Noncommutative Gauge Theories: Model for Hodge theory

Sudhaker Upadhyay\textsuperscript{a} and Bhabani Prasad Mandal\textsuperscript{b}

Department of Physics, Banaras Hindu University, Varanasi-221005, INDIA.

The nilpotent BRST, anti-BRST, dual-BRST and anti-dual-BRST symmetry transformations are constructed in the context of noncommutative (NC) 1-form as well as 2-form gauge theories. The corresponding Noether’s charges for these symmetries on the Moyal plane are shown to satisfy the same algebra as by the de Rham cohomological operators of differential geometry. The Hodge decomposition theorem on compact manifold is also studied. We show that noncommutative gauge theories are field theoretic models for Hodge theory.

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\section{I. INTRODUCTION}

In the past several years the noncommutative (NC) field theories (i.e. field theories on the Moyal plane) have been extensively studied from many different aspects \cite{1,2,3,4,5,6,7,8,9}. The motivations for NC field theories come from the string theory. The end points of the open strings trapped on the D-brane in the presence of two form background field turn out to be NC \cite{1}. The BRST symmetry in noncommutative $U(N)$ gauge theory was recently studied by determining the BRST transformations of the components of gauge and related fields \cite{8}. In commutative gauge theories the conserved current $j_{\mu}(x)$ carries according to the Noether’s theorem. However, in the case of noncommutative field theory, it was shown that the divergence of the current is equal to the Moyal $(\ast)$ product of some functions \cite{9}. One requires the physical subspace of total Hilbert space of states contains only those states that are annihilated by the nilpotent and conserved BRST charge $Q_b$, i.e. $Q_b [\text{phys}] = 0$ in the formulation of the commutative gauge theories \cite{11}. The nilpotency of the BRST charge ($Q_b^2 = 0$) and the physicality criteria ($Q_b [\text{phys}] = 0$) are the two essential ingredients of BRST quantization. However, the conserved charges in the NC gauge theories exist only for the spacelike noncommutativity.

In the language of differential geometry defined on compact, orientable Riemannian manifold, the cohomological aspects of BRST charge is realized in a simple, elegant manner. The nilpotent BRST charge is connected with exterior derivative (de Rham cohomological operator $d = dx^\mu \partial_\mu$, where $d^2 = 0$) \cite{11,16}. The conserved charge corresponding the dual-BRST transformation, which is also the symmetry of the action and leaves the gauge fixing part of the action invariant separately, is shown to be analogue of co-exterior derivative ( $d = \pm \ast d\ast$, where $\ast d = 0$ and $\ast$ is the Hodge duality operation) \cite{16}.

The structure of NC local groups satisfies the no-go theorem \cite{17}. According to this theorem, the closure condition of the gauge algebra suggests that i) the local NC $U(N)$ algebra only admits the irreducible $N \times N$ matrix representation, ii) and for any gauge group consisting of several single-group factors, the matter fields can transform under at most two NC group factors. In this work we consider pure 1-form as well as pure 2-form gauge theories in NC spacetime which satisfy the first axiom of no-go theorem. We analyse the nilpotent symmetries for the 1-form gauge theory in $n$ space-time dimensions and for the 2-form gauge theory in $(1 + 3)$ dimensional (4D) spacetime. Some attempts in the direction of nilpotent BRST symmetries have been made for the physical 4D 1-form gauge theories but the symmetry transformations turn out to be nonlocal and noncovariant \cite{18,19,20,21}. However, recently the BRST symmetry and renormalizability of the 1-form gauge theory in $(1 + 1)$ dimensions (2D) as well as in 4D have been studied thoroughly \cite{22,23}. In the present investigation, the symmetry transformations of the 2D 1-form and the 4D 2-form gauge theories are local and covariant. We construct the nilpotent BRST symmetry, anti-BRST (where the role of ghost and antighost fields are changed with some changes in coefficients), dual-BRST and anti-dual-BRST transformations in this framework. The generators of all these continuous symmetry transformation are...
shown to obey the algebra of de Rham cohomological operators of differential geometry. Hodge decomposition theorem in quantum Hilbert space of states is also discussed. The BRST and anti-dual-BRST charges are mapped with exterior derivative of differential geometry. On the other hand the anti-BRST and dual-BRST charges are shown to be analogue of co-exterior derivative. Further, we show that the NC 1-form as well as 2-form gauge theories are field theoretic models for Hodge theory.

The paper is organized as follows. In sections II and III, we discuss the $U(N)$ 1-form and 2-form gauge theories in NC spacetime respectively with their symmetry transformations. In Sec. IV, we discuss the Hodge-de Rham decomposition theorem for differential geometry. The geometrical aspects of conserved charges of NC theories are described in Sec. V. The last section is reserved for concluding remarks.

