Extended High-Gain Observers as Disturbance Estimators

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Abstract: This paper reviews recent results on the use of extended high-gain observers as disturbance estimators. It shows how the use of these observers enables recovering the performance of feedback linearization controllers. It shows also how the observer can be combined with dynamic inversion. Three experimental applications are described.

Key Words: nonlinear control, high-gain observers, disturbance estimators.

1. Introduction

Disturbance estimators are widely used in feedback control; see [1] for a comprehensive coverage of the topic; see also [2] and [3] for the use of disturbance estimators in Active Disturbance Rejection Control (ADRC). One of the common methods to estimate disturbance is to use extended observers. For a system of the form

\[ \dot{x} = f(x, u, w), \quad y = h(x) \]

with constant disturbance \( w \), the state equation can augmented with the equation \( \dot{w} = 0 \). Provided the extended system is observable, an observer is designed to estimate the extended state \((x, w)\). While such observer is designed for constant state \( w \), it will work for slowly varying disturbance under appropriate conditions. The extended high-gain observer presented in this paper is also based on extending the state vector by treating the disturbance as an additional state, but it differs from other extended observers in two aspects. First, it does not require the disturbance to be slowly varying because the observer dynamics are designed to be fast relative to the dynamics of the extended system. Second, the observer estimates a matched disturbance term, which results from model uncertainty and external disturbances.

The papers start by recalling some background information in Section 2. Section 3 presents the results of [4] on the use of extended high-gain observers to recover the performance of feedback-linearization controllers in the presence of uncertainties. Section 4 presents the results of [5], which extends the results of [4] to multi-input–multi-output systems. Section 5 presents the results of [6], which combines extended high-observers with the technique of dynamic inversion [7] to handle systems with uncertain control coefficient as well as systems that depend nonlinearly on the control signal. The results of Sections 3 and 5 have been tested experimentally with successful results. Section 6 describes applications to a DC motor with nonlinear friction, a Permanent Magnet Synchronous Motor with uncertain parameters and unknown load, and an inverted pendulum on a cart with uncertain parameters. Finally, some concluding remarks are given in Section 7.

2. Preliminaries

A single-input–single-output nonlinear system of the form

\[ \dot{x} = f(x) + g(x)u, \quad y = h(x) \]

has relative degree \( n \) in an open set \( \mathcal{R} \) if, for all \( x \in \mathcal{R} \),

\[ L_i L_j^{-1} h(x) = 0, \quad \text{for} \quad 1 \leq i \leq n - 1; \quad L_n L_j^{-1} h(x) \neq 0 \]

where \( L_i h(x) = \frac{\partial}{\partial x_i} h \) is the Lie derivative of \( h \) with respect to the vector field \( f \). Under appropriate smoothness conditions on \( f, g, \) and \( h \), there is a change of variables that transforms a relative degree \( n \) system into the normal form [8, Chapter 4]

\[ \begin{align*}
\dot{\eta} &= f_0(\eta, \xi) \\
\dot{\xi}_i &= \xi_{i+1}, & \text{for} \quad 1 \leq i \leq n-1 \\
\dot{\xi}_n &= a(\eta, \xi) + b(\eta, \xi)u \\
y &= \xi_1
\end{align*} \]

at least locally. Under some stronger conditions, the change of variables will hold globally [8, Chapter 9]. The equation \( \dot{\eta} = f_0(\eta, \xi) \) is called the internal dynamics because it represents the dynamics of the system when state feedback linearization is used to reduce the input-output map, from \( u \) to \( y \), to a chain of integrators. The equation \( \dot{\eta} = f_0(\eta, 0) \) is called the zero dynamics because it represents the dynamics of the system when the output \( y \) is identically zero. The system is minimum phase when the zero dynamics have an asymptotically stable equilibrium point. The results of this paper are presented for minimum-phase systems or systems where the internal dynamics, with input \( \xi \) and state \( \eta \), are bounded-input–bounded-state stable.

An \( m \)-input–\( m \)-output nonlinear system of the form

\[ \dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i, \quad y_i = h_i(x), \quad \text{for} \quad 1 \leq i \leq m \]

has vector relative degree \( \{n_1, n_2, \ldots, n_m\} \) in an open set \( \mathcal{R} \) if, for all \( x \in \mathcal{R} \),

\[ L_i L_j^{-1} h_k(x) = 0, \quad \text{for} \quad 1 \leq k \leq n_i - 1 \]
for all $1 \leq i \leq m$ and $1 \leq j \leq m$, and the $m \times m$ matrix
\[
\begin{bmatrix}
L_{ij}^{i-1} h_1 & \cdots & L_{ij}^{i-1} h_1 \\
L_{ij}^{j-1} h_2 & \cdots & L_{ij}^{j-1} h_2 \\
\vdots & \ddots & \vdots \\
L_{ij}^{m-1} h_m & \cdots & L_{ij}^{m-1} h_m
\end{bmatrix}
\]
is nonsingular. Under appropriate smoothness conditions on $f$, $g$, and $h$, there is a change of variables that transforms the system into the multivariable normal form [8, Chapters 5 and 9]
\[
\dot{\xi} = f_0(\eta, \xi), \quad \dot{\xi}_i = \xi_{i+1}, \quad \text{for } 1 \leq j \leq n-1, \quad 1 \leq i \leq m
\]
where $\xi^i = \text{col} (\xi^1, \xi^2, \ldots, \xi^n)$, $\xi = \text{col} (\xi^1, \xi^2, \ldots, \xi^m)$, and $n = n_1 + \cdots + n_m$.

A scalar continuous function $\alpha(r)$, defined for $r \in [0, a)$, belongs to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It belongs to class $\mathcal{K}_0$ if it is defined for all $r \geq 0$ and $\alpha(r) \to \infty$ as $r \to \infty$. A scalar continuous function $\beta(r, s)$, defined for $r \in [0, a)$ and $s \in [0, \infty)$, belongs to class $\mathcal{KL}$ if, for each fixed $s$, $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to $r$, and for each fixed $r$, $\beta(r, s)$ is decreasing with respect to $s$ and $\beta(r, s) \to 0$ as $s \to \infty$. The system $\dot{x} = f(x, u)$ is input-to-state stable (ISS) if there exist a class $\mathcal{K}$ function $\beta$ and a class $\mathcal{KL}$ function $\gamma$ such that for any $t_0 \geq 0$, any initial state $x(t_0)$, and any bounded input $u(t)$, the solution $x(t)$ exists for all $t \geq t_0$ and satisfies
\[
\|x(t)\| \leq \max \left\{ \beta(\|x(t_0)\|, t - t_0), \gamma \left( \sup_{t_0 \leq T < t} \|u(T)\| \right) \right\}
\]
for all $t \geq t_0$. If this inequality holds only on compact sets of $x$ and $u$, the system is said to be regionally input-to-state-stable; see, for example, [9] and [10].

A high-gain observer for the system
\[
\dot{x}_i = x_{i+1}, \quad \text{for } 1 \leq i \leq n-1
\]
\[
\dot{x}_n = \phi(x, u)
\]
\[
y = x_1
\]
is given by
\[
\dot{x}_i = \dot{x}_{i+1} + (\alpha_i / \varepsilon)(y - \dot{x}_1), \quad \text{for } 1 \leq i \leq n-1
\]
\[
\dot{x}_n = \phi_0(\dot{x}, u) + (\alpha_n / \varepsilon)(y - \dot{x}_1)
\]
where $\phi_0$ is a nominal model of $\phi$, $\varepsilon$ is a sufficiently small positive constant, and $\alpha_1$ to $\alpha_n$ are chosen such that the roots of
\[
s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1} s + \alpha_n = 0
\]
have negative real parts. Under appropriate smoothness conditions on $\phi$ and $\phi_0$; see for example [9], the estimation errors $\dot{x}_i = x_i - \dot{x}_i$, for $i = 1, \ldots, n$, satisfy the inequality
\[
|\dot{x}_i| \leq \max \left\{ \frac{b}{C^{n-1} \|x(0)\|} e^{-\alpha t / \varepsilon}, e^{\alpha t / \varepsilon} M \right\}
\]
for some positive constants $a$, $b$, $c$, and $M$. The first term in this inequality is due to the initial estimation error and the second term gives an ultimate bound due to the error $\phi - \phi_0$. The first term shows the peaking phenomenon of the observer where the state estimates could peak to a negative power $\varepsilon$ before they decay quickly to $O(\varepsilon)$ values. In feedback control, the effect of the observer peaking on the plant is overcome by saturating the state estimates or the control signal outside a compact set of interest; see [9].

