Higher Spin Currents

in the $\mathcal{N} = 1$ Stringy Coset Minimal Model

Changhyun Ahn

Department of Physics, Kyungpook National University, Taegu 702-701, Korea

ahn@knu.ac.kr

Abstract

This study reconsidered the $\mathcal{N} = 1$ supersymmetric extension of the $W_3$ algebra which was studied previously. This extension consists of seven higher spin supercurrents (fourteen higher spin currents in the components) as well as the $\mathcal{N} = 1$ stress energy tensor of spins ($\frac{3}{2}, 2$). Thus far, the complete expressions for the higher spin currents have not been derived.

This paper constructs them explicitly in both the $c = 4$ eight free fermion model and the supersymmetric coset model based on $(A_2^{(1)} \oplus A_2^{(1)} , A_2^{(1)})$ at level $(3,k)$. By acting with the above spin-$\frac{3}{2}$ current on the higher spin-3 Casimir current, its fermionic partner, the higher spin-$\frac{5}{2}$ current, can be generated and combined as a first higher spin supercurrent with spins ($\frac{5}{2}, 3$). By calculating the operator product expansions (OPE) between the higher spin supercurrent and itself, the next two higher spin supercurrents can be generated with spins ($\frac{7}{2}, 4$) and ($\frac{9}{2}, 5$). Moreover, the other two higher spin supercurrents with spins ($\frac{9}{2}, 5$) and ($\frac{9}{2}, 5$) can be generated by calculating the OPE between the first higher spin supercurrent with spins ($\frac{5}{2}, 3$) and the second higher spin supercurrent with spins ($\frac{7}{2}, 4$). Finally, the higher spin supercurrents, ($\frac{11}{2}, 6$) and ($6, \frac{13}{2}$), can be extracted from the right hand side of the OPE between the higher spin supercurrents, ($\frac{5}{2}, 3$) and ($\frac{9}{2}, 5$).
1 Introduction

Gaberdiel and Gopakumar proposed the duality between the $W_N (= WA_{N-1})$ minimal model in two dimensional conformal field theories and the higher spin theory of Vasiliev on the $AdS_3$ [1, 2]. The $W_N$ minimal model conformal field theory is dual, in the ’t Hooft $\frac{1}{N}$ expansion, to the higher spin coupled to one complex scalar. One of the levels for the spin-1 WZW current in the conformal field theory is fixed by 1 and the other is given by the positive integer, $k$. The motivation of [1, 2] is based on the work by Klebanov and Polyakov [3] in the context of $AdS_4$ bulk theory and three dimensional conformal theory: the $O(N)$ vector model. Therefore, it is natural to ask whether similar duality exists in the group $SO(N)$ rather than $SU(N) = A_{N-1}$. In references [4, 5], the possible generalization with the $SO(N)$ coset theory was described. All these [1, 2, 4, 5] were a truncated version of $N=2$ supergravity theory [6]. Furthermore, the full $\mathcal{N} = 2$ supersymmetric extension with higher spin $AdS_3$ supergravity was studied in [7] and in references [8, 9, 10, 11, 12, 13] where the dual conformal field theory is given by the $\mathcal{N} = 2$ $\mathbb{CP}^N$ Kazama-Suzuki model in two dimensions. Recently, in [14], the $\mathcal{N} = 1$ minimal model holography, which involves truncating the $\mathcal{N} = 2$ theory [6] to the $\mathcal{N} = 1$ theory with $SO(N)$ coset, is also described.

More general coset theories can be derived by generalizing level 1 for the spin-1 current to the arbitrary positive integer, $l$. The spin-3 current in these general cosets was constructed in reference [15] and the spin-4 current was found recently in [16]. The central charge grows like $N^2$ by defining the ’t Hooft limit with $k, l, N \to \infty$ with the appropriate relative ratios held finite [17]. A particular case, where $k = l = N$, was studied in the context of two dimensional gauge theory coupled to the adjoint fermions in reference [18].

One study [19] reported that the cosets can be supersymmetric if one of the levels is equal to $N$(i.e., $l = N$ with arbitrary $k$). Furthermore, the first model with $k = 1$ in this series (with arbitrary $l$ and $N$) has bosonic $W_N$ symmetry. The supersymmetric version of $W_N$ algebra with $k = 1, l = N$ was studied [20]. On the other hand, supersymmetric extended $W_3$ symmetry can be expected by taking $N = 3 = l$ with arbitrary $k$.

This paper reconsiders the following coset minimal model studied previously [21] previously, with arbitrary $k$ and $l = N = 3$,

\[
\frac{\hat{SU}(3)_k \oplus \hat{SU}(3)_l}{\hat{SU}(3)_{k+3}}.
\]  

(1.1)

\[1\]These general cosets are called by “stringy cosets” in [17].
The central charge can be obtained and is given by

\[ c = 4 \left[ 1 - \frac{18}{(k + 3)(k + 6)} \right], \quad k = 1, 2, \cdots \]  

(1.2)

For \( k = 1 (c = \frac{10}{3}) \) with (1.2), the explicit higher spin current as well as the superconformal generators were constructed \([22]\) and given in terms of the WZW currents of above coset (1.1). This extended algebra coincides with the “minimal” super \( W_3 \) algebra found in reference \([23]\), where there are two extra higher spin currents of spin-\( \frac{5}{2} \) and spin-3. In reference \([21]\), some of the higher spin currents were found in the \( c = 4 \) free fermion model which is equivalent to \( k \to \infty \) limit in (1.2). Moreover, those in the generic \( c < 4 \) supersymmetric coset models based on (1.1) were found. The existence of eight supercurrents via the character technique was described. In reference \([4]\), the possible application in the context of minimal model holography \([1\, 2]\) was suggested. If the discrete series for the \( \mathcal{N} = 2 \) superconformal minimal models with \( c = \frac{3m}{m+2} \) is considered, \( c = \frac{8}{3} \) (i.e., \( k = 3 \)) corresponding to \( m = 16 \).

The main item in the Casimir construction \([24\, 15]\) for the level 1 WZW models for simply laced Lie algebras is to identify the complete set of independent generating currents. One of the aims of this study was to determine the complete set of currents proposed in reference \([21]\) and some of the algebra they satisfy. The starting point is the bosonic spin-3 current found in \([24\, 15]\) with fixed \( N = 3 \) and finite or infinite \( k \). Of course, the \( \mathcal{N} = 1 \) stress energy tensor consisting of spin-2 and spin-\( \frac{3}{2} \) currents can be obtained from the usual Sugawara construction. All the higher spin currents can then be obtained from the spin-\( \frac{3}{2} \) current and spin-3 current. These two currents generate higher spin currents.

That is, by acting with the above spin-\( \frac{3}{2} \) current on the higher spin-3 current (by calculating the operator product expansion (OPE) between them), its fermionic partner, the higher spin-\( \frac{5}{2} \) current, can be generated in the right hand side of OPE, which can be combined as a first higher spin \( \mathcal{N} = 1 \) supercurrent with spins (\( \frac{5}{2}, 3 \)) \([4]\). By construction, the OPE between the composite fields can be calculated through the basic fundamental OPEs between the

\(^2\)In the first example in reference \([4]\), the levels of the \( SO(N) \) coset minimal model are given by \( (k, 2) \). The central charge behaves as \( N \) in the large \( N \) limit. A couple of conformal dimensions for the fields, in the large \( (N, k) \) ’t Hooft limit, are calculated. In the second example, where the levels are given by \( (k, 3) \) in the \( SU(N) \) coset minimal model, the higher spin current of spin \( \frac{5}{2} \) in the large \( (N, k) \) limit was found. Finding the corresponding dual theories in the \( AdS_3 \) higher spin gravity is an open problem.

\(^3\)These higher spin currents are primary under the stress energy tensor and they transform similarly under the spin-\( \frac{3}{2} \) current. Combining these two, the higher spin currents are superprimary under the \( \mathcal{N} = 1 \) stress energy tensor.

\(^4\)Although the notation will be explained in section 2, let us describe the convention used here. The fermionic current of spin \( \frac{5}{2} \) and its superpartner bosonic current of spin 3 can be written in terms of a single \( \mathcal{N} = 1 \) supercurrent, which is denoted simply by specifying the spins in this way. Similarly, one denotes the \( \mathcal{N} = 1 \) superconformal supercurrent by (\( \frac{5}{2}, 3 \)).
bosonic and fermionic WZW currents. By calculating the OPE between these two higher spin currents of spin-$\frac{5}{2}$ and spin-3 (the three nontrivial OPEs should be calculated in the component approach), the next two higher spin $\mathcal{N} = 1$ supercurrents can be generated with spins $(\frac{7}{2}, 4)$ and $(4, \frac{9}{2})$ that appear in the singular terms of the OPE. The explicit forms of these in terms of WZW currents for the $c = 4(k \to \infty)$ model were already presented in reference [21].

For the arbitrary central charge (i.e. finite $k$), the explicit forms are not known. Moreover, the other two higher spin $\mathcal{N} = 1$ supercurrents with spins, $(4, \frac{9}{2})$ and $(\frac{7}{2}, 5)$, are generated by calculating the OPE between the first higher spin supercurrent with spins $(\frac{5}{2}, 3)$ and the second higher spin supercurrent with spins $(\frac{7}{2}, 4)$ by repeating similar procedures. A range of quasi-primary field currents can be written in terms of the known higher spin currents and the $\mathcal{N} = 1$ superconformal currents. Finally, the higher spin supercurrents, $(\frac{11}{2}, 6)$ and $(6, \frac{13}{2})$, can be extracted from the right hand side of the OPE between the higher spin supercurrents, $(\frac{5}{2}, 3)$ and $(4, \frac{9}{2})$. This will produce the most complicated calculations.

How does one calculate the OPE explicitly and extract the correct primary currents in the right hand side of OPE? In general, the OPE [25, 26, 27] (See also [28, 29]) between two quasi-primary fields $\Phi_i(z)$ with spin-$h_i$ and $\Phi_j(w)$ with spin-$h_j$ (the spins are positive integer or half-integer) has the following form\(^5\)

$$
\Phi_i(z) \Phi_j(w) = \frac{1}{(z-w)^{h_i+h_j}} \gamma_{ij} + \sum_k C_{ijk} \sum_{n=0}^{\infty} \frac{1}{(z-w)^{h_i+h_j-h_k-n}} \left[ \frac{\Gamma(h_i-h_j+h_k+n)}{n!} \frac{\Gamma(2h_k)}{\Gamma(h_i-h_j+h_k) \Gamma(2h_k+n)} \right] \partial^n \Phi_k(w).
$$

(1.3)

$\gamma_{ij}$ corresponds to a metric on the space of quasi-primary fields. The quantity, $C_{ijk}$, appears in the three-point function between the quasi-primary fields, $\Phi_i(z), \Phi_j(z)$ and $\Phi_k(z)$. The index $k$ specifies all the quasi-primary fields occurring in the right hand side. The descendant fields for the quasi-primary field $\Phi_k(w)$ are (multiple) derivatives of $\Phi_k(w); \partial^n \Phi_k(w)$. Furthermore, the coefficient functions are written in terms of various Gamma functions and depend on the spins and number of derivatives. Provided $n \geq h_i + h_j - h_k$, regular terms can be obtained. Otherwise, singular terms exist when $n \leq h_i + h_j - h_k$. By noting that the Gamma function has the following property,$\gamma_{ij} = \frac{\Gamma(h_i-h_j+h_k+n)}{\Gamma(h_i-h_j+h_k)} = (h_i-h_j+h_k)(h_i-h_j+h_k+1)\cdots(h_i-h_j+h_k+n-1)$, which is a Pochhammer function, for $h_i - h_j + h_k \leq 0$ (i.e. the spin of $\Phi_j(z)$ is greater than

---

\(^5\)The OPE of the stress energy tensor with the quasi-primary field does not have a third-order pole. The primary field is also a quasi-primary field because it satisfies with this condition for the quasi-primary field. In general, the OPE between the stress energy tensor and quasi-primary field can have a nonzero singular term(s) with the order $n > 3$. Of course, for the primary field, the trivial vanishing higher order singular terms with the order $n \geq 3$ occur. This paper is restricted to the definition of a quasi-primary field as follows: what is meant by a quasi-primary field is the one that does not contain the primary field. When referring to a quasi-primary field, it is considered that the nonvanishing higher order (greater than 3) singular terms exist.
the sum of the spin of $\Phi_i(z)$ and the spin of $\Phi_k(z)$, the summation over $n$ terminates to a finite summation. For example, when $(h_i - h_j + h_k + n - 1)$ is vanishing for particular $n = n_0$, then the coefficient for $n > n_0$ always contains this vanishing value $(h_i - h_j + h_k + n_0 - 1)$. For the factor, $\frac{\Gamma(2h_k)}{\Gamma(2h_k + n)}$, there is no zero for the positive $h_k$ and $n$. Note that for different $k'$ values (different quasi-primary fields), the $h_k$ values of the corresponding spin can be equal to each other. That is, for given singular terms, several different quasi-primary fields of the same spin can coexist. This feature will be shown in the next sections.

Determining the possible quasi-primary or primary fields, $\Phi_k(w)$, in the right hand side is a nontrivial task. For lower higher spin quasi-primary or primary fields, the number of quasi-primary or primary fields is limited but the number of quasi-primary or primary fields increases with increasing spin. As mentioned before, in general, the quasi-primary fields in (1.3) are given in terms of the composite fields between the WZW currents of the integer or half-integer spins. By construction, the OPEs can be obtained from the basic fundamental OPEs between the WZW currents. All the singular terms for given spins, $h_i$ and $h_j$, can then be obtained. The most singular term should be analyzed first followed by the next lower singular terms because once the lowest quasi-primary field (or primary field) is found, then its descendant structure is fixed completely according to (1.3). After the nontrivial highest singular term is analyzed (the quasi-primary field or primary field appears in the right hand side), the next singular term contains the descendant field for the previous quasi-primary or primary fields, and the remaining terms contains the new quasi-primary fields or primary fields according to (1.3).

In general, finding these new fields is nontrivial. On the other hand, they should transform as quasi-primary or primary fields under the spin-2 current, as explained before. Therefore, the OPEs between the spin-2 and ”the remaining terms” in the given singular terms should be calculated. In general, this will include all the higher singular terms where the order is greater than 3. Therefore it is important to determine what kind of quasi-primary or primary fields occur. By taking the possible quasi-primary fields with the undetermined coefficient functions, the remaining items should be expressed in terms of these quasi-primary or primary fields. The unknown coefficient functions can be fixed using the condition that the third-order pole should vanish. Furthermore, the OPEs between the spin-$\frac{3}{2}$ current and “the remaining terms” should be calculated to observe the complete structure of the possible quasi-primary fields. For the quasi-primary fields, there is no constraint on the singular terms but for the primary fields, the OPE between the spin-$\frac{3}{2}$ current and primary current should contain either the first-order pole or the second- and first-order poles. This suggests that at least the OPE of spin-$\frac{3}{2}$ and the primary field should not contain higher order singular terms where the order
is greater than 2. The details will be seen in the next sections.

Section 2 reviews the $\mathcal{N} = 1$ superconformal algebra and the “minimal” $\mathcal{N} = 1$ super $W_3$ algebra \cite{23} and describes the higher spin currents in the $c = 4$ free fermion model.

In section 3, the higher spin currents in $c < 4$ coset model are constructed explicitly.

Section 4 summarizes the results in this paper and discusses the future directions.

In the Appendices, some OPEs relevant to the sections 2 and 3 are presented.

All the relevant works along the line of references \cite{1,2} are presented in the recent review papers \cite{17,30} and also further works are reported in \cite{31}-\cite{43}.

The mathematica package \cite{44} is used all the time.

