The Relevant Operators for the generalized time-dependent m-photon Jaynes-Cummings Hamiltonian

S. Alam, C. Bentley

Theory Group, KEK, Tsukuba, Ibaraki 305, Japan

Abstract

The m-photon Jaynes-Cummings Hamiltonian is a natural generalization of the much studied Jaynes-Cummings Hamiltonian. In this short note we give the relevant operators for the time-dependent generalized m-photon Jaynes-Cummings Hamiltonian. The dynamical equations for these operators are also given. These operators are needed and indeed are the basic building blocks for performing calculations in the context of the Maximum Entropy Formalism.

*Permanent address: Department of Physics, University of Peshawar, Peshawar, NWFP, Pakistan.

†Prairie View Texas A & M University, Texas, USA.
I. INTRODUCTION

In a series of papers [1–3], the generalized time-dependent Jaynes-Cummings Hamiltonian in the context of Maximum Entropy Principle [MEP] and group theory based methods [4] was studied. In particular, in [1] the MEP formalism was used to solve time-dependent N-level systems. A set of generalized Bloch equations, in terms of relevant operators was obtained and as an example the $N = 2$ case was solved. It was thus demonstrated in [1] that the dynamics and thermodynamics of a two-level system coupled to a classical field can be fully described in the framework of MEP and group theory based methods. Further in [2] a time-dependent generalization of the JCM was studied and by showing that the initial conditions of the operators are determined by the MEP density matrix the authors were able to demonstrate that inclusion of temperature turns the problem into a thermodynamical one. An exact solution was also presented in the time independent case. Finally in [3] more detailed analysis of the three set of relevant operators was given. These set of operators are related to each other by isomorphisms which allowed the authors to consider the case of mixed initial conditions.

As is well-known when we consider a quantum two-level system interacting with a single mode of quantized field one is led to the familiar Jaynes-Cummings Hamiltonian [JCH] [5] provided one is interested only in the difference of the population of the two levels. The JCH has been extensively used as a model Hamiltonian in fields such as quantum optics, nuclear magnetic resonance, and quantum electronics. A interesting study of the JCH is the periodic spontaneous collapse and revival due to quantum granularity of the field [6]. In the rotating wave approximation [RWA], the JCH becomes solvable and it has been broadly used in the last years [7–12].

The mean values of the field’s population, correlation functions and $n$th-order coher-
ence functions are of interest and useful in several applications. The MEP formalism allows us to describe a Hamiltonian system in terms of those, and only those, quantum operators relevant to the problem at hand. Thus, this formalism is suitable to study the Hamiltonian given in [2,3]. In [4,5] the population of each level and not their difference is considered therefore the resulting Hamiltonian is called a generalized time-dependent JCH.

Our aim is to consider the m-photon generalized time-dependent JCH. The m-photon JCH describes the interaction of a two-level system with a single quantized mode of electromagnetic field via m photon emission and absorption processes between the two-levels. A couple of forms of the m-photon JCH have been suggested [13,14] one of them being intensity dependent. This simply means that the coupling of this m-photon JCH is proportional to the square root of the number operator for the photons. As the validity of this effective Hamiltonian may be questionable under certain circumstances [15], we do not consider it in this paper. We consider the time-dependent generalized version of the simpler m-photon JCH in this paper. The question arises does at least one set of relevant operators exist for the m-photon JCM? The aims of this short note is to give such a set and to give the evolution equations for it. The layout of this paper is as follows. In the next section we recall some well-known results of the group theory based MEP formalism. In section two we give the relevant operators and the evolution equations for their expectation values i.e. we use generalized Ehrenfest theorem to obtain Bloch equations.

