Classical and new log log-theorems

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Abstract

We present a unified approach to celebrated log log-theorems of Carleman, Wolf, Levinson, Sjöberg, Matsaev on majorants of analytic functions. Moreover, we obtain stronger results by replacing original pointwise bounds with integral ones. The main ingredient is a complete description for radial projections of harmonic measures of strictly star-shaped domains in the plane, which, in particular, explains where the log log-conditions come from.

1 Introduction. Statement of results

Our starting point is classical theorems due to Carleman, Wolf, Levinson, and Sjöberg, on majorants of analytic functions.

Definition 1 A nonnegative measurable function $M$ on a segment $[a, b] \subset \mathbb{R}$ belongs to the class $L^{++}[a, b]$ if

$$\int_a^b \log^+ \log^+ M(t) \, dt < \infty.$$  

(For any real-valued function $h$, we write $h^+ = \max\{h, 0\}$, $h^- = h^+ - h$.)

Carleman was the first who remarked a special role of functions of the class $L^{++}$ in complex analysis, by proving the following variant of the Liouville theorem.

Theorem A (T. Carleman [3]) If an entire function $f$ in the complex plane $\mathbb{C}$ has the bound

$$|f(r e^{i\theta})| \leq M(\theta) \quad \forall \theta \in [0, 2\pi], \forall r \geq r_0,$$

with $M \in L^{++}[0, 2\pi]$, then $f \equiv \text{const}$.

This phenomenon appears also in the Phragmén–Lindelöf setting.

Theorem B (F. Wolf [22]) If a holomorphic function $f$ in the upper half-plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \text{Im} z > 0\}$ satisfies the condition

$$\limsup_{z \to x_0} |f(z)| \leq 1 \quad \forall x_0 \in \mathbb{R}$$
and for any $\epsilon > 0$ and all $r > R(\epsilon)$, $\theta \in (0, \pi)$, one has

$$|f(re^{i\theta})| \leq [M(\theta)]^r$$

with $M \in \mathcal{L}^+[0, \pi]$, then $|f(z)| \leq 1$ on $\mathbb{C}_+$.

The most famous statement of this type is the following local result known as the Levinson–Sjöberg theorem.

**Theorem C** (N. Levinson [13], N. Sjöberg [21], F. Wolf [23]) *If a holomorphic function $f$ in the domain $Q = \{x + iy : |x| < 1, |y| < 1\}$ has the bound

$$|f(x + iy)| \leq M(y) \quad \forall x + iy \in Q,$$

with $M \in \mathcal{L}^+[-1, 1]$, then for any compact subset $K$ of $Q$ there is a constant $C_K$, independent of the function $f$, such that $|f(z)| \leq C_K$ in $K$.*

For further developments of Theorem C, including higher dimensional variants, see [4], [5], [7], [8], [9]. Theorems A and B were extended to subharmonic functions in higher dimensions in [23].

A similar feature of majorants from the class $\mathcal{L}^+$ was discovered by Beurling in a problem of extension of analytic functions [2]. It also appears in relation to holomorphic functions from the MacLane class in the unit disk [10], [14], and in a description of non-quasi-analytic Carleman classes [6].

The next result, due to Matsaev, does not look like a log log-theorem, however (as will be seen from our considerations) it is also about the class $\mathcal{L}^+$; further results in this direction can be found in [16].

**Theorem D** (V.I. Matsaev [15]) *If an entire function $f$ satisfies the relation

$$\log |f(re^{i\theta})| \geq -Cr^\alpha|\sin \theta|^{-k} \quad \forall \theta \in (0, \pi), \forall r > 0,$$

with some $C > 0$, $\alpha > 1$, and $k \geq 0$, then it has at most normal type with respect to the order $\alpha$, that is, $\log |f(re^{i\theta})| \leq Ar^\alpha + B$.*

All these theorems can be formulated in terms of subharmonic functions (by taking $u(z) = \log |f(z)|$ as a pattern), however our main goal is to replace the pointwise bounds like (1) with some integral conditions. A model situation is the following form of the Phragmén–Lindelöf theorem.

**Theorem E** (Ahlfors [1]) *If a subharmonic function $u$ in $\mathbb{C}_+$ with nonpositive boundary values on $\mathbb{R}$ satisfies

$$\lim_{r \to \infty} r^{-1} \int_0^\pi u^+(r e^{i\theta}) \sin \theta \, d\theta = 0,$$

then $u \leq 0$ in $\mathbb{C}_+$.*

We will show that all the above theorems are particular cases of results on the class $\mathcal{A}$ defined below and that the log log-conditions appear as conditions for continuity of certain logarithmic potentials.
Definition 2 Let $\nu$ be a probability measure on a segment $[a, b]$; we will identify it occasionally with its distribution function $\nu(t) = \nu([a, t])$. Suppose $\nu(t)$ is strictly increasing and continuous on $[a, b]$, and denote by $\mu$ its inverse function extended to the whole real axis as $\mu(t) = a$ for $t < 0$ and $\mu(t) = b$ for $t > 1$. We will say that such a measure $\nu$ belongs to the class $A$ \([a, b]\) if
\[
\lim_{\delta \to 0} \sup_x \int_0^\delta \frac{\mu(x + t) - \mu(x - t)}{t} dt = 0. \tag{2}
\]

Note that this class is completely different from MacLane’s class $A$ [14] that consists of holomorphic functions in the unit disk with asymptotic values at a dense subset of the circle. MacLane’s class is however described by the condition $|f(re^{i\theta})| \leq M(r), M \in \mathcal{L}^{++}[0, 1]$.

Our results extending Theorems A–C and E are as follows.

