A solution to a conjecture on the rainbow connection number*

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Abstract

For a graph $G$, Chartrand et al. defined the rainbow connection number $rc(G)$ and the strong rainbow connection number $src(G)$ in “G. Charand, G.L. John, K.A. McKeon, P. Zhang, Rainbow connection in graphs, Mathematica Bohemica, 133(1)(2008) 85-98”. They raised the following conjecture: for two given positive integers $a$ and $b$, there exists a connected graph $G$ such that $rc(G) = a$ and $src(G) = b$ if and only if $a = b \in \{1, 2\}$ or $3 \leq a \leq b$. In this short note, we will show that the conjecture is true.

Keywords: edge-colored graph, (strong) rainbow coloring, (strong) rainbow connection number.

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1 Introduction

All graphs in this paper are finite, undirected, simple and connected. We follow the notation and terminology of [1]. Let $c$ be a coloring of the edges of a graph $G$, i.e., $c : E(G) \rightarrow \{1, 2, \ldots, k\}, k \in \mathbb{N}$. A path is called a rainbow path if no two edges of the path have the same color. The graph $G$ is called rainbow connected (with respect to $c$) if for every two vertices of $G$, there exists a rainbow path connecting them in $G$. If by coloring $c$ the graph $G$ is rainbow connected, then the coloring $c$ is called a rainbow coloring of $G$. If $k$ colors are used in $c$, then $c$ is a rainbow $k$-coloring of $G$. The minimum

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number $k$ for which there exists a rainbow $k$-coloring of $G$, is called the rainbow connection number of $G$, denoted by $rc(G)$.

Let $c$ is a rainbow coloring of a graph $G$. If for every pair $u$ and $v$ of distinct vertices of the graph $G$, the graph $G$ contains a rainbow $u$-$v$ geodesic (a shortest path in $G$ between $v$ and $u$), then $G$ is called strongly rainbow connected. In this case, the coloring $c$ is called a strong rainbow coloring of $G$. If $k$ colors are used, then $c$ is a strong rainbow $k$-coloring of $G$. The minimum number $k$ satisfying that $G$ is strongly rainbow connected, i.e., the minimum number $k$ for which there exists a strong rainbow $k$-coloring of $G$, is called the strong rainbow connection number of $G$, denoted by $src(G)$. Thus for every connected graph $G$, $rc(G) \leq src(G)$. Recall that the diameter of $G$ is defined as the largest distance between two vertices of $G$, denoted $diam(G)$. Then $diam(G) \leq rc(G) \leq src(G)$.

The following results were obtained in [2] by Chartrand et al.

**Proposition 1.1** Let $G$ be a nontrivial connected graph of size $m$. Then
1. $rc(G) = 1$ if and only if $src(G) = 1$.
2. $rc(G) = 2$ if and only if $src(G) = 2$.
3. $diam(G) \leq rc(G) \leq src(G)$ for every connected graph $G$.

Chartrand et al. also considered the problem that, given any two integers $a$ and $b$, whether there exists a connected graph $G$ such that $rc(G) = a$ and $src(G) = b$? and they got the following result.

**Theorem 1.2** Let $a$ and $b$ be positive integers with $a \geq 4$ and $b \geq (5a - 6)/3$. Then there exists a connected graph $G$ such that $rc(G) = a$ and $src(G) = b$.

Then, combining Proposition 1.1 and Theorem 1.2, they got the following result.

**Corollary 1.3** Let $a$ and $b$ be positive integers. If $a = b$ or $3 \leq a < b$ and $b \leq \frac{5a - 6}{3}$, then there exists a connected graph $G$ such that $rc(G) = a$ and $src(G) = b$.

Finally, they thought the question that whether the condition $b \leq \frac{5a - 6}{3}$ can be deleted? and raised the following conjecture:

**Conjecture 1.4** Let $a$ and $b$ be positive integers. Then there exists a connected graph $G$ such that $rc(G) = a$ and $src(G) = b$ if and only if $a = b \in \{1, 2\}$ or $3 \leq a \leq b$.

This short note is to give a confirmative solution to this conjecture.
2 Proof of the conjecture

Proof of Conjecture 1.4: From Proposition 1.1 one can see that the condition is necessary. For the sufficiency, when \(a = b \in \{1, 2\}\), from Corollary 1.3 the conjecture is true. So, we just need to consider the situation \(3 \leq a \leq b\).

Let \(n = 3b(b-a+2)\), and let \(H_n\) be the graph consisting of an \(n\)-cycle \(C_n: v_1, v_2, \ldots, v_n\) and another two vertices \(w\) and \(v\), each of which joins to every vertex of \(C_n\). Let \(G\) be the graph constructed from \(H_n\) of order \(n+2\) and the path \(P_{a-1}: u_1, u_2, \ldots, u_{a-1}\) on \(a-1\) vertices by identifying \(v\) and \(u_{a-1}\).

