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Abstract
We introduce a general framework for Markov decision problems under model uncertainty in a discrete-time infinite horizon setting. By providing a dynamic programming principle, we obtain a local-to-global paradigm, namely solving a local, that is, a one time-step robust optimization problem leads to an optimizer of the global (i.e., infinite time-steps) robust stochastic optimal control problem, as well as to a corresponding worst-case measure. Moreover, we apply this framework to portfolio optimization involving data of the S&P 500. We present two different types of ambiguity sets; one is fully data-driven given by a Wasserstein-ball around the empirical measure, the second one is described by a parametric set of multivariate normal distributions, where the corresponding uncertainty sets of the parameters are estimated from the data. It turns out that in scenarios where the market is volatile or bearish, the optimal portfolio strategies from the corresponding robust optimization problem outperform the ones without model uncertainty, showcasing the importance of taking model uncertainty into account.

Keywords
ambiguity, dynamic programming principle, Markov decision problem, portfolio optimization
1 INTRODUCTION

Suppose that today and at all future times, an agent observes the state of the surrounding world, and based on the realization of this state she decides to execute an action that may also influence future states. All actions are rewarded according to a reward function not immediately but once the subsequent state is realized. The Markov decision problem consists of finding at initial time a policy, that is, a sequence of state-dependent actions, that optimizes the expected cumulated discounted future rewards, referred to as the value of the Markov decision problem. The underlying process of states in a Markov decision problem is a stochastic process \((X_t)_{t \in \mathbb{N}_0}\) and is called Markov decision process. This process is usually modeled by a discrete-time time-homogeneous Markov process that follows a prespecified probability, which is influenced by the current state of the process and the agent’s current action. The Markov decision problem leads to an infinite horizon stochastic optimal control problem in discrete-time, which finds many applications in finance and economics, compare, for example, Bäuerle and Rieder (2011), Hambly et al. (2021), or White (1993) for an overview. It can, among a multitude of other applications, be used to learn the optimal structure of portfolios and the optimal trading behavior, see, for example, Bertoluzzo and Corazza (2012), Chang and Lee (2017), Gold (2003), Hu and Lin (2019), Xiong et al. (2018), to learn optimal hedging strategies, see, for example, Angiuli et al. (2022), Angiuli et al. (2021), Cao et al. (2021), Dixon et al. (2020), Du et al. (2020), Halperin (2020), Li et al. (2009), Schäl (2002), to optimize inventory-production systems (Uğurlu, 2017), or to study socio-economic systems under the influence of climate change as in Shuvo et al. (2020).

In most applications, the choice of the distribution, or more specifically the probability kernel, of the Markov decision process, however, is a priori unclear and hence ambiguous. For this reason, in practice, the distributions of the process often need to be estimated, compare for example, Aguirregabiria and Mira (2002), Rust (1994), and Srisuma and Linton (2012). To account for distributional ambiguity, we are, therefore, interested to study an optimization problem respecting uncertainty with respect to the choice of the underlying distribution of \((X_t)_{t \in \mathbb{N}_0}\) by identifying a policy that maximizes the expected future cumulated rewards under the worst case probability measure from an ambiguity set of admissible probability measures. This formulation allows the agent to act optimally even if adverse scenarios are realized, such as, for example, during financial crises or extremely volatile market periods in financial markets.

The recent works (Bäuerle & Glauner, 2021; Chen et al., 2019; Uğurlu, 2018; Xu and Mannor, 2012) also consider infinite horizon robust stochastic optimal control problems and follow a similar paradigm but use different underlying frameworks. More precisely, Chen et al. (2019) and Xu and Mannor (2012) assume a finite action and state space. The approach from Uğurlu (2018) assumes an atomless probability space and is restricted to so called conditional risk mappings, whereas Bäuerle and Glauner (2021) assume the ambiguity set of probability measures to be dominated. To the best of our knowledge, the generality of the approach presented in this paper has not been established so far in the literature.

Our general formulation enables to specify a wide range of different ambiguity sets of probability measures and associated transition kernels, given some mild technical assumptions are fulfilled. More specifically, we require the correspondence that maps a state-action pair to the set of transition probabilities to be nonempty, continuous, compact-valued, and to fulfill a linear growth condition; see Assumption 2.2. As we will show, these requirements are naturally satisfied. This is, for example, the case if the ambiguity set is modeled by a Wasserstein-ball around
a transition kernel or if parameter uncertainty with respect to multivariate normal distributions is considered.

To solve the robust optimization problem, we establish a dynamic programming principle that involves only a one time-step optimization problem. Via Berge's maximum theorem (see Berge (1959)) we obtain the existence of both an optimal action and a worst case transition kernel of this local one time-step problem. It turns out that the optimal action that solves this one time-step optimization problem determines also the global optimal policy of the infinite time horizon robust stochastic optimal control problem by repeatedly executing this local solution. Similarly, the global worst case measure can be determined as a product measure given by the infinite product of the worst case transition kernel of the local one time-step optimization problem. We refer to Theorem 2.7 for our main result. This local-to-global principle is in line with similar results for nonrobust Markov decision problems, compare, for example, Bäuerle and Rieder (2011, Theorem 7.1.7), where the optimal global policy can also be determined locally. Note that the local-to-global paradigm obtained in Theorem 2.7 is noteworthy, since \((X_t)_{t \in \mathbb{N}_0}\) does not need to be a time-homogeneous Markov process under each measure from the ambiguity set, as the corresponding transition kernel might vary with time. However, due to the particular setting that the set of transition probabilities is constant in time and only depends on the current state and action, and not on the whole past trajectory, we are able to derive the analog local-to-global paradigm for Markov decision processes under model uncertainty as for the ones without model uncertainty.

Eventually, we show how the discussed robust stochastic optimal control framework can be applied to portfolio optimization with real data, which was already studied extensively in the nonrobust case, for example, in Bäuerle and Rieder (2009), Moody et al. (1998), Yu et al. (2019), and Zhang et al. (2020), and in the robust case using a mean–variance approach in Blanchet et al. (2021) and Pham et al. (2022). To that end, we show how, based on a time series of realized returns of multiple assets of the S&P 500, a data-driven ambiguity set of probability measures can be derived in two cases. The first case is an entirely data-driven approach where ambiguity is described by a Wasserstein-ball around the empirical measure. In the second case, a multivariate normal distribution of the considered returns is assumed while the set of parameters for the multivariate normal distribution is estimated from observed data. Hence, this approach can be considered as semi data-driven approach. We then train neural networks to solve the (semi) data-driven robust optimization problem based on the local-to-global paradigm obtained in Theorem 2.7 and compare the trading performance of the two approaches with nonrobust approaches. It turns out that under adverse market scenarios both robust approaches outperform comparable nonrobust approaches. These results emphasize the importance of taking into account model uncertainty when making decisions that rely on financial assets.

The remainder of the paper is as follows. In Section 2.1, we present the setting and formulate the underlying distributionally robust stochastic optimal control problem. We also present our main results that include a dynamic programming principle. In Section 3, we discuss different possibilities to define ambiguity sets of probability measures and we show that these specifications meet the requirements of our setting. In Section 4, we provide a numerical routine using neural networks to approximately solve the distributionally robust stochastic optimal control problem of Section 2.3. We apply this numerical method to portfolio optimization using real financial data and compare the different ambiguity sets introduced in Section 3 also with nonrobust approaches. The proof of the main results is reported in Section 5, while the proofs of the results from Section 3 and 4 can be found in Section 6 and 7, respectively. Finally, the appendix contains several useful auxiliary known mathematical results.
2 | SETTING, PROBLEM FORMULATION, AND MAIN RESULT

We first present the underlying setting for the considered stochastic process and then formulate an associated distributionally robust optimization problem.

2.1 | Setting

We consider a closed subset $\Omega_{\text{loc}} \subseteq \mathbb{R}^d$, equipped with its Borel $\sigma$-field $\mathcal{F}_{\text{loc}}$, which we use to define the infinite Cartesian product

$$\Omega := \Omega_{\text{loc}}^{\mathbb{N}_0} = \Omega_{\text{loc}} \times \Omega_{\text{loc}} \times \cdots$$

and the $\sigma$-field $\mathcal{F} := \mathcal{F}_{\text{loc}} \otimes \mathcal{F}_{\text{loc}} \otimes \cdots$. We denote by $\mathcal{M}_1(\Omega)$ the set of probability measures on $(\Omega, \mathcal{F})$, by $d \in \mathbb{N}$ the dimension of the state space, and by $m \in \mathbb{N}$ the dimension of the control space.

On this space, we consider an infinite horizon time-discrete stochastic process. To this end, we define on $\Omega$ the stochastic process $(X_t)_{t \in \mathbb{N}_0}$ by the canonical process $X_t((\omega_0, \omega_1, \ldots, \omega_t, \ldots)) := \omega_t$ for $(\omega_0, \omega_1, \ldots, \omega_t, \ldots) \in \Omega, t \in \mathbb{N}_0$. We fix a compact set $A \subseteq \mathbb{R}^m$ and define the set of controls (also called actions) through

$$\mathcal{A} := \{ a = (a_t)_{t \in \mathbb{N}_0} \mid (a_t)_{t \in \mathbb{N}_0} : \Omega \to A; a_t \text{ is } \sigma(X_t)\text{-measurable for all } t \in \mathbb{N}_0 \}$$

$$= \{ (a_t(X_t))_{t \in \mathbb{N}_0} \mid a_t : \Omega_{\text{loc}} \to A \text{ Borel measurable for all } t \in \mathbb{N}_0 \}.$$  

For every $k \in \mathbb{N}, X \subseteq \mathbb{R}^k$, and $p \in \mathbb{N}_0$, we define the set of continuous functions $g : X \to \mathbb{R}$ with polynomial growth at most of degree $p$ via

$$C_p(X, \mathbb{R}) := \left\{ g \in C(X, \mathbb{R}) \mid \sup_{x \in X} \frac{|g(x)|}{1 + \|x\|^p} < \infty \right\},$$

where $C(X, \mathbb{R})$ denotes the set of continuous functions mapping from $X$ to $\mathbb{R}$ and $\| \cdot \|$ denotes the Euclidean norm on $\mathbb{R}^k$. We define on $C_p(\Omega_{\text{loc}}, \mathbb{R})$ the norm

$$\|g\|_{C_p} := \sup_{x \in \Omega_{\text{loc}}} \frac{|g(x)|}{1 + \|x\|^p}$$

Moreover, recall the Wasserstein $p$-topology $\tau_p$ on $\mathcal{M}_1(\Omega_{\text{loc}})$ induced by the following convergence: for any $\mu \in \mathcal{M}_1(\Omega_{\text{loc}})$ and $(\mu_n)_{n \in \mathbb{N}} \subseteq \mathcal{M}_1(\Omega_{\text{loc}})$ we have

$$\mu_n \overset{\tau_p}{\longrightarrow} \mu \text{ for } n \to \infty \Leftrightarrow \lim_{n \to \infty} \int g d\mu_n = \int g d\mu \text{ for all } g \in C_p(\Omega_{\text{loc}}, \mathbb{R}).$$

Note that for $p = 0$, the topology $\tau_0$ coincides with the topology of weak convergence. To be able to formulate a robust optimization problem, we make use of the theory of set-valued maps, also called correspondences, see also Aliprantis and Border (2006, Chapter 17) for an extensive introduction to the topic. In the following, we clarify how continuity is defined for correspondences,
compare also Lemmas A.4 and A.5, where characterizations of upper hemicontinuity and lower hemicontinuity are provided.

**Definition 2.1.** Let \( \varphi : X \rightarrow Y \) be a correspondence between two topological spaces.

(i) \( \varphi \) is called **upper hemicontinuous**, if \( \{ x \in X \mid \varphi(x) \subseteq A \} \) is open for all open sets \( A \subseteq Y \).

(ii) \( \varphi \) is called **lower hemicontinuous**, if \( \{ x \in X \mid \varphi(x) \cap A \neq \emptyset \} \) is open for all open sets \( A \subseteq Y \).

(iii) We say \( \varphi \) is continuous, if \( \varphi \) is upper and lower hemicontinuous.

Moreover, for a correspondence \( \varphi : X \rightarrow Y \), its graph is defined as

\[
\text{Gr} \varphi := \{(x, y) \in X \times Y \mid y \in \varphi(x)\}
\]

We impose the following standing assumptions\(^1\) on the process \( (X_t)_{t \in \mathbb{N}_0} \) and on the set of admissible measures, which are from now on assumed to be valid for the rest of the paper.

**Standing Assumption 2.2** (Assumptions on the set of measures). Fix \( p \in \{0, 1\} \).

(i) The set-valued map

\[
\Omega_{\text{loc}} \times A \rightarrow (\mathcal{M}_1(\Omega_{\text{loc}}), \tau_p)
\]

\[(x, a) \mapsto \mathcal{P}(x, a)\]

is assumed to be nonempty, compact-valued, and continuous.

(ii) There exists \( C_p \geq 1 \) such that for all \( (x, a) \in \Omega_{\text{loc}} \times A \) and \( \mathcal{P} \in \mathcal{P}(x, a) \) it holds

\[
\int_{\Omega_{\text{loc}}} (1 + \|y\|^p) \mathcal{P}(dy) \leq C_p (1 + \|x\|^p).
\]

Under these assumptions, we define for every \( x \in \Omega_{\text{loc}}, a \in A \) the set of admissible measures

\[
\mathfrak{P}_{x, a} := \left\{ \delta_x \otimes \mathcal{P}_0 \otimes \mathcal{P}_1 \otimes \cdots \mid \right. 
\text{for all } t \in \mathbb{N}_0 : \mathcal{P}_t : \Omega_{\text{loc}} \rightarrow \mathcal{M}_1(\Omega_{\text{loc}}) \text{ Borel-measurable, and } \mathcal{P}_t(\omega_t) \in \mathcal{P}(\omega_t, a_t(\omega_t)) \text{ for all } \omega_t \in \Omega_{\text{loc}} \left. \right\},
\]

where the notation \( \mathcal{P} = \delta_x \otimes \mathcal{P}_0 \otimes \mathcal{P}_1 \otimes \cdots \in \mathfrak{P}_{x, a} \) abbreviates\(^2\)

\[
\mathcal{P}(B) := \int_{\Omega_{\text{loc}}} \cdots \int_{\Omega_{\text{loc}}} 1_B((\omega_t)_{t \in \mathbb{N}_0}) \cdots \mathcal{P}_{t-1}(\omega_{t-1}; d\omega_t) \cdots \mathcal{P}_0(\omega_0; d\omega_1) \delta_x(d\omega_0), \quad B \in \mathcal{F}.
\]

---

\(^1\) In this paper, for any \( n \in \mathbb{N} \) and topological spaces \( X_1, \ldots, X_n \), we always endow \( X = X_1 \times \cdots \times X_n \) with the corresponding product topology.

\(^2\) We denote by \( \delta_x \) the Dirac measure centered on \( x \in \mathbb{R}^d \), that is, for any Borel set \( A \subseteq \mathbb{R}^d \) we have \( \delta_x(A) = 1 \) if \( x \in A \) and 0 else.
Remark 2.3. To ensure that the set $\mathcal{P}_{x,a}$ is nonempty, one needs to show that $\mathcal{P}$ admits a measurable selector. By Assumption 2.2, the correspondence $\Omega_{\text{loc}} \times A \ni (x, a) \mapsto \mathcal{P}(x, a)$ is closed-valued and measurable. Hence, by Kuratowski’s theorem (compare, e.g., Kuratowski (1930) and (Aliprantis and Border, 2006, Theorem 18.13)), there exists a measurable selector $\Omega_{\text{loc}} \times A \ni (x, a) \mapsto \mathcal{P}(x, a) \in \mathcal{M}_1(\Omega_{\text{loc}})$ such that $\mathcal{P}(x, a) \in \mathcal{P}(x, a)$ for all $(x, a) \in \Omega_{\text{loc}} \times A$. Since actions are by definition measurable, we also obtain that for all $(a_t)_{t \in \mathbb{N}_0} \in A$ and for all $t \in \mathbb{N}_0$ the map $\Omega_{\text{loc}} \ni \omega_t \mapsto \mathcal{P}(\omega_t, a_t(\omega_t)) =: \mathcal{P}_t(\omega_t, d\omega_{t+1})$ is measurable, as required. Then, the nonemptiness of $\mathcal{P}_{x,a}$ follows by the Ionescu–Tulcea theorem (compare, e.g., Klenke (2013, Theorem 14.32) and Ionescu Tulcea (1949)).

2.2 Problem formulation

Let $r : \Omega_{\text{loc}} \times A \times \Omega_{\text{loc}} \to \mathbb{R}$ be some reward function. We assume from now on that it fulfills the following assumptions.

Standing Assumption 2.4 (Assumptions on the reward function and the discount factor). Let $p \in \{0,1\}$ be the number fixed in Assumption 2.2.

(i) The map

$$\Omega_{\text{loc}} \times A \times \Omega_{\text{loc}} \ni (x_0, a, x_1) \mapsto r(x_0, a, x_1)$$

is continuous

(ii) There exist some $L > 0$ and moduli of continuity$^3$ $\rho_0 : [0, \infty) \to [0, \infty]$ and $\rho_A : [0, \infty) \to [0, \infty]$ such that for all $x_0, x_0', x_1 \in \Omega_{\text{loc}}$ and $a, a' \in A$ we have

$$|r(x_0, a, x_1) - r(x_0', a', x_1)| \leq L \cdot (1 + \|x_1\|^p) \cdot (\rho_0(\|x_0 - x_0'\|) + \rho_A(\|a - a'\|)).$$

(iii) There exist some $C_r \geq 1$ such that for all $x_0, x_1 \in \Omega_{\text{loc}}$ we have

$$|r(x_0, a, x_1)| \leq C_r(1 + \|x_0\|^p + \|x_1\|^p)$$

for all $a \in A$.

(iv) We fix an associated discount factor $\alpha < 1$ which satisfies

$$0 < \alpha < \frac{1}{C_P},$$

where $C_P \geq 1$ is the constant defined in Assumption 2.2 (ii).

Remark 2.5 (Discussion of the assumptions).

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$^3$ A modulus of continuity is a function $\rho : [0, \infty) \to [0, \infty]$ satisfying $\lim_{x \to 0} \rho(x) = 0 = \rho(0)$. 
(i) Note that if $r$ is Lipschitz-continuous, that is, if there exists some $L > 0$ such that for all $x_0, x'_0, x_1, x'_1 \in \Omega_{\text{loc}}$ and $a, a' \in A$ we have
\[
|r(x_0, a, x_1) - r(x'_0, a', x'_1)| \leq L (\|x_0 - x'_0\| + \|a - a'\| + \|x_1 - x'_1\|),
\]
then Equation (4) follows directly. Therefore, the requirement of Assumption 2.4 (i) and (ii) is weaker than assuming Lipschitz continuity of the reward function. In particular, if $m = d$ holds for the dimensions, then the function of the form
\[
\Omega_{\text{loc}} \times A \times \Omega_{\text{loc}} \ni (x_0, a, x_1) \mapsto r(x_0, a, x_1) := a \cdot x_1 - \lambda \cdot a^T \cdot M \cdot a
\]
for some $\lambda \geq 0$ and some $M \in \mathbb{R}^{m \times m}$ fulfills the requirement imposed in Equation (4) but is not Lipschitz continuous, unless $\Omega_{\text{loc}}$ is bounded. Compare also Section 4, where we apply portfolio optimization while taking into account a reward function of the form (6).

(ii) Note that Assumptions 2.2 (ii) and 2.4 (iii), (iv) are standard assumptions for contracting Markov decision processes (compare, e.g., Bäuerle and Rieder (2011, Definition 7.1.2 (ii)) and Bäuerle and Rieder (2011, Corollary 7.2.2)). The continuity properties required in Assumptions 2.2 (i), 2.4 (i), and 2.4 (ii) are assumptions tailored for robust Markov decision processes to ensure that the operator $\mathcal{T}$ defined in Equation (8) is a contraction even if the image of $\mathcal{P}$ is not a singleton and nondominated, compare also the proof of Theorem 2.7 (ii).

