UNITARITY OF MINIMAL W-ALGEBRAS

VICTOR G. KAC
PIERLUIGI MÖSENEDER FRAJRIA
PAOLO PAPI

Abstract. We obtain a complete classification of minimal simple unitary \( W \)-algebras.

1. Introduction

In the present paper we study unitarity of minimal \( W \)-algebras. They are the simplest conformal vertex algebras among the simple vertex algebras \( W_k(g, x, f) \), associated to a datum \((g, x, f)\) and \( k \in \mathbb{R} \). Here \( g = g_0 \oplus g_1 \) is a basic simple Lie superalgebra, i.e. its even part \( g_0 \) is a reductive Lie algebra and \( g \) carries a non-zero even invariant non degenerate supersymmetric bilinear form \( (\cdot | \cdot) \), \( x \) is an \( ad \)-diagonalizable element of \( g_0 \) with eigenvalues in \( 1/2 \mathbb{Z} \), \( f \in g_0 \) is such that \([x, f] = -f\) and the eigenvalues of \( ad_x \) on the centralizer \( g_f \) of \( f \) in \( g \) are non positive, and \( k \neq -h^\vee \), where \( h^\vee \) is the dual Coxeter number. The most important examples are provided by \( x \) and \( f \) to be part of an \( sl_2 \) triple \( \{e, x, f\} \), where \([x, e] = e, [x, f] = -f, [e, f] = x\). In this case \((g, x, f)\) is called a Dynkin datum.

We proved in [11, Lemma 7.3] that if \( \phi \) is a conjugate linear involution of \( g \) such that
\[
\phi(x) = x, \quad \phi(f) = f \quad \text{and} \quad (\phi(a) | \phi(b)) = (a | b), \quad a, b \in g,
\]
then \( \phi \) induces a conjugate linear involution of the vertex algebra \( W_k(g, x, f) \).

We also proved in [11, Proposition 7.4] that if \( \phi \) is a conjugate linear involution of \( W_k(g, x, f) \), this vertex algebra carries a non-zero \( \phi \)-invariant Hermitian form \( H(\cdot, \cdot) \) for all \( k \neq -h^\vee \) if and only if \((g, x, f)\) is a Dynkin datum; moreover, such \( H \) is unique, up to a real constant factor, and we normalize it by the condition \( H(1, 1) = 1 \). The vertex algebra is called unitary if there is a conjugate linear involution \( \phi \) such that the corresponding \( \phi \)-invariant Hermitian form \( H \) is positive definite.

For some levels \( k \) the vertex algebra \( W_k(g, x, f) \) is trivial, i.e. isomorphic to \( \mathbb{C} \); then it is trivially unitary. Another easy case is when \( W_k(g, x, f) \) “collapses” to the affine part. In both cases we will say that \( k \) is collapsing level.

Let \( g^i \) be the centralizer of the \( sl_2 \) subalgebra \( s = \text{span} \{e, x, f\} \) in \( g_0 \); it is a reductive subalgebra. If \( \phi \) satisfies the first two conditions in [11], it fixes \( e, x, f \), hence \( \phi(g^2) = g^2 \).

It is easy to see that unitarity of \( W_k(g, x, f) \) implies, when \( k \) is not collapsing, that \( \phi|_{g^2} \) is a compact involution of the reductive Lie algebra \( g^2 \).

In the present paper we consider only minimal data \((g, x, f)\), defined by the property that for the \( ad_x \) gradation \( g = \bigoplus_{j \in \frac{1}{2} \mathbb{Z}} g_j \) one has
\[
g_j = 0 \quad \text{if} \quad |j| > 1, \quad \text{and} \quad g_{-1} = \mathbb{C} f.
\]

In this case \((g, x, f)\) is automatically a Dynkin datum. The corresponding \( W \)-algebra is called minimal. The element \( f \in g \) is a root vector attached to a root \( \theta \) of \( g \), and we shall
normalize the invariant bilinear form on \( g \) by the condition \((\theta|\theta) = 2\). Recall that the dual Coxeter number \( h^\vee \) of \( g \) is half of the eigenvalue of its Casimir element of \( g \), attached to the bilinear form \((\cdot|\cdot)\). We shall denote by \( W_k^{\text{min}}(g) \) the minimal \( W \)-algebra, corresponding to \( g \) and \( k \neq -h^\vee \).

We proved in [11, Proposition 7.9] that, if \( W_k^{\text{min}}(g) \) is non-trivial unitary and \( k \) is not a collapsing level, then the parity of \( g \) is compatible with the \( ad \)-gradation i.e. the parity of the whole subspace \( g_j \) is \( 2j \mod 2 \).

It follows from [8], [9] that for each basic simple Lie superalgebra \( g \) there is at most one minimal Dynkin datum, compatible with parity, and the complete list of the \( g \) which admit such a datum is as follows:

\[
\begin{align*}
sl(2|m) & \text{ for } m \geq 3, & psl(2|2), & spo(2|m) & \text{ for } m \geq 0, \\
osp(4|m) & \text{ for } m > 2 \text{ even}, & D(2,1;a) & \text{ for } a \in \mathbb{C}, & F(4), & G(3).
\end{align*}
\]

The even part \( g_0 \) of \( g \) in this case is isomorphic to the direct sum of a reductive Lie algebra \( g^\wedge \) and \( s \cong \sl_2 \). As has been explained above, a minimal \( W \)-algebra \( W_k^{\text{min}}(g) \) with non-collapsing \( k \) can be unitary only if the conjugate linear involution \( \phi \) on \( g \) is \textit{almost compact}, according to the following definition.

**Definition 1.1.** We say that a conjugate linear involution \( \phi \) on \( g \) is \textit{almost compact} if

\begin{itemize}
  \item[(i)] \( \phi \) fixes \( e, x, f \);
  \item[(ii)] \( \phi \) is a compact conjugate linear involution of \( g^\wedge \).
\end{itemize}

Indeed (i) is equivalent to the first two requirements in (1.1), and the third requirement follows from Lemma 3.1 in Section 3.

We prove that an almost compact conjugate linear involution \( \phi \) exists for all \( g \) from the list (1.3), except that \( a \in \mathbb{R} \), and is essentially unique. So by unitarity of a minimal \( W \)-algebra we mean unitarity for \( \phi \) almost compact.

It was shown in [9] that the central charge of \( W_k^{\text{min}}(g) \) equals

\[
c(k) =\frac{k d}{k + h^\vee} - 6k + h^\vee - 4, \quad \text{where } d = \text{sdim} g.
\]

Here is another useful way to write this formula:

\[
c(k) = 7h^\vee + d - 4 - 12\sqrt{\text{--} 6\left(\frac{k + h^\vee - \sqrt{\text{--} 6}}{k + h^\vee}\right)^2}, \quad \text{where } \sqrt{\text{--}} = \sqrt{\frac{d h^\vee}{6}}.
\]

Recall that all well-known superconformal algebras in conformal field theory are the minimal \( W \)-algebras or are obtained from them by a simple modification:

\begin{itemize}
  \item[(a)] \( W_k^{\text{min}}(spo(2|2N)) \) is the Virasoro vertex algebra for \( N = 0 \), the Neveu-Schwarz vertex algebra for \( N = 1 \), the \( N = 2 \) vertex algebra for \( N = 2 \), and becomes the \( N = 3 \) vertex algebra after tensoring with one fermion; it is the Knizhnik algebra for \( N > 3 \);
  \item[(b)] \( W_k^{\text{min}}(psl(2|2)) \) is the \( N = 4 \) vertex algebra;
  \item[(c)] \( W_k^{\text{min}}(D(2,1;a)) \) tensored with four fermions and one boson is the big \( N = 4 \) vertex algebra;
\end{itemize}

The unitary Virasoro \( (N = 0) \), Neveu-Schwarz \( (N = 1) \) and \( N = 2 \) simple vertex algebras were classified in the mid 80s. Up to isomorphism, these vertex algebras depend only on the
central charge \( c(k) \), given by (1.4). Putting \( k = \frac{1}{p} - 1 \) in (1.5) in all three cases, we obtain
\begin{align*}
(1.6) \quad c(k) &= 1 - \frac{6}{p(p+1)} \quad \text{for Virasoro vertex algebra}, \\
(1.7) \quad c(k) &= \frac{3}{2} \left( \frac{1 - \frac{8}{p(p+2)}}{p(p+1)} \right) \quad \text{for Neveu-Schwarz vertex algebra}, \\
(1.8) \quad c(k) &= 3 \left( 1 - \frac{2}{p} \right) \quad \text{for } N = 2 \text{ vertex algebra}.
\end{align*}

The following theorem is a result of several papers, published in the 80s in physics and mathematics literature, see e.g. [4] for references.

**Theorem 1.2.** The complete list of unitary \( N = 0, 1, 2 \) vertex algebras is as follows: either \( c(k) \) is given by (1.6), (1.7), or (1.8) respectively for \( p \in \mathbb{Z}_{\geq 2} \) or \( c(k) \geq 1, \frac{3}{2}, 3 \) respectively.

The above three cases cover all minimal \( W \)-algebras, associated with \( g \), such that the 0th eigenspace \( g_0 \) of \( adx \) is abelian. Thus, we may assume that \( g_0 \) is not abelian. Under this assumption we obtain a complete classification of minimal simple unitary \( W \)-algebras (cf. Theorem 6.5).

