
\]

Obtaining from Eq. (17), we finally get Eq. (19) in the form

\[
\frac{d\rho_{\text{atom}}}{d\tau} = -\frac{1}{\hbar^2} \int_0^\tau \text{Tr}_{\text{fld}} \left[ \hat{V}(\tau)\hat{V}(\tau')\rho_{\text{atom}}(\tau)\rho_{\text{fld}}(0) + \rho_{\text{atom}}(\tau)\rho_{\text{fld}}(0)\hat{V}(\tau')\hat{V}(\tau) - \hat{V}(\tau')\rho_{\text{atom}}(\tau)\rho_{\text{fld}}(0)\hat{V}(\tau) \right] d\tau'.
\] (18)

Let us first consider an atom at rest, when \(\tau = t\). In the interaction representation, using Eq. (2) and the replacement described after Eq. (6), we arrive at the following equation for population of state \(|a\rangle\):

\[
\frac{d\rho_{aa}}{dt} = -\frac{\mu^2}{\hbar^2} \int_0^t dt' \left[ \left( \sum_k E_k^2 \hat{a}_k(t)\hat{a}_k^{\dagger}(t') \right) e^{i\omega(t-t')} + \{ t \leftrightarrow t' \} \right] \rho_{aa} - \left( \sum_k E_k^2 \hat{a}_k^+(t')\hat{a}_k(t) e^{i\omega(t-t')} + \{ t \leftrightarrow t' \} \right) \rho_{bb},
\] (19)

where the field operators in angular brackets can be expressed via average number of photons in the \(k\)th mode as

\[
\langle \hat{a}_k^+(t')\hat{a}_k(t) \rangle = \bar{n}_k e^{-i\nu_k(t-t')}, \quad \langle \hat{a}_k(t)\hat{a}_k^{\dagger}(t') \rangle = (1 + \bar{n}_k) e^{-i\nu_k(t-t')}.
\] (20)

After performing integration, we finally get Eq. (19) in the form

\[
\frac{d\rho_{aa}}{dt} = -\text{const} \sum_k [(1 + \bar{n}_k)\rho_{aa}(t) + \bar{n}_k\rho_{bb}(t)].
\] (21)

Its steady-state solution when \(\bar{n}_k\) is a thermal field is \(\rho_{aa}/\rho_{bb} = e^{-\hbar\omega/kT}\).

For an accelerated atom in Minkowski vacuum one can obtain a familiar Unruh result. One can also obtain a more general result for an atom accelerated through a thermal (not vacuum) background electromagnetic field with photon distribution \(n_k\):

\[
\frac{d\rho_{aa}}{dt} = -\frac{\mu^2}{\hbar^2} \int_0^t dt' \sum_k E_k^2 \langle \hat{n}_k | \hat{a}_k + \hat{a}_k^{\dagger} \rangle (\hat{a}_k + \hat{a}_k^{\dagger})^\dagger | \hat{n}_k \rangle \left[ e^{i\omega(\tau-\tau')} \rho_{aa} - e^{-i\omega(\tau-\tau')} \rho_{bb} \right] + \text{c.c.}
\]

\[
= -\frac{\mu^2}{\hbar^2} \int_0^t dt' \sum_k \left\{ \bar{n}_k e^{i\nu_k(t-t')-ik(z(t)-z(t'))} + (1 + \bar{n}_k) e^{-i\nu_k(t-t')+ik(z(t)-z(t'))} \right\}
\]

\[
\left[ e^{i\omega(\tau-\tau')} \rho_{aa} - e^{-i\omega(\tau-\tau')} \rho_{bb} \right],
\] (22)

where \(t = \tau, t' = \tau'\).

Next, we proceed following the method described in e.g. P. Milonni or J. Audretsch and R. Müller [2]. Namely, we assume that the frequency \(\nu_k\) has a small imaginary part, substitute the equations for a uniformly accelerated
where \( \mathbf{R} \) is the interaction Hamiltonian given by (9).

observer riding along with the atom. For simplicity, consider again the case of the co-propagating atom and field and where \( \text{tr} \)

\[ T = \frac{\pi \omega}{\alpha} \]

is a constant which is unimportant for a steady-state distribution of populations.

\[ β \]

is the atomic injection rate, we have a coarse grained equation of motion: 

\[ \Delta \rho = \frac{1}{e^{\frac{\nu}{kT}} - 1} \]

and \( \beta \) is a constant which is unimportant for a steady-state distribution of populations.

Equation (24) allows one to find steady-state atomic populations for a general case of an atom accelerated through a thermal field background.

**Beam of atoms accelerated through a single-mode cavity.**

In this part our goal is to evaluate the steady-state number of photons in a cavity mode as a result of interaction with a beam of atoms.

As in the quantum theory of the laser [10, 11], the (microscopic) change in the density matrix of a cavity mode due to any one atom, \( \delta \rho^i \), is small. The (macroscopic) change due to \( \Delta N \) atoms is then \( \Delta \rho = \sum_i \delta \rho^i = \Delta N \delta \rho \). Writing \( \Delta N = r \Delta t \), where \( r \) is the atomic injection rate, we have a coarse grained equation of motion: \( \Delta \rho / \Delta t = r \delta \rho \).

