QUILLEN–SUSLIN THEORY FOR CLASSICAL GROUPS:
REVISITED OVER GRADED RINGS

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Abstract: In this paper we deduce a graded version of Quillen–Suslin's Local-Global Principle for the traditional classical groups, viz. general linear, symplectic and orthogonal groups and establish its equivalence of the normality property of the respective elementary subgroups. This generalizes previous result of Basu–Rao–Khanna; cf. [3]. Then, as an application, we establish an analogue Local-Global Principle for the commutator subgroups of the special linear and symplectic groups. Finally, by using Swan-Weibel’s homotopy trick, we establish graded analogue of the Local-Global Principle for the transvection subgroups of the full automorphism groups of the linear, symplectic and orthogonal modules. This generalizes the previous result of Bak–Basu–Rao; cf. [5].

1. Introduction

In [3], the first author with Ravi A. Rao and Reema Khanna revisited Quillen-Suslin’s Local-Global Principle for the linear, symplectic and orthogonal groups to show the equivalence of the Local-Global Principle and Normality property of their Elementary subgroups. A “relative version” is also established recently by the same authors; cf. [6]. In [3], the authors have also deduced a Local-Global Principle for the commutator subgroups of the special linear groups, and symplectic groups. In this article we aim to revisit those results over commutative \( \mathbb{N} \)-graded rings with identity. We follow the line of proof described in [3].

For the linear case, the graded version of the Local-Global Principle was studied by Chouinard in [7], and by J. Gubeladze in [8], [9]. Though the analogue results are expected for the symplectic and orthogonal groups, according to our best knowledge, that is not written explicitly in any existing literatures. By deducing the equivalence, we establish the graded version of the Local-Global Principle for the above two types of classical groups.

To generalize the existing results, from the polynomial rings to the graded rings, one has to use the line of proof of Quillen–Suslin’s patching argument.
(cf. [12], [13]), and Swan–Weibel’s homotopy trick. For a nice exposition we refer to [8] by J. Gubeladze.

Though, we are writing this article for commutative graded rings, one may also consider standard graded algebras which are finite over its center. For a commutative $\mathbb{N}$-graded ring with 1, we establish:

1. (Theorem 3.8) Normality of the elementary subgroups is equivalent to the graded Local-Global Principle for the linear, symplectic and orthogonal groups.

2. (Theorem 4.8) Analogue of Quillen–Suslin’s Local-Global Principle for the commutator subgroups of the special linear and symplectic groups over graded rings.

In [5], the following was established by the first author with A. Bak and R.A. Rao:

(Local-Global Principle for the Transvection Subgroups) An analogue of Quillen–Suslin’s Local-Global Principle for the transvection subgroup of the automorphism group of projective, symplectic and orthogonal modules of global rank at least 1 and local rank at least 3, under the assumption that the projective module has constant local rank and that the symplectic and orthogonal modules are locally an orthogonal sum of a constant number of hyperbolic planes.

In this article, we observe that by using Swan-Weibel’s homotopy trick, one gets an analogue statement for the graded case. More precisely, we deduce the following fact:

3. (Theorem 5.4) Let $A = \bigoplus_{i=0}^{\infty} A_i$ be a graded ring and $Q \simeq P \oplus A_0$ be a projective $A_0$-module ($Q \simeq P \oplus \mathbb{H}(A_0)$ for symplectic and orthogonal modules). If an automorphism of $Q \otimes A_0$ is locally in the transvection subgroup, then it is globally in the transvection subgroup.

2. Definitions and Notations

Let us start by recalling the following well-known fact: Given a ring $R$ and a subring $R_0$, one can express $R$ as a direct limit of subrings which are finitely generated over $R_0$ as rings. Considering $R_0$ to be the minimal subring (i.e. the image of $\mathbb{Z}$), it follows that every ring is a direct limit of Noetherian rings. Hence we may consider $R$ to be Noetherian (cf. Pg 271 [11]).

Throughout this paper we assume $A$ to be a Noetherian, commutative graded ring with identity 1. We shall write $A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$. As we know the multiplication in a graded ring satisfies the following property: For all $i, j$, $A_i A_j \subset A_{i+j}$. An element $a \in A$ will be denoted by $a = a_0 + a_1 + a_2 + \cdots$, where $a_i \in A_i$ for each $i$, and and all but finitely many $a_i$ are zero. Let $A_+ = A_1 \oplus A_2 \oplus \cdots$. Graded structure of $A$ induces a graded structure on $M_n(A)$ (ring of $n \times n$ matrices). Let $S$ be a multiplicatively closed subset of $A_0$. Then for a non-zero divisor $s \in S$ we shall denote the localization of a matrix $\alpha \in M_n(A)$ to $M_n(A_s)$ as $\alpha_s$. Otherwise, $\alpha_i, \ i \in \mathbb{Z}$ will represent the
the $i$-th component of $\alpha$. We shall use standard notations $\text{GL}_n(A)$ and $\text{SL}_n(A)$ to denote the group of invertible matrices and its subgroup of all invertible matrices with determinant 1 respectively.

We recall the well-known “Swan-Weibel’s homotopy trick”, which is the main ingredient to handle the graded case.

**Definition 2.1.** Let $a \in A_0$ be a fixed element. We fix an element $b = b_0 + b_1 + \cdots$ in $A$ and define a ring homomorphism $\epsilon : A \to A[X]$ given by

$$
\epsilon(b) = \epsilon(b_0 + b_1 + \cdots) = b_0 + b_1X + b_2X^2 + \cdots + b_iX^i + \cdots.
$$

Then we evaluate the polynomial $\epsilon(b)(X)$ at $X = a$ and denote the image by $b^{+}(a)$, i.e. $b^{+}(a) = \epsilon(b)(a)$. Note that $(b^{+}(x))^{+}(y) = b^{+}(xy)$. Observe, $b_0 = b^{+}(0)$. We shall use this fact frequently.

