On the Optimality of Uncoded Cache Placement

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Abstract—Caching is an efficient way to reduce peak-hour network traffic congestion by storing some contents at user’s local cache without knowledge of later demands. Maddah-Ali and Niesen initiated a fundamental study of caching systems; they proposed a scheme (with uncoded cache placement and linear network coding delivery) that is provably optimal to within a factor 12. In this paper, by noticing that when the cache contents and the demands are fixed, the caching problem can be seen as an Index Coding problem, we show the optimality of Maddah-Ali and Niesen’s scheme assuming that cache placement is restricted to be uncoded and the number of users is not less than the number of files. Furthermore, this result states that further improvement to the Maddah-Ali and Niesen’s scheme in this regime can be obtained only by coded cache placement.

I. INTRODUCTION

Caching is a widely used technique to reduce traffic congestion during the peak hours. Caching can be divided into two phases: placement phase (each user stores some contents in his cache during the off peak hours without knowledge of later requests) and delivery phase (after receiving the connected users’ requests and according to their cache contents the central server broadcasts packets to users).

A fundamental study of caching systems appears in [1]: where a server has $N$ identical-length files and is connected to $K$ users. In the placement phase, users store pieces of files within their cache of size $M$. In the delivery phase, each user demands one specific file from the server. Based on the users’ demands and cache contents, packets are broadcasted over an error-free shared link from the server to all the users. The objective is minimize the number of transmission, or load, in the delivery phase for the worst-case demands. The coded caching approach of [1] uses combinatorial cache construction in the placement phase and linear network coding in the delivery phase; the scheme is shown to achieve an additional global caching gain compared to the conventional local caching gain of uncoded systems. The method in [1] uses uncoded cache placement, but examples are given to show that coded cache placement performs better in general. By using a cut-set outer bound for the min-max load, the scheme of [1] is shown to be optimal to within a factor of 12.

In [2], [3], [4] and [5], outer bounds tighter than the cut-set bound provided in [1] were proposed. An improved inner bound was proposed in [6], whose achievable load is equal to the fractional local chromatic number (described in [7]) of a directed graph formed by the users’ demands and the cache contents. In [8], it is shown that when $N \leq K$ and $MK \leq 1$ (i.e., small cache size regime) a scheme based on coded cache placement achieves the cut-set outer bound and it is thus optimal.

Variations on the basic model of [1] so as to account for decentralized cache placement, non-uniform cache sizes, non-uniform demands, non-uniform file-sizes, etc., have been considered in the literature, but we do not summarize them here for sake of space. In general, the question of the exact optimality of the achievable scheme proposed in [1] is open. This work makes progress into this direction.

Contributions. Our main result is based on the following observation. When the users’ demands and cache contents are fixed, the delivery phase can be seen as an index coding problem [7], [9] and [10]. For the index coding problem, an outer bound based on the sub-modularity of entropy is proposed in [10] Theorem 1] and loosened in [10] Corollary 1]. Although [10] Corollary 1] is not generally tight, it is fairly simple and thus often used. We exploit [10] Corollary 1] to derive a ‘converse’ for the scheme in [1]; we actually further relax the original setting in [1] by considering a constraint on the sum of the cache size of all users and on the sum of the total length of all files, in contrast to assuming that each cache size is equal and each file length is equal.

Our main result shows that under the constraint of uncoded cache placement and $N \geq K$, the minimal load of the worst case among all the possible demands is achieved by the two-phase strategy in [1], even when the system is relaxed so as to allow optimal cache size allocation among users, subject to a sum cache size constraint, and optimal file size allocation, subject to a sum file size constraint.

It is worth to mention that past work on caching has mainly focused on tightening the outer bound of [1] for $N \geq K$ rather than the inner bound. Our result shows that the inner bound [1] can not be improved, unless coded cache placement is considered, as in [8]. We also note that the inner and outer bound of [1] coincide when $M \geq N(1-1/K)$ (i.e., large cache size regime). An interesting question that emerges from these results is where the optimal load $L$ vs per-user cache size $M$ is the same when the role of $L$ and $M$ are swapped.

Paper Outline. The remainder of the paper is organized as follows. Section II presents the system model and past results from [1]. Section III shows the main result of this paper. Conclusions and the further work are discussed in Section IV.
The simplest achievable scheme in this case is: in the placement phase users cache a copy of a fraction \( M/N \) of each file, and in the delivery phase the server only sends the remaining part of the requested file to each one. With this

\[
K(1 - M/N) \geq L^*(M),
\]

where \((1 - M/N)\) is referred to as the local caching gain.

