24 RATIONAL CURVES ON K3 SURFACES

SLAWOMIR RAMS AND MATTHIAS SCHÜTT

Abstract. Given $d \in \mathbb{N}$, we prove that all smooth K3 surfaces (over any field of characteristic $p \neq 2, 3$) of degree greater than $84d^2$ contain at most 24 rational curves of degree at most $d$. In the exceptional characteristics, the same bounds hold for non-unirational K3 surfaces, and we develop analogous results in the unirational case. For $d \geq 3$, we also construct K3 surfaces of any degree greater than $4d(d+1)$ with 24 rational curves of degree exactly $d$, thus attaining the above bounds.

1. Introduction

The study of rational curves on projective K3 surfaces has a long history, starting with the result of Bogomolov and Mumford that every complex projective K3 surface contains a (possibly singular) rational curve [14]. The conjecture that every K3 surface over an algebraically closed field contains infinitely many rational curves, was recently proven in characteristic zero in [4], [5], building on previous work in [2], [3], [13].

The problem of rational curves assumes a different flavour when we consider polarized K3 surfaces of a fixed degree $2h$ (i.e. pairs $(X,H)$, such that $H \in \text{Pic}(X)$ is very ample with $H^2 = 2h$) and take the degrees of the rational curves relative to the polarization $H$ into account. Denote

$$r_d := r_d(X) := \# \{ \text{rational curves } C \subset X \text{ with } \deg(C) = d = C.H \},$$

For surfaces of small degree, the behaviour of $r_d$, especially its maximum, seems to be hard to predict in general, although the problem has a long history (cf. [6], [7], [19], [25]). In contrast, for complex K3 surfaces of high degree $2h$ (i.e. $h > 2d^2$), Miyaoka [17] applied the orbibundle Miyaoka–Yau–Sakai inequality from [16] to obtain the following bound:

$$\frac{1}{d} r_1 + \frac{2}{d} r_2 + \ldots + r_d \leq \frac{24h}{h - 2d^2}.$$

(1.1)

In particular, this implies that for $h > 50d^2$, one has

$$r_i \leq 24 \quad \forall i \leq d,$$
but it remained open to what extent this bound is sharp and which configurations of rational curves attain the maximal values \([17, \text{Rem. (2)})\]. Here we remove the weights, i.e. we consider the numbers 

\[ S_d := r_1 + \ldots + r_d = \# \{ \text{rational curves } C \subset X \text{ with } \deg(C) \leq d \}, \]

and use lattice theory to obtain characteristic-free bounds and characterize the K3 surfaces attaining them. In particular, we show that the bounds are sharp (already for smaller \( h \)).

**Theorem 1.1.** Let \( d \in \mathbb{N} \).

(i) For all \( h > 42d^2 \) and for all K3 surfaces \( X \) of degree \( 2h \) over a field \( k \) of characteristic \( p \neq 2, 3 \), one has

\[ S_d \leq 24. \]

(ii) If \( h > 46d^2 \), then the rational curves of degree at most \( d \) are fibre components of a genus one fibration.

(iii) For \( d \geq 3 \) and for all \( h \geq d(2d + 1) - 1 \), there are K3 surfaces of degree \( 2h \) with \( r_d = 24 \) over fields of characteristic \( \neq 2 \).

In characteristic \( 2, 3 \), due to the presence of quasi-elliptic fibrations we have slightly weaker bounds involving the following restricted rational curve count:

\[ S'_d = S'_d(X) := \# \left\{ \text{rational curves } C \subset X \text{ with } \deg(C) \leq d \text{ such that } p_a(C) \neq 1 \text{ if } C \text{ is cuspidal} \right\}. \]

**Theorem 1.2.** Let \( d \in \mathbb{N} \). For all K3 surfaces \( X \) of degree \( 2h \) over a field \( k \) of characteristic \( p = 2 \), one has:

(i) if \( X \) is not unirational and \( h > 42d^2 \), then \( S_d \leq 24 \).

(ii) in general, if \( h > 46.25d^2 \), then \( S'_d \leq 40 \).

For infinitely many \( d \), there are K3 surfaces of degree \( 2h \) with \( r_d = 24 \) resp. 40 for infinitely many integers \( h \) each over fields of characteristic 2 (see Remark 10.3 and Section 11.5).

**Theorem 1.3.** Let \( d \in \mathbb{N} \). For all K3 surfaces \( X \) of degree \( 2h \) over a field \( k \) of characteristic \( p = 3 \), one has:

(i) if \( h > 42d^2 \) and \( X \) is not unirational or \( X \) has Artin invariant \( \sigma > 6 \), then \( S_d \leq 24 \);

(ii) in general, if \( h > 43d^2 \), then \( S'_d \leq 30 \).

For all \( d \), there are K3 surfaces of degree \( 2h \) with \( r_d = 30 \) over fields of characteristic 3 for infinitely many \( h \) (see Remark 11.4).

**Remark 1.4.** Case (ii) in Theorems 1.2, 1.3 applies to all rational curves if \( d \leq 2 \) (as these curves are automatically smooth, we have \( S_d = S'_d \)). For \( d = 1 \) and \( p = 2 \), the estimate from Theorem 1.2 (ii) can be improved to

\[ r_1 = \# \{ \text{lines on } X \} \leq 25. \]
Of course, all bounds from Theorems 1.1, 1.2, 1.3 also apply to smooth rational curves of given degree. It may come as a surprise that even under this restriction, they are attained infinitely often – even when we consider only smooth rational curves of degree exactly $d$ (see Section 11).

Our results also relate to work of Degtyarev on complex K3 surfaces. In the paper [6] which inspired our approach, Degtyarev considers lines only, but his methods give precise maxima for the number of lines depending on the degree $2h$. In particular, Degtyarev shows that there exist arbitrary large $h \in \mathbb{N}$ such that the bound from Theorem 1.1 is never attained by lines on K3 surfaces of degree $2h$ over $\mathbb{C}$. A similar pattern persists in positive characteristic (and also for $d = 2$), but the precise analysis exceeds the scope of this paper. In contrast to the case of lines and conics, rational curves of degree 3 and higher exhibit a very regular behaviour as we show in this paper (also over fields of positive characteristic).

**Remark 1.5.** Analogous techniques apply to Enriques surfaces, see [21].

**Convention 1.6.** Since Theorems 1.1–1.3 stay valid under base extension, we will assume without loss of generality that the base field $k$ of characteristic $p \geq 0$ is algebraically closed. All rational curves are assumed to be irreducible.

2. Set-up

We consider polarized K3 surfaces of degree $2h$, i.e. pairs $(X, H)$ where $X$ is a smooth K3 surface over an algebraically closed field $k$ of characteristic $p$, and $H$ is a very ample divisor of square $H^2 = 2h$. Then, the linear system $|H|$ defines an embedding

$$X \hookrightarrow \mathbb{P}^{h+1}$$

which is an isomorphism onto its image (the latter is not contained in any hyperplane of $\mathbb{P}^{h+1}$).

Conversely, one can check whether a given divisor $H$ on $X$ with $H^2 = 2h > 0$ is very ample by the methods developed in [23]. In detail, one has:

**Criterion 2.1.** Let $p \neq 2$. A divisor $H$ on a K3 surface $X$ is very ample if

1. $H.C > 0$ for every curve $C \subset X$;
2. $H.E > 2$ for every irreducible curve $E \subset X$ of arithmetic genus 1;
3. $H^2 \geq 4$, and if $H^2 = 8$, then $H$ is not 2-divisible in $\text{Pic}(X)$.

