TOPOLOGICAL T-DUALITY, AUTOMORPHISMS AND CLASSIFYING SPACES

ASHWIN S. PANDE

Abstract. We extend the formalism of Topological T-duality to spaces which are the total space of a principal $S^1$-bundle $p : E \to W$ with an $H$-flux in $H^3(E, \mathbb{Z})$ together with an automorphism of the continuous-trace algebra on $E$ determined by $H$. The automorphism is a ‘topological approximation’ to a gerby gauge transformation of spacetime. We motivate this physically from Buscher’s Rules for T-duality. Using the Equivariant Brauer Group, we connect this problem to the $C^*$-algebraic formalism of Topological T-duality of Mathai and Rosenberg [1].

We show that the study of this problem leads to the study of a purely topological problem, namely, Topological T-duality of triples $(p, b, H)$ consisting of isomorphism classes of a principal circle bundle $p : X \to B$ and classes $b \in H^2(X, \mathbb{Z})$ and $H \in H^3(X, \mathbb{Z})$.

We construct a classifying space $R_{3,2}$ for triples in a manner similar to the work of Bunke and Schick [2]. We characterize $R_{3,2}$ up to homotopy and study some of its properties. We show that it possesses a natural self-map which induces T-duality for triples. We study some properties of this map.

1. Introduction

Topological T-duality is an attempt to study the T-duality symmetry of Type II String Theory [3] using methods from Noncommutative and Algebraic Topology [1, 2, 4, 5]. In the simplest case, T-duality states that Type II A string theory on a certain background is equivalent to Type II B string theory on another background, that is, acting on the original string theory by a canonical transformation (termed T-duality) transforms that theory into the dual one. For this to be possible, the background spacetime must carry a torus action (which need not be free). In this paper we only consider backgrounds which are principal circle bundles.

Both Type IIA and Type IIB String Theory backgrounds possess three massless bosonic fields: A graviton (associated to the metric),

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Date: May 5, 2014.

1 See Ref. [3] item (6) on page (4) and Sec. (7) for a proof.
an $H$-flux (associated to the Kalb-Ramond field $B$, $H = dB$) and a dilaton. As is well known [6], the $H$-flux is a closed integral three-form which is the gerbe curvature of a gerbe with connection form $B$ on that background. (The word ‘gerbe’ is used in the sense of Ref. [6].) It is a remarkable fact that the Topology and $H$-flux of the T-dual spacetime depend only on the Topology and $H$-flux of the original spacetime. This phenomenon is called Topological T-duality.

Let $X^m$ be a $m$-dimensional manifold which is a principal circle bundle over a base $W$. Let $X^m \times Y^{10-m}$ be the manifold which is a model for the spacetime background. Suppose that there is a type II String Theory on this spacetime background. We perform a T-duality along the circle orbits in $X^m$. Topological T-duality (see Refs. [1, 2, 4, 5]) claims that the underlying topological space of the physical T-dual background is $(X^m)^\# \times Y^{10-m}$. Here, $(X^m)^\#$ is the Topological T-dual of the underlying topological space of $X^m$.

Consider a spacetime with a $H$-flux of strength $H$. We model the spacetime by a manifold $X$ which is the total space of a principal circle bundle $p : X \to W$ and the $H$-flux by a gerbe with connection on $X$ whose gerbe curvature form is $H$. Let $V_\alpha$ be an open cover of $W = X/S^1$ and $U_\alpha$ be the lift of the cover to $X$.

For the convenience of the reader, we give the definition of a gerbe and gerbe connection here. We follow the treatment in Minasian, Ref. [6], Sec. (2.2) here.

**Def 1.1.** A gerbe on $X$ is defined by the following data:

1. A line bundle $L_{\alpha \beta}$ on each two-fold intersection $U_{\alpha \beta}$.
2. An isomorphism $L_{\alpha \beta} \simeq L_{\beta \alpha}$.
3. A smooth nowhere-zero section $f_{\alpha \beta \gamma} : X_{\alpha \beta \gamma} \to \mathbb{C}^*$ of the line bundle $L_{\alpha \beta} \otimes L_{\beta \gamma} \otimes L_{\gamma \alpha}$ on each three-fold intersection $U_{\alpha \beta \gamma}$.
4. $f_{\alpha \beta \gamma}$ satisfies the cocycle condition $(\delta f)_{\alpha \beta \gamma \delta} = f_{\alpha \beta \gamma} f_{\beta \gamma \delta} f_{\gamma \delta \alpha} f_{\delta \alpha \beta}^{-1} = 1$ on each four-fold intersection $U_{\alpha \beta \gamma \delta}$.

We now define a gerbe with connection also following Minasian:

**Def 1.2.** A gerbe with connection on $X$ is a gerbe on $X$ together with a connection $A_{\alpha \beta}$ on the line bundle $L_{\alpha \beta}$ in each $U_{\alpha \beta}$ such that the section $f_{\alpha \beta \gamma}$ is covariantly constant with respect to the induced connection on $L_{\alpha \beta} \otimes L_{\beta \gamma} \otimes L_{\gamma \alpha}$:

$$A_{\alpha \beta} + A_{\beta \gamma} + A_{\gamma \alpha} = \frac{1}{2\pi i} f_{\alpha \beta \gamma}^{-1} df_{\alpha \beta \gamma}.$$
and a two-form (the gerbe connection) \( B_\alpha \in \Omega^2(U_\alpha) \) such that \( B_\alpha - B_\beta = dA_{\alpha\beta} \) on \( U_{\alpha\beta} \).

It is clear that \( dB_\alpha = dB_\beta \) and hence the forms \( dB_\alpha \) glue into a global three-form \( H \) termed the gerbe curvature. Physically, it models the field strength of the \( B \)-field. This three-form is integral and defines a characteristic class \([H]\) in de Rham cohomology.

We now define the notion of a gauge transformation and large gauge transformation of a gerbe. We follow Ref. [14] for the definition of a gauge transformation of a gerbe (see Ref. [14]: A 1-gauge transformation\(^4\) there is termed a gauge transformation here. A 0-gauge transformation\(^5\) there would be a family of automorphisms of each of the \( L_{\alpha\beta} \), that is it would correspond to performing an independent gauge transformation on each of the \( L_{\alpha\beta} \) and Ref. [15] for the definition of a large gauge transformation of the \( B \)-field (see Ref. [15], pg. (11) before Eq. (2.17)):

**Def 1.3.** Let \( X \) and \( \{U_\alpha\} \) be as above. Let \( U_\alpha \mapsto G_\alpha \) be an assignment of one-forms to the open sets in the chart on \( X \). A gauge transformation of the gerbe on \( X \) is the following transformation of the \( B \)-field and its gauge field:

- \( A_{\alpha\beta} \mapsto A_{\alpha\beta} + (G_\alpha|_{U_{\alpha\beta}} - G_\beta|_{U_{\alpha\beta}}) \),
- \( B_\alpha \mapsto B_\alpha + dG_\alpha \).

In Ref. [15] the authors point out\(^6\), that for a \( l \)-form field strength with \((l - 1)\)-form potential \( A \), gauge transformations of the form \( A \to A + \omega \) with [\( \omega \)] nontrivial are termed ‘large’ gauge transformations. For the \( B \)-field in Type II string theories, \( l = 3 \) (see item no. (5) after Eq. (2.21) on page 13 of Ref. [15]).

Hence for large gauge transformations, \( dG_\alpha = \omega|_{U_\alpha} \), with \( \omega \) a closed two-form on \( X \). Among these, there are gauge transformations for which \( \omega = F \), with \( F \) a closed integral two-form on \( X \). These transformations are special for a geometric reason: It is possible to tensor a gerbe with a line bundle \( L \) : One sends \( L_{\alpha\beta} \mapsto L|_{U_{\alpha\beta}} \otimes L_{\alpha\beta}, A_{\alpha\beta} \mapsto A|_{U_{\alpha\beta}} + A_{\alpha\beta}, B_\alpha \mapsto B + F|_{U_\alpha} \) where \( A, F \) are the connection and curvature forms of the connection on \( L \). (See Ref. [6] after Eq. (2.10).) It is these special gauge transformations that we consider in this paper. Such a gauge transformation cannot be homotoped to zero without changing its cohomology class and is termed a large gauge transformation. These special transformations generate the gauge group.

\(^4\)See Ref. [14], Def. (2.2.5).
\(^5\)See Ref. [14], Def. (2.1.5).
\(^6\)See pg. (11) before Eq. (2.17).
Consider a gerbe with connection on $X$ with gerbe curvature form $H$ which is equivariant under the $S^1$-action on $X$: That is, for every $t \in S^1$, pulling the gerbe connection back by the $S^1$-action map $\phi_t : X \to X$ causes the gerbe connection to undergo a gauge transformation. Note that in this case the characteristic class $[H] \in H^3(X, \mathbb{Z})$ of the gerbe curvature form $H$ is invariant under pullback by $\phi_t$ for every $t \in S^1$, i.e., for every $t \in S^1$, $[\phi_t^*H] = [H]$. (See Ref. [6], before Eq. (2.3).)

In this paper we study large gauge transformations of the gerbe which are equivariant under the circle action on $X$: We had noted above that a large gauge transformation naturally gives rise to a closed integral two-form $F$ on $X$. Locally these transformations are of the form $B_\alpha \to B_\alpha + F|_{U_\alpha}$ where $F$ is the curvature of a line bundle $L$ naturally associated to the large gauge transformation as described above. Such a gauge transformation has as a characteristic class the integral cohomology class of $F$, namely $[F] \in H^2(X, \mathbb{Z})$. We require that $L$ be equivariant under the circle action on $X$: That is, we require that the circle action $\phi_t$ on $X$ acts on $L$ by bundle automorphisms such that its curvature remains invariant under pullback by $\phi_t$, $\phi_t^*(F) = F$. It is clear that (see Ref. [6] Appendix C for a treatment of equivariant line bundles on a principal bundle in another context) the Lie derivative of $F$ with respect to the circle action vanishes.

We now argue that for a smooth gerbe T-duality also T-dualizes gerbe automorphisms. We use the description of T-duality in Type II string theory using the theory of smooth gerbes given in the paper by Minasian (see Ref. [6]). After this, we will abstract this argument into a purely topological conjecture for automorphisms of gerbes on spaces and make a connection with the $C^*$-algebraic formalism of Topological T-duality.

Only for the purposes of this calculation, we use a different notation to conform with the notation of Ref. [6]. The notation will revert back to the previous notation as soon as this calculation is over. We let $\pi : X \to W$ be a smooth principal circle bundle over a smooth base manifold $W$. (See Ref. [6], Sec. (2.1), note that $n = 1$ in this paper, and $M = W$.) On $X$ we assume given a metric and a $B$-field. Let $\{U_\alpha\}$ be a cover of $W$ which we lift to a cover on $X$. We choose coordinates $(\theta, x^\alpha)$ locally on the cover $\{U_\alpha\}$ where $\theta$ is the coordinate along the circle direction. Greek subscripts on tensors now refer to the $x^\alpha$. (Hence, the connection form on $X$ is $\Theta = d\theta + A_\alpha dx^\alpha$.)
Let $\tilde{\pi} : X^\# \to W$ be the T-dual bundle. We write the metric and $B$-field on $X$ in Kaluza-Klein form as
\[
d s^2 = G_{00}(d\theta + A_\alpha dx^\alpha)^2 + g_{\alpha\beta} dx^\alpha dx^\beta \\
B = (d\theta + A_\alpha) \wedge B_\beta dx^\beta + B_{\alpha\beta} dx^\alpha \wedge dx^\beta.
\]
Here, $B_\alpha$ and $B_{\alpha\beta}$ are horizontal two-forms.

If we denote T-dual quantities by tilde superscripts, Buscher’s rules take the form
\[
\tilde{G}_{00} = 1/G_{00}, \\
\tilde{A}_\alpha = B_\alpha, \tilde{B}_\alpha = A_\alpha, \\
\tilde{g}_{\alpha\beta} = g_{\alpha\beta}, \tilde{B}_{\alpha\beta} = B_{\alpha\beta}, \\
\tilde{\Phi} = \Phi - \frac{1}{2} \ln(G_{00}).
\]
(For example, see Ref. [7], Ex. (6.12).)

Let $\Theta$ denote the connection form on the bundle $X$ and $\tilde{\Theta}$ the connection form on the T-dual bundle. From the above
\[
\Theta = d\theta + A_\alpha dx^\alpha \\
\tilde{\Theta} = d\tilde{\theta} + B_\alpha dx^\alpha
\]
where we have used Buscher’s Rules above. Now, suppose $B \to B + F$ on $X$ where $dF = 0$ and $F$ is integral. We may write $F = \Theta \wedge F_1 + F_2$ where $F_i$ are horizontal forms on $X$. The T-dual $B$-field and metric will now be
\[
d s^2 = \tilde{G}_{00}(d\tilde{\theta} + B_\alpha dx^\alpha + F_{1\alpha} dx^\alpha)^2 + g_{\alpha\beta} dx^\alpha dx^\beta \\
\tilde{B} = (d\tilde{\theta} + B_\alpha dx^\alpha + F_{1\alpha} dx^\alpha) \wedge A_\beta dx^\beta + \\
B_{\alpha\beta} dx^\alpha \wedge dx^\beta + F_{2\alpha\beta} dx^\alpha \wedge dx^\beta.
\]
It is easy to see that since $dF = 0$, $dF_1 = 0$ and, on the T-dual the form $dF_1$ is zero as well. However $F_2$ is not a closed form in general.

It can be seen directly from the definition that $F_1 = p(F)$, and so, since $F$ is integral, $F_1$ is integral as well.

Since $dF_1 = 0$, a coordinate transformation of $X^\#$ should be able to remove the term $(d\tilde{\theta} + F_{1\alpha} dx^\alpha)$ at least locally. Locally, such a transformation would only rotate the circle fiber of $X^\#$. The topological nontriviality of $F_1$ (the fact that it is integral) implies that this transformation would act in a topologically nontrivial manner on the total space of the bundle. Further, since the gerbe connection on $X$ and on the T-dual are both equivariant under the circle action, making such a
coordinate transformation would cause the T-dual $B$-field to undergo a gauge transformation.

This can be seen directly as follows: It is clear that this coordinate transformation would be a bundle automorphism (denoted $\phi$) of $\tilde{\pi} : X^\# \to W$ and so $\tilde{\pi} \circ \phi = \tilde{\pi}$. Recall that $\tilde{B}_\alpha - \tilde{B}_\beta = d\tilde{A}_{\alpha\beta}$ by definition. We have that

$$\phi^*\tilde{B}_\alpha - \phi^*\tilde{B}_\beta = \phi^*(d\tilde{A}_{\alpha\beta}) = d\phi^*\tilde{A}_{\alpha\beta}.$$ 

However, by equivariance $^7$

$$\tilde{A}_{\alpha\beta} = q^*\tilde{a}_{\alpha\beta} + \tilde{h}_{\alpha\beta}\Theta^\#$$

where $\tilde{a}_{\alpha\beta}, \tilde{h}_{\alpha\beta}$ are horizontal one and zero forms on $X^\#$. (See Ref. [6], Eq. (2.14, 2.21) and Cor. (2.1).) Remember that for a circle bundle, $m_{\alpha\beta} = 0$ so $\tilde{h}_{\alpha\beta}$ are actually the transition functions of the T-dual bundle $X^\# \to W$.

Therefore,

$$\phi^*\tilde{B}_\alpha - \phi^*\tilde{B}_\beta = \phi^*\tilde{A}_{\alpha\beta} = \phi^*\tilde{a}_{\alpha\beta} + \phi^*\tilde{h}_{\alpha\beta}\Theta^\#.$$ 

(Here $\phi^*\Theta^\# = \Theta^\#$ since $\phi$ is a bundle automorphism.) If the bundle transformation $\phi$ was topologically nontrivial, $\phi^*$ would act nontrivially on the transition functions $\tilde{h}_{\alpha\beta}$. As a result, $\phi^*\tilde{B}_\alpha$ transforms nontrivially on changing charts.

The fact that $F$ is topologically nontrivial implies that the T-dual $B$-field has undergone a large gauge transformation.

As we had said earlier, a large gauge transformation of a gerbe has a characteristic class in $H^2(X, \mathbb{Z})$. We may pick a large gauge transformation and ask for the characteristic class of the gerbe gauge transformation on the T-dual.

From now on, $X$ will refer only to a $CW$-complex which is a model for a spacetime background.

Thus, it is natural to ask the following: Given a class in $H^2(X, \mathbb{Z})$ does Topological T-duality naturally give a class in $H^2(X^\#, \mathbb{Z})$? We argued above that this phenomenon occurs for smooth gerbes. We will prove in this paper that it occurs in the $C^*$-algebraic formalism of Topological T-duality of Mathai and Rosenberg [1] and also in the classifying space formalism of Bunke and coworkers [2]. We will also attempt to obtain some information about the T-dual class. Ref. [1] suggests that given a spacetime with H-flux which also has a free $S^1$-action, one should attempt to construct an exterior equivalence class of $C^*$-dynamical systems. In the associated $C^*$-dynamical

\[ ^7 \text{See Ref. [6], Eq. (2.14).} \]
system, the continuous-trace algebra together with its associated $\mathbb{R}$-action should be viewed as a ‘topological approximation’ to the smooth gerbe on spacetime together with a circle action on it. In Ref. [8] such a $C^*$-dynamical system was naturally constructed from the data of a smooth gerbe with connection on $X$ in a large variety of examples.

Further, Ref. [1] demonstrated that the effect of the T-duality transformation on spacetime topology was given by the crossed product [9] construction in $C^*$-algebra theory in all the examples examined: The topological space underlying the T-dual spacetime was always the spectrum of the crossed-product algebra $\mathcal{A} \rtimes \mathbb{R}$. Based on this, Ref. [1] argued that it was natural to associate to the T-dual the $C^*$-dynamical system $(\mathcal{A} \rtimes \mathbb{R}, \hat{\mathbb{R}}, \alpha^\#)$.

We may use the definition of Ref. [1] and take as a model for a spacetime $X$ with $H$-flux, a continuous-trace algebra $\mathcal{A}$ with spectrum $X$ together with a lift $\alpha$ of the $S^1$-action on $X$ to a $\mathbb{R}$-action on $\mathcal{A}$ [10]. For simplicity, we restrict ourselves to $S^1$-actions which have no fixed points. Let $\mathcal{K}$ denote the compact operators on a fixed separable infinite dimensional Hilbert space. Since the circle action has no fixed points, the spacetime $X$ is a principal circle bundle $p : X \to W$. Let $\mathcal{A} = CT(X, \delta)$, $\delta \in H^3(X, \mathbb{Z})$. It can be shown that a lift $\alpha$ of the $S^1$-action on $X$ to a $\mathbb{R}$-action on $\mathcal{A}$ exists and is unique up to exterior equivalence. (See Ref. [12] Lemma (7.5) for a proof.)

We may model a large gauge transformation of the gerbe on spacetime by a locally unitary automorphism $\phi : \mathcal{A} \to \mathcal{A}$ of the associated continuous-trace algebra $\mathcal{A}$. It is well known that such automorphisms are determined up to exterior equivalence as automorphisms by the Phillips-Raeburn obstruction [13] in $H^2(X, \mathbb{Z})$.

There is a natural way to construct such a $\phi$ from the data above: We noted above that a large gauge transformation of a gerbe on $X$ was naturally associated to a line bundle $L$ on $X$. By a theorem of Phillips and Raeburn since $X = \hat{\mathcal{A}}$ and we have a locally trivial principal circle bundle (the circle bundle $P \to X$ associated to the line bundle $L \to X$), we can naturally obtain a locally unitary automorphism group $\alpha : \mathbb{Z} \to \mathcal{A}$ such that the principal bundle $p : (\mathcal{A} \rtimes \mathbb{Z})^\alpha \to X$ is isomorphic to $P$. (See Ref. [13], Thm. (3.8), and let $G = S^1$, and take $\phi$ to be the generator of the group $\hat{G} = \mathbb{Z}$.) We may take $\phi$ to be the generator of the $\mathbb{Z}$-action $\alpha$. From the proof of the above theorem, it is clear that the characteristic class of $L$ (which is the characteristic class of the large gauge transformation) will be the Phillips-Raeburn obstruction of the automorphism $\phi$. However, such an automorphism will not commute
with the $\mathbb{R}$-action on $\mathcal{A}$. In general, finding such an automorphism is not a trivial matter.

In the case when $X$ has an $S^1$-action which lifts to a $\mathbb{R}$-action on $\mathcal{A}$, the Phillips-Raeburn invariant does not take the commutation of the automorphism with the $\mathbb{R}$-action into account. We introduce a treatment based on the Equivariant Brauer Group in Sec. 2 below which fixes this problem.

See Ref. [16], Sec. (2.3) for an example of what would be needed to make such an automorphism commute with the lift of the translation action on $X$: If we take $G = \mathbb{R}$ there, the proof shows a way to obtain a $\mathbb{R}$-action on $(\mathcal{A} \rtimes \alpha \mathbb{Z})^\wedge$ making the latter into a $\mathbb{R}$-equivariant $S^1$-bundle over $\hat{A}$.

In this paper we adopt a different approach to this problem by an argument involving the Equivariant Brauer group.

