Series expansions of the density of states in SU(2) lattice gauge theory

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We calculate numerically the density of states \( n(S) \) for SU(2) lattice gauge theory on \( L^4 \) lattices. Small volume dependence are resolved for small values of \( S \). We compare \( \ln(n(S)) \) with weak and strong coupling expansions. Intermediate order expansions show a good overlap for values of \( S \) corresponding to the crossover. We relate the convergence of these expansions to those of the average plaquette. We show that when known logarithmic singularities are subtracted from \( \ln(n(S)) \), expansions in Legendre polynomials appear to converge and could be suitable to determine the Fisher’s zeros of the partition function.

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I. INTRODUCTION

Quantum Chromodynamics is a widely accepted theory of strong interactions. From a theoretical point of view, understanding the large distance behavior in terms of the weakly coupled short distance theory has been an important challenge. The connection between the two regimes can be addressed meaningfully using the lattice formulation. In the pure gauge theory (no quarks) described with the standard Wilson’s action, no phase transition between the weak and strong coupling regime has been found numerically for SU(2) or SU(3) and the theory should be in the confining phase for all values of the coupling. Recently, convincing arguments have been given \[1, 2\] in favor of the smoothness of the renormalization group flows between the two fixed points of interest, putting the confining picture on more solid mathematical ground.

The absence of phase transition discussed above suggests that it is possible to match the weak coupling and the strong coupling expansions of the lattice formulation. However, if we consider these two expansions, for instance for the average SU(2) plaquette as a function of \( \beta = 4/g^2 \), we see in Fig. 1 that there is a crossover region (approximately \( 1.5 < \beta < 2.5 \)) where none of the two expansions seem to work. This behavior is probably related to singularities in the complex \( \beta \) plane \[3, 4\] that are not completely understood. In the case of the one plaquette model \[5\], taking the inverse Laplace transform with respect to \( \beta \) (Borel transform) of the partition function yields a function that has better convergence properties. It would be interesting to know if this feature persists on \( V = L^4 \) lattices.

In this article, we study expansions of the inverse Laplace transform of the partition function (the density of states) of SU(2) lattice gauge theory on symmetric 4 dimensional lattices. The density of states is denoted \( n(S) \) and defined precisely in Sec. II. It gives a relative measure of the number of ways to get a value \( S \) of the action. Knowing \( n(S) \), we can calculate the partition function and its derivatives for any real or complex value of \( \beta \). In particular, it could be used to determine the Fisher’s zeros of the partition function \[6, 7, 8\]. The choice of SU(2) is motivated by the existence of a particular symmetry \[9\] which allows to determine the behavior of \( n(S) \) near its maximal argument without extra...

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Fig. 1: Weak and strong coupling expansions of the average plaquette \( P \) for SU(2) at various orders in the weak and strong coupling expansion compared to the numerical values.
calculation. In Sec. III we explain why \( \ln(n(S)) \) is expected to scale like the volume and can be interpreted as a "color entropy". Numerical calculations of \( n(S) \) obtained by patching plaquette distributions multiplied by the inverse Boltzmann weight at values of \( \beta \) increasing by a small increment are presented in Sec. IV. The article is focused on comparisons with numerical data on a 6\( ^4 \) lattice where finite volume effects are not too large and isplaqettes. Intermediate orders in these expansions show a good

We then show that the convergence of the new series

The numerical results are compared with expansions that can be obtained from the strong (Sec. V) and weak (Sec. VI) coupling expansions of the average plaquette. Intermediate orders in these expansions show a good overlap for values of \( S \) that correspond to the crossover. We then show that the convergence of the new series can be related empirically to those of the series for the average plaquette. The weak coupling expansion determines the logarithmic singularities of \( \ln(n(S)) \) at both boundaries. When these singularities are subtracted we obtain a bell-shaped function that can be approximated very well by Legendre polynomials (Sec. VII). We conclude with possible applications for the calculations of the Fisher’s zeros and open problems.

