Lifting of Coulomb Blockade by Alternating Voltages in Small Josephson Junctions with Electromagnetic Environment-Based Renormalization Effects

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The standard theory of Coulomb blockade \([P(E)\text{ theory}]\) in ultra-small tunnel junctions has been formulated on the basis of phase-phase correlations by several authors. It was recently extended by several experimental and theoretical works to account for novel features ranging from time-reversal asymmetry to electromagnetic environment–based renormalization effects. Despite this progress, the theory remains elusive in the case of one dimensional arrays. Here, we apply path integral formalism to derive the Cooper-pair current and the BCS quasi-particle current in single small Josephson junctions and extend it to include long Josephson junction arrays as effective single junctions. We consider renormalization effects due to the electromagnetic environment in the single junction as well as the array. As is the case in the single junction, we find that the spectrum of applied oscillating electromagnetic fields is renormalized by the same complex-valued factor \(\Xi(\omega) = |\Xi(\omega)| \exp i\eta(\omega)\) that modifies the environmental impedance in the \(P(E)\) function. This factor acts as a linear response function for applied oscillating electromagnetic fields driving the quantum circuit, leading to a mass gap in the thermal spectrum of the electromagnetic field. The mass gap can be modeled as a pair of exotic particle excitation with quantum statistics determined by the argument \(\eta(\omega)\). In the case of the array, this pair corresponds to a bosonic charge soliton/anti-soliton pair injected into the array by the electromagnetic field. Possible application of these results is in dynamical Coulomb blockade experiments where long arrays are used as electromagnetic power detectors.

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I. INTRODUCTION

Since the pioneering theoretical work by Likharev et al., small Josephson junctions have been thought of as a dual system to large Josephson junctions – the roles of current and voltage are interchanged. In the case of large Josephson junctions, their effective interaction with oscillating electromagnetic fields has been intensively studied, demonstrating their unique suitability for microwave-based applications such as the metrological standard for the Volt (in terms of voltage Shapiro steps) and other microwave-based devices. The dual system, on the other hand, holds enormous promise for complementary applications such as a metrological standard for the Ampere in terms of the current Shapiro steps. However, observation of dual phenomena in small junctions faces daunting experimental and theoretical challenges due, in part, to the lack of an approach that consistently covers both regimes. In particular, small tunnel junctions are prone to quantum and thermal fluctuations – their characteristics cannot be analyzed separate from their dissipative environment. As a consequence of Heisenberg uncertainty principle, their current-voltage \(I-V\) characteristics is highly sensitive to energy changes in the environment. Heuristically, tunneling of a single charge \(e\) across a tunnel junction of capacitance \(C\) and conductance \(1/R\) is restricted unless the maximum energy it can absorb from zero-point fluctuations in the vacuum, \(h/RC\), is sufficient to offset its own charging energy \(e^2/2C = E_c\) in the absence of other energy sources. Thus, the heuristic condition for Coulomb blockade is \(1/RC < E_c\).

Consequently, lifting of Coulomb blockade occurs when other sources of energy are present. For instance, energy is easily supplied by thermal fluctuations \(k_B T > 0\), strong coupling between pairs across the junction \(E_J > E_c\) or external constant voltages \(V_x > V_{eb} \sim E_c\) above the Coulomb blockade threshold voltage, thus resulting in current-voltage characteristics highly dependent on these environmental parameters. Formally, this implies that the action for large junctions \(iS(\phi) = \ln \left[ \prod_n \int d\phi_n \exp i \sum_n S_n(\phi, \phi_n) \right]\) is effective, meaning it emerges from tracing out environmental degrees of freedom \(\sum_n S_n(\phi, \phi_n)\) that act as energy sources for the tunnel junction. This leads to a theory of dynamical Coulomb blockade \([P(E)\text{ theory}]\) in single small Josephson junctions formulated on the basis of \(\langle \phi(t)\phi(0) \rangle = \Xi^{-1} \left[ \int D\phi [\phi(t)\phi(0)] \exp iS(\phi) \right] \) phase correlations where tunneling across the barrier is influenced by a high impedance environment treated within the Caldeira-Leggett model.

\(P(E)\) theory has successfully been tested to a great degree of accuracy in a myriad of experiments. This has lead to its widespread application in describing progressively complex tunneling processes such as dynamical Coulomb blockade in small Josephson junctions and quantum dots. Moreover, owing to significant improvement in microwave precision measurement technology such as near-quantum-limited amplification and progress in theory, recently published works suggest novel features in the \(P(E)\) framework ranging from time reversal symmetry violation and Tomonaga-Luttinger Liquid (TLL) physics to renormalization of electromagnetic quantities appearing in the \(P(E)\) function. Despite such progress, most of these works, with the possible exception of TLL physics, focus on the single junction.

Here, we apply path integral formalism to derive the Cooper-pair current and the BCS quasi-particle current in single small Josephson junctions and introduce a model which

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transforms the infinitely long array into an effective single junction. We consider environmental impedance renormalization and microwave (RF) amplitude renormalization effects in both cases when driven by oscillating voltages. In particular, as is the case for the single junction, we show that the environmental impedance renormalization factor, $\Xi(\omega) = \exp(-\beta M(\omega)) \exp(i\varepsilon_m)$ also acts as a linear response function for oscillating electromagnetic fields, and can be interpreted as the probability amplitude of exciting a particle of mass $M$ from the junction ground state by the RF field. The quantum statistics of this particle are determined by the complex valued probability amplitude for a particle, $\phi$ $\equiv$ $\exp(i\Phi)$ $\equiv$ $\exp(-\beta M)$, where $\Phi$ is the total quantum phase of the circuit excluding the junction phase $\phi_1$, $\phi_2$ is the bath degrees of freedom represented by $k$ coupled (via $\phi$) $L_n C_n$ oscillators. The effective action,

$$S' = -i \ln \int D\phi_1 \exp iS_2(\phi_1, \phi')$$

$$= \int dt \left\{ \frac{C}{2\varepsilon^2} \left( \frac{d\phi'}{dt} \right)^2 - \frac{1}{4\pi\varepsilon^2} \int ds \frac{\phi'(s)}{\sqrt{\frac{\partial Z^{-1}(t-s) \partial t}} \phi'(t)} \right\}$$

$$= \frac{2\pi}{2\varepsilon^2} \int d\omega \phi(\omega)\omega^2 Z_{\text{eff}}^{-1}(\omega)\phi(-\omega) \quad (1a)$$

requires the impedance in the $P(E)$ function to be renormalized,

$$P_*(E) = \frac{1}{2\pi} \int dt \exp \left( k^2 \mathcal{J}(t) + iEt \right) \quad (2a)$$

$$\mathcal{J}(t) = \frac{2e^2}{2\pi} \int \frac{d\omega}{\omega} \text{Re} \left[ Z_{\text{eff}}(\omega) \right] \exp(-i\omega t) - \frac{1}{1 - \exp(-\beta \omega)} \quad (2b)$$

$$Z_{\text{eff}}(\omega) = [Z^{-1}(\omega) + i\omega C]^{-1} = \Xi(\omega) Z(\omega) \quad (2c)$$

where $\beta$ is the inverse temperature, $\kappa = 1e$, $2e$ is the quasiparticle, Cooper-pair charge, $E$ is the energy exchanged between the junction, $\Xi(\omega)$ is a renormalization factor and $L_n C_n$ circuits acting as the environment, $\omega$ is the Fourier transform frequency that also plays the role of the thermal photon frequency at finite temperature.

It is known – at least since the work of Callen and Welton that the (causal) response function $\Xi(\omega)$ is given by,

$$\mathcal{L}_z = \frac{C}{2\varepsilon^2} \left( \frac{\partial \phi'}{\partial t} \right)^2 + \sum_{n=1}^k \left\{ \frac{C_n}{2\varepsilon^2} \left( \frac{\partial \phi_n}{\partial t} \right)^2 - \frac{L_n^{-1}}{2\varepsilon^2} (\phi_n - \phi')^2 \right\}$$

$e$ is the electric charge, $C = e_0 a^2 \mathcal{A}/d$ is the junction capacitance, $\phi' = \phi_1 + \phi_2 - e\Phi$ is the total quantum phase of the circuit when driven by oscillating voltages (RF field) exciting a particle with voltage characteristics that includes the RF field and renormalization effects.

The paper is organized as follows:

Section II deals with electromagnetic environment-based Renormalization effects. In the subsections, II A explains the basis for renormalization, II B introduces the finite temperature propagator and the environmental impedance renormalization procedure. II C interprets the impedance renormalization factor as a complex-valued probability amplitude for applied oscillating voltages (RF field) exciting a particle with quantum statistics given by the argument of the factor, or equivalently as a linear response function and the RF field as the external force leading the full expression for the current-voltage characteristics that includes the RF field and renormalization effects.

Finally, section III considers renormalization effects in infinitely long arrays. In the subsections, III A employs a phase– phase correlation introduced in III C The difference is an additional renormalization factor $\exp(-\Lambda^{-1})$ in the effective environmental impedance arising from considering a finite range of the electromagnetic field due to the presence of a charge soliton of length $\Lambda$ injected into the infinitely long array.

Note that, units where Planck’s constant, Swihart velocity and Boltzman constant are set to unity ($\hbar = \bar{c} = k_B = 1$) and Einstein summation convention are used through out unless otherwise stated with $\text{diag} \{\eta_{\mu\nu}\} = (1, -1, -1, -1)$ the Minkowski space-time metric and $\eta_{\nu\rho} \eta^{\rho\sigma} = \eta^{\sigma}_{\mu}$ the Kronecker delta symbol.

II. ELECTROMAGNETIC ENVIRONMENT-BASED RENORMALIZATION

A. Introduction

Within the Caldeira-Leggett model, the environment of a dissipative voltage-biased single junction shown in Fig. 1 is modeled by the action $S_z = \int dt \mathcal{L}_z$, where the Lagrangian is

![FIG. 1. A mesoscopic tunnel junction, J with capacitance C driven by a voltage source $V_0$ via an environmental impedance $Z(\omega)$ composed of infinite number of parallel $L_n C_n$ circuits. The circuit stores a flux $e \Phi = \sum_i \phi_i$ related to a topological potential $A(t)$ by $\int_{-\infty}^{\infty} ds A(s) = \Phi(t)$](image)
\[
\int_{-\infty}^{\infty} dt \theta(t) \chi(t) \exp(i\omega t) dt
\]
for a system driven by oscillating electromagnetic fields appears as the coefficient \(29\) of the black body spectrum. Consequently, this requires that the response \(V_{RF}^e(t)\) as seen by the junction \(J\) in Fig. 1 be a weighted function of \(\chi(t)\) and the applied oscillating voltage \(V_{RF}(t)\) : \(V_{RF}^e(t) = \int_{-\infty}^{\infty} ds \chi(t - s)V_{RF}(s)\). Therefore, to accurately describe the \(I-V\) characteristics of \(J\) driven by an applied oscillating voltage \(V_{RF}(t)\), it is not enough to simply renormalize the impedance \(Z(\omega)\) in the \(P(E)\) function: the amplitude and phase of the applied oscillating voltage \(V_{RF}\) has to be renormalized accordingly.

In subsequent sections, we introduce the finite temperature propagator for the junction and consider how impedance renormalization arises, and its implications for single junctions and long arrays driven by \(V_{RF}(t)\). We find that, a finite time varying flux \(\Phi(t)\) stored by the circuit simply implements the aforementioned renormalization by guaranteeing the circuit responds linearly to \(V_{RF}(t)\).

### B. Environmental Impedance Renormalization Using the Finite Temperature Green's Function and Propagator

Observe that a straightforward regularization procedure suggests \(Z_{eff}(\omega)\) plays the role of effective Green’s function of the \(P(E)\) function,

\[
Z_{eff}(\omega) = Z(\omega) + Z(\omega)[−i\omega CZ(\omega)] + Z(\omega)[−i\omega CZ(\omega)]^2 + \cdots + Z(\omega)[−i\omega CZ(\omega)]^n = \frac{Z(\omega)}{1 + i\omega CZ(\omega)} = \Xi(\omega)Z(\omega), \tag{3}
\]

analogous to the renormalization of the propagator standard Quantum Electro-Dynamics (QED) which often leads to a (Lehmann) factor \(30,31\) analogous to \(\Xi(\omega)\).

