The application of finite difference method on 2-D heat conductivity problem

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Abstract. The finite difference method is one of the numerical methods that is often used to solve partial differential equations arose in the real world physical problems. The method is approximated by Taylor series. The study considers the FDM method to calculate the heat diffusion in any point in a rectangular domain. The results show that, it has a good level of accuracy with various values of error.

1. Introduction
Many physical problems are formulated in partial differential equations. One of the most important partial differential equations in application is Laplace equation. This equation is a classic example of the elliptical equations that can model many physical phenomena in real life such as problems of heat transfer, elasticity, electrostatic and fluid mechanic.

In some cases, this equation can be solved analytically. However, for some irregular boundaries, the analytical solutions are not always possible. Therefore, some numerical techniques need to be employed in order to solve the problem. In this study, the problem that is investigated is the steady heat transfer through a plat medium with particular size.

A Finite difference method is one of the numerical methods that has been developed for many decades. The method can be applied to solve partial differential equations using Taylor series approximation.

2. Methods
The analysis is started with solving the Laplace equation analytically using separable of variations. The next stage is applying the finite difference method into the problem to generate the numerical results. Looking for a solution to the Laplace-2D equation using the variable separation method. The results are then compared to examine the level of accuracy of the numerical solutions.

3. Results and discussion

3.1. Analytical solution of 2-D Laplace equation
Using the method of separation of variables with the assumption that \( u(x,y) \) is the product of two functions that only depends on one variable respectively, we obtain the analytical solution as follows
\[ u(x, y) = \sum_{n=1}^{\infty} A_n \sin \left( \frac{n\pi}{a} \right) \sinh \left( \frac{n\pi y}{a} \right) \]

where:

\[ A_n = \frac{2}{a \sinh \left( \frac{n\pi b}{a} \right)} \int_0^a f(x) \sin \left( \frac{n\pi x}{a} \right) \, dx \]

3.2. Numerical solution with finite difference method

Applying the second order of centered difference approximation, we obtain

\[ \frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} \quad (1) \]
\[ \frac{\partial^2 u}{\partial y^2} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2} \quad (2) \]

Substitute equations (1) and (2) into 2-D Laplace -2D, gives

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2} = 0 \]
\[ u_{i+1,j} + u_{i-1,j} + \alpha^2u_{i,j+1} + \alpha^2u_{i,j-1} - 2(1 + \alpha^2)u_{i,j} = 0 \quad (3) \]

Equation (3) is the numerical solution of 2-D Laplace equation. The stencil points of the centred difference approximation can be seen on figure 1.

Figure 1. The stencil points of the centred difference approximation.

3.3. Numerical solutions

Considered a plat domain with 10 cm of its side and the temperature is 100 degrees along boundary \( y = 10 \) and 0 degrees along three other boundaries. The boundary conditions can be summarized as follows:

\[ u(0, y) = u(a, y) = 0^\circ C \quad \text{and} \quad u(x, 0) = 0^\circ C; \, u(x, b) = f(x) \]
3. From figure 2 above, we can see the plates measuring 10 x 10 cm.

3.3.1. *Numerical solutions with 5 grids.* For 5 grids division we obtain the value of $\Delta x$ and $\Delta y$ as follows

$$\Delta x = \frac{a - 0}{M - 1} = \frac{10 - 0}{5 - 1} = 2.5$$

$$\Delta y = \frac{b - 0}{N - 1} = \frac{10 - 0}{5 - 1} = 2.5$$

with $i = 0, 1, 2, ..., M - 1$ and $j = 0, 1, 2, ..., N - 1$

which implies $i = 1, 2, 3$ and $j = 1, 2, 3$, so that we obtain the Linier Algebraic Equation System

$$u_{2,1} + u_{0,1} + u_{1,2} + u_{1,0} - 4u_{1,1} = 0$$
$$u_{3,1} + u_{1,1} + u_{2,2} + u_{2,0} - 4u_{2,1} = 0$$
$$u_{4,1} + u_{2,1} + u_{3,2} + u_{3,0} - 4u_{3,1} = 0$$
$$u_{2,2} + u_{0,2} + u_{1,3} + u_{1,1} - 4u_{1,2} = 0$$
$$u_{3,2} + u_{1,2} + u_{2,3} + u_{2,1} - 4u_{2,2} = 0$$
$$u_{4,2} + u_{2,2} + u_{3,3} + u_{3,1} - 4u_{3,2} = 0$$
$$u_{2,3} + u_{0,3} + u_{1,4} + u_{1,2} - 4u_{1,3} = 0$$
$$u_{3,3} + u_{1,3} + u_{2,4} + u_{2,2} - 4u_{2,3} = 0$$
$$u_{4,3} + u_{2,3} + u_{3,4} + u_{3,2} - 4u_{3,3} = 0$$

