Averaging using normal distribution and its application in condensed matter mechanics

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Abstract. The work offers a new method of averaging of physical values. The normal distribution of Gauss accepted as a method of averaging. The proposed method of averaging allows the introduction a new commutative ring. This ring has some useful properties. It is shown that in all physical laws where averaging exists, one should switch from ordinary multiplying to multiplying proposed by authors preserving the equations form. It is shown that the shape of the motion equation changes as a result of averaging in all physics laws for condensed matter.

1. Introduction
All statistical physics in the study of condensed matter uses averaging methods. Averaging is carried out both over ensembles and over spatial and temporal variables. In addition with the improvement of the instrument and measuring base, it is discovered that real smooth curves don’t exist in Nature. Almost all objects studied in the mechanics and physics of condensed matter are affected by a field of random forces. And the more accurately the measurement is made, the more clearly their influence is manifested. For example, in electrodynamics there is a shot effect [1], in engineering and acoustics - white noise [2], in vibration mechanics - various nonlinear effects, most often leading to undesirable consequences [3-6]. Spasmodic creep was found for oriented polyethylene fibers under constant load in the mechanics of polymers [7-8].

In this study authors propose using the normal Gaussian distribution as a basic method of averaging which has a number of useful (from the point of view of mathematics) properties.

2. Fundamental principles
Let \( X(t) \) be the studied law of motion which in the simplest case is a bounded piecewise continuous function. In a more general case the theory of generalized functions can be used, for example [9]. In the study we define the smoothed law of motion as follows:
\[
\langle X(t) \rangle_t = \bar{X}(t, \tau) = \frac{1}{\sqrt{2\pi \tau}} \int_{-\infty}^{+\infty} X(\Theta) \exp \left[ -\frac{(t-\Theta)^2}{2\tau^2} \right] d\Theta = \frac{1}{\sqrt{2\pi \tau}} \exp \left[ -\frac{t^2}{2\tau^2} \right] * X(t) = U(t) * X(t),
\]

where the symbol \(*\) is an ordinary convolution of functions (multiplication in the sense of Mikusinsky [10]).

The isomorphism between an ordinary multiplication and the convolution multiplication is established using the Laplace transform. The modern operational calculus is constructed on this basis [11]. Note that if \(X(t)\) isn’t a smooth function then the resulting function \(\bar{X}(t)\) is differentiable unlimited times by the integral properties (1):

\[
U(t) = \frac{1}{\sqrt{2\pi \tau}} \exp \left[ -\frac{t^2}{2\tau^2} \right] = \frac{1}{\sqrt{4\pi \tau}} \exp \left[ -\frac{t^2}{4\tau} \right].
\]

Here \(\tau = \frac{r^2}{2}\) is an auxiliary non-negative quantity, \(\tau\) – time of averaging.

In the probability theory the most important example is the Wiener transition function:

\[
P_\tau(X, E) = \frac{1}{\sqrt{4\pi \tau}} \int_E \exp \left[ -\frac{(X-y)^2}{4\tau} \right] d y,
\]

which leads to the occurrence of the Brownian motion process [12-13]. In this case \(\rho_\tau(X, E)\) has a density with Lebesgue measure

\[
U(S, X, y) = \frac{1}{\sqrt{4\pi \tau}} \exp \left[ -\frac{(X-y)^2}{4\tau} \right] \quad S > 0
\]

Note that this function is a fundamental solution of the heat equation [14]:

\[
\frac{\partial U}{\partial S} = \frac{\partial^2 U}{\partial t^2}.
\]

Thus the smoothing operation (1) using the kernel of the integral transform (2) has a fairly deep meaning. Moreover the time of smoothing (averaging) depends on the time scale

3. Integral Transformation Properties (1)

Consider the simplest properties of transformation (1):

- Linearity.

\[
U*[\lambda_1 X_1 + \lambda_2 X_2] = \lambda_1[U* X_1 + \lambda_2 U*X_2],
\]

where \(\lambda_1\) and \(\lambda_2\) are random complex numbers.

- Transformation (1) can be considered as the action of a linear differential operator what is equal to:
\[ 
\hat{U}(\tau) \equiv \exp\left[ S D^2 \right] = \exp\left[ \frac{\tau^2}{2} D^2 \right]. 
\] 

(7)

Here \( D \equiv \frac{\partial}{\partial t}, \ S = \frac{\tau^2}{2} \). The function of the differentiation operator \( D \equiv \frac{\partial}{\partial t} \) should be understood in the sense of [13]

In addition to fairly obvious properties the smoothing operation (1) also has a number of valuable and practical properties.