Recall that given a $S^1$-action $\psi$ on $X$, there exists a lift of $\psi$ to a $\mathbb{R}$-action $\alpha$ on $\mathcal{A}$ which is unique up to exterior equivalence such that $\alpha$ induces the action $\psi$ on $X = \hat{A}$ (See Ref. [12], Lemma (7.5) for a proof.) If such a lift has been done, we may then ask if a given class in $H^2(X, \mathbb{Z})$ lifts to a unique automorphism of $\mathcal{A}$ which commutes with the $\mathbb{R}$-action on $\mathcal{A}$. It is possible to prove that a lift exists. However, the lift will in general be non-unique. The following lemma is a slight generalization of Thm. (3.1) of Ref. [17]:

**Lemma 1.1.** Let $X$ be a principal bundle $p : X \to W$. Let $\mathcal{A} = \text{CT}(X, \delta)$ for any $\delta \in H^3(X, \mathbb{Z})$. Let $\alpha_t$ be a lift of the $S^1$-action on $X$ to an $\mathbb{R}$-action on $\mathcal{A}$.

1. Let $[\lambda] \in H^2(X, \mathbb{Z})$. Then, there is a $\mathbb{R}$-action $\beta$ on $\mathcal{A}$ exterior equivalent to $\alpha$ and a spectrum-fixing $\mathbb{Z}$-action $\lambda$ on $\mathcal{A}$ which has Phillips-Raeburn obstruction $[\lambda]$ such that $\beta$ and $\lambda$ commute.

2. With the notation above, the action $\lambda$ induces a $\mathbb{Z}$-action $\bar{\lambda}$ on $\mathcal{A} \rtimes \mathbb{R}$. The induced action on the crossed product is locally unitary on the spectrum of the crossed product and is thus spectrum fixing.

**Proof.** The proof of Thm. (3.1) of Ref. [17] only uses the fact that $\mathcal{A}$ is a continuous-trace algebra. (See Ref. [17] and references therein.) The fact that $\mathcal{A}$ is $C_0(X, \mathcal{K})$ is not used anywhere in that theorem. Thus it applies even when the Dixmier-Douady invariant is nonzero. \hfill $\Box$

As a model for a gauge transformation of the gerbe associated to $\mathcal{A}$ we take a spectrum-fixing $C^*$-algebra automorphism $\phi$ on $\mathcal{A}$ which commutes with $\alpha$. We consider the $C^*$-dynamical system $(\mathcal{A}, \alpha \times \phi, \mathbb{R} \times \mathbb{R})$. 

The $C^*$-algebra automorphism $\phi$ then defines an automorphism $\psi$ of the crossed-product algebra. (See Ref. [17] and Lemma (1.1) above.) What is the Phillips-Raeburn obstruction of this automorphism? In this paper we obtain a solution to this problem using a classifying space argument. (See Thm. (6.3), Part (2).)

Thus, given any class $[\lambda] \in H^2(X; \mathbb{Z})$, there exists a (not necessarily unique) action $\alpha$ of $\mathbb{R}$ on $A$ inducing the given action of $S^1 = \mathbb{R}/\mathbb{Z}$ on $W$ and a commuting action $\lambda$ of $\mathbb{Z}$ on $A$ with Phillips-Raeburn obstruction $[\lambda]$. Also $\lambda$ passes to a locally unitary action on $E \simeq A \rtimes R$. The actions $\alpha$ and $\lambda$ are (individually) unique up to exterior equivalence, but unfortunately the pair $(\alpha, \lambda)$, as an action of $\mathbb{R} \times \mathbb{Z}$, is not necessarily unique as a $C^*$-dynamical system, so this construction is not entirely canonical. However, this non-uniqueness does not change the Phillips-Raeburn invariant of the T-dual automorphism. Thus the question asked above in the paragraph after Lemma (1.1) is well-defined.

In Section (2), we introduce a point of view based on the equivariant Brauer group, which measures precisely this lack of canonicity described above. We show that there is a natural map, $T_R$, which sends a $C^*$-dynamical system $(A, \mathbb{R} \times \mathbb{Z}, \alpha \times \phi)$ to a dual dynamical system $(A \rtimes R, \mathbb{R} \times \mathbb{Z}, \alpha^\# \times \phi^\#)$. To a $C^*$-dynamical system $(A, \mathbb{R} \times \mathbb{Z}, \alpha \times \phi)$ we associate a triple

$$(\hat{A}, \text{Phillips-Raeburn invariant of } \phi, \text{Dixmier-Douady invariant of } A).$$

We argue that there is a well-defined map of triples, $T_{3,2}$, which sends a triple

$(\text{principal bundle } p : X \to B, b \in H^2(X, \mathbb{Z}), \delta \in H^3(X, \mathbb{Z}))$

to another such triple which commutes with the map $T_R$ described above.

We then attempt to give an answer to the question raised in the paragraph after Lemma (1.1) above. In Sections (3-6) we extend the formalism of Topological T-duality to triples of the type described above. We prove the existence of the map $T_{3,2}$. Then we study some natural properties of the map $T_{3,2}$, and show that owing to the topological properties of this map, when the Dixmier-Douady invariant of $A$ is fixed, there is a partition of $H^2(X, \mathbb{Z})$ and $H^2(X^\#, \mathbb{Z})$ into cosets such that the T-dual of an automorphism with Phillips-Raeburn invariant

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8This does not contradict the statement in the previous paragraph, as there we were trying to lift a class in $H^2(X, \mathbb{Z})$ to such a dynamical system.

9We are using a different definition of dual dynamical system from Schneider's work, [18], see the last part of Sec. (2).
in a given coset is an automorphism with Phillips-Raeburn invariant in another coset. This is enough to answer the question posed in the paragraph after Lemma (1.1) for several spaces.

2. A MODEL FOR THE $H$-FLUX

Let $X$ be a manifold which is a model for a spacetime with $H$-flux $H$ as in Sec. (1) and suppose $X$ is the total space of a circle bundle with $X/S^1 \simeq W$. Let $V_\alpha$ be an open cover of $W$ and let $U_\alpha$ be the lift of the $V_\alpha$ to an open cover of $X$. Then, the $H$-flux is the curvature of a smooth $S^1$-equivariant gerbe on $X$ and $H = dB_\alpha$ locally, where $B_\alpha$ is the gerbe connection form in $U_\alpha$. As discussed in Sec. (1), changes in the $B$-field keeping $H$ fixed correspond to acting on the gerbe on spacetime with a gerbe gauge transformation. The gauge group of the gerbe is generated by the group of line bundles on $X$ with connection. (See Ref. [6], Sec. (2) for example.) That is, if $p : L \to X$ is a line bundle with curvature two-form $F$, then, under such a gauge transformation, on each patch $U_\alpha$ we have $B'_\alpha = B_\alpha + F_\alpha$ where $F_\alpha = F|_{U_\alpha}$. That is, $H = dB_\alpha$ locally, and after such a gauge transformation, $H = dB'_\alpha$ with $d(B_\alpha - B'_\alpha) = dF_\alpha = dF|_{U_\alpha} = 0$ that is, for such gauge transformations, $(B_\alpha - B'_\alpha)$ is closed. Note that $F$ is integral and hence so is $F_\alpha$. Therefore, so is $(B_\alpha - B'_\alpha)$. Since $X$ possesses a $S^1$-action (it is the total space of a principal circle bundle with $W = X/S^1$), we require $L$ to be $S^1$-equivariant.

As explained in the previous section, in the $C^*$-algebraic theory of Topological T-duality, the gerbe on spacetime is replaced by a ‘topological approximation’, a continuous-trace algebra $A$ with spectrum $X$ and Dixmier-Douady invariant equal to $H$. It is well known that spectrum-preserving automorphisms of $A$ define a cohomology class (the Phillips-Raeburn invariant) in $H^2(X, \mathbb{Z})$. (See Ref. [13], also, see Ref. [19], Lemma (4.4).)

As described in Sec. (1) above, a gerbe gauge transformation of the type described above naturally defines an automorphism $\phi$ of the associated continuous-trace algebra with the Phillips-Raeburn class of the automorphism equal to the characteristic class of the line bundle $L$ above: See Ref. [13] Thm. (3.8), take $E$ to be the circle bundle associated to $L$, $G = S^1$ and the automorphism $\phi$ to be the generator of the group $\hat{G} = \mathbb{Z}$. To study arbitrary changes in the $B$-field, that is, to study a general gerbe gauge transformation would probably require the introduction of a smooth structure and would be difficult to do in the $C^*$-algebraic picture of Topological T-duality. However, the theory of
integral changes of the $B$-field is still quite interesting mathematically, as the following sections show.

The above automorphism $\phi$ of $\mathcal{A}$ gives a $\mathbb{Z}$-action on $\mathcal{A}$. However, $\mathcal{A}$ possesses a $\mathbb{R}$-action already which is a lift of the $S^1$-action on $X$ to $\mathcal{A}$. Now, the gerbe automorphism is equivariant under the $S^1$-action on $X$, hence, $L \rightarrow X$ is an equivariant line bundle. Therefore, we require that $\phi$ commute with the $\mathbb{R}$-action on $\mathcal{A}$. It is natural to ask if we obtain a $\mathbb{Z} \times \mathbb{R}$-action on $\mathcal{A}$. The action obtained will, in general, depend on the exterior equivalence classes of the $\mathbb{R}$-action on $\mathcal{A}$ and the $\mathbb{Z}$-action induced by the automorphism $\phi$. The exterior equivalence class of $\phi$ is arbitrary, since we obtained $\phi$ by lifting the characteristic class of $L$ to an spectrum-preserving automorphism of $\mathcal{A}$. Hence, we need not obtain a unique $\mathbb{Z} \times \mathbb{R}$-action on $\mathcal{A}$. However, this will not affect anything as the following shows.

It will be useful to recall the notion of the Equivariant Brauer Group$^{10}$. Let $X$ be a second countable, locally compact, Hausdorff topological space and let $G$ be a second countable, locally compact, Hausdorff topological group acting on $X$. Following Ref. [19], let $\mathcal{B}r_G(X)$ denote the class of pairs $(\mathcal{A}, \alpha)$ consisting of continuous-trace algebras $\mathcal{A}$ on $X = \hat{\mathcal{A}}$ together with a lift $\alpha$ of the $G$-action on $X$ to $\mathcal{A}$. We say that the dynamical system $(\mathcal{A}, \alpha)$ is equivalent to $(\mathcal{B}, \beta)$ if there is a Morita equivalence bimodule $\mathcal{A}M_{\mathcal{B}}$ together with a strongly continuous $G$-action by linear transformations $\phi_s$ on $M$ such that for every $s \in G$, $\alpha_s(\langle x, y \rangle_\mathcal{A}) = \langle \phi_s(x), \phi_s(y) \rangle_\mathcal{A}$, and $\beta_s(\langle x, y \rangle_{\mathcal{B}}) = \langle \phi_s(x), \phi_s(y) \rangle_{\mathcal{B}}$. (Recall, the image of the inner product $\langle \cdot, \cdot \rangle_\mathcal{A} : M \times M \rightarrow \mathcal{A}$ is dense in $\mathcal{A}$ and similarly for $\langle \cdot, \cdot \rangle_{\mathcal{B}}$.) It is shown in Ref. [19] that this is an equivalence relation and that the quotient is a group. This group is termed the Equivariant Brauer Group $\mathcal{B}r_G(X)$ of $X$. The group operation is the $C_0(X)$-tensor product of continuous-trace algebras and group actions.

The Equivariant Brauer Group is actually a special case of the Brauer group of a groupoid applied to the transformation groupoid $G \times X$. It is shown in Ref. [20] that the Brauer group of a groupoid is isomorphic to a groupoid cohomology group. Hence the Equivariant Brauer Group is actually an abelian-group-valued functor contravariant in $G$ and in $X$.

Let $X$ be a space with a $S^1$-action $\psi$. Suppose $\mathcal{A}$ was a continuous-trace $C^*$-algebra with spectrum $X$ which possessed an $\mathbb{R}$-action $\alpha$ which induced the action $\psi$ on $X = \hat{\mathcal{A}}$. Suppose we had a lift of a class in

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$^{10}$We use Ref. [19] here.
$H^2(X, \mathbb{Z})$ to a spectrum-preserving automorphism of $\mathcal{A}$ which commuted with this $\mathbb{R}$-action on $X$. We would obtain a $\mathbb{R} \times \mathbb{Z}$-action on $\mathcal{A}$ and hence an element of $\text{Br}_{\mathbb{R} \times \mathbb{Z}}(X)$. By Lemma (1.1) above, there is an induced $\mathbb{Z}$-action on $\mathcal{A} \rtimes \alpha$. Now, the lift of a class in $H^2(X, \mathbb{Z})$ to a $\mathbb{R} \times \mathbb{Z}$-action on $\mathcal{A}$ need not be unique. Hence, the induced action on the crossed product is not unique. However, we really only care about the restriction of this action to the $\mathbb{Z}$-factor up to exterior equivalence, that is we wish to determine the Phillips-Raeburn invariant of the induced $\mathbb{Z}$-action on the crossed product $C^*$-algebra. We outline a way to approach this problem in this Section. In order to quantify this lack of uniqueness we now study $\text{Br}_{\mathbb{R} \times \mathbb{Z}}(X)$ in more detail.

In Ref. [19], the authors show (see Ref. [19], Thm. (5.1)) that there is a natural map $\xi : H^2_M(\mathbb{R} \times \mathbb{Z}, C(X, \mathbb{T})) \to \text{Br}_{\mathbb{R} \times \mathbb{Z}}(X)$, where $H^2_M$ is the Moore cohomology group mentioned in Ref. [19]. It is really only the class of the above $\mathbb{R} \times \mathbb{Z}$-action in $\text{Br}_{\mathbb{R} \times \mathbb{Z}}(X)/\text{im}(\xi)$ that we need to determine the Phillips-Raeburn invariant of the induced $\mathbb{Z}$-action on the crossed product $C^*$-algebra. We compute this group in Lemma (2.1) below.

Note that elements of $\text{Br}_{\mathbb{R} \times \mathbb{Z}}(X)$ would consist of a Morita equivalence bimodule between $\mathbb{R} \times \mathbb{Z}$-dynamical systems and an $\mathbb{R}$-equivariant automorphism of the module compatible with the $\mathbb{Z} \times \mathbb{R}$-action on the dynamical systems. Thus, by studying $\text{Br}_{\mathbb{R} \times \mathbb{Z}}(X)$ we are simply adding more structure to $\text{Br}_\mathbb{R}(X)$.

We conjecture that elements of this group are a good model for spacetimes with a possibly nonzero $B$-field. Given a dynamical system $(\mathcal{A}, \alpha \times \phi, \mathbb{R} \times \mathbb{Z})$ in $\text{Br}_{\mathbb{R} \times \mathbb{Z}}(X)$, we may forget the $\mathbb{Z}$-action to obtain a $\mathbb{R}$-dynamical system and we may forget the $\mathbb{R}$-action to obtain a $\mathbb{Z}$-dynamical system. Hence we obtain forgetful maps $F_1 : \text{Br}_{\mathbb{R} \times \mathbb{Z}}(X) \to \text{Br}_\mathbb{Z}(X)$ and $F : \text{Br}_{\mathbb{R} \times \mathbb{Z}}(X) \to \text{Br}_\mathbb{R}(X)$. We formalize the above discussion in the following

**Def 2.1.** Let $X$ be a locally compact, finite dimensional CW-complex homotopy equivalent to a finite CW-complex. Let $X$ also be a free $S^1$-space with $W = X/S^1$, so that we have a principal $S^1$-bundle $p : X \to W$. An element\textsuperscript{11} $y = [\mathcal{A}, \alpha \times \phi]$ of $\text{Br}_{\mathbb{R} \times \mathbb{Z}}(X)$ is now defined to be a model for a space with nonzero $H$-flux or zero $H$-flux and a nonzero integral $B$-field. The $H$-flux $H$ is\textsuperscript{12} $H = F(y) = [\mathcal{A}]$. If $H = 0$, the

\textsuperscript{11}Here $\alpha$ is a lift of the $S^1$-action on $X$ to a $\mathbb{R}$-action on $\mathcal{A}$, while $\phi$ is a commuting spectrum-fixing $\mathbb{Z}$-action on $\mathcal{A}$.

\textsuperscript{12}Recall that the forgetful map $\text{Br}_{\mathbb{R}}(X) \to \text{Br}(X) \simeq H^3(X, \mathbb{Z})$ is an isomorphism [19].
B-field is the unique class\textsuperscript{13} in $H^2(X, \mathbb{Z})$ which determines $F_1(y)$. This class is equal to the Phillips-Raeburn class of $\phi$.

By Ref. [19], there is a natural filtration of $\text{Br}_{\mathbb{R} \times \mathbb{Z}}(X)$ given by $0 < B_1 < \ker(F) < \text{Br}_{\mathbb{R} \times \mathbb{Z}}(X)$, where $B_1$ is the quotient $H^3_M(\mathbb{R} \times \mathbb{Z}, C(X, \mathbb{T}))/\text{im}(d_0)$. We argue below that each step in this filtration corresponds to one of the gauge fields in the problem.

We need the following characterization of the groups and maps in the above filtration. (Note that $B_1 = \ker(\eta)$ is determined in Lemma (2.1) below).

**Theorem 2.1.** Let $p : X \to W$ be as above.

1. We have a split short exact sequence
   $$0 \to \ker(F) \to \text{Br}_{\mathbb{R} \times \mathbb{Z}}(X) \xrightarrow{F} \text{Br}(X) \cong H^3(X, \mathbb{Z}) \to 0$$
   where $F : \text{Br}_{\mathbb{R} \times \mathbb{Z}}(X) \to \text{Br}(X)$ is the forgetful map.

2. We have a surjective map $\eta : \ker(F) \to H^2(X, \mathbb{Z})$.

3. We have a natural isomorphism $H^1_M(\mathbb{R}, C(X, \mathbb{T})) \cong C(W, \mathbb{R})$.

4. The group $H^2_M(\mathbb{R} \times \mathbb{Z}, C(X, \mathbb{T}))$ is connected and there is a natural surjective map $q : H^2_M(\mathbb{R} \times \mathbb{Z}, C(X, \mathbb{T})) \to C(W, \mathbb{T})_0$.

**Proof.**

1. We have a forgetful homomorphism $F_1 : \text{Br}_{\mathbb{R} \times \mathbb{Z}}(X) \to \text{Br}_{\mathbb{R}}(X)$, where $F_1 : [A, \alpha \times \phi] \to [A, \alpha]$. This map is obviously surjective, since we have a section $s : [A, \alpha] \to [A, \alpha \times \text{id}]$.

   Since $\text{Br}_{\mathbb{R}}(X) = H^3(X, \mathbb{Z})$ (by Sec. (6.1) of Ref. [19]), the kernel of $F_1$ consists of Morita equivalence classes of dynamical systems $[A, \alpha \times \phi]$ such that $\delta(A) = 0$. Thus, it actually consists of the group $\ker(F)$, where $F : \text{Br}_{\mathbb{R} \times \mathbb{Z}}(X) \to \text{Br}(X)$ is the map forgetting the group action.

2. By Thm. (5.1) of Ref. [19], we have a homomorphism $\eta : \ker(F) \to H^1_M(\mathbb{R} \times \mathbb{Z}, H^2(X, \mathbb{Z}))$. Now, by Thm. (4.2) of Ref. [1], we have that $H^1_M(\mathbb{R} \times \mathbb{Z}, M) \cong H^1(B(\mathbb{R} \times \mathbb{Z}), M)$ for any discrete $\mathbb{R} \times \mathbb{Z}$ module $M$. Also, $B(\mathbb{R} \times \mathbb{Z}) \cong S^1$, so $H^1_M(\mathbb{R} \times \mathbb{Z}, H^2(X, \mathbb{Z})) \cong H^2(X, \mathbb{Z})$.

   By Thm. (5.1) item (2) of [19], the image of $\eta$ has range which is all of $H^2(X, \mathbb{Z})$ since $H^3_M(\mathbb{R} \times \mathbb{Z}, C(X, S^1)) = 0$ by Thm. (3.1) of Ref. [17]. Hence we have a surjective homomorphism $\eta : \ker(F) \to H^2(X, \mathbb{Z})$.

3. We have the following short exact sequence of $\mathbb{R}$-modules
   $$0 \to H^0(X, \mathbb{Z}) \to C(X, \mathbb{R}) \to C(X, \mathbb{T})_0 \to 0$$

\textsuperscript{13}Note that $\text{Br}(X) \cong H^3(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z})$. 

From the associated long exact sequence for $H^*_M$, we find that $H^i_M(\mathbb{R}, H^0(X, \mathbb{Z})) \simeq 0$, $i = 1, 2$ by Cor. (4.3) of Ref. [1]; hence $H^1_M(\mathbb{R}, C(X, \mathbb{T})_0) \simeq H^1_M(\mathbb{R}, C(X, \mathbb{R}))$. By Thms. (4.5,4.6,4.7) of Ref. [1], $H^1_M(\mathbb{R}, C(X, \mathbb{R})) \simeq H^1_{\text{Lie}}(\mathbb{R}, C(X, \mathbb{R})_\infty)$. Here $C(X, \mathbb{R})_\infty$ are the $C^\infty$-vectors for the $\mathbb{R}$-action on $C(X, \mathbb{R})$ and so are the functions which are smooth along the $S^1$-orbits.

The complex computing the Lie algebra cohomology of $\mathbb{R}$ shows that this group is exactly the functions in $C(X, \mathbb{R})_\infty$ modulo derivatives of functions in $C(X, \mathbb{R})_\infty$ by the generator of the $\mathbb{R}$-action.

This group is isomorphic to $C(W, \mathbb{R})$ via the ‘averaging’ map $f \mapsto \int_{S^1} \phi_t \circ f \, dt$ where $\phi_t \circ f$ is $f$ shifted by the $S^1$-action on $X$.

