NONCOMPLEX SMOOTH 4-MANIFOLDS WITH
LEFSCHETZ FIBRATIONS

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1. Introduction

Recently, B. Ozbagci and A. Stipsicz [12] proved that there are infinitely many pairwise nonhomeomorphic 4-manifolds admitting genus-2 Lefschetz fibration over $S^2$ but not carrying any complex structure with either orientation. (For the definition of Lefschetz fibration, see [6].) Their result depends on a relation in the mapping class group of a closed orientable surface of genus 2. This relation with eight right Dehn twists was discovered by Y. Matsumoto [9] by a computer calculation, and it is the global monodromy of a Lefschetz fibration $T^2 \times S^2 \# 4\mathbb{C}P^2 \to S^2$, where $S^2$ is the 2-sphere and $T^2$ is the 2-torus.

In this paper, we generalize Matsumoto’s relation to higher genus orientable surfaces. We find a relation involving $2g + 4$ (resp., $2g + 10$) Dehn twists when the genus of the surface is even (resp., odd). Following the method of Ozbagci and Stipsicz, for every positive integer $n$, we obtain a 4-manifold $X_n$ admitting a genus-$g$ Lefschetz fibration such that the fundamental group of $X_n$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_n$ for every $g \geq 2$. We then deduce that the 4-manifold $X_n$ does not admit any complex structure. This is the main result of this paper.

Theorem 1.1. For every $g \geq 2$, there are infinitely many pairwise nonhomeomorphic 4-manifolds that admit genus-$g$ Lefschetz fibrations over $S^2$ but do not carry any complex structure with either orientation.

Our relation in the mapping class group given by Theorem 3.4 also shows that the minimal number of singular fibers in a nontrivial genus-$g$ Lefschetz fibration over $S^2$ is less than or equal to $2g + 4$ (resp., $2g + 10$) if $g$ is even (resp., odd). This result was also obtained independently by C. Cadavid [3]. By definition, a Lefschetz fibration is nontrivial if it admits singular fibers. Stipsicz proved in [14] that this minimal

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number is in fact $2g + 4$ (resp., $2g + 10$) if $g$ is even (resp., odd) and greater than or equal to 6 (resp., greater than or equal to 15) among all 4-manifolds with $b^+_2 = 1$. See [7] and [13] for the other results related to this minimal number.

Here is how we obtain our relation in the mapping class group. Let $\Sigma_g$ be a closed connected orientable surface of genus $g$. The hyperelliptic mapping class group of $\Sigma_g$ is a quotient of the braid group $B_{2g+2}$ on $2g + 2$ strings. The quotient of the hyperelliptic mapping class group with the cyclic subgroup of order 2 generated by the hyperelliptic involution is isomorphic to the mapping class group of a sphere with $2g + 2$ holes. The hyperelliptic mapping class group is equal to the mapping class group when $g = 2$. Using these facts, we lift Matsumoto’s relation to the braid group $B_6$ and generalize it to a relation in $B_{2g+2}$, although we do not say so explicitly. We then project it to the surface $\Sigma_g$ to get our relation in the mapping class group of $\Sigma_g$.

For each positive integer $n$, by considering a product of conjugates of our relation with appropriate mapping classes, we obtain a relation in the mapping class group of $\Sigma_g$ so that the fundamental group of the corresponding symplectic 4-manifold $X_n$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_n$. It follows from [12, proof of Theorem 1.3] that a symplectic manifold with fundamental group $\mathbb{Z} \oplus \mathbb{Z}_n$ admits no complex structures.

In the last section, we determine the diffeomorphism type of the 4-manifold $X$ admitting a genus-$g$ Lefschetz fibration over $S^2$ corresponding to our relation given by Theorem 3.4. If $g$ is even, then $X$ is diffeomorphic to $\Sigma_{g/2} \times S^2 \# 4\mathbb{C}P^2$. For this, we use a result of Stipsicz asserting that the only 4-manifold with $b^+_2 = 1$ which admits a (relatively minimal) Lefschetz fibration with $2g + 4$ vanishing cycles is $\Sigma_{g/2} \times S^2 \# 4\mathbb{C}P^2$.

2. Braid groups

The braid group $B_{2g+2}$ on $2g + 2$ strings admits a presentation with generators $\sigma_1, \sigma_2, \ldots, \sigma_{2g+1}$ and relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| \geq 2$$

and

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ if } 1 \leq i \leq 2g.$$  

The subgroup of $B_{2g+2}$ generated by $\sigma_1, \sigma_2, \ldots, \sigma_h$ is isomorphic to $B_{h+1}$. We identify this subgroup with $B_{h+1}$.

