PURITY RESULTS FOR SOME ARITHMETICALLY DEFINED MEASURES

PETER J. GRABNER*

Dedicated to Jörg Thuswaldner on the occasion of his 50th birthday.

Abstract. We study measures that are obtained as push-forwards of measures of maximal entropy on sofic shifts under digital maps $(x_k)_{k \in \mathbb{N}} \mapsto \sum_{k \in \mathbb{N}} x_k \beta^{-k}$, where $\beta > 1$ is a Pisot number. We characterise the continuity of such measures in terms of the underlying automaton and show a purity result.

1. Introduction

Digital representations of real numbers by infinite series

$$\sum_{k=1}^{\infty} \frac{x_k}{\beta^k}$$

with $x_k \in A$, a finite alphabet, and $\beta > 1$ have attracted attention from different points of view. The underlying dynamical system given by the map $T_\beta : x \mapsto \beta x \mod 1$ has been studied extensively since the seminal papers [26,32]. For an overview of the development we refer to [3,8,15,16]. The original study was carried out for the “canonical” digit set $A = \{0, 1, \ldots, \lceil \beta \rceil - 1\}$, but many variations have been studied. It turned out in [26] that Pisot numbers $\beta$ play a very important role in that context, as for these $\beta$ the transformation $T_\beta$ has especially nice properties. In this case the set of representations of all real numbers in $[0, 1]$ obtained by iteration of $T_\beta$ is a sofic shift (see [26]); the definition of a sofic shift will be given in Section 2. A Pisot number $\beta = \beta_1$ (of degree $r \geq 1$) is an algebraic integer all of whose Galois conjugates $\beta_2, \ldots, \beta_r$ have modulus $< 1$ (see [4]). Notice that integers $\geq 2$ are also considered as Pisot numbers. Pisot numbers have the nice property that their powers are “almost integers”, meaning that $(\beta^n \mod 1)_{n \in \mathbb{N}}$ tends to 0.

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In the present paper we will change the point of view starting with a one-sided sofic shift space $\mathcal{K}^+ \subset \mathcal{A}^\mathbb{N}$, where $\mathcal{A} \subset \mathbb{Z}$ is the underlying set of digits. Then we consider a map $\phi^+: \mathcal{K}^+ \to \mathbb{R}$ mapping $(x_k)_{k \in \mathbb{N}}$ to the series (1) for $\beta$ a Pisot number. Of course, in general nothing can be said about injectivity of this map, or even the structure of the image. Even for the full shift $\mathcal{A}^\mathbb{N}$ the structure of the image can be intricate (see [45]). If $\mathcal{K}^+$ is equipped with a shift invariant measure, then this measure is pushed forward to $\mathbb{R}$ by the map $\phi^+$. The properties of measures obtained in this way are our object of study. For the measure on $\mathcal{K}^+$ we will take the unique shift invariant measure of maximal entropy, or Parry measure, see [27, 28]. This will be discussed in Section 3.

Measures of this kind occur in different contexts. Possibly, the earliest occurrence was in two papers by Erdős [12, 13], where he proved the singularity of this measure for $\mathcal{K}$ being the full shift $\{-1, 1\}^\mathbb{N}$. This is the most studied case and goes under the name “Bernoulli convolution”. We will discuss that further in Section 4.

In the context of studying redundant expansions of integers in the context of fast multiplication algorithms used in cryptography, the precise study of the number of representations of an integer $n$ in the form

$$n = \sum_{k=0}^{K} x_k 2^k \quad \text{with } x_k \in \{0, \pm 1\}$$

and minimising

$$\sum_{k=0}^{K} |x_k|$$

led to a singular measure on $[-1, 1]$ (see [17]). Further results in this direction were obtained in [18, 19]. A more general point of view replacing the powers of 2 by the solution of a linear recurrence has been taken in [21]. Furthermore, such measures occur as spectral measures of dynamical systems related to numeration systems [20], in the study of diffraction patterns of tilings (see [1, 2]), and as spectral measures of substitution dynamical systems (see [30]). In Section 5 we will present two main results, namely the fact that the measure is pure (meaning that the Lebesgue decomposition only has one term), and a characterisation of continuity of the measure in terms of properties of the underlying automaton.

Erdős’ proof of the singularity of the measure in [12, 13] uses the fact that the Fourier transform of these measures does not tend to 0 at $\infty$, and this method was used in many other cases. This motivates the
study of the Fourier transform of the measures under consideration. The transform can be expressed in terms of infinite matrix products, which allow the computation of limits
\[ \lim_{k \to \infty} \hat{\nu}(z \beta^k) \]
for \( z \in \mathbb{Z}[\beta] \). In Section 6 we find an interpretation of these limits as Fourier coefficients of a measure on the torus given by a two sided version of the map \( \phi^+ \). A similar map has been introduced and studied in [38, 39]. This will be used to show that the vanishing of the limits (2) for all \( z \in \mathbb{Z}[\beta], z \neq 0 \) is equivalent to absolute continuity for these measures.

Very recently, similar results for self similar measures have been obtained in [5]. There for a set of linear maps \( \phi_k : x \mapsto r_k x + b_k \) with \( 0 < r_k < 1 \) (\( k = 1, \ldots, N \)) the distribution measure \( \nu \) of the map
\[(\omega_1, \omega_2, \ldots) \mapsto \lim_{n \to \infty} \phi_{\omega_n} \circ \phi_{\omega_{n-1}} \circ \cdots \circ \phi_{\omega_1}(x_0)\]
is studied, where \( \{1, \ldots, N\}^\mathbb{N} \) is equipped with the infinite product of the measures \( \mathbb{P}(\{k\}) = p_k \) (\( k = 1, \ldots, N \)) for any choice of the vector \( (p_1, \ldots, p_N) \) with \( p_k > 0 \) and \( \sum_k p_k = 1 \). It is shown (see [5, Theorem 2.3]) that \( \hat{\nu}(t) \not\to 0 \) for \( |t| \to \infty \), if and only if the contraction factors \( r_k \) are all negative integer powers of the same Pisot number. It is also shown (see [5, Theorem 2.4]) that for these measures \( \lim_{|t| \to \infty} \hat{\nu}(t) = 0 \) (called the Rajchman property) is equivalent to absolute continuity. This is a phenomenon very similar to Theorem 5 in the context of this paper.

In a final Section 7 we exhibit several simple examples as applications of our results.