The coded caching strategy of [1] can achieve an additional global caching gain as follows. Let \( M = t \frac{N}{K} \), for some positive integer \( t \in [0 : K] \). In the placement phase, each file is split into \( B(K, t) \) non-overlapping sub-files of equal size. The sub-files of \( F_i \) are denoted by \( F_i, \forall \) for \( W \subseteq [1 : K] \) where \(|W| = t\). User \( k \) stores \( F_i, \forall \) in his cache if \( k \in W \). In the delivery phase, for each subset \( S \subseteq [1 : K] \) of size \(|S| = t + 1\), the server transmits \( \sum_{s \in S} F_{d_i,s \setminus S} \). Note that user \( i \in S \) wants \( F_{d_i,S \setminus i} \) and knows \( F_{d_i,S \setminus S} \) for all \( s \neq i \), so he can recover \( F_{d_i,S \setminus i} \). As a result, the load satisfies

\[
K(1 - M/N) \frac{1}{1 + KM/N} \geq L^*(M).
\]

Comparing (2) with (1) the additional global caching gain \( \frac{1}{1 + t MK/N} \) is obtained.

When \( \frac{t}{K} < M < (t + 1) \frac{N}{K} \), time-sharing can be used between the two achievable loads for \( M = t \frac{N}{K} \) and \( M = (t + 1) \frac{N}{K} \). So the load is a piecewise linear curve.

Considering the natural multicasting opportunity in the cases \( N < K \), the load in (2) can be improved to

\[
K(1 - M/N) \cdot \min \left( \frac{1}{1 + KM/N} \frac{N}{K} \right) \geq L^*(M).
\]

In [1], the authors derived a cut-set type outer bound as well. The optimal load \( L^*(M) \) must satisfy

\[
L^*(M) \geq \max_{s \in [1 : \min(N,K)]} \left( s - \frac{s}{N/s} M \right).
\]

The load in (3) with uncoded cache placement can be improved for instance by storing linear combination of sub-files. In [8] it was shown that when \( K \geq N \) and \( M \leq \frac{1}{K} \) the outer bound in (4) can be achieved as follows. When \( M = \frac{1}{K} \), each file \( j \) is split into \( K \) disjoint parts \( F_{j,1}, F_{j,2}, \ldots, F_{j,K} \). User \( i \) stores \( \sum_{j \in [1 : K]} F_{j,i} \) in his cache. In the delivery phase, the server transmits \( F_{d_i,s} \) for each \( i \in [1 : K] \) and each \( s \in [1 : K] \setminus i \). The achievable load is

\[
L^*(M) = N(1 - M), \quad KM \leq 1, \quad K \geq N.
\]

B. Connection to Index Coding

Consider uncoded cache placement. When the cache contents and the users’ demands are given, the caching problem becomes an index coding problem. Each file is divided into sub-files and each sub-file is demanded by a new user who has the same side information as the original user who demands this sub-file. Therefore, known outer bounds for the index coding problem can be used to study the ultimate performance of uncoded cache placement.

In the index coding problem a sender wishes to communicate an independent message \( M_j, j \in [1 : N] \), uniformly
distributed over $[1 : 2^n R_j]$, to the $j$-th receiver by broadcasting a message $X^n$ of length $n$. Each receiver $j$ knows a set of messages, indicated as $A_j$. A rate vector $(R_1, \ldots, R_N)$ is achievable, for large enough $n$, if every user can restore his desired message with high possibility based on $X^n$ and his side information. The index coding problem can be represented as a directed graph $G$: each node in the graph represents one user; a directed edge connects $i$ to $j$ if user $j$ knows $M_i$.

A cut-set-type outer bound from \cite{10} is:

**Theorem 1** (\cite{10}). If $(R_1, \ldots, R_N)$ is achievable for the index coding problem represented by the directed graph $G$, then it must satisfy

$$
\sum_{j \in J} \frac{|M_j|}{n} = \sum_{j \in J} R_j \leq 1
$$

for all $J \subseteq [1 : N]$ where the sub-graph of $G$ over $J$ does not contain a directed cycle. Here $|M_j|$ indicates the length in bits of the message for receiver $j$.

### III. MAIN RESULT

Our main results are as follows.

