On inversion and connection coefficients for basic hypergeometric polynomials.

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Abstract

In this paper, we propose a general method to express explicitly the inversion and the connection coefficients between two basic hypergeometric polynomial sets. As application, we consider some $d$-orthogonal basic hypergeometric polynomials and we derive expansion formulae corresponding to all the families within the $q$-Askey scheme.

Key words. Connection coefficients, Inversion coefficients, Basic hypergeometric polynomials, $d$-orthogonal basic polynomials, $q$-Askey scheme.

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1 Introduction

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$. A polynomial sequence $\{P_n\}_{n \geq 0}$ in $\mathcal{P}$ is called a polynomial set if and only if $\text{deg} P_n = n$.

Given two polynomial sets $\{P_n\}_{n \geq 0}$ and $\{Q_n\}_{n \geq 0}$, the so-called connection problem between them asks to find the coefficients $C_m(n)$ in the expression:

$$Q_n(x) = \sum_{m=0}^{n} C_m(n) P_m(x).$$

For the particular case $Q_n(x) = x^n$ the connection problem is called inversion problem associated to $\{P_n(x)\}_{n \geq 0}$. For discrete polynomials and basic polynomials, besides the natural basis $\{x^n\}_n$, some other basis can be considered, namely, the Pochhammer basis $\{(x)_n\}_n$, the $q$-shifted factorial basis $\{(x; q)_n\}_n$ or products involving them:

$$(x)_n = \begin{cases} \prod_{k=0}^{n-1} (x + k) & \text{if } n = 1, 2, 3, \ldots \\ 1 & \text{if } n = 0 \end{cases} \quad \text{and} \quad (x; q)_n = \begin{cases} \prod_{k=0}^{n-1} (1 - x q^k) & \text{if } n = 1, 2, 3, \ldots \\ 1 & \text{if } n = 0 \end{cases}$$

The problem of connecting orthogonal polynomials is of old and recent interest. The connection coefficients play an important role in many problems in pure and applied mathematics especially in combinatorial analysis or in mathematical physics. In fact, some inversion problems have been solved as one of the steps leading to the orthogonality for the corresponding polynomial sets. Moreover, the use of inversion in order to solve connection problems was considered by Rainville [19] (Hermite, Laguerre and Legendre polynomials), by Gasper [13] (classical discrete orthogonal polynomials) and then by Area et al. [2] (polynomials within the Askey scheme and its $q$-analogue) and by Foupouagnigni et al. [11, 22] (classical continuous, classical discrete and $q$-classical orthogonal polynomials).

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The connection problem of basic polynomials have been studied by many authors. A wide variety of methods, based on specific properties of the involved polynomials, have been devised for computing the connection coefficients. For classical orthogonal polynomials the connection problem can be recurrently solved using an algorithm (the Navima-algorithm) which generates in a systematic way a linear recurrence relation in \( m \) for \( C_m(n) \). (See, for instance, [1] [3] and the reference therein). An algorithmic approach to build and solve recurrence relations for connection coefficients associated to \( q \)-classical orthogonal polynomials was also given by Lewanowicz [17] and by Foupouagnigni et al. [12].

An expansion of basic hypergeometric functions in basic hypergeometric functions called Verma formula [23] was used by Sánchez-Ruiz and Dehesa [21] and by Area et al. [2] to connect hypergeometric and basic hypergeometric polynomials. A general method based in lowering operators, dual sequences and generating functions was developed by Ben Cheikh and Chaggara to generate and compute the inversion and connection coefficients for polynomial sets with Boas-Buck generating functions [6] [7]. The same approach was implemented to Maple system to solve particular connection problem for continuous and discrete classical polynomials [10].

Our aim in this paper is to propose a simple and general method which allows us to compute the inversion and connection coefficients for basic hypergeometric polynomials. The approach we shall propose in this paper does not need particular properties of the polynomials involved in the problem.

Consider \( \{B_n\}_{n\geq 0} \) a suitable basis of polynomials. To find the coefficients \( C_m(n) \) in (1.1), we combine the inversion relation

\[
B_n(x) = \sum_{m=0}^{n} I_m(n)Q_m(x),
\]

with the explicit expression

\[
P_n(x) = \sum_{m=0}^{n} D_m(n)B_m(x),
\]

which yields, with sum manipulation, to the representation

\[
C_m(n) = \sum_{j=0}^{n-m} D_{j+m}(n)I_m(j + m) = \sum_{k=m}^{n} D_k(n)I_m(k).
\]

The connection coefficient in (1.3) can be obtained directly from the hypergeometric or the basic hypergeometric representation of \( P_n \) however the inversion coefficient will be evaluated recurrently using, for this purpose, a general result recently stated by Ben Romdhane [3].

We derive the inversion and connection coefficients for the following two classes of basic hypergeometric polynomials:

\[
r+1\phi_s\left(q^{-n}, (a_r)_{(b_s)}; q; qx\right).
\]

and

\[
r+2\phi_s\left(q^{-n}, aq^n, (a_r)_{(b_s)}; q; qx\right).
\]

The \( r\phi_s \) denotes the basic hypergeometric series or \( q \)-hypergeometric series defined by

\[
r\phi_s\left((a_r)_{(b_s)}; q; z\right) = \sum_{n=0}^{\infty} \frac{[a_r; q]_n}{[b_s; q]_n} \left((-1)^n q^{\frac{n(a-1)}{2}}\right)^{1+s-r} \frac{z^n}{(q; q)_n},
\]

where

\([a_n; q]_k = (a_1; q)_k(a_2; q)_k \cdots (a_n; q)_k, \quad k = 0, 1, 2, \ldots\).
The base $q$ will be restricted to $|q| < 1$ for non-terminating series. The contracted notation $(a_r)$ is used to abbreviate the array of $r$ parameters $a_1, \cdots a_r, \ldots$. The parameters $a$, $(a_r)$ and $(b_s)$ are assumed to be independent of $n$. The closed analytical formulae for the corresponding inversion and connection coefficients will be expressed by means of terminating basic hypergeometric functions which, in some cases, can be evaluated as a basic hypergeometric terms. By applying appropriate limit to the obtained results we derive connection coefficients for the following hypergeometric polynomials:

\[
\begin{align*}
    r+1 F_s \left( \begin{array}{c} -n, (a_r) \\ (b_s) \end{array} ; x \right), \\
    r+2 F_s \left( \begin{array}{c} -n, \lambda + n, (a_r) \\ (b_s) \end{array} ; x \right),
\end{align*}
\]

where the $_r F_s$ denotes, as usual, the generalized hypergeometric functions with $r$ numerator and $s$ denominator parameters. The polynomials defined by (1.5)-(1.9) are relevant to the study of quantum-mechanical systems and include as particular cases many known polynomial sets, we quote for instance orthogonal polynomials in Askey scheme and its $q$-analogue and their generalizations with Sobolev type orthogonality and $d$-orthogonality.

