Semilinear elliptic Schrödinger equations with singular potentials and absorption terms

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Funding information
Hellenic Foundation for Research and Innovation, Grant/Award Number: 59; Czech Science Foundation, Grant/Award Number: GA22-17403S

Abstract
Let \( \Omega \subset \mathbb{R}^N \) (\( N \geq 3 \)) be a \( C^2 \) bounded domain and \( \Sigma \subset \Omega \) be a compact, \( C^2 \) submanifold without boundary, of dimension \( k \) with \( 0 \leq k < N-2 \). Put \( L_\mu = \Delta + \mu d_\Sigma^{-2} \) in \( \Omega \setminus \Sigma \), where \( d_\Sigma(x) = \text{dist}(x, \Sigma) \) and \( \mu \) is a parameter.

We investigate the boundary value problem (P) \(-L_\mu u + g(u) = \tau \) in \( \Omega \setminus \Sigma \) with condition \( u = \nu \) on \( \partial \Omega \cup \Sigma \), where \( g : \mathbb{R} \to \mathbb{R} \) is a nondecreasing, continuous function, and \( \tau \) and \( \nu \) are positive measures. The complex interplay between the competing effects of the inverse-square potential \( d_\Sigma^{-2} \), the absorption term \( g(u) \) and the measure data \( \tau, \nu \) discloses different scenarios in which problem (P) is solvable. We provide sharp conditions on the growth of \( g \) for the existence of solutions. When \( g \) is a power function, namely \( g(u) = |u|^{p-1}u \) with \( p > 1 \), we show that problem (P) admits several critical exponents in the sense that singular solutions exist in the subcritical cases (i.e. \( p \) is smaller than a critical exponent) and singularities are removable in the supercritical cases (i.e. \( p \) is greater than a critical exponent). Finally, we establish various necessary and sufficient conditions expressed in terms of appropriate capacities for the solvability of (P).

MSC 2020
35J10, 35J25, 35J61, 35J75 (primary)
1. INTRODUCTION

1.1. Background and aim

Let $\Omega \subset \mathbb{R}^N (N \geq 3)$ be a $C^2$ bounded domain and $\Sigma \subset \Omega$ be a compact, $C^2$ submanifold in $\mathbb{R}^N$ without boundary, of dimension $k$ with $0 \leq k < N - 2$. When $k = 0$, we assume $\Sigma = \{0\} \subset \Omega$. Denote $d(x) = \text{dist}(x, \partial \Omega)$ and $d_\Sigma(x) = \text{dist}(x, \Sigma)$. For $\mu \in \mathbb{R}$, let $L_\mu$ be the Schrödinger operator with the inverse-square potential $d_\Sigma^{-2}$

$$L_\mu = L_\mu^{\Omega, \Sigma} := \Delta + \frac{\mu}{d_\Sigma^2}$$

in $\Omega \setminus \Sigma$. The study of $L_\mu$ is closely connected to the optimal Hardy constant $C_{\Omega, \Sigma}$ and the fundamental exponent $H$ given below

$$C_{\Omega, \Sigma} := \inf_{\varphi \in H^1_0(\Omega)} \frac{\int_\Omega |\nabla \varphi|^2 \, dx}{\int_\Omega d_\Sigma^{-2} \varphi^2 \, dx} \quad \text{and} \quad H := \frac{N - k - 2}{2}. \quad (1.1)$$
Obviously, \( H \leq \frac{N-2}{2} \) and \( H = \frac{N-2}{2} \) if and only if \( \Sigma \) is a singleton. It is well known that \( C_{\Omega,\Sigma} \in (0, H^2) \) (see Dávila and Dupaigne \([9, 10]\) and Barbatis, Filippas and Tertikas \([2]\)) and \( C_{\Omega,\Sigma} = (\frac{N-2}{2})^2 \). Moreover, \( C_{\Omega,\Sigma} = H^2 \) provided \(-\Delta d^2_{\Sigma} + k - N \geq 0\) in the sense of distributions in \( \Omega \setminus \Sigma \) or if \( \Omega = \Sigma_\beta \) with \( \beta \) small enough (see \([2]\)), where

\[
\Sigma_\beta := \{ x \in \mathbb{R}^N \setminus \Sigma : d_{\Sigma}(x) < \beta \}.
\]

For \( \mu \leq H^2 \), let \( \alpha_- \) and \( \alpha_+ \) be the roots of the algebraic equation \( \alpha^2 - 2H\alpha + \mu = 0 \), that is,

\[
\alpha_- := H - \sqrt{H^2 - \mu}, \quad \alpha_+ := H + \sqrt{H^2 - \mu}.
\] (1.2)

We see that \( \alpha_- \leq H \leq \alpha_+ \leq 2H \) and \( \alpha_- \geq 0 \) if and only if \( \mu \geq 0 \).

By \([9\text{, Lemma 2.4 and Theorem 2.6}]\) and \([10\text{, page 337, Lemma 7, Theorem 5}]\),

\[
\lambda_\mu := \inf \left\{ \int_{\Omega} \left( |\nabla u|^2 - \frac{\mu}{d^2_{\Sigma}} u^2 \right) dx : u \in C^1_c(\Omega), \int_{\Omega} u^2 dx = 1 \right\} > -\infty.
\]

Note that \( \lambda_\mu \) is the first eigenvalue associated to \(-L_\mu\) and its corresponding eigenfunction \( \phi_\mu \), with normalization \( \|\phi_\mu\|_{L^2(\Omega)} = 1 \), satisfies the estimate \( \phi_\mu \approx d_{\Sigma}^{\alpha_-} \) in \( \Omega \setminus \Sigma \) (see Section 2.2 for more detail). The sign of \( \lambda_\mu \) plays an important role in the study of \( L_\mu \). If \( \mu < C_{\Omega,\Sigma} \) then \( \lambda_\mu > 0 \). However, in general, this does not hold true. Under the assumption \( \lambda_\mu > 0 \), the authors of this paper obtained the existence and sharp two-sided estimates of the Green function \( G_\mu \) and Martin kernel \( K_\mu \) associated to \(-L_\mu\) (see \([14]\)) which are crucial tools in the study of the boundary value problem with measures data for linear equations involving \( L_\mu \)

\[
\begin{cases}
-L_\mu u = \tau & \text{in } \Omega \setminus \Sigma, \\
\text{tr}(u) = \nu,
\end{cases}
\] (1.3)

where \( \tau \in \mathcal{M}(\Omega; \phi_\mu) \) (i.e. \( \int_{\Omega \setminus \Sigma} \phi_\mu d|\tau| < \infty \)) and \( \nu \in \mathcal{M}(\partial \Omega \cup \Sigma) \) (i.e. \( \int_{\partial \Omega \cup \Sigma} d|\nu| < \infty \)).

In (1.3), \( \text{tr}(u) \) denotes the boundary trace of \( u \) on \( \partial \Omega \cup \Sigma \) which was defined in \([14]\) in terms of harmonic measures of \(-L_\mu\) (see Section 2.4). A highlighting property of this notion is \( \text{tr}(G_\mu[\tau]) = 0 \) for any \( \tau \in \mathcal{M}(\Omega \setminus \Sigma; \phi_\mu) \) and \( \text{tr}(K_\mu[\nu]) = \nu \) for any \( \nu \in \mathcal{M}(\partial \Omega \cup \Sigma) \),

\[
G_\mu[\tau](x) = \int_{\Omega \setminus \Sigma} G_\mu(x, y) d\tau(y), \quad \tau \in \mathcal{M}(\Omega \setminus \Sigma; \phi_\mu),
\]

\[
K_\mu[\nu](x) = \int_{\Omega \setminus \Sigma} K_\mu(x, y) d\nu(y), \quad \nu \in \mathcal{M}(\partial \Omega \cup \Sigma).
\]

Note that for a positive measure \( \tau \), \( G_\mu[\tau] \) is finite in \( \Omega \setminus \Sigma \) if and only if \( \tau \in \mathcal{M}(\Omega \setminus \Sigma; \phi_\mu) \).
It was shown in [14] that $G_{\mu}[\tau]$ is the unique solution of (1.3) with $\nu = 0$, and $K_{\mu}[\nu]$ is the unique solution of (1.3) with $\tau = 0$. As a consequence of the linearity, the unique solution to (1.3) is of the form

$$u = G_{\mu}[\tau] + K_{\mu}[\nu] \quad \text{a.e. in} \ \Omega \setminus \Sigma.$$  

Further results for linear problem (1.3) are presented in Section 2.5.

Semilinear equations driven by $L_{\mu}$ with an absorption term have been treated in some particular cases of $\Sigma$. In the free-potential case, namely $\mu = 0$ and $\Sigma = \emptyset$, the study of the boundary value problem for such equations in measure frameworks has been a research objective of numerous mathematicians, and greatly pushed forward by a series of celebrated papers of Marcus and Véron (see the excellent monograph [17] and references therein). The singleton case, namely $\Sigma = \{0\} \subset \Omega$, has been investigated in different directions, including the work of Guerch and Véron [15] on the local properties of solutions to the stationary Schrödinger equations in $\mathbb{R}^N$, interesting results by Cîrstea [7] on isolated singular solutions, and recent study of Chen and Véron [6] on the existence and stability of solutions with zero boundary condition.

In this paper, we study the boundary value problem for semilinear equation with an absorption term of the form

$$\begin{cases} 
-L_{\mu} u + g(u) = \tau & \text{in} \ \Omega \setminus \Sigma, \\
\text{tr}(u) = \nu,
\end{cases} \quad (1.4)$$

where $\Sigma$ is of dimension $0 \leq k < N - 2$, $g : \mathbb{R} \rightarrow \mathbb{R}$ is a nondecreasing continuous function such that $g(0) = 0$, $\tau \in \mathcal{M}(\Omega \setminus \Sigma; \phi_{\mu})$ and $\nu \in \mathcal{M}(\partial \Omega \cup \Sigma)$. A typical model of the absorption term to keep in mind is $g(t) = |t|^{p-1}t$ with $p > 1$.

Problem (1.4) has the following features.

- The potential $d_{\Sigma}^{-2}$ blows up on $\Sigma$ and is bounded on $\partial \Omega$. Hence, considering $\partial \Omega \cup \Sigma$ simply as the ‘whole boundary’ does not provide profound enough understanding of the effect of the potential. Therefore, we have to take care of $\partial \Omega$ and $\Sigma$ separably.
- The dimension of $\Sigma$, the value of the parameter $\mu$, and the concentration of the measures $\nu, \tau$ give rise to several critical exponents.
- Heuristically, in measure framework, the growth of $g$ plays an important role in the solvability of (1.4).

The complex interplay between the above features yields substantial difficulties and reveals new aspects of the study of (1.4). We aim to perform a profound analysis of the interplay to establish the existence, nonexistence, uniqueness, and a priori estimate for solutions to (1.4).

### 1.2 Main results

Let us assume throughout the paper that

$$\mu \leq H^2 \quad \text{and} \quad \lambda_{\mu} > 0. \quad (1.5)$$

Under the above assumption, a theory for linear problem (1.3) was developed in [14] (some results are recalled in Section 2.5), which forms a basis for the study of (1.4).
Before stating our main results, we clarify the sense of solutions we will deal with in the paper.

**Definition 1.1.** A function \( u \) is a weak solution of \((1.4)\) if \( u \in L^1(\Omega; \phi_\mu), \ g(u) \in L^1(\Omega; \phi_\mu), \) and

\[
-\int_{\Omega} u L_\mu \zeta \, dx + \int_{\Omega} g(u) \zeta \, dx = \int_{\Omega \setminus \Sigma} \zeta \, d\tau - \int_{\Omega} \mathbb{K}_\mu [v] L_\mu \zeta \, dx \quad \forall \zeta \in X_\mu(\Omega \setminus \Sigma),
\]

(1.6)

where the space of test function \( X_\mu(\Omega \setminus \Sigma) \) is defined by

\[
X_\mu(\Omega \setminus \Sigma) := \{ \zeta \in H^1_{\text{loc}}(\Omega \setminus \Sigma) : \phi^{-1}_\mu \zeta \in H^1(\Omega; \phi^2_\mu), \ \phi^{-1}_\mu L_\mu \zeta \in L^\infty(\Omega) \}.
\]

The space \( X_\mu(\Omega \setminus \Sigma) \) was introduced in [14] to study linear problem \((1.3)\). From \((1.7)\), it is easy to see that the first term on the left-hand side of \((1.6)\) is finite. By [14, Lemma 7.3], for any \( \zeta \in X_\mu(\Omega \setminus \Sigma) \), we have \( |\zeta| \lesssim \phi_\mu \), hence the second term on the left-hand side and the first term on the right-hand side of \((1.6)\) are finite. Finally, since \( \mathbb{K}_\mu[v] \in L^1(\Omega; \phi_\mu) \), the second term on the right-hand side of \((1.6)\) is also finite.

By Theorem 2.7, \( u \) is a weak solution of \((1.4)\) if and only if

\[
 u + G_\mu[g(u)] = G_\mu[\tau] + \mathbb{K}_\mu[v] \quad \text{in} \ \Omega \setminus \Sigma.
\]

We note that when \( \Sigma = \{0\} \) the study of problem \((1.4)\) with \( v = \ell \delta_0 \), for \( \ell \in \mathbb{R} \) and \( \delta_0 \) being the Dirac measure concentrated at 0, can be reduced to the study of a boundary value problem in the whole domain \( \Omega \) in the spirit of [6]. See Remark 3.2 for more detail.

**Definition 1.2.** A couple \((\tau, v) \in \mathcal{M}(\Omega \setminus \Sigma; \phi_\mu) \times \mathcal{M}(\partial \Omega \cup \Sigma)\) is called a \(g\)-good couple if problem \((1.4)\) has a weak solution. When \( \tau = 0 \), a measure \( v \in \mathcal{M}(\partial \Omega \cup \Sigma)\) is called a \(g\)-good measure if problem \((1.4)\) has a weak solution. When there is no confusion, we simply say ‘a good couple’ (resp. ‘a good measure’) instead of ‘a \(g\)-good couple’ (resp. ‘a \(g\)-good measure’).

Note that if \((\tau, v)\) is a good couple then the solution is unique.

Our first result provides a sufficient condition for a couple of measures to be good.

**Theorem 1.3.** Assume \( \mu \leq H^2 \) and \( g \) satisfies

\[
g(-G_\mu[\tau^-] - \mathbb{K}_\mu[v^-]), \ g(G_\mu[\tau^+] + \mathbb{K}_\mu[v^+]) \in L^1(\Omega; \phi_\mu).
\]

(1.8)

Then any couple \((\tau, v) \in \mathcal{M}(\Omega \setminus \Sigma; \phi_\mu) \times \mathcal{M}(\partial \Omega \cup \Sigma)\) is a \(g\)-good couple. Moreover, the solution \( u \) satisfies

\[
-G_\mu[\tau^-] - \mathbb{K}_\mu[v^-] \leq u \leq G_\mu[\tau^+] + \mathbb{K}_\mu[v^+] \quad \text{in} \ \Omega \setminus \Sigma.
\]

(1.9)

The existence part of Theorem 1.3 is based on sharp weak Lebesgue estimates on the Green kernel and Martin kernel (Theorems 2.8–2.9) and the sub and super solution theorem (see Theorem 3.4). The uniqueness is derived from Kato inequalities (see Theorem 2.7).

When \( g \) satisfies the so-called subcritical integral condition

\[
\int_1^\infty s^{-q-1} (g(s) - g(-s)) \, ds < \infty
\]

(1.10)
for suitable $q > 0$, we can show that condition (1.8) holds (see Lemma 3.5) and consequently, $(\tau, \nu)$ is a good couple.

**Theorem 1.4.** Assume $\mu < \left(\frac{N-2}{2}\right)^2$ and $g$ satisfies (1.10) with

$$q = \min \left\{ \frac{N+1}{N-1}, \frac{N - \alpha_-}{N - \alpha_- - 2} \right\},$$

where $\alpha_-$ is defined in (1.2). Then any couple $(\tau, \nu) \in \mathcal{W}(\Omega \setminus \Sigma; \phi_\mu) \times \mathcal{W}(\partial \Omega \cup \Sigma)$ is a $g$-good couple. Moreover, the solution $u$ satisfies (1.9).

The value of $q$ in condition (1.10) under which problem (1.4) with $\tau = 0$, namely problem

$$\begin{cases}
-L_\mu u + g(u) = 0 & \text{in } \Omega \setminus \Sigma, \\
\text{tr}(u) = \nu,
\end{cases}
$$

(1.11)

admits a unique solution, can be enlarged according to the concentration of the boundary measure data. The case when $\nu$ is concentrated in $\partial \Omega$ is treated in the following theorem.

**Theorem 1.5.** Assume $\mu \leq H^2$ and $g$ satisfies (1.10) with $q = \frac{N+1}{N-1}$. Then any measure $\nu \in \mathcal{W}(\partial \Omega \cup \Sigma)$ with compact support in $\partial \Omega$ is a $g$-good measure. Moreover, the solution $u$ satisfies

$$-\kappa_\mu [\nu^-] \leq u \leq \kappa_\mu [\nu^+] \quad \text{in } \Omega \setminus \Sigma.
$$

(1.12)

It is worth mentioning that, without imposing condition (1.10), one can show that any $L^1$ datum concentrated in $\partial \Omega$ is $g$-good (see Theorem 4.3 for more detail).

When $\nu$ is concentrated in $\Sigma$, it is $g$-good under the condition (1.10) with $q = \frac{N - \alpha_-}{N - \alpha_- - 2}$ if $\mu < \left(\frac{N-2}{2}\right)^2$. However, if $k = 0$ and $\mu = \left(\frac{N-2}{2}\right)^2$, which implies $\alpha_- = \frac{N-2}{2}$, condition (1.10) with $q = \frac{N+2}{N-2}$ is not enough to ensure that $\nu$ is $g$-good. In this case the condition on $g$ is modified with a logarithmic correction. This is stated in the following theorem.

**Theorem 1.6.**

(i) Assume $\mu < \left(\frac{N-2}{2}\right)^2$ and $g$ satisfies (1.10) with $q = \frac{N - \alpha_-}{N - \alpha_- - 2}$. Then any measure $\nu \in \mathcal{W}(\partial \Omega \cup \Sigma)$ with compact support in $\Sigma$ is a $g$-good measure. Moreover, the solution $u$ satisfies (1.12).

(ii) Assume $k = 0$, $\Sigma = \{0\}$, $\mu = \left(\frac{N-2}{2}\right)^2$, and $g$ satisfies

$$\int_1^\infty s^{\frac{N+2}{N-2}-1}(\ln s)^{\frac{N+2}{N-2}} g(s) ds < \infty.
$$

(1.13)

Then for any $\rho > 0$, $\nu = \rho \delta_0$ is $g$-good. Here $\delta_0$ is the Dirac measure concentrated at 0.

It is worth pointing out here that, when $k = 0$, the integral condition (1.13) and the integral condition in Theorem 1.4 coincide with integral conditions [6, (1.35) and (1.34)], respectively. However, unlike [6] where the $\Delta_2$-condition for $g$ is required, in our results, such a condition is not needed.
When \( g \) is a power function, namely \( g(t) = |t|^{p-1}t \) with \( p > 1 \), condition (1.10) with \( q = \frac{N+1}{N-1} \) is fulfilled if and only if \( 1 < p < \frac{N+1}{N-1} \), while condition (1.10) with \( q = \frac{N-\alpha_-}{N-\alpha_- - 2} \) is satisfied if and only if \( 1 < p < \frac{N-\alpha_-}{N-\alpha_- - 2} \). In these ranges of \( p \), by Theorem 1.5 and Theorem 1.6, problem (1.11) admits a unique solution. In particular, in these ranges, existence results hold when \( \nu \) is a Dirac measure. We will point out below that in case \( p \geq \frac{N+1}{N-1} \) or \( p \geq \frac{N-\alpha_-}{N-\alpha_- - 2} \) according to the concentration of the boundary data, isolated singularities are removable. This justifies the fact that the values \( \frac{N+1}{N-1} \) and \( \frac{N-\alpha_-}{N-\alpha_- - 2} \) are critical exponents.

To this purpose, let us introduce a weight function which allows to normalize the value of solutions near \( \Sigma \). Let \( \beta_0 \) be the constant in Section 2.1 and \( \eta_{\beta_0} \) be a smooth function such that \( 0 \leq \eta_{\beta_0} \leq 1 \), \( \eta_{\beta_0} = 1 \) in \( \Sigma_{\beta_0} \), and \( \text{supp} \eta_{\beta_0} \subset \Sigma_{\beta_0} \). We define

\[
W(x) := \begin{cases} 
  d\Sigma(x)^{-\alpha_+} & \text{if } \mu < H^2, \\
  d\Sigma(x)^{-H} \ln d\Sigma(x) & \text{if } \mu = H^2,
\end{cases} \quad x \in \Omega \setminus \Sigma,
\]

and

\[
W := 1 - \eta_{\beta_0} + \eta_{\beta_0}W \quad \text{in } \Omega \setminus \Sigma.
\] (1.14)

It was proved in [14] that for any \( h \in C(\partial\Omega \cup \Sigma) \), the problem

\[
\begin{cases} 
  L_\mu v = 0 & \text{in } \Omega \setminus \Sigma \\
  v = h & \text{on } \partial\Omega \cup \Sigma,
\end{cases}
\] (1.15)

admits a unique solution \( v \). Here the boundary value condition in (1.15) is understood as

\[
\lim_{x \in \Omega \setminus \Sigma, x \to y} \frac{v(x)}{W(x)} = h(y) \quad \text{uniformly with respect to } y \in \partial\Omega \cup \Sigma.
\]

**Theorem 1.7.** Assume \( \mu \leq H^2 \) and \( p \geq \frac{2 + \alpha_+}{\alpha_+} \). If \( u \in C(\overline{\Omega} \setminus \Sigma) \) is a nonnegative solution of

\[
-L_\mu u + |u|^{p-1}u = 0 \quad \text{in } \Omega \setminus \Sigma
\] (1.16)

in the sense of distributions in \( \Omega \setminus \Sigma \), such that

\[
\lim_{x \in \Omega \setminus \Sigma, x \to \xi} \frac{u(x)}{W(x)} = 0 \quad \forall \xi \in \partial\Omega.
\] (1.17)

locally uniformly in \( \partial\Omega \), then \( u \equiv 0 \).

The idea of the proof of Theorem 1.7 is to construct a function \( v \) dominating \( u \) by using to the Keller–Osserman-type estimate (see Lemma 6.1). Then, by making use of the Representation Theorem 2.3 and a subtle argument based on the maximum principle, we are able to deduce \( v \equiv 0 \), which implies \( u \equiv 0 \).
When \( \frac{N-\alpha_-}{N-\alpha_- - 2} \leq p < \frac{2 + \alpha_+}{\alpha_+} \), an additional condition on the behavior of solutions near \( \Sigma \) is required to obtain a removability result.

**Theorem 1.8.** Assume \( \mu \leq H^2 \), \( z \in \Sigma \), and \( \frac{N-\alpha_-}{N-\alpha_- - 2} \leq p < \frac{2 + \alpha_+}{\alpha_+} \). If \( u \in C(\Omega \setminus \Sigma) \) is a nonnegative solution of (1.16) in the sense of distributions in \( \Omega \setminus \Sigma \) such that

\[
\lim_{x \to \xi, x \in \Omega \setminus \Sigma} \frac{u(x)}{W(x)} = 0 \quad \forall \xi \in \partial \Omega \cup \Sigma \setminus \{z\},
\]

locally uniformly in \( \partial \Omega \cup \Sigma \setminus \{z\} \), then \( u \equiv 0 \).

The technique used in the proof of Theorem 1.8 is different from that of Theorem 1.7. In the range \( \frac{N-\alpha_-}{N-\alpha_- - 2} \leq p < \frac{2 + \alpha_+}{\alpha_+} \), by employing appropriate test functions and Keller–Osserman–type estimate (see Lemma 6.1), we can show that the solution \( u \), which may admit an isolated singularity at \( z \), belongs to \( L^p(\Omega) \). Then by using a delicate argument based on the properties of the boundary trace, we may derive that \( u \) cannot have positive mass at \( z \), which implies that the isolated singularity is removable and hence \( u \equiv 0 \).

Next, we introduce an appropriate capacity framework which enables us to obtain the solvability for

\[
\begin{cases}
-L\mu u + |u|^{p-1}u = 0 & \text{in } \Omega \setminus \Sigma \\
\operatorname{tr}(u) = \nu.
\end{cases}
\]  

(1.19)

A measure \( \nu \in \mathcal{M}(\partial \Omega \cup \Sigma) \) for which problem (1.19) admits a (unique) solution is called \( p \)-good measure.

