STRICTLY HOMOTOPY INVARIANCE OF NISNEVICH SHEAVES WITH GW-TRANSFERS

ANDREI DRUZHININ

Abstract. The strictly homotopy invariance of the associated Nisnevich sheave \( \tilde{F}_{Nis} \) of a homotopy invariant presheave \( F \) with GW-transfers (or Witt-transfers) on the category of smooth varieties over a prefect field \( k \), \( \text{char } k \neq 2 \), is proved, i.e. the isomorphism

\[ H^i_{Nis}(\mathbb{A}^1 \times X, \tilde{F}_{Nis}) \simeq H^i_{Nis}(X, \tilde{F}_{Nis}) \]

for any \( X \in \text{Sm}_k \) is obtained. This theorem is necessary for the construction of the triangulated category of GW-motives \( DM_{GW}(k) \) and Witt-motives \( DM^W(k) \) by the Voevodsky-Suslin method originally used for the construction of the category of motives \( DM(k) \).

In particular, the result of the article gives the direct prove of the strictly homotopy invariance of the Nisnevich sheaves associated to hermitian K-theory and Witt-groups (without using of the representability of these cohomology theories in the motivic homotopy category \( H_{A1}(k) \) proved by Hornbostel [14]); and on other side the strictly homotopy invariance theorem proved here and the representability criteria proved in [14] implies that cohomologies \( H^i_{nis}(-, \tilde{F}_{nis}) \) of the associated sheaf of a homotopy invariant presheave with GW-(Witt-)transfers \( F \) are representable in \( H_{A1}(k) \).

1. Introduction.

In this article we prove that the Nisnevich sheave associated to a homotopy invariant presheave with GW-transfers is strictly homotopy invariant. This result is necessarily for the construction of the category GW-motives \( DM_{GW}^W(k) \) by the Voevodsky-Suslin-method originally used for construction of the category of motives \( DM(k) \) (see [19], [20], [18], [15]). Saying Voevodsky-Suslin-method we imply that we start with some additive category of correspondences (GW-correspondences \( GW Cor_k \) of Witt-correspondences \( W Cor_k \)), and define \( DM_{GW}^W \) as \( \mathbb{G}_m \wedge 1 \)-stabilisation of the category of effective GW-motives \( DM_{GW}^W_{eff} \) and define \( DM_{GW}^W_{eff} \) as the full subcategory in derived category of the category of sheaves with GW-transfers, spanned by motivic complexes, i.e. complexes with homotopy invariant sheaf cohomology (and similarly for Witt-motives).

By definition Nisnevich sheaves with GW-transfers (Witt-transfers) are presheaves with GW-transfers (Witt-transfers) that are sheaves and presheaves with GW-transfers (Witt-transfers) are just additive presheaves on the category of GW-correspondences \( GW Cor_k \). To define in short the category of GW-correspondences (Witt-correspondences) let’s say that for affine schemes \( X, Y \) the morphism group \( GW Cor(X, Y) \) is the Grothendieck-Witt-group of the quadratic spaces \( (P, q) \), where \( P \in k[Y \times X] \) \(- mod\), that are finitely generated projective over \( k[X] \), and \( q: P \simeq Hom_{k[X]}(P, k[X]) \) is \( k[Y \times X] \)-linear isomorphism. The category Witt-correspondences is defined in the same way using Witt-groups. And in general case we replace \( k[Y \times X] \) by \( P \) by coherent sheave on \( X \times Y \) used in the definition of the category of \( K_0 \) correspondences studied by Walker in [21] and \( K \)-correspondences studied by Garkusha and Panin in [11]. Namely we consider the coherent sheave \( P \) on \( X \times Y \) that support is finite over \( X \) and that direct image on \( X \) is locally free coherent sheave of finite rank.

According to Grothendieck’s idea any category of motives plays role of the “universal” cohomology theory for some class of cohomology theories, that means that all cohomology theories of this class

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have canonical lift to functors defined on the category of motives, and the category of motives provides 'geometrical' instruments for computations of this theories. For triangulated categories of motives this means that this cohomology theories are represented and representing objects generates this triangulated category. In this sense the categories of GW-motives (or Witt-motives) are natural containers for the homotopy invariant Nisnevich excisive cohomology theories equipped with GW-transfers (or Witt-transfers), and the main examples of such theories are the higher hermitian $K$-theory $GW^i(-)$ (or the derived Witt-groups $W^i(-)$) and cohomologies of the associated Nisnevich sheaves. So this categories gives the geometrical framework for these cohomology theories, and in particular, this allows to apply the Voevodsky method of the proof of the Gersten conjecture (used originally for pretheories with transfers defined by the category $Cor$) to get the alternative proof of the Gersten conjecture for hermitian $K$-theory and Witt-groups. The categories $DM^{GW}(k)$ (and $DM^{W}(k)$) can be useful for a construction of spectral sequences converging to hermitian $K$-theory like that the categories of $K_0$-motives and $K$-motives ware used for the Grayson motivic spectral sequences [17], [10], [11].

Let’s note also that it follows from the results of [9] and [3] that the category $DM^{GW}(k)$ is rationally isomorphic to the stable motivic category $SH(k)_{\mathbb{Q}}$. The category $DM^{W}(k)$ is hypothetically equivalent to the category of Witt-motives constructed by Ananievsky, Levine, Panin in [2] via the category modules over the Witt-ring sheaf, and so $DM^{W}(k)$ is rationally equivalent to the minus part $SH^{-}(k)_{\mathbb{Q}}$ of the stable motivic homotopy category.

Since as mentioned above the category of effective GW-motives (Witt-motives) $DM^{GW}_{eff}$ ($DM^{W}_{eff}$) should satisfy the universal property for the class of homotopy invariant Nisnevich excisive cohomology theories with GW-transfers (Witt-transfers), it is natural to define the categories of GW-motives and Witt-motive as localisation of the derived category of the category of Nisnevich sheaves with GW-transfers (Witt-transfers) in respect to $\hat{A}^1$-equivalences $L_{\hat{A}^1}: D(Sh NisGW tr) \to DM^{GW}_{eff}(k)$. The important advantage of the Voevodsky-Suslin method is that in the case of a perfect base field this method provides the computation of the right adjoint functor $R_{\hat{A}^1}: DM^{GW}_{eff}(k) \to D(Sh NisGW tr)$ as the full embedding by the subcategory of motivic complexes defined above. Then the localisation functor $L_{\hat{A}^1}$ is equal to internal Hom-functor $Hom(\Delta^*, -)$ represented by the complex corresponding to infinite affine simplex $\Delta^*$. Thus this computation of the category $DM^{GW}_{eff}$ ($DM^{W}_{eff}$) and functors $L_{\hat{A}^1}$ and $R_{\hat{A}^1}$ gives an instrument for computation of Hom-groups in the category of effective GW-motives (Witt-motives) and in particular it can be useful for the computations of the mentioned cohomology theories.

The critical point in the computation of the the functor $R_{\hat{A}^1}$ according to the Voevodsky-Suslin method is the following theorem, that is the main result of the article:

**Theorem 1.1.** Nisnevich sheafification $\mathcal{F}^{nis}$ of any homotopy invariant presheaf $\mathcal{F}$ with GW-transfers is strictly homotopy invariant.

Similar to the original case of $Cor$-correspondences used in the construction of the category $DM^{-}(k)$ the proof of the theorem above is based on the computation of cohomology groups on relative affine line $A^1_U$ over a local base of the Nisnevich sheaf $\mathcal{F}^{nis}$ associated with homotopy invariant presheave with GW-transfers (lemma [8.7]):

$$\mathcal{F}^{nis}(A^1_U) \simeq \mathcal{F}(A^1_U), \quad H^{i,nis}(A^1_U, \mathcal{F}^{nis}) = 0, \quad \text{for } i > 0. \quad (1.2)$$

This equalities essentially uses transfers defined by considered category of correspondences $GW Cor_k$, and proof is based on the explicit construction of some GW-correspondences (Witt-correspondences) between etale coverings of open subschemes in relative affine. So the proof differs for different categories of correspondences, and the main innovative ingredients in the proof are geometrical constructions that allows to control 'orientation' of correspondences and to define the required quadratic forms, and this is the most essential novelty of the work.
The role of GW-correspondences for equality \([1.2]\) can be simply explained if we think about transfers on presheaves as representations of some ‘ring’ corresponding to the category of correspondences. Then using analogy between (homotopy invariant) (pre-)sheaves with transfers on the category of smooth schemes and coherent (pre-)sheaves on some scheme, we see that equality \([1.2]\) is analogously to the fact that any coherent sheaf on affine scheme is just a module over the function ring and cohomologies of coherent sheaves on affine schemes are zero. The mentioned fact about coherent sheaves relates to the existence of unit decomposition in the ring of functions. In the same sense equality \([1.2]\) relates to the existence of a \(\mathbb{A}^1\)-homotopy decomposition in the category of correspondences of the identity on the affine line, i.e. a lift along the Nisnevich covering \(U \to \mathbb{A}^1_U\). So presented here constructions of correspondences shortly speaking gives such decomposition of unit in the category of GW-(Witt-)correspondences up to \(\mathbb{A}^1\)-homotopy.

Formally proof of equality \([1.2]\) are based on the following excision and injectivity theorems:

**Theorem 1.3** (etale excision, theorem \([6.1]\)). For a homotopy invariant presheave with GW-transfers \(F\), etale morphism of essentially smooth local schemes \(\pi : V' \to V\), closed subscheme \(Z \subset V\) of codimension 1, such that \(\pi\) induces isomorphism between \(Z\) and its preimage \(Z' = \pi^{-1}(Z)\), \(\pi\) induces the isomorphism

\[
\pi^* : F(V - Z) \sim F(V) \to F(V' - Z'),
\]

**Theorem 1.4** (Zariski excision on relative affine line, theorem \([5.1]\)). For be homotopy invariant sheave with GW-transfers \(F\) Zariski open subvariety \(V \subset \mathbb{A}^1_U\) : \(V \supset V_U\) for essential smooth local scheme \(U\), restriction homomorphism induce isomorphism

\[
\frac{F(\mathbb{A}^1_U - V_U)}{F(\mathbb{A}^1_U)} \simeq \frac{F(V - 0_U)}{F(V)}
\]

**Theorem 1.5.** (theorem \([7.1]\)) Let \(F\) is be homotopy invariant sheave with GW-transfers over field \(k\) and \(K\) be geometric extension \(K/k\) (i.e. field of functions of some variety). Then for any Zariski open subschemes \(U \subset V \subset \mathbb{A}^1_k\) restriction homomorphism

\[
i^* : F(V) \to F(U)
\]

is an injective, where \(i : U \hookrightarrow V\) denotes open immersion.

**Theorem 1.6** (see \([5]\) for the case of Witt-correspondences, and theorem \([7.3]\) for the case of GW-correspondences). For any essential smooth local scheme \(U\), and closed subscheme \(Z \subset U\), restriction homomorphism

\[
i^* : F(V) \to F(U)
\]

is injective.

To prove the excision theorem above we give an explicit construction of GW-correspondences in the category of pairs \((V, V - Z) \to (V', V' - Z)\), that are inverse up to \(\mathbb{A}^1\)-homotopy to embedding morphisms \((V, V - Z) \to (V', V' - Z)\) where \(Z\) is considered closed subscheme and \(V\) is corresponding neighbourhood of \(Z\) (Nisnevich or Zarisky). To prove the injectivity theorems we give the construction of a left inverse in the category of GW-(Witt)-correspondences between pairs to the mentioned embeddings of open subschemes.

\(^1\)here we speak about the informal analogy rather then a strict mathematical notion, though indeed formally we can think, for example, about the corresponding ring spectra in the motivic homotopy.
1.1. **Overview of the text:** In the section 2 we present the definitions of the categories of GW-(Witt-)correspondences, prove the basic elementary properties and give the construction that produce GW-(Witt-)correspondences from a relative curve with trivialisation of the relative canonical class and a (good) regular function of the curve.

In the sections 3 and 4 we summarise geometric constructions used in the construction of GW-correspondences in the proofs of the excision and injectivity theorems.

In the section 5 the Zariski excision isomorphism (theorem 1.4) on the relative affine line over a local base is proved. In the section 6 the etale excision isomorphism (theorem 1.3) on the relative affine line over a local base is proved. In the section 7 the injectivity theorems on a local essential smooth scheme (theorem 1.5) and on the affine line (theorem 1.6) are proved.

The section we give 8 the main result of the article (theorem 1.1).

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1.3. **Notation.** All schemes are a separated noetherian schemes of finite type over the base, and $Sm_k$ denotes the category of smooth schemes over field $k$. We denote $Coh(X) = Coh_X$ the category of coherent sheaves on a scheme $X$, for any scheme $X$ (not only affine) to shorten denotations. For any $P\in Coh(X)$ we denote by $\text{Supp}P$ the closed subscheme in $X$ defined by the sheaf of ideals $\mathcal{I}(U) = \text{Ann} P|_U \subset k[U]$ and we denote by $\text{Supp}_{red}P$ the reduced subscheme of $\text{Supp}P$.

We write $k[X]$ for the ring of regular (global defined) functions $X \to k^1$, i.e. $k[X] = \Gamma(X, \mathcal{O}(X))$. We denote by $Z(f)$ vanish locus of $f$, for any regular function $f$ on scheme $X$, and by $Z_{red}(f)$ its reduced subscheme. Similarly, for a section $s \in \Gamma(X, \mathcal{L})$ of some line bundle on scheme $X$ we denote by $Z(s)$ the closed subscheme defined by the ideal sheaf $\{f: div f \geq div s\}$ (which is equivalent to the image of homomorphism $\mathcal{L}_{-1} \to \mathcal{O}(X)$). For an effective divisor $D$ in variety $X$ denote by $S(D)$ the closed subscheme $Z(s)$, where $s \in \Gamma(X, \mathcal{L}(D))$, $div s = D$.

2. **GW-CORRESPONDENCES**

2.1. **categories with duality** $(\mathcal{P}(Y \to X), D_X)$.

**Definition 2.1.** For a morphism of schemes $p: Y \to X$, let $Coh_{fin}(p)$ (or $Coh_{fin}(Y|_X)$) denotes the full subcategory of the category of coherent sheaves on $S$ spanned by sheaves $\mathcal{F}$ such that $\text{Supp}\mathcal{F}$ is finite over $X$; and let $\mathcal{P}(p)$ (or $\mathcal{P}(Y|_X)$) denotes the full subcategory of $Coh_{fin}(Y)$ spanned by sheaves $\mathcal{F}$ such that $p_*\mathcal{F}$ is locally free sheave on $X$.

For two schemes $X$ and $Y$ over a base scheme $S$ we denote

$$Coh_{fin}^S(X, Y) = Coh_{fin}(X \times_S Y \to X), \mathcal{P}^S(X, Y) = \mathcal{P}(X \times_S Y \to X).$$

**Remark 2.2.** In the case of affine schemes $Y, X$, $\mathcal{P}(Y \to X)$ is equivalent to the full subcategory in the category of $k[Y]$-modules consisting of modules that are finitely generated and projective over $k[X]$.

