KAM FOR HAMILTONIAN PARTIAL DIFFERENTIAL EQUATIONS WITH WEAKER SPECTRAL ASYMPTOTICS

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ABSTRACT. In this paper, we establish an abstract infinite dimensional KAM theorem dealing with normal frequencies in weaker spectral asymptotics

\[ \Omega_i(\xi) = i^d + o(i^d) + o(i^\delta), \]

where \( d > 0, \delta < 0 \), which can be applied to a large class of Hamiltonian partial differential equations in high dimensions. As a consequence, it is proved that there exist many invariant tori and thus quasi-periodic solutions for Schrödinger equations, the Klein-Gordon equations with exponential non-linearity and other equations of any spatial dimension.

1. INTRODUCTION

The KAM theory in infinite dimension requires the spectral asymptotics assumption that exhibits the asymptotic separation of the spectrum. However, H. Weyl’s theory shows that the spectrum of the eigenvalue problem

\[ \triangle \phi = -\lambda \phi, \text{ on } \Omega \]

and

\[ \phi|_{\partial \Omega} = 0 \]

has the following asymptotic behavior as \( k \to \infty \),

\[ \lambda_k \sim C_m \left( \frac{k}{V} \right)^{\frac{2}{m}}, \tag{1.1} \]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^m \), \( V \) denotes the volume of \( \Omega \), \( C_m = (2\pi)^2 B_m^{-\frac{m}{2}} \) is the Weyl constant with \( B_m = \text{volume of the unit } m\text{-ball} \).

When the dimension \( m > 2 \), it is easy to see that \( 0 < \frac{2}{m} < 1 \) from (1.1). As a result, the spectral separation disappears. Thus, a natural question is how to deal with this kind of weaker spectral asymptotics when we consider the existence of lower dimensional KAM tori which is shown for a class of nearly integrable Hamiltonian systems of infinite dimension. This is correspondent how to prove that there exist many invariant tori and thus quasi-periodic solutions for a class of Hamiltonian partial differential equations in higher dimension with weaker spectral asymptotics

\[ \Omega_i(\xi) = i^d + o(i^d) + o(i^\delta), \quad d > 0. \]

Date: June 2011.
2010 Mathematics Subject Classification. 37K55;70H08;70K43.
Key words and phrases. KAM theory; infinite-dimensional Hamiltonian systems; weaker spectral asymptotics.

This work supported by NSFC Grant No 10531050, National 973 Project of China No 2006CD805903, SRFDP Grant No 20090061110001, and the 985 Project of Jilin University.
The KAM (Kolmogorov-Arnold-Moser) theory is a very powerful tool to find periodic or quasi-periodic solution for higher dimensional Hamiltonian PDEs. There have been many remarkable results. For reader’s convenience, we refer to Kuksin [16, 17], Craig and Wayne [10, 22], Bourgain [4, 5, 6], and Pöschel [20, 21]. The first breakthrough result is made by Bourgain [6] who proved that the two dimensional nonlinear Schrödinger equations admit small-amplitude quasi-periodic solutions. Later he improved in [7] his method and proved that the higher dimensional nonlinear Schrödinger and wave equations admit small-amplitude quasi-periodic solutions.

Constructing quasi-periodic solutions of higher dimensional Hamiltonian PDEs by method from the finite dimensional KAM theory appeared later. The breakthrough of constructing quasi-periodic solutions for more interesting higher dimensional Schrödinger equation by modified KAM method was made recently by Eliasson-Kuksin. They proved in [11] that the higher dimensional nonlinear Schrödinger equations admit small-amplitude linearly-stable quasi-periodic solutions. Recently, Geng et al. [13] extended the Bourgain’s existence result [6, 8] to arbitrary finite dimensional invariant tori. They also got a nice linear normal form, which can be used to study the linear stability of the obtained solutions.

However, the corresponding KAM type theorems on higher dimensional Hamiltonian PDEs that need infinite spectrum

\[ \Omega(\xi) = (\Omega_1(\xi), \cdots, \Omega_i(\xi), \cdots)_{i \in \mathbb{Z}_+} \]

is generally assumed to be separated as \( i \to \infty \), namely, there exits \( d \geq 1, \delta < 0 \) such that

\[ \Omega_i(\xi) = i^d + o(i^d) + o(i^\delta), \xi \in \Pi \) (a bounded region in \( \mathbb{R}^n \)). \]

Usually, at each KAM step, one must remove the frequencies not satisfying Diophantine conditions. The spectral asymptotics guarantee that there exists a Cantor set \( \Pi_\gamma \subset \Pi \), for all \( \xi \in \Pi_\gamma \), the tori with frequency \( \omega(\xi) \) survive and the measure \( |\Pi \setminus \Pi_\gamma| \to 0 \) as \( \gamma \to 0 \).

Now what happens to the persistence for weaker spectral asymptotics, due to the spectral in higher dimensional PDEs might destroy the separation by Weyl’s asymptotics formula. The motivation leads to whether one can relax the spectral asymptotics condition to the case \( d > 0 \) or not. In the present paper, just at this weaker spectral asymptotics, we prove the Melnikov persistence for infinite-dimensional Hamiltonian systems.

In the usual KAM formulism to PDEs, one can make the perturbation \( P \) getting smaller by only truncating the order of the angle variable \( x \). However, to deal with the weaker spectral asymptotics, we not only truncate the angle variable \( x \) but also truncate the normal variables \( z, \bar{z} \). This results in that we are forced to modify the usual Diophantine condition. According to this new difficulty, we introduce a KAM iteration that is similar to Nash-Moser iteration and somewhat different from previous ones.

We consider the Hamiltonian system

\[ H = \langle \omega(\xi), y \rangle + \sum_{j \geq 1} \Omega_j(\xi)z_j \bar{z}_j + P(x, y, z, \bar{z}; \xi), \]

where \( (x, y, z, \bar{z}) \) lies in the complex neighborhood

\[ D_{a,p}(s,r) = \{(x, y, z, \bar{z}) : |\text{Im } x| < r, |y| < s^2, |z|^{a-p}, |\bar{z}|^{a-p} < s \} \]
of $T^n \times \{0\} \times \{0\} \times \{0\}$, $T^n$ is an $n$-torus, and $a \geq 0$, $p \geq 0$, with the norm $|z|^{a,p} = \sqrt{\sum_{i \geq 1} |z_i|^{2i^2p+2ai}}$. The frequencies $\omega$ and $\Omega$ depend on $\xi \in \Pi \subset \mathbb{R}^n$, $\Pi$ is a closed bounded region.

With respect to the symplectic form

$$\sum_{i=1}^n dx_i \wedge dy_i + \sum_{j=1}^n dz_j \wedge d\bar{z}_j,$$

the associated unperturbed motion is simply described by

$$\begin{cases}
\dot{x} = \omega(\xi), \\
\dot{y} = 0, \\
\dot{z} = \Omega(\xi)\bar{z}, \\
\dot{\bar{z}} = -\Omega(\xi)z.
\end{cases}$$

Assume that:

**A1) Nondegeneracy.** The map $\xi \mapsto \omega(\xi)$ is a homeomorphism and Lipschitz continuous in both directions. Moreover, for all integer vectors $(k, l) \in \mathbb{Z}^n \times \mathbb{Z}^\infty$ with $1 \leq |l| \leq 2$,

$$\text{meas}\{\xi : \langle k, \omega(\xi) \rangle + \langle l, \Omega(\xi) \rangle = 0\} = 0$$

and

$$\langle l, \Omega(\xi) \rangle \neq 0, \ \forall \xi \in \Pi,$$

where $|l| = \sum_{j \geq 1} |l_j|$ for integer vectors.

**A2) Spectral Asymptotics.** The components $\Omega_i \neq \Omega_j$ if $i \neq j$, and there exist $d > 0$ and $\delta < 0$ such that

$$\Omega_j(\xi) = j^d + o(j^d) + o(j^{\delta}), \ j \to \infty,$$

where the dots stand for lower order terms in $j$, allowing also negative exponents.

Moreover,

$$|\omega|_\Pi^\mathcal{C} + |\Omega|_{-\delta,\Pi}^\mathcal{C} \leq M < \infty,$$

where

$$|\omega|_\Pi^\mathcal{C} = \sup_{\xi \neq \varepsilon} \frac{|\omega(\xi) - \omega(\varepsilon)|}{|\xi - \varepsilon|},$$

$$|\Omega|_{-\delta,\Pi}^\mathcal{C} = \sup_{\xi \neq \varepsilon} \sup_{i \geq 1} \frac{|\Omega_i(\xi) - \Omega_i(\varepsilon)|i^{-\delta}}{|\xi - \varepsilon|},$$

for all $\xi \in \Pi$.

**A3) Regularity.** The perturbation $P \in \mathcal{F}_{a,p}^{\bar{a},\bar{p}}$, that is, $P$ is real analytic on the space coordinate and Lipschitz on the parameters, and for each $\xi$, Hamiltonian vector space field $X_P = (P_y, -P_x, P_\bar{z}, -P_z)^T$ defines a real analytic map

$$X_P : \mathbb{P}^{a,p} \to \mathbb{P}^{\bar{a},\bar{p}},$$

where $\bar{a} > a > 0$, $\bar{p} \geq p \geq 0$, and $\mathbb{P}^{a,p} = T^n \times \mathbb{R}^n \times \mathcal{L}^{a,p} \times \mathcal{L}^{a,p}$.
Suppose that $H = N + P$ satisfies A1) – A3). Then we have

**Theorem A.** For given $r, s, a > 0$, if there exits a sufficiently small $\mu = \mu(a, p, \bar{p}, r, s, n, \tau) > 0$ (or equivalently $\mu_* = \mu_*(a, p, \bar{p}, r, s, n, \tau) > 0$) such that the perturbation $P$ satisfies

$$|X_P|_{a, \bar{p}} \leq \gamma \mu,$$

$$|X_P|_{0} \leq M \mu,$$

for all $(x, y, z, \bar{z}) \in D_{a, \bar{p}}(r, s)$, $\xi \in \Pi$. Then there exits a Cantor set $\Pi_\gamma \subset \Pi$ with $|\Pi \setminus \Pi_\gamma| \to 0$, as $\gamma \to 0$, and a family of $C^2$ symplectic transformations

$$\Psi_\xi : D_{a + \frac{\gamma}{M} \bar{p} \frac{r}{2} \frac{s}{2}} \to D_{a, \bar{p}}(r, s), \quad \xi \in \Pi_\gamma,$$

which are Lipschitz continuous in parameter $\xi$ and $C^2$ uniformly close to the identity, such that for each $\xi \in \Pi_\gamma$, corresponding to the unperturbed torus $T_\xi$, the associated perturbed invariant torus can be described as

$$\begin{align*}
\dot{x} &= \omega_\xi \xi, \\
\dot{y} &= 0, \\
\dot{z} &= \Omega^* \xi \bar{z}, \\
\dot{\bar{z}} &= -\Omega^* \xi \bar{z},
\end{align*}$$

where

$$|\omega_\xi - \omega| + \frac{\gamma}{M} |\omega_\xi - \omega| \leq c\mu,$$

$$|\Omega^* - \Omega| + \frac{\gamma}{M} |\Omega^* - \Omega| \leq c\mu.$$ 

The perturbation $P^* = P \circ \Phi_\xi$ is real analytic on variables $x$, $C^2$ on variables $(y, z, \bar{z})$ Lipschitz continuous on the parameters, and

$$X_{P^*}|_{(y, z, \bar{z}) = (0)} = 0,$$

for all $x \in \mathbb{T}^n$, $\xi \in \Pi_\gamma$. Namely, the unperturbed torus $T_\xi = \mathbb{T}^n \times \{0\} \times \{0\} \times \{0\}$ associated to the frequency $\omega(\xi)$ and $z\bar{z}$-space frequency $\Omega(\xi)$ persists and gives rise to an analytic, Diophantine, invariant torus of the perturbed system with the frequency $\omega_\xi(\xi)$ and $z\bar{z}$-space frequency $\Omega^*(\xi)$. Moreover, these perturbed tori form a Lipschitz continuous family.

**Remark I** When the infinite spectrum $\Omega$ is multiple, we first give some notations.

For given $\rho \in N_+$, and $\{i_1, \ldots, i_n\} \subset \mathbb{Z}^\rho$, denote $Z_1^\rho = \mathbb{Z}^\rho \setminus \{i_1, \ldots, i_n\}$, and without loss of generality, set $0 \in \{i_1, \ldots, i_n\}$.

Throughout this paper, we set

$$\mathcal{L}^{a, p} = \{z : z = (\cdots, z_{i_1}, \cdots)_{i \in Z_1^\rho}\}$$

with the norm

$$|z|^{a, p} = \sqrt[2p+2^a]{\sum_{i \in Z_1^\rho} |z_i|^2 |i|^{2p+2^a}},$$

where $i = (i_1, \cdots, i_\rho)$, $|i| = |i_1| + \cdots + |i_\rho|$, for fixed $a \geq 0, p \geq 0$. For $\rho = 1$, we set $Z_1^1 = \{1, 2, \cdots\}$ and the corresponding norm is $|z|^{a, p} = \sqrt[2p]{\sum_{i \geq 1} |z_i|^2 |i|^{2p+2^a}}$. 


Consider the function family \( F^a_{a,p} \). Let \( F : \mathcal{P}^{a,p} \to \mathbb{R} \), where \( \mathcal{P}^{a,p} = \{(x, y, z, \bar{z}) \in \mathbb{T}^n \times \mathbb{R}^n \times L^{a,p} \times L^{a,p}\} \). We say \( F \in F^a_{a,p} \) if \( F \) has the following properties:

1) \( F \) is a real analytic function in \((x, y, z, \bar{z})\) and Lipschitz continuous in parameter \( \xi \);

2) For fixed \( \xi \), \( F \) can be expanded as:

\[
F(x, y, z, \bar{z}, \xi) = \sum_{k \in \mathbb{Z}^n, m \in \mathbb{N} : a, q} F_{kmqq}(\xi) y^m z^q \bar{z}^q e^{\sqrt{-1}(k,x)},
\]

where \( q = (\cdots, q_i, \cdots) \in \mathbb{Z}_1^r, \bar{q} = (\cdots, \bar{q}_i, \cdots) \in \mathbb{Z}_1^r \) have finitely many non-zero positive components; for the case \( \rho = 1 \), we set \( q = (q_1, q_2, \cdots) \). The term \( z^q \bar{z}^q \) denotes \( \prod_i z_i^{q_i} \bar{z}_i^{q_i} \);

3) \( X_F \) is finite in the following weighted norm:

\[
|X_F|_{s,D_{a,p}(r,s) \times \Pi} = |F_y|_{D_{a,p}(r,s) \times \Pi} + \frac{1}{s^2} |F_x|_{D_{a,p}(r,s) \times \Pi} + \frac{1}{s} |F_z|_{D_{a,p}(r,s) \times \Pi} + \frac{1}{s} |F_{\overline{z}}|_{D_{a,p}(r,s) \times \Pi} < \infty,
\]

where \( a, \overline{p} \) are fixed positive numbers;

4) The Lipschitz semi-norm

\[
|X_F|_{C_{a,p}} = \sup_{\xi \neq \varepsilon} \frac{|\Delta_{\xi \varepsilon} X_F|_{s,D_{a,p}(r,s)}}{|\xi - \varepsilon|} < \infty,
\]

where \( \Delta_{\xi \varepsilon} = X_F(\cdot, \xi) - X_F(\varepsilon, \cdot) \), and the superior is taken over \( \Pi \).

Consider the following infinite dimensional Hamiltonian system with small perturbation

\[
(1.5) \quad H = \langle \omega(\xi), y \rangle + \sum_{i \in \mathbb{Z}_1^n} \Omega_i(\xi) z_i \bar{z}_i + P(x, y, z, \bar{z}; \xi),
\]

where \((x, y, z, \bar{z})\) lies in the complex neighborhood

\[
D_{a,p}(s, r) = \{(x, y, z, \bar{z}) : |\text{Im} x| < r, |y| < s^2, |z|^{a,p}, |\bar{z}|^{a,p} < s\}
\]

of \( \mathbb{T}^n \times \{0\} \times \{0\} \times \{0\} \) with the norm \( \|F\|_{1.5} \). The frequencies \( \omega \) and \( \Omega \) depend on \( \xi \in \Pi \subset \mathbb{R}^n \), \( \Pi \) is a closed bounded region.