First, we will show \(\text{rc}(G) = a\). Because \(\text{diam}(G)=a\), by Proposition 1.1 we have \(\text{rc}(G) \geq a\). It remains to show \(\text{rc}(G) \leq a\). Note that \(n = 3b(b-a+2) \geq 18\). Define a coloring \(c\) for the graph \(G\) by the following rules:

\[
\begin{align*}
c(e) = \begin{cases} 
i & \text{if } e = u_iu_{i+1} \text{ for } 1 \leq i \leq a-2, \\
a-1 & \text{if } e = v_iv \text{ and } i \text{ is odd,} \\
a & \text{if } e = v_iv \text{ and } i \text{ is even,} \\
a & \text{if } e = v_iw \text{ and } 1 \leq i \leq n \\
1 & \text{otherwise.}
\end{cases}
\end{align*}
\]

Since \(c\) is a rainbow \(a\)-coloring of the edges of \(G\), it follows that \(\text{rc}(G) \leq a\). This implies \(\text{rc}(G) = a\).

Next, we will show \(\text{src}(G) = b\). We first show \(\text{src}(G) \leq b\), by giving a strong rainbow \(b\)-coloring \(c\) for the graph \(G\) as follows:

\[
\begin{align*}
c(e) = \begin{cases} 
i & \text{if } e = u_iu_{i+1} \text{ for } 1 \leq i \leq a-2, \\
a-2+i & \text{if } e = v_{3(i-1)+j}v \text{ for } 1 \leq i \leq b-a+2 \text{ and } 1 \leq j \leq 3b, \\
i & \text{if } e = v_{3(j-1)+k}w \text{ for } 1 \leq j \leq b-a+2 \text{ and } 1 \leq i \leq b \\
 & \text{and } 1 \leq k \leq 3, \\
1 & \text{if } e = v_{3(i-1)+1}v_{3(i-1)+2} \text{ for } 1 \leq i \leq b(b-a+2), \\
2 & \text{if } e = v_{3(i-1)+2}v_{3(i-1)+3} \text{ for } 1 \leq i \leq b(b-a+2), \\
3 & \text{otherwise}
\end{cases}
\end{align*}
\]

It remains to show \(\text{src}(G) \geq b\). By contradiction, suppose \(\text{rc}(G) < b\). Then there exists a strong rainbow \((b-1)\)-coloring \(c: E(G) \to \{1, 2, \ldots, b-1\}\). For every \(v_i (1 \leq i \leq n)\), \(d(v_i, u_1) = a-1\), and the path \(v_ivu_a\cdots u_1\) is the only path of length \(a-1\) connecting \(v_i\) and \(u_1\), and so \(v_ivu_a\cdots u_1\) is a rainbow path. Without loss of generality, suppose \(c(u_2u_1) = 1, c(u_3u_2) = 2, \ldots, c(u_{a-1}u_a) = a-2\). Then \(c(v_i) \in \{a-1, a, \ldots, b\}\), for \(1 \leq i \leq n\). We first consider the set of edges \(A = \{v_iv, 1 \leq i \leq n\}\), and so \(|A| = n\). Thus there exist at least \(\lceil \frac{n}{b-a+1} \rceil \geq 3b+1\) edges in \(A\) colored the same. Suppose there exist \(m\) edges \(v_{j_1}v, \ldots, v_{j_m}v, (1 \leq j_1 < j_2 < \cdots < j_m \leq n)\) colored the same and
Second, we consider the set of edges $B = \{v_{j_1}w, \ldots, v_{j_m}w\}$. Since $c(v_{j_i}w) \in \{1, 2, \ldots, b - 1\}$, for $1 \leq i \leq m$, then there exist at least $\left\lceil \frac{m}{b-1} \right\rceil \geq \left\lceil \frac{3b+1}{b-1} \right\rceil \geq 4$ edges colored the same. Thus from $B$ we can choose 4 edges of the same color. Since $n \geq 18$, from the corresponding vertices on the cycle $C_n$ of the four edges chosen above, we can get two vertices such that their distance on the cycle $C_n$ is more than 3. Without loss of generality, we assume that the two vertices are $v_1', v_2'$ and their distance in graph $G$ is 2. Then the geodesic between $v_1'$ and $v_2'$ in graph $G$ is either $v_1'wv_2'$ or $v_1'vv_2'$. However, neither $v_1'wv_2'$ nor $v_1'vv_2'$ is a rainbow path. Thus the coloring $c$ is not a strong rainbow coloring of $G$, a contradiction. Therefore $src(G) \leq b$ and so $src(G) = b$. The proof is thus complete.

References

[1] J.A. Bondy, U.S.R. Murty. Graph Theory, Springer, Heidelberg, 2008.

[2] G. Chartrand, G.L. Johns, K.A. McKeon, P. Zhang. Rainbow connection in graphs. Math. Bohem., 133(1)(2008) 85-98.