EXCLUDED HOMEOMORPHISM TYPES FOR DUAL COMPLEXES OF SURFACES

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Abstract. We study an obstruction to prescribing the dual complex of a strict semistable degeneration of an algebraic surface. In particular, we show that if ∆ is a complex homeomorphic to a 2-dimensional manifold with negative Euler characteristic, then ∆ is not the dual complex of any semistable degeneration. In fact, our theorem is somewhat more general and applies to some complexes homotopy equivalent to such a manifold. Our obstruction is provided by the theory of tropical complexes.

The dual complex of a semistable degeneration is a combinatorial encoding of the combinatorics of the components of the special fiber. In recent years, it has been studied because of connections to tropical geometry [HK12], non-Archimedean analytic geometry [Ber99], and birational geometry [dFKX12, BF14]. In this paper, we study obstructions to realizing arbitrary complexes as dual complexes of degenerations of surfaces.

We let $R$ be any rank 1 valuation ring with algebraically closed residue field and we will consider a degeneration over $R$ to be a flat, proper scheme $X$ over $\text{Spec} \, R$ which is strictly semistable in the sense of [GRW14, Sec. 3]. Specifically, we require that $X$ is covered by open sets which admit étale morphisms over $R$ to $\text{Spec} \, R[x_0, \ldots, x_n]/\langle x_0 \cdots x_m - \pi \rangle$ for some $m \leq n$ and some $\pi$ in the maximal ideal of $R$. The dual complex of $X$ is a $\Delta$-complex with one vertex for each irreducible component of the special fiber and higher-dimensional simplices for each connected component where irreducible components intersect.

Since semistability implies that the special fiber is normal crossing, the dimension of the dual complex is at most the relative dimension of the family $X$. In dimension 1, any graph is the dual complex of some degeneration of curves [Bak08, Cor. B.3]. However, in this paper, we show that the analogous statement is not true in dimension 2.

Theorem 1. There is no strict semistable degeneration over a rank 1 valuation ring $R$, whose general fiber is a smooth, geometrically irreducible surface and such that the dual complex of the special fiber is homeomorphic to a topological surface $\Sigma$ with $\chi(\Sigma) < 0$.

We conjecture that Theorem 1 can be strengthened to replace “homeomorphic” with “homotopy equivalent.” In fact, we can prove a strengthening in this direction which applies to $\Delta$-complexes formed from a manifold with negative Euler characteristic by attaching additional simplices in a controlled way.
way. First, what we call “fins” are allowed so long as they don’t change the homotopy type and where the gluing is along a subset that’s not too complicated. Second, arbitrary complexes may be attached to the manifold, so long as the gluing is along a finite set. These complexes are collectively the “ornaments” in the following definition.

**Definition 2.** We say that a 2-dimensional $\Delta$-complex $\Delta$ is a manifold with fins and ornaments if there exists a subcomplex $\Sigma$, the manifold, subcomplexes $F_1, \ldots, F_n$, the fins, and a subcomplex $O$, the ornaments, such that:

1. We have a decomposition $\Delta = \Sigma \cup F_1 \cup \ldots \cup F_n \cup O$.
2. $\Sigma$ is homeomorphic to a connected 2-dimensional topological manifold.
3. For any $i$, $F_i$ is contractible and $F_i \cap \Sigma$ is a path.
4. For any $i > j$, $F_j \cap F_i$ is a subset of the endpoints of the path $F_i \cap \Sigma$.
5. The intersection $O \cap (\Sigma \cup F_1 \cup \ldots \cup F_n)$ is a finite set of points.

If the manifold $\Sigma$ has negative Euler characteristic, we call $\Delta$ a hyperbolic manifold with fins and ornaments and if the subcomplex $O$ is empty, then we call $\Delta$ a manifold with fins.

**Theorem 3.** If $\Delta$ is a hyperbolic manifold with fins and ornaments, then there is no degeneration with dual complex $\Delta$.

The obstruction to having the dual complex of a degeneration be a hyperbolic manifold with fins and ornaments is in lifting the dual complex to a tropical complex. A tropical complex is a $\Delta$-complex, together with the intersection numbers of the 1-dimensional strata inside the 2-dimensional strata, which are called the structure constants of the tropical complex [Car13].

**Theorem 4.** If $\Delta$ is a hyperbolic manifold with fins and ornaments, then there is no 2-dimensional tropical complex with $\Delta$ as its underlying topological space.

