Total weight choosability of $d$-degenerate graphs

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October 6, 2015

Abstract

A graph $G$ is $(k, k')$-choosable if the following holds: For any list assignment $L$ which assigns to each vertex $v$ a set $L(v)$ of $k$ real numbers, and assigns to each edge $e$ a set $L(e)$ of $k'$ real numbers, there is a total weighting $\phi : V(G) \cup E(G) \rightarrow R$ such that $\phi(z) \in L(z)$ for $z \in V \cup E$, and $\sum_{e \in E(u)} \phi(e) + \phi(u) \neq \sum_{e \in E(v)} \phi(e) + \phi(v)$ for every edge $uv$. This paper proves the following results: (1) If $G$ is a connected $d$-degenerate graph, and $k > d$ is a prime number, and $G$ is either non-bipartite or has two non-adjacent vertices $u, v$ with $d(u) + d(v) < k$, then $G$ is $(1, k)$-choosable. As a consequence, every planar graph with no isolated edges is $(1, 7)$-choosable, and every connected 2-degenerate non-bipartite graph other than $K_2$ is $(1, 3)$-choosable. (2) If $d + 1$ is a prime number, $v_1, v_2, \ldots, v_n$ is an ordering of the vertices of $G$ such that each vertex $v_i$ has back degree $d^{-}(v_i) \leq d$, then there is a graph $G'$ obtained from $G$ by adding at most $d - d^{-}(v_i)$ leaf neighbours to $v_i$ (for each $i$) and $G'$ is $(1, 2)$-choosable. (3) If $G$ is $d$-degenerate and $d + 1$ a prime, then $G$ is $(d, 2)$-choosable. In particular, 2-degenerate graphs are $(2, 2)$-choosable. (4) Every graph is $(\lceil \frac{\text{mad}(G)}{2} \rceil + 1, 2)$-choosable. In particular, planar graphs are $(4, 2)$-choosable, planar bipartite graphs are $(3, 2)$-choosable.

Key words: Total weighting; $(k, k')$-choosable graphs; permanent; $d$-degenerate graphs.

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1 Introduction

A total weighting of a graph $G$ is a mapping $\phi : V(G) \cup E(G) \to R$. A total weighting $\phi$ is proper if for any edge $uv$ of $G$,

$$\sum_{e \in E(u)} \phi(e) + \phi(u) \neq \sum_{e \in E(v)} \phi(e) + \phi(v),$$

where $E(v)$ is the set of edges incident to $v$. Total weighting of graphs has attracted considerable recent attention [9, 11, 16, 6, 11, 12, 13, 19, 21].

The well-known 1-2-3 conjecture, proposed by Karoński, Lučak and Thomason [9], asserts that every graph with no isolated edge has a proper total weighting $\phi$ with $\phi(v) = 0$ for every vertex and $\phi(e) \in \{1, 2, 3\}$ for every edge $e$. The conjecture has been studied by many authors [11, 2, 16] and the current best result is that the conjecture would be true if instead of $\{1, 2, 3\}$, every edge $e$ can have weight $\phi(e) \in \{1, 2, 3, 4, 5\}$ [11]. The 1-2 conjecture, proposed by Przybyło and Woźniak in [12], asserts that every graph $G$ has a proper total weighting $\phi$ with $\phi(z) \in \{1, 2\}$ for all $z \in V(G) \cup E(G)$. The best result on this conjecture is that every graph $G$ has a proper total weighting $\phi$ with $\phi(v) \in \{1, 2\}$ for $v \in V(G)$ and $\phi(e) \in \{1, 2, 3\}$ for $e \in E(G)$ [10].

Total weighting of graphs is naturally extended to the list version, independently by Przybyło and Woźniak [13] and by Wong and Zhu [19]. Suppose $\psi : V(G) \cup E(G) \to \{1, 2, \ldots, \}$ is a mapping which assigns to each vertex and each edge of $G$ a positive integer. A $\psi$-list assignment of $G$ is a mapping $L$ which assigns to $z \in V(G) \cup E(G)$ a set $L(z)$ of $\psi(z)$ real numbers. Given a total list assignment $L$, a proper $L$-total weighting is a proper total weighting $\phi$ with $\phi(z) \in L(z)$ for all $z \in V(G) \cup E(G)$. We say $G$ is total weight $\psi$-choosable if for any $\psi$-list assignment $L$, there is a proper $L$-total weighting of $G$. We say $G$ is $(k, k')$-choosable if $G$ is $\psi$-total weight choosable, where $\psi(v) = k$ for $v \in V(G)$ and $\psi(e) = k'$ for $e \in E(G)$.

As strengthenings of the 1-2-3 conjecture and the 1-2 conjecture, it was conjectured in [19] that every graph with no isolated edges is $(1,3)$-choosable and every graph is $(2,2)$-choosable. Some special graphs are shown to be $(1,3)$-choosable, such as complete graphs, complete bipartite graphs, trees [2], Cartesian product of an even number of even cycles, of a path and an even cycle, of two paths [17]. Some special graphs are shown to be $(2,2)$-choosable, such as complete graphs, generalized theta graphs, trees [19], subcubic graphs, Halin graphs [20], complete bipartite graphs [18].

It was shown in [21] that every graph is $(2,3)$-choosable. However, it is unknown whether there is a constant $k$ such that every graph with no isolated edge is $(1,k)$-choosable, and whether there is a constant $k$ such that every graph is $(k,2)$-choosable.

For graphs $G$ of maximum degree $k$ with no isolated edges, it was proved by Seamone [14] that $G$ is $(1,2k+1)$-choosable, by Wang and Yan [15] that $G$ is $(1,\lceil \frac{2k+1}{3} \rceil)$-choosable, and recently, it is proved in [8] that $G$ is $(1,k+1)$-choosable. In this paper, we first
consider connected $d$-degenerate graphs $G$. We prove that if $k > d \geq 2$ and either $G$ is non-bipartite or $G$ is bipartite and there are two non-adjacent vertices $u, v$ with $d(u) + d(v) < k$, then $G$ is $(1, k)$-choosable. As a consequence, every planar graph with no isolated edges is $(1, 7)$-choosable, and every connected 2-degenerate non-bipartite graph other than $K_2$ is $(1, 3)$-choosable. Next we prove that if $d + 1$ is a prime number and $G$ is a $d$-degenerate graph, $v_1, v_2, \ldots, v_n$ is an ordering of the vertices of $G$ such that each vertex $v_i$ has back degree $d - d^-(v_i) \leq d$, then there is a graph $G'$ obtained from $G$ by adding at most $d - d^-(v_i)$ leaf neighbours to $v_i$ (for each $i$) and $G'$ is $(1, 2)$-choosable. In particular, if $d + 1$ is a prime clique $K$ of $G$, there is a graph $G'$ obtained from $G$ by adding at most $j$ leaf neighbours to the $j$th vertex of $K$ so that the resulting graph is $(1, 2)$-choosable.

