Current Algebra in the Path Integral framework

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Abstract

In this letter we describe an approach to the current algebra based in the Path Integral formalism. We use this method for abelian and non-abelian quantum field theories in 1+1 and 2+1 dimensions and the correct expressions are obtained. Our results show the independence of the regularization of the current algebras.

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Abelian bosonization and current algebras play an important role in the description of two-dimensional quantum field theories as non-perturbative methods and they are an important ingredient in order to show the equivalence between different (two-dimensional) models [1–3] (for a complete review of the most important references in the field see [4]).

The non-abelian extension of the bosonization is, however, a more technical problem that was solved in [5]. Essentially, the solution given by Witten for \( N \) free fermionic fields was to show the equivalence with a Wess-Zumino-Witten [6] theory with the current algebra describing a \( SU(N) \) Kac-Moody algebra.

In the abelian or non-abelian bosonization, the current commutators are normally computed using a point splitting regularization plus the Bjorken-Johnson-Low (BJL) limit, in order to have an equal-time commutator. Although it seems a technical point, the computation of the current-current commutator using different regularizations could shed some light on the independence of the regularization of the current algebra [8].

The purpose of this paper is to present an explicit calculation of the current algebra in two and three dimensions based in the path integral approach. This procedure allows translating the definition of the product of two operators at the same point, to a regularization of a functional determinant where many other regularizations are available.

In order to compute the current algebra, let us start considering a massless fermion in 1+1 dimensions coupled to a gauge field \( A_\mu \)

\[
L = \bar{\psi} i \not{D} \psi, \tag{1}
\]

where \( D_\mu = \partial_\mu + A_\mu \).

The gauge field \( A_\mu \) is an external auxiliary field that can be set equal to zero at the end of the calculation.

Thus, from the euclidean partition function

\[
Z [A] = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \left[ - \int d^2 x \bar{\psi} i \not{D} \psi \right], \tag{2}
\]

one can compute the current-current correlator by means

\[
\langle J^\sigma(z) J^\rho(w) \rangle = \frac{\delta^2 Z}{\delta A_\sigma(z) \delta A_\rho(w)} \bigg|_{A=0}. \tag{3}
\]

However, as \( Z = \det(i\not{D}) \), the calculation of (3) requires a regularization prescription for the fermionic determinant. This is the main advantage of our method, because one could use different regularizations for the determinant and, as a consequence, the commutator

\[
\langle [J^\sigma(z), J^\rho(w)] \rangle \equiv \left( \frac{\delta^2 Z}{\delta A_\sigma(z) \delta A_\rho(w)} - \frac{\delta^2 Z}{\delta A_\rho(w) \delta A_\sigma(z)} \right) \bigg|_{A=0}, \tag{4}
\]

showing the universability of the current algebra.

There is, however, a delicate point; the commutator must be computed at equal times implying one additional regularization, namely the BJL limit [9,10]. Below we show explicitly these calculations for several examples.

In order to prove our previous assertions, let us start considering the 1+1 abelian case where the fermionic determinant is exact.

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In the 1+1 dimensional case considered above, the fermionic determinant becomes

\[ Z[A] = \det(iD) = \exp \left[ -\frac{1}{4\pi} \int d^2x F_{\mu\nu} \Box^{-1} F^{\mu\nu} \right], \]  \hspace{1cm} (5)

where the determinant have been regularized using the \( \zeta \)-function method.

From (5) it is easy to obtain

\[ \frac{\delta^2 Z}{\delta A_{\sigma}(z) \delta A_{\rho}(w)} = -\frac{Z[A]}{\pi} \int \frac{d^2p}{(2\pi)^2} \left\{ \delta^{\rho\sigma} - \frac{(p^{\rho} \cdot p^{\sigma})}{p^2} \right\} \exp(-ip(w - z)), \]  \hspace{1cm} (6)

where the terms proportional to \( A \) have been omitted. By interchanging \( (\rho, \sigma) \) with \( (w, z) \), one finds the covariant commutator

\[ \langle [J_{\rho}(z), J_{\sigma}(w)] \rangle_{A=0} = \frac{2i}{\pi} \int \frac{d^2p}{(2\pi)^2} \left\{ \delta^{\rho\sigma} - \frac{(p^{\rho} \cdot p^{\sigma})}{p^2} \right\} \sin(p(w - z)). \]  \hspace{1cm} (7)

