Exact Duality Relations in Correlated Electron Systems

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Using gauge transformations on electron bond operators, we derive exact duality relations between various order parameters for correlated electron systems. Applying these transformations, we find two duality relations in the generalized two-leg Hubbard ladder at arbitrary filling. The relations show that unconventional density-wave orders such as staggered flux or circulating spin current are dual to conventional density-wave orders and there are direct mappings between dual phases. Several exact results on the phase diagram are also concluded.

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Unconventional density-wave orders such as staggered flux [which is equivalently called as $d$-density wave (DDW) or orbital antiferromagnets] and circulating spin current were first proposed in the context of excitonic insulators\cite{1}, and later discussed in high-$T_c$ superconductors\cite{2,3,4}, but the appearance of these orders was not established at that time. Recently, several experimental results have led to a resurgence of interest in the possibility of the exotic orders. A circulating spin-current state\cite{4} and DDW state\cite{2} were proposed as an origin of hidden order in the heavy-fermion compound URu$_2$Si$_2$. A DDW state has also been discussed to appear in underdoped region of high-$T_c$ superconductors\cite{5}, where a pseudogap was observed\cite{6}, and in the quasi-two-dimensional organic conductor $\alpha$-(ET)$_2$KHg(SCN)$_4$\cite{7}.

From a theoretical viewpoint, the possibility of the unconventionally ordered phases in microscopic models of correlated electrons has been the focus of interest. In particular, the two-leg ladder system is attracting attention as a minimal model for showing the exotic orders. Though the two-leg Hubbard and $t$-$J$ ladders do not show any order\cite{8,9,10}, both strong coupling and weak coupling analyses revealed that the generalized two-leg Hubbard ladder including nearest-neighbor interactions exhibits various ordered phases at half-filling and doped cases\cite{10,11}. Large-scale numerical calculations also reported the DDW phase\cite{12} at half-filling and charge-density-wave (CDW)\cite{13} and DDW\cite{14} phases for less than half-filling.

In this Letter, introducing gauge transformations on electron bond operators, we derive two exact duality relations between various exotic phases in correlated electron systems. One duality relates conventional density waves to unconventional density waves (or currents) and the other relates charge-density degrees of freedom to spin-current ones. These transformations also show duality relations between $s$- and $d$-wave superconductivity (SSC and DSC), as well as between $s$- and $d$-wave Mott-insulating (S-Mott and D-Mott) states. Applying these transformations, we show two exact duality relations in the generalized two-leg Hubbard ladder (or similarly the Hubbard model with doubly degenerate orbitals). The duality relations among various phases are summarized in Fig. 1, where we use density-wave terminology\cite{15}. Starting from previously established phases, we can reveal the appearance of various exotic phases with and without spin symmetry breaking. The transformations give parameter mapping between dual phases and fix boundaries between them on self-dual lines. These exact relations enable us to clarify the nature of the phase diagram and find the parameter space of unconventionally ordered phases.

Before discussing the duality transformations, we list operators and order parameters considered in this work. We use ladder geometry in the following for simplicity, but one can extend to other lattices adopting appropriate bonds. Regarding charge degrees of freedom, we consider CDW, staggered-flux [or equivalently DDW], staggered dimer [or $p$-density-wave (PDW)], and diagonal current [or $f$-density-wave (FDW)] operators on rungs (or pla-

![FIG. 1: Duality relations (I) and (II) between various states; (a) CDW, DDW, SDW, and DSDW states, (b) PDW, FDW, PSDW, and FSDW states, and (c) s- and d-wave Mott-insulating and superconducting states.](image-url)
CDW, DDW, PDW, and FDW operators, respectively.

The gauge transformation of antibonding operators given by

\[ \mathcal{O}_{\text{DDW}}(j) = \frac{1}{2} \sum_{\sigma} (c_{j,1,\sigma} c_{j,2,\sigma} - \text{H.c.}) \]

characterize the DSC and SSC states. We also consider wave (PSDW) operators, i.e., DDW, DSDW, FDW, and PSDW phases, turn out to be dual to density-wave phases, i.e., CDW, SDW, PDW, and PSDW phases, respectively. In other words, this transformation changes the bond parity and hence \( s^- (p^-) \) wave orders are converted to \( d^- (f^-) \) wave ones. The pairing operators and the Mott states are transformed in the similar way,

\[ \mathcal{O}_{\text{DSC}} = -\mathcal{O}_{\text{SSC}}, \quad \mathcal{O}_{\text{SSC}} = -\mathcal{O}_{\text{DSC}}, \]

\[ [\text{D-Mott}] = [\text{S-Mott}], \quad [\text{S-Mott}] = [\text{D-Mott}]. \]

We note that the unitary operator \( U_1(\theta) \) gives a continuous transformation between dual operators, e.g., \( U_1(\theta) \mathcal{O}_{\text{CDW}} U_1(\theta)^{-1} = \mathcal{O}_{\text{CDW}} \cos \theta - \mathcal{O}_{\text{DDW}} \sin \theta \).

