Chern-Simons Theories
in the AdS/CFT Correspondence

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Abstract

We consider the AdS/CFT correspondence for theories with a Chern-Simons term in three dimensions. We find the two-point functions of the boundary conformal field theories for the Proca-Chern-Simons theory and the Self-Dual model. We also discuss particular limits where we find the two-point function of the boundary conformal field theory for the Maxwell-Chern-Simons theory. In particular our results are consistent with the equivalence between the Maxwell-Chern-Simons theory and the Self-Dual model.

PACS numbers: 11.10.Kk 11.25.Mf
Keywords: AdS/CFT Correspondence, Chern-Simons Theory

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1 Introduction

Since the proposal of Maldacena’s conjecture that the large N limit of a certain conformal field theory (CFT) in a d-dimensional space is a boundary theory of string/M-theory on $AdS_{d+1} \times K$ (where $K$ is a suitable compact space) [1], an intensive work has been devoted to understand all of its implications. In particular, a precise form to the conjecture has been given in [2][3]. Their suggestion is that the partition function for a field theory on $AdS_{d+1}$, considered as the functional of the asymptotic value of the field on the boundary, is the generating functional for the correlation functions in the CFT on the boundary. Schematically,

$$Z_{AdS}[\phi_0] = \int_{\phi_0} D\phi \exp (-I[\phi]) \equiv Z_{CFT}[\phi_0] = \left \langle \exp \left ( \int_{\partial \Omega} d^d x O \phi_0 \right ) \right \rangle$$

where $\phi_0$ is the boundary value of $\phi$ which couples to the boundary CFT operator $O$. This allows us to obtain the correlation functions of the boundary CFT theory in d dimensions by calculating the partition function on the $AdS_{d+1}$ side.

The AdS/CFT correspondence has been studied for scalar fields [4], massive vector fields [5][6], spinor fields [5][7][8], Rarita-Schwinger field [9], classical gravity [10] and type IIB string theory [11][12]. In this work we discuss the $AdS_3/CFT_2$ correspondence for vector field theories including a Chern-Simons term. In section 2 we deal with the Proca-Chern-Simons theory. An explicit expression for the two-point function of the boundary CFT is obtained. We then study the massless limit to obtain the two-point function for the Maxwell-Chern-Simons theory [13]. In section 3 we deal with the Self-Dual model [14]. Since the standard Self-Dual action has at most first order derivatives it vanishes on-shell. Also the variational principle requires a surface term in order to have a stationary action [15]. As in the case involving spinors [5][7][8][9] the only contribution to the boundary CFT comes from a surface term. An expression for the two-point function on the border is then obtained. There is a well known equivalence between the Self-Dual model and the Maxwell-Chern-Simons theory [14]. We find that the resulting two-point functions of the corresponding boundary CFT’s are consistent with this equivalence. Finally section 4 presents our conclusions.
2 The Proca-Chern-Simons Theory

Since we are going to consider the Euclidean version of $AdS_3$ we start with the Euclidean signature action for the Proca-Chern-Simons theory which is given by

$$I_{PCS} = - \int d^3x \sqrt{g} \left( \frac{1}{8} F_{\mu\nu} F^{\mu\nu} + \frac{1}{4} m^2 A_\mu A^\mu + \frac{1}{\sqrt{g}} \frac{i \mu}{8} \epsilon^{\mu\nu\alpha} F_{\nu\alpha} A_\mu + c.c. \right),$$

(2)

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $\epsilon^{\mu\nu\alpha}$ is the Levi–Civita tensor density with $\epsilon^{012} = 1$. Using the variational principle to obtain the field equations a surface term is generated

$$\int d^3x \partial_\mu \left( - \frac{1}{2} \sqrt{g} F^{\mu\nu} \delta A_\nu + \frac{i \mu}{8} \epsilon^{\mu\nu\alpha} A_\nu \delta A_\alpha + c.c. \right).$$

(3)

We will choose coordinates such that the Minkowski border of $AdS_3$ is situated at $x_0 = 0$. Then the boundary term Eq.(3) will depend only on variations of the spatial components $A_i$ of the vector potential. Choosing boundary conditions only on the $A_i$'s makes the boundary term to vanish so that no further surface terms need to be added to the action Eq.(2). This kind of consideration will play a fundamental role in the next section.

