Measurement of the yaw moment of inertia of a go-kart

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Abstract. A computational model of the dynamic behaviour of a go-kart has been implemented as the first step in the development of an electrical vehicle advanced control system and the lateral dynamics model needed its yaw moment of inertia as input value. Therefore, this study presents experimental measurements of the yaw moment of inertia of a recreational go-kart using two similar methodologies: the ‘three cables’ (trifilar) swinging torsion pendulum and the ‘four cables’ (quadrifilar) swinging torsion pendulum. Parallel to the experimental tests carried out, yaw moment of inertia calculation was done by the traditional methodology of mechanical engineering. Also, the measurement uncertainty was calculated for each method. The trifilar method presented, in this test, the lowest measurement uncertainty.

1. Introduction
Quantities of the same nature, in a given system of magnitudes, have the same dimension [1]. However, quantities of the same dimensions are not necessarily of the same nature. The momentous magnitudes of a force are not, by convention, considered of the same nature, although they have the same dimension. The moment of inertia of area, magnitude used in static calculations, should not be confused with the moment of mass inertia of a wheel, used in dynamic calculations. Although both are referred to as 'moments of inertia', they are not of the same nature, as described in the International Vocabulary of Metrology. The moment of inertia (MI) referenced in this study is the mass moment of inertia and expresses the difficulty in changing the state of motion of a rotating body, whose unit in the International System of Units (SI) is kg.m².

The object to having its mass MI estimated is a go-kart model M2 manufactured by the company MINI, a four-wheel single-seater small vehicle which can be used in recreational and professional races. Vehicles of this type are not the target of many academic studies compared to the traditional automobile, but recently some studies carried out at PUC-Rio have resulted in computerized mathematical models of longitudinal and lateral dynamics for this type of vehicle [2]. Thus, to develop a mathematical model of the lateral dynamic of a go-kart, the yaw MI is fundamental. In this work, two experimental methods were applied and both have the same principle: the go-kart is suspended by three cables (the trifilar method) or four cables (the quadrifilar method) with fixed attachment lengths and positions, the torsion oscillation period is measured and the mathematical equation corresponding to each method is applied.

However, all methods for obtaining MI values incur measurement uncertainties. The sources of uncertainties of measurement calculated in this study consider the evaluations of type A uncertainty by the statistical analysis of series of observations, which in this case were the time measurements, and also evaluations of type B for means other than the statistical analysis of series, which in this case were the
measurements of length and weight. Combined standard uncertainty was applied as different quantities (time, weight and length) are included in the mathematical equations for obtaining the MI.

This study is structured as follows: in section 2 the methods for obtaining a go-kart yaw moment of inertia are presented. Section 3 presents the results of the measurements made in a real go-kart. In section 4, the measurement uncertainties are presented. Finally, section 5 presents the manuscript conclusions.

2. Moment of Inertia Estimation Methods

2.1. Trifilar method

The vehicle is suspended by three cables, forming a triangle whose central point must coincide with its centre of gravity (CG). It is then twisted around 20 degrees in the horizontal plane and released, performing harmonic oscillations around its CG. Some assumptions must be observed: the oscillatory movements, mass and axial flexibility of the cables are disregarded. According to [3], the equation corresponding to the trifilar method of obtaining the moment of inertia is

\[ J_z = \frac{r^2mg\tau^2}{4\pi^2L}, \]

where \( J_z \) is the moment of inertia in the horizontal plane, \( r \) is the distance between the kart CG and the fixing cables, \( m \) is the mass of the vehicle, \( g \) is the acceleration of gravity, \( \tau \) is the period of torsion oscillation and \( L \) is the height between the horizontal plane containing the CG of the vehicle and the parallel horizontal plane of cable attachment.

Figure 1 illustrates the test and shows a photograph of the test being performed.

![Figure 1. Test to obtain the moment of inertia with trifilar arrangement.](image)

To measure the period of oscillation \( \tau \), a manual digital timer was used, combined with the visual observation of the stops that characterize the change of direction of oscillation of the front tip of the kart, indicating the end of one period and the immediate beginning of another.
2.2. Quadrifilar method

