A simple method of determining the Hubble constant

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Abstract

Bidirectional relativistic proper motions of radio components of nearby extragalactic sources give a strong constraint on the determination of the Hubble constant $H_0$. Under the assumption that the real velocity of radio components of extragalactic sources is not less than that of Galactic sources, the value of $H_0$ can be estimated at a high level of accuracy. The assumption is reasonable due to the general belief that the activity in the core of galaxies must be more powerful than that of stars. This method is simple and with only one uncertainty — the real velocity of components. This uncertainty is related to the value of the real velocity of components of Galactic sources and the latter is always well-determined (note that the determination is independent of $H_0$ and the distance of Galactic sources can be directly measured at a rather high level of accuracy). Hopefully the method will play an important role in future research to fix the value of $H_0$. With the data of the three sources available so far and the assumption that the real velocity of components of at least one of the sources is not less than a known velocity of components of a Galactic source, the constant is estimated to be within $27.08\,\text{km}\,\text{s}^{-1}\text{Mpc}^{-1} < H_0 \leq 53.15\,\text{km}\,\text{s}^{-1}\text{Mpc}^{-1}$ with this method.

The real value of the Hubble constant has been a hot topic for a long time. In the past few years, some advances have been achieved (for more details, see Trimble and Aschwanden 1999, Trimble and McFadden 1998). An exciting result of observation by HST led to $H_0 = 80 \pm 17\,\text{km}\,\text{s}^{-1}\text{Mpc}^{-1}$

\textsuperscript{1}This essay received an “honorable mention” in the 1999 Essay Competition of the Gravity Research Foundation.
(Freedman et al. 1994). However, with the same data, some people got a much smaller value, e.
g., \(H_0 = 40 \text{kms}^{-1}\text{Mpc}^{-1}\) (Sandage et al. 1994). Among the many methods, that taking the peak
brightness of Type Ia supernovae as a distance indicator is generally used. It gave a small value of
the constant, \(H_0 = 61 \pm 10 \text{kms}^{-1}\text{Mpc}^{-1}\), by Brach (1992).

Bidirectional relativistic motions of extragalactic radio sources can be used to estimate the
constant (Marscher and Broderick 1982). Recently there were some sound works using this method
published. The most successful one was done by Taylor et al. (1997). However, the method they
used is somewhat complicated. And it does not tell how the uncertainties of the measurements
used affect the estimation of \(H_0\), and what one should do when the data of several sources are
available.

In the following, we illustrate a simple method of determining the Hubble constant using
bidirectional relativistic proper motions of extragalactic radio sources.

The apparent transverse velocities of components of a source along an axis at an angle \(\theta\) to the
line of sight at a velocity \(\beta c\) can be expressed as (Rees 1966, 1967)

\[
(\beta_{\text{app}})_a = \frac{\mu_a D_L}{c(1+z)} = \frac{\beta \sin \theta}{1 - \beta \cos \theta},
\]

\[
(\beta_{\text{app}})_r = \frac{\mu_r D_L}{c(1+z)} = \frac{\beta \sin \theta}{1 + \beta \cos \theta},
\]

where \(a\) and \(r\) stand for the motions of the approaching and receding components, with \(\mu_a\) and \(\mu_r\)
being the corresponding proper motions, respectively.

These two equations give

\[
\frac{D_L}{c(1+z)} = \frac{1}{2\mu_a \mu_r} \sqrt{\beta^2 (\mu_a + \mu_r)^2 - (\mu_a - \mu_r)^2}.
\]

For small redshift sources, the following is maintained for all kinds of the universe within the
framework of the FRW cosmology,

\[
\frac{D_L}{1+z} \simeq \frac{cz}{H_0},
\]

where \(H_0\) is the Hubble constant of the universe.

From Equation (3) one has

\[
H_0 \simeq \frac{2\mu_a \mu_r z}{\sqrt{\beta^2 (\mu_a + \mu_r)^2 - (\mu_a - \mu_r)^2}}.
\]
$\beta$ is known, the Hubble constant would be well determined. This relation provides a very strong constrain on the determination of $H_0$.