The change \( \delta \rho^i \) due to an atom injected at time \( \tau_i \) in the atomic rest frame is

\[ \delta \rho^i = -\frac{1}{\hbar^2} \int_{\tau_i}^{\tau_i + \tau'} \text{Tr}_{\text{atom}} \times \]

\[ \times \left[ \hat{V}(\tau'), \left[ \hat{V}(\tau''), \rho_{\text{atom}}(\tau_i) \otimes \rho(t(\tau_i)) \right] \right] d\tau' d\tau'' , \]

where \( \text{tr}_{\text{atom}} \) denotes the trace over atom states. The time \( \tau \) is the atomic proper time, i.e., the time measured by an observer riding along with the atom. For simplicity, consider again the case of the co-propagating atom and field and the interaction Hamiltonian given by (9).

In the case of random injection times, the equation of motion for the density matrix of the field is

\[ \frac{d \rho_{n,n}}{dt} = -R_2 \left[ (n + 1) \rho_{n,n} - n \rho_{n-1,n-1} \right] \]

\[ -R_1 \left[ n \rho_{n,n} - (n + 1) \rho_{n+1,n+1} \right] , \]

where \( R_{1,2} \) are defined in the following. If \( R_1 > R_2 \), there is a steady state solution which is thermal [10]

\[ \rho_{n,n} = e^{-\hbar \nu/k_B T} \left( 1 - e^{-\hbar \nu/k_B} \right) , \]

\[ \bar{n} = \sum_n n \rho_{n,n} = \frac{1}{e^{\hbar \nu/k_B T} - 1} , \quad e^{-\hbar \nu/k_B} = \frac{R_2}{R_1} , \]

where an effective temperature of the field in the cavity is \( T = \hbar \nu/k_B \ln [R_1/R_2] \). Thus, spontaneous emission of randomly injected ground state atoms in the cavity results in thermal statistics of the mode excitation. Note, that the thermal statistics of the atomic excitation in the standard Unruh effect in free space is due to spontaneous emission into a vacuum field reservoir with a continuous spectrum of modes.

Absorption and emission coefficients \( R_{1,2} = r |g_{1,2}|^2 \) are determined by the amplitudes \( g_{1,2} = \pm \int_{\tau_i}^{\tau_i + T} V_{1,2} d\tau \) of the matrix elements \( V_1 = \langle a, 0 | \hat{V} | b, 1 \rangle \) and \( V_2 = \langle a, 1 | \hat{V} | b, 0 \rangle \) of the interaction Hamiltonian [9], respectively, and their explicit form is given by Eq. (11). Using the results of the previous section, we get the same result that in the limit \( \nu, \omega \gg \alpha \) the emission/absorption ratio is \( R_2/R_1 \approx \alpha/(2\pi \omega) \), which is an enhancement by many orders of magnitude as compared to the exponentially small value \( R_2/R_1 = \exp(-2\pi \omega/\alpha) \).
After the substitution of variables \( x = \frac{\omega}{\alpha} e^{-\alpha \tau} \), the absorption and emission amplitudes can be expressed via incomplete gamma-functions:

\[
I_{a,e}(\omega) = \frac{i}{\nu} \left( \frac{\alpha}{\nu} \right) \frac{\pi i}{2} e^{\frac{\pi i}{2}} e^{-\frac{\omega}{2}} \left[ \Gamma\left( \xi, u e^{-\alpha (\tau_e - \tau)} \right) - \Gamma\left( \xi, u \right) \right],
\]

where \( \xi = 1 \mp \frac{i\pi}{\alpha} \), \( u = -\frac{i\pi e^{-\alpha \tau_e}}{\alpha} \), and \( \Gamma(\xi, u) = \int_{a}^{\infty} e^{-x} x^{-\xi + 1} dx \) is the incomplete gamma function.

In principle, expressions (24) can be fully analyzed because the properties and asymptotic behavior of incomplete gamma-functions are well known. Some representative graphs of the emission and absorption amplitudes as functions of the field frequency will be shown below in Figs. 2, 3, 4. However, it is more instructive and transparent to directly calculate the asymptotic of the integral (11) by applying integration by parts and the method of stationary phase.

In particular, we consider the most realistic case \( \nu, \omega \gg \alpha \) and apply the stationary phase method that can be summarized as

\[
\int_{a}^{b} F(\tau) e^{i A f(\tau)} d\tau = B + S,
\]

where

\[
B = \left. \frac{F(\tau) e^{i A f(\tau)}}{i A f'(\tau)} \right|_{a}^{b} + \sum_{n=1}^{N} \frac{1}{i A^{n+1}} \left( -\frac{1}{d} \frac{d}{f'(\tau)} \right)^{n} \left. \frac{F(\tau) e^{i A f(\tau)}}{f'(\tau)} \right|_{a}^{b} + o(A^{-N})
\]

is the contribution from integration boundaries obtained by integration by parts,

\[
S = \sqrt{\frac{2\pi i}{Af'(\tau_s)}} (F(\tau_s) + O(A^{-1})) e^{i A f(\tau_s)}
\]

is the contribution from a stationary point \( \tau_s \) such that \( f'(\tau_s) = 0, f''(\tau_s) \neq 0 \), obtained by expanding \( f(\tau) \) in Taylor series around \( \tau_s \). It is assumed that \( A \gg 1 \). We will also consider separately the case when the stationary point approaches one of the integration boundaries; see Eq. (24) below.

