Duality for symmetric second rank tensors. (I) : the massive case.

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A generalization of duality transformations for arbitrary Lorentz tensors is presented, and a systematic scheme for constructing the dual descriptions is developed. The method, a purely Lagrangian approach, is based on a first order parent Lagrangian, from which the dual partners are generated. In particular, a family of theories which are dual to the massive spin two Fierz-Pauli field $h_{\mu\nu}$, both free and coupled to an external source, is constructed in terms of a $T_{(\mu\nu)\sigma}$ tensor.

PACS numbers: 11.10.-z, 11.90.+t, 02.90.+p

I. INTRODUCTION

There usually is a great deal of freedom in the choice of variables for the description of a physical system. Different choices of variables are considered equivalent when they are able to describe the same system, and any difference is mainly due to subjective choices. However, there might be practical reasons to prefer a given description to others. For example, in some cases it might be desirable to have a formulation where some symmetries are made explicit in the Lagrangian. This usually requires the use of a redundant set of variables to describe the system configurations, as in the case of gauge theories. Conversely, in other situations it is more convenient to choose a minimal, non-redundant, set of variables. Another interesting example of this freedom is the bosonization of fermionic systems. The actual proof of the equivalence between different descriptions is usually a non-trivial task.

Duality, in its wider meaning, refers to two equivalent descriptions for a physical system using different fields. One of the simplest cases is the scalar-tensor duality. It corresponds to the equivalence between a free massless scalar field $\phi$, with field strength $f_\mu = \partial_\mu \phi$, and a massless antisymmetric field $B_{\mu\nu}$, the Kalb-Ramond field, with field strength $H_{\mu\nu\rho} = \partial_{[\mu}B_{\nu\rho]}$, and $\partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$. Another example is in fact a predecessor of the modern approach to duality, the electric-magnetic symmetry $(E + i B) \rightarrow e^{i\phi}(E + i B)$ of the free Maxwell equations. When there are charged sources this symmetry can be maintained by introducing magnetic monopoles. This transformation provides a connection between weak and strong couplings via the Dirac quantization condition. At the level of Yang-Mills theories with spontaneous symmetry breaking this kind of duality is expected, due to the existence of topological dyon-type solitons. The extension of electromagnetic duality to $SL(2, Z)$ is usually referred to as S-duality, and plays an important role in the non-perturbative study of field and string theories.

These basic ideas have been subsequently generalized to arbitrary forms in arbitrary dimensions. Well known dualities are the ones between massless $p$-form and $(d-p-2)$-form fields and between massive $p$ and $(d-p-1)$-forms in $d$ dimensional space-time. These dualities among free fields have been proved by using the method of parent Lagrangians as well as the canonical formalism. They can be extended to include source interactions.

The above duality among forms can be understood as a relation between fields in different representations of the Lorentz group. The origin of this equivalence can be traced using the little group technique for constructing the representations of the Poincaré group in $d$ dimensions. A detailed discussion of this observation suggests the possibility of generalizing the duality transformations among $p$-forms to tensorial fields with arbitrary Young symmetry types. Consistent massless free, interacting, and massive theories of mixed Young symmetry tensors were constructed in the past, but the attempts to prove a dual relation between these descriptions did not lead to a positive answer. Additional interest in this type of theories arises from the recent formulation of $d = 11$ dimensional supergravity as a gauge theory for the osp(32|1) superalgebra. It includes a totally antisymmetric fifth-rank Lorentz tensor one form $b_{\mu \nu \rho \sigma \tau}$, whose mixed symmetry piece does not have any related counterpart in the standard $d = 11$ supergravity theory.

In this paper we present a general scheme to construct dual theories based on a Lagrangian approach. The method, sketched in a preceding article, has been originally motivated by the relationship between field representations corresponding to associated Young diagrams. Here we fully develop this approach on a purely Lagrangian basis, and apply it to the case of a massive spin-2 theory.
Let us consider a simple example, the scalar field $\phi$, in order to illustrate our general procedure for constructing dual theories. The starting point is the corresponding second order Lagrangian
\[ L(\phi) = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 + J \phi , \] (1)
where $m$ is the mass parameter of $\phi$ and $J$ is a source coupled to the field. As the first step, we construct a first order Lagrangian, using a generalization of a procedure used in Ref. [17]. We are interested in a particular Lagrangian structure, which we will call the standard form
\[ L(\phi, L^\mu) = L^\mu \partial_\mu \phi - \frac{1}{2} L^\mu L_\mu - \frac{1}{2} m^2 \phi^2 + J \phi . \] (2)
This standard form is defined by the kinetic term. It contains the derivative of the original field times a new auxiliary variable, which we call, in a rather loose way, the field strength of the original theory. In this first order formulation the original field and the auxiliary field we have just introduced define the configuration space. The equations of motion are
\[ m^2 \phi = -\partial_\mu L^\mu + J, \quad L_\mu = \partial_\mu \phi . \] (3)

The key recipe to construct the dual theory is to introduce a point transformation in the configuration space for the auxiliary variable
\[ L^\mu = \epsilon^{\mu\nu\sigma\tau} H_{\nu\sigma\tau} , \] (4)
which leads to a new first order Lagrangian
\[ L(\phi, H_{\nu\sigma\tau}) = H_{\nu\sigma\tau} \epsilon^{\mu\nu\sigma\tau} \partial_\mu \phi + 3 H_{\nu\sigma\tau} H^{\nu\sigma\tau} - \frac{1}{2} m^2 \phi^2 + J \phi . \] (5)
This turns out to be the parent Lagrangian from which both dual theories can be obtained. In fact, using the equation of motion for $H_{\nu\sigma\tau}$ we obtain $H_{\nu\sigma\tau}(\phi)$ which takes us back to our starting action (1) after it is substituted in Eq. (5). On the other hand, we can also eliminate the field $\phi$ from the Lagrangian using its equation of motion
\[ m^2 \phi = -\partial_\mu \epsilon^{\mu\nu\sigma\tau} H_{\nu\sigma\tau} + J . \] (6)
In such a way we now obtain the new theory
\[ L(H_{\nu\sigma\tau}) = \frac{1}{2} (\epsilon^{\mu\nu\sigma\tau} \partial_\mu H_{\nu\sigma\tau})^2 + 3 m^2 H_{\nu\sigma\tau} H^{\nu\sigma\tau} - J \epsilon^{\mu\nu\sigma\tau} \partial_\mu H_{\nu\sigma\tau} + \frac{1}{2} J^2 , \] (7)
which is equivalent to the original one through the transformation (6). In this form we have obtained a theory dual to (1).

For a massless theory, $m = 0$, we lose the connection between the original field $\phi$ and the new one $H_{\nu\sigma\tau}$. In this case Eq. (6) becomes a constraint on $H_{\nu\sigma\tau}$
\[ \partial_\mu \epsilon^{\mu\nu\sigma\tau} H_{\nu\sigma\tau} = J . \] (8)
Out of the sources, the above equation tells us that the field $H_{\nu\sigma\tau}$ can be considered as a field strength with an associated potential.

Another paradigmatic example of dualization is the standard S-duality for electrodynamics with a $\theta$ term. In the following we describe a method to deal with this case which, together with the previous example, will serve as a motivation for the general scheme to be presented in the next section. Let us consider the Euclidean Lagrangian
\[ L = \frac{1}{8\pi} \left( \frac{4\pi}{g^2} F_{\mu\nu} F^{\mu\nu} + i \frac{\theta}{2\pi} \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} \right) . \] (9)
Using the notation
\[ \tau = i \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} \quad \bar{\tau} = \frac{\theta}{2\pi} - \frac{4\pi i}{g^2} , \] (10)
the standard Euclidean S-dualization recipe
\[ F \rightarrow \tilde{F}, \quad \tilde{F} \rightarrow +F, \quad \tau \rightarrow \frac{1}{\tilde{\tau}}, \quad \bar{\tau} \rightarrow \frac{1}{\bar{\tau}} \]

leads to a new Lagrangian

\[ \tilde{L} = -\frac{1}{8\pi} \left( \frac{4\pi}{g^2 \tau \tilde{\tau}} \tilde{F}^{\mu\nu} \tilde{F}_{\mu\nu} - \frac{i}{\tau \tilde{\tau}} \frac{1}{2\pi \tau 2} \epsilon_{\mu\nu\rho\sigma} \tilde{F}^{\mu\nu} \tilde{F}^{\rho\sigma} \right). \]

For the purpose of our discussion it is more convenient to write the initial Lagrangian (9) as

\[ L = \frac{1}{8\pi} \left( a F^{\mu\nu} F_{\mu\nu} + ib \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} \right), \]

where

\[ a = \frac{4\pi}{g^2}, \quad b = \frac{\theta}{2\pi}, \quad \tau \tau = a^2 + b^2. \]

In this notation, the Euclidean dual is

\[ \tilde{L} = -\frac{1}{8\pi} \left( \frac{a}{a^2 + b^2} \tilde{F}^{\mu\nu} \tilde{F}_{\mu\nu} - \frac{ib}{a^2 + b^2} \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \tilde{F}^{\mu\nu} \tilde{F}^{\rho\sigma} \right). \]

Now we will show how to go from Lagrangian (9) to Lagrangian (12) using the basic ideas of our approach. To begin with, we construct a first order Lagrangian for (13), introducing the Lagrange multiplier

\[ L(F, A, G) = \frac{1}{8\pi} \left( a F^{\mu\nu} F_{\mu\nu} + ib \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} \right) - \frac{1}{4\pi} (G^{\mu\nu} F_{\mu\nu} - G^{\mu\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu)). \]

The Euler-Lagrange equation for \( F_{\mu\nu} \)

\[ a F^{\mu\nu} + ib \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} = G_{\mu\nu} \]

leads to

\[ F^{\alpha\beta} = \frac{1}{(a^2 + b^2)} \left( aG^{\alpha\beta} - ib \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} G^{\mu\nu} \right) \]

by a purely algebraic manipulation. This allows us to eliminate this field from Lagrangian (16), obtaining

\[ L(A, G) = -\frac{1}{8\pi} \frac{1}{(a^2 + b^2)} \left( aG^{\alpha\beta} - ib \frac{1}{2} \epsilon^{\alpha\beta\mu\nu} G_{\mu\nu} \right) G_{\alpha\beta} + \frac{1}{4\pi} G^{\mu\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu) \]

which identifies \( G^{\mu\nu} \) as the field strength of \( A^\mu \). The above first order Lagrangian is equivalent to the second order Lagrangian (13). This can be verified via the solution

\[ G^{\alpha\beta} = a (\partial^\alpha A^\beta - \partial^\beta A^\alpha) + ib \frac{1}{2} (\partial_\rho A_\sigma - \partial_\sigma A_\rho) \epsilon^{\rho\sigma\alpha\beta} \]

of the equation of motion for \( G_{\mu\nu} \), together with the definition (18). The variation of \( A_\mu \) in Lagrangian (19) produces the remaining equation

\[ \partial_\mu G^{\mu\nu} = 0. \]

Now we define the dual field \( H_{\alpha\beta} \)

\[ H^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} H_{\rho\sigma}. \]