The field equations which follow from Eq.(2) are

$$\nabla_\mu F^{\mu\nu} - m^2 A^\nu - i \mu \frac{1}{\sqrt{g}} \epsilon^{\mu\nu\alpha} \partial_\alpha A_\beta = 0,$$

(4)

which implies

$$\nabla_\mu A^\mu = 0.$$

(5)

Solving Eqs.(4,5) in the $AdS_3$ background is difficult due to the presence of the Levi–Civita tensor density. However it can be eliminated in the following way. Using Eq.(5) in Eq.(4) we get

$$\left( \nabla^2 - m^2 - \frac{R}{3} \right) A^\mu - i \mu \epsilon^{\mu\nu\alpha} F_{\nu\alpha} = 0,$$

(6)

where $*F^\mu = \frac{1}{2} \sqrt{g} \epsilon^{\mu\nu\alpha} F_{\nu\alpha}$ and $R$ is the scalar curvature of $AdS_3$. Now multiplying Eq.(4) by the Levi–Civita tensor density and using again Eq.(5) we get

$$\left( \nabla^2 - m^2 - \mu^2 - \frac{R}{3} \right) *F^\mu + i \mu m^2 A^\mu = 0.$$

(7)
Finally eliminating \( *F^\mu \) from Eqs. (6,7) we arrive at

\[
\left( \nabla^2 - m_+^2 - \frac{R}{3} \right) \left( \nabla^2 - m_-^2 - \frac{R}{3} \right) A^\mu = 0, \tag{8}
\]

where

\[
m_\pm^2 (m, \mu) = \left[ \left( m^2 + \frac{\mu^2}{4} \right)^{\frac{1}{2}} \pm \frac{\mu}{2} \right]^2. \tag{9}
\]

In the flat space limit this is an indication that the Proca-Chern-Simons theory describes two excitations with masses \( m_\pm \). We notice that the solutions of Eq. (8) must satisfy

\[
\left( \nabla^2 - m_+^2 - \frac{R}{3} \right) A^\mu = 0, \tag{10}
\]

or

\[
\left( \nabla^2 - m_-^2 - \frac{R}{3} \right) A^\mu = 0. \tag{11}
\]

Therefore the general solution of Eq. (8) is a superposition of solutions of the Proca theory with masses \( m_+ \) and \( m_- \). The Proca theory has been analyzed in \cite{5}, and we now follow closely the derivation in that paper.

We take the usual representation of the \( AdS_3 \) described by the half space \( x_0 > 0, \ x_i \in \mathbb{R} \) with metric

\[
ds^2 = \frac{1}{x_0^2} \sum_{\mu=0}^2 dx_\mu^2, \tag{12}
\]

for which the curvature scalar is \( R = -6 \). As in \cite{5} we also introduce vector potentials with Lorentz indices \( \tilde{A}_\mu \) using the vielbein of \( AdS_3 \)

\[
\tilde{A}_\mu = x_0 A_\mu. \tag{13}
\]

The solutions which are regular at \( x_0 \to \infty \) can be written as

\[
\tilde{A}_\mu = \frac{1}{2} (\tilde{A}_\mu^+ + \tilde{A}_\mu^-), \tag{14}
\]

where

\[
\tilde{A}_0^\pm (x) = \int \frac{d^2k}{(2\pi)^2} e^{-i\vec{k} \cdot \vec{x}} x_0^2 a_0^\pm (\vec{k}) K_{m_\pm} (k x_0), \tag{15}
\]

\[
\tilde{A}_i^\pm (x) = \int \frac{d^2k}{(2\pi)^2} e^{-i\vec{k} \cdot \vec{x}} x_0 \left( a_i^\pm (\vec{k}) K_{m_\pm} (k x_0) + i a_0^\pm (\vec{k}) \frac{k_i}{k} x_0 K_{m_\pm+1} (k x_0) \right), \tag{16}
\]
\( \vec{x} = (x^1, x^2), \quad k = | \vec{k} |, \quad K_{m\pm} \) are the modified Bessel functions, and from now on \( m_{\pm} \) is to be understood as \( | m_{\pm} |. \) The normalization in Eq. (14) has been chosen so that it reproduces the results in [3] in the particular case \( \mu = 0 \) (and hence \( \tilde{A}^+ = \tilde{A}^- \)). Inserting Eqs. (15, 16) in the original equations of motion Eq. (4) gives the following relations among the coefficients \( a^\pm \)