3. Single-Input–Single-Output Systems

Consider a single-input–single-output nonlinear system in the globally defined normal form:
\[
\dot{y} = f_0(\eta, \xi, w)
\]
\[
\dot{\xi}_i = \xi_{i+1}, \quad \text{for } 1 \leq i \leq n - 1
\]
\[
\dot{\xi}_n = a(\eta, \xi, w) + b(\eta, \xi, w) u
\]
\[
e = \xi_1
\]
where $\eta \in \mathbb{R}^p$ and $\xi = \text{col} (\xi^1, \ldots, \xi^n) \in \mathbb{R}^n$ are the state variables, $u$ is the control input, $e$ is the measured regulation error, and $w \in \mathbb{R}^r$ is an exogenous input. The functions $a$, $b$, and $f_0$ and the signal $w$ could be unknown.

Assumption 1 $w(t)$ and $\dot{w}(t)$ are bounded and $w(t)$ belongs to a known compact set $W \subset \mathbb{R}^r$.

Assumption 2 The functions $a$ and $b$ are continuously differentiable with locally Lipschitz derivatives, $f_0$ is locally Lipschitz, and $b(\eta, \xi, w) \geq b_0 > 0$, for all $(\eta, \xi, w) \in \mathbb{R}^p \times \mathbb{R}^r \times W$.

Assumption 3 There is a continuously differentiable function $V_0(\eta)$, class $\mathcal{K}_0$ functions $\gamma_1$ and $\gamma_2$, and a nonnegative continuous non-decreasing function $\chi$ such that
\[
\gamma_1(\|\eta\|) \leq V_0(\eta) \leq \gamma_2(\|\eta\|)
\]
\[
\frac{\partial V_0}{\partial \eta} f_0(\eta, \xi, w) \leq 0, \quad \forall \|\eta\| \geq \chi(\|\eta\|) + \|w\|
\]
for all $(\eta, \xi, w) \in \mathbb{R}^p \times \mathbb{R}^r \times W$.

Assumption 3 ensures that the system $\dot{\eta} = f_0(\eta, \xi, w)$, with input $(\xi, w)$, is bounded-input–bounded-state stable. It is satisfied whenever $\dot{\eta} = f_0(\eta, \xi, w)$ is ISS because ISS is equivalent to the existence of an ISS Lyapunov function $V_0(\eta)$ that satisfies the foregoing inequality with $\|V_0/\partial \eta\| f_0(\eta, \xi, w) \leq -\gamma_0(\|\eta\|)$ for some class $\mathcal{K}$ functions $\gamma_0$ [11]. Assumption 3, however, is less restrictive than ISS because it does not require the origin of $\eta = f_0(\eta, 0, 0)$ to be asymptotically stable.

The goal is to design an output feedback controller to asymptotically regulate the output $e(t)$ to zero while meeting certain requirements on the transient response. Had $(\eta, \xi, w)$ been available for feedback and the functions $a$ and $b$ been known, we could have used the feedback linearization control
\[
u = -a(\eta, \xi, w) w + \frac{b(\eta, \xi, w)}{b(\eta, \xi, w)} v
\]
to reduce the input-output map to
\[
\dot{\xi} = A\xi + Bv, \quad e = C\xi
\]
where the triple $(A, B, C)$ represents a chain of $n$ integrators; that is,
\[
A = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
1
\end{bmatrix}, \quad C = \begin{bmatrix}
1 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]
The auxiliary control $v$ could then be designed as a feedback function of $\xi$ to stabilize the origin $\xi = 0$ and meet the transient response specifications. Let $v = -K\dot{\xi}$ where $K$ is chosen using a linear control design method to make $(A - BK)$ Hurwitz and shape the transient response. Hence, the origin of

$$\dot{\xi} = (A - BK)\xi \quad (7)$$

is exponentially stable and $V_1(\xi) = \xi^TP_1\xi$ is a Lyapunov function, where $P_1 = P_1^T > 0$ is the solution of the Lyapunov equation $P_1(A - BK)^T + (A - BK)P_1 = -Q_1$, for some $Q_1 = Q_1^T > 0$. Let

$$\Omega = \{V_0(\eta) \leq c_0 \times \{V_1(\xi) \leq c\}$$

where $c > 0$ and $c_0 \geq \|\mathbb{Q}_2(\xi(k_\eta + c_\xi))\|$, in which $k_\eta = \max_{\xi \in V_1(\xi) \leq c}\|\xi\|$ and $k_\xi = \max_{\omega \in \|w\|}$। The set $\Omega$ is compact and positively invariant with respect to the system

$$\dot{\eta} = f_0(\eta, \xi, w), \quad \dot{\xi} = (A - BK)\xi$$

The constants $c$ and $c_0$ can be chosen arbitrarily large so that any compact set of $R^n$ can be put in the interior of $\Omega$.

Let $\hat{a}(\xi)$ and $\hat{b}(\xi)$ be twice continuously differentiable, globally bounded functions that model $a(\eta, \xi, w)$ and $b(\eta, \xi, w)$, respectively. It is allowed to take $\hat{a} = 0$ and $\hat{b} > 0$ as a constant. The $\hat{\xi}_n$-equation can be written as

$$\dot{\hat{\xi}}_n = \sigma + \hat{a}(\xi) + \hat{b}(\xi)u$$

Augmenting $\sigma$ as an additional state to the chain of integrators (3)–(5), a high-gain observer for the extended system is taken as

$$\begin{align*}
\dot{\hat{\xi}}_i &= \hat{\xi}_{i+1} + (a_i/\varepsilon')(e - \hat{\xi}_1), \quad \text{for } 1 \leq i \leq n - 1 \\
\dot{\hat{\xi}}_n &= \hat{\sigma} + \hat{a}(\hat{\xi}) + \hat{b}(\hat{\xi})u + (a_n/\varepsilon')(e - \hat{\xi}_1) \\
\hat{\sigma} &= (a_{n+1}/\varepsilon')\varepsilon' \quad (10)
\end{align*}$$

where $a_i$ to $a_{n+1}$ are chosen such that the roots of

$$s^{n+1} + a_1s^n + \cdots + a_{n+1} = 0$$

have negative real parts and $\varepsilon > 0$ is a small parameter. Global boundedness of $\hat{a}$ and $\hat{b}$ is required to overcome the peaking phenomenon of high-gain observers. It does not exclude linear or unbounded functions because global boundedness can be achieved by saturating the control outside a compact set.

