Semiclassical Boltzmann theory of spin Hall effects in giant Rashba systems

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For the spin Hall effect arising from strong band-structure spin-orbit coupling, a semiclassical Boltzmann theory reasonably addressing the intriguing disorder effect called side-jump is still absent. In this paper we describe such a theory of which the key ingredient is the spin-current-counterpart of the semiclassical side-jump velocity (introduced in the context of the anomalous Hall effect). Applying this theory to spin Hall effects in a two-dimensional electron gas with giant Rashba spin-orbit coupling, we find largely enhanced spin Hall angle in the presence of magnetic impurities when only the lower Rashba band is partially occupied.

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I. INTRODUCTION

It is now generally accepted that three mechanisms – intrinsic, side-jump and skew scattering – contribute to both the spin Hall effect (SHE) and anomalous Hall effect (AHE) [1, 2]. Among the three mechanisms, the side-jump mechanism is of special interest because it originates from scattering but can, in some simple cases [1–5], be independent of both the disorder density and scattering strength. In particular, when the SHE or AHE arises from strong spin-orbit coupling in the band-structure, the side-jump belongs to the category of disorder-induced interband-coherence effect which has recently been an important topic in condensed matter physics [13–16].

In investigating transport phenomena in solids, the semiclassical Boltzmann approach is appealing due to its conceptual intuition [10]. In the study of SHE-AHE, the renewed semiclassical theory addressing this issue has proven useful in obtaining physical pictures [6]. In such a theory, the quantum mechanical information on side-jump is coded in the expressions of gauge-invariant classical concepts such as the coordinate-shift and side-jump velocity [6]. On the other hand, in the field of SHE when the spin is not conserved due to strong spin-orbit coupling in the band structure, such as in a Rashba two-dimensional electron gas (2DEG), a semiclassical description to side-jump SHE is still absent [11, 12]. Although the modified Boltzmann equation [6] developed in studying the AHE can be directly applied to SHE, the spin-current-counterpart of the side-jump velocity in this case has not been addressed before.

In the present paper we formulate a semiclassical Boltzmann framework of SHE when the spin is not conserved due to strong band-structure spin-orbit coupling. This semiclassical theory takes into account interband-coherence effects induced by both the dc uniform electric field and weak static disorder. We work out the spin-current-counterpart of the side-jump velocity based on scattering-induced modifications of conduction-electron states. When the electric field turns on, this quantity contributes one part of the side-jump SHE.

As applications we consider the SHE in a 2DEG with giant Rashba spin-orbit coupling and short-range impurities. We focus on the enhancement of spin Hall angle when the Fermi energy is tuned down towards and below the band crossing point in giant Rashba 2DEGs with magnetic disorder. The spin Hall angle which measures the generation efficiency of the transverse spin current from the longitudinal electric current is the figure of merit of the SHE. Giant Rashba spin-orbit coupling energy comparable to or even larger than the Fermi energy is possible in the polar semiconductor BiTeX (X=Cl, Br and I) family and related surfaces and interfaces [13–15]. Thus these systems are promising to realize efficient conversion of charge current into spin current.

The paper is organized as follows. In Sec. II we outline the semiclassical formulation of SHE, Section III introduces the Rashba model and calculates the SHE. Section IV concludes the paper.

II. SEMICLASSICAL FORMULATION

Considering the linear response of the spin current polarized in one particular direction (z direction is chosen in the following) to a weak dc uniform electric field E in non-degenerate multiband electron systems in the weak disorder regime, one has the semiclassical formula

\[ j^z = \sum_l f_l j^z_l, \]

(1)

where \( j^z_l \) is the amount of spin current carried by the conduction-electron state denoted by \( l \), \( f_l \) is the semiclassical distribution function.