Our main problem consists, for every initial value $x \in \Omega_{\text{loc}}$, in maximizing the expected value of $\sum_{t=0}^{\infty} \alpha^t r(X_t, a_t, X_{t+1})$ under the worst case measure from $\mathcal{P}_{x,a}$ over all possible actions $a \in A$. More precisely, we introduce the value function
\[
\Omega_{\text{loc}} \ni x \mapsto V(x) := \sup_{a \in A} \inf_{\mathcal{P} \in \mathcal{P}_{x,a}} \left( \mathbb{E}_{\mathcal{P}} \left[ \sum_{t=0}^{\infty} \alpha^t r(X_t, a_t, X_{t+1}) \right] \right).
\]

**Definition 2.6.** We call $(X_t)_{t \in \mathbb{N}_0}$ a Markov decision process under model uncertainty on state space $\Omega_{\text{loc}} \subseteq \mathbb{R}^d$ with corresponding set of transition probabilities $\mathcal{P}$, and we call the problem defined in Equation (7) a Markov decision problem under model uncertainty.

### 2.3 Main result: The dynamic programming principle

In this section, we provide the main results of the paper, which comprise a dynamic programming principle which in particular allows to solve the optimization problem (7) by solving a related one-step fixed point equation.

To this end, we define the space of one-step actions
\[
\mathcal{A}_{\text{loc}} := \{a_{\text{loc}} : \Omega_{\text{loc}} \to A \text{ measurable}\},
\]
and, we define for every $a_{\text{loc}} \in \mathcal{A}_{\text{loc}}$ the set of kernels
\[
\mathcal{P}_{a_{\text{loc}}} := \{\mathbb{P}_0 : \Omega_{\text{loc}} \to \mathcal{M}_1(\Omega_{\text{loc}}) \text{ measurable} \mid \mathbb{P}_0(x) \in \mathcal{P}(x, a_{\text{loc}}(x)) \text{ for all } x \in \Omega_{\text{loc}}\}.
\]
Moreover, we define on $C_p(\Omega_{\text{loc}}, \mathbb{R})$ the operator $\mathcal{T}$, which for every $v \in C_p(\Omega_{\text{loc}}, \mathbb{R})$ is defined by

$$
\Omega_{\text{loc}} \ni x \mapsto \mathcal{T}v(x) := \sup_{a \in A} \inf_{p \in P(x, a)} \mathbb{E}_p[r(x, a, X_1) + \alpha v(X_1)].
$$

(8)

Our main findings are collected in the subsequent theorem.

**Theorem 2.7.** Assume that Assumptions 2.2 and 2.4 hold true. Then the following holds.

(i) For every $v \in C_p(\Omega_{\text{loc}}, \mathbb{R})$, there exists $P^*_0 : \Omega_{\text{loc}} \times A \to M_1(\Omega_{\text{loc}})$ such that for all $(x, a) \in \Omega_{\text{loc}} \times A$, we have $P^*_0(x, a) \in P(x, a)$ and

$$
\mathbb{E}_{P_0^*(x, a)}[r(x, a, X_1) + \alpha v(X_1)] := \int_{\Omega_{\text{loc}}} r(x, a, \omega_1) + \alpha v(\omega_1) \mathbb{P}_0^*(x, a; d\omega_1)
$$

$$
= \inf_{p_0 \in P(x, a)} \mathbb{E}_{P_0}[r(x, a, X_1) + \alpha v(X_1)].
$$

(9)

Moreover, there exists $a^*_{\text{loc}} \in A_{\text{loc}}$ such that for every $x \in \Omega_{\text{loc}}$, we have

$$
\inf_{P_0 \in P(x, a^*_{\text{loc}}(x))} \mathbb{E}_{P_0}[r(x, a^*_{\text{loc}}(x), X_1) + \alpha v(X_1)]
$$

$$
= \sup_{a_{\text{loc}} \in A_{\text{loc}}} \inf_{P_0 \in P(x, a_{\text{loc}}(x))} \mathbb{E}_{P_0}[r(x, a_{\text{loc}}(x), X_1) + \alpha v(X_1)].
$$

(10)

Furthermore, let $P^*_\text{loc} : \Omega_{\text{loc}} \to M_1(\Omega_{\text{loc}})$ be defined by

$$
P^*_\text{loc}(x) := P^*_0(x, a^*_{\text{loc}}(x)), \quad x \in \Omega_{\text{loc}}.
$$

Then $P^*_\text{loc} \in P^*_{a^*_{\text{loc}}}$ and for every $x \in \Omega_{\text{loc}}$, it holds that

$$
\mathcal{T}v(x) = \sup_{a_{\text{loc}} \in A_{\text{loc}}} \inf_{p_0 \in P^*_\text{loc}} \mathbb{E}_{P_0(x)}[r(x, a_{\text{loc}}(x), X_1) + \alpha v(X_1)]
$$

$$
= \inf_{p_0 \in P^*_\text{loc}} \mathbb{E}_{P_0(x)}[r(x, a^*_{\text{loc}}(x), X_1) + \alpha v(X_1)]
$$

$$
= \mathbb{E}_{P^*_\text{loc}(x)}[r(x, a^*_{\text{loc}}(x), X_1) + \alpha v(X_1)].
$$

(11)

(ii) We have that $\mathcal{T}(C_p(\Omega_{\text{loc}}, \mathbb{R})) \subseteq C_p(\Omega_{\text{loc}}, \mathbb{R})$, that is, $\mathcal{T}_v \in C_p(\Omega_{\text{loc}}, \mathbb{R})$ for all $v \in C_p(\Omega_{\text{loc}}, \mathbb{R})$ and for all $v, w \in C_p(\Omega_{\text{loc}}, \mathbb{R})$, the following inequality holds true

$$
\|\mathcal{T}_v - \mathcal{T}_w\|_{C_p} \leq \alpha C_p\|v - w\|_{C_p}.
$$

(12)

In particular, there exists a unique $v \in C_p(\Omega_{\text{loc}}, \mathbb{R})$ such that $\mathcal{T}v = v$. Moreover, for every $v_0 \in C_p(\Omega_{\text{loc}}, \mathbb{R})$, we have $v = \lim_{n \to \infty} \mathcal{T}^nv_0$.

(iii) Let $v \in C_p(\Omega_{\text{loc}}, \mathbb{R})$ satisfy $\mathcal{T}_v = v$ and let $a^*_{\text{loc}} \in A_{\text{loc}}$, $P^*_\text{loc} \in P^*_{a^*_{\text{loc}}}$ be defined as in (i). Define $a^* := (a^*_{\text{loc}}(X_0), a^*_{\text{loc}}(X_1), \ldots) \in A$ and for all $x \in \Omega_{\text{loc}}$, $P^*_x := \delta_x \otimes P^*_\text{loc} \otimes P^*_\text{loc} \otimes \ldots$
Then, for all \( x \in \Omega_{\text{loc}} \), we have that

\[
\mathbb{E}_{\mathbb{P}^x} \left[ \sum_{t=0}^{\infty} \alpha_t r(X_t, a_{\text{loc}}^*(X_t), X_{t+1}) \right] = \inf_{\mathbb{P} \in \mathfrak{P}_{x,a^*}} \mathbb{E}_{\mathbb{P}} \left[ \sum_{t=0}^{\infty} \alpha_t r(X_t, a_{\text{loc}}^*(X_t), X_{t+1}) \right] = V(x) = v(x).
\]

**Remark 2.8.** Note that the local-to-global paradigm obtained in Theorem 2.7 is noteworthy, since \( (X_t)_{t \in \mathbb{N}_0} \) does not need to be a (time homogeneous) Markov process under each \( \mathbb{P} \in \mathfrak{P}_{x,a^*} \), as the corresponding transition kernel might vary with time. However, due to the particular setting that the set of transition probabilities \( (x, a) \mapsto \mathbb{P}(X_{t+1} = \cdot | X_t = x, A_t = a) \) is constant in time and only depends on the current state and action, and not on the whole past trajectory, we are able to derive the analog local-to-global paradigm for Markov decision processes under model uncertainty as for the ones without model uncertainty. Moreover, if \( (x, a) \mapsto \mathbb{P}(X_{t+1} = \cdot | X_t = x, A_t = a) \) is single-valued, then \( (X_t)_{t \in \mathbb{N}_0} \) is a Markov decision process in the classical sense, compare, for example, Bäuerle and Rieder (2011). This justifies to call \( (X_t)_{t \in \mathbb{N}_0} \) a Markov decision process under model uncertainty on state space \( \Omega_{\text{loc}} \subseteq \mathbb{R}^d \) with respect to \( \mathbb{P} \). Moreover, note that a posteriori, we see that \( (X_t)_{t \in \mathbb{N}_0} \) is a time-homogeneous Markov process under the worst-case measure, and the optimal strategy only depends on the current state of the process, and not on time, as observed for classical Markov decision problems.

## 3 CAPTURING DISTRIBUTIONAL UNCERTAINTY

In this section, we present different approaches that enable to capture uncertainty with respect to the choice of the underlying probability measure. We focus on ambiguity sets of probability measures, which can be constructed from observed data. More precisely, in the first case, we consider an entirely data-driven approach where the ambiguity set is described by a Wasserstein-ball around, for example, the empirical measure. In the second case, we follow a semi data-driven approach by considering a parametric family of distributions as ambiguity set, where the set of feasible parameters can be estimated from observed data. Finally, we also demonstrate how the two approaches can be generalized to the case where the state process \( (X_t)_{t \geq 0} \) describes an auto-correlated time series. We show that all of the presented approaches fulfill the requirements of the setting presented in Section 2.1.

### 3.1 Uncertainty expressed through the Wasserstein distance

The first example involves the case when distributional uncertainty is captured through the \( q \)-Wasserstein-distance \( W_q(\cdot, \cdot) \) for some \( q \in \mathbb{N} \). For any \( \mathbb{P}_1, \mathbb{P}_2 \in \mathcal{M}_1(\Omega_{\text{loc}}) \) let \( W_q(\mathbb{P}_1, \mathbb{P}_2) \) be defined as

\[
W_q(\mathbb{P}_1, \mathbb{P}_2) := \left( \inf_{\pi \in \Pi(\mathbb{P}_1, \mathbb{P}_2)} \int_{\Omega_{\text{loc}} \times \Omega_{\text{loc}}} \|x - y\|^q d\pi(x, y) \right)^{1/q},
\]
where \( \| \cdot \| \) denotes the Euclidean norm on \( \mathbb{R}^d \), and where \( \Pi(\mathbb{P}_1, \mathbb{P}_2) \) denotes the set of joint distributions of \( \mathbb{P}_1 \) and \( \mathbb{P}_2 \), compare also, for example, Villani (2009, Definition 6.1).

We fix some \( q \in \mathbb{N} \) and specify \( p := 0 \) in Assumptions 2.2 and 2.4. Further, we assume that there exists a continuous map

\[
\Omega_{\text{loc}} \times A \to (\mathcal{M}_1(\Omega_{\text{loc}}), \tau_q)
\]

\[(x, a) \mapsto \hat{\mathbb{P}}(x, a)\]

(14)
such that \( \hat{\mathbb{P}}(x, a) \) has finite \( q \)th moments for all \((x, a) \in \Omega_{\text{loc}} \times A\). Then, we define for any \( \varepsilon > 0 \) the set-valued map

\[
\Omega_{\text{loc}} \times A \ni (x, a) \mapsto \mathcal{P}(x, a) := B^q_\varepsilon(\hat{\mathbb{P}}(x, a)) := \left\{ \mathbb{P} \in \mathcal{M}_1(\Omega_{\text{loc}}) \mid W_q(\mathbb{P}, \hat{\mathbb{P}}(x, a)) \leq \varepsilon \right\}
\]

(15)

where \( B^q_\varepsilon(\hat{\mathbb{P}}(x, a)) \) denotes the \( q \)-Wasserstein-ball (or Wasserstein-ball of order \( q \)) with \( \varepsilon \)-radius and center \( \hat{\mathbb{P}}(x, a) \).

**Proposition 3.1.** Let \( \Omega_{\text{loc}} \times A \ni (x, a) \mapsto \hat{\mathbb{P}}(x, a) \in (\mathcal{M}_1(\Omega_{\text{loc}}), \tau_q) \) be continuous with finite \( q \)th moments. Then, the set-valued map \( \Omega_{\text{loc}} \times A \ni (x, a) \mapsto \mathcal{P}(x, a) \) defined as in Equation (15) fulfills the requirements of Assumption 2.2 with \( p = 0 \).

### 3.2 Knightian uncertainty in parametric models

Next, we consider a parametric approach, taking into account the so called *Knightian uncertainty*, which is named after the American economist Frank Knight (see Knight (1921)) and which describes the unquantifiable risk of having chosen the wrong model to determine probabilities for future events. Following this paradigm, we aim at describing a class of parametric models that model future events, but in order to take into account Knightian uncertainty and to avoid a misspecification by choosing a *wrong* model, we allow for a range of possible parameters. To this end, we consider a set-valued map of the form

\[
\Omega_{\text{loc}} \times A \ni (x, a) \mapsto \Theta(x, a) \subseteq \mathbb{R}^{\mathfrak{D}}, \quad \text{for some } \mathfrak{D} \in \mathbb{N}.
\]

(16)

The set \( \Theta(x, a) \) refers to the set of parameters that are admissible in dependence of \((x, a) \in \Omega_{\text{loc}} \times A\). The underlying parametric probability distribution is described by

\[
\{(x, a, \theta) \mid (x, a) \in \Omega_{\text{loc}} \times A, \theta \in \Theta(x, a)\} \to (\mathcal{M}_1(\Omega_{\text{loc}}), \tau_p)
\]

\[(x, a, \theta) \mapsto \hat{\mathbb{P}}(x, a, \theta),\]

(17)

which enables us to define the ambiguity set of probability measures by

\[
\Omega_{\text{loc}} \times A \ni (x, a) \mapsto \mathcal{P}(x, a) := \left\{ \hat{\mathbb{P}}(x, a, \theta) \mid \theta \in \Theta(x, a) \right\} \subseteq (\mathcal{M}_1(\Omega_{\text{loc}}), \tau_p).
\]

(18)
Proposition 3.2. Let \((x, a) \mapsto \Theta(x, a)\), as defined in Equation (16), be nonempty, compact-valued, and continuous, let \((x, a, \theta) \mapsto \hat{P}(x, a, \theta)\), as defined in Equation (17), be continuous. Then \((x, a) \mapsto \mathcal{P}(x, a)\), as defined in Equation (18), is nonempty, compact-valued, and continuous.

3.3 Uncertainty in autocorrelated time series

Next, we consider the case where the state process \((X_t)_{t \in \mathbb{N}_0}\) is given by an autocorrelated time series. More precisely, we assume that at time \(t \in \mathbb{N}_0\) the past \(m \in \mathbb{N}\) observations \((Y_{t-m+1}, \ldots, Y_t)\) of a time series \((Y_t)_{t \in \{-m, \ldots, 0, 1, \ldots\}}\) may have an influence on the next value of the state process. In this case, we have for all \(t \in \mathbb{N}\) a representation of the form

\[ X_t := (Y_{t-m+1}, \ldots, Y_t) \in \Omega_{\text{loc}} := Z^m \subseteq \mathbb{R}^{D \cdot m}, \text{ with } Z \subseteq \mathbb{R}^D \text{ closed}, \text{ for some } D \in \mathbb{N}. \]

To define the ambiguity set of measures, we first consider a set-valued map of the form

\[ \Omega_{\text{loc}} \times A \ni (x, a) \mapsto \tilde{\mathcal{P}}(x, a) \subseteq (\mathcal{M}_1(Z), \tau_p). \tag{19} \]

We consider the projection \( \Omega_{\text{loc}} \ni (x_1, \ldots, x_m) \mapsto \pi((x_1, \ldots, x_m)) := (x_2, \ldots, x_m) \in Z^{m-1} \) that projects onto the last \(m-1\) components, and define a set-valued map \( \mathcal{P} \), in dependence of \( \tilde{\mathcal{P}} \), by

\[ \Omega_{\text{loc}} \times A \ni (x, a) \mapsto \mathcal{P}(x, a) := \{ \delta_{\pi(x)} \otimes \mathbb{P} | \mathbb{P} \in \tilde{\mathcal{P}}(x, a) \} \subseteq (\mathcal{M}_1(\Omega_{\text{loc}}), \tau_p). \tag{20} \]

This means, by considering \( \mathcal{P} \), we take into account uncertainty with respect to the evolution of the next value of the time series. However, we do not want to consider uncertainty with respect to the \(m-1\) preceding values of the time series, as they constitute of the already observed realizations.

Proposition 3.3. Let \((x, a) \mapsto \tilde{\mathcal{P}}(x, a)\), as defined in Equation (19), be nonempty, compact-valued, and continuous. Then \((x, a) \mapsto \mathcal{P}(x, a)\), as defined in Equation (20) is nonempty, compact-valued, and continuous.

4 APPLICATION TO PORTFOLIO OPTIMIZATION

In this section, we discuss a finance-related application of the presented robust stochastic optimal control problem of Section 2.1. In particular, we compare different specifications to measure uncertainty with respect to the choice of the underlying probability measure.

4.1 Setting

We present a setting that can be applied to the robust optimization of financial portfolios. Compare among many others also Boyd et al. (2017), Dixon et al. (2020, Chapter 10), and Filos (2019), where alternative approaches to portfolio optimization relying on the optimal control of Markov decision processes are discussed. Let \(D \in \mathbb{N}\) denote the number of assets that are taken into account for
portfolio optimization. Then, the underlying asset returns in the time period between \( t-1 \) and \( t \) are given by

\[
R_t := (R^i_t)_{i=1}^{D} := \left( \frac{S^i_t - S^i_{t-1}}{S^i_{t-1}} \right)_{i=1}^{D} \in Z \subseteq \mathbb{R}^D, \quad t \in \{-m+1, \ldots, 0, 1, \ldots, \}
\]

where \( S^i_t \in (0, \infty) \) denotes the time \( t \)-value of asset \( i \in \{1, \ldots, D\} \), \( m \in \mathbb{N} \), and \( Z \subseteq \mathbb{R}^D \) closed.

To take into account the autocorrelation of the time series, we want to base our portfolio allocation decisions not only on the current portfolio allocation and the present state of the financial market, but also on the past \( m \in \mathbb{N} \) observed returns. Thus, we consider at every time \( t \in \mathbb{N}_0 \) realized returns \( (R_{t-m+1}, \ldots, R_t) \in Z^m \). Then, the underlying stochastic process \((X_t)_{t \in \mathbb{N}_0}\) is modeled as

\[
X_t := (R_{t-m+1}, \ldots, R_t) \in \Omega_{\text{loc}}, \quad t \in \mathbb{N}_0, \quad (21)
\]

with

\[
\Omega_{\text{loc}} := Z^m \subseteq \mathbb{R}^{D \times m}.
\]

Next, we introduce the compact set

\[
A := \{a = (a^i)_{i=1}^{D} \in [-C, C]^D\},
\]

for the possible values of the controls, which corresponds to the monetary investment in the \( D \) stocks, where \( C > 0 \) relates to a budget constraint when investing. Then, we define the reward function by

\[
\Omega_{\text{loc}} \times A \times \Omega_{\text{loc}} \ni (X_t, a_t, X_{t+1}) \mapsto r(X_t, a_t, X_{t+1}) := \sum_{i=1}^{D} a^i_t \cdot R^i_{t+1} - \lambda \cdot (a^T_t \cdot \Sigma \cdot a_t), \quad (22)
\]

for some risk-aversion parameter \( \lambda \geq 0 \) and a covariance matrix \( \Sigma \in \mathbb{R}^{D \times D} \) associated to the asset returns. The reward function in Equation (22) expresses the cumulated gain from trading in the period between \( t \) and \( t+1 \), where risky positions are additionally penalized by a risk measure expressed in terms of the variance of the cumulated gain from trading between \( t \) and \( t+1 \) multiplied with a risk-aversion parameter \( \lambda \). This approach is similar to the approaches presented in Boyd et al. (2017, Section 4.2) or Dixon et al. (2020, Chapter 10, Section 5.6). We will specify a data-driven estimate for the covariance matrix \( \Sigma \) in the next subsection.