In the group $B_{2g+2}$, let us define the words $\Delta_k = \sigma_1 \sigma_2 \cdots \sigma_k$ and $\bar{\Delta}_k = \sigma_k \cdots \sigma_2 \sigma_1$ for each $k = 1, 2, \ldots, 2g + 1$. We define $\Delta_0 = 1$ for
the convention. For each \( k = 0, 1, \ldots, g \), let us also define
\[
\beta_k = \bar{\Delta}_k \Delta_{2g+1-k} \Delta_{2g-k}^{-1} \bar{\Delta}_k^{-1}
\]
and
\[
\beta = \bar{\Delta}^{g+1}.
\]
We have, for example, \( \beta_0 = \Delta_{2g+1} \Delta_1^{-1} \), \( \beta_1 = \bar{\Delta}_1 \Delta_{2g} \Delta_{2g-1}^{-1} \bar{\Delta}_1^{-1} \), and \( \beta_g = \bar{\Delta}_g \Delta_{g+1} \Delta_g^{-1} \bar{\Delta}_g^{-1} \). Note that \( \beta_k \) is the conjugate of \( \sigma_{2g+1-k} \) with the element \( \Delta_k \Delta_{2g-k} \).

The following lemma follows easily from the defining relations of the braid group.

**Lemma 2.1.** The following relations hold in the group \( B_{2g+2} \).

(a) \( \sigma_k \Delta_m = \Delta_m \sigma_k \) and \( \sigma_k^{-1} \Delta_m = \Delta_m \sigma_k^{-1} \) if \( 1 < k \leq m \);
(b) \( \sigma_k \bar{\Delta}_m = \bar{\Delta}_m \sigma_{k+1} \) and \( \sigma_k^{-1} \bar{\Delta}_m = \bar{\Delta}_m \sigma_{k+1} \) if \( 1 \leq k < m \);
(c) \( \sigma_k \Delta_m = \Delta_m \sigma_k \) and \( \sigma_k \bar{\Delta}_m = \bar{\Delta}_m \sigma_k \) if \( k > m + 1 \); and
(d) \( \Delta_g^k = \bar{\Delta}_g^{-1} \Delta_g^{-1} \sigma_g \sigma_{k+1} \) and \( \bar{\Delta}_g^k = \sigma_g^{-1} \Delta_g^{-1} \bar{\Delta}_g^{-1} \) if \( 1 \leq k \leq g \).

**Lemma 2.2.** In the braid group \( B_{2g+2} \), we have the following:

(a) The element \( \beta \) is equal to \( \bar{\Delta}_g \Delta_g \bar{\Delta}_g^{-1} \); and
(b) The element \( \bar{\Delta}_g \Delta_g \) is in the centralizer of \( B_g \); in particular, it commutes with \( \Delta_g \).

**Proof.** We claim that \( \bar{\Delta}_g^k = \Delta_k \bar{\Delta}_g^{-k} \Delta_g^k \) for every \( 0 \leq k \leq g \). First of all, the claim holds trivially for \( k = 0 \). Suppose by induction that \( \bar{\Delta}_g^k = \Delta_k \bar{\Delta}_g^{-k} \Delta_g^k \). By Lemma 2.1 (d), we have \( \Delta_g^{-k} = \sigma_{k+1} \Delta_g^{-k-1} \Delta_g \). Then we have
\[
\bar{\Delta}_g^k = \Delta_k \bar{\Delta}_g^{-k} \Delta_g \Delta_g^{-1} \Delta_g^{-k} \Delta_g = \Delta_k \sigma_{k+1} \bar{\Delta}_g^{-k-1} \Delta_g^{-k} \Delta_g \Delta_g^{-1} = \Delta_{k+1} \bar{\Delta}_g^{-k-1} \Delta_g^{-k} \Delta_g^{-1}.
\]
In particular, \( \bar{\Delta}_g^g = \Delta_g \bar{\Delta}_g^{-g} \). The proof of (a) follows.

For \( k < g \), by Lemma 2.1 we have \( \bar{\Delta}_g \Delta_g \sigma_k = \bar{\Delta}_g \sigma_{k+1} \Delta_g = \sigma_k \bar{\Delta}_g \Delta_g \). Now the proof of (b) follows. \( \Box \)

**Lemma 2.3.** For every \( 0 \leq k \leq g-1 \), we have
\[
\Delta_{2g-k}^{-1} \bar{\Delta}_k^{-1} \Delta_{k+1} \Delta_{2g-k} \bar{\Delta}_k^{-1} \Delta_{2g-k-1} = \bar{\Delta}_k \Delta_{2g-k} \gamma_k,
\]
where \( \gamma_k = \bar{\Delta}_k \Delta_{2g-k-1} \Delta_{2g-k-2} \bar{\Delta}_k^{-1} \).
Theorem 2.4. In the braid group \( B_{2g+2} \), we have the relation
\[
\beta_0\beta_1\beta_2\cdots\beta_g\beta^2 = \Delta_{2g+1}\Delta_{2g}\cdots\Delta_3\Delta_2\Delta_1.
\]

