On a reconstruction theorem for holonomic systems

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Abstract: Let $X$ be a complex manifold. The classical Riemann-Hilbert correspondence associates to a regular holonomic system $\mathcal{M}$ the $\mathbb{C}$-constructible complex of its holomorphic solutions. Let $t$ be the affine coordinate in the complex projective line. If $\mathcal{M}$ is not necessarily regular, we associate to it the ind-$\mathbb{R}$-constructible complex $G$ of tempered holomorphic solutions to $G \boxtimes D\mathcal{C}^t$. We conjecture that this provides a Riemann-Hilbert correspondence for holonomic systems. We discuss the functoriality of this correspondence, we prove that $\mathcal{M}$ can be reconstructed from $G$ if $\dim X = 1$, and we show how the Stokes data are encoded in $G$.

Key words: Riemann-Hilbert problem; holonomic $\mathcal{D}$-modules; ind-sheaves; Stokes phenomenon.

Introduction. Let $X$ be a complex manifold. The Riemann-Hilbert correspondence of [2] establishes an anti-equivalence

$$D^b_{\text{hol}}(\mathcal{D}_X) \overset{\Phi}{\rightarrow} D^b_{\text{C, c}}(\mathcal{C}_X)$$

between regular holonomic $\mathcal{D}$-modules and $\mathbb{C}$-constructible complexes. Here, $\Phi^0(\mathcal{L}) = R\text{Hom}_{\mathcal{D}_X}(\mathcal{L}, \mathcal{O}_X)$ is the complex of holomorphic solutions to $\mathcal{L}$, and $\Psi^0(\mathcal{L}) = \mathcal{T}\text{Hom}(\mathcal{L}, \mathcal{O}_X) = R\text{Hom}(\mathcal{L}, \mathcal{O}_X^\mathbb{C})$ is the complex of holomorphic functions tempered along $L$. Since $\mathcal{L} \simeq \Psi^0(\Phi^0(\mathcal{L}))$, this shows in particular that $\mathcal{L}$ can be reconstructed from $\Phi^0(\mathcal{L})$.

We are interested here in holonomic $\mathcal{D}$-modules which are not necessarily regular.

The theory of ind-sheaves from [6] allows one to consider the complex $\Phi^t(\mathcal{M}) = R\text{Hom}_{\mathcal{D}_X}^t(\mathcal{M}, \mathcal{O}_X^\mathbb{C})$ of tempered holomorphic solutions to a holonomic module $\mathcal{M}$. The basic example $\Phi^t(D\mathcal{C}^1/z)$ was computed in [7], and the functor $\Phi^t$ has been studied in [10,11]. However, since $\Phi^t(D\mathcal{C}^{1/z}) \simeq \Phi^t(D\mathcal{C}^{1/z})$, one cannot reconstruct $\mathcal{M}$ from $\Phi^t(\mathcal{M})$.

Set $\Phi(\mathcal{M}) = \Phi^t(\mathcal{M} \boxtimes D\mathcal{P}^t)$, for $t$ the affine variable in the complex projective line $\mathcal{P}$. This is an ind-$\mathbb{R}$-constructible complex in $X \times \mathcal{P}$. The arguments in [1] suggested us how $\mathcal{M}$ could be reconstructed from $\Phi(\mathcal{M})$ via a functor $\Psi$, described below ($\S$3).

We conjecture that the contravariant functors

$$D^b(\mathcal{D}_X) \overset{\Psi}{\leftarrow} D^b(\mathcal{IC}_X \times \mathcal{P})$$

between the derived categories of $\mathcal{D}_X$-modules and of ind-sheaves on $X \times \mathcal{P}$, provide a Riemann-Hilbert correspondence for holonomic systems.

To corroborate this statement, we discuss the functoriality of $\Phi$ and $\Psi$ with respect to proper direct images and to tensor products with regular objects ($\S$4). This allows us to reduce the problem to the case of holonomic modules with a good formal structure.

