Abstract

We give a very short proof of the Melvin-Morton conjecture relating the colored Jones polynomial and the Alexander polynomial of knots. The proof is based on explicit evaluation of the corresponding weight systems on primitive elements of the Hopf algebra of chord diagrams which, in turn, follows from simple identities between four-valent tensors on the Lie algebra $sl_2$ and the Lie superalgebra $gl(1|1)$. This shows that the miraculous connection between the Jones and Alexander invariants follows from the similarity (supersymmetry) between $sl_2$ and $gl(1|1)$.

1 Introduction

It is well known that the Jones polynomial $V$ of knots can be constructed by working with the standard two-dimensional representation of the quantum group $sl_2$. An arbitrary, $d$-dimensional, representation of $sl_{2,q}$ gives rise to a generalization of $V$, the so-called colored Jones invariant $V^d$ (see [10, 19] for more details). Morton and Strickland [17] proved that $V^d$ is determined by all the cabling of the Jones polynomial thus showing that $V^d(K, t)$ can be computed combinatorially by formulas generalizing the famous skein relation for $V(K, t)$.

The Alexander-Conway polynomial is another famous knot invariant that can be computed by using a skein relation. In [14] Melvin and Morton studied pieces of $V^d(K, t)$ appearing in a certain power series expansions and conjectured that the Alexander polynomial of a knot is determined by its colored Jones invariant. They proved that the coefficients $a_{in}$ of the power series expansion

$$V^d(K, e^z) = \sum_{i, n \geq 0} a_{in}(K)d^iz^n$$

(1)

of $V^d$ in variables $d$ and $z = \log t$ are Vassiliev knot invariants of order $\leq n$ and that $a_{in} = 0$ if $i > 2n$. They formulated the following Melvin-Morton conjecture (later Morton [16] proved it for torus knots).
Theorem 1.1

(i) The coefficient “matrix” \((a_{in})\) of the expansion (1) is upper triangular, i.e.
\[ a_{in} = 0 \quad \text{for} \quad i > n \] (2)

(ii) the leading (“diagonal”) term
\[ V_0(K, z) = \sum_{n \geq 0} a_{nn}(K) z^n \] (3)
is the inverse of the renormalized Alexander polynomial of \(K\), i.e.
\[ V_0(K, z) \cdot \frac{z}{e^{z/2} - e^{-z/2}} \Delta(K, e^z) = 1. \] (4)

Rozansky \cite{20} derived relations (2) and (4) from Witten’s \cite{24} path integral interpretation of the Jones invariant using quantum field theory tools. Bar-Natan and Garoufalidis \cite{4} showed that it is enough to establish (2) and (4) on the level of Vassiliev weight systems and proved the Melvin-Morton conjecture by finding combinatorial expressions for the weight systems corresponding to the leading terms \(a_{nn}\) of the expansion (1) and the coefficients of the Alexander-Conway polynomial. In \cite{12} A. Kricker gave another proof based on the characterization \cite{11} of the subspace generated by the Alexander-Conway weight systems in terms of the cabling operations. In \cite{21} we gave a different, simpler proof using the recursion relations from \cite{6} and \cite{7} for computing the Jones and the Alexander-Conway weight systems.

All these proofs are to various extent based on some miraculous cancellations or coincidences occurring in computations of the Jones and the Alexander-Conway weight systems. This leaves one puzzled wondering whether Theorem 1.1 could have been discovered without the prior experimental work, wishing for a more conceptual explanation of the miracles.

In this note we attempt to address this question. We give a very short proof of the identities (2) and (4) on the level of weight systems that does not use any involved computations. The idea is to restrict everything to primitive elements of the Hopf algebra of chord diagrams and to use the fact that the colored Jones and the Alexander-Conway invariants are the universal Vassiliev invariants coming from the Lie algebra \(sl_2\) and the Lie superalgebra \(gl(1|1)\) (see \cite{7}). Among other things this allows us to ignore the difference between the framed and unframed versions of the colored Jones invariant. All that is needed to compute the corresponding weight systems on the primitive subspace are simple identities \cite{30} and \cite{36} between invariant tensors of order four on \(sl_2\) and \(gl(1|1)\). The superalgebra \(gl(1|1)\) is the fermionic counterpart of \(sl_2\) (or rather \(gl_2 = sl_2 \oplus C\)) which explains the similarity between relations \cite{30} and \cite{36}. However, algebraically \(gl(1|1)\) is much simpler than \(sl_2\), it is solvable and not simple as a Lie superalgebra. This is precisely the reason why the Alexander invariant is weaker than the colored Jones invariant being equivalent to its leading part \cite{3}.

The outline of the paper is as follows. In Section 2 we review the basic facts on Vassiliev invariants and the Hopf algebra \(A\) of chord diagrams. In Section 3 we establish some facts on primitive elements of the algebra \(A\). A construction assigning to a Lie superalgebra with

\footnote{In \cite{3} and \cite{4} these identities were used to derive recursive formulas for the universal weight systems with values in \(U(sl_2)\) and \(U(gl(1|1))\) respectively.}
an invariant inner product a series of Vassiliev weight systems is recalled in Section 4. In Section 5 the Melvin-Morton conjecture is reduced to two statements on the values of the Jones and the Alexander-Conway weight system on the primitive elements of $\mathcal{A}$ which are proved in Sections 6 and 7 respectively.

