Mean-payoff Games with Partial-Observation*  
(Extended abstract)

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Abstract. Mean-payoff games are important quantitative models for open reactive systems. They have been widely studied as games of perfect information. In this paper we investigate the algorithmic properties of several subclasses of mean-payoff games where the players have asymmetric information about the state of the game. These games are in general undecidable and not determined according to the classical definition. We show that such games are determined under a more general notion of winning strategy. We also consider mean-payoff games where the winner can be determined by the winner of a finite cycle-forming game. This yields several decidable classes of mean-payoff games of asymmetric information that require only finite-memory strategies, including a generalization of perfect information games where positional strategies are sufficient. We give an exponential time algorithm for determining the winner of the latter.

1 Introduction

Mean-payoff games (MPGs) are two-player, infinite duration, turn-based games played on finite edge-weighted graphs. The two players alternately move a token around the graph; and one of the players (Eve) tries to maximize the (limit) average weight of the edges traversed, whilst the other player (Adam) attempts to minimize the average weight. Such games are particularly useful in the field of verification of models of reactive systems, and the perfect information versions of these games have been extensively studied [4, 9, 10, 13]. One of the major open questions in the field of verification is whether the following decision problem, known to be in the intersection of the classes NP and coNP [13], can be solved in polynomial time: Given a threshold $\nu$, does Eve have a strategy to ensure a mean-payoff value of at least $\nu$?

In game theory the concepts of imperfect, partial and limited information indicate situations where players have asymmetric knowledge about the state of the game. In the context of verification games this partial knowledge is reflected in one or both players being unable to determine the precise location of the token amongst several equivalent vertices, and such games have also been extensively studied [2, 3, 12, 16, 22]. Adding partial-observation to verification games results in an enormous increase in complexity, both algorithmically and in terms of strategy synthesis. For example, it was shown in [12] that for MPGs with partial-observation, when the mean payoff value is defined

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1 From results in [24] and [14] it follows that the problem is also in UP $\cap$ coUP.
using $\lim \sup$, the analogue of the above decision problem is undecidable; and whilst memoryless strategies suffice for MPGs with perfect information, infinite memory may be required. The first result of this paper is to show that this is also the case when the mean payoff value is defined using the stronger $\lim \inf$ operator, closing two open questions posed in [12]. As a consequence, we generalize a result from [8] which uses the undecidability result from [12] to show several classical problems for mean-payoff automata are also undecidable.

These unfavourable results motivate the main investigation of this paper: identifying classes of MPGs with partial-observation where determining the winner is decidable and where strategies with finite memory, possibly memoryless, are sufficient.

To simplify our definitions and algorithmic results we initially consider a restriction on the set of observations which we term limited-observation. In games of limited-observation the current observation contains only those vertices consistent with the observable history, that is the observations are the belief set of Eve (see, e.g. [7]). This is not too restrictive as any MPG with partial-observation can be realized as a game of limited-observation via a subset construction. In Section 9 we consider the extension of our definitions to MPGs with partial-observation via this construction.

Our focus for the paper will be on games at the observation level, in particular we are interested in observation-based strategies for both players. Whilst observation-based strategies for Eve are usual in the literature, observation-based strategies for Adam have not, to the best of our knowledge, been considered. Such strategies are more advantageous for Adam as they encompass several simultaneous concrete strategies. Further, in games of limited-observation there is guaranteed to be at least one concrete strategy consistent with an observation-based strategy. Our second result is to show that although MPGs with partial-observation are not determined under the usual definition of strategy, they are determined when Adam can use an observation-based strategy.

In games of perfect information one aspect of MPGs that leads to simple (but not quite efficient) decision procedures is their equivalence to finite cycle-forming games. Such games are played as their infinite counterparts, however when the token revisits a vertex the game is stopped. The winner is determined by a finite analogue of the mean-payoff condition on the cycle now formed; that is, Eve wins if the average weight of the edges traversed in the cycle exceeds a given threshold. Ehrenfeucht and Mycielski [13] and Björklund et al. [4] used this equivalence to show that positional strategies are sufficient to win MPGs with perfect information and this leads to a $\text{NP} \cap \text{coNP}$ procedure for determining the winner. Critically, a winning strategy in the finite game translates directly to a winning strategy in the MPG, so such games are especially useful for strategy synthesis.

We extend this idea to games of partial-observation by introducing a finite, perfect information, cycle-forming game played at the observation level. That is, the game finishes when an observation is revisited (though not necessarily the first time). In this reachability game winning strategies can be translated to finite-memory winning strategies in the MPG. This leads to a large, natural subclass of MPGs with partial-observation, forcibly terminating games, where determining the winner is decidable and finite memory observation-based strategies suffice.

\footnote{A recent result of Aminof and Rubin [11] corrects some errors in [4].}
Unfortunately, recognizing if an MPG is a member of this class is undecidable, and although determining the winner is decidable, we show that this problem is complete (under polynomial-time reductions) for the class of all decidable problems. Motivated by these negative algorithmic results, we investigate two natural refinements of this class for which winner determination and class membership are decidable. The first, *forcibly first abstract cycle* games (forcibly FAC games, for short), is the natural class of games obtained when our cycle-forming game is restricted to simple cycles. Unlike the perfect information case, we show that winning strategies in this finite simple cycle-forming game may still require memory, though this memory is at most exponential in the size of the game. The second refinement, *first abstract cycle* (FAC) games, is a further structural refinement that guarantees a winner in the simple cycle-forming game. We show that in this class of games positional observation-based strategies suffice.

The subclasses of MPGs with limited-observation we study then give rise to subclasses of MPGs with partial-observation. For the class membership problem we show there is, as expected, an exponential blow-up in the complexity, however for the problem of determining the winner the algorithmic cost is significantly better.

Table 1 summarizes the results of this paper. For space reasons the full details of most proofs can be found in the appendices.

| Memory | Class membership | Winner determination |
|--------|------------------|----------------------|
| Finite | Undecidable      | R-complete           |
| Exponential | PSPACE-complete | EXP-complete |
| 2-Exponential | EXPSPACE-hard, in EXPSPACE | EXP-complete |
| Positional | coNP-complete | EXP-complete |
| CoNEXP-complete | EXP-complete | EXP-complete |

Table 1. Summary of results for the classes of games studied.

2 Preliminaries

Mean-payoff games. A mean-payoff game (MPG) with partial-observation is a tuple $G = \langle Q, q_I, \Sigma, \Delta, w, Obs \rangle$, where $Q$ is a finite set of states, $q_I \in Q$ is the initial state, $\Sigma$ is a finite set of actions, $\Delta \subseteq Q \times \Sigma \times Q$ is the transition relation, $w : \Delta \rightarrow \mathbb{Z}$ is the weight function, and $Obs \in \text{Partition}(Q)$ is a set of observations. We assume $\Delta$ is total, that is, for every $(q, \sigma) \in Q \times \Sigma$ there exists $q' \in Q$ such that $(q, \sigma, q') \in \Delta$. We say that $G$ is a *mean-payoff game with limited-observation* if additionally, $Obs$ satisfies the following:

1. $\{q_I\} \in Obs$, and
2. For each $(o, \sigma) \in Obs \times \Sigma$ the set $\{q' \mid \exists q \in o \text{ and } (q, \sigma, q') \in \Delta\}$ is a union of elements of $Obs$. 
Note that condition (2) is equivalent to saying that if $q \in o$, $q' \in o'$ and $(q, \sigma, q') \in \Delta$ then for every $r' \in o'$ there exists $r \in o$ such that $(r, \sigma, r') \in \Delta$. If every element of $Obs$ is a singleton, then we say $G$ is a mean-payoff game with perfect information. For simplicity, we denote by $\text{post}_o(s) = \{q' \in Q \mid \exists q \in s : (q, \sigma, q') \in \Delta\}$ the set of $\sigma$-successors of a set of states $s \subseteq Q$.

Figure 1 gives an example of an MPG with limited-observation, with $\Sigma = \{a, b\}$ and $Obs = \{\{q_0\}, \{q_1, q_2\}, \{q_3\}\}$. In this work, unless explicitly stated otherwise, we depict states from an MPG with partial-observation as circles and transitions as arrows labelled by an action-weight pair: $\sigma, w$. Observations are represented by dashed boxes.

**Abstract and concrete paths.** A concrete path in an MPG with partial-observation is a sequence $q_0 \sigma q_1 \sigma_1 \ldots$ where for all $i \geq 0$ we have $q_i \in Q$, $\sigma_i \in \Sigma$ and $(q_i, \sigma_i, q_{i+1}) \in \Delta$. An abstract path is a sequence $o_0 o_1 o_2 \sigma_1 \ldots$ where $o_i \in Obs$, $\sigma_i \in \Sigma$ and for all $i$ there exists $q_i \in o_i$ and $q_{i+1} \in o_{i+1}$ with $(q_i, \sigma_i, q_{i+1}) \in \Delta$. Given an abstract path $\psi$, let $\gamma(\psi)$ be the (possibly empty) set of concrete paths that agree with the observation and action sequence. In other words $\gamma(\psi) = \{q_0 o_0 o_1 o_2 \sigma_1 \ldots \mid \forall i \geq 0 : q_i \in o_i \text{ and } (q_i, \sigma_i, q_{i+1}) \in \Delta\}$. Note that in games of limited-observation this set is never empty. Also, given abstract (respectively concrete) path $\rho$, let $\rho[...n]$ represent the prefix of $\rho$ up to the $(n+1)$-th observation (state), which we express as $\rho[n]$; similarly, we denote by $\rho[...n]$ the suffix of $\rho$ starting from the $l$-th observation (state) and by $\rho[...n]$ the finite subsequence starting and ending in the aforementioned locations.

An abstract (respectively concrete) cycle is an abstract (concrete) path $\chi = o_0 o_1 o_2 \ldots o_n$ where $o_n = o_0$. We say $\chi$ is simple if $o_i \neq o_j$ for $0 \leq i < j < n$. Given $k \in \mathbb{N}$ define $\chi^k$ to be the abstract (concrete) cycle obtained by traversing $\chi$ $k$ times. That is, $\chi^k = o_0 o_1 o_2 \ldots o_n$ where for all $j$, $o'_j = o_j \mod n$ and $\sigma'_j = \sigma_j \mod n$. A cyclic permutation of $\chi$ is an abstract (concrete) cycle $o_0 o_1 o_2 \ldots o_n$ such that $o'_j = o_{j+k} \mod n$ and $\sigma'_j = \sigma_{j+k} \mod n$ for some $k$. If $\chi' = o'_0 o'_1 o'_2 \ldots o'_n$ is a cycle with $o'_0 = o_i$ for some $i$, the interleaving of $\chi$ and $\chi'$ is the cycle $o_0 o_1 o_2 \ldots o_n$ where $o'_i = o_i$ and $\sigma'_i = \sigma_i$.

Given a concrete path $\pi = q_0 \sigma q_1 \sigma_1 \ldots$, the payoff up to the $(n+1)$-th element is given by

$$w(\pi[...n]) = \sum_{i=0}^{n-1} w(q_i, \sigma_i, q_{i+1}).$$

If $\pi$ is infinite, we define two mean-payoff values $\underline{MP}$ and $\overline{MP}$ as:

$$\underline{MP}(\pi) = \liminf_{n \to \infty} \frac{1}{n} w(\pi[...n]) \quad \overline{MP}(\pi) = \limsup_{n \to \infty} \frac{1}{n} w(\pi[...n])$$

**Remark 1.** It was shown in [12] that in MPGs with partial-observation where finite memory strategies suffice Eve wins the $\underline{MP}$ version of the game if and only if she wins the $\overline{MP}$ version. As the majority of games considered in this paper only require finite memory, we can take either definition. For simplicity and consistency with Section 3 we will use $\underline{MP}$.

**Plays and strategies.** A play in an MPG with partial-observation $G$ is an infinite abstract path starting at $o_1 \in Obs$ where $q_1 \in o_1$. Denote by $\mathcal{Plays}(G)$ the set of all plays and
by \( \text{Prefs}(G) \) the set of all finite prefixes of such plays ending in an observation. Let \( \gamma(\text{Plays}(G)) \) be the set of concrete paths of all plays in the game, and \( \gamma(\text{Prefs}(G)) \) be the set of all finite prefixes of all concrete paths.

An observation-based strategy for Eve is a function from finite prefixes of plays to actions, i.e. \( \lambda_\exists : \text{Prefs}(G) \to \Sigma \). A play \( \psi = o_0\sigma_0\sigma_1\ldots \) is consistent with \( \lambda_\exists \) if \( \sigma_i = \lambda_\exists(\psi[..i]) \) for all \( i \). An observation-based strategy for Adam is a function \( \lambda_\forall : \text{Prefs}(G) \times \Sigma \to \text{Obs} \) such that for any prefix \( \pi = o_0\sigma_0\ldots\sigma_n \in \text{Prefs}(G) \) and action \( \sigma \), \( \lambda_\forall(\pi,\sigma) \cap \text{post}_\pi(\{\pi[n]\}) \neq \emptyset \). A play \( \psi = o_0\sigma_0\sigma_1\ldots \) is consistent with \( \lambda_\forall \) if \( \lambda_\forall(\psi[..i],\sigma_i) = o_{i+1} \) for all \( i \). A concrete strategy for Adam is a function \( \mu_\forall : \gamma(\text{Prefs}(G)) \times \Sigma \to Q \) such that for any concrete prefix \( \pi = q_0\sigma_0\ldots\sigma_n \in \gamma(\text{Prefs}(G)) \) and action \( \sigma \), \( \mu_\forall(\pi,\sigma) \in \text{post}_\pi(\{\pi[n]\}) \). A play \( \psi = o_0\sigma_0\sigma_1\ldots \) is consistent with \( \mu_\forall \) if there exists a concrete path \( \pi \in \gamma(\psi) \) such that \( \mu_\forall(\pi[..i],\sigma_i) = \pi[i+1] \) for all \( i \).

We say an observation-based strategy for Eve \( \lambda_\exists \) has memory \( m \) if there is a set \( M \) with \( |M| = m \), an element \( m_0 \in M \), and functions \( \alpha_u : M \times \text{Obs} \to M \) and \( \alpha_o : M \times \text{Obs} \to \Sigma \) such that for any play prefix \( \rho = o_0\sigma_0\ldots\sigma_n \) we have \( \lambda_\exists(\rho) = \alpha_u(m_n, o_n) \), where \( m_n \) is defined inductively by \( m_{i+1} = \alpha_u(m_i, o_i) \) for \( i \geq 0 \). An observation-based strategy for Adam \( \lambda_\forall \) has memory \( m \) if there is a set \( M \) with \( |M| = m \), an element \( m_0 \in M \), and functions \( \alpha_u : M \times \text{Obs} \times \Sigma \to M \) and \( \alpha_o : M \times \text{Obs} \times \Sigma \to \text{Obs} \) such that for any play prefix ending in an action \( \rho = o_0\sigma_0\ldots\sigma_n \) we have \( \lambda_\forall(\rho) = \alpha_o(m_n, o_n, \sigma_n) \), where \( m_n \) is defined inductively by \( m_{i+1} = \alpha_u(m_i, o_i, \sigma_i) \).