2 The higher spin currents in the $c = 4$ eight free fermion model

2.1 The $\mathcal{N} = 1$ superconformal algebra: review

Let us describe the $\mathcal{N} = 1$ supersymmetric extension of Virasoro algebra. The $\mathcal{N} = 1$ superconformal algebra is generated by the $\mathcal{N} = 1$ super stress energy tensor of spin-$\frac{3}{2}$ \cite{45},

$$\hat{T}(Z) = \frac{1}{2} G(z) + \theta \ T(z),$$

(2.1)

where $Z = (z, \theta)$ is a complex supercoordinate, $T(z)$ is the usual bosonic stress energy tensor of spin-2 and $G(z)$ is its fermionic superpartner of spin-$\frac{3}{2}$. The superconformal algebra, in components, is summarized by the three OPEs as follows. The OPE between the fermionic field of spin-$\frac{3}{2}$ and itself can be expressed as

$$G(z) \ G(w) = \frac{1}{(z-w)^3} \left[ \frac{2}{3} \frac{c}{(z-w)^2} 2T(w) + \cdots \right],$$

(2.2)

where the right hand side of this OPE contains the bosonic stress energy tensor and the central term. The equation contains no second-order singular term because there is no spin-1 field. The standard OPE between the bosonic stress energy tensor and itself (i.e., Virasoro algebra) is given by

$$T(z) \ T(w) = \frac{1}{(z-w)^4} \left[ \frac{c}{2} \frac{1}{(z-w)^2} 2T(w) + \frac{1}{(z-w)} \partial T(w) + \cdots \right],$$

(2.3)

and finally, the fermionic field is primary with respect to $T(w)$, i.e.,

$$T(z) \ G(w) = \frac{1}{(z-w)^2} \left[ \frac{3}{2} \frac{c}{2} G(w) + \frac{1}{(z-w)} \partial G(w) + \cdots \right].$$

(2.4)
Of course, the OPE $G(z) T(w)$ can be obtained from (2.4) in the standard way. The $\mathcal{N} = 1$ superconformal algebra is represented by (2.2), (2.3) and (2.4) or its $\mathcal{N} = 1$ single OPE with (2.1). These OPEs can also be expressed using the (anti)commutator relations for the modes of $T(z)$ and $G(z)$, as usual. Ramond algebra is used for the integer mode of $G(z)$ and Neveu-Schwarz algebra is used for the half-integer mode of $G(z)$. According to the definition of quasi-primary field, the stress energy tensor, $T(z)$, is a quasi-primary field. The coefficients $2$ in (2.2), $2$ in (2.3) and $3$ in (2.4) play the role of $C_{ijk}$ in (1.3) and the central terms in (2.2) and (2.3) correspond to $\gamma_{ij}$ in (1.3). Moreover, the relative coefficients, $\frac{1}{2}$ in (2.3) and $\frac{1}{3}$ in (2.4), appearing in the first-order singular term coincide with the general expression given in (1.3). Appendix A presents more details of the coefficient functions.

2.2 The “minimal” $\mathcal{N} = 1$ super $W_3$ algebra where $c = \frac{10}{7}$: review

The simplest extension of the previous $\mathcal{N} = 1$ superconformal algebra is to add a single higher spin superprimary current of spin $\frac{5}{2}$

$$\hat{W}(Z) = \frac{1}{\sqrt{6}} U(z) + \theta W(z),$$

(2.5)

where $W(z)$ is a bosonic spin-3 current and $U(z)$ is a fermionic spin $\frac{5}{2}$ current. These are primary fields with respect to the stress energy tensor, $T(z)$, such as (2.4). Furthermore, the spin $\frac{3}{2}$ current $G(z)$ transforms $U(w)$ into $W(w)$ and vice versa (fermion goes to the boson and the boson goes to the fermion).

$$G(z) U(w) = \frac{1}{(z-w)} \sqrt{6} W(w) + \cdots,$$

(2.6)

and

$$G(z) W(w) = \frac{1}{(z-w)^2} \frac{5}{\sqrt{6}} U(w) + \frac{1}{(z-w)} \frac{1}{\sqrt{6}} \partial U(w) + \cdots.$$  

(2.7)

This suggests that once any component field of (2.5) is found, its superpartner can be determined automatically from (2.6) or (2.7). Again, the relative coefficient $\frac{1}{5}$ showing the first-order singular term in (2.7) comes from the general expression given in (1.3). The role of the spin $\frac{3}{2}$ current $G(z)$ is very important and this property will be used continually.

By assuming that the OPE between the additional supercurrent (2.5) and itself does not generate any new superprimary current (i.e. the “minimal” extension), the possible structures in the right hand side of the OPE can be written. The unknown coefficient functions can be determined using the so-called Jacobi identities for normal ordered graded commutators of the supercurrents $\hat{T}(Z)$ and $\hat{W}(Z)$. The “minimal” $\mathcal{N} = 1$ super $W_3$ algebra is associative for
\[ c = \frac{10}{7}, \text{ where the unitary representation exists and } c = -\frac{5}{2} \text{ with a nonunitary representation } \text{[23]. For } c = \frac{10}{7}, \text{ the three OPEs in the components are summarized as follows. The OPE between the bosonic spin-3 current and itself leads to the following} \]

\[
W(z) W(w) = \frac{1}{(z-w)^6} \frac{10}{21} + \frac{1}{(z-w)^4} 2T(w) + \frac{1}{(z-w)^3} \partial T(w) \\
+ \frac{1}{(z-w)^2} \left[ \frac{3}{10} \partial^2 T + \frac{56}{51} \left( T^2 - \frac{3}{10} \partial^2 T \right) \right] (w) \\
+ \frac{1}{(z-w)} \left[ \frac{1}{15} \partial^3 T + \frac{3}{2} \frac{56}{51} \partial \left( T^2 - \frac{3}{10} \partial^2 T \right) \right] (w) + \cdots, \tag{2.8}
\]

which is precisely the same as the Zamolodchikov’s \( W_3 \) algebra for \( c = \frac{10}{7} \), as expected. The coefficient \( \frac{1}{7} \) for the descendant field with spin-5 associated with the quasi-primary field of spin 4 in (2.8) is derived from the general expression in (1.3). The coefficients, \( \frac{1}{7}, \frac{3}{20} \) and \( \frac{1}{30} \), appearing in the descendant fields of the stress energy tensor can be obtained similarly \[3\].

The OPE between the spin-3 and spin-\( \frac{5}{2} \) can be summarized as

\[
W(z) U(w) = \frac{1}{(z-w)^4} \frac{3}{\sqrt{6}} G(w) + \frac{1}{(z-w)^3} \frac{2}{3} \frac{3}{\sqrt{6}} \partial G(w) \\
+ \frac{1}{(z-w)^2} \left[ \frac{3}{4} \frac{3}{\sqrt{6}} \partial^2 G + \frac{77\sqrt{6}}{187} \left( GT - \frac{1}{8} \partial^2 G \right) \right] (w) \\
+ \frac{1}{(z-w)} \left[ \frac{3}{15} \frac{3}{\sqrt{6}} \partial^3 G + \frac{4}{7} \frac{77\sqrt{6}}{187} \partial \left( GT - \frac{1}{8} \partial^2 G \right) \right] (w) \\
+ \frac{4\sqrt{6}}{17} \left( \frac{4}{3} T \partial G - G \partial T - \frac{4}{15} \partial^3 G \right) (w) + \cdots. \tag{2.9}
\]

The coefficients \( \frac{2}{5}, \frac{1}{4} \) and \( \frac{1}{15} \), for the descendant fields for the spin-\( \frac{3}{2} \) field were written down intentionally in the right hand side (2.9) to emphasize that they can be determined from (1.3). The spin-\( \frac{7}{2} \) and spin-\( \frac{9}{2} \) has two quasi-primary fields. The coefficient \( \frac{1}{7} \) appearing in the descendant field for the former can be obtained from the general formula. The first-order singular term consists of the descendant field, \( \partial^3 G(w) \) for \( G(w) \), the descendant field \( \partial(GT - \frac{1}{8} \partial^2 G)(w) \) for spin-\( \frac{7}{2} \) quasi-primary field \( (GT - \frac{1}{8} \partial^2 G)(w) \) and the quasi-primary field \( (\frac{4}{3} T \partial G - G \partial T - \frac{4}{15} \partial^3 G)(w) \) of spin-\( \frac{9}{2} \). In other words, three independent terms, which are characterized by \( \partial^3 G(w) \), \( \partial GT(w) \) and \( G \partial T(w) \) can be rewritten in terms of the above three terms \[6\].

\[6\] The quasi-primary field has the following OPE with the stress energy tensor, \( T(z) (TT - \frac{1}{10} \partial^2 T)(w) = \frac{1}{(z-w)^4} \left( \frac{22}{5} + c \right) T(w) + \mathcal{O}((z-w)^{-2}) \), where there is no third-order singular term.

\[7\] The following OPEs can be obtained easily to determine if they are really quasi-primary fields. They are \( T(z) (GT - \frac{1}{8} \partial^2 G)(w) = \frac{1}{(z-w)^3} \frac{37}{8} G(w) + \mathcal{O}((z-w)^{-2}) \) and \( T(z) (\frac{4}{3} T \partial G - G \partial T - \frac{4}{15} \partial^3 G)(w) = \)
Finally, the spin-$\frac{5}{2}$ and spin-$\frac{5}{2}$ OPE can be expressed as

$$U(z) \ U(w) = \frac{1}{(z-w)^5} \bigg[ \frac{4}{7} + \frac{1}{(z-w)^3} \ 2T(w) + \frac{1}{(z-w)^2} \ \partial T(w) \bigg] + \bigg[ \frac{3}{10} \ \partial^2 T + \frac{63}{68} \left( T^2 - \frac{3}{10} \ \partial^2 T \right) + P_{4}^{uu} \bigg] (w) + \cdots. \tag{2.10}$$

The relative coefficients, $\frac{1}{2}$ and $\frac{3}{20}$, appearing in the descendant fields for the stress energy tensor, $T(w)$, can be analyzed previously and provide the correct values. In the first-order singular term, there is a quasi-primary field $(T^2 - \frac{3}{10} \ \partial^2 T)(w)$ and primary field of spin-$4$ \cite{22} given as

$$P_{4}^{uu}(w) = \frac{21}{17} \left[ -\frac{7}{10} \ \partial^2 T + \frac{7}{12} \left( T^2 - \frac{3}{10} \ \partial^2 T \right) + G \partial G \right] (w), \tag{2.11}$$

where the central charge, $c = \frac{10}{7}$, is used. For the general $c$, the coefficient, $\frac{7}{12}$, should be replaced by $\frac{17}{(22+60)}$. Note that this primary field is not an additional field because this can be obtained from the currents, $T(w)$ and $G(w)$. In this example, at the first-order singular term of (2.10), there are two types of quasi-primary fields. In the language of (1.3), for a fixed $h_k$, there are two degeneracies. More precisely, one is a spin-$4$ quasi-primary field and the other is a spin-$4$ primary field (2.11). This primary field (2.11) has an unusual OPE with an above spin-$\frac{3}{2}$ current $G(z)$ in the next subsection.

### 2.3 The $c = 4$ free fermion model

A consistent generalization of the above minimal extension of $W_3$ algebra for an arbitrary central charge is needed. This subsection will consider the particular supersymmetric coset model introduced in the introduction. Before going into detail, this section first describes its particular limit where all the algebraic structures are observed.

Consider the eight fermion fields $\psi^a$ of spin-$\frac{1}{2}$ where the $SU(3)$ adjoint index $a$ runs from 1 to 8($= 3^2 - 1$). In this paper, $N$ is fixed by 3. The OPE of this fermion field is given by

$$\psi^a(z) \ \psi^b(w) = -\frac{1}{(z-w)} \ \frac{1}{2} \delta^{ab} + \cdots. \tag{2.12}$$

Define the spin-1 Kac-Moody current $J^a(z)$ as

$$J^a(z) = f^{abc} \psi^b \psi^c(z). \tag{2.13}$$

$$-\frac{1}{(z-w)} \ \frac{33}{3} \ G(w) - \frac{1}{(z-w)^2} \left( -\frac{1}{3} \ \frac{33}{3} \ \partial G(w) + O((z-w)^{-2}) \right). \text{ The coefficient, } -\frac{1}{3}, \text{ coincides with the general expression (1.3) by substituting } h_i = 2, h_j = \frac{9}{2}, h_k = \frac{4}{3}, \text{ and } n = 1. \text{ The minus sign is because } h_i - h_j + h_k = -1. \text{ Appendices } B \text{ and } C \text{ presents other quasi-primary fields and their OPEs between } T(z) \text{ or } G(z).$$
Then it is easy to calculate the OPE \[ J^a(z) J^b(w) = -\frac{1}{(z-w)^2} \frac{3}{2} \delta^{ab} + \frac{1}{(z-w)} f^{abc} J^c(w) + \cdots, \] (2.14)
where the level is given by 3. Similarly,
\[ \psi^a(z) J^b(w) = \frac{1}{(z-w)} f^{abc} \psi^c(w) + \cdots. \] (2.15)

In the context of (1.3), the above OPEs (2.12), (2.14) and (2.15) can be described easily.

For a given \( N = 1 \) super Kac-Moody algebra where \( \hat{Q}^a(Z) = \sqrt{3} \psi^a(z) + \theta J^a(z) \), characterized by (2.12), (2.14) and (2.15), the \( N = 1 \) superconformal algebra is realized by the spin-2 current

\[ T(z) = \psi^a \partial \psi^a(z) \] (2.16)

and the spin-\( \frac{3}{2} \) current

\[ G(z) = -\frac{2}{3\sqrt{3}} \psi^a J^a(z). \] (2.17)

They satisfy (2.2), (2.3) and (2.4) for \( c = 4 \). The normalizations in (2.16) and (2.17) are fixed automatically.

Now we are ready to construct the higher spin currents. In reference [24], the spin-3 current is described by the third order Casimir operator for \( A^{(1)}_2 = \hat{SU}(3) \),

\[ W(z) = \sqrt{\frac{2}{1215}} i d^{abc} J^a J^b J^c(z), \] (2.18)

where \( d^{abc} \) is a completely symmetric traceless \( SU(3) \) invariant tensor of rank 3 and the spin-1 current is defined as (2.13). As mentioned previously, its fermionic superpartner of spin-\( \frac{5}{2} \) can be determined from the relation (2.7) with (2.17) and (2.18)

\[ U(z) = \sqrt{\frac{2}{375}} i d^{abc} \psi^a J^b J^c(z). \] (2.19)

*More precisely, one can start by writing \<< OPEdefs.m \> in the mathematica notebook. The operators Fermionic[psi[n_]], Bosonic[K[n_]] are defined for the model in this section (and Bosonic[\( K_1, K_2, \cdots, K_8 \)] for the coset model in section 3). The singular OPEs between the basic operators are then given by OPE[psi[n_], psi[m_]] = MakeOPE[{-\frac{i}{2} One * Delta[m, n]}; OPE[K[n_], K[m_]] = MakeOPE[{-\frac{i}{2} One * Delta[m, n], Sum[f[m, n, p] * K[p], \{p, 1, 8\}]}; Finally, the structure constants \( d \) and \( f \) symbols \( f[1, 2, 3] = 1; f[1, 4, 7] = \frac{1}{2}; \cdots; d[1, 1, 8] = \frac{1}{\sqrt{3}}; d[2, 2, 8] = \frac{1}{\sqrt{3}}; \cdots \) should be defined. The command, OPESimplify[OPE[A, B], Factor] or OPESimplify[OPEPole[3][A, B], Factor] can be used for any operators \( A \) and \( B \). All the detailed descriptions are given in reference [44].
Thus far, the currents are given by the $\mathcal{N} = 1$ superconformal generators and $\mathcal{N} = 1$ $W_3$ supercurrent. On the other hand, additional higher spin currents can be observed once the OPEs between these currents are calculated.