With respect to the symplectic form

\[
\sum_{i=1}^n dx_i \wedge dy_i + \sum_{i \in \mathbb{Z}_1^n} dz_i \wedge d\bar{z}_i,
\]

the associated unperturbed motion of \( (1.5) \) is simply described by

\[
\begin{align*}
\dot{x} &= \omega(\xi), \\
\dot{y} &= 0, \\
\dot{z}_i &= -\Omega_i(\xi) \bar{z}_i, \\
\dot{\bar{z}}_i &= \Omega_i(\xi) z_i.
\end{align*}
\]

Hence, for each \( \xi \in \Pi \), there is a family of invariant \( n \)-dimensional torus \( \mathcal{T}_\xi = \mathbb{T}^n \times \{0\} \times \{0\} \times \{0\} \) with fixed frequency \( \omega(\xi) \).

We also make the following assumptions for \( (1.5) \).

**A1’ Nondegeneracy.** The map \( \xi \to \omega(\xi) \) is a homomorphism between \( \Pi \) and its image, and Lipschitz continuous in both directions.
\textbf{A2')} Spectral Asymptotics. There exist \( d > 0 \) and \( \delta < 0 \) such that for all \( i \in \mathbb{Z}^n_+ \),
\begin{align}
\Omega_i(\xi) & \neq 0, \ i \in \mathbb{Z}^n_+, \ \xi \in \Pi, \\
\Omega_i(\xi) & = \tilde{\Omega}_i + \bar{\Omega}_i, \quad \tilde{\Omega}_i = |i|^d + o(|i|^d), \quad \bar{\Omega}_i = o(|i|^\delta), \\
\end{align}
and
\begin{equation}
\tilde{\Omega}_i - \bar{\Omega}_j = |i|^d - |j|^d + o(|j|^{-\delta}), \ |j| \leq |i|.
\end{equation}
Moreover, \( \omega(\xi), \ \Omega(\xi) \) are Lipschitz continuous in \( \xi \), and
\begin{equation}
|\omega|_\Pi^C + |\Omega|_{\delta, \Pi}^C \leq M < \infty,
\end{equation}
where
\begin{align}
|\omega|_\Pi^C & = \sup_{\xi \neq \bar{\varepsilon}} \frac{|\omega(\xi) - \omega(\bar{\varepsilon})|}{|\xi - \bar{\varepsilon}|}, \\
|\Omega|_{\delta, \Pi}^C & = \sup_{\xi \neq \bar{\varepsilon}} \sup_{|i| \geq 1} \frac{|\Omega_i(\xi) - \Omega_i(\bar{\varepsilon})||i|^{-\delta}}{|\xi - \bar{\varepsilon}|}.
\end{align}

\textbf{A3')} Regularity. The perturbation \( P \in \mathcal{F}_{a, p}^{a, \bar{p}}, \) that is, \( P \) is real analytic in the space coordinate and Lipschitz in the parameters, and for each \( \xi \), Hamiltonian vector space field \( X_P = (P_y, -P_z, P_x, -P_\bar{z})^T \) defines a real analytic map
\[ X_P : \mathcal{P}^{a, p} \to \mathcal{P}^{a, \bar{p}}, \]
where \( \bar{a} > a > 0, \ \bar{p} \geq p \geq 0. \)

\textbf{A4')} Special form of the perturbation. The perturbation \( P \in \mathcal{A} \) where
\[ \mathcal{A} = \{ P \in \mathcal{F}_{a, p}^{a, \bar{p}} : \ P_{kmqq} = 0 \text{ if } \sum_{j=1}^n k_j t_j + \sum_{i \in \mathbb{Z}^n_+} (q_i - \bar{q}_i)i \neq 0 \}. \]

**Theorem A'.** Suppose that \( (1.3) \) satisfies A1)' - A4)'. For given \( r, s > 0, \) if there exists a sufficiently small \( \mu = \mu(a, p, \bar{p}, r, s, n, \tau) > 0, \) or equivalently \( \mu_* = \mu_* (a, p, \bar{p}, r, s, n, \tau) > 0, \) such that
\begin{align}
|X_P|^{a, \bar{p}} & \leq \gamma \mu, \\
|X_P|^C_{a, p} & \leq M \mu,
\end{align}
for all \( (x, y, z, \bar{z}) \in D_{a, p}(r, s), \ \xi \in \Pi. \) Then there exist a Cantor set \( \Pi_\gamma \subset \Pi \) with \( |\Pi \setminus \Pi_\gamma| \to 0, \) as \( \gamma \to 0, \) and a family of \( C^2 \) symplectic transformations
\[ \Psi_\xi : D_{a + \frac{-a-n}{2}, p + \frac{r}{2}} \to D_{a, p}(r, s), \ \xi \in \Pi_\gamma, \]
which are Lipschitz continuous in parameter \( \xi \) and \( C^2 \) uniformly close to the identity, such that for each \( \xi \in \Pi_\gamma, \) corresponding to the unperturbed torus \( T_{\xi}, \) the associated perturbed invariant torus can be described as
\[ \begin{align*}
\dot{x} & = \omega_*(\xi), \\
\dot{y} & = 0, \\
\dot{z}_i & = -\Omega_i^*(\xi)\bar{z}_i, \\
\dot{\bar{z}}_i & = \Omega_i^*(\xi)z_i,
\end{align*} \]
where
\[ |\omega_* - \omega| + \frac{\gamma}{M}|\omega_* - \omega|^C \leq \epsilon \mu_*. \]
For details, \( (1.12) \)

Remark II

not needed as in Theorem A. \( A_k \) make the measure estimates simpler, and for the non-multiple case, \( A_k \) containing explicitly the space and the time variables do have the special form. In [12]. The Hamiltonian systems deriving from the PDEs in high dimension not

Remark III

In the following, one can find that we only need a decreasing sequence \( \{a\} \) such that

\[
a < a + \frac{\bar{a} - a}{2} \leq a_{\nu+1} < a_{\nu} < \bar{a},
\]

Moreover the perturbation \( P^* = P \circ \Phi_\xi \) is real analytic in phase variables \( x \), \( C^2 \) in variables \( (y, z, \bar{z}) \), Lipshcitz continuous in the parameters, and

\[ X|_{(y, z, \bar{z}) = (0)} = 0, \]

for all \( x \in \mathbb{T}^n \), \( \xi \in \Pi_\gamma \). Namely, the unperturbed torus \( T_\xi = T^n \times \{0\} \times \{0\} \times \{0\} \) associated to the frequency \( \omega(\xi) \) persists and gives rise to an analytic, Diophantine, invariant torus of the perturbed system with the frequency \( \omega(\xi) \). Moreover, these perturbed tori form a Lipshcitz continuous family.

The first three ones are the same as \( A1 - A3 \), and the fourth one is introduced in [12]. The Hamiltonian systems deriving from the PDEs in high dimension not containing explicitly the space and the time variables do have the special form. \( A_k \) make the measure estimates simpler, and for the non-multiple case, \( A_k \) is not needed as in Theorem A.

Remark II

The case \( d \geq 2 \), \( \rho > 1 \) or \( d > 1 \), \( \rho = 1 \) has been proved in [20]. According to the weaker spectral asymptotics, we consider the following modified Diophantine condition:

\[
(1.12) \quad \{ \xi : \langle (k, \omega(\xi)) + (l, \Omega(\xi)) \rangle \geq \frac{\gamma}{A_{k,l}}, (k, l) \in \mathbb{Z} \}.
\]

For details,

\[
\langle (k, \omega(\xi)) \rangle \geq \frac{\gamma}{1 + |k|^\tau}, \quad \xi \in \Pi, \quad |l| = 0,
\]

\[
\langle (k, \omega(\xi)) + (l, \Omega(\xi)) \rangle \geq \frac{\gamma \rho d}{1 + |k|^\tau}, \quad \xi \in \Pi, \quad l \in \Lambda_+,
\]

\[
\langle (k, \omega(\xi)) + (l, \Omega(\xi)) \rangle \geq \frac{\gamma}{|l|^\tau (1 + |k|^\tau)}, \quad \xi \in \Pi, \quad l \in \Lambda_-,
\]

where \( l \in \Lambda_- \) denotes that \( l \) contains two opposite components, \( \Lambda_+ \) denotes others expect \( |l| = 0 \). And \( i \) is the site of the positive components of \( l \), \( c(\rho) \geq c_1(\rho) + \rho \), \( \tau \geq n + \frac{c(\rho) + 2}{d} + 4 \), \( c_1(\rho) \) is constant only depending on \( \rho \), and \( \langle l \rangle_d = \sum_i l_i |i|^d \).

When \( \rho = 1 \), we set \( c(\rho) = \frac{\gamma}{2} \) and for \( l \in \Lambda_- \), we have

\[
\langle (k, \omega(\xi)) + (l, \Omega(\xi)) \rangle \geq \frac{\gamma}{|l|^\tau (1 + |k|^\tau)}, \quad \xi \in \Pi, \quad l \in \Lambda_-.
\]

One can find, when \( l \in \Lambda_- \), the frequencies satisfying the modified Diophantine condition are more than before and the the other case is the same. The definition is intrinsical. Roughly speaking, the parameter \( \frac{\gamma}{A_{k,l}} \) describes the measure of the removed region for fixed \( (k, l) \), and the spectral asymptotics describe the cardinal number of the eliminated regions for fixed \( k \). Since the spectrum can’t be separable, the cardinal number of the removed regions get more, naturally, the measure of the removed region must get smaller to guarantee the measure \( |\Pi \setminus \Pi_\gamma| \to 0 \) as \( \gamma \to 0 \).
for all $\nu = 0, 1, \cdots$. Without loss of generality, we change A3) to
\[ X_P : \mathcal{P}^{\bar{a}, \bar{p}} \rightarrow \mathcal{P}^{a, p}, \]
where $0 < a < \bar{a}$, $0 \leq p \leq \bar{p}$ for simplicity.

This paper is organized as follows. The section 2 produces the KAM steps where we introduce a new diophantine conditions and truncate the normal variables $z$ and $\bar{z}$. In section 3, we derive an iteration Lemma. And in section 4, we give measure estimates and complete the proof of the main theorem. In the final section, we apply our results to nonlinear Schrödinger equations and the Klein-Gordon equations with exponential nonlinearity, which possess weaker spectral asymptotics. Then we obtain the existence of quasi-periodic solutions for both the Klein-Gordon equations with exponential nonlinearity and Schrödinger equations in higher dimension.

2. KAM steps

In this section, we outline an iterative scheme for the Hamiltonian (1.3) in one KAM cycle, say, from $\nu$th KAM step to the $(\nu + 1)$th-step. Set
\[
\begin{align*}
& r_0 = r, \gamma_0 = \gamma, \mu_0 = \mu, M_0 = M, \Pi_0 = \Pi, \\
& \omega_0 = \omega, \Omega_0 = \Omega, H_0 = H, a_0 = a, \\
& N_0 = \langle \omega(\xi), y \rangle + \sum_{i \in \mathbb{Z}_1} \Omega_i(\xi)z_i\bar{z}_i, P_0 = P,
\end{align*}
\]
where $H_0$ satisfies the assumptions A1) - A3) for Theorem A (or A1') for Theorem A for all $(x, y, z, \bar{z}) \in D_{\bar{x}, x}(r_0, s_0)$, $\xi \in \Pi_0$. As usual, we construct a sympletic transformation
\[
\Phi_1 : D_{\bar{x}, x}(r_1, s_1) \times \Pi_1 \rightarrow D_{\bar{x}, x}(r_0, s_0) \times \Pi_1
\]
where $r_1 < r_0, s_1 < s_0$. Our aim is to make the perturbation much smaller on a smaller domain $D_{\bar{x}, x}(r_1, s_1) \times \Pi_1$, which will be defined clearly below.

We also have a sequence of sympletic transformations and domains as follows
\[
\Phi_i : D_{\bar{x}, x}(r_i, s_i) \times \Pi_i \rightarrow D_{\bar{x}, x}(r_{i-1}, s_{i-1}) \times \Pi_i
\]
for all $i = 1, \cdots, \nu$. And we arrive at the following real analytic Hamiltonian:
\[
\begin{align*}
H_\nu &= H \circ \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_\nu = N_\nu + P_\nu, \\
N_\nu &= c_\nu(\xi) + \langle \omega_\nu(\xi), y \rangle + \sum_{n \in \mathbb{Z}_1} \Omega_i(\xi)z_i\bar{z}_i,
\end{align*}
\]
where
\[
\begin{align*}
|X_{P_\nu}|^{a, p} &\leq \gamma_\nu \mu_\nu, |X_{P_\nu}|^{\bar{a}, \bar{p}} \leq M_\nu \mu_\nu, \\
|\omega_\nu|^{\bar{a}, \bar{p}} + |\Omega_\nu|^{\bar{a}, \bar{p}} &\leq M_\nu < \infty,
\end{align*}
\]
(2.3)
for all $(x, y, z, \bar{z}) \in D_{\bar{x}, x}(r_\nu, s_\nu)$ and $\xi \in \Pi_\nu \subset \Pi_0$.

In the following, we try to find a sympletic transformation
\[
\Phi_{\nu+1} : D_{\bar{x}, x}(r_{\nu+1}, s_{\nu+1}) \times \Pi_{\nu+1} \rightarrow D_{\bar{x}, x}(r_\nu, s_\nu) \times \Pi_{\nu+1},
\]
such that $P_{\nu+1}$ is the new perturbation with the similar estimates on $D_{\bar{x}, x}(r_{\nu+1}, s_{\nu+1}) \times \Pi_{\nu+1}$.

For simplicity, we shall omit index for all quantities of the present KAM step (the $\nu$th-step) and index all quantities (Hamiltonian, normal form, perturbation,
transformation, and domains, etc) in the next KAM step (the \((\nu + 1)\)-th step) by 
"+". All constants \(c_i, c\) below are positive and independent of the iteration process.
To simplify the notations, we shall omit the scripts like \(a, \bar{p}, r, D(r, s)\), and only mark the changes.

Define

\[
\begin{align*}
 r_+ &= \frac{r}{2} + \frac{r_0}{4}, \\
 a_+ &= \frac{a}{2} + \frac{a_0}{4}, \\
 \gamma_+ &= \frac{\gamma}{2} + \frac{\gamma_0}{4}, \\
 M_+ &= \frac{M}{2} + M_0, \\
 s_+ &= \eta s, \quad \eta = \mu^{\frac{3}{2}}, \quad \mu = s^{\frac{1}{2}}, \\
 K_+ &= (\lceil \log \frac{1}{\mu} \rceil + 1)^{3\alpha_1}, \quad I_+ = (\lceil \log \frac{1}{\mu} \rceil + 1)^{3\alpha_2}, \\
 D_{\bar{z}, \bar{p}; i} &= D_{\bar{z}, \bar{p}}(r_+ + \frac{i-1}{8}(r - r_+), i\eta s), \quad i = 1, 2, \cdots, 8, \\
 D(\beta) &= \{ y \in \mathbb{C}^d : |y| < \beta \}, \quad \beta > 0, \\
 \hat{D}_{\bar{z}, \bar{p}}(\beta) &= D_{\bar{z}, \bar{p}}(r_+ + 7 \frac{7}{8}(r - r_+), \beta), \quad \beta > 0, \\
 \tilde{D}_{\bar{z}, \bar{p}}(\beta) &= D_{\bar{z}, \bar{p}}(r_+ + 3 \frac{3}{4}(r - r_+), \beta), \quad \beta > 0, \\
 D_+ &= D_{\bar{z}, \bar{p}; 1}, \quad D = D(r, s), \\
 \Pi_+ &= \{ \xi \in \Pi : |\langle k, \omega(\xi) \rangle + \langle l, \Omega(\xi) \rangle| > \frac{\gamma}{A_k l} \}, \\
 \Gamma(r - r_+) &= \sum_{0 < |k| \leq K_+} |k|^{4\tau + 4} e^{-|k|^r}, \\
 C(a - a_+) &= r \sum_{i \in \mathbb{Z}^n} |i|^{2c(\rho) + 2e^{-(a-a_+)i}|i|},
\end{align*}
\]

where \(\alpha_1, \alpha_2\) are constants to be specified.

2.1. **Truncation.**

We write \(P\) in the Taylor-Fourier series:

\[
P(x, y, z, \bar{z}) = \sum_{m, q, \bar{q}} \sum_{k \in \mathbb{Z}^n} P_{k m q \bar{q}} y^m \bar{q}^\bar{q} e^{\sqrt{-1}k \cdot x},
\]

where \(k \in \mathbb{Z}^n, m \in \mathbb{N}^n, q, \bar{q}\) are multi-index infinite vectors with finitely many nonzero components of positive integers.