2. Regular languages and sofic shifts

Let \( \mathcal{A} \subset \mathbb{Z} \) be a finite alphabet, and denote by \( \mathcal{A}^* \) the set of finite words over \( \mathcal{A} \), i.e. \( \mathcal{A}^* = \{\epsilon\} \cup \bigcup_{n \in \mathbb{N}} \mathcal{A}^n \), where \( \epsilon \) denotes the empty word. Let \( G = (V, E) \) be a finite directed graph (see [9]) equipped with a labelling \( \ell : E \to \mathcal{A} \). Then the pair \( (G, \ell) \) is called a labelled graph. A finite automaton is a quadruple \( \mathbf{A} = (G, \ell, I, T) \), where \( I \) (initial states) and \( T \) (terminal states) are subsets of the set of vertices (also called states in this context). A path of length \( n \) in the graph \( G \) is a sequence of edges \( p = e_1 e_2 \ldots e_n \), such that for every \( j = 1, \ldots, n-1 \) the edges \( e_j = (i(e_j), t(e_j)) \) satisfy \( t(e_j) = i(e_{j+1}) \); the terminal vertex \( t(e_j) \) of every edge coincides with the initial vertex \( i(e_{j+1}) \) of the consecutive edge. We say that \( p \) connects \( i(e_1) \) and \( t(e_n) \). The language \( \mathcal{L} = \mathcal{L}(\mathbf{A}) \)
recognised by the automaton $A$ is given by

\begin{equation}
\mathcal{L}_n = \{ \ell(e_1)\ell(e_2)\ldots\ell(e_n) \mid e_1e_2\ldots e_n \text{ a path in } G, i(e_1) \in I, t(e_n) \in T \}
\end{equation}

\begin{equation*}
\mathcal{L} = \{ \epsilon \} \cup \bigcup_{n=1}^{\infty} \mathcal{L}_n,
\end{equation*}

where $\epsilon$ denotes the empty word, which by definition has length 0; $\mathcal{L}_n$ is the set of words of length $n$. A subset of $A^*$ is called a regular language, if it is recognised by a finite automaton $A$. A language $\mathcal{L}$ is called irreducible, if for any $w_1, w_2 \in \mathcal{L}$ there exists a $w \in A^*$ such that $w_1ww_2 \in \mathcal{L}$. This is equivalent to the underlying graph being strongly connected; for any two vertices $v_1, v_2 \in V$ there is a path connecting $v_1$ with $v_2$. The language is called primitive, if there is an $N \in \mathbb{N}$ such that for any $w_1, w_2 \in \mathcal{L}$ and any $n \geq N$ there exists a word $w \in A^*$ of length $n$ such that $w_1ww_2 \in \mathcal{L}$.

From now on we assume that all languages are primitive. For a comprehensive introduction to the theory of formal languages we refer to [10, 22, 34].

For a given automaton $A$ we study the sets of one- and two-sided infinite words recognised by $A$

\begin{equation}
\mathcal{K}_I^+ = \{ (\ell(e_k))_{k \in \mathbb{N}} \mid \forall n \in \mathbb{N} : e_1e_2\ldots e_n \text{ is a path in } G, i(e_1) \in I \}
\end{equation}

\begin{equation}
\mathcal{K}^+ = \{ (\ell(e_k))_{k \in \mathbb{N}} \mid \forall n \in \mathbb{N} : e_1e_2\ldots e_n \text{ is a path in } G \}
\end{equation}

\begin{equation}
\mathcal{K} = \{ (\ell(e_k))_{k \in \mathbb{Z}} \mid \forall m < n : e_me_{m+1}\ldots e_n \text{ is a path in } G \}.
\end{equation}

The set $\mathcal{K}$ is called the (two-sided) sofic shift associated to $A$ (see [24]), $\mathcal{K}^+$ is the one-sided sofic shift, both spaces are closed under the according shift transformation

\begin{equation*}
\sigma^+ : \mathcal{K}^+ \rightarrow \mathcal{K}^+ : (x_k)_{k \in \mathbb{N}} \mapsto (x_{k+1})_{k \in \mathbb{N}}
\end{equation*}

\begin{equation*}
\sigma : \mathcal{K} \rightarrow \mathcal{K} : (x_k)_{k \in \mathbb{Z}} \mapsto (x_{k+1})_{k \in \mathbb{Z}}.
\end{equation*}

The space $\mathcal{K}_I^+$, which can be seen as an extension of $\mathcal{L}$ to infinite words, is in general not closed under $\sigma^+$, but the relation

\begin{equation*}
\mathcal{K}^+ = \bigcup_{n=0}^{N} (\sigma^+)^n \mathcal{K}_I^+
\end{equation*}

holds for some $N \in \mathbb{N}$ as a consequence of the strong connectedness of the graph underlying $A$. Notice that $\sigma$ is bijective, whereas $\sigma^+$ is not.
3. Measures on shift spaces

We equip the spaces $K^+_I$ and $K^+$ with a “canonical” measure that we will define now. We define the cylinder set

$$[\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k] = \{(x_n)_{n \in \mathbb{N}} \in K^+_I \mid x_1 = \varepsilon_1, \ldots, x_k = \varepsilon_k\}$$

for $\varepsilon_i \in \mathcal{A}$ for $i = 1, \ldots, k$. The cylinder sets generate a topology on $K^+_I$ and also the $\sigma$-algebra of Borel sets for this topology. We define

$$(5) \quad \mu_I^+([\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k]) = \lim_{n \to \infty} \frac{\#([\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k] \cap L_n)}{\#L_n};$$

the existence of the limit will become obvious from the following discussion.

For $a \in \mathcal{A}$ define the $a$-transition matrix by

$$(6) \quad (M_a)_{ij} = \begin{cases} 1 & \text{if } (i, j) \in E \text{ and } \ell((i, j)) = a \\ 0 & \text{otherwise} \end{cases}$$

and

$$(7) \quad M = \sum_{a \in \mathcal{A}} M_a.$$  

Furthermore, set $v_I$ and $v_T$ the indicator vectors of the sets $I$ and $T$, respectively. Then for $n \geq k$ we have

$$\#([\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k] \cap L_n) = v_I^T M_{\varepsilon_1} \cdots M_{\varepsilon_k} M^{n-k} v_T.$$  

By the assumption that the language $\mathcal{L}$ is primitive which is equivalent to the fact that $M$ is primitive, the Perron-Frobenius theorem (see [35]) implies that there is a dominating eigenvalue $\lambda > 0$ such that

$$(8) \quad M^n = \lambda^n v_R v_L^T + o(\lambda^n),$$

where $v_L^T$ is a left eigenvector of $M$ for the eigenvalue $\lambda$, and $v_R$ is a right eigenvector with

$$v_L^T v_R = 1.$$  

With this we can write the quantity under the limit in (5) as

$$v_I^T M_{\varepsilon_1} \cdots M_{\varepsilon_k} (\lambda^{n-k} v_R v_L^T + o(\lambda^n)) v_T$$

$$= v_I^T (\lambda^n v_R v_L^T + o(\lambda^n)) v_T,$$

which shows that the limit exists and equals

$$(9) \quad \mu_I^+([\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k]) = \lambda^{-k} \frac{v_I^T M_{\varepsilon_1} \cdots M_{\varepsilon_k} v_R}{v_I^T v_R}.$$  

