Charged Particle Dynamics in the Field of a Slowly Rotating Compact Star

Babur M. Mirza*
Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan. 45320
August 31, 2004

Abstract
We study the dynamics of a charged particle in the field of a slowly rotating compact star in the gravitoelectromagnetic approximation to the geodesic equation. The star is assumed to be surrounded by an ideal, highly conducting plasma (taken as a magnetohydrodynamic fluid) with a stationary, axially symmetric electromagnetic field. The general relativistic Maxwell equations are solved to obtain the effects of the background spacetime on the electromagnetic field in the linearized Kerr spacetime. The equations of motion are then set up and solved numerically to incorporate the gravitational as well as the electromagnetic effects. The analysis shows that in the slow rotation approximation the frame dragging effects on the electromagnetic field are absent. However the particle is directly effected by the rotating gravitational source such that close to the star the gravitational and electromagnetic field produce contrary effects on the particle trajectories.

1 Introduction
The study of accretion dynamics of charged particles in the vicinity of a compact gravitational source is of observational as well as theoretical interest; particularly for the information it reveals about the nature and influence of the background spacetime on the various physical processes occurring in the star’s vicinity. Whereas a compact star is formed largely of a degenerate plasma\(^1\), matter surrounding the star exists in a highly ionized, though a less dense, state\(^2\). In this region the charged particles are more free to move while not accreted to relativistic velocities, and the dynamics of these charged particles is governed not only by the gravitational but also by the electromagnetic field of the star. However the electromagnetic field is itself effected by the geometry of the background spacetime. Therefore it is of importance and of astrophysical

*E-mail: bmmirza2002@yahoo.com
relevance that the two-fold effects; namely, the coupling of the gravitational
and the electromagnetic field of the star, as well as the direct effects of the
background spacetime on the motion of a charged particle, must be addressed
together\textsuperscript{3,4}.

Here we present an investigation of these effects on the dynamics of an accreted charged particle lying in the star’s plasma atmosphere. To pose the problem quantitatively we consider a slowly rotating isolated compact star as the source of gravitational field which defines the geometry of the background spacetime. An axially symmetric, highly conducting plasma with a stationary, axially symmetric electromagnetic field is assumed to constitute the star’s atmosphere. We regard the electromagnetic field as perturbed by the background spacetime, however the back-reaction of the electromagnetic field on the spacetime curvature is neglected and the standard Kerr metric (in its linearized form) is used for the fixed background spacetime. For the plasma surrounding the star the magnetohydrodynamic (MHD) approximation is assumed, as such the plasma is considered as an ideal highly conducting MHD fluid\textsuperscript{5–7}. The formulation of the electromagnetic field equations is based on a generalized definition of the electromagnetic field tensor for an ideal MHD fluid in a curved spacetime\textsuperscript{8,9}. The direct effects of the spacetime dragging on the particle dynamics are discussed in the gravitoelectromagnetic (GEM) approximation to the geodesic equation\textsuperscript{10–12}. The GEM approximation, which assumes the charged particle motion to be slow, is generally valid in the case of stable compact stars (i.e. non-collapsing objects). Also for the case of a magnetized compact star with a dipole field the GEM approximation is applicable except at or very close to the poles where particle velocities can be relativistic due to intense magnetic field. Generally the accretion occurs well below the relativistic limit and so the GEM approximation can be used. Throughout we take the velocity four vector as defined by a comoving observer ZAMO (zero angular momentum observer). The ZAMO is a useful class of observers, relevant to astrophysics, circling the gravitational source with a given angular velocity at a fixed radial distance and polar angle\textsuperscript{13,14}.