\[
\mu m_{\pm}^2 a_i^\pm (\vec{k}) = \mp i\mu m_{\pm} \epsilon^{0ij} a_j^\pm (\vec{k}), \quad (17)
\]

\[
\mu m_{\pm} a_0^\pm (\vec{k}) (\mp \epsilon^{0ij} k_j - ik_i) = \mp i\mu k^2 \epsilon^{0ij} a_j^\pm (\vec{k}). \quad (18)
\]

From Eq. (5) we also find

\[
\epsilon^{0ij} a_i^\pm (\vec{k}) = m_{\pm} a_0^\pm (\vec{k}), \quad (19)
\]

which is consistent with Eq. (18). We consider first the case \( \mu \neq 0, m \neq 0 \) and rewrite Eqs. (17) and (18) as

\[
a_i^\pm (\vec{k}) = \mp i\epsilon^{0ij} a_j^\mp (\vec{k}), \quad (20)
\]

\[
m_{\pm} a_0^\pm (\vec{k}) \left( \pm ik_i + \epsilon^{0ij} k_j \right) = \mp k^2 a_i^\pm (\vec{k}). \quad (21)
\]

In order to capture the effect of the Minkowski boundary of the \( AdS_3 \), situated at \( x_0 = 0 \), we first consider a Dirichlet boundary value problem on the boundary surface \( x_0 = \epsilon > 0 \) and then take the limit \( \epsilon \to 0 \). The potential at the near boundary surface will be denoted by \( \tilde{A}_{\epsilon,\mu} \). Imposing the near boundary condition on Eqs. (15, 16) and using Eqs. (20, 21) allow us to find the coefficients \( a^\pm \) in terms of the Fourier transform of the fields \( \tilde{A}_{\epsilon,\mu} \)

\[
a_i^\pm (\vec{k}) = \pm \frac{\epsilon^{0ij} \omega_i^\pm (\vec{k}) \tilde{A}_{\epsilon,j}(\vec{k})}{\epsilon^{0ij} \omega_i^\mp (\vec{k}) \omega_j^\pm (\vec{k})}, \quad (22)
\]

\[
a_i^\pm (\vec{k}) = \epsilon^{-1} \left( \tilde{A}_{\epsilon,i}(\vec{k}) \mp i\epsilon^{0ij} \tilde{A}_{\epsilon,j}(\vec{k}) \right) + K_{m\pm}(k\epsilon) + \frac{k_i \mp i\epsilon^{0ij} k_j}{k} \frac{i\epsilon}{2K_{m\pm}(k\epsilon)} \frac{\epsilon^{0kl} \tilde{A}_{\epsilon,l}(\vec{k})}{\epsilon^{0rs} \omega_r^-(\vec{k}) \omega_s^+(\vec{k})} \times \\
\times \left[ \omega_k^+(\vec{k}) K_{m_{\pm}+1}(k\epsilon) - \omega_k^- (\vec{k}) K_{m_{\pm}+1}(k\epsilon) \right], \quad (23)
\]

where

\[
\omega^\pm_i (\vec{k}) = \frac{i\epsilon}{2k^2} \left[ m_{\pm} \left( k_i \mp i\epsilon^{0ij} k_j \right) K_{m\pm}(k\epsilon) + k_i k_\epsilon K_{m_{\pm}-1}(k\epsilon) \right]. \quad (24)
\]
From Eqs. (13) and (16) we get
\[ \tilde{F}_{\epsilon,0}\left(\vec{x}\right) = (1 - m_{\pm}) \frac{1}{\epsilon} \tilde{A}_{\epsilon,i}\left(\vec{x}\right) - \int \frac{d^2k}{(2\pi)^2} e^{-ik\cdot\vec{x}} \tilde{a}_{i}^{\pm}(\vec{k}) k\epsilon K_{m_{\pm}-1}(k\epsilon). \] (25)