II. THE DENSITY OF STATES

We consider the standard pure gauge partition function

with the Wilson action

and \( \beta \equiv 2N/g^2 \). We use a \( D \) dimensional cubic lattice with periodic boundary conditions. For a symmetric lattice with \( L^D \) sites, the number of plaquettes is

In the following, we restrict the discussion to the group \( SU(2) \) and \( D = 4 \). For \( SU(2) \), one can show [4] that the maximal value of \( S \) is \( 2N_p \). We define the average plaquette:

Inserting 1 as the integral of delta function over the numerical values \( S \) of \( \overline{S} \) in \( Z \), we can write

with

We call \( n(S) \) the density of states. A more general discussion for spin models [10] or gauge theories [11] can be found in the literature where the density of states is sometimes called the spectral density. From its definition, it is clear that \( n(S) \) is positive. Assuming that the Haar measure for the links is normalized to 1, the partition function at \( \beta = 0 \) is 1 and consequently we can normalize \( n(S) \) as a probability density.

A first idea regarding the convergence properties of various expansions can be obtained from the single plaquette model [8]. In that case, we have

The large \( \beta \) behavior of the partition function is determined by the behavior of \( n(S) \) near \( S = 0 \). In this example, \( n(S) \sim \sqrt{S} \) for small \( S \), implies that \( Z \sim \beta^{-3/2} \) at leading order. Successive subleading corrections can be calculated by expanding the remaining factor \( \sqrt{2-S} \) in powers of \( S \) and integrating over \( S \) from 0 to \( \infty \). If we factor out the leading behavior, we obtain a power series in \( 1/\beta \). The large order behavior of this power series is determined by the large order behavior of the expansion of \( \sqrt{2-S} \), itself dictated by the branch cut at \( S = 2 \). One can see [5] that the \( S \)-integration over the whole positive real axis converts an expansion with a finite radius of convergence into one with a zero radius of convergence. On the other hand, if the \( S \)-integration is carried over the interval [0, 2], the resulting series converges but the coefficients need to be expressed in terms of the incomplete gamma function. From this example, one may believe that it is easier to approximate \( n(S) \) than the corresponding partition function. However, it is not clear that these considerations will survive the infinite volume limit. Note also that the behavior of \( n(S) \) near \( S = 2 \) can be probed by taking \( \beta \to -\infty \) in agreement with the common wisdom that the large order behavior of weak coupling series can be understood in terms of the behavior at small negative coupling.

It was showed [8] that if the lattice has even number of sites in each direction and if the gauge group contains \(-I\), that it is possible to change \( \beta \text{ReTr} U_p \) into \(-\beta \text{ReTr} U_p \) by a change of variables \( U_1 \to -U_1 \) on a set of links such that for any plaquette, exactly one link of the set belongs to that plaquette. This implies

This symmetry implies that

In the following, we will be working exclusively with \( SU(2) \) which contains \(-I\) and lattices with even numbers of sites in every direction. We will thus assume that Eq. (9) is satisfied and we only need to know \( n(S) \) for \( 0 \leq S \leq N_p \).
III. VOLUME DEPENDENCE

In this section, we discuss the volume dependence of the density of state. We make this dependence explicit by writing \( n(S, N_p) \). Given the density of states, we can always write

\[
f(x, N_p) \equiv \ln(n(xN_p, N_p))/N_p .
\]

The function is nonzero only if \( 0 \leq x \leq 2 \). The symmetry implies that

\[
f(x, N_p) = f(2 - x, N_p)
\]

In the statistical mechanics interpretation of the partition function (where \( \beta \) is an inverse temperature), \( f(x, N_p) \) can be interpreted as a density of entropy. The existence of the infinite volume limit requires that

\[
\lim_{N_p \to \infty} f(x, N_p) = f(x) ,
\]

with \( f(x) \) volume independent. In the same limit, the integral \( \int f(x) \) can be evaluated by the saddle point method. The maximization of the integrand requires

\[
f'(x) = \beta .
\]

We believe that \( f(x) \) is strictly increasing for \( 0 < x < 1 \) with an absolute maximum at \( x = 1 \). By symmetry, this would imply that \( f(x) \) is strictly decreasing for \( 1 < x < 2 \). We also believe that \( f'(x) \) is strictly decreasing and that Eq. \( \text{13} \) has a unique solution (with positive \( \beta \) if \( 0 < x < 1 \) and negative \( \beta \) if \( 1 < x < 2 \)).