Proof. We use Lemma 2.1 several times:
\[
\Delta_{2g-k}^{-1} \bar{\Delta}_k^{-1} \Delta_{k+1} \Delta_{2g-k} \bar{\Delta}_{k+1} \Delta_{2g-k-1}
\]
\[
= \sigma_{2g-k}^{-1} \cdots \sigma_{k+2g-k}^{-1} \Delta_k^{-1} \bar{\Delta}_k^{-1} (\bar{\Delta}_{k+1} \Delta_{2g-k} \bar{\Delta}_{k+1} \Delta_{2g-k-1})
\]
\[
= \sigma_{2g-k}^{-1} \cdots \sigma_{k+2g-k}^{-1} (\bar{\Delta}_{k+1} \Delta_{2g-k} \bar{\Delta}_{k+1} \Delta_{2g-k-1}) \Delta_k^{-1} \bar{\Delta}_k^{-1}
\]
\[
= \sigma_{2g-k}^{-1} \cdots \sigma_{k+3g-k+2}^{-1} \bar{\Delta}_k \Delta_{2g-k} \bar{\Delta}_{k+1} \Delta_{2g-k-1} \Delta_k^{-1} \bar{\Delta}_k^{-1}
\]
\[
= \Delta_k \sigma_{2g-k}^{-1} \cdots \sigma_{k+3g-k+2}^{-1} \bar{\Delta}_k \Delta_{2g-k} \bar{\Delta}_{k+1} \Delta_{2g-k-1} \Delta_k^{-1} \bar{\Delta}_k^{-1}
\]
\[
= \bar{\Delta}_k \Delta_{2g-k} \bar{\Delta}_k \Delta_{2g-k-1} \Delta_{2g-k-2} \bar{\Delta}_k^{-1}
\]
\[
= \bar{\Delta}_k \Delta_{2g-k} \gamma_k.
\]

The main result of this section is the next theorem.

Theorem 2.4. In the braid group \( B_{2g+2} \), we have the relation
\[
\beta_0\beta_1\beta_2\cdots\beta_g\beta^2 = \Delta_{2g+1}\Delta_{2g}\cdots\Delta_3\Delta_2\Delta_1.
\]

Proof. Recall that, for any \( h \leq 2g+2 \), we identify the group \( B_h \) with the subgroup of \( B_{2g+2} \) generated by the elements \( \sigma_1, \sigma_2, \ldots, \sigma_{h-1} \).

The proof of the theorem is by induction on \( g \). Suppose that \( g = 0 \). In the group \( B_2 \), \( \beta_0 = \Delta_1 \Delta_0^{-1} \) and \( \beta = \Delta_0 \). Thus \( \beta_0\beta^2 = \Delta_1 \). Hence, the conclusion of the theorem holds for \( g = 0 \).

In the subgroup \( B_{2g} \) of \( B_{2g+2} \), let us define
\[
\gamma_k = \bar{\Delta}_k \Delta_{2g-k} \Delta_{2g-k-1} \bar{\Delta}_k^{-1}, \quad 0 \leq k \leq g - 1
\]
and
\[
\gamma = \bar{\Delta}_{g-1}.
\]

Then, by the induction hypothesis
\[
\gamma_0\gamma_1\gamma_2\cdots\gamma_{g-1}\gamma^2 = \Delta_{2g-1}\Delta_{2g-2}\cdots\Delta_3\Delta_2\Delta_1.
\]

Let us also define \( \gamma_g = 1 \) for the convention.

In the group \( B_{2g+2} \), we claim that
\[
\beta_k\beta_{k+1} \cdots \beta_g \beta^2 = \bar{\Delta}_k \Delta_{2g+1-k} \bar{\Delta}_k \Delta_{2g-k} \gamma_k \gamma_{k+1} \cdots \gamma_{g-1} \gamma_g \gamma^2.
\]
The proof of this claim is by induction on $g - k$. We start with the following computation:

$$
\beta_g \beta^2 = \Delta_{g+1} \Delta_{g}^{-1} \Delta_{g}^{-1} (\Delta_{g} \Delta_{g} \Delta_{g-1})^2 \\
= \Delta_{g+1} \Delta_{g}^{-1} \Delta_{g} \Delta_{g} \Delta_{g-1} \\
= \Delta_{g+1} \Delta_{g} \Delta_{g} ^2 \Delta_{g}^{-1} \\
= \Delta_{g+1} \Delta_{g} \Delta_{g} \Delta_{g} \gamma \gamma^2.
$$

Hence, the claim holds for $k = g$. Suppose inductively that

$$
\beta_{k+1} \beta_{k+2} \cdots \beta_g \beta^2 = \Delta_{k+1} \Delta_{2g-k} \Delta_{k+1} \Delta_{2g-k-1} \gamma_{k+1} \gamma_{k+2} \cdots \gamma_g \gamma^2.
$$

Then by Lemma 2.3 we get

$$
\beta_k \beta_{k+1} \cdots \beta_g \beta^2 \\
= \Delta_k \Delta_{2g+1-k} (\Delta_{2g-k} \Delta_{k+1} \Delta_{2g-k} \Delta_{k+1} \Delta_{2g-k-1} \gamma_{k+1} \gamma_{k+2} \cdots \gamma_g \gamma^2) \\
= \Delta_{k} \Delta_{2g+1-k} \Delta_{k} \Delta_{2g-k} \gamma_{k} \gamma_{k+1} \gamma_{k+2} \cdots \gamma_g \gamma^2.
$$

