QCD Sum Rules and $1/N_c$ expansion

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The $1/N_c$ arguments are developed to classify the hadronic states in the correlators. Arguments applied to the $\sigma$ meson correlator enable to separate the instanton, glueball, and, in particular, the $\pi\pi$ scattering states by $1/N_c$ from both 2q and 4q correlators. The bare resonance pole with no mixing effects are analyzed with the QCD sum rules (QSR). The results suggest the existence of nontrivial correlation for the mass reduction of 4q system.

§1. $1/N_c$ classification of hadronic states

The QCD based description of the exotic hadrons is one of the current problems to study the nonperturbative properties of QCD beyond those manifested in the usual hadron spectra. The $\sigma$ meson with $I = J = 0$ is a typical example with almost all important ingredients of the exotics. The $\sigma$ meson could include not only usual $q\bar{q}$ (2q) state but hadronic states beyond the simple constituent quark picture: glueball, $\pi\pi$ molecule, and $qq\bar{q}\bar{q}$ (4q) states, and is good laboratory to study the properties of these hadronic states and their interplay. The studies of the $\sigma$ meson, however, are not straightforward. Since it can be described as admixture of several hadronic states, it is difficult to identify which hadronic states are responsible to which part of the $\sigma$ meson properties. Therefore, it is important to theoretically clarify the properties of each hadronic state without mixing effects.

For this purpose, we introduce the $1/N_c$ classifications of hadronic states in the correlators, $\Pi(q) = \int d^4x e^{iq\cdot x} \langle T J(x)J(0) \rangle$. One of the largest virtues of $1/N_c$ expansion is that it relates the quark-gluon dynamics to the hadronic states order by order in terms of $1/N_c$, reflecting their qualitative difference through the difference in $1/N_c$ counting. We will identify and separate the instanton, glueball, and, in particular, $\pi\pi$ scattering states, to concentrate on the remaining bare "2q" and "4q" states (concrete definition will be given later) without the complication from the mixing effects. The "2q" and "4q" are dynamically studied through the QCD sum rules (QSR) which has the firm theoretical ground in large $N_c$ limit.

The operators used in the correlators are as follows: The 2q interpolating fields are described as $J_M^F = \overline{q}_F \Gamma_M q$, where Dirac matrix $\Gamma_M = (1, \gamma_\mu)$ and $\tau_F$ ($F = 1, 2, 3$) are the Pauli matrices acting on $q = (u, d)^T$. The 4q operators with the $\sigma$ quantum number are given (assuming the ideal mixing for the $\sigma$ meson) by $J_{MM}(x) = \sum_{F=1}^{3} J_M^F(x)J_M^F(x)$ as products of meson operators.

Here we give the inductive definition of "2q" and "4q" states based on the $1/N_c$ counting for the 2q and 4q correlators. Since the large $N_c$ 2q correlators are known to have the 2 meson scattering, glueball, and instanton only in the quark-gluon graphs of higher order of $1/N_c$, we can regard the leading $N_c$ contributions as the sum of
the bare "2q" states.\(^2\)\(^3\)\(^4\)\(^5\) Next we consider the 4q correlator, incorporating the 4q participating diagrams from the beginning. Here we give the inductive definition of the "4q" state following the \(1/N_c\) based orthogonal condition: a) "4q" is not generated in the leading \(N_c\) 2q correlator, b) "4q" can appear in the 4q correlators even in the absence of the 2-meson states. Thus the dynamical origin of "4q" is different from "2q" and 2-meson molecule states. The glueball and instanton are easily verified as the higher order effects in \(1/N_c\) than those we will consider, thus will not be discussed further in the following.

The studies of the "4q" component in the \(\sigma\) meson require systematic steps beyond the leading \(1/N_c\) arguments for the 4q correlators, since leading order \(O(N_c^2)\) quark-gluon diagrams include only 2 planar loops (Fig. 1a), which are interpreted as irrelevant free 2-meson scattering. Thus we must proceed to the next leading order of \(1/N_c\), \(O(N_c)\) diagrams which could include the "2q" and "4q" states. To discuss 2-point correlators in compact description, we classify the overlap strength of the operator \(\mathcal{J}_{MM}\) with hadronic states by \(1/N_c\). The explicit examination of quark-gluon diagrams in the 3-point correlator among \(\mathcal{J}_{MM}\) and two separated \(\mathcal{J}_{M'}\) (Fig. 1 d-f) indicates that the overlap strength of 4q field with 2-meson states \(|M'|\) is \(O(N_c^{1/2})\). On the other hand, the overlap strength with "2q" and "4q" states cannot be deduced from \(1/N_c\) arguments only, and are assumed to be \(\langle 0|\mathcal{J}_{MM}|R\rangle = O(N_c^{1/2})\), \((R = "2q"\) or "4q") which will be assured after the dynamical calculations. Following our \(1/N_c\) orthogonal condition a), "4q" will be identified by examining the quantitative difference of poles in 2q and 4q correlators, which is found to be large enough to distinguish the "4q" and "2q" states. The coupling of \(R\) to two mesons is estimated by 3-point function in the same way to obtain the meson couplings performed in Ref.\(^3\) and is found to be \(O(N_c^{-1/2})\).