In reference [21], extra higher spin currents were given. Let us present them in $\mathcal{N} = 1$ superspace with their components. Their spins are denoted in the subscript and the prime notation is used to describe the different field content with the same spin

\[
\hat{W}(Z) = \frac{1}{\sqrt{6}} U(z) + \theta W(z),
\]

\[
\hat{O}_\frac{7}{2}(Z) = \hat{O}_\frac{7}{2}(z) + \theta O_4(z),
\]

\[
\hat{O}_4(Z) = \hat{O}_4(z) + \theta \hat{O}_\frac{4}{3}(z),
\]

\[
\hat{O}_4'(Z) = \hat{O}_\frac{4}{3}(z) + \theta O_5(z),
\]

\[
\hat{O}_4''(Z) = \hat{O}_\frac{4}{3}(z) + \theta O_6(z),
\]

\[
\hat{O}_4'''(Z) = \hat{O}_\frac{4}{3}(z) + \theta O_6(z).\]

(2.20)

The normalization factor $\frac{1}{\sqrt{2}\hbar^{\frac{1}{2}}}$ can also be introduced in front of the $\theta$-independent term, like $\hat{W}(Z)$. Some of the currents were explicitly found in reference [21]. The higher spin currents will be calculated in terms of eight fermion fields $\psi^a(z)$. In particular, some OPEs between $\hat{W}(Z)$, $\hat{O}_\frac{7}{2}(Z)$ and $\hat{O}_4(Z)$ will be calculated. There is no reason why $\hat{O}_4(Z)$ was considered instead of $\hat{O}_4(Z)$. In these computations, unknown higher spin currents arise in the right hand side of the OPE. Twelve higher spin currents in terms of $\psi^a(z)$ were constructed explicitly.

Consider the lower higher spin currents first.

- Construction of higher spin supercurrents, $\hat{O}_\frac{7}{2}(Z)$ and $\hat{O}_4(Z)$

First consider the OPE between the spin-$\frac{7}{2}$ currents. All the singular terms can be obtained using the defining equation (2.12), (2.15) and (2.19). The OPE can be expressed as

\[
U(z) U(w) = \frac{1}{(z-w)^2} \hbar^{\frac{1}{2}} + \frac{1}{(z-w)^2} \frac{8}{2} T(w) + \frac{1}{(z-w)^2} \frac{1}{2} \partial T(w) + \frac{1}{(z-w)^2} \frac{1}{2} \partial^2 T(w)
\]

\[
+ \frac{1}{(z-w)^2} \left[ \frac{3}{10} \partial T + \frac{9}{14} \left( T^2 - \frac{3}{10} \partial^2 T \right) + P_4^{uu} + P_4^{uu} \right] (w) + \cdots, (2.21)
\]

where the spin-4 primary field is given by

\[
P_4^{uu}(z) = \frac{75}{407} \left[ \frac{7}{10} \partial T + \frac{17}{42} \left( T^2 - \frac{3}{10} \partial^2 T \right) + G \partial G \right] (z) \]

(2.22)
which is identical to the one in (2.11) with \( c = 4 \). Note that the new spin-4 primary field, compared to the OPE (2.10), arises, as reported elsewhere [22]

\[
P^{uu}_4(z) = -\frac{18}{407} \psi^a \partial^3 \psi^a(z) + \text{other lower order derivative terms}, \tag{2.23}
\]

where only eight out of 156 terms are presented. In the context of (1.3), at the first-order singular term, there are triplet degeneracies for given spin-4 (quasi)primary fields. One way to see this extra new primary field (2.23) is to subtract the \( \frac{3}{10} \partial^2 T(w) \)-term, \((TT - \frac{3}{10} \partial^2 T)(w)\)-term with an arbitrary coefficient, and the spin-4 term (2.22) with an undetermined coefficient, from the first-order singular term. These two unknown coefficients \( \frac{9}{14} \) and \( \frac{75}{407} \) were fixed by the (quasi)primary condition. This spin-4 field (2.23) will provide some component field of \( \mathcal{N} = 1 \) superprimary field \( \hat{O}_7^2(Z) \) or \( \hat{O}_4(Z) \).

Let us move on to the next OPE between the spin-3 current (2.18) and spin-\( \frac{5}{2} \) current (2.19)

\[
W(z) U(w) = \frac{1}{(z-w)^4} \frac{3}{\sqrt{6}} G(w) + \frac{1}{(z-w)^3} \left( \frac{2}{3} \right) \frac{3}{\sqrt{6}} \partial G(w)
\]

\[
+ \frac{1}{(z-w)^2} \left[ \left( \frac{1}{4} \right) \frac{3}{\sqrt{6}} \partial^2 G + \frac{11 \sqrt{6}}{37} \left( GT - \frac{1}{8} \partial^2 G \right) + O_{\frac{7}{2}} \right](w)
\]

\[
+ \frac{1}{(z-w)} \left[ \left( \frac{1}{15} \right) \frac{3}{\sqrt{6}} \partial^2 G + \left( \frac{4}{7} \right) \frac{11 \sqrt{6}}{37} \partial \left( GT - \frac{1}{8} \partial^2 G \right) + \left( \frac{4}{7} \right) \partial O_{\frac{7}{2}} \right] (w) + \cdots . \tag{2.24}
\]

Compared to the minimal extension in previous subsection, there are two new primary fields [22]. One is the spin-\( \frac{7}{2} \) field

\[
O_{\frac{7}{2}}(z) = -\frac{\sqrt{2}}{37} f^{abc} \psi^a \psi^b \partial^2 \psi^c(z) + \text{other first-order derivative terms} \tag{2.25}
\]

where only the highest derivative terms of 99 terms is presented. The other is a spin-\( \frac{9}{2} \) field

\[
O_{\frac{9}{2}}(z) = -\frac{4 \sqrt{2}}{231} f^{abc} \psi^a \psi^b \partial^3 \psi^c(z) + \text{other lower order derivative terms} \tag{2.26}
\]

which consists of 270 terms. The structure constant, \( \frac{11 \sqrt{6}}{37} \) and \( \frac{4 \sqrt{6}}{17} \), in (2.24) are determined by the (quasi)primary condition as before. Generally, these are given in terms of the central charge, which are different from those in (2.9) because the \( c = 4 \) model is considered. The advantage of the \( c = 4 \) model is that because the explicit form for the WZW currents is known, one can always calculate the OPE and determine the singular terms. On the other
hand, the construction in a minimal extension is based on the assumption that there are some extended generators whose realizations are unknown. Therefore, the possible structures with unknown coefficient functions should be expressed in the right hand side of the OPE and the Jacobi identities should be used to fix them. This subsection and next section arranges the known singular terms and extracts all the possible (quasi)primary fields using (1.3) or the expression in Appendix A, (A.1) and (A.2).

For the spin-3 current (2.18) and spin-3 current OPE,

\[
W(z) W(w) = \frac{1}{(z-w)^6} + \frac{1}{(z-w)^4} 2T(w) + \frac{1}{(z-w)^3} \partial T(w)
\]

\[
+ \frac{1}{(z-w)^2} \left[ \left( \frac{3}{10} \right) \partial^2 T + \frac{16}{21} \left( T^2 - \frac{3}{10} \partial^2 T \right) + P_{4w}^w + P_{w4}^w \right] (w)
\]

\[
+ \frac{1}{(z-w)} \left[ \left( \frac{1}{15} \right) \partial^3 T + \left( \frac{1}{2} \right) \frac{16}{21} \partial \left( T^2 - \frac{3}{10} \partial^2 T \right) + \frac{1}{2} \partial P_{4w}^w + \frac{1}{2} \partial P_{w4}^w \right] (w)
\]

\[
+ \cdots,
\]

(2.27)

where the primary field of spin-4 is

\[
P_{4w}^w(z) = -\frac{48}{407} \left[ -\frac{7}{10} \partial^2 T + \frac{17}{42} \left( T^2 - \frac{3}{10} \partial^2 T \right) + G \partial G \right] (z)
\]

(2.28)

and the new primary spin-4 field [22] is

\[
P_{4w}^w(z) = \frac{2}{1221} \psi^a \partial^3 \psi^a (z) + \text{other lower order derivative terms}
\]

(2.29)

whose number of terms is the same as that for spin-4 field in (2.23). Note that these two currents do not appear in (2.8). The spin-4 field (2.29) will provide some component of the superprimary field \( \hat{O}_2^Z (Z) \) or \( \hat{O}_4^Z (Z) \). Although the spin-4 field (2.22) or (2.28) is primary under the stress energy tensor, the OPE with a spin-\( 3/2 \) current exhibits unusual behavior as mentioned previously. If the primary field is one of the components in the superprimary field with a given spin, it should transform like (2.6) or (2.7). On the other hand, the OPE of this spin-4 field and the spin-\( 3/2 \) current has third-order and fourth-order singular terms. This suggests that there is no superpartner for this spin-4 field.

Thus far, the spin-3 current and spin-\( 5/2 \) current are given in (2.18) and (2.19), whereas the spin-\( 7/2 \) field and spin-\( 9/2 \) field are found in (2.25) and (2.26), respectively. These are located in the first three supercurrents in the list (2.20). Other higher spin currents, the superpartners of \( O_2^Z (z) \) and \( O_2^Z (z) \), should be found in terms of eight fermion fields.

\[\text{More explicitly, the OPE is obtained } G(z) \left( TT + \frac{42}{17} G \partial G - \frac{69}{34} \partial^2 T \right)(w) = -\frac{1}{(z-w)^4} \left( \frac{1221}{68} G(w) - \frac{1}{(z-w)^4} \right) + \mathcal{O}((z-w)^{-2}).\]
How should the other higher spin currents corresponding to $O_4(z)$ or $O_4'(z)$ be determined? Consider the OPE between the spin-$\frac{3}{2}$ current $G(z)$ and spin-$\frac{5}{2}$ current $O_{\frac{7}{2}}(w)$. This OPE was calculated because there are explicit forms in (2.17) and (2.25), respectively. The results showed that this OPE leads to the following first-order singular term with 
\[-\frac{1}{\sqrt{6}}P_{4w}^{uu} + \sqrt{6}P_{4w}^{ww}\](w), where the spin-4 currents are given in (2.23) and (2.29) as before. This suggests that the following current of spin-4, due to $\mathcal{N} = 1$ supersymmetry, can be constructed

$$O_4(z) = \left( -\frac{1}{\sqrt{6}}P_{4}^{uu} + \sqrt{6}P_{4}^{ww} \right)(z). \quad (2.30)$$

That is,

$$G(z)O_{\frac{7}{2}}(w) = \frac{1}{(z-w)^2}O_4(w) + \cdots. \quad (2.31)$$

Steps should be taken to ensure that the OPE $G(z)$ with $O_4(w)$ leads to expected singular terms with $O_{\frac{7}{2}}(w)$ by $\mathcal{N} = 1$ supersymmetry.

$$G(z)O_{\frac{7}{2}}(w) = \frac{1}{(z-w)^2} 7O_4(w) + \frac{1}{(z-w)} \partial O_4(w) + \cdots. \quad (2.32)$$

Furthermore, the OPE between the spin-$\frac{3}{2}$ current $G(z)$ and spin-$\frac{9}{2}$ current $O_{\frac{9}{2}}(w)$ (2.26) can be calculated to determine the superpartner with a spin-4 field. Similar to (2.7),

$$G(z)O_{\frac{9}{2}}(w) = \frac{1}{(z-w)^2} 8O_4'(w) + \frac{1}{(z-w)} \partial O_4'(w) + \cdots, \quad (2.33)$$

where the superpartner of $O_{\frac{7}{2}}(z)$ is given by

$$O_4'(z) = \frac{1}{8} \left( \frac{16}{7} P_{4}^{uu} - \frac{4}{7} \sqrt{6}P_{4}^{ww} \right)(z). \quad (2.34)$$

In general, the coefficient in the second-order pole in (2.33) is equal to 2 times the spin of the current appearing in that singular term. In (2.33), $8 = 2 \times 4$ whereas in (2.32), $7 = 2 \times \frac{7}{2}$.

Similar to (2.6),

$$G(z)O_4'(w) = \frac{1}{(z-w)} O_{\frac{7}{2}}(w) + \cdots. \quad (2.35)$$

Therefore, the supercurrents, $\hat{O}_{\frac{7}{2}}(z)$ and $\hat{O}_4(z)$, in (2.20) are determined completely.\(^{10}\)

\(^{10}\)The OPEs can be written conveniently as $G(z) P_{4w}^{uu}(w) = \frac{1}{(z-w)^2} \sqrt{6}O_{\frac{7}{2}}(w) + \frac{1}{(z-w)^2} \left( \frac{\sqrt{2}}{6} \partial O_{\frac{7}{2}} + 2\sqrt{6}O_{\frac{7}{2}} \right)(w) + \cdots$ and $G(z) P_{4w}^{ww}(w) = \frac{1}{(z-w)^2} 4\sqrt{6}O_{\frac{7}{2}}(w) + \frac{1}{(z-w)^2} \left( \frac{4}{3} \sqrt{6} \partial O_{\frac{7}{2}} + \sqrt{2}O_{\frac{7}{2}} \right)(w) + \cdots$. This is why the primary fields $P_{4w}^{uu}(z)$ and $P_{4w}^{ww}(w)$ are unsuitable for the $\mathcal{N} = 1$ supercurrents.
The supercurrent $\hat{O}_2(Z)$ was constructed with (2.31) and (2.32) and $\hat{O}_4(Z)$ with (2.33) and (2.35) in the list of (2.20). Indeed, these were reported in reference [21]. Their component fields are given by (2.33), (2.34) and (2.26). In the $N=1$ notation, the above superfusion rule between the supercurrent $\hat{W}(Z)$ and itself can be rewritten as $[\hat{W}] [\hat{W}] = \hat{I} + [\hat{O}_2] + [\hat{O}_4]$. The right hand side of this OPE contains two higher spin superprimary fields.

The OPE between $\hat{W}(Z_1)$ and $\hat{O}_2(Z_2)$ which is the next lower higher spin supercurrent could be considered.

- The construction of higher spin supercurrents $\hat{O}_4(Z)$ and $\hat{O}_9(Z)$

The next lower higher spin currents in (2.20) can be obtained. The OPE between the spin-$\frac{5}{2}$ current $U(z)$ given in (2.19) and the spin-$\frac{7}{2}$ current $O_7^2(w)$ given in (2.25) can be calculated. The results showed that

$$
U(z) O_7^2(w) = \frac{1}{(z-w)^3} \frac{36}{185} W(w) + \frac{1}{(z-w)^2} \left[ \left( \frac{1}{3} \right) \frac{36}{185} \partial W - \frac{6\sqrt{6}}{481} \left( GU - \frac{\sqrt{6}}{3} \partial W \right) + O_4' \right] (w) + \frac{1}{(z-w)} \left[ \left( \frac{1}{14} \right) \frac{36}{185} \partial^2 W - \left( \frac{3}{8} \right) \frac{6\sqrt{6}}{481} \partial \left( GU - \frac{\sqrt{6}}{3} \partial W \right) + \left( \frac{3}{8} \right) \partial O_4' \right] + \frac{764}{8325} \left( TW - \frac{3}{14} \partial^2 W \right) + \frac{287}{5550\sqrt{6}} \left( G\partial U - \frac{5}{7} \partial GU - \frac{\sqrt{6}}{3} \partial^2 W \right) + O_5 \right] (w) + \cdots.
$$

(2.36)

For the primary field $W(w)$ with the structure constant, $\frac{36}{185}$, in the right hand side, the relative coefficients for its descendant fields appearing in various singular terms can be read off from (1.3). See also Appendix A for the detailed coefficients in the structure constants. Consider the second-order singular terms. The first term originating from $W(w)$ is fixed. Therefore, how can the next quasi-primary or primary field be observed? From the second-order pole, the OPE between $T(z)$ and the second order-pole subtracted by $\left( \frac{1}{3} \right) \frac{36}{185} \partial W(w)$ can be calculated to extract the possible quasi-primary fields (i.e. the exact expressions and number of quasi-primary fields). $T(z) \left\{ U O_7^2 \right\}_{-2} - \left( \frac{1}{3} \right) \frac{36}{185} \partial W \right\} (w) = + O((z-w)^{-2})$ can then be obtained. In other words, it transforms as a primary field. On the other hand, $G(z) \left\{ U O_7^2 \right\}_{-2} - \left( \frac{1}{3} \right) \frac{36}{185} \partial W \right\} (w) = \frac{1}{(z-w)^3} \frac{2\sqrt{6}}{37} U(w) + O((z-w)^{-2})$ can be calculated.