By substitution in Eq. (13) we obtain

\[ L = -\frac{1}{8\pi} \frac{1}{(a^2 + b^2)} \left( aH_{\kappa\lambda} - ib \frac{1}{2} \epsilon_{\kappa\lambda\rho\sigma} H^{\rho\sigma} \right) H^{\kappa\lambda} + \frac{1}{4\pi} \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} H^{\mu\nu} (\partial_\rho A_\sigma - \partial_\sigma A_\rho), \]

where

\[ \epsilon_{\mu\nu\rho\sigma} = \epsilon_{\nu\mu\rho\sigma}, \quad \epsilon_{\mu\nu\rho\sigma} = -\epsilon_{\mu\rho\nu\sigma} = \epsilon_{\nu\rho\mu\sigma}. \]
which is the correspondent parent Lagrangian. The variation of this last Lagrangian with respect to $A^\mu$ produces a Bianchi identity for $H_{\rho\sigma}$

$$\epsilon^{\nu\rho\sigma} \partial_{\nu} H_{\rho\sigma} = 0,$$

which implies

$$H_{\rho\sigma}(B) = \partial_{\rho} B_{\sigma} - \partial_{\sigma} B_{\rho},$$

where $H_{\rho\sigma}$ is identified as the dual field strength. Using this property in Eq. (23) leads to the second order Lagrangian

$$L(B) = -\frac{1}{8\pi} \frac{1}{(a^2 + b^2)} \left( a H_{\kappa \lambda} - ib \frac{1}{2} \epsilon^{\kappa \lambda \rho \sigma} H_{\rho \sigma} \right) H^{\kappa \lambda},$$

the dual version of the original one. The above Lagrangian is precisely (13) with the notation $H^\mu_\nu = \tilde{F}^\mu_\nu$. The relation with the original theory appears at the level of the potentials and is given by

$$a \left( \partial^\alpha A^\beta - \partial^\beta A^\alpha \right) + ib \left( \partial_\rho A_\sigma - \partial_\sigma A_\rho \right) \epsilon^{\rho\sigma\alpha\beta} = \frac{1}{2} \epsilon^{\alpha\beta\rho\sigma} \left( \partial_\rho B_\sigma - \partial_\sigma B_\rho \right).$$

The case $b = 0$ reduces to standard Electrodynamics and leads to

$$L(A) = \frac{a}{8\pi} F_{\mu\nu} F^{\mu\nu}, \quad F_{\rho\sigma} = (\partial_\rho A_\sigma - \partial_\sigma A_\rho),$$

$$L(B) = -\frac{1}{8\pi a} H_{\kappa \lambda} H^{\kappa \lambda}, \quad H_{\rho\sigma} = (\partial_\rho B_\sigma - \partial_\sigma B_\rho) = a \tilde{F}_{\rho\sigma},$$

together with the relation

$$a \left( \partial^\alpha A^\beta - \partial^\beta A^\alpha \right) = \frac{1}{2} \epsilon^{\alpha\beta\rho\sigma} \left( \partial_\rho B_\sigma - \partial_\sigma B_\rho \right).$$

This paper focuses on the construction of a dual theory for a massive spin-2 field in four dimensions. It is organized as follows. In Section II we formulate the general scheme for dualization pursued here. Section III contains the construction of an auxiliary first order Lagrangian which is equivalent to the usual one in terms of the standard Fierz-Pauli field $h_{\mu\nu}$ for a massive spin-2 particle. The general method for constructing such an auxiliary Lagrangian is briefly reviewed in Appendix A. An explicit proof of the equivalence between this auxiliary Lagrangian and the massive Fierz-Pauli Lagrangian is given in Appendix B. Section IV contains the definition of the dual field $T_{(\mu\nu)\sigma}$ together with the construction of the parent Lagrangian. In Section V the duality transformations arising from the parent Lagrangian are derived. The dual Lagrangian, in terms of $T_{(\mu\nu)\sigma}$, is obtained in Section VI together with the corresponding equations of motion and the set of Lagrangian constraints. These contraints allow us to make sure that we have obtained the correct number of degrees of freedom. Most of the calculations in this section are relegated to Appendix C. In Section VII we discuss the example of a fixed point mass $m$ whose field is calculated in each of the dual theories, thus allowing the explicit verification of the duality trasformations. Finally we close with Section VIII which contains a summary of the work together with some comments regarding preliminary work in the zero mass limit of the present approach. A complete discussion of the massless case is deferred to a forthcoming publication.

II. THE DUALIZATION PROCEDURE

In general terms, the method applied to the previous examples can be summarized as follows, assuming that there is no external source, for simplicity. We start from a second order theory for the free field $\Phi$ of a given tensorial character, which can be schematically presented as

$$L(\Phi) = \frac{1}{2} \partial \Phi \partial \Phi - \frac{M^2}{2} \Phi \Phi.$$

Next, we introduce an auxiliary field $W$ to construct a first order formulation in the standard form

$$L(\Phi, W) = (\partial \Phi) W - \frac{1}{2} WW - \frac{M^2}{2} \Phi \Phi,$$
as explained in Appendix A. This identifies $W$ as the field strength of $\Phi$, with the equation of motion
\[ \partial W + M^2 \Phi = 0. \] (33)

Now, we introduce $\tilde{W}$ as the field strength dual to $W$ via the change of variables
\[ W = \epsilon \tilde{W}, \] (34)
and substitute in the first order action (32) to obtain the Lagrangian
\[ \tilde{L}(\Phi, \tilde{W}) = (\partial \Phi) \epsilon \tilde{W} - \frac{1}{2} \epsilon \tilde{W} \epsilon \tilde{W} - \frac{M^2}{2} \Phi \Phi. \] (35)

In this way we obtain the parent Lagrangian (35) which generates the pair of dual theories. In fact, the field $\tilde{W}$ can always be eliminated from Lagrangian (35) to recover the initial Lagrangian (31).

The Euler-Lagrange equation for $\Phi$ is
\[ \epsilon \partial \tilde{W} + M^2 \Phi = 0. \] (36)

If $M \neq 0$, or more generally if it is a regular matrix, Eq. (36) allows the algebraic elimination of the field $\Phi$ in Lagrangian (35), yielding a second order Lagrangian for $\tilde{W}$
\[ \tilde{L} \left( \Phi = -\frac{1}{M^2} \epsilon \partial \tilde{W}, \tilde{W} \right) \propto \frac{1}{2} \epsilon \partial \tilde{W} \epsilon \tilde{W} - \frac{M^2}{2} \epsilon \tilde{W} \epsilon \tilde{W}, \] (37)
which is the dual to the original $L(\Phi)$.

If $M = 0$, the parent Lagrangian reduces to
\[ \tilde{L}(\Phi, \tilde{W}) = (\partial \Phi) \epsilon \tilde{W} - \frac{1}{2} \epsilon \tilde{W} \epsilon \tilde{W}, \] (38)
with the equations of motion
\[ \epsilon \partial \tilde{W} = 0, \] (39)
\[ \epsilon \tilde{W} - \epsilon (\partial \Phi) = 0, \] (40)
preventing the algebraic solution for $\Phi$. Nevertheless, Eq. (39) is a Bianchi identity for $\tilde{W}$ whose solution can be written symbolically as
\[ \tilde{W} = \partial B. \] (41)
That is to say, the dual field $\tilde{W}$ is a field strength and can be written in terms of a new potential $B$. The dual Lagrangian results from substituting Eq. (41) into Lagrangian (38) and is
\[ L(B) = \frac{1}{2} \epsilon \partial B \epsilon \partial B, \] (42)
where we have explicitly used the Bianchi identity in the second term of the RHS of Eq. (38). Finally, the relation
\[ \epsilon \partial B \epsilon - \epsilon \partial \Phi = 0, \] (43)
obtained from Eq. (40), provides the connection between the dual theories.

### III. MASSIVE FIERZ-PAULI LAGRANGIAN

The Lagrangian for the massive Fierz-Pauli field is
\[ \mathcal{L} = -\partial_{\mu} \partial^{\mu} \partial_{\alpha} \partial^{\alpha} - \frac{1}{2} \partial_{\mu} \partial^{\mu} \partial_{\alpha} \partial^{\alpha} h_{\mu} h_{\alpha} - \frac{1}{2} \partial_{\mu} \partial_{\alpha} \partial^{\mu} \partial^{\alpha} h_{\nu} h_{\nu} - \frac{M^2}{2} (h_{\mu \nu} h^{\mu \nu} - h_{\mu} h_{\nu} h_{\mu} h_{\nu}) + h_{\mu \nu} \Theta^{\mu \nu}, \] (44)
where $\Theta^{\mu\nu}$ is the source described by a symmetric tensor, not necessarily conserved in contrast to the massless case. The kinetic part of Lagrangian (44) is just the linearized Einstein Lagrangian. The equations of motion are

\[
\partial^\alpha \partial_\alpha h_{\mu\nu} + \partial_\mu \partial_\nu h_{\alpha}^\alpha - g_{\mu\nu} (\partial^\beta \partial_\beta h_{\alpha}^\alpha - \partial_\alpha \partial_\beta h^{\alpha\beta}) - (\partial_\mu \partial_\nu h^{\alpha}_{\alpha} + \partial_\mu \partial_\alpha h_{\nu}^{\alpha}) + M^2 (h_{\mu\nu} - g_{\mu\nu} h_{\alpha}^{\alpha}) = \Theta_{\mu\nu} .
\] (45)

Taking the trace and the divergence of this equation we have

\[
h_{\alpha}^{\alpha} = - \frac{1}{3M^2} \left( \Theta_{\alpha}^\alpha - \frac{2}{M^2} \partial_\alpha \partial_\beta \Theta^{\alpha\beta} \right) ,
\] (46)

\[
\partial^\mu h_{\mu\nu} = \frac{1}{M^2} \left( \partial_\mu \Theta^\mu_\nu - \frac{1}{3} \partial_\nu \Theta^\alpha_\alpha + \frac{2}{3M^2} \partial_\nu \partial_\alpha \partial_\beta \Theta^{\alpha\beta} \right) ,
\] (47)

which show that the trace and the divergence of $h_{\mu\nu}$ do not propagate, vanishing outside the sources, as expected for a pure spin-2 theory. Eq. (45) now becomes

\[
(\partial^\alpha \partial_\alpha + M^2) h_{\mu\nu} = \tilde{\Theta}_{\mu\nu} ,
\] (48)

where $\tilde{\Theta}_{\mu\nu}$ is a source dependent term that can be expressed in terms of $\Theta_{\mu\nu}$ and its derivatives.

Following the procedure sketched in Appendix A, we can construct an equivalent first order Lagrangian in the standard form. This Lagrangian is not unique, because of the freedom in the choice of the auxiliary fields. Alternatively, we can construct a Lagrangian having the standard form with arbitrary coefficients, which are subsequently adjusted to obtain the original Lagrangian when the auxiliary fields are eliminated. In the present case the last approach is simpler, and we will follow it. Therefore, we start by proposing a field strength $K^{\alpha(\beta\sigma)}$ satisfying the following symmetry properties

\[
K^{\alpha(\beta\sigma)} = K^{\alpha(\sigma\beta)} ,
\] (49)

\[
K^{\alpha(\beta\sigma)} + K^{\beta(\sigma\alpha)} + K^{\sigma(\alpha\beta)} = 0 .
\] (50)

These symmetry properties greatly simplify the manipulations and, as it will become evident in the following, they are consistent with the degrees of freedom of the spin-2 massive field. With this auxiliary field we construct the first order Lagrangian

\[
L = -\frac{1}{6} a K^{\alpha(\beta\sigma)} K_{\alpha(\beta\sigma)} + \frac{1}{8} q K^{\beta}_{\beta} K_{\beta} - \frac{2}{9} e^{\gamma\delta\lambda\alpha} K^{\gamma(\lambda}_{\gamma(\beta\sigma)} K^{\gamma(\sigma\beta)} - \frac{e}{\sqrt{2}} K^{\alpha(\beta\sigma)} \partial_\alpha h_{\beta\sigma} - \frac{M^2}{2} (h_{\mu\nu} h^{\mu\nu} - h^{\mu}_{\mu} h^{\nu}_{\nu}) + \Theta_{\mu\nu} h_{\mu\nu} + \Lambda_{\alpha(\beta\sigma)} \left( K^{\alpha(\beta\sigma)} + K^{\beta(\sigma\alpha)} + K^{\sigma(\alpha\beta)} \right) ,
\] (51)

where $K_{\alpha} = K_{\alpha(\lambda)}$. This Lagrangian has the most general mass term for the field $K^{\alpha(\beta\sigma)}$ with the symmetry properties (49-50). Here $K^{\alpha(\beta\sigma)}$ is identified as the field strength of $h_{\beta\sigma}$. The constraint (51) is enforced by the Lagrange multiplier $\Lambda_{\alpha(\beta\sigma)} = \Lambda_{\alpha(\sigma\beta)}$. In Appendix B we show that the elimination of $K^{\alpha(\beta\sigma)}$ and $\Lambda_{\alpha(\sigma\beta)}$ in (51) leads effectively to the Fierz-Pauli Lagrangian when the coefficients satisfy

\[
4r^2 = a (e^2 - a) ,
\] (52)

\[
3q = 2a + e^2 .
\] (53)

In such a case both theories are equivalent and Lagrangian (51) is the first order standard Lagrangian for the Fierz-Pauli massive field. From conditions (52-53) only two independent coefficients in Lagrangian (51) remain, one of them being the normalization of the auxiliary field.

