Jump and variational inequalities for rough operators

Yong Ding
School of Mathematical Sciences
Beijing Normal University
Laboratory of Mathematics and Complex Systems (BNU)
Ministry of Education of China
Beijing 100875, China
E-mail: dingy@bnu.edu.cn

Guixiang Hong
School of Mathematics and Statistics
Wuhan University
Wuhan 430072, China
and
Instituto de Ciencias Matemáticas
CSIC-UAM-UC3M-UCM
Consejo Superior de Investigaciones Científicas
Madrid 28049, Spain
E-mail: guixiang.hong@icmat.es

Honghai Liu
School of Mathematics and Information Science,
Henan Polytechnic University,
Jiaozuo, Henan, 454003, China
E-mail: hhliu@hpu.edu.cn

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Abstract

In this paper, we systematically study jump and variational inequalities for rough operators, whose research have been initiated by Jones et al. More precisely, we show some jump and variational inequalities for the families $T := \{T_\varepsilon\}_{\varepsilon > 0}$ of truncated singular integrals and $M := \{M_t\}_{t > 0}$ of averaging operators with rough kernels, which are defined respectively by

$$T_\varepsilon f(x) = \int_{|y| > \varepsilon} \frac{\Omega(y')}{|y|^n} f(x - y) dy$$

and

$$M_t f(x) = \frac{1}{t^n} \int_{|y| < t} \Omega(y') f(x - y) dy,$$

where the kernel $\Omega$ belongs to $L \log^+ L(S^{n-1})$ or $H^1(S^{n-1})$ or $G_\alpha(S^{n-1})$ (the condition introduced by Grafakos and Stefanov). Some of our results are sharp in the sense that the underlying assumptions are the best known conditions for the boundedness of corresponding maximal operators.

1 Introduction

The jump and variational inequalities have been the subject of many recent articles in probability, ergodic theory and harmonic analysis. The first variational inequality was proved by Lépingle [28] for martingales (see [33] for a simple proof). Bourgain [3] is the first one using Lépingle’s result to obtain similar variational estimates for the ergodic averages, and then directly deduce pointwise convergence results without previous knowledge that pointwise convergence holds for a dense subclass of functions, which are not available in some ergodic models. In particular, Bourgain’s work [3] has inaugurated a new research direction in ergodic theory and harmonic analysis. In their papers [22] [21] [23] [6], Jones and his collaborators systematically studied jump and variational inequalities for ergodic averages and truncated singular integrals (mainly of homogeneous type). Since then many other publications came to enrich the literature on this subject (cf. e.g. [15], [27], [11], [25], [31], [32], [18]). Recently, several works on weighted and vector-valued jump and variational inequalities in ergodic theory and harmonic analysis have also appeared (cf. e.g. [30], [26], [21], [19], [20]).

Let us first recall some definitions and known results, then state our results.

1. $q$-variation norm $\|a\|_{V_q}$ of a family $a$ of complex numbers

Given a family of complex numbers $a = \{a_t : t \in \mathbb{R}\}$ and $q \geq 1$, the $q$-variation norm of the family $a$ is defined by

$$(1.1) \quad \|a\|_{V_q} = \sup \left( |a_{t_0}| + \sum_{k \geq 1} |a_{t_k} - a_{t_{k-1}}|^q \right)^{\frac{1}{q}},$$

where the supremum runs over all increasing sequences $\{t_k : k \geq 0\}$. It is trivial that

$$(1.2) \quad \|a\|_{L^\infty(\mathbb{R})} := \sup_{t \in \mathbb{R}} |a_t| \leq \|a\|_{V_q} \quad \text{for} \quad q \geq 1.$$
2. Strong $q$-variation function $V_q(\mathcal{F})(x)$ of a family $\mathcal{F}$ of functions

Via the definition (1.1) of the $q$-variation norm of a family of numbers, one may define the strong $q$-variation function $V_q(\mathcal{F})$ of a family $\mathcal{F}$ of functions. Given a family of Lebesgue measurable functions $\mathcal{F} = \{F_t : t \in \mathbb{R}\}$ defined on $\mathbb{R}^n$, for fixed $x \in \mathbb{R}^n$, the value of the strong $q$-variation function $V_q(\mathcal{F})$ of the family $\mathcal{F}$ at $x$ is defined by

$$V_q(\mathcal{F})(x) = \|\{F_t(x)\}_{t \in \mathbb{R}}\|_{V_q}, \quad q \geq 1.$$  (1.3)

Usually, the measurability of the strong $q$-variation function is not automatically available. However, all the strong $q$-variation function considered in the present paper are measurable, see e.g. [6] or [25] for some explanations.

Suppose $\mathcal{A} = \{A_t\}_{t > 0}$ is a family of operators on $L^p(\mathbb{R}^n)$ ($1 \leq p \leq \infty$). The strong $q$-variation operator is simply defined as

$$V_q(\mathcal{A}f)(x) = \|\{A_t(f)(x)\}_{t > 0}\|_{V_q}, \quad \forall f \in L^p(\mathbb{R}^n).$$

It is easy to observe from the definition of $q$-variation norm that for any $x$ if $V_q(\mathcal{A}f)(x) < \infty$, then $\{A_t(f)(x)\}_{t > 0}$ converges when $t \to 0$ or $t \to \infty$. In particular, if $V_q(\mathcal{A}f)$ belongs to some function spaces such as $L^p(\mathbb{R}^n)$ or $L^{p,\infty}(\mathbb{R}^n)$, then the sequence converges almost everywhere without any additional condition. This is why mapping property of strong $q$-variation operator is so interesting in ergodic theory and harmonic analysis.

3. $\lambda$-jump function $N_\lambda(\mathcal{F})$ of a family $\mathcal{F}$ of functions

Given a family of Lebesgue measurable functions $\mathcal{F} = \{F_t : t \in \mathbb{R}\}$ defined on $\mathbb{R}^n$, for $\lambda > 0$ and $x \in \mathbb{R}^n$, the value of the $\lambda$-jump function $N_\lambda(\mathcal{F})$ at $x$ is defined by

$$N_\lambda(\mathcal{F})(x) = \sup \{N \in \mathbb{N} : \exists s_1 < t_1 \leq s_2 < t_2 \leq \ldots \leq s_N < t_N \text{ such that } |F_{t_k}(x) - F_{t_k}(x)| > \lambda\}.$$  (1.4)

By [3], for a function family $\mathcal{F} = \{F_t : t \in \mathbb{R}\}$, $\lambda > 0$ and $q \geq 1$, the $\lambda$-jump function $N_\lambda(\mathcal{F})$ is pointwisely controlled by the strong $q$-variation $V_q(\mathcal{F})$ in the following sense,

$$\lambda(N_\lambda(\mathcal{F})(x))^{1/q} \leq C_q V_q(\mathcal{F})(x), \quad x \in \mathbb{R}^n,$$
where $C_q = 2^{1+1/q}$. On the other hand, in 2008, Jones, Seeger and Wright gave the following result.

**Lemma 1.1.** ([25]) Let $p_0 < \rho < p_1$ and $\mathcal{A} = \{A_t\}_{t > 0}$ be a family of operators. If for all $p_0 < p < p_1$,

$$\sup_{\lambda > 0} \|\lambda(N_\lambda(\mathcal{A}f))^{1/p}\|_p \leq C_p \|f\|_p,$$

then for all $p_0 < p < p_1$ and for all $q > \rho$, $\|V_q(\mathcal{A}f)\|_p \leq C_{p,q} \|f\|_p$.  

This lemma gives a converse inequality of \((1.4)\) in some sense.

4. Family \( \mathcal{T} \) of truncated singular integral operators

Recall the Calderón-Zygmund singular integral operator \( T \) with homogeneous kernel is defined by

\[
T(f)(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(y')}{|y'|^n} f(x - y) dy,
\]

where \( \Omega \in L^1(S^{n-1}) \) satisfies the cancelation condition

\[
\int_{S^{n-1}} \Omega(y') d\sigma(y') = 0.
\]

Let \( \mathcal{T} = \{T_\varepsilon\}_{\varepsilon > 0} \), where \( T_\varepsilon \) is the truncated operator of \( T \) defined by

\[
T_\varepsilon f(x) = \int_{|y| > \varepsilon} \frac{\Omega(y')}{|y'|^n} f(x - y) dy.
\]

The famous Hilbert transform \( H \), which is defined by

\[
H(f)(x) = \text{p.v.} \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(y)}{x-y} dy,
\]

is the example of the homogeneous singular integral operator \( T_\Omega \) when the dimension \( n = 1 \).

In 2000, Campbell et al [6] first considered the \( L^p(\mathbb{R}) \) \((1 < p < \infty)\) boundedness of the strong \( q \)-variation operator of the family of the truncated Hilbert transforms denoted by \( \mathcal{H} := \{H_\varepsilon\}_{\varepsilon > 0} \). As a corollary, the authors of [6] obtained the boundedness of \( \lambda \)-jump function for Hilbert transform associated to \( q \) with \( q > 2 \). The proof depends on a special property of Hilbert transform \( H \). That is, \( H_\varepsilon \) can be written as a combinations of certain convolution operators, which in turn can be written as combinations of differential operators.

In 2002, Campbell et al [7] gave the \( L^p(\mathbb{R}^n) \) boundedness of the strong \( q \)-variation operator of \( \mathcal{T} \), the family of homogenous singular integrals with \( \Omega \in L \log^+ L(S^{n-1}) \) and \( n \geq 2 \).

**Theorem A.** (7) Suppose \( \Omega \) satisfies \((1.6)\) and \( \Omega \in L \log^+ L(S^{n-1}) \). If \( q > 2 \), then for \( 1 < p < \infty \) and \( f \in L^p(\mathbb{R}^n) \), the strong \( q \)-variation function \( V_q(T f)(x) \in L^p(\mathbb{R}^n) \). In particular,

\[
\|V_q(T f)\|_{L^p(\mathbb{R}^n)} \leq C_{p,q,n} \|f\|_{L^p(\mathbb{R}^n)},
\]

where and in the sequel, the constant \( C_{p,q,n} > 0 \) depends only on \( q, p, n \).

The basic idea of proving Theorem A is classical, that is, the authors applied the Calderón-Zygmund rotation method to reduce the desired variational inequalities to the one dimensional results. The \( \lambda \)-jump inequalities associated to \( q > 2 \) were obtained as a corollary by \((1.4)\).
In 2008, using the Fourier transform and square function estimates given in [13], Jones, Seeger and Wright [25] developed a general method, which allows one to obtain some jump inequalities for the family of truncated singular integral operators $T = \{T_\varepsilon\}_{\varepsilon > 0}$ and the other integral operators arising from harmonic analysis.

**Theorem B.** ([25]) Suppose $\Omega$ satisfies (1.6) and $\Omega \in L^r(S^{n-1})$ for $r > 1$. Then $\lambda$-jump inequality $\sup_{\lambda > 0} \|\lambda \sqrt{N_\lambda(T f)}\|_{L^p(\mathbb{R}^n)} \leq C_{p,n}\|f\|_{L^p(\mathbb{R}^n)}$ $(1 < p < \infty)$ holds, where and in the sequel, the constant $C_{p,n} > 0$ depends only on $p$ and $n$.

Notice the following well known inclusion relations between some function spaces on $S^{n-1}$:

(1.9) $L^\infty(S^{n-1}) \subseteq L^r(S^{n-1}) (1 < r < \infty) \subseteq L \log^+ L(S^{n-1}) \subseteq H^1(S^{n-1}) \subseteq L^1(S^{n-1})$,

where and in the sequel, $H^1(S^{n-1})$ denotes the Hardy space on $S^{n-1}$, the definition and some facts on $H^1(S^{n-1})$ can be found in [8], [29] and [34].

Thus, (1.9) inspires us to consider the question: whether the conclusions of Theorem A and Theorem B are still true if $\Omega \in H^1(S^{n-1})$? The first main result in this paper gives an answers of the above question.

**Theorem 1.2.** Let $T$ be the family of truncated singular integral operators given in (1.7) and $\Omega$ satisfies (1.6).

