An Operator Theoretical Proof Of The Fundamental Theorem Of Algebra

Ali Taghavi
Department of Mathematics, Damghan University of Basic Science, Iran
alitghv@yahoo.com, taghavi@dubs.ac.ir

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Abstract

We give a proof of the fundamental theorem of algebra with operator theoretical approach

Introduction:

The fundamental theorem of algebra states every polynomial equation $P(z) = 0$ with complex coefficients has at least one complex root. Various methods have been used for a proof of this historical theorem. Topological methods, complex variable methods (using Liouville theorem) and algebraic methods are examples of such arguments. In spite of its algebraic nature, almost all proofs of this theorem involve topology or analysis. In this note, we present a proof with an operator theoretical view point.

First we present some preliminaries:

Preliminaries:

A Banach space is a vector space $X$ over the real or complex numbers with a norm $||.||$ such that every Cauchy sequence (with respect to the metric $d(x, y) = ||x - y||$) in $X$ has a limit in $X$. A Hilbert space, is a vector space $H$ with an inner product $< . >$, such that $H$ with the norm which induced by $< . >$ is a Banach space.

Assume $X$ is a Banach space, a continuous linear transformation from $X$ to $X$ is called a bounded operator. Here the word bounded signifies that any such operator maps the unit disk of $X$ to a bounded set of $X$. We denote by $B(X)$, the linear space of all bounded operators on $X$. Consider the standard
norm on $B(X)$ as follows: for a given $S$ in $B(X)$, define

$$||S|| = \sup_{|z|=1} |S(z)|.$$ This norm induce a topology on $B(X)$, which is called the norm topology.

By an isometry on $X$ we mean a linear operator on $X$ which preserves the norm of $X$.

$l^2$ is the Hilbert space of all sequences of complex numbers $(a_i)$ with convergence series $\sum |a_i|^2$. $l^2$ is equipped with the inner product $\sum_{i=1}^{\infty} a_i \overline{b_i}$. We say $(a_n)$ is an eventually zero sequence if there exist $k$ such that $a_i = 0$ for all $i > k$

By $i$-shift operator on $l^2$, we mean the operator which sends $(a_i)$ to $(0, 0, \ldots, a_1, a_2, \ldots)$, namely this operator adds $i$ zeroes at the first terms of the sequence $(a_n)$.

$Hol(\mathbb{C})$ is the space of all entire maps from $\mathbb{C}$ to $\mathbb{C}$. Using Taylor expansion, we embed $Hol(\mathbb{C})$ in $l^2$ as a dense linear subspace, as follows:

Let $f$ be an entire map with Taylor expansion $f(z) = \sum a_n z^n$, then the sequens $(a_n)$ is an element of $l^2$, because convergence of $\sum a_n z^n$ for all $z$ with $|z| > 1$ implies that the series is absolutely converge for $z = 1$, then $\sum |a_n|$ converges, and this implies that $\sum |a_n|^2$ converges, too. Note that this embedding sends a polynomial $P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0$ to the sequence $(a_0, a_1, \ldots, a_n, 0, 0, \ldots)$, this sequens is eventually zero. As we prove in the following lemma 1, the space of all sequence which are eventually zero is a dense linear subspace of $l^2$. So we consider $Hol(\mathbb{C})$ as a dense linear subspace of $l^2$, since $Hol(\mathbb{C})$ contains all polynomials in one variable.

**Lemma 1:** The space of all eventually zero sequences is a dense linear subspace of $l^2$.

**Proof of lemma 1:** Assume $(a_n)$ is an element of $l^2$, and $\varepsilon$ is given, we give an eventually zero sequence $(b_n)$ such that $|(a_n) - (b_n)|_2 \leq \varepsilon$. With the Cauchy criteria for the series $\sum |a_n|^2$, we have an integer $k$ such that $\sum_{i=k+1}^{\infty} |a_i|^2 < \varepsilon^2$, now put $(b_n) = (a_1, a_2, \ldots, a_k, 0, 0, \ldots)$, then $|(a_n) - (b_n)|_2 = \sqrt{\sum_{i=k+1}^{\infty} |a_i|^2} \leq \varepsilon$.
We assign to a polynomial $P(z)$, a bounded operator on $l^2$ which restriction to $Hol(\mathbb{C})$ is equal to the map of multiplication by $P(z)$, i.e. the map which sends $f$ to $Pf$.

To prove this, it suffices to assign a bounded operator on $l^2$ correspond to monomial $z^i$, since any polynomial is a linear combination of monomials $z^i$.

Assume $(a_n)$ is the coefficients of Taylor series of an entire map $f(z)$, then the coefficients of Taylor expansion of $z^if(z)$ is $(0, 0, \ldots, 0, a_1, a_2, \ldots)$, where $a_1$ is placed at $i+1$-th term. So multiplication by $z^i$ as a linear operator on $Hol(\mathbb{C})$ can be extended to $i$-shift operator on $l^2$.