In matrix form: $AU = B$

$$A = \begin{bmatrix}
-4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & -4 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -4 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\end{bmatrix}$$

$$U = \begin{bmatrix}
u_{1,1} \\
u_{2,1} \\
u_{3,1} \\
u_{1,2} \\
u_{2,2} \\
u_{3,2} \\
u_{1,3} \\
u_{2,3} \\
u_{3,3} \end{bmatrix}$$

$$B = \begin{bmatrix}0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
-1 \\
-1 \\
-1 \end{bmatrix}$$

We solve the equation using Ms Excel. The results are displayed in table 1.
Table 1. Numerical solutions of 2-d Laplace Equation with 5 grids.

| $u_{i,j}$ | Numerical solution | Analytical solution | Error (%) |
|-----------|--------------------|---------------------|-----------|
| $u_{1,1}$ | 7.14285714         | 6.7713              | 5.48723499 |
| $u_{2,1}$ | 9.82142857         | 9.5761              | 2.561883976 |
| $u_{3,1}$ | 7.14285714         | 6.7713              | 5.48723499 |
| $u_{1,2}$ | 18.75              | 17.934              | 4.550016728 |
| $u_{2,2}$ | 25                 | 25.3624             | 1.428886856 |
| $u_{3,2}$ | 18.75              | 17.934              | 4.550016728 |
| $u_{1,3}$ | 42.8571429         | 40.7243             | 5.237273218 |
| $u_{2,3}$ | 52.6785714         | 57.5971             | 8.539542045 |
| $u_{3,3}$ | 42.8571429         | 40.7243             | 5.237273218 |

The graph of the solution can be seen on figure 3-9 bellow.

![Simulation 5 Different Grids](image)

Figure 3. Solution graph $u(x,y)$ for $M = N = 5$. 
Figure 4. Solution graph $u(x, y)$ for $M = N = 6$.

Figure 5. Solution graph $u(x, y)$ for $M = N = 7$.

Figure 6. Solution graph $u(x, y)$ for $M = N = 8$. 
3.4. Interpretation

The results show some interesting phenomena such as:

- For any point \((x,y)\) approaches \((5,5)\) the error of the solution tends to zero.
As the number of finite difference grids increases, the error of the solution in any point \((x,y)\) approaches the line \(y=0\) decreases.

As the number of finite difference grids increases, the error of the solution in any point \((x,y)\) approaches the line \(y=10\) increases.

The error of the solutions in any point \((x,y)\) approaches \((0,10)\) and \((0,10)\) increases.

The value of \(u(x,y)\) approaches 100 as the point \((x,y)\) approaches \((5,10)\).

The higher the number of finite difference grids, the heat is transferred in parabolic form.

The error of the solution near boundary \(u=100\) is greater than solutions near the boundary conditions \(u=0\).

The mean of error increases consistently as the number of grids increases.

The results are in line with has been obtained by previous research [1-4].

4. Conclusion

Things that can be concluded from the study are:

- Analytical solution of the 2-D Laplace Equation is
  \[
  u(x, y) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a}
  \]
  with,
  \[
  A_n = \frac{2}{a \sinh \frac{n\pi b}{a}} \int_{0}^{a} f(x) \sin \frac{n\pi x}{a} \, dx
  \]

- Numerical solution for 2-D Laplace Equation by applying Finite Difference Method is
  \[
  u_{i+1,j} + u_{i-1,j} + \alpha^2 u_{i,j+1} + \alpha^2 u_{i,j-1} - 2(1 + \alpha^2)u_{i,j} = 0
  \]
  With boundary conditions
  \[
  u(0, y) = u(a, y) = 0^\circ C
  \]
  \[
  u(x, 0) = 0^\circ C \text{ and } u(x, 10) = 100^\circ C
  \]

- Error produced by solutions in the domain is higher in points approach the boundary of \(u=100\) compared to error produced in points approach other boundaries with \(u=0\). The results are in line with the results obtained by previous research [1-4].

5. Recommendation

It is important to work out mathematically why the solution in the interior points produced greater error for points approaches the boundary \(u = 100\) compared to the points approaches to other boundaries with \(u=0\). The results can also be compared with the generated results produced by other numerical methods such as Boundary element method and finite volume method.

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