We take \(K_+, I_+\) such that

\[
\begin{align*}
 H1) & \int_{K_+}^{\infty} t^{n+4} e^{-t(r-r_+)} dt \leq \mu, \\
 H2) & \int_{I_+}^{\infty} t^{\rho+4} e^{-t(a-a_+)} dt \leq \mu.
\end{align*}
\]

Then, in a usual manner we have
Lemma 2.1. ∀ (m, q, q), ξ ∈ Π₀, there is a constant c₁ such that
\[ |X_{P - R}|^{α, ς}_{7q₃, Dₚ, p, τ} \leq c₁γμ², \]
\[ |X_{P - R}|^{ξ, ω}_{7q₃, Dₚ, p, τ} \leq c₁Mμ², \]
where \( R = R⁺ + R⁻ + R² \) as follows:
\[ R⁺ = \sum_{0 < |k| \leq K⁺, |m| \leq 1} P_{km00} y^m e^{-T(k,x)}, \]
\[ R⁻ = \sum_{|k| \leq K⁺} (\langle P_{k010}, z \rangle + \langle P_{k001}, z \rangle) e^{-T(k,x)} \]
\[ = \sum_{k,i} P_{k10} z_i x e^{-T(k,x)} + \sum_{k,i} P_{k10} z_i x e^{-T(k,x)}, \]
\[ R² = \sum_{|k| \leq K⁺} (\langle P_{k020}, z, z \rangle + \langle P_{k011}, z, z \rangle + \langle P_{k002}, z, z \rangle) e^{-T(k,x)} \]
\[ = \sum_{|k| + |i| - |j| \neq 0} P_{ij}^{k010} z_i z_j x e^{-T(k,x)} + \sum_{|k| + |i| - |j| \neq 0} P_{ij}^{k020} z_i z_j x e^{-T(k,x)} + \sum_{|k| + |i| - |j| \neq 0, |i| \leq I⁺} P_{ij}^{k11} z_i z_j x e^{-T(k,x)}, \]
and \( P_{k10} = P_{k0q0} \) with \( q = e_i, q = 0; P_{k010} = P_{k0qq} \) with \( q = 0, \bar{q} = 0; P_{ij}^{k11} = P_{k0qq} \)
with \( q = e_i, \bar{q} = e_j; P_{ij}^{k010} = P_{k0qq} \) with \( q = e_i + e_j, \bar{q} = 0; P_{ij}^{k020} = P_{k0qq} \) with
\( q = 0, \bar{q} = e_i + e_j; \) here \( e_i \) denotes the vector with the \( i \)-th component being 1 and
other components being zero.

Proof. Let \( P - R = I + I⁺ + I⁻ \), where
\[ I = \sum_{|k| \geq K⁺} P_{km00} y^m z_q z_q e^{-T(k,x)}, \]
\[ I⁺ = \sum_{k \leq K⁺} \sum_{|m| + |q| \leq 2} P_{km00} y^m z_q z_q e^{-T(k,x)}, \]
\[ I⁻ = \sum_{i \geq I⁺} P_{ij}^{k11} z_i z_j, \]
As previous, \( H1 \) guarantees
\[ |X_{7q₃, Dₚ, p, τ}|^{α, ς}_{Tₗγ₃, Dₚ, p, τ} \leq c₁γμ², \]
\[ |X_{7q₃, Dₚ, p, τ}|^{ξ, ω}_{Tₗγ₃, Dₚ, p, τ} \leq c₁Mμ². \]
Then we only consider the term \( I⁻ \) where \( P^{11} = \sum_{k,i,j} P_{ij}^{k11} e^{-T(k,x)}, \)
\( P_{ij}^{k11} = k P_{ij}^{k11} e^{-T(k,x)}. \)
Since \( P^{11} = \partial_x \partial_z P \), by \( H11 \), we have
\[ \| P^{11} \| \leq \sup_{|z| \leq \bar{q}_p = 1} |P^{11} z|^{α, ς}_{Tₗγ₃, p, D(s)} = \sup_{|z| \leq \bar{q}_p = s} \frac{1}{s} |P^{11} z|^{α, ς}_{Tₗγ₃, p, D(s)} \leq γμ, \]
where \( \| \cdot \| \) means operator norm.
For any \( z^j = (\cdots, e^{-\frac{1}{p} |j|^{p}}|j|^{-p}, \cdots) \), that is, all components are zero except for the \( j \)-th,
\[ P^{11} z^j = (\cdots, P_{ij}^{k11} e^{-\frac{1}{p} |j|^{p}}|j|^{-p}, \cdots). \]
Moreover, \( |z_j|^{\bar{a} + \bar{p}} = 1 \) yields
\[
\left( |P_{ij}^{11} z_j|^{\bar{a} + \bar{p}} \right)^2 = \sum_{i \in Z_1^p} |P_{ij}^{11} e^{\bar{4} |j| |i|} e^{2a |i| |j|^{\bar{p}}} |^2 \\
= e^{-\bar{4} |j| |i|^{\bar{p}}} \sum_{i \in Z_1^p} |P_{ij}^{11} e^{2a |i| |j|^{\bar{p}}} |^2 \leq \gamma \mu,
\]
i.e. for all \( i, j \in Z_1^p \), we have
\[
|P_{ij}^{11}| \leq \gamma \mu e^{-a |i| |j|^{\bar{p}}} e^{4 |j| |i|^{\bar{p}}} |j|^p.
\]
Then any \( |z_j|^{\bar{a} + \bar{p}} \leq s \), i.e. \( |z_j| \leq se^{-\bar{4} |j| |i|^{\bar{p}}} |j|^p \) yields
\[
\sum_{j \in Z_1^p} |P_{ij}^{11} z_j|^2 \leq \gamma^2 \mu^2 e^{-2a |i| |j|^{\bar{p}}} e^{-2 |j|^{\bar{p}} |i|} \sum_{j \in Z_1^p} |z_j| e^{2a |i| |j|^{\bar{p}}} |j|^2p
\]
\[
\leq \gamma^2 \mu^2 s^2 e^{-2a |i| |j|^{\bar{p}}} |j|^2p.
\]
Then
\[
(IH_1 z_{II}) \leq \sqrt{\sum_{|i| \geq I_+} \sum_{j \in Z_1^p} \left| P_{ij}^{11} z_j \right|^2 e^{2a |i|^{\bar{p}}} |j|^2p} \\
\leq s \gamma \mu \sum_{|i| \geq I_+} e^{-(a-a_-) i} \leq s \gamma \mu \int_{I_+}^{\infty} t^\rho e^{-(a-a_-) t} dt \\
\leq s \gamma \mu^2.
\]
The same is for \( II_1 \) and the Lipschitz semi-norm. 

For the proof of Theorem A, H2) shows
\[
\int_{I_+}^{\infty} \tilde{t}^5 e^{-t(a-a_-)} dt \leq \mu,
\]
and the truncation \( R^2 \) has the form:
\[
R^2 = \sum_{|k| \leq K_{+,i,j}} P_{ij}^{k20} z_i z_j e^{|\sqrt{-1} T(k,x)} + \sum_{|k| \leq K_{+,i,j}} P_{ij}^{k02} z_i z_j e^{|\sqrt{-1} T(k,x)} \\
+ \sum_{|k| \leq K_{+,i} \leq I_+, j} P_{ij}^{k11} z_i z_j e^{|\sqrt{-1} T(k,x)}.
\]
We choose \( \rho = 1 \), \( c(\rho) = \frac{2}{3} \), the results and the proof also hold.

2.2. Linear Equations. We induce the time-1 map \( \phi_t^F|_{t=1} \) generated by the Hamiltonian \( F \):
\[
F = \sum_{0 < |k| \leq K_+, |m| \leq 1} F_{km00} \bar{y}^m e^{|\sqrt{-1} T(k,x)} \\
+ \sum_{|k| \leq K_+, i} \left( \langle F_{i10}^{k10}, z_i \rangle + \langle F_{i01}^{k01}, \bar{z}_i \rangle \right) e^{|\sqrt{-1} T(k,x)} \\
+ \sum_{|k| \leq K_+, |k|+|i|-|j| \neq 0} \left( \langle F_{ij}^{k20}, z_i z_j \rangle + \langle F_{ij}^{k02}, z_i \bar{z}_j \rangle \right) e^{|\sqrt{-1} T(k,x)} \\
+ \sum_{|k| \leq K_+, |k|+|i|-|j| \neq 0, |i| \leq I_+} \langle F_{ij}^{k11}, z_i \bar{z}_j \rangle e^{|\sqrt{-1} T(k,x)}.
\]
where the coefficients depending on $\xi$ are to be determined.

Let

$$[R] = \sum_{|m|+|q|=1} P_{mqq} y^m z^q z^q + \sum_{i,j} P_{0,i,j} z_i \bar{z}_j.$$ 

Substituting $F, R, [R]$ into

$$(2.7) \quad \{N, F\} + R - [R] = 0,$$

we have the following equations for $|k| < K_+, i, j \in Z^p_1$:

$$(2.8) \quad \langle k, \omega \rangle F_{km00} = -1P_{km00}, \quad k \neq 0, \quad |m| \leq 1,$$

$$(2.9) \quad \langle (k, \omega) - \Omega_i \rangle F_{k10} = -1P_{k10}^k,$$

$$(2.10) \quad \langle (k, \omega) + \Omega_i \rangle F_{k10} = -1P_{k10}^k,$$

$$(2.11) \quad \langle (k, \omega) - \Omega_i - \Omega_j \rangle F_{k20} = -1P_{k20}^k, \quad |k| + ||i| - |j|| \neq 0,$$

$$(2.12) \quad \langle (k, \omega) + \Omega_i + \Omega_j \rangle F_{k20} = -1P_{k20}^k, \quad |k| + ||i| - |j|| \neq 0,$$

$$(2.13) \quad \langle (k, \omega) + \Omega_i - \Omega_j \rangle F_{k11} = -1P_{k11}^k, \quad |k| + ||i| - |j|| \neq 0, \quad |i| \leq I_+.$$

For Theorem A', note that the term $\sum_{i,j} \rho_{ij}^0 z_i \bar{z}_j$ can be decomposed into $\sum_{i \in Z^p_1} \rho_{ii}^1 z_i \bar{z}_i$ and $\sum_{i,j} \rho_{ij}^0 z_i \bar{z}_j$. Since $P \in \mathcal{A}$, the last term is absent.

Consider $\xi \in \Pi_+$, which satisfies (1.12), then we have

**Lemma 2.2.** $F$ is uniquely determined by (2.7) and there exits a constant $c_2$ such that on $\hat{D}_{\hat{\omega},p}(s) \times \Pi_+$,

$$|X_F| \leq c_2 \gamma \mu \Gamma(r-r_+) C(a-a_+),$$

Moreover, $F \in \mathcal{A}$ in Theorem A'.

**Proof.** First, we consider the case $l \in \Lambda_-$, i.e.,

$$|\langle k, \omega(\xi) \rangle + \Omega_k - \Omega_j| \geq \frac{\gamma}{(1+|k|^r)|j|e^{2|j|}}, \quad |k| + ||i| - |j|| \neq 0, \quad |i| \leq I_+.$$

Since $P^{11} = \partial_z \partial_{\bar{z}} P$, by (1.10), we have

$$\|P^{11}\| \leq \sup_{|z|_{\hat{\omega},p} = 1} |P^{11} z|_{\hat{\omega},\bar{z}} = \sup_{|z|_{\hat{\omega},p} = s} \frac{1}{\rho-s} \|P^{11} z|_{\hat{\omega},\bar{z},D(s)} \leq \gamma \mu,$$

where $\|\cdot\|$ means operator norm.

Expanding $P^{11}$ into its Fourier series, we have

$$\|P^{k11}\| \leq c_1 \gamma \mu e^{-|k|^r},$$

for all $k \in \mathbb{Z}^n$.

By lemma 2.1, choose $z^j = (\cdots, e^{-\hat{\omega}|j||j|^{-p}, \cdots}$, that is, all components are zero except for the $j$-th. For all $i, j \in Z^p_1$, we have

$$|P^{k11}_{ij}| \leq \gamma \mu e^{-a|z^j|} |z^j|||j|^{-p}e^{-2|k||r},$$

Then any $|z|_{\hat{\omega},p} \leq s$, i.e., $|z_j| \leq se^{-\frac{\gamma}{|j|} e^{2|j|}}, |z^j| \leq 2se^{-\frac{\gamma}{|j|} e^{2|j|}},$ yields

$$\sum_{j \in Z^p_1} |P^{k11}_{ij} z_j|^2 \leq \gamma^2 \mu^2 e^{-2a|z^j|} |z^j|^{-2|k||r} \sum_{j \in Z^p_1} |z_j|^2 e^{2|j||j|^p}$$

$$\leq \gamma^2 \mu^2 s^2 e^{-2a|z^j|} |z^j|^{-2|k||r}.$$
Denote $\delta_{k,i,j} = (k, \omega) + \Omega_i - \Omega_j$. According to (2.13) and (2.16), we have

$$F_{ij}^{k11} = \sqrt{-1} \frac{P_{ij}^{k11}}{\delta_{k,i,j}},$$

$$|F_{ij}^{k11}z_j| \leq \frac{(1 + |k|^\gamma)|\epsilon|^2(\rho)}{\gamma} |P_{ij}^{k11}z_j|.$$

This together with (2.18) implies that

$$| \sum_{j \in Z_i^r} F_{ij}^{k11} z_j | \leq \sum_{j \in Z_i^r} |F_{ij}^{k11} z_j|^2 \leq \frac{(1 + |k|^\gamma)|\epsilon|^2(\rho)}{\gamma^2} \sum_{j \in Z_i^r} |P_{ij}^{k11} z_j|^2 \leq (1 + |k|^\gamma)^2 |s|^2 e^{-2|k||\tau|} e^{-2|\delta|}|\epsilon|^{-2\beta + 2\epsilon(\rho)},$$

and

$$\left( |F_{ij}^{k11} z_j|^{a+\beta}, D \times \Pi_+ \right)^2 \leq \sum_{j \in Z_i^r} |F_{ij}^{k11} z_j|^2 e^{2a+|\tau|}|\epsilon|^{2\beta} \leq \mu s^2 |k|^2 e^{-2|k|\tau} C(a - a_+).$$

Then we have

$$\frac{1}{s} |F_{ij}^{k11} z_j|^{a+\beta} \leq \frac{1}{s} \sum_{k \leq K_+} |F_{ij}^{k11} z_j|^{a+\beta} e^{k(r+\frac{\gamma}{2}(r-r_+))} \leq c \mu \sum_{0 < k \leq K_+} (1 + |k|^\gamma) e^{-|k|r-r_+} C(a - a_+) \leq c \mu \Gamma(r-r_+) C(a - a_+).$$

The Lipschitz estimate almost follows the same idea. First, we decompose the following into two parts:

$$\Delta F_{k,i,j}^{11} z_j = \frac{P_{ij}^{k11}(\xi)z_j}{\delta_{k,i,j}(\xi)} - \frac{P_{ij}^{k11}(\xi)z_j}{\delta_{k,i,j}(\xi)} = \frac{P_{ij}^{k11}(\xi)z_j - P_{ij}^{k11}(\xi)z_j}{\delta_{k,i,j}(\xi)} + P_{ij}^{k11}(\xi)z_j \left( \frac{1}{\delta_{k,i,j}(\xi)} - \frac{1}{\delta_{k,i,j}(\xi)} \right) = \frac{\Delta P_{ij}^{k11}(\xi)z_j}{\delta_{k,i,j}(\xi)} + P_{ij}^{k11}(\xi)z_j \Delta \delta_{k,i,j}^{-1}.$$

Since

$$-\Delta \delta_{k,i,j}^{-1} = \frac{\Delta \delta_{k,i,j}}{\delta_{k,i,j}(\xi)} \delta_{k,i,j}(\xi) = \frac{(k, \Delta \omega) + \Delta \Omega_i - \Delta \Omega_j}{\delta_{k,i,j}(\xi)\delta_{k,i,j}(\xi)},$$

we obtain

$$|\Delta \delta_{k,i,j}^{-1}| \leq \frac{(1 + |k|^\gamma)^2 |\epsilon|^{2\epsilon(\rho)}}{\gamma^2} (|k||\Delta \omega| + |\Delta \Omega| - \delta) \leq c (1 + |k|^\gamma)(1 + |k|^2 |\epsilon|^{2\epsilon(\rho)}) |M| |\xi - \varepsilon|.$$

For fixed $\xi, \varepsilon$, $\Delta \varepsilon$ is a bounded linear operator, in the operator norm below:

$$\| \Delta \varepsilon P^{11} \| = \sup_{|z| \leq \bar{z}} \| \Delta \varepsilon P^{11} z \|_{a, \bar{p}, D} = \sup_{|z| \leq \bar{z}} \| \frac{1}{s} \Delta \varepsilon P^{11} z \|_{a, \bar{p}, D} \leq M \mu |\xi - \varepsilon|,$$
and for all $k$,

$$\| \Delta \xi \| P^{k11} \| \leq \epsilon \| \Delta \xi \| P^{11} \leq M \mu |\xi| e^{-|k|r}.$$  

