SUPERSYMMETRIC $SO(N)$ FROM A PLANCK-SCALE STATISTICAL PICTURE

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Several refinements are made in a theory which starts with a Planck-scale statistical picture and ends with supersymmetry and a coupling of fundamental fermions and bosons to $SO(N)$ gauge fields. In particular, more satisfactory treatments are given for (1) the transformation from the initial Euclidean form of the path integral for fermionic fields to the usual Lorentzian form, (2) the corresponding transformation for bosonic fields (which is much less straightforward), (3) the transformation from an initial primitive supersymmetry to the final standard form (containing, e.g., scalar sfermions and their auxiliary fields), (4) the initial statistical picture, and (5) the transformation to an action which is invariant under general coordinate transformations.

Keywords: supersymmetry, $SO(N)$ gauge theory

1. Introduction

This paper contains several refinements of ideas proposed earlier, in the context of a theory which starts with a statistical picture at the Planck scale and ultimately results in a supersymmetric $SO(N)$ gauge theory. The present treatment supersedes previous versions.

2. Transformation to Lorentzian path integral: fermions

We begin with the following low-energy action for the (initially massless) fundamental fermions and bosons, which follows from essentially the same arguments as in Refs. 1 and 2, within a Euclidean picture (as in Eq. (6.17) of Ref. 1) but with all the components of the vierbein real (as in Eq. (3.45) of Ref. 2):

\begin{align}
S &= S_f + S_b \\
S_f &= \int d^4x \, \psi_f^\dagger (x) \, e^\mu_{\alpha} i \sigma^\alpha D_\mu \psi_f (x) \\
S_b &= \int d^4x \, \psi_b^\dagger (x) \, e^\mu_{\alpha} i \sigma^\alpha D_\mu \psi_b (x) \\
D_\mu &= \partial_\mu - i A^\mu_{\mu} t_i
\end{align}
in an obvious notation (which is defined in Refs. 1 and 2). The transformation of $S_b$ to the standard form for scalar bosons will be treated in the next section, and here we consider $S_f$ only.

A key point is that the low-energy operator $e^\mu_i \sigma^\mu D_\mu$ in $S_f$ is automatically in the correct Lorentzian form, even though the initial path integral is in Euclidean form. It is this fact which permits the following transformation to a Lorentzian path integral: Within the present theory, neither the fields nor the operators (nor the meaning of the time coordinate) need to be modified in performing this transformation.

In a locally inertial coordinate system, the Hermitian operator within $S_f$ can be diagonalized to give

$$S_f = \int d^4x \psi^\dagger_f(x) i\sigma^\mu D_\mu \psi_f(x)$$

$$= \sum_s \bar{\psi}^*_f (s) a(s) \bar{\psi}^*_f (s)$$

where

$$\psi_f(x) = \sum_s U(x,s) \bar{\psi}^*_f (s) \quad , \quad \bar{\psi}^*_f (s) = \int d^4x U^\dagger (s,x) \psi_f(x)$$

with

$$i\sigma^\mu D_\mu U(x,s) = a(s) U(x,s)$$

$$\int d^4x U^\dagger (s,x) U(x,s') = \delta_{ss'} \quad , \quad \sum_s U(x,s) U^\dagger (s,x') = \delta(x-x')$$

so that

$$\int d^4x U^\dagger (s,x) i\sigma^\mu D_\mu U(x,s') = a(s) \delta_{ss'} .$$

$U(x,s)$ is a multicomponent eigenfunction (which could also be written $U_s(x)$ or $\langle s | x \rangle$, with $U^\dagger (s,x)$ written as $U^\dagger_s (x)$ or $\langle x | s \rangle$). Alternatively, $U(x,s)$ is a unitary matrix which transforms $\bar{\psi}^*_f (s)$ into $\psi_f(x)$. There is an implicit inner product in

$$U^\dagger (s,x) \psi_f(x) = \sum_r U^\dagger_r (s,x) \psi_{f,r}(x)$$

$$U^\dagger (s,x) U_r (x,s) = \sum_{r,a} U^\dagger_{r,a} (s,x) U_{r,a} (x,s)$$

with the $2N$ components of $\psi_f(x)$ labeled by $r = 1, ..., N$ (spanning all components of all gauge representations) and $a = 1, 2$ (labeling the components of Weyl spinors), and with $s$ and $x, r$ each formally regarded as having $2N$ values.