From the properties of high-gain observers, it is anticipated that the terms $(a_i/\varepsilon')(e - \hat{\xi}_1)$ for $1 \leq i \leq n$ will be $O(\varepsilon)$ after a short transient period. Then, equations (8)–(9) can be viewed as a perturbation of the system

$$\dot{\hat{\xi}} = L\hat{\xi} + B(\hat{\sigma} + \hat{a}(\hat{\xi}) + \hat{b}(\hat{\xi})u) \quad (11)$$

which can be made to coincide with the target system (7) by taking

$$u = -\hat{\sigma} - \hat{a}(\hat{\xi}) - K\hat{\xi} \quad \text{let } \phi(\hat{\xi}, \hat{\sigma})$$

To protect the system from peaking in the observer’s transient response, the control is saturated outside the compact set $\Omega$. Let $M > \max_{(\eta, \xi) \in \Omega \times W} \|\alpha(\eta, \xi, w) - \hat{K}\hat{\xi}\|/\|\hat{b}(\eta, \xi, w)\| \quad (13)$

Saturating the foregoing expression of $u$ at $\pm M$ using the saturation function $\text{sat}(\cdot)$ yields the output feedback controller

$$u = M\text{sat}\left(\frac{\phi(\hat{\xi}, \hat{\sigma})}{M}\right) \quad (14)$$

where $\text{sat}(y) = \min\{1, |y|\}\text{sign}(y)$ and $\psi$ is defined by (12).

Let

$$k_u = \max_{(\eta, \xi) \in \Omega \times W} \|\hat{b}(\eta, \xi, w) - \hat{b}(\xi)/\|\hat{b}(\xi)\|$$

And $||H||_\infty = \max_{\omega \in ||w||}$ Hurwitz and $||H||_\infty \geq 1$. It can be verified that $||H||_\infty = 1$ if all the poles of $H(s)$ are real.

**Theorem 1** Consider the closed-loop system formed of the plant (2)–(5), the observer (8)–(10) and the controller (14). Suppose Assumptions 1 to 3 are satisfied and

$$k_u < \frac{1}{||H||_\infty} \quad (15)$$

Let $\Omega_0$ be a compact set in the interior of $\Omega$ and $S$ a compact subset of $R^n$. Then, for all initial conditions $(\eta(0), \xi(0)) \in \Omega_0$ and $(\hat{\xi}(0), \hat{\sigma}(0)) \in S$,

- there exists $\varepsilon_1 > 0$ such that for every $0 < \varepsilon \leq \varepsilon_1$, the trajectories of the closed-loop system are bounded for all $t \geq 0$;
- given any $\mu > 0$ there exist $\varepsilon_2 > 0$, dependent on $\mu$, such that for every $0 < \varepsilon \leq \varepsilon_2$,

$$||\xi(t) - \hat{\xi}(t)|| \leq \mu, \quad \forall t \geq 0 \quad (16)$$

where $\hat{\xi}_i$ is the solution of the target system (7) with $\xi_i(0) = \xi(0)$.
- given any $\mu > 0$ there exist $\varepsilon_3 > 0$ and $T_1 > 0$, both dependent on $\mu$, such that for every $0 < \varepsilon \leq \varepsilon_3$,

$$||\xi(t)|| \leq \mu, \quad \forall t \geq T_1 \quad (17)$$

The proof of Theorem 1 is given in [4].

**Remark 1** The second bullet shows performance recovery because the trajectories of $\xi$ under output feedback can be made arbitrarily close to the trajectories of the target system by choosing $\varepsilon$ small enough. The third bullet shows that the controller achieves practical regulation because the ultimate bound on $||\xi||$ can be made arbitrarily small by choosing $\varepsilon$ small enough.

**Remark 2** Condition (15) can be always satisfied by appropriate choice of $\hat{b}$ and the observer eigenvalues if an upper bound on $b(x, z, w)$ is known. Let $b_m \geq \max_{(\eta, \xi) \in \Omega \times W} b(\eta, \xi, w)$. Taking $\hat{b} > b_m$ ensures that $k_u < 1$. Assigning the observer eigenvalues to be real results in $||H||_\infty = 1$; hence $k_u||H||_\infty < 1$. 

\[ \diamond \]
Remark 3 There is interplay between $\varepsilon$ and the constants $c$ and $c_0$ that define $\Omega$. The larger are $c$ and $c_0$, the larger is the saturation level $M$, which, in general, requires a smaller $\varepsilon$ to achieve enough separation of times scales between the observer and the plant. This interplay can be interpreted the other way around by saying that a smaller $\varepsilon$ would lead to a larger region of attraction. 

Theorem 1 ensures only practical regulation. The next theorem shows that if $w$ is constant, then, under additional conditions on the zero dynamics, the controller provides integral action that ensures regulation of $e(t)$ (in fact the whole vector $\xi(t)$) to zero even when the model uncertainty does not vanish at $\xi = 0$. The following assumption states the additional requirements on $\eta = f_0(\eta, \xi, w)$ when $w$ is constant.

Assumption 4

- $f_0(\eta, 0, w)$ is continuously differentiable in $\eta$;
- for each $w \in \mathcal{W}$, the system $\eta = f_0(\eta, 0, w)$ has a unique equilibrium point $\eta_{ss} = f_0(\eta, 0, w) \in \{V_0(\eta) \leq c_0\}$.
- With $\xi = \eta - \eta_{ss}$, there is a continuously differentiable Lyapunov function $V_0(\xi)$, possibly dependent on $w$, and class $\mathcal{K}$ functions $\gamma_1$ to $\gamma_4$, independent of $w$, such that
  \[ \gamma_1(\|\xi\|) \leq V_0(\xi) \leq \gamma_2(\|\xi\|); \quad \|\xi\| \geq \gamma_3(\|\xi\|) \]
  for all $(\eta, \xi, w) \in \Omega \times \mathcal{W}$.
- the origin of $\dot{\xi} = f_0(\xi, \eta_{ss}, w)$ is exponentially stable.

Theorem 2 Consider the closed-loop system formed of the plant (2)–(5), the observer (8)–(10) and the controller (14), under the assumptions of Theorem 1. Suppose $w$ is constant and Assumption 4 is satisfied. Then, there exists $\tilde{\varepsilon} > 0$ such that for $\varepsilon \in (0, \tilde{\varepsilon})$, $\xi(t)$ converges to zero as $t \to \infty$. Furthermore, if $f_0(0, 0, w) = 0$, then the origin $\xi = 0$ is exponentially stable equilibrium point and $\Omega_0 \times \Sigma$ is a subset of the region of attraction. 

The proof of Theorem 2 is given in [4].

4. Multi-Input–Multi-Output Systems

The globally-defined normal form

\[
\dot{\eta} = f_0(\eta, \xi, w)
\]
\[
\dot{\xi}^i = \xi_{i+1}, \quad i = 1, \ldots, n_i - 1
\]
\[
\dot{\xi}_n = f_0(\eta, \xi, w) + G_1(\eta, \xi, w)u, \quad 1 \leq i \leq m
\]
\[
e_i = \xi_i, \quad 1 \leq i \leq m
\]
represents a nonlinear system with vector relative degree $\{n_1, \ldots, n_m\}$, where $\xi^2 = \text{col}(\xi_1^2, \xi_2^2, \ldots, \xi_m^2)$, $\xi = \text{col}(\xi^1, \xi^2, \ldots, \xi^n)$, $n = n_1 + \cdots + n_m$, $\eta \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$ are the state variables, $u \in \mathbb{R}^m$ is the control input, $e_i$ to $e_m$ are the measured regulation errors, and $w \in \mathbb{R}^r$ is an exogenous input. Equations (19)–(21) represents $m$ chains of integrators in the $i$th chain. The system can be equivalently represented by the equations

\[
\dot{\eta} = f_0(\eta, \xi, w)
\]
\[
\dot{\xi} = A\xi + B(\gamma(\eta, \xi, w) + G(\eta, \xi, w)u)
\]
\[
e = Cx
\]
where

\[
A = \text{block diag}[A_1, \ldots, A_m]
\]
\[
B = \text{block diag}[B_1, \ldots, B_m]
\]
\[
C = \text{block diag}[C_1, \ldots, C_m]
\]

The functions $f_0$, $f$, and $G$ and the signal $w$ could be unknown. Suppose $w$ satisfies Assumption 1.

