On the explicit blowup solutions for 3D incompressible Magnetohydrodynamics equations

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July 3, 2018

Abstract

This paper concerns with the explicit blowup phenomenon for 3D incompressible MHD equations in \( \mathbb{R}^3 \). More precisely, we find two family of explicit blowup solutions for 3D incompressible MHD equations in \( \mathbb{R}^3 \). One family of solutions admit the smooth initial data, and the initial data of another family of solutions are not smooth. The energy of those solutions is infinite. Moreover, our results tell us that the blowup phenomenon of 3D incompressible MHD can only take place in the velocity field of the fluid, but no blowup for the magnetic field.

1 Introduction and main results

The 3D incompressible Magnetohydrodynamics equations (MHD) describes the dynamics of electrically conducting fluids arising from plasmas or some other physical phenomena. We are concerned with the blowup phenomena of smooth solutions to the MHD equations in \( \mathbb{R}^3 \):

\[
\begin{align*}
\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla P &= \nu \Delta \mathbf{v} + (\nabla \times \mathbf{H}) \times \mathbf{H}, \\
\partial_t \mathbf{H} &= \nu \Delta \mathbf{H} + \nabla \times (\mathbf{v} \times \mathbf{B}), \\
\nabla \cdot \mathbf{v} &= 0, \quad \nabla \cdot \mathbf{H} = 0,
\end{align*}
\]

where \( (t, x) \in \mathbb{R} \times \mathbb{R}^3 \), \( \mathbf{v} \) denotes the 3D velocity field of the fluid, \( P \) stands for the pressure in the fluid, \( \mathbf{H} \) is the the magnetic field, \( \nu \geq 0 \) denotes the viscosity constant. The divergence free condition in second equations of (1.1) guarantees the incompressibility of the fluid. In particularly, if \( \nu = 0 \), equations (1.1) is called the ideal MHD equations.

It is easy to check that solutions of 3D incompressible MHD equations (1.1) admits the scaling invariant property, that is, let \( (\mathbf{v}, \mathbf{H}, P) \) be a solution of (1.1), then for any constant \( \lambda > 0 \) and \( \alpha \in \mathbb{R} \), the functions

\[
\begin{align*}
\mathbf{v}_{\lambda, \alpha}(t, x) &= \lambda^\alpha \mathbf{v}(\lambda^{\alpha+1}t, \lambda x), \\
\mathbf{H}_{\lambda, \alpha}(t, x) &= \lambda^\alpha \mathbf{H}(\lambda^{\alpha+1}t, \lambda x), \\
P_{\lambda, \alpha}(t, x) &= \lambda^{2\alpha} P(\lambda^{\alpha+1}t, \lambda x),
\end{align*}
\]

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are also solutions of 3D incompressible MHD equations (1.1). Here the initial data \((v_0(x), H_0(x))\) is changed into \((\lambda^\alpha v_0(\lambda x), \lambda^\alpha H_0(\lambda x))\).

The MHD equations (1.1) is a combination of the Navier-Stokes equations of fluid dynamics and Maxwell’s equations of electromagnetism. As we known, the question of finite time singularity/global regularity for 3D incompressible Navier-Stokes equations is the most important open problems in mathematical fluid mechanics [9]. In 1934, Leray [19] showed that the 3D incompressible Navier-Stokes equations admit global-forward-in-time weak solution of the initial value problem. After that, there are many papers concerns with the well-posedness of weak solutions or blowup of solution for this problem. One can see [2, 7, 12, 14, 23] for more details. So it is also natural important problem for the 3D incompressible MHD equations. There are some numerical results to approach the singularity of this kind problem [10].

The Beale-Kato-Majda’s blowup criterion for incompressible MHD was obtained in [3, 6]. Chae [8] excluded the scenario of the apparition of finite time singularity in the form of self-similar singularities. In this paper, we will give two family of explicit blowup solutions for 3D incompressible MHD equations (1.1). One can see that the ideal MHD equations also has the same explicit blowup solutions with equations (1.1). This means the viscosity constant \(\nu\) can not effect the blowup phenomenon.