For \( \alpha \in \mathbb{R} \) we defined the Bessel kernel of order \( \alpha \) by \( B_{d,\alpha}(\xi) := \mathcal{F}^{-1}((1 + |\xi|^2)^{-\frac{\alpha}{2}})\), where \( \mathcal{F} \) is the Fourier transform in space \( S'(\mathbb{R}^d) \) of moderate distributions in \( \mathbb{R}^d \). For \( \kappa > 1 \), the Bessel space \( L_{\alpha,\kappa}(\mathbb{R}^d) \) is defined by

\[
L_{\alpha,\kappa}(\mathbb{R}^d) := \{ f = B_{d,\alpha} \ast g : g \in L^\kappa(\mathbb{R}^d) \},
\]

with norm

\[
\|f\|_{L_{\alpha,\kappa}} := \|g\|_{L^\kappa} = \|B_{d,-\alpha} \ast f\|_{L^\kappa}.
\]

The Bessel capacity \( \text{Cap}_{\alpha,\kappa}^{\mathbb{R}^d} \) is defined for compact subsets \( K \subset \mathbb{R}^d \) by

\[
\text{Cap}_{\alpha,\kappa}^{\mathbb{R}^d}(K) := \inf\{\|f\|_{L_{\alpha,\kappa}}^\kappa, f \in S'(\mathbb{R}^d), f \geq 1_K\}.
\]

See Section 8 for further discussion on the Bessel spaces and capacities.

**Definition 1.9.** Let \( \nu \in \mathcal{M}^+(\partial \Omega \cup \Sigma) \). We say that \( \nu \) is absolutely continuous with respect to the Bessel capacity \( \text{Cap}_{\alpha,\kappa}^{\mathbb{R}^d} \) if

\[
\forall E \subset \partial \Omega \cup \Sigma, E \text{ Borel}, \text{Cap}_{\alpha,\kappa}^{\mathbb{R}^d}(E) = 0 \implies \nu(E) = 0.
\]
When \( \frac{N-\alpha}{N-\alpha-2} \leq p < \frac{2+\alpha_+}{\alpha_+} \) and \( \nu \) is concentrated in \( \Sigma \), a sufficient condition expressed in terms of a suitable Bessel capacity for a measure to be \( p \)-good is provided in the next theorem.

**Theorem 1.10.** Assume \( k \geq 1 \), \( \mu \leq H^2 \), \( \frac{N-\alpha}{N-\alpha-2} \leq p < \frac{2+\alpha_+}{\alpha_+} \), and \( \nu \in \mathcal{M}^+(\partial \Omega \cup \Sigma) \) with compact support in \( \Sigma \). Put

\[
\varrho := \frac{2 - (p-1)\alpha_+}{p}.
\]

(1.20)

If \( \nu \) is absolutely continuous with respect to \( \text{Cap}_{\varrho, p'}^k \), where \( p' = \frac{p}{p-1} \), then \( \nu \) is \( p \)-good.

A pivotal ingredient in the proof of Theorem 1.10 is a sophisticated potential estimate on the Martin kernel (see Theorem 8.2) inspired by [18], which allows us to implement an approximation procedure to derive the existence of a solution to (1.19).

In case \( p \geq \frac{N+1}{N-1} \) and \( \nu \) is concentrated in \( \partial \Omega \), we show that the absolute continuity of \( \nu \) with respect to a suitable Bessel capacity is not only a sufficient condition, but also a necessary condition for \( \nu \) to be \( p \)-good.

**Theorem 1.11.** Assume \( \mu \leq H^2 \), \( p \geq \frac{N+1}{N-1} \), and \( \nu \in \mathcal{M}^+(\partial \Omega \cup \Sigma) \) with compact support in \( \partial \Omega \). Then \( \nu \) is a \( p \)-good measure if and only if it is absolutely continuous with respect to \( \text{Cap}_{\frac{N-1}{2}, p'} \).

**Organization of the paper**

In Section 2, we present main properties of the submanifold \( \Sigma \) and recall important facts about the first eigenpair, Green kernel and Martin kernel of \( -L_{\mu} \). In Section 3, we prove the sub and super solution theorem (see Theorem 3.4), which is an important tool in the proof of Theorem 1.3 and Theorem 1.4. Section 4 and Section 5 are devoted to the proof of Theorem 1.5 and Theorem 1.6, respectively. Next we establish Keller–Osserman estimates in Section 6, which is a crucial ingredient in the proof of Theorem 1.7 and Theorem 1.8 in Section 7. Then we prove Theorems 1.10–1.11 in Section 8. Finally, in the Appendix, we construct a barrier function and demonstrate some useful estimates.

### 1.3 | Notations

We list below notations that are frequently used in the paper.

- Let \( \phi \) be a positive continuous function in \( \Omega \setminus \Sigma \) and \( \kappa \geq 1 \). Let \( L^\kappa(\Omega; \phi) \) be the space of functions \( f \) such that

\[
\|f\|_{L^\kappa(\Omega; \phi)} := \left( \int_\Omega |f|^\kappa \phi \, dx \right)^{\frac{1}{\kappa}} < \infty.
\]

The weighted Sobolev space \( H^1(\Omega; \phi) \) is the space of functions \( f \in L^2(\Omega; \phi) \) such that \( \nabla f \in L^2(\Omega; \phi) \). This space is endowed with the norm

\[
\|f\|^2_{H^1(\Omega; \phi)} := \int_\Omega |f|^2 \phi \, dx + \int_\Omega |\nabla f|^2 \phi \, dx.
\]

The closure of \( C^\infty(\Omega) \) in \( H^1(\Omega; \phi) \) is denoted by \( H^1_0(\Omega; \phi) \).
Denote by $𝔐(Ω; 𝜙)$ the space of Radon measures $𝜏$ in $Ω$ such that

$$
\|𝜏\|_{𝔐(Ω; 𝜙)} := \int_{Ω} 𝜙\, d|𝜏| < \infty,
$$

and by $𝔐^+(Ω; 𝜙)$ its positive cone. Denote by $𝔐(\partial Ω \cup Σ)$ the space of finite measure $ν$ on $\partial Ω \cup Σ$, namely

$$
\|ν\|_{𝔐(\partial Ω \cup Σ)} := |ν|(\partial Ω \cup Σ) < \infty,
$$

and by $𝔐^+(\partial Ω \cup Σ)$ its positive cone.

- For a measure $ω$, denote by $ω^+$ and $ω^-$ the positive part and negative part of $ω$, respectively.
- For $β > 0$, let $Ω_β := \{x ∈ Ω : d(x) < β\}$ and $Σ_β := \{x ∈ \mathbb{R}^N \setminus Σ : d_Σ(x) < β\}$.
- We denote by $c, c_1, C$ ... the constant which depends on initial parameters and may change from one appearance to another.
- The notation $A \geq B$ (resp. $A \leq B$) means $A \geq c B$ (resp. $A \leq c B$) where the implicit $c$ is a positive constant depending on some initial parameters. If $A \geq B$ and $A \leq B$, we write $A \approx B$. Throughout the paper, most of the implicit constants depend on some (or all) of the initial parameters such as $N, Ω, Σ, k, μ$, and we will omit these dependencies in the notations (except when it is necessary).
- For $a, b ∈ \mathbb{R}$, denote $a \wedge b = \min\{a, b\}$, $a \vee b = \max\{a, b\}$.
- For a set $D ⊂ \mathbb{R}^N$, $1_D$ denotes the indicator function of $D$.

## 2  PRELIMINARIES

### 2.1  Assumptions on $Σ$

Throughout this paper, we assume $Σ ⊂ Ω$ is a $C^2$ compact submanifold in $\mathbb{R}^N$ without boundary, of dimension $k$, $0 ≤ k < N − 2$. When $k = 0$ we assume $Σ = \{0\} ⊂ Ω$.

For $x = (x_1, ..., x_k, x_{k+1}, ..., x_N) ∈ \mathbb{R}^N$, we write $x = (x', x'')$ where $x' = (x_1, ..., x_k) ∈ \mathbb{R}^k$ and $x'' = (x_{k+1}, ..., x_N) ∈ \mathbb{R}^{N-k}$. For $β > 0$, we denote by $B^k(x', β)$ the ball in $\mathbb{R}^k$ with center at $x'$ and radius $β$. For any $ξ ∈ Σ$, we set

$$
Σ_β := \{x ∈ \mathbb{R}^N \setminus Σ : d_Σ(x) < β\},
$$

$$
V(ξ, β) := \{x = (x', x'') : |x' − ξ'| < β, |x_i − Ψ_i(x')| < β, ∀i = k + 1, ..., N\},
$$

for some functions $Γ_i^ξ : \mathbb{R}^k → \mathbb{R}$, $i = k + 1, ..., N$.

Since $Σ$ is a $C^2$ compact submanifold in $\mathbb{R}^N$ without boundary, we may assume the existence of a real number $β_0 > 0$ such that the followings hold.

- $Σ_{6β_0} ⊂ Ω$ and for any $x \in Σ_{6β_0}$, there is a unique $ξ ∈ Σ$ that satisfies $|x − ξ| = d_Σ(x)$.
- $d_Σ ∈ C^2(Σ_{4β_0})$, $|∇d_Σ| = 1$ in $Σ_{4β_0}$ and there exists $η ∈ L^∞(Σ_{4β_0})$ such that

$$
Δd_Σ(x) = \frac{N - k - 1}{d_Σ(x)} + η(x) \quad in Σ_{4β_0}.
$$

(See [20, Lemma 2.2] and [11, Lemma 6.2].)
For any $\xi \in \Sigma$, there exist $C^2$ functions $\Gamma^i_\xi \in C^2(\mathbb{R}^k; \mathbb{R})$, $i = k + 1, \ldots, N$, such that (upon relabeling and reorienting the coordinate axes if necessary), for any $\beta \in (0, 6\beta_0)$, $V(\xi, \beta) \subset \Omega$ and

$$V(\xi, \beta) \cap \Sigma = \{x = (x', x'') : |x' - \xi'| < \beta, x_i = \Gamma^i_\xi(x'), \forall i = k + 1, \ldots, N\}. \quad (2.3)$$

There exist $m_0 \in \mathbb{N}$ and points $\xi^j \in \Sigma$, $j = 1, \ldots, m_0$, and $\beta_1 \in (0, \beta_0)$ such that

$$\Sigma_{2\beta_1} \subset \bigcup_{j=1}^{m_0} V(\xi^j, \beta_0) \Subset \Omega. \quad (2.4)$$

Now for $\xi \in \Sigma$, set

$$d^\xi_{\Sigma}(x) := \left(\sum_{i=k+1}^{N} |x_i - \Gamma^i_\xi(x')|^2\right)^{1/2}, \quad x = (x', x'') \in V(\xi, 4\beta_0). \quad (2.5)$$

Then we see that there exists a positive constant $C = C(N, \Sigma)$ such that

$$d(x) \leq d^\xi_{\Sigma}(x) \leq C\|\Sigma\|_{C^2} d^\xi_{\Sigma}(x), \quad \forall x \in V(\xi, 2\beta_0), \quad (2.6)$$

where $\xi^j = ((\xi^j)', (\xi^j)'') \in \Sigma$, $j = 1, \ldots, m_0$, are the points in (2.4) and

$$\|\Sigma\|_{C^2} := \sup\{||\Gamma^i_\xi||_{C^2(B_{\frac{1}{2}\beta_0}((\xi^j)'))) : i = k + 1, \ldots, N, j = 1, \ldots, m_0\} < \infty. \quad (2.7)$$

Moreover, $\beta_1$ can be chosen small enough such that for any $x \in \Sigma_{\beta_1}$,

$$B(x, \beta_1) \subset V(\xi, \beta_0), \quad (2.8)$$

where $\xi \in \Sigma$ satisfies $|x - \xi| = d(x)$.

### 2.2 Eigenvalue of $-L_\mu$

Recall that $H$ is defined in (1.1) and $\alpha_-$ and $\alpha_+$ are defined in (1.2). We summarize below main properties of the first eigenfunction of the operator $-L_\mu$ in $\Omega \setminus \Sigma$ from [9, Lemma 2.4 and Theorem 2.6] and [10, p. 337, Lemma 7, Theorem 5].

(i) For any $\mu \leq H^2$, it is known that

$$\lambda_\mu := \inf \left\{ \int_{\Omega} \left( |\nabla u|^2 - \frac{\mu}{d^2_{\Sigma}} u^2 \right) dx : u \in H^1_0(\Omega), \int_{\Omega} u^2 dx = 1 \right\} > -\infty. \quad (2.9)$$

(ii) If $\mu < H^2$, there exists a minimizer $\phi_\mu$ of (2.9) belonging to $H^1_0(\Omega)$. Moreover, it satisfies $-L_\mu \phi_\mu = \lambda_\mu \phi_\mu$ in $\Omega \setminus \Sigma$ and $\phi_\mu \approx d_{\Sigma}^{-\alpha_-}$ in $\Sigma_{\beta_0}$.
(iii) If $\mu = H^2$, there is no minimizer of (2.9) in $H_0^1(\Omega)$, but there exists a nonnegative function $\phi_{H^2} \in H^1_{\text{loc}}(\Omega)$ such that $-L_{H^2}\phi_{H^2} = \lambda_{H^2}\phi_{H^2}$ in the sense of distributions in $\Omega \setminus \Sigma$ and $\phi_{H^2} \approx d^{-H}_\Sigma$ in $\Sigma_{\beta_0}$. In addition, the function $d^{-H}_\Sigma \phi_{H^2} \in H^1_0(\Omega; d^{-2H}_\Sigma)$. From (ii) and (iii) we deduce that
\[ \phi_\mu \approx d^{-\alpha_-}_\Sigma \quad \text{in } \Omega \setminus \Sigma. \] (2.10)

### 2.3 Estimates on Green kernel and Martin kernel

Recall that throughout the paper, we always assume that (1.5) holds. Let $G_\mu$ and $K_\mu$ be the Green kernel and Martin kernel of $-L_\mu$ in $\Omega \setminus \Sigma$, respectively. Let us recall two-sided estimates on Green kernel.

**Proposition 2.1** [14, Proposition 4.1].

(i) If $\mu < (\frac{N-2}{2})^2$, then for any $x, y \in \Omega \setminus \Sigma$, $x \neq y$,
\[
G_\mu(x, y) \approx |x - y|^{2-N} \left( 1 \wedge \frac{d(x)d(y)}{|x - y|^2} \right) \left( 1 \wedge \frac{|x - y|}{d(x)} + 1 \right)^{\alpha_-} \left( 1 \wedge \frac{|x - y|}{d(y)} + 1 \right)^{\alpha_-}
\]
\[
\approx |x - y|^{2-N} \left( 1 \wedge \frac{d(x)d(y)}{|x - y|^2} \right) \left( 1 \wedge \frac{d(x)d(y)}{|x - y|^2} \right)^{-\alpha_-}. \tag{2.11}
\]

(ii) If $k = 0$, $\Sigma = \{0\}$ and $\mu = (\frac{N-2}{2})^2$, then for any $x, y \in \Omega \setminus \Sigma$, $x \neq y$,
\[
G_\mu(x, y) \approx |x - y|^{2-N} \left( 1 \wedge \frac{d(x)d(y)}{|x - y|^2} \right) \left( 1 \wedge \frac{|x - y|}{d(x)d(y)} + 1 \right)^{\frac{N-2}{2}}
\]
\[
+ \left( |x| |y| \right)^{-\frac{N-2}{2}} \left| \ln \left( 1 \wedge \frac{|x - y|^2}{d(x)d(y)} \right) \right|^\frac{N-2}{2}
\]
\[
\approx |x - y|^{2-N} \left( 1 \wedge \frac{d(x)d(y)}{|x - y|^2} \right) \left( 1 \wedge \frac{|x| |y|}{|x - y|^2} \right)^{-\frac{N-2}{2}}
\]
\[
+ \left( |x| |y| \right)^{-\frac{N-2}{2}} \left| \ln \left( 1 \wedge \frac{|x - y|^2}{d(x)d(y)} \right) \right|. \tag{2.12}
\]

The implicit constants in (2.11) and (2.12) depend on $N, \Omega, \Sigma, \mu$.

**Proposition 2.2** [14, Theorem 1.2].

(i) If $\mu < (\frac{N-2}{2})^2$, then
\[
K_\mu(x, \xi) \approx \begin{cases} 
\frac{d(x)d_\Sigma(x)^{-\alpha_-}}{|x - \xi|^N} & \text{if } x \in \Omega \setminus \Sigma, \ \xi \in \partial \Omega \\
\frac{d(x)d_\Sigma(x)^{-\alpha_-}}{|x - \xi|^{N-2-2\alpha_-}} & \text{if } x \in \Omega \setminus \Sigma, \ \xi \in \Sigma.
\end{cases} \tag{2.13}
\]
(ii) If \( k = 0, \Sigma = \{ 0 \} \) and \( \mu = (\frac{N-2}{2})^2 \), then
\[
K_\mu(x, \xi) \approx \begin{cases} 
  d(x)|x|^{-\frac{N-2}{2}} & \text{if } x \in \Omega \setminus \{0\}, \xi \in \partial \Omega \\
  d(x)|x|^{-\frac{N-2}{2}} \ln \frac{|x|}{D_\Omega} & \text{if } x \in \Omega \setminus \{0\}, \xi = 0,
\end{cases}
\]
where \( D_\Omega := 2 \sup_{x \in \Omega} |x| \).

The implicit constant depends on \( N, \Omega, \Sigma, \mu, p \).

The Green operator and Martin operator are, respectively,
\[
G_\mu[\tau](x) = \int_{\Omega \setminus \Sigma} G_\mu(x, y) \, d\tau(y), \quad \tau \in \mathcal{M}(\Omega \setminus \Sigma; \phi_\mu), \quad (2.15)
\]
\[
K_\mu[\nu](x) = \int_{\partial \Omega \cup \Sigma} K_\mu(x, y) \, d\nu(y), \quad \nu \in \mathcal{M}(\partial \Omega \cup \Sigma). \quad (2.16)
\]

Next we recall the Representation theorem.

**Theorem 2.3** [14, Theorem 1.3]. For any \( \nu \in \mathcal{M}^+(\partial \Omega \cup \Sigma) \), the function \( K_\mu[\nu] \) is a positive \( L_\mu \)-harmonic function (i.e. \( L_\mu K_\mu[\nu] = 0 \) in the sense of distributions in \( \Omega \setminus \Sigma \)). Conversely, for any positive \( L_\mu \)-harmonic function \( u \) (i.e. \( L_\mu u = 0 \) in the sense of distribution in \( \Omega \setminus \Sigma \)), there exists a unique measure \( \nu \in \mathcal{M}^+(\partial \Omega \cup \Sigma) \) such that \( u = K_\mu[\nu] \).

### 2.4 Notion of boundary trace

Let \( z \in \Omega \setminus \Sigma \) and \( h \in C(\partial \Omega \cup \Sigma) \) and denote \( L_{\mu, z}(h) := v_h(z) \) where \( v_h \) is the unique solution of the Dirichlet problem
\[
\begin{cases}
  L_\mu v = 0 & \text{in } \Omega \setminus \Sigma \\
  v = h & \text{on } \partial \Omega \cup \Sigma.
\end{cases} \quad (2.17)
\]

Here the boundary value condition in (2.17) is understood in the sense that
\[
\lim_{\text{dist}(x,F) \to 0} \frac{v(x)}{W(x)} = h \quad \text{for every compact set } F \subset \partial \Omega \cup \Sigma.
\]

The mapping \( h \mapsto L_{\mu, z}(h) \) is a linear positive functional on \( C(\partial \Omega \cup \Sigma) \). Thus, there exists a unique Borel measure on \( \partial \Omega \cup \Sigma \), called \( L_\mu \)-harmonic measure in \( \partial \Omega \cup \Sigma \) relative to \( z \) and denoted by \( \omega^z_{\Omega \setminus \Sigma} \), such that
\[
v_h(z) = \int_{\partial \Omega \setminus \Sigma} h(y) \, d\omega^z_{\Omega \setminus \Sigma}(y).
\]
Let \( x_0 \in \Omega \setminus \Sigma \) be a fixed reference point. Let \( \{\Omega_n\} \) be an increasing sequence of bounded \( C^2 \) domains such that

\[
\overline{\Omega}_n \subset \Omega_{n+1}, \quad \cup_n \Omega_n = \Omega, \quad \mathcal{H}^{N-1}(\partial \Omega_n) \to \mathcal{H}^{N-1}(\partial \Omega),
\]

where \( \mathcal{H}^{N-1} \) denotes the \((N-1)\)-dimensional Hausdorff measure in \( \mathbb{R}^N \). Let \( \{\Sigma_n\} \) be a decreasing sequence of compact sets such that for each \( n \in \mathbb{N}, \Sigma_n = \Sigma_n^O \cup \partial \Sigma_n^o \), where \( \Sigma_n^O \) is a \( C^2 \) domain, and

\[
\Sigma \subset \Sigma_n^O \subset \Sigma_{n+1}^O \subset \Sigma_n^O \subset \Sigma_n \subset \Omega_n, \quad \cap_n \Sigma_n = \Sigma.
\]

For each \( n \in \mathbb{N} \), set \( O_n = \Omega_n \setminus \Sigma_n \) and assume \( x_0 \in O_1 \). Such a sequence \( \{O_n\} \) will be called a \( C^2 \) exhaustion of \( \Omega \setminus \Sigma \). If \( \Sigma = \{0\} \subset \Omega \), one may choose \( \Sigma_n = B(0, \frac{1}{n}) \) for \( n \) large enough.

Then \(-L_\mu\) is uniformly elliptic and coercive in \( H^1_0(O_n) \) and its first eigenvalue \( \lambda^{O_n}_\mu \) in \( O_n \) is larger than its first eigenvalue \( \lambda_\mu \) in \( \Omega \setminus \Sigma \).

For \( h \in C(\partial O_n) \), the following problem

\[
\left\{ \begin{array}{ll}
-L_\mu v = 0 & \text{in } O_n \\
v = h & \text{on } \partial O_n,
\end{array} \right.
\]

admits a unique solution which allows to define the \( L_\mu \)-harmonic measure \( \omega_{O_n}^{x_0} \) on \( \partial O_n \) by

\[
v(x_0) = \int_{\partial O_n} h(y) \, d\omega_{O_n}^{x_0}(y).
\]

Let \( G_\mu^{O_n}(x, y) \) be the Green kernel of \(-L_\mu\) on \( O_n \). Then \( G_\mu^{O_n}(x, y) \uparrow G_\mu(x, y) \) for \( x, y \in \Omega \setminus \Sigma, x \neq y \).

We recall below the definition of boundary trace which is defined in a dynamic way.

**Definition 2.4** (Boundary trace). A function \( u \in W^{1, \kappa}_{\text{loc}}(\Omega \setminus \Sigma) \) for some \( \kappa > 1 \), possesses a boundary trace if there exists a measure \( \nu \in \mathcal{M}(\partial \Omega \cup \Sigma) \) such that for any \( C^2 \) exhaustion \( \{O_n\} \) of \( \Omega \setminus \Sigma \), there holds

\[
\lim_{n \to \infty} \int_{\partial O_n} \phi u \, d\omega_{O_n}^{x_0} = \int_{\partial \Omega \cup \Sigma} \phi \, d\nu \quad \forall \phi \in C(\overline{\Omega}).
\]

The boundary trace of \( u \) is denoted by \( \text{tr}(u) \).

**Proposition 2.5** [14, Proposition 1.8].

(i) For any \( \nu \in \mathcal{M}(\partial \Omega \cup \Sigma), \text{tr}(\kappa_\mu [\nu]) = \nu \).

(ii) For any \( \tau \in \mathcal{M}(\Omega \setminus \Sigma; \phi_\mu), \text{tr}(G_\mu [\tau]) = 0 \).

### 2.5 Boundary value problem for linear equations

**Definition 2.6.** Let \( \tau \in \mathcal{M}(\Omega \setminus \Sigma; \phi_\mu) \) and \( \nu \in \mathcal{M}(\partial \Omega \cup \Sigma) \). We will say that \( u \) is a weak solution of

\[
\left\{ \begin{array}{ll}
-L_\mu u = \tau & \text{in } \Omega \setminus \Sigma, \\
\text{tr}(u) = \nu, &
\end{array} \right.
\]

(2.23)
if \( u \in L^1(\Omega \setminus \Sigma; \phi, \mu) \) and \( u \) satisfies

\[
- \int_\Omega u L_\mu \xi \, dx = \int_{\Omega \setminus \Sigma} \xi \, d\tau - \int_\Omega \kappa_\mu [v] L_\mu \xi \, dx \quad \forall \xi \in X_\mu(\Omega \setminus \Sigma).
\]

**Theorem 2.7** [14, Theorem 1.8]. Let \( \tau, \rho \in \mathcal{M}(\Omega \setminus \Sigma; \phi, \mu) \), \( v \in \mathcal{M}(\partial \Omega \cup \Sigma) \), and \( f \in L^1(\Omega; \phi, \mu) \).

