Membrane topology and matrix regularization

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Abstract

The problem of membrane topology in the matrix model of M-theory is considered. The matrix regularization procedure, which makes a correspondence between finite-sized matrices and functions defined on a two-dimensional base space, is reexamined. It is found that the information of topology of the base space manifests itself in the eigenvalue distribution of a single matrix. The precise manner of the manifestation is described. The set of all eigenvalues can be decomposed into subsets whose members increase smoothly, provided that the fundamental approximations in matrix regularization hold well. Those subsets are termed as eigenvalue sequences. The eigenvalue sequences exhibit a branching phenomenon which reflects Morse-theoretic information of topology.

Furthermore, exploiting the notion of eigenvalue sequences, a new correspondence rule between matrices and functions is constructed. The new rule identifies the matrix elements directly with Fourier components of the corresponding function, evaluated along certain orbits. The rule has semi-locality in the base space, so that it can be used for all membrane topologies in a unified way. A few numerical examples are studied, and consistency with previously known correspondence rules is discussed.

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1 Introduction

Topological properties of a system are often important in investigating the dynamics of the system. It seems certain that M-theory\cite{1} has membranes as dynamical degrees of freedom. Furthermore, the only existing proposal for formulation of M-theory, namely the matrix model of M-theory\cite{2}, can be considered as an attempt to define quantum membrane theory \cite{3, 4, 5}. More explicitly stated, it is a regularized version of membrane theory in lightcone gauge, dynamical variables becoming $N \times N$ matrices instead of functions defined on two-dimensional worldspace.

But, at present, the topological properties of membranes in M-theory are not known. The concern of this paper is membrane topology \footnote{We use the word membrane topology to express the topology of a configuration of membranes in a time-slice. The topology is not of a single membrane but of a totality of membranes. Thus, membrane topology is classified by the numbers of membranes $n_{i}$ which has genus $i = 0, 1, \cdots$.} in the matrix model. It is believed that the matrix model can describe membranes of arbitrary topologies. However, there has been a problem: we do not know whether and how the information of the topology manifests itself in the matrix model. The cause for this problem lies in the manner in which the correspondence between matrices and functions has been given. There has been no unique rule that can deal with all membrane topologies. Instead, we have many different rules for different topologies \cite{3, 4, 5, 6, 7}, interrelationships between those rules being unclear.

In this paper, we address this problem by reexamining the regularization procedure, the so-called matrix regularization. We shall show that the information indeed manifests itself in the eigenvalue distribution of a single matrix. The precise manner of the manifestation will be described. Moreover, we have constructed a new correspondence rule between functions and matrices which can be applied to all membrane topologies in a unified way.

We start by discussing relevant aspects of matrix regularization, in section 2. We take the simple view that, classically, the matrix regularization is an approximation of continuum theory by a discretized theory. \footnote{In quantum theory, at the same time, the matrix regularization is considered as a definition of continuum theory by a non-trivial limit of discrete theories. This is the reason why we should first treat the finite-$N$ theory carefully.} This approximation between two theories is based solely on some fundamental large-$N$ approximation formulae (2.7)-(2.9). They play a vital role in this paper. We also recall the well-known mathematical analogy between the matrix regularization and canonical quantization of systems with one degree of freedom, which will be our main tool in subsequent discussions.

Then, in section 3, we turn to the investigation of membrane topology in the matrix model. Our basic observation is that, in order to study membrane topology, it suffices to consider the two-dimensional base space, which we shall term as the $\sigma$-space, not the shape.
of the membranes in the target space. This observation greatly simplifies the analysis, since it enables us to deal with only a single matrix, not many matrices.

We base the discussion on the analog of the Bohr-Sommerfeld quantization condition. We shall show that, in the case where the fundamental approximations hold well, the eigenvalue distribution of a matrix has a particular structure. Namely, the set of all eigenvalues can be decomposed into subsets characterized by the following property: the eigenvalues in one of the subsets, when sorted, increase smoothly. We call these subsets as eigenvalue sequences. The grouping of the eigenvalues into sequences reveals a branching phenomenon of sequences. We find that the branching phenomenon, in turn, reflects certain Morse-theoretic information of topology of the $\sigma$-space. This is our answer to the above problem. Thus, the information of topology manifests itself, in the world of matrices, as a branching phenomenon of eigenvalue sequences.

Furthermore, the notion of eigenvalue sequences enables us to construct a new correspondence rule between matrices and functions, which is the subject in section 4. The matrix elements are approximately equal to Fourier components of the corresponding function, calculated along appropriate orbits on the $\sigma$-space. The rule is analogous to the correspondence noticed by Heisenberg when he created Matrix Mechanics pursuing Bohr’s correspondence principle[8]. There, the matrix elements of an observable, in the basis which makes the Hamiltonian diagonal, are equal to the classical Fourier components of the observable along the appropriate classical orbits on the phase space. We shall show that the fundamental approximation formulae hold well if the new correspondence rule holds. The correspondence rule contains the above-mentioned analog of the Bohr-Sommerfeld condition. This justifies the use of it in section 3.

The new rule is semi-local in the $\sigma$-space, and consequently can be applied for all membrane topologies uniformly, in marked contrast with the previously known rules. This, in particular, enables one to construct functions corresponding to given matrices when the approximations are good. Using previous rules, one could only do the reverse, namely, to construct matrices corresponding to given functions. This is because one could not know the topology corresponding to the given matrices, and therefore could not choose the rule to be used.

Apart from the unified treatment for all topologies, the new rule has the virtue that the identification of the matrix elements with Fourier components is direct, and so that the geometrical meanings of the matrix elements are clear. Our arguments are also relevant to the matrix model of type IIB string theory [9], since the same regularization is involved. Further, the same kind of mathematics as that of matrix regularization appears in such subjects as bound states of D-branes or non-commutative field theory. Ideas in this paper may find some applications in those subjects.

A few illustrative numerical examples are given in section 5. The consistency between
our new rule and previous rules is checked by studying them. Finally, we conclude with some
discussions in section 6.

2 Matrix regularization

Let us briefly recall the matrix regularization procedure from our viewpoint. Although it
is supermembrane theory in eleven dimension [10] that is relevant to M-theory, we consider
bosonic membrane theory for simplicity of presentation.