The conduction-electron state may be modified by the electric field and static impurity scattering, \( j^z_l \) can thus deviate from the customary pure-band value [10, 11]:

\[ (j^z_l)^0 \equiv \langle l | j^z l \rangle \] with \( j^z \) the spin-current operator. Here \( | l \rangle = | \eta k \rangle = | k \rangle | u^z_k \rangle \) is the Bloch state, \( \eta \) is the band index, \( k \) is the crystal momentum, \( | k \rangle \) and \( | u^z_k \rangle \) are the plane-wave and periodic parts of \( | \eta k \rangle \), respectively. Following the recipe based on the quantum-mechanical perturbation theory for the electric-field modified Bloch
state and the Lippmann-Schwinger equation for the scattering modified conduction-electron state in Ref. 9, in the weak disorder regime nontrivial corrections caused by interband-coherence effects to \((\hat{j}_l^i)^0\) read:

\[
\hat{j}_l^i = (\hat{j}_l^i)^0 + \delta^{\text{in}} \hat{j}_l^i + \delta^{\text{ex}} \hat{j}_l^i.
\] (2)

The intrinsic correction \(\delta^{\text{in}} \hat{j}_l^i = 2 \text{Re} \langle l | \hat{j}^z | g \rangle \langle g | l \rangle \) arises from the interband-virtual-transition (electron charge \(e\))
\[
|\delta \hat{E}_{l}^g| = -ieE \sum_{\eta' \neq \eta} \langle \eta' k | \hat{v} | u_k \rangle / \left( \epsilon_k - \epsilon_{\eta' k} \right)^2
\]
duced by the electric field \([11]\), with \(\epsilon_l \equiv \epsilon_{\eta k}\) the energy of Bloch state \(|\eta k\rangle\) and \(\hat{V}\) the velocity operator. Thus
\[
\delta^{\text{in}} \hat{j}_l^i = \hbar e \sum_{\eta' \neq \eta} \frac{2 \text{Im} \langle \eta k | \hat{v} | \eta' k \rangle \langle \eta' k | \hat{v} | u_k \rangle}{\left( \epsilon_k - \epsilon_{\eta' k} \right)^2}
\] (3)
is an electric-field-induced interband-coherence effect.

The extrinsic correction \(\delta^{\text{ex}} \hat{j}_l^i\) originates from the interband-coherence during the elastic electron-impurity scattering process. The scattering-induced modification to conduction-electron states is captured by the Lippmann-Schwinger equation describing the scattering state \(|l^*\rangle = |l\rangle + \left( \epsilon_l - H_0 + i\epsilon \right)^{-1} \hat{T} |l\rangle\) with the T-matrix \(\hat{T} |l\rangle = \hat{V} |l^*\rangle\) related to the disorder potential \(\hat{V}\) and disorder-free Hamiltonian \(\hat{H}_0\).

\[
\delta^{\text{ex}} \hat{j}_l^i = \hbar e \sum_{\eta' \neq \eta} \sum_{\eta' k'}
\]
\[
\times \frac{\langle \eta k | \hat{V} | \eta' k' \rangle \langle \eta' k' | \hat{v} | \eta k \rangle}{\epsilon_k - \epsilon_{\eta' k'}} \times \frac{\langle \eta' k' | \hat{v} | \eta' k' \rangle}{\epsilon_{\eta' k'} - \epsilon_{\eta' k'} - \hbar \omega_{\text{sc}}}
\]
\[
+ 2 \text{Re} \sum_{\eta' \neq \eta} \sum_{\eta' k'}
\]
\[
\times \frac{\langle \eta k | \hat{V} | \eta' k' \rangle \langle \eta' k' | \hat{v} | \eta k \rangle}{\epsilon_k - \epsilon_{\eta' k'}} \times \frac{\langle \eta' k | \hat{v} | \eta' k' \rangle}{\epsilon_{\eta' k'} - \epsilon_{\eta' k'}}.
\] (4)

It has been shown \([3]\) that the side-jump velocity \(\hat{v}_l^i\)
which is an important ingredient in the semiclassical theory of AHE \([3]\) can also be obtained in this way \(\delta^{\text{ex}} \hat{v}_l = \hat{v}_l^i\) and thus shares the same origin. \(\delta^{\text{ex}} \hat{j}_l^i\) can therefore be deemed as the spin–current-counterpart of the side-jump velocity in the case of band-structure spin-orbit coupling.