4.2 Data-driven ambiguity set and Wasserstein uncertainty

We rely on the setting elaborated above.

As exposed in Section 3.1, we may capture distributional uncertainty by considering a Wasserstein-ball around some kernel

\[
\Omega_{\text{loc}} \times A \ni (x, a) \mapsto \hat{\mathbb{P}}(x, a) \in \mathcal{M}_1(Z),
\]
for $Z \subseteq \mathbb{R}^D$ closed. We consider a time series of past realized returns

$$(\mathcal{R}_1, \ldots, \mathcal{R}_N) \in Z^N,$$ for some $N \in \mathbb{N} \cap [2, \infty).$ (23)

Compare also Figure 1, where we illustrate the relation between this time series and the time series of future returns.

Relying on the time series from Equation (23), we aim at constructing an ambiguity set $\mathcal{P}$. To this end, we define $\hat{\mathbb{P}}$ through a sum of Dirac-measures given by

$$\Omega_{loc} \ni X_t = (\mathcal{R}_{t-m+1}, \ldots, \mathcal{R}_t) \mapsto \hat{\mathbb{P}}(X_t)(dx) := \sum_{s=m}^{N-1} \pi_s(X_t) \cdot \delta_{\mathcal{R}_{s+1}}(dx) \in \mathcal{M}_1(Z),$$ (24)

where $\pi_s(X_t) \in [0, 1]$, $s = m, \ldots, N - 1$ with $\sum_{s=m}^{N-1} \pi_s(X_t) = 1$. We want to weight the distance between the past $m$ returns before $\mathcal{R}_{t+1}$ and the $m$ returns before $\mathcal{R}_{s+1}$, while assigning higher probabilities to more similar sequences of $m$ returns. This means, the measure $\hat{\mathbb{P}}$ relies its prediction for the next return on the best fitting sequence of $m$ consecutive returns that precede the prediction. To this end, we set for some (small) constant $\tilde{\varepsilon} > 0$

$$\Omega_{loc} \ni X_t = (\mathcal{R}_{t-m+1}, \ldots, \mathcal{R}_t) \mapsto \pi_s(X_t) := \left( \frac{\text{dist}_s(X_t) + \tilde{\varepsilon}}{\sum_{\ell=m}^{N-1} \text{dist}_\ell(X_t) + \tilde{\varepsilon}} \right)^{-1},$$

with

$$\text{dist}_s(X_t) := \left\| (\mathcal{R}_{s-m+1}, \ldots, \mathcal{R}_s) - X_t \right\| = \left\| (\mathcal{R}_{s-m+1}, \ldots, \mathcal{R}_s) - (\mathcal{R}_{t-m+1}, \ldots, \mathcal{R}_t) \right\|,$$

for all $s = m, \ldots, N - 1$. Then, we define for any fixed $\varepsilon > 0$ and $q \in \mathbb{N}$ the ambiguity set of probability measures on $\mathcal{M}_1(\Omega_{loc})$ via the set-valued map

$$\Omega_{loc} \ni x \mapsto \mathcal{P}(x) := \left\{ \delta_\pi(x) \otimes \mathbb{P} \mid \mathbb{P} \in \mathcal{E}_\varepsilon^{(q)}(\hat{\mathbb{P}}(x)) \right\} \subseteq \left( \mathcal{M}_1(\Omega_{loc}), \tau_\mathcal{P} \right)$$ (25)

4 Note that $x \mapsto \hat{\mathbb{P}}(x)$ does not depend on $a \in A$.

5 The constant $\tilde{\varepsilon} > 0$ is merely a technical requirement, which is considered to avoid division by zero in the case $\text{dist}_s(X_t) = 0$ for some indices $s \in \{1, \ldots, N\}, t \in \mathbb{N}_0$, that is, in the case that a sequence of $m$ random returns equals a sequence of past realized returns. Hence, in practice, $\tilde{\varepsilon}$ can be set to be a negligible small positive real number.

6 Note that $\mathcal{P}$ does not depend on $a \in A$, and recall that $\Omega_{loc} \ni (x_1, \ldots, x_m) \mapsto \pi((x_1, \ldots, x_m)) := (x_2, \ldots, x_m) \in \mathcal{T}^{m-1}$ denotes the projection onto the last $m - 1$ components.
that takes into account a $q$-Wasserstein-ball around $\hat{P}$ for the next future return. Relying on the time series of realized returns in Equation (23), we estimate a covariance matrix $\Sigma_R \in \mathbb{R}^{D \times D}$ by

$$
\Sigma_R := \frac{1}{N-1} \sum_{s=1}^{N} (\mathcal{R}_s - \text{ER}) \cdot (\mathcal{R}_s - \text{ER})^T
$$

for $\text{ER} := \frac{1}{N} \sum_{i=1}^{N} \mathcal{R}_s \in \mathbb{R}^D$. Hence, this choice for $\Sigma_R$ specifies the reward function from Equation (22).

**Proposition 4.1.** Let $Z \subseteq \mathbb{R}^D$ be compact, and let $p = 0$. Then, the set-valued map $\mathcal{P}$, defined in Equation (25), satisfies Assumption 2.2. Moreover, the reward function $r$, defined in Equation (22), satisfies Assumption 2.4.

### 4.3 Parametric uncertainty

Next, we introduce a parametric approach in which we assume that the asset returns follow a multivariate normal distribution with unknown parameters.\(^7\)

To this end, we build on the setting exposed in Section 4.1, where $m > 1$, and where we choose $Z = \mathbb{R}^D$, and $p = 1$. Moreover, we consider the following unbiased estimators of mean and covariance

$$
m : (\mathbb{R}^D)^m \to \mathbb{R}^D
$$

$$
x = (x_1, \ldots, x_m) \mapsto \frac{1}{m} \sum_{i=1}^{m} x_i,
$$

and

$$
c : (\mathbb{R}^D)^m \to \mathbb{R}^{D \times D}
$$

$$
x = (x_1, \ldots, x_m) \mapsto \frac{1}{m-1} \sum_{i=1}^{m} (x_i - m(x)) \cdot (x_i - m(x))^T.
$$

Let $\varepsilon > 0$. To define the set of admissible parameters, we consider the following set-valued maps:

$$
\Omega_{\text{loc}} \ni x \mapsto \hat{\mu}(x) := \{ \mu \in \mathbb{R}^D \mid \| \mu - m(x) \| \leq \varepsilon \},
$$

$$
\Omega_{\text{loc}} \ni x \mapsto \hat{\Sigma}(x) := \{ \Sigma \in \mathbb{R}^{D \times D} \mid \Sigma = c(y) \text{ for some } y \in \Omega_{\text{loc}} \text{ with } \| y - x \| \leq \varepsilon \},
$$

$$
\Omega_{\text{loc}} \ni x \mapsto \Theta(x) := \{ (\mu, \Sigma) \in \mathbb{R}^D \times \mathbb{R}^{D \times D} \mid \mu \in \hat{\mu}(x), \Sigma \in \hat{\Sigma}(x) \}.\footnote{We say that $X \in \mathbb{R}^D$ has a $D$-dimensional multivariate normal distribution with mean $\mu \in \mathbb{R}^D$ and covariance matrix $\Sigma \in \mathbb{R}^{D \times D}$, which is symmetric and positive semidefinite if the characteristic function of $X$ is of the form $\mathbb{R}^D \ni u \mapsto \varphi_X(u) := \exp(\text{iu}^T \mu - \frac{1}{2} \mu^T \Sigma \mu)$, compare, for example, Gut (2009, p. 124). We write $X \sim \mathcal{N}_D(\mu, \Sigma)$.}
We define an ambiguity set related to $D$-dimensional multivariate normal distributions by

$$
\Omega_{loc} \ni x \mapsto \tilde{P}(x) := \{N_D(\mu, \Sigma) \mid (\mu, \Sigma) \in \Theta(x)\} \subseteq (\mathcal{M}_1(\mathbb{R}^D), \tau_1).
$$

As in Section 3.3, we denote by $\Omega_{loc} \ni (x_1, \ldots, x_m) \mapsto \pi((x_1, \ldots, x_m)) := (x_2, \ldots, x_m) \in \mathbb{R}^{D(m-1)}$ the projection onto the last $m - 1$ components. These definitions allow us to define the ambiguity set on $\mathcal{M}_1(\Omega_{loc})$ by

$$
\Omega_{loc} \times A \ni (x, a) \mapsto P(x, a) := \{\delta_{\pi(x)} \otimes P \mid P \in \tilde{P}(x)\} \subseteq (\mathcal{M}_1(\Omega_{loc}), \tau_1).
$$

(28)

This means, we consider as an ambiguity set for the next return a set of multivariate normal distributions with unknown mean and covariance, where the set of admissible means and covariances is specified by the estimators $\mu$ and $\Sigma$ as well as by the degree of ambiguity specified through $\varepsilon$.

**Proposition 4.2.** Let $Z = \mathbb{R}^D$ and $p = 1$. Then, the ambiguity set $P$, as defined in Equation (28), fulfills the requirements from Assumption 2.2. Moreover, the reward function $r$, defined in Equation (22), satisfies Assumption 2.4.

4.4 | Numerics

We present an explicit numerical algorithm that can be applied to approximately compute the optimal value function $V$ and to determine an optimal policy $a^* \in A$. We emphasize the need of an algorithm, which produces approximate optimal solutions, since even the one time-step optimization problem in Equation (8) leading to an optimal policy according to Theorem 2.7 involves the fixed point $Tv = v$, which cannot be derived explicitly. Hence, one cannot expect to obtain an explicit optimal policy.

4.4.1 | Value iteration

Theorem 2.7 directly provides an algorithm for the computation of the optimal value $V(x)$ to which we refer as the value iteration algorithm.

In this algorithm, we start with an arbitrary $V^{(0)} \in C_p(\Omega_{loc}, \mathbb{R})$ and then compute recursively $V^{(n+1)} := TV^{(n)}$ for all $n \in \mathbb{N}_0$. According to Theorem 2.7, we then have

$$
\lim_{n \to \infty} TV^{(n)} = V.
$$

(29)

Compare also, for example, Bäuerle and Rieder (2011, Section 7), where this algorithm (in a nonrobust setting) is discussed in detail.

---

Note that $P$ does not depend on $a \in A$, and recall that $\Omega_{loc} \ni (x_1, \ldots, x_m) \mapsto \pi((x_1, \ldots, x_m)) := (x_2, \ldots, x_m) \in \mathbb{R}^{D(m-1)}$ denotes the projection onto the last $m - 1$ components.
With Algorithm 1, we present a pseudocode of the methodology that can be applied to compute both the optimal value function and the optimal policy numerically. The algorithm relies on the value iteration principle. This means we solve Equation (29) by approximating the value function...
\( V \) through neural networks\(^9\) and by repeatedly applying the recursion
\[
V^{(n+1)} = TV^{(n)}.
\]
Note that Algorithm 1 approximates an optimal one-step action \( a^*_{\text{loc}} \in A_{\text{loc}} \). According to Theorem 2.7, an approximation of the optimal policy can then be obtained as
\[
a^* := (a^*_{\text{loc}}(X_0), a^*_{\text{loc}}(X_1), \ldots) \in A.
\]
We also remark that, to approximate the minimum among all measures from the ambiguity set, we sample a finite amount \( N \in \mathbb{N} \) of measures from the ambiguity set in each iteration step. In the case that the ambiguity set is given by a Wasserstein-ball, we apply the algorithm presented in Neufeld et al. (2022, Algorithm 2) ensuring that the sampled measures lie within the Wasserstein-ball around the corresponding reference measure.

### 4.5 Numerical experiments

In the sequel, we solve the portfolio optimization problem that was discussed in Section 4.1 by applying the numerical method based on Theorem 2.7 that is elaborated in Appendix 4.4 to real financial data. In particular, we compare the different approaches to capture distributional uncertainty outlined in Sections 4.2 and 4.3, respectively, and evaluate how these approaches perform under different market scenarios.

#### 4.5.1 Implementation

To apply the numerical method from Appendix 4.4, we use the following hyperparameters: number of measures \( N_P = 10 \); batch size \( B = 2^8 \); Monte-Carlo sample size \( N_{\text{MC}} = 2^3 \); discount factor \( \alpha = 0.45 \); number of epochs \( E = 50 \); number of iterations for \( a \): \( \text{Iter}_a = 10 \); number of iterations for \( v \): \( \text{Iter}_v = 10 \). The neural networks that approximate \( a \) and \( v \) constitute of two layers with 128 neurons each possessing \( \text{ReLu} \) activation functions in each layer, except for the output layer of \( a \), which possesses a \( \text{tanh} \) activation function in order to constraint the output. The learning rate used to optimize the networks \( a \) and \( v \) when applying the Adam optimizer (Kingma and Ba, 2014) is 0.001. Further details of the implementation and the code used in this work can be found under https://github.com/juliansester/Robust-Portfolio-Optimization.

#### 4.5.2 Data

To train and test the performance of the portfolio optimization approach, we consider the price evolution of \( d = 5 \) constituents\(^10\) of the S&P 500 between 2010 and 2021. We consider a look-back period of \( m = 10 \) days, that is, the prediction of the optimal trading execution relies on the previously realized \( m = 10 \) returns. This choice of \( m \) is in line with empirical findings showing that most of the autocorrelation of daily stock returns is contained in the past 10 stock returns (compare, e.g., Ding et al. (1993, tab. 3.1)). Moreover, \( m = 10 \) also provides a numerically tractable number of returns that can be taken into account without unnecessarily bloating the state space. We split the data into a training period ranging from January 2010 until September 2018, and

\(^9\) For a general introduction to neural networks, we refer the reader to LeCun et al. (2015), for applications of neural networks in finance, see Dixon et al. (2020), for a proof of the universal approximation property of neural networks, compare Hornik (1991).

\(^10\) The constituents are Apple, Microsoft, Google, Ebay, and Amazon.
Figure 2 The left panel of the figure shows the normalized evolution (with initial value = 1) of the stocks of $d = 5$ constituents of the S&P 500. Further, we divide the data into a training phase (blue) and three testing periods thereafter that are highlighted with different colors. The right panel of the figure shows the normalized evolution (with initial value = 1 for each of the assets) of the considered stocks in the three testing periods. [Color figure can be viewed at wileyonlinelibrary.com]

three different testing periods thereafter. The normalized evolution of the asset values is depicted in Figure 2.

The testing periods are illustrated in detail in the right panel of Figure 2, and they comprise three different market scenarios. While the first testing period covers an overall declining market phase, the second testing period is a volatile period without a clear trend. Eventually, the third period is a bullish market with a strong upward trend.

4.5.3 Results

We consider the reward function from Equation (22) with the particular choices $\lambda = 0$ and $\lambda = \frac{1}{2}$, respectively. We recall that choosing $\lambda > 0$ penalizes a large variance of the cumulated one-period gains. We follow the discussion from Boyd et al. (2017, Section 4.2) and choose in the following $\lambda = \frac{1}{2}$ and compare the results with the case $\lambda = 0$. Note that the parameter $\lambda$ could be adjusted for more risk-seeking or more risk-averse agents. In Tables 1–6, respectively, we depict the results of the numerical method from Section 4.4 applied to the presented data in the three testing periods with different radii $\varepsilon$ for both of the considered approaches to define ambiguity sets, see Sections 4.2 and 4.3. Note also that we consider different ranges of values for $\varepsilon$ for the two considered approaches. In Figures 3–5, we depict the cumulated trading profits of the trained strategies in the respective testing periods.

Remark 4.3 (Computational time). It turns out that training a robust trading strategy in the Wasserstein approach is slightly faster than training a strategy in the parametric approach: for training 50 epochs with the parameters specified in Section 4.5.1, the Wasserstein approach needs 15.07 min whereas the parametric approach needs 23.12 min on a standard computer.\textsuperscript{11}

\textsuperscript{11} We used for the computations a Gen Intel(R) Core(TM) i7-1165G7, 2.80-GHz processor with 40-GB RAM.
**Table 1** The table shows the results of the Wasserstein-ball approach (Section 4.2) and the parametric approach (Section 4.3) in the first testing period where we set $\lambda = 0$ in Equation (22).

|                      | Overall profit | Average profit | % of profitable trades | Sharpe ratio | Sortino ratio |
|----------------------|----------------|----------------|------------------------|--------------|---------------|
| **Wasserstein approach, $\lambda = 0$** |                |                |                        |              |               |
| $\varepsilon = 0$    | $-1.035019$    | $-0.015221$    | 47.06                  | $-0.172211$  | $-0.207249$   |
| $\varepsilon = 0.01$ | 0.221663       | 0.003260       | 58.82                  | 0.035932     | 0.052785      |
| $\varepsilon = 0.05$ | 0.080857       | 0.001189       | 55.88                  | 0.015079     | 0.022100      |
| $\varepsilon = 0.1$  | $-0.154961$    | $-0.002279$    | 44.12                  | $-0.028555$  | $-0.041425$   |
| $\varepsilon = 0.3$  | 0.005442       | 0.000080       | 48.53                  | 0.035702     | 0.055826      |
| **Parametric approach, $\lambda = 0$** |                |                |                        |              |               |
| $\varepsilon = 0$    | $-0.378669$    | $-0.005569$    | 50.00                  | $-0.067022$  | $-0.083380$   |
| $\varepsilon = 0.005$| $-0.165537$    | $-0.002434$    | 42.65                  | $-0.127499$  | $-0.146742$   |
| $\varepsilon = 0.025$| $-0.143357$    | $-0.002108$    | 38.24                  | $-0.198093$  | $-0.225493$   |
| $\varepsilon = 0.05$ | $-0.006977$    | $-0.000103$    | 50.00                  | $-0.035993$  | $-0.043762$   |
| $\varepsilon = 0.15$ | 0.003136       | 0.000046       | 58.82                  | 0.053274     | 0.066614      |

**Table 2** The table shows the results of the Wasserstein-ball approach (Section 4.2) and the parametric approach (Section 4.3) in the first testing period where we set $\lambda = \frac{1}{2}$ in Equation (22).

|                      | Overall profit | Average profit | % of profitable trades | Sharpe ratio | Sortino ratio |
|----------------------|----------------|----------------|------------------------|--------------|---------------|
| **Wasserstein approach, $\lambda = \frac{1}{2}$** |                |                |                        |              |               |
| $\varepsilon = 0$    | $-0.564527$    | $-0.008302$    | 44.12                  | $-0.087936$  | $-0.120798$   |
| $\varepsilon = 0.01$ | $-0.197820$    | $-0.002909$    | 47.06                  | $-0.035237$  | $-0.048919$   |
| $\varepsilon = 0.05$ | $-0.549781$    | $-0.008085$    | 50.00                  | $-0.113071$  | $-0.137965$   |
| $\varepsilon = 0.1$  | $-0.307764$    | $-0.004526$    | 42.65                  | $-0.084528$  | $-0.112735$   |
| $\varepsilon = 0.3$  | $-0.007034$    | $-0.000103$    | 42.65                  | $-0.034035$  | $-0.065102$   |
| **Parametric approach, $\lambda = \frac{1}{2}$** |                |                |                        |              |               |
| $\varepsilon = 0$    | $-0.569345$    | $-0.008373$    | 47.06                  | $-0.097639$  | $-0.117886$   |
| $\varepsilon = 0.005$| $-0.165356$    | $-0.002432$    | 44.12                  | $-0.163848$  | $-0.188656$   |
| $\varepsilon = 0.025$| $-0.083480$    | $-0.001228$    | 41.18                  | $-0.124146$  | $-0.151676$   |
| $\varepsilon = 0.05$ | $-0.008617$    | $-0.000127$    | 51.47                  | $-0.050311$  | $-0.066198$   |
| $\varepsilon = 0.15$ | $-0.008218$    | $-0.000121$    | 38.24                  | $-0.129144$  | $-0.152631$   |