For the existence of explicit solutions of incompressible fluids, Landau [17] gave a family of explicit stationary solutions for 3D incompressible stationary Navier-Stokes equations, those solutions now are called Landau’s solutions [18]. Sverák [24] proved that there is only a Landau solution for a nontrivial smooth solutions of 3D incompressible stationary Navier-Stokes equations if the solution satisfying the scaling invariant \(v(x) = \lambda v(\lambda x)\) for each constant \(\lambda > 0\). Kapitanskiy [15, 16] found the first nontrivial example of hidden symmetries connected with reduction of PDEs for the Navier-Stokes equations. Constantin [5] found a class of smooth, mean zero initial data for which the solution of 3D Euler equations becomes infinite in finite time, meanwhile, he gave an explicit formulas of solutions for the 3D Euler equations by reducing this equations into a local conservative Riccati system in two-dimensional basic square. Recently, Yan [26] constructed two family of explicit blowup solutions for 3D incompressible Navier-Stokes equations in \(\mathbb{R}^3\). One family of explicit blowup solutions admit smooth data and infinite energy, another family of explicit blowup solutions have non-smooth data. Due to space limitations, here we can not list all of interesting results.

To the author’s knowledgement, there is few result on the explicit meaningful solution of incompressible MHD equations (1.1) except the famous Alfvén waves [1]. One can see [3, 11] for the result on nonlinear stability of Alfvén waves.

Here we state the main result of this paper.

**Theorem 1.1.** Let constant \(T^* > 0\) be maximal existence time and the viscosity constant \(\nu \geq 0\). The 3D incompressible MHD equations (1.1) admits two family of explicit blowup solutions as follows
• One family of explicit blowup solutions are

\[ \mathbf{v}(t, x) = \left( v_1(t, x), v_2(t, x), v_3(t, x) \right)^T, \quad (t, x) \in [0, T^*) \times \mathbb{R}^3, \]

\[ \mathbf{H}(t, x) = \left( H_1(t, x), H_2(t, x), H_3(t, x) \right)^T, \quad (t, x) \in [0, T^*) \times \mathbb{R}^3, \]

where

\[ v_1(t, x) := \frac{ax_1}{T^* - t} + \frac{kx_2}{x_1^2 + x_2^2}, \]

\[ v_2(t, x) := \frac{ax_2}{T^* - t} - \frac{kx_1}{x_1^2 + x_2^2}, \]

\[ v_3(t, x) := -2ax_3 \]

and

\[ H_1(t, x) := \bar{a}x_1 + \frac{2\bar{a}kx_2(T^* - t)}{(2a + 1)(x_1^2 + x_2^2)}, \]

\[ H_2(t, x) := \bar{a}x_2 - \frac{2\bar{a}kx_1(T^* - t)}{(2a + 1)(x_1^2 + x_2^2)}, \]

\[ H_3(t, x) := -2\bar{a}x_3, \]

with the initial data

\[ \mathbf{v}(0, x) = \left( \frac{ax_1}{T^*}, \frac{kx_2}{x_1^2 + x_2^2}, \frac{ax_2}{T^*} - \frac{kx_1}{x_1^2 + x_2^2}, \frac{2ax_3}{T^*} \right)^T, \]

\[ \mathbf{H}(0, x) = \left( \bar{a}x_1 + \frac{2\bar{a}kT^*x_2}{(2a + 1)(x_1^2 + x_2^2)}, \bar{a}x_2 - \frac{2\bar{a}kx_1T^*}{(2a + 1)(x_1^2 + x_2^2)}, -2\bar{a}x_3 \right)^T, \]

where constants \( k, \bar{a} \in \mathbb{R}/\{0\} \) and \( a \in \mathbb{R}/\{-\frac{1}{2}, 0\} \).

• Another family of explicit blowup solutions are

\[ \mathbf{v}(t, x) = \left( v_1(t, x), v_2(t, x), v_3(t, x) \right)^T, \quad (t, x) \in [0, T^*) \times \mathbb{R}^3, \]

\[ \mathbf{H}(t, x) = \left( H_1(t, x), H_2(t, x), H_3(t, x) \right)^T, \quad (t, x) \in [0, T^*) \times \mathbb{R}^3, \]

where

\[ v_1(t, x) := \frac{ax_1}{T^* - t} + kx_2(T^* - t)^{2a}, \]

\[ v_2(t, x) := \frac{ax_2}{T^* - t} - kx_1(T^* - t)^{2a}, \]

\[ v_3(t, x) := -2ax_3 \]

and

\[ H_1(t, x) := \bar{a}x_1 + \frac{2\bar{a}kx_2x_3(T^* - t)^{2a+1}}{4a + 1}, \]

\[ H_2(t, x) := \bar{a}x_2 - \frac{2\bar{a}kx_1x_3(T^* - t)^{2a+1}}{4a + 1}, \]

\[ H_3(t, x) := -2\bar{a}x_3, \]
On one hand, it follows from (1.2) that

\[ v(0, x) = \left( \frac{ax_1}{T^*} + kx_2(T^*)^2a, \frac{ax_2}{T^*} - kx_1(T^*)^2a, -2ax_3, \frac{-2ax_3}{T^*} \right), \]

\[ H(0, x) = \left( \frac{\bar{a}x_1 + 2\bar{a}kx_2x_3(T^*)^{2a+1}}{4a + 1}, \frac{\bar{a}x_2 - 2\bar{a}kx_1x_3(T^*)^{2a+1}}{4a + 1}, -2\bar{a}x_3 \right)^T, \]

where constants \( k, \bar{a} \in \mathbb{R}/\{0\} \) and \( a \in \mathbb{R}/\{-\frac{1}{4}, 0\} \).