**IV. DUAL FIELD AND PARENT LAGRANGIAN**

Now that we have identified the field strength $K^{\alpha(\beta\sigma)}$ for $h_{\mu\nu}$ and the corresponding first order theory, we can implement the transformation

\[
K^{\alpha(\beta\sigma)} \rightarrow \Omega^{\alpha(\beta\sigma)}_{(\mu\nu\xi)} ; \quad K^{\alpha(\beta\sigma)} \rightarrow \epsilon^{\alpha\mu\nu\xi} \Omega^{(\beta\sigma)}_{(\mu\nu\xi)}
\] (54)
that leads to the dual theory. Substituting this transformation in Eq. (51), we obtain the parent Lagrangian

\[
L = a\Omega_{(\mu\nu)}^{\{\beta\gamma\}}\Omega_{(\mu\nu)}^{\{\alpha\beta\}} + \frac{2}{3}\sqrt{a} (e^2 - a) e^{\mu\nu\lambda\gamma}\Omega_{(\mu\nu)}^{(\alpha\beta)}\Omega_{(\mu\nu)}^{(\gamma\lambda)}
\]

\[
-\frac{e}{\sqrt{2}} e^{\mu\nu\lambda\gamma}\Omega_{(\mu\nu)}^{\{\alpha\beta\}} \partial_{\mu\nu} h_{\alpha\beta} - \frac{M^2}{2} (h_{\mu\nu} h^{\mu\nu} - h_{\mu}^{\mu} h_{\nu}^{\nu}) + \Theta_{\mu\nu} h^{\mu\nu}
\]

\[
+ 2\alpha^\mu_{\nu\alpha\beta} \left( e^{\mu\nu\xi} \Omega_{(\mu\nu)}^{\{\beta\gamma\}} + e^{\mu\nu\xi} \Omega_{(\mu\nu)}^{(\sigma\alpha)} + e^{\mu\nu\xi} \Omega_{(\mu\nu)}^{(\alpha\beta)} \right)
\]

(55)

where

\[
\Omega_{(\mu\nu)} = g_{\alpha\beta} \Omega_{(\mu\nu)}^{(\alpha\beta)}.
\]

(56)

The dual theory is derived by eliminating \( h_{\alpha\beta} \). Alternatively, by eliminating \( \Omega_{(\mu\nu)}^{(\alpha\beta)} \) from Eq. (55) we recover the Fierz-Pauli theory. The field \( \Omega_{(\beta\gamma)} \) satisfies the constraint

\[
e^{\mu\nu\xi} \Omega_{(\mu\nu)}^{\{\beta\gamma\}} + e^{\mu\nu\xi} \Omega_{(\mu\nu)}^{(\sigma\alpha)} + e^{\mu\nu\xi} \Omega_{(\mu\nu)}^{(\alpha\beta)} = 0,
\]

(57)

as a consequence of Eq. (56). A simple way to warrant this constraint is to express \( \Omega_{(\rho\sigma\tau)}^{\{\beta\gamma\}} \) in terms of a tensor \( T_{(\rho\sigma)} = -T_{(\sigma\rho)} \), as follows:

\[
\Omega_{(\rho\sigma\tau)}^{\{\beta\gamma\}} = \frac{1}{3\sqrt{2}} \left( g_{\beta}^{\gamma} T_{(\rho\sigma)} - g_{\rho}^{\gamma} T_{(\sigma\tau)} + g_{\sigma}^{\gamma} T_{(\tau\rho)} \right)
\]

(58)

This expression identically satisfies the constraint, and avoids the necessity of its explicit use throughout the remaining manipulations. The duality transformation (54) now reads

\[
K^{\alpha(\beta\gamma)} = -\frac{1}{\sqrt{2}} \left( T_{(\mu\nu)}^{\sigma} e^{\mu\nu\alpha\beta} + T_{(\mu\nu)}^{\beta} e^{\mu\nu\alpha\sigma} \right)
\]

(59)

with

\[
K^{\alpha} = -\frac{1}{2} K^{\alpha(\beta\gamma)} = -\sqrt{2} e^{\mu\nu\alpha\beta} T_{(\mu\nu)}^{\beta}.
\]

(60)

The trace of \( T_{(\mu\nu)}^{\beta} \) does not contribute to the expression (58). Thus, we will take \( T_{(\mu\nu)}^{\beta} \) to be traceless and impose this constraint by means of a Lagrange multiplier. The analysis in Section VI will show that this choice is indeed compatible with the dynamics of the Fierz-Pauli field.

Using the identities

\[
e^{\mu\nu}_{\alpha\beta} T_{(\mu\nu)}^{\sigma} \left( T^{(\alpha\beta)} + 2 T^{(\sigma\alpha)} \right) = -\frac{2}{3} T_{(\mu\nu)}^{\sigma} e^{\alpha\beta\gamma\mu} \left( T_{(\alpha\beta)}^{\gamma} + T_{(\gamma\alpha)}^{\beta} + T_{(\beta\gamma)}^{\alpha} \right)
\]

(61)

\[
e^{\mu\nu\alpha\beta} T_{(\mu\nu)}^{\sigma} T_{(\alpha\beta)}^{\sigma} = -2 e^{\mu\nu\alpha\beta} T_{(\mu\nu)}^{\sigma} T_{(\alpha\beta)}^{\sigma} = -4 e^{\mu\nu\alpha\beta} T_{(\mu\nu)}^{\sigma} T_{(\alpha\beta)}^{\sigma}
\]

(62)

which follows from the antisymmetry of \( T^{(\mu\nu)} \) and the null trace property \( T^{(\mu\nu)} = 0 \), we rewrite the parent Lagrangian (53) as

\[
L = \frac{1}{3} (2a - 1 - e^2) T_{(\mu\nu)}^{\sigma} T_{(\mu\nu)}^{\sigma} + \frac{1}{3} (2a + e^2) T_{(\mu\nu)}^{\beta} T_{(\mu\nu)}^{(\beta\gamma)} + \frac{1}{3} \sqrt{a} (e^2 - a) e^{\mu\nu\alpha\lambda} T_{(\mu\nu)}^{\sigma} T_{(\kappa\lambda)}^{\sigma}
\]

\[
+ e T_{(\mu\nu)}^{\sigma} e^{\mu\nu\alpha\beta} \partial_{\alpha} h_{\beta\gamma} - \frac{M^2}{2} (h_{\mu\nu} h^{\mu\nu} - h_{\mu}^{\mu} h_{\nu}^{\nu}) + \Theta_{\mu\nu} h^{\mu\nu} + \lambda_{\beta} T^{(\beta\gamma)}
\]

(63)

The above Lagrangian is equivalent to Eq. (12) in Ref. [16] when \( \Theta_{\mu\nu} = 0 \).
V. DUALITY TRANSFORMATIONS

Varying $h^{\mu \nu}$ in the parent Lagrangian we obtain the Euler-Lagrange equation

$$M^2 (h_\alpha^\sigma g^{\mu \nu} - h^{\mu \nu}) + \Theta^{\mu \nu} - \frac{e}{2} \left( \partial_\sigma T_{(\alpha \beta)}^{\mu \nu} \epsilon^{\alpha \beta \sigma \nu} + \partial_\sigma T_{(\alpha \beta)}^{\nu \sigma} \epsilon^{\alpha \beta \sigma \mu} \right) = 0 .$$

(64)

From here we compute the trace

$$h_\alpha^\alpha = \frac{1}{3M^2} \left( e \epsilon^{\alpha \beta \sigma \nu} \partial_\sigma T_{(\alpha \beta)}^{\mu \nu} - \Theta_\alpha^\alpha \right) ,$$

(65)

which allows to solve for $h^{\mu \nu}$

$$h^{\mu \nu} = - \frac{e}{2M^2} \left( \epsilon^{\alpha \beta \sigma \nu} \partial_\sigma T_{(\alpha \beta)}^{\mu \nu} + \epsilon^{\alpha \beta \sigma \mu} \partial_\sigma T_{(\alpha \beta)}^{\nu \sigma} \right) + \frac{1}{3M^2} g^{\mu \nu} \left( e \epsilon^{\alpha \beta \rho \kappa} \partial_\rho T_{(\alpha \beta) \kappa} - \Theta_\alpha^\alpha \right) + \frac{1}{M^2} \Theta^{\mu \nu} ,$$

(66)

giving the first duality relation $h^{\mu \nu} = h^{\mu \nu}(T)$. Varying $T^{(\mu \nu) \sigma}$ we derive the equation

$$\frac{1}{3} (4a - e^2) T^{(\mu \nu) \sigma} + \frac{1}{3} (2a + e^2) \left( T^{(\mu \sigma) \nu} - T^{(\nu \sigma) \mu} \right) + \sqrt{a (e^2 - a)} e^{\mu \nu \alpha \beta} T_{(\alpha \beta)}^{(\sigma) \sigma} + e \epsilon^{\mu \nu \alpha \beta} \partial_\alpha h_\beta^\sigma + \frac{1}{2} (\lambda^\mu g^{\nu \sigma} - \lambda^\nu g^{\mu \sigma}) = 0 .$$

(67)

From here, contracting with the metric and the Levi-Civita tensors we have

$$T^{(\mu \nu)} = - \frac{1}{2a - e^2} \frac{e}{\sqrt{a (e^2 - a)}} \left( \partial_\sigma h^{\sigma \mu} - \partial_\mu h^\sigma \right) + \frac{3}{4} e \lambda^\mu ,$$

(68)

$$T^{(\mu \nu)} + T^{(\sigma \mu)} + T^{(\nu \sigma)} = \frac{1}{2a - e^2} e^{\mu \nu \sigma \lambda} \left[ e (\partial_\sigma h_\lambda^\sigma - \partial_\lambda h^\sigma) + \frac{3}{2} \sqrt{a (e^2 - a)} \lambda^\sigma \right] ,$$

(69)

$$a e_{\mu \nu \kappa} T^{(\mu \nu) \sigma} = 2 \sqrt{a (e^2 - a)} T_{(\kappa \lambda)}^{(\sigma \lambda)} + e (\partial_\sigma h_\lambda^\sigma - \partial_\lambda h^\sigma) + \frac{1}{2} e_\sigma^\kappa \lambda^\mu h^\sigma$$

$$- \frac{1}{3} \frac{2a + e^2}{2a - e^2} (g_\sigma^\rho g_\lambda^\rho - g_\sigma^\rho g_\rho^\lambda) \left[ e (\partial_\sigma h_\rho^\sigma - \partial_\rho h^\sigma) + \frac{3}{2} \sqrt{a (e^2 - a)} \rho^\sigma \right] ,$$