From Equation (5), the law of $\beta < 1$ leads to

$$H_0 > z \sqrt{\mu_a \mu_r}$$

for any sources. Therefore, among many values of the lower limit of $H_0$, calculated from various sources, the largest one would be the closest value to the limit. If $\mu_a$ and $\mu_r$ are presented in the form $\mu \pm \Delta \mu$ for a source, then the lower limit of $H_0$ determined by the source should be

$$H_{0, \min} = z \sqrt{(\mu_a - \Delta \mu_a)(\mu_r - \Delta \mu_r)} ,$$

with $H_{0, \min}$ satisfying

$$H_{0, \min} < H_0. \quad (7)$$

In this way, the largest value of $H_{0, \min}$ among those calculated from various sources should be taken as the best estimation of the lower limit of $H_0$.

Let

$$\alpha \equiv 1 - \beta. \quad (9)$$

Since $0 \leq \beta < 1$, then $0 < \alpha \leq 1$. For a small value of $\alpha$, Equation (5) gives

$$H_0 \simeq z \sqrt{\mu_a \mu_r}[1 + \frac{(\mu_a + \mu_r)^2}{4 \mu_a \mu_r} \alpha]. \quad (10)$$

Considering the case where $\alpha$ is known to the extent of $\alpha \leq \alpha_{\max}$ for a source, the upper limit of the Hubble constant would be determined by the source in the way

$$H_0 \leq H_{0, \max}, \quad (11)$$

where

$$H_{0, \max} = z \sqrt{(\mu_a + \Delta \mu_a)(\mu_r + \Delta \mu_r)}[1 + \frac{(\mu_a + \Delta \mu_a + \mu_r + \Delta \mu_r)^2}{4(\mu_a + \Delta \mu_a)(\mu_r + \Delta \mu_r)} \alpha_{\max}]. \quad (12)$$

In determination of the range of $H_0$, there is a reasonable demand that the ranges of $H_0$ estimated from different sources should be overlapped. This demand is consistent with the above principle of choosing the lower limit of $H_0$. When determining the upper limit of $H_0$ from (12), the requirement can be realized by adopting various values of $\alpha_{\max}$ for different sources.

If among these sources, there is at least one source satisfying $\alpha \leq \alpha_{\max}$ for a given $\alpha_{\max}$, the largest value of $H_{0, \max}$ calculated with this value of $\alpha_{\max}$ for all the sources should be taken as the
best estimation of the upper limit of $H_0$. In this way, while the given value of $\alpha_{\text{max}}$ is assigned to the source of the largest value of $H_{0,\text{max}}$, some larger values of $\alpha_{\text{max}}$ should be assigned to other sources, so that the estimated ranges of $H_0$ may be overlapped.

Since the value of $\alpha_{\text{max}}$ for Galactic sources can be calculated at a rather high level of accuracy, that for extragalactic sources can then be well settled by assuming that it would not be less than that for Galactic sources. This assumption is reasonable due to the general belief that the activity in the core of galaxies must be more powerful than that of stars.

Recently, several Galactic sources were found to have bidirectional relativistic proper motions of radio components. Among them, the largest and well calculated value of $\beta$ is $0.92 \pm 0.02$ for GRO J1655-40 (Hjellming and Rupen 1995). This corresponds to $\beta_{\text{min}} = 0.9$ and $\alpha_{\text{max}} = 0.1$ for the source. Therefore, at present, it is reasonable to take $\alpha_{\text{max}} = 0.1$ for extragalactic sources according to the assumption.