Firstly, we shall describe the continuum theory. We parametrize the membranes by three
parameters \((\tau, \sigma^1, \sigma^2) = (\tau, \sigma)\). Then, the geometrical shape of membranes in spacetime is
described by the coordinate functions \(x^\mu(\tau, \sigma)\). In lightcone gauge formalism, \(\tau\) is chosen to
be equal to \(x^+\), and \(\sigma\) is chosen so as to make the area of a domain in the \(\sigma\)-space proportional
to total \(p^+\) contained in the domain. Here, we denote the momentum density vector of the
membranes by \(p^\mu\). The canonical variables of the system are transverse coordinates and
momenta (which are functions defined on the \(\sigma\)-space) as well as zero modes,

\[ x^\alpha(\sigma), p^\alpha(\sigma); X^-, -P^+ . \]  

The Hamiltonian is given by

\[ H = [\sigma] \int \frac{(p^\alpha)^2 + \frac{1}{2}(\{x^\alpha, x^\beta\})^2}{2P^+} d^2 \sigma, \]  

with Lie brackets

\[ \{f, g\} = \frac{\partial f}{\partial \sigma^1} \frac{\partial g}{\partial \sigma^2} - \frac{\partial f}{\partial \sigma^2} \frac{\partial g}{\partial \sigma^1}, \]  

where \(f\) and \(g\) are functions on the \(\sigma\)-space. We have also introduced a conventional constant
\([\sigma]\) which is the total area of the \(\sigma\)-space, \([\sigma] = \int d^2 \sigma\). The remaining ingredients of the
theory are the phase space constraints

\[ \{x^\alpha, p^\alpha\}(\sigma) = 0, \]  

and its global version. They correspond to the local symmetry of the lightcone gauge theory
under reparametrization by area-preserving diffeomorphism (APD) on the \(\sigma\)-space. This is
a local symmetry, because one can perform reparametrization by different APD for different
\(\tau\).

Secondly, we shall give the regularized theory. The canonical variables are \(N \times N\) matrices
as well as zero modes,

\[ \hat{x}^\alpha, \hat{p}^\alpha; X^-, -P^+ . \]
The Hamiltonian is given by

\[ H = N \text{Tr} \left( \hat{p}^\alpha \right)^2 - \frac{1}{\hbar} (2\pi [\hat{x}^\alpha, \hat{x}^\beta])^2/2P^+ \],

and the constraints are,

\[ [\hat{x}^\alpha, \hat{p}^\alpha] = 0, \]

where \([ , ]\) is a commutator of matrices.

Now, we turn to the explanation of the matrix regularization. The following fact is essential: there exists a correspondence between appropriate functions on the \(\sigma\)-space \(f(\sigma), g(\sigma), \cdots\) and matrices \(\hat{f}, \hat{g}, \cdots\) such that the fundamental approximation formulae

\[ \frac{1}{|\sigma|} \int f(\sigma) d^2 \sigma \approx \frac{1}{N} \text{Tr} \hat{f} \]

\[ \hat{f} \hat{g} \approx \hat{f} \hat{g} \]

\[ \{\hat{f}, \hat{g}\} \approx -i \frac{2\pi N}{|\sigma|} [\hat{f}, \hat{g}] \]

hold. Here, we denote by \(\{\hat{f}, \hat{g}\}\) and \(\hat{f} \hat{g}\) the matrices which correspond to the functions \(\{f, g\}(\sigma)\) and \(f(\sigma) g(\sigma)\), respectively.\(^3\) The larger is \(N\), the better is the approximation. From these formulae it follows that the continuum theory, defined by (2.1)-(2.3) can be approximated by a regularized theory defined by (2.4)-(2.6). We stress the importance of above formulae. They are almost the definition of the matrix regularization.

Since Lie brackets and matrix commutators both obey the Jacobi identity and antisymmetry, the important advantage of matrix regularization follows. Namely, the regularized theory has local symmetry under the transformation

\[ x^{\alpha'}(\tau) = U(\tau)x^{\alpha}(\tau)U(\tau)^{-1}, \quad p^{\alpha'}(\tau) = U(\tau)p^{\alpha}(\tau)U(\tau)^{-1}, \]

where \(U(\tau)\) is an arbitrary matrix which is a function of \(\tau\), corresponding to the APD symmetry in continuum theory.

The matrix regularization procedure is analogous to the quantization of a system which has one degree of freedom, as is well known. The analogy can be summarized as,

| Canonical quantization | Matrix regularization |
|------------------------|-----------------------|
| \((x, p)\)             | \((\sigma^1, \sigma^2)\) |
| Canonical transformation | Area-preserving diffeomorphism |
| \(\{, \}_\text{P.B.} \to -i \frac{1}{\hbar} [ , \] \) | \(\{, \} \to -i \frac{2\pi N}{|\sigma|} [ , \] \) |

\(^3\)Maybe we should add the linearity of the correspondence, \(\hat{f} + \hat{g} = \hat{f} + \hat{g}\), for the sake of completeness. We have omitted it since it holds trivially in all our discussions.
where \( \{ , \} \) is the usual Poisson brackets. We shall motivate our discussion by this analogy in section 3.

We conclude this section with discussions on the previously known correspondence rules. We first recall the general manner the rules are formulated. We must, first of all, fix topology of the \( \sigma \)-space. After that, we consider a basis in the vector space of all functions defined on the \( \sigma \)-space. Then, we define an appropriate basis in the vector space of all \( N \times N \) matrices, and postulate a correspondence between it and the basis in the space of the functions appropriately truncated. The rules are, finally, justified by checking that the fundamental approximations (2.7)-(2.9) hold well for large \( N \) by them.

This manner has made difficult to consider whether and how membrane topology manifests itself in the matrix model. In particular, one can expand an arbitrary matrix by basis referring to any particular topology. This fact, at first sight, seems to suggest that a configuration of matrix model could be interpreted as membranes of arbitrary topology, and there would be, therefore, no information of topology in the matrix model.

This is not necessarily true. Even if one can formally expand some matrices by a basis referring to a particular topology, the fundamental approximations may not work at all. In our perspective, that the matrix regularization is an approximation scheme, we cannot, then, interpret the matrices as membranes of the particular topology. Information of topology may be hidden in the matrices in this way. Through sections 3 and 4, we shall see indeed that, provided that the approximations are good, the information reflects in the eigenvalue distribution.

### 3 Membrane topology and matrix regularization

In this section, we show that the information of membrane topology manifests itself in the matrix model. Before explicit description of the manner of the manifestation, let us give some basic observations.

If one wishes to specify the complete shape of membranes in the target space, one needs...

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4This may be expected from the previously known rules. Let us, for example, imagine a smooth function defined on a torus. One can construct the corresponding matrix using the basis for torus topology. One can then expand the matrix by the basis (in the space of matrices) corresponding to the topology of a sphere, and construct a function defined on a sphere. We expect that the resulting function would have discontinuity or, in any case, some singularity (see subsection II. C of [11]). This implies that the function varies considerably in a very small length scale. Therefore the approximations may well be no good, since, in general, the smaller the length scale of the variation of functions, the larger must be \( N \) in order that the approximations are good.

However, it is difficult to characterize precisely, using only previously known correspondence rules, when the approximations break down. Hence, it has not been clear if this picture is indeed right.
information of many (that is, roughly speaking, as many as the dimension of the target space) functions. This would imply that one should study many matrices in the matrix model. However, the information of membrane topology, or at least the information of topology of the $\sigma$-space, is contained in one generic function defined on the $\sigma$-space, as is strongly suggested by Morse theory. We choose, as our basic strategy, to consider the latter information. Then, we shall seek in a single matrix the information of topology of the $\sigma$-space.

