Subset synchronization of DFAs and PFAs, and some other results

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Abstract

This paper contains results which arose from the research which led to arXiv:1801.10436 but which did not fit in arXiv:1801.10436. So arXiv:1801.10436 contains the highlight results, but there are more results which are interesting enough to be shared.

1 Introduction

For an NFA (nondeterministic finite automaton) $A = (Q, \Sigma, \cdot)$, $Q$ and $\Sigma$ are sets, and $\cdot : Q \times \Sigma \to 2^Q$. A complete NFA is an NFA $A = (Q, \Sigma, \cdot)$ for which $\cdot : Q \times \Sigma \to 2^Q \setminus \emptyset$. Here, the members of $Q$ are called states, and $\Sigma$ is called an alphabet of letters or symbols. $\cdot$ is called the transition function in some sources, and sometimes the symbol $\delta$ is used for it.

A PFA (partial finite automaton) is in fact an NFA for which $\cdot$ maps to sets of size at most 1, but we will use a different syntax, namely $\cdot : Q \times \Sigma \to Q \cup \{\perp\}$. A DFA (deterministic finite automaton) is a PFA which is in fact also a complete NFA, so we have $\cdot : Q \times \Sigma \to Q$ in that case.

$\cdot$ is left-associative, and we will omit it mostly. We additionally define $\cdot : 2^Q \times \Sigma \to 2^Q$ for NFAs $A = (Q, \Sigma, \cdot)$, by $Sa = \bigcup_{s \in S} sa$. For PFAs (and DFAs), we do this as well, but in a totally different way, namely by $Sa = \bigcup_{s \in S} \{sa\}$ if $sa \neq \perp$ for all $s \in S$, and $Sa = \emptyset$ otherwise. Notice that both definitions agree with each other for DFAs, but not for proper PFAs.

We define $qw$ inductive as follows for states $q \in Q$, subsets $S \subseteq Q$, and words $w \in \Sigma^*$:

$$q\lambda = q \quad q(xw) = (qx)w \quad S\lambda = S \quad S(xw) = (Sx)w$$

Here, $\lambda$ is the empty word, $x$ is the first letter of the word $xw$ and $w$ is the rest of $xw$.

We say that a complete NFA $A = (Q, \Sigma, \cdot)$ is synchronizing (in $l$ steps), if there exists a $w \in \Sigma^*$ (of length $l$), such that $Qw$ has size 1. We say that a PFA $B = (Q, \Sigma, \cdot)$ is carefully synchronizing (in $l$ steps), if there exists a $w \in \Sigma^*$ (of length $l$), such that $Qw$ has size 1. We say that a DFA is synchronizing (in $l$ steps) if it is carefully synchronizing as a PFA (in $l$ steps), or equivalently, if it is synchronizing as a complete NFA (in $l$ steps).

In [1], there are many results about DFAs and PFAs in connection with synchronization, each of which is considered a relatively important result by at
least one of the authors. But there are more results, and although they may be less important, they are still important enough to be shared. For many of them, we need to extend the definition of (careful) synchronization to subsets of the state set.

We say that a complete NFA \( A = (Q, \Sigma, \cdot) \) is synchronizing on \( S \subseteq Q \) (in \( l \) steps), if there exists a \( w \in \Sigma^* \) (of length \( l \)), such that \( Sw \) has size 1. We say that a PFA \( B = (Q, \Sigma, \cdot) \) is carefully synchronizing on \( S \subseteq Q \) (in \( l \) steps), if there exists a \( w \in \Sigma^* \) (of length \( l \)), such that \( Sw \) has size 1. We say that a DFA is synchronizing on \( S \subseteq Q \) (in \( l \) steps) if it is carefully synchronizing on \( S \) as a PFA (in \( l \) steps), or equivalently, if it is synchronizing on \( S \) as a complete NFA (in \( l \) steps).

In section 2, we give all possible maximum subset synchronization lengths for all PFAs and all DFAs with up to 5 states, and all subsets of these states. These results arise from a combination of computations and reasoning. The actual computations can be found along with the code of [1]. The computations by itself are already sufficient to find all possible maximum subset synchronization lengths for all synchronizing DFAs with up to 6 states and to prove Cardoso’s conjecture for 6 states. Cardoso’s conjecture is Conjecture 7 of [5], and was already proved for up to 5 states in [5].

In section 3, we give the maximum synchronization lengths for state subsets of complete NFAs. More precisely, we show that we can force that all subsets which are not proper supersets of the start subset need to be traversed to synchronize a specific (start) subset of size at least 2. This extends a result of [2] to state subsets. The alphabet size of our construction is less than \( \frac{1}{2}n^2 \). Furthermore, we give the maximum synchronization lengths for state subsets of size at most 3 of PFAs. More precisely, we show that we can force that all subsets of size 3 and 2 need to be traversed to synchronize a specific subset of size 3. The Cerny automaton has the property that all subsets of size 2 need to be traversed to synchronize a specific subset of size 2.

In section 4, we give asymptotic lower bounds for the maximum subset synchronization lengths of PFAs and DFAs (where the size of the subset is not arbitrary, but chosen to obtain the best lower bound). With this, we also take transitivity of the automaton into account. A PFA \( B = (Q, \Sigma, \cdot) \) (in particular a DFA) is transitive, if for every pair \((q, q') \in Q^2\), there exists a \( w \in \Sigma^* \) such that \( qw = q' \). We start with considering PFAs and DFAs with an arbitrary number of symbols. The results about them are formulated as propositions instead of theorems, because they follow more or less directly from techniques by others. Next, we consider binary PFAs and binary DFAs. The result for binary PFAs is obtained by adapting the construction of [1]; the number of states is reduced and there is a finishing symbol instead of a start symbol. The result for binary DFAs is obtained by combining the techniques of the result for binary PFAs with some of those in [3]. [8] contains lower bounds for the maximum subset synchronization lengths of (transitive) binary DFAs as well, but the lower bounds in this paper are better.