**Theorem 1.3.** The simple minimal \( W \)-algebra \( W_{-k}^{\text{min}}(g) \) with \( k \neq h^\vee \) and \( g_0 \) non-abelian is non-trivial unitary if and only if
\begin{enumerate}
\item \( g = sl(2|m), m > 3, k = 1; \)
\item \( g = psl(2|2), k \in \mathbb{N} + 1; \)
\item \( g = spo(2|3), k \in \frac{1}{4}(\mathbb{N} + 2); \)
\item \( g = spo(2|m), m > 3, k = \frac{1}{2}(\mathbb{N} + 1); \)
\item \( g = D(2,1;-\frac{m}{m+n}), k = \frac{mn}{m+n}, \text{ where } m, n \in \mathbb{N}, m + n > 2; \)
\item \( g = F(4), k \in \frac{1}{2}(\mathbb{N} + 1); \)
\item \( g = G(3), k \in \frac{3}{2}(\mathbb{N} + 1). \)
\end{enumerate}

The case (1) is just the free boson vertex algebra. The results (2) and (3) of Theorem 2.7 are consistent with the results of [2] and [12] respectively.

By Theorem 6.3 (b) the vertex algebra \( W_{k}^{\text{min}}(g) \), with \( k \neq -h^\vee \) and non-collapsing, has a non-trivial unitary module only when \( W_{k}^{\text{min}}(g) \) is a unitary vertex algebra. We are planning to classify its unitary modules and compute their characters in a subsequent publication.

Throughout the paper the base field is \( \mathbb{C} \), and \( \mathbb{Z}_+ \) and \( \mathbb{N} \) stand for the set of non negative and positive integers, respectively.

**Acknowledgements.** Victor Kac is partially supported by the Bert and Ann Kostant fund and the Simons collaboration grant.

2. Setup

2.1. Basic simple Lie superalgebras. Let \( g = g_0 \oplus g_1 \) be a basic simple finite-dimensional Lie superalgebra over \( \mathbb{C} \) as in (1.3). Choose a Cartan subalgebra \( h \) of \( g_0 \). It is a maximal \( ad \)-diagonalizable subalgebra of \( g \), for which the root space decomposition is of the form
\begin{equation}
g = h \oplus \bigoplus_{\alpha \in \Delta} g_{\alpha},
\end{equation}
where \( \Delta \subset h^* \setminus \{0\} \) is the set of roots. In all cases, except for \( g \cong psl(2|2) \), the root spaces have dimension 1. In the case \( g = psl(2|2) \) one can achieve this property by embedding
in \(pgl(2|2)\) and replacing \(\phi\) by the root space decomposition with respect to a Cartan subalgebra of \(pgl(2|2)\), which we will do.

Let \(\Delta^+\) be a subset of positive roots and \(\Pi = \{\alpha_1, \ldots, \alpha_r\}\) be the corresponding set of simple roots. Set \(\Pi_1\) be the set of simple odd roots. For each \(\alpha \in \Delta^+\) choose \(X_\alpha \in \mathfrak{g}_\alpha\) and \(X_{-\alpha} \in \mathfrak{g}_{-\alpha}\) such that \((X_\alpha|X_{-\alpha}) = 1\), and let \(h_\alpha = [X_\alpha|X_{-\alpha}]\). Let \(e_i = X_{\alpha_i}, f_i = X_{-\alpha_i}, i = 1, \ldots, r\). The set \(\{e_i, f_i, h_{\alpha_i} \mid i = 1, \ldots, r\}\) generates \(\mathfrak{g}\), and satisfies the following relations
\[
\begin{align*}
[e_i, f_j] &= \delta_{ij}h_{\alpha_i}, \\
[h_{\alpha_i}, e_j] &= (\alpha_i|\alpha_j)e_j, \\
[h_{\alpha_i}, f_j] &= -(\alpha_i|\alpha_j)f_j.
\end{align*}
\]

The Lie superalgebra \(\tilde{\mathfrak{g}}\) on generators \(\{e_i, f_i, h_{\alpha_i} \mid i = 1, \ldots, r\}\) subject to relations \(\Pi\) is a (infinite-dimensional) \(\mathbb{Z}\)-graded Lie algebra, where the grading is defined by \(\deg h_{\alpha_i} = 0, \deg e_i = -\deg f_i = 1\), with a unique \(\mathbb{Z}\)-graded maximal ideal, and \(\mathfrak{g}\) is the quotient of \(\tilde{\mathfrak{g}}\) by this ideal. Note that \((\alpha_i|\alpha_j) \in \mathbb{R}\) for any \(\alpha_i, \alpha_j \in \Pi\).

### 2.2. Conjugate linear involutions and real forms.

In the above setting, given a collection of complex numbers \(\Lambda = \{\lambda_1, \ldots, \lambda_r\}\) such that \(\lambda_i \in \sqrt{-1}\mathbb{R}\) if \(\alpha_i\) is an odd root and \(\lambda_i \in \mathbb{R}\) if \(\alpha_i\) is an even root, we can define an antilinear involution \(\omega_\Lambda : \mathfrak{g} \to \mathfrak{g}\) setting
\[
\omega_\Lambda(e_i) = \lambda_i f_i, \quad \omega_\Lambda(f_i) = \lambda_i^{-1} e_i, \quad \omega_\Lambda(h_{\alpha_i}) = -h_{\alpha_i}, \quad 1 \leq i \leq r.
\]

Since \(\omega_\Lambda\) preserves relations \(\Pi\), it induces an antilinear involution of \(\tilde{\mathfrak{g}}\), and, since \(\omega_\Lambda\) preserves the \(\mathbb{Z}\)-grading of \(\tilde{\mathfrak{g}}\), it preserves its unique maximal ideal, hence it induces an antilinear involution of \(\mathfrak{g}\).

Set \(\sigma_\alpha = -1\) if \(\alpha\) is an odd negative root and \(\sigma_\alpha = 1\) otherwise, so that \((X_\alpha|X_{-\alpha}) = \sigma_\alpha\).

Let
\[
\xi_\alpha = \begin{cases} 
\text{sgn}(\alpha|\alpha) & \text{if } \alpha \text{ is an even root,} \\
1 & \text{if } \alpha \text{ is an odd root.}
\end{cases}
\]

Then in [5 (4.13), (4.15)] it is proven (using results from [6]), that one can choose root vectors \(X_\alpha\) in such a way that
\[
\omega_\Lambda(X_\alpha) = -\sigma_\alpha \xi_\alpha \lambda_\alpha X_{-\alpha},
\]
where
\[
\lambda_\alpha = \prod_i (-\xi_\alpha \lambda_i)^{n_i} \text{ for } \alpha = \sum_{i=1}^r n_i \alpha_i.
\]

We shall call this a good choice of root vectors.

### 2.3. Invariant Hermitian forms on vertex algebras.

Let \(V\) be a conformal vertex algebra with conformal vector \(L = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}\) (see [11] for the definition and undefined notation). Let \(\phi\) be a conjugate linear involution of \(V\). A Hermitian form \(H(\cdot, \cdot)\) on \(V\) is called \(\phi\)-invariant if, for all \(a \in V\), one has \(\Pi\)
\[
H(v, Y(a, z)u) = H(Y(A(z)a, z^{-1}v)u), \quad u, v \in V.
\]

Here the linear map \(A(z) : V \to V((z))\) is defined by
\[
A(z) = e^{zL_1} z^{-2L_0} g,
\]
where
\[
g(a) = e^{-\pi \sqrt{-1} (\frac{1}{2} p(a) + \Delta_a)} \phi(a), \quad a \in V,
\]
\( \Delta_0 \) stands for the \( L_0 \)-eigenvalue of \( a \), and
\[
p(a) = \begin{cases} 
0 \in \mathbb{Z} & \text{if } a \in \mathfrak{g}_0, \\
1 \in \mathbb{Z} & \text{if } a \in \mathfrak{g}_1.
\end{cases}
\]

**Definition 2.1.** We say that a conformal vertex algebra \( V \) is unitary if there exists a conjugate linear involution \( \phi \) of \( V \) and a \( \phi \)-invariant positive definite Hermitian form on \( V \).

3. **The almost compact conjugate linear involution of \( \mathfrak{g} \)**

From now on we let \( \mathfrak{g} \) be a basic simple finite-dimensional Lie superalgebra such that
\[
\mathfrak{g}_0 = \mathfrak{s} \oplus \mathfrak{g}^\natural.
\]
where \( \mathfrak{s} \cong \mathfrak{sl}_2 \) and \( \mathfrak{g}^\natural \) is the centralizer of \( \mathfrak{s} \) in \( \mathfrak{g} \).

This corresponds to consider \( \mathfrak{g} \) as in Table 2 of [2]. We will also assume that \( \mathfrak{g}^\natural \) is not abelian; this condition rules out \( \mathfrak{g} = \mathfrak{spo}(2|m) \), \( m = 0, 1, 2 \). The explicit list is given in the leftmost column of Table 1. Note that \( \mathfrak{sl}(2|1) \) and \( \mathfrak{osp}(4|2) \) are missing there since \( \mathfrak{sl}(2|1) \cong \mathfrak{spo}(2|2) \) and \( \mathfrak{osp}(4|2) \cong D(2, 1; -\frac{1}{2}) \).

First, we prove the simple lemma mentioned in the Introduction, which implies the first two conditions of (1.1) imply the third one.

**Lemma 3.1.** Let \( \mathfrak{g} \) be a simple Lie superalgebra with an invariant supersymmetric bilinear form \( (\cdot|\cdot) \), let \( x \in \mathfrak{g} \), and let \( \phi \) be a conjugate linear involution of \( \mathfrak{g} \), such that
\[
(3.1) \quad (x|x) \text{ is a non-zero real number, and } \phi(x) = x.
\]
Then
\[
(3.2) \quad (\phi(a)|\phi(b)) = (a|b), \text{ for all } a, b \in \mathfrak{g}.
\]

**Proof.** Note that \( (\phi(a)|\phi(b)) \) is an invariant supersymmetric bilinear form as well, hence it is proportional to \( (a|b) \) since \( \mathfrak{g} \) is simple. Due to (3.2) these two bilinear forms coincide. \( \square \)

**Proposition 3.2.** For any \( \mathfrak{sl}_2 \)-triple \( \{e, x, f\} \), such that (3.1) holds for \( \mathfrak{s} = \text{span}\{e, x, f\} \), an almost compact involution exists.