The paper is organized as follows. In the first part of our analysis, comprising section II, we obtain exact solutions of the general relativistic Maxwell equations using the generalized definition of the electromagnetic field tensor for an ideal, highly conducting MHD fluid in linearized Kerr spacetime. The solution obtained shows that in the slow rotation approximation the effects of rotation on the electromagnetic field are absent. Then, in section III, assuming the star to be a sphere of homogeneous mass density, we study numerically the dynamics of the charged particle using the GEM approximation. In this approximation the direct effects of rotation (i.e. via frame-dragging) on particle trajectories are found to oppose that of the electromagnetic field, hence the particle motion is a result of balancing and counter-balancing of these effects. Lastly, in section IV, we give a discussion and a summary of the main conclusions of the paper.
2 The Electromagnetic Field

2.1 General Relativistic Maxwell Equations

The exterior spacetime to a slowly rotating compact gravitational source of mass $M$ is described by the linearized Kerr metric. In the Boyer-Lindquist coordinates the metric can be written as the general line element:

$$ds^2 = -e^{2\Phi(r)}dt^2 - 2\omega(r)r^2 \sin^2 \theta dt d\varphi + r^2 \sin^2 \theta d\varphi^2 + e^{-2\Phi(r)}dr^2 + r^2 d\theta^2,$$

where

$$e^{2\Phi(r)} = (1 - \frac{2M}{r}),$$

and

$$\omega(r) \equiv \frac{d\varphi}{dt} = -\frac{g_{r\varphi}}{g_{\varphi\varphi}},$$

is the angular velocity of a free falling frame brought into rotation by the frame dragging of the spacetime. Here and in what follows the Greek indices run as $t, r, \theta,$ and $\varphi$ respectively. Also throughout we assume the gravitational units in which $G = 1 = c$.

The general relativistic form of the Maxwell equations is

$$F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0,$$

$$\left( \sqrt{-g} F^{\alpha\beta} \right)_{,\beta} = 4\pi \sqrt{-g} J^\alpha,$$

where $g$ represents the determinant of the metric tensor $g_{\alpha\beta}$ given by the Einstein field equations. Here $F_{\alpha\beta}$ is the generalized electromagnetic field tensor for an ideal MHD fluid given by a unique tensorial expression:

$$F_{\alpha\beta} = u_\alpha E_\beta - u_\beta E_\alpha + \eta_{\alpha\beta\gamma\delta} u_\gamma B_\delta,$$

and $J^\alpha$ is current four vector. In general the current four vector is the sum of two terms corresponding to a convection and to a conduction current:

$$J^\alpha = \epsilon u^\alpha + \sigma u_\beta F^{\beta\alpha},$$

where $\epsilon$ is the proper charge density, $\sigma$ is the conductivity of the fluid, and $u^\alpha$ is the unit velocity 4-vector. The volume element 4-form $\eta_{\alpha\beta\gamma\delta}$ and its dual $\eta^{\alpha\beta\gamma\delta}$ are defined by

$$\eta_{\alpha\beta\gamma\delta} = \sqrt{-g} \epsilon_{\alpha\beta\gamma\delta}, \quad \eta^{\alpha\beta\gamma\delta} = -\frac{1}{\sqrt{-g}} \epsilon^{\alpha\beta\gamma\delta},$$

where $\epsilon_{\alpha\beta\gamma\delta}$ is the Levi-Civita symbol, which is $+1, -1,$ or $0$ for an even, odd, or non-permutation of $\alpha\beta\gamma\delta$ respectively. Also the four vectors $E_\alpha$ and $B_\alpha$, denote
the electric and magnetic field components in the four dimensional spacetime. For a ZAMO $u_r$ and $u_\theta$ vanish, and using $u^\alpha u_\alpha = -1$, the components of the four velocity vector are:

$$u^\alpha = e^{-\Phi(r)}(1, 0, 0, \omega(r)), \quad u_\alpha = e^{\Phi(r)}(-1, 0, 0, 0).$$

Here we employ the often made assumption of a perfectly conducting plasma surrounding the star (cf e.g. [5], [14] for a discussion of this assumption with respect to the generalized Ohm’s law). Since for plasma the condition of neutrality also holds thus $J^\alpha$ must vanish identically. The Maxwell equations (4) give:

$$(\sqrt{-g}u^r B^r)_{,r} + (\sqrt{-g}u^\theta B^\theta)_{,\theta} = 0,$$

$$(u_t E_r)_{,r} = 0,$$

$$(u_t E_\theta)_{,\theta} = 0,$$

$$(u_t E_\theta - \sqrt{-g} u^r B^r)_{,r} - (u_t E_r - \sqrt{-g} u^\theta B^\theta)_{,\theta} = 0,$$

whereas from expression (5) we obtain:

$$(\sqrt{-g}u^r E^r)_{,r} + (\sqrt{-g}u^\theta E^\theta)_{,\theta} = 0,$$

$$(u_t B_r)_{,r} = 0,$$

$$(u_t B_\theta)_{,\theta} = 0,$$

$$(u_t B_\theta - \sqrt{-g} u^r E^r)_{,r} - (u_t B_r - \sqrt{-g} u^\theta E^\theta)_{,\theta} = 0.$$