On $K^+$ we define the measure $\mu^+$ by

$$(10) \quad \mu^+([\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k]) = \lambda^{-k} v_L^T M_{\varepsilon_1} \cdots M_{\varepsilon_k} v_R.$$
This measure is shift invariant by the observation
\[
\mu^+((\sigma^+)^{-1}[\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k]) = \sum_{a \in A} \mu^+([a, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k])
\]
\[
= \sum_{a \in A} \lambda^{-k-1} v_L^T M a M \varepsilon_1 \cdots M \varepsilon_k v_R = \lambda^{-k-1} v_L^T M M \varepsilon_1 \cdots M \varepsilon_k v_R
\]
\[
= \lambda^{-k} v_L^T M \varepsilon_1 \cdots M \varepsilon_k v_R = \mu^+([\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k]).
\]
A similar computation shows that the measure is well defined by Kolmogorov’s consistency theorem.

Assuming that \( L \) is primitive, the dynamical system \((\mathcal{K}^+, \mu^+, \sigma^+)\) is strongly mixing and thus ergodic (see [7, 44]):
\[
\lim_{n \to \infty} \mu^+\left([\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k] \cap (\sigma^+)^{-n}[\delta_1, \ldots, \delta_s]\right)
\]
\[
= \lim_{n \to \infty} \lambda^{-n-s} v_L^T M \varepsilon_1 \cdots M \varepsilon_k M^n \delta_1 \cdots M \delta_s v_R
\]
\[
= \lambda^{-k} v_L^T M \varepsilon_1 \cdots M \varepsilon_k v_R \lambda^{-s} v_L^T M \delta_1 \cdots M \delta_s v_R
\]
\[
= \mu^+([\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k]) \mu^+([\delta_1, \ldots, \delta_s]).
\]
The measures \( \mu^+_I \) and \( \mu^+|_{\mathcal{K}_I^+} \) (restriction to \( \mathcal{K}_I^+ \)) are equivalent. The measure \( \mu^+ \) is the unique shift invariant measure of maximal entropy on \( \mathcal{K}^+ \), also called the Parry measure [27,28]. Maximality and uniqueness follow from [27, Theorems 6 and 7] together with [27, Theorem 10].

Similarly, we define a measure on \( \mathcal{K} \) by
\[
\mu([\varepsilon_1, \ldots, \varepsilon_k]_m) = \lambda^{-k} v_L^T M \varepsilon_1 \cdots M \varepsilon_k v_R,
\]
where
\[
[\varepsilon_1, \ldots, \varepsilon_k]_m = \{ (x_n)_{n \in \mathbb{Z}} \in \mathcal{K} \mid x_{m+1} = \varepsilon_1, \ldots, x_{m+k} = \varepsilon_k \}
\]
for \( m \in \mathbb{Z} \). The measure \( \mu \) is \( \sigma \)-invariant by definition.

4. Bernoulli convolutions

In [12] Erdős studied the distribution measure of the random series
\[
\sum_{k=1}^{\infty} \frac{X_k}{\beta^k},
\]
where \((X_k)_{k \in \mathbb{N}}\) is a sequence of i.i.d. random variables taking the values \( \pm 1 \) with equal probability \( \frac{1}{2} \), and \( \beta = \frac{1+\sqrt{5}}{2} \). He showed that the distribution is purely singular continuous. Later [13] he extended this result for \( \beta \) an irrational Pisot number. The Pisot property plays an important rôle in the proof of singularity, as it allows to show that the
Fourier transform of the measure does not tend to 0 along the sequence \((\beta^k)_{k \in \mathbb{N}}\). This argument will be elaborated later.

In the meantime the set of \(\beta > 1\), for which the measure constructed as above is singular continuous has been studied further. Solomyak [40] could prove that for almost all \(\beta \in (1, 2)\) the measure is absolutely continuous. This result was refined by Shmerkin [36], who proved that the exceptional set has Hausdorff dimension 0. It is still open, whether Pisot numbers are the only exceptions. For a survey on the development until the year 2000 we refer to [29]. For more recent developments and results in this direction we refer to [33, 37].

A newer development in the study of Bernoulli convolutions was initiated with the proof that ergodic invariant measures on the full shift \(\mathcal{A}^\mathbb{N}\) are projected to exact dimensional measures by iterated function systems (see [14]). More precisely, a measure \(\nu\) is called exact dimensional with dimension \(\alpha\), if

\[
\lim_{r \to 0} \frac{\log \nu((x - r, x + r))}{\log r} = \alpha
\]

for \(\nu\)-almost all \(x\). Of course, absolutely continuous measures on \(\mathbb{R}\) have dimension 1, whereas the opposite implication is not true. This allowed for the proof that the dimension of the distribution measure of (12) has dimension 1 for all transcendental \(\beta \in (1, 2)\) (see [42]). For further results in that direction we refer to [6, 41, 43].

5. Generalised Erdős measures

From now on we assume that the alphabet \(\mathcal{A}\) is a subset of \(\mathbb{Z}\). For an automaton \(A\) and a Pisot number \(\beta\) of degree \(r \geq 1\) we introduce the maps

\[
\phi_+^+: \mathcal{L} \to \mathbb{R} : (x_k)_{k=1}^n \mapsto \sum_{k=1}^n \frac{x_k}{\beta^k}
\]

(13)

\[
\phi_+^+: \mathcal{K}_+^+ \to \mathbb{R} : (x_k)_{k \in \mathbb{N}} \mapsto \sum_{k=1}^\infty \frac{x_k}{\beta^k}
\]

\[
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\]

The measures

\[
\nu_I = (\phi_+^+)_* (\mu_+^+)
\]

(14)

\[
\nu = (\phi_+^+)_* (\mu^+)
\]
are analogues of the Erdős measures studied in [12, 13]. Here and throughout this paper we denote by $f_*(\mu)$ the push-forward measure on $Y$ given by a map $f : X \to Y$ and a measure $\mu$ on $X$; $f_*(\mu)(A) = \mu(f^{-1}(A))$. The properties of the measures $\nu$ and $\nu_I$ will be the subject of the remaining part of this paper. By definition, $\nu_I$ is absolutely continuous with respect to $\nu$. Furthermore, by the definition of $\mu^+_I$, $\nu_I$ is given by

$$\nu_I = \lim_{n \to \infty} \frac{1}{\# L_n} \sum_{w \in L_n} \delta_{\phi_+^I(w)},$$

where $\delta_x$ denotes a unit point mass in $x$.