**Theorem 2.** For the setting of \cite{10}, the load in (2) attains the outer bound in (4) when $\frac{N(K-1)}{K} \leq M \leq N$.

**Proof:** From the cut-set bound in (4) with $s = 1$ we have

$$L^*(M) \geq 1 - \frac{M}{N}.$$  

The bound in (4) contains the points $(M, L) = (\frac{N(K-1)}{K}, 1)$ and $(M, L) = (N, 0)$. The point $(M, L) = (N, 0)$ is trivially achievable (each cache can store all files). When $M = \frac{N(K-1)}{K}$ the load in (2) equals $\frac{1}{K}$. Thus the scheme in \cite{10} is optimal for $\frac{N(K-1)}{K} \leq M \leq N$.

**Theorem 3.** The minimal load of the worst case among all the possible demands under the constraint of uncoded cache placement and $N \geq K$ for the case where the total file size and the total cache size are fixed, is achieved by letting each user have the same cache size and each file have the same length, then using the coded caching in (4) with the load in (2).

The rest of the section is devoted to the proof of Theorem 3. We note that for the case $N < K$ a sub-file may be demanded by more than one user; so the messages represented by the nodes in the index coding graph are not independent; as a result, Theorem 3 from \cite{10} Corollary 1 can not be used. Before we present the actual proof, we give an example to highlight the key ideas behind this proof.

**A. Example for $N = K = 3$**

Assume that the server has $N = K = 3$ files ($F_1, F_2, F_3$). The total file length is $\sum_{i \in [1 : 3]} |F_i| \geq N = 3$. The total cache size is $\sum_{i \in [1 : 3]} |M_i| \leq KM = 3M$, for some $M \in [0, N] = [0, 3]$. Each file $F_i$ is divided into $2^K = 2^3 = 8$ disjoint parts, denoted as $F_i, \forall W \in 2^{|W|}$ where $2^{|W|}$ indicates the power set $2^{|W|} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.

Fig. 2: Directed graph for the equivalent index coding problem in Example 1 with demand vector $(1, 2, 3)$.

$F_{i,W}$ is only known by the users in $W$. For simplicity in the following we omit the braces when we indicate sets, i.e., $F_{1,12}$ represents $F_{1,\{1,2\}}$.

Consider the demand vector $d = (d_1, d_2, d_3) \in [1 : 3]^3$, where $d_i \neq d_j$ for all $i \neq j$. According to $d$, user 1, 2, 3 require $F_{d_1}$, $F_{d_2}$, $F_{d_3}$, respectively. For each demand vector $d$ we generate an index coding problem with $K2^{K-1} = 12$ independent messages, each of which represents a sub-file demanded by a user in the caching system who does not have it his cache. For this index coding problem, we can generate a directed graph including 12 nodes as follows. Each node corresponds to one different sub-file/message. We denote the user in the caching system who wants sub-file $j$ by $P_j$. There is a directed edge from node $i$ to node $j$ if user $P_j$ knows sub-file $i$.

In order to apply Theorem 3 in the constructed graph, we want to find the sets that do not containing a cycle. Nobody knows $F_{1,0}, F_{2,0}, F_{3,0}$ so there is no outgoing edge from $F_{1,0}, F_{2,0}, F_{3,0}$ to any other nodes. Therefore, $F_{1,0}, F_{2,0}, F_{3,0}$ are always in the such sets $J$ when we evaluate (6). For clarity of representation, we do not draw $F_{1,0}, F_{2,0}, F_{3,0}$ in the directed graph representing the index coding problem. In Fig. 2 we draw such a graph for $d = (1, 2, 3)$.

For a demand vector $d$, consider now permutations $u = (u_1, u_2, u_3)$ of $\{1, 2, 3\}$. For each $u$, a set of nodes not containing a cycle is as follows: $F_{d_{u_1},W_1}$ for all $W_1 \subseteq [1 : 3]\{u_1\}$, and $F_{d_{u_2},W_2}$ for all $W_2 \subseteq [1 : 3]\{u_1, u_2\}$, and $F_{d_{u_3},W_3}$ for all $W_3 \subseteq [1 : 3]\{u_1, u_2, u_3\} = \emptyset$. For example, when
\[d = (1, 2, 3) \text{ and } u = (1, 2, 3),\]
\[d_{u_1} = d_1 = 1; W_1 \subseteq [1 : 3 \setminus \{u_1\} = [1 : 3 \setminus \{3\} = \{2, 3\},\]
\[d_{u_2} = d_3 = 3; W_2 \subseteq [1 : 3 \setminus \{u_1, u_2\} = [1 : 3 \setminus \{3, 1\} = \{2\},\]
\[d_{u_3} = d_2 = 2; W_3 \subseteq [1 : 3 \setminus \{u_1, u_2, u_3\} = \emptyset,\]