When $X$ is a curve and $\mathcal{M}$ is holonomic, we prove that the natural morphism $\mathcal{M} \rightarrow \Psi(\Phi(\mathcal{M}))$ is an isomorphism ($\S$6). Thus $\mathcal{M}$ can be reconstructed from $\Phi(\mathcal{M})$.

Recall that irregular holonomic modules are subjected to the Stokes phenomenon. We describe with an example how the Stokes data of $\mathcal{M}$ are encoded topologically in the ind-$\mathbb{R}$-constructible sheaf $\Phi(\mathcal{M})$ ($\S$7).

In this Note, the proofs are only sketched. Details will appear in a forthcoming paper. There, we will also describe some of the properties of the essential image of holonomic systems by the functor $\Phi$. Such a category is related to a construction of [13].

1. Notations. We refer to [3–6].

Let $X$ be a real analytic manifold.

Denote by $D^b(\mathcal{C}_X)$ the bounded derived category of sheaves of $\mathbb{C}$-vector spaces, and by

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\(D_{\mathbb{R},c}^b(C_X)\) the full subcategory of objects with \(R\)-
constructible cohomologies. Denote by \(\otimes, RH\text{Hom}, \ f^{-1}, Rf_!, Rf_*\), \(f\) the six Grothendieck operations for
sheaves. (Here \(f:X \rightarrow Y\) is a morphism of real
analytic manifolds.)

For \(S \subset X\) a locally closed subset, we denote by \(C_S\) the zero extension to \(X\) of the constant sheaf
on \(S\).

Recall that an ind-sheaf is an ind-object in the
category of sheaves with compact support. Denote by \(D^b(\mathbb{IC}_X)\) the bounded derived category of ind-
sheaves, and by \(D^b_{\mathbb{IC}_X}(\mathbb{IC}_X)\) the full subcategory of objects with ind-\(R\)-
constructible cohomologies. Denote by \(\otimes, RH\text{Hom}, \ f^{-1}, Rf_!, Rf_*\), \(f\) the six Grothendieck operations for ind-sheaves.

Denote by \(\alpha\) the left adjoint of the embedding of sheaves into ind-sheaves. One has \(\alpha(\lim^{\infty} F_i) = \lim F_i\). Denote by \(\beta\) the left adjoint of \(\alpha\).

Denote by \(D_{\mathbb{R},c}^b(C_X)\) the ind-\(R\)-
constructible sheaf of tempered distributions.

Let \(X\) be a complex manifold. We set for short
\(d_X = \dim X\).

Denote by \(O_X\) and \(D_X\) the rings of holomorphic functions and of differential operators. Denote by \(\Omega_X\) the invertible sheaf of differential forms of top
degree.

Denote by \(D^b(D_X)\) the bounded derived category of \(L\) \(D_X\)-modules, and by \(D^b_{\text{hol}}(D_X)\) and \(D^b_{\text{hol}}(\mathbb{IC}_X)\) the full subcategory of objects with holonomic and regular holonomic cohomologies, respectively. Denote by \(\otimes^D, Df^{-1}, Df_!\) the operations for \(D\)-modules. (Here \(f:X \rightarrow Y\) is a morphism of complex manifolds.)

Denote by \(D_M\) the dual of \(M\) (with shift such that
\(DO_X \cong \Omega_X\)).

For \(Z \subset X\) a closed analytic subset, we denote by \(RF[Z]M\) and \(M(+Z)\) the relative algebraic
cohomologies of a \(D_X\)-module (with shift).

Denote by \(ss(M) \subset X\) the singular support of
\(M\), that is the set of points where the characteristic
varieties does not reduce to the zero-section.

Denote by \(O_X \subset D^b_{\mathbb{IC}_X}(\mathbb{IC}_X)\) the complex of
tempered holomorphic functions. Recall that \(O_X\) is the Dolbeault complex of \(D_{\mathbb{R},c}^b(C_X)\) and that it has a structure of \(\beta D_X\)-module. We will write for short \(RH\text{Hom}_{D_X}(M, O_X)\) instead of \(RH\text{Hom}_{D_X}(\beta M, O_X)\).

