Scattering Matrix of the SU(n) Gauge Theory with Explicit Gauge Mass Term

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Abstract

Based on the renormalisability of the SU(n) theory with massive gauge bosons, we start with the path integral of the generating functional for the renormalized Green functions and develop a method to construct the scattering matrix so that the unitarity is evident. By using as basical variables the renormalized field functions and defining the unperturbed Hamiltonian operator $H_0$ that, under the Lorentz condition, describes the free particles of the initial and final states in scattering processes, we form an operator description with which the renormalized Green functions can be expressed as the vacuum expectations of the time ordered products of the Heisenberg operators of the renormalized field functions, that satisfy the usual equal time commutation or anticommutation rules. From such an operator description we find a total Hamiltonian $\tilde{H}$ that determine the time evolution of the Heisenberg operators of the renormalized field functions. The scattering matrix is nothing but the matrix of the operator $U(\infty, -\infty)$, which describes the time evolution from $-\infty$ to $\infty$ in the interaction picture specified by $\tilde{H}$ and $H_0$, respect to a base formed by the physical eigen states of $H_0$. We also explain the asymptotic field viewpoint of constructing the scattering matrix within our operator description. Moreover, we find a formular to express the scattering matrix elements in terms of the truncated renormalized Green functions.

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I. Introduction

Although the SU(n) theory with a mass term of the gauge fields was considered to be in general nonrenormalisable for a long time, it was repeatedly studied (see for example Refs. [1-8]). In particular, we have proven the renormalisability of the theory under the original expression of the mass term of the gauge fields [8]. Therefore it becomes possible to to deeply studied the scattering matrix of such theories. In the present paper, instead of checking the negative arguments concerning the unitarity we will develop a method to construct the scattering matrix with the renormalized field functions as basical variables so that the unitarity is evident.

Owing to the renormalisability of theory, one can start with the path integral of the generating functional for the renormalized Green functions and form an operator description with the renormalized field functions as basical variables. This means that the Heisenberg operators of the renormalized field functions satisfy the usual equal time commutation or anticommutation rules and the renormalized Green functions can be expressed as the vacuum expectations of the time ordered products of such Heisenberg field operators. On the other hand starting from the path integral of the generating functional for the unrenormalized Green functions, one can form another operator description with the bare field functions as basical variables. It should be emphasized that such two operator descriptions are of different types. In particular, the field operators belonging to the two operator descriptions operate on different state spaces and can not be transformed into each other by unitary transformations or by scale transformations.

According to the very idea of renormalisation we decide to choose the first kind of operator description. With the renormalized field functions and the renormalized parameters we will define a non-interacting Lagrangian $L_0$ which, under the Lorentz condition, describes the free particles of the initial and final states in scattering processes. Next we will find an unperturbed Hamiltonian $H_0$ operation in a way used in the Gupta-Bleuler quantization and define a Quasi-Gupta-Bleuler subspace ($QGB$) which is spanned by all the physical eigen states of $H_0$. By giving an operator expression for the generating functional for the Green functions of the free fields, we will able to do the same for the generating functional for the renormalized full Green functions. Then we can express the renormalized Green functions as the vacuum expectations of the time ordered products and find the total Hamiltonian $\tilde{H}$ that determine the time evolution of the Heisenberg operators of the renormalized field functions. With the $QGB$ subspace and the operator $U(\infty, -\infty)$, which describes the time evolution from $-\infty$ to $\infty$ in the interaction picture specified by $\tilde{H}$ and $H_0$, we can easily define the scattering matrix and make the unitarity evident.

The operator description will be given in section 2. In section 3 we will explain the definition and
the unitarity of the scattering matrix and briefly describe the asymptotic field viewpoint based on our operator description. In section 4 a formula will be found to express the scattering matrix elements in terms of the truncated renormalized Green functions. Concluding remarks will be given in the final section.

II. Operator description with renormalized field functions as basic variables

Assume that the theory has been renormalized under a proper renormalization scheme so that the renormalized parameters such as the masses represent the physical ones. The generating functional for the renormalized Green functions can be expressed as

\[ Z[R,J,\chi,\bar{\chi},\eta,\bar{\eta}] = \frac{1}{N} \int D[A,\omega,\omega,\psi,\bar{\psi}] \prod_{a',x'} \delta(\partial^\mu A_{a'\mu}(x')) \exp\left\{ i \left( I_{\text{eff}} + I_s \right) \right\} , \quad (2.1) \]

where \( A_{a\mu}, \omega_a, \bar{\omega}_a \) and \( \psi, \bar{\psi} \) stand for the renormalized gauge fields, ghost fields and material fields, all the parameters are also renormalized ones, \( N \) is a constant to make \( Z[R,0,0,0,0] \) equal to 1, \( I_s \) is the source term

\[ I_s = \int d^4x \left[ J_a^\mu(x) A_{a\mu}(x) + \bar{\chi}_a(x) \omega_a(x) + \bar{\omega}_a(x) \chi_a(x) \right. \]
\[ \left. + \overline{\psi}_a(x) \psi_a(x) + \overline{\eta}_a(x) \eta_a(x) \right] . \]

The action \( I_{\text{eff}} \) contains the counterterm and is formed by the following Lagrangian

\[ I_{\text{eff}} = -\frac{1}{4} F_{a\mu\nu} F_{a}^{\mu\nu} + \frac{1}{2} M^2 A_{a\mu} A_{a}^{\mu} + \mathcal{L}_\psi + \mathcal{L}_{\psi A} \]
\[ + \left( - \partial_{\mu} \bar{\omega}_a(x) \right) D_{a\mu}^b \omega_b(x) + \mathcal{L}_{\text{count}} . \quad (2.2) \]