(70)

after some algebraic manipulations. Using these relationships we solve for $T^{(\mu \nu) \sigma}$ in Eq. (67), obtaining

$$T^{(\mu \nu) \sigma} = - \frac{1}{2e} e^{\mu \nu \alpha \beta} \partial_\sigma h_\beta^\alpha + \frac{1}{2a - e^2} \frac{2a + e^2}{6e} e^{\mu \nu \sigma \lambda} (\partial_\sigma h_\lambda^\sigma - \partial_\lambda h^\sigma) - \frac{1}{2a - e^2} \sqrt{a (e^2 - a)} (\partial_\mu h^{\sigma \nu} - \partial_\nu h^{\sigma \mu})$$

$$+ \frac{1}{6a^2} (2a + e^2) \sqrt{a (e^2 - a)} [g^{\sigma \mu} (\partial_\kappa h^{\kappa \nu} - \partial_\nu h^{\kappa \mu}) - g^{\sigma \nu} (\partial_\kappa h^{\kappa \mu} - \partial_\mu h^{\kappa \nu})]$$

$$+ \frac{1}{2a - e^2} \frac{1}{a^2} (2a + e^2) e^{\mu \nu \sigma \lambda} \lambda^\lambda - \frac{e^2}{4a} (2a - e^2) (\lambda^\mu g^{\nu \sigma} - \lambda^\nu g^{\mu \sigma}) ,$$

(71)

which constitutes the second duality relation $T^{(\mu \nu) \sigma} = T^{(\mu \nu) \sigma}(h, \lambda)$. Only the Lagrange multiplier $\lambda^\mu$ remains to be varied, which imposes the null trace constraint on $T^{(\mu \nu) \sigma}$. Summarizing, using only algebraic manipulations and without any mixing between the results of different variations, the Lagrangian equations of motion are and together with

$$T^{(\mu \nu) \sigma} = 0 .$$

(72)

Eqs. and are those to be used in eliminating either $h^{\sigma \mu}$ or $T^{(\mu \nu) \sigma}$ from the parent Lagrangian, to obtain the corresponding Lagrangians for $T^{(\mu \nu) \sigma}$ or $h^{\sigma \mu}$ respectively.

In fact, the degrees of freedom of $h^{\sigma \mu}$ are mapped into the traceless part of $T^{(\mu \nu) \sigma}$, $\hat{T}^{(\mu \nu) \sigma}$. The relationship between the Fierz-Pauli field and $T^{(\mu \nu) \sigma}$ can be obtained in a straightforward way as follows. The null trace condition imposes

$$\lambda^\mu = - \frac{4}{3e} \sqrt{a (e^2 - a)} (\partial_\sigma h^{\sigma \mu} - \partial_\mu h) .$$

(73)
Using this constraint to eliminate the Lagrange multiplier in Eq. (71), we get
\[
\hat{T}^{(\mu\nu)^{\sigma}} = -\frac{1}{2e} \epsilon^{\mu\nu\alpha\beta} \partial_\alpha h_\beta^\sigma + \frac{1}{2e} \epsilon^{\mu\nu\lambda} (\partial_\sigma h_\lambda^\nu - \partial_\lambda h_\nu^\sigma) - \frac{1}{2ae} \frac{\sqrt{a(e^2 - a)}}{(e^2 - a)} \left( \partial^\mu h^{\nu\sigma} - \partial^\nu h^{\mu\sigma} \right)
+ \frac{1}{6ae} \frac{\sqrt{a(e^2 - a)}}{(e^2 - a)} [g^{\sigma\mu} (\partial_\sigma h^{\nu\lambda} - \partial_\lambda h^{\nu\sigma}) - g^{\sigma\nu} (\partial_\sigma h^{\mu\lambda} - \partial_\lambda h^{\mu\sigma})],
\]
which is the final expression for the duality transformation \(\hat{T}^{(\mu\nu)^{\sigma}} = \hat{T}^{(\mu\nu)^{\sigma}}(h)\).

All previous relations greatly simplify on shell. Under this circumstance the condition
\[
(\partial^\alpha h_\alpha^\lambda - \partial^\lambda h_\alpha^\sigma) = \frac{1}{M^2} \partial_\sigma \Theta^{\eta\lambda}
\]
for the trace and divergence of \(h^{\sigma\lambda}\) is obtained, using Eqs. (66) and (67). The remaining on shell constraints are
\[
T_{(\mu\nu)^{\mu}} = 0, \quad \partial_\beta T_{(\mu\nu)^{\beta}} = -\frac{\sqrt{a(e^2 - a)}}{3aeM^2} (\partial_\mu \partial_\alpha \Theta^{\alpha}_{\nu} - \partial_\nu \partial_\alpha \Theta^{\alpha}_{\mu}),
\]
\[
e^{\lambda\mu\nu\beta} T_{(\mu\nu)^{\lambda\beta}} = \frac{2}{eM^2} \partial_\beta \Theta^{\mu\lambda},
\]
\[
\partial_\kappa T^{(\kappa\lambda)^{\sigma}} + \frac{1}{2a} \sqrt{a(e^2 - a)} \epsilon^{\mu\nu\kappa\lambda} \partial_\kappa T^{(\mu\nu)^{\sigma}} = -\frac{(2a + e^2)}{6aeM^2} \epsilon_{\lambda\mu}^{\kappa\lambda} \partial^\mu \partial_\beta \Theta^{\nu\beta}
+ \frac{1}{6aeM^2} (-\partial^\sigma \partial_\mu \Theta^{\alpha\lambda} + g^{\alpha\lambda} \partial_\beta \partial_\kappa \Theta^{\alpha\beta}).
\]

Using the constraint equations for both fields the on shell duality relations become
\[
h^{\mu\nu} = -\frac{e}{2M^2} \left( \epsilon^{\alpha\beta\gamma\nu} \partial_\alpha T_{(\alpha\beta)^{\mu}} + \epsilon^{\alpha\beta\gamma\nu} \partial_\beta T_{(\alpha\beta)^{\mu}} \right) + \frac{1}{3M^2} \partial^\mu \partial_\lambda \Theta^{\gamma\lambda} - \Theta^{\gamma\mu}
+ \frac{1}{M^2} \Theta^{\mu\nu},
\]
\[
T_{(\mu\nu)^{\beta}} = -\frac{1}{2e} \epsilon^{\alpha\mu\nu} \partial_\alpha h_\sigma^\beta - \frac{\sqrt{a(e^2 - a)}}{2ae} (\partial_\mu h_\nu^\beta - \partial_\nu h_\mu^\beta)
\]
\[-\frac{1}{2eM^2} \epsilon^{\mu\nu\beta} \partial_\eta \Theta^{\eta\beta} - \frac{\sqrt{a(e^2 - a)}}{6aeM^2} (g_{\beta\gamma} \partial_\eta \Theta^{\eta\beta} - g_{\mu\beta} \partial_\eta \Theta^{\eta\beta}).
\]

In the particular case where \(a = e^2\) the duality transformations acquire the usual form
\[
T_{(\mu\nu)^{\beta}} = -\frac{1}{2e} \epsilon^{\alpha\mu\nu} \partial_\alpha h_\sigma^\beta - \frac{1}{2eM^2} \epsilon^{\mu
u\beta} \partial_\eta \Theta^{\eta\beta},
\]
involving only the Levi-Civita tensor. The constraint (73) reduces to \(\partial_\kappa T^{(\kappa\lambda)^{\sigma}} = 0\) out of sources, which means that the field \(T^{(\kappa\lambda)^{\sigma}}\) contains purely transversal degrees of freedom. Otherwise, if \(a \neq e^2\), the degrees of freedom of \(h^{\alpha\beta}\) are also mapped in the longitudinal components \((\partial_\kappa T^{(\kappa\lambda)^{\beta}} + \partial_\kappa T^{(\kappa\beta)^{\lambda}})\).

VI. DUAL THEORY

The substitution of \(h_{\sigma\beta}(T)\), given by Eq. (66), in the parent Lagrangian (63) leads to the following Lagrangian for \(T_{(\mu\nu)^{\sigma}}\)
\[
L = \frac{1}{3} \left( 2a - \frac{e^2}{2} \right) T_{(\mu\nu)^{\sigma}} T^{(\mu\nu)^{\sigma}} + \frac{1}{3} \left( 2a + e^2 \right) T_{(\mu\nu)^{\beta}} T^{(\mu\nu)^{\beta}} + \frac{1}{2} \sqrt{a(e^2 - a)} \epsilon^{\mu
u\lambda\sigma} T_{(\mu\nu)^{\sigma}} T^{(\kappa\lambda)^{\sigma}}
+ \frac{1}{2} h_{\sigma\beta}(T) \left( -e \epsilon^{\mu
u\alpha\beta} \partial_\alpha T_{(\mu\nu)^{\beta}} + \Theta^{\sigma\beta} \right) + \lambda_\beta T^{\beta}.
\]
\[
L = \frac{4}{9} F_{(\alpha \beta \gamma) \mu} F^{(\alpha \beta \gamma) \nu} + \frac{2}{3} F_{(\alpha \beta \gamma) \mu} F^{(\alpha \beta \gamma) \nu} - F_{(\alpha \beta \mu) \nu} F^{(\alpha \beta \nu) \mu} \\
- \frac{2 M^2}{3 e^2} \left[ \left( 2a - \frac{1}{2} e^2 \right) T_{(\mu \nu) \sigma} T^{(\mu \nu) \sigma} + (2a + e^2) T_{(\mu \nu) \sigma} T^{(\mu \sigma) \nu} + \frac{3}{2} \sqrt{a (e^2 - a)} \varepsilon^{\mu \nu \alpha \beta} T_{(\mu \nu) \sigma} T_{(\alpha \beta) \sigma} \right] \\
+ \lambda T^{(\alpha \beta \mu)} J^{(\alpha \beta)} \mu.
\] (84)

Here we have introduced the field strength
\[
F_{(\alpha \beta \gamma) \mu} = \partial_\alpha T_{(\beta \gamma) \mu} + \partial_\beta T_{(\gamma \alpha) \mu} + \partial_\gamma T_{(\alpha \beta) \mu},
\] (85)

and the source term is given as a function of the traceless field \(T_{\alpha \beta \mu} \) by
\[
J^{(\alpha \beta) \mu} = \frac{2}{e} \left( \frac{1}{3} \varepsilon^{\alpha \beta \mu \nu} \partial_\rho \Theta^\rho_\alpha - \varepsilon^{\alpha \beta \sigma \nu} \partial_\sigma \Theta^\nu_\mu \right).
\] (86)

Note that the new source \(J^{(\alpha \beta) \mu} \) is traceless
\[
J^\alpha = J^{(\alpha \beta)} \beta = \frac{2}{e} \left( \frac{1}{3} \varepsilon^{\alpha \beta \mu \nu} \partial_\mu \Theta^\mu_\alpha - \varepsilon^{\alpha \beta \sigma \nu} \partial_\sigma \Theta^\nu_\beta \right) = 0,
\] (87)

and also satisfies
\[
\varepsilon_{\alpha \beta \mu \nu} J^{(\alpha \beta) \mu} = -\frac{4}{e} \partial_\mu \Theta^\mu_\kappa, \quad \partial_\alpha J^{(\alpha \beta) \mu} = 0, \quad \partial_\mu J^{(\alpha \beta) \mu} = -\frac{2}{e} \varepsilon^{\alpha \beta \sigma \nu} \partial_\sigma \partial_\mu \Theta^\nu_\mu.
\] (88)

