Exceptional points in Faddeev scattering problem

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Abstract

Exceptional points are values of the spectral parameter for which the homogeneous Faddeev scattering problem has a non-trivial solution. We show that these points coincide with the values of the parameter where a certain self-adjoint family of operators has a non-trivial kernel. This allows us to introduce a counting function for those points and obtain other results. In particular, we show that problems with absorbing potentials do not have exceptional points. We study small perturbations of conductive potentials of arbitrary shape. It is also shown in the paper that each value of the spectral parameter in the transmission scattering problem is an exceptional point for infinitely many values of energy.

Key words: Scattering problem, exceptional points, Faddeev’s Green function, conductive potentials, absorbing potentials

1 Introduction

The paper concerns the 2-D Faddeev scattering problem where incident waves grow exponentially at infinity. This problem is used often for solving inverse problems [7–9]. Let us recall the statement of the Faddeev problem. Let $O$ be an open bounded domain in $\mathbb{R}^2 = \mathbb{R}_x \times \mathbb{R}_y$, $z = (x, y)$, with $C^2$ boundary $\partial O$ and the outward normal $\nu$.

Let $\zeta = (\zeta_1, \zeta_2) \neq 0$ be a vector with complex components $\zeta_i \in \mathbb{C}$ and $\zeta_1^2 + \zeta_2^2 = E \geq 0$. First, we will consider the case when the energy $E$ is equal to zero, i.e., $\zeta = (k, ik)$, $k \in \mathbb{C} \setminus 0$. Then we will extend the results to the case of an arbitrary positive energy.

If $E = 0$, then $u = u(z, \zeta)$ is a solution of the Faddeev scattering problem if

$$-\Delta u - nu = 0, \quad z \in \mathbb{R}^2, \quad (1)$$

$$u(z, \zeta) = e^{i\zeta \cdot z} + u_{\text{out}}, \quad e^{-i\zeta \cdot z} u_{\text{out}} \in W^{1,p}(\mathbb{R}^2), \quad p > 2, \quad (2)$$

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where \( \zeta \cdot z = \zeta_1 x + \zeta_2 y \) and the potential \( n = n(z) \) is bounded in \( \mathcal{O} \) and vanishes outside \( \mathcal{O} \).

The main object of our study is the set \( \mathcal{E} \) of exceptional points \( 0 \neq k \in \mathbb{C} \) such that the homogeneous problem \((1), (2)\) with \( \zeta = (k, ik) \) has a nontrivial solution. This homogeneous problem has the form

\[
- \Delta v - nv = 0, \quad z \in \mathbb{R}^2; \quad e^{-ik \cdot z} v \in W^{1,p}(\mathbb{R}^2), \quad p > 2.
\]

We will say that an exceptional point has multiplicity \( m \) if the dimension of the solution space of problem \((3)\) is \( m \). Certain inverse methods based on the Faddeev scattering require the set \( \mathcal{E} \) to be empty, see \([8],[9],[1]\), and this makes the study of \( \mathcal{E} \) particularly important.

The incident waves in the classical scattering problem (with positive energy) have the form \( e^{i(k_1 x + k_2 y)} \). The exceptional set in this case is empty due to the absence of eigenvalues imbedded into continuous spectrum (a very simple proof can be found in \([14]\)). Similar arguments do not work for the Faddeev scattering problem since solutions of \((3)\) do not decay at infinity.

There exists a very limited number of results concerning the existence or non-existence of exceptional points for the Faddeev scattering problem. The first example of a problem with exceptional points was constructed in \([2]\). It is known that the exceptional points do not exist in the case of sufficiently small potentials \([8]\) and conductive potentials \([7]\). The latter potentials have the form \( n = -q^{-\frac{1}{2}} \Delta q^\frac{1}{2} \), where \( q \) is smooth, non-negative, and \( q - 1 \) vanishes outside \( \mathcal{O} \). In this case, equation \((1)\) can be reduced to the equation \( \nabla q \nabla v = 0 \) by the substitution \( u = \sqrt{q} v \). Sign definite perturbations of conductive potentials were studied in \([6],[12]\) under the condition that the potential and its perturbation are spherically symmetric. The existence or non-existence of exceptional points in this case depends on the sign of the perturbation.

One can find solution \( u(z,k) \) of \((1), (2)\) by reducing the problem to the Lipmann-Schwinger equation which leads to (see e.g. \([8]\))

\[
u(z,k) = (I + S_k(F_n - F_0))^{-1} e^{i \zeta \cdot z}, \quad z \in \partial \mathcal{O}. \quad (4)\]

Here \( F_n \) is the Dirichlet-to-Neumann map for the equation \((-\Delta - n)u = 0 \) in \( \mathcal{O} \), \( F_0 = F_n|_{n=0}, S = S_k \) is the single layer operator on the boundary with Faddeev’s Green function \( G_k(z) \):

\[
S_k : H^{-\frac{1}{2}}(\partial \mathcal{O}) \to H^{\frac{1}{2}}(\partial \mathcal{O}); \quad S_k \sigma(z) = \int_{\partial \mathcal{O}} G_k(z - z') \sigma(z') dl_{z'}, \quad z \in \partial \mathcal{O}, \quad (5)\]

where \( dl \) is the element of the length and

\[
G_k(z) = \frac{1}{2\pi^2} e^{i \zeta \cdot z} \int_{\mathbb{R}^2} \frac{e^{i(\xi_1 x + \xi_2 y)}}{\xi^2 + 2k\xi} d\xi_1 d\xi_2, \quad \xi = \xi_1 + i \xi_2. \quad (2.8)\]
Function $G_k$ is real-valued (e.g., [13, Part 3.1.1]). Indeed, the second condition in (3) can be written in the form $|e^{i\zeta \cdot z}|u_{\text{out}} \in W^{1,p}(\mathbb{R}^2)$. From here it follows that $\Re G_k$ is the Green function, while $\Im G_k = 0$ due to Nachman’s uniqueness result [7].

Equation (4) provides the standard basis for studying the exceptional set $E$: exceptional points can be defined as the values of the parameter $k \in \mathbb{C}\{0\}$ for which the non-self-adjoint family of operators

$$I + S_k(F_n - F_0)$$

has a non-trivial kernel. It is natural to consider (3) as operator in $L^2(\partial \Omega)$ or $H^4(\partial \Omega)$. The second term in (6) is a compact operator in any Sobolev space, and the kernel of operator (6) does not depend on the choice of the space.

**Description of the results obtained below.** Our progress in study of $E$ is based on establishing a connection between exceptional values of $k$ and kernels of a family of self-adjoint operators, which are much easier to control than kernels of non-selfadjoint operator (6). Our first result (Section 2) states that a point $k \in \mathbb{C}\{0\}$ is exceptional if and only if the kernel of the self-adjoint family of operators $F_n - F_{\text{out}}(k)$ is nontrivial. Moreover, the dimension of the kernel coincides with the multiplicity of the exceptional point (which is the dimension of the solution space of problem (3)). Here $F_{\text{out}}(k)$ is the Dirichlet-to-Neumann map for the Faddeev scattering problem in $\mathbb{R}^2 \setminus \Omega$.

As a simple consequence, this implies the absence of exceptional points in the case of absorbing potentials. This reduction to the family of self-adjoint operators also allows us (Section 4) to generalise results of [6], [12] on perturbations of conducive potentials to non-spherically symmetrical problems.

More profound results are obtained (Section 5) if the potential has a non-vanishing jump on $\partial \Omega$ (or the potential and its normal derivatives of orders $j < m$ are continuous, and the normal derivative of order $m$ has a jump on $\partial \Omega$). Then $F_n - F_{\text{out}}$ can be replaced by another self-adjoint family of operators $P(k) = -\sigma [(F_n - F_0)^{-1} + S_k]$, which are elliptic pseudo-differential operators with positive principal symbol. Here $\sigma = \pm 1$ depends on the sign of the jump. Denote by $n^-(k)$ the number of negative eigenvalues of $P(k)$. Obviously, $n^-(k) < \infty$. We show (Section 5) that if $n^-(k) \neq n^-(\hat{k})$ for a couple of points $k, \hat{k} \in \mathbb{C}\{0\}$, then there exists at least one exceptional point on every analytical path connecting $k$ and $\hat{k}$. Moreover, the number of exceptional points on the path with their multiplicity taken into account can be estimated from below by $|n^-(k) - n^-(\hat{k})|$. The same result holds (Section 6) in the case of positive energy.