Evaluation of the Euclidean path integral (a Gaussian integral with Grassmann
variables) is then trivial for fermions: as usual,

\[ Z_f = \int \mathcal{D} \psi_f^\dagger (x) \mathcal{D} \psi_f (x) \ e^{-S_f} \]

\[ = \prod_{x,ra} \int d\psi_{f,ra}^* (x) \int d\psi_{f,ra} (x) \ e^{-S_f} \]

\[ = \prod_s z_f (s) \]

(13)

(14)

(15)

with

\[ z_f (s) = \int d\tilde{\psi}_f^* (s) \int d\tilde{\psi}_f (s) \ e^{-\tilde{\psi}_f^* (s) a(s) \tilde{\psi}_f (s)} \]

\[ = a (s) \]

(16)

(17)

since the Jacobian \( J \) of the transformation in the path integral is unity:

\[ d\psi_f (x) = \sum_s U (x, s) d\tilde{\psi}_f (s) \quad , \quad d\psi_f^\dagger (x) = \sum_s d\tilde{\psi}_f^* (s) U^\dagger (s, x) \]

(18)

which gives

\[ J = \det (U) \det (U^\dagger) = \det (UU^\dagger) = 1. \]

(19)

Now let

\[ \tilde{Z}_f = \int \mathcal{D} \tilde{\psi}_f^* (s) \mathcal{D} \tilde{\psi}_f (s) \ e^{iS_f} \]

(20)

with the notation in this context now meaning that

\[ \tilde{Z}_f = \prod_s \tilde{z}_f (s) \]

(21)

where

\[ \tilde{z}_f (s) = i \int d\tilde{\psi}_f^* (s) \int d\tilde{\psi}_f (s) \ e^{i\tilde{\psi}_f^* (s) a(s) \tilde{\psi}_f (s)} \]

\[ = a (s) \]

(22)

(23)

so that

\[ Z_f = \tilde{Z}_f. \]

(24)

This is the path integral for an arbitrary time interval (with the fields, operator, and meaning of time left unchanged), so the Lorentzian path integral \( \tilde{Z}_f \) will give the same results as the Euclidean path integral \( Z_f \) for any physical process. The same is true of more general path integrals derived from more general operators, as long as they can be put into Gaussian form.

When the inverse transformation from \( \tilde{\psi}_f \) to \( \psi_f \) is performed, we obtain

\[ Z_f = \int \mathcal{D} \psi_f^\dagger (x) \mathcal{D} \psi_f (x) \ e^{iS_f} \]

(25)

with \( S_f \) having its form \[ 5 \] in the coordinate representation.
3. Transformation to Lorentzian path integral: bosons

For bosons we can again perform the transformation (17) to obtain

\[ S_b = \sum_s \bar{\psi}_b^s (s) a(s) \psi_b^s (s) \]  \hspace{1cm} (26)

The formal expression for the Euclidean path integral is

\[ Z_b = \int \mathcal{D} \psi_b^0 (x) \mathcal{D} \bar{\psi}_b (x) e^{-S_b} \]
\[ = \prod_s \int_{-\infty}^\infty d(\text{Re} \psi_{b,s} (x)) \int_{-\infty}^\infty d(\text{Im} \psi_{b,s} (x)) e^{-S_b} \]
\[ = \prod_s z_b (s) \] \hspace{1cm} (27)

with

\[ z_b (s) = \int_{-\infty}^\infty d(\text{Re} \bar{\psi}_b (s)) \int_{-\infty}^\infty d(\text{Im} \bar{\psi}_b (s)) e^{-S_b}. \] \hspace{1cm} (28)

We will now show that this action can be put into a form which corresponds to scalar bosonic fields plus their auxiliary fields. First, if the gauge potentials \( A_\mu^i \) were zero, we would have

\[ i\sigma^\mu \partial_\mu U_0 (x, s) = a_0 (s) U_0 (x, s). \] \hspace{1cm} (29)

Then

\[ U_0 (x, s) = \mathcal{V}^{-1/2} u (s) e^{ip_{\nu} x} , \quad p_\nu \cdot x = \eta_{\mu\nu} p_\nu x^\nu , \quad \eta_{\mu\nu} = \text{diag} \, (-1, 1, 1, 1) \] \hspace{1cm} (30)

(with \( \mathcal{V} \) a four-dimensional normalization volume) gives

\[ -\eta_{\mu\nu} \sigma^\mu p_\nu U_0 (x, s) = a_0 (s) U_0 (x, s) \] \hspace{1cm} (31)

where \( \sigma^\mu \) implicitly multiplies the identity matrix for the multicomponent function \( U_0 (x, s) \). A given 2-component spinor \( u_r (s) \) has two eigenstates of \( p_s^k \sigma^k \):

\[ p_s^k \sigma^k u_r^+ (s) = |\vec{p}_s| \sigma^k u_r^+ (s) , \quad p_s^k \sigma^k u_r^- (s) = -|\vec{p}_s| u_r^- (s) \] \hspace{1cm} (32)

where \( \vec{p}_s \) is the 3-momentum and \( |\vec{p}_s| = (p_s^k p_s^k)^{1/2} \). The multicomponent eigenstates of \( i\sigma^\mu \partial_\mu \) and their eigenvalues \( a_0 (s) = p_s^0 \pm |\vec{p}_s| \) thus come in pairs, corresponding to opposite helicities.