The families (1.6) and (1.7) extend the hypergeometric polynomials given by (1.8) and (1.9). It is clear that

\[
\lim_{q \to 1^-} \frac{(q^n ; q)_k}{(1 - q)_k} = (a)_k,
\]

hence

\[
\lim_{q \to 1^-} \phi_s \left( q^{a_1}, \cdots, q^{a_r} ; q^{b_1}, \cdots, q^{b_s} ; q ; (1 - q)^{s+1-r}x \right) = _r F_s \left( (a_r) ; x \right).
\]

We apply the obtained results to some $d$-orthogonal basic hypergeometric polynomials (Big $q$-Laguerre type, Little $q$-Laguerre type, $q$-Laguerre type and $q$-Meixner type) as well as to the orthogonal basic hypergeometric polynomials of the $q$-analogue of Askey scheme (Askey-Wilson, $q$-Racah, Continuous dual $q$-Hahn, Continuous $q$-Hahn, Big $q$-Jacobi,...).

The content of this paper is organized as follows:

1. In Section 2 we prove our main result Theorem 2.1 as well as some useful consequences.

2. In Section 3 we apply our results to many generalized basic hypergeometric polynomial sets studied in the framework of $d$-orthogonality.

3. In Section 4 the explicit inversion and connection coefficients between orthogonal polynomials of the $q$-Askey scheme are summarized in Tables 1-2.

### 2 Connection coefficients between basic Hypergeometric polynomials

**Theorem 2.1.** The inversion and connection formulae for basic hypergeometric polynomials (1.6) are given by

\[
x^n = \frac{[b_r; q]_n}{[a_s; q]_n} \left( (-1)^n q^{\frac{m(m-1)}{2}} \right)^{r+1-s} \sum_{m=0}^{n} \frac{n}{m} \left( -1 \right)^m q^{\frac{m(m-1)}{2}} \left( q^{m+1}; q \right)_{n-m} \left( aq^m; q \right)_m \left( b_s; q \right)_n \phi_s \left( q^{-m}, aq^m, (a_r) ; q, qx \right),
\]

(2.1)
and
\[
\phi_s\left(q^{-n}, aq^n, (a_r)_{(b_s)}; q; qx \right) = \sum_{m=0}^{n} \binom{n}{m} (-1)^{m(s+t-r-h)} \binom{aq^n; q}_m \binom{a_r; q}_m \binom{d_h; q}_m \binom{cq^m; q}_m \binom{ci; q}_m
\]
\[
\times r + h + 2\phi_{s+t+1} \left( q^{m-n}, aq^{m+n}, (a_r q^m), (d_h q^m) ; q; q^{1+m(s+t-r-h)} \right)
\]
\[
\times q^{m(n-n)} q^{m(m-n)} \phi_{h}(q^{m-n}, c_q^m, (c_i) ; q; qx), \quad (2.2)
\]

where, the \( q \)-binomial coefficient \( \binom{n}{k}_q \) is defined by \( \binom{n}{k}_q = \frac{(q; q)_n}{(q; q)_k(q; q)_{n-k}} \).

Starting point for the proof of this theorem is the following result.

**Lemma 2.2.** \( \triangleright \) **Theorem 2.1** Let \( \{P_n\}_{n \geq 0} \) be a monic polynomial set expanded in a given basis \( \{B_n\}_{n \geq 0} \) by

\[
P_n(x) = \sum_{k=0}^{n} A_k(n) B_{n-k}(x).
\]

Then the following inversion formula holds

\[
\Phi_n(x) = \sum_{m=0}^{n} b_m(n, 0) P_{n-m}(x),
\]

where

\[
\begin{aligned}
b_0(n, 0) &= 1, \\
b_0(n, k) &= 0, \\
b_{m+1}(n, k) &= b_m(n, k+1) - b_m(n, 0) A(n - m, k + 1) \quad \text{if} \quad 1 \leq k, \\
b_{m+1}(n, k) &= b_m(n, k+1) - b_m(n, 0) A(n - m, k) \quad \text{if} \quad 0 \leq m \leq n - 1, 0 \leq k \leq n - m - 1.
\end{aligned}
\]

**Proof.** In order to get the explicit expression of the inversion coefficient \( b_m(n, 0) \) in (2.4), we need to compute the coefficients \( b_m(n, k) \) for each \( m \) and \( k \). For this, we use the recurrence relation (2.5) to compute some initial values and to guess the resulting term. Then we proceed by induction to give the proof.

Let \( \tilde{P}_n(x) \) be the monic basic hypergeometric polynomial defined by

\[
\tilde{P}_n(x) = \frac{(-1)^n[b_s; q]_n}{(aq^n; q)_n(a_r; q)_n} \left( -1 \right)^n q^{-\frac{n(n-1)}{2}} \sum_{k=0}^{n} (-1)^k q^{-\frac{k(k+1)}{2}} \binom{k}{k^n-k-1} \binom{aq^n-k; q}_{k} \binom{a_r q^n-k; q}_{k} \binom{b_s q^n-k; q}_{k} x^{n-k}.
\]

It follows

\[
\begin{aligned}
b_1(n, k) &= -A(n, k+1) \\
&= (-1)^k \left[ \binom{n}{k+1} q^{\frac{(k+1)(2n-k-2)}{2}} \right] \sum_{r-s+1}^{r-s-1} q_{r-k(s-r+1)} \cdot q_{k(k-1)} \cdot q_{k^n-k-1} \cdot q_{k+1}
\end{aligned}
\]

\[
\times \frac{(aq^n-k; q)_{n-k-1}}{(aq^n; q)_n} \cdot \frac{b_s q^n-k; q}_{k+1} \cdot \frac{a_r q^n-k; q}_{k+1}.
\]

For $b_2(n, k)$, we have

$$b_1(n, 0) A(n - 1, k + 1) = (-1)^{k+1} \binom{n}{k+2} \frac{(q^{k+2}; q)_1}{(q; q)_1} \left( -1 \right)^k q^{\frac{(k+2)(2n-k-3)}{2}} \frac{r-1}{q^{k+1}} - \frac{1}{q^{k+1}}$$

$$\times \frac{[b_s q^{n-k-2}; q]_{k+2} (aq^n; q)_{n-k-2}}{[a_s q^{n-k-2}; q]_{k+2} (aq^n; q)_n}.$$