**Theorem 1.** The measures $\nu_I$ and $\nu$ are pure in the sense that they are either absolutely continuous with respect to Lebesgue measure, purely singular continuous, or purely atomic. The last case can only occur, if the image $\phi_+^I(K^+) \subset A$ is finite. The number of atoms is bounded by the number of vertices in $A$.

**Proof.** The Jessen-Wintner theorem [23, Theorem 35] (for a more modern formulation see also [11, Lemma 1.22 (ii)]) is concerned with the distribution measure of a random series

$$\sum_{n=1}^{\infty} X_n,$$

where $(X_n)_{n \in \mathbb{N}}$ is a sequence of independent discrete random variables and the series

$$\sum_{n=1}^{\infty} \mathbb{E}(X_n) \text{ and } \sum_{n=1}^{\infty} \mathbb{V}(X_n)$$

converge. It states that this measure is either absolutely continuous with respect to Lebesgue measure, purely singular continuous, or purely atomic.

For our purposes we need a more general version, which allows for some dependence between the random variables $X_n$.

**Lemma 1.** Let $K^+ \subset A^\mathbb{N}$ be a shift invariant subset equipped with a shift invariant measure $\mu$ such that the shift is ergodic with respect to $\mu$. Let

$$X = \sum_{n=1}^{\infty} X_n$$

be a series of random variables $X_n$, where $X_n$ only depends on the $n$-th coordinate of the argument. Also assume that the series converges $\mu$-almost everywhere. Then the distribution of $X$ is either purely discrete,
or purely singular continuous, or absolutely continuous with respect to Lebesgue measure.

Proof of Lemma 1. The proof of this lemma is just the observation that the proof of the Jessen-Wintner theorem only uses the fact that the measure on the product space satisfies a 0-1-law. The proof follows the lines of proof given for [11, Lemma 1.22(ii)]. Let \( \nu = X_\ast(\mu) \) be the distribution measure of \( X \). Then \( \nu \) has a Lebesgue decomposition

\[
\nu = a_1 \nu_1 + a_2 \nu_2 + a_3 \nu_3,
\]

where \( a_1 + a_2 + a_3 = 1 \) and \( \nu_i \) (\( i = 1, 2, 3 \)) are probability measures, where \( \nu_1 \) is purely atomic, \( \nu_2 \) is absolutely continuous, and \( \nu_3 \) is singular continuous. Let \( D \) be the support of \( \nu_1 \). Then \( X_\ast(D) \) is a shift invariant subset of \( K^+ \), which has measure 0 or 1 by ergodicity. Thus \( a_1 \) is either 0 or 1. Similarly, let \( S \) be the set of Lebesgue measure 0 that supports \( \nu_3 \). Then again \( X_\ast(S) \) is shift invariant, and again has measure 0 or 1. Thus the decomposition has exactly one term. \( \square \)

The statement for \( \nu_I \) follows from the absolute continuity of \( \nu_I \) with respect to \( \nu \). The assertion about the number of atoms will be proved in the proof of Theorem 2. \( \square \)

Lemma 2. Let \( P = \{ x \in \mathbb{R} \mid \nu(\{ x \}) > 0 \} \). Then \( P \) is a finite subset of \( \mathbb{Q}(\beta) \).

Proof. For \( x \in P \) set \( A_x = (\phi^+)^{-1}(\{ x \}) \). Then \( \phi^+(A_x) = \{ x \} \) and

\[
\phi^+(\sigma^{-m}(A_x)) \subseteq \left\{ \beta^{-m} \left( x + \sum_{k=0}^{m-1} \ell(e_{m-k})\beta^k \right) \mid e_1 \ldots e_m \text{ a path in } A \right\}.
\]

Since \( \mu^+(A_x) > 0 \) there is an \( n \geq 1 \) such that \( A_x \cap \sigma^{-n}(A_x) \neq \emptyset \). This implies that

\[
x \in \left\{ \beta^{-n} \left( x + \ell(e_n) + \cdots + \ell(e_1)\beta^{n-1} \right) \mid e_1 \ldots e_n \text{ a path in } A \right\},
\]

from which it follows that \( x \in \mathbb{Q}(\beta) \). Then there exists \( N \in \mathbb{N} \) such that \( x \in \frac{1}{N}\mathbb{Z}[\beta] \).

Now fix \( x \in P \) and thus \( N \). Let \( y \in P \). Then there exists an \( n \geq 1 \) such that \( A_x \cap \sigma^{-n}(A_y) \neq \emptyset \) again by ergodicity. This implies that there exists a path \( e_1 \ldots e_n \) in \( A \) such that

\[
y = \beta^n x - \sum_{k=0}^{n-1} \ell(e_{n-k})\beta^k,
\]
which shows that $y \in \mathbb{N} \mathbb{Z}[\beta]$. Thus $P \subset \mathbb{N} \mathbb{Z}[\beta]$. Now by definition $P$ is bounded. Applying the conjugations $\beta \mapsto \beta_q$ $(q = 2, \ldots, r)$ gives
\[
|y_q| \leq |\beta_q|^n |x_q| + \sum_{k=0}^{m-1} |\ell(e_{m-k})||\beta_q|^k \leq |x_q| + \frac{M}{1 - |\beta_q|},
\]
if $|a| \leq M$ for all $a \in \mathcal{A}$. Thus $NP$ is a set of algebraic integers all of whose conjugates are bounded. Thus $P$ is finite. \hfill \square

Lemma 3. Let $\beta$ be a Pisot number. Then the set of words
\[
\mathcal{L}_0 = \left\{ (x_1, x_2, \ldots, x_n) \in \mathcal{A}^* \left| \sum_{k=1}^n \frac{x_k}{\beta^k} = 0 \right. \right\}
\]
is recognisable by a finite automaton. As a consequence the set
\[
(15) \quad \mathcal{K}_0^+ = \left\{ (x_1, x_2, \ldots) \in \mathcal{A}^\mathbb{N} \left| \sum_{k=1}^\infty \frac{x_k}{\beta^k} = 0 \right. \right\}
\]
is a space $\mathcal{K}_0^+$ for that automaton and an appropriate initial state $I$ (labelled by 0).