For nonzero \( A_\mu^i \), the eigenvalues \( a (s) \) will also come in pairs, with one growing out of \( a_0 (s) \) and the other out of its partner \( a_0 (s') \) as the \( A_\mu^i \) are turned on. To see this, first write (25) as

\[ (i\partial_0 + A_0^i t_i) U (x, s) + \sigma^k (i\partial_k + A_k^i t_i) U (x, s) = a (s) U (x, s) \] \hspace{1cm} (33)

or

\[ (i\partial_0 + A_0^i t_i)_{rr'} U_{rr'} (x, s) + P_{rr'} U_{rr'} (x, s) - a (s) \delta_{rr'} U_{rr'} (x, s) = 0 \] \hspace{1cm} (34)

\[ P_{rr'} \equiv \sigma^k (i\partial_k + A_k^i t_i)_{rr'} \] \hspace{1cm} (35)
with the usual implied summations over repeated indices. At fixed \( r,r' \) (and \( x,s \)), apply a unitary matrix \( u \) which will diagonalize the \( 2 \times 2 \) matrix \( P_{r,r'} \), bringing it into the form \( p_{r,r'} \sigma^3 + \overline{p}_{r,r'} \sigma^0 \), where \( p_{r,r'} \) and \( \overline{p}_{r,r'} \) are 1-component operators, while at the same time rotating the 2-component spinor \( U_{r'} \):

\[
u P_{r,r'} u^\dagger = P_{r,r'}' = p_{r,r'} \sigma^3 + \overline{p}_{r,r'} \sigma^0, \quad U_{r'}' = u U_{r'} , \quad u u^\dagger = 1 \quad (38)\]

\[
\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (39)\]

But \( P_{r,r'} \) is traceless, and the trace is invariant under a unitary transformation, so \( \overline{p}_{r,r'} = 0 \). Then the second term in \( (36) \) becomes \( u^\dagger p_{r,r'} \sigma^3 U_{r'}' (x,s) \). The two independent choices

\[
U_{r'}' (x,s) \propto \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \sigma^3 U_{r'}' (x,s) = U_{r'}' (x,s) \quad (40)\]

\[
U_{r'}' (x,s) \propto \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \sigma^3 U_{r'}' (x,s) = -U_{r'}' (x,s) \quad (41)\]

give \( \pm u^\dagger p_{r,r'} U_{r'}' (x,s) \). Now use \( u^\dagger U_{r'}' = U_{r'} \) to obtain for \( \sigma^3 \) reduces to two \( N \times N \) eigenvalue equations with solutions

\[
(i \partial_0 + A_{r}^0 t_0) U (x,s) + \sigma^k (i \partial_k + A_{r}^k t_k) U (x,s) = a (s) U (x,s) \quad (42)\]

\[
a (s) = a_1 (s) + a_2 (s) \quad (43)\]

\[
(i \partial_0 + A_{r}^0 t_0) U (x,s') + \sigma^k (i \partial_k + A_{r}^k t_k) U (x,s') = a (s') U (x,s') \quad (44)\]

\[
a (s') = a_1 (s) + a_2 (s) \quad (45)\]

where these equations define \( a_1 (s) \) and \( a_2 (s) \). Notice that letting \( \sigma^k \to -\sigma^k \) in \( (33) \) reverses the signs in \( (42) \), and changes the eigenvalue of \( U (x,s) \) to \( a (s') = a_1 (s) - a_2 (s) \).