Using this we can finally calculate the value of the classical action in the near boundary surface using the action Eq. (2). After an integration by parts and using the equations of motion we find that there is only a contribution from the boundary
\[ I_{PCS} = -\frac{1}{4} \int d^3x \partial_\mu (\sqrt{g} F^{\mu\nu} A_\nu) + \text{c.c.}, \] (26)
which evaluated on the near boundary surface gives
\[ I_{PCS} = -\frac{1}{4} \int d^2x \epsilon^{-2} \tilde{A}_{\epsilon,i} \left( -\tilde{A}_{\epsilon,i} + \epsilon \tilde{F}_{\epsilon,0}\right) + \text{c.c.} \] (27)

Using Eqs. (20,23,25,27), keeping only the relevant terms in the series expansion of the Bessel functions, and integrating over the momenta we get
\[ I_{PCS} = I^{+} + I^{-} + \cdots, \] (28)
where the dots stand for either contact terms or higher order terms in \( \epsilon \) and
\[ I^{\pm} = \frac{m_{\pm}(m,\mu)}{8} \int d^2x \epsilon^{-2} \left[ \tilde{A}_{\epsilon,i}\left(\vec{x}\right) \tilde{A}_{\epsilon,i}\left(\vec{x}\right) + \text{c.c.} \right] \]
\[ - \frac{1}{4} \tilde{c}_{\pm}(m,\mu) \tilde{\Delta}_{\pm}(m,\mu) \int d^2xd^2y \left[ \tilde{A}_{\epsilon,i}\left(\vec{x}\right) \tilde{A}_{\epsilon,i}\left(\vec{y}\right) + \text{c.c.} \right] \frac{\epsilon^{2[m_{\pm}(m,\mu)-1]}}{\left| \vec{x} - \vec{y} \right|^{2\tilde{\Delta}_{\pm}(m,\mu)}} \]
\[ + \tilde{c}_{\pm}(m,\mu) \tilde{\Delta}_{\pm}(m,\mu) \int d^2xd^2y \left[ \tilde{A}_{\epsilon,i}\left(\vec{x}\right) \tilde{A}_{\epsilon,i}\left(\vec{y}\right) - \tilde{A}_{\epsilon,i}\left(\vec{x}\right) \tilde{A}_{\epsilon,i}\left(\vec{y}\right) \right] \frac{\epsilon^{2[m_{\pm}(m,\mu)-1]}}{\left| \vec{x} - \vec{y} \right|^{2\tilde{\Delta}_{\pm}(m,\mu)}} \frac{(x-y)_i(x-y)_j}{\left| \vec{x} - \vec{y} \right|^2}, \] (29)

\[ \tilde{\Delta}_{\pm}(m,\mu) = m_{\pm}(m,\mu) + 1, \] (30)
\[ \tilde{c}_{\pm}(m,\mu) = \frac{m_{\pm}(m,\mu)}{\pi}. \] (31)

Here \( \tilde{A}_{\epsilon,i} \) denotes the real (imaginary) part of \( \tilde{A}_{\epsilon,i} \). Since the metric is singular in the border the action is divergent and the limit \( \epsilon \to 0 \) has to be taken carefully \[18\].

In order to have a finite action we take the limit
\[ \lim_{\epsilon \to 0} \epsilon^{m_{\pm}(m,|\mu|-1)} \tilde{A}_{\epsilon,i}\left(\vec{x}\right) = \tilde{A}_{0,i}\left(\vec{x}\right). \] (32)
Then we use the equivalence AdS/CFT in the form
\[
\exp (-I_{AdS}) \equiv \left\langle \exp \left( \int d^2x \, J_i(x) \, A_{0,i}(x) \right) \right\rangle. \tag{33}
\]
When \( \mu < 0 \) we have \( m_-(m, |\mu|) - 1 = m_+(m, \mu) - 1 \), the relevant part of \( I^- \) vanishes and the only contribution to the two-point function of the conformal field \( J_{PC}^i \) on the boundary CFT comes from \( I^+ \). When \( \mu > 0 \) we have \( m_-(m, |\mu|) - 1 = m_-(m, \mu) - 1 \), and the only contribution to the two-point function of \( J_{PC}^i \) comes from \( I^- \). In both cases we find the following two-point function
\[
\langle J_{PC}^i(x) \, J_{PC}^j(y) \rangle = \tilde{c}_{PC} \Delta_{PC} \left[ \delta_{ij} - 2 \frac{(x-y)_i(x-y)_j}{|x-y|^2} \right] |x-y|^{-2\Delta_{PC}}. \tag{34}
\]
where
\[
\tilde{\Delta}_{PC} = \tilde{\Delta}_-(m, |\mu|), \tag{35}
\]
and
\[
\tilde{c}_{PC} = \tilde{c}_-(m, |\mu|), \tag{36}
\]
so that \( J_{PC}^i \) has conformal dimension \( \tilde{\Delta}_{PC} \). It is important to note that the identification Eq.(32) agrees with the requirement that the isometries of \( AdS_3 \) correspond to the conformal isometries in \( CFT_2 \) [18].