Finally, we consider the transmission problem with the potential of the form $E_n$ (Section 7) and show that each value of the spectral parameter $\lambda$ is an exceptional point for infinitely many values $E = E_i$ of energy, and these values of energy grow linearly as $E \to \infty$.

The paper also contains a couple of important technical results concerning the asymptotic behavior of operator $F_{\text{out}}(k)$ as $k \to 0$ (Section 3) and an estimate of operator $S_k$ as $k \to \infty$ (Section 5).
2 Reduction to self-adjoint operators. Zero energy case

We consider the Faddeev scattering problem (1), (2) in this section. Without loss of the generality we can assume that the equation \(-\Delta v - nv = 0\) in \(\mathcal{O}\) does not have non-trivial solutions vanishing at the boundary (i.e., zero is not an eigenvalue of the interior Dirichlet problem). In other words, the operator \(F_n\) is well defined (on the whole space). We can make sure that this condition holds by extending the domain \(\mathcal{O}\) slightly while preserving the function \(n\).

Consider the exterior Dirichlet problem

\[-\Delta u = 0, \quad z \in \mathbb{R}^2 \setminus \mathcal{O}; \quad u|_{\partial \mathcal{O}} = f \in H^{\frac{1}{2}}(\partial \mathcal{O}); \quad e^{-ik \cdot z}u \in W^{1,p}(\mathbb{R}^2), \quad p > 2.\] (7)

By \(F_{\text{out}}^n(k) : H^{\frac{1}{2}}(\partial \mathcal{O}) \rightarrow H^{-\frac{1}{2}}(\partial \mathcal{O})\) we denote the operator that maps the Dirichlet data \(f\) into the outward (with respect to \(\mathcal{O}\)) normal derivative \(u_\nu\) of the solution of the problem (7) at the boundary \(\partial \mathcal{O}\). Denote by \(E_D\) the set of exceptional values of \(k \in \mathbb{C} \setminus \{0\}\) for which the homogeneous problem (7) has a non-trivial solution (the subindex \(D\) here stands for the Dirichlet).

**Definition.** We will call a set \(\{k = k_1 + ik_2\} \subset \mathbb{C}\) a 1-D real analytic variety if the set of corresponding points \((k_1, k_2)\) \( \in \mathbb{R}^2\) is an intersection of a 1-D analytic variety in \(\mathbb{C}^2 = \mathbb{C} k_1, k_2\) with the Euclidean space \(\mathbb{R}^2\). Let us stress the meaning of the notation \(k\). It will be used for points \(k = k_1 + ik_2\) of the complex plane. When we need to think about these points as vectors in the Euclidean space \(\mathbb{R}^2\), we will use notation \((k_1, k_2)\) instead of \(k\).

Let operator \(\hat{S}_k : H^{-\frac{1}{2}}(\partial \mathcal{O}) \rightarrow W^{1,p}(\mathbb{R}^2 \setminus \mathcal{O})\), \(p > 2\), be the extension of the single layer operator defined by the same formula as in (5), but for all \(z \in \mathbb{R}^2 \setminus \mathcal{O}\). We will use the same notation when the extension is considered on the whole space \(\mathbb{R}^2\).

The following lemma concerns the exterior Faddeev problem (7).

**Lemma 2.1.** 1) The set \(E_D \subset \mathbb{C}\) is a real analytic variety and coincides with the set \(K\) of values of \(0 \neq k \in \mathbb{C}\) for which the operator \(S_k\) has a non-trivial kernel. The operator \(S_k\) is onto when \(k \notin E_D \cup \{0\}\).

2) When \(k \notin E_D\), the Dirichlet-to-Neumann map \(F_{\text{out}}(k)\) of the exterior Faddeev problem (7) exists, is analytic in each of the variables \(k_1, k_2\), and is a self-adjoint elliptic pseudo-differential operator of the first order with negative symbol.

**Proof.** We will start with a study of invertibility of operator \(S_k\) defined by (5). In particular, we are going to prove that the set \(K\) is a real analytic variety. Let

\[G^0_k(z) = -\frac{1}{2\pi} \ln |z| - \frac{\gamma}{2\pi} - \frac{1}{2\pi} \ln |k|,\]

and let \(S^0_k : H^{-\frac{1}{2}}(\partial \mathcal{O}) \rightarrow H^\frac{1}{2}(\partial \mathcal{O})\) be the single layer operator (similar to (5)) defined by the kernel \(G^0_k(z)\):

\[S^0_k \sigma(z) = -\frac{1}{2\pi} \int_{\partial \mathcal{O}} G^0_k(z - z') \sigma(z') dl_{z'}, \quad z \in \partial \mathcal{O}.\] (8)
Note (eg [3]) that $G_k - G_k^0$ and its derivatives tend to zero uniformly when $k \to 0$. Thus $\|S_k - S_k^0\| \to 0$ as $k \to 0$. We will use notations $\hat{S}_k, \tilde{S}_k^0$ when the corresponding single layer operator is extended to either of the domains $\mathbb{R}^2$, $\mathcal{O}$, or $\mathbb{R}^2 \setminus \mathcal{O}$.

Operator $S_k^0$ is an elliptic PDO of order $-1$ on the compact manifold $\partial \mathcal{O}$, and therefore it has zero index. Then the same is true for the operator $S_k$, since function $G_k - G_k^0$ is infinitely smooth in $(z, k)$ and analytic in $k_1, k_2$ where $k = k_1 + ik_2 \neq 0$ (eg [6]). Thus $S_k$ is a Fredholm family of operators analytic in $k_1, k_2$. Hence, if $S_k$ is invertible at one point $k \neq 0$, then the set of values of $k \in \mathbb{C}\{0\}$ for which $S_k$ has a non-trivial kernel form a 1-D real analytic variety (see [5]). We will prove below that $\mathcal{K} = \mathcal{E}_D$, but first we will show that operator $S_k$ is invertible for small $|k| > 0$.

We represent spaces $H^{-\frac{1}{2}}(\partial \mathcal{O})$ and $H^{\frac{1}{2}}(\partial \mathcal{O})$ as direct sums of the one dimensional space of constants and their orthogonal complements $H^{-\frac{1}{2}, \perp}(\partial \mathcal{O})$ and $H^{\frac{1}{2}, \perp}(\partial \mathcal{O})$, respectively. Then operator $S_k^0$ can be written as the following matrix

$$S_k^0 = \begin{pmatrix} -\frac{\alpha}{2\pi} - \frac{1}{2\pi} \ln |k| & b_1 \\ b_2 & B \end{pmatrix},$$

where $B : H^{-\frac{1}{2}, \perp}(\partial \mathcal{O}) \to H^{\frac{1}{2}, \perp}(\partial \mathcal{O})$ is the single layer operator (similar to ([8])) with the kernel $\frac{-1}{2\pi} \ln |z - z'|$. Operator $B$ is a pseudo-differential operator of order $-1$. Operators $b_1, b_2, B$ are bounded and $k$-independent. From the standard potential theory, it follows that operator $B^{-1}$ is bounded. Obviously, matrix $S_k^0$ is invertible, and therefore, operator $S_k$ is invertible for small $|k| > 0$. Thus $\mathcal{K}$ is a 1-D real analytic variety.

Let us show that $\mathcal{E}_D = \mathcal{K}$. Consider the problem

$$-\Delta u = 0, \ z \in \mathbb{R}^2 \setminus \mathcal{O}; \ u|_{\partial \mathcal{O}} = f \in H^{\frac{1}{2}}(\partial \mathcal{O}); \quad e^{-i\zeta \cdot z} u \in W^{1,p}(\mathbb{R}^2), \ p > 2.$$ \hspace{1cm} (10)

Let $k = k' \notin \mathcal{K}$. Then operator $S_k$ is onto, and there is a function $\mu \in H^{-\frac{1}{2}}(\partial \mathcal{O})$ such that $S_k \mu = f$. Thus, $u = \hat{S}_k \mu$ is a solution of (10). If this solution is unique, then operator $F_{\text{out}}^{\text{out}}$ is well defined (by $F_{\text{out}}^{\text{out}} f = u|_{\partial \mathcal{O}}$) and $k' \notin \mathcal{E}_D$. Assume that there exists a non-trivial solution $u$ of the homogeneous problem (11) when $k = k'$. Denote by $v$ the extension of $u$ by zero in $\mathcal{O}$. Then

$$-\Delta v = \alpha \delta(\partial \mathcal{O}), \ z \in \mathbb{R}^2; \quad e^{-i\zeta \cdot z} v \in W^{1,p}(\mathbb{R}^2), \ p > 2,$$

where $\delta(\partial \mathcal{O})$ is the delta-function on $\partial \mathcal{O}$ and $\alpha = u|_{\partial \mathcal{O}}$. From the Nachman uniqueness result [7, Lemma 1.3], it follows that $\alpha \neq 0$ (otherwise $v \equiv 0$) and that $v = \hat{S}_{k'} \alpha \delta(\partial \mathcal{O})$. Thus $0 = u|_{\partial \mathcal{O}} = \hat{S}_{k'} \alpha$. This contradicts the assumption that $k' \notin \mathcal{K}$. Thus $k' \notin \mathcal{E}_D$.