The action for a single eigenvalue \( a (s) \) and its partner \( a (s') \) is

\[
\bar{s}_b (s) = \bar{\psi}_b (s) a (s) \bar{\psi}_b (s) + \bar{\psi}_b (s') a (s') \bar{\psi}_b (s') \quad (46)\]

\[
= \bar{\psi}_b (s) (a_1 (s) + a_2 (s)) \bar{\psi}_b (s) + \bar{\psi}_b (s') (a_1 (s) - a_2 (s)) \bar{\psi}_b (s') \quad (47)\]

There are 4 cases: For \( a_1 (s) > 0 \) and \( a_2 (s) > 0 \), let

\[
\bar{\psi}_b (s') = a (s) \bar{\psi}_b (s) = (a_1 (s) + a_2 (s))^{1/2} \bar{\psi}_b (s) \quad (48)\]

\[
\bar{\psi}_b (s) = a (s)^{-1/2} \bar{F}_b (s) = (a_1 (s) + a_2 (s))^{-1/2} \bar{F}_b (s) \quad (49)\]

and for \( a_1 (s) > 0 \) and \( a_2 (s) < 0 \)

\[
\bar{\psi}_b (s') = a (s') \bar{\psi}_b (s) = (a_1 (s) - a_2 (s))^{1/2} \bar{\psi}_b (s) \quad (50)\]

\[
\bar{\psi}_b (s) = a (s')^{-1/2} \bar{F}_b (s) = (a_1 (s) - a_2 (s))^{-1/2} \bar{F}_b (s) \quad (51)\]
so that for both of these first two cases
\[ \tilde{s}_b(s) = \tilde{\phi}_b^*(s) \tilde{\alpha}(s) \tilde{\phi}_b(s) + \tilde{F}_b^*(s) \tilde{F}_b(s), \quad a_1(s) > 0 \] (53)
where
\[ \tilde{\alpha}(s) = a(s) a(s') = a_1(s)^2 - a_2(s)^2. \] (54)

For \( a_1(s) < 0 \) and \( a_2(s) < 0 \), let
\[ \tilde{\psi}_b(s') = (-a(s))^{1/2} \tilde{\phi}_b(s) = (-a_1(s) - a_2(s))^{1/2} \tilde{\phi}_b(s) \] (55)
\[ \tilde{\psi}_b(s) = (-a(s))^{-1/2} \tilde{F}_b(s) = (-a_1(s) - a_2(s))^{-1/2} \tilde{F}_b(s) \] (56)
and for \( a_1(s) < 0 \) and \( a_2(s) > 0 \)
\[ \tilde{\psi}_b(s) = (-a(s'))^{1/2} \tilde{\phi}_b(s) = (-a_1(s) + a_2(s))^{1/2} \tilde{\phi}_b(s) \] (57)
\[ \tilde{\psi}_b(s') = (-a(s'))^{-1/2} \tilde{F}_b(s) = (-a_1(s) + a_2(s))^{-1/2} \tilde{F}_b(s) \] (58)
so for each of these last two cases
\[ \tilde{s}_b(s) = -\left[ \tilde{\phi}_b^*(s) \tilde{\alpha}(s) \tilde{\phi}_b(s) + \tilde{F}_b^*(s) \tilde{F}_b(s) \right], \quad a_1(s) < 0. \] (59)

Then we have
\[ S_b = \sum_{s} \tilde{s}_b(s) \] (60)
\[ = \sum_{a_1(s) > 0} \left[ \tilde{\phi}_b^*(s) \tilde{\alpha}(s) \tilde{\phi}_b(s) + \tilde{F}_b^*(s) \tilde{F}_b(s) \right] \]
\[ - \sum_{a_1(s) < 0} \left[ \tilde{\phi}_b^*(s) \tilde{\alpha}(s) \tilde{\phi}_b(s) + \tilde{F}_b^*(s) \tilde{F}_b(s) \right] \]
where a prime on a summation or product over \( s \) means that only one member of an \( s, s' \) pair (as defined in (43)-(46)) is included, so that there are only \( N \) terms rather than \( 2N \).

All of the transformations above from \( \tilde{\psi}_b \) to \( \tilde{\phi}_b \) and \( \tilde{F}_b \) have the form
\[ \tilde{\psi}_b(s_1) = A(s)^{1/2} \tilde{\phi}_b(s), \quad \tilde{\psi}_b(s_2) = A(s)^{-1/2} \tilde{F}_b(s) \] (61)
so that
\[ d\tilde{\psi}_b(s_1) = A(s)^{1/2} d\tilde{\phi}_b(s), \quad d\tilde{\psi}_b(s_2) = A(s)^{-1/2} d\tilde{F}_b(s) \] (62)
and the Jacobian is
\[ J' = \prod_s A(s)^{1/2} A(s)^{-1/2} = 1. \] (63)