(4) We use the spectral sequence calculation of Ref. [17], Thm. (3.1) to note that this group is isomorphic to $H^1_M(\mathbb{R}, H^1_M(\mathbb{Z}, C(X, \mathbb{T})))$. Since $\mathbb{Z}$ is discrete and acts trivially on $C(X, \mathbb{T})$, we have $H^1_M(\mathbb{Z}, C(X, \mathbb{T})) \simeq H^1(\mathbb{Z}, C(X, \mathbb{T})) \simeq C(X, \mathbb{T})$. Hence we need to calculate $H^1_M(\mathbb{R}, C(X, \mathbb{T}))$.

We have the following short exact sequence of $\mathbb{R}$-modules

$$0 \to C(X, \mathbb{T})_0 \to C(X, \mathbb{T}) \to H^1(X, \mathbb{Z}) \to 0$$

where $C(X, \mathbb{T})_0$ is the connected component of $C(X, \mathbb{T})$ containing the constant maps.

This gives us a long exact sequence

$$H^0_M(\mathbb{R}, C(X, \mathbb{T})) \to H^0_M(\mathbb{R}, H^1(X, \mathbb{Z})) \to H^1_M(\mathbb{R}, C(X, \mathbb{T})_0) \to H^1_M(\mathbb{R}, C(X, \mathbb{T})) \to H^1_M(\mathbb{R}, H^1(X, \mathbb{Z})) \to \ldots$$

(1)

Also, by Cor. (4.3) of Ref. [1], we have that $H^1_M(\mathbb{R}, H^1(X, \mathbb{Z})) \simeq 0$. Again, by Cor. (4.3) of Ref. [1], we find that $H^0_M(\mathbb{R}, H^1(X, \mathbb{Z})) \simeq H^1(X, \mathbb{Z})$ (since $B\mathbb{R}$ is contractible). Also $H^0_M(\mathbb{R}, C(X, \mathbb{T}))$ consists of the $\mathbb{R}$-invariant functions in $C(X, \mathbb{T})$ and hence is naturally isomorphic to $C(W, \mathbb{T})$.

Hence we find an exact sequence

$$C(W, \mathbb{T}) \to H^1(X, \mathbb{Z}) \to H^1_M(\mathbb{R}, C(X, \mathbb{T})_0) \to H^1_M(\mathbb{R}, C(X, \mathbb{T})) \to 0.$$

(2)

The map $C(W, \mathbb{T}) \to H^1(X, \mathbb{Z})$ is the composite $C(W, \mathbb{T}) \to H^1(W, \mathbb{Z}) \xrightarrow{p^*} H^1(X, \mathbb{Z})$. Its cokernel is $H^1(X, \mathbb{Z})/p^*(H^1(W, \mathbb{Z}))$ which is the image of $p_1 : H^1(X, \mathbb{Z}) \to H^0(W, \mathbb{Z})$ by the Gysin sequence. (The image can only be 0 or $\mathbb{Z}$ if $X$ is connected.) Using the isomorphism mentioned in the previous item of this
lemma, we see that we need to find the connecting map $\text{im}(p_i) \to H^1_M(\mathbb{R}, C(X, \mathbb{T})) \simeq C(W, \mathbb{R})$. This map sends any class in $H^0(W, \mathbb{Z})$ to a constant $\mathbb{Z}$-valued function on $W$.

The above exact sequence now becomes

\[ 0 \to \text{im}(p_i) \to C(W, \mathbb{R}) \to H^1_M(\mathbb{R}, C(X, \mathbb{T})) \to 0. \]

(3)

So $H^1_M(\mathbb{R}, C(X, \mathbb{T}))$ is isomorphic to the quotient of $C(W, \mathbb{R})$ by $\text{im}(p_i)$. It surjects onto the quotient of $C(W, \mathbb{R})$ by all of $H^0(W, \mathbb{Z})$ which is isomorphic to $C(W, \mathbb{T})_0$.

□

Lemma 2.1. We have a commutative diagram with exact rows and columns

\[
\begin{array}{ccccccccc}
0 & \downarrow & & & & & & & \\
& & H^2_M(\mathbb{R} \times \mathbb{Z}, C(X, \mathbb{T}))/\text{im}(d'_2) & \downarrow & & & & & \\
& & 0 \to \ker(F) \to \text{Br}_{\mathbb{R} \times \mathbb{Z}}(X) \overset{F}{\longrightarrow} \text{Br}_\mathbb{R}(X) \to 0 & & & & & \\
& & \downarrow{\eta} & & & & & \\
& & H^2(X, \mathbb{Z}) & & & & & \\
& & 0 & & & & & \\
\end{array}
\]

where $d'_2$ is defined in Ref. [19], Thm. (5.1). The group $\ker(\eta) = H^2_M(\mathbb{R} \times \mathbb{Z}, C(X, \mathbb{T}))/\text{im}(d'_2)$ is isomorphic to a quotient of $C(W, \mathbb{R})$.

Proof. The vertical and horizontal short exact sequences above are of the form $0 \to B_i \to B_{i+1} \to B_{i+1}/B_i \to 0$ where the $B_i$ are the groups in the filtration of $\text{Br}_{\mathbb{R} \times \mathbb{Z}}(W)$ described in the unnumbered Theorem on page (153) of Ref. [19].

The horizontal short exact sequence is the sequence of part (1) of Thm. (2.1). The map $\eta$ is the map of part (2) of Thm. (2.1). We are interested in the group $\ker(\eta)$. This is the collection of $C^*$-dynamical systems of the form $(C_0(X, \mathcal{K}), \alpha)$ with $\alpha$ inner. (Note that the Mackey obstruction of $\alpha$ may be nonzero.) By Part (2) of Thm. (2.1), above, the map $\eta$ in Thm. (5.1) of Ref. [19] is the map $\eta : \ker(F) \to H^2(X, \mathbb{Z})$ above. By part (3) of Thm. (5.1) in Ref. [19], we have natural homomorphisms $\xi : H^2_M(\mathbb{R} \times \mathbb{Z}, C(X, \mathbb{T})) \to \text{Br}_{\mathbb{R} \times \mathbb{Z}}(X)$, and $d'_2 : H^2(X, \mathbb{Z}) \to H^2_M(\mathbb{R} \times \mathbb{Z}, C(X, \mathbb{T}))$. Also, from that theorem it
follows that \( \ker(\eta) \) is identical to \( \text{im}(\xi) \subseteq \text{Br}_{\mathbb{R} \times \mathbb{Z}}(X) \) and\(^1\) we have the identifications \( \text{im}(\xi) \simeq H^2(\mathbb{R} \times \mathbb{Z}, C(T, \mathbb{T})) / \ker(\xi), \ker(\xi) = \text{im}(d''_2) \). Hence, it follows that \( \ker(\eta) = H^2_M(\mathbb{R} \times \mathbb{Z}, C(X, \mathbb{T})) / \text{im}(d''_2) \).

We need to calculate this group. By the previous theorem (in particular, Eq. (2) above), \( H^2_M(\mathbb{R} \times \mathbb{Z}, C(X, \mathbb{T})) \) is isomorphic to \( H^1(\mathbb{R}, C(X, \mathbb{T})) \) which is isomorphic to \( C(W, \mathbb{R}) / \text{im}(p_1) \). Therefore, \( B_1 \) should be isomorphic to a quotient of \( C(W, \mathbb{R}) \).

The maps \( F, \eta, q \) were defined in the previous lemma. \( \square \)

We now make the following dictionary
\[ \begin{align*}
\bullet & \ y \in \text{Br}_{\mathbb{R} \times \mathbb{Z}}(X), \ y \not\in \ker(F) \iff \text{Space } X \text{ with } H \neq 0. \text{ Here, } H = F(y). \\
\bullet & \ \text{Element } y \in \ker(F) \subseteq \text{Br}_{\mathbb{R} \times \mathbb{Z}}, \ y \not\in H^2_M(\mathbb{R} \times \mathbb{Z}, C(X, \mathbb{T})) \iff \text{Space } W \text{ with } H = 0, B \neq 0. \text{ Here, } B = \eta(y). \\
\bullet & \ y \in H^2_M(\mathbb{R} \times \mathbb{Z}, C(X, \mathbb{T})) / \text{im}(d''_2) \subseteq \text{Br}_{\mathbb{R} \times \mathbb{Z}}(X) \iff \text{Space } X \text{ with } H = 0, B = 0, A \neq 0. \text{ We obtain an element of } C(W, \mathbb{T}), \text{ however, this may instead be viewed as a collection of elements in } C(X, \mathbb{R}) \text{ which are constant on the } S^1\text{-orbits of } X. \text{ Any two of these elements differ by an element of } C(X, \mathbb{Z}) \simeq H^0(X, \mathbb{Z}). \\
\text{This should be compared with the allowed gauge transformations of the } B\text{-field in string theory.} \\
\bullet & \ C_0(X, \mathcal{K}) \text{ with the lift of the } \mathbb{R}\text{-action and the trivial } \mathbb{Z}\text{-action} \\
\text{\iff Space } X \text{ with } H = 0, B = 0, A = 0. \\
\end{align*} \]

Note that the last item above is exactly the \( C^*\)-dynamical system assigned to a space \( X \) with zero \( H\)-flux in Ref. [1].

Suppose we had a string background which was a principal bundle \( X \) with \( X/S^1 \simeq W \) together with a continuous-trace algebra \( \mathcal{A} \) with spectrum \( X \) satisfying the conditions of Def. (2.1) above. By Def. (2.1) we would obtain an element \( y = [\mathcal{A}, \alpha \times \phi, \mathbb{R} \times \mathbb{Z}] \) in \( \text{Br}_{\mathbb{R} \times \mathbb{Z}}(X) \) (see Ref. [19]). We remarked earlier that \( \mathcal{A} \) should be viewed as a ‘topological approximation’ to the gerbe on spacetime. If \( y \in \text{Br}_{\mathbb{R} \times \mathbb{Z}}(X) \), we have
\[
[H] = F(y). \text{ Then } F^{-1}([H]) \text{ is the coset } y \circ \ker(F) \text{ and we note that we may change } y \text{ by an element } x \text{ of } \ker(F) \text{ without changing the } H\text{-flux.} \\
\text{That is, by definition of the Brauer group (see Ref. [19] Prop. (3.3)), } \\
\text{corresponding to the element } yx \in \text{Br}_{\mathbb{R} \times \mathbb{Z}}(X), \text{ we obtain a new } C^*\text{-dynamical system} \\
[\mathcal{A}, \alpha \times \psi] \circ [\mathcal{A}, \alpha \times \phi] \simeq [\mathcal{A} \otimes C_0(X), (\alpha \times \psi) \circ (\alpha \times \phi)] \\
\simeq [\mathcal{A}, \alpha \times (\phi \circ \psi)]. \text{ Here, the } \mathbb{R}\text{-action obtained from the composite dynamical system} \\
[\mathcal{A}, (\alpha \times (\phi \circ \psi))] \text{ is still the lift of the circle action on } X, \text{ but the} \]
\(^1\)Note that the translation action on \( X \) acts trivially on \( H^2(X, \mathbb{Z}) \) and that the \( \mathbb{R} \times \mathbb{Z}\)-action covers this translation action.
Phillips-Raeburn invariant of the $\mathbb{Z}$-action has changed, it has shifted by $\eta(x)$.

Given the above, if we act on the gerbe on spacetime by a gerbe gauge transformation $x$, the Phillips-Raeburn invariant of the $\phi$ factor of $y = [\mathcal{A}, \alpha \times \phi]$ should be shifted by $\eta(x)$.

We may view this action as affecting the $H$-flux on spacetime by shifting the $B$-field, that is, if $H = dB$ locally, we act by a gerbe automorphism to obtain $H = dB'$ locally. As argued at the beginning of this section, we are restricting ourselves to large gauge transformations of the $B$-field so $(B - B')$ is actually a global quantity, in fact, it is a closed, integral two-form on $X$.

Hence, we only allow shifts of $B$ by elements of $H^2(X, \mathbb{Z}) \hookrightarrow H^2(X, \mathbb{R})$, where $i$ is the canonical inclusion. Also, physically, $i \circ \eta(x)$ can only equal $[(B' - B)]$.

Similarly, if $y \in \ker(F)$, then $H = 0, B = \eta(y)$ and changing $y$ by an element $z$ of $B_1 = H^2_\mathbb{Z}(\mathbb{R} \times \mathbb{Z}, C(X, \mathbb{T}))$ doesn’t change $B$ but might correspond to making a change in the $A$-field, the gauge field of the $B$-field.

We have argued here that it is natural to associate the group $\text{Br}_{\mathbb{R} \times \mathbb{Z}}(X)$ to a spacetime $X$ with $H$-flux since it captures properties of the $H$-flux on that spacetime. The assignment $X \to \text{Br}_{\mathbb{R} \times \mathbb{Z}}(X)$ is a functor $\mathcal{R}$ on a certain category defined below (see paragraph before Lemma 2.3 below).

We now construct the functor $\mathcal{R}$ above. We also define several other functors of interest which we will need later. In the rest of this section we will study these functors in more detail. For circle bundles, we obtain the functor of Bunke et al (see Ref. [2]) from the $C^*$-algebraic theory by using a construction based on the Equivariant Brauer Group (before Lemma 2.5 below). We also explain how these constructions help answer the question raised in the paragraph after Lemma (1.1) above.

As was discussed before Def. (2.1) above, the Equivariant Brauer Group of $X$ is a special case of the Brauer Group of a groupoid applied to the transformation groupoid $G \times X$. It is shown in Ref. [20] that the Brauer Group of a groupoid is isomorphic to a groupoid cohomology group. Hence, the Equivariant Brauer Group is an abelian-group-valued functor contravariant in $G$ and $X$. The following lemma is then obvious:

**Lemma 2.2.** Let $Y \to Z$ be a principal bundle of $CW$-complexes. Let $f : W \to Z$ be a continuous map. Let $X \to W$ be the pullback bundle and $\phi : X \to Y$ be the map induced by pullback. Consider $\text{Br}_{\mathbb{R} \times \mathbb{Z}}(X)$
as described above. Then, pullback of $C^*$-algebras induces the following maps

1. $\lambda : Br_{R}(Y) \to Br_{R}(X)$.
2. $\kappa : Br_{R \times Z}(Y) \to Br_{R \times Z}(X)$.

Consider the category of principal $S^1$-bundles over $CW$-complexes with morphisms pullback maps. Let $X \to W$ be a principal $S^1$-bundle over a $CW$-complex $W$. Let $(\mathcal{A}, \alpha, R \times Z)$ be a $R \times Z$ $C^*$-dynamical system with spectrum $X$ with $\alpha$ a lift of the $S^1$-action on $X$ to a $R \times Z$ action on $\mathcal{A}$ as described in the previous paragraph. If $Y \to Z$ is another such principal $S^1$-bundle, then $(\mathcal{A}, \alpha, R \times Z)$ pulls back to a $C^*$-dynamical system on $Y$. This induces a pullback morphism $\phi : Br_{R \times Z}(X) \to Br_{R \times Z}(Y)$. Under pullback, $Br_{R \times Z}$ gives an abelian-group valued functor on the category of principal $S^1$-bundles of $CW$-complexes with arrows pullback squares of bundles: Given a pullback square

$$
\begin{array}{ccc}
Y & \xrightarrow{j} & X \\
\downarrow & & \downarrow \\
Z & \xrightarrow{f} & W
\end{array}
$$

the functor assigns $Br_{R \times Z}(X)$ to the any object $P \to W$ and the pullback map $\phi$ to the above pullback square.

Lemma 2.3. Let $X, Y, Z, W$ be as described in the previous paragraph. Let $\phi$ be as above. The map $\phi$ preserves the groups in the natural filtration on $Br_{R \times Z}(X)$ in Ref. [19].

Proof. The natural filtration on $Br_{R \times Z}(X)$ is

$$H^2(R \times Z, C(X, T))/\text{im}(d_2') \subseteq \text{ker}(F) \subseteq Br_{R \times Z}(X)$$

(see Lemma (2.1) above). Under pullback, as shown above, $Br_{R \times Z}$ maps naturally. Under pullback, the Dixmier-Douady invariant also pulls back, so, if an element in $\text{ker}(F) \subseteq Br_{R \times Z}(Y)$ corresponded to a dynamical system of the form $(C_0(Y, \mathcal{K}), \alpha \times \beta, R \times Z)$ its pullback would be of the form $(C_0(X, \mathcal{K}), \alpha' \times \beta', R \times Z)$. Also, a locally unitary map pulls back to another locally unitary map. Thus, under pullback an element of $\text{ker}(F)$ maps to another element of $\text{ker}(F)$. By Lemma (2.1) above, $H^2(R \times Z, C(X, T))/\text{im}(d_2')$ is a quotient of $C(W, R)$ and pullback of continuous-trace algebras induces a natural map to $H^2(R \times Z, C(Y, T))/\text{im}(d_2')$ (which is a quotient of $C(Z, R)$). □
Let \( X \to W \) be a principal circle bundle as described just before Lemma (2.3) above. For any \( i \), let \( G_i \) be any of the following contravariant abelian-group-valued functors\(^{15}\) which assign to \( X \) the value \( \text{Br}_R(X), \text{Br}_{\mathbb{R} \times \mathbb{Z}}(X) \), any of the groups in the filtration on \( \text{Br}_{\mathbb{R} \times \mathbb{Z}}(X) \), any of the cohomology groups \( H^q(X, \mathbb{Z}), H^q_M(\mathbb{R} \times \mathbb{Z}, C(X, \mathbb{T})) \) or \( H^p(X, \mathbb{Z}) \oplus H^q(X, \mathbb{Z}) \). Let \( \eta : G_1 \to G_2 \) be a natural transformation between the \( G_i \).

**Theorem 2.2.** Let \( W \) be a CW-complex. Choose a principal \( S^1 \)-bundle \( E_p \) on \( W \) for every \( [p] \in H^2(W, \mathbb{Z}) \). Let \( G, G_i \) be as described before this theorem and let \( \eta : G_1 \to G_2 \) be a natural transformation as described above. Then we have the following

1. The group \( \text{Aut}(E_p) \) acts on \( G(E_p) \) for every \( E_p \).
2. For any of the choices of \( G \) listed before this theorem, let \( Y_p \) denote the set \( G(E_p)/\text{Aut}(E_p) \). Let \( P_G(W) = \prod_{[p] \in H^2(W, \mathbb{Z})} Y_p \). Then, the set \( P_G(W) \) doesn’t depend on the choice of \( E_p \).
3. Let \( \text{Set} \) be the category whose objects are sets and whose morphisms are functions. \( P_G(W) \) is a contravariant \( \text{Set} \)-valued functor\(^{16}\) on CW-complexes for every choice of \( G \).
4. The natural transformation \( \eta : G_1 \to G_2 \) yields a natural transformation \( P_\eta : P_{G_1} \to P_{G_2} \).
5. Let \( \eta : G_1 \to G_2 \) be the natural transformation mentioned above. If the map \( \eta(E_p) : G_1(E_p) \to G_2(E_p) \) is always surjective for every principal bundle \( E_p \to W \) for every CW-complex \( W \), then the induced map \( P_\eta(W) : P_{G_1}(W) \to P_{G_2}(W) \) is surjective for every CW-complex \( W \). Similarly if \( \eta(E_p) \) is injective or bijective then so is \( P_\eta(W) \) for every CW-complex \( W \).

**Proof.**

1. For \( \text{Br}_R \) and \( \text{Br}_{\mathbb{R} \times \mathbb{Z}} \) this follows from Lemma (2.2) above and functoriality. For the groups in the filtration on \( \text{Br}_{\mathbb{R} \times \mathbb{Z}} \) this follows from Lemma (2.3) above. For \( H^*, H^*_M, H^* \oplus H^* \) this is obvious.
2. Suppose we pick another bundle \( E_p' \) for each \( p \in H^2(W, \mathbb{Z}) \). Since \( E_p, E_p' \) have the same characteristic class, they are all isomorphic. Let \( \phi_p : E_p' \to E_p \) be choices of isomorphisms one for each \( p \in H^2(W, \mathbb{Z}) \). Each \( \phi_p \) induces isomorphisms \( \text{Aut}(E_p) \to \text{Aut}(E_p') \). Let \( \lambda \) be any element of \( \text{Aut}(E_p) \) and

\(^{15}\)By naturality, (See Ref. [19]), these are the groups that would appear in the discussion of \( \mathbb{R} \times \mathbb{Z} \)-actions on \( CT(X, \delta) \) which on \( X \) cover the \( S^1 \)-action on \( X \).

\(^{16}\)See also the definitions of \( \text{Par}(B) \) after Rem. (2.2) and \( \text{Dyn}(E, B) \) after Prop. (2.8) in Ref. [18]
\( \lambda' \) the corresponding element of \( \text{Aut}(E'_p) \). Then, we have that the following diagram commutes

\[
\begin{array}{ccc}
E_p & \xrightarrow{\phi} & E'_p \\
\downarrow{\lambda} & & \downarrow{\lambda'} \\
E_p & \xrightarrow{\phi} & E'_p.
\end{array}
\]

Applying \( G \) to each element of the above diagram shows that the sets \( G(E_p)/\text{Aut}(E_p) \) are in bijection. Changing the isomorphisms \( \phi_p \) doesn’t affect the result.

(3) It is clear that \( P_G(W) \) is always a set. All the \( G \) described above are contravariant abelian group valued functors. If \( f : V \rightarrow W \) is a continuous map, pulling back \( E_p \rightarrow W \) along \( f \) induces a bundle \( f^*(E_p) \rightarrow V \). Hence, there is a well-defined natural map \( P_G(f) : P_G(W) \rightarrow P_G(V) \) which is induced by pullback. This map doesn’t depend on the choices of the \( E_p \) as in the previous part.