Then there exists a unique weak solution \( u \in L^1(\Omega; \phi, \mu) \) of (2.23). Furthermore,

\[
u = G_\mu[\tau] + \kappa_\mu[v]
\]

and for any \( \zeta \in X_\mu(\Omega \setminus \Sigma) \), there holds

\[
\|u\|_{L^1(\Omega; \phi, \mu)} \leq \frac{1}{\lambda_\mu} \|\tau\|_{\mathcal{M}(\Omega \setminus \Sigma; \phi, \mu)} + C \|v\|_{\mathcal{M}(\partial \Omega \cup \Sigma)},
\]

where \( C = C(N, \Omega, \Sigma, \mu) \). In addition, if \( d\tau = f \, dx + d\rho \), then, for any \( 0 \leq \zeta \in X_\mu(\Omega \setminus \Sigma) \), the following estimates are valid:

\[
- \int_\Omega |u| L_\mu \xi \, dx \leq \int_\Omega \text{sign}(u)f \xi \, dx + \int_{\Omega \setminus \Sigma} \xi \, d|\rho| - \int_\Omega \kappa_\mu [v] L_\mu \xi \, dx,
\]

\[
- \int_\Omega u^+ L_\mu \xi \, dx \leq \int_\Omega \text{sign}^+(u)f \xi \, dx + \int_{\Omega \setminus \Sigma} \xi \, d|\rho^+| - \int_\Omega \kappa_\mu [v^+] L_\mu \xi \, dx.
\]

### 2.6 Weak Lebesgue estimates on Green kernel and Martin kernel

In this subsection, we present sharp weak Lebesgue estimates for the Green kernel and Martin kernel.

We first recall the definition of weak Lebesgue spaces (or Marcinkiewicz spaces). Let \( D \subset \mathbb{R}^N \) be a domain. Denote by \( L^\kappa_w(D; \tau) \), \( 1 \leq \kappa < \infty \), \( \tau \in \mathcal{M}^+(D) \), the weak Lebesgue space (or Marcinkiewicz space) defined as follows: A measurable function \( f \) in \( D \) belongs to this space if there exists a constant \( c \) such that

\[
\lambda_f(a; \tau) := \tau\{|x \in D : |f(x)| > a\} \leq ca^{-\kappa}, \quad \forall a > 0.
\]

The function \( \lambda_f \) is called the distribution function of \( f \) (relative to \( \tau \)). For \( \kappa \geq 1 \), denote

\[
L^\kappa_w(D; \tau) := \{f \text{ Borel measurable} : \sup_{a > 0} a^\kappa \lambda_f(a; \tau) < \infty\},
\]

\[
\|f\|_{L^\kappa_w(D; \tau)}^* := \left(\sup_{a > 0} a^\kappa \lambda_f(a; \tau)\right)^{\frac{1}{\kappa}}.
\]

The \( \|\cdot\|_{L^\kappa_w(D; \tau)}^* \) is not a norm, but for \( \kappa > 1 \), it is equivalent to the norm

\[
\|f\|_{L^\kappa_w(D; \tau)} := \sup \left\{ \frac{\int_A |f| \, d\tau}{\tau(A)^{1-\frac{1}{\kappa}}} : A \subset D, A \text{ measurable}, 0 < \tau(A) < \infty \right\}.
\]
More precisely,
\[ \|f\|_{L^\kappa_w(D;\tau)}^\kappa \leq \|f\|_{L^\kappa_w(D;\tau)} < \frac{\kappa}{\kappa - 1} \|f\|_{L^\kappa_w(D;\tau)}. \] (2.32)

When \( d\tau = \varphi \, dx \) for some positive continuous function \( \varphi \), for simplicity, we use the notation \( L^\kappa_w(D;\varphi) \). Notice that
\[ L^\kappa_w(D;\varphi) \subset L^r(D;\varphi) \quad \text{for any} \quad r \in [1, \kappa). \] (2.33)

From (2.30) and (2.32), one can derive the following estimate which is useful in the sequel. For any \( f \in L^\kappa_w(D;\varphi) \), there holds
\[ \int_{\{x \in D : |f(x)| \geq s\}} \varphi \, dx \leq s^{-\kappa} \|f\|_{L^\kappa_w(D;\varphi)}^\kappa. \] (2.34)

Recall that \( \alpha_- \) is defined in (1.2). Put
\[ p_{\alpha_-} := \min \left\{ \frac{N - \alpha_-}{N - 2 - \alpha_-}, \frac{N + 1}{N - 1} \right\}. \] (2.35)

Notice that if \( \mu > 0 \), then \( \alpha_- > 0 \), hence \( p_{\alpha_-} = \frac{N + 1}{N - 1} \).

**Theorem 2.8** [13, Theorem 3.8 and Theorem 3.9]. There holds
\[ \left\| G_\mu[\tau] \right\|_{L^p_w(\Omega;\Sigma;\phi_\mu)} \lesssim \|\tau\|_{\mathcal{M}(\Omega;\Sigma;\phi_\mu)}, \quad \forall \tau \in \mathcal{M}^+(\Omega;\Sigma;\phi_\mu). \] (2.36)
The implicit constant depends on \( N, \Omega, \Sigma, \mu \).

**Theorem 2.9** [13, Theorem 3.10].

I. Assume \( \mu \leq H^2 \) and \( \nu \in \mathcal{M}(\partial \Omega \cup \Sigma) \) with compact support in \( \partial \Omega \). Then,
\[ \left\| K_\mu[\nu] \right\|_{L^p_w(\Omega;\Sigma;\phi_\mu)} \lesssim \|\nu\|_{\mathcal{M}(\partial \Omega)}. \] (2.37)

II. Assume \( \nu \in \mathcal{M}(\partial \Omega \cup \Sigma) \) with compact support in \( \Sigma \).
   (i) If \( \mu < \left( \frac{N - 2}{2} \right)^2 \), then
\[ \left\| K_\mu[\nu] \right\|_{L^{p_{\alpha_-}}(\Omega;\Sigma;\phi_\mu)} \lesssim \|\nu\|_{\mathcal{M}(\Sigma)}. \] (2.38)
   (ii) If \( k = 0, \Sigma = \{0\} \) and \( \mu = \left( \frac{N - 2}{2} \right)^2 \), then for any \( 1 < \theta < \frac{N + 2}{N - 2} \),
\[ \left\| K_\mu[\nu] \right\|_{L^\theta_w(\Omega;\Sigma;\phi_\mu)} \lesssim \|\nu\|_{\mathcal{M}(\Sigma)}. \] (2.39)

In addition, for \( \lambda > 0 \), set
\[ A_\lambda(0) := \left\{ x \in \Omega \setminus \{0\} : K_\mu[\delta_0](x) > \lambda \right\}, \quad m_\lambda := \int_{A_\lambda(0)} d(x)|x|^{-\frac{N - 2}{2}} \, dx, \] (2.40)
where $\delta_0$ is the Dirac measure concentrated at 0. Then,

$$m_\lambda \lesssim (\lambda^{-1} \ln \lambda)^{\frac{N+2}{N-2}}, \quad \forall \lambda > e.$$  \hfill (2.41)

The implicit constant depends on $N$, $\Omega$, $\Sigma$, $\mu$, and $\sigma$.

### 3 \hspace{1em} BOUNDARY VALUE PROBLEM FOR SEMILINEAR EQUATIONS

In the sequel, we assume $g : \mathbb{R} \to \mathbb{R}$ is a nondecreasing continuous function such that $g(0) = 0$.

#### 3.1 \hspace{1em} Sub and super solutions theorem

We start with the definition of subsolutions and supersolutions of (1.4).

**Definition 3.1.** A function $u$ is a weak subsolution (resp. supersolution) of (1.4) if $u \in L^1(\Omega; \phi_\mu)$, $g(u) \in L^1(\Omega; \phi_\mu)$, and

$$- \int_{\Omega} u L_{\mu} \phi^\xi \, dx + \int_{\Omega} g(u) \phi^\xi \, dx \leq \text{(resp. \geq)} \int_{\Omega \setminus \Sigma} \phi^\xi \, d\tau - \int_{\Omega} \kappa_{\phi_\mu} [v] L_{\mu} \phi^\xi \, dx \quad \forall 0 \leq \phi \in X_\mu(\Omega \setminus \Sigma).$$  \hfill (3.1)

A function $u$ is a weak solution of (1.4) if $u$ is a subsolution and a supersolution of (1.4).

**Remark 3.2.** We note that when $\Sigma = \{0\}$ the study of problem (1.4) with $v = \ell \delta_0$, for $\ell \in \mathbb{R}$ and $\delta_0$ being the Dirac measure concentrated at 0, can be reduced to the study of a boundary value problem in the whole domain $\Omega$ in the spirit of [6]. More precisely, using [14, Lemma 5.4 and Theorem 1.2], we may show that if $u$ is a weak solution of (1.4), then $u \in L^1(\Omega; |x|^{-\alpha-1})$. Therefore, for any $\xi \in C^{1,1}_0(\Omega)$, we may take $\phi = |x|^{-\alpha} \xi$ as a test function in (1.6) to deduce that

$$- \int_{\Omega} u L_{\mu}^n \phi^\xi |x|^{-\alpha-} \, dx + \int_{\Omega} g(u) \phi^\xi |x|^{-\alpha-} \, dx = \int_{\Omega \setminus \Sigma} \phi^\xi |x|^{-\alpha-} \, d\tau - \int_{\Omega} \kappa_{\phi_\mu} [\delta_0] L_{\mu}^n \phi^\xi |x|^{-\alpha-} \, dx.$$  \hfill (3.2)

In the above formula, the operator $L_{\mu}^n$ is defined (as in [5, Theorem 1.1]) by

$$L_{\mu}^n := \Delta - \frac{2\alpha}{|x|^2} x \cdot \nabla.$$

In addition, thanks to [14, Lemma 5.6 and Proposition 6.6], we have

$$\kappa_{\phi_\mu} [\delta_0] (x) = K_{\phi_\mu} (x, 0) = \frac{\phi_{\phi_\mu} (x)}{\phi_{\phi_\mu} (x_0)},$$  \hfill (3.3)

where $\phi_{\phi_\mu} \in C^2(\Omega \setminus \{0\}) \cap C(\overline{\Omega} \setminus \{0\})$ solves

$$L_{\mu} \phi_{\phi_\mu} = 0 \quad \text{in} \ \Omega \setminus \{0\}, \quad \phi_{\phi_\mu} = 0 \quad \text{on} \ \partial \Omega, \quad \lim_{|x| \to 0} \frac{\phi_{\phi_\mu} (x)}{W(x)} = 1,$$
where

\[
W(x) = \begin{cases} 
|x| - \frac{N-2}{2} - \sqrt{\frac{(N-2)^2}{4} - \mu} & \text{if } \mu < \left(\frac{N-2}{2}\right)^2 \\
|x| - \frac{N-2}{2} \ln |x| & \text{if } \mu = \left(\frac{N-2}{2}\right)^2
\end{cases}
\quad x \in \Omega \setminus \{0\}.
\]

From [5, Theorem 1.2], we see that

\[
-\int_{\Omega} K_\mu(x,0)L^*_\mu |x|^{-\alpha-} \, dx = \frac{c_\mu}{\Phi^{\Omega}(x_0)} \xi(0),
\]

where \(c_\mu\) is the constant defined in [5, (1.9)].

Combining (3.2), (3.3), and (3.4) leads to

\[
-\int_{\Omega} u L^*_\mu |x|^{-\alpha-} \, dx + \int_{\Omega} g(u) \xi |x|^{-\alpha-} \, dx = \int_{\Omega \setminus \Sigma} \xi |x|^{-\alpha-} \, d\tau + \frac{c_\mu}{\Phi^{\Omega}(x_0)} \xi(0).
\]

Therefore, the study of problem (1.4) with \(\nu = \ell \delta_0\) can be reduced to that of

\[
\begin{cases} 
-L_\mu u + g(u) = \tau + \ell \Phi^{\Omega}(x_0) \delta_0 \\
u = 0
\end{cases}
\quad \text{in }\Omega \\
u = \tau = \nu
\quad \text{on }\partial\Omega,
\]

in the spirit of [6].

**Lemma 3.3.**

(i) Let \(u \in L^1(\Omega; \phi_\mu)\) be a weak supersolution of (1.4). Then there exist \(\tau_u \in \mathcal{M}^+(\Omega \setminus \Sigma; \phi_\mu)\) and \(\nu_u \in \mathcal{M}^+(\partial\Omega \cup \Sigma)\) such that \(u\) is a weak solution of

\[
\begin{cases} 
-L_\mu u + g(u) = \tau + \tau_u \\
\text{tr}(u) = \nu + \nu_u
\end{cases}
\quad \text{in }\Omega \setminus \Sigma,
\]

(ii) Let \(u \in L^1(\Omega; \phi_\mu)\) be a weak subsolution of (1.4). Then there exist \(\tau_u \in \mathcal{M}^+(\Omega \setminus \Sigma; \phi_\mu)\) and \(\nu_u \in \mathcal{M}^+(\partial\Omega \cup \Sigma)\) such that \(u\) is a weak solution of

\[
\begin{cases} 
-L_\mu u + g(u) = \tau - \tau_u \\
\text{tr}(u) = \nu - \nu_u
\end{cases}
\quad \text{in }\Omega \setminus \Sigma.
\]
Proof.  
(i) Let $w$ be the unique solution of
\begin{equation}
\begin{aligned}
-L_\mu w + g(w) = \tau & \quad \text{in } \Omega \setminus \Sigma, \\
\text{tr}(w) = v.
\end{aligned}
\end{equation}
(3.7)

Then
\begin{equation}
-\int_\Omega (w - u)L_\mu \zeta \, dx \leq 0 \quad \forall 0 \leq \zeta \in X_\mu(\Omega \setminus \Sigma).
\end{equation}
(3.8)

Let $\eta \in X_\mu(\Omega \setminus \Sigma)$ be such that $-L_\mu \eta = \text{sign}^+(w - u)\phi_\mu$. Then by using $\eta$ as a test function in (3.8), we obtain that $w \leq u$ in $\Omega \setminus \Sigma$.

Set $v = u - w$ then $v \geq 0$ in $\Omega \setminus \Sigma$ and $-L_\mu v \geq 0$ in the sense of distributions in $\Omega \setminus \Sigma$. This implies the existence of a nonnegative Radon measure $\tau_u$ in $\Omega \setminus \Sigma$ such that $-L_\mu v = \tau_u$ in the sense of distribution. By [17, Corollary 1.2.3], $v \in W^{1,\infty}_{loc}(\Omega \setminus \Sigma)$ for some $\kappa > 1$. Let $\{O_n\}$ be a smooth exhaustion of $\Omega \setminus \Sigma$ and $\zeta_n$ be the weak solution of
\begin{equation}
\begin{aligned}
-L_\mu \zeta_n = 0 & \quad \text{in } O_n, \\
\zeta_n = v & \quad \text{on } \partial O_n.
\end{aligned}
\end{equation}
(3.9)

Therefore, $v = G_\mu^O[\tau_u] + \zeta_n$. Since $\tau_u, \zeta_n$ are nonnegative and $G_\mu^O(x, y) \geq G_\mu(x, y)$ for any $x \neq y$ and $x, y \in \Omega \setminus \Sigma$, we obtain $0 \leq G_\mu[\tau_u] \leq v$ a.e. in $\Omega \setminus \Sigma$. In particular, $0 \leq G_\mu[\tau_u](x^*) \leq v(x^*)$ for some point $x^* \in \Omega \setminus \Sigma$. This, together with the estimate $G_\mu(x^*, \cdot) \geq \phi_\mu$ a.e. in $\Omega$, implies $\tau_u \in \mathcal{M}(\Omega \setminus \Sigma; \phi_\mu)$.

Moreover, we observe from above that $v = G_\mu^O[\tau_u]$ is a nonnegative $L_\mu$-harmonic function in $\Omega \setminus \Sigma$. Thus by Theorem 2.3, there exists a unique $\nu_u \in \mathcal{W}^+(\partial \Omega \cup \Sigma)$ such that $v - G_\mu[\tau_u] = \kappa_\mu[\nu_u]$ a.e. in $\Omega \setminus \Sigma$. This, together with $w + G_\mu[g(u)] = G_\mu[\tau + \tau_u] + \kappa_\mu[\nu + \nu_u]$, yields
\begin{equation*}
u + G_\mu[g(u)] = G_\mu[\tau + \tau_u] + \kappa_\mu[\nu + \nu_u],
\end{equation*}
which means that $u$ is a weak solution of (3.5).

(ii) The proof is similar to that of (i) and we omit it. \)

The main result of this subsection is the following sub and super solution theorem.

**Theorem 3.4.** Assume $\tau \in \mathcal{M}(\Omega \setminus \Sigma; \phi_\mu)$ and $\nu \in \mathcal{W}(\partial \Omega \cup \Sigma)$. Let $v, w \in L^1(\Omega; \phi_\mu)$ be weak sub-solution and supersolution of (1.4), respectively, such that $v \leq w$ in $\Omega \setminus \Sigma$ and $g(v), g(w) \in L^1(\Omega; \phi_\mu)$. Then problem (1.4) admits a unique weak solution $u \in L^1(\Omega; \phi_\mu)$ which satisfies $v \leq u \leq w$ in $\Omega \setminus \Sigma$.

**Proof.** Uniqueness. If $u_1$ and $u_2$ are two solutions of (1.4), then $u_1 - u_2$ satisfies
\begin{equation}
\begin{aligned}
-L_\mu(u_1 - u_2) + g(u_1) - g(u_2) = 0 & \quad \text{in } \Omega \setminus \Sigma, \\
\text{tr}(u_1 - u_2) = 0.
\end{aligned}
\end{equation}

Then by using (2.27) with \( u = u_1 - u_2, f = -(g(u_1) - g(u_2)), \rho = 0, \) and \( \nu = 0, \) we have

\[
- \int_{\Omega} |u_1 - u_2| L_{\mu} \zeta \, dx + \int_{\Omega} \text{sign}(u_1 - u_2)(g(u_1) - g(u_2)) \zeta \, dx \leq 0.
\]

Choosing \( \zeta = \phi_\mu, \) and keeping in mind that \( g \) is nondecreasing, we obtain from the above estimate that \( u_1 = u_2 \) in \( \Omega \setminus \Sigma. \)

**Existence.** We follow some ideas of the proof of [17, Theorem 2.2.4]. Define

\[
g_n(t) := \max\{-n, \min\{g(t), n\}\}. \tag{3.10}
\]

Set

\[
\tilde{g}_n(x) := \begin{cases} 
  g_n(w(x)) & \text{if } z(x) \geq w(x), \\
  g_n(z(x)) & \text{if } v(x) < z(x) < w(x), \\
  g_n(v(x)) & \text{if } z(x) \leq v(x).
\end{cases}
\]

Let \( u \in L^1(\Omega; \phi_\mu) \) and denote by \( \Upsilon(u) \) the unique solution of

\[
\begin{aligned}
- L_{\mu} \varphi + \tilde{g}_n(u) &= \tau & \text{in } \Omega \setminus \Sigma, \\
\text{tr}(\varphi) &= \nu.
\end{aligned} \tag{3.11}
\]

Then \( \Upsilon(u) \in L^1(\Omega; \phi_\mu) \) and

\[
\Upsilon(u) = -G_{\mu}[\tilde{g}_n(u)] + G_{\mu}[\tau] + K_{\mu}[\nu]. \tag{3.12}
\]

By [14, Remark 5.5], \( G_{\mu}[1](x) \lesssim d(x)d_\Sigma(x)^{\min\{\alpha_-0\}} \) for a.e. \( x \in \Omega \setminus \Sigma. \) Therefore, there exists a constant \( C = C(\Omega, \Sigma, N, \mu) > 0 \) such that

\[
|\Upsilon(u)| \lesssim Cnd d_\Sigma^{\min\{\alpha_-0\}} + G_{\mu}[|\tau|] + K_{\mu}[|\nu|]. \tag{3.13}
\]

By Theorems 2.8–2.9, estimate (2.33) (with \( D = \Omega \setminus \Sigma \) and \( \varphi = \phi_\mu \), estimate (2.10), and the above inequality, we can show that there exists \( C_1 = C_1(\Omega, \Sigma, N, \mu) > 0 \) such that

\[
\|\Upsilon(u)\|_{L^1(\Omega; \phi_\mu)} \lesssim C_1(n + \|\tau\|_{\text{Gr}(\Omega \setminus \Sigma; \phi_\mu)} + \|\nu\|_{\text{Gr}(\partial \Omega \cup \Sigma)}). \tag{3.14}
\]

We will use the Schauder fixed point theorem to prove the existence of a fixed point of \( \Upsilon \) by examining the following criteria.

The operator \( \Upsilon : L^1(\Omega; \phi_\mu) \to L^1(\Omega; \phi_\mu) \) is continuous. Indeed, let \( \{\varphi_m\} \) be a sequence such that \( \varphi_m \to \varphi \) in \( L^1(\Omega; \phi_\mu) \) as \( m \to \infty. \) Since \( g_n \) is continuous and bounded, we can easily show \( \tilde{g}_n(\varphi_m) \to \tilde{g}_n(\varphi) \) in \( L^1(\Omega; \phi_\mu), \) which implies \( \Upsilon(\varphi_m) \to \Upsilon(\varphi) \) as \( m \to \infty \) in \( L^1(\Omega; \phi_\mu), \) by (3.12) and (2.36).

The operator \( \Upsilon \) is compact. Indeed, let \( \{\varphi_m\} \) be a sequence in \( L^1(\Omega; \phi_\mu), \) then by (3.14) and [17, Theorem 1.2.2], \( \{\Upsilon(\varphi_m)\} \) is uniformly bounded in \( W^{1,\kappa}(D) \) for any \( 1 < \kappa < \frac{N}{N-1} \) and any open set \( D \subset \Omega \setminus \Sigma. \) Therefore, there exist \( \psi \in W^{1,\kappa}_{\text{loc}}(\Omega \setminus \Sigma) \) and a subsequence still denoted by \( \{\Upsilon(\varphi_m)\} \)
such that \( T(\varphi_m) \to \psi \) in \( L^\infty_{\text{loc}}(\Omega \setminus \Sigma) \) and a.e. in \( \Omega \setminus \Sigma \). By (3.13) and the dominated convergence theorem, we deduce that \( T(\varphi_m) \to \psi \) in \( L^1(\Omega; \phi_\mu) \).

Now set
\[
A := \{ \varphi \in L^1(\Omega; \phi_\mu) : \| \varphi \|_{L^1(\Omega; \phi_\mu)} \leq C_1(n + \| \tau \| \gamma(\Omega \setminus \Sigma; \phi_\mu) + \| \nu \| \gamma(\partial \Omega \cup \Sigma)) \}.
\]

Then \( A \) is a closed, convex subset of \( L^1(\Omega; \phi_\mu) \) and \( T(A) \subset A \). Thus, we can apply Schauder fixed point theorem to obtain the existence of a function \( u_n \in A \) such that \( T(u_n) = u_n \). This means \( u_n \) satisfies
\[
\begin{cases}
-L_\mu u_n + \tilde{g}_n(u_n) = \tau \quad \text{in } \Omega \setminus \Sigma, \\
\text{tr}(u_n) = \nu.
\end{cases}
\]

Then
\[
|u_n| = |-G_\mu[\tilde{g}_n(u)] + G_\mu[\tau] + \kappa_\mu[\nu]| \leq G_\mu[|g(w)| + |g(v)|] + G_\mu[|\tau|] + \kappa_\mu[|\nu|],
\]
which implies
\[
\|u_n\|_{L^1(\Omega; \phi_\mu)} \leq C_2(\|g(w)\|_{L^1(\Omega; \phi_\mu)} + \|g(v)\|_{L^1(\Omega; \phi_\mu)} + \|\tau\|_{\gamma(\Omega \setminus \Sigma; \phi_\mu)} + \|\nu\|_{\gamma(\partial \Omega \cup \Sigma)}),
\]
for some positive constant \( C_2 = C_2(\Omega, \Sigma, N, \mu) \).