For $(k, 2)$-choosability, we prove that if $G$ is $d$-degenerate and $d + 1$ a prime, then $G$ is $(d, 2)$-choosable. In particular, 2-degenerate graphs are $(2, 2)$-choosable. In the last section, we prove that every graph is $(\lceil \text{mad}(G)/2 \rceil + 1, 2)$-choosable. In particular, planar graphs are $(4, 2)$-choosable, planar bipartite graphs are $(3, 2)$-choosable.

2 \hspace{1em} (1, k)$-choosability

This section proves the following result.

**Theorem 1** Assume $G$ is a connected $d$-degenerate graph, $k > d \geq 2$ is a prime number and one of the following holds:

- $G$ is non-bipartite.
- $G$ is bipartite, and there are two non-adjacent vertices $u, v$ with $d(u) + d(v) < k$.

Then $G$ is $(1, k)$-choosable.

For each $z \in V(G) \cup E(G)$, let $x_z$ be a variable associated to $z$. Fix an arbitrary orientation $D$ of $G$. Consider the polynomial

$$P_G(\{x_z : z \in V(G) \cup E(G)\}) = \prod_{uv \in E(D)} \left( \left( \sum_{e \in E(u)} x_e + x_u \right) - \left( \sum_{e \in E(v)} x_e + x_v \right) \right).$$

Assign a real number $\phi(z)$ to the variable $x_z$, and view $\phi(z)$ as the weight of $z$. Let $P_G(\phi)$ be the evaluation of the polynomial at $x_z = \phi(z)$. Then $\phi$ is a proper total weighting of $G$ if and only if $P_G(\phi) \neq 0$. The question is under what condition one can find an assignment $\phi$ for which $P_G(\phi) \neq 0$.

An **index function** of $G$ is a mapping $\eta$ which assigns to each vertex or edge $z$ of $G$ a non-negative integer $\eta(z)$. An index function $\eta$ of $G$ is **valid** if $\sum_{z \in V \cup E} \eta(z) = |E|$. Note that
$|E|$ is the degree of the polynomial $P_G(\{x_z : z \in V(G) \cup E(G)\})$. For a valid index function $\eta$, let $c_\eta$ be the coefficient of the monomial $\prod_{z \in V \cup E} x_z^{\eta(z)}$ in the expansion of $P_G$. It follows from the Combinatorial Nullstellensatz \cite{3, 5} that if $c_\eta \neq 0$, and $L$ is a list assignment which assigns to each $z \in V(G) \cup E(G)$ a set $L(z)$ of $\eta(z) + 1$ real numbers, then there exists a mapping $\phi$ with $\phi(z) \in L(z)$ such that

$$P_G(\phi) \neq 0.$$ 

An index function $\eta$ of $G$ is called non-singular if there is a valid index function $\eta' \leq \eta$ (i.e., $\eta'(z) \leq \eta(z)$ for all $z \in V(G) \cup E(G)$) such that $c_{\eta'} \neq 0$.

The main result of this section, Theorem \[1\] follows from Theorem \[2\]

**Theorem 2** Assume $G$ is a connected $d$-degenerate graph, $k > d \geq 2$ is a prime number and one of the following holds:

- $G$ is non-bipartite.
- $G$ is bipartite, and there are two non-adjacent vertices $u, v$ with $d(u) + d(v) < k$.

Then $G$ has a non-singular index function $\eta$ with $\eta(v) = 0$ for $v \in V(G)$ and $\eta(e) \leq k - 1$ for $e \in E(G)$.

We write the polynomial $P_G(\{x_z : z \in V(G) \cup E(G)\})$ as

$$P_G(\{x_z : z \in V(G) \cup E(G)\}) = \prod_{e \in E(D)} \sum_{z \in V(G) \cup E(G)} A_G[e, z] x_z.$$ 

It is straightforward to verify that for $e \in E(G)$ and $z \in V(G) \cup E(G)$, if $e = (u, v)$ (oriented from $u$ to $v$), then

$$A_G[e, z] = \begin{cases} 
1 & \text{if } z = v, \text{ or } z \neq e \text{ is an edge incident to } v, \\
-1 & \text{if } z = u, \text{ or } z \neq e \text{ is an edge incident to } u, \\
0 & \text{otherwise}.
\end{cases}$$ 

Now $A_G$ is a matrix, whose rows are indexed by edges of $G$ and the columns are indexed by edges and vertices of $G$. Given a vertex or an edge $z$ of $G$, let $A_G(z)$ be the column of $A_G$ indexed by $z$. As observed in \[10\], for an edge $e = uv$ of $G$, we have

$$A_G(e) = A_G(u) + A_G(v). \quad (1)$$

For an index function $\eta$ of $G$, let $A_G(\eta)$ be the matrix, each of its column is a column of $A_G$, and each column $A_G(z)$ of $A_G$ occurs $\eta(z)$ times as a column of $A_G(\eta)$. For $e \in E(G)$ and $z \in E(G) \cup V(G)$ with $\eta(z) \geq 1$, $A_G[e, z]$ denote the entry of $A_G(\eta)$ at row $e$ and column...
z, and \( A_G[\eta, z] \) denotes the matrix obtained from \( A_G(\eta) \) by deleting the row indexed by \( e \) and a column indexed by \( z \).

It is known \(^{14}\) and easy to verify that for a valid index function \( \eta \) of \( G \), \( c_\eta \neq 0 \) if and only if per\((A_G(\eta))\) \( \neq 0 \) (here per\((A_G(\eta))\) denotes the permanent of \( A_G(\eta) \)). Thus a valid index function \( \eta \) of \( G \) is non-singular if and only if per\((A_G(\eta))\) \( \neq 0 \).