This suggests that the remaining terms in the second-order pole contain a primary field with unusual behavior with $G(z)$. This can be obtained explicitly. By subtracting $GU(w)$ plus the

\footnote{A simplified notation here $\{ U O_7^2 \}_{-2}(w)$ was used for the second order pole of (2.36) in the spirit of [24] [15] [27].}
other derivative term with unknown coefficient into the above second-order singular terms, the unwanted third-order pole, which is proportional to \(U(w)\), can be removed by choosing the correct coefficient. This is because when the OPE \(G(z)\) with \(GU(w)\) is calculated, the \(U(w)\) term in the third-order singular term can be derived via (2.2). Therefore, \((GU - \sqrt{\frac{7}{4}}\partial W)(w)\) might be a possible candidate for the primary field that needs to be subtracted. The coefficient \(-\frac{6\sqrt{7}}{481}\) in (2.36) in front of this field was fixed by requiring that there should be no third-order pole from the superprimary condition. Then we are left with the following primary field. The spin-4 primary field can be expressed as

\[
O_{4'}(z) = \frac{384}{13} \sqrt{\frac{2}{5}} i \psi^1 \psi^2 \psi^3 \psi^4 \psi^5 \psi^6 \psi^7 \psi^8(z) + \text{other derivative terms. (2.37)}
\]

Consider the last first-order singular terms. The first line of these terms in (2.36) describes the descendant field for the spin-3 current and two descendant fields for the spin-4 primary fields. As stated before, the difference between the whole first-order singular terms and those three descendant terms were calculated to ensure that there are two quasi-primary fields presented in the second line of the first-order singular terms in (2.36). The OPE \(T(z)\) with \[\{UO_7\}_{-1} - \text{first line}\] \((w)\) leads to \(\frac{1}{(z-w)^3} \frac{162}{259} W(w) + \mathcal{O}((z-w)^{-2})\). This suggests that the extra quasi-primary field, \(T W(w)\), plus derivative terms, should be considered to cancel the fourth-order term \(\frac{162}{259} W(w)\) for the superprimary condition. Furthermore, the OPE between \(G(z)\) and \[\{UO_7\}_{-1} - \text{first line}\] \((w)\) should be also calculated, which leads to \(\frac{1}{(z-w)^3} \frac{33}{518} \sqrt{\frac{3}{2}} U(w) + \frac{1}{(z-w)^3} (-\frac{1}{5}) \frac{33}{518} \sqrt{\frac{3}{2}} \partial U(w) + \mathcal{O}((z-w)^{-2})\). This suggests that the extra quasi-primary field consisting of \(G \partial U(w), \partial G U(w)\) and other derivative terms should be considered to remove the higher singular terms above. Finally, the consistent coefficients, \(\frac{764}{8325}\) and \(\frac{287}{5500\sqrt{6}}\), in the second line of the first-order pole in (2.36) are fixed from the above analysis (i.e. superprimary condition) and the following spin-5 primary current remains

\[
O_5(z) = -\frac{8}{225} \sqrt{\frac{2}{15}} i f^{abc} d^{ade} \psi^b \psi^c \psi^d \partial^3 \psi^e(z) + \text{other lower order derivative terms. (2.38)}
\]

Therefore, two primary currents (2.37) and (2.38) were obtained where the former is the \(\theta\)-independent component field of \(\hat{O}_{4'}(Z)\) and the latter is the \(\theta\)-dependent component field of \(\hat{O}_{5}(Z)\).

Let us calculate the OPE of spin-\(\frac{3}{2}\) current (2.19) and spin-4 current (2.30) to determine the superpartners corresponding to the above two primary fields. Each singular term, starting

\[\footnote{One obtains the OPE \(G(z) (GU - \sqrt{\frac{7}{4}}\partial W)(w) = \frac{1}{(z-w)^3} \frac{15}{3} U(w) + \mathcal{O}((z-w)^{-2})\). See also Appendix C. In the \(\mathcal{N} = 1\) supercurrent, this primary field originates from \(TW(Z_2)\).}
from fourth-order singular term can be obtained. The final result is presented first, which explains how this result can be obtained explicitly

\[
U(z) \cdot O_4(w) = \frac{1}{(z-w)^4} U(w) + \frac{1}{(z-w)^3} \left( \frac{1}{5} \frac{6 \sqrt{6}}{37} \partial U(w) \right) \\
+ \frac{1}{(z-w)^2} \left[ \left( \frac{1}{30} \right) \frac{6 \sqrt{6}}{37} \partial^2 U + \left( \frac{1}{3} \right) \frac{1596}{12025} \partial \left( G W - \frac{1}{6 \sqrt{6}} \partial^2 U \right) \right] \\
+ \frac{1}{(z-w)} \left[ \left( \frac{1}{210} \right) \frac{6 \sqrt{6}}{37} \partial^3 U + \left( \frac{1}{3} \right) \frac{1596}{12025} \partial \left( G W - \frac{1}{6 \sqrt{6}} \partial^2 U \right) \right] \\
+ \left( \frac{1}{3} \right) \partial P_{\frac{9}{2}} + Q_{\frac{11}{2}} \right) (w) + \cdots \\
\] (2.39)

The structure of the right hand side appears similar to (2.36) by changing \( W(w) \) to \( U(w) \) and vice versa. Once the structure constant, \( \frac{6 \sqrt{6}}{37} \) in the highest order singular term is found, the other relative coefficients in the lower singular terms, which are associated with the spin-\( \frac{5}{2} \) current \( U(w) \), are determined automatically using the formula (1.3). Therefore, the other terms in the right hand side of (2.39) should be determined. Consider the nontrivial second-order singular terms. One should ensure that there are three extra quasi-primary fields including the last primary field with the right structure constants. As performed before, the OPE between \( T(z) \) and \( \left\{ UO_4 \right\}_{-2} - \left( \frac{1}{30} \right) \frac{6 \sqrt{6}}{37} \partial^2 U \) \( (w) \) was calculated. This OPE has \( \frac{18 \sqrt{6}}{37} U(w) \) in the fourth-order singular term. This shows that the extra quasi-primary field should contain a \( TU(w) \) term. Moreover, the OPE between \( G(z) \) and \( \left\{ UO_4 \right\}_{-2} - \left( \frac{1}{30} \right) \frac{6 \sqrt{6}}{37} \partial^2 U \) \( (w) \) should be calculated. This OPE has the following third-order singular term, \( \frac{1}{(z-w)^3} \frac{27 \sqrt{6}}{180} W(w) \). Then \( GW(w) \) can be considered a quasi-primary field with some derivative term. The final new spin-\( \frac{9}{2} \) primary field can be obtained by subtracting these two candidates from the second-order singular terms,

\[
P_{\frac{9}{2}}(z) = \frac{1}{13} \sqrt{\frac{2}{5}} \epsilon^{abc} f^{bde} f^{efg} \psi^a \psi^d \psi^e \psi^f \partial^2 \psi^g (z) + \text{other first-order derivative terms},
\] (2.40)

which will play the role of the component field of some superprimary field.

Let us focus on the next first-order singular term. Now all possible descendant fields originating from the quasi-primary fields of spin-\( \frac{5}{2} \) and of spin-\( \frac{9}{2} \) can be written with the correct coefficient functions. Then the following quasi-primary field of spin-\( \frac{11}{2} \) remains

\[
Q_{\frac{11}{2}}(z) = - \frac{584}{7215} \sqrt{\frac{2}{3}} \left( T \partial U - \frac{5}{4} \partial TU - \frac{1}{7} \partial^3 U \right) (z) \\
+ \frac{10}{481} \left( G \partial W - 2 \partial GW - \frac{1}{21} \sqrt{\frac{2}{3}} \partial^3 U \right) (z) + \frac{3}{4} \left( G \partial O_2 - \frac{2}{9} \partial O_2 \right) (z). \] (2.41)
This expression is expressed in terms of the various WZW currents and it is difficult to express this in terms of (2.41). Note that there is a new primary field, $O_{9}^{2}(z)$, in the right hand side of (2.41). From the newly obtained spin-4 primary field in (2.37), the OPE of $G(z)$ with $O_{9}^{2}(w)$ can be calculated, which leads to a nonzero first-order singular term that is equal to the above $O_{9}^{2}(w)$ where

$$O_{9}^{2}(z) = -\frac{8}{65} \sqrt{\frac{2}{5}} \epsilon^{abc} f^{bde} f^{c} g \psi^{a} \psi^{d} \psi^{f} \partial^{2} \psi^{g}(z) + \text{other first-order derivative terms} \tag{2.42}$$

The OPE between $G(z)$ and the field $O_{9}^{2}(w)$ should be calculated for a double check. This provides a consistent result. That is, the second-order singular term has $8O_{9}^{2}(w)$, whereas the first-order singular term is given by $\partial O_{9}^{2}(w)$. These are combined into a single superprimary field, $\hat{O}_{9}^{2}(Z)$, as shown in (2.20). The remaining current, $O_{9}^{2}(z)$, should be determined. The quasi-primary property of (2.41) can be checked by the following OPE with $T(z)$

$$T(z) Q_{\frac{3}{2}}^{\frac{1}{2}}(w) = \frac{1}{(z-w)^{5}} \frac{72}{259} \sqrt{6} U(w) + \frac{1}{(z-w)^{4}} \left(-\frac{1}{5}\right) \frac{72}{259} \sqrt{6} \partial U(w)$$

$$+ \mathcal{O}((z-w)^{-2}). \tag{2.43}$$

No third-order pole exists. Furthermore, the following OPE can be obtained with the spin-$\frac{3}{2}$ current

$$G(z) Q_{\frac{3}{2}}^{\frac{1}{2}}(w) = \frac{1}{(z-w)^{4}} \frac{48}{1295} W(w)$$

$$+ \frac{1}{(z-w)^{2}} \left[\left(-\frac{1}{6}\right) \frac{48}{1295} \partial W + \frac{16}{3} O_{9}^{2} + \frac{32}{481} \sqrt{6} \left(GU - \frac{\sqrt{6}}{3} \partial W\right)\right](w)$$

$$+ \mathcal{O}((z-w)^{-2}). \tag{2.44}$$

Note that in (2.41), three quantities specified by the bracket can be checked to ensure that they are quasi-primary fields. Appendices B and C present some properties of the various quasi-primary fields described in this paper. According to the definition of quasi-primary field introduced in the introduction, any linear combination of quasi-primary fields leads to another quasi-primary field that can be written in terms of the known currents.

Consider the next OPE between the spin-3 current (2.18) and the spin-$\frac{7}{2}$ current (2.25) to complete the OPE between $W(Z_{1})$ and $\hat{O}_{\frac{7}{2}}(Z_{2})$. The result is as follows:

$$W(z) Q_{\frac{7}{2}}^{\frac{1}{2}}(w) = \frac{1}{(z-w)^{4}} \frac{6}{37} U(w) + \frac{1}{(z-w)^{3}} \left(\frac{2}{3}\right) \frac{6}{37} \partial U(w)$$

$$+ \frac{1}{(z-w)^{2}} \left[\left(\frac{1}{10}\right) \frac{6}{37} \partial^{2} U + \frac{412}{7215} \left(T U - \frac{1}{4} \partial^{2} U\right)\right] + \frac{116 \sqrt{6}}{12025} \left(GW - \frac{1}{6 \sqrt{6}} \partial^{2} U\right) + \frac{1}{\sqrt{6}} P_{\frac{7}{2}}$$

17
\[
\left[ 1 \frac{\sqrt{6}}{2} \right] (w) + \frac{1}{(z-w)} \left[ \left( \frac{2}{105} \right) \frac{6}{37} \partial^3 U + \left( \frac{4}{9} \right) \frac{412}{7215} \partial \left( TU - \frac{1}{4} \partial^2 U \right) \right] + \left( \frac{4}{9} \right) \frac{116\sqrt{6}}{12025} \partial \left( GW - \frac{1}{6\sqrt{6}} \partial^2 U \right) + \left( \frac{4}{9} \right) \frac{1}{\sqrt{6}} \partial P^2 + \left( \frac{4}{9} \right) \frac{1}{\sqrt{6}} \partial O^2 + Q^2 \right] (w) + \cdots \quad (2.45)
\]

This appears similar to the previous OPE (2.39). As stated before, once the structure constant appearing in front of \( U(w) \) in the right hand side of (2.45) is found from the corresponding singular term with the WZW currents, the relevant coefficients associated with its descendant fields in the second-order and first-order poles are known from (1.3). The next step is to look at the next nontrivial lower singular terms to determine if there is a new primary field or not. If not, the singular terms should be expressed in terms of the known quasi-primary fields or new unknown quasi-primary fields. The calculated OPE between \( T(z) \) and \( \{ WQ \}_{-2} - \left( \frac{4}{9} \right) \frac{2}{37} \partial^2 U \) (w) becomes

\[
- \frac{1}{(z-w)^2} \frac{9}{37} U(w) - \frac{1}{(z-w)^2} \left( \frac{2}{9} \right) \frac{9}{37} \partial U(w) + \mathcal{O}((z-w)^{-2}).
\]

On the other hand, the OPE with \( G(z) \) leads to

\[
- \frac{1}{\sqrt{3}} \frac{9}{185} W(w)\quad (2.46)
\]

All the field contents for this quasi-primary field are given in terms of the previously determined quasi-primary fields. Each quasi-primary field in (2.46) also appears in (2.41). The only difference is the relative coefficients between them. As shown in (2.43) and (2.44), the following OPE can be derived

\[
T(z) Q^{\frac{11}{2}}(w) = \frac{1}{(z-w)^5} \frac{80}{711} U(w) + \frac{1}{(z-w)^4} \frac{1}{5} \frac{80}{711} \partial U(w) + \mathcal{O}((z-w)^{-2}). \quad (2.47)
\]

No third-order pole exists. Similarly,

\[
G(z) Q^{\frac{11}{2}}(w) = \frac{1}{(z-w)^4} \frac{144}{1295} \sqrt{6} W(w)
\]

18
\[ + \frac{1}{(z-w)^3} \left[ (\frac{1}{6}) \frac{144}{1295} \partial W + \frac{8}{3} \sqrt{2} O_{4'} + \frac{32}{481} \left( GU - \frac{\sqrt{6}}{3} \partial W \right) \right] (w) \]
\[ + \mathcal{O}((z-w)^{-2}). \]  

(2.48)

Of course, the field contents in the right hand side of (2.47) and (2.48) are the same as those in (2.43) and (2.44), as expected. Thus far, two primary fields, (2.40) and (2.42), were found, which will play the role of the undetermined spin-9/2 current \( O_{9/2}(z) \).

The OPE between spin-3 current (2.18) and spin-4 current (2.30) was considered. Therefore,

\[ W(z) O_4(w) = \frac{1}{(z-w)^4} \frac{48\sqrt{6}}{185} W(w) \]
\[ + \frac{1}{(z-w)^3} \left[ \left( \frac{1}{3} \right) \frac{48\sqrt{6}}{185} \partial W + \sqrt{2} O_{4'} + \frac{12}{481} \left( GU - \frac{\sqrt{6}}{3} \partial W \right) \right] (w) \]
\[ + \frac{1}{(z-w)^2} \left[ \left( \frac{1}{14} \right) \frac{48\sqrt{6}}{185} \partial^2 W + \left( \frac{3}{8} \right) \sqrt{2} \partial O_{4'} + \left( \frac{3}{8} \right) \frac{12}{481} \partial \left( GU - \frac{\sqrt{6}}{3} \partial W \right) \right] \]
\[ + 5 \sqrt{\frac{2}{3}} O_5 + Q_5 \] (w)
\[ + \frac{1}{(z-w)} \left[ \left( \frac{1}{84} \right) \frac{48\sqrt{6}}{185} \partial^3 W + \left( \frac{1}{12} \right) \sqrt{2} \partial^2 O_{4'} + \left( \frac{1}{12} \right) \frac{12}{481} \partial^2 \left( GU - \frac{\sqrt{6}}{3} \partial W \right) \right] \]
\[ + \left( \frac{2}{3} \right) 5 \sqrt{2} \partial O_5 + \left( \frac{2}{3} \right) \partial Q_5 + Q_6 \] (w).  