Assumption 5 The functions $f$ and $G$ are continuously differentiable with locally Lipschitz derivatives, $f_0$ is locally Lipschitz, and the matrix $G(\eta, \xi, w)$ is nonsingular with bounded inverse, for all $(\eta, \xi, w) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{W}$.

Similar to the previous section, a target system is taken as

\[
\dot{\xi} = (A - BK)\xi
\]

where $K$ is designed to make $(A - BK)$ Hurwitz and shape the transient response of (25). A Lyapunov function for (25) can be taken as $V_1(\xi) = \xi^T P_1 \xi$, where $P_1 = P_1^T > 0$ is the solution of the Lyapunov equation $P_1(A - BK) + (A - BK)^T P_1 = -Q_1$, for some $Q_1 = Q_1^T > 0$. Let $\Omega = \{V_0(\eta) \leq c_0\} \times \{V_1(\xi) \leq c\}$, where $c$ and $c_0$ are positive constants and $c_0$ is chosen as in the previous section. The set $\Omega$ is compact and positively invariant with respect to the system

\[
\dot{\eta} = f_0(\eta, \xi, w), \quad \dot{\xi} = (A - BK)\xi
\]

Let the $m$-dimensional function $f(\xi)$ and the $m \times m$ matrix $G(\xi)$ be twice continuously differentiable, globally bounded models of $\gamma(\eta, \xi, w)$ and $G(\eta, \xi, w)$, respectively, where $\hat{G}(\xi)$ be nonsingular with bounded inverse. It is allowed to take $\tilde{f} = 0$ and $\hat{G}$ as a constant matrix. The $\hat{\xi}$-equation can be written as

\[
\dot{\hat{\xi}} = A\hat{\xi} + B(\sigma + \hat{f}(\hat{\xi}) + \hat{G}(\hat{\xi})u), \quad e = C\hat{\xi}
\]
Consider the closed-loop system formed of the extended system is taken as

$$\dot{\bar{e}}_i = \xi_j,$$

where $\bar{e}_i$ is the $i$th row of $\bar{G}$. Augmenting $\bar{e}_i$ as an additional state to the $i$th chain of integrators, a high-gain observer of the extended system is taken as

$$\dot{\tilde{\xi}}_j = \tilde{\xi}_{j+1} + (\alpha'_i e_{\xi}')(e_i - \tilde{\xi}_j), \quad 1 \leq j \leq n_1 - 1$$

$$\dot{\tilde{\xi}}_{n_1} = \sigma_i + \tilde{f}_i(\tilde{\xi}) + \bar{G}_i(\tilde{\xi})\bar{e}_i,$$

$$e_i = \tilde{\xi}_1$$

for $1 \leq i \leq m$, where $\alpha'_i$ to $\alpha'_{n_1+1}$ are chosen such that the polynomial

$$s^{n_1+1} + \alpha'_1 s^n + \cdots + \alpha'_{n_1+1}$$

is Hurwitz, and $e > 0$ is a small parameter. Let

$$\psi(\bar{e}_i, \sigma_i) = \bar{G}^{-1}(\bar{e}_i)[-\bar{f}(\bar{e}_i) - \sigma_i - K \bar{e}_i]$$

$h_i(\eta, \xi, w)$ be the $i$th component of $G^{-1}(\eta, \xi, w)[-f(\eta, \xi, w) - K \bar{e}_i]$, and

$$M_i > \max_{(\eta, \xi, w) \in W} |h_i(\eta, \xi, w)|$$

The output feedback controller is taken as

$$u_i = M_i \left( \psi(\hat{\xi}_i, \hat{\sigma}_i) / M_i \right), \quad \text{for } 1 \leq i \leq m$$

Let

$$k_n = \max_{1 \leq i \leq 1} \left\{ \max_{(\eta, \xi, w) \in W} \left| |G(\eta, \xi, w) - \bar{G}(\xi)| D \bar{G}^{-1}(\xi)| \right| \right\}$$

where $D = \text{diag}(d_1, \ldots, d_m)$,

$$H(s) = \text{diag}[H_1(s), \ldots, H_m(s)]$$

$$H_i(s) = \frac{\alpha'_{n_1+1}}{s^{n_1+1} + \alpha'_1 s^n + \cdots + \alpha'_{n_1+1}}$$

and $||H||_{\infty} = \max_{1 \leq i \leq 1} \max_{\omega} |H_i(j \omega)|$. Because $H_i(0) = 1$, $||H||_{\infty} \geq 1$. It can be verified that $||H||_{\infty} = 1$ if for each $1 \leq i \leq m$, all the poles of $H_i(s)$ are real. The following theorem reads almost the same as Theorem 1.

Theorem 3 Consider the closed-loop system formed of the plant (18)–(21), the observer (26)–(28) and the controller (31). Suppose Assumptions 1, 3 and 5 are satisfied and

$$k_n < \frac{1}{||H||_{\infty}}$$

Let $\Omega_0$ be a compact set in the interior of $\Omega$ and $S$ a compact subset of $R^{n_m}$. Then, for all initial conditions $(\eta(0), \xi(0)) \in \Omega_0$ and $(\xi(0), \sigma(0)) \in S$,

- there exists $\varepsilon^*_1 > 0$ such that for every $0 < \varepsilon \leq \varepsilon^*_1$, the trajectories of the closed-loop system are bounded for all $t \geq 0$.

- given any $\mu > 0$ there exist $\varepsilon_2 > 0$, dependent on $\mu$, such that for every $0 < \varepsilon \leq \varepsilon_2$,

$$||\xi(t) - \xi(t)|| \leq \mu, \quad \forall t \geq 0$$

(34)

where $\xi_t$ is the solution of the target system (25) with $\xi_t(0) = \xi(0)$.

- given any $\mu > 0$ there exist $\varepsilon_3^* > 0$ and $T_1 > 0$, both dependent on $\mu$, such that for every $0 < \varepsilon \leq \varepsilon_3^*$,

$$||\xi(t)|| \leq \mu, \quad \forall t \geq T_1$$

(35)

The main part of the proof of Theorem 3 is given in [5]; the rest follows the proof of Theorem 1.

Remark 4 Because the observer can be designed to have $||H||_{\infty} = 1$, it is sufficient to require $k_n < 1$.

The next theorem extends Theorem 2 to the multi-input–multi-output case.

Theorem 4 Consider the closed-loop system formed of the plant (18)–(21), the observer (26)–(28) and the controller (31), under the assumptions of Theorem 3. Suppose $w$ is constant and Assumption 4 is satisfied. Then, there exists $\varepsilon > 0$ such that for $\varepsilon \in (0, \varepsilon)$, $\xi(t)$ converges to zero as $t \to \infty$. Furthermore, if $f_0(0, 0, w) = 0$, then the origin $(\eta, \xi, \varphi) = (0, 0, 0)$ is an exponentially stable equilibrium point of the closed-loop system and $\Omega_0 \times S$ is a subset of the region of attraction.