\textbf{Duality II: density and spin current.}—Next, we consider a gauge transformation given by the unitary operator \( U_{11}(\theta) = \prod_j \exp[-i\theta(d_{j,\uparrow}^{\dagger} d_{j,\uparrow} + d_{j,\downarrow}^{\dagger} d_{j,\downarrow})] \). This operator with \( \theta = \pi/2 \) gives the duality transformation (II) as

\[ \tilde{d}_{j,\uparrow,\uparrow} = id_{j,\downarrow}, \quad \tilde{d}_{j,\downarrow,\uparrow} = id_{j,\downarrow}, \quad \tilde{d}_{j,\downarrow,\downarrow} = d_{j,\downarrow,\downarrow}, \quad \tilde{d}_{j,\downarrow,\downarrow} = d_{j,\downarrow,\downarrow}. \]

This transformation converts the operators as

\[ \mathcal{O}_{\text{CDW}} = -\mathcal{O}_{\text{DDW}}, \quad \mathcal{O}_{\text{DDW}} = -\mathcal{O}_{\text{CDW}}, \]

\[ \mathcal{O}_{\text{PDW}} = \mathcal{O}_{\text{FSDW}}, \quad \mathcal{O}_{\text{FSDW}} = -\mathcal{O}_{\text{PDW}}. \]

One can see that density waves of charges are transformed into spin currents while density waves of spins into charge currents. This transformation thus exchanges density and current as well as spin and charge degrees of freedom. Similarly to \( U_1(\theta) \), the unitary operator \( U_{11}(\theta) \) gives a continuous transformation between dual operators such as \( U_{11}(\theta) \mathcal{O}_{\text{CDW}} U_{11}(\theta)^{-1} = \mathcal{O}_{\text{CDW}} \cos \theta + \mathcal{O}_{\text{DDW}} \sin \theta \). We note that under this transformation the pairing operators and Mott states are invariant except for phase factors.


**Duality between spin and charge.**— Combining the duality transformations (11) and (9), we obtain the transformation

\[
\tilde{d}_{j,+} = d_{j,+},
\]
\[
\tilde{d}_{j,-} = id_{j,-}.
\]

From the duality relations (9) and (10), it follows that this transformation exchanges spin and charge degrees of freedom, e.g., density-wave operators are transformed to spin-density-wave operators.

To elucidate efficiency of these transformations, we apply them to the generalized two-leg Hubbard ladder with intra-rung couplings. The Hamiltonian is defined by

\[
H = -\sum_{j,\sigma} \left( t \sum_{\mu=1,2} c_{j,\mu,\sigma}^\dagger c_{j+1,\mu,\sigma} + t_\perp c_{j,1,\sigma}^\dagger c_{j,2,\sigma} + \text{H.c.} \right) + \sum_{j} \left[ U \sum_{\mu=1,2} n_{j,\mu,\sigma} n_{j,\mu,\bar{\sigma}} + V_{\perp} n_{j,1} n_{j,2} \right.
\]
\[
+J_{\perp} (S_{j,1}^x S_{j,2}^x + S_{j,1}^y S_{j,2}^y) + J_{\perp}^z S_{j,1}^z S_{j,2}^z
\]
\[
+ t_{\text{pair}} (c_{j,1,\uparrow}^\dagger c_{j,1,\downarrow} c_{j,2,\uparrow} c_{j,2,\downarrow} + \text{H.c.}) \right),
\]

(12)

where \( n_{j,\mu} = \sum_{\sigma} n_{j,\mu,\sigma} \) and \( S_{j,\mu} = \sum_{\sigma,\sigma'} c_{j,\mu,\sigma}^\dagger c_{j,\mu,\sigma'} \). When \( t_\perp = 0 \), this Hamiltonian includes the Hubbard model with doubly degenerate orbitals (10). We first apply the transformation (1) [Eq. (7)] to the Hamiltonian. It can be shown that this transformation maps the model onto the same Hubbard model with different coupling parameters. The total charge density, the total magnetization, and the kinetic energy terms are invariant under the transformation. However, the coupling terms are mixed by the transformation. Here, it is easy to rewrite the couplings in terms of bonding and antibonding operators as

\[
A(d_{j,+}^\dagger d_{j,-}+t_\perp d_{j,+}^\dagger d_{j,-}+ \text{H.c.})
\]
\[
+ B(d_{j,+}^\dagger d_{j,-}+t_\perp d_{j,+}^\dagger d_{j,-}+ \text{H.c.}) + C \sum_{\sigma} n_{j,+,\sigma} n_{j,+,\bar{\sigma}}
\]
\[
+ D \sum_{\alpha=\pm} n_{j,\alpha,\sigma} n_{j,\alpha,\bar{\sigma}} + E \sum_{\alpha=\pm} n_{j,\alpha,+,\sigma} n_{j,\alpha,+,\bar{\sigma}}
\]