Theorem 1 Let a subharmonic function $u$ in the complex plane satisfy
\[
\int_0^{2\pi} u^+(te^{i\theta}) \, d\nu(\theta) \leq V(t) \quad \forall t \geq t_0, \tag{3}
\]
with $\nu \in A[0, 2\pi]$ and a nondecreasing function $V$ on $\mathbb{R}_+$. Then there exist constants $c > 0$ and $A \geq 1$, independent of $u$, such that
\[
u(te^{i\theta}) \leq cV(At) \quad \forall t \geq t_0. \tag{4}
\]

Theorem 2 If a subharmonic function $u$ in the upper half-plane $\mathbb{C}_+$ satisfies the conditions
\[
\limsup_{z \to x_0} u(z) \leq 0 \quad \forall x_0 \in \mathbb{R}
\]
and
\[
\lim_{t \to \infty} t^{-1} \int_0^{\pi} u^+(te^{i\theta}) \, d\nu(\theta) = 0
\]
with $\nu \in A[0, \pi]$, then $u(z) \leq 0 \ \forall z \in \mathbb{C}_+$.

Theorem 3 Let a subharmonic function $u$ in $Q = \{x + iy : |x| < 1, |y| < 1\}$ satisfy
\[
\int_{-1}^1 u^+(x + iy) \, d\nu(y) \leq 1 \quad \forall x \in (-1, 1) \tag{5}
\]
with $\nu \in A[-1, 1]$. Then for each compact set $K \subset Q$ there is a constant $C_K$, independent of the function $u$, such that $u(z) \leq C_K$ on $K$.

Relation of these results to the log log-theorems becomes clear by means of the following statement.
Definition 3 Denote by \( L^{-}[a,b] \) the class of all nonnegative integrable functions \( g \) on the segment \([a,b]\), such that
\[
\int_a^b \log^{-} g(s) \, ds < \infty.
\] (6)

Proposition 1 If the density \( \nu' \) of an absolutely continuous increasing function \( \nu \) belongs to the class \( L^{-}[a,b] \), then \( \nu \in \mathcal{A}[a,b] \). Consequently, if a holomorphic function \( f \) has a majorant \( M \in L^{++} \), then \( \log |f| \) has the corresponding integral bound with the weight \( \nu \in \mathcal{A} \) with the density \( \nu'(t) = \min\{1, 1/M(t)\} \).

We recall that positive measures \( \nu \) on the unit circle with \( \nu' \in L^{-}[0,2\pi] \) are called Szegő measures. Proposition 1 states, in particular, that absolutely continuous Szegő measures belong to the class \( \mathcal{A}[0,2\pi] \).

An integral version of Theorem D has the following form.

Theorem 4 Let a function \( u \), subharmonic in \( \mathbb{C} \) and harmonic in \( \mathbb{C} \setminus \mathbb{R} \), satisfy the inequality
\[
\int_{-\pi}^{\pi} u^{-1}(re^{i\theta})\Phi(|\sin \theta|) \, d\theta \leq V(r) \quad \forall r \geq r_0,
\] (7)
where \( \Phi \in L_{-}[0,1] \) is nondecreasing and the function \( V \) is such that \( r^{-1-\delta}V(r) \) is increasing in \( r \) for some \( \delta > 0 \). Then there are constants \( c > 0 \) and \( A \geq 1 \), independent of \( u \), such that
\[
u(\{z \in \mathbb{C} : |z| = r\}) \leq cV(Ar) \quad \forall r \geq r_1 = r_1(u).
\]

Our proofs of Theorems 1-4 rest on a presentation of measures of the class \( \mathcal{A}[0,2\pi] \) as radial projections of harmonic measures of star-shaped domains. Let \( \Omega \) be a bounded Jordan domain containing the origin. Given a set \( E \subset \partial \Omega \), \( \omega(z, E, \Omega) \) will denote the harmonic measure of \( E \) at \( z \in \Omega \), i.e., the solution of the Dirichlet problem in \( \Omega \) with the boundary data 1 on \( E \) and 0 on \( \partial \Omega \setminus E \). The measure \( \omega(0, E, \Omega) \) generates a measure on the unit circle \( T \) by means of the radial projection \( \zeta \mapsto \zeta/|\zeta| \). It is convenient for us to consider it as a measure on the segment \([0, 2\pi]\), so we put
\[
\hat{\omega}_\Omega(F) = \omega(0, \{\zeta \in \partial \Omega : \arg \zeta \in F\}, \Omega)
\] (8)
for each Borel set \( F \subset [0, 2\pi] \).

The inverse problem is as follows. Given a probability measure on the unit circle \( T \), is it the radial projection of the harmonic measure of any domain \( \Omega \)?

For our purposes we specify \( \Omega \) to be strictly star-shaped, i.e., of the form
\[
\Omega = \{re^{i\theta} : r < r_\Omega(\theta), \ 0 \leq \theta \leq 2\pi\}
\] (9)
with \( r_\Omega \) a positive continuous function on \([0, 2\pi]\), \( r_\Omega(0) = r_\Omega(2\pi) \).
Theorem 5 A continuous probability measure $\nu$ on $[0, 2\pi]$ is the radial projection of the harmonic measure of a strictly star-shaped domain if and only if $\nu \in A[0, 2\pi]$.

Corollary 4 Every absolutely continuous measure from the Szegő class on the unit circle is the radial projection of the harmonic measure of some strictly star-shaped domain.

Theorem 5 is proved by a method originated by B.Ya. Levin in theory of majorants in classes of subharmonic functions [11].

Theorems 1–3 and 5 (some of them in a slightly weaker form) were announced in [18] and proved in [19] and [20]. The main objective of the present paper, Theorem 4, is new. Since its proof rests heavily on Theorem 5, we present a proof of the latter as well, having in mind that the papers [19] and [20] are not easily accessible. Moreover, we include the proofs of Theorems 1–3, too, motivated by the same accessibility reason as well as by the idea of showing the whole picture.