It is well-known (and follows easily from the definition) that the permanent of a matrix is multi-linear on its column vectors (as well as its row vectors): If a column \( C \) of \( A \) is a linear combination of two columns vectors \( C = \alpha C' + \beta C'' \), and \( A' \) (respectively, \( A'' \)) is obtained from \( A \) by replacing the column \( C \) with \( C' \) (respectively, with \( C'' \)), then

\[
\text{per}(A) = \alpha \text{per}(A') + \beta \text{per}(A'').
\]

(2)

Assume \( A \) is a square matrix whose columns are linear combinations of columns of \( A_G \). Define an index function \( \eta_A : V(G) \cup E(G) \to \{0, 1, \ldots, 1\} \) as follows:

For \( z \in V(G) \cup E(G) \), \( \eta_A(z) \) is the number of columns of \( A \) in which \( A_G(z) \) appears in the linear combinations with nonzero coefficient.

Note that the columns of \( A_G \) are not linearly independent. There are different ways of expressing the columns of a same matrix \( A \) as linear combination of columns of \( A_G \). So \( \eta_A \) is not uniquely determined by the matrix \( A \) itself, instead it depends on how its columns are expressed as linear combinations of columns of \( A_G \). For simplicity, we use the notation \( \eta_A \), and each time the function \( \eta_A \) is used, it refers to an explicit expression of the columns of \( A \) as linear combinations of columns of \( A_G \). In particular, for an index function \( \eta \) of \( G \), we may write a column of \( A_G(\eta) \) as a linear combination of other columns of \( A_G \), and \( \eta_A(\eta) \) may become another index function of \( G \).

To prove that a graph is \((1, k)\)-choosable, it suffices to find a square matrix \( A \) with \( \text{per}(A) \neq 0 \) whose columns are linear combinations of columns of \( A_G \) such that for each \( v \in V(G) \), \( \eta_A(v) = 0 \), and for each edge \( e \) of \( G \), \( \eta_A(e) \leq k - 1 \).

**Lemma 1** Assume \( G \) is a connected \( d \)-degenerate graph, \( k > d \geq 2 \) is a prime number and one of the following holds:

- \( G \) is non-bipartite.
- \( G \) is bipartite and there are two non-adjacent vertices \( u, v \) with \( d(u) + d(v) < k \).

Then there is a matrix \( A \) whose columns are integral linear combinations (i.e., linear combination with integer coefficients) of edge columns of \( G \) such that \( \text{per}(A) \neq 0 \) (mod \( k \)).

Before proving Lemma 1, we first show that Theorem 2 follows from Lemma 1. Assume there is a matrix \( A \) whose columns are linear combinations of edge columns of \( G \) such that
per(A) \neq 0 \pmod k. By repeatedly using (2), we know that there is a matrix \(A'\) whose columns are edge columns of \(G\) and per(\(A') \neq 0 \pmod k\). If each edge column occurs at most \(k - 1\) times in \(A'\), then we are done. If there is an edge column which appears \(k'\) times for some \(k' \geq k\), then per(\(A')\) is a multiple of \(k'!\), and hence per(\(A') = 0 \pmod k\), contrary to our choice of \(A'\). This proves that Theorem 2 follows from Lemma 1.

**Proof of Lemma** First we consider the case that \(G\) is non-bipartite. Since \(G\) is a \(d\)-degenerate graph, there is an ordering \(v_1, v_2, \ldots, v_n\) of the vertices such that for each \(i\), vertex \(v_i\) has \(d^-(v_i) \leq d\) neighbours \(v_j\) with \(j < i\). Let \(A\) be the square matrix which consists of \(d^-(v_i)\) copies of \(2A_G(v_i)\). It can be proved easily by induction on \(n\) that \(|\per(A)| = 2^n \prod_{i=1}^n d^-(v_i)!\). As \(G\) is non-bipartite, we know that \(d \geq 2\) and hence \(k > 2\). Also by our hypothesis, \(d^-(v_i) \leq d < k\) for each \(i\). Hence per(\(A) \neq 0 \pmod k\).

It suffices to show that each column of \(A\) is an integral linear combination of edge columns of \(G\). In other words, for each vertex \(v\) of \(G\), \(2A_G(v)\) can be written as an integral linear combination of edge columns of \(G\).

By assumption \(G\) is connected and has an odd cycle \((u_0, e_0, u_1, e_1, \ldots, u_{2q}, e_{2q}, u_0)\). If \(v\) is on the cycle, say \(v = u_0\), then \(2A_G(u_0) = A_G(e_0) - A_G(e_1) + A_G(e_2) - \ldots + A_G(e_{2q})\). If \(v\) is not on the odd cycle, then let \((v_0, e'_0, v_1, e'_1, \ldots, e'_{n-1}, w_0)\) be a path connecting \(v\) to \(u_0\), say \(v_0 = v\) and \(w_0 = u_0\). Then \(2A_G(v_0) = 2A_G(e'_0) - 2A_G(e'_1) + 2A_G(e'_2) - \ldots + \pm 2A_G(e'_{n-1})\mp 2A_G(w_0)\), and then write \(2A_G(w_0)\) as an integral linear combination of edge columns of \(G\), we are done. This prove the non-bipartite case of Lemma 1.

Assume \(G\) is bipartite, and \(u, v\) are the two specified vertices, and \(d' = d(u) + d(v)\). Similarly as above, there is an ordering \(v_1, v_2, \ldots, v_{n-2}\) of the vertices of \(G - \{u, v\}\) such that for each \(i\), vertex \(v_i\) has \(d^-(v_i) \leq d\) neighbours \(v_j\) with \(j < i\). Let \(u = v_{n-1}, v = v_n\). Let \(A\) be the matrix which consists of \(d^-(v_i)\) copies of \(A_G(v_i) \pm A_G(v)\) for \(i = 1, 2, \ldots, n-2\) and \(d'\) copies of \(A_G(u) \pm A_G(v)\), where the \(\pm\) is determined by the distance between the two involved vertices: if the distance is odd, then choose +, and otherwise choose −. It is easy to verify that \(|\per(A)| = (\prod_{i=1}^{n-2} d^-(v_i)!d')!\). Hence per(\(A) \neq 0 \pmod k\).