Proof. Assume that $\mathcal{L}_0 \neq \emptyset$; otherwise the assertion is trivial. We define the set
\[
E = \left\{ \sum_{k=1}^{m-s} \frac{x_{k+s}}{\beta^k} \left| 0 \leq s < m, (x_1, \ldots, x_m) \in \mathcal{L}_0 \right. \right\}.
\]
This will be the set of states. Between two elements $x, y \in E$ there is a transition, if and only if
\[
a = \beta x - y \in \mathcal{A};
\]
this transition will then be marked with $a$. Since by definition of $E$ every element of $E$ can be obtained from $0 \in E$ by finitely many transitions $x \mapsto \beta x - a$ with $a \in \mathcal{A}$. This shows that every element of $E$ can be expressed in the form
\[
- \sum_{k=0}^{m} x_k \beta^k \quad \text{with} \quad x_0, \ldots, x_m \in \mathcal{A}.
\]
This shows that all elements of $E$ are algebraic integers. The original definition of $E$ shows that all elements of $E$ are bounded by $\frac{M}{\beta - 1}$, if $|a| \leq M$ for all $a \in \mathcal{A}$. Furthermore, all conjugates of elements of $E$ are bounded by
\[
\left| - \sum_{k=0}^{m} x_k \beta_q^k \right| \leq \frac{M}{1 - |\beta_q|}.
\]
Lemma 4. Let \( (x, \varepsilon, a) \) be such that \( \phi(x) > 0 \). Then the set 
\( \phi(x) \leq 1 \) is open.

Proof. Since the set \( P \) of atoms of \( \phi \) is finite, 
\( (\phi^+)^{-1}(\{x\}) = (\phi^+)^{-1}((x-\varepsilon, x+\varepsilon)) \) for small enough \( \varepsilon > 0 \). The continuity of \( \phi^+ \) implies that 
\( (\phi^+)^{-1}(\{x\}) \) is open. □

Theorem 2. The set \( \phi^+(\mathcal{K}^+) \) is either finite or perfect and thus uncountable. In the first case the measure \( \phi^+\phi^+(\mu^+) \) is atomic, in the second case it is continuous.

Proof. The set \( \phi^+(\mathcal{K}^+) \) is compact as the continuous image of a compact set.

Assume that there exists a vertex \( v \in V \) and two paths \( e_1e_2\ldots e_{L_1} \) and \( f_1f_2\ldots f_{L_2} \) both connecting \( v \) to itself such that

\[
\frac{1}{1 - \beta^{L_1}} \sum_{k=1}^{L_1} \frac{\ell(e_k)}{\beta^k} \neq \frac{1}{1 - \beta^{L_2}} \sum_{k=1}^{L_2} \frac{\ell(f_k)}{\beta^k}.
\]

Then we will show that the set \( \phi^+(\mathcal{K}^+) \) is perfect. For this purpose we choose \( x \in \phi^+(\mathcal{K}^+) \) and show that there is a sequence \( (x_n)_{n \in \mathbb{N}} \) of points in \( \phi^+(\mathcal{K}^+) \) with \( x = \lim_{n \to \infty} x_n \) and \( x_n \neq x \) for all \( n \).

Let \( x = \phi^+((\ell(a_1), \ell(a_2), \ldots)) \). For \( n \in \mathbb{N} \) we set

\[
\xi_n = \phi^+((\ell(a_1), \ell(a_2), \ldots, \ell(a_n), \ell(b_1), \ell(e_1), \ldots, \ell(e_{L_1})))
\]

\[
\eta_n = \phi^+((\ell(a_1), \ell(a_2), \ldots, \ell(a_n), \ell(b_1), \ell(f_1), \ldots, \ell(f_{L_2}))),
\]

where \( b_1, \ldots, b_k \in E \) are chosen so that \( \ell(a_1) \ldots \ell(a_n) \ell(b_1) \ldots \ell(b_k) \in \mathcal{L} \) and \( t(b_k) = v \). There exists an integer \( k \leq \#V \) with this property for every \( n \). By (16) \( \xi_n \neq \eta_n \), and thus at least one of these two values is different from \( x \). We take \( x_n \) to be this value. Then \( \lim_{n \to \infty} x_n = x \) showing that \( x \) is not isolated.

By Lemma 4 the preimage of any atom of the measure \( \phi^+(\mu^+) \) would contain a cylinder set. This would contradict the fact that we have just proved that the images of all cylinder sets are uncountable. This shows that \( \nu \) is continuous in this case.
Assume on the contrary that for all \( v \in V \) there exists a value \( c(v) \in \mathbb{R} \) such that for all paths \( e_1e_2\ldots e_n \) connecting \( v \) to itself \( (17) \)

\[
c(v) = \frac{1}{1 - \beta^{-n}} \sum_{k=1}^{n} \ell(e_k) \beta^k.
\]

In this case every infinite path \( e_1e_2\ldots \) starting at \( v \) yields the value \( c(v) \) for \( \phi^+ \): assume that \( j \) is chosen to be the minimal index so that \( w = t(e_j) \) is visited infinitely often by the path. If \( j = 1 \), the value of \( \phi^+ \) given by the path is \( c(v) \) by definition. For \( j > 1 \) we decompose the path

\[ e_1\ldots e_{k_1}e_{k_1+1}\ldots e_{k_2}e_{k_2+1}\ldots e_{k_3}\ldots, \]

where \( k_1 = j \) and \( t(e_{k_m}) = w \). Then by our assumption \( (17) \) every path \( e_{k_{m+1}}\ldots e_{k_{m+1}} \) can be replaced by a path \( f_1\ldots f_s f_{s+1}\ldots f_{s+q} \) with \( i(f_1) = w \), \( t(f_s) = i(f_{s+1}) = v \), and \( t(f_{s+q}) = w \) without changing the value of \( \phi^+ \). The new path visits \( v \) infinitely often, and thus assigns the value \( c(v) \) to \( \phi^+ \). Thus \( \phi^+ \) only takes the values \( \{c(v) \mid v \in V\} \).

Each of these values is assigned a positive mass. This proves that the number of atoms of \( \phi^+_+(\mu) \) is bounded by the number of vertices of \( A \). This shows the last assertion of Theorem 1. \( \square \)

The following result is a consequence of the proof of Theorem 2.

**Theorem 3.** The set \( \phi^+(\mathcal{K}^+) \) is finite, if and only if for every vertex \( v \in V \) there exists a value \( c(v) \in \mathbb{R} \) such that for all paths \( e_1e_2\ldots e_n \) connecting \( v \) to itself \( (17) \) holds.

**Remark 1.** Notice that the measure \( \phi^+_+(\mu^+) \) can only be absolutely continuous, if \( \beta \leq \lambda \) (recall that \( \lambda \) is the Perron-Frobenius eigenvalue associated to the automaton \( A \)). Otherwise the set \( \phi^+(\mathcal{K}^+) \) has Hausdorff dimension \( \leq \frac{\log \lambda}{\log \beta} < 1 \) and cannot support an absolutely continuous measure.

6. **Fourier transforms and matrix products**

The Fourier transforms of the measures \( \nu \) and \( \nu_I \) are given by

\[
\widehat{\nu_I}(t) = \int_{-\infty}^{\infty} e^{-2\pi ixt} d\nu_I(x) \quad \widehat{\nu}(t) = \int_{-\infty}^{\infty} e^{-2\pi ixt} d\nu(x). \tag{18}
\]
In order to derive expressions for \( \hat{\nu} \) and \( \hat{\nu}_I \), we introduce the weighted transition matrix for \( \phi^+ \) and the underlying automaton \( A \)

\[
W(t) = \frac{1}{\lambda} \sum_{a \in A} e(-at)M_a,
\]

where we use the notation \( e(x) = e^{2\pi ix} \).