Let $H$ be a Hilbert space, and $T$ be a bounded operator on $H$, which range is a closed subspace of $H$, we say $T$ is a Fredholm operator if both kernel and co-kernel of $T$ are finite dimensional linear space where co-kernel of $T$ is the quotient space $H/rang(T)$.

By definition, the Fredholm index of $T$ is dim of kernel of $T$ minus dim of co-kernel of $T$.

In the following we state, without proofs, some essential properties of Fredholm operators, (for more information about Fredholm operator theory see [1]) or [2]:

The space of all Fredholm operators on $H$ is denoted by $Fred(H)$. $Fred(H)$ is an open subset of the space of all bounded operators and index is a continuous map from $Fred(H)$ to integers, $\mathbb{Z}$. Thus if $T$ is a Fredholm operator, there is a neighborhood of $T$ in $B(H)$, with the norm topology, which elements are Fredholm operators of the same index as index of $T$. In fact Fredholm index is a continuous map which is locally constant. This property is called "Invariance of Fredholm index with small perturbation".

In the next part, we present a complete proof for the Fundamental theorem of algebra.

**The proof**

Assume $P(z) = z^n + a_{n-1}z^{n-1} + \ldots$ has no root, then $Q(z) = \varepsilon^n P(z/\varepsilon^n)$ does not have any root, too. We have $Q(z) = z^n + \varepsilon a_{n-1}z^{n-1} + \varepsilon^2 a_{n-2}z^{n-2} + \ldots + \varepsilon^n a_0$.

We define a bijective operator on $Hol(\mathbb{C})$ by multiplication by $Q(z)$, since $Q(z)$ has no root. This operator can be extended to a bounded linear operator.
on \( l^2 \), as we explained above. We call this extension, \( Q \), again.

Now \( Q(z) \) is the perturbation of \( n - \text{shift} \) operator, that is, \( Q \) is sufficiently close to \( n - \text{shift} \) operator if \( \varepsilon \) is sufficiently small. Further it can be easily proved that \( n - \text{shift} \) operator satisfies in the following two conditions:

1) The Fredholm index of \( n - \text{shift} \) operator is \(-n\).

2) \( n - \text{shift} \) operator is an isometry.

Since \( Q \) is a perturbation of \( n - \text{shift} \) operator, it is a Fredholm operator of index \(-n\), this is because of invariance of Fredholm index with small perturbation. Further we use the following lemma 2 to prove that \( Q \) is a one to one, Fredholm operator of index \(-n\) and satisfies the inequality \(|Q(z)| > k|z|\) for some constant \( k \).

Obviously any \( Q \) with such properties can not be a surjective operator because the kernel of \( Q \) has zero dimension, so the codimension of the range of \( Q \) is \( n \). The fact that \( Q \) is not surjective contradicts to the following lemma 3, and this would complete the proof of the fundamental theorem of algebra.

**Lemma 2**: Let \( H \) be a Banach space and \( T \) be an isometry on \( H \), then there is a neighborhood \( W \) of \( T \) in the space of bounded operators on \( H \), such that for every \( S \) in \( W \), we have \(|S(z)| > k|z|\), for a constant \( k \) depend on \( S \).

**Proof of lemma 2**: Assume \(|T - S| \leq 1/2\), then the inequality \(|z| = |T(z)| \leq |T(z) - S(z)| + |S(z)|\) implies \(|S(z)| \geq |z|/2\).

**Lemma 3**: Let \( F \) be a dense subspace of a Banach space \( E \), and \( T \) is a bounded linear operator on \( E \), which maps \( F \) onto \( F \), further \(|Tz| > k|z|\) for some \( k \), then \( T \) is a surjective operator onto \( E \).

**Proof of lemma 3**: If \( T \) is not surjective, then there is an element \( e \) of \( E \) which is not in the range of \( T \). Since \( F \) is dense, there is a sequence \( f_n \) of elements of \( F \) which converge to \( e \) and \( f_n = T(b_n) \) for some \( b_n \) in \( F \). The condition \(|Tz| > k|z|\) implies that the pre-image of a Cauchy sequence under \( T \) is a Cauchy sequence. So \( b_n \) is a Cauchy sequence, then converges to some \( b_* \) in \( E \). From continuity of \( T \) we have \( T(b_*) = e \), that is \( e \) is in the image of \( T \), which contradicts to our first assumption. This completes the proof of lemma 3.
References

[1] John B. Conway, *A Course In Functional Analysis*, Springer-Verlag, 1985.

[2] John B. Conway, *A Course In Operator Theory*, American Mathematical Society, Providence, RI, 2000.