Note that when we construct the tropical complex from a degeneration, we do not incorporate valuations from the defining equations, in contrast with both the suggestion from the introduction of [Car13] and the construction in [GRW14, Ex. 3.10], where edges of the dual complex have lengths coming from the value group of $R$. We believe that for many applications, such metric information will be essential, but in this paper, we intentionally ignore it for the purpose of being able to use the results from [Car15], where the edges implicitly all have length 1. Philosophically, we think of our approach as taking a deformation in the category of “metric tropical complexes” from the complex which encodes the valuations to a tropical complex with all edges of length 1, where we can apply Theorem 4.

There are two characteristics of the special fiber of a degeneration which are incorporated into the axioms of a tropical complex. The first is that the special fiber is principal which gives a relationship among the intersection
numbers with a fixed curve. The second is that Hodge index theorem, which restricts the possible intersection matrices of a fixed surface in the special fiber.

Both axioms of a tropical complex are necessary in the proof of the obstruction. Without the condition coming from the Hodge index theorem, the object would only be a weak tropical complex, and any $\Delta$-complex lifts to a weak tropical complex. For example, if $\Delta$ is homeomorphic to a topological manifold, then choosing all structure constants equal to 1 gives a weak tropical complex, but this will not be a tropical complex if $\chi(\Delta) < 0$, as explained in Example 6.

On the other hand, Kollár has shown that any finite $n$-dimensional simplicial complex is realizable as the dual complex of a simple normal crossing divisor [Kol14, Thm. 1], but such a divisor would not give a tropical complex because the divisor is not necessarily principal. However, when connected, such a divisor can be realized as the exceptional locus of the resolution of a normal, isolated singularity [Kol14, Thm. 2]. Thus, we see Theorem 3 as an example of how the global geometry of a smooth algebraic variety is more restricted than the local geometry of a singularity, in line with [KK14].

We also note that unlike the cases in Theorem 1, topological surfaces with non-negative Euler characteristic are all possible as homeomorphism types of degenerations. In particular, the 2-sphere, the real projective plane, the torus, and the Klein bottle appear as degenerations of K3 surfaces, Enriques surfaces, Abelian surfaces, and bielliptic surfaces respectively. In fact, a partial converse is possible in that the dual complexes of such degenerations have been classified by results of Kulikov, Persson, Pinkham, and Morrison [Kul77, Per77, PP81, Mor81]. Note that topological surfaces of non-negative Euler characteristic all arose from degenerations of varieties of Kodaira dimension 0. However, these classification results would already suffice to prove Theorem 1 if we assumed that the general fiber had Kodaira dimension 0.

1. Tropical complexes and tropical surfaces

We begin by recalling the definition of tropical complexes, as introduced in [Car13] and their properties, as studied in [Car15]. Unlike those papers, we additionally assume that the underlying $\Delta$-complex is regular, meaning that the faces of any fixed simplex are distinct. All of our combinatorial results also hold without a regularity assumption, but the dual complex of a strictly semistable degeneration always is always regular, so that case is sufficient for our applications. In addition, we will only work with 2-dimensional tropical complexes, which we will call tropical surfaces. We will refer to the simplices of dimensions 0, 1, and 2 in a 2-dimensional $\Delta$-complex as its vertices, edges, and facets, respectively.

**Definition 5.** A weak tropical surface is a finite, connected, regular $\Delta$-complex whose cells have dimension at most 2, together with integers $\alpha(v, e)$
for every endpoint \( v \) of an edge \( e \), such that for each edge \( e \), we have an equality:

\[
\alpha(v, e) + \alpha(w, e) = \text{deg}(e),
\]

where \( v \) and \( w \) are the endpoints of \( e \) and \( \text{deg}(e) \) is the number of 2-dimensional facets containing \( e \).

At a vertex \( v \) of a weak tropical surface \( \Delta \), the local intersection matrix \( M_v \) is a symmetric matrix whose rows and columns are indexed by edges containing \( v \) and such that the entry corresponding to edges \( e \) and \( e' \) is:

\[
(M_v)_{e,e'} = \begin{cases} 
\# \{ \text{facets containing both } e \text{ and } e' \} & \text{if } e \neq e' \\
-\alpha(w, e) & \text{if } e = e',
\end{cases}
\]

where \( w \) is the endpoint of \( e \) other than \( v \). A tropical surface is a weak tropical surface \( \Delta \) such that for every vertex \( v \) of \( \Delta \), \( M_v \) has exactly one positive eigenvalue.