The free evolution of the physical particles of the initial and final states in scattering processes should be described by combining the Lorentz condition with the Lagrangian

\[ \mathcal{L}_0(x) = -\frac{1}{4} F_{a\mu\nu}^{[0]} F_{a}^{[0]\mu\nu} + \frac{1}{2} M^2 A_{a\mu}(x) A_{a}^{\mu}(x) + \mathcal{L}_\psi(x) + \left( - \partial_{\mu} \bar{\omega}_a(x) \right) D_{a\mu}^b \omega_b(x) , \quad (2.3) \]

where

\[ F_{a\mu\nu}^{[0]} = \partial_{\mu} A_{a\nu} - \partial_{\nu} A_{a\mu} . \]

In order to describe such free particles by an operator method, we replace (2.3) by

\[ \tilde{\mathcal{L}}_0(x) = \mathcal{L}_{GB}(x) + \frac{1}{2} M^2 A_{a\mu}(x) A_{a}^{\mu}(x) + \mathcal{L}_\psi(x) + \left( - \partial_{\mu} \bar{\omega}_a(x) \right) D_{a\mu}^b \omega_b(x) , \quad (2.4) \]
where $L_{GB}(x)$ is a Gupta–Bleuler Lagrangian

$$L_{GB}(x) = -\frac{1}{4} F_{a\mu\nu}^0 F_a^{0\mu\nu} - \frac{1}{2} \left( \partial^\mu A_{a\mu} \right)^2.$$  

Then we perform a canonical quantization with $\tilde{L}_0(x)$ and define the corresponding Hamiltonian operator $H_0$ and field operators. Moreover we have to choose the so-called physical states and define a physical initial-final state subspace according to Lorentz condition and the residual symmetry. This subspace is spanned by all the physical eigen states of $H_0$ and can be formed in a similar method used in the Gupta–Bleuler quantization although there are differences caused by the gauge field mass term (see the Appendix or Ref. [9]). We will simply call such a subspace as Quasi-Gupta-Bleuler or QGB subspace.

If we can find a total Hamiltonian which determines the time evolution of the Heisenberg operators of the renormalized field functions, we will be able to construct the scattering matrix with the help of the QGB subspace and the time evolution operators $U(\infty,-\infty)$ which describes the time evolution from $-\infty$ to $+\infty$ in the interaction picture defined by the total Hamiltonian and $H_0$. It will be seen that these tentative ideas are correct. However, we will have to introduce the $\xi-$gauge fixing term and construct an appropriate operator expression for the generating function for the renormalized Green functions.

With the help of the the gauge fixing term $L_{GF} = \frac{1}{2\xi} \left( \partial^\mu A_{a\mu} \right)^2$ one can express (2.1) as

$$Z^{[R]}[J,\chi,\chi,\eta,\eta] = \lim_{\xi \to 0} Z^{[R]}_\xi [J,\chi,\chi,\eta,\eta],$$  

(2.5)

where

$$Z^{[R]}_\xi [J,\chi,\chi,\eta,\eta] = \frac{1}{N_\xi} \int D[A,\omega,\omega,\psi,\psi] \prod_a \delta(\partial^\mu A_{a\mu}(x')) \exp \left\{ i \left( I_{\text{eff}} + I_s \right) \right\},$$  

(2.6)

where $N_\xi$ is a constant to make $Z^{[R]}_\xi [0,0,0,0,0]$ equal to 1, and

$$I_{\text{eff}} = I_{\text{eff}} + \int d^4x L_{GF}(x).$$  

(2.7)

Needles to say, quantities defined by using $Z^{[R]}_\xi [J,\chi,\chi,\eta,\eta]$ depend on $\xi$, but this parameter will often be omitted.

Let $\tilde{Z}^0[J,\chi,\chi,\eta,\eta]$ be the generating functional for the Green functions defined by the effective Lagrangian $\tilde{L}_0$, namely

$$\tilde{Z}^0[J,\chi,\chi,\eta,\eta] = \frac{1}{N_0} \int D[A,\omega,\omega,\psi,\psi] \prod_{a',x'} \delta(\partial^\mu A_{a'\mu}(x')) \exp \left\{ i \left( \int d^4x \tilde{L}_0(x) + I_s \right) \right\},$$  

where $N_0$ is a constant to make $\tilde{Z}^0_0 [0,0,0,0,0]$ equal to 1. One thus has

$$\tilde{Z}^0[J,\chi,\chi,\eta,\eta] = \langle 0 | \exp \left\{ i \int d^4x \left[ J^\mu a(x) A_{a\mu}(x) ight] ight. \left. + \overline{\chi}_a(x) \omega_a(x) + \overline{\eta}_a(x) \gamma_a(x) + \overline{\psi}_a(x) \psi_a(x) \right\} | 0 \rangle,$$  

(2.8)
where \( A_{\mu}(x), \omega_{\alpha}(x), \psi_{\alpha}(x), \ldots \) are field operators satisfying the usual equal time commutation or anticommutation rules and their time evolution are determined by \( H_0 \). The state \( |0\rangle \) is the eigen state of \( H_0 \) belonging to the lowest eigenvalue (assigned to be zero). Define \( \tilde{H}_I \) so that

\[
I_{\text{eff}} = - \int d^4x \tilde{H}_I(x) + \int d^4x \tilde{L}_0(x). \tag{2.9}
\]