Now we can classify the hadronic states in 2-point correlators \(\langle J_{MM}J_{M'M'}\rangle\) based on \(1/N_c\) (See, Fig. 1 a-c): (i) If \(M = M'\), \(O(N_c^2)\) quark-gluon graphs include only the free 2M scattering states in the region \(E \geq 2m_M\). Otherwise, the contributions from these quark-gluon diagrams vanish, indicating the absence of 2 meson scattering states. (ii) \(O(N_c)\) graphs include the 2M or 2M' scattering and possible resonance, "2q" and/or "4q". Note that in the case of \(M, M' \neq P\) or \(A\), 2\(\pi\) scattering states are not included up to \(O(N_c)\) diagrams, and then the resonance peaks (if exist) below 2\(m_M\) are isolated and have no width since the decay channels are absent. Therefore, now we can reduce the \(\sigma\) spectrum in the 4q correlator into peak(s) plus continuum if we retain only diagrams up to \(O(N_c)\).
§2. QCD Sum Rules for the reduced spectra

This separate investigation of the $O(N_c^2)$ and $O(N_c)$ part enables to perform the step by step analyses for the each hadronic state. In the application of the QSR, we perform the operator product expansion (OPE) for the correlators in deep Euclidean region ($q^2 = -Q^2$), then relate them, term by term of $1/N_c$, to the integral of the hadronic spectral function through the dispersion relation:

$$
\Pi_{N_c}^{\text{ope}}(-Q^2) = \int_0^\infty ds \, \frac{1}{\pi} \frac{\text{Im}\Pi_{N_c}^h(s)}{s + Q^2} \quad (n = 2, 1).
$$

(2.1)

Now we emphasize the practical aspects of $1/N_c$ expansion in the application of the QSR. First, the higher dimension condensates in the OPE, whose values have not been well-known despite of their importance, can be factorized into the products of known condensates, $\langle \bar{q}q \rangle$, $\langle G^2 \rangle$, and $\langle \bar{q}g_s \sigma G q \rangle$. To keep this merit, we will deduce the final $O(N_c)$ results from the off diagonal correlator $\langle J_{VV}^\dagger J_{SS}^\dagger \rangle$, whose leading order is $O(N_c)$ thus without the factorization violations at the $O(N_c)$ OPE.

Secondly, the lowest resonance in the reduced $O(N_c)$ spectra $\text{Im}\Pi_{N_c}^h(s)$ for "2q" and "4q" states, can be described as the sharp peak because of the absence of the decay channel. Applying the usual quark-hadron duality approximations to the higher excited states, $\pi\text{Im}\Pi_{N_c}^h(s) = \lambda^2 \delta(s - m_h^2) + \Theta(s - s_{th})\text{Im}\Pi_{N_c}^{\text{ope}}(s)$, and after the Borel transformation for the Eq.(2.1), we can express the effective mass as

$$
m_h^2(M^2; s_{th}) = \frac{\int_{s_{th}}^{s_{th}} ds \, e^{-s/M^2} s \, \text{Im}\Pi_{N_c}^{\text{ope}}(s)}{\int_{s_{th}}^{s_{th}} ds \, e^{-s/M^2} \text{Im}\Pi_{N_c}^{\text{ope}}(s)}.
$$

(2.2)

$s_{th}$ can be uniquely fixed to satisfy the least sensitivity of the expression (2.2) against the variation of $M$, since the physical peak should not depend on the unphysical expansion parameter $M$. This criterion is justified only when the peak is very narrow, and our $1/N_c$ reduction of spectra is essential for its application to allow the QSR framework to determine all physical parameters ($m_h, \lambda, s_{th}$) in self-contained way.