The above ring homomorphism $\epsilon$ induces a group homomorphism at the $\text{GL}_n(A)$ level for every $n \geq 1$, i.e. for $\alpha \in \text{GL}_n(A)$ we get a map $\epsilon : \text{GL}_n(A) \to \text{GL}_n(A[X])$ defined by

$$
\alpha = \alpha_0 \oplus \alpha_1 \oplus \alpha_2 \oplus \cdots \mapsto \alpha_0 \oplus \alpha_1X \oplus \alpha_2X^2 \cdot \cdot ,
$$

where $\alpha_i \in \text{GL}_n(A_i)$. As above for $a \in A_0$, we define $\alpha^{+}(a)$ as $\alpha^{+}(a) = \epsilon(\alpha)(a)$.

Now we are going to recall the definitions of the traditional classical groups, viz. the general linear groups, the symplectic and orthogonal groups (of even size) and their type subgroups, viz. elementary (symplectic and orthogonal elementary resp.) subgroups.

Let $e_{ij}$ be the matrix with 1 in the $ij$-position and 0’s elsewhere. The matrices of the form $\{E_{ij}(\lambda) : \lambda \in A \mid i \neq j\}$, where

$$
E_{ij}(\lambda) = I_n + \lambda e_{ij}
$$

are called the elementary generators.

**Definition 2.2.** The subgroup generated by the set $\{E_{ij}(\lambda) : \lambda \in A \mid i \neq j\}$, is called the elementary subgroup of the general linear group and is denoted by $E_n(A)$. Observe that $E_n(A) \subseteq \text{SL}_n(A) \subseteq \text{GL}_n(A)$.

Let $\sigma$ be a permutation defined by: For $i \in \{1, \ldots, 2m\}$

$$
\sigma(2i) = 2i - 1 \quad \text{and} \quad \sigma(2i - 1) = 2i.
$$

With respect to this permutation, we define two $2m \times 2m$ forms (viz.) $\psi_m$ and $\tilde{\psi}_m$ as follows: For $m > 1$, let

$$
\psi_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and for } m > 1, \quad \psi_m = \begin{bmatrix} \psi_{m-1} & 0 \\ 0 & I_2 \end{bmatrix},
$$

$$
\tilde{\psi}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and for } m > 1, \quad \tilde{\psi}_m = \begin{bmatrix} \tilde{\psi}_{m-1} & 0 \\ 0 & I_2 \end{bmatrix}.
$$

Using above two forms, we define the following traditional classical groups:
Definition 2.3. A matrix \( \alpha \in \text{GL}_{2m}(A) \) is called symplectic if it fixes \( \psi_m \) under the action of conjugation, i.e.
\[
\alpha^t \psi_m \alpha = \psi_m.
\]
The group generated by all symplectic matrices is called the **symplectic group** and is denoted by \( \text{Sp}_{2m}(A) = \text{Sp}_n(A) \), where \( n = 2m \).

Definition 2.4. A matrix \( \alpha \in \text{GL}_{2m}(A) \) is called orthogonal if it fixes \( \tilde{\psi}_m \) under the action of conjugation, i.e.
\[
\alpha^t \tilde{\psi}_m \alpha = \tilde{\psi}_m.
\]
The group generated by all orthogonal matrices is called the **orthogonal group** and is denoted by \( \text{O}_{2m}(A) = \text{O}_n(A) \), where \( n = 2m \).

Definition 2.5. The matrices of the form
\[
\{se_{ij}(z) \in \text{GL}_{2m}(A) : z \in A \mid i \neq j \},
\]
where
\[
se_{ij}(z) = I_{2m} + ze_{ij} \text{ if } i = \sigma(j)
\]
or
\[
se_{ij}(z) = I_{2m} + ze_{ij} - (-1)^{i+j}ze_{\sigma(j)\sigma(i)} \text{ if } i \neq \sigma(j) \text{ and } i < j.
\]
are called **symplectic elementary generators**. The subgroup generated by symplectic elementary generators is called the **symplectic elementary group**.

Definition 2.6. The matrices of the form \( \{oe_{ij}(z) : z \in A \mid i \neq j \} \), where
\[
oe_{ij}(z) = I_{2m} + ze_{ij} - ze_{\sigma(j)\sigma(i)} \text{ if } i \neq \sigma(j) \text{ and } i < j.
\]
are called **orthogonal elementary generators**. The subgroup generated by orthogonal elementary generators is called the **orthogonal elementary group**.

Remark 2.7. It is a well known fact that the elementary subgroups are normal subgroups of the respective classical groups; for \( n \geq 3 \) in the linear case, \( n = 2m \geq 4 \) in the symplectic case, and \( n = 2m \geq 6 \) in the orthogonal case. The linear case was due to A. Suslin (cf. [15]), symplectic was proved by V. Kopeiko (cf. [15]). Finally the orthogonal case was proved by Suslin–Kopeiko (cf. [14]).

We use the following notations to treat above three groups uniformly:
\[
\text{G}(n, A) \text{ will denote GL}_n(A) \text{ or Sp}_{2m}(A) \text{ or O}_{2m}(A).
\]
\[
\text{E}(n, A) \text{ will denote E}_n(A) \text{ or EO}_{2m}(A) \text{ or ESp}_{2m}(A).
\]
\[
\text{S}(n, A) \text{ will denote SL}_n(A) \text{ or SO}_{2m}(A), \text{ and}
\]
ge\(e_{ij}\) will denote the elementary generators of E\(_n\)(A) or ESp\(_n\)(A) or EO\(_n\)(A).

Here \( \text{SO}_n(A) \) is the subgroup of \( \text{O}_n(A) \) with determinant 1. Throughout the paper we will assume \( n = 2m \) to treat the non-linear cases. We shall assume
$n \geq 3$ while treating the linear case, and $n \geq 6$; i.e. $m \geq 3$ while treating the symplectic and orthogonal cases. Let

$$\tilde{v} = \begin{cases} v \psi_m & \text{in the symplectic case,} \\ v \tilde{\psi}_m & \text{in the orthogonal case.} \end{cases}$$

**Definition 2.8.** Let $v, w \in A^n$ be vectors of length $n$, then we define the inner product $\langle v, w \rangle$ as follows:

1. $\langle v, w \rangle = v^t w$ in the linear case,
2. $\langle v, w \rangle = \tilde{v}w$ in the symplectic and orthogonal cases.