Then one gets from (2.6)

\[
Z^{[R]}_{\xi}[J, \overline{\chi}, \chi, \overline{\eta}, \eta] \propto \exp \left\{ -i \int d^4x \tilde{H}_I \left( \frac{\delta}{\delta J(x)}, \frac{\delta}{\delta \overline{\chi}(x)}, \frac{\delta}{\delta \chi(x)}, \frac{\delta}{\delta \overline{\eta}(x)}, \frac{\delta}{\delta \eta(x)} \right) \right\} \tilde{Z}_0[J, \overline{\chi}, \chi, \overline{\eta}, \eta]
\]

\[
\propto \langle 0 | \mathbf{T} \left\{ \exp \left\{ i \int d^4x \left[ J^\mu(x) A_{\mu\alpha}(x) + \overline{\chi}_{\alpha}(x) \omega_{\alpha}(x) + \overline{\psi}_{\alpha}(x) \eta_{\alpha}(x) \right] \right\} \right\} | 0 \rangle. \tag{2.10}
\]

where \( \tilde{H}_I \left( \frac{\delta}{\delta J(x)}, \frac{\delta}{\delta \overline{\chi}(x)}, \frac{\delta}{\delta \chi(x)}, \frac{\delta}{\delta \overline{\eta}(x)}, \frac{\delta}{\delta \eta(x)} \right) \) is a differential operator obtained from \( \tilde{H}_I(x) \) by replacing the field functions \( A_{\mu\alpha}(x), \omega_{\alpha}(x), \overline{\chi}_{\alpha}(x), \psi_{\alpha}(x) \) and \( \overline{\psi}_{\alpha}(x) \) with \( \frac{\delta}{\delta J^\mu(x)}, \frac{\delta}{\delta \overline{\chi}_{\alpha}(x)}, \frac{\delta}{\delta \chi_{\alpha}(x)}, \frac{\delta}{\delta \overline{\eta}_{\alpha}(x)}, \frac{\delta}{\delta \eta_{\alpha}(x)} \) and \( \frac{\delta}{\delta \overline{\eta}_{\alpha\beta}(x)} \) respectively. The operator \( \tilde{H}_I \) appearing in the last line is obtained from the classical quantity by replacing the field functions with operators \( A_{\mu\alpha}(x), \omega_{\alpha}(x), \overline{\chi}_{\alpha}(x), \psi_{\alpha}(x) \) and \( \overline{\psi}_{\alpha}(x) \) and then taking the normal products with respect to \( |0\rangle \). Each of the two proportion coefficients in (4.10) should make \( Z^{[R]}_{\xi}[0, 0, 0, 0, 0] \) equal to 1. It is evident that the time evolution of \( \tilde{H}_I \) is determined by \( H_0 \):

\[
\tilde{H}_I(t) = e^{iH_0 t} \tilde{H}_I(0) e^{-iH_0 t}. \tag{2.11}
\]

Therefore, the operator

\[
\mathbf{T} \exp \left\{ -i \int d^4x \tilde{H}_I(x) \right\}
\]

is the time evolution operator \( U(\infty, -\infty) \) in the interaction picture defined by \( H_0 \) and the interaction \( \int d^3x \tilde{H}_I(0) \), namely

\[
\mathbf{T} \exp \left\{ -i \int d^4x \tilde{H}_I(x) \right\} = U(\infty, -\infty), \tag{2.12}
\]

and

\[
U(t, t_0) = e^{iH_0 t} e^{-i\tilde{H}(t-t_0)} e^{-iH_0 t_0}, \tag{2.13}
\]

\[
\tilde{H} = H_0 + \int d^3x \tilde{H}_I(0). \tag{2.14}
\]

It follows that (2.10) can be written as

\[
Z^{[R]}_{\xi}[J, \overline{\chi}, \chi, \overline{\eta}, \eta] \propto \langle 0 | \mathbf{T} \exp \left\{ i \int d^4x \left[ J^\mu(x) A_{\mu\alpha}(x) + \overline{\chi}_{\alpha}(x) \omega_{\alpha}(x) + \overline{\psi}_{\alpha}(x) \eta_{\alpha}(x) \right] \right\} U(0, -\infty) | 0 \rangle. \tag{2.15}
\]
The proportion coefficient should make $Z_{0}^{[R]}[0, 0, 0, 0, 0]$ equal to 1. The time evolution of the operators $A_{a \mu}^{h}, \cdots$ are determined by $\tilde{H}$:

\[
A_{a \mu}(t) = e^{i\tilde{H}t}A_{a \mu}(0)e^{-i\tilde{H}t},
\]

\[
\omega_{a}^{h}(t) = e^{i\tilde{H}t}\omega_{a}(0)e^{-i\tilde{H}t},
\]

\[
\overline{\omega}_{a}(t) = e^{i\tilde{H}t}\overline{\omega}_{a}(0)e^{-i\tilde{H}t},
\]

\[
\psi_{a}^{h}(t) = e^{i\tilde{H}t}\psi_{a}(0)e^{-i\tilde{H}t},
\]

\[
\overline{\psi}_{a}(t) = e^{i\tilde{H}t}\overline{\psi}_{a}(0)e^{-i\tilde{H}t}.
\]

Therefore $\tilde{H}$ is the total Hamiltonian operator and $A_{a \mu}^{h}(x), \omega_{a}^{h}(x), \overline{\omega}_{a}(x), \psi_{a}^{h}(x)$ and $\overline{\psi}_{a}(x)$ stand for the Heisenberg operators of the renormalized field functions. Thus (2.15) means that the renormalized Green functions can be expressed as the vacuum expectations of the time ordered products of the Heisenberg operators of the renormalized field functions. It should be emphasized that these field operators satisfy the usual equal time commutation or anticommutation rules.