## 2.2 Solution to the Maxwell Equations

The electromagnetic field outside the plasma surrounding the compact star is now determined by equation (10) to (17). To solve this system of equations let us assume the following separation ansatz for the magnetic field components $B^r$ and $B^\theta$:

$$B^r(r, \theta) = R^r_B(r) \Theta^r_B(\theta), \quad B^\theta(r, \theta) = R^\theta_B(r) \Theta^\theta_B(\theta),$$

and similarly for the electric field components $E^r$ and $E^\theta$:

$$E^r(r, \theta) = R^r_E(r) \Theta^r_E(\theta), \quad E^\theta(r, \theta) = R^\theta_E(r) \Theta^\theta_E(\theta).$$

Substituting from equation (18) into equation (10) and simplifying we obtain

$$\frac{1}{\Theta^\theta_B} \frac{d \Theta^r_B}{d \theta} + \cot \theta = -\left(\frac{\Theta^r_B}{\Theta^\theta_B}\right) \frac{1}{r^2 u^r R^r_B} \frac{d (r^2 u^r R^r_B)}{dr}.$$ 

For separation we require that $\Theta^r_B = \Theta^\theta_B = \Theta_B$ and obtain
\[
\frac{1}{\Theta_B} \frac{d\Theta_B}{d\theta} + \cot \theta = k_1, \tag{21}
\]

and

\[
- \frac{1}{r^2 u^t R_B^\theta} \frac{d(r^2 u^t R_B^\theta)}{dr} = k_1, \tag{22}
\]

where \( k_1 \) is the separation constant. Solving for \( \Theta_B \) the equation (21) we obtain

\[ A_1 \exp k_1 \theta (\csc \theta). \]

Notice that because of the exponential function the field at coincident points \( \theta = \pi/2 \) and \( \pi/2 + 2\pi \) is not identical. We therefore require single valued solution \( k_1 = 0 \). The solutions to equations (21) and (22) are now given by:

\[
\Theta_B = \frac{A_1}{\sin \theta}, \tag{23}
\]

and

\[
R_B^r = \frac{A_2}{r^2 u^t}, \tag{24}
\]

where \( A_1 \) and \( A_2 \) are constants. Therefore we have

\[
B'(r, \theta) = \frac{A}{r^2 u^t \sin \theta}, \quad B^\theta(r, \theta) = R_B^\theta(r) \frac{A_1}{\sin \theta}, \tag{25}
\]

where \( A = A_1 A_2 \). And similarly from equation (13) and (18) we obtain

\[
\Theta_E = \frac{C_1}{\sin \theta}, \quad R_E^r = \frac{C_2}{r^2 u^t}, \tag{26}
\]

and hence

\[
E'(r, \theta) = \frac{C}{r^2 u^t \sin \theta}, \quad E^\theta(r, \theta) = R_E^\theta(r) \frac{C_1}{\sin \theta}, \tag{27}
\]

where \( C = C_1 C_2 \) and \( \Theta_E = \Theta_E^\theta = \Theta_E^\theta \).

Now to determine the functions \( R_E^r \), we substitute from (25) and (27) into equation (13). After some simplification we obtain in a separated form:

\[
\frac{d(u r^2 R_E^\theta)}{dr} = \frac{\sin \theta}{C_1} \left( A \frac{d\omega}{dr} - \frac{C \cot \theta}{r^2 \sin \theta} \right), \tag{28}
\]

As before we have

\[
\frac{d(u r^2 R_E^\theta)}{dr} = k_2, \tag{29}
\]

and

\[
A \frac{d\omega}{dr} \sin \theta - \frac{C}{r^2 \sin \theta} \cos \theta - k_2 C_1 = 0. \tag{30}
\]
where \( k_2 \) is a constant. Since (30) holds for every value of \( \theta \), on comparing coefficients of \( \sin \theta \), \( \cos \theta \), and unity we obtain \( C = 0 = k_2 \) and \( A = 0 \). Here are now two possibilities for non-trivial solutions depending on either \( A_1 \neq 0 \) or \( A_2 \neq 0 \) (i.e. \( A_1 = 0 \)).