An observation-based strategy (for either player) with memory 1 is positional.

Note that for any concrete strategy \( \mu \) of Adam there is a unique observation-based strategy \( \lambda_\mu \) such that all plays consistent with \( \mu \) are consistent with \( \lambda_\mu \). Conversely there may be several, but possibly no, concrete strategies that correspond to a single observation-based strategy. In games of limited-observation there is guaranteed to be at least one concrete strategy for every observation-based strategy.

Given a threshold \( \nu \in \mathbb{R} \), we say a play \( \psi \) is winning for Eve if \( \mu P(\pi) \geq \nu \) for all concrete paths \( \pi \in \gamma(\psi) \), otherwise it is winning for Adam. Given \( \nu \), one can construct an equivalent game in which Eve wins if and only if \( \mu P(\pi) \geq 0 \) if and only if she wins the original game, so without loss of generality we will assume \( \nu = 0 \). A strategy \( \lambda \) is winning for a player if all plays consistent with \( \lambda \) are winning for that player. We say that a player wins \( G \) if (s)he has a winning strategy.

Reachability games. A reachability game \( G = (Q, q_I, \Sigma, \Delta, T_\exists, T_\forall) \) is a tuple where \( Q \) is a (not necessarily finite) set of states; \( \Sigma \) is a finite set of actions; \( \Delta \subseteq Q \times \Sigma \times Q \) is a finitary transition function (that is, for any \( q \in Q \) and \( \sigma \in \Sigma \) there are finitely many \( q' \) such that \( (q, \sigma, q') \in \Delta \)); \( q_I \in Q \) is the initial state; and \( T_\exists, T_\forall \subseteq Q \) are the terminating states. The game is played as follows. We place a token on \( q_I \in Q \) and start the game. Eve chooses an action \( \sigma \in \Sigma \) and Adam chooses a \( \sigma \)-successor of the current location as determined by \( \Delta \). The process is repeated until the game reaches a state in \( T_\exists \) or \( T_\forall \). In the first case we declare Eve as the winner whereas the latter corresponds to Adam winning the game. Notice that the game, in general, might not terminate, in which case neither player wins. Notions of plays and strategies in the reachability game follow from the definitions for mean-payoff games, however we extend plays to include finite paths that end in \( T_\exists \cup T_\forall \).
3 Undecidability of Liminf Games

Mean-payoff games with partial-observation were extensively studied in [12], however there were few results for the mean-payoff condition used in this paper ($MP$). In that paper the authors show that there are games with partial-observation where Eve wins the $\bar{MP}$ version of the game but does not win the $\overline{MP}$ version, contrary to the perfect information where she wins one if and only if she wins the other. They show that, with the mean payoff condition defined using $\bar{MP}$ (and $>$), determining whether Eve has a winning strategy is undecidable and strategies with infinite memory may be necessary. The analogous, and more general, questions using $\overline{MP}$ (and $\geq$) were left open. In this section we answer these questions, showing that both results still hold.

**Proposition 1.** There exist MPGs with partial-observation for which Eve requires infinite memory observation-based strategies to ensure $MP \geq 0$.

We sketch here the main idea of the proof and defer the details to Appendix A.

Consider the game in Figure 2 and consider the strategy of Eve that plays (regardless of location) $aba^2ba^3ba^4b\ldots$ As $b$ is played infinitely often in this strategy, the only concrete paths consistent with this strategy are $\pi = q_0q_1^kq_2^l$ and $\pi = q_0q_1^kq_2^lq_3$ for non-negative integers $k, l$. In both cases $\bar{MP} \geq 0$, so the strategy is winning.

Against a finite memory strategy of Eve, Adam plays to ensure the game remains in $\{q_1, q_2\}$. As Eve’s strategy has finite memory, her choice of actions must be ultimately periodic. Now there are two cases, if she plays a finite number of $b$’s then Adam has a concrete winning strategy which consists in guessing when she will play the last $b$ and moving to $q_2$. If, on the other hand, she plays $b$ infinitely often then Adam can choose to stay in $q_1$ and again win the game.

**Theorem 1.** Let $G$ be an MPG with partial-observation. Determining whether Eve has an observation-based strategy to ensure $MP \geq 0$ is undecidable.

The proof of this result is based on a similar construction to Theorem 5, so we defer the proof to Appendix E.

In [8], the authors present a reduction from blind MPGs to mean-payoff automata. This reduction, together with the undecidability result from [12], imply several classical automata-theoretical problems for mean-payoff automata are also undecidable. In [8], the authors study the non-strict $\geq$ relation between quantitative languages. It follows from the undecidability result presented above, that even when one considers the strict order, $>$, these problems remain undecidable.

**Corollary 1.** The strict quantitative universality, and strict quantitative language inclusion problems are undecidable for non-deterministic and alternating mean-payoff automata.

4 Observable Determinacy

One of the key features of MPGs with perfect information is that they are determined, that is, it is always the case that one player has a winning strategy. This is not true in
games of partial or limited-observation as can be seen in Figure 1. Any concrete strategy of Adam reveals to Eve the successor of \( q_0 \) and she can use this information to play to \( q_3 \). Conversely Adam can defeat any strategy of Eve by playing to whichever of \( q_1 \) or \( q_2 \) means the play returns to \( q_0 \) on Eve’s next choice (recall Eve cannot distinguish \( q_1 \) and \( q_2 \) and must therefore choose an action to apply to the observation \( \{ q_1, q_2 \} \)). This strategy of Adam can be encoded as an observation-based strategy: “from \( \{ q_1, q_2 \} \) with action \( a \) or \( b \), play to \( \{ q_0 \} \)”. It transpires that, under an assumption about large cardinals, any such counter-play by Adam is always encodable as an observable strategy.

**Theorem 2 (Observable determinacy).** Assuming the existence of a measurable cardinal, one player always has a winning observation-based strategy in an MPG with limited-observation.

The existence of a measurable cardinal implies \( \Sigma^1_1 \)-Determinacy [17] – a weak form of the “Axiom of Determinacy”. The observable determinacy of MPGs with limited-observation then follows from the following result:

**Lemma 1.** The set of plays that are winning for Eve in an MPG with limited-observation is co-Suslin.

## 5 Strategy Transfer

In this section we will construct a reachability game from an MPG with limited-observation in which winning strategies for either player are sufficient (but not necessarily necessary) for winning strategies in the original MPG.

Let us fix a mean-payoff game with limited-observation \( G = \langle Q, q_1, \Sigma, \Delta, \text{Obs}, w \rangle \). We will define a reachability game on the weighted unfolding of \( G \).

Let \( \mathcal{F} \) be the set of functions \( f : Q \to \mathbb{Z} \cup \{ +\infty, \bot \} \). Our intention is to use functions in \( \mathcal{F} \) to keep track of the minimum weight of all concrete paths ending in the given vertex. A function value of \( \bot \) indicates that the given vertex is not in the current observation, and intuitively a function value of \( +\infty \) is used to indicate to Eve that the token is not located at such a vertex. The added complication permits our winning condition to include games where Adam wins by ignoring paths going through the given vertex. The support of \( f \) is \( \text{supp}(f) = \{ q \in Q \mid f(q) \neq \bot \} \). We say that \( f' \in \mathcal{F} \) is a \( \sigma \)-successor of \( f \in \mathcal{F} \) if:

\( ^3 \) This assumption is independent of the theory of ZFC.
memory observation-based winning strategy in wins the reachability game then she can transform her strategy into one that plays definitely by returning, whenever the play reaches ensures that all concrete paths consistent with the play have non-negative mean-payoff \( T \).

We define a family of partial orders, \( \preceq_k \) \( (k \in \mathbb{N}) \), on \( \mathcal{F} \) by setting \( f \preceq_k f' \) if \( \text{supp}(f) = \text{supp}(f') \) and \( f(q) + k \leq f'(q) \) for all \( q \in \text{supp}(f) \) (where \( +\infty + k = +\infty \)).

Denote by \( \mathcal{G} \) the set of all sequences 
\[ f_0 \sigma_0 f_1 \ldots \sigma_{n-1} f_n \in (\mathcal{F} \cdot \Sigma)^* \mathcal{F} \] 

such that for all \( 0 \leq i < n \), \( f_{i+1} \) is a \( \sigma_i \)-successor of \( f_i \). Observe that for each function-action sequence \( \rho = f_0 \sigma_0 \ldots f_n \in \mathcal{G} \) there is a unique abstract path 
\[ \text{supp}(\rho) = \sigma_0 \sigma_0 \ldots o_n \] 

such that \( o_i = \text{supp}(f_i) \) for all \( i \). Conversely for each abstract path \( \psi = o_0 \sigma_0 \ldots o_n \) there may be many corresponding function-action sequences in \( \text{supp}^{-1}(\psi) \).

The reachability game associated with \( G \), i.e. \( \Gamma_G = \langle \Pi_G, \Sigma, f_1, \delta, T_3, T_\psi \rangle \), is formally defined as follows: \( f_1 \in \mathcal{F} \) is the function for which \( f(q) \rightarrow 0 \) if \( q = q_1 \) and \( f(q) \rightarrow -\infty \) otherwise. \( \Pi_G \) is the subset of \( \mathcal{G} \) where for all \( f_0 \sigma_0 f_1 \ldots \sigma_{n-1} f_n \in \Pi_G \) we have \( f_0 = f_1 \) and for all \( 0 \leq i < j < n \) we have \( f_i \nleq_0 f_j \) and \( f_j \nleq_1 f_i \); \( \delta \) is the natural transition function, that is, if \( x \) and \( x \cdot \sigma \cdot f \) are elements of \( \Pi_G \) then \( (x, \sigma, x \cdot \sigma \cdot f) \in \delta \); \( T_3 \) is the set of all \( f_0 \sigma_0 f_1 \ldots \sigma_{n-1} f_n \in \Pi_G \) such that for some \( 0 \leq i < n \) we have \( f_i \nleq_0 f_n \); and \( T_\psi \) is the set of all \( f_0 \sigma_0 f_1 \ldots \sigma_{n-1} f_n \in \Pi_G \) such that for some \( 0 \leq i < n \) we have \( f_n \nleq_1 f_i \) and \( f_i(q) \neq +\infty \) for some \( q \in \text{supp}(f_i) \).

Note that the directed graph defined by \( \Pi_G \) and \( \delta \) is a tree, but not necessarily finite. To gain an intuition about \( \Gamma_G \), let us say an abstract cycle \( \rho \) is good if there exists \( f_0 \sigma_0 \ldots f_n \in \text{supp}^{-1}(\rho) \) such that \( f_i(q) \neq +\infty \) for all \( q \) and all \( i \) and \( f_0 \nleq_0 f_n \). Let us say \( \rho \) is bad if there exists \( f_0 \sigma_0 \ldots f_n \in \text{supp}^{-1}(\rho) \) such that \( f_0(q) \neq +\infty \) for some \( q \in \text{supp}(f_0) \) and \( f_n \nleq_1 f_0 \). Then it is not difficult to see that \( \Gamma_G \) is essentially an abstract cycle-forming game played on \( G \) which is winning for Eve if a good abstract cycle is formed and winning for Adam if a bad abstract cycle is formed.

**Theorem 3.** Let \( G \) be an MPG with limited-observation and let \( \Gamma_G \) be the associated reachability game. If Adam (Eve) has a winning strategy in \( \Gamma_G \) then (s)he has a finite-memory observation-based winning strategy in \( G \).

The idea behind the strategy for the mean-payoff game is straightforward. If Eve wins the reachability game then she can transform her strategy into one that plays indefinitely by returning, whenever the play reaches \( T_3 \), to the natural previous position — namely the position that witnesses the membership of \( T_3 \). By continually playing her winning strategy in this way Eve perpetually completes good abstract cycles and this ensures that all concrete paths consistent with the play have non-negative mean-payoff value. Likewise if Adam has a winning strategy in the reachability game, he can continually play his strategy by returning to the natural position whenever the play reaches \( T_\psi \). By doing this he perpetually completes bad abstract cycles and this ensures that there is a concrete path consistent with the play that has strictly negative mean-payoff value. The finiteness of the size of the memory required for this strategy follows from the following result.

**Lemma 2.** If \( \lambda \) is a winning strategy for Adam or Eve in \( \Gamma_G \), then there exists \( N \in \mathbb{N} \) such that for all plays \( \pi \) consistent with \( \lambda \), \( |\pi| \leq N \).
Although the following results are not used until Section 7, they give an intuition toward the correctness of the strategies described above.

**Lemma 3.** Let \( \rho \) be an abstract cycle.

(i) If \( \rho \) is good (bad) then an interleaving of \( \rho \) with another good (bad) cycle is also good (bad).

(ii) If \( \rho \) is good then for all \( k \) and all concrete cycles \( \pi \in \gamma(\rho^k) \), \( w(\pi) \geq 0 \).

(iii) If \( \rho \) is bad then \( \exists k \geq 0, \pi \in \gamma(\rho^k) \) such that \( w(\pi) < 0 \).

**Corollary 2.** No cyclic permutation of a good abstract cycle is bad.

## 6 Forcibly Terminating Games

The reachability game defined in the previous section gives a sufficient condition for determining the winner in an MPG with limited-observation. However, as there may be plays where no player wins, such games are not necessarily determined. The first subclass of MPGs with limited-observation we investigate is the class of games where the associated reachability game is determined.

**Definition 1.** An MPG with limited-observation is **forcibly terminating** if in the corresponding reachability game \( \Gamma_G \) either Adam has a winning strategy to reach locations in \( T_A \) or Eve has a winning strategy to reach locations in \( T_E \).

It follows immediately from Theorem 3 that finite memory strategies suffice for both players in forcibly terminating games. Note that an upper bound on the memory required is the number of vertices in the reachability game restricted to a winning strategy, and this is exponential in \( N \), the bound obtained in Lemma 2.

**Theorem 4 (Finite-memory determinacy).** One player always has a winning observation-based strategy with finite memory in a forcibly terminating MPG.

We now consider the complexity of two natural decision problems associated with forcibly terminating games: the problem of recognizing if an MPG is forcibly terminating and the problem of determining the winner of a forcibly terminating game. Both results follow directly from the fact that we can accurately simulate a Turing Machine with an MPG with limited-observation.

**Theorem 5.** Let \( M \) be a Deterministic Turing Machine. Then there exists an MPG with limited-observation \( G \), constructible in polynomial time, such that Eve wins \( \Gamma_G \) if and only if \( M \) halts in the accept state and Adam wins \( \Gamma_G \) if and only if \( M \) halts in the reject state.

**Corollary 3 (Class membership).** Let \( G \) be an MPG with limited-observation. Determining if \( G \) is forcibly terminating is undecidable.

**Corollary 4 (Winner determination).** Let \( G \) be a forcibly terminating MPG. Determining if Eve wins \( G \) is \( \mathcal{R} \)-complete.