II. NONCOMMUTATIVE 1-FORM GAUGE THEORY

A. BRST and anti-BRST symmetries

We start with the BRST invariant action for $n$ dimensional 1-form gauge theory (in manifestly covariant gauge) in NC spacetime as

$$ S_B = \int d^n x \, \mathcal{L}_B, $$

with Lagrangian density

$$ \mathcal{L}_B = Tr \left[ -\frac{1}{4} F_{\mu\nu} \ast F^{\mu\nu} + B \ast \partial \cdot A + \frac{1}{2} B \ast B - i \partial_\mu \tilde{c} \ast D^\mu c \right], $$

where $B, c,$ and $\tilde{c}$ are the Nakanishi-Lautrup auxiliary field, ghost field and anti-ghost field respectively and $\ast$ is the Moyal star product. Here the trace is taken over the $N \times N$ matrices. The field strength tensor ($F_{\mu\nu}$) and covariant derivative ($D_\mu$) of $c$ are defined as

$$ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu]_\ast, $$

$$ D_\mu c = \partial_\mu + g A_\mu \ast c, $$

with the definition of star commutator

$$ [A(x), B(x)]_\ast = A(x) \ast B(x) - B(x) \ast A(x). $$

The connection $A_\mu$ takes the values in the algebra of $U(N)$, with generators $T^a$ satisfying following commutation and anticommutation relations

$$ [T^a, T^b] = i f^{abc} T^c, $$

$$ \{T^a, T^b\} = \delta^{abc}, $$

where $f^{abc}$ and $\delta^{abc}$ are totally antisymmetric and symmetric in nature respectively. The gauge field $(A_\mu)$, ghost field $(c)$, anti-ghost field $(\tilde{c})$ and Nakanishi-Lautrup field $(B)$ are described in the forms of components fields as

$$ A_\mu(x) = A^a_\mu(x) T^a, $$

$$ c(x) = c^a(x) T^a, $$

$$ \tilde{c}(x) = \tilde{c}^a(x) \ast T^a, $$

$$ B(x) = B^a(x) T^a, $$

where group index $a$ has following values, $a \equiv 0, 1, ..., N^2 - 1$. The Lagrangian density $\mathcal{L}_B$ remains invariant under following off-shell nilpotent BRST transformation $(s_b)$:

$$ s_b A_\mu = D_\mu c, \quad s_b c = -\frac{1}{2} g (c \ast c), \quad s_b \tilde{c} = i B, $$

$$ s_b B = 0. $$

The conserved current $J_\mu$ in NC field theory is calculated by considering the divergence of the current is equal to the Moyal product of the certain functions as

$$ \partial^\mu J_\mu = [f(x), g(x)]_\ast, $$

where the functions $f(x)$ and $g(x)$ are, specific to the symmetry and the star commutator, defined as

$$ [f(x), g(x)]_\ast = \left( e^{i \theta^i \sigma_i} \tilde{c}^i \partial_i c - e^{-i \theta^i \sigma_i} \partial_i \tilde{c}^i \right) \ast f(x_1) g(x_2) |_{x_1=x_2=x}. $$

The nilpotent conserved charge for the above BRST transformation in NC spacetime is calculated as

$$ Q_b = \int d^{n-1} x \, j_0 = \int d^{n-1} x \, [B \ast D_0 c - B \ast c + \frac{1}{2} i q \tilde{c} \ast c \ast c]. $$

Here we note that the BRST transformation leads to a conserved charge only for the space like non-commutativity i.e. $\theta^0 = 0$ and our whole analysis is therefore restricted to this case.

To calculate the absolutely anticommuting BRST and anti-BRST transformation, one needs
to introduce another Nakanishi-Lautrup type auxiliary field ($\tilde{B}$) in the Lagrangian density given in Eq. (2) as

$$\mathcal{L}_{\tilde{B}} = Tr \left[ -\frac{1}{4} F_{\mu\nu}^* F^{\mu\nu} + B \ast \partial \cdot A + \frac{1}{2} (B \ast B \ast \tilde{B} \ast \tilde{B}) - i \partial_{\mu} \tilde{c} \ast D^\mu \tilde{c} \right], \quad (11)$$

The BRST and anti-BRST ($s_{ab}$) symmetry transformations under which above Lagrangian density remains invariant are

$$s_{ab}A_\mu = D_\mu c, \quad s_{ab}c = -\frac{1}{2} g c \ast c, \quad s_{ab} \tilde{c} = iB,$n

$$s_{ab}B = 0, \quad s_{ab} \tilde{B} = g \tilde{B} \ast c,$n

$$s_{ab} A_\mu = D_\mu \tilde{c}, \quad s_{ab} \tilde{c} = -\frac{1}{2} g \tilde{c} \ast \tilde{c}, \quad s_{ab} \tilde{B} = 0,$n

where the auxiliary fields $B$ and $\tilde{B}$ are restricted to satisfy the following Curci-Ferrari (CF) type condition \[25, 26\]

$$B + \tilde{B} = ig(c \ast \tilde{c}). \quad (14)$$

The nilpotent charge corresponding to the anti-BRST transformation ($s_{ab}$), using Noether’s theorem, is calculated as

$$Q_{ab} = \int d^{n-1}x \left[ \tilde{B} \ast \tilde{c} - \tilde{B} \ast D_0 \tilde{c} - \frac{1}{2} ig \tilde{c} \ast \tilde{c} \right]. \quad (15)$$

### B. Dual-BRST and anti-dual-BRST symmetries

In this subsection we develop two more nilpotent symmetry transformations known as dual-BRST and anti-dual-BRST transformations which leave the gauge-fixing part of the 2D Lagrangian density in Eq. (2) invariant separately. We linearize the kinetic part of the Lagrangian density by introducing an extra auxiliary field $\mathcal{B}$. The linearized Lagrangian density can then be written as

$$\mathcal{L}_{\mathcal{B}} = Tr \left[ \mathcal{B} \ast E - \frac{1}{2} \mathcal{B} \ast \mathcal{B} \ast \partial \cdot A + \frac{1}{2} \mathcal{B} \ast B \ast \mathcal{B} \ast \mathcal{B} - i \partial_{\mu} \mathcal{c} \ast D^\mu \mathcal{c} \right], \quad (16)$$

where $E$ is the electric field.