By the useful identities,

$$\binom{n}{m}_q \binom{n-m}{k}_q = \sum_{j=1}^{n} (q^{n+1}; q)_{m} (aq; q)_n (a; q)_n \frac{1 - aq^n}{1 - a}$$

and $(a; q)_n + (a; q)_m = (a; q)_m (aq^m; q)_{n-m}$, we find

$$b_2(n, k) = (-1)^k \binom{n}{k+2} q^{\frac{(k+2)(2n-k-3)}{2}} \frac{r-1}{q^{k+1}} - \frac{1}{q^{k+1}}$$

$$\times \frac{(aq^n; q)_{n-k-2}}{[a_s q^{n-k-2}; q]_{k+2} (aq^n; q)_n}.$$  

Now, we can suggest the following form of $b_m(n, k)$:

$$b_m(n, k) = (-1)^k \binom{n}{k+m} q^{\frac{(k+m)(2n-k-m-1)}{2}} \frac{r-1}{q^{k+m}} - \frac{1}{q^{k+m}}$$

$$\times \frac{(aq^n; q)_{n-k-m}}{[a_s q^{n-k-m}; q]_{k+m} (aq^n; q)_n}.$$  

Assuming the formula to hold for $m$. Substituting $b_m(n, 0), b_m(n, k+1)$ and $A(n - m, k + 1)$ with their expressions in $b_{m+1}(n, k)$ given by (2.3), it follows that the assumption is valid for $m + 1$. Thus, up taking $k = 0$, we obtain:

$$b_m(n, 0) = \binom{n}{m}_q (-1)^m (q; q)_n \frac{m(m-1)}{2} - \frac{q^m}{q^{m+1}}$$

That leads to (2.1).

According to the basic hypergeometric representation given by (1.6), to the associated inversion formula (2.1) and to the composition formula (1.3) and by the relation

$$(q^{-n}; q)_m = \frac{(-1)^m (q; q)_n}{(q; q)_{n-m} q^{\frac{m(m-1)}{2}}} q^{-nm}$$

we get

$$C_m(n) = (-1)^m \frac{(q^{-n}; q)_m (aq^n; q)_m [a_r; q]_m}{[b_s; q]_m} q^m \frac{[d_h; q]_m}{[c_l; q]_m (cq^m; q)_m} \frac{m(m-1)}{2}$$

$$\times \sum_{j=0}^{n-m} \binom{j + m}{m}_q \frac{(q^{m-n}; q)_j (aq^{n+m}; q)_j [a_r; q]_m}{(q; q)_m [b_s q^m; q]_j}$$

$$\times \frac{[d_h q^m; q]_j}{[cq^{2m+1}; q]_j [cq^m; q]_j} q^j \left( -1 \right)^{m+j} q^{\frac{(m+j)(m+j-1)}{2}} - \frac{1}{q^{m+j}}$$

$$= q^m (q; q)_n (aq^n; q)_m [a_r; q]_m [d_h; q]_m \frac{m(m-1)}{2}$$

$$\sum_{j=0}^{n-m} \frac{(q^{m-n}; q)_j (aq^{n+m}; q)_j [a_r; q]_m [d_h q^m; q]_j}{(q; q)_{n-m} (q; q)_m [b_s; q]_m [c_l; q]_m (cq^m; q)_m}$$

$$\times \sum_{j=0}^{n-m} \frac{(q^{m-n}; q)_j (aq^{n+m}; q)_j [a_r; q]_m [d_h q^m; q]_j}{(c q^{2m+1}; q)_j [cq^m; q]_j [c q^m; q]_j [b_s q^m; q]_j}$$

$$\times \frac{(-1)^{m+j} q^{\frac{(m+j)(m+j-1)}{2}}}{q^{m+j}}.$$  

Then (2.2) follows.
If we put $a = c = 0$ in (2.1) and (2.2) we obtain the inversion and connection formula for basic hypergeometric polynomials (1.5).

**Corollary 2.3.** The inversion and connection formulae associated to (1.5) are given by:

$$x^n = \binom{b_s}{a_r} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{q^{k(k-1)/2}}{\phi_{k,q}} \binom{n-k}{r+1} \phi_{k,q} \left( q^{-k}, (a_r)_{(b_s)} : q;q^x \right), \quad (2.6)$$

and

$$r+1 \phi_s \left( q^{-n}, (a_r)_{(b_s)} : q;q^x \right) = \sum_{m=0}^{n} \binom{n}{m} (-1)^m \frac{q^{m(m-1)/2}}{\phi_{m,q}} \binom{m}{s} \frac{q^{s(s+1)/2}}{\phi_{s,q}} \phi_{r+1+\phi_{s+q}} \left( q^{m-n}, (a_r,q^m)_{(b_s,q^m)} : q^{1+m(s+1-r-h)} \right) \times \frac{\sum_{r=0}^{q^{m-n}} (a_r,q^m)_{(b_s,q^m)} : q^{1+m(s+1-r-h)} \right) \times \phi_{r+2 \phi_{s+1}} \left( q^{-m}, (c_r)_{(d_s)} ; q^x \right). \quad (2.7)$$

The following two corollaries provide inversion and connection formulae for generalized hypergeometric polynomials given by (1.9) and (1.8). These results can be obtained as a limit cases of Eqs. (2.1), (2.2), (2.6) and (2.7) when $q \to 1^-$ and by using (1.10).

**Corollary 2.4.** The inversion and connection formulae associated to (1.9) are given by

$$x^n = \binom{b_s}{a_r} \sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^m}{\lambda + m + 1} \frac{\phi_{r+1+\phi_{m}}}{\phi_{r+1+\phi_{m}}} \left( q^{-m}, \lambda + m, (a_r)_{(b_s)} ; x \right), \quad (2.8)$$

and

$$r+1 \phi_{s+1} \left( q^{-n}, \lambda + n, (a_r)_{(b_s)} ; x \right) = \sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^m}{\lambda + m + 1} \frac{\phi_{r+1+\phi_{m}}}{\phi_{r+1+\phi_{m}}} \left( q^{-m}, \lambda + n + m, (a_r + m), (d_s + m)_{(b_s + m), (c_r + m)} ; 1 \right) \times \phi_{r+1+\phi_{s+1}} \left( q^{-m}, \lambda + n + m, (a_r + m), (d_s + m)_{(b_s + m), (c_r + m)} ; 1 \right) \times \phi_{r+1+\phi_{s+1}} \left( q^{-m}, \lambda + n + m, (a_r + m), (d_s + m)_{(b_s + m), (c_r + m)} ; 1 \right). \quad (2.9)$$