Choose $z^j = (\cdots, e^{-\frac{3}{2}|i|}j^{-p}, \cdots)$. Similar argument as above implies

$$|\Delta P^{k11}_{ij}|^2 \leq c M^2 \mu^2 |\xi - \zeta|^2 e^{-2a|i|} |i|^{2p} e^{-2|k|r},$$

then, we have

$$| \sum_{z \in Z_k} \frac{\Delta P^{k11}_{ij}}{\delta_{k,ij}} z_j |^2 \leq \sum_{z \in \mathbb{Z}^d} \frac{|\Delta P^{k11}_{ij} z_j|^2}{|\delta_{k,ij}|^2} + \frac{(1 + |k|^\gamma)^2}{\gamma^2} \sum_{z \in \mathbb{Z}^d} |\Delta P^{k11}_{ij} z_j|^2 \leq c \frac{(1 + |k|^\gamma)^2}{\gamma^2} e^{-2|k|r} M^2 \mu^2 s^2 |\xi - \zeta|^2 e^{-2a|i|} |i|^{2p},$$

$$\| \sum_{z \in \mathbb{Z}^d} P^{k11}_{ij} z_j \Delta \delta_{k,ij}^{-1} |^2 \leq \sum_{z \in \mathbb{Z}^d} |P^{k11}_{ij} z_j|^2 |\delta_{k,ij}^{-1}|^2 + c \frac{(1 + |k|^\gamma)^4 |k|^2}{\gamma^4} |\xi - \zeta|^2 M^2 \sum_{z \in \mathbb{Z}^d} |P^{k11}_{ij} z_j|^2.$$ 

Hence

$$\left( |\Delta F^{k11}_{|s,D \times \Pi_+}|^{a+\bar{p}} \right)^2 = \sum_{|i| \leq L_s} \left( \sum_{z \in \mathbb{Z}^d} \frac{|\Delta P^{k11}_{ij} z_j|}{\delta_{k,ij}} + R^{k11}_{ij} z_j \Delta \delta_{k,ij}^{-1} \right)^2 e^{2a+b |i|} |i|^{2\bar{p}} \leq 2 \sum_{|i| \leq L_s} \sum_{z \in \mathbb{Z}^d} \left( \sum_{z \in \mathbb{Z}^d} \frac{|\Delta P^{k11}_{ij} z_j|}{\delta_{k,ij}} + |P^{k11}_{ij} z_j \Delta \delta_{k,ij}^{-1} |^2 \right) e^{2a+b |i|} |i|^{2\bar{p}} \leq c \frac{(1 + |k|^\gamma)^4 |k|^2}{\gamma^4} M^2 \mu^2 s^2 |\xi - \zeta|^2 e^{-2|k|r} e^{-2a|i|} |i|^{4b} |e(r). \right.$$ 

Since $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$, $a, b > 0$, we have

$$\frac{1}{s} |F^{k11}_{|s,D \times \Pi_+}|^{a+b} = \sup_{\xi \neq \zeta} \frac{|\Delta F^{k11}_{|s,D \times \Pi_+}|^{a+\bar{p}} |\xi - \zeta|^2}{|\xi - \zeta|^2} \leq c \frac{1}{s} \mu M (1 + |k|^\gamma |e^{-|k|r} C(a - a_+),$$

and

$$\frac{1}{s} |F^{k11}_{|s,D \times \Pi_+}|^{a+b} = \frac{1}{s} \sum_{0 < |k| \leq K_+} |F^{k11}_{|s,D \times \Pi_+}|^{a+b} e^{k(r + \frac{1}{\gamma}(r - r_+))} C(a + a_+ + a_+) \leq c \frac{M}{\gamma} \mu (1 + |k|^\gamma |e^{-|k| a_+} C(a - a_+)).$$
The estimates of \( |F_{11}^{a_+ \bar{p}}|, |F_{11}^{a_+ \bar{p}}| \) show all differences from the previous KAM steps. Then we give the outline estimates of \( F_{k10} \). Since (1.10), we have
\[
\frac{1}{r} |P_{10}^{a_+ \bar{p}}(r,D(r,s))| \leq \gamma \mu,
\]
and
\[
|P_{k10}^{a_+ \bar{p}}(r,D(r,s))| \leq c r \gamma \mu \gamma |k| r,
\]
where \( c \) is a constant. Then, according to the corresponding Diophantine condition
\[
|\langle k, \omega \rangle + \Omega_j| \geq \gamma_j \frac{d}{A_k} \geq \gamma \frac{d}{A_k},
\]
and
\[
\sqrt{-1} F_{k,j}^{10} \leq \frac{R_{k,j}^{10}}{\langle k, \omega \rangle + \Omega_j}, \quad j \geq 1,
\]
the following hold:
\[
\frac{1}{r} |F_{10}^{a_+ \bar{p}}(k,\omega)| \leq c \mu r(r-r_+)C(a-a_+),
\]
\[
\frac{1}{r} |F_{10}^{a_+ \bar{p}}(k,\omega)| \leq c \mu r(r-r_+)C(a-a_+).
\]
All the other estimates are similar or even simpler and admit the same results, for details, see [12], [20]. Moreover, in Theorem A’, \( P \in A \) implies \( F \in A \).

Above all, we fulfill the proof of the Lemma.

Let \( \Phi_+ \) denote the time-1 map generated by \( F \). Then it is a canonical transformation and
\[
H \circ \Phi_+ = N + [R] + \int_0^1 \{ R_t, F \} \circ \phi_t dt + (P - R) \circ \Phi_+ = N_+ + P_+,
\]
where \( R_t = (1 - t) \{(R), F \} + R \), and
\[
(2.22) \quad e_+ = e + P_{0000},
\]
\[
(2.23) \quad \omega_+ = \omega + P_{0100},
\]
\[
(2.24) \quad \Omega_+ = \Omega + P_{0011}.
\]

For Theorem A, we induct \( F \) as
\[
F = \sum_{0 < |k| \leq K_+ \, |m| \leq 1} F_{k200} y^m e^{\sqrt{-1} \langle k, x \rangle}
\]
\[
+ \sum_{k,i} F_{k,i}^{10} z_i e^{\sqrt{-1} \langle k, x \rangle}
\]
\[
+ \sum_{k,i,j} F_{k,i,j}^{20} z_i z_j e^{\sqrt{-1} \langle k, x \rangle}
\]
\[
+ \sum_{k,i,j} F_{k,i,j}^{11} z_i z_j e^{\sqrt{-1} \langle k, x \rangle},
\]
where \( F_{k200}, F_{k,i}^{10}, F_{k,i}^{01}, F_{k,i,j}^{20}, F_{k,i,j}^{02}, F_{k,i,j}^{11} \) are functions of \( \xi \).
According to (2.7), we get the following equations for \( |k| \leq K_+ \):
\[
(k, \omega) F_{km00} = \sqrt{-1} P_{km00}, \quad k \neq 0, \quad |m| \leq 1,
\]
Then the following hold:

\[(k, \omega) + \Omega_i F_{k10}^{i} = \sqrt{1-P_{k10}^{i}},\]

\[(k, \omega) - \Omega_i F_{k01}^{i} = \sqrt{1-P_{k01}^{i}},\]

\[(k, \omega) + \Omega_i + \Omega_j F_{k20}^{ij} = \sqrt{1-P_{k20}^{ij}},\]

\[(k, \omega) - \Omega_i - \Omega_j F_{k20}^{ij} = \sqrt{1-P_{k20}^{ij}},\]

\[(k, \omega) + \Omega_i - \Omega_j F_{k11}^{ij} = \sqrt{1-P_{k11}^{ij}}, i \leq I_+ ,\]

(2.26)

for all \( \xi \in \Pi_+ = \{ \xi \in \Pi : |\langle k, \omega(\xi) \rangle + \langle l, \Omega(\xi) \rangle | \geq \frac{\bar{c}}{A_{k,l}} \} \).

Replace \( c(\rho) = \frac{\rho}{2}, \rho = 1, |i| = i, C(a-a+) = \sum_{0<i \leq I_+} \bar{c}^i e^{-(a-a^+)}i \), we have the lemma:

**Lemma 2.3.** The linearized equation \( L \) has a unique solution \( F \) normalized by \( [F] = 0 \), and there exists a constant \( c_2 \) such that on \( \mathcal{D}_{\pi,p}(s) \times \Pi_+ \),

\[ |X_F|^{a^+} \leq c_2 \mu \Gamma(r-r_+)C(a-a^+), \]

\[ |X_F|^{\epsilon_{a^+}} \leq \frac{M}{\gamma} \mu \Gamma(r-r_+)C(a-a^+). \]

2.3. Coordinate Transformation.

**Lemma 2.4.** Assume the following

\[ H_3 \] \( c_2 \mu \Gamma(r-r_+)C(a-a^+) < \frac{1}{8}(r-r_+) \);

\[ H_4 \] \( c_2 s^2 \mu \Gamma(r-r_+)C(a-a^+) < 5s^2 \);

\[ H_5 \] \( c_2 s \mu \Gamma(r-r_+)C(a-a^+) < s^2 \).

Then the following hold:

1) \( \phi_t^F : \mathcal{D}_{\pi,p} : \mathcal{D}_{\pi,p} \rightarrow \mathcal{D}_{\pi,p} \), \( \forall 0 < t \leq 1 \),

so \( \phi_t^F \) is well defined.

2) \( \Phi_+ : D_{\pi,p} \rightarrow \mathcal{D}_{\pi,p} \).

3) There exists a positive constant \( c_2 \) such that on \( \mathcal{D}_{\pi,p} \times \Pi_+ \),

\[ |\phi_t^F - id|^{a^+} \leq c_3 \mu \Gamma(r-r_+)C(a-a^+), \]

\[ |D\phi_t^F - Id|^{a^+} \leq c_3 \mu \Gamma(r-r_+)C(a-a^+), \]

and \( \phi_t^F \) is Lipschitz continuous on parameter \( \xi \), i.e.,

\[ |\phi_t^F - id|^{\epsilon_{a^+}} \leq \frac{M}{\gamma} \mu \Gamma(r-r_+)C(a-a^+), \]

for all \( 0 \leq |t| \leq 1 \).

4) On \( \mathcal{D}_{\pi,p} \times \Pi_+ \)

\[ |\Phi_+ - id|^{a^+} \leq c_3 \mu \Gamma(r-r_+)C(a-a^+), \]

\[ |D\Phi_+ - Id|^{a^+} \leq c_3 \mu \Gamma(r-r_+)C(a-a^+), \]

and \( \Phi_+ \) is Lipschitz continuous on parameter \( \xi \), i.e.,

\[ |\Phi_+ - id|^{\epsilon_{a^+}} \leq \frac{M}{\gamma} \mu \Gamma(r-r_+)C(a-a^+). \]
Proof. 1) Denote \( \phi_F = (\phi_{F1}, \phi_{F2}, \phi_{F3}, \phi_{F4})^T \), where \( \phi_{F1}, \phi_{F2}, \phi_{F3}, \phi_{F4} \) are components in \( x, y, z, \bar{z} \) planes respectively. Note that

\[
\phi_F' = id + \int_0^t X_F \circ \phi_F^\mu du.
\]

For any \((x, y, z, \bar{z}) \in D_{\Phi, p; 3}\), let \( t_* = \sup\{t \in [0, 1] : \phi_F(x, y, z, \bar{z}) \in D_{\Phi, p; 3} \}. \) Since \( 0 < \frac{a}{2} < a_+ < a \),

\[
|u|^\Phi \leq |w|^{a_+}, \quad \forall w \in P^{\Phi, p},
\]

it follows from H2)-H5) \((2.14)\) and \((2.15)\), that for any \( 0 \leq t \leq t_* \),

\[
|\phi_{F1}(x, y, z, \bar{z})||D_{\Phi, p; 3} \leq |x| + \int_0^t F_{\psi} \circ \phi_F^\mu du \leq |x| + |F_{\psi}|_{D_{\Phi, p; 3} \times \Pi_+}
\]

\[
\leq r_+ + \frac{2}{8}(r - r_+) + c_2 \mu \Gamma(r - r_+)C(a - a_+)
\]

\[
\leq r_+ + \frac{3}{8}(r - r_+),
\]

\[
|\phi_{F2}(x, y, z, \bar{z})||D_{\Phi, p; 3} \leq |y| + \int_0^t F_{\psi} \circ \phi_F^\mu du \leq |y| + |F_{\psi}|_{D_{\Phi, p; 3} \times \Pi_+}
\]

\[
\leq 9s_+^2 + c_2 s_+^2 \mu \Gamma(r - r_+)C(a - a_+) \leq 16s_+^2,
\]

\[
|\phi_{F3}(x, y, z, \bar{z})||D_{\Phi, p; 3} \leq |z|^{\Phi} + \int_0^t F_{\psi} \circ \phi_F^\mu du
\]

\[
\leq |z|^{\Phi} + |F_{\psi}|^{\Phi} + |z|^{\Phi} + |F_{\psi}|_{D_{\Phi, p; 3} \times \Pi_+}
\]

\[
\leq 3s_+ + c_2 s_+ \mu \Gamma(r - r_+)C(a - a_+) \leq 4s_+,
\]

\[
|\phi_{F4}(x, y, z, \bar{z})||D_{\Phi, p; 3} \leq |z|^{\Phi} + \int_0^t F_{\psi} \circ \phi_F^\mu du
\]

\[
\leq |z|^{\Phi} + |F_{\psi}|^{\Phi} + |z|^{\Phi} + |F_{\psi}|_{D_{\Phi, p; 3} \times \Pi_+}
\]

\[
\leq 3s_+ + c_2 s_+ \mu \Gamma(r - r_+)C(a - a_+) \leq 4s_+,
\]

i.e., \( \phi_F(x, y, z, \bar{z}) \in D_{\Phi, p; 4}, \forall 0 \leq t \leq t_* \). Thus, \( t_* = 1 \) and \((2.21)\) holds.

2) Since \( \Phi_* = \phi_{F1} \), the result follows from 1).

3) Using Lemma 2.2 and \((2.28)\), on \( D_{\Phi, p; 3} \times \Pi_+ \), we have

\[
|\Phi_* - id|^{a_+} \leq |\int_0^1 X_F \circ \phi_F^\mu du| \leq |X_F|^{a_+} \leq c_3 \mu \Gamma(r - r_+)C(a - a_+).
\]

Since

\[
D\Phi = Id + \int_0^1 J(D^2 F)D\phi_F^\mu dt,
\]

by Cauchy’s inequality, we have

\[
|D\Phi_* - Id|^{a_+} \leq 2|D^2 F| \leq c_3 \mu \Gamma(r - r_+)C(a - a_+).
\]

The Lipshitz semi-norm is the same as the first argument of 3):

\[
|\Phi_* - id|^{a_+} \leq |\int_0^1 X_F \circ \phi_F^\mu du| \leq |X_F|^{a_+} \leq c_3 \frac{M}{\gamma} \mu \Gamma(r - r_+)C(a - a_+).
\]

The estimates on the higher-order partial derivatives of \( \Phi \) follows a similar arguments.
4) is immediately follows from 3).

For the proof of Theorem A, we have the same hypotheses H3)-H5) with \( c(\rho) = \frac{5}{2} \), then for Hamiltonian (1.3) and the coordinate transformation \( \phi^t \) generated by the Hamiltonian function (2.24), lemma 2.4 also holds.

2.4. The New Normal Form.

**Lemma 2.5.** There is a constant \( c_4 \) such that on \( \Pi_+ \),

\[
|\omega+ - \omega|_{\Pi_+} \leq c_4 \gamma \mu, \quad |\omega+ - \omega|_{\Pi_+}^c \leq c_4 M \mu, \\
|\Omega+ - \Omega|_{\Pi_+} \leq c_4 \gamma \mu, \quad |\Omega+ - \Omega|_{\Pi_+}^c \leq c_4 M \mu.
\]

**Proof.** It immediately follows from (1.10), (1.11), (2.17), (2.23) and (2.24). Then,

\[
|\omega+|_{\Pi} + |\Omega+|_{\Pi} \leq |\omega|_{\Pi} + |\Omega|_{\Pi} + |\omega+ - \omega|_{\Pi} + |\Omega+ - \Omega|_{\Pi} \\
\leq M + 2c_4 M \mu.
\]

2.5. The New Frequency.

**Lemma 2.6.** Assume that

\[
H_6) \quad 2c_4 M \mu \leq M_0 - \frac{M_0}{2}.
\]

Then

\[
|\omega+|_{\Pi}^c + |\Omega+|_{\Pi}^c \leq \frac{M + M_0}{2} \leq M_+.
\]

2.7. Assume that

\[
H_7) \quad c_5 \mu \left( 1 + K_+ \right)^T K_+ I^c(\rho) \leq \gamma - \gamma_+.
\]

Then, we have for all \( |k| \leq K_+ \), \( |l| \leq 2 \),

\[
|\langle k, \omega+ \rangle| \geq \frac{\gamma}{1 + |k|^a}, \quad 0 < |k| \leq K_+, \quad l = 0,
\]

\[
|\langle k, \omega+ \rangle + \langle l, \Omega+ \rangle| \geq \frac{\gamma+ \langle l \rangle d}{1 + |k|^a}, \quad |k| \leq K_+, \quad l \in \Lambda_+,
\]

\[
|\langle k, \omega+ \rangle + \langle l, \Omega+ \rangle| \geq \frac{\gamma+ \langle l \rangle d}{(1 + |k|^a)|i| c(\rho)} , \quad |k| \leq K_+, \quad l \in \Lambda_-, \quad |i| < I_+,
\]

where \( i \) is the site of the positive component of \( l \).