To elucidate this, consider the Green’s function of the environment \(G(\omega) = −e^2i\omega^{-1}Z(\omega)\), the Fourier transform of the effective phase \(\theta'\) of the Fourier transform of the action \(S_z^{\text{int}}\) given in eq. (S17a),

\[
S_z^{\text{int}}|_{\hbar=0} = \frac{(2\pi)^2}{4\pi} \int_{-\infty}^{+\infty} d\omega e^{\imath \phi'(\omega)} G^{-1}(\omega)\phi'(-\omega) \tag{4}
\]

when we neglect the junction impedance \(1/i\omega C\). Motivated by eq. (3), we can assume that the Fourier transform of the finite temperature propagator is given by

\[
D^*(\omega) = \frac{-\kappa^2}{2\pi i} \left[ G(\omega) \langle aa^\dagger \rangle_{\omega} + G(−\omega) \langle a^\dagger a \rangle_{−\omega} \right] = \frac{-\kappa^2}{2\pi i} \left[ G(−\omega) \langle a^\dagger a \rangle_{−\omega} + G(\omega) \langle aa^\dagger \rangle_{\omega} \right] \tag{5}
\]

where \(\kappa e = 1e, 2e\) is the Cooper pair, BCS quasi-particle charge, \(a, a^\dagger\) is the photon annihilation, creation operator and the prefactor \(\kappa^2\) included for completeness. Negative frequencies are allowed since we are interested in energy differences due to single photon emission and absorption processes. For instance, processes where a photon is created before annihilation are related to processes where a photon is annihilated before creation by reversing the sign of the frequency \(\omega\) and re-ordering the \(a^\dagger, a\) operators appropriately [eq. (5)]. In fact, the operators \(a, a^\dagger\) can be redefined into a single operator function that is frequency dependent using the Heaviside function \(\theta(\omega),\)

\[
\begin{align*}
  a(\omega) &= a^\dagger, \quad (6a) \\
  a(-\omega) &= a, \quad (6b) \\
  [a(\omega), a'(\omega')] &= [\theta(\omega') - \theta(\omega)] \delta_{\omega,\omega'}, \quad (6c)
\end{align*}
\]

which clearly displays the roles of positive and negative frequencies. Consequently, processes with \(D^*(\omega) = \frac{-\kappa^2}{2\pi i} \left[ G(−\omega) \langle aa^\dagger \rangle_{−\omega} + G(\omega) \langle a^\dagger a \rangle_{\omega} \right]\) are forbidden since represent actual negative frequency photons.

We proceed to include the effect of \(S_z^0(\phi')\) from a finite tunnel junction impedance by introducing the potential \(U(\omega) =\)
\( e^{-2(\alpha^2 C - \sum_n L_n)} = -e^{-2i\omega}(\Omega). \) Assuming the effective propagator is given by the sum of all the single photon interaction is given by

\[
U = \sum_{\alpha} \frac{e^{i\omega\alpha}}{\omega - \alpha}.
\]

Thus, the e\( \text{fff} \)ffective ac- tance due to \( U(\omega) \) and is proportional to \( \langle aa' \rangle \) for positive and \( \langle a^i a^j \rangle \) for negative frequencies [eq. (5)], we find

\[
D_{\text{eff}}(\omega) = \frac{-k^2}{2\pi i} \{ G(\omega) + G(\omega)U(\omega)G(\omega) + ... \} \langle aa' \rangle \omega + \frac{-k^2}{2\pi i} \{ G(-\omega) + G(-\omega)U(-\omega)G(-\omega) + ... \} \langle a^i a^j \rangle_{-\omega}.
\]

The term proportional to \( [U(\pm\omega)G(\pm\omega)]^n \) is the finite temperature propagator for the photon interacting \( n \) times with the junction impedance and the perturbation series

\[
\sum_{n=0}^{+\infty} [U(\pm\omega)G(\pm\omega)]^n = [1 + \gamma(\pm\omega)Z(\pm\omega)]^{-1} = 1 - \gamma(\pm\omega)Z_{\text{eff}}(\pm\omega) \equiv \Xi(\pm\omega)
\]

is computed by analytic continuation of the series \( 1 + x + x^2 \cdots + x^n \to [1 - x]^{-1} \) as represented in Fig. (2). Comparing eq.(5) to (7a), we find that \( G(\omega) \) is renormalized to \( G_{\text{eff}}(\omega) = [G^{-1}(\omega) - U(\omega)]^{-1} \). Consequently, the effective action is given by

\[
S'_{\text{eff}} \bigg|_{t=0} = \frac{(2\pi)^2}{4\pi} \int_{-\infty}^{+\infty} d\omega \phi'(\omega)G_{\text{eff}}^{-1}(\omega)\phi'(-\omega).
\]

Indeed we recover \( J(t) = e^2 \left[ D_{\text{eff}}^*(t) - D_{\text{eff}}(0) \right] \), where \( D_{\text{eff}}^*(t) = \int d\omega D_{\text{eff}}^*(\omega) \exp(-i\omega t) \) is the Fourier transform of \( D_{\text{eff}}(\omega) \), by substituting \( \langle aa' \rangle_{-\omega} = n(\omega) + 1 = [1 - \exp(-\beta\omega)]^{-1} \) and \( \langle a^i a^j \rangle_{-\omega} = n(\omega) + 1 = [1 - \exp(-\beta\omega)]^{-1} \) in eq. (5) and (8), where the analytic continuation

\[
\sum_{n=0}^{+\infty} \exp(\pi\beta n|\omega|) = \frac{1}{1 - \exp(\pi\beta|\omega|)}
\]

has been used to regularize the divergent sum when calculating \( \langle \cdots \rangle_{-\omega} \) for negative frequencies.

\[ C. \text{Vacuum Excitation and Photon amplitudes} \]

Notice that in eq. (7a), the factor \( [1 - U(\omega)G(\omega)]^{-1} = \Xi(\omega) \equiv |\Xi(\omega)| \exp \{ i\eta(\omega) \} = |\Xi(\omega)| \exp \{ i\eta_\text{m} \} \) either renormalizes \( G(\omega) \) or the photon number thermal averages \( \langle \cdots \rangle \), implying that the photon number states are modified by the junction impedance. Rearranging, we find

\[
G_{\text{eff}}(\omega) \langle a(-\omega)a(\omega) \rangle_{\omega} = G(\omega)|\Xi(\omega)| \langle a(-\omega)a(\omega) \rangle_{\omega} = G(\omega) \langle |\Xi(\omega)| \exp \{ i\eta(\omega) \} a(-\omega)a(\omega) \rangle_{\omega}
\]

\[
= G(\omega)Z_{\text{ph}}^{-1} \sum_{n=0}^{+\infty} \langle n|a(-\omega)a(\omega) \rangle \exp \left\{ -\beta\omega \left[ a(-\omega)a(\omega) + \frac{\eta(\omega)}{2} \right] + \ln|\Xi(\omega)| \right\} |n\rangle_{\omega}
\]

\[
\equiv G(\omega)Z_{\text{ph}}^{-1} \sum_{n=0}^{+\infty} \langle n|a(-\omega)a(\omega) \rangle \exp \left\{ -\beta \left[ \omega[a(-\omega)a(\omega) + \frac{\eta(\omega)}{2}] + M - i\eta_\text{m} \right] \right\} |n\rangle_{\omega}, \quad (10)
\]

where \( \text{sgn}(\omega) \) is the sign function, \( M - i\eta_\text{m} = -\beta^{-1} \ln|\Xi(\omega)| \) is a gap in the electromagnetic energy spectrum. Taking \( M(\omega) \) to be positive definite leads to \( |\Xi(\omega)|^2 \leq 1 \).

Thus, \( M \) is the energy of a particle excited from the vacuum by the electromagnetic field where the probability that the vacuum will be excited is given by the Boltzmann distribution \( |\Xi(\omega)|^2 = \exp(-2\beta M) \). Since this excitation carries electromagnetic energy, \( |\Xi(\omega)|^2 \) gives the fraction of electro-
magnetic power absorbed via excitations at the capacitor C by the junction.\textsuperscript{23} We note that the complex nature of $\beta^{-1} \ln \Xi(\omega)$ is not a concern since we are already accustomed to shifting our frequencies or energies by an infinitesimal [e.g. $\omega$ to $\omega + i \delta$ in eq. S17b]. Taking in eq. (7b) the impedance $Z(\omega) = R$ to be real and $y(\omega) = i\omega C$, corresponding to eq. S11, yields

$$\arctan \omega RC = \eta(\omega) = \epsilon_m.$$ \hfill (11)

We can determine the statistics of the particle by identifying $\epsilon_m = (2m + 1)\pi$ or $\epsilon_m = 2\pi m$ as the fermionic or bosonic Matsubara frequency\textsuperscript{24} respectively where $m \in \mathbb{Z}$. This requires the oscillation period $2\pi/\omega$ of the electromagnetic field to greatly exceed the relaxation time $RC$ of the circuit, $2\pi/\omega \gg 2\pi RC$. Thus, when this condition is not satisfied, it leaves the possibility for anyons\textsuperscript{34} with exotic statistics. Consequently, we can take $\Xi(\omega)$ as the amplitude\textsuperscript{35} that a photon of frequency $\omega$ is absorbed by the junction creating a particle of mass $M$ and statistics according to the Matsubara frequency $\epsilon_m$. This realization, together with the fact that the field theory introduced in Sec. II B lives in $1 + 2$ dimensions, suggests that anyonic excitations cannot be ignored.\textsuperscript{34}

Finally, notice that the exponent can be re-written conveniently as

$$G^{-1}_e(M, i\epsilon_m) + \frac{1}{2} \coth(\beta \omega/2)$$

$$= G^{-1}_e(M, i\epsilon_m) + \sum_{m=-\infty}^{\infty} \frac{1}{2\pi m - \beta \omega}$$ \hfill (12a)

with $G^{-1}_e(M, i\epsilon_m) = \beta(M - i\epsilon_m)$ and the statistics of the excitations computed as,

$$\sum_{m=-\infty}^{\infty} G_e(M, i\epsilon_m) = \sum_{m=-\infty}^{\infty} \beta^{-1} \frac{1}{M - i\epsilon_m}$$ \hfill (12b)

where eq. (12b) is the inverse thermal Green’s function of the particle with mass $M$.

1. Renormalization of Applied Alternating Voltage Amplitude and Time Reversal Symmetry Violation

In Sec. II C, we have established that the fraction of photons absorbed by a tunnel junction is $|\Xi(\omega)|^2 = [1 + y(\omega)\Xi(\omega)]^{-2}$. A straightforward way to experimentally measure $|\Xi(\omega)|^2$ is applying an external oscillating electric field in the form of ac voltage $\tilde{V}(t)$ supplying power $P \propto \int T/2 dt \tilde{V}^2(t)$ where $T$ is the oscillation period. Whether the ac power is efficiently transferred to the junction from this ac source ought to depend on $\Xi(\omega)$. We proceed to formally express the explicit form of the effective alternating voltage at the junction.

This can be done by substituting $\Delta \phi(t) = \Delta \phi(t - T - 0^+) = \int_0^T V_e(\tau) d\tau$ in eq. (S7) where $V_e = V + \tilde{V}(t)$ and $\tilde{V}(t)$ is an alternating voltage corresponding to the effect of the RF field given by

$$\tilde{V}(t) = \int_{-\infty}^{+\infty} \tilde{V}(\omega) \exp(-i\omega t) d\omega = V_{ac} \cos \{\Omega t\}$$ \hfill (13a)$$

and

$$V(\omega) = \frac{V_{ac}}{2} \left[(\delta(\Omega - \omega) + \delta(\Omega + \omega))\right]$$ \hfill (13b)

where $\tilde{V}(\omega)$, $V_{ac}$ and $\Omega$ is the spectrum, the amplitude and frequency of the RF field respectively and the significance of the $+$ sign is that $0^+$ is experimentally, not computationally equal to 0; it is the limit $0^+ \equiv t \to 0$. An even spectrum guarantees that the voltage $\tilde{V}(t)$ (electric field) is invariant under time-reversal meaning that, $t \to -t$ should not alter the polarity of the voltage, $\tilde{V}(t) = \tilde{V}(t)$.

Proceeding, the applied power $P$ of the RF field is the mean-square value of $\tilde{V}(t)$ given by

$$P = \frac{1}{\pi/\Omega} \int_{-\pi/\Omega}^{\pi/\Omega} \frac{|\tilde{V}(\omega)|^2}{Z_0} d\omega = \frac{V_{ac}^2}{Z_0}$$ \hfill (14)

where $Z_0$ is the characteristic impedance typically chosen to be 50 $\Omega$ for RF and microwave applications.