The vehicle is suspended by four cables, forming a rectangle whose central point must coincide with its center of gravity (CG). It is then twisted in about 20 degrees in the horizontal plane and released, performing simple harmonic oscillations around its CG. Again, some assumptions must be observed: any damping of the oscillatory movement, the mass and the axial flexibility of the cables are disregarded. According to [4], the equation corresponding to the quadrifilar method of obtaining the moment of inertia is

\[
J_z = \frac{m \times g \times D^2 \times T^2}{16 \times \pi^2 \times L} \tag{2}
\]

where \( J_z \) is the moment of inertia in the horizontal plane, \( m \) is the mass of the vehicle, \( g \) is the acceleration of gravity, \( D \) is the distance between the cables (major edge of the rectangle), \( T \) is the period of the torsional oscillation and \( L \) is the height between the horizontal plane containing the vehicle CG and the parallel horizontal plane of cable attachment. Figure 2 shows a photograph of the test being performed. The variables of the equation must be known to obtain the value of the inertia moment (\( J_z \)) of the kart yaw and the procedures for obtaining them are similar as in the trifilar method.

Figure 2. Test to obtain the moment of inertia with quadrifilar arrangement.
2.3. Traditional Calculation

The classical mechanics presents the following equation to obtain the moment of inertia \( (J) \) for \( n \) point masses, \( m_i \) being the mass of each body displaced and \( r_i \) the distance between the position of each mass and the center of rotation:

\[
J = \sum_{i=1}^{n} m_i \cdot r_i^2 .
\]  

In a vehicle, the center of rotation is at the center of gravity. Figure 3 shows the distances between elements with masses that are relevant to the composition of the kart CG, and the CG itself. The mass values of these elements are listed in table 1.

Table 1. Mass value of elements with relevant mass.

| Element                          | Ref. | Mass (kg) |
|----------------------------------|------|-----------|
| Structural Frame                 | Fra  | 53.0      |
| Front Right Wheel                | Frw  | 1.8       |
| Front Left Wheel                 | Flw  | 1.8       |
| Rear Left Wheel                  | Rlw  | 2.3       |
| Rear Right Wheel                 | Rrw  | 2.3       |
| Gasoline Tank (within 3/4 cap)   | Gas  | 4.5       |
| Engine                           | Eng  | 35.0      |
| Transmission                     | Tra  | 6.0       |
| **Free total mass**              |      | **106.7** |

Figure 3. Distance between the relevant mass elements and the kart Center of Gravity.
3. Experimental Results

3.1. Trifilar method results

The results of the oscillation period measurements are shown in Table 2. To verify the existence of outliers among the values measured, the Grubbs criterion was applied where \( n = 50 \) and t-student = 4.45 thus yielding \( \eta_{\text{critical}} = 3.75 \), identifying two outliers. The statistical results presented in Table 2 already consider the rejection of the measurements disapproved by the Grubbs criterion.

### Table 2. Oscillation periods measured in the trifilar arrangement test

| Period (s) | Grubbs (\( \eta \)) |
|------------|-------------------|
| 2.51       | 2.40              |
| 2.67       | 0.82              |
| 2.61       | 0.39              |
| 2.48       | 3.01              |
| 2.74       | 2.22              |
| 2.65       | 0.41              |
| 2.56       | 1.62              |
| 2.65       | 0.41              |
| 2.66       | 0.62              |
| 2.70       | 1.42              |

Average of 50 samples 2.63
Standard deviation 0.05
Relative standard uncertainty (type A) 2%

### Table 3. Oscillation periods measured in the quadrifilar arrangement test

| Period (s) | Grubbs (\( \eta \)) |
|------------|-------------------|
| 3.43       | 1.10              |
| 3.29       | 0.09              |
| 3.18       | 1.03              |
| 3.18       | 1.03              |
| 3.14       | 1.36              |
| 3.32       | 0.16              |
| 3.19       | 0.94              |
| 3.15       | 1.28              |
| 3.23       | 0.60              |
| 3.25       | 0.43              |

Mean of 50 samples 3.30
Standard deviation 0.12
Relative uncertainty 3.6%

Considering the parameters \( r = 0.85 \text{ m}, m = 107 \text{ kg}, g = 9.81 \text{ m/s}^2, L = 3.02 \text{ m}, \text{ and } \tau = 2.63 \text{ s} \), applying the trifilar method corresponding equation, the result is \( J_z = 44 \text{ kg.m}^2 \).

3.2. Quadrifilar method results

The results of the oscillation period measurements are shown in Table 3. As in the trifilar method, the values of \( \eta \) of the Grubbs criterion were calculated, with the critical \( \eta = 3.75 \). No outliers were found in Table 3.