**Remark 1.1.** If the 3D velocity field of the fluid \( \mathbf{v} \) and the magnetic field \( \mathbf{H} \) are given by (1.2), respectively, by direct computations, the pressure

\[ P(t, x) = -\frac{1}{2} \left( \frac{a(a + 1)(x_1^2 + x_2^2)}{(T^* - t)^2} + \frac{k^2}{x_1^2 + x_2^2} + \frac{ax_3^2(1 - 2a)}{(T^* - t)^2} \right), \]

where constants \( k \in \mathbb{R}/\{0\} \) and \( a \in \mathbb{R}/\{-\frac{1}{2}, 0\} \).

If the 3D velocity field of the fluid \( \mathbf{v} \) and the magnetic field \( \mathbf{H} \) are given by (1.4), respectively, by direct computations, the pressure

\[ P(t, x) = \frac{x_1^2 + x_2^2}{2} \left( k^2(T^* - t)^4a - \frac{a(a + 1)}{(T^* - t)^2} - \frac{8\bar{a}^2k^2x_3^2(T^* - t)^{2(2a+1)}}{(4a + 1)^2} \right) \]

\[ + x_3^2 \left( \frac{a(1 - 2a)}{(T^* - t)^2} - \frac{2\bar{a}^2k^2x_3^2(T^* - t)^{2(2a+1)}}{(4a + 1)^2} \right), \]

where constants \( k, \bar{a} \in \mathbb{R}/\{0\} \) and \( a \in \mathbb{R}/\{-\frac{1}{4}, 0\} \).

From (1.6), we can see if the blowup phenomenon only takes place in the velocity field, then the pressure \( P \) depends on two parameters \( a, k \). But if the blowup phenomenon takes place in both the velocity field and the magnetic field, it follows from (1.7) that the pressure \( P \) depends on three parameters \( a, k, \bar{a} \), where the parameter \( \bar{a} \) comes from the magnetic field.

**Remark 2.** For the velocity field of the fluid \( \mathbf{v} \), it follows from (1.2) and (1.4) that there is self-similar singularity in \( x_3 \) direction, that is, \( -\frac{2ax_3}{T^* - t} \) for \( a \in \mathbb{R}/\{0\} \). Moreover, on one hand, by (1.2), we find there is only blowup for the velocity field of the fluid \( \mathbf{v} \), the magnetic field does not blowup as \( t \to (T^*)^- \). On the other hand, by (1.4), there are not only blowup for velocity field of the fluid \( \mathbf{v} \), but also blowup for the magnetic field \( \mathbf{H} \) with constant \( a < -\frac{1}{2} \) as \( t \to (T^*)^- \).

**Remark 3.** On one hand, it follows from (1.2) that

\[ \nabla v_1(t, x) = \left( \frac{a}{T^* - t} - \frac{2kx_1x_2}{(x_1^2 + x_2^2)^2}, \frac{k(x_1^2 - x_2^2)}{(x_1^2 + x_2^2)^2}, 0, \frac{-2a}{T^* - t} \right)^T, \]

\[ \nabla v_2(t, x) = \left( \frac{k(x_1^2 - x_2^2)}{(x_1^2 + x_2^2)^2}, \frac{a}{T^* - t} + \frac{2kx_1x_2}{(x_1^2 + x_2^2)^2}, 0, \frac{-2a}{T^* - t} \right)^T, \]

\[ \nabla v_3(t, x) = \left( 0, 0, -\frac{2a}{T^* - t} \right)^T, \]

which means that

\[ \text{div}(v_1)|_{x=x_0} = \infty, \quad \text{as} \quad t \to (T^*)^-, \]
for a fixed point $x_0 \in \mathbb{R}^3$. There is no blowup in the magnetic field $H$. Here one can see the initial data is not smooth from (1.3).