Hence, the claim is proved. For $k = 0$, in particular, we obtain

$$
\beta_0 \beta_1 \beta_2 \cdots \beta_g \beta^2 = \Delta_0 \Delta_{2g+1} \Delta_0 \Delta_{2g} \gamma_0 \gamma_1 \cdots \gamma_g \gamma^2 \\
= \Delta_{2g+1} \Delta_{2g} \gamma_0 \gamma_1 \cdots \gamma_g \gamma^2 \\
= \Delta_{2g+1} \Delta_{2g} \Delta_{2g-1} \cdots \Delta_2 \Delta_1.
$$

This finishes the proof of the theorem. 

3. Mapping class groups

Let $\Sigma_g$ be a closed connected orientable surface of genus $g$ embedded in $\mathbb{R}^3$ such that it is invariant under the involution $J(x, y, z) = (-x, y, -z)$ (cf. Fig. 4). Notice that $J$ is the rotation about $y$-axis by $\pi$. We orient $\Sigma_g$ so that the unit normal vectors are pointing outward. Let $j$ be the isotopy class of $J$. Let us denote by $M_g$ the mapping class group of $\Sigma_g$ which is the group of isotopy classes of orientation-preserving diffeomorphisms of $\Sigma_g$. The hyperelliptic mapping class group is defined as the centralizer $C_{M_g}(j)$ of $j$ in $M_g$, the subgroup consisting of those the mapping classes that commute with $j$.

Throughout this paper, we use functional notation. That is, for any two mapping classes $f$ and $g$, the multiplication $fg$ means that $g$ is applied first.

For a simple closed curve $a$ on the oriented surface $\Sigma_g$, by the abuse of notation, a right Dehn twist about $a$ and its isotopy class is denoted by $t_a$.

Let us consider the simple closed curves $A_1, A_2, \ldots, A_{2g+1}$ on $\Sigma_g$ defined as follows: $A_{2k} = b_k, A_1 = a_1, A_{2k-1} = a_k a_{k+1}^{-1}$, and $A_{2g+1} = a_g$. 


where \(a_k\) and \(b_k\) are the curves shown in Fig. 4. Let \(t_k\) denote the right Dehn twist about \(A_k\). The simple closed curves \(A_k\) are invariant under \(J\). It follows that Dehn twists \(t_1, t_2, \ldots, t_{2g+1}\) commute with \(j\). Hence, they are contained in the hyperelliptic mapping class group.

The involution \(J\) has \(2g+2\) fixed points. Hence, we have a branched covering \(p : \Sigma_g \rightarrow S^2\) branching over \(2g+2\) points. Let us mark these \(2g+2\) points on \(S^2\), and let \(\Sigma_{0,2g+2}\) be the resulting surface. Notice that the interior of each \(p(A_k)\) is an embedded arc on \(\Sigma_{0,2g+2}\) connecting two distinct marked points and that it is disjoint from \(p(A_l)\) for \(k \neq l\). Let us denote by \(w_k\) the isotopy class of a right half twist about \(p(A_k)\). Thus, if we orient the arc \(p(A_k)\) arbitrarily, then \(w_k(p(A_k))\) (defined up to isotopy) is isotopic to the arc \(p(A_k)^{-1}\). Therefore, \(w_k^2\) is the right Dehn twist about the boundary component of a regular neighborhood of \(p(A_k)\). It is well-known that the half twists \(w_1, w_2, \ldots, w_{2g+1}\) generate the mapping class group \(M_{0,2g+2}\) of \(\Sigma_{0,2g+2}\) (cf. \[4\], Theorem 4.5]). Here, the group \(M_{0,2g+2}\) is defined to be the group of the isotopy classes of the orientation-preserving diffeomorphisms of \(\Sigma_{0,2g+2}\) that preserve the marked points setwise. The isotopies are assumed to fix each marked point.

**Theorem 3.1.** The hyperelliptic mapping class group \(C_{M_0}(j)\) is generated by the Dehn twists \(t_1, t_2, \ldots, t_{2g+1}\), the function given by \(\Psi(t_k) = w_k\) on the generators defines a surjective homomorphism
\[
\Psi : C_{M_0}(j) \rightarrow M_{0,2g+2},
\]
and the kernel of \(\Psi\) is \(\langle j \rangle\), which is a subgroup of order 2.

Theorem 3.1 was proved by J. Birman and H. Hilden \[2\]. They also obtained a presentation of the hyperelliptic mapping class group. Since \(t_it_j = t_jt_i\) for \(|i-j| \geq 2\) and \(t_it_{i+1}t_i = t_{i+1}t_it_{i+1}\) in the group \(C_{M_0}(j)\), the fact that \(t_1, t_2, \ldots, t_{2g+1}\) generate \(C_{M_0}(j)\) implies that \(\sigma_k \mapsto t_k\) defines a surjective homomorphism \(B_{2g+2} \rightarrow C_{M_0}(j)\).