(i) If $\Omega \in H^1(S^{n-1})$, then for all $q > 2$ and $1 < p < \infty$, the following strong $q$-variation inequality holds

(1.10) $\|V_q(T f)\|_{L^p(\mathbb{R}^n)} \leq C_{p,q,n}\|f\|_{L^p(\mathbb{R}^n)}$.

(ii) If $\Omega \in L \log^+ L(S^{n-1})$, then the following $\lambda$-jump inequality holds

(1.11) $\sup_{\lambda > 0} \|\lambda \sqrt{N_\lambda(T f)}\|_{L^p(\mathbb{R}^n)} \leq C_{p,n}\|f\|_{L^p(\mathbb{R}^n)}$ for all $1 < p < \infty$.

**Remark 1.3.** From the relation (1.9), it is easy to see that the conclusion (i) of Theorem 1.2 is an essential improvement of Theorem A by (1.9). Similarly, the conclusion (ii) is also an essential improvement of Theorem B.

**Remark 1.4.** It is well-known that $\Omega \in H^1(S^{n-1})$ is the best condition for the boundedness of maximal singular integral operator. However, we would like to point out that it is still not clear whether the estimate (1.11) holds for $\Omega \in H^1(S^{n-1})$ up to now. So, this remains an open problem.
In 1998, Grafakos and Stefanov [16] introduced alternative conditions on $\Omega$ to study $L^p$ boundedness of maximal singular integral operators. More precisely, they considered the family of conditions

$$G_\alpha(S^{n-1}) = \left\{ \Omega : \sup_{\xi \in S^{n-1}} \int_{S^{n-1}} |\Omega(\theta)| (\log \frac{1}{|\theta \cdot \xi|})^{1+\alpha} d\theta < \infty \right\}, \quad \alpha > 0. \tag{1.12}$$

It is known from [16] and [17] that

$$\bigcup_{r > 1} L^r(S^{n-1}) \subsetneq \bigcap_{\alpha > 0} G_\alpha(S^{n-1}),$$

$$\bigcap_{\alpha > 0} G_\alpha(S^{n-1}) \not\subseteq L \log L(S^{n-1}) \not\subseteq \bigcap_{\alpha > 0} G_\alpha(S^{n-1}),$$

$$\bigcap_{\alpha > 0} G_\alpha(S^{n-1}) \not\subseteq H^1(S^{n-1}).$$

Hence, it is of interest to ask whether the jump and variation inequalities hold for the family $\mathcal{T}$ of truncated singular integral operators if $\Omega \in G_\alpha$ for some $\alpha > 0$? This constitutes the second result of this paper.

**Theorem 1.5.** Let $\mathcal{T}$ be the family of truncated singular integral operators given in (1.7), $\Omega$ satisfies (1.6) and $\Omega \in G_\alpha(S^{n-1})$ for some $\alpha > 1$. Then the following $\lambda$-jump inequality

$$\sup_{\lambda > 0} \|\lambda \sqrt{N_\lambda(Tf)}\|_{L^p(\mathbb{R}^n)} \leq C_{p,n} \|f\|_{L^p(\mathbb{R}^n)} \tag{1.13}$$

holds for all $$(3 + \alpha)/(1 + \alpha) < p < (3 + \alpha)/2.$$ In particular, if $\Omega \in \bigcap_{\alpha > 1} G_\alpha(S^{n-1})$, then (1.13) holds for all $1 < p < \infty$.

The associated strong $q$-variation inequality is an immediate consequence of Theorem 1.5 by Lemma 1.1.

**Corollary 1.6.** Under the same conditions as Theorem 1.5, the strong $q$-variation inequality

$$\|V_q(Tf)\|_{L^p(\mathbb{R}^n)} \leq C_{p,q,n} \|f\|_{L^p(\mathbb{R}^n)}$$

holds for all $q > 2$ and $(3 + \alpha)/(1 + \alpha) < p < (3 + \alpha)/2$. In particular, the above estimate holds for $\Omega \in \bigcap_{\alpha > 1} G_\alpha(S^{n-1})$ and all $1 < p < \infty$.

**Remark 1.7.** In [14], the author showed the associated maximal singular integral operator, that is the strong $\infty$-variation operator, is $L^p$-bounded for $(1 + 2\alpha)/(2\alpha) < p < 1 + 2\alpha$ and $\alpha > 1/2$. So we think the scope of $p$ in Corollary 1.6 is not optimal, and it is very interesting to enlarge the scope of $p$ depending on $q$.  

The main sketch of proving Theorem 1.2 and Theorem 1.5 are taken from [7] and [25]. That is, we first reduce the $\lambda$-jump estimate to short 2-variation estimate and dyadic $\lambda$-jump estimate (see the beginning of the next section for related definitions and statement); then use the rotation method or the vector-valued singular integral operator theory to deal with short 2-variation operators; and use Fourier transform and square function estimates to obtain dyadic $\lambda$-jump estimate.

However, the underlying details are substantially different due to the kernels being quite rough. For instance, firstly when $\Omega \in L\log^+ L(S^{n-1})$, it follows from [13] that unlike the case $\Omega \in L^r(S^{n-1})$ ($r > 1$) the decay of Fourier transform of associated kernel is not available any more, and we have to decompose $\Omega$ into pieces having polynomial decay as done in [2]. In order to finally sum all the pieces to conclude the desired result, we exploit some subtle calculations to get sharp bounds for each piece in obtaining dyadic $\lambda$-jump estimate.

Secondly when $\Omega \in G_\alpha(S^{n-1})$, to obtain the short 2-variation estimate, first of all, the rotation method seems not to work here. Instead, we appeal to the vector-valued singular integral operator theory, which has been made use of by Jones et al. [25] for averaging over spheres and curves. But in our case, the kernel being rough brings us a lot of difficulties in both obtaining the $L^2$ estimate and verifying the Hörmander condition.

5. Family $\mathcal{M}$ of the averaging operators with rough kernel

The third aim of the present paper is to establish some jump and variational inequalities for the family $\mathcal{M} = \{M_t\}_{t > 0}$. Here $M_t$ denotes the averaging operator with rough kernels defined by

$$M_t f(x) = \frac{1}{t^n} \int_{|y| < t} \Omega(y') f(x - y) dy,$$

where $\Omega \in L^1(S^{n-1})$.

The motivation of considering the family $\mathcal{M} = \{M_t\}_{t > 0}$ is two-fold. Firstly, in the case $\Omega \equiv 1$, the jump and variational inequalities for the family $\mathcal{M} = \{M_t\}_{t > 0}$ have been well studied in [6] and [23], which were based on the 1-dimensional results [4] and [22]. Secondly, the maximal operator associated with $\mathcal{M} = \{M_t\}_{t > 0}$

$$M^*(f)(x) = \sup_{t > 0} \frac{1}{t^n} \int_{|y| < t} |\Omega(y') f(x - y)| dy,$$

plays a very important role in studying rough singular integral operators (see [35], [13], [9], [12] or [29] for more details).

**Theorem 1.8.** Suppose the family $\mathcal{M} = \{M_t\}_{t > 0}$ is defined in (1.14).
(i) If $\Omega \in H^1(S^{n-1})$ or $L(\log^+ L)^{1/2}(S^{n-1})$ and $1 < p < \infty$, then the $\lambda$-jump inequality for the family $\mathcal{M}$ holds. That is, there exists $C_{p,n} > 0$ so that

$$\sup_{\lambda > 0} \| \lambda \sqrt{N_{\lambda}(\mathcal{M}f)} \|_{L^p(\mathbb{R}^n)} \leq C_{p,n} \| f \|_{L^p(\mathbb{R}^n)}.$$  

(ii) If $\Omega \in L^1(S^{n-1})$, then for $q > 2$ and $1 < p < \infty$, the strong $q$-variation inequality for the family $\mathcal{M}$ holds. That is, there exists $C_{p,q,n} > 0$ so that

$$\| V_q(\mathcal{M}f) \|_{L^p(\mathbb{R}^n)} \leq C_{p,q,n} \| f \|_{L^p(\mathbb{R}^n)}.$$  

The idea of the proof of Theorem 1.8 is similar as that for Theorem 1.2. That is, we use the rotation method to show (1.16) and discrete Marcinkiewicz integrals to deal with associated dyadic $\lambda$-jump estimate.

Remark 1.9. Note that both $H^1(S^{n-1})$ and $L(\log^+ L)^{1/2}(S^{n-1})$ contain $L \log L(S^{n-1})$, but they do not contain each other (see e.g. [1]). It seems difficult to get (1.15) using only the method in the present paper for $\Omega \in L^1(S^{n-1})$, since which is not sufficient for boundedness of Marcinkiewicz integrals.

The proofs of Theorem 1.2, Theorem 1.5 and Theorem 1.8 will be given in Sections 2, 3 and 4, respectively.

2 Proof of Theorem 1.2

We shall mainly show Theorem 1.2(ii), the $\lambda$-jump estimate (1.11), while Theorem 1.2(i) will be obtained in the course of the proof. Let us start with recalling the strategy taken by Jones et al [25]. Let $\mathcal{F} = \{F_t\}_{t \in \mathbb{R}^+}$ be a family of functions. For $j \in \mathbb{Z}$ and $x \in \mathbb{R}^n$, define

$$V_{2,j}(\mathcal{F})(x) = \left( \sup_{[t_1, t_2, \ldots, t_N] \subset [2^j, 2^{j+1}]} \sum_{l=1}^{N-1} |F_{t_{l+1}}(x) - F_{t_l}(x)|^2 \right)^{1/2},$$

and the short $2$-variation operator

$$S_2(\mathcal{F})(x) = \left( \sum_{j \in \mathbb{Z}} \left| V_{2,j}(\mathcal{F})(x) \right|^2 \right)^{1/2}.$$  

We also define dyadic $\lambda$-jump function as follows:

$$N_\lambda^d(\mathcal{F})(x) = \sup_{N \in \mathbb{N}} \{ \exists j_1 < k_1 \leq j_2 < k_2 \leq \cdots \leq j_N < k_N, \text{s.t.} |F_{2k_j}(x) - F_{2j_i}(x)| > \lambda \}. $$

Then the following pointwise comparison holds.
Lemma 2.1. (see [25, Lemma 1.3])
\[
\lambda \sqrt{N_{\lambda}(\mathcal{F})(x)} \leq C \left( S_{2}(\mathcal{F})(x) + \lambda \sqrt{N_{\lambda/3}(\mathcal{F})(x)} \right)
\]
uniformly in \( \lambda > 0 \).

Lemma 2.1 reduces the desired estimate
\[
\| \lambda \sqrt{N_{\lambda}(A f)} \|_{L^p(\mathbb{R}^n)} \leq C_p \| f \|_{L^p(\mathbb{R}^n)}
\]
for any fixed \( 1 < p < \infty \) to
\[
\| \lambda \sqrt{N_{\lambda}(A f)} \|_{L^p(\mathbb{R}^n)} \leq C_p \| f \|_{L^p(\mathbb{R}^n)} \tag{2.1}
\]
and
\[
\| S_{2}(A f) \|_{L^p(\mathbb{R}^n)} \leq C_p \| f \|_{L^p(\mathbb{R}^n)} \tag{2.2}
\]
for any family of linear operators \( A \).

Let \( \mathcal{T} \) be a family of truncated singular integrals of homogeneous type. In [7], the authors established estimate (2.2) for \( \Omega \in L \log^+ L(\mathbb{S}^{n-1}) \) using the rotation method. In the following subsection, we observe that the rotation method works also in the case \( \Omega \in H^1(\mathbb{S}^{n-1}) \). Whence we obtain the strong \( q \)-strong estimate (1.10), which is Theorem 1.2(i).

While in [25], the authors showed estimate (2.1) for \( \Omega \in L^r(\mathbb{S}^{n-1}) \) making use of the polynomial decay of Fourier transform of the measure \( \nu \) defined by
\[
\langle \nu, f \rangle = \int_{1 \leq |x| \leq 2} \frac{\Omega(|x|/|x|)}{|x|^n} f(x) dx.
\]
In the second subsection, we show that actually estimate (2.1) is still true for \( \Omega \in L \log^+ L(\mathbb{S}^{n-1}) \) even though the polynomial decay of Fourier transform of the measure \( \nu \) is not available as explained in the Introduction. Thus we obtain (1.11), which is Theorem 1.2(ii).