It suffices to show that each column of \(A\) can be written as an integral linear combination of edge columns of \(G\). This is so, because if \(x, y\) are two vertices connected by a path of odd length \((u_0, e_0, u_1, e_1, \ldots, u_{2q}, e_{2q}, u_{2q+1})\), say \(x = u_0, y = u_{2q+1}\), then \(A_G(x) + A_G(y) = A_G(e_0) - A_G(e_1) + \ldots + A_G(e_{2q})\). If \(x, y\) are two vertices connected by a path of even length \((u_0, e_0, u_1, e_1, \ldots, u_{2q-1}, e_{2q-1}, u_{2q})\), say \(x = u_0, y = u_{2q}\), then \(A_G(x) - A_G(y) = A_G(e_0) - A_G(e_1) + \ldots - A_G(e_{2q-1})\). This completes the proof of Lemma 1.

**Corollary 1** If \(G\) is \(d\)-degenerate, non-bipartite graph, then \(G\) is \((1, 2d - 3)\)-choosable.

**Proof.** Using the Bertrand Theorem that for \(d > 3\), there is a prime \(p\) such that \(d < p < 2d - 2\).

**Corollary 2** If \(G \neq K_2\) is a tree, or a 2-tree, then \(G\) is \((1, 3)\)-choosable. If \(G\) is a 3-tree, then \(G\) is \((1, 5)\)-choosable. If \(d \geq 4\) and \(G\) is a \(d\)-tree, then \(G\) is \((1, 2d - 3)\)-choosable.
Proof. All these follow easily from Theorem 1 and Corollary 1.

The result that trees are (1, 3)-choosable was proved in [6], however, the proof is different from the one presented here.

Corollary 3 Every planar graph with no isolated edges is (1, 7)-choosable.

Proof. We may assume $G$ is connected, for otherwise, we consider components of $G$ separately. It is well-known that every planar graph is 5-degenerate. If $G$ is non-bipartite, then we are done by Theorem 1. If $G$ is bipartite, then $G$ is triangle free. By Euler formula $G$ has minimum degree $\delta(G) \leq 3$. If $\delta(G) = 3$, then it follows from Euler formula that $G$ has at least 8 vertices of degree 3, and hence there are non-adjacent vertices $u$ and $v$ with $d(u) + d(v) < 7$. In case $\delta(G) = 1$ or 2, it is also easy to see that there are two non-adjacent vertices $u, v$ with $d(u) + d(v) < 7$. So the conclusion again follows from Theorem 1.

3 Almost (1, 2)-choosability

In this section, we prove the following result.

Theorem 3 Assume $d + 1$ is a prime number and $G$ is a $d$-degenerate graph. Let $v_1, v_2, \ldots, v_n$ be an ordering of the vertices of $G$ such that each vertex $v_i$ has $d^{-}(v_i) \leq d$ backward neighbours. Then there is a (1, 2)-choosable graph $G'$ obtained from $G$ by adding at most $d - d^{-}(v_i)$ leaf neighbours to $v_i$ (i.e., neighbours of degree 1).

Prior to this paper, all the known (1, 2)-choosable graphs are bipartite graphs. As a consequence of this lemma, every graph $G$ is a subgraph of a (1, 2)-choosable graph $G'$.

Before proving Theorem 3, we shall first prove that if $G$ is $d$-degenerate and each vertex of $G$ has backdegree “almost” $d$, then $G$ is “almost” (1, 2)-choosable.

Lemma 2 Assume $G$ is a graph and $\eta$ is a non-singular index function of $G$, and $E'$ is a subset of edges of $G$. If $\eta(e) = 0$ for every $e \in E'$, then $\eta$ is a non-singular index function of $G - E'$.

Proof. Let $G' = G - E'$. As $\eta(e) = 0$ for every $e \in E'$, $A_{G'}(\eta)$ is the matrix obtained from $A_G(\eta)$ by deleting the rows indexed by edges $e \in E'$. Since $\text{per}(A_G(\eta)) \neq 0$, one can delete some columns from $A_G(\eta)$ to obtain a square matrix with nonzero permanent. I.e., there is a valid index function $\eta'$ of $G'$ such that $\eta' \leq \eta$, and $\text{per}(A_{G'}(\eta')) \neq 0$. Thus $\eta$ is a non-singular index function of $G'$.
Theorem 4 Assume \( d+1 \) is a prime number, \( G \) is a \( d \)-degenerate graph, and \( v_1, v_2, \ldots, v_n \) is an ordering of the vertices such that for each \( i \), vertex \( v_i \) has \( d^-(v_i) \leq d \) backward neighbours. Let \( G' \) be obtained from \( G \) by adding \( d - d^-(v_i) \) leaf neighbours to \( v_i \) for \( i = 1, 2, \ldots, n \). Let \( \eta \) be the index function of \( G' \) defined as \( \eta(v_i) = d - d^-(v_i) \) and \( \eta(e) = 1 \) for each edge \( e \) of \( G \), and \( \eta(z) = 0 \) for each added vertex and edge. Then \( \text{per}(A_{G'}(\eta)) \neq 0 \) \((\text{mod } d+1)\).

Proof. Let \( M_0 = A_{G'}(\eta) \). For \( i = 1, 2, \ldots, n \), let \( M_i \) be obtained from \( M_{i-1} \) as follows: For each edge \( e = v_iv_j \in E(G) \) with \( j < i \), replace the edge column \( A_{G'}(e) \) with \( A_{G'}(v_i) \).

Claim 1 For any \( j \leq i \), \( M_i \) contains exactly \( d \) copies of the column \( A_{G'}(v_j) \) and \( \text{per}(M_i) = \text{per}(M_{i-1}) \) \((\text{mod } d+1)\).

First we prove that \( M_i \) contains exactly \( d \) copies of the column \( A_{G'}(v_j) \) for any \( j \leq i \). This is certainly true for \( i = 1 \), because \( M_1 = M_0 \) and \( \eta(v_1) = d \). Assume this is true for \( M_{i-1} \). By the rule above, \( d^-(v_i) \) copies of \( A_{G'}(v_i) \) are used to replace \( d^-(v_i) \) edge columns.

