Normal-mode analysis for collective neutrino oscillations

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Abstract. In an interacting neutrino gas, collective modes of flavor coherence emerge that can be propagating or unstable. We derive the general dispersion relation in the linear regime that depends on the neutrino energy and angle distribution. The essential scales are the vacuum oscillation frequency $\omega = \Delta m^2/(2E)$, the neutrino-neutrino interaction energy $\mu = \sqrt{2}G_F n_\nu$, and the matter potential $\lambda = \sqrt{2}G_F n_e$. Collective modes require non-vanishing $\mu$ and may be dynamical even for $\omega = 0$ (“fast modes”), or they may require $\omega \neq 0$ (“slow modes”). The growth rate of unstable fast modes can be fast itself (independent of $\omega$) or can be slow (suppressed by $\sqrt{|\omega/\mu|}$). We clarify the role of flavor mixing, which is ignored for the identification of collective modes, but necessary to trigger collective flavor motion. A large matter effect is needed to provide an approximate fixed point of flavor evolution, while spatial or temporal variations of matter and/or neutrinos are required as a trigger, i.e., to translate the disturbance provided by the mass term to seed stable or unstable flavor waves. We work out explicit examples to illustrate these points.

Keywords: neutrino theory, supernova neutrinos, supernovas

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1 Introduction

The early universe, collapsing stellar cores, or neutron-star mergers provide environments where neutrinos are so dense that they can have a strong refractive impact on the dynamics of flavor evolution. In contrast to ordinary matter, neutrinos tend to develop coherence between different flavors, producing a flavor-off-diagonal refractive effect which in turn can cause large flavor conversion [1, 2]. The interplay between flavor coherence and flavor conversion can become self-accelerating, leading to self-induced flavor conversion. For appropriate neutrino distributions, such run-away solutions can exist even in the absence of neutrino mixing (“fast flavor conversion”), assuming the instability is triggered by a suitable seed.

On the level of a kinetic treatment, the neutrino distribution as a function of momentum \( p \) and flavor is described by the usual occupation numbers, generalized to a matrix structure.
$\varrho_p$ in flavor space [3, 4]. In this mean-field description, coherence between different $p$ modes is assumed not to build up, whereas flavor coherence for a given momentum is followed explicitly in the form of the off-diagonal elements of $\varrho_p$. The diagonal elements are the usual occupation numbers $f_{\nu_\ell}^{\nu_p}$ for the flavors $\nu_\ell = \nu_e, \nu_\mu$ and $\nu_\tau$. In general, the matrix $\varrho_p$ can be larger, to include additional degrees of freedom such as spin and charge parity to encode spin, spin-flavor, or neutrino-antineutrino correlations [5–11], but here we stick to the simplest case of flavor correlations alone.

In the limit of ultra-relativistic neutrinos, the space-time evolution of the $\varrho_p$ matrices is governed by a Boltzmann kinetic equation of the form [11–17]

$$ (\partial_t + \hat{p} \cdot \partial_x) \varrho_p = -i[H_p, \varrho_p] + C[\varrho], \quad (1.1) $$

where $\hat{p}$ is a unit vector in the direction of $p$. Moreover, $\partial_x$ is understood as a gradient vector with regard to the spatial variables. In the Liouville operator on the left-hand side (lhs) we have ignored a term causing a drift of momenta by external forces such as cosmic or gravitational redshift or deflection. On the right-hand side (rhs), the collision term includes source and sink terms by charged-current or pair processes as well as collisions between neutrinos and particles of the ambient medium or among neutrinos. The collision term includes all momentum modes, so on this level the equation of motion (EOM) is generically nonlinear. Coherent flavor evolution is engendered by the commutator term on the r.h.s. where the Hamiltonian matrix $H_p$ depends on the matrix $M^2$ of squared neutrino masses, on the density and distribution of background particles, and on the $\varrho_p$ matrices themselves. It is these $\varrho_p$ inside the Hamiltonian, that arise on the refractive level, that are the source of nonlinearity with which we are concerned in studies of collective neutrino oscillations.

Equation (1.1) is surprisingly hard to solve in situations of practical astrophysical interest. Even without the flavor oscillation term, neutrino transport in numerical core-collapse or neutron-star merger calculations is the most expensive part and requires various approximations. Flavor oscillations themselves involve very fast time scales, so a combined brute-force numerical solution is out of the question. Even post-processing flavor evolution on a given astrophysical background model defies numerical solutions. One problem is the emergence of unstable collective modes that can break the symmetries of the original system. Moreover, depending on the numerical scheme, spurious unstable modes can appear and can dominate the numerical result [18, 19]. Moreover, even in a stationary background model we are not assured of the stationarity of the solutions for $\varrho_p$ [20–22].

If there were no nonlinearity, the flavor dynamics would be independent for each $p$ and could be found by solving the corresponding one-dimensional problem. In this case, flavor evolution, signified by the accumulation of a relative phase between the mass eigenstate components of a neutrino mode $\varrho_p$, occurs exactly proportional to its displacement along its trajectory in space. It is due to the nonlinearity that flavor and kinematic degrees of freedom evolve differently and the dispersion relation for the flavor waves, obtained via normal-mode analysis, is different from the kinematic one.

To develop a deeper understanding of possible space-time dependent solutions of equation (1.1), and in particular to identify run-away modes, one approach is to study the linearized version of this equation and look for its normal modes. Depending on wavevector and frequency, these modes can simply propagate like waves or show spatial or temporal instabilities. While a normal-mode analysis of such an equation would seem like the first thing one should do, it is only recently that one has begun to look at collective flavor oscillations from this perspective [23, 24].
For a long time one pictured collective flavor conversion as a stationary phenomenon. Neutrinos were taken to be emitted from a source, typically the neutrino sphere of a supernova core, and one asked for the evolution as a function of distance [25, 26]. However, going beyond this simple “bulb model” of neutrino emission with its identical angle distributions for all flavors reveals the existence of “fast flavor modes,” i.e., collective behavior on a scale defined by the neutrino-neutrino interaction energy $\mu \sim \sqrt{2} G_F n_\nu$, rather than the vacuum oscillation frequency $\omega = \Delta m^2 / (2E)$ [27–32]. Moreover, quite generically one needs to worry about situations where neutrinos stream in all directions, so flavor evolution is not trivially a problem of evolution along some pre-defined spatial direction. Therefore, one is motivated to consider an interacting neutrino gas, more or less homogeneous on the scales of perhaps a few meters, and consider possible forms of flavor evolution irrespective of a simple boundary condition — on such scales the picture of a “neutrino sphere” makes no physical sense. Therefore, one is naturally led to worry about general space-time dependent collective solutions of the EOM and the question of how they would be triggered.

So far this dispersion-relation approach was applied in the two-flavor context to the relatively simpler case of fast flavor conversion, where neutrino masses and mixing are irrelevant except for providing seeds for fast flavor instabilities. Here we extend this approach to include “slow oscillations,” driven by neutrino masses, as well as to a three-flavor framework. We will see that all previously studied stability-analysis examples are special cases of such a more general mode analysis. In a simple one-dimensional model of the interacting neutrino gas (“colliding beams”) we will work out explicit examples for the interplay between fast and slow modes. Moreover, we will address the question of seeding coherent flavor motion by the mass term, which is the only source of flavor violation and thus the only possible source of collective flavor waves. We clarify the role of non-uniformity of the matter and/or neutrino density to couple general flavor waves to the mass term, which by itself is perfectly symmetric and thus has no overlap with the inhomogeneous flavor modes.

2 Equation of motion

As a first step we set up the different elements entering the EOM (1.1) and note once more that we work in the limit of ultra-relativistic neutrinos. In this case masses enter only in the vacuum oscillation term, whereas otherwise we can take neutrinos to move with the speed of light [17]. We denote the neutrino velocity vector as $\mathbf{v} = \hat{\mathbf{p}} = \mathbf{p} / |\mathbf{p}|$ and define a velocity four-vector as $v^\mu = (1, \mathbf{v})$ for every $\mathbf{p}$ mode. Ignoring henceforth the collision term, we may thus write the EOM for flavor oscillations in the form

$$v^\alpha \partial_\alpha \varrho_\mathbf{p} = -i[H_\mathbf{p}, \varrho_\mathbf{p}],$$

where a summation over $\alpha = 0, \ldots, 3$ is implied. The Hamiltonian matrix $H_\mathbf{p}$ has the usual contributions from neutrino masses, background matter, and from other neutrinos.