The following lemma is easy to prove (cf. Fig. 4 (a)).

**Lemma 3.2.** In the group \(M_{0,2g+2}\), the element \((w_k \cdots w_2 w_1)^{k+1}\) is equal to the right Dehn twist about the boundary component of a regular neighborhood of \(p(A_1) \cup p(A_2) \cup \cdots \cup p(A_k)\).

In order to state the main result of this section, let us consider the simple closed curves \(B_k, a, b,\) and \(c\) illustrated in Fig. 4. Note that \(a\) and \(b\) are defined for odd \(g\), and \(c\) is defined for even \(g\).

**Lemma 3.3.** The following relations hold in the mapping class group:
\[\begin{align*}
(a) & \ t_c = (t_g \cdots t_2 t_1)^{2g+1} \text{ if } g \text{ is even}; \text{ and}\\
(b) & \ t_at_b = (t_g \cdots t_2 t_1)^{g+1} \text{ if } g \text{ is odd}.
\end{align*}\]
Proof. Suppose that \( g \) is even. Consider the branched covering \( p : \Sigma_g \to S^2 \). Notice that \( p(c) \) is a simple closed curve on \( \Sigma_{0,2g+2} \). The projection \( \Psi(t_c) \) of \( t_c \) to \( \Sigma_{0,2g+2} \) is the square of the Dehn twist about \( p(c) \). This can be seen geometrically as follows. Consider the arcs \( p(A_1), p(A_2), \ldots, p(A_{2g+1}) \). Since the surface obtained by cutting \( \Sigma_{0,2g+2} \) along these arcs is a disc without any marked points in the interior, in order to show that \( \Psi(t_c) = t^2_{p(c)} \), it is enough to check that the actions of \( \Psi(t_c) \) and \( t^2_{p(c)} \) on these arcs are the same (up to isotopy). To see the action of \( \Psi(t_c) \) on an arc \( A' \), lift \( A' \) to \( \Sigma_g \), apply \( t_c \), and then project it down to \( \Sigma_{0,2g+2} \) (cf. Fig. 3). Since \( t^2_{p(c)} = (w_g \cdots w_2 w_1)^{g+1} \) by Lemma 3.2, we conclude that \( \Psi(t_c) = \Psi((t_g \cdots t_2 t_1)^{2(g+1)}) \). Hence, \( (t_g \cdots t_2 t_1)^{2(g+1)} \) is equal to either \( t_c \) or \( j t_c \). We rule out the latter possibility as follows. It is easily checked that \( (t_g \cdots t_2 t_1)^{2(g+1)} \) acts trivially on the first homology group \( H_1(\Sigma_g; \mathbb{Z}) \). The action of \( t_c \) is also trivial since \( c \) is null homologous. On the other hand, \( j \) acts as the minus identity on \( H_1(\Sigma_g; \mathbb{Z}) \).

The part (b) is proved similarly. \( \square \)
Figure 3. Projection of the Dehn twist $t_c$.

**Theorem 3.4.** In the mapping class group $M_g$, the following relations between right Dehn twists hold:

(a) $(t_{B_0} t_{B_1} t_{B_2} \cdots t_{B_g} t_c)^2 = 1$ if $g$ is even;
(b) $(t_{B_0} t_{B_1} t_{B_2} \cdots t_{B_g} t_a^2 t_b^2)^2 = 1$ if $g$ is odd.

**Proof.** In the mapping class group $M_g$ of $\Sigma_g$, for each $k = 0, 1, 2, \ldots, g$, we define $\Delta_k$, $\bar{\Delta}_k$, $\beta_k$, and $\beta$ as in Section 2 by replacing $\sigma_i$ by $t_i$. Recall that $t_i$ is the (right) Dehn twist about the simple closed curve $A_i$. Hence,

$$\beta_k = (\bar{\Delta}_k \Delta_{2g-k}) t_{2g+1-k} (\bar{\Delta}_k \Delta_{2g-k})^{-1}.$$ 

It is easy to see that $\bar{\Delta}_i \Delta_{2g-i}(A_{2g+1-i}) = B_i$. Since $f t_e f^{-1} = t_{f(e)}$ for any $f \in M_g$ and for any simple closed curve $e$, we conclude that $\beta_k = t_{B_k}$. Also, by Lemma 3.3, $\beta^2 = t_c$ if $g$ is even and $\beta^2 = t_a^2 t_b^2$ if $g$ is odd. Let us define the word

$$W = \begin{cases} 
(t_{B_0} t_{B_1} t_{B_2} \cdots t_{B_g} t_c)^2 & \text{if } g \text{ is even,} \\
(t_{B_0} t_{B_1} t_{B_2} \cdots t_{B_g} t_a^2 t_b^2)^2 & \text{if } g \text{ is odd.}
\end{cases}$$