(2.49)

By identifying the highest singular term from the explicit expression of the OPE, the descendant fields can be expressed with correct coefficient functions for the spin-3 current, \( W(w) \), on the right hand side. The first nontrivial third-order term can be analyzed. The OPE between \( T(z) \) and \( \{ W O_4 \}_{-3} - \left( \frac{1}{3} \right) \frac{48\sqrt{6}}{185} \partial W \) (w) can be calculated, and this OPE does not produce any higher order ( \( \geq 2 \) ) singular terms. This leads to the appearance of a primary field. What of the OPE between \( G(z) \) and \( \{ W O_4 \}_{-2} - \left( \frac{1}{3} \right) \frac{48\sqrt{6}}{185} \partial W \) (w)? The third-order term of this OPE contains \( \frac{1}{37} U(w) \). Therefore, the primary field containing \( GU(w) \) can be extracted from the third-order pole. The spin-4 current (2.37) remains as described previously. Because the third-order singular terms are determined completely, let us move on the second-order term. According to (1.3), the first line of the second-order terms in (2.49) can be extracted from the entire second-order pole. The OPE can then be calculated with \( T(z) \) and \( \{ W O_4 \}_{-2} \) (first line) (w). The nontrivial part of this OPE contains \( \frac{1248\sqrt{6}}{1295} W(w) \) at the fourth-order pole. For the OPE between \( G(z) \) and \( \{ W O_4 \}_{-2} \) (first line) (w), there
are $\frac{573}{578} U(w)$ in the fourth-order term and $(-\frac{1}{3})\frac{573}{578} \partial U(w)$ in the third-order term. Therefore, the two quasi-primary fields corresponding to $TW(w)$ (and derivative term) and $G\partial U(w)$ (and other terms) respectively, were subtracted. The spin-5 primary field (2.38) was obtained after subtracting these two quasi-primary fields properly in the second-order pole. Moreover, the following spin-5 quasi-primary field can be obtained

$$Q_5(z) = \frac{458}{1665} \sqrt{\frac{2}{3}} \left( TW - \frac{3}{14} \partial^2 W \right)(z) - \frac{19}{3330} \left( G\partial U - \frac{5}{3} \partial GU - \frac{\sqrt{6}}{7} \partial^2 W \right)(z)$$

(2.50)
and its OPEs with $T(z)$ and $G(z)$ are given by

$$T(z) Q_5(w) = \frac{1}{(z-w)^3} \frac{1248 \sqrt{6}}{1295} W(w) + \mathcal{O}((z-w)^{-2}),$$

$$G(z) Q_5(w) = -\frac{1}{(z-w)^3} \frac{573}{2590} U(w) - \frac{1}{(z-w)^3} \left( -\frac{5}{3} \frac{573}{2590} \partial U(w) \right) + \mathcal{O}((z-w)^{-2}).$$

(2.51)

The first equation of (2.51) does not have a third-order pole. The entire structure of the second-order pole is fixed, and the last first-order pole can be described. By subtracting the three descendant fields (coming from spin-3 primary field and two spin-4 quasi-primary fields) and the remaining two descendant fields coming from the spin-5 quasi-primary fields appearing in the second-order pole, there is a spin-6 quasi-primary field that can be expressed as follows:

$$Q_6(z) = \frac{32}{12025} \left( G\partial^2 U - 4\partial G\partial U + \frac{5}{2} \partial^2 GU - \frac{1}{2} \sqrt{3} \partial^2 W \right)(z)$$

$$- \frac{192}{12025} \sqrt{6} \left( T\partial W - \frac{3}{2} \partial TW - \frac{1}{8} \partial^3 W \right)(z) + \frac{1}{2} \sqrt{\frac{3}{2}} \left( TO_{4'} - \frac{1}{6} \partial^2 O_{4'} \right)(z)$$

$$- \frac{1}{4} \sqrt{\frac{3}{2}} \left( GO_{2'} - \frac{1}{9} \partial^2 O_{2'} \right)(z).$$

(2.52)

The following OPEs can be calculated easily

$$T(z) Q_6(w) = \frac{1}{(z-w)^5} \frac{288}{925} \sqrt{6} W(w)$$

$$+ \frac{1}{(z-w)^4} \left[ (-\frac{1}{6}) \frac{288}{925} \sqrt{6} \partial W + 4 \frac{288}{481} \left( GU - \sqrt{6} \partial W \right) \right](w)$$

$$+ \mathcal{O}((z-w)^{-2}),$$

$$G(z) Q_6(w) = \frac{1}{(z-w)^5} \frac{64}{185} U(w) + \frac{1}{(z-w)^4} \left( -\frac{2}{5} \frac{64}{185} \partial U(w) \right)$$

$$+ \frac{1}{(z-w)^3} \left[ \frac{1}{30} \frac{64}{185} \partial^2 U + \frac{32}{481} \sqrt{6} \left( GW - \frac{1}{6 \sqrt{6}} \partial^2 U \right) \right]$$

$$- \frac{64}{481} \left( TU - \frac{1}{4} \partial^2 U \right) - \frac{8}{3} \frac{2}{3} \left( O_{4'} \right)(w) + \mathcal{O}((z-w)^{-2}).$$

(2.53)
The first equation of (2.53) does not have a third-order pole.

Therefore, the supercurrent, \( \hat{O}_4(Z) \) and \( \hat{O}_2(Z) \), in the list of (2.20) was constructed. Originally, the expression, \( \hat{O}_4(Z) \), was reported by [21]. Their component fields can be expressed as (2.37), (2.42) and (2.38). Furthermore, the remaining component field can be expressed as

\[
O_{p'}(z) = O_{p'}(z) + \frac{8}{5} P_p(z), \tag{2.54}
\]

where the component fields are given in (2.42) and (2.40). The OPE between \( O \) and \( G \) can be checked to determine if (2.54) is the right superpartner of the current \( O_5(z) \). The results showed that the first-order pole provides the spin-5 as with (2.31). Moreover, the OPE between the \( T \) part of the next singular terms. As stated before, this descendant field in the second-order determined and the formula (1.3) with this numerical value gives the explicit form for some

\[
\text{The first equation of (2.53) does not have a third-order pole.}
\]

\[
\text{Therefore, the supercurrent, } \hat{O}_4(Z) \text{ and } \hat{O}_2(Z) \text{, in the list of (2.20) was constructed. Originally, the expression, } \hat{O}_4(Z) \text{, was reported by [21]. Their component fields can be expressed as (2.37), (2.42) and (2.38). Furthermore, the remaining component field can be expressed as}
\]

\[
O_{p'}(z) = O_{p'}(z) + \frac{8}{5} P_p(z), \tag{2.54}
\]

\[
\text{where the component fields are given in (2.42) and (2.40). The OPE between } G(z) \text{ and } O_{p'}(w) \text{ can be checked to determine if (2.54) is the right superpartner of the current } O_5(z). \]

\[
\text{The results showed that the first-order pole provides the spin-5 as with (2.31). Moreover, the OPE between } G(z) \text{ and } O_5(w) \text{ provides the correct singular terms where the second order pole has } 9O_{p'}(w) \text{ and the first-order pole has } \partial O_{p'}(w), \text{ as expressed in (2.32) [13]. The following superfusion rule } [\hat{W}][\hat{O}_2] = [\hat{W}] + [\hat{O}_4] + [\hat{O}_2] \text{ can be considered. The OPE between } \hat{W}(Z_1) \text{ and } \hat{O}_4(Z_2) \text{ can be calculated to find the remaining higher spin currents.}
\]

\[
\text{\enspace \bullet \text{ The construction of } \hat{O}_5(Z) \text{ and } \hat{O}_6(Z)}
\]

\[
\text{This section describes the OPE between the spin-5 current (2.19) and spin-4 current (2.37).}
\]

\[
U(z) O_{p'}(w) = \frac{1}{(z - w)^3} \frac{888}{65} O_2(w) + \frac{1}{(z - w)^2} \left[ \frac{2}{7} \frac{888}{65} \partial O_2 - \frac{888}{65} \frac{1}{7} \partial^2 O_2 \right] (w)
\]

\[
\quad + \frac{1}{(z - w)} \left[ \frac{3}{56} \frac{888}{65} \partial^2 O_2 - \frac{3}{56} \frac{888}{65} \partial O_2 + \frac{288}{61} \left( TO_2 - \frac{3}{16} \partial^2 O_2 \right) \right.
\]

\[
\quad + \frac{58 \sqrt{6}}{793} \left( GP_{w'w} - \frac{4 \sqrt{6}}{9} \partial O_2 - \frac{\sqrt{6}}{56} \partial^2 O_2 \right)
\]

\[
\quad - \frac{5493}{3965} \frac{3}{2} \left( GP_{w'w} - \frac{2}{9} \frac{\sqrt{3}}{3} \partial O_2 - \frac{1}{14} \frac{\sqrt{3}}{3} \partial^2 O_2 \right) + O_{p'} \right) (w) + \cdots . \tag{2.55}
\]

\[
\text{From the highest singular term in (2.55), the structure constant for the spin-}\frac{5}{2}\text{ current can be determined and the formula (1.3) with this numerical value gives the explicit form for some part of the next singular terms. As stated before, this descendant field in the second-order singular term can be extracted and the remaining term can be expressed in terms of the spin-}\frac{9}{2}\text{ current (2.26). The OPE between the } T(z) \text{ and } (\{ UO_{p'} \}_{-1} - (\frac{3}{56}) \frac{888}{65} \partial^2 O_2 + (\frac{1}{3}) \frac{888}{65} \partial O_2) \text{ can be calculated to determine the nontrivial first-order singular term completely. This leads to the nontrivial fourth-order pole with } \frac{888}{65} O_2(w) \text{, the quasi-primary field containing } TO_2 \text{ can be extracted from the first-order pole. Furthermore, the OPE between } G(z) \text{ and } (\{ UO_{p'} \}_{-1} - (\frac{3}{56}) \frac{888}{65} \partial^2 O_2 + (\frac{1}{3}) \frac{888}{65} \partial O_2) \text{ has a nontrivial third-order singular term}
\]

\[
\text{\textsuperscript{13}The OPE } G(z) P_2(w) = -\frac{1}{(z - w)^5} 5O_{p'}(w) + \frac{1}{(z - w)^4} \frac{1}{2} (O_5 - \partial O_{p'}) (w) + \cdots \text{ can be obtained conveniently. Therefore, one should consider equation (2.54) to remove the unwanted terms } O_{p'}(w) \text{ and its descendant field.}
\]
with \((-\frac{439}{195}\sqrt{\frac{2}{3}}P_{4u}^{uu} + \frac{526}{65}\sqrt{\frac{2}{3}}P_{4w}^{ww})\)(w). The two quasi-primary fields containing \(GP_{4u}^{uu}(w)\) and \(GP_{4w}^{ww}(w)\) can be considered. Finally, the new spin-\(\frac{11}{2}\) current can be derived by rearranging the first-order terms as done previously,

\[
O_{\frac{11}{2}}(z) = \frac{3}{1586\sqrt{2}} f^{abc} \psi^a \psi^b \partial^4 \psi^c(z) + \text{other lower order derivative terms},
\]

which is the \(\theta\)-independent term of \(\hat{O}_{\frac{11}{2}}(Z)\).

Consider the spin-\(\frac{5}{2}\) current (2.19) and the spin-\(\frac{9}{2}\) current (2.42) to determine other unknown higher spin currents. The result can be expressed as

\[
U(z)O_{\frac{9}{2}}(w) = \frac{1}{(z-w)^3} \left[ \frac{116}{65} \sqrt{\frac{2}{3}} P_{4u}^{uu} - \frac{584\sqrt{6}}{65} P_{4w}^{ww} \right](w)
\]

\[
+ \frac{1}{(z-w)^2} \left[ \left( \frac{1}{4} \right) \frac{116}{65} \sqrt{\frac{2}{3}} \partial P_{4u}^{uu} - \left( \frac{1}{4} \right) \frac{584\sqrt{6}}{65} \partial P_{4w}^{ww} \right](w)
\]

\[
+ \frac{792}{65} \left( GO_{\frac{9}{2}} + \frac{1}{4\sqrt{6}} \partial P_{4u}^{uu} - \frac{\sqrt{6}}{4} \partial P_{4w}^{ww} \right)(w)
\]

\[
+ \frac{1}{(z-w)} \left[ \left( \frac{1}{24} \right) \frac{116}{65} \sqrt{\frac{2}{3}} \partial^2 P_{4u}^{uu} - \left( \frac{1}{24} \right) \frac{584\sqrt{6}}{65} \partial^2 P_{4w}^{ww} \right]
\]

\[
+ \frac{3}{10} \frac{792}{65} \partial \left( GO_{\frac{7}{2}} + \frac{1}{4\sqrt{6}} \partial P_{4u}^{uu} - \frac{\sqrt{6}}{4} \partial P_{4w}^{ww} \right)
\]

\[
+ \frac{41524\sqrt{6}}{91195} \left( TP_{4u}^{uu} - \frac{1}{6} \partial^2 P_{4u}^{uu} \right) - \frac{328993\sqrt{6}}{91195} \left( TP_{4w}^{ww} - \frac{1}{6} \partial^2 P_{4w}^{ww} \right)
\]

\[
- \frac{43068}{19825} \left( G\partial O_{\frac{7}{2}} - \frac{7}{3} \partial G O_{\frac{7}{2}} + \frac{1}{9\sqrt{6}} \partial^2 P_{4u}^{uu} - \frac{1}{3} \sqrt{\frac{2}{3}} \partial^2 P_{4w}^{ww} \right)
\]

\[
- \frac{231}{299} \left( GO_{\frac{7}{2}} - \frac{2}{63} \sqrt{\frac{2}{3}} \partial^2 P_{4u}^{uu} + \frac{1}{21\sqrt{6}} \partial^2 P_{4w}^{ww} \right) + P_6 \right](w) + \cdots 
\]

(2.57)

In this case, the first nontrivial primary fields in the right hand side are given by (2.23) and (2.29). How can the quasi-primary field be extracted in the second-order pole? Actually, there exists a primary field of spin-5 but its OPE with \(G(z)\) exhibits unusual behavior. See Appendices B and C. A third-order pole exists, which is expressed as \(\frac{4884}{65}O_{\frac{7}{2}}(w)\). The primary field can be written as the one in the third term of second-order pole in (2.57). By taking the derivative to these second-order terms with the appropriate coefficients from (1.3) and subtracting them, the nontrivial terms remain in the first-order singular terms. Therefore, it is important to determine if there are four quasi-primary fields and a single primary field right. To accomplish this, the OPE between \(T(z)\) and \(\{UO_{\frac{9}{2}}\}_{-1} - \)}
first three descendant fields)\((w)\) should be calculated. The fourth-order term of this OPE is given as \(\frac{406}{65} \sqrt{2} P_{4^u}^{uu}(w) - \frac{2044}{65} \sqrt{6} P_{4^w}^{uw}(w)\). The quasi-primary field containing \(TP_{4^u}^{uu}(w)\) and the quasi-primary field \(TP_{4^w}^{uw}(w)\) with possible derivative terms can be considered. Moreover, the OPE \(G(z)\) with \((\{ UO_4^\gamma \}_{-1} - \text{first three descendant fields})(w)\) produces \(-\frac{3108}{325} \sqrt{2} (w)\) at the fourth-order term and \((-\frac{1}{7}) \frac{3108}{325} \partial O_{\frac{7}{2}}(w) - \frac{2464}{195} O_{\frac{7}{2}}(w)\) at the third-order pole. This allows the two additional quasi-primary fields to be taken, as shown in (2.57). Finally, subtracting the above four quasi-primary fields properly (with correct coefficients) leaves the following new spin-6 primary field:

\[
P_6(z) = \frac{17}{8418 \sqrt{6}} \psi^a \partial^5 \psi^a(z) + \text{other lower order derivative terms.} \tag{2.58}\]

This will play the role of the final spin-6 current that this study is interested in. Thus far, two primary fields (2.56) and (2.58) were found.