Section 5 is about the property of D3-directing for NFAs. We discuss Lemma 3 of [4], which states that the maximum length of a D3-directing word of an NFA with \( n \) states is the same as that of a PFA with \( n \) states. Furthermore, we discuss Theorem 1 of [3], which states that the maximum length of a D3-directing word of a complete NFA with \( n \) states is the same as that of a DFA with \( n \) states. The proof of Lemma 3 of [4] is actually too short, and the proof
of Theorem 1 of [3] is actually too long. We give a combined proof of which the length is in between. Furthermore, we compute the minimum alphabet size for D3-directing NFAs which take the maximum number of steps, up to 7 states. Again, the actual computations can be found along with the code of [1]. Section 5 does not discuss the subset variant of D3-directing and (careful) synchronization, but the results in it can be generalized to subset variants in a straightforward manner.

In section 6, we show that the prime number construction for careful synchronization of PFAs in [6] by Martyugin yields a synchronization length which is strictly between polynomial and exponential in the number of states. Proving this was not the only reason to redo the complexity estimation in [6]. The other reason is that the estimation is not so accurate on the first point, and overly accurate on other points, to be not so accurate over all. Our estimate is more accurate on the first point, and not more accurate than needed on other points, to be more accurate over all.

2 Subset synchronization up to 5 states

In our algorithm for computing slowly synchronizing automata, the synchronization estimate for a subset $S$ is determined roughly as follows. First, the length of a path from $S$ to a smaller subset $S'$ is estimated. Here $S'$ does not need to be determined; even the size of $S'$ may be undetermined. Next, the length of a synchronizing path from $S'$ is estimated recursively.

Here, it is assumed that $S'$ does indeed synchronize. But with subset synchronization, $S'$ does not need to synchronize. To overcome this problem, we just assume that subsets of size less than that of $S$, which we denote by $|S|$, synchronize.

The DFA algorithm for computing slowly synchronizing automata is constructed in such a way, that it only gives solutions for which all pairs synchronize, even if we start with a strict subset of the states. So all given solutions are synchronizing automata, and $S'$ above will synchronize all the time.

The PFA algorithm for computing slowly synchronizing automata on subset $S$ will give all solutions in which subsets of size less than $|S|$ synchronize, but it may possibly give some other solutions as well.

2.1 DFAs up to 5 states

We found the following subset synchronizations lengths for fully synchronizing DFAs:

| $|S|$ | $n = 3$ | $n = 4$ | $n = 5$ | $n = 6$ |
|------|--------|--------|--------|--------|
| 2    | 3      | 6      | 10     | 15     |
| 3    | 4      | 8      | 13     | 20     |
| 4    | 9      | 15     | 22     |        |
| 5    |        | 16     | 24     |        |
| 6    |        |        |        | 25     |
If you see a pattern in these values, then take special attention to the value 20 for \( n = 6 \) and \(|S| = 3\). Actually, there is a pattern, namely the known formula

\[
(n - 1)^2 \left( \left\lceil \frac{n}{|S|} \right\rceil - 1 \right) \left( 2n - |S| \left\lceil \frac{n}{|S|} \right\rceil - 1 \right)
\]

for the synchronization lengths of the Cerny automata. A conjecture of Ângela Cardoso asserts that the subset synchronization lengths of the Cerny automata are the best possible for synchronizing DFAs, see [5]. So we have proved this conjecture up to 6 states. The conjecture is trivially true for \(|S| = 2\), because in order to synchronize the hardest pair in a Cerny automaton, all pairs must be traversed.

**Theorem 2.1.** Suppose that \( n \leq 5 \) and that subset \( S \) of the \( n \) states synchronizes. If our DFA is not synchronizing, then the length of the minimum synchronizing word for \( S \) is less than the value for \( n \) and \(|S|\) in the above table.

In particular, the above table yields the subset synchronization lengths for DFAs with \( n \leq 5 \) states.

**Proof.** If \(|S| = 2\), then the values in the table indicate that the corresponding synchronization paths visit all subsets of size 2. In particular, our DFA is synchronizing.

If \(|S| = n\), then our DFA is synchronizing as well. So three cases remain.

- **\( n = 4 \) and \(|S| = 3\).**

  Assume that our DFA is not synchronizing. Then there is a subset of size 2 which does not synchronize, say \( \{3, 4\} \) does not synchronize. Then there are at most \( \binom{4}{2} - 1 = 5 \) subsets of size 2 which do synchronize.

  Since \( \{1, 3, 4\} \) and \( \{2, 3, 4\} \) do not synchronize either, there are at most \( \binom{4}{3} - 2 = 2 \) subsets of size 3 which do synchronize. This leaves \( 2 + 5 = 7 \) subsets for the synchronization path, which is less than the value 8 for \( n \) and \(|S|\) in the above table.

- **\( n = 5 \) and \(|S| = 4\).**

  Assume that our DFA is not synchronizing. Then there is a subset of size 2 which does not synchronize, say \( \{4, 5\} \) does not synchronize. We distinguish three subcases.

  - **\( \{4, 5\} \) is the only subset of size 2 which does not synchronize.**

    Then every symbol of our DFA acts as a permutation on \( \{4, 5\} \). So the number of states of the set \( \{4, 5\} \) in our synchronization path will be increasing. We can count the number of subsets \( S' \) for given values of \(|S'|\) and \(|S' \cap \{4, 5\}|\)

      | \(|S' \cap \{4, 5\}|\) | \(|S'| = 4\) | \(|S'| = 3\) | \(|S'| = 2\) |
      |---|---|---|---|
      | 0 | 0 | 1 | 3 |
      | 1 | 2 | 6 | 6 |

    From this, we deduce that the synchronization path has length at most \( 2 + 6 + 6 = 14 \), which is less than the value 15 for \( n = 5 \) and \(|S| = 4\) in the table.
\(-\{3, 4\}\) and \(\{4, 5\}\) do not synchronize.

Then one can count that at most one subset of size 4 synchronizes, at most 5 subsets of size 3 synchronize, and at most 8 subsets of size 2 synchronize. From this, we deduce that the synchronization path has length at most \(1 + 5 + 8 = 14\), which is less than the value 15 for \(n = 5\) and \(|S| = 4\) in the table.