**Proposition 1.** The Fourier transforms of the measures \( \nu \) and \( \nu_I \) can be expressed as

\[
\hat{\nu}(t) = \mathbf{v}_L^T \prod_{n=1}^{\infty} W(\beta^{-n}t) \mathbf{v}_R
\]

and

\[
\hat{\nu}_I(t) = \frac{1}{\mathbf{v}_I^T \mathbf{v}_R} \mathbf{v}_I^T \prod_{n=1}^{\infty} W(\beta^{-n}t) \mathbf{v}_R,
\]

where the infinite matrix product is interpreted so that the factors are ordered from left to right.

**Proof.** We set

\[
\phi_n^+((x_k)_{k \in \mathbb{N}}) = \sum_{k=1}^{n} \frac{x_k}{\beta^k}.
\]

Then \( \lim_{n \to \infty} \phi_n^+((x_k)_{k \in \mathbb{N}}) = \phi^+((x_k)_{k \in \mathbb{N}}) \) holds uniformly on \( K^+ \). We set \( \nu_n = (\phi_n^+) \ast (\mu^+) \) and observe that \( \nu_n \rightharpoonup \nu \). Then

\[
\hat{\nu}_n(t) = \mathbf{v}_L^T \prod_{k=1}^{n} W(\beta^{-k}t) \mathbf{v}_R.
\]

The limit relation \( \lim_{n \to \infty} \hat{\nu}_n = \hat{\nu} \) and equation (20) then follow by weak convergence. A similar reasoning gives (21).

**Remark 2.** As pointed out in Section 4 Erdős [12, 13] proved the singularity of the distribution measure \( \nu \) of the random series

\[
\sum_{n=1}^{\infty} \frac{X_n}{\beta^n}
\]

by showing that \( (\hat{\nu}(\beta^k))_{k \in \mathbb{N}} \) does not tend to 0. Of course the fact that \( \lim_{|t| \to \infty} \hat{\nu}(t) = 0 \) does not suffice in general to prove absolute continuity, as there are singular measures, so called Rajchman measures, whose Fourier transform vanishes at \( \infty \) (see [31]).
Using the Pisot property of $\beta$ we define the map

$$\Phi : \mathcal{K} \rightarrow \mathbb{T}, (x_k)_{k \in \mathbb{Z}} \mapsto \left( \sum_{k=\infty}^{\infty} x_k \beta^{-k+m} \mod 1 \right)_{m=0}^{r-1},$$

where we use the notation $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Notice that the series for $k \leq 0$ converges $\mod 1$ by the fact that

$$\beta^m + \beta^m_2 + \cdots + \beta^m_r \in \mathbb{Z}$$

and $|\beta_2|, \ldots, |\beta_r| < 1$.

**Remark 3.** A similar map was studied in [38, 39]. In [38] this map was used to give a dynamic proof of the singularity of the Erdős measure introduced in [12]. In [39] conditions were given under which this map between the two-sided $\beta$-shift $X_\beta$ and $\mathbb{T}^d$ is almost bijective. In the present situation, where $\beta$ and the underlying language $\mathcal{L}$ are not necessarily related, nothing can be said about (almost) bijectivity of this map.

**Theorem 4.** Let $z = m_0 + m_1 \beta + \cdots + m_{r-1} \beta^{r-1} \in \mathbb{Z}[\beta]$ for $\beta$ a Pisot number of degree $r$ and $\nu$ be the measure given by (14). Then the limit

$$\hat{\psi}(m_0, \ldots, m_{r-1}) = \lim_{k \rightarrow \infty} \hat{\nu}(z \beta^k)$$

exists. These values are the Fourier coefficients of the measure $\psi = \Phi_*(\mu)$ on $\mathbb{T}^r$.

**Proof.** We define the maps

$$\Phi_n((x_k)_{k \in \mathbb{Z}}) = \left( \sum_{k=-n}^{\infty} x_k \beta^{-k+m} \mod 1 \right)_{m=0}^{r-1}.$$

Then $(\Phi_n)_{n \in \mathbb{N}}$ converges to $\Phi$ uniformly on $\mathcal{K}$. Let $\psi_n = (\Phi_n)_*(\mu)$. Then $\psi_n \rightharpoonup \psi$ and

$$\hat{\psi}_n(m_0, \ldots, m_{r-1}) = \hat{\nu}(z \beta^n).$$

The limit relation (23) then follows by weak convergence and the fact that $(\beta^n \mod 1) \rightarrow 0$. $\square$

**Remark 4.** The shift on $\mathcal{K}$ is conjugate via $\Phi$ to the hyperbolic toral endomorphism

$$B : \mathbb{T}^r \rightarrow \mathbb{T}^r, \left( \begin{array}{c} t_0 \\ t_1 \\ \vdots \\ t_{r-1} \end{array} \right) \mapsto \left( \begin{array}{c} t_1 \\ \vdots \\ t_{r-1} \end{array} \right) + \left( \begin{array}{c} a_0 t_0 + \cdots + a_{r-1} t_{r-1} \end{array} \right) \mod 1,$$
where
\[ \beta^r = a_{r-1}\beta^{r-1} + \cdots + a_1\beta + a_0 \]
is the minimal equation of \( \beta \). The measure \( \psi \) is then a \( B \)-invariant measure on \( T^r \), and \( B \) is ergodic with respect to \( \psi \).

**Theorem 5.** The measure \( \nu \) given by (14) is absolutely continuous, if and only if for all \( z \in \mathbb{Z}[\beta] \setminus \{0\} \)
\[ \lim_{k \to \infty} \hat{\nu}(z\beta^k) = 0. \]  