**Proof.** \( \mathcal{R} \)-hardness follows from Theorem 5. For decidability, Lemma 2 implies that an alternating Turing Machine simulating a play on \( \Gamma_G \) will terminate.
7 Forcibly First Abstract Cycle Games

In this section and the next we consider restrictions of forcibly terminating games in order to find subclasses with more efficient algorithmic bounds. The negative algorithmic results from the previous section largely arise from the fact that the abstract cycles required to determine the winner are not necessarily simple cycles. Our first restriction of forcibly terminating games is the restriction of the abstract cycle-forming game to simple cycles.

More precisely, let $G$ be an MPG with limited-observation and $\Gamma_G$ be the associated reachability game. Define $\Pi'_G \subseteq \Pi_G$ as the set of all sequences $x = f_0\sigma_0f_1\sigma_1 \ldots f_n \in \Pi_G$ such that $\text{supp}(f_i) \neq \text{supp}(f_j)$ for all $0 \leq i < j < n$ and denote by $\Gamma'_G$ the reachability game $\langle \Pi'_G, \Sigma, f_I, \delta', T_\exists, T_\forall \rangle$ where $\delta'$ is $\delta$ restricted to $\Pi'_G$, $T'_\exists = T_\exists \cap \Pi'_G$, and $T'_\forall = T_\forall \cap \Pi'_G$.

Definition 2. An MPG with limited-observation is forcibly first abstract cycle (or forcibly FAC) if in the associated reachability game $\Gamma'_G$, either Adam has a winning strategy to reach locations in $T'_\forall$ or Eve has a winning strategy to reach locations in $T'_\exists$.

One immediate consequence of the restriction to simple abstract cycles is that the bound in Lemma 2 is at most $|\text{Obs}|$. In particular an alternating Turing Machine can, in linear time, simulate a play of the reachability game and decide which player, if any, has a winning strategy. Hence the problems of deciding if a given MPG with partial-observation is forcibly FAC and deciding the winner of a forcibly FAC game are both solvable in PSPACE. The next results show that there is a matching lower bound for both these problems.

Theorem 6 (Class membership). Let $G$ be an MPG with limited-observation. Determining if $G$ is forcibly FAC is PSPACE-complete.

We sketch the argument, the full details of the proof are in Appendix F.

PSPACE-hardness follows from a reduction from the satisfiability of quantified boolean formulas. The construction is similar to the construction used to prove PSPACE-hardness for Generalized Geography in [19]. That is, the game proceeds through diamond gadgets – the choice of each player on which side to go through corresponds to the selection of the value for the quantified variable. The (abstract) play then passes through a gadget for the formula in the obvious way (Adam choosing for $\land$ and Eve choosing for $\lor$), returning to a diamond gadget when a variable is reached. If the variable has been seen before the cycle is closed and the game ends, otherwise the play proceeds to the bottom of the diamond gadget which has been seen before, thus ending the game one step later. We set up the concrete paths within the observations in such a way that if the cycle closes at the variable then it is good (and thus Eve wins) and if it closes at the bottom of the gadget then it is not good. Corollary 2 implies that the cycle closed is never bad, so either Eve wins and the game is forcibly FAC, or neither player wins and it is not forcibly FAC.

We can slightly modify the above construction in such a way that if the game does not finish when the play returns to a variable then Adam can close a bad cycle. This results in a forcibly FAC game that Eve wins if and only if the formula is satisfied. Hence,
Theorem 7 (Winner determination). Let $G$ be a forcibly FAC MPG. Determining if Eve wins $G$ is \text{PSPACE}-complete.

It also follows from the $|\text{Obs}|$ upper bound on plays in $\Gamma'_G$ that there is an exponential upper bound on the memory required for a winning strategy for either player. Furthermore, we can show this bound is tight – the games constructed in the proof of Theorem 7 can be used to show that there are forcibly FAC games that require exponential memory for winning strategies.

Theorem 8 (Exponential memory determinacy). One player always has a winning observation-based strategy with exponential memory in a forcibly FAC MPG. Further, for any $n \in \mathbb{N}$ there exists a forcibly FAC MPG, of size polynomial in $n$, such that any winning strategy has memory at least $2^n$.

8 First Abstract Cycle Games

We now consider a structural restriction that guarantees $\Gamma'_G$ is determined.

Definition 3. An MPG with limited-observation is a first abstract cycle game (FAC) if in the associated reachability game $\Gamma'_G$ all leaves are in $T'_G \cup \bar{T}_G$.

Intuitively, in an FAC game all simple abstract cycles (that can be formed) are either good or bad. It follows then from Corollary 2 that any cyclic permutation of a good cycle is also good and any cyclic permutation of a bad cycle is also bad. Together with Lemma 3, this implies the abstract cycle-forming games associated with FAC games can be seen to satisfy the following three assumptions: (1) A play stops as soon as an abstract cycle is formed, (2) The winning condition and its complement are preserved under cyclic permutations, and (3) The winning condition and its complement are preserved under interleavings. These assumptions correspond to the assumptions required in [1] for positional strategies to be sufficient for both players. That is,

Theorem 9 (Positional determinacy). One player always has a positional winning observation-based strategy in an FAC MPG.

As we can check in polynomial time if a positional strategy is winning in an FAC MPG, we immediately have:

Corollary 5 (Winner determination). Let $G$ be an FAC MPG. Determining if Eve wins $G$ is in $\text{NP} \cap \text{coNP}$.

A path in $\Gamma'_G$ to a leaf not in $T'_G \cup \bar{T}_G$ provides a short certificate to show that an MPG with limited-observation is not FAC. Thus deciding if an MPG is FAC is in coNP. A matching lower bound can be obtained using a reduction from the complement of the \textsc{Hamiltonian Cycle} problem. The full construction is shown in Appendix G.

Theorem 10 (Class membership). Let $G$ be an MPG with limited-observation. Determining if $G$ is FAC is coNP-complete.

\footnote{These conditions supercede those of [4] which were shown in [1] to be insufficient for positional strategies.}
9 MPGs with Partial-Observation

In the introduction it was mentioned that an MPG with partial-observation can be transformed into an MPG with limited-observation. This translation allows us to extend the notions of FAC and forcibly FAC games to the larger class of MPGs with partial-observation. In this section we will investigate the resulting algorithmic effect of this translation on the decision problems we have been considering.

The idea behind the translation is to take subsets of the observations and restrict transitions to those that satisfy the limited-observation requirements. The full details can be found in the appendix.

We say an MPG with partial-observation is \((forcibly) \text{ first belief cycle}, \) or FBC, if the corresponding MPG with limited-observation is \((forcibly) \text{ FAC}\).

9.1 FBC and Forcibly FBC MPGs

Our first observation is that FBC MPGs generalize the class of visible weight games studied in [12]. An MPG with partial-observation is considered a visible weights game if its weight function satisfies the condition that all \(\sigma\)-transitions between any pair of observations have the same weight. We base some of our results for FBC and forcibly FBC games on lower bounds established for problems on visible weights games.

**Lemma 4.** Let \(G\) be a visible weights MPG with partial-observation. Then \(G\) is FBC.

We now turn to the decision problems we have been investigating throughout the paper. Given the exponential blow-up in the construction of the game of limited-observation, it is not surprising that there is a corresponding exponential increase in the complexity of the class membership problem.

**Theorem 11 (Class membership).** Let \(G\) be an MPG with partial-observation. Determining if \(G\) is FBC is \(\text{coNEXP}\)-complete and determining if \(G\) is forcibly FBC is in \(\text{EXPSPACE}\) and \(\text{NEXP}\)-hard.

Somewhat surprisingly, for the winner determination problem we have an \(\text{EXP}\)-time algorithm to match the \(\text{EXP}\)-hardness lower bound from visible weights games. This is in contrast to the class membership problem in which an exponential increase in complexity occurs when moving from limited to partial-observation.

**Theorem 12 (Winner determination).** Let \(G\) be a forcibly FBC MPG. Determining if Eve wins \(G\) is \(\text{EXP}\)-complete.

**Corollary 6.** Let \(G\) be an FBC MPG. Determining if Eve wins \(G\) is \(\text{EXP}\)-complete.
References

1. B. Aminof and S. Rubin. First cycle games. In *Strategic Reasoning*, volume 146, pages 83–90, 2014.
2. D. Berwanger, K. Chatterjee, L. Doyen, T. A. Henzinger, and S. Raje. Strategy construction for parity games with imperfect information. In *CONCUR’08*, pages 325–339, 2008.
3. D. Berwanger and L. Doyen. On the power of imperfect information. In *FSTTCS’08*, pages 73–82, 2008.
4. H. Björklund, S. Sandberg, and S. Vorobyov. Memoryless determinacy of parity and mean payoff games: a simple proof. *TCS*, 310(1):365–378, 2004.
5. K. Chatterjee. Concurrent games with tail objectives. *TCS*, 388(1):181–198, 2007.
6. K. Chatterjee and L. Doyen. The complexity of partial-observation parity games. In *LPAR’10*, pages 1–14. Springer, 2010.
7. K. Chatterjee and L. Doyen. Partial-observation stochastic games: How to win when belief fails. In *LICS’12*, pages 175–184. IEEE Computer Society, 2012.
8. K. Chatterjee, L. Doyen, H. Edelsbrunner, T. A. Henzinger, and P. Rannou. Mean-payoff automaton expressions. In *CONCUR’10*, pages 269–283, 2010.
9. K. Chatterjee, L. Doyen, and T. A. Henzinger. Quantitative languages. In *CSL’08*, pages 385–400, 2008.
10. K. Chatterjee, L. Doyen, T. A. Henzinger, and J.-F. Raskin. Generalized mean-payoff and energy games. In *FSTTCS’10*, pages 505–516, 2010.
11. K. Chatterjee, L. Doyen, S. Nain, and M. Y. Vardi. The complexity of partial-observation stochastic parity games with finite-memory strategies. In *FoSSaCS’14*, pages 242–257, 2014.
12. A. Degorre, L. Doyen, R. Gentilini, J.-F. Raskin, and S. Toruńczyk. Energy and mean-payoff games with imperfect information. In *CSL’10*, pages 260–274, 2010.
13. A. Ehrenfeucht and J. Mycielski. Positional strategies for mean payoff games. *International Journal of Game Theory*, 8:109–113, 1979.
14. M. Jurdziński. Deciding the winner in parity games is in UP∩coUP. *Information Processing Letters*, 68(3):119–124, 1999.
15. A. S. Kechris. *Classical descriptive set theory*, volume 156. Springer-Verlag New York, 1995.
16. O. Kupferman and M. Y. Vardi. Synthesis with incomplete information. *Advances in Temporal Logic*, 16:109–127, 2000.
17. D. A. Martin and J. R. Steel. Projective determinacy. *Proceedings of the National Academy of Sciences of the United States of America*, 85(18):6582, 1988.
18. M. L. Minsky. *Computation: Finite and Infinite Machines*. Prentice-Hall, 1967.
19. C. H. Papadimitriou. *Computational complexity*. John Wiley and Sons Ltd., 2003.
20. C. H. Papadimitriou and M. Yannakakis. A note on succinct representations of graphs. *Information and Control*, 71(3):181 – 185, 1986.
21. D. Perrin and J.-E. Pin. *Infinite words: automata, semigroups, logic and games*, volume 141. Academic Press, 2004.
22. J. H. Reif. The complexity of two-player games of incomplete information. *Journal of Computer and System Sciences*, 29(2):274–301, 1984.
23. L. J. Stockmeyer and A. R. Meyer. Word problems requiring exponential time. In *ACM symposium on Theory of computing*, pages 1–9. ACM, 1973.
24. U. Zwick and M. Paterson. The complexity of mean payoff games on graphs. *TCS*, 158(1):343–359, 1996.
A Proofs from Section 3

Proposition 1. There exist MPGs with partial-observation for which Eve requires infinite memory observation-based strategies to ensure $MP \geq 0$.

Proof. Consider the game $G$ in Figure 2. We will show that Eve has an infinite memory observation-based strategy to win this game, but no finite memory observation-based strategy.

Consider the strategy that plays (regardless of location) $aba^2ba^3ba^4b \ldots$. As $b$ is played infinitely often in this strategy, the only concrete paths consistent with this strategy are $\pi = q_0q_1^0q_2^1q_3^2q_4^3q_5^4 \ldots$ for non-negative integers $k, l$. In the first case we see that $\frac{1}{n}w(\pi[..n]) \to 0$ as $n \to \infty$, and for all paths matching the second case we have $\frac{1}{n}w(\pi[..n]) \to 1$ as $n \to \infty$. Thus $MP \geq 0$ and so the strategy is winning.

Now suppose Eve has a finite memory observation-based winning strategy for $G$. Consider the observation-based strategy of Adam that ensures the game remains in $\{q_1, q_2\}$. The resulting play can now be seen as choosing a word $w \in \{a, b\}^\omega$, but as Eve’s strategy has finite memory, this word must be ultimately periodic, that is $w = w_0v^\omega$ for words $w_0, v \in \{a, b\}^*$. But then Adam has a concrete winning strategy as follows. If $w$ contains finitely many $b$’s then Adam moves to $q_2$ on the final $b$ and $\frac{1}{n}w(\pi[..n]) \to -1$ as $n \to \infty$. Otherwise Adam remains in $q_1$ and $\frac{1}{n}w(\pi[..n]) \to -\frac{m}{|v|}$ as $n \to \infty$ where $m$ is the number of $b$’s in $v$.

B Proof of Observable determinacy

Theorem 2 (Observable determinacy). Assuming the existence of a measurable cardinal, one player always has a winning observation-based strategy in an MPG with limited-observation.

To prove this we recall the definition of Borel and Suslin sets. For a detailed description of both hierarchies and their properties we refer the reader to [15].

Definition 4 (Borel and Projective hierarchies.). For a (possibly infinite) alphabet $A$, let $A^\omega$ and $A^*$ denote the set of infinite and finite words on $A$, respectively. The Borel hierarchy is inductively defined as follows. $\Sigma^0_1 = \{W \cdot A^\omega \mid W \subseteq A^*\}$ is the set of open sets. For all $n \geq 1$, $\Pi^0_n = \{A^\omega \setminus L \mid L \in \Sigma^0_n\}$ consists of the complement of sets in $\Sigma^0_n$. For all $n \geq 1$, $\Sigma^0_{n+1} = \{\bigcup_{i \in \mathbb{N}} L_i \mid \forall i \in \mathbb{N} : L_i \in \Pi^0_n\}$ is the set obtained by countable unions of sets in $\Pi^0_n$.

The first level of the Projective hierarchy consists of $\Sigma^1_1$ (Suslin) sets, which are those whose preimage is a Borel set, i.e. all sets that can be defined as a projection of a Borel set, and $\Pi^1_1$ (co-Suslin) sets: those sets whose complement is the image of a Borel set.