The internal hom-functor $D_X = \mathcal{H}om(-, \mathcal{O}(X))$ on the category of coherent sheaves on $X$ can be naturally lifted to a functor

$$D_X: Coh_{fin}(Y_X)^{op} \to Coh_{fin}(Y_X)^{op},$$

for any morphism of schemes $Y \to X$.

Indeed, firstly let’s note that we can define the required functor $D_X$ locally along $X$, i.e. it is enough to define the functor $D_X$ in a natural way for affine schemes $X$. Next let’s note that if $X$ is affine and $Y \to X$ is finite morphism, then $Coh_{fin}(Y \to X) \simeq Coh(Y) \simeq k[Y] - mod$, and the
required functor $D_X$ is equivalent to the functor $\text{Hom}(-, k[X]): k[Y] - \text{mod} \to k[Y] - \text{mod}$. In general case of a morphism $Y \to X$ we have

$$\text{Coh}_{fin}(Y \to X) = \lim_{Z} \text{Coh}(Z \to X),$$

where $Z$ ranges over the set of closed subschemes in $Y$ finite over $X$; and hence the functor $D_X$ on $\text{Coh}_{fin}(Y \to X)$ can be defined as a direct limit of functors $D_X$ defined on $\text{Coh}_{fin}(Z \to X) = \text{Coh}(Z)$ that are just defined by the above.

Next let’s note that since the functor $\text{Coh}_{fin}(Y \to X) \to \text{Coh}(X)$ is conservative and $D_X$ on $\text{Coh}(X)$ defines the duality on the subcategory category $\mathcal{P}(X)$ spanned by locally free coherent sheaves, it follows that the functor $D_X$ on $\text{Coh}_{fin}(X)$ defines the duality on the category $\mathcal{P}(Y \to X)$. In addition for any morphisms of schemes $X_1 \to X_2 \to X_1$ the tensor product of modules (coherent sheaves) defines a functor of categories with duality

$$(2.3) \quad - \circ - : (\mathcal{P}^X_{X_2}, D_{X_2}) \times (\mathcal{P}^X_{X_1}, D_{X_1}) \to (\mathcal{P}^X_{X_1}, D_{X_1}),$$

which is natural along $X_3, X_2, X_1$ and satisfies the associativity in that sense that for any three morphisms $X_4 \to X_3 \to X_2 \to X_1$ and $P_3 \in \mathcal{P}(X_4, X_3), P_2 \in \mathcal{P}(X_3, X_2), P_2 \in \mathcal{P}(X_2, X_1)$, there is a natural isomorphism $\xi: P_3 \circ (P_2 \circ P_1) \simeq (P_3 \circ P_2) \circ P_1$, such that $(\xi_{1,2}) \circ (\xi_{1,3,4}) = (\xi_{1,2,3}) \circ (\xi_{1,3,4})$, and $(\eta_{1,2}) \circ (\eta_{1,3}) \circ D(\xi) = D(\xi)^{-1} \circ (\eta_{1,2,3}) \circ (\eta_{1,2,3})$, where $\eta_{i,j}: D_{X_i}((- \circ - )) \to D_{X_j}((- \circ - )) (i, j = 1, \ldots, 4)$ denotes the structure morphism of the functor of categories with duality.

**Remark 2.4.** The functor $D_X$ is represented by $p^!(O(X))$, i.e. $D_X(F) = \text{Hom}(F, p^!(O(X)))$ where $\text{Hom}$ denotes internal homomorphism functor in $\text{Coh}(Y)$.

**2.2. Categories $GW\text{Cor}_S, \text{WCor}_{S}$**

**Definition 2.5.** The category $Q\text{Cor}_S$ is the category with objects being smooth schemes over $S$, morphism groups being defined as

$$Q\text{Cor}_S(X, Y) = Q(\mathcal{P}^S(X, Y), D_X)$$

where the symbol $Q$ denotes the set of isomorphism classes of quadratic spaces in the category with duality the composition is induced by the functor $(2.3)$, and identity morphism

$$1d_X = [(O(\Delta), 1)],$$

where $\Delta$ denotes diagonal in $X \times_S X$.

The categories $GW\text{Cor}_S$ and $\text{WCor}_S$ are the additive categories with the same objects and such that

$$GW\text{Cor}_S(X, Y) = GW(\mathcal{P}^S(X, Y), D_X), \text{WCor}_S(X, Y) = W(\mathcal{P}^S(X, Y), D_X),$$

where $GW$ denotes the Grothendieck-Witt-group of the exact category with duality, i.e. the group completion of the groupoid (up to direct sums) of non-degenerate quadratic spaces $(P, q): P \in \mathcal{P}^S(X, Y), q: P \simeq D_X(P)$, and $W$ are Witt group of the exact category with duality (see Balmer [4]).

**Remark 2.6.** Equivalently to the definition above we can say that the category $GW\text{Cor}_S$ is the additivisation of the category $Q\text{Cor}_S$, and the category $\text{WCor}_S$ is the factor-category of $GW\text{Cor}_S$ such that classes of metabolic spaces defines the zero morphism.

**Definition 2.7.** Let’s define a functor $Sm_S \to GW\text{Cor}_S$,

$$f \in Mor_{Sm_S}(X, Y) \to [(O(\Gamma_f), 1)],$$

where $\Gamma_f$ denotes graph of morphism $f$, that is closed subscheme in $Y \times X$ isomorphic to $X$, and $1$ denotes unit quadratic form on a free coherent sheave of a rank one. The composition with factorisation $GW\text{Cor}_S \to W\text{Cor}_S$ gives us the functor $Sm_S \to W\text{Cor}_S$. 
Remark 2.8. For any $\Phi \in GWCor$ and regular maps $f : X' \to X$ and $g : Y \to Y'$,
$$\Phi \circ f = (id_Y \times f)^*(\Phi), \quad g \circ \Phi = (g \times id_X)_*(\Phi),$$
where $(id_Y \times f)^*$ denotes inverse image along morphism $id_Y \times f : Y \times X' \to Y \times X$, and $(g \times id_X)_*$ denotes direct image along morphism $g \times id_X : Y \times X' \to Y \times X$.

The following definitions and lemmas can be given in the same manner for $GW$-correspondences and Witt-correspondences.

Definition 2.9. A presheave on $Sm_S$ is an additive functor $F : Sm_S \to Ab$; a presheave with $GW$-transfers over a base $S$ is an additive functor $F : GWCor_S \to Ab$.

A presheave $F$ on $Sm_S$ is called homotopy invariant is the natural homomorphism $F(X) \simeq F(k^1 \times X)$ is an isomorphism for any $X \in Sm_S$; a presheave $F$ with $GW$-transfers is homotopy invariant if it is homotopy invariant as a presheave on $Sm_S$ via the functor from definition 2.7.

Lemma 2.10. Suppose $S$ is a scheme and $s \in S$ is a point; then there is an embedding functor $GWCor_{S,s} \to pro-GWCor_S$ (and $WC_{Cor,S,s} \to pro-WC_{Cor_S}$). So consequently any presheave with $GW$-transfers over $S$ defines in canonical way a presheave with $GW$-transfers over $S_s$.

Proof. We omit the full prove to shortify text. Let’s note only that the claim follows from the following points: 1) any scheme of finite type over $S_s$ is projective limit of schemes over Zariski neighbourhoods $s \in U_i \subset S$; 2) any quadratic space can be defined by finite the set of data (regular functions and equations); the isomorphism of quadratic spaces and the property of a quadratic space to be metabolic can be defined by finite set of data (regular functions and equalities). Note that for this statement it is essential that schemes considered in definition 2.5 are schemes of finite type over the base.

Definition 2.11. An algebra $R/k$ is called geometric extension of the base field $k$ if $R$ is isomorphic to the local ring $k[X_s]$ for some smooth variety $X$ over $k$ and point $x \in X$.

Corollary 2.12. A presheave with $GW$-transfers over $k$ defines in canonical way a presheave with $GW$-transfers over $R$ for any geometric extension $R/k$.

Definition 2.13. We define additive category of $GW$-correspondences between pairs $GWCor_{pair}^S$ over base $S$, as follows: objects of $GWCor_{pair}^S$ are pairs $(X, U)$ of smooth scheme $X$ over $S$ and open subscheme $U \subset X$, and the group of morphisms

$$GWCor_{pair}^S((X, U), (Y, V)) = HGWCor(X, V) \xrightarrow{d_0} GWCor(U, V) \oplus GWCor(X, Y) \xrightarrow{d_1} GWCor(U, Y),$$

where $d_0 = (- \circ i, j \circ -)$, $d_1 = (j \circ -, - \circ i)$, $i : U \hookrightarrow X$, $j : V \hookrightarrow Y$, and $H$ denotes cohomology in the middle term, i.e. $\text{Ker}(d_1)/\text{Im}(d_0)$.

Equivalent $GWCor_{pair}^S$ can be defined as the factor category in the full subcategory of the category of arrows in $GWCor_S$ spanned by open embeddings and factorised by the ideal consisting of morphisms $(\Phi, \Phi) : (X, U) \to (Y, V)$ such that there is a lift $\Theta : X \to V$ in $GWCor_S$.

\[
\begin{array}{ccc}
X & \xrightarrow{\Phi} & Y \\
\downarrow & & \downarrow \\
U & \xrightarrow{\delta} & V
\end{array}
\]

Remark 2.14. For a homotopy invariant presheave with $GW$-transfers $F$ the formula

$$GWCor_{pair}^S \to Ab \to \xrightarrow{(Y, U)} \text{Coker}(F(Y) \to F(U))$$
defines a homotopy invariant presheave on the category $GWCor^pair$.

**Lemma 2.15.** If $Q = (P,q) \in Q(\mathcal{P}(X, Y))$ is a quadratic space, and $j: Y' \hookrightarrow Y$ is an open immersion such that $\text{Supp } P = Y' \times_Y \text{Supp } P$, then there is a unique quadratic space $Q' \in Q(\mathcal{P}(X, Y'))$, such that $j \circ Q' = Q$.

**Proof.** The uniqueness follows from that the functor $(j_X)_*$, where $j_X: j \times id_X: Y' \times X \hookrightarrow Y \times X$, is faithful. The existence follows from that if $\text{Supp } P = Y' \times_Y \text{Supp } P$, then $P$ defines the element in $\mathcal{P}(X, Y')$, and $q$ defines the quadratic form on it.

More precisely let $j_X = j \times id_X: Y' \times X \hookrightarrow Y \times X$, and $(P',q') = j_X^*(P,q)$, where by definition $P' \in \text{Coh}(Y' \times X)$ and $q: P' \rightarrow D_X(P')$. Then $\text{Supp } P' = \text{Supp } P$ is finite over $X$ and since $(j_X)_*(j_X^*(P)) = P$, we have $pr'_*(P') = pr_*(j_X)_*(j_X^*(P)) = pr_*(P) \in \text{calc}(X)$, where $pr': Y' \times X \rightarrow X$ and $pr: Y \times X \rightarrow X$.

□

**Lemma 2.16.** Let $j^X: X' \hookrightarrow X$, $j^Y: Y' \hookrightarrow Y$ be open immersions, $Z = Y \setminus Y'$, and $Q = (P,q) \in Q(\mathcal{P}(X, Y))$ be a quadratic space such that

$$Z \times_Y \text{Supp } P \times_X X' = \emptyset;$$

then $Q$ defines in a canonical way an element in $\Phi \in GWCor((X, X'), (Y, Y'))$, such that $\Phi = \left(\left[\left(\left[Q\right], \left[Q'\right]\right]\right]\right)$ and $(Y^Z)_*(Q') = (j^X)^*(Q)$. We will denote such element simply by $[Q] \in GWCor((X, X'), (Y, Y'))$.

If moreover

$$Z \times_Y \text{Supp } P = \emptyset,$$

then $\Phi = 0 \in GWCor((X, X'), (Y, Y'))$.

**Proof.** Both statements immediately follows from lemma 2.15 applied for the first one to $(j^X)^*(Q)$ and for the second to $Q$. □

2.3. Construction of quadratic space from a function on a curve.

**Definition 2.17.** A curve $C$ over a base scheme $S$ a scheme $C$ over $S$ such that dimension of all fibres of $C$ over $S$ is one, and we say that a curve $C$ is relative smooth if canonical morphism $C \rightarrow S$ is smooth.

An orientation on smooth relative curve $C$ any trivialisation of its canonical class, i.e. an isomorphism $\omega_S(C) \simeq \mathcal{O}(C)$.

We say that a regular function $f$ on relative curve $pr: C \rightarrow S$ is relatively finite if morphism $C \xrightarrow{(f, pr)} \mathbb{A}^1 \times S$ is finite.

So an oriented relative curve $C$ with a relatively finite function $f$ with support $Z$ a set $(C, \mu, f, Z)$, where $C \rightarrow S$ is a smooth relative curve, $\mu$ is an orientation, $f$ is a relatively finite function, $Z = Z(f)$. For a given finite scheme $Z$ over a scheme $S$, denote by $\text{OrCur}_Z^S$ the set of isomorphism classes of oriented curves with relatively finite function over $S$.

**Proposition 2.18.** There is map

\[
\langle \cdot \rangle: \text{OrCur}_Z^S \rightarrow Q(\mathcal{P}(Z \rightarrow S))
\]

\[
(C, \mu, f, Z) \mapsto (k[Z], q),
\]

that is natural in respect to the base changes. In other words this map takes any smooth oriented curve with relatively finite regular function $C$ over $S$, with orientation $\mu: \omega_C \otimes \omega_S^{-1} \simeq \mathcal{O}(C)$ and relatively finite function $f \in k[C]$ with vanish locus $Z$ to an invertible function $q$ in $k[Z]$ in a natural way over $S$. 
Proof. Consider the regular map \( F = (f, pr_S): C \to \mathbb{A}^1_S \). By assumption \( F \) is finite morphism of smooth schemes over \( S \), and since \( \mathbb{A}^1_S \) is affine then \( C \) is affine too and \( F \) is flat (by corollary V.3.9. and theorem II.4.7 [1]). So we can apply proposition 2.1 from [16] and get isomorphism

\[
\omega(C) \simeq \text{Hom}_{k[A^1_S]}(k[C], k[A^1_S])
\]

that respects base changes. Then using base change along the zero section \( i_0: S \to A^1_S \) we get the isomorphism

\[
q: k[Z] \simeq \text{Hom}_{k[S]}(k[Z], k[S])
\]

that defines required quadratic space. \( \square \)

Lemma 2.19. Suppose \( \mathcal{C} \to S \) is a relative projective curve over a local base \( S \), \( \mathcal{L} \) is a very ample invertible sheaf on \( \mathcal{C} \), and \( s, d \in \Gamma(\mathcal{C}, \mathcal{L}) \) are sections such that \( Z(d) \cap E \neq \emptyset \), for each irreducible component \( E \) of the closed fibre of \( \mathcal{C} \), \( Z(s) \neq \emptyset \), \( Z(s) \cap Z(d) = \emptyset \); then the function \( s/d: \mathcal{C} - Z(d) \to \mathbb{A}^1 \) is relatively finite.