Assume now that $k = k' \in \mathcal{K}$. Then there exists a non-trivial $\mu$ such that $S_{k'} \mu = 0$. Function $v = \hat{S}_{k'} \alpha \delta(\partial \mathcal{O})$ is a solution of (11). Function $v$ vanishes in $\mathcal{O}$ since $v$ is harmonic there and $v = 0$ on $\partial \mathcal{O}$. Since the jump of the normal derivative of $v$ is proportional to $\mu$, function $v$ is not identically equal to zero. Thus $v$ is a non-trivial solution of homogeneous ($f = 0$) problem (10). Thus $k' \in \mathcal{E}_D$. Hence $\mathcal{K} = \mathcal{E}_D$. To complete the proof of the first statement of Lemma [2.1], it remains to recall that operator $S_k$ has zero index, and therefore it is onto when the kernel is trivial.
Let us prove the second statement of the Lemma. The following simple formula from the potential theory is valid:

$$(F_0 - F_{\text{out}})S_k = I.$$  \hfill (12)

This formula implies that

$$F_0 - F_{\text{out}} = (S_k)^{-1}, \quad k \notin \mathcal{E}_D.$$  \hfill (13)

Since the right-hand side is analytic in $k_1, k_2$ and $F_0$ does not depend on $k$, operator $F_{\text{out}}$ is analytic in $k_1, k_2$ when $k \notin \mathcal{E}_D$. Consider the standard Dirichlet-to-Neumann map $F'$ defined using the bounded solutions of the exterior problem. The difference $F_{\text{out}} - F'$ is an infinitely smoothing operator (the integral kernel of the difference is an infinitely smooth function). Thus, $F_{\text{out}}$ is an elliptic pseudo-differential operator of the first order with negative symbol. It is self-adjoint since the other two operators in (13) have this property.

In order to obtain an alternative definition of exceptional set $\mathcal{E}$, we reduce system (1), (2) to the boundary:

$$ \left\{ \begin{array}{l}
  u = u_{\text{out}} + e^{i\zeta \cdot z}, \quad z \in \partial \mathcal{O}, \\
  F_n u = F'_{\text{out}}u_{\text{out}} + F_0 e^{i\zeta \cdot z}, \quad z \in \partial \mathcal{O}.
\end{array} \right.$$  \hfill (14)

This system immediately implies the following representation (which is equivalent to (4)) of function $u$ at the boundary $\partial \mathcal{O}$:

$$u = (F_n - F'_{\text{out}})^{-1}(F_0 - F'_{\text{out}})e^{i\zeta \cdot z}.$$  \hfill (15)

The equivalence of (15) and (4) can be easily justified using the equality $F_n - F_0 = (F_n - F'_{\text{out}}) + (F'_{\text{out}} - F_0)$ and (12).

So, now we get a simple but important alternative definition of exceptional set $\mathcal{E}$.

**Theorem 2.2.** Let operator $F_n$ be well defined. Then a point $k \neq 0$ is exceptional if and only if the operator $F_n - F_{\text{out}}(k)$ has a non-trivial kernel. Moreover, the multiplicity of the exceptional point (i.e., the number of linearly independent solutions of (3)) is equal to the dimension of $\text{Ker}(F_n - F_{\text{out}}(k))$.

**Remark.** If $k' \in \mathcal{E}_D$, i.e., the operator $F_n - F_{\text{out}}(k)$ has a singularity at $k = k'$, then the kernel is the set of functions on which both the singular and principle parts of the operator vanish. To be more rigorous, a function $\sigma$ belongs to the kernel of the operator if $\lim[F_n - F_{\text{out}}(k)]\sigma = 0$ when $k \to k'$, $k \notin \mathcal{E}_D$.

**Proof.** Let $k \in \mathcal{E}$ and let $\sigma = v|_{\partial \mathcal{O}}$, where $v$ is a non-trivial solution of (3). From the assumption on $F_n$ it follows that $\sigma \neq 0$, and equation (3) implies that $F_n \sigma = F_{\text{out}} \sigma$. Thus $F_n - F_{\text{out}}$ has a non-trivial kernel that includes $\sigma$. Conversely, assume that $\sigma \neq 0$ belongs to the kernel of $F_n - F_{\text{out}}$ for some $k = k_0$. We define a non-trivial solution $v$ of (3) as follows. In $\mathcal{O}$, it is defined as the solution of the Dirichlet problem that is equal to
σ at the boundary (recall that zero is not an eigenvalue of the interior Dirichlet problem). In \( R^2 \setminus \mathcal{O} \), it is defined as \( F^{\text{out}} \sigma \) if \( k_0 \notin \mathcal{E}_D \). Otherwise, it is defined as \( \lim_{k \to k_0} F^{\text{out}} \sigma \). The existence of the limit follows from the Remark above.

The following statement is a simple consequence of Theorem 2.2.

**Theorem 2.3.** Let \( n(x) \) be absorbing, i.e., \( \Im n(x) > 0 \) on \( \mathcal{O} \) or \( \Im n(x) < 0 \) on \( \mathcal{O} \). Then there are no exceptional points.

**Proof.** The Green formula implies that the quadratic form

\[
(\Im F_n u, u) = \Im \int_{\partial \mathcal{O}} \frac{\partial u}{\partial \nu} \pi dl = \Im \int_{\mathcal{O}} \Delta u \pi dS = -\int_{\mathcal{O}} \Im n(x) |u(x)|^2 dS
\]

is sign definite for non-zero \( u \), and therefore, the operator \( \Im (F_n - F^{\text{out}}(k)) = \Im F_n \) is sign definite. Hence the kernel of operator \( F_n - F^{\text{out}}(k), k \in \mathbb{R}^2 \setminus \{0\} \), is trivial.

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3 Asymptotics for small \( k \)

The lemma below is important for studying the exceptional points for small \( k \). It is based on the decomposition of Faddeev’s Green function at the origin \( k = 0 \).

**Lemma 3.1.** \( F^{\text{out}}(k) \) is negative for small \( |k| \neq 0 \). The largest eigenvalue of \( F^{\text{out}}(k) \) tends to zero and the corresponding eigenfunction tends to a constant as \( |k| \to 0 \).

**Proof.** Let us show that operator \( -F^{\text{out}} \) is positive for small \( |k| > 0 \). It was shown in the proof of Lemma 2.1 that a small deleted neighborhood of the origin does not contain points of \( \mathcal{E}_D \). Thus, due to [13], the positivity of \( -F^{\text{out}} \) for small \( |k| > 0 \) is equivalent to the positivity of \( (S_k)^{-1} - F_0 \) for the same set of \( k \).

We will use matrix representations for all operators similar to [14]. For shortness of notations, we introduce \( \varepsilon = [-\frac{\gamma}{2\pi} - \frac{1}{2\pi} \ln |k|]^{-1} \). One can easily show that matrix \( (S_k^0)^{-1} \) for small \( \varepsilon > 0 \) has the form

\[
(S_k^0)^{-1} = \begin{pmatrix} \varepsilon & 0 \\ 0 & B^{-1} \end{pmatrix} + \begin{pmatrix} O(\varepsilon^2) & O(\varepsilon) \\ O(\varepsilon) & O(\varepsilon) \end{pmatrix},
\]

where the notation \( O(\varepsilon^j) \) is used for different operators with norms of specified order as \( \varepsilon \to 0 \). Since the elements of \( S_k \) differ from the corresponding elements of \( S_k^0 \) by operators of order \( O(|k|) \) and \( |k| = O(\varepsilon^n) \) for every \( n > 0 \), the matrix \( (S_k)^{-1} \) has the same representation as the matrix \( (S_k^0)^{-1} \). We also take into account that only the lower right element of the matrix representation of the operator \( F_0 \) is non-zero. We will preserve the same notation \( F_0 \) for this element. Then

\[
F^{\text{out}}(k) = F_0 - (S_k)^{-1} = \begin{pmatrix} -\varepsilon & 0 \\ 0 & F_0 - B^{-1} \end{pmatrix} + \begin{pmatrix} O(\varepsilon^2) & O(\varepsilon) \\ O(\varepsilon) & O(\varepsilon) \end{pmatrix}.
\]
Let us denote by $\tilde{F}_{\text{out}}$ the Dirichlet-to-Neumann operator that maps the Dirichlet data on $\partial O$ into the normal derivative $u_\nu$ of the corresponding bounded solution $u$. We denote by $\tilde{S}$ the single layer operator with the kernel $-\frac{1}{2\pi} \ln|z - z'|$. Similarly to (12), we have that $(F_0 - \tilde{F}_{\text{out}})\tilde{S} = I$ on functions orthogonal to constants. From here it follows that

$$F_0 - \tilde{F}_{\text{out}} = B^{-1}$$
onumber on $H^{\frac{1}{2}},(\partial O)$.