Thus by [17, Theorem 1.2.2], \( \{u_n\} \) is uniformly bounded in \( W^{1,\kappa}(D) \) for any \( 1 < \kappa < \frac{N}{N-1} \) and any open set \( D \Subset \Omega \setminus \Sigma \). Therefore, there exist \( u \in W^{1,\kappa}_{\text{loc}}(\Omega \setminus \Sigma) \) and a subsequence still denoted by \( \{u_n\} \) such that \( u_n \to u \) in \( L^\infty_{\text{loc}}(\Omega \setminus \Sigma) \) and a.e. in \( \Omega \setminus \Sigma \). By (3.12) and the dominated convergence theorem, we deduce that \( u_n \to u \) in \( L^1(\Omega; \phi_\mu) \). Taking into account that \( |\tilde{g}_n(u_n)| \leq |g(w)| + |g(v)| \), we can easily show \( \tilde{g}_n(u_n) \to \tilde{g}(u) \) in \( L^1(\Omega; \phi_\mu) \), where
\[
\tilde{g}(u(x)) = \begin{cases} 
g(w(x)) & \text{if } u(x) \geq w(x), \\
g(u(x)) & \text{if } u(x) \leq w(x) \leq v(x), \\
g(v(x)) & \text{if } u(x) \leq v(x). \end{cases}
\]

Combining all above we deduce that \( u \) is a weak solution of
\[
\begin{cases}
-L_\mu u + \tilde{g}(u) = \tau \quad \text{in } \Omega \setminus \Sigma, \\
\text{tr}(u) = \nu.
\end{cases}
\]

Since \( w \) is a supersolution of (1.4), by Lemma 3.3 there exist measures \( \tau_w \in \mathcal{M}^+(\Omega \setminus \Sigma; \phi_\mu) \) and \( \nu_w \in \mathcal{M}^+(\partial \Omega \cup \Sigma) \) such that \( w \) is a weak solution of
\[
\begin{cases}
-L_\mu w + g(w) = \tau + \tau_w \quad \text{in } \Omega \setminus \Sigma, \\
\text{tr}(w) = \nu + \nu_w.
\end{cases}
\]
From (3.19) and (3.20), we deduce
\[
\begin{cases}
-L_\mu(u - w) = -\tilde{g}(u) - g(w) - \tau_w & \text{in } \Omega \setminus \Sigma, \\
\text{tr}(u - w) = -\nu_w.
\end{cases}
\tag{3.21}
\]
Applying (2.28) for (3.21) yields
\[
-\int_\Omega (u - w) + L_\mu \zeta \, dx \leq -\int_\Omega \text{sign}^+(u - w)(\tilde{g}(u) - g(w)) \zeta \, dx \quad \forall \zeta \in X_\mu(\Omega \setminus \Sigma).
\]
By taking $\zeta = \phi_\mu$ and taking into account the definition of $\tilde{g}(u)$ in (3.18), we derive that $\int_\Omega (u - w)^+ \phi_\mu \, dx \leq 0$, which implies $u \leq w$.

Similarly we can show $u \geq v$ in $\Omega \setminus \Sigma$. Therefore, $\tilde{g}(u) = g(u)$ and thus $u$ is a weak solution of (1.4). \hfill \Box

### 3.2 Sufficient conditions for existence

We first prove Theorem 1.3.

**Proof of Theorem 1.3.** Put $U_1 = -\mathcal{G}_{\mu}[\tau^-] - \kappa_{\mu}[\nu^-]$ and $U_2 = \mathcal{G}_{\mu}[\tau^+] + \kappa_{\mu}[\nu^+]$. By Theorems 2.8–2.9 and (2.33) (with $D = \Omega \setminus \Sigma$ and $\varphi = \phi_\mu$), $U_1, U_2 \in L^1(\Omega; \phi_\mu)$, and by the assumption, $g(U_1), g(U_2) \in L^1(\Omega; \phi_\mu)$. Moreover, we see that $U_1$ and $U_2$ are subsolution and supersolution of (1.4) respectively. Therefore, by Theorem 3.4, there exists a unique solution $u$ of (1.4) which satisfies (1.9). The proof is complete. \hfill \Box

In order to prove Theorem 1.4, we need the following result.

**Lemma 3.5** [13, Lemma 5.1]. Assume
\[
\int_1^\infty s^{-q-1}(\ln s)^m(g(s) - g(-s)) \, ds < \infty
\tag{3.22}
\]
for $q, m \in \mathbb{R}$, $q > 1$, and $m \geq 0$. Let $v$ be a function defined in $\Omega \setminus \Sigma$. For $s > 0$, set
\[
E_s(v) := \{ x \in \Omega \setminus \Sigma : |v(x)| > s \} \quad \text{and} \quad e(s) := \int_{E_s(v)} \phi_\mu \, dx.
\]
Assume that there exists a positive constant $C_0$ such that
\[
e(s) \leq C_0 s^{-q}(\ln s)^m, \quad \forall s > e^{\frac{2m}{q}}.
\tag{3.23}
\]
Then for any $s_0 > e^{\frac{2m}{q}}$, there hold
\[
\| g(|v|) \|_{L^1(\Omega; \phi_\mu)} \leq \int_{(\Omega \setminus \Sigma) \setminus E_{s_0}(v)} g(|v|) \phi_\mu \, dx + C_0 q \int_{s_0}^{\infty} s^{-q-1}(\ln s)^m g(s) \, ds, \tag{3.24}
\]
\[
\| g(-|v|) \|_{L^1(\Omega; \phi_\mu)} \leq -\int_{(\Omega \setminus \Sigma) \setminus E_{s_0}(v)} g(-|v|) \phi_\mu \, dx - C_0 q \int_{s_0}^{\infty} s^{-q-1}(\ln s)^m g(-s) \, ds.
\tag{3.25}
\]
We are ready to demonstrate Theorem 1.4 and Theorem 1.6.

Proof of Theorem 1.4. Let \( U_1 \) and \( U_2 \) as in Theorem 1.3. Then by Theorem 2.8 and Theorem 2.9, \( U_1, U_2 \in L^p_{\alpha-}(\Omega \setminus \Sigma; \phi_\mu) \) (recall that \( \alpha_- \) is defined in (2.35)). Applying Lemma 3.5 for \( q = \frac{N+1}{N-1} \) and \( m = 0 \), we deduce \( g(U_1), g(U_2) \in L^1(\Omega; \phi_\mu) \). Finally, due to Theorem 1.3, there exists a unique solution \( u \) of (1.4) which satisfies (1.9). The proof is complete.

\[ \square \]

4 | BOUNDARY DATA CONCENTRATED IN \( \partial \Omega \)

In this section, we consider the following problem:

\[
\begin{cases}
-\mu u + g(u) = 0 & \text{in } \Omega \setminus \Sigma, \\
\text{tr}(u) = \nu,
\end{cases}
\]

where \( \nu \) is concentrated in \( \partial \Omega \).

4.1 | Poisson kernel and \( L_\mu \)-harmonic measure on \( \partial \Omega \)

The following result asserts the existence of the Poisson kernel and its properties.

Proposition 4.1. For any \( x \in \Omega \setminus \Sigma \), \( G_\mu(x, \cdot) \in C^{1, \gamma}(\Omega \setminus (\Sigma \cup \{x\})) \cap C^2(\Omega \setminus (\Sigma \cup \{x\})) \) for all \( \gamma \in (0, 1) \). Let \( P_\mu \) be the Poisson kernel defined by

\[
P_\mu(x, y) := -\frac{\partial G_\mu(x, y)}{\partial n}, \quad x \in \Omega \setminus \Sigma, \ y \in \partial \Omega,
\]

where \( n \) is the unit outer normal vector of \( \partial \Omega \). Let \( x_0 \in \Omega \setminus \Sigma \) be a fixed reference point.

(i) There holds

\[
P_\mu(x, y) = P_\mu(x_0, y)K_\mu(x, y) \approx \frac{d(x)d_\Sigma(x)^{-\alpha_-}}{|x - y|^N}, \quad x \in \Omega \setminus \Sigma, \ y \in \partial \Omega.
\]

(ii) For any \( h \in L^1(\partial \Omega \cup \Sigma; d\omega^{x_0}_{\Omega \setminus \Sigma}) \) with compact support in \( \partial \Omega \), there holds

\[
\int_{\partial \Omega} h(y) d\omega^{x_0}_{\Omega \setminus \Sigma}(y) = P_\mu[h](x_0).
\]

Here

\[
P_\mu[h](x) = \int_{\partial \Omega} P_\mu(x, y)h(y) dS(y).
\]

where \( S \) is the \((N - 1)\)-dimensional surface measure on \( \partial \Omega \).
Consequently, if \( h \in L^1(\partial \Omega \cup \Sigma; d\omega_x) \) with compact support in \( \partial \Omega \), then \( h \in L^1(\partial \Omega) \). In particular, for any Borel set \( E \subset \partial \Omega \), there holds

\[
\omega_{x_0}^\Sigma(E) = P_\mu[1_E](x_0). \tag{4.6}
\]

Proof. For any \( x \in \Omega \setminus \Sigma \), the regularity of \( G_\mu(x, \cdot) \) follows from the standard elliptic theory. Also, we note that \( P_\mu(\cdot, y) \) is \( L_\mu \)-harmonic in \( \Omega \setminus \Sigma \) and

\[
\lim_{x \in \Omega, x \to \xi} \frac{P_\mu(x, y)}{W(x)} = 0 \quad \forall \xi \in \partial \Omega \cup \Sigma \setminus \{y\}.
\]

By the uniqueness of kernel functions with pole at \( y \) and basis at \( x_0 \) [14, Proposition 6.6], we deduce the first equality in (4.3). This, together with (2.13), implies the asymptotic behavior of the Poisson kernel in (4.3).

Now, let \( \{\Sigma_n\} \) be a decreasing sequence of compact smooth domains as in (2.19). We denote by \( \phi_* \) the unique solution of

\[
\begin{cases}
-L_\mu u = 0 & \text{in } \Omega \setminus \Sigma \\
u = 1 & \text{on } \partial \Omega \\
u = 0 & \text{on } \Sigma.
\end{cases}
\]

Then by [14, Lemma 5.6], there exist constants \( c_1 = c_1(\Omega, \Sigma, \Sigma_n, \mu) \) and \( c_2 = c_2(\Omega, \Sigma, N, \mu) \) such that \( 0 < c_1 \leq \phi_*(x) \leq c_2 d_\Sigma(x)^{-\alpha^+} \) for all \( x \in \Omega \setminus \Sigma_n \). By the standard elliptic theory, \( \phi_* \in C^2(\Omega \setminus \Sigma) \cap C^{1,\gamma}(\Omega \setminus \Sigma) \) for any \( 0 < \gamma < 1 \).

Let \( \tilde{\xi} \in C(\partial \Omega \cup \Sigma_n) \), we consider the problem

\[
\begin{cases}
-L_\mu v = 0 & \text{in } \Omega \setminus \Sigma_n \\
v = \tilde{\xi} & \text{on } \partial \Omega \cup \partial \Sigma_n.
\end{cases} \tag{4.7}
\]

We observe that \( v \) satisfies (4.7) if and only if \( w = v/\phi_* \) satisfies

\[
\begin{cases}
-\text{div}(\phi_*^2 \nabla w) = 0 & \text{in } \Omega \setminus \Sigma_n \\
w = \frac{\tilde{\xi}}{\phi_*} & \text{on } \partial \Omega \cup \partial \Sigma_n.
\end{cases} \tag{4.8}
\]

We note that for any \( \tilde{\xi} \in C(\partial \Omega \cup \partial \Sigma_n) \), there exists a unique solution of (4.8). From the above observation, we deduce that there exists a unique solution of (4.7). Thus, for any \( n \) and \( x \in \Omega \setminus \Sigma \), there exists \( L_\mu \)-harmonic measure \( \omega_n^\xi \) on \( \partial \Omega \cup \partial \Sigma_n \). Denote by \( v_n \) the solution of (4.7), then

\[
v_n(x) = \int_{\partial \Omega \cup \partial \Sigma_n} \tilde{\xi}(y) d\omega_n^x(y). \tag{4.9}
\]

For any \( \xi \in C(\partial \Omega) \), we set \( \tilde{\xi} = \xi \) if \( x \in \partial \Omega \) and \( \tilde{\xi} = 0 \) otherwise. In view of the proof of [14, Proposition 6.12] and (4.9), we may deduce that \( v_n(x) \to v(x) = \int_{\partial \Omega \cup \Sigma} \xi(y) d\omega_{\Omega \setminus \Sigma}(y). \)
On the other hand, for any \( n \in \mathbb{N} \), the Green function of \(-L_\mu\) in \( \Omega \setminus \Sigma_n \) exists, denoted by \( G^n_\mu \). We see that \( G^n_\mu(x, y) \neq G_\mu(x, y) \) for any \( x \neq y \) and \( x, y \in \Omega \setminus \Sigma \).

Denote the Poisson kernel of \(-L_\mu\) in \( \Omega \setminus \Sigma_n \) by

\[
P^n_\mu(x, y) = -\frac{\partial G^n_\mu(x, y)}{\partial n}, \quad x \in \Omega \setminus \Sigma_n, y \in \partial \Omega \cup \partial \Sigma_n,
\]

where \( n \) is the unit outer normal vector of \( \partial \Omega \cup \partial \Sigma_n \). Then we have the representation

\[
u_n(x) = \int_{\partial \Omega \cup \partial \Sigma_n} P^n_\mu(x, y) \xi(y) dS(y), \quad (4.10)
\]

where \( S \) is the \((N - 1)\)-dimensional surface measure on \( \partial \Omega \cup \partial \Sigma_n \). From (4.9) and (4.10) and using the fact that \( \xi \) has compact support in \( \partial \Omega \), we obtain

\[
\int_{\partial \Omega} \xi(y) d\omega^n_\mu(x) = \int_{\partial \Omega} P^n_\mu(x, y) \xi(y) dS(y).
\]

Put \( \beta = \frac{1}{2} \min\{d(x), \text{dist}(\partial \Omega, \Sigma)\} \). Let \( \Omega_\beta = \{x \in \Omega : d(x) < \beta\} \). Then \( \{G^n_\mu(x, \cdot)\}_n \) is uniformly bounded with respect to \( W^{2,\infty}(\Omega_\beta) \)-norm for any \( \kappa > 1 \). Thus, by compact embedding, there exists a subsequence, still denoted by \( \{G^n_\mu(x, \cdot)\}_n \), which converges to \( G_\mu(x, \cdot) \) in \( C^1(\Omega_\beta) \) as \( n \to \infty \). In particular \( P^n_\mu(x, \cdot) \to P_\mu(x, \cdot) \) uniformly on \( \partial \Omega \) as \( n \to \infty \).

Therefore, by letting \( n \to \infty \) in (4.11), we obtain

\[
\int_{\partial \Omega} \xi(y) d\omega^n_\mu(x) = \lim_{n \to \infty} \int_{\partial \Omega} \xi(y) d\omega^n_\mu(x) = \lim_{n \to \infty} \int_{\partial \Omega} P^n_\mu(x, y) \xi(y) dS(y) = \int_{\partial \Omega} P_\mu(x, y) \xi(y) dS(y).
\]

Since \( \inf_{y \in \partial \Omega} P_\mu(x_0, y) > 0 \) and (4.12) holds for any \( \xi \in C(\partial \Omega) \), we have that (4.6) is valid, which implies (4.4). The proof is complete. \( \square \)

**Proposition 4.2.**

(i) For any \( h \in L^1(\partial \Omega \cup \Sigma; d\omega^{x_0}_{\Omega \setminus \Sigma}) \) with support on \( \partial \Omega \), there holds

\[
- \int_{\Omega} K_\mu[h d\omega^{x_0}_{\Omega \setminus \Sigma}] L_\mu \eta dx = - \int_{\partial \Omega} \frac{\partial \eta}{\partial n}(y) h(y) dS(y), \quad \forall \eta \in X_\mu(\Omega \setminus \Sigma).
\]

(ii) For any \( \nu \in M(\partial \Omega \cup \Sigma) \) with support on \( \partial \Omega \), there holds

\[
- \int_{\Omega} K_\mu[\nu] L_\mu \eta dx = - \int_{\partial \Omega} \frac{\partial \eta}{\partial n}(y) \frac{1}{P_\mu(x_0, y)} d\nu(y), \quad \forall \eta \in X_\mu(\Omega \setminus \Sigma),\]

where \( P_\mu(x_0, y) \) is defined in (4.2) and \( X_\mu(\Omega \setminus \Sigma) \) is defined by (1.7).

**Proof.**

(i) Let \( \{\Sigma_n\} \) be as in (2.19). Let \( \eta \in X_\mu(\Omega \setminus \Sigma) \), \( \zeta \in C(\partial \Omega \cup \partial \Sigma_n) \) with compact support in \( \partial \Omega \) and \( \nu_n \) be the solution of (4.7).
In view of the proof of Proposition 4.1, \( v_n \in C(\Omega \setminus \Sigma_n) \) and

\[
v_n(x) = \int_{\partial \Omega} \zeta(y) \, d\omega_n^x(y) = \int_{\partial \Omega} P_n^\mu(x,y) \zeta(y) \, dS(y).
\]

Put

\[
v(x) = \int_{\partial \Omega} \zeta(y) \, d\omega^x(y) \quad \text{and} \quad w(x) = \int_{\partial \Omega} |\zeta(y)| \, d\omega^x(y).
\]

Then \( v_n(x) \to v(x) \) and \( |v_n(x)| \leq w(x) \). By [17, Proposition 1.3.7],

\[- \int_{\Omega \setminus \Sigma_n} v_n L_\mu Z \, dx = - \int_{\partial \Omega} \zeta \frac{\partial Z}{\partial n} \, dS, \quad \forall Z \in C^2(\Omega \setminus \Sigma_n).\]

By approximation, the above equality is valid for any \( Z \in C^1,\gamma(\Omega \setminus \Sigma_n) \), for some \( \gamma \in (0,1) \) and \( \Delta Z \in L^\infty \). Hence, we may choose \( Z = \eta_n \), where \( \eta_n \) satisfies

\[
\begin{cases}
-L_\mu \eta_n = -L_\mu \eta & \text{in } \Omega \setminus \Sigma_n \\
\eta_n = 0 & \text{on } \partial \Omega \cup \partial \Sigma_n,
\end{cases}
\]

we obtain

\[- \int_{\Omega \setminus \Sigma_n} v_n L_\mu \eta_n \, dx = - \int_{\partial \Omega} \zeta \frac{\partial \eta_n}{\partial n} \, dS.\]

We note that \( \eta_n \to \eta \) a.e. in \( \Omega \setminus \Sigma \) and in \( C^1(\Omega \setminus \Sigma_1) \). Therefore by the dominated convergence theorem, we obtain

\[- \int_{\Omega} v L_\mu \eta \, dx = - \int_{\partial \Omega} \zeta \frac{\partial \eta}{\partial n} \, dS. \quad (4.15)\]

Now let \( h \in L^1(\partial \Omega \setminus \Sigma; d\omega_0^x_{\Omega \setminus \Sigma}) \) with support on \( \partial \Omega \) and \{\( h_n \)\} be a sequence of functions in \( C(\partial \Omega \setminus \Sigma) \) with support on \( \partial \Omega \) such that \( h_n \to h \) in \( L^1(\partial \Omega \cup \Sigma; d\omega_0^x_{\Omega \setminus \Sigma}) \), that is,

\[
\lim_{n \to \infty} \int_{\partial \Omega} |h_n(y) - h(y)| \, d\omega_0^x_{\Omega \setminus \Sigma}(y) = 0. \quad (4.16)
\]

This, together with (4.4) with \( h \) replaced by \( |h_n - h| \) and the fact \( P_\mu(x_0, \cdot) \in C(\partial \Omega) \), yields

\[
\lim_{n \to \infty} \int_{\partial \Omega} P_\mu(x_0,y) |h_n(y) - h(y)| \, dS(y) = \lim_{n \to \infty} \int_{\partial \Omega} |h_n(y) - h(y)| \, d\omega_0^x_{\Omega \setminus \Sigma}(y) = 0.
\]

As a consequence, \( h_n \to h \) in \( L^1(\partial \Omega) \) due to the fact that \( \inf_{y \in \partial \Omega} P_\mu(x_0, y) > 0 \).

Put

\[
u_n(x) = \int_{\partial \Omega} K_\mu(x,y) h_n(y) \, d\omega_0^x_{\Omega \setminus \Sigma}(y), \quad x \in \Omega \setminus \Sigma.
\]

By (4.16) and the fact that \( K_\mu(\cdot, y) \) is bounded in any compact subset of \( \Omega \setminus \Sigma \) (the bound depends on the distance from the compact subset to \( \partial \Omega \) and \( \Sigma \)), we deduce that \( u_n \to u \) locally
uniformly in $\Omega \setminus \Sigma$, where

$$u(x) = \int_{\partial \Omega} K_\mu(x,y) h(y) \, d\omega_{\Omega \setminus \Sigma}(y).$$

Therefore, up to a subsequence, $u_n \to u$ in $\Omega \setminus \Sigma$.

Again, since $K_\mu(x, \cdot), h_n \in C(\partial \Omega)$, by (4.4), we derive

$$u_n(x) = \int_{\partial \Omega} K_\mu(x,y) P_\mu(x_0,y) h_n(y) \, dS(y).$$

By Theorem 2.9 and (2.33) and the fact that $0 < \max_{y \in \partial \Omega} P_\mu(x_0,y) < \infty$ and $\|h_n\|_{L^1(\partial \Omega)} \leq C\|h\|_{L^1(\partial \Omega)}$, we deduce that for any $1 < \kappa < \frac{N+1}{N-1}$, there exists a positive constant $C = C(N, \Omega, \Sigma, \mu, \kappa)$ such that $\|u_n\|_{L^\kappa(\Omega; P_\mu)} \leq C\|h\|_{L^1(\partial \Omega)}$ for all $n \in \mathbb{N}$. This in turn implies that \{u_n\} is equi-integrable in $L^1(\Omega; \phi_\mu)$. Therefore, by Vitali’s convergence theorem, up to a subsequence, $u_n \to u$ in $L^1(\Omega; \phi_\mu)$.

Next applying (4.15) with $v = u_n$ and $\zeta = h_n$, we obtain

$$-\int_{\Omega} u_n L_\mu \eta \, dx = -\int_{\partial \Omega} h_n \frac{\partial \eta}{\partial n} \, dS. \quad (4.17)$$

Since $u_n \to u$ in $L^1(\Omega; \phi_\mu)$, $h_n \to h$ in $L^1(\partial \Omega)$ and $|\frac{\partial \eta}{\partial n}|$ is bounded on $\partial \Omega$, by letting $n \to \infty$ in (4.17), we conclude (4.13).

(ii) Let \{h_n\} be a sequence in $C(\partial \Omega)$ converging weakly to $\nu$, that is,

$$\int_{\partial \Omega} \zeta h_n \, dS \to \int_{\partial \Omega} \zeta \, d\nu \quad \forall \zeta \in C(\partial \Omega), \quad (4.18)$$

and $\|h_n\|_{L^1(\partial \Omega)} \leq C\|\nu\|_{\mathcal{H}(\partial \Omega)}$ for every $n \geq 1$. Put

$$u_n(x) = \int_{\partial \Omega} K_\mu(x,y) \frac{h_n(y)}{P_\mu(x_0,y)} \, d\omega_{\Omega \setminus \Sigma}(y).$$

Since $P_\mu(x_0, \cdot), K_\mu(x, \cdot) \in C(\partial \Omega)$ and $\inf_{y \in \partial \Omega} P_\mu(x_0,y) > 0$, by (4.4) and (4.18), we have

$$u_n(x) = \int_{\partial \Omega} K_\mu(x,y) h_n(y) \, dS(y) \to \int_{\partial \Omega} K_\mu(x,y) \, d\nu(y) = u(x).$$

Therefore $u_n \to u$ a.e. in $\Omega \setminus \Sigma$.

On the other hand, by Theorem 2.9 and (2.33), for any $1 < \kappa < \frac{N+1}{N-1}$, there exists a positive constant $C = C(N, \Omega, \Sigma, \mu, \kappa)$ such that $\|u_n\|_{L^\kappa(\Omega; P_\mu)} \leq C\|\nu\|_{\mathcal{H}(\partial \Omega)}$. By a similar argument as in the proof of (i), we can show $u_n \to u$ in $L^1(\Omega; \phi_\mu)$. Hence by applying (4.15) with $v = u_n$ and $\zeta = h_n/P_\mu(x_0, \cdot)$, and then letting $n \to \infty$, we conclude (4.14). \qed

### 4.2 | Existence and uniqueness

We start with a result on the solvability in $L^1$ setting.
Theorem 4.3. Assume $\mu \leq H^2$ and $h \in L^1(\partial \Omega \cup \Sigma; d\omega^{x_0}_{\Omega \cup \Sigma})$ with compact support in $\partial \Omega$. Then there exists a unique weak solution of (1.11) and $d\nu = h \, d\omega^{x_0}_{\Omega \cup \Sigma}$. Furthermore, there holds

$$- \int_{\Omega} u \mathcal{L}_\mu \eta \, dx + \int_{\Omega} g(u) \eta \, dx = - \int_{\partial \Omega} \frac{\partial \eta}{\partial n}(y) h(y) \, dS(y), \quad \forall \eta \in X_\mu(\Omega \setminus \Sigma)$$

(4.19)

and

$$u + \mathcal{G}_\mu [g(u)] = \kappa_\mu [h \, d\omega^{x_0}_{\Omega \cup \Sigma}] = \mathcal{P}_\mu [h],$$

(4.20)

where $\mathcal{P}_\mu (x, y)$ is defined in (4.5).

Proof. The uniqueness is obtained by a similar argument as in the proof of Theorem 3.4.