Now we prove \( \text{per}(M_i) = \text{per}(M_{i-1}) \) \((\text{mod } d+1)\). For each edge \( e = v_iv_j \) with \( j < i \), we write the column \( A_{G'}(e) \) in \( M_{i-1} \) as \( A_{G'}(v_i) + A_{G'}(v_j) \). Apply (2) to expand \( \text{per}(M_{i-1}) \) as the sum of a family of permanents. Then \( \text{per}(M_i) \) is one of the permanents. For each of the other permanents \( M' \), there is an index \( j < i \) such that \( M' \) contains at least one more column of \( A_{G'}(v_j) \) than \( M_{i-1} \), and hence contains at least \( d + 1 \) copies of the column \( A_{G'}(v_j) \). Therefore \( \text{per}(M') = 0 \) \((\text{mod } d+1)\). Therefore \( \text{per}(M_i) = \text{per}(M_{i-1}) \) \((\text{mod } d+1)\). This completes the proof of the claim.

Let \( \eta' \) be the index function defined as \( \eta'(v) = d \) for \( v \in V(G) \), \( \eta'(v) = 0 \) for \( v \in V(G') \setminus V(G) \) and \( \eta'(e) = 0 \) for each edge \( e \) of \( G' \). By Claim \( \blacksquare \) \( M_n = A_{G'}(\eta') \). As each vertex \( v \in V(G) \) has back degree exactly \( d \), we conclude that \( |\text{per}(M_n)| = (dl)^n \neq 0 \) \((\text{mod } d+1)\). Therefore \( \text{per}(A_{G'}(\eta)) \neq 0 \) \((\text{mod } d+1)\). So \( \eta \) is a non-singular index function of \( G' \). By Lemma \( \blacksquare \) \( \eta \) is a non-singular index function of \( G \).

If \( d+1 \) is prime, \( G \) is \( d \)-degenerate and almost every vertex has back degree exactly \( d \), then \( G \) is “almost” \((1, 2)\)-choosable. For example, we have the following corollary.

Corollary 4 If \( d+1 \) is a prime number and \( G \) is a \( d \)-tree, then \( G \) is almost \((1, 2)\)-choosable, except that the first \( d \) vertices require lists of sizes \( d+1, d, \ldots, 2 \), respectively. In particular, if \( G \) is a tree, and \( v \) is an arbitrary vertex of \( G \), then \( G \) is \((1, 2)\)-choosable, except that \( v \) needs a list of size 2. If \( G \) is \( 2 \)-degenerate, and every vertex except the first 2 vertices have back-degree exactly 2, then \( G \) is almost \((1, 2)\)-choosable, except that for the first two vertices \( v_1, v_2 \) need a list of size 3.
Now we are ready to prove Theorem 3. For a graph $G$, let $B_G = A_G(\eta)$, where $\eta(e) = 1$ for each edge $e$, and $\eta(v) = 0$ for each vertex $v$.

**Lemma 3** Assume $\eta$ is an index function of a graph $G$ and $X$ is a set of leaves of $G$ for which the following hold:

1. For each edge $e$, $\eta(e) = 0$ if $e$ is incident to a vertex in $X$ and $\eta(e) = 1$ otherwise.
2. For each vertex $v$, $\eta(v) = |N_G(v) \cap X|$.

If $\per(A_G(\eta)) \neq 0$, then there is a subset $Y$ of $X$ such that $G - Y$ is $(1,2)$-choosable.

**Proof.** Assume the lemma is not true and $G$ is a minimum counterexample.

For each vertex $v$ of $G$, let $N_G(v) \cap X = \{v'_j : 1 \leq j \leq \eta(v)\}$ and let $e_{v,j} = vv'_j$. Take the matrix $B_G$, and for each edge $e_{v,j}$, write $A_G(e_{v,j})$ as the sum $A_G(v) + A_G(v'_j)$. By repeatedly using (2), $\per(B_G)$ can be written as the summation of the permanents of many matrices. To be precise, $\per(B_G) = \sum_{\eta' \in \Gamma} A_G(\eta')$, where $\Gamma$ consists of all the index functions $\eta'$ such that

1. $\eta'(e) = \eta(e)$ for each edge $e$.
2. For $v'_j \in X$, $\eta'(v'_j) = 0$ or $1$.
3. For each vertex $v$, $\eta'(v) = \eta(v) - |\{v'_j : \eta'(v'_j) = 1\}|$.

Observe that $\eta \in \Gamma$.

**Claim 2** If $\eta' \in \Gamma$ and $\eta' \neq \eta$, then $\per(A_G(\eta')) = 0$.

**Proof.** Assume to the contrary that there exists $\eta' \in \Gamma$, $\eta' \neq \eta$, $\per(A_G(\eta')) \neq 0$.

Let $Z = \{v'_j \in X : \eta'(v'_j) = 1\}$. As $\eta' \neq \eta$, $Z \neq \emptyset$. The column $A_G(v'_j)$ has only one entry equals 1, namely the entry at the row indexed by $e'_{v,j}$, and all the other entries are 0. Therefore, $\per(A_{G-Z}(\eta')) = \per(A_G(\eta'))$, where in $\per(A_{G-Z}(\eta'))$, $\eta'$ denotes its restriction to $G - Z$. As $\per(A_{G-Z}(\eta')) \neq 0$, $G' = G - Z$ together with $\eta'$ and $X' = X - Z$ satisfy the condition of Lemma 3. By the minimality of $G$, there is a subset $Y'$ of $X'$, such that $G' - Y'$ is $(1,2)$-choosable. Let $Y = Y' \cup Z$, we have $G - Y$ is $(1,2)$-choosable, a contradiction. This completes the proof of Claim 2.

Now Claim 2 implies that $\per(B_G) = \per(A_G(\eta)) = 0$, and hence $G$ itself is $(1,2)$-choosable, a contradiction.

Theorem 3 follows from Theorem 4 and Lemma 3.
Corollary 5 If $d + 1$ is a prime number, $G$ is $d$-tree, and $K$ is a $d$-clique in $G$, then there is a $(1, 2)$-choosable graph which is obtained from $G$ by adding $k_1, k_2, \ldots, k_d$ leaf neighbours to the $d$ vertices of $K$ respectively, for some $k_j \leq j$.

Proof. The vertices of $G$ can be ordered as $v_1, v_2, \ldots, v_n$ so that $K = \{v_1, v_2, \ldots, v_d\}$ and $v_j$ has $j - 1$ backward neighbours for $j \leq d$, and each other vertex has $d$ backward neighbours. The conclusion then follows from Theorem 3.