On the other hand, it follows from (1.4) that

$$
\nabla v_1(t, x) = \left( \frac{a}{T^* - t}, k(T^*)^{2a}, 0 \right)^T,
$$

$$
\nabla v_2(t, x) = \left( -k(T^*)^{2a}, \frac{a}{T^* - t}, 0 \right)^T,
$$

$$
\nabla v_3(t, x) = \left( 0, 0, -\frac{2a}{T^* - t} \right)^T,
$$

$$
\nabla H_1(t, x) = \left( \bar{a}, \frac{2\bar{a}kx_3(T^* - t)^{2a+1}}{4a+1}, \frac{2\bar{a}kx_2(T^* - t)^{2a+1}}{4a+1} \right)^T,
$$

$$
\nabla H_2(t, x) = \left( -\frac{2\bar{a}kx_3(T^* - t)^{2a+1}}{4a+1}, \bar{a}, -\frac{2\bar{a}kx_1(T^* - t)^{2a+1}}{4a+1} \right)^T,
$$

$$
\nabla H_3(t, x) = \left( 0, 0, -2\bar{a} \right)^T,
$$

which means that

$$
div(v_i)|_{x=x_0} = \infty, \quad \text{as} \quad t \to (T^*)^-,
$$

and for $a < -\frac{1}{2}$, there is

$$
div(H_i)|_{x=x_0} = \infty, \quad \text{as} \quad t \to (T^*)^-,
$$

for a fixed point $x_0 \in \mathbb{R}^3$. Here one can see the initial data is smooth from (1.3). But the initial data goes to infinity as $x \to \infty$.

In conclusion, our results tell us that the blowup phenomenon of 3D incompressible MHD (1.1) can only take place in the velocity field of the fluid $v$, but no blowup for the magnetic field $H$. It is easy to see the blowup solutions (1.2) and (1.4) independent of viscosity constant $\nu$, so our results also hold for 3D ideal incompressible MHD. Hence those two family of blowup solutions (1.2) and (1.4) are not finite energy solutions.

**Remark 1.4.** When the magnetic field $H \equiv 0$, equations (1.1) is reduced into 3D incompressible Navier-Stokes equations. Then corresponding explicit blowup solutions given in (1.2) and (1.4) are also explicit blowup solutions for 3D incompressible Navier-Stokes equations [26].

## 2 Proof of Theorem 1.1

We first recall a result on the existence of explicit blowup axisymmetric solutions for 3D incompressible Navier-Stokes equations.

Let $e_r$, $e_\theta$ and $e_z$ be the cylindrical coordinate system,

$$
e_r = \left( \frac{x_1}{r}, \frac{x_2}{r}, 0 \right)^T,
$$

$$
e_\theta = \left( \frac{x_2}{r}, -\frac{x_1}{r}, 0 \right)^T,
$$

$$
e_z = \left( 0, 0, 1 \right)^T,
$$

(2.8)
where $r = \sqrt{x_1^2 + x_2^2}$ and $z = x_3$.

The following result is taken from [26].

**Proposition 2.1.** Let $T^* > 0$ be a constant, the kinematic viscosity $\nu > 0$, and $e_r, e_\theta, e_z$ are defined in (2.8), $r = \sqrt{x_1^2 + x_2^2}$ and $z = x_3$. The 3D incompressible Navier-Stokes equations admits two family of explicit blow up axisymmetric solutions:

- One family of explicit blowup axisymmetric solutions are
  \[ v(t, x) = v^r(t, r, z)e_r + v^\theta(t, r, z)e_\theta + v^z(t, r, z)e_z, \quad (t, x) \in [0, T^*) \times \mathbb{R}^3, \]  
  where
  \[ v^r(t, r, z) = \frac{ar}{T^* - t}, \]
  \[ v^\theta(t, r, z) = \frac{k}{r}, \]
  \[ v^z(t, r, z) = -\frac{2az}{T^* - t}, \]
  where constants $a, k \in \mathbb{R}/\{0\}$.

- Another family of explicit blowup axisymmetric solutions are
  \[ v(t, x) = v^r(t, r, z)e_r + v^\theta(t, r, z)e_\theta + v^z(t, r, z)e_z, \quad (t, x) \in [0, T^*) \times \mathbb{R}^3, \]  
  where
  \[ v^r(t, r, z) = \frac{ar}{T^* - t}, \]
  \[ v^\theta(t, r, z) = kr(T^* - t)^2, \]
  \[ v^z(t, r, z) = -\frac{2az}{T^* - t}, \]
  where constants $a, k \in \mathbb{R}/\{0\}$.