**Proof.** Choose a Cartan subalgebra \( t \) of \( \mathfrak{g}_0 \). We observe that if we prove the existence of an almost compact involution \( \phi \) for a special choice of \( \{e, x, f\} \), then an almost compact involution exists for any choice of the \( \mathfrak{sl}_2 \)-triple. Indeed, if \( \{e', x', f'\} \) is another \( \mathfrak{sl}_2 \)-triple, then there is an inner automorphism \( \psi \) of \( \mathfrak{s} \) mapping \( \{e, x, f\} \) to \( \{e', x', f'\} \), which extends to an inner automorphism of \( \mathfrak{g} \). Therefore \( \phi' = \psi \phi \psi^{-1} \) is an almost compact involution for \( \{e', x', f'\} \). The construction of \( \{e, x, f\} \) and \( \phi \) and the verification of properties (i)-(iii) in Definition 1.1 will be done in four steps:

1. make a suitable choice of positive roots for \( \mathfrak{g} \) with respect to \( t \);
2. define \( \phi \) by specializing (2.3);
3. construct \( \{e, f, x\} \) and verify that \( \phi(f) = f, \phi(x) = x, \phi(e) = e \);
4. check that \( \phi \) is a compact involution for \( \mathfrak{g}^\natural \);

**Step 1.** We need some preparation. Let \( \Delta^\natural \) be the set of roots of \( \mathfrak{g}^\natural \) with respect to the Cartan subalgebra \( t \cap \mathfrak{g}^\natural \). Let \( \{\pm \theta\} \) be the \( t \cap \mathfrak{s} \)-roots of \( \mathfrak{s} \). Then \( R_0 = \{\pm \theta\} \cup \Delta^\natural \) is the set of roots of \( \mathfrak{g}_0 \) with respect to \( t \).

Let \( R \) be the set of roots of \( \mathfrak{g} \) with respect to \( t \), let \( R^+ \) be subset of positive roots whose corresponding set of simple roots \( S = \{\alpha_1, \ldots, \alpha_r\} \) is displayed in Table 1. Note that \( \theta \) is the
Consider a good choice of root vectors $\phi$. If $\theta$ corresponds to the fact that $\alpha$ fixes $e, f, x$. One checks directly that $\{e, f, x\}$ is an $sl_2$-triple.

Step 4. Endow $\mathfrak{g}$ with the $\mathbb{Z}$-grading

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$$

which assigns degree 0 to $h \in \mathfrak{t}$ and to $e_i$ and $f_i$ if $\alpha_i$ is even, and degree 1 to $e_i$ and degree $-1$ to $f_i$, if $\alpha_i$ is odd.

A direct check on Table 1 shows that $\mathfrak{q}_0 = \mathfrak{g}^\theta$. Recall from [5, Proposition 4.5] that the fixed points of $\phi$ in $\mathfrak{q}_0$ are a compact form of $\mathfrak{q}_0$ if and only if $\lambda_i(\alpha_i | \alpha_i) < 0$ for all $\alpha_i \in S \setminus S_1$. Step 4 now follows from (3.4).

### 4. Explicit expressions for almost compact forms

In this section we exhibit explicitly an almost compact involution $\phi$ in each case and discuss its uniqueness. If $\phi$ is an almost compact involution of $\mathfrak{g}$, we denote by $\mathfrak{g}^{ac}$ the corresponding real form (the fixed point set of $\phi$). We can define $\mathfrak{g}_0^{ac}$ by specifying a real form $\mathfrak{g}_0^{ac}$ of $\mathfrak{g}_0$ and a real form $\mathfrak{g}_1^{ac}$ of $\mathfrak{g}_1$. 
(1) $\mathfrak{g} = spo(2|m)$. Then $\mathfrak{g}_0 = sl_2 \oplus so_m$ and $\mathfrak{g}_1 = \mathbb{C}^2 \otimes \mathbb{C}^m$ as $\mathfrak{g}_0$-module. We set
\[
\mathfrak{g}_0^{ac} = sl_2(\mathbb{R}) \oplus so_m(\mathbb{R}), \quad \mathfrak{g}_1^{ac} = \mathbb{R}^2 \otimes \mathbb{R}^m.
\]
Explicitly, let $B$ be a non-degenerate $\mathbb{R}$-valued bilinear form of the superspace $\mathbb{R}^{2|m}$ with matrix $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & I_m \end{pmatrix}$. Then for $\mathfrak{g} = spo(2|m)$ we have:
\[
\mathfrak{g}^{ac} = \{ A \in sl(m|n; \mathbb{R}) \mid B(Au, v) + (-1)^p(A)p(u)B(u, Av) = 0 \}.
\]
(2) $\mathfrak{g} = psl(2|2)$. Let $H$ be a $\mathbb{C}$-valued non-degenerate sesquilinear form on the superspace $\mathbb{C}^{2|2}$ whose matrix is $\text{diag}(\sqrt{-1}, -\sqrt{-1}, 1, 1)$. Set
\[
\tilde{\mathfrak{g}}^{ac} = \{ A \in sl(2|2; \mathbb{C}) \mid H(Au, v) + (-1)^p(A)p(u)H(u, Av) = 0 \}.
\]
Then
\[
\mathfrak{g}^{ac} = \tilde{\mathfrak{g}}^{ac}/R\sqrt{-1}I.
\]
Explicitly, we have $\mathfrak{g}_0 = sl_2 \oplus sl_2$ and $\mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \mid B, C \in M_{2,2}(\mathbb{C}) \right\}$ as a $\mathfrak{g}_0$-module. Then
\[
\tilde{\mathfrak{g}}^{0c} = \left\{ \begin{pmatrix} A \\ D \end{pmatrix} \mid A \in su(1,1), D \in su_2 \right\},
\]
\[
\tilde{\mathfrak{g}}^{1c} = \left\{ \begin{pmatrix} u & v \\ \sqrt{-1}u \bar{v} - \bar{u}v & 0 \end{pmatrix} \mid u, v \in \mathbb{C}^2 \right\}.
\]
(3) $\mathfrak{g} = D(2,1;a)$. Then $\mathfrak{g}_0 = sl_2 \oplus sl_2 \oplus sl_2 = so(4,\mathbb{C}) \oplus sl_2$ and $\mathfrak{g}_1 = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^4 \otimes \mathbb{C}^2$ as $\mathfrak{g}_0$-module. We set
\[
\mathfrak{g}_0^{ac} = so(4,\mathbb{R}) \oplus \text{span}_\mathbb{R}\{e, f, x\}, \quad \mathfrak{g}_1^{ac} = \mathbb{R}^4 \otimes \mathbb{R}^2.
\]
To get an explicit realization, consider the contact Lie superalgebra (see [7] for more details)
\[
K(1,4) = \mathbb{C}[t, \xi_1, \xi_2, \xi_3, \xi_4]
\]
where $t$ is an even variable and $\xi_i$, $1 \leq i \leq 4$, are odd variables. Introduce on the associative superalgebra $K(1,4)$ a $\mathbb{Z}$-grading by letting
\[
\deg' t = 2, \quad \deg' \xi_i = 1,
\]
and the bracket
\[
\{f, g\} = (2 - \sum_{i=1}^{4} \xi_i \partial_i) f \partial_i g - \partial_i f (2 - \sum_{i=1}^{4} \xi_i \partial_i) g + \sum_{i=1}^{4} (-1)^{p(f)} \partial_i f \partial_i g
\]
where $\partial_i = \partial_{\xi_i}$. This is a $\mathbb{Z}$-graded Lie superalgebra with compatible grading $\deg f = \deg' f - 2$. We have
\[
K(1,4) = \bigoplus_{j \geq -2} K(1,4)_j,
\]
where
\[
K(1,4)_{-2} = \mathbb{C}1, \quad K(1,4)_{-1} = \text{span}_\mathbb{C}(\xi_i \mid 1 \leq i \leq 4), \quad K(1,4)_{0} = \text{span}_\mathbb{C}(\xi_i \xi_j \mid 1 \leq i, j \leq 4), \quad K(1,4)_{1} = \mathfrak{g}_1^{ac}, \quad K(1,4)_{2} = \mathfrak{g}_1^{ac}.
\]
Here
\[
\mathfrak{g}_1' = \text{span}_\mathbb{C}(t \xi_i \mid 1 \leq i \leq 4), \quad \mathfrak{g}_1'' = \text{span}_\mathbb{C}(\xi_i \xi_j \xi_k \mid 1 \leq i, j, k \leq 4).
\]
Note that $\text{span}_\mathbb{C}(\xi_1, \xi_2, \xi_3, \xi_4) = \Lambda^2 \mathbb{C}^4 \cong \text{so}(4, \mathbb{C})$, that $\mathfrak{g}_1'$ is isomorphic to the standard representation $\mathbb{C}^4$ of $\text{so}(4, \mathbb{C})$ and that $\mathfrak{g}_1''$ is isomorphic to $\Lambda^3 \mathbb{C}^4$, so that $K(1, 4)_1 = \mathbb{C}^4 \oplus \mathbb{C}^4$ as $\text{so}(4, \mathbb{C})$-module. Also notice that $\{\mathfrak{g}_1', \mathfrak{g}_1''\} = \mathbb{C}^2$, $\{\mathfrak{g}_1', \mathfrak{g}_1''\} = 0$. Fix now a copy $\hat{\mathfrak{g}}_b$ of an $\text{so}(4, \mathbb{C})$-module $\mathbb{C}^4$ in $\mathbb{C}^4 \oplus \mathbb{C}^4$, depending on a constant $b \in \mathbb{R}$, as follows. Set, for $1 \leq i \leq 4$,

$$a_i = t \xi_i + \hat{\xi}_i,$$

where $\hat{\xi}_i = (-1)^{i+1} \prod_{j \neq i} \xi_j$.

