Wellposedness and scattering for the generalized Boussinesq equation

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Abstract. In this paper, we show the local well-posedness of the generalized Boussinesq equation (gBQ) in $L^2(\mathbb{R}^d), H^1(\mathbb{R}^d)$ and obtain the global well-posedness, finite-time blowup and small initial data scattering of gBQ in energy space $H^1(\mathbb{R}^d)$. Moreover, we obtain the large radial initial data scattering of defocusing case for $d \geq 3$ by using the method of Dodson-Murphy [10].

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1 Introduction

We study the Cauchy problem of the generalized Boussinesq equation (gBQ)

$$\begin{cases}
\partial_t^2 u - \Delta u + \Delta^2 u = \beta \Delta |u|^{\alpha-1} u, & (t, x) \in \mathbb{R} \times \mathbb{R}^d, \\
u(0, x) = u_0(x), & u_t(0, x) = u_1(x),
\end{cases} \quad (1.1)$$

where $u$ is real, $\beta = \pm 1, \alpha > 1$. The equation has two conservation laws:

$$E(t) := \frac{1}{2} \int_{\mathbb{R}^d} \left| (-\Delta)^{-\frac{1}{2}} u_t \right|^2 + u^2 + |\nabla u|^2 + \frac{2\beta}{\alpha + 1} |u|^\alpha + 1 dx = E(0),$$

$$M(t) := \int_{\mathbb{R}^d} \left| (-\Delta)^{-\frac{1}{2}} u_t \right| \nabla \left| (-\Delta)^{-\frac{1}{2}} u \right| dx = M(0), \quad (1.2)$$

and it is called focusing if $\beta = -1$, while it called defocusing if $\beta = 1$.

The generalized Boussinesq equation is raised and studied in Bona et al. [1] and is a generalization of the so called “good” Boussinesq equation

$$\begin{cases}
\partial_t^2 u - \Delta^2 u + \partial_x^4 u + \partial^2_x u^2 = 0, & (t, x) \in \mathbb{R} \times \mathbb{R} \\
u(0, x) = u_0(x), & u_t(0, x) = u_1(x),
\end{cases}$$

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which models the phenomenon of nonlinear strings and is studied in Kishimoto [19]. Mckean [20] and so on. There are several results about gBQ. Bona et al. [11] studied the local and global well-posedness and stability of solitary-wave solution in one dimension. Linares [21] researched the local well-posedness of one dimension in $L^2$ and $H^1$ by using the $L^p$-$L^q$ estimates. Liu [22, 23, 24] studied the local and global well-posedness, scattering, instability of solitary waves in one dimension. Cho et al. [6] investigated the existence and scattering of global small amplitude solutions for all dimensions. Farah [12] researched the local well-posedness of gBQ for all dimensions by applying the Strichartz estimates of Schrödinger equation (1.11). Farah [11, 13] studied the asymptotic behavior of solutions for gBQ and the spirit of the results is somewhat similar with those in [6].

As far as we know, the results about the global well-posedness, finite-time blowup for gBQ in higher dimensions haven’t been obtained yet, although Liu [24] proved the preliminary Lemmas for all dimensions and the proof scheme of one dimension works for higher dimensions as well. What is needed is the local wellposed solution $(u, u_t) \in H^1(\mathbb{R}^d) \times H^-1(\mathbb{R}^d)$ that is suited for the conservation law (1.2). Wang et al. [31, 32] researched the global wellposedness, finite-time blowup, however their results depend on the damped terms $-\alpha \Delta n_1 + \gamma \Delta^2 n_1$, $\alpha > 0, \gamma > 0$ and $-\gamma \Delta n_1, \gamma > 0$ respectively. The local wellposed solution in Farah [12] is $(u, (-\Delta)^{-\frac{1}{2}} u_t) \in H^s(\mathbb{R}^d) \times H^s(\mathbb{R}^d), s \geq 0$ and their solution don’t match the results of Liu [24]. See more discussion in Appendix A.

Inspired by Gustafson, et al., [17], Kishimoto [19], we write (1.1) into

\[
\left( i \partial_t - \sqrt{(-\Delta)(1-\Delta)} \right) \left( i \partial_t + \sqrt{(-\Delta)(1-\Delta)} \right) u = -\beta \Delta |u|^{\alpha-1} u. \tag{1.4}
\]

and introduce

\[
v = u + i(-\Delta)^{-\frac{1}{2}} (1-\Delta)^{-\frac{1}{2}} \partial_t u = u + i \mathcal{B}^{-1} \partial_t u, \tag{1.5}
\]

where $\mathcal{B} := (-\Delta)^{-\frac{1}{2}} (1-\Delta)^{-\frac{1}{2}}$. Then, (1.4) is transformed into

\[
\begin{cases}
    i \partial_t v - \mathcal{B} v - \beta |\mathfrak{M}| \text{Re}(v)^{\alpha-1} \text{Re}(v) = 0, \\
v(0, x) = v_0(x) = u_0 + i \mathcal{B}^{-1} u_1,
\end{cases} \tag{1.6}
\]

where

\[
\mathcal{B} := \sqrt{-\Delta(1-\Delta)}, \quad \mathfrak{M} := \mathcal{F}_\xi^{-1} \sqrt{|\xi|^2/(1+|\xi|^2)} \mathcal{F}_x = \sqrt{\frac{-\Delta}{1-\Delta}}. \tag{1.7}
\]

\[
u = \frac{1}{2} (v + \bar{v}) = \text{Re}(v), \quad (-\Delta)^{-\frac{1}{2}} (1-\Delta)^{-\frac{1}{2}} \partial_t u = \frac{1}{2} (\bar{v} - v) = \text{Im}(v). \tag{1.8}
\]

By Duhamel principle, (1.6) can be written as following integral equation,

\[
v(t) = e^{-it\mathcal{B}} v_0 - i \beta \int_0^t e^{-i(t-\tau)\mathcal{B}} \mathfrak{M} |\text{Re}(v)|^{\alpha-1} \text{Re}(v) d\tau, \tag{1.9}
\]

where

\[
e^{-it\mathcal{B}} = \mathcal{F}_\xi^{-1} e^{-it|\xi| \sqrt{|\xi|^2+1}} \mathcal{F}_x. \tag{1.10}
\]

The form of (1.6) is similar with the Cauchy problem of the nonlinear Schrödinger equation

\[
\begin{cases}
i \partial_t z + \Delta z + \beta |z|^{\alpha-1} z = 0, & t \geq 0, x \in \mathbb{R}^d \\
z(0, x) = z_0(x),
\end{cases} \tag{1.11}
\]
where $\alpha > 1, \beta = \pm 1$. The nonlinear Schrödinger equation (1.11) is one of the most famous dispersive equations and there are numerous results and many mature theories about it. We refer to Cazenave [4] for a brief introduction of (1.11). Naturally, we shall try to derive results that are analogous to those of (1.11). The result in Gustafson et al. [17] has given the Strichartz type estimates for operator (1.10), which is presented in Section 2, and this give us the start point and an effective tool to research the results in this paper. The local wellposedness is directly obtained by following the classic results studying the Schrödinger equation and the global wellposedness, finite-time blowup thus follows by basing on the analysis of Liu [24] and Wang et al. [32].

There are rare results about the scattering of gBQ. Liu [25] researched the small initial data scattering in one dimension. Cho et al. [6] researched the small initial data scattering in higher dimensions. Muñoz et al. [27] studied the small initial data scattering in one dimension as well, and their method is much different from the classic ones studying the Schrödinger equation, wave equation etc. Actually, their way is more likely as those investigating stability of soliton solutions. It is not difficult to study small initial data scattering by using the transformation (1.5) and the Strichartz type estimates in Lemma 2.1. However, for the large initial data scattering, it is quite nontrivial. The $\Delta$ in the nonlinear term $\Delta(|u|^{\alpha-1}u)$ makes it difficult to calculate the classic Morawetz type estimate studying the Schrödinger equation. Specifically, if we research on (1.6), the operator $M$ before the nonlinear term $|\text{Re}(v)|^{\alpha-1}\text{Re}(v)$ makes it hard to construct Morawetz type estimate perfect as those for the Schrödinger equation. Inspired by Dodson et al. [10], we calculate the Morawetz-virial type estimate related to (1.1). We remark that the calculation relies heavily on the radial Sobolev embedding inequality and the non-radial large data scattering is still open.

The main results in this paper are as follows.

**Theorem 1.1** Local wellposedness

The initial value problem (1.1) with $(u, (-\Delta)^{-\frac{1}{2}}u_t)$ in $H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)$ is locally well-posed for

\begin{align*}
    s &= 0, \quad \text{if } 1 < \alpha \leq 1 + \frac{4}{d}, \\
    s &= 1, \quad \text{if } 1 < \alpha < \infty, d = 1, 2 \text{ or } 1 < \alpha \leq \frac{d+2}{d-2}, d \geq 3.
\end{align*}

(1.12)

Before introducing Theorem 1.2, we bring in some definitions used in Liu [24] and the notations may be slightly different from those in [24].

\begin{align*}
    E(u) &= \frac{1}{2} \|u\|_{H^1(\mathbb{R}^d)}^2 - \frac{1}{\alpha+1} \|u\|_{L^{\alpha+1}(\mathbb{R}^d)}^{\alpha+1}, \\
    R(u) &= \|u\|_{H^1(\mathbb{R}^d)}^2 - \|u\|_{L^{\alpha+1}(\mathbb{R}^d)}^{\alpha+1}, \\
    \eta_d &= \min \left\{ E(u) \mid 0 \neq u \in H^1(\mathbb{R}^d), R(u) = 0 \right\},
\end{align*}

(1.13)

$\varphi_d$ is the ground state solution of

\[-\Delta \varphi_d + \varphi_d - |\varphi_d|^{\alpha-1} \varphi_d = 0, \quad x \in \mathbb{R}^d.
\]

(1.14)

Define

\[ C_* = C_*(\alpha, d) := C_{\alpha,d} \]

(1.15)

as the best constant for Sobolev inequality $\|u\|_{L^{\alpha+1}(\mathbb{R}^d)} \leq C_{\alpha,d} \|u\|_{H^1(\mathbb{R}^d)}$ and it is determined in Proposition 2.10. If $\beta = -1$, then $E(t) = E(u) + \frac{1}{2} \|u_t\|_{H^{-1}(\mathbb{R}^d)}^2$. 

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\textbf{Theorem 1.2} Global wellposedness and finite-time blowup

(i) Assume $E(0)$ is a finite positive constant and $\alpha$ satisfies

\[
\begin{cases}
1 - \alpha < \infty, & \text{if } d = 1, 2, \\
1 - \alpha < \frac{d + 2}{2}, & \text{if } d \geq 3.
\end{cases}
\] (1.16)

Then, the initial value problem (1.1) with $(u, (-\Delta)^{-\frac{1}{2}} u_t)$ in $H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)$ is globally wellposed for $\beta = 1$ or

\[
\beta = -1, \quad E(0) < \frac{\alpha - 1}{2(\alpha + 1)} C_* \frac{2(\alpha + 1)}{\alpha + 1}, \quad \|u_0\|_{H^1} < C_* \frac{\alpha + 1}{\alpha - 1}. \quad (1.17)
\]

(ii) Assume $\alpha$ satisfies (1.16), $\beta = -1$, $(u_0, u_1) \in H^1(\mathbb{R}^d) \times \dot{H}^{-1}(\mathbb{R}^d)$, $u_0 \in \dot{H}^{-1}(\mathbb{R}^d)$ and

\[
E(0) < \frac{\alpha - 1}{2(\alpha + 1)} C_* \frac{2(\alpha + 1)}{\alpha + 1}, \quad \|u_0\|_{H^1} > C_* \frac{\alpha + 1}{\alpha - 1}, \quad (1.18)
\]

then the local wellposed solution of (1.1) $u \in H^1(\mathbb{R}^d)$ blows up in some finite time, i.e. the maximum lifespan $T_{\text{max}} < \infty$, and $u$ satisfies

\[
\lim_{t \to T_{\text{max}}} \|u(t)\|_{H^1(\mathbb{R}^d)} = \lim_{t \to T_{\text{max}}} \|u(t)\|_{L^{s+1}(\mathbb{R}^d)} = \infty.
\]

(iii) Let $\varphi_d \in H^1(\mathbb{R}^d)$ be the ground state of (1.13) and assume $u_1 = 0, \beta = -1$. For any $\delta > 0$, there exists initial data $u_0 \in H^1(\mathbb{R}^d)$ with $\|u_0 - \varphi_d\|_{H^1} < \delta$, such that the local solution of (1.1) in $H^1(\mathbb{R}^d) \times \dot{H}^{-1}(\mathbb{R}^d)$ satisfies

\[
\lim_{t \to T^-} \|u(t)\|_{H^1} = \infty
\]

for some $0 < T < \infty$.