If we consider \( id : W \rightarrow W \), the pullback of \( E_p \) is naturally isomorphic to \( E_p \) and so the induced natural map \( P_G(id) : P_G(W) \rightarrow P_G(W) \) is the identity. This is independent of the choice of \( E_p \) for the same reason as in the previous part.

It is clear that if \( f : U \rightarrow V \) and \( g : V \rightarrow W \), we have that \( f^*(g^*(E_p)) \simeq (f \circ g)^*(E_p) \). Hence, the induced map \( P_G(g \circ f) \) satisfies \( P_G(g \circ f) = P_G(f) \circ P_G(g) \) independently of the choice of \( E_p \).

(4) Suppose we had a natural transformation \( N : G_1 \rightarrow G_2 \). We have, for every continuous map \( f : V \rightarrow W \), and every commutative square

\[
\begin{array}{ccc}
E'_p & \xrightarrow{F} & E_p \\
\downarrow & & \downarrow \\
V & \xrightarrow{f} & W
\end{array}
\]

maps \( N(E_p), N(E'_p) \) such that the following diagram commutes

\[
\begin{array}{ccc}
G_1(E_p) & \xrightarrow{N(E_p)} & G_2(E_p) \\
G_1(F) \downarrow & & \downarrow G_2(F) \\
G_1(E'_p) & \xrightarrow{N(E'_p)} & G_2(E'_p).
\end{array}
\]
Let \( Y_p^1 = G_1(E_p)/\text{Aut}(E_p) \) and \( Y_p^2 = G_2(E_p)/\text{Aut}(E_p) \). Then, we have that \( N(E_p) \) induces maps \( Y_p^1 \to Y_p^2 \). This, in turn, induces maps \( P_N(W) : P_{G_1}(W) \to P_{G_2}(W) \).

Similarly, from the commutative diagram in Part (2) above, elements of \( \text{Aut}(E_p) \) give rise to elements of \( \text{Aut}(E_p') \) by composition. Hence, the above diagram remains commutative when we quotient each \( E_p \) by \( \text{Aut}(E_p) \).

Hence, for every \( W \) we have that the following diagram commutes for every \( f : V \to W \):

\[
\begin{array}{c}
P_{G_1}(W) \\
p_{G_1}(f)
\end{array} \xrightarrow{P_N(W)} \begin{array}{c}P_{G_2}(W) \\
p_{G_2}(f)
\end{array} \xrightarrow{P_{G_1}(V)} \begin{array}{c}P_{G_1}(V) \\
p_{G_1}(f)
\end{array} \xrightarrow{P_N(W)} \begin{array}{c}P_{G_2}(V) \\
p_{G_2}(f)
\end{array}
\]

Hence, we have a natural transformation \( P_N : P_{G_1} \to P_{G_2} \).

(5) Suppose \( N(E_p) : G_1(E_p) \to G_2(E_p) \) was always surjective for every principal bundle \( E_p \to W \) for every \( CW \)-complex \( W \). Then, the induced map \( Y_p^1 \to Y_p^2 \) is always surjective (since quotienting both sides of \( N(E_p) : G_1(E_p) \to G_2(E_p) \) by \( \text{Aut}(E_p) \) will give a surjective map). Hence, the induced map \( P_N(W) : P_{G_1}(W) \to P_{G_2}(W) \) would be surjective as well.

The proof in the case \( N(E_p) \) is injective or bijective is obvious.

Using the previous Theorem, let \( P, P_2, P_3 \) be the functors associated to \( \text{Br}, H^2, H^3 \) respectively. Similarly, let \( P_{3,2}, \mathcal{R} \) be the functors associated to \( H^2 \oplus H^3, \text{Br}_{\mathbb{R} \times \mathbb{Z}} \) respectively.

**Lemma 2.4.** The functor \( P_3 \) above is the functor of Ref. [2].

**Proof.** This is obvious: Both functors take the same value on \( CW \)-complexes and, for any \( f : V \to X \), both act on objects via pullback of pairs as defined in Ref. [2].

We will study the functors \( P_2, P_3, P_{3,2} \) in more detail in Sec. (3).

**Corollary 2.1.** Suppose \( W, E_p \) were as in Thm. (2.2).

1. The natural isomorphism \( F : \text{Br}_{\mathbb{R}}(E_p) \to H^3(E_p, \mathbb{Z}) \) induces a natural transformation \( P \to P_3 \).
2. The forgetful map (see Def. (2.1) above) \( F : \text{Br}_{\mathbb{R} \times \mathbb{Z}}(E_p) \to \text{Br}_{\mathbb{R}}(E_p) \) induces a natural transformation \( F : \mathcal{R} \to P_3 \).
3. There is a natural surjective map \( \text{Br}_{\mathbb{R} \times \mathbb{Z}}(E_p) \to H^2(E_p, \mathbb{Z}) \oplus H^3(E_p, \mathbb{Z}) \). This induces a natural transformation \( \pi : \mathcal{R} \to P_{3,2} \).
For every CW-complex $W$, the induced map $\mathcal{R}(W) \to P_{3,2}(W)$ is surjective.

Proof.

(1) By Ref. [1], Sec. (4.1), the forgetful map induces a natural isomorphism $F : \text{Br}(E_p) \to H^3(E_p, \mathbb{Z})$. By the previous Theorem, this induces a natural transformation $P \to P_3$.

(2) This follows from Thm. (2.2) items (3,4) applied to the forgetful map (see Def. (2.1) above) $F : \text{Br}_R \times \mathbb{Z}(E_p) \to \text{Br}(E_p)$.

(3) By Thm. (2.1) item (1), we have a split short exact sequence

$$0 \to \ker(F) \to \text{Br}_R \times \mathbb{Z}(E_p) \xrightarrow{F} \text{Br}(E_p) \to 0.$$ 

Thus, we have a natural isomorphism

$$\text{Br}_R \times \mathbb{Z}(E_p) \simeq H^3(E_p, \mathbb{Z}) \oplus \ker(F)$$

where we have used the fact that $\text{Br}(E_p) \simeq H^3(E_p, \mathbb{Z})$. By Thm. (2.1) item (2), there is a natural surjective map $\eta : \ker(F) \to H^2(E_p, \mathbb{Z})$.

Hence, we have a natural map $\text{Br}_R \times \mathbb{Z}(E_p) \to H^3(E_p, \mathbb{Z}) \oplus H^2(E_p, \mathbb{Z})$. Since $F, \eta$ are both surjective, the above map is surjective. By Thm. (2.2) items (3,4) applied to the above map, this induces a natural transformation $\pi : \mathcal{R} \to P_{3,2}$. By the same Theorem, item (4), for every CW-complex $W$, the functor $\pi$ induces a surjective map $\mathcal{R}(W) \to P_{3,2}(W)$.

□

For a given string background $p : E_p \to W$, $P_3(W)$ encodes the data important for T-duality, namely the $H$-flux on $E_p$ and the characteristic class of $E_p$.

As we noted at the beginning of this section, for a given string background if we fix a closed three-form $H$ such that $[H]$ is the $H$-flux and a two-form field $B$ such that $H = dB$ then a large gauge transformation of the $H$-flux will change $B$ to $B'$ such that $H = dB'$. By definition, $d(B - B') = 0$, i.e., $(B - B')$ is a cohomology class in $H^2_{\text{de Rham}}(X)$. In the discussion at the beginning of this section, it has been pointed out that for a large gauge transformation, this class is actually integral.

To the data encoded in $P_3(W)$, we may add, in addition, the characteristic class of a large gauge transformation of the $H$-flux. For any CW-complex $W$, this is encoded in $P_{3,2}(W)$: Here to $W$ we associate the triple $([p], b, H)$. The class $b$ parametrizes large gauge transformations of the $H$-flux and should not be identified with the physical Kalb-Ramond field unless $H = 0$. 

\[\square\]
With this caveat in mind we will refer to the class \( b \) as ‘the \( B \)-field’ or ‘the \( B \)-class’ in what follows. It should be clear, however, that this should not, in fact, be interpreted as the physical Kalb-Ramond field, but should be viewed as a shift in the Kalb-Ramond field \( B \) that is, as the characteristic class of a large gauge transformation of the \( H \)-flux.

We now point out some relations among the functors \( P_2, P_3, P_{3,2} \) described above. We point out the action of the transformations induced by Topological T-duality on these functors. We show how these indicate a way to answer the question posed in Sec. (1) in the paragraph after Lemma (1.1) (see Eq. (8) below). We close with a comparison to the work of Schneider (see Ref. [18]).

Let \( W \) be as above. Consider the functor \( P_2 \) above. As pointed out above after Lemma (2.5), for \( W, E_p \) as above, by considering \( \text{Br}_{\mathbb{R} \times \mathbb{Z}}(E_p)/B_1 \), we are led to consider ‘triples’ of the form (principal bundle \( E_p \), Class in \( H^2(E_p, \mathbb{Z}) \), Class in \( H^3(E_p, \mathbb{Z}) \)). The isomorphism classes of such triples over \( W \) are \( P_{3,2}(W) \) where \( P_{3,2} \) is the functor defined before Cor. (2.1) above. Also, by the discussion in Cor. (2.1) item (3) above, there is a natural transformation \( \pi : R \to P_{3,2} \).

In Sec. (6) we prove that there is a well-defined map \( T_{3,2} : R_{3,2} \to R_{3,2} \) inducing a natural transformation \( T_{3,2}(W) : P_{3,2}(W) \to P_{3,2}(W) \) by an argument involving the classifying space of triples \( R_{3,2} \). We also show that this map induces the following commutative diagram.

\[
\begin{array}{ccc}
T_{3,2}(W) & \xrightarrow{T_{3,2}(W)} & T_{3,2}(W) \\
\pi(W) \downarrow & & \pi(W) \downarrow \\
T_3(W) & \xrightarrow{T(W)} & T_3(W)
\end{array}
\]

i.e., \( \pi \circ T_{3,2} = T \circ \pi \) as natural transformations.

Given a \( C^* \)-dynamical system \((\mathcal{A}, \mathbb{R} \times \mathbb{Z}, \alpha)\) corresponding to a class \( a \in \text{Br}_{\mathbb{R} \times \mathbb{Z}}(E_p) \), adding an element of \( B_1 \) to \( a \) will not change the Phillips-Raeburn invariant of \( \alpha|_{\mathbb{Z}} \) or the \( H \)-flux \( F(a) \). We showed in Ref. [17] that under T-duality the Phillips-Raeburn invariant of the \( \mathbb{Z} \)-action on \( \mathcal{A} \times R \) associated to \( T(a) \) only depends on the \( H \)-flux and the Phillips-Raeburn invariant of the dynamical system associated to \( a \). It doesn’t depend on the lift of these data to a \( \mathbb{R} \times \mathbb{Z} \)-action on \( \mathcal{A} \). Hence, the following diagram commutes

\[
\begin{array}{ccc}
\mathcal{R}(W) & \xrightarrow{T_R(W)} & \mathcal{R}(W) \\
\pi(W) \downarrow & & \pi(W) \downarrow \\
P_{3,2}(W) & \xrightarrow{T_{3,2}(W)} & P_{3,2}(W)
\end{array}
\]
Also, by Cor. (2.1), item(3), the map \( \pi(W) : \mathcal{R}(W) \to P_{3,2}(W) \) induced by the functor \( \pi \) in Cor. (2.1) item (3), is always surjective. Thus we may infer properties of \( T_R \) from those of \( T_{3,2} \) but it should be clear that they are not the same. In this paper we mainly study \( P_{3,2} \) and \( T_{3,2} \). This is sufficient to answer the question we raised in Section (1) in the paragraph after Lemma (1.1).

Let \( W \) be as above. Consider the functor \( P_{3,2} \) above. In Ref. [2], the authors show that there is a classifying space \( R_{3,2} \) for the functor \( P_{3,2} \) and \( P_3(W) = [W, R_3] \). Further, Ref. [2] shows that that T-duality is a natural transformation from \( P_3 \) to itself giving a map we denote as \( T_3(W) : P_3(W) \to P_3(W) \).

By Cor. (2.1) above, there is a natural transformation \( P \to P_3 \) induced by the isomorphism \( Br_R(E_p) \to H^3(E_p, \mathbb{Z}) \) which by Part (5) of Thm. (2.2) induces a natural isomorphism \( P(W) \to P_3(W) \) for every \( CW \)-complex \( W \).

Hence, for every \( CW \)-complex \( W \), we have an isomorphism of sets

\[
\prod_{[p] \in H^2(W,\mathbb{Z})} H^3(E_p,\mathbb{Z})/Aut(E_p) \cong \prod_{[p] \in H^2(W,\mathbb{Z})} Br_R(E_p)/Aut(E_p).
\]

Therefore, an isomorphism class of a pair over \( W \) (in the sense of Ref. [2]) determines and is determined by an element of \( P \). In particular, for any \( CW \) complex \( W \), we have that \( [W, R_3] = P(W) \) as well. Thus, the Topological T-duality functor of Bunke et al may be easily derived from the \( C^* \)-algebraic theory of Ref. [1] for circle bundles (see also Ref. [18] Prop. (2.8)).

Let \( R \) be as defined above. By the above, this is a set-valued functor on the category of unbased \( CW \)-complexes. Then, we have the following theorem

**Lemma 2.5.** There is a well-defined map \( T_R : \mathcal{R}(W) \to \mathcal{R}(W) \) induced by the crossed product.

**Proof.** We need the following well-known fact. Let \( \mathcal{A}, \mathcal{B} \) be \( C^* \)-algebras with \( G \)-action \( \alpha, \beta \) respectively. Let \( C_c(G, \mathcal{A}) \) the \( \alpha \)-twisted convolution algebra of \( \mathcal{A} \)-valued functions on \( G \) which are of compact support on \( G \). Similarly, let \( C_c(G, \mathcal{B}) \) be the \( \beta \)-twisted convolution algebra of \( \mathcal{B} \)-valued functions on \( G \) which are of compact support on \( G \). Give \( C_c(G, \mathcal{A}) \) and \( C_c(G, \mathcal{B}) \) the inductive limit topology\(^{17}\).

**Theorem 2.3.** Suppose that \( (\mathcal{A}, G, \alpha) \) and \( (\mathcal{B}, G, \beta) \) are dynamical systems and \( \phi : \mathcal{A} \to \mathcal{B} \) is an equivariant homomorphism. Then, there is a homomorphism \( \phi \times \text{id} : \mathcal{A} \times G \to \mathcal{B} \times G \) mapping \( C_c(G, \mathcal{A}) \) into \( C_c(G, \mathcal{B}) \) such that \( \phi \times \text{id}(f)(s) = \phi(f(s)) \).

\(^{17}\)See Ref. [9], Corollary 2.48.
From the proof of this theorem, it is clear that the extension $\phi \times \text{id}$ is unique.

Here, $\mathcal{A} = \mathcal{B}$ and $G = \mathbb{R}$. Suppose $y \in \text{Br}_{\mathbb{R} \times \mathbb{Z}}(W)$, and we pick a representative $x = (\mathcal{A}, \alpha \times \phi)$ of $y$. We may define $T(y)$ to be the dynamical system $(\mathcal{A} \times \mathbb{R}, \alpha^\# \times \phi^\#)$ where $\phi^\#$ is the map induced on the crossed product by $\phi^\#(f)(t) = \phi \times \text{id}(f)(t)$. It is clear that it is unique and commutes with the $\mathbb{R}$-action.

Changing the representative to a Morita equivalent one $y = (\mathcal{A}', \alpha' \times \phi')$ will not change the Morita equivalence class of the answer because, by Lemma (3.1) of Ref. [19], $\alpha' \times \phi'$ is outer conjugate to $\alpha \times \phi$. Hence, there is an isomorphism $\Phi : \mathcal{A} \to \mathcal{A}$ such that $\Phi \circ (\alpha \times \phi)\Phi^{-1}$ is exterior equivalent to $(\alpha' \times \phi')$. Hence, there is a continuous map $u : \mathbb{R} \times \mathbb{Z} \to UM(\mathcal{A})$ such that $u_{gh} = u_g(\alpha \times \phi)_g(u_h)$ such that $(\alpha' \times \phi')_g = \text{Ad} u_g \circ (\alpha \times \phi)_g$. Then the suspension $u'(t) = u, \forall t \in \mathbb{R}$ gives an exterior equivalence between $\alpha^\# \times \phi^\#$ and $\alpha'^\# \times \phi'^\#$.

We now remark on some properties of the functor $P_{3,2}$ described above. Apart from the construction above, there is another reason to look at $P_{3,2}(W)$ : By Thm. (2.1) above, $\text{Br}_{\mathbb{R} \times \mathbb{Z}}(E_p)$ has a large continuous part, namely the elements of $H^2_{\mathbb{Z}}(\mathbb{R} \times \mathbb{Z}, C(E_p, \mathbb{T}))$. If we quotient $\text{Br}_{\mathbb{R} \times \mathbb{Z}}(E_p)$ by this continuous part, we obtain the group $H^2(E_p, \mathbb{Z}) \oplus H^3(E_p, \mathbb{Z})$ associated to $E_p$. Alternatively, we may consider $\text{im}(F_1)$ where $F_1 : \text{Br}_{\mathbb{R} \times \mathbb{Z}}(E_p) \to \text{Br}_\mathbb{Z}(E_p)$ is the forgetful map of Def. (2.1). In either case, for any CW-complex $W$, we have to consider the functor $P_{3,2}(W) =$ (principal bundle, closed integral two-form, H-flux) over $W$. We have the following property of $P_{3,2}$ :

**Lemma 2.6.** Let $T, T_3$ be as defined above. The map $F : \text{Br}_{\mathbb{R} \times \mathbb{Z}}(X) \to \text{Br}_\mathbb{Z}(X)$ in Def. (2.1) above gives a natural transformation $F$ between the functors $\mathcal{R}$ and $P_3$ defined above. We have that $F \circ T = T_3 \circ F$.

**Proof.** By Part (2) of Cor. (2.1) above, the map $F : \text{Br}_{\mathbb{R} \times \mathbb{Z}}(X) \to \text{Br}_\mathbb{Z}(X)$ gives a natural transformation between the functors $\mathcal{R}$ and $P_3$. By construction, we know that if we ‘forget’ the $\mathbb{Z}$-action on $\mathcal{A}$, the crossed product by the $\mathbb{R}$-action doesn’t change in either $H$-flux or topology. That is, if we have two T-dual (in the sense above) $C^*$-dynamical systems $(\mathcal{A}, \mathbb{R} \times \mathbb{Z}, \alpha)$ and $(\mathcal{B}, \hat{\mathbb{R}} \times \mathbb{Z}, \beta)$ then the dynamical systems $(\mathcal{A}, \mathbb{R}, \alpha|_\mathbb{R})$ and $(\mathcal{B}, \hat{\mathbb{R}}, \beta|_{\hat{\mathbb{R}}})$ are T-dual in the sense of Ref. [1]. Hence, the following diagram commutes.
This implies that $F \circ T = T_3 \circ F$. \hfill \Box

Now, by the discussion in Cor. (2.1) item (3) above, there is a natural transformation $\pi : \mathcal{R} \to P_{3,2}$.

In Sec. (6) we prove that there is a classifying space for $P_{3,2}$ denoted $R_{3,2}$ such that $P_{3,2}(W) = [W, R_{3,2}]$. We show that there is a well-defined map $T_{3,2} : R_{3,2} \to R_{3,2}$ inducing a natural transformation $T_{3,2}(W) : P_{3,2}(W) \to P_{3,2}(W)$. We also show that this map induces the following commutative diagram.

\[
\begin{array}{ccc}
R(W) & \xrightarrow{T(W)} & R(W) \\
F(W) \downarrow & & \downarrow F(W) \\
P_{3}(W) & \xrightarrow{T_3(W)} & P_{3}(W).
\end{array}
\]

This implies $P_3(W) = P_{3,2}(W)$.

Also, by Cor. (2.1), item(3), the map $\pi(W) : \mathcal{R}(W) \to P_{3,2}(W)$ induced by the functor $\pi$ in Cor. (2.1) item (3), is always surjective. Thus we may infer properties of $T_{R}$ from those of $T_{3,2}$ but it should be clear that they are not the same. In this paper we mainly study $P_{3,2}$ and $T_{3,2}$. This is sufficient to answer the question we raised in Section (1) in the paragraph after Lemma (1.1).
In a recent thesis by Schneider [18] the author defines the collection of equivalence classes of $C^*$-dynamical systems $(\mathcal{A}, G, \alpha)$ whose spectrum is a principal $G/N$-bundle over $W$ (denoted $\operatorname{Dyn}^+(W)$). He then defines a T-duality map induced by the crossed product between equivalence classes of such dynamical systems. The resulting dynamical system is $(\mathcal{A} \rtimes \hat{G}, \hat{G}, \alpha^\#)$ and has spectrum a principal $\hat{G}/N^\perp$-bundle over $W$.

Thus the duality map in Ref. [18] would map systems of the form $(\mathcal{A}, \mathbb{R} \times \mathbb{Z}, \alpha)$ with spectrum a principal $S^1$-bundle over $W$ to those of the form $(\mathcal{A}^\#, \mathbb{R} \times \mathbb{T}, \hat{\alpha})$ with a spectrum-fixing $\mathbb{T}$-action and spectrum a principal $S^1$-bundle over $W$.