III. The scattering matrix

One can choose to specify the initial condition at $t = -\infty$ or at $t = +\infty$ for state vectors in the interaction picture and constructing the scattering theory. Therefore for an arbitrary state $|\Psi_{GB}\rangle$ in the $QGB$ subspace there is a state $|\Psi_{GB}'\rangle$ in this subspace so that

\[
U(t, \infty)|\Psi_{GB}'\rangle = U(t, -\infty)|\Psi_{GB}\rangle. \tag{3.1}
\]

Conversely, for an arbitrary state $|\Psi_{GB}'\rangle \in QGB$, there is a state $|\Psi_{GB}\rangle \in QGB$ that satisfies this equation. Next, states $U(t, \pm\infty)|\Psi_{GB}\rangle$ are normalized just like $|\Psi_{GB}\rangle$ one thus has

\[
U(\pm\infty, t)U(t, \pm\infty)|\Psi_{GB}\rangle = |\Psi_{GB}\rangle. \tag{3.2}
\]

This and (3.1) also indicate that the operators $U(\pm\infty, \mp\infty)$ conserve invariant the $QGB$ subspace. We therefore express the elements of the scattering matrix as

\[
S_{fi} = \lim_{\xi \to 0} \frac{\langle f|U(\infty, -\infty)|i\rangle}{\langle 0|U(\infty, -\infty)|0\rangle}, \tag{3.3}
\]

where $|i\rangle$ and $|f\rangle$ are within the QGB subspace and are the eigen states of $H_{0}$.

From (3.1) and (3.2), for an arbitrary state $|\Psi_{GB}\rangle \in QGB$ one has

\[
U(0, -\infty)|\Psi_{GB}\rangle = U(0, \infty)|\Psi_{GB}'\rangle,
\]
\[ U(\infty, -\infty) |\Psi_{GB}\rangle = |\Psi'_{GB}\rangle, \]
\[ U(-\infty, \infty) |\Psi'_{GB}\rangle = |\Psi_{GB}\rangle. \]

Thus,
\[ U(-\infty, \infty) U(\infty, -\infty) |\Psi_{GB}\rangle = U(-\infty, \infty) |\Psi'_{GB}\rangle = |\Psi_{GB}\rangle. \]

Similarly for arbitrary \( |\Psi'_{GB}\rangle \in Q_{GB} \),
\[ U(\infty, -\infty) U(-\infty, \infty) |\Psi'_{GB}\rangle = |\Psi'_{GB}\rangle. \]

These ensure the unitarity of the scattering matrix.

With the help of the time evolution operator in the interaction picture we can also define the operators of so-called asymptotic fields which will be denoted by \( A_{a\mu,\text{in}}(x) \), \( A_{a\mu,\text{out}}(x) \), \( \psi_{a,\text{in}}(x) \) and \( \psi_{a,\text{out}}(x) \), \( \cdots \).

These are free fields, and when \(-t\) is very large each in-operator proportional to the Heisenberg operator of the renormalized field function. Actually each proportion coefficient should be equal to 1 in order not to destroy the usual equal time commutation or anticommutation rules. On the other hand when \( t \) is very large each out-operator is equal to the Heisenberg operator of the renormalized field function. Using
\[ A^h_{a\mu}(x) = U(0, t) A_{a\mu}(x) U(t, 0), \]
\[ \psi^h_{a}(x) = U(0, t) \psi_{a}(x) U(t, 0), \]
\[ \omega^h_{a}(x) = U(0, t) \omega_{a}(x) U(t, 0), \]

and noticing that when \(|t|\) is large enough, \( U(0, t) \) and \( U(t, 0) \) are independent of \( t \), one gets
\[ A_{a\mu,\text{in}}(x) = U(0, -\infty) A_{a\mu}(x) U(-\infty, 0), \] \hspace{1cm} (3.4)
\[ \psi_{a,\text{in}}(x) = U(0, -\infty) \psi_{a}(x) U(-\infty, 0), \] \hspace{1cm} (3.5)
\[ A_{a\mu,\text{out}}(x) = U(0, \infty) A_{a\mu}(x) U(\infty, 0), \] \hspace{1cm} (3.6)
\[ \psi_{a,\text{out}}(x) = U(0, \infty) \psi_{a}(x) U(\infty, 0). \] \hspace{1cm} (3.7)

Since the asymptotic fields are free ones, these are valid for any \( t \). The formulae for the other fields are similar.