**Example 6.** Let \( \Delta \) be a triangulated manifold, meaning a regular \( \Delta \)-complex which is homeomorphic to a 2-dimensional, connected topological manifold. Then, every edge \( e \) is contained in two facets, so a symmetric choice for the structure constants satisfying the constraint (1) on \( e \) is to set \( \alpha(v, e) = \alpha(w, e) = 1 \) for both endpoints \( v \) and \( w \). This gives us a weak tropical surface.

At any vertex \( v \), the link is a cycle and so, for example, if this cycle has length 5, the local intersection matrix is:

\[
M_v = \begin{pmatrix}
-1 & 1 & 0 & 0 & 1 \\
1 & -1 & 1 & 0 & 0 \\
0 & 1 & -1 & 1 & 0 \\
0 & 0 & 1 & -1 & 1 \\
1 & 0 & 0 & 1 & -1
\end{pmatrix}
\]

An \( m \times m \) matrix of this form always has a positive eigenvalue of 1 for the eigenvector \((1, \ldots, 1)^T\). However, if \( m > 6 \), then there exist at least two other positive eigenvalues. Therefore, \( \Delta \) is a tropical complex if and only if each vertex is contained in at most 6 edges. A simple counting argument shows that if each vertex is contained in at most 6 edges, then the Euler characteristic \( \chi(\Delta) \) is non-negative. Thus, we’ve verified the special case of Theorem 4 that there is no tropical surface with all structure constants equal to 1 for which the underlying topological space is a hyperbolic manifold.

In this paper, the main purpose of the structure constants \( \alpha(v, e) \) is to define a sheaf of linear functions on a weak tropical surface \( \Delta \). These linear functions are the tropical analogue of non-vanishing regular functions in algebraic geometry.

**Definition 7** ([Const. 4.2 and Def. 4.3 in Car13]). Let \( \Delta \) be a weak tropical surface and \( e \) an edge of \( \Delta \). We define \( N_e \) to be the simplicial complex consisting of \( e \) with a 2-dimensional simplex attached for each facet of \( \Delta \).
containing $e$. Then, there exists a natural map $\pi_e : N_e \to \Delta$ which is an open inclusion on $N^o_e$, which we define to be the union of interior of $e$ and of the interiors of the facets of $N_e$.

We also define a map $\phi_e : N_e \to \mathbb{R}^{d+2}/\mathbb{R}$, where $d$ is the number of facets of $\Delta$ containing $e$ and the quotient is by the line in $\mathbb{R}^{d+2}$ generated by the vector $(1, \ldots, 1, -\alpha(v, e), -\alpha(w, e))$. The map $\phi_e$ sends each vertex of $N_e$ to the image $e_i$ of the $i$th basis vector of $\mathbb{R}^{d+2}$, with $v$ and $w$ going to $e_{d+1}$ and $e_{d+2}$ respectively. We then extend $\phi_e$ linearly to all of $N_e$. The vectors $e_1, \ldots, e_{d+2}$ generate a lattice inside $\mathbb{R}^{d+2}/\mathbb{R}$ and we say that a function $\ell : \mathbb{R}^{d+2}/\mathbb{R} \to \mathbb{R}$ has linear slopes if $\ell(e_i) - \ell(e_j)$ is an integer for any two vectors $e_i$ and $e_j$.

Finally, we say that a continuous function $\phi$ on an open subset $U \subset \Delta$ is linear if the following two conditions hold. First, if we identify the interior of any facet $f$ meeting $U$ with a unimodular simplex in $\mathbb{R}^2$, then $\phi$ is an affine linear function with integral slopes on the interior of $f$. Second, on any edge $e$ meeting $U$, $\phi|_{U \cap N^o_e} = \ell \circ \phi_e$ for some linear function $\ell : \mathbb{R}^{d+2}/\mathbb{R} \to \mathbb{R}$ with integral slopes. We write $\mathcal{A}$ for the sheaf of linear functions on $\Delta$.

Note that the image of the map $\phi_e$ in Definition 7 lies in the affine linear space consisting of the linear combinations $\sum c_i e_i$ such that $c_1 + \cdots + c_{d+2} = 1$, which is a proper subset of $\mathbb{R}^{d+2}/\mathbb{R}$ because of relation (1) in the definition of a weak tropical surface. Roughly speaking, the quotient in Definition 7 means that linearity on a weak tropical surface imposes one condition beyond linearity on each facet of $\Delta$. More precisely, we have the following:

**Lemma 8.** Let $e$ be an edge of a tropical surface $\Delta$ and let $N^o_e$ be the union of the interiors of $e$ and of the facets containing $e$, as in Definition 4. If $\phi$ is a linear function on $N^o_e$, and is constant on all but one of these facets, then $\phi$ is constant.