Next we prove the existence. First we assume $h \in C(\partial \Omega)$ and $h \geq 0$ on $\partial \Omega$. Let $g_n$ be the function defined in (3.10) then $g_n \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})$. Put $v_h = \mathcal{K}_\mu [h \, d\omega^{x_0}_{\Omega \cup \Sigma}]$, by Theorem 2.9 and (2.33), $v_h \in L^1(\Omega; \phi \mu)$. Moreover, by Proposition 4.1 and Proposition 2.2, for $x \in \Omega \setminus \Sigma$,

$$0 \leq v_h(x) = \int_{\partial \Omega} K_\mu(x, y) P_\mu(x_0, y) h(y) \, dS(y) \lesssim \|h\|_{L^\infty(\partial \Omega)} dS(x) - \int_{\partial \Omega} d(y) |x - y|^\alpha \, dS(y) \lesssim d_2(x)^{-\alpha}.$$  

(4.21)

Since $v_h$ and 0 are supersolution and subsolution of (4.1) with $g = g_n$ and $d\nu = h \, d\omega^{x_0}_{\Omega \cup \Sigma}$ and 0, respectively, by Theorem 3.4, there exists a unique weak solution $u_n \in L^1(\Omega; \phi \mu)$ of

$$\begin{cases} 
- L_\mu u + g_n(u) = 0 & \text{in } \Omega \setminus \Sigma, \\
\text{tr}(u) = h \, d\omega^{x_0}_{\Omega \cup \Sigma}, 
\end{cases}$$

(4.22)

such that $0 \leq u_n \leq v_h$ in $\Omega \setminus \Sigma$. By Proposition 4.2(i), $u_n$ satisfies

$$- \int_{\Omega} u_n L_\mu \eta \, dx + \int_{\Omega} g_n(u_n) \eta \, dx = - \int_{\Omega} v_h L_\mu \eta \, dx = - \int_{\partial \Omega} \frac{\partial \eta}{\partial n} h \, dS, \quad \forall \eta \in X_\mu(\Omega \setminus \Sigma).$$

(4.23)

By applying (2.27) with $\zeta = \phi \mu$, $f = -g_n(u_n)$, $\rho = 0$, $d\nu = h \, d\omega^{x_0}_{\Omega \cup \Sigma}$, and using Theorem 2.9 and (2.33), we assert that

$$\|u_n\|_{L^1(\Omega; \phi \mu)} + \|g_n(u_n)\|_{L^1(\Omega; \phi \mu)} \lesssim \|h\|_{L^1(\partial \Omega \cup \Sigma; d\omega^{x_0}_{\Omega \cup \Sigma})}.$$  

(4.24)

Owing to standard local regularity, $\{u_n\}$ is uniformly bounded in $W^{1, \kappa}(D)$ for any $1 < \kappa < \frac{N}{N-1}$ and any open $D \subset \Omega \setminus \Sigma$. By a compact embedding, there exist a subsequence, say $\{u_n\}$, and a nonnegative function $u$ such that $u_n \rightrightarrows u$ a.e. in $\Omega \setminus \Sigma$. Since $|u_n| \leq v_h \in L^1(\Omega; \phi \mu)$, by the dominated convergence theorem, we have that $u_n \rightrightarrows u \in L^1(\Omega; \phi \mu)$. We also note that $g_n(u_n) \rightrightarrows g(u)$ and $0 \leq g_n(u_n) \leq g(v_h)$ a.e. in $\Omega \setminus \Sigma$. From (4.21), we see that $g(v_h) \in L^1(\Omega \setminus \Sigma; \phi \mu)$ for every $\beta \in (0, \beta_0)$. Therefore, by the dominated convergence theorem, we derive $g_n(u_n) \rightrightarrows g(u)$ in $L^1(\Omega \setminus \Sigma; \phi \mu)$ for every $\beta \in (0, \beta_0)$. By (4.24) and Fatou’s lemma, $g(u) \in L^1(\Omega; \phi \mu)$. In addition, by letting $n \to \infty$ in (4.23), we derive that (4.19) holds true for all $\eta \in X_\mu(\Omega \setminus \Sigma)$ with $\text{supp } \eta \subset \overline{\Omega} \setminus \Sigma$. 
We note that $u + G_\mu[g(u)]$ is a nonnegative $L_\mu$-harmonic function in $\Omega \setminus \Sigma$, hence by Theorem 2.3, there exists a unique measure $\nu \in \mathcal{M}^+(\partial \Omega \cup \Sigma)$ such that

$$u + G_\mu[g(u)] = \kappa_\mu[\nu].$$

(4.25)

This, combined with the fact that $g(u) \in L^1(\Omega; \phi_\mu)$ and Proposition 2.5, implies $\text{tr}(u) = \nu$.

By choosing $\phi \in C(\bar{\Omega})$ such that $0 \leq \phi \leq 1$ in $\Omega$, $\phi = 0$ in $\Omega_{\rho_0}$ and $\phi = 1$ in $\Sigma_{\rho_0}$ in Definition 2.4, we deduce

$$\lim_{n \to \infty} \int_{\partial \Sigma_n} u \, d\omega_{O_n} = \int_{\Sigma} d\nu = \nu(\Sigma).$$

(4.26)

Here we choose the sequence $\{\Sigma_n\}$ such that $\text{dist}(\Sigma_n, \Sigma) = \frac{1}{n}$.

Next we show that $\nu$ has compact support in $\partial \Omega$. Suppose by contradiction $\nu(\Sigma) > 0$. If $\mu < H^2$, then from the estimate $u(x) \leq v_h(x) \leq C d_\Sigma(x)^{-\alpha-}$ for any $x \in \Omega \setminus \Sigma$, the definition of $\bar{W}$ in (1.14) and [14, Proposition 6.12] (with $\phi$ chosen as above) and (4.26), we have

$$\int_{\Sigma} d\omega_{\Omega \setminus \Sigma}(x) = \lim_{n \to \infty} \int_{\partial \Sigma_n} d_\Sigma(x)^{-\alpha_+} \, d\omega_{O_n}^{x_0}(x)$$

$$= \lim_{n \to \infty} n^{\alpha_+ - \alpha_-} \int_{\partial \Sigma_n} d_\Sigma(x)^{-\alpha_-} \, d\omega_{O_n}^{x_0}(x)$$

$$\geq \lim_{n \to \infty} n^{\alpha_+ - \alpha_-} \int_{\partial \Sigma_n} u(x) \, d\omega_{O_n}^{x_0}(x) = +\infty,$$

which yields a contradiction since $\omega_{\Omega \setminus \Sigma}^{x_0} \in \mathcal{M}^+(\partial \Omega \cup \Sigma)$ (note that $\alpha_+ - \alpha_- > 0$). If $\mu = H^2$, then by a similar argument, we obtain

$$\int_{\Sigma} d\omega_{\Omega \setminus \Sigma}(x) = \lim_{n \to \infty} \int_{\partial \Sigma_n} d_\Sigma(x)^{-H} |\ln d_\Sigma(x)| \, d\omega_{O_n}^{x_0}(x)$$

$$= \lim_{n \to \infty} \ln(n) \int_{\partial \Sigma_n} d_\Sigma(x)^{-H} \, d\omega_{O_n}^{x_0}(x)$$

$$\geq \lim_{n \to \infty} \ln(n) \nu(\Sigma) = +\infty,$$

which is a contradiction. Therefore, $\nu$ has compact support in $\partial \Omega$.

Since $u$ satisfies (4.25), by using Proposition 4.2(ii), we obtain

$$- \int_{\Omega} uL_\mu \eta \, dx + \int_{\Omega} g(u) \eta \, dx = - \int_{\Omega} \kappa_\mu[\nu]L_\mu \eta \, dx = - \int_{\partial \Omega} \frac{\partial \eta}{\partial n}(y) \frac{1}{P_\mu(x_0, y)} \, d\nu(y),$$

(4.27)

for all $\eta \in X_\mu(\Omega \setminus \Sigma)$. Combining (4.19) (which holds for all $\eta \in X_\mu(\Omega \setminus \Sigma)$ with supp $\eta \in \bar{\Omega} \setminus \Sigma$) and (4.27) yields

$$- \int_{\partial \Omega} \frac{\partial \eta}{\partial n}(y) \frac{1}{P_\mu(x_0, y)} \, d\nu(y) = - \int_{\partial \Omega} \frac{\partial \eta}{\partial n}(y) h(y) \, dS(y),$$

(4.28)

for all $\eta \in X_\mu(\Omega \setminus \Sigma)$ with supp $\eta \in \bar{\Omega} \setminus \Sigma$. 
Let $\eta \in X_\mu(\Omega \setminus \Sigma)$ and $\phi$ be the cut-off function above (4.26). Using the test function $\bar{\eta} = (1 - \phi)\eta$ in (4.28), we can show that (4.28) holds for all $\eta \in X_\mu(\Omega \setminus \Sigma)$. This in turn implies that (4.19) holds for any $\eta \in X_\mu(\Omega \setminus \Sigma)$. Combining (4.19) and Proposition 4.2(i), we deduce that

$$-\int_{\Omega} u L_\mu \eta \, dx + \int_{\Omega} g(u) \eta \, dx = -\int_{\Omega} \kappa_\mu [hd\omega^x_0] L_\mu \eta \, dx,$$

which means $u$ is a weak solution of (4.1) with $d\nu = hd\omega^x_0$. 

Next we still assume $h \in C(\partial\Omega)$, but drop the assumption that $h \geq 0$ on $\partial\Omega$. Let $u_n$ and $\bar{u}_n$ are weak solutions of (4.22) with boundary datum $hd\omega^x_0$ and $h|d\omega^x_0$, respectively. Then by (2.28), $|u_n| \leq \bar{u}_n$ in $\Omega \setminus \Sigma$. Moreover, by local regularity results, $\{u_n\}$ is uniformly bounded in $W^{1,\kappa}(D)$ for any $1 < \kappa < \frac{N}{N-1}$ and $D \Subset \Omega \setminus \Sigma$. By the compact embedding, up to a subsequence, $u_n \to u$ a.e. in $\Omega \setminus \Sigma$ as a consequence, $g_n(u_n) \to g(u)$ a.e. in $\Omega \setminus \Sigma$ and $|g_n(u_n)| \leq g_n(\bar{u}_n) - g_n(-\bar{u}_n)$ a.e. in $\Omega \setminus \Sigma$. Therefore, $u_n \to u$ and $g_n(u_n) \to g(u)$ in $L^1(\Omega; \phi_\mu)$. Consequently $u$ is a weak solution of (4.1) with $d\nu = hd\omega^x_0$. 

Formula (4.19) follows from formula (1.6) with $d\nu = hd\omega^x_0$ and Proposition 4.2(i). 

The first equality in (4.20) follows from (2.25) with $d\nu = hd\omega^x_0$. The second equality in (4.20) follows from Proposition 4.3.

Proof of Theorem 1.5. Put $U_1 = -\kappa_\mu [v^-]$ and $U_2 = \kappa_\mu [v^+]$. Then by Theorem 2.9, $U_1, U_2 \in L^1(\Omega; \phi_\mu)$. Moreover, from Theorem 2.9 and Lemma 3.5 with $m = 0$ and $q = \frac{N+1}{N-1}$, we have $g(U_1), g(U_2) \in L^1(\Omega; \phi_\mu)$. We also note that $U_1$ and $U_2$ are subsolution and supersolution with $U_1 \leq 0 \leq U_2$. By applying Theorem 3.4, we deduce that there exists a unique weak solution $u$ of (4.1) which satisfies (1.12).

5 \quad BOUNDARY DATA CONCENTRATED IN $\Sigma$

In this section, we consider the case where the measure data are concentrated in $\Sigma$. Below is a regularity result in weak Lebesgue spaces.

Lemma 5.1. Assume $1 \leq k < N - 2$ and $S_\Sigma$ is the $k$-dimensional surface measure on $\Sigma$.

(i) If $\mu < H^2$, then $\kappa_\mu [S_\Sigma] \approx d(x)dS_\Sigma^{\alpha_+} \in L^\frac{N-k-\alpha}{N-1} (\Omega \setminus \Sigma; \phi_\mu).$
(ii) If \( \mu = H^2 \), then \( \mathcal{K}_\mu[S_\Sigma] \approx d(x)d^{-H}_\Sigma | \ln \frac{d(x)}{d\Sigma} | \mathcal{I}_\Omega \mathcal{S} \gamma \), for all \( 1 < \theta < \frac{N-k+2}{N-k-2} \), where \( D_\Omega = 2 \sup_{x \in \Omega} |x| \). In addition, for \( \lambda > 0 \), set

\[
\tilde{A}_\lambda(0) := \{ x \in \Omega \setminus \{0\} : \mathcal{K}_\mu[S_\Sigma](x) > \lambda \}, \quad \tilde{m}_\lambda := \int_{\tilde{A}_\lambda(0)} d(x)|x|^{-\frac{N-2}{2}} \, dx.
\]

Then

\[
\tilde{m}_\lambda \lesssim (\lambda^{-1} \ln \lambda)^{\frac{N+k+2}{N+k-2}}, \quad \forall \lambda > e.
\]

The implicit constant depends on \( N, \Omega, \Sigma, \mu, \) and \( \theta \).

**Proof.** By (2.13), we have, for \( x \in \Omega \setminus \Sigma \),

\[
\mathcal{K}_\mu[S_\Sigma](x) = \int_{\Sigma} K_\mu(x,y) dS_\Sigma(y) \approx d(x)d_\Sigma(x)^{-\alpha} \int_{\Sigma} |x-y|^{-\gamma} dS_\Sigma(y).
\]

(i) If \( \mu < H^2 \), then \( \alpha_- < H \). From (2.5), (2.6), and (5.1), we obtain \( \mathcal{K}_\mu[S_\Sigma] \approx d(x)d_\Sigma^{-\alpha} \) in \( \Omega \setminus \Sigma \).

Then we can proceed as in the proof of [13, Theorem 3.5(i)] to derive \( \mathcal{K}_\mu[S_\Sigma] \in L^w_{\alpha-\alpha+}(\Omega \setminus \Sigma; \phi_\mu) \).

(ii) If \( \mu = H^2 \), then \( \alpha_- = H \). By (2.5), (2.6), and (5.1), we obtain \( \mathcal{K}_\mu[S_\Sigma] \approx d(x)d_\Sigma^{-H} | \ln \frac{d(x)}{D_\Omega} | \). Then by proceeding as in the proof of [13, Theorem 3.6], we can derive the desired result. \( \square \)

**Theorem 5.2.**

(i) Assume \( \mu < H^2 \) and \( g \) satisfies (3.22) with \( q = \frac{N-k-\alpha_-}{\alpha_+} \) and \( m = 0 \). Then for any \( h \in L^1(\partial \Omega \cup \Sigma; dS_\Sigma) \) with compact support in \( \Sigma \), problem (4.1) with \( d\nu = h dS_\Sigma \) admits a unique weak solution.

(ii) Assume \( \mu = H^2 \) and \( g \) satisfies (3.22) with \( q = m = \frac{N+k+2}{N-k-2} \). Then for any \( h \in L^1(\partial \Omega \cup \Sigma; dS_\Sigma) \) with compact support in \( \Sigma \), problem (4.1) with \( d\nu = h dS_\Sigma \) admits a unique weak solution.

**Proof.** Let \( h \in L^1(\partial \Omega \cup \Sigma; dS_\Sigma) \) with compact support in \( \Sigma \). Let \( \{h_n\} \subset L^\infty(\partial \Omega \cup \Sigma) \) with compact support in \( \Sigma \) be such that \( h_n \to h \) in \( L^1(\Sigma; dS_\Sigma) \). For each \( n \), set \( U_{n,1} = -\mathcal{K}_\mu[(h_n)^-] \) and \( U_{n,2} = \mathcal{K}_\mu[(h_n)^+] \).

(i) Assume \( \mu < H^2 \) and \( g \) satisfies (3.22) with \( q = \frac{N-k-\alpha_-}{\alpha_+} \) and \( m = 0 \). For \( i = 1, 2 \), by Lemma 5.1, (2.34), and Lemma 3.5 for \( q = \frac{N-k-\alpha_-}{\alpha_+} \) and \( m = 0 \), we have \( g(U_{n,i}) \in L^1(\Omega; \phi_\mu) \), \( i = 1, 2 \). Moreover, we see that \( U_{n,1} \) and \( U_{n,2} \) are, respectively, subsolution and supersolution of (4.1) with \( \nu = h_n \) with \( U_{n,1} < U_{n,2} \) in \( \Omega \setminus \Sigma \). Therefore, by Theorem 3.4, there exists a unique solution \( u_n \) of (4.1) with \( \nu = h_n \) which satisfies \( U_{n,1} \leq u_n \leq U_{n,2} \) in \( \Omega \setminus \Sigma \). Furthermore \( |u_n|^p \in L^1(\Omega; \phi_\mu) \) and there holds

\[
-\int_{\Omega} u_n L_\mu \zeta \, dx + \int_{\Omega} |u_n|^{p-1} u_n \zeta \, dx = \int_{\Omega \setminus \Sigma} \zeta \, d\tau - \int_{\Omega} \mathcal{K}_\mu[h_n] L_\mu \zeta \, dx, \quad \forall \zeta \in \mathcal{X}_\mu(\Omega \setminus \Sigma).
\]
In addition, by using a similar argument leading to (4.29) and Proposition 5.1, we can show that there exists a positive constant $C$ such that
\[ \|u_n - u_l\|_{L^1(\Omega; \phi)} + \|g(u_n) - g(u_l)\|_{L^1(\Omega; \phi)} \leq C\|h_n - h_l\|_{L^1(\Sigma; d\sigma)}. \]
The result follows by using the above inequality and argument following (4.29).

The proof of (ii) is similar and we omit it.

Remark 5.3. Let $u$ be the unique solution of problem (4.1) with $d\nu = d\sigma$ in Theorem 5.2, then $0 \leq u \leq K_{\mu}[\Sigma]$ in $\Omega \setminus \Sigma$. This and Lemma 5.1 imply
\[ u(x) \lesssim \begin{cases} d(x)d(x)^{\alpha_+} & \text{if } \mu < H^2, \\ d(x)d(x)^{-H}\ln \frac{d(x)}{D} & \text{if } \mu = H^2, \end{cases} \quad x \in \Omega \setminus \Sigma. \]
Similarly we can show the following:

\textbf{Theorem 5.4.}

(i) Assume $\mu < H^2$ and $g$ satisfies (3.22) with $q = \frac{N-k-\alpha_-}{\alpha_+}$ and $m = 0$. Then for any $h \in L^1(\delta\Omega \cup \Sigma; \omega_{\Omega \setminus \Sigma})$ with compact support in $\Sigma$, problem (4.1) with $d\nu = h d\omega_{\Omega \setminus \Sigma}$ admits a unique weak solution.

(ii) Assume $\mu = H^2$ and $g$ satisfies (3.22) with $q = m = \frac{N+k+2}{N-k-2}$. Then for any $h \in L^1(\delta\Omega \cup \Sigma; \omega_{\Omega \setminus \Sigma})$ with compact support in $\Sigma$, problem (4.1) with $d\nu = h d\omega_{\Omega \setminus \Sigma}$ admits a unique weak solution.

\textbf{Proof.} By [14, Lemma 5.6], we have that
\[ K_{\mu}[\omega_{\Omega \setminus \Sigma}] \approx \begin{cases} d(x)d(x)^{\alpha_+} & \text{if } \mu < H^2, \\ d(x)d(x)^{-H}\ln \frac{d(x)}{D} & \text{if } \mu = H^2. \end{cases} \]

By the same arguments as in the proof of Theorem 5.2, we may deduce the desired result.

\textbf{Proof of Theorem 1.6.}

(i) The proof is similar to that of Theorem 1.5 with some minor modification and hence we omit it.

(ii) Without loss of generality we assume $\nu \geq 0$. Put $U_1 = 0$ and $U_2 = K_{\mu}[\nu]$. By (2.41) and Lemma 3.5 with $q = m = \frac{N+2}{N-2}$, we have that $g(U_2) \in L^1(\Omega; \phi)$. Proceeding as in the proof of Theorem 1.5, we can obtain the desired result.

\section{KELLER–OSSERMAN ESTIMATES IN THE POWER CASE}

In this section, we prove Keller–Osserman–type estimates on nonnegative solutions to equations with a power nonlinearity.
Lemma 6.1. Assume $p > 1$. Let $u \in C(\overline{\Omega} \setminus \Sigma)$ be a nonnegative solution of

$$-L_\mu u + |u|^{p-1} u = 0$$

(6.1)

in the sense of distributions in $\Omega \setminus \Sigma$. Assume

$$\lim_{x \in \Omega, x \to \xi} u(x) = 0, \quad \forall \xi \in \partial \Omega.$$ 

Then there exists a positive constant $C = C(\Omega, \Sigma, \mu, p)$ such that

$$0 \leq u(x) \leq Cd(x)d_\Sigma(x)^{-\frac{2}{p-1}}, \quad \forall x \in \Omega \setminus \Sigma.$$  

(6.2)

Proof. Let $\beta_0$ be as in Section 2.1 and $\eta_{\beta_0} \in C^\infty_c(\mathbb{R}^N)$ such that

$$0 \leq \eta_{\beta_0} \leq 1, \quad \eta_{\beta_0} = 1 \text{ in } \Sigma_{\beta_0}^4 \quad \text{and} \quad \text{supp}(\eta_{\beta_0}) \subset \Sigma_{\beta_0}^2.$$ 

Let $\varepsilon \in (0, \frac{\beta_0}{16})$, we define

$$V_\varepsilon := 1 - \eta_{\beta_0} + \eta_{\beta_0}(d_\Sigma - \varepsilon)^{-\frac{2}{p-1}} \text{ in } \Omega \setminus \Sigma_\varepsilon.$$ 

Then $V_\varepsilon \geq 0$ in $\overline{\Omega} \setminus \Sigma_\varepsilon$. It can be checked that there exists $C = C(\Omega, \Sigma, \beta_0, \mu, p) > 1$ such that the function $W_\varepsilon := CV_\varepsilon$ satisfies

$$-L_\mu W_\varepsilon + W_\varepsilon^p = C(-L_\mu V_\varepsilon + V_\varepsilon^p) \geq 0 \text{ in } \Omega \setminus \Sigma_\varepsilon.$$  

(6.3)

Since $u \in C(\Omega \setminus \Sigma)$ is a nonnegative solution of Equation (6.1), by standard regularity results, $u \in C^2(\Omega \setminus \Sigma)$. Combining (6.1) and (6.3) yields

$$-L_\mu (u - W_\varepsilon) + u^p - W_\varepsilon^p \leq 0 \text{ in } \Omega \setminus \Sigma_\varepsilon.$$  

(6.4)

We see that $(u - W_\varepsilon)^+ \in H^1_0(\Omega \setminus \Sigma_\varepsilon)$ and $(u - W_\varepsilon)^+$ has compact support in $\Omega \setminus \Sigma_\varepsilon$. By using $(u - W_\varepsilon)^+$ as a test function for (6.4), we deduce that

$$0 \geq \int_{\Omega \setminus \Sigma_\varepsilon} |\nabla (u - W_\varepsilon)^+|^2 dx - \mu \int_{\Omega \setminus \Sigma_\varepsilon} \frac{[(u - W_\varepsilon)^+]^2}{d_\Sigma^2} dx + \int_{\Omega \setminus \Sigma_\varepsilon} (u^p - W_\varepsilon^p)(u - W_\varepsilon)^+ dx$$

$$\geq \int_{\Omega \setminus \Sigma_\varepsilon} |\nabla (u - W_\varepsilon)^+|^2 dx - \mu \int_{\Omega \setminus \Sigma_\varepsilon} \frac{[(u - W_\varepsilon)^+]^2}{d_\Sigma^2} dx \geq \lambda_\mu \int_{\Omega \setminus \Sigma_\varepsilon} |(u - W_\varepsilon)^+|^2 dx.$$ 

This and the assumption $\lambda_\mu > 0$ imply $(u - W_\varepsilon)^+ = 0$, whence $u \leq W_\varepsilon$ in $\Omega \setminus \Sigma_\varepsilon$. Similarly we can show $-W_\varepsilon \leq u$ in $\Omega \setminus \Sigma_\varepsilon$. Thus, $u \leq W_\varepsilon$ in $\Omega \setminus \Sigma_\varepsilon$. Letting $\varepsilon \to 0$, we obtain

$$u \leq Cd_\Sigma^{-\frac{2}{p-1}} \text{ in } \Omega \setminus \Sigma.$$  

(6.5)
Let \(0 < \delta_0 < \frac{1}{4}\text{dist}(\partial \Omega, \Sigma)\). Then by (6.5), \(u \leq C(\delta_0, p)\) in \(\Omega_{\delta_0}\). As a consequence, by standard elliptic estimates, there exists a constant \(C\) depending only on \(\delta_0\) and the \(C^2\) characteristic of \(\Omega\) such that

\[
u \leq Cd \quad \text{in} \quad \Omega_{\delta_0}.
\]  

(6.6)

Combining (6.5) and (6.6) gives (6.2).

In case of lack of boundary condition on \(\partial \Omega\), by adapting the above argument, we can show \(u \leq Cd^{-\frac{2}{p-1}}\) in \(\Omega_{\delta_0}\). Combining (6.5) and (6.6) leads to the following result whose proof is omitted.