2 Vassiliev invariants and weight systems

All necessary information on Vassiliev invariants, chord diagrams and weight systems can be found, for example, in [3, 9, 1]. The purpose of this section is to establish notation and introduce some lesser standard terminology.

Let $\mathcal{V}_n$ denote the space of Vassiliev knot invariants of order $\leq n$ with values in a field $k$ (we usually assume that $k = \mathbb{C}$). The space $\mathcal{V} = \bigcup \mathcal{V}_n$ of all Vassiliev invariants is a filtered Hopf algebra with the comultiplication induced by the operation of connected sum of knots and the antipode given by the orientation reversal.

A chord diagram of order $n$ is an oriented circle (called the Wilson line) with $n$ disjoint pairs of points (chords) up to an orientation preserving diffeomorphism of the circle. Denote by $\mathcal{A}_n$ the space generated by the set $\mathcal{D}_n$ of all chord diagrams with $n$ chords modulo the following four-term relations

$$
\begin{align*}
\begin{array}{c}
\gamma_1 - \gamma_2 + \gamma_3 - \gamma_4 = 0.
\end{array}
\end{align*}
$$

(5)

The space $\langle \mathcal{D}_n \rangle$ modulo (5) and the following one-term relations

$$
\begin{align*}
\begin{array}{c}
\circ \otimes \circ = 0
\end{array}
\end{align*}
$$

(6)

will be denoted by $\bar{\mathcal{A}}_n$.

The spaces

$$
\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{A}_n \quad \text{and} \quad \bar{\mathcal{A}} = \bigoplus_{n \geq 0} \bar{\mathcal{A}}_n
$$

have structures of $\mathbb{Z}$-graded commutative and co-commutative Hopf algebras with the product given by the operation of the connected sum of diagrams (which is well-defined modulo four-term relations) and the coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ by the formula

$$
\Delta(D) = \sum_{E \subset D} E \otimes (D \setminus E).
$$

The projection $\mathcal{A} \to \bar{\mathcal{A}}$ splits and $\bar{\mathcal{A}}$ can be embedded into $\mathcal{A}$ as a Hopf subalgebra so that

$$
\mathcal{A} = \bar{\mathcal{A}}[\Theta],
$$

where $\Theta$ is the unique chord diagram with one chord.
Elements of the space $W_n = A_n^*$ (resp. $\overline{W}_n = \overline{A}_n^*$) are called weight systems (resp. strong weight systems) of order $n$. We will view weight systems (resp. strong weight systems) of order $n$ as linear functions on $\langle D_n \rangle$ vanishing on the combinations of diagrams in four-term relations (5) (resp. one- and four-term relations). The spaces

$$W = \oplus W_n \quad \text{and} \quad \overline{W} = \oplus \overline{W}_n$$

are also $\mathbb{Z}$-graded Hopf algebras. The product of $W_1 \in W_m$ and $W_2 \in W_n$ is defined by

$$(W_1 \cdot W_2)(D) = \sum_{E \subset D, |E| = m} W_1(E)W_2(D \setminus E), \quad \text{where } D \in D_{m+n}.$$

The fundamental result of the theory of Vassiliev invariants is the theorem of Kontsevich [9, 3] that the Hopf algebra $\overline{W}$ is canonically isomorphic to the adjoint graded algebra of the filtered Hopf algebra $V$. Let $V^f$ denote the space of Vassiliev invariants of framed knots. Kontsevich proved that

$$V_n / V_{n-1} \simeq \overline{W}_n, \quad \text{and} \quad V_n^f / V_{n-1}^f \simeq W_n$$

by explicitly constructing splitting maps

$$Z : W_n \to V_n^f \quad \text{and} \quad \overline{Z} : \overline{W}_n \to V_n. \quad (7)$$

The embedding $\overline{A}_n \hookrightarrow A_n$ gives a canonical deframing projection

$$W_n \to \overline{W}_n \quad (8)$$

allowing to construct Vassiliev invariants of unframed knots from weight system that give a priori only invariants of framed knots.

Many knot invariants such as the Alexander and Jones polynomials are not Vassiliev invariants, whereas their coefficients (after an appropriate change of variables) are. Therefore, we can associate with such invariants not just one weight system, but a sequence of weight systems $w_0, w_1, \ldots, w_n, \ldots$ where $w_n \in W_n$. We call elements of

$$\hat{W} = \prod_{n \geq 0} W_n$$

Vassiliev series and write them as formal sums $W = w_0 + w_1 + w_2 + \ldots$ (or sometimes as formal power series $W = \sum w_n z^n$, where $z$ is a formal parameter). Vassiliev series can be viewed as linear functionals on the space generated by chord diagrams $D = \bigoplus_{n \geq 0} (D_{n})$. For $D \in D_n$ and $W \in W$ we define $W(D) = w_n(D)$. The space $\hat{W}$ also has an algebra structure

$$(W \cdot W')(D) = \sum_{E \subset D} W(E)W'(D \setminus E). \quad (9)$$

A Vassiliev series $W \in \hat{W}$ is called multiplicative if

$$W(D_1 \cdot D_2) = W(D_1) \cdot W(D_2) \quad \text{for any } D_1, D_2 \in D.$$
3 Feynman diagrams and primitive elements of the Hopf algebra of diagrams