There is another point we would like to discuss. It is most natural to identify functions which are transformed into each other by APD transformations. We shall identify those matrices which are transformed into each other by similarity transformations, since (2.9) tells us that the counterpart of the APD transformation is the similarity transformation. This has some non-triviality, since it may happen that the identification is only allowed approximately. Nonetheless, we shall carry out the identification, because that the APD symmetry survives as (2.10) is the most important advantage of the matrix regularization. This identification and our strategy, to consider the topology of the $\sigma$-space, act together to greatly simplify the analysis. Since one can always diagonalize a single matrix, we can concentrate on the eigenvalue distribution of the matrix.

Having explained our basic strategy, we shall now proceed to investigate the manner of the manifestation of membrane topology in the eigenvalue distribution.

First, let us consider how one can read off the information of topology from a function in an APD invariant way. We choose an arbitrary generic function $f(\sigma)$. It could be one of the transverse coordinates, for instance. The function is fixed, throughout our discussion, as a kind of reference. Thus, we shall use $f$, shortly below, as both an analog of the Hamiltonian in canonical quantization and a Morse function. As a natural APD-invariant concept with a given function $f(\sigma)$, we introduce an ordinary differential equation (ODE)

$$\frac{d}{dt}\sigma = \{\sigma, f\}, \quad (3.1)$$

drawing analogy to the Hamiltonian equation of motion with a given Hamiltonian function $H(x, p)$, which is invariant under canonical transformations. This ODE governs the motion of points of the $\sigma$-space. Thus, we envisage an auxiliary Hamiltonian-like dynamical system with the $\sigma$-space as its phase space and with $f$ as its Hamiltonian. Since $f$ is conserved along the motion by the identity $\{f, f\} = 0$, an orbit of this equation is a part of an equal-$f$ line in the $\sigma$-space. It will form a closed loop because of the compactness of the $\sigma$-space.

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5 We use the word generic in the sense of Morse theory: we avoid degenerate functions, constant functions for instance, which can be changed into generic functions by arbitrarily small perturbations.

6 We note that the independent variable of the ODE, $t$, is just a mathematical tool to substantiate the analogy to canonical quantization. It has nothing to do with physical time coordinates of membrane theory.
Figure 1: The $\sigma$-space which has topology of a torus. The height in the figure is the reference function. Some orbits of the ODE $d\sigma/dt = \{\sigma, f\}$ are drawn, which form closed loops that are, in turn, parts of the equal-$f$ lines. If one gradually increases the value of $f$, one observes a branching phenomenon of the orbits: appearing, branching, merging, and disappearing processes at the points A, B, C, and D, respectively. Depicted is essentially the $\sigma$-space, so that the horizontal directions of the figure have rather arbitrary meanings. If one wishes, one can also give definite meanings to the horizontal directions by interpreting this figure as the geometrical shape of a membrane in the target space, and the reference function as one of the coordinate functions.

Then, if we scan the $\sigma$-space by gradually increasing the value of $f$, we will observe branching processes of these orbits. There are four types of these branching processes: appearing, disappearing, branching and merging. Let us consider, for a typical example, the situation depicted in Fig. 1. The membrane topology is that of a torus. The reference function $f$ is chosen to be the height in the figure, and some orbits of (3.1) are drawn. In this example, at the points A, B, C, D, the orbits appear, branch, merge, disappear, respectively. We can read off the information of topology from these processes. This is just the well-known idea of Morse theory. In particular, we obtain the Euler number of the $\sigma$-space, by subtracting the total number of the branching and merging processes from the total number of the appearing and disappearing processes.

Now, we shall show that this analysis of topology in the world of functions has a counterpart in the world of matrices. The analogy of the matrix regularization to canonical
quantization is useful here. In the latter, the Bohr-Sommerfeld quantization condition determines the eigenvalues of the Hamiltonian operator $\hat{H}$ from the classical Hamiltonian function $H(x, p)$ defined on the phase space $(x, p)$. Namely, we draw classical orbits in $(x, p)$ space, that are parts of equal-$H$ lines, so that the areas of the domains between two adjacent orbits are equal to $2\pi\hbar$. Then eigenvalues of $\hat{H}$ are given by the values of $H$ at these orbits. Here, we shall exploit the analogy, which is summarized in (2.11), and state the analog of the Bohr-Sommerfeld condition. Namely, we draw orbits of (3.1) in the $\sigma$-space so that the areas of the domains $^7$ between two adjacent orbits are equal to $[\sigma]/N$. Since $[\sigma]$ is the total area of the $\sigma$-space, this simply means that we divide the $\sigma$-space into $N$ parts of equal area. Eigenvalues of $\hat{f}$ are then given by the values of $f$ at these orbits. We assume this rule to hold. We shall justify the assumption in section 4.

If we apply this rule to the case in Fig. 1, then the eigenvalues of $\hat{f}$ can be grouped into four subsets each of which corresponds to the family of the orbits belonging to (i) the region from the point A to the point B, (ii) the left branch of the torus from the point B to the point C, (iii) the right branch of the torus from the point B to the point C, (iv) the region from the point C to the point D, respectively. We call these subsets as eigenvalue sequences. For large enough $N$, eigenvalues belonging to each sequence have the following property. If we sort the eigenvalues contained in a sequence in increasing order of their values, and make a graph plotting the values of them versus their order, then the plotted points can be linearly approximated locally. To put it short, the eigenvalues in a sequence increase smoothly. It should be clear that, in general, if we do not group the eigenvalues properly, then the graph become zigzag-shaped and the above property is lost. In section 4, we see that this linear approximation is essential in order the fundamental approximations (2.7)-(2.9) to hold.

The eigenvalue sequences should exhibit the same branching phenomenon as that of the orbits. For the example of Fig. 1, the sequence (i) appears and then branches into the sequences (ii) and (iii). They merge into the sequence (iv), and finally (iv) disappears. It is clear that all these considerations work the same in general cases other than that of Fig. 1. Thus, the information of membrane topology manifests itself in the branching phenomenon (which consists of appearing, branching, merging, disappearing processes) of eigenvalue sequences. A few examples, including the case similar to the situation in Fig. 1, are given in section 5.

$^7$The area of a domain means here area in the $\sigma$-space not in the target space. Its physical meaning is the total $p^+$ contained in the domain, apart from a conventional factor, by the gauge choice made in the lightcone gauge formalism.
4 The new correspondence rule

In this section, we present a new correspondence rule between matrices and functions, and then show that the fundamental approximations (2.7)-(2.9) stem from the rule.

We choose an arbitrary generic function \( f \) and fix it as a reference, as in section 3. The rule is formulated in such a way that the representation of matrices is so chosen that the matrix \( \hat{f} \), corresponding to the function \( f \), is diagonal. For simplicity of notation, we shall consider the case where only one eigenvalue sequence is present. We explain the generalization later in this section.