Proof. Consider the regular map \( f = ([s]: [d], c): \mathcal{C} \to \mathbb{A}^1 \times S \), where \( c: \mathcal{C} \to S \) denotes the canonical morphism. The morphism \( f \) is projective and \( f = (s/d, c): \mathcal{C} - Z(d) \to \mathbb{A}^1 \times S \) is the base change of \( f \) along the immersion \( \mathbb{A}^1 \times S \to \mathbb{P}^1 \times \mathbb{A}^1 \), hence \( f \) is projective.

On the other hand since \( d \) is section of very ample invertible sheaf \( \mathcal{L} \), it follows that \( C - Z(d) \) is affine. Hence \( f \) is affine morphism. Thus \( f \) is finite. \( \square \)

3. Quillen’s Trick and Compactification

Here we summarise some some technical facts and geometric construction providing compactifications with ample bundles for some relative curves used in the next sections. We start with following standard facts:

Lemma 3.1. Suppose \( \mathcal{O}(1) \) is an ample invertible sheaf on a scheme \( X \), \( Z \subset X \) is a closed subscheme, then there is an integer \( L \) such that for all \( l > L \), the restriction homomorphism \( \Gamma(X, \mathcal{O}(l)) \to \Gamma(Z, \mathcal{O}(l)|_Z) \) is surjective.

Lemma 3.2. Suppose \( U \) is a local scheme, \( C \to U \) is a morphism of schemes, \( \mathcal{O}(1) \) is an ample invertible sheaf on \( C \), and \( Z_1, Z_2 \subset X \) are closed subschemes such that \( Z_2 \) is finite over \( U \) and \( Z_1 \cap Z_2 = \emptyset \); then for some integer \( L \), for all \( l > L \), there is a section \( s \in \Gamma(C, \mathcal{O}(l)) \) such that \( s \big|_{Z_1} = 0 \), and \( s \big|_{Z_2} \) is invertible.

Proof. The claim of the first lemma is corollary of the Serre theorem [13, ch. 3, theorem 5.2]. The second lemma follows form the first. \( \square \)

Definition 3.3. Suppose \( Z \) is a closed subscheme of a scheme \( X \) over some base \( S \). A Nisnevich neighbourhood \( \pi: (X', Z) \to (X, Z) \) is an etale morphism \( \pi: X' \to X \) and closed subscheme \( Z' \subset X' \) such that \( \pi \) induces an isomorphism \( Z' \simeq Z \). For a pair of Nisnevich neighbourhoods \( \pi_1: (X'_1, Z'_1) \to (X_1, Z_1) \) and \( \pi_2: (X'_2, Z'_2) \to (X_2, Z_2) \), a morphism of Nisnevich neighbourhoods \( \pi_1 \to \pi_2 \) is a set of four morphisms \( X'_1 \to X'_2 \) and \( Z'_1 \to Z'_2 \tos X_1 \to X_2 \) and \( Z_1 \to Z_2 \), such that corresponding cube is commutative (i.e. \( \pi_1 \to \pi_2 \) is a morphism in the category of arrows of the category of arrows of the category of schemes).

Definition 3.4. A good relative compactification of quasi-finite morphism of curves \( \pi: X' \to X \) over a base scheme \( S \) the following set of data:
1) a finite morphism \( \pi: \overline{X'} \to \overline{X} \) of projective schemes over \( S \) with commutative diagram in the category of schemes over \( S \):

\[
\begin{array}{ccc}
\overline{X'} & \xrightarrow{\pi} & \overline{X} \\
\downarrow{j'} & & \downarrow{j} \\
X' & \xrightarrow{\pi} & X,
\end{array}
\]

where \( j \) and \( j' \) are open immersions such that \( \overline{X'} \setminus X' \) and \( \overline{X} \setminus X \) are finite over \( S \);

2) a very ample invertible sheaf \( \mathcal{O}(1) \) on \( \overline{X} \), such that \( \pi^*(\mathcal{O}(1)) \) is very ample too, and a pair of sections \( d' \in \Gamma(X', \pi^*(\mathcal{O}(1))), d \in \Gamma(X, \mathcal{O}(1)) \), such that \( \overline{X'} - Z(d') = X' \) and \( \overline{X} - Z(d) = X \).

**Lemma 3.5.** Suppose \( k^d_k \) is an affine spaces of dimension \( d \) over an infinite field \( k \), \( z \in k^d_k \) is a point, and \( B, B_1 \subset k^d_k \) are closed subschemes such that \( B \neq k^d_k \), \( \text{codim } B_1 \geq 2 \). Then there is a linear projection \( pr: k^d_k \to k^{d-1}_k \) such that the induced projection \( B \to k^{d-1}_k \) is finite, and \( pr^{-1}(pr(z)) \cap B_2 = \emptyset \).

**Proof.** Both of the required conditions on the projection \( pr \) defines Zariski open suschmes in the projective space of linear projections \( k^{d} \to k^{d-1} \), which is equal to the infinity subspace \( \mathbb{P}^{d-1} \) in \( \mathbb{P}^{d} \). Namely, \( U_1 \) is a complement to the intersection \( \mathbb{P}^{d-1} \cap B \) And let’s denote the second one as \( U_2 \).

Then \( U_1 \neq \emptyset \), since \( \dim(\mathbb{P}^{d-1} \setminus U_1) = \dim(\mathbb{P}^{d-1} \cap B) \leq \dim B - 1 \leq d - 2 \). Since the case field \( k \) is infinite, to prove that \( U_2 \neq \emptyset \) it is enough to consider the case of rational point \( z \in k^d_k \). In such case \( \mathbb{P}^{d-1} \setminus U_2 \) is image of \( B_1 \) under the projection to the infinity subspace \( \mathbb{P}^{d-1} \) with the center at the point \( z \), and hence \( \dim(\mathbb{P}^{d-1} \setminus U_2) \leq \dim B_2 \leq d - 2 \).

\( \square \)

Also we use following corollary of the Zariski main theorem ([12] theorem 8.12.6).

**Proposition 3.6.** For any etale morphism \( e: U \to Y \) there is a decomposition \( U \xrightarrow{u} X \xrightarrow{p} Y \), \( p \circ u = e \) with \( u \) dense open immersion and \( p \) finite.

**Proof.** By the Zariski main theorem there is a decomposition \( U \xrightarrow{\tilde{u}} \tilde{X} \xrightarrow{\tilde{p}} Y \) such that \( \tilde{u} \) is open immersion and \( \tilde{p} \) is finite. Then if we put \( X = Cl_{\tilde{X}}(U) \) (i.e. closure of \( U \) in \( \tilde{X} \)) and define \( u: U \to X \) and \( p = \tilde{p}|_X \). Then \( u \) is dense open immersion and \( p \) is finite, since it is composition of a closed embedding and a finite morphism.

\( \square \)

**Lemma 3.7.** Suppose \( \pi: (X', Z') \to (X, Z) \) is Nisnevich neighbourhood with smooth \( X' \) and \( X \) and suppose \( z' \in Z', z \in Z \) are points such that \( \pi(z') = z \); then there are a (relative) Nisnevich neighbourhood \( \varpi: (\overline{X}, \overline{Z}') \to (X, Z) \) over an essential smooth local base \( S \), a morphism \( \varpi \to \pi \) and a good relative compactification \( \overline{\varpi} \to \overline{\pi} \)

\[
\begin{array}{ccc}
X' & \xrightarrow{v} & \overline{X'} \\
\downarrow{\pi} & & \downarrow{\overline{\pi}} \\
X & \xrightarrow{\varpi} & \overline{X} \\
\downarrow{\overline{\varpi}} & & \downarrow{\overline{\pi}} \\
S & & S
\end{array}
\]

such that :

1) \( v \) and \( v' \) are pro-limits of open immersions,
2) there are lifts of \( z' \) and \( z \) in \( \overline{X'} \) and \( \overline{X} \),
3) \( \overline{Z'} = v^{-1}(Z), \overline{Z} = v'^{-1}(Z') \) and both schemes are finite over \( S \).
4) \( \overline{X'} \) and \( \overline{X} \) are smooth over \( S \) and the relative canonical classes of both schemes are trivial.
5) \( \overline{X'} \setminus X' \) and \( \overline{X} \setminus X \) are finite over \( S \).
Proof. Firstly let’s replace $X$, $X'$ and $Z$, $Z'$ to some Zariski neighbourhoods of points $z$ and $z'$, in such way that canonical classes of new $X$ and $X'$ are trivial.

Let $e: X \to \mathbb{A}^d$ ($d = \dim X$) be any etale morphism (which exists since $X$ is smooth). Then using proposition 3.6 we find a decomposition of the morphism $e$ into $X \overset{\sim}{\to} \mathbb{X} \overset{\pi}{\to} \mathbb{A}^d$ such that $\pi$ is dense open immersion and $p$ is finite. Again applying proposition 3.6 we find a decomposition of the morphism $u \circ \pi$ into $X' \overset{\sim}{\to} \mathbb{X}' \overset{\pi'}{\to} \mathbb{A}^d$, where $\pi'$ is dense open immersion and $\pi$ is finite.

Let’s denote $B = \mathbb{X} \setminus X$, $B' = \mathbb{X}' \setminus X'$, $B_1 = \mathbb{Z} \cap B$, $B'_1 = \mathbb{Z}' \cap B'$, where $\mathbb{Z}$ denotes the closure of $Z$ in $\mathbb{X}$ and similarly for $Z'$. Then since $X$ and $X'$ are dense in $\mathbb{X}$ and $\mathbb{X}'$, it follows that $\dim B, \dim B' < d$ and $\dim B_1, \dim B'_1 \leq \dim \mathbb{Z} - 1 \leq d - 2$. Now applying lemma 3.4 to the closed subsets $P = p(\pi(B')) \cup B \cup \pi(\mathbb{Z}')$, $P_1 = p(B_1 \cup B_2)$ and the point $p(z) \in \mathbb{A}^d$, we get a projection $pr: \mathbb{A}^d \to \mathbb{A}^{d-1}$, such that restriction to $P \to \mathbb{A}^{d-1}$ is finite and $pr^{-1}(pr(e(z))) \cap P_1 = \emptyset$.

Now consider base changes of $e$ and $\pi$ over local scheme $S = \mathbb{A}^d_S$, $s = pr(p(z)) \in \mathbb{A}^{d-1}_S$. We get etale morphisms $e_S: X \to \mathbb{A}^{d-1}_S$ and $\pi_S: X' \to X$ that are finite over open subscheme $V = \mathbb{A}^{d-1}_S \setminus P_S$, where $P_S = P \times_{\mathbb{A}^{d-1}_S} S$ is finite over $S$. And more over $Z' = Z \times_{\mathbb{A}^{d-1}_S} S$ and $Z' = Z' \times_{\mathbb{A}^{d-1}_S} S$ are finite over $S$ (since $Z, Z'$ are finite over $\mathbb{A}^d$ and since $\mathbb{Z} \setminus Z = \mathbb{Z}' \setminus Z'$ don’t intersect with closed fibres).

Consider immersion $\mathbb{A}^d_S \hookrightarrow \mathbb{X}^1_S$ and twice applying proposition 3.6 we get decomposition $X \overset{j}{\to} \mathbb{X} \overset{\pi}{\to} \mathbb{A}^d$, and $X' \overset{j'}{\to} \mathbb{X}' \overset{\pi'}{\to} \mathbb{A}^d$.

Then $X \times_{\mathbb{A}^d} V$ is dense in $\mathbb{X} \times_{\mathbb{A}^d} V$, and in the same time it is finite over $V$; hence $X \times_{\mathbb{A}^d} V = \mathbb{X} \times_{\mathbb{A}^d} V$. Whence $\mathbb{X} \setminus X \subset \mathbb{X} \times_{\mathbb{A}^d} (\mathbb{P}^1_S \setminus V) = \mathbb{X} \times_{\mathbb{A}^d} (\mathbb{P}^1_S \cup P_S)$, and consequently $\mathbb{X} \setminus X$ is finite over $S$. Similarly $\mathbb{X}' \setminus \mathbb{X}'$ is finite over $S$.

Thus we construct diagram (3.8) satisfying the required properties, and it is enough to find very ample sheave $O(1)$ and sections required in definition 3.4.

Since $\mathbb{X}$ is projective over the local scheme $S$, there is an ample sheave $O(1)$ on $\mathbb{X}$. Then since $\mathbb{X}$ is finite, $\mathbb{X}(O(1))$ is ample sheave on $\mathbb{X}$. Then for some $l$ sheaves $O(l)$ and $\mathbb{X}^l(O(l))$ are very ample, and let’s replace $O(1)$ by $O(l)$.

Now by lemma 3.11 for some integer $l_0$ there is a section $r \in \Gamma(\mathbb{X}, O(l_0))$, $Z(r) \cap (\mathbb{X} - X) = \emptyset$. Again by lemma 3.1 there is an integer L such that for all $l > L$ there is a section $d \in \Gamma(\mathbb{X}, O(l_1))$ such that $d|_{\mathbb{X} - X} = 0$, $d(z) \neq 0$, $d|_{Z(r)}$ is invertible. Then on the one side $Z(d) \subset \mathbb{X} - Z(r)$, and hence it is affine. And on the other side $Z(d)$ is a closed subscheme in the projective scheme $\mathbb{X}$ over $S$, and hence $Z(d)$ is projective over $S$. Thus $Z(d)$ is finite over $S$.

Next similarly there is an integer $l'$ such that for all $l > l'$ there is a section $d' \in \Gamma(\mathbb{X}^l, O(l_1))$ such that $d'|_{\mathbb{X}^l - X'} = 0$, $d'(z) \neq 0$, $Z(d')$ is finite over $S$. Finally replacing $O(1)$ to $O(l)$ for any $l > L, L'$ and replacing $X$ to $X - Z(d)$, and $X'$ by $X' - Z(d')$, we get the claim.

\[\square\]

4. Constructions of functions on relative curves

In this section we present the constructions of relative curves with trivialisation of the relative canonical class ans a regular functions on the curves, which are used in the proofs of excision and injectivity theorems.
**Lemma 4.1.** Suppose $\pi : X' \to X$ is a finite morphism of projective curves over an infinite field $k$, and $\mathcal{O}(1)$ is a very ample line bundle on $X'$; suppose $z_i \in X'$ is a closed point, $Y \subset X'$ is a closed subscheme, $z \notin Y$, $X' - Y$ is smooth; then for any invertible section $s_Y \in \Gamma(Y, \mathcal{O}(1))$ there is an integer $L$ such that for all $l > L$ there is a global section $s \in \Gamma(X', \mathcal{O}(l))$ such that

1) $s(z) = 0$ and $s|_Y = s_Y$,

2) $\text{div } s$ hasn’t multiple points (or equivalently $Z(s)$ is reduced),

3) $\pi$ induces the closed injection $Z(s) \to X$.

**Proof.** In short the claim follows from that for a big $l$ the first condition on $s$ defines some non-empty affine subspace $\Gamma$ in the affine space of global sections of $\mathcal{O}(l)$ and $s(z)$, and the second and third conditions defines non-empty open subsets in $\Gamma$. and for big $l$ this conditions define subscheme of codimension at least 1.