Now we consider the particular cases \( m = 0 \) and \( \mu = 0 \). In order to get the boundary CFT associated to the Maxwell-Chern-Simons theory we take \( m = 0 \) in Eq.(9), which gives
\[
m_{\pm}(0, \mu) = \frac{1}{2} (|\mu| \pm \mu). \tag{37}
\]
When \( \mu > 0 \) Eq.(18) implies \( a_-^1(\vec{k}) = a_2^-(\vec{k}) = 0 \) and the only contribution to the action comes from \( I^+ \), whereas when \( \mu < 0 \) Eq.(18) fixes \( a_1^+ (\vec{k}) = a_2^+ (\vec{k}) = 0 \) and the only contribution comes from \( I^- \). So the actions corresponding to the cases \( \mu > 0 \) and \( \mu < 0 \) read
\[
I_{MCS}^{[\mu|=\pm\mu} = \frac{|\mu|}{8} \int d^2x \, \epsilon^{-2} \left[ \tilde{A}_{\epsilon,i}(\vec{x}) \, \tilde{A}_{\epsilon,i}(\vec{x}) + c.c. \right] - \frac{1}{4} \tilde{c}_{MCS} \Delta_{MCS} \int d^2x d^2y \left[ \tilde{A}_{\epsilon,i}(\vec{x}) \, \tilde{A}_{\epsilon,i}(\vec{y}) + c.c. \right] \frac{\epsilon^{2|\mu|-1}}{|x-y|^{2\Delta_{MCS}}} + \tilde{c}_{MCS} \Delta_{MCS} \int d^2x d^2y \left[ \tilde{A}_{\epsilon,i}^R(\vec{x}) \tilde{A}_{\epsilon,i}^R(\vec{y}) - \tilde{A}_{\epsilon,i}^I(\vec{x}) \tilde{A}_{\epsilon,i}^I(\vec{y}) \right] 6
\[ + \epsilon^{0il} \left( \tilde{A}_{\xi l}(\vec{x}) \tilde{A}_{\xi j}^{R}(\vec{y}) + \tilde{A}_{\xi l}^{R}(\vec{x}) \tilde{A}_{\xi j}^{R}(\vec{y}) \right) \frac{\epsilon^{2[\mu|-1]} |x-y)_i(x-y)_j}{|\vec{x}-\vec{y}|^{2\Delta_{MCS}}} |\vec{x}-\vec{y}|^2 \]

where

\[ \tilde{\Delta}_{MCS} = |\mu| + 1, \]  

and

\[ \tilde{c}_{MCS} = \frac{|\mu|}{\pi}. \]

We take

\[ \lim_{\epsilon \to 0} \epsilon^{1|-1} \tilde{A}_{\xi i}(\vec{x}) = \tilde{A}_{0i}(\vec{x}), \]

and use again the AdS/CFT correspondence Eq.(33), so that in both cases, \( \mu > 0 \) and \( \mu < 0 \), we get the following two-point function for the boundary conformal field \( J_{i}^{MCS} \)

\[ \langle J_{i}^{MCS}(\vec{x}) J_{j}^{MCS}(\vec{y}) \rangle = \tilde{c}_{MCS} \tilde{\Delta}_{MCS} \left( \delta_{ij} - 2 \frac{(x-y)_i(x-y)_j}{|\vec{x}-\vec{y}|^2} \right) |\vec{x}-\vec{y}|^{-2\tilde{\Delta}_{MCS}}, \]