If the in–states and out–states are defined by
\[ |0, \text{in}\rangle = \frac{U(0, -\infty) |0\rangle}{\sqrt{\langle 0 | U(\infty, -\infty) |0\rangle}}, \]
\[ |0, \text{out}\rangle = \frac{U(0, \infty) |0\rangle}{\sqrt{\langle 0 | U(\infty, -\infty) |0\rangle}}. \]
\[
|i, \text{in}\rangle = \frac{U(0, -\infty)|i\rangle}{\sqrt{\langle 0|U(\infty, -\infty)|0\rangle}},
\]
\[
|i, \text{out}\rangle = \frac{U(0, \infty)|i\rangle}{\sqrt{\langle 0|U(\infty, -\infty)|0\rangle}},
\]
then (3.9) can be written as
\[
S_{fi} = \lim_{\xi \to 0} \langle f, \text{out}|i, \text{in}\rangle = \lim_{\xi \to 0} \langle f, \text{in}|U(0, -\infty)U(\infty, 0)|i, \text{in}\rangle. \tag{3.8}
\]

**IV. An expression for the scattering matrix elements in terms of the truncated renormalized Green functions**

In the following we will derive a formula to express the elements of the scattering matrix in terms of the truncated renormalized Green functions. Define the following operator
\[
U(J, \chi, \tilde{\chi}, \eta, \tilde{\eta}) = T \exp \left\{ i \int d^4x \left[ J_\mu^a(x) A_{a\mu}(x) + \chi_a(x) \omega_a(x) + \bar{\omega}_a(x) \chi_a(x) + \eta_a(x) \psi_a(x) + \bar{\psi}_a(x) \eta_a(x) \right] \right\}. \tag{4.1}
\]
We thus have
\[
U(0, 0, 0, 0, 0) = U(\infty, -\infty),
\]
and
\[
(0|U(J, \chi, \tilde{\chi}, \eta, \tilde{\eta})|0)/(0|U(\infty, -\infty)|0) = Z^{[\text{eff}]}_{\infty}[J, \chi, \tilde{\chi}, \eta, \tilde{\eta}]. \tag{4.2}
\]
Analogous to (2.10), we also have
\[
U(J, \chi, \tilde{\chi}, \eta, \tilde{\eta}) = \exp \left\{ -i \int d^4x \tilde{\mathcal{H}}_I \left( \frac{\delta}{\delta J(x)}, \frac{\delta}{\delta \chi(x)}, \frac{\delta}{\delta \tilde{\chi}(x)}, \frac{\delta}{\delta \eta(x)}, \frac{\delta}{\delta \tilde{\eta}(x)} \right) \right\} \tilde{U}^{(0)}(J, \chi, \tilde{\chi}, \eta, \tilde{\eta}), \tag{4.3}
\]
where
\[
\tilde{U}^{(0)}(J, \chi, \tilde{\chi}, \eta, \tilde{\eta}) = T \exp \left\{ i \int d^4x \left[ J_\mu^a(x) A_{a\mu}(x) + \chi_a(x) \omega_a(x) + \bar{\omega}_a(x) \chi_a(x) + \eta_a(x) \psi_a(x) + \bar{\psi}_a(x) \eta_a(x) \right] \right\}.
\]
Define \( \mathcal{H}_I(x) \) and \( \mathcal{L}'_{GF}(x) \) so that
\[
I_{\text{eff}} = -\int d^4x \mathcal{H}_I(x) + \int d^4x \left\{ \mathcal{L}_{GF}(x) + \mathcal{L}_0(x) \right\},
\]
\[
\mathcal{L}'_{GF} = \mathcal{L}_{GF} + \frac{1}{2} (\partial^\mu A_{a\mu})^2 = \left( \frac{1}{2} - \frac{1}{2\xi} \right) (\partial^\mu A_{a\mu})^2.
\]
We can write (4.3) as

$$U(J, \overline{\chi}, \chi, \overline{\eta}, \eta) = \exp \left\{ -i \int d^4x \mathcal{H}_t \left( \frac{\delta}{\delta \overline{J}(x)} - \frac{\delta}{\delta \chi(x)} \right) \right\} U(0)(J, \overline{\chi}, \chi, \overline{\eta}, \eta), \quad (4.4)$$

where

$$U(0)(J, \overline{\chi}, \chi, \overline{\eta}, \eta) = \exp \left\{ i \int d^4x \mathcal{L}_{GF} \left( \frac{\delta}{\delta \overline{J}(x)} \right) \right\} \tilde{U}(0)(J, \overline{\chi}, \chi, \overline{\eta}, \eta). \quad (4.5)$$

Noticed that $\langle 0 | U(0)(J, \overline{\chi}, \chi, \overline{\eta}, \eta) | 0 \rangle$ is proportional to the generating functional for the Green functions defined by the effective Lagrangian $\mathcal{L}_0 + \mathcal{L}_{GF}$ and the latter can be explicitly expressed as

$$\mathcal{Z}_\xi^{(0)} [J, \overline{\chi}, \chi, \overline{\eta}, \eta] = \exp \left\{ - \frac{i}{2} \int d^4x \int d^4y J_{\mu}^\alpha(x) D_{\mu \nu}^{ab}(x-y) J_{\nu}^\beta(y) \right\} \times \exp \left\{ - i \int d^4x \int d^4y \overline{\chi}_a(x) C_{ab}(x-y) \chi_b(y) \right\} \times \exp \left\{ - i \int d^4x \int d^4y \overline{\eta}_a(x) S_{ab}(x-y) \eta_b(y) \right\},$$

where $iD_{\mu \nu}^{ab}(x-y)$, $iC_{ab}(x-y)$ and $iS_{ab}(x-y)$ are the propagators. Particularly, one has

$$D_{\mu \nu}^{ab}(k) = \frac{-1}{k^2 - M^2 + i\epsilon} \left\{ g_{\mu \nu} + (\xi - 1) \frac{k_\mu k_\nu}{k^2 - \xi M^2 + i\epsilon} \right\} \delta_{ab}, \quad (4.6)$$

$$[D(k)^{-1}]_{\mu \nu}^{ab} = \left\{ -(k^2 - M^2 + i\epsilon) g_{\mu \nu} + (1 - \frac{1}{\xi}) k_\mu k_\nu \right\} \delta_{ab}. \quad (4.7)$$