This map is not the same as the T-duality map we are considering here, since we map equivalence classes of $C^*$-dynamical systems of the form $(\mathcal{A}, \mathbb{R} \times \mathbb{Z}, \alpha)$ with spectrum a principal $S^1$-bundle over $W$ and a spectrum-fixing $\mathbb{Z}$-action on $\mathcal{A}$ to those of the form $(\mathcal{A}^\#, \mathbb{R} \times \mathbb{Z}, \hat{\alpha})$. Here $\mathcal{A}^\#$ has spectrum a principal $S^1$-bundle over $W$ but there is a spectrum-fixing $\mathbb{Z}$-action on $\mathcal{A}^\#$.

3. The Classifying Space of $k$-pairs

In this section, we study the functors $P_2, P_{3,2}$ defined above. We show that they are representable and possess classifying spaces $R_2, R_{3,2}$ respectively. We calculate some properties of these classifying spaces.

Let $\mathcal{SET}$ be the category of sets with functions as morphisms. Let $\mathcal{C}$ be the category of unbased CW complexes with unbased homotopy classes of continuous maps as morphisms. Let $\mathcal{C}_0$ be the category of CW complexes which are finite subcomplexes of some fixed countably infinite dimensional standard simplicial complex. $(\mathcal{C}, \mathcal{C}_0)$ is a homotopy category in the sense of Ref. [21] (see Thm. (2.5) in [21]).

Let $W$ be a fixed CW complex. We define a $k$-pair over $W$ to consist of a principal $S^1$-bundle $p : E \to W$ together with a cohomology class $b \in H^k(W, \mathbb{Z})$. We denote a $k$-pair as $([p], b)$. (Here the space $W$ is understood from the context as is the value of $k$.) Note that a ‘pair’ in the sense of Ref. [2] would be a termed a 3-pair here.

**Def 3.1.** We declare two $k$-pairs (same $k$) $([p], b)$ and $([q], b')$ over $W$ equivalent if
• We are given two principal $S^1$-bundles $p : E \to W$ and $q : E' \to W$ such that
\[
\begin{array}{ccc}
E & \xrightarrow{\phi} & E' \\
p & & \downarrow q \\
W & \xrightarrow{id} & W
\end{array}
\]
commutes.

• We also require that $b' = \phi^*(b)$.

It is clear that the collection of equivalence classes of $k$-pairs over a fixed space $W$ (denoted $P_k(W)$) is a set. For all $W$, we have a distinguished pair consisting of the trivial $S^1$-bundle over $W$ with the zero class in $H^k(W, \mathbb{Z})$. Thus, $P_k(W)$ is actually a pointed set.

**Def 3.2.** Let $W$ and $Y$ be two CW-complexes and let $f : W \to Y$ be a continuous map. Let $([p], b) \in P_k(Y)$ be represented by a principal $S^1$-bundle $p : E \to Y$ and a class $b \in H^k(Y, \mathbb{Z})$. We define the pullback of $([p], b)$ via $f$, denoted $f^*([p], b)$, to be the following data

• The unique principal $S^1$-bundle $f^*p : f^*E \to W$ such that the following diagram commutes
\[
\begin{array}{ccc}
f^*E & \xrightarrow{\phi_f} & E \\
f^*p & & \downarrow p \\
W & \xrightarrow{f} & Y.
\end{array}
\]

• The cohomology class $\phi_f^*(b)$ in $H^k(f^*E, \mathbb{Z})$.

That is, we define $f^*([p], b) = (f^*[p], \phi_f^*(b))$.

**Lemma 3.1.** Let $f_0, f_1 : W \to Y$ be freely homotopic. For any pair $([p], b) \in P_2(Y)$, $f_0^*([p], b)$ is equivalent to $f_1^*([p], b)$.

**Proof.** Let $p : E \to Y$ be a principal $S^1$-bundle. We have pullback squares for $i = 0, 1$
\[
\begin{array}{ccc}
f_i^*E & \xrightarrow{\phi_i} & E \\
f_i^*p & & \downarrow p \\
W & \xrightarrow{f_i} & Y.
\end{array}
\]

Then, by Ref. [22], Cor. (1.8), the pullback bundles $f_0^*p : f_0^*E \to W$ and $f_1^* : f_1^*E \to W$ are isomorphic. Further, by the same lemma, this isomorphism is implemented by a map $\psi : f_0^*E \to f_1^*E$. This map induces isomorphisms on the cohomology groups such that $\psi \circ f_0^* = f_1^*$. As a result, by the above definition, $f_0^*([p], b) = f_1^*([p], b)$. \qed
Hence, \( P_k(W) \) is a pointed set depending only on the homotopy type of \( W \). Given a map \( f : W \to Y \), define \( P_k(f) : P_k(Y) \to P_k(W) \) to be the map induced by pullback of pairs. It is clear that \( P_k(1) = \text{Id} \). (This is just the condition that two pairs be equivalent). Hence, \( P_k \) extends to a functor (also denoted \( P_k \)) \( P_k : \mathcal{C} \to \text{SET} \).

**Theorem 3.1.** For every \( k \), the functor \( P_k \) above satisfies the conditions of the Brown Representability Theorem. Hence, for every \( k \), there exists a classifying space \( R_k \) for \( P_k \).

**Proof.** There are two conditions we need to prove.

1. Consider an arbitrary family \( \{W_\mu\}, \mu \in I \) of objects in \( \mathcal{C} \). Let \( Y = \bigsqcup_{\mu \in I} W_\mu \). Let \( h_\mu : W_\mu \to \bigsqcup_{\mu \in I} W_\mu \) be the inclusion maps.

   We have a pullback square (for every \( \mu \in I \))
   \[
   \begin{array}{ccc}
   h_\mu^* E & \xrightarrow{\tilde{h}_\mu} & E \\
   p_\mu \downarrow & & \downarrow p \\
   W_\mu & \xrightarrow{h_\mu} & Y.
   \end{array}
   \]
   Here \( p_\mu = h_\mu^* p_\mu \). Since \( H^2(W, \mathbb{Z}) \simeq \prod_{\mu \in I} H^2(W_\mu, \mathbb{Z}) \), we have that \([p] = ([p_\mu]), \mu \in I\).

   Let \( h_\mu^* E = E_\mu \), then, we also have that \( E = \bigsqcup_{\mu \in I} E_\mu \) and \( H^k(E, \mathbb{Z}) \simeq \prod_{\mu \in I} H^k(E_\mu, \mathbb{Z}) \). Hence, every class \( b \in H^k(E, \mathbb{Z}) \) may be written as \( (b_\mu), \mu \in I \) with \( b_\mu = \tilde{h}_\mu^*(b) \). Hence, we have an isomorphism
   \[
   \Pi_\mu P(h_\mu) : P(\bigsqcup_{\mu} W_\mu) \simeq \Pi_\mu P(W_\mu).
   \]

2. Suppose we are given CW complexes \( A, W_1, W_2 \) and continuous maps \( f_i : A \to W_i, g_i : W_i \to Z, i = 1, 2 \) such that
   \[
   \begin{array}{ccc}
   A & \xrightarrow{f_1} & W_1 \\
   \downarrow f_2 & & \downarrow g_1 \\
   W_2 & \xrightarrow{g_2} & Z
   \end{array}
   \]
   commutes up to homotopy and is a pushout square in \( \mathcal{C} \). We may take \( f_i \) to be inclusions into the \( W_i \) and \( Z \) the result of gluing \( W_1 \) to \( W_2 \) along \( A \). Suppose \( u_i \in P(W_i) \) satisfy \( P(f_1)u_1 = P(f_2)u_2 \). For \( i = 1, 2 \), let \( u_i \) correspond to the pair \( ([p_i], b_i) \) over \( W_i \), where \( p_i : E_i \to W_i \) are principal \( S^1 \)-bundles. Then, since \( P(f_1)u_1 = P(f_2)u_2, f_1^*E_i \simeq f_2^*E_2 \). This implies that the
restrictions of $f_i^*E_i$ to $A$ are the same. Hence, these two bundles may be glued into a unique bundle $p : E \to Z$. Note that $G_i = f_i^*E_i \subset E, i = 1, 2$ and $G_1 \cup G_2 = E$. We have a pullback square

$$
\begin{array}{ccc}
G_i & \xrightarrow{\tilde{g}_i} & E \\
p_i & \downarrow & \downarrow p \\
W_i & \xrightarrow{g_i} & Z
\end{array}
$$

By the Mayer-Vietoris theorem, we have

$$H^k(E, \mathbb{Z}) \to H^k(G_1, \mathbb{Z}) \oplus H^k(G_2, \mathbb{Z}) \to H^k(G_1 \cap G_2, \mathbb{Z})$$

Now $f_1^*(b_1) = f_2^*(b_2)$ and so the image of $(b_1, b_2)$ via the second map above is zero. Hence, by exactness, there is an element in $c \in H^k(E, \mathbb{Z})$ such that $\tilde{g}_i(c) = b_i, i = 1, 2$. Thus, we define an element $v \in P(Z)$ by $v = ([p], c)$. It is clear that $P(g_i)v = u_i, i = 1, 2$.

As a result, for every $k$, there is a CW complex $R_k$ such that isomorphism classes of $k$-pairs over a space $W$ correspond to unbased homotopy classes of maps from $W \to R_k$.

Similarly we may define a $k_1, k_2, \ldots, k_n$-tuple $(k_i \in \mathbb{N})$ over a space $W$ to consist\textsuperscript{18} of a principal $S^1$-bundle $p : E \to W$ together with cohomology classes $b_1, \ldots, b_n$ such that $b_i \in H^{k_i}(E, \mathbb{Z})$. Exactly as above we may define the notion of equivalent tuples and a set valued functor $P_{k_1, k_2, \ldots, k_n}(W)$. As above, such a functor is representable and has a representation space denoted $R_{k_1, k_2, \ldots, k_n}$. Note that there are natural maps $R_{k_1, \ldots, k_n} \to K(\mathbb{Z}, 2)$ and $R_{k_1, \ldots, k_n} \to K(\mathbb{Z}, k_i - 1)$ given by sending $([p], b_1, \ldots, b_n) \to [p]$ and $([p], b_1, \ldots, b_n) \to p_i(b_i)$ respectively. In addition, the canonical $S^1$-bundle $U$ over $K(\mathbb{Z}, 2)$ defines a unique pair $(U, 0)$ over $R_k$ for every $k$ and hence the natural map $R_k \to K(\mathbb{Z}, 2)$ is naturally split. This implies that the cohomology ring of $R_k$ contains $\mathbb{Z}[\alpha], \alpha \in H^2(R_k, \mathbb{Z})$.

Given a 3, 2-tuple $([p], b, H)$ over $W$, we obtain a unique 2-pair $([p], b)$ and 3-pair $([p], H)$ over $W$. Similarly, given a 2-pair $([p], b)$ and a 3-pair $([p], H)$ (same $[p]$) over $W$, we obtain a unique triple $([p], b, H)$ over $W$. Thus, $R_{3,2}$ is a fiber product $R_3 \times R_2$. Similarly $R_{k_1, \ldots, k_n} \simeq R_{k_1} \times \cdots \times R_{k_n}$. There are natural fibrations $q_{k_i} : R_{k_1, \ldots, k_n} \to R_{k_i}$.

For the sake of completeness we note the following

\textsuperscript{18}Obviously the ordering of the $k_i$ is irrelevant.
Lemma 3.2. Suppose $W$ is $k$-connected, $k \geq 2$. Then, $P_{k+1}(W) \simeq H^{k+1}(W, \mathbb{Z})$.

Proof. Since $W$ is at least 2-connected, all principal $S^1$-bundles on $W$ are trivial. Further, $H^{k+1}(W \times S^1, \mathbb{Z}) \simeq H^{k+1}(W, \mathbb{Z})$ and hence the result. □

We will only consider 2-tuples, 3-tuples and 3,2-tuples in this paper. We will also consider $P_2, P_3, P_{3,2}$ and the corresponding classifying spaces $R_2, R_3, R_{3,2}$.

Bunke et al. [2] have considered the case $k = 3$. We denote their classifying space $R_3$ here. For the remainder of this section and the next we work with $k = 2$. We abbreviate 2-pair to ‘pair’.

A priori, $R_2$ is an unbased CW complex. We now arbitrarily pick a basepoint $r_0$ in $R_2$.

Lemma 3.3. Let $W$ be any CW complex. Pick a basepoint $x_0 \in W$. Any unbased map $f : W \to R_2$ may be freely homotoped to a based map $g : (W, x_0) \to (R_2, r_0)$.

Proof. Suppose $W$ was any CW complex, and $f : W \to R_2$ an unbased map. By the Lemma that follows, we know that $R_2$ is a fibration of a connected space over a connected base space. Hence $R_2$ is connected. Pick a basepoint $x_0 \in W$. Pick a path $q : I \to R_2$ connecting $f(x_0)$ to $r_0$. Extend the data $f, q$ to a free homotopy $H : W \times I \to R_2$. Then, $g = H(1, .) : W \to R_2$ is map such that $g(x_0) = r_0$. The map $g$ classifies the same pair that $f$ does, since $R_2$ is an unbased classifying space. □

Lemma 3.4. There is a fibration $K(\mathbb{Z}, 2) \to R_2 \to K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 1)$.

Proof. Given a pair $([p], b)$ over $W$, we obtain two natural cohomology classes $[p] \in H^2(W, \mathbb{Z})$ and $p_1(b) \in H^1(W, \mathbb{Z})$. As a result, there is a natural map $\phi \times \psi : R_2 \to K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 1)$. Given a pair $([p], b)$ over any space $W$, classified by $f : W \to R_2$, the map $f \mapsto \phi \circ f$ corresponds to the map $([p], b) \mapsto [p]$. Similarly, $f \mapsto \psi \circ f$ corresponds to the map $([p], b) \mapsto p_1(b)$. We pick a basepoint in $K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 1)$ such that $\phi \times \psi$ is a based map. Suppose $f : W \to R_2$ classified a pair $([p], b)$ over $W$. Pick a basepoint $x_0 \in W$. By Lemma (3.3) above, $f$ may be freely homotoped to a based map $g : (W, x_0) \to (R_2, r_0)$. Suppose $g$ was in the homotopy fiber of $\phi \times \psi$. Then, we would obtain a pair $([p], b)$ over $W$ such that $p_1(b) = 0, [p] = 0$. This would correspond to the trivial bundle $W \times S^1 \to W$ equipped with the cohomology class $1 \times a, a \in H^2(W, \mathbb{Z})$. Hence we would get a natural based map $W \to K(\mathbb{Z}, 2)$. Conversely,
given a class $a$ in $H^2(W, \mathbb{Z})$, we could obtain a pair $(0, 1 \times a)$ over $W$ which would have $[p] = 0$ and $p_!(1 \times a) = 0$. By Lemma (3.3), this pair would be classified by a based map $g : (W, x_0) \to (R_2, r_0)$. Obviously, $(\phi \times \psi) \circ g$ would be nullhomotopic.

Thus, the homotopy fiber of $\phi \times \psi$ is $K(\mathbb{Z}, 2)$. \hfill \square

**Lemma 3.5.** The homotopy groups of $R_2$ are as follows

- $\pi_1(R_2) = \mathbb{Z}$,
- $\pi_2(R_2) \simeq \mathbb{Z}^2$,
- $\pi_i(R_2) = 0$, $i > 2$.

*Proof.* We had picked a basepoint for $R_2$. Hence, we may calculate $\pi_i(R_2)$ from the long exact sequence of the fibration in Lemma (3.4).

We find that the nonzero part of the sequence is

$$0 \to \mathbb{Z} \to \pi_2(R_2) \to \mathbb{Z} \to 0 \to \pi_1(R_2) \to \mathbb{Z} \to 0.$$ 

Thus, $\pi_1(R_2) = \mathbb{Z}$, $\pi_2(R_2) \simeq \mathbb{Z}^2$, and $\pi_i(R_2) = 0$, $i > 2$. \hfill \square

We may characterize $R_2$ as follows

**Lemma 3.6.** Let $c : K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 1) \to K(\mathbb{Z}, 3)$ be the based map which induces the cup product. Then $R_2$ is the homotopy fiber of $c$.

*Proof.* Let $f : W \to R_2$ be a map inducing the pair $(\{p\}, b)$ over $W$. Fixing a basepoint $x_0 \in W$, we may replace $f$ by a based map $g : (W, x_0) \to (R_2, r_0)$ by Lemma (3.3). It is clear that we may take $\phi \times \psi$ to be based. Then we have a principal $S^1$-bundle $p : E \to W$. By the Gysin sequence of this bundle we have that $[p] \cup p_!(b) = 0$. This implies that $c \circ (\phi \times \psi) \circ f$ is nullhomotopic via a based homotopy, since $c$ is exactly the based map which gives the cup product.

Conversely, suppose we are given a based map $f : W \to K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 1)$ such that $c \circ f$ is nullhomotopic. Then, this corresponds to a class $a \in H^1(W, \mathbb{Z})$ and a class $[p] \in H^2(W, \mathbb{Z})$ such that $[p] \cup a = 0$. Pick a principal $S^1$-bundle $p : E \to W$ with characteristic class $[p]$. From the Gysin sequence of this bundle we see that $[p] \cup a = 0$ implies that $a = p_!(b)$ for some $b \in H^2(E, \mathbb{Z})$. Thus, we obtain a pair $(\{p\}, b)$ over $W$ and hence an unbased map $g : W \to R_2$. By the above argument, we may replace it with a based map $h$ classifying the same pair over $W$. Obviously, $(\phi \times \psi) \circ h = f$ as a based map.

Hence, $R_2$ is the homotopy fiber of the based map $c$ in the category of based CW complexes with basepoint preserving homotopy classes of maps between them. There is a forgetful functor from this category to the category $\mathcal{C}$. We take the image of the homotopy fiber of $c$ via this functor. This determines $R_2$ up to homotopy equivalence in $\mathcal{C}$. \hfill \square
Since $\pi_1(R_2) \neq 0$, the choice of basepoints might be important. Indeed, we have the following

**Lemma 3.7.** The space $R_2$ is not simple.

**Proof.** Suppose $R_2$ was simple: Then, from Postnikov theory, we see that $R_2$ would be homotopy equivalent to the product $K(\mathbb{Z}, 1) \times K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 2)$ via a based homotopy. Given $f : W \to R_2$ by Lemma (3.3), we could obtain a based map $g : W \to R_2$ classifying the same pair over $W$ as $f$. Hence we would obtain based maps $W \to K(\mathbb{Z}, 2)$, $W \to K(\mathbb{Z}, 2)$ and $W \to K(\mathbb{Z}, 1)$. The pair would then be determined by classes $[p], a \in H^2(W, \mathbb{Z})$ and $p_!(b) \in H^1(W, \mathbb{Z})$. Here $[p]$ would be the characteristic class of a principal $S^1$-bundle $p : E \to W$. This would imply in turn that $b$ would be determined by $p_!(b)$ and $a$ and hence that the Gysin sequence for $p : E \to W$ would split at degree two for any principal $S^1$-bundle $E$ over $W$. Since $W, E$ were arbitrary, this is obviously impossible. □

**Lemma 3.8.** The cohomology of $R_2$ up to degree 3 is

- $H^0(R_2, \mathbb{Z}) \cong \mathbb{Z}$,
- $H^1(R_2, \mathbb{Z}) \cong \mathbb{Z}$,
- $H^2(R_2, \mathbb{Z}) \cong \mathbb{Z}$,
- $H^3(R_2, \mathbb{Z}) \cong \mathbb{Z}$.

**Proof.** Consider the fibration $K(\mathbb{Z}, 2) \to R_2 \to K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 1)$. We have that $H^*(K(\mathbb{Z}, 2), \mathbb{Z}) \cong \mathbb{Z}[a]$ where $a$ is a generator of $H^2(K(\mathbb{Z}, 2), \mathbb{Z}) \cong \mathbb{Z}$. This ring has no automorphisms apart from $a \to -a$. Since the fibration is oriented, the generator of $\pi_1(K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 1))$ acts trivially on the cohomology of $K(\mathbb{Z}, 2)$. As a result, we may use the Serre spectral sequence using cohomology with untwisted coefficients to calculate $H^*(R_2, \mathbb{Z})$.

We note that the above fibration is pulled back from the path-loop fibration over $K(\mathbb{Z}, 3)$ via the map $c : K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 1) \to K(\mathbb{Z}, 3)$ inducing the cup product. Suppose $\mu$ was a generator of $H^1(K(\mathbb{Z}, 1), \mathbb{Z})$ and that $\lambda$ was a generator of $H^2(K(\mathbb{Z}, 2), \mathbb{Z})$. Let $\bar{\mu} = (\phi \times \psi)^*(\mu)$, and $\bar{\lambda} = (\phi \times \psi)^*(\lambda)$. Then we have that $\bar{\mu} \cup \bar{\lambda} = 0$. This shows that the transgression $E_2^{0, 3} \to E_2^{2, 3}$ must be a map $k : H^2(K(\mathbb{Z}, 2), \mathbb{Z}) \simeq \mathbb{Z} \to H^2(K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 1), \mathbb{Z}) \cong \mathbb{Z}$ sending $a \to a(\mu \cup \lambda)$.

From the spectral sequence table, we see that $H^0(R_2, \mathbb{Z}) \cong \mathbb{Z}$, $H^1(R_2, \mathbb{Z}) \cong \mathbb{Z}$, $H^2(R_2, \mathbb{Z}) \cong \mathbb{Z}$ and $H^3(R_2, \mathbb{Z}) \cong \mathbb{Z}$. □

Note that the canonical bundle over $K(\mathbb{Z}, 2)$ gives rise to a unique pair $(1, 0)$ over $K(\mathbb{Z}, 2)$. This is classified by a map $K(\mathbb{Z}, 2) \to R_2$ which
is a section of the natural map $R_2 \to K(\mathbb{Z}, 2)$. Hence, the cohomology ring of $R_2$ contains $\mathbb{Z}[b], b \in H^2(R_2, \mathbb{Z})$.