\textbf{Theorem 1.3} Small initial data scattering

Suppose $(u_0, u_1) \in H^1(\mathbb{R}^d) \times \dot{H}^{-1}(\mathbb{R}^d)$ small enough and $\alpha$ satisfying

\[
\begin{cases}
4/d + 1 \leq \alpha < \infty, & \text{if } d = 1, 2, \\
4/d + 1 \leq \alpha < \frac{d + 2}{2}, & \text{if } d \geq 3,
\end{cases}
\] (1.19)

then the $H^1$ solution of (1.1) $(u, u_t) \in H^1(\mathbb{R}^d) \times \dot{H}^{-1}(\mathbb{R}^d)$ scatters. More precisely, there exists unique linear solution of (1.1) $(u_0^\pm, u_1^\pm) \in H^1(\mathbb{R}^d) \times \dot{H}^{-1}(\mathbb{R}^d)$ such that

\[
\lim_{t \to \pm \infty} \|(u(t), u_1(t)) - (u_0^\pm, u_1^\pm)\|_{\mathcal{H}_1} = 0, \quad (1.20)
\]

where $\mathcal{H}_1 := H^1 \times \dot{H}^{-1}$, $\|(w_1, w_2)\|_{\mathcal{H}_1} := (\|w_1\|_{H^1}^2 + \|w_2\|_{\dot{H}^{-1}}^2)^{\frac{1}{2}} = \|w_1 + i\mathcal{B}^{-1}w_2\|_{H^1}$.

\textbf{Theorem 1.4} Large radial initial data scattering of defocusing case

Suppose $(u_0, u_1)$ is radial symmetric, $\beta = 1$, $4/d + 1 \leq \alpha < (d + 2)/(d - 2)$ and $d \geq 3$, then the $H^1$ solution of (1.1) scatters.

The rest of the paper is organized as follows. We research the local and global wellposedness, finite-time blowup in Section 2 and derive the small initial data scattering in Section 3. The large
initial data scattering for radial defocusing case is obtained in Section 4. Some more discussion about the properties of the Boussinesq operator is given in Section A.

We end up with notations used in this paper. $F_x u$ and $\hat{u}$ denote the Fourier transform of the function $u(t,x)$ with respect to $x$, and $\mathcal{F}_x^{-1} u, \hat{u}(x)$ the inverse Fourier transformation to $\xi$:

$$\mathcal{F}_x u = \hat{u}(\xi) := \int_{\mathbb{R}^d} u(x)e^{-2\pi i x \cdot \xi} dx, \quad \mathcal{F}_x^{-1} u = u(\xi) := \hat{u}(-x).$$

Define $|x| := \sqrt{\sum_i x_i^2}$ and $D^n := \mathcal{F}^{-1} |\xi|^n(\mathcal{F} u)(x), \eta \in \mathbb{R}, \xi \in \mathbb{R}^d$, then $D = (-\Delta)^{\frac{1}{2}}$. Denote $\langle \cdot \rangle := \sqrt{1 + |\cdot|^2}$, $D := \sqrt{1 + D^2}$, $\langle \nabla \rangle := 1 + \nabla$, then we have $\langle D \rangle \sim \langle \nabla \rangle$. Write $A \lesssim B, A \lesssim C(a)B$ as $A \lesssim B, A \lesssim C$ respectively, where $C, C(a)$ are positive constants and $C(a)$ depends on $a$. The Rezis operator is denoted as $R_j := -\mathcal{F}_x^{-1} i\xi_j \mathcal{F}_x$. The Lebesgue, Sobolev and Besov norms are standard

$$\|u\|_{L^p(\mathbb{R}^d)} = \left( \int_{\mathbb{R}^d} |u(x,t)|^p dx \right)^{\frac{1}{p}}, \quad \|u\|_{L^q([0,T];L^r(\mathbb{R}^d))} = \left( \int_0^T \|u\|_{L^q(\mathbb{R}^d)}^r dt \right)^{\frac{1}{r}},$$

$$\|u\|_{W^{k,q}(\mathbb{R}^d)} = \|\langle D \rangle^k u\|_{L^q(\mathbb{R}^d)}$, \quad \|u\|_{H^s(\mathbb{R}^d)} = \|\langle D \rangle^s u\|_{L^2(\mathbb{R}^d)}$, \quad \|u\|_{\dot{H}^s(\mathbb{R}^d)} = \|\langle D \rangle^s u\|_{L^2(\mathbb{R}^d)},$$

$$\|u\|_{B^{s,p,r}_r} = \left( \sum_{j \in \mathbb{Z}} 2^{js} \|\hat{u}_j\|_{L^p}^{\frac{1}{r}} \right)^{\frac{1}{s}}, \quad \hat{u}_j u = \varphi (2^{-j} D) u, \quad \sum_{j \in \mathbb{Z}} \varphi (2^{-j} \xi) = 1, \forall \xi \in \mathbb{R}^d \setminus \{0\},$$

and $\varphi$ is the Littlewood-Paley function, which is radial symmetry and supported in the annulus $\{ \xi \in \mathbb{R}^d | 3/4 \leq |\xi| \leq 8/3 \}$. If there is no special explanation, the norm without domain subscript is always acting on the whole space $\mathbb{R}^d$.

## 2 Local and global wellposedness, finite-time blowup

### 2.1 Local wellposedness

The Strichartz type estimates for operator $\Box_0$ in Gustafson et al. [17] are as follows.

**Lemma 2.1** Let $d \geq 1$ be integer.

(i) For $2 \leq q \leq \infty$, we have

$$\left\| e^{-it \Box_0} \varphi \right\|_{B^{0,2}_{q,2}} \lesssim |t|^{-d \sigma} \left\| M^{(d-2)\sigma} \varphi \right\|_{B^{0,2}_{q',2}}$$

where $\frac{1}{q} + \frac{1}{q'} = 1, \sigma = 1/2 - 1/q$.

(ii) Suppose $j = 1, 2 \leq p_j, q_j \leq \infty, 0 < p_j + d/q_j = d/2, s_j = \frac{d-2}{2}(1/2 - 1/q_j)$, but $(p_j, q_j) \neq (2, \infty)$, then

$$\left\| e^{-it \Box_0} \varphi \right\|_{L^p B^{s_j} \dot{B}^{0,2}_{q,2}} \lesssim \left\| M_{t^j} \varphi \right\|_{L^2},$$

$$\left\| \int_{-\infty}^t e^{-i(t-s) \Box} f(s) ds \right\|_{L^p B^{s_j} \dot{B}^{0,2}_{q,2}} \lesssim \left\| M_{t^j} f \right\|_{L^2 B^{s_j} \dot{B}^{0,2}_{q,2}},$$

where $M$ is defined as [17].
Analogous to the Strichartz estimates for (1.11), we call \((p, q)\) is admissible, if it satisfies

\[
\frac{2}{p} = d \left(\frac{1}{2} - \frac{1}{q}\right),
\]

\[
\begin{cases}
2 \leq q \leq \infty, & \text{if } d = 1, \\
2 \leq q < \infty, & \text{if } d = 2, \\
2 \leq q \leq 2d/(d - 2), & \text{if } d \geq 3.
\end{cases}
\]

By the definition of \(s_i\) and the relationship between Sobolev space and Besov space, one can follow (2.2), (2.3) to get

\[
\left\| e^{-i \xi \cdot \mathbf{x}} \right\|_{L^p L^q} \lesssim \| \mathcal{M}^{s_1} \varphi \|_{L^2},
\]

\[
\left\| \int_{-\infty}^{t} e^{-i(t - s) \xi \cdot \mathbf{x}} f(s) ds \right\|_{L^p L^q} \lesssim \left\| \int_{-\infty}^{t} e^{-i(t - s) \xi \cdot \mathbf{x}} \mathcal{M}^{s_1 + s_2} f(s) ds \right\|_{L^p L^q} \lesssim \| \mathcal{M}^{s_1 + s_2} f \|_{L^p L^q},
\]

(2.4)

\[
2 \leq q_i < \infty, 1 < q'_i \leq 2, q_i = 1, 2. \text{ One can see (2.5), (2.6) and the Strichartz estimates are very similar in form and (2.5), (2.6) are essentially better. Indeed, formally, there is}
\]

\[
|\mathcal{F}(\mathcal{M}^s g(x))| \leq |\hat{g}|, \ s > 0
\]

and for the multiplier of operator \(\mathcal{M}^s, s > 0\), namely \((\frac{|\xi|^2}{1 + |\xi|^2})^s\), it is easy to calculate that

\[
|\mathcal{Q}_{\xi}^s \left(\frac{|\xi|^2}{1 + |\xi|^2}\right)| \lesssim A|\xi|^{-|\alpha|}
\]

for \(|\alpha| \leq \left[\frac{d}{2}\right] + 1\) and some constant \(A\), thus the Hörmander–Mihlin Multiplier Theorem shows \((\frac{|\xi|^2}{1 + |\xi|^2})^s\) is a \(L^p, 1 < p < \infty\) multiplier and \(\|\mathcal{M}^s f\|_{L^p} \lesssim \|f\|_{L^p}\). On the other hand, for the high frequency, there is

\[
|\mathcal{F}(\mathcal{M}^s g(x))| \sim |\hat{g}|,
\]

and it seems that we can’t expect results for (1.6) that are better than those of (1.11) within the standard method for discussing local wellposedness for (1.11). Thus, the classification for (1.11)

according to exponent \(\alpha\) of nonlinear term, such as mass critical case \(\alpha = 1 + 4/d\) and energy critical case \(\alpha = 1 + 4/(d - 2), d \geq 3\), may be still used to sort through (1.6) or (1.1). The local wellposedness results for (1.6) is presented as below.

**Lemma 2.2** \(1 < \alpha < 1 + 4/d\) case in \(L^2(\mathbb{R}^d)\)

If \(1 < \alpha < 1 + 4/d\), then for every \(v_0 \in L^2(\mathbb{R}^d)\), there exist \(T = T(\|v_0\|_{L^2}, d, \alpha) > 0\) and a unique solution \(v\) of the integral equation (1.9) in the time interval \([-T, T]\) with

\[
v \in C([-T, T]; L^2(\mathbb{R}^d)) \cap L^r([-T, T]; L^{\alpha+1}(\mathbb{R}^d)),
\]

where \(r = \frac{4(\alpha+1)}{4(\alpha-1)}\). Moreover, for every \(T' < T\) there exists a neighborhood \(V\) of \(v_0\) in \(L^2(\mathbb{R}^d)\) such that

\[
F : V \mapsto C([-T', T']; L^2(\mathbb{R}^d)) \cap L^r([-T', T']; L^{\alpha+1}(\mathbb{R}^d)), \quad v_0 \mapsto \tilde{v}(t)
\]

is Lipschitz.
Lemma 2.3 Critical case, \( \alpha = 1 + 4/d \) in \( L^2(\mathbb{R}^d) \)

If \( \alpha = 1 + 4/d \), then for every \( v_0 \in L^2(\mathbb{R}^d) \) there exist \( T = T(v_0, \alpha) > 0 \) and a unique solution \( v \) of the integral equation (1.9) in the time interval \([-T, T] \) with

\[
v \in C \left( [-T, T]; L^2(\mathbb{R}^d) \right) \cap L^\sigma \left( [-T, T]; L^\sigma(\mathbb{R}^d) \right),
\]

where \( \sigma = 2 + 4/d \). Moreover, for every \( T' < T \) there exists a neighborhood \( V \) of \( v_0 \) in \( L^2(\mathbb{R}^d) \) such that

\[
F : V \mapsto C \left( [-T', T']; L^2(\mathbb{R}^d) \right) \cap L^\sigma \left( [-T', T']; L^\sigma(\mathbb{R}^d) \right), \quad \tilde{v}_0 \mapsto \tilde{v}(t)
\]

is Lipschitz.

Lemma 2.4 Local theory in \( H^1(\mathbb{R}^d) \)

If \( \alpha \) satisfying

\[
\begin{cases}
1 < \alpha < \frac{d+2}{2}, & \text{if } d \geq 3 \\
1 < \alpha < \infty, & \text{if } d = 1, 2,
\end{cases}
\]

(2.7)

then for every \( v_0 \in H^1(\mathbb{R}^d) \), there exist \( T = T(\|v_0\|_{H^1}, d, \alpha) > 0 \) and a unique solution \( v \) of the integral equation (1.9) in the time interval \([-T, T] \) with

\[
v \in C \left( [-T, T]; H^1(\mathbb{R}^d) \right) \cap L^p \left( [-T, T]; W^{1,q}(\mathbb{R}^d) \right),
\]

where \( (p, q) = \left( \frac{4(\alpha+1)}{d(\alpha-1)}, \alpha + 1 \right) \) for \( d \geq 3 \), and \( (p, q) \) satisfies (2.4) for \( d = 1, 2 \).

Moreover, \( v \) depends continuously on \( v_0(x) \) as follows. Let \((v_{0,n})_{n \geq 1} \subset H^1(\mathbb{R}^d)\) such that \( v_{0,n} \rightarrow v_0 \) in \( H^1(\mathbb{R}^d) \) as \( n \rightarrow \infty \), and let \( v_n \) be the maximal solution of 1.10 corresponding to the initial value \( v_{0,n} \), then there exists \( 0 < T' < T \) depending on \( \|v_0\|_{H^1} \) such that \( v_n \) is defined on \([-T', T'] \) for \( n \) large enough and \( v_n \rightarrow v \) in \( C \left( [-T', T']; H^1(\mathbb{R}^d) \right) \) as \( n \rightarrow \infty \).