By the structure theorem on commutative and co-commutative Hopf algebras \cite{[15]}, the algebra \( \mathcal{A} \) of diagrams is isomorphic to the polynomial algebra \( S(\mathcal{P}) \), where \( \mathcal{P} \) is the space of primitive elements

\[ \mathcal{P} = \{ a \in \mathcal{A} : \Delta(a) = 1 \otimes a + a \otimes 1 \}. \]

The primitive space \( \mathcal{P} \) of \( \mathcal{A} \) can be very conveniently described in terms of Feynman diagrams.

A Feynman diagram of order \( n \) is a graph with \( 2n \) vertices of degrees 1 or 3 with a cyclic ordering on the set of its univalent (external) vertices (or legs) and on each set of three edges meeting at a trivalent (internal) vertex. Drawing Feynman diagrams we put legs on a circle (Wilson line) and assume everything oriented in the counterclockwise direction.

**Remark 3.1** Feynman diagrams are called Chinese Character diagrams in \cite{[3]}, but they are indeed Feynman diagrams arising in the perturbative Chern-Simons-Witten quantum field theory \cite{[24], [2]}.

Let \( \mathcal{F}_n \) denote the set of all Feynman diagrams with \( 2n \) vertices (up to the natural equivalence of graphs with orientations) having at least one external vertex on each connected component. The set \( \mathcal{D}_n \) of chord diagrams with \( p \) chords is a subset of \( \mathcal{F}_n \).

**Proposition 3.2** \((\cite{3})\) The embedding \( \mathcal{D}_n \hookrightarrow \mathcal{F}_n \) induces an isomorphism of the space \( \mathcal{A}_n \) of chord diagrams with \( n \) chords modulo four-term relations \((\cite{3})\) and the space \( (\mathcal{F}_n)/\langle D_Y - D_{||} + D_X \rangle \) of Feynman diagrams modulo the following three-term relations

\[ \begin{align*}
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{relation10.png}
\end{array}
\end{align*} \]

In addition, the following local relations hold for internal vertices of Feynman diagrams

\[ \begin{align*}
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{relation11.png}
\end{array}
\end{align*} \]

(Drawing relations between Feynman diagrams we always assume that all diagrams entering the same relation are identical except for the depicted fragments.)
This proposition implies, in particular, that the space $\langle F_n \rangle / \langle D_Y - D_|| + D_X \rangle$ is canonically dual to $W_n$ and that any weight system $W : D_n \to k$ extends to $F_n$ by the rule

$$W(D_Y) = W(D_||) - W(D_X).$$

The realization of the algebra $A$ in terms of Feynman diagrams endows it with a natural filtration consistent with the grading

$$A_n^{(0)} \subset A_n^{(1)} \subset \ldots \subset A_n^{(\ell)} \subset \ldots = A_n^{(2n)} = A_n,$$

where $A_n^{(\ell)}$ is the subspace of $A_n$ generated by Feynman diagrams with $\leq \ell$ legs.

From the definition of the co-product in $A$ it follows (see e.g. [3]) that the primitive subspace $P \subset A$ is spanned by connected Feynman diagrams and that the homogeneous component $P_n$ of the primitive space of $\tilde{A}$ coincides with $P_n$ for $n \neq 1$, and $P_1 = \langle \Theta \rangle$. However, $P_1 = 0$ whereas $P_1 = \langle \Theta \rangle$.

Denote by $P_n^{(\ell)}$ the space $P \cap A_n^{(\ell)}$.

The following proposition leaves us with fewer diagrams in the spanning set. Let us call a subgraph $T$ of a graph $F$ a hanging tree if $T$ is a tree with more than one edge and becomes disconnected from $F$ after removal of one of its leaves (terminal vertices). In particular, a hanging tree can be a connected component of $F$.

**Theorem 3.3**

(i) The space $P$ has a basis consisting of connected Feynman diagrams without hanging trees.

(ii) For $n \geq 2$ the filtration

$$0 \subset \ldots \subset P_n^{(l-1)} \subset P_n^{(l)} \subset \ldots \subset P_n$$

stabilizes at the term $P_n^{(n)}$, i.e.

$$P_n^{(n)} = P_n^{(n+1)} = \ldots = P_n.$$

(iii) The leading quotient space $P_n^{(n)} / P_n^{(n-1)}$ of (13), $n \geq 2$, is generated by the $n$-spoke wheel diagram

$$w_n =$$

First we will prove several simple lemmas.
Lemma 3.4 The following identities hold in the algebra $A$ of diagrams

\[
\begin{align*}
\includegraphics[width=0.15\textwidth]{example_11} &= 2 \\
\includegraphics[width=0.15\textwidth]{example_12} &= 2 \includegraphics[width=0.15\textwidth]{example_13} = \includegraphics[width=0.15\textwidth]{example_14}.
\end{align*}
\] (15) and (16)