The numerical study of Sec. \( \text{IV} \) is in agreement with these statements, but we are not aware of mathematical proofs. Assuming that Eq. \( \text{13} \) has a unique solution, the infinite volume solution should be \( x = P \) the average plaquette defined above. We can then convert an expansion for \( P \) into an expansion of \( f \). If we want to include the volume dependence, the distribution has a finite width, and we should expand about the saddle point and perform the integration. In the following, we will work at large but finite volume and residual volume dependence in \( f \) will be kept implicit in equations.

The behavior of \( f(x) \) for small \( x \), can be probed by studying the model at large positive \( \beta \) (weak coupling expansion discussed in Sec. \( \text{VII} \)). On the other hand, at small values of \( \beta \) (strong coupling expansion discussed in Sec. \( \text{V} \)), the partition function is dominated by the behavior of \( f(x) \) near its peak value \( x = 1 \). For convenience, we introduce notations suitable for the study of the density of state near \( x = 1 \)

\[
g(y) \equiv f(1 + y) .
\]

\( g(y) \) is then an even function defined for \(-1 < y < 1 \).

IV. NUMERICAL CALCULATION OF \( n(S) \)

To find \( n(S) \) numerically we will use a Monte Carlo simulation to create configurations of SU(2) for different values of \( \beta \). In the following example we will follow the steps we will use to find \( n(S) \) for a volume of \( 6^4 \). We will start with 550 different sets of data ranging from \( \beta = 0.02 \) to \( \beta = 11.00 \) in steps of 0.02 and with sizes of \( 10^5 \) configurations. To join the data from different values of \( \beta \) we will first create histograms of each set of data, each of these histograms is roughly Gaussian in shape. We then filter out the data that has statistics that are lower than half of the maximum bin. We can then remove the beta dependence by multiplying the height of each bin by \( e^{\beta S} \). We will be left with a series of arches which when overlayed on each other form the curve \( n(s) \). To create this overlay we will start with the lowest \( \beta \), which will correspond to the peak of \( n(s) \), and then take the logarithm of this. We will then look at the neighboring \( \beta \) and do the same thing but then shifting it up or down so that the average distance in the bins overlapping with the first is zero. We will then continue in this manner until the supply of datasets has been exhausted. A portion of this process can be seen in Fig. \( 2 \). We then average the points for each bin together and divide both the bin width and height by \( N_p \) and shift the top of the curve to zero to make the final output, which can be seen in Fig. \( 3 \) for both \( 4^4 \) and \( 6^4 \). We see that they overlap well.

We now consider the difference between two different volumes, as shown in Fig. \( 4 \) We can see in Fig. \( 5 \) that as we get closer to \( S/N_p = 1 \) this difference turns into noise, and as we get closer to \( S/N_p = 0 \) we see a volume dependence growing. The results reported here correspond to the difference between \( 6^4 \) and \( 4^4 \). We have also studied the difference between \( 8^4 \) and \( 6^4 \) and found consistent results. Calculation at larger volumes are much more computationally expensive and require many more sets of data because of the narrow width of the distributions.
SU(2) Patching Results

FIG. 3: Results of patching for $4^4$ and $6^4$.

SU(2) Diff $4^4$ and $6^4$

FIG. 4: The difference between $\ln(n(S))/N_p$ for $4^4$ and $6^4$.

SU(2) Diff $4^4$ and $6^4$

FIG. 5: The difference between $\ln(n(S))/N_p$ for $4^4$ and $6^4$ with $\beta > 0.5$ (close up).

SU(2) Diff $4^4$ and $6^4$

FIG. 6: The difference between $\ln(n(S))/N_p$ for $4^4$ and $6^4$ with $\beta < 0.5$ (close-up).

SU(2) Diff $(4^4-6^4)/\ln S/N_p$

FIG. 7: The difference between $\ln(n(S))/N_p$ for $4^4$ and $6^4$ divided by $\ln(S/N_p)$.