and define

$$\hat{\mathfrak{g}}_b = \sum_{i=1}^4 \mathbb{C} a_i.$$

Let $b \in \mathbb{R}$. Note that, setting $\xi = \xi_1 \xi_2 \xi_3 \xi_4$, we have

$$\{t \xi_i + \hat{\xi}_i, t \xi_j + \hat{\xi}_j\} = \delta_{ij}(-t^2 + 2b \xi).$$

Hence, if we set

$$e = -t^2 + 2b \xi, \quad f = -1, \quad x = t/2,$$

then $\{e, x, f\}$ is an $\mathfrak{sl}_2$-triple. Set

$$\mathfrak{g}^{ac} = \mathbb{R} \cdot 1 \oplus \left( \sum_{i=1}^4 \mathbb{R} \xi_i \right) \oplus \left( \sum_{i,j=1}^4 \mathbb{R} \xi_i \xi_j \oplus \mathbb{R} \frac{t}{2} \right) \oplus \left( \sum_{i=1}^4 \mathbb{R} a_i \right) \oplus \mathbb{R}(-t^2 + 2b \xi).$$

Then $\mathfrak{g}^{ac}$ is an almost compact form of $D(2, 1; \frac{1+b}{1-b})$. To prove this, it suffices to calculate the Cartan matrix for a choice of Chevalley generators of the complexification of $\mathfrak{g}^{ac}$. Fix a Cartan subalgebra in $\mathfrak{g}^2 = \text{so}(4, \mathbb{C})$ as the span of $v_2 = -\sqrt{-1} \xi_1 \xi_2, v_3 = -\sqrt{-1} \xi_3 \xi_4$. Set $v_1 = t$; then $\{v_1, v_2, v_3\}$ is a basis of a Cartan subalgebra of $\mathfrak{g}$. Let $\{e_1, e_2, e_3\}$ the dual basis to $\{v_1, v_2, v_3\}$. One can choose $\{\alpha_1 = e_2 - e_1, \alpha_2 = e_1 - e_3, \alpha_3 = e_1 + e_3\}$ as a set of simple roots. The associated Chevalley generators are

$$e_1 = -\sqrt{-1} a_1 + a_2, \quad e_2 = \xi_1 \xi_3 + \xi_2 \xi_4 + \sqrt{-1}(\xi_1 \xi_4 - \xi_2 \xi_3), \quad e_3 = \xi_1 \xi_3 - \xi_2 \xi_4 - \sqrt{-1}(\xi_1 \xi_4 + \xi_2 \xi_3),$$

$$f_1 = \sqrt{-1} \xi_1 + \xi_2, \quad f_2 = \xi_1 \xi_3 + \xi_2 \xi_4 - \sqrt{-1}(\xi_1 \xi_4 - \xi_2 \xi_3), \quad f_3 = \xi_1 \xi_3 - \xi_2 \xi_4 + \sqrt{-1}(\xi_1 \xi_4 + \xi_2 \xi_3),$$

$$h_1 = -2v_1 + 2v_2 + 2b v_3, \quad h_2 = 4v_1 - 4v_3, \quad h_3 = 4v_1 + 4v_3,$$

and the corresponding Cartan matrix, normalized as in [7], is

$$
\begin{pmatrix}
0 & 1 & \frac{1+b}{1-b} \\
-1 & 2 & 0 \\
-1 & 0 & 2
\end{pmatrix}
$$

Hence $a = \frac{1+b}{1-b}$

and therefore all $a \neq -1$ occur in this construction. Since this subalgebra is 17-dimensional, it is isomorphic to $D(2, 1; a)$.

**Remark 4.1.** Note that $a = 0$ for $b = -1$. In this case, $D(2, 1; 0)$ contains a 11-dimensional solvable ideal generated by $f_1$, which is spanned by $h_1$ and the root vectors relative to roots having $\alpha_1$ in their support. If we replace $a_i$ by $a_i/b$ and $h_1$ by $h_1/b$, and tend $b$ to $+\infty$, we recover also the Lie superalgebra of derivations of $\mathfrak{psl}(2|2)$, and its almost compact real form.

(4) $\mathfrak{g} = G(3)$. Then $\mathfrak{g}_0 = \mathfrak{sl}_2 \oplus G_2$ and $\mathfrak{g}_1 = \mathbb{C}^2 \otimes L_{\text{min}}$, where $L_{\text{min}}$ is the 7-dimensional over $\mathbb{C}$ irreducible representation of $G_2$, and we let

$$\mathfrak{g}_0^{ac} = \mathfrak{sl}_2(\mathbb{R}) \oplus G_{2,0}, \quad \mathfrak{g}_1^{ac} = \mathbb{R}^2 \otimes L_{\text{min},0}.$$
4.1. Uniqueness of the almost compact involution.

**Proposition 4.2.** An almost compact involution is uniquely determined up to a sign by its action on \( \mathfrak{g}_0 \), provided that the \( \mathfrak{g}_0 \)-module \( \mathfrak{g}_{1/2} \) is irreducible.

**Proof.** If there are two different extensions of the compact involution, then their ratio \( \psi \), say, is identical on \( \mathfrak{g}_0 \), hence, by Schur’s lemma, \( \psi \) acts as a scalar on \( \mathfrak{g}_{-1/2} \). Since \( \phi(f) = f \), we conclude that this scalar is \( \pm 1 \). \( \square \)

It remains to discuss the cases \( \mathfrak{g} = sl(2|m) \), \( m \geq 3 \), and \( psl(2|2) \), since in all other cases of Table 1 the \( \mathfrak{g}_0 \)-module \( \mathfrak{g}_{1/2} \) is irreducible. In this cases \( \mathfrak{g} \) is of type I, that is \( \mathfrak{g}_1 = \mathfrak{g}_1^+ \oplus \mathfrak{g}_1^- \) where \( \mathfrak{g}_1^\pm \) are contragredient irreducible \( \mathfrak{g}_0 \)-modules and \( [\mathfrak{g}_1^+, \mathfrak{g}_1^-] = 0 \). Let \( \delta_\lambda \) be the linear map on \( \mathfrak{g} \) defined by setting

\[
\delta_\lambda|_{\mathfrak{g}_0} = Id, \quad \delta_\lambda|_{\mathfrak{g}_1^+} = \lambda Id, \quad \delta_\lambda|_{\mathfrak{g}_1^-} = \lambda^{-1} Id. \tag{4.1}
\]

Then \( \delta_\lambda \) is an automorphism of \( \mathfrak{g} \) for any \( \lambda \in \mathbb{C} \). Suppose that \( \phi' \) is another conjugate almost compact linear involution such that \( \phi'|_{\mathfrak{g}_0} = \phi \). Then \( \phi' = \phi \circ \gamma \) with \( \gamma \) an automorphism of \( \phi \) such that \( \gamma|_{\mathfrak{g}_0} = Id \). If \( \mathfrak{g} = sl(2|m) \), by [13] Lemmas 1 and 2], we have \( \gamma = \delta_\lambda \). Since \( \phi(\mathfrak{g}_1^+) = \mathfrak{g}_1^- \) and \( (\phi')^2 = Id \) we have that \( \lambda \in \mathbb{R} \).

Consider the following symmetric bilinear form on \( \mathfrak{g}_{-1/2} \)

\[
\langle u, v \rangle = (e[u, v]). \tag{5.1}
\]

We want to prove the following

**Proposition 5.1.** We can choose an almost compact involution such that \( \langle \cdot, \cdot \rangle \) is positive definite on \( \mathfrak{c} = \{u \in \mathfrak{g}_{-1/2} \mid \phi(u) = -u\} \).

The proof requires a detailed analysis of the action of an almost compact involution on \( \mathfrak{g}_{-1/2} \). Define structure constants \( N_{\alpha, \beta} \) for a good choice of root vectors (see Subsection 2.2) by the relation

\[
[X_\alpha, X_\beta] = N_{\alpha, \beta} X_{\alpha + \beta}.
\]

Observe that \( \{X_\theta, X_{\theta}, \frac{1}{2} h_\theta\} \) is a \( sl_2 \)-triple in \( \mathfrak{s} \). Let

\[
\mathfrak{g} = \mathbb{C} X_\theta \oplus \mathfrak{g}_{1/2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1/2} \oplus \mathbb{C} X_{-\theta}
\]

(5) \( \mathfrak{g} = F(4) \). Then \( \mathfrak{g}_0 = sl_2 \oplus so_7 \) and \( \mathfrak{g}_1 = \mathbb{C}^2 \otimes \text{spin}_7 \), and we let

\[
\mathfrak{g}_0^{ac} = sl_2(\mathbb{R}) \oplus so_7(\mathbb{R}), \quad \mathfrak{g}_1^{ac} = \mathbb{R}^2 \otimes \text{spin}(\mathbb{R}^7).
\]

It is proved in [7, Proposition 5.3.2] that in both cases (4) and (5) \( \mathfrak{g}^{ac} = \mathfrak{g}_0^{ac} \oplus \mathfrak{g}_1^{ac} \) is an almost compact form of \( \mathfrak{g} \).
be the decomposition into \( ad \frac{h}{2} \theta \) eigenspaces. By \( sl_2 \) theory, \( ad X_{\pm \theta} : \tilde{g}_{1/2} \to \tilde{g}_{1/2} \) is an isomorphism of \( g \)-modules. Moreover, by our choice of \( R^+ \), the roots of \( \tilde{g}_{1/2} \) (resp. \( \tilde{g}_{-1/2} \)) are precisely the negative (resp. positive) odd roots. In particular, the map \( \alpha \mapsto -\theta + \alpha \) defines a bijection between the positive and negative odd roots. We shall need the following properties.