2.1 Proof of Theorem 1.2(i)

As observed in [7], the rotation method allows us to obtain simultaneously short 2-variation estimate and strong \( q \)-variation estimate since both estimates for the family of truncated Hilbert transforms have been established in [6]. In [7], they provided the proof of strong \( q \)-variation estimate. In the present paper, we give a sketch of the proof of short 2-variation.

Proposition 2.2. Let \( \mathcal{T} \) be given as in (1.7). \( \Omega \) satisfies (1.6) and \( \Omega \in H^1(\mathbb{S}^{n-1}) \). Then
\[
\| S_{2}(\mathcal{T} f) \|_{L^p(\mathbb{R}^n)} \leq C_p \| f \|_{L^p(\mathbb{R}^n)} \text{ for } 1 < p < \infty.
\]

The same inequality holds for the strong \( q \)-variation operator \( V_q(\mathcal{T} f) \) with \( q > 2 \).
To prove Proposition 2.2 by the Calderón-Zygmund rotation method, we need the following known result in one dimension case.

Lemma 2.3. ([8, Theorem 1.4]) Let $H = \{H_\varepsilon\}_{\varepsilon > 0}$ be the family of truncated Hilbert transforms. Then

$$
\|S_2(Hf)\|_{L^p(\mathbb{R})} \leq C_p\|f\|_{L^p(\mathbb{R})} \quad \text{for } 1 < p < \infty.
$$

Similar statement holds true for $V_q(Hf)$ with $q > 2$.

Proof. Let us turn to the proof of Proposition 2.2. Since $\Omega = H^{1}(S^{n-1})$ and satisfies (1.6), it is easy to check that we may write $\Omega = \Omega_\omega + \Omega_\varepsilon$, where $\Omega_\omega$ is odd and $\Omega_\varepsilon$ is even, and both of them belonging to $H^{1}(S^{n-1})$ as well as satisfying cancelation condition (1.6). Thus, we need only to prove Lemma 2.2 for $\Omega$ is odd and even, respectively. The following proof is based on the idea of [7], we cite some estimates there for the sake of completeness.

**Case 1.** $\Omega$ is odd. For an interval $I \subset (0, \infty)$, let

$$
H^1_I f(x) = \int_{|s| \in I} \frac{f(x-s)}{s} ds,
$$

where $1 = (1, 0, \cdots, 0)$. For simplicity, we denote $H^{1}_x$ by $H^1_x$, and set $H^1 = \{H^1_x\}_{\varepsilon > 0}$. Let $y' = y/|y| \in S^{n-1}$ and $\sigma$ be the rotation on $\mathbb{R}^n$. We define the rotation of a function by $(R_\sigma f)(x) = f(\sigma x)$. Let $d\sigma$ denote Haar measure on $SO(n)$, normalized so that $\int_{SO(n)} d\sigma = |S^{n-1}|$, the Lebesgue measure of $S^{n-1}$. Then by [30, p.222], we have

$$
\int_{|y| \in I} \frac{\Omega(y')}{|y|^n} f(x - y) dy = \frac{1}{2} \int_{SO(n)} (R_{\sigma^{-1}} H^1_x R_\sigma f)(x)\Omega(\sigma 1) d\sigma(y').
$$

For fixed $j \in \mathbb{Z}$, there is a partition $\{I_{i,j}\} = \{I_{i,j}(x)\}$ of $[2^j, 2^{j+1}]$ such that (see [7])

$$
V_{2,j}(Tf)(x) \leq 2 \left[ \sum_i \frac{1}{2} \int_{SO(n)} (R_{\sigma^{-1}} H^1_{i,j} R_\sigma f)(x)\Omega(\sigma 1) d\sigma(y') \right]^{1/2}.
$$

Further, by Minkowski’s inequality, we get the following estimate

$$
S_2(Tf)(x) \leq \int_{SO(n)} R_{\sigma^{-1}} S_2(H^1 R_\sigma f)(x)\Omega(\sigma 1) d\sigma(y').
$$

For $1 < p < \infty$, using Minkowski’s inequality and Lemma 2.3, we have

$$
\|S_2(Tf)\|_{L^p(\mathbb{R}^n)} \leq \int_{SO(n)} \left( \int_{\mathbb{R}^n} R_{\sigma^{-1}} |S_2(H^1 R_\sigma f)(x)|^p dx \right)^{1/p} |\Omega(\sigma 1) d\sigma(y')
$$

$$
\leq C_p \int_{SO(n)} \|R_\sigma f\|_{L^p(\mathbb{R}^n)} |\Omega(\sigma 1) d\sigma(y')|
$$

$$
\leq C_p \|\Omega\|_{L^1(S^{n-1})} \|f\|_{L^p(\mathbb{R}^n)}.
$$
Case 2. \( \Omega \) is even. It is well known that \( f = -\sum_{i=1}^{n} R_{i}^{2} f \) for any Schwartz function \( f \), where \( R_{i} \) denotes the Riesz transform. For fixed \( f \), we denote \( -R_{i} f \) by \( g_{i} \). Let \( \Phi \) be a infinitely differentiable function on \( \mathbb{R} \) such that \( \Phi(t) = 0 \) if \( t < \frac{1}{4} \), and \( \Phi(t) = 1 \) if \( t > \frac{3}{4} \). Moreover, \( 0 \leq \Phi(t) \leq 1 \) for all \( t \in \mathbb{R} \). Define

\[
M_{i}(x) = \lim_{\varepsilon \to 0} \int_{|x-y| > \varepsilon} \frac{\Omega(y) \Phi(|y|)}{|y|^{n}} \frac{x_{i} - y_{i}}{|x - y|^{n+1}} dy.
\]

By [5, p.302], we know that

\[
\int_{\mathbb{R}^{n}} \frac{\Omega(x-y)}{|x-y|^{n}} \Phi\left(\frac{|x-y|}{\varepsilon}\right) f(y) dy = \frac{1}{\varepsilon^{n}} \int_{\mathbb{R}^{n}} \sum_{i=1}^{n} M_{i}\left(\frac{x-y}{\varepsilon}\right) g_{i}(y) dy.
\]

Hence we have

\[
T_{\varepsilon} f(x) = \sum_{i=1}^{n} \frac{1}{\varepsilon^{n}} \int_{\mathbb{R}^{n}} M_{i}\left(\frac{x-y}{\varepsilon}\right) g_{i}(y) dy - \int_{|x-y| < \varepsilon} \frac{\Omega(x-y)}{|x-y|^{n}} \Phi\left(\frac{|x-y|}{\varepsilon}\right) f(y) dy
\]

\[
= \sum_{i=1}^{n} M_{i,\varepsilon} * g_{i}(x) - T_{\varepsilon 0} f(x).
\]

Note that \( S_{2} \) is subadditive, then

\[
S_{2}(T f)(x) \leq \sum_{i=1}^{n} S_{2}(\{M_{i,\varepsilon} * g_{i}\})(x) + S_{2}(\{T_{\varepsilon 0} f\})(x).
\]

To estimate \( S_{2}(\{M_{i,\varepsilon} * g_{i}\}) \) \( (i = 1, 2, \cdots, n) \), it suffices to consider the case \( i = 1 \) and the other cases can be treated similarly. Define

\[
N(x) = \lim_{\varepsilon \to 0} \int_{|x-y| > \varepsilon} \frac{\Omega(y)}{|y|^{n}} \frac{x_{1} - y_{1}}{|x - y|^{n+1}} dy.
\]

Write

\[
M_{1,\varepsilon} * g_{1}(x) = \frac{1}{\varepsilon^{n}} \int_{|y| > \varepsilon} N\left(\frac{y}{\varepsilon}\right) g_{1}(x-y) dy + \frac{1}{\varepsilon^{n}} \int_{|y| \leq \varepsilon} N\left(\frac{y}{\varepsilon}\right) \Phi\left(\frac{y}{\varepsilon}\right) g_{1}(x-y) dy + \frac{1}{\varepsilon^{n}} \int_{|y| \leq \varepsilon} \left[ M_{1}\left(\frac{y}{\varepsilon}\right) - N\left(\frac{y}{\varepsilon}\right) \Phi\left(\frac{y}{\varepsilon}\right) \right] g_{1}(x-y) dy + \frac{1}{\varepsilon^{n}} \int_{|y| > \varepsilon} \left[ M_{1}\left(\frac{y}{\varepsilon}\right) - N\left(\frac{y}{\varepsilon}\right) \right] g_{1}(x-y) dy =: N_{\varepsilon} * g_{1}(x) + N_{\Phi} * g_{1}(x) + \Delta_{\varepsilon} * g_{1}(x) + D_{\varepsilon} * g_{1}(x).
\]

Hence, we are reduced to estimate the \( L^{p} \) norm of the following terms

\[
S_{2}(\{N_{\varepsilon} * g_{1}\})(x), \; S_{2}(\{N_{\varepsilon}^{\Phi} * g_{1}\})(x), \; S_{2}(\{\Delta_{\varepsilon} * g_{1}\})(x), \; S_{2}(\{D_{\varepsilon} * g_{1}\})(x), \; S_{2}(\{T_{\varepsilon 0} f\})(x).
\]

The estimate of \( S_{2}(\{N_{\varepsilon} * g_{1}\}) \) depends on the following result.
Lemma 2.4. ([17]) Suppose \( \Omega \) satisfies (1.6) and belongs to \( H^1(S^{n-1}) \), \( N(x) \) is given in (2.6). Then \( N \) satisfies the following properties:

(i) \( N(x) \) is a homogeneous function of order \(-n\) on \( \mathbb{R}^n \);

(ii) \( N(-x) = -N(x), \quad x \in \mathbb{R}^n \);

(iii) \( \int_{S^{n-1}} |N(x')| \, d\sigma(x') \leq C \| \Omega \|_{H^1(S^{n-1})} \).

By the result showed in Case 1, Lemma 2.4 and the \( L^p \) \((1 < p < \infty)\) boundedness of Riesz transform, we obtain the estimate

\[ \| S_2(\{N_\varepsilon \ast g_1\})\|_{L^p(\mathbb{R}^n)} \leq C_p \| g_1 \|_{L^p(\mathbb{R}^n)} \leq C_p \| f \|_{L^p(\mathbb{R}^n)}. \]

To estimate the other four terms, we give a lemma.

Lemma 2.5. For any \( u \in S^{n-1} \), let \( \varphi_u \) be a function defined on \([0, \infty)\) and \( F_u(s) = \{ \frac{1}{\varepsilon} \varphi_u(\frac{s}{\varepsilon}) \} \varepsilon > 0 \). If there exists a constant \( C_p \) independent of \( u \) such that

\[ \| S_2(F_u \ast h)\|_{L^p(\mathbb{R})} \leq C_p \| h \|_{L^p(\mathbb{R})}, \quad 1 < p < \infty, \]

then

\[ \| S_2(G_f)\|_{L^p(\mathbb{R}^n)} \leq C_p \| f \|_{L^p(\mathbb{R}^n)}, \quad 1 < p < \infty, \]

where \( G_f = \{ G_\varepsilon f \} \varepsilon > 0 \) and

\[ G_\varepsilon f(x) = \int_{S^{n-1}} \Omega(u) \int_0^\infty \varphi_u(t) f(x - \varepsilon tu) \, dt \, d\sigma(u) \]

with \( \Omega \in L^1(S^{n-1}) \).