**4.5.4 Discussion of the results**

Note that in contrast to the third bullish testing period, the first and second testing periods comprise scenarios that did not occur in similar form during the training period. Hence, it cannot be guaranteed that a nonrobust trading strategy that was trained in periods where such scenarios never occurred can trade profitably during testing period 1 and testing period 2. Indeed, as the numerical results reveal, applying a trained nonrobust trading strategy not respecting any distributional ambiguity (i.e., $\varepsilon = 0$) leads to significant losses in the adverse scenarios considered in testing periods 1 and 2, while robust trading strategies, which did encounter also adverse scenar-
Table 3 The table shows the results of the Wasserstein-ball approach (Section 4.2) and the parametric approach (Section 4.3) in the second testing period where we set $\lambda = 0$ in Equation (22).

| Wasserstein approach, $\lambda = 0$ | Overall profit | Average profit | % of profitable trades | Sharpe ratio | Sortino ratio |
|------------------------------------|----------------|----------------|------------------------|--------------|---------------|
| $\varepsilon = 0$                  | $-0.319652$    | $-0.003229$    | 51.52                  | $-0.068955$  | $-0.091312$   |
| $\varepsilon = 0.01$              | $-0.086229$    | $-0.000871$    | 56.57                  | $-0.018503$  | $-0.025662$   |
| $\varepsilon = 0.05$              | $-0.128121$    | $-0.001294$    | 52.53                  | $-0.036148$  | $-0.044057$   |
| $\varepsilon = 0.1$               | $-0.108500$    | $-0.001096$    | 50.51                  | $-0.054067$  | $-0.074651$   |
| $\varepsilon = 0.3$               | $0.001415$     | $0.000014$     | 52.53                  | $0.027997$   | $0.040511$    |

| Parametric approach, $\lambda = 0$ | Overall profit | Average profit | % of profitable trades | Sharpe ratio | Sortino ratio |
|-----------------------------------|----------------|----------------|------------------------|--------------|---------------|
| $\varepsilon = 0$                 | $-0.065187$    | $-0.000658$    | 48.48                  | $-0.014956$  | $-0.020914$   |
| $\varepsilon = 0.005$             | $-0.094396$    | $-0.000953$    | 50.51                  | $-0.063190$  | $-0.079788$   |
| $\varepsilon = 0.025$             | $0.027990$     | $0.000283$     | 55.56                  | $0.030187$   | $0.053250$    |
| $\varepsilon = 0.05$              | $0.022454$     | $0.000227$     | 49.49                  | $0.079555$   | $0.277870$    |
| $\varepsilon = 0.15$              | $-0.001429$    | $-0.000014$    | 49.49                  | $-0.048469$  | $-0.061703$   |

Table 4 The table shows the results of the Wasserstein-ball approach (Section 4.2) and the parametric approach (Section 4.3) in the second testing period where we set $\lambda = \frac{1}{2}$ in Equation (22).

| Wasserstein approach, $\lambda = \frac{1}{2}$ | Overall profit | Average profit | % of profitable trades | Sharpe ratio | Sortino ratio |
|-----------------------------------------------|----------------|----------------|------------------------|--------------|---------------|
| $\varepsilon = 0$                            | $-0.267468$    | $-0.002702$    | 47.47                  | $-0.058955$  | $-0.082091$   |
| $\varepsilon = 0.01$                         | $-0.026789$    | $-0.000271$    | 53.54                  | $-0.006770$  | $-0.009480$   |
| $\varepsilon = 0.05$                         | $0.344366$     | $0.003478$     | 52.53                  | $0.099127$   | $0.180309$    |
| $\varepsilon = 0.1$                          | $0.246400$     | $0.002489$     | 43.43                  | $0.092752$   | $0.181330$    |
| $\varepsilon = 0.3$                          | $0.003219$     | $0.000033$     | 49.49                  | $0.041700$   | $0.076582$    |

| Parametric approach, $\lambda = \frac{1}{2}$ | Overall profit | Average profit | % of profitable trades | Sharpe ratio | Sortino ratio |
|-----------------------------------------------|----------------|----------------|------------------------|--------------|---------------|
| $\varepsilon = 0$                            | $-0.257053$    | $-0.002596$    | 47.47                  | $-0.063529$  | $-0.084109$   |
| $\varepsilon = 0.005$                        | $-0.080422$    | $-0.000812$    | 50.51                  | $-0.064610$  | $-0.086200$   |
| $\varepsilon = 0.025$                        | $0.017278$     | $0.000175$     | 45.45                  | $0.020485$   | $0.035732$    |
| $\varepsilon = 0.05$                         | $0.032120$     | $0.000324$     | 44.44                  | $0.090080$   | $0.578979$    |
| $\varepsilon = 0.15$                         | $0.000581$     | $0.000006$     | 51.52                  | $0.022708$   | $0.037737$    |

ios during training, clearly outperform the nonrobust strategy for both considered approaches, namely the Wasserstein-ball approach and the parametric approach.

As the third testing period comprises a bullish market period, occurring in similar form in the training data, the nonrobust approach turns out to be the most profitable approach in this period, since it is the approach that is best adjusted to this scenario. However, when choosing the right level of $\varepsilon$, the robust approach still can trade profitably in this period.

In general, it turns out to be important to correctly identify the appropriate size of the ambiguity set (here encoded by the radius $\varepsilon$) which needs to be adjusted for each optimization under consideration. Due to the formulation of the robust optimization problem as a worst-case approach,
TABLE 5 The table shows the results of the Wasserstein-ball approach (Section 4.2) and the parametric approach (Section 4.3) in the third testing period where we set $\lambda = 0$ in Equation (22).

|                | Overall profit | Average profit | % of profitable trades | Sharpe ratio | Sortino ratio |
|----------------|----------------|----------------|------------------------|--------------|---------------|
| **Wasserstein approach, $\lambda = 0$** |                |                |                        |              |               |
| $\varepsilon = 0$ | 0.845952       | 0.010846       | 52.56                  | 0.166630     | 0.335301      |
| $\varepsilon = 0.01$ | -0.160733      | -0.002061      | 48.72                  | -0.031032    | -0.040478     |
| $\varepsilon = 0.05$ | 0.358808       | 0.004600       | 52.56                  | 0.096431     | 0.154537      |
| $\varepsilon = 0.1$  | 0.087670       | 0.001124       | 52.56                  | 0.037463     | 0.053469      |
| $\varepsilon = 0.3$  | -0.017214      | -0.000221      | 43.59                  | -0.199209    | -0.231336     |
| **Parametric approach, $\lambda = 0$** |                |                |                        |              |               |
| $\varepsilon = 0$ | 0.682794       | 0.008754       | 64.10                  | 0.124028     | 0.170558      |
| $\varepsilon = 0.005$ | 0.176339       | 0.002261       | 61.54                  | 0.054989     | 0.069144      |
| $\varepsilon = 0.025$ | 0.043324       | 0.000555       | 64.10                  | 0.014557     | 0.017213      |
| $\varepsilon = 0.05$ | -0.047627      | -0.000611      | 52.56                  | -0.064649    | -0.075748     |
| $\varepsilon = 0.15$ | -0.001740      | -0.000022      | 51.28                  | -0.039336    | -0.049270     |

respecting for more distributional ambiguity means to consider more bad scenarios, and therefore may eventually result in a more careful, less volatile trading behavior with smaller returns, as it clearly can be seen in the case $\varepsilon = 0.3$ for the Wasserstein-ball approach and in the case $\varepsilon = 0.15$ for the parametric approach, respectively. In contrast, insufficiently accounting for uncertainty comes at the cost of not being well equipped when adverse scenarios occur, an observation that was already made in similar empirical studies that use different approaches to solve robust optimization problems, compare, for example, Lütkebohmert et al. (2022) and Neufeld et al. (2022). Hence, choosing an intermediate level of ambiguity seems to be an appropriate choice. Supporting this rationale, our numerical results show that for both cases $\lambda = 0$ and $\lambda = \frac{1}{2}$ choosing a level

TABLE 6 The table shows the results of the Wasserstein-ball approach (Section 4.2) and the parametric approach (Section 4.3) in the third testing period where we set $\lambda = \frac{1}{2}$ in Equation (22).

|                | Overall profit | Average profit | % of profitable trades | Sharpe ratio | Sortino ratio |
|----------------|----------------|----------------|------------------------|--------------|---------------|
| **Wasserstein approach, $\lambda = 1/2$** |                |                |                        |              |               |
| $\varepsilon = 0$ | 0.418353       | 0.005364       | 56.41                  | 0.094879     | 0.144220      |
| $\varepsilon = 0.01$ | 1.467143       | 0.018810       | 69.23                  | 0.339216     | 0.793886      |
| $\varepsilon = 0.05$ | 0.685678       | 0.008791       | 58.97                  | 0.174976     | 0.310182      |
| $\varepsilon = 0.1$  | 0.356257       | 0.004567       | 52.56                  | 0.194070     | 0.417113      |
| $\varepsilon = 0.3$  | 0.025321       | 0.000325       | 57.69                  | 0.179535     | 0.757265      |
| **Parametric approach, $\lambda = 1/2$** |                |                |                        |              |               |
| $\varepsilon = 0$ | 0.442670       | 0.005675       | 58.97                  | 0.086985     | 0.117434      |
| $\varepsilon = 0.005$ | 0.138669       | 0.001778       | 62.82                  | 0.048293     | 0.058150      |
| $\varepsilon = 0.025$ | 0.026678       | 0.000342       | 61.54                  | 0.011119     | 0.013274      |
| $\varepsilon = 0.05$ | -0.042799      | -0.000549      | 60.26                  | -0.071591    | -0.083011     |
| $\varepsilon = 0.15$ | -0.003295      | -0.000042      | 56.41                  | -0.056704    | -0.065249     |
The figures show the cumulated profit of the trained strategies in testing period 1. In (a), we report the results when setting $\lambda = 0$ in Equation (22) and where in (b) we set $\lambda = \frac{1}{2}$. The left panel of both figures illustrates the profit when applying a Wasserstein-ball approach, whereas the right panel illustrates the profit under a parametric approach. Note that the $x$-axis describes the trading days, whereas the $y$-axis describes the cumulated profit. [Color figure can be viewed at wileyonlinelibrary.com]

This provides evidence that taking into account distributional uncertainty may be of particular importance in volatile and crisis-like market scenarios, which did not occur previously in a similar form. Our results also show that taking into account a penalization of large variances of cumulated gains by setting $\lambda = \frac{1}{2}$ in Equation (22) leads to slightly changed outcomes while the above discussed observations remain true.

A practitioner may be in the situation to decide whether to use a robust approach or a nonrobust approach for portfolio optimization: one can derive from our results that the robust portfolio optimization approach takes into account that an assumed or estimated underlying model may be misspecified, that is, realized scenarios are not appropriately covered by an assumed model. Hence, if the practitioner sees no risk of misspecification and believes strongly in his model, then it is advisable for him to use a nonrobust approach as the empirical results provide evidence that if there is indeed no misspecification, nonrobust approaches slightly outperform robust approaches.
Testing Period 2

![Cumulated Profit of Trained Strategy in Testing Period 2, Wasserstein Approach](image1)

(a) $\lambda = 0$

![Cumulated Profit of Trained Strategy in Testing Period 2, Parametric Approach](image2)

(b) $\lambda = \frac{1}{2}$

**FIGURE 4** Testing period 2. The figure shows the cumulated training profit of the trained strategies in testing period 2 where in (a), we report the results when setting $\lambda = 0$ in Equation (22) and where in (b), we set $\lambda = \frac{1}{2}$. The left panel of both figures illustrates the profit when applying a Wasserstein-ball approach, whereas the right panel illustrates the profit under a parametric approach. Note that the x-axis describes the trading days, whereas the y-axis describes the cumulated profit. [Color figure can be viewed at wileyonlinelibrary.com]

If, however, the practitioner wants to take into account the (difficult to determine) risk of having chosen a wrong model and wants to be robust against adverse scenarios, then our results imply that she should rather rely on a distributionally robust approach.

4.5.5 Comparison with other robust portfolio optimization approaches

The literature on robust portfolio optimization is rapidly growing. We refer to, for example, Blanchet et al. (2021) and Pham et al. (2022) for alternative approaches to robust portfolio optimization, to Bartl (2019), Bartl et al. (2021), Bartl et al. (2019), Blanchard and Carassus (2018), Neufeld and Nutz (2018), Neufeld and Šikić (2019) for the fast growing literature on the related problem of robust utility optimization in nondominated settings in discrete time, and to Biagini and Pinar (2017), Chau and Rásonyi (2019), Denis and Kervarec (2013), Fouque et al. (2016), Guo et al. (2022), Ismail and Pham (2019), Liang and Ma (2020), Lin and Riedel (2021), Lin et al. (2020),
Testing Period 3

Fig. 5 Testing period 3. The figure shows the cumulated training profit of the trained strategies in testing period 3 where in (a), we report the results when setting $\lambda = 0$ in Equation (22) and where in (b), we set $\lambda = \frac{1}{2}$. The left panel of both figures illustrates the profit when applying a Wasserstein-ball approach, whereas the right panel illustrates the profit under a parametric approach. Note that the $x$-axis describes the trading days, whereas the $y$-axis describes the cumulated profit. [Color figure can be viewed at wileyonlinelibrary.com]

Matoussi et al. (2015), Neufeld and Nutz (2018), Pham et al. (2022), Pun (2021), Tevzadze et al. (2013), Yang et al. (2019) for robust utility maximization in continuous time. Let us compare our contribution with articles that recently analyzed how different choices of (the size of) ambiguity sets affect the performance of the portfolio optimization problem.

In the distributionally robust optimization approach proposed in Blanchet et al. (2021), the ambiguity set is specifically given by a Wasserstein ambiguity set and their specific objective function relates to Markowitz’s mean-variance portfolio optimization problem. Their setting is reduced to this ambiguity specification but, therefore, allows to apply a duality result, which transfers the distributionally robust optimization problem into a nonrobust regularized optimization problem, which can be solved explicitly. Moreover, the authors provide a data-driven methodology to determine the size of the ambiguity set relying on results from Blanchet et al. (2019).

In a recent paper, Pham et al. (2022) study in a continuous-time framework portfolio optimization under drift and correlation ambiguity. Their framework allows to determine different optimal allocation decisions in dependence of the degree of ambiguity specified, where more ambiguity
leads to less investment in risky assets. This result can be seen in line with our finding that a higher degree of ambiguity leads to more careful trading decisions.

In the same spirit, the results from Obłój and Wiesel (2021), where the authors consider a portfolio optimization approach under Wasserstein-ambiguity in an one-period market, imply that the size of the ambiguity set corresponds to the risk attitude of the market participant.

In Du et al. (2020), the authors also rely their approach on Wasserstein-ambiguity sets and analyze the mean-CVar portfolio optimization problem where CVar refers to the conditional value at risk. Their empirical study that is carried out on Chinese and American stock markets again confirms the previously discussed findings that a higher radius of the Wasserstein-ball corresponds to a higher degree of risk aversion, which leads to different, more careful allocation results.

5 | PROOF OF THEOREM 2.7

Proof of Theorem 2.7 (i). Let \( v \in C_p(\Omega_{\text{loc}}, \mathbb{R}) \). We define the map

\[
F : GrP = \{(x, a_0, P_0) \mid x \in \Omega_{\text{loc}}, a_0 \in A, P_0 \in \mathcal{P}(x, a_0) \} \to \mathbb{R}
\]

\[
(x, a_0, P_0) \mapsto \int_{\Omega_{\text{loc}}} r(x, a_0, \omega_1) + \alpha v(\omega_1) P_0(\text{d}\omega_1).
\]

We claim that the map \( F \) is continuous (or equivalently sequentially continuous). To this end, let \( (x, a_0, P_0) \in GrP \) and \( (x^{(n)}, a_0^{(n)}, P_0^{(n)}) \subseteq GrP \) be a sequence with \(^{12} (x^{(n)}, a_0^{(n)}, P_0^{(n)}) \to (x, a_0, P_0) \) for \( n \to \infty \). Then,

\[
|F(x^{(n)}, a_0^{(n)}, P_0^{(n)}) - F(x, a_0, P_0)|
\leq |F(x^{(n)}, a_0^{(n)}, P_0^{(n)}) - F(x, a_0, P_0^{(n)})| + |F(x, a_0, P_0^{(n)}) - F(x, a_0, P_0)|. \tag{31}
\]

The second summand \( |F(x, a_0, P_0^{(n)}) - F(x, a_0, P_0)| \) vanishes for \( n \to \infty \) since the integrand \( \omega_1 \mapsto r(x, a_0, \omega_1) + \alpha v(\omega_1) \) is an element of \( C_p(\Omega_{\text{loc}}, \mathbb{R}) \). For the first summand we obtain, by using Assumption 2.4 (ii), that

\[
\lim_{n \to \infty} \left| F(x^{(n)}, a_0^{(n)}, P_0^{(n)}) - F(x, a_0, P_0^{(n)}) \right|
\leq \lim_{n \to \infty} \int_{\Omega_{\text{loc}}} \left| r(x^{(n)}, a_0^{(n)}, \omega_1) - r(x, a_0, \omega_1) \right| P_0^{(n)}(\text{d}\omega_1)
\leq \lim_{n \to \infty} \int_{\Omega_{\text{loc}}} L \cdot (1 + \|\omega_1\|^p) \cdot \left( \rho_0 \left( \left\| x^{(n)} - x \right\| \right) + \rho_A \left( \left\| a_0^{(n)} - a_0 \right\| \right) \right) P_0^{(n)}(\text{d}\omega_1)
= \lim_{n \to \infty} L \cdot \left( \rho_0 \left( \left\| x^{(n)} - x \right\| \right) + \rho_A \left( \left\| a_0^{(n)} - a_0 \right\| \right) \right) \cdot \int_{\Omega_{\text{loc}}} (1 + \|\omega_1\|^p) P_0(\text{d}\omega_1) = 0,
\]

where we use in the last equality that due to the convergence \( P_0^{(n)} \overset{\tau_p}{\longrightarrow} P_0 \) also the \( p \)th moments converge, see, for example, Villani (2009, Definition 6.8. (i), and Theorem 6.9). Thus, \( F \) is

\(^{12} \) We highlight that \( GrP \subseteq \Omega_{\text{loc}} \times A \times (\mathcal{M}_1(\Omega_{\text{loc}}), \tau_p) \) is endowed with the product topology, see also Section 2.1.
continuous. Therefore, we may apply Berge’s maximum theorem (Theorem A.2) and obtain that the map

$$G : \Omega_{\text{loc}} \times A \to \mathbb{R}$$

$$G((x, a_0)) = \inf_{\mathcal{P}_0 \in \mathcal{P}(x, a_0)} \int_{\Omega_{\text{loc}}} r(x, a_0, \omega_1) + \alpha v(\omega_1) \mathcal{P}_0(d\omega_1)$$

(32)

is continuous, and the set of minimizers is nonempty for each \((x, a_0) \in \Omega_{\text{loc}} \times A\). According to the measurable maximum theorem (Theorem A.3), there exists a measurable selector \(\Omega_{\text{loc}} \times A \ni (x, a_0) \mapsto \mathcal{P}_0^*(x, a_0) \in \mathcal{P}(x, a_0)\), where \(\mathcal{P}_0^*(x, a_0)\) minimizes the integral in Equation (32) for each \((x, a_0) \in \Omega_{\text{loc}} \times A\). This shows the assertion in Equation (9). Next, we use that \(\Omega_{\text{loc}} \times A \ni (x, a_0) \mapsto G(x, a_0)\) is continuous, and apply again Berge’s maximum theorem to the constant compact-valued correspondence \(\Omega_{\text{loc}} \ni x \mapsto A \subset \mathbb{R}^m\) and to \(G\). This yields that the map

$$H : \Omega_{\text{loc}} \to \mathbb{R}$$

$$x \mapsto \sup_{a_0 \in A} G(x, a_0).$$

(33)

is continuous and that the set of maximizers of \(G(x, \cdot)\) is nonempty for each \(x \in \Omega_{\text{loc}}\). Moreover, by the measurable maximum theorem (Theorem A.3), there exists a measurable selector \(\Omega_{\text{loc}} \ni x \mapsto a_{\text{loc}}^*(x) \in A\), where \(a_{\text{loc}}^*(x)\) maximizes \(G(x, \cdot)\) for each \(x \in \Omega_{\text{loc}}\). This shows Equation (10). Next, we define the map \(\mathcal{P}_{\text{loc}}^* \in \mathcal{P}_{a_{\text{loc}}^*}\) by

$$\mathcal{P}_{\text{loc}}^* : \Omega_{\text{loc}} \to \mathcal{M}(\Omega_{\text{loc}})$$

$$x \mapsto \mathcal{P}_0^*(x, a_{\text{loc}}^*(x)).$$

Then, by Equations (32), (33), the definition of \(\Omega_{\text{loc}} \ni x \mapsto a_{\text{loc}}^*(x) \in A\), and the definition of \(\Omega_{\text{loc}} \times A \ni (x, a_0) \mapsto \mathcal{P}_0^*(x, a_0) \in \mathcal{M}(\Omega_{\text{loc}})\), we obtain for each \(x \in \Omega_{\text{loc}}\) that

$$\mathcal{T} v(x) = H(x) = \sup_{a_0 \in A} G(x, a_0)$$

$$= \sup_{a_0 \in A} \inf_{\mathcal{P}_0 \in \mathcal{P}(x, a_0)} \int_{\Omega_{\text{loc}}} r(x, a_0, \omega_1) + \alpha v(\omega_1) \mathcal{P}_0(d\omega_1)$$

$$= \inf_{\mathcal{P}_0 \in \mathcal{P}(x, a_0)} \int_{\Omega_{\text{loc}}} r(x, a_{\text{loc}}^*(x), \omega_1) + \alpha v(\omega_1) \mathcal{P}_0(d\omega_1)$$

$$= \int_{\Omega_{\text{loc}}} r(x, a_{\text{loc}}^*(x), \omega_1) + \alpha v(\omega_1) \mathcal{P}_{\text{loc}}^*(x; d\omega_1)$$

$$= \inf_{\mathcal{P} \in \mathcal{P}_{a_{\text{loc}}^*}} \int_{\Omega_{\text{loc}}} r(x, a_{\text{loc}}^*(x), \omega_1) + \alpha v(\omega_1) \mathcal{P}(x; d\omega_1)$$

$$= \sup_{a_0 \in A} \inf_{\mathcal{P} \in \mathcal{P}_{a_{\text{loc}}^*}} \int_{\Omega_{\text{loc}}} r(x, a_{\text{loc}}^*(x), \omega_1) + \alpha v(\omega_1) \mathcal{P}(x; d\omega_1).$$

(34)
This shows Equation (11) and, therefore, completes the proof of (i).