The properly modified steady-state linearized Boltzmann equation in the presence of weak static disorder has been proposed as \([6]\):

\[
eE \cdot \hat{v}_l \frac{\partial f_0}{\partial \epsilon_l} = -\sum_{\nu} \omega_{\nu l} \left[ f_l - f_{\nu} - \frac{\partial f_0}{\partial \epsilon_l} eE \cdot \delta \hat{r}_{\nu l} \right],
\] (5)

where \(\hat{v}_l^0 = \partial \epsilon_l / \partial \hat{c} k\) is the band velocity, \(f_0\) is the Fermi distribution function, \(\delta \hat{r}_{\nu l}\) is the coordinate-shift in the scattering process \((l \rightarrow l')\) \([3]\) and \(\omega_{\nu l}\) is the semiclassical scattering rate \((l' \rightarrow l)\). Up to the linear order of the electric field one has the decomposition \([3, 17]\)

\[
f_l = f_0 + g_l^0 + \hat{g}_l^0,
\] (6)

with \(g_l^0\) the normal part of the out-of-equilibrium distribution function satisfying the Boltzmann equation in the absence of \(\delta \hat{r}_{\nu l}\) and \(\hat{g}_l^0\) the anomalous distribution function related to \(\delta \hat{r}_{\nu l}\). It is now clear \([6]\) that \(\delta \hat{r}_{\nu l}\) is a disorder-induced interband-coherence effect and so is \(g_l^0\).

Given that the semiclassical formulation is relevant in the weak disorder regime, Eq. (1) reduces to \([6]\)

\[
\hat{j}_l^i = \sum_l f_l (\hat{j}_l^i)^0 + \sum_l g_l^{2s} \delta^{\text{ex}} \hat{j}_l^i + \sum_l f_l^0 \delta^{\text{in}} \hat{j}_l^i,
\] (7)

up to the zeroth order of total impurity density and scattering strength, \(g_l^{2s}\) represents the value of \(g_l^0\) in the lowest Born order \([3]\). In higher Born orders, some additional contributions to \(g_l^0\) appear and are responsible for the transverse transport due to the breakdown of the principle of microscopic detailed balance. The analysis of these higher-Born-order contributions has been detailed in Ref. [17]. Here we only mention that there is an interband-coherence scattering effect called “intrinsic-skew-scattering induced side-jump” appearing in the third Born order under the Gaussian disorder. Below we set \(\hat{j}_l^{\text{in}} = \sum_l g_l^{2s} \delta^{\text{in}} \hat{j}_l^i\) which is just the intrinsic contribution to the spin current independent of the disorder \([11]\), and \(\hat{j}_l^{\text{ex}} = \sum_l g_l^{2s} \delta^{\text{ex}} \hat{j}_l^i\) because it is related to the spin–current-counterpart of the side-jump velocity. In general case of SHE induced by strong band-structure spin-orbit coupling, \(\hat{j}_l^{\text{ex}}\) is just one part of the side-jump SHE arising from disorder-induced interband-coherence effects. Other two semiclassical contributions to the side-jump SHE (from the anomalous distribution function \(g_l^0\) and the intrinsic-skew-scattering induced side-jump) \([18]\) and the skew scattering SHE arising from non-Gaussian disorder are all included in the first term of Eq. (7) \([3, 17]\).