We will revisit this criterion in Section 10 in order to apply it to certain divisors on K3 surfaces with genus one fibrations. Then any irreducible curve serves either as a fibre component $\Theta$ or as a multisection $D'$ of index $d' = F.D'$ where $F$ denotes any fibre. This structure simplifies the analysis of assumptions of Criterion 2.1 substantially, especially since $p_a(D') > 0$ implies $d' > 1$. 
3. Preparations

Given a K3 surface $X$ of degree $2h$, consider the set

$$\Gamma = \{ \text{rational curves } C \subset X \text{ with } \deg(C) \leq d \}$$

which we can also interpret as a graph without loops with multiple edges corresponding to the intersection number $C.C'$ for $C, C' \in \Gamma$. For ease of exposition, we say that an effective divisor $D$ is supported on $\Gamma$ if there is a subgraph $\Gamma' \subset \Gamma$ such that

$$\text{supp}(D) = \bigcup_{C \in \Gamma'} C.$$

For each curve (or vertex) $C \in \Gamma$, we take two values into consideration: the square $C^2$ and the degree $d_C = C.H$ of the corresponding curve.

Together these rational curves generate the formal group

$$M := \mathbb{Z}\Gamma \subset \text{Div}(X),$$

equipped with the intersection pairing extending linearly that on the single curves. We emphasize that $M$ may be degenerate (in shorthand $\ker(M) \neq 0$) as it falls into the following three cases:

1. elliptic – $M \otimes \mathbb{R}$ is negative-definite ($\ker(M) = 0$);
2. parabolic – $M \otimes \mathbb{R}$ is negative semi-definite, but not elliptic ($\ker(M) \neq 0$);
3. hyperbolic – $M \otimes \mathbb{R}$ has a one-dimensional positive-definite subspace and none of greater dimension.

The last condition comes from the Hodge index theorem which will also enter crucially in the next sections.

The squares $C^2 \geq -2$ of the curves in $\Gamma$ play a fundamental role in distinguishing these three cases. First we discuss the elliptic case.

4. Elliptic case

The results in this section and the next will be independent of $h$, so they can be used to construct K3 surfaces with up to 24 rational curves for arbitrary degrees (see Section 10).

If $\Gamma$ is elliptic, then clearly $C^2 = -2$ for any $C \in \Gamma$. Moreover, if $C.C' > 0$ for $C, C' \in \Gamma$, then $C.C' = 1$ for else $(C + C')^2 \geq 0$, and $\Gamma$ would not be elliptic. In summary,

$$C.C' \in \{0, 1\} \quad \text{for all } C \neq C' \in \Gamma.$$

Lemma 4.1. If $M$ is elliptic, then it is an orthogonal sum of finitely many Dynkin diagrams (ADE-type).

Proof. By assumption, the lattice $M$ is negative-definite and non-degenerate. Since it embeds into $\text{Pic}(X)$ which is hyperbolic of rank $\rho(X) \leq 22$ (or $\rho(X) \leq 20$ if $p = 0$, see Remark 4.3), we find that

$$\text{rank } M \leq 21.$$
The claim now follows from (4.1) and the classification of (negative-definite) root lattices. □

We can now derive the first step towards Theorem 1.1:

**Corollary 4.2.** If $M$ is elliptic, then $\#\Gamma \leq \text{rank } M \leq 21$.

**Proof.** Since $M$ is elliptic we have $\#\Gamma = \text{rank } M$. Thus the claim follows from (4.2). □

**Remark 4.3.** Equality can only be attained in characteristics $p \leq 19$ by [26]. Over $\mathbb{C}$, one even has the bound $\#\Gamma \leq 19$, since $\rho(X) \leq 20$ by Lefschetz’ (1,1)-theorem, equality being attained, for instance, by extremal elliptic K3 surfaces (cf. [15], [27]). The condition in Theorem 1.1 (ii) can thus be improved to $S_d > 19$.

5. Parabolic case

If $\Gamma$ is parabolic, then this implies as before that $C^2 \leq 0$ for all $C \in \Gamma$. Moreover, for any isotropic divisor $D \in M$, we obtain

$D.C = 0 \quad \text{for all} \quad C \in \Gamma,$

since else $(2D \pm C)^2 > 0$. In particular, this applies to all curves $C \in \Gamma$ with $C^2 = 0$: $C \in \ker(M)$. All other curves $C \in \Gamma$ satisfy $C^2 = -2$. Arguing as before, we find

$C.C' \in \{0, 1, 2\} \quad \text{for all} \quad C \neq C' \in \Gamma.$

**Lemma 5.1.** If $M$ is parabolic, then it is an orthogonal sum of finitely many Dynkin diagrams, finitely many extended Dynkin diagrams ($\tilde{A}\tilde{D}\tilde{E}$-type) and isotropic vertices. Moreover, at least one summand is either an isotropic vertex or an extended Dynkin diagram.

**Remark 5.2.** In the parabolic case, $\Gamma$ may a priori involve infinitely many isotropic vertices, cf. the quasi-elliptic case in Section 9.

**Proof.** Assume that $M$ is parabolic. If there is $C \in \Gamma$ with $C^2 = 0$, then $C \in \ker(M)$ as noticed above. This leads to an orthogonal decomposition $M = M_z \oplus M_r$ with all vertices $C \in \Gamma \cap M_z$ satisfying the condition $C^2 = 0$, and $\Gamma \cap M_r$ comprising exactly the roots. Moreover, we have $C.C' \in \{0, 1, 2\}$ for any $C \neq C' \in \Gamma \cap M_r$ by (5.2).

If $C, C' \in \Gamma \cap M_r$ satisfy $C.C' = 2$, then $D = C + C' \in M_r$ is isotropic and we get the extended Dynkin diagram $\tilde{A}_1$, that defines an orthogonal summand of $M_r$ by (5.1). Since each such summand gives $A_1 \hookrightarrow \text{Pic}(X)$, we can find only finitely many such summands in $M_r$. Away from these summands, we may assume that $C.C' \in \{0, 1\}$ for any $C \neq C' \in \Gamma \cap M_r$ by (5.2). The classification of (negative semi-definite) root lattices shows that $M_r$ further decomposes into (extended) Dynkin diagrams as claimed. Since $M/\ker(M)$ gives a negative-definite sublattice of $\text{Pic}(X)$, both the number and the rank of these summands are bounded. □
Lemma 5.3. If $M$ is parabolic, then there is a genus one fibration
\begin{equation}
X \to \mathbb{P}^1
\end{equation}
such that
}\Gamma = \{ \text{rational fibre components } \Theta \text{ of } (5.3) \text{ with } \deg(\Theta) \leq d \}.