**Definition 2.9.** Let $v, w \in A^n$, we define the map $M : A^n \times A^n \to M_n(A)$ as follows:

1. $M(v, w) = vw^t$ in the linear case,
2. $M(v, w) = \tilde{v}w + w\tilde{v}$ in the symplectic case,
3. $M(v, w) = \tilde{v}w - w\tilde{v}$ in the orthogonal case.

By $[G, G]$, we denote the commutator subgroup of $G$ i.e. group generated by elements of the form $ghg^{-1}h^{-1}$, for $g, h \in G$.

**Definition 2.10.** A row $(v_1, \ldots, v_n) \in A^n$ is called unimodular row of length $n$ if there exist $b_1, \ldots, b_n \in A$ such that $\Pi_{i=1}^n v_i b_i = 1$; i.e. If $I$ is the ideal generated by $\langle v_1, \ldots, v_n \rangle$, then $(v_1, \ldots, v_n)$ is unimodular if and only if $I = A$. In the above case we say $A$ has $n$-cover.

### 3. Local-Global Principle for classical groups

Before discussing the main theorem we recall following standard facts.

**Lemma 3.1 (Splitting Lemma).** The elementary generators of $n \times n$ matrices satisfy the following property:

$$ge_{ij}(x + y) = ge_{ij}(x) ge_{ij}(y)$$

for all $x, y \in A$ and for all $i, j = 1, \ldots, n$ with $i \neq j$.

**Proof.** Standard. (cf. [3], §3, Lemma 3.2).

**Lemma 3.2.** Let $G$ be a group, and $a_i, b_i \in G$ for $i = 1, \ldots, r$. Let $J_k = \prod_{j=1}^k a_j$. Then

$$\prod_{i=1}^r a_i b_i = \prod_{i=1}^r J_i b_i J_i^{-1} \prod_{i=1}^r a_i.$$

**Proof.** cf. [3], §3, Lemma 3.4.

Following structural lemma pays a key role in the proof of main theorem. It is well-known for the polynomial rings; cf. [3] (§3, Lemma 3.6). Here, we deduce the analogue for the graded rings.
Lemma 3.3. Let $G(n, A, A_+)$ denote the subgroup of $G(n, A)$ which is equal to $I_n$ modulo $A_+$. Then the group $G(n, A, A_+)$ is generated by the elements of the type $\epsilon ge_{ij}(A_+)\epsilon^{-1}$ for some $\epsilon \in E(n, A_0)$.

Proof. Let $\alpha \in E(n, A) \cap G(n, A, A_+)$. Then we can write

$$\alpha = \prod_{k=1}^{r} ge_{ik,jk}(a_k)$$

for some elements $a_k \in A$, $k = 1, \ldots, r$. As $a_k = (a_0)_k + (a_+)_k$ for some $(a_0)_k \in A_0$ and $(a_+)_k \in A_+$. Using the splitting lemma (Lemma 3.1) we can rewrite the expression as:

$$\alpha = \prod_{k=1}^{r} (ge_{ik,jk}(a_0)_k)(ge_{ik,jk}(a_+)_k).$$

We write $\epsilon_t = \prod_{k=1}^{r} ge_{ik,jk}((a_0)_k)$ for $t \in \{1, 2, \ldots, r\}$. Observe that $\epsilon_r = I_n$, as $\alpha \in G(n, A, A_+)$. Then

$$\alpha = \left( \prod_{k=1}^{r} \epsilon_k ge_{ik,jk}((a_+)_k) \epsilon_k^{-1} \right) \left( \prod_{k=1}^{r} ge_{ik,jk}((a_0)_k) \right) = AB \text{ (say)} ,$$

where $A = \left( \prod_{k=1}^{r} \epsilon_k ge_{ik,jk}((a_+)_k) \epsilon_k^{-1} \right)$ and $B = \left( \prod_{k=1}^{r} ge_{ik,jk}((a_0)_k) \right)$. Now go modulo $A_+$. Let $\bar{A} = \{ (\alpha_0)_k \}$ and $\bar{B} = \{ (\alpha_+)_k \}$. Let $\bar{\alpha} = \bar{\alpha}_n = \bar{A}\bar{B} = \bar{I}_n \bar{B} = \bar{I}_n \longrightarrow \bar{B} = \bar{I}_n,$

as $\alpha \in G(n, A, A_+)$. Since entries of $B$ are in $A_0$, it follows that $B = I_n$.

$$\alpha = \prod_{k=1}^{r} \epsilon_k \left( ge_{ik,jk}((a_+)_k) \right) \epsilon_k^{-1};$$

as desired. \qed

Now we prove a variant of “Dilation Lemma” mentioned in the statement of (3) of Theorem 3.8

Lemma 3.4. Assume the “Dilation Lemma” (Theorem 3.8 - (4)) to be true. Let $\alpha_s \in E(n, A_+)$, with $\alpha^+(0) = I_n$. Then one gets

$$\alpha^+(b + d)\alpha^+(d)^{-1} \in E(n, A)$$

for some $s, d \in A_0$ and $b = s^l, l \geq 0$.

Proof. We have $\alpha_s \in E(n, A_+)$. Hence $\alpha^+_s(X) \in E(n, A_+).$ Let $\beta^+(X) = \alpha^+(X + d)\alpha^+(d)^{-1}$, where $d \in A_0$. Then $\beta^+_s(X) \in E(n, A_+|X]$ and $\beta^+(0) = I_n$. Hence by Theorem 3.8 - (4) there exists $\beta(X) \in E(n, A[X])$ such that $\beta(X) = \beta^+_s(bX)$. Putting $X = 1$, we get the required result. \qed

Following is a very crucial result we need for our method of proof. There are many places we use this fact in a very subtle way. In particular, we mainly use this lemma for the step $(4) \Rightarrow (3)$ of 3.8.
Lemma 3.5. (cf. [10], Lemma 5.1) Let $R$ be a Noetherian ring and $s \in R$. Then there exists a natural number $k$ such that the homomorphism $G(n, s^kR) \rightarrow G(n, R_s)$ (induced by localization homomorphism $R \rightarrow R_s$) is injective. Moreover, it follows that the induced map $E(n, R, s^kR) \rightarrow E(n, R_s)$ is injective.