the corresponding set not containing a cycle is \((F_{1, \emptyset}, F_{1, 2}, F_{1, 3}, F_{2, 3}, F_{3, 2}, F_{2, \emptyset})\), as it can be easily verified by inspection of Fig. 2 from (6) we have that this set implies the bound
\[n \geq |F_{1, \emptyset}| + |F_{1, 2}| + |F_{1, 3}| + |F_{2, 3}| + |F_{3, 2}| + |F_{2, \emptyset}|.\]

In general, we can find such a bound for all possible pairs \((d, u) \in [3]^2\) (here \([3]\) denotes the set of all permutations of the integers \([1 : 3]\); there are 3! elements in the set \([3]\)).

We then sum all the \((3!)^2\) inequalities and get
\[n(3!)^2 \geq \sum_{d \in [3]} \sum_{u \in [3]} \sum_{j \in [3]} \sum_{W_j \subseteq [2 : 3 \setminus W_j]} |F_{d_j, W_j}|\]
\[= (3!)^2 \sum_{i \in [0:3]} x_i \frac{1 - i/3}{1 + i},\]
\[\iff n \geq 3 \cdot x_0 + 1 \cdot \frac{1}{3} \cdot x_1 + \frac{1}{9} \cdot x_1 + x_2 + 0 \cdot x_3, \quad (8)\]
where
\[0 \leq x_i := \sum_{j \in [1:3]} \sum_{W_i \subseteq [1:K]} |F_{j, W_i}|, \quad t \in [0 : K], \quad (9)\]
is the total length of the sub-files that are known by subsets of \(t\) users, and where the proof of why the \(x_i\)'s are multiplied by \(\frac{1 - i/3}{1 + i}\) will be given in the next sub-section.

We also have the sum file size constraint
\[3 \leq \sum_{j \in [1:3]} \sum_{W_i \subseteq [1:3]} |F_{j, W_i}|\]
\[\iff 3 \leq x_0 + x_1 + x_2 + x_3, \quad (10)\]
and the sum cache size constraint
\[\sum_{j \in [1:3]} \sum_{W_i \subseteq [1:3]} |F_{j, W_i}| \leq 3M\]
\[\iff 0 \cdot x_0 + 1 \cdot x_1 + 2 \cdot x_2 + 3 \cdot x_3 \leq 3M. \quad (11)\]

The bounds in (8)-(11) provide an outer bound for the load \(n\) with uncoded cache placement. Since we have many bounds/inequalities in four unknowns, we proceed to eliminate \((x_0, x_1, x_2, x_3)\) in the system of inequalities in (8)-(11). By doing so, we obtain
\[n \geq \frac{2}{3} M + \frac{5}{3}, \quad (13)\]
\[n \geq \frac{1}{3} M + 1. \quad (14)\]

The maximum among the right-hand sides of (12), (13) and (14) give a piecewise linear curve with corner points \((0,3), (1,1), (2,\frac{1}{2}), (3,0)\). Since these corner points are achieved by \(|\{3\}\) and for all possible demand vectors the loads of this scheme are the same, it is optimal for the example problem to firstly let each user have the same cache size and let each file have the same length, then to use the two-phase strategy in [1].

B. General Proof of Theorem 3

The general case \(N \geq K\) is proved by a similar method as in the previous example. Firstly we consider the case where the file demanded by each user is different. Note that for this case there are \(P(N, K)\) demand vectors, each of which is \(d = (d_1, d_2, \ldots, d_K)\) where \(d_i \in [1 : N]\) and \(d_i \neq d_j\) for all \(i \neq j\). We divide file \(F_j\) into \(2^K\) disjoint parts, each of which is denoted by \(F_{j, W}\) such that \(\sum_{W \subseteq [1:K]} |F_{j, W}| = |F_j|\). \(F_{j, W}\) is only known by the users in \(W\). For each demand vector, we generate a directed graph with \(2^{K-1}\) nodes as the same method claimed in the previous example.

We construct cycles in the directed graph by the following Lemma.