2 Radial projections of harmonic measures (Proofs of Theorem 5 and Proposition 1)

Measures from the class $A$ have a simple characterization as follows.

Proposition 2 Let $\mu$ and $\nu$ be as in Definition 2. Then the function

$$N(x) = \int_0^1 \log |x - t| \, d\mu(t)$$

is continuous on $[0, 1]$ if and only if $\nu \in A[a, b]$.

Proof. The function $N(x)$ is continuous on $[0, 1]$ if and only if for any $\epsilon > 0$ one can choose $\delta \in (0, 1)$ such that

$$I_x(\delta) = \int_{|t - x| < \delta} \log |x - t| \, d\mu(t) > -\epsilon$$

for all $x \in [0, 1]$. Integrating $I_x$ by parts, we get

$$|I_x(\delta)| = \int_0^\delta r_x(t) \frac{dt}{t} + r_x(\delta)|\log \delta|,$$

where $r_x(t) = \mu(x + t) - \mu(x - t)$. Therefore, continuity of $N(x)$ implies (2). On the other hand, since $r_x(t)$ increases in $t$, we have

$$r_x(\delta)|\log \delta| = 2r_x(\delta) \int_0^\delta \frac{dt}{t} \leq 2 \int_0^\delta \frac{\sqrt{\delta}}{t} \frac{r_x(t)}{t} \, dt,$$

which gives the reverse implication. □

In the proof of Theorem 5 we will use this property in the following form.
Proposition 3  Let $\mu$ and $\nu$ be as in Definition 2 for the class $A[0, 2\pi]$. Then the function
\[ h(z) = \int_0^{2\pi} \log |e^{i\theta} - z| \, d\mu(\theta/2\pi) \]
is continuous on $\mathbb{T}$ if and only if $\nu \in A[0, 2\pi]$.

Proof of Theorem 5  1) First we prove the sufficiency: every $\nu \in A[0, 2\pi]$ has the form $\nu = \hat{\omega}_\Omega$ for some strictly star-shaped domain $\Omega$. In particular, for any compact set $K \subset \Omega$ there is a constant $C(K)$ such that
\[ \omega(z, E, \Omega) \leq C(K) \nu(\arg E) \quad \forall z \in E \quad (10) \]
for every Borel set $E \subset \partial \Omega$, where $\arg E = \{ \arg \zeta : \zeta \in E \}$.

Let $u(z) = \frac{1}{\pi} \int_0^{2\pi} \log |e^{i\theta} - z| \, d\mu(\theta/2\pi)$ with $\mu$ the inverse function to $\nu \in A[0, 2\pi]$. The function $u$ is subharmonic in $\mathbb{C}$ and harmonic outside the unit circle $\mathbb{T}$. By Proposition 3 it is continuous on $\mathbb{T}$ and thus, by Evans’ theorem, in the whole plane. Let $v$ be a harmonic conjugate to $u$ in the unit disk $\mathbb{D}$, which is determined uniquely up to a constant. Since $u \in C(\overline{\mathbb{D}})$, radial limits $v^*(e^{i\psi})$ of $v$ exist a.e. on $\mathbb{T}$. Let us fix such a point $e^{i\psi_0}$ and choose the constant in the definition of $v$ in such a way that $v^*(e^{i\psi_0}) = \psi_0$.

Consider then the function $w(z) = z \exp\{-u(z) - iv(z)\}$, $z \in \mathbb{D}$. By the Cauchy-Riemann condition, $\partial v/\partial \phi = r \partial u/\partial r$, which implies
\[
\text{arg } w(re^{i\psi}) = \psi - v(re^{i\psi_0}) - \int_{\psi_0}^{\psi} \frac{\partial v(re^{i\phi})}{\partial \phi} \, d\phi = \psi_0 - v(re^{i\psi_0})
\]
\[ + \frac{1}{2\pi} \int_{\psi_0}^{\psi} \int_0^{2\pi} \left[ 1 - \frac{2r^2 - 2r \cos(\theta - \phi)}{|r - e^{i(\theta - \phi)}|^2} \right] \, d\mu(\theta/2\pi) \, d\phi \]
\[ = \psi_0 - v(re^{i\psi_0}) + \frac{1}{2\pi} \int_{\psi_0}^{\psi} \int_0^{2\pi} \frac{1 - r^2}{|r - e^{i(\theta - \phi)}|^2} \, d\mu(\theta/2\pi) \, d\phi. \]

By changing the integration order and passing to the limit as $r \to 1$, we derive that for each $\psi \in [0, 2\pi]$ there exists the limit
\[ \lim_{r \to 1} \text{arg } w(re^{i\psi}) = \mu(\psi/2\pi) - \mu(\psi_0/2\pi). \]

Therefore the function $\text{arg } w$ is continuous up to the boundary of the disk; in particular, we can take $\psi_0 = 0$. Since $|w|$ is continuous in $\overline{\mathbb{D}}$ as well, so is $w$.

By the boundary correspondence principle, $w$ gives a conformal map of $\mathbb{D}$ onto the domain
\[ \Omega = \{ re^{i\theta} : r < \exp\{-u(\exp\{2\pi i\nu(\theta)\})\}, \quad 0 \leq \theta \leq 2\pi \}. \quad (11) \]
It is easy to see that the domain $\Omega$ is what we sought. Let $f$ be the conformal map of $\Omega$ to $\mathbb{D}$, inverse to $w$. For $z \in \Omega$ and $E \subset \partial \Omega$, we have

$$\omega(z, E, \Omega) = \omega(f(z), f(E), U) = \frac{1}{2\pi} \int_{\Gamma} \frac{1 - |f(z)|^2}{|f(z) - e^{it}|^2} dt$$

$$= (1 - |f(z)|^2) \int_{\partial E} \frac{d\nu(s)}{|f(z) - e^{2\pi i \nu(s)}|^2},$$

which proves the claim.