Indeed, let’s choose some section $d \in \mathcal{O}(1)$ that is invertible on $z \cup Y$, fix some lift of $s_Y$ to an invertible section $\pi$ of $\mathcal{L}(n)$ on $Y \cup Z(d)$ and consider for each $l$ the affine subspace

$$\Gamma = \{ s \in \Gamma(X', \mathcal{O}(l)) : s|_{Y \cup Z(d)} = \pi, s(z) = 0 \}.$$ 

Let $s_\Gamma \in \Gamma(X' \times \Gamma, \mathcal{O}(l))$ be the universal section. Consider the closed subschemes

$$B_1 = Z(s_\Gamma) \cap Z(d(s_\Gamma/d^l)) \subset X' \times \Gamma,$$

where $d(s_\Gamma/d^l)$ denotes differential of regular function $s_\Gamma/d^l$ on $X' - Y$ that is section of $\omega(X')$ (image is closed since $B_1 \subset Z(s_\Gamma)$ that is finite over $\Gamma$), and

$$B_2 = \text{Supp Coker}\{\pi^*(\mathcal{O}(X')) \otimes \mathcal{O}(X') \xrightarrow{(\varepsilon, s_\Gamma)} \pi \mathcal{O}(l)]\},$$

where $p: Z(s_\Gamma) \to X \times \Gamma$ be composition of injection into $X' \times \Gamma$ and $\pi \times \text{id}_\Gamma$ and $\varepsilon_p$ denotes unit of the adjunction $(p^*, p_*)$ (image is closed since $B_2 \subset (\pi \times \text{id}_\Gamma)_*(Z(s_\Gamma))$ is finite over $\Gamma$).

Then the subset of such sections $s$ that the zero divisor $Z(s)$ has multiple points is contained in the image of $B_1$ under the projection $pr': X' \times \Gamma \to \Gamma$. And subset of such sections $s$ that $\pi|_{Z(s)}$ isn’t injection, is contained in image of $B_2$ under the projection $pr: X \times \Gamma \to \Gamma$. So to prove the claim it is enough to find at least one rational point in $\Gamma - (pr'(B_1) \cup pr(B_2))$.

Since base filed $k$ is infinite, to get the claim it is enough to show that dimensions of $pr'(B_1)$ and $pr(B_2)$ are less then $\dim \Gamma$, and since base changes along filed extensions doesn’t change dimension, it is enough to consider the case of algebraically closed filed $k$.

Let’s note that for $l$ larger some $L$ for each pair of points $x_1, x_2 \in X'$ the restriction homomorphism

$$r_{x_1, x_2, l}: \Gamma(X', \mathcal{O}(l)) \to \Gamma(S(x_1 + x_2 + z + Y + \text{div } d), \mathcal{O}(l))$$

is surjective. Indeed, for each $l$ surjectivity of $r_{x_1, x_2, l}$ is open condition on the pair $(x_1, x_2)$, and lemma [4.1] implies that for each pair of points $(x_1, x_2)$ there is some $L_{(x_1, x_2)}$ such that for all $l > L_{(x_1, x_2)}$, $r_{(x_1, x_2, l)}$ is surjective.

Then for $l > L$,

$$\Gamma \to \{ s \in \Gamma(S(x_1 + x_2 + z + Y + \text{div } d), \mathcal{O}(l)) : s|_{Y \cup Z(d)} = \pi, s(z) = 0 \} \simeq k^2$$

and for any pair $x_1, x_2 \in X' - (Y \cup Z(d))$

$$\text{codim}_\Gamma\{ s \in \Gamma | \text{div } s \geq x_1 + x_2 \} = \begin{cases} 2, & \text{if } x_1, x_2 \neq z, \\ 1, & \text{otherwise}. \end{cases}$$

Hence for almost all points $x \in X'$, the dimension of the fibre of $B_1$ over $x$ is at least 2, and for all $x \in X$, the dimension of the fibre of $B_2$ over $x$ is at lest 2. Thus

$$\text{codim}_\Gamma pr'_k(B_1) \geq 2 - \dim X = 1, \quad \text{codim}_\Gamma pr_k(B_2) \geq 2 - \dim X' > 1.$$ 

□
Lemma 4.2. Suppose \(\varpi: (X', Z') \rightarrow (X, Z)\) is a Nisnevich neighbourhood over a local base scheme \(S\) over an infinite field \(k\) (see def. 3.3) such that \(Z\) is finite over \(S\), and \(\varpi: \overline{X'} \rightarrow \overline{X}\) is a good relative compactification (see def. 3.4); suppose \(z \in X'\) is closed point and \(\Delta\) is primitive divisor in \(X\) finite over \(S\), such that closed fibre of \(\Delta\) is \(z\); then there is an integer \(L\) such that for any \(l > L\) there is a section \(s\) in \(\Gamma(\mathcal{O}(l))\) such that

1) \(Z(s|_{\varpi^{-1}((Z^l))}) \simeq \pi|_{Z'}^{-1}(i_Z^* (\Delta))\), where \(i_Z: Z \rightarrow X\) denotes the closed injection (so the term \(\pi|_{Z'}^{-1}(i_Z^* (\Delta))\) is equal to \(\Delta \cap Z\) considered as subscheme in \(\overline{X'}\) using the isomorphism \(Z' \simeq Z\).)

2) \(Z(s) \subset X', Z(s)\) is reduced and irreducible components \(Z(s)\) don’t intersect to each other.

3) \(Z(s) \rightarrow X\) is closed injection.

Proof. Since \(Z\) is finite over the local scheme \(S\), it is semi-local. Hence any line bundle on \(Z\) is trivial. Since \(\varpi\) induce the isomorphism \(Z' \simeq Z\), for each integer \(l\) there is some isomorphism (4.3)

\[\mathcal{O}(l)|_Z \simeq \varpi^*(\mathcal{L}(\Delta)|_Z).\]

Let \(\delta \in \Gamma(Z, \mathcal{L}(\Delta))\) be a section such that \(\text{div} \ \delta = \Delta\), and since \(\Delta\) is primitive then \(Z(\delta) = \Delta\). Then using isomorphism (4.3) for each \(l\) we get some section

\[\delta' \in \Gamma(Z', \mathcal{O}(l)); \ Z(\delta') \simeq \pi|_{Z'}^{-1}(i_Z^* (\Delta)).\]

Let \(\pi: (X', Z') \rightarrow (X, Z)\) and \(\pi: \overline{X'} \rightarrow \overline{X}\) denotes the fibre of \(\varpi\) over the closed point of \(S\). Let \(z'\) denotes preimage of \(z\) in \(Z'\). Using lemma 4.1 for all \(l\) bigger some \(L\) we find a section \(s_{cf} \in \Gamma(\overline{X'}, \mathcal{O}(l))\) such that

1) \(s_{cf}|_{Z'} = \delta'|_{Z'}, s_{cf}|_{\varpi^{-1}(Z) - Z'}\) is invertible,

2) \(Z(s_{cf}) \subset X', \text{div} \ s_{cf}\) hasn’t multiple points (or equivalently \(Z(s_{cf})\) is reduced),

3) \(\pi\) induce closed injection \(Z(s_{cf}) \rightarrow X\).

(By (4.3) we set \(z = z', Y = (Z' - z') \cup (\overline{X'} \setminus X')\), and \(s_Y\) is any lift of \(\delta'\) to invertible section of \(Y\).)

Let \(s\) be any lift of \(s_{cf}\) and \(\delta'\) to a global section on \(\overline{X'}\), that exists by lemma 3.1 since \(s_{cf}\) and \(\delta'\) are agreed on \(Z'\).

Since \(s_{cf}|_{\varpi^{-1}(Z) - Z'}\) is invertible, it follows that \(s|_{\varpi^{-1}(Z) - Z'}\) is invertible too. Hence \(Z(s|_{\varpi^{-1}(Z)}) = Z(s_{cf})|_{\varpi^{-1}(Z)} = Z(\delta') = \pi|_{Z'}^{-1}(i_Z^* (\Delta))\), and so the first claim of the lemma on \(s\) holds.

By Nakayama’s lemma if the closed fibre of the intersection of two subschemes of a scheme over a local base is empty, then the intersection if these schemes is empty at all. So the second claim on \(s\) follows that \(\text{div} \ s_{cf}\) hasn’t multiple points.

And since by Nakayama’s lemma again, injectivity of \(\pi|_{Z(s_{cf})}: Z(s_{cf}) \rightarrow X\) implies that \(\varpi\) induce injection \(Z(s) \rightarrow X\), it follows that the claim of the third point of lemma holds.

\[\square\]

Lemma 4.4. Let \(\pi: X' \rightarrow X\) be a morphism of relative projective curves over a local base \(U\), with a good compactification (\(\pi: \overline{X'} \rightarrow \overline{X}, \mathcal{O}(l))\) (see def. 3.4), let \(Z \subset X', Z' \subset X'\) be closed subschemes, and \(\Delta \subset X\) be a primitive divisor such that closed fibre of \(\Delta\) is a point \(z \in X\) and

(4.5)

\[Z \times_X \Delta \times_U (U - Z) = \emptyset.\]

Then there are regular functions \(f_0, f_1 \in k[X]\) and \(f \in k[X \times \mathbb{A}^1]\) and a regular map \(l: Z(f_0) \rightarrow X'\) such that

1) \(f_0, f_1\) and \(f\) are relatively finite over \(U\) and \(U \times \mathbb{A}^1\) (see def. 2.17),

2) \(i_{0*}(f) = f_0, i_{1*}(f) = f_1\) where \(i_0, i_1: U \rightarrow U \times \mathbb{A}^1\) denotes zero and unit sections,

3) \(\overline{Z} \times_X \overline{Z}(f) \times_U (U - Z) = \emptyset,

4) \(Z(f_1) = \Delta \cap (Z(f_1) - \Delta), Z \times_X (Z(f_1) - \Delta) = \emptyset\)

5) \(l\) is a lift of the canonical injection \(i_{Z(f_0)}: Z(f_0) \hookrightarrow X, i.e. \ \varpi \circ l = i_{Z(f_0)},\)
6) \( Z' \times X', l(Z(f_0)) \times_U (U - Z_z) = \emptyset \).

**Proof.** Denote \( D = \mathcal{X} \setminus X', D' = \mathcal{X}' \setminus X' \), and denote \( \Delta' = \varpi|_{Z'}^{-1}(i_Z^{-1}(\Delta)) \), where \( i_Z : Z \to X \) denotes the canonical injection (so \( \Delta' \subset X' \) is the image of \( \Delta \cap Z \) after the identification \( Z' \simeq Z \)).

Applying lemma 4.2, we find \( l_0 \) such that for any \( l > L_0 \) there is a global section \( s' \in \Gamma(X', \mathcal{O}(l)) \) such that the restriction \( \varpi|_{Z(s')} : Z(s') \to X' \) is injective, \( Z(s'|\mathcal{O}(l)) = \Delta' \), and such that \( Z(s') \) is reduced and it is the disjoint union of irreducible components.

Hence \( Z(s') \) is disjoint union of irreducible closed subschemes in \( \mathcal{X}' \) of codimension one. The direct image effective divisor \( \text{div } s' \) along \( \varpi \) is in the linear system of the line bundle \( \mathcal{O}(ln) \), where \( n = \deg \varpi \). Then there is a global section
\[
s_0 \in \Gamma(\mathcal{X}, \mathcal{O}(ln)), \text{ div } s_0 = \varpi_*(\text{div } s),
\]
, and then \( Z(s_0) \) is equal to the image of \( Z(s') \) under the injection defined by \( \varpi \). Moreover since \( Z(s'|\mathcal{O}(ln)) = i_Z^{-1}(\Delta) \), it follows that \( Z(s_0)|Z = i_Z^{-1}(\Delta) \).

Thus for each \( l > L_0 \) we have find
\[
(4.6) \quad Z_0 \subset X, l: Z_0 \to X', s_0 \in \Gamma(\mathcal{X}, \mathcal{O}(ln)): Z(s) = Z_0, Z \cap Z_0 = Z \cap \Delta, \pi \circ l = i_Z.
\]

Next by lemma 3.1 there is \( L_1 \) such that for all \( l > L_1 \) we can find a section
\[
s_1 \in \Gamma(\mathcal{X}, \mathcal{O}(ln)): s_1|_{D\cup Z} = s_0|_{D\cup Z}, s_1|_\Delta = 0.
\]

Define
\[
s = ts_1 + (1 - t)s_0 \in \Gamma(\mathcal{X}, \mathcal{O}(ln)),
\]
\[
f_0 = s_0/d^n \in k[X], f_1 = s_1/d^n \in k[X], f = s/d^n \in k[X \times X'].
\]

By lemma 2.19 functions \( f_0, f_1, f \) are relatively finite over \( S \) (see def. 2.17), so the point 1 holds. The point 2 follows from the definition.

The point 5 and 6 follows from (4.6) and from (4.5). The point 3 follows from that \( s|_Z = s_0|_Z \) and from (4.6) and (4.5) too.

To get the claim it is enough to check the point 4. Let \( \delta \in \Gamma(\mathcal{X}, \mathcal{L}(\Delta)) : \text{div } \delta = \Delta \). Since by definition \( s_1|_\Delta = 0 \), it follows that \( s_1/\delta \in \Gamma(\mathcal{X}, \mathcal{L}(-\Delta)(l)) \) is regular section. Then since \( Z(s_1|_\Delta) = Z \cap \Delta, s_1/\delta|_Z \) is invertible, and since the closed fibre of \( \Delta \) is point \( z \) contained in \( Z \), the above implies that \( s_1/\delta|_\Delta \) is invertible too, and hence we get \( Z(s_1/\delta) \cap (Z \cup \Delta) = \emptyset \). Thus
\[
Z(s_1) = Z \cup Z(s_1/\delta), Z \cap Z(s_1/\delta) = \emptyset.
\]

\[\square\]

**Lemma 4.7.** Let \( U \) be a local scheme and \( \pi: (U', Z') \to (U, Z) \) be a Nisnevich neighbourhood of a closed subscheme \( Z' \subset U \). Let \( \mathcal{X} \) be a projective curve over \( U \) with a very ample bundle \( \mathcal{O}(1) \), let \( d \in \Gamma(\mathcal{X}, \mathcal{O}(1)) \), \( X = \mathcal{X} - \mathcal{Z}(d) \), and let \( Z \subset X \) be a closed subscheme finite over \( U \). Let \( \mathcal{X}' = \mathcal{X} \times_U U' \), \( X' = X \times_U U' \), \( Z' = Z \times_U U' \):

![Diagram](image)

Let \( \Delta' \subset X' \) be a primitive divisor finite over \( U' \), and \( \delta \in k[Z] \) be a regular function such that
\[
(4.8) \quad (Z' \cap \Delta') \times_{U'} (U' - Z') = \emptyset, (Z \cap Z(\delta)) \times_U (U - Z) = \emptyset, Z(\delta) \times_U U' = Z' \cap \Delta' \cdot
\]
Then there are a relatively finite (over $U$) regular function $f' \in k[\mathcal{X}]$ a relatively finite (over $U'$ and $U' \times \mathbb{A}^1$) and functions $f_0, f_1 \in k[\mathcal{X}']$, $f \in k[\mathcal{X}' \times \mathbb{A}^1]$ such that

1) $\pi^*_U(f') = f_0$, $\iota^*_h(f) = f_0$, $\iota^*_h(f) = f_1$,
2) $Z' \times _X Z(f) \times _{U'} (U' - Z') = \emptyset$
3) $Z' \times _X Z(f') \times _U (U - Z) = \emptyset$
4) $Z(f_1) = \Delta' \amalg (Z(f_1) - \Delta)$, $(Z') \times _X (Z(f_1) - \Delta') = \emptyset$

Proof. Using lemma 3.1 we choose $l$ such that

\[ (4.9) \quad \Gamma(\mathcal{O}(l)) \rightarrow \Gamma(D \amalg Z, \mathcal{O}(l)), \quad \Gamma(\mathcal{O}(l), L(-\Delta')(l)) \rightarrow \Gamma(D \amalg Z', L(-\Delta')(l)). \]

Schemes $D$ and $Z$ are finite over the local scheme $U$, hence any line bundle restricted to these schemes is trivial. Let’s fix an invertible sections $w \in \Gamma(D, \mathcal{O}(l))$, $e \in \Gamma(Z, \mathcal{O}(l))$ and fix $\delta \in \Gamma(L(\Delta'))$ such that $Z(\delta) = \Delta'$.