The off-shell nilpotent dual-BRST transformation for above Lagrangian density is given by

$$s_d A_\mu = -\epsilon_{\mu \nu} \partial^\nu \tilde{c}, \quad s_d c = -i\mathcal{B}, \quad s_d \tilde{c} = 0,$n

$$s_d \mathcal{B} = 0, \quad s_d \tilde{\mathcal{B}} = 0, \quad (17)$$

where $\epsilon_{\mu \nu}$ is a Levi-Civita tensor of rank-2. The conserved charge for above dual-BRST is then calculated using Noether’s theorem, as

$$Q_d = \int dx \left[ \mathcal{B} \ast \mathcal{c} - \mathcal{B} \ast \tilde{\mathcal{B}} \ast \mathcal{c} + ig \tilde{c} \ast \partial_0 \tilde{\mathcal{c}} \ast \mathcal{c} \right]. \quad (18)$$

Here we note that the sapcelike noncommutativity in the present 2D reflects the Moyal star products in the ordinary product and therefore theory turnout to be in commutative world.

To write the absolutely anticommuting dual and anti-dual-BRST transformation we introduce one more auxiliary field $\mathcal{B}$. Then the Lagrangian density given in Eq. (12) is written as

$$\mathcal{L}_{\mathcal{B}} = Tr \left[ \mathcal{B} \ast E - \frac{1}{2} \mathcal{B} \ast \mathcal{B} \ast \partial \cdot A + \frac{1}{2} \mathcal{B} \ast \mathcal{B} \ast \mathcal{B} \ast \mathcal{B} - i \partial_{\mu} \mathcal{c} \ast D^\mu \mathcal{c} \right]. \quad (19)$$

The off-shell nilpotent anti-dual-BRST transformation ($s_{ad}$), under which the Lagrangian density $\mathcal{L}_{\mathcal{B}}$ remains invariant, is given by

$$s_{ad} A_\mu = -\epsilon_{\mu \nu} \partial^\nu \mathcal{c}, \quad s_{ad} c = 0, \quad s_{ad} \mathcal{c} = i\mathcal{B},$$n

$$s_{ad} \mathcal{B} = 0, \quad s_{ad} \tilde{\mathcal{B}} = 0. \quad (20)$$

Using Noether’s theorem, the nilpotent and conserved charge for anti-dual-BRST transformation is calculated as

$$Q_{ad} = \int dx \left[ \mathcal{B} \ast \mathcal{c} - \mathcal{B} \ast \tilde{\mathcal{B}} \ast \mathcal{c} + ig \mathcal{c} \ast \partial_0 \tilde{\mathcal{c}} \ast \mathcal{c} \right]. \quad (21)$$

In the above expression of charge the Moyal product behaves as a ordinary product.

### C. Bosonic symmetry transformation

We construct the bosonic symmetry transformations ($s_b$ and $s_d$) for the noncommutative 1-form gauge theory. The BRST ($s_b$), anti-BRST ($s_{ab}$), dual-BRST ($s_d$) and anti-dual-BRST ($s_{ad}$) symmetry operators which are constructed in the previous subsections satisfy the following algebra

$$\{s_d, s_{ad}\} = 0, \quad \{s_b, s_{ab}\} = 0,$n

$$\{s_b, s_d\} = 0, \quad \{s_d, s_{ab}\} = 0,$n

$$\{s_b, s_d\} = s_\omega, \quad \{s_{ab}, s_{ad}\} = s_\omega. \quad (22)$$
The above scale transformation leads to the following symmetry transformation for the field variables is given by

\[ Q_\omega = \int dx \left[ \mathcal{B} \star \dot{B} - B \star D_0 B - ig(\mathcal{B} \star \dot{c}) - \partial_\mu \dot{c} \star B \right] \star c \right]. \]

(24)

On the other hand, the bosonic symmetry transformation \( s_\omega \) for the field variables is given by

\[ s_\omega \dot{c} = 0, \quad s_\omega c = 0, \quad s_\omega B = 0, \quad s_\omega \mathcal{B} = 0, \]

\[ s_\omega A_\mu = -i[D_0 B + \epsilon_{\mu \nu \rho} \partial^\rho B - ig(\mathcal{B} \star \dot{c}) - \partial_\mu \dot{c} \star B] \star c \right]. \]

(25)

Using the Noether’s theorem we calculate the generator of this symmetry transformation \( s_\omega \) as

\[ Q_\omega = \int dx \left[ \mathcal{B} \star \dot{B} - B \star D_0 B + ig(\mathcal{B} \star \dot{c}) - \dot{B} \star \partial_\mu c \right] \star c \right]. \]

(26)

Here we note that both bosonic transformations (\( s_\omega \) and \( s_\omega \)) are not independent as their conserved charges are equivalent on the CF type restricted surface (14).

D. Ghost symmetry

The Lagrangian density has yet another symmetry namely ghost scaling symmetry. The Lagrangian density as well as the ghost part of it remain invariant under the following scale transformation for the ghost fields

\[ c \rightarrow e^{-\tau} c, \quad \bar{c} \rightarrow e^{\tau} \bar{c}, \]

(27)

where \( \tau \) is a real scale parameter. The ghost number of the ghost field (\( c \)) and anti-ghost field (\( \bar{c} \)) are 1 and -1 respectively. Rest of the fields in the action, whose ghost number is zero, do not change,

\[ A_\mu \rightarrow A_\mu, \quad B \rightarrow B, \quad \bar{B} \rightarrow \bar{B}. \]

(28)

The above scale transformation leads to the following conserved ghost charge

\[ Q_g = -i \int dx \left[ \bar{B} \star \dot{c} - \bar{B} \star D_0 \bar{c} - \frac{1}{2} ig \dot{c} \star \bar{c} \right] \]

(29)

All these charges constructed in this section will be shown to satisfy the algebra satisfied by the de Rham cohomological operators in section IV.