From the Green formula, it follows that $\tilde{F}_{\text{out}} < 0$ on $H^{\frac{1}{2}},(\partial O)$. Thus $F_0 - B^{-1} < 0$ on $H^{\frac{1}{2}},(\partial O)$. The latter operator does not depend on $k$. It is an elliptic pseudo-differential operator of the first order (it is a restriction of $F_{\text{out}}$ to a subspace of co-dimension one), and therefore its eigenvalues tend to infinity. Thus from the negativity of $F_0 - B^{-1}$ it follows that $F_0 - B^{-1} < 0$ on $H^{\frac{1}{2}},(\partial O)$. Hence, the quadratic form of the operator (16) on each vector $(x_1, x_2)$, $x_1 = \text{const}$, $x_2 \in H^{\frac{1}{2}},(\partial O)$, is bounded from above by

$$-\varepsilon \|x_1\|^2 - \delta \|x_2\|^2 + (\varepsilon^2 \|x_1\|^2 + \varepsilon \|x_1\| \|x_2\| + \varepsilon \|x_2\|^2).$$

The latter expression is negative for non-zero $(x_1, x_2)$ if $\varepsilon > 0$ is small enough.

4 Sign-definite perturbations of conductive potentials

Let $n$ be a conductive potential vanishing outside $O$. It means that

$$n = -q^{-\frac{1}{2}}\Delta q^{\frac{1}{2}},$$

where $q \in C^2(\mathbb{R}^2)$ is a smooth positive function and $q - 1$ vanishes outside $O$. Nachman proved [7] that there are no exceptional points for such potentials. Perturbations $n_\lambda = n(z) + \lambda \omega(z)$ of conductive potentials were considered in [6], where $\omega$ is supported on $O$. Under the assumptions that the potential is radial, i.e., $n_\lambda = n_\lambda(|z|)$, and

$$\mu = \int_{\partial O} \omega q dS > 0,$$ (17)

the authors of [6] proved that the exceptional set is empty for small negative $\lambda$, and there exists an exceptional set for positive small $\lambda$. It was shown that the exceptional set is a circle of radius $e^{-\frac{1}{\mu \lambda}(1+o(1))}$, $\lambda \to +0$.

Our approach allows us to extend this result to the case of non-radial potentials. The exceptional set in this case is not a circle anymore, but it approaches the circle as $\lambda \to +0$. Consider the variables $\varepsilon = -\ln|k|$, $\varphi = \arg k$, $\varphi \in [0, 2\pi)$.

Theorem 4.1. Let $n$ be a conductive potential vanishing outside $O$ and let (17) hold.

If $\lambda < 0$ is small enough, then the exceptional set $E$ is empty. If $\lambda > 0$ is small enough, then exceptional points exist only in a neighbourhood of the origin and the exceptional set is given by the equation $\varepsilon = \mu \lambda(1+o(1))$, $\lambda \to +0$, where the remainder depends smoothly on $\lambda$ and $\varphi$. 

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Proof. Since $\|S\|_{L^2(\partial\mathcal{O})} \to 0$ as $|k| \to \infty$ (see Lemma 5.3 below) and operator $F_n - F_0$ is a pseudo-differential operator of order at most $-1$ (see [3, Lemma 1.1]), there exists $K_0 > 0$ such that

$$\|I + S(k)(F_n - F_0)\| > \frac{1}{2}, \quad |k| > K_0.$$ 

Then a similar estimate with $1/4$ on the right is valid if $n$ is replaced by $n_\lambda$, $\lambda \ll 1$. Thus operator $F_n$ does not have a pole when $|k| > K_0$, and the region $|k| > K_0$ is free of points $k \in \mathcal{E}$ if $|\lambda|$ is small enough.

Operator $F_n$ is a pseudo-differential operator of the first order with a positive principal symbol. Due to Lemma 2.1 operator $F^{out}(k), k \neq 0$, is a pseudo-differential operator of the first order with a negative principal symbol. Hence, $F_n - F^{out}(k)$ is an elliptic operator of the first order, and therefore, its eigenvalues tend to infinity. We take additionally into account that the kernel of the operator $F_n - F^{out}(k)$ is trivial for all $k \neq 0$ due to Theorem 2.2. This implies that the norm of the operator $F_n - F^{out}(k)$ is separated from zero for each fixed $k \neq 0$. From the analyticity in $k_1, k_2$ it follows that a lower bound for the norm can be chosen uniformly in $k$ on each region of the form $K_0 \geq |k| \geq \delta > 0$. Then the same is true if $n$ is replaced by $n_\lambda$ with small enough $|\lambda|$. Hence Theorem 2.2 implies that the exceptional points for the problem with the perturbed potential $n_\lambda$ and sufficiently small $|\lambda|$ can appear only in a small neighborhood of $k = 0$.

Now let us study the structure of the set $\mathcal{E}$ in a neighborhood of the origin $k = 0$. Since the substitution $u = \sqrt{q} v$ reduces equation (11) with a conductive potential to the equation $\nabla q \nabla v = 0$, the D-t-N maps for these equations coincide. Hence, the kernel and co-kernel of operator $F_n$ are one dimensional spaces of constants. The norm of the restriction of $F_n$ on the space $L^{2,1}$ of functions orthogonal to constants is greater than some positive constant.

Consider the operator $A(\lambda, k) := F_{n_\lambda} - F^{out}(k)$. From Lemma 3.1 and the properties of $F_n$ established above, it follows that $A(0, 0)$ has zero eigenvalue with constant eigenfunction, and all the other eigenvalues are greater than some positive constant $\delta > 0$. Operator $F_{n_\lambda}$ is analytic in $\lambda$, and operator $F^{out}(k)$ is an infinitely smooth function of $\varepsilon = [-\frac{\gamma}{2\pi} - \frac{1}{2\pi} \ln |k|]^{-1}$ at $\varepsilon = 0$ with all the derivatives at $\varepsilon = 0$ independent of the polar angle of $k$. The properties of $F^{out}(k)$ can be easily obtained by a slight extension of the arguments used to derive (16). Hence operator $A(\lambda, k)$ with small enough $|\lambda| + |k|$ has an eigenvalue $\xi$ of the form

$$\xi(\lambda, \varepsilon, \varphi) = a\lambda + b\varepsilon + O(\lambda^2 + \varepsilon^2)$$

with all the other eigenvalues being greater than $\delta/2 > 0$.

Let us find constants $a$ and $b$. Denote by $e$ a constant function on $\partial\mathcal{O}$ such that $\|e\| = 1$. Operator $A(\lambda, k) := F_{n_\lambda} - F^{out}(k)$ is self-adjoint, and therefore

$$a = \left( \frac{\partial}{\partial \lambda} A(0, 0) e, e \right) = \left( \frac{\partial}{\partial \lambda} F_{n_\lambda} e, e \right) |_{\lambda = 0} = -\mu,$$
where $\mu$ is given by (17). The details of this calculations can be found in the proof of Lemma 5.1 in the paragraph containing formula (22).