Since $Z(e^*(\delta)) = Z' \cap \Delta' = Z(\delta'|Z')$, it follows that the rational section $e^*(\delta)/\delta'|Z'$ is an invertible regular section in $\Gamma(Z', \mathcal{L}(-\Delta')(l))$. Using (4.9) let’s choose sections

\[ (4.10) \quad s' \in \Gamma(\mathcal{O}(l)), \quad s'|_0 = w, \quad s'|_Z = e\delta, \]
\[ r \in \Gamma(\mathcal{L}(-\Delta')(l)), \quad r|_D = e^*(w)/\delta'|_D, \quad r|_Z = e^*(\delta)/\delta'|_Z. \]

Then from the above we get $Z(r) \cap (D \cup Z') = \emptyset$, and since (4.8) implies that the closed fibre of $\Delta$ is contained in $Z'$, then

\[ (4.11) \quad Z(r) \cap (D \cup Z' \cup \Delta') = \emptyset. \]

Now let’s put

\[ s_0 = e^*(s') \in \Gamma(\mathcal{O}(l)), \quad s_1 = r\delta \in \Gamma(\mathcal{L}(l)), \quad s = (1 - t)s_1 + ts_0 \in \Gamma(\mathcal{O}(l)) \]
\[ f' = s'/d \in k[\mathcal{X}], \quad f_0 = s_0/d \in k[\mathcal{X}], \quad f_1 = s_1/d \in k[\mathcal{X}], \quad f = s/d \in k[\mathcal{X}' \times \mathbb{A}^1]. \]

Then by definition these functions satisfy the claim of the point 1 of the lemma, and by lemma 2.19 $f'$ and $f_0$, $f_1$ and $f$ are relatively finite over $U$ and $U' \times \mathbb{A}^1$. Next from (4.11) we get

\[ Z(f_1) = \Delta' \amalg (Z(f_1) - \Delta), \quad (Z') \times _X (Z(f_1) - \Delta') = \emptyset \]

that is the claim of the point 4).

So to get the claim of the lemma it is enough to check points 2 and 3. So the claim follows from that $Z(s'|_Z) = Z(\delta) Z(s|_Z) = Z' \cap \Delta$ and from (4.8).

\[ \square \]

5. Zariski excision on affine line

Theorem 5.1. Suppose $\mathcal{F}$ is homotopy invariant sheaf with $GW$-transfers over a field $k$ and $U$ is a local essential smooth scheme $(U = S_s, \ s \in S, \ S \in Sm_L)$; then for any Zariski open subscheme $V \subset \mathcal{A}^n_U$ such that $0_U \subset \mathcal{A}^n_U$, the restriction homomorphism

\[ i^*: \frac{\mathcal{F}(\mathcal{A}^n_U - 0_U)}{\mathcal{F}(\mathcal{A}^n_U)} \rightarrow \frac{\mathcal{F}(V - 0_U)}{\mathcal{F}(V)} \]

is an isomorphism, where $i: V \hookrightarrow \mathcal{A}^n_U$ denotes the open immersion.

Proposition 5.2. Suppose $i: V \hookrightarrow \mathcal{A}^n_U$ is a morphism as in theorem, then there is a morphism $\Phi_i \in GWoCor((\mathcal{A}^n_U, 0_U), (V, V - 0_U))$ such that

\[ i \circ \Phi_i \sim id_{(\mathcal{A}^n_U, 0_U)} \in GWoCor((\mathcal{A}^n_U, 0_U), (\mathcal{A}^n_U, 0_U)). \]
Proof. Consider the following divisors in the relative projective line over $\mathbb{A}_U$:

$$T = \mathbb{P}^1_{\mathbb{A}_U} \setminus A^1_{\mathbb{A}_U}, \quad D = \mathbb{P}^1_{\mathbb{A}_U} \setminus (V \times_U A^1_U), \quad Z = 0 \times A^1_U, \quad \Delta = (\mathbb{A}^1_U \hookrightarrow \mathbb{P}^1_U) \subset \mathbb{P}^1_{\mathbb{A}_U \times U}.$$  

Then let’s fix sections

$$\mu, \nu, \delta \in \Gamma(\mathbb{P}^1_{\mathbb{A}_U}, \mathcal{L}(1)):\ div_0 \mu = T, \ div_0 \nu = 0_{\mathbb{A}_U}, \ div_0 \delta = \Delta, \ \nu|_T = \delta|_T.$$  

Since $0_U \subset V$ and consequently $Z \cap D = 0$, it follows by lemma 3.1 that for a sufficiently large $l$ there is a section

$$s_0 \in \Gamma(\mathbb{P}^1_{\mathbb{A}_U}, \mathcal{L}(l)) : s_0|_D = \nu^1|_D, \ s_0|_Z = \delta \mu^{l-1}|_Z$$

$$g \in \Gamma(\mathbb{P}^1_{\mathbb{A}_U}, \mathcal{L}(l-1)) : g|_\Delta = \mu^{l-1}|_\Delta, \ g|_Z = \mu^{l-1}|_Z, \ g|_T = \nu^{l-1}. $$

Let $s = s_0 \cdot (1-t) + \delta g \cdot t \in \Gamma(\mathbb{P}^1_{\mathbb{A}_U}, \mathcal{L}(l))$. Then

$$s|_{Z \times A^1} = \delta \mu^{l-1}, \ s|_{T \times A^1} = \delta|_{T \times A^1}, \nu^{l-1}|_{T \times A^1} = \nu^{l-1}. $$

By lemma 2.10 functions $s_0/\mu^l \in k[A^1_{\mathbb{A}_U}]$ and $s/\mu^l \in k[A^1_{\mathbb{A}_U \times A^1}]$ are relatively finite, then we can apply construction from proposition 2.18 and put

$$Q_0 = \langle dy, s_0/\mu^l \rangle = \langle k[Z_0], s_0 \rangle \in Q(\mathcal{P}(Z_0 \to A^1_U)), \ Q = \langle dy, s/\mu^l \rangle = \langle k[Z], s \rangle \in Q(\mathcal{P}(Z \to A^1_U \times A^1)),$$

where

$$Z_0 = Z(s_0) \subset A^1_{\mathbb{A}_U}, \quad Z = Z(s) \subset A^1_{\mathbb{A}_U \times A^1}.$$  

Again since $0_U \subset V$, it follows that $s_0|_D = \nu^1|_D$ is invertible, and hence $Z_0 \subset V \times_U A^1_U$. Then since $s|_{T \times A^1} = \nu^1|_{T \times A^1}$ is invertible, it follows that $Z \subset A^1_U \times A^1_U$. Let’s denote by

$$i_0 : Z_0 \to V \times_U A^1_U, \quad i_Z : Z \to A^1_U \times A^1_U = A^1_{\mathbb{A}_U}$$

the canonical closed injections.

Next since $Z(\delta) = \Delta$ and so $\delta$ is invertible on $0_U \times_U (V - 0_U) = 0 \times (V - 0_U)$ and consequently $s_0$ and $s$ are invertible on $0 \times (A^1_U - 0_U)$ and $0 \times (A^1_U - 0_U)$ respectively, it follows that

$$0_U \times V \times_U (A^1 - 0_U) = 0, \quad 0 \times A^1_U \times_U (A^1_U - 0_U) = 0.$$  

Hence by lemma 2.10 the quadratic spaces $Z_0 \circ Q_0$ and $i_Z \circ Q$ defines GW-corrrespondences between pairs

$$\Phi = [i_0 \circ Z_0 \circ Q_0] \in GWCor((A^1_U, A^1_U - 0_U), (V, V - 0_U)), \quad \Theta = [i_Z \circ Q] \in GWCor((A^1_U \times A^1, (A^1 - 0_U) \times A^1), (A^1_U, A^1_U - 0_U)).$$

Thus

$$i \circ \Phi = [i \circ i_0 \circ (dy, s_0/\mu^l)] = [i \circ i_0 \circ i_0 \circ (\langle dy, s/\mu^l \rangle)] = [i_0 \circ \langle dy, s/\mu^l \rangle \circ i_0] = \tilde{\Theta} \circ i_0.$$  

On other side, since $g|_\Delta = \mu^{l-1}$ then $Z(\delta g) = \Delta \Pi_1(g)$, and since $g|_Z = \mu^{l-1}$ then $Z(g) \subset A^1_U - 0_U$, using lemma 2.13 we get that

$$\tilde{\Theta} \circ i_1 = [i_Z(\delta g) \circ \langle dy, \delta g \rangle] = [(k[\Delta], u)] \in GWCor((A^1_U, A^1_U - 0_U), (A^1_U, A^1_U - 0_U)), $$

for some invertible $u \in k[A^1_U]$*. Thus if we put

$$\Phi = \Phi \circ [(u^{-1})], \quad \Theta = \tilde{\Theta} \circ [(u^{-1})],$$

then

$$\Theta \circ i_0 = i \circ \Phi, \quad \Theta \circ i_1 = [(k[\Delta], 1)] = id(A^1_U, A^1_U - 0_U),$$

and so $i \circ \Phi \cong id(A^1_U, A^1_U - 0_U)$.  

\hfill \square
Proposition 5.3. Suppose \( i: V \hookrightarrow A^1_U \) is a morphism as in theorem [5.7]; then there is a morphism 
\[ \Phi \in \text{GWCor}((A^1_U, A^1_U - 0_U), (V, V - 0_U)) \]

\[ \Phi \circ i \cong \text{id}_{(V, V - 0_U)} \in \text{GWCor}((V, V - 0_U), (V, V - 0_U)). \]

Proof. Similarly as in the proof of proposition [5.2] let’s denote:

\[ T = \mathbb{P}^1_{A_U^1} \setminus A^1_U, \quad T' = \mathbb{P}^1_{A_U^1} \setminus A^1_U, \]

\[ D = (\mathbb{P}^1_U \setminus V) \times U A^1_U \subset \mathbb{P}^1_{A_U^1}, \quad Z = 0 \times A^1_U \subset \mathbb{P}^1_{A_U^1}, \quad D' = D \times A^1_U, \quad V \subset \mathbb{P}^1_V, \quad Z' = 0 \times V \subset \mathbb{P}^1_V, \]

\[ \Delta = \Gamma(A^1_U \hookrightarrow \mathbb{P}^1_U) \subset \mathbb{P}^1_{A^1_U \times U}, \quad \Delta' = \Gamma(V \hookrightarrow \mathbb{P}^1_U) \subset \mathbb{P}^1_V, \]

and fix some sections

\[ \mu, \nu, \delta \in \Gamma(\mathbb{P}^1_{A^1_U}, \mathcal{L}(1)): \text{div}_0 \mu = T, \quad \text{div}_0 \nu = Z, \quad \text{div}_0 \delta = \Delta, \quad \nu|_T = \delta|_T. \]

Also let’s denote by \( v: \mathbb{P}^1_U \hookrightarrow \mathbb{P}^1_{A^1_U} \) the immersion defined by the base change.

Since \( D \cap V = \emptyset \), and consequently \( \Delta \cap D' = \emptyset \), then \( \delta \) is invertible on \( D_V \). Let’s denote the inverse section by \( \delta^{-1} \in \Gamma(\mathcal{L}(-1)|_{D'}) \). Next by lemma [3.1] for sufficiently large \( l \) there exist sections

\[ s' \in \Gamma(\mathcal{L}(l), \mathbb{P}^1_{A^1_U}); \quad s'|_{D} = \nu', \quad s'|_{Z} = \mu^{l-1} \cdot \delta \]

\[ g \in \Gamma(\mathcal{L}((l - 1), \mathbb{P}^1 \times V)): \quad g|_{D_V} = \nu^{l} \cdot \delta^{-1}, \quad g|_{\Delta} = \mu^{l-1}, \quad g|_{0_U \times V} = \mu^{l-1}. \]

Then we can define sections

\[ s_0 \in \Gamma(\mathcal{L}(l), \mathbb{P}^1 \times V): \quad s_0 = v^*(s') \]

\[ s_1 \in \Gamma(\mathcal{L}(l), \mathbb{P}^1_U): \quad s_1 = g \cdot \delta \]

\[ s \in \Gamma(\mathcal{L}(l), \mathbb{P}^1_{V \times A^1}): \quad s = s_0 \cdot (1-t) + s_1 \cdot t. \]

Then

\[ s|_{D' \times A^1} = s_0|_{D'} = s_1|_{D'} = \nu'|_{D}, \quad s|_{Z' \times A^1} = s_0|_{Z'} = s_1|_{Z'} = \mu^{l-1} \cdot \delta, \quad \text{div} s_1 = \text{div} g \prod \Delta. \]

By lemma [2.19] functions \( s' / \nu' \in k[A^1_U] \) and \( s / \nu \in k[A^1_U \times A^1] \) are relatively finite, then we can apply construction from proposition [2.15] and put

\[ Q' = \langle dy, s_0 / \nu' \rangle = \langle k[Z'], q' \rangle \in Q(\mathbb{P}(Z' \to A^1_U)), \quad Q = \langle dy, s / \nu \rangle = \langle k[Z], q \rangle \in Q(\mathbb{P}(Z \to V \times A^1)), \]

where \( Z' = Z(s') \subset A^1_U, \quad Z = Z(s) \subset A^1_U \times A^1 \).

Since \( 0_U \subset V \), it follows that \( s'|_D = \nu'|_D \) is invertible, and consequently \( Z' \subset V \times U A^1_U \). Since \( s'|_{T \times A^1} = \nu'|_{T \times A^1} \) is invertible, it follow that \( Z \subset V \times U \). Let’s denote by

\[ i_{Z'}: Z' \hookrightarrow V \times U A^1_U, \quad i_Z: Z \hookrightarrow V \times U \times V \times A^1, \quad i_{Z_0}: Z' = Z(s_0) \hookrightarrow V \times U \times V, \quad i_{Z_1}: Z(s_1) \hookrightarrow V \times U \times V \]

the canonical closed injections.