**Lemma 1.** Let \(u = (u_1, u_2, \ldots, u_K)\) be a permutation of \([1 : K]\). A set of nodes not containing a cycle in the directed graph of the corresponding index coding problem contains sub-file \(F_{d_{u_i}, W_i}\) for all \(i \in [1 : K]\) and all \(W_i \subseteq [1 : K] \setminus \{u_1, \ldots, u_i\}\).

**Proof:** For a \(u = (u_1, u_2, \ldots, u_K)\), we say that sub-files/nodes \(F_{d_{u_i}, W_i}\) for all \(W_i \subseteq [1 : K] \setminus \{u_1, \ldots, u_i\}\) are in level \(i\). It is easy to see each node in level \(i\) only knows the sub-files \(F_{j, W}\) such that \(u_i \in W\). So each node in level \(i\) knows neither the sub-files in the same level, nor the sub-files in the higher levels. As a result, in the proposed set there is no subset containing a directed cycle. \(\blacksquare\)

According to (6) and Lemma 1 in order to recover all the desired sub-files for each user, the number of broadcast bits \(n\) needs to satisfy
\[n \geq \sum_{W_i \subseteq [1:K] \setminus \{u_1\}} |F_{d_{u_1}, W_i}| + \ldots + \sum_{W_i \subseteq [1:K] \setminus \{u_{i+1}, \ldots, u_K\}} |F_{d_{u_{i+1}}, W_i}|. \quad (15)\]

Considering all the possible demands vectors where \(d_i \neq d_j\) if \(i \neq j\) and all the \(u\) for each demand vector, we can list all the inequalities in the form of (15). There are \(P(N, K) P(K, K)\) such inequalities. Because of symmetry, for each \(i \in [0 : K]\) on the right side of the sum of all the \(P(N, K) P(K, K)\) inequalities the coefficients of the term \(|F_{j, W_j}|\), for \(j \in [1 : N]\) and \(|W| = i\) are equal.
In [15] there are $B(K-1, i) + B(K-2, i) + \ldots + B(i, i)$ terms with $|W| = i$ whose coefficient is 1. Since there are totally $B(K, i)N$ sub-files with $|W| = i$, in the sum expression the coefficient of each $F_j, W$ with $|W| = i$ is $P(N, K) P(K, K) \frac{B(K-1, i) + \ldots + B(i, i)}{B(K, i)N}$. As a result we have

$$n \geq \sum_{i=0}^{K} \frac{B(K-1, i) + \ldots + B(i, i)}{B(K, i)N} x_i. \quad (16)$$

From the Pascal’s triangle we have

$$B(K-1, i) + \ldots + B(i, i) = B(K, i + 1), \quad (17)$$

thus we rewrite (16) as

$$n \geq \sum_{i=0}^{K} \frac{B(K, i+1)}{B(K, i)N} x_i = \sum_{i=0}^{K} \frac{K-i}{(i+1)N} x_i \quad (18)$$

Since the total size of all files is

$$x_0 + x_1 + \ldots + x_K \geq N \quad (19)$$

and the total cache size is

$$x_1 + 2x_2 + \ldots + ix_i + \ldots + Kx_K \leq KM \quad (20)$$

we obtained the desired bound on $n$.

In Appendix, we prove the by combining the derived bound we can write: for each $q \in [1 : K]$,

$$n \geq -\frac{(K+1)KM}{Nq(q+1)} + \frac{2K-q+1}{q+1} + \sum_{i=0}^{K} Z(N, K, i, q)x_i \quad (21)$$

$$Z(N, K, i, q) = \frac{(K+1)(i-q+1)(q-i)}{qN(q+1)(i+1)}. \quad (22)$$

Note that $x_i$ depends on $M$ and for each $M$ we need not strictly find the maximum of the right side of (21) among all $q$. Instead, for each $M$ we prove that the right side is achievable with a $q \in [1 : K]$. As a result, this load is the minimum for such $M$. From (21), for one $q \in [1 : K]$ the outer bound in (21) becomes linear in terms of $M$. We focus our attention on $\frac{q-1}{K}N \leq M \leq \frac{qN}{K}$.