2. Exponential \(D\)-modules. Let \(X\) be a
complex analytic manifold, let \(D \subset X\) be a hypersurface, and set \(U = X \setminus D\). For \(\varphi \in O_X(+D)\), we set

\[\mathcal{E}_{D_X}^{\varphi} = \mathcal{E}_X / \{ P; Pe^\varphi = 0 \text{ on } U \},\]
\[\mathcal{E}_{D_X}^{\varphi} = (\mathcal{D}_X e^\varphi)(+D).\]

As an \(O_X(+D)\)-module, \(\mathcal{E}_{D_X}^{\varphi}\) is generated by \(e^\varphi\).

Note that \(ss(\mathcal{E}_{D_X}^{\varphi}) = D\), and \(\mathcal{E}_{D_X}^{\varphi}\) is holonomic. It is regular if \(\varphi \in O_X\), since then \(\mathcal{E}_{D_X}^{\varphi} = \mathcal{O}_X(+D)\).

One easily checks that \((\mathcal{D}_\mathcal{E}_{D_X}^{\varphi})(+D) \cong \mathcal{E}_{D_X}^{\varphi}\).

**Proposition 2.1.** If \(\dim X = 1\), and \(\varphi\) has an effective pole at every point of \(D\), then \(\mathcal{D}_\mathcal{E}_{D_X}^{\varphi} \cong \mathcal{E}_{D_X}^{\varphi}\).

Let \(P\) be the complex projective line and denote by \(t\) the coordinate on \(C = P \setminus \{\infty\}\).

For \(c \in R\), we set for short
\[\{ Re \varphi < c \} = \{ x \in U; Re \varphi(x) < c \},\]
\[\{ Re(t + \varphi) < c \} = \{ (x, t): x \in U, t \in C, \ Re(t + \varphi(x)) < c \}.\]

Consider the ind-\(R\)-constructible sheaves on \(X\) and on \(X \times P\), respectively,
\[C_{\{ Re \varphi < c \}} = \lim_{c \to -\infty}\ C_{\{ Re \varphi < c \}},\]
\[C_{\{ Re(t + \varphi) < c \}} = \lim_{c \to -\infty}\ C_{\{ Re(t + \varphi) < c \}}.\]

The following result is analogous to
[1, Proposition 7.1]. Its proof is simpler than loc.
cit., since \(\varphi\) is differentiable.

**Proposition 2.2.** One has an isomorphism
in \(D^b(D_X)\)
\[\mathcal{E}_{D_X}^{\varphi} \cong RH\text{Hom}_{D_X}(\varphi_1^*=\varphi_2^*),\]
\[RH\text{Hom}_{D_X}(C_{(Re(t + \varphi) < c)}(O_X + P)),\]
for \(q\) and \(p\) the projections from \(X \times P\).

The following result is analogous to
[7, Proposition 7.3].

**Lemma 2.3.** Denote by \((u, v)\) the coordinates in \(C^2\). There is an isomorphism in \(D^b(\mathbb{IC}_C^2)\)
\[RH\text{Hom}_{D_X}(\mathcal{E}_{\{ Re(u/v) < c \}}(O_X + P)),\]
\[RH\text{Hom}(C_{\{ Re(u/v) < c \}}).\]

**Proposition 2.4.** There is an isomorphism
in \(D^b(D_X)\)
\[RH\text{Hom}_{D_X}(\mathcal{E}_{\{ Re(u/v) < c \}}(O_X + P)),\]
\[RH\text{Hom}(C_{\{ Re(u/v) < c \}}, C_{\{ Re(u/v) < c \}}).\]