Next since \( Z(\delta) = \Delta \) and so \( \delta \) is invertible on \( 0_U \times U (V - 0_U) \neq 0 \times (V - 0_U) \) and consequently \( s_0 \) and \( s \) are invertible on \( 0 \times (A^1_U - 0_U) \) and \( 0 \times (V - 0_U) \times A^1 \) respectively, it follows that

\[ 0_U \times V (Z' \times A^1 (A^1_U - 0_U)) = \emptyset, \quad 0 \times A^1, Z \times (V - 0_U) = \emptyset. \]

Hence by lemma [2.16] the quadratic spaces \( i_{Z'} \circ Q' \) and \( i_Z \circ Q \) define GW-correspondences between pairs

\[ \tilde{\Phi} = [i_{Z'} \circ Q'] \in \text{GWCor}((A^1_U, A^1_U - 0_U), (V, V - 0_U)), \]

\[ \tilde{\Theta} = [i_Z \circ Q] \in \text{GWCor}(((V \times A^1), (V - 0_U) \times A^1), (V, V - 0_U)). \]

Then

\[ \tilde{\Phi} \circ i = [i_{Z'} \circ (dy, i^*(s / \nu'))] = [i_{Z_0} \circ (dy, s_0 / \nu')] = [i_Z \circ (dy, i^0(s / \nu'))] = [i_Z \circ (dy, s / \nu') \circ i_0] = \tilde{\Theta} \circ i_0. \]
and using lemma 2.16 we get that
\[ \Theta \circ i_1 = [\{Z \circ (dy, s_1/v')\} = ([k[\Delta'], u')] + ([k[Z_1 - \Delta'], q_1]) = ([k[\Delta'], u']), \]
for some invertible \( u' \in k[\Delta']^* \). Thus if we put
\[ \Phi = ([u^{-1}]) \circ \Phi', \Theta = ([u^{-1}]) \circ \Theta, \]
then \( \Theta \circ i_0 = \Phi \circ i, \Theta \circ i_1 = ([k[\Delta'], 1]) = \text{id}_{(V,V'-0)}, \) and so \( i \circ \Phi \sim \text{id}_{(V,V'-0)}. \)

**Proof of the theorem 5.1.** As noted in remark 2.14, for homotopy invariant presheaf with GW-transfers \( F \), formula
\[ GWCor_{\text{pair}}(Y,U) \implies \text{Coker}(F(Y) \to F(U)) \]
defines homotopy invariant presheaf on the category \( GWC_{\text{pair}} \). Hence the injectivity of the homomorphism \( i^* \) follows from proposition 5.2, and the surjectivity follows from proposition 5.3.

**Theorem 5.4.** Let \( F \) be a homotopy invariant presheaf with GW-transfers over field \( k \) and \( K \) be a geometric extension \( K/k \). Then for any Zariski open subschemes \( U \subset V \subset K \) and point \( z \in U \), the restriction homomorphism
\[ i^*: F(V - z)/F(V) \to F(U - z)/F(U) \]
is an isomorphism, where \( i: U \hookrightarrow V \) denotes the open immersion.

**Proof.** The proof is similar to the proof of theorem 5.1, but we should change the section
\[ \nu \in \Gamma(\mathbb{P}^1_k, O(1)): div_0 \nu = 0 \times \mathbb{A}^1_k \]
in the construction of the inverse morphisms by a section of the line bundle with zero divisor being any divisor of the form \( x \times \mathbb{A}^1_k \), where \( x \in V \) is a rational point over \( K \). Such point always exists if \( k \) is infinite.

**Corollary 5.5.** Let \( F \) be a homotopy invariant presheaf with GW-transfers over field \( k \) and \( K \) be a geometric extension \( K/k \). Then for any Zariski open subschemes \( U \subset V \subset K \) and point \( z \in U \), the restriction homomorphism
\[ i^*: F(V - z)/F(V) \to F(U - z)/F(U) \]
is an isomorphism, where \( i: U \hookrightarrow V \) denotes the open immersion.

**Proof.** The claim follows by isomorphisms \( F(V - z)/F(V) \simeq F(\mathbb{A}^1_k - z)/F(\mathbb{A}^1_k) \simeq F(U - z)/F(U) \).

6. **Étale excision**

**Theorem 6.1.** Let \( F \) be a homotopy invariant presheaf with GW-transfers and \( \pi: X' \to X \) be an étale morphism of smooth schemes over a geometric extension \( K/k \); let \( Z \subset X \) be a reduced closed subscheme of codimension one such that \( \pi \) induces the isomorphism of \( Z \) and the preimage \( Z' = \pi^{-1}(Z) \); and let \( z \in Z \) and \( z' \text{prime} \in Z' \) are points such that \( \pi(z') = z \).

Then \( \pi \) induces the isomorphism
\[ \pi^*: \frac{F(X_z - Z_z)}{F(X_z)} \to \frac{F(X'_z - Z'_z)}{F(X'_z)}. \]
Proposition 6.2. Suppose \( \pi, X, X', Z, Z', z, z' \), as in theorem [6.1], then there is a morphism \( \Phi_1 \in \text{GWCor}((X, Z)_z, (X', Z')) \) such that \( \pi \circ \Phi_1 = i_2 \) \( i_2 \in \text{GWCor}((X, Z)_z, (X, Z)) \), where \( i_2 : (X, Z)_z \hookrightarrow (X, Z) \) denotes the canonical morphism of pairs and \( \pi \) is considered as a morphism \( \pi : (X', Z') \rightarrow (X, Z) \).

Proof. In terms of def. [3.3] the assumptions of the theorem [6.1] give us a Nisnevich neighbourhood \( (X', Z') \rightarrow (X, Z) \), such that \( \text{codim} Z' = \text{codim} Z = 1 \) and points \( z' \in X' \) and \( z \in X \), \( \pi(z') = z \).

Using lemma [3.7] we modify it to (a relative) Nisnevich neighbourhood \( \varpi : (\tilde{X}', \tilde{Z}') \rightarrow (\tilde{X}, \tilde{Z}) \) over some essential smooth local base \( S \) equipped with a good relative compactification \( \overline{\varpi} : \overline{\tilde{X}} \rightarrow \overline{\varpi} \).

Lemma [5.6] implies in addition that \( X' \) and \( X \) are smooth over \( S \), there are trivialisations of relative canonical classes \( \mu' : \omega_S(X') \simeq \mathcal{O}(X') \) and \( \mu : \omega_S(X) \simeq \mathcal{O}(X) \), and there is a very ample bundle \( \mathcal{O}(1) \) on \( \overline{X} \), such that \( \overline{\varpi} (\mathcal{O}(1)) \) is very ample too. To shortify notations let’s denote \( \overline{\varpi} (\mathcal{O}(1)) \) by the same symbol \( \mathcal{O}(1) \).

Denote \( U = X_z \) and consider the base change along \( X_z \rightarrow S \) (see the first digram of (6.6)).

Set \( (X, Z) = (\tilde{X}, \tilde{Z}) \times_S U, \overline{X} = X \times_S U, (X', Z') = (X', \tilde{X}') \times_S U, \overline{X}' = X' \times_S U \) (note that \( Z = Z \times_S U, Z' = Z' \times_S U \)).

Denote by \( \Delta \) the graph of the canonical embedding \( U = X_z = \tilde{X}_z \hookrightarrow \tilde{X} \) considered as a closed subset in \( \overline{X} \), and let \( \Delta' = \varpi|_{Z'}^{-1}(i_Z^{-1}(\Delta)) \), where \( i_Z : Z \rightarrow X \) denotes the canonical injection (so \( \Delta' \subset X' \) is the image of \( \Delta \cap \tilde{Z} \) after the identification \( Z' \simeq Z \)).

Then \( Z \cap \Delta \) is equal to the diagonal in \( Z_z \times_S Z_z \) and so \( Z \times_{X} \Delta \times_U (U - Z) = \emptyset \). Hence we can apply lemma [4.4] and find relatively finite (over \( X_z \)) regular functions \( f_0, f_1 \in k[X] \) and \( f \in k[A^1] \) (see def. [2.17]) such that

1) \( i_0(f) = f_0, i_1(f) = f_1 \), where \( i_0, i_1 : U \ightarrow U \times A^1 \) denote zero and unit sections;
2) and the following conditions holds:

\[
\begin{align*}
(6.3) & \quad Z(f) \times_{U \times A^1} (U \times A^1 - Z_z \times A^1) = (X - Z) \times_X Z(f) \times_{U \times A^1} (U \times A^1 - Z_z \times A^1) \\
(6.4) & \quad Z(f_1) = \Delta \cup (Z(f_1) - \Delta), (Z(f_1) - \Delta) = (X - Z) \times_X (Z(f_1) - \Delta)
\end{align*}
\]

3) there is a lift \( l : Z(f_0) \rightarrow X' \) of the canonical injection \( i_{Z(f_0)} : Z(f_0) \hookrightarrow X \) such that

\[
(6.5) \quad l(Z(f_0)) \times_U (U - Z_z) = (X' - Z') \times_X l(Z(f_0)) \times_U (U - Z_z).
\]

The inverse images of the trivialisation \( \mu \) define the trivialisations \( \mu_X : \omega_X(X) \simeq \mathcal{O}(X) \), \( \mu_{X_z \times A^1} : \omega_{X_z \times A^1} \simeq \mathcal{O}(X_z \times A^1) \), so the base changes along the zero and unit sections of \( X_z \times A^1 \) give us morphisms of orientated relative curves with relative finite functions (see the following diagram and see def. [2.17] for the notion of oriented curve with relative finite functions).

Then applying the construction from proposition [2.18] we get the quadratic spaces

\[
\begin{align*}
(k[Z(f_0)], q_0) = (X, \mu_X, f_0) & \in Q(\mathcal{P}(Z(f_0) \rightarrow X_z)), \\
(k[Z(f_1)], q_1) = (X, \mu_X, f_1) & \in Q(\mathcal{P}(Z(f_1) \rightarrow X_z)), \\
(k[Z(f)], q) = (X, \mu_{X_z \times A^1}, f) & \in Q(\mathcal{P}(Z(f) \rightarrow X_z \times A^1)),
\end{align*}
\]
where the second and third equalities follows from lemma 2.16 and (6.3). Thus if we replace $\Phi$ and $\Theta$ by $\Phi \circ (u^{-1})$ and $\Theta \circ (u^{-1})$, the claim follows.
Proposition 6.7. Suppose $\pi, X, X', Z, Z', z, z'$ satisfy the assumptions as in theorem 6.4 then there is a morphism $\Phi_\pi \in GWCor((X_z, Z_z), (X', Z'))$ such that

$$\Phi \circ \pi^1 \sim i^1_z \in GWCor((X'_{z'}, Z'_{z'}), (X', Z')),$$

where $i_z : (X_z, Z_z) \hookrightarrow (X, Z)$ denotes the canonical morphism of pairs, and $\pi_z : (X'_{z'}, Z'_{z'}) \rightarrow (X_z, Z_z)$ is a morphism of Nisnevich neighbourhoods of local schemes induced by $\pi$.

Proof. We start by the same way as in proof of proposition 6.2 this means that using lemma 3.7 we modify Nisnevich neighbourhood $(X', Z') \rightarrow (X, Z)$, to a (relative) Nisnevich neighbourhood $\varpi : (\tilde{X}', \tilde{Z}') \rightarrow (\tilde{X}, \tilde{Z})$ over some essential smooth local base $S$ equipped with good relative compactification $\varpi : \tilde{X} \rightarrow X$, such that $X'$ and $X$ are smooth over $S$ and there are trivialisations of relative canonical classes $\omega : \omega_S(X') \simeq \mathcal{O}(X')$ and $\omega : \omega_S(X) \simeq \mathcal{O}(X)$, and such that there is a very ample bundle $\mathcal{O}(1)$ on $\tilde{X}$, such that $\varpi^*(\mathcal{O}(1))$ is very ample. Similarly as in proof of the proposition 6.2 to shortify notations let’s denote $\varpi^*(\mathcal{O}(1))$ by the same symbol $\mathcal{O}(1)$.

Denote $U = X_z, U' = X'_{z'}, \pi_z : U' \rightarrow U$ and let’s shrink $U'$ in such way that $Z'_{z'} = \pi^{-1}_z(Z_z)$. Then consider the base changes of $X' \rightarrow S$ along $U \rightarrow S$ and $U' \rightarrow S$ (see diagram 6.12), and denote

$$(\mathcal{X}', \mathcal{Z}') = (\tilde{X}', \tilde{X}') \times_S U, (\mathcal{X}'', \mathcal{Z}'') = (\tilde{X}', \tilde{X}') \times_S U'.$$

Denote by $\Delta' \subset X'$ the graph of the morphism of the canonical classes $U' = X'_{z'} = \tilde{X}'_{z'} \rightarrow \tilde{X}'$. Then $Z' \cap \Delta'$ is equal to the diagonal $\Delta Z' \subset Z' \times_S Z'_{z'}$, i.e. the closed subscheme in $\tilde{X}'$ that is the graph of the composition $Z'_{z'} \rightarrow U'_{z'} = \tilde{X}'_{z'} \rightarrow \tilde{X}'$.

Next we want to apply lemma 3.7 to the morphism $\pi_z : (U', Z') \rightarrow (U, Z)$, the curves $X'$ over $U$ and $X''$ over $U'$, the closed subschemes $Z' \subset X'$, $Z'' \subset X''$, and the divisor $\Delta' \subset X''$. To do this we should find a regular function $\delta \in k[Z]$, such that $Z(\delta)$ is equal to the subscheme $\Delta Z'$ that is the graph of the composition $Z_z \simeq Z'_{z'} \rightarrow U'_{z'} = \tilde{X}'_{z'} \rightarrow \tilde{X}'$ (i.e. to the diagonal in $Z_z \times S Z_z$). Since $Z_z = Z \times_S U \simeq Z \times_S U \rightarrow U \times_S U$, the scheme $Z'$ can be identified with a closed subscheme in $U \times_S U$. Let $\delta$ denote the inverse image to $Z'$ of some section of the line bundle $\mathcal{L}(\Delta U)$ on $U \times_S U$ that zero divisor is diagonal. Since the scheme $Z'$ is finite over the local scheme $U$, and consequently any line bundle on $Z'$ is trivial, the section $\delta$ can be considered as a regular function that gives us the required function $\delta$. Thus using lemma 3.7 we find a regular function $f' \in k[\mathcal{X}']$ that is relatively finite over $U$ and functions $f \in k[\mathcal{X}' \times \mathcal{A}]$, $f_0, f_1 \in k[\mathcal{X}'']$ that are relatively finite over $U'$, such that

$$(6.8) \quad \pi_z^1(f') = f_0, i_z^0(f') = f_0, i_z^1(f') = f_1,$$

and such that

$$(6.9) \quad Z' \times_X, Z(f) \times_U, (U' \rightarrow Z'_{z'}) = \emptyset$$

$$(6.10) \quad Z' \times_X, Z(f') \times_U, (U' \rightarrow Z'_{z'}) = \emptyset$$

$$(6.11) \quad Z(f_1) = \Delta' \cap Z(f_1) - \Delta, Z' \times_X, (Z(f_1) - \Delta') = \emptyset.$$

Now similarly as in proposition 6.2 define the base changes of the trivialisation $\mu : \mu_U : \omega_U(X') \simeq \mathcal{O}(\mathcal{X}')$, $\mu_U \times \mathcal{A} : \omega_U \times \mathcal{A}(X' \times \mathcal{A}) \simeq \mathcal{O}(\mathcal{X}' \times \mathcal{A})$. Then we get the first one of three following diagrams, which is the diagram of oriented relative curves with finite functions (see def. 2.17).