**Proof.** If $\phi_e$ is as in Definition 4, then $\phi$ factors as $\ell \circ \phi_e$ for some affine linear function $\ell$. We also let $d$ and $e$ be as in Definition 4, and we can assume that the facets containing $e$ are numbered so that $\phi$ is constant on all but the first one. Then, we have the linear relation

$$e_1 = -e_2 - \cdots - e_d + \alpha(v, e)e_{d+1} + \alpha(w, e)e_{d+2},$$

where the sum of the coefficients is 1 by the relation (1), and so $\ell(e_2) = \cdots = \ell(e_{d+2})$ implies that $\ell(e_1) = \ell(e_2) = \cdots = \ell(e_{d+2})$. Therefore, $\ell$ is constant on $\phi_e(N^o_e)$, which is what we wanted to show. \qed

**Example 9.** As in Example 6 suppose we have a triangulated manifold with $\alpha(v, e) = 1$ for all endpoints $v$ of all edges $e$. Then, in Definition 4, the interior of $e$ and of the two facets containing $e$ is identified with the interior of a unit square in $\mathbb{R}^2$, with the square triangulated along a diagonal. Linear functions on this neighborhood of $e$ are equivalent to affine linear functions on the square. Thus, in this case, Lemma 8 amounts to the observation that affine linear functions on $\mathbb{R}^2$ are determined by their restriction to on any open set.
Any constant function on a weak tropical surface is linear, so the sheaf of locally constant \( \mathbb{R} \)-values functions is a subsheaf of \( \mathcal{A} \). If we denote the quotient sheaf \( \mathcal{A}/\mathbb{R} \) by \( \mathcal{D} \), then we have a long exact sequence in sheaf cohomology \cite[Sec. 3]{Car15}:

\[
0 \to H^0(\Delta, \mathbb{R}) \to H^0(\Delta, \mathcal{A}) \to H^0(\Delta, \mathcal{D}) \to H^1(\Delta, \mathbb{R}) \to \cdots
\]

One of the main results from \cite{Car15} is the following:

**Theorem 10** (Thm. 4.7 in \cite{Car15}). If \( \Delta \) is a tropical surface which is locally connected through codimension 1, the \( \mathbb{R} \)-span of the image of the morphism \( H^0(\Delta, \mathcal{D}) \to H^1(\Delta, \mathbb{R}) \) has codimension at most 1 in \( H^1(\Delta, \mathbb{R}) \).

In Theorem 10, locally connected through codimension 1 means that the link of each vertex is connected.

If linear functions on weak tropical surfaces are taken to be analogous to non-vanishing regular functions, then the analogues of rational functions on an algebraic variety come from relaxing the linearity condition in codimension 1. More precisely, suppose that \( d_1, \ldots, d_m \) are closed line segments, each in one facet of a weak tropical surface \( \Delta \), such that \( d_i \cap d_j \) is finite for distinct \( i \) and \( j \). Then we say a function \( \phi \) on an open subset \( U \subset \Delta \) is a piecewise linear function whose divisor is supported in \( d_1 \cup \cdots \cup d_m \) if \( \phi \) is continuous and \( \phi \) is linear on \( U \setminus (d_1 \cup \cdots \cup d_m) \). Although we will not need it in this paper, we can justify our terminology with:

**Proposition 11** (Prop. 4.5 in \cite{Car13}). Let \( d_1, \ldots, d_m \) be line segments in a weak tropical surface \( \Delta \) as above. If \( U \) is an open set meeting all of the \( d_i \), then there exists a homomorphism from the group of piecewise linear functions \( \phi \) whose divisor is supported in \( d_1 \cup \cdots \cup d_m \) under addition to formal sums of the \( d_i \), known as the divisor of \( \phi \). The maximal open subset of \( U \) on which such a function \( \phi \) is linear is the complement of those \( d_i \) with non-zero coefficient in the divisor of \( \phi \).

Finally, we have an analogue of the maximum modulus principle from complex analysis. The result in \cite{Car15} also applies to piecewise linear functions whose divisor has non-negative coefficients, but we’ll only need it for linear functions, i.e. when the divisor is trivial.

**Proposition 12** (Prop. 2.11 in \cite{Car15}). Let \( \Delta \) be a tropical surface which is connected through codimension 1. If \( \phi \) is a linear function on a connected open set \( U \) which achieves its maximum on \( U \), then \( \phi \) is constant.