III. NONCOMMUTATIVE FREE 4D ABELIAN 2-FORM GAUGE THEORY

The purpose of this section is to extend the results of the previous section in the case of 2-form gauge theory. In this section, we discuss the absolutely anticommuting BRST and anti-BRST transformations of Abelian 2-form gauge in noncommutative plane. Dual BRST, anti-dual BRST, bosonic and ghost symmetry transformations for such theory are also constructed.

A. BRST and anti-BRST symmetry transformations

The coupled Lagrangian densities for 2-form gauge theory in 4D [27], which remains unchanged under nilpotent and absolutely anticommuting BRST and anti-BRST symmetry transformations, in noncommutative plane are given by

\[ \mathcal{L}_{(\beta, \mathcal{B})} = \text{Tr} \left[ \frac{1}{2} \partial_\mu \varphi_2 \star \partial^{\mu} \varphi_2 - \frac{1}{2} \mathcal{B}^\mu \star \epsilon_{\mu \nu \rho \tau} \partial^\rho \mathcal{B}^\nu \right] \]

\[ - \frac{1}{2} (\mathcal{B}_\mu \star \mathcal{B}^\mu + \bar{\mathcal{B}}_\mu \star \bar{\mathcal{B}}^\mu) - \beta^\mu \star \partial^\nu \mathcal{B}^\nu \mu + \frac{1}{2} (\bar{\beta}_\mu \star \beta^\mu + \bar{\beta}_\mu \star \beta^\mu) - \frac{1}{2} \partial_\mu \varphi_1 \star \partial^\mu \varphi_1 + \partial_\mu \bar{\sigma} \star \partial^\mu \sigma + (\partial_\mu \rho^\nu - \partial_\nu \rho^\mu) \star \partial^\mu \rho^\nu + (\partial_\mu \rho^\mu - \chi) \star \chi + (\partial_\mu \bar{\rho}^\mu + \bar{\chi}) \star \chi + L^\mu \star (\beta_\mu - \bar{\beta}_\mu - \partial_\mu \varphi_2) + M^\mu \star (\mathcal{B}_\mu - \bar{B}_\mu - \partial_\mu \varphi_2) \right], \]

(30)

\[ \mathcal{L}_{(\bar{\beta}, \bar{\mathcal{B}})} = \text{Tr} \left[ \frac{1}{2} \partial_\mu \varphi_2 \star \partial^{\mu} \varphi_2 - \frac{1}{2} \bar{\mathcal{B}}^\mu \star \epsilon_{\mu \nu \rho \tau} \partial^\rho \bar{\mathcal{B}}^\nu \right] \]

\[ - \frac{1}{2} (\mathcal{B}_\mu \star \mathcal{B}^\mu + \bar{\mathcal{B}}_\mu \star \bar{\mathcal{B}}^\mu) - \bar{\beta}^\mu \star \partial^\nu \mathcal{B}^\nu \mu + \frac{1}{2} (\beta_\mu \star \beta^\mu + \bar{\beta}_\mu \star \beta^\mu) - \frac{1}{2} \partial_\mu \varphi_1 \star \partial^\mu \varphi_1 + \partial_\mu \bar{\sigma} \star \partial^\mu \sigma + (\partial_\mu \bar{\rho}^\nu - \partial_\nu \rho^\mu) \star \partial^\mu \rho^\nu + (\partial_\mu \rho^\mu - \chi) \star \chi + (\partial_\mu \rho^\nu - \bar{\chi}) \star (\partial_\mu \bar{\rho}^\nu - \bar{\chi}) \star \chi + L^\mu \star (\beta_\mu - \bar{\beta}_\mu - \partial_\mu \varphi_2) + M^\mu \star (\mathcal{B}_\mu - \bar{B}_\mu - \partial_\mu \varphi_2) \right], \]

(31)

where the Lorentz vectors \( L_\mu \) and \( M_\mu \) are the Lagrange multiplier fields and \( \mathcal{B}_\mu, \bar{\mathcal{B}}_\mu, \beta_\mu, \bar{\beta}_\mu \) are
the Nakanishi-Lautrup type auxiliary vector fields. The fields $\rho_\mu$ and $\bar{\rho}_\mu$ are anticommuting vector fields, fields $\chi$ and $\bar{\chi}$ are anticommuting scalar fields and fields $\sigma, \varphi_1$, and $\bar{\sigma}$ are commuting scalar fields. These fields are described in the component form as

\begin{align*}
B_{\mu\nu}(x) &= B_{\mu\nu}^\alpha(x)T^\alpha, \quad \rho_\mu(x) = \rho^\alpha_\mu(x)T^\alpha, \\
\bar{\rho}_\mu(x) &= \bar{\rho}_\mu^\alpha(x)T^\alpha, \quad \sigma(x) = \sigma^\alpha(x)T^\alpha, \\
\bar{\sigma}(x) &= \bar{\sigma}^\alpha(x)T^\alpha, \quad \chi(x) = \chi^\alpha(x)T^\alpha, \\
\bar{\chi}(x) &= \bar{\chi}^\alpha(x)T^\alpha, \quad \beta_\mu(x) = \beta^\alpha_\mu(x)T^\alpha, \\
\bar{\beta}_\mu(x) &= \bar{\beta}^\alpha_\mu(x)T^\alpha, \quad B_\mu(x) = B^\alpha_\mu(x)T^\alpha,
\end{align*}