Proof. By Lemma 5.1 $M$ contains an isotropic vertex $v_0 \in \Gamma$ or an extended Dynkin diagram. Either yields an isotropic divisor $D$ of Kodaira type, i.e. a nodal or cuspidal curve of arithmetic genus one or a configuration of smooth rational curves (with multiplicities) which appears in Kodaira’s and Tate’s classification of singular fibres of elliptic surfaces [12], [30]. The linear system $|D|$ gives the claimed genus one fibration (5.3). By construction, $D$ is a fibre of (5.3), and all vertices of $\Gamma$ feature as fibre components of the fibration by (5.1). Conversely, any rational component $\Theta$ of a singular fibre of degree at most $d$ appears in $\Gamma$ while multisections would force $\Gamma$ to be hyperbolic. \qed

We shall now discuss how Lemma 5.3 fits together with the bound from Theorem 1.1 (i). If the general fibre of the genus one fibration (5.3) is smooth, then the number of rational fibre components can be bounded by topological arguments. Indeed, in terms of the Euler–Poincaré characteristic, we have
\begin{equation}
24 = e(X) = \sum_{t \in \mathbb{P}^1} e(F_t) + \delta_t.
\end{equation}
Here $\delta_t \geq 0$ accounts for the wild ramification (which only occurs for certain additive fibre types in characteristics 2 and 3, see [24]). Moreover, for a singular fibre $F_t$ with $m_t$ irreducible components, one has
\begin{equation}
e(F_t) = \begin{cases} 
m_t, & \text{if } F_t \text{ is multiplicative (type I}_n, n > 0), 
m_t + 1, & \text{if } F_t \text{ is additive}. \end{cases}
\end{equation}
By [29], the general fibre can be non-smooth only in characteristics 2, 3, so outside those characteristics we have
\begin{equation}
\# \Gamma \leq \{ \text{rational fibre components of } (5.3) \} \leq 24.
\end{equation}
This settles Theorem 1.1 (i) in the parabolic case.

In characteristics 2 and 3, the genus one fibration (5.3) may also be quasi-elliptic, i.e. the general fibre is a cuspidal cubic (as excluded in the curve count $S'_d$). Note that this automatically implies the surface to be unirational, so if $X$ is not unirational, Theorem 1.2 (i) follows from what we have seen above. Moreover, quasi-elliptic K3 surfaces in characteristic 3 automatically satisfy $\sigma \leq 6$ (just compute the sublattice generated by fibre components and the curve of cusps), so Theorem 1.3 (i) follows as well.

To conclude the proofs, let (5.3) be quasi-elliptic, but remove all cuspidal rational curves with $p_a = 1$ from consideration. Hence Lemma 5.3 leads to \[\Gamma' = \{ \text{smooth rational fibre components } \Theta \text{ of } (5.3) \text{ with } \deg(\Theta) \leq d \}.\]
That is, we only have to inspect the reducible fibres. Since the general fibre $F$ has $e(F) = 2$, the formula for the Euler-Poincaré characteristic of a quasi-elliptic fibration reads

$$24 = e(X) = 4 + \sum_{t \in \mathbb{P}_1} (e(F_t) - 2).$$

(5.7)

Since all fibres are additive, (5.5) gives

$$\# \Gamma' \leq 20 + \# \{\text{reducible fibres}\}.$$ 

By (5.7), there are at most 20 reducible fibres, so the estimate from Theorem 1.2 (ii) for $p = 2$ follows readily (and is attained by the fibrations from [22], see [11.5]). If $p = 3$, then the only possible reducible fibre types of a quasi-elliptic fibration are IV, IV*, II*, so $e(F_t) - 2 \geq 2$ for a reducible fibre, and (5.7) allows for at most 10 of them. Hence $\# \Gamma' \leq 30$ as stated in Theorem 1.3 (ii). To see that the bound is attained (for certain $h$), confer 11.4.

6. Intrinsic polarization

It remains to study the case where $M$ is hyperbolic and where finally the degree of the K3 surface $X$ plays a role. Extending on [6], we consider the hyperbolic lattice

$$L = M/\ker(M)$$

and try to equip $L \otimes \mathbb{Q}$ with an intrinsic polarization $H_\Gamma \in L \otimes \mathbb{Q}$ obtained by solving for

$$C.H_\Gamma = d_C \quad \forall C \in \Gamma.$$ 

Note that $H_\Gamma$ need not exist at all, since the system of equations tends to be overdetermined. The intrinsic polarization has to be compared to the canonical way of enhancing $M$ by the polarization $H$ with $H^2 = 2h$ where we set

$$C.H = d_C \quad \forall C \in \Gamma.$$ 

This leads to the non-degenerate lattice $L_H = (Z\Gamma + ZH)/\ker(Z\Gamma + ZH)$.

Lemma 6.1. If $L_H$ is hyperbolic, then $L$ embeds into $L_H$.

Proof. It suffices to verify that

$$\ker(Z\Gamma) \subset \ker(Z\Gamma + ZH).$$

Assume to the contrary that there is some $w \in \ker(Z\Gamma) \setminus \ker(Z\Gamma + ZH)$. Picking some positive vector $x \in Z\Gamma$, we obtain an auxiliary rank 3 lattice $L' = Zw + Zx + ZH$ with intersection form

$$Q = \begin{pmatrix} 0 & 0 & w.H \\ 0 & x^2 & x.H \\ w.H & x.H & H^2 \end{pmatrix}$$

This has determinant $\det(Q) = -x^2(w.H)^2 < 0$. On the other hand, $L'$ is hyperbolic by the Hodge index theorem, so $\det(Q) > 0$ gives the required contradiction. \qed
We are now in the position to formulate and prove the following key reduction result:

**Proposition 6.2.** If $L_H$ is hyperbolic, then $H_\Gamma$ exists and $2h \leq H_\Gamma^2$.

The proof of Proposition 6.2 follows the ideas of [6] (which also states the converse implication). For completeness, we provide a direct argument.

**Proof.** By Lemma 6.1, the lattice $L$ embeds into $L_H$ with corank 0 or 1. The underlying vector spaces thus admit an orthogonal decomposition

$$L_H \otimes \mathbb{Q} = (L \otimes \mathbb{Q}) \perp (L^\perp \otimes \mathbb{Q}).$$

Here $L^\perp$ is either zero or negative-definite, since both lattices $L$ and $L_H$ are hyperbolic by assumption. Express $H$ uniquely as

$$H = H_\Gamma + H_\perp,$$

where $H_\Gamma \in L \otimes \mathbb{Q}$, $H_\perp \in L^\perp \otimes \mathbb{Q}$.

Hence $H_\Gamma$ exists, and

$$H^2 = H_\Gamma^2 + (H_\perp)^2 \leq H_\Gamma^2$$

as stated. $\square$

**Remark 6.3.** The above arguments also apply to any hyperbolic subgraph $\Gamma_0 \subset \Gamma$. This will be used in the sequel, for instance in Example 7.6.

Proposition 6.2 forms a cornerstone of our argument due to the following consequence:

**Corollary 6.4.** If $\Gamma$ is hyperbolic, then it can be realized by rational curves on $K3$ surfaces of degree $2h$ only for a finite number of integers $h$.