However, we are interested in the power absorbed by the junction $P_j$ instead since it modifies the $I - V$ characteristics. In Sec. II C, we argued that this power $P_j$ is proportional to $|\Xi(\Omega)|^2$ in the presence of bosonic, anyonic or fermionic excitations.

2. Linear Response

Within the context of linear response theory\textsuperscript{36}, this means that the applied voltage $\tilde{V}(t)$ acts as an external force, and the effective voltage as a linear response $\tilde{V}'(t)$ of the circuit,

$$\tilde{V}'(t) = \int_{-\infty}^{\infty} \chi(t - s) \tilde{V}(s) ds,$$ \hfill (15a)$$

$$\tilde{V}'(\omega) = \Xi(\omega) \tilde{V}(\omega)$$ \hfill (15b)$$

$$\Xi(\omega) = \int_0^{\infty} \chi(t) \exp(i\omega t) dt,$$ \hfill (15c)

where $\chi(t - s)$ is the response function, making $\Xi(\omega)$ the susceptibility. For the special case discussed in eq. (11), we have $\chi(t) = (1/RC) \exp(-t/RC)$. Thus, the RF spectrum above gets modified to

$$\tilde{V}'(\omega) = \frac{V_{ac}}{2} \Xi(\omega) \left[(\delta(\Omega - \omega) + \delta(\Omega + \omega))\right],$$ \hfill (16a)$$

$$\tilde{V}'(t) = \int_{-\infty}^{+\infty} \tilde{V}(\omega) \exp(-i\omega t) d\omega,$$

$$= |\Xi(\Omega)| V_{ac} \cos \{\Omega t + \epsilon_m\},$$ \hfill (16b)

and $P_j$ is given by

$$P_j = \frac{1}{\pi/\Omega} \int_{-\pi/\Omega}^{\pi/\Omega} \frac{|\tilde{V}'(\omega)|^2}{Z_0} d\omega = \frac{V_{ac}^2}{Z_0} |\Xi(\Omega)|^2 = P|\Xi(\Omega)|^2,$$ \hfill (17)
where we have used eq. (13), (14) and (16). Rearranging, we find

$$V_{ac}^{\text{eff}} = \sqrt{P |\Xi(\Omega)|^2} Z_0 = V_{ac} |\Xi(\Omega)|, \quad (18a)$$
$$Z_0 = Z_0 |\Xi(\Omega)|^2. \quad (18b)$$

We conclude that the renormalization effect $P \geq P_1$ leads to a load mismatch, $Z_0' \geq Z_0$. As expected, anyons ($\epsilon_m \neq m\pi$) violate time reversal symmetry of the oscillating electric field we sought to guarantee with the even voltage spectrum.

### 3. Unitarity and the topological potential

Defining matrices $V_x$ and $U_{RF}$ and two quantum states $\psi_1$ and $\psi_2$ of the junction and the environment respectively,

$$V_x = \frac{1}{2} \{ Vx_0 + V_{ac} U_{RF} \}, \quad (19a)$$
$$U_{RF} = \left( \begin{array}{c} \Xi(\Omega) e^{i\delta} \\ \sqrt{1 - |\Xi(\Omega)|^2} e^{-i\delta} \\ \end{array} \right), \quad (19b)$$
$$V'_x = \text{tr} \{ V_x \} = V + \int_{-\infty}^{\infty} \tilde{V}(\omega) \exp(-i\omega t) d\omega, \quad (19d)$$
$$\text{det} \{ U_{RF} \} = 1, \quad (19e)$$
$$U_{RF}' U_{RF} = U_{RF} U_{RF}' = 1, \quad (19f)$$

where $\sigma_0$ is the $2 \times 2$ identity matrix, we find that these states $\psi_1 = |1\rangle$, $\psi'_1 = |1'\rangle$ and $\psi_2 = |0\rangle$, $\psi'_2 = |0'\rangle$ are normalized $\langle 0|0\rangle = \langle 0'|0'\rangle = 1$, $\langle 1|1\rangle = \langle 1'|1'\rangle = 1$ and orthogonal to each other, $\langle 0|1\rangle = \langle 0'|1'\rangle = 0$, while orthogonality is preserved under the unitary transformation $U_{RF}$ leading to a renormalized external voltage $V_x = V + V_{ac} \cos(\Omega t)$ yielding $V'_x = \text{tr} \{ V'_x \} = V_x + A(t)$. This requires the topological flux $\Delta \Phi(t) \neq 0$ in eq. (20) not vanish in the presence of oscillating electromagnetic fields. Solving for the topological potential $A(t)$, we find

$$A(t) = V'_x - V_x = V_{ac} \{ \text{v}(\Omega) \sin \Omega t - u(\Omega) \cos \Omega t \} \quad (20a)$$
$$\Gamma(\Omega) = u(\Omega) + i\text{v}(\Omega) \quad (20b)$$
$$\Xi(\Omega) = 1 - \Gamma(\Omega) = 1 - u(\Omega) - i\text{v}(\Omega) \quad (20c)$$

with eq. (20c) relating the impedance renormalization factor $\Xi(\Omega)$ to the topological potential amplitude factors $u(\Omega), v(\Omega)$.

Thus, the purpose of the topological potential is to implement the renormalization scheme above. By eq. (20), we find that the topological flux $\Phi(t) = \int_0^\infty A(\tau) d\tau$ is ill-defined for $\tau = -\infty$ since $\sin(\omega \infty)$ and $\cos(\omega \infty)$ both oscillate rapidly without converging. Nonetheless, this poses no problem since it is the flux difference $\Delta \Phi(t) = \int_0^\infty A(t) d\tau$ that appears in the correlation function in eq. (20) rendering the $I-V$ characteristics in eq. (20) perfectly well-defined.

Moreover, by eq. (7b), we find that

$$\Xi(\Omega) = \frac{y^{-1}(\Omega)}{\pi \Gamma(\Omega)^{\gamma(\Omega)}} = \frac{1}{1 + \pi \Gamma(\Omega)^{\gamma(\Omega)}} \quad (21)$$
$$\Gamma(\Omega) = \frac{\pi \Gamma(\Omega)^{\gamma(\Omega)}}{\pi \Gamma(\Omega)^{\gamma(\Omega)}} = \pi \Gamma(\Omega)^{\gamma(\Omega)} \quad (22)$$

are ratios of impedances. Thus, in the simple model in eq. (S11), power renormalization is negligible ($\Xi(\Omega) \approx 1$) only for extremely low frequencies satisfying $1/RC \ll \Omega$. However, for samples exhibiting Coulomb blockade that satisfy the Lorentzian-delta function approximation $\text{Re} \{ Z_{RF} \} = R/(1 + \Omega^2 C^2 R^2) \sim \pi C^{-1} \delta(\Omega)$, the conductance $R^{-1}$ is extremely small ($1/RC \ll \Omega$) and thus we should expect power renormalization for virtually all applied frequencies.

### 4. Current–Voltage Characteristics with finite RF Field

Finally, substituting $V'_x$ in eq. (24) yields,

$$I(V) = \sum_{n=-\infty}^{\infty} J_n \left( \frac{2eV_{ac}^{\text{eff}}}{\Omega} \right) I_1 \left( V - \frac{ne\Omega}{e} \right) + \sum_{n=-\infty}^{\infty} J_n \left( \frac{2eV_{ac}^{\text{eff}}}{\Omega} \right) I_2 \left( V - \frac{ne\Omega}{2e} \right). \quad (23)$$

Here, $I_{1,2}$ are the quasi-particle, Cooper pair RF-free $I-V$ characteristics given in eq. (24), $J_n(x)$ are Bessel functions of the first kind where the order $n$ is the number of actual photons absorbed by the junction, $V_{ac}$ is the amplitude of the alternating voltage, $\Omega$ is the energy quantum of individual photons, $V$ and $I$ respectively are the applied dc voltage and tunneling current of the junction. The total current is shifted by the number of photons reflecting energy conservation and is proportional to the square of the Bessel function reflecting the modification of density of states. This equation neglects higher harmonics derived in 19 which we can assume are largely suppressed.

### III. JOSEPHSON JUNCTIONS ARRAYS AS EFFECTIVE SINGLE JUNCTIONS

#### A. Phase-Phase Correlation Approach

It is prudent to highlight the ingredients that went into deriving the $I-V$ characteristics of the single small Josephson junction given in eq. (24):

1) The Caldeira-Leggett action $S_\Phi(\Phi)$ that is varied with respect to $\phi = \phi_1 + \phi_2$ to obtain the equation of motion for the single large Josephson junction; 2) the correlation $\langle \cos[\Delta \Phi(t)] \rangle$ calculated with respect to $\phi_2$; 3) The condition $\sum_i \phi_i = e\Phi$ enforced by the circuit.

In the case of a one dimensional array of $N_0$ small Josephson junctions, it is clear that constraint 3) has to include all the phases $\phi_{i=1} \cdots \phi_{i=N_0}$ of the junctions along the array, and the
quantum average in 2) taken over each phase where the action in 1) is the sum of the action of individual junctions in the array. To simplify the calculation, one assumes all the junctions have the same structure constant \( \alpha_k(t) \) and calculates the quantum average \( \langle \sin [k\Delta \phi_{1}(t)] \rangle_{\phi_{1} \sim \phi_{0}} \) [step 2]). Since the same condition 3) as before. Hence, treating the other junctions as environments each with an effective action of the form \( S'_{\phi}(\phi_{i}) \), we have,

\[
I_{A}(V) = \frac{ie}{2} \sum_{x=1}^{N} \int_{-\infty}^{+\infty} dt \alpha_{x}(t) \langle \sin [k\Delta \phi_{1}(t)] \rangle_{\phi_{1} \sim \phi_{0}}
\]

\[
= -e\ell \sum_{x=1}^{N} \int_{-\infty}^{+\infty} dt \alpha_{x}(t) \sin \left[ ke \int_{0}^{t} A(\tau)d\tau + k\Delta \phi_{x}(t) \right] \int \prod_{x=1}^{N} D\phi_{x} \exp is'_{\phi}(\phi_{x})[\cos k\Delta \phi_{x}(t)],
\]

where,

\[
- \langle \sin [k\Delta \phi_{1}(t)] \rangle_{\phi_{1} \sim \phi_{0}} = \langle \cos [k\Delta \phi_{x}(t)] \rangle_{\phi_{x}} \langle \sin [\sum_{j=2}^{N} \Delta \phi_{j}(t) + ke \int_{0}^{t} A(\tau)d\tau + k\Delta \phi_{x}(t)] \rangle_{\phi_{x} \sim \phi_{0}},
\]

\[
= \prod_{x'}^{N} \langle \cos [k\Delta \phi_{x'}(t)] \rangle_{\phi_{x'}} \sin \left[ ke \int_{0}^{t} A(\tau)d\tau + k\Delta \phi_{x'}(t) \right] \exp \left[ -\frac{k^{2}}{2} \sum_{x'}^{N} \langle \Delta \phi_{x'}(t) \rangle_{\phi_{x'}} \right] \sin \left[ ke \int_{0}^{t} A(\tau)d\tau + k\Delta \phi_{x'}(t) \right],
\]

\[
= \exp \kappa^{2} \sum_{x'}^{N} J_{x'}(t) \sin \left[ ke \int_{0}^{t} A(\tau)d\tau + k\Delta \phi_{x'}(t) \right] \prod_{x'}^{N} P_{x'}(t) \sin \left[ ke \int_{0}^{t} A(\tau)d\tau + k\Delta \phi_{x'}(t) \right].
\]

Evidently, the current depends on the product

\[
P_{x}^{A}(t) = \prod_{x'}^{N} P_{x'}^{A}(t)
\]

as expected, since, it is comprised of individual tunneling events at each junction. Since, \( 1/2\pi \int dt \prod_{x}^{N} P_{x}^{A}(t) \exp i\beta t \) is the probability that the array will absorb energy \( E \) from the environment, we discover that the tunnel current obeys the product rule of probabilities. For identical junctions of capacitance \( C \) and impedance \( Z(\omega) \),

\[
\langle \phi_{x}(t) \phi_{x}(0) \rangle = \langle \phi_{x'}(t) \phi_{x'}(0) \rangle = \cdots = \langle \phi_{x'}(n_{0}(t)) \phi_{x'}(n_{0}(s)) \rangle,
\]

we have \( \prod_{x'}^{N} P_{x'}^{A}(t) = [P_{x}(t)]^{N_{0}} \) where

\[
[P_{x}(t)]^{N_{0}} = \exp \left( N_{0}k^{2} J_{x}(t) \right) \quad (25)
\]

where \( Z_{\text{eff}}^{A}(\omega) \) in \( J_{x}(t) = \langle \{ \phi_{x}(t) - \phi_{x}(0) \} \phi_{x}(0) \rangle_{\phi_{x}} \) may differ from \( Z_{\text{eff}}(\omega) \) in eq. (25a) due to possible interaction terms. Neglecting these interactions by setting \( J_{x}(t) \approx J(t) \), eq. (25) implies that at zero temperature, Cooper pair Coulomb blockade threshold voltage \( V_{\text{ct}}^{A} = N_{0}V_{\text{cb}} \) for the array is a factor \( N_{0} \) larger than for the single junction.