Proof. In the same way as the proof of (2.4), we have

\[ S_2(G_f)(x) \leq \int_{SO(n)} |\Omega(\sigma 1)| R_{\sigma^{-1}} S_2(T^1 R_{\sigma} f)(x) \, d\sigma(u), \]

where \( \sigma \) is the rotation such that \( \sigma 1 = u \),

\[ T^1 f(x) = \{ I^1_{\varepsilon} f(x) \} \varepsilon > 0 \quad \text{and} \quad I^1_{\varepsilon} f(x) = \int_0^\infty \frac{1}{\varepsilon} \varphi_1(r/\varepsilon) f(x - r 1) \, dr. \]

By Minkowski’s inequality and hypothesis, we have

\[ \| S_2(G_f)\|_{L^p(\mathbb{R}^n)} \leq \int_{SO(n)} |\Omega(\sigma 1)| \left( \int_{\mathbb{R}^n} |S_2(T^1 R_{\sigma} f)(x)|^p \, dx \right)^{1/p} \, d\sigma(u) \]

\[ \leq C_p \| \Omega \|_{L^1(S^{n-1})} \| f \|_{L^p(\mathbb{R}^n)}. \]
Now we continue the proof of Proposition 2.2. It was proved that all the kernels of $T^0_\varepsilon$, $N^\Phi_\varepsilon$, $\Delta_\varepsilon$ and $D_\varepsilon$ have a representation that satisfies the hypothesis of Lemma 3.4 in [7, p.2124], and
\[
\int_0^1 t|\varphi_f'(t)|dt < \infty.
\]
Thus, we get
\[
\|S_2(F_u * h)\|_{L^p(\mathbb{R})} \leq C_p \|h\|_{L^p(\mathbb{R})}
\]
for $1 < p < \infty$ according to Lemma 2.4 in [6]. Therefore, Lemma 2.5 and the $L^p$ boundedness of Riesz transforms imply that
\[
\|S_2(N^\Phi_\varepsilon * g_1)\|_{L^p(\mathbb{R}^n)} + \|S_2(\Delta_\varepsilon * g_1)\|_{L^p(\mathbb{R}^n)} + \|S_2(D_\varepsilon * g_1)\|_{L^p(\mathbb{R}^n)} + \|S_2(T^0_\varepsilon f)\|_{L^p(\mathbb{R}^n)}
\]
\[
\leq C_p \|f\|_{L^p(\mathbb{R}^n)}.
\]
This completes the proof of Proposition 2.2 and (1.10) follows.

2.2 Proof of Theorem 1.2(ii)

As explained at the beginning of this section, it suffices to deal with dyadic $\lambda$-jump estimate.

Proposition 2.6. Let $T$ be given as in (1.7) with $\Omega \in L \log^+ L(S^{n-1})$ satisfying (1.6). Then
\[
\left\|\lambda \sqrt{N^\Omega(Tf)}\right\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}.
\]

Proof. For $j \in \mathbb{Z}$, define a measure $\nu_j$ by
\[
\nu_j * f(x) = \int_{2^j \leq |y| < 2^{j+1}} \frac{\Omega(y)}{|y|^n} f(x - y)dy.
\]
Obviously, for $k \in \mathbb{Z}$,
\[
T_{2^k} f(x) = \int_{|y| \geq 2^k} \frac{\Omega(y)}{|y|^n} f(x - y)dy = \sum_{j \geq k} \nu_j * f(x).
\]
Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be a radial function such that $\tilde{\phi}(\xi) = 1$ for $|\xi| \leq 2$ and $\tilde{\phi}(\xi) = 0$ for $|\xi| > 4$. We have the following decomposition
\[
T_{2^k} f = \phi_k * Tf - \phi_k * \sum_{l < 0} \nu_{k+l} * f + \sum_{s \geq 0} (\delta_0 - \phi_k) * \nu_{k+s} * f
\]
\[
:= T^1_k f - T^2_k f + T^3_k f,
\]
where $\phi_k$ satisfies $\hat{\phi}_k(\xi) = \hat{\phi}(2^k \xi)$ and $\delta_0$ is the Dirac measure at 0. $\mathcal{S}^i f$ denotes the family $\{T_k^i f\}_{k \in \mathbb{Z}}$ for $i = 1, 2, 3$. Obviously, to show Proposition 2.6 it suffices to prove the following inequalities:

$$\|\lambda [N_{\lambda/3}^d(\mathcal{S}^i f)]^{1/2}\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}, \quad 1 < p < \infty, \quad i = 1, 2, 3. \tag{2.7}$$

**Estimate of (2.7) for $i = 1$.** We need the following result, which has been essentially included in Theorem 1.1(i) of [25]. However some details there were omitted, we add these details for completeness.

**Lemma 2.7.** Let $\phi_k$ be given above, set $\mathcal{G} f = \{\phi_k * f\}_k$. Then for $1 < p < \infty$,

$$\|\lambda \sqrt{N_{\lambda}^\mathcal{G} f}\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)} \tag{2.8}$$

holds uniformly in $\lambda > 0$.

If we accept Lemma 2.7 for a moment and combine it with the $L^p$-boundedness of $T$ (see [5]), we can get the following estimate easily

$$\|\lambda [N_{\lambda}^d(\mathcal{S}^1 f)]^{1/2}\|_{L^p(\mathbb{R}^n)} = \|\lambda [N_{\lambda}(\{\phi_k * T f\})]^{1/2}\|_{L^p(\mathbb{R}^n)} \leq C_p \|T f\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}.$$

Hence, to estimate (2.7) for $i = 1$ it remains to prove Lemma 2.7.

**Proof.** We borrow some notations and results from [25, pp.6724]. For $j \in \mathbb{Z}$ and $\beta = (m_1, \ldots, m_n) \in \mathbb{Z}^n$, we denote the dyadic cube $\prod_{k=1}^n (m_k2^j, (m_k + 1)2^j]$ in $\mathbb{R}^n$ by $Q_{\beta}^j$, and the set of all dyadic cubes with side-length $2^j$ by $D_j$. The conditional expectation of a local integrable $f$ with respect to $D_j$ is given by

$$E_j f(x) = \sum_{Q \in D_j} \frac{1}{|Q|} \int_Q f(y)dy \cdot \chi_Q(x)$$

for all $j \in \mathbb{Z}$. Note that $N_{\lambda}$ is subadditive, then

$$N_{\lambda}(\mathcal{G} f) \leq N_{\lambda/2}(\mathcal{D} f) + N_{\lambda/2}(\mathcal{E} f),$$

where

$$\mathcal{D} f = \{\phi_k * f - E_k f\}_k \quad \text{and} \quad \mathcal{E} f = \{E_k f\}_k.$$ 

The following inequality is the $\lambda$-jump estimate for dyadic martingales (see, for instance, [33]),

$$\|\lambda \sqrt{N_{\lambda/2}(\mathcal{E} f)}\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}, \quad 1 < p < \infty,$$
uniformly in \( \lambda \). Next, observe that
\[
\lambda \sqrt{N_{\lambda/2}}(\nabla f) \leq C \left( \sum_{k \in \mathbb{Z}} |\phi_k * f - E_k f|^2 \right)^{1/2} := S f.
\]

On the other hand, for \( x \in \mathbb{R}^n \) and \( j \in \mathbb{Z} \), let \( \nabla_j f(x) = E_j f(x) - E_{j-1} f(x) \). Thus, by \( f \in L^p(\mathbb{R}^n) \) and the Lebesgue differential theorem we see that
\[
f(x) = - \sum_j \nabla_j f(x) \quad \text{a.e. } x \in \mathbb{R}^n.
\]

By Lemma 3.2 in [25], we have
\[
\|S f\|_{L^2(\mathbb{R}^n)} \leq \left( \sum_k \left( \sum_j \|\phi_k * \nabla_j f(x) - E_k \nabla_j f\|_{L^2(\mathbb{R}^n)}^2 \right) \right)^{1/2} \leq C_{\theta} \|f\|_{L^2(\mathbb{R}^n)}
\]
for some \( \theta > 0 \).

Jones et al have proved the following weak-type \((1,1)\) estimate for \( S \),
\[
\alpha \{ x \in \mathbb{R}^n : S f(x) > \alpha \} \leq C \|f\|_{L^1(\mathbb{R}^n)},
\]
which is (33) in [25]. By interpolation, (2.9), (2.10), imply all the \( L^p \) bounds for \( 1 < p \leq 2 \).

For \( 2 < p < \infty \), by Hölder’s inequality, we have
\[
\left\| \left( \sum_{k \in \mathbb{Z}} |\phi_k * f - E_k f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} = \sup_{\|\{h_k\}\|_{L^{p'}(\mathbb{R}^n)} \leq 1} \left| \int_{\mathbb{R}^n} \sum_k [\phi_k * f(x) - E_k f(x)] h_k(x) dx \right|
\]
\[
= \sup_{\|\{h_k\}\|_{L^{p'}(\mathbb{R}^n)} \leq 1} \left| \int_{\mathbb{R}^n} \sum_k [\phi_k * h_k(y) - E_k h_k(y)] f(y) dy \right|
\]
\[
\leq \sup_{\|\{h_k\}\|_{L^{p'}(\mathbb{R}^n)} \leq 1} \|\sum_k [\phi_k * h_k - E_k h_k]\|_{L^{p'}(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}.
\]

It suffices to show that
\[
(2.11) \quad \|\sum_k [\phi_k * h_k - E_k h_k]\|_{L^{p'}(\mathbb{R}^n)} \leq C \|\{h_k\}\|_{L^{p'}(\mathbb{R}^n)}, \quad 1 < p' \leq 2.
\]

Clearly, the following estimate is a consequence \( L^2 \) boundedness of \( S \) by duality
\[
(2.12) \quad \|\sum_k [\phi_k * h_k - E_k h_k]\|_{L^2(\mathbb{R}^n)} \leq C \|\{h_k\}\|_{L^2(\mathbb{R}^n)}.
\]
Therefore, we just need to prove that, if \( \{h_k\} \in L^1(l^2) \), then

\[
\| \{x \in \mathbb{R}^n : \sum_k [\phi_k * h_k(x) - \mathbb{E}_kh_k(x)] > \alpha \} \| \leq \frac{C}{\alpha} \|\{h_k\}\|_{L^1(l^2)},
\]

where \( \alpha > 0 \) and \( C \) is independent of \( \alpha \) and \( \{h_k\} \). In fact, by interpolation between (2.12) and (2.13), we get (2.11).

For \( \alpha > 0 \), we perform Calderón-Zygmund decomposition of \( \|\{h_k\}\|_{l^2} \) at height \( \alpha \), then there exists \( \Lambda \subseteq \mathbb{Z} \times \mathbb{Z}^n \) such that the collection of dyadic cubes \( \{Q^j_{\beta}\}_{(j, \beta) \in \Lambda} \) are disjoint and the following hold:

(i) \( |\bigcup_{(j, \beta) \in \Lambda} Q^j_{\beta}| \leq \alpha^{-1}\|\{h_k\}\|_{L^1(l^2)} \);

(ii) \( \|\{h_k(x)\}\|_{l^2} \leq \alpha \), if \( x \not\in \bigcup_{(j, \beta) \in \Lambda} Q^j_{\beta} \);

(iii) \( \frac{1}{|Q^j_{\beta}|} \int_{Q^j_{\beta}} \|\{h_k(x)\}\|_{l^2} dx \leq 2^n \alpha \) for each \( (j, \beta) \in \Lambda \).