II. SUMMARY OF THE MEP FORMALISM

It is instructive to summarize the principal concepts of the MEP [1,3,16,17]. Given the expectation values $\langle \hat{O}_j \rangle$ of the operators $\hat{O}_j$, the statistical operator $\hat{\rho}(t)$ is defined by

$$\hat{\rho}(t) = \exp \left( -\lambda_0 \hat{I} - \sum_{j=1}^{L} \lambda_j \hat{O}_j \right),$$

(1)
where \( L \) is a natural number or infinity, and the \( L+1 \) Lagrange multipliers \( \lambda_j \), are determined to fulfill the set of constraints

\[
<\hat{O}_j> = \text{Tr} \left[ \hat{\rho}(t) \hat{O}_j \right], \quad j = 0, 1, \ldots, L ,
\]

(\( \hat{O}_0 = \hat{I} \) is the identity operator) and the normalization in order to maximize the entropy, defined (in units of the Boltzmann constant) by

\[
S(\hat{\rho}) = -\text{Tr} \left[ \hat{\rho} \ln \hat{\rho} \right].
\]

Eq. (1) is a generalization of the more familiar density operator. For e.g. in open system, where we have Grand Canonical Ensemble there are two Lagrange multipliers, \( \beta = \frac{1}{k_B T} \) and \( \mu \) are present, and we write the density operator as

\[
\hat{\rho}(t) = \exp \left( \beta \Omega(T, V, \mu) - \beta \hat{H} + \beta \mu \hat{N} \right),
\]

As is well-known the dynamics are governed by the time evolution of the statistical operator. The time evolution of the statistical operator is given by

\[
i\hbar \frac{d\hat{\rho}}{dt} = [ \hat{H}(t), \hat{\rho}(t) ].
\]

The essence of the MEP formalism in conjunction with the group theory method is to find the relevant operators entering Eq. (1) so as to guarantee not only that \( S \) is maximum, but also is a constant of motion. Introducing the natural logarithm of Eq. (1) into Eq. (3) it can be easily verified that the relevant operators are those that close a semi-Lie algebra under commutation with the Hamiltonian \( \hat{H} \), i.e.

\[
[ \hat{H}(t), \hat{O}_j ] = i\hbar \sum_{i=0}^{L} g_{ij}(t) \hat{O}_i.
\]

Thus the relevant operators may be defined as those satisfying the above equation. Eq. (6) defines an \( L \times L \) matrix \( G \) and constitutes the central requirement to be fulfilled by the
operators entering in the density matrix. The Liouville Eq. [3] can be replaced by a set of coupled equations for the mean values of the relevant operators or the Lagrange multipliers as follows [19]:

\[
\frac{d}{dt} \langle \hat{O}_j \rangle_t = -\sum_{i=0}^{L} g_{ij} \langle \hat{O}_i \rangle, \quad j = 0, 1, \ldots, L, \tag{7}
\]

\[
\frac{d\lambda_j}{dt} = \sum_{i=0}^{L} \lambda_i g_{ji}, \quad j = 0, 1, \ldots, L. \tag{8}
\]

In the MEP formalism, the mean value of the operators and the Lagrange multipliers belongs to dual spaces which are related by [17]

\[
\langle \hat{O}_j \rangle = -\partial\lambda_0 \over \partial\lambda_j. \tag{9}
\]

III. THE RELEVANT OPERATORS AND EVOLUTION EQUATIONS

The generalized time-dependent m-photon JCH in the RWA takes the form

\[
\hat{H} = E_1 \hat{b}^\dagger_1 \hat{b}_1 + E_2 \hat{b}^\dagger_2 \hat{b}_2 + \omega \hat{a}^\dagger \hat{a} + T(t) \left( \gamma \hat{a}^m \hat{b}^\dagger_1 \hat{b}_2^\dagger + \gamma^* \hat{b}^\dagger_2 \hat{b}_1^\dagger \hat{a}^m \right), \tag{10}
\]

\( (\hbar = 1) \), where \( \gamma \) is the coupling constant between the system and the external field, \( E_j \) and \( \omega \) are the energies of the levels and the field, respectively, \( \hat{a}^\dagger, \hat{a} \), are boson operators, \( \hat{b}^\dagger_j \) and \( \hat{b}_j \) are fermion operators and \( T(t) \) is an arbitrary function of time. On setting \( m = 1 \) we recover the generalized time-dependent JCH of Gruver et al., [2].