**Lemma 5.2.** For a positive odd root \( \alpha \) we have

\[
\begin{align*}
N_{-\theta, \alpha} N_{\theta, -\alpha} &= 1, \\
N_{-\theta, \alpha}^2 &= 1.
\end{align*}
\]

In particular \( N_{\theta, \alpha} \) is real.

_Proof._ Relation (5.2) is proven in [5, Lemma 4.3]. Equation (5.3) follows from [5, (4.8)], noting that the \( -\theta \)-string through \( \alpha \) has length 1. \( \square \)

Arguing as in Proposition 3.2, we can assume in the proof of Proposition 5.1 that \( \{ e, x, f \} \) is the \( sl_2 \)-triple defined in (3.5); \( ad x \) defines on \( g \) a minimal grading

\[
(5.4) \quad g = \mathbb{C} f \oplus g_{-1/2} \oplus g_0 \oplus g_{1/2} \oplus \mathbb{C} e.
\]

Set, for an odd root \( \alpha \in R^+ \)

\[
(5.5) \quad u_\alpha = X_\alpha + \sqrt{-1} N_{-\theta, \alpha} X_{\alpha - \theta}.
\]

Note that

\[
[x, u_\alpha] = \sqrt{-1} [X_\alpha - X_{-\theta}, X_\alpha + \sqrt{-1} N_{-\theta, \alpha} X_{\alpha - \theta}] = \frac{1}{2} N_{-\theta, \alpha} N_{\theta, \alpha} X_\alpha - \sqrt{-1} N_{-\theta, \alpha} X_{\alpha - \theta} = -\frac{1}{2} u_\alpha,
\]

hence \( \{ u_\alpha \mid \alpha \in R^+, \alpha \text{ odd} \} \) is a basis of \( g_{-1/2} \).

**Lemma 5.3.** If \( \alpha \) is a positive odd root then

\[
(5.6) \quad \phi(u_\alpha) = N_{-\theta, \alpha} u_{\theta - \alpha}.
\]

_Proof._ By (2.4), \( \phi(X_\alpha) = \sqrt{-1} X_{-\alpha} \) if \( \alpha \) is an odd root, hence, by (5.3), since \( N_{-\theta, \alpha} \) is real,

\[
(5.7) \quad \phi(u_\alpha) = \phi(X_\alpha + \sqrt{-1} N_{-\theta, \alpha} X_{\alpha - \theta}) = \sqrt{-1} X_{-\alpha} + N_{-\theta, \alpha} X_{-\alpha} = N_{-\theta, \alpha} (X_{-\theta + \alpha} + \sqrt{-1} N_{-\theta, \alpha} X_{-\alpha}).
\]

Note that, since \( \phi(x) = x, \phi(u_\alpha) \) has to belong to \( g_{-1/2} \). This forces

\[
(5.8) \quad N_{-\theta, \alpha} N_{-\theta, -\alpha} = 1,
\]

and (5.7) becomes (5.6). \( \square \)

_Proof of Proposition 5.1._ Set \( v_\alpha = \frac{1}{2} (u_\alpha - \phi(u_\alpha)) + \frac{\sqrt{-1}}{2} (u_\alpha + \phi(u_\alpha)) \), where \( \alpha \) runs over the positive odd roots. It is clear that \( v_\alpha \in \mathfrak{r} \). We want to prove that the vectors \( v_\alpha \) form an orthogonal basis of \( \mathfrak{r} \). We need two auxiliary computations:

\[
(5.9) \quad [e, u_\alpha] = \sqrt{-1} X_\alpha + N_{-\theta, \alpha} X_{\alpha - \theta},
\]

\[
(5.10) \quad \langle u_\alpha, u_\beta \rangle = -(N_{-\theta, \alpha} + N_{-\theta, \beta}) \delta_{\alpha - \beta}.
\]
To prove (5.9) use (5.2):
\[
[e, u_\alpha] = \frac{1}{2} [X_\theta + X_\alpha + \sqrt{-1} h_\theta, X_\alpha + \sqrt{-1} N_{-\theta,\alpha} X_{\alpha - \theta}] = \frac{1}{2} [X_\theta + X_\alpha + \sqrt{-1} h_\theta, X_\alpha + \sqrt{-1} N_{-\theta,\alpha} X_{\alpha - \theta}]
\]
\[
= \frac{1}{2} (\sqrt{-1} N_{-\theta,\alpha} N_{\theta,\alpha - \theta} X_\alpha + N_{-\theta,\alpha} X_{\alpha - \theta} + \sqrt{-1} X_\alpha + N_{-\theta,\alpha} X_{\alpha - \theta}) = \sqrt{-1} X_\alpha + N_{-\theta,\alpha} X_{\alpha - \theta}.
\]
To prove (5.10) use (5.9)
\[
\langle u_\alpha, u_\beta \rangle = \langle e[u_\alpha, u_\beta] \rangle = \langle e, u_\alpha \rangle \langle u_\beta \rangle
\]
\[
= \langle \sqrt{-1} X_\alpha + N_{-\theta,\alpha} X_{\alpha - \theta} | X_\alpha + \sqrt{-1} N_{-\theta,\beta} X_{\beta - \theta} \rangle = \sigma_{\alpha - \theta} N_{-\theta,\alpha} \delta_{-\alpha, \beta} - \sigma_{\alpha} N_{-\theta,\beta} \delta_{\alpha, \beta}
\]
\[
= -(N_{-\theta,\alpha} + N_{-\theta,\beta}) \delta_{\alpha, \beta}.
\]
Set
\[
M_{\alpha, \beta} = -(N_{-\theta,\alpha} + N_{-\theta,\beta}).
\]
Then, using (5.10)
\[
\langle v_\alpha, v_\beta \rangle
\]
\[
= \langle 1+\frac{\sqrt{-1}}{2} u_\alpha - \frac{1-\sqrt{-1}}{2} N_{-\theta,\alpha} u_{\theta - \alpha}, 1+\frac{\sqrt{-1}}{2} u_\beta - \frac{1-\sqrt{-1}}{2} N_{-\theta,\beta} u_{\theta - \beta} \rangle
\]
\[
= \frac{\sqrt{-1}}{2} \langle u_\alpha, u_\beta \rangle - \frac{1}{2} N_{-\theta,\alpha} \langle u_{\theta - \alpha}, u_\beta \rangle - \frac{1}{2} N_{-\theta,\beta} \langle u_\alpha, u_{\theta - \beta} \rangle - \frac{\sqrt{-1}}{2} N_{-\theta,\alpha} N_{-\theta,\beta} \langle u_\alpha, u_\alpha \rangle
\]
\[
= \frac{\sqrt{-1}}{2} M_{\alpha, \beta} \delta_{\alpha, \beta} - \frac{1}{2} N_{-\theta,\alpha} M_{\theta - \alpha, \beta} \delta_{\alpha, \beta} - \frac{1}{2} N_{-\theta,\beta} M_{\alpha, \theta - \beta} \delta_{\alpha, \beta}
\]
\[
- \frac{\sqrt{-1}}{2} N_{-\theta,\alpha} N_{-\theta,\beta} M_{\theta - \alpha, \theta - \beta} \delta_{\alpha, \beta}.
\]
Therefore by (5.2) and (5.8)
\[
\langle v_\alpha, v_\beta \rangle = 2 \delta_{\alpha, \beta}.
\]
In particular, the restriction of \( \langle \cdot, \cdot \rangle \) to \( \mathfrak{r} \) is positive definite. □

## 6. Unitarity

Let \( \{e, x, f\} \) be the \( sl_2 \) triple constructed in Proposition 3.2. Let \( \mathfrak{h}^\perp \subset \mathfrak{g}^\perp \) be a Cartan subalgebra of \( \mathfrak{g}^\perp \) and fix
\[
\mathfrak{h} = \mathbb{C} x \oplus \mathfrak{h}^\perp
\]
as a Cartan subalgebra of \( \mathfrak{g} \). Let \( \Delta \) be the set of roots of \( \mathfrak{g} \) with respect to \( \mathfrak{h} \). Let \( \{\pm \theta\} \) be the set of roots of \( \mathfrak{s} \). Fix a set \( \Delta^+ \) of positive roots on \( \Delta \) in such a way that \( \theta \in \Delta^+ \) and \( e \in \mathfrak{g}_\theta \) (hence \( f \in \mathfrak{g}_{-\theta} \)). Note that \( x = \frac{1}{2} h_\theta \). Note that \( \langle \cdot, \cdot \rangle_\mathfrak{s} \) is non-degenerate, hence \( (x|x) \neq 0 \) and therefore we have the orthogonal direct sum of reductive subalgebras
\[
(6.1) \quad \mathfrak{g}_0 = \mathbb{C} x \oplus \mathfrak{g}^\perp.
\]

Let \( \mathfrak{g}^\perp = \bigoplus_{i \geq 0} \mathfrak{g}^\perp_i \) be the decomposition of \( \mathfrak{g}^\perp \) into the direct sum of ideals, where \( \mathfrak{g}^\perp_i \) is simple for \( i > 0 \) and \( \mathfrak{g}^\perp_0 \) is the center. Let \( \mathfrak{h}^\vee \) be the dual Coxeter number of \( \mathfrak{g} \), and denote by \( \tilde{\mathfrak{h}}_i^\vee \) half of the eigenvalue of the Casimir element of \( \mathfrak{g}^\perp_i \) with respect to \( \langle \cdot, \cdot \rangle_{\mathfrak{g}^\perp_i \times \mathfrak{g}^\perp_i} \), when acting on \( \mathfrak{g}^\perp_i \). Note that \( \tilde{\mathfrak{h}}_0^\vee = 0 \).