We first give the rule to determine the diagonal matrix \( \hat{f} \). To this end, we set up some notations. We again consider ODE (3.1)

\[
\frac{d\sigma}{dt} = \{\sigma, f\}.
\]

A solution of this ODE is periodic, the point of the \( \sigma \)-space circulating on a loop which is part of an equal-\( f \) line. We shall denote its period, as a function of \( f \), by \( T(f) \). We sort the eigenvalues of \( \hat{f} \) in increasing order, and call them \( \hat{f}_n, \cdot\cdot\cdot \leq \hat{f}_{n-1} \leq \hat{f}_n \leq \hat{f}_{n+1} \leq \cdot\cdot\cdot \). (4.1)

To be specific, we choose the representation such that

\[
\hat{f} = \text{diag}(\cdot\cdot\cdot, \hat{f}_{n-1} \quad \hat{f}_n \quad \hat{f}_{n+1} \quad \cdot\cdot\cdot).
\]

(4.2)

The relation between the function \( f \) and the matrix elements \( \hat{f}_n \) is the analog of the Bohr-Sommerfeld quantization condition stated in section 3. If \( N \) is sufficiently large, the rule can be formulated as,

\[
\hat{f}_m - \hat{f}_n \approx (m - n) \frac{[\sigma]}{N} \frac{1}{T(\hat{f}_m + \hat{f}_n)}.
\]

(4.3)

when \( |m - n| \) is small. We have used that for two nearby loops, one at \( f \) and the other at \( f + \delta f \), the area \( \delta S \) between them can be approximated by

\[
\delta S = \int |\text{grad} f| \delta s = T(f + \frac{\delta f}{2}) \delta f.
\]

(4.4)

We can construct \( \hat{f}_m \) satisfying (4.3) directly by the following method. We first define \( S(f) = \int^f (1/T(f)) df \). The value of \( S(f) \) runs from 0 to \([\sigma]\) in this case where there is only one eigenvalue sequence. We then consider the inverse function \( f(S) \), and set \( \hat{f}_m = f(S_m) \),
where $S_m$ are determined by $S_{m+1} - S_m = [\sigma]/N$ up to a constant shift. The shift should be of order $1/N$ for consistency.  

Having stated the correspondence rule for the reference function, we next turn to the correspondence rule for an arbitrary function $g$. We denote the matrix elements of the corresponding matrix $\hat{g}$ by $\hat{g}_{mn}$. When $|m-n|$ is small, $\hat{g}_{mn}$ is equal to the Fourier component of order $m-n$ of the function $g(\sigma(t))$. Here, $\sigma(t)$ denotes the solution of (3.1) along which the function $f(\sigma)$ takes the (constant) value $(\hat{f}_m + \hat{f}_n)/2$. To obtain explicit formulae, we define the Fourier components $g_s(f)$ by

$$g(\sigma(t)) = \sum_{s=-\infty}^{+\infty} g_s(f) e^{i(\frac{2\pi}{N} f)} t,$$

(4.5)

where the parameter $f$ denotes the value of the function $f(\sigma)$ along the solution $\sigma(t)$. We then set,

$\hat{g}_{mn} = g_m-n(\frac{\hat{f}_m + \hat{f}_n}{2})$.

(4.6)

We also require that when $|m-n|$ gets larger, the value of $g_{mn}$ falls off rapidly. This condition naturally conforms with (4.6), provided that the function $g$ is sufficiently smooth and $N$ is sufficiently large.

A comment to the rule (4.6) is in order. We have freedom to change the orbit $\sigma(t)$ by translation of $t$. The amount of translation is a function of $f$, which we denote by $\Delta t(f)$. By this transformation, $g_s(f)$ becomes

$$e^{iS_{\sigma}(f)} g_s(f).$$

(4.7)

Therefore, $\hat{g}_{mn}$ changes into

$$e^{i(m-n)\Delta t(\frac{\hat{f}_m + \hat{f}_n}{2})} \hat{g}_{mn}.$$

(4.8)

This freedom has a counterpart in the world of matrices. Namely, we can change $m$-th eigenvector by a phase factor $e^{i\delta_m}$. By this transformation, $\hat{g}_{mn}$ becomes

$$e^{i(\delta_n - \delta_m)} \hat{g}_{mn}.$$

(4.9)

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8 We can determine the shift by setting $S_1 = [\sigma]/(2N)$ for an eigenvalue sequence beginning with an appearing process. This is analogous to the $1/2$ in the Bohr-Sommerfeld condition $\oint pdq = (n + (1/2))2\pi\hbar$. The justification for the above rule comes from the fact that (4.11) holds at one more higher order in $1/N$ by this rule. In other words, the rule is just the midpoint rule for numerical integration. Similar rule exists for an eigenvalue sequence ending with a disappearing process.

9 This relation is the direct analog of the correspondence, in semi-classical region, between quantum matrix elements and Fourier components along classical orbits, first introduced in [8]. Also, formulae which bear some resemblance to ours appear in [12], where a correspondence between membrane theory and Matrix String Theory is considered. See also [13].
Comparing (4.8) and (4.9), we find that if

$$\delta_n - \delta_m \approx (m - n) \frac{2\pi}{T} \Delta t \left( \frac{\hat{f}_m + \hat{f}_n}{2} \right)$$

(4.10)

holds, then the two transformations are approximately identical. We can construct \(\delta_m\) satisfying (4.10) from given \(\Delta t(f)\), provided that \(\Delta t(f)\) is sufficiently smooth and \(N\) is sufficiently large.

Equations (4.3) and (4.6) constitute, then, our new correspondence rule. We shall now deduce the fundamental approximations (2.7)-(2.9) from the new rule.

By (4.3), we have divided the \(\sigma\)-space into \(N\) domains around orbits along which \(f(\sigma)\) takes the values \(\hat{f}_1, \cdots, \hat{f}_N\). We can evaluate the integral of an arbitrary function \(\int g(\sigma) d^2\sigma\) approximately, by summing up the average values of \(g(\sigma)\) on these loops multiplied by the areas of each domains. Since (4.5) and (4.6) tell us that the average value of \(g\) on the orbit along which \(f(\sigma(t)) = \hat{f}_n\) is \(\hat{g}_{nn}\), and since each area is equal to \(\sigma/\sigma N\), we obtain

$$\int g(\sigma) d^2\sigma \approx \sum_n g_{nn} \frac{\sigma}{\sigma N},$$

(4.11)

which is nothing but (2.7).

We next consider multiplication of matrices constructed by (4.6)

$$(\hat{g} \hat{h})_{mn} = \sum_l \hat{g}_{ml} \hat{h}_{ln} = \sum_l g_{m-l} \left( \frac{\hat{f}_m + \hat{f}_n}{2} \right) h_{l-n} \left( \frac{\hat{f}_l + \hat{f}_n}{2} \right).$$

(4.12)

Since \(g_{m-l}\) and \(g_{l-n}\) fall off rapidly when \(|m-l|\) and \(|l-n|\) are large, respectively, the terms in which \(l\) is not far away from \(m\) or \(n\) dominate the summation. Then, by (4.3), neglecting higher order terms in \(1/N\), we can replace both \((\hat{f}_m + \hat{f}_l)/2\) and \((\hat{f}_l + \hat{f}_n)/2\) by \((\hat{f}_m + \hat{f}_n)/2\). We have, therefore,

$$(\hat{g} \hat{h})_{mn} \approx \sum_l g_{m-l} \left( \frac{\hat{f}_m + \hat{f}_n}{2} \right) h_{l-n} \left( \frac{\hat{f}_m + \hat{f}_n}{2} \right) = (g h)_{m-n} \left( \frac{\hat{f}_m + \hat{f}_n}{2} \right) = \tilde{g} h_{mn},$$

(4.13)

where the second equality is the convolution law of Fourier series. Thus, the matrix corresponding to the multiplication of the two functions approximately coincides with the multiplication of matrices corresponding to the functions. We have derived (2.8).