Working in the context of the generalized time-dependent JCH Gruver et al., [2] found that the relevant operators can be presented in three different but equivalent forms, each of them having different physical interpretations. These sets are connected via isomorphisms which allows one to go from one set into another. As noted in [2] the advantage of this multiple representation comes from the fact that when partial information in any set is
known, for instance the initial values only for some operators are known, it is possible to complete the missing information via the isomorphisms, if the complementary data in any other set is known, i.e. mixed initial conditions [17]. We find that working in the framework of the Hamiltonian 10 the same arguments go through. From the form of the Hamiltonian 10, we can guess by looking at the level’s population and the structure of the interaction terms that a basic set of relevant operators satisfying Eq. 6 is

\[ \hat{N}_1 = \hat{b}_1^\dagger \hat{b}_1, \]

\[ \hat{N}_2 = \hat{b}_2^\dagger \hat{b}_2, \]

\[ \hat{\Delta} = \hat{a}^\dagger \hat{a}, \]

\[ \hat{I}^m = \gamma \hat{a}^m \hat{b}_1 \hat{b}_2^\dagger + \gamma^* \hat{b}_2 \hat{b}_1^\dagger \hat{a}^\dagger m, \]

\[ \hat{F}^m = i(\gamma \hat{a}^m \hat{b}_1 \hat{b}_2^\dagger - \gamma^* \hat{b}_2 \hat{b}_1^\dagger \hat{a}^\dagger m), \]

\[ \hat{N}_{2,1} = \hat{b}_2^\dagger \hat{b}_2 \hat{b}_1^\dagger \hat{b}_1. \] (16)

These fundamental operators appear in the three possible sets of relevant operators outlined below. The above operators possess a simple physical interpretation. \( \hat{N}_1 \) is the number or population operator for level one, \( \hat{N}_2 \) is the number operator for level two. \( \hat{\Delta} \) is the familiar number operator for the photon/external field. The operator \( \hat{I}^m \) may be considered as representing the interaction energy between the levels and the external the field. The particle’s current between levels is governed by \( \hat{F}^m \). \( \hat{N}_{2,1} \) is the double occupation number operator. The operators 11, 12 and 14, 15 can be considered as the m-photon quantum counter parts of the operators obtained for the semiclassical 2-level system studied in ref. 1.

The simplest set which closes a semi-Lie algebra with the Hamiltonian, Eq. 6, is found to be,

\[ \hat{N}_1^n = (\hat{a}^\dagger)^n \hat{N}_1 (\hat{a})^n, \] (17)
\[ \hat{N}^n_2 = (\hat{a}^\dagger)^n \hat{N}_2 (\hat{a})^n, \] (18)

\[ \hat{\Delta}^n = (\hat{a}^\dagger)^n \hat{\Delta} (\hat{a})^n, \] (19)

\[ \hat{I}^{n,m} = (\hat{a}^\dagger)^n \hat{I}^m (\hat{a})^n, \] (20)

\[ \hat{F}^{m,n} = (\hat{a}^\dagger)^n \hat{F}^m (\hat{a})^n, \] (21)

\[ \hat{N}^n_{2,1} = (\hat{a}^\dagger)^n \hat{N}_{2,1} (\hat{a})^n, \] (22)

For \( n = 0,1,\ldots \) For \( n = 0 \) Eqs. (17-22) reduce to the fundamental set of operators given in Eqs. (11) through (16). This set of relevant operators is suitable for numerical simulation as it provides the simplest form of the system of differential equations for the evolution of their mean values. Eqs. (17-22) are the fundamental operators sandwiched between powers of creation \( \hat{a}^\dagger \) and destruction \( \hat{a} \) photon operators. This leads us to consider the operators with \( n > 1 \) as a measure of virtual transitions due to the absorption of more than one photon followed by the emission of the extra photons in a transition between the levels.