Lemma 2.5 Critical case, \( \alpha = (d+2)/(d-2), d \geq 3 \), in \( H^1(\mathbb{R}^d) \)

Let \( d \geq 3 \), \( \alpha = (d+2)/(d-2) \), \( p = \alpha + 1 = 2d/(d-2) \) and \( q = \frac{2d(\alpha+1)}{(d-2)} = 2d^2/(d^2-2d+4) \), then for every \( v_0 \in H^1(\mathbb{R}^d) \), there exist \( T = T(v_0, d, \alpha) > 0 \) and a unique solution \( v \) to 1.10 satisfying

\[
v \in C \left( [-T, T]; H^1(\mathbb{R}^d) \right) \cap L^p_{\text{loc}} \left( [-T, T]; W^{1,q}(\mathbb{R}^d) \right).
\]

Moreover, \( v \) depends continuously on \( v_0(x) \) as follows. Let \((v_{0,n})_{n \geq 1} \subset H^1(\mathbb{R}^d)\) such that \( v_{0,n} \rightarrow v_0 \) in \( H^1(\mathbb{R}^d) \) as \( n \rightarrow \infty \), and let \( v_n \) be the maximal solution of 1.10 corresponding to the initial value \( v_{0,n} \), then exists \( 0 < T' < T \) depending on \( v_0 \) such that \( v_n \) is defined on \([-T', T'] \) for \( n \) large enough and \( v_n \rightarrow v \) in \( L^\infty \left( [−T', T'], H^1(\mathbb{R}^d) \right) \), as \( n \rightarrow \infty \).

\[
\square
\]
Both of \cite{3} and \cite{22} have given the local wellposedness for (1.11) of energy critical case in \(H^1(\mathbb{R}^d)\). The method of Theorem 4.5.1 in \cite{3} obtaining the results is an iteration in Strichartz spaces and those of Theorem 5.5 in \cite{22} is Kato’s method. However, as clarified in Tao et al. \cite{30}, it seems if the dimension \(d\) is large enough, we can’t get the Lipschitz continuity of solution \(z(t,x)\) for (1.11) of energy critical case in \(H^1(\mathbb{R}^d)\) that are stated in Theorem 5.5 of \cite{22}. Here, the proof method for Lemma 2.5 is a combination of those in \cite{3} and \cite{22}. We will use Kato’s method and prove the continuity for the solution \(v\) in the sense as those in \cite{3}.

We recall the Kato-Ponce type inequalities to deal with the nonlinear terms.

**Lemma 2.6** \cite{14} Let \(\frac{1}{2} < r < \infty, 1 < p_1, p_2, q_1, q_2 \leq \infty\) satisfy \(\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}\). Given \(s > \max(0, \frac{d}{2} - d)\) or \(s \in 2\mathbb{N}\), there exists \(\mathcal{C} = \mathcal{C}(d, s, r, p_1, q_1, p_2, q_2) < \infty\) such that for all \(f, g \in \mathcal{S}(\mathbb{R}^d)\), we have

\[
\|D_\delta^s(fg)\|_{L_x^r(\mathbb{R}^d)} \lesssim \|D_\delta^s f\|_{L_{p_1}(\mathbb{R}^d)} \|g\|_{L_{q_1}^r(\mathbb{R}^d)} + \|f\|_{L_{p_2}(\mathbb{R}^d)} \|D_\delta^s g\|_{L_{q_2}^r(\mathbb{R}^d)},
\]

(2.8)

\[
\|D_\delta^s(fg)\|_{L_x^r(\mathbb{R}^d)} \lesssim \|f\|_{L_{p_1}^1(\mathbb{R}^d)} \|D_\delta^s g\|_{L_{q_2}^1(\mathbb{R}^d)} + \|D_\delta^s f\|_{L_{p_2}^1(\mathbb{R}^d)} \|g\|_{L_{q_1}^1(\mathbb{R}^d)}.
\]

(2.9)

We also need following Lemma to deal with the initial data term.

**Proposition 2.7** Let \((p, q)\) satisfy (2.11). Given \(u_0 \in L^2(\mathbb{R}^d)\) and \(\varepsilon > 0\), there exist \(\delta > 0\) and \(T > 0\) such that if \(\|v_0 - u_0\|_2 < \delta\), then

\[
\left( \int_0^T \|e^{-it\partial_x^2}v_0\|_q^p dt \right)^{1/p} < \varepsilon.
\]

(2.10)

Noticing that that \(\{e^{-it\partial_x^2}\}\) defines a unitary group in \(H^s(\mathbb{R}^d)\), Proposition 2.7 is essentially the same with Proposition 5.1 of \cite{22}.

**Proof of Lemma 2.5**

Without loss of generality, we set \(t \geq 0\). Define space as

\[
X_T := \{v \in L^\infty([0, T]; H^1(\mathbb{R}^d)) \cap L^p([0, T]; W^{1,s}) \mid \sup_{[0, T]} \|v(t) - e^{-it\partial_x^2}v_0\|_{H^1} + \|v\|_{L^p([0, T]; W^{1,s})} \leq M\},
\]

and distance norm of \(X_T\) as

\[
\partial(v, w) = \sup_{[0, T]} \|v - w\|_{L^2} + \|v - w\|_{L^p([0, T]; L^q)}.
\]

By Theorem 1.2.5 of Cazenave \cite{3} and following corresponding discussion in Theorem 4.4.1 of \cite{3}, we know \((X_T, \partial)\) is a complete metric space. Thus, one only needs to prove

\[
\mathcal{F} : v \mapsto e^{-it\partial_x^2}v_0 - i\beta \int_0^t e^{-i(t-\tau)\partial_x^2}\mathcal{R}(\text{Re}(v))^{n-1}\text{Re}(v) d\tau,
\]

(2.11)

is a contraction map on \((X_T, \partial)\).

As \(v_0 \in H^1(\mathbb{R}^d)\), we can let the \(u_0, v_0\) in Proposition 2.7 be \(v_0 = u_0\) and find small enough \(T > 0\) such that

\[
\left( \int_0^T \|e^{-it\partial_x^2}v_0\|_{W^{1,q}}^p dt \right)^{1/p} < \varepsilon.
\]
for any given small $\varepsilon > 0$, where $p = \alpha + 1 = 2d/(d-2)$ and $q = \frac{2d(\alpha+1)}{d(\alpha+1)-d} = 2d^2/(d^2 - 2d + 4)$ is admissible. Let $r_1 = d(\alpha + 1)(\alpha - 1)/4 = 2d^2/(d^2 - 2d + 4)^2$, then the Sobolev embedding inequality

$$
\frac{1}{q'} = \frac{1}{q} + \frac{\alpha - 1}{r_1}, \quad \text{for } W^{1,q}(\mathbb{R}^d) \hookrightarrow L^{r_1}(\mathbb{R}^d),
$$

by nonlinear estimate \((2.14)\), Strichartz estimates \((2.25)\), \((2.26)\) and Hölder inequality show

$$
\|\mathcal{T}(v)\|_{L^p([0,T];W^{1,q})} \leq C \|e^{-it_2B}v_0\|_{L^p([0,T];W^{1,q})} + C \|\mathfrak{G}^{1+2s_1}|\text{Re}(v)|^{\alpha-1}\text{Re}(v)\|_{L^p([0,T];W^{1,q})}
$$

$$
\leq C\varepsilon + C\left\|\|v\|_1^{\alpha-1}\right\|_{L^p([0,T];W^{1,q})}^{\frac{\alpha}{r_1}}
$$

$$
\leq C\varepsilon + C\|v\|_1^{\alpha-1}, \quad (2.12)
$$

where $s_1 = \frac{d-2}{2}(1/2 - 1/q)$ is determined as those in Lemma \((2.1)\). By Bootstrap argument and the smallness of $\varepsilon$, we know $\|v\|_{L^p([0,T];W^{1,q})} \leq C(\varepsilon)$ is small and can set $M$ small enough such that

$$
CM^{\alpha-1} \leq 1/2, \quad C\varepsilon + CM^\alpha \leq M.
$$

Therefore,

$$
\|\mathcal{T}(v)\|_{L^p([0,T];W^{1,q})} \leq C\varepsilon + CM^\alpha \leq M
$$

and similar discussion with \((2.14)\) gives

$$
\sup_{[0,T]} \|\mathcal{T}(v) - e^{-it_2B}v_0\|_{H^1} \leq C \|\mathfrak{G}^{1+2s_1}|\text{Re}(v)|^{\alpha-1}\text{Re}(v)\|_{L^p([0,T];W^{1,q})}
$$

$$
\leq CM^\alpha \leq M,
$$

$$
\sup_{[0,T]} \|\mathcal{T}(v) - \mathcal{T}(w)\|_{L^2(\mathbb{R}^d)} + \|\mathcal{T}(v) - \mathcal{T}(w)\|_{L^p([0,T];L^q(\mathbb{R}^d))}
$$

$$
\leq C \left\|\|v - w\|_{L^2} \left(\|v\|_1^{\alpha-1} + \|w\|_1^{\alpha-1}\right)\right\|_{L^p([0,T])}
$$

$$
\leq CM^{\alpha-1}\|v - w\|_{L^p([0,T];L^q)} \leq \frac{1}{2}\|v - w\|_{L^p([0,T];L^q)}, \quad (2.15)
$$

As a consequence, we obtain that

$$
\mathcal{T}(X_T) \subset X_T, \quad \mathcal{T}(v), \mathcal{T}(w) \leq \frac{1}{2}B(\mathcal{T}(v), \mathcal{T}(w))
$$

and $\mathcal{T}(v)$ is a contraction map on $(X_T, \mathfrak{d})$. Uniqueness follows a similar process.

Next, we prove the continuity by utilizing the method in \((2.3)\). Since $\|v_{n,0} - v_0\|_{H^1} \to 0$ and $\|v_{n,0}\|_{H^1} \leq 2\|v_0\|_{H^1}$ for $n \to \infty$, we can utilize Proposition \((2.7)\) and the analysis in \((2.12)\) to find $N_1$ and small $T' > 0$ such that, when $n \geq N_1$, one has

$$
\left(\int_0^{T'} \left|e^{-it_2B}v_{0,n}\right|^p dt\right)^{1/p} < \varepsilon, \quad \left(\int_0^{T'} \left|e^{-it_2B}v_{0,n}\right|^q dt\right)^{1/q} < \varepsilon
$$

$$
\|\mathcal{T}(v)\|_{L^p([0,T'];W^{1,q})} \leq M, \quad \|\mathcal{T}(v_n)\|_{L^p([0,T'];W^{1,q})} \leq M,
$$

where $M$ is the one set as in \((2.13)\). Noticing that

$$
v_n(t) - v(t) = e^{-it_2B}(v_{0,n} - v_0) - i\beta \int_0^t e^{-i(t-\tau)2B}\mathfrak{G}(\text{Re}(v_n))^{\alpha-1}\text{Re}(v_n) - |\text{Re}(v)|^{\alpha-1}\text{Re}(v)\,d\tau,
$$

(2.16)
Proof of Theorem 1.1:

Actually, if (2.19) fails, there exist a $\epsilon > 0$ and a subsequence $\{v_{n_j}\}_{j \geq 1}$ such that

$$\left\| (|\text{Re}(v_{n_j})|^{\alpha - 1} - |\text{Re}(v)|^{\alpha - 1}) \nabla \text{Re}(v) \right\|_{L^p([0,T'];L^q)} \geq \epsilon_1 \text{ for } j \geq 1.$$  

(2.20)

However, we can use (2.17) to extract a subsequence, which is still denoted by $\{v_{n_j}\}_{j \geq 1}$, such that as $j \to \infty$, $v_{n_j} \to v$ a.e. on $[0,T'] \times \mathbb{R}^d$ and $\text{Re}(v_{n_j}) \to \text{Re}(v)$ a.e. on $[0,T'] \times \mathbb{R}^d$. Since $\left(||\text{Re}(v_{n_j})|^{\alpha - 1} - |\text{Re}(v)|^{\alpha - 1}\|\nabla\text{Re}(v) \in L^p([0,T'];L^q)\right)$, we obtain from the dominated convergence a contradiction with (2.20).

\[\square\]

**Proof of Theorem 1.1**

Recall

$$u = \frac{1}{2} (v + \bar{v}), \quad (-\Delta)^{-\frac{1}{2}} (1 - \Delta)^{-\frac{1}{2}} \partial_t u = \frac{i}{2} (\bar{v} - v),$$  

(2.21)

thus, the transformation

$$(v, \nabla) \mapsto (u, (-\Delta)^{-\frac{1}{2}} \partial_t)$$

$C([0,T];H^s) \times C([0,T];H^s) \to C([0,T];H^s) \times C([0,T];H^{s-1})$

is Lipschitz continuous and Lemma 2.2.2.5 can yield the results.

\[\square\]
2.2 Global wellposedness and finite-time blowup

As mentioned in the Introduction, despite only giving the conclusion in dimension one, Liu [24] proved the preliminary Lemmas for all dimensions and the proof of global wellposedness, finite blowup and instability doesn’t depend on the dimension. Thus, the conclusion can be applied to higher dimensions. Here, we give a proof based on the analysis of Wang et al. [32], Liu [24]. Compared with those of Liu [24], our proof is more concise and the results of conclusion (i), (ii) are better. We firstly present the preliminary Lemmas.