To be more clear we can consider the case of randomly distributed scalar pointlike Gaussian disorder with density \(n_0\) and average strength \(V_0\). Then \(g_l^{2s} \sim n_0 \delta \hat{V}_0^2\), \(\delta^{\text{ex}} \hat{j}_l^i \sim n_0 V_0^2\), \(\hat{g}_l^0 \sim n^0 V_0^2\) and the third-Born-order contribution to \(g_l^0\) behaves as \(\sim n^0 V_0^2\) (thus is called the intrinsic-skew-scattering \([3, 17]\)). In this case the side-jump SHE may consist of three semiclassical contributions in the zeroth order of both the impurity density and scattering strength: \(\hat{j}_l^{\text{in}}, \sum_l g_l^{2s} (\hat{j}_l^i)^0\) and the intrinsic-skew-scattering induced side-jump.
III. MODEL CALCULATION

A. Model

The model Hamiltonian of a Rashba 2DEG is \( H_0 = \frac{\hbar^2 k^2}{2m} + \alpha_R \hat{\sigma} \cdot (k \times \hat{\pi}) \), where \( k \) is the 2D wavevector, \( m \) is the effective mass, \( \hat{\sigma} \) is the vector of Pauli matrices, and \( \alpha_R \) is the Rashba coefficient. The internal eigenstates read \( |n_k \rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1, -i\eta \exp(i\phi) \end{bmatrix} \), where \( \eta = \pm \) label the two bands \( \epsilon_{\pm k} = \hbar^2 k^2/(2m) + \eta \alpha_R k \), and \( \phi = k_y/k_x \).

For \( \epsilon > 0 \) the corresponding wave number in \( \eta \) band is given as \( k_{\eta} (\epsilon) = -\eta k_R + k_0 (\epsilon) \). Here \( k_R = m \alpha_R / \hbar^2 = \frac{1}{2} (k_+ (\epsilon) - k_-(\epsilon)) \) measures the momentum splitting of two Rashba bands, whereas \( k_0 (\epsilon) \equiv \alpha_R^{-1} \sqrt{\epsilon + 2\eta \hbar^2} = \frac{1}{2} \sum_{\eta} k_{\eta} (\epsilon) \). The density of states of \( \eta \) band takes the form \( D_{\eta} (\epsilon) = D_{0} \frac{k_{\eta} (\epsilon)}{k_0 (\epsilon)} \), with \( D_0 = m / 2 \hbar^2 \).

For \( \epsilon > \epsilon_R \), the iso-energy surface slices the spectrum two rings of radii \( k_{-\nu} (\epsilon) = k_R + (-1)^{\nu-1} k_0 (\epsilon) \), where \( \nu = 1, 2 \) denote the two monotonic segments (Fig. 1), \( k_0 (\epsilon) \equiv \alpha_R^{-1} \sqrt{\epsilon + 2\eta \hbar^2} = \frac{1}{2} (k_+ (\epsilon) - k_-(\epsilon)) \). The density of states of \( -\nu \) branch reads \( D_{-\nu} (\epsilon) = D_0 \frac{k_{-\nu} (\epsilon)}{k_0 (\epsilon)} \).

The conventional definition of the spin current as an anti-commutator of velocity and spin is employed: \( \hat{j}^z = \sum_{\nu} \{ \hat{\sigma}_z, \hat{\pi} \} = \frac{\hbar}{2} \{ \hat{\sigma}_z, \hat{\pi} \} \). It is purely off-diagonal in band-index space in this model: \( \langle \hat{j}^z \rangle^0 = 0 \), thus the SHE in Eq. 17 is determined only by \( \hat{j}^z = \hat{j}^{z, in} + \hat{j}^{z, sj} \).

The Boltzmann equation can be conveniently solved by using variables \( l = (\epsilon, \eta, \phi) \) for \( \epsilon > 0 \) and \( l = (\epsilon, -\nu, \phi) \) for \( \epsilon > \epsilon_R / 2 \). Correspondingly, \( \sum_{\nu} = \sum_{\eta(l)} \int d\epsilon D_{\nu} (\epsilon) \int d\phi \frac{d\epsilon}{\hbar} \) for \( \epsilon > 0 \) \( (\epsilon > -\epsilon_R / 2) \). If \( \epsilon > 0 \), in the lowest Born order the energy-integrated elastic scattering rate is \( \omega_{\nu,\nu'} = \int d\epsilon \Delta' D_{\nu} (\epsilon) \omega_{\nu,\nu'}^2 \).