The above Lagrangian densities for 2-form gauge fields $B_{\mu\nu}$ and $\beta_\mu$ and $\bar{\beta}_\mu$ are commuting scalar fields and $\bar{\sigma}, \varphi_1$ and $\bar{\sigma}$ are commuting scalar fields. These fields satisfy the following algebra

\begin{align*}
Q_{ad} &= \int d^3x \left[ (\partial_0 \rho_\mu - \partial_\mu \rho_0) \ast \bar{B}^\nu \right. \\
&\quad - \epsilon^{\mu
u\kappa} \beta_\nu \ast (\partial_\kappa \rho_\mu) \ast \bar{\chi} \ast \partial_0 \varphi_2 \\
&\quad - (\partial_0 \rho_\mu - \partial_\mu \rho_0) \ast \varphi_1 \\
&\quad - \left. \chi \ast \partial_0 \bar{\sigma} + \bar{\chi} \ast M_0 \right], \tag{40}
\end{align*}

The Noether's charges for above dual BRST and anti-dual-BRST symmetries are calculated as

\begin{align*}
Q_b &= \int d^3x \left[ (\partial_0 \bar{\rho}_\mu - \partial_\mu \bar{\rho}_0) \ast \partial^\nu \partial_\nu \sigma \right. \\
&\quad - \epsilon^{\mu
u\kappa} (\partial_\nu \rho_\kappa) \ast B_\kappa - \bar{\chi} \ast \partial_0 \sigma \\
&\quad - (\partial_0 \rho_\nu - \partial_\nu \rho_0) \ast \beta^\nu \ast \chi \ast \partial_0 \varphi_1 \\
&\quad - \left. \chi \ast L_0 \right], \tag{36}
\end{align*}

\begin{align*}
Q_{ab} &= \int d^3x \left[ - (\partial_0 \rho_\nu - \partial_\nu \rho_0) \ast \partial^\nu \bar{\sigma} \right. \\
&\quad - \epsilon^{\mu
u\kappa} (\partial_\nu \bar{\rho}_\kappa) \ast \bar{B}_\kappa - \chi \ast \partial_0 \bar{\sigma} \\
&\quad - (\partial_0 \bar{\rho}_\nu - \partial_\nu \bar{\rho}_0) \ast \bar{\beta}^\nu \ast \bar{\chi} \ast \partial_0 \varphi_1 \\
&\quad - \left. \bar{\chi} \ast L_0 \right]. \tag{37}
\end{align*}

These charges will be used in section IV.

B. Dual and anti-dual-BRST symmetries

The dual and anti-dual-BRST transformations are also the symmetries of the effective action for 2-form gauge theory. Further these transformations leave the gauge fixing term invariant independently. The nilpotent and absolutely anticommuting dual-BRST and anti-dual-BRST transformations, which leave this noncommutative 2-form Lagrangian density invariant, are calculated as follows,

\begin{align*}
s_d B_{\mu\nu} &= - \epsilon_{\mu\nu\kappa} \partial^\kappa \bar{\rho}^\rho, \quad s_d \rho_\mu = - \partial_\mu \bar{\sigma}, \\
s_d \bar{\rho}_\mu &= - B_\mu, \quad s_d \varphi_2 = - \bar{\chi}, \quad s_d \sigma = - \chi, \\
s_d \bar{B}_\mu &= \partial_\mu \bar{\chi}, \quad s_d M_\mu = - \partial_\mu \bar{\chi}, \quad s_d \varphi = 0, \quad (\varphi \equiv \bar{\chi}, \chi, \bar{\sigma}, \varphi_1, B_\mu, \beta_\mu, \bar{\beta}_\mu, L_\mu), \tag{38}
\end{align*}

\begin{align*}
s_{ad} &= - \epsilon_{\mu\nu\kappa} \partial^\kappa \rho^\rho, \quad s_{ad} \rho_\mu = \partial_\mu \sigma, \\
s_{ad} \bar{\rho}_\mu &= \bar{B}_\mu, \quad s_{ad} \varphi_2 = - \chi, \quad s_{ad} \sigma = \bar{\chi}, \\
s_{ad} B_\mu &= - \partial_\mu \lambda, \quad s_{ad} M_\mu = - \partial_\mu \chi, \quad s_{ad} \varphi = 0, \quad (\varphi \equiv \bar{\chi}, \chi, \sigma, \varphi_1, B_\mu, \beta_\mu, \bar{\beta}_\mu, L_\mu). \tag{39}
\end{align*}

The Noether's charges for above dual BRST and anti-dual-BRST symmetries are calculated as

\begin{align*}
Q_d &= \int d^3x \left[ (\partial_0 \rho_\nu - \partial_\nu \rho_0) \ast B^\nu \right. \\
&\quad - \epsilon^{\mu
u\kappa} \beta_\nu \ast (\partial_\kappa \rho_\mu) - \bar{\chi} \ast \partial_0 \varphi_2 \\
&\quad - (\partial_0 \rho_\nu - \partial_\nu \rho_0) \ast \varphi_1 \\
&\quad - \left. \chi \ast \partial_0 \bar{\sigma} + \bar{\chi} \ast M_0 \right], \tag{41}
\end{align*}