Proof. As \(\mathcal{E}_{\{ Re(u/v) < c \}}(O_X + P) \cong \mathcal{E}_{\{ Re(u/v) < c \}}(C^2),\) Lemma 2.3 gives
\[\Omega^1_{C^2} \otimes_{D_X} \mathcal{E}_{\{ Re(u/v) < c \}}(O_X + P)) \cong \mathcal{E}_{\{ Re(u/v) < c \}}(O_X + P),\]
\[RH\text{Hom}(C_{\{ Re(u/v) < c \}}, C_{\{ Re(u/v) < c \}}).\]
Write $\varphi = a/b$ for $a,b \in \mathcal{O}_X$ such that $b^{-1}(0) \subset D$, and consider the map

$$f = (a,b): X \rightarrow \mathbb{C}^2.$$ 

As $D_f^{-1}\mathcal{E}^{\varphi}_{\{t=0\}} \simeq \mathcal{E}^{\varphi}_{D[X]}$ [6, Theorem 7.4.1] implies

$$\Omega^1_X \otimes D_X \mathcal{E}^{\varphi}_{D[X]}[-d_X] \simeq RHom(C_U, C_{(Re \varphi < 0)}).$$ 

Finally, one has

$$\Omega^1_X \otimes D_X \mathcal{E}^{\varphi}_{D[X]}[-d_X] \simeq RHom_{\mathcal{D}_X}(D\mathcal{E}^{\varphi}_{D[X]}, \Omega^1_X). \quad \square$$

3. A correspondence. Let $X$ be a complex analytic manifold. Consider the contravariant functors

$$\mathcal{D}^b_X(\mathcal{O}_X) \rightarrow \mathcal{D}^b_X(\mathcal{I}\mathcal{C}_X \times \mathbb{P})$$

defined by

$$\Phi(M) = RHom_{\mathcal{D}_X}(\mathcal{M} \otimes D\mathcal{E}^{\varphi}_{\mathcal{O}_X \times \mathbb{P}}, \mathcal{O}_{X \times \mathbb{P}}),$$

$$\Psi(F) = Rq_* RHom_{\mathcal{D}_P}(p^{-1}\mathcal{E}^{\varphi}_{\mathbb{P}}, RHom(F, \mathcal{O}(\mathbb{P}_X \times \mathbb{P}))),$$

for $q$ and $p$ the projections from $X \times \mathbb{P}$.

We conjecture that this provides a Riemann-Hilbert correspondence for holonomic systems:

**Conjecture 3.1.**

(i) The natural morphism of endofunctors of

$$\mathcal{D}^b_X(\mathcal{D}_X)$$

$$(3.1) \quad id \rightarrow \Psi \circ \Phi$$

is an isomorphism on $\mathcal{D}^b_{\text{hol}}(\mathcal{D}_X).$

(ii) The restriction of $\Phi$

$$\Phi|_{\mathcal{D}^b_{\text{hol}}(\mathcal{D}_X)}: \mathcal{D}^b_{\text{hol}}(\mathcal{D}_X) \rightarrow \mathcal{D}^b_X(\mathcal{I}\mathcal{C}_X \times \mathbb{P})$$

is fully faithful.

Let us prove some results in this direction.

4. Functorial properties. The next two Propositions are easily deduced from the results in [6].

**Proposition 4.1.** Let $f: X \rightarrow Y$ be a proper map, and set $f_P = f \times \text{id}_P$. Let $\mathcal{M} \in \mathcal{D}^b_{\text{hol}}(\mathcal{D}_X)$ and $F \in \mathcal{D}^b_{\text{hol}}(\mathcal{I}\mathcal{C}_X \times \mathbb{P})$. Then

$$\Phi(Df_*\mathcal{M}) \simeq Rf_P^*\Phi(M),$$

$$\Psi(Rf_P^*F) \simeq \mathcal{D}f_\ast \Psi(F).$$

For $\mathcal{L} \in \mathcal{D}^b_{\text{hol}}(\mathcal{D}_X)$, set

$$\Phi^0(\mathcal{L}) = RHom_{\mathcal{D}_X}(\mathcal{L}, \mathcal{O}_X).$$

Recall that $\Phi^0(\mathcal{L})$ is a $\mathbb{C}$-constructible complex of sheaves on $X$. 