These transformations lead to the formal result
\[ Z_b = \prod_{a_1(s) > 0} f_b(s) \cdot \prod_{a_1(s) < 0} f_b(s) \] (64)
\[ z_b (s) = \int_{-\infty}^{\infty} d(\text{Re} \, \phi_b (s)) \int_{-\infty}^{\infty} d(\text{Im} \, \phi_b (s)) \int_{-\infty}^{\infty} d(\text{Re} \, \bar{F}_b (s)) \int_{-\infty}^{\infty} d(\text{Im} \, \bar{F}_b (s)) \times e^{-\bar{\alpha}(s) \left[ (\text{Re} \, \phi_b (s))^2 + (\text{Im} \, \phi_b (s))^2 \right]} e^{-\left[ (\text{Re} \, \bar{F}_b (s))^2 + (\text{Im} \, \bar{F}_b (s))^2 \right]} , \quad a_1 (s) > 0 \] (65)

\[ z_b (s) = \int_{-\infty}^{\infty} d(\text{Re} \, \phi_b (s)) \int_{-\infty}^{\infty} d(\text{Im} \, \phi_b (s)) \int_{-\infty}^{\infty} d(\text{Re} \, \bar{F}_b (s)) \int_{-\infty}^{\infty} d(\text{Im} \, \bar{F}_b (s)) \times e^{\bar{\alpha}(s) \left[ (\text{Re} \, \phi_b (s))^2 + (\text{Im} \, \phi_b (s))^2 \right]} e^{\left[ (\text{Re} \, \bar{F}_b (s))^2 + (\text{Im} \, \bar{F}_b (s))^2 \right]} , \quad a_1 (s) < 0 \] (66)

At this point we encounter a difficulty which is not present for fermions, since the integral over Grassmann variables is well-defined for both positive and negative \( a (s) \), whereas the corresponding integrals above, over ordinary commuting variables, are divergent for the states with either \( \bar{\alpha} (s) < 0 \) or \( a_1 (s) < 0 \). This divergence results from the approximate linearization that led to (3), and will ultimately be controlled by various nonlinear effects, beginning with a self-interaction term involving \( (\psi^\dagger_b (x) \psi_b (x))^2 \) which is present in the original theory, but also including gauge interactions and various other complications which certainly lie beyond the simple treatment given here. We will therefore omit these states, which have a different status and require special treatment, in the expansions of \( \phi_b (x) \) and \( F_b (x) \):

\[ \phi_b (x) = \sum_{s > 0} \bar{U} (x, s) \phi_b (s) , \quad \bar{\phi}_b (s) = \int d^4 x \bar{U}^* (s, x) \phi_b (x) \] (67)

\[ F_b (x) = \sum_{s > 0} \bar{U} (x, s) \bar{F}_b (s) , \quad \bar{F}_b (s) = \int d^4 x \bar{U}^* (s, x) F_b (x) \] (68)

where \( s > 0 \) means that \( a_1 (s) > 0 \) and \( \bar{\alpha} (s) > 0 \).

Here \( \bar{U} \) is an \( N \times N \) matrix (since there is no longer a spinor index \( a \), and the number of values of \( s \) has also been reduced by a factor of 2) which satisfies

\[ \eta^{\mu \nu} D_\mu D_\nu \bar{U} (x, s) = \left[ a_1 (s)^2 - a_2 (s)^2 \right] \bar{U} (x, s) = \bar{\alpha} (s) \bar{U} (x, s) \] (69)

\[ \int d^4 x \bar{U}^* (s, x) \bar{U} (x, s') = \delta_{ss'} . \] (70)

I.e., we assume the existence of basis functions \( \bar{U} (x, s) \) which are the appropriate solutions of (69). In cases where an appropriate set of such basis functions does not exist, these bosons will exhibit further nonstandard behavior.

In the case of free fields (i.e. with \( A^I_\mu = 0 \)), we have

\[ \bar{U} (x, s) = \mathcal{V}^{-1/2} e^{ip_\mu x} \] (71)

\[ a_1 (s) = \omega \equiv p^0_s , \quad a_2 (s) = \pm |\vec{p}_s| , \quad |\vec{p}_s| \equiv (p^k_s p^k_s)^{1/2} \] (72)

\[ \bar{\alpha} (s) = (p^0_s \pm |\vec{p}_s|) (p^0_s + |\vec{p}_s|) = (p^0_s)^2 - |\vec{p}_s|^2 \] (73)

and

\[ \eta^{\mu \nu} D_\mu D_\nu \bar{U} (x, s) = \eta^{\mu \nu} \partial_\mu \partial_\nu \bar{U} (x, s) = \left[ (p^0_s)^2 - |\vec{p}_s|^2 \right] \bar{U} (x, s) . \] (74)
Also, \( s > 0 \) then means that \( \omega > 0 \) and
\[
\omega > |\vec{p}|.
\] (75)