C. Bosonic symmetry transformation

Now we construct the bosonic symmetry transformations out of these nilpotent BRST symmetries for this theory. The BRST ($s_b$), anti-BRST ($s_{ab}$), dual-BRST ($s_d$) and anti-dual-BRST ($s_{ad}$) symmetry operators satisfy the following algebra

\begin{align*}
\{ s_b, s_{ab} \} &= 0, \quad \{ s_b, s_d \} = 0, \\
\{ s_d, s_{ab} \} &= 0, \quad \{ s_d, s_{ad} \} = 0, \\
\{ s_b, s_d \} &= s_{\omega}, \quad \{ s_{ab}, s_{ad} \} = s_{\omega}. \tag{42}
\end{align*}
The last two anticommutators define the bosonic transformations under which the fields transform as

\[ s_\omega B_{\mu\nu} = \partial_\nu B_\mu - \partial_\mu B_\nu + \varepsilon_{\mu\nu\rho\sigma} \partial^\rho \beta^\sigma, \]

\[ s_\omega \rho_\mu = \partial_\mu \rho - \varepsilon_{\mu\nu\rho\sigma} \partial^\rho \rho^\sigma, \]

\[ s_\omega \bar{\rho}_\mu = \partial_\mu \bar{\rho}, \]

\[ s_\omega \bar{\beta}_\mu = \partial_\mu \bar{\beta}, \]

\[ s_\omega \bar{\beta}_\mu = \partial_\mu \bar{\beta}, \]

\[ L_\mu, M_\mu. \] (43)

The nilpotent and conserved charge for \( s_\omega \), using Noether’s theorem, is calculated as

\[ Q_\omega = \int d^3 x [\varepsilon_{\mu\nu\rho \tau} \{ (\partial^\rho B^\nu) \ast B^\tau + (\partial^\tau B^\rho) \ast \beta^\nu \} + \partial_\nu (\partial^\rho \rho^\nu - \partial^\nu \rho^\rho) \ast \rho^\sigma \] + \[ (\partial_\nu \bar{\rho} - \partial_\nu \bar{\rho}) \ast \bar{\rho}^\nu \] - \[ (\partial_\nu \rho - \partial_\nu \rho) \ast \bar{\rho}^\nu \] + \[ (\partial_\nu \beta - \partial_\nu \beta) \ast \beta^\nu \]. (45)

**D. Ghost and discrete symmetry**

Now, we would like to mention yet another symmetry of this system, namely, the ghost symmetry. We introduce a scale transformation of the ghost field, under which the effective action for NC 2-form gauge theory is invariant, as

\[ s_g \sigma = 2 \tau \sigma, \quad s_g \bar{\sigma} = -2 \tau \bar{\sigma}, \quad s_g c_\mu = \tau \rho_\mu, \]

\[ s_g \rho_\mu = -\tau \rho_\mu, \quad s_g \bar{\rho}_\mu = \tau \bar{\rho}_\mu, \]

\[ s_g \bar{\beta}_\mu = \tau \bar{\beta}_\mu, \quad s_g \bar{\beta}_\mu = \tau \bar{\beta}_\mu, \quad s_g \bar{\beta}_\mu = \tau \bar{\beta}_\mu, \]

\[ s_g \bar{\beta}_\mu = \tau \bar{\beta}_\mu, \quad s_g \bar{\beta}_\mu = \tau \bar{\beta}_\mu, \]

\[ \varepsilon \equiv \{ \bar{B}_\mu, \varphi_1, \varphi_2, \sigma, \bar{\sigma}, \bar{\beta}_\mu, \bar{\beta}_\mu, \bar{\beta}_\mu, L_\mu, M_\mu \}. \] (46)

where \( \tau \) is an arbitrary scale parameter.

The conserved charge for the above symmetry transformations is

\[ Q_g = \int d^3 x \left[ 2 \sigma \ast \partial^\rho \sigma - 2 \bar{\sigma} \ast \partial_\rho \sigma \right. \]

\[ + \left. (\partial^\rho \rho^\nu - \partial^\nu \rho^\rho) \ast \bar{\rho}_\nu + (\partial^\rho \bar{\rho}^\nu - \partial^\nu \bar{\rho}^\rho) \ast \rho_\nu \right. \]

\[ + \left. \rho_\nu \ast \bar{\chi} - \bar{\rho}^\nu \ast \chi \right]. \] (47)

Further, the Lagrangian densities given in Eqs (40) and (41) remain invariant under following discrete symmetry transformations

\[ B_{\mu\nu} \rightarrow \pm i \varepsilon_{\mu\nu\rho} B^{\rho \kappa}, \quad \rho_\mu \rightarrow \pm i \bar{\rho}_\mu, \]

\[ \bar{\rho}_\mu \rightarrow \pm i \rho_\mu, \quad \sigma \rightarrow \pm i \bar{\sigma}, \quad \bar{\sigma} \rightarrow \mp i \sigma, \]

\[ \varphi_1 \rightarrow \pm i \varphi_2, \quad \varphi_2 \rightarrow \mp i \varphi_1, \quad \bar{\chi} \rightarrow \mp i \chi, \]

\[ \chi \rightarrow \mp i \bar{\chi}, \quad L_\mu \rightarrow \mp i M_\mu, \quad M_\mu \rightarrow \pm i L_\mu, \]

\[ \beta_\mu \rightarrow \pm i \bar{\beta}_\mu, \quad \bar{\beta}_\mu \rightarrow \pm i \beta_\mu, \]

\[ \bar{E}_\mu \rightarrow \mp i \bar{E}_\mu. \] (48)

The above symmetry transformations play very important role in establishing a connection between the symmetries on the one hand and some key concepts of the differential geometry on the other. For instance, these discrete symmetry transformations are the analogue of the Hodge duality operator \((\ast)\) of differential geometry. It is interesting to point out the following relations under the two successive \( \ast \) operations on the fields

\[ \ast (\ast B) = B, \quad B \equiv \{ B_{\mu\nu}, \beta_\mu, \bar{\beta}_\mu, \bar{\beta}_\mu, \bar{\beta}_\mu, L_\mu, M_\mu, \sigma, \bar{\sigma}, \}; \]

\[ \ast (\ast F) = -F, \quad F \equiv \{ \rho_\mu, \bar{\rho}_\mu, \bar{\chi}, \chi \}, \] (49)

where \( \ast \) corresponds to the discrete symmetry transformations given in Eq. (48).