The OPE between the spin-3 current (2.18) and spin-4 (2.37) current are described as

\[
W(z) O_{4^\nu}(w) = \left[ \frac{1}{(z-w)^3} \right] \left[ -\frac{328}{195} P_{4^u}^{uu} + \frac{304}{65} P_{4^w}^{uw} \right] (w)
\]

\[
\frac{1}{(z-w)^2} \left[ -\frac{3}{8} \frac{328}{195} \partial P_{4^u}^{uu} + \frac{3}{8} \frac{304}{65} \partial P_{4^w}^{uw} + \frac{132 \sqrt{6}}{65} \left( GO_{\frac{7}{2}} + \frac{1}{4} \sqrt{6} \partial P_{4^u}^{uu} - \sqrt{6} \partial P_{4^w}^{uw} \right) \right] (w)
\]

\[
\frac{1}{(z-w)} \left[ -\frac{1}{12} \frac{328}{195} \partial^2 P_{4^u}^{uu} + \frac{1}{12} \frac{304}{65} \partial^2 P_{4^w}^{uw} + \frac{2}{5} \frac{132 \sqrt{6}}{65} \partial \left( GO_{\frac{7}{2}} + \frac{1}{4} \sqrt{6} \partial P_{4^u}^{uu} - \sqrt{6} \partial P_{4^w}^{uw} \right) \right]
\]

\[
-\frac{16896}{91195} \left( TP_{4^u}^{uu} - \frac{1}{6} \partial^2 P_{4^u}^{uu} \right) - \frac{4736 \sqrt{6}}{10675} \left( TP_{4^w}^{uw} - \frac{1}{6} \partial^2 P_{4^w}^{uw} \right)
\]

\[
-\frac{466 \sqrt{6}}{1495} \left( GO_{\frac{7}{2}} - \frac{2}{3} \partial^2 P_{4^u}^{uu} + \frac{1}{21 \sqrt{6}} \partial^2 P_{4^w}^{uw} \right) + P_6 \right] (w) + \cdots. \tag{2.59}\]

The right hand side appears similar to the OPE (2.57). For given structure constants on the two spin-4 fields, its descendant field terms are fixed completely and the OPE between \(T(z)\) with \((\{ WO_{4^\nu} \}_{-2} - \text{two descendant fields})(w)\) are calculated, which means there is no higher order singular term (order greater than 2) suggesting that the extra terms should correspond to the primary field. On the other hand, the OPE between \(G(z)\) and \((\{ WO_{4^\nu} \}_{-2} - \text{two descendant fields})(w)\) provides a nontrivial third-order pole with \(\frac{814 \sqrt{7}}{63} O_{\frac{7}{2}}(w)\). The corresponding primary field can be expressed in terms of \(GO_{\frac{7}{2}}(w)\) plus other derivative terms in (2.59). Based on these results for the second-order pole, three derivative terms can be obtained correctly in the first-order singular terms (coming from the second-order pole). The
OPE between $T(z)$ and $\{WO_6\}_1$ – three descendant fields $(w)$ can be calculated, which will determine the possible quasi-primary fields that need to be considered. A fourth-order pole exists in this OPE, which is expressed as $-\frac{\mathcal{O}_6}{354} P^{wmu}(w) + \frac{222}{345} P^{ww}(w)$. This suggests that the quasi-primary field should contain $TP^{wmu}_d(w)$ and $TP^{ww}_d(w)$, respectively. For the OPE $G(z)$ $(\{WO_6\}_1$ – three descendant fields) $(w)$, there is $\frac{225}{345} O_6^\pm(w)$ at the fourth-order pole and $(-\frac{2}{7}) \frac{225}{345} \partial O_6^\pm(w)$ at the third-order pole. Finally, after extracting the new four quasi-primary fields from the first-order pole, the following new spin-6 primary field can be derived

$$P_6(z) = \frac{121}{820755} \psi^a \partial^5 \psi^a(z) + \text{other lower order derivative terms.}$$

This is another candidate for the spin-6 current in the list $2.20$.

As the spin-$\frac{11}{2}$ current $2.56$ has been found, its superpartner $O_6(z)$ current should be determined. By calculating the OPE $G(z)$ with $O_6(w)$, which should generate $O_6(w)$, it can be seen that it consists of a linear combination of the previous spin-6 fields $2.58$ and $2.60$

$$O_6(z) = -P_6(z) + \sqrt{6} P_6^\prime(z).$$

Similarly, the OPE between $G(z)$ and the current $O_6(w)$ can be calculated. The results show that the second order pole has $11 O_6^\pm(w)$ and the first-order pole has $\partial O_6^\pm(w)$, as expected. Therefore, two unknown spin-$\frac{11}{2}$ currents and its superpartner spin-6 current remain. On the other hand, the last spin-6 current can be obtained from the previous independent spin-6 currents. Effectively, one is left with the highest spin-$\frac{11}{2}$ current in $2.20$.

Consider the last most complicated OPE between the spin-3 current $2.18$ and spin-$\frac{9}{2}$ current $2.42$ to determine the last unknown spin-$\frac{13}{2}$ current:
The independent fields in the right hand side up to the second-order singular terms are the same as that in the OPE (2.55). First of all, the third-order pole in (2.62) should be checked to determine if it contains a primary field of a spin-$\frac{9}{2}$ current (2.26) after subtracting the descendant field for $O_2^z(w)$. For the second-order pole, new structures should be identified after subtracting the right quasi-primary fields with the correct coefficients. Calculate the OPE between $T(z)$ and $(\{WO_2^z\})_{-2} - \left(\frac{3}{56}\right) \frac{444\sqrt{6}}{65} \partial^2 O_2^z + \left(\frac{1}{3}\right) \frac{88}{13} \sqrt{\frac{2}{3}} \partial O_2^z(w)$. This OPE contains \(\frac{3441}{65} \sqrt{\frac{2}{3}} O_2^z(w)\) at the fourth-order pole. The OPE between $G(z)$ and $(\{WO_2^z\})_{-2} - \left(\frac{3}{56}\right) \frac{444\sqrt{6}}{65} \partial^2 O_2^z + \left(\frac{1}{3}\right) \frac{88}{13} \sqrt{\frac{2}{3}} \partial O_2^z(w)$ produces \(\frac{8887}{588} P_{4w}(w) - \frac{14126}{195} P_{4w}(w)\) at the third-order pole. The correct three quasi-primary fields with the correct coefficients can then be subtracted from the second-order pole, leaving the spin-$\frac{13}{2}$ current (2.56). The complete structure of the first-order singular terms can now be determined. The OPE between the $T(z)$ and $(\{WO_2^z\})_{-1} \partial T(z)(w)$ contains \(\frac{8288\sqrt{6}}{715} O_2^z(w)\) at the fifth-order pole and \(\left(-\frac{1}{7}\right) \frac{24912}{715} \partial O_2^z(w) - \frac{6166\sqrt{6}}{65} O_2^z(w)\) at the fourth-order pole. Similarly, from the OPE $G(z)$ with $(\{WO_2^z\})_{-1} \partial T(z)(w)$, \(-\frac{896}{115} P_{4w}(w) - \frac{24192}{715} P_{4w}(w)\) and \(-\left(-\frac{1}{8}\right) \frac{896}{115} \partial P_{4w}(w) - \left(-\frac{1}{8}\right) \frac{24192}{715} \partial P_{4w}(w)\) can be observed as the four-th order and third-order pole, respectively. From this analysis, the four possible quasi-primary fields with fixed coefficients can be expressed. The properties of the currents $T(z)$ or $G(z)$ and the quasi-primary fields are reported in Appendices B and C. The general structures of quasi-primary fields for given known quasi-primary fields, $\Phi_i(z)$ and $\Phi_j(z)$ found by [26] are explained. Interestingly, once the OPE $\Phi_i(z)$ $\Phi_j(z)$ is found, the quasi-primary fields containing the derivatives of these two quasi-primary fields are determined completely. The following spin-$\frac{13}{2}$ primary current, which is the highest spin current in the list of (2.20), can be derived by subtracting the above terms in the first-order singular terms,

\[
O_{\frac{13}{2}}(z) = \frac{16\sqrt{2}}{4485} f^{abc} \psi^a \psi^b \partial^5 \psi^c(z) + \text{other lower order derivative terms.} \tag{2.63}
\]

The OPE $G(z)$ with the spin-$\frac{13}{2}$ current (2.63) should be calculated to determine its super-
partner. Finally, its correct superpartner is

\[ O_6'(z) = P_6(z) + \frac{7}{2} \sqrt{\frac{3}{2}} P_6'(z). \]  

(2.64)

Furthermore, the OPE between \( G(z) \) and \( O_6'(w) \) leads to the first-order pole, \( O_{13}(w) \), as expected. Therefore, the supercurrents, \( \hat{O}_{11}(Z) \) and \( \hat{O}_6(Z) \), are constructed in the list of (2.20). The former consists of (2.56) and (2.61) and the latter is given by (2.64) and (2.63). The \( \mathcal{N} = 1 \) superfusion rule is given by

\[ \hat{W} \hat{O}_4 = \hat{O}_7 + \hat{O}_4 + \hat{O}_{11} + \hat{O}_6. \]

In summary, the OPEs described thus far are given by (2.21), (2.24), (2.27), (2.36), (2.39), (2.45), (2.49), (2.55), (2.57), (2.59), and (2.62). In principle, the other OPEs not calculated in this paper can be determined. During these computations, the 12 higher spin primary currents, where the spins are greater than 3, can be found. Moreover, there are four quasi-primary fields and a primary field but these can be obtained from the known primary currents \( T(z), G(z), U(z), W(z), O_4'(w) \) or \( O_6'(w) \). The OPEs found in \( \mathcal{N} = 1 \) superspace should be written down. All these computations are based on the \( c = 4 \) eight free fermion model. In the next section, this model is generalized to the \( c < 4 \) coset model.

3 Higher spin currents in the \( \mathcal{N} = 1 \) supersymmetric coset minimal model (1.2)

Consider the perturbations of the \( k \to \infty \) model described in the previous subsection. Two spin-1 currents exist, \( J^a(z) \) and \( K^a(z) \) of level 3 and \( k \), which generate the algebra \( (A_2^{(1)} \oplus A_2^{(1)}) \). The OPE between the spin-1 currents, \( J^a(z) \), is given in (2.14) and the corresponding OPE for the spin-1 current \( K^a(z) \) is

\[ K^a(z) K^b(w) = -\frac{1}{(z-w)2} \frac{k}{2} \delta^{ab} + \frac{1}{(z-w)} f^{abc} K^c(w) + \cdots. \]  

(3.1)

The diagonal subalgebra \( A_2^{(1)} \) generates the spin-1 current \( J^a(z) + K^a(z) \) with level \( k + 3 \).

The coset Virasoro algebra is generated using the following Sugawara stress energy tensor

\[ T(z) = -\frac{1}{6} J^a J^a(z) - \frac{1}{(k + 3)} K^a K^a(z) + \frac{1}{(k + 6)} (J^a + K^a)(J^a + K^a)(z), \]  

(3.2)

which commutes with the above spin-1 current \( J^a(z) + K^a(z) \), as expected. As the \( k \to \infty \), the above (3.2) becomes (2.16). The OPE of this spin-2 current and itself is given by (2.3), using 14Of course, the OPEs between \( G(z) \) and \( P_6(w) \) (or \( P_6'(w) \)) can be obtained from the standard results for the OPEs between \( G(z) \) and \( O_6(w) \) (and \( O_6'(w) \)) with (2.61) and (2.64).
the OPEs (2.14) and (3.1), where the coset central charge is characterized by the following function of $k$

\[
c = 4 \left[ 1 - \frac{18}{(k + 3)(k + 6)} \right], \quad k = 1, 2, \cdots.
\] (3.3)

When $k = 1$, this central charge reduces to the one in the minimal extension given in subsection 2.2.

By requiring that the spin-$\frac{3}{2}$ current should commute with the diagonal spin-1 current and should transform as a primary field under the stress energy tensor (3.2),

\[
G(z) = -\frac{2k}{3\sqrt{3(k + 3)(k + 6)}} \psi^a \left( J^a - \frac{9}{k} K^a \right) (z),
\] (3.4)

which satisfies (2.2) with the central charge (3.3), is derived, as reported in [21]. In addition, this current reduces to (2.17) as $k$ approaches $\infty$.

Similarly, the higher spin-3 current can be fixed using above regularity condition under the diagonal spin-1 current and primary condition using the stress energy tensor $T(z)$. In addition, the highest singular term should behave as $\frac{c}{3}$. Therefore, it can be expressed as [21]

\[
W(z) = \frac{2i}{9(k + 3)(k + 6)\sqrt{30(2k + 3)(2k + 15)}} d^{abc} \left[ k(k + 3)(2k + 3)J^a J^b J^c 
- 18(k + 3)(2k + 3)J^a J^b K^c + 162(k + 3)J^a K^b K^c - 162K^a K^b K^c \right] (z),
\] (3.5)

which reduces to the previous expression (2.18) for $k \rightarrow \infty$. As performed previously, its fermionic superpartner can be obtained from the spin-$\frac{3}{2}$ current (3.4), which leads to the derivation reported in reference [21]

\[
U(z) = \frac{2\sqrt{6i}}{15\sqrt{10(k + 3)(k + 6)(2k + 3)(2k + 15)}} d^{abc} \left[ k(2k + 3)\psi^a J^b J^c 
- 15(2k + 3)\psi^a J^b K^c + 90\psi^a K^b K^c \right] (z),
\] (3.6)

which also becomes (2.19) for $k \rightarrow \infty$.

The 12 higher spin currents in (2.20) can be constructed for the $c < 4$ coset model.

• The construction of $\hat{O}_7^{(2)}(Z)$ and $\hat{O}_4(Z)$

The OPE between the spin-$\frac{3}{2}$ current (3.6) and itself can be calculated. The only difference between the $c = 4$ model and $c < 4$ model is the $k$-dependence in front of (3.6) and there is an extra current $K(z)$ dependence. Therefore, the calculations are more involved. Nevertheless,

\[
U(z) U(w) = \frac{1}{(z - w)^6} \frac{8k(9 + k)}{5(3 + k)(6 + k)} + \frac{1}{(z - w)^3} 2T(w) + \frac{1}{(z - w)^2} \partial T(w)
\]
The algebraic structure is the same as (2.21) except that the $k$-dependence occurs in many places. The stress energy tensor is given by (3.2). The $k$-dependent primary spin-4, which is a generalization of (2.22), is given by \[ P_{4u}(z) = \frac{3(3+k)(6+k)(498+225k+25k^2)}{(-42+99k+11k^2)(378+333k+37k^2)} \times \left[ -\frac{7}{10} \partial^2 T + \frac{17(3+k)(6+k)}{6(66+63k+7k^2)} \left( T^2 - \frac{3}{10} \partial^2 T \right) + G\partial G \right] (z). \] (3.8)

In addition, the $k$-dependent generalization of (2.23) can be obtained as
\[ P_{4u'}(z) = -\frac{2(-1+k)k(9+k)(18+17k+9k^2)}{(6+k)(-42+99k+11k^2)(378+333k+37k^2)} \psi^a \partial^3 \psi^a (z) + \text{other 1970 terms}. \] (3.9)

Only the $\psi^a(z)$ dependent terms have a $(k-1)$ factor in (3.9). The presence of this current was reported in reference [21], where there is no explicit form for this current. Note that the $k$-dependence in front of the quasi-primary field occurs, whereas there is no $k$-dependence in front of the stress energy tensor and its descendant fields in (3.7). Of course, the central term in the highest singular term of (3.7) is the usual expression, $\frac{2}{5} c$, where $c$ is given by (3.3). The results appear to be different from the original equation (4.16) in reference [21] but they are the same by manipulating the singular terms appropriately. The quasi-primary field can be obtained from (3.7) rather than from old one. Moreover, (3.7) reduces to (2.21) as $k \to \infty$. The OPE between $T(z)$ and the quasi-primary field $(T^2 - \frac{3}{10} \partial^2 T)(w)$ has a nontrivial $k$-dependence and this OPE is presented in Appendix D. Also see Appendix E.