The existence of a measurable cardinal implies $\Sigma^1_1$-Determinacy [12] – a weak form of the “Axiom of Determinacy”. This in turn implies Gale-Stewart games defined by Suslin or co-Suslin sets are determined [15]. The observable determinacy of MPGs with incomplete information then follows from the following result:
Lemma 1. The set of plays that are winning for Eve in an MPG with limited-observation is co-Suslin.

Proof. Let $G$ be a MPG with incomplete information and $W'$ be the set of all concrete plays in $G$ for which Eve wins, namely $W' = \{ \pi \in \gamma(\text{Plays}(G)) \mid MP(\pi) \geq 0 \}$, and $\overline{W'}$ its complement. Note that $W' \subseteq \gamma(\text{Plays}(G))$ is the payoff-set defined by the MPG winning condition ($MP \geq 0$) and is therefore in the class $\Pi_0^3$ of Borel sets [5].

Let $W$ be the set of all abstract plays such that Eve wins. Formally, we have $W = \{ \psi \in \text{Plays}(G) \mid \forall \pi \in \gamma(\psi) : \pi \in W' \}$.

To show that $W$ is in $\Pi_1^1$, we adapt the proof from [21] (to prove that an infinite tree is co-Suslin) and consider the set $U = \{ (\psi, \pi) \in \text{Plays}(G) \times \overline{W'} \mid \exists \pi' \in \gamma(\psi) : \pi' = \pi \}$ which is in the class $\Pi_0^2$ of the Borel hierarchy. To demonstrate this, let $U_n$ be the set of pairs $(\psi, \pi)$ satisfying the following property: there exists $\pi' \in \gamma(\psi)$ such that $\pi'[:n] = \pi[:n]$, where $n \in \mathbb{N}$. Then $U = \bigcap_{n \geq 0} U_n$, which proves $U$ is in the class $\Pi_0^2$ since the sets $U_n$ are open ($\Sigma_0^1$). Finally, we observe the projection of $U$ on its first component is the complement of $W$, which is thus co-Suslin.

C Proofs from Section 5

The following definitions will be used throughout this section.

Definition 5 (Proper $\sigma$-successor). A emphproper $\sigma$-successor of a function $f$ is a $\preceq_0$-minimal $\sigma$-successor of $f$.

Note that $\sigma$-successors that are not proper are defined to be $+\infty$ somewhere.

We observed earlier that for an abstract play $\psi = o_0o_0o_1\ldots$ there may be many function-action sequences in $\text{supp}^{-1}(\psi)$, however we observe that for every $f$ with $\text{supp}(f) = o_0$ there is a unique (pointwise) $\preceq_0$-minimal sequence in $\text{supp}^{-1}(\psi)$ starting at $f$ obtained by taking proper successors with appropriate supports. We denote this sequence by $\xi(\psi, f)$.

For convenience given a finite function-action sequence $\rho = f_0o_0\ldots f_n$, let $f_\rho$ denote $f_n$.

We will repeatedly use the next result which follows by induction immediately from the definition of a $\sigma$-successor.

Lemma 5. Let $\rho = f_0o_0\ldots f_n \in \mathfrak{S}_G$ be a sequence such that $f_{i+1}$ is a proper $\sigma_i$-successor of $f_i$. Then for all $q \in \text{supp}(f_n)$

$$f_n(q) = \min\{ f_0(\pi[0]) + w(\pi) \mid \pi \in \gamma(\text{supp}(\rho)) \text{ and } \pi[n] = q \}.$$

The following simple facts about $\preceq_\sigma$ will also be useful:

Lemma 6. For any $f_1, f_2 \in \mathcal{F}$ with $f_1 \preceq_k f_2$: 
(i) For all $k' \leq k$, $f_1 \preceq_{k'} f_2$.
(ii) For all $k' \geq 0$, if $f_2 \preceq_{k'} f_3$ for some $f_3 \in F$ then $f_1 \preceq_{k + k'} f_3$, and
(iii) If $f_1'$ is a proper $\sigma$-successor of $f_1$ and $f_2'$ is a $\sigma$-successor of $f_2$ with $\text{supp}(f_2') = \text{supp}(f_1')$, then $f_1' \preceq_k f_2'$.

Proof. (i) and (ii) are trivial. For (iii), let $d_{i,j} = w(q_i, \sigma, q_j)$ for $q_i \in \text{supp}(f_1)$ and $q_j \in \text{supp}(f_2')$ where such a transition exists and $+\infty$ otherwise. We now observe that as $f_1'$ is $\preceq_0$-minimal, $f_1'(q_j)$ can be defined as $\min\{f_1(q_i) + d_{i,j} | q_i \in \text{supp}(f_1)\}$ for all $q_j \in \text{supp}(f_2')$. As $f_1(q_i) \leq f_2(q_i) - k$ for any $q_i \in \text{supp}(f_1)$, it follows that

$$f_1'(q_j) \leq \min\{f_2(q_i) + d_{i,j} | q_i \in \text{supp}(f_1)\} - k \leq f_2'(q_j) - k,$$

where the second inequality follows from the definition of a $\sigma$-successor. Thus $f_1' \preceq_k f_2'$.

Lemma 2. If $\lambda$ is a winning strategy for Adam or Eve in $\Gamma_\lambda$, then there exists $N \in \mathbb{N}$ such that for all plays $\pi$ consistent with $\lambda$, $|\pi| \leq N$.

Proof. Let $\Gamma_\lambda$ be the restriction of $\Gamma_G$ to plays that are consistent with $\lambda$. Suppose there is no bound on the length of paths in $\Gamma_\lambda$. As $\Gamma_G$, and hence $\Gamma_\lambda$, is acyclic, it follows that $\Gamma_\lambda$ contains infinitely many states. However, as $\Gamma_G$ is finitely-branching, it follows from König’s lemma that there exists an infinite path in $\Gamma_\lambda$. As this path is not winning for either player and it is consistent with $\lambda$, this contradicts the fact that $\lambda$ is a winning strategy.

Lemma 3. Let $\rho$ be an abstract cycle.

(i) If $\rho$ is good (bad) then an interleaving of $\rho$ with another good (bad) cycle is also good (bad).
(ii) If $\rho$ is good then for all $k$ and all concrete cycles $\pi \in \gamma(\rho^k)$, $w(\pi) \geq 0$.
(iii) If $\rho$ is bad then $\exists k \geq 0, \pi \in \gamma(\rho^k)$ such that $w(\pi) < 0$.

Proof. (i) This follows from Lemma 6.

(ii) Let $f_0 \sigma_0 \ldots f_n \in \text{supp}^{-1}(\rho)$ be such that $f_i(q) \neq +\infty$ for all $i$ and $q$ and $f_0 \preceq_0 f_n$. In particular this means that $f_{i+1}$ is a proper $\sigma_i$-successor of $f_i$. Now fix $k$ and let $\chi \in \gamma(\rho^k)$ be a concrete cycle. From Lemma 6 we have, for all $0 \leq i < k$,

$$w(\chi[ni..n(i+1)]) \geq f_n(\chi[n(i+1)]) - f_0(\chi[ni])$$

and

$$f_n(\chi[n(i+1)]) - f_0(\chi[ni]) \geq f_0(\chi[n(i+1)]) - f_0(\chi[ni]).$$

Hence

$$w(\chi) = \sum_{i=1}^{k} w(\chi[ni..n(i+1)]) \geq f_0(\chi[nk]) - f_0(\chi[0]) = 0.$$

(iii) Let $f_0 \sigma_0 \ldots f_n \in \text{supp}^{-1}(\rho)$ and $q_0 \in \text{supp}(f_0)$ be such that $f_0(q_0) \neq +\infty$ and $f_n \preceq_1 f_0$. It follows that $f_n(q_0) < +\infty$. From the definition of a $\sigma$-successor, it follows that there exists $r \in \text{supp}(f_{n-1})$ such that $f_{n-1}(r) < +\infty$, and there is
an edge from \( r \) to \( q_0 \) with weight \( f_n(q_0) - f_{n-1}(r) \). Proceeding this way inductively we find there is a \( q_1 \in \text{supp}(f_0) \) with \( f_0(q_1) < +\infty \) and a concrete path \( \pi_0 \in \gamma(\rho) \) from \( q_1 \) to \( q_0 \) with \( w(\pi_0) = f_n(q_0) - f_0(q_1) \). As \( f_0(q_1) < +\infty \) and \( f_n \geq 1 \) \( f_0 \) we have \( f_n(q_1) \leq f_0(q_1) - 1 < +\infty \). Repeating the argument yields a sequence of states \( q_0, q_1, \ldots \) such that there is a concrete path \( \pi_i \in \gamma(\rho) \) from \( q_{i+1} \) to \( q_i \) with

\[
w(\pi_i) = f_n(q_i) - f_0(q_{i+1}) \leq f_0(q_i) - f_0(q_{i+1}) - 1.
\]

As \( Q \) is finite it follows that there exists \( i < j \) such that \( q_i = q_j \). Then the concrete path \( \pi = \pi_j \cdot \pi_{j-1} \cdots \pi_{i+1} \in \gamma(\rho^{j-i}) \) is a concrete cycle with

\[
w(\pi) = \sum_{k=i+1}^{j} w(\pi_k) \leq f_0(q_i) - f_0(q_j) - (j - i) < 0.
\]

### C.1 Proof of Theorem 3

We note that as a play prefix in \( G \) is completely described by the last vertex in the sequence we observe that it suffices to consider positional strategies for both players.

**Strategy transfer for Eve.** Let us first assume that Eve has a (positional) winning strategy in \( G \). Let \( \gamma : \Pi \to \Sigma \) be a strategy for both players. We will show shortly that \( \gamma \) is a winning strategy for Eve in \( G \).

We will define a strategy with memory \( |M| \), \( \gamma^* \), for Eve in \( G \). Given a memory state \( \mu = f_0 \sigma_0 f_1 \cdots \sigma_{n-1} f_n \in M \) let

\[
\mu' = \begin{cases} 
\text{the proper prefix of } \mu \text{ such that } f_{\mu'} \preceq_0 f_\mu & \text{if } \mu \in T_3 \\
\mu & \text{otherwise.}
\end{cases}
\]

The initial memory state is \( \mu_0 := f_I \). We define the output function \( \alpha_o : M \times \text{Obs} \to \Sigma \) as \( \alpha_o(\mu, o) = \lambda(\mu') \). Finally we define the update function \( \alpha_u : M \times \text{Obs} \to M \) as \( \alpha_u(\mu, o) = \mu' \cdot \lambda(\mu') : f \) where \( f \) is the proper \( \lambda(\mu') \)-successor of \( f_{\mu'} \) with \( \text{supp}(f) = o \). Observe that we maintain the invariant that the current observation is \( \text{supp}(f_\mu) \), consequently the \( \text{Obs} \) input to \( \alpha_o \) is not used.

We will show shortly that \( \gamma^* \) is a winning strategy for Eve in \( G \). First we require some definitions and a result about finite prefixes of plays consistent with \( \gamma^* \). Given a play \( \rho = o_0 \sigma_0 o_1 \cdots \) consistent with \( \gamma^* \), let \( \mu^i_\rho \) denote the \( i \)-th memory state reached in the generation of \( \rho \). That is, \( \mu^i_\rho = \mu_i \) and \( \mu^i_{\rho+1} = \alpha_u(\mu^i_\rho, \sigma_i) \); so \( \alpha_o(\mu^i_\rho, o_i) = \sigma_i \).

Recall the definition of \( \xi(\cdot, \cdot) \) from the start of the section. For convenience, let \( \xi^i_\rho = \xi(\rho[\cdot, i], f_I) \).

**Lemma 7.** Let \( \rho \in \text{plays}(G) \) be a play consistent with \( \gamma^* \). Then for all \( i \), \( f_{\mu_i} \preceq_0 f_{\xi_i} \).

**Proof.** We prove this by induction. Let \( \rho = o_0 \sigma_0 o_1 \cdots \). For \( i = 0 \) we have

\[
f_{\mu^0_\rho} = f_{\mu_0} = f_I = f_{\xi(\rho_0, f_I)} = f_{\xi^0_\rho}.
\]
Now suppose \( f_{\mu'} \preceq f_{\xi^0 \cdot \mu'} \). Let \( f' = f_{\xi^0+i} \), so \( f' \) is the proper \( \sigma_{\tau} \)-successor of \( f_{\xi^0} \) with \( \text{supp}(f') = o_{i+1} \). Assume first that \( \mu_i^\alpha \notin T_\beta \). Then \( \mu_{i+1}^\alpha = \alpha_u(\mu_i^\alpha, o_i) = \mu_i^\alpha \cdot \sigma_i \cdot f \), where \( f \) is the proper \( \sigma_{\tau} \)-successor of \( f_{\mu_i^\alpha} \) with \( \text{supp}(f) = o_{i+1} \). Then, by Lemma 6 (iii) we have \( f_{\mu_{i+1}^\alpha} = f \preceq f' \).

Now assume \( \mu_i^\alpha \in T_\beta \), and let \( \mu' \) denote the proper prefix of \( \mu_i^\alpha \) such that \( f_{\mu'} \preceq f_{\mu_i^\alpha} \). Then \( \mu_{i+1}^\alpha = \alpha_u(\mu_i^\alpha, o_i) = \mu' \cdot \sigma_i \cdot f \) where \( f \) is the proper \( \sigma_{\tau} \)-successor of \( f_{\mu'} \) with \( \text{supp}(f) = o_{i+1} \). From Lemma 6 (ii) we have \( f_{\mu'} \preceq f_{\xi^0} \), so by Lemma 6 (iii) we have \( f_{\mu_{i+1}^\alpha} = f \preceq f' \) as required.

**Lemma 8.** Let \( G \) be a mean-payoff game with limited-observation and let \( \Gamma_G \) be the associated reachability game. If Eve has a winning strategy in \( \Gamma_G \) then she has a finite memory winning strategy in \( G \).

**Proof.** We will show that \( \lambda^* \) described above is a winning strategy for Eve. Let \( \rho = o_0, o_1, \ldots \in \text{Players}(G) \) be any play consistent with \( \lambda^* \). We will show that there exists a constant \( \beta \in \mathbb{R} \) such that for all concrete paths \( \pi \in \gamma(\rho) \) and all \( n \geq 0 \), \( w(\pi[n..n]) \geq \beta \). It follows that \( MP(\pi) \geq 0 \), and so \( \rho \) is winning for Eve.

Let \( W = \{ f_{\mu}(q) \mid \mu \in M, q \in Q, \text{ and } f_{\mu}(q) \neq \bot \} \). Note that \( W \) is finite because \( M \) and \( Q \) are finite, and non-empty because \( f_{\mu_0}(q_1) = 0 \in W \). Let \( \beta = \min W \). As \( 0 \in W \), \( \beta < +\infty \).