In [S] the authors introduced (as a special case of a more general construction) the universal \textit{minimal} \( W \)-algebra \( W^k_{\text{min}}(\mathfrak{g}) \), whose simple quotient is \( W^k_{\text{min}}(\mathfrak{g}) \), attached to the grading (5.1).
This is a vertex algebra strongly and freely generated by elements \( L, J^v \) where \( v \) runs over a basis of \( g^\ast \), \( G^v \) where \( u \) runs over a basis of \( g_{-1/2} \), with the following \( \lambda \)-brackets: \( L \) is a Virasoro element (conformal vector) with central charge \( c(k) \) given by \( (1.4) \), \( J^v \) are primary of conformal weight 1, \( G^v \) is primary of conformal weight \( \frac{k}{2} \) and \( (9) \) (Theorem 5.1)

\[
\begin{align*}
[J^v, L G^u] &= G^{[v,u]} \\
[J^v, L J^w] &= J^{[v,u]} + \lambda \beta_k(v, w)
\end{align*}
\]

for \( u \in g_{-1/2}, v \in g^\ast \), \( v, w \in g^\ast \).

Here

\[
\beta_k(u, v) = \delta_{i,j}(k + \frac{h^\vee - h^\vee}{2})(u|v), \quad u \in g_i^\ast, v \in g_j^\ast.
\]

The most explicit formula for the \( \lambda \)-bracket between the \( G^u \) is given in [10] Proposition 5.8. It turns out that a crucial role is played by a certain monic quadratic polynomial \( p(k) \), introduced in [1] Table 4] and thoroughly investigated in [10]. (In Remark 6.3 we define it explicitly.) The following relation will be relevant in the sequel (see [1] Theorem 3.2):

\[
G^{(u)}(2) G^{(v)} = 4(e_g|[u, v]) p(k).
\]

The following proposition is a special case of [11] Lemma 7.3], in view of Lemma 3.1.

**Proposition 6.1.** Let \( \phi \) be a conjugate linear involution on \( g \) such that \( \phi(f) = f, \phi(x) = x, \phi(e) = e \). Then the map

\[
\phi(J^u) = J^{\phi(u)}, \quad \phi(G^v) = G^{\phi(v)}, \quad \phi(L) = L, \quad u \in g^\ast, \quad v \in g_{-1/2}
\]

extends to a conjugate linear involution of the vertex algebra \( W_k^{min}(g) \).

By Proposition 3.2 there is a conjugate linear involution \( \phi \) on \( g \) such that \( \phi(x) = x, \phi(f) = f \) and \( (g^\ast)^\phi \) is a compact real form of \( g^\ast \), hence, by Proposition 6.1 \( \phi \) induces a conjugate linear involution of the vertex algebra \( W_k^{min}(g) \), and descends to a conjugate linear involution \( \phi \) of its unique simple quotient \( W_k^{min}(g) \), which we again denote by \( \phi \).

By [11] Proposition 7.4 (b]), \( W_k^{min}(g) \) admits a unique \( \phi \)-invariant Hermitian form \( H(\cdot, \cdot) \) such that \( H(1, 1) = 1 \). Recall that if \( k + h^\vee \neq 0 \) then the kernel of \( H(\cdot, \cdot) \) is the unique maximal ideal of \( W_k^{min}(g) \), hence \( H(\cdot, \cdot) \) descends to define a \( \phi \)-invariant Hermitian form on \( W_k^{min}(g) \), which we again denote by \( H(\cdot, \cdot) \).

In the following we investigate the unitarity of \( W_k^{min}(g) \) (cf. Definition 2.1] when \( g \) is as in Section 3. With a slight abuse of terminology, we also say that \( W_k^{min}(g) \) is unitary if \( H(\cdot, \cdot) \) is positive semidefinite. It is clear that for \( k \neq -h^\vee \), \( W_k^{min}(g) \) is unitary if and only if \( W_k^{min}(g) \) is unitary.

We need to fix notation for affine vertex algebras. Let \( a \) be a Lie superalgebra equipped with a nondegenerate invariant supersymmetric bilinear form \( B \). The universal affine vertex algebra \( V^B(a) \) is the universal enveloping vertex algebra of the Lie conformal superalgebra \( R = (\mathbb{C}[T] \otimes a) \oplus \mathbb{C} \) with \( \lambda \)-bracket given by

\[
[a_{\lambda} b] = [a, b] + \lambda B(a, b), \quad a, b \in a.
\]

In the following, we shall say that a vertex algebra \( V \) is an affine vertex algebra if it is a quotient of some \( V^B(a) \). If \( a \) is simple Lie algebra, we denote by \( (\cdot | \cdot)^a \) the normalized invariant bilinear form on \( a \), defined by the condition \( (a|\alpha)^a = 2 \) for a long root \( \alpha \). Then \( B = k(\cdot | \cdot)^a \), and we simply write \( V^a(a) \). If \( k \neq -h^\vee \), then \( V^a(a) \) has a unique simple quotient, which will be denoted by \( V_k(a) \).
Consider the universal affine vertex algebra $V^{\alpha_k}(g_0)$ with $\lambda$-bracket (see [9 (5.16)])

$$[a, b] = [a, b] + \lambda \alpha_k(a, b),$$

where

$$\alpha_k(a, b) = ((k + h^\vee)(a|b) - \frac{1}{2} \kappa_{g_0}(a, b)),\tag{6.5}$$

and where $\kappa_{g_0}$ denotes the Killing form of $g_0$. Note that

$$\alpha_k(a, b) = \delta_{i, j}(k + h^\vee - \bar{h}^\vee_i)(a|b)$$

if $a \in g_i^\perp$, $b \in g_j^\perp$.

Let $\psi$ be a conjugate linear involution of $g$ such that $(\psi(x)|\psi(y)) = (x|y)$. By [11 §5.3] there exists a unique $\psi$-invariant Hermitian form $H$ on $V^k(g)$. If $k \neq -h^\vee$, then the kernel of $H$ is the maximal ideal of $V^k(g)$, hence $H$ can be pushed down to $V_k(g)$.

Recall that we assume that $g^\perp$ is not abelian. Let $\theta_i$ be the highest root of $g_i^\perp$ for $i > 0$. Set

$$z_i(k) = \frac{2}{u_i} \left( k + \frac{h^\vee - \bar{h}^\vee_i}{2} \right),$$

where

$$u_i = \begin{cases} 2 & \text{if } i = 0, \\ (\theta_1|\theta_i) & \text{if } i > 0. \end{cases}$$

Let $(\cdot|\cdot)_i^\perp$ denote the normalized invariant bilinear form on $g_i^\perp$ for $i > 0$ and let $(\cdot|\cdot)_0^\perp = (\cdot|\cdot)_{g_0^\perp \times g_0^\perp}$. Note that, for $i > 0$, $(a|b)^\perp_i = \delta_{i, j}(\theta_1|\theta_i)(a|b)$, hence,

$$\beta_k(a, b) = \delta_{i, j} z_i(k)(a|b)^\perp_i$$

for $a \in g_i^\perp$, $b \in g_j^\perp$.

Moreover we have

$$\alpha_k(a, b) = \delta_{i, j} \frac{2}{(\theta_1|\theta_i)} \left( k + h^\vee - \bar{h}^\vee_i \right) (a|b)^\perp_i = \delta_{i, j} (z_i(k) + \chi_i)(a|b)^\perp_i$$

for $a \in g_i^\perp$, $b \in g_j^\perp$,

where

$$\chi_i = \frac{h^\vee - \bar{h}^\vee_i}{u_i}, \quad i \geq 0.\tag{6.8}$$

The relevant data for computing $\chi_i$, for $i \geq 0$ are collected in Table 2, where the explicit value of $z_i(k)$ for $i \geq 0$ is also displayed. Note that $z_0(k) = k + \frac{1}{2} h^\vee$.

Recall from [11] that a level $k$ is collapsing for $W_k^{\min}(g)$ if $W_k^{\min}(g)$ is either $\mathbb{C}$ or the simple affine vertex algebra over $g^\perp$. The main result of [11] states that $k$ is collapsing if and only if $p(k) = 0$.

| $g$   | $g^\perp$ | $u_i$ | $h^\vee$ | $\bar{h}^\vee_i$ | $z_i(k)$ | $\chi_i$ |
|-------|-----------|-------|----------|-----------------|-----------|---------|
| $sl(2|m)$, $m \geq 3$ | $\mathbb{C} \oplus sl_m$ | 2, -2 | 2 - $m$ | 0, $-m$ | $k - \frac{m-2}{2}$, $-k - 1$ | 1 - $m/2$, -1 |
| $psl(2|2)$ | $sl_2$ | -2 | 0 | -2 | $-k - 1$ | -1 |
| $osp(4|m)$, $m \geq 2$ | $sl_2 \oplus sp_m$ | 2, -4 | 2 - $m$ | 2, $-m - 2$ | $k - \frac{m}{2}$, $-\frac{1}{2}k - 1$ | -$m/2$, -1 |
| $spo(2|3)$ | $sl_2$ | -1/2 | 1/2 | -1/2 | $-k - 2$ | -2 |
| $spo(2|m)$, $m \geq 3$ | $so_m$ | -1 | 2 - $m/2$ | 1 - $m/2$ | $-2k - 1$ | -1 |
| $D(2,1;a)$ | $sl_2 \oplus sl_2$ | -2 - 2$a$, 2$a$ | 0 | -2 - 2$a$, 2$a$ | $-\frac{1}{1+a} k - 1$, $\frac{1}{1+a} k - 1$ | -1, -1 |
| $F(4)$ | $so_7$ | -4/3 | -3/2 | -3 | $-\frac{2}{3} k - 1$ | -1 |
| $G(3)$ | $G_2$ | -2/3 | -3/2 | -3 | $-\frac{2}{3} k - 1$ | -1 |

Table 2.
We summarize in the following result the content of Theorem 3.3 and Proposition 3.4 of [11] relevant to our setting.

**Theorem 6.2.** Let \( g \) be a basic simple Lie superalgebra from Table 2. Assume \( k \neq -h^\vee \).

1. If \( g^\circ \) is simple then \( k \) is collapsing if and only if \( z_1(k) = 0 \) or \( k = -\frac{h^\vee}{2} - 1 \).
2. If \( g^\circ \) is not simple then \( k \) is collapsing if and only if there is \( i \) such that \( z_i(k) = 0 \).