Turning first to the matter effect, we use a local four-fermion current-current description of weak interactions that is relevant for the low-energy environments in collapsed stellar cores or in neutron-star mergers. This approximation is not sufficient in the early universe where the background medium is nearly matter-antimatter symmetric so that the dominant contribution derives from gauge-boson propagator effects [33]. Moreover, we assume neutral-current interactions to be independent of flavor, i.e., we ignore radiative corrections [34, 35]. We finally assume that the medium is not spin-polarized, but we take into account possible convective currents. With these assumptions and working in the weak-interaction basis, the
ordinary matter term depends only on the charged-current contribution of charged leptons. It takes on the form 
\[ H_{\text{matter}}^p = \sqrt{2} G_F v_\alpha F^\alpha_\ell \]
with the matrix of charged-lepton fluxes
\[ F^\alpha_\ell = \int d\mathbf{p} \begin{pmatrix} v^\alpha_e (f_{e,p} - \bar{f}_{e,p}) & 0 & 0 \\ 0 & v^\alpha_\mu (f_{\mu,p} - \bar{f}_{\mu,p}) & 0 \\ 0 & 0 & v^\alpha_\tau (f_{\tau,p} - \bar{f}_{\tau,p}) \end{pmatrix}, \tag{2.2} \]
where \( \int d\mathbf{p} = \int d^3\mathbf{p}/(2\pi)^3 \) and \( v^\alpha_e = (1, \mathbf{v}_e) \) with \( \mathbf{v}_e = \mathbf{p}/(\mathbf{p}^2 + m^2_e)^{1/2} \), and similar for \( \mu \) and \( \tau \) leptons. Moreover, \( f_{e,p} \) and \( \bar{f}_{e,p} \) are the occupation numbers of electrons and positrons with momentum \( \mathbf{p} \), respectively, and analogous for the other charged leptons. Realistically, of course, in compact stars there are no \( \tau \) leptons, but there will be some population of muons \([36]\). If the medium is isotropic, the spatial components vanish and \( F^0_\ell \) is simply the matrix of net charged lepton densities (leptons minus antileptons).

The neutrino-neutrino refractive term has the analogous form
\[ H_{\nu\nu}^p = \sqrt{2} G_F v_\alpha F^\alpha_\nu \]
with the neutrino flux matrix
\[ F^\alpha_\nu = \int d\mathbf{p} v^\alpha (\varrho_p - \bar{\varrho}_p), \tag{2.3} \]
where \( v^\alpha = (1, \hat{\mathbf{p}}) \) and \( \varrho_p \) and \( \bar{\varrho}_p \) are the occupation number matrices for neutrinos and antineutrinos, respectively. Thus the Hamiltonian matrix is
\[ H_p = \frac{M^2}{2E} + \sqrt{2} G_F v_\alpha (F_\ell + F_\nu)^\alpha, \tag{2.4} \]
where \( E = |\mathbf{p}| \). The EOM for the antineutrino matrices \( \bar{\varrho}_p \) is the same with a sign change of the vacuum oscillation term. The refractive term does not depend on \( E \), but only on \( v \). The \( E \)-independence of the refractive term owes to the current-current approximation of the electroweak interaction.

### 3 Linearization

#### 3.1 Small deviation from flavor states

A flavor-dependent neutrino population, for example in a compact astrophysical object, depends on charged-current production and interaction processes, typically involving electrons and positrons and perhaps some muons. Therefore, neutrinos are produced in flavor eigenstates so that the \( \varrho \) matrices are diagonal in the flavor basis and modified by the subsequent effect of flavor conversion. Neutrino mixing angles are large so that off-diagonal elements of the \( \varrho \) matrices quickly develop in vacuum. However, we are interested in astrophysical environments where matter effects are large, i.e., the in-medium effective mixing angles are small and neutrinos remain essentially pinned to the flavor basis except for the possible effect of adiabatic conversion (MSW effect) as they pass a resonance region. We here explicitly exclude such scenarios so that the off-diagonal \( \varrho \) elements remain small unless something new happens in the form of self-induced flavor conversion caused by neutrino-neutrino interactions.

In addition, we assume that the system is stationary apart from the possible effect of neutrino-neutrino interactions, i.e., the occupation numbers of both neutrinos and charged leptons at a given location remain constant except for the possible effect of flavor conversion. To linear order in the small off-diagonal \( \varrho \)-elements, the diagonal elements thus remain conserved and we only ask about the evolution of the off-diagonals which encode flavor coherence \([37]\). Actual flavor conversion, i.e., a non-trivial evolution of the diagonal \( \varrho \)-elements,
becomes visible only at second order in the off-diagonals and thus becomes relevant once the evolution turns nonlinear.

Therefore, we may define an overall matter effect as an external parameter that is caused by both charged leptons and neutrinos as

$$ H_{\text{matter}} = v_\alpha \Lambda^\alpha, $$

where $\Lambda^\alpha$ is the diagonal part of $\sqrt{2} G_F (F_\ell + F_\nu)^\alpha$. Explicitly

$$ \Lambda^\alpha = \text{diag}(\Lambda^\alpha_e, \Lambda^\alpha_\mu, \Lambda^\alpha_\tau), $$

and

$$ \Lambda^\alpha_\ell = \int d\mathbf{p} \left[ v^\alpha_\ell^2 \left( f_{\ell,\mathbf{p}} - \bar{f}_{\ell,\mathbf{p}} \right) + v^\alpha \left( f_{\nu_\ell,\mathbf{p}} - \bar{f}_{\nu_\ell,\mathbf{p}} \right) \right]. $$

Here $v^\alpha_\ell = (1, v_\ell)$ with $v_\ell = p/(p^2 + m^2_\ell)^{1/2}$ is the charged-lepton four-velocity and $v^\alpha = (1, \hat{p})$ the neutrino four-velocity. In other words, $\Lambda^\alpha_\ell$ is the four-current of lepton number $\ell$, carried by both charged leptons and neutrinos.

### 3.2 Vanishing flavor mixing

We are aiming at an EOM and mode analysis for the off-diagonal elements of $\varrho$, an approach that makes strict sense only if the system has a stationary fixed point in the absence of neutrino-neutrino interactions. However, some degree of flavor coherence develops by the effect of $M^2$ alone and indeed in vacuum this effect is large because neutrino mixing angles are large. On the other hand, matter effects strongly suppress vacuum flavor conversion, so in our context, the main effect of the flavor off-diagonal $M^2$ elements will be to trigger instabilities. For the moment we simply take $M^2$ to be diagonal, meaning that we ignore flavor mixing, but not neutrino masses. In the fast-flavor context, $M^2$ was taken to vanish, whereas here only its off-diagonals are taken to vanish.

In the limit of vanishing neutrino mixing, the linearized EOMs for the three off-diagonal elements of $\varrho_\mathbf{p}$ (and their complex conjugates) decouple, leading to equations of the form

$$ i v^\alpha \partial_\alpha \varrho^\mu_\mathbf{p} = \left[ \frac{M^2_{\nu e} - M^2_{\nu_\mu}}{2E} + v_\alpha (\Lambda_e - \Lambda_\mu)^\alpha \right] \varrho^\mu_\mathbf{p} $$

$$ - \sqrt{2} G_F \left( f_{\nu_\tau,\mathbf{p}} - f_{\nu_\mu,\mathbf{p}} \right) v^\alpha \int d\mathbf{p'} v^\alpha_{\mathbf{p'}} \left( \varrho^\mu_{\mathbf{p'}} - \bar{\varrho}^\mu_{\mathbf{p'}} \right) $$

and analogous for the other pairs of flavors. Therefore, in this approach the three-flavor system corresponds to three independent two-flavor cases.

Of course, there are three nontrivial cases only if the distributions of the three flavors are different. In the supernova context, it was often assumed that $\nu_\mu$ and $\nu_\tau$ had identical distributions, but realistically charged muons exist in this environment, implying non-negligible differences between all flavors.

### 3.3 Two-flavor system

The two-flavor EOM further simplifies when we recognize that all flavor coherence effects depend only on the difference of the original neutrino distributions. The commutator structure of the EOM implies that the diagonal parts of all matrices in flavor space drop out. In particular, we may write the neutrino matrices of occupation numbers in the form

$$ \varrho^\mu_\mathbf{p} = \frac{f_{\nu_e,\mathbf{p}} + f_{\nu_\tau,\mathbf{p}}}{2} \mathbb{1} + \frac{f_{\nu_\mu,\mathbf{p}} - f_{\nu_\tau,\mathbf{p}}}{2} \begin{pmatrix} s_p & s_p \\ s_p & -s_p \end{pmatrix}, $$

where

$$ s_p = \int d\mathbf{p'} v^\alpha \left( \varrho^\mu_{\mathbf{p'}} - \bar{\varrho}^\mu_{\mathbf{p'}} \right) \frac{d\mathbf{p'}}{(2\pi)^3} $$
where \( s_p \) is a real number, \( S_p \) a complex one, and \( s_p^2 + |S_p|^2 = 1 \). To linear order, \( s_p = 1 \), so in our linearized system we ask for the space-time evolution of \( S_p \) alone which holds all the information concerning flavor coherence.

Defining the two-flavor matter effect through \( \Lambda^\alpha = (\Lambda_e - \Lambda_\mu)^\alpha \) and the vacuum oscillation frequency through \( \omega_E = (M_{ee}^2 - M_{\mu\mu}^2)/(2E) \), the EOM of equation (3.3) becomes

\[
i \upsilon^\alpha \partial_\alpha S_p = (\omega_E + \upsilon^\alpha \Lambda^\alpha) S_p - \upsilon^\alpha \int dp' v'_\alpha (S_{p'} g_{p'} - S_p g_{p'}) .
\]

(3.5)

An analogous equation applies to the antineutrino flavor coherence \( \bar{S}_p \) with a sign change of \( \omega_E \). Here we use the spectrum \( g_p = \sqrt{2} G_F (f_{\nu_e, p} - f_{\nu_\mu, p}) \) and \( \bar{g}_p = \sqrt{2} G_F (f_{\bar{\nu}_e, p} - f_{\bar{\nu}_\mu, p}) \), where we have absorbed \( \sqrt{2} G_F \) for notational convenience.