SU(2) Diff $(4^4-6^4)/\ln S/N_p$

V. STRONG COUPLING EXPANSION

In this section, we discuss the strong coupling expansion of the logarithm of the density of state. We will work with the shifted function $g(y)$ defined in Eq. (14). The strong coupling expansion of $P$ can be extracted from the expansion of $\ln Z$ given in Ref. [12, 13] using appropriate rescalings (for instance the $\beta$ used there is one half of the $\beta$ used here). The expansion is of the form

$$P(\beta) \simeq 1 + \sum_{m=1} a_{2m-1} \beta^{2m-1}. \quad (15)$$

The values of the coefficients are given in Table I.

With periodic boundary conditions, the low order coefficients are volume independent. This can be understood from the exact translation invariance for the low order strong coupling graphs that provides a multiplicity that cancels exactly the $1/N_p$ in Eq. (14). Volume dependence may appear for graphs wrapping around the torus. The
TABLE I: Strong coupling expansion coefficients defined in the text.

| m  | $a_{2m-1}$ | $g_{2m}$ | $h_{2m}$ |
|----|------------|----------|----------|
| 1  | $-\frac{1}{4}$ | $-2$     | $-\frac{1}{4}$ |
| 2  | $\frac{1}{96}$ | $-\frac{3}{4}$ | $-\frac{7}{72}$ |
| 3  | $-\frac{7}{12}$ | $\frac{20}{9}$ | $\frac{1}{2}$ |
| 4  | $\frac{21}{16384}$ | $-\frac{1}{16}$ | $-\frac{1}{2}$ |
| 5  | $-\frac{441}{1677721}$ | $\frac{16816}{2025}$ | $\frac{8100}{256}$ |
| 6  | $\frac{264883}{7845166496}$ | $\frac{8505}{23040}$ | $\frac{66049}{1486356480}$ |
| 7  | $\frac{403651}{6094562907}$ | $\frac{163150033}{1190700}$ | $\frac{1109700}{68491306598400}$ |
| 8  | $\frac{1826017873}{68491306598400}$ | $\frac{255150}{8847360}$ | $\frac{4082400}{96}$ |

The coefficient $g$ is obtained coefficients that are showed in Table I for $A = 3/4$. The coefficients $g_{2m}$ and $h_{2m}$ are also shown on a logarithmic scale in Fig. | at successive orders. For reference, we show the estimated numerical error on $P$.

![FIG. 8: Logarithm of the absolute value of $g_{2m}$ and $h_{2m}$](image)

![FIG. 9: Logarithm of the absolute value of the difference between the numerical data and the strong coupling expansion of $P$ at successive orders.](image)

![FIG. 10: Numerical value of $f(x)$ compared to the strong coupling expansion at successive orders.](image)

The simplest such graph is a straight line that closes into itself due to the periodic boundary condition. It appears at order $\beta^2 L$ and has a reduced translation multiplicity since translation along the graph does not generate a new graph. This type of graphs produce $1/L$ corrections that to the best of our knowledge have not been studied quantitatively. In the following, we will ignore such effects, but a study of the contribution of graphs with a nontrivial topology would certainly be interesting.

We will plug the expansion of $P$ in the expansion

$$g(y) \simeq \sum_{m=0} g_{2m} y^{2m} . \quad (16)$$

At lowest order we have $y \simeq a_1 \beta$ and the saddle point Eq. (11) yields $2g_2 y \simeq 2g_2 a_1 \beta \simeq \beta$ which implies $g_2 = 1/(2 a_1)$. This procedure can be followed order by order in $\beta$. The results are shown in Table I.

Since $n(S)$ is zero for $S = 0$ and $2N_F$, we expect logarithmic singularities at $x = 0$ and 2 for $f(x)$ and $y = \pm 1$ for $g(y)$. This singularities will cause the strong coupling series to diverge when $|y| \geq 1$. Consequently, we define the subtracted function

$$h(y) \equiv g(y) - A (\ln(1 - y^2)) . \quad (17)$$

The coefficient $A$ will be calculated using the weak coupling expansion in Sec. VI. In the infinite volume limit, we have $A = 3/4$. Expanding

$$h(y) \simeq \sum_{m=0} h_{2m} y^{2m} , \quad (18)$$

we obtained coefficients that are showed in Table I for $A = 3/4$. The coefficients $g_{2m}$ and $h_{2m}$, are also shown on a logarithmic scale in Fig. 8. This graph shows that the two types of coefficients become rapidly of the same order, which indicates singularities in the complex $y$ plane for smaller values of $|y|$ than the ones at $\pm 1$.