**Proof.** Let \( \beta_2, \ldots, \beta_r \) denote the Galois conjugates of \( \beta \) and assume that \( \beta, \beta_2, \ldots, \beta_r \in \mathbb{R} \) and \( \beta_{s+1}, \beta_{s+2}, \ldots, \beta_{s+t}, \beta_{s+t+1}, \beta_{s+t+2}, \ldots, \beta_{s+2t} \in \mathbb{C} \setminus \mathbb{R} \); then \( r = s + 2t \). Then the map
\[ \tilde{\Phi} : K \to \mathbb{R}^s \times \mathbb{C}^t, (x_k)_{k \in \mathbb{Z}} \mapsto \left( \sum_{k=1}^{\infty} x_k\beta^{-k}, \sum_{k=0}^{\infty} x_k\beta_2^k, \ldots, \sum_{k=0}^{\infty} x_k\beta_{s+t}^k \right) \]
is continuous and \( \tilde{\Phi}(K) \) is compact. Together with the map
\[ \rho : \mathbb{R}^s \times \mathbb{C}^t \to \mathbb{R}^r, (y_1, \ldots, y_s, z_{s+1}, \ldots, z_{s+t}) \mapsto \left( \beta^my_1 - (\beta_2^my_2 + \cdots + \beta_s^my_s) - 2\Re(\beta_{s+1}^mz_{s+1} + \cdots + \beta_{s+t}^mz_{s+t}) \right)_{m=0}^{r-1} \]
we have
\[ \Phi = \rho \circ \tilde{\Phi} \pmod{1}. \]
The map \( \rho \) is linear and \( \rho \pmod{1} \) is finite to one on \( \tilde{\Phi}(K) \). Now the measure \( \psi = (\pmod{1}) \circ \rho \circ \tilde{\Phi} \circ \mu \) is equal to the Lebesgue measure, if and only if \( \hat{\psi}(m_0, \ldots, m_{r-1}) = 0 \) for all \( (m_0, \ldots, m_{r-1}) \in \mathbb{Z}^r \setminus \{0\} \). If \( \psi \) is Lebesgue measure, then the measure \( \nu = P_* \circ \tilde{\Phi} \circ \mu \) is absolutely continuous, where \( P \) denotes the projection to the first coordinate. On the other hand, if \( \psi \) is not the Lebesgue measure, then \( \nu \) cannot be absolutely continuous by (23). \[ \square \]

**7. Examples**

**Example 1.** Let \( K \subset \{0, \pm 1\}^\mathbb{N} \) be given by the automaton in Figure 1 and take \( \beta = \frac{1+\sqrt{5}}{2} \). Then the map \( \phi^+ \) takes the values
\[ \begin{align*}
0 & \quad \nu(\{0\}) = \frac{1}{\gamma^2} \\
\pm 1 & \quad \nu(\{1\}) = \nu(\{-1\}) = \frac{1}{2} \left( \frac{1}{\gamma} - \frac{1}{\gamma^2} \right) \\
\pm \frac{1}{\beta} & \quad \nu \left( \left\{ \frac{1}{\beta} \right\} \right) = \nu \left( \left\{ -\frac{1}{\beta} \right\} \right) = \frac{1}{\gamma^3}
\end{align*} \]
Figure 1. The automaton recognising all expansions of 0 in base $\frac{1+\sqrt{5}}{2}$ with digits $\{0, \pm 1\}$

![Automaton diagram]

Figure 2. The automaton recognising all greedy expansions in base $\frac{1+\sqrt{5}}{2}$ with digits $\{0, 1\}$

with the indicated probabilities, where $\gamma$ is the positive solution of

$$x^3 = x^2 + 2.$$ 

The value $\gamma$ is the Perron-Frobenius eigenvalue of the adjacency matrix of the automaton given by Figure 1.

Example 2. Let $\mathcal{K} \subset \{0, 1\}^\mathbb{N}$ be the set of sequences of 0 and 1 with no two consecutive 1s (given by the automaton in Figure 2). These are the digital representations of all real numbers in $[0, 1]$ obtained by iteration of the $\beta$-transformation $T_\beta : x \mapsto \beta x \mod 1$ for $\beta = \frac{1+\sqrt{5}}{2}$ (see [26]). Then the measure $\phi^+(\mu^+)$ is Lebesgue measure on $[0, 1]$.

Example 3. Take $\mathcal{K} = \{0, 1, 2, 3\}^\mathbb{N}$ and $\mu$ the infinite product measure assigning probability $\frac{1}{4}$ to each letter. Take $\beta = 2$ and consider the map $\phi^+$ as above. Then the corresponding measure on $[0, 3]$ is given by the density

$$h(x) = \begin{cases} 
\frac{1}{2}x & \text{for } 0 \leq x \leq 1 \\
\frac{1}{2} & \text{for } 1 \leq x \leq 2 \\
\frac{1}{2}(3-x) & \text{for } 2 \leq x \leq 3 \\
0 & \text{otherwise.}
\end{cases}$$

Projecting this measure mod 1 gives Lebesgue measure, because in this case $r = 1$ and the map $\Phi$ maps to $\mathbb{T}$. 


Figure 3. The automaton recognising all expansions in base $\beta^2 = \frac{3+\sqrt{5}}{2}$ with digits \{0, 1, 2\}

Figure 4. The automaton recognising expansions in base $2$ with digits \{0, ±1\}

**Example 4.** Let $\mathcal{K}^+ \subset \{0, 1, 2\}^\mathbb{N}$ be the set of sequences recognised by the automaton in Figure 3. These are all expansions of real numbers expressed in base $\beta^2$, where $\beta = \frac{1+\sqrt{5}}{2}$. This is similar to Example 3, where expansions in base $\beta^2$ are interpreted in base $\beta$. In this case the measure is singular, though. This can be seen by computing

$$\lim_{n \to \infty} \hat{\nu}(\beta^n) = 0.0608424\ldots + 0.0208583\ldots i.$$ numerically.

**Example 5.** This example is taken from [17]. Take $\mathcal{K} \subset \{0, \pm1\}^\mathbb{N}$ to be the set of sequences recognised by the automaton in Figure 4. The automaton with initial state $I$ recognises all expansions of integers in the form

$$n = \sum_{k=0}^{K} x_k 2^k \quad \text{with} \quad x_k \in \{0, \pm1\}$$

which minimise the weight

$$w(n) = \sum_{k=0}^{K} |x_k|.$$ This is motivated by the fact that this weight is the number of additions/subtractions in computing $nP$ by a Horner-type scheme, where $P$ is a point on an elliptic curve. The measures $\phi_+^*(\mu)$ and $\phi_+^*(\mu_I)$ are singular in this case, as was proved in [17].

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REFERENCES

1. M. Baake and U. Grimm, *Aperiodic order. Vol. 1*, Encyclopedia of Mathematics and its Applications, vol. 149, Cambridge University Press, Cambridge, 2013, A mathematical invitation, With a foreword by Roger Penrose.

2. , *Fourier transform of Rauzy fractals and point spectrum of 1D Pisot inflation tilings*, Doc. Math. 25 (2020), 2303–2337.

3. M.-P. Béal and D. Perrin, *Symbolic dynamics and finite automata*, Handbook of formal languages, Vol. 2, Springer, Berlin, 1997, pp. 463–505.

4. M.-J. Bertin, A. Decomps-Guilloux, M. Grandet-Hugot, M. Pathiaux-Delefosse, and J.-P. Schreiber, *Pisot and Salem numbers*, Birkhäuser Verlag, Basel, 1992, With a preface by David W. Boyd.