Moreover we have \( W^\text{min}_k(g) = \mathbb{C} \) if and only if \( g^\circ \) is simple and \( z_1(k) = 0 \).

**Remark 6.3.** We say that an ideal in \( g^\circ \) is a component of \( g^\circ \) if it is simple or 1-dimensional. It follows from [11, Lemma 3.1], [11, Theorem 5.9] that, up to a constant factor

\[
p(k) = \begin{cases} 
z_1(k)z_2(k) & \text{if } g^\circ \text{ has two components,} 
z_1(k)(k + \frac{h^\vee}{2} + 1) & \text{otherwise.}
\end{cases}
\]

The roots of \( p(k) \) are the collapsing levels defined in the Introduction.

**Theorem 6.4.** Let \( g \) be a Lie superalgebra from Table 2 with \( g^\circ \) non abelian and let \( W^\text{min}_k(g) \) be the corresponding minimal simple \( W \)-algebra. Assume that \( k \neq -h^\vee \).

(a) \( W^\text{min}_k(g) \) is unitary precisely in the following cases:

1. \( g \) is semisimple and \( z_i(k) \in \mathbb{Z}_+ \) for all \( i > 0 \).
2. \( g = sl(2|m), m \geq 3, k = -1 \).

(b) If there exists a unitary module for \( W^\text{min}_k(g) \), then \( W^\text{min}_k(g) \) is unitary.

**Proof.** Assume first that \( g^\circ \) is semisimple. We first prove that, if \( z_i(k) \in \mathbb{Z}_+ \) for all \( i > 0 \) then \( W^\text{min}_k(g) \) is unitary.

Recall from [11, Theorem 5.2] that there is a vertex algebra homomorphism \( \Psi : W^k_{\text{min}}(g) \rightarrow V^{\alpha_k}(g_0) \otimes F(A_{\beta}) \). We equip \( W^k_{\text{min}}(g) \) and \( V^{\alpha_k}(g_0) \otimes F(A_{\beta}) \) with their invariant Hermitian forms. Using the explicit description of the embedding given in [9], by (6.4), one can check that \( \Psi \) preserves the forms.

If \( a \) is a quadratic abelian subalgebra of \( g_0 \), and \( B = t(\cdot | \cdot) |_{a \times a} \) we denote \( V^B(a) \) by \( V^t(a) \). By (5.7)

\[
V^{\alpha_k}(g_0) = \left( \bigotimes_{i \geq 1} V^{z_i(k) + \chi_i(g^\circ)} \right) \otimes V^{k + h^\vee}(\mathbb{C} x).
\]

Since \( \phi \) from Proposition 3.2 corresponds to a compact real form of \( g^\circ \), if \( z_i(k) + \chi_i \in \mathbb{Z}_+ \), the \( \phi \)-invariant Hermitian form on \( V^{z_i(k) + \chi_i(g^\circ)} \) is positive semidefinite.

Recall from [11] §5.2] that if \( a \) is an even abelian Lie algebra with a symmetric bilinear form \( B \) and \( \eta \) is a conjugate linear involution such that \( B(\eta(a), \eta(b)) = B(a, b), a, b \in a \), then letting \( \eta \) be the unique extension of \( \eta \) to \( V^B(a) \), the unique \( \eta \)-invariant Hermitian form on \( V^B(a) \) is positive definite if and only if \( B |_{a \times a_R} \) is positive definite, where \( a_R = \{ a \in a | \eta(a) = -a \} \).

Applying this remark to \( a = C x, \eta = \phi \) and \( B = k + h^\vee(\cdot, \cdot) \), since \( \phi(x) = x \) and \( (x|x) = \frac{1}{2}(\phi(\theta) = \frac{1}{2} \), the \( \phi \)-invariant Hermitian form on \( V^{k + h^\vee}(\mathbb{C} x) \) is positive definite if and only if

\[
(k + h^\vee)(x|x) < 0 \iff k + h^\vee < 0.
\]

Finally we check the unitarity of \( F(A_{\beta}) = F(g_{1/2}) \). We have to check that, according to [11 §5.1], the restriction of \( \langle \cdot, \cdot \rangle \) to \( \{ u \in g_{-1/2} | \phi(u) = -u \} \) is positive definite. This is Proposition 5.4.
It follows that, if \( z_i(k) + \chi_i \in \mathbb{Z}_+ \) and \( k + h^\vee < 0 \), then \( V^{\alpha k}(g_0) \otimes F(A_m) \) is unitary, hence \( W_k^{\text{min}}(g) \) is. By looking at Table 2 one sees that \( \chi_i \in -\mathbb{Z}_+ \) and checks that \( z_i(k) \geq 0 \) implies \( k + h^\vee < 0 \). Thus, if \( z_i(k) \geq -\chi_i \), then \( W_k^{\text{min}}(g) \) is unitary.

Now we look at the missing cases \( 0 \leq z_i(k) < -\chi_i \). Assume first \( g^\dagger \) simple. If \( \chi_1 = -1 \) then the only value not covered by the above condition is \( z_1(k) = 0 \), so, by Theorem 6.2 \( W_k^{\text{min}}(g) = \mathbb{C} \). In the case of \( \text{spo}(2|3) \) one should also consider the cases \( z_1(k) = 1, z_1(k) = 0 \):

by Theorem 6.2 in the former case \( k \) is collapsing with \( W_k^{\text{min}}(\text{spo}(2|3)) = V_1(sl_2) \), whereas in the latter case \( k = -h^\vee \). If \( g^\dagger \) is semisimple but not simple, then \( g \) is either \( \text{osp}(4|m) \) or \( D(2,1;a) \). For \( g = D(2,1;a) \) we have to consider only the case in which either \( z_1(k) \) or \( z_2(k) \) is zero. By Theorem 6.2 they correspond to collapsing levels. For \( g = \text{osp}(4|m) \), our condition reduces to

\[
m/2 \leq k < m, \quad k \leq -2, \quad \text{or} \quad m/2 \leq k, \quad -4 < k \leq -2
\]

which is never satisfied.

We now prove that our condition is also necessary, if \( g^\dagger \) is semisimple. Assume that there is a conjugate linear involution \( \psi \) of \( W_k^{\text{min}}(g) \) such that the \( \psi \)-invariant Hermitian form is positive definite. Recall that the map \( a \mapsto J^{(a)} \) extends to an embedding \( V^{\beta(k)}(g^\dagger) \subset W_k^{\text{min}}(g) \). In particular the image of \( V^{\beta(k)}(g^\dagger) \) in \( W_k^{\text{min}}(g) \) is unitary and simple.

By (6.6),

\[
V^{\beta(k)}(g^\dagger) = \bigotimes_{i \geq 1} V^{z_i(k)}(g_i^\dagger),
\]

hence we have an embedding

\[
\bigotimes_{z_i(k) \neq 0} V^{z_i(k)}(g_i^\dagger) \subset W_k^{\text{min}}(g).
\]

By §5.3 of [11], we see that \( \psi \) restricted to \( \bigoplus_{z_i(k) \neq 0} g_i^\dagger \) defines a compact real form and, if \( z_i(k) \neq 0 \), then \( z_i(k) \in \mathbb{N} \). Therefore \( z_i(k) \in \mathbb{Z}_+ \) for all \( i \).

We now discuss the case when \( g \) is not semisimple. Note that \( g_0^\dagger \neq \{0\} \) only when \( g = sl(2|m) \). In this case \( g_0^\dagger = \mathbb{C} \varpi \), where \( \varpi = \left( \begin{array}{cc} m/2 & 0 & 0 \\ 0 & -m/2 & 0 \\ 0 & 0 & I_m \end{array} \right) \), and \( (a|b) = str(ab) \).

By Theorem 6.2, the collapsing levels are \( k = -1 \) and \( k = -m/2 - 1 \). If \( k = -1 \) then \( z_0(-1) = -m/2, z_1(-1) = 0 \) and \( W_k^{\text{min}}(g) \) is the Heisenberg vertex algebra \( M(\mathbb{C} \varpi) = V^{-m/2}(\mathbb{C} \varpi) = V_{-m/2}(\mathbb{C} \varpi) \) and this vertex algebra is unitary. If \( k = -m/2 - 1 \) then \( z_0(m/2 - 1) = 0, z_1(m/2 - 1) = -m/2 \) and \( W_k^{\text{min}}(sl(2|m)) = V_{-m/2}(sl(m)) \) which is not unitary. Assume now that \( k \) is not collapsing. Let \( \psi \) be a conjugate linear involution of \( W_k^{\text{min}}(sl(2|m)) \) such that the \( \psi \)-invariant Hermitian form \( H \) is positive definite. In particular there is an embedding

\[
V^{k-m/2-1}(\mathbb{C} \varpi) \otimes V_{-k-1}(sl(m)) \hookrightarrow W_k^{\text{min}}(sl(2|m))
\]

Hence \( \psi \) induces by restriction a conjugate linear involution of \( \mathbb{C} \varpi \oplus sl(m) \), thus \( \psi(\varpi) = \zeta \varpi, |\zeta| = 1 \). Moreover, \( \psi|_{sl(m)} \) corresponds to a compact real form of \( sl(m) \) and \( -k-1 \in \mathbb{Z}_+ \). Let \( (b)_{ev} \) denote the expectation value of \( b \), i.e. the coefficient of the projection of \( b \) on the vacuum vector \( 1 \). Using the formulas given in [11] §5.3 we have

\[
0 < H(J^{(\varpi)}, J^{(\varpi)}) = \langle -J_1^{(\psi(\varpi))} J_{-1}^{(\varpi)} \rangle_{ev} = -(k-m/2-1)\zeta(\varpi|\varpi) = -\zeta(k-m/2-1)(m^2/2-m).
\]
Therefore \( \zeta = 1 \), so that

\[
\psi(\varpi) = \varpi.
\]  

Observe now that by [11], since \( k \) is not collapsing, the image of \( G^{(u)} \) in \( W^\text{min}_k(g) \) is non-zero if \( u \neq 0 \). Note that, from the relation \( [J^{(a)}, G^{(u)}] = J^{([a,u])} \), \( a \in g_0, u \in g_{-1/2} \), it follows that \( \psi \) can be extended to \( g_{-1/2} \), in such a way that \( \psi([a,u]) = [\psi(a), \psi(u)] \).