It is important to emphasize, as pointed out in [2] that the assumption of RWA made at the beginning introduces in a natural way the set of correlation functions. For example if one works out the commutator of the particle’s current between levels, \( \hat{F}^m \), and the Hamiltonian one obtains correlation operators involving the population operators of levels with the field operators, see Eq. (25) below. Specializing Eq. (25) to the simplest case, \( n = 0, m = 1 \), we have averages of terms such as \( \hat{N}^1_1 = \hat{a}^\dagger \hat{N}_1 \hat{a} \), which represent correlation between the field (\( \hat{a} \) and \( \hat{a}^\dagger \)) and population of level one (\( \hat{N}_1 \)). Clearly if one did not impose RWA, the structure of the algebra would be very different. Thus the algebra of the relevant operators carry the very blueprint of the physical assumptions we make. In short as is evident from Eq. (6) that the generator of the algebra, the Hamiltonian, generates a set of operators which are closely related to the physics of the problem and in this sense we mean that the set is physically relevant, thus it is not surprising that any physical assumptions that are put into
the Hamiltonian will be “transferred” to the operators which close the algebra with the Hamiltonian.

The other two sets of relevant operators which satisfy Eq. 6 are

\[
\left\{ \frac{1}{2} \left[ \hat{O}_i (\hat{a}^\dagger)^n (\hat{a})^n + (\hat{a}^\dagger)^n (\hat{a})^n \hat{O}_i \right] \right\}_{n=0}^\infty
\]

and

\[
\left\{ \frac{1}{2} \left[ \hat{O}_i (\hat{a}^\dagger \hat{a})^n + (\hat{a}^\dagger \hat{a})^n \hat{O}_i \right] \right\}_{n=0}^\infty.
\]

Here \( \hat{O}_i \) are the fundamental operators given by Eq. 11 through Eq. 16. This once again generalizes the \( m = 1 \) case considered by [2]. As pointed out in [2] the first set can be interpreted as the correlation functions between the fundamental operators and \( (\hat{a}^\dagger)^n (\hat{a})^n \).\( (\hat{a}^\dagger)^n (\hat{a})^n \) are proportional to the \( n \)th-order coherence function of the field, [3]. The operators included in the second set are proportional to the correlations between the fundamental operators and the energy of the field.

The dynamical equations for the operators given in Eq. 17-22 can be obtained using the Ehrenfest theorem (Eq. 7), and are given by

\[
\frac{d \left< \hat{N}_1^n \right>}{dt} = T(t) \left< \hat{F}^{n,m} \right> + T(t)[nC_1m \left< \hat{F}^{n-1,m} \right> + nC_2m(m-1) \left< \hat{F}^{n-2,m} \right> + \ldots + nC_m(m-1)(m-2) \ldots \left< \hat{F}^{0,m} \right>],
\]

(23)

\[
\frac{d \left< \hat{N}_2^n \right>}{dt} = -T(t) \left< \hat{F}^{n,m} \right>,
\]

(24)

\[
\frac{d \left< \hat{F}^{n,m} \right>}{dt} = -\alpha \left< \hat{I}^{n,m} \right> + 2|\gamma|^2T(t)[- \left< \hat{N}_1^{n+m} \right> + \left< \hat{N}_2^{n+m} \right> + n+mC_1m(\left< \hat{N}_2^{n+m-1} \right> - \left< \hat{N}_{2,1}^{n+m-1} \right>) + n+mC_2m(m-1)(\left< \hat{N}_2^{n+m-1} \right> - \left< \hat{N}_{2,1}^{n+m-1} \right>) + \ldots],
\]