Similarly, from (16) it follows that

$$b = (A_\varepsilon(0,0)e,e) = -(\partial_{\varepsilon}F^{\text{out}}(k)e,e)_{|\varepsilon=0} = 1.$$ 

Thus

$$\xi(\lambda,\varepsilon,\varphi) = -\mu\lambda + \varepsilon + O(\lambda^2 + \varepsilon^2), \quad |\lambda| + |k| \ll 1, \quad \mu > 0. \quad (18)$$

Since the set $E$ is located in a small neighborhood of the origin $k = 0$, from Theorem 2.2 it follows that $E$ is defined by the relations $\xi(\lambda,\varepsilon,\varphi) = 0$, $0 < \varepsilon \ll 1$. Since $\varepsilon > 0$, statements of the theorem follow immediately from (18).

\[\square\]

5 Counting function for exceptional points

Now we present a method that allows one to estimate the number of exceptional points on an arbitrary path $\gamma \subset \mathbb{C}$ that is analytic in $k_1, k_2$ by making certain measurements at the end points of $\gamma$. We assume that $n(z)$ is smooth enough and that either $n(z)$ does not vanish at the boundary $\partial \mathcal{O}$ or $n(z)$ and its normal derivatives up to a certain order vanish identically at the boundary, while the next derivative does not vanish at the boundary. So either

$$n(z) \neq 0, \quad z \in \partial \mathcal{O},$$

and we say that $m = 0$ in this case, or for some $m \in \mathbb{N}$

$$\frac{\partial^i n}{\partial \nu^i}(z) = 0, \quad i \leq m - 1, \quad z \in \partial \mathcal{O}, \quad \text{and} \quad \frac{\partial^m n}{\partial \nu^m}(z) \neq 0, \quad z \in \partial \mathcal{O}. \quad (19)$$

Then (see [3, Lemma 1.1 and Remark 4]) $F_n - F_0$ is an elliptic PDO of order $-1 - m$ and, moreover, the sign of its principal symbol coincides with $-\sigma$, where

$$\sigma = \text{sgn} n(z) \quad \text{when} \quad z \in \mathcal{O}, \quad 0 < \text{dist}(z, \partial \mathcal{O}) < < 1. \quad (20)$$

Additionally, we assume in this section that the kernel of $F_n - F_0$ is trivial. It is not a very restrictive assumption due to the following statement.

**Lemma 5.1.** The kernel of operator $F_{n_\lambda} - F_0$ is trivial for generic perturbations $n_\lambda$ of an arbitrary potential $n$ by real-valued smooth functions.

**Proof.** Consider perturbations $n_\lambda$ of the form $n_\lambda := n + \lambda \omega$.

Assume that $F_n - F_0$ has a non-trivial kernel. Since $F_{n_\lambda} - F_0$ is an elliptic operator, the kernel has a finite dimension (i.e., the corresponding eigenvalue has a finite multiplicity),
and therefore, \[11, \text{Th. XII.13}\] the eigenfunctions of \(F_{n \lambda} - F_0\) are holomorphic for \(\lambda \ll 1\). For the sake of transparency of the proof, we assume that the kernel is one-dimensional.

The lemma will be proved if we show the existence of a function \(h(z)\) such that condition

\[
\int_{\partial} \omega h dx dy \neq 0 \quad (21)
\]
guarantees that \(F_{n \lambda} - F_0\) does not have a kernel for small \(\lambda \neq 0\).

Since operator \(F_{n \lambda} - F_0\) is self-adjoint and depends analytically on \(\lambda\), it has (see \[11\]) a family of normalized eigenfunctions \(\varphi(\lambda)\) with eigenvalues \(\mu(\lambda), \mu(0) = 0\), that are analytic in \(\lambda\), \(|\lambda| \ll 1\). The lemma will be proved if we show that \((21)\) implies that \(\mu'(\lambda)|_{\lambda=0} \neq 0\). Since \((\varphi(\lambda), \varphi(\lambda)) \equiv 1\) and \(\mu(0) = 0\), we have

\[
\mu(\lambda)|_{\lambda=0} = ((F_{n \lambda} - F_0)\varphi(\lambda), \varphi(\lambda))|_{\lambda=0} = ((F_{n \lambda} - F_0)|_{\lambda=0}\varphi(0), \varphi(0)). \quad (22)
\]

Denote by \(u(\lambda)\) the solution of \((-\Delta - n\lambda)u(\lambda) = 0\) with the Dirichlet data \(\varphi(\lambda)\). Then

\[
(-\Delta - n\lambda)u'(\lambda) = \omega u(\lambda),
\]

and from the Green formula we get that

\[
\mu'(0) = ((F_{n \lambda})|_{\lambda=0}\varphi(0), \varphi(0)) = -\int_{\partial} \omega |u(0)|^2 dx dy.
\]

Hence, one can take \(h = |u(0)|^2\).

Let us denote by \(P(k)\) the operator

\[
P(k) = -\sigma[(F_n - F_0)^{-1} + S(k)], \quad k \neq 0, k \in \mathbb{C}, \quad (23)
\]

which is obtained by multiplication of \(F_n - F^{\text{out}}(k)\) by \(-\sigma(F_n - F_0)^{-1}\) from the left and by \(S_k\) from the right:

\[
-\sigma(F_n - F_0)^{-1}(F_n - F^{\text{out}}(k))S_k = -\sigma(F_n - F_0)^{-1}[(F_n - F_0) + (F_0 - F^{\text{out}}(k))]S_k = P(k). \quad (24)
\]

Relation \((12)\) was used to obtain the last equality above. In fact, \((12)\) for \(k = k' \in \mathcal{E}_D\) must be understood in the sense of the limit as \(k \to k', k \notin \mathcal{E}_D\). The same is true for the definition of the kernel of \(F_n - F^{\text{out}}(k)\) (see Remark followed Theorem \[2,2\]) as well as for arguments in the proof of the following lemma.

**Lemma 5.2.** Let operator \(F_n\) be well defined and let the kernel of operator \(F_n - F_0\) be trivial. Then

1) Operator \(P(k)\) is an elliptic self-adjoint pseudo-differential operator of order \(1 + m\). It is analytic in \(k_1, k_2 \in \mathbb{R}^2 \setminus \{0\}\) and has a positive principal symbol.

2) A point \(k \neq 0\) is exceptional if and only if the operator \(P(k)\) has a non-trivial kernel, and the dimension of the kernel of \(P(k)\) coincides with the multiplicity of the exceptional point \(k\).
**Proof.** The first statement is obvious. Let us prove the second one. Let \( \phi \) belong to the kernel of \( P(k') \). Note that both operator \( F_n - F_0 \) and its inverse are well defined (recall that we perturbed \( \mathcal{O} \) in order to make sure that \( F_n \) is well defined). Thus from (23) it follows that \( \psi = S_{\nu'} \phi \neq 0 \), and (24) implies that \( \psi \) belongs to the kernel of \( F_n - F_{\text{out}}(k') \). Conversely, if \( \psi \neq 0 \) belongs to the kernel of \( F_n - F_{\text{out}}(k') \), then
\[
0 = [F_n - F_{\text{out}}(k')]\psi = [F_n - F_0 + F_0 - F_{\text{out}}(k')]\psi.
\]
Since \( F_n - F_0 \) does not have a non-trivial kernel, it follows that \( \phi := [F_0 - F_{\text{out}}(k')]\psi \neq 0 \). Then (12) implies that \( \psi = S_{\nu'} \phi \), and from (24) it follows that \( \phi \neq 0 \) belongs to the kernel of \( P(k') \). Hence the one-to-one correspondence between kernels of \( F_n - F_{\text{out}}(k) \) and \( P(k) \) is established.

We can introduce a function that counts the negative eigenvalues \( \varphi_i(k) \) of \( P(k) \):
\[
n^-(k) = \# \{ i \mid \varphi_i(k) < 0 \}.
\]
Since the principal symbol of \( P(k) \) is positive, operator \( P(k) \) has at most a finite number of negative eigenvalues, i.e., \( n^-(k) < \infty \).

**Theorem 5.3.** Let operator \( F_n \) be well defined and let the kernel of operator \( F_n - F_0 \) be trivial. Let \( k, \hat{k} \in \mathbb{C}\setminus\{0\} \) be arbitrary points such that \( n^-(k) \neq n^-(\hat{k}) \). Then every analytic path \( \gamma \) connecting points \( k \) and \( \hat{k} \) contains at least one point of the set \( \mathcal{E} \). Moreover, the sum of all the multiplicities of exceptional points on the path is at least \( |n^-(k) - n^-(\hat{k})| \).