Proof. These identities follow immediately from the three-term relations (10) and (11). $\square$

Lemma 3.5 Let $F$ be a Feynman diagram with $\ell$ legs and a planar hanging tree $T$ with $d$ legs that does not intersect the edges of $F \setminus T$. Then

\[ F = 2^{-(d-1)} F', \]

where $F'$ is the diagram with $\ell - d + 1$ legs obtained from $F$ by replacing the hanging tree $T$ with the chain of $d - 1$ “bubbles”. For example,

\[
\begin{align*}
\includegraphics[width=0.15\textwidth]{example_21} &= \frac{1}{4} \\
\includegraphics[width=0.15\textwidth]{example_22}.
\end{align*}
\]

Proof. The statement follows from Lemma 3.4 by induction on $d$. $\square$

Lemma 3.6 A Feynman diagram $F$ with $\ell$ legs and a hanging tree belongs to $A^{(\ell-1)}$.

Proof. Repeatedly applying the three-term relation (10) and the relation (11), we can transform $F$ to a Feynman diagram $\tilde{F}$ with a planar hanging tree not intersecting the rest of $\tilde{F}$ so that $F = \pm \tilde{F} + G$, where $G$ is a linear combination of diagrams with $\leq \ell - 1$ legs. Therefore $G \in A^{(\ell-1)}$ and, by 3.5, $\tilde{F} \in A^{(\ell-1)}$. $\square$

Lemma 3.7 Let $F$ be a connected Feynman diagram of order $n$ with $\ell$ legs. Then the first Betti number of $F$ is

\[ h_1(F) = n - \ell + 1. \]
Proof. The diagram $F$ has $2n - \ell$ trivalent vertices and, consequently, $e = \frac{1}{2}(\ell + 3(2n - \ell)) = 3n - \ell$ edges. Since $F$ is connected, its Euler characteristic is $\chi(F) = 1 - h^1(F) = 2n - e = \ell - n$, and $h^1(F) = n - \ell + 1$. 

Proof of Theorem 3.3. Choose a basis $B^{(\ell)}$ in $P^{(\ell)}_n$ represented by connected Feynman diagrams with $\leq \ell$ legs such that $B^{(\ell)} \subset B^{(\ell + 1)}$. If $F \in B^{(\ell + 1)} \setminus B^{(\ell)}$ then it has exactly $\ell$ legs and, by Lemma 3.6, cannot have hanging trees. Therefore, $\bigcup_{\ell \geq 0} B^{(\ell)}$ is a basis of $P_n$ satisfying (i).

Let $m$ be the largest $\ell$ for which $P^{(\ell)}_n \neq P^{(\ell - 1)}_n$ and $F \in B^{(m)}_n \setminus B^{(m - 1)}_n$. Since $F$ is a connected diagram with $m$ legs and without hanging trees and $n \geq 2$, the diagram $F$ cannot be a tree and its first Betti number $h^1(F)$ must be at least 1. By Lemma 3.7 $h^1(F) = n - m + 1 \geq 1$ from where $m \leq n$. This gives part (ii) of the theorem.

In the case $F \in B^{(n)}$ the diagram $F$ has exactly one cycle and, since $F$ does not contain hanging trees, all the $n$ legs of $F$ are adjacent to this cycle. Therefore, $F$ is an $n$-spoked wheel diagram with a possibly different ordering of legs than that of the standard wheel diagram $w_n$. By applying relations (10) and (11) we can change the ordering of legs of $F$ at the cost of adding extra diagrams with less than $n$ legs. Therefore $F = \pm w_n$ modulo $P^{(n - 1)}_n$ which proves (iii). 

Remark 3.8 It follows from (iii) that $\dim P^{(n)}_n / P^{(n - 1)}_n \leq 1$. Chmutov and Varchenko proved in [6] a stronger statement that this dimension is equal to one if and only if $n$ is even, but we will not need this for the proof of the Melvin-Morton conjecture.

4 Weight systems coming from Lie algebras

Here we recall a construction that assigns a Vassiliev series to every Lie (super)algebra with an invariant inner product. (For more details see [22].)

Let $L$ be a Lie (super)algebra with an $L$-invariant inner product $b : L \otimes L \to k$. To each Feynman diagram $F$ with $m$ univalent vertices we assign a tensor

$$T_{L,b}(F) \in L^\otimes m$$

as follows.

The Lie bracket $[\ ,\ ] : L \otimes L \to L$ can be considered as a tensor in $L^* \otimes L^* \otimes L$. The inner product $b$ allows us to identify the $L$-modules $L$ and $L^*$, and therefore $[\ ,\ ]$ can be considered as a tensor $f \in (L^*)^\otimes 3$ and $b$ gives rise to an invariant symmetric tensor $c \in V \otimes V$.