For \( k \in \mathbb{Z} \), we set

\[
g^{(k)}(x) = \begin{cases} h_k(x), & \text{if } x \not\in \bigcup_{(j, \beta) \in \Lambda} Q^j_{\beta}, \\ \frac{1}{|Q^j_{\beta}|} \int_{Q^j_{\beta}} h_k(y) dy, & \text{if } x \in Q^j_{\beta}, (j, \beta) \in \Lambda. \end{cases}
\]

and

\[
b^{(k)}(x) = \sum_{(j, \beta) \in \Lambda} [h_k(x) - \mathbb{E}_j h_k(x)] \chi_{Q^j_{\beta}}(x) := \sum_{(j, \beta) \in \Lambda} b^{(k)}_{j, \beta}(x).
\]

First we have \( \|\{g^{(k)}\}\|_{L^2(l^2)}^2 \leq 2\alpha \|\{h_k\}\|_{L^1(l^2)} \). In fact, by (ii),(iii) and Minkowski’s inequality,

\[
\|\{g^{(k)}\}\|_{L^2(l^2)}^2 = \int_{\bigcup_{(j, \beta) \in \Lambda} Q^j_{\beta}} \|\{h_k(x)\}\|_{l^2}^2 dx + \sum_{(j, \beta) \in \Lambda} \int_{Q^j_{\beta}} \sum_k \left| \frac{1}{|Q^j_{\beta}|} \int_{Q^j_{\beta}} h_k(y) dy \right|^2 dx \\
\leq \alpha \int_{\bigcup_{(j, \beta) \in \Lambda} Q^j_{\beta}} \|\{h_k(x)\}\|_{l^2} dx + 2^n \alpha \sum_{(j, \beta) \in \Lambda} \int_{Q^j_{\beta}} \|h_k(x)\|_{l^2} dx \\
\leq 2^n \alpha \|\{h_k\}\|_{L^1(l^2)}.
\]

On the other hand, it is easy to see that

\[
\int_{\mathbb{R}^n} b^{(k)}_{j, \beta}(x) dx = 0 \quad \text{for all } k \in \mathbb{Z}, (j, \beta) \in \Lambda,
\]

and \( \|\{b^{(k)}\}\|_{L^1(l^2)} \leq 2\|\{h_k\}\|_{L^1(l^2)} \). In fact, by (iii) and Minkowski’s inequality,

\[
\|\{b^{(k)}\}\|_{L^1(l^2)} = \sum_{(j, \beta) \in \Lambda} \int_{Q^j_{\beta}} \left( \sum_k |h_k(x) - \mathbb{E}_j h_k(x)|^2 \right)^{1/2} dx \\
\leq \sum_{(j, \beta) \in \Lambda} \int_{Q^j_{\beta}} \left( \sum_k |h_k(x)|^2 \right)^{1/2} dx + \sum_{(j, \beta) \in \Lambda} \int_{Q^j_{\beta}} \left( \sum_k \left| \frac{1}{|Q^j_{\beta}|} \int_{Q^j_{\beta}} h_k(y) dy \right|^2 \right)^{1/2} dx \\
\leq 2 \|\{h_k\}\|_{L^1(l^2)}.
\]
This completes the proof of Lemma 2.7.

\[ \alpha^2 \{ x \in \mathbb{R}^n : | \sum_k (\phi_k * g^{(k)}(x) - E_k g^{(k)}(x)) | \geq \alpha \} \leq C \| \sum_k (\phi_k * g^{(k)} - E_k g^{(k)}) \|_{L^2(\mathbb{R}^n)}^2 \leq C \| g^{(k)} \|_{L^2(I^2)} \leq C \alpha \| \{ h_k \} \|_{L^1(I^2)}. \]

Thus, for above \( \alpha \), by (2.12),

\[ \alpha^2 \{ x \in \mathbb{R}^n : | \sum_k (\phi_k * g^{(k)}(x) - E_k g^{(k)}(x)) | > \alpha \} \leq C \| \sum_k (\phi_k * g^{(k)} - E_k g^{(k)}) \|_{L^2(\mathbb{R}^n)}^2 \leq C \| g^{(k)} \|_{L^2(I^2)} \leq C \alpha \| \{ h_k \} \|_{L^1(I^2)}. \]

So, we get

\[ \{ x \in \mathbb{R}^n : | \sum_k (\phi_k * g^{(k)}(x) - E_k g^{(k)}(x)) | > \alpha \} \leq C \alpha \| \{ h_k \} \|_{L^1(I^2)}. \]

Let \( \tilde{Q}_j^i \) be the cube concentric with \( Q_j^i \) and with side length 4 times that of \( Q_j^i \). It is obvious that

\[ (2.14) \quad \left| \bigcup_{(j, \beta) \in \Lambda} \tilde{Q}_j^i \right| \leq C \sum_{(j, \beta) \in \Lambda} |Q_j^i| \leq C \alpha \| \{ h_k \} \|_{L^1(I^2)}. \]

Note that \( E_k b_{j, \beta}^{(k)} \) is supported in \( Q_j^i \) when \( k \leq j \) and \( E_k b_{j, \beta}^{(k)} \) vanishes everywhere when \( k \geq j \).

\[ \alpha \left| \left\{ x \notin \bigcup \tilde{Q}_j^i : | \sum_k (\phi_k * b^{(k)}(x) - E_k b^{(k)}(x)) | > \alpha \right\} \right| \leq C \sum_{(j, \beta) \in \Lambda} \int_{(\tilde{Q}_j^i)^c} |\phi_k * b_{j, \beta}^{(k)}(x)| dx.
\]

By [25] (34) and (35), p.6726, the inequality

\[ \int_{(\tilde{Q}_j^i)^c} |\phi_k * b_{j, \beta}^{(k)}(x)| dx \leq C 2^{-\delta j - k} \int_{\mathbb{R}^n} |b_{j, \beta}^{(k)}(x)| dx \]

holds for some \( \delta > 0 \). Applying Hölder’s inequality, we get

\[ \sum_{(j, \beta) \in \Lambda} \sum_k \int_{(\tilde{Q}_j^i)^c} |\phi_k * b_{j, \beta}^{(k)}(x)| dx \leq C \sum_{(j, \beta) \in \Lambda} \sum_k 2^{-\delta j - k} \int_{\mathbb{R}^n} |b_{j, \beta}^{(k)}(x)| dx \]

\[ \leq C \sum_{(j, \beta) \in \Lambda} \int_{\mathbb{R}^n} \left( \sum_k |b_{j, \beta}^{(k)}(x)|^2 \right)^{1/2} dx \]

\[ \leq C \sum_{(j, \beta) \in \Lambda} \| b_{j, \beta} \|_{L^1(I^2)} \leq C \| \{ h_k \} \|_{L^1(I^2)}. \]

This completes the proof of Lemma 2.7. \( \square \)

**Estimate of (2.7) for \( i = 3 \).** First, let us give a decomposition of \( \Omega \in L \log^+ L(S^{n-1}) \) satisfying (L3), which can be found in [2] or [11]. For \( m \in \mathbb{N} \), denote

\[ E_m = \{ y' \in S^{n-1} : 2^{m-1} \leq |\Omega(y')| < 2^m \} \]
and

\[ \Omega_m = \|\Omega\|_{L^1(E_m)}^{-1} \left[ \Omega \chi_{E_m} - \frac{1}{|S^{n-1}|} \int_{E_m} \Omega d\sigma \right]. \]

Set

\[ \Gamma = \{ m \in \mathbb{N} : \sigma(E_m) > 2^{-4m} \} \quad \text{and} \quad \Omega_0 = \Omega - \sum_{m \in \Gamma} \|\Omega\|_{L^1(E_m)} \Omega_m. \]

**Lemma 2.8.** (see [2] or [1]) Suppose that \( \Omega \in L \log^+ L(S^{n-1}) \) satisfying (1.0). Then \( \Omega_0 \) and \( \Omega_m (m \in \Gamma) \) defined above satisfy the following properties:

(i) \( \int_{S^{n-1}} \Omega_m d\sigma = 0, \|\Omega_m\|_{L^1(S^{n-1})} \leq 2 \) and \( \|\Omega_m\|_{L^2(S^{n-1})} \leq C 2^m \), where \( C = C(n) \) is independent of \( m \);

(ii) \( \int_{S^{n-1}} \Omega_0 d\sigma = 0 \) and \( \Omega_0 \in L^2(S^{n-1}) \);

(iii) \( \sum_{m \in \Gamma} m \|\Omega\|_{L^1(E_m)} \leq \|\Omega\|_{L \log^+ L(S^{n-1})} \).

We define measure \( \nu_j^{(m)} \) by

\[ \nu_j^{(m)} f(x) = \int_{2^j < |y| \leq 2^{j+1}} \frac{\Omega_m(y)}{|y|^n} f(x - y) dy \]

for \( j \in \mathbb{Z} \) and \( m \in \{0\} \cup \Gamma \). Then we have the following pointwise estimate

\[ \lambda \left[ \mathcal{N}^d_\chi (\mathcal{T}^3 f)(x) \right]^{1/2} \leq \sum_{s \geq 0} \left( \sum_k \|[(\delta_0 - \phi_k) \ast \nu^{(0)}_{k+s}] \ast f(x)\|^2 \right)^{1/2} \]

\[ + \sum_{s \geq 0} \sum_{m \in \Gamma} \|\Omega\|_{L^1(E_m)} \left( \sum_k \|[(\delta_0 - \phi_k) \ast \nu^{(m)}_{k+s}] \ast f(x)\|^2 \right)^{1/2} \]

\[ : = \sum_{s \geq 0} G^{(0)}_s f(x) + \sum_{m \in \Gamma} \|\Omega\|_{L^1(E_m)} \sum_{s \geq 0} G^{(m)}_s f(x). \]

We are reduced to establish estimate of \( \|G^{(m)}_s f\|_{L^p(\mathbb{R}^n)} \) as sharp as possible so that we are able to sum up over \( s \geq 0 \) and \( m \in \{0\} \cup \Gamma \). Let us start with the case \( m \in \Gamma \). We first prove a rapid decay estimate of \( \|G^{(m)}_s f\|_{L^2(\mathbb{R}^n)} \).

For \( m \in \Gamma \), Al-Salman and Pan [2] proved that

\[ |\hat{\nu}_s^{(m)}(\xi)| \leq C \min \left\{ \|\Omega_m\|_{L^1(S^{n-1})}, \|\Omega_m\|_{L^2(S^{n-1})} [2^s |\xi|]^{-1/3}, \Omega_m \|_{L^1(S^{n-1})} 2^s |\xi| \right\}. \]

By (i) of Lemma 2.8 and interpolation, we have

\[ (2.15) \quad |\hat{\nu}_s^{(m)}(\xi)| \leq C \min \{ [2^s |\xi|]^{-\frac{1}{3m}}, 2^s |\xi| \}. \]
It is trivial that $|(1 - \hat{\phi})(\xi)| \leq \min\{1, |\xi|\}$. To dominate $\|G_s^{(m)} f\|_{L^2(\mathbb{R}^n)}$, we use above two estimates and get

$$
\sum_k |(1 - \hat{\phi})(2^k \xi)|^2 |\nu^{(m)}(2^{s+k} \xi)|^2
\leq \sum_{2^k \leq |\xi|} (2^{s+k} |\xi|)^2 (2^k |\xi|)^2 + \sum_{|\xi| / 2^k < 2^k \leq |\xi|} (2^{s+k} |\xi|)^{-\frac{2}{3m}} (2^k |\xi|)^2 + \sum_{2^k \geq |\xi|} (2^{s+k} |\xi|)^{-\frac{2}{3m}}
\leq 2^{-2s} + 2^{\frac{3m}{2}} (2^{-\frac{3m}{2}} - 2^{-2s}) + 2^{-\frac{3m}{2}}
\leq C 2^{\frac{3m}{2}} 2^{-\frac{3m}{2}}.
$$

Plancherel’s theorem implies that

$$
(2.16) \quad \|G_s^{(m)} f\|_{L^2(\mathbb{R}^n)} \leq C 2^{\frac{3m}{2}} 2^{-\frac{3m}{2}} \|f\|_{L^2(\mathbb{R}^n)}.
$$

To proceed with the estimate of $\|G_s^{(m)} f\|_{L^p(\mathbb{R}^n)}$, we need three well known or easily checked lemmas below, which will be used in the following proof.

**Lemma 2.9.** ([2] p. 157) Let $\{\sigma_j\}_{j \in \mathbb{Z}}$ be a sequence of finite Borel measures. Suppose that there are $\gamma \in (0, 1]$ and $p_0 \in (2, \infty)$ such that

(i) $\|\sigma_j\| \leq C$;

(ii) $|\hat{\sigma}_j(\xi)| \leq C \min\{|2^j \xi|^\gamma, |2^j \xi|^{-\gamma}\}$;

(iii) $\|(\sum_j |\sigma_j * \hat{g}_j|^2)^{1/2}\|_{L^{p_0}} \leq C \|(\sum_j |\hat{g}_j|^2)^{1/2}\|_{L^{p_0}}$.