5. Dynamic Inversion

Consider a multi-input–multi-output system represented by the equations

$$\dot{\eta} = f_0(\eta, \xi, w)$$

$$\dot{\xi}_j = \xi_{j+1}, \quad \text{for } 1 \leq j \leq n_1 - 1, \ 1 \leq i \leq m$$

$$\dot{\xi}_{n_1} = f_1(\eta, \xi, w), \quad \text{for } 1 \leq i \leq m$$

$$e_i = \xi_{n_1+1}^i, \quad \text{for } 1 \leq i \leq m$$

(36)

(37)

(38)

(39)

where $\xi^i = (\xi^{i_1}, \xi^{i_2}, \ldots, \xi^{i_m})$, $\xi = (\xi^1, \xi^2, \ldots, \xi^m)$, $n = n_1 + \cdots + n_m$, $\eta \in R^n$ and $\xi \in R^p$ are the state variables, $u \in R^m$ is the control input, $e_i$ to $e_m$ are the measured regulation errors, and $w \in R^z$ is an exogenous input. The system is defined for $\eta \in D_\eta \subset R^n$ and $\xi \in D_\xi \subset R^p$ for some domains $D_\eta$ and $D_\xi$. The system can be equivalently represented by the equations

$$\dot{\tilde{\eta}} = f_0(\eta, \xi, w)$$

$$\dot{\tilde{\xi}} = A \xi + B f(\eta, \xi, w, u)$$

$$e = C x$$

(40)

(41)

(42)

where the block diagonal matrices $[A, B, C]$ are defined in the previous section. Equations (36)–(39) are similar to equations (18)–(21) except that the right-hand side of (38) depends nonlinearly on $u$.

Suppose $w$ satisfies Assumption 1 and $f$ and $f_0$ satisfy the following assumption.
Assumption 6 The function $f$ is continuously differentiable with locally Lipschitz derivatives and $f_0$ is locally Lipschitz, for all $(\eta, \xi, w) \in D_\eta \times D_\xi \times \mathcal{W}$.

Consider the target system

$$\dot{\xi} = (A - BK)\xi$$

(43)

where $(A - BK)$ is Hurwitz. The goal is to design an output feedback controller to asymptotically regulate the regulation error $e$ to zero while bringing $\xi(t)$ of the closed-loop system arbitrarily close to the response of the target system when both systems start from the same initial states.

Assumption 7 For $(\eta, \xi, w) \in D_\eta \times D_\xi \times \mathcal{W}$, the equation

$$f(\eta, \xi, w, u) = -K\xi$$

(44)

has a unique solution $u = \phi(\eta, \xi, w)$ and $\phi$ is continuously differentiable.

It follows that $\dot{\phi}$ is bounded on compact sets of $(\eta, \xi)$.

Assumption 8 There is a known, locally Lipschitz, $m \times m$ non-singular matrix $P(\xi)$ such that

$$z^TP(\xi)[f(\eta, \xi, w, z + \phi) - f(\eta, \xi, w, \phi)] \geq \beta|z|^2$$

(45)

for all $(\eta, \xi, w, z) \in D_\eta \times D_\xi \times \mathcal{W} \times \mathbb{R}^m$ where $\beta$ is a positive constant.

Remark 5 When $f(\eta, \xi, w, u) = h(\eta, \xi, w) + G(\eta, \xi, w)u$, then (45) is satisfied if

$$P(\xi)G(\eta, \xi, w) + (P(\xi)G(\eta, \xi, w))^T \geq 2\beta I$$

(46)

Let $V_1(\xi) = \xi^TP_1\xi$, where $P_1 = P_1^T > 0$ is the solution of the Lyapunov equation $P_1(A - BK) + (A - BK)^TP_1 = -Q_1$, for some $Q_1 = Q_1^T > 0$. Let $\epsilon$ be a positive constant such that $V_1(\xi) \leq \epsilon$. Suppose Assumption 3 is satisfied. Then, the compact set $\Omega = \{V_0(\eta) \leq \epsilon\}$ is positively invariant with respect to the system

$$\dot{\eta} = f_0(\eta, \xi, w), \quad \dot{\xi} = (A - BK)\xi$$

(47)

Let $\hat{f}(\xi, u)$ be a twice continuously differentiable, globally bounded function that models $f(\eta, \xi, w, u)$. The choice $\hat{f} = 0$ is allowed. Set $\sigma = f(\eta, \xi, w, u) - \hat{f}(\xi, u)$. If $\xi$ and $\sigma$ were available for feedback, the dynamic inversion algorithm could have been taken as

$$\mu\sigma = -P(\xi)[\hat{f}(\xi, u)] + \sigma + K\xi$$

and $z = u - \phi(\eta, \xi, w)$ would satisfy the equation

$$\mu\sigma \leq -P(\xi)[\hat{f}(\eta, \xi, w, z + \phi) - f(\eta, \xi, w, \phi)] - \mu\phi$$

Then, from Assumption 8,

$$\mu\sigma \leq -\beta|z|^2 + \mu|\sigma||\phi|$$

For all $(\eta, \xi, w) \in \Omega \times \mathcal{W}$ and $|u| \leq r_0$, $\phi$ is bounded. Therefore, the set $||\sigma|| \leq r$, for any $r > 0$, is positively invariant for sufficiently small $\mu$. Taking $r \geq r_0 + \max_{(\eta, \xi, w) \in \Omega \times \mathcal{W}}||\phi(\eta, \xi, w)||$ ensures that $||\sigma(t)|| \leq r$ for all $|u(t)| \leq r_0$.

An estimate $\hat{\sigma}$ of $\sigma$ is provided by the extended high-gain observer

$$\frac{\hat{\sigma}_j}{\hat{\sigma}_{j+1}} = \hat{\sigma}_j + (\alpha_j/e^\epsilon)(\hat{\sigma}_j), \; 1 \leq j \leq n_i - 1$$

$$\hat{\sigma}_m = \hat{\sigma}_1 + f_1(\xi, u) + (\alpha_m/e^\epsilon)(\hat{\sigma}_m - \hat{\sigma}_j)$$

$$\hat{\sigma}_j = (\alpha_j/e^\epsilon)(\hat{\sigma}_j + \hat{\sigma}_j)$$

(48)

(49)

(50)

for $1 \leq i \leq m$, where $\alpha_i$ to $\alpha_{n+1}$ are chosen such that the polynomial

$$s^{n+1} + \alpha_1s^n + \ldots + \alpha_n$$

is Hurwitz, and $e > 0$ is a small parameter. Let

$$M_i > \max_{\xi, \epsilon \in \mathcal{W}} |K_i\xi|$$

where $K_i$ is the $i$th row of $K_i$. Suppose Assumptions 1, 3, 6, 7 and 8 are satisfied. Let $\Omega_0$ be a compact set in the interior of $\Omega$, $U$ a compact subset of $\mathbb{R}^m$ and $S$ a compact subset of $\mathbb{R}^{m\times m}$. Then, for all initial conditions $(\eta(0), \xi(0)) \in \Omega_0$, $u(0) \in U$, and $(\hat{\xi}(0), \hat{\sigma}(0)) \in S$,

- there exist $\mu_1 > 0$ and $\lambda_i > 0$ such that for each $\mu \in (0, \mu_1]$ and $e/\mu \in (0, \lambda_i)$, the trajectories of the closed-loop system are bounded for all $t \geq 0$;

Theorem 5 Consider the closed-loop system formed of the plant (36)–(39), the observer (48)–(50) and the dynamic inversion algorithm (51). Suppose Assumptions 1, 3, 6, 7 and 8 are satisfied. Let $\Omega_0$ be a compact set in the interior of $\Omega$, $U$ a compact subset of $\mathbb{R}^m$ and $S$ a compact subset of $\mathbb{R}^{m\times m}$. Then, for all initial conditions $(\eta(0), \xi(0)) \in \Omega_0$, $u(0) \in U$, and $(\hat{\xi}(0), \hat{\sigma}(0)) \in S$,

$$\mu\sigma = -P(\xi)[\hat{f}(\xi, u)] + \sigma + K\xi$$

(51)
• Given any $\zeta > 0$ there exist $\mu^* > 0$ and $\lambda^*_2 > 0$ such that for each $\mu \in (0, \mu^*)$ and $\varepsilon/\mu \in (0, \lambda^*_2)$,
\[ \|\xi(t) - \xi(t)\| \leq \zeta, \quad \forall t \geq 0 \] 
(52)

where $\xi_t$ is the solution of the target system (43) with $\xi_t(0) = \xi(0)$. 