However, the rest of the array \( \phi_{x'}(\cdots \phi_{x'}(n_{0})) \) acts as the environment for the single junction \( \phi_{x}(s) \) thus introducing interaction terms. In particular, the single junction interacts with the rest of the array electromagnetically. Since, in the presence of Cooper pair solitons\(^{37,38} \) and the Meissner effect, the electromagnetic field has a finite range within an infinite long array \( (N_{0} \gg 1) \), the array has a cut-off number of junctions beyond which no electromagnetic interactions occur. Consequently, the effective number of junctions \( N_{c} \leq N_{0} - 1 \) acting as the environment will be determined by the range of the electromagnetic field. \( N_{c}(\Lambda) \) is independent of \( H \) when the superconducting islands are shorter than the penetration depth of the magnetic field. It can be evaluated by equating the Coulomb blockade voltage \( V_{\text{cb}} \) estimated by replacing \( N_{0} \) with \( N_{c}(\Lambda) \) and setting \( \text{Re} \left[ Z_{\text{ct}}^{A}(\omega) \right] \approx \text{Re} \left[ Z_{\text{eff}}(\omega) \right] \) in eq. (25) to the standard expression for the soliton threshold voltage\(^{22} \) of the array, \( eV_{\text{ct}}^{A} = eV_{\text{cb}} \approx 2e|\Lambda|^{-1} \), leading to

\[
N_{0} \rightarrow N_{c}(\Lambda) = \frac{1}{\text{exp}(\Lambda^{-1}) - 1},
\]

which approaches the soliton length \( N_{c}(\Lambda) \rightarrow \Lambda \) when \( \Lambda \gg 1 \). However, the infinite array effectively has

\[
N_{c}(\Lambda) + 1 = \frac{1}{1 - \exp(\Lambda^{-1})},
\]

junctons. This means that \( N_{0} \) in eq. (25) is instead renormalized to \( N_{c}(\Lambda) + 1 \). Since \( V_{\text{ct}}^{A}(\Lambda) \) should be invariant under the transformation \( N_{c}(\Lambda) \rightarrow N_{c}(\Lambda) + 1 \), we find

\[
\lim_{R \rightarrow +\infty} \text{Re} \left[ Z_{\text{ct}}^{A}(\omega) \right] = \lim_{R \rightarrow +\infty} \text{Re} \left[ Z_{\text{eff}}(\omega) \right] \rightarrow \lim_{R \rightarrow +\infty} \text{Re} \left[ Z_{\text{eff}}^{A}(\omega) \right] = \exp(\Lambda^{-1}) \lim_{R \rightarrow +\infty} \text{Re} \left[ Z_{\text{eff}}(\omega) \right],
\]

(27a)
we discover that switching on electromagnetic interactions leads to impedance renormalization and a renormalized response function given by \( \Xi(\omega) \rightarrow \Xi(\omega) = \exp(-\Lambda^{-1}) \Xi(\omega) \).

IV. DISCUSSION

a. Implication In the case of the single Josephson junction, the renormalization of the amplitude of applied oscillating electromagnetic fields is implemented by linear response. This entails the excitation of particles appearing as a mass gap \( M - i\varepsilon_m = -\beta^{-1}\ln \Xi(\omega) \) in the thermal radiation spectrum, where \( \varepsilon_m \) is the Matsubara frequency. \( \Xi(\omega) = [1 + \gamma(\omega)Z(\omega)]^{-1} \) is the linear response function, \( Z(\omega) \) is the environmental impedance of the junction and \( \gamma(\omega) \approx i\omega C \) is the impedance of the junction.

When an infinitely long array is modeled as a single junction interacting with the rest of the array, we find an additional renormalization factor \( |\Xi_\alpha| = \exp(-\Lambda^{-1}) \) by considering a finite electric field interaction range \( \langle N_\beta \rangle = N_\alpha(\Lambda) + 1 = [1 - \exp(-\Lambda^{-1})]^{-1} \) of the single junction with the rest of the array as an effective environment. This finite range arises due to the presence of charge solitons in the array, suggesting that their role is to endow the photon with a mass \( \sim [N_\alpha(\Lambda) + 1]^{-1} \). This is dual to the Meissner effect where the Cooper-pair order parameter leads to a finite range of the magnetic field.

A Josephson junction circuit that exhibits a large Coulomb blockade voltage is ideal for the observation of the renormalization effect. In particular, for the single junction, power renormalization is negligible (\( \Xi(\Omega) \approx 1 \)) only for extremely low microwave frequencies satisfying \( 1/RC \gg \Omega \). However, for samples exhibiting Coulomb blockade that also satisfy the Lorentzian-delta function approximation \( \Re[\{\Xi(\Omega)\} = R/(1 + \Omega^2C^2R^2) \approx \pi C^{-1}\delta(\Omega) \), the conductance \( R^{-1} \) is extremely small \( (1/RC \ll \Omega) \) and thus we should expect power renormalization for virtually all applied frequencies. In the case of long arrays \( \langle N_\beta \rangle \gg \Lambda \) with \( \Xi(\Omega) \approx 1 \), RF amplitude renormalization should be readily observed due to the additional factor \( |\Xi_\alpha(\Omega)| \approx \exp(-\Lambda^{-1}) \). Finally, a list of the electromagnetic quantities and their renormalization formulae is displayed in TABLE I.

b. Summary We have employed path integral formalism to derive the Cooper-pair current and the BCS quasiparticle current in small Josephson junctions and introduced a model which transforms the infinitely long array into an effective single junction with a renormalized environmental impedance, \( Z_{\text{eff}} = exp(-\Lambda^{-1})\Xi(\omega)Z(\omega) \), where \( Z(\omega) \) is the environmental impedance as seen by a single junction in the array. As is the case for the single junction, we expect that \( \Xi(\omega) = \exp[-\beta(M(\omega) - \Lambda^{-1})] \exp i\varepsilon_m \) also acts as a linear response function for oscillating electromagnetic fields, and can be interpreted as the probability amplitude of exciting a particle of mass \( M + \beta^{-1}\Lambda^{-1} \) from the junction ground state by the electromagnetic field. The quantum statistics of this particle are determined by the complex phase \( \varepsilon_m \) identified as the Matsubara frequency. In the case of the infinite array, this particle corresponds to a bosonic charge soliton injected into the array.

c. Application Since the quasi-particle current naturally reduces to the normal current and the supercurrent vanishes when the superconducting gap goes to \( \Delta = 0 \), the final expression of the tunnel current eq. (23) is essentially the same for high averaged current result previously proposed in ref.19.

In the classical limit when the RF frequency \( \Omega \) is low compared to the amplitude of the alternating voltage \( \sqrt{v_{\text{ac}}^2}/h\Omega \gg 1 \), multi-photon absorption occurs. Setting \( keV_{\text{ac}}\sin \theta = \pi h\Omega \), the sum over photon number can be approximated by an integral formula that corresponds to a classical detection of the RF field,

\[
I(V) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} 0 (V - |\Xi(\Omega)|v_{\text{ac}}\sin \theta) d\theta.
\]

where \( I_0(V) \) is given by eq. (S24). This result offers a way to measure the magnitude of the renormalization factor \( \Xi(\Omega) \), where \( (\Xi(\Omega)) \) is the sensitivity of the detector to RF power. Conversely, this implies that our results are indispensable in dynamical Coulomb blockade experiments where long arrays are used as detectors of oscillating electromagnetic fields.

V. ACKNOWLEDGEMENT

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TABLE I. Electromagnetic quantities and their renormalization formulae.

| Quantity                              | Expression | Renormalized Expression |
|---------------------------------------|------------|-------------------------|
| Renormalization factor                | 1          | $\Xi(\omega) = [1 + i\omega C Z(\omega)]^{-1}$ |
| Environmental impedance              | $Z(\omega)$ | $Z_{\text{eff}}(\omega) = \Xi(\omega) Z(\omega)$ |
| Coulomb blockade voltage (single junction) | $V_{\text{cb}} = 0$ | $V_{\text{cb}} = \kappa e / C$ |
| Microwave amplitude (single junction) | $V_{ac}$   | $V_{\text{ac}} = \Xi(\Omega) V_{ac}$ |
| Microwave amplitude (long array)      | $V_{ac}$   | $V_{\text{ac}} = \exp(-\Lambda^{-1}) \Xi(\Omega) V_{ac}$ |
| Effective number of junctions (long array) | $\langle N_0 \rangle = [1 - \exp(-\Lambda^{-1})]^{-1}$ | $N_{\text{eff}}(\Lambda) = \exp(-\Lambda^{-1}) \langle N_0 \rangle$ |
| Coulomb blockade voltage (long array) | $V_{\text{cb}}^A = \langle N_0 \rangle V_{\text{cb}}$ | $V_{\text{cb}}^A(\Lambda) = N_{\text{eff}}(\Lambda) V_{\text{cb}}$ |

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Appendix B

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Appendix A: Causal Linear Response

Appendix A is meant for the skimming reader, who wants a quick reference to linear response (and its relevance to microwave power renormalization). Thus, we do not strive to introduce the entire subject of linear response and its subtleties. (For a comprehensive introduction to the subject, see ref. 36.)
Within Linear Response Theory, the response $\tilde{R}(t)$ of a system is related to the driving force, $\tilde{F}(t)$ by the central causal relation

$$\tilde{R}(t) = \int_{-\infty}^{t} \chi(t-s)\tilde{F}(s)ds,$$

where $\chi(t)$ is the response function. The system variable, $\tilde{R}(t)$ obeys some equation of motion,

$$f(\partial/\partial t)\tilde{R}(t) = \tilde{F}(t),$$

with $f(\partial/\partial t)$ a function of $\partial/\partial t$. Introducing the Green’s function of the system, $G_{\tilde{R}}(t)$, satisfying,

$$f(\partial/\partial t)G_{\tilde{R}}(t-s) = \delta(t-s),$$

we see that,

$$f(\partial/\partial t)\tilde{R}(t) = \int_{-\infty}^{t} f(\partial/\partial t)G_{\tilde{R}}(t-s)\tilde{F}(s)ds = \int_{-\infty}^{t} \delta(t-s)\tilde{F}(s)ds = \int_{0}^{t} \delta(\tau)\tilde{F}(t-\tau)d\tau = \int_{-\infty}^{t} \delta(\tau)\theta(\tau + 0^+)\tilde{F}(t-\tau)d\tau = \theta(0^+)\tilde{F}(t) = \tilde{F}(t)$$

with $\theta(\tau)$ the Heaviside function. Thus, we can equate the Green’s function to the response function: $\chi(t-s) = G_{\tilde{R}}(t-s)$. Substituting the Fourier transforms of $\tilde{R}(t) = \int d\omega \tilde{R}(\omega)\exp(-i\omega t)$ and $\tilde{F}(t) = \int d\omega \tilde{F}(\omega)\exp(-i\omega t)$ into eq. (A1),

$$\int d\omega \tilde{R}(\omega)\exp(-i\omega t) = \int_{-\infty}^{\infty} \int \tilde{F}(\omega)\exp(-i\omega s)\chi(t-s)d\omega ds \int \left\{ \tilde{F}(\omega) \int_{0}^{\infty} \chi(\tau)\exp(i\omega\tau)d\tau \right\} \exp(-i\omega t)d\omega = \int \left[ \tilde{F}(\omega) \Xi(\omega) \right] \exp(-i\omega t)d\omega,$$

where $\tau = t-s$ and,

$$\tilde{R}(\omega) = \Xi(\omega)\tilde{F}(\omega),$$

$$\Xi(\omega) = \int_{0}^{\infty} \chi(\tau)\exp(i\omega\tau)d\tau = \int_{-\infty}^{\infty} \theta(\tau)\chi(\tau)\exp(i\omega\tau)d\tau,$$

where $\tau = t-s$ and,

$$\Xi(\omega) = \int_{0}^{\infty} \chi(\tau)\exp(i\omega\tau)d\tau = \int_{-\infty}^{\infty} \theta(\tau)\chi(\tau)\exp(i\omega\tau)d\tau,$$

Finally, that $\Xi(\omega) = \pm \exp(-\beta M)\exp i\eta(\omega)$ acts as the response function to the applied oscillating electromagnetic field is to be understood as the result of the arguments in Sec. II C, and not necessarily the converse. This leaves the possibility that linear response is violated in complicated circuits, where novel physics may lurk.