### 3.4 Flavor-isospin convention

The structure of these equations becomes both more compact and more physically transparent in the “flavor isospin convention” where we interpret antiparticles as particles with negative energy and describe their spectrum with negative occupation numbers. In the ultrarelativistic limit, neutrino modes are thus described by \(-\infty < E < +\infty\) and their direction of motion \( \nu \) with \( p = |E| \nu \) and \( \upsilon^\alpha = (1, \nu) \). The two-flavor spectrum is

\[
g_{E, \nu} = \sqrt{2} G_F \begin{cases} f_{\nu_e, p} - f_{\nu_\mu, p} & \text{for } E > 0, \\ f_{\bar{\nu}_\mu, p} - f_{\bar{\nu}_e, p} & \text{for } E < 0. \end{cases}
\]

(3.6)

There is no sign-change in the definition of \( S \). The EOM thus reads

\[
i \upsilon^\alpha \partial_\alpha S_{E, \nu} = (\omega_E + \upsilon^\alpha \Lambda^\alpha) S_{E, \nu} - \upsilon^\alpha \int d\Gamma' v'_\alpha g_{E', \nu'} S_{E', \nu'} ,
\]

(3.7)

where the phase-space integration is

\[
\int d\Gamma = \int_{-\infty}^{+\infty} \frac{E^2 dE}{2\pi^2} \int \frac{d\nu}{4\pi} ,
\]

(3.8)

with \( d\nu \) an integral over the unit surface, i.e., over all polar angles of \( p \). The vacuum oscillation frequency \( \omega_E \), in this convention, automatically changes sign for antineutrinos. For positive \( E \), it is positive for inverted mass ordering \((M_{ee}^2 > M_{\mu\mu}^2)\) and negative for the normal mass ordering \((M_{ee}^2 < M_{\mu\mu}^2)\).

The equations become even more compact if one uses \( \omega_E \) as a parameter to describe the neutrino energy and express the spectrum \( g_{E, \nu} \) instead as \( g_{\omega, \nu} \). However, as we here make the connection to fast flavor conversion which corresponds to \( M^2 \to 0 \), it would be cumbersome to take this limit in the \((\omega, \nu)\) language.

### 3.5 Normal-mode analysis

For a linear EOM it is natural to seek solutions in Fourier space, i.e., to look for its normal modes and associated dispersion relation. We thus seek space-time dependent solutions of equation (3.7) that can be written in the form

\[
S_{\Gamma, r} = Q_{\Gamma, K} e^{-i(K_0 t - K \cdot r)} ,
\]

(3.9)
where \( \Gamma = \{ E, v \} \), \( r = (t, r) \) and \( K = (K_0, K) \). The quantity \( Q_{\Gamma,K} \) is the eigenvector in \( \Gamma \)-space for a given eigenvalue \( K \).

To find these normal modes of the EOM we insert the ansatz of equation (3.9) into equation (3.7) and find

\[
(v_\alpha k^\alpha - \omega_E) Q_{\Gamma,k} = v_\alpha A_k^\alpha \quad \text{with} \quad A_k^\alpha = -\int d\Gamma v^\alpha g_\Gamma Q_{\Gamma,k} \tag{3.10}
\]

and \( k^\alpha = K^\alpha - \Lambda^\alpha \). Fully analogous to the fast-flavor case, we have shifted the original four wavevector, \( K^\mu = (K_0, K) \), to the redefined four wavevector, \( k^\mu = (k_0, k) \), by subtracting the matter-effect four vector \( \Lambda^\mu \). Solving the EOMs in Fourier space is thus completely independent of the matter effect which has been “rotated away” by shifting the origin in the four wavevector space.

In the absence of neutrino-neutrino interactions, the r.h.s. of equation (3.10) vanishes and nontrivial solutions require \( v_\alpha k^\alpha - \omega_E = 0 \), i.e., the purely kinematical dispersion relation \( k_0 = \omega_E + v \cdot k \) where each neutrino mode labelled by \( \{ E, v \} \) evolves independently. In the presence of neutrino-neutrino interactions, collective oscillations become possible where this dispersion relation changes. Therefore, we consider solutions with \( v_\alpha k^\alpha - \omega_E \neq 0 \) for all \( \{ E, v \} \) so that equation (3.10) implies

\[
Q_{\Gamma,k} = \frac{v_\alpha A_k^\alpha}{v_r k^\gamma - \omega_E}. \tag{3.11}
\]

Inserting this form on both sides of equation (3.10) yields

\[
v_\alpha A_k^\alpha = -v^\alpha A_k^\beta \int d\Gamma' g_{\Gamma'} \frac{v'_\alpha v'_\beta}{v_r' k^\gamma' - \omega_E}. \tag{3.12}
\]

In more compact notation this can be written in the form

\[
v_\alpha \Pi_k^{\alpha\beta} A_{k,\beta} = 0 \quad \text{with} \quad \Pi_k^{\alpha\beta} = h^{\alpha\beta} + \int d\Gamma g_{\Gamma} \frac{v^\alpha v^\beta}{v_r k^\gamma - \omega_E}, \tag{3.13}
\]

where \( h^{\alpha\beta} = \text{diag}(+, -, -, -) \) is the metric tensor. This equation must hold for any \( v^\alpha \) and thus amounts to four independent equations \( \Pi_k^{\alpha\beta} A_{k,\beta} = 0 \). Nontrivial solutions require

\[
\det \Pi_k^{\alpha\beta} = 0, \tag{3.14}
\]

establishing a connection between the components of \( k = (k_0, k) \), i.e., the dispersion relation of the system. It depends only on the neutrino flavor spectrum \( g(E, v) \), which itself contains the neutrino density, and the energy-dependent vacuum oscillation frequency \( \omega_E \).

Notice that the eigenfunctions \( Q_k(E, v) \) of equation (3.11) for collective motions are not a complete set of solutions of the original EOM. There can be other solutions which are such that \( v_\alpha k^\alpha - \omega_E = 0 \) for some \( \{ E, v \} \), i.e., there exists at least one mode \( \{ E, v \} \) for which the propagation remains purely kinematical. In the context of our colliding beams model we will encounter an explicit example of a system that maintains kinematical modes in the presence of neutrino-neutrino interactions. However, purely kinematical modes will not be unstable, so solving equation (3.14) is expected to turn up all solutions where \( k_0 \) or \( k \) has an imaginary part.
3.6 Fast flavor limit

Equation (3.14) with the definition of $\Pi_{\alpha\beta}^k$ given in equation (3.13) is the master equation for all cases of linear stability analysis performed in the past literature. The case of fast flavor oscillations corresponds to strictly massless neutrinos so that $\omega_E \to 0$. In this case the energy integral in equation (3.13) can be performed on the distribution function alone. We may thus define

$$G_v = \int_{-\infty}^{+\infty} \frac{E^2 dE}{2\pi^2} g_{E,v} = \sqrt{2} G F \int_0^{\infty} \frac{E^2 dE}{2\pi^2} (f_{\nu_e,p} - f_{\bar{\nu}_e,p} - f_{\nu_\mu,p} + f_{\bar{\nu}_\mu,p}),$$

(3.15)

with $p = E v$. The $\Pi$ tensor thus becomes

$$\Pi_{\alpha\beta}^k = h_{\alpha\beta} + \int d\Gamma \frac{v_{\alpha\beta}}{\gamma} k^\gamma,$$

(3.16)

in agreement with earlier results [23].

In the fast flavor limit of $\omega \to 0$ the EOM is the same for all modes $S_{E,v}$ with equal $v$ but different $E$. In reference [23] the discussion was formulated as if one only needed an EOM for $S_v$, taken to be the same for all values of $E$. However, this assumption is not justified because the different modes $S_{E,v}$ need not evolve the same for all $E$ just because they obey the same EOM — their evolution also depends on the initial conditions that could be different for different $E$, depending on what triggers a nontrivial evolution.

3.7 Multi-angle matter effect

The discussion so far appears to imply that background matter plays no role in collective flavor conversion because it was removed by the shift $K^\alpha \to k^\alpha = K^\alpha - \Lambda^\alpha$. On the other hand, in the previous literature it appeared that the multi-angle matter effect would suppress collective flavor conversion in many cases of practical astrophysical interest. This apparent discrepancy is resolved if we remember that in much of the past literature it was assumed that a stationary astrophysical background model implied time-independent solutions for collective flavor conversion. Therefore, only time-independent solutions were considered, which in our terminology implies $K_0 = 0$. With this assumption, the polarization tensor is

$$\Pi_{\alpha\beta}^k = h_{\alpha\beta} - \int dT g_T \frac{v_{\alpha\beta}}{v \cdot K + v_\gamma \Lambda^\gamma + \omega_E}.$$ 

(3.17)

On this basis we can look for eigenvalues $K$ and their imaginary parts. This will lead to the same results as those found in reference [38].

However, solutions which are not stationary in this sense may have much larger growth rates. Therefore, the role of the matter effect depends on the temporal behavior of the solution [20–22]. Therefore, it is crucial to look at collective flavor conversion as a space-time dependent effect, not just one that by fiat depends only on time alone or only on space alone.