For \( A_1 \neq 0 \) equation (29) gives

\[
R_E^\theta = \frac{A_3}{u_\phi r^2}. \tag{31}
\]

where \( A_3 \) is a constant. From equations (25), (27) and (17) we obtain in a similar manner

\[
R_B^\theta = \frac{C_3}{u_\phi r^2}. \tag{32}
\]

Hence

\[
B^\theta(r, \theta) = 0, \quad B^\phi(r, \theta) = \frac{A_1 A_3}{u_\phi r^2 \sin \theta}, \tag{33}
\]

and

\[
E^\theta(r, \theta) = 0, \quad E^\phi(r, \theta) = \frac{C_1 C_3}{u_\phi r^2 \sin \theta}. \tag{34}
\]

Also from equations (11), (12) and (15), (16) we obtain

\[
B^\varphi = \frac{A_4}{u_\phi r^2 \sin^2 \theta}, \tag{35}
\]

\[
E^\varphi = \frac{C_4}{u_\phi r^2 \sin^2 \theta}. \tag{36}
\]

For \( A_1 = 0 \) (and similarly for \( C_1 = 0 \)), there is no poloidal component to the magnetic (electric) field and the only non-vanishing components to the electromagnetic field are given by equations (35) and (36). Clearly in both cases the electromagnetic field is independent of \( \omega(r) \).

In Figures (1) and (2) we give plots for the magnetic (electric) field components \( B^\theta (E^\theta) \) and \( B^\varphi (E^\varphi) \) for \( A_1 A_3 = 1 \) (\( = C_1 C_3 \)) and \( A_4 = 1 \) (\( = C_4 \)).

3 Trajectory of a Charged Particle

According to the general theory of relativity the trajectory of a particle in the field of a massive object is determined by the geodesic equation:

\[
d^2 x^\alpha \over ds^2 + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{ds} \frac{dx^\gamma}{ds} = 0; \tag{37}
\]

where \( \Gamma^\alpha_{\beta\gamma} \) is the Christoffel symbol. To linearize the geodesic equation we assume that the metric tensor can be expressed as:

\[
g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \tag{38}
\]
where $\eta_{\alpha\beta} = \text{diag}(-1, -1, -1, 1)$ is the Minkowski metric tensor and $h_{\alpha\beta}$ is a small perturbation to the spacetime metric such that $h_{\alpha\beta} \ll 1$. We can take time $t$ for the affine parameter $s$. Further requiring $v = |d\mathbf{r}/dt| \ll 1$ for a ‘slow’ moving particle we obtain

$$\frac{d^2\mathbf{r}}{dt^2} = (\mathbf{G} + \mathbf{v} \times \mathbf{H})$$

(39)

where

$$\mathbf{G} = -\nabla \varphi, \quad \mathbf{H} = \nabla \times 4\mathbf{a}$$

(40)

and

$$\varphi = -\iiint \frac{\rho}{r} dV, \quad \mathbf{a} = \iiint \frac{\partial \mathbf{v}}{r} dV$$

(41)

$\rho$ being the mass density and $V$ is the volume of the gravitational source. This approximation, due to its formal analogy to the classical electromagnetic theory, is referred to as the gravitoelectromagnetic (GEM) approximation to the general theory of relativity (see reference [16] and [17] for different interpretations). Thus in expression (40) $\mathbf{G}$ is called the gravitoelectric (GE) force per unit mass whereas the term $m\mathbf{v} \times \mathbf{H}$ is called the gravitomagnetic (GM) force for a unit mass. In particular the GM force, being independent of the choice of a particular coordinate system used, is general relativistically significant.