The $d = 1$ case of Corollary 5 was proved in [7], where it is shown that trees with an even number of edges are $(1, 2)$-choosable.

4 (k, 2)-choosability

By applying Theorem 4, we prove in this section that when $d + 1$ is a prime, then $d$-degenerate graphs are $(d, 2)$-choosable. In particular, 2-degenerate graphs are $(2, 2)$-choosable.

Theorem 5 Assume $d + 1$ is a prime number, $G$ is a $d$-degenerate graph. Then $G$ is $(d, 2)$-choosable.

Proof. Assume Theorem 5 is not true, and $G$ is a connected $d$-degenerate graph which is not $(d, 2)$-choosable. Let $v_1, v_2, \ldots, v_n$ be an ordering of the vertices of $G$ such that $1 \leq d^-(v_i) \leq d$ for $2 \leq i \leq n$ (note that $d^-(v_1) = 0$). Let $G'$ be obtained from $G$ by adding $d - d^-(v_i)$ leaf neighbours to $v_i$ for $i = 1, 2, \ldots, n$.

Let $\eta$ be the index function of $G'$ such that $\eta(e) = 1$ for every edge $e$ of $G$, $\eta(v_i) = d - d^-(v_i)$ for every vertex of $G$, and $\eta(z) = 0$ for all the added vertices and edges $z$. Since $1 \leq d^-(v_i) \leq d$ for $i > 1$, hence $\eta(v_i) \leq d - 1$ for every vertex of $G$ except that $\eta(v_1) = d$.

Since $|E(G')| = |E(G)| + \sum_{i=1}^{n}(d - d^-(v_i))$, $A_{G'}(\eta)$ is a square matrix. By Theorem 4 we have $\text{per}(A_{G'}(\eta)) \neq 0 \pmod{d + 1}$.

It follows from Lemma 2 that there is a non-singular index function $\eta'$ of $G$ with $\eta'(z) \leq \eta(z)$ for $z \in V(G) \cup E(G)$. In the following, we shall further prove that there is such an index function $\eta'$ for which $\eta'(v_1)$ is strictly less than $\eta(v_1)$. Hence $\eta'(z) \leq d - 1$ for all $z \in V(G)$ and $\eta'(z) \leq 1$ for all $z \in E(G)$ and hence $G$ is $(d, 2)$-choosable, which is in contrary to our assumption.

We define a comb-plus subgraph of $G'$ as a subgraph indicated in Figure 1, where $(w_1, w_2, \ldots, w_p)$ is a path in $G$, $w_p$ adjacent to $w_s$ for some $1 \leq s \leq p - 2$, and $e'_j = w_jw_j \in E(G') - E(G)$ for $j = 1, 2, \ldots, p$.

Claim 3 There is a comb-plus subgraph of $G'$ as in Figure 7 for which the following hold:
Figure 1: The comb-plus subgraph

- \( \eta(w_1) = d \) and \( \eta(w_j) = d - 1 \) for \( 2 \leq j \leq p \) and \( \eta(e_j) = 1 \) for \( 1 \leq j \leq p \), where \( e_j = w_jw_{j+1} \) for \( 1 \leq j \leq p - 1 \) and \( e_p = w_pw_s \).

- For \( 0 \leq i \leq p \), \( \text{per}(A_{H_0}(\eta_i)) \neq 0 \) (mod \( d + 1 \)), where \( H_i = G' - \{ e'_1, e'_2, \ldots, e'_i \} \), and \( \eta_i = \eta \), except that \( \eta_i(e_j) = 0 \) for \( 1 \leq j \leq i \).

**Proof.** We choose the vertices \( w_1, w_2, \ldots, w_p \), and hence the edges \( e'_1, e'_1, e'_2, \ldots, e'_p, e_p \), recursively. Initially let \( w_1 = v_1 \). Let \( e'_1 = w_1u_1 \) be an added edge incident to \( w_1 \) (recall that \( v_1 \) is incident to \( d \) added edges). Note that \( H_0 = G' \) and \( \eta_0 = \eta \).

Calculating \( \text{per}(A_{H_0}(\eta_0)) \) by expanding along the row indexed by \( e'_1 \), we conclude that there is a column of \( A_{H_0}(\eta_0) \) indexed by \( z \in V(G) \cup E(G) \) such that

\[
A_{H_0}(\eta_0)[e'_1, z] \neq 0 \pmod{d + 1} \quad \text{and} \quad \text{per}(A_{H_0}(\eta_0)[e'_1, \overline{z}]) \neq 0 \pmod{d + 1}.
\]

As \( A_{H_0}(\eta_0)[e'_1, z] \neq 0 \), we know that either \( z = w_1 \) or \( z \) is an edge of \( G \) incident to \( w_1 \).

Note that \( H_1 = H_0 - e'_1, e'_0, e'_1, \ldots, e'_p, e_p \), hence

\[
A_{H_0}(\eta_0)[e'_1, \overline{z}] = A_{H_1}(\eta_1)
\]

where \( \eta_1 \) agrees with \( \eta_0 \), except that \( \eta_1(z) = \eta_0(z) - 1 \).

If \( z = w_1 \), then \( \eta_1(w_1) = d - 1 \). It follows from Lemma \( \text{Lemma} \) 2 that there is a non-singular index function \( \eta' \) for which \( \eta'(z) \leq d - 1 \) for all \( z \in V(G) \) and \( \eta'(z) \leq 1 \) for all \( z \in E(G) \), and hence \( G \) is \( (d, 2) \)-choosable, contrary to our assumption.

Assume \( z \) is an edge of \( G \) incident to \( w_1 \). Let \( w_2 \) be the other end vertex of \( z \), and let \( e_1 = z = w_1w_2 \). If \( \eta(w_2) \leq d - 2 \), then write the column \( A_{H_1}(w_1) \) of \( A_{H_1}(\eta_1) \) as \( A_{H_1}(e_1) - A_{H_1}(w_2) \). By this expression of the matrix \( A_{H_1}(\eta_1) \), we have \( \eta_{A_{H_1}(\eta_1)}(z) \leq d - 1 \).
for all \( z \in V(G) \) and \( \eta_{A_{H_i}(\eta_i)}(z) \leq 1 \) for all \( z \in E(G) \) and \( \eta_{A_{H_i}(\eta_i)}(z) = 0 \) for all \( z \notin V(G) \cup E(G) \). As \( \text{per}(A_{H_i}(\eta_i)) \neq 0 \), by Lemma 2, \( G \) is \((d,2)\)-choosable, contrary to our assumption.