Proof of Theorem 2.7 (ii). The continuity of $\Omega_{\text{loc}} \ni x \mapsto \mathcal{T}v(x)$ follows from the continuity of $H$ defined in Equations (33) and (34). By the growth conditions on $r$ and $X_1$ (Assumption 2.4 (iii)), we obtain for all $x \in \Omega_{\text{loc}}$ that

$$
\mathcal{T}v(x) \leq \sup_{a_{\text{loc}} \in A_{\text{loc}}} \inf_{P_0 \in \mathcal{P}_{a_{\text{loc}}}} \mathbb{E}_{P_0} \left[ C_r (1 + \|x\|^p + \|X_1\|^p) + \alpha \|v\|_{C_p} (1 + \|X_1\|^p) \right]
\leq C_r (\|x\|^p + C_p (1 + \|x\|^p)) + \alpha \|v\|_{C_p} C_p (1 + \|x\|^p)
\leq \left( C_r + C_r C_p + \alpha \|v\|_{C_p} C_p \right) (1 + \|x\|^p).
$$

Hence, $\mathcal{T}v \in C_p(\Omega_{\text{loc}}, \mathbb{R})$. Note that for every nonempty set $Q$ and for all $G, H : Q \to \mathbb{R}$ we have that

$$
\left| \inf_{Q \in Q} G(Q) - \inf_{Q \in Q} H(Q) \right| \leq \sup_{Q \in Q} |G(Q) - H(Q)|.
$$

Therefore, we obtain for every $v, w \in C_p(\Omega_{\text{loc}}, \mathbb{R}), x \in \Omega_{\text{loc}}$, by using Equation (3), that

$$
\left| \mathcal{T}v(x) - \mathcal{T}w(x) \right| \leq \alpha \sup_{a \in A} \sup_{P_0 \in \mathcal{P}(x,a)} \mathbb{E}_{P_0} \left[ |v(X_1) - w(X_1)| \right]
\leq \alpha \sup_{a \in A} \sup_{P_0 \in \mathcal{P}(x,a)} \mathbb{E}_{P_0} \left[ |v(X_1) - w(X_1)| \right]
\leq \alpha \sup_{a \in A} \sup_{P_0 \in \mathcal{P}(x,a)} \mathbb{E}_{P_0} \left[ \|v - w\|_{C_p} (1 + \|X_1\|^p) \right]
\leq \alpha C_p \|v - w\|_{C_p} \cdot (1 + \|x\|^p).
$$

Hence, we obtain Equation (12). Now, let $v_0 \in C_p(\Omega_{\text{loc}}, \mathbb{R})$. Then, since by Assumption 2.4 (iv), we have $0 < \alpha C_p < 1$, $\mathcal{T}$ is a contraction on $C_p(\Omega_{\text{loc}}, \mathbb{R})$. Hence, Banach’s fixed point theorem (Theorem A.1) implies existence and uniqueness of a fixpoint $v \in C_p(\Omega_{\text{loc}}, \mathbb{R})$ such that $v = \mathcal{T}v = \lim_{n \to \infty} \mathcal{T}^n v_0$. □

Proof of Theorem 2.7 (iii). We first show the inequality $v(x) \leq V(x)$.

Let $v \in C_p(\Omega_{\text{loc}}, \mathbb{R})$ be the fixed point of $\mathcal{T}$ whose existence and uniqueness were proved in part (ii), that is, we have that $v = \mathcal{T}v$. Let $a_{\text{loc}}^* \in A_{\text{loc}}$ be the minimizer from part (i) and write $a^* : = (a_{\text{loc}}^*(X_0), a_{\text{loc}}^*(X_1), ...) \in A$, and let $x \in \Omega_{\text{loc}}$. First, we claim that it holds for all $n \in \mathbb{N}$ with $n \geq 2$ that

$$
\mathcal{T}^n v(x) \leq \inf_{P \in \Psi_{x,a^*}} \mathbb{E}_P \left[ \sum_{t=0}^{n-1} \alpha^t r(X_t, a_{\text{loc}}^*(X_t), X_{t+1}) + \alpha^n v(X_n^+) \right].
$$

(36)
We prove the claim (36) inductively. To that end, note that by part (i), we can write

\[ \mathcal{T} v(x) = \inf_{P_0 \in \mathcal{P}_{a_{loc}^*}} \mathbb{E}_{P_0(x)}[r(x, a_{loc}^*(x), X_1) + \alpha v(X_1)] \]

\[ = \inf_{P_0 \in \mathcal{P}_{a_{loc}^*}} \int_{\Omega_{loc}} r(x, a_{loc}^*(x), \omega_1) + \alpha v(\omega_1) P_0(x; d\omega_1). \]

This implies for all \( t \in \mathbb{N}_0 \) and \( \omega = (\omega_t)_{t \in \mathbb{N}_0} \in \Omega \) that

\[ \mathcal{T} v(X_t(\omega)) = \inf_{P_0 \in \mathcal{P}_{a_{loc}^*}} \int_{\Omega_{loc}} r(X_t(\omega), a_{loc}^*(X_t(\omega)), \omega_t) + \alpha v(\omega_1) P_0(X_t(\omega); d\omega_1) \]

\[ = \inf_{P_1 \in \mathcal{P}_{a_{loc}^*}} \int_{\Omega_{loc}} r(\omega_t, a_{loc}^*(\omega_t), \omega_{t+1}) + \alpha v(\omega_{t+1}) P_1(\omega_t; d\omega_{t+1}), \]

(37)

where we just used the definition of the canonical process and relabeled the variables \( P_0 \) and \( \omega_1 \).

For \( n = 2 \), as \( \mathcal{T} v = v \), we thus have

\[ \mathcal{T}(\mathcal{T} v)(x) = \mathcal{T} v(x) = \inf_{P_0 \in \mathcal{P}_{a_{loc}^*}} \int_{\Omega_{loc}} r(x, a_{loc}^*(x), \omega_1) + \alpha v(x) P_0(x; d\omega_1) \]

\[ = \inf_{P_0 \in \mathcal{P}_{a_{loc}^*}} \int_{\Omega_{loc}} r(x, a_{loc}^*(x), \omega_1) + \alpha \mathcal{T} v(x) P_0(x; d\omega_1) \]

\[ = \inf_{P_0 \in \mathcal{P}_{a_{loc}^*}} \int_{\Omega_{loc}} [r(x, a_{loc}^*(x), \omega_1) + \alpha \mathcal{T} v(x) P_0(x; d\omega_1)] P_0(x; d\omega_1) \]

\[ \leq \inf_{P_1 \in \mathcal{P}_{a_{loc}^*}} \int_{\Omega_{loc}} \sum_{t=0}^{1} \alpha^t r(\omega_t, a_{loc}^*(\omega_t), \omega_{t+1}) + \alpha^2 v(\omega_2) P_1(\omega_1; d\omega_2) P_0(x; d\omega_1) \]

\[ = \inf_{P \in \mathcal{P}_{x,a^*}} \mathbb{E}_P \left[ \sum_{t=0}^{1} \alpha^t r(X_t, a_{loc}^*(X_t), X_{t+1}) + \alpha^2 v(X_2) \right], \]

where we use Equation (37) and the structure of the measures \( \mathcal{P}_{x,a^*} \). The general case for arbitrary \( n \) follows with analog arguments. Indeed, let the claim in Equation (36) be true for \( n - 1 \), then it...
follows by the same argument as in Equation (38) and by the structure of every \( \mathbb{P} \in \mathfrak{P} \omega_1, a^* \), that

\[
\mathcal{T}^n v(x) = \mathcal{T} (\mathcal{T}^{n-1} v)(x)
\]

\[
\leq \inf_{\mathbb{P}_0 \in \mathfrak{P}_{a_{\text{loc}}^*}} \int_{\Omega_{\text{loc}}} [r(x, a_{\text{loc}}^*(x), \omega_1) + \alpha \inf_{\mathbb{P} \in \mathfrak{P}_{a_{\text{loc}}^*}} \int_{\Omega} \left\{ \sum_{t=1}^{n-1} \alpha^{t-1} r(\omega_t, a_{\text{loc}}^*(\omega_t), \omega_{t+1}) + \alpha^{n-1} v(\omega_n) \right\} \mathbb{P}(d\omega)] \mathbb{P}_0(x; d\omega_1)
\]

\[
= \inf_{\mathbb{P}_0 \in \mathfrak{P}_{a_{\text{loc}}^*}} \int_{\Omega_{\text{loc}}} [r(x, a_{\text{loc}}^*(x), \omega_1) + \alpha \inf_{\mathbb{P}_1 \in \mathfrak{P}_{a_{\text{loc}}^*}} \int_{\Omega_{\text{loc}}} \cdots \int_{\Omega_{\text{loc}}} \left\{ \sum_{t=1}^{n-1} \alpha^{t-1} r(\omega_t, a_{\text{loc}}^*(\omega_t), \omega_{t+1}) \right\} \mathbb{P}_1(\omega_1; d\omega_2) \mathbb{P}_n-1(\omega_{n-1}; d\omega_n) \mathbb{P}_0(x; d\omega_1)
\]

\[
\leq \inf_{\mathbb{P}_1 \in \mathfrak{P}_{a_{\text{loc}}^*}} \int_{\Omega_{\text{loc}}} \cdots \int_{\Omega_{\text{loc}}} \left\{ \sum_{t=0}^{n-1} \alpha^t r(\omega_t, a_{\text{loc}}^*(\omega_t), \omega_{t+1}) \right\} \mathbb{P}_1(\omega_1; d\omega_2) \mathbb{P}_n-1(\omega_{n-1}; d\omega_n) \mathbb{P}_0(x; d\omega_1)
\]

\[
= \inf_{\mathbb{P} \in \mathfrak{P}_{x,a^*}} \mathbb{E}_{\mathbb{P}} \left[ \sum_{t=0}^{n-1} \alpha^t r(X_t, a_{\text{loc}}^*(X_t), X_{t+1}) + \alpha^n v(X_n) \right].
\]

According to Equation (36), we have for all \( n \in \mathbb{N} \) that

\[
v(x) = \mathcal{T} v(x) = \mathcal{T}^n v(x) \leq \inf_{\mathbb{P} \in \mathfrak{P}_{x,a^*}} \mathbb{E}_{\mathbb{P}} \left[ \sum_{t=0}^{n-1} \alpha^t r(X_t, a_{\text{loc}}^*(X_t), X_{t+1}) + \alpha^n v(X_n) \right]. \tag{39}
\]

Further, we obtain for all \( \mathbb{P} = \delta_x \otimes \mathbb{P}_0 \otimes \mathbb{P}_1 \cdots \in \mathfrak{P}_{x,a^*} \) and \( n \in \mathbb{N} \) by Equations (1) and (3) that

\[
\mathbb{E}_{\mathbb{P}}[|v(X_n)|] \leq \mathbb{E}_{\mathbb{P}} \left[ ||v||_{C_p} (1 + ||X_n||^p) \right]
\]

\[
= ||v||_{C_p} \int_{\Omega_{\text{loc}}} \cdots \int_{\Omega_{\text{loc}}} (1 + ||\omega_n||^p) \mathbb{P}_{n-1}(\omega_{n-1}; d\omega_n) \cdots \mathbb{P}_1(\omega_1; d\omega_2) \mathbb{P}_0(x; d\omega_1)
\]

\[
\leq ||v||_{C_p} \int_{\Omega_{\text{loc}}} \cdots \int_{\Omega_{\text{loc}}} C_p (1 + ||\omega_{n-1}||^p) \mathbb{P}_{n-2}(\omega_{n-2}; d\omega_{n-1}) \cdots \mathbb{P}_1(\omega_1; d\omega_2) \mathbb{P}_0(x; d\omega_1)
\]

\[
= ||v||_{C_p} \mathbb{E}_{\mathbb{P}}[C_p (1 + ||X_{n-1}||^p)].
\]

Therefore, by repeating this argument, we obtain that

\[
\mathbb{E}_{\mathbb{P}}[|v(X_n)|] \leq ||v||_{C_p} \mathbb{E}_{\mathbb{P}}[C_p (1 + ||X_{n-1}||^p)] \leq \cdots \leq ||v||_{C_p} C_p^n (1 + ||x||^p).
\]
Note that Assumption 2.4 implies $0 < C_P \cdot \alpha < 1$. When letting $n \to \infty$, we thus have for all $\mathbb{P} \in \mathcal{P}_{x,a}$ that
\[
0 \leq \limsup_{n \to \infty} \mathbb{E}_{\mathbb{P}}[\alpha^n | v(X_n) |] \leq \|v\|_{C_P} (1 + \|x\|^p) \cdot \limsup_{n \to \infty} (C_P \cdot \alpha)^n = 0. \quad (40)
\]

Moreover, note that by the growth condition on $r$ in Assumption 2.4 (iii), we have for each $n \in \mathbb{N}$ that
\[
\sum_{t=0}^{n-1} \alpha^t r(X_t, a_{loc}^*(X_t), X_{t+1}) \leq \sum_{t=0}^{\infty} \alpha^t C_r (1 + \|X_t\|^p + \|X_{t+1}\|^p). \quad (41)
\]

Furthermore, for all $\mathbb{P} \equiv \delta_x \otimes \mathbb{P}_0 \otimes \mathbb{P}_1 \cdots \in \mathcal{P}_{x,a^*}$, by using Equation (3) and Beppo Levi’s theorem, we have that
\[
\begin{align*}
\mathbb{E}_{\mathbb{P}} \left[ \sum_{t=0}^{\infty} \alpha^t C_r (1 + \|X_t\|^p + \|X_{t+1}\|^p) \right] \\
= \sum_{t=0}^{\infty} \mathbb{E}_{\mathbb{P}} \left[ \alpha^t C_r (\|X_t\|^p + 1 + \|X_{t+1}\|^p) \right] \\
= \sum_{t=0}^{\infty} \int_{\Omega_{loc}} \cdots \int_{\Omega_{loc}} \alpha^t C_r (\|\omega_t\|^p + 1 + \|\omega_{t+1}\|^p) \mathbb{P}_t (\omega_t; d\omega_{t+1}) \cdots \mathbb{P}_1 (\omega_1; d\omega_2) \mathbb{P}_0 (x; d\omega_1) \\
\leq \sum_{t=0}^{\infty} \int_{\Omega_{loc}} \cdots \int_{\Omega_{loc}} \alpha^t C_r (\|\omega_t\|^p + C_P (1 + \|\omega_t\|^p)) \mathbb{P}_{t-1} (\omega_{t-1}; d\omega_t) \cdots \mathbb{P}_1 (\omega_1; d\omega_2) \mathbb{P}_0 (x; d\omega_1) \\
= \sum_{t=0}^{\infty} \mathbb{E}_{\mathbb{P}} \left[ \alpha^t C_r (\|X_t\|^p + C_P (1 + \|X_t\|^p)) \right].
\end{align*}
\]
Recall that $\alpha \cdot C_P < 1$ according to Assumption 2.4 (iv). Therefore, by repeating the same arguments using Equation (3), we obtain that

$$
\mathbb{E}_p \left[ \sum_{t=0}^{\infty} \alpha^t C_r(1 + ||X_t||^p + ||X_{t+1}||^p) \right] \\
\leq \sum_{t=0}^{\infty} \alpha^t C_r(1 + C_P) \mathbb{E}_p[(1 + ||X_t||^p)] \\
= \sum_{t=0}^{\infty} \alpha^t C_r(1 + C_P) \int_{\Omega_{loc}} \cdots \int_{\Omega_{loc}} (1 + ||\omega_t||^p) \mathbb{P}_{t-1}(\omega_{t-1}; d\omega_t) \cdots \mathbb{P}_1(\omega_1; d\omega_2) \mathbb{P}_0(x; d\omega_1) \\
\leq \sum_{t=0}^{\infty} \alpha^t C_r(1 + C_P) C_p \mathbb{E}_p[(1 + ||X_{t-1}||^p)] \\
= \sum_{t=0}^{\infty} \alpha^t C_r(1 + C_P) C_p \mathbb{E}_p[(1 + ||x||^p)] \\
\leq \sum_{t=0}^{\infty} \alpha^t C_r(1 + C_P) C_p \mathbb{E}_p[(1 + ||x||^p)] \\
= \frac{C_r(1 + C_P)(1 + ||x||^p)}{1 - \alpha C_P} < \infty.
$$

Hence the dominating function in Equation (41) is integrable and we obtain, by using the dominated convergence theorem and Equation (40), that

$$
v(x) \leq \limsup_{n \to \infty} \inf_{\mathbb{P} \in \mathcal{P}_{x,a}^*} \mathbb{E}_p \left[ \sum_{t=0}^{n-1} \alpha^t r(X_t, a_{loc}^*(X_t), X_{t+1}) + \alpha^n v(X_n) \right] \\
\leq \inf_{\mathbb{P} \in \mathcal{P}_{x,a}^*} \limsup_{n \to \infty} \mathbb{E}_p \left[ \sum_{t=0}^{n-1} \alpha^t r(X_t, a_{loc}^*(X_t), X_{t+1}) \right] + \limsup_{n \to \infty} \mathbb{E}_p[\alpha^n |v(X_n)|] \\
= \inf_{\mathbb{P} \in \mathcal{P}_{x,a}^*} \mathbb{E}_p \left[ \sum_{t=0}^{\infty} \alpha^t r(X_t, a_{loc}^*(X_t), X_{t+1}) \right] \leq V(x).
$$