In order to satisfy the microcausality principle one must, additionally, impose the limit \( t \to t' \). For the commutator \( \langle [A(x), B(y)] \rangle \) this limit means

\[ \langle [A(x), B(y)] \rangle = \lim_{\tau \to +0} \left( \langle A(\tau, x) B(0, y) \rangle - \langle A(-\tau, x) B(0, y) \rangle \right). \]  \hspace{1cm} (8)

Using this fact and equation (8), we find

\[ \langle [J_0(z), J_0(w)] \rangle = \langle [J_1(z), J_1(w)] \rangle = 0, \]  \hspace{1cm} (9)

\[ \langle [J_0(z), J_1(w)] \rangle = \frac{-2i}{\pi} \int \frac{d^2p}{(2\pi)^2} \frac{p_0 p_1}{p^2} \lim_{\tau \to +0} \{ \sin(p_0 \tau) \} \exp(-ip_1(z - w)), \]  \hspace{1cm} (10)

where \( \tau = t - t' \).

The integrals in (10), can be evaluated using the well known trick \( a^{-1} = \int_0^\infty dx \exp(-ax) \) and after integrating in \( p_0 \) we obtain

\[ \langle [J_0(z), J_1(w)] \rangle = \frac{-i}{\pi} \int \frac{dp_1}{(2\pi)^2} p_1 \exp(-ip_1(z - w)) \sqrt{\pi} \lim_{\tau \to +0} \tau (4p_1^2/\tau^2)^{1/4} K_{1/2}(\sqrt{\tau^2p_1^2}). \]  \hspace{1cm} (11)

The limit \( \tau \to +0 \) is computed using

\[ K_{1/2}(z) \to \frac{1}{2} \frac{\sqrt{2\pi}}{z} \quad \text{when} \quad z << 1, \]

and the commutator (11) becomes

\[ \langle [J_0(z), J_1(w)] \rangle = \frac{-i}{2\pi} \int \frac{dp_1}{(2\pi)^2} p_1 \exp(-ip_1(z - w)) \]  \hspace{1cm} (12)

\[ = \frac{1}{2\pi} \delta'(z - w), \]

that is the expected Schwinger term [11].
One can extend the above result for the 2+1 abelian case where the fermionic determinant in powers of $1/m$ is also available. Indeed, using a Pauli-Villars regulator the partition function becomes

$$Z[A] = \exp \left[ -\frac{m}{16\pi|m|} \int d^3x e^{\mu\alpha\beta} A_\mu F_{\alpha\beta} + O\left(\frac{1}{m}\right) \right].$$ (13)

Using again equation (3), one finds

$$\langle J^\rho(z) J^\sigma(w) \rangle = -\frac{1}{4\pi|m|} e^{\rho\alpha\mu} \partial_\mu \delta^{(3)}(z-w),$$ (14)

and the current-current commutator reads

$$\langle [J^\rho(z), J^\sigma(w)]_0 \rangle = -\frac{1}{2\pi|m|} \epsilon^{\mu\rho\sigma} \partial_\mu \delta^{(3)}(z-w).$$ (15)

Using the identity

$$\delta^{(3)}(z-w) = \int \frac{d^3p}{(2\pi)^3} \exp(-ip(z-w)),
$$

the commutator (15) after taking the BJL limit becomes

$$\langle [J^\rho(z), J^\sigma(w)] \rangle \big|_{t=t'} = -\frac{1}{4\pi|m|} e^{\rho\alpha\mu} \partial_\mu \delta^{(3)}(z-w).$$

To solve these integrals a converging factor $\exp(-\alpha p_0)$ is added to the integrand, and we take the limit $\alpha \to 0$ at the end. We explicitly find

$$\langle [J^0(z), J^1(w)] \rangle = -\frac{1}{4\pi|m|} \partial_2 \delta^{(2)}(z-w) \lim_{\tau \to +0} \frac{1}{\tau},$$

$$\langle [J^0(z), J^2(w)] \rangle = -\frac{1}{4\pi|m|} \partial_1 \delta^{(2)}(z-w) \lim_{\tau \to +0} \frac{1}{\tau},$$

$$\langle [J^1(z), J^2(w)] \rangle = -\frac{1}{6\pi|m|} \delta^{(2)}(z-w) \lim_{\tau \to +0} \frac{1}{\tau^2},$$ (16)

where, in the last expression, $\delta(\tau)$ was written using the representation $\delta(\tau) = \pi^{-1} \lim_{\alpha \to 0} \alpha/(\tau^2 + \alpha^2)$.