2. Degenerations

In this section, we construct weak tropical surfaces from strictly semistable degenerations. The case of regular semistable degenerations over discrete valuation rings was treated in \cite[Sec. 2]{Car13}, but here we want to work over possibly non-discrete valuation rings. The data of a weak tropical surface only depends on the special fiber as a simple normal crossing scheme over
the residue field, and not on the valuation ring, so we can use the same construction as [Car13, Sec. 2], which we now recall.

We let \( \mathcal{X} \) be a strictly semistable degeneration over \( R \) and, as stated in the introduction, the dual complex \( \Delta \) has one \( k \)-dimensional simplex for each stratum of dimension \( 2 - k \) in the special fiber of \( \mathcal{X} \). Thus, if \( e \) is an edge with endpoints \( v \) and \( w \), then we let \( C_e \) and \( C_w \) denote the curve and surface corresponding to \( e \) and \( w \) respectively. We set \( \alpha(v, e) = -C_e^2 \), where \( C_e^2 \) denotes the self-intersection number of \( C_e \) in \( C_w \). Even for regular strictly semistable degenerations over discrete valuation rings, this data may only give a weak tropical surface without an additional technical condition of robustness in dimension 2 [Car13, Prop. 2.7]. Over non-discrete valuation rings, we also get weak tropical surfaces

**Proposition 13.** The special fiber of any degeneration \( \mathcal{X} \) yields a weak tropical surface \( \Delta \) such that the local intersection matrix \( M_v \) has at most one positive eigenvalue for each vertex \( v \) of \( \Delta \).

**Proof.** For \( \Delta \) to be a weak tropical surface, we need to check that for any edge \( e \) with endpoints \( v \) and \( w \), we have the equality (4):

\[
\alpha(v, e) + \alpha(w, e) = \deg e.
\]

Let \( C_e \) be the curve corresponding to \( e \) in the special fiber of \( \mathcal{X} \), and by our semistability condition, on a Zariski open neighborhood meeting \( C_e \), there is an étale map to \( \text{Spec} R[x, y, z]/(xy - \pi) \) for some element \( \pi \) in the maximal ideal of \( R \). Then, the principal Cartier divisor defined by \( \pi \) can be written, at least in a formal open neighborhood of \( C_e \), as the union of Cartier divisors, each of which is supported on an irreducible component of the special fiber of \( \mathcal{X} \). For example, in the above chart, the functions \( x \) and \( y \) pull back to give defining equations for each of the components containing \( C_e \).

Thus, using linearity of the intersection product [Gub03, Prop. 5.9(b)], we can split up the intersection of the principal divisor defined by \( \pi \) with the curve \( C_e \) into terms coming from the components of the special fiber of \( \mathcal{X} \). For components of the special fiber which don’t contain \( C_e \), if we pull back to \( C_e \) we get a Cartier divisor equal to the points of intersection, with multiplicities equal to 1. Thus, the degree of the intersection of such a Cartier divisor with \( C_e \) is equal to the number of points of intersection by the projection formula [Gub03, Prop. 5.9(c)]. The total degree for all components which don’t contain \( C_e \) gives \( \deg e \), which is the right-hand side of (4).

Now consider the two components \( C_v \) and \( C_w \) containing \( C_e \). If we pull back the Cartier divisor supported on \( C_v \) to \( C_w \) then we get the divisor \( C_e \) on \( C_w \). The self-intersection of \( C_e \) is \( -\alpha(v, e) \) by the definition of the structure constants. Thus, using the projection formula again, the components containing \( C_e \) contribute a cycle of degree equal to \( -\alpha(v, e) - \alpha(w, e) \), so the desired equality (4) follows because \( \pi \) obviously defines a principal divisor.
Finally, we claim that the local intersection matrix $M_v$ records the intersection theory on the surface of the special fiber corresponding to $v$, restricted to curves of the special fiber. For the diagonal entries of the local intersection matrix (2), this follows immediately from our definition of the structure constants. The off-diagonal entries of the local intersection matrix count facets containing two edges $e$ and $e'$, which are in bijection with the number of reduced points in the intersection of corresponding curves $C_e$ and $C_{e'}$, and thus equal to the intersection number $C_e \cdot C_{e'}$. Therefore, by the Hodge index theorem, the local intersection matrix $M_v$ can have at most one positive eigenvalue.