With a return to the general case, (64) becomes
\[
Z_\beta = \prod_{s > 0} z_\beta(s)
\] (76)
where
\[
z_\beta(s) = \frac{\pi}{a(s)} \cdot \frac{\pi}{1} \quad \text{for } s > 0.
\] (77)

Now let
\[
\tilde{Z}_\beta = \int \mathcal{D}\bar{\phi}_\beta(s) \mathcal{D}\phi_\beta(s) \mathcal{D}\bar{F}_\beta(s) \mathcal{D}F_\beta(s) \, e^{iS_\beta}
\] (78)
\[
= \prod_{s > 0} \tilde{z}_\beta(s)
\] (79)
with
\[
\tilde{z}_\beta(s) \equiv -\int_{-\infty}^\infty d(\text{Re} \, \tilde{\phi}_\beta(s)) \int_{-\infty}^\infty d(\text{Im} \, \tilde{\phi}_\beta(s)) \int_{-\infty}^\infty d(\text{Re} \, \bar{F}_\beta(s)) \int_{-\infty}^\infty d(\text{Im} \, F_\beta(s)) e^{i\tilde{a}(s)}[(\text{Re} \, \tilde{\phi}_\beta(s))^2 + (\text{Im} \, \tilde{\phi}_\beta(s))^2] e^{i[(\text{Re} \, \bar{F}_\beta(s))^2 + (\text{Im} \, F_\beta(s))^2]} (80)
\]
\[
= \frac{\pi}{a(s)} \cdot \frac{\pi}{1} \quad \text{for } s > 0
\] (81)
since \( \int_{-\infty}^\infty dx \int_{-\infty}^\infty dy \exp \left( i a (x^2 + y^2) \right) = i\pi/a \). (Nuances of Lorentzian path integrals are discussed in, e.g., Peskin and Schroeder.4)

We have then obtained \( Z_\beta = \tilde{Z}_\beta \), or after a transformation to the coordinate representation via (67) and (68),
\[
Z_\beta = \int \mathcal{D}\phi_\beta(x) \mathcal{D}F_\beta(x) \, e^{iS_\beta}
\] (82)
\[
S_\beta = \int d^4x \left[ \phi_\beta^\dagger(x) \eta^{\mu\nu} D_\mu D_\nu \phi_\beta(x) + F_\beta^\dagger(x) F_\beta(x) \right].
\] (83)

Again, this is the path integral for an arbitrary time interval, so the Lorentzian path integral \( \tilde{Z}_\beta \) will give the same results as the Euclidean path integral \( Z_\beta \) for any physical process, and the same is true for more general path integrals derived from more general operators.

Recall, however, that the states with \( s < 0 \) have been omitted from the expansion of \( \phi_\beta(x) \) and \( F_\beta(x) \), so these bosonic fields should exhibit nonstandard behavior, and this feature may provide the most testable new prediction of the present theory.
4. Supersymmetry

The total action for fermions and bosons is

\[
S = S_f + S_b
\]

\[
= \int d^4x \left[ \psi_f^\dagger(x) i \sigma^\mu D_\mu \psi_f(x) + \phi_b^\dagger(x) \eta^{\mu\nu} D_\mu D_\nu \phi_b(x) + F_b^\dagger(x) F_b(x) \right]
\]

which in a general coordinate system becomes

\[
S = \int d^4x e \left[ \psi_f^\dagger(x) i e_\alpha^\mu \sigma^\alpha \tilde{D}_\mu \psi(x) - g^{\mu\nu} \left( \tilde{D}_\mu \phi(x) \right)^\dagger \tilde{D}_\nu \phi(x) + F^\dagger(x) F(x) \right]
\]

where \( g_{\mu\nu} \) is the metric tensor, \( e = \det e_\mu^a = (- \det g_{\mu\nu})^{1/2} \), \( \tilde{D}_\mu = D_\mu + e^{-1/2} \partial_\mu e^{1/2} \),

\[
\psi(x) = e^{-1/2} \psi_f(x), \quad \phi(x) = e^{-1/2} \phi_b(x), \quad F(x) = e^{-1/2} F_b(x).
\]

We thus obtain the standard basic form for a supersymmetric action, where the fields \( \phi, F \), and \( \psi \) respectively consist of 1-component complex scalar bosonic fields, 1-component complex scalar auxiliary fields, and 2-component spin 1/2 fermionic fields \( \psi \). These fields span the various physical representations of the fundamental gauge group, which must be \( SO(N) \) (e.g., \( SO(10) \)) in the present theory. I.e., \( \psi \) includes all the Standard Model fermions and the Higgsinos, and \( \phi \) includes the sfermions and Higgses.