**Lemma 2.8** [32] Assume that \( y = y(t) \) is a continuous function. The constants \( s, C_1, C_2 \) satisfy \( s > 1, C_2 > 0, \) and \( 0 < C_1 < \frac{1}{s} \left( \frac{1}{C_2s} \right)^{\frac{1}{s-1}} \) are constants. If \( 0 \leq y(t) \leq C_1 + C_2y(t)^s \), for any \( t \geq 0 \), then there exist constants \( y_1, y_2 \), satisfying

\[
0 \leq y(t) \leq y_1, \quad \text{if} \quad y(0) < y_0 = \left( \frac{1}{C_2s} \right)^{\frac{1}{s-1}} \tag{2.22}
\]

or

\[
y_2 \leq y(t) < \infty, \quad \text{if} \quad y(0) > y_0 = \left( \frac{1}{C_2s} \right)^{\frac{1}{s-1}} \tag{2.23}
\]

for any \( t \geq 0 \).

**Lemma 2.9** [20, 32] Suppose that for \( t \geq t_0, t_0 \) is a positive constant, a nonnegative, twice-differentiable function \( \phi(t) \) satisfies the inequality

\[
\phi'' \phi - (\eta + 1) (\phi')^2 \geq 0 \tag{2.24}
\]

where \( \eta > 0 \) is a constant. If \( \phi(t_0) > 0 \) and \( \phi'(t_0) > 0 \), then \( \phi(t) \to \infty \) as \( t \to t_1 \leq T_0 = \phi(t_0) / (\eta \phi'(t_0)) + t_0 \).

**Proposition 2.10** Let \( \eta_d, E(u), \varphi_d, C_\alpha \) be the notation defined in (1.13), (1.14), (1.15), \( \alpha \) satisfies (1.16), then

\[
\eta_d = E(\varphi_d), \tag{2.25}
\]

\[
C_\alpha = C_{\alpha,d} = \| \tilde{\varphi}_d \|_{H^1(\mathbb{R}^d)}. \tag{2.26}
\]

Proposition 2.10 is the combination of Theorem 2.6 and Corollary 2.5 in Liu [24].

**Proof of Theorem 1.2** Obviously, due to (1.2), the solution \( (u, u_t) \in H^1(\mathbb{R}^d) \times \dot{H}^{-1}(\mathbb{R}^d) \) in Theorem 1.2 is global if \( \beta = 1 \). When \( \beta = 1 \), we let

\[
y(t) = \| u \|_{H^1}^2,
\]

then (1.2) shows

\[
y(t) \leq 2E(0) + \frac{2}{\alpha + 1} \| u \|_{L^{\alpha+1}}^{\alpha+1}
\]

\[
\leq 2E(0) + \frac{2}{\alpha + 1} C_{\alpha+1} y(t)^{\frac{\alpha+1}{\alpha}}.
\]
where \( C_* \) is the best constant for \( \| u \|_{L^{\alpha+1}} \leq C_* \| u \|_{H^1} \). Combining with (1.17) \( \mathcal{E}(0) < \frac{\alpha-1}{2(\alpha+1)} C_*^{-\frac{2}{\alpha+1}} \), \( \| u_0 \|_{H^1} < C_* \) and Lemma 2.8 one has
\[
\| u(t) \|_{H^1}^2 < C_*^{-\frac{2}{\alpha+1}},
\] (2.27)
for \( t > 0 \). It follows that
\[
\frac{1}{\alpha + 1} \| u \|_{L^{\alpha+1}}^{\alpha+1} \leq \frac{1}{\alpha + 1} C_*^{\alpha+1} \| u \|_{H^1}^{\alpha+1}
= \frac{1}{\alpha + 1} C_*^{\alpha+1} \| u \|_{H^1}^{\alpha+1} \| u \|_{H^1}^2 < \frac{1}{\alpha + 1} \| u \|_{H^1}^2,
\] (2.28)
which combined with (1.2) shows
\[
\frac{\alpha - 1}{2(\alpha + 1)} \| u \|_{H^1}^2 + \frac{1}{2} \| (\Delta)^{-\frac{1}{2}} u_t \|_{L^2}^2 \leq \mathcal{E}(0).
\] (2.29)
This validates \( T_{\text{max}} = \infty \) under premise (1.17).

Next, we discuss the finite-time blowup in conclusion (ii) and we’ll prove the argument by contradiction. Suppose that \( T_{\text{max}} = +\infty \), and for some \( 0 < T < \infty \) we can set
\[
\phi(t) = \left\| (\Delta)^{-\frac{1}{2}} u \right\|_{L^2}^2,
\] (2.30)
then,
\[
\phi'(t) = 2 \left( (\Delta)^{-\frac{1}{2}} u_t, (\Delta)^{-\frac{1}{2}} u_t \right), \quad \forall t \in [0, T],
\] (2.31)
and
\[
\frac{1}{4} |\phi'(t)|^2 \leq \phi(t) \left\| (\Delta)^{-\frac{1}{2}} u_t \right\|_{L^2}^2, \quad \forall t \in [0, T].
\] (2.32)
It follows the equation (1.1) that
\[
\phi''(t) = 2 \left( (-\Delta)^{-1} u_t, u \right)_{H^{-1}, H^1} + 2 \left\| (\Delta)^{-\frac{1}{2}} u_t \right\|_{L^2}^2
= -2 \| u \|_{H^1}^2 + 2 \int_{\mathbb{R}^d} |u|^{\alpha+1} dx + 2 \left\| (\Delta)^{-\frac{1}{2}} u_t \right\|_{L^2}^2.
\] (2.33)
Noticing that (1.2) can be written into
\[
2 \int_{\mathbb{R}^d} |u|^{\alpha+1} dx = -2(\alpha + 1)\mathcal{E}(0) + (\alpha + 1) \| u \|_{H^1}^2 + (\alpha + 3) \left\| (\Delta)^{-\frac{1}{2}} u_t \right\|_{L^2}^2,
\]
one has
\[
\phi''(t) = (\alpha - 1) \| u \|_{H^1}^2 - 2(\alpha + 1)\mathcal{E}(0) + (\alpha + 3) \left\| (\Delta)^{-\frac{1}{2}} u_t \right\|_{L^2}^2.
\] (2.34)
By Lemma 2.8 and premise (1.18), we know there exists a constant \( y_2 > 0 \) such that
\[
\| u(t) \|_{H^1}^2 \geq y_2 > C_*^{-\frac{2}{\alpha+1}},
\] (2.35)
and
\[
\phi''(t) \geq (\alpha - 1)(\| u \|_{H^1}^2 - C_*^{-\frac{2}{\alpha+1}}).
\]
\[ \geq (\alpha - 1)(y_2 - C_\star)^{2(n+1)/\alpha-1}t > 0, \]

where
\[ E(0) < \frac{\alpha - 1}{2(\alpha + 1)} C_\star^{2(n+1)/\alpha-1}. \]

Therefore,
\[ \phi'(t) > \phi'(0) + (\alpha - 1)(y_2 - C_\star)^{2(n+1)/\alpha-1}t. \] (2.36)

For sufficiently large \( t_0 \) satisfying \( t_0 < T \), we have \( \phi'(t_0) > 0 \) and as \( \phi(t_0) \geq 0 \), we have \( \phi(t) > 0 \) on \( [t_0, T] \).

On the other hand, (2.32) and (2.34) show
\[
0 < \phi(t)\phi''(t) - \frac{1}{4} (\alpha + 3) \phi'(t)^2
= \phi(t)\phi''(t) - \left(1 + \frac{1}{4}(\alpha - 1)\right) \phi'(t)^2.
\] (2.37)

As a consequence, the Lemma 2.30 validates that there exists \( t_1 \leq T_0 = \frac{\phi(t_0)}{4(\alpha - 1)\phi'(t_0)} + t_0 \) such that
\[
\lim_{t \to t_1} \phi(t) = +\infty.
\] (2.38)

As
\[
\|(-\Delta)^{-\frac{1}{2}}u\|_{L^2} \leq \|(-\Delta)^{-\frac{1}{2}}u_0\|_{L^2} + \int_0^t \|(-\Delta)^{-\frac{1}{2}}u_\tau\|_{L^2} d\tau, \quad \text{a.e. } t \in [0, T].
\]
we have
\[
\int_0^{t_1} \|(-\Delta)^{-\frac{1}{2}}u_\tau\|_{L^2} d\tau = \infty
\]
and this implies there exists a sequence \( \{t_n\} \), \( 0 < t_n < t_1 \), such that
\[
\lim_{t_n \to t_1} \|(-\Delta)^{-\frac{1}{2}}u_\tau\|_{L^2} = \infty,
\]
which contradicts to the assumption \( T_{\max} = \infty \). Thus, the solution \( u \) will blowup under premise (1.18) and
\[
\lim_{t \to T_{\max}} (\|u\|_{H^1} + \|(-\Delta)^{-\frac{1}{2}}u_\tau\|_{L^2}) = \infty,
\]
which combined with (1.2) gives
\[
\lim_{t \to T_{\max}} \|u\|_{L^{p+1}} = \lim_{t \to T_{\max}} \|u\|_{H^1} = \infty.
\]

Conclusion (iii) follows Theorem 4.3 and Lemma 4.4 of Liu [24], as the proof method doesn’t depend on the dimension. Thus, we omit the proof. □
Remark 2.11  The same conclusion of (i), (ii) in Theorem 1.2 can be obtained if \( \beta = -1 \), \( \mathcal{E}(0) < \eta_d \), \( R(u_0) > 0 \) and \( \mathcal{E}(0) < \eta_d \), \( R(u_0) < 0 \) respectively. These correspond to Theorem 3.4, 4.2 of Liu [27]. We remark that our results are better, as we have

\[
\eta_d = \frac{\alpha - 1}{2(\alpha + 1)} C_* \alpha \frac{2(\alpha + 1)}{\alpha}. \tag{2.39}
\]

If \( \mathcal{E}(0) < \eta_d \), \( R(u_0) > 0 \), then \( \mathcal{E}(0) < \frac{\alpha - 1}{2(\alpha + 1)} C_* \alpha \frac{2(\alpha + 1)}{\alpha} \| u_0 \|_{H^1} < C_* \alpha \frac{\alpha + 1}{\alpha}. \tag{2.40} \)

If \( R(u_0) < 0 \), then \( \| u_0 \|_{H^1} > C_* \alpha \frac{\alpha + 1}{\alpha}. \tag{2.41} \)

In fact, multiplying \( \alpha \) with \( \varphi_d \) and integrating on \( \mathbb{R}^d \) gives \( \| \varphi_d \|_{H^1}^2 = \| \varphi_d \|_{L^{\alpha+1}}^\alpha \). Then, \( \| \varphi_d \|_{H^1}^2 \) follows immediately by \( \mathcal{E}(0) < \eta_d \) and \( \mathcal{E}(0) < \frac{\alpha - 1}{2(\alpha + 1)} C_* \alpha \frac{2(\alpha + 1)}{\alpha} \). For \( \mathcal{E}(0) < \frac{\alpha - 1}{2(\alpha + 1)} C_* \alpha \frac{2(\alpha + 1)}{\alpha} \) and \( R(u_0) > 0 \), i.e. \( -\| u_0 \|_{H^1}^2 < -\| u_0 \|_{L^{\alpha+1}}^\alpha \), give

\[
\frac{\alpha - 1}{2(\alpha + 1)} C_* \alpha \frac{2(\alpha + 1)}{\alpha} > \frac{1}{2} \| u_0 \|_{H^1}^2 - \frac{1}{\alpha + 1} \| u_0 \|_{L^{\alpha+1}}^\alpha > \frac{\alpha - 1}{2(\alpha + 1)} \| u_0 \|_{H^1}^2,
\]

thus one has \( \| u_0 \|_{H^1}^2 < C_* \alpha \frac{2(\alpha + 1)}{\alpha} \). \( \tag{2.41} \) follows immediately by \( R(u_0) < 0 \) and the definition of \( C_* \)

\[
\| u_0 \|_{H^1}^2 < \| u_0 \|_{L^{\alpha+1}}^\alpha \leq C_* \alpha \frac{\alpha + 1}{\alpha} \| u_0 \|_{H^1}^\alpha,
\]

i.e. \( \| u_0 \|_{H^1} > C_* \alpha \frac{\alpha + 1}{\alpha} \).