**Proposition 4.2.** Let $\mathcal{L} \in \mathcal{D}^b_{\text{hol}}(\mathcal{D}_X)$, $\mathcal{M} \in \mathcal{D}^b_{\text{hol}}(\mathcal{D}_X)$ and $F \in \mathcal{D}^b_{\mathbb{R}\text{hol}}(\mathcal{I}\mathcal{C}_X \times \mathbb{P})$. Then

$$\Phi(D(\mathcal{L} \otimes\mathcal{D}_X \mathcal{M})) \simeq RHom(q^{-1}\Phi^0(\mathcal{L}), \Phi(\mathcal{M})),$$

$$\Psi(F \otimes q^{-1}\Phi^0(\mathcal{L})) \simeq \Psi(F) \otimes\mathcal{D}_X \mathcal{L}.$$ 

Noticing that

$$\Phi(\mathcal{O}_X) \simeq \mathbb{C} \otimes RHom(\mathcal{C}_{[p<\infty]}, \mathcal{C}_{[Re \varphi < 0]}),$$

one checks easily that $\Psi(\Phi(\mathcal{O}_X)) \simeq \mathcal{O}_X$. Hence, Proposition 4.2 shows:

**Theorem 4.3.**

(i) For $\mathcal{L} \in \mathcal{D}^b_{\text{hol}}(\mathcal{D}_X)$, we have

$$\Phi(\mathcal{L}) \simeq q^{-1}\Phi^0(\mathcal{L}) \otimes \Phi(\mathcal{O}_X) \simeq \Phi^0(\mathcal{L}) \otimes RHom(\mathcal{C}_{[p<\infty]}, \mathcal{C}_{[Re \varphi < 0]}).$$

(ii) The morphism (3.1) is an isomorphism on $\mathcal{D}^b_{\text{hol}}(\mathcal{D}_X)$.

(iii) For any $\mathcal{L}, \mathcal{L}' \in \mathcal{D}^b_{\text{hol}}(\mathcal{D}_X)$, the natural morphism

$$\text{Hom}_{\mathcal{D}_X}(\mathcal{L}, \mathcal{L}') \rightarrow \text{Hom}(\Phi(\mathcal{L}'), \Phi(\mathcal{L}))$$

is an isomorphism.

Therefore, Conjecture 3.1 holds true for regular holonomic $\mathcal{D}$-modules.

5. Review on good formal structures.

Let $D \subset X$ be a hypersurface. A flat meromorphic connection with poles at $D$ is a holonomic $\mathcal{D}_X$-module $\mathcal{M}$ such that $ss(\mathcal{M}) = D$ and $\mathcal{M} \simeq M(\mathcal{D}_X)$. We recall here the classical results on the formal structure of flat meromorphic connections on curves. (Analogous results in higher dimension have been obtained in [8,9,12].)

Let $X$ be an open disc in $\mathbb{C}$ centered at $0$.

For $\mathcal{F}$ an $\mathcal{O}_X$-module, we set

$$\mathcal{F}_{\text{loc}} = \hat{\mathcal{O}}_{X,0} \otimes_{\mathcal{O}_X} \mathcal{F},$$

where $\hat{\mathcal{O}}_{X,0}$ is the completion of $\mathcal{O}_{X,0}$.

One says that a flat meromorphic connection $\mathcal{M}$ with poles at $0$ has a good formal structure if

$$\mathcal{M}_{\text{loc}} \simeq \bigoplus_{i \in I} \left( \mathcal{L}_i \otimes \mathcal{D}_X^{\varphi_i}_{0,\mathcal{X}} \right)_{\text{loc}}$$

as $(\hat{\mathcal{O}}_{X,0} \otimes_{\mathcal{O}_X} \mathcal{D}_{X,0})$-modules, where $I$ is a finite set, $\mathcal{L}_i$ are regular holonomic $\mathcal{D}_X$-modules, and $\varphi_i \in \mathcal{O}_X(\{0\})$.