This ring has generators $a, b, c$ in degrees 1, 2 and 3 respectively such that

(9) $a \cdot b = 0, b^n \neq 0$ for any $n$.

We now determine the action of $\pi_1(R_2)$ on $\pi_2(R_2)$.

**Theorem 3.2.** The action of the generator $S$ of $\pi_1(R_2) \cong \mathbb{Z}$ on $\pi_2(R_2)$ is given by

$$S(a, b) = (a + b, b).$$

**Proof.** From the long exact sequence of homotopy groups of the fibration in Lemma (3.4), we see that we have a sequence $0 \to \mathbb{Z} \to \mathbb{Z}^2 \to \mathbb{Z} \to 0$. Now, $\pi_1(R_2)$ acts on each term of this sequence by $\mathbb{Z}$-module automorphisms with the trivial action on the first and last $\mathbb{Z}$ factors and by an action $\theta$ on the middle factor.

This implies that $\theta$ may be taken to be the homomorphism induced by the matrix

$$\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}.$$ 

We claim that

$$\theta \simeq \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$ 

We have a fibration $K(\mathbb{Z}, 2) \to R_2 \to K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 1)$. We could view this as a fibration over the $K(\mathbb{Z}, 1) \cong S^1$ factor $K(\mathbb{Z}^2, 2) \to R_2 \to S^1$. From the long exact sequence of a fibration, it is clear that $K(\mathbb{Z}^2, 2)$ is the universal cover $\tilde{R}_2$ of $R_2$. Now $\pi_1(S^1)$ acts on $\tilde{R}_2$ by deck transformations. Hence, using the Cartan-Leray spectral sequence (see Ref. [23] Ch. XVI Sec. (9) for details), we have a spectral sequence with $E_2^{pq} = H^p(\mathbb{Z} = \pi_1(S^1), H^q(K(\mathbb{Z}^2, 2), \mathbb{Z})) \Rightarrow H^*(R_2, \mathbb{Z})$ (here $H^*(\mathbb{Z}, M)$ denotes the group cohomology of $\mathbb{Z}$ with coefficients in a module $M$).

This sequence collapses at the $E_2$ term itself, since $E_2^{pq} \simeq 0$ for $p \geq 2$. Thus $\mathbb{Z} \cong H^2(R_2, \mathbb{Z}) \cong H^0(\mathbb{Z}, H^2(K(\mathbb{Z}^2, 2), \mathbb{Z})) \cong H^0(\mathbb{Z}, \mathbb{Z}^2)$, and so the fixed points of $\theta$ on $\mathbb{Z}^2$ are $\mathbb{Z} \neq \mathbb{Z}^2$. Hence the action $\theta$ is not trivial and $R_2$ is not simple.

Now, $\mathbb{Z} \cong H^3(R_2, \mathbb{Z}) \cong H^1(\mathbb{Z}, H^2(K(\mathbb{Z}^2, 2), \mathbb{Z})) \cong H^1(\mathbb{Z}, \mathbb{Z}^2)$. If $\mathbb{Z}$ acts on $\mathbb{Z}^2$ with an action $\theta$, $H^*(\mathbb{Z}, \mathbb{Z}^2)$ is the cohomology of the complex $\mathbb{Z}^2 \xrightarrow{\theta - 1} \mathbb{Z}^2$. Hence, $\mathbb{Z}^2/(\theta - 1)\mathbb{Z}^2 \cong \mathbb{Z}$ here, and using the above form for $\theta$,

$$\theta \cong \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$
Note that since $R_2$ was defined in the unbased category, for any CW-complex $W$, pairs over $W$ are classified by unbased maps from $W$ to $R_2$. Since the space of unbased maps from $W$ to $R_2$ is the quotient of the space of based maps from $W$ to $R_2$ by the action of $\pi_1(R_2)$, we see that the non-trivial action of $\pi_1(R_2)$ does not affect our results. We simply have to be careful to use unbased maps in all our constructions.

We can see an example of this when we try to determine all the pairs over $S^2$. If pairs were classified by based maps, then $P_2(S^2)$ would be a group. However, we have the following

**Lemma 3.9.** $P_2(S^2)$ is not a group.

**Proof.** For any CW complex $W$, we have a natural map $\phi : P_2(W) \to H^2(W, \mathbb{Z})$ which sends a pair $([p], b)$ over $W$ to $[p]$. Now, for every $a \in H^2(S^2, \mathbb{Z}) \simeq \mathbb{Z}$, we claim that the set $\phi^{-1}(a)$ has cardinality $|a|$. To see this, it is enough to note that if $D_p \to S^2$ is a principal $S^1$-bundle of Chern class $[p]$, then $H^2(D_p, \mathbb{Z}) \simeq \mathbb{Z}_p$. This implies that $P_2(S^2) \to H^2(S^2, \mathbb{Z})$ is not a group homomorphism, and hence that $P_2(S^2)$ is not a group. \hfill \Box

In fact, $P_2(S^2)$ is the quotient of the group $\pi_2(R_2) \simeq \mathbb{Z}_2$ by the action of $\pi_1(R_2)$ calculated in Thm. (3.2) above.

**Lemma 3.10.** Consider the principal $S^1$-bundle $p : E_2 \to R_2$ whose characteristic class is $b$ the generator of $H^2(R_2, \mathbb{Z})$. Its cohomology groups are

- $H^0(E_2, \mathbb{Z}) \simeq \mathbb{Z}$,
- $H^1(E_2, \mathbb{Z}) \simeq \mathbb{Z}a$,
- $H^2(E_2, \mathbb{Z}) \simeq \mathbb{Z}w$,
- $H^3(E_2, \mathbb{Z}) \simeq \mathbb{Z}c$.

where $\bar{a}, \bar{c}$ are the images of the generators $a, c$ of $H^i(R_2, \mathbb{Z}), i = 1, 3$ in $H^*(E_2, \mathbb{Z})$. In addition $p_i(w) = a, \bar{a} = p^*(a), \bar{c} = p^*(c)$.

**Proof.** The Serre Spectral Sequence table is

| $\mathbb{Z}z$ | $\mathbb{Z}az$ | $\mathbb{Z}bz$ | $\mathbb{Z}cz$ |
|-------------|-------------|-------------|-------------|
| $\mathbb{Z}$ | $\mathbb{Z}a$ | $\mathbb{Z}b$ | $\mathbb{Z}c$ |

The transgression map $d^2 : E^{0,1}_2 \to E^{2,0}_2$ has to be $dz = b$. Hence using the ring structure, the fact that $ab = 0$ in $H^*(R_2, \mathbb{Z})$, and the fact that $b$ generates a copy of $\mathbb{Z}[b]$ inside $H^*(E_2, \mathbb{Z})$ gives the cohomology groups shown. In addition, note that $w$ is the image of $a.z$ in $H^2(E_2, \mathbb{Z})$. 

\
Then, an inspection of the Gysin sequence for $E_2$ shows that the elements $\bar{a}, \bar{c}$ must be images of $a, c$ under $p^*$.

This bundle acts as a universal bundle for pairs: Given a principal $S^1$-bundle $p : D \to W$ and $b \in H^2(D, \mathbb{Z})$ we have a classifying map $f : W \to R_2$. This bundle pulls back along $f$ to give the bundle $p : D \to W$ while the generator of $H^2(E_2, \mathbb{Z})$ pulls back to the $b$-class.

We hope to study the map $T$ of Section (1) using the classifying space $R_2$ studied above.

4. T-duality for automorphisms is not involutive

By the proof of Thm. (3.1) in Ref. [17], we know that the T-dual of an automorphism even with $H$-flux is always unique. However, T-duality for automorphisms is not involutive: If we perform two successive T-dualities we may not get the automorphism we started with. For example, if $W = S^2$ with 1 unit of $H$-flux on $S^2 \times S^1$, the T-dual is $S^3$ with no $H$-flux. Since $H^2(S^2 \times S^1, \mathbb{Z}) \simeq \mathbb{Z}$, but $H^2(S^3, \mathbb{Z}) \simeq 0$, every locally unitary (but not necessarily unitary) automorphism of $CT(S^2 \times S^1, 1)$ dualizes to a unitary automorphism of $C(S^3, \mathfrak{K})$. From the proof of Thm. (3.1) of Ref. [17], it is clear that the T-dual of a unitary automorphism is unitary. Hence, taking one more T-dual gives a unitary automorphism of $CT(S^2 \times S^1, 1)$.

At the level of triples $([p], b, H)$ we conjecture that the T-duality map of Section (1) should have the following properties:

**Lemma 4.1.** Let $W$ be connected and simply connected. Let $p : E \to W$ be a principal $S^1$-bundle with $H$-flux $H$ and $b \in H^2(E, \mathbb{Z})$. Let $q : E^# \to W$ be the T-dual principal $S^1$-bundle with $H$-flux $H^#$ and $b^# \in H^2(E^#, \mathbb{Z})$ where $b^# = T(b)$. Then, for all $b \in H^2(E, \mathbb{Z}), \forall l, m \in \mathbb{Z}$, the Gysin sequence induces a natural isomorphism $H^2(E/\mathbb{Z})/<p^*p_!(H)> \simeq H^2(E^#/\mathbb{Z})/<q^*q_!(H^#)>$.

**Proof.** The Gysin sequence of $q : E^# \to W$ is

$$
\mathbb{Z} \xrightarrow{[q]} H^2(W, \mathbb{Z}) \xrightarrow{q^*} H^2(E^#, \mathbb{Z}) \xrightarrow{q_!} H^1(Z, \mathbb{Z}) \to \cdots
$$

The Gysin sequence of $p : E \to W$ is

$$
\mathbb{Z} \xrightarrow{[p]} H^2(W, \mathbb{Z}) \xrightarrow{p^*} H^2(E, \mathbb{Z}) \xrightarrow{p_!} H^1(Z, \mathbb{Z}) \to \cdots
$$

Since $H^1(W, \mathbb{Z}) = 0$, $p^*, q^*$ are surjective by the Gysin sequence. Suppose $b = p^*(a)$ and $b^# = q^*(a^#)$ for $a, a^# \in H^2(W, \mathbb{Z})$. 


The kernel of $p^*$ is the subgroup $\langle [p] \rangle$. Similarly, the kernel of $q^*$ is the subgroup $\langle [q] \rangle$. Let $G = \langle [p], [q] \rangle = \langle p_!(H), q_!(H^\#) \rangle$. Since $H^1(W, \mathbb{Z}) = 0$, $p^*, q^*$ are surjective by the Gysin sequence. Note that $p^*G \simeq (p^*p_!(H))$ and $q^*G \simeq (q^*q_!(H^\#))$. Thus, we have isomorphisms $H^2(W, \mathbb{Z})/G \simeq H^2(E, \mathbb{Z})/(p^*p_!(H)) \simeq H^2(E^\#, \mathbb{Z})/(q^*q_!(H^\#))$. \qed

The cosets corresponding to $b$ and $b^\#$ may be written as $(l, m \in \mathbb{Z})$

$$\{b + lp^*p_!(H)\} \text{ and } \{b^\# + mq^*q_!(H^\#)\}.$$  

We conjecture that each coset is precisely the collection of $b$-fields with the same $T$-dual (even when $X$ is not simply connected). As support for this, note the following: Suppose $E^\# = W \times S^1$, with $k$ units of $H$-flux. Let $q = \pi : W \times S^1 \to W$ be the projection map. Then, if $H^1(W, \mathbb{Z}) \neq 0$, the above theorem would not be expected to hold: For one thing, $\text{im}(\pi^*)$ would not be all of $H^2(W \times S^1, \mathbb{Z})$. It is strange then, that $H^2(E, \mathbb{Z})/(p^*p_!(H))$ is isomorphic to $H^2(E^\#, \mathbb{Z})/(q^*q_!(H^\#))$ in all the following cases in many of which $H^1(W, \mathbb{Z}) \neq 0$ (I use Ref. [4] for the examples):

1. $W = T^2$: For $E^\# = W \times S^1$, $H^0(E^\#, \mathbb{Z}) = \mathbb{Z}$, $H^1 = \mathbb{Z}^3$, $H^2 = \mathbb{Z}^3$, $H^3 = \mathbb{Z}$. The $H$-flux is a class $[p] \times z \in H^2(T^2) \otimes H^1(S^1) \simeq H^3(T^2 \times S^1) \simeq \mathbb{Z}$. The T-dual is the nilmanifold $p : N \to T^2$ whose cohomology is $H^0 = \mathbb{Z}$, $H^1 = \mathbb{Z}^2$, $H^2 = \mathbb{Z}^2 \oplus \mathbb{Z}_p$, $H^3 = \mathbb{Z}$. It is clear that $\mathbb{Z}^3/p\mathbb{Z} \simeq \mathbb{Z}^2 \oplus \mathbb{Z}_p$.

2. $W = M$, an orientable surface of genus $g > 1$: The cohomology of $W \times S^1$ is $H^0 = \mathbb{Z}$, $H^1 = \mathbb{Z}^{2g+1}$, $H^2 = \mathbb{Z}^{2g+1}$, $H^3 = \mathbb{Z}$. The $H$-flux is a class $j \times z \in H^3 \simeq H^2(M) \otimes H^1(S^1) \simeq \mathbb{Z}$. The T-dual is a bundle $j : E \to M$ with $H^0 = \mathbb{Z}$, $H^1 = \mathbb{Z}^{2g}$, $H^2 = \mathbb{Z}^{2g} \oplus \mathbb{Z}_j$, $H^3 = \mathbb{Z}$. Here, $\mathbb{Z}^{2g+1}/j\mathbb{Z} \simeq \mathbb{Z}^{2g} \oplus \mathbb{Z}_j$.

3. $W = \mathbb{RP}^2$: The cohomology of $W \times S^1$ is $H^0 = \mathbb{Z}$, $H^1 = \mathbb{Z}$, $H^2 = \mathbb{Z}_2 \simeq H^2(W) \otimes H^0(S^1)$, $H^3 \simeq H^2(W) \otimes H^1(S^1) \simeq \mathbb{Z}_2$. The $H$-flux is the class $1 \times z \in H^2(W) \otimes H^1(S^1)$. The T-dual is a bundle $k : E \to \mathbb{RP}^2$ with $H^0 = \mathbb{Z}$, $H^1 = \mathbb{Z}$, $H^2 = 0$, $H^3 = \mathbb{Z}_2$. Once again, $\mathbb{Z}_2/1\mathbb{Z}_2 \simeq 0$.

4. $W = \mathbb{RP}^3$. The cohomology of $W \times S^1$ is $H^0 = \mathbb{Z}$, $H^1 = \mathbb{Z}$, $H^2 = \mathbb{Z}_2$, $H^3 = \mathbb{Z} \oplus \mathbb{Z}_2$, $H^4 = \mathbb{Z}$. Then, $H^3 \simeq H^2(W) \otimes H^1(S^1) \oplus H^3(W) \otimes H^0(S^1)$. The $H$-flux is $1 \times z + k \times 1$. Now, $\pi^*\pi_!(H) = 1 \times 1$. The T-dual is $q : S^1 \times S^3 \to \mathbb{RP}^3$ with cohomology $H^0 = \mathbb{Z}$, $H^1 = \mathbb{Z}$, $H^3 = \mathbb{Z}$, $H^4 = \mathbb{Z}$. The T-dual has no $B$-class and $H$-flux $k \in \mathbb{Z}$.

\[\langle a_1, a_2, \ldots, a_n \rangle\] denotes the subgroup generated by $a_1, a_2, \ldots, a_n$. The ambient group is understood from context.
\( W = \mathbb{RP}^{2m}(m > 1) \): The cohomology of \( W \times S^1 \) is \( H^0 = \mathbb{Z}, H^1 = \mathbb{Z}, H^q = \mathbb{Z}_2, q = 2, \ldots, m - 1, H^{2m+1} = \mathbb{Z}_2 \). The \( H \)-flux is the class \( 1 \times z \in H^2(\mathbb{RP}^{2m}) \otimes H^1(S^1) \). The T-dual is a bundle \( q : E \to \mathbb{RP}^{2m} \) with cohomology \( H^0 = \mathbb{Z}, H^1 = \mathbb{Z}, H^{2m+1} = \mathbb{Z}_2 \). Here, \( \mathbb{Z}_2/\mathbb{Z}_2 \cong 0 \).

\( W = \mathbb{RP}^{2m+1}(m > 1) \): The cohomology of \( W \times S^1 \) is \( H^0 = \mathbb{Z}, H^1 = \mathbb{Z}, H^q = \mathbb{Z}_2, q = 2, \ldots, m - 1, H^{2m+1} = \mathbb{Z}_2, H^{2m+2} = \mathbb{Z} \). Note \( H^3 = \mathbb{Z}_2 \); we have that \( H^3 \cong H^2(\mathbb{RP}^{2m+1}) \otimes H^1(S^1) \). The \( H \)-flux is the class \( 1 \times z \in H^3 \cong \mathbb{Z}_2 \). The T-dual is a principal bundle \( q : S^1 \times S^{2m+1} \to \mathbb{RP}^{2m+1} \) with cohomology \( H^0 = \mathbb{Z}, H^1 = \mathbb{Z}, H^{2m+1} = \mathbb{Z}, H^{2m+2} = \mathbb{Z} \). The T-dual has no second cohomology as expected.

\( W = \mathbb{CP}^2 \): The cohomology of \( W \times S^1 \) is \( H^0 = \mathbb{Z}, H^1 = \mathbb{Z}, H^2 = \mathbb{Z}, H^3 = \mathbb{Z}, H^4 = \mathbb{Z}, H^5 = \mathbb{Z} \). We have \( H^3 \cong H^2(W) \oplus H^1(S^1) \cong \mathbb{Z} \). The \( H \)-flux is the class \( j \times z \in H^3 \). The T-dual is the Lens space \( L(2,j) \to \mathbb{CP}^2 \) if \( j \neq 0 \). It has cohomology \( H^0 = \mathbb{Z}, H^2 = \mathbb{Z}_j, H^4 = \mathbb{Z}_j, H^5 = \mathbb{Z} \). Once again, \( \mathbb{Z}/j\mathbb{Z} \cong \mathbb{Z}_j \).

5. Properties of \( R_{3,2} \)

We noted in Sec. (1) that from an element of \( S \) we may obtain a *triple* over \( W \) consisting of a principal \( S^1 \)-bundle \( p : E \to W \) a class \( b \in H^2(E, \mathbb{Z}) \) and a class \( H \in H^3(E, \mathbb{Z}) \). As in Sec. (3), the map \( W \to (\text{Triples over } W) \) is a set-valued functor on the category of unbased CW complexes. It has a classifying space denoted \( R_{3,2} \). We will show below that this classifying space has a canonical bundle over it which classifies triples.

We noted above that \( R_{3,2} \) is a fiber product \( R_2 \times_{K(\mathbb{Z},2)} R_3 \). As in Sec. (3), this classifying space is unbased. However, one may choose a basepoint for it as in that section. We pick basepoints \( r_i \in R_i \) and a basepoint \( r_{3,2} \in R_{3,2} \) such that \( q_2(r_{3,2}) = r_2 \) and \( q_3(r_{3,2}) = r_3 \). (That is, the \( q_i \) may be taken to be based maps.)

**Theorem 5.1.**

1. The homotopy groups of \( R_{3,2} \) are
   - \( \pi_1(R_{3,2}) \cong \mathbb{Z} \),
   - \( \pi_2(R_{3,2}) \cong \mathbb{Z}^3 \),
   - \( \pi_3(R_{3,2}) \cong \mathbb{Z} \),
   - \( \pi_i(R_{3,2}) \cong 0, i > 3 \).

2. \( R_{3,2} \) is not simple

**Proof.** Given an unbased map \( f : W \to R_{3,2} \), classifying a triple \( ([p], b, H) \) over \( W \), we pick a basepoint \( x_0 \) for \( W \). We use the argument of
Lemma (3.3), to homotope $f$ to a based map. Note that $q_3 \circ f$ is a pair over $W$, the pair $([p], H)$. If $q_3 \circ f$ is nullhomotopic, then $[p] = 0, H = 0$ and hence $b$ defines classes in $H^2(W \times S^1, \mathbb{Z})$ and $H^1(W \times S^1, \mathbb{Z})$. That is, we obtain a based map $W \to K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 1)$ and hence the homotopy fiber of $q_3$ is $K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 1)$. Similarly we see that the homotopy fiber of $q_2$ is $K(\mathbb{Z}, 3) \times K(\mathbb{Z}, 2)$.

(1) We know that $R_{3,2}$ may be taken to be a based fibration of the form

$$K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 1) \to R_{3,2} \to R_3$$

and also a based fibration of the form

$$K(\mathbb{Z}, 3) \times K(\mathbb{Z}, 2) \to R_{3,2} \to R_2.$$  

This gives two long exact sequences for the homotopy groups of $R_{3,2}$: $0 \to \pi_3 \to \mathbb{Z} \to \pi_2 \to \mathbb{Z}^2 \to \mathbb{Z} \to \pi_1 \to 0$ and $0 \to \mathbb{Z} \to \pi_3 \to 0 \to \mathbb{Z} \to \pi_2 \to \mathbb{Z}^2 \to 0 \to \pi_1 \to \mathbb{Z}$. From these sequences it is clear that $\pi_i(R_{3,2}) \simeq 0$ if $i > 3$. Further, from the second sequence $\pi_3 \simeq \mathbb{Z}$, and $\pi_2 \simeq \mathbb{Z}^3$. Then it follows that $\pi_1 \simeq \mathbb{Z}$.