**Proof.** Note that \( \delta < 0 \). Denote \( \omega+ = \omega + \hat{\omega} \), \( \Omega+ = \Omega + \hat{\Omega} \). We have

\[
|\langle k, \hat{\omega}(\xi) \rangle + \langle l, \hat{\Omega}(\xi) \rangle| \leq |k| \hat{\omega}|_{\Pi+} + |l| \hat{\Omega}|_{\Pi+} \leq c_5 \gamma \mu |k||l| d.
\]

Combining with H7), as usual, we have

\[
c_5 \mu \left( 1 + K_+ \right)^T K_+ \leq \gamma - \gamma_+,
\]

and

\[
|k, \omega+ \rangle + \langle l, \Omega+ \rangle \geq \frac{\gamma+ \langle l \rangle d}{1 + |k|^a}.
\]
For $l \in \Lambda_-$, $|l|_\delta \leq 2$, 
\[
|\langle k, \hat{\omega}(\xi) \rangle + (l, \hat{\Omega}(\xi))| \leq |k|\hat{\omega}|_{\Pi^+} + 2|\hat{\Omega}|_{\Pi^+}^{a+\beta} \leq c_6 \gamma \mu |k|.
\]
Then we have 
\[
|\langle k, \omega_+(\xi) \rangle + (l, \Omega^+(\xi))| \geq |\langle k, \omega(\xi) \rangle + (l, \Omega(\xi))| - |\langle k, \hat{\omega}(\xi) \rangle + (l, \hat{\Omega}(\xi))|
\]
\[
\geq \frac{\gamma}{1 + |k|^\tau |l|^{c(\rho)}} - \frac{|k|(\gamma - \gamma_+)}{(1 + K_+)^\tau K_+ I_+^{c(\rho)}},
\]
\[
\geq \frac{\gamma_+}{1 + |k|^\tau |l|^{c(\rho)}}.
\]

For the proof of Theorem A, we have Lemma 2.7: Assume that 
\[
H7') \ c_6 \mu (1 + K_+^2)K_+ I_+^2 \leq \gamma - \gamma_+.
\]
Then we have that for all $|k| \leq K_+$, $|l| \leq 2$, 
\[
|\langle k, \omega_+(\xi) \rangle | \geq \frac{\gamma}{1 + |k|^\tau}, \quad 0 < |k| \leq K_+, \quad l = 0,
\]
\[
|\langle k, \omega_+(\xi) \rangle + (l, \Omega^+(\xi))| \geq \frac{\gamma_+ (l)}{1 + |k|^\tau}, \quad |k| \leq K_+, \quad l \in \Lambda_+,
\]
\[
|\langle k, \omega_+(\xi) \rangle + (l, \Omega^+(\xi))| \geq \frac{\gamma_+}{1 + |k|^\tau |l|^{c(\rho)}}, \quad |k| \leq K_+, \quad l \in \Lambda_-, \quad 0 < i < I_+.
\]

2.6. The New Perturbation. In this subsection, we consider $| \cdot |^{a+\beta}$, and we denote $D_{\hat{D},p}^{\pm}, \hat{D}_{\hat{D},p}(s)$ by $D$, $\hat{D}$ respectively for short.

Recall the new perturbation 
\[
P_+ = \int_0^1 \{R_t, F\} \circ \phi_t^F dt + (P - R) \circ \Phi_+,
\]
with $R_t = (1 - t)|R| + t R$. Then the new perturbed vector field is 
\[
X_{P_+} = X_{(P - R) \circ \phi_t^F} + \int_0^1 (\phi_t^F)^* [X_{R_t}, X_F] dt.
\]

We have 
\[
|X_{(P - R) \circ \phi_t^F}|^{a+\beta}_{s+D_1} \leq |X_{(P - R)}|^{a+\beta}_{\Pi^+,D_4} \leq c_6 \gamma \mu^2,
\]
\[
|X_{(P - R) \circ \phi_t^F}|^{\xi+\rho}_{s+D_4} \leq |X_{(P - R)}|^{\xi+\rho}_{\Pi^+,D_4} \leq c_6 M \mu^2.
\]

Lemma 2.8. Let $\phi_t^F$ be the time-$t$ map of the flow generated by Hamiltonian $F$. Then there exists a positive constant $c$ such that 
\[
|\phi_t^F|^{a+\beta}_{\Pi^+,D_2} \leq c |Y|^{a+\beta}_{\Pi^+,D_4},
\]
\[
|\phi_t^F|^{\xi+\rho}_{\Pi^+,D_2} \leq c |Y|^{\xi+\rho}_{\Pi^+,D_4} + \frac{c}{(r - r_+)^2} |Y|^{a+\beta}_{\Pi^+,D_4} |X_F|^{\xi+\rho}_{s,D},
\]
for all $0 \leq t \leq 1$.

See section 3 of [20] for the detail proof.

In the following, we estimate the commutator $[X_{R_t}, X_F]$ on the domain: 
\[
|[X_{R_t}, X_F]|_s \leq |DX_{R_t} \cdot X_F|_s + |DX_F \cdot X_{R_t}|_s.
\]
Using the generalized Cauchy estimate, lemma 3.1 and lemma 3.2, we obtain

\[ \|X_{R_t}, X_F\|_{s, \tilde{D}} \leq (r - r_+)^{-1}|X_F|_{s, \tilde{D}}|X_F|_{r, \tilde{D}} \]

\[ \leq c(r - r_+)^{-1}\gamma \mu^2 \Gamma(r - r_+)C(a - a_+). \]

Similarly,

\[ \|X_{R_t}, X_F\|^2_{s, \tilde{D}} \leq |DX_{R_t}|^2_{s, \tilde{D}}|X_F|_{s, \tilde{D}} + |DX_{R_t}|_{s, \tilde{D}}|X_F|^2_{s, \tilde{D}} + |DX_F|_{s, \tilde{D}}|X_{R_t}|_{s, \tilde{D}} + |DX_F|_{s, \tilde{D}}|X_{R_t}|_{s, \tilde{D}} \]

\[ \leq (r - r_+)^{-1}(|X_F|^2_{s, \tilde{D}}|X_F|_{s, \tilde{D}} + |X_F|_{s, \tilde{D}}|X_F|^2_{s, \tilde{D}}) \]

\[ \leq c(r - r_+)^{-1}\mu^2 \Gamma(r - r_+)C(a - a_+). \]

Finally, we have \[|Y|^*_{s} \leq c\eta^{-2}|Y|^*_{s} \] for any vector field \( Y \), where \(| \cdot |^* \) stands for either \(| \cdot |_s \) or \(| \cdot |^2_{s, \tilde{D}} \).

Combining the above, there exists a constant \( c_0 \) such that

\[ |(\phi^t_F)^*[X_{R_t}, X_F]|_{\tilde{D}, \tilde{D}, D_2} \leq c_0(r - r_+)^{-1}\eta^{-2}\mu^2 \Gamma(r - r_+)C(a - a_+), \]

and

\[ |(\phi^t_F)^*[X_{R_t}, X_F]|^2_{\tilde{D}, \tilde{D}, D_2} \]

\[ \leq c_0(r - r_+)^{-1}\eta^{-2}\mu^2 \Gamma(r - r_+)C(a - a_+) + +c_0(r - r_+)^{-2}\eta^{-4}\mu^2 \Gamma^2(r - r_+)C^2(a - a_+) + c_0\mu^2 \leq M_+ \mu_+. \]

**Lemma 2.9.** Assume that

- **H8** \( c_0(\gamma \mu^2 + (r - r_+)^{-1}\eta^{-2}\mu^2 \Gamma(r - r_+))C(a - a_+) \leq \gamma + \mu_+ \),
- **H9** \( c_0(r - r_+)^{-1}\eta^{-2}\mu^2 \Gamma(r - r_+)^2C(a - a_+) + c_0(r - r_+)^{-2}\eta^{-4}\mu^2 \Gamma^2(r - r_+)^2C^2(a - a_+) + c_0\mu^2 \leq M_+ \mu_+. \)

Then we have

\[ |X_{P^t_a}|_{s, s, D_+} \leq \gamma + \mu_+, \]

\[ |X_{P^t_a}|^2_{s, s, D_+} \leq M_+ \mu_+. \]

**Proof.** Note that

\[ |X_{P^t_a}|^2_{s, s, D_+} \leq \left| X_{(P - R) \cdot \phi^t_F}_{s, s, D_+} \right|^2 + \int_0^1 (\phi^t_F)^*[X_{R_t}, X_F]dl \]

\[ \leq \left| X_{(P - R) \cdot \phi^t_F}_{s, s, D_+} \right|^2 + \left| (\phi^t_F)^*[X_{R_t}, X_F] \right|^2 \]

\[ \leq \gamma + \mu_+, \]

and

\[ |X_{P^t_a}|^2_{s, s, D_+} \leq \left| X_{(P - R) \cdot \phi^t_F}_{s, s, D_+} \right|^2 + \left| (\phi^t_F)^*[X_{R_t}, X_F] \right|^2 \]

\[ \leq M_+ \mu_+. \]

For Theorem A, the proof is the same as \( C(a - a_+) = \sum_{0 < i < \ell_+} \int_0^1 e^{-i(a - a_+)} \).
In subsection 1.1, for Theorem A’, we assume that the perturbation $P$ has the special form. So to accomplish one KAM step, we need to prove the new perturbation $P_+ \in A$.

**Lemma 2.10.** If the function $G_1(y, x, z, \bar{z}), G_2(y, x, z, \bar{z}) \in A$, then $G_1 \pm G_2, \{G_1, G_2\} \in A$.

**Proof.** See [12] for details. \qed

Note that

$$P_+ = P - R + \frac{1}{2!}\{(N, F), F\} + \frac{1}{2!}\{(P, F)F\} + \cdots ,$$

$$\{N, F\} = P_{0000} + \langle P_{0100}, y \rangle + \sum_{i} P_{i}^{011}z_{i}\bar{z}_{i}.$$

It is easy to see $P, R, F, \{N, F\} \in A$, according to the lemma above, we know $P_+ \in A$.

### 3. Proof of Main Theorem

#### 3.1. Iteration Lemma.

In this subsection, we will summarize section 3 and choose iteration sequences.

Set

$$r_\nu = r_0(1 - \sum_{i=1}^{\nu} \frac{1}{2i+1}),$$

$$a_\nu = a_0(1 - \sum_{i=1}^{\nu} \frac{1}{2i+1}),$$

$$\gamma_\nu = \gamma_0(1 - \sum_{i=1}^{\nu} \frac{1}{2i+1}),$$

$$M_\nu = M_0(2 - 2^{\nu+1}),$$

$$s_\nu = \eta_{\nu-1}s_{\nu-1}, \eta_\nu = \mu_\nu^{\frac{1}{2}}, \mu_\nu = \nu_{\nu-1}^{\frac{1}{2}},$$

$$K_\nu = (\log \frac{1}{\mu_{\nu-1}} + 1)^{3\alpha_1}, \ I_\nu = (\log \frac{1}{\mu_{\nu-1}} + 1)^{3\alpha_2},$$

$$D_{\nu, p, i}^{\nu-1}(\beta) = D_{\nu, p}(r_\nu + \frac{i-1}{8}(r_{\nu-1} - r_\nu), \frac{1}{\nu-1}s_{\nu-1}), \ i = 1, 2, \cdots , 8,$$

$$D_{\nu, p}^{\nu, (\beta)} = D_{\nu, p}(r_\nu + \frac{7}{8}(r_{\nu-1} - r_\nu), \beta), \ \beta > 0,$$

$$D_{\nu, p}^{\nu, (\beta)} = D_{\nu, p}(r_\nu + \frac{3}{4}(r_{\nu-1} - r_\nu), \frac{\beta}{2}), \ \beta > 0,$$

for all $\nu = 1, 2, \cdots$.

**Lemma 3.1.** If $\mu_0 = \mu_0(a, p, \bar{p}, r, s, m, \tau)$, or equivalently, $\mu_* = \mu_*(a, p, \bar{p}, r, s, m, \tau)$, is sufficiently small, then the KAM step described in Section 3 is valid for all $\nu = 0, 1, \cdots$. Consider the sequences:

$$\Lambda_\nu, H_\nu, N_\nu, e_\nu, \omega_\nu, M_\nu, P_\nu, \Phi_\nu,$$

$\nu = 1, 2, \cdots$. The following properties hold:
1) \( \Phi_\nu : D_{\frac{2}{\nu+3}} \times \Pi_{\nu+1} \rightarrow D_{\frac{2}{\nu+3}} \times \Pi_\nu \) is symplectic, and is of class \( C^2 \) smooth and depends Lipschitz-continuously on parameter,
\[
|D^i(\Phi_{\nu+1} - \text{id})|_{D_{\frac{2}{\nu+3}} \times \Pi_{\nu+1}} \leq \frac{\mu_\ast}{2^\nu},
\]
where \( i = 0, 1 \), \( \mu_\ast = \mu^{1-\sigma}, \sigma \in \left( \frac{3}{4}, 1 \right) \).

2) On \( \hat{D}_{\frac{2}{\nu+3}} \times \Pi_\nu \),
\[
H_{\nu+1} = H_\nu \circ \Phi_{\nu+1} = N_{\nu+1} + P_{\nu+1},
\]
where
\[
|\omega_{\nu+1} - \omega_\nu|_{\Pi_{\nu+1}} \leq \gamma_0 \frac{\mu_\ast}{2^\nu+1},
\]
\[
|\omega_{\nu+1} - \omega_0|_{\Pi_{\nu+1}} \leq \gamma_0 \mu_\ast,
\]
\[
|\Omega^{\nu+1} - \Omega^0|_{-6, \Pi_{\nu+1}} \leq \gamma_0 \frac{\mu_\ast}{2^\nu+1},
\]
\[
|\Omega^{\nu+1} - \Omega^0|_{-8, \Pi_{\nu+1}} \leq \gamma_0 \mu_\ast,
\]
\[
|\omega_{\nu+1} - \omega_\nu|_{\Pi_{\nu+1}} \leq M_0 \frac{\mu_\ast}{2^\nu+1},
\]
\[
|\omega_{\nu+1} - \omega_\nu|_{\Pi_{\nu+1}} \leq M_0 \mu_\ast,
\]
\[
|\Omega^{\nu+1} - \Omega^0|_{-8, \Pi_{\nu+1}} \leq M_0 \frac{\mu_\ast}{2^\nu+1},
\]
\[
|\omega_{\nu+1} - \omega_\nu|_{\Pi_{\nu+1}} + |\Omega^{\nu+1}|_{-8, \Pi_{\nu+1}} \leq M_{\nu+1},
\]
\[
|X_{\nu+1}|_{s_{\nu+1}} \leq \mu_{\nu+1},
\]
\[
|X_{\nu+1}|_{s_{\nu+1}} \leq M_{\nu+1} \mu_{\nu+1}.
\]

3)
\[
\Pi_{\nu+1} = \Pi_\nu \setminus \bigcup_{K_\nu < |k| \leq K_{\nu+1}, 0 < |l| \leq 2} R_{k,l}^{\nu+1},
\]
where
\[
\tilde{R}_{k,l}^{\nu+1} = \bigcup_{K_\nu < |k| \leq K_{\nu+1}, 0 < |l| \leq 2} R_{k,l}^{\nu+1} = \{ \xi \in \Pi_\nu : |\langle k, \omega_\nu(\xi) \rangle + \langle l, \Omega_\nu(\xi) \rangle| \leq \frac{\gamma_\nu}{A_{k,l}} \}.
\]

Proof. We first verify H1)-H9) for all \( \nu = 0, 1 \cdots \) to guarantee the KAM cycle in section 3. Generally, we choose \( \mu_0 \) sufficiently small so that H1)-H9) hold for \( \nu = 1 \) and let \( r_0, \gamma_0 = 1 \) for simplicity. Note that
\[
\mu_\nu = \mu_0 (\frac{7}{6})^\nu, \ s_\nu = s_0 (\frac{7}{6})^\nu, \ r_\nu - r_{\nu+1} = \frac{1}{2^{\nu+2}}, \ a_\nu - a_{\nu+1} = \frac{1}{2^{\nu+2}}.
\]
Let \( c_0 = \max\{ c_1, \cdots, c_0 \} \).