We now derive the 3D incompressible MHD equations (1.1) with axisymmetric velocity field in the cylindrical coordinate (e.g. see [20]). The 3D velocity field $v(t, x)$ and magnetic field $H(t, x)$ are called axisymmetric if they can be written as

\[ v(t, x) = v^r(t, r, z)e_r + v^\theta(t, r, z)e_\theta + v^z(t, r, z)e_z, \]
\[ H(t, x) = H^r(t, r, z)e_r + H^\theta(t, r, z)e_\theta + H^z(t, r, z)e_z, \]
\[ P(t, x) = P(t, r, z), \]

where $(v^r, v^\theta, v^z)$, $(H^r, H^\theta, H^z)$ and $P(t, r, z)$ do not depend on the $\theta$ coordinate.

Note that the Lorentz force term

\[ (\nabla \times H) \times H = H \cdot \nabla H - \nabla \frac{|H|^2}{2}. \]

Then the 3D MHD equations (1.1) with axisymmetric velocity field in the cylindrical coordinates can be reduced into a system as follows

\[ \partial_t v^r + v^r \partial_r v^r + v^z \partial_z v^r - \frac{1}{r} (v^\theta)^2 + \partial_r \bar{P} = \nu (\Delta - \frac{1}{r^2}) v^r + H^r \partial_r H^r + H^z \partial_z H^r - \frac{1}{r} (H^\theta)^2, \]

(2.11)
\[ \partial_t v^\theta + v^r \partial_r v^\theta + v^z \partial_z v^\theta + \frac{1}{r} v^r v^\theta = \nu(\Delta - \frac{1}{r^2})v^\theta + H^r \partial_r H^\theta + H^z \partial_z H^\theta + \frac{1}{r} H^\theta H^r, \]  

\[ \partial_t v^z + v^r \partial_r v^z + v^z \partial_z v^z + \partial_z \theta = \nu(\Delta - \frac{1}{r^2})v^z + H^r \partial_r v^z + H^z \partial_z v^z, \]  

\[ \partial_t H^r + v^r \partial_r H^r + v^z \partial_z H^r = \nu(\Delta - \frac{1}{r^2})H^r + H^r \partial_r v^r + H^z \partial_z v^r, \]  

\[ \partial_t H^z + v^r \partial_r H^z + v^z \partial_z H^z = \nu(\Delta - \frac{1}{r^2})H^z + H^r \partial_r v^z + H^z \partial_z v^z, \]  

where the pressure is given by

\[ P = P + \frac{|H|^2}{2}. \]  

The incompressibility condition becomes

\[ \partial_r (r v^r) + \partial_z (r v^z) = 0, \]

\[ \partial_r (r H^r) + \partial_z (r H^z) = 0. \]  

The following result gives two family of explicit self-similar blowup solutions for system (2.11)-(2.17) with the incompressibility condition (2.18).

**Proposition 2.2.** Let \( T^* > 0 \) be a constant and \( \nu > 0 \). System (2.11)-(2.17) with the incompressibility condition (2.18) admits two family of explicit blowup solutions:

- One family of explicit blowup solutions takes the form

  \[ v^r(t, r, z) = \frac{ar}{T^* - t}, \]

  \[ v^\theta(t, r, z) = \frac{k}{r}, \]

  \[ v^z(t, r, z) = -\frac{2az}{T^* - t}, \]

  \[ H^r(t, r, z) = \bar{a}r, \]

  \[ H^\theta(t, r, z) = \frac{2k\bar{a}(T^* - t)}{(2a + 1)r}, \]

  \[ H^z(t, r, z) = -2\bar{a}z, \]  

  where constants \( \bar{a}, k \in \mathbb{R}/\{0\} \) and \( a \in \mathbb{R}/\{-\frac{1}{2}, 0\} \).

- Another family of explicit blowup solutions is

  \[ v^r(t, r, z) = \frac{ar}{T^* - t}, \]

  \[ v^\theta(t, r, z) = kr(T^* - t)^{2a}, \]

  \[ v^z(t, r, z) = -\frac{2az}{T^* - t}, \]

  \[ H^r(t, r, z) = \bar{a}r, \]

  \[ H^\theta(t, r, z) = \frac{2\bar{a}kr(T^* - t)^{2a+1}}{4a + 1}, \]

  \[ H^z(t, r, z) = -2\bar{a}z, \]
Proof of Proposition 2.2. The idea of finding explicit blowup solutions for system (2.11)-(2.17) with the incompressibility condition (2.18) comes from [25, 26, 27]. This is based on the observation on the structure of system (2.11)-(2.17) and incompressibility condition (2.18). We notice that if the magnetic field $H = 0$, equations (1.1) is reduced into 3D incompressible Navier-Stokes equations, so the explicit blowup solutions (2.9) and (2.10) of Navier-Stokes equations should be a part of solutions for the corresponding MHD equations.