\(-\{2, 3\}\) and \(\{4, 5\}\) do not synchronize.

Then there is no subset of size 4 which synchronizes. This contradicts that \(|S|\) synchronizes.

- \(N = 5\) and \(|S| = 3\).

Just as in the previous case, we assume that \(\{4, 5\}\) does not synchronize, and we distinguish the same three subcases.

- \(\{4, 5\}\) is the only subset of size 2 which does not synchronize.

In a similar manner as in the corresponding subcase of the case above, we deduce that the synchronization path has length at most \(1 + 6 + 6 = 13\), which is exactly the value 13 for \(n = 5\) and \(|S| = 3\) in the table.

Suppose that length 13 is indeed possible. We derive a contradiction. We can deduce that \(S = \{1, 2, 3\}\). Let \(a\) be the first symbol of the shortest synchronizing word for \(S\), and let \(S' = Sa\). Then \(|S' \cap \{4, 5\}| = 1\), say that \(S' \cap \{4, 5\} = \{4\}\).

The preimage \(\{4\} a^{-1}\) of \(\{4\}\) under \(a\) contains exactly one element of \(\{1, 2, 3\}\) and exactly one element of \(\{4, 5\}\). There are 2 subsets of size 3, which contain \(\{4\} a^{-1}\), but not \(\{4, 5\}\). Both subsets are mapped by \(a\) to a subset of size 2 which synchronizes. This contradicts that there are 7 subsets of size 3 in our synchronization path.

- \(\{3, 4\}\) and \(\{4, 5\}\) do not synchronize.

In a similar manner as in the corresponding subcase of the case above, we deduce that the synchronization path has length at most \(5 + 8 = 13\), which is exactly the value 13 for \(n = 5\) and \(|S| = 3\) in the table.

Suppose that length 13 is indeed possible. We derive a contradiction. Notice that \(\{1, 2\}\) is in the synchronization path, say \(Sw = \{1, 2\}\). Since either \(S \cap \{3, 4\} \neq \emptyset\) or \(S \cap \{4, 5\} \neq \emptyset\), we deduce that either \(\{3, 4\} w \cap \{1, 2\} \neq \emptyset\) or \(\{4, 5\} w \cap \{1, 2\} \neq \emptyset\). So there is another subset of size 2 which does not synchronize. Contradiction.

- \(\{2, 3\}\) and \(\{4, 5\}\) do not synchronize.

Then one can count that at most 4 subsets of size 3 synchronize, and at most 8 subsets of size 2 synchronize. From this, we deduce that the synchronization path has length at most \(4 + 8 = 12\), which is less than the value 13 for \(n = 5\) and \(|S| = 3\) in the table.

\textbf{PFAs up to 5 states}

We found the following subset synchronizations lengths for PFAs, for which subsets of size less than \(|S|\) synchronize, and possibly other PFAs.
Notice that the value 22 for \( n = 5 \) and \(|S| = 4\) is bigger than the value 21 for \( n = 5 = |S|\). Hence the automaton for the case \( n = 5 \) and \(|S| = 4\) is not synchronizing.

Just as for the DFAs, we can take the Cerny automata if \(|S| = 2\). If \(|S| = n\), then we take the PFAs in [I]. Otherwise, we take the following ternary PFAs:

\[
\begin{array}{cccc}
|S| & n = 3 & n = 4 & n = 5 \\
2 & 3 & 6 & 10 \\
3 & 4 & 10 & 20 \\
4 & 10 & 22 & \\
5 & 21 & & \\
\end{array}
\]

Theorem 2.2. Suppose that \( n \leq 5 \) and that subset \( S \) of the \( n \) states synchronizes. If our PFA does not synchronize all subsets of size less than \(|S|\), then the length of the minimum synchronizing word for \( S \) is less than the values for \( n \) and \(|S|\) in the above table.

In particular, the above table yields the subset synchronization lengths for PFAs with \( n \leq 5 \) states.

Proof. If \(|S| \leq 3\), then the values in the table indicate that the corresponding synchronization paths visit all subsets of size at least 2 and at most \(|S|\). In particular, all subsets of size less than \(|S|\) synchronize.

If \(|S| = n\), then our PFA synchronizes, and so do all subsets of size less than \(|S|\).

So the only case which remains is \( n = 5 \) and \(|S| = 4\).

So assume we have a PFA with \( n = 5 \) states, which synchronizes a subset \( S \) of size 4. Now suppose that there is a subset \( S' \) of size less than 4, which does not synchronize. Then we can choose such an \( S' \) of size 3. Say that \( \{3, 4, 5\} \) does not synchronize. Then \( \{1, 3, 4, 5\} \) and \( \{2, 3, 4, 5\} \) do not synchronize either. So there are at most

\[
\left( \binom{3}{2} - 2 \right) + \left( \binom{3}{2} - 1 \right) + \binom{3}{2} = 3 + 9 + 10 = 22
\]

subsets of size 2, 3 or 4 which do synchronize.

It follows that a synchronization path of minimum length at least 22 for \( S \) contains all subsets of size 2 and 3, except \( S' \). This is however not possible. Indeed, let \( a \) be the symbol in the synchronization path which is applied on the last subset of size 4, say \( \{q_1, q_2, q_3, q_4\} \). Then \( \{q_1, q_2, q_3, q_4\}a \) has size less than 4. Say that \( q_1a = q_2a \).
Then \( \{q_1, q_2, q_4\}a \) and \( \{q_1, q_2, q_4\}a \) have size less than 3. Since our synchronization path contains all subsets of size 2, we deduce that \( \{q_1, q_2, q_4\} \neq S' \neq \{q_1, q_2, q_4\} \). As our synchronization path does not require both \( \{q_1, q_2, q_4\} \) and \( \{q_1, q_2, q_4\} \), there is another subset of size 3 besides \( S' \) which can be excluded from the synchronization path. Contradiction, so our PFA synchronizes all subsets of size less than \(|S|\).

\[\square\]

3 Subset synchronization with any number of states

3.1 Complete NFAs

We construct a complete NFA with \( O(n^2) \) symbols in which every subset of size at least 2 must be traversed before a singleton can be reached. But first, we construct a complete NFA with only \( O(n) \) symbols in which half of the subsets is traversed in a synchronization path.