For a Feynman diagram $F$ denote by $T$ the set of its trivalent vertices, by $U$ the set of its univalent (exterior) vertices, and by $E$ the set of its edges. Taking $|T|$ copies of the tensor $f$ and $|E|$ copies of the tensor $c$ we construct a new tensor

$$\tilde{T}_L(F) = \left( \bigotimes_{v \in T} f_v \right) \otimes \left( \bigotimes_{\ell \in E} c_\ell \right)$$
which is an element of the tensor product
\[ \mathcal{L}^F = \left( \bigotimes_{v \in T} (L^*_{v,1} \otimes L^*_{v,2} \otimes L^*_{v,3}) \right) \otimes \left( \bigotimes_{\ell \in E} (L_{\ell,1} \otimes L_{\ell,2}) \right), \]
where \((v, i), i = 1, 2, 3\), mark the three edges meeting at the vertex \(v\) (consistently with the cyclic ordering of these edges), and \((\ell, j), j = 1, 2\), denote the endpoints of the edge \(\ell\).

Since \(c\) is symmetric and \(f\) is completely antisymmetric, the tensor \(\tilde{T}_L(F)\) does not depend on the choices of orderings.

If \((v, i) = \ell\) and \((\ell, j) = v\), there is a natural contraction map \(L^*_{v,i} \otimes L_{\ell,j} \rightarrow k\). Composition of all such contractions gives us a map
\[ \mathcal{L}^F \rightarrow \bigotimes_{u \in U} L = L^\otimes m, \text{ where } m = |U|. \]

The image of \(\tilde{T}_L(F)\) in \(L^\otimes m\) is denoted by \(T_{L,b}(F)\) (or usually just by \(T_L(F)\)).

(In the case of Lie superalgebras we also have to take special care of signs. See [22] for details.)

For example, for the diagrams
\[ C = \quad \text{and} \quad K = \quad (17) \]
we have
\[ T_L(C) = \sum_{ij} b^{ij} e_i \otimes e_j = c, \]
the Casimir element corresponding to the inner product \(b\),
\[ T_L(B) = \sum b^{is} b^{lj} b^{kp} b^{lq} f_{skl} f_{pq} e_i \otimes e_j, \]
the tensor in \(L \otimes L\) corresponding to the Killing form on \(L\) under the identification \(L^* \simeq L\), and
\[ T_L(K) = \sum b^{is} b^{jp} b^{gr} b^{kt} f_{npq} f_{tqr} e_i \otimes e_j \otimes e_k \otimes e_{\ell}, \]
where \(f_{jk} = \sum b^{is} f_{sjk}\) are the structure constants of \(L\) in a basis \(e_1, e_2, \ldots\).

Tensor \(T_L(F)\) is invariant with respect to the \(L\)-action on \(L^\otimes m\) and its image \(V_L(F)\) in the universal enveloping algebra \(U(L)\) belongs to the center \(Z(U(L)) = U(L)^L\) and does not depend on the place where we cut the Wilson line to obtain a linear ordering of the external vertices of \(F\).

Therefore, for every Lie algebra \(L\) with an invariant inner product there exists a natural Vassiliev series
\[ V_L : \mathcal{D} \rightarrow Z(U(L)) \]
which is called the universal weight system corresponding to \(L\) and \(b\). It is universal in the sense that any Vassiliev series \(V_{L,R}\) constructed using a representation \(R\) of the Lie algebra \(L\) (see [3]) is an evaluation of \(V_L\):
\[ V_{L,R}(D) = \text{Tr}_R(V_L(D)). \]
Proposition 4.1 Both the universal Vassiliev series $V_L$ and its deframing $\overline{V}_L$ are multiplicative, i.e.

$$V_L(D_1 \cdot D_2) = V_L(D_1)V_L(D_2) \quad \text{and} \quad \overline{V}_L(D_1 \cdot D_2) = \overline{V}_L(D_1)\overline{V}_L(D_2).$$

Proof. Multiplicativeness of $V_L$ follows immediately from its construction. Deframing $\overline{W}$ of a multiplicative Vassiliev series $W$ is again multiplicative, since the embedding $\mathcal{A}_n \hookrightarrow \mathcal{A}_n$ is a homomorphism of algebras and $W$ and $\overline{W}$ coincide on all primitive Feynman diagrams $F$ in $\mathcal{P}$ but one

$$\overline{W}(F) = \begin{cases} W(F) & \text{if } F \neq \Theta, \\ 0 & \text{if } F = \Theta. \end{cases}$$

Therefore $\overline{V}_L$ is also multiplicative. \qed

Corollary 4.2 The colored Jones and the Alexander-Conway weight system are multiplicative.

Proof. This is so because the deframed universal $sl_2$ and $gl(1|1)$ weight systems are the Vassiliev series corresponding to the colored Jones and Alexander-Conway invariants respectively (see Sections 5 and 7). \qed

5 Reduction of the Melvin-Morton conjecture to Feynman diagrams

There are two constructions of knot invariants from a semi-simple Lie algebra $L$ with an invariant inner product $b$ and a representation $R$. The Reshetikhin-Turaev construction \cite{19} based on quantum groups gives invariants $I_{L,R}$ and $\overline{I}_{L,R}$ (of framed and unframed knots, resp.) with values in $\mathbb{Z}[t, t^{-1}]$. The second way is to apply Kontsevich’s construction to the Vassiliev series $V_{L,R} = \text{Tr}_R(V_L)$ and $\overline{V}_{L,R}$ discussed in the previous section.