Hence, $W = (\beta_0 \beta_1 \cdots \beta_g \beta^2)^2$. Let $\Delta = \Delta_{2g+1} \Delta_{2g} \cdots \Delta_1$. Since $t_i t_j = t_j t_i$ for $|i - j| \geq 2$ and $t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$ by Theorem 2.4, we obtain $W = \Delta^2$. Now the element $\Psi(\Delta)$ is of order 2 in $M_{0,2g+2}$ (cf. Fig. 4 (b) for $g = 6$), where $\Psi$ is the epimorphism in Theorem 3.1. Hence, either $\Delta^2 = 1$ or $\Delta^2 = J$. It is easy to verify that $\Delta^2$ acts trivially on the first homology group $H_1(\Sigma_g; \mathbb{Z})$ but $J$ does not. Therefore, $W = 1$.

This finishes the proof of the theorem. $\square$
4. Noncomplex genus-

In this section, we prove the main result of this paper. We assume from now on that $g \geq 2$. We construct a 4-manifold $X_n$ a admitting genus-$g$ Lefschetz fibration with fundamental group $\mathbb{Z} \oplus \mathbb{Z}_n$ for every positive integer $n$. Then we conclude the main result of this paper from the proof of the main result of [12]. As the model for a closed connected oriented surface $\Sigma_g$, we will consider the one embedded in $\mathbb{R}^3$ as shown in Fig. 4.

For any two elements $x$ and $y$ in a group, we denote $yx y^{-1}$ and $x y x^{-1} y^{-1}$ by $xy$ and $[x, y]$, respectively.

Let us consider the word $W$ in $M_g$ defined by

$$W = \left\{ \begin{array}{ll} (t_{B_0} t_{B_1} t_{B_2} \cdots t_{B_g} t_c)^2 & \text{if } g \text{ is even}, \\ (t_{B_0} t_{B_1} t_{B_2} \cdots t_{B_n} t_{a_b}^2)^2 & \text{if } g \text{ is odd}. \end{array} \right.$$ 

By Theorem 3.1, $W = 1$ in $M_g$. Since the conjugation of a right Dehn twist with an element of $M_g$ is again a right Dehn twist, the word $W^f$ is a product of right Dehn twists for any mapping class $f$. For each positive integer $n$, we define

$$W_n = \left\{ \begin{array}{ll} W W^{t_{a_1}} W^{t_{a_2}} \cdots W^{t_{a_r-1}} W^{t_{a_r}} W^{t_{b_r+2}} W^{t_{b_r+3}} \cdots W^{t_{b_g}} & \text{if } g = 2r, \\ W W^{t_{c_3}} W^{t_{c_2}} W^{t_{c_1}} \cdots W^{t_{c_{r+1}}} W^{t_{b_r+2}} W^{t_{b_r+3}} \cdots W^{t_{b_g}} & \text{if } g = 2r + 1, \end{array} \right.$$ 

where $a_i$ and $b_i$ are simple closed curves given in Fig. 4 (considered up to isotopy). Note that $W_n$ is a product of $g(2g + 4)$ (resp., $g(2g + 10)$) right Dehn twists if $g$ is even (odd).

The word $W_n$ is equal to the identity in the mapping class group $M_g$. Let $X$ and $X_n$ be the smooth 4-manifolds that admit the genus-$g$ Lefschetz fibrations over $S^2$ whose global monodromies are $W$ and $W_n$, respectively. Thus, $X_n$ is the fiber sum of $g$ copies of $X$.

**Theorem 4.1.** The fundamental group $\pi_1(X_n)$ of $X_n$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_n$.

**Proof.** Let $a_k$ and $b_k$ be the standard generators of $\pi_1(\Sigma_g)$ illustrated in Fig. 4. By the theory of Lefschetz fibrations, $\pi_1(X_n)$ is isomorphic to the quotient of $\pi_1(\Sigma_g)$ by the normal subgroup generated by the vanishing cycles.

Suppose first that $g = 2r$. It is easy to check that up to conjugation the following equalities hold in $\pi_1(\Sigma_g)$:

- $B_0 = b_1 b_2 \cdots b_g$;
- $B_{2k-1} = a_k b_k b_{k+1} \cdots b_{g+1-k} c_{g+1-k} a_{g+1-k} a_{g+1-k}$, $1 \leq k \leq r$;
- $B_{2k} = a_k b_{k+1} b_{k+2} \cdots b_{g-k} c_{g-k} a_{g+1-k}$, $1 \leq k \leq r - 1$;
- $B_g = B_{2r} = a_r c_r a_{r+1}$;
- $c = c_r = [a_1, b_1][a_2, b_2] \cdots [a_r, b_r]$. 