Thus we may assume that \( \eta(w_2) = d - 1 \).

Assume \( i \geq 1 \), and we have chosen distinct vertices \( w_1, w_2, \ldots, w_i \), edges \( e'_1, e'_2, \ldots, e'_i \) and \( e_1 = w_1w_2, e_2 = w_2w_3, \ldots, e_i = w_iw_{i+1} \), for which the following hold:

- \( \eta(w_1) = d \) and \( \eta(w_j) = d - 1 \) for \( 2 \leq j \leq i + 1 \) and \( \eta(e_j) = 1 \) for \( 1 \leq j \leq i \).
- \( \eta_j = \eta_{j-1} \) except that \( \eta_j(e_j) = \eta_{j-1}(e_j) - 1 = 0 \) for \( 1 \leq j \leq i \).
- For \( 0 \leq j \leq i \), \( \text{per}(A_{H_j}(\eta_j)) \neq 0 \) \((\text{mod } d + 1)\).

If \( w_{i+1} = w_s \) for some \( 1 \leq s \leq i - 2 \), then let \( p = i \), and the claim is proved. Assume \( w_{i+1} \neq w_j \) for any \( 1 \leq j \leq i - 2 \). Since \( \eta(w_{i+1}) = d - 1 \), \( d - d^-(w_{i+1}) = d - 1 \) and there is an edge \( e'_{i+1} = w_{i+1}w_{i+1} \in E(G') - E(G) \). As \( w_{i+1} \neq w_j \) for any \( 1 \leq j \leq i - 2 \), we have \( e'_{i+1} \in E(H_i) \).

Calculating \( \text{per}(A_{H_i}(\eta_i)) \) by expanding along the row indexed by \( e'_{i+1} \), we conclude that there is a column of \( A_{H_i}(\eta_i) \) indexed by \( z \in V(G) \cup E(G) \) such that

\[
A_{H_i}(\eta_i)[e'_{i+1}, z] \neq 0 \pmod{d + 1}, \quad \text{per}(A_{H_i}(\eta_i)[e'_{i+1}, z]) \neq 0 \pmod{d + 1}.
\]

Similarly, \( A_{H_i}(\eta_i)[e'_{i+1}, z] \neq 0 \) implies that either \( z = w_{i+1} \) or \( z \) is an edge of \( G \) incident to \( w_{i+1} \).

As \( H_{i+1} = H_i - e'_{i+1} \), we have

\[
A_{H_i}(\eta_i)[e'_{i+1}, z] = A_{H_{i+1}}(\eta_{i+1})
\]

where \( \eta_{i+1} \) is an index function which agrees with \( \eta_i \), except that \( \eta_{i+1}(z) = \eta_i(z) - 1 \).

If \( z = w_{i+1} \), then \( \eta_{i+1}(w_{i+1}) = d - 2 \). In \( A_{H_{i+1}} \),

\[
A_{H_{i+1}}(w_1) = A_{H_{i+1}}(e_1) - A_{H_{i+1}}(e_2) + A_{H_{i+1}}(e_3) - \ldots + (-1)^{i-1} A_{H_{i+1}}(e_i) + (-1)^{i} A_{H_{i+1}}(w_{i+1}).
\]

By this expression of the columns of \( A_{H_{i+1}}(\eta_{i+1}) \), the column \( A_{H_{i+1}}(z) \) occurs at most \( d - 1 \) times for each \( z \in V(G) \) and the column \( A_{H_{i+1}}(z) \) occurs at most once for each \( z \in E(G) \). For each \( z \notin V(G) \cup E(G) \), the column \( A_{H_{i+1}}(z) \) does not occur. By Lemma 2, \( G \) is \((d,2)\)-choosable, contrary to our assumption.

Assume \( z \) is an edge of \( G \) incident to \( w_{i+1} \). Let \( w_{i+2} \) be the other end vertex of \( z \) and let \( e_{i+1} = z = w_{i+1}w_{i+2} \). If \( \eta_{i+1}(w_{i+2}) \leq d - 2 \), then in \( A_{H_{i+1}}(\eta_{i+1}) \),

\[
A_{H_{i+1}}(w_1) = A_{H_{i+1}}(e_1) - A_{H_{i+1}}(e_2) + A_{H_{i+1}}(e_3) - \ldots + (-1)^{i} A_{H_{i+1}}(e_{i+1}) + (-1)^{i+1} A_{H_{i+1}}(w_{i+2}),
\]

which again leads to a contradiction. Thus \( \eta_{i+1}(w_{i+2}) = d - 1 \).
This process of finding new vertices \( w_j \) will eventually stop (as \( G \) is finite), and at the end we obtain the required comb-plus subgraph. This completes the proof of Claim 3.

Assume first that \( s = 1 \), and hence \( C = (w_1, w_2, \ldots, w_p) \) is a cycle. By definition, \( \eta_p(w_1) = d, \eta_p(w_i) = d - 1 \) for \( 2 \leq i \leq p \) and \( \eta_p(e_i) = 0 \) for \( 1 \leq i \leq p \).

**Claim 4** Let \( \eta_p' = \eta_p \) except that \( \eta_p'(w_2) = \eta_p(w_2) - 1 = d - 2 \) and \( \eta_p'(e_1) = \eta_p(e_1) + 1 = 1 \).

\[
\text{per}(A_{H_p}(\eta_p')) \neq 0 \pmod{d + 1}.
\]

**Proof.** To prove this claim, we write the column \( A_{H_p}(e_1) \) as \( A_{H_p}(w_1) + A_{H_p}(w_2) \). By linearity of permanent with respect to columns,

\[
\text{per}(A_{H_p}(\eta_p')) = \text{per}(A') + \text{per}(A''),
\]

where \( A' \) is the matrix obtained from \( A_{H_p}(\eta_p) \) by replacing the column \( A_{H_p}(e_1) \) with \( A_{H_p}(w_1) \), and \( A'' \) is the matrix obtained from \( A_{H_p}(\eta_p) \) by replacing the column \( A_{H_p}(e_1) \) with \( A_{H_p}(w_2) \). Thus \( A'' = A_{H_p}(\eta_p) \) and \( A' \) contains \( d + 1 \) copies of the column \( A_{H_p}(w_1) \). Hence \( \text{per}(A') = 0 \pmod{d + 1} \). Therefore, \( \text{per}(A_{H_p}(\eta_p')) = \text{per}(A'') \pmod{d + 1} = \text{per}(A_{H_p}(\eta_p)) \pmod{d + 1} \neq 0 \pmod{d + 1} \). This completes the proof of Claim 4.