3.8 Discrete modes

The derivation of the dispersion relation of equation (3.14) leaves a number of questions open, in particular concerning the completeness of solutions. An alternative approach, in particular for numerical studies, is to consider a discrete mode spectrum $g_j = g_{E_j,v_j}$ with $j = 1, \ldots, N$.
so that the phase-space integral $\int dE \, d\nu$ turns into a summation. The initial EOM of equation (3.7) is a set of $N$ equations and the normal-modes $Q_{j,k}$ become $N$-dimensional vectors of complex numbers fulfilling the equation

$$
\sum_{i=1}^{N} \left[ (v_{j,\alpha} k^\alpha - \omega_j) \delta_{ij} + v_{j,\alpha} v_{i,\alpha} g_{ij} \right] Q_{i,k} = \sum_{i=1}^{N} M_{ij,k} Q_{i,k} = 0. \tag{3.18}
$$

The dispersion relation follows from the condition

$$
D(k_0, k) = \det M_k = 0. \tag{3.19}
$$

This is a purely polynomial equation and thus provides us with $N$ solutions. In this approach one need not divide by the potentially vanishing term $(v_{j,\alpha} k^\alpha - \omega_j)$ and thus one will not miss any solutions. The condition of equation (3.19) is equivalent to equation (3.14) for the non-singular cases owing to the degeneracy (separability) of the “kernel” of the summation (or integral) equation presently at hand.

Typically a discrete neutrino distribution $g_j$ will be used, for example in numerical studies, to represent a continuous distribution $g_E, \nu$. The virtue of finding a complete set of solutions in the discrete case can now become the vice of “too many” solutions which have been dubbed “spurious modes” [18, 19]. Of course, they are spurious only in the sense that they may not appear in the continuous case that we wish to represent. Recently it was shown in a specific example that in the limit $N \to \infty$ the spurious solutions of $D(k_0, k) = 0$ merge to form a branch cut of this complex function [19]. However, it is not assured that this will always be the case, i.e., conceivably the discrete-mode approach in the $N \to \infty$ limit might reveal “non-spurious” solutions that were missed by the $\det \Pi^\beta_k = 0$ criteria, although no such example appears to exist in the literature.

4 Slow and fast modes

In order to illustrate existence of slow and fast modes, the simultaneous action of slow and fast modes, and later to study the role of flavor mixing to trigger instabilities, we set up a simple model that has proven useful as a theoretical laboratory.

4.1 Colliding beams model

We consider a homogeneous system that is essentially one-dimensional, i.e., all neutrino modes are along or opposite the $z$-direction (“colliding beams”). Moreover, we only consider solutions that vary in time and/or in the $z$-direction. We use altogether 4 modes, with Nos. 1 and 3 having $v_z = +1$ and Nos. 2 and 4 having $v_z = -1$. For the moment we leave the four energies $E_j$ ($j = 1, \ldots, 4$) unspecified, giving us four vacuum oscillation frequencies $\omega_j = \Delta m^2 / (2E_j)$. We work in the flavor isospin convention, so negative energies will signify antineutrinos. The EOM for the four complex flavor coherence functions $S_j(t, z)$ following from equation (3.7) are then explicitly

$$
i(\partial_t + \partial_z) S_1 = (\omega_1 + \Lambda_0 - \Lambda_z) S_1 - \mu_2 S_2 - \mu_4 S_4, \tag{4.1a}
$$

$$
i(\partial_t - \partial_z) S_2 = (\omega_2 + \Lambda_0 + \Lambda_z) S_2 - \mu_1 S_1 - \mu_3 S_3, \tag{4.1b}
$$

$$
i(\partial_t + \partial_z) S_3 = (\omega_3 + \Lambda_0 - \Lambda_z) S_3 - \mu_2 S_2 - \mu_4 S_4, \tag{4.1c}
$$

$$
i(\partial_t - \partial_z) S_4 = (\omega_4 + \Lambda_0 + \Lambda_z) S_4 - \mu_1 S_1 - \mu_3 S_3. \tag{4.1d}
$$

We look for solutions of the form $S_j(t,z) = Q_j e^{-i(K_0 t - K_z z)}$, leading to $(K_0 - K_z)Q_1 = (\omega_1 + \Lambda_0 - \Lambda_z)Q_1 - \mu_2 Q_2 - \mu_4 Q_4$ for the first equation, and analogous for the others. We

| Mode No. | Particle | $\omega_j$ | Direction | $v_z$ | $\mu_j$ |
|----------|----------|------------|-----------|-------|--------|
| 1        | $\nu_e$  | $+\omega$  | $\rightarrow$ | +1 | $+g_1\mu$ |
| 2        | $\bar{\nu}_e$ | $-\omega$  | $\leftarrow$ | -1 | $-g_2\mu$ |
| 3        | $\bar{\nu}_\mu$ | $-\omega$  | $\rightarrow$ | +1 | $-g_3\mu$ |
| 4        | $\nu_\mu$  | $+\omega$  | $\leftarrow$ | -1 | $+g_4\mu$ |

Table 1. Properties of the four modes in our colliding beams model. Here $\omega = \Delta m^2 / 2|E|$ is positive for inverted mass ordering and negative otherwise. The mode occupations $g_j$ are positive numbers if there is an excess of $\nu_e$ and $\bar{\nu}_e$ compared to $\nu_\mu$ and $\bar{\nu}_\mu$, respectively, for all modes.

The neutrino-neutrino interaction potentials are $\mu_j = 2\sqrt{2}G_F(n_j^{\nu_e} - n_j^{\nu_\mu})$ with $n_j^{\nu_e}$ the number densities of flavor $\nu_e$ or $\nu_\mu$ in mode $j$. For antineutrino modes (depending on the sign of $E_j$), we have instead $\mu_j = 2\sqrt{2}G_F(n_j^{\bar{\nu}_e} - n_j^{\bar{\nu}_\mu})$. The factor 2 arises from $\nu^\alpha v'^\alpha_\nu$ of the two modes, which is 0 for parallel and 2 for anti-parallel velocities. In this way in equation (4.1) only the even-numbered modes influence the odd-numbered ones and the other way around. The neutrino part of the matter potential is $\Lambda_{\nu,0} = (\mu_1 + \mu_2 + \mu_3 + \mu_4)/2$ and $\Lambda_{\nu,z} = (\mu_1 - \mu_2 + \mu_3 - \mu_4)/2$.

In order to cover all simple examples we consider a monochromatic system, meaning that $|E_j|$ is the same for all modes. We take modes No. 1 and No. 4 to consist of neutrinos, so we use $\omega \equiv \omega_1 = \omega_4$, and Nos. 2 and 3 to consist of antineutrinos, so $\omega_2 = \omega_3 = -\omega$, as shown in table 1. Moreover, for inverted mass ordering we have $\omega > 0$, otherwise $\omega < 0$, i.e., we define $\Delta m^2 = m_1^2 - m_2^2$ to be positive for inverted mass ordering.

We also introduce dimensionless mode occupations $g_j$ such that $\mu_j = g_j \mu$ with the common positive scale factor,

$$
\mu = \frac{2\sqrt{2}G_F}{4} \left( \sum_{j=1,4} |n_j^{\nu_e} - n_j^{\nu_\mu}| + \sum_{j=2,3} |n_j^{\bar{\nu}_e} - n_j^{\bar{\nu}_\mu}| \right),
$$

(4.2)

Separating the overall neutrino-neutrino interaction strength $\mu$ from the “spectrum” $g_j$ is not unique, but it is instructive to separate the overall scale from the detailed relative mode occupations. The normalization condition is chosen as $\sum_{j=1}^4 |g_j| = 4$. If we assume an initial excess of electron-flavored neutrinos and antineutrinos over their muon-flavored counterparts, the mode occupations $g_j$ are positive with $\sum_{j=1}^4 g_j = 4$.

The diversity and complexity of collective neutrino oscillations derives from the dependence on initial conditions encoded in the mode occupations $g_i$. Different choices of $g_i$ lead to qualitatively different evolutions. To illustrate this point, we will first consider that all mode occupations are equal, say $g_i = g$. We dub this highly symmetric case as the “flavor pendulum.” Then we will consider the more generic case, where the various $g_i$ are not all equal, which we will refer to as the general colliding beams model.

### 4.2 Dispersion relation and classification of modes

We look for solutions of the form $S_j(t,z) = Q_j e^{-i(K_0 t - K_z z)}$, leading to $(K_0 - K_z)Q_1 = (\omega_1 + \Lambda_0 - \Lambda_z)Q_1 - \mu_2 Q_2 - \mu_4 Q_4$ for the first equation, and analogous for the others. We
shift the wave vector by virtue of \((k_0, k_z) = (K_0, K_z) - (\Lambda_0, \Lambda_z)\), so our equations become

\[
\begin{pmatrix}
(\omega - k_0 + k_z) & g_2 \mu & 0 & -g_4 \mu \\
-g_1 \mu & (-\omega - k_0 - k_z) & g_3 \mu & 0 \\
0 & g_2 \mu & (-\omega - k_0 + k_z) & -g_4 \mu \\
-g_1 \mu & 0 & g_3 \mu & (\omega - k_0 - k_z)
\end{pmatrix}
\begin{pmatrix}
Q_1 \\
Q_2 \\
Q_3 \\
Q_4
\end{pmatrix} = 0.
\] (4.3)

Non-trivial solutions, and hence the dispersion relation, follow from the requirement that the determinant of the 4\times4 matrix in this equation vanishes.