As with Lemma 7 let \( \xi^0_{\pi} = \xi(\rho[n..n], f_{\mu}) \). As \( \text{supp}(f_{\mu}) = \{ q_1 \} \) and \( f_{\mu}(q_1) = 0 \), Lemma 5 implies for all \( q \in \text{supp}(f_{\xi^0_{\pi}}), f_{\xi^0_{\pi}}(q) \neq +\infty \). Hence, for all concrete paths \( \pi \in \gamma(\rho) \) we have:

\[
\begin{align*}
 w(\pi[n..n]) &\geq f_{\xi^0_{\pi}}(\pi[n]) - f_{\xi^0_{\pi}}(\pi[0]) \quad \text{from Lemma 5} \\
 &= f_{\xi^0_{\pi}}(\pi[0]) \\
 &\geq f_{\xi^0_{\pi}}(\pi[n]) \quad \text{from Lemma 7} \\
 &\geq \beta
\end{align*}
\]

as required.

**Strategy transfer for Adam.** To complete the proof of Theorem 3 we now show how to transfer a winning strategy for Adam in \( \Gamma_G \) to a winning strategy in \( G \). So let us assume \( \lambda : \Pi_G \times \Sigma \to \Pi_G \) is a (positional) winning strategy for Adam in \( \Gamma_G \). The finite-memory based strategy for Adam is similar to that for Eve in that it perpetually plays \( \lambda \), returning to a previous position whenever the play reaches \( T_\gamma \). However, the proof of correctness is more intricate because we need to handle the \(+\infty\) function values.

Formally, the finite-memory strategy \( \lambda^* \) is given as follows. As before, let \( M = \Pi'_{\lambda^*} \) and \( \mu_0 = f_{\pi} \). Given \( \mu \in M \), let

\[
\mu' = \begin{cases} 
\text{the proper prefix of } \mu \text{ such that } f_{\mu} \preceq f_{\mu'} \text{ if } \mu \in T_\gamma \\
\mu \text{ otherwise.}
\end{cases}
\]

The output function \( \alpha_o : M \times \text{Obs} \times \Sigma \to \text{Obs} \) is defined as: \( \alpha_o(\mu, o, \sigma) = \text{supp}(\lambda(\mu', \sigma)) \).

The update function \( \alpha_u : M \times \text{Obs} \times \Sigma \to M \) is defined as: \( \alpha_u(\mu, o, \sigma) = \mu' \cdot \sigma \cdot \lambda(\mu', \sigma) \). Note that as the current observation is stored in the memory state, the Obs input to \( \alpha_o \) and \( \alpha_u \) is obsolete.
To show that $\lambda^*$ is winning for Adam in $G$ we require an analogue to Lemma 7. Given a play $\rho = o_0\sigma_0\ldots$ in $G$ consistent with $\lambda^*$, let $\mu_i^r$ be the $i$-th memory state reached in the generation of $\rho$. That is, $\mu_i^0 = \mu_0$ and $\mu_{i+1}^r = \alpha(o_i(\mu_i^r, \sigma_i), \alpha)$, so $\alpha(o_i(\mu_i^r, \sigma_i)) = o_{i+1}$. Let $r_{i}^0$ denote the number of times the memory is “reset” in the first $i$ steps. That is, $r_{i}^0 = 0$, and if $\mu_i^r \in T_r$ then $r_{i+1}^0 = 1 + r_{i}^r$, otherwise $r_{i+1}^0 = r_{i}^r$. Rather than relate $f_{\mu_i^r}$ with functions in $\xi(\rho, f_i)$, we need to consider more general sequences which are $\preceq_{1}$-minimal after some point. Let us say a function-action sequence $f_0\sigma_0f_1\ldots \in \Gamma_G$ is ultimately proper from $k$ if for all $i \geq k$, $f_{i+1}$ is a proper $\sigma_i$-successor of $f_i$.

**Lemma 9.** Let $\rho \in \text{Plays}(G)$ be a play consistent with $\lambda^*$ and let $\zeta = z_0\sigma_0z_1\ldots \in \text{supp}^{-1}(\rho)$ be ultimately proper from $k$. If $z_k \preceq_r f_{\mu_k^r}$ for some $r$, then for all $i \geq k$, $z_i \preceq_{r_i} f_{\mu_i^r}$ where $r_i = r + r_{i}^0 - r_{i}^k$.

**Proof.** We prove this by induction on $i$. For $i = k$ the result clearly holds. Now suppose $i \geq k$ and $z_i \preceq_{r_i} f_{\mu_i^r}$ where $r_i = r + r_{i}^0 - r_{i}^k$. We consider two cases depending on whether $\mu_i^r \in T_r$. If $\mu_i^r \notin T_r$ then $\mu_i^r = \mu_{i+1}^r \sigma_i \cdot f$ where $f$ is a $\sigma_i$-successor of $f_{\mu_i^r}$ with $\text{supp}(f) = o_{i+1}$, and $r_{i+1}^0 = r_{i}^0$. Then, by Lemma 6 (iii) we have $z_{i+1} \preceq_{r_i} f = f_{\mu_{i+1}^r}$ and $r_i + 1 = r + r_{i}^0 - r_{i}^k + 1 = r + r_{i}^0 - r_{k}^0 = r_{i+1}^0$ as required.

Otherwise if $\mu_i^r \in T_r$, let $\mu'$ be the proper prefix of $\mu_i^r$ such that $f_{\mu_i^r} \preceq_1 f_{\mu'}$. From Lemma 6 (ii) we have $z_i \preceq_{r_i+1} f_{\mu_i^r}$. We also have $\mu_{i+1}^r = \mu' \cdot \sigma_{i+1} \cdot f$ where $f$ is a $\sigma_{i+1}$-successor of $f_{\mu_i^r}$ with $\text{supp}(f) = o_{i+1}$. So by Lemma 6 (iii) we have $z_{i+1} \preceq_{r_i+1} f = f_{\mu_{i+1}^r}$, and as $r_i + 1 = r + r_{i}^0 - r_{k}^0 + 1 = r + r_{i}^0 - r_{k}^0 = r_{i+1}^0$ the result holds for $i + 1$.

**Lemma 10.** Let $G$ be a mean-payoff game with limited-observation and let $\Gamma_G$ be the associated reachability game. If Adam has a winning strategy in $\Gamma_G$ then he has a finite memory winning strategy in $G$.

**Proof.** We will show that the strategy $\lambda^*$ constructed above is winning for Adam. Let $\rho = o_0\sigma_0\ldots$ be any play consistent with $\lambda^*$. As $M$ is finite, there exists $\mu \in M$ and an infinite set $\mathcal{I} \subseteq \mathbb{N}$ of indices such that for all $i \in \mathcal{I}$, $\mu_i^r = \mu$. We will show that this implies there exists $\pi \in \gamma(\rho)$ such that $\mathcal{M}(\pi) < 0$. As $\mathcal{M}(\pi) \geq \mathcal{M}(\pi)$ the result follows. For convenience, given $n \in \mathbb{N}$, let $s_{\mathcal{I}}(n) = \min\{n \in \mathcal{I} \mid i > n\}$. Also, let $o = \{q \in \text{supp}(f_\rho) \mid f_\rho(q) \neq +\infty\}$. Note that from the definition of $\mathcal{T}_r$ it follows that $o$ is non-empty.

We will use a function-action sequence to find a concrete path where the weights of the prefixes can be identified and seen to be strictly decreasing. In Lemma 6 the sequence $\xi(\rho, f_i)$ fulfilled this role. However, to handle $+\infty$ values, which correspond to irrelevant paths, here we require a sequence more complex than $\xi(\rho, f_i)$. The sequence we construct will be piecewise proper in the sense that for all $i \in \mathcal{I}$ the sequence will consist of proper successors in the interval $[i, s_{\mathcal{I}}(i))$. When the sequence reaches an element of $\mathcal{I}$ we “reset” the values of the vertices not in $o$ to $+\infty$. More formally, the required sequence, $\zeta = z_0\sigma_0\ldots \in \text{supp}^{-1}(\rho)$, is constructed inductively as follows. Initially, let $z_0 = z_0' = f_j$. For $i \geq 0$, let $z_{i+1}'$ be the proper $\sigma_i$-successor of $z_i$ with $\text{supp}(z_{i+1}) = o_{i+1}$. If $i \notin \mathcal{I}$ then $z_i = z_i'$. Otherwise,

$$ z_i(q) = \begin{cases} +\infty & \text{if } q \notin o \\ z_i'(q) & \text{otherwise.} \end{cases} $$
We claim that for all \( i \in \mathbb{N} \): \( z_i \preceq_{r^\mu} f_{\mu^i} \). From Lemma 5 it follows that we only need to show that for all \( i \in \mathcal{I} \): \( z_i \preceq_{r^\mu} f_{\mu} \). Induction and Lemma 9 imply that for all \( i \in \mathcal{I} \) we have \( z_i' \preceq_{r^\mu} f_{\mu} \). As \( z_i \) differs from \( z_i' \) only on states where \( f_{\mu} \) is equal to \(+\infty\), we therefore have \( z_i' \preceq_{r^\mu} f_{\mu} \) as required.

We will now show that there is an infinite concrete path \( q_0\sigma_0 \ldots \) consistent with \( \rho \) such that \( q_i \in o \) for all \( i \in \mathcal{I} \). To do this we will show for any \( i \in \mathcal{I} \) and any \( q \in o \) there is a concrete path, consistent with \( \rho[i..s_\mathcal{I}(i)] \), that ends in \( q \) and starts at some state in \( o \). The result then follows by induction. Let us fix \( i \in \mathcal{I} \), \( q \in o \), and let \( j = s_\mathcal{I}(i) \).

As \( z_j'(q) \neq +\infty \), we have that \( z_j(q) \neq +\infty \). From Lemma 5 there is a concrete path \( \pi = q_0\sigma_0 \ldots q_n \) from \( q_0 \in o_i \) ending at \( q_n = q \) such that \( z_j'(q) = z_j(q_0) + \omega(\pi) \).

As \( z_j'(q) \neq +\infty \) it follows that \( z_i(q_0) \neq +\infty \), and as \( z_i(q_0) = +\infty \) if and only if \( f_{\mu}(q_0) = +\infty \), it follows that \( q_0 \in o \). Note that Lemma 5 implies for all \( k \leq |\pi| \):

\[
\omega(\pi[k]) = z_{i+k}(q_k) - z_i(q_0) = z_{i+k}(q_k) - z_i(q_0).
\]

In particular \( \omega(\pi) = z_j(q) - z_i(q_0) \).

Now let \( \pi = q_0\sigma_0 \ldots \) be the infinite path implied by the above construction and for convenience for \( i \in \mathcal{I} \) let \( \pi_i = q_0\sigma_i \ldots q_j \) where \( j = s_\mathcal{I}(i) \). To show \( \overline{\mathcal{MP}}(\pi) < 0 \) we need to show \( \limsup_{n \to \infty} \frac{1}{n} \omega(\pi[i..n]) < 0 \). To prove this, we will show there exists a constant \( \beta < 0 \) such that for all sufficiently large \( n \): \( \omega(\pi[i..n]) \leq \beta \cdot n \).

For convenience, let \( i_0 = \min(\mathcal{I}) \) and let \( i_\infty = \max\{ i \in \mathcal{I} \mid i \leq n \} \). From Lemma 5 and the construction of \( \pi_i \) we have for all \( n \):

\[
w(\pi[i..n]) = w(\pi[i..i_0]) + w(\pi[i_0..n]) + \sum_{i < n} w(\pi_i)
\]

\[
= w(\pi[i..i_0]) + (z_n(q_n) - z_{i_\infty}(q_{i_\infty}))
\]

\[
+ \sum_{i < n} \left( z_{s_\mathcal{I}(i)}(q_{s_\mathcal{I}(i)}) - z_i(q_i) \right)
\]

\[
= w(\pi[i..i_0]) + z_n(q_n) - z_{i_\infty}(q_{i_\infty})
\]

\[
\leq w(\pi[i..i_0]) + f_{\mu^i}(q_{i_0}) - r_{i_0}^\mu - z_{i_\infty}(q_{i_\infty})
\]

There are only finitely many values for \( f_{\mu^i}(q_{i_0}) \) and from Lemma 2 \( r_{i_0}^\mu \geq \lceil \frac{\alpha}{\beta'} \rceil \). Hence

\[
w(\pi[i..n]) \leq \alpha - \beta' \cdot n
\]

for constants \( \alpha \) and \( \beta' > 0 \). Thus there exists \( \beta < 0 \) such that for sufficiently large \( n \) we have \( w(\pi[i..n]) \leq \beta \cdot n \). Hence \( \overline{\mathcal{MP}}(\pi) \leq \overline{\mathcal{MP}}(\pi) < 0 \).

D Proof of Theorem 5

**Theorem 5.** Let \( M \) be a Deterministic Turing Machine. Then there exists an MPG with limited-observation \( \mathcal{G} \), constructible in polynomial time, such that Eve wins \( \Gamma_{\mathcal{G}} \) if and only if \( M \) halts in the accept state and Adam wins \( \Gamma_{\mathcal{G}} \) if and only if \( M \) halts in the reject state.
Proof. We will in fact show how to simulate a (deterministic) four-counter machine (4CM). The standard reduction from Turing Machines to 4CMs, via finite state machines with two stacks (see e.g. [18]), is readily seen to be constructible in polynomial time.

A Counter Machine (CM) $M$ consists of a finite set of control states $S$, an initial state $s_I \in S$, a final accepting state $s_A \in S$, a final rejecting state $s_R$, a set $C$ of integer-valued counters and a finite set $\delta_M$ of instructions manipulating the counters. $\delta_M$ contains tuples $(s, \text{instr}, c, s')$ where $s, s' \in S$ are source and target states respectively, the action $\text{instr} \in \{\text{INC}, \text{DEC}\}$ applies to counter $c \in C$. It also contains tuples of the form $(s, 0\text{CHK}, c, s', s^0)$ where $s', s^0$ are two target states, one of which will be chosen depending on the value of counter $c$ at the moment the instruction is “executed”. Without loss of generality we may assume $M$ is deterministic in the sense that for every state $s \in S$ there is exactly one instruction of the form $(s, 0\text{CHK}, \cdot, \cdot, \cdot)$ in $\delta_M$ or one of the form $(s, \cdot, \cdot, \cdot, \cdot)$. We also assume that DEC instructions are always preceded by 0CHK instructions so that counter values never go below 0.

A configuration of $M$ is a pair $(s, v)$ where $s \in S$ and $v : C \to \mathbb{N}$ is a valuation of the counters. A valid run of $M$ is a finite sequence $(s_0, v_0)\delta_0(s_1, v_1)\delta_1 \ldots \delta_{n-1}(s_n, v_n)$ where $\delta_i \in \delta_M$ is either $(s_i, \text{instr}_i, c_i, s'_i)$ or $(s_i, \text{instr}_i, c_i, s'_i, s^0_i)$ and $(s_i, v_i)$ are configurations of $M$ such that $s_0 = s_I$, $v_0(c) = 0$ for all $c \in C$, and for all $0 \leq i < n$ we have that:

- $v_{i+1}(c) = v_i(c)$ for $c \in C \setminus \{c_i\}$;
- if $\text{instr}_i = \text{INC}$ then $v_{i+1}(c_i) = v_i(c_i) + 1$ and $s_{i+1} = s'_i$;
- if $\text{instr}_i = \text{DEC}$ then $v_{i+1}(c_i) = v_i(c_i) - 1$ and $s_{i+1} = s'_i$;
- if $\text{instr}_i = 0\text{CHK}$ then $v_{i+1}(c_i) = v_i(c_i)$ and if $v_i(c_i) = 0$ we have $s_{i+1} = s^0_i$.