Remark 2.12  Consider the “good” Boussinesq equation for all dimensions

\[
\begin{cases}
\partial_t^2 u - \Delta u + \Delta^2 u + \Delta(u^2) = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d \\
u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x),
\end{cases}
\tag{2.42}
\]

which has conservation law

\[
\mathcal{E}_1(t) := \frac{1}{2} \int_{\mathbb{R}^d} \left( |(-\Delta)^{-\frac{1}{2}} u_t|^2 + u^2 + |\nabla u|^2 \right) - \frac{1}{3} \int_{\mathbb{R}^d} u^3 = \mathcal{E}_1(0).
\tag{2.43}
\]

Though the sign for term \( \frac{1}{3} \int_{\mathbb{R}^d} u^3 \) in \( \tag{2.43} \) is undetermined, the “good” Boussinesq equation \( \tag{2.42} \) is somewhat similar with the focusing generalized Boussinesq equation \( \tag{2.1} \) with \( \alpha = 2 \) and it is easy to verify that Theorem 1.1 and the proof in Subsection 2.2 work for \( \tag{2.42} \) as well. For instance, if we can obtain

\[
\lim_{t \to T_{\text{max}}} (\| u \|_{H^1} + |(-\Delta)^{-\frac{1}{2}} u_t|_{L^2}) = \infty,
\]

then \( \tag{2.43} \) shows

\[
\lim_{t \to T_{\text{max}}} \frac{1}{3} \int_{\mathbb{R}^d} u^3 = \lim_{t \to T_{\text{max}}} \left[ \frac{1}{2} \int_{\mathbb{R}^d} \left( |(-\Delta)^{-\frac{1}{2}} u_t|^2 + u^2 + |\nabla u|^2 \right) - \mathcal{E}_1(0) \right] = \infty,
\]
which combined with \( \int_{\mathbb{R}^d} |u|^3 d^3 u \leq C_{2,d}^3 |u_0|^3_{H^1} \) gives
\[
\lim_{t \to T_{\text{max}}} \|u\|_{L^3} = \lim_{t \to T_{\text{max}}} \|u\|_{H^1} = \infty.
\]
So we have following theorem.

**Theorem 2.13**
(i) The initial value problem (2.42) with \((u, (-\Delta)^{-\frac{1}{2}} u_t)\) in \(H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)\) is locally well-posed for
\[
s = 0, \quad \text{if } d \leq 4,
\]
\[
s = 1, \quad \text{if } d \leq 6.
\]
(ii) Suppose \((u_0, u_t) \in H^1(\mathbb{R}^d) \times \dot{H}^{-1}(\mathbb{R}^d), d \leq 5\), then the initial value problem (2.42) with \((u, u_t)\) in \(H^1(\mathbb{R}^d) \times \dot{H}^{-1}(\mathbb{R}^d)\) is globally well-posed if
\[
E_1(0) < \frac{1}{6C_{2,d}^6}, \quad \|u_0\|_{H^1} < C_{2,d}^{-3},
\]
and blows up in finite time if
\[
u_0 \in \dot{H}^{-1}(\mathbb{R}^d), \quad E(0) < \frac{1}{6C_{2,d}^6}, \quad \|u_0\|_{H^1} > C_{2,d}^{-3}.
\]

3 Small initial data scattering

Similar with the arguments of local wellposedness, we can follow corresponding theory of (1.11) to arrive at the results of small initial data scattering.

**Proof of Theorem 1.3**
Define
\[
\|v\|_{S^1((0,t) \times \mathbb{R}^d)} := \sup_{(p,q) \text{ admissible}} \|\langle \nabla \rangle^s v\|_{L^p_t L^q_x((0,t) \times \mathbb{R}^d)}
\]
and let \(q_1' = (d + 2)/2, p_1' = 2(d + 2)/(d + 6), s_c = d/2 - 2/(\alpha - 1)\). Noticing that the assumption gives
\[
\|v_0\|_{H^1} \leq \|u_0\|_{H^1} + \|(-\Delta)^{-\frac{1}{2}} (1 - \Delta)^{-\frac{1}{2}} u_1\|_{H^1}
\]
\[
\quad = \|u_0\|_{H^1} + \|u_1\|_{\dot{H}^{-1}} \leq \varepsilon,
\]
where \(\varepsilon\) is a small enough constant, then by (2.11), (2.13), (2.19) and Sobolev embedding inequality, one has
\[
\|\mathcal{F} v\|_{S^1((0,t) \times \mathbb{R}^d)} \leq C \|v_0\|_{H^1} + C \|v_0\|_{L^p_t H^1_x((0,t) \times \mathbb{R}^d)}
\]
\[
\leq C\varepsilon + C \|v\|_{L^p_t H^1_x} \|v_0\|_{L^p_t H^1_x((0,t) \times \mathbb{R}^d)}
\]
\[
\leq C\varepsilon + C \|v\|_{L^p_t H^1_x} \|v_0\|_{L^p_t H^1_x((0,t) \times \mathbb{R}^d)}
\]
\[
\leq C\varepsilon + C \|v\|_{S^1((0,t) \times \mathbb{R}^d)}.
\]
where \( l_1 = \frac{2(\alpha - 1) d + 3}{(\alpha - 1) d + 2}, \) \((\alpha - 1)(d + 2)/2, l_1)\) is admissible, \( \alpha \) satisfies (1.19), \( 0 \leq s_c \leq 1 \) and we used Sobolev embedding relationship \( W^{1, l_1}(\mathbb{R}^d) \hookrightarrow W^{s_c, l_1}(\mathbb{R}^d) \hookrightarrow L^{\frac{(\alpha - 1)(d + 2)}{2}}(\mathbb{R}^d) \) (see [3]). Then, standard bootstrap argument shows

\[
\|v\|_{S^1(\mathbb{R} \times \mathbb{R}^d)} \leq C. \tag{3.4}
\]

Let

\[
v^+ = v_0 - i\beta \int_0^{+\infty} e^{it\mathcal{B}[\Re v]^{\alpha - 1}} \Re dt', \tag{3.5}
\]

it follows (1.9) that

\[
v(t) - e^{-it\mathcal{B}} v^+ = i\beta \int_{-\infty}^{+\infty} e^{-i(t-t')\mathcal{B}[\Re v]^{\alpha - 1}} \Re dt', \tag{3.6}
\]

and Strichartz estimate (2.5), (2.6), process of (3.3) that

\[
\|v(t) - e^{-it\mathcal{B}} v^+\|_{H^1} \lesssim \|v\|_{L^\infty_t(\mathbb{R} \times W^{s_c, l_1})} \lesssim \|v\|_{L^\infty_t H^1} \|v\|_{L^{\frac{(\alpha - 1)(d + 2)}{2}}(\mathbb{R} \times W^{s_c, l_1})} \lesssim \|v\|_{S^1(\mathbb{R} \times \mathbb{R}^d)} \leq C, \tag{3.7}
\]

and thus

\[
\lim_{t \to \infty} \|v(t) - e^{-it\mathcal{B}} v^+\|_{H^1} = 0. \tag{3.8}
\]

Similarly,

\[
\lim_{t \to -\infty} \|v(t) - e^{-it\mathcal{B}} v^-\|_{H^1} = 0 \tag{3.9}
\]

for

\[
v^- = v_0 - i\beta \int_{-\infty}^0 e^{it\mathcal{B}[\Re v]^{\alpha - 1}} \Re dt'. \tag{3.10}
\]

Simple calculation on (3.3), (3.10), (1.6) gives

\[
\Re(e^{-it\mathcal{B}}v^\pm) = \cos(t\mathcal{B}) \Re(v_0) + \sin(t\mathcal{B}) \Im(v_0) + \beta \int_{I^\pm} \sin((t - t')\mathcal{B}) \Re [\Re v]^{\alpha - 1} \Re dt',
\]

\[
\Im(e^{-it\mathcal{B}}v^\pm) = -\sin(t\mathcal{B}) u_0 + \cos(t\mathcal{B}) \mathcal{B}^{-1} u_1 + \beta \int_{I^\pm} \cos((t - t')\mathcal{B}) \Re [\Re v]^{\alpha - 1} \Re dt',
\]

where \( I^+ = [0, \infty], I^- = [-\infty, 0] \) and

\[
\cos(t\mathcal{B}) = \mathcal{F}^{-1}_\xi \cos \left( t|\xi| \sqrt{\xi^2 + 1} \right) \mathcal{F}_x, \quad \sin(t\mathcal{B}) = \mathcal{F}^{-1}_\xi \sin \left( t|\xi| \sqrt{\xi^2 + 1} \right) \mathcal{F}_x. \tag{3.11}
\]

Define \( u_0^\pm := \Re(e^{-it\mathcal{B}} v^\pm), u_1^\pm := \Im(e^{-it\mathcal{B}} v^\pm), \) then (3.8), (3.9) show (1.20):

\[
\lim_{t \to \pm \infty} \| (u(t), u(t)) - B(t) (\Re(v^\pm), \Im(v^\pm)) \|_{H^1} = 0, \tag{3.12}
\]

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where $\mathcal{H}^1 = H^1 \times \dot{H}^{-1}$, $B(t) = \left(\begin{array}{cc} \cos(t\mathcal{B}) & \sin(t\mathcal{B}) \\ -\sin(t\mathcal{B}) & \cos(t\mathcal{B}) \end{array} \right)$ and $(u(t), \mathcal{B}^{-1}u(t)) = B(t)(\tilde{u}_0, \mathcal{B}^{-1}\tilde{u}_1)$ solves the linear part of gBQ, i.e.
\[
\begin{cases}
\partial_t^2 u - \Delta u + \Delta^2 u = 0, \\
u(0, x) = \tilde{u}_0(x), \quad u_t(0, x) = \tilde{u}_1(x).
\end{cases}
\]

Remark 3.1 The mass critical case $\alpha = 1 + 4/d$ in Theorem 1.3 can be improved as
\[
\lim_{t \to \pm \infty} \left\| (u(t), u_t(t)) - (u_0^\pm, u_1^\pm) \right\|_{L^2} = 0,
\]
for given $(u_0, (-\Delta)^{-\frac{1}{2}}u_1) \in L^2(\mathbb{R}^d) \times H^{-1}(\mathbb{R}^d)$ small enough. Here, $\| (u_1, u_2) \|_{L^2} := (\|u_1\|_{L^2}^2 + \|(-\Delta)^{-\frac{1}{2}}u_2\|_{H^{-1}}^2)^{\frac{1}{2}}$, $(u_0^\pm, (-\Delta)^{-\frac{1}{2}}u_1^\pm) \in L^2(\mathbb{R}^d) \times H^{-1}(\mathbb{R}^d)$ is the linear solution of (1.1). In fact, we can follow the arguments of Theorem 1.17 in Dodson [7] to obtain that (1.10) scatters in $L^2(\mathbb{R}^d)$ if $\|v_0\|_{L^2(\mathbb{R}^d)}$ small enough. Then, similar deduction with (3.12) shows the conclusion. Likewise, following Theorem 3.1 of [9], the energy critical case $\alpha = 1 + 4/(d - 2), d \geq 3$ can be improved as
\[
\lim_{t \to \pm \infty} \left\| (u(t), u_t(t)) - (u_0^\pm, u_1^\pm) \right\|_{\dot{H}^1} = 0,
\]
for given $(u_0, u_1) \in \dot{H}^1(\mathbb{R}^d) \times H^{-1}(\mathbb{R}^d)$ small enough, where $\| (u_1, u_2) \|_{\dot{H}^1} := (\|u_1\|_{\dot{H}^1}^2 + \|u_2\|_{H^{-1}}^2)^{\frac{1}{2}}$ and $(u_0^\pm, u_1^\pm) \in \dot{H}^1(\mathbb{R}^d) \times H^{-1}(\mathbb{R}^d)$ is the linear solution of (1.1).

4 Large initial data scattering for radial defocusing case

To show larger initial data scattering, it is classical to establish Morawetz-virial type estimate. Similar to the wave and Klein-Gordon equation, we define
\[
M_a(t) := -\int_{\mathbb{R}^d} \left(-\Delta\right)^{-\frac{1}{2}}u_t \left(\nabla a \cdot \nabla \left(-\Delta\right)^{-\frac{1}{2}}u + \frac{1}{2} \Delta a \left(-\Delta\right)^{-\frac{1}{2}}u \right) dx,
\]
where $a$ will be clarified later. See also [28] for more general form of the definition of such quality. Combining the bootstrap argument and Morawetz-virial type estimate, Dodson-Murphy [10] give a new proof of the scattering result in [18] about the cubic Schrödinger equation in dimension three, which can avoid the use of concentration compactness argument. See also [10, 5] for similar argument for other dispersive models. The new ingredient here is that we need to deal carefully with the operator $(-\Delta)^{-\frac{1}{2}}$. Explicitly, we need Lemma 4.2 to estimate the commutator between $(-\Delta)^{\frac{1}{2}}$ and some good enough functions.

Proposition 4.1 Space-time estimate
Suppose $4/d + 1 \leq \alpha \leq (d + 2)/(d - 2), d \geq 3$ and the $H^1(\mathbb{R}^d)$ solution $u$ of (1.1) is globally wellposed, then $u$ satisfies
\[
\int_{I} \int_{\mathbb{R}^d} |u|^\alpha + 1 dx dt \leq \begin{cases} 
C |I|^{\frac{\alpha}{\alpha - 1}}, & \text{if } d \geq 4, \\
C |I|^{\frac{\alpha}{\log |I|}}, & \text{if } d = 3 \text{ and } |I| \geq 2,
\end{cases}
\]
where $\theta$ is a positive constant less than 1 and $I$ is any time interval in $\mathbb{R}$. 