In the slow rotation approximation the deformation to the star due to the effects of rotation are small compared to the radially attractive GE force; hence we take the star to be a slowly rotating sphere of homogeneous mass density. Then using GEM analogy, we obtain:

$$\mathbf{G} = -\frac{M}{r^2} \hat{r}$$

(42)

and

$$\mathbf{H} = -\frac{12}{5} \frac{MR^2}{r^5} \left( \Omega \frac{r}{r^3} - \frac{1}{3} \frac{\Omega}{r^2} \right)$$

(43)

where $M$ is the mass, $R$ is the radius and $\Omega$ is the angular velocity of the star. Thus in the field of a slowly rotating compact star, the net force acting on a particle of mass $m$ and charge $q$ is the sum of the gravitational and the electromagnetic forces:

$$m\frac{d^2\mathbf{r}}{dt^2} = m(\mathbf{G} + \mathbf{v} \times \mathbf{H}) + q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

(44)

To specify the physical situation we study the motion of the charged particle in a Cartesian coordinate system $(x, y, z)$ and take $\mathbf{H}$ to be along the positive $z$-axis. Furthermore, since now the effective component of the GM force lies in the $XY$-plane, we assume that the $XY$-plane is the equatorial plane of the star and the particle’s motion is confined to this plane only.
Then for the case of non-vanishing poloidal magnetic field we transform expressions (33)-(36) and (42)-(43) into the Cartesian coordinates, and obtain the equations of motion in the equatorial plane (i. e. for $\theta = \pi/2$):

\[
\frac{d^2x}{dt^2} = -\frac{M}{x^2+y^2} + \frac{yE_0}{\sqrt{(x^2+y^2)^2 - 2M(x^2+y^2)^{3/2}}} + \left(\frac{\mu}{(x^2+y^2)^{3/2}} - \frac{B_0}{\sqrt{x^2+y^2 - 2M\sqrt{x^2+y^2}}}\right)\frac{dy}{dt},
\]

(45)

\[
\frac{d^2y}{dt^2} = -\frac{M}{x^2+y^2} - \frac{yE_0}{\sqrt{(x^2+y^2)^2 - 2M(x^2+y^2)^{3/2}}} - \left(\frac{\mu}{(x^2+y^2)^{3/2}} - \frac{B_0}{\sqrt{x^2+y^2 - 2M\sqrt{x^2+y^2}}}\right)\frac{dx}{dt},
\]

(46)

where $B_0 = qA_1A_2/m$, $E_0 = qC_4/m$ and $\mu = 4MR^2\Omega/5$. Notice that here the case of vanishing poloidal magnetic field is obtained for $B_0 = 0$ whereas requiring $C_1 = 0$ does not alter the equations of motion (45) and (46). We numerically solve the equations of motion (45) and (46) for the initial conditions $x(0) = 10 = y(0)$ and $\frac{dx}{dt} |_{t=0} = 0 = \frac{dy}{dt} |_{t=0}$, and plot (in Figure (3) to (6)) the solutions for various values of the parameters $M$, $\mu$, $E_0$, and $B_0$.

4 Discussion and Conclusions

We have considered the dynamics of a charged particle in a highly, conducting plasma surrounding a slowly rotating compact gravitational source. The effects of background spacetime have been discussed, both directly on the particle’s motion and indirectly via an axially symmetric electromagnetic field around the star.

We found (equations (33) to (36)) that when slow rotation is assumed for the compact source the spacetime dragging does not modify or reduce the electromagnetic field of the star. Hence the field remains, in this approximation, the same as for the Schwarzschild spacetime. However the motion of the particle is directly influenced via gravitational field of the star.

Our numerical analysis of the effects of gravitational and electromagnetic field shows (Fig. (3) to (6)) that whereas the electromagnetic effects dominate the overall dynamics of the charged particle at sufficiently large distances, the gravitational field also becomes important closer to the surface of the star especially if the star is rotating fast enough. Firstly we observe that the effects of the electric field is to lift a charge particle from close to the surface of the star (Fig. (3)) and make them move along the magnetic field lines. Here are now two possibilities: one, if the magnitude of the magnetic field is larger than the magnitude of the electric field the particle following a helical trajectory falls into...
the star (case (a), (b), and (c) in Fig. (4)); two, if the magnetic field strength is weaker or is even comparable to the electric field strength, then the electrical effects become dominant and the particle escapes from falling into the star (case (d), and (e) in Fig.(4)).