5. Higher-derivative terms in the initial bosonic action

It was mentioned below Eq. (3.21) in Ref. 2 that higher-derivative terms are required in the initial bosonic action in order for the action in the internal space to be finite. It is easy to revise the treatment in Ref. 3 between Eqs. (73) and (87) to obtain the lowest-order such term. First (for better-defined statistical counting) we choose the length scale \( a \) in external space to be the same as the original fundamental length scale \( a_0 \) and rewrite Eq. (73) of Ref. 3 as

\[
\mathcal{S} = S_0 + \sum_{\pi,k} a \left\langle \Delta \rho_k \right\rangle a_0^D - \sum_{\pi,k} b \left[ \left\langle \Delta \rho_k \right\rangle^2 + \left\langle (\delta \rho_k)^2 \right\rangle \right] \left( a_0^D \right)^2.
\]

We then retain the second-order term in \( \delta \rho_k \):

\[
\delta \rho_k = \frac{\partial \Delta \rho_k}{\partial \bar{x}^M} \delta x + \frac{1}{2} \frac{\partial^2 \Delta \rho_k}{\partial \bar{x}^M \partial \bar{x}^M} \left( \delta x \right)^2.
\]

With \( \partial \Delta \rho_k / \partial \bar{x}^M = \partial (\rho_k - \bar{\rho}) / \partial \bar{x}^M = \partial \rho_k / \partial \bar{x}^M \), it follows that

\[
\left\langle (\delta \rho_k)^2 \right\rangle = \sum_M \left[ \left( \frac{\partial \rho_k}{\partial \bar{x}^M} \right)^2 \left( a_0^D \right)^2 + \left( \frac{1}{2} \frac{\partial^2 \rho_k}{\partial \bar{x}^M \partial \bar{x}^M} \right)^2 \left( a_0^D \right)^4 \right]
\]

\[
= \sum_M \rho_k a_0^2 \left[ \left( \frac{\partial \phi_k}{\partial \bar{x}^M} \right)^2 + \frac{a_0^2}{16} \left( \frac{\partial^2 \phi_k}{\partial \bar{x}^M \partial \bar{x}^M} \right)^2 \right]
\]
with the higher-order term involving the first derivative neglected. In the continuum limit, \( \sum x a^D_0 \rightarrow \int_{a_0}^{\infty} d^D x \), this leads to

\[ \mathcal{S} = S_0' + \int_{a_0}^{\infty} d^D x \sum_k \left\{ \frac{\mu}{m} \phi_k^2 - \frac{1}{2m^2} \sum_M \left[ \left( \frac{\partial \phi_k}{\partial x^M} \right)^2 + \frac{a_0^2}{16} \left( \frac{\partial^2 \phi_k}{\partial (x^M)^2} \right)^2 \right] \right\} \]

with the lower limit \( a_0 \) automatically providing an ultimate ultraviolet cutoff. Eq. (87) of Ref. 3 is then replaced by

\[ \mathcal{S}_b = \int_{a_0}^{\infty} d^D x \left\{ \frac{1}{2m^2} \left[ \frac{\partial \Psi_b^\dagger}{\partial x^M} \frac{\partial \Psi_b}{\partial x^M} + \frac{a_0^2}{16} \frac{\partial^2 \Psi_b^\dagger}{\partial (x^M)^2} \frac{\partial^2 \Psi_b}{\partial (x^M)^2} \right] \right. \]

\[ -\mu \Psi_b^\dagger \Psi_b + i \tilde{V} \Psi_b^\dagger \Psi_b \}. \]

Ordinarily we can let \( a_0 \rightarrow 0 \), but both the nonzero lower limit and the higher-derivative term in the action can be relevant in the internal space, where the length scales can be comparable to \( a_0 \), which may itself be regarded as comparable to the Planck length. Finally, we emphasize that the randomly fluctuating imaginary potential \( i \tilde{V} \) is a separate postulate of the theory. As mentioned below, the present theory is based on both statistical counting and these stochastic fluctuations, as well as the specific symmetry-breaking or “geography” of our universe.