One approach to obtaining a tropical surface instead of a weak tropical surface is Proposition 2.10 in [Car13], which shows that for degenerations with projective components, robustness can be obtained by appropriate blow-ups. Rather than adapting this proposition to the case of non-discrete valuations, while also keeping track of the effect on the underlying topological space, it is more convenient to perform the modification combinatorially:

**Lemma 14.** Let $\Delta$ be a weak tropical surface and suppose that for each vertex $v$ of $\Delta$, the local intersection matrix $M_v$ has at most one positive eigenvalue. Then, there exists a tropical surface $\Delta'$ such that the underlying topological space of $\Delta'$ is formed by attaching a finite number of 2-simplices to edges of $\Delta$.

**Proof.** We suppose that $v$ is a vertex of $\Delta$ such that $M_v$ has no positive eigenvalues, i.e. it is negative semidefinite. Let $e$ be an edge containing $v$ and let $w$ be the other endpoint of $e$. We attach an additional 2-simplex onto $e$ and label the new vertex $v'$, with the new edges $e_v'$ and $e_w'$. We use $v'$, $w'$ and $e'$ to denote the representatives of $v$, $w$, and $e$ in the new weak tropical surface $\Delta'$. We assign the coefficients on $\Delta'$ to be the same as on $\Delta$, except that:

$$
\begin{align*}
\alpha(w', e') &= \alpha(w, e) \\
\alpha(w', e_w') &= 0 \\
\alpha(v', e_v') &= 2 \\
\alpha(v', e'_{w}) &= 1
\end{align*}
\begin{align*}
\alpha(v', e'_{w}) &= \alpha(v, e) + 1 \\
\alpha(v', e'_{w}) &= 1 \\
\alpha(u', e'_{v}) &= -1
\end{align*}

Then we claim the local intersection matrices $M_{u'}$ and $M_{v'}$ each have exactly one positive eigenvalue and that the number of positive eigenvalues of $M_w$ is the same as that of $M_w$. Once we show the claim, then we can repeat the above construction at each vertex $v$ whose local intersection matrix $M_v$ is negative semidefinite to get the desired tropical surface.

The first part of the claim is that the local intersection matrices

$$
M_{u'} = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} \quad \text{and} \quad M_{v'} = \begin{pmatrix} \star & \cdots & \star & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \star & \cdots & -\alpha(w, e) & 1 \\ 0 & \cdots & 1 & 1 \end{pmatrix}
$$
have exactly one positive eigenvalue each, where * indicates the parts of $M_v'$ that coincide with $M_v$. For $M_u'$, there is exactly one positive eigenvalue because it has determinant $-1$. For $M_v'$, we use the change of coordinates:

$$NM_v'N^T = \begin{pmatrix} * & \cdots & * & 0 \\ \vdots & \ddots & \vdots & \vdots \\ * & \cdots & -\alpha(w,e) - 1 & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

where $N = \begin{pmatrix} 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & -1 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$

to get a block diagonal matrix whose upper left block is $M_v$ minus a negative semidefinite diagonal matrix, and is thus negative semidefinite. Moreover, the lower right block of $NM_v'N^T$ is a single positive entry, so $NM_v'N^T$ has exactly one positive eigenvalue, as in the first part of the claim.

For the second part of our claim, we want to show that $M_w'$ has the same number of positive eigenvalues as $M_w$, where * denotes parts of the matrix which coincide with $M_w$. We again use a change of coordinates to a block diagonal matrix:

$$PM_w'P^T = \begin{pmatrix} * & \cdots & * & 0 \\ \vdots & \ddots & \vdots & \vdots \\ * & \cdots & -\alpha(v,e) - 1 & 1 \\ 0 & \cdots & 1 & -1 \end{pmatrix}$$

where $P = \begin{pmatrix} 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 1 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$.

The upper left block of $PM_w'P^T$ agrees with $M_w$ and the lower right block adds a single negative eigenvalue. Thus, $M_w'$ has the same number of positive eigenvalues as $M_w$, which completes the proof of the claim. □

3. Proof of the main theorems

The crux of Theorem 4 and thus of Theorem 3 is the following lemma:

Lemma 15. Let $\Delta$ be a tropical surface whose underlying $\Delta$-complex is a manifold with fins and is connected through codimension 1. If $s$ is a facet contained in the manifold subcomplex of $\Delta$, and $U_s$ denotes the interior of $s$, then the restriction map

$$H^0(\Delta, D) \to H^0(U_s, D) \cong \mathbb{Z}^2$$

is injective.