### Table 3. Oscillation periods measured in the quadrifilar arrangement test

| Period (s) | Grubbs (\( \eta \)) |
|------------|-------------------|
| 3.30       | 0.12              |
| 3.34       | 0.33              |
| 3.37       | 0.59              |
| 3.38       | 0.67              |
| 3.39       | 0.76              |
| 3.33       | 0.25              |
| 3.22       | 0.69              |
| 3.21       | 0.77              |
| 3.22       | 0.42              |
| 3.19       | 0.94              |

Mean of 50 samples 3.30
Standard deviation 0.12
Relative uncertainty 3.6%

Considering the parameters \( D = 1.32 \text{ m}, L = 3.02 \text{ m}, m = 107 \text{ kg}, \text{ and } \tau = 3.30 \text{ s} \), applying the quadrifilar method corresponding equation, the result is \( J_z = 42 \text{ kg.m}^2 \).
3.3. Traditional calculation results

The application of the classical mechanics equation to obtain MI, using the variables $m$ of Table 1 and $r$ of Figure 3, results in $J = 40 \text{ kg.m}^2$.

4. Measurement Uncertainties

According to the Guide to the Expression of Uncertainty in Measurement (GUM) [5], the combined standard uncertainty $u_c(y)$ is the positive square root of the combined variance $u^2_c(y)$, assuming uncorrelated input quantities, which is given by

$$u^2_c(y) = \sum_{i=1}^{N} \left( \frac{\partial f}{\partial x_i} \right)^2 u^2(x_i). \quad (4)$$

Therefore, the equations for obtaining the moment of inertia of the trifilar and quadrifilar methods were considered as the functions $f$ for the purpose of obtaining the respective combined variances.

To measure length parameters, a common millimeter resolution tape measure was used, whose reading was rounded to the nearest centimeter. Thus, considering a smallest significant digit of 0.01 m, following GUM’s guidelines,

$$u(r) = \frac{0.01}{\sqrt{12}}. \quad (5)$$

The mass measurements were done with a Toledo digital scale with resolution of 10 grams. However, during the measurement the value recorded in the field was rounded to values in kilograms due to oscillations of the values on the scale display. Therefore, the smallest significant digit considered was 1.00 kg. Thus, for the mass, standard deviation due to the resolution of this device is

$$u(m_i) = \frac{1}{\sqrt{12}}. \quad (6)$$

For time measurements, $n = 50$ successive measurements were performed with a common digital clock, so for $u(\tau)$ the standard uncertainty Type A was considered. Type B uncertainties were not considered for the digital clock.

According to GUM, for a single magnitude estimated by the arithmetic mean of $n$ independent observations, the number of degrees of freedom $v$ is equal to $n-1$ and, for a quantity obtained from $N$ other quantities from the combined uncertainty of measurement, the effective number of degrees of freedom is given by the Welch-Satterthwaite formula, given by

$$v_{eff} = \frac{u^2_c(y)}{\sum_{i=1}^{N} \frac{u^2_i(y)}{v_i}}. \quad (7)$$

Once the combined standard uncertainties and the effective degrees of freedom numbers were calculated, a confidence level of 95.45 % was defined, defining the coverage factors that are then multiplied by the combined standard uncertainties to calculate the expanded uncertainties in each case studied.

4.1. Trifilar method uncertainties

Table 4 presents the calculations to obtain the measurement uncertainty of the trifilar method.
Table 4. Trifilar method measurement uncertainty of the yaw moment of inertia

| $x_i$ | $\frac{df}{dx_i}$ | $C(x_i) = \frac{\partial f}{\partial x_i}$ | $C(x_i)^2$ | Smallest significant digit | $u(x_i)$ | $u^2(x_i)$ | $C(x_i)^2 \cdot u^2(x_i)$ | Uncertainty type | Degrees of Freedom |
|-------|-------------------|------------------------------------------|-------------|-----------------------------|--------|-------------|--------------------------|------------------|-------------------|
| (r)   | $\frac{mgR^2}{2\pi^2L}$ | 103.52 | 10717.39 | 0.01 | 2.88x10$^3$ | 8.33x10$^6$ | 8.93x10$^2$ | B | $\infty$ |
| (m)   | $\frac{r^2gt^2}{4\pi^2L}$ | 0.41 | 0.17 | 1.00 | 2.88x10$^4$ | 8.33x10$^2$ | 1.40x10$^2$ | B | $\infty$ |
| (l)   | $-\frac{r^2mgt^2}{4\pi^2L^2}$ | -14.57 | 212.25 | 0.01 | 2.88x10$^3$ | 8.33x10$^6$ | 1.76x10$^2$ | B | $\infty$ |
| (g)   | $\frac{r^2mgR^2}{2\pi^2L^2}$ | 33.46 | 1119.48 | - | 0.05 | 2.50x10$^3$ | 2.79 | A | 49 |

$u_i^2(y) = \sum_{i=1}^{N} \left( \frac{\partial f}{\partial x_i} \right)^2 \cdot u_i^2(x_i)$

Thus, the trifilar method presented a yaw moment of inertia with a measurement uncertainty equals to 3.41. Therefore, the result is shown as $J_z = 44 (\pm 3)$ kg.m$^2$.