The run is accepting if $s_n = s_A$ and it is rejecting if $s_n = s_R$.

Given a 4CM $M$, we now show how to construct an MPG with limited-observation $G_M$ in which Eve wins $\Gamma_{G_M}$ if and only if $M$ has an accepting run, and Adam wins $\Gamma_{G_M}$ if and only if $M$ has a rejecting run. Plays in $G_M$ correspond to executions of $M$.

As we will see, the tricky part is to make sure that “zero check” instructions are faithfully simulated by one of the players. Initially, both players will be allowed to declare how many instructions the machine needs to execute in order to reach an accepting or rejecting state. Either player can bail out of this initial “pumping phase” and become the Simulator. The Simulator is then responsible for the faithful simulation of $M$ and the opponent will be monitoring the simulation and punish him if the simulation is not executed correctly. Let us now go into the details.

First, the control structure of the machine $M$ is encoded in the observations of our game, i.e. to each location of the machine, there will correspond at most three observations in the game. We require two copies of each such observation since, in order to punish Adam or Eve (whoever plays the role of Simulator), existential and universal gadgets have to be set up in a different manner. For technical reasons that will be made clear below, we also need two additional observations. Formally, the observations in our game contain observations $\{b^+, b^0, b^-\}$, $\{a^+, a^-\}$ and $\{q_I\}$, which do not correspond to instructions from the 4CM but they are used in gadgets that will make sure that zero test are faithfully executed.
Second, the values of counters will be encoded using the weights of traversed edges that reach designated states. We will associate to each observation (so to each location in the 4CM) two states for each counter: $c_i^+$ and $c_i^-$, $i \in \{1, 2, 3, 4\}$. If the value of counter $c_i$ along an execution simulated by an abstract path with concrete paths $\Psi_c$ is $v$, then the payoff over all the concrete paths that reach $c_i^+$ divided by the number of concrete paths that reach $c_i^+$ is equal to $v$, and for those that reach $c_i^-$ this value is $-v$.

More formally, let $\Psi_{c_i}$ denote the set of concrete paths consistent with an abstract path $\psi$ ending in $c_i^+$, then the value of counter $c_i$ is given by the following invariant:

$$v(c_i) = \frac{1}{|\Psi_{c_i}|} \sum_{\pi \in \Psi_{c_i}} w(\pi)$$

(1)

$G_M$ starts in $\{\{q_I\}\}$ and $\Delta$ contains $\sigma$-transitions (for all $\sigma \in \Sigma'$) from $q_I$ to $b^+, b^0, b^-$. This observation represents the pumping phase of the simulation. From here each player will be allowed to declare how many steps they require to reach a halting state that will accept or reject. If Adam bails, we go to the initial instruction of $M$ on the universal side of the construction ($s_{\forall I}$), if Eve does so then we go to the analogue in the existential side ($s_{\exists I}$). $\Sigma'$ contains a symbol $\text{bail}$ which represents Eve choosing to leave the gadget and try simulating an accepting run of $M$, that is $\Delta \ni (b^+, \text{bail}, (s_{f_i}^3, \alpha^-)), (b^-, \text{bail}, (s_{f_i}^3, \alpha^+)), (b^0, \text{bail}, (s_{f_i}^3, c))$ where $c \in \{c_i^+, c_i^-\}$ for all $i$. For all other actions in $\Sigma'$, self-loops are added on the states $b^+, b^0, b^-$ with weights $+1, 0, -1$ respectively. Meanwhile, Adam is able to exit the gadget at any moment – via non-deterministic transitions $(b^+, \sigma, (s_{f_i}^3, \alpha^-)), (b^-, \sigma, (s_{f_i}^3, \alpha^+)), (b^0, \sigma, (s_{f_i}^3, c))$ where $c \in \{c_i^+, c_i^-\}$ for all $i$ and $\sigma \in \Sigma' \setminus \{\text{bail}\}$ – to the universal side of the construction, i.e. he will try to simulate a rejecting run of the machine. Bailing transitions (transitions going to states $(s_{f_i}^3, \cdot)$ or $(s_{f_i}^3, \cdot)$) have weight 0.

Note that after this initial transitions the value of all the counters is 0. Indeed, this corresponds to the beginning of a simulation of $M$ starting from configuration $(s_I, v)$ where $v(c) = 0$ for all $c \in C$; thus the invariant (1) initially holds.

Fig. 3. Observation gadget for $(s, INC, c_i, s')$ instruction. For $(s, q) \in Q$ only the $q$ component is shown.
Fig. 4. Existential observation gadget for \((s, 0CHK, c_i, s', s^0)\) instruction. Transitions to \(s'\)-observation not shown.

Fig. 5. Universal observation gadget for \((s, 0CHK, c_i, s', s^0)\) instruction. Transitions to \(s', s^0\) observations are weighted as with the existential observation gadget.
Let us now explain how Eve simulates increments of counter values using this encoding (decrements are treated similarly). The gadget we explain below actually works the same in both sides of the construction, i.e. the universal and existential gadgets for increments and decrements are identical. For that, consider Figure 3, the upper part of the figure is related to the location (instruction) \( s \) of \( M \), while the bottom part is related to the location \( s' \) of \( M \), and assume that \( (s, INC, c_i, s') \in \delta_M \).

As can be seen in the figure, the observation related to the instruction \( s \) contains: the states \( c_i^+, c_i^- \) are used for the encoding of the value of counter \( c_i \), using the encoding described above by Equation (1). The additional states \( \alpha^+, \alpha^- \) are used to encode the number of steps in the simulation (again one positive ending in \( \alpha^+ \) and one negative encoding in \( \alpha^- \)). Now, let us consider the transitions of the gadget. The increment of the counter \( c_i \) from location \( s \) to location \( s' \) is encoded using the weights on the transitions that go from the observation \( s \) to the observation \( s' \). As you can see, the weight on the edge between the copy of state \( c_i^+ \) of observation \( s \) to the copy of this state in observation \( s' \) is equal to \(+1\), while the weight on the edge between the copy of state \( c_i^- \) of observation \( s \) to the copy of this state in observation \( s' \) is equal to \(-1\). It is easy to see that the invariant described by equation (1) is maintained by this construction and that the value encoded for counter \( c_i \) when reaching observation \( s' \) is incremented. As you can also see from the figure, when going from location \( s \) to location \( s' \), we also increment the additional counter that keeps track of the number of steps in the simulation of \( M \). As the machine is deterministic there is no choice for Eve in observation \( s \), since only an increment can be executed, this is why, regardless of the action chosen from \( \Sigma' \), the same transition is taken.

Now, let us turn to the gadget of Figure 4 that is used to simulate “zero check” instructions. We first focus on the case in which it is the duty of Eve to reveal if the counter has value zero or not, by forcibly choosing the next letter to play in \( \{z, \overline{z}\} \subseteq \Sigma' \).

In the observation that corresponds to the location \( s \) of \( M \), Eve decides to declare that the counter \( c_i \) is equal to zero (by issuing \( z \)) or not (by issuing \( \overline{z} \)), then Adam resolves non-determinism as follows. If Eve does not cheat then Adam should let the simulation to continue to either \( s^0 \) or \( s' \) depending on Eve’s choice (the figure only depicts the branching to \( s^0 \), the branching to \( s' \) is similar). Now if Eve has cheated, then Adam should have a way to retaliate: we allow him to do so by branching to observation \( \{q_f\} \) from state \( (s, c_i^-) \) with weight 0 in case \( z \) has been issued and the counter \( c_i \) is not equal to zero and with weight \(-1\) in case \( \overline{z} \) has been issued and the counter \( c_i \) is equal to zero. It should be clear that in both cases Adam closes a bad abstract cycle.

A similar trick is used for the gadget from Figure 5 where Adam is forced to simulate a truthful “zero check” or lose \( \Gamma_{GM} \). Since Adam can control non-determinism and not the action chosen, we have transitions going from \( (s, \cdot) \) to states in both \( (s_{z\cdot}) \) and \( (s_{\overline{z}\cdot}) \) with weight 0 and all actions in \( \Sigma' \). Eve is then allowed to branch back to \( \{q_f\} \) as follows. If Adam does not cheat, then Eve will play any action in \( \Sigma' \setminus \{\text{bail}\} \) and transitions, with weights similar to those used in the existential check gadget, will take the play from \( (s_{z\cdot}) \) to \( (s^0, \cdot) \) and from \( (s_{\overline{z}\cdot}) \) to \( (s', \cdot) \). Now if Adam has cheated by taking the play to \( (s_{z\cdot}) \) when \( c_i \) was not zero, then Eve – by playing \( \text{bail} \) – can go from \( (s_{z\cdot}, c_i^+) \) to the initial observation with weight \(-1\) and close a good abstract cycle. If Adam cheated by taking the play to \( (s_{\overline{z}\cdot}) \) when \( c_i \) was indeed zero, Eve can go (with
the same action) from \((s_c, c^-_i)\) to the initial observation with weight 0 again and close a good abstract cycle.

It should be clear also from the gadgets, that the opponent of Simulator has no incentive to interrupt the simulation if there is no cheat. Doing so is actually beneficial to Simulator, who can get a function-action sequence which makes him win \(\Gamma_{G_M}\).

Finally, \(\Delta\) also contains self-loops at all \((s_A, \cdot)\) with all \(\Sigma'\) and with 0 weights and at all \((s_R, \cdot)\) with all \(\Sigma'\) and with \(-1\) weights. Thus, if the play reaches the observation representing state \(s_{F}\) or \(s_R\) from \(M\) then Simulator will be able to force function-action sequences which allow him to win \(\Gamma_{G_M}\).

All that is left is to explain the idea behind observation gadget \(\{a^+, a^-\}\) and to show how we allow the opponent of Simulator to stop the simulated run of \(M\) in case Simulator exhausts the number of instructions he initially declared would be used to accept or reject. Note that Adam could break Eve’s simulation of an accepting run by declaring the value of functions from \(\Gamma_{G_M}\), which are actually our means of encoding the values of the counters, to be \(+\infty\) (or at least some subset of the values of the functions). We describe how we obtain the final set of actions for the constructed game and mention the required transitions from every observation in the game so that Adam is unable to do so – he is able to, but in doing so allows Eve to win the reachability game. Denote by \((o, q_i)\) the \(i\)-th state in observation \(o\). Observe that in our construction we need at most 10 states per observation: two copies of every counter state and two additional step counters. \(\Sigma = \Sigma' \cup \{q_i \mid 0 \leq i < 6\} \cup \{ex\}\). For every observation \(o\) in \(G_M\) we add the transitions \(((o, q_i), q_i, a^-), ((o, q_j), q_i, a^+)\) for all \(q_i, q_j\) in \(o\) where \(q_i \neq q_j\).

The \(ex\) action is used in transitions \(((o, \alpha^+), ex, \{q_I\}) \in \Delta\) for all observation gadgets \(o\) in the universal side of the construction. This allows Eve to stop Adam (who is playing Simulator) in case he tries to simulate more steps than he said were required for \(M\) to reject. Similarly, in the existential part of the construction, we add a transition \(((o, \alpha^-), \sigma, \{q_I\})\) for all observation gadgets \(o\) and all \(\sigma \in \Sigma\), which lets Adam stop Eve’s simulation if she tries to cheat in the same way.

To finish, we add the self-loops \((a^+, \sigma, a^+)\) and \((a^-, \sigma, a^-)\) are part of \(\Delta\) for all \(\sigma \in \Sigma\). Clearly, Adam cannot choose anything other than proper \(\sigma\)-successors in \(\Gamma_{G_M}\) or he gives Eve enough information for her to win the game. To have the game be limited-observation we let all missing \(\sigma\)-transitions on the existential (resp. universal) side of the simulation go to a sink state in which Adam (Eve) wins.

Now, let us prove the correctness of the overall construction. Assume that \(M\) has an accepting or rejecting run. Then, Simulator, by simulating faithfully the run of \(M\) has a strategy that allows him to force abstract paths which induce good or bad abstract cycles depending on who is simulating. Clearly, in this case even if the opponent decides to interrupt the simulation \(M\) at a “zero check” gadget, he will only be helping Simulator.

If \(M\) has no accepting or rejecting run, then by simulating the machine faithfully, Simulator will be generating cycles in the control state of the machine and such abstract paths are “mixed” because of concrete paths between corresponding \(\alpha^-, \alpha^+\) states. Cheating does not help him either since after the opponent catches him cheating and restarts the simulation of the machine (by returning to the initial observation), the corresponding paths is losing for him.
E  Modifications for Theorem 1

To prove Theorem 1 we reduce from the non-terminating problem for two counter machines using a construction similar to the construction above. Given a two-counter machine $M$, we construct a game $G_M$ as in the proof of Theorem 5 with the following adjustments:

- We only consider the universal side of the simulation;
- The observation corresponding to the accept state of $M$ is a sink state winning for Adam;
- The $\alpha^-$ states are replaced with $\beta$ states which have transitions to other $\beta$ states of weight 0 except in one case specified below;
- The pumping gadget has self loops of weights 0, 0, $-1$ and the transition from $b^+$ to $\beta$ has weight $-1$ if Eve exits and weight 0 if Adam exits;
- The reset transition also goes from $\beta$ states to $q_I$.

Suppose the counter machine halts in $N$ steps. The strategy for Adam is as follows. Exit the pumping gadget after $N$ steps and faithfully simulate the counter machine. Suppose Eve can beat this strategy. If she allows a faithful simulation for $N$ steps then Adam reaches a sink state and wins, so Eve must play reset within $N$ steps of the simulation. Let us consider each cycle of at most $2N$ steps. If she waits for Adam to exit the pumping gadget then the number of steps in the simulation is less than the number of steps in the pumping gadget, so a negative cycle is closed. On the other hand if she exits the pumping gadget before $N$ steps then the cycle through the $\beta$ vertices has negative weight. In both cases, a negative cycle is closed in at most $2N$ steps, so the limit average is bounded above by $-\frac{1}{2}$. Thus this strategy is winning for Adam.

Now suppose the counter machine does not halt. The (infinite memory) strategy for Eve is as follows. For increasing $n$, exit the pumping gadget after $n$ steps and faithfully simulate (i.e. call any, and only, cheats of Adam) the counter machine for $n$ steps. Then play reset and increase $n$. Cheating in the simulation does not benefit Adam, so we can assume Adam faithfully simulates the counter machine. Likewise, if Eve always waits until the number of steps in the simulation exceeds the number of steps in the pumping gadget, then there is no benefit for Adam to exit the pumping gadget. However if the play proceeds as Eve intends then the weight of the path through the $\alpha^+$ states is non-negative and although the weight through the $\beta$ states is negative, the limit average is 0. Thus the strategy is winning for Eve.