We have just seen that, to the leading order, the multiplication of the matrices is commutative, since \(\tilde{g} h = h \tilde{g}\), as a matter of course. Incorporating one more higher order terms in \(1/N\), we shall evaluate the non-commutativity of the matrices. Thus, from (4.12), we have

$$(\hat{g} \hat{h})_{mn} \approx \sum_l \left( g_{m-l} \left( \frac{\hat{f}_m + \hat{f}_n}{2} \right) + \frac{\hat{f}_l - \hat{f}_n}{2} g_{m-l} \right) \left( h_{l-n} \left( \frac{\hat{f}_m + \hat{f}_n}{2} \right) + \frac{\hat{f}_l - \hat{f}_m}{2} h_{l-n} \right).$$

(4.14)
Here, we set \( g_s'(f) = dg_s/df \). \(^{10}\) We have omitted the value of \( f \) at which \( g' \) or \( h' \) is evaluated, since that does not affect the results to the order we are working. By (4.3), it follows that

\[
(\hat{g} \hat{h})_{mn} \approx \sum_{u+v=m-n} \left( g_u \left( \hat{f}_m + \frac{\hat{f}_n}{2} \right) + \frac{[\sigma]}{N} \frac{1}{T} \left( \frac{u}{2} \right) g'_u \right) \left( h_v \left( \hat{f}_m + \frac{\hat{f}_n}{2} \right) + \frac{[\sigma]}{N} \frac{1}{T} \left( -\frac{u}{2} \right) h'_v \right),
\]

where we have introduced new dummy indices \( u = m - l, v = l - n \). Then, finally, we have

\[
([\hat{g}, \hat{h}])_{mn} \approx \frac{[\sigma]}{N} \frac{1}{T} \sum_{u+v=m-n} \left( -(ug_u'h'_v + g'_u(vh_v)) \right).
\]

In order to understand the relation of the last expression to the function \( \{g, h\}(\sigma) \), it is instructive to consider the special case \( h = f \). Namely, we consider the case in which one of the functions is the reference function. In that case we have by (4.3),

\[
[\hat{g}, \hat{f}]_{mn} = \hat{g}_{mn}(\hat{f}_n - \hat{f}_m) \approx \frac{[\sigma]}{N} \frac{1}{T} (n - m)g_{m-n},
\]

which is the special case of (4.15). On the other hands, the Lie brackets between \( g \) and \( f \) can be expressed by a solution of (3.1) as,

\[
\{g, f\}(\sigma) = \left( \frac{d}{dt} g(\sigma(t)) \right) \bigg|_{\sigma(t)=\sigma},
\]

where the total derivative with respect to \( t \) is taken at the point where the Lie bracket is calculated. Then, from the definition of the Fourier component \( g_s \), (4.5), we get

\[
\{\hat{g}, \hat{f}\}_{mn} = i(m - n)\frac{2\pi}{T}g_{m-n}.
\]

Comparing with (4.16), we obtain

\[
\{g, f\}_{mn} \approx -i\frac{2\pi N}{[\sigma]} [\hat{g}, \hat{f}]_{mn},
\]

the special case of (2.9).

\(^{10}\) We choose the orbits in (4.5) smoothly, so that \( dg_s/df \) is well-defined.
The last expression in (4.15) and the above derivation of (4.19) suggest the natural generalization. We reinterpret the independent parameter of the ODE, $t$, as a function defined locally on the $\sigma$-space. Then, from (4.17), we find
\[
\{t, f\} = \frac{dt}{dt} = 1,
\tag{4.20}
\]
which means that we can consider that $(t, f)$ as canonically conjugate variables in terms of the analogous canonical formalism. It follows that,
\[
\{g, h\} = \left(\frac{\partial g}{\partial t}\right)_f \left(\frac{\partial h}{\partial f}\right)_t - \left(\frac{\partial g}{\partial f}\right)_t \left(\frac{\partial h}{\partial t}\right)_f.
\tag{4.21}
\]
The definition of the Fourier components $g_s$, (4.5), is now interpreted as the representation of $g$ as a function of $(t, f)$
\[
g(t, f) = \sum_{s=\infty}^{+\infty} g_s(f)e^{(\frac{2\pi}{T}s)t},
\tag{4.22}
\]
Substituting (4.22) and the similar formula for $h$ into (4.21) we get, \footnote{Technically, that $t$ is defined only locally poses a problem. However, we can cope with it easily by introducing patches on each of which $t$ is well-defined, and considering the relation between the patches.}
\[
\left\{g, h\right\}_s = \sum_{u+v=s} (i\frac{2\pi}{T}ug_u)h'_v - g'_u(i\frac{2\pi}{T}vh_v).
\tag{4.23}
\]
(Terms in which $(\partial/\partial f)_t$ acts on $1/T(f)$ cancel out.) By comparing this expression with (4.15), we finally prove (2.9),
\[
\{\hat{g}, \hat{h}\}_{mn} \approx -i\frac{2\pi N}{|\sigma|}[\hat{g}, \hat{h}]_{mn}.
\tag{4.24}
\]
Up to this point, our derivation has been confined to the case where there is only one eigenvalue sequence. The extension to the general case where there are several eigenvalue sequences is easy. Namely, we apply (4.3) and (4.6) within each sequences separately. They determine the matrix elements between eigenvectors belonging to the same sequence. We then set remaining matrix elements, that is, matrix elements between eigenvectors which belong to different sequences, to zero. Above derivations of the fundamental approximations work just the same.

This argument means that we can concentrate on the behaviour of functions on one branch of the $\sigma$-space, ignoring the behaviour on other branches. Also, since in our arguments the matrix element $\hat{g}_{mn}$ falls off rapidly when $|m - n|$ gets larger, we can ignore the
behaviour of the functions at the place differing much in the value of \( f \). These properties render our new rule a semi-local nature. That is, both the rule and the approximations work locally in the direction \( f \) changes. This situation is somewhat reminiscent of the uncertainty principle in the analogous quantum mechanical case. We have chosen the representation to make \( \hat{f} \) diagonal. This choice achieves minimum uncertainty in \( f \), and, at the same time, makes the conjugate variable \( t \) maximally uncertain. That our rule can be applied to any topology may be considered as a direct consequence of this semi-locality.

The linear approximation (4.3) has been essential in the machinery of the derivations of (2.7)-(2.9). Therefore, it seems that the linear approximation, hence the existence of the eigenvalue sequences is necessary in order that the approximations are good. Also the use of the analog of the Bohr-Sommerfeld condition in section 3 is justified, since the condition is nothing but (4.3).