Next applying construction from proposition 2.13 we get quadratic spaces

$$\langle k[Z(f)], q' \rangle = (\mathcal{X}', \mu_U, f') \in Q(P(Z(f) \rightarrow U)), (k[Z(f)], q_0) = (\mathcal{X}'', \mu_U, f_0) \in Q(Z_1 \rightarrow X_z),$$

$$(k[Z], q) = (\mathcal{X}'', \mu_U \times \mathcal{A}, f) \in Q(P(Z(f) \rightarrow U' \times \mathcal{A})),$$

$$(k[Z(f)], q_1) = (\mathcal{X}'', \mu_U, f_1) \in Q(P(Z_1 \rightarrow X_z)).$$
and \( \Theta \) by \( \Phi \circ (u^{-1}) \) and \( \Theta \circ (u^{-1}) \), where \( u \in k[U]^* \) is any invertible regular function such that \( u|_{Z_0} = u'|_{Z_0'} \), (we use that \( Z' \simeq Z \)).
To finish the proof of the proposition it is enough to prove that

\[ j \circ (u'/u) \sim j \in WCor((X', X' - Z'), (V, V - Z)), \]

where \( j: (U', U' - Z'_{\text{rel}}) \to (X', X' - Z') \). Consider affine Zariski neighbourhood of \( z' \) in \( X' \) with a lift of the function \( u \) to a regular invertible function \( \bar{u} \), and consider two-degree covering \( c: W = \text{Spec}k[V]/(w^2 - \bar{u}) \to V \), which is etale in some neighbourhood of \( Z \), since \( u'/u(z') = 1 \) and \( \text{char } k \neq 2 \). Let’s shrink \( V \) and \( W \) in such way that \( c \) becomes etale, and denote by \( Z'' \subset U'' \) the closed subscheme defined by the ideal \( (w - 1) \) in \( k[Z'][w]/(w^2 - 1) \). Then \( c: (U'', z'') \to (U', z') \) is Nisnevich neighbourhood.

Since the inverse image of \( \bar{u} \) in \( k[W] \) is equal to the square function \( w^2 \), it follows that \( \left[ (\bar{u}) \circ \text{id}_{(V,V-Z)} \circ d \right] = \left[ (w^2) \circ \text{id}_{(V,V-Z)} \circ d \right] = \left[ (1) \circ \text{id}_{(V,V-Z)} \circ d \circ \Phi \right] \in \text{GWCor}((W, W - Z), (V, V - Z)) \). By proposition \([6.2]\) there is a GW-correspondence \( \Phi \in \text{GCor}((U', U' - Z'_{\text{rel}}), (W, W - Z)) \) left inverse to the class of morphism \( c \). Hence if we denote \( j_V^X : (V, V - Z) \to (X', X' - Z') \), \( j_V^U : (U', U' - Z'_{\text{rel}}) \to (V, V - Z) \),

\[
[j \circ (u) \circ \text{id}_{(U', U' - Z'_{\text{rel}})}] = [j_V^X \circ (\bar{u}) \circ j_V^U] = [j_V^X \circ (\bar{u}) \circ \text{id}_{(V,V-Z)} \circ c \circ \Phi] = [j_V^X \circ (1) \circ \text{id}_{(V,V-Z)} \circ c \circ \Phi] = [j_V^X \circ j_V^U] = [j] \in \text{GWCor}((U', U' - Z'_{\text{rel}}), (X', X' - Z'))
\]

\( \square \)

**Proof of the theorem \([6.1]\)*** As noted in remark \([2.14]\) for homotopy invariant presheave with GW-transfers \( \mathcal{F} \), the formula

\[
\text{GWCor}^{\text{pair}}(Y, U) \quad \rightarrow \quad \text{Ab} \quad \text{Coker}(\mathcal{F}(Y) \to \mathcal{F}(U))
\]

defines the homotopy invariant presheave on the category \( \text{GWCor}^{\text{pair}} \), and since the injective limit functor is exact,

\[
\text{Coker}(\mathcal{F}(X_z) \to \mathcal{F}(X_z - Z_z)) = \lim_{U: z \in U \subset X} \text{Coker}(\mathcal{F}(U) \to \mathcal{F}(U - Z)).
\]

Hence the injectivity of the homomorphism \( \pi^* \) follows from proposition \([6.2]\) and the surjectivity from proposition \([6.7]\) \( \square \)

7. **Injectivity**

**Theorem 7.1.** Let \( \mathcal{F} \) be a homotopy invariant presheave with GW-transfers over field \( k \) and \( K \) be a geometric extension \( K/k \) (i.e. field of functions of some variety). Then for any Zariski open subschemes \( U \subset V \subset K \) the restriction homomorphism

\[
i^*: \mathcal{F}(V) \to \mathcal{F}(U)
\]

is injective, where \( i: U \hookrightarrow V \) denotes the open immersion.

**Proposition 7.2.** For a morphism \( i: U \hookrightarrow V \) satisfying the assumptions of theorem \([7.1]\) there is a morphism \( \Phi \in \text{GWCor}(V, U) \) such that

\[
i \circ \Phi \sim \text{id}_U \in \text{GWCor}(V, V).
\]

**Proof.** Lemma \([2.10]\) implies that is is enough to consider the case \( K = k \). Denote divisors on relative projective line \( \infty V = \mathbb{P}^1, T = \mathbb{A}_k^1 \setminus \mathbb{A}_k^1 \times V \subset \mathbb{P}^1, D = V \times V \setminus U \times V \subset \mathbb{P}^1, \Delta = \Gamma(V \hookrightarrow \mathbb{P}^1) \), i.e. \( \Delta \) is diagonal in \( V \times V \), and let’s fix sections

\[
\mu, \nu, \delta \in \Gamma(\mathbb{P}^1, \mathcal{L}(1)): d\mu = \infty V, d\nu = 0_V, d\delta = \Delta, \nu|_{\infty V} = \delta|_{\infty V}.
\]
Proposition 7.4. For any essential smooth local scheme $U$, and a closed subscheme $Z \subset U$, the restriction homomorphism

$$i^* : \mathcal{F}(V) \to \mathcal{F}(U)$$

is injective.

Theorem 7.3. For any essential smooth local scheme $U$, and a closed subscheme $Z \subset U$, the

$$i^* : \mathcal{F}(V) \to \mathcal{F}(U)$$

is injective.
Proof. In the same way as in lemma 3.7 we change \( X \) to a relative smooth curve with good compactification. Let’s repeat this construction in this situation: Firstly, shrink \( X \) in such way that canonical clase of \( X \) becomes trivial. Next, consider decomposition \( X \xrightarrow{u} \mathbb{X} \xrightarrow{p} \mathbb{A}^d \), of the etale morphism \( e: X \to \mathbb{A}^d \) (\( d = \text{dim} \ X \)), where \( u \) is dense open immersion and \( p \) is finite, given by proposition 3.6. Then using lemma 3.3 find projection \( pr: \mathbb{A}^d \to \mathbb{A}^{d-1} \) such that the restrictions \( p\mathbb{X} \setminus X \to \mathbb{A}^{d-1} \) and \( p(Z) \to \mathbb{A}^{d-1} \) are finite.

Now using the base change along the projection \( U = X_z \to \mathbb{A}^{d-1} \) we get the curve \( \overline{\mathbb{X}} = U \times_{\mathbb{A}^{d-1}} \mathbb{X} \) with the finite morphism \( \pi: \overline{\mathbb{X}} \to \mathbb{A}^1 \) and the smooth open subscheme \( \mathcal{X} = U \times_{\mathbb{A}^{d-1}} X \) such that \( p \) is etale on \( \mathcal{X} \) and \( \overline{\mathbb{X}} \setminus \mathcal{X} \) is finite over \( U \).

Next we consider immersion \( \mathbb{A}^1 \times \mathcal{X} \to \mathbb{P}^1_U \) and again applying Zariski main theorem (proposition 3.6) we get decomposition

\[
\begin{array}{ccc}
\mathbb{A}^1_U & \xrightarrow{\pi} & \overline{\mathbb{X}} \\
\downarrow & & \downarrow \ev \circ \iota \\
\mathbb{P}^1_U & \xrightarrow{\pi} & \mathbb{X}
\end{array}
\]

Let \( \Delta_z = \Gamma(z \to X) \in \mathcal{X} \) be a closed point (the diagonal in \( \mathbb{A} \times \mathcal{X} \)). Since \( \mathbb{P} \) is finite, then \( \mathcal{O}(1) \) is an ample bundle on \( \mathcal{X} \). Using lemma 3.1 and replacing \( \mathcal{O}(1) \) by a some power we can assume that there is \( d \in \Gamma(\mathbb{P}^1_U, \mathcal{O}(1)) : Z(d) \supset \overline{\mathcal{X}} \setminus \mathcal{X}, Z(d) \not\supset z \), and let’s redenote \( \mathcal{X} = \overline{\mathcal{X}} - Z(d) \). And let’s redenote \( \overline{\mathcal{X}} \) by \( \overline{\mathcal{X}} \).

Now consider the closed subschemes \( \Delta, Z \subset \overline{\mathcal{X}} \):

\[
U \simeq \Delta = \Gamma(U \to X), \ Z = U \times_{\mathbb{A}^{d-1}} \overline{\mathcal{X}}.
\]

(\( Z \) is a closed subscheme in \( \overline{\mathcal{X}} \), since \( Z \) is finite over \( U \))

By lemma 3.1 for some sufficiently large \( l \), there are sections

\[
g \in \Gamma(\overline{\mathcal{X}}, \mathcal{L}(\Delta)^{-1} \otimes \mathcal{O}(l)) : Z(g) \cap (D \cup \Delta \cup Z) = \emptyset,
\]

\[
s_0 \in \Gamma(\overline{\mathcal{X}}, \mathcal{O}(l)) : s_0 \mid D = \delta g, \ Z(s_0) \cap Z = \emptyset.
\]

Define \( s = s_0(1-t) + \delta gt \in \Gamma(\overline{\mathcal{X}} \times \mathbb{A}^1, \mathcal{O}(l)) \),

\[
i_{Z_0} : Z_0 = Z(s_0) \to \mathcal{X} - Z, \ i_{Z(g)} : Z(g) \to \mathcal{X} - Z, \ i_{\Delta} : \Delta \to \mathcal{X},
\]

\[
Q_0 = \langle s_0, d^l \rangle \in Q(\mathbb{P}(Z_0 \to U)), \ Q = \langle s, d^l \rangle \in Q(\mathbb{P}(Z \to U)), \ Q_1 = \langle \delta g, d^l \rangle \in Q(\mathbb{P}(Z(\delta g) \to U)).
\]

Since \( Z(\delta g) = \Delta \amalg Z(g) \), it follows that

\[
Q_1 = (k|\Delta|, u) \oplus Q_{Z(g)}, \ Q_{Z(g)} = (k|Z(g)|, q_{Z(g)}).
\]

Define

\[
\tilde{\phi} = [ev \circ i_{Z_0} \circ Q_0] - [ev \circ i_{Z(g)} \circ Q_{Z(g)}] \in GWCor(U, X - Z),
\]

\[
\tilde{\gamma} = [ev \circ i_Z \circ Q] - [ev \circ i_{Z(g)} \circ Q_{Z(g)} \circ \overline{pr}] \in GWCor(U, X),
\]

where \( pr : U \times \mathbb{A}^1 \to U \). Then

\[
\tilde{\Theta} \circ \iota_0 = \tilde{\phi} \circ \iota_1 = [(k|\Delta|, u)] = [i \circ (u)],
\]

where \( i_0, i_1 : U \to U \times \mathbb{A}^1 \) denotes zero and unit sections.

Hence the morphisms \( \Phi = \tilde{\phi} \circ \langle (u^{-1}) \rangle, \ \tilde{\gamma} \circ \langle (u^{-1}) \rangle \) give the required 'right inverse' GW-correspondence up to a homotopy. \( \Box \)
8. Strictly homotopy invariance of associated sheaves

**Theorem 8.1.** Let \( \mathcal{F} \) be a homotopy invariant presheave with GW-transfers, then

\[
\tilde{\mathcal{F}}_{\text{Nis}}|_{\mathbb{A}^1_K} \simeq \mathcal{F}|_{\mathbb{A}^1_K}, \quad h^i_{\text{Nis}}(\tilde{\mathcal{F}}_{\text{Nis}})|_{\mathbb{A}^1_K} \simeq 0, \forall i > 0,
\]

for any geometric extension \( K/k \).

**Proof.** Let’s denote \( X = \mathbb{A}^1_K \). Consider the exact sequence of sheaves

\[
0 \to \tilde{\mathcal{F}}_{\text{Nis}} \to \eta_* \mathcal{F}(\eta) \to \bigoplus_{z \in \mathbb{A}^1_K} \mathbb{Z} \left( \frac{\mathcal{F}(X^h_2 - z)}{\mathcal{F}(X^h_2)} \right) \to 0,
\]

where \( z \) ranges over the set of closed points of \( \mathbb{A}^1_K \). The sequence \( \mathbb{Z} \) gives a flasque resolvent of \( \tilde{\mathcal{F}}_{\text{Nis}} \). (Note that \( \mathcal{F}(X^h_2 - z) = \mathcal{F}(\text{eta}) \) since \( \dim X = 1 \).

Let’s compute cohomology presheaves of \( \tilde{\mathcal{F}}_{\text{Nis}} \) on \( \mathbb{A}^1_K \) using this resolvent. Let \( U \subset \mathbb{A}^1_K \) be an open subscheme, then

\[
\tilde{\mathcal{F}}_{\text{Nis}} = H^0(U, \tilde{\mathcal{F}}_{\text{Nis}}) = \text{Ker}(\mathcal{F}(\eta) \to \bigoplus_{z \in U} \mathbb{Z} \left( \frac{\mathcal{F}(X^h_2 - z)}{\mathcal{F}(X^h_2)} \right)),
\]

So \( H^0(U, \tilde{\mathcal{F}}_{\text{Nis}}) \) is the subset in \( \mathcal{F}(\eta) \) consisting of elements \( a \) that has a lift to a germ at each closed point of \( U \).