(25)
\[
\frac{d <\hat{I}^{n,m}>}{dt} = \alpha <\hat{F}^{n,m}> ,
\]
\[
\frac{d <\hat{\Delta}^{n}>}{dt} = +T(t)[^{n+1}C_1m <\hat{F}^{n,m}> +^{n+1}C_2m(m-1) <\hat{F}^{n-1,m}> +.... +^{n+1}C_m(m-1)(m-2)... <\hat{F}^{0,m}>],
\]
\[
\frac{d <\hat{N}_{2,1}^{n}>}{dt} = 0,
\]
\[
n = 0, 1, \ldots, \text{ where } \alpha = E_2 - E_1 - \omega \text{ and } ^{n}C_m = \frac{n!}{(n-m)!m!}.
\]
Eqs. 23-28 are the exact dynamical evolution equations of the relevant operators for the \textit{generalized time-dependent m-photon JCH}. They can be thought of as a kind of generalized Bloch equations for the quantum field case. As can be seen, the different order correlations are connected via the operators \(\hat{N}_1^n\) and \(\hat{F}^{n,m}\) , see Eq. 23 and 25. As a check, if set \(m = 1\) in Eqs. 23 through 28 we find that these reduce to the equations resulting from the single photon generalized time-dependent JCH considered by Gruver et al., [2], see their Eqs. 22-27.

Comparing our fundamental set of operators, viz Eq. 11 through Eq. 16 with the one given in [2] we note that the operators which have not changed are \(\hat{N}_1, \hat{N}_1, \hat{\Delta}_1\) and \(\hat{N}_{2,1}\). Thus one would expect that [although in our case we have more involved correlations between \(\hat{N}_1^n\) and \(\hat{F}^{n,m}\)],
\[
\left\{ <(\hat{a}^\dagger)^n \hat{N}_1 (\hat{a})^n > + <(\hat{a}^\dagger)^n \hat{N}_2 (\hat{a})^n > - <(\hat{a}^\dagger)^{n-1} \Delta (\hat{a})^{n-1} > \right\}_{n=0}^\infty ,
\]
and
\[
\left\{ <(\hat{a}^\dagger)^n \hat{N}_{2,1} (\hat{a})^n > \right\}_{n=0}^\infty ,
\]
are constants of the motion. Indeed we find this to be the case as can be seen from Eqs. 23.
It is easy to see that \( 31 \) holds by directly looking at Eq. 28. To check the validity of 29 we first rewrite Eq. 27 for \( n - 1 \)

\[
\frac{d \langle \hat{\Delta}^{n-1} \rangle}{dt} = +T(t)\left[ nC_1m < \hat{F}^{n-1,m} > + nC_2m(m - 1) < \hat{F}^{n-2,m} > + \ldots \\
+ nC_m m(m - 1)(m - 2)\ldots < \hat{F}^{0,m} > \right]
\]

Adding Eqs. 23 and 24 and subtracting 31 we immediately see that 29 holds. This provides an excellent check on our manipulations which are tedious. From the expressions 29 and 30 it follow that the particle’s current between levels is equal to the photon’s flux. In the case, \( n = 0 \) we end-up with the conservation of the level’s population. For \( n > 0 \) we obtain a restriction for the correlations.

It is clear from Eq. 29 that the mean value of the operators are not be independent. This restricts the choice of the initial conditions. It is therefore necessary to choose a formalism which respects the restriction on the initial conditions. Following ref. 4 one may settle this issue by using the MEP density matrix given in Eq. 1. As is known in the MEP formalism the mean values and the Lagrange multipliers live in the dual spaces. As mentioned in 2 the Lagrange multipliers are numbers that can be freely chosen. Now once the restriction Eq. 29 on the mean values is implemented the equivalent restriction for the Lagrange multipliers are automatically satisfied when the density operator is diagonalized. In the MEP formalism used in 2 lack of knowledge on the mean value of one operator is equivalent to setting its Lagrange multiplier equal to zero.

**IV. CONCLUSIONS**

We have presented a generalized version of the m-photon \( JCH \) giving a description in terms of physical relevant operators. The temporal evolution equations have been worked out for one of the set which provides the simplest form of the system of differential equations
for the evolution of the mean values of the operators. Since an arbitrary function of time has been included, this formalism allows us to study the system even when the coupling is time dependent. Our work is a simple extension of the work of Gruver et al. [2] to the m-photon case. One advantage, as mentioned in [2], of giving a description of the system in terms of the three sets of physical relevant operators is that it allows one to treat the case of mixed boundary conditions.

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