Unfortunately, it seems that our new rule does not apply to the following exceptional quantities: matrix elements near branching and merging processes. Our rule is essentially a WKB approximation. In the immediate vicinity of the branching and merging processes, there should be tunneling effects which make the WKB approximation unreliable. Consider an analog problem in quantum mechanics, that is, the motion of a particle in the double-well potential. It is possible to deal with each well separately semi-classically, for sufficiently small \( \hbar \), and for generic energy levels. Indeed, tunneling amplitudes between the wells in general are negligibly small, behaving like \( \exp(-O(1)/\hbar) \). However, for those rare energy levels which have energy close to the value of the potential at the local maximum, the tunneling amplitudes are not negligible. The break down of our rule could also be expected from a more direct argument. The solution of (3.1), \( \sigma(t) \), in the vicinity of the branching and merging processes, spends most of the time near the branching point, moving very slowly. Then, even if \( g(\sigma) \) is a smooth function, \( g(\sigma(t)) \) might develop singularity. Then, the validity of the condition used in our argument, that \( \hat{g}_{mn} \) is negligible for large \(|m-n|\), might be questioned.

5 Examples

In this section we shall present three examples. In the first example, by an analytical calculation, we show the equivalence between our new rule and the previously known rules. Both diagonal and off-diagonal matrix elements are compared. In the remaining two examples, our purpose is mainly to illustrate the notion of eigenvalue sequences. We calculate numerically eigenvalues of matrices constructed by the previously known rules. We represent the resulting eigenvalue distribution in a method such that the structure discussed in section 3, namely the eigenvalue sequences and their branching phenomenon, can be easily seen. We confirm that the branching phenomenon of the eigenvalue sequences coincides with that of
the orbits of the ODE (3.1). We further numerically compute the eigenvalues by our rule (4.3), and compare them with those calculated by the previously known rules.

**Example 1** We consider the $\sigma$-space which has topology of a sphere. We represent the $\sigma$-space as an unit sphere in $\xi, \eta, \zeta$-space,

$$\xi^2 + \eta^2 + \zeta^2 = 1$$

(5.1)

with the area element given by

$$dS = \sin \theta d\theta d\phi,$$

(5.2)

where $\theta$ and $\phi$ are polar coordinates defined by $\zeta = \cos \theta, \xi = \sin \theta \cos \phi, \eta = \sin \theta \sin \phi$. Then the Lie brackets are $\{\xi, \eta\} = \zeta, \cdots$. We choose the simple reference function $f = \zeta$. Fig. 2 represents the $\sigma$-space and the reference function. The orbits of (3.1) appear at the point A and disappear at the point B.

We first construct the matrix $\hat{\zeta}$ corresponding to the function $\zeta$, by our new rule. The area of the domain $\zeta \leq \zeta'$ is given by

$$\frac{\zeta' + 1}{2} 4\pi.$$  

(5.3)
Then, by the analog of the Bohr-Sommerfeld quantization condition, or (4.3), we obtain

\[ (\hat{\zeta}_1, \cdots, \hat{\zeta}_N) = (-1 + \frac{1}{N}, -1 + \frac{3}{N}, \cdots, 1 - \frac{1}{N}). \] (5.4)

We further construct the matrices \( \hat{\xi} \) and \( \hat{\eta} \), corresponding to the functions \( \xi \) and \( \eta \). The solutions to the ODE (3.1) can be explicitly written as,

\[ (\xi + i\eta)(t) = \sqrt{1 - \zeta^2} e^{it} \] (5.5)

\[ (\xi - i\eta)(t) = \sqrt{1 - \zeta^2} e^{-it}. \]

Then, from (4.6), the only non-zero matrix elements is,

\[ (\hat{\xi} + i\hat{\eta})_{m+1, m} = \sqrt{1 - \left( \frac{\hat{\zeta}_m + \hat{\zeta}_n}{2} \right)^2} = \sqrt{1 - \frac{4}{N^2 - 1} \left( m - \frac{N}{2} \right)^2} = (\hat{\xi} - i\hat{\eta})_{m, m+1}. \] (5.6)

We shall now compare these results with those obtained from the previously known rules. The rule for the spherical topology reads \([3, 4, 5]\),

\[ \hat{\xi} = \sqrt{\frac{4}{N^2 - 1}} \hat{l}_x, \quad \hat{\eta} = \sqrt{\frac{4}{N^2 - 1}} \hat{l}_y, \quad \hat{\zeta} = \sqrt{\frac{4}{N^2 - 1}} \hat{l}_z, \] (5.7)

where \( \hat{l}_x, \hat{l}_y, \hat{l}_z \) are generators of the representation of \( SU(2) \) with spin \( l = (N - 1)/2 \). Since eigenvalues of \( \hat{l}_z \) are \( \{-l, -l + 1, \cdots, l\} \), we have,

\[ (\hat{\zeta}_1, \cdots, \hat{\zeta}_N) = \left( -\sqrt{\frac{4}{N^2 - 1}} \frac{N - 1}{2}, -\sqrt{\frac{4}{N^2 - 1}} \frac{N + 1}{2}, \cdots, \sqrt{\frac{4}{N^2 - 1}} \frac{N - 1}{2} \right), \] (5.8)

which coincides, for large \( N \), with the result of our new rule, (5.4). Further, it is well known that in the basis where \( \hat{l}_z \) is diagonalized, \( \hat{l}_x \) and \( \hat{l}_y \) have matrix elements only between the eigenvectors corresponding to adjacent eigenvalues. The expression for the non-zero matrix elements are,

\[ < l_z' + 1 | (\hat{l}_x + i\hat{l}_y) | l_z' >= \sqrt{l(l + 1) - l_z'(l_z' + 1)} = < l_z' | (\hat{l}_x - i\hat{l}_y) | l_z' + 1 > \] (5.9)

where we have denoted by \( | l_z' > \) the eigenvectors of \( \hat{l}_z \) belonging to the eigenvalue \( l_z' \). Thus, the result of the previously known rule is,

\[ (\hat{\xi} + i\hat{\eta})_{m+1, m} = \sqrt{1 - \frac{4}{N^2 - 1} \left( m - \frac{N - 1}{2} \right) \left( m - \frac{N + 1}{2} \right)} = (\hat{\xi} - i\hat{\eta})_{m, m+1}. \] (5.10)

\[ \text{See also footnote 8.} \]
These matrix elements are also approximately equal to (5.6). The agreements of our rule with the previously known rule for the simple functions $\xi, \eta, \zeta$ imply agreements for more general functions which can be constructed by multiplying $\xi, \eta, \zeta$ finite (much less than $N$) times. The reason for this is that the approximate equality between multiplication of functions and that of matrices, (2.8), is valid for both rules.  

The eigenvalues $\hat{\zeta}_i$ and the difference $\hat{\zeta}_{i+1} - \hat{\zeta}_i$ of the eigenvalues are given in Fig. 3. We see that the eigenvalues consist of one eigenvalue sequence. The sequence appears at the point A and disappears at the point B in Fig. 3. They correspond to the branching points of orbits A, B in Fig. 2.