2) Now we prove the necessity: if $\omega$ is of the form (9), then $\hat{\omega}_\Omega \in A[0, 2\pi]$.

We use an idea from the proof of [11, Theorem 2.4]. Let $w$ be a conformal map of $\mathbb{D}$ to $\Omega$, $w(0) = 0$. Since $\Omega$ is a Jordan domain, $w$ extends to a continuous map from $\overline{\mathbb{D}}$ to $\overline{\Omega}$, and we can specify it to have $\arg w(1) = 0$. Define

$$f(z) = u(z) + iv(z) = \log \frac{w(z)}{z} \text{ for } |z| \leq 1, \quad f(z) = f(|z|^{-2}z) \text{ for } |z| > 1.$$ 

It is analytic in $\mathbb{D}$ and continuous in $\mathbb{C}$. Define then the function

$$\lambda(z) = u(z) + \frac{1}{\pi} \int_0^{2\pi} \log |e^{i\psi} - z| \, dv(e^{i\psi}), \quad (12)$$

$\delta$-subharmonic in $\mathbb{C}$ and harmonic in $\mathbb{C} \setminus \mathbb{T}$. Let us show that it as actually harmonic (and, hence, continuous) everywhere. To this end, take any function $\alpha \in C(\mathbb{T})$ and a number $r < 1$, and apply Green’s formula for $u(z)$ and $A(z) = |z| \alpha(z/|z|)$ in the domain $D_r = \{r < |z| < r^{-1}\}$:

$$\int_{D_r} (A \Delta u - u \Delta A) = \left[ \frac{\rho}{2\pi} \int_0^{2\pi} \left( \rho \alpha(e^{i\psi}) \frac{\partial u(\rho e^{i\psi})}{\partial \rho} - u(\rho e^{i\psi}) \alpha(e^{i\psi}) \right) d\psi \right]_{\rho=r}^{\rho=R}. \quad (13)$$

Using the definition of the function $f$ outside $\mathbb{D}$ and the Cauchy-Riemann equations $\partial v/\partial \phi = \rho \partial u/\partial \rho$ if $\rho < 1$ and $\partial v/\partial \phi = -\rho \partial u/\partial \rho$ if $\rho > 1$ (which follows from the definition of $f$), we can write the right hand side of (13) as

$$\frac{r + r^{-1}}{2\pi} \int_0^{2\pi} \alpha(e^{i\psi}) d\phi v(re^{i\psi}) + \frac{r - r^{-1}}{2\pi} \int_0^{2\pi} u(re^{i\psi}) \alpha(e^{i\psi}) d\psi.$$ 

When $r \to 1$, (13) takes the form

$$\int_{\mathbb{T}} \alpha \Delta u = -\frac{1}{\pi} \int_{\mathbb{D}} \alpha(e^{i\psi}) \, dv(e^{i\psi}),$$

which implies the harmonicity of the function $\lambda(z)$ (12) in the whole plane.
Now we recall that \( v(e^{i\psi}) = \arg w(e^{i\psi}) - \psi \). Since the harmonic measure of the \( w \)-image of the arc \( \{ e^{i\theta} : 0 < \theta < \psi \} \) equals \( \psi/2\pi \), we have
\[
\hat{\omega}_\Omega(\arg w(e^{i\psi})) = \psi/2\pi
\]
and thus \( \arg w(e^{i\psi}) = \mu(\psi/2\pi) \) with \( \mu \) the inverse function to \( \hat{\omega}_\Omega(\psi) \). Therefore, \( v(e^{i\psi}) = \mu(\psi/2\pi) - \psi \).

Consider, finally, the function
\[
\gamma(z) = \frac{1}{\pi} \int_0^{2\pi} \log |e^{i\psi} - z| \, d\mu(\psi/2\pi) = \lambda(z) - u(z) + \frac{1}{\pi} \int_0^{2\pi} \log |e^{i\psi} - z| \, d\psi.
\]
Since it is continuous on \( \mathbb{T} \), Proposition 3 implies \( \hat{\omega}_\Omega \in \mathcal{A}[0, 2\pi] \), and the theorem is proved. \( \square \)

Note that all the dilations \( t\Omega \) of \( \Omega \) \((t > 0)\) represent the same measure from \( \mathcal{A}[0, 2\pi] \), and \( \Omega \) with a given projection \( \hat{\omega}_\Omega \) is unique up to the dilations.

Now we prove Proposition 1 that presents a wide subclass of \( \mathcal{A} \) with a more explicit description.

**Proof of Proposition 1.** Let \( \nu : [0, 1] \to [0, 1] \) be an absolutely continuous, strictly increasing function, \( \nu' \in L^{-}[0, 1] \). Since mes \( \{ t : \nu'(t) = 0 \} = 0 \), its inverse function \( \mu \) is absolutely continuous (\[17\], p. 297), so
\[
\mu(t) = \int_0^t g(s) \, ds
\]
with \( g \) a nonnegative function on \([0, 1]\). We have
\[
\infty > \int_0^1 \log^{-} \nu'(t) \, dt = \int_0^1 \log^{-} \frac{1}{\nu'(t)} \, d\mu(t) = \int_0^1 g(t) \log^{+} g(t) \, dt,
\]
so \( g \) belongs to the Zygmund class \( L \log L \).