**Lemma 6.2.** Let \(u \in C(\overline{\Omega} \setminus \Sigma)\) be a nonnegative solution of (6.1) in the sense of distributions in \(\Omega\). Then there exists a positive constant \(C = C(\Omega, \Sigma, \mu, p)\) such that

\[
u(x) \leq C(\min\{d(x), d_\Sigma(x)\})^{-\frac{2}{p-1}}, \quad \forall x \in \Omega \setminus \Sigma.
\]

7 | REMOVABLE SINGULARITIES

In this section, we show that singularities are removable in supercritical cases.

**Proof of Theorem 1.7.** Assume \(\mu < H^2\) and \(p = \frac{2+\alpha_+}{\alpha_+}\). Let \(u\) be a nonnegative solution of (1.16) satisfying (1.17). Denote \(O_n = \Omega \setminus \Sigma_{\frac{1}{n}}\) and

\[
V(x) = 2C\text{diam}(\Omega) \int_{\Sigma} K_\mu(x, y) d\omega^{x_0}_{\Omega \setminus \Sigma}(y) = 2C\text{diam}(\Omega) K_\mu[\chi_{\overline{\Omega \setminus \Sigma}}](x),
\]

where \(C\) is the constant in (6.2). Then by [14, estimate (5.29)], there exists \(\bar{\rho} > 0\) such that

\[
V(x) \geq C\text{diam}(\Omega) d_\Sigma(x)^{-\alpha_+}, \quad \forall x \in \Sigma_{\bar{\rho}}.
\]  

(7.1)

Let \(n_0 \in \mathbb{N}\) be large enough such that \(\frac{1}{n} \leq \frac{\bar{\rho}}{2}\) for any \(n \geq n_0\). Let \(u_n\) be the solution of

\[
\begin{align*}
-L_\mu O_n u_n + u_n^p & = 0 \quad \text{in} \quad O_n, \\
u_n & = 0 \quad \text{on} \quad \partial \Omega, \\
u_n & = V \quad \text{on} \quad \partial \Sigma_{\frac{1}{n}}.
\end{align*}
\]

(7.2)

Then by (6.2), we have \(0 \leq u \leq u_n\) in \(O_n\). Furthermore, \(\{u_n\}\) is a nonincreasing sequence. Let \(G_\mu^{O_n}\) and \(P_\mu^{O_n}\) be the Green function and Poisson kernel of \(-L_\mu\) in \(O_n\). Denote by \(G_\mu^{O_n}\) and \(P_\mu^{O_n}\) the corresponding Green operator and Poisson operator. We extend \(V\) by zero on \(\partial \Omega\) and use the same notation for the extension. Then, we deduce from (7.2) that

\[
u_n + G_\mu^{O_n}[u_n^p] = P_\mu^{O_n}[V] = V \quad \text{in} \quad O_n.
\]  

(7.3)
This implies \( v_n \leq V \) in \( O_n \) for any \( n \in \mathbb{N} \). Therefore, \( v_n \downarrow v \) locally uniformly and in \( L^1(\Omega; \mu) \).

Using the fact that \( G_{\mu}^{O_n} \uparrow G_\mu \) and Fatou’s Lemma, by letting \( n \to \infty \) in (7.3), we obtain \( v + G_\mu[v^p] \leq V \) in \( \Omega \setminus \Sigma \), which implies \( v \in L^p(\Omega; \mu) \).

Since \( v + G_\mu[v^p] \) is a nonnegative \( L_\mu \) harmonic in \( \Omega \setminus \Sigma \), by the Representation Theorem 2.3 and the fact that \( v + G_\mu[v^p] \leq V \), there exists \( v \in \mathcal{W}^+(\partial \Omega \cup \Sigma) \) with compact support in \( \Sigma \) such that

\[
v + G_\mu[v^p] = \kappa_\mu[v] \quad \text{in} \quad \Omega \setminus \Sigma. \tag{7.4}\]

Let \( \tilde{O}_n = \Omega_n \setminus \Sigma_n \) be a smooth exhaustion of \( \Omega \setminus \Sigma \). We denote by \( \tilde{v}_n \) the solution of

\[
\begin{cases}
-L_{\tilde{\mu}} \tilde{v}_n + \tilde{v}_n^p = 0 & \text{in} \quad \tilde{O}_n \\
\tilde{v}_n = 2v & \text{on} \quad \partial \tilde{O}_n.
\end{cases} \tag{7.5}
\]

Then \( \tilde{v}_n \leq 2v \leq 2V \) in \( \tilde{O}_n \), since \( 2v \) is a supersolution of (7.5). Hence, there exist a function \( \tilde{\vartheta} \) and a subsequence, still denoted by \( \{\tilde{v}_n\} \), such that \( \tilde{v}_n \to \tilde{\vartheta} \) a.e. in \( \Omega \setminus \Sigma \). Let \( G_{\tilde{\mu}}^{\tilde{O}_n} \) and \( P_{\tilde{\mu}}^{\tilde{O}_n} \) be the Green function and Poisson kernel of \( -L_{\tilde{\mu}} \) in \( \tilde{O}_n \). Denote by \( G_{\mu}^{\tilde{O}_n} \) and \( P_{\mu}^{\tilde{O}_n} \) the corresponding Green operator and Poisson operator. From (7.5), we have that

\[
\tilde{\vartheta}_n + G_{\mu}^{\tilde{O}_n}[\tilde{v}_n^p] = 2P_{\mu}^{\tilde{O}_n}[v] \quad \text{in} \quad \tilde{O}_n. \tag{7.6}
\]

By (7.4), we obtain

\[
P_{\mu}^{\tilde{O}_n}[v](x) = \int_{\partial \tilde{O}_n} v \, d\omega^x_{\tilde{O}_n} = - \int_{\partial \tilde{O}_n} G_{\mu}[v^p] \, d\omega^x_{\tilde{O}_n} + \kappa_\mu[v](x).
\]

Since \( \text{tr}(G_{\mu}[v^p]) = 0 \) (see Proposition 2.5), we derive from Definition 2.4 and the above expression that \( P_{\mu}^{\tilde{O}_n}[v] \to \kappa_\mu[v] \) a.e. in \( \Omega \setminus \Sigma \). Since \( \tilde{v}_n \leq 2v \in L^p(\Omega; \mu) \), by dominated convergence theorem, we have \( G_{\mu}^{\tilde{O}_n}[\tilde{v}_n^p] \to G_{\mu}[v^p] \) in \( \Omega \setminus \Sigma \). Letting \( n \to \infty \) in (7.6) yields

\[
\tilde{\vartheta} + G_{\mu}[\tilde{v}^p] = 2\kappa_\mu[v] \quad \text{in} \quad \Omega \setminus \Sigma.
\]

On the other hand, since \( 0 \leq \tilde{\vartheta} \in C^2(\Omega \setminus \Sigma) \) satisfies \( -L_{\tilde{\mu}} \tilde{\vartheta} + \frac{2+\alpha_+}{\alpha_+} \tilde{\vartheta} = 0 \), we deduce from Lemma 6.1 that \( \tilde{\vartheta}(x) \leq Cd(x)d_\Sigma(x)^{-\alpha_+} \) for all \( x \in \Omega \setminus \Sigma \). This and (7.1) implies \( \tilde{\vartheta}(x) \leq V(x) \) for all \( x \in \partial \Sigma \). By the maximum principle, \( \tilde{\vartheta} \leq v_n \) in \( O_n \). Since \( v_n \downarrow v \) locally uniformly in \( \Omega \setminus \Sigma \), we derive that \( \tilde{\vartheta} \leq v \) in \( \Omega \setminus \Sigma \). Consequently, \( 2v = \text{tr}(\tilde{\vartheta}) \leq \text{tr}(v) = v \), thus \( v \equiv 0 \) and hence, by (7.4), \( v \equiv 0 \). Thus \( u \equiv 0 \).

When \( p > \frac{2+\alpha_+}{\alpha_+} \) or \( p = \frac{2+\alpha_+}{\alpha_+} \) if \( \mu = H^2 \), the proof is similar to the above case, hence we omit it.

**Proof of Theorem 1.8.** Without loss of generality, we may assume \( z = 0 \). Let \( \zeta : \mathbb{R} \to [0, \infty) \) be a smooth function such that \( 0 \leq \zeta \leq 1 \), \( \zeta(t) = 0 \) for \( |t| \leq 1 \) and \( \zeta(t) = 1 \) for \( |t| > 2 \). For \( \varepsilon > 0 \), we set \( \zeta_\varepsilon(x) = \zeta(\frac{|x|}{\varepsilon}) \).

Since \( u \in C(\Omega \setminus \Sigma) \) by standard elliptic theory we have that \( u \in C^2(\Omega \setminus \Sigma) \) and hence

\[
L_{\mu}(\zeta_\varepsilon u) = u \Delta \zeta_\varepsilon + \zeta_\varepsilon u^p + 2\nabla \zeta_\varepsilon \nabla u \quad \text{in} \quad \Omega \setminus \Sigma.
\]
Step 1: We show $L_\mu(\zeta_\varepsilon u) \in L^1(\Omega; \phi_\mu)$.

We first see that

$$
\int_\Omega |L_\mu(\zeta_\varepsilon u)|\phi_\mu \, dx \leq \int_\Omega \zeta_\varepsilon u^p \phi_\mu \, dx + \int_\Omega u|\Delta \zeta_\varepsilon|\phi_\mu \, dx + 2 \int_\Omega |\nabla \zeta_\varepsilon| |\nabla u|\phi_\mu \, dx. 
$$  
(7.7)

We note that there exists a constant $C > 0$ that does not depend on $\varepsilon$ such that

$$
|\nabla \zeta_\varepsilon|^2 + |\Delta \zeta_\varepsilon| \leq C \varepsilon^{-2} 1_{|x| \leq 2\varepsilon}.
$$

This, together with (A.17), (A.18), (2.10), the estimate $\int_{\Sigma_\beta} d\Sigma(x)^{-\alpha} \, dx \lesssim \beta^{N-\alpha}$ for $\alpha < N-k$, and the assumption $p \geq \frac{N-\alpha}{N-\alpha-2}$, yields

$$
\int_\Omega \zeta_\varepsilon u^p \phi_\mu \, dx \lesssim \varepsilon^{-\frac{2p}{p+1}+\alpha_-} \int_\Omega \beta_\mu(x)^{-(p+1)\alpha_-} \, dx \lesssim \varepsilon^{-\frac{2p}{p+1}-\alpha_-} p,
$$

$$
\int_\Omega u|\Delta \zeta_\varepsilon|\phi_\mu \, dx \leq \varepsilon^{-\frac{2}{p+1}+\alpha_-} \int_{\Omega \cap \{|x| < 2\varepsilon\}} d\Sigma(x)^{-2\alpha_-} \, dx \lesssim \varepsilon^{\frac{N-2}{p+1}-\alpha_-} \lesssim 1,
$$

$$
\int_\Omega |\nabla \zeta_\varepsilon| |\nabla u|\phi_\mu \, dx \lesssim \varepsilon^{-\frac{2}{p+1}+\alpha_-} \int_{\Omega \cap \{|x| < 2\varepsilon\}} d\Sigma(x)^{-2\alpha_-} \, dx \lesssim \varepsilon^{\frac{N-2}{p+1}-\alpha_-} \lesssim 1.
$$  
(7.8)

Estimates (7.7) and (7.8) yield $L_\mu(\zeta_\varepsilon u) \in L^1(\Omega; \phi_\mu)$.

Step 2: We will show $u \in L^p(\Omega; \phi_\mu)$.

By [14, Lemma 7.4], we have

$$
- \int_\Omega \zeta_\varepsilon u L_\mu \eta \, dx = - \int_\Omega (u \Delta \zeta_\varepsilon + \zeta_\varepsilon u^p + 2 \nabla \zeta_\varepsilon \nabla u) \eta \, dx, \quad \forall \eta \in X_\mu(\Omega \setminus \Sigma).
$$

Taking $\eta = \phi_\mu$, we obtain

$$
\lambda_\mu \int_\Omega \zeta_\varepsilon u \phi_\mu \, dx + \int_\Omega \zeta_\varepsilon u^p \phi_\mu \, dx = - \int_\Omega (u \Delta \zeta_\varepsilon + 2 \nabla \zeta_\varepsilon \nabla u) \phi_\mu \, dx.
$$

By the last two lines in (7.8), we have

$$
\lambda_\mu \int_\Omega \zeta_\varepsilon u \phi_\mu \, dx + \int_\Omega \zeta_\varepsilon u^p \phi_\mu \, dx \leq C.
$$

By Fatou’s lemma, letting $\varepsilon \to 0$, we deduce that

$$
\lambda_\mu \int_\Omega u \phi_\mu \, dx + \int_\Omega u^p \phi_\mu \, dx \leq C. 
$$  
(7.9)

This implies $u \in L^p(\Omega; \phi_\mu)$.

Step 3: End of proof. Let $\{O_n\}$ be a smooth exhaustion of $\Omega \setminus \Sigma$. From Step 2, we see that $u + G_\mu[u^p]$ is a nonnegative $L_\mu$ harmonic function and by the Representation Theorem, there exists $\rho > 0$ such that

$$
u + G_\mu[u^p] = \rho K_\mu(\cdot, 0) \quad \text{in} \ \Omega \setminus \Sigma.
$$  
(7.10)
We will show \( \rho = 0 \). Suppose by contradiction \( \rho > 0 \). Let \( n_0 \in \mathbb{N} \) large enough such that \( \frac{1}{n} \leq \frac{\rho_0}{10} \) for any \( n \geq n_0 \). For \( 1 < M \in \mathbb{N} \), let \( u_{M,n} \) be the positive solution of

\[
\begin{cases}
-L^{O_n}_{\mu} u_{M,n} + u^{p}_{M,n} = 0 & \text{in } O_n \\
u_{M,n} = Mu & \text{on } \partial O_n.
\end{cases}
\] (7.11)

Then \( u \leq v_{M,n} \leq Mu \) in \( O_n \), since \( Mu \) is a supersolution of (7.11). Furthermore, by (6.2), there exist a function \( v_M \) and a subsequence, still denoted by the same notation, such that \( v_{M,n} \to v_M \) locally uniformly in \( \Omega \setminus \Sigma \). Moreover, from (7.11), we have

\[
v_{M,n}(x) + G^{O_n}_{\mu}[v^{p}_{M,n}](x) = p^{O_n}_{\mu}[Mu](x) = \int_{\partial O_n} Mu \, d\omega^x_{O_n} = h_n(x), \quad \forall x \in O_n.
\] (7.12)

Now, by (7.10),

\[
h_n(x) = \int_{\partial O_n} Mu \, d\omega^x_{O_n} = -M \int_{\partial O_n} G_{\mu}[u^p] \, d\omega^x_{O_n} + M\rho K_{\mu}(x,0).
\]

Since \( \text{tr}(G_{\mu}[u^p]) = 0 \), by Definition 2.4 (with \( \phi = 1 \)), it follows that \( h_n(x) \to M\rho K_{\mu}(x,0) \) as \( n \to \infty \). By dominated convergence theorem, letting \( n \to \infty \) in (7.12), we obtain

\[
v_M(x) + G_{\mu}[v^{p}_{M}](x) = M\rho K_{\mu}(x,0).
\] (7.13)

We observe that \( \{v_M\}_{M=1}^{\infty} \) is nondecreasing and by (A.17), it is locally uniformly bounded from above. Therefore, \( v_M \to v \) locally uniformly in \( \Omega \setminus \Sigma \) as \( M \to \infty \). For each \( M > 1 \), we have \( v_M \leq Mu \) in \( \Omega \setminus \Sigma \), which implies that \( v_M \) satisfies (1.18). Therefore, by using an argument similar to the one leading to (7.9), we deduce that \( \{v_M\} \) is uniformly bounded in \( L^p(\Omega \setminus \Sigma; \phi_{\mu}) \). By the monotonicity convergence theorem, we deduce that \( v_M \to v \) in \( L^p(\Omega \setminus \Sigma; \phi_{\mu}) \), whence \( G_{\mu}[v^{p}_{M}] \to G_{\mu}[v^p] \) a.e. in \( \Omega \setminus \Sigma \). Therefore, by letting \( M \to \infty \) in (7.13), we derive \( \lim_{M \to \infty} (v_M(x) + G_{\mu}[v^{p}_{M}](x)) = \infty \), which is a contradiction. Thus \( \rho = 0 \) and hence by (7.10), \( u \equiv 0 \) in \( \Omega \setminus \Sigma \). The proof is complete. 

\[ \square \]

8 \hspace{1cm} GOOD MEASURES

In this section, we investigate the problem

\[
\begin{cases}
-L_{\mu} u + |u|^{p-1} u = 0 & \text{in } \Omega \setminus \Sigma, \\
\text{tr}(u) = \nu,
\end{cases}
\] (8.1)

where \( p > 1 \) and \( \nu \in \mathcal{M}(\partial \Omega \cup \Sigma) \). Recall that a measure is called a \( p \)-good measure if problem (8.1) admits a (unique) solution.

Let us first remark that if \( 1 < p < \min\{\frac{N+1}{N-1}, \frac{N-\alpha_{-}}{N-\alpha_{-}-2}\} \), then by Theorem 1.4, problem (8.1) admits a unique solution for any \( \nu \in \mathcal{M}(\partial \Omega \cup \Sigma) \). Furthermore, if \( \nu \) has compact support in \( \partial \Omega \) and \( 1 < p < \frac{N+1}{N-1} \) (resp. \( \nu \) has compact support in \( \Sigma \) and \( 1 < p < \frac{N-\alpha_{-}}{N-\alpha_{-}-2} \)), then (8.1) admits a unique weak solution by Theorem 1.5 (resp. by Theorem 1.6).
In order to characterize $p$-good measures, we make use of appropriate capacities. We recall below some notations concerning Besov space (see, e.g., [1, 19]). For $\sigma > 0$, $1 \leq \kappa < \infty$, we denote by $W^{\sigma,\kappa}(\mathbb{R}^d)$ the Sobolev space over $\mathbb{R}^d$. If $\sigma$ is not an integer the Besov space $B^{\sigma,\kappa}(\mathbb{R}^d)$ coincides with $W^{\sigma,\kappa}(\mathbb{R}^d)$. When $\sigma$ is an integer, we denote
\[ \Delta_{x,y}f := f(x+y) + f(x-y) - 2f(x) \]
and
\[ B^{1,\kappa}(\mathbb{R}^d) := \left\{ f \in L^\kappa(\mathbb{R}^d) : \frac{\Delta_{x,y}f}{|y|^{1+d/\kappa}} \in L^\kappa(\mathbb{R}^d \times \mathbb{R}^d) \right\}, \]
with norm
\[ \|f\|_{B^{1,\kappa}} := \left( \|f\|_{L^\kappa}^\kappa + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\Delta_{x,y}f|^\kappa}{|y|^{\kappa+d}} \, dx \, dy \right)^{1/\kappa}. \]
Then
\[ B^{m,\kappa}(\mathbb{R}^d) := \{ f \in W^{m-1,\kappa}(\mathbb{R}^d) : D_\alpha f \in B^{1,\kappa}(\mathbb{R}^d) \ \forall \alpha \in \mathbb{N}^d \ \text{such that} |\alpha| = m - 1 \}, \]
with norm
\[ \|f\|_{B^{m,\kappa}} := \left( \|f\|_{W^{m-1,\kappa}}^\kappa + \sum_{|\alpha| = m-1} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|D_\alpha \Delta_{x,y}f|^\kappa}{|y|^{\kappa+d}} \, dx \, dy \right)^{1/\kappa}. \]
These spaces are fundamental because they are stable under the real interpolation method developed by Lions and Petree. For $\alpha \in \mathbb{R}$ we defined the Bessel kernel of order $\alpha$ in $\mathbb{R}^d$ by $B_{d,\alpha}(\xi) := \mathcal{F}^{-1}((1 + |.|^2)^{-\alpha/2})(\xi)$, where $\mathcal{F}$ is the Fourier transform in the space $S'(\mathbb{R}^d)$ of moderate distributions in $\mathbb{R}^d$. For $\kappa > 1$, the Bessel space $L^{\alpha,\kappa}(\mathbb{R}^d)$ is defined by
\[ L_{\alpha,\kappa}(\mathbb{R}^d) := \{ f = B_{d,\alpha} \ast g : g \in L^\kappa(\mathbb{R}^d) \}, \]
with norm
\[ \|f\|_{L_{\alpha,\kappa}} := \|g\|_{L^\kappa} = \|B_{d,-\alpha} \ast f\|_{L^\kappa}. \]
It is known that if $1 < \kappa < \infty$ and $\alpha > 0$, $L_{\alpha,\kappa}(\mathbb{R}^d) = W^{\alpha,\kappa}(\mathbb{R}^d)$ if $\alpha \in \mathbb{N}$. If $\alpha \notin \mathbb{N}$, then the positive cone of their dual coincide, that is, $(L_{-\alpha,\kappa}(\mathbb{R}^d))^+ = (B^{-\alpha,\kappa'}(\mathbb{R}^d))^+$, always with equivalent norms.

The Bessel capacity is defined for compact subsets $K \subset \mathbb{R}^d$ by
\[ \text{Cap}_{\kappa,\alpha}^{\text{mod}}(K) := \inf\{\|f\|_{L_{\alpha,\kappa}}^\kappa : f \in S'(\mathbb{R}^d), f \geq 1_K \}. \]

**Lemma 8.1.** Let $k \geq 1$, $\max\{1, \frac{N-k-\alpha}{N-2-\alpha} \} < p < \frac{2+\alpha_+}{\alpha_+}$, and $\nu \in \mathcal{M}^+(\mathbb{R}^k)$ with compact support in $B^\nu(0, \frac{R}{2})$ for some $R > 0$. Let $\mathcal{S}$ be as in (1.20). For $x \in \mathbb{R}^{k+1}$, we write $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^k$. Then
there exists a constant $C = C(R, N, k, \mu, p) > 1$ such that

\[
C^{-1} \| \nu \|_{B^{-\frac{p}{p}, p}(\mathbb{R}^k)}^p \\
\leq \int_{B^k(0,R)} \int_0^R x_1^{-N-k-1-(p+1)\alpha_-} \left( \int_{B^k(0,R)} (|x_1| + |x' - y'|)^{-(N-2\alpha_-)} d\nu(y') \right)^p dx_1 \ dx' \ (8.2)
\]

\[
\leq C \| \nu \|_{B^{-\frac{p}{p}, p}(\mathbb{R}^k)}^p.
\]

**Proof.** The proof is inspired by the idea in [3, Proposition 2.8].

**Step 1:** We will prove the upper bound in (8.2).

Let $0 < x_1 < R$ and $|x'| < R$. In view of the proof of [1, Lemma 3.1.1], we obtain

\[
\int_{B^k(0,R)} (x_1 + |x' - y'|)^{(N-2\alpha_- - 2)} d\nu(y') \leq \int_{B^k(x', 2R)} (x_1 + |x' - y'|)^{(N-2\alpha_- - 2)} d\nu(y')
\]

\[
= (N - 2\alpha_- - 2) \left( \int_0^{2R} \frac{\nu(B^k(x', r))}{x_1 + r} \frac{dr}{r} + \frac{\nu(B^k(x', 2R))}{(x_1 + 2R)^{N-2\alpha_- - 2}} \right)
\]

\[
\leq \int_0^{3R} \frac{\nu(B^k(x', r))}{(x_1 + r)^{N-2\alpha_- - 2}} \frac{dr}{x_1 + r} \leq \int_{x_1}^{4R} \frac{\nu(B^k(x', r))}{r^{N-2\alpha_- - 2}} \frac{dr}{r}.
\]

It follows that

\[
\int_0^R x_1^{-N-k-1-(p+1)\alpha_-} \left( \int_{B^k(0,R)} (|x_1| + |x' - y'|)^{-(N-2\alpha_-)} d\nu(y') \right)^p dx_1
\]

\[
\leq \int_0^R x_1^{-N-k-1-(p+1)\alpha_-} \left( \int_{x_1}^{4R} \frac{\nu(B^k(x', r))}{r^{N-2\alpha_- - 2}} \frac{dr}{r} \right)^p dx_1.
\]

Since $p < \frac{2+\alpha_+}{\alpha_+} < \frac{N-k-\alpha_-}{\alpha_-}$, it follows that $N - k - (p + 1)\alpha_- > 0$. Let $\varepsilon$ be such that $0 < \varepsilon < N - k - (p + 1)\alpha_-$. By Hölder’s inequality and Fubini’s theorem, we have

\[
\int_0^R x_1^{-N-k-1-(p+1)\alpha_-} \left( \int_{x_1}^{4R} \frac{\nu(B^k(x', r))}{r^{N-2\alpha_- - 2}} \frac{dr}{r} \right)^p dx_1
\]

\[
\leq \int_0^R x_1^{-N-k-1-(p+1)\alpha_-} \left( \int_{x_1}^{\infty} \frac{\nu(B^k(x', r))}{r^{N-2\alpha_- - 2}} \frac{dr}{r} \right)^p \int_{x_1}^{4R} \frac{\nu(B^k(x', r))}{r^{N-2\alpha_- - 2}} \frac{dr}{r} dx_1
\]

\[
= C(p, \varepsilon) \int_0^R x_1^{-N-k-1-(p+1)\alpha_- - \varepsilon} \int_{x_1}^{4R} \frac{\nu(B^k(x', r))}{r^{N-2\alpha_- - 2 - \frac{\varepsilon}{p}}} \frac{dr}{r} dx_1
\]

\[
\leq C(p, \varepsilon, N, k, \alpha_-, R) \int_0^{4R} \left( \frac{\nu(B^k(x', r))}{r^{N-2\alpha_- - 2 - \frac{N-k-(p+1)\alpha_-}{p}}} \right)^p \frac{dr}{r}.
\]

From the assumption on $p$ and the definition of $\vartheta$ in (1.20), we see that $0 < \vartheta < k$. Moreover,

\[
N - 2\alpha_- - 2 - \frac{N - k - (p + 1)\alpha_-}{p} = k - \vartheta.
\]
We have
\[
\int_{0}^{4R} \left( \frac{\nu(B^k(x', r))}{r^{k-\delta}} \right)^p \frac{dr}{r} = \sum_{n=0}^{\infty} \int_{2^{-n+1}R}^{2^{-n+2}R} \left( \frac{\nu(B^k(x', r))}{r^{k-\delta}} \right)^p \frac{dr}{r}
\]
\[
\leq \ln 2 \sum_{n=0}^{\infty} 2^{p(n-1)(k-\delta)} \left( \frac{\nu(B^k(x', 2^{-n+2}R))}{R^{k-\delta}} \right)^p
\]
\[
\leq \ln 2 \left( \sum_{n=0}^{\infty} 2^{(n-1)(k-\delta)} \nu(B^k(x', 2^{-n+2}R)) \right)^p
\]
\[
\leq 2^{p(k-\delta)} (\ln 2)^{-(p-1)} \left( \sum_{n=0}^{\infty} \int_{2^{-n+2}R}^{2^{-n+3}R} \nu(B^k(x', r)) \frac{dr}{r} \right)^p
\]
\[
= 2^{p(k-\delta)} (\ln 2)^{-(p-1)} \left( \int_{0}^{8R} \nu(B^k(x', r)) \frac{dr}{r} \right)^p.
\]

Set
\[
\mathcal{W}_{\delta, 8R}[\nu](x') := \int_{0}^{8R} \nu(B^k(x', r)) \frac{dr}{r} \quad \text{and} \quad \mathcal{B}_{k, \delta}[\nu](x') := \int_{\mathbb{R}^k} B_{k, \delta}(x' - y') d\nu(y').
\]

Then
\[
\int_{B^k(0, R)} \int_{0}^{R} x_1^{N-k-1-(p+1)\alpha_-} \left( \int_{B^k(0, R)} (x_1 + |x' - y'|)^{-(N-2\alpha_- - 2)} d\nu(y') \right)^p \, dx_1 \, dx'
\]
\[
\leq \int_{\mathbb{R}^k} \mathcal{W}_{\delta, 8R}[\nu](x')^p \, dx' \leq \int_{\mathbb{R}^k} \mathcal{B}_{k, \delta}[\nu](x')^p \, dx',
\]
where in the last inequality we have used [4, Theorem 2.3]. Note that the assumption on \( p \) ensures that [4, Theorem 2.3] can be applied.