One family of explicit blowup solutions. When we set
\begin{align*}
v^r(t, r, z) &= \frac{ar}{T^* - t}, \\
v^\theta(t, r, z) &= \frac{k}{r}, \\
v^z(t, r, z) &= -\frac{2az}{T^* - t},
\end{align*}
(2.21)
be a part of solutions for (2.11)-(2.16), where constants $a, k \in \mathbb{R}/\{0\}$.

Substituting (2.21) into equations (2.12) and (2.14)-(2.16), we get
\begin{align*}
\partial_t H^r + \frac{ar}{T^* - t} \partial_r H^r - \frac{2az}{T^* - t} \partial_z H^r &= \nu(\Delta - \frac{1}{r^2}) H^r + \frac{a}{T^* - t} H^r, \\
\partial_t H^\theta + \frac{ar}{T^* - t} \partial_r H^\theta - \frac{2az}{T^* - t} \partial_z H^\theta + \frac{2k}{r^2} H^r &= \nu(\Delta - \frac{1}{r^2}) H^\theta + \frac{a}{T^* - t} H^\theta, \\
\partial_t H^z + \frac{ar}{T^* - t} \partial_r H^z - \frac{2az}{T^* - t} \partial_z H^z &= \nu \Delta H^z - \frac{2a}{T^* - t} H^z.
\end{align*}
(2.22, 2.23, 2.24)

We observe the the structure of incompressibility condition (2.18) on the magnetic field, and we find it is better to set
\begin{align*}
H^r(t, r, z) &= \frac{\bar{a}r}{(T^* - t)^{\alpha}}, \\
H^z(t, r, z) &= -\frac{2\bar{a}z}{(T^* - t)^{\alpha}},
\end{align*}
(2.25, 2.26)
where $\bar{a} \neq 0, \alpha$ are two unknown constants.

It is easy to see $H^r(t, r, z)$ and $H^z(t, r, z)$ given in (2.26) satisfies the incompressibility condition (2.18) on the magnetic field.

Note that $(\Delta - \frac{1}{r^2})r = 0$. Substituting the $H^r$ in (2.26) into (2.23), we get
\begin{align*}
\alpha &= 0.
\end{align*}
which gives that
\begin{align*}
H^r(t, r, z) &= \bar{a}r, \\
H^z(t, r, z) &= -2\bar{a}z,
\end{align*}
(2.27)
We now find \( H^\theta(t, r, z) \). Assume that

\[
H^\theta(t, r, z) = \frac{\bar{k} r^p z^q}{(T^* - t)^\beta},
\]

(2.28)

where \( \bar{k} \neq 0 \) and \( p, q, \beta \) are unknown constants.

We substitute (2.27)-(2.28) into (2.22) and (2.24), respectively, there are

\[
p - 2q + 1 = 0,
\]

(2.29)

and

\[
\frac{\beta \bar{k} r^p z^q}{(T^* - t)^{\beta+1}} + \frac{ap \bar{k} r^p z^q}{(T^* - t)^{\beta+1}} - \frac{2a \bar{k} r^p z^q}{r} + \frac{2k\bar{a} r^p z^q}{(T^* - t)^{\beta+1}} = \frac{a \bar{k} r^p z^q}{(T^* - t)^{\beta+1}}.
\]

(2.30)

Equation (2.30) means that

\[
\beta = -1, \quad p = -1, \quad q = 0,
\]

and

\[
\bar{k} = \frac{2k\bar{a}}{2a + 1}, \quad \text{for} \quad a \neq -\frac{1}{2}.
\]

Thus we get

\[
H^\theta = \frac{2k\bar{a}(T^* - t)}{(2a + 1)r}.
\]

(2.31)

Meanwhile, it is easy to check that (2.25) holds.

Hence by (2.27) and (2.31), we conclude

\[
H^r(t, r, z) = \bar{a} r,
\]

\[
H^\theta(t, r, z) = \frac{2k\bar{a}(T^* - t)}{(2a + 1)r},
\]

\[
H^z(t, r, z) = -2\bar{a} z,
\]

which combining with (2.21) gives a family of solutions of system (2.11)-(2.17) with the incompressibility condition (2.18). Here constants \( \bar{a}, k \in \mathbb{R}/\{0\} \) and \( a \in \mathbb{R}/\{-\frac{1}{2}, 0\} \).