**Theorem 3.1.** Let \( n \geq 2 \). There exists a complete NFA with state set \( Q = \{1, 2, \ldots, n\} \) and \( 2n - 2 \) symbols, which synchronizes \( Q \) in \( 2n - 2 \) steps and \( \{n - 1, n\} \) in \( 2n - 2 \) steps.

More precisely, the shortest path from \( Q \) to \( \{1\} \) traverses all subsets of \( Q \) except \( \emptyset \) in reverse lexicographic order. Furthermore, the NFA is transitive on the set of nonempty state subsets.

**Proof.** To get from a subset without 1 to its successor, we define

\[ (n, n - 1, \ldots, 1) = (\{n\}, \{n - 1\}, \ldots, \{i + 1\}, \{1, 2, \ldots, i - 1\}, Q, \ldots, Q) \]

for each \( i \in \{2, 3, \ldots, n\} \). The successor of a subset with 1 is obtained by removing 1, for which we use symbols defined by

\[ (n, n - 1, \ldots, 1) = (\{n\}, \{n - 1\}, \ldots, \{2\}, \{i\}) \]

for each \( i \in \{2, 3, \ldots, n\} \). The last claim follows from \( \{1\}a_2 = Q \)

\[\square\]

**Theorem 3.2.** Let \( n \geq 3 \). There exists a complete NFA with state set \( Q = \{1, 2, \ldots, n\} \) and \( \frac{n}{2} + 1 \) symbols, in which the shortest path from \( Q \) to \( \{3\} \) first traverses all subsets of size at least 2 of \( Q \) in reverse lexicographic order, and next traverses \( \{2, 1\}, \{n\}, \{n - 1\}, \ldots, \{3\} \), in that order. Furthermore, the NFA is transitive on the set of nonempty state subsets.

**Proof.** Just as before, to get from a subset of size at least 2 without 1 to its successor, we define

\[ (na, n - 1, \ldots, 1) = (\{n\}, \{n - 1\}, \ldots, \{i + 1\}, \{1, 2, \ldots, i - 1\}, Q, \ldots, Q) \]

but only for each \( i \in \{2, 3, \ldots, n - 1\} \). For the successor of a subset of size at least 2 with 1, we use symbols \( b_{ij} \) and \( b_{j1} \) with \( 3 \leq i < j \leq n \), defined by

\[ (nb_{j1}, n - 1, b_{ij}, \ldots, 1b_{ij}) = (\{n\}, \{n - 1\}, \ldots, \{2\}, \{i, j\}) \]

and symbols defined by \( ic_i = \{i - 1\} \) and

\[ (nc_i, (n - 1)c_i, \ldots, 1c_i) = (Q, Q, Q, \{i - 1\}, Q, \ldots, Q, \{1, 2, \ldots, i - 2\}) \]
for each $i \in \{3, 4, \ldots, n\}$. If $i \geq 4$, then symbol $c_i$ sends $\{i\}$ to its successor, too. Since $c_3$ sends $\{2\}$ to its successor, we only need a symbol which sends $\{1\}$ to its successor. For that purpose, we define $c_1$ by $ic_1 = \{n\}$ if $i = 1$ and $ic_1 = Q$ otherwise. The last claim follows from $\{3\}c_1 = Q$.

**Corollary 3.3.** The maximum synchronization length for state subsets of size $|S| \geq 2$ in complete NFAs with $n$ states is

$$2^n - n - 2^{n-|S|}$$

**Proof.** Let $|S|$ be a subset of the $n$ states of an NFA. In a synchronization path of $|S|$, the $2^{n-|S|} - 1$ proper supersets of $S$ do not occur. $\emptyset$ does not occur either, and singleton sets only occur at the end. This reduces the first claim to the last claim. The last claim follows by taking $S = \{n - |S| + 1, n - |S| + 2, \ldots, n\}$ in the above theorem.

In [7], careful synchronization is defined for NFAs in general, not just PFAs. For that type of synchronization, we have the same results as in the above corollary. This is because we can turn an NFA which synchronizes some subset $S$ carefully into a complete NFA as follows: we replace each $ix = \emptyset$ by $ix = Q$, where $i$ is any state, $x$ is any symbol, and $Q$ is the subset of all states.

### 3.2 PFAs and subsets of size 3

The following theorem yields the lengths of 3-set synchronization for PFAs.

**Theorem 3.4.** Let $n \geq 3$. Then there exists a PFA with $n$ states and $2n - 3$ symbols, say with states $1, 2, \ldots, n$, such that the shortest path from $\{n - 2, n - 1, n\}$ to $\{n\}$ traverses all subsets of size 1, 2 and 3.

**Proof.** We traverse the subsets of size 1 and 2 in lexicographic order. For that purpose, we define

$$(1a_i, 2a_i, \ldots, na_i) = \begin{cases} (1, \ldots, i - 1, i + 1, \perp, \ldots, \perp, i + 2) & 1 \leq i \leq n - 2 \\ (1, \ldots, i - 1, i + 1, i) & i = n - 1 \end{cases}$$

Traversing the subsets of size 3 in the same order is impossible, since it would yield $n - 3$ shortcuts for the subsets of size 2: subset $\{i, n\}$ can be skipped for all $i \leq n - 3$. The order of the subsets of size 3 will be lexicographic as well, but first, we negate the first state of the 3-set. So subset $\{n - 2, n - 1, n\}$ is ordered by way of $\{2 - n, n - 1, n\}$ and is the first 3-set, and subset $\{1, n - 1, n\}$ is ordered by way of $\{-1, n - 1, n\}$ and is the last 3-set.