The dispersion relation for our colliding beams system following from the condition that the determinant of the matrix in equation (4.3) vanishes is given by the general expression

\[
(k_0^2 - k_z^2) [k_0^2 - k_z^2 + (g_1 - g_3)(g_2 - g_4)\mu^2] = 2 \left[(g_1 g_4 - g_2 g_3)k_0 + (g_3 g_4 - g_1 g_2)k_z\right] \mu^2 \omega
\]
\[+ \left[2 (k_0^2 + k_z^2) + (g_1 + g_3)(g_2 + g_4)\mu^2\right] \omega^2
\]
\[- \omega^4.
\] (4.4)

A general solution of this quartic equation is complicated and not informative, but we can explicitly consider several limiting cases.

Unless each of the two colliding beams consists of only neutrinos or only antineutrinos, a special case that one may consider, each beam consists of two modes. Therefore, generally there are four nontrivial solutions, corresponding to the four modes of our system. These modes typically have different symmetries and it is a different question if they get equally excited by a given perturbation in flavor space that may favor one symmetry over another. To address these questions one also needs to study the properties of eigenvectors corresponding to the four eigenvalues.

In the limit \(\mu \to 0\) where neutrinos do not interact with each other, we get four branches of the dispersion relation \(k_0 = \pm k_z \pm \omega\). These solutions are not dynamical in that they correspond to a perturbation on a given mode to simply drift along with the velocity of that mode. We call these “inert” or “kinematical” solutions.

For nonzero \(\mu\) one typically finds “dynamical” modes. These are solutions that do not correspond to a perturbation simply drifting along the velocity. Further, if a mode is dynamical in the limit \(\omega \to 0\), we call it a “fast mode,” and otherwise a “slow mode.” Obviously, the latter are inert in the limit that neutrinos are massless.

We are ultimately interested in modes that can grow exponentially with time or space. The dispersion relation for dynamical propagating modes, i.e., those with both \(k_0\) and \(k_z\) being real, do not cross, so typically there will be “forbidden” ranges of \(k_z\) where \(k_0\) has an imaginary part or the other way round. These can be classified as absolute or convective instabilities [24], with the imaginary parts determining the growth rate. If the growth rate does not depend on \(\omega\), but requires only a nonvanishing \(\mu\), we say that the mode has a “fast instability.” If a nonvanishing \(\omega\) is required, so that the growth rate vanishes in the \(\omega \to 0\) limit, we say that the mode has a “slow instability.” Note that fast modes can have fast or slow instabilities, whereas slow modes can only have slow instabilities. The former happens when fast and slow modes mix with each other, as demonstrated explicitly in section 4.3.3.

In our colliding beams model, fast modes are the solutions which arise when the r.h.s. of equation (4.4) vanishes. In this limit of \(\omega \to 0\) there exist two purely kinematical solutions with \(k_0 = \pm k_z\) and two dynamical ones \(k_0 = \pm \sqrt{k_z^2 + m^2}\), where \(m^2 = (g_1 - g_3)(g_4 - g_2)\mu^2\). The latter are either analogous to a particle with mass \(|m|\) or a tachyon with imaginary mass.
For tachyonic solutions one finds imaginary $k_0$, signaling an instability. In this case, the group velocity $dk_0/dk_z$ is superluminal as opposed to the particle-like solutions. However, this behavior does not imply a violation of causality because signals still propagate only within the light cone, i.e., the group velocity is not a reliable measure for signal propagation. The presence of imaginary frequencies for some range of real wave numbers prevents a localized perturbation to propagate with superluminal speed [41]. Other modes require $\omega \neq 0$ to be dynamical and their possible instabilities have the same requirement. From linear perturbation for small $\omega$ one can easily show that in this case the growth rate scales with $\sqrt{|\omega|}$, which is characteristic for slow instabilities.

4.3 Modes in the colliding beams model

Now we will explore how different modes can arise in our colliding beams model, depending on the values of $g_i$ and $\omega$. In particular, we will see that it is possible that a solution that is inert in the limit $\omega \to 0$, after including a non-vanishing $\omega$ becomes dynamical and acquires a slow instability.

4.3.1 Only slow modes for the flavor pendulum

The flavor pendulum, i.e., an isotropic gas of equal $\nu$ and $\bar{\nu}$ densities corresponds in our 1D model to $g_1 = g_2 = g_3 = g_4 = 1$, leading to the explicit dispersion relation

$$k_0^2 = \omega^2 + k_z^2 \pm 2\sqrt{\omega^2 (k_z^2 + \mu^2)}.$$  \hfill (4.5)

Obviously they are all slow modes. We show these solutions for propagating modes (real $k_0$ vs. real $k_z$) in the lower left panel of figure 1. There are four solutions. One pair of branches shows a frequency gap when for real $k_z$ we get $k_0$ with nonzero imaginary part. The other pair shows a gap in wavenumbers. So there are “forbidden” bands with imaginary frequencies or wavenumbers, i.e., possibly unstable solutions. These instabilities can be further classified as absolute vs. convective following reference [24] for fast modes. We show the imaginary parts, relevant in the gaps, in the top and right panels.

It is instructive to consider specifically the homogeneous solutions ($k_z = 0$) and the corresponding eigenvectors. We may write and group them as

$$\text{Even: } k_0 = \pm \sqrt{\omega^2 - 2\mu\omega} \quad \text{and} \quad Q \propto \begin{pmatrix} \mu \\ \mu - \omega \pm \sqrt{\omega^2 - 2\mu\omega} \\ \mu - \omega \pm \sqrt{\omega^2 - 2\mu\omega} \\ \mu \end{pmatrix},$$  \hfill (4.6a)

$$\text{Odd: } k_0 = \pm \sqrt{\omega^2 + 2\mu\omega} \quad \text{and} \quad Q \propto \begin{pmatrix} -\mu \\ -\mu + \omega \mp \sqrt{\omega^2 + 2\mu\omega} \\ -\mu + \omega \mp \sqrt{\omega^2 + 2\mu\omega} \\ \mu \end{pmatrix}. $$  \hfill (4.6b)

Recall that the eigenvector $Q$ has its components ordered by the mode numbers in table 1. For the even solutions, neutrinos of the two opposing beams contained in mode nos. 1 and 4 behave the same, as do antineutrinos of the two beams contained in mode nos. 2 and 3. In the odd solutions, the same particles of opposing beams have opposite amplitudes. Assuming $\mu > |\omega|/2$ and for $\omega > 0$ (inverted mass ordering) the even solution has an imaginary $k_0$ and thus is unstable, whereas the odd solution has an oscillating one. For $\omega < 0$ (normal ordering), it is the odd solution that is unstable. Therefore, the different branches in figure 1 can be classified according to these symmetry properties that apply at $k_z = 0$. 

4.3.2 Fast modes

In general, with generic values of $g_i$ one finds both fast and slow modes. We recall that by “fast modes” we mean those showing dynamical behavior in the $\omega \to 0$ limit. The general dispersion relation of equation (4.4) reads in this case

$$\left( k_0^2 - k_z^2 \right) \left( k_0^2 - k_z^2 - m^2 \right) = 0 \quad \text{where} \quad m^2 = (g_1 - g_3)(g_4 - g_2)\mu^2.$$  \hfill (4.7)

Figure 2 shows representative solutions for the tachyonic case, i.e., $m^2 = (g_1 - g_3)(g_4 - g_2) < 0$. Two solutions (dashed lines) are inert, while the other two correspond to dynamical fast modes with a fast instability. Note that the neutrino mass ordering is irrelevant in this limit.

The net occupation $g_+ \equiv g_1 - g_3$ is proportional to the number densities of $\nu_e - \bar{\nu}_e - \nu_\mu + \bar{\nu}_\mu$ in the $v_z = +1$ beam. If initially the $\nu_\mu$ and $\bar{\nu}_\mu$ densities are the same, this is the electron lepton number (ELN) carried by the $v_z = +1$ beam. Likewise, $g_- \equiv g_4 - g_2$ is the ELN carried by the $v_z = -1$ beam. Therefore, the dispersion relation is particle-like for $g_+g_- > 0$, i.e., both beams carry an excess of ELN. In the opposite situation with “a crossing of the angle-spectrum,” i.e., when $g_+g_- < 0$ the dispersion relation is tachyron-like. In this case the system shows exponential growth in time for some range of wavenumbers.
Figure 2. Dispersion relation of equation (4.7). There are both fast modes (continuous lines) and slow modes (dashed lines) in the limit of vanishing $\omega$. The fast modes exhibit an instability for small values of $k_z$, as evident from the upper left panel. The real and imaginary parts are understood as in figure 1, with all frequencies shown in units of $\mu$. Note the change of scale for the imaginary axes.