so that \( J_{i}^{MCS} \) has conformal dimension \( \tilde{\Delta}_{MCS} \). As it is well known the Maxwell-Chern-Simons theory describes a particle with mass \( \mu \) \cite{13} and this fact is reflected in the conformal dimension Eq.(39). Furthermore, our result is consistent with the holographic principle since the mass \( m_{\pm}(0, |\mu|) = 0 \) is not physical in the bulk \cite{13} and does not contribute to the border two-point function.

The Proca theory has been considered in \cite{5} and we derive here the main results for completeness. Making \( \mu = 0 \) on Eq.(9) gives \( m_{\pm}(m, 0) = m \) so that \( A_{\mu}^{\pm} = A_{\mu}^{\mp} = A_{\mu} \). Eqs.(17,18) vanish identically and the field \( A_{\mu} \) is real. The action reads

\[ I_{P} = \frac{m}{2} \int d^2x \epsilon^{2} \tilde{A}_{\xi i}(\vec{x}) \tilde{A}_{\xi i}(\vec{x}) \]

\[ - \tilde{c}_{P} \tilde{\Delta}_{P} \int d^2x d^2y \tilde{A}_{\xi i}(\vec{x}) \tilde{A}_{\xi i}(\vec{y}) \frac{2^{m-1} |x-y)_i(x-y)_j}{|\vec{x}-\vec{y}|^{2\tilde{\Delta}_{P}}} \left( \delta_{ij} - 2 \frac{(x-y)_i(x-y)_j}{|\vec{x}-\vec{y}|^2} \right) \]

\[ + \cdots, \]  

where

\[ \tilde{\Delta}_{P} = m + 1, \]  

and

\[ \tilde{c}_{P} = \frac{m}{\pi}. \]
Taking
\[ \lim_{\epsilon \to 0}\epsilon^{m-1} \tilde{A}_{\epsilon,i}(\vec{x}) = \tilde{A}_{0,i}(\vec{x}), \] (46)
and using the AdS/CFT correspondence Eq.(33) we get
\[ \langle J^P_i(\vec{x}) J^P_j(\vec{y}) \rangle = 2\tilde{c}_P \tilde{\Delta}_P \left( \delta_{ij} - 2 \frac{(x-y)_i(x-y)_j}{|\vec{x} - \vec{y}|^2} \right) \frac{1}{|\vec{x} - \vec{y}|^{-2\tilde{\Delta}_P}}, \] (47)
so that the field \( J^P_i \) has conformal dimension \( \tilde{\Delta}_P \).

3 The Self-Dual Model

We now start with the Euclidean signature action
\[ I^0_{SD} = -\int d^3x \sqrt{g} \left( \frac{i\kappa}{\sqrt{g}} \epsilon^{\mu\nu\alpha} F_{\mu\nu} A_\alpha + \frac{1}{4} M^2 A_\mu A^\mu + \text{c.c.} \right), \] (48)
for the Self-Dual model \[14\]. In order to have a stationary action we must supplement the action Eq.(48) with a surface term which cancels its variation \[15\]. The variational principle generates a boundary term
\[ -\kappa \int d^2x \epsilon^{0ij} \left[ A^R_{i}(\vec{x}) \delta A^I_J(\vec{x}) + A^I_i(\vec{x}) \delta A^R_J(\vec{x}) \right], \] (49)
which is written in terms of the real and imaginary parts of the vector potential. Since the field equations derived from Eq.(48) are first order differential equations we can not choose boundary conditions which fix simultaneously the real and imaginary parts of the \( A_i \)'s. Then we choose boundary conditions on the \( A^R_i \)'s leaving a non-vanishing term proportional to the \( \delta A^I_i \)'s in the boundary term Eq.(49). So we add to the action Eq.(48) a surface term of the form
\[ I^\text{surface}_{SD} = \frac{\kappa}{2} \int d^2x \epsilon^{0ij} A^R_i(\vec{x}) A^I_j(\vec{x}), \] (50)
and the action
\[ I_{SD} = I^0_{SD} + I^\text{surface}_{SD}, \] (51)
is now stationary.