Let \( \Delta(t) \) denote the modulus of continuity of the function \( \mu \). Note that it can be expressed in the form
\[
\Delta(t) = \int_0^t h(s) \, ds
\]
where \( h \) is the nonincreasing equimeasurable rearrangement of \( g \). Then
\[
\int_0^1 \frac{\Delta(t)}{t} \, dt = \int_0^1 t^{-1} \int_0^1 h(s) \, ds \, dt = \int_0^1 h(s) \log s^{-1} \, ds
\]
\[
= \int_{E_1 \cup E_2} h(s) \log s^{-1} \, ds,
\]
where \( E_1 = \{ s \in (0, 1) : h(s) > s^{-1/2} \} \), \( E_2 = (0, 1) \setminus E_1 \). Since \( h \in L \log L [0, 1] \),
\[
\int_{E_1} h(s) \log s^{-1} \, ds \leq 2 \int_{E_1} h(s) \log h(s) \, ds < \infty.
\]
Besides,
\[ \int_{E_2} h(s) \log s^{-1} \, ds \leq \int_{E_2} s^{-1/2} \log s^{-1} \, ds < \infty. \]

Therefore,
\[ \int_{0}^{1} \frac{\Delta(t)}{t} \, dt < \infty \]
and thus
\[ \lim_{\delta \to 0} \int_{0}^{\delta} \frac{\Delta(t)}{t} \, dt = 0, \]
which gives (2). \qed

Corollary 4 follows directly from the definition of the Szegö class, Theorem 5 and Proposition 1.

3 Proofs of Theorems 1 and 2

Here we show how the integral variants of Carleman’s and Wolf’s theorems can be derived from Theorem 5.

We will need an elementary

Lemma 5 Let \( r(\theta) \in C[0, 2\pi], 1 < r_1 \leq r(\theta) \leq r_2, \) let \( \nu \) be a positive measure on \([0, 2\pi]\) and \( V(t) \) be a nonnegative function on \([0, \infty]\). If a nonnegative function \( v(te^{i\theta}) \) satisfies
\[ \int_{0}^{2\pi} v(te^{i\theta}) \, d\nu(\theta) \leq V(t) \quad \forall t \geq t_0, \]
then for any \( R_2 > R_1 \geq t_0, \)
\[ \int_{R_1}^{R_2} \int_{0}^{2\pi} v(t r(\theta)e^{i\theta}) \, d\nu(\theta) \, dt \leq r_1^{-1} \int_{r_1 R_1}^{r_2 R_2} V(t) \, dt. \]

Proof of Lemma 5 is straightforward:
\[ \int_{R_1}^{R_2} \int_{0}^{2\pi} v(t r(\theta)e^{i\theta}) \, d\nu(\theta) \, dt = \int_{0}^{2\pi} \int_{R_1 r(\theta)}^{R_2 r(\theta)} v(te^{i\theta}) \, dt \, \frac{d\nu(\theta)}{r(\theta)} \]
\[ \leq r_1^{-1} \int_{0}^{2\pi} \int_{R_1 r_1}^{R_2 r_2} v(te^{i\theta}) \, dt \, d\nu(\theta) \leq r_1^{-1} \int_{R_1 r_1}^{R_2 r_2} V(t) \, dt. \] \qed

Proof of Theorem 2. By Theorem 5 there exists a domain \( \Omega \) of the form (10) that contains \( \mathbb{D} \) such that
\[ \omega(z, E, \Omega) \leq c_1 \nu(\text{arg} \, E), \quad \forall z \in \mathbb{D}, \; E \subset \partial \Omega, \quad (14) \]
with a constant $c_1 > 0$, see (10). Let $r_1 = \min r(\theta)$. By the Poisson–
Jensen formula applied to the function $v_t(z) = u^+(tz)$ ($t > 0$) in the domain $s\Omega$
($s > 1$) we have, due to (14),
\[
v_t(z) \leq \int_{\partial s\Omega} v_t(\zeta) \omega(z, d\zeta, s\Omega) = \int_{\partial\Omega} v_t(s\zeta) \omega(s^{-1}z, d\zeta, \Omega)
\leq c_1 \int_0^{2\pi} v_t(s \rho(\theta)e^{i\theta}) \, d\nu(\theta), \quad z \in \overline{D}.
\]
The integration of this relation over $s \in [1, R]$ ($R > 1$) gives, by Lemma 5,
\[
(R - 1)v_t(z) \leq c_1 \int_1^R \int_0^{2\pi} v_t(s \rho(\theta)e^{i\theta}) \, d\nu(\theta) \, ds \leq c_2 t^{-1} r_1^{-1} \int_{tr_1}^{tr_2 R} V(s) \, ds
\]
for each $t \geq t_0$. So,
\[
u(t)e^{i\theta}) \leq c(R)V(tr_2R), \quad t \geq t_0,
\]
which proves the theorem. □

Remarks. 1. It is easy to see that the constant $A$ in (11) can be chosen arbitrarily
close to $r_2/r_1 \geq 1$.

2. Note that we have used inequality (3) in the integrated form only, so the
following statement is actually true: If a subharmonic function $u$ on $\mathbb{C}$ satisfies
\[
\int_{t_0}^t \int_0^{2\pi} u^+(se^{i\theta}) \, d\nu(\theta) \, ds \leq W(t) \quad \forall t \geq t_0
\]
with $\nu \in \mathcal{A}[0, 2\pi]$ and a nondecreasing function $W$, then there are constants $c > 0$
and $A \geq 1$, independent of $u$, such that $u(te^{i\theta}) \leq ct^{-1}W(At)$ for all $t \geq t_0$.

Now we prove Theorem 2 as a consequence of Theorem 1.

Proof of Theorem 2. The function $v$ equal to $u^+$ in $\mathbb{C}_+$ and 0 in $\mathbb{C} \setminus \mathbb{C}_+$ is a
subharmonic function in $\mathbb{C}$ satisfying the condition
\[
\int_0^{2\pi} v^+(te^{i\theta}) \, d\nu(\theta) \leq V_1(t)
\]
with $\nu \in \mathcal{A}[0, 2\pi]$ and $V_1(t) = o(t)$, $t \to \infty$. Therefore, it satisfies the conditions of
Theorem 1 with the majorant $V(t) = \sup\{V_1(s) : s \leq t\}$. So, $\sup \rho u^+(te^{i\theta}) = o(t)$ as
$t \to \infty$, and the conclusion holds by the standard Phragmén–Lindelöf theorem. □

4 Proof of Theorem 3

The integral version of the Levinson–Sjöberg theorem will be proved along the same
lines as Theorem 1 however the local situation needs a more refined adaptation.