The next two lemma’s will be used in intermediaries to prove the equivalent conditions mentioned in Theorem 3.8. We state it without proof.

Lemma 3.6. Let $v = (v_1, v_2, \ldots, v_n)$ be a unimodular row over over a commutative semilocal ring $R$. Then the row $(v_1, v_2, \ldots, v_n)$ is completable; i.e. $(v_1, v_2, \ldots, v_n)$ elementarily equivalent to the row $(1, 0, \ldots, 0)$; i.e. there exists $\epsilon \in E_n(R)$ such that $(v_1, v_2, \ldots, v_n)\epsilon = (1, 0, \ldots, 0)$.

Proof. cf. [4], Lemma 1.2.21. \hfill \Box

Lemma 3.7. Let $A$ be a ring and $v \in E(n, A)e_1$. Let $w \in A^n$ be a column vector such that $\langle v, w \rangle = 0$. Then $I_n + M(v, w) \in E(n, A)$.

Proof. cf. [4], Lemma 2.2.4. \hfill \Box

Theorem 3.8. The followings are equivalent for any graded ring $A = \oplus_{i=0}^{\infty} A_i$ for $n \geq 3$ in the linear case and $n \geq 6$ otherwise.

1. (Normality): $E(n, A)$ is a normal subgroup of $G(n, A)$.
2. If $v \in Um_n(A)$ and $\langle v, w \rangle = 0$, then $I_n + M(v, w) \in E(n, A)$.
3. (Local-Global Principle): Let $\alpha \in G(n, A)$ with $\alpha^+(0) = I_n$. If for every maximal ideal $m$ of $A_0$, $\alpha_m \in E(n, A_m)$, then $\alpha \in E(n, A)$.
4. (Dilation Lemma): Let $\alpha \in G(n, A)$ with $\alpha^+(0) = I_n$ and $\alpha_s \in E(n, A_s)$ for some non-zero divisor $s \in A_0$. Then there exists $\beta \in E(n, A)$ such that

\[ \beta_s^+(b) = \alpha_s^+(b) \]

for some $b = s^l; l \gg 0$. i.e. $\alpha_s^+(s^l)$ will be defined over $A$ for $l \gg 0$.

5. If $a \in E(n, A)$, then $\alpha^+(a) \in E(n, A)$, for every $a \in A_0$.
6. If $v \in E(n, A)e_1$, and $\langle v, w \rangle = 0$, then $I_n + M(v, w) \in E(n, A)$.

Proof. (5) $\Rightarrow$ (6): Let $\alpha = \prod_{k=1}^{t} \left(I_n + a \cdot M(e_{i_k}, e_{j_k})\right)$ where $a \in A$, and $t \geq 1$, a positive integer. Then

\[ \alpha^+(b) = \prod_{k=1}^{t} \left(I_n + a^+(b) \cdot M(e_{i_k}, e_{j_k})\right), \]

where $b \in A_0$. Take $v = e_i$ and $w = a^+(b)e_j$, Then $\alpha = I_n + M(v, w)$ and $\langle v, w \rangle = 0$. Indeed,

Linear case:

\[ M(v, w) = vw^t = e_i(a^+(b)e_j)^t = a^+(b)e_i^te_j^t = a^+(b)M(e_i, e_j), \]

\[ \langle v, w \rangle = v^tw = e_i^t(a^+(b)e_j)^t = a^+(b)e_i^te_j = a^+(b)\langle e_i, e_j \rangle = 0. \]
Symplectic case:

\[
\begin{align*}
M(v, w) &= vw^t \psi_m + wv^t \psi_m = e_i (a^+(b) e_j)^t \psi_m + a^+(b) e_j e_i^t \psi_m \\
&= a^+(b) e_i e_j^t \psi_m + a^+(b) e_j e_i^t \psi_m = a^+(b) M(e_i, e_j),
\end{align*}
\]

\[
\langle e_i, e_j \rangle = e_i^t \psi_m e_j = (e_i^t \psi_m e_i) (e_i^t e_j) = \psi_m (e_i^t e_j) = 0 \text{ hence,}
\]

\[
\langle v, w \rangle = v^t \psi_m w = e_i^t \psi_m a^+(b) e_j = a^+(b) e_i e_j = a^+(b) \langle e_i, e_j \rangle = 0.
\]

Orthogonal case:

\[
\begin{align*}
M(v, w) &= vw^t \tilde{\psi}_m - wv^t \tilde{\psi}_m = e_i (a^+(b) e_j)^t \tilde{\psi}_m - a^+(b) e_j e_i^t \tilde{\psi}_m \\
&= a^+(b) e_i e_j^t \tilde{\psi}_m - a^+(b) e_j e_i^t \tilde{\psi}_m = a^+(b) M(e_i, e_j),
\end{align*}
\]

\[
\langle e_i, e_j \rangle = e_i^t \tilde{\psi}_m e_j = (e_i^t \tilde{\psi}_m e_i) (e_i^t e_j) = \tilde{\psi}_m (e_i^t e_j) = 0 \text{ hence,}
\]

\[
\langle v, w \rangle = v^t \tilde{\psi}_m w = e_i^t \tilde{\psi}_m a^+(b) e_j = a^+(b) e_i e_j = a^+(b) \langle e_i, e_j \rangle = 0.
\]

Hence applying \((\ref{equality3})\) over ring \(A\), we have \(\alpha^+(b) \in E(n, A)\). Therefore \(\alpha^+(b) \in E(n, A)\) for \(b \in A_0\).

\((\ref{equality1}) \Rightarrow \text{Orthogonal case}\): Since \(\alpha_s \in E(n, A_s)\) with \((\alpha_{0,s}) = I_n\), the diagonal entries of \(\alpha\) are of the form \(1 + g_{ii}\), where \(g_{ii} \in (A_+)\) and off-diagonal entries are of the form \(g_{ij}\), where \(i \neq j\) and \(g_{ij} \in (A_+)^2\).