**Proof.** Let \( k = k(s), \ 0 \leq s \leq 1, \) be an analytic parametrization of \( \gamma \). Due to the first statement of Lemma 5.2, operator \( P(k(s)) \) is analytic in \( s \) and its spectrum consists of eigenvalues of finite multiplicities. Thus (see [11, Th. XII.13]) its eigenvalues are analytic in \( s \). Therefore, function \( n^-(k(s)) \) can change its value only if an eigenvalue \( \mu(s) \) of \( P \) passes through the point \( \mu = 0 \). If this happens at \( s = s' \), then \( k(s') \in \mathcal{E} \) due to the second statement of Lemma 5.2. This lemma also allows one to estimate from below the sum of multiplicities of the exceptional points on \( \gamma \). □

The following theorem allows one to evaluate \( n^-(k) \) for large \( k \) using a simple operator that is not related to the Faddeev scattering problem.

**Theorem 5.4.** Let operator \( F_n \) be well defined and let the kernel of operator \( F_n - F_0 \) be trivial. Then the value of \( n^-(k) \) for \( |k| \gg 1 \) coincides with the number of negative eigenvalues of \( -\sigma(F_n - F_0) \).

**Consequence.** The existence of exceptional points will be justified if a point \( k_0 \neq 0 \) is found such that \( n^-(k_0) \) is different from the number of negative eigenvalues of \( -\sigma(F_n - F_0) \).

The validity of Theorem 5.4 follows immediately from (23) and the decay of \( \|S_k\| \) as \( k \to \infty \). Thus it is enough to prove the following lemma.
Lemma 5.5. The norm of $S_k$ as operator from $L^2(\partial \Omega)$ to $L^2(\partial \Omega)$ tends to zero as $|k|$ goes to infinity.

Proof. Consider the single layer operator $S_k^g$ on $\partial \Omega$ defined by the Green function

$$g_k(z) := e^{-iKz}G_k(z) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i(\xi_1 x + \xi_2 y)} \frac{d\xi_1 d\xi_2}{|\xi|^2 + 2k\xi}, \quad \xi = \xi_1 + i\xi_2.$$ 

Then

$$S_k = C_k S_k^g C_k^{-1},$$

where $C_k$ is the operator of multiplication by $e^{ikz}$.

We will show below (see also [9, (3.18)]) that

$$|g_k(z)| \leq \frac{c}{\sqrt{|k||z|}}, \quad c > 0.$$  

(26)

The Young inequality implies that

$$\|S_k^g\|_{H^0} \leq C\|g(\cdot)\|_{L^1}.$$ 

From here and (26) it follows that the norm of $S_k^g$ is bounded by $\alpha(k) := c_1 / \sqrt{|k|}, c_1 > 0$. Thus the spectrum of $S_k^g$ is located in the disk of radius $\alpha(k)$. Due to (25), operators $S_k$ and $S_k^g$ have the same eigenvalues. Since $S_k$ is self-adjoint, its eigenvalues belong to the segment $[-\alpha(k), \alpha(k)]$ and its norm does not exceed $\alpha(k)$. Thus the lemma will be established as soon as (26) is proved.

An estimate similar to (26) with the right-hand side $c/|k||z|$ can be found in [9, (3.18)]. The arguments there can be modified to justify (26). However, we will provide an independent proof. We write $g_k(z)$ in the form

$$g_k(z) = \frac{1}{2\pi^2} \int_{\mathbb{R}^2} \frac{e^{i(\xi_1 x' + \xi_2 y')}}{\xi(\xi + 2)} d\xi_1 d\xi_2 = g^{(1)}(z') + g^{(2)}(z'),$$

(27)

where the components of the vector $z' = (x', y') \in \mathbb{R}^2$ are the real and imaginary parts of $(k_1 + ik_2)(x + iy)$ and $g^{(i)}(z')$ coincide with the integral above where additional factors $\alpha_i(\xi)$ are added to the integrand. Here $\alpha^{(i)} \in C^\infty(\mathbb{R}^2)$, $\alpha^{(1)} = 1$ when $|\xi| < 3$, $\alpha^{(2)} = 1 - \alpha^{(1)}$ when $|\xi| > 4$. $\alpha^{(2)} = 1 - \alpha^{(1)}$. To obtain the first equality in (27), one needs to make the substitution $\xi_1 + i\xi_2 \rightarrow (k_1 - ik_2)(\xi_1 + i\xi_2)$ in the integral that defines $g_k(z)$.

We will estimate $g^{(1)}$, $g^{(2)}$ separately for $|z'| \leq 1$ and for $|z'| > 1$. Since $\alpha^{(1)}/|\xi(\xi + 2)| \in L^1$, function $|g^{(1)}(z')|$ is uniformly bounded. In order to estimate $|g^{(2)}(z')|$, we represent the pre-exponential factor in the integral as $\frac{\alpha^{(1)}}{\xi(\xi + 2)} = \frac{\alpha^{(1)}}{|\xi|^2} + \alpha^{(1)}O(|\xi|^{-3}), \quad |\xi| \rightarrow \infty$. The second term is integrable, and therefore its contribution to $g^{(2)}(z')$ is uniformly bounded. The contribution of the first term differs from the inverse Fourier transform of $1/|\xi|^2$ (which is proportional to $\ln|z'|$) by the inverse Fourier transform of a compactly supported function $\alpha^{(1)}/|\xi|^2$. The latter inverse Fourier transform is analytic in $z'$, and therefore it is bounded.
when $|z'| \leq 1$. Thus $g_k(z)$ has a logarithmic estimate when $|z'| = |k||z| \leq 1$, which implies (20) when $|k||z| \leq 1$.

Since $\nabla \frac{\alpha^{(2)}}{\xi(\xi + 2)} \in L^1$, integration by parts implies that $g^{(2)}$ can be estimated from above by $C/|z'|$. In order to obtain a similar estimate on $g^{(1)}$, we separate the singularities of the integrand:

$$\frac{\alpha^{(1)}}{\xi(\xi + 2)} = \frac{\alpha^{(1)}}{2\xi} - \frac{\alpha^{(1)}}{2(\xi + 2)} + \beta'(\xi'), \quad \nabla \beta \in L^1.$$ 

Obviously, the contribution of $\beta$ to $g^{(1)}$ can be estimated from above by $C/|z'|$. Similarly, integration by parts implies that the inverse Fourier transform of $\frac{\alpha^{(2)}}{2\xi}$ is a homogeneous function of order $-1$, and the inverse Fourier transform of $\frac{1}{2(\xi + 2)}$ is a homogeneous function of order $-1$ multiplied by $e^{2ix'}$, it follows that $g^{(1)}$ can be estimated from above by $C/|z'|$. This leads to (20) when $|k||z| > 1$.

6 The case of positive energy

All the previous results remain valid when energy is positive. Equation (1) should be replaced by

$$-\Delta u - Eu - nu = 0, \quad z = (x, y) \in \mathbb{R}^2,$$

where $n$ is smooth enough in $\overline{O}$ and $n = 0$ outside $\overline{O}$. The following parametrization (see [9]) of $\zeta \in \mathbb{C}^2$, $\zeta^2 = E > 0$, will be used instead of $\zeta = (k, ik)$:

$$\zeta = \left(\frac{(\lambda + \frac{1}{\lambda})\sqrt{E}}{2}, \frac{(\frac{1}{\lambda} - \lambda)i\sqrt{E}}{2}\right), \quad |\lambda| \neq 1.$$ 

The fundamental solution that corresponds to outgoing waves has the form

$$G_{\zeta}(z) = e^{ikz} \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i(x', z')\cdot z'} dz',$$

and condition (2) will be replaced by the representation of the solution $u$ through $G$:

$$u(z, \zeta) = e^{ikz} + \int_{\partial O} G_{\zeta}(z - w)\mu_{\zeta}(w) dl_w, \quad \mu \in H^{-\frac{1}{2}}(\partial O), \quad z \in \mathbb{R}^2 \setminus O.$$ 

Definition. A point $\lambda \in \mathbb{C} \setminus \{0\}, |\lambda| \neq 1$, will be called exceptional if problem (28), (29) has a nontrivial solution. The multiplicity of an exceptional point is defined by the number of linearly independent solutions of (28), (29). The set of all exceptional points will be denoted by $\mathcal{E}(E)$. 

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Let $E > 0$ be fixed. Suppose that either $n(z)$ does not vanish at the boundary or it vanishes but the first derivative does not vanish, etc.

\[ \frac{\partial^n}{\partial \nu^i}(z) = 0, \quad i \leq m - 1 \quad \text{and} \quad \frac{\partial^m n}{\partial \nu^m}(z) \neq 0, \quad z \in \partial O. \]

Constant $\sigma$ (see (20)) will be defined now by

\[ \sigma = \text{sgn} n(z) \quad \text{when} \quad z \in O, \quad 0 < \text{dist}(z, \partial O) \ll 1, \quad (30) \]

and operator $P(k)$ (see (23)) will be replaced by

\[ P(\lambda, E) = -\sigma[(F_n - F_0)^{-1} + S(\lambda)], \quad (31) \]

where $F_n = F_n(E)$ is the Dirichlet-to-Neumann map for equation (28).