**Appendix B: Charge Solitons and the Mass Gap in the Electromagnetic Thermal Spectrum**

Consider the charge soliton Lagrangian of the array,

$$L_{sol} = \frac{1}{2\pi} \int dx \left\{ \frac{1}{2} \left( \frac{\partial \chi}{\partial t} \right)^2 - \frac{1}{2} \left( \frac{\partial \chi}{\partial x} \right)^2 - \frac{2}{\chi^2} \sin^2(\chi/2) \right\},$$

where the co-ordinates $x \equiv x/a$, $t \equiv v_0 t/a$ are dimensionless, and $a = 1$ is the length of the islands (lattice constant), $v_0 = 1$ is the velocity of electromagnetic radiation along $x$. The Euler-Lagrange equations of eq. (B1a) yield the solution,

$$\chi = 4 \arctan \exp \left\{ \Lambda^{-1} \gamma(x \pm vt) \right\} + 2\pi n,$$

with $\gamma = 1/\sqrt{1-v^2}$ the Lorentz factor. Plugging in eq. (B1b) into eq. (B1a), we can eliminate $\partial/\partial t$ in favor of $\partial/\partial x$,

$$\left( \frac{\partial \chi}{\partial t} \right)^2 = v^2 \left( \frac{\partial \chi}{\partial x} \right)^2,$$

$$L_{sol} = -\frac{1}{\pi} \int dx \left\{ \frac{1}{4\gamma^2} \left( \frac{\partial \chi}{\partial x} \right)^2 + \Lambda^{-2} \sin^2(\chi/2) \right\}.$$

Observing that since the integrand is quadratic, we can apply the Bogomol’nyi inequality \((A^2 + B^2 \geq 2|AB|)\) to evaluate the mass, \(M\) given by,

\[
\mathcal{L}_{sol} \geq M = \frac{1}{\pi} \int dx \left| \frac{1}{2\gamma \Lambda} \left( \frac{\partial \chi}{\partial x} \right) \sin(\chi/2) \right| = \frac{1}{\gamma \Lambda \pi} \int dx \frac{\partial \cos(\chi/2)}{\partial x} = \frac{1}{\gamma \Lambda \pi} \left| \cos(\chi/2) \right|_{1}^{\infty} = \frac{1}{\gamma \Lambda} \left[ \cos(\pi) - \cos(0) \right] = \frac{2}{\gamma \Lambda} \equiv E_{0}
\]

with \(E_0 = 1/\alpha \pi\). Treating the solitons as charged dust of mass density, \(M/V\) where \(u^\mu = dx^\mu/d\tau = (\gamma, \pm v\gamma, 0, 0)\) is the four-velocity, \(V = A l\) is the volume and \(l\) the length of the array, we can introduce the energy-momentum tensor,

\[
T^{\mu\nu} = \frac{2}{V} \mu u^\mu u^\nu + \varepsilon_0 \varepsilon_r \left\{ F^{\mu\sigma} F_{\sigma\nu} - \frac{1}{4} F^{\sigma\rho} F_{\sigma\rho} \eta^{\mu\nu} \right\}.
\]

The total energy for a single photon frequency mode \(\omega\) is given by,

\[
E = \int_V d^3x \langle T^{00} \rangle = \frac{1}{V} \int_V d^3x \langle M u^0 u^0 \rangle + \frac{1}{V} \sum_{s=\pm 1} \int_V d^3x \omega \left( \langle a_s(\omega) a_s(-\omega) \rangle + \frac{\text{sgn}(\omega)}{2} \right),
\]

with \(s \pm 1\) the photon polarization. Note that,

\[
\langle M u^0 u^0 \rangle = \frac{E_0}{\Lambda} \langle \gamma \rangle \approx \frac{E_0}{\Lambda} + \frac{E_0}{\Lambda} \langle \gamma^2 \rangle = \frac{1}{\Lambda} \left( E_0 + \frac{1}{2\beta} \right),
\]

where \(\langle \cdots \rangle \equiv \left( \int_{-\infty}^{+\infty} dv \exp(-\beta E_0 v^2/2) \right)^{-1} \int_{-\infty}^{+\infty} dv (\cdots \exp(-\beta E_0 v^2/2))\) is the Boltzmann average for a gas of (anti-)solitons in 1 + 1 dimensions. Comparing eq. \((B4)\) to eq. \((10)\) (neglecting the vacuum energy \(E_0/\Lambda\) and considering only one photon polarization mode), we conclude that the array is an effective single junction with \(\Xi = \exp(-\Lambda^{-1}) \exp(2\pi ni)\).

Finally, \(N_c\) [eq. \((26b)\)] is given by [confer: eq. \((12b)\)],

\[
N_c(\Lambda) = \sum_{m=-\infty}^{+\infty} \Lambda^{-1} - \frac{1}{2\pi ni} - \frac{1}{2} = \frac{1}{2} \coth \left( \frac{1}{2\Lambda} \right) - \frac{1}{2}.
\]

This result is not surprising, since we have determined the \(I-V\) characteristics of the array by treating it as a single junction with the rest of the array acting as its environment. This means that the \(N_0 - 1 = k\) junctions themselves act as anyonic excitations whose (average) number \(\langle k \rangle = N_c\) determine the electromagnetic cut-off number, which is also the effective number of junctions that can be approximated as the environment of the effective single junction.
Supplementary Material: The Electromagnetic Environment in Large and Small Josephson Junctions

I. INTRODUCTION

Despite the existence of excellent reviews on the subject, the authors found much of the techniques and prior concepts useful in following the arguments in the main paper are scattered in various literature through the years. Thus, this supplementary material is included here for completeness and compactness.

This supplementary material is organized as follows:

Section II considers how the effects of the electromagnetic environment arises via a normal current in large junctions. In the subsections, II A introduces a 2-spinor and Pauli matrices that act on the spinor to derive the well-known Josephson equations. We proceed in II B to introduce the effect of the environment as a normal current proportional to the electric field in the tunneling direction and a fluctuating noise current whose degrees of freedom we introduce in II C as a Caldeira-Leggett heat bath.

Section III tackles the environment in small junctions. In the subsections, III A introduces the total Hamiltonian of the Josephson junction including the environment and derives an expression for the tunneling current. III B expands this expression into a perturbation series and explicitly calculates the Cooper-pair kernel using the Pauli matrices introduced in II A while III C uses path integral formalism to calculate phase correlation functions in the $P(E)$ function. The expression for the Cooper-pair and quasi-particle tunneling current at finite temperature is derived in III D.

Note that, units where Planck’s constant, Swihart velocity and Boltzman constant are set to unity ($\hbar = \bar{c} = k_b = 1$) and Einstein summation convention are used throughout unless otherwise stated with $\text{diag} \{\eta_{\mu\nu}\} = (1, -1, -1, -1)$ the Minkowski space-time metric and $\eta_{\sigma\rho} \eta^{\sigma\nu} = \delta_\mu^\nu$ the Kronecker delta symbol.

II. JOSEPHSON EFFECT AND THE ELECTROMAGNETIC ENVIRONMENT

A. The Josephson Effect

The physics of Josephson junctions is described by the well known Josephson equations

\[ I_S = 2eE_1 \sin 2\phi_x(t) \]  
\[ \frac{d\phi_x(t)}{dt} = eV_s \]

Here, $\phi_x(t)$ denotes the phase difference $\phi_2(t) - \phi_1(t)$ across the junction and $E_1$ is the Josephson coupling energy. The simplest derivation of eq. (S1) follows from the real and imaginary parts of these two coupled equations

With $I_S/2e = \frac{\partial n_1}{\partial t} - \frac{\partial n_2}{\partial t}$, the Josephson coupling energy is given by $E_1 = 2m_0 \sqrt{n_1 n_2}$ and wave functions by $\psi_1 = \sqrt{n_1} \exp(-i2\phi_1(t))$, $\psi_2 = \sqrt{n_2} \exp(-i2\phi_2(t))$ respectively where we have assumed the junction is biased symmetrically.

Setting $\mu = (\mu_1 + \mu_2)/2 = 0$ allows the introduction of a 2-spinor $\psi^T = \psi_1(1, 0) + \psi_2(0, 1)$ and the Schrödinger equation

\[ i\frac{\partial \psi}{\partial t} = H_{cp} \psi \]

where $\sigma_z (a = 1, 2, 3)$ and $\sigma_0$ are the three Pauli matrices and the $2 \times 2$ identity matrix given by

\[ \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

These matrices will prove extremely useful when we introduce the Cooper-pair tunneling Hamiltonian for small Josephson junctions. Finally, taking the Fourier transform of eq. (S3), we find

\[ eV_s \sigma_3 + \omega \sigma_0 + m_0 \sigma_1 = \hat{0} \]

\[ \rightarrow eV_s \sigma_1 - i\omega \sigma_2 - m_0 \sigma_3 = \hat{0} \]

\[ \rightarrow \hat{W} \cdot \sigma = \hat{0} \]

where $\hat{W} = (eV_s, -i\omega, -m_0)$, $\hat{0}$ annihilates $\psi$, $\sigma$ is the Pauli vector and $\sigma_0$ is the $2 \times 2$ identity matrix.
B. Sources of the Electromagnetic Field as the Josephson Junction Environment

Eq. (S5) represents a closed algebra for the junction. It only considers the superconducting current and thus neglects the environment that lead to effects such as Coulomb blockade. The environment consists of all sources of the electromagnetic field (including the field itself) which couple to the Cooper-pair wavefunction via the phase difference thus altering the $I - V$ characteristics in eq. (S1). Specifically, the environment arises from processes such as the alternating currents and voltages, thermal fluctuations in the form of Johnson-Nyquist noise and coupled high impedance circuit environments.$^{56,351}$

Using eq. (S1), one can define a conserved energy by treating the junction as a capacitance

$$E = Q_x^2/2C - E_1 \cos(2 \phi_x)$$  \hspace{0.5cm} (S6a)

$$Q_x = -CV_x$$  \hspace{0.5cm} (S6b)

$$\frac{\partial Q_x}{\partial t} = I_S$$  \hspace{0.5cm} (S6c)

where $C$ is the capacitance of the junction. Modifying the last equation in (S6) to

$$\frac{\partial Q_x}{\partial t} = \sum_a I_a,$$  \hspace{0.5cm} (S7)

one can then include all the environmental sources of energy in the form of currents. In fact, to arrive at eq. (S7), the phase-difference needs to couple to the electromagnetic field (Maxwell equations) in a straight-forward manner

$$\frac{\partial \phi_x}{\partial x^\mu} = e \epsilon_{\text{eff}} N^\mu F_{\mu \nu} \equiv e \epsilon_{\text{eff}} F_\mu$$  \hspace{0.5cm} (S8a)

$$N^\mu N_\nu = -\sum_{ij} N_i N_j \delta_{ij} = -1$$  \hspace{0.5cm} (S8b)

$$\frac{\partial F_{\mu \nu}}{\partial x^\mu} = \frac{1}{\epsilon_0 \epsilon_{\text{eff}}} \sum_a J_a$$  \hspace{0.5cm} (S8c)

Here, $F_{\mu \nu} = \partial A_\mu/\partial x_\nu - \partial A_\nu/\partial x_\mu = -F_{\nu \mu}$ is the electromagnetic tensor, $\epsilon_{\text{eff}}$ is the thickness of the barrier and $N^\mu = (0, N^i)$ points in the direction $N^i$ normal to the tunnel barrier. We have used Einstein notation where only the Greek indices are summed over and the Minkowski space-time signature is diag($\eta_{\mu \nu}$) = (+, −, −, −).