Now move the following OPE between the spin-3 current (3.5) and spin-$\frac{5}{2}$ current (3.6),
\[ W(z) U(w) = \frac{1}{(z-w)^4} \frac{3}{\sqrt{6}} G(w) + \frac{1}{(z-w)^3} \frac{2}{3} \frac{3}{\sqrt{6}} \partial G(w) + \frac{1}{(z-w)^2} \left[ \frac{1}{4} \frac{3}{\sqrt{6}} \partial^2 G + \frac{11\sqrt{6}(3+k)(6+k)}{(378+333k+37k^2)} \left( GT - \frac{1}{8} \partial^2 G \right) + O(\frac{1}{z^2}) \right] (w). \]
\begin{align*}
+ \frac{1}{(z-w)} \left[ \left( \frac{1}{15} \right) \frac{3}{\sqrt{6}} \partial^3 G + \left( \frac{1}{7} \right) \frac{11\sqrt{6}(3+k)(6+k)}{378 + 333k + 37k^2} \partial \left( GT - \frac{1}{8} \partial^2 G \right) + \left( \frac{1}{7} \right) \partial O_z \right] + \frac{4\sqrt{6}(3+k)(6+k)}{7(-42 + 99k + 11k^2)} \left( \frac{4}{3} T \partial G - G \partial T - \frac{4}{15} \partial^3 G \right) + O_{\frac{3}{2}} \right] (w) + \cdots. \tag{3.10}
\end{align*}

The \( k \)-dependence occurs in front of the quasi-primary field. \( (3.10) \) leads to \( (2.24) \) as \( k \to \infty \). The \( k \) dependent spin-\( \frac{3}{2} \) (the corresponding \( k \to \infty \) limit was given in \( (2.25) \)) has the following form

\begin{align*}
O_{\frac{3}{2}}(z) &= -\frac{\sqrt{2}(1+k)(9+k)(9+2k)}{(15+2k)\sqrt{18 + 9k + k^2}(378 + 333k + 37k^2)} f^{abc} \psi^a \psi^b \partial^2 \psi^c (z) \\
&\quad + \text{other 746 terms}, \tag{3.11}
\end{align*}

where only \( \psi^a(z) \)-dependent terms have the \((k-1)\) factor in \((3.11)\). \( K^a \)-dependent terms and mixed terms exist between \( \psi^a(z) \) and \( K^a(z) \). Furthermore, the spin-\( \frac{3}{2} \) current, which generalizes the previous expression \((2.26)\), has

\begin{align*}
O_{\frac{3}{2}}(z) &= \frac{8\sqrt{2}(1+k)(1+k)(9+k)}{21(15+2k)\sqrt{18 + 9k + k^2}(-42 + 99k + 11k^2)} f^{abc} \psi^a \psi^b \partial^3 \psi^c (z) \\
&\quad + \text{other 3624 terms}. \tag{3.12}
\end{align*}

Only \( \psi^a(z) \) dependent terms have a \((k-1)\) factor in \((3.12)\). The OPEs between the \( T(z) \) and quasi-primary fields appearing in \((3.10)\) contain the \( k \)-dependence, and these OPEs are given in Appendix \( D \). Similarly, the OPEs between the \( G(z) \) and those quasi-primary fields are given in Appendix \( E \) where the \( k \)-dependence can be found explicitly.

The generalization of \((2.27)\) can be obtained and the spin-3 current OPE is given by

\begin{align*}
W(z) W(w) &= \frac{1}{(z-w)^6} \frac{4k(9+k)}{3(3+k)(6+k)} + \frac{1}{(z-w)^4} \frac{2T(w)}{3} + \frac{1}{(z-w)^3} \partial T(w) \\
&\quad + \frac{1}{(z-w)^2} \left[ \frac{1}{10} \partial^3 T + \frac{1}{3} \frac{16(3+k)(6+k)}{66 + 63k + 7k^2} \left( T^2 - \frac{3}{10} \partial^2 T \right) + P^{ww}_{1} + P^{ww}_{4} \right] (w) \\
&\quad + \frac{1}{(z-w)} \left[ \frac{1}{15} \partial^2 T + \frac{1}{2} \frac{16(3+k)(6+k)}{66 + 63k + 7k^2} \partial \left( T^2 - \frac{3}{10} \partial^2 T \right) \\
&\quad + \frac{1}{2} \partial P^{ww}_{4} + \frac{1}{2} \partial P^{ww}_{4'} \right] (w) + \cdots, \tag{3.13}
\end{align*}

where the \( k \)-dependent generalization of \((2.28)\), spin-4 primary field, has the following form

\begin{align*}
P^{ww}_{4}(z) &= -\frac{48(1+k)(3+k)(6+k)(10+k)}{(-42 + 99k + 11k^2)(378 + 333k + 37k^2)} \times \left[ \frac{7}{10} \partial^2 T + \frac{17(3+k)(6+k)}{6(66 + 63k + 7k^2)} \left( T^2 - \frac{3}{10} \partial^2 T \right) + G \partial G \right] (z). \tag{3.14}
\end{align*}
Owing to the \((k-1)\) factor in \((3.14)\), this current vanishes at the “minimal” extension described before. Another spin-4 primary current is obtained, which generalizes equation \((2.29)\)

\[
P^{ww}_4(z) = \frac{4(-1 + k)^2 k(9 + k)(27 + k)}{3(15 + 2k)(-42 + 99k + 11k^2)(378 + 333k + 37k^2)} \psi^a \partial^3 \psi^a(z) + \text{other 1818 terms,}
\]

where only \(\psi^a(z)\) dependent terms have a \((k-1)\) factor in \((3.15)\). The central term \(c_3\) in \((3.13)\) can be derived easily.

Therefore, as for the infinite \(k\) case, the two primary currents \((3.11)\) and \((3.12)\) can be derived. The new primary currents \((2.30)\) and \((2.34)\) are constructed from the other two primary fields, \((3.9)\) and \((3.15)\). In other words, four independent currents in \((2.20)\) are found while calculating the OPEs \((3.7)\), \((3.10)\) and \((3.13)\). The OPEs between the spin-\(\frac{3}{2}\) current and the above four independent currents are the same as those in \((2.31)\), \((2.32)\), \((2.33)\) and \((2.35)\).

- The construction of \(\hat{O}_4'(Z)\) and \(\hat{O}_9(Z)\)

Consider the OPE between the spin-\(\frac{5}{2}\) current \((3.6)\) and spin-\(\frac{7}{2}\) current \((3.11)\)

\[
U(z) \quad O_7^{\frac{7}{2}}(w) = \frac{1}{(z - w)^3} c_{uow} W(w)
+ \frac{1}{(z - w)^2} \left[ \frac{1}{3} c_{uow} \partial W - c_{uogu} \left( GU - \frac{\sqrt{6}}{3} \partial W \right) + O_{4'} \right] (w)
+ \frac{1}{(z - w)} \left[ \frac{1}{14} c_{uow} \partial^2 W - \frac{3}{8} c_{uogu} \partial \left( GU - \frac{\sqrt{6}}{3} \partial W \right) + \frac{3}{8} \partial O_{4'} \right]
+ c_{uotw} \left( TW - \frac{3}{14} \partial^2 W \right) + c_{uogu'} \left( G \partial U - \frac{5}{3} \partial GU - \frac{\sqrt{6}}{7} \partial^2 W \right) + O_5 \right] (w)
+ \cdots.
\]

Here the structure constants can be written as

\[
c_{uow} = \frac{36(-1 + k)(10 + k)(9 + 2k)^2}{5(3 + 2k)(15 + 2k)(378 + 333k + 37k^2)}.
\]

\[
c_{uogu} = \frac{6\sqrt{6}(-1 + k)(3 + k)(6 + k)(10 + k)(9 + 2k)^2}{(3 + 2k)(15 + 2k)(90 + 117k + 13k^2)(378 + 333k + 37k^2)^\frac{1}{4}}.
\]

\[
c_{uotw} = \frac{4(3 + k)(6 + k)(9 + 2k)^2(198 + 1719k + 191k^2)}{45(3 + 2k)(15 + 2k)(74 + 45k + 5k^2)(378 + 333k + 37k^2)}.
\]

\[
c_{uogu'} = \frac{30\sqrt{6}(3 + 2k)(15 + 2k)(74 + 45k + 5k^2)(378 + 333k + 37k^2)^\frac{1}{4}}{(3 + k)(6 + k)(9 + 2k)^2(5562 + 2583k + 287k^2)}.
\]

Note that the OPE between the current \(G(z)\) and spin-4 primary field in \((3.16)\) appearing in Appendix \(E\) is used to determine the complete coefficient functions in the right hand side of
Moreover, the OPEs between the spin-2 current \( T(z) \) or \( G(z) \) and the quasi-primary fields of spin 5 can be obtained from Appendices \( D \) or \( E \).  \((k-1)\) factors exist in the first two structure constants in \((3.17)\). Therefore, how can the coefficient \( c_{uogw} \) be obtained explicitly?  The algebraic structure in \((2.36)\) could explain an infinite \( k \). Because the coefficient, \( c_{uow} \), is fixed from the third-order pole, the first term in the second-order pole is determined. By introducing the undetermined coefficient \( c_{uogu} \) in front of the primary field of spin-4, the OPE \( G(z) \) with \( \left( \{ UO_{\frac{5}{2}} \}_{-2} - \frac{1}{3} c_{uow} \partial W + c_{uogw} (GU - \frac{\sqrt{6}}{3} \partial W) \right) (w) \) can be calculated, where \( c_{uow} \) is given in \((3.17)\). The requirement that the third-order pole should vanish (i.e. primary condition) then determines the constant \( c_{uogw} \) explicitly, leaving the spin-4 primary field, which is given by

\[
O_4' (z) = \frac{384i{\sqrt{2}}(-1+k)(1+k)(9+2k)\sqrt{45+36k+4k^2}}{(3+k)(3+2k)(15+2k)(90+117k+13k^2)} \psi^1 \psi^2 \psi^3 \psi^4 \psi^5 \psi^6 \psi^7 \psi^8 (z) + \text{other 1376 terms},
\]

\((3.18)\)

where only \( \psi^a(z) \) dependent terms have a \((k-1)\) factor in \((3.18)\).

What happens in the next-order pole? Because the algebraic structure is known completely except for the \( k \)-dependent coefficient functions, the OPEs between \( T(z)(G(z)) \) using \( \left( \{ UO_{\frac{5}{2}} \}_{-1} - \text{three descendant terms} - \text{two quasi-primary terms} \right) (w) \) can be calculated using the undetermined coefficients \( c_{uotw} \) and \( c_{uogw} \). This should transform very specially. In other words, there should be no higher order terms where the order is greater than 2 (once again the primary condition). This enables the above two constants to be fixed, as in \((3.17)\). Therefore, the following spin-5 primary current remains

\[
O_5(z) = -\frac{2i{\sqrt{2}}(-1+k)(9+k)(9+2k)^2(9+4k)}{45(3+k)(6+k)(15+2k)\sqrt{45+36k+4k^2}(74+45k+5k^2)} f^{abcd} \phi \psi^a \phi \psi^b \phi \psi^d \partial^3 \phi \psi^e (z) + \text{other 9603 terms},
\]

\((3.19)\)

where only \( \psi^a(z) \) dependent terms have a \((k-1)\) factor in \((3.19)\). The corresponding \( k \to \infty \) limit expressions are \((2.37)\) and \((2.38)\), respectively.

The OPE between the spin-\( \frac{5}{2} \) current and spin-4 current can be expressed as follows:

\[
U(z) O_4(w) = \frac{1}{(z-w)^4} c_{uow} U(w) + \frac{1}{(z-w)^3} \frac{1}{5} c_{uow} \partial U(w)
\]

\[
+ \frac{1}{(z-w)^2} \left[ \frac{1}{30} c_{uow} \partial^2 U + c_{uogw} \left( GW - \frac{1}{6\sqrt{6}} \partial^2 U \right) + c_{uotu} \left( TU - \frac{1}{4} \partial^2 U \right) + P_\frac{5}{2} \right] (w)
\]

\[
+ \frac{1}{(z-w)} \left[ \frac{1}{210} c_{uow} \partial^3 U + \frac{1}{3} c_{uogw} \partial \left( GW - \frac{1}{6\sqrt{6}} \partial^2 U \right) + \frac{1}{3} c_{uotu} \partial \left( TU - \frac{1}{4} \partial^2 U \right)
\]

\[
+ \frac{1}{3} \partial P_\frac{5}{2} + Q_\frac{5}{2} \right],
\]

\((3.20)\)
where the correct spin-4 current is the sum of the previous spin-4 currents \(^1\) with (3.9) and (3.15)

\[
O_4(z) = \left( -\frac{1}{\sqrt{6}} P^{uu}_{4'} + \sqrt{6} P^{ww}_{4'} \right)(z).
\] (3.21)

This is identical to (2.30). The structure constants in (3.20) are given by

\[
c_{uou} = \frac{6\sqrt{6}(-1 + k)(10 + k)(9 + 2k)^2}{(3 + 2k)(15 + 2k)(378 + 333k + 37k^2)},
\] (3.22)

\[
c_{uogw} = \frac{12(-1 + k)(3 + k)(6 + k)(10 + k)(9 + 2k)^2(1290 + 1197k + 133k^2)}{5(3 + 2k)(15 + 2k)(74 + 45k + 5k^2)(90 + 117k + 13k^2)(378 + 333k + 37k^2)},
\]

\[
c_{uotu} = \frac{8\sqrt{2}(-1 + k)(3 + k)(6 + k)(10 + k)(9 + 2k)^2(90 + 261k + 29k^2)}{(3 + 2k)(15 + 2k)(74 + 45k + 5k^2)(90 + 117k + 13k^2)(378 + 333k + 37k^2)},
\]

where all of these have \((k-1)\) factors in their expressions. Of course, these constants (3.22) reduce to the ones appearing in (2.39) for an infinite \(k\) limit.

Once again, the coefficients, \(c_{uogw}\) and \(c_{uotu}\), appearing in the second-order pole are determined by evaluating the OPEs between \(T(z)(\text{and similarly } G(z))\), and the whole second-order pole terms subtract the first three terms where the coefficient, \(c_{uou}\) is known from the higher order terms. The disappearance of the higher order singular terms, where the order is greater than 2, fixes the above unknown two coefficients, which are given in (3.22), leaving the primary field in this singular terms. The spin-\(\frac{9}{2}\) primary current, which generalizes the previous expression (2.40), can be obtained by the following:

\[
P_{\frac{9}{2}}(z) = \frac{n_1}{d_1} d_{abc} f^{bde} f^{cfg} \psi^a \psi^d \psi^e \partial^2 \psi^g(z) + \text{other 4671 terms},
\] (3.23)

where the intermediate \(k\)-dependent expressions are

\[
n_1 = 2i \sqrt{\frac{2}{5}}(-1 + k)(9 + 2k)(79110 + 149883k + 86489k^2 + 19367k^3 + 1741k^4 + 50k^5),
\] (3.24)

\[
d_1 = 5(3 + k)(15 + 2k)(74 + 45k + 5k^2)(90 + 117k + 13k^2)\sqrt{810 + 1053k + 441k^2 + 72k^3 + 4k^4}.
\]

In this case, only \(\psi^a(z)\) dependent terms have a \((k-1)\) factor in (3.23). Of course, expression (3.22) reduces to the numerical coefficient in (2.40).