It follows from (3.1) that
\[
\log(n+4)! + (\nu+6)(n+4) \log 2 + 3\alpha_1 n \log (\log \frac{1}{\mu_\nu}) + 1
\]
\[
- \frac{K_{\nu+1}}{2^{\nu+2}} - \log \mu_\nu
\]
\[
\leq \log(n+4)! + (\nu+6)(n+4) \log 2 + 3\alpha_1 n \log (\log \frac{1}{\mu_\nu} + 2)
\]
Choose $\mu_0$ and $\alpha_1$ such that $(7/6)^{3\alpha_1-1} \geq 8$, so
\[
\int_{K_{\nu+1}}^\infty t^{n+3} e^{-\frac{2^n+2}{2^n+2} t} dt \leq (n+4)!2^{(n+3)(n+3)+2} e^{-K^{n+1}/2^n+2} \leq \mu_\nu,
\]
i.e., H1) holds. Similarly, choose $\alpha_2$ such that
\[
\int_{I_{\nu+1}}^\infty t^{\rho+4} e^{-\frac{2^{\rho+4}}{2^{\rho+4}} t} dt \leq (\rho+5)!2^{(\rho+5)(\rho+5)+2} e^{-K^{\rho+1}/2^{\rho+2}} \leq \mu_\nu,
\]
so H2) holds.
Then we consider H7),
\[
(3.2) \quad c_0 \mu_\nu (K_{\nu+1} + K_{\nu+1}^{\tau+1}) I^{c(\rho)}_{\nu+1}
\leq 2 c_0 \mu_\nu (\log \frac{1}{\mu_\nu} + 2)^{3\alpha_1(\tau+1)} (\log \frac{1}{\mu_\nu} + 2)^{3\alpha_2 c(\rho)}
\leq 2 c_0 (2 \log \frac{1}{\mu_0})^{3\alpha_1(\tau+1)+3\alpha_2 c(\rho)} \mu_0^{\frac{2}{7} \nu} (7/6)^{3\alpha_1(\tau+1)+3\alpha_2 c(\rho)}
\leq 2 c_0 (2 \log \frac{1}{\mu_0})^{3\alpha_1(\tau+1)+3\alpha_2 c(\rho)} \mu_0^{\frac{2}{7} \nu} (7/6)^{3\alpha_1(\tau+1)+3\alpha_2 c(\rho)}
\]
since $\mu_0^{\frac{2}{7} \nu} \leq \mu_0^{\frac{1}{2} \nu}$. Choose $\mu_0$ such that
\[
\mu_0^{\frac{2}{7} \nu} (7/6)^{3\alpha_1(\tau+1)+3\alpha_2 c(\rho)} \leq \frac{1}{3}
\]
So H7) holds. Also
\[
4 c_0 \mu_0^{\frac{1}{2} \nu} < \frac{1}{2^{\nu+2}}
\]
implies H6).
One can find that, in the proof above, choosing $\rho = 1$, $c(\rho) = \frac{5}{2}$, H1), H2), H6), H7)(or H7)') also hold.
Note that
\[
\Gamma_\nu = \Gamma(r_\nu - r_{\nu+1}) \leq \sum_{0 < k \leq K_{\nu+1}} |k|^{4\tau+n+4} e^{-|k|/2^{\nu+6}}
\leq \int_1^\infty \lambda^{4\tau+n+4} e^{-\frac{\lambda}{2^{\nu+6}}} d\lambda
\leq (4[\nu] + n + 4)!2^{(n+6)(\nu+4)},
\]
and
\[
C_\nu = C(a_\nu - a_{\nu+1}) \leq \sum_{0 < |i| \leq I_{\nu+1}} |i|^{c(\rho)+10} e^{-(a_\nu - a_{\nu+1})|i|}
\leq \int_1^\infty \lambda^{c(\rho)+10} e^{\frac{\lambda}{2^{\nu+6}}} d\lambda
\leq (c(\rho+10))!2^{(\nu+2)(c(\rho)+10)}.
\]
Let
\[
a^* = (4[\nu] + n + 4)!((c(\rho) + 10))2^{(4\tau+n+6)+2(c(\rho)+10)},
\]
\[ b^* = 4r + n + c(\rho) + 14. \]

Then
\[ \Gamma_\nu C_\nu \leq a^* 2^{b^* \nu}. \]

It is clear that, H3)-H5) and H8)-H9) are equivalent to
\[ \begin{align*}
32c_0 a^* \mu_\nu 2^{(b^* + 1)\nu} &\leq 1, \\
\frac{1}{5} c_0 a^* \frac{b^*}{6} 2^{b^* \nu} &\leq 1.
\end{align*} \]

Observing H8), H9) yields
\[ \begin{align*}
c_0 \frac{\gamma_\nu \mu_\nu^2}{\gamma_{\nu+1} \mu_{\nu+1}} + c_0 \frac{\gamma_\nu \mu_\nu^2 \Gamma_\nu C_\nu}{(r_\nu - r_{\nu+1}) \eta_\nu^2 \gamma_{\nu+1} \mu_{\nu+1}} &\leq 2c_0 \frac{b^*}{6} + 8c_0 2^{(b^* + 1)\nu}, \\
\frac{M_\nu \mu_\nu^2}{M_{\nu+1} \mu_{\nu+1}} + c_0 \frac{M_\nu \mu_\nu^2 \Gamma_\nu C_\nu}{(r_\nu - r_{\nu+1}) \eta_\nu^2 M_{\nu+1} \mu_{\nu+1}} + c_0 \frac{M_\nu \mu_\nu^2 \Gamma_\nu C_\nu}{(r_\nu - r_{\nu+1}) \eta_\nu^2 M_{\nu+1} \mu_{\nu+1}} &\leq \frac{1}{3}.
\end{align*} \]

Combining the above, we only prove
\[ \begin{align*}
8c_0 2^{(b^* + 1)\nu} &\leq 1, \\
\end{align*} \]

In fact
\[ 8c_0 a^* \mu_\nu 2^{(b^* + 1)\nu} \leq 8c_0 a^* (\mu_0^\frac{1}{6}) (1 + (\nu/6)) 2^{(b^* + 1)\nu} \leq 8c_0 a^* \mu_0^\frac{1}{6} 2^{b^* + 1)\nu}. \]

Choose \( \mu_0 \) sufficiently small such that (3.7) is verified. And by (3.3)-(3.6), we know that H3)-H5), H8), H9) hold.

From (3.7), we have
\[ c_0 \mu_\nu \Gamma_\nu \leq \frac{1}{10} \mu_\nu^{(5/6)} \leq \frac{\mu_*}{2^{\nu}}, \]

where \( \mu_* = \mu_0^{1-\sigma}, \sigma \geq \frac{1}{2}, \) so
\[ c_0 \mu_\nu \Gamma_\nu \leq \frac{\mu_*}{2^{\nu}}, \quad c_0 \frac{M_\nu}{\gamma_\nu} \mu_\nu \Gamma_\nu \leq \frac{M_\nu \gamma^\sigma \mu_*}{\gamma_\nu} 2^{\nu+1}, \]

for all \( \nu = 0, 1 \cdots. \)

We now proceed the iterative schemes. First, it is clear to see the part 1) and 2) of the lemma hold for \( \nu = 1. \) Then assume that for some \( \nu_*, 1) \) and 2) are true for all \( \nu = 1, \cdots, \nu_* \). Then by Lemmas in section 2 and arguments above, we claim that the KAM iteration is valid for \( \nu = \nu_* + 1. \)

For the proof of Theorem A, we choose \( c(\rho) = \frac{5}{2}, \) and
\[ \begin{align*}
C_\nu &\leq \int_1^\infty \lambda^6 e^{-\lambda \rho^\alpha} d\lambda \leq 6\lambda^6 \rho^{6\nu + 12}, \\
a^* &\leq (4[\tau] + n + 4)! 6! 2^{6(4r+n+2)},
\end{align*} \]
\[ b^* = 4\tau + n + 12. \]

The other proof is the same.

It is easy to see that 3) holds for \( \nu = 1 \). For \( \nu > 1 \), from lemma 2.7 we have

\[ \Pi \nu = \{ \xi \in \Pi : |\langle k, \omega \nu(\xi) \rangle + \langle l, \Omega \nu(\xi) \rangle | > \gamma \nu \}, \]

for all \( 0 < |k| \leq K_\nu, \ 0 < |l| \leq 2 \} \).

Set

\[ \tilde{R}_{\nu+1} = \{ \xi \in \Pi : |\langle k, \omega \nu(\xi) \rangle + \langle l, \Omega \nu(\xi) \rangle | \leq \gamma \nu \}, \]

\( K_\nu \leq |k| \leq K_{\nu+1}, \ 0 < |l| \leq 2 \} \).

Then

\[ \Pi_{\nu+1} = \{ \xi \in \Pi : |\langle k, \omega \nu(\xi) \rangle + \langle l, \Omega \nu(\xi) \rangle | > \gamma \nu \}, \]

for all \( 0 < |k| \leq K_{\nu+1} \ 0 < |l| \leq 2 \} \)

\[ = \Pi_\nu \backslash \tilde{R}_{\nu+1}. \]

The lemma is now complete. \( \square \)

### 3.2. Convergence

In this section, we prove the convergence of the sequences from subsection 3.1. Let

\[ \Psi \nu = \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_\nu : D_{\bar{a}, p} 	imes \Pi_{\nu+1} \to D_0, \]

\[ H \circ \Psi \nu = H_\nu = N_\nu + P_\nu, \]

for \( \nu = 0, 1, \cdots \), which satisfy all properties described in Lemma 3.1. By the iteration lemma, we claim that \( \Psi \nu \) is convergent to a function \( \Psi^\infty \in C^1(D(\frac{r_0}{2}, 0) \times \Pi_\nu) \), in \( C^2(D(\frac{r_0}{2}, 0) \times \Pi_\nu) \), where

\[ \Pi_\nu = \cap_{\nu=0}^\infty \Pi_\nu. \]

Then \( \Pi_\nu \) is a Cantor set, and \( \{ \Psi_\xi : \xi \in \Pi_\nu \} \) is a Lipschitz continuous family of real analytic, symplectic transformations on \( D(\frac{r_0}{2}, 0) \).

By part 2) of lemma 3.1, it is easy to see that \( \omega \nu, \Omega \nu \) converge uniformly on \( \Pi_\nu \).

We denote \( \omega_\ast, \Omega_\ast \) as their limits, respectively. Then, we have

\[ |\omega_\ast - \omega_0|_\Pi_\ast = O(\gamma_0 \mu_\ast), \]

\[ |\Omega_\ast - \Omega^0|_{-\delta, \Pi_\ast} = O(\gamma_0 \mu_\ast), \]

\[ |\omega_\ast - \omega_0|_{\Pi_\ast} = O(M_0 \mu_\ast), \]

\[ |\Omega_\ast - \Omega^0|_{-\delta, \Pi_\ast} = O(M_0 \mu_\ast). \]

Thus, on \( G_\ast, |N_\nu - N^\infty_\ast| \) converges uniformly to \( 0, \) and

\[ P_\nu = H \circ \Psi \nu - N_\nu, \]

converges uniformly to

\[ P^\infty = H \circ \Psi^\infty - N^\infty. \]

Clearly, these limits above are uniformly Lipschitz continuous in \( \xi \in \Pi_\ast \) and real analytic in \( (x, y, z, \bar{z}) \in D_{\bar{a}, p}(\frac{r_0}{2}, \frac{r_0}{2}). \)

Note that

\[ |X_{P_\nu}|_{a^\nu, \bar{p}} \leq \gamma_\nu \mu_\nu, \]

\[ |X_{P_\nu}|_{c^\nu, \bar{p}} \leq M_\nu \mu_\nu. \]
It means that for any $\xi \in \Pi_\nu$, $m \in \mathbb{Z}_+^n$, $q, \bar{q} \in \mathbb{Z}_+^n$ with $2|j| + |q + \bar{q}| \leq 2$,\hspace{1cm} (3.9)
\begin{equation}
|\partial_\gamma^n \partial_{\xi}^2 \partial_{\bar{\xi}}^2 P_{\nu}|_{D_\nu} \leq \gamma_{\nu} \mu_{\nu}.
\end{equation}
Since the right hand side of the above converges to 0 as $\nu \to \infty$, we have that
\begin{equation}
\partial_{\xi}^2 \partial_{\bar{\xi}}^2 P_{\infty} |_{(y, x, z) = 0} = 0
\end{equation}
for all $x \in \mathbb{T}^n$, $\xi \in \Pi_\nu$, $m \in \mathbb{Z}_+^n$, $q, \bar{q} \in \mathbb{Z}_+^n$ with $2|j| + |q + \bar{q}| \leq 2$.

4. Measure Estimates

It remains to show the measure estimate. We induct a lemma first. Recall that
\begin{align}
|\omega_\nu|^C + |\Omega^\nu|_{-\delta, \Pi_\nu} & \leq M_\nu \leq 2M_\nu, \\
|\omega_\nu - \omega_0|, |\Omega^\nu - \Omega^0|_{-\delta, \Pi_\nu} & \leq \gamma_0 \mu_\nu, \\
R_{kl}^{\nu+1} = \{\xi \in \Pi_\nu : |\langle k, \omega_\nu(\xi) \rangle + \langle l, \Omega_\nu(\xi) \rangle| \leq \frac{\gamma_\nu}{A_{k,l}}\}.
\end{align}

Lemma 4.1. Define $R_{kl}(\alpha_{k,ij}) := \{\xi \in \Pi_\nu : |\langle k, \omega_\nu(\xi) \rangle + \langle l, \Omega_\nu(\xi) \rangle| \leq \alpha_{k,ij}\}$, where $k \in \mathbb{Z}^n \setminus \{0\}$, $i, j$ denote the sites of non-zero components of $l$, and $\alpha_{k,ij}$ is a positive constant depending on $k, i, j$.

If $|\Omega^\nu|_{-\delta, \Pi_\nu} \leq M$ and $|k| \geq 16M$, then for fixed $l \in \Lambda$, we have
\begin{equation}
|R_{kl}(\alpha_{k,ij})| \prec c(n) \frac{\alpha_{k,ij}}{|k|},
\end{equation}
where $\prec$ means $\cdot \leq c\cdot$, $c$ is a positive constant independent of any parameters, and $c(n) > 0$ depends only on $n$.

Proof. Fix $\varpi \in \{-1, 1\}^n$ such that $|k| = k \cdot \varpi$, and write $\omega_\nu = a\varpi + \rho$, $\varpi \perp \rho$. Let $a$ be a variable. Then for $t > s$,
\begin{align}
\langle k, \omega_\nu \rangle &= \langle k, \omega_\nu \rangle |_s - \langle k, \omega_\nu - \omega_0 \rangle |_s \geq |k|(t - s) - \frac{1}{2} |k|(t - s) = \frac{1}{2} |k|(t - s), \\
|\langle l, \Omega^\nu \rangle| &\leq |l|_\delta |\Omega^\nu|_{-\delta, \Pi_\nu}(t - s) \leq 2M |l|_\delta |t - s| \leq \frac{1}{4} |k|(t - s),
\end{align}
since $|l|_{-\delta} \leq 2$. Hence,
\begin{align}
\langle k, \omega_\nu \rangle + \langle l, \Omega^\nu \rangle &\geq \frac{1}{4} |k|(t - s).
\end{align}
This can deduce (4.3). The details can be seen in lemma 5 of [20] or lemma 5.6 of [21].

Lemma 4.2. There exists a positive constant $c_\nu$ such that
\begin{equation}
|\langle l, \Omega^\nu \rangle| \geq \begin{cases} 
\frac{c_\nu |l|}{d}, & l \in \Lambda_+; \\
|c_\nu |l| - 2\gamma_0 \mu_\nu|, & l \in \Lambda_-,
\end{cases}
\end{equation}
for all $\xi \in \Pi_\nu$.

Proof. Consider the case $l \in \Lambda$. By A1) and A2) we have
\begin{equation}
\frac{|\langle l, \Omega^0 \rangle|}{|l|_d} \to 1,
\end{equation}
uniformly in $\xi \in \Pi_0$, i.e., there is a small positive $\varepsilon$,
\begin{equation}
|\langle l, \Omega^0 \rangle| \geq (1 - \varepsilon)|l|_d.
\end{equation}
For $l \in \Lambda_+$, since $|l|_\delta \leq \langle l \rangle_d$,
\[ |\langle l, \Omega'^0 - \Omega^0\rangle| \leq |l|_\delta |\Omega' - \Omega^0|_{-\delta, \pi} \leq \langle l \rangle_d \gamma_0 \mu_*, \]
and
\[ |\langle l, \Omega'^0\rangle| \geq |\langle l, \Omega^0\rangle| - |\langle l, \Omega' - \Omega^0\rangle| \\
\geq (1 - \varepsilon - \gamma_0 \mu_*) \langle l \rangle_d. \]
Choosing $\varepsilon$ such that $1 - \varepsilon - \gamma_0 \mu_* > c_7$, we prove the first part of the result.