As stated previously \(T_{(\mu \nu) \sigma} = -T_{(\nu \mu) \sigma} \) and therefore the field \(T_{(\mu \nu) \sigma} \) has 24 components. But not all of them are true degrees of freedom, because there are cyclic variables. This becomes clear by defining
\[
T_{(\mu \nu) \sigma} = \hat{T}_{(\mu \nu) \sigma} - \frac{1}{3} (g_{\sigma \mu} T_{\nu} - g_{\sigma \nu} T_{\mu})
\] (89)

where \(T_{\mu} \equiv T_{(\mu \beta)}^\beta \), and \(\hat{T}_{(\mu \nu) \sigma} \) is a traceless field, \(\hat{T}_{(\mu \nu) \nu} = 0 \). Next, we rewrite Lagrangian \([4]\) in terms of \(\hat{T}_{(\lambda \psi) \sigma} \) and \(T_{\mu} \) and we further use the Euler-Lagrangian equation for \(T_{\mu} \) to eliminate this variable from the Lagrangian. The resulting Lagrangian contains linear and bilinear terms in \(\lambda \beta \). Finally, using the corresponding Euler-Lagrangian equation for \(E_{\beta} \) we can also eliminate this variable. In such a way we obtain an alternative version of the dual Lagrangian for \(\hat{T}_{(\mu \nu) \sigma} \)
\[
L = \frac{4}{9} \hat{F}_{(\alpha \beta \gamma) \mu} \hat{F}^{(\alpha \beta \gamma) \nu} + \frac{2}{3} \hat{F}_{(\alpha \beta \gamma) \mu} \hat{F}^{(\alpha \beta \gamma) \nu} - \hat{F}_{(\alpha \beta \mu) \nu} \hat{F}^{(\alpha \beta \nu) \mu} + \hat{T}_{(\alpha \beta) \mu} J^{(\alpha \beta)}
- \frac{2 M^2}{3 e^2} \left[ \left( 2a - \frac{1}{2} e^2 \right) \hat{T}_{(\mu \nu) \sigma} \hat{T}^{(\mu \nu) \sigma} + (2a + e^2) \hat{T}_{(\mu \nu) \sigma} \hat{T}^{(\mu \sigma) \nu} + \frac{3}{2} \sqrt{a (e^2 - a)} \varepsilon^{\mu \nu \alpha \beta} \hat{T}_{(\mu \nu) \sigma} \hat{T}_{(\alpha \beta) \sigma} \right].
\] (90)

This clearly shows that the degrees of freedom are in the traceless field \(\hat{T}_{(\mu \nu) \sigma} \), as has already been assumed in Section IV. When varying Lagrangian \([4]\) it is necessary to take into account that not all the components of \(\hat{T}_{(\mu \nu) \sigma} \) are independent, because of the traceless condition, and this is rather cumbersome.

Consequently, in order to study the properties of the dual field \(T_{(\mu \nu) \sigma} \) it is more convenient to go back to Lagrangian \([4]\), because there we have to impose only the antisymmetry constraint. In this way, starting from this Lagrangian we obtain the equations of motion
\[
E^{(\beta \gamma) \nu} : = \frac{2}{3} \partial_\alpha \left[ 2 F^{(\alpha \beta \gamma) \nu} + \left( F^{(\alpha \beta \nu) \gamma} + F^{(\gamma \alpha \nu) \beta} + F^{(\beta \gamma \nu) \alpha} \right) \right] - \partial_\nu F^{(\beta \gamma) \nu} \kappa \\
- \partial_\alpha \left( g^{\gamma \nu} F^{(\alpha \beta \kappa) \kappa} + g^{\beta \nu} F^{(\gamma \alpha \kappa) \kappa} \right) + \frac{M^2}{e^2} \sqrt{a (e^2 - a)} \varepsilon^{\beta \gamma \nu \lambda} T_{(\kappa \lambda)} \nu \\
+ \frac{2 M^2}{3 e^2} \left[ \left( 2a - \frac{1}{2} e^2 \right) T^{(\beta \gamma) \nu} + \frac{1}{2} (2a + e^2) T^{(\alpha \gamma) \nu} - \frac{1}{2} \lambda T^{(\alpha \gamma) \nu} \right] \\
- \frac{1}{4} (\lambda T^{(\beta \gamma) \nu} - \lambda T^{(\beta \gamma) \nu}) - \frac{1}{2} J^{(\beta \gamma) \nu} = 0,
\] (91)

\[
T^{(\mu \nu) \nu} = 0.
\] (92)
In order to make explicit the Lagrangian constraints arising from Eq. (94), which determine the number of propagating degrees of freedom associated to the massive field $T^{(\mu\nu)\rho}$, we consider the case of zero sources. In Appendix C we provide some details of the derivation and the results are summarized here. The constraints are

$$T^{(\mu\alpha)}_{\alpha} = 0, \quad T^{(\alpha\beta)\alpha} + T^{(\beta\alpha)\mu} + T^{(\beta\mu)\alpha} = 0, \quad \partial_{\beta} T^{(\alpha\beta)\theta} = 0, \quad (93)$$

$$0 = \left( \partial_{\beta} T^{(\beta\gamma)\nu} + \partial_{\beta} T^{(\beta\nu)\gamma} \right) + \frac{(e^2 - a)}{a} \left( \epsilon^{\beta\gamma\kappa\lambda} \partial_{\beta} T_{(\kappa\lambda)}^{(\gamma)\nu} + \epsilon^{\beta\nu\kappa\lambda} \partial_{\beta} T_{(\kappa\gamma)}^{(\lambda)\nu} \right) := S^{(\gamma)\nu}, \quad (94)$$

where we observe that the antisymmetric part of $\partial_{\beta} T^{(\beta\gamma)\nu}$ turns out to be zero, as shown in the sourceless version of Eq. (164) of Appendix C. The count of the number of independent degrees of freedom goes as follows. The field $T^{(\mu\nu)\rho}$ has 24 independent components. Eqs. (93-94) provide $4 + 4 + 6 = 14$ constraints respectively, thus leaving $24 - 14 = 10$ independent variables up to this level. Because of the symmetry $S^{(\gamma)\nu} = S^{(\nu)\gamma}$, the remaining Eq. (94) provides only 10 relations. Nevertheless, among them we find 5 additional identities: 4 arising from $\partial_{\nu} S^{(\gamma)\nu} = 0$ and 1 arising from $\partial_{\nu} S^{(\gamma)\nu} = 0$, leaving only 5 additional independent constraints. Thus, Eq. (94) reduces to $10 - 5 = 5$ the previous 10 independent degrees of freedom, as appropriate for a massive spin-2 system.

Taking into account the constraints (93-94), the equation of motion with zero sources becomes

$$\partial^{2} T^{(\beta\gamma)\nu} + \partial_{\alpha} \partial^{\gamma} T^{(\alpha\beta)\nu} - \partial_{\alpha} \partial^{\gamma} T^{(\alpha\gamma)\nu} + \frac{M^{2}}{2e^{2}} \sqrt{a} \left( \epsilon^{\beta\gamma\kappa\lambda} \partial_{\alpha} T_{(\kappa\lambda)}^{(\gamma)\nu} + \epsilon^{\alpha\nu\kappa\lambda} \partial_{\alpha} T_{(\kappa\gamma)}^{(\lambda)\nu} \right) = 0. \quad (95)$$

Furthermore, from Eq. (94) we can write

$$\partial_{\alpha} \partial_{\gamma} T^{(\alpha\beta)\nu} - \partial_{\alpha} \partial_{\gamma} T^{(\alpha\nu)\beta} = - \frac{D}{4a} \left[ \partial_{\gamma} \left( \epsilon^{\alpha\beta\kappa\lambda} \partial_{\alpha} T_{(\kappa\lambda)}^{(\gamma)\nu} + \epsilon^{\alpha\nu\kappa\lambda} \partial_{\alpha} T_{(\kappa\gamma)}^{(\lambda)\nu} \right) - \partial_{\beta} \left( \epsilon^{\alpha\gamma\kappa\lambda} \partial_{\alpha} T_{(\kappa\lambda)}^{(\gamma)\nu} + \epsilon^{\alpha\nu\kappa\lambda} \partial_{\alpha} T_{(\kappa\gamma)}^{(\lambda)\nu} \right) \right]. \quad (96)$$

The above equation and its dual imply

$$\left( \partial^{\gamma} \partial_{\alpha} T^{(\alpha\beta)\nu} - \partial^{\beta} \partial_{\alpha} T^{(\alpha\nu)\gamma} \right) = - \frac{D^{2}}{a e^{2}} \partial^{2} T^{(\beta\gamma)\nu} + \frac{D}{2 e^{2}} \partial^{2} \epsilon^{\beta\gamma\kappa\lambda} T_{(\kappa\lambda)}^{(\gamma)\nu}. \quad (97)$$

Hence, the equation of motion (93) and its dual can be written as

$$\frac{a}{e^{2}} \partial^{2} T + \frac{D}{e^{2}} \partial^{2} T^{*} + \frac{a M^{2}}{e^{2}} T + \frac{D M^{2}}{e^{2}} T^{*} = 0, \quad (98)$$

$$\frac{a}{e^{2}} \partial^{2} T^{*} - \frac{D}{e^{2}} \partial^{2} T + \frac{a M^{2}}{e^{2}} T^{*} - \frac{D M^{2}}{e^{2}} T = 0, \quad (99)$$

where we have omitted the indices of $T^{(\beta\gamma)\nu}$ and $T^{*}$ means the dual of $T$. Finally, solving for $T^{(\beta\gamma)\nu}$ we get

$$\left( \partial_{2} + M^{2} \right) T^{(\beta\gamma)\nu} = 0. \quad (100)$$

The simpler case $e^{2} = a$ is reminiscent of the standard duality transformations and the constraint equations simplify to

$$T^{(\mu\delta)}_{\delta} = 0, \quad T^{(\mu\alpha\beta)} = 0, \quad \partial_{\beta} T^{(\mu\nu)\theta} = 0, \quad \partial_{\beta} T^{(\theta\mu)\nu} = 0. \quad (101)$$

**VII. POINT MASS SOURCE**

As an illustration let us discuss a simple example: the field generated by a point mass $m$ at rest. The equation of motion in the massive Fierz-Pauli theory, arising from Eqs. (44), (46), and (47), is

$$\left( \partial^{\alpha} \partial_{\alpha} + M^{2} \right) h_{\mu\nu} = \Theta_{\mu\nu} - \frac{1}{3} \left( g_{\mu\nu} + \frac{1}{M^{2}} \partial_{\mu} \partial_{\nu} \right) \Theta_{\alpha}^{\alpha}. \quad (102)$$

The corresponding source is the energy-momentum tensor of a point mass at rest and has the components

$$T^{\mu\nu} = \frac{1}{c^{2}} \Theta_{\mu\nu}, \quad T^{\mu\nu} = \frac{1}{c^{2}} \Theta_{\mu\nu}, \quad \Theta_{\mu\nu} = \frac{1}{c^{2}} \Theta_{\mu\nu}.$$
\[ \Theta_{00} = 16\pi m \delta(r), \quad \Theta_{0i} = \Theta_{ji} = 0. \] (103)

The resulting field configuration is
\[ h_{00} = \frac{8m}{3}r e^{-Mr}, \quad h_{ij} = \frac{1}{2} \delta_{ij} h_{00} - \frac{1}{2M^2} \partial_i \partial_j h_{00}, \quad h_{0i} = 0. \] (104)

The last term in \( h_{ij} \) is irrelevant when the field is coupled to a conserved source.