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Proof of Proposition 4.1

Denote \( w = (-\Delta)^{-\frac{1}{2}} u \), \( w_j = \partial x_j w \), \( w_{kl} = \partial x_k \partial x_l w \), \( a_{jk} = \partial x_j \partial x_k a \), then \( \partial x_j w = -R_j u \) and

\[
M_a(t) = \int_{\mathbb{R}^d} a_{jk} (2w_j w_{kl} + w_j w_k) \, dx
\]

\[
+ \int_{\mathbb{R}^d} \frac{1}{4} (\Delta^3 a - \Delta^2 a) \, w^2 - \frac{1}{2} \Delta^2 a |\nabla w|^2 - \frac{1}{2} \Delta a_{jk} w_j w_k + a_{jk} w_j w_k \, dx
\]

(4.2)

It follows Hardy inequality that

\[
\int_{\mathbb{R}^d} |u|^{a-1} u (\Delta)^{-\frac{1}{2}} u + \frac{1}{2} \Delta a_R (\Delta)^{-\frac{1}{2}} u \, dx
\]

For case \( d \geq 4 \), we choose \( a(x) = R^2 a_0(x/R), R \geq 1 \) to define the Morawetz quality \( M_R \), where \( a_0(x) = \tilde{a}_0(|x|) \) and \( \tilde{a}_0(r) \in C^\infty \) is a smooth function in \( \mathbb{R} \) such that

\[
\tilde{a}_0(r) = \begin{cases} 
  r^2, & r < 1/2 \\
  \gamma_1 r + \gamma_2, & r > 1
\end{cases}
\]

(4.3)

for constants \( \gamma_1 > 0, \gamma_2 \in \mathbb{R} \), and

\[
\tilde{a}_0'(r) \geq 0, \quad \tilde{a}_0''(r) \geq 0, \quad \left( \partial_r^2 + \frac{d-1}{r} \partial_r \right) \tilde{a} \leq 0, \quad \forall r > 0.
\]

(4.4)

Thus, one has

\[
\Delta^2 a = 0 \quad \text{for} \quad |x| < R/2, \quad \Delta^2 a \leq 0 \quad \text{for} \quad R/2 < |x| \leq R,
\]

\[
|\partial^\alpha a(x)| \lesssim a |x|^{-|\alpha| + 1} \quad \text{for} \quad |\alpha| \geq 1, \ R/2 \leq |x|.
\]

Note that it is easy to prove there is no nontrivial \( \tilde{a}_0(r) \) in \( d = 3 \) that satisfies both (4.3) and (4.4).

Since \( \{(a_R)_{jk}\} \) is a nonnegative matrix, we have

\[
M_R'(t) \geq \int_{\mathbb{R}^d} |u|^{a-1} u (\Delta)^{-\frac{1}{2}} u + \frac{1}{2} \Delta a_R (\Delta)^{-\frac{1}{2}} u \, dx
\]

\[
- \int_{\mathbb{R}^d} \frac{1}{4} \Delta^3 a_R \, w^2 + \frac{1}{2} \left[ \Delta^2 a_R + \sum_{j,k} \Delta (a_R)_{jk} \right] |\nabla w|^2 \, dx
\]

\[
- \int_{\mathbb{R}^d} \sum_{j,k,l} |(a_R)_{jk}| |w_j| w_{kl} \, dx
\]

(4.5)

\[
\geq \int_{|x| \geq \frac{R}{2}} \frac{R}{|x|^5} |w|^2 + \frac{R}{|x|^3} |\nabla w|^2 + \frac{R}{|x|^2} |\nabla w||\nabla^2 w| \, dx.
\]

It follows Hardy inequality that

\[
\int_{|x| \geq \frac{R}{2}} \frac{R}{|x|^5} |w|^2 \lesssim \frac{1}{R^2} \int_{\mathbb{R}^d} |\nabla w|^2 \lesssim \frac{1}{R^2} \|w\|_{L^2}^2.
\]

Then, Hölder inequality and (4.5) give

\[
M_R'(t) \geq \int_{\mathbb{R}^d} |u|^{a-1} u (\Delta)^{-\frac{1}{2}} u + \frac{1}{2} \Delta a_R (\Delta)^{-\frac{1}{2}} u \, dx - \frac{C}{R} \|u\|_{H^1}^2.
\]

(4.6)
For case $d = 3$, we can set $a(x)$ as those in $d \geq 4$ and $\tilde{a}_0$ as

$$
\tilde{a}_0(r) = \begin{cases} 
  r^2, & r < 1/2 \\
  2r, & r > 1, 
\end{cases}
$$

(4.7)

also

$$
\tilde{a}_1'(r) \geq 0, \quad \tilde{a}_1''(r) \geq 0, \quad \forall r > 0.
$$

As $\Delta^2 a = 0$ for $|x| < R/2$ or $|x| > R$, the same analysis as those in $d \geq 4$ shows

$$
M_R(t) \geq \int_{R^3} |u|^{\alpha-1} u(-\Delta)^{\frac{\theta}{2}} \left( \nabla a_R \cdot \nabla (-\Delta)^{-\frac{1}{2}} u + \frac{1}{2} \Delta a_R (-\Delta)^{-\frac{1}{2}} u \right) \, dx
- \frac{C}{R^2} \| u \|_{H^1(\mathbb{R}^3)}^2 - \frac{C}{R^2} \int_{R/2 \leq |x| \leq R} |u|^2 \, dx.
$$

(4.8)

If one can derive for $d \geq 3$ and for some constant $c > 0, \theta > 0$ that

$$
\int_{R^d} |u|^{\alpha-1} u(-\Delta)^{\frac{\theta}{2}} \left( \nabla a_R \cdot \nabla (-\Delta)^{-\frac{1}{2}} u + \frac{1}{2} \Delta a_R (-\Delta)^{-\frac{1}{2}} u \right) \, dx
\geq c \int_{R^d} |u|^{\alpha+1} \, dx - \frac{C}{R^2} \left( \| u \|_{H^1}^2 + \| u \|_{H^2}^{\alpha+1} \right),
$$

(4.9)

we can utilize (4.6) and (4.8) to get (4.11). Actually, for case $d \geq 4$, (4.6) and (4.9) show

$$
M_R(t) \geq c \int_{R^d} |u|^{\alpha+1} \, dx - \frac{C}{R^2} \left( \| u \|_{H^1}^2 + \| u \|_{H^2}^{\alpha+1} \right),
$$

(4.10)

and the conservation law (1.2), the definition of $a_R$ and Cauchy-Schwarz inequality give $|M_R(t)| \lesssim R \| (-\Delta)^{\frac{\theta}{2}} u \|_{L^2} \| u \|_{H^1} \lesssim R \mathcal{E}(0)$. Integrating (4.10) on any time interval $I$, we have

$$
\int_I \int_{R^d} |u|^{\alpha+1} \, dx \, dt \leq CR + \frac{CI}{R^\theta}
$$

If $|I| > 1$, taking $R = |I|^{1/\theta}$, we obtain (4.11). If $|I| \leq 1$, the (4.11) is trivial by the conservation law (1.2). For case $d = 3$, we can use (4.8) and (4.9) to get

$$
M_R(t) \geq c \int_{R^d} |u|^{\alpha+1} \, dx - \frac{C}{R^2} \left( \| u \|_{H^1(\mathbb{R}^3)}^2 + \| u \|_{H^2(\mathbb{R}^3)}^{\alpha+1} \right) - \frac{C}{R^2} \int_{R/2 \leq |x| \leq R} |u|^2 \, dx
\geq c \int_{R^d} |u|^{\alpha+1} \, dx - \frac{C}{R^2} - \frac{C}{R^2} \int_{R/2 \leq |x| \leq R} |u|^2 \, dx.
$$

(4.11)

Integrating (4.11) on any time interval $I$ and similar discussion as before give

$$
\int_I \int_{R^3} |u|^{\alpha+1} \, dx \, dt \leq C (R + R^{-\theta} |I|) + C \int_{I \cap R/2 \leq |x| \leq R} R^{-2} |u|^2 \, dx dt,
$$

(4.12)

where the $C$ in the right hand side is independent of $R$. Assume $|I| \geq 2$, we can sum (4.12) over $R$ with $R = 2^N \leq |I|^{1/(1+\theta)}$, $N \in \mathbb{N}^*$ and use Hardy inequality to get

$$
\sum_{N \geq 2^N \leq |I|^{1/(1+\theta)}} \int_I \int_{R^3} |u|^{\alpha+1} \, dx \, dt \leq C |I|^{1+\theta} + C |I| + C \int_{I} \int_{R^3} \frac{|u(x)|^2}{|x|^2} \, dx dt
\leq C |I|^{1+\theta} + C |I|
$$

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By Hölder inequality, Sobolev embedding inequality, we have
\[ \int_{\mathbb{R}^3} |u|^{\alpha+1} dx dt \leq C \frac{|I|}{\log |I|}, \]
which is (4.1).

Now, we only need to prove (4.9). Directly calculation shows that
\[
\begin{align*}
\int_{\mathbb{R}^d} \left| u^{\alpha-1} u (-\Delta)^{\frac{1}{2}} \left( \nabla a_R \cdot \nabla (-\Delta)^{\frac{1}{2}} u + \frac{1}{2} \Delta a_R (-\Delta)^{\frac{1}{4}} u \right) \right| dx \\
= \int_{\mathbb{R}^d} (-\Delta)^{-\frac{1}{2}} \left| \left| u \right|^{\alpha-1} u \right| (-\Delta) \left( \partial_j a_R \partial_j (-\Delta)^{\frac{1}{2}} u + \frac{1}{2} \partial_j \partial_j a_R (-\Delta)^{-\frac{1}{2}} u \right) dx \\
= - \int_{\mathbb{R}^d} (-\Delta)^{-\frac{1}{2}} \left( \partial_j (-\Delta)^{-\frac{1}{2}} u \right) u \partial_j a_R dx \\
- 2 \int_{\mathbb{R}^d} (-\Delta)^{-\frac{1}{2}} \left( \left| u \right|^{\alpha-1} u \right) \partial_j a_R \partial_j (-\Delta)^{-\frac{1}{2}} u dx \\
- \int_{\mathbb{R}^d} \frac{1}{2} (-\Delta)^{-\frac{1}{2}} \left| \left| u \right|^{\alpha-1} u \right| \Delta a_R (-\Delta)^{-\frac{1}{2}} u dx \\
+ \int_{\mathbb{R}^d} (-\Delta)^{-\frac{1}{2}} \left| \left| u \right|^{\alpha-1} u \right| \left( -2 \Delta \partial_j a_R \partial_j (-\Delta)^{-\frac{1}{2}} u - \frac{1}{2} \Delta^2 a_R (-\Delta)^{-\frac{1}{2}} u \right) dx \\
:= I + II + III + IV.
\end{align*}
\]

By Hölder inequality, Sobolev embedding inequality \( \|v\|_{L^p(\mathbb{R}^d)} \lesssim \|\nabla v\|_{L^p(\mathbb{R}^d)} \) and interpolation inequality, we have
\[
\begin{align*}
|IV| &\lesssim \int_{|x| \geq \frac{R}{2}} \left| (-\Delta)^{-\frac{1}{2}} \left| \left| u \right|^{\alpha-1} u \right| \left( \frac{R}{|x|^2} \left| \nabla (-\Delta)^{\frac{1}{2}} u \right| + \frac{R}{|x|^3} \left| (-\Delta)^{-\frac{1}{2}} u \right| \right) \right| dx \\
&\lesssim \frac{1}{R} \left\| (-\Delta)^{-\frac{1}{2}} \left| \left| u \right|^{\alpha-1} u \right| \right\|_{L^2} \left( \left\| \nabla (-\Delta)^{\frac{1}{2}} u \right\|_{L^2} + \left\| \frac{1}{|x|} (-\Delta)^{-\frac{1}{2}} u \right\|_{L^2} \right) \\
&\lesssim \frac{1}{R} \left\| \left| \left| u \right|^{\alpha-1} u \right| \right\|_{L^\infty} \left\| u \right\|_{L^2} \\
&\lesssim \frac{1}{R} \left\| u \right\|_{H^1} \left\| u \right\|_{L^2} \lesssim \frac{1}{R},
\end{align*}
\]
where we have used \( \frac{2d-1}{d+2} \leq \frac{2d-2}{d+2} \) or \( \alpha \leq (d+2)/(d-2) \).

To dispose the commutation between \((-\Delta)^{\frac{1}{2}}\) and terms of \(a_R\) in I, II, III, we need the following important lemma from [2], where we give a short clarification in Appendix [3].

**Lemma 4.2** Assume \( \varphi \in C^\infty, \nabla \varphi \in L^\infty, 1 < p < \infty. \) Then,
\[
\left\| \left( (-\Delta)^{\frac{1}{2}}, \varphi \right) f \right\|_{L^p(\mathbb{R}^d)} \lesssim \left\| \nabla \varphi \right\|_{L^\infty(\mathbb{R}^d)} \left\| f \right\|_{L^p(\mathbb{R}^d)}.
\]

We also need the radial Sobolev embedding inequality.