**Corollary 2.5.** The inversion and connection formulae associated to (1.8) are given by

$$x^n = \binom{b_s}{a_r} \sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^m}{\lambda + m + 1} \frac{\phi_{r+1+\phi_{m}}}{\phi_{r+1+\phi_{m}}} \left( q^{-m}, \lambda + m, (a_r)_{(b_s)} ; x \right), \quad (2.10)$$

and

$$r+1 \phi_{s+1} \left( q^{-n}, (a_r)_{(b_s)} ; x \right) = \sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^m}{\lambda + m + 1} \frac{\phi_{r+1+\phi_{m}}}{\phi_{r+1+\phi_{m}}} \left( q^{-m}, (a_r + m), (d_s + m)_{(b_s + m), (c_r + m)} ; 1 \right) \times \phi_{r+1+\phi_{s+1}} \left( q^{-m}, (a_r + m), (d_s + m)_{(b_s + m), (c_r + m)} ; 1 \right) \times \phi_{r+1+\phi_{s+1}} \left( q^{-m}, (a_r + m), (d_s + m)_{(b_s + m), (c_r + m)} ; 1 \right). \quad (2.11)$$
The inversion and connection formulae (2.8)–(2.11) were already given in [7] via generating functions manipulations. Note here that, as in the classical Laguerre case [20], the generalized hypergeometric polynomial:
\[ \tilde{L}_n((\alpha_r);(\beta_s))(x) := \left[ \beta_s \right]_n \left[ \alpha_r \right]_n x^{r+1} F_s\left( -n, (\alpha_r); (\beta_s); x \right) \]
is self inverse. That is to say, polynomial with same expression of the coefficients in the direct and in the inverse formulae. This can be translated in the Umbral calculus context by:
\[ \tilde{L}_n((\alpha_r);(\beta_s))(\tilde{L}_n((\alpha_r);(\beta_s))(x)) = x^n. \]

3. **Connection coefficients between \( d \)-orthogonal basic polynomials**

Next we apply the obtained results to some \( d \)-orthogonal basic hypergeometric polynomials. The notion of \( d \)-orthogonal polynomials generalize the standard orthogonal polynomials in that they satisfy a \( d \) orthogonality conditions and they obey a higher-order recurrence relation [18]. This kind of orthogonality appears as a special case of the general multiple orthogonality. In fact, the \( d \)-orthogonal polynomials correspond to multiple orthogonal polynomials near the diagonal [14]. The concept of \( d \)-orthogonality has been the subject of numerous investigations and applications. In particular, it is connected with the study of vector Padé approximants, vectorial continued fractions, resolution of higher-order differential equations and spectral study of multi-diagonal non-symmetric operators.

Most of the known explicit examples of \( d \)-orthogonal polynomial sets were introduced by solving a characterization problem that consists in finding all \( d \)-orthogonal polynomials, satisfying a given property.

The basic hypergeometric \( d \)-orthogonal polynomials that will be considered here generalize, firstly, the known \( q \)-Meixner, big \( q \)-Laguerre, little \( q \)-Laguerre and \( q \)-Laguerre orthogonal polynomials and, on the other hand, can be viewed as \( q \)-analogs of the \( d \)-orthogonal polynomials of Meixner and Laguerre type.

The inversion coefficients for \( d \)-orthogonal polynomials were used to derive the \( d \)-dimensional functional vectors ensuring their \( d \)-orthogonality [9]. The \( q \) analogs of the \( d \)-orthogonal polynomial sets were introduced and investigated in details in [16].

\( d \)-orthogonal little \( q \)-Laguerre type

The \( d \)-orthogonal of little \( q \)-Laguerre type polynomials are defined by the basic hypergeometric sum [16]:
\[ p_n(x, (b_s)/q) = d+1 \phi_s\left( q^{-n}, 0, \ldots, 0; b_s; qx \right). \] (3.1)

By (2.6) and (2.7), the inversion and connection formulae are given by
\[ x^n = \sum_{m=0}^{n} \left[ \begin{array}{c} n \\ m \end{array} \right]_q \left( -1 \right)^n q^{m(\frac{n-1}{2})} \frac{d-s}{d} [b_s; q]_n (-1)^m q^{\frac{m(m-1)}{2}} p_n(x, (b_s)/q), \] (3.2)

and
\[ p_n(x, (b_s)/q) = \sum_{m=0}^{n} \left[ \begin{array}{c} n \\ m \end{array} \right]_q q^{m(m-n)} [b_s; q]_m^{\frac{1}{2}} \phi_s\left( q^{m-n}, (b_s q^m); q \right) p_m(x, (b_s)/q). \] (3.3)
**d-orthogonal q-Meixner type**

The d-orthogonal polynomials of q-Meixner type are defined by \[16\]

\[ M_n(q^{-x}; (b_d), c; q) = 2\phi_d \left( q^{-n}, q^{-x} \right) \left( \frac{q^n - 1}{c} \right) \]  

(3.4)

The following inversion and connection relations are valid:

\[ (q^{-x}; q)_n = \sum_{m=0}^{n} [b_d; q]_n \left( (-1)^{n} q^{\frac{m(n-1)}{2}} \right)^{1-d} (-1)^{m+n} q^{\frac{m(m-1)}{2} - mn} (c)_n \left[ \frac{n}{m} \right]_q M_m(q^{-x}; (b_d), c; q), \]  

(3.5)

and

\[ M_n(q^{-x}; (b_d), c; q) = \sum_{m=0}^{n} \left( \frac{c}{\gamma} \right)^m \left[ \frac{n}{m} \right]_q \left[ \frac{\beta_d; q}_m \right]_q \]  

\[ \times d+1 \phi_d \left( q^{m-n}, (\beta_d q^m) \right) \left( \frac{\gamma}{c} q^{1+n-m} \right) M_m(q^{-x}; (\beta_d), \gamma; q). \]  

(3.6)

**d-orthogonal big q-Laguerre type**

The d-orthogonal polynomial of big q-Laguerre type has the following q-hypergeometric representation \[16\]:

\[ P_n(x; (b_{d+1}); q) = d+2\phi_{d+1} \left( \frac{q^{-n}, 0, \ldots, 0, x}{(b_{d+1})}; q \right). \]  

(3.7)

This polynomial set fulfil the following inversion ans connection formulae:

\[ (x; q)_n = \sum_{m=0}^{n} [b_{d+1}; q]_n q^{\frac{m(m-1)}{2}} \left[ \frac{n}{m} \right]_q P_m(x; (b_{d+1}); q), \]  

(3.8)

and

\[ P_n(x; (b_{d+1}); q) = \sum_{m=0}^{n} \left[ \frac{n}{m} \right]_q q^{m(m-n)} \left[ \frac{\beta_{d+1}; q}_m \right]_q \]  

\[ \times d+2\phi_{d+1} \left( q^{m-n}, (\beta_{d+1} q^m) \right) \left( \frac{\gamma}{c} q^{1+n-m} \right) P_m(x; (\beta_{d+1}); q) \]  