The eigenvectors of these four modes can be explicitly determined and written in the following form

\[ k_0 = +k_z \quad \text{and} \quad Q \propto \begin{pmatrix} g_3 \\ 0 \\ g_1 \\ 0 \end{pmatrix}, \quad (4.8a) \]

\[ k_0 = -k_z \quad \text{and} \quad Q \propto \begin{pmatrix} 0 \\ g_4 \\ 0 \\ g_2 \end{pmatrix}, \quad (4.8b) \]

\[ k_0 = \pm \sqrt{k_z^2 + m^2} \quad \text{and} \quad Q \propto \begin{pmatrix} \mp \sqrt{k_z^2 + m^2 - k_z} \\ (g_1 - g_3)\mu \\ \mp \sqrt{k_z^2 + m^2 - k_z} \\ (g_1 - g_3)\mu \end{pmatrix}. \quad (4.8c) \]

The first two are inert solutions, corresponding to the eigenvalues shown using dashed lines in figure 2, where the neutrinos and antineutrinos of one beam have amplitudes such that the last two terms exactly cancel in the r.h.s. of equation (4.1) and do not influence the
Figure 3. Dispersion relation following from equation (4.4). Slow modes (dashed lines) that were inert for vanishing \( \omega \) now become dynamical. This is clear not only near the origin, as evident from the inset in the upper right panel, but also at large \( k_z \) due to mixing between fast and slow modes. Such a behavior is generic except for some very special parameter choices. The real and imaginary parts are understood as in figure 1 with all frequencies shown in units of \( \mu \).

opposite-moving beam. The dynamical modes, on the other hand, are such that neutrinos and antineutrinos of a given beam can indeed be grouped together and are described by the same coherence function. Notice however that these eigenvectors, while being linearly independent, are not mutually orthogonal except for the two inert modes.

4.3.3 Mixing of slow and fast modes

Next we include nonvanishing neutrino masses and thus take \( \omega \neq 0 \). Now the previously inert modes also become dynamical and, more generally, the previous four solutions get mixed with each other. In figure 3 we show an example for the dispersion relation, corresponding to the same model parameters as the previous section, but now including a small nonzero \( \omega > 0 \).

Near the crossing region of the previously inert modes when at the origin, where \( |k_z| < |\omega| \), these modes become dynamical and for the chosen parameters develop a gap in \( k_z \), i.e., they are tachyonic in this region (see the inset in figure 3). We can see this behavior analytically starting from the general dispersion relation in equation (4.4) and finding the corrections to inert modes introduced due to non-zero \( \omega \). After a first perturbative iteration, keeping terms up to quadratic powers of \( \omega \) and \( k_z \), the dispersion relation are

\[
\begin{align*}
k_0^2 & \simeq k_z^2 + \frac{2(g_3 g_4 - g_1 g_2)}{(g_1 - g_3)(g_2 - g_4)} \omega k_z + \frac{(g_1 + g_3)(g_2 + g_4)}{(g_1 - g_3)(g_2 - g_4)} \omega^2 - \frac{(k_z^2 - \omega^2)^2}{(g_1 - g_3)(g_2 - g_4)\mu^2}.
\end{align*}
\]
For sufficiently large $|k_z|$ the inert modes $k_0 = \pm k_z$ and the dynamical ones $k_0 = \pm \sqrt{k_z^2 + m^2}$ get so close to each other that the small perturbation provided by non-vanishing $\omega$ can be enough to mix them strongly. Quantitatively, this happens for

$$|k_z| \gtrsim \left| \frac{(g_1 - g_3)^2(g_2 - g_4)\mu^2}{8(g_1 + g_3)\omega} \right|,$$

(4.10)

providing the roots after mixing

$$k_0 \simeq k_z \pm \sqrt{\omega^2 - \frac{(g_1 + g_3)(g_2 - g_4)\omega\mu^2}{2k_z}}.$$

(4.11)

In our example, $(g_1 + g_3)(g_2 - g_4) > 0$ so that for $\omega > 0$ and provided that equation (4.10) holds true, the square-root becomes imaginary for a range of values of $k_z$ (it stays real for $k_z \to \infty$), leading to the non-vanishing $\text{Im}(k_0)$ shown in the upper panel of figure 3. Likewise, for a chosen real $k_0$ in this region, $k_z$ develops an imaginary part. For negative $k_z$, the modes do not mix but get pushed apart, never becoming asymptotically close. Changing the sign of $\omega$ is equivalent to changing the sign of $k_z$, i.e., depending on the sign of $\omega$ (the mass ordering) this plot flips horizontally.

### 4.4 Section summary

Except for very special choices of parameters that preserve certain symmetries, e.g., as in the isotropic limit of our colliding beams model discussed in section 4.3.1, one generically finds fast collective flavor modes, i.e., they are dynamical even for vanishing neutrino masses when $\omega = 0$. In keeping with previous findings, the dispersion relation is tachyonic if the beams carry opposite ELN, otherwise it is particle-like.

Including neutrino masses in the form of non-vanishing $\omega$ would naively seem to suggest only a small correction. However, in addition to the fast dynamical modes there exist inert ones that had been ignored in previous two-beam studies [23, 24]. When these additional modes cross each other or when they become asymptotically close to the fast dynamical ones, even a small $\omega$ is enough to mix them and can provide new instabilities. These new instabilities however are slow, i.e., possible growth rates scale with $\sqrt{|\omega/\mu|}$ compared with fast instabilities.

The two-beam examples in references [23, 24] were constructed somewhat differently in that the two “beams” represented two different cones of azimuth-integrated modes with different zenith angle streaming from a source. In this case the velocities represent the radial motion from the source and as such are less than the speed of light. Further, the two inert modes and the two fast dynamical ones have different asymptotes for large $k_z$, i.e., they do not get asymptotically close and do not get mixed by a small $\omega$. However, there are several intersection points, providing “bubbles” of slow instabilities similar to what happens near the origin of figure 3.

### 5 Triggering of modes

In this section we discuss how the flavor mixing term in the vacuum Hamiltonian and non-uniform matter density are responsible for exciting the various normal-modes we found using the dispersion relation.
5.1 Flavor mixing

Thus far our discussion has assumed the absence of flavor mixing so that the mass matrix was diagonal in the weak-interaction basis. In this way, diagonal \( \varrho \) matrices represent a fixed point of the EOM and small perturbations in space and time lead to stable or unstable wave solutions. Of course, in the absence of any source of flavor violation, no stable or unstable perturbations can be excited except perhaps by quantum fluctuations. On the other hand, the observed \( M^2 \) matrix is far from diagonal and it is the only known source of leptonic flavor violation. We here always ignore hypothetical flavor-violating non-standard interactions.

Even though the off-diagonal \( M^2 \) elements are large, initially the off-diagonal \( \varrho \) elements are small so that we may still linearize the EOM. Equation (3.3), for the off-diagonal element \( \varrho^{e\mu} \), then acquires the following additional terms on the r.h.s.

\[
- \frac{M^2_{e\mu}}{2E} (\varrho^{ee} - \varrho^{p\mu}) + \frac{M^2_{\tau\mu}}{2E} \varrho^{p\mu} - \frac{M^2_{e\mu}}{2E} \varrho^{e\tau}.
\]

Therefore, the linearized three-flavor EOMs for the three coherences \( \varrho^{e\mu}, \varrho^{\mu\tau} \) and \( \varrho^{\tau e} \) no longer separate as independent two-flavor systems — flavor coherence among any two flavors is communicated to the others. We here do not pursue such three-flavor effects any further and return to a two-flavor discussion.

In a two-flavor system, for which we symbolically use the \( e\mu \) system, the relevant components of the matrix of squared neutrino masses are

\[
\omega_E^c = \frac{M^2_{ee} - M^2_{\mu\mu}}{2E} = \frac{\Delta m^2}{2E} \cos 2\theta,
\]

\[
\omega_E^s = \frac{M^2_{e\mu}}{2E} = \frac{\Delta m^2}{2E} \sin 2\theta,
\]

where \( \Delta m^2 = m_1^2 - m_2^2 \) is the difference of the squared mass eigenvalues and \( \theta \) is the mixing angle. These frequencies are positive for positive \( E \) and for inverted mass ordering. The EOM in flavor-isospin convention of equation (3.7) thus becomes

\[
i \nu^a \partial_a S_{E,\nu} = -\omega^a_E + (\omega^c_E + v^a \Lambda_0) S_{E,\nu} - v^a \int d\Gamma' \nu'_a g_{E',\nu'} S_{E',\nu'}',
\]

and thus has acquired a constant term, i.e., the first term on the r.h.s. Initially \( S_{E,\nu} = 0 \), so it is this new term which starts any possible motion of the system.

5.1.1 Flavor pendulum

As an explicit example of triggering of modes, we consider the model of section 4.1 in the isotropic limit in one dimension, i.e., the number of neutrinos and antineutrinos going in opposite directions are equal (\( g_1 = g_2 \) and \( g_2 = g_3 \)), and with equal densities of neutrinos and antineutrinos, i.e., the flavor pendulum where only slow modes appear. In this case the EOM is

\[
i \delta_t \begin{pmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{pmatrix} = \begin{pmatrix} -\omega^c \\ \omega^c \\ \omega^c \\ -\omega^c \end{pmatrix} + \begin{pmatrix} \Lambda_0 + \omega^c - \Lambda_z & \mu & 0 & -\mu \\ -\mu & \Lambda_0 - \omega^c + \Lambda_z & \mu & 0 \\ 0 & \mu & \Lambda_0 - \omega^c - \Lambda_z & -\mu \\ -\mu & 0 & \mu & \Lambda_0 + \omega^c + \Lambda_z \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{pmatrix},
\]

\[\text{(5.4)}\]
where $\omega^c = \omega \cos 2\theta$ and $\omega^s = \omega \sin 2\theta$. Notice that even though we have specialized to the homogeneous solution, we keep a possible matter current $\Lambda_z$ that breaks the isotropy of the background medium even though the neutrinos themselves are taken to be isotropic.