- Any smoothed function \( \tilde{X}(t, S) \) by virtue of relations (5) and (7) is a solution of the heat equation:

\[
\frac{\partial \tilde{X}(S, t)}{\partial S} = \frac{\partial^2 \tilde{X}(S, t)}{\partial t^2}
\]

with initial condition \( \tilde{X}(t, S)|_{S=0} = X(t) \).

- Submits to the switching relation. For a random analytic operator function \( \phi(t, D) \)

\[
\hat{U}(\tau)\phi(\tau) = \phi(t + 2SD, D)\hat{U}(\tau)
\]

(9)

- The law of dispersion addition is satisfied:

\[
\hat{U}(S + \lambda) = \hat{U}(\lambda) \cdot \hat{U}(S) \cdot X(t) = \hat{U}(\lambda) \cdot \tilde{X}(t, S).
\]

(10)

- Multiplicativeness:

\[
\hat{U}(S) \left[ X_1(t) \cdot X_2(t) \cdot X_3(t) \ldots \right] = \tilde{X}_1(t) \circ \tilde{X}_2(t) \circ \tilde{X}_3(t) \ldots
\]

(11)

Expression (11) allows to introduce a new commutative ring which will be denoted below as \( R_{\circ} \).

The reason for replacing classical multiplication with multiplication in the sense of (11) is the fact that in the most macroscopic laws of physics there are already averaged quantities.

4. Ring \( R_{\circ} \) properties

Consider an associative commutative ring where along with naturally defined addition we define the multiplication as follows. Let for two analytic functions \( \tilde{X}(t) \) and \( \tilde{Y}(t) \)

\[
\tilde{Z}(t, S) = \tilde{X}(t) \circ \tilde{Y}(t) = \sum_{n=0}^{\infty} \frac{\tau^{2n}}{n!} \frac{\partial^n \tilde{X}(t)}{\partial t^n} \frac{\partial^n \tilde{Y}(t)}{\partial t^n} = \sum_{n=0}^{\infty} \frac{(2S)^n}{n!} \tilde{X}^{(n)}(t) \cdot \tilde{Y}^{(n)}(t).
\]

(12)

Let us prove this statement. \( \tilde{X}(t) = \hat{U}(S) \cdot X(t) \) and \( \tilde{Y}(t) = \hat{U}(S) \cdot Y(t) \) obey the heat equation (8) with the initial conditions \( \tilde{X}_{S=0} = X(t) \) and \( \tilde{Y}_{S=0} = Y(t) \).

We show that also obeys equation (8):
\[ \tilde{Z}_s'(t,S) = 2\sum_{n=1}^{\infty} (2S)^{n-1} (n-1)! \tilde{X}^{(n)}(t,S) \cdot \tilde{y}^{(n)}(t,S) + \]
\[ + \sum_{n=0}^{\infty} (2S)^n (n)! \left[ \tilde{X}^{(n)}(t,S) \cdot \tilde{y}^{(n)}(t,S) + \tilde{X}^{(n)}(t,S) \cdot \tilde{y}_S^{(n)}(t,S) \right] = \]
\[ = \sum_{n=0}^{\infty} (2S)^n (n)! \left\{ \tilde{X}^{(n+2)}(t,S) \tilde{y}^{(n)}(t,S) + \tilde{X}^{(n)}(t,S) \tilde{y}^{(n+2)}(t,S) + 2\tilde{X}^{(n+1)}(t,S) \tilde{y}^{(n+1)}(t,S) \right\} = \]
\[ = D^2 \left[ \tilde{X}(t,S) \circ \tilde{y}(t,S) \right] \]
with the initial conditions
\[ \tilde{Z}(t,S)\big|_{S=0} = X(t) \cdot y(t). \]

By the uniqueness theorem for the heat equation we have
\[ \tilde{Z}(t,S) = U(S) \cdot [X(t) \cdot y(t)]. \]

The statement is proved.

The definition of commutative multiplication in the sense of (12) is one of its possible representations. Note that if the smoothing time tends to zero, then the multiplication (12) turns into the usual multiplication. Thus, \( C_f \) is a ring of functions with natural multiplication.

Note that \( R_o \) is a unit ring with \( R(t) \equiv 1 \). The ring properties are considered in more detail in [14-16].