These two constructions are equivalent. The coefficients of the power series expansions of $\overline{J}_{L,R}(z) = \overline{I}_{L,R}(e^z)$ are Vassiliev invariants, and the corresponding series of weight systems coincide with $\overline{V}_{L,R}$ (see \cite{4, 8}). Conversely, Kontsevich’s splitting maps \cite{6} applied to $\overline{V}_{L,R}$ and $V_{L,R}$ gives the sequences of the coefficients of $\overline{J}_{L,R}$ and $J_{L,R}$ (see \cite{3, 13}).

A Vassiliev invariant is called canonical if it belongs to the image of Kontsevich’s map $Z : \mathcal{W}_n \to \mathcal{V}_n$. A formal power series $\sum_{n \geq 0} a_n z^n \in \mathcal{V}[[z]]$ is called canonical if every coefficient $a_n$ is a canonical Vassiliev invariant of order $\leq n$.

The colored Jones invariant is the canonical invariant $Z(\chi_{sl_2,R_d})$ corresponding to the $d$-dimensional representation of $sl_2$.

Bar-Natan and Garoufalidis \cite{4} proved that

$$\hat{\Delta}(z) = \frac{z}{e^{z/2} - e^{-z/2}} \Delta(e^z),$$

(18)
where $\Delta(K, t)$ is the Alexander polynomial of knot $K$, is a canonical series whose Vassiliev series coincides with the series $C$ of the Alexander-Conway polynomial.

A canonical invariant is uniquely determined by its weight system and the product of two canonical invariants or Vassiliev series $Z(W_1)$ and $Z(W_2)$ is again canonical with the weight system equal to $W_1 \cdot W_2$ (see [4]). Therefore, to prove the Melvin-Morton conjecture it is enough to establish the relations (2) and (4) on the level of weight systems.

Theorem 1.1 is thus reduced to the following relations between weight systems.

**Proposition 5.1** Let

$$\hat{W}_{sl} = \frac{1}{d} \sum_{n \geq 0} \mathbb{W}_{sl_2, R_d, n} = \sum_{i, n \geq 0} w_{in} d^i$$  

be the deframed Vassiliev series coming from the $d$-dimensional representation of $sl_2$ with the standard metric $\langle x, y \rangle = \text{Tr}(xy)$ normalized by dividing by $d$. Then

(i) $w_{in} = 0$ for $i > n$  

and

(ii) $(\sum_{n \geq 0} w_{nn}) \cdot C = \varepsilon,$  

where $C = \sum C_n z^n \in \hat{W}$ is the Vassiliev series of the Alexander-Conway invariant and $\varepsilon$ is the Vassiliev series

$$\varepsilon(D) = \begin{cases} 1 & \text{if } |D| = 0, \\ 0 & \text{if } |D| > 0. \end{cases}$$  

We will derive (20) and (21) from Theorem 3.3 and the following two propositions.

Let $V_{sl}$ be the universal Vassiliev weight system corresponding to the Lie algebra $sl_2$ with the standard invariant form $\langle x, y \rangle = \text{Tr}(xy)$. We consider $V_{sl}$ as a homomorphism from the algebra of diagrams to $C[c] = Z(U(sl_2))$, where $c$ is the quadratic Casimir. Since $V_L$ respects the filtrations in $A$ and $U(L)$, we have

$$\deg_c V_{sl}(F) \leq \left[ \frac{\ell}{2} \right]$$  

for $F \in \mathcal{P}^{(\ell)}$.  

**Proposition 5.2** Let $w_{2n}$ be the wheel diagram with $2n$ spokes, $n \geq 2$. Then

$$V_{sl}(w_{2n}) = 2^{n+1} c^n + \text{lower order terms in } c$$

This proposition will be proved in Section 6.

**Proposition 5.3** Let $C$ be the Alexander-Conway weight system. Then for a connected Feynman diagram $F$ of order $n \geq 2$ we have

$$C(F) = \begin{cases} 0 & \text{if } F \in \mathcal{P}_n^{(n-1)}, \\ -2 & \text{if } F = w_n \text{ and } n \text{ is even}, \\ 0 & \text{if } F = w_n \text{ and } n \text{ is odd}. \end{cases}$$  

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This proposition is proved in Section 7.

Proof of the Melvin-Morton conjecture (Proposition 5.1). Let
\[ V_{sl}(c) = \sum_{n \geq 0} V_n(c) = \sum_{n,i \geq 0} v_{in} c^i, \]
be the deframed universal sl2 Vassiliev series considered as a function on \( c \) and \( V_n \) be its order \( n \) component.

The quadratic Casimir \( c \in Z(\mathcal{U}(sl_2)) \) in the \( d \)-dimensional irreducible representation acts as a multiplication by \((d^2 - 1)/2\). Therefore,
\[ \frac{1}{d} d W_n(d) = V_n \left( \frac{d^2 - 1}{2} \right), \tag{24} \]
and equations (20) and (21) become equivalent to
\[ \begin{align*}
(i) & \quad \overline{v}_{in} = 0 \text{ for } i > n/2 \tag{25} \\
(ii) & \quad \overline{V} \cdot C = \varepsilon, \tag{26}
\end{align*} \]
where \( \overline{V} = \sum_{n \geq 0} 2^{-n} v_{n,2n} \).