By [1, Corollaries 3.6.3 and 4.1.6], we obtain
\[
\int_{\mathbb{R}^k} \mathcal{B}_{k, \delta}[\nu](x')^p \, dx' \leq C(\delta, k, p)\|\nu\|_{p, B^{-\delta, p}(\mathbb{R}^k)}^p.
\]

Combining (8.3) and (8.4), we obtain the upper bound in (8.2).

**Step 2:** We will prove the lower bound in (8.2).

Let \( 0 < x_1 < R \) and \( |x'| < R \). Then by [1, Lemma 3.1.1], we have
\[
\int_{B^k(0, R)} (x_1 + |x' - y'|)^{-(N-2\alpha_- - 2)} d\nu(y') = (N - 2\alpha_- - 2) \int_{x_1}^{\infty} \frac{\nu(B^k(x', r - x_1))}{r^{N-2\alpha_- - 2}} \frac{dr}{r}
\]
\[
\geq (N - 2\alpha_- - 2) \int_{2x_1}^{\infty} \frac{\nu(B^k(x', \frac{r}{2}))}{r^{N-2\alpha_- - 2}} \frac{dr}{r}
\]
\[
\geq C(N, \alpha_-) \int_{x_1}^{\infty} \frac{\nu(B^k(x', r))}{r^{N-2\alpha_- - 2}} \frac{dr}{r}.
\]
It follows that
\[
\int_0^R x_1^{N-k-1-(p+1)\alpha-1} \left( \int_{B_k(0,R)} \left( x_1 + |x'-y'| \right)^{-2(N-2\alpha-2)} d\nu(y') \right)^p dx_1 \\
\geq \int_0^R x_1^{N-k-1-(p+1)\alpha-1} \left( \int_{x_1}^\infty \frac{\nu(B^k(x',r))}{r^{N-2\alpha-2}} \frac{dr}{r} \right)^p dx_1 \\
\geq \int_0^R x_1^{N-k-1-(p+1)\alpha-1} \left( \int_{x_1}^{2x_1} \frac{\nu(B^k(x',r))}{r^{N-2\alpha-2}} \frac{dr}{r} \right)^p dx_1 \\
\geq \int_0^R \left( \frac{\nu(B^k(x',x_1))}{x_1^{k-\vartheta}} \right)^p dx_1/x_1.
\]

For \(0 < r < \frac{R}{2}\), we obtain
\[
\int_0^R \left( \frac{\nu(B^k(x',x_1))}{x_1^{k-\vartheta}} \right)^p dx_1/x_1 \geq \int_r^{2r} \left( \frac{\nu(B^k(x',x_1))}{x_1^{k-\vartheta}} \right)^p dx_1/x_1 \geq \left( \frac{\nu(B^k(x',r))}{r^{k-\vartheta}} \right)^p,
\]
which implies
\[
\int_0^R \left( \frac{\nu(B^k(x',x_1))}{x_1^{k-\vartheta}} \right)^p dx_1/x_1 \geq \left( \sup_{0 < r < \frac{R}{2}} \frac{\nu(B^k(x',r))}{r^{k-\vartheta}} \right)^p.
\]

Set
\[
M_{\vartheta, \frac{R}{2}}(x') := \sup_{0 < r < \frac{R}{2}} \frac{\nu(B^k(x',r))}{r^{k-\vartheta}}.
\]

Then, since \(\nu\) has compact support in \(B(0, \frac{R}{2})\),
\[
\int_{B^k(0,R)} \int_0^R x_1^{N-k-1-(p+1)\alpha-1} \left( \int_{B^k(0,R)} \left( x_1 + |x'-y'| \right)^{-2(N-2\alpha-2)} d\nu(y') \right)^p dx_1 dx'
\geq \int_{B^k(0,R)} M_{\vartheta, \frac{R}{2}}(x')^p dx' = \int_{\mathbb{R}^k} M_{\vartheta, \frac{R}{2}}(x')^p dx'.
\]

By [4, Theorem 2.3] and [1, Corollaries 3.6.3 and 4.1.6],
\[
\int_{\mathbb{R}^k} M_{\vartheta, \frac{R}{2}}(x')^p dx' \geq \int_{\mathbb{R}^k} B_{k, \vartheta}[\nu](x')^p dx' \geq \|\nu\|_{B^{-\vartheta, p}(\mathbb{R}^k)}^p.
\]

Combining (8.5)–(8.6), we obtain the lower bound in (8.2). \(\square\)

**Theorem 8.2.** Let \(k \geq 1\), \(\max\{1, \frac{N-k-\alpha}{N-\alpha-2}\} < p < \frac{2+\alpha_+}{\alpha_+}\), and \(\nu \in \mathfrak{M}^+(\partial \Omega \cup \Sigma)\) with compact support in \(\Sigma\). Then there exists a constant \(C = C(\Omega, \Sigma, \mu) > 1\) such that
\[
C^{-1} \|\nu\|_{B^{-\vartheta, p}(\Sigma)} \leq \|K_{\mu}[\nu]\|_{L_p(\Omega; h_\mu)} \leq C \|\nu\|_{B^{-\vartheta, p}(\Sigma)},
\]
where \(\vartheta\) is given in (1.20).
Proof. By (2.4), there exists $\xi^j \in \Sigma, j = 1, 2, \ldots, m_0$ (where $m_0 \in \mathbb{N}$ depends on $N, \Sigma$), and $\beta_1 \in \left(0, \frac{\hat{\beta}_0}{4}\right)$ such that $\Sigma_{\beta_1} \subset \bigcup_{j=1}^{m_0} V(\xi^j, \frac{\hat{\beta}_0}{2}) \subset \Omega$.

**Step 1:** We establish local 2-sided estimates.

Assume $\nu \in \mathfrak{M}^+(\partial \Omega \cup \Sigma)$ with compact support in $\Sigma \cap V(\xi^j, \frac{\hat{\beta}_0}{2})$ for some $j \in \{1, \ldots, m_0\}$. We write

$$\int_{\Omega} \phi_\mu K_\mu [\nu]^p \, dx = \int_{\Omega \setminus V(\xi^j, \frac{\hat{\beta}_0}{2})} \phi_\mu K_\mu [\nu]^p \, dx + \int_{V(\xi^j, \frac{\hat{\beta}_0}{2})} \phi_\mu K_\mu [\nu]^p \, dx. \quad (8.8)$$

On one hand, by (2.10) and Proposition 2.2, we have

$$\int_{\Omega \setminus V(\xi^j, \frac{\hat{\beta}_0}{2})} \phi_\mu K_\mu [\nu]^p \, dx$$

$$\approx \int_{\Omega \setminus V(\xi^j, \frac{\hat{\beta}_0}{2})} d(x) d_x(x)^{-\alpha_-} \left( \int_{\Sigma \cap V(\xi^j, \frac{\hat{\beta}_0}{2})} \frac{d(x) d_x(x)^{-\alpha_-}}{|x-y|^{N-2-2\alpha_-} \, d\nu(y)} \right)^p \, dx$$

$$\lesssim \nu(\Sigma \cap V(\xi^j, \frac{\hat{\beta}_0}{2}))^p \int_{\Omega \setminus \Sigma} d_x(x)^{-(p+1)\alpha_-} \, dx \lesssim \nu(\Sigma \cap V(\xi^j, \frac{\hat{\beta}_0}{2}))^p. \quad (8.9)$$

In the last estimate, we have used estimate $\int_{\Omega} d_x(x)^{-(p+1)\alpha_-} \, dx \lesssim 1$ since $(1+p)\alpha_- < N-k$.

On the other hand, again by (2.10) and Proposition 2.2, we have

$$\int_{V(\xi^j, \frac{\hat{\beta}_0}{2})} \phi_\mu K_\mu [\nu]^p \, dx$$

$$\approx \int_{V(\xi^j, \frac{\hat{\beta}_0}{2})} d(x) d_x(x)^{-\alpha_-} \left( \int_{\Sigma \cap V(\xi^j, \frac{\hat{\beta}_0}{2})} \frac{d(x) d_x(x)^{-\alpha_-}}{|x-y|^{N-2-2\alpha_-} \, d\nu(y)} \right)^p \, dx$$

$$\gtrsim \nu(\Sigma \cap V(\xi^j, \frac{\hat{\beta}_0}{2}))^p \int_{V(\xi^j, \frac{\hat{\beta}_0}{2})} d_x(x)^{-(p+1)\alpha_-} \, dx \gtrsim \nu(\Sigma \cap V(\xi^j, \frac{\hat{\beta}_0}{2}))^p. \quad (8.10)$$

Combining (8.8)–(8.9) yields

$$\int_{\Omega} \phi_\mu K_\mu [\nu]^p \, dx \approx \int_{V(\xi^j, \frac{\hat{\beta}_0}{2})} \phi_\mu K_\mu [\nu]^p \, dx. \quad (8.10)$$

For any $x \in \mathbb{R}^N$, we write $x = (x', x'')$ where $x' = (x_1, \ldots, x_k)$ and $x'' = (x_{k+1}, \ldots, x_N)$, and define the $C^2$ function

$$\Phi(x) := (x', x_{k+1} - \Gamma_{k+1}^{\xi^j}(x'), \ldots, x_N - \Gamma_N^{\xi^j}(x')).$$

By (2.3), $\Phi : V(\xi^j, \frac{\hat{\beta}_0}{2}) \to B^k(0, \frac{\hat{\beta}_0}{2}) \times B^{N-k}(0, \frac{\hat{\beta}_0}{2})$ is $C^2$ diffeomorphism and $\Phi(x) = (x', 0_{\mathbb{R}^{N-k}})$ for $x = (x', x'') \in \Sigma$. In view of the proof of [1, Lemma 5.2.2], there exists a measure $\nu \in \mathfrak{M}^+(\mathbb{R}^k)$ with compact support in $B^k(0, \frac{\hat{\beta}_0}{2})$ such that for any Borel $E \subset B^k(0, \frac{\hat{\beta}_0}{2})$, there holds $\nu(E) = \nu(\Phi^{-1}(E \times \{0_{\mathbb{R}^{N-k}}\}))$. 


Set \( \psi = (\psi', \psi'') = \Phi(x) \) then \( \psi' = x' \) and \( \psi'' = (x_{k+1} - \Gamma^x_{j+1}(x'), \ldots, x_N - \Gamma^x_{N}(x')) \). By (2.6), (2.10), and (2.13), we have

\[
\phi_{\mu}(x) \approx |\psi''|^{\alpha_-}, \\
K_{\mu}(x, y) \approx |\psi''|^{\alpha_-} (|\psi''| + |\psi' - y'|)^{-(N-2\alpha_- - 2)}, \forall x \in V(\xi^j, \beta_0) \setminus \Sigma, \\
\forall y = (y', y'') \in V(\xi^j, \beta_0) \cap \Sigma.
\]

Therefore,

\[
\int_{V(\xi^j, \beta_0)} \phi_{\mu}^p K_{\mu}^p [\nu] \, dx \\
\approx \int_{B^k(0, \beta_0)} \int_{B^{N-k}(0, \beta_0)} |\psi''|^{-(p+1)\alpha_-} \left( \int_{B^k(0, \beta_0)} (|\psi''| + |\psi' - y'|)^{-(N-2\alpha_- - 2)} \, d\nu(y') \right)^p \, d\psi'' \, d\psi' \\
= C(N, k) \int_{B^k(0, \beta_0)} \int_0^{\beta_0} r^{N-1-(p+1)\alpha_-} \left( \int_{B^k(0, \beta_0)} (r + |\psi' - y'|)^{-(N-2\alpha_- - 2)} \, d\nu(y') \right)^p \, dr \, d\psi'.
\]

(8.11)

Since \( \nu \mapsto \nu \circ \Phi^{-1} \) is a \( C^2 \) diffeomorphism between \( \mathcal{M}^+(\Sigma \cap V(\xi^j, \beta_0)) \cap B^{-\delta, p}(\Sigma \cap V(\xi^j, \beta_0)) \) and \( \mathcal{M}^+(B^k(0, \beta_0)) \cap B^{-\delta, p}(B^k(0, \beta_0)) \), using (8.10), (8.11), and Lemma 8.1, we derive

\[
C^{-1} \|\nu\|_{B^{-\delta, p}(\Sigma)} \leq \|K_{\mu}^p [\nu]\|_{L^p(\Omega; \phi_{\mu})} \leq C \|\nu\|_{B^{-\delta, p}(\Sigma)},
\]

**Step 2:** We will prove global two-sided estimates.

If \( \nu \in \mathcal{M}^+(\partial \Omega \cup \Sigma) \) with compact support in \( \Sigma \), we may write \( \nu = \sum_{j=1}^{m_0} \nu_j \), where \( \nu_j \in \mathcal{M}^+(\partial \Omega \cup \Sigma) \) with compact support in \( V(\xi^j, \beta_0/2) \). On one hand, by step 1, we have

\[
\|K_{\mu} [\nu]\|_{L^p(\Omega; \phi_{\mu})} \leq \sum_{j=1}^{m_0} \|K_{\mu} [\nu_j]\|_{L^p(\Omega; \phi_{\mu})} \leq C \sum_{j=1}^{m_0} \|\nu_j\|_{B^{-\delta, p}(\Sigma)} \leq C m_0 \|\nu\|_{B^{-\delta, p}(\Sigma)}.
\]

(8.12)

On the other hand, we deduce from step 1 that

\[
\|K_{\mu} [\nu]\|_{L^p(\Omega; \phi_{\mu})} \geq m_0^{-1} \sum_{j=1}^{m_0} \|K_{\mu} [\nu_j]\|_{L^p(\Omega; \phi_{\mu})} \geq (C m_0)^{-1} \sum_{j=1}^{m_0} \|\nu_j\|_{B^{-\delta, p}(\Sigma)} \geq (C m_0)^{-1} \|\nu\|_{B^{-\delta, p}(\Sigma)}.
\]

This and (8.12) imply (8.7). The proof is complete.

Using Theorem 8.2, we are ready to prove Theorem 1.10.

**Proof of Theorem 1.10.** If \( \nu \) is a positive measure which vanishes on Borel sets \( E \subset \Sigma \) with \( \text{Cap}_{\delta, p}^{R^k} \) capacity zero, there exists an increasing sequence \( \{\nu_n\} \) of positive measures in \( B^{-\delta, p}(\Sigma) \) which converges weakly to \( \nu \) (see [8, 12]). By Theorem 8.2, we have \( K_{\mu} [\nu_n] \in L^p(\Omega \setminus \Sigma; \phi_{\mu}) \), hence we may apply Theorem 3.4 with \( w = K_{\mu} [\nu_n], \nu = 0, \) and \( g(t) = |t|^{p-1} t \) to deduce that there exists a unique nonnegative weak solution \( u_n \) of (8.1) with \( \text{tr}(u_n) = \nu_n \).
Since \{νₙ\} is an increasing sequence of positive measures, by Theorem 2.7, \{uₙ\} is increasing and its limit is denoted by \(u\). Moreover,

\[
- \int_{\Omega} u_n L_{\mu} \zeta \, dx + \int_{\Omega} u_{n}^P \zeta \, dx = - \int_{\Omega} \kappa_{\mu} [νₙ] L_{\mu} \zeta \, dx \quad \forall \zeta \in X_{\mu}(\Omega \setminus \Sigma).
\]  
(8.13)

By taking \(\zeta = \phi_{\mu}\) in (8.13), we obtain

\[
\int_{\Omega} (\lambda_{\mu} u_n + u_{n}^P) \phi_{\mu} \, dx = \lambda_{\mu} \int_{\Omega} \kappa_{\mu} [νₙ] \phi_{\mu} \, dx,
\]

which implies that \{uₙ\} and \{u_{n}^P\} are uniformly bounded in \(L^1(\Omega \setminus \Sigma; \phi_{\mu})\). Therefore, \(u_n \to u\) in \(L^1(\Omega; \phi_{\mu})\) and in \(L^p(\Omega; \phi_{\mu})\). By letting \(n \to \infty\) in (8.13), we deduce

\[
\int_{\Omega} -u L_{\mu} \zeta \, dx + \int_{\Omega} u_{n}^P \zeta \, dx = - \int_{\Omega} \kappa_{\mu} [ν] L_{\mu} \zeta \, dx \quad \forall \zeta \in X_{\mu}(\Omega \setminus \Sigma).
\]

This means \(u\) is the unique weak solution of (8.1) with \(\text{tr}(u) = ν\). □

Next we demonstrate Theorem 1.11.

**Proof of Theorem 1.11.**

1. Suppose \(u\) is a weak solution of (8.1) with \(\text{tr}(u) = ν\). Let \(\beta > 0\). Since

\[
\phi_{\mu}(x) \approx d(x) \quad \text{and} \quad K_{\mu}(x, y) \approx d(x)|x - y|^{-N} \quad \forall (x, y) \in (\Omega \setminus \Sigma_{\beta}) \times \delta \Omega,
\]

proceeding as in the proof of [16, Theorem 3.1], we may prove that \(ν\) is absolutely continuous with respect to the Bessel capacity \(\text{Cap}_{\frac{N-1}{2}p', p} \Omega\).

2. We assume \(ν \in \mathfrak{M}^+(\delta \Omega) \cap B_{\frac{2}{p-1}p} \Omega\). Then by (8.14), we may apply [16, Theorem A] to deduce that \(\kappa_{\mu}[ν] \in L^p(\Omega \setminus \Sigma_{\beta}; \phi_{\mu})\) for any \(\beta > 0\). Denote \(g_{n}(t) = \max\{\min\{|t|^{p-1}, n\}, -n\}\). Then by applying Theorem 3.4 with \(w = \kappa_{\mu}[ν], v = 0\), and \(g = g_{n}\), we find that there exists a unique weak solution \(vₙ \in L^1(\Omega; \phi_{\mu})\) of

\[
\begin{cases}
-L_{\mu} vₙ + g_{n}(vₙ) = 0 & \text{in } \Omega \setminus \Sigma, \\
\text{tr}(vₙ) = ν,
\end{cases}
\]

such that \(0 \leq vₙ \leq \kappa_{\mu}[ν]\) in \(\Omega \setminus \Sigma\). Furthermore, by (2.28), \{vₙ\} is nonincreasing. Denote \(v = \lim_{n \to \infty} vₙ\) then \(0 \leq v \leq \kappa_{\mu}[ν]\) in \(\Omega \setminus \Sigma\).

We have

\[
- \int_{\Omega} vₙ L_{\mu} \zeta \, dx + \int_{\Omega} g_{n}(vₙ) \zeta \, dx = - \int_{\Omega} \kappa_{\mu} [νₙ] L_{\mu} \zeta \, dx \quad \forall \zeta \in X_{\mu}(\Omega \setminus \Sigma).
\]

By taking \(\phi_{\mu}\) as test function, we obtain

\[
\int_{\Omega} (\lambda_{\mu} vₙ + g_{n}(vₙ)) \phi_{\mu} \, dx = \lambda_{\mu} \int_{\Omega} \kappa_{\mu} [ν] \phi_{\mu} \, dx,
\]

(8.15)
which, together with by Fatou’s Lemma, implies \( v, v^p \in L^1(\Omega; \phi) \) and
\[
\int_\Omega \left( \lambda v + v^p \right) \phi \, dx \leq \lambda \int_\Omega \kappa \left[ v \right] \phi \, dx.
\]
Hence \( v + G_\mu[v^p] \) is a nonnegative \( L^1 \) harmonic. By Representation Theorem 2.3, there exists a unique \( \overline{v} \in \mathcal{W}^1(\Omega \cup \Sigma) \) such that \( v + G_\mu[v^p] = \kappa_\mu[\overline{v}] \). Since \( v \leq \kappa_\mu[v] \), by Proposition 2.5(i), \( \overline{v} = \text{tr}(v) \leq \text{tr}(\kappa_\mu[v]) = v \) and hence \( \overline{v} \) has compact support in \( \partial \Omega \).

Let \( \zeta \in X_{\kappa_\mu}(\Omega \setminus \Sigma) \) and \( \beta > 0 \) be small enough such that \( \Omega_{4\beta} \cap \Sigma = \emptyset \) (recall that \( \Omega_{\beta} \) is defined in Notations). We consider a cut-off function \( \psi_\beta \in C^\infty(\mathbb{R}^N) \) such that \( 0 \leq \psi_\beta \leq 1 \) in \( \mathbb{R}^N \), \( \psi_\beta = 1 \) in \( \Omega_{\beta} \), and \( \psi_\beta = 0 \) in \( \Omega \setminus \Omega_\beta \). Then the function \( \psi_\beta \zeta = \psi_\beta \zeta \in X_{\mu}(\Omega \setminus \Sigma) \) has compact support in \( \overline{\Omega}_{\beta} \). Hence, by (4.14) and the fact that \( \frac{\partial \psi_\beta \zeta}{\partial n} = \frac{\partial \zeta}{\partial n} \) on \( \partial \Omega \), we obtain
\[
\int_\Omega \left( -v \mu \psi_\beta \zeta + v^p \psi_\beta \zeta \right) \phi \, dx = -\int_{\partial \Omega} \frac{\partial \zeta}{\partial n} P_{\mu}(x_0, y) \, d\nu(y) = -\int_\Omega \kappa_\mu[\overline{v}] \mu \zeta \phi \, dx. \tag{8.16}
\]

Also,
\[
\int_\Omega \left( -v_n \mu \psi_\beta \zeta + g_n(v_n) \psi_\beta \zeta \right) \phi \, dx = -\int_{\partial \Omega} \frac{\partial \zeta}{\partial n} P_{\mu}(x_0, y) \, d\nu(y) = -\int_\Omega \kappa_\mu[v] \mu \zeta \phi \, dx. \tag{8.17}
\]
Since \( v \leq v_n \leq \kappa_\mu[v] \) and \( \kappa_\mu[v] \in L^p(\Omega_{4\beta}; \phi) \), by letting \( n \to \infty \) in (8.17), we obtain by the dominated convergence theorem that
\[
\int_\Omega \left( -v \mu \psi_\beta \zeta + v^p \psi_\beta \zeta \right) \phi \, dx = -\int_\Omega \kappa_\mu[v] \mu \zeta \phi \, dx. \tag{8.18}
\]
From (8.16) and (8.18), we deduce that
\[
\int_\Omega \kappa_\mu[\overline{v}] \mu \zeta \phi \, dx = \int_\Omega \kappa_\mu[v] \mu \zeta \phi \, dx, \quad \forall \zeta \in X_{\mu}(\Omega \setminus \Sigma).
\]
Since \( \kappa_\mu[\overline{v}], \kappa_\mu[v] \in C^2(\Omega \setminus \Sigma) \), by the above inequality, we can easily show \( \kappa_\mu[\overline{v}] = \kappa_\mu[v] \), which implies \( \overline{v} = v \) by Proposition 2.5.