The proof of Theorem 5 is given in [6].

Remark 7 Since $\lim_{t \to \infty} \xi_t(t) = 0$, (52) implies that for every $\zeta > 0$ there is $T > 0$ such that for sufficiently small $\mu$ and $\varepsilon/\mu$, $\|\xi(t)\| \leq \zeta$ for all $t \geq T$. 

As in Theorem 2, when $w$ is constant the dynamic inversion controller provides integral action and stabilization under appropriate conditions. These results are stated in Theorem 6.

Theorem 6 Consider the closed-loop system formed of the plant (36)–(39), the observer (48)–(50) and the dynamic inversion algorithm (51), under the assumptions of Theorem 5. Suppose $w$ is constant and Assumption 4 is satisfied. Then, there exist $\bar{\mu} > 0$ and $\bar{\lambda} > 0$ such that for each $\mu \in (0, \bar{\mu})$ and $\varepsilon/\mu \in (0, \bar{\lambda})$, $\xi(t)$ converges to zero as $t \to \infty$. Furthermore, if $f(0, 0, \phi(0, 0, w)) = 0$, $\dot{f}(0, \phi(0, 0, w)) = 0$, and $f_0(0, 0, w) = 0$, then the origin $(\eta, \xi, \zeta, \varphi) = (0, 0, 0, 0)$, where $z = u - \phi(\eta, \xi, \zeta, \varphi)$, is an exponentially stable equilibrium point and $\Omega_0 \times U \times S$ is a subset of the region of attraction.

6. Experimental Testbeds

The theory described in the foregoing sections has been experimentally tested in three experiments, which are described in this section. The DC motor with nonlinear friction is presented in [4], the Permanent Magnet Synchronous Motor in [12], and the inverted pendulum on a cart in [13]. We describe the controller derivation in each case, but the reader is referred to the afore-mentioned papers to see the experimental results. In all three cases the experiments confirm the theory. The extended high-gain observer in all cases is a third-order observer with all the observer eigenvalues at $-1/\varepsilon$. In the first two experiments, $\varepsilon = 0.01$ and the sampling frequency is 10 kHz. In the third experiment $\varepsilon = 0.005$ and the sampling frequency is 1.667 kHz.

6.1 DC Motor with Friction

The experimental set-up is a DC-motor with an attached rigid arm, equipped with a tacho generator servo amplifier and an incremental encoder whose resolution is 1024 imp/turn. Matlab/Simulink in combination with a dSPACE board are used to obtain a discrete-time version of the controller and implement it in real time with the sampled period of 0.1 ms. The system is modeled by
\[ J\ddot{\theta} = k_c u - F \]
where $J$ is the moment of inertia, $k_c$ is the motor constant, $u$ is the control signal, and $F$ is an unknown disturbance, mostly due to friction torque, which is modeled by the dynamic LuGre model
\[ \mu_0 \ddot{\theta} = \sigma_1 \frac{|\ddot{\theta}|}{g(\theta)} - \sigma_2 \theta - F = z + (\sigma_1 + \sigma_2)\dot{\theta} - \sigma_2 \frac{|\ddot{\theta}|}{g(\theta)} \dot{\theta} - \sigma_2 \frac{|\ddot{\theta}|}{g(\theta)} \theta \]
where $\mu_0 > 0$, $\sigma_1 > 0$, and $\sigma_2 > 0$ are positive constants and $g(v)$ is the Striebeck curve, modeled by
\[ g(v) = \begin{cases} F_{c+} + (F_{s+} - F_{c+}) e^{-|v|/v_s}, & v > 0 \\ F_{c-} + (F_{s-} - F_{c-}) e^{-|v|/v_s}, & v < 0 \end{cases} \]
with $F_{c+}$ and $F_{s+}$ being the Coulomb and static friction values, and $v_s$ being the Striebeck velocity.

The goal of the experiment is to have $\theta$ asymptotically track a smooth reference signal $r(t)$, which has bounded derivatives up to the third order. The system is transformed into the normal form by the change of variables
\[ x_1 = \theta - r(t), \quad x_2 = \dot{\theta} - \dot{r}(t), \quad w_1 = \dot{r}(t), \quad w_2 = \ddot{r}(t) \]
which results in
\[ \dot{x}_1 = \dot{x}_2, \quad \dot{x}_2 = \dot{a} u + b(x_1, z, w), \quad \dot{z} = f_0(x_1, z, w), \quad y = x_1 \]
where $a = k_c/J, b(x_1, z, w) = -w_1 - h(x_1, z, w)/J, h(x_1, z, w) = z + (\sigma_1 + \sigma_2)(x_2 + w_2) - \sigma_1 \frac{|x_2 + w_2|}{g(x_2 + w_2)} z, f_0(x_1, z, w) = \frac{1}{z_0} \left( (x_2 + w_2) - \frac{|x_2 + w_2|}{g(x_2 + w_2)} \right) z$. 

The equation $\dot{z} = f_0(x_1, z, w)$ is not ISS because the origin $z = 0$ is not asymptotically stable when $x_2 + w_2 = 0$. However, Assumption 3 is satisfied with $V_0(z) = z^2/2$ because $V_0(0) \leq 0$ for $|z| > g(x_2 + w_2)$. With $\dot{a}$ as the nominal value of $k_c/J$, the extended observer is taken as
\[ \dot{x}_1 = \dot{x}_2 + (y - x_1) / e \]
\[ \dot{x}_2 = \dot{a} u - \dot{r} + \dot{\sigma} + (y - \dot{x}_1) / e^2 \]
\[ \dot{\sigma} = (y - \dot{x}_1) / e^3 \]

With $K = [25 \ 7]$, the control is given by
\[ u = M \text{sat}((-\dot{\sigma} + \dot{r} - K \dot{x}_1)/(\dot{a} M)) \]
where $M = 0.5$ is determined by simulation.

6.2 Permanent Magnet Synchronous Motor with Uncertain Parameters and Unknown Load

The mathematical model of a Permanent Magnet Synchronous Motor (PMSM) in the rotor frame of reference is given by
\[ \frac{d\dot{\theta}}{dt} = -Ri_d + n_p L_\omega i_q + u_d \]
\[ \frac{d\dot{i}_d}{dt} = -Ri_d - n_p L_\omega i_q - k_{oa} \omega + u_q \]
\[ \frac{d\dot{\omega}}{dt} = k_{oa} i_q - D\omega - T_L \]
where $\theta$ is the rotor position, $\omega$ the rotor speed, $i_d$ and $i_q$ are the direct axis current and voltage, $i_q$ and $u_q$ are the quadrature axis current and voltage, and $T_L$ is a load torque. The parameters $L, R, n_p, k_{oa}, J,$ and $D$ are the stator inductance, stator resistance, number of pole pairs, rotor magnetic flux linkage, moment of inertia, and damping coefficient, respectively. The goal is to design a feedback controller, using only measurement of $\theta, i_d, \dot{\theta}$, and $i_q$, to regulate the speed $\omega$ to a reference signal $\omega_{ref}$ in
the presence of both unknown bounded load $T_L$ and parameter uncertainty. This is to be done while shaping the speed transient response and meeting the constraints $u_d^2 + u_q^2 \leq V^2_{\text{max}}$ and $i_d^2 + i_q^2 \leq I^2_{\text{max}}$.