We choose \(l\) to be large enough such that \(s^l\) is greater than the common denominator of all \(g_{ii}\) and \(g_{ij}\). Then using \((\ref{equality5})\), we get

\[
\alpha_s^+(s^l) \in E(n, A_s).
\]

Since that \(\alpha^+(s^l)\) permits a natural pullback (as denominators are cleared), we have \(\alpha^+(s^l) \in E(n, A)\).

\((\ref{equality4}) \Rightarrow \text{Symplectic case}\):

Since \(\alpha_m \in E(n, A_m)\), we have an element \(s \in A_0 - m\) such that \(\alpha_s \in E(n, A_s)\). Let \(s_1, s_2, \ldots, s_r \in A_0\) be non-zero divisors with \(s_i \in A_0 - m\) such that \(\langle s_1, s_2, \ldots, s_r \rangle = A\). From \((\ref{equality4})\) we have \(\alpha^+(b_i) \in E(n, A)\) for some \(b_i = s_i^l\) with \(b_1 + \cdots + b_r = 1\). Now consider \(\alpha_{s_1 s_2 \cdots s_r}\), which is the image of \(\alpha\) in \(A_{s_1 \cdots s_r}\).

Due to Lemma \(\ref{lemma3}\), \(\alpha \mapsto \alpha_{s_1 s_2 \cdots s_r}\) is injective and hence we can perform our calculation in \(A_{s_1 \cdots s_r}\) and then pull it back to \(A\).

\[
\begin{align*}
\alpha_{s_1 s_2 \cdots s_r} &= \alpha_{s_1 s_2 \cdots s_r}^+(b_1 + b_2 \cdots + b_r) \\
&= ((\alpha_{s_1})_{s_2 s_3 \cdots} + (b_1 + \cdots + b_r))(\alpha_{s_1})_{s_2 s_3 \cdots} + (b_2 + \cdots + b_r)^{-1} \cdots \\
&\quad \cdot ((\alpha_{s_1})_{s_1 \cdots s_i \cdots s_r} + (b_i + \cdots + b_r))(\alpha_{s_1})_{s_1 \cdots s_i \cdots s_r} + (b_i+1 + \cdots + b_r)^{-1} \\
&\quad \cdots ((\alpha_{s_1})_{s_1 s_2 \cdots s_{r-1}} + (b_{r-1}))(\alpha_{s_1})_{s_1 s_2 \cdots s_{r-1}} + (0)^{-1}
\end{align*}
\]
Observing that each
\[(\alpha_{s_i})_{s_1s_2\ldots s_i\ldots s_r}^+ (b_i + \cdots + b_r)(\alpha_{s_i})_{s_1\ldots s_i\ldots s_r}^+ (b_i+1 + \cdots + b_r)^{-1} \in E(n, A)\]
due to Lemma 3.4 (here \(\hat{s}_i\) means we omit \(s_i\) in the product \(s_1 \ldots \hat{s}_i \ldots s_r\)), we have \(\alpha_{s_1\ldots s_r} \in E(n, A_{s_1\ldots s_r})\) and hence \(\alpha \in E(n, A)\).

**Remark 3.9.** Following is a commutative diagram (here we are assuming \(\langle s_i, s_j \rangle = A\)):

\[
\begin{array}{ccc}
A & \rightarrow & A_{s_i} \\
\downarrow & & \downarrow \\
A_{s_j} & \rightarrow & A_{s_i s_j}
\end{array}
\]

Let \(\theta_i = \alpha_{s_i}^+ (b_i + \cdots + b_r)\alpha_{s_i}^+ (b_i+1 + \cdots + b_r) \in A_{s_i}\) and \(\theta_{ij}\) be the image of \(\theta_i\) in \(A_{s_i s_j}\) and similarly \(\pi_j = \alpha_{s_j}^+ (b_j + \cdots + b_r)\alpha_{s_j}^+ (b_j+1 + \cdots + b_r) \in A_{s_j}\) and \(\pi_{s_i, s_j}\) be its image in \(A_{s_i s_j}\). Then due to Lemma 3.4, the product \((\theta_{s_i s_j})(\pi_{s_i s_j})\) can be identified with the product \(\theta_i \pi_j\).

\(3 \Rightarrow 2\): Since polynomial rings are special case of graded rings, the result follows by using 3 \(\Rightarrow 2\) in \(3\), §3.

\(2 \Rightarrow 1 \Rightarrow 6\): The proof goes as in \(3\). ($3, 2 \Rightarrow 1 \Rightarrow 7 \Rightarrow 6$).

\[\square\]

### 4. Local-Global Principle for commutator subgroup

In this section we deduce the analogue of Local-Global Principle for the commutator subgroup of the linear and the symplectic group over graded rings. Unless mentioned otherwise, we assume \(n \geq 3\) for the linear case and \(n \geq 6\) for the symplectic case.

Let us begin with the following well-known fact for semilocal rings.

**Lemma 4.1.** Let \(A\) be a semilocal commutative ring with identity. Then for \(n \geq 2\) in the linear case and \(n \geq 4\) in the symplectic case, one gets
\[S(n, A) = E(n, A)\.

**Proof.** cf. Lemma 1.2.25 in [4]. \[\square\]

**Remark 4.2.** If \(\alpha = (\alpha_{ij}/1) \in G(n, A_{s})\), then \(\alpha\) has a natural pullback \(\beta = (\alpha_{ij}) \in G(n, A)\) such that \(\beta_{s} = \alpha\). If \(\alpha_{s} \in S(n, A_{s})\) such that it admits a natural pullback \(\beta \in G(n, A)\), then \(\beta \in S(n, A)\).