5. J. Brémont, *Self-similar measures and the Rajchman property*, Ann. H. Lebesgue 4 (2021), 973–1004.

6. E. Breuillard and P. P. Varjú, *On the dimension of Bernoulli convolutions*, Ann. Probab. 47 (2019), no. 4, 2582–2617.

7. I. P. Cornfeld, S. V. Fomin, and Ya. G. Sinai, *Ergodic theory*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 245, Springer-Verlag, New York, 1982, Translated from the Russian by A. B. Sosinskiĭ.

8. K. Dajani and C. Kraaikamp, *Ergodic theory of numbers*, Carus Mathematical Monographs, vol. 29, Mathematical Association of America, Washington, DC, 2002.

9. R. Diestel, *Graph theory*, fifth ed., Graduate Texts in Mathematics, vol. 173, Springer, Berlin, 2017.

10. S. Eilenberg, *Automata, languages, and machines. Vol. A*, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York, 1974, Pure and Applied Mathematics, Vol. 58.

11. P. D. T. A. Elliott, *Probabilistic number theory. I, mean-value theorems*, Grundlehren der Mathematischen Wissenschaften, vol. 239, Springer-Verlag, New York, 1979.

12. P. Erdős, *On a family of symmetric Bernoulli convolutions*, Amer. J. Math. 61 (1939), 974–976.

13. , *On the smoothness properties of a family of Bernoulli convolutions*, Amer. J. Math. 62 (1940), 180–186.

14. D.-J. Feng and H. Hu, *Dimension theory of iterated function systems*, Comm. Pure Appl. Math. 62 (2009), no. 11, 1435–1500.

15. C. Frougny, *Representations of numbers and finite automata*, Math. Systems Theory 25 (1992), no. 1, 37–60.

16. , *Numeration systems*, ch. 7, pp. 230–268, vol. 90 of Encyclopedia of Mathematics and its Applications [22], 2002.

17. P. J. Grabner and C. Heuberger, *On the number of optimal base 2 representations of integers*, Des. Codes Cryptogr. 40 (2006), 25–39.

18. P. J. Grabner, C. Heuberger, and H. Prodinger, *Distribution results for low-weight binary representations for pairs of integers*, Theor. Comput. Sci. 319 (2004), 307–331.
PURITY RESULTS FOR SOME ARITHMETICALLY DEFINED MEASURES

19. , Counting optimal joint digit expansions, Integers 5 (2005), no. 3, A09, 19 pages (electronic).

20. P. J. Grabner, P. Liardet, and R. F. Tichy, Spectral disjointness of dynamical systems related to some arithmetic functions, Publ. Math. Debrecen 66 (2005), 213–244.

21. P. J. Grabner and W. Steiner, Redundancy of minimal weight expansions in Pisot bases, Theor. Comput. Sci 412 (2011), 6303–6315.

22. M. A. Harrison, Introduction to formal language theory, Addison-Wesley Publishing Co., Reading, Mass., 1978.

23. B. Jessen and A. Wintner, Distribution functions and the Riemann zeta function, Trans. Amer. Math. Soc. 38 (1935), 48–88.

24. D. Lind and B. Marcus, An introduction to symbolic dynamics and coding, Cambridge University Press, Cambridge, 1995.

25. M. Lothaire, Algebraic combinatorics on words, Encyclopedia of Mathematics and its Applications, vol. 90, Cambridge University Press, Cambridge, 2002.

26. W. Parry, On the β-expansions of real numbers, Acta Math. Acad. Sci. Hung. 11 (1960), 401–416.

27. W. Parry, Intrinsic Markov chains, Trans. Amer. Math. Soc. 112 (1964), 55–66.

28. Symbolic dynamics and transformations of the unit interval, Trans. Amer. Math. Soc. 122 (1966), 368–378.

29. Y. Peres, W. Schlag, and B. Solomyak, Sixty years of Bernoulli convolutions, Fractal geometry and stochastics, II (Greifswald/Koserow, 1998), Progr. Probab., vol. 46, Birkhäuser, Basel, 2000, pp. 39–65.

30. M. Queffélec, Substitution dynamical systems—spectral analysis, second ed., Lecture Notes in Mathematics, vol. 1294, Springer-Verlag, Berlin, 2010.

31. A. Rajchman, Une classe de séries trigonométriques qui convergent presque partout vers zéro, Math. Ann. 101 (1929), no. 1, 686–700.

32. A. Rényi, Representation for real numbers and their ergodic properties, Acta Math. Acad. Sci. Hung. 8 (1957), 477–493.

33. S. Saglietti, P. Shmerkin, and B. Solomyak, Absolute continuity of non-homogeneous self-similar measures, Adv. Math. 335 (2018), 60–110.

34. J. Sakarovitch, Elements of automata theory, Cambridge University Press, Cambridge, 2009, Translated from the 2003 French original by Reuben Thomas.

35. E. Seneta, Nonnegative matrices and Markov chains, second ed., Springer Series in Statistics, Springer-Verlag, New York, 1981.

36. P. Shmerkin, On the exceptional set for absolute continuity of Bernoulli convolutions, Geom. Funct. Anal. 24 (2014), no. 3, 946–958.

37. P. Shmerkin and B. Solomyak, Absolute continuity of self-similar measures, their projections and convolutions, Trans. Amer. Math. Soc. 368 (2016), no. 7, 5125–5151.

38. N. Sidorov and A. Vershik, Ergodic properties of the Erdös measure, the entropy of the golden shift, and related problems, Monatsh. Math. 126 (1998), no. 3, 215–261.

39. N. A. Sidorov, Bijective and general arithmetic codings for Pisot toral automorphisms, J. Dynam. Control Systems 7 (2001), no. 4, 447–472.

40. B. Solomyak, On the random series ∑ ±λn (an Erdös problem), Ann. of Math. (2) 142 (1995), no. 3, 611–625.
41. P. P. Varjú, *Absolute continuity of Bernoulli convolutions for algebraic parameters*, J. Amer. Math. Soc. 32 (2019), no. 2, 351–397.

42. ———, *On the dimension of Bernoulli convolutions for all transcendental parameters*, Ann. of Math. (2) 189 (2019), no. 3, 1001–1011.

43. Péter P. Varjú, Recent progress on Bernoulli convolutions, European Congress of Mathematics, Eur. Math. Soc., Zürich, 2018, pp. 847–867.

44. P. Walters, *Ergodic theory*, Springer, Berlin, 1982.

45. R. Winkler, *The order theoretic structure of the set of P-sums of a sequence*, Publ. Math. Debrecen 58 (2001), no. 3, 467–490.

Institut für Analysis und Zahlentheorie, Technische Universität Graz, Kopernikusgasse 24/II, 8010 Graz, Austria

Email address: peter.grabner@tugraz.at