Thus, we note that the fermionic and bosonic fields of the theory transform in a different manner under the successive operations of the discrete transformations. This important observation leads to the following operator relationships:

\[ s_d = \pm \ast s_b \ast, \quad s_{ab} = \pm \ast s_{ab} \ast. \] (50)

**IV. HODGE-DE RHAM DECOMPOSITION THEOREM AND DIFFERENTIAL OPERATORS**

The de Rham cohomological operators (exterior derivative \( d \), co-exterior derivative \( \delta \) and Laplace-Beltrami operator \( \Delta \)) of differential geometry obey the following algebra

\[ d^2 = \delta^2 = 0, \quad \Delta = (d + \delta)^2 = d\delta + \delta d = \{ d, \delta \} \]

\[ [\Delta, \delta] = 0, \quad [\Delta, d] = 0. \] (51)

The operators \( d \) and \( \delta \) are adjoints of duals of each other and \( \Delta \) is self-adjoint operator. It is well-known that the exterior derivative raises the degree of a form by one when it operates on it (i.e. \( df_n \sim f_{n+1} \)). On the other hand, the dual-exterior derivative lowers the degree of a form by one when it operates on forms (i.e. \( \delta f_n \sim f_{n-1} \)).
Let \( M \) be a compact, orientable Riemannian manifold, then an inner product on the vector space \( E^n(M) \) of \( n \)-forms on \( M \) can be defined as
\[
(\alpha, \beta) = \int_M \alpha \wedge \ast \beta,
\]
for \( \alpha, \beta \in E^n(M) \) and \( \ast \) is the Hodge duality operator. Suppose that \( \alpha \) and \( \beta \) are forms of degree \( n \) and \( n + 1 \) respectively. Then following relation for inner product will be satisfied
\[
(d\alpha, \beta) = (\alpha, d\beta).
\]
Similarly, if \( \beta \) is form of degree \( n - 1 \), then we have the relation \( (d\alpha, d\beta) = (d\delta, \beta) \). Thus the necessary and sufficient condition for \( \alpha \) to be closed is that it should be orthogonal to all co-exact forms of degree \( n \). The form \( \omega \in E^n(M) \) is called harmonic if \( \Delta \omega = 0 \). Now let \( \beta \) be a \( n \)-form on \( M \) and if there exists another \( n \)-form \( \alpha \) such that \( \Delta \alpha = \beta \), then for a harmonic form \( \gamma \in H^n \),
\[
(\beta, \gamma) = (\Delta \alpha, \gamma) = (\alpha, \Delta \gamma) = 0,
\]
where \( H^n(M) \) denote the subspace of \( E^n(M) \) of harmonic forms on \( M \). Therefore, if a form \( \alpha \) exist with the property that \( \Delta \alpha = \beta \), then Eq. (51) is necessary and sufficient condition for \( \beta \) to be orthogonal to the subspace \( H^n \). This reasoning lead to the idea that \( E^n(M) \) can be partitioned in to three distinct subspaces \( \Lambda^n_0 \), \( \Lambda^n_\delta \) and \( H^n \) which are consistent with exact, co-exact and harmonic forms respectively. The Hodge-de Rham decomposition theorem can be stated as [28]:

A regular differential form of degree \( n \) may be uniquely decomposed into a sum of the form
\[
\alpha = \alpha_H + \alpha_\delta + \alpha_d,
\]
where \( \alpha_H \in H^n, \alpha_\delta \in \Lambda^n_\delta \) and \( \alpha_d \in \Lambda^n_d \).

A. Hodge-de Rham decomposition theorem and conserved charges

In this subsection we study the analogy between the de Rham cohomological operators and the conserved charges for symmetry transformations for noncommutative gauge theories. In particular we draw the similarity between the algebras obeyed by de Rham cohomological operators and the conserved charges.

The constructed nilpotent symmetry transformations (in earlier sections) for NC 2-form theory pursue the following algebra
\[
s^2_b = 0, \quad s^2_{ab} = 0, \quad \{s_b, s_{ad}\} = 0 = \{s_d, s_{ad}\},
\]
\[
s_\omega = \{s_b, s_d\} = \{-s_{ad}, s_{ad}\}, \quad [s_\omega, s_{\delta}] = 0,
\]
\[
[s_g, s_b] = s_b, \quad [s_g, s_d] = -s_d, \quad [s_g, s_{ad}] = s_{ad}, \quad [s_g, s_{ab}] = -s_{ab}, \quad [s_r, s_{\delta}] = s_{ab}, \quad [s_r, s_{ad}] = s_{ad}, \quad [s_r, s_{g}] = 0.
\]

With the Eqs. (51) and (55), we draw the following two to one mappings
\[
(s_b, s_{ad}) \rightarrow d, \quad (s_d, s_{ad}) \rightarrow \delta,
\]
\[
\{s_b, s_d\} = -\{s_{ab}, s_{ad}\} \rightarrow \Delta.
\]