With the above symbols, we can already move subsets of size 3 to their successors, provided the first state does not need to be changed for that. To move other 3-sets to their successors and to move the last subset of size 2 and 3 to the first subset of size 1 and 2 respectively, we add the following symbols.

$$(0b_i, 1b_i, \ldots, nb_i) = \begin{cases} (i + 1, \perp, \ldots, \perp, i, i) & i = 1 \\ (1, \ldots, i - 2, \perp, i + 1, \perp, \ldots, \perp, i, i - 1) & 2 \leq i \leq n - 2 \end{cases}$$

It is a straightforward exercise to verify that no shortcuts are possible.
Corollary 3.5. The maximum synchronization length for state subsets of size 3 in PFAs with \( n \) states is
\[
\binom{n}{3} + \binom{n}{2}
\]
This maximum cannot be obtained for PFAs which synchronize a state subset of size 4.

Proof. The first claim follows from the above theorem. To prove the last claim, let \( a \) be a symbol which reduces a subset of size 4, say \( \{q_1, q_2, q_3, q_4\} \), to a smaller subset. Say that \( q_1a = q_2a \). Then both \( \{q_1, q_2, q_3\}a \) and \( \{q_1, q_2, q_4\}a \) have size less than 3. Hence not all subsets of size 3 need to be traversed.

Notice that the above corollary proves that the maximum synchronization length \( p(4) \) of a PFA with 4 states is at most 10, and that \( p(4) = 10 \).

4 Asymptotics for subset synchronization of automata

We start with some results for an arbitrary number of symbols.

Proposition 4.1. There exists a transitive PFA \( M \) with \( n \) states, which has a subset which synchronizes carefully in
\[
\Omega\left(\frac{2^n}{\sqrt{n}}\right) = \Omega(1.9999^n)
\]

steps.

Proof. Let \( S_1, S_2, \ldots, S_b \) be the subsets of size \( \lfloor n/2 \rfloor \) of the \( n \) states. Take for each \( i < b \) a symbol \( a_i \) such that \( S_i a_i = S_{i+1} \) and \( a_i \) is undefined outside \( S_i \). Take \( a_b \) such that \( S_b a_b \) is a singleton and \( a_b \) is undefined outside \( S_b \). The word \( a_1 a_2 \ldots a_b \) is carefully synchronizing for subset \( S_1 \), and is a prefix of every other such word. For even \( n \), the estimate \( \Theta\left(\frac{2^n}{\sqrt{n}}\right) \) for \( b \) can be found on the internet. For odd \( n \), \( b \) is half of the value it would have if \( n \) was 1 larger.

Take \( a_0 \) such that \( S_b a_0 = S_1 \) and \( a_0 \) is undefined outside \( S_b \). Then the word \( a_1 a_2 \ldots a_b a_0 \) is maintained as the shortest carefully synchronizing word. The orbit of a state \( q \) has size \( > \lfloor n/2 \rfloor \), because it intersects with every subset of size \( \lfloor n/2 \rfloor \). So this orbit contains a subset of \( \lfloor n/2 \rfloor \) states, hence it contains all states. So \( M \) is transitive.

The first claim of the following result can be found in [2].

Proposition 4.2. There exists a DFA \( M \) with \( n \) states, which has a subset which synchronizes in
\[
\Omega\left(\frac{2^n}{\sqrt{n}}\right) = \Omega(1.9999^n)
\]
steps. Furthermore, the DFA is transitive up to 2 sink states.

There exists a transitive DFA \( M \) with \( n \) states, which has a subset which synchronizes in
\[
\Omega\left(\frac{2^{n/2}}{\sqrt{n}}\right) = \Omega(1.4142^n)
\]
steps.
Proof. The first claim follows from proposition 4.1 above and Lemma 1 of \[8\]. The second claim follows from the first claim and Lemma 4 of \[8\].

Since $1.4142^n > 7.9995^{n/6} > 3^{n/6}$, the last result in the above proposition improves the third result in Proposition 1 of \[8\]. In Theorem 16 of \[1\], we improve the second results of Theorem 1 and Proposition 1 of \[8\]:

$$\Omega\left(\frac{2^{n/3}}{(n\sqrt{n})}\right) = \Omega(1.9999^{n/3}) = \Omega(1.2599^n)$$

The first results of Theorem 1 and Proposition 1 of \[8\] will be improved by theorem 4.5 and theorem 4.4 below, respectively.

### 4.1 Binary PFAs

We adapt some techniques of \[1\] to construct binary PFAs with large subset synchronization.

**Theorem 4.3.** There exists a transitive PFA $M$ with $n$ states and only 2 symbols, which has a subset which synchronizes carefully in

$$\Omega\left((3 - \epsilon)^{n/3}\right) = \Omega(1.4422^n)$$

steps.

**Proof.** We first construct a PFA $M$ with $n = 3k$ states and 3 symbols. The state set is $\{A_i, X_i, B_i \mid i \in \mathbb{Z}/(k\mathbb{Z})\}$. We leave out the states $C_i$ and the starting symbol $s$. Instead, we add a finishing symbol $f$.

$$(A_i c, X_i c, B_i c) = (A_{i+1}, X_{i+1}, B_{i+1})$$

$$(A_i r, X_i r, B_i r) = \begin{cases} (\bot, \bot, A_i), & \text{if } i = 1, 2, \ldots, h \\ (X_i, B_i, \bot), & \text{if } i = h + 1 \\ (A_i, X_i, B_i), & \text{if } i = h + 2, h + 3, \ldots, k \end{cases}$$

$$(A_i f, X_i f, B_i f) = \begin{cases} (A_i, X_i, B_k), & \text{if } i = 1 \\ (\bot, \bot, \bot), & \text{if } i = 2 \\ (A_i, X_i, B_i), & \text{if } i = 3, 4, \ldots, k \end{cases}$$

The subset we start with is $\{A_{h+1}, A_{h+2}, \ldots, A_k\}$, so $k - h$ groups are represented with a state. Using techniques of \[1\], we can prove that it takes $\Omega((3 - \epsilon)^{n/3})$ r-steps to reduce to $k - h - 1$ such groups.