**Proof of Theorem 1.1 (ii).** Assume that $\# \Gamma > 21$. By Corollary 4.2, $\Gamma$ cannot be elliptic. We claim that it is not hyperbolic, either. Otherwise, there is some elliptic or parabolic $\Gamma_0 \subset \Gamma$ and a single curve $C \in \Gamma$ such that $\Gamma' = \Gamma_0 \cup \{C\}$ is hyperbolic. But then there are only finitely many possibilities for $\Gamma'$, since the shape of $\Gamma_0$ is limited by Lemmas 4.1 and 5.1 while the Hodge Index Theorem gives

$$C^2 \leq \frac{(C.H)^2}{H^2} \leq \frac{d^2}{2h} \leq \frac{d^2}{2}$$

and

$$C.C' \leq d_{C'd'C'} \quad \forall C' \in \Gamma.$$

Corollary 6.4 thus gives an upper bound for $h$ when $\Gamma$ is hyperbolic. For $h \gg 0$, $\Gamma$ therefore is parabolic, and Lemma 5.3 proves the claim. $\square$

**Remark 6.5.** For $h \gg 0$, this gives a quick proof of the bounds in Theorems 1.1-1.3 based on the results from Section 5. Previously, this has been made effective only for the case $d = 1$ over the field $k = \mathbb{C}$, see [6].
7. Preparations for the hyperbolic case

In this section, we explain how the above ideas lead to an effective constraint on the degree of the K3 surface in the hyperbolic case. Throughout this section, we assume that

\[ h > 42d^2 \]

as in Theorem 1.1 (i), 1.2 (i) and 1.3 (i). Certainly (6.2) implies

\[ C^2 \in \{0, -2\} \quad \forall C \in \Gamma, \]

and (6.3) can be improved drastically to \( C.C' \leq d_Cd_{C'}/h \) in case \( C^2, C'^2 \geq 0 \). Thus we get

\[ C.C' = 0 \quad \forall \text{isotropic } C, C' \in \Gamma, \]

so any two such curves are fibres of the same genus one fibration (given by \( |C| = |C'| \)), and as such they are linearly equivalent.

We can directly extend these ideas to isotropic divisors. Given an effective (thus nef) isotropic divisor \( D \neq 0 \) with \( \deg(D) \leq 6d \), we claim that

\[ D.C = 0 \quad \forall C \in \Gamma. \]

To see this, consider the Gram matrix of \( D, C \) and \( H \). By the Hodge Index Theorem, its determinant is non-negative (cf. the proof of Lemma 6.1), whereas (7.1) and the main assumption of this section continue to hold. This implies directly that \( D.C \) vanishes (for another argument, see Example 7.5).

Similarly, one has

\[ C.C' \leq 2 \quad \forall C, C' \in \Gamma \quad \text{with } C^2 = C'^2 = -2. \]

Recall that by a divisor \( D \) of Kodaira type, we mean a nodal or cuspidal curve of arithmetic genus one or a configuration of smooth rational curves (with multiplicities) which appears as a singular fibre of some elliptic surface. Given a divisor \( D = \sum_i n_i C_i \) of Kodaira type, one defines its weight

\[ \text{wt}(D) := \sum_i n_i. \]

In practice, \( \text{wt}(I_n) = n, \text{wt}(I_n^*) = 2n + 6, \text{wt}(IV^*) = 12, \text{wt}(III^*) = 18, \text{wt}(II^*) = 30. \) Note that, if \( D \) is supported on \( \Gamma \), then

\[ \deg(D) \leq \text{wt}(D)d. \]

The following result will prove very useful in the sequel:

**Lemma 7.1.** If \( \Gamma \) is not elliptic, then it supports a divisor of Kodaira type.

**Proof.** If there is an isotropic \( C \in \Gamma \), we’re done, so we may assume that \( C^2 = -2 \) for all \( C \in \Gamma \) by (7.1). Then the statement follows from the classification of Dynkin diagrams: any simple graph that is not a Dynkin diagram contains an extended one. Its fundamental cycle gives the claimed divisor of Kodaira type. \( \square \)
Remark 7.2. The statement of [7.1] can also be verified directly in our situation. To this end, pick a (minimal) $\Gamma_0 \subset \Gamma$ which is elliptic, together with a curve $C_0 \in \Gamma$ such that $\Gamma_0 \cup \{C_0\}$ is not elliptic anymore. If there is $C \in \Gamma$ such that $C_0.C = 2$, then $C_0 + C$ has Kodaira type $I_2$ or $III$. Otherwise, $C_0.C \leq 1$ for all $C \in \Gamma$ by (7.4). By what we have seen before, $\Gamma_0$ is an orthogonal sum of root lattices, and an easy case-by-case analysis confirms the claim.

The consequence for the hyperbolic case is immediate:

**Corollary 7.3.** If $\Gamma$ is hyperbolic, then it supports a divisor $D$ of Kodaira type, and any such divisor has degree $\deg(D) > 6d$. In particular, there are no divisors of Kodaira types $I_n$ ($n \leq 6$), $\Pi$, $\Pi$, $\Pi$, $I_0^*$ supported on $\Gamma$, and all curves in $\Gamma$ are smooth rational.

**Proof.** The existence of $D$ follows from Lemma 7.1. But then $|D|$ induces a genus one fibration, and if $\deg(D) \leq 6d$, then any $C \in \Gamma$ is a fibre of this fibration by (7.3). Hence $\Gamma$ is parabolic, and we obtain a contradiction.

The statement about the Kodaira types follows from the degree bound (7.5) in terms of the weight. \qed

Remark 7.4. Corollary 7.3 implies that in the hyperbolic case, one has $S_d = S'_d$, so we do not have to limit ourselves to the restricted count in the exceptional cases from Theorems 1.2, 1.3.

Another restriction on the possible hyperbolic graphs $\Gamma$ comes from considering hyperbolic subgraphs $\Gamma_0$ (cf. Remark 6.3). Arguing with $L_0 = \mathbb{Z}\Gamma_0/\ker \hookrightarrow \text{Pic}(X)$, one shows as in Proposition 6.2 (see (6.1)) that the intrinsic polarisation $H_0 \in L_0 \otimes \mathbb{Q}$ (if it exists) satisfies

\begin{equation}
2h \leq H_0^2.
\end{equation}

We illustrate the use of this bound by two examples, the second of which will become important soon.

**Example 7.5.** If $0 \neq D \in \text{Pic}(X)$ is nef and isotropic, assume that $|D|$ admits some multisection $C \in \Gamma$. Then applying the above argument to $\langle D, C \rangle$ exactly recovers the bound $\deg(D) > 6d$ from (7.3).

Before coming to the second example, we introduce a general idea how to bound $H_0^2$ from above. To this end, fix a basis of $L_0$ in $\Gamma$ with Gram matrix $G$. The intrinsic polarization

$$H_0 = G^{-1}\vec{d}$$

depends on the degree vector $\vec{d}$ (i.e. the coordinates $d_i$ of $\vec{d}$ are the degrees of the elements of the basis $L_0$), and estimating $H_0^2$ may amount to a non-trivial optimization problem. However, since the degrees are positive, there
is a rough bound in terms of the entries $g_{ij}$ of $G^{-1}$ by

\[(7.7) \quad H_0^2 \leq \sum_{i,j} \max(0, g_{ij})d^2.\]

Indeed, we have $H_0^2 = H^T_0 G H_0 = \vec{d}^T G^{-1} \vec{d} = \sum_{i,j} d_i g_{ij} d_j$, so (7.4) follows from the inequalities $\sum_{i,j} d_i g_{ij} d_j \leq \sum_{i,j} \max(0, g_{ij})d_i d_j \leq \sum_{i,j} \max(0, g_{ij})d^2$.