A ramification at $0$ is a map $X \rightarrow X$ of the form $x \mapsto x^m$ for some $m \in \mathbb{N}$.

The Levelt-Turrittin theorem asserts:
Theorem 5.1. Let $\mathcal{M}$ be a meromorphic connection with poles at 0. Then there is a
ramification $f: X \to X$ such that $Df^{-1}\mathcal{M}$ has a good formal structure at 0.
Assume that $\mathcal{M}$ satisfies (5.1). If $\mathcal{M}$ is regular, then $\varphi_i \in \mathcal{O}_X$ for all $i \in I$, and (5.1) is induced by an
isomorphism
$$\mathcal{M}_0 \simeq \bigoplus_{i \in I} \left( \mathcal{L}_i \otimes \mathcal{E}_{0,X}^\circ \right)_0.$$ However, such an isomorphism does not hold in general.
Consider the real oriented blow-up
\begin{equation}
\pi: B = \mathbb{R} \times S^1 \to X, \quad (\rho, \theta) \mapsto \rho e^{i\theta}.
\end{equation}
Set $V = \{ \rho > 0 \}$ and let $Y = \{ \rho \geq 0 \}$ be its closure.
If $W$ is an open neighborhood of $(0, \theta) \in \partial Y$, then \(\pi(W \cap V)\) contains a germ of open sector around
the direction $\theta$ centered at 0.
Consider the commutative ring
$$A_Y = R\mathrm{Hom}_{\mathbb{Z}^{-1}R} (\pi^{-1}\mathcal{O}_X, R\mathrm{Hom}(\mathbb{C}_Y, \mathcal{D}^b_{\mathcal{H}})),$$
where $\overline{X}$ is the complex conjugate of $X$. To a $\mathcal{D}_X$-module $\mathcal{M}$, one associates the $A_Y$-module
$$\pi^*\mathcal{M} = A_Y \otimes_{\mathbb{Z}^{-1}R} \pi^{-1}\mathcal{M}.$$ The Hukuara-Turrittin theorem states that (5.1) can be extended to germs of open sectors:

**Theorem 5.2.** Let $\mathcal{M}$ be a flat meromorphic connection with poles at 0. Assume that $\mathcal{M}$ admits the good formal structure (5.1). Then for any
$(0, \theta) \in \partial Y$ one has
\begin{equation}
(\pi^*\mathcal{M})_{(0,\theta)} \simeq \left( \bigoplus_{i \in I} \pi^i (\mathcal{E}_{0,X}^\circ)^{m_i} \right)_{(0,\theta)},
\end{equation}
where $m_i$ is the rank of $\mathcal{L}_i$.
(Note that only the ranks of the $\mathcal{L}_i$’s appear here, since $x^i (\log x)^m$ belongs to $A_Y$ for any $\lambda \in \mathbb{C}$
and $m \in \mathbb{Z}_{\geq 0}$.)

One should be careful that the above isomorphism depends on $\theta$, giving rise to the Stokes
phenomenon.

We will need the following result:

**Lemma 5.3.** If $\mathcal{M}$ is a flat meromorphic connection with poles at 0, then
$$R\pi_*(\pi^*\mathcal{M}) \simeq \mathcal{M}.$$ 

6. Reconstruction theorem on curves.
Let $X$ be a complex curve. Then Conjecture 3.1 (i) holds true:

**Theorem 6.1.** For $\mathcal{M} \in \mathcal{D}^b_{\mathcal{H}}(\mathcal{D}_X)$ there is a
functorial isomorphism
\begin{equation}
\mathcal{M} \simeq \Psi(\Phi(\mathcal{M})).
\end{equation}

**Sketch of proof.** Since the statement is local, we can assume that $X$ is an open disc in $\mathbb{C}$ centered
at 0, and that $\text{ss}(\mathcal{M}) = \{ 0 \}$.
By devissage, we can assume from the beginning that $\mathcal{M}$ is a flat meromorphic connection with
poles at 0.
Let $f: X \to X$ be a ramification as in Theorem 5.1, so that $Df^{-1}\mathcal{M}$ admits a good formal structure at 0.