6. Gravity and cosmological constant

According to (86), the coupling of matter to gravity is very nearly the same as in standard general relativity. However, if \( S \) is written in terms of the original fields \( \psi_f \) and \( \phi_b \), there is no factor of \( e \). In other words, in the present theory the original action has the form

\[ S = \int d^4 x \mathcal{L} \]

whereas in standard physics it has the form

\[ S = \int d^4 x e \mathcal{L}. \]

For an \( \mathcal{L} \) corresponding to a fixed vacuum energy density, there is then no coupling to gravity in the present theory, and the usual cosmological constant vanishes. This point was already made in Ref. 1, where the “cosmological constant” was defined to be the usual contribution to the stress-energy tensor from a constant vacuum Lagrangian density \( \mathcal{L}_0 \), which results from the factor of \( e \). However, as was also pointed out in this 1996 paper, “There may be a much weaker term involving \( \delta \mathcal{L}_0/\delta g^{\mu\nu} \), but this appears to be consistent with observation.”

This much weaker term we now interpret to be a “diamagnetic response” of vacuum fields to changes in both the vierbein and gauge fields, which results from a shifting of the energies of the vacuum states when fields are applied, just as the energies of the electrons in a metal are shifted by the application of a magnetic
field. We postulate that this effect produces contributions to the action which are consistent with the general coordinate invariance and gauge symmetry of the present theory. The lowest-order such contributions are, of course, the Maxwell-Yang-Mills and Einstein-Hilbert actions, plus a relatively weak cosmological constant arising from this same mechanism:

\[ L_g = -\frac{1}{4} g_0^{-2} e F_{\mu
u}^i F_{\rho\sigma}^i g^{\mu\rho} g^{\nu\sigma}, \quad L_G = e \Lambda + \ell_P^2 e^{(4) R}. \]  

(96)

Here \( g_0 \) is the coupling constant for the fundamental gauge group (e.g. \( SO(10) \)), \( \Lambda \) is a constant, and \( \ell_P^2 = 16\pi G \). These terms are analogous to the usual contributions to the free energy from Landau diamagnetism in a metal.

The actions for gauginos and gravitinos are postulated to have a similar origin, as the vacuum responds to these fields. Particle masses and Yukawa couplings are postulated to arise from supersymmetry breaking and radiative corrections.

As pointed out in Ref. 1, the above gauge and gravitational curvatures require that the order parameter contain a superposition of configurations with topological defects (without which there could be no curvature). Here we do not attempt to discuss these defects in detail, but we now interpret them as 1-dimensional defect lines in 4-dimensional spacetime, analogous to vortex lines in a superfluid.

There is clearly a lot of work remaining to be done – including actual predictions for experiment – but the theory is relatively close to real-world physics, and the following arise as emergent properties from a fundamental statistical picture: Lorentz invariance, the general form of Standard-Model physics, an \( SO(N) \) fundamental gauge theory (with e.g. \( SO(10) \) permitting coupling constant unification and neutrino masses), supersymmetry, a gravitational metric with the form \((- , + , + , +)\), the correct coupling of matter fields to gravity, vanishing of the usual cosmological constant, and a mechanism for the origin of spacetime and fields.

The new predictions of the present theory appear to be subtle, but include Lorentz violation at very high energies and nonstandard behavior of scalar bosons.

7. Conclusion

For a theory to be viable, it must be mathematically consistent, its premises must lead to testable predictions, and these predictions must be consistent with experiment and observation. The theory presented here appears to satisfy these requirements, although it is still very far from complete.

Experiment should soon confront theory with more stringent constraints. For example, supersymmetry, fundamental scalar bosons, and \( SO(N) \) grand unification seem to be unavoidable consequences of the theory presented here, but there is as yet no direct evidence for any of these extensions of established physics.

The present theory starts with a picture which is far from that envisioned in more orthodox approaches: There are initially no laws, and instead all possibilities are realized with equal probability. The observed laws of Nature are emergent phenomena, which result from statistical counting and stochastic fluctuations, together
with the specific symmetry-breaking (or “geographical features”) of our universe.

It is reassuring that such an unconventional picture ultimately leads back to both established physics and standard extensions like the three mentioned above. Perhaps this fact helps to demonstrate the robustness and naturalness of these extensions, and the importance of experimental searches for supersymmetric partners, dark matter, Higgs bosons, the various consequences of grand unification, and related phenomena in cosmology and astrophysics.

The present theory shares several central concepts with string theory – namely supersymmetry, higher dimensions, and topological defects – perhaps indicating that these elements may be inescapable in a truly fundamental theory.

8. Acknowledgements

I have benefitted greatly from many discussions with Seiichirou Yokoo and Zorawar Wadiasingh. In particular, Seiichirou Yokoo obtained a determinantal transformation for free fields which was a precursor of the explicit transformation of fields in Eqs. (49)-(52) and (55)-(58).

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