We start with two elementary statements close to Lemma 5.
Lemma 6 Let a nonnegative integrable function $v$ in the square $Q = \{|x|, |y| < 1\}$ satisfy (5) with a continuous strictly increasing function $\nu$. Then for any $d \in (0, 1)$ there exists a constant $M_1(d)$, independent of $u$, such that for each $y_0 \in (-1, 1)$ one can find a point $y_1 \in (-1, 1) \cap (y_0 - d, y_0 + d)$ with

$$\int_{-1}^{1} v(x + iy_1) \, dx < M_1(d).$$

Proof. Assume $y_0 \geq 0$, then

$$\int_{y_0-d}^{y_0} \int_{-1}^{1} v(x + iy) \, dx \, d\nu(y) = \int_{-1}^{1} \int_{y_0-d}^{y_0} v(x + iy) \, d\nu(y) \, dx \leq 2.$$  

Therefore for some $y_1 \in (y_0 - d, y_0)$,

$$\int_{-1}^{1} v(x + iy_1) \, dx \leq 2[\nu(y_0) - \nu(y_0 - d)]^{-1} \leq 2[\Delta_*(\nu, d)]^{-1}$$

with $\Delta_*(\nu, d) = \inf\{\nu(t) - \nu(t - d) : t \in (0, 1)\} > 0$. □

Lemma 7 Let a function $v$ satisfy the conditions of Lemma 6, a function $r$ be continuous on a segment $[a, b] \subset [-1, 1]$, $0 < r_1 = \min r(y) \leq \max r(y) = r_2 < 1$, and $\delta \in (0, 1 - r_2)$. Then there exists $t \in (0, \delta)$ such that

$$\int_{a}^{b} v(t + r(y) + iy) \, d\nu(y) < M_2(\delta)$$

with $M_2(\delta)$ independent of $v$.

Proof. We have

$$\int_{0}^{\delta} \int_{a}^{b} v(t + r(y) + iy) \, d\nu(y) = \int_{a}^{b} \int_{r(y)}^{\delta + r(y)} v(s + iy) \, ds \, d\nu(y) \leq \int_{r_1}^{\delta + r_2} \int_{a}^{b} v(s + iy) \, d\nu(y) \, ds \leq \delta + r_2 - r_1.$$

Thus one can find some $t \in (0, \delta)$ such that

$$\int_{a}^{b} v(t + r(y) + iy) \, d\nu(y) < \delta^{-1}(\delta + r_2 - r_1).$$

□

Proof of Theorem 3 Consider the measure $\nu_1$ on $[-i, i]$ defined as

$$\nu_1(E) = \nu(-iE), \quad E \subset [-i, i].$$
The conformal map \( f(z) = \exp\{z\pi/2\} \) of the strip \( \{|\text{Im} \, z| < 1\} \) to the right half-plane \( \mathbb{C}_r \) pushes the measure \( \nu_1 \) forward to the measure \( f^* \nu \) on the semicircle \( \{e^{i\theta} : -\pi/2 \leq \theta \leq \pi/2\} \), producing a measure of the class \( \mathcal{A}[-\pi/2, \pi/2] \); we extend it to some measure \( \nu_2 \in \mathcal{A}[-\pi, \pi] \). By Theorem 5, there is a strictly star-shaped domain \( \Omega \supset \mathbb{D} \) such that the radial projection of its harmonic measure at 0 is the normalization \( \nu_2/\nu_2([-\pi, \pi]) \) of \( \nu_2 \).

Let \( \Omega_1 = \Omega \cap \mathbb{C}_r \), then for every Borel set \( E \subset \Gamma = \partial \Omega_1 \cap \mathbb{C}_r \) and any compact set \( K \subset \Omega_1 \),

\[
\omega(w, E, \Omega_1) \leq C_1(K) \nu_2(\arg E) \quad \forall w \in K.
\]

The pre-image \( \Omega_2 = f^{-1}(\Omega_1) \) of \( \Omega_1 \) has the form

\[
\Omega_2 = \{z = x + iy : x < \varphi(y), \; y \in (0, 1)\}
\]

with some function \( \varphi \in C[-1, 1] \). Let

\[
\Gamma_2 = \{x + iy : x = \varphi(y), \; y \in (0, 1)\},
\]

then for every Borel \( E \subset \Gamma_2 \) and any compact subset \( K \) of \( \Omega_2 \),

\[
\omega(z, E, \Omega_2) \leq C_2(K) \nu(\text{Im} \, E) \quad \forall z \in K. \tag{16}
\]

For the domain

\[
\Omega_3 = \{z = x + iy : x > -\varphi(y), \; y \in (0, 1)\}
\]

we have, similarly, the relation

\[
\omega(z, E, \Omega_3) \leq C_3(K) \nu(\text{Im} \, E) \quad \forall z \in K \tag{17}
\]

for each \( E \subset \Gamma_3 = \{x + iy : x = -\varphi(y), \; y \in (0, 1)\} \) and compact set \( K \subset \Omega_3 \).

Let now \( K \) be an arbitrary compact subset of the square \( Q \). We would be almost done if we were able to find some reals \( h_2(K) \) and \( h_3(K) \) such that

\[
K \subset \{\Omega_2 + h_2(K)\} \cap \{\Omega_3 + h_3(K)\} \subset \{\Omega_2 + h_2(K)\} \cap \{\Omega_3 + h_3(K)\} \subset Q.
\]

However this is not the case for any \( K \) unless \( \varphi \equiv \text{const} \). That is why we need partition.