Suppose for definiteness that \( \nu \geq \omega \) and \( \tau_i = 0 \). When \( \nu - \omega \geq \sqrt{\omega \nu} \), the stationary point \( \tau_s = \frac{1}{\nu} \log \frac{\nu}{\omega} \) of the absorption integral \( I_a \) in Eq. (14) is within the integration limits and far enough from the boundaries. Therefore, \( I_a \) can be evaluated as a sum of the boundary contribution

\[
I_a^{(b)} \approx \exp \left[ \frac{i}{\alpha} \left( \frac{\omega}{\alpha} \right) - 1 \right] + \frac{1}{i(n - \omega)}
\]

and the contribution from the stationary point \( \tau_s \):

\[
I_a^{(s)} \approx \sqrt{\frac{2\pi}{\alpha \omega}} \nu \exp \left[ \frac{\nu}{\alpha} \log \frac{i}{\alpha} \right].
\]

It is clearly seen that the contribution from the stationary point dominates in the absorption integral \( I_a \). The same result can be of course obtained directly from Eq. (24) after moving the integration boundaries to \( \pm \infty \) and considering the resulting expression

\[
I_1(\omega) = \frac{i}{\nu} \left( \frac{\alpha}{\nu} \right) e^{\frac{\omega}{2}} e^{-i \frac{\pi}{4}} \left( 1 - i \frac{\omega}{\alpha} \right)
\]

in the asymptotic limit of a large complex argument of the gamma-function.

The emission integral \( I_e \), which originates from the counter-rotating term \( \propto \hat{a}^+ \hat{\sigma}^+ \) in the interaction Hamiltonian, does not have a stationary point within the integration limits. Therefore, its value is solely determined by the boundary contribution, and \( I_e(\omega) \sim I_a^{(b)}(\omega) \). If we further assume long enough interaction time, \( \alpha \tau_e \gg 1 \), the second term on the right-hand side of (11) is much greater than the first term, and we obtain

\[
\frac{P(1_k, a)}{P(0_k, a)} = \frac{\alpha \nu^2}{2\pi \omega (\nu + \omega)^2},
\]
which is equal to \( \frac{\alpha^2}{2\pi\omega} \) for \( \nu \gg \omega \).

Exactly at resonance, \( \nu - \omega = \sqrt{\omega\alpha} \), the stationary point coincides with the lower integration limit \( \tau = 0 \). In this case one can show that the main contribution again comes from the stationary point, and the value of the integral \( I_{1e} \) is two times smaller than \( \frac{\alpha^2}{2\pi\omega} \). The resulting ratio of probabilities is equal to

\[
\frac{P(1_k,a)}{P(0_k,a)} = \frac{\alpha}{2\pi\omega}. \tag{37}
\]

The above analysis can be readily generalized for an arbitrary value of \( \sqrt{\omega\alpha} \). In this case the stationary phase method gives an additional term in the integrals \( I_{1,2} \) that contains the error function \( \text{erf}[z] \) of detuning:

\[
I^{(s)}_a \simeq \sqrt{\frac{2\pi}{|\alpha\omega|}} \frac{\omega}{2\nu} \exp \left( i\frac{\omega - \nu}{\alpha} + i\frac{\omega}{\alpha} \log \frac{\nu}{\omega} + i\frac{\pi}{4} \right) \left[ 1 + \text{erf} \left( \frac{\omega - \nu}{\sqrt{2\alpha\omega}} e^{-i\pi/4} \right) \right]. \tag{38}
\]

The function \( |I^{(s)}_a|^2 \) gives the spectral profile of the absorption line.

The ratio \( \alpha \) or \( \frac{\alpha^2}{2\pi\omega} \) is surprisingly large; in fact, it is exponentially larger than the value

\[
e^{-2\pi\omega/\alpha} = e^{-\frac{h\omega}{k_B T_u}}
\]

one would expect to obtain on the basis of studies of the Unruh effect. Here

\[
k_B T_u = \frac{\hbar\alpha}{2\pi} \tag{39}
\]

is Unruh temperature. In our case the effective temperature of radiation in the vacuum state of the cavity mode is determined from

\[
e^{-\frac{h\omega}{k_B T}} = \frac{\alpha}{2\pi\omega},
\]

which gives

\[
k_B T = \frac{\hbar\omega}{\log \frac{2\pi\omega}{\alpha}}. \tag{40}
\]