Proof. Note that the isomorphism $H^0(U_s, D) \cong \mathbb{Z}^2$ holds because affine linear functions on $U_s$ are equivalent to affine linear functions with integral slopes on a unimodular simplex in $\mathbb{R}^2$ by definition. Thus, $A$ restricted to
We start with $\omega$ with integral slope on this set. As above, $\omega$ would contradict Definition 2. In particular, $f$ will be contained in $\Sigma$. Then, Definition 7 identifies the union of the interiors of $V$ will be contained in $\Sigma$.

Let $\tilde{\omega}$ denote by $\tilde{\omega}$ maximum or minimum, respectively, on $F$. Then, it would have its long exact sequence of cohomology associated to the quotient $D$ isomorphic to the locally constant sheaf with values in $R$.

First suppose that there exists an edge $e$ in the boundary of $V$ such that $e$ is not contained in any of the fins. Let $f$ denote the 2-simplex bordering $e$ whose interior is in $V$ and let $f'$ denote the 2-simplex on the other side of $e$. Then, Definition 7 identifies the union of the interiors of $f$, $f'$, and $e$ with an open subset of $R^2$, and the sections of $A$ are exactly affine linear functions with integral slope on this set. As above, $A$ and $D$ are therefore locally constant sheaves with values in $R \times \mathbb{Z}^2$ and $\mathbb{Z}^2$ respectively. Thus, we can expand $V$ to include the interiors of $e$ and $f'$, where $\omega$ is also zero.

Second, we assume that every edge in the boundary of $V$ is contained in some $\Sigma \cap F_i$. Let $i$ be the maximal index such that $F_i$ intersects the boundary of $V$. Then, the entire path $\Sigma \cap F_i$ must be in the boundary of $V$ or else there would be a fin $F_j$ with $j < i$ intersecting $F_i$ not at its endpoint, which would contradict Definition 2. In particular, $\omega$ must vanish along $\Sigma \cap F_i$.

Let $\tilde{A}_i$ be the sheaf of piecewise linear functions on $\Delta$ whose divisors are supported on $\Sigma \cap F_i$. If we let $\tilde{D}_i$ denote the quotient sheaf $\tilde{A}_i/R$, then we can give a global section $\tilde{\omega}_i$ of $\tilde{D}_i$ defined piecewise such that $\tilde{\omega}_i$ vanishes on $\Delta \setminus F_i$ and it agrees with $\omega$ on $F_i \setminus \Sigma$. This defines a valid section of $\tilde{D}_i$ because, as we noted, $\omega$ vanishes along $\Sigma \cap F_i$, and $\tilde{\omega}_i$ is clearly a section of $D$ away from $\Sigma \cap F_i$, on which we only require continuity. Consider the long exact sequence of cohomology associated to the quotient $\tilde{D}_i$, analogous to (3):

$$0 \to H^0(\Delta, \mathbb{R}) \to H^0(\Delta, \tilde{A}_i) \to H^0(\Delta, \tilde{D}_i) \to H^1(\Delta, \mathbb{R}) \to$$

Since $\tilde{\omega}_i$ is only non-trivial on $F_i$, which is contractible, the image of $\tilde{\omega}_i$ in $H^1(\Delta, \mathbb{R})$ is trivial, so $\tilde{\omega}_i$ lifts to an element of $H^0(\Delta, \tilde{A}_i)$, which we also denote by $\tilde{\omega}_i$ and we choose the representative such that $\tilde{\omega}_i$ is zero on $\Sigma$.

If $\tilde{\omega}_i$ is non-constant, then it must have a maximum value strictly greater than zero or minimum value strictly less than zero. Then, it would have its maximum or minimum, respectively, on $F_i \setminus \Sigma$. We apply Proposition 12 to the linear function $\tilde{\omega}_i|_{F_i \setminus \Sigma}$ or to its negative to show that $\tilde{\omega}_i|_{F_i \setminus \Sigma}$ must be constant, and so $\tilde{\omega}_i$ is zero everywhere. Thus, $\omega$ is identically zero on $F_i$, and so we can expand $V$ to include $F_i \setminus \Sigma$.

We’ve now shown that for each edge $e$ of $\Sigma \cap F_i$, the section $\omega$ is zero on all but one simplex containing $e$, namely the simplex in $\Sigma$ on the other side from $V$. As in Definition 2 we let $N^0_e$ denote the union interior of $e$ with
the interiors of the facets containing $e$. Since $N_e^o$ is simply connected, we can lift $\omega|_{N_e^o}$ to a linear function $\phi$ on $N_e^o$. Then, by Lemma 8, $\phi$ is constant on $N_e^o$, and so $\omega$ vanishes on $N_e^o$. Therefore, we can further expand $V$ to also include the interiors of all 2-simplices meeting $F_i$ and the path $\Sigma \cap F_i$.