Note that

\[
[\varpi,u] = -\frac{m}{2} u, \quad u \in g_{-1/2}.
\]

Take \( u \in g_{-1/2} \). Compute, using (6.10) and (6.9)

\[
\langle \psi(u)|u\rangle = -\frac{m}{2} \langle \psi(u)||\varpi,u\rangle = \frac{2}{m} \langle \varpi,\psi(u)||u\rangle
\]

\[
= \frac{2}{m} \langle \psi(\varpi) , \psi(u)||u\rangle = \frac{2}{m} \langle \psi(\varpi) , u||u\rangle = -\langle \psi(u)|u\rangle,
\]

so \( \langle \psi(u)|u\rangle = 0 \). But this contradicts positivity of \( H \), since, by [11] (7.6) and (6.3)

\[
H(G^{(u)},G^{(u)}) = H(G^{-3/2}_1,G^{-3/2}_1) = \langle G^{(u)}_3 G^{(u)}_3 \rangle_{ev} = \langle G^{(u)}(2) G^{(u)}_1 \rangle_{ev} = 4p(k) \langle \psi(u)|u\rangle.
\]

For (b) observe that, if there is a unitary module for \( W^\text{min}_k(g) \) then also \( V_{z_1(k)}(g^\natural) \) admits a unitary module, but then \( z_1(k) \in \mathbb{Z}_{\geq 0} \).

**Theorem 6.5.** Let \( g \) be a basic simple Lie superalgebra from Table 1. Let \( V = W^\text{min}_k(g) \) with \( k \neq h^\vee \). Assume that \( g \neq spo(2|m) \), \( m = 0, 1, 2 \), which correspond to the well-understood cases of Virasoro, Neveu-Schwarz and \( N = 2 \) superconformal algebra. Then \( V \) is a non-trivial unitary vertex algebra if and only if

1. \( g = sl(2|m), m \geq 3, k = 1 \);
2. \( g = psl(2|2), k \in \mathbb{N} + 1 \);
3. \( g = spo(2|3), k \in \frac{1}{4}(\mathbb{N} + 2) \);
4. \( g = spo(2|2m), m > 3, k = \frac{1}{2}(\mathbb{N} + 1) \);
5. \( g = D(2,1; -\frac{m}{m+n}), k = \frac{m}{m+n}, m, n \in \mathbb{N}, m + n > 2 \);
6. \( g = F(4), k \in \frac{2}{3}(\mathbb{N} + 1) \);
7. \( g = G(3), k \in \frac{2}{3}(\mathbb{N} + 1) \).

**Proof.** The result is obtained by a direct application of Theorem 6.4 using the data displayed in Table 2; one should take care, using Theorem 6.2 to exclude (collapsing) levels for which \( V \) is trivial. Combining the conditions coming from Theorems 6.3 and 6.2 we obtain that \( z_1(-k) \) has to be a positive integer if \( g^\natural \) is simple: this covers cases (2), (3), (4), (6), (7). For the remaining cases of Table 1, \( sl(2|m) \) is handled directly by Theorem 6.4 (2). If \( g = osp(4|m) \), then the conditions \( z_1(-k) \in \mathbb{Z}_+, z_2(-k) \in \mathbb{Z}_+ \) are not compatible, hence in this case \( V \) is never unitary. It remains to deal with \( g = D(2,1; a) \). Then by Theorem 6.4 we have \( -\frac{1}{a} k = N + 1 \), hence

\[
k = -a(N + 1), \quad N \in \mathbb{Z}_+.
\]

and \( \frac{1}{1+a} k - 1 = M \in \mathbb{Z}_+ \), hence, substituting (1.14), we have

\[
-\frac{a}{1+a}(N + 1) - 1 = M,
\]
hence \( a(M + N + 2) + M + 1 = 0 \), and

\[
a = -\frac{M + 1}{M + N + 2} = -\frac{m}{m + n},
\]

where \( m = M + 1, \ n = N + 1 \in \mathbb{N} \)

Substituting in (6.11), we get

\[
k = \frac{mn}{m + n}.
\]

The case \( m = n = 1 \) is excluded since in this case \( c(k) = 0 \), hence \( W_k^{\min}(g) \) is trivial, since it is unitary. \( \square \)

**Remark 6.6.** Replacing \( N \) by 0 in cases (2)-(4), (6) and (7) of Theorem 6.5 we obtain \( k \), for which \( W_{-k}^{\min}(g) = \mathbb{C} \). In case (5) this happens for \( m = n = 1 \).

**Remark 6.7.** It has been proved in [11] that if \( g \) does not appear in Table 1, then \( W_k^{\min}(g) \) can be unitary only if \( k \) is collapsing. Then \( W_k^{\min}(g) \) is non trivial unitary precisely in the following cases.

1. \( W_{-1}^{\min}(sl(m|n)) \cong M(\mathbb{C}), \ m \neq n, n + 1, n + 2, m \geq 2 \), where \( M(\mathbb{C}) \) is the Heisenberg vertex algebra with central charge \( c = 1 \);
2. \( W_{4/3}^{\min}(G_2) \cong V_1(sl(2)) \) with central charge \( c = 1 \);
3. \( W_{-2}^{\min}(osp(m|n)) \cong V_{m-n-8}(sl(2)), m - n \geq 10, m \) and \( n \) even, with central charge

\[
c = \frac{3(m-n-8)}{m-n-4}.
\]

**Remark 6.8.** Unitary representations for the \( N = 3 \) and \( N = 4 \) superconformal algebras have been studied in [12], and [2], [3], [4], respectively. The same central charges as in cases (2) and (3) of Theorem 6.5 appear in [4, (6)], [12, (2.3.1)], respectively.

**Remark 6.9.** It follows easily from [9, §6, (2.6)] for any \( W_k(g, x, f) \) that, if \( dh^\vee \neq 0 \), there are precisely two values \( k_1, k_2 \) of \( k \) which afford the same central charge \( c(k) \), and they are related by

\[
(k_1 + h^\vee)(k_2 + h^\vee) = \frac{dh^\vee}{12(x|x)}.
\]

This follow from (1.3) by writing \( c(k_1) - c(k_2) = 0 \). A direct check shows that, except for \( g = spo(2|m) \) with \( m = 0 \) or 1, if \( k_1 \) is in the non-trivial unitary range, then \( k_2 \) is not. In the case of \( spo(2|m) \) this can be proved using that the right hand side of (6.12) equals

\[
\frac{(2 - m)(3 - m)(4 - m)}{24}.
\]

**References**

[1] D. Adamović, V. G. Kac, P. Möseneder Frajria, P. Papi, O. Perše, **Conformal embeddings of affine vertex algebras in minimal W-algebras I: Structural results**, J. Algebra, 500 (2018), 117–152.
[2] T. Eguchi, A. Taormina, **Unitary representations of the N = 4 superconformal algebra**. Phys. Lett. B 196 (1987), no. 1, 75–81.
[3] T. Eguchi, A. Taormina, **Character formulas for the N=4 superconformal algebra**. Phys. Lett. B 200 (1988), no. 3, 315–322.
[4] T. Eguchi, A. Taormina, **On the unitary representations of N=2 and N=4 superconformal algebras**. Phys. Lett. B 210 (1988), no. 1-2, 125–132.
[5] M. Gorelik, V. G. Kac, P. Möseneder Frajria, P. Papi, **Denominator identities for finite-dimensional Lie superalgebras and Howe duality for compact dual pairs**, Japan. J. Math., 7 (2012), 41–134
[6] K. Iohara, Y. Koga, **Central extension of Lie superalgebras**, Comment. Math. Helv. 76 (2001), 110–154.
[7] V. G. Kac, **Lie superalgebras**, Advances in Math. 26 (1977), no. 1, 8–96.
[8] V. G. Kac, S. Roan, M. Wakimoto, *Quantum reduction for affine superalgebras*. Comm. Math. Phys. 241 (2003), no. 2-3, 307–342.

[9] V. G. Kac, M. Wakimoto, *Quantum reduction and representation theory of superconformal algebras*, Adv. in Math. 185 (2004), 400–458.

[10] V. G. Kac, Möseneder Frajria, P. Papi, *Yangians versus minimal W-algebras: A surprising coincidence*, Comm. Contemp. Math., (2020), DOI: 10.1142/S0219199720500364

[11] V. G. Kac, Möseneder Frajria, P. Papi, *Invariant Hermitian forms on vertex algebras*, arXiv:2008.13178

[12] K. Miki, *The representation theory of the SO(3) invariant superconformal algebra*. Internat. J. Modern Phys. A 5 (1990), no. 7, 1293–1318.

[13] V. Serganova, *Automorphisms of simple Lie superalgebras*, Math. USSR Izvestiya Vol. 24, (1985), no. 3, 539–551.

V.K.: Department of Mathematics, MIT, 77 Mass. Ave, Cambridge, MA 02139; kac@math.mit.edu

P.MF.: Politecnico di Milano, Polo regionale di Como, Via Anzani 4, 22100, Como, Italy; pierluigi.moseneder@polimi.it

P.P.: Dipartimento di Matematica, Sapienza Università di Roma, P.le A. Moro 2, 00185, Roma, Italy; papi@mat.uniroma1.it