Now in \( A_{H_p}(\eta_p') \), we re-write the columns as follows:

\[
\begin{align*}
A_{H_p}(w_3) &= A_{H_p}(e_2) - A_{H_p}(w_2), \\
A_{H_p}(w_4) &= A_{H_p}(e_3) - A_{H_p}(w_3), \\
&\vdots \\
A_{H_p}(w_p) &= A_{H_p}(e_{p-1}) - A_{H_p}(w_{p-1}), \\
A_{H_p}(w_1) &= A_{H_p}(e_p) - A_{H_p}(w_p).
\end{align*}
\]

By using these expressions, in the matrix \( A_{H_p}(\eta_p') \), the column \( A_{G'}(z) \) occurs at most \( d - 1 \) times for each \( z \in V(G) \) and the column \( A_{G'}(z) \) occurs at most once for each \( z \in E(G) \). For each \( z \notin V(G) \cup E(G) \), the column occurs 0 times. By Lemma 2, \( G \) is \((d,2)\)-choosable.

Assume next that \( s \geq 2 \). Then the path \( P' = (w_1, w_2, \ldots, w_s) \) connect \( w_1 \) to a cycle \( C = (w_s, w_{s+1}, \ldots, w_p) \). In \( A_{H_p}(\eta_p) \), write one copy of \( A_{H_p}(w_1) \) as

\[
A_{H_p}(e_1) - A_{H_p}(e_2) + \ldots + (-1)^s A_{H_p}(e_{s-1}) + (-1)^{s+1} A_{H_p}(w_s).
\]

By using this expression and by linearity of permanent with respect to columns, we obtain an index function \( \eta' \) of \( H_p \) in which \( \eta'(z) \leq d - 1 \) for all \( z \in V(G) \) except that possibly \( \eta'(w_s) = d \), and \( \eta'(z) \leq 1 \) for all \( z \in E(G) \), such that \( \text{per}(A_{H_p}(\eta')) \neq 0 \pmod{d + 1} \). Moreover, for this index function \( \eta' \), we have \( \eta'(w_i) = d - 1 \) for \( s + 1 \leq i \leq p \), \( \eta'(e_i) = 0 \) for \( s \leq i \leq p \). This is the same as the \( s = 1 \) case, and the proof is complete.

**Corollary 6** Every 2-degenerate graph is \((2,2)\)-choosable.
5 Graphs with bounded maximum average degree

The average degree \( \overline{d}(G) \) of \( G \) is \( \overline{d}(G) = \frac{2|E(G)|}{|V(G)|} \). The maximum average degree of \( G \), denoted by \( \text{mad}(G) \), is defined as \( \text{mad}(G) = \max\{\overline{d}(H) : H \subseteq G\} \). This section proves that if \( \text{mad}(G) \leq 2k \) for some integer \( k \), then \( G \) is \((k + 1, 2)\)-choosable.

**Lemma 4** Assume \( D \) is an orientation of a graph \( G \), and \( \eta \) is the index function defined as \( \eta(v) = d_D^+(v) \) for every vertex \( v \) and \( \eta(e) = 1 \) for every edge \( e \). Then \( \eta \) is a non-singular index function of \( G \).

**Proof.** First we prove that the lemma is true if \( D \) is an acyclic orientation. In this case, we prove that the index function \( \eta \) defined as \( \eta(v) = d_D^+(v) \) for each vertex \( v \) and \( \eta(e) = 0 \) for each edge \( e \) is a valid index function with \( \text{per}(A_G(\eta)) \neq 0 \).

Assume this is not true and \( G \) is a minimum counterexample. As \( D \) is acyclic, there is a source vertex \( v \). By the minimality of \( G \), the restriction \( \eta' \) of \( \eta \) to \( G - v \) is a non-singular index function of \( G - v \). We extend the matrix \( A_{G-v}(\eta') \) to \( A_G(\eta) \) by adding \( d_G(v) \) rows indexed by edges incident to \( v \), and adding \( d_G(v) = d_D^+(v) \) copies of the column \( A_G(v) \). Then \( \text{per}(A_G(\eta)) = d_G(v)! \text{per}(A_{G-v}(\eta')) \neq 0 \).

Next we consider the case that \( D \) is an arbitrary orientation. Let \( D' \) be an acyclic orientation of \( G \). Let \( \eta' \) be the index function defined as \( \eta'(v) = d_{D'}^+(v) \) for each vertex \( v \) and \( \eta'(e) = 0 \) for each edge \( e \). By the previous paragraph, \( \text{per}(A_G(\eta')) \neq 0 \). For each directed edge \( e = (u, v) \) of \( D' \) that is oriented differently in \( D \), we replace a copy of the column \( A_G(u) \) by the linear combination \( A_G(e) - A_G(v) \). Note that the matrix is not changed, because \( A_G(u) = A_G(e) - A_G(v) \). However, in such linear combinations of the columns of \( A_G(\eta') \), for each edge \( e \), \( A_G(e) \) occurs at most once, and for each vertex \( v \), \( A_G(v) \) occurs at most \( d_D^+(v) \) times. Therefore the index function defined as \( \eta(v) = d_D^+(v) \) for every vertex \( v \) and \( \eta(e) = 1 \) for every edge \( e \) is a non-singular index function of \( G \).

**Corollary 7** If \( \text{mad}(G) \leq 2k \), then \( G \) is \((k + 1, 2)\)-choosable. In particular, planar graphs are \((4, 2)\)-choosable and planar bipartite graphs are \((3, 2)\)-choosable.

**Proof.** It is well-known that if \( G \) has maximum average degree at most \( 2k \), then \( G \) has an orientation with maximum out-degree at most \( k \). Therefore the index function \( \eta \) defined as \( \eta(v) = k \) for every vertex \( v \) and \( \eta(e) = 1 \) for every edge \( e \) is a non-singular index function of \( G \). It follows from the argument in the introduction that \( G \) is \((k + 1, 2)\)-choosable.

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