Then, for $p \in (p_0', p_0)$, there exists a constant $C_p$ such that

$$
\|(\sum_j |\sigma_j * f|^2)^{1/2}\|_{L^p} \leq C_p 2^{\alpha_p \gamma} \frac{2^{\alpha_p \gamma} + 1}{2^{\alpha_p \gamma} - 1} \|f\|_{L^p},
$$

where

$$
\alpha_p = \begin{cases} 
\left(\frac{1}{p} - \frac{1}{p_0}\right) / \left(\frac{1}{2} - \frac{1}{p_0}\right), & p > 2, \\
\left(\frac{1}{p} - \frac{1}{p_0}\right) / \left(\frac{1}{2} - \frac{1}{p_0}\right), & p \leq 2.
\end{cases}
$$

**Lemma 2.10.** ([13] p. 544) Suppose that $\{\sigma_j\}_{j \in \mathbb{Z}}$ is a sequence of finite Borel measures. If the maximal operator $\sigma^*(f) = \sup_j ||\sigma_j^* f||_{L^p_j}$ is bounded on $L^{p_0}$ for $1 < p_0 \leq \infty$, then

$$
\|(\sum_j |\sigma_j * g_j|^2)^{1/2}\|_{L^p} \leq C \|(\sum_j |g_j|^2)^{1/2}\|_{L^p},
$$

where $p$ satisfies $\frac{1}{2p_0} = \frac{1}{2} - \frac{1}{p}$ and $C$ depends on the $L^{p_0}$ norm of $\sigma^*$.

**Lemma 2.11.** Let $a > 1$, $t \in (0, 1]$ and $\theta \in (0, 1]$. Then $t^\theta a^{2\theta t} \leq C_{a, \theta}(a^{\theta t} - 1)$.
Now we estimate \( \|G^{(m)}_s f\|_{L^p(\mathbb{R}^n)} \). For \( 1 < p < \infty \) and \( p \neq 2 \), there exists a \( \theta_1 \in (0, 1) \) and \( 1 < p_1 < \infty \) such that \( 1/p = (1 - \theta_1)/p_1 + \theta_1/2 \). For fixed \( p_1 \), we choose \( p_0 \in (2, \infty) \) such that \( p_1 \in (p_0', p_0) \). Further, choose \( p_2 \in (1, \infty) \) so that \( 1/(2p_2) = |1/2 - 1/p_0| \). It is trivial that

\[
\left\| \left( \delta_0 - \phi_k \right) * \nu^{(m)}_{k+s} \right\| \leq 2\|\nu^{(m)}_s\| \leq C\|\Omega_m\|_{L^1(\mathbb{R}^{n-1})} \leq C.
\]

On the other hand, note that \( a \geq 1 \),

\[
\left\| \left( \delta_0 - \phi_k \right) * \nu^{(m)}_{k+s} \right\| \leq C \min \{1, 2^k \xi\} \min \{|2^k + s|, 1, 2^k + s|^{-\frac{1}{m}}\}
\leq C \min \{2^k, 1, 2^k|^{-\frac{1}{m}}\}
\leq C \min \{2^k|^{-\frac{1}{m}}, 2^k|^{-\frac{1}{m}}\}.
\]

By the \( L^p \) boundedness of maximal function with rough kernel (see [5]) and Lemma 2.8, we have

\[
\left\| \sup_k \left[ \left( \delta_0 - \phi_k \right) * \nu^{(m)}_{k+s} \right] \right\|_{L^{p_2}(\mathbb{R}^n)} \leq C_{p_2} \|\Omega_m\|_{L^1(\mathbb{R}^{n-1})} \|f\|_{L^{p_2}(\mathbb{R}^n)} \leq C_{p_2} \|f\|_{L^{p_2}(\mathbb{R}^n)},
\]

which, together with Lemma 2.11 and Lemma 2.9 implies

\[
\|G^{(m)}_s f\|_{L^{p_1}(\mathbb{R}^n)} = \left\| \left( \sum_k \left[ \left( \delta_0 - \phi_k \right) * \nu^{(m)}_{k+s} \right] \right)^2 \right\|^{1/2}_{L^{p_1}(\mathbb{R}^n)}
\leq C_{p_1} 2^\alpha_{p_1}\left( \frac{2^\alpha_{p_1} + 1}{2^\alpha_{p_1} - 1} \right) \|f\|_{L^{p_1}(\mathbb{R}^n)}
\leq C'_{p_1} \left( \frac{2^\alpha_{p_1} - 1}{2^\alpha_{p_1} - 1} \right) \|f\|_{L^{p_1}(\mathbb{R}^n)}
\leq C''_{p_1} \|f\|_{L^{p_1}(\mathbb{R}^n)},
\]

the last inequality was obtained by Lemma 2.11 with \( a = 2^{\alpha_{p_1}/m}, \theta = 1 \) and \( t = 1/m \).

Now by interpolation between (2.16) and (2.17), we obtain

\[
\|G^{(m)}_s f\|_{L^p(\mathbb{R}^n)} \leq C_{p_1} m^{1 - \theta_1} 2^\alpha_{\theta_1} 2^{-\alpha_{\theta_1} - \alpha_{p_1}/m} \|f\|_{L^p(\mathbb{R}^n)}.
\]

Applying Lemma 2.11 with \( a = 2^{\alpha_{\theta_1} + 1}, \theta = \theta_1 \) and \( t = 1/m \), we conclude that

\[
\left\| \sum_{s \geq 0} G^{(m)}_s f \right\|_{L^p(\mathbb{R}^n)} \leq C_{p_1} m^{1 - \theta_1} 2^{\alpha_{\theta_1}} \|f\|_{L^p(\mathbb{R}^n)}
\leq C_{p_1} m^{1 - \theta_1} 2^{\alpha_{\theta_1}} \|f\|_{L^p(\mathbb{R}^n)} \leq C_{p_1} m m^{1 - \theta_1} \frac{2^{\alpha_{\theta_1}}}{2^{\alpha_{\theta_1}} + 1} \|f\|_{L^p(\mathbb{R}^n)} \leq C_{p_1} m \|f\|_{L^p(\mathbb{R}^n)}.
\]

In the same way, we can deal with \( G^{(0)}_s f \) for \( s \geq 0 \) and obtain

\[
\left\| \sum_{s \geq 0} G^{(0)}_s f \right\|_{L^p(\mathbb{R}^n)} \leq C_p m \sum_{s \geq 0} 2^{-\alpha_{\theta_1} - \alpha_{p_1}/m} \|f\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}.
\]
Finally, using Lemma 2.8 we get

\[ \| \lambda [N^d_A(\mathcal{T}^2 f)]^{1/2} \|_{L^p(\mathbb{R}^n)} \leq \| \sum_{s \geq 0} G_s^{(0)} f \|_{L^p(\mathbb{R}^n)} + \sum_{m \in \Gamma} \| \Omega \|_{L^1(E_m)} \| \sum_{s \geq 0} G_s^{(m)} f \|_{L^p(\mathbb{R}^n)} \]

\[ \leq C_p [1 + \sum_{m \in \Gamma} \| \Omega \|_{L^1(E_m)}] \| f \|_{L^p(\mathbb{R}^n)} \]

\[ \leq C_p [1 + \| \Omega \|_{L^1(\mathbb{R}^n)}] \| f \|_{L^p(\mathbb{R}^n)} \]

**Estimate of (2.7) for \( i = 2 \).** Similarly, we have the following pointwise estimate

\[ \lambda [N^d_A(\mathcal{T}^2 f)(x)]^{1/2} \]

\[ \leq \sum_{l < 0} \left( \sum_k \left( |\phi_k * \nu_{k+l}^{(m)}| * f(x) |^2 \right)^{1/2} \right)^{1/2} \]

\[ \:= \sum_{l < 0} G_l^{(0)} f(x) + \sum_{m \in \Gamma} \| \Omega \|_{L^1(E_m)} \sum_{l < 0} G_l^{(m)} f(x). \]

For \( m \in \Gamma \) and \( l < 0 \), we first estimate the \( L^2 \) bounds of \( G_l^{(m)} f \). Note that \( |\hat{\phi}(\xi)| \leq 2 |\xi|^4 \). By Plancherel’s theorem and (2.15),

\[ \| \lambda [N^d_A(\mathcal{T}^2 f)](x) \|_{L^2(\mathbb{R}^n)} \sup_{\xi} \left( \sum_k |\hat{\phi}(2^k \xi)|^2 |\nu_{l+k}^{(m)}(2^{l+k} \xi)|^2 \right)^{1/2} \]

\[ \leq C \| f \|_{L^2(\mathbb{R}^n)} \sup_{\xi} \left( \sum_{2^k \xi} (2^{l+k} |\xi|)^2 \right)^{1/2} \leq C 2^l \| f \|_{L^2(\mathbb{R}^n)}. \]

Next, we consider \( \| G_l^{(m)} f \|_{L^p(\mathbb{R}^n)} \), where \( p \) is that in case \( i = 3 \). Obviously, we have

\[ \| \phi_k * \nu_{k+l}^{(m)} \| \leq C \text{ and } \| \phi_k * \nu_{k+l}^{(m)} \| \leq C \min\{|2^k \xi|, |2^k \xi|^{-1}\}. \]

In the same way, \( \| \sup_k \| \phi_k * \nu_{k+l}^{(m)} \| * f \|_{L^p(\mathbb{R}^n)} \leq C \| f \|_{L^p(\mathbb{R}^n)}. \) The following inequality is a consequence of Lemma 2.9 and Lemma 2.10

\[ (2.18) \quad \| G_l^{(m)} f \|_{L^{p_1}(\mathbb{R}^n)} \leq C_{p_1} \| f \|_{L^{p_1}(\mathbb{R}^n)}. \]

Interpolating between above \( L^p \) estimate and \( L^2 \) estimate, and summing over \( l \), we get

\[ \| \sum_{l < 0} G_l^{(m)} f \|_{L^p(\mathbb{R}^n)} \leq C_p \sum_{l < 0} 2^{l \theta_1} \| f \|_{L^p(\mathbb{R}^n)} \leq C_p \| f \|_{L^p(\mathbb{R}^n)}. \]

The quantity \( \| \sum_{l < 0} G_l^{(0)} f \|_{L^p(\mathbb{R}^n)} \) can be treated in the same way. Finally, using Lemma 2.8 we get

\[ \| \lambda [N^d_A(\mathcal{T}^2 f)]^{1/2} \|_{L^p(\mathbb{R}^n)} \leq C_p [1 + \| \Omega \|_{L^1(\mathbb{R}^n)}] \| f \|_{L^p(\mathbb{R}^n)}. \]
3 Proof of Theorem 1.5

By Lemma 2.1 to show Theorem 1.5 it suffices to prove the following two propositions.

**Proposition 3.1.** Let $T$ be given as in (1.7), $\Omega$ satisfies (1.6) and $\Omega \in G_\alpha(S^{n-1})$ for some $\alpha > 0$. Then for 
\[
\frac{(2\alpha+1)(1+\alpha)}{2\alpha^2+\alpha+1/2} < p < \frac{2(1+\alpha)(2\alpha+1)}{4\alpha+1},
\]

\[
\|\lambda \sqrt{N^d_\lambda(Tf)}\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}
\]

uniformly in $\lambda > 0$.

**Proposition 3.2.** Let $T$ and $\Omega$ be given as in Proposition 3.1 but with $\alpha > 1$. Then for 
\[
\frac{3 + \alpha}{1+\alpha} < p < \frac{3 + \alpha}{2},
\]

\[
\|S_2(Tf)\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}.
\]

3.1 Proof of Proposition 3.1

We use again Fourier transform and square function estimates. Note that in the present case, the Fourier transform of the measure $\nu$ defined at the beginning of Section 2 has logarithmic decay (see (9) in [16]), which is better than the case $\Omega \in L_\log L(S^{n-1})$, but worse than the case $\Omega \in L^r(S^{n-1})$ ($r > 1$). Some estimates from [14] and [17] are taken for granted here. Let us start the proof.