At the end, we will have that $\omega$ is zero on an open set $V$ which contains the interior of every 2-simplex in $\Delta$ and since affine linear functions are continuous by definition, this means that $\omega$ is zero, which finishes the proof of the lemma.

We use Lemma 15 to prove the following strengthening of Theorem 4.

**Theorem 16.** If $\Delta$ is a hyperbolic manifold with fins and ornaments, then there is no weak tropical surface, with $\Delta$ as its underlying topological space, and such that for every vertex $v$ of $\Delta$, $M_v$ has at most one positive eigenvalue.

**Proof.** Suppose that $\Delta$ is a weak tropical surface whose underlying $\Delta$-complex is as in the theorem statement. We can assume that when decomposing $\Delta$ as in Definition 2, the subcomplex of ornaments $O$ is maximal, so that if we let $\Delta'$ denote the subcomplex consisting of just the manifold and fins, then $\Delta'$ is locally connected through codimension 1. Then, taking the restriction of the structure constants from $\Delta$, we get that $\Delta'$ has the structure of a weak tropical surface, because the local structure around each edge of $\Delta'$ is unchanged. Moreover, at each vertex $v$ of $\Delta'$, the local intersection matrix $M'_v$ is a block of the block diagonal matrix $M_v$ for $\Delta$. Therefore, $M'_v$ also has at most one positive eigenvalue.

Next, we apply Lemma 14 to transform $\Delta'$ into a tropical surface $\Delta''$ by gluing simplices onto edges of $\Delta'$. Whenever we glue a simplex onto an edge $e$ which is contained in one of the fins $F_i \subset \Delta'$, we can include that simplex in the fin, which remains contractible and its intersection with the manifold $\Sigma$ is unchanged. If we glue a simplex onto an edge $e$ contained in the manifold $\Sigma$, then the simplex forms a new fin $F_{n+1}$, numbered after all the other fins. Since $F_{n+1} \cap \Sigma$ is a single edge, the intersection of $F_{n+1}$ with any other fin will be a subset of the endpoints of this edge. Thus, $\Delta''$ is still a hyperbolic manifold with fins.

Finally, suppose that the manifold $\Sigma \subset \Delta''$ is not orientable. Then $\Sigma$ has a 2-to-1 orientable cover, corresponding to an index 2 subgroup of its fundamental group [Hat02, Prop. 3.25]. Since $\Sigma \subset \Delta''$ is a homotopy equivalence, the oriented cover extends to a cover of $\Delta''$, which we call $\tilde{\Delta}''$. Each fin $F_i$ of $\Delta''$ is attached along a path of $\Sigma$, and so the preimage of $F_i$ in $\tilde{\Delta}''$ is two disjoint fins, which we number $F_{2i-1}$ and $F_{2i}$ to get a manifold with fins. Moreover, the Euler characteristic is multiplicative when taking covers, so $\chi(\tilde{\Delta}'')$ is again negative. Therefore, we can replace $\Delta''$ with $\tilde{\Delta}''$ and so from now on we assume that the manifold $\Sigma \subset \Delta''$ is orientable.

By the classification of compact topological surfaces [GX13, Thm. 6.3], the Euler characteristic of a compact oriented surface is even, and so $\chi(\Delta'') \leq$
Therefore, $\dim_{\mathbb{R}} H^1(\Delta'', \mathbb{R}) = 2 - \chi(\Delta'') \geq 4$. By Theorem 10, the $\mathbb{R}$-span of the image of $H^0(\Delta'', D)$ has codimension at most 1 in $H^0(\Delta'', \mathbb{R})$, so the rank of $H^0(\Delta'', D)$ as an Abelian group is at least $\dim H^1(\Delta'', \mathbb{R}) - 1 \geq 3$. On the other hand, by Lemma 15, $H^0(\Delta'', D)$ is a subgroup of $\mathbb{Z}^2$, and so a free Abelian group of rank at most 2. Therefore, we have a contradiction, so the weak tropical surface $\Delta$ cannot exist. □

Proof of Theorem 3. Suppose $X$ is a strict semistable degeneration whose dual complex $\Delta$ is a hyperbolic manifold with fins and ornaments. Then, by Proposition 13, $\Delta$ has the structure of a weak tropical complex such that the matrix $M_v$ has at most one positive eigenvalue for every vertex $v$. However, by Theorem 16, such a weak tropical complex cannot exist, so we conclude that $X$ cannot exist. □

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