**Example 2** We treat another case of spherical topology, which exhibits a more interesting branching phenomenon of eigenvalue sequences than the previous example. Perturbing the reference function considered there, we here consider the reference function of the form

$$f(\sigma) = a\zeta + b\xi + c\xi^2.$$  

The reference function and the $\sigma$-space are schematically depicted in Fig. 4. Orbits of (3.1) appear at the point A and then branch into two families at the point B. Then, the orbits

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For torus topology, similar argument as in this example, using simple functions such as $\cos \sigma^1$ or $\sin \sigma^2$ (see example 3 for definitions), has a tricky aspect since these simple functions are degenerate functions in the sense of Morse theory.
Figure 5: The plot of the eigenvalues and their difference, of the matrix corresponding to the reference function (5.11) given in Fig. 4. The eigenvalue distribution is calculated both by the previously known rule (open squares), and our new rule, namely, the analog of the Bohr-Sommerfeld condition (crosses). They agree almost completely. The branching phenomenon for eigenvalue sequences is the same as that of the orbits in Fig. 4. Horizontal lines signify critical values of $f$ at which the processes in the branching phenomenon take place, calculated directly from $f$. 
belonging to the right branch disappear at the point C, and finally the orbits belonging to the left branch disappear at the point D.

The corresponding matrix $\hat{f}$ is given by

$$\hat{f} = a\sqrt{\frac{4}{N^2 - 1}}\hat{l}_z + b\sqrt{\frac{4}{N^2 - 1}}\hat{l}_x + c\left(\sqrt{\frac{4}{N^2 - 1}}\hat{l}_x\right)^2,$$

(5.12)

if one uses the previously known correspondence rule (5.7). We have computed numerically its eigenvalues, in the case $a = 1, b = 2, c = 6$, with $N = 40$. We have also obtained the eigenvalue distribution from our new rule. To this end, we have computed numerically the area of the $\sigma$-space as a function of the height $f$ for each branches of the $\sigma$-space. Then, by the analog of the Bohr-Sommerfeld condition (4.3), we have calculated the eigenvalues of $\hat{f}$.

We represent the eigenvalues by the following method to see the information of membrane topology. Firstly, we sort the eigenvalues in increasing order,

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N.$$

(5.13)

They are given in Fig. 5. In order to see the branching phenomenon clearly, it is useful to plot also the difference of the eigenvalues $\lambda_{i+1} - \lambda_i$. By the plot one finds that from the point B to the point C, the plot of $\lambda_i$ is zig-zag shaped. Thus, the plot of $\lambda_i$ gives a juxtaposition of four eigenvalue sequences. We see the same branching phenomenon of the sequences as that of the orbits in Fig. 4. The agreement between our new rule and the previously known rule is remarkable.

**Example 3** We consider the $\sigma$-space which has topology of a torus. The $\sigma$-space can be represented by $[0, 2\pi) \times [0, 2\pi)$, where periodic boundary conditions are understood. We choose the reference function to be

$$f(\sigma) = a \cos \sigma^1 + b \cos \sigma^2.$$

(5.14)

We assume that $a \neq b$, $a \neq 0$, $b \neq 0$, in order to avoid degenerate reference functions. To be specific we choose $0 < a < b$. The reference function is represented in Fig. 6. It has essentially the same feature as the reference function in Fig. 1. At the points A, B, C, D the function $f$ takes the critical values $-a - b, a - b, -a + b, a + b$, respectively.

In the previously known correspondence rule for torus topology, one postulates[6],

$$e^{i\sigma^1} = h_1, \quad e^{i\sigma^2} = h_2,$$

(5.15)

[14]See also footnote 8.
Figure 6: (a) Contour plot of the reference function $f = a \cos \sigma^1 + b \cos \sigma^2$, with $0 < a < b$. (b) Schematic picture of the $\sigma$-space and the reference function. The contours, i.e., the orbits of (3.1) appear, branch, merge and disappear at the points A, B, C, and D, respectively. In (a) the orbits are so written that the areas of the domains between two adjacent orbits are $[\sigma]/N$.

where $h_1$ and $h_2$ are the well-known $N \times N$ matrices which satisfy the relation $h_1 h_2 = h_2 h_1 \exp (i2\pi/N)$. Then, it follows that

$$\hat{f} = \frac{a}{2} (h_1 + h_1^\dagger) + \frac{b}{2} (h_2 + h_2^\dagger).$$

(5.16)

We have computed the eigenvalue distribution of this matrix numerically, in the case $a = 1, b = 3$, with $N = 30$. We can also calculate them by the new rule as we have done in the previous example. The results by the two methods are given in Fig. 7. They agree well, except at the vicinity of the branching process at the point B. The reason for the discrepancy is noted at the end of the previous section: our new rule should not be trusted in the vicinity of branching processes. We can trust the previously known rule, on the other hand, since the fundamental approximations (2.7)-(2.9) are guaranteed by the rule (5.15), irrespectively of the branching phenomenon.

In Fig. 7, we see a sequence, which appears at the point A and branches into two sequences at the point B. Then, the two sequences merge at the point C, and finally the last sequence

$^{15}$Due to the poor knowledge in the vicinity of the merging and branching processes discussed at the end of section 4, we have two (or rather one due to the symmetry of the present example) undetermined parameters of order $1/N$ mentioned in footnote 8. We have fixed the order $1/N$ parameter by comparison to the result of the previously known rule.
Figure 7: Same as Fig. 5 but for the reference function (5.14) given in Fig. 6. The topology of the $\sigma$-space is that of a torus. The eigenvalues calculated by the new rule and the previously known rule agree well except at the immediate vicinity of the point B or the point C. The branching phenomenon of the eigenvalue sequences is the same as that of the orbits in Fig. 6. The (approximate) degeneracy of eigenvalues from the point B to the point C is only accidental, being result of the symmetry of the $f$. 
disappears at the point D. These branching processes of sequences directly correspond to the branching processes of the orbits A, B, C, D in Fig. 6.

6 Conclusions and discussions

In this paper, we have clarified some elementary but unknown features of the matrix regularization procedure. We have worked under the simple view that it is an approximation of a continuum theory by a discrete theory. The approximation between two theories is based solely on the fundamental approximation formulae (2.7)-(2.9). We have constructed a new geometrical correspondence rule between matrices and functions. We have shown the validity of the rule directly by deriving the fundamental approximations from it. The new rule is semi-local in the $\sigma$-space, and, as a consequence, can be applied to all membrane topologies in a unified way, in marked contrast with previously known rules. Using our rule, one can construct functions corresponding to given matrices such that the fundamental approximations hold well, provided that these functions exist. Whether these functions exist for given matrices can be also determined. As a physical application, for given matrices $\hat{x}^\alpha, \hat{p}^\alpha$ of the matrix model, one can construct the geometrical shape and the momentum densities of the membranes.