In Fig. 9 we show the error made at successive order of the strong coupling expansion of the plaquette. We then show successive approximation of $f(x)$ (Fig. 10) and the corresponding errors (Fig. 11).
We then assume the behavior \( f \). Using the saddle point Eq. (13), and using the leading convergence. Consequently, it seems like the convergence properties of the two expansions are the same (finite radius of convergence). Similarly, the larger order errors for \( f \) cross for \( x \) between 0.5 and 0.6 which are approximately the values of \( P \) in the \( \beta \) interval of crossing. Consequently, it seems like the convergence properties of the two expansions are the same (finite radius of convergence).

VI. WEAK COUPLING EXPANSION

In this section, we discuss the weak coupling expansion of \( f(x) \). The starting point is the expansion of \( P \) in inverse powers of \( \beta \)

\[
P(\beta) \simeq \sum_{m=1} \frac{b_m \beta^{-m}}{m!}.
\]  

We then assume the behavior

\[
f(x) \simeq A \ln(x) + \sum_{m=0} f_m x^m.
\]

Using the saddle point Eq. (13), and using the leading large \( \beta \) and small \( x \) terms we find

\[
\beta \simeq A/x \simeq A/(b_1/\beta),
\]

which implies \( A = b_1 \) at infinite volume. The procedure can be pursued order by order without difficulty. The result for the two lowest orders is

\[
f_1 = b_2/b_1
\]

\[
f_2 = (b_3b_1 - b_2^2)/(2b_1^2)
\]

Numerical experiments indicate that the two series have the same type of growth (power or factorial). Note that \( f_0 \) cannot be fixed by the saddle point equation. The overall height of \( f \) depends on the behavior near \( x = 1 \) (if we insist on normalizing \( n(S) \) as probability density) and it seems unlikely that it can be found by a weak coupling expansion.

At finite volume, the saddle point calculation of \( P \) should be corrected in order to include \( 1/V \) effects (\( V \) the number of sites, \( L^D \) for a symmetric lattice). If we perform the Gaussian integration of the quadratic fluctuations, and use the \( V \) dependent value of \( b_1 \) given in Eq. (23) below, we find after a short calculation that the coefficient \( A \) of \( \ln(x) \) is

\[
A = (3/4) - (5/12)(1/V).
\]

This leading coefficient correction, predicts a difference of \(-0.0013/ln(x) \) for the difference between \( f(x) \) for a \( 4^4 \) and \( 6^4 \) and is roughly consistent with Fig. 7.

Our next task is to find the values of \( b_m \). A closed form expression can be found \([15,16]\) for \( b_1 \). For the case \( N_c = 2 \) and \( D = 4 \), we obtain

\[
b_1 = (3/4)(1 - 1/(3V)).
\]

The \((-1/(3V)) \) comes from the absence of zero mode \((-1/V) \) in a sum calculated in \([15]\) plus the contribution of the zero mode with periodic boundary conditions \((+2/(3V)) \) calculated in \([16]\). Numerical values for \( b_2 \) can be found in Ref. \([15]\) and for \( b_3 \) in Ref. \([17]\). In these Refs., several sums are calculated numerically at particular volumes that do not include \( 6^4 \). Rough extrapolations from the existing data indicate that for \( V = 6^4 \) uncertainties are less than 0.0002 for \( b_2 \) and 0.0008 for \( b_3 \). For \( \beta \geq 3 \), these effects are close to the numerical errors for \( P \). In the following, we use the approximate values \( b_2 = 0.1511 \) and \( b_3 = 0.1427 \) for \( V = 6^4 \).