As for the operator $U(0)(J, \overline{\chi}, \chi, \overline{\eta}, \eta)$, one has

$$U(0)(J, \overline{\chi}, \chi, \overline{\eta}, \eta) \propto \exp \left\{ - \int d^4x \int d^4y \left( A_{ab}(x) [D(x-y)^{-1}]_{\mu \nu}^{ab} \frac{\delta}{\delta J_{\nu}^\beta(y)} \right. \right.$$

$$\left. + \overline{\psi}_b(x) [S(x-y)^{-1}]_{b \beta, a \alpha} \frac{\delta}{\delta \eta_a(x)} + [S(x-y)^{-1}]_{b \beta, a \alpha} \psi_a(y) \frac{\delta}{\delta \eta_b(x)} \right) \right.$$

$$\left. + \overline{\omega}_b(x) [C(x-y)^{-1}]_{b \alpha, a \alpha} \frac{\delta}{\delta \chi_a(y)} + [C(x-y)^{-1}]_{b \alpha, a \alpha} \omega_a(y) \frac{\delta}{\delta \chi_b(x)} \right) \right\} : \mathcal{Z}_\xi^{(0)} [J, \overline{\chi}, \chi, \overline{\eta}, \eta].$$

Substituting this in (4.4) we get

$$U(J, \overline{\chi}, \chi, \overline{\eta}, \eta) \langle (0 \langle U(\infty, -\infty) | 0 \rangle \rangle$$

$$= \exp \left\{ - \int d^4x \int d^4y \left( A_{ab}(x) [D(x-y)^{-1}]_{\mu \nu}^{ab} \frac{\delta}{\delta J_{\nu}^\beta(y)} \right. \right.$$

$$\left. + \overline{\psi}_b(x) [S(x-y)^{-1}]_{b \beta, a \alpha} \frac{\delta}{\delta \eta_a(x)} + [S(x-y)^{-1}]_{b \beta, a \alpha} \psi_a(y) \frac{\delta}{\delta \eta_b(x)} \right) \right.$$

$$\left. + \overline{\omega}_b(x) [C(x-y)^{-1}]_{b \alpha, a \alpha} \frac{\delta}{\delta \chi_a(y)} + [C(x-y)^{-1}]_{b \alpha, a \alpha} \omega_a(y) \frac{\delta}{\delta \chi_b(x)} \right) \right\} : \mathcal{Z}_\xi^{(0)} [J, \overline{\chi}, \chi, \overline{\eta}, \eta]. \quad (4.8)$$
Denoting by $iD_{\mu\nu}(x-y)$, $iC_{ab}(x-y)$ and $iS_{ab}(x-y)$ the full propagators defined by $Z_{\xi}^{[R]}[J,\bar{\chi},\chi,\bar{\eta},\eta]$, and changing the sources in (4.8) into

$$
J'_{\nu}^{\alpha}(x) = -\int d^4x' J_{\mu\nu}(x') i [D(x'-x)^{-1}]_{\mu\nu}^{ab}, \\
\bar{\eta}'_{aa}(x) = -\int d^4x' \bar{\eta}_{a'a'}(x') i [S(x'-x)^{-1}]_{a'a'a}, \\
\eta_{aa}(x) = -\int d^4x' \eta_{a'a'}(x') i [S(x'-x)^{-1}]_{a'a'a'a}, \\
\chi'_{a}(x) = -\int d^4x' \chi_{a}(x') i [C(x'-x)^{-1}]_{a'a}, \\
\chi_{a}(x) = -\int d^4x' \chi_{a}(x') i [C(x'-x)^{-1}]_{a'a},
$$

we can find, in the limit as $\xi \to 0$, that

$$
S_{fi} = \langle f | : \exp \left\{ \int d^4x \left( -A_{\mu\nu}(x) \frac{\delta}{\delta J_{\mu\nu}(x)} + \bar{\psi}_{aa}(x) \frac{\delta}{\delta \bar{\eta}_{aa}(x)} - \psi_{aa}(x) \frac{\delta}{\delta \eta_{aa}(x)} \right) + \bar{\omega}_{a}(x) \frac{\delta}{\delta \bar{\chi}_{a}(x)} - \omega_{a}(x) \frac{\delta}{\delta \chi_{a}(x)} \right\} : | i \rangle X^{[R]}[J,\bar{\chi},\chi,\bar{\eta},\eta] \mid J=\bar{\chi}=\chi=\bar{\eta}=\eta=0 \rangle,
$$

where $X^{[R]}[J,\bar{\chi},\chi,\bar{\eta},\eta]$ stands for the generating functional for truncated renormalized Green functions [10]. This is the formular we need. In the above manipulation, we have taken into account the fact that the momenta of the particles in the initial or final states are on the mass shell and for such momenta the renormalized full propagators become the free propagators.