**Example 7.6.** Let $D$ be a divisor of Kodaira type $I^*_2$, corresponding to an extended Dynkin diagram $\tilde{D}_6 \subset \Gamma$. Assume that there are 3 disjoint $(−2)$-curves of degree at most $d$ on $X$, serving as sections for the fibration induced by $|D|$, and meeting different components of $D$. The corresponding vertices in $\Gamma$ connect to different monovalent vertices of $\tilde{D}_6$. This gives a rank 10 hyperbolic lattice $L_0$. For its Gram matrix $G$ one has $\sum_{i,j} \max(0, g_{ij}) = \frac{78}{21}$, so (7.6) and (7.7) contradict our assumption $h > \frac{42}{d^2}$, i.e. this configuration is impossible.

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### 8. Proof for non-exceptional hyperbolic case

We are now in the position to make our previous ideas effective. To this end, in this section we make the following

**Assumption 8.1.** $\Gamma$ is hyperbolic with $\#\Gamma > 24$.

Observe that for characteristic $p \neq 2, 3$ the first assumption follows from the second (by Corollary 4.2 and (5.6)). Recall that by Corollary 7.3, all curves in $\Gamma$ are smooth rational, and as hinted in Remark 7.4, we can treat all characteristics almost alike (see the next section for the few subtleties remaining). Note also that

\[C.C' \leq 1 \quad \forall C, C' \in \Gamma\]

by (7.4), since the case $C.C' = 2$ would lead to a divisor of Kodaira type $I_2$ or III of degree $\leq 2d$, contradicting Corollary 7.3.

By Lemma 7.1, $\Gamma$ supports a divisor $D$ of Kodaira type (with $\deg(D) > 6d$) by Corollary 7.3. We proceed with two reduction steps.

**Lemma 8.2** (First reduction step). Given a divisor $D$ of Kodaira type supported on $\Gamma$, assume that $|D|$ is not quasi-elliptic. Then there are at least 3 multisections of $|D|$ in $\Gamma$.

**Proof.** Assume to the contrary that $\Gamma$ contains at most 2 multisections for $|D|$. In consequence, $\Gamma$ contains at least 23 fibre components. Regardless of there being sections or not, any orthogonal sum of root lattices $L_0$ embedding into the fibres embeds into $\text{Pic}(X)$ with orthogonal complement hyperbolic indefinite. (This is just like in the jacobian case, where the orthogonal complement contains the hyperbolic plane $U$.) In particular,

\[(8.1) \quad \text{rank } L_0 \leq 20.\]

In case there are at most two singular fibres completely supported on $\Gamma$ (i.e. $\Gamma$ contains the corresponding extended Dynkin diagrams), omitting a single
component of each of these yields \( L_0 \) of rank 21, contradicting (8.1). Thus \( \Gamma \) has to support at least 3 singular fibres completely. If there were four or more of them, then connecting any multisection in \( \Gamma \) with one fibre component in each fibre would produce a divisor of Kodaira type \( I_0^* \) supported on \( \Gamma \), contradicting Corollary 7.3. (This case can also be ruled out by considering the Euler–Poincaré characteristic.) Hence there are exactly three singular fibres completely supported on \( \Gamma \), and we get rank \( L_0 = 20 \). Comparing Euler–Poincaré characteristic and degree, one of the divisors \( D \) of these fibres has \( \deg(D) \leq 12d \). The analogue of (7.3) then shows that

\[
D.C \leq 1 \quad \forall C \in \Gamma.
\]

That is, all curves in \( \Gamma \) are either fibre components or sections of the fibration induced by \( |D| \). In particular, the fibration is jacobian, and by the Shioda–Tate formula, the rank of \( L_0 \) implies that \( \text{MW}(|D|) \) is finite. These (quasi)-elliptic surfaces are called extremal, and they are very rare. Indeed, the classification of extremal elliptic K3 surfaces by Ito in [10], [11] reveals that the only possibility with three singular fibres allowed by Corollary 7.3 has configuration \( I_7, I_7, \Pi^* \) and \( \text{MW} = \{O\} \), in characteristic \( p = 7 \) only, so \( \#\Gamma = 7 + 7 + 9 + 1 = 24 \), contradiction. \( \square \)

Remark 8.3. In the quasi-elliptic case, there are a few further configurations with 3 singular fibres supported on \( \Gamma \) (denoted by extended Dynkin diagrams) and two sections only (see e.g. [26, Table QE]):

1. \( p = 3 \), configuration \( 3\tilde{E}_6 + A_2 \) with two sections from \( \text{MW}(|D|) \cong \mathbb{Z}/3\mathbb{Z} \);
2. \( p = 2 \), configuration \( 3\tilde{D}_6 + 2A_1 \) with \( \text{MW}(|D|) \supset \mathbb{Z}/2\mathbb{Z} \);
3. \( p = 2 \), configuration \( 2\tilde{E}_7 + \tilde{D}_6 \) with \( \text{MW}(|D|) = \mathbb{Z}/2\mathbb{Z} \).

There are two other configurations which are a priori possible by [26, Table QE], \( 2\tilde{E}_6 + \tilde{E}_8 \) and \( 3\tilde{E}_6 + A_2 \), but both have \( \text{MW} = \{O\} \), so \( \#\Gamma \leq 24 \).

In the proof of the second reduction step below, the notion of weight of a divisor of Kodaira type and the inequality (7.5) again play an important role.

**Lemma 8.4 (Second Reduction step).** \( \Gamma \) supports a divisor of Kodaira type \( I_1^*, I_2^* \) or \( IV^* \).

**Proof.** By Corollary 7.3 \( \Gamma \) supports a divisor \( D \) of Kodaira type. Assume that \( D \) does not have Kodaira type \( I_1^*, I_2^* \) or \( IV^* \). This means that the exceptional configurations from Remark 8.3 cannot occur, so by Lemma 8.2 the fibration induced by \( |D| \) has at least 3 multisections in \( \Gamma \). If one of them were not a section, then either it would meet two irreducible components of \( D \), thus giving a cycle of weight less than \( \text{wt}(D) \), or it would meet a multiple component of \( D \). For \( I_n^* \) \((n > 2)\), this results in a divisor of type \( I_m^* \) with \( m < n \), while \( \Pi^* \) may also give \( III^* \) or \( IV^* \), and \( III^* \) may also give \( IV^* \). In any case, the weight drops or we get one of the stated types. So we
may assume that all non-fibre components in Γ are sections of the fibration induced by |D| and proceed with a case-by-case analysis.

If D has type II*, then the 3 sections meet one and the same (simple) component, and we would get I3 (if two of them meet) or I0, both of which are excluded by Corollary 7.3.

In what follows we will often suppress those cases ruled out in the same fashion.

If D has type III*, the 3 sections lead to cases as above or we get I3 of weight 12 < 18 = wt(III*).

If D has type IΓn (n > 2), we get I1 or IV* (since two of the sections meet simple fibre components which connect through a single fibre component, and they are disjoint by Corollary 7.3).