(3.9)

**d-orthogonal q-Laguerre type**

The explicit expression of the d-orthogonal polynomial of q-Laguerre type is as follows \[16\]:

\[ L_n^{(b_1, \ldots, b_d)}(x; q) = d\phi_d \left( \frac{q^{-n}}{(b_d)}; q; q^n x \right) \]  

(3.10)

The d-orthogonal q-Laguerre polynomials satisfy the following inversion and connection formulae:

\[ x^n = [b_d; q]_n \sum_{m=0}^{n} \left[ \frac{n}{m} \right]_q \left( (-1)^n q^{\frac{m(n-1)}{2}} \right)^{-d} \frac{q^{\frac{m(m-1)}{2} - mn}}{c} (-1)^m L_m^{(b_1, \ldots, b_d)}(x; q), \]  

(3.11)

and

\[ L_n^{(b_1, \ldots, b_d)}(x; q) = \sum_{m=0}^{n} \left[ \frac{\beta_d; q}_m \right]_q \left[ \beta_d; q^{m+1} \right]_q d+1 \phi_d \left( \frac{q^{m-n}, (\beta_d q^m)}{(b_d q^m)} \right) \left( \frac{\gamma}{c} q^{1+n-m} \right) L_m^{(\beta_1, \ldots, \beta_d)}(x; q). \]  

(3.12)
4 Connection relation for basic orthogonal hypergeometric polynomials

In this section we deal with families of basic hypergeometric orthogonal polynomials appearing in the $q$-Askey scheme. We give the corresponding inversion and connection coefficients without using the orthogonality property. In this way, the obtained formulae are still valid outside the range of orthogonality of the parameters.

The $q$-Askey scheme is a $q$-analogue of the Askey scheme. It contains specific orthogonal polynomials which can be written in terms of basic hypergeometric functions starting in the top with Askey-Wilson polynomials and $q$-Racah polynomials and ending in the bottom with continuous and discrete $q$-Hermite polynomials and Stieltjes-Wigert polynomials. The inversion and connection formulae for Askey-Wilson and $q$-Racah polynomials follow from (2.1) and (2.2) and they will be given in the sequel of this section. The corresponding expansions for the remainder polynomials within the $q$-Askey scheme are quoted in Tables 1 and 2. They can be obtained either as limit cases and specialization process from Askey-Wilson and $q$-Racah polynomials or by mean of the general inversion and connection formulae obtained in Section 2. The polynomial basis considered in Table 1 are suggested by the basic hypergeometric representation of each family (see [15], for more details).

Askey-Wilson

The Askey-Wilson polynomial set is a family of orthogonal polynomials introduced by Askey and Wilson in [5] as $q$-analogs of the Wilson polynomials. They include many of other orthogonal polynomials as special or limiting cases. The Askey-Wilson polynomials belong to the so-called classical orthogonal polynomials on a non uniform lattice which are known to satisfy a particular divided-difference equation [15, Chapter14]. The Askey-Wilson polynomials are defined by [5]

$$P_n(x; a, b, c, d/q) = \frac{(ab, ac, dq)_n}{a^n} 5\phi_3\left(\begin{array}{c}
q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\
ab, ac, ad
\end{array}; q^2, x = \cos(\theta), \right)$$

(4.1)

where $(ab, ac, dq)_n = (ab)_n(ac)_n(dq)_n$.

The following expansion formulae of Askey-Wilson basis in terms of Askey-Wilson polynomials is valid:

$$(ae^{i\theta}; q)_n(a^{-i\theta}; q)_n = \sum_{m=0}^{n} \binom{n}{m} q^{m(m-1)/2} (-a)^m \frac{(abq^m, acq^m, adq^m; q)_{n-m}}{(abcdq^{m-1}; q)_m(abcdq^{2m}; q)_{n-m}} P_m(x; a, b, c, d/q).$$

(4.2)

The connection formula between two Askey-Wilson polynomials, with first common parameter, is given by

$$P_n(x, a, b, c, d/q) = \sum_{m=0}^{n} a^{m-n} q^{m(m-n)} \frac{(ab, ac, ad, q; q)_n}{(q; q)_{n-m}} \frac{(abcdq^{n-1}; q)_m}{(ab, ac, ad, q; q)_m(a\beta\gamma\delta q^{m-1}; q)_m}$$

$$\times 5\phi_4\left(\begin{array}{c}
q^{-m-n}, a\beta q^m, a\gamma q^m, a\delta q^m, abcdq^{n+m-1} \\
abq^m, acq^m, adq^m, a\beta\gamma\delta q^{2m}
\end{array}; q, q\right) P_m(x, \alpha, \beta, \gamma, \delta/q).$$

(4.3)

Note that, the inversion and connection problems for Askey-Wilson polynomials were already studied by many authors: Askey and Wilson used orthogonality assumption [5], Area et al. used Verma Formula [2]. However, Foupouagnigni et al. solved this problem recurrently by computer algebra tools [11].
\(q\)-Racah

The \(q\)-Racah polynomials is a set of orthogonal polynomials that generalize the Racah coefficients or \(6-j\) symbols. They were introduced in [3]. Their hypergeometric representation is given by

\[
R_n(\nu(x); \alpha, \beta, \delta; \gamma/q) = 4\phi_3 \left( \begin{array}{c}
q^{-n}, \alpha\beta q^{n+1}, q^{-x}, \delta q^{-\nu+1} \\
\alpha q, \beta \delta q, \gamma q
\end{array} ; q^\nu q \right), \quad n = 0, 1, \ldots, N,
\]

where

\[
\nu(x) = q^{-x} + \delta q^{x+1},
\]

and \(\alpha q = q^{-N}\) for some integer \(N\).

Clearly, \(R_n(\nu(x); \alpha, \beta, \delta; \gamma/q)\) is a polynomial of degree \(n\) in \(\nu(x)\) and the case \(q \to 1\) gives the Racah polynomials.