As mentioned earlier, this flavor pendulum case is easier to understand by combining the neutrino and antineutrino modes each in a symmetric and antisymmetric solution through $S_{1\pm4} = (S_1 \pm S_4)/2$ and $S_{2\pm3} = (S_2 \pm S_3)/2$ so that

$$
\begin{pmatrix}
S_{1+4} \\
S_{2+3} \\
S_{1-4} \\
S_{2-3}
\end{pmatrix} = 
\begin{pmatrix}
-\omega^s & \mu & -\Lambda_z & 0 \\
0 & -\mu & 0 & \Lambda_z \\
-\Lambda_z & 0 & -\mu & \mu \\
0 & \Lambda_z & -\mu & 0
\end{pmatrix}
\begin{pmatrix}
S_{1+4} \\
S_{2+3} \\
S_{1-4} \\
S_{2-3}
\end{pmatrix}.
$$

(5.5)

If there is no matter current ($\Lambda_z = 0$) the symmetric and antisymmetric solutions decouple from each other, each forming a closed set of equations.

**Symmetric Solution**  Assuming $\Lambda_z = 0$ and with our initial conditions $S_j(0) = 0$, the antisymmetric solution will stay in this fixed point because it is not disturbed by the mass term which has even parity. The symmetric solution, on the other hand, gets triggered. If there is no neutrino-neutrino interaction ($\mu = 0$), the explicit solutions are

$$
S_{1,2}(t) = \frac{\omega^s}{\omega^c \pm \Lambda_0} \left(1 - e^{-i(\Lambda_0 \pm \omega^c) t}\right),
$$

(5.6)

where $S_4 = S_1 = S_{1+4}$ and $S_3 = S_2 = S_{2+3}$. These are the usual neutrino oscillations in matter, here expressed in the limit of small off-diagonal $\theta$ elements. The trajectory of these solutions in the complex plane are circles of radius $|\omega^s/(\omega^c \pm \Lambda_0)|$ and centers displaced from the origin by the same amount, i.e., they are circles passing through the origin.

When discussing the dispersion relation of the unperturbed system we performed a shift $K^\mu \to k^\mu = K^\mu - \Lambda^\mu$ which we can here mimic by going to a rotating frame. In this spirit we consider the rotating solutions $\tilde{S}_{1,2}(t) = S_{1,2}(t) e^{i\Lambda_0 t}$ that are explicitly

$$
\tilde{S}_{1,2}(t) = \frac{\omega^s}{\omega^c \pm \Lambda_0} \left(e^{i\Lambda_0 t} - e^{\mp i\omega^c t}\right).
$$

(5.7)

If we assume the limit of a large matter effect, $|\Lambda_0| \gg |\omega^c|$, we may average over the fast motion and obtain

$$
\langle \tilde{S}_{1,2} \rangle(t) = \mp \frac{\omega^s}{\Lambda_0} e^{\mp i\omega^c t},
$$

(5.8)

where $\langle \ldots \rangle$ means quantities averaged over a cycle of the fast oscillation. These solutions correspond to a rotation in the complex plane around the origin (which is the weak-interaction direction) with the projected vacuum oscillation frequency $\omega^c = (\Delta m^2/2|E|) \cos 2\theta$ and initial conditions $\langle \tilde{S}_{1,2} \rangle(0) = \mp \omega^s/\Lambda_0$.

Next we include neutrino-neutrino interactions with $\mu > 0$ With the notation $\kappa = \sqrt{2(\mu - \omega^c)\omega^c}$ the full solution is

$$
S_1(t) = \frac{\omega^s}{\omega^c} \left[-\kappa + i\omega^c \left(\frac{\omega^s}{\omega^c} - \frac{2(\kappa - i\Lambda_0)}{2(\kappa + i\Lambda_0)} \right) e^{\kappa t} - \frac{\kappa + i\omega^c}{\kappa^2 + \Lambda_0^2} e^{i\Lambda_0 t} + \frac{\kappa^2 + \Lambda_0^2 \omega^c}{\kappa^2 + \Lambda_0^2} e^{i\Lambda_0 t}\right] e^{-i\Lambda_0 t}.
$$

(5.9)

If we assume inverted mass ordering ($\omega^c > 0$) and $\mu > \omega^c/2$, the quantity $\kappa$ is real and positive, so the first term is an exponentially growing solution. This corresponds to the traditional flavor pendulum solution [39].
One peculiar case is the limit $\omega_c \to 0$ that obtains for maximal mixing when $\cos 2\theta \to 0$. In this case $\kappa \to 0$, i.e., there is no unstable solution. We may expand the exponentials $e^{\pm \kappa t}$ to lowest order in $\kappa$. For $\kappa \to 0$ we thus find the limiting solutions $S_{1,2}(t) = (1 - e^{-i\Lambda_0 t})\omega^s/\Lambda_0$, corresponding to free in-medium oscillations as if there were no neutrino-neutrino interaction. The limit of maximal two-flavor mixing implies that the diagonal elements of the $M^2$ matrix are equal so that indeed the stability analysis reveals the absence of instabilities.

Returning to the case $\omega_c > 0$ we now assume a hierarchy of scales $\Lambda_0 > \mu \gg |\omega_c|$ so that $\Lambda_0 \gg \kappa \gg |\omega_c|$. The asymptotic solution for sufficiently long times for the growing modes is

$$S_{1+4}(t) \simeq S_{2+3}(t) \simeq -i \frac{\omega_c \kappa}{2\omega_c \Lambda_0} e^{\kappa t} e^{-i\Lambda_0 t}. \quad (5.10)$$

One finds the same asymptotic solutions if instead of solving the equations (5.4) with vanishing initial conditions, one solves the corresponding equations without the constant term on the r.h.s. but with nonzero initial coherences $S_1(0) = S_4(0) = -\omega^s/\Lambda_0$ and $S_2(0) = S_3(0) = \omega^s/\Lambda_0$. These initial conditions correspond to equation (5.8) at $t = 0$, i.e., effectively we are solving the unstable solutions for quantities that are time-averaged over the fastest scale $\Lambda_0$.

**Antisymmetric solution.** The neutrino mass term cannot trigger the antisymmetric flavor pendulum, corresponding to $S_{1-4}$ and $S_{2-3}$. However, if the background medium has a non-vanishing current $\Lambda_z$, we see from equation (5.5) that the odd and even solutions are coupled. Therefore, after the mass has triggered the even solution, this motion is communicated to the odd one.

To see this more explicitly we now assume normal mass ordering ($\omega < 0$) where the even solution is stable, whereas the odd one grows exponentially. We assume that $\Lambda_z$ is small so that at first the even solution develops without backreaction from the odd one. In the limit $\Lambda_0 > \mu \gg |\omega_c|$ we find the approximate solution

$$S_{1+4}(t) \simeq S_{2+3}(t) \simeq -i \frac{\gamma \omega^s}{\Lambda_0 \omega_c} \sin(\gamma t) e^{-i\Lambda_0 t}, \quad (5.11)$$

where $\gamma = \sqrt{-\omega_c(2\mu - \omega_c)}$ is real for $\omega_c < 0$. The even solution is essentially a flavor pendulum with a small excursion and as such a harmonic oscillator with frequency $\gamma$.

We can now solve the EOM for the odd solution, the lower right block diagonal of equation (5.5), using the even solution in equation (5.11) as a driving force. With a hierarchy of scales $\Lambda_0 > \mu \gg |\omega_c|$, the asymptotic solution after enough exponential growth is

$$S_{1-4}(t) \simeq -S_{2-3}(t) \simeq -i \frac{\omega_c \Lambda_z}{4\omega_c \Lambda_0} e^{\kappa t} e^{-i\Lambda_0 t}, \quad (5.12)$$

with $\kappa = \sqrt{-\omega_c(2\mu + \omega_c)} \simeq \sqrt{-2\omega_c \mu} \simeq \gamma$.