If D has type IΓn (n > 6), we either get a cycle of length at most n + 3 < n or a divisor of type IΓm (the precise value m ≤ n/3 does not matter) to which we then apply the previous step. □

8.1. Proof of Theorems 1.1 (i), 1.2 (i) and 1.3 (i). We continue to assume h > 42d2. By Lemma 8.4, it remains to rule out configurations with #Γ > 24 involving a divisor D of Kodaira type IΓ1, IΓ2 or IV* inducing an elliptic fibration – like we already did for a special configuration in Example 7.6.

If D has type IΓ1, then the 3 sections either lead to a cycle of length at most 6 (which is impossible by Corollary 7.3), or the sections are disjoint, and together they support a divisor D′ of type IΓ2 or IV*, but with one component of IΓ1 now forming a bisection of the fibration induced by |D′|, contradicting (8.2).

If D has type IΓ2, we distinguish whether the 3 sections are pairwise disjoint. In the disjoint case, two of the sections have to meet the same component (since otherwise Example 7.6 gives a contradiction), so Γ supports IΓ1 as well, and the previous case gives a contradiction. If the sections are not all disjoint, then IΓ2 extended by two sections which meet supports a divisor D′ of type III* while the remaining fibre component of IΓ2 serves as a trisection of the fibration induced by |D′|. Since deg(D′) ≤ 18d, this contradicts the analogue of (7.3), (8.2).

If D has type IV*, either some of the sections meet, giving a configuration of type I3 which we ruled out before, or I7 which, with 3 sections attached, leads on to IΓ1 or IΓ2, or the sections are disjoint. The latter case leads to IΓ2 (so that the previous considerations give a contradiction) or to a single central vertex extended by 3 disjoint A3 configurations.

For this last case, we analyse more closely the possible configurations in Γ. Namely, if there are more than 3 sections, then we are automatically in the cases with I7 or IΓ2 above, so we may assume that Γ contains exactly three sections of the fibration induced by |D|. Hence there are at least 22 fibre components in Γ.
If there are less than 3 fibres supported completely on $\Gamma$, then we again obtain an extremal fibration which only leads to exceptional cases (see Section 9).

Assume that there are 3 fibres completely supported on $\Gamma$, one of them being $D$. Here any elliptic configuration has an $I_n$ fibre supported on $\Gamma$ (with $n > 6$ by Corollary 7.3), since else the Euler–Poincaré characteristic reveals that there can be at most 21 fibre components in $\Gamma$. But then any given section connects with $I_n$ and the other two fibres supported on $\Gamma$ to give a divisor of type $I^*$ as treated before.

In summary, if $h > 42d^2$ and we are outside the exceptional cases, then no K3 surface of degree $2h$ admits a configuration $\Gamma$ of more than 24 rational curves of degree at most $d$ such that $\Gamma$ is hyperbolic. Thus, by Corollary 4.2 and (5.6), we obtain

$$S_d \leq 24$$

which completes the proof of Theorems 1.1 (i), 1.2 (i) and 1.3 (i). \hfill $\Box$

9. Proofs of Theorems 1.2 and 1.3

To complete the proofs of Theorems 1.2 and 1.3, it remains to cover the exceptional cases in characteristics 2 and 3. In the argument from 8.1 there are the following exceptional cases occurring (continuing the numbering from Remark 8.3). First with less than 3 fibres completely supported on $\Gamma$:

1. $p = 2$, elliptic fibration with configuration $\tilde{A}_{11} + \tilde{E}_6 + A_3$ and $\text{MW} = \mathbb{Z}/3\mathbb{Z}$ by [10], [11];
2. $p = 3$, quasi-elliptic fibration with configuration $2\tilde{E}_6 + E_6 + A_2$ and $\text{MW} = \mathbb{Z}/3\mathbb{Z}$.

The only other a priori possible configuration $2\tilde{E}_6 + 4A_2$ ($p = 3$, quasi-elliptic) is ruled out as follows. The three sections are 3-torsion (as enforced by quasi-elliptic fibrations). Hence, for the height pairing from [28] to evaluate as zero, any two of them have to meet exactly three out of the four IV fibres in different components. In particular, this implies that some section meets two of the $A_2$ summands. But then it connects with these two and with the two IV$^*$ fibres to a divisor of type $I^*_3$, so we obtain a contradiction (see Corollary 7.3).

If there are 3 fibres completely supported on $\Gamma$, one of them being $D$, then the quasi-elliptic case only allows for

1. $p = 3$, configurations $3\tilde{E}_6 + A_1$ or $3\tilde{E}_6 + A_2$ as in (11), but now with all three sections contained in $\Gamma$.

9.1. Degree bound $h > 43d^2$ in characteristic 3. One easily verifies that each exceptional case in characteristic $p = 3$ (i.e. (11), (5), (6)) features a divisor of Kodaira type $I^*_3$ with 4 disjoint sections, one meeting each simple fibre component (the monovalent vertices in the corresponding extended Dynkin diagram). Let $G$ denote the Gram matrix.
Lemma 9.1. In the box $[0, d]^{12}$, the product $\bar{x}G^{-1}\bar{x}^\top$ is maximized by $\bar{x}_{\text{max}} = (d, \ldots, d)$.

Proof. The inverse $G^{-1}$ of the Gram matrix $G$ has only a few negative entries, occurring in $2 \times 2$ blocks of the shape $A = \frac{1}{3} \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}$. We define the auxiliary matrix

$$
G_0 = \begin{pmatrix}
A & 0 & 0 & 0 & -A & 0 \\
0 & A & 0 & 0 & 0 & -A \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-A & 0 & 0 & 0 & A & 0 \\
0 & -A & 0 & 0 & 0 & A \\
\end{pmatrix}
$$

where each entry stands for a $2 \times 2$ block. This results in the decomposition

$$
G^{-1} = G_0 + G_+
$$

where all entries of $G_+$ are non-negative. Moreover $G_0$ is negative-semidefinite with $\bar{x}_{\text{max}}$ in its kernel, so $\bar{x}_{\text{max}}$ maximizes $\bar{x}G_0\bar{x}^\top$. Obviously, it also optimizes $\bar{x}G_+\bar{x}^\top$, and the claim follows.

Proof of Theorem 1.3 (ii). Let $\Gamma_0 \subset \Gamma$ be given by the 12 smooth rational curves in the above configuration (i.e. a divisor of Kodaira type $I_3^* \times I_3^*$ with 4 disjoint sections, one meeting each simple fibre component). We estimate the square of the intrinsic polarization $H_0$. Since $G_0$ has zero sum of entries, $G^{-1}$ and $G_+$ have the same sum of entries 86. Arguing as in (7.7), we find $H_0^2 \leq 86d^2$, so this configuration is excluded as soon as $h > \frac{43}{4}d^2$.

9.2. Degree bound $h > 46.25d^2$ in characteristic 2. Each exceptional case in characteristic $p = 2$ (i.e. (2), (3), (4)) features a divisor of Kodaira type $IV^*$ extended by three disjoint $A_2$ configurations (or a single central vertex extended by 3 disjoint $A_4$ configurations). The inverse of the Gram matrix has few negative entries and sum of entries 92.5. The proof of Theorem 1.2 (ii) is similar to the above; the details are left to the reader.

10. K3 surfaces with 24 rational curves

In this section, we prove Theorem 1.1 (iii) fixing $d \geq 3$. We need the following auxiliary result.