Next, we show the inequality $v(x) \geq V(x)$. To this end, let $\mathbb{P}_0^* : \Omega_{loc} \times A \to \mathcal{M}_1(\Omega_{loc})$ be defined as in part (i) with respect to the unique fixed point $v \in C_{\mathcal{P}_1}(\Omega_{loc}, \mathbb{R})$ of $\mathcal{T}$. Moreover, for every $a = (a_t)_{t \in \mathbb{N}_0} \in A$ let $\mathbb{P}_{x,a}^* := \delta_x \otimes \mathbb{P}_0^* \otimes \mathbb{P}_{a_1}^* \otimes \cdots$, where for $t \in \mathbb{N}$, we define $\mathbb{P}_{a_t}^* : \Omega_{loc} \ni \omega \mapsto \mathbb{P}_{0}^*(\omega_{t}, a_t(\omega_t)) \in \mathcal{P}(\omega_t, a_t(\omega_t))$. Thus, we have, by using the dominated convergence theorem with...
the same dominating function as in Equation (41), that

\[
V(x) = \sup_{a \in A} \inf_{P \in \mathbb{P}_x,a} \mathbb{E}_P \left[ \sum_{t=0}^{\infty} \alpha^t r(X_t, a_t(X_t), X_{t+1}) \right]
\]

\[
\leq \sup_{a \in A} \mathbb{E}_{P^*_x,a} \left[ \sum_{t=0}^{\infty} \alpha^t r(X_t, a_t(X_t), X_{t+1}) \right]
\]

\[
= \sup_{a \in A} \sum_{t=0}^{\infty} \mathbb{E}_{P^*_x,a} [\alpha^t r(X_t, a_t(X_t), X_{t+1})]
\]

\[
= \sup_{a \in A} \sum_{t=0}^{\infty} \left( \alpha^t \mathbb{E}_{P^*_x,a} [r(X_t, a_t(X_t), X_{t+1}) + \alpha v(X_{t+1})] - \mathbb{E}_{P^*_x,a} [\alpha^{t+1} v(X_{t+1})] \right)
\]

\[
= \sup_{a \in A} \sum_{t=0}^{\infty} \left( \alpha^t \int_{\Omega_{loc}} \cdots \int_{\Omega_{loc}} r(\omega_t, a_t(\omega_t), \omega_{t+1}) + \alpha v(\omega_{t+1}) \mathbb{P}^*_0(\omega_t, a_t(\omega_t); d\omega_{t+1}) \right.
\]

\[
- \mathbb{E}_{P^*_x,a} [\alpha^{t+1} v(X_{t+1})]
\]

(45)

Moreover, by using the results from part (i), we have for all \( \omega_t \in \Omega_{loc} \)

\[
\int_{\Omega_{loc}} r(\omega_t, a_t(\omega_t), \omega_{t+1}) + \alpha v(\omega_{t+1}) \mathbb{P}^*_0(\omega_t, a_t(\omega_t); d\omega_{t+1})
\]

\[
= \inf_{P_0 \in \mathcal{P}(a_t(\omega_t))} \int_{\Omega_{loc}} r(\omega_t, a_t(\omega_t), \omega_{t+1}) + \alpha v(\omega_{t+1}) \mathbb{P}^*_0(\omega_t, a_t(\omega_t); d\omega_{t+1})
\]

\[
\leq \sup_{a_{loc} \in A_{loc}} \inf_{P_0 \in \mathcal{P}(\omega_t, a_{loc}(\omega_t))} \int_{\Omega_{loc}} r(\omega_t, a_{loc}(\omega_t), \omega_{t+1}) + \alpha v(\omega_{t+1}) \mathbb{P}^*_0(\omega_t, a_{loc}(\omega_t); d\omega_{t+1})
\]

\[
= \int_{\Omega_{loc}} r(\omega_t, a_{loc}^*(\omega_t), \omega_{t+1}) + \alpha v(\omega_{t+1}) \mathbb{P}^*_0(\omega_t, a_{loc}^*(\omega_t); d\omega_{t+1}) = T v(\omega_t) = v(\omega_t).
\]

(46)

Hence, we obtain with Equations (45) and (46) that

\[
V(x) \leq \sup_{a \in A} \sum_{t=0}^{\infty} \left( \alpha^t \int_{\Omega_{loc}} v(\omega_t) \mathbb{P}_0^*(\omega_{t-1}, a_{t-1}(\omega_{t-1}); d\omega_t) \cdots \mathbb{P}_0^*(x, a_0(x); d\omega_1) \right.
\]

\[
- \mathbb{E}_{P^*_x,a} [\alpha^{t+1} v(X_{t+1})]
\]

\[
= \sup_{a \in A} \sum_{t=0}^{\infty} \left( \alpha^t \mathbb{E}_{P^*_x,a} [v(X_t)] - \alpha^{t+1} \mathbb{E}_{P^*_x,a} [v(X_{t+1})] \right)
\]

\[
= \sup_{a \in A} v(x) = v(x).
\]

This shows \( V(x) = v(x) \). Eventually, to see that the first line of Equation (13) holds, we compute by using Equation (46), the definition \( \mathbb{P}_x^* := \delta_x \otimes \mathbb{P}_{loc}^* \otimes \mathbb{P}_{loc}^* \otimes \cdots \) as well as the dominated
convergence theorem that

\[ v(x) = \sum_{t=0}^{\infty} \left( \alpha^t E_{\mathbb{P}_x^t}[v(X_t)] - \alpha^{t+1} E_{\mathbb{P}_x^t}[v(X_{t+1})] \right) \]

\[ = \sum_{t=0}^{\infty} \left( \alpha^t E_{\mathbb{P}_x^t}[r(X_t, a_{loc}(X_t), X_{t+1}) + \alpha v(X_{t+1})] - \alpha^{t+1} E_{\mathbb{P}_x^t}[v(X_{t+1})] \right) \]

\[ = \sum_{t=0}^{\infty} E_{\mathbb{P}_x^t}[\alpha^t r(X_t, a_{loc}(X_t), X_{t+1})] = E_{\mathbb{P}_x^t}\left[ \sum_{t=0}^{\infty} \alpha^t r(X_t, a_{loc}(X_t), X_{t+1}) \right]. \]

Moreover, by Equation (44), as we have shown that \( V = v \), we obtain that

\[ V(x) = \inf_{\mathbb{P} \in \mathcal{P}_{x,a}} E_{\mathbb{P}}\left[ \sum_{t=0}^{\infty} \alpha^t r(X_t, a_{loc}(X_t), X_{t+1}) \right]. \] (47)

\[ \square \]

6 PROOF OF RESULTS IN SECTION 3

6.1 Proof of results in Section 3.1

Proof of Proposition 3.1. Let \( x \in \Omega_{\text{loc}}, a \in A \).

We see that \( P(x, a) \) is nonempty since the measure \( \hat{\mathbb{P}}(x, a) \) is contained in \( P(x, a) \) by definition of the \( q \)-Wasserstein-ball.

The compactness of \( B_{2q}^q(\hat{\mathbb{P}}(x, a)) \) with respect to \( \tau_0 \), which is the topology induced by the weak convergence of measures, follows from, for example, Yue et al. (2020, Theorem 1), where we use the assumption that \( \hat{\mathbb{P}}(x, a) \) has finite \( q \)th moments.

To show the upper hemicontinuity of \( P \), we apply Lemma A.4. Let \( (x^{(n)}, a^{(n)})_{n \in \mathbb{N}} \subseteq \Omega_{\text{loc}} \times A \) such that \( (x^{(n)}, a^{(n)}) \rightarrow (x, a) \in \Omega_{\text{loc}} \times A \) for \( n \rightarrow \infty \). Further, consider a sequence \( (\mathbb{P}^{(n)})_{n \in \mathbb{N}} \) such that \( \mathbb{P}^{(n)} \in B_{2q}^q(\hat{\mathbb{P}}(x^{(n)}, a^{(n)})) \) for all \( n \in \mathbb{N} \), that is, we have \( ((x^{(n)}, a^{(n)}), \mathbb{P}^{(n)})_{n \in \mathbb{N}} \subseteq \text{Gr} P \).

Let \( (\delta_n)_{n \in \mathbb{N}} \subseteq (0, 1) \) with \( \lim_{m \rightarrow \infty} \delta_n = 0 \). Note that, since \( \Omega_{\text{loc}} \times A \ni (x, a) \mapsto \hat{\mathbb{P}}(x, a) \) is, by assumption, continuous in \( \tau_q \), we have \( \lim_{n \rightarrow \infty} W_q(\hat{\mathbb{P}}(x, a), \hat{\mathbb{P}}(x^{(n)}, a^{(n)})) = 0 \). Hence, there exists a subsequence \( (\hat{\mathbb{P}}(x^{(n_k)}, a^{(n_k)}))_{k \in \mathbb{N}} \) such that

\[ W_q\left( \hat{\mathbb{P}}(x, a), \hat{\mathbb{P}}(x^{(n_k)}, a^{(n_k)}) \right) < \delta_k \cdot \varepsilon \text{ for all } k \in \mathbb{N}. \] (48)

This implies for each \( \mathbb{P}^{(n_k)}, k \in \mathbb{N} \), that

\[ W_q\left( \hat{\mathbb{P}}(x, a), \mathbb{P}^{(n_k)} \right) \leq W_q\left( \hat{\mathbb{P}}(x, a), \hat{\mathbb{P}}(x^{(n_k)}, a^{(n_k)}) \right) + W_q\left( \hat{\mathbb{P}}(x^{(n_k)}, a^{(n_k)}), \mathbb{P}^{(n_k)} \right) \leq \delta_k \cdot \varepsilon + \varepsilon \leq 2\varepsilon. \]

Hence, \( \mathbb{P}^{(n_k)} \in B_{2\varepsilon}^q(\hat{\mathbb{P}}(x, a)) \) for all \( k \in \mathbb{N} \). By the compactness of \( B_{2\varepsilon}^q(\hat{\mathbb{P}}(x, a)) \) in \( \tau_0 \), there exists a subsequence \( (\mathbb{P}^{(n_{k\ell})})_{\ell \in \mathbb{N}} \) such that \( \mathbb{P}^{(n_{k\ell})} \xrightarrow{\tau_0} \mathbb{P} \) as \( \ell \rightarrow \infty \) for some \( \mathbb{P} \in B_{2\varepsilon}^q(\hat{\mathbb{P}}(x, a)) \). In par-
ticular, since by assumption \( \hat{P}(x, a) \) possesses finite \( q \)th moments, \( P \) has also finite \( q \)th moments, see Yue et al. (2020, Lemma 1). It remains to prove that \( P \in B_{\varepsilon}^{(q)}(\hat{P}(x, a)) \). To that end, define for each \( k \in \mathbb{N} \)

\[
\hat{P}(n_k) := (1 - \delta_k) \cdot P(n_k) + \delta_k \cdot \hat{P}(x(n_k), a(n_k)).
\] (49)

Then, for each \( k \in \mathbb{N} \), we have

\[
\begin{align*}
W_q \left( \hat{P}(x(n_k), a(n_k)), \hat{P}(n_k) \right) & = W_q \left( (1 - \delta_k) \cdot \hat{P}(x(n_k), a(n_k)) + \delta_k \cdot \hat{P}(x(n_k), a(n_k)), (1 - \delta_k) \cdot P(n_k) + \delta_k \cdot \hat{P}(x(n_k), a(n_k)) \right) \\
& = (1 - \delta_k) \cdot W_q \left( \hat{P}(x(n_k), a(n_k)), P(n_k) \right) \leq (1 - \delta_k) \cdot \varepsilon.
\end{align*}
\] (50)

Therefore, by Equations (48) and (50), we have for each \( \ell \in \mathbb{N} \) that

\[
\begin{align*}
W_q \left( \hat{P}(x, a), \hat{P}(n_k \ell) \right) & \leq W_q \left( \hat{P}(x, a), \hat{P}(n_k \ell) \right) + W_q \left( \hat{P}(x(n_k \ell), a(n_k \ell)), \hat{P}(n_k \ell) \right) \\
& \leq \delta_{k \ell} \cdot \varepsilon + (1 - \delta_{k \ell}) \cdot \varepsilon = \varepsilon.
\end{align*}
\] (51)

Furthermore, we have by Equation (49) that

\[
\lim_{\ell \to \infty} \hat{P}(n_k \ell) = \lim_{\ell \to \infty} P(n_k \ell) = P \text{ in } \tau_0.
\] (52)

Since \( \mu \mapsto W_q (\hat{P}(x, a), \mu) \) is lower semicontinuous in \( \tau_0 \), see [?], Corollary 5.3], we obtain from Equations (51) and (52) that

\[
W_q \left( \hat{P}(x, a), P \right) \leq \liminf_{\ell \to \infty} W_q \left( \hat{P}(x, a), \hat{P}(n_k \ell) \right) \leq \varepsilon,
\]

and hence \( P \in B_{\varepsilon}^{(q)}(\hat{P}(x, a)) \). The assertion that \( P \) is upper hemicontinuous follows now with the characterization of upper hemicontinuity provided in Lemma A.4.

To show the lower hemicontinuity of \( P \), we first define the set-valued map

\[
\hat{P} : \Omega_{\text{loc}} \times A \ni (x, a) \mapsto B_{\varepsilon}^{(q)}(\hat{P}(x, a)) := \left\{ P \in M_1(\Omega_{\text{loc}}) \mid W_q(P, \hat{P}(x, a)) < \varepsilon \right\}
\]

and conclude the lower hemicontinuity of \( \hat{P} \) with Lemma A.5. To this end, we consider a sequence \( (x(n), a(n))_{n \in \mathbb{N}} \subset \Omega_{\text{loc}} \times A \) such that \( (x(n), a(n)) \to (x, a) \in \Omega_{\text{loc}} \times A \) for \( n \to \infty \), and we consider some \( P \in \hat{P}(x, a) = B_{\varepsilon}^{(q)}(\hat{P}(x, a)) \). Note that since \( B_{\varepsilon}^{(q)}(\hat{P}(x, a)) \) is defined as an open ball with respect to \( \tau_q \), there exists some \( 0 < \delta < \varepsilon \) such that \( P \in B_{\varepsilon - \delta}^{(q)}(\hat{P}(x, a)) \). We define for \( n \in \mathbb{N} \) the measure

\[
P(n) := \begin{cases} 
\hat{P}(x(n), a(n)), & \text{if } W_q \left( \hat{P}(x(n), a(n)), \hat{P}(x, a) \right) \geq \delta \\
\mu, & \text{else.}
\end{cases}
\]
Then, we claim that $\mathbb{P}^{(n)} \in \hat{\mathbb{P}}((x^{(n)}, a^{(n)}))$ for all $n \in \mathbb{N}$. Indeed, if $W_q(\hat{P}(x^{(n)}, a^{(n)}), \hat{P}(x, a)) \geq \delta$, this follows by definition of $\mathbb{P}^{(n)}$, whereas if $W_q(\hat{P}(x^{(n)}, a^{(n)}), \hat{P}(x, a)) < \delta$, then $\mathbb{P}^{(n)} = \mathbb{P}$, and hence by the triangle inequality

$$W_q\left(\mathbb{P}, \hat{P}(x^{(n)}, a^{(n)})\right) \leq W_q\left(\mathbb{P}, \hat{P}(x, a)\right) + W_q\left(\hat{P}(x, a), \hat{P}(x^{(n)}, a^{(n)})\right) < (\varepsilon - \delta) + \delta = \varepsilon.$$ 

By the continuity of $(x, a) \mapsto \hat{P}(x, a)$ in $\tau_q$, we have that $\hat{P}(x^{(n)}, a^{(n)}) \xrightarrow{\tau_q} \hat{P}(x, a)$ as $n \to \infty$. Thus, there exists some $N \in \mathbb{N}$ such that we have $\mathbb{P}^{(n)} = \mathbb{P}$ for all $n \geq N$ and thus, in particular $\mathbb{P}^{(n)} \to \mathbb{P}$ weakly for $n \to \infty$, which concludes the lower hemicontinuity of $\hat{P}$ with Lemma A.5. Next, we claim that the $\tau_0$-closure of $\hat{B}_x^{(q)}(\hat{P}(x, a))$, denoted by $\text{cl}_{\tau_0}(\hat{B}_x^{(q)}(\hat{P}(x, a)))$, coincides with $B_x^{(q)}(\hat{P}(x, a))$. Indeed, the inclusion $B_x^{(q)}(\hat{P}(x, a)) \subseteq \text{cl}_{\tau_0}(\hat{B}_x^{(q)}(\hat{P}(x, a)))$ holds, since $\text{cl}_{\tau_0}(\hat{B}_x^{(q)}(\hat{P}(x, a)))$ is closed in $\tau_0$ and hence also in $\tau_q$. To show the reverse inclusion $\text{cl}_{\tau_0}(\hat{B}_x^{(q)}(\hat{P}(x, a))) \subseteq B_x^{(q)}(\hat{P}(x, a))$, let $\mathbb{P} \in \text{cl}_{\tau_0}(\hat{B}_x^{(q)}(\hat{P}(x, a)))$. Then, there exists a sequence $(\hat{P}^{(n)})_{n \in \mathbb{N}} \subseteq \hat{B}_x^{(q)}(\hat{P}(x, a))$ with $\mathbb{P}^{(n)} \xrightarrow{\tau_0} \mathbb{P}$ as $n \to \infty$. Hence, by using the lower semicontinuity of $\mu \mapsto W_q(\mu, \hat{P}(x, a))$ with respect to $\tau_0$, we obtain

$$W_q\left(\mathbb{P}, \hat{P}(x, a)\right) \leq \liminf_{n \to \infty} W_q\left(\mathbb{P}^{(n)}, \hat{P}(x, a)\right) \leq \varepsilon.$$ 

Hence, $\text{cl}_{\tau_0}(\hat{B}_x^{(q)}(\hat{P}(x, a))) = B_x^{(q)}(\hat{P}(x, a))$ and Aliprantis and Border (2006, Lemma 17.22) implies that the set-valued map $\mathcal{P} : \Omega_{\text{loc}} \times A \ni (x, a) \mapsto \text{cl}_{\tau_0}(\hat{B}_x^{(q)}(\hat{P}(x, a)))$ is lower hemicontinuous.

Eventually, since $p = 0$, the growth constraint (3) is automatically fulfilled.

\[ \square \]

6.2 Proof of results in Section 3.2

Proof of Proposition 3.2. Let $(x, a) \in \Omega_{\text{loc}} \times A$.

The nonemptiness of $\mathcal{P}(x, a)$ follows directly since $\Theta$ is nonempty.

To show the compactness of $\mathcal{P}(x, a)$, let $(\mathbb{P}^{(n)})_{n \in \mathbb{N}} \subseteq \mathcal{P}(x, a)$, that is, for all $n \in \mathbb{N}$, we have $\mathbb{P}^{(n)} = \hat{P}(x, a, \Theta^{(n)})$ for some $\Theta^{(n)} \in \Theta(x, a)$. The compactness of $\Theta(x, a)$ implies the existence of a subsequence $(\Theta^{(n_k)})_{k \in \mathbb{N}} \subseteq \Theta(x, a)$ such that $\Theta^{(n_k)} \to \Theta \in \Theta(x, a)$ for $k \to \infty$. Hence, since $\hat{P}$ is continuous, it follows $\hat{P}(x, a, \Theta^{(n_k)}) \to \hat{P}(x, a, \Theta) \in \mathcal{P}(x, a)$ in $\tau_p$ for $k \to \infty$.