Furthermore, we compute the pressure \( P \). We substitute (2.19) into (2.11) and (2.13), there are

\[
\partial_r \bar{P} = \left( \bar{a}^2 - \frac{a(1 + a)}{(T^* - t)^2} \right) r - \frac{k^2 \left( 4\bar{a}^2(T^* - t)^2 - (2a + 1)^2 \right)}{r^3(2a + 1)^2},
\]

and

\[
\partial_z \bar{P} = 2z \left( 2\bar{a}^2 + \frac{a(1 - 2a)}{(T^* - t)^2} \right).
\]

Note that

\[
|H|^2 = \bar{a}^2 r^2 + \frac{4k^2 \bar{a}^2 (T^* - t)^2}{(2a + 1)^2 r^2} + 4\bar{a}^2 z^2.
\]

Thus by (2.17), direct computations give the pressure

\[
P(t, r, z) = -\frac{1}{2} \left( \frac{a(a + 1)r^2}{(T^* - t)^2} + \frac{k^2}{r^2} \right) + \frac{az^2(1 - 2a)}{(T^* - t)^2}.
\]
Another family of explicit blowup solutions. When we set

$$v^r(t, r, z) = \frac{ar}{T^* - t},$$

$$v^\theta(t, r, z) = kr(T^* - t)^{2a},$$

$$v^z(t, r, z) = -\frac{2az}{T^* - t},$$

(2.32)

be a part of solutions for (2.11)-(2.16), where constants $a, k \in \mathbb{R}/\{0\}$.

Substituting (2.32) into equations (2.12) and (2.14)-(2.16), we get

$$H^r \partial_r H^0 + H^z \partial_z H^\theta + \frac{1}{r} H^0 H^r = 0,$$

(2.33)

$$\partial_t H^r + \frac{ar}{T^* - t} \partial_r H^r - \frac{2az}{T^* - t} \partial_z H^r = \nu(\Delta - \frac{1}{r^2}) H^r + \frac{a}{T^* - t} H^r,$$

(2.34)

$$\partial_t H^\theta + \frac{ar}{T^* - t} \partial_r H^\theta - \frac{2az}{T^* - t} \partial_z H^\theta + 2k(T^* - t)^{2a} H^r = \nu(\Delta - \frac{1}{r^2}) H^\theta + \frac{a}{T^* - t} H^\theta,$$

(2.35)

$$\partial_t H^z + \frac{ar}{T^* - t} \partial_r H^z - \frac{2az}{T^* - t} \partial_z H^z = \nu \Delta H^z - \frac{2a}{T^* - t} H^z,$$

(2.36)

which compare with system (2.22)-(2.25), we find there is only one equations different. This causes we have a new $H^\theta(t, r, z)$. It is easy to check that $H^r(t, r, z)$ and $H^z(t, r, z)$ given in (2.27)-(2.28) are the solution of (2.34) and (2.36) with

$$p - 2q + 1 = 0.$$

(2.37)

We substitute (2.27)-(2.28) into (2.35), there is

$$\frac{\beta kr^p z^q}{(T^* - t)^{\beta + 1}} + \frac{ap\bar{k}r^p z^q}{(T^* - t)^{\beta + 1}} - \frac{2\bar{a}kr^p z^q}{(T^* - t)^{\beta + 1}} + 2\bar{a}r(T^* - t)^{2a} = \frac{a\bar{k}r^p z^q}{(T^* - t)^{\beta + 1}},$$

(2.38)

which gives that

$$\alpha = -2a - 1, \quad p = 1, \quad q = 1,$$

and

$$\bar{k} = \frac{2\bar{a}k}{4a + 1}.$$

Thus we get

$$H^\theta(t, r, z) = \frac{2\bar{a}krz(T^* - t)^{2a+1}}{4a + 1}.$$

(2.39)

In conclusion, by (2.27) and (2.39), we obtain

$$H^r(t, r, z) = \bar{a}r,$$

$$H^\theta(t, r, z) = \frac{2\bar{a}krz(T^* - t)^{2a+1}}{4a + 1},$$

$$H^z(t, r, z) = -2\bar{a}z,$$

which combining with (2.21) gives another family of solutions of system (2.11)-(2.17) with the incompressibility condition (2.18). Here constants $\bar{a}, k \in \mathbb{R}/\{0\}$ and $a \in \mathbb{R}/\{-\frac{1}{4}, 0\}$. 