Since both \( \overline{V}_{sl} \) and \( C \) are multiplicative Vassiliev series, it is enough to verify (25) and (26) on primitive elements of \( \mathcal{A} \). On diagrams of degree 0 or 1 these equations are trivially satisfied.

For \( n \geq 2 \) by Theorem 3.3.(ii) the primitive space \( P_n \) is spanned by Feynman diagrams with \( \leq n \) legs. Therefore by (22) \( \deg_v V_{sl}(F) \leq n/2 \) for any \( F \in P_n \) and \( \overline{v}_{in}(F) = 0 \) for \( i > n/2 \). This gives (25) and the first part of the Melvin-Morton conjecture.

Since \( V_{sl} \) and \( \overline{V}_{sl} \) coincide on \( P_n \) for \( n \geq 2 \), we have by (22) and Proposition 5.2
\[ \overline{V}(F) = \begin{cases} 0 & \text{if } F \in P_n^{(n-1)}, \\ 2 & \text{if } F = w_n \text{ and } n \text{ is even}, \\ 0 & \text{if } F = w_n \text{ and } n \text{ is odd}. \end{cases} \tag{27} \]

If \( F \in P_n \), then by the definition of a primitive element
\[ (\overline{V} \cdot C)(F) = (\overline{V} \otimes C)\Delta(F) = (\overline{V} \otimes C)(F \otimes 1 + 1 \otimes F) = \overline{V}(F) + C(F). \tag{28} \]

If \( F \in P_n^{(n-1)} \) then both terms in (28) vanish due to (23) and (27). Therefore, it only remains to check that \( \overline{V}(F) + C(F) = 0 \) for \( F \in P_n \setminus P_n^{(n-1)} \). In this case according to Theorem 3.3.(iii) we can assume that \( F = w_n \) and again by (23) and (27) we get \( \overline{V}(F) = -C(F) \). \( \Box \)
6 Proof of Proposition 5.2

Proposition 6.1  The universal $sl_2$ weight system $V_{sl}$ satisfies the relations

\[(i) \quad V_{sl}\left(\begin{array}{c}
\end{array}\right) = 4V_{sl}\left(\begin{array}{c}
\end{array}\right) \tag{29}\]

and

\[(ii) \quad V_{sl}\left(\begin{array}{c}
\end{array}\right) = 2V_{sl}\left(\begin{array}{c}
\end{array}\right) \tag{30}\]

Proof. The standard invariant metric on $L = sl_2$ is $\langle x, y \rangle = \text{Tr}_Y(xy)$. This metric is equal to the one fourth of the Killing form which is the tensor represented by the ‘bubble’ diagram in (29). This gives (i).

The relation (30) is a relation between invariant tensors in $L \otimes^4$ which is equivalent to the following classical identity for the bracket and the inner product of $L$

\[\langle [a, b], [c, d] \rangle = 2\langle a, d \rangle \cdot \langle b, c \rangle - 2\langle a, c \rangle \langle b, d \rangle \quad \text{for} \quad a, b, c, d \in sl_2. \tag{31}\]

This relation is better known in the $so_3$ realization of the Lie algebra $L = sl_2$. Under the isomorphism $sl_2 \simeq so_3$, the inner product and the Lie bracket on $sl_2$ become respectively the scalar and the vector product in the three-dimensional space and (31) becomes the classical Lagrange’s identity:

\[\lbrack a \times b \rbrack \cdot \lbrack c \times d \rbrack = (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c) \tag{32}\]

which is equivalent to the better known fundamental relation of vector calculus

\[\lbrack a \times \lbrack b \times c \rbrack \rbrack = (a \cdot c)b - (a \cdot b)c.\]

\[\square\]

Remark 6.2  Identity (30) allows us to compute the universal $sl_2$ framed and unframed weight systems recursively. See [6] and [21] for the recursion formulas.

Proof of Proposition 5.2. Applying (30) to the wheel diagram $w_{2n}$ and using the three-term relation we get

\[V_{sl}(w_{2n}) = V_{sl}\left(2\Theta \cdot w_{2n-2} - 2T + 2w_{2n-1}\right),\]

*In fact, the identity (32) goes back to Euler’s theory of the motion of a rigid body. In the notation of 20th century physics it looks like $\varepsilon_{\alpha\beta\gamma}\varepsilon_{\rho\gamma\sigma} = \delta_{\alpha\rho}\delta_{\beta\sigma} - \delta_{\alpha\rho}\delta_{\beta\sigma}$, where $\delta$ is the Kronecker delta, and $\varepsilon$ is the standard completely antisymmetric tensor in $\mathbb{R}^3$.\]
where $T$ is a planar tree with $2n$ legs. According to Lemma 3.5, $T$ is proportional to a diagram with only 2 legs, therefore both $V_{sl}(T)$ and $V_{sl}(w_{2n-1})$ are polynomials in $c$ of degree less than $n$ and do not contribute to the degree $n$ term in $V_{sl}(w_{2n})$. Therefore, keeping track of only highest degree terms we get

$$V_{sl}(w_{2n}) = 2cV_{sl}(w_{2n-2}) + \ldots = (2c)^{n-1}V_{sl}(w_{2}) = 2^{n+1}c^n,$$

since by (29) $V_{sl}(w_{2}) = 4c$. \hfill $\square$

7 Alexander-Conway weight system

Here we prove Proposition 5.3. It follows from special relations satisfied by the Alexander-Conway weight system $C$.