For $M = \frac{q-1}{K}N$ we have

$$n \geq -\frac{(K+1)KM}{Nq(q+1)} + \frac{2K-q+1}{q+1} = \frac{K-q+1}{q} = K(1 - \frac{1}{N}) \frac{M}{1+KM/N},$$

and for $M = \frac{qN}{K}$ we have

$$n \geq -\frac{(K+1)KM}{Nq(q+1)} + \frac{2K-q+1}{q+1} = \frac{K-q+1}{q} = K(1 - \frac{1}{N}) \frac{M}{1+KM/N}. \quad (23)$$

At last we take (25) into (19), we have

$$n \geq \sum_{i=0}^{K} \frac{K-i}{(i+1)N} x_i = \frac{K-q+1}{N} x_{q-1} + \frac{K-q}{N(q+1)} x_q + \sum_{i \in [0:K]:i \neq q-1,q} \frac{K-i}{(i+1)N} x_i \quad (24)$$

IV. CONCLUSION AND FURTHER WORK

In this paper, we considered the cache problem of [1], but where we assumed a total cache size constraint and a total file size constraint. By leveraging an outer bound for the index coding problem, we proved that under the constraint of uncoded cache placement and $N \geq K$, for minimizing the worst-case load it is optimal to let each user have the same cache size and each file have the same length, then to use the coded caching proposed in [1]. Our results show that the only way to improve on the load of [1] is by coded cache placement.

Further work will be in two directions: study the case where $N < K$ and study achievable loads for coded cache placement.

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APPENDIX

Proof: For each $q \in [1 : K]$, we want to eliminate $x_q$ and $x_{q-1}$ in (18) by the help of (19) and (20).

From (19), we have

$$\frac{2K-q+1}{N(q+1)} (x_{q-1} + x_q) \geq \frac{2K-q+1}{N(q+1)} (N - \sum_{i \in [0:K]:i \neq q-1,q} x_i) \quad (23)$$

From (20), we have

$$-\frac{(K+1)}{Nq(q+1)} (q-1)x_{q-1} - \frac{K+1}{Nq(q+1)} q x_q \geq \frac{-(K+1)}{Nq(q+1)} K M + \frac{K+1}{Nq(q+1)} \sum_{i \in [0:K]:i \neq q-1,q} i x_i \quad (24)$$

Then we sum (23) and (24).

$$\frac{K-q+1}{N} x_{q-1} + \frac{K-q}{N(q+1)} x_q \geq 2 \frac{K-q+1}{N(q+1)} (N - \sum_{i \in [0:K]:i \neq q-1,q} x_i) + \frac{-(K+1)}{Nq(q+1)} K M + \frac{K+1}{Nq(q+1)} \sum_{i \in [0:K]:i \neq q-1,q} i x_i$$

$$= \frac{-(K+1)K M}{Nq(q+1)} + \frac{2K-q+1}{q+1} + \sum_{i \in [0:K]:i \neq q-1,q} (-\frac{2K-q+1}{Nq(q+1)} + \frac{(K+1)i}{Nq(q+1)}) x_i \quad (25)$$

For $M = \frac{q-1}{K}N$ we have

$$n \geq -\frac{(K+1)KM}{Nq(q+1)} + \frac{2K-q+1}{q+1} = \frac{K-q+1}{q} = K(1 - \frac{1}{N}) \frac{M}{1+KM/N},$$

and for $M = \frac{qN}{K}$ we have

$$n \geq -\frac{(K+1)KM}{Nq(q+1)} + \frac{2K-q+1}{q+1} = \frac{K-q+1}{q} = K(1 - \frac{1}{N}) \frac{M}{1+KM/N}.$$

At last we take (25) into (19), we have

$$n \geq \sum_{i=0}^{K} \frac{K-i}{(i+1)N} x_i = \frac{K-q+1}{N} x_{q-1} + \frac{K-q}{N(q+1)} x_q + \sum_{i \in [0:K]:i \neq q-1,q} \frac{K-i}{(i+1)N} x_i$$

IV. CONCLUSION AND FURTHER WORK

In this paper, we considered the cache problem of [1], but where we assumed a total cache size constraint and a total file size constraint. By leveraging an outer bound for the index coding problem, we proved that under the constraint of uncoded cache placement and $N \geq K$, for minimizing the worst-case load it is optimal to let each user have the same cache size and each file have the same length, then to use the coded caching proposed in [1]. Our results show that the only way to improve on the load of [1] is by coded cache placement.

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where $Z(N, K, i, q) = \frac{(K+1)(i-q+1)(i-q)}{qN(q+1)(i+1)}$. This concludes the proof.

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