5. Relations for elementary functions
Using the transformation (1) over averaged quantities, all algebraic operations can be performed. We give some simple relations for a number of elementary functions:

- \( \dot{U}(S) \cdot \exp[\lambda t] = \exp\left[ \frac{\lambda^2 t^2}{2} \right] \cdot \exp[\lambda t] = \exp[S\lambda^2] \cdot \exp[\lambda t]; \)

- \( \dot{U}(S) \cdot \exp[\lambda t^2] = \frac{1}{\sqrt{1-4S\lambda}} \exp\left[ -\frac{\lambda t^2}{1-4S\lambda} \right]; \)

- \( U(S) \cdot \cos \omega t = A \exp\left[ -\frac{(\tau w)^2}{2} \right] \cos \omega t = \tilde{A} \cos \omega t, \)

those the amplitude of the oscillations is reduces in \( \exp\left[ -\frac{(\tau w)^2}{2} \right] \) times. In particular if \( \tau = T \) is the period of oscillations, then \( \tilde{A} = A \exp[-2\pi^2]. \)

Thus, as a result of smoothing the functions represented by the Fourier series, high frequencies are cut off:

- A polynomial of any degree becomes a polynomial of the same degree. In particular for the Chebyshev – Hermite polynomial [17]
  \[ H_n(t) = (-1)^n \exp[t^2] D^n \exp[-t^2] = (2t-D)^n \cdot 1. \]
With consideration to (7) we have

$$\tilde{H}_n(t) = \left(1 - \tau^2 \right)^{n/2} H_n \left[ \frac{t}{\left(1 - \tau^2 \right)^{1/2}} \right],$$

as a result the Chebyshev – Hermite polynomial has only time scale changes.

In some cases an inverse transformation may be required, i.e. \( \hat{U}^{-1} = \exp\left[-\tau^2 / 2\right]D^2 \). In this case the known averaged function \( \bar{X}(t) \) can be expanded in a series of orthogonal polynomials, for example Chebyshev-Hermite polynomials

$$X(x) = \sum_{n=0}^{\infty} K_n H_n(t)$$

$$K_n = \frac{1}{2^n n! \sqrt{\pi}} \left[ H_n(\Theta) \exp\left[-\Theta^2\right] \bar{X}(\Theta) d\Theta \right]$$

$$\|H_n\| = n! 2^n \sqrt{\pi}$$

Then

$$X(t) = \sum_{n=0}^{\infty} \left(1 + \tau^2 \right)^{n/2} K_n H_n \left[ \frac{t}{\left(1 + \tau^2 \right)^{1/2}} \right].$$

(13)

Here are the simplest formulas for commutative multiplication \( \circ \):

- \( \exp[\alpha t] \circ \exp[\beta t] = \exp[\alpha \beta \tau^2] \exp[(\alpha + \beta)t] \):

- \( \cos w_1 t \circ \cos w_2 t = \frac{1}{2} \left[ \exp\left[w_1 w_2 \tau^2\right] \cos\left[(w_1 - w_2)t\right] + \exp\left[-w_1 w_2 \tau^2\right] \cos\left[(w_1 + w_2)t\right] \right] \).

This last expression should be noted since a quantity \( \exp\left[w_1 w_2 \tau^2\right] \) can take on very large values.

And when averaging trigonometric functions this should be remembered.

6. Sample #1. The kinetic energy change theorem

Due to the fact that most physical laws deal with macroscopic quantities, for the correct representation of laws we should go from ordinary multiplication to multiplication in the sense of “\( \circ \)”. The last one can very significantly change the usual form of physical laws. As a justification of the proposed hypothesis we consider the kinetic energy theorem.

We first introduce the following definitions.

Let the functions \( X(t) \) and \( Y(t) \) \( \in R \) and \( Z(t) = X(t) \circ Y(t) \).

We will call an uncertainty \( Z \) the value \( \Delta Z \) equal to

$$\Delta Z = X(t) \circ Y(t) - X(t) \cdot Y(t).$$

(14)

Define the average kinetic energy of a material point according to (11):

$$\bar{W}_{\text{ave}} = \frac{m \bar{v}^2}{2} = \frac{m}{2} \bar{\phi}(t) \circ \bar{v}(t),$$

(15)
where \( \bar{\omega}(t) \) is the average speed determined by relation (1). We determine the rate of change of the average kinetic energy:

\[
\frac{d}{dt} \left[ \frac{m \bar{\omega} \cdot \bar{\omega}}{2} \right] = \bar{\omega} \cdot m \ddot{\bar{\omega}} = \bar{\omega} \cdot \ddot{\bar{F}} = \bar{N} = \ddot{\bar{K}} (t) \cdot \bar{F} (t) - \frac{\tau^2}{m} \left( \bar{F} \cdot \hat{\bar{F}} \right) + \frac{1}{m} \left( \frac{\tau^4}{2!} \left( \hat{\bar{F}} \cdot \hat{\bar{F}} \right) + \ldots \right)
\]