For $l \in \Lambda_-, |l|_\delta \leq 2$, we have
\[ |\langle l, \Omega'^0 - \Omega^0\rangle| \leq |l|_\delta |\Omega' - \Omega^0|_{-\delta, \pi} \leq 2 \gamma_0 \mu_*, \]
and
\[ |\langle l, \Omega'^0\rangle| \geq |\langle l, \Omega^0\rangle| - |\langle l, \Omega' - \Omega^0\rangle| \\
\geq (1 - \varepsilon) \langle l \rangle_d - 2 \gamma_0 \mu_. \]

**Lemma 4.3.** If $R^{\nu+1}_{kl} \neq \emptyset$, and $\gamma_0 \leq \frac{c_7}{2}$, then there exists a positive constant $c_8$, such that
\[ \langle l \rangle_d \leq c_8|k|, \]
for $l \in \Lambda$.

**Proof.** If $R^{\nu+1}_{kl}$ is not empty, then for $l \in \Lambda_+$ or $l \in \Lambda_-$, there exists some $\xi \in \Pi_\nu$, such that $|\langle k, \omega_\nu \rangle + \langle l, \Omega' \rangle| < \gamma_0 \langle l \rangle_d$ or $|\langle k, \omega_\nu \rangle + \langle l, \Omega'^0 \rangle| < \gamma_0$, respectively.

Consider $l \in \Lambda_+$,
\[ |k|_{\omega_\nu} \geq |\langle l, \Omega'^0\rangle| - \gamma_0 \langle l \rangle_d \geq \frac{c_7}{2} \langle l \rangle_d, \]
i.e.,
\[ \langle l \rangle_d \leq \frac{4}{c_7} |k|_{\omega_0|\Pi}, \]
with $c_8 \geq \frac{4}{c_7} |\omega_0|_{\Pi}$.

For $l \in \Lambda_-$, we have
\[ |k|_{\omega_\nu} + \gamma_0 \geq |\langle l, \Omega' \rangle| \geq |c_7 \langle l \rangle_d - 2 \gamma_0 \mu_*|, \]
that is
\[ \langle l \rangle_d \leq \frac{1}{c_7} (\gamma_0 (2 \mu_* + 1) + 2 |k|_{\omega_0|\Pi}) \leq \frac{2}{c_7} \gamma_0 + |k|_{\omega_0|\Pi}, \]
\[ \leq \frac{2}{c_7} (\gamma_0 |k| + |k|_{\omega_0|\Pi}), \]
\[ \leq \frac{2}{c_7} (\gamma_0 + |\omega_0|_{\Pi}) |k|, \]
with $c_8 \geq \frac{2}{c_7} (\gamma_0 + |\omega_0|_{\Pi})$. \qed

Then we decompose $\tilde{R}_{\nu+1}$ into two parts
\[ \tilde{R}_{\nu+1} = \bigcup_{K_\nu \leq K_{\nu+1}} (\bigcup_{l \in \Lambda_-} R^{\nu+1}_{kl} + \bigcup_{l \in \Lambda_+} R^{\nu+1}_{kl}) \]
\[ = \bigcup_{K_\nu \leq K_{\nu+1}} I + \bigcup_{K_\nu \leq K_{\nu+1}} II. \]
Case I: \( l \in \Lambda_\cdot \). From lemma 4.1, we have
\[
|R_{kl}^{\nu+1}| < c(n) \frac{\gamma_\nu}{|k|(1 + |k|^{\nu+1})^{\nu}} < c(n) \frac{\gamma_0}{|k|(1 + |k|^{\nu+1})^{\nu}}
\]
for \( |k| \geq 16M, \ l \in \Lambda_\cdot \).

**Lemma 4.4.** The following holds for \( |k| \) sufficiently large,
\[
|I| < c(n) \frac{\gamma_0 |k|^{c_1(\rho)}}{1 + |k|^\tau},
\]
where \( c_1(\rho) \) is a constant depending only on \( \rho \).

**Proof.** Since \( l \in \Lambda_\cdot, \ (l)_d = |i|^d - |j|^d \), by lemma 4.3, we have
\[
| |i|^d - |j|^d | \leq c_8 |k|, \ |k| \geq 16M.
\]
For any fixed \( i \in \mathbb{Z}_1^d \), we have
\[
\text{card}\{j : ||i|^d - |j|^d | \leq c_8 |k|\} \\
\leq \text{card}\{j : |j| \leq |i|\} + \text{card}\{j : |j|^d \leq c_8 |k| + |i|^d\} \\
\leq i + \text{card}\{j : j^d \leq (c_8 + 1)|k|^d\} \leq |i|^{c_1(\rho)} + c|k|^{c_1(\rho)}|i|^{c_1(\rho)} \\
\leq (c + 1)|k|^d |i|^{c_1(\rho)}.
\]
Then for the fixed \( i \),
\[
|R_{kl}^{\nu+1}| \leq |R_{kl}^{\nu+1}| \text{card}\{j : ||i|^d - |j|^d | \leq c_8 |k|\} \leq c(n) \frac{\gamma_0 |k|^{c_1(\rho)} |i|^{c_1(\rho)}}{(1 + |k|^\tau)|i|^{c_1(\rho)}},
\]
and
\[
|I| \leq \sum_{i \in \mathbb{Z}_1^d} |R_{kl}^{\nu+1}| \leq c(n) \frac{\gamma_0 |k|^{c_1(\rho)}}{(1 + |k|^\tau)} \sum_{i \in \mathbb{Z}_1^d} \frac{1}{|i|^{c_1(\rho)} - c_1(\rho)}
\]
Choose \( c(\rho) \geq c_1(\rho) + \rho \) such that \( \sum_{i \in \mathbb{Z}_1^d} \frac{1}{|i|^{c_1(\rho)} - c_1(\rho)} \) converges to a constant.

Case I: \( l \in \Lambda_+ \). By lemma 4.1, we have
\[
|l| \leq |l|^{\nu+1} |< c(n) \frac{\gamma_\nu}{1 + |k|^\tau} < c(n) \frac{\gamma_0}{(1 + |k|^\tau)},
\]
for \( |k| \geq 16M, \ l \in \Lambda_+ \).

**Lemma 4.5.** The following holds
\[
|II| < c(n) \frac{|k|^{2c_1(\rho)}}{1 + |k|^\tau}
\]
for \( |k| \) sufficiently large.

**Proof.** Since \( l \in \Lambda_+, \) we have
\[
(l)_d = |i|^d \text{ or } |j|^d \text{ or else.}
\]
Then
\[
\text{card}\{l : (l)_d \leq c_8^{-\frac{\tau}{d}} |k|\} \\
\leq \text{card}\{l : |i| \leq c_8^{-\frac{\tau}{d}} |k|^{\frac{1}{d}}\} + \text{card}\{l : |j| \leq c_8^{-\frac{\tau}{d}} |k|^{\frac{1}{d}}\} \\
\leq 3c_8^{-\frac{\tau}{d}} |k|^{2c_1(\rho)}.
\]
\[
\square
\]
hence
\[ |II| < c(n) \gamma_0 \frac{|k|^{2+\frac{1}{2}(\rho)}}{1 + |k|^{\tau}}. \]

\[ \square \]

Combining the two cases, we have
\[ |\cup_{K_{\nu}<|k|<K_{\nu+1}} (I + II)| \leq c\gamma_0 \sum_{K_{\nu}<|k|<K_{\nu+1}} \frac{|k|^{2+\frac{1}{2}(\rho)}}{1 + |k|^{\tau}}. \]

From lemma 3.1 3), we know that
\[ \Pi \setminus \Pi_* \subset \cup_{\nu=0}^{\infty} \tilde{R}_{\nu+1}. \]
Then there exists a \( \nu_* \) such that \( K_{\nu_*} \) sufficiently large and by choosing \( \tau \), we have
\[ |\Pi \setminus \Pi_*| \leq c\gamma_0 \sum_{\nu=0}^{\infty} \sum_{K_{\nu}<|k|<K_{\nu+1}} \frac{|k|^{2+\frac{1}{2}(\rho)}}{1 + |k|^{\tau}} \]
\[ \leq O(\gamma_0), \]

i.e. \( |\Pi \setminus \Pi_*| \to 0 \), as \( \gamma_0 \to 0 \).

For the proof of Theorem A, we only show the lemmas and the key point.

**Lemma 4.6.** There exists a positive constant \( c_9 \) such that
\[ \langle l, \Omega' \rangle \geq \begin{cases} c_9 |l|_d, & l \in \Lambda_+; \\ c_9 |l|_d - 2\gamma_0 \mu_*, & l \in \Lambda_- , \end{cases} \]
for all \( \xi \in \Pi_\nu \).

**Lemma 4.7.** If \( R_{kl}^{\nu+1} \neq \emptyset \), and \( \gamma_0 \leq \frac{c_9}{2} \), then there exists a positive constant \( c_{10} \), such that
\[ \langle l \rangle_d \leq c_{10} |k|, \]
for \( l \in \Lambda \).

Consider the case \( l \in \Lambda_- \). From lemma 4.1, we have
\[ |R_{kl}^{\nu+1}| < c(n) \frac{\gamma_\nu}{|k|(1 + |k|^{\tau+1})^{\frac{1}{2}}} < c(n) \frac{\gamma_0}{|k|(1 + |k|^{\tau+1})^{\frac{1}{2}}} \]
for \( |k| \geq 16M \), \( l \in \Lambda_- \).

**Lemma 4.8.** The following holds
\[ |I| < c(n) \frac{\gamma_0}{1 + |k|^{\tau}} \]
for fixed \( k \geq 16M \).

Proof. Since \( l \in \Lambda_- \), \( \langle l \rangle_d = |i^d - j^d| \), by lemma 4.7, we have
\[ |i^d - j^d| \leq c_{10} |k|, \quad |k| \geq 16M. \]

For any fixed \( i \in \mathbb{N}_+ \), we have
\[ \text{card}\{j : |i^d - j^d| \leq c_{10} |k|\} \]
\[ \leq \text{card}\{j : j \leq i\} + \text{card}\{j : j^d \leq c_{10} |k| + i^d\} \]
\[ \leq i + \text{card}\{j : j^d \leq (c_{10} + 1)|k|^d\} \leq i + c|k|^\frac{d}{2} \]
\[ \leq (c_{10} + 1)|k|^\frac{d}{2}. \]
Then for fixed $i$,

$$|R_{kl(i)}^{\nu+1}| \leq |R_{kl}^{\nu+1}| \text{card}\{j : |i^d - j^d| \leq c_8|k|\} \leq c(n) \frac{\gamma_0|k|^\frac{d}{2}}{(1 + |k|^\tau)^{\frac{d}{2}}}$$

and

$$|I| \leq \sum_{i \geq 1} |R_{kl(i)}^{\nu+1}| \prec c(n) \frac{\gamma_0}{(1 + |k|^\tau)^{\frac{d}{2}}} \sum_{i \geq 1} \frac{1}{i^2} \prec c(n) \frac{\gamma_0}{(1 + |k|^\tau)^{\frac{d}{2}}}.$$  

The case $l \in \Lambda_+$ is the same, we omit the details.

Then for the case $\rho = 1$, we set $c(\rho) = \frac{5}{2}$, and the measure estimate is the following:

$$|\Pi \setminus \Pi_*| \leq \sum_{\nu_*} \tilde{R}_{\nu+1} + \sum_{\nu_*} \sum_{K_{\nu_*} < |k| < K_{\nu+1}} \tilde{R}_{\nu+1} \leq O(\gamma_0),$$

i.e. $|\Pi \setminus \Pi_*| \to 0$, as $\gamma_0 \to 0$. $\square$

5. Applications to PDEs

In this section, we shall give two applications of our result: 1. The nonlinear Schrödinger equations with Dirichlet boundary condition. 2. The Klein-Gordon equations with exponential nonlinearity subject to periodic boundary condition. Due to the work of [19], we can see that both equations possess weaker spectral asymptotics in higher dimension. We remark that those results can be extended to the nonlinear Schrödinger equations (or the Klein-Gordon equations with exponential nonlinearity) with other boundary conditions.

5.1. Nonlinear Schrödinger Equations. Consider the Schrödinger equations

$$\sqrt{-1} u_t - \Delta u + V(\xi)u = \frac{\partial F}{\partial \bar{u}}, \quad x \in \Omega, \quad t \in \mathbb{R}$$

with the linear boundary condition

$$u|_{\partial \Omega} = 0,$$

where $\Omega \subset \mathbb{R}^m$ is a bounded domain with smooth boundary $\partial \Omega$, $V$ is a real analytic function on $\xi$, $F$ is real analytic and $\frac{\partial F}{\partial \bar{u}} = f(|u|^2)u$ with $f(0) = 0$, $f'(0) \neq 0$, $\xi$ is defined as before.

Consider the eigenvalues problem as following:

$$-\Delta \phi = \lambda \phi,$$

Weyl's asymptotic formula asserts

$$\lambda_j^* \sim C_m \left(\frac{j}{|\Omega|}\right)^{2/m}, \quad k \to \infty,$$

where $|\Omega|$ is the volume of $\Omega$, $C_m = (2\phi)^2 B_m^{-2/m}$ is the Weyl constant. See [19].

For fixed $\xi \in \Pi$, the eigenvalues of the operator $-\Delta + V(\xi)$ has the following asymptotic behavior, as $j \to \infty$,

$$\lambda_j \sim cj^{2/m} + o(j^{2/m}), \quad j \to \infty,$$
where \( c = \frac{C_m}{(1/x_{max})^2} \).

Denote that, the eigenvalues of operator \( A = -\Delta + V(x, \xi) \) under the boundary condition satisfy

\[
\hat{\omega}_j = \lambda_j(\xi), \ j = \{1, \cdots, n\} \\
\hat{\Omega}_j = \lambda_j(\xi), \ j \in N \equiv \mathbb{N} \setminus \{1, \cdots, n\}.
\]

Rewrite (5.1) as

\[
u_t = \sqrt{-1} \frac{\partial H}{\partial \bar{\nu}},
\]

with the associated Hamiltonian

\[
(5.4) \quad H = \langle Au, u \rangle + \int_{\Omega} F(u) \, dx,
\]

where \( \langle \cdot, \cdot \rangle \) denote the inner product in \( L^2 \).

Consider \( u(t, x) = \sum_{j \geq 1} q_j(t) \phi_j(x) \), where \( \phi_j(x) \) is the eigenvector corresponding to eigenvalue \( \lambda_j \). Then (5.1) can be transformed as

\[
(5.5) \quad \dot{q}_j = \sqrt{-1}(\lambda_j q_j + \frac{\partial G}{\partial \bar{q}_j}), \ j \geq 1,
\]

where \( G = \int_{\Omega} F(u) \, dx \), and (5.4) becomes

\[
(5.6) \quad H = \sum_{j \geq 1} \lambda_j q_j \bar{q}_j + G(q, \bar{q}, \phi, \bar{\phi}).
\]

To apply Theorem A, we need to consider the regularity of the nonlinearity \( G \).

Introduce the Hilbert space \( l^2 = (\cdots, q_{-j}, \cdots, q_j, \cdots) \) denotes all bi-infinite square summable sequences with complex coefficients. \( L^2 \) denotes all square-integrable complex-valued functions on \( \mathbb{T}^n \), by the inverse Fourier transform

\[
F : q \rightarrow \mathcal{F}q = \frac{1}{2\sqrt{2\pi}} \sum_j q_j \phi_j(x).
\]

Let \( a \geq 0, \ p \geq 0 \). The subspace \( l^{a,p} \subset l \) consist of all bi-infinite sequence with finite form

\[
((q|^{a,p})^2 = |q_0|^2 + \sum_j |q_j||j|^{2p}e^{2a|j|} < \infty.
\]

Denote function space \( W^{a,p} \subset L^2 \) normed by \( |\mathcal{F}q|^{a,p} = |q|^{a,p} \). For \( a > 0 \), the space \( W^{a,p} \) may be analytic functions bounded in the complex strip \( |\text{Im}x| < a \) with trace functions on \( |\text{Im}x| = a \) belonging to the usual Sobolev space \( W^p \). (see [17])

**Lemma 5.1.** For any fixed \( a \geq 0, \ p \geq 0 \), the gradient \( G_q \) is real analytic as a map in a neighborhood of the origin with

\[
|G_q|^{a,p} \leq c(|q|^{a,p})^3.
\]

**Proof.** Consider a function \( u \in W^{a,p} \) with the norm \( |u|^{a,p} = |q|^{a,p} \). The function \( f(|u|^2)u \) also belongs to \( W^{a,p} \) with

\[
|f(|u|^2)u|^{a,p} \leq c(|u|^{a,p})^3,
\]

in a sufficiently small neighbourhood of the origin, for details, see [17]. \( \square \)
Consider the subspace $L^{2\alpha-a,2\rho-p} \subset L^{\alpha,\rho} \subset L^{a,p}$, and the corresponding function subspaces $W^{2\alpha-a,2\rho-p} \subset W^{\alpha,\rho} \subset W^{a,p} \subset L^{2}$, where $0 < a < \bar{a}$, $0 \leq p \leq \bar{p}$.