In practical application of the QSR, it is essential to reduce the error of the finite order truncation of the OPE and of the quark-hadron duality approximation. Thus Eq.(2.2) must be treated in the window ($M_{\min}^2, M_{\max}^2(s_{th})$) to achieve the conditions: good OPE convergence for $M_{\min}$ (highest dimension terms $\leq 10\%$ of whole OPE) and sufficient ground state saturation for $M_{\max}$ (pole contribution $\geq 50\%$ of the total). Without $M^2$ constraint, we are often stuck with the pseudo-peak artifacts often seen in the multiquark SRs. Thus we carry out the OPE up to dimension 12 to find the reasonable $M^2$ window.

The $N_c$ dependence of the quantities entering the OPE is summarized as follows. The gauge coupling constant behaves like $O(N_c^{-1/2})$, and the condensates, $\langle O \rangle = \langle \bar{q}q \rangle, \langle \alpha_s G^2 \rangle, \langle \bar{q}g_s \sigma G q \rangle$ are $O(N_c)$. Here we additionally put the simple $N_c$ scaling assumption on these values, $\alpha_s(N_c) = 3\alpha_s/N_c, \langle O \rangle|_{N_c} = \langle O \rangle|_{N_c} = \langle O \rangle N_c / 3$. We take the values with errors for $N_c = 3$ case, $\alpha_s(1\text{GeV}) = 0.4, \langle \alpha_s G^2 / \pi \rangle = (0.33 \text{ GeV})^4$. 


The large $N_c$ 2q correlator results for the scalar and vector mesons (left), the $O(N_c)$ part of the 4q correlators (middle), and their mass relation for the various condensate values (right).

$$\langle \bar{q}q \rangle = -(0.25 \pm 0.03 \text{ GeV})^3,$$

and

$$m_0^2 = \frac{\langle \bar{q}q \sigma Gq \rangle}{\langle \bar{q}q \rangle} = (0.8 \pm 0.1) \text{ GeV}^2,$$ respectively. The results shown below will use the central value.

First we show in the left panel of Fig.2 the results of the large $N_c$ 2q correlators (expanded up to dimension 6) for vector meson as a reference and scalar meson as the "2q" state in the $\sigma$ meson. The downarrow (upperarrow) indicates the values of $M_{\text{min}}^2$ ($M_{\text{max}}^2(s_{\text{th}})$). Following the $E_{\text{th}}$ ($=\sqrt{s_{\text{th}}}$) fixing criterion, we fix $E_{\text{th}}$ to 1.0 (1.4) GeV for vector (scalar) mesons, and determine the mass as 0.65 (1.10) GeV.

Now we turn to the $O(N_c)$ part of the 4q correlator, $\langle J_{VV}^1 J_{SS}^1 \rangle$, to investigate "2q" and "4q" states. Shown in the middle panel of Fig.2 are the effective masses for $E_{\text{th}}$=1.0, 1.2, and 1.4 GeV. We select the $E_{\text{th}}$=1.2 GeV case and evaluate its mass as $\sim$0.90 GeV, which is obviously lower than that of "2q" scalar meson case, $\sim$1.10 GeV in large $N_c$ limit, and thus is considered as the mass of "4q" state. We have also investigated $\langle J_{SS}^1 J_{SS}^1 \rangle$ ($\langle J_{VV}^1 J_{VV}^1 \rangle$), and obtained the almost same mass 0.80 (0.90) GeV although they could suffer from the factorization violation coming from $O(N_c^2)$ OPE. This 3-independent correlator analyses consistently suggest the existence of "4q" state lighter than "2q" state. Finally, in the right panel of Fig.2 we show the $\langle \bar{q}q \rangle$ and $m_0^2$ dependence of "2q" vector, scalar meson masses, and of the "4q" mass deduced from $\langle J_{VV}^1 J_{SS}^1 \rangle$. The inequality $m_\rho < m_{4q} < m_\sigma$ holds irrespective of the condensate values.

In conclusion, the $1/N_c$ expansion is useful to classify the hadronic states in the correlators, especially of the multiquark operators. The qualitative difference between several hadronic states is related to the difference in $1/N_c$ counting. A novel consequence of this approach is that we can investigate the bare 4q state which is generated from the genuine 4q dynamics, not factorized into the 2-meson dynamics. The results obtained here suggest that the $\sigma$ meson has the "4q" component, which includes the nontrivial correlation needed for the considerable mass reduction.

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