V. Concluding Remarks

Starting from the path integral of the generating functional for the renormalized Green functions with the renormalized field functions as basic variables and using the unperturbed Hamiltonian operator that, under the Lorentz condition, describes the free particles of the initial and final states in scattering processes, we have formed a satisfying operator description and found the total Hamiltonian which determine the time evolution of the Heisenberg operators of the renormalized field functions. With the help of the time evolution operator $U(\infty, -\infty)$ in the interaction picture and of the Quasi-Gupta-Bleuler subspace formed by the physical initial-final states, we have been able to easily explain the definition and the unitarity of the scattering matrix and found a formular to express the matrix elements in terms of the truncated renormalized Green functions.

We have also briefly described the asymptotic field viewpoint based on our operator description. It would be interesting to compare such a viewpoint with that of the traditional asymptotic theory [11–14].
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Appendix The Gupta-Bleuler Subspace of Initial-Final States

As was pointed out in section 2, the particles in the initial or final states are described by the following Lagrangian and Lorentz condition:

\[
\mathcal{L}_0 = -\frac{1}{4} F_{\mu \nu}^{(0)} F^{(0) \mu \nu} + \frac{1}{2} M^2 A_{\mu \nu} A^{\mu \nu} + \mathcal{L}_\psi - \left( \partial_\mu \varphi_a(x) \right) \partial^\mu \omega_a(x) , \tag{A1}
\]

\[
\partial^\mu A_{\mu \nu} = 0 , \tag{A2}
\]

According to the operator description in the text of this paper we should keep the ghost term in equation (A1) and form the initial-final state subspace by following the Gupta-Bleuler quantization method. To this end we first use the Lorentz and modify (A1) into

\[
\tilde{\mathcal{L}}_0(x) = \mathcal{L}_{GB}(x) + \frac{1}{2} M^2 A_{\mu \nu}(x) A^{\mu \nu}(x) + \mathcal{L}_\psi(x) - \left( \partial_\mu \varphi_a(x) \right) \partial^\mu \omega_a(x) , \tag{A3}
\]

where the Gupta-Bleuler Lagrangian \( \mathcal{L}_{GB}(x) \) can be written as

\[
\mathcal{L}_{GB}(x) = -\frac{1}{2} \left( \partial_\mu A_{\mu \nu} \right) \left( \partial^\nu A^{\mu \nu}_a \right) .
\]

Then we disregard the Lorentz condition and perform a canonical quantization with \( \tilde{\mathcal{L}}_0(x) \) and define the corresponding Hamiltonian operator \( \mathcal{H}_0 \) and field operators. Moreover we have to choose the so-called physical states and define a physical initial-final state subspace with the help of the Lorentz condition and the residual symmetry. We certainly expect the ghost particles to be excluded from appearing in the initial-final states.

Assume that the operators of \( \omega_a(x) \) are anti-hermitian and therefore the operators of \( \varphi_a(x) \) is hermitian. Let \( \omega_a^{(1)}(x) \), \( \omega_a^{(2)}(x) \) stand for \( \varphi_a(x) \), \( i\omega_a(x) \) and \( \Pi_a^{(1)}, \Pi_a^{(2)} \) stand for the canonical momenta conjugate to \( \omega_a^{(1)}(x), \omega_a^{(2)}(x) \), \( A^{\mu \nu}_a \) respectively. Thus the operators of these quantities in the Schrodinger
picture are

\[ A_{\mu\nu}(x) = \int \frac{d^3k}{\sqrt{2\Omega(2\pi)^4}} \left[ a_{\mu\nu}(k) e^{ik\cdot x} + a_{\mu\nu}^\dagger(k) e^{-ik\cdot x} \right], \]  
\[ \Pi_{\mu\nu}(x) = \int \frac{id^3k}{\sqrt{(2\pi)^3}} \left[ a_{\mu\nu}(k) e^{ik\cdot x} - a_{\mu\nu}^\dagger(k) e^{-ik\cdot x} \right], \]  
\[ \omega_a^{(1)}(x) = \int \frac{d^3k}{\sqrt{2\Omega(2\pi)^3}} \left[ \overline{\mathcal{C}}_a(k) e^{ik\cdot x} + \mathcal{C}_a^\dagger(k) e^{-ik\cdot x} \right], \]  
\[ \Pi_a^{(1)}(x) = \int \frac{(-i)d^3k}{\sqrt{(2\pi)^3}} \left[ C_a(k) e^{ik\cdot x} - C_a^\dagger(k) e^{-ik\cdot x} \right], \]  
\[ \omega_a^{(2)}(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3}} \left[ \mathcal{C}_a(k) e^{ik\cdot x} - \overline{\mathcal{C}}_a(k) e^{-ik\cdot x} \right], \]  
\[ H_0 = -g^{\mu\nu} \int d^3k \Omega(|k|) a_{\mu\nu}^\dagger(k) a_{\mu\nu}(k) + \int d^3k |k| \left\{ \mathcal{C}_a(k) C_a^\dagger(k) + \overline{\mathcal{C}}_a(k) \mathcal{C}_a^\dagger(k) \right\} + H_0^0, \]  

where

\[ \Omega(|k|) = \sqrt{M^2 + |k|^2}, \]
\[ \left[ a_{\mu\nu}(k), a_{\mu\nu}^\dagger(k') \right] = -g_{\mu\nu} \delta(k - k'), \]
\[ \left[ a_{\mu\nu}(k), a_{\mu\nu}^\dagger(k') \right] = \left[ a_{\mu\nu}^\dagger(k), a_{\mu\nu}(k') \right] = 0, \]
\[ \left[ \mathcal{C}_a(k), \mathcal{C}_b^\dagger(k') \right]_+ = \left[ C_a(k), C_b^\dagger(k') \right]_+ = \delta_{ab} \delta^3(k - k'), \]
\[ \left[ C_a(k), \mathcal{C}_b(k') \right]_+ = \left[ \mathcal{C}_a(k), C_b(k') \right]_+ = \left[ C_a(k), C_b^\dagger(k') \right]_+ = \left[ \mathcal{C}_a(k), \mathcal{C}_b^\dagger(k') \right]_+ = 0. \]