The control algorithm consists of three parts, fast inner current loops, speed and disturbance estimation using the measured position via an extended high-gain observer, and speed regulation via feedback linearization. The fast current loops are made possible by the smallness of the electrical time constant $L/R$ and the use of PI controllers. Define the current tracking errors $e_d$ and $e_q$ by

$$e_d = i_{d,\text{ref}} - i_d, \quad e_q = i_{q,\text{ref}} - i_q$$

where $i_{d,\text{ref}}$ and $i_{q,\text{ref}}$ are the direct and quadrature current reference signals, respectively. The control inputs $u_d$ and $u_q$ are taken as

$$u_d(t) = k_p e_d(t) + k_i \int_0^t e_d(\lambda) d\lambda$$
$$u_q(t) = k_p e_q(t) + k_i \int_0^t e_q(\lambda) d\lambda$$

where $k_p$ and $k_i$ are the proportional and integral gains, respectively. By substitution of $u_d$ and $u_q$ in the motor equations, it can be seen that $e_d$ and $e_q$ satisfy the equations

$$\tau \frac{d e_d}{dt} = \tau \frac{d i_{d,\text{ref}}}{dt} + \frac{R}{R + k_p} i_{d,\text{ref}} - e_d$$
$$- \tau n_p \omega i_{q,\text{ref}} - e_q = \frac{1}{R + k_p} x_d$$
$$\tau \frac{d e_q}{dt} = \tau \frac{d i_{q,\text{ref}}}{dt} + \frac{R}{R + k_p} i_{q,\text{ref}} - e_q$$
$$+ \tau n_p \omega i_{d,\text{ref}} - e_d = \frac{1}{R + k_p} x_q + \frac{k_m}{R + k_p} \omega$$

where $\tau = L/(R + k_p)$, $x_d = k_i \int e_d$, and $x_q = k_i \int e_q$. By proper choice of the gains $k_p$ and $k_i$, we can meet the voltage constraint while making the time constant $\tau$ small enough so that $e_d$ and $e_q$ become much faster than the other state variables. By singular perturbation theory [14], we can reduce the order of the model by replacing $e_d$ and $e_q$ by their quasi-steady-state values, obtained by setting $\tau = 0$ in the foregoing equation. The resulting reduced-order model is given by

$$\frac{dx_d}{dt} = \frac{k_i}{R + k_p} (R i_{d,\text{ref}} - x_d)$$
$$\frac{dx_q}{dt} = \frac{k_i}{R + k_p} (R i_{q,\text{ref}} + k_m \omega - x_q)$$
$$\frac{da}{dt} = \alpha i_{q,\text{ref}} - \gamma \omega + \mu x_q - \frac{1}{J} T_L$$
$$\frac{d\hat{\theta}}{dt} = \omega$$

where $i_{d,\text{ref}}$ and $i_{q,\text{ref}}$ are viewed as the control inputs, and the parameters $\alpha$, $\gamma$, and $\mu$ are defined by

$$\alpha = k_p \mu, \quad \gamma = k_m + \frac{D}{J}, \quad \mu = \frac{k_m}{J(R + k_p)}$$

The variables $x_d$ and $x_q$ are available for feedback because they are integrations of $e_d$ and $e_q$, which are defined in terms of the measured currents $i_d$ and $i_q$. Since $i_{d,\text{ref}}$ does not affect the speed regulation, we take it to be zero; the $\dot{x}_d$-equation shows that $x_d(t)$ converges to zero. So we are left with the design of a feedback controller for $i_{q,\text{ref}}$ to regulate the speed $\omega$ to $\omega_{\text{ref}}$. Since the load $T_L$ is unknown and the parameters $\alpha$, $\gamma$, and $\mu$ are uncertain, we rewrite the $\dot{\omega}$-equation as

$$\frac{d\omega}{dt} = \hat{\alpha} i_{q,\text{ref}} - \hat{\gamma} \omega + \hat{\mu} x_q + \sigma$$

where $\hat{\alpha}$, $\hat{\gamma}$, and $\hat{\mu}$ are nominal values of $\alpha$, $\gamma$, and $\mu$, respectively, and

$$\sigma = (\alpha - \hat{\alpha}) i_{q,\text{ref}} - (\gamma - \hat{\gamma}) \omega + (\mu - \hat{\mu}) x_q = \frac{1}{J} T_L$$

Had $\omega$ and $\sigma$ been available for feedback we could have used feedback linearization to regulate the speed tracking error $e_\omega = \omega_{\text{ref}} - \omega$ to zero while shaping its transient response. In particular, taking

$$i_{q,\text{ref}} = \frac{1}{\alpha} \left[ \frac{d\omega_{\text{ref}}}{dt} + \hat{\gamma} \omega_{\text{ref}} + (k_m - \hat{\gamma}) e_\omega - \hat{\mu} x_q - \sigma \right]$$

yields the target system

$$\frac{de_\omega}{dt} = -k_p e_\omega$$

whose transient response is shaped by choosing the positive gain $k_p$. The choice of $k_p$ is limited by the current constraint. Now, the extended high-gain observer

$$\frac{d\hat{\theta}}{dt} = \dot{\omega} + \frac{3}{e}(\theta - \hat{\theta})$$
$$\frac{d\omega}{dt} = \hat{\alpha} i_{q,\text{ref}} - \hat{\gamma} \omega + \hat{\mu} x_q + \frac{3}{e^2}(\theta - \hat{\theta})$$
$$\frac{d\hat{\omega}}{dt} = \frac{1}{e}(\theta - \hat{\theta})$$

is used to estimate $\omega$ and $\sigma$ by $\dot{\omega}$ and $\dot{\sigma}$, respectively. The condition (15) is satisfied provided $|\alpha - \hat{\alpha} |/\alpha | < 1$. With $\omega$ and $\sigma$ replaced by their estimates, the output feedback controller is given by

$$i_{q,\text{ref}} = \frac{1}{\alpha} \left[ \frac{d\omega_{\text{ref}}}{dt} + \hat{\gamma} \omega_{\text{ref}} + (k_m - \hat{\gamma}) \hat{\omega} - \hat{\mu} x_q - \hat{\sigma} \right]$$

where $\hat{\omega} = \omega_{\text{ref}} - \dot{\omega}$. Analysis of the closed-loop system shows that $|e_\omega(t) - e_\omega(t)|$ can be made arbitrarily small for all $t \geq 0$ by choosing $e$ and $\tau$ sufficiently small. The need for a small ratio $\tau/e$ is expected because of the model order reduction that is performed before the observer design. The nominal value of $\tau$ is $2.145 \times 10^{-4}$ seconds and $e$ is taken as 0.01, so that $\tau/e = 2.145 \times 10^{-2}$. The controller was tested experimentally with 10 kHz sampling frequency and 2500 PPR incremental encoder mounted on the motor’s shaft. The electric currents are measured with Hall Effect sensors. The control algorithm is implemented using a National Instruments’ Real-Time system.

6.3 Inverted Pendulum on a Cart in the Presence of Uncertainties

The mathematical model of the inverted pendulum on a cart is given by

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = f_1(x_1, x_2, F)$$
$$\dot{\theta}_1 = \theta_2, \quad \dot{\theta}_2 = f_2(x_1, x_2, F)$$
where $x_1$ is the position of the cart, $x_2$ its velocity, $\alpha_1$ is the angular displacement of the pendulum measured clockwise from the vertical position, $\alpha_2$ is the angular velocity of the pendulum, and $F$ is the force applied to the cart. The functions $f_1$ and $f_2$ are given by

\[
\begin{align*}
f_1 &= \frac{F}{m_p + m_c - m_p \cos^2 \alpha_1} + G_x \\
f_2 &= -\frac{F \cos \alpha_1}{\ell (m_p + m_c - m_p \cos^2 \alpha_1)} + G_u
\end{align*}
\]

where

\[
\begin{align*}
G_x &= \frac{\ell m_p \alpha_1^2 \sin \alpha_1 - m_p g \cos \alpha_1 \sin \alpha_1}{m_p + m_c - m_p \cos^2 \alpha_1} \\
G_u &= \frac{g \sin \alpha_1 - G_x \cos \alpha_1}{\ell}
\end{align*}
\]

$m_p$ and $m_c$ are the masses of the pendulum and the cart, respectively, $g$ is the acceleration due to gravity, and $\ell$ is the length of the pendulum. The angle $\alpha_1$ is restricted to the domain $-\pi/2 < \alpha_1 < \pi/2$. The goal is to stabilize the pendulum at the origin using measurements of $x_1$ and $\alpha_1$, despite uncertainties in the system parameters.