(13)

where \( n_{j,\alpha,\sigma} = d_{j,\alpha,\sigma}^\dagger d_{j,\alpha,\sigma} \), and

\[
A = (U - V_{\perp} + t_{\text{pair}})/2 + (2J_{\perp} + J_{\perp}^z)/8,
\]
\[
B = (U - V_{\perp} - t_{\text{pair}})/2 - (2J_{\perp} - J_{\perp}^z)/8,
\]
\[
C = V_{\perp} + J_{\perp}^z/4,
\]
\[
D = (U + V_{\perp} + t_{\text{pair}})/2 - (2J_{\perp} + J_{\perp}^z)/8,
\]
\[
E = (U + V_{\perp} - t_{\text{pair}})/2 + (2J_{\perp} - J_{\perp}^z)/8.
\]

It is easily found in Eq. (13) that the transformation (7) changes only the sign of the \( A \)-term, but keeps the rest of terms invariant. This leads to an exact duality relation in the two-leg Hubbard ladder. The model with a parameter \( A \) is dual to the model with \(-A\) and the model is self-dual in the space \( A = 0 \), i.e.,

\[
U - V_{\perp} + t_{\text{pair}} + (2J_{\perp} + J_{\perp}^z)/4 = 0.
\]

(14)

The mapping of the original coupling parameters is as follows:

\[
\tilde{U} = \frac{1}{2} (U + V_{\perp} - t_{\text{pair}}) - \frac{1}{8} (2J_{\perp} + J_{\perp}^z),
\]
\[
\tilde{V}_{\perp} = \frac{1}{4} (U + 3V_{\perp} + t_{\text{pair}}) + \frac{1}{16} (2J_{\perp} + J_{\perp}^z),
\]
\[
\tilde{J}_{\perp} = -U + V_{\perp} - t_{\text{pair}} + \frac{1}{4} (2J_{\perp} - J_{\perp}^z),
\]
\[
\tilde{J}_{\perp}^z = -U + V_{\perp} - t_{\text{pair}} - \frac{1}{4} (2J_{\perp} - 3J_{\perp}^z),
\]
\[
\tilde{t}_{\text{pair}} = \frac{1}{2} (U - V_{\perp} + t_{\text{pair}}) - \frac{1}{8} (2J_{\perp} - J_{\perp}^z).
\]

(15)

Note that spin isotropy \( (J_{\perp} = J_{\perp}^z) \) is conserved through this parameter mapping as \( \tilde{J}_{\perp} = \tilde{J}_{\perp}^z \). From the mapping (14) and the duality relation (8), one can conclude that if a density-wave order, e.g., CDW or PDW order, appears in a certain parameter region, a dual current order, i.e., DDW or FDW order, respectively, exists in a corresponding dual region of the parameter space. Because of this duality relation, all phase boundaries must be symmetric with respect to the self-dual space. Indeed, the transition line between the CDW and DDW phases at half-filling derived in the weak- and strong-coupling limits (14) coincides with the self-dual line (13). We stress that our result holds in general cases, regardless of the coupling strength, filling, and system size.

Next we discuss the duality transformation (II) [Eq. (16)], which gives another parameter mapping. Similarly to the transformation (I), the total charge density, the total magnetization, and the kinetic energy terms are invariant under the transformation (16), but the coupling terms are mixed up. From Eq. (16), one can see that the transformation (16) changes only the sign of the \( B \)-term. We thus find that the present model has another duality: the model with a parameter \( B \) is dual to the model with \(-B\) under the transformation (16) and the model is self-dual in the space \( B = 0 \), i.e.,

\[
U - V_{\perp} - t_{\text{pair}} - (2J_{\perp} - J_{\perp}^z)/4 = 0.
\]

(16)