### 5.1.2 Colliding beams model

As another example of mode-triggering, we consider the homogeneous modes with $\partial_z S_j = 0$ in the general colliding beams model. In this case we now have fast modes and the dynamical eigenvectors of equation (4.8c) suggest that we should combine the neutrinos and antineutrinos of each beam to a common mode, i.e., to consider the coherence functions...
\[ S_{1\pm 3} = (S_1 \pm S_3)/2 \] and \[ S_{2\pm 4} = (S_2 \pm S_4)/2. \] With this transformation the EOM for homogeneous background current is explicitly

\[
\begin{pmatrix}
S_{1+3} \\
S_{2+4} \\
S_{1-3} \\
S_{2-4}
\end{pmatrix}
= \begin{pmatrix}
0 & (g_2 - g_1) \mu & 0 & (g_2 + g_4) \mu \\
(\mu g_3 - \mu g_1) & \Lambda_0 & (g_1 + g_3) \mu & 0 \\
\omega_0 & 0 & 0 & 0 \\
\omega_0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
S_{1+3} \\
S_{2+4} \\
S_{1-3} \\
S_{2-4}
\end{pmatrix}. \tag{5.13}
\]

The upper-left block-diagonal part of this EOM represents the fast modes, which however do not have a source term from the mass matrix. However, they are coupled to the odd modes which are excited by the mass term. We can mimic the fast-mode-only case by using \( \omega_0 = 0 \) while \( \omega_0 \neq 0 \), a case corresponding to maximal mixing. In this case the odd modes are not dynamical and they are not influenced by the even ones, so their simple solution is

\[ S_{1-3}(t) = -S_{2-4}(t) = \frac{\omega_0}{\Lambda_0} (1 - e^{-i\Lambda_0 t}) . \tag{5.14} \]

These odd modes then seed the even modes which do not couple to the mass term directly. For the parameters chosen in section 4.3.2 (i.e., \( g_1 = g_2 = 3/2 \) and \( g_3 = g_4 = 1/2 \)) we find

\[ S_{1+3}(t) \simeq \frac{(1 + i) \omega_0}{\Lambda_0 + i \mu} e^{-i\Lambda_0 t} e^{\mu t} \quad \text{and} \quad S_{2+4}(t) \simeq \frac{(1 - i) \omega_0}{\Lambda_0 + i \mu} e^{-i\Lambda_0 t} e^{\mu t} \tag{5.15} \]

for the large-\( t \) asymptotic behavior.

In this example, the unstable modes could not be excited directly by the mass term, but the disturbance was communicated through the inert modes. The eigenmodes of the neutrino system were not orthogonal, so the excitation by the mass term of one set of modes was communicated to those without direct overlap with the mass term. In this case a matter current was not needed to trigger these modes.

### 5.2 Matter inhomogeneity

The main take-away message from the previous subsection is that asymmetries in the background medium are inherited by the flavor evolution. The only term responsible to kick off the evolution is the flavor-violating part of the vacuum Hamiltonian, but it can only trigger homogeneous modes by itself. Inhomogeneous modes get triggered only when we break homogeneity of the background medium by introducing some non-uniformity in the matter density.

To see this effect explicitly we need to consider the time evolution of inhomogeneous modes separately, and for that we need to look at the Fourier transform of the EOMs over space in the presence of time-independent small density fluctuations \( \Lambda_0 + \delta \Lambda_0(r) \), where \( \Lambda_0 \gg \delta \Lambda_0(r) \) is the homogeneous matter density and for simplicity we assume that the background current \( \mathbf{A} = 0 \). We denote the amplitude of each \( \mathbf{k} \) mode at time \( t \) by \( S_{E, \nu}(\mathbf{k}, t) \), so the transformed EOMs are

\[
i\partial_t S_{E, \nu}(\mathbf{k}, t) - (\mathbf{v} \cdot \mathbf{k}) S_{E, \nu}(\mathbf{k}, t) = -\omega_0^E \delta(\mathbf{k}) + \Lambda_0 S_{E, \nu}(\mathbf{k}, t) + \int d\mathbf{k'} \delta \Lambda_0(\mathbf{k} - \mathbf{k'}) S_{E, \nu}(\mathbf{k'}, t) + \omega_0^E S_{E, \nu}(\mathbf{k}, t) - v^\alpha \int d\Gamma' \nu' g_{E', \nu'} S_{E', \nu'}(\mathbf{k}, t), \tag{5.16}
\]

where \( \delta \Lambda_0(\mathbf{k}) \) are the Fourier modes of the small density variation.
It is clear from this equation that matter variations couple different \( k \) modes which makes it possible to trigger non-zero \( k \) modes. We now drop the self interaction term which is a good approximation for small \( t \). We also omit the subscript \( \{E,v\} \) from now onwards. Now we can find the solution to this initial-value problem for small times perturbatively by writing the transformed EOMs as
\[
e^{i\nu k t} S(k, t) = i \omega_E^s \delta(k) t \\
- i \left[ \int_0^t dt' e^{i k v t'}(\Lambda_0 + \omega^c) S(k, t') + \int_0^t dt' e^{i k v t'} \int d k' \delta \Lambda_0(k - k') S(k', t') \right]. \tag{5.17}
\]
Initially the \( S \)-functions are zero, hence only the homogeneous term is excited which then couples to the rest of the modes and excites them too. Given that the coupling between different \( k \) modes is small we can safely assume that there is no feedback. So, to the zeroth order
\[
S(k, t) = \frac{\omega^s}{\Lambda_0 + \omega^c} \delta(k). \tag{5.18}
\]
Putting this back into equation (5.17) for \( k \ne 0 \) we find
\[
S(k, t) = \frac{\omega^s}{\Lambda_0 + \omega^c} \delta(k) \int_0^t dt' e^{i \nu k(t'-t)}(1 - e^{-i(\Lambda_0 + \omega^c)t'}). \tag{5.19}
\]
which shows how the variations in background matter density couple the instability in a homogeneous mode to the inhomogeneous modes.

5.3 Section summary

These explicit examples show that the logic behind the linearized stability analysis remains justified for non-vanishing neutrino mixing, implying non-vanishing off-diagonal elements of the mass matrix. However, we need a large matter effect which implies that without neutrino-neutrino interactions, all polarization vectors remain in a narrow cone around the flavor direction. The fixed-point solution is actually fuzzy and not a fixed point, but a small environment around \( S_j = 0 \). It is this fuzziness of the "fixed point," caused by the off-diagonals of \( M^2 \), which triggers the unstable solutions.

However, the mass term is perfectly symmetric (static, translational invariant, isotropic, parity even, etc.) and as such can only affect collective modes with the same symmetries. On the other hand, it is enough that the background medium through the matter effect violates these symmetries to couple the mass term to other unstable modes. It is ultimately the interplay of the mass term with the background medium that triggers collective modes of neutrino flavor coherence.

6 Conclusion

Starting from the usual kinetic equation for the flavor content of a locally homogeneous neutrino gas, we have studied the EOM for the two-flavor coherence functions in the linear regime. Under the assumption of vanishing neutrino mixing, there is a fixed point corresponding to vanishing flavor coherence, serving as the starting point for the linearization. Based on the flavor-dependent neutrino energy and angle distributions we have derived the dispersion relation for collective modes that emerge in the presence of neutrino-neutrino interactions measured by \( \mu = \sqrt{2 G_F n_{\nu}} \). Including neutrino mass differences (but still ignoring
flavor mixing) the dispersion relation can show fast modes, which are dynamical even for \( \Delta m^2 = 0 \) and thus \( \omega = \Delta m^2/(2E) = 0 \), and/or slow modes which require \( \omega \neq 0 \). Either type of mode typically shows unstable behavior for some range of frequencies (imaginary wavevector) and/or some range of wavevectors (imaginary frequency). The imaginary part can be of the order of \( \mu \) (fast instability) or \( \sqrt{|\omega|} \) (slow instability). Slow modes can only have slow instabilities, whereas fast modes can have fast or slow instabilities.

Up to this point we have studied a wave equation without sources. Within standard physics, the only source of flavor coherence is flavor mixing by the neutrino mass term. It destroys the fixed point of the original EOM, so we need to include a large matter effect which essentially de-mixes neutrinos in that propagation eigenstates nearly coincide with flavor eigenstates. Therefore, without neutrino-neutrino interactions the solutions are flavor oscillations with a very small amplitude. Therefore, our linearization of the EOM remains strictly correct, whereas the original fixed point becomes fuzzy in that modes with different \( E \) precess with different frequencies and around different center points in the complex plane of flavor coherence. Unstable modes are thus seeded by the mass term and, after sufficient growth, coincide with the unstable modes derived from the dispersion relation.

We have studied explicit examples and general expressions for the triggering of unstable modes. If they have non-vanishing wavenumber they require fluctuations of the matter term to mediate between the homogeneous and isotropic mass term and unstable modes that break these symmetries. We have derived an explicit expression for the seed amplitude of a given unstable mode based on the given amplitude of matter fluctuations. In this context we have stressed that the spectrum of collective modes is not complete — there can be modes that remain non-dynamical (inert) even in the presence of neutrino-neutrino interactions, although our dispersion relation captures all possible unstable modes. Moreover, the propagating and/or unstable collective modes, as well as the inert ones, need not be orthogonal. The inert modes can be crucial for mediating the disturbance caused by the mass term to the exponentially growing modes.

The key limitation of our analysis is that it only predicts if and when an instability can occur, but not what the final outcome will be. Specifically, this depends on what seeds the instability and this information is perhaps system-dependent. Also, the analysis is limited to a local volume inside which conditions are taken to be static and homogeneous (except due to the effect of instabilities under consideration) but extending our analysis to the situation where the external conditions are varying requires new ideas. Other ingredients, e.g., including collisions, external forces such as gravity, and effects of time-of-flight, can be challenging to implement within this framework. On the other hand, extending the analysis presented in this paper to include the spin, spin-flavor, and neutrino-antineutrino correlations, is perhaps more straightforward.

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