By (2.1),(2.2), the following inversion and connection formulae of \(q\)-Racah polynomials are valid:

\[
(q^{-x}, \delta q^{x+1}; q)_n = \sum_{m=0}^{n} (-1)^m \binom{n}{m} \frac{(\alpha q, \beta q^{m+1}, q^{-x}; q)_m}{(\alpha q^m; q)_m} \frac{R_m(\nu(x); \alpha, \beta, \delta; \gamma/q)}{n-m},
\]

and

\[
R_n(\mu(x); \alpha, \beta, \delta, \gamma/q) = \sum_{m=0}^{n} \binom{n}{m} \frac{(\alpha q^{m+1}, \beta q, \gamma q; q)_n}{(\alpha q^{m+1}; q)_n} \frac{R_m(\mu(x); \alpha, \beta, \delta, \gamma/q)}{n-m}.
\]

provided that \(\gamma \delta = cd\).

| Polynomial set | Polynomial sets basis | Inversion coefficients \(I_m(n)\) |
|----------------|-----------------------|----------------------------------|
| Askey-Wilson: \(P_n(x; a, b, c, d/q)\) | \((ae^{i\theta}, ae^{-i\theta}; q)_n\) | \[
\sum_{m=0}^{n} \binom{n}{m} \frac{q^{-m}}{n-m} \left(\frac{(-a)^m}{(abq^{m+1}; q)_m} - \frac{q^{-m+1}}{n-m}\right)
\]
| \(q\)-Racah: \(R_n(\nu(x); \alpha, \beta, \delta; \gamma/q)\) | \((q^{-x}, \delta q^{x+1}; q)_n\) | \[
\sum_{m=0}^{n} \binom{n}{m} \frac{q^{-m}}{n-m} \left(\frac{(-a \alpha q^m \beta \delta q; q)_n}{(abdq^{m+1}; q)_m} - \frac{q^{-m+1}}{n-m}\right)
\]
| Big \(q\)-Jacobi: \(P_n(x; a, b, c, d/q)\) | \((x; q)_n\) | \[
\sum_{m=0}^{n} \binom{n}{m} \frac{q^{-m}}{n-m} \left(\frac{(-a \alpha q^m \beta \delta q; q)_n}{(abdq^{m+1}; q)_m} - \frac{q^{-m+1}}{n-m}\right)
\]
| \(q\)-Hahn: \(Q_n(q^{-x}; \alpha, \beta, N/q)\) | \((q^{-x}; q)_n\) | \[
\sum_{m=0}^{n} \binom{n}{m} \frac{q^{-m}}{n-m} \left(\frac{(-a \alpha q^m \beta \delta q; q)_n}{(abdq^{m+1}; q)_m} - \frac{q^{-m+1}}{n-m}\right)
\]
| Dual \(q\)-Hahn: \(R_n(\nu(x); \gamma, \delta, N/q)\) | \((q^{-x}, \gamma \delta q^{x+1}; q)_n\) | \[
\sum_{m=0}^{n} \binom{n}{m} \frac{q^{-m}}{n-m} \left(\frac{(-a \alpha q^m \beta \delta q; q)_n}{(abdq^{m+1}; q)_m} - \frac{q^{-m+1}}{n-m}\right)
\]
| Al-Salam-Chihara: \(Q_n(x; a, b/q)\) | \((ae^{i\theta}, ae^{-i\theta}; q)_n\) | \[
\sum_{m=0}^{n} \binom{n}{m} \frac{q^{-m}}{n-m} \left(\frac{(-a \alpha q^m \beta \delta q; q)_n}{(abdq^{m+1}; q)_m} - \frac{q^{-m+1}}{n-m}\right)
\]
| \(q\)-Meixner-Pollaczek: \(P_n(x; a/q)\) | \((ae^{i\theta}, ae^{-i\theta}; q)_n\) | \[
\sum_{m=0}^{n} \binom{n}{m} \frac{q^{-m}}{n-m} \left(\frac{(-a \alpha q^m \beta \delta q; q)_n}{(abdq^{m+1}; q)_m} - \frac{q^{-m+1}}{n-m}\right)
\]
| Continuous \(q\) Jacobi: \(P_n^{(\alpha, \beta)}(x/q)\) | \((\frac{1}{2} \alpha + \frac{1}{2} \beta e^{i\theta}, \frac{1}{2} \alpha + \frac{1}{2} \beta e^{-i\theta}; q)\) | \[
\sum_{m=0}^{n} \binom{n}{m} \frac{q^{-m}}{n-m} \left(\frac{(-a \alpha q^m \beta \delta q; q)_n}{(abdq^{m+1}; q)_m} - \frac{q^{-m+1}}{n-m}\right)
\]