Define the momentum of force overtime \( t \) as \( K(t) = \int_{0}^{t} \bar{F} \, dt \). Thus

\[
\frac{1}{m} \dot{K}(t) \cdot \bar{F} (t) = \frac{1}{m} \dot{\bar{K}} (t) \cdot \bar{F} (t) + \frac{\tau^2}{m} \bar{F} (t) \cdot \hat{\bar{F}} (t) + \ldots
\]

Comparing relations (16) and (17), we note that

\[
\frac{d}{dt} \left[ \frac{m \bar{\omega}^2}{2} \right] = \bar{\omega} \cdot \ddot{\bar{\omega}} + \frac{1}{m} \Delta \left( \bar{K} \circ \bar{F} \right) = \bar{\omega} \cdot \ddot{\bar{\omega}} + \frac{1}{2m} \Delta \frac{d}{dt} \left[ \bar{K}^2 (t) \right] + \ldots
\]

Thus the theorem on the change in the average kinetic energy can be formulated as follows: the rate of change of the average kinetic energy is equal to the power of external forces and the rate of change of the uncertainty of the square of the momentum of the force referred to twice the mass of the body.

It follows from equation (18) that even in the case when the average external force, additional terms appearing on the right side of the equation that responsible for the change in the average kinetic energy.

This suggests that when the average external force acting on the body is zero the average kinetic energy can change. Additional terms on the right-hand side of equation (18) can be interpreted as the flow of thermal energy.

7. Sample #2.

Let consider a problem of condensed matter mechanics. All the forces between the individual particles of the system are potential as is commonly believed. Then where do the dissipative forces come from?

Consider a system with \( N \) degrees of freedom. Let \( \{q_1,\ldots,q_n,\ldots,q_N\} \) be the vector of generalized coordinates and \( \{\dot{q}_1,\ldots,\dot{q}_n,\ldots,\dot{q}_N\} \) be the vector of generalized velocities.

We assume that tem all forces are potential in the original sys, therefore the original autonomous system in the general case has the form \( N \) equations of motion:

\[
\ddot{q}_n (t) = f_n (\bar{q}), n \in \mathbb{N}
\]

Let \( \bar{q}_n (t) \) be the smoothed average dependence using transformation (1) with the macroscopic averaging time \( \tau \). It is easy to show that the averaged equations of motion have the same form but with a different force function namely

\[
\ddot{\bar{q}}_n (t) = F_n (\bar{q}, \dot{\bar{q}})
\]

We find a new form of the force function \( \bar{F} \). It can be proved that the force function \( F_n \) obeys the following system of equations
\[
\frac{\partial F_n}{\partial S} = m \sum_{m=1}^{N} \left\{ \frac{\partial^2 F_n}{\partial q_m \partial q_k} \dot{q}_m \dot{q}_k + 2 \frac{\partial^2 F_n}{\partial q_m \partial \dot{q}_k} \dot{q}_m \ddot{q}_k + \frac{\partial^2 F_n}{\partial \dot{q}_m \partial \dot{q}_k} F_m F_k \right\}
\]

Equation (21) shows that the appearance of terms depending on speed occurs during the transition from \( k \) macroscopic times, i.e. in real systems, when measuring (averaged values are always measured) forces depend on the speed.

We note that if \( \phi(\dot{q}, \ddot{q}) \) is some physical quantity (angular momentum, energy, etc.) expressed in terms of generalized coordinates, then its averaged value already changes the form of the dependence and obeys a similar equation of type (21), namely

\[
\frac{\partial \psi}{\partial S} = \sum_{m=1}^{N} \left\{ \frac{\partial^2 \psi}{\partial q_k \partial q_m} \dot{q}_k \dot{q}_m + 2 \frac{\partial^2 \psi}{\partial q_k \partial \dot{q}_m} \dot{q}_k F_m + \frac{\partial^2 \psi}{\partial \dot{q}_k \partial \dot{q}_m} F_k F_m \right\}
\]

8. Conclusion

- Classical mechanics is based on a hypothesis according to which the laws of motion are described using smooth dependencies like most of the observed physical phenomena.
- The choice of smooth curves is due to the convenience of their subsequent description.
- Moreover, the most accurate measurements show that absolutely smooth curves are not observed in Nature.
- The authors proposed a method of averaging physical quantities by the normal distribution method. The proposed averaging method allows the introduction of a new commutative ring with a number of valuable properties.
- The use of the proposed commutative ring is shown by the example of the theorem on the change in kinetic energy. The presented example shows that even in the case when the average external force, additional terms appear on the right side of the kinetic energy change equation which can be interpreted as a heat energy flux. A thermodynamic expression is obtained that takes into account the emerging dissipative forces in complex systems.
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