It is interesting to observe how the zero mass limit discontinuity (the van Dam-Veltman-Zakharov discontinuity [18-19]) manifests itself here. In the limit \( M \to 0 \), \( h_{00} \) converges to \( \frac{4}{3} \) of the Newtonian potential \( \frac{m}{r} \). Besides, the component \( h_{ij} \) has a divergent term plus \( \frac{1}{4} h_{00} \delta_{ij} \). This has to be contrasted with the massless spin-2 theory in the Lorentz gauge, where the non-zero fields are
\[ \tilde{h}_{00} = \frac{2m}{r}, \quad \tilde{h}_{ij} = \tilde{h}_{00} \delta_{ij} + \partial_i \partial_j f(r), \] (105)

where the last term in \( \tilde{h}_{ij} \) accounts for a remaining gauge freedom associated to a time independent spatial rotation.

Next we consider the corresponding theory for \( T_{(\alpha\beta)\gamma} \), in the smaller case of \( a = e^2 \). We refer the reader to Appendix C for the notation. Here the dual source is
\[
\Theta^{00} = 16\pi m \delta(r), \quad \Theta^{0i} = \Theta^{ji} = 0.
\]

The resulting field configuration is
\[
h^{00} = \frac{8m}{3}r e^{-Mr}, \quad h^{ij} = \frac{1}{2} \delta_{ij} h^{00} - \frac{1}{2M^2} \partial_i \partial_j h^{00}, \quad h^{0i} = 0.
\]

It is straightforward to verify that both solutions, \( h^{\mu\nu} \) and \( T^{(\alpha\beta)\gamma} \), are in fact related by the duality transformations. Out of the source the on-shell duality relations are
\[
T^{(\alpha\beta)\gamma} = -\frac{32m}{e} \left( g^{\mu0} \delta^{\alpha\beta\rho} - \frac{1}{3} \epsilon^{\alpha\beta\rho} \right) \partial_\mu \delta(r),
\]
\[
J^{(\alpha\beta)\mu} = \frac{32m}{e} \left( g^{\mu0} \epsilon^{\alpha\beta\rho} + g^{\alpha0} \epsilon^{\beta\rho\mu} - 2 g^{\beta\rho} \epsilon^{\alpha\mu} \right) \partial_\mu \delta(r).
\]

Our conventions are \( \epsilon^{0123} = \epsilon_{123} = +1 \). The equation of motion, arising from Eq. (172) of Appendix C, becomes
\[
(\partial_\alpha \partial^\alpha + M^2) T^{(\beta\gamma)\nu} = -\frac{1}{4} J^{(\beta\gamma)\nu} + \frac{1}{4} J^{(\gamma\beta)\nu} + \frac{1}{6M^2} \partial_\alpha \partial^\alpha J^{(\gamma\beta)\nu},
\]

with the constraints
\[
M^2 T^{(\beta\gamma)\nu} = -\frac{1}{2} J^{(\beta\gamma)\nu}, \quad M^2 \partial_\theta T^{(\mu\nu)\theta} = 0, \quad M^2 \partial_\beta T^{(\beta\gamma)\nu} = 0.
\]

From here, the non zero components of \( T_{(\alpha\beta)\gamma} \) are
\[
T^{(0i)j} = \frac{2m}{3e} (1 + Mr) e^{-Mr} \epsilon_{ijk} x_k,
\]
\[
T^{(ij)0} = \frac{4m}{3e} (1 + Mr) e^{-Mr} \epsilon_{ijk} x_k.
\]

We can now compare both theories. In terms of the massive Fierz-Pauli solution, the solution for \( T_{(\mu\nu)\sigma} \) can be written as
\[
T^{(0i)j} = -\frac{1}{4e} \epsilon_{ijk} \partial_k h^{00},
\]
\[
T^{(ij)0} = +\frac{1}{2e} \epsilon_{ijk} \partial_k h^{00}.
\]

It is straightforward to verify that both solutions, \( h^{\mu\nu} \) and \( T^{(\alpha\beta)\gamma} \), are in fact related by the duality transformations. Out of the source the on-shell duality relations are
\[
h_{\alpha\beta} = -\frac{e}{2M^2} \left( \epsilon_{\gamma\rho\alpha} \partial^\rho T^{(\gamma\delta)}_\beta + \epsilon_{\gamma\rho\beta} \partial^\rho T^{(\gamma\delta)}_\alpha \right),
\]
\[
T^{(\alpha\beta)\gamma} = -\frac{1}{2e} \epsilon_{\alpha\beta} \partial_\rho h^{\delta\gamma}.
\]

From Eq. (115) and Eqs. (111) (111) we obtain Eqs. (112) and (113). Conversely, applying Eq. (114) to the expressions (112) and (113) we recover (110) and (111). It is interesting to observe that the term that diverges in the zero mass limit in (104) does not contribute to \( T^{(\alpha\beta)\gamma} \), which remains non divergent in this limit. Thus the description in terms of \( T^{(\alpha\beta)\gamma} \) seems more suitable for studying the massless limit.

The analysis of how the massless limit and the van Dam-Veltman-Zakharov discontinuity appears in the dual theory requires the discussion of duality in the case of \( M = 0 \). We postpone the discussion of this interesting point to a forthcoming paper.
VIII. SUMMARY AND FINAL REMARKS

In this article we have shown that a generalization of duality transformations applicable to arbitrary Lorentz tensors is possible, and we have developed a scheme to construct such dual descriptions. The main idea of this scheme is the use of a first order parent Lagrangian from which either the original theory or the dual one can be obtained, by means of permissible substitutions arising from the algebraic solutions of the corresponding equations of motion. This procedure guarantees the equivalence of both theories and thus provides an adequate dualization for arbitrary spin, either in the free or in the coupled to an external source case, in contrast with previous proposals \cite{14}. In such a way one is able to construct new non trivial and more general actions to describe a given physical system, which might show some advantages over the standard ones.

Given a Lagrangian to be dualized, we first construct an equivalent auxiliary first order Lagrangian written in standard form, which can always be done by using the method of Lagrange multipliers in the manner described in Refs. \cite{17} and \cite{20}. This first order Lagrangian is not unique and provides the identification of what we have called the field strength of the original field. Dualization occurs at this level, through the introduction of the dual tensor defined by the contraction of the Levi-Civita tensor with the field strength. Different possibilities might arise at this level which will produce alternative dual theories.

Substitution of the field strength in terms of the dual tensor in the auxiliary first order Lagrangian produces the parent Lagrangian, which is a functional of the original field configuration together with the new dual field. On the one hand, the elimination of the dual field from this Lagrangian, via its equations of motion, always takes us back to the original second order theory. On the other hand, the elimination of the original field from the parent Lagrangian defines the dual theory.

This dual tensor plays different roles in the massive and the massless cases, because the duality transformation is singular in the limit $M \to 0$. The mapping between dual theories is also very different according to these cases. For $M \neq 0$ the dual tensor turns out to be the basic configuration variables of the dual theory and its definition in terms of the original field strength provides the relation among the resultant theories. Here the dual field is interpreted as a potential. For $M = 0$ the equation of motion of the dual field becomes a constraint (a Bianchi identity) on the dual variable, which implies that it can be written in terms of a potential. Hence the dual field can be interpreted as a new field strength. The connection between both theories is now given by a relation between the original and dual potentials which usually involves derivatives. Summarizing, for massive theories duality relates field strengths and potentials, while for massless theories it relates the corresponding potentials.

We have applied this scheme to the massive spin-2 field coupled to external sources, obtaining a family of dual theories. The starting point is the symmetric massive Fierz-Pauli field $h_{\mu \nu}$ with its standard Lagrangian. The corresponding first order auxiliary Lagrangian, which has two independent parameters, is written in terms of $h_{\mu \nu}$ plus the field strength $K_{\alpha \beta \gamma}$. The latter satisfies additional symmetry properties. We have explicitly shown that the elimination of the auxiliary field leads to the original massive Fierz-Pauli Lagrangian. At this stage there is some freedom in the election of the dual field $\Omega$ and we have chosen the relation $K_{\alpha \beta \gamma} = \epsilon^{\alpha \mu \rho \sigma} \Omega^{\beta \gamma}_{\mu \rho \sigma}$. In order to partially fulfill the induced symmetry properties of $T^{\alpha \beta \gamma}_{\mu \nu \rho}$ we have introduced the auxiliary tensor $T^{\alpha \beta \gamma}$ which is required only to be antisymmetric in the first two indices, and which serves as the basic dual field in the sequel. This field is reminiscent of what is called the Fierz tensor in Ref. \cite{21}. Our approach for the massive case is different from the latter reference because we take $T^{\alpha \beta \gamma}$ as the basic variable for the massive situation. The Lagrangian for this field is subsequently constructed by eliminating $h^{\mu \nu}$ from the parent Lagrangian. By construction this dual theory is equivalent to the initial Fierz-Pauli description, and the connection between both is established. The correct number of degrees of freedom in the dual theory is verified by identifying the Lagrange constraints arising from the equations of motion.

Finally, we have discussed the case of the massive field generated by a point mass $m$ at rest, which is described using both the original and the dual theory. This simple example suggests that the description in terms of $T^{(\alpha \beta \gamma)}$ behaves continuously in the limit $M \to 0$, in contrast with that in terms of $h^{\mu \nu}$. The latter theory develops a singularity in the $M \to 0$ case, while the components of the dual field remain finite.

We postpone for a separate publication a detailed discussion of the $M = 0$ case. This situation is directly related to the problem of dualizing linearized gravity, which has been the subject of recent investigations \cite{22}, \cite{23}, \cite{24}. Our preliminary work on this subject shows some interesting features: (i) the zero mass limit of the dual Lagrangian for $T^{(\alpha \beta \gamma)}$, given in Eq. \cite{24}, has no arbitrary parameters so that one would expect it to be completely determined by a set of gauge symmetries to be determined. (ii) the Dirac analysis of the constraints leads to the count of two degrees of freedom per space point, in contrast with the results in Ref. \cite{14}. The analysis of the gauge structure that arises in the approach pursued here should be of some interest, together with the discussion of the Van Dam-Veltman-Zakharov discontinuity in the dual theory.
ACKNOWLEDGMENTS

This work was partially supported by CONICET-Argentina and CONACYT-México. LFU acknowledges support from DGAPA-UNAM project IN-117000, as well as CONACYT project 32431-E. He also thanks the program CERN-CONACYT for additional support.