The current algebra (16) could be used as starting point in applications to condensed matter physics as was done in [12]. However, (16) depends on the regulator and we do not find agreement with [12]. The same algebra (16) was derived also in [13] using different methods.

The extensions to the non-abelian case is performed as follows; Firstly, let us consider the 1+1 dimensional case. One can proceed along the same lines because the fermionic
Following the same lines described above, we find the Kac-Moody algebra given by

\[ S_{WZW} = \frac{1}{2} \int d^2 x \text{tr}(\partial_\mu g^{-1} \partial^\mu g) + \frac{i}{8\pi} \int d^3 \xi e^{abc} \text{tr}(g^{-1} \partial_\sigma gg^{-1} \partial_\phi gg^{-1} \partial_\sigma g), \tag{17} \]

where \( g \) is the \( SU(N) \) matrix. The auxiliary field \( A_\mu = A_i^\mu \sigma_i \) written in light cone coordinates is represented in terms of \( g \) as \( A_+ = g^{-1} \partial_+ g \), and in the light cone gauge \( A_- = 0 \). Thus, the partition function is

\[ Z[g] = \exp[-S_{WZW}], \tag{18} \]

where locally, (17) was written as

\[ S_{WZW} = \int_0^1 d\alpha (1 - \alpha) \int d^2 x \text{tr}(\partial_- \phi e^{- \alpha \phi} \partial_+ \phi e^{\alpha \phi}), \]

with \( g = e^{i\phi} \) with \( \phi \) an antihermitian matrix [7].

Then we can write \( \partial_+ \phi = -iA_+ \) and \( \partial_- \phi = -i \int dy^+ \partial_- A_+(y^+, x^-) \) and using (3) we obtain

\[ \frac{\delta^2 Z[A]}{\delta A_+^a(z) \delta A_+^b(w)} \bigg|_{A=0} = -\frac{1}{2} \int dx_+ dx_- \text{tr} \left( \frac{\delta(\partial_- \phi)}{\delta A_+^a(z)} \frac{\delta(\partial_+ \phi)}{\delta A_+^b(w)} \right), \]

\[ = -\frac{1}{2} \delta_{ab} \delta'(w_- - z_-), \]

where we imposed \( \phi = 0 \), so that \( g = 1 \) and \( A_+ = 0 \) (here the normalization \( tr(\sigma_a \sigma_b) = 2\delta_{ab} \) has been used).

In analogy with the previous case we obtain

\[ \langle [J_a(x_-), J_b(y_-)] \rangle = i f_{abc} J_c(x_-) \delta(x_- - y_-) + \delta_{ab} \delta'(x_- - y_-), \tag{19} \]

where (19) is the \( SU(N) \) Kac-Moody algebra of level \( k = 1 \). We should note that the use of light cone coordinates and the light cone gauge simultaneously allowed us to obtain expression (19) at equal times without taking the BJL limit that was needed before (see [3]).

The non-abelian extension to 2+1 dimensions, can be performed as follows. Start from the well-known expression for the partition function

\[ Z[A] = \exp \left[ -\frac{1}{2} m \int \frac{m}{|m|} \int d^3 x \left( A_\mu^* F_\mu - \frac{i}{3} \varepsilon^{\mu \alpha \beta} A_\mu A_\alpha A_\beta \right) + O \left( \frac{1}{m} \right) \right]. \tag{20} \]

Since we must take \( A_\mu = 0 \) at the end of the calculation, the second term in the argument of (20) does not contribute. Then, we work with an action similar to (13) but written in terms of the non-abelian field \( A_\mu \) and the stress tensor \( F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \).

Following the same lines described above, we find the Kac-Moody algebra

\[ [J_a^\rho(x), J_b^\sigma(y)] = i f_{abc} J_c^\tau(x) \gamma^0 \gamma^\rho \delta(x - y) + \frac{\delta_{ab}}{4\pi |m|} \varepsilon^{\rho \mu \alpha} \partial_\mu \delta^{(3)}(x - y). \tag{21} \]
The BJL limit is taken in analogy with (14).

In conclusion we have described an approach based in the path integral formalism, where the current algebra can be computed for quantum field theories in 1+1 and 2+1 dimensions. This approach could be considered as an alternative method which permits obtaining the current algebra in different regularizations.

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