The next lemma deduces an analogue result of “Dilation Lemma” (Theorem 3.8-(3)) for \(S(n, A)\); the special linear (resp. symplectic) group.
Lemma 4.3. Let $\alpha = (\alpha)_{ij} \in S(n, A)$. Then $\alpha^+(a) \in S(n, A)$, where $a \in A_0$. Hence if $\alpha \in G(n, A)$ with $\alpha^+(0) = I_n$ and $\alpha_s \in S(n, A_s)$ then $\alpha^+_s(b) \in S(n, A)$ (after identifying $\alpha^+_s(b)$ with its pullback by using Lemma 3.5), where $s \in A_0$ is a non-zero divisor and $b = s^l$ with $l \gg 0$, for $n \geq 1$ in linear case and $n \geq 2$ in symplectic case.

Proof. Since $\det : A \to A^*$ (units of $A$) $\subset A_0$, and $A_i, A_j \subset A_{i+j}$, the non-zero component of $\alpha$ doesn’t contribute for the value of the determinant and hence

$$\det \alpha = \det \alpha^+(0).$$

Therefore, if $\beta = \alpha^+(a)$, then

$$\det \beta = \det \beta^+(0) = \det (\alpha^+(a))^+(0) = \det \alpha^+(a \cdot 0) = \det \alpha^+(0) = 1.$$

Hence $\alpha^+(a) \in S(n, A)$.

Since $\alpha_s \in S(n, A_s)$ with $\alpha^+(0) = I_n$, the diagonal entries are of the form $1 + g_{ii}$, and off diagonal entries are $g_{ij}$, where $g_{ij} \in (A_+)_s$ for all $i, j$. We choose $b \in (s)$ such that $b = s^l$ with $l \gg 0$, so that $b$ can dilute the denominator of each entries. Then $\alpha^+_s(b) \in S(n, A)$. \hfill \Box

The next lemma gives some structural information about commutators.

Lemma 4.4. Let $\alpha, \beta \in S(n, A)$ and $A_0$ be a commutative semilocal ring. Then the commutator subgroup

$$[\alpha, \beta] = [\alpha \alpha_0^{-1}, \beta \beta_0^{-1}] E(n, A).$$

Proof. Since $A_0$ is semilocal, we have $S(n, A_0) = E(n, A_0)$ by Lemma 4.1. Hence $\alpha^+(0), \beta^+(0) \in E(n, A_0)$. Let $a = \alpha \alpha^+(0)^{-1}$ and $b = \beta \beta^+(0)^{-1}$. Then

$$[\alpha, \beta] = [\alpha \alpha^+(0)^{-1} \alpha^+(0), \beta \beta^+(0)^{-1} \beta^+(0)]$$

$$= \alpha \alpha^+(0)b \beta^+(0) \alpha^+(0)^{-1} a^{-1} \beta^+(0)^{-1} b^{-1}$$

$$= (aba^{-1}b^{-1})(bab^{-1}\alpha^+(0)ba^{-1}b^{-1})(ba\beta^+(0)\alpha^+(0)^{-1}a^{-1}b^{-1})(b\beta^+(0)^{-1}b^{-1}).$$

Since $E(n, A)$ is a normal subgroup of $S(n, A)$, the elements $bab^{-1}\alpha^+(0)ba^{-1}b^{-1}$, $ba\beta^+(0)\alpha^+(0)^{-1}a^{-1}b^{-1}$ and $b\beta^+(0)^{-1}b^{-1}$ are in $E(n, A)$.

\hfill \Box

Corollary 4.5. Let $\alpha \in [S(n, A), S(n, A)]$ with $\alpha^+(0) = I_n$, and let $A_0$ be a semilocal commutative ring. Then using the normality property of the elementary (resp. elementary symplectic) subgroup $\alpha$ can be written as

$$\prod_{k=1}^t [\beta_k, \gamma_k] \epsilon,$$

for some $t \geq 1$, and $\beta_k, \gamma_k \in S(n, A)$, with $\beta_k^+(0) = \gamma_k^+(0) = I_n$, and $\epsilon \in E(n, A)$ with $\epsilon^+(0) = I_n$.  

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Proof. Since $\alpha \in [S(n, A), S(n, A)]$, $\alpha = \prod_{k=1}^{t} [a_k, b_k]$ for some $t \geq 1$. Using Lemma 4.4 we identify $\beta_k$ with $a_k a_k^+ (0)^{-1}$ and $\gamma_k$ with $b_k b_k^+ (0)^{-1}$. This gives

$$\alpha = \prod_{k=1}^{t} [\beta_k, \gamma_k] \epsilon.$$ 

Then it follows that $\epsilon^+ (0) = I_n$, as $\alpha^+ (0) = I_n$. \hfill $\square$

The next lemma gives a variant of “Dilation Lemma” (Lemma 4.3) for $S(n, A)$; the special linear (resp. symplectic) group.

**Lemma 4.6.** If $\alpha \in S(n, A_s)$ with $\alpha^+ (0) = I_n$, then

$$\alpha^+ (b + d) \alpha^+ (d)^{-1} \in S(n, A),$$

where $s$ is a non-zero divisor and $b, d \in A_0$ with $b = s^l$ for some $l \gg 0$.

**Proof.** Let $\alpha^+(X) \in G(n, A[X])$, then $\alpha^+_s (X) \in E(n, A_s [X])$. Let $\beta^+(X) = \alpha^+ (X + d) \alpha^+ (d)^{-1}$. Then $\beta^+_s (X) \in S(n, A_s [X])$. Hence by Lemma 4.3

$$\beta^+ (bX) \in S(n, A).$$

Putting $X = 1$, we get result. \hfill $\square$

The following lemma makes use of Lemma 4.3 to deduce “Dilation Lemma” for the commutator subgroup $[S(n, A), S(n, A)]$.

**Lemma 4.7.** Let $\alpha_s = \prod_{i=1}^{t} [\beta_{is}, \gamma_{is}] \epsilon_s$ for some $t \geq 1$, such that $\beta_{is}$ and $\gamma_{is} \in S(n, A_s)$ and $\epsilon_s \in E(n, A_s)$ with $\gamma_{is}^+ (0) = \beta_{is}^+ (0) = \epsilon_s^+ (0) = I_n$. Then (by identifying with its pullback by using Lemma 4.3)

$$\alpha_s^+ (b + d) \alpha_s^+ (d)^{-1} \in [S(n, A), S(n, A)],$$

where $b, d \in A_0$ with $b = s^l$ for some $l \gg 0$.