(2) From the long exact sequence of the fibration over $R_2$, we have a $\pi_1(R_2)$ equivariant sequence $0 \to \pi_2(K(\mathbb{Z}, 3) \times K(\mathbb{Z}, 2)) \to \pi_2 \to \mathbb{Z}^2 \to 0 \to \pi_1 \to \mathbb{Z}$. From this, it is clear that the generator of $\pi_1(R_{3,2})$ maps isomorphically to the generator of $\pi_1(R_2)$. In addition, this generator acts on the degree two part of the sequence. Since $K(\mathbb{Z}, 3) \times K(\mathbb{Z}, 2)$ is simply connected and since every action of $\pi_1$ factors through the action of the fundamental group, the generator of $\pi_1(R_{3,2})$ acts trivially on $\pi_2(K(\mathbb{Z}, 3) \times K(\mathbb{Z}, 2))$. In the above sequence, this generator acts nontrivially on $\pi_2(R_2) \simeq \mathbb{Z}^2$ and hence must act nontrivially on $\pi_2(R_{3,2})$ since the sequence is $\pi_1$-equivariant. Hence $R_{3,2}$ is not simple.

We determine the action of $\pi_1(R_{3,2})$ on $\pi_2(R_{3,2})$ below (see Eq. (12) below and surrounding text).

The T-duality map should be a map $T_{3,2} : R_{3,2} \to R_{3,2}$. It might be hoped that this map is a fiber product of two maps $R_3 \to R_3$ and $R_2 \to R_2$. Unfortunately, $T_{3,2}$ cannot be of this form: Note that under T-duality a pair $([p], b)$ maps into a triple $(0, b^\#([p] \times z))$. To determine $T$, we first need to study the structure of $R_{3,2}$. Since $R_{3,2}$ is a fibration over $R_3$, and $R_2$ is a mapping torus (see previous section), it might be hoped that $R_{3,2}$ is also a mapping torus. This is indeed the case.
Lemma 5.1. The universal cover \( \tilde{R}_{3,2} \) of \( R_{3,2} \) is homotopy equivalent to \( R_3 \times K(\mathbb{Z}, 2) \). Hence \( R_{3,2} \) is a mapping torus of a map \( \psi : R_3 \times K(\mathbb{Z}, 2) \to R_3 \times K(\mathbb{Z}, 2) \).

Proof. Let \( p : \tilde{R}_{3,2} \to R_{3,2} \) be the covering projection. Pick \( \tilde{r}_{3,2} \in \tilde{R}_{3,2} \) such that \( p(\tilde{r}_{3,2}) = r_{3,2} \). Consider the map \( \phi : \pi_2(\tilde{R}_{3,2}) \to \pi_2(R_3) \) given by \( \phi = q_2 \circ p_* : \pi_2(\tilde{R}_{3,2}) \to \pi_2(R_3) \). Note that \( p_* \) is an isomorphism by definition. I claim \( \phi \) is onto: We have a fibration \( K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 1) \to R_{3,2} \xrightarrow{\pi_1} R_3 \). The long exact sequence of homotopy groups associated to this fibration is

\[
0 \longrightarrow \mathbb{Z} \xrightarrow{\lambda} \mathbb{Z} \xrightarrow{\mu} \mathbb{Z} \xrightarrow{\kappa} \mathbb{Z}^3 \\
\phi \quad \mathbb{Z}^2 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0.
\]

Note that all the groups in the sequence are torsion-free. Here, \( \lambda \) is an injective map \( \mathbb{Z} \to \mathbb{Z} \) and hence \( \lambda \) is an isomorphism and \( \ker(\mu) \simeq \mathbb{Z} \). Hence by exactness and the absence of torsion \( \mu = 0, \kappa \) is injective and \( \phi \) onto.

Consider the map \( \phi = q_3 \circ p : \tilde{R}_{3,2} \to R_3 \). By the arguments in Lemma (3.3) we may take it to be based. Let \( W \) be its homotopy fiber. Then the l.e.s. of the fibration \( W \to \tilde{R}_{3,2} \to R_3 \) is

\[
0 \longrightarrow \pi_3(W) \xrightarrow{\alpha} \mathbb{Z} \xrightarrow{\beta} \mathbb{Z} \xrightarrow{\gamma} \pi_2(W) \\
\mu \quad \mathbb{Z}^2 \xrightarrow{\kappa} \mathbb{Z}^2 \xrightarrow{\nu} \pi_1(W) \longrightarrow 0 \longrightarrow 0.
\]

We know \( \kappa \) is onto and hence \( \nu = 0 \) and \( \pi_1(W) = 0 \). Since \( \kappa \) is onto, \( \ker(\kappa) \simeq \mathbb{Z} \), and hence, \( \text{im}(\mu) \simeq \mathbb{Z} \).

Now, consider the map \( \phi : \pi_3(\tilde{R}_{3,2}) \to \pi_3(R_3) \) given by \( \phi = q_3 \circ p_* \) in degree 3. Note that \( p_* \) is an isomorphism by definition. By the l.e.s. Eq. (11), \( \beta = \lambda \circ p_* \). By the argument given above both \( \lambda \) and \( p_* \) are isomorphisms, and so is \( \beta \). Hence, \( \text{im}(\alpha) = 0 \) and \( \pi_3(W) \simeq 0 \).

Also \( \ker(\gamma) \simeq \mathbb{Z} \) and so \( \gamma = 0 \). Hence \( \pi_2(W) \simeq \mathbb{Z} \) by counting ranks. Its clear that \( \pi_i(W) \simeq 0 \) if \( i > 3 \). Therefore, \( W \) is homotopy equivalent to \( K(\mathbb{Z}, 2) \).

Now, \( W \to \tilde{R}_{3,2} \to R_3 \) is a based fibration with \( W \simeq K(\mathbb{Z}, 2) \). Also, \( R_{3,2} \) and \( R_3 \) are spaces of finite type since their fundamental groups are finitely generated (by Ref. [24]) and the action of \( \pi_1(R_3) \) on \( W \) is zero since \( \pi_1(R_3) \simeq 0 \). By Lemma 8.28 of Ref. [25], this implies that the fibration is principal. Hence, \( \tilde{R}_{3,2} \simeq K(\mathbb{Z}, 2) \times R_3 \) because by Ref. [2] \( H^3(R_3, \mathbb{Z}) = 0 \) and so the classifying map \( R_3 \to K(\mathbb{Z}, 3) \) is always trivial.
Since $R_3 \times K(\mathbb{Z}, 2)$ is homotopy equivalent to the universal cover of $R_{3,2}$, we have a commutative diagram of spaces

$$
\begin{array}{ccc}
R_3 \times K(\mathbb{Z}, 2) & \xrightarrow{\tilde{q}_2} & K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 2) \\
p_{3,2} & & p_2 \\
\downarrow & & \downarrow \\
R_{3,2} & \xrightarrow{q_2} & R_2 \\
\end{array}
$$

which gives a diagram of homotopy groups and maps equivariant under the action of $\pi_1(K(\mathbb{Z}, 1)) \simeq \mathbb{Z}$. It is clear that since $R_2$ is the mapping torus of $K(\mathbb{Z}^2, 2)$ under the lift of this action, $R_{3,2}$ is the mapping torus of $R_3 \times K(\mathbb{Z}, 2)$ under the lift of this action (by the commutativity of the diagram). The generator of this action acts on $R_3 \times K(\mathbb{Z}, 2)$ as the generator $\psi$ of the deck transformation group.

It is interesting to note that we have a sequence of mapping tori

$$R_{3,2} \rightarrow R_2 \rightarrow K(\mathbb{Z}, 1)$$

such that there are isomorphisms $\pi_1(R_{3,2}) \simeq \pi_1(R_2) \simeq \pi_1(K(\mathbb{Z}, 1))$ and $\pi_1(K(\mathbb{Z}, 1))$ acts on the homotopy groups of each of these spaces.

Note that $\tilde{R}_2$ is a trivial $K(\mathbb{Z}, 2)$ fibration over $K(\mathbb{Z}, 2)$ (since $R_2$ is). Also, $\tilde{R}_3 \simeq R_3$ is a fibration over $K(\mathbb{Z}, 2)$ as well (since $R_3$ is). It then follows from the definition that $\tilde{R}_{3,2}$ is the fiber product of $\tilde{R}_3$ and $\tilde{R}_2$ over $K(\mathbb{Z}, 2)$.

We have a commutative diagram

$$
\begin{array}{ccc}
R_3 \times K(\mathbb{Z}, 2) & \xrightarrow{\tilde{q}_2} & K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 2) \\
p_{3,2} & & p_2 \\
\downarrow & & \downarrow \\
R_{3,2} & \xrightarrow{q_2} & R_2 \\
\end{array}
$$

We note that $\pi_2(R_{3,2}) \simeq \pi_2(\tilde{R}_{3,2})$. We choose generators for $\pi_2(\tilde{R}_{3,2}) \simeq \mathbb{Z}^3$ such that the projection $(a, b, c) \mapsto c$ is induced by the map $R_3 \times K(\mathbb{Z}, 2) \rightarrow K(\mathbb{Z}, 2)$. Also the projection $(a, b, c) \mapsto (a, b)$ is induced by the map $R_3 \times K(\mathbb{Z}, 2) \rightarrow R_3$. With these choices of generators, the map $\tilde{q}_2$ is given by $(a, b, c) \mapsto (a, c)$.

In Sec. (3) we had noted that the deck group of $\tilde{R}_2$ acts on pairs by multiplication by a matrix of the form

$$
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
$$

The map $\tilde{q}_2^*\pi_2$ must be equivariant under the action of the deck group since $q_2^*\pi_1$ is equivariant under the action of $\pi_1$ and the maps $p_{3,2}^*\pi_1$. 

□
and $p_1^*$ are isomorphisms on $\pi_2$. This implies that the deck group acts on $\pi_2(\tilde{R}_{3,2})$ by multiplication by the matrix
\[(12) \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

Using the isomorphisms $p_{3,2}^*$ and $p_2^*$ above, this is also the action of $\pi_1(R_{3,2})$ on $\pi_2(R_{3,2})$. We may also determine the deck transformation explicitly as follows:

**Lemma 5.2.** The deck transformation $\psi : \tilde{R}_{3,2} \to \tilde{R}_{3,2}$ has the form $\psi = \pi_1 \times f$ where $\pi_1 : R_3 \times K(\mathbb{Z}, 2) \to R_3$ is the projection onto the first factor and $f : R_3 \times K(\mathbb{Z}, 2) \to K(\mathbb{Z}, 2)$ is determined below.

**Proof.** Consider a triple over a simply connected space $W$. This determines a classifying map $f : W \to R_{3,2}$. Using the argument of Lemma (3.3) we may take it to be based. Since $W$ is simply connected, this map lifts to $\tilde{R}_{3,2}$. This implies that a triple $(p,b,H)$ over $W$ is determined by a pair $([p],H)$, $p : E \to W$, and a class $b$ in $H^2(W,\mathbb{Z})$. From the Gysin sequence, since $H^1(W,\mathbb{Z}) \cong 0$ we see that $\mathbb{Z} \cong H^0(W,\mathbb{Z}) \xrightarrow{\cup p} H^2(W,\mathbb{Z}) \to H^2(E,\mathbb{Z}) \to 0$ is exact. Thus, every class in $H^2(E,\mathbb{Z})$ is the image of some class in $H^2(W,\mathbb{Z})$ from the Gysin sequence. This is the class $b$ above. Further two such classes differ by an integral multiple of $p$.

It is also clear that if $\psi : \tilde{R}_{3,2} \to \tilde{R}_{3,2}$ is a deck transformation, then, since $p_{3,2} \circ \psi = \psi$, $\psi \circ f$ represents the same triple over $W$. Thus, $\psi \circ f$ is also representable as $((p,H),b')$ for some $b'$. If $\tilde{f}$ is the lift of a map $f : W \to R_{3,2}$ representing a triple $(p,b,H)$ to $\tilde{R}_{3,2}$ then it is clear that all possible lifts may be represented as $((p,H), y + mp), m \in \mathbb{Z}$ where $y \in H^2(W,\mathbb{Z})$ and $p^*(y) = b$. By the above, the action of the deck transformation is to shift $((p,H), y)$ to $((p,H), y + mp)$. Thus, $\psi = \pi_1 \times f$ where $f : R_3 \times K(\mathbb{Z}, 2) \to K(\mathbb{Z}, 2)$ is the map defined by $((p,H), y) \to y + mp, m \in \mathbb{Z}$, and $\pi_1 : R_3 \times K(\mathbb{Z}, 2) \to R_3$ be the projection onto the first factor. \[\qed\]

It is clear that the map $f$ above defines a class in $H^2(R_3 \times K(\mathbb{Z}, 2), \mathbb{Z})$.

As an illustration of this, consider all triples over $S^2$. For any principal circle bundle $E_p \to S^2$, we have that $H^3(E_p,\mathbb{Z}) \cong \mathbb{Z}$. Hence, for each pair $([p],b)$, the $H$-flux can have countably many values.

**Lemma 5.3.** The set of unbased homotopy classes of maps $S^2 \to R_3 \times K(\mathbb{Z}, 2)$ modulo the action of the the deck transformation group on $\tilde{R}_{3,2}$ is exactly the set of triples over $S^2$. 

Proof. Clearly, the set of unbased homotopy classes of maps $S^2 \to R_3 \times K(\mathbb{Z}, 2)$ may be made based since both $R_3$ and $K(\mathbb{Z}, 2)$ are connected and simply connected: This set is exactly $\pi_2(R_3 \times K(\mathbb{Z}, 2))$.

Hence, this set modulo the action of the deck transformation group on $\tilde{R}_{3,2}$ is the same as $\pi_2(\tilde{R}_{3,2})$ modulo the action of $\pi_1(\tilde{R}_{3,2})$. By Thm. (5.1), this is the quotient of $\mathbb{Z}^3$ under the action of the matrix

\[
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

From the previous section (in particular the discussion before and after Thm. (3.9)) we know that the quotient of $\mathbb{Z}^2$ by the matrix

\[
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
\]

is exactly the collection of pairs over $S^2$. The action on $\mathbb{Z}^3$ leaves the last coordinate fixed. Hence, for each isomorphism class of pairs $([p], b)$ over $S^2$, we obtain a countable number of triples as required. \hfill \Box

Note that at least for $S^2$, the only action of the deck group is to shift a pair $([p], H)$ to an equivalent pair.

6. The T-duality Mapping

We note that the maps $q_2 : R_{3,2} \to R_2$ and $q_3 : R_{3,2} \to R_3$ have natural sections. Thus these maps are injective on cohomology. In particular (from the Theorem below) $H^2(R_{3,2}, \mathbb{Z}) \simeq \mathbb{Z}^2 \simeq H^2(R_3, \mathbb{Z}) \simeq \mathbb{Z}a_1 \oplus \mathbb{Z}a_2$. Similarly, $H^2(R_2, \mathbb{Z}) \hookrightarrow H^2(R_{3,2}, \mathbb{Z})$. The latter map may be taken as the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Z} \oplus \mathbb{Z}$ into the first factor. Thus we may write $a_1 = q_2^*(b)$. Further, from the Theorem below, we see that $H^1(R_{3,2}, \mathbb{Z}) \simeq \mathbb{Z}l$. Therefore $l = q_2^*(a)$ and so $a_1 \cdot l = 0$ since $b \cdot a = 0$ in $H^*(R_2, \mathbb{Z})$ (see Eq. (9)).

In addition since $R_{3,2} \to R_3$ is fiber-preserving over $K(\mathbb{Z}, 2)$, and the natural map $R_{3,2} \to K(\mathbb{Z}, 2)$ has sections, we see that $\mathbb{Z}[c] \simeq H^*(K(\mathbb{Z}, 2), \mathbb{Z}) \hookrightarrow H^*(R_{3,2}, \mathbb{Z})$.

**Theorem 6.1.** The cohomology groups of $R_{3,2}$ are

- $H^0(R_{3,2}, \mathbb{Z}) \simeq \mathbb{Z}$,
- $H^1(R_{3,2}, \mathbb{Z}) \simeq \mathbb{Z}l$,
- $H^2(R_{3,2}, \mathbb{Z}) \simeq \mathbb{Z}a_1 \oplus \mathbb{Z}a_2$,
- $H^3(R_{3,2}, \mathbb{Z}) \simeq \mathbb{Z}a_2$,
- $H^4(R_{3,2}, \mathbb{Z}) \simeq \mathbb{Z}a_1^2 \oplus \mathbb{Z}a_2^2 \oplus \mathbb{Z}x$

where $a_1 \cdot a_2 = 0$ and $a_1 \cdot l = 0$. Also, $a_1 = q_2^*(b)$. 

Proof. $R_{3,2}$ is connected, being a fiber product of two connected spaces over a connected space. Hence $H^0(R_{3,2}, \mathbb{Z}) \simeq \mathbb{Z}$. We know that the universal cover of $R_{3,2}$ is $R_3 \times K(\mathbb{Z}, 2)$ with deck transformation group $\mathbb{Z}$. Note that by the Hurewicz theorem, $H^2(\tilde{R}_{3,2}, \mathbb{Z}) \simeq \pi_2(\tilde{R}_{3,2})$. Further, since $\tilde{R}_{3,2}$ is the universal cover, $\pi_2(\tilde{R}_{3,2}) \simeq \pi_2(R_{3,2}) \simeq \mathbb{Z}^3$. We calculated the action of $\pi_1(R_{3,2})$ on $\pi_2(R_{3,2})$ above. Using this we see that the action of $\pi_1(R_{3,2})$ on $H^2(\tilde{R}_{3,2}, \mathbb{Z})$ is multiplication by the same matrix. The cohomology ring of $\tilde{R}_{3,2}$ is given by

- $H^0(\tilde{R}_{3,2}, \mathbb{Z}) \simeq \mathbb{Z}$,
- $H^1(\tilde{R}_{3,2}, \mathbb{Z}) \simeq 0$,
- $H^2(\tilde{R}_{3,2}, \mathbb{Z}) \simeq (\mathbb{Z}a_1 \oplus \mathbb{Z}a_2) \oplus \mathbb{Z}c$,
- $H^3(\tilde{R}_{3,2}, \mathbb{Z}) \simeq 0$,
- $H^4(\tilde{R}_{3,2}, \mathbb{Z}) \simeq \mathbb{Z}a_1^2 \oplus \mathbb{Z}a_2^2 \oplus \mathbb{Z}a_1c \oplus \mathbb{Z}a_2c \oplus \mathbb{Z}c^2$.

The action of the deck group on $H^2$ may be used to calculated the action of the deck group on $H^4$. This is multiplication by the matrix (in the basis $(a_1^2, a_2^2, a_1c, a_2c, c^2)$)

$$
\phi = \begin{pmatrix}
1 & 0 & 2 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

We apply the Cartan-Leray spectral sequence to this universal cover to compute the cohomology groups of $R_{3,2}$. Since $\mathbb{Z}$ is cohomologically one dimensional, the sequence collapses at the $E_2$ page itself. In particular only $E_2^{2,1}$ and $E_2^{2,2}$ are nonzero. We find that $H^0(\mathbb{Z}, H^q(\tilde{R}_{3,2}, \mathbb{Z})) \simeq H^q(\tilde{R}_{3,2}, \mathbb{Z})$ and $H^1(\mathbb{Z}, H^q(\tilde{R}_{3,2}, \mathbb{Z})) \simeq H^q(\tilde{R}_{3,2}, \mathbb{Z})/(\phi-1)H^q(\tilde{R}_{3,2}, \mathbb{Z})$. Using this action we obtain the cohomology groups shown.

Here $a_i$ are the pullbacks of the generator of $H^2(R_3, \mathbb{Z})$ and so $a_1 \cdot a_2 = 0$. In addition $l$ is the pullback of the generator $a$ of $H^1(R_2, \mathbb{Z})$ via $q_2^*$. Also, $a_1 = q_2^*(b)$ and so $^{20}a_1 \cdot l = q_2^*(b \cdot a) = 0$. Also, $x$ is a new generator in degree 3.

Let $p_3 : E_3 \to R_3$ be the classifying bundle $E$ of Ref. [2] Sec. (2.4). By Sec. (3), there is a classifying bundle $p_2 : E_2 \to R_2$. By the isomorphisms on $H^2$ discussed above, it is clear that $q_2^*E_2 \simeq q_3^*E_3$ and we denote this bundle by $E_{3,2}$. This bundle has characteristic class $a_1 \in H^2(R_{3,2}, \mathbb{Z})$. Let $p : E_{3,2} \to R_{3,2}$ be the bundle map.

**Theorem 6.2.** The cohomology groups of $E_{3,2}$ are

---

[^20]: See Eq. (9).
\begin{itemize}
  \item $H^0(E_{3,2}, \mathbb{Z}) \simeq \mathbb{Z}$,
  \item $H^1(E_{3,2}, \mathbb{Z}) \simeq \mathbb{Z}y$,
  \item $H^2(E_{3,2}, \mathbb{Z}) \simeq \mathbb{Z}p^*(a_2) \oplus \mathbb{Z}b$,
  \item $H^3(E_{3,2}, \mathbb{Z}) \simeq \mathbb{Z}p^*(a_2l) \oplus \mathbb{Z}h$.
\end{itemize}

Here $y = p^*(l)$ so $p_l(y) = 0$, $p_l(b) = l$ and $p_l(h) = a_2$.