The reason for such a large effective temperature is apparently the sudden turn-on of the interaction of an atom with a cavity mode. If we eliminate the nonadiabatic switching effect by letting \( \tau_i \to -\infty \) and \( \tau_e \to \infty \), the integrals in Eq. (29) are reduced to

\[
I_{a,e} = \frac{i}{\nu} e^{-i\frac{\pi}{2}} \left( \frac{\alpha}{\nu} \right)^{\frac{\pi}{2} + i\frac{\pi}{4}} e^{i\frac{\pi}{4}} \Gamma \left( 1 + i\frac{\omega}{\alpha} \right) \Gamma \left( 1 - i\frac{\omega}{\alpha} \right).
\]

Using the equality

\[
\Gamma \left( 1 - \frac{i\omega}{\alpha} \right) \Gamma \left( 1 + \frac{i\omega}{\alpha} \right) = \frac{\frac{-i\omega}{\alpha}}{\sinh \frac{2\omega}{\alpha}}, \tag{41}
\]

we arrive at the Unruh-type result

\[
\frac{P(1_k,a)}{P(0_k,a)} = e^{-2\pi\omega/\alpha}. \tag{42}
\]

V. ANGULAR DEPENDENCE OF EMISSION/ABSORPTION PROBABILITIES

The above conclusion does not depend on our assumption of interaction with a single co-propagating cavity mode and can be generalized for the case of an electromagnetic mode with an arbitrary \( k \)-vector. Similarly to Sec. III, we calculate the probability \( P(1_k,a) \) of excitation of an atom with simultaneous photon emission into the \( k \)th mode assuming that the field was initially in the vacuum state. Then we calculate The probability \( P(0_k,a) \) of photon
absorption from the $k$th mode by a ground-state atom, when there is only photon in this mode. The arguments of the $P$ functions denote the final state of the field and atom. The ratio of these probabilities is given by

$$
\frac{P(1_k, a)}{P(0_k, a)} = \frac{I_k(-\omega)}{I_k(\omega)},
$$

where

$$I_k(\omega) = \int_{\tau_i}^{\tau_e} \frac{k_z}{k} \exp[i\nu t(\tau) - i k_z z(\tau) - i \omega \tau - \alpha \tau] \, d\tau,$$

where $k = |k| = \nu/c$. The probability of emission by an atom into all electromagnetic modes is proportional to $\int I_k \, d^3k$.

We will be interested in evaluating the ratio (43). Using equations for the trajectory of a uniformly accelerated atom, we arrive at

$$I_k(\omega) = \frac{k_z}{k} e^{i\nu \tau_i + \frac{ik_z}{\nu}} \int_{\tau_i}^{\tau_e} \exp \left[ \frac{i \nu}{\alpha} \left( \sinh \alpha \tau - \frac{k_z}{k} \cosh \alpha \tau \right) - i \omega \tau - \alpha \tau \right] \, d\tau. \tag{45}$$

As in the case of co-propagating mode, the above integral can be calculated exactly in the infinite limits and can be evaluated approximately by the method of stationary phase in finite limits.

For the infinite integration limits, $\tau_i, \tau_e \to \infty$, it was shown in [6] that it is convenient to change the integration variable to

$$\beta = \alpha \tau - \eta,$$

where $\tanh \eta = k_z/k$. Then the integral in (45) can be written as

$$\int_{-\infty}^{\infty} e^{ik_\perp \sinh \beta - i\xi \eta - \frac{i\xi^2}{2}} K_\xi(k_\perp), \tag{46}$$

where $k_\perp = k_\perp c/\alpha$, $\xi = 1 + i\omega/\alpha$, and $K_\xi(k_\perp)$ is McDonald function. Using the above result in (43), we obtain

$$\frac{P(1_k, a)}{P(0_k, a)} = e^{-\frac{2\pi \omega}{\alpha} \left| \frac{K_1 - i\omega/\alpha(k_\perp)}{K_1 + i\omega/\alpha(k_\perp)} \right|^2}, \tag{47}$$

which is “almost” Unruh factor in the limit $\omega/\alpha \gg 1$, since $K_{-\rho}(x) = K_{\rho}(x)$. The extra factor of 1 in $\xi$ is due to the fact that we are dealing with photons that have spin 1. This introduced an additional term in the integral as a result of Lorentz transformation of the field to the atom frame. For the scalar (spin 0) field we would have exactly the thermal Unruh factor.

To evaluate the integrals in finite limits, let us suppose again for definiteness that $\nu - \omega \gg \sqrt{\alpha \omega}$, $\tau_i = 0$, and $\alpha \tau_e \gg 1$.

The “counter-rotating” integral $I_k(-\omega)$ does not have stationary points, and its value is

$$|I_k(-\omega)|^2 \sim \frac{1}{(\nu + \omega)^2}. $$

The integral $I_k(\omega)$ is dominated by contribution from the stationary point $\tau_s$ defined by

$$\cosh \alpha \tau_s - \frac{k_z}{k} \sinh \alpha \tau_s = \frac{\omega}{\nu}. \tag{48}$$

It is easy to find that

$$I_k(\omega) \sim \frac{k_z}{k} \sqrt{\frac{2\pi}{\alpha \sqrt{\omega^2 - k_\perp^2 c^2}}} e^{i \frac{\sqrt{\omega^2 - k_\perp^2 c^2}}{\alpha} + \frac{ik_z}{\nu} \omega \tau_s - \alpha \tau_s + \frac{i\pi}{4}}. \tag{49}$$

As in the co-propagating mode case, the ratio (43) is anomalously large: it is not exponentially small but linear with respect to $\alpha/\omega$. 