The conserved charges of all the symmetry transformations are shown to satisfy the following algebra
\[
Q^2_b = 0, \quad Q^2_{ab} = 0, \quad Q^2_\delta = 0, \quad Q^2_d = 0, \quad \{Q_b, Q_{ab}\} = 0, \quad \{Q_d, Q_{ad}\} = 0, \quad \{Q_d, Q_{ab}\} = 0,
\]
\[
i[Q_g, Q_{\delta}] = Q_{\delta}, \quad i[Q_g, Q_d] = -Q_d, \quad i[Q_g, Q_{ab}] = -Q_{ab}, \quad i[Q_\omega, Q_r] = 0.
\]

This algebra is reminiscent of the algebra satisfied by the de Rham cohomological operators of differential geometry given in Eq. (51). Comparing (51) and (58) we obtain following mappings
\[
(Q_b, Q_{ad}) \rightarrow d, \quad (Q_d, Q_{ab}) \rightarrow \delta, \quad Q_\omega \rightarrow \Delta.
\]

Let \( n \) be the ghost number associated with a particular state \( |\psi\rangle_n \) defined in the total Hilbert space of states, i.e.,
\[
Q_g |\psi\rangle_n = n |\psi\rangle_n
\]
Then it is easy to verify the following relations
\[
iQ_g Q_b |\psi\rangle_n = (n + 1)Q_b |\psi\rangle_n, \quad iQ_g Q_{ad} |\psi\rangle_n = (n + 1)Q_{ad} |\psi\rangle_n,
\]
\[
iQ_g Q_d |\psi\rangle_n = (n - 1)Q_d |\psi\rangle_n, \quad iQ_g Q_{ab} |\psi\rangle_n = (n - 1)Q_{ab} |\psi\rangle_n,
\]
\[
iQ_g Q_\omega |\psi\rangle_n = nQ_\omega |\psi\rangle_n
\]
which imply that the ghost numbers of the states \( Q_b |\psi\rangle_n \), \( Q_d |\psi\rangle_n \) and \( Q_\omega |\psi\rangle_n \) are \( (n + 1), (n - 1) \) and \( n \) respectively. The states \( Q_{ab} |\psi\rangle_n \) and \( Q_{ad} |\psi\rangle_n \) have ghost numbers \( (n - 1) \) and \((n + 1)\) respectively. The properties of \( d \) and \( \delta \) are mimicked by sets \( (Q_b, Q_{ad}) \) and \( (Q_d, Q_{ab}) \) respectively. It is evident from Eq. (61) that the set \( (Q_b, Q_{ad}) \) raises the ghost number of a state by one and on the other
hand the set \((Q_d, Q_{ab})\) lowers the ghost number of the same state by one. These observations, keeping the analogy with the Hodge-de Rham decomposition theorem, enable us to express any arbitrary state \(|\psi\rangle_n\) in terms of the sets \((Q_b, Q_d, Q_\omega)\) and \((Q_{ad}, Q_{ab}, Q_\omega)\) as

\[
|\psi\rangle_n = |\omega\rangle_n + Q_b |\chi\rangle_{n-1} + Q_d |\phi\rangle_{n+1},
\]

(62)

\[
|\psi\rangle_n = |\omega\rangle_n + Q_{ad} |\chi\rangle_{n-1} + Q_{ab} |\phi\rangle_{n+1},
\]

(63)

where the most symmetric state is the harmonic state \(|w\rangle_n\), which satisfies the following relations,

\[
Q_\omega |\omega\rangle_n = 0, \quad Q_{(a)b} |\omega\rangle_n = 0, \quad Q_{(a)d} |\omega\rangle_n = 0,
\]

(64)

analogous to the Eq. [54].

V. CONCLUSION

We have considered the NC \(U(N)\) 1-form as well as \(U(N)\) 2-form gauge theories on Moyal plane which satisfies a no-go theorem by restricting the gauge fields to have \(N \times N\) matrix representation. We have studied the BRST symmetry transformation for these theories on Moyal plane. Further, we have shown that the dual of BRST transformation, which is the symmetry of the effective Lagrangian density and leaves the gauge-fixing part invariant separately, also exists for such theories. Interchanging the role of ghost and antighost field with some coefficients anti-BRST and anti-dual-BRST symmetry transformations have also been constructed. We have noted that all the conserved charges for such symmetry transformations exist only in the case of spacelike noncommutativity \(\theta_{ik} = 0\). However, we have observed that the Moyal star product in the expressions of the conserved charges, in the case of 2-dimensional NC 1-form gauge theory, behaves like an ordinary product. The nilpotent BRST symmetry transformation is turned out to be the analogue of the exterior derivative as the kinetic term remains invariant under this. In the similar fashion we have shown that the dual-BRST symmetry transformation is also linked with the co-exterior derivative. The anti-commutator of either BRST and the dual-BRST symmetry generators or anti-BRST and anti-dual-BRST symmetry generators leads to a bosonic symmetry in the theory which turns out to be the analogue of the Laplacian operator. Further, the effective theory has a non-nilpotent ghost symmetry transformation which is also the symmetry of the ghost terms of the effective action. We have shown that the algebra satisfied by the nilpotent charges is exactly same as the de Rham cohomological operator. These results are shown for both NC 1-form and 2-form gauge theories. Thus, the NC 1-form as well as 2-form gauge theories have been realized as the field theoretic models for Hodge theory. It will be interesting to see that whether no-go theorem on NC spacetime puts more restrictions on the matter sector of the NC gauge theories in the context of Hodge theorem.

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[30] One should not confuse the Moyal star product (*) with the Hodge duality operation (⋆).