Note that $\mathcal{D}f, Df^{-1}\mathcal{M} \simeq \mathcal{M} \oplus \mathcal{N}$ for some
$\mathcal{N}$. If (6.1) holds for $Df^{-1}\mathcal{M}$, then it holds for $\mathcal{M} \oplus \mathcal{N}$ by Proposition 4.1, and hence it also holds for
$\mathcal{M}$.
We can thus assume that $\mathcal{M}$ admits a good formal structure at 0.
Consider the real oriented blow-up (5.2).
By Lemma 5.3, one has $\mathcal{M} \simeq R\pi_* \pi^*\mathcal{M}$. Hence
Proposition 4.1 (or better, its analogue for $\pi$) implies that we can replace $\mathcal{M}$ with $\pi^*\mathcal{M}$.

By Proposition 5.2, we finally reduce to prove
$$\mathcal{E}_{0,X}^\circ \simeq \Psi(\Phi(\mathcal{E}_{0,X}^\circ)).$$
Set $D' = \{ x = 0 \} \cup \{ t = \infty \}$ and $U' = (X \times \mathbb{P}) \setminus D'$. By Proposition 2.1,
$$\mathcal{D}^b_{\mathcal{H}}(D' \times \mathbb{P}) \simeq \mathcal{D}^b_{\mathcal{H}}(\mathcal{E}_{0,X}^\circ \otimes \mathcal{E}_{\infty}^\circ P) \simeq \mathcal{E}_{D' \times \mathbb{P}}^{t-\varphi}.$$ By Proposition 2.4, we thus have
$$\Phi(\mathcal{E}_{0,X}^\circ) \simeq R\mathcal{H}om(\mathcal{C}_{U'}, \mathcal{C}_{[\text{Re}(t+\varphi)<\subset]}).$$ Noticing that $\Phi(\mathcal{E}_{0,X}^\circ) \otimes \mathcal{C}_{D'} \in \mathcal{D}^b_{\mathcal{H}}(\mathcal{C}_{X \times \mathbb{P}})$, one checks that $\Psi(\Phi(\mathcal{E}_{0,X}^\circ) \otimes \mathcal{C}_{D'}) \simeq 0$.
Hence, Proposition 2.2 implies
$$\Psi(\Phi(\mathcal{E}_{0,X}^\circ)) \simeq \Psi(\mathcal{C}_{[\text{Re}(t+\varphi)<\subset]} \simeq \mathcal{E}_{0,X}^\circ.$$ 

**Example 6.2.** Let $X = \mathbb{C}$, $\varphi(x) = 1/x$ and
$\mathcal{M} = \mathcal{E}_{0,X}^\circ$. Then we have
$$H^k \Phi(\mathcal{M}) = \begin{cases}
\mathcal{C}_{[\text{Re}(t+\varphi)<\subset]}, & \text{for } k = 0, \\
\mathcal{C}_{[t = 0, \varphi(\infty) \supset]} \oplus \mathcal{C}_{[t \neq 0, t \leftarrow \infty]}, & \text{for } k = 1, \\
0, & \text{otherwise}. 
\end{cases}$$

7. Stokes phenomenon. We discuss here
an example which shows how, in our setting, the
Stokes phenomenon arises in a purely topological
fashion.
Let $X$ be an open disc in $\mathbb{C}$ centered at $0$. (We will shrink $X$ if necessary.) Set $U = X \setminus \{0\}$.