VI. COUNTER-RESONANT GAIN AND PARAMETRIC AMPLIFICATION

Remarkably, not only enhanced spontaneous emission but also laser gain and parametric gain are possible in cavity QED via counter-resonant emission by ground state atoms even with random injection times. The gain is reached when $|I_e/I_a|^2 > 1$, or, more exactly, an excess in $|I_e/I_a|^2$ over 1 should be greater than the normalized cavity losses. For the gain to occur, the time of flight $T$ should be within a certain range to ensure that the atom emits into the cavity mode more energy than it takes away, $R_2 > R_1$.

In the case of uniformly accelerated atoms, we find that for a co-propagating wave the gain is possible only when the acceleration is large enough: $\alpha > \omega$. In the opposite case the ratio $R_2/R_1$ approaches the asymptotic value of $\alpha/2\pi\omega$, as was shown in previous sections. Below we plot the ratio $R_2/R_1$ for both cases using the incomplete gamma-function representation of emission and absorption integrals. Instead of varying the time of flight $T$, we plot the gain spectrum as a function of the electromagnetic field frequency $\nu$ for the fixed values of $T$ and the atomic frequency $\omega$.

As is seen from Fig. 2, when $\omega/\alpha > 1$, the emission to absorption ratio drops down to almost zero due to a large absorption near the resonance frequency $\nu = \omega$ and then approaches the asymptotic value $\alpha/2\pi\omega$ in the oscillatory way. When $\omega/\alpha < 1$, there are strong peaks of a large ratio $R_2/R_1 \gg 1$ at frequencies corresponding to minima of the absorption probability; see Fig. 3. Note that the minima of the emission rate are shifted with respect to the minima in the absorption. At large field frequencies the envelope of the emission to absorption rate peaks approaches the asymptotic value $2(2\alpha/\pi\omega)^2$.

The counter-propagating mode is more favorable for the amplification due to sharp dips in the absorption spectrum. As is illustrated in Fig. 4, even in the limit $\omega \gg \alpha$ the gain spectrum has sharp maxima larger than 1 at the points corresponding to nearly vanishing absorption. The case $\omega < \alpha$ is qualitatively similar to that of a co-propagating mode.

Note that the peaks of large gain in Figs. 3,4 are not due to maxima of the emission integral but due to minima of the absorption probability that are shifted with respect to the minima of the emission spectrum. Absolute values of both integrals are small. This is illustrated in the insets to Figs. 3,4 where the emission and absorption spectra are shown on the same plot.

In the optimal regime for amplification, when $\omega \sim \alpha$, $\nu \gg \alpha$, and $e^{-\alpha T} \ll 1$ where the time of flight $T \sim L/c$, one needs to use a longitudinal cavity mode $\Omega_n = n\pi c/L$ with index $n > 1$. For example, if $\alpha T \approx \alpha L/c = 10$, to provide $\nu = \Omega_n = n\pi c/L = 10\alpha$ one needs $n \approx 3$. The multimode regime is possible. It is expected to give the same qualitative results.

The effects originated from counter-rotating terms are in fact not uncommon. Two well-known examples are parametric resonance and anomalous Doppler effect. In all “counter-rotating” processes, an atom can emit a photon and simultaneously make a transition from ground to excited state. The required energy is provided by the
work done by an external force that sustains the center-of-mass motion of an atom along a given trajectory. However, an important difference between the nonadiabatic processes considered in this paper and the anomalous Doppler effect is that the latter does not require any time-changing parameters.

It is clear from the above derivation of the emission and absorption probabilities that the enhancement of the acceleration radiation is related to a strong nonadiabatic effect at the cavity boundaries. Evidently, this effect should exist for an arbitrary trajectory of an atom and in particular, for an atom moving with constant velocity. Of course, the presence of acceleration leads to both qualitative and quantitative changes in the excitation rate and emission/absorption probabilities by allowing the atom to pass through the resonance between the transition frequency of the atom and the Doppler-shifted frequency of the field.
For a ground-state atom moving through a cavity with a constant velocity and interacting with a co-propagating wave, it is straightforward to obtain the analytic expressions for $R_2$ and $R_1$:

$$R_1 = g^2 \left( \frac{1}{\nu' - \omega} \right)^2 \left| 1 - e^{-i(\nu'-\omega)T} \right|^2,$$

(50)

$$R_2 = g^2 \left( \frac{1}{\nu' + \omega} \right)^2 \left| 1 - e^{-i(\nu'+\omega)T} \right|^2 \nu' = \nu \left( \frac{\nu - k \cdot v}{\nu + k \cdot v} \right)^{1/2}.$$