An analogue of Lemma 5.2 is valid for operator (31):

**Lemma 6.1.** Let $E$ be such that operator $F_n(E)$ is well defined and the kernel of the operator $F_n(E) - F_0(E)$ is trivial. Then

1) Operator $P(\lambda, E)$ is an elliptic self-adjoint pseudo-differential operator of order $1 + m$ with a positive principal symbol.

2) A point $\lambda \neq 0$ is exceptional ($\lambda \in \mathcal{E}(E)$) if and only if the operator $P(\lambda, E)$ has a non-trivial kernel. Moreover, the dimension of the kernel of $P(\lambda, E)$ coincides with the multiplicity of $\lambda$ as an exceptional point.

The proof of Lemma 6.1 is exactly the same as the proof of Lemma 5.2. In the proof, one needs to use the fact that the integral kernel of operator $S$ is real-valued. If $E > 0$, the latter follows from formulas (2.4), (2.5) of [10].

Theorem 5.3 also remains valid when $E > 0$. Denote by $n^{-}_E(\lambda)$ the number of negative eigenvalues of $P(\lambda, E)$.

**Theorem 6.2.** Let $E$ be such that operator $F_n(E)$ is well defined and the kernel of the operator $F_n(E) - F_0(E)$ is trivial. Let $\lambda, \hat{\lambda} \in \mathbb{C}\backslash\{0\}$ be arbitrary points such that $n^{-}_E(\lambda) \neq n^{-}_E(\hat{\lambda})$. Then every analytic path $\gamma$ connecting points $\lambda$ and $\hat{\lambda}$ contains at least one point of the set $\mathcal{E}(E)$. Moreover, the sum of all the multiplicities of exceptional points on the path is at least $|n^{-}_E(\lambda) - n^{-}_E(\hat{\lambda})|$.

7 **The case of the potential $En(z)$**

Consider the transmission scattering problem when the potential $n(z)$ is replaced by $En(z)$. Equation (28) now takes the form

\[ -\Delta u - Enu = 0, \quad z = (x, y) \in \mathbb{R}^2, \quad (32) \]

where $n > 0$ is smooth enough in $\overline{O}$, and $n = 1$ when $x \notin \overline{O}$. 

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Theorem 7.1. Let \( n(z) \neq 1 \) on \( \partial \mathcal{O} \) and \( \int_{\mathcal{O}} (n^2(z) - 1)dz \neq 0 \). Then for each fixed \( \lambda \neq 0, |\lambda| \neq 1 \), there exist infinitely many positive energies \( E \) such that \( \lambda \in \mathcal{E}(E) \). Moreover,

\[
\#\{ E : 0 < E < s, \lambda \in \mathcal{E}(E) \} \geq \frac{s}{4\pi} \left| \int_{\mathcal{O}} (n^2(z) - 1)dz \right| + o(s), \quad s \to \infty.
\]

Proof. Consider the operator (compare with (31))

\[
P(\lambda, E) = -\sigma[(F_{En} - F_E)^{-1} + S],
\]

where \( F_{En} \) is the Dirichlet-to-Neumann map for equation (32). Since \( \lambda \) is fixed, we will omit it in all the notations below. It was already discussed earlier that the assumption \( n(z) \neq 0, z \in \partial \mathcal{O} \), implies that \( -\sigma(F_{En} - F_E) \) is an elliptic PDO of order \(-1\) with a positive principal symbol (see [3, Lemma 1.1 and Remark 4]) and that \( S \) is also an operator of order \(-1\). Thus \( P \) (for each \( E > 0 \)) is an elliptic operator of the first order with a positive symbol and therefore, for each \( E > 0 \), it has at most a finite number \( n^-(E) \) of negative eigenvalues \( \mu = \mu_j(E), \; 1 \leq j \leq n^-(E) \). The proof of Theorem 7.1 will be based on studying the properties of this important quantity \( n^-(E) \).

Since \( P \) is an elliptic operator of the first order, it is convenient to consider \( P \) as an operator from \( H^1(\partial \mathcal{O}) \) to \( L^2(\partial \mathcal{O}) \). Unlike the way it was done with operator (31), now we will need to study operator (34) for all values of \( E \). Note that operators \( F_{En}, F_E : L^2(\partial \mathcal{O}) \to H^1(\partial \mathcal{O}) \) have poles at points \( E \) where the corresponding interior homogeneous Dirichlet problem has a non-trivial solution (zero is an eigenvalue), i.e., \( F_{En} - F_E \) is meromorphic in \( E \in \mathbb{R} \). From (39) and (40) it follows that \( F_{En} - F_E \) is invertible for small \( E > 0 \). Thus (see [3, Lemma 1.1]) operators \( (F_{En} - F_E)^{-1} \) and \( P(\lambda, E) \) are meromorphic in \( E \). As earlier (see Remark after Theorem 2.2), the kernel of \( P \) at a pole is defined as the set of all elements that belong to the kernels of both the principal part and the regular part of the operator.

The following extension of Lemma 6.1 is valid.

Lemma 7.2. Operator \( P(\lambda, E) \) has a non-trivial kernel at \( (\lambda, E), \lambda \neq 0, 1, \) if and only if \( \lambda \in \mathcal{E}(E) \). Moreover, the dimension of the kernel of \( P(\lambda, E) \) coincides with the multiplicity of \( \lambda \) as an exceptional point.

Proof. Due to Lemma 6.1, one needs to prove Lemma 7.2 only when \( E = E' \) is a pole of either \( P \) or one of the operators \( F_{En}, F_E \). Let \( \phi \neq 0 \) belong to the kernel of \( P(\lambda, E') \) and let \( \psi = S\phi \). Then (34) implies that

\[
\lim_{E \to E'} (F_{En} - F_E)^{-1}\phi = \psi.
\]

From here it follows that \( \lim_{E \to E'} (F_{En} - F_E)\psi = \phi \), and therefore \( \psi \neq 0 \), since \( \phi \neq 0 \). Hence, the relation \( P(\lambda, E)\phi = 0 \) and (34) imply that \( \psi \neq 0 \) belongs to the kernel of operator

\[
I + S(F_{En} - F_E)
\]
which is an analogue of (6). The dimension of the kernel of operator (35) coincides with the multiplicity of the exceptional point \( \lambda \in \mathcal{E}(E) \). Conversely, if \( \psi \neq 0 \) belongs to the kernel of operator (35), then \( \phi := (F_n - F_0) \psi \neq 0 \), and \( \phi \) belongs to the kernel of \( P(\lambda, E) \). Hence the one-to-one correspondence between the kernels of operators (35) and \( P(\lambda, E) \) is established.

We return now to the proof of Theorem 7.1. Consider the change in \( n^{-}(E) \) when \( E \) changes from zero to some value \( s, s \to \infty \). Operator \( P \) depends on \( E \) meromorphically with poles only at points \( E \) where the operator \( F_n - F_0 \) has a non-trivial kernel. Hence the eigenvalues \( \mu_j(E) \) are meromorphic in \( E \) and may enter/exit the negative semi-axis \( \mathbb{R}^{-} = \{ \mu : \mu < 0 \} \) only through the end points of the semi-axis. Thus we can split \( n^{-}(s) - n^{-}(0) \) as

\[
n^{-}(s) - n^{-}(0) = n_1(s) + n_2(s), \tag{36}
\]

where \( n_1(s) \) is the number of eigenvalues \( \mu_j(E) \) that enter/exit the negative semi-axis \( R_\mu^- = \{ \mu : \mu < 0 \} \) through the point \( \mu = -\infty \) (when \( E \) changes from 0 to \( s > 0 \)) and \( n_2(s) \) is the number of eigenvalues \( \mu_j(E) \) that enter/exit the negative semi-axis \( R_\mu^- \) through the point \( \mu = 0 \). These numbers are positive if the eigenvalues enter the semi-axis \( \mathbb{R}^{-} \), and they are negative if the eigenvalues exit the semi-axis.