The transformed coupling parameters are given by

\[
\tilde{U} = \frac{1}{2} (U + V_{\perp} + t_{\text{pair}}) + \frac{1}{8} (2J_{\perp} - J_{\perp}^z),
\]
\[
\tilde{V}_{\perp} = \frac{1}{4} (U + 3V_{\perp} + t_{\text{pair}}) - \frac{1}{16} (2J_{\perp} + J_{\perp}^z),
\]
\[
\tilde{J}_{\perp} = U - V_{\perp} - t_{\text{pair}} + \frac{1}{4} (2J_{\perp} + J_{\perp}^z),
\]
\[
\tilde{J}_{\perp}^z = -U + V_{\perp} + t_{\text{pair}} + \frac{1}{4} (2J_{\perp} + 3J_{\perp}^z),
\]
\[
\tilde{t}_{\text{pair}} = \frac{1}{2} (U - V_{\perp} + t_{\text{pair}}) - \frac{1}{8} (2J_{\perp} - J_{\perp}^z).
\]

(17)
This duality relation leads to the conclusion that, if CDW or DDW order, for example, exists in a certain parameter region, spin current (DSDW) or SDW order exists in a dual region, respectively. Note that even if we start from a spin isotropic model the dual model is spin anisotropic and hence spin symmetry breaking can occur in the dual model. The phase diagram must be symmetric with the self-dual space (13) and the transitions between dual phases, if ever, locate exactly on the self-dual space.

The spin-charge duality relation associated with the transformation (11) can be obtained from the combination of two transformations discussed above. Thereby one can arrive at a direct duality relation between orders in the charge and spin sectors. The self-dual space is the intersection of Eqs. (14) and (16), i.e., $U - V_1 + \frac{1}{2} J_1^z = 0$ and $\frac{1}{2} J_1 + t_{\text{pair}} = 0$.

In the self-dual spaces (14) and (16), the model Hamiltonian is invariant under the continuous transformations with the unitary operators $U_1(\theta)$ and $U_{\text{II}}(\theta)$, respectively, and hence U(1) symmetric. [This can be easily seen in Eq. (13)]. Because of the U(1) symmetry, a rigorous theorem [20] concludes that the dual orders that are continuously transformed to each other disappear on the self-dual models in one dimension, in general [21], and the phase transition between the dual phases, if ever, is of second order. The $A$- and $B$-terms in the interaction (13) serve as symmetry-breaking perturbations for the U(1) symmetry of $U_1(\theta)$ and $U_{\text{II}}(\theta)$, respectively. When a symmetry-breaking perturbation is relevant, it induces order and a gap determining the criticality of the phase transition.

In summary, we have developed duality transformations for electron systems, which lead to duality relations between various exotic phases shown in Fig. 11. These duality relations have an analogy with the spin-chirality duality transformation we and co-workers introduced for the spin ladder [22, 23], but are applicable to a much wider variety of systems. The transformations clarify for the generalized two-leg Hubbard ladder that the stability of unconventional density-wave orders such as staggered flux (DDW) and circulating spin current (DSDW) is equal to that of conventional density-wave orders in the dual spaces. Recent large-scale numerical analyses [12, 18] reported the appearance of CDW and DDW phases under doping. Application of the duality relations to these results immediately concludes that SDW and DSDW phases stably exist under doping in dual parameter spaces. Further arguments and possible applications of the duality transformations will be reported elsewhere.

Finally we stress that the duality transformations (13) and (16) can be applied to a variety of systems in which bond degrees of freedom are important, since they are based only on simple gauge transformations of bond operators. Applications to double-layer systems, and two- and three-dimensional systems with orbital degeneracy are of interest.

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[54x-3130]erators. Applications to double-layer systems, and two-
[54x-2892]ble applications of the duality transformations will be
[54x-2510]dual parameter spaces. Further arguments and possi-
[54x-2392]SDW and DSDW phases stably exist under doping in
[54x-2034]analyses[12, 18] reported the appearance of CDW and
[54x-1796]as staggered flux (DDW) and circulating spin current
[54x-1677]clarify for the generalized two-leg Hubbard ladder that
[54x-1439]to a variety of systems in which bond degrees of freedom are important, since they are
[54x-1319]are continuously transformed to each other disappear on
[54x-1201]duced for the spin ladder[22, 23], but are applicable to
[54x-1081]chirality duality transformation we and co-workers intro-
[54x-962]These duality relations have an analogy with the spin-
[54x-843]tions between various exotic phases shown in Fig. 1.
[54x-724]tions for electron systems, which lead to duality rela-
[54x-700]tion of two transformations discussed above. Thereby
[54x-462]a symmetry-breaking perturbation is relevant, it induces
[54x-330]of second order. The
[54x-199]the phase transition between the dual phases, if ever, is
[54x39]are continuously transformed to each other disappear on
[54x277]seen in Eq. (13)]. Because of the U(1) symmetry, a rig-
[54x408]with the unitary operators
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