Using Theorem 9 of \[1\], we reduce $\{f, c, rc\}$ to only 2 symbols. Condition (1) of this theorem is not fulfilled for $s = f$, but this is not a problem because we may choose the subset we start with. By removing up to two states from $Q'$, we can obtain binary automata for every number of states.

### 4.2 Binary DFAs

We combine theorem 4.3 with techniques of \[8\] to construct binary DFAs with exponential subset synchronization.
Theorem 4.4. There exists a DFA \( M \) with \( n \) states and only 2 symbols, which has a subset which synchronizes carefully in
\[
\Omega((3 - \epsilon)n/3) = \Omega(1.4422^n)
\]
steps. Furthermore, the DFA is transitive up to 2 sink states.

Proof. We first construct a PFA \( M \) with \( n = 3k + 2 \) states and 3 symbols. The state set is \( \{A_i, X_i, B_i \mid i \in \mathbb{Z}/(k\mathbb{Z})\} \cup \{D, \bar{D}\} \). With two exceptions, we take the definitions in the proof of theorem 4.3 of \( c, r, f \) for all \( i \in \mathbb{Z}/(k\mathbb{Z}) \), and extend them by imposing that \( D \) and \( \bar{D} \) are sink states.

The first exception is that we replace \( \perp \) by \( \bar{D} \) in all definitions. The second exception is that we replace \( B_1 f = B_k \) by \( B_1 f = D \). Just as above, the reduction to only 2 symbols can be done without affecting the estimate. The subset we start with is \( \{A_h + 1, A_h + 2, \ldots, A_k, D\} \).

Theorem 4.5. There exists a transitive DFA \( M \) with \( n \) states and only 2 symbols, which has a subset which synchronizes in
\[
\Omega((3 - \epsilon)n/6) = \Omega(1.2009^n)
\]
Proof. We first construct a PFA \( M \) with \( n = 6k + 4 \) states and 3 symbols. The state set is
\[
\{A_i, \bar{A}_i, X_i, \bar{X}_i, B_i, \bar{B}_i \mid i \in \mathbb{Z}/(k\mathbb{Z})\} \cup \{D, \bar{D}, E, \bar{E}\}
\]
By way of swap congruence, we define how symbols act on states with bars above them. So we only need to describe how symbols act on states without bars above them. Symbols \( c \) and \( r \) are now defined by \( Ec = E \), \( Er = E \), and their definitions in the proof of theorem 4.3. We define symbol \( f \) as follows
\[
Df = E \quad E f = B_k \\
(A_i f, X_i f, B_i f) = \begin{cases}
(E, \bar{E}, D), & \text{if } i = 1 \\
(E, \bar{E}, \bar{E}), & \text{if } i = 2 \text{ or } i = h + 1 \\
(A_i, X_i, B_i), & \text{if } i = 3, 4, \ldots, h, h + 2, h + 3, \ldots, k
\end{cases}
\]
The subset we start with is \( \{A_{h+1}, A_{h+2}, \ldots, A_k, D\} \) or \( \{A_{h+1}, A_{h+2}, \ldots, A_k, D, E\} \). Getting both a state and its corresponding swap state in the subset makes synchronization impossible. Just as in theorem 4.3 \( f \) can only be applied near the end, because otherwise we get both \( E \) and \( \bar{E} \) in our subset. Just as before, the reduction to only 2 symbols can be done without affecting the estimate. \( \square \)

5 D3-directing NFAs

We say that an NFA \( A = (Q, \Sigma, \cdot) \) is D3-directing, if there exists a word \( w \in \Sigma^* \), such that \( \bigcap_{q \in Q} qw \neq \emptyset \). We denote the length of the shortest such \( w \) by \( d_3(A) \).

For symbols \( a, b \) of an NFA \( A = (Q, \Sigma, \cdot) \), we say that \( a = b \) if \( qa = qb \) for all \( q \), and \( a \leq b \) if \( qa \subseteq qb \) for all \( q \in Q \). Furthermore, we say that \( a < b \) if \( a \leq b \) and \( a \neq b \).
We say that a symbol $a$ of an NFA $A = (Q, \Sigma, \cdot)$ is a PFA-symbol, if $|qa| \leq 1$ for all $q \in Q$, and define

$$\text{Split}(A) := (Q, \{x \text{ is a PFA-symbol } | \text{there is an } y \in \Sigma \text{ such that } x \leq y \}, \cdot)$$

Notice that $\text{Split}(A)$ is defined as an NFA which is actually a PFA. For an NFA $A = (Q, \Sigma, \cdot)$, we define

$$\text{Basic}(A) := (Q, \{x \in \Sigma \mid x \not\in \text{Id}_Q \text{ and } x \not\leq y \text{ for all } y \in \Sigma \}, \cdot)$$

**Lemma 5.1.** $d_3(B) = d_3(\text{Basic}(B))$.

**Proof.** Since the symbols of $\text{Basic}(B)$ are a subset of those of $B$, $d_3(B) \leq d_3(\text{Basic}(B))$ follows.

Suppose that $w$ is a D3-directing word of $B$. If all letters of $w$ are symbols of $\text{Basic}(B)$, then we are done, so assume that $w$ has a symbol $x$ which is not a symbol of $\text{Basic}(B)$.

- If $x \leq \text{Id}_Q$, then we can remove all occurrences of $x$ in $w$.
- If $x < y$ for some symbol $y$ of $B$, then we can replace all occurrences of $x$ by $y$ in $w$.

We cannot repeat the above forever, so we have a procedure to change $w$ into a D3-directing word with only symbols of $\text{Basic}(B)$, which is not longer than $w$. Hence $d_3(B) \geq d_3(\text{Basic}(B))$. □

**Theorem 5.2.** $d_3(A) = d_3(\text{Split}(A))$.

**Proof.** Adopt a total ordering on the state set $Q$ of $A$. Let $w = w_1w_2 \cdots w_l$ be a word of $A$, and suppose that $w$ is D3-directing. Say that $r \in qw$ for all $q \in Q$.