For any two elements $x$ and $y$ in a group, we denote $xy y^{-1}$ and $x y x^{-1} y^{-1}$ by $xy$ and $[x, y]$, respectively.
The vanishing cycles corresponding to $W^t_{ak}$ are the set
\[ \{ a_kB_0, \ldots, a_kB_{2k-1}, B_{2k}, \ldots, B_g, c \}, \ 1 \leq k \leq r - 1. \]
Similarly, the vanishing cycles corresponding to $W^t_{ar}$ and $W^t_{bg+1-k}$ are
\[ \{ a^n_rB_0, \ldots, a^n_rB_{g-1}, B_g, c \}, \ 1 \leq k \leq r - 1, \]
and
\[ \{ B_0, \ldots, B_{2k-2}, b_{g+1-k}^{-1}B_{2k-1}, b_{g+1-k}^{-1}B_{2k}, B_{2k+1}, \ldots, B_g, c \}, \ 1 \leq k \leq r - 1, \]
respectively. It follows that the fundamental group of $X_n$ has a presentation with generators $a_1, b_1, a_2, b_2, \ldots, a_g, b_g$ and relations
- $\Pi_{k=1}^g [a_k, b_k] = 1$;
- $B_0 = B_1 = B_2 = \cdots = B_g = c = 1$;
- $a_1 = a_2 = \cdots = a_{r-1} = a^n_1 = b_{r+2} = b_{r+3} = \cdots = b_g = 1$.

It is easy to see that this presentation is equivalent to the presentation with generators $a_r, b_r$ and relations $a^n_r = [a_r, b_r] = 1$.

Suppose now that $g = 2r + 1$. A similar argument as in the case of even $g$ shows that the fundamental group of $X_n$ has a presentation with generators $a_1, b_1, a_2, b_2, \ldots, a_g, b_g$ and relations
- $\Pi_{k=1}^g [a_k, b_k] = 1$;
- $B_0 = B_1 = B_2 = \cdots = B_g = a = b = 1$;
- $(a_2a_1^{-1})^n = a_3a_2^{-1} = a_4a_3^{-1} = \cdots = a_{r+1}a_r^{-1} = b_{r+2} = b_{r+3} = \cdots = b_g = 1$.

Since $a = a_{r+1}$, this presentation is equivalent to the presentation with generators $a_1, b_1$ and relations $a^n_1 = [a_1, b_1] = 1$.

This completes the proof of the theorem.

\[ \text{Figure 4. Generators of the fundamental group.} \]

The following theorem can be concluded from [12, proof of Theorem 1.3].

**Theorem 4.2.** Let $M$ be an orientable 4-manifold such that $b_2^+(M) \geq 1$ and $\pi_1(M) = \mathbb{Z} \oplus \mathbb{Z}_n$. Then $M$ does not carry any complex structure.
Proof of Theorem 1.1. The smooth 4-manifold $X_n$ admits a genus-$g$ Lefschetz fibration for every positive integer $n$. Since $X_n$ is symplectic by a result of R. Gompf [6], we have $b_2^+(X_n) \geq 1$. We showed above that $\pi_1(X_n) = \mathbb{Z} \oplus \mathbb{Z}_n$. Hence, the manifold $X_n$ is not homeomorphic to $X_m$ for $n \neq m$. By Theorem 1.2, the manifold $X_n$ and the manifold obtained from it by reversing the orientation do not admit any complex structure.

5. The 4-manifold admitting a Lefschetz fibration with global monodromy $W$.

In this section, we determine the 4-manifold corresponding to the word $W$ given in Section 3.

The case of even $g$. Notice that simple closed curves $B_0, B_1, \ldots, B_g$, and $c$ are invariant under the involution $J$. Hence, the genus-$g$ Lefschetz fibration $X \to S^2$ with global monodromy $W$ is is hyperelliptic.

**Theorem 5.1 ([8],[9],[4],[11]).** Let $M$ be a 4-manifold that admits a hyperelliptic Lefschetz fibration of genus $g$ over $S^2$. Let $m$ and $s = \sum_{h=1}^{[g/2]} s_h$ be the numbers of nonseparating and separating vanishing cycles in the global monodromy of this fibration, respectively, where $s_h$ denotes the number of separating vanishing cycles that separate the genus-$g$ surface into two surfaces one of which has genus $h$. Then the signature of $M$ is

$$\sigma(M) = -\frac{g+1}{2g+1} m + \sum_{h=1}^{[g/2]} \left( \frac{4h(g-h)}{2g+1} - 1 \right) s_h.$$  

Since there are $2g + 4$ vanishing cycles, Euler characteristic of $X$ is $\chi(X) = 2(2 - 2g) + 2g + 4 = 8 - 2g$. There are only two separating vanishing cycles, and they bound a surface of genus $g/2$ on both sides. Hence, the signature of $X$ is

$$\sigma(X) = -\frac{g+1}{2g+1} (2g + 2) + \left( \frac{4g(g - g/2)}{2g + 1} - 1 \right) 2 = -4.$$  

The group $\pi_1(X)$ has a presentation with generators $a_1, b_1, \ldots, a_g, b_g$ and relations $\Pi_{k=1}^g [a_k, b_k] = B_0 = B_1 = B_2 = \cdots = B_g = c = 1$. It is now easy to see that $H_1(X; \mathbb{Z}) = \mathbb{Z}^g$. In particular, $b_1(X) = g$. It follows from $\chi(X) = 8 - 2g$ and $\sigma(X) = -4$ that $b_2^+(X) = 1$. By [14, Remark 4.5(a)], $X$ is diffeomorphic to $\Sigma_{g/2} \times S^2 \# 4CP^2$ if $g \geq 6$. 