F  Proofs from Section 7

Theorem 6 (Class membership). Let $G$ be an MPG with limited-observation. Determining if $G$ is forcibly FAC is PSPACE-complete.

Proof. For PSPACE membership we observe that a linear bounded alternating Turing Machine can decide whether one of the players can force to reach $T^*_{\exists}$ or $T^*_{\forall}$ in $\Gamma_G$. To show hardness we use a reduction from the True Quantified Boolean Formula (TQBF) problem. Given a fully quantified boolean formula $\Psi = \exists x_0 \forall x_1 \ldots Q x_{n-1}(\Phi)$, where
\( Q \in \{ \exists, \forall \} \) and \( \Phi \) is a boolean formula expressed in \textit{conjunctive normal form} (CNF), the \textit{TQBF} problem asks whether \( \Psi \) is true or false. The \textit{TQBF} problem is known to be \textit{PSPACE-complete} [23].

This problem is often rephrased as a game between Adam and Eve. In this game the two players alternate choosing values for each \( x_i \) from \( \Phi \). Eve wins if the resulting evaluation of \( \Phi \) is true while Adam wins if it is false. We simulate such a game with the use of “diamond” gadgets that allow Eve to choose a value for existentially quantified variables by letting her choose the next observation. Similarly, the same gadget – except for the labels on the transitions, which are completely non-deterministic in the following case – allow Adam to choose values for variables that are universally quantified.

We construct a game \( G_\Phi = \langle Q, q_I, \Sigma, Obs, w \rangle \) in which there are no concrete negative cycles, hence it follows from Lemma[3] that there are no bad cycles. The game will thus be forcibly FAC if and only if Eve is able to force good cycles. If Eve is unable to prove the QBF is true, Adam will be able to avoid such plays. For this purpose, the “diamond” gadgets employed have two states per observation. This will allow two disjoint concrete paths to go from the initial state \( q_I \) through the whole arena and form a simple abstract cycle that is either good or not good depending on where the cycle started from.

Concretely, let \( x_1 \) be a universally quantified variable from \( \Psi \). We add a gadget to \( G_\Phi \) consisting of eight states grouped into four observations: \{\( b_0^1, b_0^2 \), \( \overline{x}_1, \overline{z}_1 \), \( x_1, z_1 \), \( b_1^1, b_1^2 \)\}. We also add the following transitions:

- from \( b_0^1 \) to \( \overline{x}_1 \) and \( x_1, b_0^2 \) to \( \overline{z}_1 \) with all \( \Sigma \) and weight 0;
- from \( \overline{x}_1 \) and \( x_1 \) to \( b_1^1, \overline{z}_1 \) and \( z_1 \) to \( b_1^2 \), with all \( \Sigma \) and the first two with weight \(-1\) while the last two have weight 0.

Figure 6 shows the universal “diamond” gadget just described. The observation \{\( x_1, z_1 \)\} corresponds to the variable being given a false valuation, whereas the \{\( x_1, z_1 \)\} observation models a true valuation having been picked. Observe that the choice of the next observation from \{\( b_0^1, b_0^2 \)\} is completely non-deterministic, i.e. Adam chooses the valuation for this variable.

For existentially quantified variables, the first set of transitions from the gadget is slightly different. Let \( \bar{x}_i \) be an existentially quantified variable in \( \Psi \), then the upper part of the gadget includes transitions from \( \overline{b}_i \) to \( \overline{x}_i \) and from \( b_i^1 \) to \( \overline{z}_i \) with action symbol \( \neg x_i \) and weight 0; as well as transitions from \( b_i^2 \) to \( x_i \) and from \( b_i^1 \) to \( z_i \) with action symbol \( x_i \) and weight 0.

A play in \( G_\Phi \) traverses gadgets for all the variables from the QBF and eventually gets to the observation \{\( b_{m-1}^1, b_{n-1}^0 \)\} where the assignment of values for every variable has been simulated. At this point we want to check whether the valuation of the variables makes \( \Phi \) true. We do so by allowing Adam to choose the next observation (corresponding to one of the clauses from the CNF formula \( \Phi \)) and letting Eve choose a variable from the clause (which might be negated). Let \( x_i \) (resp. \( \bar{x}_i \)) be the variable chosen by Eve, in \( G_\Phi \) the next observation will correspond to closing a good abstract cycle if and only if the chosen valuation of the variables for \( \Psi \) assigns to \( x_i \) a true (false) value. For this part of the construction we have \( 2 \cdot m \) states grouped in \( m \) observations, where \( m \) is the number of clauses in the formula. The lower part of figure 6 shows the clause observations we just described.
Denote by \( \{ c_i, c_i^0 \} \) the observation associated to clause \( c_i \). The game has transitions from \( c_i \) to \( x_i \) (or \( \overline{x_i} \)) and from \( c_i^0 \) to \( z_i \) (or \( \overline{z_i} \)) with action symbol \( x_i \) (or \( \overline{x_i} \)) and weight \( n - i \) for the first, 0 for the latter, if and only if the clause \( c_i \) includes the (negated) variable \( x_i \).

Fig. 6. Corresponding game for QBF \( \exists x_0 \forall x_1 \ldots (\overline{x_0}) \land (x_1) \ldots \)

After Eve and Adam have chosen values for all variables (and the game reaches observation \( \{ b_{n-1}, b_0 \} \)) there are two concrete paths corresponding with the current play: one with payoff 0 and one with payoff \( -n \). When Adam has chosen a clause and Eve chooses a variable \( x_i \) from the clause, the next observation is reached with both concrete paths having payoffs 0 and \( -i \). Observe, however, that if we consider the suffix \( 5 \) All missing transitions for \( G_{\Psi} \) to be complete go to a dummy state with a negative and 0-valued non-deterministic transitions.
of said concrete paths starting from \{x_i, z_i\} or \{\overline{x_i}, \overline{z_i}\} – depending on which valuation the players chose – both payoffs are 0. Indeed, if the observation was previously visited, i.e. Eve has proven the clause to be true, then a good cycle is closed. On the other hand, if the observation has not been visited previously, then Eve has no choice but to keep playing. We note that traversing the lower part of our “diamond” gadgets results in a mixed payoff of −1 and 0 and since \{\overline{b_i}, b_i\} must have already been visited, a cycle is closed that is not good.

Therefore, if \(\Psi\) is true then Eve has a strategy to close a good cycle, so \(G_{\Psi}\) is forcibly FAC. Conversely, if \(\Psi\) is false then Adam has a strategy to force Eve to close a cycle that is not good. Hence \(G_{\Psi}\) is not forcibly FAC.

F.1 Adjustments for Theorem 7

Consider the following modifications to the construction described above:

– First, we augment every observation with \(2 \cdot n\) states corresponding to variables from \(\Phi\) and their negation (say, \(y_i\) and \(\overline{y_i}\) for \(0 \leq i < n\)).
– We then add transitions from every new state \(y_i\) to its counterpart in the next observation so as to form \(2 \cdot n\) new disjoint cycles going from \(q_I\) through the whole construction – up to this point \(\text{Play}(G_{\Psi})\) remains unchanged. These transitions all have weight zero except for a few exceptions:
  – the transition corresponding to the lower part of the gadget which represents the variable itself, i.e. the transition from augmented observation \{\overline{x_i}, \overline{z_i}, \ldots\} to \{\overline{b_i}, b_i, \ldots\} (resp. \{\overline{x_i}, \overline{z_i}, \ldots\} to \{\overline{b_i}, b_i, \ldots\}) now has weight of +1 for the \(y_i\)-transition (\(\overline{y_i}\)-transition);
  – outgoing transitions from clause observations have weight −1 on the \(y_i\)-transition going to the \(x_i\)-gadget.
  – Finally, at every \{\overline{x_i}, \overline{z_i}, \ldots\} and \{\overline{x_i}, \overline{z_i}, \ldots\} augmented observation, Adam is allowed to resolve non-determinism by going back to \(q_I\) – i.e. in these observations we add a transition from \(y_i\) and \(\overline{y_i}\), respectively, back to the initial state.

In this new game, we see that when the play reaches \{\overline{x_i}, \overline{z_i}, \ldots\} (or \{\overline{x_i}, \overline{z_i}, \ldots\}) after Eve has chosen a variable from a clause then the concrete path ending at \(y_i\) (resp. \(\overline{y_i}\)) has weight 0 if the observation was previously visited, and weight −1 if it was not. The concrete paths ending at all the other new states have weight 0 or +1 depending on the choices made by the players. Thus if the observation was previously visited then the cycle closed is good as before, and if the observation was not previously visited then Adam can choose to play to \(q_I\) and close a bad cycle. Note that if Adam chooses to play to \(q_I\) before the clause gadgets are reached then he will only be closing good cycles. Following the same argument as before, if \(\Psi\) is true then Eve has a winning strategy and if \(\Psi\) is false then Adam has a winning strategy. So \(G_{\Psi}\) is forcibly FAC and Eve wins if and only if \(\Psi\) is true.

F.2 Proof of Theorem 8

**Theorem 8 (Exponential memory determinacy).** One player always has a winning observation-based strategy with exponential memory in a forcibly FAC MPG. Further,
for any \( n \in \mathbb{N} \) there exists a forcibly FAC MPG, of size polynomial in \( n \), such that any winning strategy has memory at least \( 2^n \).

**Proof.** For the upper bound we observe that plays in \( \Gamma'_G \) are bounded in length by \( |\text{Obs}| \). It follows that the strategy constructed in Theorem 3 has memory at most \( |\Sigma||\text{Obs}| \).

For the lower bound, consider the forcibly FAC game \( G_n \) constructed in the proof of Theorem 7 for the formula

\[
\varphi_n = \forall x_1 \forall x_2 \ldots \forall x_n \exists y_1 \ldots \exists y_n \bigwedge_{i=1}^{n} (x_i \lor \neg y_i) \land (\neg x_i \lor y_i).
\]

As \( \varphi_n \) is satisfied, Eve wins \( G_n \). Now consider any strategy of Eve with memory \(< 2^n \). As there are \( 2^n \) possible assignments for the values of \( x_1, \ldots, x_n \) it follows there are at least two different assignments of values such that Eve makes the same choices in the game. Suppose these two assignments differ at \( x_i \) and assume WLOG that Eve’s choice is at \((n + i)\)-th gadget to play to \( y_i \). Then Adam can win the game by choosing values for the universal variables that correspond to the assignment which sets \( x_i \) to \text{false}, and then playing to the clause \((x_i, \lor \neg y_i)\). Thus any winning strategy for Eve must have size at least \( 2^n \).

In a similar way the game defined by the formula \( \neg \varphi_n \) is won by Adam, but any winning strategy must have size at least \( 2^n \).

### G Proof of Theorem 10

**Theorem 10 (Class membership).** Let \( G \) be an MPG with limited-observation. Determining if \( G \) is FAC is coNP-complete.

**Proof.** For coNP membership, one can guess a large enough simple abstract cycle \( \psi \) and (in polynomial time with respect to \( Q \)) check that it is neither good nor bad. To show coNP-hardness we use a reduction from the HAMILTONIAN CYCLE problem.

Given graph \( G = \langle V, E \rangle \) where \( V \) is the set of vertices and \( E \subseteq V \times V \) the set of edges. We construct a directed weighted graph with limited-observation \( G = \langle Q, q_I, \Sigma, \Delta, \text{Obs}, w \rangle \) where:

- \( Q = V \cup \{q_I, q_+, q_-\} \);
- \( \text{Obs} = \{\{v\} \mid v \in V\} \cup \{\{q_-\}, \{q_I\}\} \);
- \( \Sigma = V \cup \{\tau\} \);
- \( \Delta \) contains transitions \((u, v, v)\) such that \((u, v) \in E \) and self-loops \((u, v', u)\) for all \((u, v') \notin E \), transitions (with all \( \sigma \)) from \( q_I \) to both \( q_+ \) and \( q_- \) and from these last two to all states \( v \in V \), as well as \( \tau \)-transitions from every state \( v \in V \) to \( q_+ \) and \( q_- \);
- \( w \) is such that all outgoing transitions from \( q_+ \) and \( q_- \) have weight \( 1 - |V| \), \((u, v, v)\) transitions where \((u, v) \in E \) have weight \( +1 \), \( \tau \)-transitions from states \( v \in V \) have weight \(-1 \) and all other transitions have weight \( 0 \).

Notice that the only non-deterministic transitions in \( G \) are those incident on and outgoing from the states \( q_+, q_- \). Clearly, the only way for a simple abstract cycle to be not
good and not bad (thus making $G$ not FAC) is if there is a path from $\{q_-, q_+\} \in \text{Obs}$ that traverses $|V|$ unique observations and ends with a $\tau$-transition back at $\{q_-, q_+\}$. Such a path corresponds to a Hamiltonian cycle in $G$. If there is no Hamiltonian cycle in $G$ then for any play $\pi$ in $G$, a bad cycle will be formed (hence, $G$ is FAC).

H Proofs from Section 9

Given an MPG with partial-observation $G = \langle Q, \Sigma, \Delta, q_I, w, \text{Obs} \rangle$ we construct an MPG with limited-observation $G' = \langle Q', \Sigma, \Delta', q'_I, w', \text{Obs}' \rangle$ where:
- $Q' = \{(q, K) \in Q \times 2^Q \mid q \in K$ and $K \subseteq o \in \text{Obs}\}$,
- $q'_I = (q_I, \{q_I\})$,
- $\text{Obs}' = \{(q, K) \mid q \in K \mid K \subseteq o$ for some $o \in \text{Obs}\}$,
- $\Delta'$ contains the transitions $((q, K), \sigma, (q', K'))$ such that $(q, \sigma, q') \in \Delta$ and $K' = \text{post}_\sigma(K) \cap o$ for some $o \in \text{Obs}$, and
- $w'((q, K), \sigma, (q', K')) \mapsto w(q, \sigma, q')$ for all $((q, K), \sigma, (q', K')) \in \Delta'$.

**Theorem 13 (Equivalence).** Let $G$ be an MPG with partial-observation and $G'$ be the corresponding MPG with limited-observation as constructed above. Eve has a winning observation-based strategy in $G$ if and only if she has a winning observation-based strategy in $G'$.

**Proof.** For any $\psi = o_0\sigma_0... \in \text{Prefs}(G) \cup \text{Plays}(G)$ let $\kappa(\psi) = o'_0\sigma_0...$ be the (possibly infinite) sequence, where $o'_{j+1} = o_{j+1} \times \text{post}_\sigma(o_j)$ for all $j \geq 0$. $\kappa(\psi)$ is in fact the sequence of belief sets Eve observes throughout $\psi$ paired with the action she played at every turn. By construction of $G'$ we have that for all $j \geq 0 : o'_j \in \text{Obs}'$.