We consider the impurity potential is produced by randomly distributed short-range scatters at \( R_0 \), i.e., \( V (r) = \sum_{i,m} V_i \delta (r - R_i) \sigma_\mu \), and \( \sigma_\mu \) the unit matrix in spin space \( [20] \). Here the short-range potential is approximated by the delta-potential. We assume Gaussian disorder approximation and isotropic magnetic scattering \( [20, 21] \). \( n_0 \) and \( n_m \) are the concentrations of nonmagnetic and magnetic impurities, respectively. \( V_0 \) and \( V_m \) are the average strengths for the nonmagnetic and magnetic scattering, respectively. The external electric field is applied in x direction.

1. Nonmagnetic impurities

When \( \epsilon > 0 \), straightforward calculation leads to the spin-current-counterpart of the side-jump velocity

\[ \delta^{ex} (\hat{j}_l^{z, nm}) = \frac{1}{\tau_0} \frac{\eta}{\hbar} = \cos \phi, \]

with \( \tau_0 = \frac{1}{2\pi n_0 V_0^2 D_0} \). The transport time reads \( [19] \) \( \tau_0 (\epsilon) = \tau_0 (\epsilon) / D_0 \), and then the side-jump spin Hall current is

\[ j_l^{z, sj} = \sum_{l} g_l D_{\nu} (\epsilon) \delta \cdot \int \hat{j}_l^{z, nm} = \frac{e}{S_m} E_x, \]

which completely cancels out the intrinsic spin Hall current \( j_l^{z, sj} = \frac{e}{S_m} E_x \). This just reproduces the well-known \( [1] \) vanishing spin Hall current \( j_l^{z, sj} = 0 \) in the semiclassical Boltzmann theory for the first time.
When $0 > \epsilon_F > -\epsilon_R/2$, the intrinsic spin Hall current is $j_{y}^{z,\text{in}} = \frac{k_0(\epsilon_F)}{k_R} \frac{\epsilon}{8\pi} E_x$. Meanwhile the spin-current-counterpart of the side-jump velocity reads
\[
\delta^{ex}(\mathbf{j}_{y}^{\text{in}}) = \frac{1}{\tau_0} \frac{1}{\tau_0(\epsilon)} \frac{\hbar}{8} \cos \phi.
\] (15)
and thus the side-jump spin Hall current
\[
j_{y}^{z,sj} = \sum_l g_l 2^{2\epsilon} \delta^{ex} (\mathbf{j}_{y}^{\text{in}}) = \frac{k_0(\epsilon_F)}{k_R} \frac{\epsilon}{8\pi} E_x
\] (16)
again cancels out the intrinsic one. This also coincides with the zero SHE obtained by the Kubo formula [22].

2. Magnetic impurities

For isotropic delta-like magnetic impurity potential, since the contributions from $V_y^0$ and $V_y^1$ cancel out in Eq. (4), the spin–current-counterpart of the side-jump velocity is given by
\[
\delta^{ex}(\mathbf{j}_{y}^{\text{in}}) = \frac{1}{\tau_0} \frac{1}{\tau_0(\epsilon)} \frac{\hbar}{8} \cos \phi.
\] (17)

The transport time is given by
\[
\frac{\tau^{\text{tr}}_m(\epsilon)}{\tau_m} = \frac{8k_0(\epsilon) - k_\gamma(\epsilon)}{7k_0(\epsilon)}
\] (18)
for $\epsilon > 0$, and
\[
\frac{\tau^{\text{tr}}_{\nu}(\epsilon)}{\tau_m} = \frac{k_0(\epsilon) 8k_R - k_{-\nu}(\epsilon)}{7k_R}
\] (19)
for $0 > \epsilon > -\epsilon_R/2$, with $\tau_m = \left(\frac{\pi m V^2 \delta}{h}\right)^{-1}$. When both Rashba bands are partially occupied, the side-jump spin Hall current
\[
j_{y}^{z,sj} = \sum_l g_l 2^{2\epsilon} \delta^{ex} (\mathbf{j}_{y}^{\text{in}}) = \frac{1}{\tau_0} \frac{1}{\tau_0(\epsilon)} \frac{\hbar}{8} \cos \phi.
\] (20)
enhances the total spin Hall current to $j_{y}^{z} = j_{y}^{z,\text{in}} + j_{y}^{z,sj} = \frac{8}{7} j_{y}^{z,\text{in}}$. This $j_{y}^{z}$ is the same as the weak-disorder-limit value of that obtained by Kubo diagrammatic calculations [21, 22]. The longitudinal electric current is
\[
j_{x} = \frac{e^2}{\pi m \tau m} \frac{\tau}{3} E_x
\]
the spin Hall angle is therefore
\[
\alpha_{sH} = \frac{-2\hbar}{\tau_m \epsilon_R} \frac{1}{3 + \frac{7\epsilon}{\epsilon_R}}.
\] (21)