The field equations which follow from the action Eq.(51) are
\[ i\kappa \frac{1}{\sqrt{g}} \epsilon^{\mu\alpha\beta} \partial_\alpha A_\beta + M^2 A^\nu = 0. \] (52)
It implies again

$$\nabla_\mu A^\mu = 0. \quad (53)$$

Using the equations of motion we find that

$$A^I_j = -\frac{\kappa}{2M^2} \sqrt{g} \epsilon^{i\alpha\beta} F^R_{\alpha\beta}, \quad (54)$$

so that $I_{SD}^{\text{surface}}$ is rewritten as

$$I_{SD}^{\text{surface}} = -\frac{\kappa^2}{2M^2} \int d^3 x \partial_\mu (\sqrt{g} F^R_{\mu\nu} A^R_\nu)$$

$$= -\frac{\kappa^2}{2M^2} \int d^2 x \epsilon^{-2} \tilde{A}^R_{\epsilon,i} \left( -\tilde{A}^R_{\epsilon,i} + \epsilon F^R_{\epsilon,0i} \right), \quad (55)$$

and depends only on the $\tilde{A}^R_{\epsilon,i}$'s.

As in the case of the Proca-Chern-Simons theory we can eliminate the Levi–Civita tensor density by increasing the order of the equations of motion. We then get

$$\left( \nabla^2 - \frac{M^4}{\kappa^2} - \frac{R}{3} \right) A^\mu = 0. \quad (56)$$

Proceeding as before we find the solution

$$\tilde{A}_0(x) = \int \frac{d^2 k}{(2\pi)^2} e^{-i\vec{k}\cdot\vec{x}} x_0^2 b_0(\vec{k}) K_{\frac{M^2}{|\kappa|}}(kx_0), \quad (57)$$

and

$$\tilde{A}_i(x) = \int \frac{d^2 k}{(2\pi)^2} e^{-i\vec{k}\cdot\vec{x}} x_0 \left( b_i(\vec{k}) K_{\frac{M^2}{|\kappa|}}(kx_0) + ib_0(\vec{k}) \frac{k_i}{k} k x_0 K_{\frac{M^2}{|\kappa|}+1}(kx_0) \right). \quad (58)$$

From Eq.(53) we get

$$ik_i b_i(\vec{k}) = \frac{M^2}{|\kappa|} b_0(\vec{k}). \quad (59)$$

As before we would like to express the coefficients $b$ in terms of the Fourier components of the vector field at the near boundary surface $x_0 = \epsilon$. It should be noted that since the bulk term of the action Eq.(51) vanishes on-shell all the contributions to the two-point function on the boundary CFT come from the surface term Eq.(55) and depend only on the real components $\tilde{A}^R_{\epsilon,i}$'s. Using that Eq.(54) applies separately
to the real and imaginary parts of $A_\mu$ we find for the relevant parts of the coefficients $b$ (i.e., those which contain the real components of the $\tilde{A}_{\epsilon,i}$’s) the following expressions

\begin{align}
  b_0(\vec{k}) &= -\frac{i\epsilon^{-1} \tilde{A}^R_{\epsilon,i}(\vec{k})k_i}{M^2_{[\kappa]} K\frac{M^2}{M^2_{[\kappa]}} (k\epsilon) + k\epsilon M^2_{\frac{M^2}{[\kappa]}} - 1(k\epsilon)}, \\
  b_i(\vec{k}) &= \frac{\epsilon^{-1} \tilde{A}^R_{\epsilon,i}(\vec{k})}{K\frac{M^2}{M^2_{[\kappa]}} (k\epsilon)} - \frac{k_i k_j}{k} \frac{\tilde{A}^R_{\epsilon,j}(\vec{k}) K\frac{M^2}{M^2_{[\kappa]}} + 1(k\epsilon)}{K\frac{M^2}{M^2_{[\kappa]}} (k\epsilon) + k\epsilon M^2_{\frac{M^2}{[\kappa]}} (k\epsilon) K\frac{M^2}{M^2_{[\kappa]}} - 1(k\epsilon)}. \tag{60}
\end{align}