3. If \( v \in \mathcal{W}^+(\partial \Omega) \) vanishes on Borel sets \( E \subset \partial \Omega \) with zero \( \text{Cap}_{\mu_{p, p'}(\mathbb{R}^N)} \)-capacity, there exists an increasing sequence \( \{v_n\} \) of positive measures in \( B_{\mu_{p, p'}}(\partial \Omega) \) which converges to \( v \) (see [8, 12]). Let \( u_n \) be the unique weak solution of (8.1) with boundary trace \( v_n \). Since \( \{v_n\} \) is increasing, by (2.28), \( \{u_n\} \) is increasing. Moreover, \( 0 \leq u_n \leq \kappa_\mu[v_n] \leq \kappa_\mu[v] \). Denote \( u = \lim_{n \to \infty} u_n \). By an argument similar to the one leading to (8.15), we obtain
\[
\int_\Omega \left( \lambda u_n + u_n^p \right) \phi \, dx = \lambda \int_\Omega \kappa[v] \phi \, dx,
\]
it follows that \( u, u^p \in L^1(\Omega; \phi) \). By the dominated convergence theorem, we derive
\[
\int_\Omega \left( -u \mu \zeta + u^p \zeta \right) \phi \, dx = -\int_\Omega \kappa_\mu[v] \mu \zeta \phi \, dx \quad \forall \zeta \in X_{\mu}(\Omega \setminus \Sigma),
\]
and thus \( u \) is the unique weak solution of (8.1). \( \square \)
APPENDIX: A PRIORI ESTIMATES

**Proposition A.1.** There exists $R_0 \in (0, \beta_0)$ such that for any $z \in \Sigma$ and $0 < R \leq R_0$, there is a supersolution $w := w_{R,z}$ of (6.1) in $\Omega \cap B(z, R)$ such that

$$w \in C(\Omega \cap B(z, R)), \quad w = 0 \text{ on } \Sigma \cap B(z, R),$$

$$w(x) \to \infty \text{ as } \text{dist}(x, F) \to 0, \text{ for any compact subset } F \subset (\Omega \setminus \Sigma) \cap \partial B(z, R).$$

More precisely, for $\gamma \in (\alpha_-, \alpha_+)$, $w$ can be constructed as

$$w(x) = \begin{cases} 
\Lambda (R^2 - |x-z|^2)^{-b} \sigma (x)^{-\gamma} & \text{if } \mu < H^2, \\
\Lambda (R^2 - |x-z|^2)^{-b} \sigma (x)^{-H} \sqrt{\ln \left( \frac{eR}{\sigma (x)} \right)} & \text{if } \mu = H^2,
\end{cases}$$

with $b \geq \max\{\frac{2}{p-1}, \frac{N-2}{2}, 1\}$ and $\Lambda > 0$ large enough depending only on $R_0, \gamma, N, b, p$ and the $C^2$ characteristic of $\Sigma$.

**Proof.** Without loss of generality, we assume $z = 0 \in \Sigma$.

**Case 1:** $\mu < H^2$. Set

$$w(x) := \Lambda (R^2 - |x|^2)^{-b} \sigma (x)^{-\gamma} \text{ for } x \in B(0, R),$$

where $\gamma > 0$, $b$ and $\Lambda > 0$ will be determined later on. Then, by straightforward computation with $r = |x|$ and using (2.2), we obtain

$$-I_\mu w + w^p = \Lambda (R^2 - r^2)^{-b-2} \sigma (x)^{-2} (I_1 + I_2 + I_3 + I_4), \quad (A.1)$$

where

$$I_1 := \Lambda^{p-1} (R^2 - r^2)^{-(p-1)b+2} \sigma^{-(p-1)\gamma+2},$$

$$I_2 := -(R^2 - r^2)^2 (-\gamma \eta \sigma - \gamma (N - k - 2 - \gamma) + \mu),$$

$$I_3 := -2b \sigma^2 (N R^2 + (2b + 2 - N) r^2),$$

$$I_4 := 4b \gamma \sigma (R^2 - r^2) x \nabla \sigma.$$

If we choose $b \geq \frac{N-2}{2}$, then

$$-I_3 \leq 4b(b+1)R^2 \sigma^2 \text{ and } |I_4| \leq 4b |\gamma| R (R^2 - r^2) \sigma. \quad (A.2)$$

Next we choose $\gamma \in (\alpha_-, \alpha_+)$, then $-\alpha_+ (N - k - 2) + \mu < -\gamma (N - k - 2 - \gamma) + \mu < 0$. In addition, there exist $\epsilon_0 > 0$ and $\delta_0 > 0$ such that if $\sigma \leq \delta_0$, then

$$-\alpha_+ (N - k - 2) + \mu < -\gamma \eta \sigma - \gamma (N - k - 2 - \gamma) + \mu < -\epsilon_0.$$

It follows that if $\sigma \leq \delta_0$, then

$$I_2 \geq \epsilon_0 (R^2 - r^2)^2. \quad (A.3)$$
We set

\[ \mathcal{A}_1 := \left\{ x \in \Omega \cap B_R(0) : d_\Sigma(x) \leq c_1 \frac{R^2 - r^2}{R} \right\}, \]

where \( c_1 = \frac{e_0}{16b(|\gamma| + 1)} \),

\[ \mathcal{A}_2 := \left\{ x \in \Omega \cap B_R(0) : d_\Sigma(x) \leq \delta_0 \right\}, \quad \mathcal{A}_3 := \{ x \in \Omega : d_\Sigma(x) \geq \delta_0 \} \]

In \( \mathcal{A}_1 \cap \mathcal{A}_2 \), by (A.2) and (A.3), for \( b \geq \max\left\{ \frac{N-2}{2}, 1 \right\} \), we have

\[ I_2 + I_3 + I_4 \geq \frac{e_0(R^2 - r^2)^2}{2}. \]  

(A.4)

In \( \mathcal{A}_1 \cap \mathcal{A}_2 \), \( d_\Sigma \geq c_1 \frac{R^2 - r^2}{R} \). If we choose \( b > \frac{2}{p-1} \), then there exists \( \Lambda \) large enough depending on \( p, R_0, \delta_0, N, b, \gamma \) such that the following estimate holds:

\[ I_1 \geq 2 \max\{4b(b + 1)R^2 d_\Sigma^2, 4b|\gamma| d_\Sigma R(R^2 - r^2)\}. \]  

(A.5)

This, together with (A.5), yields

\[ I_1 + I_3 + I_4 \geq 0. \]  

(A.6)

In \( \mathcal{A}_3 \), \( d_\Sigma \geq \delta_0 \). Therefore, we can show that there exists \( c_2 > 0 \) depending on \( N, \gamma, b, \|\eta\|_{L^\infty(\Sigma_{4\delta_0})}, \delta_0, p \) such that if \( \Lambda \geq c_2 \), then, in \( \mathcal{A}_3 \),

\[ I_1 \geq 3 \max\{|\gamma \eta|d_\Sigma(R^2 - r^2)^2, 4d_\Sigma^2 b(b + 1)R^2, 4bd_\Sigma R(R^2 - r^2)\}. \]

It follows that

\[ I_1 + I_2 + I_3 + I_4 \geq 0. \]  

(A.7)

Combining (A.1), (A.3), (A.4), (A.6), and (A.7), we deduce that for \( \gamma \in (0, \alpha_+) \), \( b \geq \max\left\{ \frac{2}{p-1}, \frac{N-2}{2}, 1 \right\} \), and \( \Lambda > 0 \) large enough, there holds

\[ -L_\mu w + w^p \geq 0 \quad \text{in } \Omega \cap B(0, R). \]  

(A.8)

**Case 2:** \( \mu = H^2 \). Set

\[ w(x) := \Lambda(R^2 - r^2)^{-b}d_\Sigma^{\frac{1}{2}} \left( \ln \frac{eR_0}{d_\Sigma} \right)^{\frac{1}{2}}, \quad \text{for } |x| < R, \]

where \( b \) and \( \Lambda \) will be determined later. Then, by straightforward calculations we have

\[-L_\mu w + w^p = \Lambda(R^2 - r^2)^{-b-2}d_\Sigma^{-H-2} \left( \ln \frac{eR}{d_\Sigma} \right)^{-\frac{1}{2}} (I_1 + I_2 + I_3 + I_4), \]

(A.9)

where

\[ I_1 := (R^2 - r^2)^2 \left[ \frac{1}{2} \eta d_\Sigma \left( 2H \left( \ln \frac{eR}{d_\Sigma} \right)^2 + \left( \ln \frac{eR}{d_\Sigma} \right) + \frac{1}{4} \right) \right], \]
\[ I_2 := 2b(R^2 - r^2)\left[ 2H \left( \ln \frac{eR}{d\Sigma} \right)^2 + \left( \ln \frac{eR}{d\Sigma} \right) \right] x \nabla d\Sigma, \]

\[ I_3 := -2bd^2 \left( \ln \frac{eR}{d\Sigma} \right)^2 [NR^2 + (2b + 2 - N)r^2], \]

\[ I_4 := \Lambda^{p-1}(R^2 - r^2)^{-b(p-1)+2}d\Sigma^{-H(p-1)+2} \left( \ln \frac{eR}{d\Sigma} \right)^{\frac{1}{2}(p-1)+2}. \]

Notice that \( \frac{eR}{d\Sigma} \geq e \), whence

\[ (2H + 1) \left( \ln \frac{eR}{d\Sigma} \right)^2 \leq 2H \left( \ln \frac{eR}{d\Sigma} \right)^2 \leq (2H + 1) \left( \ln \frac{eR}{d\Sigma} \right)^2. \]  

(A.10)

If we choose \( b \geq \frac{N-2}{2} \), then

\[ |I_2| \leq 4b(b+1)(R^2 - r^2)(\ln \frac{eR}{d\Sigma})^2 d\Sigma R, \]

\[ |I_3| \leq 4b(b+1)(\ln \frac{eR}{d\Sigma})^2 d^2 R^2. \]  

(A.11)

From (A.10), we deduce that there exist \( \varepsilon_0 > 0 \) and \( \delta_0 > 0 \) such that if \( d\Sigma \leq \delta_0 \), then

\[ \frac{1}{2} \eta d\Sigma \left( 2H \left( \ln \frac{eR}{d\Sigma} \right)^2 + \left( \ln \frac{eR}{d\Sigma} \right) \right) + \frac{1}{4} \geq \varepsilon_0. \]

Therefore if \( d\Sigma \leq \delta_0 \), then

\[ \tilde{I}_1 \geq \varepsilon_0 (R^2 - r^2)^2. \]  

(A.12)

Denote

\[ \tilde{A}_1 := \left\{ x \in \Omega \cap B_R(0) : d\Sigma(x) \leq \tilde{c}_1 \frac{R^2 - r^2}{R(\ln \frac{eR}{d\Sigma})^2} \right\} \quad \text{where} \quad \tilde{c}_1 = \frac{\varepsilon_0}{16b(b+1)}, \]

\[ \tilde{A}_2 := \left\{ x \in \Omega \cap B_R(0) : d\Sigma(x) \leq \delta_0 \right\}, \quad \tilde{A}_3 := \{ x \in \Omega : d\Sigma(x) \geq \delta_0 \}. \]

In \( \tilde{A}_1 \cap \tilde{A}_2 \), for \( b \geq \max\{\frac{N-2}{2}, 1\} \), we have

\[ I_1 + I_2 + I_3 \geq \frac{(R^2 - r^2)^2}{16}. \]  

(A.13)

In \( \tilde{A}_1^c \cap \tilde{A}_2 \), we have \( d\Sigma \geq \tilde{c}_1 \frac{R^2 - r^2}{R(\ln \frac{eR}{d\Sigma})^2} \). If \( b > \frac{2}{p-1} \), then we can choose \( \Lambda \) large enough depending on \( p, R_0, k, \delta_0, N, b \) such that

\[ I_4 \geq 2 \max \left\{ 4b(b+1)(R^2 - r^2) \left( \ln \frac{eR}{d\Sigma} \right)^2 d\Sigma R, 4b(b+1) \left( \ln \frac{eR}{d\Sigma} \right)^2 d_{\Sigma}^2 R^2 \right\}. \]
This and (A.11) imply
\begin{equation}
\bar{I}_2 + \bar{I}_3 + \bar{I}_4 \geq 0.
\end{equation}
(A.14)

In \( A_3 \), \( d_\Sigma \geq \delta_0 \). Similarly as in Case 1, we can choose \( \Lambda \) large enough depending on \( p, R_0, \delta_0, N, k, b \) such that
\begin{equation}
\bar{I}_1 + \bar{I}_2 + \bar{I}_3 + \bar{I}_4 \geq 0.
\end{equation}
(A.15)

Combining (A.9), (A.12), (A.13), (A.14), and (A.15), we obtain (A.8). \( \square \)

We recall here that \( \bar{W} \) has been defined in (1.14).

**Proposition A.2.** Let \( 1 < p < \frac{2+\alpha_-}{\alpha_-} \) if \( \alpha_- > 0 \) or \( p < \infty \) if \( \alpha_- \leq 0 \). Assume \( F \subset \Sigma \) is a compact subset of \( \Sigma \) and denote by \( d_F(x) = \text{dist}(x,F) \). There exists a constant \( C = C(N, \Omega, \Sigma, \mu, p) \) such that if \( u \) is a nonnegative solution of (6.1) in \( \Omega \setminus \Sigma \) satisfying
\begin{equation}
\lim_{x \in \Omega \setminus \Sigma, x \to \xi} \frac{u(x)}{\bar{W}(x)} = 0 \quad \forall \xi \in (\partial \Omega \cup \Sigma) \setminus F, \quad \text{locally uniformly in} \ \Sigma \setminus F,
\end{equation}
then
\begin{equation}
u(x) \leq C d(x) d_\Sigma(x)^{-\alpha_-} d_F(x)^{-\frac{2}{p-1}+\alpha_-} \quad \forall x \in \Omega \setminus \Sigma,
\end{equation}
\begin{equation}|
\nabla u(x)| \leq C \frac{d(x)}{\min(d(x), d_\Sigma(x))} d_\Sigma(x)^{-\alpha_-} d_F(x)^{-\frac{2}{p-1}+\alpha_-} \quad \forall x \in \Omega \setminus \Sigma.
\end{equation}
(A.17) (A.18)

**Proof.** The proof is in the spirit of [17, Proposition 3.4.3]. Let \( \xi \in \Sigma \setminus F \) and put \( d_{F,\xi} = \frac{1}{2} d_F(\xi) < 1 \). Denote
\begin{equation}
\Omega_{\xi} := \frac{1}{d_{F,\xi}} \Omega = \{ y \in \mathbb{R}^N : d_{F,\xi} y \in \Omega \} \quad \text{and} \quad \Sigma_{\xi} = \frac{1}{d_{F,\xi}} \Sigma = \{ y \in \mathbb{R}^N : d_{F,\xi} y \in \Sigma \}.
\end{equation}

If \( u \) is a nonnegative solution of (6.1) in \( \Omega \setminus \Sigma \), then the function
\begin{equation}
u_{\xi}(y) := \frac{1}{d_{F,\xi}^{-2}} u(d_{F,\xi} y), \quad y \in \Omega_{\xi} \setminus \Sigma_{\xi}
\end{equation}
is a nonnegative solution of
\begin{equation}
-\Delta \nu_{\xi} - \frac{\mu}{|\text{dist}(y,\Sigma_{\xi})|^2} \nu_{\xi} + \left( \nu_{\xi} \right)^p = 0
\end{equation}
in \( \Omega_{\xi} \setminus \Sigma_{\xi} \).

As \( d_{F,\xi} \leq 1 \), the \( C^2 \) characteristic of \( \Omega \) (respectively \( \Sigma \)) is also a \( C^2 \) characteristic of \( \Omega_{\xi} \) (respectively, \( \Sigma_{\xi} \)), therefore this constant \( C \) can be taken to be independent of \( \xi \). Let \( R_0 = \beta_0 \) be the constant in Proposition A.1. Set \( r_0 = \frac{3R_0}{4} \), and let \( w_{r_0,\xi} \) be the supersolution of (A.19) in \( B(\frac{1}{d_{F,\xi}} \xi, r_0) \cap (\Omega_{\xi} \setminus \Sigma_{\xi}) \) constructed in Proposition A.1 with \( R = r_0 \) and \( z = \frac{1}{d_{F,\xi}} \xi \). By a similar
argument as in the proof of Lemma 6.1, we can show

$$u^\xi(y) \leq w_{r_0, \xi}(y) \quad \forall y \in B\left(\frac{1}{d_{F, \xi}}, r_0\right) \cap (\Omega^\xi \setminus \Sigma^\xi).$$

Thus, $u^\xi$ is bounded from above in $B\left(\frac{1}{d_{F, \xi}}, \frac{3r_0}{5}\right) \cap (\Omega^\xi \setminus \Sigma^\xi)$ by a constant $C$ depending only on $N, k, \mu, p$, and the $C^2$ characteristic of $\Omega$ and $\Sigma$.

Now we note that $u^\xi$ is a nonnegative $L_\mu$ subharmonic function and by the last inequality satisfies, for any $y \in (\alpha_-, \alpha_+)$,

$$u^\xi(y) \leq C \begin{cases} d_{\Sigma^\xi}(y)^{-\gamma} & \text{if } \mu < H^2, \\ d_{\Sigma^\xi}(y)^{-H} \sqrt{\ln \left(\frac{e^R}{d_{\Sigma^\xi}(y)}\right)} & \text{if } \mu = H^2, \end{cases}$$

for any $y \in B\left(\frac{1}{d_{F, \xi}}, r_0\right) \cap (\Omega^\xi \setminus \Sigma^\xi)$, where $C$ is a positive constant depending only on $R_0, \gamma, N, \beta, p$, and the $C^2$ characteristic of $\Sigma$. Hence,

$$\lim_{y \to P \atop y \in \Omega^\xi} u^\xi(y) = 0 \quad \forall P \in B\left(\frac{1}{d_{F, \xi}}, \frac{3r_0}{5}\right) \cap \Omega^\xi \setminus \Sigma^\xi,$$

where

$$W^\xi(y) = 1 - \eta \frac{\bar{\beta}_0}{d_{F, \xi}} + \eta \frac{\bar{\beta}_0}{d_{F, \xi}} W^\xi(y) \quad \text{in } \Omega^\xi \setminus \Sigma^\xi,$$

and

$$W^\xi(x) = \begin{cases} d_{\Sigma^\xi}(x)^{-\alpha_+} & \text{if } \mu < H^2, \\ d_{\Sigma^\xi}(x)^{-H} \ln d_{\Sigma^\xi}(x) & \text{if } \mu = H^2, \end{cases} \quad x \in \Omega^\xi \setminus \Sigma^\xi.$$
and $d_F(x) \leq \frac{2(8+2\beta_0)}{8+\beta_0} d_F(x)$. This, combined with (A.22), (A.23), and the fact that $p < \frac{2+\alpha_-}{\alpha_-}$, yields

$$u(x) \leq Cd_\Sigma(x)^{-\alpha_-} d_{F,(\xi)}^{-\frac{2}{p-1}+\alpha_-} \leq Cd_\Sigma(x)^{-\alpha_-} d_F(x)^{-\frac{2}{p-1}+\alpha_-}.$$ 

If $d_\Sigma(x) > \frac{\beta_0}{8+2\beta_0} d_F(x)$, then by (6.2) and the assumption $p < \frac{2+\alpha_-}{\alpha_-}$, we obtain

$$u(x) \leq Cd_\Sigma(x)^{-\alpha_-} \leq Cd_\Sigma(x)^{-\alpha_-} d_F(x)^{-\frac{2}{p-1}+\alpha_-}.$$ 

Thus, (A.17) holds for every $x \in \Sigma \frac{\beta_0}{2}$ such that $d_F(x) < \frac{4+\beta_0}{2+\beta_0}$.

**Case 2:** $d_F(x) \geq \frac{4+\beta_0}{2+\beta_0}$. Let $\xi$ be the unique point in $\Sigma \setminus F$ such that $|x - \xi| = d_\Sigma(x)$. Since $u$ is an $L_\mu$-subharmonic function in $B(\xi, \frac{\beta_0}{4}) \cap (\Omega \setminus \Sigma)$.

By (A.16) and [14, Lemma 3.3 and estimate (2.10)], we deduce that

$$u(x) \leq Cd_\Sigma(x)^{-\alpha_-} \leq Cd_\Sigma(x)^{-\alpha_-} d_F(x)^{-\frac{2}{p-1}+\alpha_-} \quad \forall x \in B\left(\xi, \frac{\beta_0}{2}\right) \cap (\Omega \setminus \Sigma).$$

In view of the proof of A.21, we may show that $C$ depends only on $\beta_0, \gamma, N, \beta, p$, and the $C^2$ characteristic of $\Sigma$.

(ii) Let $x_0 \in \Omega \setminus \Sigma$. Put $\ell' = \text{dist}(x_0, \Omega \setminus \Sigma) = \min\{d(x_0), d_\Sigma(x_0)\}$ and

$$(\Omega \setminus \Sigma)^\ell := \frac{1}{\ell'}(\Omega \setminus \Sigma) = \{y \in \mathbb{R}^N : \ell'y \in \Omega \setminus \Sigma\}, \quad d_{(\Omega \setminus \Sigma)^\ell}(y) := \text{dist}(y, \partial(\Omega \setminus \Sigma)^\ell).$$

If $x \in B(x_0, \frac{\ell'}{2})$, then $y = \ell'^{-1}x$ belongs to $B(y_0, \frac{1}{2})$, where $y_0 = \ell'^{-1}x_0$. Also we have that $\frac{1}{2} \leq d_{(\Omega \setminus \Sigma)^\ell}(y) \leq \frac{3}{2}$ for each $y \in B(y_0, \frac{1}{2})$. Set $v(y) = u(\ell'y)$ for $y \in B(y_0, \frac{1}{2})$, then $v$ satisfies

$$-\Delta v - \frac{\mu}{d_{(\Omega \setminus \Sigma)^\ell}^2} v + \ell'^2 |v|^p = 0 \quad \text{in} \ B\left(y_0, \frac{1}{2}\right).$$

By standard elliptic estimate, we have

$$\sup_{y \in B(y_0, \frac{1}{4})} |\nabla v(y)| \leq C \left( \sup_{y \in B(y_0, \frac{1}{4})} |v(y)| + \sup_{y \in B(y_0, \frac{1}{4})} \ell'^2 |v(y)|^p \right).$$

This, together with the equality $\nabla v(y) = \ell'\nabla u(x)$, estimate (A.17), and the assumption on $p$, implies

$$|\nabla u(x_0)| \leq C \ell'^{-1} \left( d(x_0) d_\Sigma^{-\alpha_-} (x_0) d_F(x_0)^{-\frac{2}{p-1}+\alpha_-} + \ell'^2 d(x_0)^p d_\Sigma(x_0)^{-\alpha_-} d_F(x_0)^{p \left( -\frac{2}{p-1}+\alpha_- \right)} \right)$$

$$\leq C \frac{d(x_0)}{\min\{d(x_0), d_\Sigma(x_0)\}} d_\Sigma(x_0)^{-\alpha_-} d_F(x_0)^{-\frac{2}{p-1}+\alpha_-} \left[ 1 + \left( \frac{d_\Sigma(x_0)}{d_F(x_0)} \right)^{2-(p-1)\alpha_-} \right]$$

$$\leq C \frac{d(x_0)}{\min\{d(x_0), d_\Sigma(x_0)\}} d_\Sigma(x_0)^{-\alpha_-} d_F(x_0)^{-\frac{2}{p-1}+\alpha_-}.$$ 

Therefore, estimate (A.18) follows since $x_0$ is an arbitrary point. The proof is complete. \(\square\)
ACKNOWLEDGEMENTS
K. T. Gkikas acknowledges support by the Hellenic Foundation for Research and Innovation (H.F.R.I.) under the “2nd Call for H.F.R.I. Research Projects to support postdoctoral researchers” (Project Number: 59). P.-T. Nguyen was supported by Czech Science Foundation, Project GA22-17403S.

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