We apply Lemma A.4 to show the upper hemicontinuity of $\mathcal{P}$. To this end, consider a sequence $(x^{(n)}, a^{(n)})_{n \in \mathbb{N}} \subseteq \Omega_{\text{loc}} \times A$ with $\lim_{n \to \infty} (x^{(n)}, a^{(n)}) = (x, a)$ and a sequence $(\mathbb{P}^{(n)})_{n \in \mathbb{N}}$ with $\mathbb{P}^{(n)} \in \mathcal{P}(x^{(n)}, a^{(n)})$ for all $n \in \mathbb{N}$. We have a representation $\mathbb{P}^{(n)} = \hat{P}(x^{(n)}, a^{(n)}, \Theta^{(n)})$ for some $\Theta^{(n)} \in \Theta(x^{(n)}, a^{(n)})$ for all $n \in \mathbb{N}$. Then, since $\Theta$ is upper hemicontinuous, there exists a subsequence $(\Theta^{(n_k)})_{k \in \mathbb{N}}$ with $\Theta^{(n_k)} \in \Theta(x^{(n_k)}, a^{(n_k)})$ for all $k \in \mathbb{N}$ such that $\Theta^{(n_k)} \to \Theta$ for $k \to \infty$ for some $\Theta \in \Theta(x, a)$. Hence with the continuity of $\hat{P}$ it follows $\mathbb{P}^{(n_k)} \to \mathbb{P} := \hat{P}(x, a, \Theta) \in \mathcal{P}(x, a)$ in $\tau_p$ for $k \to \infty$.

To show the lower hemicontinuity, we let $(x^{(n)}, a^{(n)})_{n \in \mathbb{N}} \subseteq \Omega_{\text{loc}} \times A$ with $\lim_{n \to \infty} (x^{(n)}, a^{(n)}) = (x, a)$ and $\mathbb{P} := \hat{P}(x, a, \Theta) \in \mathcal{P}(x, a)$ for some $\Theta \in \Theta(x, a)$. Then, the lower hemicontinuity of $\Theta$ implies the existence of a subsequence $(x^{(n_k)}, a^{(n_k)})_{k \in \mathbb{N}}$ and of a sequence $(\Theta^{(k)})_{k \in \mathbb{N}}$ with $\Theta^{(k)} \in \Theta(x^{(n_k)}, a^{(n_k)})$ for all $k \in \mathbb{N}$ such that $\Theta^{(k)} \to \Theta$ for $k \to \infty$. Hence, it follows with the continuity
of $\hat{P}$ that $P(x^{(n_k)}, a^{(n_k)}) \ni P^{(k)} := \hat{P}(x^{(n_k)}, a^{(n_k)}, \theta^{(k)}) \to P$ for $k \to \infty$, implying with Lemma A.5 the lower hemicontinuity of $P$.

\[ \square \]

6.3 Proof of results in Section 3.3

Before reporting the proof of Proposition 3.3, we establish the following lemma.

**Lemma 6.1.** Let $D \in \mathbb{N}$ and let $Z \subseteq \mathbb{R}^D$ be closed. Moreover, let $D := \{\delta_x \ | \ x \in Z^{m-1}\} \subseteq (\mathcal{M}_1(Z^{m-1}), \tau_p)$ be the closed subset consisting of all Dirac measures on $Z^{m-1}$. Then, for any $p \in \{0, 1\}$, the map

$$\varphi : (D, \tau_p) \times (\mathcal{M}_1(Z), \tau_p) \to (\mathcal{M}_1(Z^m), \tau_p)$$

$$(\delta_x, P) \mapsto \delta_x \otimes P$$

is continuous.

**Proof.** We show the sequential continuity of the map $\varphi$. First, we consider the case $p = 0$. Let $\delta_x \in D$ for some $x \in Z^{m-1}$ and $P \in \mathcal{M}_1(Z)$, and let $(\delta_x^{(n)})_{n \in \mathbb{N}} \subseteq D$ and $(P^{(n)})_{n \in \mathbb{N}} \subseteq \mathcal{M}_1(Z)$ with $\delta_x^{(n)} \xrightarrow{\tau_0} \delta_x \in D$ and $P^{(n)} \xrightarrow{\tau_0} P \in \mathcal{M}_1(Z)$ for $n \to \infty$. Now, let $f : Z^m \to \mathbb{R}$ be Lipschitz continuous with Lipschitz constant $L > 0$. Then we have

$$\lim_{n \to \infty} \left| \int_{Z^m} f(y, z) \delta_x^{(n)}(dy) \otimes P^{(n)}(dz) - \int_{Z^m} f(y, z) \delta_x(dy) \otimes P(dz) \right| = 0,$$

where the second summand in Equation (53) vanishes due to $P^{(n)} \xrightarrow{\tau_0} P$. By Jacod and Protter (2003, Theorem 18.7) we conclude that $\varphi$ is (sequential) continuous.

Now, we consider the case $p = 1$. Let $\delta_x \in D$ for some $x \in Z^{m-1}$ and $P \in \mathcal{M}_1(Z)$, and let $(\delta_x^{(n)})_{n \in \mathbb{N}} \subseteq D$ and $(P^{(n)})_{n \in \mathbb{N}} \subseteq \mathcal{M}_1(Z)$ with $\delta_x^{(n)} \xrightarrow{\tau_1} \delta_x \in D$ and $P^{(n)} \xrightarrow{\tau_1} P \in \mathcal{M}_1(Z)$ for $n \to \infty$. Since convergence in $\tau_1$ implies convergence in $\tau_0$, we obtain, by the already considered case $p = 0$, that $\delta_x^{(n)} \otimes P^{(n)} \xrightarrow{\tau_0} \delta_x \otimes P$ for $n \to \infty$. It remains to show that the convergence also follows with respect to $\tau_1$. To conclude the convergence in $\tau_1$ it suffices, by Villani (2009, Theorem 6.9), to show that

$$\lim_{n \to \infty} \int_{Z^m} \|y, z\| \delta_x^{(n)}(dy) \otimes P^{(n)}(dz) = \int_{Z^m} \|y, z\| \delta_x(dy) \otimes P(dz).$$
To see this, note that

\[
\lim_{n \to \infty} \left| \int_{Z_m} (y, z) \| \delta_{\chi_n}(dy) \otimes \mathbb{P}(n)(dz) - \int_{Z_m} (y, z) \| \delta_{\chi}(dy) \otimes \mathbb{P}(dz) \right| = \lim_{n \to \infty} \left| \int_{Z} (\chi_n, z) \| \mathbb{P}(n)(dz) - \int_{Z} (\chi, z) \| \mathbb{P}(dz) \right| \\
\leq \lim_{n \to \infty} \left( \int_{Z} (\chi_n, z) \| - \| (\chi, z) \| \mathbb{P}(n)(dz) + \int_{Z} (\chi, z) \| \mathbb{P}(n)(dz) - \int_{Z} (\chi, z) \| \mathbb{P}(dz) \right) \\
\leq \lim_{n \to \infty} \left( (\chi_n - \chi) + \int_{Z} (\chi, z) \| \mathbb{P}(n)(dz) - \int_{Z} (\chi, z) \| \mathbb{P}(dz) \right) = 0,
\]

where we use that \( Z \ni y \mapsto \| (x, y) \| \in C_1(Z, \mathbb{R}) \) and \( \mathbb{P}(n) \xrightarrow{\tau_1} \mathbb{P} \) for \( n \to \infty \). 

\[ \square \]

**Proof of Proposition 3.3.** Let \((x, a) \in \Omega_{\text{loc}} \times A\).

It is immediate that \( \mathcal{P}(x, a) \neq \emptyset \), since \( \overline{\mathcal{P}}(x, a) \neq \emptyset \) by assumption.

To show the compactness of \( \mathcal{P}(x, a) \), we consider a sequence \((\mathbb{P}_{n})_{n \in \mathbb{N}} \subseteq \mathcal{P}(x, a)\), where for all \( n \in \mathbb{N} \), we have \( \mathbb{P}_n = \delta_{\pi(x)} \otimes \overline{\mathbb{P}}(n) \) for some \( \overline{\mathbb{P}}(n) \in \overline{\mathcal{P}}(x, a) \). Then, by the compactness of \( \overline{\mathcal{P}}(x, a) \), there exists a subsequence \((\overline{\mathbb{P}}(n_k))_{k \in \mathbb{N}} \) such that \( \overline{\mathbb{P}}(n_k) \rightarrow \overline{\mathbb{P}} \in \overline{\mathcal{P}}(x, a) \) in \( \tau_\pi \) as \( k \rightarrow \infty \). Now, let \( g \in C_p(\Omega_{\text{loc}}, \mathbb{R}) \). Then the map \( Z \ni y \mapsto g(\pi(x), y) \) is contained in \( C_p(Z, \mathbb{R}) \), and hence

\[
\lim_{k \to \infty} \int_{\Omega_{\text{loc}}} g(z)\overline{\mathbb{P}}(n_k)(dz) = \lim_{k \to \infty} \int_{Z} g(\pi(x), y)\overline{\mathbb{P}}(n_k)(dy) = \int_{Z} g(\pi(x), y)\overline{\mathbb{P}}(dy) = \int_{\Omega_{\text{loc}}} g(z)\mathbb{P}(dz)
\]

for \( \mathbb{P} := \delta_{\pi(x)} \otimes \overline{\mathbb{P}} \in \mathcal{P}(x, a) \), which proves the compactness of \( \mathcal{P}(x, a) \).

To show the upper hemicontinuity of \( \mathcal{P} \), let \((x^{(n)}, a^{(n)}) \subseteq \Omega_{\text{loc}} \times A \) with \((x^{(n)}, a^{(n)}) \rightarrow (x, a)\) for \( n \to \infty \), and let \( \mathbb{P}^{(n)} \in \mathcal{P}(x^{(n)}, a^{(n)}) \) for all \( n \in \mathbb{N} \). Then, we have a representation \( \mathbb{P}^{(n)} = \delta_{\pi(x^{(n)})} \otimes \overline{\mathbb{P}}(n) \) with \( \overline{\mathbb{P}}(n) \in \overline{\mathcal{P}}(x^{(n)}, a^{(n)}) \) for all \( n \in \mathbb{N} \). By the upper hemicontinuity of \( \overline{\mathcal{P}} \), there exists, according to Lemma A.4, a subsequence \((\overline{\mathbb{P}}(n_k))_{k \in \mathbb{N}} \) with \( \overline{\mathbb{P}}(n_k) \rightarrow \overline{\mathbb{P}} \in \overline{\mathcal{P}}(x, a) \) in \( \tau_\pi \) as \( k \rightarrow \infty \). Moreover, \( \delta_{\pi(x^{(n_k)})} \rightarrow \delta_{\pi(x)} \) as \( n \to \infty \).

We apply Lemma 6.1 and obtain that \( \delta_{\pi(x^{(n_k)})} \otimes \overline{\mathbb{P}}(n_k) \rightarrow \delta_{\pi(x)} \otimes \overline{\mathbb{P}} \in \mathcal{P}(x, a) \), and hence the upper hemicontinuity follows with Lemma A.4.

To prove the lower hemicontinuity of \( \mathcal{P} \), we consider again a sequence \((x^{(n)}, a^{(n)}) \subseteq \Omega_{\text{loc}} \times A \) with \((x^{(n)}, a^{(n)}) \rightarrow (x, a)\) for \( n \to \infty \), and some \( \mathbb{P} \in \mathcal{P}(x, a) \) with a representation \( \mathbb{P} = \delta_{\pi(x)} \otimes \overline{\mathbb{P}} \) for \( \overline{\mathbb{P}} \in \overline{\mathcal{P}}(x, a) \). By the lower hemicontinuity of \( \overline{\mathcal{P}} \), there exists, according to Lemma A.5, a subsequence \((x^{(n_k)}, a^{(n_k)})_{k \in \mathbb{N}} \) and \( \overline{\mathbb{P}}(n_k) \in \overline{\mathcal{P}}(x^{(n_k)}, a^{(n_k)}) \) for all \( k \in \mathbb{N} \) such that \( \overline{\mathbb{P}}(n_k) \rightarrow \overline{\mathbb{P}} \) in \( \tau_\pi \). Then, we set \( \mathbb{P}^{(n_k)} := \delta_{\pi(x^{(n_k)})} \otimes \overline{\mathbb{P}}(n_k) \in \mathcal{P}(x^{(n_k)}, a^{(n_k)}) \) for all \( k \in \mathbb{N} \), and we conclude \( \mathbb{P}^{(n_k)} \rightarrow \mathbb{P} \) in \( \tau_\pi \) for \( k \to \infty \) with Lemma 6.1. Hence, the lower hemicontinuity follows with Lemma A.5. 

\[ \square \]
7 | PROOF OF RESULTS IN SECTION 4

7.1 | Proof of results in Section 4.2

Proof of Proposition 4.1. Since Assumption 2.2 (ii) is automatically fulfilled for \( p = 0 \), the fulfillment of Assumption 2.2 follows from Propositions 3.1 and 3.3, once we have shown that \( \Omega_{\text{loc}} \ni x \mapsto \hat{P}(x) \in \mathcal{M}_1(Z) \) is continuous in \( \tau_q \) and possesses finite \( q \)th moments.

To show the (sequential) continuity of \( \hat{P} \), let \( X_t \in \Omega_{\text{loc}} \) and let \( (X_t^{(n)})_{n \in \mathbb{N}} \subseteq \Omega_{\text{loc}} \) be a sequence with \( X_t^{(n)} \to X_t \in \Omega_{\text{loc}} \) for \( n \to \infty \). By construction \( \Omega_{\text{loc}} \ni x \mapsto \pi_s(x) \in [0, 1] \) is continuous for all \( s = m, \ldots, N-1 \), which implies for all \( g \in C^q(Z, \mathbb{R}) \)

\[
\lim_{n \to \infty} \int_Z g(y)\hat{P}(X_t^{(n)}; dy) = \lim_{n \to \infty} \sum_{s=m}^{N-1} \pi_s(X_t^{(n)}) g(\mathcal{R}_{s+1}) = \sum_{s=m}^{N-1} \pi_s(X_t) g(\mathcal{R}_{s+1}) = \int_Z g(y)\hat{P}(X_t; dy).
\]

Moreover, the existence of the \( q \)th moment follows by

\[
\int_Z \|y\|^q \hat{P}(X_t; dy) = \sum_{s=m}^{N-1} \pi_s(X_t) \cdot \|\mathcal{R}_{s+1}\|^q < \infty.
\]

Now, to verify Assumption 2.4 note that \( r \) is continuous and that the compactness of \( Z \) and of \( A \) imply that \( r \) is bounded, and thus Assumption 2.4 (i) and (iii) are fulfilled. Next, let \( X_t, X_t' \in \Omega_{\text{loc}}, X_{t+1} = (R_{t-m+2}, \ldots, R_{t+1}) \in \Omega_{\text{loc}}, \) and let \( a_t, a_t' \in A \). Then, by the Cauchy–Schwarz inequality, we see that

\[
|r(X_t, a_t, X_{t+1}) - r(X_t', a_t', X_{t+1})| = \left| \sum_{i=1}^{D} (a_i^T \cdot \Sigma_R \cdot a_t - a_i^T \cdot \Sigma_R \cdot a_t') \right|
\]

\[
\leq \|R_{t+1}\| \cdot \|a_t - a_t'\| + \lambda \cdot \left( |a_i^T \cdot \Sigma_R \cdot (a_t - a_t')| + |(a_t - a_t')^T \cdot \Sigma_R \cdot a_t'| \right)
\]

\[
\leq \max_{z \in Z} \|z\| \cdot \|a_t - a_t'\| + \lambda \cdot 2 \cdot \max_{b \in A} \|b\| \cdot \|\Sigma_R\|_F \cdot \|a_t - a_t'\|
\]

\[
= \left( \max_{z \in Z} \|z\| + \lambda \cdot 2 \cdot \max_{b \in A} \|b\| \cdot \|\Sigma_R\|_F \right) \cdot \|a_t - a_t'\|
\]

where \( \| \cdot \|_F \) denotes the Frobenius-norm. Hence, Assumption 2.4 (ii) is fulfilled.

7.2 | Proof of results in Section 4.3

Proof of Proposition 4.2. To show Assumption 2.2 (ii), let

\[
C_p := 1 + \sqrt{\frac{\varepsilon^2 + \frac{1}{m} + 4 \cdot \frac{\varepsilon^2 + \frac{1}{m}}{m-1}}{m}}
\]
Now we consider some \((x, a) \in \Omega_{\text{loc}} \times A\) and some \(p \in \mathcal{P}(x, a)\). Then, we have a representation of the form \(p = \delta_{\pi(x)} \otimes \bar{p}\) for some \(\bar{p} \in \bar{P}(x)\), where \(\bar{P} \sim \mathcal{N}_D(\mu, \Sigma)\) with \((\mu, \Sigma) \in \mathbb{R}^D \times \mathbb{R}^{D \times D}\) fulfilling \(\|\mu - \mathbf{m}(x)\| \leq \varepsilon\) and \(\Sigma = \mathcal{E}(y)\) for some \(y \in \Omega_{\text{loc}}\) with \(\|y - x\| \leq \varepsilon\). Therefore, we have with Jensen’s inequality that

\[
\int_{\Omega_{\text{loc}}} 1 + \|y\|p(dy) = 1 + \int_{\mathbb{R}^D} \|\pi(x), z\|\bar{p}(dz) \\
\leq 1 + \int_{\mathbb{R}^D} \|\pi(x)\| + \|z\|\bar{p}(dz) \\
\leq 1 + \|x\| + \sqrt{\int_{\mathbb{R}^D} \|z\|^2\bar{p}(dz)}.
\]

(55)

Moreover, for \(Z = (Z_1, \ldots, Z_D) \sim \mathcal{N}_D(\mu, \Sigma)\) we have

\[
\int_{\mathbb{R}^D} \|z\|^2\bar{p}(dz) = \mathbb{E}[\|Z\|^2] = \mathbb{E}\left[\sum_{i=1}^{D} Z_i^2\right] \\
= \sum_{i=1}^{D} \mathbb{E}[Z_i]^2 + \sum_{i=1}^{D} \text{Var}(Z_i) \\
= \|\mu\|^2 + \text{trace}(\Sigma).
\]

(56)

In the next step, we write \(x = (x_{ij})_{i=1,\ldots,m}^{j=1,\ldots,D}\), use the Cauchy–Schwarz inequality, and compute

\[
\|\mu\|^2 \leq (\|\mu - \mathbf{m}(x)\| + \|\mathbf{m}(x)\|)^2 \\
\leq \left(\varepsilon + \sqrt{\sum_{j=1}^{D} \left(\frac{1}{m} \sum_{i=1}^{m} x_{ij}\right)^2}\right)^2 \\
\leq \left(\varepsilon + \sqrt{\sum_{j=1}^{D} \left(\frac{1}{m} \sum_{i=1}^{m} (x_{ij})^2\right)}\right)^2 \\
= \left(\varepsilon + \frac{1}{\sqrt{m}}\|x\|\right)^2 \leq \left(\varepsilon^2 + \frac{1}{m}\right)(1 + \|x\|^2).
\]

(57)
Further, we write
\[ y = (y_i^{(j)})_{i=1,\ldots,m} , \]
and obtain with the Cauchy–Schwarz inequality that
\[
\| m(y) - m(x) \|^2 = \frac{1}{m^2} \sum_{j=1}^{D} \left( \sum_{i=1}^{m} (y_i^{(j)} - x_i^{(j)}) \right)^2 \\
\leq \frac{1}{m} \sum_{j=1}^{D} \sum_{i=1}^{m} (y_i^{(j)} - x_i^{(j)})^2 = \| y - x \|^2 \leq \frac{\varepsilon^2}{m} .
\]

The above inequality (58), and \( \| m(x) \| \leq \frac{1}{\sqrt{m}} \| x \| \) (see also Equation 57) imply together with the Cauchy–Schwarz inequality that
\[
\text{trace}(\Sigma) = \frac{1}{m-1} \sum_{i=1}^{m} \text{trace}((y_i - m(y))(y_i - m(y))^T) \\
= \frac{1}{m-1} \sum_{i=1}^{m} \sum_{j=1}^{D} (y_i^{(j)} - m(y)^{^{(j)})}^2 \\
\leq \frac{2}{m-1} \sum_{i=1}^{m} \sum_{j=1}^{D} \left( (y_i^{(j)})^2 + (m(y)^{^{(j)})^2 \right) \\
= \frac{2}{m-1} \| y \|^2 + \frac{2m}{m-1} \| m(y) \|^2 \\
\leq \frac{2}{m-1} (\| y - x \| + \| x \|)^2 + \frac{2m}{m-1} (\| m(y) - m(x) \| + \| m(x) \|)^2 \\
\leq \frac{2}{m-1} (\varepsilon + \| x \|)^2 + \frac{2m}{m-1} \left( \frac{\varepsilon}{\sqrt{m}} + \frac{1}{\sqrt{m}} \| x \| \right)^2 \\
\leq \frac{2}{m-1} \left( \varepsilon^2 + 1 \right) (1 + \| x \|^2) + \frac{2m}{m-1} \cdot \frac{\varepsilon^2 + 1}{m} \cdot (1 + \| x \|^2) \\
= 4 \cdot \frac{\varepsilon^2 + 1}{m-1} \cdot (1 + \| x \|^2).
\]

Hence, by combining Equations (55)–(57) and (59), we have
\[
\int_{\Omega_{\text{loc}}} 1 + \| y \| \, dP(dy) \leq 1 + \| x \| + \sqrt{\left( \varepsilon^2 + \frac{1}{m} \right) (1 + \| x \|^2) + 4 \cdot \frac{\varepsilon^2 + 1}{m-1} \cdot (1 + \| x \|^2)} \\
\leq \left( 1 + \sqrt{\varepsilon^2 + \frac{1}{m} + 4 \cdot \frac{\varepsilon^2 + 1}{m-1}} \right) \cdot (1 + \| x \|) \\
= C_P \cdot (1 + \| x \|),
\]
as required in Assumption 2.2 (ii).
Since Assumption 2.2 (ii) is fulfilled, the fulfillment of Assumption 2.2 follows now with an application of Proposition 3.3. Thus, to verify the assumptions of Proposition 3.3, we need to show that \( \Omega_{\text{loc}} \ni x \mapsto \hat{\mathcal{P}}(x) \mapsto (\mathcal{M}_1(\mathbb{R}^D), \tau_1) \) is nonempty, compact-valued, and continuous. This, in turn follows from Proposition 3.2 once we have shown that \( \Omega_{\text{loc}} \ni x \mapsto \Theta(x) \subseteq \mathbb{R}^D \times \mathbb{R}^{D \times D} \) is nonempty, compact-valued, and continuous and that

\[
\{ (x, \mu, \Sigma) \mid x \in \Omega_{\text{loc}}, (\mu, \Sigma) \in \Theta(x) \} \rightarrow (\mathcal{M}_1(\mathbb{R}^D), \tau_1)
\]

is continuous.