Proceeding as before we find

\begin{align}
  I_{SD} &= \frac{|\kappa|}{2} \int d^2 x \epsilon^{-2} \tilde{A}^R_{\epsilon,i}(\vec{x}) \tilde{A}^R_{\epsilon,i}(\vec{x}) \\
  &- \tilde{c}_{SD} \tilde{\Delta}_{SD} \int d^2 x d^2 y \tilde{A}^R_{\epsilon,i}(\vec{x}) \tilde{A}^R_{\epsilon,j}(\vec{y}) \frac{e^{\frac{M^2}{[\kappa]} - 1}}{|\vec{x} - \vec{y}|^{2 \Delta_{SD}}} \left( \delta_{ij} - \frac{2(x - y)_i (x - y)_j}{|\vec{x} - \vec{y}|^2} \right) \\
  &+ \ldots, \tag{62}
\end{align}

where

\begin{align}
  \tilde{\Delta}_{SD} &= \frac{M^2}{[\kappa]} + 1, \tag{63}
\end{align}

and

\begin{align}
  \tilde{c}_{SD} &= \frac{|\kappa|}{\pi}. \tag{64}
\end{align}

Now taking

\begin{align}
  \lim_{\epsilon \to 0} \epsilon^{\frac{M^2}{[\kappa]} - 1} \tilde{A}^R_{\epsilon,i}(\vec{x}) = A_{0,i}(\vec{x}), \tag{65}
\end{align}

and using the AdS/CFT correspondence Eq.(33) we find the two-point function of the conformal field $J^{SD}_i$ coupled to the field $\tilde{A}_i$ on the boundary

\begin{align}
  \langle J^{SD}_i(\vec{x}) J^{SD}_j(\vec{y}) \rangle &= 2\tilde{c}_{SD} \tilde{\Delta}_{SD} \left( \delta_{ij} - \frac{2(x - y)_i (x - y)_j}{|\vec{x} - \vec{y}|^2} \right) |\vec{x} - \vec{y}|^{-2\tilde{\Delta}_{SD}}. \tag{66}
\end{align}

We then find that the field $J^{SD}_i$ has conformal dimension $\tilde{\Delta}_{SD}$.

Comparing Eqs.(53) and (54) we see that the conformal dimensions of the conformal fields corresponding to the Maxwell-Chern-Simons theory and the Self-Dual model are the same for $\frac{M^2}{[\kappa]} = |\mu|$ in agreement with the equivalence between those models [16].
4 Conclusions

As expected from the holographic principle the conformal dimensions depend only on the masses of the corresponding theories in the bulk. Although the solutions in the bulk expressed in terms of the boundary values have a complicated form the boundary two-point functions are very simple as dictated by conformal invariance. In fact for each theory we could have just solved the corresponding Proca equations and used this solution to find the two-point function on the border. The final result is insensitive to the detailed structure in the bulk.

Another manifestation of the holographic principle is the fact that in the massless limit the Proca-Chern-Simons theory gives rise to only one massive excitation of mass $|\mu|$ and the massless mode becomes unphysical. This is reflected in the border CFT where the two-point function Eq.(12) has a contribution only from the massive mode of the bulk theory.

The equivalence between the Self-Dual model and the Maxwell-Chern-Simons theory in flat space-time is well known [16] and it can easily be shown to be true also in curved space-time either at the level of the equations of motion or by defining a master Lagrangian in curved space-time. The fact that we obtain the same conformal dimension for the corresponding CFT’s in the border is in support of the holographic principle. Not only the conformal dimensions are the same but the coefficients $\tilde{c}$ of the two-point functions can be made the same by an appropriate normalization of the Self-Dual action. Since we started with two independent parameters in Eq.(18) we can now choose $M = |\kappa|$ so that the model describes a particle with mass $M$. Now our results have an universal form in which the conformal dimension and the two-point function coefficient can be written as $\tilde{\Delta} = m + 1$ and $\tilde{c} = m/\pi$ respectively, where $m$ is the mass of the bulk theory.

5 Acknowledgements

It is a pleasure to thank Carlos Núñez for encouragement and very useful conversations. P.M. acknowledges the support by CAPES. V.O.R. is partially supported by CNPq and acknowledges a grant by FAPESP.

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