Till now, there are only a few extragalactic sources with measured values of $\mu_a$ and $\mu_r$ found in literature. Excluding those with high redshifts or uncertain values of proper motions, there are only three sources suitable for our study. They are: 1146+596 (NGC 3894), $z = 0.01085$, $\mu_a = 0.126 \pm 0.05 \text{mas yr}^{-1}$ and $\mu_r = 0.19 \pm 0.05 \text{mas yr}^{-1}$ (Taylor et al. 1998); 0316+413 (3C 84), $z = 0.0172$, $\mu_a = 0.58 \pm 0.12 \text{mas yr}^{-1}$ and $\mu_r \leq 0.28 \text{mas yr}^{-1}$ (Marr et al. 1989, Vermeulen et al. 1994); 1946+708, $z = 0.101$, $\mu_a = 0.117 \pm 0.020 \text{mas yr}^{-1}$ and $\mu_r = 0.053 \pm 0.020 \text{mas yr}^{-1}$ (Taylor and Vermeulen 1997).

For the lower limit of $H_0$, the first and the third sources give $H_{0,\text{min}} = 8.82 \text{km s}^{-1} \text{Mpc}^{-1}$ and $27.08 \text{km s}^{-1} \text{Mpc}^{-1}$, respectively from (7), while the second source gives no lower limit of $H_0$. According to the above principle of choosing $H_{0,\text{min}}$, the best estimation of the lower limit of $H_0$ from these data should be $H_{0,\text{min}} = 27.08 \text{km s}^{-1} \text{Mpc}^{-1}$.

For the given value of $\alpha_{\text{max}} = 0.1$, the three sources give $H_{0,\text{max}} = 15.45 \text{km s}^{-1} \text{Mpc}^{-1}$, $40.51 \text{km s}^{-1} \text{Mpc}^{-1}$, and $53.15 \text{km s}^{-1} \text{Mpc}^{-1}$, respectively from (12). Assuming that there is at least one source satisfying $\alpha \leq \alpha_{\text{max}}$ for $\alpha_{\text{max}} = 0.1$, then according to the above requirement we choose $H_{0,\text{max}} = 53.15 \text{km s}^{-1} \text{Mpc}^{-1}$ as the best estimation of the upper limit of $H_0$.

Therefore, we obtain the range of $27.08 \text{km s}^{-1} \text{Mpc}^{-1} < H_0 \leq 53.15 \text{km s}^{-1} \text{Mpc}^{-1}$ for the Hubble constant from the data of the three sources.

In practice, the observable bidirectional relativistic proper motions of radio components of a
source are always those moving almost perpendicular to the line of sight, and then the values of their \( \mu_a \) and \( \mu_r \) are close (see, e.g., Taylor et al. 1998). Therefore \( \frac{(\mu_a + \Delta \mu_a + \mu_r + \Delta \mu_r)^2}{4(\mu_a + \Delta \mu_a)(\mu_r + \Delta \mu_r)} \approx 1 \) and \( \frac{(\mu_a + \Delta \mu_a + \mu_r + \Delta \mu_r)^2}{4(\mu_a + \Delta \mu_a)(\mu_r + \Delta \mu_r)} \alpha_{\text{max}} \approx 0.1 \) for \( \alpha_{\text{max}} = 0.1 \). It shows that taking \( \alpha_{\text{max}} = 0.1 \) will only produce about 10% uncertainty for \( H_{0,\text{max}} \). When more such sources have been observed, the expected value of \( \alpha_{\text{max}} \) and the corresponding uncertainty will be smaller.

This method depends on only one assumption and it concerns only one uncertainty — the real velocity of components. This uncertainty concerns the value of the real velocity of components of Galactic sources and the latter is always well determined (note that the determination is independent of \( H_0 \) and the distance of Galactic sources can be directly measured at a rather high level of accuracy). The assumption is weak due to the general belief that the activity in the core of galaxies must be more powerful than that of stars. Also, the method is simple. To determine \( H_0 \) at a high level, one only needs to measure \( \mu_a \) and \( \mu_r \) at a high level of accuracy and finds a sufficient number of such sources (say, 10 or more). The method is then hopeful to play an important role for finally fixing the value of the Hubble constant in future researches.

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