Lemma 10.1. Assume that $\text{char}(k) \neq 2$ and let $c \in 2\mathbb{Z}$. Then there is a family of $K3$ surfaces over $k$ with generic Picard lattice

$$
\text{Pic} = \begin{pmatrix} 0 & d \\ d & c \end{pmatrix}.
$$

Proof. We will obtain the desired K3 surfaces by deforming certain other K3 surface $Y_r$. To set up the K3 surfaces, fix $c_0 \in \{2, 4, \ldots, 2d\}$ such that $c \equiv -c_0 \mod 2d$. 
Write $c_0 - 2$ as twice the sum of at most $r$ squares:

$$c_0 = 2(1 + n_2^2 + \ldots + n_r^2), \quad 1 \leq r \leq 5, \quad n_i \in \mathbb{N}.$$

Consider a K3 surface $Y_r$ admitting an elliptic fibration with zero section $O$ and $r$ fibres of type $I_2$. Independent of the characteristic, it is a consequence of Tate’s algorithm [30] that $Y_r$ can be given in Weierstrass form

$$y^2 + a_2xy + a_6 + g_r y = x^3 + a_4x^2 + a_8 + a_{12 - 2r}g_r^2$$

where $a_i, b_j, g_r \in k[t]$ with the subscript indicating the degree and $g_r$ square-free. One verifies that the family of such elliptic K3 surfaces depends on $(18 - r)$ parameters, so the generic member will have $\rho = 2 + r$. Here we shall work with a general member $Y_r$ of this family which has no other reducible singular fibres while being non-supersingular. Thus there is a primitive embedding

$$U \oplus A^1_r \hookrightarrow \text{Pic}(Y_r)$$

where the sublattice is generated by the fibre $F$, the zero section $O$ and the fibre components $\Theta_1, \ldots, \Theta_r$ not meeting $O$ (additional generators of $\text{Pic}(Y_r)$ are given by sections by [28]). Let $N \in \mathbb{N}$ and consider the divisor

$$H = NF + dO - \Theta_1 - n_2\Theta_2 - \ldots - n_5\Theta_5.$$

For $N > 2d$, one directly verifies using Criterion 2.1 that $H$ is very ample.

By [8, Prop. 1.5], the K3 surface $Y_r$ deforms together with the divisor classes $H, F$ in an 18-dimensional family over $k$. By construction, the generic member has Picard lattice isometric to the stated one. \hfill \Box

**Remark 10.2.** For complex K3 surfaces Lemma 10.1 follows immediately from the general theory developed by Nikulin ([18, Cor. 1.12.3], see also [9, Cor. 14.3.1]). Over fields of positive characteristic, however, lattice theory no longer suffices to prove the existence of K3 surface with a given Picard group (see e.g. [9, Remark 14.3.2]).

**Proof of Theorem 1.1 (iii).** We continue with the family of K3 surfaces from Lemma 10.1. Applying an isometry of Pic, we may assume that $c \in \{-2, 0, 2, \ldots, 2d - 4\}$. Denote the generators of Pic by $D, C$. By Riemann–Roch, the isotropic vector $D$ is either effective or anti-effective, so let us assume the former by adjusting the signs of $D$ and $C$, if necessary. Then $|D|$ may still involve some base locus which can be eliminated by the composition $\sigma$ of a finite number of reflections. The resulting divisor $E = \sigma(D)$ is a fibre of a genus one fibration. We choose $X$ general in the family of K3 surfaces such that $X$ is not supersingular and all singular fibres of the genus one fibration are nodal cubics (so they are 24 in number by (5.4)).

Turning to the divisor $B = \sigma(C)$, it is effective, again by Riemann–Roch (and since $E.B = d > 0$). Since the fibre class $E$ is nef, every irreducible component $B'$ of the support of $B$ satisfies

$$0 \leq B'.E \leq d,$$
where the left inequality becomes equality if and only if $B'$ itself is a fibre. Arguing with all components of the support of $B$ and with all other multisections of the genus one fibration, one verifies using Criterion 2.1 that $NE + B$ is very ample for $N > 2d$.

Applying this procedure separately to all values $c \in \{-2, 0, 2, \ldots, 2d - 4\}$, we find K3 surfaces of degree $2h$ with 24 rational curves of degree exactly $d$ (the images of the singular fibres) for all $h \geq d(2d + 1) - 1$. □

Remark 10.3. In characteristic 2, the same construction can be carried out to produce K3 surfaces with ample divisors $H$ by the Nakai–Moishezon criterion. Then $mH$ is very ample for all $m \geq m_0$ for a certain $m_0 \in \mathbb{N}$, so we get projective models of K3 surfaces with 24 rational curves of degree $md$ (and infinitely many such polarizations fixing $md$, because we can always add positive multiples of the nef divisor $E$ to $mH$ to obtain further very ample divisors).

11. K3 surfaces with 24 smooth rational curves

This section aims to exhibit explicit projective models of K3 surfaces attaining the bounds from Theorems 1.1–1.3 for infinitely many degrees $H^2 = 2h$ although we restrict to smooth rational curves exclusively. Throughout we fix an integer $d$ and only consider K3 surfaces with smooth rational curves of degree exactly $d$ to simplify the exposition. By the discussion of the hyperbolic case in 8.1, once $h > \frac{42d^2}{d + 1}$, all curves have to be fibre components of some genus one fibration. In the non-quasi-elliptic case (e.g. outside characteristics 2, 3), comparing (5.4) and (5.5) shows that all singular fibres have to be multiplicative and reducible (Kodaira type $I_n$, $n > 1$); since $F.H = dn$ is fixed, they all have the same type. This gives three cases,

\[(11.1) \quad 12 \times I_2, \quad 8 \times I_3, \quad 6 \times I_4\]

which we will study in detail in what follows. (There is an additional case $4 \times I_6$ in characteristic 2 while the other combinatorial cases are ruled out by the Shioda–Tate formula [28, Cor. 5.3].)

We shall start with models covering the minimal degree $d = 1$. This obviously rules out the first configuration from (11.1) since then each pair of fibre components would meet in two points which is impossible for lines.

11.1. Fermat surface ($6 \times I_4$). Assume that $\text{char}(k) \neq 2$. Let $X_4$ be the Fermat quartic surface, defined by

\[X_4 = \{x_0^4 - x_1^4 + x_2^4 - x_3^4 = 0\} \subset \mathbb{P}^3.\]

This has 48 lines over $k$ (112 in characteristic 3, see [11.4]), and the signs were chosen for 8 lines such as

\[(11.2) \quad \{x_0 \pm x_1 = x_2 \pm x_3 = 0\}\]
to be defined over the prime field. As noted in [1], the morphism
\[ \pi : \mathbb{X}_4 \to \mathbb{P}^1 \]
\[ [x_0, x_1, x_2, x_3] \mapsto [x_0^2 - x_1^2, x_2^2 + x_3^2] \]
defines a genus one fibration with 6 fibres of Kodaira type $I_4$, each comprising four of the lines (for instance, the fibre at $[0, 1]$ is just the 4-cycle of lines from (11.2)). The other lines serve as bisections, and over $\mathbb{C}$ or fields of characteristic $p \equiv 1 \mod 4$, one can show that there are no sections. The next fact, just like the ones to follow, can easily be checked using Criterion 2.1, so we omit the details.