Given \( K \) compactly supported in \( Q \), choose a positive \( \lambda \sim (4 \text{ dist}(K, \partial Q))^{-1} \) and then \( \tau \in (0, \lambda) \) such that the modulus of continuity of \( \varphi \) at \( 4\tau \) is less than \( \lambda \). Take a finite covering of \( K \) by disks \( B_j = \{z : |z - z_j| < \tau\} \), \( z_j \in K \), \( 1 \leq j \leq n \). To prove the theorem, it suffices to estimate the function \( u \) on each \( B_j \).

Let \( Q_j = \{z \in Q : |\text{Im} \, (z - z_j)| < 2\tau\} \), then \( B_j \subset Q_j \) and \( \text{dist}(B_j, \partial Q_j) = \tau \). Take also

\[
\Omega^{(j)}_2 = \Omega_2 \cap Q_j, \quad \Gamma^{(j)}_2 = \Gamma_2 \cap \overline{\Omega^{(j)}_2} = \{x + iy : x = \varphi(y), \; a_j \leq y \leq b_j\}.
\]
Now we can find reals \( h_2^{(j)} \) and \( h_3^{(j)} \) such that
\[
\Gamma_2^{(j)} + h_2^{(j)} = \{ x + iy : x = r_2^{(j)}(y) \} \subset Q_j \cap \{ x + iy : 1 - 4\lambda < x < 1 < 2\lambda \}
\]
and
\[
\Gamma_3^{(j)} + h_3^{(j)} = \{ x + iy : x = r_3^{(j)}(y) \} \subset Q_j \cap \{ x + iy : -1 + 2\lambda < x < -1 + 4\lambda \}.
\]
Furthermore, by Lemma 7, there exist \( t_2^{(j)} \in (0, \lambda) \) and \( t_3^{(j)} \in (-\lambda, 0) \) such that
\[
\int_{a_j}^{b_j} u^+(t_k^{(j)} + r_k^{(j)}(y) + iy) \, d\nu(y) < M_2(\lambda), \quad k = 2, 3. \tag{18}
\]
Finally we can find, due to Lemma 6, \( y_1^{(j)} \in (a_j, a_j + \tau) \) and \( y_2^{(j)} \in (b_j - \tau, b_j) \) such that
\[
\int_{-1}^{1} u^+(x + iy_m) \, dx < M_1(\tau), \quad m = 1, 2. \tag{19}
\]
Denote
\[
\Omega^{(j)} = \{ x + iy : r_3^{(j)}(y) + t_3^{(j)} < x < r_2^{(j)}(y) + t_2^{(j)}, \ y_1^{(j)} \leq y \leq y_2^{(j)} \}.
\]
Since \( B_j \subset \Omega^{(j)} \), relations (16) and (17) imply
\[
\omega(z, E, \Omega^{(j)}) \leq C(B_j) (\nu(\text{Im} E)) \quad \forall z \in B_j \tag{20}
\]
for all \( E \) in the vertical parts of \( \partial \Omega^{(j)} \). For \( E \) in the horizontal parts of \( \partial \Omega^{(j)} \), we have, evidently,
\[
\omega(z, E, \Omega^{(j)}) \leq C(B_j) \text{mes} \, E \quad \forall z \in B_j. \tag{21}
\]
Now we can estimate \( u(z) \) for \( z \in B_j \). By (18)–(21),
\[
u(z) \leq \int_{\partial \Omega^{(j)}} u^+(\zeta) \omega(z, d\zeta, \Omega^{(j)}) \leq C(B_j) \sum_{k=2}^{3} \int_{a_j}^{b_j} u^+(t_k^{(j)} + r_k^{(j)}(y) + iy) \, d\nu(y)
+ C(B_j) \sum_{m=1}^{2} \int_{-1}^{1} u^+(x + iy_m) \, dx \leq 2C(B_j)(M_1(\tau) + M_2(\lambda)),
\]
which completes the proof. \( \square \)
5 Proof of Theorem 4

By Theorem 1 and Proposition 1 it suffices to prove

**Proposition 4** If a function \( u \) satisfies the conditions of Theorem 4, then there exists a function \( f \in L[-\pi, \pi] \) and a constant \( c_1 > 0 \), the both independent of \( u \), such that
\[
\int_{-\pi}^{\pi} u^{+}(re^{i\theta})f(\theta)\,d\theta \leq c_1 V(r) \quad \forall r > r_0.
\] (22)

**Proof.** What we will do is a refinement of the arguments from the proof of the original Matsaev’s theorem (see [15], [12]). Let
\[
D_{r,R,a} = \{ z \in \mathbb{C} : r < |z| < R, \, |\arg z - \pi/2| < \pi(1/2 - a) \}, \quad 0 < a < 1/4,
\]
b = \( (1 - 2a)^{-1} \), \( S(\theta, a) = \sin b(\theta - a\pi) \). Carleman’s formula for the function \( u \) harmonic in \( D_{r,R,a} \) has the form
\[
2bR^{-b} \int_{\pi a}^{\pi - \pi a} u(Re^{i\theta})S(\theta, a)\,d\theta - b(r^{-b} + r^b R^{-2b}) \int_{\pi a}^{\pi - \pi a} u(Re^{i\theta})S(\theta, a)\,d\theta
\]
\[\quad -(r^{-b+1} - r^{b+1} R^{-2b}) \int_{-\pi a}^{\pi a} u'_i(Re^{i\theta})S(\theta, a)\,d\theta
\]
\[\quad + b \int_{r}^{R} \left[ u(xe^{i\pi a}) + u(xe^{i\pi(1-a)}) \right] (x^{-b-1} - x^{b-1} R^{-2b}) \,dx = 0.
\]