We are not aware of any calculation of \( b_m \) for \( m \geq 4 \) for \( SU(2) \). In the case of \( SU(3) \), calculations up to order \( 10 \) \([18]\) and \( 16 \) \([19]\) are available and show remarkable regularities. Using the assumption \([4]\) that \( \partial P/\partial \beta \) has a logarithmic singularity in the complex \( \beta \) plane and integrating, we obtained \([20]\) the approximate form

\[
\sum_{m=1} b_m \beta^{-k} \approx C(Li_2(\beta^{-1}/(\beta^{-1} + i\Gamma))) + h.c.,
\]

with

\[
Li_2(x) = \sum_{k=1} x^k/k^2.
\]

We believe that at zero temperature, the new parameter \( \Gamma \) which measures the (small) distance from the singularity to the real axis in the \( 1/\beta \) plane stabilizes at a nonzero value in the infinite volume limit. For reasons not fully understood, this parametrization of the series turns out to work very well for \( SU(3) \). For instance, by fixing the value of \( \Gamma \) in the middle of the allowed range.
and using the values of $b_9$ and $b_{10}$, we obtain values of
the lower order coefficients with a relative error of 0.2
percent for $b_8$ and that increases up to 5 percent for $b_3$.
In the limit $\Gamma = 0$, the parametrization provides simple
predictions for instance $b_3/b_2 \simeq (4/9)\beta_m$. The location
of the Fisher’s zeros for $SU(2)$ [5] suggests $\beta_m = 2.18$.
This implies $b_3/b_2 \simeq 0.969$ in good agreement with our
numerical estimate $b_3/b_2 \simeq 0.944$. In the following we
use the values $\beta_m = 2.18, \Gamma = 0.18/\beta_m^2 \simeq 0.038$ (see [5])
and we fixed $C = 0.0062$ in order to reproduce $b_3$. The
numerical values of $b_m$ and the corresponding values of
$f_m$ are displayed in Table II.

We have compared the weak coupling expansion of $P$
with numerical values in the case $V = 6^4$. The results
are shown in Fig. 12. In the region where the curves are
smooth, the error decrease with the order and appears to
accumulate. This is very similar to the case of $SU(3)$[20].
However, it is clear that more reliable estimates for $m \geq 4$
would be desirable for $SU(2)$. It should be noted that for
large $\beta$, the noise in the error is at the same level as the
numerical error on $P$. This would not be the case if we
had not included the contribution of the zero mode to $b_1$
as shown in the second part of Fig. 12.

We have compared the weak coupling expansion of $f(x)$
with numerical values in the case $V = 6^4$. The results
are shown in Fig. 13. The differences are resolved
in Fig. 14. In these graphs we have taken $f_0 = -0.14663$
which maximizes the length of the accumulation line on
the left of Fig. 14.

|   | $b_m$   | $f_m$  |
|---|---------|--------|
| 1 | 0.7498  | 0.2015 |
| 2 | 0.1511  | 0.0999 |
| 3 | 0.1427  | 0.0796 |
| 4 | 0.1747  | 0.0791 |
| 5 | 0.2435  | 0.0908 |
| 6 | 0.368   | 0.1156 |
| 7 | 0.5884  | 0.1597 |
| 8 | 0.98    | 0.2351 |
| 9 | 1.6839  | 0.3643 |
|10 | 2.9652  | 0.5883 |
|11 | 5.326   | 0.9828 |
|12 | 9.7234  | 1.6883 |
|13 | 17.995  | 2.9683 |
|14 | 33.690  | 5.3207 |
|15 | 63.702  | 9.6945 |

TABLE II: Weak coupling coefficients defined in Sec. VII. The
choice of $b_1$ corresponds to $V = 6^4$. 

FIG. 12: Logarithm of the absolute value of the difference
between the numerical data and the weak coupling expansion
of $P$ at successive orders (above). For reference, we give the
estimated numerical error on $P$. The graph below is the same
except that we have not included the zero mode in $b_1$.