Let $\mathcal{M}$ be a flat meromorphic connection with poles at $0$ such that

$$\mathcal{M} \big|_0 \simeq (E_{0,X}^s \oplus E_{0,X}^{\omega}) \big|_0, \quad \varphi, \psi \in \mathcal{O}_X(\ast 0).$$

Assume that $\psi - \varphi$ has an effective pole at $0$.

The Stokes curves of $E_{0,X}^s \oplus E_{0,X}^{\omega}$ are the real analytic arcs $\ell_i, i \in I$, defined by

$$\{\text{Re}(\psi - \phi) = 0\} = \bigcup_{i \in I} \ell_i.$$

(Here we possibly shrink $X$ to avoid crossings of the $\ell_i$’s and to ensure that they admit the polar coordinate $\rho > 0$ as parameter.)

Since $E_{0,X}^s \simeq E_{0,X}^{\omega}\varphi_0$ for $\varphi_0 \in \mathcal{O}_X$, the Stokes curves are not invariant by isomorphism.

The Stokes lines $L_i$ defined as the limit tangent half-lines to $\ell_i$ at $0$, are invariant by isomorphism.

The Stokes matrices of $\mathcal{M}$ describe how the isomorphism (5.3) changes when $\theta$ crosses a Stokes line.

Let us show how these data are encoded in $\Phi(\mathcal{M})$.

Set $D' = \{x = 0\} \cup \{t = \infty\}$ and $U' = (X \times \mathbb{P}) \setminus D'$. Set

$$F = C_{\{\text{Re}(t + \psi) < c\}}, \quad G = C_{\{\text{Re}(t + \varphi) < c\}},$$

$$F = C_{\{\text{Re}(t + \varphi) < \zeta\}}, \quad G = C_{\{\text{Re}(t + \psi) < \zeta\}}.$$

By Proposition 2.4 and Theorem 5.2,

$$\Phi(\mathcal{M}) \simeq \text{RTHom}(C_{U'}, H),$$

where $H$ is an ind-sheaf such that

$$H \otimes C_{q^{-1}S} \simeq (F \oplus G) \otimes C_{q^{-1}S}$$

for any sufficiently small open sector $S$.

Let $b^\pm$ be the vector space of upper/lower triangular matrices in $M_2(\mathbb{C})$, and let $t = b^+ \cap b^-$ be the vector space of diagonal matrices.

**Lemma 7.1.** Let $S$ be an open sector, and $v$ a vector space, which satisfy one of the following conditions:

(i) $v = b^\pm$ and $S \subset \{\pm \text{Re}(\psi - \varphi) > 0\}$,

(ii) $v = t$, $S \supset L_i$ for some $i \in I$ and $S \cap L_j = \emptyset$ for $i \neq j$.

Then, for $c' > c$, one has

$$\text{Hom}(F_i \oplus G_i)_{q^{-1}S}, (F_{c'} \oplus G_{c'})_{q^{-1}S} \simeq v.$$

In particular,

$$\text{End}(F \oplus G) \otimes C_{q^{-1}S} \simeq v.$$

This proves that the Stokes lines are encoded in $H$. Let us show how to recover the Stokes matrices of $\mathcal{M}$ as glueing data for $H$.

Let $S_i$ be an open sector which contains $L_i$ and is disjoint from $L_j$ for $i \neq j$. We choose $S_i$ so that $\bigcup_{i \in I} S_i = U$.

Then for each $i \in I$, there is an isomorphism

$$\alpha_i : H \otimes C_{q^{-1}S_i} \simeq (F \oplus G) \otimes C_{q^{-1}S_i}.$$

Take a cyclic ordering of $I$ such that the Stokes lines get ordered counterclockwise.

Since $\{S_i\}_{i \in I}$ is an open cover of $U$, the ind-sheaf $H$ is reconstructed from $F \oplus G$ via the glueing data given by the Stokes matrices

$$A_i = \alpha_{i+1}^{-1}\alpha_i|_{q^{-1}(S_i \cup S_{i+1})} \in b^\pm.$$

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