The new rule includes the linear approximation (4.3), which is the analog of the Bohr-Sommerfeld condition. The linear approximation has lead us to the particular structure of the eigenvalue distribution, namely the branching phenomenon of the eigenvalue sequences. The eigenvalue sequences, which we have introduced in this paper, are subsets of the all eigenvalues whose members can be linearly approximated locally. From the analog of the Bohr-Sommerfeld condition, we have shown that the branching phenomenon reflects the information of topology.

Thus, we have clarified the manner the information of topology manifests itself in the eigenvalue distribution. It is natural to further ask the question: “How completely can we read off the information of topology from given matrices?”. We shall give here some observations which are essential to this question. In the first place, our argument implies that there is no information of topology in such ill-behaved matrices for which the fundamental approximation formulae (2.7)-(2.9) do not hold well. Indeed, it is only for the case (2.7)-(2.9) work, that the linear approximation (4.3) should hold. Hence, even the existence of the eigenvalue sequences is not guaranteed for those ill-behaved matrices. Secondly, there occurs overlapping of topologies when we consider the interaction of membranes. For a typical example, let us consider process shown in Fig. 8. At first there are two spheres. Then, these spheres approach each other and the distance $\Delta$ (in the target space) between two spheres reduces to zero gradually, and finally the two spheres merge into a sphere. This overlapping
is also present in the eigenvalue distribution. The process from the viewpoint of eigenvalue distribution is as follows. In Fig. 9 the eigenvalue distribution of the matrix corresponding to the height in Fig. 8 is shown. The eigenvalue distribution consists of two eigenvalue sequences. If the distance $\Delta$ between the two eigenvalue sequences gradually reduces to zero, then we cannot distinguish the eigenvalue distribution from that of a matrix corresponding to one sphere. It is interesting to treat topology changing processes of membranes by the matrix model in this way.

Our discussion in this paper has been of purely kinematical nature. To explore the dynamical implication of our rule is also clearly important. For example, our consideration has made clear the distinction between the configurations of matrices which approximate membranes well and which do not. It is interesting to consider whether and how the former configurations dominate in the path integral of the matrix model.

We would like to comment on the issue of the membrane instability[14]. Let us consider a configuration of membranes which has a spike-like portion whose area is less than $1/N$. If we simply apply the analog of the Bohr-Sommerfeld condition, we should fail to include the information of the spike into the matrices. Stated more appropriately, our argument tells us that the configuration cannot be well approximated by $N \times N$ matrices. We want to stress that this spike has an essential difference to the spike which is considered in the membrane instability. One uses the word spike for a portion of a surface when its linear dimension is large, and at the same time its area is small. The difference between the spike in our context and the spike in the instability context lies in the meaning of the area. In the former, the area means area in the $\sigma$-space, that is essentially $p^+$. In the latter, the area means area in the target space, or the energy of the spike. This difference is meaningful. Indeed, there are membranes which have portions that have small energy but large $p^+$ or vice versa. In
particular, one can construct configurations of the matrix model which approximate well the membranes with spikes in the sense of the membrane instability.

As a direction of further investigation of the matrix regularization procedure itself, we recall the discussion at the end of section 4. Our rule in the present form does not include tunneling effects between sequences. Although our rule gives correct overall behaviour of the matrix elements, we could not trust the rule in the present form to investigate the matrix elements in the immediate vicinity of the merging and branching processes. Concrete examples of the processes are the points B, C in example 3 in section 5. \(^{16}\) It is an important task to extend our rule to incorporate the tunneling effects[15]. One possible strategy would be to revisit the analog problem, namely the quantum mechanics of a particle in a double-well potential. We can construct a formula to relate the semi-classical wave functions in both wells, which is valid even for the energy level near the local maximum, extending the ordinary argument in the WKB approximation using Airy functions. Another interesting question is the uniqueness of the correspondence which gives the fundamental approximations. Although we cannot, at present, provide the proof, we suspect that the correspondence rule from which (2.7)-(2.9) can be derived is unique up to similarity transformation. Indeed, examples studied in section 5 suggest that the previously known rules and our new rule are the same up to similarity transformation. An immediate consequence of the uniqueness is that a change of the reference function should amount to a similarity transformation. \(^{17}\)

It is believed that the matrix regularization can be extended for general even dimensional base spaces on which the Lie brackets can be defined. The correspondence rule between matrices and functions can be easily constructed by using tensor product of matrices, when the topology of the base space are given by direct product of some two dimensional spaces. These extensions are important in the matrix model of M-theory, in order to incorporate longitudinal 5-branes. To extend our analysis to study topological properties of these higher dimensional objects is also an interesting problem. We believe that the analysis analogous to the WKB approximation, used throughout in this paper, will also be useful for the higher dimensional case. However, it would be a challenging task, since the WKB approximation itself is not fully understood for generic non-integrable Hamiltonian systems on four or more dimensional phase spaces, compared to that for the necessarily integrable systems on two dimensional phase spaces considered in this paper.

We conclude with three possible applications of the new correspondence rule.

(1) One can construct various interesting configurations of the matrix model by our rule. A particular merit of our rule in this respect is that it can be applied to a membrane which

\(^{16}\)We can trust our rule near appearing or disappearing processes, such as the point C in example 2 of section 5. The jump in the plot of the difference of eigenvalues is a natural consequence of our rule.

\(^{17}\)This property implies that the fundamental approximations are no good for a configuration of matrices which consists of matrices corresponding to different topologies.
has genus higher than two, with no more difficulty than to a membrane having topology of a sphere or a torus.

(2) Our rule may be useful when investigating the Lorentz symmetry of the matrix model. It has been tried to regularize the Lorentz generators of the continuum theory in order to construct those of the matrix model [16]. However, since some of the Lorentz generators are not built up by simple multiplications or integrations or Lie brackets, one has been inclined to use the basis expansion of the previously known correspondence rules. Since there are many different expansions for different topologies, it has been difficult to define the Lorentz generators in a unique way. Since our rule can be applied uniformly to all topologies, it is a promising tool in constructing definitions of the Lorentz generators of the matrix model in a unique way.

(3) The geometrical interpretation of matrix elements in our rule may make the problem of the large $N$ limit of the matrix model accessible. The problem can be interpreted as a renormalization of the membrane theory. We hope that our rule, by determining the short-distance (or rather, the small-area) degrees of freedom, enables us to construct a block-spin transformation of the matrix model.

I would like to thank, first of all, Prof. T. Yoneya for discussions, encouragements and careful reading of the manuscript. I would like to thank, for discussions and encouragements, other members and former members of Komaba particle theory group, especially Y. Aisaka, S. Dobashi and T. Sato. Also, I would like to thank W. Taylor, S. Iso and T. Shimada for discussions and encouragements, and J. Hoppe for helpful and encouraging correspondence.

Note Added: This paper is an extended version of the author’s master thesis [17] submitted to University of Tokyo on April 2002, where the basic results were preliminarily reported. The author has recently noticed that Hyakutake has constructed matrices corresponding to axial symmetric membrane configurations [18], which are special cases of the general prescription given in section 4 of this paper.

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