When only the lower Rashba band is partially occupied, the side-jump and the total spin Hall currents are
\[
\begin{align*}
\alpha_{sH} &= \frac{-2\hbar}{\tau_m \epsilon_R} \frac{1}{3 + \frac{7\epsilon}{\epsilon_R}}
\end{align*}
\]
(22)
respectively. Here we define $\tau^{-1}_{S,0} \equiv \tau^{-1}_{0} + \tau^{-1}_{m}$, Tuning the ratio $\tau_0/\tau_m$ one can find that $\alpha_{sH}$ changes monotonically and continuously from the scalar-disorder-dominated case to the magnetic-disorder-dominated regime.
IV. DISCUSSION AND SUMMARY

Before concluding this paper, we comment on some important issues not mentioned in above sections.

First, the simple form of the semiclassical Boltzmann equation (5) is exactly valid only for isotropic bands and isotropic scattering [17]. In our opinion, in the presence of anisotropy a more generic and complicated form of the Boltzmann equation may be necessary, we refer the readers to Ref. 27 for detailed discussions.

Second, the recently highlighted “coherent skew scattering” under the Gaussian disorder beyond the non-crossing approximation [28] is also included in the first term of Eq. (7). This additional contribution is also in the zeroth order of both the impurity density and scattering strength in the weak disorder limit in the presence of only one type of disorder, like the side-jump contribution, but is not an interband-coherence scattering effect [18]. Thus how to place this contribution into the classification of AHE-SHE mechanisms suggested in Refs. 1 and 2 is still an open question. Therefore, in presenting our theory we avoid this issue. Fortunately, in the Rashba model considered in Sec. III the first term of Eq. (7) vanishes.

Besides, we should remind the interested readers that this so-called “coherent skew scattering” has actually already been proposed sixty years ago by Kohn and Luttinger [27, 29]. We will provide a comprehensive description of a semiclassical Boltzmann theory going beyond the non-crossing approximation in a future publication.

Finally, in the presence of spin-orbit coupling the electron spin is not conserved thus the spin current is not uniquely defined. The conventionally defined spin current adopted in this study is not a conserved transport current. A physically attracting definition of the conserved spin current has been suggested by Shi et al. [30] by introducing the torque dipole moment. However, disorder effects on the torque dipole spin current [31] in the Bloch representation are hard to deal with under the uniform external electric field in the Boltzmann theory. We reserve these for future studies.

In summary, we have formulated a semiclassical Boltzmann framework of spin Hall effects induced by strong band-structure spin-orbit coupling in non-degenerate multiband electron systems in the weak disorder regime. We worked out the absent ingredient in previous semiclassical theories, i.e., the spin-current-counterpart of the semiclassical side-jump velocity. This gauge-invariant quantity arises from the interband-coherence during the elastic electron-impurity scattering, and contributes one part of the side-jump spin Hall effect.

Applying this theory to a 2DEG with giant Rashba spin-orbit coupling, we showed an enhanced spin Hall angle when only the lower Rashba band is partially occupied in the presence of magnetic impurities. We note that this energy regime below the band crossing point in Rashba systems and similar systems is of intense theoretical interest also from the standpoint of enhanced efficiency of spin-orbit torque and of Edelstein effect [32–34], as well as enhanced thermoelectric conversion efficiency [13, 19].