Taking the total derivative $\eta^{\mu \nu} \partial / \partial x^\nu$ of eq. (S8a) (with $\eta_{\mu \nu} \eta^{\nu \alpha} = \delta_\mu^\alpha$) and using $\partial N^\mu / \partial x^\nu = 0$, we arrive at

$$\eta^{\mu \nu} \frac{\partial^2 \phi_x}{\partial x^\mu \partial x^\nu} = \frac{\partial^2 \phi_x}{\partial x^2} + \frac{e \epsilon_{\text{eff}}}{\epsilon_0 \epsilon_{\text{eff}}} \sum_a N_\mu J_a^\mu = w J$$  \hspace{0.5cm} (S9)

which is eq. (S7) in disguise. Note that $F_{\mu \nu} = E_i$ and $F_{ij} = \sum_k e^{ik} B_k$ where $E_i$ and $B_i$ are the $x, y, z$ components of the electric and magnetic fields respectively and $\eta_{ijk}$ is the Levi-Civita symbol. eq. (S9) is the sourced Klein-Gordon equation with $J = \sum_a N_\mu J_a^\mu$ the source and $w = e \epsilon_{\text{eff}}/\epsilon_0 \epsilon_{\text{eff}}$ the coupling constant. The vector $N_\mu$ and the anti-symmetry of the electromagnetic tensor $F_{\mu \nu}$ guarantees that, unlike Maxwell equations, the coupled Klein-Gordon equation lives in $2 + 1$ dimensions instead of $1 + 3$. For instance, when the tunnel barrier is aligned to the $y$-$z$ direction, $N^\mu = (0, 1, 0, 0)$ and eq. (S8a) and by extension eq. (S9) become independent of $x$.

Furthermore, taking the limit for small junctions which corresponds to taking the area of the barrier $A$ to be small such that the phase neither varies with $y$ nor $z$, we arrive at eq. (S7)

$$\frac{\partial^2 \phi_x}{\partial t^2} = -4E_1 E_1 \sin(2 \phi_x) - \frac{1}{RC} \frac{\partial \phi_x}{\partial t} - 2E_c I_F/e$$  \hspace{0.5cm} (S11a)

$$\sum_a J_a^1 = J_0^1 + J_N^1 + J_F^1$$  \hspace{0.5cm} (S11b)

$$\langle I_F(t) I_F(s) \rangle = \frac{4\beta^{-1}}{R} \delta(t - s)$$  \hspace{0.5cm} (S11c)

with $J_0^1 = J_{cp} \sin 2 \phi_x$ the supercurrent, $J_N^1 = \sigma_{xx} F_0$ the normal current and $\sigma_{xx}$ the effective conductivity of the barrier along the $x$ direction. Here, we have used the cross-sectional area of the junction, $A$ to define $J_a^1 = J_a^1 A = I_a^1$, the junction capacitance $C = \epsilon_0 \epsilon_{\text{eff}} A/d_{\text{eff}}$, the charging energy $E_c$ = $e^2/2C$ and the junction conductance $1/R = \sigma_{xx} A/d_{\text{eff}}$. Finally, $\beta^{-1} = k_B T$ is the inverse temperature and we have assumed the fluctuation current $J_F^1 A = I_F$ is Gaussian-correlated over a bath (B stands for bath or Boltzmann distribution) with the thermal correlation function given by eq. (S11c).

C. Generalized Impedance Environment and the Thermal Bath

It is straightforward to generalize the condactance $1/R$ in eq. (S11) using the spectral function $K(\omega) = Z^{-1}(\omega)/2\pi$, where $\text{Re} Z(\omega) = \text{Re} Z(-\omega)$ and $\text{Im} Z(\omega) = -\text{Im} Z(-\omega)$, which describes the macroscopic physics of the microscopic degrees of freedom of the system undergoing Brownian motion due to a heat bath comprising $k$ harmonic oscillators.$^{546}$

![FIG. S2. Schematic of the single large Josephson junction displaying the electromagnetic components ($E_x, B_x, B_z$) given by eq. (S8)](image-url)
\[ \sum_{n=1}^{k} H_B(\omega_n) = \sum_{n=1}^{k} \left\{ \frac{Q_n^2}{2C_n} \left( \phi_n - \phi_s \right)^2 \right\} \]

\[ K(t) = \sum_{n=1}^{k} L_n^{-1} \cos(\omega_n t) = \int_{-\infty}^{+\infty} d\omega K(\omega) \exp(-i\omega t) \]

The average is over the thermal bath degrees of freedom. For the fluctuation current we have

\[ \langle I_F(t)I_F(s) \rangle = \sum_{n=1}^{k} 2L_n^{-1} \langle H_B(\omega_n) \rangle \cos(\omega_n(t-s)) \]

where the fluctuation current is given by

\[ I_F(t) = \sum_{n=1}^{k} \left( \omega_n Q_n \sin(\omega_n t) + e^{-1} L_n^{-1} (\phi_n - \phi_s) \cos(\omega_n t) \right) \]

The average is over the thermal bath degrees of freedom. For the Ohmic conductance above, we have

\[ Z^{-1}(\omega) = 1/R \quad \text{and} \quad \langle H_B(\omega) \rangle = \beta^{-1} \]

where the continuous, large \( k \) limit

\[ \lim_{k \to \infty} \sum_{n=1}^{k} L_n^{-1} \times \int_{-\infty}^{+\infty} d\omega K(\omega) \times \]

is taken in accordance with eq. (S12b) thus recovering eq. (S11c). The fluctuation current density certainly satisfies the Green-Kubo relation

\[ \sigma_{xx} = \frac{\beta}{4} \int d^4x V^N V^N \langle J_{F}(t)J_{Fp}(0) \rangle = \frac{\beta}{4} \mathcal{A}^{-1} d_{eff} \int dt \langle I_F(t)I_F(s) \rangle = \frac{d_{eff}}{R_{eff}} \]

where we have used eq. (S11c) in the last line. Note that to obtain the correct equation of motion, integration by parts of the second term in eq. (S13) should be performed after applying the Euler-Lagrange equations, then the boundary term is dropped

\[ \frac{1}{e^2} \int_{-\infty}^{+\infty} \frac{\partial}{\partial S} [K(s-t)\phi_s(s)] ds = 0 \]

III. COULOMB BLOCKADE AND THE ELECTROMAGNETIC ENVIRONMENT

A. Hamiltonian

Consider a mesoscopic tunnel junction with capacitance \( C \) driven by a voltage source \( V_s \) via an environmental impedance

![Z(\omega)](image)

FIG. S3. A mesoscopic tunnel junction, \( J \) with capacitance \( C \) driven by a voltage source \( V_s \) via an environmental impedance \( Z(\omega) \) composed of infinite number of parallel \( L_n \), \( C_n \) circuits. The circuit stores a topological flux \( e\Phi = \sum_n \phi_n \) related to a topological electromagnetic scalar potential \( \int J A(\tau)d\tau = \Phi(\tau) \)

\[ Z(\omega) \]

Each circuit element is characterized by a phase \( \phi_n \) related to the voltage drop \( V_n \) of the element in the circuit by

\[ \phi_n(t) = \int_{-\infty}^{t} e V_n(\tau)d\tau, \] where the subscript \( n = J, x \) or \( z \) corresponds to the junction, voltage source and environmental impedance and \( ke = 2e, e \) corresponds to Cooper pair, quasi-particle charge respectively. The circuit can store a topological flux \( e\Phi = \sum_n \phi_n \) related to a topological potential \( \int J A(\tau)d\tau = \Phi(\tau) \), which we will find out leads to RF power renormalization when present.

The total Hamiltonian, \( \mathcal{H} \) of the circuit (Fig. S3) is given by the expression

\[ \mathcal{H} = \sum_{k=1}^{2} H_k + H_j + H_z \]

Here, \( \sum_{k=1}^{2} H_k = H_1 + H_2 \) where the Cooper-pair Hamiltonian

\[ H_z = \mu \psi^\dagger \psi + \psi^\dagger \left( \frac{\partial}{\partial \tau} \right) + iH_{cp} \psi \]

depends on the chemical potential \( \mu \) and the 2-spinor \( \psi \) introduced in Sec. II A and the quasi-particle Hamiltonian \( H_1 \) is given by

\[ H_1 = H_L + H_R = \sum_{p,q} \epsilon_{pq} \gamma^p_{pq} \gamma^q_{pq} + \sum_{q,\sigma} \epsilon_{q,\sigma} \gamma^q_{q,\sigma} \]

where \( \gamma^p_{pq} \) or \( \gamma^q_{pq} \) are the annihilation and creation operators respectively of a quasi-particle state with energy \( \epsilon_{pq} \) or \( \epsilon_{q,\sigma} \), momentum \( p \) or \( q \) and spin \( \sigma \) in the left or right electrode,

\[ H_j = \sum_{k=1}^{2} \Theta_k \exp(-ik\phi_j) + h.c. \]

is the tunneling Hamiltonian, where

\[ \Theta_1 = \sum_{p,q,\sigma} M_{pq} \gamma_p^q \gamma_{pq}^\dagger, \]

\[ \Theta_2 = \frac{E_{J}}{2} (\sigma_1 - i\sigma_2), \]
\( \sigma_{1,2} \) are the \( x, y \) Pauli matrices acting on the 2-spinor, \( M_{pq} \) is a dimensionful spin-conserving complex-valued quasi-particle tunneling matrix, \( p \neq q \) enforces the condition \( [H_L, H_R] = 0 \) and \( E_J \) is the Josephson coupling energy\(^{550}\),

\[
H_z = \left( \frac{Q_t + C V_x - CA}{2C} \right)^2 + \sum_{n=1} \left\{ \frac{Q_n^2}{2C_n} + e^{-z} (\phi_n - \phi - \phi_n - e\Phi)^2 \right\} \tag{S5}
\]

is the Hamiltonian describing the environmental impedance \( Z(\omega) \) and junction capacitance \( C \), where \( Z(\omega) \) is characterized by an infinite number of parallel \( L_nC_n \) circuits coupled linearly to the tunnel junction, and \( Q_t = Q - CV_x \), \( Q_n \) are the conjugate variables to \( \phi = \phi - \phi_n \), \( \phi_n \) satisfying the charge-phase commutation relation

\[
[\phi_n, Q_0] = i\delta_{ab}\epsilon \tag{S6}
\]

\( T \) is the time ordering operator with the property given by

\[
T \Theta_x(t) \Theta_x^\dagger(0) = \Theta_x(t) \Theta_x^\dagger(0), \quad T \Theta_x(t) \Theta_y^\dagger(0) = \Theta_y(t) \Theta_x^\dagger(0) \tag{S7}
\]

and the average \( \langle \cdots \rangle \) is over the quasi-particle equilibrium states, whose density matrix is given by \( \rho_a = \rho_{L,R} = Z^{-1}_1 \exp(-\beta H_I) \) with \( Z_1 = L \times R = \prod_{p=1} \left( 1 + \exp(-\beta e_{pe}) \right) \times \prod_{q=1} \left( 1 + \exp(-\beta e_{pq}) \right) \), and the environment \( \rho_{env} = Z^{-1}_{env} \exp(-\beta H_z) \) where \( \beta = 1/k_B T \) is the inverse temperature while the trace (tr) is over the Pauli matrices.

B. Perturbation Expansion

We can then expand eq. (S8) as a perturbation series in the tunneling Hamiltonian \( H_I(t) \),

\[
I = \langle I_I(t) - i \int_{-\infty}^{+\infty} [I_I(t), H_I(t)] dt + O(H_I^2) \rangle \tag{S9}
\]

where \( \delta_{ab} \) is the Kronecker delta. Note that \( \phi_I \) and \( \phi \) are related by a suitable unitary transformation \( U \) of the Hamiltonian\(^{511}\)

\[
\mathcal{H}' = i\frac{d}{dt} U + U \mathcal{H} U^\dagger \tag{S7}
\]

where \( \mathcal{H}' = H_I' + H_z + H + H_x, \quad H = \sum_{p=q,\sigma} \epsilon_\sigma \gamma_\sigma \gamma_\sigma + \sum_{p=\sigma,\sigma'} \epsilon_\sigma \gamma_\sigma \gamma_{\sigma'} \) and \( \epsilon_\sigma = \epsilon_{\sigma'} + eV_x \). Operators, \( \mathcal{O}(t) \) in the Heisenberg picture are related to the ones in the Schrödinger picture, \( \mathcal{O}(0) \) by \( \mathcal{O}(t) = U(t) \mathcal{O}(0) U(t)^\dagger \) with the unitary evolution operator \( U(t) \) given by \( U(t) = \exp \left\{ -i \sum_{k=1} H_{kt} \right\} \) in the absence of tunneling. In what follows, we assume the Cooper-pair ground state energy \( \mu = 0 \).