It implies the inequality
\[
\int_{\pi a}^{\pi - \pi a} u^+(Re^{i\theta})S(\theta, a)\,d\theta \leq c(r, u)R^{b} + \int_{\pi a}^{\pi - \pi a} u^-(Re^{i\theta})S(\theta, a)\,d\theta
\]
\[\quad + R^{b} \int_{r}^{R} \left[ u^-(xe^{i\pi a}) + u^-(xe^{i\pi(1-a)}) \right] (x^{-b-1} - x^{b-1} R^{-2b}) \,dx.
\] (23)

Fix some \( \tau \in (0, 1/4) \) such that
\[
\beta := (1 - 2\tau)^{-1} < 1 + \delta
\] (24)

with \( \delta \) as in the statement of Theorem 1. Inequality (23) gives us the relation
\[
I_0 := \int_{0}^{\tau} \Phi(\sin \pi a) \int_{\pi a}^{\pi - \pi a} u^+(Re^{i\theta})S(\theta, a)\,d\theta\,da
\]
\[\leq c(r, u) \int_{0}^{\tau} R^{b}\Phi(\sin \pi a)\,da + \int_{\pi a}^{\pi - \pi a} \Phi(\sin \pi a) \int_{\pi a}^{\pi - \pi a} u^-(Re^{i\theta})S(\theta, a)\,d\theta\,da
\]
\[+ \int_{0}^{\tau} \Phi(\sin \pi a) \int_{r}^{R} \left[ u^-(xe^{i\pi a}) + u^-(xe^{i\pi(1-a)}) \right] R^{b}x^{-b-1}\,dx\,da
\]
\[= I_1 + I_2 + I_3.
\] (25)
We can represent $I_0$ as

$$I_0 = \int_0^{\pi} u^+ (Re^{i\theta}) \Psi(\theta) \, d\theta$$

with

$$\Psi(\theta) = \int_{\lambda(\theta)/2}^{\lambda(\theta)} S(\theta, a) \Phi(\sin \pi a) \, da$$  \hspace{1cm} (26)$$

and

$$\lambda(\theta) = \min\{\theta/\pi, 1 - \theta/\pi, \tau\}.$$  \hspace{1cm} (27)$$

Note that $S(\theta, a) \geq 0$ when $a \leq \lambda(\theta)$, and $S'_a(\theta, a) \leq 0$ for all $a < 1/4$. Since $\Phi(t)$ is nondecreasing, this implies the bound

$$\Psi(\theta) \geq \int_{\lambda(\theta)/2}^{\lambda(\theta)} S(\theta, a) \Phi(\sin \pi a) \, da \geq f(\theta) = \lambda^2(\theta) \Phi\left(\sin \frac{\pi \lambda(\theta)}{2}\right)$$

and thus,

$$I_0 \geq \int_0^{\pi} u^+ (Re^{i\theta}) f(\theta) \, d\theta$$  \hspace{1cm} (28)$$

with $f \in \mathcal{L}^-[0, \pi]$.

Let us now estimate the right hand side of (25). We have

$$I_1 \leq c(r, u) R^3 \int_0^\tau \Phi(\sin \pi a) \, da \leq c_1(r, \tau, u) R^3;$$  \hspace{1cm} (29)$$

$$I_2 = \int_0^{\pi} u^-(Re^{i\theta}) \Psi(\theta) \, d\theta \leq \int_0^{\pi} u^- (Re^{i\theta}) \Phi(\sin \theta) \, d\theta;$$  \hspace{1cm} (30)$$

$$I_3 \leq \int_0^\tau \int_r^R \Phi(\sin \pi a) \left[u^-(xe^{i\pi\alpha}) + u^-(xe^{i\pi(1+a)})\right] \left(\frac{R}{x}\right)^\beta x^{-1} \, dx \, da
\leq R^\beta \int_r^R x^{-\beta-1} \left[\int_0^{\pi} + \int_{\pi(1-\tau)}^{\pi}\right] u^-(xe^{i\theta}) \Phi(\sin \theta) \, d\theta \, dx
\leq R^\beta \int_r^R x^{-\beta-1} \int_0^{\pi} u^- (xe^{i\theta}) \Phi(\sin \theta) \, d\theta \, dx.$$  \hspace{1cm} (31)$$

We insert (28)–(31) into (25):

$$\int_0^{\pi} u^+ (Re^{i\theta}) f(\theta) \, d\theta \leq c_1(r, \tau, u) R^3 + \int_0^{\pi} u^- (Re^{i\theta}) \Phi(\sin \theta) \, d\theta$$

$$+ R^\beta \int_r^R x^{-\beta-1} \int_0^{\pi} u^- (xe^{i\theta}) \Phi(\sin \theta) \, d\theta \, dx
= J_1(R) + J_2(R) + J_3(R).$$  \hspace{1cm} (32)$$
By the choice of $\beta$ (24), $J_1(R) = o(V(R))$ as $R \to \infty$. Condition (7) implies $J_2(R) \leq V(R)$, $R > r_0$. As to the term $J_3$, take any $\epsilon \in (0, 1 + \delta - \beta)$, then

$$J_3(R) \leq R^\beta \int_r^R x^{-\beta - 1} V(x) \, dx = R^\beta \int_r^R x^{-\beta - \epsilon} V(x) x^{\epsilon - 1} \, dx \leq R^\beta R^{-\beta - \epsilon} V(R) \int_r^R x^{\epsilon - 1} \, dx \leq \epsilon^{-1} V(R).$$

These bounds give us

$$\int_0^\pi u^+(Re^{i\theta}) f(\theta) \, d\theta \leq c_2 V(R) \quad \forall R > r_1(u).$$

Absolutely the same way, we get a similar inequality in the lower half-plane and, as a result, relation (22).

Remark. We do not know if condition (7) can be replaced by a more general one in terms of the class $\mathcal{A}$.

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