(51)

Clearly, the factors $1/(\nu' - \omega)^2$ and $1/(\nu' + \omega)^2$ have the same origin as the nonadiabatic boundary contribution to the emission and absorption probabilities given by Eq. (33). When we are far from resonance $\nu' = \omega$, the magnitudes of $R_{1,2}$ in the cases of constant velocity and constant acceleration are similar and are proportional to the above factors. Thus, when the acceleration shifts the frequency $\nu$ far away from the resonance (e.g. when $\nu < \omega$ for the counter-propagating wave or when $\nu > \omega$ for the co-propagating wave), the emission-to-absorption ratio is increasing. In this case the effect of acceleration results in the increase of the steady-state number of photons in the cavity as compared to the constant velocity case. This tendency is of course reversed when the frequency is shifted towards the resonance by acceleration. At the same time, in the case of a constant acceleration we can also have the situation when the atom starts far from resonance, then passes through resonance in the course of acceleration, and finally ends up far from the resonance. In this case the ratio $R_2/R_1$ can be quite large and given by $\alpha/\omega$, while for an atom moving with a constant velocity and close to resonance $|\nu' - \omega| \ll \nu$, the ratio $R_2/R_1$ is very small due to a strongly enhanced absorption. Thus, depending on the initial conditions, acceleration can lead to either increase or decrease in the emission-to-absorption ratio.

The right-hand side of Eqs. (33) strongly depends also on the interference factors $e^{-i(\nu'+\omega)T}$ that are defined by the time of flight $T$, i.e. the phase an atom accumulates relative to the cavity mode while passing through the cavity. The ratio $R_2/R_1$ can be even greater than one. To achieve $R_2/R_1 > 1$, one can tune the time of flight to get the proper interference factors: $e^{-i(\nu'-\omega)T} \rightarrow 1$, $|e^{-i(\nu'+\omega)T} - 1| \sim 1$. A similar time of flight tuning is used in some electronic devices, e.g., in klystrons. The above requirements define a set of the time-of-flight values, with the maximum gain corresponding to

$$\begin{align*}
(\nu - kv + \omega)T &= (2n_1 - 1)\pi; \\
(\nu - kv - \omega)T &= 2n_2\pi,
\end{align*}$$

(52)

where $n_{1,2}$ are integer numbers. For the particular case $n_1 = 0$, $n_2 = -1$ one obtains $2\omega T = \pi$. The monochromaticity of the beam should satisfy the condition

$$\begin{align*}
\frac{\Delta \nu}{\nu} &\sim \frac{\Delta T}{T} \ll \frac{\pi}{2\omega T} \sim \frac{v}{4c L},
\end{align*}$$

where $\lambda = 2\pi c/\omega$, $L$ is a cavity length, and we assumed $\omega \sim \nu \gg kv$. For $v \sim 1$ km/s and $L \sim \lambda$ one gets $\Delta \nu/\nu \ll 10^{-6}$, which is tough but possible to satisfy.

The counter-propagating mode is more favorable for the gain since the absorption can then be anomalously small while the gain remains as large as for the co-propagating mode.

Similar interference effects, obviously, are present in the case of a constant acceleration according to Eqs. (29), (30), (33), and (34), as can be seen in Figs. 1-3. They can also lead to the net gain, as we have already discussed.

In the case of a parametric resonance, consider an atom moving along an oscillating trajectory $z = z_0 + A \cos \omega_0 t$, $t = \tau$. The photon absorption and emission probabilities by a ground-state atom are given by

$$R_{1,2} = g^2 \left| \int_0^{\tau_\pi} e^{-ik_z z + i\nu t + i\omega t - i\gamma t} dt \right|^2,$$

(53)

where we introduced a small factor $\gamma$ describing the atomic decay. Using

$$e^{ikA \cos \omega_0 t} = \sum_{p=-\infty}^{\infty} i^p J_p(k_z A) e^{ip\omega_0 t},$$

the above probabilities can be written as

$$R_{1,2} = \left| \sum_{p=-\infty}^{\infty} \frac{gJ_p(k_z A)}{p\omega_0 + \omega + \nu + i\gamma} \right|^2,$$

(54)
where $J_p(x)$ is Bessel’s function. Evidently, the probabilities are sharply peaked close to parametric resonance, where $p\omega_0 \pm \nu \pm k \approx 0$. Resonance for emission corresponds to $\nu + \omega = p\omega_0$, while the absorption resonance is at $\nu - \omega = p\omega_0$. When $\omega = \nu$, absorption is always stronger than emission. Indeed, resonance in absorption exists for $p = 0$, while parametric resonance in emission requires $p \geq 1$. Therefore, in this case $R_2/R_1 \sim J_0^2(k_\nu A)/J_0^2(k_\gamma A) < 1$ for $p \geq 1$.