**Fact 11.1.** Let \( F \) denote a fibre of \( \pi \), \( H_0 \) a hyperplane section of \( \mathbb{X}_4 \subset \mathbb{P}^3 \) and \( N \in \mathbb{N}_0 \). Then \( H = NF + dH_0 \) is very ample.

For any \( h = 2d(2N + d) \) we thus obtain a degree-2 model of \( \mathbb{X}_4 \) containing 24 smooth rational curves of degree \( d \) (the images of the fibre components).

**Remark 11.2.** In characteristic \( p \equiv -1 \mod 4 \), \( \mathbb{X}_4 \) is supersingular, and the fibration \( \pi \) has sections, accounting for the jump of the Picard number. The sections allow us to find polarizations of \( \mathbb{X}_4 \) for further values for \( H^2 \) with 24 smooth rational curves of degree \( d \).

11.2. **Hesse pencil** \((8 \times I_3)\). Assume that \( \text{char}(k) \neq 3 \). Let \( S \) be the rational elliptic surface defined by the Hesse pencil
\[ S : x^3 + y^3 + z^3 = 3txyz. \]
Then \( S \) has 4 singular fibres of type \( I_3 \) at \( \infty \) and the third roots of unity. The Mordell–Weil group of \( S \) consists of the nine base points of the pencil, \( P_1, \ldots, P_9 \).

**Lemma 11.3.** Let \( F' \) denote a fibre of \( S \). Then the class \( D' = F' + P_1 + \ldots + P_9 \) is 3-divisible in \( \text{Pic}(S) \).

**Proof.** Picking \( P_9 \) as zero of the group law, say, the theory of Mordell–Weil lattices [28] gives an isomorphism
\[ \text{MW}(S) = \text{Pic}(S) / (\text{trivial lattice generated by fibre components and } P_9). \]
Presently this yields
\[ P_1 + \ldots + P_8 = 8P_9 + 3 \sum_{i=1}^{4} \Theta_i - 4F', \]
where the \( \Theta_i \) denote the rational fibre components met by \( P_9 \). Adding \( P_9 \) and \( F' \) on both sides, we derive the claimed divisibility. \( \square \)

Consider the base change \( X_3 \) of \( S \) by a separable quadratic morphism \( \mathbb{P}^1 \to \mathbb{P}^1 \) which is unramified at the singular fibres. Then \( X_3 \) is an elliptic K3 surface with eight fibres of type \( I_3 \) and the said nine sections (but now featuring as \((-2)\)-curves). Let \( F \) denote a fibre of \( X_3 \). By pull-back, the divisor \( D = 2F + P_1 + \ldots + P_9 \in \text{Pic}(X_3) \) is 3-divisible.
Fact 11.4. Let $N \in \mathbb{N}$. Then $H = NF + \frac{d}{3}D \in \text{Pic}(X_3)$ is very ample.

This gives degree-2$h$ models of $X_3$ with 24 smooth rational curves of degree $d$ for $h = d(3N + d)$.

Remark 11.5. As in Remark 10.3 for infinitely many $d$, we obtain non-unirational projective K3 surfaces with infinitely many polarizations of degree $2h$, containing exactly 24 smooth rational degree-$d$ curves in characteristic 2.

11.3. Configuration $12 \times I_2$. Assume that $\text{char}(k) \neq 2$ and consider square-free polynomials $f, g \in k[t]$ of degree 4 such that $f - g$ is also squarefree of the same degree. Then the extended Weierstrass form
\[ y^2 = x(x - f)(x - g) \]
defines an elliptic K3 surface $X_2$ over $\mathbb{P}^1_t$ with twelve singular fibres of type $I_2$ at the zeroes of $f, g$ and $f - g$. Generically, one has $\text{MW}(X_2) \cong (\mathbb{Z}/2\mathbb{Z})^2$ with disjoint sections
\[ P_1 = (0, 0), \quad P_2 = (0, f), \quad P_3 = (0, g) \]
and $P_0$ the point at $\infty$.

Fact 11.6. Assume that $d$ is even. Let $F$ denote a fibre and $N > d$. Then $H = NF + \frac{d}{2}(P_0 + P_1 + P_2 + P_3)$ is very ample.

Therefore, we obtain degree-2$h$ models of $X_2$ for $h = d(2N - d)$ (and $d$ even) with 24 smooth rational curves of degree $d$ in characteristic $\neq 2$.

Remark 11.7. Assuming $X_2$ to arise from a rational elliptic surface with six fibres of type $I_2$, one can endow $X_2$ with two additional independent sections. This also allows one to realize polarizations $H^2 \equiv d^2 \mod 4d$.

11.4. Extra bound in characteristic 3. The Fermat quartic $X_4 \subset \mathbb{P}^3$ admits further elliptic fibrations; one can be obtained by fixing any line $\ell \subset X_4$ and considering the pencil of hyperplanes containing $\ell$. In characteristic 3, the resulting fibration is quasi-elliptic with 10 fibres of type IV (compare [20] for the special role of this surface among quartics in characteristic 3). As before, denote a fibre of the fibration in question by $F$.

Fact 11.8. If $N > 2d/3$, then the divisor $H = NF + d\ell$ is very ample.

Thus we obtain degree-2$h$ models of $X_4$ with $h = 3dN - d^2$ which contain 30 smooth rational curves of degree $d$ in characteristic 3.

11.5. Bounds in characteristic 2. We have already seen in Remarks 10.3, 11.5 how the bound from Theorem 1.2 (i) can be attained in characteristic 2. It remains to establish the same statement for the bound in Theorem 1.2 (ii). To this end, let $d \geq 2$ and consider a quasi-elliptic K3 surface $X$ with 20 fibres of type III as in [22]. Then the curve of cusps $C$ can be regarded as a smooth rational bisection which meets each component of a reducible fibre with multiplicity one. Denoting a fibre by $F$, it follows from
the Nakai–Moishezon criterion that the divisor $H = NF + dC$ is ample for any $N > d$.

For $m \gg 0$, we thus obtain projective models of $X$ with infinitely many different polarizations $mH + nF$ ($n \in \mathbb{N}_0$), each of which contains exactly 40 smooth rational curves of degree $md$.

The bound (1.2) from Remark 1.4 arises from considering quasi-elliptic K3 surfaces $X'$ with 5 fibres of type $I_0^\ast$. We conjecture that (1.2) is sharp (at least for large $h$). Indeed, let us consider the surface $X'$. The curve of cusps $C$ and the double fibre components $\Theta_1, \ldots, \Theta_5$ allow us to define the ample divisor

$$H = NF + d(3C + \Theta_1 + \ldots + \Theta_5)$$

for $N > d/2$. Since $H$ meets all the requirements of Criterion 2.1 for $N \geq 2$, we conjecture that, at least for $N \gg 0$, it is very ample and thus yields projective models of $X'$ with 25 lines.

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Institute of Mathematics, Jagiellonian University, ul. Lojasiewicza 6, 30-348 Kraków, Poland
Email address: slawomir.rams@uj.edu.pl

Institut für Algebraische Geometrie, Leibniz Universität Hannover, Welfengarten 1, 30167 Hannover, Germany

Riemann Center for Geometry and Physics, Leibniz Universität Hannover, Appelstrasse 2, 30167 Hannover, Germany
Email address: schuett@math.uni-hannover.de