**Lemma 4.3** [7] Let \( d \geq 2 \) and \( 1/2 \leq \varsigma < 1 \), then there exists \( C \) such that for all radial \( u \in H^1 \)
\[
\sup_{x \in \mathbb{R}^d \setminus \{0\}} |x|^{\frac{d}{2} - \varsigma} |u(x)| \leq C(d, \varsigma) \| u \|_{L^2(\mathbb{R}^d)} \| \nabla u \|_{L^\infty(\mathbb{R}^d)}.
\]
Later in this paper, we always set $\zeta = 1/2$. By Lemma 4.2 and (4.16), we have

$$III = -\frac{1}{2} \int \Delta u \left(\Delta a_R \Delta \left|u^{\alpha-1} u\right|\right) dx = -\frac{1}{2} \int \Delta a_R |u|^{\alpha+1} dx + \frac{1}{2} \int \left[\Delta u \Delta a_R \left(\Delta \left|u^{\alpha-1} u\right|\right)\right] dx$$

$$\geq -d \int_{|x| \leq \frac{R}{2}} |u|^{\alpha+1} dx - C \int_{|x| \geq \frac{R}{2}} |u|^{\alpha+1} dx + C \left|\nabla \Delta a_R \right|_{L^\infty} \left\|\left(-\Delta\right)^{\frac{1}{2}} |u^{\alpha-1} u|\right\|_{L^2} \|u\|_{L^2}$$

(4.17)

$$\geq -d \int_{|x| \leq \frac{R}{2}} |u|^{\alpha+1} dx - C \left\|u\right\|_{L^{(2-l)}\left(|x| \geq \frac{R}{2}\right)} \|u\|_{L^2}^2 - \frac{C}{R} \|u\|^{\alpha+1}_{H^1}$$

$$\geq -d \int_{|x| \leq \frac{R}{2}} |u|^{\alpha+1} dx - C \left(R^{-\frac{d-1}{4} (\alpha-1)} + R^{-1}\right).$$

Similarly,

$$II = -2 \int \nabla \partial_k \left(\Delta \left|u^{\alpha-1} u\right|\right) dx - 2 \int \nabla \partial_k \nabla \partial_k \left(\Delta \left|u^{\alpha-1} u\right|\right) dx$$

$$\geq 2d \int_{|x| \leq \frac{R}{2}} |u|^{\alpha+1} dx - C \int_{|x| \geq \frac{R}{2}} |u| \left|\partial_k \partial_k \left(\Delta \left|u^{\alpha-1} u\right|\right)\right| dx$$

(4.18)

$$\geq 2d \int_{|x| \leq \frac{R}{2}} |u|^{\alpha+1} dx - C \left\|u\right\|_{L^{(2-l)}\left(|x| \geq \frac{R}{2}\right)} \|u\|_{L^2} \left\|\left(-\Delta\right)^{\frac{1}{2}} |u^{\alpha-1} u|\right\|_{L^2}$$

$$\geq 2d \int_{|x| \leq \frac{R}{2}} |u|^{\alpha+1} dx - C \left(R^{-\frac{d-1}{4} (\alpha-1)} + R^{-1}\right),$$

where $l = 4/(d + 4), (d - 1)l/2(2 - l) = \frac{(d+2)(d-1)}{2d(d+1)}$.

Noticing that $\left(\Delta \left|\nabla \partial_k \left(\Delta \left|u^{\alpha-1} u\right|\right)\right|\right) \nabla \partial_k \left(\Delta \left(\nabla \partial_k \left(\Delta \left|u^{\alpha-1} u\right|\right)\right)\right) = R_j$, one has

$$I = -\int \left(\Delta \left|u^{\alpha-1} u\right|\right) dx - \int \left(\Delta \left|u^{\alpha-1} u\right|\right) dx$$

$$= -\int \left(\Delta \left|u^{\alpha-1} u\right|\right) dx - 2 \int \nabla \partial_k \left(\Delta \left|u^{\alpha-1} u\right|\right) dx$$

(4.19)

Directly calculation gives

$$I_1 = \frac{2\alpha d}{\alpha + 1} \int |u|^{\alpha+1} dx, \quad I_2 = -2 \int |u|^{\alpha+1} dx.$$ (4.20)
Integrating by parts, we obtain

\[ I_4 = -\int_{\mathbb{R}^d} (1 - \chi_R) u (\partial_j a_R - 2x_j) \partial_j |u|^{\alpha - 1} u dx \]

\[ - \int_{\mathbb{R}^d} (1 - \chi_R) u \left[ (-\Delta)^{\frac{\alpha}{2}} (\partial_j a_R - 2x_j) \right] \partial_j (-\Delta)^{-\frac{\alpha}{2}} |u|^{\alpha - 1} u dx \]

\[ = \frac{\alpha}{\alpha + 1} \int_{\mathbb{R}^d} \partial_j ((1 - \chi_R) (\partial_j a_R - 2x_j)) |u|^{\alpha + 1} dx \]

\[ - \int_{\mathbb{R}^d} (1 - \chi_R) u \left[ (-\Delta)^{\frac{\alpha}{2}} (\partial_j a_R - 2x_j) \right] \partial_j (-\Delta)^{-\frac{\alpha}{2}} |u|^{\alpha - 1} u dx \]

\[ \geq C \left| \left| x \right|^{-\frac{\alpha + 1}{4}} d\right| - C \left( R^{-\frac{\delta}{2}} + R^{-\frac{\delta}{2}} \right), \]

and

\[ I_3 = -\int_{\mathbb{R}^d} (-\Delta)^{\frac{\alpha}{2}} (\chi_R u) (\partial_j a_R - 2x_j) \partial_j (-\Delta)^{-\frac{\alpha}{2}} |u|^{\alpha - 1} u dx \]

\[ \geq C \left| \left| x \right|^{-\frac{\alpha}{2}} (\chi_R u) \right|_{L^2(|x| \geq R/2)} \left| \left| \partial_j (-\Delta)^{-\frac{\alpha}{2}} \right| \left( |u|^{\alpha - 1} u \right) \right|_{L^2}. \]

Given \( x \in \mathbb{R}^d, |x| \geq R/2 \), we have

\[ (-\Delta)^{\frac{\alpha}{2}} (\chi_R u)(x) = \frac{1}{(2\pi)^{\frac{\alpha}{2}}} \int_{\mathbb{R}^d} \frac{\chi_R u(\xi) e^{ix\xi}}{\xi} d\xi \]

\[ = \frac{1}{(2\pi)^{\frac{\alpha}{2}}} \int_{\mathbb{R}^d} \chi(M\xi) |\chi_R u(\xi)| e^{ix\xi} d\xi \]

\[ + \frac{1}{(2\pi)^{\frac{\alpha}{2}}} \int_{\mathbb{R}^d} \left( 1 - \chi(M\xi) \right) |\chi_R u(\xi)| e^{ix\xi} d\xi \]

\[ := A_1 + A_2. \]

\( A_1 \) can be estimated directly

\[ |A_1| \lesssim M^{-\frac{\alpha}{2}} \| \chi_R u \|_{L^2} \lesssim M^{-\frac{\alpha}{2}}. \]

Integrating by parts, we obtain

\[ |A_2| = \left| \frac{1}{(2\pi)^{\frac{\alpha}{2}}} \int_{\mathbb{R}^d} \chi(M\xi) \int_{\mathbb{R}^d} \chi_R u(y) e^{ix(y-x)} \xi d\xi \right| \]

\[ \leq \left| \frac{1}{(2\pi)^{\frac{\alpha}{2}}} \int_{\mathbb{R}^d} \Delta^N \left( (1 - \chi(M\xi)) |\xi| \chi_R u(\xi) \right) e^{ix\xi} d\xi \right| \]

\[ \lesssim |x|^{-2N} M^{2N - d - 1} \| \chi_R u \|_{L^1} \lesssim |x|^{-2N} M^{2N - d - 1} R^{\frac{d}{2}}, \]

where \( N \in \mathbb{N}^* \). Let \( M = |x|^{2N/(2N-d)/2} R^{-d/(4N-d)} \), then

\[ \left| \left| x \right|^{-\frac{\alpha}{2}} \chi_R u \right|_{L^2(|x| \geq R/2)} \lesssim \left| \left| x \right|^{-2N(d+2)/(4N-d)} R^{d(d+2)/(8N-2d)} \right|_{L^{2-2l}(|x| \geq R/2)} \]

\[ \lesssim R^{-\frac{(d+2)^2}{4d}}, \quad \text{where } l = 4/(d+4). \]

Thus,

\[ I_3 \gtrsim -R^{-\frac{d+2}{4d}}. \]
Overall, we have obtained
\[
\int_{\mathbb{R}^d} |u|^{\alpha-1} u(-\Delta)^{\frac{\alpha}{2}} \left( \nabla a_R \cdot \nabla (-\Delta) u + \frac{1}{2} \Delta a_R (-\Delta)^{\frac{\alpha}{2}} u \right) \, dx \\
\geq (-d + 2 + \frac{2ad}{\alpha + 1} - 2) \int_{\mathbb{R}^d} |u|^\alpha \, dx
\]
\[
- \left( \frac{1}{R} + R^{\frac{d}{2}} \left( \frac{d}{\alpha - 1} + \frac{d}{\alpha + 1} \right) + R^{-\frac{d}{2}} \left( \frac{d}{d - 1} \right) \right)
\]
\[
\geq (d + \frac{2ad}{\alpha + 1} - 2) \int_{\mathbb{R}^d} |u|^\alpha \, dx - CR^{-\theta},
\]
where
\[
\theta := \min \left\{ 1, \frac{d-1}{2}(\alpha - 1), \frac{(d+2)(d-1)}{d(d+4)} \right\}.
\]
This is (4.26).

\[\square\]

**Corollary 4.4** For any \( n \in \mathbb{Z}^+ \), we can find \( T_n \geq n \) such that
\[
\|u\|_{L^{\alpha+1}(T_n, n+T_n; L^{\alpha+1})} \leq \frac{1}{2n}.
\]

**Proof:**

If not, then for some \( n_0 \in \mathbb{Z}^+ \) and case \( d \geq 4 \), one has
\[
\|u\|_{L^{\alpha+1}(kn_0, (k+1)n_0; L^{\alpha+1})} > \frac{1}{2n_0}, \quad \forall \, k \geq 1.
\]

For any \( K \in \mathbb{Z}^+ \), it follows (4.4) that
\[
\frac{K}{2(\alpha+1)n_0} < \int_{kn_0}^{(K+1)n_0} \int_{\mathbb{R}^d} |u(t, x)|^\alpha \, dx \, dt \leq CK \frac{1}{n_0^{\frac{d}{d+4}}}.
\]
which is a contradiction when \( K \) is sufficiently large. Similar discussion can show the results for case \( d = 3 \).

\[\square\]

Now, we can be able to obtain the space-time bound.

**Proposition 4.5** The \( H^1 \) wellposed solutions to (1.6) satisfy
\[
\|\langle \nabla \rangle^{sc} v\|_{L_t^q(L_x^2)} \lessapprox \|v_0\|_{H^1}, \quad 1,
\]
where \( sc = d/2 - 2/(\alpha - 1), \quad q = 2(d+2)/d, \quad 4/d + 1 \leq (d+2)/(d-2) \) and \( d \geq 3 \).

**Proof:**

(1.9) can be written as
\[
v(t) = e^{-it\Delta} v_0 - i \int_0^{T_n} e^{-i(t-\tau)\Delta} \mathcal{V}[R(\tau)] \Re(v(t-\tau))^{\alpha-1} \Re(v(t-\tau)) \, d\tau
\]
\[
- i \int_{n+T_n}^{n+T_{n+1}} e^{-i(t-\tau)\Delta} \mathcal{V}[R(\tau)] \Re(v(t-\tau))^{\alpha-1} \Re(v(t-\tau)) \, d\tau
\]
\[
- i \int_{T_n}^t e^{-i(t-\tau)\Delta} \mathcal{V}[R(\tau)] \Re(v(t-\tau))^{\alpha-1} \Re(v(t-\tau)) \, d\tau
\]
\[
:= e^{-it\Delta} v_0 + D_1 + D_2 + D_3.
\]

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Denote $\|v\|_{X(T)} = \|(\nabla)^s v\|_{L^q_t(L^r_x)}$, where $0 \leq s < 1$. For any $T > n + T_n$, it follows Strichartz estimates (2.5), (2.6) that

$$
\|D_3\|_{X([n+T_n, T])} \lesssim \|(\nabla)^s v\|_{L^p_t([n+T_n, T])} L^r_x \left(\frac{1}{\|v\|_{L^q_t([n+T_n, T])}} \right)
$$

(4.30)

where $p'_q = q' / a$, $r'_q = 2d(d+2)/(d^2 + 6d + 8 - 2ad)$, $r = (\alpha - 1)(d+2)/(4 + (3 - \alpha)d)$.