Take any $q \in Q$ and define $p_{q0} = q$. Assume by induction that $p_{qi} \in qw_1w_2 \cdots w_i$, and that $r \in p_{qi}w_{i+1}w_{i+2} \cdots w_l$. Then there exists a $p_{q(i+1)} \in p_{qi}w_{i+1}$, such that $r \in p_{qi}w_{i+1}w_{i+2}w_{i+3} \cdots w_l$. Choosing $p_{qi+1}$ to be as small as possible with respect to the ordering of $Q$, yields an inductive definition of $p_{qi}$ for all $q \in Q$ and all $i \in \{1, 2, \ldots, l\}$. Furthermore, $p_{ql} = r$ for all $q \in Q$.

Since we used the ordering of $Q$, we have

$$p_{qi} = p_{q' i} \implies p_{qi+1} = p_{q'(i+1)}$$

Consequently, the following is a proper definition of $w'_i$ for all $i \in \{1, 2, \ldots, l\}$, where $P_i = \{p_{qi} \mid q \in Q\}$.

$$p_{qi}w'_i = \{p_{q(i+1)}\} \quad (q \in Q)$$

$$qw'_i = \emptyset \quad (q \in Q \setminus P_i)$$

Let $w' = w'_1w'_2 \cdots w'_l$. Then $qw' = \{p_{ql}\} = \{r\}$ for all $q \in Q$. As $w'_i$ is a PFA-symbol and $w'_i \leq w_i$ for all $i$, we see that $w'$ is a D3-directing word of $\text{Split}(A)$.

So $d_3(A) \geq d_3(\text{Split}(A))$. Conversely, if $w' = w'_1w'_2 \cdots w'_l$ is a D3-directing word, say that $r \in qw'$ for all $q \in Q$, and $w'_i \leq w_i$ for all $i$, then $r \in qw$ for all $q \in Q$, where $w = w_1w_2 \cdots w_l$. So $d_3(A) \leq d_3(\text{Split}(A))$. □
Let
\[
\begin{align*}
\quad d_3(n) & := \max\{d_3(A) \mid A \text{ is an NFA}\} \\
\quad cd_3(n) & := \max\{d_3(A) \mid A \text{ is a complete NFA}\} \\
\quad p(n) & := \max\{d_3(A) \mid A \text{ is a PFA}\} \\
\quad d(n) & := \max\{d_3(A) \mid A \text{ is a DFA}\}
\end{align*}
\]

For a PFA, D3-directing is the same as careful synchronization, which is just synchronization in case of a DFA. So \(p(n)\) and \(d(n)\) are just the maximum (careful) synchronization lengths for PFAs and DFAs respectively of \(n\) states.

**Corollary 5.3.** \(d_3(n) = p(n)\) and \(cd_3(n) = d(n)\).

**Proof.** Clearly, \(d_3(n) \geq p(n)\) and \(cd_3(n) \geq d(n)\). From the above lemma and theorem, it follows that it suffices to show that \(\text{Basic}(\text{Split}(A))\) is actually a DFA for every complete NFA \(A\). This is straightforward. \(\Box\)

In [3], in which \(cd_3(n) = d(n)\) is proved as well, there is a description of \(\text{Split}\) which is more restrictive in choosing symbols in the following sense: a PFA symbol \(x \leq y\) is only selected if for all states \(q, qx = \emptyset\) if and only if \(qy = \emptyset\). Due to this, their \(\text{Split}\) maps complete NFAs directly to DFAs. But their description of \(\text{Split}\) is kind of algorithmic and takes pages. This is not preferable compared to an algebraic description of only one line.

Lemma 3 of [4] implies that \(d_3(n) = p(n)\). Propositions 2 and 10 in [4] however indicate that \(cd_3(n) = d(n)\) was missed by the authors. The proof of Lemma 3 in [4] follows that of theorem 5.2 above more or less, except for adopting a total ordering on the state set, which is \(Q\) in theorem 5.2 above and \(S\) in Lemma 3 in [4]. Using a total ordering on \(S\) is needed to make that the partial functions \(\rho_1, \rho_2, \ldots, \rho_r\) in the proof of Lemma 3 of [4] are properly defined, i.e. do not depend on the state \(s \in S\).

Let \(A, C\) be NFAs. We say that \(C \leq A\) if for every symbol \(x\) of \(C\), there is a symbol \(y\) of \(A\) such that \(x \leq y\). We say that \(C < A\) if \(C \leq A\) and \(A \not\leq C\). Notice that
\[
C \leq A \implies d_3(A) \geq d_3(C)
\]
Furthermore,
\[
\text{Split}(A) \leq A \quad \text{and} \quad \text{Basic}(A) \leq A
\]
and \(\text{Split}(A) < A\), if and only if \(A\) is not actually a PFA.

Let \(B\) be an NFA, with alphabet \(\Sigma\). Let \(P\) be a partition of \(\Sigma\) into \(p\) subsets. Then we can merge the symbols of each of the subsets of \(P\), to obtain an NFA \(C\) with at most \(p\) symbols. We call \(C\) a **partitional symbolic merge** of \(B\). Notice that \(B \leq C\).

**Proposition 5.4.** Let \(A, B\) be NFAs, such that \(B \leq A\). Then there exists a partitional symbolic merge \(C\) of \(B\) for which \(B \leq C \leq A\), such that the number of symbols of \(C\) does not exceed that of \(A\).

Furthermore, \(d_3(C) = d_3(B) = d_3(A)\) if \(d_3(B) \leq d_3(A)\).