The restriction of the Lorentz condition on a physical state \(|\Psi_{ph}\rangle\) can be expressed as

\[ k^\mu a_{\mu\nu}(k)|\Psi_{ph}\rangle = 0 \quad (k^0 = \Omega(|k|)). \]  

For a single particle state with the polarization vector \(\varepsilon^\mu(k)\)

\[ \varepsilon^\mu(k, \lambda) a_{\mu\nu}^\dagger(k)|0\rangle \]  

Eq. (16) gives

\[ k_{\mu} \varepsilon^\mu(k, \lambda) = 0. \]

According to this condition two transversal polarization states \((\lambda = 1, 2)\) and a longitudinal polarization state \((\lambda = 3)\) are allowed to be present for each \(k\).
Since $\partial^\mu \partial_\mu \omega_a(x) = 0$ the theory we are treating is invariant under the infinitesimal transformation
\[
\delta A_{a\mu}(x) = \delta \zeta \partial_\mu \omega_a(x), \quad \delta \omega_a(x) = 0, \quad \delta \bar{\omega}_a(x) = 0, \quad \delta \psi(x) = 0, \quad (A19)
\]
where $\delta \zeta$ is an infinitesimal fermionic real constant. Similarly, since $\partial^\mu \partial_\mu \omega_a(x) = 0$ the theory is also invariant under the transformation
\[
\delta A_{a\mu}(x) = \delta \zeta \partial_\mu \omega_a(x), \quad \delta \omega_a(x) = 0, \quad \delta \bar{\omega}_a(x) = 0, \quad \delta \psi(x) = 0. \quad (A20)
\]
This is the residual symmetry mentioned above. Under such a kind of transformations a physical state should become a equivalent state. Particularly $\varepsilon^x(k, \lambda)\delta a_{a\mu}^\dagger(k)|0\rangle$ should be equivalent to zero, where $\delta a_{a\mu}^\dagger(k)$ is determined by
\[
\delta A_{a\mu}(x) = \delta \zeta \partial_\mu \omega_a \big|_{t=0}, \quad (A21)
\]
or by
\[
\delta A_{a\mu}(x) = \delta \zeta \partial_\mu \bar{\omega}_a \big|_{t=0}. \quad (A22)
\]
From (A21), one has
\[
\delta a_{a\mu}^\dagger(k) = i\delta \zeta \tilde{k}_\mu \sqrt{\Omega/|k|} C^\dagger_a(k). \quad (A23)
\]
where the components of $\tilde{k}_\mu$ are
\[
\tilde{k}_0 = |k|; \quad \tilde{k}_l = k_l. \quad (A24)
\]
Similarly from (A22), one has
\[
\delta a_{a\mu}^\dagger(k) = i\delta \zeta \tilde{k}_\mu \sqrt{\Omega/|k|} C^\dagger_a(k). \quad (A25)
\]
It follows that $C^\dagger_a(k)|0\rangle$ and $\bar{C^\dagger_a}(k)|0\rangle$ should be equivalent to zero.

In conclusion, the initial-final state subspace is formed by all the physical states. A physical state can only contain material particles, transversal polarization or longitudinal polarization gauge Bosons and not ghost particles.
References

[1] G.Curci and R.Ferrari, Nuovo Cim. 32, 151(1976).
[2] I.Ojima, Z. Phys. C13,173(1982).
[3] A.Blasi and N.Maggiore, het-th/9511068; Mod. Phys. Lett.A11, 1665(1996).
[4] R.Delbourgo and G.Thompson, Phys. Rev. Lett. 57, 2610(1986).
[5] M.Carena and C.Wagner, Phys. Rev. D37, 560(1988).
[6] A.Burnel, Phys. Rev. D33, 2981(1986);D33, 2985(1986).
[7] T.Fukuda, M.Monoa, M.Takeda and K.Yokoyama,
    Prog. Theor. Phys. 66,1827(1981);67,1206(1982);70,284(1983).
[8] Yang Ze-sen, Zhou Zhining, Zhong Yushu and Li Xianhui, hep-th/9912046, 7 Dec 1999.
[9] Z.S.Yang, X.H.Li and X.L.Chen, hep-ph/0007007, 3 Jul 2000.
[10] Z.N.Zhou, Y.S.Zhong and X.H.Li, Chin.Phys.lett.12, 1(1995).
[11] H.Lehmann, K.Symanzik and W.Zimmermann, Nuovo Cimento, 1, 205 (1955).
[12] J.D.Bjorken and S.D.Drell, ”Relativistic Quantum Field”, McGraw-Hill, New York, 1965.
[13] C.Itzykson and J.B.Zuber, ”Quantum Field Theory”, McGraw-Hill Inc., New York, 1980.
[14] Michio Kaku,”Quantum Field Theory”, Oxford University Press, Oxford, 1993.