Noticing that $\|D_2\|_{X([n+T_n, T])} \lesssim \|v(n+T_n)\|_{H^s} + \|v(T_n)\|_{H^s} \lesssim \mathcal{E}(0)$ (4.31) and the Gagliardo–Nirenberg inequality, (4.27), Strichartz estimates (2.5), (2.6) to give

$$
\|D_2\|_{X([n+T_n, T])} \lesssim \|(\nabla)^{s_0} v\|_{L^q_t([n+T_n, T])} \|D_2\|_{L^p_t([n+T_n, T])} \lesssim \|u\|_{L^2([n+T_n, T])} \left(\frac{1}{\|v\|_{L^q_t([n+T_n, T])}} \right)
$$

(4.32)

To dispose $D_1$, we need the radial Strichartz estimates. For radial symmetry $\phi \in L^2(\mathbb{R}^d), d \geq 2$, the (3.16) in Corollary 3.4. of Z. Guo et al. [15] gives

$$
\|e^{-it\mathbf{B}} \phi\|_{L^q_t L^r_x} \lesssim \|\phi\|_{L^q_x}, \quad d \geq 2
$$

(4.33)

where $(r, q)$ satisfy $q (1/2 - 1/r) = 1/(d-1)$, $2/(2d-1) \leq q (1/2 - 1/r)$, and it follows that

$$
\|D_1\|_{L^q([0, T_n] \cup [T_n, T])} \lesssim \|D_1(0)\|_{L^q} \lesssim 1. \quad (4.34)
$$

Assume $(2 - \alpha)/2 \leq 1/r < (d - 2)/2d, 2 < r < \infty$, then

$$
\|D_1\|_{L^q_t([n+T_n, T])} \lesssim \int_0^{T_n} \int_0^T (T_n + n - s)^{-d(\frac{1}{2} - \frac{1}{r})} \|u(s)\|_{L^q}^\alpha ds
$$

(4.35)

where we used $\|e^{-it\mathbf{B}} f\|_{L^r} \leq c|t|^{-\alpha(d/2 - 1/r)} \|f\|_{L^r}$.

If $r \geq 4 - 4/d$, it follows interpolation inequality, Hölder inequality and Strichartz estimates (4.5), (4.6) that

$$
\|D_1\|_{L^q_t([n+T_n, T])} \lesssim \|D_1\|_{L^q_t([n+T_n, T])} \|D_1\|_{L^q_t([n+T_n, T])} \lesssim \|u\|_{L^q_t([n+T_n, T])} \lesssim \|u\|_{L^q_t([n+T_n, T])}
$$

(4.36)

where $\theta_1 = (d-2)r/((d-1)(d+2) (r-2)), \theta_2 = (dr - 4d+4)/((d-1)(d+2) (r-2)), 1 - \theta_1 - \theta_2 = d/(d+2)$. If $r \leq 4 - 4/d$, one has

$$
\|D_1\|_{L^q_t([0, T_n])} \lesssim \|D_1\|_{L^q_t([0, T_n])} \|D_1\|_{L^q_t([0, T_n])} \|D_1\|_{L^q_t([0, T_n])} \lesssim \|u\|_{L^q_t([0, T_n])}
$$

(4.37)
Thus, we obtain
\[
\theta_1 = 2/(d + 2), \theta_2 = d(4d - rd - 4)/((d - 2)(d + 2)r).
\]
Similarly as (1.31), one has
\[
\|\langle \nabla \rangle D_1\|_{L^1_t L^2_x} \lesssim \varepsilon(0),
\]
thus
\[
D_1 \|_{X([n + T_n, T])} \lesssim \|\langle \nabla \rangle D_1\|_{L^1_t L^2_x([n + T_n, T]; L^2_x)}^{1-s_\alpha}
\]
(4.38)

For any given \( \varepsilon > 0 \), since \( \|e^{-it\Delta}v(0)\|_X \lesssim \|v(0)\|_{H^{s_\alpha}} < \infty \), we can choose \( n \) sufficient large such that \( \|e^{-it\Delta}v_0\|_X \leq \varepsilon \) and follow (4.32), (4.38) to get
\[
\|D_1\|_{X([n + T_n, T])} + \|D_2\|_{X([n + T_n, T])} \leq \varepsilon.
\]
Thus, we obtain
\[
\|v\|_{X([n + T_n, T])} \leq 2\varepsilon + C\|v\|_{X([n + T_n, T])}^{1+\alpha}, \quad \forall T > 0.
\]
By standard bootstrap argument, we arrive at \( v \in X(\mathbb{R}) \).

\[\square\]

**Corollary 4.6** The \( H^1 \) wellposed solutions to (1.6) satisfies
\[
\|v\|_{S^1(R \times \mathbb{R}^d)} \lesssim \|v_0\|_{H^1}, \quad (4.39)
\]
where \( S^1(R \times \mathbb{R}^d) \) is defined as (3.1) and the \( \alpha \) in (1.6) satisfies \( 4/d + 1 \leq \alpha < (d + 2)/(d - 2) \) and \( d \geq 3 \).

Combining with (4.28) and follow the standard analysis in Proposition 3.31 of T.Tao [29], we can obtain the (4.39), thus can prove Theorem 1.4. In fact, for \( t_1 < t_2 \), we have
\[
\left\|e^{it_1\Delta}v(t_1) - e^{it_2\Delta}v(t_2)\right\|_{H^1} \sim \left\|\int_{t_1}^{t_2} e^{i\tau\Delta} \langle \nabla \rangle \text{Re}(v)^{\alpha - 1}\text{Re}(v) d\tau\right\|_{L^2}
\]
(4.40)
\[
\lesssim \|\langle \nabla \rangle v\|_{L^2_t((t_1, t_2] ; L^2_x)}^{\alpha - 1}
\]
\[
\lesssim \|\langle \nabla \rangle v\|_{L^2_t((t_1, t_2] ; L^2)}\|v\|_{L^2_t((t_1, t_2) ; L^2_x)}^{\alpha - 1}
\]
\[
\lesssim \|v\|_{S^1((t_1, t_2) \times \mathbb{R}^d)} \to 0, \quad \text{as } t_1, t_2 \to \infty,
\]
where \( q, p_2, r_2, r \) is defined as in (4.30). The rest is as those in Section 3.

**Remark 4.7** One of the surprising points of Theorem 1.4 is that we can prove scattering for the mass critical defocusing case \( \alpha = 1 + d/4, d \geq 3 \) with large radial initial data, while we do not use of the concentration compactness argument. This is different from those of defocusing Schrödinger equation (1.11). By inequality (A.2), we know that the low frequency of linear estimate of \( gBQ \) is better than that of Schrödinger equation (1.11). Then, we can get radial Strichartz estimate as (4.33) (see Guo et al. [15]) and thus can interpolate between (4.33) and energy estimate to obtain the smallness of \( D_1 \) in \( X([n + T_n, T]) \). Note that Strichartz estimate as (4.33) is impossible for Schrödinger equation (1.6) within the analysis in Guo et al. [25, 15].
A \ Analysis without transformation

We can prove Theorem 1.1 without using the transform (1.5) in the Introduction. The Duhamel principle shows (1.1) is equivalent to the integral equation,

\[ u(t) = \cos(t\mathcal{B})u_0 + \sin(t\mathcal{B})\mathfrak{M}^{-1}u_1 - \beta \int_0^t \sin((t-s)\mathcal{B})\mathfrak{M}(|u|^\alpha u) \, ds, \quad (A.1) \]

where \( \mathcal{B}, \mathfrak{M}, \cos(t\mathcal{B}), \sin(t\mathcal{B}) \) are defined as (1.7), (1.11) respectively. As \( \cos(t\mathcal{B})f, \sin(t\mathcal{B})f \) are the real and imaginary part of \( e^{-it\mathcal{B}}f \) respectively, the Strichartz estimates (25), (2.6) may be still applied to \( \cos(t\mathcal{B}), \sin(t\mathcal{B}) \). One can obtain local wellposedness results that are analogous to those in Section 2 and (A.1) is equivalent to (1.1). For instance, by (28), (2.5), (2.6), we have

\[ \|u\|_{L_p^p([0,T];W^{s,a}(\mathbb{R}^d))} \lesssim \|u_0\|_{H^s(\mathbb{R}^d)} + \|\mathcal{B}^{-1}u_1\|_{H^s(\mathbb{R}^d)} + \|\|\mathfrak{M}\|_{L_p^p([0,T];W^{s,a})}} + \|\mathfrak{M}\|_{L_p^p([0,T];W^{s,a})}}, \]

\[ \leq \|u_0\|_{H^s(\mathbb{R}^d)} + \|(-\Delta)^{s/2}u_1\|_{L_p^p(\mathbb{R}^d)} + \|\|\mathfrak{M}\|_{W^{s,a}(\mathbb{R}^d)}} + \|\mathfrak{M}\|_{L_p^p([0,T];W^{s,a})}}, \]

\[ \leq \|u_0\|_{H^s(\mathbb{R}^d)} + \|(-\Delta)^{s/2}u_1\|_{L_p^p(\mathbb{R}^d)} + \|\mathfrak{M}\|_{L_p^p([0,T];W^{s,a})}}. \]

A crucial tool to obtain these results is the stationary phase estimate derived in Gustafson et al. [17], Cho et al. [6]

\[ \sup_{x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} e^{i(x \cdot \xi + t|\xi| \sqrt{1+|\xi|^2})} \varphi \left( \frac{\xi}{N} \right) d\xi \right| \lesssim \frac{N|N|^2}{(N|^2)^{3/2}}, \quad (A.2) \]

where \( N \in \mathbb{Z}^d, \varphi(\cdot) \) is the Littlewood-Paley function. This immediately gives the the Strichartz estimates (2.1), (2.2), (2.3) and improves the decay estimate introduced in

\[ \sup_{x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} e^{i(x \cdot \xi + t|\xi| \sqrt{1+|\xi|^2})} d\xi \right| \leq C \left| t \right|^{-1/2} + \left| t \right|^{-1/3}, \]

which is based on the Van der Corput lemma, thus improves the Strichartz estimates in Liu [21] and the \( L_p-L_q \) estimates in Linares [25], which is a special case of (2.1), (2.6) in one dimension. Wang et al. [31] obtained the local wellposedness for the generalized Boussinesq equation by adding damped terms \(-\alpha \Delta u_t + \gamma \Delta^2 u, \alpha \geq 0, \gamma > 0, \) thus to give exponential decay term \( e^{-\gamma^2/2(t+|\xi|)} \) in the integral equation and obtain the Strichartz type estimates. Here, we can get corresponding results of [31] just within (A.2) and (2.1), (2.3).

B \ Proof of Lemma 4.2

It is a corollary of Theorem 2 of Calderón [2], i.e.

**Theorem B.1** Let \( h(x) \) be homogeneous of degree \(-d-1\) and locally integrable in \(|x| > 0\). Let \( b(x) \) have first-order derivatives in \( L^r, 1 < r \leq \infty \). Then, if \( 1 < p < \infty, 1 < q < \infty, q^{-1} = p^{-1} + r^{-1}, h(x) \) is an even function and

\[ C_r(f) = \int_{|x-y| > \varepsilon} h(x-y)[b(x) - b(y)]f(y)dy. \]
Then, we can see (4.15) is the simplest case of Theorem B.1. For a fixed $C$, $\|C(f)\|_q \leq c\|\text{grad} b\|_r \|f\|_p \int |f(x)| \, dx$, where the integral is extended over $|x| = 1$, $dv$ denotes the surface area of $|x| = 1$, and $c$ depends on $p$ and $r$ but not on $\varepsilon$. Furthermore, $\varepsilon$ tends to zero $C(\varepsilon)$ converges in norm in $L^q$.

In fact, denote $c_{d,\alpha} := \int_{R^d} |x|^{-\alpha} e^{-\varepsilon|x|^2/2} \, dx$, then

$$D = \sum_{j=1}^d R_j \partial_j, \quad \mathcal{F}[\cdot]^{-1} = c_1 |\cdot|^{-d+1/c_{d-1}},$$

$$-\varepsilon^{-1} \mathcal{F}^{-1} \mathcal{F}^{-1}[\varepsilon] = -\partial_j \mathcal{F}^{-1}[\varepsilon]^{-1} = -c_1 (-d+1)|\cdot|^{-d-1} x_j/c_{d-1}, \quad \text{for } d \geq 3,$$

$$R_j f(x) = c \lim_{\varepsilon \to 0^+} \int_{B(0,\varepsilon)c} \frac{y_j f(x-y)}{|y|^{d+1}} \, dy, \quad c = c_1 (d-1)/c_{d-1}.$$

For $f \in C_0^\infty (R^d), \varphi \in C^{1,\alpha}(R^d)$, one has

$$[D, \varphi] f = D(\varphi f) - \varphi D f = \sum_{j=1}^d R_j (\partial_j \varphi f) + R_j (\varphi \partial_j f) - \varphi R_j \partial_j f \quad \text{for } x \neq y$$

$$= \sum_{j=1}^d R_j (\partial_j \varphi f) + \sum_{j=1}^d \lim_{\varepsilon \to 0^+} \int_{B(0,\varepsilon)c} \frac{y_j (\varphi(x-y) - \varphi(x)) \partial_j f(x-y)}{|y|^{d+1}} \, dy$$

$$= c \lim_{\varepsilon \to 0^+} \int_{B(0,\varepsilon)c} \frac{-1}{|x|^{d+1}} (\varphi(x-y) - \varphi(x)) f(x-y) \, dy.$$

Replace $x - y$ with $x$, we have

$$[D, \varphi] f(x) = c \lim_{\varepsilon \to 0^+} \int_{|x-y| > \varepsilon} \frac{1}{|x-y|^{d+1}} (\varphi(x) - \varphi(x-y)) f(y) \, dy. \quad (B.1)$$

Then, we can see (4.15) is the simplest case of Theorem B.1.

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