Let measure $\nu_j$ and operator $T_{jk}$ be defined as in the proof of Lemma 2.6 with $\Omega \in G_\alpha(S^{n-1})$. Let $\phi$ be a Schwartz function such that $\hat{\phi}(\xi) = 1$ for $|\xi| \leq 2$ and $\hat{\phi}(\xi) = 0$ for $|\xi| > 4$. We have the following decomposition

\[
T_{jk}f = \phi_k * T\Omega f - \phi_k * \sum_{l<0} \nu_{k+l} * f + \sum_{s \geq 0} (\delta_0 - \phi_k) * \nu_{k+s} * f
\]

\[
= T^1_k f - T^2_k f + T^3_k f,
\]

where $\phi_k$ satisfies $\hat{\phi_k}(\xi) = \hat{\phi}(2^k \xi)$ and $\delta_0$ is the Dirac measure at 0. $\mathcal{T}^i f$ denotes the family $\{T^i_k f\}_{k \in \mathbb{Z}}$ for $i = 1, 2, 3$. Clearly, we need to prove the inequalities below

\[
\|\lambda [N^d_\lambda(\mathcal{T}^i f)]^{1/2}\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)},
\]

for

\[
\frac{(2\alpha+1)(1+\alpha)}{2\alpha^2+\alpha+1/2} < p < \frac{2(1+\alpha)(2\alpha+1)}{4\alpha+1}.
\]

The case $i = 1$ of estimate (3.3) is just a combination of Lemma 2.7 and Theorem 1 in [14],

\[
\|\lambda [N^d_\lambda(\mathcal{T}^1 f)]^{1/2}\|_{L^p(\mathbb{R}^n)} \leq C_p \|T\Omega f\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)},
\]
for all \((2 + 2\alpha)/(1 + 2\alpha) < p < 2 + 2\alpha\).

Next, we consider the case \(i = 3\) of estimate (3.3). For \(\frac{(2\alpha+1)(1+\alpha)}{2\alpha^2+\alpha+1/2} < p < \frac{2(1+\alpha)(2\alpha+1)}{4\alpha+1}\), there are \(\theta_p \in \left(\frac{2}{2\alpha+1}, 1\right]\) and \(p_1 \in \left(\frac{2+2\alpha}{1+2\alpha}, 2 + 2\alpha\right]\) such that

\[
\frac{1}{p} = \frac{\theta_p}{2} + \frac{1 - \theta_p}{p_1}.
\]

The following known result can be found in [14, p.80] and [16, p.461],

\[
(3.4) \quad \|G_s f\|_{L^2(\mathbb{R}^n)} \leq C(1 + s)^{-\alpha-1/2}\|f\|_{L^2(\mathbb{R}^n)}, \quad s > 0,
\]

where

\[
G_s f(x) = \left(\sum_k |[\hat{\delta}_0 - \hat{\phi}_k] * \nu_{k+s}] * f(x)|^2\right)^{1/2}.
\]

Note that \(\|\hat{\delta}_0 - \hat{\phi}\|^2(\xi) \leq C\min\{1, |\xi|^\}, \quad |\hat{\nu}_s(\xi)| \leq C, \quad |\hat{\nu}_s(\xi)| \leq C|2^s\xi|\) when \(|2^s\xi| \leq 2\) and \(|\hat{\nu}_s(\xi)| \leq C\ln |2^s\xi|^{-1-\alpha}\) when \(|2^s\xi| \geq 2\). Then we get

\[
|\hat{\delta}_0 - \hat{\phi}_k - \hat{\nu}_{k+s}]^2(\xi) \leq \begin{cases} 
C|2^k\xi|, & \text{for } |2^k\xi| \leq 2, \\
C(\ln |2^k\xi|)^{-1-\alpha}, & \text{for } |2^k\xi| \geq 2,
\end{cases}
\]

uniformly in \(s > 0\). Similarly, for \(1 < p_2 < \infty\)

\[
\|\sup_k |[\hat{\delta}_0 - \hat{\phi}_k] * \nu_{k+s}] * f\|_{L^{p_2}(\mathbb{R}^n)} \leq C\|f\|_{L^{p_2}(\mathbb{R}^n)}.
\]

By using the same argument in [16, p.461] or [14, p.77], we have uniformly in \(s > 0\)

\[
\|G_s f\|_{L^{p_1}(\mathbb{R}^n)} \leq C_{p_1}\|f\|_{L^{p_1}(\mathbb{R}^n)},
\]

which, interpolated with (3.4), implies

\[
\|G_s f\|_{L^p(\mathbb{R}^n)} \leq C_p(1 + s)^{-\theta_p(\alpha+1/2)}\|f\|_{L^p(\mathbb{R}^n)}.
\]

Finally, we sum the above \(L^p\) estimates over \(s > 0\) and get

\[
\|\lambda[N^d_X(\mathcal{F}^3 f)]^{1/2}\|_{L^p(\mathbb{R}^n)} \leq \sum_{s>0} \|G_s f\|_{L^p(\mathbb{R}^n)} \leq C_p \sum_{s>0} (1 + s)^{-\theta_p(\alpha+1/2)}\|f\|_{L^p(\mathbb{R}^n)} \leq C_p\|f\|_{L^p(\mathbb{R}^n)}.
\]

The case \(i = 2\) of estimate (3.3) can be treated similarly. For \(l \leq 0\), we set

\[
G_l f(x) = \left(\sum_k |[\hat{\phi}_k] * \nu_{k+l}] * f(x)|^2\right)^{1/2}.
\]

Note that \(\hat{\phi}\) vanishes for \(|\xi| \leq 2\) and \(l \leq 0\). Using above estimates of \(\hat{\nu}_s\), we have

\[
\sum_k \left|\tilde{\phi}(2^k\xi)\right|^2 \left|\hat{\nu}(2^{l+k}\xi)\right|^2 \leq \sum_{|\xi| < 2^k \leq \frac{1}{2}|\xi|} (2^{l+k}|\xi|)^2(2^k|\xi|)^{-4} + \sum_{2^k \leq \frac{1}{2}|\xi|} 2(2^{l+k}|\xi|)^2 \leq C2^{|l|}.
\]
Then, \( \|G_t f\|_{L^2(\mathbb{R}^n)} \leq C t^d \|f\|_{L^2(\mathbb{R}^n)} \). \( \|G_t f\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)} \) can be proved as (2.18). Interpolating and summing over \( t \leq 0 \),

\[
\|\lambda\left[N_t^\lambda(T^2 f)\right]\|_{L^p(\mathbb{R}^n)}^{1/2} \leq \sum_{t \leq 0} \|G_t f\|_{L^p(\mathbb{R}^n)} \leq \sum_{t \leq 0} 2^d \|f\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}
\]

for \( 1 < p < \infty \).

### 3.2 Proof of Proposition 3.2

In the present case, the rotation method seems not to work. Instead we appeal to the vector-valued singular integral operator theory. That the underlying kernel is rough brings a lot of trouble, but homogeneity of the kernel is taken advantage of in getting \( L^2 \)-estimate and Hörmander condition. Let us start the proof.

For \( t \in [1, 2] \), we define \( \nu_{0, t} \) as

\[
\nu_{0, t}(x) = \frac{\Omega(x')}{|x|^n} \chi_{\{t \leq |x| \leq 2\}}(x)
\]

and \( \nu_{j, t}(x) = 2^{-jn} \nu_{0, t}(2^{-j} x) \) for \( j \in \mathbb{Z} \). For \( k \in \mathbb{Z} \), we define \( \Phi_{j, k} f \) by \( \Phi_{j, k} f(\xi) = \varphi(2^{j-k} \xi) \hat{f}(\xi) \), where \( \varphi \) is a Schwartz function such that \( \text{supp} (\hat{\varphi}) \subset \{ 1/2 \leq |\xi| \leq 2 \} \) and \( \sum_{k \in \mathbb{Z}} \varphi(2^{-k} \xi) = 1 \) for all \( \xi \in \mathbb{R}^n \setminus \{0\} \). Observe that \( V_{2, j}(T f)(x) \) is just the strong 2-variation function of the family \( \{ \nu_{j, t} * f(x) \}_{t \in [1, 2]} \), hence

\[
S_2(T f)(x) = \left( \sum_{j \in \mathbb{Z}} |V_{2, j}(T f)(x)|^2 \right)^{1/2} = \left( \sum_{j \in \mathbb{Z}} \| \nu_{j, t} * f(x) \|_{L^2}^2 \right)^{1/2} \leq \sum_{k \in \mathbb{Z}} \left( \sum_{j \in \mathbb{Z}} \| \nu_{j, t} * \Phi_{j, k} f(x) \|_{L^2}^2 \right)^{1/2} := \sum_{k \in \mathbb{Z}} S_{2, k}(T f)(x).
\]

The desired estimate (3.2) will follow from the following two estimates by interpolation and summation over \( k \),

\[
\|S_{2, k}(T f)\|_{L^2(\mathbb{R}^n)} \leq C (1 + |k|) \frac{\|f\|_{L^2(\mathbb{R}^n)}}{t^{1+\alpha}}
\]

and

\[
\|S_{2, k}(T f)\|_{L^p(\mathbb{R}^n)} \leq C_p (1 + |k|) \|f\|_{L^p(\mathbb{R}^n)},
\]

for all \( 1 < p < \infty \).

To deal with (3.5), we borrow the fact \( \|a\|_{V^2} \leq \|a\|_{L^2}^{1/2} \|a'\|_{L^2}^{1/2} \), where \( a' = \left\{ \frac{d}{dt} a_t : t \in \mathbb{R} \right\} \). It is a special case of (39) in [25]. Then,

\[
[S_{2, k}(T f)(x)]^2 \leq \sum_{j \in \mathbb{Z}} \| \nu_{j, t} * \Phi_{j, k} f(x) \|_{L^2([1, 2])} \| \frac{d}{dt} \nu_{j, t} * \Phi_{j, k} f(x) \|_{L^2([1, 2])},
\]

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By Cauchy-Schwarz inequality, we have
\[ \| S_{2,k}(Tf) \|_{L^2(\mathbb{R}^n)}^2 \leq \left( \sum_{j \in \mathbb{Z}} \| \nu_{j,t} \ast \Phi_{j,k}f \|_{L^2_t([1,2])}^2 \right)^{1/2} \left( \sum_{j \in \mathbb{Z}} \| \frac{d}{dt} \nu_{j,t} \ast \Phi_{j,k}f \|_{L^2_t([1,2])}^2 \right)^{1/2}. \]

To deal with the first term on the right-hand side, we need the following estimates
\begin{equation}
(3.7) \quad |\hat{\nu}_{j,t}(\xi)| \leq C \left\{ \begin{array}{ll}
\ln(2^j |\xi|)^{-1} & \text{if } 2^j |\xi| \geq 2 \\
2^{|\xi|} & \text{if } 2^j |\xi| \leq 2
\end{array} \right.
\end{equation}
uniformly in \( t \in [1, 2] \), which have been essentially proved in [16]. By Plancherel’s theorem,
\[ \left( \sum_{j \in \mathbb{Z}} \| \nu_{j,t} \ast \Phi_{j,k}f \|_{L^2_t([1,2])}^2 \right)^{1/2} = \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^n} |\hat{\nu}_{j,t}(\xi)| |\hat{\varphi}(2^j \xi)\hat{f}(\xi)|^2 d\xi dt \leq C(1 + |k|)^{-2(1+\alpha)} \| f \|_{L^2(\mathbb{R}^n)}^2. \]

In order to treat the second term, we need an elementary fact. That is, for any Schwartz function \( h \),
\begin{equation}
(3.8) \quad |(\frac{d}{dt} \nu_{j,t} \ast h)(x)| \leq C \|\Omega\|_{L^1(S^{n-1})} \| \hat{h}(\xi) \|
\end{equation}
uniformly in \( t \in [1, 2] \). Indeed, by spherical coordinate transformation, a trivial calculation shows
\[ \frac{d}{dt} \nu_{j,t} \ast h(x) = \frac{d}{dt} \left[ \int_{2^jt < |y| < 2^{j+1}} \frac{\Omega(y')}{|y|^n} h(x-y) dy \right] = \frac{d}{dt} \left[ \int_{S^{n-1}} \Omega(y') \int_{2^jt}^{2^{j+1}} \frac{1}{r} h(x-ry') dr d\sigma(y') \right] = \frac{1}{t} \int_{S^{n-1}} \Omega(y') h(x - 2^j ty') d\sigma(y'). \]
Note that \( t \in [1, 2] \),
\[ |(\frac{d}{dt} \nu_{j,t} \ast h)(x)| \leq C \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} \int_{S^{n-1}} \Omega(y') h(x - 2^j ty') d\sigma(y') dx \leq C \int_{S^{n-1}} |\Omega(y')| \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} h(x - 2^j ty') dx d\sigma(y') \leq C \|\Omega\|_{L^1(S^{n-1})} \| \hat{h}(\xi) \|. \]