However, when an atom is not at resonance with the field, one can have parametric resonance in emission but no resonance in absorption, which results in the parametric gain. The energy is drawn from the external force causing an atom to follow an oscillating trajectory, and the high efficiency of this energy transfer is due to a non-stationary, strongly nonadiabatic character of the atomic center-of-mass motion. In the case of Unruh effect, i.e. a uniformly accelerated atom in free space, it is also nonadiabaticity that drives simultaneous excitation of the atom and the field. However, the efficiency is much lower due to much slower change in the atomic velocity. For an atom entering the cavity, a sudden nonadiabatic switch-on of the interaction causes a stronger excitation.

VII. NONADIABATIC NATURE OF ACCELERATION RADIATION

The above calculations clearly show that the mechanism of simultaneous excitation of both field and atom is the same as for the Unruh effect in free space, namely nonadiabatic transition due to the counter-rotating term $\hat{\sigma}_-\hat{\sigma}_+^*$ in the interaction Hamiltonian \( \mathcal{H}_a \), i.e. $V_2$. The reason for an enhanced excitation in the cavity is the relatively large amplitude for a quantum transition $|b, 0\rangle \rightarrow |a, 1\rangle$ due to the sudden nonadiabatic switching on of the interaction, whereas for the Unruh effect in free space the emission is exponentially small due to a slow switching on. However, in both cases there is quite a real emission of a photon accompanied by the excitation of an atom – not just dressing of the ground state of an atom as a result of interaction.

We will now illustrate the above statement by explicit derivation of both the Unruh factor and the enhanced excitation factor as a probability of the nonadiabatic transition from the dressed ground state to the dressed excited state. Consider first our case of a sudden turn on of the interaction in cavity QED. As a result of the interaction, the initial state $|b, 0\rangle$ is no longer an eigenstate of the Hamiltonian. Now, a linear superposition of the excited states of the atom and field makes up the dressed ground state of the interacting system $\psi_0 = |b, 0\rangle - \frac{\langle \tau |\hat{\sigma}_-\hat{\sigma}_+^*|a, 1\rangle}{\sqrt{\nu + \omega}} |a, 1\rangle$ as well as the dressed excited state $\psi_1 = |a, 1\rangle + \frac{\langle \tau |\hat{\sigma}_-\hat{\sigma}_+^*|b, 0\rangle}{\sqrt{\nu + \omega}} |b, 0\rangle$.

In particular, the amplitude of the bare excited state $|a, 1\rangle$ in $\psi_0$ is of the order of $C \sim \mu E'/\hbar(\omega + \nu')$. It is easy to calculate that the latter corresponds to the atomic excitation probability $\rho_{a,a}^{\text{atom}} = |C|^2 \sim |\mu E'/\hbar(\omega + \nu)|^2 \sim |gI_a|^2$, where the emission integral $I_a$ is defined above. This result can be also obtained directly from the density matrix equation for the atom, via the atomic counterpart of Eq. \( \ref{eq:68} \) with a trace over the photon states instead of the transition amplitudes. This probability has the same origin and value as the well-known Bloch-Siegert shift of a two-level atomic transition \( \psi_0 \). $\Delta\omega/\omega = (\mu E'/\hbar(\omega + \nu))^2$, due to counter-rotating terms in the interaction Hamiltonian.

The counter-rotating term in Eq. \( \ref{eq:68} \) represents the contribution from boundaries to the nonadiabatic transition amplitudes. In the absence of the boundary contributions, the emission integral $I_a(\omega) = I_a(-\omega)$ in Eq. \( \ref{eq:68} \) becomes exponentially small $\exp(-\pi\omega/\alpha)$ for the small parameter $\alpha/2\pi\omega \ll 1$ since there are no stationary phase points in the integration interval. The absorption integral $I_a$ does have a point of stationary phase when the atomic frequency $\omega$ is brought into resonance with the field due to the time-dependent Doppler shift of the mode frequency $\nu'\tau = \nu \exp(-\alpha\tau)$. This fact explains why the related exponential factor effectively disappears from the absorption integral \( \ref{eq:68} \) when $\alpha \ll 2\pi\omega$. As a result, if there are no edge effects, we obtain the same excitation factor $R_2/R_1 = \exp(-2\pi\omega/\alpha)$ as in the Unruh effect (in free space). This means that in order to observe the standard Unruh effect one has to extend the mode profile $g(z)$ near the boundaries, i.e., eliminate nonadiabatic boundary contributions.