4.2. Quadrifilar method uncertainties

Table 5 presents the calculations to obtain the measurement uncertainty of the quadrifilar method.

Table 5. Quadrifilar method measurement uncertainty of the yaw moment of inertia

| $x_i$ | $\frac{df}{dx_i}$ | $C(x_i) = \frac{\partial f}{\partial x_i}$ | $C(x_i)^2$ | Smallest significant digit | $u(x_i)$ | $u^2(x_i)$ | $C(x_i)^2 \cdot u^2(x_i)$ | Uncertainty type | Degrees of Freedom |
|-------|-------------------|------------------------------------------|-------------|-----------------------------|--------|-------------|--------------------------|------------------|-------------------|
| (D)   | $\frac{mgD^4}{8n^2L^2}$ | 63.278 | 4004.163 | 0.01 | 2.88x10$^3$ | 8.33x10$^6$ | 3.33x10$^2$ | B | $\infty$ |
| (m)   | $\frac{gd^2y^2}{16n^2L}$ | 0.390 | 0.152 | 1.00 | 2.88x10$^4$ | 8.33x10$^2$ | 1.27x10$^2$ | B | $\infty$ |
| (l)   | $-\frac{mgd^2T^2}{16n^2L^2}$ | -13.829 | 191.243 | 0.01 | 2.88x10$^3$ | 8.33x10$^6$ | 1.59x10$^2$ | B | $\infty$ |
| (t)   | $\frac{mgd^2T}{8n^2L}$ | 25.311 | 640.666 | - | 0.12 | 1.44x10$^2$ | 9.22 | A | 49 |

$u_i^2(y) = \sum_{i=1}^{N} \left( \frac{\partial f}{\partial x_i} \right)^2 \cdot u_i^2(x_i)$

Thus, the quadrifilar method presented a yaw moment of inertia with a measurement uncertainty equal to 6.09. Therefore, the result is shown as $J_z = 42 (\pm 6)$ kg.m$^2$.

4.3. Traditional calculation uncertainties

The traditional calculation method also incurs in measurement uncertainties since the distances and weight measurements contain uncertainties. Although it would be possible to calculate an associated uncertainty, the purpose of obtaining the yaw moment of inertia by means the traditional manner was only...
to check the order of magnitude of the experimental results. That is, it served only as a ballpark value, being not necessary to calculate its uncertainty. Table 6 presents the consolidated results.

Table 6. Yaw moment of inertia results

| Method                  | Trifilar ($J_z$) | Quadrifilar ($J_z$) | Traditional calculation |
|-------------------------|------------------|---------------------|-------------------------|
| MI (kg.m²)              | $44 \pm 3$       | $42 \pm 6$          | $40$                     |

5. Conclusions

The values obtained with different methods resulted in compatible values as $J_z1 - J_z2 \leq \sqrt{U(J_z1)^2 - U(J_z2)^2}$, with a difference of approximately 10% without considering the measurement uncertainties, and up to 15% considering the measurement uncertainties.

The trifilar method demonstrates lowest measurement uncertainty in this test and therefore is considered the best result despite its bigger difference from the value of the traditional calculation, which actually is also inaccurate and can’t be considered as a reference value.

The mass of the driver body is very significant when it comes to a go-kart, more significant than in general automobiles. However, the kart seat, where almost the entire mass of the driver lies, is very close to the kart center of gravity. Being so, the influence of the driver mass in the yaw moment of inertia is not as relevant as it might seem. According to [2], the insertion of a 60 kg mass in the kart seat changes the moment of inertia value around 3% (without considering measurement uncertainties).

In order to apply these methods to obtain the yaw moment of inertia of their own kart, the results presented here can be used as general reference, but specific tests should be performed with the driver on the vehicle, to obtain accurate value. It is important to note that not only the mass of the driver is variable but also all other elements of relevant mass are variable, especially the go-kart engine. The go-kart is characterized by being a semi-craft vehicle whose geometry of mass distribution is quite variable.

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