**Proof.** For each symbol \(x\) of \(B\), we choose a symbol \(y\) of \(A\) such that \(x \leq y\). We make \(C\) from \(B\) by merging symbols \(x\) for which we chose the same symbol \(y\) of \(A\). The last claim follows from \(d_3(B) \geq d_3(C) \geq d_3(A)\). \(\Box\)
For \( n = 2, 3, 4, 5, 6, 7 \), we scanned all PFAs with maximum careful synchronization lengths \( p(n) \) which were minimal as such with respect to \( \leq \), and tried every partitional symbolic merge of it (up to straightforward pruning by way of merges which decrease the length of the D3-directing word). This yielded several NFAs with \( n - 1 \) symbols which were D3-directing in \( p(n) \) steps, but no such NFAs with fewer symbols. From the above proposition, we infer that NFAs with \( n \leq 7 \) states and fewer than \( n - 1 \) symbols are not D3-directing in the maximum number of \( p(n) \) steps.

Below on the left, there is a PFA with 4 states and 4 symbols, of which \( a' \) is the identity symbol, which takes \( p(4) = 10 \) steps to synchronize carefully. It remains D3-directing in exactly 10 steps if the symbols \( a \) and \( a' \) are merged. This merge yields an NFA with 4 states and 3 symbols.

Above on the right, there is a PFA with 5 states and 6 symbols, which takes \( p(5) = 21 \) steps to synchronize carefully. It remains D3-directing in exactly 21 steps if the symbols \( a \) and \( a' \) are merged, and the symbols \( d \) and \( d' \) are merged. These merges yield an NFA with 5 states and 4 symbols.

Above, there is a PFA with 6 states and 7 symbols, of which \( a' \) is the identity symbol, which takes \( p(6) = 37 \) steps to synchronize carefully. It remains D3-directing in exactly 37 steps if the symbols \( a \) and \( a' \) are merged, and the symbols \( e \) and \( e' \) are merged. These merges yield an NFA with 6 states and 5 symbols.

Above, there is a PFA with 7 states and 8 symbols, which takes \( p(7) = 63 \) steps to synchronize carefully. It remains D3-directing in exactly 63 steps if the
symbols $a$ and $a'$ are merged, and the symbols $f$ and $f'$ are merged. These merges yield an NFA with 7 states and 6 symbols.

For $n = 5$ and $n = 7$ states, the minimum number of $n - 1$ symbols is not possible if Split contains the identity symbol. For $n = 2, 3, 4, 6$ states, Split may contain the identity symbol without affecting the minimum number of $n - 1$ symbols. Up to the identity symbol $a'$ in case of 4 or 6 states, the displayed PFAs are the same as those in [1].

Since the symbols of an NFA (which is actually a DFA) can be merged in other ways than by way of a partitional symbolic merge (the partition may be replaced by any cover), the above techniques do not help if you want to count the number of (basic) NFAs with $n$ states which are D3-directing in the maximum number of steps. In [3], the basic complete NFAs with $n$ states which are D3-directing in the maximum number of steps are counted for all $n \leq 7$, and for $n \geq 8$ under the assumption that the Černý automaton with $n$ states is the only automaton which synchronizes in at least $(n - 1)^2$ steps.

I have made the same counts and can confirm the results of [3]. In the case of 2 states, there are some subtleties with symmetry to take into account: there are 33 basic complete NFAs $A$ for which Basic(Split($A$)) is one of the 6 basic DFAs which synchronize, 27 basic complete NFAs $A$ for which Basic(Split($A$)) is one of the 4 symmetrically different basic DFAs which synchronize, and 20 symmetrically different basic complete NFAs $A$ for which Basic(Split($A$)) synchronizes. Synchronization always takes 1 step.

6 An estimate on both sides of the prime number constructions of Martyugin

My co-author Henk Don of [1] reinvented the prime number construction in [6] of ternary PFAs with superpolynomial synchronization, which is as follows.

![Diagram]

The shortest synchronizing word is $bq^{p-1}c$, where $p$ is the product of the prime numbers which are represented by a group of states.

In [6], there is a prime number construction of binary PFAs with superpolynomial synchronization as well, which is slightly modified below. The difference is that $a$ maps the node right from the lowest of a prime group to the sink state.
here, instead of the node left from it. This improves the synchronization length with 2.

The shortest synchronizing word is \( b^2(ab)^{p-1}a^2 = b(ba)p a \), where \( p \) is the product of the prime numbers which are represented by a group of states.

In [4], the author Martyugin found distinct lower bounds for the synchronization lengths of the above ternary and binary construction, but a closer look reveals that the bounds are actually the same, and strictly weaker than \( 2^\Omega(\sqrt{n}) \).

We improve this lower bound and also obtain an upper bound which is subexponential, by showing that the number of required steps is

\[
2^{\Theta(\sqrt{n \log n})} = n^{\Theta(\sqrt{n \log n})}
\]

We can follow the proof of Martyugin to some extend. Suppose that the first \( r \) primes are represented by a group in one of the above constructions. Then the \( i \)-th prime is \( \Theta(i \log i) \), which is \( O(i \log r) \) and \( \Omega((i^2 \log r)/r) \) if \( i \leq r \), because \( i \mapsto (\log i)/i \) is decreasing beyond Euler’s number. Hence

\[
n = \sum_{i=1}^{r} O(i \log r) = \sum_{i=1}^{r} O(i \log n) = O(r^2 \log n)
\]

So \( r = \Omega(\sqrt{n/\log n}) \) and \( \log r = \Omega(\log n) \). Hence

\[
n = \sum_{i=1}^{r} \Omega((i^2 \log r)/r) = \sum_{i=1}^{r} \Omega((i^2 \log n)/r) = \Omega(r^2 \log n)
\]

So \( r = O(\sqrt{n/\log n}) \). Hence \( r = \Theta(\sqrt{n/\log n}) \). Furthermore,

\[
r! = \sqrt[\prod_{i=1}^{r} (i \cdot (r + 1 - i))} \geq \sqrt[\prod_{i=1}^{r} r] \geq r^{r/2}
\]

so

\[
p \geq r! = r^{\Omega(r)} = 2^{\Omega(r \log r)} = 2^{\Omega(\sqrt{n/\log n \log n})}
\]

and

\[
p \leq n^r = n^{\Omega(r)} = 2^{\Omega(r \log n)} = 2^{\Omega(\sqrt{n/\log n \log n})}
\]

which yields the estimate.
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