Table 1: Inversion coefficients in the \(q\)-Askey scheme
| Type                                  | Formula                                                                                                                                                                                                 |
|---------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Continuous $q$-Ultraspherical/        | $C_n(x; \beta/q) = \left( \beta^x e^{\theta}, \beta^{-x}; q \right)_n \frac{\beta^n}{q^{m(m-1)/2} \left( \frac{q}{q^2} \right)^{x/2}}$ |
| Rogers:                               | $C_n(x; \beta/q) = \left( \beta^x e^{\theta}, \beta^{-x}; q \right)_n \frac{\beta^n}{q^{m(m-1)/2} \left( \frac{q}{q^2} \right)^{x/2}}$ |
| Continuous $q$-Legendre: $P_n(x/q)$   | $P_n(x/q; a, b; \theta) = \frac{q^{n(n-1)/2} \left( \frac{q}{q^2} \right)^{x/2}}{n!} \sum_{m=0}^{n} \frac{(a)_m (b)_m}{(\theta)_m} \frac{q^{m(m-1)/2}}{(q^2)^{x/2}} x^m$ |
| Big $q$-Laguerre: $P_n(x/a, b, q)$    | $(x;q)_n \frac{(aq, bq; q)_n}{(q^2)^{x/2}} \sum_{m=0}^{n} \frac{(a)_m (b)_m}{(q^2)^{x/2}} q^{-m(m+1)/2}$ |
| Little $q$-Jacobi: $P_n(x/a, b/q)$    | $x^n \frac{(aq, bq; q)_n}{(aq, bq; q)_n} \sum_{m=0}^{n} \frac{(a)_m (b)_m}{(q^2)^{x/2}} q^{-m(m+1)/2}$ |
| Little $q$-Legendre: $P_n(x/q)$       | $x^n \frac{(aq, bq; q)_n}{(aq, bq; q)_n} \sum_{m=0}^{n} \frac{(a)_m (b)_m}{(q^2)^{x/2}} q^{-m(m+1)/2}$ |
| $q$-Meixner: $M_n(x^{q^{-x}}, b, c; q)$ | $(x^{q^{-x}}; q)_n \frac{(aq, bq; q)_n}{(aq, bq; q)_n} \sum_{m=0}^{n} \frac{(a)_m (b)_m}{(q^2)^{x/2}} q^{-m(m+1)/2}$ |
| Quantum $q$-Krawtchouk               | $K_n(q^{-x}; p, -N; q) = \left( q^{-x}; q \right)_n \frac{(p)_n}{(q^2)^{x/2}} \sum_{m=-N}^{N} \frac{(pq)_m}{(pq^2)_m} q^{-m(m+1)/2}$ |
| $q$-Krawtchouk: $K_n(q^{-x}; p, N; q)$ | $(q^{-x}; q)_n \frac{(p)_n}{(q^2)^{x/2}} \sum_{m=-N}^{N} \frac{(pq)_m}{(pq^2)_m} q^{-m(m+1)/2}$ |
| Affine $q$-Krawtchouk                 | $(q^{-x}; q)_n \frac{(p)_n}{(q^2)^{x/2}} \sum_{m=-N}^{N} \frac{(pq)_m}{(pq^2)_m} q^{-m(m+1)/2}$ |
| Dual $q$-Krawtchouk                   | $(q^{-x}, cq^{N}; q)_n \frac{(p)_n}{(q^2)^{x/2}} \sum_{m=-N}^{N} \frac{(pq)_m}{(pq^2)_m} q^{-m(m+1)/2}$ |
| Continuous Big $q$-Hermite: $H_n(x/aq)$ | $(ae^{\theta}, ae^{-\theta}; q)_n \frac{(-a)_n}{(q^2)^{x/2}} \sum_{m=0}^{n} \frac{(a)_m (e^{-\theta})_m}{(q^2)^{x/2}} q^{-m(m+1)/2}$ |
| Continuous $q$-Laguerre: $P^{(n)}_n(x/q)$ | $\left( \frac{q^2 + q^{-1}}{q^2 + q^{-1}} x^{q^{n+1}} \frac{e^{\theta}}{q^{-x}; q}_n \frac{(-1)^m}{q^{m(m-1)/2}} \frac{(x^{q^{-x}}; q)_n}{(x^{q^{-x}}; q)_n} \sum_{m=0}^{n} \frac{(a)_m (b)_m}{(q^2)^{x/2}} q^{-m(m+1)/2} x^m$ |
| Little $q$-Laguerre/Wall: $P_n(x/a/q)$ | $(aq; q)_n \frac{(-a)_n}{(q^2)^{x/2}} \sum_{m=0}^{n} \frac{(a)_m (b)_m}{(q^2)^{x/2}} q^{-m(m+1)/2} x^m$ |
| $q$-Laguerre: $L^{(n)}_n(x/q)$       | $x^n \frac{q^{-n(n+1)/2} \frac{(q^2)^{x/2}}{q^{-n(n+1)/2}} \sum_{m=0}^{n} \frac{(a)_m (b)_m}{(q^2)^{x/2}} q^{-m(m+1)/2}}{x^m}$ |
| Alternative $q$-Charlier: $K_n(x/a, q)$ | $x^n \frac{(-1)^n}{q^{m(m-1)/2}} \frac{(-aq^{2m+1}; q)_m}{(-aq^{2m+1}; q)_n} \sum_{m=0}^{n} \frac{(a)_m (b)_m}{(q^2)^{x/2}} q^{-m(m+1)/2} x^m$ |
| $q$-Charlier: $C_n(q^{-x}; a; q)$    | $(q^{-x}; q)_n \frac{(-1)^n a^n}{q^{m(m-1)/2}} \frac{(q^2)^{x/2}}{q^{-n(n+1)/2}} \sum_{m=0}^{n} \frac{(a)_m (b)_m}{(q^2)^{x/2}} q^{-m(m+1)/2} x^m$ |
| Al-Salam-Carlitz I: $U^{(n)}_n(x/q)$ | $x^n (x^{-1}; q)_n \frac{(-1)^n}{q^{m(m-1)/2}} \frac{(-aq^{2m+1}; q)_n}{(-aq^{2m+1}; q)_n} \sum_{m=0}^{n} \frac{(a)_m (b)_m}{(q^2)^{x/2}} q^{-m(m+1)/2} x^m$ |
| Al-Salam-Carlitz II: $V^{(n)}_n(x/q)$ | $(x; q)_n \frac{(-1)^n}{q^{m(m-1)/2}} \frac{(-aq^{2m+1}; q)_n}{(-aq^{2m+1}; q)_n} \sum_{m=0}^{n} \frac{(a)_m (b)_m}{(q^2)^{x/2}} q^{-m(m+1)/2} x^m$ |
Continuous $q$-Hermite: $H_n(x/q) \quad \frac{e^{-\Im \theta}}{q^{-\frac{m}{2}}q^{-\frac{m-1}{2}}q^{-\frac{n}{2}}q^{-\frac{n-1}{2}}}$

Stieltjes-Wigert: $S_n(x/q) \quad x^n \quad (q; q)_{m-1} q^{-\frac{m-1}{2}} q^{-\frac{n-1}{2}}$

Discrete $q$-Hermite I: $h_n(x;q) \quad x^n (x^{-1}; q)_n \quad (-1)^{m+n} q^{-\frac{m}{2}} q^{-\frac{n}{2}}$

Discrete $q$-Hermite II: $\tilde{h}_n(x;q) \quad (ix; q)_n \quad (-i)^m q^{n(n-1)/2} q^{-\frac{m}{2}} q^{-\frac{n}{2}}$

| Polynomials sets $\{P_n\}_{n \geq 0}$ | Connection coefficients $C_m(n)$ |
|----------------------------------------|----------------------------------|
| Askey-Wilson: $P_n(x; a, b, c, d/q) \rightarrow P_m(x; a, b, \gamma, \delta/q)$ | $q^{m(n-m)}(q; q)_n (ab, ac, ad; q)_m (abc, a\beta, a\gamma, a\delta; q)_m$ |
| $\times \phi_3$ | $a^{m-n} q^{m-m-n} (ab, ac, ad; q)_m (abc, a\beta, a\gamma, a\delta; q)_m$ |

$q$-Racah:

| $R_n(\mu(x); \alpha, \beta, \gamma, \delta/q) \rightarrow R_m(\mu(x); a, b, c, d/q)$ | $q^{m(m-n)}(q; q)_n (\alpha, \beta, \gamma, \delta, \mu; q)_m$ |
| $\times \phi_4$ | $q^{m(m-n)}(q; q)_n (\alpha, \beta, \gamma, \delta; \mu; q)_m$ |

provided that $\gamma \delta = cd$

Continuous dual $q$-Hahn:

| $P_n(x; a; b, c/q) \rightarrow P_m(x; a, \beta, \gamma/q)$ | $q^{m(m-n)}(q; q)_n (ab, ac; q)_m q^{m-n} a^{m-n} e^{(m-n)}$ |
| $\times \phi_3$ | $q^{m(m-n)}(q; q)_n (ab, ac; q)_m q^{m-n} a^{m-n} e^{(m-n)}$ |