Similary to what we did for the sudden turn-on case, let us now demonstrate the nonadiabatic nature of the Unruh effect by the following explicit derivation of the Unruh factor as a probability of the nonadiabatic transition $\psi_0 \rightarrow \psi_1$ from the dressed ground state. The Schrödinger equation $i\hbar\psi/d\tau = \mathcal{H}\psi$ in the two-level case $\psi = c_0\psi_0 + c_1\psi_1$ yields $dc_1/d\tau + (iE_1/\hbar + \langle \psi_1 |\psi_1\rangle)c_1 = -c_0\langle \psi_0 |\psi_1\rangle$. The difference between the eigenenergies is, to the first order, $E_1 - E_0 = \hbar(\omega + \nu')$. For small nonadiabatic coupling $\langle \psi_0 |\psi_1\rangle = \frac{d}{d\tau} \left( \frac{g(\nu')}{\omega + \nu'} \right) \ll \omega + \nu'$, the perturbation solution is $|c_1|^2 = |\int_{1/T_\tau}^\infty \exp[i \int_{1/T_\tau}^x (\nu' + \omega)dx'''] \frac{d}{dx''} \left( \frac{g(x''')}{\omega + \nu'} \right) dx'''|^2$. If we now make the assumption of an adiabatic switching (on and off) of the interaction $g(\tau)$ as in standard Unruh effect treatments, then after integration by parts the latter integral is reduced to the integral $I_a(\omega) = I_a(-\omega)$ in Eqs. \( \ref{eq:6}\) but in the infinite limits, i.e. without edge effects. This yields the standard Unruh factor $|c_1|^2 \propto \exp(-2\pi\omega/\alpha)$. This derivation clearly shows the dramatic effect of boundary contributions leading to a large amplitude $\sim g(\tau)/(\omega + \nu')$ of the atomic excited state $|a\rangle$. Only if we eliminate the edge effects by adiabatic switching of the interaction do we retrieve the exponentially small excitation factor.

Note that in the cavity the excitation factor $\exp(-\hbar\nu/k_BT) \equiv R_2/R_1 = \alpha/2\pi\omega$ is determined by the first power
of the same nonadiabaticity parameter $\alpha/2\pi \omega$. The reason for this effect is the existence of a true resonance, i.e., a stationary-phase point, in the absorption coefficient. As mentioned earlier, this yields a resonance between the atomic transition frequency and the Doppler-shifted frequency of the field seen by the atom.

VIII. CONCLUSIONS

Our simple model clearly demonstrates that the ground state atoms accelerated through a vacuum-state cavity radiate real photons. For relatively small acceleration $a < 2\pi \omega c$, the excitation Boltzman factor $\exp(-\hbar \nu/k_B T) \sim \alpha/2\pi \omega$ is much larger than the standard Unruh factor $\exp(-2\pi \omega/\alpha)$. The physical origin of the field energy in the cavity and of the real internal energy in the atom is, of course, the work done by an external force driving the center-of-mass motion of the atom against the radiation reaction force. Both the present effect (in a cavity) and standard Unruh effect (in free space) originate from the transition of the ground state atom to the excited state with simultaneous emission of photon due to the counter-rotating term $\hat{a}_b^+ \hat{a}^+$ in the time-dependent Hamiltonian (9). Thus, these effects have essentially the same counter-resonant, nonadiabatic mechanism. We emphasize that there is emission of real photons in both cases; however the emission probability is exponentially small for the standard Unruh condition of the absence of boundaries and slow turn-on of the interaction — simply because the nonadiabatic effect is very small in the latter case. The enhanced rate of emission into the cavity mode comes from the enhanced nonadiabatic transition at the cavity boundaries; the standard Unruh excitation comes from the nonadiabatic transition in free space due to the time dependence of the Doppler-shifted field frequency $\nu' = \nu e^{-\alpha \tau}$, as seen by the atom in the course of acceleration.

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Figure Captions

Fig. 1. (a) Atoms or ions in their ground state $|b\rangle$ are accelerated through a single-mode microwave or optical cavity. (b) Resonant absorption or emission: an atom is excited (deexcited) as it simultaneously absorbs (emits) a photon. (c) Counter-resonant absorption or emission processes that are usually neglected in the “rotating wave approximation”: an atom is excited (deexcited) as it simultaneously emits (absorbs) a photon. (d) the energy for counter-resonant emission is drawn from work done by a force accelerating an atom.

Fig. 2. (a) Absorption rate $|I_a|^2$ (thin line) and emission rate $|I_e|^2$ (thick line) as functions of the field-to-acceleration frequency ratio $\nu/\alpha$ for the atomic frequency $\omega = 3\alpha$ and co-propagating wave. Integrals $I_{a,e}$ are given by Eq. (11). (b) The ratio of emission and absorption rates shown in (a). At large $\nu/\alpha$ the curve reaches the asymptotic value $\alpha/2\pi\omega$.

Fig. 3. (a) Absorption rate $|I_a|^2$ (thin line) and emission rate $|I_e|^2$ (thick line) as functions of the field-to-acceleration frequency ratio $\nu/\alpha$ for the atomic frequency $\omega = \alpha/3$ and co-propagating wave. Integrals $I_{a,e}$ are given by Eq. (11). The inset shows the tails in more detail. (b) The ratio of emission and absorption rates shown in (a). At large $\nu/\alpha$ the curve reaches the asymptotic value $(2\alpha/\pi\omega)^2/2 = 8$.

Fig. 4. (a) Absorption rate $|I_a|^2$ (thin line) and emission rate $|I_e|^2$ (thick line) as functions of the field-to-acceleration frequency ratio $\nu/\alpha$ for the atomic frequency $\omega = 3\alpha$ and counter-propagating wave. Integrals $I_{a,e}$ are given by Eq. (11). The inset shows the tails in more detail. (b) The ratio of emission and absorption rates shown in (a).