The tunneling current \( I(V) \) at the junction is given by

\[
I(V, s) = tr \langle T \left\{ U^\dagger I_I(0) U \right\} \rangle \tag{S8}
\]

Here, \( U = U_0 + U_{int} \) where

\[
U_{int} = \exp \left\{ -i \int_{-s}^{s} dt H_S(t) \right\} \exp \left\{ -i \int_{-s}^{s} dt H_I(t) \right\} = \exp \left\{ i \int_{0}^{s} dt H_I(t) \right\} \exp \left\{ -i \int_{0}^{s} dt H_I(t) \right\} = \exp \left[ i \int_{0}^{s} dt \phi_I(t) \right] \tag{S9}
\]

Using \( T \Theta_x(t) \Theta_x^\dagger(0) = \Theta_x(t) \Theta_x^\dagger(0), \quad T \Theta_x(t) \Theta_y^\dagger(0) = \Theta_y(t) \Theta_x^\dagger(0), \quad \langle \Theta_x(t) \rangle = 0 \) and \( \langle \Theta_y(t) \Theta_y^\dagger(0) \rangle = \langle \Theta_y(t) \Theta_y^\dagger(0) \rangle = \alpha_x(t) \delta_{xy} \), we find that

\[
I \approx \left\langle \int_{-\infty}^{+\infty} dt \left[ H_I(t), [Q_I(t), H_I(0)] \right] \right\rangle = i e^2 \sum_{k=1} \int_{-\infty}^{+\infty} dt \left\{ \alpha_x(t) \left\langle \sin \left[ \kappa \Delta \phi_I(t) \right] \right\rangle_\phi \right\} \tag{S10}
\]

where \( \Delta \phi_I(t) = \phi_I(t) - \phi_I(0) \). Thus, the particle degrees of freedom \( \alpha_x(t) \) and the environment are decoupled and the trace over the environment \( \langle \cdots \rangle_{env} \) has been re-written as \( \langle \cdots \rangle_{\phi} \) in terms of the junction phase \( \phi_I \) degree of freedom. (Sec. III C)

For the quasi-particle current, the kernel \( \alpha_1(t) \) scales with the dimensionless tunneling conductance \( e^{-2} R_T^\dagger \) but its functional form depends on the gap, reflecting the corresponding structures in the quasi-particle \( I - V \) characteristics. It can be computed by taking the continuous limit, \( 2\pi e^2 R_T N_k(0) N_\phi(0) M_{pq} M_{sp} \to 1 \).
\[ \alpha_1(t) = \left\langle \Theta_1(t)\Theta_1^\dagger(0) \right\rangle = \sum_{p\neq q} \sum_{p'\neq q'} M_{pq} M_{p'q'}^* \langle R, s|L, sl\gamma_{pq}(t)\gamma_{p'q'}^\dagger(t)\gamma_{p'q'}(0)\gamma_{pq}(0)\rho_1|L, s\rangle\langle R, s \rangle \]

\[ = 2 \sum_{p, q} \sum_{p' \neq q} M_{pq} M_{p'q'}^* \langle R\gamma_{pq}(t)L\gamma_{p'q'}^\dagger(t)\gamma_{p'q'}(0)\rho_1|L, s\rangle = 2 \sum_{p, q} f(e_p) \exp\left[ -i\epsilon_p t \right] \sum_{p' \neq q} M_{pq} M_{p'q'}^* \langle R\gamma_{pq}(t)\gamma_{p'q'}^\dagger(0)\rho_1\rangle \langle R\gamma_{pq}(t)\gamma_{p'q'}^\dagger(0)\rho_1\rangle \]

\[ = 2 \sum_{p, q} f(e_p) \left[ 1 - f(e_q) \right] \exp\left[ i(\epsilon_q - \epsilon_p) t \right] \sum_{p' \neq q} M_{pq} M_{p'q'}^* \delta_{qq'} \delta_{pp'} \sum_{p \neq q} f(e_p) \left[ 1 - f(e_q) \right] M_{pq} M_{p'q'}^* \exp\left[ i(\epsilon_q - \epsilon_p) t \right] \]

\[ \frac{1}{\pi e^2 R_1} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\epsilon_p d\epsilon_q \frac{N_L(\Delta)}{N_L(0)} \frac{N_R(\Delta)}{N_R(0)} f(e_p) \left( 1 - f(e_q) \right) \exp\left[ i(\epsilon_q - \epsilon_p) t \right]. \quad (S12) \]

Here, \( e^2 R_1 \) is the dimensionless tunnel resistance, \( f(E) = [1 + \exp(\beta E)]^{-1} \) is the Fermi-Dirac function and \( N_L(\Delta), N_R(\Delta) \) is the left, right BCS density of states which reduce to the electron density of states \( N_L(0), N_R(0) \) when the superconducting gap \( \Delta = 0 \) vanishes.

\[ \frac{dE_p}{d\epsilon_p} = \frac{N_L(\Delta)}{N_L(0)} \frac{N_R(\Delta)}{N_R(0)} \quad (S13a) \]

\[ E_p = \sqrt{\epsilon_p^2 - \Delta^2}, \quad E_q = \sqrt{\epsilon_q^2 - \Delta^2} \quad (S13b) \]

where \( E_p = \epsilon_p^2 / 2m, E_q = \epsilon_q^2 / 2m \) is the kinetic energy of the electrons above the Fermi sea.

Likewise, calculating \( \alpha_2(t) \), we find,

\[ \alpha_2 = \left\langle \Theta_2^\dagger(0)\Theta_2(0) \right\rangle = \left\langle \Theta_2^\dagger(0)\Theta_2(0) \right\rangle = \left( \frac{E_1}{2} \right)^2 \text{tr} \left[ (\sigma_1 + i\sigma_2)(\sigma_1 - i\sigma_2) \right] = \frac{E_1^2}{4} \text{tr} \left[ 2\sigma_0 + i[\sigma_2, \sigma_1] \right] = \frac{E_1^2}{2} \text{tr} \left[ \sigma_0 + \sigma_3 \right] = E_3^2. \quad (S14) \]

We discover that, unlike \( \alpha_1(t) \), \( \alpha_2(t) = \alpha_2(0) = E_3^2 \) is time independent and only depends on the strength of Cooper pair tunneling, \( E_3 \).

**C. Path Integrals and Phase Correlations**

To calculate the remaining average over \( \phi_j \) in eq. (S11), we work in Minkowski time at zero temperature (thus by-passing a rigorous but otherwise tedious Wick rotation to Euclidean time) since the finite temperature propagator is trivially related to the zero temperature result [eq. (S22b) for the trivial relation and Appendix IV for the formalism].

In this formalism, given an observable \( O(\phi) \), its average at zero temperature is given by the functional/path integral,

\[ \lim_{\beta \to \infty} \left\langle O(\phi) \right\rangle = \mathcal{Z}^{-1} \int \prod_{n=1}^{k} D\phi_n D\phi_1 O(\phi_1) \exp iS_z(\phi_n, \phi_1), \quad (S15) \]

where \( \mathcal{Z} = \int \prod_{n=1}^{k} D\phi_n D\phi_1 \exp iS_z(\phi_n, \phi_1) \) is the partition function normalizing eq. (S15) and the Lagrangian in the action for the environment \( S_z(\phi_1, \phi_2) \) is given by the (inverse) Legendre transform of the environment Hamiltonian in eq. (S5)

\[ S_z = \int L_z dt = \int \left\{ (Q_1 + Q_2 - CA) \frac{\partial H_z}{\partial Q_1} - H_z \right\} dt, \quad (S16a) \]

\[ C \frac{\partial \phi_n(t)}{\partial t} = eQ_n, \quad C \frac{\partial \phi_1(t)}{\partial t} = eQ_1, \quad \frac{\partial H(t)}{\partial t} = A(t). \quad (S16b) \]

The effective action \( S_z(\phi) \) resulting from performing first the functional integral product over \( \phi_n \) is given by

\[ S_z(\phi - e\Phi) = S_z^0(\phi - e\Phi) + S_z^\text{int}(\phi - e\Phi) = \frac{C}{2e^2} \int \int_{-\infty}^{+\infty} \left( \frac{\partial(\phi(t) - e\Phi(t))}{\partial t} \right)^2 dt - \frac{1}{2e^2} \int_{-\infty}^{+\infty} \frac{[\phi(t) - e\Phi(t)]^2}{\sum_{n} \sum_{n'} M_{nn'} L_{nn'}} dt \]

\[ - \frac{1}{4\pi^2 e^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{[\phi(t) - e\Phi(t)]}{\sum_{n} \sum_{n'} M_{nn'} L_{nn'}} \left[ \frac{\partial Z^{-1}(s - t)}{\partial s} [\phi(t) - e\Phi(t)] ds dt + \frac{1}{e} \int_{-\infty}^{+\infty} I_0(t)[\phi(t) - e\Phi(t)] dt \right] \quad (S17a) \]

with a fluctuation current \( I_k(t) = 0 \) and \( \phi_1 + \phi = \phi_2 \). Here, \( Z^{-1}(\omega) \) is the Fourier transform of a generalized admittance function \( Z^{-1}(\omega) \) given by

\[ Z^{-1}(\omega) = \sum_{n=1}^{k} \frac{\omega_n}{i\omega L_n} \left( \frac{\omega_n}{(\omega + i\epsilon)^2 - \omega_n^2} \right) = \sum_{n=1}^{k} \frac{\omega_n}{i\omega L_n} \left( \frac{1}{\omega - \omega_n + i\epsilon} - \frac{1}{\omega + \omega_n + i\epsilon} \right) \quad (S17b) \]
\[ \frac{1}{\omega + \omega_n \pm i\varepsilon} = \mp i\pi \delta(\omega - \omega_n) + \text{p.p.} \left( \frac{1}{\omega + \omega_n} \right), \quad (S17c) \]

where eq. (S17c) is the Sokhotski-Plemelj formula and p.p. stands for Cauchy principal part. Eq. (S17b) is related to the spectral function \( K(\omega) = [Z^{-1}(\omega) + Z^{-1}(-\omega)]/(2\pi) \) given in Sec. IIIC where \( \varepsilon \) is the infinitesimal satisfying \( i\varepsilon = \varepsilon \) and the nilpotent condition \( \varepsilon^2 = 0 \). Note, the spectral function is the sum of negative and positive frequency impedance accounting for emission and absorption processes respectively.