On the other hand the effects of gravitational field can be regarded as opposing that of the electromagnetic field. In Figure (5) we notice that the GE force attracts the particle towards the star’s surface against the electric field. Furthermore it is clear from Figure (6) that the gyroradius of the helical trajectory of the particle is increased due to the GM force; hence the GM force weakens the effects of the magnetic field.

Summing up these observations we note that the accretion of a charged particle in vicinity of a compact star is mainly due to the GE and magnetic effects. These effects are especially dominant at a sufficiently large distances from the star. However close to the star the particle trajectory is also effected by the GM force as well as electric field which cause accerted charged particles to follow open lines of force. For a sufficiently fast rotating star and an electric field comparable to the magnetic field the above mechanism (GM + electric) may contribute to opposing the in-fall of the plasma surrounding the compact star.

Acknowledgments

Useful comments of Dr. H. Saleem and Dr. D. V. Ahluwalia are gratefully acknowledged.

References

[1] N. K. Glendenning, Compact Stars (Springer-Verlag, New York, 1997).

[2] W. K. Rose, Advanced Stellar Astrophysics (Cambridge University Press, New York, 1998).

[3] A. R. Prasana and S. Sengupta, Phys. Lett. A 193, 25 (1994).

[4] S. Sengupta, Int. J. Mod. Phys. D 6, 591 (1997).

[5] A. R. Choudhuri, The Physics of Fluids and Plasmas (Cambridge University Press, New York,1998).

[6] S. Nitta, M. Takahashi and A. Tomimatsu, Phys. Rev. D 44, 2295 (1991).

[7] K. Elsässer, Phys. Rev. D 62, 044007 (2000).

[8] A. Lichnerowicz, Relativistic Hydrodynamics and Magnetohydrodynamics (Benjamin Press, New York,1967).

[9] G. F. R. Ellis, in Cargese Lectures in Physics 6, ed. E. Schatzman (Gordon, Breach, 1973).

[10] M. L. Ruggiero and A. Tartaglia, Nuovo Cim. 117B, 743 (2002).
[11] W. Rindler, *Phys. Lett. A* **233**, 25 (1997).

[12] B. M. Mirza, *Inter. J. Mod. Phys. D* **13**, 327 (2004).

[13] B. Punsly, *ApJ* **498**, 640 (1998).

[14] L. Rezzolla, B. J. Ahmedov and J. C. Miller, *Mon. Not. R. Astr. Soc.* **322**, 723 (2001).

[15] W. Rindler, *Essential Relativity* (Springer-Verlag, New York, 1979).

[16] D. V. Ahluwalia, *Gen. Rel. Grav.* **29**, 1491 (1997).

[17] R. T. Jantzen and G. Mac Keiser, ed. *The Seventh Marcel Grossmann Meeting on General Relativity*, Part A, (World Scientific, New Jersey, 1996).

**Figure Captions**

1) Plot for the magnetic (electric) field component $B^\theta$ ($E^\theta$) as a function of the radial distance $r$ and polar angle $\theta$ for $A_1 A_3 = 1 (= C_1 C_3)$.

2) Plot for the magnetic (electric) field component $B^\varphi$ ($E^\varphi$) as a function of the radial distance $r$ and polar angle $\theta$ for $A_4 = 1 (= C_4)$.

3) Trajectory of the charged particle for electric force of varying strength with $E_0 = 0.1, 1, 5, 10, 20$, and $M = 1$, $\mu = 0.1$, $B_0 = 10$ in gravitational units.

4) Trajectory of the charged particle for magnetic force of varying strength with $B_0 = 0, 10, 30, 50, 70$, and $M = 1$, $\mu = 0.1$, $E_0 = 1$ in gravitational units.

5) Trajectory of the charged particle for GE force of varying strength with $M = 1, 2, 3, 5, 10$, and $\mu = 0.1$, $E_0 = 1$, $B_0 = 10$ in gravitational units.

6) Trajectory of the charged particle for GM force of varying strength with $\mu = 0.1, 0.2, 0.3, 0.5, 0.8$, and $M = 1$, $E_0 = 1$, $B_0 = 20$ in gravitational units.
Figure 3

- a) $E_0 = 20$
- b) $E_0 = 10$
- c) $E_0 = 5$
- d) $E_0 = 1$
- e) $E_0 = 0.1$
Figure 5
Figure 6