It was shown in [4], [5] that \( n_1(s) \) is equal to the signed counting function for the interior transmission eigenvalue problem with the potential \( n(z) + 1 \) and, importantly, that it has the following estimate

\[
|n_1(s)| \geq \frac{1}{4\pi} \left| \int_{\mathcal{O}} (n'(z) - 1) dz \right| + o(s), \quad s \to \infty. \tag{37}
\]

It will be proved below that

\[
n^{-}(s) = o(s), \quad s \to \infty. \tag{38}
\]

Lemma 7.2 implies that \( \lambda \) is an exceptional point \((\lambda \in \mathcal{E}(E'))\) if an eigenvalue of \( P(\lambda, E) \) passes through the point \( \mu = 0 \) when \( E = E' \) while \( E \) moves from \( E = 0 \) to \( E = s \). Hence the left-hand side in (33) can be estimated from below by \(|n_2(s)|\). If we also take into account that \( n^{-}(0) \) is a constant, then the statement of Theorem 7.1 becomes an immediate consequence of (36), (37), and (38). Thus the proof of the theorem will be completed as soon as (38) is established.

The proof of (38) requires studying the behavior of operator (34) as \( E \to \infty \). The latter study will be based on the following facts concerning the terms in formula (34). Consider operator \( F_n - F_0 \). It was shown in [3] Lemma 1.1] (by calculating the symbol of the operator) that

\[
-\sigma(F_n - F_0) = EM + Q(E), \tag{39}
\]

where \( M \) is a positive operator of order \(-1\) and \( Q(E) \) has a smaller order. A more important fact was established in [4] lemma 3.1: one can take \( M = -\sigma(F_n - F_0)'_{|E=0} \).
Then \( Q(E) \) is an operator of order \(-3\), and the following estimate holds

\[
\|Q(E)\phi\|_{H^{5/2}} \leq C \left( \frac{E^2}{\text{dist}(E,Sp)} + E \right)^2 \|\phi\|_{H^{-1/2}},
\]

(40)

where \( Sp \) is the union of the spectra of the Dirichlet problems for equation (32) with potentials \( E_n \) and \( E \). Moreover, one can reduce the set \( Sp \) in the estimate (40) by dropping those eigenvalues of the Dirichlet problems for which \( \phi \) is orthogonal to the Neumann data of the corresponding eigenfunctions.

We will also need an estimate on the norm of the second term in (31) as \( E \to \infty \). It can be obtained using the same arguments as in the proof of Lemma 5.5 with the only difference that the following estimate

\[
|g(z)| \leq \frac{C}{\sqrt{z|E|^{1/4}}}, \quad C = C(\lambda),
\]

takes the place of (26) (see also [9, prop.3.1]). This leads to the following estimate on operator \( S \):

\[
\|S\|_{H^0(\partial\Omega)} \leq \frac{C}{E^{1/4}}, \quad E \to \infty.
\]

(41)

Let us justify (38). Recall that operator \( M \) has a positive symbol. Thus we can add a finite-dimensional operator to \( M \) to make it positive and subtract the corresponding term from \( Q \). This will not affect the validity of (40). Thus, without loss of the generality, we may assume that operator \( M \) is positive.

From the mini-max principle, it follows that (38) will be justified if we show that the quadratic form of the operator \( P(\lambda,E) \) is non-negative on a subspace of co-dimension \( o(E) \), \( E \to \infty \). Thus it is enough to show the existence of a subspace of co-dimension \( o(E) \), \( E \to \infty \), on which

\[
(EM + Q)^{-1} \geq \frac{1}{2} (EM)^{-1} \quad \text{and} \quad \frac{1}{2} (EM)^{-1} + S \geq 0.
\]

(42)

These inequalities are understood in the sense of the quadratic forms on the corresponding subspace.

It is enough to prove (42) on a dense set of smooth functions in the subspace, and therefore one can multiply the first relation in (42) from the left and right by \( M^{1/2} \) and replace the first relation in (42) by

\[
(1 + M^{-1/2}E^{-1}QM^{-1/2})^{-1} \geq \frac{1}{2}
\]

on a subspace of co-dimension \( o(E) \), \( E \to \infty \).

First we are going to show that there exists a subspace \( D = D(E) \) of co-dimension \( o(E) \), \( E \to \infty \), such that

\[
\|RM^{-1/2}E^{-1}QM^{-1/2}R\| \leq \frac{1}{2}.
\]

(44)
where $R$ is the orthogonal projection on $D$. Indeed, the Weyl law for the Dirichlet problems for equation (32) with potentials $En$ and $E$ implies that the dimension of the space spanned by the eigenfunctions of these two problems with eigenvalues within the interval $L := (E - E^p, E + E^p)$, $\frac{1}{2} < p < 1$, does not exceed $CE^p$ as $E \to \infty$. Let $D_0 = D_0(E)$ be the space spanned by the Neumann data of these eigenfunctions. Define $D_1 = M^{-\frac{1}{2}}D_0$. From (40) it follows that
\[ \| M^{-\frac{5}{2}}Q(E)M^{-\frac{1}{2}}\phi \|_{L^2(\partial \Omega)} \leq c_0 E^{4-2p} \| \phi \|_{L^2(\partial \Omega)}, \quad \phi \in D_1^+, \quad E \to \infty. \]
Thus
\[ |(M^{-\frac{1}{2}}E^{-1}QM^{-\frac{1}{2}}\phi, \phi)| = |(M^{-\frac{5}{2}}E^{-1}QM^{-\frac{1}{2}}\phi, M^2\phi)| \leq c_0 E^{3-2p} \| \phi \|_{L^2(\partial \Omega)} \| M^2\phi \|_{L^2(\partial \Omega)}, \quad \phi \in D_1^+, \quad E \to \infty. \quad (45) \]
The counting function for eigenvalues of $M^{-2}$ has the following asymptotics at infinity
\[ N_{M^{-2}}(\mu) = O(\mu^{\frac{1}{2}}), \quad \mu \to \infty. \]
Thus the dimension of the space $D_2 = D_2(E)$ spanned by the eigenfunctions of $M^{-2}$ with the eigenvalues smaller than or equal to $2c_0E^{3-2p}$ does not exceed $C(E^{3-2p})^\frac{1}{2}$, $E \to \infty$. Since $\|M^2\| \leq (2c_0E^{3-2p})^{-1}$ on $D_1^+$, estimate (45) implies the validity of (44) on the space $D = D_1^+ \cap D_2^+$ of co-dimension $o(E)$, $E \to \infty$.

Let us now justify (13). Recall that $R$ is the orthogonal projection on $D = D_1^+ \cap D_2^+$. Denote
\[ A := (1 + M^{-\frac{1}{2}}E^{-1}QM^{-\frac{1}{2}})^{-1} = (1 + RM^{-\frac{1}{2}}E^{-1}QM^{-\frac{1}{2}}R)^{-1} + K =: A_1 + K, \]
where $K$ is a finite-dimensional operator of rank $o(E)$. We are going to prove the validity of (13) on the space $G = \{ \phi \}$ that consists of functions $\phi = A_1\psi$ such that $K\psi = 0$. From (14) it follows that $\|A_1\| \leq \frac{1}{2}$, and therefore the co-dimension of $G$ coincides with the rank of $K$ and has order $o(E)$, $E \to \infty$. Since $A\psi = A_1\psi = \phi$ for $\phi \in G$, we have
\[ (A^{-1}\phi, \phi) = (A_1^{-1}\phi, \phi) \geq \frac{1}{2} \| \phi \|, \quad \phi \in G, \]
for those $E$ where operator $A$ has the trivial kernel. The latter inequality is an immediate consequence of (14). The proof of (13) is complete if $E$ does not belong to the set $S$ where $A$ has a non-trivial kernel. The latter set is discrete (see the paragraph before the statement of Lemma 7.2).

Let us prove the second relation in (42). Consider the space $H$ of functions orthogonal to the eigenfunctions of $M^{-1}$ whose eigenvalues do not exceed $E^\gamma$, $3/4 < \gamma < 1$. From the Weyl law for the operator $M^{-1}$, it follows that the co-dimension of $H$ has order $O(E^\gamma)$, $E \to \infty$. Since $M^{-1} > E^\gamma$ on $H$, the validity of the second relation in (42) on space $H$ follows immediately from (11). Hence both relations (42) hold on $G \cap H$ when $E \to \infty$ without taking values at the set $S$. This leads to the validity of (38) under the
restriction that $s \notin S$. The latter implies (33) for $s \notin S$. Thus (33) holds for all $s \to \infty$ since the left-hand side there is monotonic in $s$ and $S$ is discrete.

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