We find,

\[ \text{where eq. (S17b) is related to the spectral function } K(\omega) = [Z^{-1}(\omega) + Z^{-1}(-\omega)]/(2\pi) \text{ given in Sec. IIIC where } \varepsilon \text{ is the infinitesimal satisfying } i\varepsilon = \varepsilon \text{ and the nilpotent condition } \varepsilon^2 = 0. \text{ Note, the spectral function is the sum of negative and positive frequency impedance accounting for emission and absorption processes respectively. Thus, eq. (S17) differs slightly from eq. (S17a) where the real-valued spectral function } K(t) \text{ in the classical Lagrangian gets replaced with the complex valued admittance } Z^{-1}(\omega)/(2\pi) \text{ in the quantum case.} \]

Introducing the Dirac delta function \( \delta(x) \) for functional integrals with the property

\[ \int Dx f(x)\delta(x - y) = f(y) \quad (S18) \]

for any functional \( f(x) \), we may proceed to insert \( \int D\phi_x \delta(\phi_x + \phi_y + \phi_z - e\Phi) = 1 \) into eq. (S15) thus introducing the constraint \( \sum_n \phi_n = \phi_x + \phi_y + \phi_z - e\Phi \) guaranteed by the circuit in Fig. (S3). Consequently, the average in eq. (S11) is now taken over both \( \phi_x \) and \( \phi_z \):

\[ \lim_{\beta \to +\infty} \langle \sin \{k\Delta\phi_x(t)\} \rangle_{\phi, \Phi} = Z^{-1} \int D\phi_x \int D\phi_z \delta \left( \sum_n \phi_x - e\Phi \right) \sin \{k\Delta\phi_x(t)\} \exp iS''_x(\phi - e\Phi). \quad (S19) \]

We find,

\[ - \langle \sin \{k\Delta\phi_x(t)\} \rangle_{\phi, \Phi} = \langle \sin \{k\Delta\phi_x(t) + \kappa e \int_0^\tau A(\tau)d\tau + k\Delta\phi_x(t)\} \rangle_{\phi_x} \]

\[ = \langle \sin \{k\Delta\phi_x(t)\} \rangle_{\phi_x} \cos \left[ k\Delta\phi_x(t) + \kappa e \int_0^\tau A(\tau)d\tau \right] \]

\[ + \langle \cos \{k\Delta\phi_x(t)\} \rangle_{\phi_x} \sin \left[ k\Delta\phi_x(t) + \kappa e \int_0^\tau A(\tau)d\tau \right], \quad (S20) \]

with \( \Delta\Phi(t) = e \int_0^\tau A(\tau)d\tau. \) We have assumed Fubini’s theorem for interchange of integration order applies and thus performed first the integral over \( \phi_x. \) Using the fact that \( S'_x \) is quadratic, the resulting functional integral over \( \phi_x \) in eq. (S20) is Gaussian resulting in \( \langle \sin \{k\Delta\phi_x(t)\} \rangle_{\phi_x} = 0 \) term vanishing. Likewise, \( \langle \cos \{k\Delta\phi_x(t)\} \rangle_{\phi_x} \) satisfies Wick’s theorem

\[ \langle \cos \{k\Delta\phi_x(t)\} \rangle_{\phi_x} = \exp \left( \kappa^2 \langle \phi_x(t) - \phi_x(0) \rangle^2 \right), \quad (S21a) \]

\[ \int D\phi_x \exp iS''_x(\phi_x, I_F) = \exp iS''_x(I_F), \quad (S21b) \]

\[ S''_x(I_F) = \frac{2\pi}{2\beta^2} \int_{-\infty}^{+\infty} I_F(-\omega)G_{\text{eff}}(\omega)I_F(\omega)d\omega \quad (S21c) \]

\[ G_{\text{eff}}(\omega) = -e^2i\omega^{-1}Z_{\text{eff}}(\omega), \quad (S21d) \]

\[ Z_{\text{eff}}(\omega) = \frac{1}{Z^{-1}(\omega) + y(\omega)} \quad (S21e) \]

where \( y(\omega) = i\omega C - i\omega^{-1}\sum_n L_n^{-1}. \)

We introduce the zero temperature propagator \( D_{+\infty}(t) \) given by

\[ D_{+\infty}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{d\omega}{\omega} \exp -i\omega t \left| Z_{\text{eff}}(\omega) + n.f. \right|, \quad (S22a) \]

where \( n.f. \) stands for negative frequency. The finite temperature propagator is related to \( D_{+\infty}(t) \) by a sum over the photon number states

\[ D_{+\infty}(t) \rightarrow D_{+\beta}(t) = \sum_{n=0}^{+\infty} D_{+\infty}(t - in\beta) \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{d\omega}{\omega} \exp -i\omega t \left| Z_{\text{eff}}(\omega) + n.f. \right| \quad (S22b) \]

Thus, computing the phase–phase correlation function, we find

\[ \langle \phi_x(s)\phi_x(t) \rangle_{\phi_x} = 2 \times \left| \frac{Z^{-1}\delta^2 \exp iS''_x(I_F)}{e^{-2i\delta I_F(s)\delta I_F(t)} \mid \delta I_F|=0, Z=1 \right| \]

\[ = e^2 D_{+\infty}(s - t) \rightarrow e^2 D_{+\beta}(s - t), \quad (S23) \]

which satisfies the well-known fluctuation-dissipation theorem.\(^7\) The factor of two is included to account for two photon polarization states.

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Finally, plugging in results (S20) and (S22b) in eq. (S11), and using \( \Delta\phi_x(t) = \int_0^\tau V(\tau)d\tau = Vt \) where \( V_x = V \) is a constant external voltage and \( \Delta\Phi(t) = 0 \), the total \( I - V \) characteristics is given by
\[ I_0(V) = I_1(V) + I_2(V) \]
\[ = e^{-1}R_1^1 \int_{-\infty}^{\infty} dE_p dE_q \frac{N(\epsilon_p)N(\epsilon_q)}{N^2(0)} f(\epsilon_p) \left\{ P_1(\epsilon_q - \epsilon_p + eV) - P_1(\epsilon_q - \epsilon_p - eV) \right\} \]
\[ + e\pi E_1^2 \left\{ P_2(2eV) - P_2(-2eV) \right\} \quad (S24) \]

where we have introduced the so called \( P \) function\(^{511} \)

\[ P_s(E) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dt \ exp\left( t^2 \right) \exp(iEt), \quad (S25a) \]
\[ e^{-2} f(t) = D_p(t) - D_p(0) \quad (S25b) \]

\[ I(V)_{\Delta=0} = e^{-1}R_1^1 \int_{-\infty}^{\infty} dE_p dE_q f(\epsilon_p) \left\{ P_1(\epsilon_q - \epsilon_p + eV) - P_1(\epsilon_q - \epsilon_p - eV) \right\} \]
\[ = e^{-1}R_1^1 \int_{-\infty}^{\infty} dE \frac{E}{1 - \exp(-\beta E)} \left\{ P_1(-E + eV) - P_1(-E - eV) \right\} \quad (S26) \]

when the superconducting gap vanishes \( \Delta = 0 \), since \( E_f(\Delta = 0) = 0, \) \( N(\Delta = 0)/N(0) = 1 \) and \( E_p = \epsilon_p, E_q = \epsilon_q. \)

### IV. PATH INTEGRAL FORMALISM WITH GAUSSIAN FUNCTIONAL INTEGRAL

For completeness, this section summarizes how to compute correlation functions with Gaussian functional integrals such as the ones used in Sec. III C in the derivation of the propagator \( D_{+\infty}(t) \) in eq. (S22). Our approach differs from typical procedures with imaginary time\(^{584}. \) We work with real time instead since the finite temperature propagator is trivially related to the zero temperature propagator [eq. (S22b)].

Consider a quadratic action \( S(X, Y) \) with \( X \) as the coordinate variable, \( Y \) as a fluctuation force, \( a \) as a mass term and \( g \) a coupling constant. The computation procedure is then as follows:

1. Take the Fourier transform of the action by substituting the Fourier or inverse Fourier transforms

\[ X(t) = \int d\omega X(\omega) \exp(-i\omega t), \quad X(\omega) = \frac{1}{2\pi} \int dt X(t) \exp(i\omega t), \quad Y(t) = \int d\omega Y(\omega) \exp(-i\omega t), \quad Y(\omega) = \frac{1}{2\pi} \int dt Y(t) \exp(-i\omega t) \]

in the action,

\[ S(X, Y) = \int dt \left\{ \frac{a}{2} \left( \frac{\partial X(t)}{\partial t} \right)^2 - \frac{a}{2} \omega_0^2 X^2(t) + gX(t)Y(t) \right\} = 2\pi \int d\omega \left[ \frac{1}{2} X(\omega)G_X^{-1}(\omega)X(-\omega) + gX(\omega)Y(-\omega) \right], \quad (S1) \]

where \( a^{-1}G_X^{-1}(\omega) = (\omega + i\epsilon)^2 - \omega_0^2 \) and \( G_X(t) = \int d\omega G_X(\omega) \exp(-i\omega t); \)

2. Perform the functional integral \( \int DX \exp(iS(X, Y)) \propto \exp iS'(Y) \) emulating a typical Gaussian integral

\[ \int d\chi \exp\left[ \frac{a}{2} \chi^2 + g\chi \right] \propto \exp\left[ \frac{(ig)^2 \chi^2}{2a} \right] = \exp\left[ -\frac{g^2 \chi^2}{2a} \right] \]
\[ \rightarrow S'(Y) = 2\pi \int d\omega \left[ \frac{(ig)^2}{2} Y(\omega)G_X(\omega)Y(-\omega) \right] = \frac{(ig)^2}{2 \times 2\pi} \int dt ds Y(s)G_X(s-t)Y(t); \quad (S2) \]
3. Compute the correlation functions with the quadratic part of the action as follows,

\[ \langle X(t_1) \cdots X(t_n) \rangle = Z^{-1} \int DX [X(t_1) \cdots X(t_n)] \exp iS(X, Y = 0) \]

\[ = \left[ \frac{1}{(ig)^n} \frac{\delta}{\delta Y(t_n)} \cdots \frac{\delta}{\delta Y(t_1)} Z^{-1} \int DX \exp iS(X, Y \neq 0) \right]_{Y=0, Z=1} \]

\[ = \left[ \frac{1}{(ig)^n} \frac{\delta}{\delta Y(t_n)} \cdots \frac{\delta}{\delta Y(t_1)} \exp iS'(Y) \right]_{Y=0} = \left[ \frac{1}{(ig)^n} \frac{\delta}{\delta Y(t_n)} \cdots \frac{\delta}{\delta Y(t_1)} \sum_{m=0}^{\infty} \frac{iS'(Y)^m}{m!} \right]_{Y=0}. \tag{S3} \]

We require the variation \( \delta / \delta Y(t) \) and the delta function \( \delta(t) \) to satisfy

\[ \frac{\delta}{\delta Y(t)} \frac{\delta}{\delta Y(s)} + \frac{\delta}{\delta Y(s)} \frac{\delta}{\delta Y(t)} = 0, \tag{S4a} \]

\[ \frac{\delta}{\delta Y(t)} = \delta(t - s). \tag{S4b} \]

Note that the anti-commutation rule in eq. (S4a) accounts for time ordering. Since \( S'(Y) \) is quadratic in \( Y \), the integral vanishes for odd number of variables \( n = 2N - 1 \) where \( N \) is a positive integer. For even number of variables \( n = 2N \) we have the continuation,

\[ \langle X(t_1) \cdots X(t_n) \rangle = \left[ \frac{\delta}{\delta Y(t_n)} \cdots \frac{\delta}{\delta Y(t_1)} S^{2N}(Y) \right]_{Y=0, n=2N} \]

\[ = \frac{(ig)^{2N}}{(ig)^{2N}} \frac{\delta}{\delta Y(t_n)} \cdots \frac{\delta}{\delta Y(t_1)} \prod_{m=1}^{n} ds_{2m-1} ds_{2m} Y(s_{2m-1} - s_{2m}) Y(s_{2m}); \tag{S5} \]

4. For illustration, we compute the case \( N = 1 \),

\[ \langle X(t_1) X(t_2) \rangle_{t_1 \neq t_2} = \frac{i}{2 \times 2\pi} \frac{\delta}{\delta Y(t_2)} \int ds_1 ds_2 Y(s_1) G_X(s_1 - s_2) Y(s_2) \]

\[ = \frac{i}{2 \times 2\pi} \int ds_1 ds_2 \frac{\delta Y(s_1)}{\delta Y(t_2)} G_X(s_1 - s_2) \frac{\delta Y(s_2)}{\delta Y(t_1)} - \frac{i}{2 \times 2\pi} \int ds_1 ds_2 \frac{\delta Y(s_1)}{\delta Y(t_1)} G_X(s_1 - s_2) \frac{\delta Y(s_2)}{\delta Y(t_2)} \]

\[ = \frac{i}{2 \times 2\pi} [G_X(t_2 - t_1) - G_X(t_1 - t_2)] = \frac{i}{2 \times 2\pi} \int d\omega [G_X(\omega) - G_X(-\omega)] \exp(-i\omega t) \tag{S6} \]

and

\[ \langle X(0) X(0) \rangle \equiv \langle X(t_1) X(t_2) \rangle_{t_1 = t_2} = \frac{i}{2 \times 2\pi} [G_X(0^+) - G_X(0^-)] = \frac{i}{2 \times 2\pi} \int d\omega [G_X(\omega) - G_X(-\omega)] \neq 0, \tag{S7} \]

where \( t = t_2 - t_1 \) and we have used eq. (S4a). Note that, after Fourier transforming the action given in eq. (S17a) and the delta functional integral (S18) performed in eq. (S20), we simply have \( X(t) \rightarrow \phi_x(t) / \sqrt{2} \) and \( G_X(\omega) \rightarrow G_{\text{eff}}(\omega) = -e^2 i\omega^{-1} Z_{\text{eff}}(\omega) \) to yield eq. (S23), where \( \sqrt{2} \) necessarily counts two photon polarization states.