\textbf{Proof.} Consider the Gysin sequences associated to $E_{3,2}, E_3$ and $E_2$. By naturality, there are morphisms from the sequences associated to $E_3$ and $E_2$ to the sequence associated to $E_{3,2}$ induced by the maps $q_2, q_3$. Further, the maps $q_i$ have natural sections and hence $q_i^*$ are injective on cohomology.

\begin{itemize}
  \item **Degree 0** $E_{3,2}$ is a fibration of a connected space ($S^1$) over a connected base ($R_{3,2}$). Hence it is connected and $H^0(E_{3,2}, \mathbb{Z}) \simeq \mathbb{Z}$.
  \item **Degree 1** From the Serre spectral sequence, we find that $H^1(E_{3,2}, \mathbb{Z}) \simeq \mathbb{Z}y$. From the Gysin sequence for $p : E_{3,2} \to R_{3,2}$ we have
    \[ \mathbb{Z} \xrightarrow{p^*} \mathbb{Z}y \to \mathbb{Z}. \]
    Now, since $p^*$ is injective, by exactness, and the absence of torsion $y = p^*(l)$ and $p_l(y) = 0$.
  \item **Degree 2** From the Gysin sequence associated to $p : E_{3,2} \to R_{3,2}$ beginning at $H^0(R_{3,2}, \mathbb{Z})$ we have
    \[ \mathbb{Z} \xrightarrow{\phi_1=\cup a_1} \mathbb{Z}a_1 \oplus \mathbb{Z}a_2 \xrightarrow{p^*} H^2(E_{3,2}, \mathbb{Z}) \xrightarrow{p_l} \mathbb{Z}l \xrightarrow{\cup a_2} \mathbb{Z}a_2 l \]
    From the previous part of the sequence $\phi_1$ is injective. Therefore, by exactness $H^2(E_{3,2}, \mathbb{Z}) \simeq \mathbb{Z}p^*(a_2) \oplus \mathbb{Z}h$. Note that the generator $w$ of $H^2(E_2, \mathbb{Z})$ pulls back to $H^2(E_{3,2}, \mathbb{Z})$ as a generator $b$. Then, we have $p_l(b) = l \in H^1(R_{3,2}, \mathbb{Z})$. Under pullback by a classifying map $W \to R_{3,2}$, $b$ pulls back to the $B$-class of the triple $([p], b, H)$ being classified.
  \item **Degree 3** Consider the Gysin sequence for $p : E_{3,2} \to R_{3,2}$
    \[ \mathbb{Z}l \xrightarrow{\cup a_2} \mathbb{Z}a_2 l \xrightarrow{p^*} H^3(E_{3,2}, \mathbb{Z}) \xrightarrow{p_l} \mathbb{Z}a_1 \oplus \mathbb{Z}a_2 \xrightarrow{\cup a_1} \mathbb{Z}a_1^2 \oplus \mathbb{Z}a_2^2 \oplus \mathbb{Z}x. \]
    We have that $a_1 \cdot l = 0$ (see Eq. (9)) hence $p^*$ is injective, also the last map has kernel $\mathbb{Z}a_2 \subset \mathbb{Z}a_1 \oplus \mathbb{Z}a_2$. By exactness, $H^3(E_{3,2}, \mathbb{Z}) \simeq p^*(a_2l) \oplus \mathbb{Z}h$. Note that $H^3(E_{3,2}, \mathbb{Z})$ contains a $\mathbb{Z}$-subgroup which is $\text{im}(\lambda_2) \simeq H^3(E_3, \mathbb{Z}) \simeq h\mathbb{Z}$ such that $p_l(h) = a_2$. This class pull back along the classifying map $W \to R_{3,2}$ to the $H$-flux of the triple being classified.
\end{itemize}
Thus over $R_{3,2}$ we have the canonical triple $([p], b, h)$ corresponding to the bundle $E_{3,2}$ above.

**Lemma 6.1.** The pullback of the fibration $q_3 : R_{3,2} \to R_3$ along the T-duality map $T_3 : R_3 \to R_3$ is $R_{3,2}$.

**Proof.** We have a commutative diagram

$$T_{3,2}^*R_{3,2} \xrightarrow{T_{3,2}} R_{3,2}$$

$$\downarrow q_3 \downarrow q_3$$

$$R_3 \xrightarrow{T_3} R_3$$

where we define $T_{3,2}$ as the map on $R_{3,2}$ induced by the pullback and, by definition,

$$T_{3,2}^*R_{3,2} = \{(x, y) \in R_3 \times R_{3,2} | T_3(x) = q_3(y)\}.$$

Given $f : W \to T_{3,2}^*R_3$, $f(w) = (f_1(w), f_2(w))$ such that $T_3 \circ f_1 = q_3 \circ f_2$, we get $T_3 \circ f_1 : W \to R_3$ and $q_2 \circ f_2 : W \to R_2$. Obviously these maps define pairs $([p], b)$ and $([p], H)$ with the same $[p]$. Conversely, given two such pairs, we obtain maps $h_1 : W \to R_3$, $h_2 : W \to R_2$, such that $\mu \circ h_1 = \mu \circ h_2$ where $\mu : R_3 \to K(\mathbb{Z}, 2)$ is the map sending a pair to the class of the principal bundle in the pair. We define $g_1 = T_3^{-1} \circ h_1 : W \to R_3$. By the universal property of the fiber product, there is a map $g_2 : W \to R_{3,2}$. Then $q_3 \circ g_2 = h_1 = T_3 \circ g_1$ by definition. Hence, $(g_1, g_2)$ give a well-defined map $W \to T_{3,2}^*R_{3,2}$. Therefore $T_{3,2}^*R_{3,2}$ also classifies triples of the form $([p], b, H)$ since a 2-pair and a 3-pair with the same $[p]$ define and are defined by the same triple. Hence, by uniqueness of the classifying space of triples, we get a homeomorphism $\phi : T_{3,2}^*R_{3,2} \to R_{3,2}$. \qed

Note that if $X$ is a CW-complex, and $f : X \to R_{3,2}$ is a classifying map which corresponds to a triple $(p, b, H)$ on $X$ the pair associated to this triple is $(p, H)$. The triple associated to $T_{3,2} \circ f$ is $(p^\#, b^\#, H^\#)$ and the associated pair is $(p^\#, b^\#)$. Thus, we see that $q_2 \circ T_{3,2} \neq q_2$. However, we always require $q_2 = q_2 \circ \phi$, i.e., we require that $a_1 = q_2(b)$ in $H^*(R_{3,2}, \mathbb{Z})$ and in $H^*(T^*R_{3,2}, \mathbb{Z})$ as well. In addition we require that $a_1 \cdot \lambda = a_1 \cdot a_2 = 0$ with no such relation on $a_2$ in both spaces.

Note that $q_3 \circ T_{3,2} = T_3 \circ q_3$ and hence $T_{3,2}^*$ acts on $H^2(R_{3,2}, \mathbb{Z})$ by interchanging $a_1$ and $a_2$.

**Lemma 6.2.** Let $\hat{E}_{3,2}$ be the bundle over $R_{3,2}$ of characteristic class $a_2 \in H^2(R_{3,2}, \mathbb{Z})$. The cohomology groups of $\hat{E}_{3,2}$ are

- $H^0(\hat{E}_{3,2}, \mathbb{Z}) \simeq \mathbb{Z}$,
Consider the Gysin sequence for \( \hat{E}_{3,2} \) beginning at \( H^1(\hat{E}_{3,2}, \mathbb{Z}) \)

\[
\begin{align*}
H^1(\hat{E}_{3,2}, \mathbb{Z}) &\simeq \mathbb{Z}y, \\
H^2(\hat{E}_{3,2}, \mathbb{Z}) &\simeq \mathbb{Z}p^*(a_1), \\
H^3(\hat{E}_{3,2}, \mathbb{Z}) &\simeq \mathbb{Z}\hat{h},
\end{align*}
\]

and \( \hat{p}_1(\hat{y}) = 0, \hat{y} = \hat{p}^*(l), \hat{p}_1(\hat{h}) = a_1 \).

**Proof.** The calculation for \( H^0 \) and \( H^1 \) is exactly the same as for \( H^0(E_{3,2}, \mathbb{Z}) \) and \( H^1(E_{3,2}, \mathbb{Z}) \) with \( y \) replaced by \( \hat{y} \).

- **Degree 2** Consider the Gysin sequence for \( \hat{E}_{3,2} \) beginning at \( H^0(\hat{E}_{3,2}, \mathbb{Z}) \)

\[
\begin{align*}
\mathbb{Z} &\xrightarrow{\phi_1 = \cup a_2} \mathbb{Z}a_1 \oplus \mathbb{Z}a_2 \xrightarrow{\hat{p}^*} H^2(\hat{E}_{3,2}, \mathbb{Z}) \xrightarrow{\hat{p}_1} \mathbb{Z}l \xrightarrow{\cup a_2} \mathbb{Z}a_2 l
\end{align*}
\]

The map \( \phi_1 \) is injective because of the previous part of the sequence. The last map is injective and surjective since \( a_2 \cdot l \neq 0 \) because we chose \( q_2 = q_2 \circ \phi \) above\(^{21}\). Hence, by exactness, the map \( \hat{p}_1 \) is zero and \( H^2(\hat{E}_{3,2}, \mathbb{Z}) \simeq \mathbb{Z}\hat{p}^*(a_1) \).

- **Degree 3** Consider the Gysin sequence for \( \hat{E}_{3,2} \) beginning at \( H^1(\hat{E}_{3,2}, \mathbb{Z}) \)

\[
\begin{align*}
\mathbb{Z}l &\xrightarrow{\phi_1 = \cup a_2} \mathbb{Z}a_2 l \xrightarrow{\hat{p}^*} H^3(\hat{E}_{3,2}, \mathbb{Z}) \xrightarrow{\hat{p}_1} \mathbb{Z}a_1 \oplus \mathbb{Z}a_2 \xrightarrow{\cup a_2} \mathbb{Z}a_1^2 \oplus \mathbb{Z}a_2^2 \oplus \mathbb{Z}x
\end{align*}
\]

The map \( \phi_1 \) is an isomorphism and hence \( \hat{p}^* = 0 \). By exactness, \( H^3(\hat{E}_{3,2}, \mathbb{Z}) \simeq \mathbb{Z}\hat{h} \) and \( \hat{p}_1(\hat{h}) = a_1 \).

\( \square \)

**Theorem 6.3.**

1. The bundle \( \hat{p} : \hat{E}_{3,2} \to R_{3,2} \) described above and cohomology classes \( \hat{b} \in H^2(\hat{E}_{3,2}, \mathbb{Z}) \) and \( \hat{h} \in H^2(\hat{E}_{3,2}, \mathbb{Z}) \) are such that \( T_{3,2} \) classifies the triple \( ([\hat{p}], \hat{b}, \hat{h}) \) over \( R_{3,2} \).

2. Let \( \tilde{T} : E_{3,2} \to E_{3,2} \) be the map covering \( T_{3,2} : R_{3,2} \to R_{3,2} \), then \( \tilde{T} \) acts on the generators of the low-dimensional cohomology of \( H^*(E_{3,2}, \mathbb{Z}) \) as follows

\[
\begin{align*}
\tilde{T}^*(y) &= 0  \\
\tilde{T}^*(p^*(a_2)) &= \hat{p}^*(a_1)  \\
\tilde{T}^*(p^*(a_2l)) &= 0  \\
\tilde{T}^*(\hat{h}) &= \hat{h}
\end{align*}
\]

- The \( T \)-dual of \( b \) is of the form \( kp^*(a_1) \) for some \( k \). Hence, the \( T \)-dual of a triple \( (p, b, H) \) is a triple of the form \( (\hat{p}, p^*(y), \hat{H}) \).

3. \( T_{3,2}^2 \) cannot be homotopic to the identity, i.e., \( T \)-duality for triples is not involutive.

\(^{21}\)See paragraph after the proof of the previous Lemma.
Proof.  

(1) We consider the triple \((\tilde{\mathcal{p}}, \hat{b}, \hat{h})\) over \(R_{3,2}\). This triple is classified by a map \(f : R_{3,2} \rightarrow R_{3,2}\). Then \(f^*(a_1) = a_2\). It is clear that \(f^*E_{3,2} = \tilde{E}_{3,2}\). In addition if we forget the \(B\)-class, the action of \(f\) on the pair \([(p), h]\) is to transform it into \((\tilde{\mathcal{p}}, \hat{h})\). This implies that \(q_3 \circ f = T_3 \circ q_3\). Hence \(f\) is the homotopic to \(T_{3,2}\) by the uniqueness of the pullback square described in Thm. (6.1).

(2) Now, in the above two sequences, \(T_{3,2}^*\) induces a natural map \((a_1, a_2) \rightarrow (a_2, a_1)\) on \(H^2(R_{3,2}, \mathbb{Z})\), further the induced maps on the remaining cohomology groups except \(H^2(E_{3,2}, \mathbb{Z})\) are the identity.

We have that \(\hat{\mathcal{p}}^* \circ T_{3,2}^* = \tilde{T}^* \circ p^*\). As a result, \(\tilde{T}(p^*(a_2)) = \hat{\mathcal{p}}^*(a_1)\). Also, we have \(T_{3,2}^* \circ p = \hat{\mathcal{p}} \circ \tilde{T}^*\). Hence \(\hat{\mathcal{p}} \circ \tilde{T}^*(b) = T_{3,2}^*(l)\).

However, from the Gysin sequence for \(\tilde{E}_{3,2}\), we have

\[
\mathbb{Z}\hat{\mathcal{p}}^*(a_1) \xrightarrow{\hat{\mathcal{p}}} \mathbb{Z}l \xrightarrow{\cup a_2} \mathbb{Z}a_2l \rightarrow \mathbb{Z}h.
\]

Now the map \(H^1(R_{3,2}, \mathbb{Z}) \rightarrow H^3(R_{3,2}, \mathbb{Z})\) is an isomorphism and hence \(\hat{\mathcal{p}}^*\) is zero. As a result, \(T_{3,2}^*(l) = 0\), and hence \(\tilde{T}^*(y) = 0\).

Consider the above pair of Gysin sequences beginning at \(H^3(R_{3,2}, \mathbb{Z})\): We have for \(E_{3,2}\)

\[
\mathbb{Z}l \xrightarrow{\cup a_2} \mathbb{Z}a_2l \xrightarrow{p^*} \mathbb{Z}p^*(a_2l) \oplus \mathbb{Z}h \xrightarrow{\hat{\mathcal{p}}} \mathbb{Z}a_1 \oplus \mathbb{Z}a_2 \xrightarrow{\cup a_2} \mathbb{Z}a_1^2 \oplus \mathbb{Z}a_2^2 \oplus \mathbb{Z}x
\]

and for \(\tilde{E}_{3,2}\)

\[
\mathbb{Z}l \xrightarrow{\cup a_2} \mathbb{Z}a_2l \xrightarrow{\tilde{T}^*} \mathbb{Z}h \xrightarrow{\hat{\mathcal{p}}} \mathbb{Z}a_1 \oplus \mathbb{Z}a_2 \xrightarrow{\cup a_2} \mathbb{Z}a_1^2 \oplus \mathbb{Z}a_2^2 \oplus \mathbb{Z}x
\]

From the above action of \(T_{3,2}^*\) on \(H^2(R_{3,2}, \mathbb{Z})\) it is clear that \(\tilde{T}^*(h) = \hat{h}\). Also, since \(p^*\) is an isomorphism and \(\hat{\mathcal{p}}^*\) is zero, it is clear that \(\tilde{T}^*(p^*(a_2l)) = 0\). From the above, we have

- \(\tilde{T}^*(y) = 0\)
- \(\tilde{T}^*(p^*(a_2)) = \hat{\mathcal{p}}^*(a_1)\)
- \(\tilde{T}^*(p^*(a_2l)) = 0\)
- \(\tilde{T}^*(h) = \hat{h}\).

This is clear since, by the above, we know that \(\hat{\mathcal{p}}(\tilde{T}(b)) = 0\). The result for the form of the T-dual triple is obvious.

(3) From the above, it is clear that

- \(T_{3,2}^*(l) = 0\)
- \(T_{3,2}^*(a_1, 0) = (0, a_2)\) and \(T_{3,2}^*(0, a_2) = (a_1, 0)\).
- \(T_{3,2}^*(a_2l) = 0\)

Therefore \(T_{3,2}^*\) cannot be homotopic to the identity.

\(\square\)
Theorem 6.4.  
(1) There is a natural map \( \gamma : R_3 \to R_{3,2} \) and hence there is a natural triple associated to a pair \((p, H)\) over any space, namely the triple \((p, p^*(pH), H)\).

(2) The T-dual of the triple in the previous item is the triple \((\hat{p}, \hat{p^*}(\hat{pH}), \hat{H})\), that is, \(T_{3,2} \circ \gamma = \gamma \circ T_3\).

(3) Each class \(\alpha\) in \(H^2(K(\mathbb{Z}^2, 2), \mathbb{Z})\) gives a map \(\gamma_\alpha : R_3 \to R_{3,2}\). Each of these maps is an inverse to the forgetful map \(R_{3,2} \to R_3\) which ‘forgets’ the class \(b\).

Proof.  
(1) Note that we have a natural map \(R_3 \to K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 2)\) which is given by sending \((p, H)\) to \((p, p_0(H))\). We have a natural identification (up to homotopy) of \(K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 2)\) with \(R_2\) and, as a result, there is a natural map \(\gamma : R_3 \to R_2\). We may naturally associate to any 3-pair \((p, H)\) over a space \(X\) a 2-pair \((p, p^*pH)\) over that space. Note that the map \(\gamma\) commutes with the natural maps \(\mu : R_3 \to K(\mathbb{Z}, 2)\) and \(\nu : R_2 \to K(\mathbb{Z}, 2)\). Hence, there is a natural lift \(\eta\) of \(\gamma\) to \(R_{3,2}\) given by \(\eta : R_3 \to R_{3,2}\) such that \(\eta(a) = (a, \gamma(a)) \in R_3 \times R_2\). This gives a natural triple associated to any pair \((p, H)\) : To \((p, H)\) we associate \((p, p^*pH, H)\).

(2) We know that under T-duality, the bundle \(E_3\) on \(R_3\) is mapped to the bundle \(\hat{E}_3\) on \(R_3\). If we examine the cohomology of these bundles, we see that the T-dual of the triple \((p, p^*pH, h)\) should be a triple of the form \((q, k, q^*q^*h, \hat{h})\), \(k \in \mathbb{Z}\). By the above theorem, the T-dual is exactly the triple \((q, q^*q^*h, \hat{h})\). This implies that the map \(\gamma\) commutes with \(T_{3,2}\) and \(T_3\). That is, \(T_{3,2} \circ \gamma = \gamma \circ T_3\).

(3) Note that each map \(f : K(\mathbb{Z}^2, 2) \to K(\mathbb{Z}^2, 2)\) up to homotopy gives a lift of the form above, if the map is such that \(f(a, b) = (a, \alpha(a, b))\). Thus, each element \(\alpha\) in \(H^2(K(\mathbb{Z}^2, 2), \mathbb{Z})\) gives such a lift. It is clear that each of these lifts correspond to triples of the form \((p, k, p^*(a_1), h)\), \(k \in \mathbb{Z}\) on \(R_3\). Note that each of these lifts is an inverse to the forgetful map \(R_{3,2} \to R_3\) which ‘forgets’ the element \(b\) of a triple.

\[\square\]

We may now prove the result we conjectured in Sec. (4):

Corollary 6.1. Let \(W\) be a CW-complex and \(p : E \to W\) a principal \(S^1\)-bundle over \(W\). Let \(b \in H^2(E, \mathbb{Z})\) and \(H \in H^3(E, \mathbb{Z})\) be the B-class and H-flux on the bundle. Let \([(q), b\# , H\# )\] be the T-dual triple. Topological T-duality induces a bijection between the sets \(\{b + lp^*_p(H)\}\) and \(\{b\# + mq^*_q(H\# )\}\).
Proof. Consider the universal bundle $E_{3,2}$ over $R_{3,2}$ and the T-dual bundle $\hat{E}_{3,2}$ over $R_{3,2}$. The result is obvious for the universal bundle since $b^\#$ is always of the form $k \cdot q^* q(H^\#), k \in \mathbb{Z}$. By pullback, the result is true for $X$. 

We now return to the question raised in the paragraph after Lemma (1.1) in Sec. (1). By Lemma (1.1) and Sec. (2), for a given space $X$, at fixed $H$ this is equivalent to knowing the image of the map $T_{3,2}$ studied above. The above Corollary indicates that all automorphisms with Phillips-Raeburn invariant in a given coset map into automorphisms with Phillips-Raeburn invariant in another coset. This constrains the map and determines it in a large variety of cases as was seen previously in Sec. (4).

7. ACKNOWLEDGEMENTS

I thank Professor Jonathan Rosenberg, University of Maryland, College Park, for encouraging me to work on this problem and for many useful discussions. I acknowledge financial aid from a Research Assistantship supported by the NSF Grant DMS-0504212 during a portion of this work.

I thank Professor Peter Bouwknegt, ANU, for useful discussions on the physical meaning of the triples studied here. I acknowledge financial aid from a Postdoctoral Fellowship supported by the ARC Discovery Project ‘Generalized Geometries and their Applications’ during a portion of this work.

I thank the Department of Mathematics, Harish-Chandra Research Institute, Allahabad, for support from a postdoctoral fellowship during a portion of this work.

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E-mail address: ashwin@hri.res.in