Let \( a \) be such an element in \( \mathcal{F}(\eta) \). Then for some open \( V \subset U \) there is a section \( a' \in \mathcal{F}(V) \), \( \eta^*(a') = a \), and using injectivity in the Zariski and etale excision on \( \mathbb{A}^1_K \) (corollary 5.5, theorems 6.1) we can consequently attach points of the complement \( U \setminus V \) to \( U \) and find an element \( \tilde{a} \in \mathcal{F}(U) : \eta^*(\tilde{a}) = a \). Thus homomorphism \( \mathcal{F}(U) \to \tilde{\mathcal{F}}_{\text{Nis}} \) is surjective.

On other hand, the injectivity on affine line (theorem 7.1) implies that the composition \( \mathcal{F}(U) \to \tilde{\mathcal{F}}_{\text{Nis}} \) is isomorphism.

Since the length of the resolvent \( \mathbb{Z} \) is 2, it follows that \( H^i_{\text{Nis}}(U, \tilde{\mathcal{F}}_{\text{Nis}}) = 0 \) for \( i > 1 \). Now let’s prove that

\[
H^1(U, \tilde{\mathcal{F}}_{\text{Nis}}) = \text{Coker}(\mathcal{F}(\eta) \to \bigoplus_{z \in U} \mathbb{Z} \left( \frac{\mathcal{F}(X^h_2 - z)}{\mathcal{F}(X^h_2)} \right)) = 0.
\]

To do this it is enough to show that for any finite set of elements \( a_i \in \mathcal{F}(X^h_2 - z_i)/\mathcal{F}(X^h_{2i}) \), \( i = 1 \ldots n \) there is an element \( b \in \mathcal{F}(U - \{z_1, \ldots z_n\}) \), such that \( z^*(b) = a_i \), where

\[
z^* : \mathcal{F}(U - \{z_1, \ldots z_n\})/\mathcal{F}(U - \{z_1, \ldots z_i \ldots z_n\}) \to \mathcal{F}(X^h_{2i} - z_i)/\mathcal{F}(X^h_{2i}).
\]

The claim follow from the surjectivity of the excision homomorphisms in corollary 5.5 and theorem 6.1.

\[\square\]

**Theorem 8.3.** The Nisnevich sheaf \( \tilde{\mathcal{F}}_{\text{Nis}} \) associated with homotopy invariant presheave with GW-transfers is homotopy invariant.

**Remark 8.4.** For Witt-correspondences this was proved in [7].

**Proof.** Let \( X \in \text{Sm}_k \) and \( K = k(X) \). The theorem states that for any \( X \in \text{Sm}_k \), \( i^*_X : \tilde{\mathcal{F}}_{\text{Nis}}(\mathbb{A}^1_K) \to \tilde{\mathcal{F}}_{\text{Nis}}(X) \) is isomorphism (for zero section \( i_X : X \to \mathbb{A}^1_K \)).

Since the projection \( \mathbb{A}^1_K \to X \) is right inverse for \( i_X \), the homomorphism \( i^*_X \) is surjective.
To prove injectivity consider the commutative square

\[
\begin{array}{ccc}
\tilde{F}_{\text{Nis}}(A_1 X) & \longrightarrow & \tilde{F}_{\text{Nis}}(A_1 K) \\
\downarrow \psi_X & & \downarrow \psi_K \\
\tilde{F}_{\text{Nis}}(X) & \longrightarrow & \tilde{F}_{\text{Nis}}(K)
\end{array}
\]

where \( K = k(X) \). Theorem 7.3 yields that the horizontal arrows are injections, and the right vertical arrow is isomorphism by theorem 8.1. The claim follows. \( \square \)

Now we prove the main result of the article:

**Theorem 8.5.** For any homotopy invariant presheave with GW-transfers \( F \), the presheaves of Nisnevich cohomologies of associated sheaf \( h_{\text{Nis}}(\tilde{F}_{\text{Nis}}) \) are homotopy invariant for all \( i \geq 0 \), i.e.

\[ H^*_{\text{Nis}}(A_1 \times X, \tilde{F}_{\text{Nis}}) = H^*_{\text{Nis}}(X, \tilde{F}_{\text{Nis}}) \]

for \( X \in \text{Sm}_k \).

The deduction of this theorem from excision theorems 5.1 and 6.1 is similar to the original proof in the case of Cor-correspondences. We start with the case of \( X = \text{Spec} \, K \) for a geometric extension \( K/k \):

**Lemma 8.6.** For a homotopy invariant presheave with GW-transfers \( F \) and any geometric extension \( K/k \)

\[ \tilde{F}_{\text{Nis}}|_{A_1 K} \simeq F|_{A_1 K}, \quad H^*_{\text{Nis}}(A_1 K, \tilde{F}_{\text{Nis}}) \simeq 0, \quad i > 0. \]

**Proof.** This is particular case of the theorem 8.1. \( \square \)

The next step is the case of essential smooth local scheme \( X \):

**Lemma 8.7.** For homotopy invariant presheave with GW-transfers \( F \) and essential smooth local henselian scheme \( X \)

\[ \tilde{F}_{\text{Nis}}(A_1 X) \simeq F(A_1 X), \quad H^*_{\text{Nis}}(A_1 X, \tilde{F}_{\text{Nis}}) \simeq 0, \quad i > 0. \]

**Remark 8.8.**

To prove of the last lemma we use the following notations:

**Definition 8.9.** For any presheave (with GW-transfers) let’s denote

\[ F_{-1}(-) = \text{Coker}(F(-) \rightarrow F(- \times G_m)) \]
\[ F(- \times A^1/0) = \text{Coker}(F(-) \rightarrow F(- \times A^1)) \]

This defines the presheaves with GW-transfers. (Note that we consider this just as notation, though these presheave are in fact internal-hom-presheaves represented by \( \mathbb{Z}(G_m)/\mathbb{Z}(1) \) and \( \mathbb{Z}(A^1)/\mathbb{Z}(0) \).)

**Lemma 8.10.** The functors \( F \mapsto F_{-1} \) and \( F \mapsto F(- \times A^1/0) \) are exact in the category of presheaves (with GW-transfers).

If \( F \) is homotopy invariant or if it is a Nisnevich sheaf, then presheaves \( F_{-1} \) and \( F(- \times A^1/0) \) are of such type too.

**Proof.** The claim follows immediate from that the canonical projections \( G_m \rightarrow pt \) and \( A^1 \rightarrow pt \) have splitting by unit sections, and so the homomorphisms \( F(-) \rightarrow F(- \times G_m) \) and \( F(-) \rightarrow F(- \times A^1) \) are surjective and splitting. \( \square \)
Lemma 8.11. Let $U$ be an open subscheme of an essentially smooth local henselian scheme $X$, such that $Z = X \setminus U$ is essentially smooth and $H^{i-1}(Z \times \mathbb{A}^1/0, \mathcal{F}) = 0$ for some $i > 0$. Then the restriction homomorphism
\[ H^i_{\text{nis}}(X \times \mathbb{A}^1/0, \tilde{\mathcal{F}}_{\text{nis}}) \to H^i_{\text{nis}}(U \times \mathbb{A}^1/0, \tilde{\mathcal{F}}_{\text{nis}}) \]
is injective.

Proof of lemma 8.11. Denote by $i_Y : (Z \times Y)_{\text{nis}} \hookrightarrow (X \times Y)_{\text{nis}}, j : (U \times Y)_{\text{nis}} \hookrightarrow (X \times Y)_{\text{nis}}$, the morphisms of small Nisnevich sites, for any $Y \in Sm_k$.

The Zariski excision on the relative affine line over the local base (theorems 6.1 and the etale excision (theorem 5.1) yields the following:

Sublemma 8.12. There is the following natural isomorphism of sheaves
\[ \text{Coker}_{\text{Sh,Nis}}(\mathcal{F} \xrightarrow{\phi} j_*(j^*(\mathcal{F}_{|X \times Y})))_{\text{nis}} \simeq i_*((\mathcal{F}_{-1})_{|Z \times Y}) \]
for any $Y \in Sm_k$.

Proof. Since $X$ is an essentially smooth local henselian scheme, then there is an isomorphism $(f, p) : X \cong (Z \times \mathbb{A}^1)^{\Delta_x} \times Z$ where $Z$ is considered as the subscheme of $Z \times \mathbb{A}^1$ via the zero section.

Let $V \in X_{\text{nis}}$. Denote by $v : V \to X$, and let $Z' = V \times Z^{\Delta_x}$. Next consider the Nisnevich neighbourhood $V'$ of $Z'$ in $V$ defined as $V' = Z' \times Z \to \Delta_{Z'}$, where $\Delta_{Z'}$ denotes the diagonal in $Z' \times Z'$. The sequence of etale morphisms
\[ (V, Z') \hookrightarrow (V', Z') \xrightarrow{p \times id_{Z'}} (Z' \times \mathbb{A}^1, Z') \]
induce homomorphisms
\[ \mathcal{F}(V' - Z')/\mathcal{F}(V) \to \mathcal{F}(V' - Z')/\mathcal{F}(V') \leftarrow \mathcal{F}((Z' \times \mathbb{A}^1)_z - Z')/\mathcal{F}((Z' \times \mathbb{A}^1)). \]

Since (8.13) is natural in $V$, this defines homomorphisms of presheaves
\[ \text{Coker}_{\text{Pre,Sh}}(\mathcal{F} \xrightarrow{\phi} j_*(j^*(\mathcal{F}_{|X \times Y})))_{\text{nis}} \hookrightarrow \mathcal{E} \to i_*(\mathcal{F}_{-1})_{|Z \times Y}). \]

Now consider the groups of germs at the point $x \in V$, i.e. substitute the henselisation $V^h_x$ in the equality (8.13). If $x \notin Z'$, then all three germs are trivial. Otherwise the first homomorphism becomes isomorphism by the definition of the Nisnevich neighbourhood $V'$; hence consequently applying theorems 6.1 and 5.1 we get that the second homomorphism in (8.13) is isomorphism too:
\[ \mathcal{F}(V^h_x - Z^h_x)/\mathcal{F}(V^h_x) \cong \mathcal{F}((Z' \times \mathbb{A}^1)_z - Z')/\mathcal{F}((Z' \times \mathbb{A}^1)) \simeq \mathcal{F}((Z' \times \mathbb{A}^1)_x - Z')/\mathcal{F}((Z' \times \mathbb{A}^1)). \]

Thus for any $Y \in Sm_k$ we get the short exact sequence of Nisnevich sheaves $\mathcal{F}_{|X \times Y} \xrightarrow{\phi} j_*(j^*(\mathcal{F}_{|X \times Y})) \to i_*(\mathcal{F}_{-1})_{|Z \times Y})$. Hence there is the same sequence for any $Y$ in the idempotent completion of $Sm_k$. Substituting $Y = \mathbb{A}^1/0$ we get the long exact sequence of Nisnevich cohomology groups:
\[ \cdots \to H^{i-1}(Z \times \mathbb{A}^1/0, \mathcal{F}_{-1}) \to H^i_{\text{nis}}(X \times \mathbb{A}^1/0, \mathcal{F}) \to H^i_{\text{nis}}(U \times \mathbb{A}^1/0, \mathcal{F}) \to \cdots \]
The claim follows.

Proof of the lemma 8.7. Using denotation 8.3 the claim is to prove that $H^i_{\text{nis}}(X \times \mathbb{A}^1/0, \mathcal{F}) = 0$.

Let’s prove this by the induction in respect to $i$. The base of the induction, i.e. the case $i = 0$, is theorem 8.3. Suppose that the statement of theorem holds for all homotopy invariant presheaves for all $i < n$, for some $n$.

Let $a \in H^i_{\text{nis}}(X \times \mathbb{A}^1/0, \mathcal{F})$. By lemma 8.6 $H^i_{\text{nis}}(\eta \times \mathbb{A}^1/0, \mathcal{F}) = 0$, where $\eta$ denotes the generic point of $X$, and hence $a|_{U \times \mathbb{A}^1/0} = 0$ for some open affine subscheme $U \subset X$. 

\[ H^{i-1}(Z \times \mathbb{A}^1/0, \mathcal{F}_{-1}) \to H^i_{\text{nis}}(X \times \mathbb{A}^1/0, \mathcal{F}) \to H^i_{\text{nis}}(U \times \mathbb{A}^1/0, \mathcal{F}) \to \cdots \]

The claim follows.
Since $k$ is perfect the generic point $\eta_Z$ of subscheme $Z = X \setminus U$ is smooth and by lemma 8.11 we have $\partial_{\eta_Z} \times \mathbb{A}^1/0 = 0$. Hence $\alpha|_{U_1 \times \mathbb{A}^1/0} = 0$ for some open subscheme $U_1 \subset X$, $U_1 \ni \eta_Z$ (we use here that $X$ is local). Denote $Z_1 = X \setminus U_1$, then $\dim Z_1 < \dim Z$. Now consequently applying lemma 8.11 we can increase the dimension of the closed subscheme $Z_1$ to zero, so the claim follows.

Proof of the theorem 8.5. Let $F$ be an $\mathbb{A}^1$-homotopy invariant presheave with GW-transfers. Consider the Leray spectral sequence for the morphism of small Nisnevich sites $pr: (X \times \mathbb{G}_m)_{nis} \to X_{nis}$:

$$H^p_{nis}(X, R^q pr(\tilde{F}|_{X \times \mathbb{G}_m})) \Rightarrow H^{p+q}_{nis}(X \times \mathbb{A}^1, \tilde{F}_{nis}).$$

Lemma 8.7 implies $R^i pr(F|_{U \times \mathbb{A}^1})(U_x) = 0$ for $i > 0$ and $R^0 pr(F|_{U \times \mathbb{A}^1})(U_x) = pr_*(F|_{U \times \mathbb{A}^1})(U_x) = F(U_x \times \mathbb{A}^1)$. So the spectral sequence degenerates and

$$H^i(X, \tilde{F}_{nis}) \simeq H^i(X, pr_*(\tilde{F}_{nis}|_{U \times \mathbb{A}^1})) \simeq H^i(X, F_{nis}|U).$$

Remark 8.15. Since there is a functor $GW Cor \to WC or$ and proofs are based on explicit constructions of correspondences, then this yields the similar results for presheaves with Witt-transfers.

Corollary 8.16. The Nisnevich cohomology presheaves of the associated sheave of a homotopy invariant presheaves with GW-transfers are representable in the motivic homotopy category $H^A(k)$.

Proof. Consider the simplicial presheave $I_s$ corresponding (via the Dold-Kan correspondence) to the injective resolvent $I^\bullet$ of the associated sheave $\tilde{F}_{nis}$ for a homotopy invariant presheave with GW-transfers $F$. Then since $I^\bullet$ is a bounded above complex of injective Nisnevich sheaves, it follows that $I_s$ fulfills the Nisnevich-Mayer-Viertoris, and theorem 8.5 implies that $I_s$ fulfills homotopy invariance. Thus the claim follows from theorem 3.1 of [11].

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Chebyshev Laboratory, St. Petersburg State University, 14th Line V.O., 29B, Saint Petersburg 199178 Russia
E-mail address: andrei.druzh@gmail.com