By Plancherel’s theorem and (3.8), we have
\[ \left( \sum_{j \in \mathbb{Z}} \| \frac{d}{dt} \nu_{j,t} \ast \Phi_{j,k}f \|_{L^2_t([1,2])}^2 \right)^{1/2} \leq C \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^n} |\varphi(2^j \xi)\hat{f}(\xi)|^2 d\xi \leq C \| f \|_{L^2(\mathbb{R}^n)}^2. \]
Then we get \( L^2 \) estimate (3.5).
To achieve the estimate (3.6), we use the vector-valued singular integral theory for convolution operators. For \( t \in [1, 2] \) and \( j \in \mathbb{Z} \), we define \( K_k(x) = \{ \nu_{j,t} \ast \tilde{\varphi}_{j-k}(x) \}_{j,t} \), where \( \tilde{\varphi}_{j-k}(y) = 2^{n(k-j)} \tilde{\varphi}(2^{k-j}y) \) and \( K_k \) takes value in the Banach space

\[
B = \{ a(j,t) : \|a\|_B := (\sum_j \|a\|^2_{V_j})^{1/2} < \infty \}.
\]

Obviously, \( S_{2,k}(Tf) = \|K_k \ast f\|_B \). By the vector-valued singular integral operator theory, we have the \( L^p \) estimates

\[
\|S_{2,k}(Tf)\|_{L^p(\mathbb{R}^n)} \leq C_p(M_k + N_k)\|f\|_{L^p(\mathbb{R}^n)}, \quad 1 < p < \infty,
\]

where \( M_k = (1 + |k|)^{-\frac{\alpha}{2}} \) is the \( L^2 \)-bound in (3.5) and \( N_k \) is any upper bound for

\[
\sup_{y \in \mathbb{R}^n} \int_{|x| > 2|y|} \|K_k(x - y) - K_k(x)\|_B \, dx.
\]

To get the desired upper bound \( C(1 + |k|) \) for above integral, we need the following facts

\[
\|a\|_B \leq (\sum_j \|a\|^2_{V_j})^{1/2} \leq \sum_j \int_1^2 \frac{d}{dt}a(j,t) \, dt
\]

and

\[
\frac{d}{dt}(\nu_{j,t} \ast \tilde{\varphi}_{j-k})(x) = \int_{S^{n-1}} \frac{\Omega(y')}{t} \tilde{\varphi}_{j-k}(x - 2^j ty') \, d\sigma(y').
\]

Therefore,

\[
\int_{|x| > 2|y|} \|K_k(x - y) - K_k(x)\|_B \, dx \\
\leq \sum_j \int_1^2 \int_{|x| > 2|y|} \left| \frac{d}{dt}(\nu_{j,t} \ast \tilde{\varphi}_{j-k})(x - y) - \frac{d}{dt}(\nu_{j,t} \ast \tilde{\varphi}_{j-k})(x) \right| \, dx \, dt \\
\leq \int_1^2 \int_{S^{n-1}} |\Omega(u')| \sum_j \int_{|x| > 2|y|} \left| \tilde{\varphi}_{j-k}(x - y - 2^j tu') - \tilde{\varphi}_{j-k}(x - 2^j tu') \right| \, dx \, d\sigma(u') \, dt.
\]

To complete the proof, it suffices to show

\[
\sum_j \int_{|x| > 2|y|} \left| \tilde{\varphi}_{j-k}(x - y - 2^j tu') - \tilde{\varphi}_{j-k}(x - 2^j tu') \right| \, dx \leq C(1 + |k|)
\]

uniformly in \( k, t \) and \( u' \). Without loss of generality, we assume \( t = 1 \) and \( u' = 1 \). For any fixed \( y \in \mathbb{R}^n \) and \( k \in \mathbb{Z} \), we divide the sum into two parts \( I := \sum_{2^j+1 < |y|} \) and \( II := \sum_{2^j+1 \geq |y|} \). For
the first part, we treat it as follows

\[ I = \sum_{2^{j+1} < |y|} \int_{|x| > \frac{|y|}{2^{j-k}}} |\tilde{\varphi}(x - y + \frac{2^j}{2^{j-k}}) - \tilde{\varphi}(x - 2^k 1)| \, dx \]

\[ \leq 2 \sum_{2^{j+1} < |y|} \int_{|x| > \frac{|y|}{2^{j-k+1}}} |\tilde{\varphi}(x)| \, dx \leq 2 \int_{|x| > 2^k} |\tilde{\varphi}(x)| [\sum_{2^k |y| < 2^{j+1} < |y|} 1] \, dx \]

\[ \leq 2 \int_{|x| > 2^k} |\tilde{\varphi}(x)| \log \frac{|x|}{2^{k-1}} \, dx \leq C. \]

For the second part, we use the mean value theorem

\[ II = \sum_{2^{j+1} \geq |y|} \int_{|x| > \frac{|y|}{2^{j-k}}}|\tilde{\varphi}(x - y + \frac{2^j}{2^{j-k}}) - \tilde{\varphi}(x - 2^k 1)| \, dx \]

\[ \leq C \sum_{2^{j+1} \geq |y|} \min\{1, 2^k - j, |y|\} \leq C(1 + |k|). \]

This completes the proof of (3.6).

4 Proof of Theorem 1.8

As in the proof of Theorem 1.2 to show Theorem 1.3, it suffices to prove dyadic \(\lambda\)-jump estimate and short 2-variation estimate respectively. Also in the course of proving strong \(q\)-variation estimate \((q > 2)\) by the rotation method, we can get short 2-estimate. Let us start with the dyadic \(\lambda\)-jump estimate.

**Proposition 4.1.** Let \(\Omega \in H^1(S^{n-1})\) or \(L(\log^+ L)^{1/2}(S^{n-1})\). Then

\[ \|\lambda \sqrt{N_X(Mf)}\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}, \quad 1 < p < \infty, \]

uniformly in \(\lambda > 0\).

**Proof.** We assume \(\Omega\) satisfies the cancelation condition (1.6). Otherwise, we write

\[ \Omega(x') = [\Omega(x') - \frac{1}{\omega_n} \int_{S^{n-1}} \Omega(y') \, d\sigma(y')] + \frac{1}{\omega_n} \int_{S^{n-1}} \Omega(y') \, d\sigma(y') \]

\[ := \Omega_0(x') + C(\Omega, n), \]

where \(\omega_n\) denotes the area of \(S^{n-1}\). Thus,

\[ M_t f(x) = \frac{1}{t^n} \int_{|y| < t} \Omega_0(y') f(x - y) \, dy + C(\Omega, n) \frac{1}{t^n} \int_{|y| < t} f(x - y) \, dy. \]
has been established in [22] with Ω = 1. Define \( \sigma_{2k}(y) = 2^{-kn}\chi_{\{|y|<2^k\}}(y)\Omega(y') \), then \( M_{2k}f = \sigma_{2k} * f \). The pointwise domination

\[
\lambda^2 N^d_\lambda(Mf) = \lambda^2 N_\lambda(\{M_{2k}f\}_k) \leq \sum_k |f * \sigma_{2k}|^2
\]

reduces the desired estimate to

\[
\|\left(\sum_k |f * \sigma_{2k}|^2\right)^{1/2}\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)},
\]

(4.3) can be showed in the similar way as that in [10, p.597] for \( \Omega \in H^1(S^{n-1}) \) and [1, p.698] for \( \Omega \in L(\log^+ L)^{1/2}(S^{n-1}) \). We omit those details for simplicity.

Remark 4.2. The square function having appeared in the proof of Proposition 4.1 is actually a discrete analogue of the Marcinkiewicz integral with rough kernels. In [37], Walsh proved that Marcinkiewicz integrals may not be \( L^2(\mathbb{R}^n) \)-bounded if the kernel \( \Omega \in L(\log^+ L)^{1/2-\varepsilon}(S^{n-1}) \) for \( 0 < \varepsilon < 1/2 \). Hence it is hopeless to obtain (4.1) by using the pointwise domination (4.2) with \( \Omega \in L^1(S^{n-1}) \).

To apply the rotation method, we need the following result in one dimension.

Lemma 4.3. Let \( \mathcal{M} = \{\mathcal{M}_t\}_{t>0} \) with \( \mathcal{M}_t \) defined as

\[
\mathcal{M}_t f(x) = \frac{1}{t^n} \int_0^t f(x - s)s^{n-1}ds.
\]

Then for \( 1 < p < \infty \),

\[
\|\lambda \sqrt{N_\lambda(\mathcal{M} f)}\|_{L^p(\mathbb{R})} \leq C_p \|f\|_{L^p(\mathbb{R})},
\]

uniformly in \( \lambda > 0 \). Whence \( \|V_q(\mathcal{M} f)\|_{L^p(\mathbb{R})} \leq C_p \|f\|_{L^p(\mathbb{R})} \) for \( q > 2 \) and \( 1 < p < \infty \).

Proof. Note that \( \mathcal{M}_t f(x) = \mu_t * f(x) \), where \( \mu_t(s) = \frac{1}{t^n}\mu(\frac{x}{t^n}) \) and \( \mu(s) = \chi_{\{0 \leq s \leq 1\}}(s)_s^{n-1} \). Vander Corput’s lemma implies that

\[
|\hat{\mu}(\xi)| \leq \left| \int_0^1 e^{-2\pi ir \xi r^{n-1}} dr \right| \leq C|\xi|^{-1}.
\]

By Theorem 1.1 in [25], we obtain the following dyadic \( \lambda \)-jump inequality

\[
\|\lambda \sqrt{N^d_\lambda(\mathcal{M} f)}\|_{L^p(\mathbb{R})} \leq C_p \|f\|_{L^p(\mathbb{R})},
\]

uniformly in \( \lambda > 0 \). Further, by Lemma 6.1 of [25], we get the \( L^p \) boundedness of the short 2-variation operator

\[
\|S_2(\mathcal{M} f)\|_{L^p(\mathbb{R})} \leq C_p \|f\|_{L^p(\mathbb{R})}, 1 < p < \infty.
\]
Above two estimates and Lemma 2.1 imply our desired \( \lambda \)-jump inequality. Clearly, the \( q \)-variation inequality is a trivial consequence of Lemma 1.1.

Finally, let us turn to the proof of Theorem 1.8.

Proof. We first prove Theorem 1.8(ii) by using the rotation method applied in Section 2. We define another family of averaging operators \( \mathcal{M}^t = \{M^t_1\}_{t>0} \) with

\[
M^t_1 f(x) = \frac{1}{t^n} \int_0^t f(x - s\mathbf{1}) s^{n-1} ds,
\]

where \( \mathbf{1} = (1, 0, \cdots, 0) \). Recall the rotation of a function is defined as \( (R_\sigma f)(x) = f(\sigma x) \). Then, we have

\[
M_t f(x) = \int_{S^{n-1}} \Omega(y') \frac{1}{t^n} \int_0^t f(x - sy') s^{n-1} ds d\sigma(y') = \int_{SO(n)} (R_{\sigma^{-1}} M_t R_\sigma f)(x) \Omega(\sigma \mathbf{1}) d\sigma.
\]

Using Minkowski’s inequality and Proposition 4.3, we get for \( 1 < p < \infty \)

\[
\|V_q(\mathcal{M}f)\|_{L^p(\mathbb{R}^n)} \leq \int_{SO(n)} \|R_{\sigma^{-1}} V_q(\mathcal{M}^t R_\sigma f)\|_{L^p(\mathbb{R}^n)} \Omega(\sigma \mathbf{1}) d\sigma \leq C_p \|f\|_{L^p(\mathbb{R}^n)}.
\]

By Lemma 2.1 and Proposition 4.1, our desired result Theorem 1.8(i) follows from

\[
\|S_2(\mathcal{M}f)\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)},
\]

which is consequence of (4.4) by rotation method.

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