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[1] J. Sinova, S. O. Valenzuela, J. Wunderlich, C. H. Back, and T. Jungwirth, Spin Hall effects, Rev. Mod. Phys. 87, 1213 (2015).
[2] N. Nagaosa, J. Sinova, S. Onoda, A. H. MacDonald, and N. P. Ong, Anomalous Hall effect, Rev. Mod. Phys. 82, 1539 (2010).
[3] A. A. Kovalov, J. Sinova, and Y. Tserkovnyak, Anomalous Hall Effect in Disordered Multiband Metals, Phys. Rev. Lett. 105, 036601 (2010).
[4] D. Culcer, E. M. Hankiewicz, G. Vignale, and R. Winckler, Side jumps in the spin Hall effect: Construction of the Boltzmann collision integral, Phys. Rev. B 81, 125332 (2010).
[5] S. A. Yang, H. Pan, Y. Yao, and Q. Niu, Scattering universality classes of side jump in the anomalous Hall effect, Phys. Rev. B 83, 125122 (2011).
[6] N. A. Sinitsyn, semiclassical theories of the anomalous Hall effect, J. Phys.: Condens. Matter 20, 023201 (2008).
[7] D. Hou, G. Su, Y. Tian, X. Jin, S. A. Yang, and Q. Niu, Multivariable Scaling for the Anomalous Hall Effect, Phys. Rev. Lett. 114, 217203 (2015).
[8] D. Culcer, A. Sekine, and A. H. MacDonald, Interband coherence response to electric fields in crystals: Berry-phase contributions and disorder effects, Phys. Rev. B 96, 035106 (2017).
[9] C. Xiao and Q. Niu, semiclassical theory of spin-orbit torques in disordered multiband electron systems, preprint, arXiv: 1705.07947, to be published in Phys. Rev. B.
[10] J. M. Ziman, Electrons and Phonons (Clarendon, Oxford, 1960).
[11] S. Zhang and Z. Yang, Intrinsic Spin and Orbital Angular Momentum Hall Effect, Phys. Rev. Lett. 94, 066602 (2005).
[12] For instance, one can verify that the semiclassical framework described in Ref. 11 cannot produce any extrinsic contribution to the SHE of the conventional spin current polarized in z direction in a disordered Rashba 2DEG. (Equations (2) – (7) in that paper).
[13] S. V. Eremeev, I. A. Nechaev, Yu. M. Koroteev, P. M. Echenique, and E. V. Chulkov, Ideal two-dimensional electron systems with a giant Rashba-type spin splitting in real materials: Surfaces of Bismuth tellurohalides, Phys. Rev. Lett. 108, 246802 (2012).
[14] M. Sakano, M. S. Bahramy, A. Katayama, T. Shimojima, H. Murakawa, Y. Kaneko, W. Malaeb, S. Shin, K. Ono, Zhongshui Ma for insightful discussions.
H. Kumigashira, R. Arita, N. Nagaosa, H. Y. Hwang, Y. Tokura, and K. Ishizaka, Strongly spin-orbit coupled two dimensional electron gas emerging near the surface of polar semiconductors, Phys. Rev. Lett. 110, 107204 (2013).

[15] L. Wu, J. Yang, S. Wang, P. Wei, J. Yang, W. Zhang, and L. Chen, Thermopower enhancement in quantum wells with the Rashba effect, Appl. Phys. Lett. 105, 202115 (2014).

[16] In the present paper we do not consider side-jump induced by spin-orbit scattering. A semiclassical treatment of this case can be found in P. M. Levy, H. Yang, M. Chshiev, and A. Fert, Spin Hall effect induced by Bi impurities in Cu: Skew scattering and side-jump, Phys. Rev. B 88, 214432 (2013).

[17] C. Xiao, D. Li, and Z. Ma, Role of band-index-dependent transport relaxation times in anomalous Hall effect, Phys. Rev. B 95, 035426 (2017).