Continuous $q$-Hahn:

| $P_n(x; a, b, c, d/q) \rightarrow P_m(x; a, b, \gamma, \delta/q)$ | $q^{m(m-n)}(q; q)_n (ab, ac, ad; q)_m q^{m(n-m)}$ |
| $\times \phi_4$ | $q^{m(m-n)}(q; q)_n (ab, ac, ad; q)_m q^{m(n-m)}$ |

Big $q$-Jacobi:

| $P_n(x; a, b, c; q) \rightarrow P_m(x; a, \beta, \gamma; q)$ | $q^{m(m-n)}(q; q)_n (ab, a\beta; q)_m q^{m-n} a^{m-n} e^{(m-n)}$ |
| $\times \phi_3$ | $q^{m(m-n)}(q; q)_n (ab, a\beta; q)_m q^{m-n} a^{m-n} e^{(m-n)}$ |

$q$-Hahn:

| $Q_n(q^{-x}; \alpha, \beta, N/q) \rightarrow Q_m(q^{-x}; \alpha_1, \beta_1, N_1/q)$ | $q^{m(m-n)}(q; q)_n (\alpha, \beta; q)_m q^{m-n} e^{(m-n)}$ |
| $\times \phi_3$ | $q^{m(m-n)}(q; q)_n (\alpha, \beta; q)_m q^{m-n} e^{(m-n)}$ |

Table 2: connection coefficients in $q$-Askey scheme
Dual $q$-Hahn:
\[ R_n(\mu(x); \gamma, \delta, N/q) \rightarrow R_n(\mu(x); \gamma_1, \delta_1, N_1/q) \]
\[
\times_3 \phi_2 \left( \frac{q^{m(m-n)}(q; q)_n (\gamma q, q^{-N_1}; q)_m}{(q; q)_m (q; q)_{n-m} (\gamma q, q^{-N_1}; q)_m}; q; q \right)
\]

Al-Salam Chihara:
\[ Q_n(a, b/q) \rightarrow Q_m(x; a/b; q) \]
\[
\times_3 \phi_2 \left( \frac{q^{m(m-n)}(q; q)_n (a q^{-1}, b q^{-1}; q)_m}{(q; q)_m (q; q)_{n-m} (a q^{-1}, b q^{-1}; q)_m}; q; q \right)
\]

Continuous $q$-Jacobi
\[ P_n^{(\alpha, \beta)}(x/q) \rightarrow P_m^{(\alpha, \beta_1)}(x/q) \]
with $n = \alpha + \beta + 1$, $\lambda = \alpha + \beta_1 + 1$
\[
\times_3 \phi_2 \left( \frac{q^{m(m-n)}(q; q)_n (\alpha q, \beta q^{-1}; q)_m}{(q; q)_m (q; q)_{n-m} (\alpha q, \beta q^{-1}; q)_m}; q; q \right)
\]

Big $q$-Laguerre
\[ P_n(x; a, b/q) \rightarrow P_m(x; a/b; q) \]
\[
\times_3 \phi_2 \left( \frac{q^{m(m-n)}(q; q)_n (a q, b q^{-1}; q)_m}{(q; q)_m (q; q)_{n-m} (a q, b q^{-1}; q)_m}; q; q \right)
\]

Little $q$-Jacobi
\[ P_n(x; a, b/q) \rightarrow P_m(x; a/b; q) \]
\[
\times_3 \phi_2 \left( \frac{(q; q)_n (\alpha q, \beta q^{-1}; q)_m}{(q; q)_m (q; q)_{n-m} (\alpha q, \beta q^{-1}; q)_m}; q; q \right)
\]

$q$-Meixner
\[ M_n(q^{-x}; b, c; q) \rightarrow M_m(q^{-x}; \beta, \gamma; q) \]
\[
\times_2 \phi_1 \left( \frac{(q; q)_n (q^{-N}; q)_m}{(q; q)_m (q; q)_{n-m} (q^{-N}; q)_m}; q; q \right)
\]

Quantum $q$-Krawtchouk
\[ K_n(q^{-x}; p, -N; q) \rightarrow K_m(q^{-x}; p_1, -N_1; q) \]
\[
\times_3 \phi_2 \left( \frac{(q; q)_n (-p q^{-N}; q)_m}{(q; q)_m (q; q)_{n-m} (-p q^{-N}; q)_m}; q; q \right)
\]

$q$-Krawtchouk
\[ K_n(q^{-x}; p, N; q) \rightarrow K_m(q^{-x}; p_1, N_1; q) \]
\[
\times_3 \phi_2 \left( \frac{q^{m(m-n)}(-p q^{-N}; q)_n (\alpha q, q^{-N}; q)_m}{(q; q)_m (q; q)_{n-m} (-p q^{-N}; q)_m}; q; q \right)
\]

Little $q$-Laguerre/Wall
\[ P_n(x; a/q) \rightarrow P_m(x; \alpha/q) \]
\[
\times_2 \phi_1 \left( \frac{(q; q)_n (\alpha q; q)_m}{(q; q)_m (q; q)_{n-m} (\alpha q; q)_m}; q; q \right)
\]

$q$-Laguerre
\[ L_n^{(\alpha)}(x; q) \rightarrow L_m^{(\beta)}(x; q) \]
\[
\times_2 \phi_1 \left( \frac{(q; q)_n}{(q; q)_m}; q; q \right)
\]
Alternative $q$-Charlier

\[ K_n(x; a; q) \rightarrow P_m(x; a; q) \]

\[ \frac{(q^2 q)_n q^m(m-n)^n(-aq^n q)_m}{(q^m q)_n q^m(-aq^n q)_m} \times \frac{(1+aq^{m-n})}{(1+a)} \frac{\alpha}{a} \frac{m-n+1}{2} q^{m-n+1} q_n \]

$q$-Charlier

\[ C_n(q^{-x}; a; q) \rightarrow C_m(q^{-x}, \alpha; q) \]

\[ \frac{(q^2 q)_n}{(q^m q)_n q^m(-aq^{n-m})} \times \frac{\alpha}{a} \frac{m-n+1}{2} q^{m-n+1} q_n \]

Al-Salam Carlitz I

\[ U_n^{(a)}(x; q) \rightarrow U_m^{(a)}(x; q) \]

\[ \frac{(q^a q)_n}{(q^a q)_n q^m(-aq^{n-m})} \times \frac{\alpha}{a} \frac{m-n+1}{2} q^{m-n+1} q_n \]

Al-Salam Carlitz II

\[ V_n^{(a)}(x; q) \rightarrow V_m^{(a)}(x; q) \]

\[ \frac{(q^a q)_n}{(q^a q)_n q^m(-aq^{n-m})} \times \frac{\alpha}{a} \frac{2^{m-n}}{2} q^{2^{m-n}} q_n \]

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