10
Furthermore, we compute the pressure $P$. We substitute (2.21) into (2.11) and (2.13), there are
\[ \partial_r \bar{P} = \left( \bar{a}^2 + k^2 (T^* - t)^{4a} - \frac{a(1 + a)}{(T^* - t)^2} - \frac{4 \bar{a}^2 k^2 z^2 (T^* - t)^{2(2a + 1)}}{(4a + 1)^2} \right) r, \]
and
\[ \partial_z \bar{P} = 2z \left( \frac{2 \bar{a}^2 + a(1 - 2a)}{(T^* - t)^2} \right). \]
Note that
\[ |H|^2 = \bar{a}^2 r^2 + \frac{4k^2 \bar{a}^2 r^2 z^2 (T^* - t)^{2(2a + 1)}}{(4a + 1)^2} + 4\bar{a}^2 z^2. \]
Thus by (2.17), direct computations give the pressure
\[ P(t, r, z) = \frac{r^2}{2} \left( k^2 (T^* - t)^{4a} - \frac{a(1 + 1)}{(T^* - t)^2} - \frac{8 \bar{a}^2 k^2 z^2 (T^* - t)^{2(2a + 1)}}{(4a + 1)^2} \right) + z^2 \left( \frac{a(1 - 2a)}{(T^* - t)^2} - \frac{2 \bar{a}^2 k^2 r^2 (T^* - t)^{2(2a + 1)}}{(4a + 1)^2} \right). \]

Proof of Proposition 2.1. Since

\[ \mathbf{v}(t, x) = v^r(t, r, z)\mathbf{e}_r + v^\theta(t, r, z)\mathbf{e}_\theta + v^z(t, r, z)\mathbf{e}_z, \]
\[ \mathbf{H}(t, x) = H^r(t, r, z)\mathbf{e}_r + H^\theta(t, r, z)\mathbf{e}_\theta + H^z(t, r, z)\mathbf{e}_z, \]

we can obtain a family of explicit blowup axisymmetric solutions for 3D incompressible MHD equations by noticing that $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z$ are defined in (2.8), $r = \sqrt{x_1^2 + x_2^2}$ and $z = x_3$, and

\[ v^r(t, r, z) = \frac{ar}{T^* - t}, \]
\[ v^\theta(t, r, z) = \frac{k}{r}, \]
\[ v^z(t, r, z) = -\frac{2az}{T^* - t}, \]
\[ H^r(t, r, z) = \bar{a}r, \]
\[ H^\theta(t, r, z) = \frac{2k\bar{a}(T^* - t)}{(2a + 1)r}, \]
\[ H^z(t, r, z) = -2\bar{a}z, \]

where constants $\bar{a}, k \in \mathbb{R}/\{0\}$ and $a \in \mathbb{R}/\{-\frac{1}{2}, 0\}$, or

\[ v^r(t, r, z) = \frac{ar}{T^* - t}, \]
\[ v^\theta(t, r, z) = k\frac{r}{(T^* - t)^{2a}}, \]
\[ v^z(t, r, z) = -\frac{2az}{T^* - t}, \]
\[ H^r(t, r, z) = ar, \]
\[ H^\theta(t, r, z) = \frac{2k\bar{a}(T^* - t)^{2a + 1}}{4a + 1}, \]
\[ H^z(t, r, z) = -2\bar{a}z, \]
where constants $\bar{a}, k \in \mathbb{R}/\{0\}$ and $a \in \mathbb{R}/\{-\frac{1}{4}, 0\}$.

Furthermore the vorticity vector $\omega$ is

$$\omega(t, x) = \omega^r(t, r, z)e_r + \omega^\theta(t, r, z)e_\theta + \omega^z(t, r, z)e_z,$$

where

$$\omega^r(t, r, z) = -\partial_z v^\theta = 0,$$

$$\omega^\theta(t, r, z) = \partial_z v^r - \partial_r v^z = 0,$$

$$\omega^z(t, r, z) = \frac{1}{r} \partial_r (rv^\theta) = 0,$$

or

$$\omega^r(t, r, z) = -\partial_z v^\theta = 0,$$

$$\omega^\theta(t, r, z) = \partial_z v^r - \partial_r v^z = 0,$$

$$\omega^z(t, r, z) = \frac{1}{r} \partial_r (rv^\theta) = 2k(T^* - t)^{2a}.$$

**Proof of Theorem 1.1.** By directly computations, we can obtain two family of explicit blowup solutions from [2.9] and [2.10]. Moreover, the vorticity vector

$$\omega(t, x) = \nabla \times \mathbf{v} = 0,$$

or

$$\omega(t, x) = \nabla \times \mathbf{v} = 2k(T^* - t)^{2a}.$$

**Acknowledgments.**

The author expresses his sincerely thanks to the BICMR of Peking University and Professor Gang Tian for constant support and encouragement. The author also expresses his sincerely thanks to Prof. V. Sverák for informed the paper [5, 15, 16] and his suggestions. The author is supported by NSFC No 11771359.

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