Taking

\[
F = (m_p + m_c - m_p \cos^2 \alpha_1)(u - G_x)
\]

with

\[
u = g \tan \alpha_1 + \frac{[\beta_1(\alpha_1 - \alpha_2) + \beta_2 \alpha_2]}{\cos \alpha_1}
\]

where $\beta_1$ and $\beta_2$ are positive constants and $\alpha_1$ is a reference trajectory for $\alpha_1$, results in

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= g \tan \alpha_1 + \left(\frac{\ell}{\cos \alpha_1}\right) [\beta_1(\alpha_1 - \alpha_2) + \beta_2 \alpha_2] \\
\dot{\alpha}_1 &= \alpha_2 \\
\dot{\alpha}_2 &= -\beta_1(\alpha_1 - \alpha_2) - \beta_2 \alpha_2
\end{align*}
\]

The control strategy is to choose the controller parameters to make the pendulum dynamics much faster than the cart dynamics so that $\alpha_1$ and $\alpha_2$ converge quickly to $\alpha_1$ and $0$. Then, the cart dynamics can be represented by the reduced model

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= g \tan \alpha_2
\end{align*}
\]

With

\[
\alpha_2 = \tan^{-1}\left(\frac{-\gamma_1 x_1 - \gamma_2 x_2}{g}\right)
\]

the cart dynamics are given by

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -\gamma_1 x_1 - \gamma_2 x_2
\end{align*}
\]

With positive gains $\gamma_1$ and $\gamma_2$, $x_1$ and $x_2$ converge to zero; consequently, $\alpha_2$ converges to zero, showing that the whole state vector converges to the origin. The two time-scale behavior is achieved by taking $\gamma_1 = \varepsilon_1^2 k_1$ and $\gamma_2 = \varepsilon_1 k_2$, where $k_1$, $k_2$, and $\varepsilon_1$ are positive constants with $\varepsilon_1 \ll 1$.

In the presence of parameter uncertainties, $f_1$ and $f_2$ are unknown. These functions are estimated using extended high-gain observers. In preparation for that, dynamic inversion is used to compute $F$ and $u$ in terms of $f_1$ and $f_2$. Noting that $F$ and $u$ satisfy the equations $f_1 - u = 0$ and $f_2 + \beta_1(\alpha_1 - \alpha_2) + \beta_2 \alpha_2 = 0$, respectively, the dynamic inversion algorithm is taken as

\[
\begin{align*}
\varepsilon_2^{-1} F &= -f_1 + u \\
&= \frac{\dot{f}_2}{f_2} \left(\dot{\alpha}_1(\alpha_1 - \alpha_2) + \dot{\nu} \alpha_2\right)
\end{align*}
\]

where $0 < \varepsilon_2 \ll 1$. Let $\dot{f}_2(\alpha_1, \alpha_2, F)$ and $\dot{\nu}(\alpha_1, \alpha_2, F)$ be nominal models of $f_2$ and $\nu$. Define the uncertain terms $\sigma_\alpha$ and $\sigma_{\nu_\alpha}$ by $\sigma_\alpha = \dot{f}_2 - \dot{f}_2^* \text{ and } \sigma_{\nu_\alpha} = \dot{\nu} - \dot{\nu}^*$. The velocities $x_2, \alpha_2$ and the uncertain terms $\sigma_\alpha, \sigma_{\nu_\alpha}$ are estimated by the extended high-gain observers

\[
\begin{align*}
\dot{\hat{x}}_1 &= \dot{\hat{x}}_2 + (3/\varepsilon_3)(x_1 - \hat{x}_1) \\
\dot{\hat{x}}_2 &= \dot{\hat{f}}_2(\hat{\alpha}_1, \hat{\alpha}_2, F) + \sigma_\alpha(3/\varepsilon_3)(x_1 - \hat{x}_1) \\
\dot{\hat{\sigma}}_\alpha &= (1/\varepsilon_3^2)(x_1 - \hat{x}_1) \\
\dot{\hat{x}}_1 &= \hat{\alpha}_2 + (3/\varepsilon_2)(\alpha_1 - \hat{\alpha}_1) \\
\dot{\hat{\alpha}}_2 &= \hat{f}_2(\hat{\alpha}_1, \hat{\alpha}_2, F) + \sigma_{\nu_\alpha}(3/\varepsilon_3^2)(\alpha_1 - \hat{\alpha}_1) \\
\dot{\hat{\sigma}}_{\nu_\alpha} &= (1/\varepsilon_3^2)(\alpha_1 - \hat{\alpha}_1)
\end{align*}
\]

The observer parameter $\varepsilon_3$ is chosen such that $\varepsilon_3/\varepsilon_2 \ll 1$ to make the observer dynamics faster than the dynamic inversion.

The output feedback controller is given by

\[
\begin{align*}
\varepsilon_2^{-1} F &= \frac{\dot{f}_2}{f_2} \left[-f_1 + u + \frac{\dot{\hat{f}}_2}{f_2} \left(\hat{\alpha}_1(\alpha_1 - \alpha_2) + \dot{\nu} \alpha_2\right)\right]
\end{align*}
\]

where

\[
\begin{align*}
\dot{\hat{f}}_2 &= \hat{f}_2(\alpha_1, M_1 \hat{\alpha}_2, M_1 \hat{f}_2) + M_2 \hat{\sigma}_\alpha(\hat{\alpha}_2, M_1) \\
\dot{\hat{f}}_2 &= \hat{f}_2(\alpha_1, M_1 \hat{\alpha}_2, M_1 \hat{f}_2) + M_3 \hat{\sigma}_{\nu_\alpha}(\hat{\alpha}_2, M_3)
\end{align*}
\]

The estimates $\hat{\alpha}_2, \hat{\sigma}_\alpha, \text{ and } \hat{\sigma}_{\nu_\alpha}$ are saturated outside the compact set of operation with the saturation levels $M_1, M_2, \text{ and } M_3$, respectively. The estimate $\hat{\alpha}_2$ is not saturated because it enters through the globally bounded arctan function.

In the experimental setup, a 6V-DC motor with a planetary gearhead (reduction ratio 3.71 : 1) drives the cart on a track. The angle of the pendulum and the position of the cart are measured by optical encoders with resolution 1024 lines per revolution. The experimental hardware was interfaced with dSPACE board and the controller was implemented in the MATLAB/Simulink environment with sampling period 0.6 ms. To reduce the effect of measurement noise, the encoder signals were passed through a low-pass filter with 1 kHz cutoff frequency. The parameters $\varepsilon_1 \text{ to } \varepsilon_3$ were taken as $\varepsilon_1 = 0.2, \varepsilon_2 = 0.01, \varepsilon_3 = 0.005$.

7. Conclusions

This paper reviewed some recent results on the use of extended high-gain observers as disturbance estimators. Three experimental testbeds are described. High-gain observers are known to be sensitive to measurement noise [15]. The experimental results presented here show that successful results can be obtained without using very large gains (equivalently, very small $\varepsilon$). In all three experiments, the observer’s order was three. For higher-order observers, the issue of measurement noise could be challenging. More experimental testing is needed to determine how high the observer’s order could be and still achieve successful results.

\footnote{See [13] for the other controller parameters.}
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