To that end, let \( x \in \Omega_{\text{loc}} \).

The nonemptiness of \( \Theta(x) \) follows by definition.

To show the compactness of \( \Theta(x) \), let \( (\mu^{(n)}(x), \Sigma^{(n)})_{n \in \mathbb{N}} \subseteq \Theta(x) \). Then, we have \( \| \mu^{(n)} - \mu(x) \| \leq \varepsilon \) for all \( n \in \mathbb{N} \) and \( \Sigma^{(n)}(x) = \varepsilon(y^{(n)}) \) for some \( y^{(n)} \in \Omega_{\text{loc}} \) with \( \| y^{(n)} - x \| \leq \varepsilon \). Then, according to the Bolzano–Weierstrass theorem, there exists a subsequence \( (\mu^{(nk)}, y^{(nk)})_{k \in \mathbb{N}} \subseteq \mathbb{R}^D \times \Omega_{\text{loc}} \) such that \( y^{(nk)} \rightarrow y \in \Omega_{\text{loc}} \) with \( \| y - x \| \leq \varepsilon \), and \( \mu^{(nk)} \rightarrow \mu \in \mathbb{R}^D \) with \( \| \mu - \mu(x) \| \leq \varepsilon \) for \( k \rightarrow \infty \). Since \( \varepsilon \) is continuous, we obtain that \( (\mu^{(nk)}, \Sigma^{(nk)}) \rightarrow (\mu, \Sigma) := (\mu, \varepsilon(y)) \in \Theta(x) \) for \( k \rightarrow \infty \).

To show the upper hemicontinuity of \( \Theta \), let \( (x^{(n)})_{n \in \mathbb{N}} \subseteq \Omega_{\text{loc}} \) with \( (x^{(n)}) \rightarrow x \in \Omega_{\text{loc}} \) for \( n \rightarrow \infty \) as well as \( (\mu^{(n)}, \Sigma^{(n)})_{n \in \mathbb{N}} \subseteq \Theta(x^{(n)}) \) for all \( n \in \mathbb{N} \). We have for all \( n \in \mathbb{N} \) that \( \| \mu^{(n)} - m(x^{(n)}) \| \leq \varepsilon \) and that \( \Sigma^{(n)} = \varepsilon(y^{(n)}) \) with \( \| y^{(n)} - x^{(n)} \| \leq \varepsilon \) for some \( y^{(n)} \in \Omega_{\text{loc}} \). Therefore, since \( \| \mu^{(n)} - m(x) \| \leq \| \mu^{(n)} - m(x^{(n)}) \| + \| m(x^{(n)}) - m(x) \| \), the continuity of \( m \) ensures for every \( n \) large enough that \( \| \mu^{(n)} - m(x) \| \leq 2\varepsilon \). Hence, there exists according to the Bolzano–Weierstrass theorem a subsequence \( (\mu^{(nk)}, \Sigma^{(nk)})_{k \in \mathbb{N}} \) with \( (\mu^{(nk)} \rightarrow \mu \in \mathbb{R}^D \) for \( k \rightarrow \infty \) for some \( \mu \in \mathbb{R}^D \). Therefore, since \( m \) is continuous, we obtain

\[
\| \mu - m(x) \| = \lim_{k \rightarrow \infty} \| \mu^{(nk)} - m(x^{(nk)}) \| \leq \varepsilon.
\]

Analogously, we have that \( \| y^{(n)} - x \| \leq \| y^{(n)} - x^{(n)} \| + \| x^{(n)} - x \| < 2\varepsilon \) for every \( n \) large enough. This implies the existence of a subsequence \( (\mu^{(nk)}, \Sigma^{(nk)})_{k \in \mathbb{N}} \subseteq \Omega_{\text{loc}} \) converging against some \( y \in \Omega_{\text{loc}} \) with \( \| y - x \| = \lim_{k \rightarrow \infty} \| y^{(nk)} - x^{(nk)} \| \leq \varepsilon \). Then, for \( \Sigma := \varepsilon(y) \), we have \( (\mu, \Sigma) \in \Theta(x) \) and \( (\mu^{(nk)}, \Sigma^{(nk)}) \rightarrow (\mu, \Sigma) \) for \( k \rightarrow \infty \). Thus, the upper hemicontinuity follows with Lemma A.4.

To show the lower hemicontinuity of \( \Theta \), we consider a sequence \( (x^{(n)})_{n \in \mathbb{N}} \subseteq \Omega_{\text{loc}} \) with \( x^{(n)} \rightarrow x \in \Omega_{\text{loc}} \) for \( n \rightarrow \infty \) and some \( (\mu, \Sigma) \in \Theta(x) \). We have by definition \( \| \mu - m(x) \| \leq \varepsilon \) as well as \( \Sigma = \varepsilon(y) \) for some \( y \in \mathbb{R}^D \) with \( \| y - x \| \leq \varepsilon \). We define for every \( n \in \mathbb{N} \)

\[
\mu^{(n)} := \left( 1 - \frac{1}{n} \right) \mu + \frac{1}{n} m(x^{(n)}).
\]

Then, due to the convergence \( m(x^{(n)}) \rightarrow m(x) \), there exists a subsequence \( (x^{(nk)})_{k \in \mathbb{N}} \) such that for every \( k \in \mathbb{N} \), we have \( \| m(x^{(nk)}) - m(x) \| \leq \varepsilon/(n_k - 1) \). This implies for all \( k \in \mathbb{N} \) that

\[
\| \mu^{(nk)} - m(x^{(nk)}) \| = \left( 1 - \frac{1}{n_k} \right) \| \mu - m(x^{(nk)}) \|
\]

\[
\leq \left( 1 - \frac{1}{n_k} \right) \left( \| \mu - m(x) \| + \| m(x) - m(x^{(nk)}) \| \right) \leq \left( 1 - \frac{1}{n_k} \right) \left( \varepsilon + \frac{\varepsilon}{n_k - 1} \right) = \varepsilon.
\]
Next, we define for all $n \in \mathbb{N}$

$$y^{(n)} := \left(1 - \frac{1}{n}\right)y + \frac{1}{n}x^{(n)}, \quad \Sigma^{(n)} := c\left(y^{(n)}\right).$$  \hfill (63)

We obtain by the convergence $x^{(n)} \to x$ for $n \to \infty$ the existence of a subsequence $x^{(n_k)}$ such that $\|x^{(n_k)} - x\| < \varepsilon/(n_k - 1)$ for all $l \in \mathbb{N}$. This implies

$$\|y^{(n_k)} - x^{(n)}\| = \left(1 - \frac{1}{n_k}\right)\|y - x^{(n)}\|$$

$$\leq \left(1 - \frac{1}{n_k}\right)\left(\|y - x\| + \|x - x^{(n)}\|\right) \leq \left(1 - \frac{1}{n_k}\right)\left(\varepsilon + \frac{\varepsilon}{n_k - 1}\right) = \varepsilon. \hfill (64)$$

Hence, with Equations (62)–(64), we have shown the existence of a subsequence $(\mu^{(n_k)}, \Sigma^{(n_k)})_{k \in \mathbb{N}}$ with $(\mu^{(n_k)}, \Sigma^{(n_k)}) \in \Theta(x^{(n_k)})$ for all $l \in \mathbb{N}$ and such that, by the continuity of $r$, $(\mu^{(n_k)}, \Sigma^{(n_k)}) \to (\mu, \Sigma)$ for $l \to \infty$. This implies the lower hemicontinuity of $\Theta$ by Lemma A.5.

It remains to show that the map defined in Equation (60) is continuous with respect to $\tau_1$.

To that end, consider a sequence $(x^{(n)})_{n \in \mathbb{N}} \subseteq \Omega_{\text{loc}}$ as well as a sequence $(\mu^{(n)}, \Sigma^{(n)})_{n \in \mathbb{N}}$ with $(\mu^{(n)}, \Sigma^{(n)}) \in \Theta(x^{(n)})$ for all $n \in \mathbb{N}$ and such that $(x^{(n)}, \mu^{(n)}, \Sigma^{(n)}) \to (x, \mu, \Sigma) \in \Omega_{\text{loc}} \times \Theta(x)$ for $n \to \infty$. Then we write $\mathbb{P}^{(n)} := \mathcal{N}_{\mathbb{R}^D}(\mu^{(n)}, \Sigma^{(n)}) \in \mathcal{M}_1(\mathbb{R}^D)$ for $n \in \mathbb{N}$ as well as $\mathbb{P} := \mathcal{N}_{\mathbb{R}^D}(\mu, \Sigma) \in \mathcal{M}_1(\mathbb{R}^D)$. The characteristic function of $\mathbb{P}^{(n)}$, denoted by

$$\mathbb{R}^D \ni u \mapsto \varphi_{\mathbb{P}^{(n)}}(u) := \exp\left(iu^T\mu^{(n)} - \frac{1}{2}u^T\Sigma^{(n)}u\right)$$

converges for $n \to \infty$ pointwise against

$$\mathbb{R}^D \ni u \mapsto \varphi_{\mathbb{P}}(u) := \exp\left(iu^T\mu - \frac{1}{2}u^T\Sigma u\right),$$

which is the characteristic function of $\mathbb{P}$, and hence by Lévy’s continuity theorem (see, e.g., Jacod and Protter (2003, Theorem 19.1)), we have $\mathbb{P}^{(n)} \to \mathbb{P}$ weakly, that is, in $\tau_0$ for $n \to \infty$.

The convergence of $\mathbb{P}^{(n)} \to \mathbb{P}$ with respect to $\tau_1$ now follows with, for example, Bogachev (1998, Example 3.8.15), since $(\mathbb{P}^{(n)})_{n \in \mathbb{N}}$ and $\mathbb{P}$ are Gaussian.

To verify Assumption 2.4, first note that $r$ is continuous, and hence Assumption 2.4 (i) is fulfilled. Let $X_t, X'_t \in \Omega_{\text{loc}}, X_{t+1} = (R_{t-m+2}, \ldots, R_{t+1}) \in \Omega_{\text{loc}}$, and let $a_t, a'_t \in A$. Then, the Cauchy–Schwarz inequality implies

$$|r(X_t, a_t, X_{t+1}) - r(X'_t, a'_t, X_{t+1})| = \left|\sum_{i=1}^D (a^i_t - a^i'_t)R^i_{t+1} - \lambda \cdot (a^T_t \cdot \Sigma_R \cdot a_t - a'^T_t \cdot \Sigma_R \cdot a'_t)\right|$$

$$\leq \|R_{t+1}\| \cdot \|a_t - a'_t\| + \lambda \cdot \left(\|a^T_t \cdot \Sigma_R \cdot (a_t - a'_t)\| + \|(a_t - a'_t)^T \cdot \Sigma_R \cdot a'_t\|\right)$$

$$\leq \|X_{t+1}\| \cdot \|a_t - a'_t\| + \lambda \cdot 2 \cdot \max_{b \in A} \|b\| \cdot \|\Sigma_R\|_F \cdot \|a_t - a'_t\|$$
\[ = \left( \|X_{t+1}\| + \lambda \cdot 2 \cdot \max_{b \in A} \|b\| \cdot \|\Sigma_R\|_F \right) \cdot \|a_t - a_t'\|, \]

where \( \| \cdot \|_F \) denotes the Frobenius-norm. This implies Assumption 2.4 (ii). Moreover, we have by using the Cauchy–Schwarz inequality that

\[
|r(X_t, a_t, X_{t+1})| = \left| \sum_{i=1}^D a_i^T R_{t+1}^i - \lambda \cdot (a_t^T \Sigma_R \cdot a_t) \right| \\
\leq \|a_t\| \cdot \|R_{t+1}\| + \lambda \|a_t\|^2 \|\Sigma_R\|_F \\
\leq \max_{b \in A} \|b\| \cdot \|X_{t+1}\| + \lambda \max_{b \in A} \|b\|^2 \|\Sigma_R\|_F \\
\leq \left( \max_{b \in A} \|b\| + \lambda \max_{b \in A} \|b\|^2 \|\Sigma_R\|_F \right) \cdot (1 + \|X_{t+1}\|),
\]

as required in Assumption 2.4 (iii).

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DATA AVAILABILITY STATEMENT

The details of the implementation and the code used in this work can be found under https://github.com/juliansester/Robust-Portfolio-Optimization.

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**APPENDIX A: SUPPLEMENTARY RESULTS**

The first auxiliary result is Banach’s fixed point theorem, compare, for example, Bäuerle and Rieder (2011, Theorem A 3.5.), or any standard monograph on analysis or functional analysis.

**Theorem A.1** (Banach’s Fixed Point Theorem). Let $M$ be a complete metric space with metric $d(x, y)$ and let $\mathcal{T} : M \to M$ be an operator such that there exists a number $\beta \in (0, 1)$ such that $d(\mathcal{T} v, \mathcal{T} w) \leq \beta d(v, w)$ for all $v, w \in M$. Then, we have that

(i) $\mathcal{T}$ has a unique fixed point $v^*$ in $M$, that is, $\mathcal{T} v^* = v^*$.

(ii) $\lim_{n \to \infty} \mathcal{T}^n v = v^*$ for all $v \in M$.

(iii) For $v \in M$, we obtain

$$d(v^*, \mathcal{T}^n v) \leq \frac{\beta^n}{1 - \beta} d(\mathcal{T} v, v).$$

The following result, Berge’s maximum theorem, can for example be found in Aliprantis and Border (2006, Theorem 17.31).

**Theorem A.2** (Berge’s maximum theorem). Let $\varphi : X \to Y$ be a upper and lower hemicontinuous correspondence between topological spaces with nonempty compact values, and suppose that $f : \{ (x, y) \in X \times Y \mid y \in \varphi(x) \} \to \mathbb{R}$ is continuous. Then the following holds.

(i) The function

$$m : X \to \mathbb{R}$$

$$x \mapsto \max_{y \in \varphi(x)} f(x, y)$$

is continuous.

(ii) The correspondence

$$c : X \to Y$$

\[ x \mapsto \{ y \in \varphi(x) \mid f(x, y) = m(x) \} \]

has nonempty, compact values.

(iii) If \( Y \) is Hausdorff, then \( c \) is upper hemicontinuous.

We also provide the assertion of the measurable maximum theorem\(^{13}\), see, for example, Aliprantis and Border (2006, Theorem 18.19).

**Theorem A.3** (Measurable maximum theorem). Let \( X \) be a separable metrizable space and \((S, \Sigma)\) be a measurable space. Let \( \varphi : S \rightarrow X \) be a weakly measurable correspondence with nonempty compact values, and suppose \( f : S \times X \rightarrow \mathbb{R} \) is a Carathéodory function. Define the value function \( m : S \rightarrow \mathbb{R} \) by

\[ m : S \rightarrow \mathbb{R} \]

\[ s \mapsto \max_{x \in \varphi(s)} f(s, x), \]

and the correspondence of maximizers by

\[ \mu : S \rightarrow X \]

\[ s \mapsto \{ x \in \varphi(s) \mid f(s, x) = m(s) \}, \]

Then the following holds.

(i) The value function \( m \) is measurable.
(ii) The argmax correspondence \( \mu \) has nonempty and compact values.
(iii) The argmax correspondence \( \mu \) is measurable and admits a measurable selector.

The following two lemmas provide characterizations of upper and lower hemicontinuity, respectively\(^ {14}\). The results can be found, for example, in Aliprantis and Border (2006, Theorem 17.20) and Aliprantis and Border (2006, Theorem 17.21).

**Lemma A.4** (Upper hemicontinuity). Assume that the topological space \( X \) is first countable and that \( Y \) is metrizable. Then, for a correspondence \( \varphi : X \rightarrow Y \), the following statements are equivalent.

(i) The correspondence \( \varphi \) is upper hemicontinuous and \( \varphi(x) \) is compact for all \( x \in X \).
(ii) For any \( x \in X \), if a sequence \((x^{(n)}, y^{(n)})_{n \in \mathbb{N}} \subseteq \text{Gr}\varphi \) satisfies \( x^{(n)} \rightarrow x \) for \( n \rightarrow \infty \), then there exists a subsequence \((y^{(n_k)})_{k \in \mathbb{N}} \) with \( y^{(n_k)} \rightarrow y \in \varphi(x) \) for \( k \rightarrow \infty \).

\(^{13}\) Note that every upper hemicontinuous correspondence is (weakly) measurable, see Aliprantis and Border (2006, Lemma 17.4, Definition 18.1 and Lemma 18.2). Moreover, if \( S \) is a topological space and \( \Sigma \) its Borel \( \sigma \)-field, then every continuous function \( \Psi : S \times X \rightarrow Z \) is a Carathéodory function, see Aliprantis and Border (2006, Definition 4.50).

\(^{14}\) To illustrate the notions of lower and upper hemicontinuity, we also refer to Aliprantis and Border (2006, Example 17.3) where examples are provided for correspondences that are upper hemicontinuous but not lower hemicontinuous and vice versa.
Lemma A.5 (Lower hemicontinuity). For a correspondence $\varphi : X \rightarrow Y$ between first countable topological spaces, the following statements are equivalent.

(i) The correspondence $\varphi$ is lower hemicontinuous.

(ii) For any $x \in X$, if $x^{(n)} \rightarrow x$ for $n \rightarrow \infty$, then for each $y \in \varphi(x)$, there exists a subsequence $(x^{(n_k)})_{k \in \mathbb{N}}$ and elements $y^{(k)} \in \varphi(x^{(n_k)})$ for each $k \in \mathbb{N}$ such that $y^{(k)} \rightarrow y$ for $k \rightarrow \infty$. 