FIG. 13: Numerical value of $f(x)$ compared to the weak
coupling expansion at successive orders.
VII. EXPANSION IN LEGENDRE POLYNOMIALS

We now consider the function \( h(y) \), which is \( g(y) \) with the logarithmic singularity subtracted as defined in Eq. (17). This is a bell shaped even function defined on the interval \([-1, 1]\) and shown on Fig. 16. We can expand this function in terms of the even Legendre polynomials.

\[
h(y) = \sum_{m=0} q_{2m} P_{2m}(y) \tag{26}
\]

The \( q_{2m} \) can be determined from the orthogonality relations with interpolated values of \( h(y) \) to perform the integral. A minor technical difficulty is that we do not have numerical data all the way down to \( y = -1 \). This is because as \( y \to -1^+ \), or in other words \( x \to 0^+ \), \( \beta \to +\infty \) where the plaquette distribution becomes infinitely narrow. Consequently there is a small gap in the numerical data that needs to be filled. Fortunately, this is precisely where the weak coupling expansion works well. Using the weak coupling expansion (including the overall constant), subtracting \( A \ln(x(2-x)) \) and shifting to the \( y \) coordinate, we obtained the approximate behavior near \( y = -1 \) for the \( 6^4 \) data:

\[
h(y) \simeq 0.2145 + 1.2961y + 0.5261y^2 + 0.1109y^3 \tag{27}
\]

In order to estimate the error associated with this approximation we have compared with an extrapolation of a quadratic fit of the leftmost part of the data. In order to give an idea of the volume effects, we have also used the second method on a \( 4^4 \) lattice. The results are shown in Table III. This indicates that the variations are small, increase with the order in relative magnitude and that the volume effects are stronger than the dependence on the extrapolation procedure. The logarithm of the coefficients is shown in Fig. 15, which illustrate the exponential decay of the coefficients.

The expansion provides excellent approximation of \( h(y) \) shown in Fig. 16. The errors are resolved in Fig. 17. It is also possible to calculate \( P(\beta) \) by solving the saddle point Eq. (13) using successive approximations for \( h \). This is shown in Figs. 18 and 19. The spikes in the error graphs correspond to change of sign of the errors. It is important to notice that the quality of the approximations improves with the order in all region of the interval.

VIII. CONCLUSIONS

We have calculated the density of states for \( SU(2) \) lattice gauge theory. The intermediate orders in weak and strong coupling agree well in an overlapping region of action values as shown in Fig. 20. However, the large or-
FIG. 16: $h(y)$ together with the expansion in Legendre polynomials up to order 20.

FIG. 17: Logarithm of the absolute value of the difference between the numerical data for $h(y)$ and expansions in Legendre polynomials at successive orders.

FIG. 18: $P$ together with the expansion in Legendre polynomials up to order 20.

FIG. 19: Logarithm of the absolute value of the difference between the numerical data for $P$ and expansions in Legendre polynomials at successive orders.

FIG. 20: Weak and strong coupling expansion of $f$ at a few intermediate orders.

der behaviors of these expansions appear to be similar to the corresponding ones for the plaquette. Volume effects can be resolved well for small actions values. Corrections to the saddle point estimate need to be developed systematically. Approximation of a subtracted quantity by Legendre polynomials looks very promising and works well uniformly. We plan to use this approximate form to look for Fisher’s zeros.

The density of states can be calculated in more general situations. For instance,

$$Z(\beta, \{\beta_i\}) = \int_0^{2N_p} dS \, n(S, \{\beta_i\}) e^{-\beta S},$$  \hspace{1cm} (28)
with
\[ n(S, \{\beta_i\}) = \prod_i \int dU_i \delta(S - \sum_p (1 - (1/N) Re Tr(U_p))) \] (29)
\[ \times e^{-\sum_{i} \beta_{i}(1-\chi_{i}(U_{p})/d_{i})}, \] (30)
\[ (31) \]
and $\chi_i$ a complete set of SU(2) characters. This is a type of action which naturally arises in the RG studies of SU(N) lattice gauge theories. It is possible to apply exact renormalization group transformation [1, 2] or the MCRG procedure [21] to the partition function in order to define the couplings. Following the analogy between $f' = \beta$ and $V' = J$ for the effective potential $V$ in presence of a source $J$ in scalar models, it would be interesting to study finite size effects from this point of view.

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