APPENDIX A: FIRST ORDER LAGRANGIAN IN THE STANDARD FORM

Here we show how to construct a first order Lagrangian equivalent to a given second order Lagrangian using the method of Lagrange multipliers. This approach has been presented in the framework of classical mechanics for regular systems in Ref. [17], to construct a Hamiltonian formalism without the use of a Legendre transformation. Ref. [20] deals with its application to singular systems. To give a general idea of the method, we simply sketch it for the case of regular field theories. Consider a given regular Lagrangian

\[ L = L(\psi^a, \psi^{\alpha,\mu}) . \]  

(116)

Let us assume that we want to introduce a set of new functions \( f^a_{\mu} = f^a_{\mu}(\psi^b, \lambda^a) \), and treat them as new independent fields. To do this it must be possible to solve \( \psi^a,\mu \) in terms of \( f^a_{\mu} \), i.e. \( |\frac{\partial f^a_{\mu}}{\partial \psi^a}| \neq 0 \). Under such a condition, we can write a new Lagrangian of the form

\[ L = L(\psi^a, f^a_{\mu}) , \]  

(117)

and impose the constraints \( f^a_{\mu} - f^a_{\mu}(\psi^b, \lambda^a) = 0 \). This can be done in a consistent way by introducing a set of Lagrange multipliers \( \lambda^a_{\mu} \)

\[ \tilde{L} = L(\psi^a, f^a_{\mu}) + \lambda^a_{\mu} (f^a_{\mu}(\psi^b, \lambda^a) - f^a_{\mu}) . \]  

(118)

The new formulation is clearly equivalent to the original one. The auxiliary functions \( f^a_{\mu} \) exclusively appear as algebraic variables, without derivatives. Therefore, they can be eliminated by using their equations of motion

\[ \frac{\partial \tilde{L}}{\partial f^a_{\mu}} = \frac{\partial L}{\partial f^a_{\mu}} - \lambda^a_{\mu} = 0 . \]  

(119)

The solution of this equation is a set of functions \( f^a_{\mu} = f^a_{\mu}(\psi^a, \lambda^a) \), and the resulting Lagrangian has the form

\[ \tilde{L} = L(\psi^a, f^a_{\mu}) + \lambda^a_{\mu} (f^a_{\mu}(\psi^b, \lambda^a) - f^a_{\mu}) . \]  

(120)

The first term of the above Lagrangian shows that \( \lambda^a_{\mu} \) define the field strength of \( \psi^b \). Their relationship with the configuration variables is given by their equation of motion in the first order theory

\[ f^a_{\mu}(\psi^b, \lambda^a) = f^a_{\mu}(\psi^a, \lambda^a) - \lambda^a_{\mu} \frac{\partial f^b_{\mu}}{\partial \lambda^a} + \lambda^a_{\mu} \frac{\partial L}{\partial \lambda^a} = 0 . \]  

(121)

This definition of the field strength is not unique, because it depends on the choice of the functions \( f^a_{\mu}(\psi^b, \lambda^a) \).

APPENDIX B: EQUIVALENCE BETWEEN SECOND ORDER AND FIRST ORDER LAGRANGIANS

In this Appendix we show that Lagrangian (51), where the coupling constants satisfy Eqs. (52) and (53), is a first order Lagrangian for the massive Fierz-Pauli theory [14]. From Lagrangian (51) we obtain the equations of motion for \( K^a_{\alpha(\beta\sigma)} \) and \( \Lambda^a_{\alpha(\beta\sigma)} \)

\[ -\frac{1}{3} a K_{\alpha(\beta\sigma)} + \frac{1}{4} q g_{\beta\sigma} K_{\alpha} - \frac{2}{9} \left( \epsilon^{\delta\alpha\beta\gamma} K_{\gamma(\delta\sigma)} + \epsilon^{\gamma\delta\alpha\sigma} K_{\gamma(\delta\beta)} \right) - \frac{e}{\sqrt{2}} \partial_\alpha h_{\beta\sigma} + \Phi_{\alpha(\beta\sigma)} = 0 , \]  

(122)

\[ K^{\alpha(\beta\sigma)} + K^{\beta(\sigma\alpha)} + K^{\sigma(\alpha\beta)} = 0 , \]  

(123)

where
\[ \Phi_{\alpha\beta\sigma} = \Lambda_{\alpha\beta\sigma} + \Lambda_{\beta\sigma\alpha} + \Lambda_{\sigma\alpha\beta}, \]  

is a completely symmetric tensor. Eq. (124) leads to a constraint between the two possible contractions of the indices of \( K^{\alpha\beta\sigma} \)

\[ K_\alpha + 2K^{\beta}_{\{\beta\alpha\}} = 0. \]  

First, we solve the field strength \( K_{\alpha(\beta\gamma)} \) in terms of the potential field \( h_{\sigma\beta} \). We take the two possible traces in Eq. (122), thus obtaining

\[ \frac{1}{3} (3q - a) K_\alpha - \frac{e}{\sqrt{2}} \partial_{\alpha} h_{\beta}^\beta + \Phi_{\alpha\beta}^\beta = 0, \]  

\[ \frac{1}{12} (3q + 2a) K_\alpha - \frac{e}{\sqrt{2}} \partial_{\alpha} h_{\beta}^\alpha + \Phi_{\alpha\beta}^\beta = 0, \]  

where we have used Eq. (125). Therefore

\[ K_\alpha = \frac{2\sqrt{2}e}{(2a - 3q)} (\partial_{\alpha} h_{\beta}^\beta - \partial_\alpha h_{\beta}^\beta) . \]  

Performing a cyclic permutation of the indices in Eq. (122) and adding the results we get

\[ \Phi_{\alpha\beta\sigma} = \frac{e}{3\sqrt{2}} (\partial_{\alpha} h_{\beta\sigma} + \partial_{\beta} h_{\alpha\sigma} + \partial_{\sigma} h_{\alpha\beta}) - \frac{q}{12} (g_{\sigma\beta} K_\alpha + g_{\beta\alpha} K_\alpha + g_{\alpha\sigma} K_\beta) . \]  

The contraction of Eq. (122) with \( \epsilon^{\alpha\beta}_{\mu\nu} \) gives

\[ -\frac{1}{3} a e^{\gamma\delta}_{\mu\nu} K_{\gamma(\delta\sigma)} + \frac{1}{4} q e^{\alpha\beta}_{\mu\nu} K_{\alpha\beta} + \frac{2}{3} r (K_{\mu(\nu\sigma)} - K_{\nu(\mu\sigma)}) + \frac{1}{3} r (g_{\mu\sigma} K_{\nu} - g_{\nu\mu} K_{\sigma}) - \frac{e}{\sqrt{2}} \epsilon^{\gamma\delta}_{\mu\nu} h_{\delta\sigma} = 0 . \]  

Combining this last equation with Eq. (122) to eliminate terms proportional to \( \epsilon^{\gamma\delta}_{\mu\nu} K_{\gamma(\delta\sigma)} \) we obtain the following expression for \( K_{\alpha(\beta\sigma)} \) in terms of \( h_{\alpha\beta} \)

\[ K_{\alpha(\beta\sigma)} = -\sqrt{2} e \left[ A P_{\alpha(\beta\sigma)} + B Q_{\alpha(\beta\sigma)} + C R_{\alpha(\beta\sigma)} \right] , \]  

where

\[ P_{\alpha(\beta\sigma)} = g_{\alpha\sigma} (\partial^\gamma h_{\gamma\beta} - \partial_\beta h_{\gamma\sigma}) + g_{\alpha\beta} (\partial^\gamma h_{\gamma\sigma} - \partial_\sigma h_{\gamma\beta}) - 2g_{\beta\sigma} (\partial^\gamma h_{\gamma\alpha} - \partial_\alpha h_{\gamma\beta}) , \]  

\[ Q_{\alpha(\beta\sigma)} = \partial_{\beta} h_{\alpha\sigma} + \partial_{\sigma} h_{\alpha\beta} - 2 \partial_\alpha h_{\beta\sigma} , \]  

\[ R_{\alpha(\beta\sigma)} = \epsilon^{\gamma\delta}_{\alpha\beta} \partial_{\gamma} h_{\delta\sigma} + \epsilon^{\gamma\delta}_{\alpha\sigma} \partial_{\gamma} h_{\delta\beta} , \]  

together with the coefficients

\[ A = e^2 \left( \frac{2}{3} r^2 + \frac{1}{4} a q \right) \left( a - \frac{3}{4} q \right)^{-1} (a^2 + 4r^2)^{-1} , \]  

\[ B = -\frac{1}{2} a e^2 (a^2 + 4r^2)^{-1} , \]  

\[ C = -re^2 (a^2 + 4r^2)^{-1} . \]  

All three tensors appearing on the right hand side of Eq. (131) have a vanishing cyclic sum. The Lagrangian in terms of \( h_{\alpha\beta} \) can be obtained by replacing expression (131) in the first order Lagrangian (51), or more simply, noting that the contribution from the mass terms for \( K^{\alpha(\beta\sigma)} \) is \((-1/2)\) of the contribution from the interaction term \(-e\sqrt{2} K^{\alpha(\beta\sigma)} \partial_{\alpha} h_{\beta\sigma} \). Half of this last contribution gives the kinetic terms of the \( h_{\alpha\beta} \) Lagrangian. The interaction term gives

\[ (2A + 2B) \partial_{\mu} h^{\mu\nu} \partial_\nu h_{\alpha}^\alpha + (-2B) \partial_{\mu} h^{\mu\nu} \partial^\alpha h_{\alpha\nu} + (-4A) \partial_{\mu} h^{\mu\nu} \partial_\nu h_{\alpha}^\alpha + (2A) \partial_{\mu} h^{\mu\nu} \partial^\alpha h_{\alpha\nu} , \]  

after substituting Eq. (131). Let us observe that the term proportional to \( C \) gives no contribution. Comparing this expression with the kinetic part of the original Lagrangian (44) we obtain \( A = (-1/2) \), \( B = (-1/2) \). From here the relations (52) and (53) follow.
We start from Eqs. (91) and (92). The zero trace condition implies
\[ F^{(\alpha\beta\theta)}_{\theta} = \partial_\theta T^{(\alpha\beta)}_{\theta}. \] (139)

We will also use the property
\[ \epsilon_{\alpha\beta\gamma\delta} F^{(\beta\gamma\delta)}_{\psi} = 3\epsilon_{\alpha\beta\gamma\delta} \partial^{\delta} T^{(\gamma\delta)}_{\psi}, \] (140)
together with the notation
\[ T^{(\mu\nu\rho)}_{\mu} = T^{(\mu\nu)}_{\rho} + T^{(\nu\rho)}_{\mu} + T^{(\rho\mu)}_{\nu}, \] (141)
and
\[ D = \sqrt{a (e^2 - a)}. \] (142)

Useful relations to determine the Lagrangian constraints are obtained according to the following manipulations.

- \( g_{\gamma\nu} E^{(\beta\gamma)}_{\nu} = 0 \) implies
  \[ \lambda^\beta = \frac{4DM^2}{3e^2} \epsilon^{\beta\sigma\kappa\tau} T_{(\kappa\tau)\sigma}. \] (143)

- \( \partial_\beta E^{(\beta\gamma)}_{\nu} = 0 \) leads to
  \[ \begin{align*}
  & DM^2 \epsilon^{\beta\gamma\kappa\lambda} \partial_\beta T_{(\kappa\lambda)^\nu} - \frac{e^2}{4} (g^{\gamma\nu} \partial_\beta \lambda^\beta - \partial^\nu \lambda^\gamma) \\
  & + \frac{2}{3} M^2 \partial_\beta \left[ \left( 2a - \frac{1}{2} e^2 \right) T^{(\beta\gamma)}_{\nu} + \frac{1}{2} (2a + e^2) \left( T^{(\beta\nu)}_{\gamma} - T^{(\gamma\nu)}_{\beta} \right) \right] = 0.
  \end{align*} \] (144)

This expression can be decomposed in the symmetric and antisymmetric part
\[ \begin{align*}
  & M^2 \partial_\beta \left( T^{(\beta\gamma)}_{\nu} + T^{(\beta\nu)}_{\gamma} \right) = - \frac{DM^2}{2a} \left( \epsilon^{\beta\gamma\kappa\lambda} \partial_\beta T_{(\kappa\lambda)^\nu} + \epsilon^{\beta\nu\kappa\lambda} \partial_\beta T_{(\kappa\lambda)^\gamma} \right), \\
  & + \frac{e^2}{4a} \left( g^{\gamma\nu} \partial_\beta \lambda^\beta - \frac{1}{2} (\partial^\nu \lambda^\gamma + \partial^\gamma \lambda^\nu) \right),
  \end{align*} \] (145)
\[ \begin{align*}
  & M^2 (a - e^2) \partial_\beta \left( T^{(\beta\gamma)}_{\nu} - T^{(\beta\nu)}_{\gamma} \right) = - \frac{3DM^2}{2} \left( \epsilon^{\beta\gamma\kappa\lambda} \partial_\beta T_{(\kappa\lambda)^\nu} - \epsilon^{\beta\nu\kappa\lambda} \partial_\beta T_{(\kappa\lambda)^\gamma} \right) \\
  & + M^2 \partial_\beta \left( 2a + e^2 \right) T^{(\gamma\nu)}_{\beta} + \frac{3e^2}{8} (\partial^\gamma \lambda^\nu - \partial^\nu \lambda^\gamma).
  \end{align*} \] (146)