Proposition 7.1 The Alexander-Conway Vassiliev series $C$ satisfies the following relations

(i) \quad $C\left(\begin{array}{c}\vline \hline \hline \end{array}\right) = -2C\left(\begin{array}{c}\vline \hline \hline \end{array}\right), \quad (33)$

(ii) \quad $C\left(\begin{array}{c}\vline \hline \hline \end{array}\right) = 0, \quad (34)$

(iii) \quad $C\left(\begin{array}{c}\vline \hline \hline \end{array}\right) = 0, \quad (35)$

(iv) \quad $C\left(\begin{array}{c}\vline \hline \hline \end{array}\right) = \frac{1}{2}C\left(\begin{array}{c}\vline \hline \hline \end{array}\right) + C\left(\begin{array}{c}\vline \hline \hline \end{array}\right) - C\left(\begin{array}{c}\vline \hline \hline \end{array}\right) - C\left(\begin{array}{c}\vline \hline \hline \end{array}\right). \quad (36)$

Proof of Proposition 5.3. Let $F \in P_{n}^{(n-1)}$ be a connected Feynman diagram with $\ell \leq n - 1$ legs. Since the number of trivalent vertices in $F$ adjacent to the Wilson line cannot exceed the number of legs and $\ell < n < 2n - \ell$, there is at least one trivalent vertex in $F$ adjacent to only trivalent vertices. Therefore $F$ contains a fragment shown on (34) or (35), and by Proposition 7.1(ii) and (iii), $C(F) = 0$. \hfill $\square$
To evaluate $C$ on the wheels $w_n$ we apply (iv) to $w_n$ and taking into account (i), (ii) and (16) we obtain

$$C(w_n) = C(w_{n-2}) = \ldots = \begin{cases} C(w_2) = -2 & \text{if } n \text{ is even}, \\ C(w_3) = 0 & \text{if } n \text{ is odd}. \end{cases}$$

\[\square\]

**Proof of Proposition 7.1.** The Lie superalgebra $L = gl(1|1)$ of endomorphisms of the $(1|1)$-dimensional superspace has an invariant inner product $\langle x, y \rangle = str(xy)$.

Therefore, we can consider the universal weight system $V_{gl(1|1)}$ with values in $Z(U(gl(1|1))) = \mathbb{C}[h, c]$, where $h = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in gl(1|1)$ and $c$ is the quadratic Casimir.

As it was proved in [7], the deframed Vassiliev series $V_L$ coincides with the Alexander-Conway weight system. Namely, for $F \in \mathcal{F}_n$

$$V_L(F) = C(F)h^{n/2},$$

in particular, $V$ vanishes on $\mathcal{F}_n$ for odd $n$.

In [7] we proved the following analogs of the fundamental $sl_2$ relations (29) and (30) between invariant tensors on $L = gl(1|1)$ defined by the diagrams $B$ and $K$ (see (17))

$$V_L\left(\begin{array}{c}
\end{array}\right) = -2h^2$$

and

$$V_L\left(\begin{array}{c}
\end{array}\right) = \frac{1}{2}V_L(M),$$

where

$$M = \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} - \begin{array}{c}
\end{array} - \begin{array}{c}
\end{array}.$$

These identities give relations (33) and (36) for the Alexander-Conway weight system.

The identity (37) implies (34), since $h$ is a central element in $gl(1|1)$. Finally, the relation (35) follows from (34) and (36). Algebraically it is equivalent to the well-known fact that $gl(1|1)$ is solvable of depth three, i.e. that

$$[[[a, b], [c, d]], g] = 0, \text{ for any } a, b, c, d, g \in gl(1|1).$$

Equation (39) is obvious from a glance at the multiplication table for the bracket on $gl(1|1)$ which shows that $[[a, b], [c, d]]$ is always a multiple of the central element $h$. 

\[\square\]
Remark 7.2 A direct combinatorial proof of the key Proposition 7.1 which is not based on the relation between the Alexander-Conway polynomial and Lie superalgebra $gl(1|1)$ follows from our lemmas in Section 4 of [7]. A different proof of Proposition 7.3 can be derived from the characterization of the space generated by the Alexander-Conway weight systems in [11].

Acknowledgments. S. Chmutov informed me that he has also found a proof of the Melvin-Morton conjecture based on similar ideas. The main difference is that instead of using the connection between the Alexander-Conway polynomial and $gl(1|1)$ he gives a combinatorial proof of Proposition 7.1 similar to the proof of our lemmas in Section 4 of [7]. I would like to thank him for sending me a preliminary version of his preprint [5].

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