**Proof.** Without loss of generality (as we would conclude by the end of the proof), we can assume

$$\alpha_s = [\beta_s, \gamma_s] \epsilon_s = \beta_s \gamma_s \beta_s^{-1} \gamma_s^{-1} \epsilon_s.$$ 

Since $\beta^+_s (0) = I_n$, by Lemma 4.3 $\beta^+_s (b), \gamma^+_s (b) \in S(n, A)$ for some $b = s^l$ with $l \gg 0$. Also, $\epsilon^+_s (b) \in E(n, A)$ by (5) of Theorem 3.8. Hence

$$\alpha_s^+ (b + d) \alpha_s^+ (d)^{-1} = \beta_s^+ (b + d) \gamma_s^+ (b + d) \beta_s^+ (b + d)^{-1} \gamma_s^+ (b + d)^{-1}$$

$$\epsilon_s^+ (b + d) \epsilon_s^+ (d)^{-1} \gamma_s^+ (d) \beta_s^+ (d)^{-1} \gamma_s^+ (d)^{-1} = \beta_s^+ (b + d) \beta_s^+ (d)^{-1} \gamma_s^+ (d)^{-1}.$$ 

Since $E(n, A_s)$ and $S(n, A_s)$ are normal subgroups in $G(n, A_s)$, by rearranging (using $\epsilon_s^+ (b + d) \epsilon_s^+ (d)^{-1}$ as intermediary) we can consider $\gamma_s^+ (b + d) \gamma_s^+ (d)^{-1}$ and $\beta_s^+ (b + d) \beta_s^+ (d)^{-1}$ together. Now by using Lemma 4.6 we get $\gamma_s^+ (b + d) \gamma_s^+ (d)^{-1}$
and \(\beta^+_{s}(b + d)^{-1} \in S(n, A)\). Hence (by identifying with its pullback by using Lemma 3.5) it follows that

\[
\alpha^+_s(b + d)\alpha^+_s(d)^{-1} \in [S(n, A), S(n, A)].
\]

Now we deduce the graded version of the Local-Global Principle for the commutators subgroups.

\textbf{Theorem 4.8.} Let \(\alpha \in S(n, A)\) with \(\alpha^+(0) = I_n\). If

\[
\alpha_p \in [S(n, A_p), S(n, A_p)] \text{ for all } p \in \text{Spec}(A_0),
\]

then

\[
\alpha \in [S(n, A), S(n, A)].
\]

\textbf{Proof.} Since \(\alpha_p \in [S(n, A_p), S(n, A_p)] \subset [S(n, A_m), S(n, A_m)]\), we have for a maximal ideal \(m \supseteq p, s \in A_0 - m\) such that

\[
\alpha_s \in [S(n, A_s), S(n, A_s)],
\]

hence \(\alpha_s\) can be decomposed as

\[
\alpha_s = \prod_{i=1}^{t} [\beta_{s_i}, \gamma_{s_i}] \epsilon_{s_i}
\]

by Corollary 4.5, where \(\beta_{s_i}, \gamma_{s_i} \in S(n, A_s)\) and \(\epsilon_{s} \in E(n, A_s)\). Now using the “Dilation Lemma” (Lemma 4.3 for \(S(n, A_s)\) and theorem 3.8 (4) for \(E(n, A_s)\)) on each of these elements, we have for \(b = s^i, i \geq 0\)

\[
\alpha^+_s(b) = \prod_{i=1}^{t} [\beta^+_{s_i}(b), \gamma^+_{s_i}(b)] \epsilon^+_{s_i}(b) \in [S(n, A), S(n, A)]E(n, A).
\]

Since \(E(n, A_s) \subseteq [S(n, A_s), S(n, A_s)]\), we have \(\alpha_s \in [S(n, A_s), S(n, A_s)]\). Let \(s_1, \ldots, s_r \in A_0\) non-zero divisors such that \(s_i \in A_0 - m\) and \(\langle s_1, \ldots, s_r \rangle = A_0\). Then it follows that \(b_1 + \cdots + b_r = 1\) for suitable (as before) \(b_i \in (s_i)\); and \(i = 1, \ldots, r\). Now

\[
\alpha_{s_1s_2 \cdots s_r} = \alpha^+_{s_1s_2 \cdots s_r}(1) = \alpha^+_{s_1s_2 \cdots s_r}(b_1 + b_2 + \cdots + b_r)
\]

\[
= ((\alpha_{s_1})_{s_2s_3 \cdots}^+(b_1 + \cdots + b_r)((\alpha_{s_1})_{s_2s_3s_4 \cdots}^+) \cdots \times (\alpha_{s_{r-1}})_{s_{r-2}s_r}^+(b_{r-1} + \cdots + b_r)^{-1}
\]

Using Lemma 3.5 the product is well defined (see Remark 3.9) and using Lemma 4.7 we conclude that \(\alpha \in [S(n, A), S(n, A)]\). \[\square\]
5. Auxiliary Result for Transvection subgroup

Let $R$ be a commutative ring with 1. We recall that a finitely generated projective $R$-module $Q$ has a unimodular element if there exists $q \in Q$ such that $qR \simeq R$ and $Q \simeq qR \oplus P$ for some projective $R$-module $P$.

In this section, we shall consider three types of (finitely generated) classical modules; viz. projective, symplectic and orthogonal modules over graded rings. Let $A = \bigoplus_{i=0}^{\infty} A_i$ be a commutative graded Noetherian ring with identity. For definitions and related facts we refer [5]; §1, §2. In that paper, the first author with A. Bak and R.A. Rao has established an analogous Local-Global Principle for the elementary transvection subgroup of the automorphism group of projective, symplectic and orthogonal modules of global rank at least 1 and local rank at least 3. In this article we deduce an analogous statement for the above classical groups over graded rings. We shall assume for every maximal ideal $m$ of $A_0$, the symplectic and orthogonal module $Q_m \simeq (A_0^{2m})_m$ with the standard form (for suitable integer $m$).