[18] The reason why the side-jump AHE-SHE induced by band-structure spin-orbit coupling is defined as the sum of these three semiclassical contributions was detailed in Ref. 2. Simply speaking, there are at least two motivations: one is the equivalence described in Ref. [26], the other is that all these three contributions belong to the disorder-induced interband-coherence effect (see Refs. [2, 3, 7, and 9].

[19] C. Xiao, D. Li, and Z. Ma, Unconventional thermoelectric behaviors and enhancement of figure of merit in Rashba spintronic systems, Phys. Rev. B 93, 075150 (2016).

[20] H.-Z. Lu and S.-Q. Shen, Extrinsic anomalous Hall conductivity of a topologically nontrivial conduction band, Phys. Rev. B 88, 081304(R) (2013).

[21] J.-I. Inoue, T. Kato, Y. Ishikawa, H. Itoh, G. E.W. Bauer, and L.W. Molenkamp, Vertex Corrections to the Anomalous Hall Effect in Spin-Polarized Two-Dimensional Electron Gases with a Rashba Spin-Orbit Interaction, Phys. Rev. Lett. 97, 046604 (2006).

[22] C. Grimaldi, E. Cappelluti, and F. Marsiglio, Off-Fermi surface cancellation effects in spin-Hall conductivity of a two-dimensional Rashba electron gas, Phys. Rev. B 73, 081303(R) (2006).

[23] P. Wang, Y.-Q. Li, and X. Zhao, Nonvanishing spin Hall currents in the presence of magnetic impurities, Phys. Rev. B 75, 075326 (2007).

[24] K Chadova, S. Wimmer, H. Ebert, and D. Kodderitzsch, Tailoring of the extrinsic spin Hall effect in disordered metal alloys, Phys. Rev. B 92, 235142 (2015).

[25] A. Fert and P. M. Levy, Spin Hall Effect Induced by Resonant Scattering on Impurities in Metals, Phys. Rev. Lett. 106, 157208 (2011).

[26] In the context of AHE induced by band-structure spin-orbit coupling, it has been established that the disorder-induced interband-coherence contribution (side-jump) calculated in the semiclassical Boltzmann theory is equivalent to the ladder vertex correction in the weak disorder limit to the bubble of the anomalous Hall conductivity in the non-chiral basis ($\sigma_z$ basis for the Rashba model). The present calculations suggest that this equivalence is also valid for the spin Hall conductivity of the conventionally defined spin current. In fact, this equivalence has been employed in the statement of Ref. [4].

[27] W. Kohn and J. M. Luttinger, Quantum Theory of Electrical Transport Phenomena, Phys. Rev. 108, 590 (1957).

[28] I. A. Ado, I. A. Dmitriev, P. M. Ostrovsky, and M. Titov, Anomalous Hall effect with massive Dirac fermions, Europhys. Lett. 111, 37004 (2015).

[29] J. M. Luttinger, Theory of the Hall Effect in Ferromagnetic Substances, Phys. Rev. 112, 739 (1958).

[30] J. Shi, P. Zhang, D. Xiao, and Q. Niu, Proper definition of spin current in spin-orbit coupled systems, Phys. Rev Lett. 96, 076604 (2006).

[31] N. Sugimoto, S. Onoda, S. Murakami, and N. Nagaosa, Spin Hall effect of a conserved current: Conditions for a nonzero spin Hall current, Phys. Rev. B 73, 113305 (2006).

[32] K. Tsutsui and S. Murakami, Spin-torque efficiency enhanced by Rashba spin splitting in three dimensions, Phys. Rev. B 86, 115201 (2012).

[33] C. Xiao, D. Li, and Z. Ma, Thermoelectric response of spin polarization in Rashba spintronic systems, Front. Phys. 11, 117201 (2016).

[34] N. Zhang, Y. Wang, J. Berakdar and C. Jia, Giant spin-orbit torque and spin current generation in carriers at oxide interfaces, New J. Phys. 18, 093034 (2016).