The case of odd $g$. Suppose that $g$ is at least 3 and odd. Let $X \to S^2$ be the genus-$g$ Lefschetz fibration with global monodromy $W$. Since there are $2g+10$ singular fibers, the Euler characteristic of $X$ is $\chi(X) = 2(2 - 2g) + 2g + 10 = 14 - 2g$. The fundamental group $\pi_1(X)$ of $X$ has a presentation with generators $a_1, b_1, \ldots, a_g, b_g$ and relations $\prod_{k=1}^g [a_k, b_k] = B_0 = B_1 = B_2 = \cdots = B_g = a = b = 1$. It is now easy to see that $H_1(X; \mathbb{Z}) = \mathbb{Z}^{g-1}$. In particular, $b_1(X) = g - 1$. Hence,

$$14 - 2g = 2 - 2b_1(X) + b_2(X) = 2 - 2(g - 1) + b_2(X);$$

that is, $b_2(X) = 10$.

The manifold $X$ is symplectic. Since $1 - b_1 + b_2^+$ is even for any symplectic manifold, we conclude that $b_2^+(X)$ is odd and is between 1 and 9. Hence, $b_2^-(X)$ is also odd and is between 1 and 9.

We now determine the signature of $X$. A handlebody decomposition for $X$ is obtained as follows. Start with $\Sigma_g \times D^2$, where $D^2$ is the 2-disc. Its boundary is $\Sigma_g \times S^1$. Attach a 2-handle along each vanishing cycle (by counting its multiplicity) with the $-1$ framing relative to the product framing. The cores of the first two 2-handles attached along $a$ gives us a $(-2)$-sphere $S_1$. Denote the class of $S_1$ in $H_2(X; \mathbb{R})$ by $[S_1]$. Similarly, the cores of the second and the third 2-handles, and the third and the fourth 2-handles give two $(-2)$-spheres $S_2$ and $S_3$. Note that $[S_1][S_3] = 0$. Orient each $S_i$ so that $[S_1][S_2] = [S_2][S_3] = -1$. Similarly, the four 2-handles glued along $b$ give three $(-2)$-spheres $S_4, S_5, S_6$ with $[S_4][S_5] = [S_5][S_6] = -1$ and $[S_4][S_6] = 0$. Since $a$ and $b$ are disjoint, $[S_i][S_j] = 0$ for $1 \leq i \leq 3$ and $4 \leq j \leq 6$. The first handles attached along $a$ and $b$ give a surface $S_7$ of genus $(g - 1)/2$ such that $[S_7]^2 = -2$, $[S_7][S_1] = [S_7][S_4] = -1$, and $[S_7][S_i] = 0$ for $i = 2, 3, 5, 6$.

The homology classes $[S_1], \ldots, [S_7]$ are linearly independent. Hence, they form a basis for a subspace $V$ of $H_2(X; \mathbb{R})$ of dimension 7. The matrix of the intersection form restricted to $V$ in the above basis is the matrix $-A$, where

$$A = \begin{pmatrix}
2 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 2 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 2 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 2 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 2
\end{pmatrix}$$

It is easily check that the matrix $A$ is positive definite. Hence, the restriction of the intersection form to $V$ is negative definite.
On the other hand, we have \([F] \neq 0, [F]^2 = 0\) and \([F] \in V^\perp\), where \(F\) is a generic fiber and \(V^\perp\) is the orthogonal complement of \(V\). Since the restriction of the intersection form to \(V^\perp\) is nondegenerate, there is a class \([S_8] \in V^\perp\) with \([S_8]^2 < 0\). Thus, the restriction of the intersection form to the 8-dimensional subspace generated by \([S_1], \ldots, [S_8]\) is negative definite. Therefore, \(b_2^-(X)\) is at least 8, and hence \(b_2^+(X) = 9\), since it is also odd. Consequently, we have \(b_2^+(X) = 1\) and \(\sigma(X) = -8\).

**Remarks 5.2.** (1) Stipsicz [14] points out that when \(g\) is odd, the manifold \(X\) that admits a Lefschetz fibration over \(S^2\) with global monodromy \(W\) is diffeomorphic to \(\Sigma_{(g-1)/2} \times S^2 \# 8\mathbb{CP}^2\).

(2) R. Fintushel and R. Stern [5] constructed infinite classes of simply connected homeomorphic but nondiffeomorphic symplectic manifolds, all of which admit Lefschetz fibrations of a fixed fiber genus.

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