Remark 5.1. By definition the global rank or simply rank of a finitely generated projective $R$-module (resp. symplectic or orthogonal $R$-module) is the largest integer $k$ such that $\bigoplus^k R$ (resp. $\perp^k \mathbb{H}(R)$) is a direct summand (resp. orthogonal summand) of the module. Here $\mathbb{H}(R)$ denotes the hyperbolic plane.

Let $Q$ denote a projective, symplectic or orthogonal $A_0$-module of global rank $\geq 1$, and total (or local) rank $r+1$ in the linear case and $2r+2$ otherwise and let $Q_1 = Q \otimes A_0 A$.

We use the following notations to deal with the above three classical modules uniformly.

(1) $G(Q_1) :=$ the full automorphism group of $Q_1$.
(2) $T(Q_1) :=$ the subgroup generated by transvections of $Q_1$.
(3) $ET(Q_1) :=$ the subgroup generated by elementary-transvections of $Q_1$.
(4) $\tilde{E}(r, A) := E(r + 1, A)$ for linear case and $E(2r + 2, A)$ otherwise.

In [5], the first author with Ravi A. Rao and A. Bak established the “Dilation Lemma” and the “Local-Global Principle” for the transvection subgroups of the automorphism group over polynomial rings. In this section, we generalize their results over graded rings. For the statements over polynomial rings we request the reader to look at Proposition 3.1 and Theorem 3.6 in [5].

Following is the graded version of the “Dilation Lemma”.

Proposition 5.2 (Dilation Lemma). Let $Q$ be a projective $A_0$-module, and $Q_1 = Q \otimes A_0 A$. Let $s$ be a non-nilpotent element in $A_0$. Let $\alpha \in G(Q_1)$ with
\[ \alpha^+(0) = I_n. \] Suppose
\[
\alpha_s \in \begin{cases} 
E(r + 1, (A_0)_s) & \text{in the linear case,} \\
E(2r + 2, (A_0)_s) & \text{otherwise.}
\end{cases}
\]

Then there exists \( \tilde{\alpha} \in ET(Q_1) \) and \( l \gg 0 \) such that \( \tilde{\alpha} \) localizes at \( \alpha^+(b) \) for some \( b = (s^l) \) and \( \alpha^+(0) = I_n. \)

**Proof.** Let \( \alpha \in G(Q_1) \) and \( \alpha(X) := \alpha^+(X) \in G(Q_1[X]) \). Since \( \alpha_s \in \widehat{E}(r, A_s) \), \( \alpha_s(X) \in \widehat{E}(r, A_s[X]) \). Since \( \alpha(0) := \alpha^+(0) = I_n \), we can apply “Dilation Lemma” for the ring \( A[X] \) (cf. [3], Proposition 3.1) and hence there exists \( \tilde{\alpha}(X) \in ET(Q_1[X]) \) such that \( \tilde{\alpha}(X) = \alpha_s^+(bX) \) for some \( b = (s^l) \), \( l \gg 0 \). Substituting \( X = 1 \), we get \( \tilde{\alpha} \in ET(Q_1) \) and \( \tilde{\alpha}_s := \tilde{\alpha}_s(1) = \alpha_s(b) = \alpha_s^+(b) \). This proves the proposition.

**Lemma 5.3.** If \( \alpha \in G(Q_1) \) with \( \alpha^+(0) = I_n \) such that \( \alpha_s \in E(n, A_s) \), then \( \alpha^+(b + d)\alpha^+(d)^{-1} \in ET(Q_1) \) for some \( b = (s^l) \) with \( l \gg 0 \), and \( d \in A_0 \).

**Proof.** Let \( \beta = \alpha_s^+(b + d)\alpha_s^+(d)^{-1} \). Since \( \alpha_s \in E(n, A_s) \) implies \( \alpha^+(a) \in E(n, A_s) \) for \( a \in A_s \) and hence \( \beta \in E(n, A_s) \). Hence by Theorem 5.2, their exists a \( \tilde{\beta} \in ET(Q_1) \) such that \( \tilde{\beta}_s^+(b) = \beta \). Hence the lemma follows.

**Theorem 5.4** (Local-Global Principle). Let \( Q \) be a projective \( A_0 \)-module, and \( Q_1 = Q \otimes_{A_0} A \). Suppose \( \alpha \in G(Q_1) \) with \( \alpha^+(0) = I_n \). If
\[
\alpha_m \in \begin{cases} 
E(r + 1, (A_0)_m) & \text{in the linear case,} \\
E(2r + 2, (A_0)_m) & \text{otherwise.}
\end{cases}
\]

for all \( m \in \max(A_0) \), then \( \alpha \in ET(Q_1) \subseteq T(Q_1) \).

**Proof.** Let \( \alpha \in G(Q_1) \) with \( \alpha^+(0) = I_n \). Since \( \alpha_m \in \widehat{E}(r, A_m) \), Hence their exists a non-nilpotent \( s \in A_0 - m \), such that \( \alpha_s \in \widehat{E}(r, A_s) \). Hence by the above “Dilation Lemma” there exists \( \tilde{\alpha} \in ET(Q_1) \) such that \( \tilde{\alpha}_s = \alpha_s^+(b) \), for some \( b = s^l \) with \( l \gg 0 \). For each maximal ideal \( m_i \) we can find a suitable \( b_i \). Since \( A \) is Noetherian, it follows that \( b_1 + \cdots + b_r = 1 \) for some positive integer \( r \). Now we observe that \( \alpha^+(b_1 + \cdots + b_r)\alpha^+(b_{i+1} + \cdots + b_r)^{-1} \in ET(Q_1) \) and hence calculating in the similar manner as we did it in Remark 3.9, we get
\[
\alpha = \alpha^+(1) = \alpha^+(b_1 + \cdots + b_r)\alpha^+(b_2 + \cdots + b_r)^{-1} \cdots \alpha^+(b_i + \cdots + b_r)
\]
\[
\alpha^+(b_{i+1} + \cdots + b_r)^{-1} \cdots \alpha^+(0)^{-1} \in ET(Q_1) \subseteq T(Q_1);
\]

as desired.

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