As $\kappa$ is clearly a bijection between the sets $\text{Prefs}(G) \cup \text{Plays}(G)$ and $\text{Prefs}(G') \cup \text{Prefs}(G')$, we can define, for any strategy $\lambda_3$ of Eve in $G$, a unique strategy $\lambda'_3$ for her in $G'$, and vice versa. This implies that fixing a strategy for her in either game defines a unique set of plays in both $G$ and $G'$ which are consistent with said strategy. It is easy to show that for any play $\psi \in \text{Plays}(G)$ consistent with this strategy, the projection of a concrete play in $\gamma(\kappa(\psi))$ on its first component is a concrete play in $\gamma(\psi)$. Furthermore, by construction of $G'$ for any $\pi \in \gamma(\psi)$ with its corresponding $\pi' \in \gamma(\kappa(\psi))$ we have that $w(\pi[..i]) = w'(\pi'[..i])$ for all $i \geq 0$. Thus, if all plays consistent with the fixed strategy are winning for Eve in $G$ they must also be winning for her in $G'$ and vice versa as well.

H.1 Proof of Theorem 11

**Theorem 11 (Class membership).** Let $G$ be an MPG with partial-observation. Determining if $G$ is FBC is coNEXP-complete and determining if $G$ is forcibly FBC is in EXPSPACE and NEXP-hard.

**Upper bounds.** Membership of the relevant classes is straightforward, they follow directly from the upper bounds for MPGs with limited-observation and the (at worst) exponential blow-up in the translation from games of partial-observation to games of limited-observation.
Lower bound for FBC games  For coNEXP-hardness we reduce from the complement of the Succinct Hamiltonian Cycle problem: Given a Boolean circuit $C$ with $2^N$ inputs, does the graph on $2^N$ nodes with edge relation encoded by $C$ have a Hamiltonian cycle? This problem is known to be NEXP-complete [20].

The idea is that we simulate a traversal of the succinct graph in our MPG: if we make $2^N$ valid steps without revisiting a vertex of the succinct graph then that guarantees a Hamiltonian cycle. To do this, we start with a transition of weight $-2^N$ and add 1 to all paths every time we make a valid transition. We include a pair of transitions back to the initial state with weights 0 and $-1$ and ensure this is the only transition that can be taken that results in paths of different weight. The resulting game then has a mixed lasso if and only if we can make $2^N$ valid transitions. If we encode the succinct graph vertex in the knowledge set then the definition of an FAC game will give us an automatic check if we revisit a vertex. In fact, we store several pieces of information in the knowledge sets of the observations: the current (succinct) graph vertex, the potential successor, and the evaluation of the edge-transition circuit up to a point. We now describe the construction in detail.

Let us assume inputs of the circuit $C$ are labelled $x_1, \ldots, x_{2^N}$ and that it has $k$ gates $G_1, \ldots, G_k$ numbered in an order that respects the circuit graph, so $G_j$ has inputs from $\{x_i, \neg x_i : 1 \leq i < 2^N + j\}$ where, for convenience, $x_{2^N+j}$ indicates the output of gate $G_j$. We may assume each gate has two inputs and (as we are allowing negated inputs) we may assume we only have AND and OR gates. The overall (i.e. observation-level) structure of the game is shown in Figure 7.

![Fig. 7. Overall structure of the game for Succinct Hamiltonian Cycle](image-url)
ternal transitions, and the edge weights indicate the weight of all transitions between observations.

Our game proceeds in several stages:

1. The transition from $S$ to $O_1$ sets the initial (succinct) vertex (stored in a subset of the states of $O_1$) and initializes the vertex counter to $-2^N$. 
2. Internal transitions in $G_0$ select the next vertex, the transition from $O_1$ to $G_0$ initializes this procedure.
3. For $i > 0$, internal transitions in $G_i$ evaluate gate $i$, incoming transitions initialize this by passing on the previous evaluations (including the current and next vertices).
4. Internal transitions in $Chk$ test if the circuit evaluates to 1.
5. The next succinct vertex (chosen in $G_0$) is passed to $O_2$, where there is an implicit check that this vertex has not been visited before, and the counter is incremented.
6. The play can return to $S$, generating a mixed lasso if and only if the vertex counter is 0, i.e. $2^N$ vertices have been correctly visited, or return to $O_1$ with a new current succinct vertex.

The weights on the incoming transitions to an observation are designed to impose a penalty that can only be nullified if the correct sequence of internal transitions is taken.

We observe that if there is a penalty that is not nullified then the game can never enter a mixed lasso (as the vertex counter will still be negative when a vertex is necessarily revisited). We now describe the structure of the observations.

$O_1$: $O_1$ contains $2N$ states: $\{x_i, \overline{x}_i \mid 1 \leq i \leq N\}$. For convenience we will use the same labels across different observations, using observation membership to distinguish them. There are $\sigma$-transitions from $S$ to $\{x_i \mid 1 \leq i \leq N\}$ with weight $-2^N$.

$O_2$: $O_2$ contains $2N + 1$ states: $\{x_i, \overline{x}_i \mid 1 \leq i \leq N\} \cup \{\bot\}$. There are $\sigma$-transitions from each state in $O_2$ other than $\bot$ to its corresponding state in $O_1$ with weight 0.

There is a $\sigma'$-transition from each state in $O_2$ other than $\bot$ to $S$ with weight 0, and a $\sigma'$-transition from $\bot$ to $S$ with weight $-1$.

$G_0$: $G_0$ contains $5N$ states: $\{x_i, \overline{x}_i \mid 1 \leq i \leq 2N\} \cup \{y_i \mid N < i \leq 2N\}$. There is a $\sigma$-transition from each state in $O_1$ to its corresponding state in $G_0$ of weight $-N$ and in addition, $\sigma$-transitions from every state in $O_1$ to $\{y_i \mid N < i \leq 2N\}$ also of weight $-N$. For $N < j \leq 2N$ there is a $\tau_j^+$ transition of weight 1 from $y_j$ to $x_j$ and a $\tau_j^-$ transition of weight 1 from $y_j$ to $\overline{x}_j$. For all states in $G_0$ other than $y_j$ there is a $\tau_j^+$ and $\tau_j^-$ loop of weight 1. Figure 8 shows the construction.

$G_j$ ($j > 0$): The observation corresponding to gate $j$ contains $4N + 2j + 8$ states: $\{x_i, \overline{x}_i \mid 1 \leq i \leq 2N + j\} \cup \{v_m, \overline{v}_m \mid 0 \leq m \leq 3\}$. Recall gate $j$ has inputs from $\{x_i, \overline{x}_i \mid 1 \leq i < 2N + j\}$. Suppose these inputs are $y_i \in \{x_i, \overline{x}_i\}$ and $y_r \in \{x_r, \overline{x}_r\}$, and for convenience let $\overline{y}_i$ and $\overline{y}_r$ denote the other member of the pair (i.e. the complement of the input). We have a $\sigma$-transition of weight $-1$ from $\{x_i, \overline{x}_i \mid 1 \leq i < 2N + j\} \subseteq G_{j-1}$ to the corresponding vertex in $G_j$. In addition we have $\sigma$-transitions of weight $-1$ from $y_i, \overline{y}_i, y_r, \overline{y}_r \in G_{j-1}$ to $v_0, \overline{v}_0, v_1, \overline{v}_1 \in G_j$ respectively. We have the following internal transitions:

- $\tau^+$ (weight 0): $v_1$ to $v_2$, $\overline{v}_1$ to $\overline{v}_3$, $v_0$ to $x_{2N+j}$, $\overline{v}_0$ to $\overline{x}_{2N+j}$ if gate $j$ is an AND gate, $\overline{v}_0$ to $x_{2N+j}$ if it is an OR gate,
For all other states in $G_j$ these transitions loop with the same weight (i.e. $\chi$ loops have weight 1, $\tau^{\pm}$ loops have weight 0).

Figure 8 shows an example of the construction of $G_j$ for the gate $x_l \land \neg x_r$ (self-loops not shown).

Chk: Chk contains $4N + 2$ states: $\{x_i, \overline{x}_i \mid 1 \leq i \leq 2N\} \cup \{y, z\}$. There is a $\sigma$-transition of weight $-1$ from $\{x_i, \overline{x}_i \mid 1 \leq i \leq 2N\} \subseteq G_k$ to their corresponding states in Chk, and a $\sigma$-transition of weight $-1$ from $x_{2N+k} \in G_k$ to $y$. There is a $\chi$-transition of weight 1 from $y$ to $z$ and for all other states in Chk there is a $\chi$-loop of weight 1. There is a $\sigma$-transition of weight 1 from all states in Chk to $\bot \in O_2$ and for $N < i \leq 2N$ there is a $\sigma$-transition of weight 1 from $x_i \in Chk$ to $x_{i-N} \in O_2$ and from $\overline{x}_i \in Chk$ to $\overline{x}_{i-N} \in O_2$.

Lower bound for forcibly FBC games Suppose we make the following adjustments to the construction:

- Remove $\bot$ from $O_2$;
- Replace remaining $\tau, \chi, \sigma'$ loops that do
- Remove the transitions from Chk to $\bot \in O_2$,
- Change the weights of incoming transitions to $G_i$ ($i > 0$) to $-5$ and the weights of all internal $\tau$-transitions to 1,
- Change the weight of the $\sigma'$-transition from $\bot \in O_2$ to $S$ to 0,
- Add a new vertex $\bot$ to all observations other than $S$ (and $O_2$),
- Add a $\sigma$-transition of weight $2^N$ from $S$ to $\bot \in O_1$, and
- Whenever there is a transition of weight $w$ from $x_i \in o$ to $x_j \in o'$ ($o, o'$ and $i, j$ possibly the same) add a transition of weight $-w$ from $\bot \in o$ to $\bot \in o'$.

Then the only possible non-mixed lasso in the resulting graph is one that would correspond to a successful traversal of a Hamiltonian cycle. Eve can force the play to this cycle if and only if the succinct graph has a Hamiltonian cycle.

### H.2 Proof of Theorem 12

**Theorem 12 (Winner determination).** Let $G$ be a forcibly FBC MPG. Determining if Eve wins $G$ is EXP-complete.

**Proof.** The lower bound follows from the fact that forcibly FBC games are a generalization of visible weights games (see Lemma 4), shown to be EXP-complete in [12]. For the upper bound, rather than working on $I'_G$, which is doubly-exponential in the size of $G$, we instead reduce the problem of determining the winner to that of solving a safety game which is only exponential in the size of $G$. Given an MPG

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6 assuming dead-ends go to a dummy state
with partial-observation $G$, let $G'$ be the corresponding limited-observation game. Let $\mathcal{E} = \{-1, 2 \cdot W \cdot |\text{Obs}'|\} \cup \{\perp\}$ where $W = \max\{|w(e)| \mid e \in \delta\}$, and let $\mathcal{F}' \subseteq \mathcal{F}$ be the set of functions $f : Q \rightarrow \mathcal{E}$.

The safety game will be played on $\mathcal{F}'$ with the transitions defined by $\sigma$-successors. The idea is that a given position $f \in \mathcal{F}'$ of the safety game corresponds to being in an observation of $G'$, namely $\text{supp}(f)$. Similar to before (in $I_G$), the non-negative integer values of $f$ give a lower bound for the minimum weights of the concrete paths ending in the given state (see Lemma 5), that is: if $f(q) \neq \perp$ and $f(q) \geq 0$ then the minimum weight over all concrete paths starting at $q_I$ and ending at $q$ is at least $f(q) + W \cdot |\text{Obs}'|$; whereas if $f(q) = -1$ then there is a concrete path of weight at most $-W \cdot |\text{Obs}'| - 1$. As the winner of a forcibly FAC game can be resolved in at most $|\text{Obs}'|$ transitions it turns out that this is sufficient information to determine the winner.

Formally, the safety game is $S_G = (\mathcal{F}', f_1', \Sigma, \Delta_{\text{succe}})$ where $f_1'(q_I) = W \cdot |\text{Obs}'|$ and $f_1'(q) = \perp$ for all other $q \in Q$; $(f, \sigma, f') \in \Delta_{\text{succe}}$ if $f'$ is a proper $\sigma$-successor of $f$ where we let

$$a + b = \begin{cases} \perp & \text{if } a = \perp \text{ or } b = \perp, \\ -1 & \text{if } a = -1, b = -1, \\ \min\{a + b, 2 \cdot W \cdot |Q|\} & \text{otherwise.} \end{cases}$$

$\mathcal{F}_{neg}'$ is the set of all functions $f \in \mathcal{F}'$ such that $f(q) = -1$ for some $q \in \text{supp}(f)$. The game is played similar to the reachability game $I_G$, i.e. Eve chooses an action $\sigma$ and Adam resolves non-determinism by selecting a proper $\sigma$-successor. In this case, however, Eve’s goal is to avoid visiting any function in $\mathcal{F}_{neg}'$.

The above observation that non-negative values give lower bounds for concrete paths ending at the given vertex implies that if Eve has a strategy to always avoid $\mathcal{F}_{neg}'$ then $\lim inf \frac{f_i(q)}{n} \geq 0$ for all concrete paths $\pi$ consistent with the play. That is, if Eve has a winning strategy in $S_G$ then she has a winning strategy in $G$.

Now suppose Eve has a winning strategy in $G$. It follows from the determinacy of forcibly FAC games and Theorem 3 that she has a winning strategy $\lambda$ in $I_{G'}$. Let $\lambda'$ be the translation of $\lambda$ to $G'$ as per Theorem 3 and let $M$ denote the set of memory states required for $\lambda'$. Clearly $\lambda'$ induces a strategy in $S_G$. We claim this induced strategy is winning in $S_G$. Let $\rho = f_0\sigma_0\.\.\.$ be any play in $S_G$ consistent with $\lambda'$, and let $\mu_i$ denote the $i$-th memory state obtained in the generation of $\rho$ (as in Lemma 7). Then, with a slight adjustment to the proof of Lemma 7 to account for function values not exceeding $2 \cdot W \cdot |\text{Obs}'|$ we have for all $i$ and all $q$:

$$f_i(q) - W \cdot |\text{Obs}'| \geq f_{\mu_i}(q)$$

$$= \min\{w(\pi) \mid \pi \in \gamma(\text{supp}(\mu_i)) \text{ and } \pi \text{ ends at } q\}$$

$$\geq -W \cdot |\text{Obs}'|$$

because $|\mu_i| \leq |\text{Obs}'|$ from the definition of $I_{G'}$. Thus $f_i(q) \geq 0$ for all $i$. Hence $\rho$ does not reach $\mathcal{F}_{neg}'$ and is winning for Eve. Thus $\lambda'$ is a winning strategy for Eve.

\footnote{from Lemma 5}
So to determine the winner of $G$, it suffices to determine the winner of $S_G$. This is just the complement of alternating reachability, known to be decidable in polynomial time (see e.g. [19]). As

$$|S_G| = O(|F'|^2) = O((2 \cdot W \cdot |Obs'| + 1)^{|Q|}) = 2^{O(|Q|^2)},$$

determining the winner of $S_G$, and hence $G$, is in EXP.