Let $u = \sum_{j} q_{j} \phi_{j}(x)$. If $u \in W^{\alpha,\rho} \subset W^{a,\bar{p}}$, then according to Lemma 5.3, we have

\[ |G_{\bar{q}}|^{\alpha,\rho} \leq c(|q|^{a,\bar{p}})^{3}, \]

and

\[ |q|^{2\alpha-a,2\rho-p} \leq C < \infty. \]

By the Schwarz inequality,

\[ (|q|^{\alpha,\rho})^{2} = \sum_{j} |q_{j}|^{2} |j|^{2\rho} e^{2\bar{\alpha}|j|} = \sum_{j} (|q_{j}|^{2p} e^{a|j|})(|q_{j}|^{2\rho-p} e^{(\alpha-a)|j|}) \]

\[ \leq \sqrt{\sum_{j} |q_{j}|^{2} |j|^{2p} e^{2\bar{\alpha}|j|}} \sqrt{\sum_{j} |q_{j}|^{2} |j|^{2\rho-p} e^{(2\alpha-a)|j|}} \leq C|q|^{a,p}. \]

Combining with (5.7) yields

\[ |G_{\bar{q}}|^{\alpha,\rho} \leq c(|q|^{a,p})^{3}, \]

where $c$ depends on $a$, $\bar{a}$, $p$, $\bar{p}$. The regularity of $X_{\bar{G}}$ follows from the regularity of $\bar{G}_{\bar{q}}$, $G_{\bar{q}}$. Similarly, its Lipschitz semi-norm with respect to $\xi$ follows. Choose $s$ such that the perturbation satisfies A3).

In the following, we introduce the usual action-angle variables $(\varphi, I) \in T^{n} \times \mathbb{R}^{n}$ and infinite-dimensional vector $z$, $\bar{z} \in \mathcal{L}^{a,p} \times \mathcal{L}^{a,p}$ (see [12]). Then, systems (5.10) becomes

\[
\left\{
\begin{array}{l}
\dot{\varphi}_{j} = \bar{\omega}_{j}(\xi) + G_{\bar{j}l}, \\
\dot{I}_{j} = -G_{\bar{j}l}, \\
\dot{\bar{z}}_{j} = (\bar{\Omega}_{j}(\xi) \bar{z}_{j} + G_{\bar{z}j}), \\
\dot{\bar{z}}_{j} = -\bar{\Omega}_{j}(\xi) \bar{z}_{j} + G_{\bar{z}j},
\end{array}
\right.
\]

associated to the Hamiltonian

\[ H = \langle \bar{\omega}(\xi), I \rangle + \sum_{j \in \mathbb{N}} \bar{\Omega}_{j}(\xi) \bar{z}_{j} \bar{z}_{j} + G(I, \varphi, z, \bar{z}), \]

where $(x, y, z, \bar{z})$ lie in complex neighborhood

\[ D_{a,p}(s, r) = \{ (x, y, z, \bar{z}) : \| x \| < r, \| y \| < s^{2}, \| z \|^{a,p}, \| \bar{z} \|^{a,p} < s \} \]

of $T^{n} \times \{ 0 \} \times \{ 0 \} \subset T^{n} \times \mathbb{R}^{n} \times \mathcal{L}^{a,p} \times \mathcal{L}^{a,p}$, and with respect to the symplectic form

\[ \sum_{i=1}^{n} dx_{i} \wedge dy_{i} + \sum_{j \in \mathbb{N}} dz_{j} \wedge d\bar{z}_{j}. \]

Note that the coefficient of $j^{2/m}$ can always be normalized to one by rescaling. It is easily to prove that the frequences $\bar{\omega}(\xi)$, $\bar{\Omega}_{j}(\xi)$ satisfy A1), A2) after rescaling. Using Theorem A, we get the following result:

**Theorem 5.1** For any $0 < \gamma \leq 1$, there exists a Cantor set $\Pi_{\gamma} \subset \Pi$, with $|\Pi \setminus \Pi_{\gamma}| = O(\gamma)$, such that for any $\xi \in \Pi_{\gamma}$, Schrödinger equations (5.7) subjected to the boundary condition (5.4) admits a family of small amplitude quasi-periodic solutions $u_{\ast}(t, x)$ with respect to time $t$. Moreover $u_{\ast}(t, x) \in W^{a,p}_{0}$ for fixed $t$. 

5.2. the Klein-Gordon equations with exponential nonlinearity. In this subsection, using Theorem A, we show that the existence of quasi-periodic solutions for the Klein-Gordon equations with exponential nonlinearity subject to periodic boundary conditions

\[
\begin{align*}
  u_{tt} - \Delta u + M_\xi u &= u e^{\alpha u^2}, \quad x \in T^d, \quad t \in \mathbb{R}, \\
  u(t, x_1 + 2\pi, x_2, \ldots, x_d) &= \cdots = u(t, x_1, \ldots, x_d + 2\pi) \\
  &= u(t, x_1, \ldots, x_d),
\end{align*}
\]

(5.8)

where \(M_\xi\) is a real Fourier multiplier,

\[
M_\xi \sin jx = \xi_j \sin jx, \quad \xi_j \in \mathbb{R}^\rho.
\]

Exponential-type nonlinearities have been considered in several physical models (see, e.g., [13] on a model of self-trapped beams in plasma). S. Ibrahim, M. Majdoub and N. Masmoudi [14] has obtained the existence of global solutions for the Klein-Gordon equations with exponential nonlinearity in two dimension. Recently, S. Ibrahim, M. Majdoub, N. Masmoudi and K. Nakanishi [15] investigated the existence and asymptotic completeness of the wave operators for the nonlinear Klein-Gordon equation with a defocusing exponential nonlinearity in two space dimensions. In this paper, we will prove the existence of quasi-periodic solutions for the Klein-Gordon equations with exponential nonlinearity.

Let

\[
\frac{\partial F}{\partial u} = u(e^{\alpha u^2} - 1).
\]

Then we can rewrite the Klein-Gordon equations with exponential nonlinearity as

\[
(5.9) \quad u_{tt} - \Delta u + (M_\xi - 1)u = \frac{\partial F}{\partial u},
\]

In order to obtain the existence of quasi-periodic solutions, we need assume:

1) Then operator \(A = -\Delta + M_\xi - 1\) under the boundary condition (5.8) admits the spectrum

\[
\omega_j = \lambda_j |j| + \xi_j - 1, \quad 1 \leq j \leq n,
\]

\[
\Omega_j = \lambda_j |j| + o(|j|^{-1}), \quad j \in \mathbb{Z}^d \setminus \{i_1, \cdots, i_n\}.
\]

The corresponding eigenfunctions \(\phi_j = \frac{1}{\sqrt{\omega_j}} e^{-\langle j, x \rangle} \) form a basis.

Without loss of generality, we assume 0 \(\in \{i_1, \cdots, i_n\}\).

Introduce \(v = u_t\), and (5.9) reads

\[
\begin{align*}
  \left\{ \begin{array}{l}
    u_t = v, \\
    v_t = -Au - \frac{\partial F}{\partial u},
  \end{array} \right.
\end{align*}
\]

with the associated to the Hamiltonian

\[
H = \frac{1}{2} \langle v, v \rangle + \langle Au, u \rangle + \int_{T^d} F(u)dx.
\]

Consider the coordinate transformation:

\[
\begin{align*}
  u &= \sum_{j \in \mathbb{Z}^d} \frac{q_j}{\sqrt{\lambda_j}} \phi_j(x), \quad v = \sum_{j \in \mathbb{Z}^d} \sqrt{\lambda_j} p_j \phi_j(x),
\end{align*}
\]
where \( \phi_j(x) \) is the eigenvector corresponding to eigenvalue \( \lambda_j \). Then (5.9) can be transformed to

\[
(5.10) \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}, \quad \dot{q}_j = \frac{\partial H}{\partial p_j}, \quad j \in \mathbb{Z}^p,
\]

and the corresponding Hamiltonian is

\[
H = \frac{1}{2} \sum_j \lambda_j (p_j^2 + q_j^2) + \int_{\mathbb{R}^p} F(\sum_j q_j) \, dx.
\]

Introduce complex variables \( w_j = \frac{1}{\sqrt{2}}(q_j + \sqrt{-1}p_j), \bar{w}_j = \frac{1}{\sqrt{2}}(q_j - \sqrt{-1}p_j) \). Then the perturbation of equation (5.10) reads

\[
(5.11) \quad G(w, \bar{w}) = \int_{\mathbb{R}^p} F(\sum_j w_j \dot{\phi}_j + \bar{w}_j \bar{\dot{\phi}}_j) \, dx.
\]

Next using usual action-angle variables \((\varphi, I) \in T^p \times \mathbb{R}^p \) and infinite-dimensional vector \( w = z_j, \bar{w} = \bar{z}_j, \) \( j \neq t_1, \cdots, t_n \). Then, systems (5.10) becomes

\[
\begin{align*}
\dot{\varphi}_j &= \omega_j(\xi) + G_{\xi_j}, \\
\dot{z}_j &= \Omega_j(\xi)z_j + G_{z_j}, \\
\end{align*}
\]

associated to the Hamiltonian

\[
(5.12) \quad H = \langle \omega(\xi), I \rangle + \sum_{j \in Z^p_1} \Omega_j(\xi)z_j \bar{z}_j + G(I, \varphi, z, \bar{z}),
\]

where \((x, y, z, \bar{z})\) lies in complex neighborhood

\[
D_{a,p}(s, r) = \{(x, y, z, \bar{z}) : |\text{Im} \, x| < r, |y| < s^2, |z|^{a,p}, |\bar{z}|^{a,p} < s\}
\]

of \( \mathbb{T}^n \times \{0\} \times \{0\} \subset \mathbb{T}^n \times \mathbb{R}^n \times \mathcal{L}^{a,p} \times \mathcal{L}^{a,p} \), and with respect to the symplectic form

\[
\sum_{i=1}^n dx_i \wedge dy_i + \sum_{j \in Z^p_1} dz_j \wedge d\bar{z}_j.
\]

To apply our main theorem, we need to verify the perturbation \( P \) satisfies A3)' and A4). (see section 1)

Introduce the Hilbert space \( l^2 = (\cdots, l_n, \cdots)_{n \in \mathbb{Z}^p} \) and \( L^2 \) denotes all square-integrable complex-valued functions on \( \mathbb{T}^p \), by the inverse Fourier transform

\[
\mathcal{F} : \quad q \rightarrow \mathcal{F}q = \frac{1}{\sqrt{2\pi^p}} \sum_j q_j \phi_j(x).
\]

Let \( a \geq 0, \quad p \geq 0 \). The subspace \( l^{a,p} \subset l^2 \) consist of sequences with finite form

\[
(|q|^{a,p})^2 = |q_0|^2 + \sum_j |q_j|^2 |j|^{2p} e^{2|j|a} < \infty.
\]

Denote function space \( W^{a,p} \subset L^2 \) normed by \( |\mathcal{F}q|^{a,p} = |q|^{a,p} \). For \( a > 0 \), the space \( W^{a,p} \) may be analytic functions bounded in the complex strip \( |\text{Im}x| < a \) with trace functions on \( |\text{Im}x| = a \) belonging to the usual Sobolev space \( W^p \). (see [17])

Consider the subspace \( l^{2a-a,p} \subset l^{a,p} \subset l^{a,p} \), and \( u \in W^{2a-a,p+c} \subset W^{a,p+c} \subset W^{a,p+c} \) respectively with the norm above, where \( a > 0, \quad p \geq 0 \).
Lemma 5.2. For any fixed a ≥ 0, p ≥ 0, the gradient $G_q$ is real analytic as a map in a neighborhood of the origin with

$$|G_q|^a p + c \leq c(|q|^a p)^3, \tag{5.13}$$

where $c \leq \frac{1}{2}$. 

Proof. Let $q \in l^{a,p}$. Then $u = \sum q_j \phi_j$ is in $W^{a,p+c}$ with $|u|^{a,p+c} = |q|^{a,p}$. One can see [21] for details. □

Lemma 5.3. If $u \in W^{2\bar{a}-a,p+c} \subset W^{\bar{a},p+c}$, we have

$$|G_q|^\bar{a} p + c \leq c(|q|^a p)^\frac{1}{2}, \tag{5.14}$$

where c depends on a, $\bar{a}$, p, $\bar{p}$. 

Proof. According to Lemma 5.1 and the Schwarz inequality,

$$\left(|q|^{a,p}\right)^2 = \sum_j |q_j|^2 |j|^2 p e^{2j} \leq \sum_j |q_j|^2 |j|^2 p e^{(2\bar{a}-a)|j|} \leq \left(\sum_j |q_j|^2 |j|^2 p e^{(2\bar{a}-a)|j|}\right) \cdot \sqrt{\sum_j |q_j|^2 |j|^2 p e^{(2\bar{a}-a)|j|}}.$$

Since $q \in l^{2\bar{a}-a,p}$, we have

$$\sqrt{\sum_j |q_j|^2 |j|^2 p e^{(2\bar{a}-a)|j|}} \leq c(a, \bar{a}, p, \bar{p}),$$

then (5.14) is verified. □

Remark The regularity of perturbation $X_G$ depends not only on the order of $u$ but also on the weight of the function space $W^{a,p}$. In previous, one assumed $u \in W^{a,\bar{p}}$ and proved that the quasi-periodic solution $u_*$ of the perturbed system has the finite $a, \bar{p}$-norm for fixed $t$. In our paper, we demand $u \in W^{2\bar{a}-a,\bar{p}}$, and prove that the perturbed quasi-periodic solution $u_*$ is in $W^{a,\bar{p}}$. The more rigorous regularity of the perturbation is to overcome the continuous spectrum. This is just intrinsic.

The regularity of $X_G$ follows from the regularity of $G_q$, $G_q$. Similarly, its Lipschitz semi-norm follows. Choosing the parameter $s$, A3) is verified.

Then we show that $P$ has the special form. Since $e^{au^2}$ is real analytic in $a$, making use of $q(x) = \sum_{j \in Z^d} q_n \phi_n(x)$, $F$ can be rewritten as

$$F(w, \bar{w}) = \sum_{\alpha, \beta} F_{\alpha, \beta} w^{\alpha} \bar{w}^{\beta},$$

hence

$$G(w, \bar{w}) = \sum_{\alpha, \beta} G_{\alpha, \beta} \int_{T^d} e^{\sqrt{-1}(\sum_{j \in Z^d} \alpha_j - \beta_j)x_j)} \, dx,$$

that is

$$G_{\alpha, \beta} = 0, \text{ if } \sum_j (\alpha_j - \beta_j) j \neq 0.$$
Denote \( k = (k_1, \cdots, k_n) \), \( k_j = \alpha_j - \beta_j \), \( 1 \leq j \leq n \). Then

\[
(5.15) \quad G = \sum_{j \in \mathbb{Z}^n} \sum_{(\alpha_j - \beta_j)j \neq 0} G_{\alpha \beta} w^\alpha \tilde{w}^\beta = \sum_{j=1}^{n+1} (\alpha - \beta)_j \sum_{j \in \mathbb{Z}^n} (\alpha_j - \beta_j)j \neq 0 \sum_{j=1}^{n} \alpha_j \epsilon_j q_j \tilde{q}_j - \sum_{j=1}^{n} \beta_j \epsilon_j \equiv P,
\]

that is, \( P \in \mathcal{A} \). Therefore, Hamiltonian PDEs in \( \mathbb{T}^n \) not containing explicitly the space and time variable do have the special form. The above argument is similar to the paper [12].

Thus we have verified A1')-A4'). Then we have the following result for the Klein-Gordon equations with exponential nonlinearity.

**Theorem 5.2** Under the assumptions I), for any \( 0 < \gamma \leq 1 \), there exists a Cantor set \( \Pi \subset \Pi \), with \( |\Pi \setminus \Pi| = O(\gamma) \), such that for any \( \xi \in \Pi \), wave equations (5.9) subjected to the boundary condition (5.3) admits a family of small amplitude quasi-periodic solutions \( u_*(t, x) \) with respect to time \( t \). Moreover \( u_*(t, x) \in W^{\alpha, \beta}_k \) for fixed \( t \).

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