Applying \( \partial_\nu \) to Eq. (144) we have
\[ \begin{align*}
  & 4M^2 \left[ De^{\beta\gamma\kappa\lambda} \partial_\nu \partial_\beta T_{(\kappa\lambda)^\nu} + 2a \partial_\nu \partial_\beta T^{(\beta\gamma)}_{\nu} \right] = e^2 \partial_\nu (\partial^\gamma \lambda^\nu - \partial^\nu \lambda^\gamma).
  \end{align*} \] (147)

- From \( \epsilon_{\beta\gamma\nu\psi} E^{(\beta\gamma)}_{\nu} = 0 \) we obtain
  \[ \begin{align*}
  & 2 \epsilon_{\beta\gamma\nu\psi} \partial_\alpha F^{(\beta\gamma)}_{\alpha} - 6M^2 \epsilon_{\beta\gamma\nu\psi} T^{(\beta\gamma)}_{\nu} - 3 \epsilon_{\beta\gamma\nu\psi} J^{(\beta\gamma)}_{\nu} = 0.
  \end{align*} \] (148)

Contracting the above equation with \( \epsilon^{\beta\sigma\tau\psi} \) leads to
\[ \partial_\alpha F^{(\beta\gamma)}_{\alpha} - M^2 T^{(\beta\gamma)}_{\nu} = \frac{1}{2} J^{(\beta\gamma)}_{\nu}. \] (149)
\[ (\partial^\nu \lambda^\mu - \partial^\mu \lambda^\nu) = \frac{4DM^2}{e^2} e^{\mu\nu\kappa\lambda} \partial_\kappa T_{(\kappa\lambda)}^\theta + \frac{8aM^2}{e^2} \partial_\theta T^{(\mu\nu)^\theta} + \frac{8D^2M^2}{3ae^2} \partial_\theta T^{(\theta\mu\nu)}. \quad (150) \]

This equation contains \( \partial_\theta T^{(\mu\nu)^\theta} \) and its dual \( \frac{1}{2} e^{\alpha\beta\mu\nu} \partial_\theta T^{(\mu\nu)^\theta} \). We now show that this relation leads to a solution of the combination

\[ \Lambda_{\theta\beta} = (\partial_\theta \lambda_\beta - \partial_\beta \lambda_\theta). \quad (151) \]

Taking into account that the constraint (143) gives for \( ^*\Lambda_{\mu\nu} = \frac{1}{2} e^{\mu\nu\theta\beta} (\partial_\theta \lambda_\beta - \partial_\beta \lambda_\theta) \) the expression

\[ ^*\Lambda_{\theta\beta} = \frac{8DM^2}{3e^2} \partial_\theta T^{(\theta\mu\nu)}, \quad (152) \]

we can rewrite Eq. (150) in terms of \( \Lambda^{\mu\nu} \) and its dual as

\[ \Lambda^{\mu\nu} + \frac{D}{a} (\Lambda^{\mu\nu}) = \frac{4DM^2}{e^2} e^{\mu\nu\kappa\lambda} \partial_\kappa T_{(\kappa\lambda)}^\theta - \frac{8aM^2}{e^2} \partial_\theta T^{(\mu\nu)^\theta}. \quad (153) \]

The dual of the above equation together with the property \( ^*\Lambda = -\Lambda \) produce a second independent equation

\[ ^*\Lambda^{\mu\nu} - \frac{D}{a} \Lambda^{\mu\nu} = \frac{8DM^2}{e^2} \partial_\theta T^{(\mu\nu)^\theta} - \frac{4aM^2}{e^2} e^{\mu\nu\alpha\beta} \partial_\theta T^{(\alpha\beta)^\theta}. \quad (154) \]

Solving the system we are left with

\[ \Lambda^{\mu\nu} = -\frac{8M^2a}{e^2} \partial_\theta T^{(\mu\nu)^\theta}, \quad (155) \]

\[ ^*\Lambda^{\mu\nu} = -\frac{4M^2a}{e^2} e^{\mu\nu\alpha\beta} \partial_\theta T^{(\alpha\beta)^\theta}. \quad (156) \]

These expressions are consistent with the duality relationship. Eq. (157) directly gives

\[ M^2 \partial_\theta T^{(\mu\nu)^\theta} = -\frac{e^2}{8a} \Lambda^{\mu\nu}. \quad (157) \]

Taking now the divergence of Eq. (158) we obtain

\[ \partial_\mu (\partial^\mu \lambda^\nu - \partial^\nu \lambda^\mu) = -\frac{8aM^2}{e^2} \partial_\theta \partial_\theta T^{(\mu\nu)^\theta}. \quad (158) \]

The comparison of the above relation with Eq. (147) gives

\[ 4M^2 e^{\beta\nu\kappa\lambda} \partial_\kappa \partial_\lambda T_{(\kappa\lambda)}^\sigma = 0. \quad (159) \]

Contracting the free index of this last equation with a Levi-Civita tensor we obtain

\[ 4M^2 \partial^\theta F_{(\alpha\beta\kappa)\theta} = 0, \quad (160) \]

which together with Eq. (149) implies

\[ M^2 T^{(\beta\gamma\nu)} = -\frac{1}{2} J^{(\beta\gamma\nu)}. \quad (161) \]

Taking the divergence of this equation respect to one of the antisymmetric indices we get

\[ M^2 \partial_\beta \left( T^{(\beta\gamma\nu)} - T^{(\beta\nu)^\gamma} \right) = -M^2 \partial_\beta T^{(\gamma\nu)^\beta} - \frac{1}{2} \partial_\beta J^{(\gamma\nu)^\beta}. \quad (162) \]
Using Eq. (161) in Eq. (143) we obtain a relationship between the Lagrange multiplier $\lambda_\sigma$ and the equation of motion is

$$\chi^\beta = -\frac{2D}{9e^2}e^{\beta\kappa\tau}J_{(\kappa\tau\sigma)}. \quad (163)$$

Finally, Eqs. (162) and (157) imply

$$M^2\partial_\beta \left( T^{(\beta\gamma)\nu} - T^{(\beta\nu)\gamma} \right) = \frac{e^2}{8a} \gamma^{\gamma\nu} - \frac{1}{2} \partial_\alpha J^{(\gamma\nu)\alpha}. \quad (164)$$

From the above results the following independent Lagrangian constraints arise:

$$\lambda_\sigma = -\frac{2D}{9e^2}e_{\sigma\beta\gamma\nu}J^{(\beta\gamma)\nu}, \quad (165)$$

$$M^2T^{(\beta\gamma)\nu} = -\frac{1}{2} J^{(\beta\gamma)\nu}, \quad (166)$$

$$M^2\partial_\theta T^{(\mu\nu)\theta} = -\frac{e^2}{8a} \Lambda^{\mu\nu}, \quad (167)$$

$$M^2\partial_\gamma T^{(\beta\gamma)\nu} = \frac{1}{16a} \left[ e^2 \gamma^{\gamma\nu} - 4a \partial_\alpha J^{(\gamma\nu)\alpha} \right] + \frac{e^2}{8a} \left[ g^{\gamma\nu}\partial_\rho \lambda^\rho - \frac{1}{2} \left( \partial^{\mu}\gamma^\nu \partial_\alpha \gamma^\nu - \frac{2}{e^2} \partial_\alpha \left( e^{\beta\gamma\kappa}\lambda^{(\kappa\lambda)} \gamma + e^{\beta\nu\kappa}\lambda^{(\kappa\lambda)} \gamma \right) \right) - \frac{2DM^2}{e^2} \partial_\beta \left( e^{\beta\gamma\kappa}\lambda^{(\kappa\lambda)} \gamma + e^{\beta\nu\kappa}\lambda^{(\kappa\lambda)} \gamma \right). \quad (168)$$

and the equation of motion is

$$2\partial_\alpha \partial^{\alpha} T^{(\beta\gamma)\nu} + \frac{D}{2a} \partial_\gamma \left[ \partial^{\beta} \left( e^{\gamma\kappa\lambda}T^{(\kappa\lambda)\nu} + e^{\nu\kappa\lambda}T^{(\kappa\lambda)\gamma} \right) - \partial^\gamma \left( e^{\beta\gamma\kappa}\lambda^{(\kappa\lambda)} \gamma + e^{\nu\kappa\lambda}\lambda^{(\kappa\lambda)} \beta \right) \right] + \frac{M^2}{e^2} \sqrt{a} (e^2 - a) e^{\beta\gamma\kappa}\lambda^{(\kappa\lambda)} \nu \right] + \frac{2aM^2}{e^2} T^{(\beta\gamma)\nu} = \frac{1}{2} J^{(\beta\gamma)\nu} + \frac{2a + e^2}{6e^2} J^{(\gamma\beta)\nu} + \frac{1}{6M^2} \left( 2\partial_\alpha a J^{(\gamma\beta)\nu} + 2\partial_\nu \partial_\alpha J^{(\beta\gamma)\nu} + \partial^\gamma \partial_\alpha J^{(\gamma\beta)\nu} + \partial^\gamma \partial_\alpha J^{(\beta\gamma)\nu} \right) \right] \right) \right] - \frac{1}{4} \left( g^{\gamma\nu}\lambda^\beta - g^{\beta\nu}\lambda^\gamma \right) + \frac{e^2}{4M^2a} \partial^{\beta} \left( g^{\gamma\nu}\partial_\rho \lambda^\rho - \frac{1}{2} \partial^{\mu}\gamma^\nu \partial_\alpha \gamma^\nu \right) - \frac{e^2}{4M^2a} \left( g^{\beta\nu}\partial_\gamma - g^{\gamma\nu}\partial_\beta \right) \partial_\rho \lambda^\rho \right. \right] \right] - \frac{e^2}{24aM^2} \left( 8\partial^{\mu}\lambda^{\beta\gamma} + \partial^\beta\lambda^{\nu\gamma} + \partial^\gamma\lambda^{\nu\beta} \right) \right] \right] - \frac{e^2}{8aM^2} \partial_\alpha \left( g^{\gamma\nu}\lambda^{\alpha\beta} + g^{\beta\nu}\lambda^{\alpha\gamma} \right). \quad (169)$$

In the case $a = e^2$, we have

$$\lambda^\beta = 0, \quad M^2T^{(\beta\gamma)\nu} = -\frac{1}{2} J^{(\beta\gamma)\nu}, \quad (170)$$

$$M^2\partial_\theta T^{(\mu\nu)\theta} = 0, \quad M^2\partial_\gamma T^{(\beta\gamma)\nu} = -\frac{1}{4} \partial_\alpha J^{(\gamma\nu)\alpha}, \quad (171)$$

$$(\partial_\alpha \partial^{\alpha} + M^2) T^{(\beta\gamma)\nu} = \frac{1}{4} J^{(\beta\gamma)\nu} + \frac{1}{4} J^{(\gamma\beta)\nu} + \frac{1}{12M^2} \left( 2\partial_\alpha \partial^{\alpha} J^{(\gamma\beta)\nu} + 2\partial^{\nu} \partial_\alpha J^{(\beta\gamma)\nu} + \partial^\beta \partial_\alpha J^{(\gamma\nu)\alpha} + \partial^\gamma \partial_\alpha J^{(\beta\nu)\alpha} \right). \quad (172)$$

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