What about the first-order singular terms? Because the second-order terms are determined, their descendant fields can be found with the known coefficient functions. By introducing the arbitrary three coefficient functions in (2.41), the equation can be solved in such another spin-4 current exists \(O_4(z) = \frac{1}{2} \left( \frac{16}{7} \sqrt{2} P^{uu}_{4'} - \frac{4}{7} \sqrt{6} P^{ww}_{4'} \right)(z)\), which is equal to the relationship of (2.34) with (3.9) and (3.15).
a way that the whole first-order terms subtracted from above four known descendant field terms is equal to the quasi-primary field $Q_{\frac{11}{2}}(w)$. This provides all the information for the three unknown coefficient functions that were introduced. Therefore, the general expression containing (2.41) can be obtained

$$Q_{\frac{11}{2}}(z) = - \frac{8\sqrt{\frac{2}{3}}(3 + k)(6 + k)(9 + 2k)^2(18 + 657k + 73k^2)}{15(3 + 2k)(15 + 2k)(90 + 117k + 13k^2)(378 + 333k + 37k^2)}$$

$$\times \left( T\partial U - \frac{5}{4} T\partial^2 U - \frac{1}{7} \partial^3 U \right) (z) + \frac{2(3 + k)(6 + k)(9 + 2k)^2(498 + 225k + 25k^2)}{5(3 + 2k)(15 + 2k)(90 + 117k + 13k^2)(378 + 333k + 37k^2)}$$

$$\times \left( G\partial W - 2\partial GW - \frac{1}{21} \sqrt{\frac{2}{3}} \partial^3 U \right) (z) + \frac{3(3 + k)(6 + k)}{(3 + 2k)(15 + 2k)} \left( GO_{\psi'} - \frac{2}{9} \partial O_{\psi'} \right) (z). \quad (3.25)$$

Compared to (2.41), $k$-dependent coefficient functions are in front of the three independent quasi-primary fields in (3.25). The OPEs can be calculated in a similar way as that used in (2.43) and (2.44), and they will show an explicit $k$-dependence in the right hand side of the OPEs. Compared to the previous OPE (3.16), there was no need to calculate the OPEs between the $T(z)$ (or $G(z)$) and some terms in the first-order pole. This is because there are no other quasi-primary fields in the first-order terms. Here, the generalization of (2.42) can be given by

$$O_{\psi'}(z) = - \frac{8i\sqrt{\frac{2}{3}}(-1 + k)k(1 + k)(9 + k)(9 + 2k)^2\sqrt{45 + 36k + 4k^2}}{5(3 + k)(3 + 2k)(15 + 2k)\sqrt{18 + 9k + k^2}(90 + 117k + 13k^2)}$$

$$\times d^{abc} f^{bde} f^{cij} \psi^a \psi^d \psi^e \psi^j \partial^2 \psi^9(z) + \text{other 4430 terms}. \quad (3.26)$$

Only the $\psi^a(z)$ dependent terms have $(k - 1)$ factors in (3.26).

Consider the OPE (2.45) when $k$ is finite

$$W(z) O_{\frac{11}{2}}(w) = \frac{1}{(z - w)^3} c_{wou} U(w) + \frac{1}{(z - w)^4} \frac{2}{5} c_{wou} \partial U(w)$$

$$+ \frac{1}{(z - w)^2} \left[ \frac{1}{10} c_{wou} \partial^2 U + c_{wotu} \left( TU - \frac{1}{4} \partial^2 U \right) + c_{wogw} \left( GW - \frac{1}{6}\sqrt{6} \partial^2 U \right) + c_{wop} P_{\frac{11}{2}} \right] (w) + \frac{1}{(z - w)} \left[ \frac{2}{105} c_{wou} \partial^2 U + \frac{4}{9} c_{wotu} \partial \left( TU - \frac{1}{4} \partial^2 U \right) \right.$$

$$\left. + \frac{4}{9} c_{wogw} \partial \left( GW - \frac{1}{6}\sqrt{6} \partial^2 U \right) + \frac{4}{9} c_{wop} \partial P_{\frac{11}{2}} + \frac{4}{9} c_{wou'} \partial O_{\psi'} + Q_{\psi'} \right] (w) + \cdots. \quad (3.27)$$
The three structure constants in (3.27) that depend on \( k \) explicitly can be written in terms of
\[
\begin{align*}
\mathcal{C}_{\text{wou}} &= \frac{6(-1 + k)(10 + k)(9 + 2k)^2}{(3 + 2k)(15 + 2k)(378 + 333k + 37k^2)}, \\
\mathcal{C}_{\text{wop}} &= \frac{1}{\sqrt{6}}, \\
\mathcal{C}_{\text{woo}} &= \frac{1}{\sqrt{6}}, \\
\mathcal{C}_{\text{wotu}} &= \frac{4(-1 + k)(3 + k)(6 + k)(10 + k)(9 + 2k)^2(846 + 927k + 103k^2)}{(3(3 + 2k)(15 + 2k)(74 + 45k + 5k^2)(90 + 117k + 13k^2)(378 + 333k + 37k^2)}, \\
\mathcal{C}_{\text{wogw}} &= \frac{4}{\sqrt{6}}(-1 + k)(3 + k)(6 + k)(10 + k)(9 + 2k)^2(90 + 261k + 29k^2)
\end{align*}
\]

In particular, they contain the \((k - 1)\) factor. These terms in (3.28) become those in (2.45). Because the algebraic structure is known for an infinite \( k \), four undetermined coefficients, \( \mathcal{C}_{\text{wotu}}, \mathcal{C}_{\text{wogw}}, \mathcal{C}_{\text{wop}}, \text{and } \mathcal{C}_{\text{woo}} \) are taken in front of two quasi-primary fields and two primary fields (given by (3.23) and (3.26)), respectively. Note that the constant \( \mathcal{C}_{\text{wou}} \) can be fixed from the higher order terms. The second order terms can then be expressed in a similar manner to that in (3.27). On the other hand, the explicit second-order pole from WZW currents is known. By equating these two, the unknown four coefficient functions can be obtained as given in (3.28).

Furthermore, the spin \( \frac{11}{2} \) quasi-primary field can be obtained. As done in the OPE (3.20), because there is no other quasi-primary field except this spin-\( \frac{11}{2} \) current, the explicit form for this field can be derived as follows:
\[
Q_{\frac{11}{2}}(z) = -\frac{2\sqrt{\frac{2}{3}}(3 + k)(6 + k)(9 + 2k)^2(18 + 657k + 73k^2)}{15(3 + 2k)(15 + 2k)(90 + 117k + 13k^2)(378 + 333k + 37k^2)}
\times \left( G\partial W - 2\partial G W - \frac{1}{21}\sqrt{\frac{2}{3}}\partial^3 U \right)(z)
\]
\[
+ \frac{16(3 + k)(6 + k)(9 + 2k)^2(738 + 9k + k^2)}{135(3 + 2k)(15 + 2k)(90 + 117k + 13k^2)(378 + 333k + 37k^2)}
\times \left( T\partial U - \frac{5}{4}\partial T U - \frac{1}{7}\partial^3 U \right)(z)
\]
\[
+ \frac{\sqrt{\frac{3}{4}}(3 + k)(6 + k)}{(3 + 2k)(15 + 2k)} \left( GO_{4\nu} - \frac{2}{9}\partial O_{4\nu} \right)(z). \\
\]

As stated before, the OPEs can also be calculated as in (2.47) and (2.48). Even for the \( k \)-dependent coefficients, this field (3.29) is a quasi-primary field because the three terms are quasi-primary fields.

For the OPE between the spin-3 current and the spin-4 current (3.21), the following \( k \)-dependent expression can be obtained, which appeared in (2.49)
\[
W(z) O_4(w) = \frac{1}{(z - w)^4} \mathcal{C}_{\text{wou}} W(w)
\]
\[
\begin{align*}
+ \frac{1}{(z-w)^3} & \left[ \frac{1}{3} c_{\text{wow}} \partial W + c_{\text{woo}} O_4 + c_{\text{wogu}} \left( GU - \frac{\sqrt{6}}{3} \partial W \right) \right] (w) \\
+ \frac{1}{(z-w)^2} & \left[ \frac{1}{14} c_{\text{wow}} \partial^2 W + \frac{3}{8} c_{\text{woo}} \partial O_4 + \frac{3}{8} c_{\text{wogu}} \partial \left( GU - \frac{\sqrt{6}}{3} \partial W \right) + c_{\text{woo}}' O_5 + Q_5 \right] (w) \\
+ \frac{1}{(z-w)} & \left[ \frac{1}{84} c_{\text{wow}} \partial^3 W + \frac{1}{12} c_{\text{woo}} \partial^2 O_4 + \frac{1}{12} c_{\text{wogu}} \partial^2 \left( GU - \frac{\sqrt{6}}{3} \partial W \right) \right. \\
& \left. + \frac{2}{5} c_{\text{woo}}' \partial O_5 + \frac{2}{5} \partial Q_5 + Q_6 \right] (w) + \cdots.
\end{align*}
\]

The structure constants can be expressed in terms of

\[
\begin{align*}
c_{\text{wow}} &= \frac{48\sqrt{6}(-1+k)(10+k)(9+2k)^2}{5(3+2k)(15+2k)(378 + 333k + 37k^2)}, & c_{\text{woo}} &= \sqrt{\frac{2}{3}}, & c_{\text{woo}}' &= 5\sqrt{\frac{2}{3}}, \\
c_{\text{wogu}} &= \frac{12(-1+k)(3+k)(6+k)(10+k)(9+2k)^2}{(3+2k)(15+2k)(90 + 117k + 13k^2)(378 + 333k + 37k^2)}. \tag{3.31}
\end{align*}
\]

As stated before, the coefficient, \( c_{\text{wow}} \), can be fixed easily from the fourth-order pole. Because the algebraic structure is determined from the infinite \( k \) result, two unknown coefficients, \( c_{\text{woo}} \) and \( c_{\text{wogu}} \), are placed in the two primary fields, respectively. One of the primary fields was given in (3.18). The two coefficients can then be fixed without difficulty. Now the focus shifts to the next order singular terms. The easiest way to obtain the quasi-primary fields in (3.30) can be seen in the following example. Once the second-order pole in (3.30) is found, the arbitrary coefficient function \( c_{\text{woo}}' \) and two additional coefficients can be placed in the quasi-primary field in (2.50). Then all the coefficients can be fixed, as in (3.31) and the spin-5 quasi-primary field with \( k \)-dependent coefficients can be expressed as

\[
Q_5(z) = \frac{2\sqrt{\frac{2}{3}}(9+2k)^2(-168156 + 101412k + 104013k^2 + 20610k^3 + 1145k^4)}{45(3+2k)(15+2k)(74 + 45k + 5k^2)(378 + 333k + 37k^2)} \\
\times \left( TW - \frac{3}{14} \partial^2 W \right) (z) \\
- \frac{(3+k)(6+k)(9+2k)^2(-5166 + 855k + 95k^2)}{90(3+2k)(15+2k)(74 + 45k + 5k^2)(378 + 333k + 37k^2)} \\
\times \left( G\partial U - \frac{5}{3} \partial GU - \frac{\sqrt{6}}{7} \partial^2 W \right) (z). \tag{3.32}
\]

Similarly, the spin-6 quasi-primary field can be analyzed in a similar way to that done in (3.32) using the four unknown coefficients in (2.52). The OPEs can be constructed in a similar way to that in (2.51). These can be obtained by equating the first-order pole to the above expressions with four unknown coefficients. One has the following spin-6 quasi-primary field

\[
Q_6(z) = \frac{32(-1+k)(3+k)(6+k)(10+k)(9+2k)^2}{25(3+2k)(15+2k)(90 + 117k + 13k^2)(378 + 333k + 37k^2)}
\]

35
\[
\times \left( G \partial^2 U - 4 \partial U \partial G + \frac{5}{2} \partial^2 GU - \frac{1}{2\sqrt{6}} \partial^3 W \right) (z)
\]
\[
- \frac{192\sqrt{6}(-1 + k)(3 + k)(6 + k)(10 + k)(9 + 2k)^2}{25(3 + 2k)(15 + 2k)(90 + 117k + 13k^2)(378 + 333k + 37k^2)}
\]
\[
\times \left( T \partial W - \frac{3}{2} \partial TW - \frac{1}{8} \partial^3 W \right) (z)
\]
\[
+ \frac{\sqrt{6}(3 + k)(6 + k)}{(3 + 2k)(15 + 2k)} \left( TO_{\nu} - \frac{1}{6} \partial^2 O_{\nu} \right) (z)
\]
\[
- \frac{\sqrt{2}(3 + k)(6 + k)}{(3 + 2k)(15 + 2k)} \left( GO_{\nu} - \frac{1}{9} \partial^2 O_{\nu} \right) (z).
\]

(3.33)

The OPEs between the spin-2 current (or the spin-\(\frac{3}{2}\) current) and the current (3.33) can be calculated in a similar way to that in (2.53).

As in the infinite \(k\) case (2.54), the following relation with (3.23) and (3.26) exists

\[
O_{\nu} (z) = O_{\nu}(z) + \frac{8}{9} P_{\nu}(z).
\]

(3.34)

Therefore, two primary currents (3.18) and (3.19) can be found for the infinite \(k\) case. From the other two primary fields (3.23) and (3.26), the primary current (3.34) can be constructed. In other words, the four independent currents in (2.20) are found when calculating the OPEs (3.16), (3.20), (3.27) and (3.30). Four quasi-primary fields, which can be written in terms of known higher spin currents as well as the stress energy tensor and its superpartner can be found.

- The construction of \(\hat{O}_{11}\) (\(Z\)) and \(\hat{O}_6(Z)\)

The OPE between the spin-\(\frac{3}{2}\) current (3.6) and spin-4 current (3.18), corresponding to the infinite \(k\) result (2.55), can be calculated

\[
U(z) O_\nu(w) = \frac{1}{(z - w)^5} c_{uoo} O_{\frac{3}{2}}(w) + \frac{1}{(z - w)^2} \left[ \frac{2}{7} c_{uoo} \partial O_{\frac{3}{2}} + c_{woo} O_{\frac{9}{2}} \right] (w)
\]
\[
+ \frac{1}{(z - w)} \left[ \frac{3}{56} c_{woo} \partial^2 O_{\frac{3}{2}} - \frac{1}{3} c_{woo} \partial O_{\frac{1}{2}} + \partial^{2} O_{\frac{1}{2}} \right]
\]
\[
+ c_{uogp} \left( GP_{-}^{w} - \frac{4\sqrt{6}}{9} \partial O_{\frac{3}{2}} - \sqrt{6} \frac{56}{9} \partial^2 O_{\frac{3}{2}} \right) - c_{uogp} \left( GP_{w}^{w} - \frac{2}{9} \sqrt{3} \partial O_{\frac{1}{2}} - \frac{1}{14} \sqrt{3} \partial^2 O_{\frac{1}{2}} \right)
\]
\[
+ O_{11} (w) + \cdots .
\]

(3.35)

The structure constants in (3.35) can be obtained

\[
c_{uoo} = \frac{24(1 + k)(8 + k)(378 + 333k + 37k^2)}{5(3 + k)(6 + k)(90 + 117k + 13k^2)} .
\]
\[ c_{uogw} = -\frac{2(9 + 2k)^2(-42 + 99k + 11k^2)}{5(3 + k)(6 + k)(90 + 117k + 13k^2)}, \]
\[ c_{uoto} = \frac{288(1 + k)(8 + k)(846 + 585k + 65k^2)}{5(90 + 117k + 13k^2)(954 + 549k + 61k^2)}, \]
\[ c_{uogp} = \frac{n_1}{d}, \quad c_{uogp'} = \frac{n_2}{d}, \]  
(3.36)

where the numerators and denominator in the last two coefficients are functions of \( k \)

\[ n_1 \equiv 2\sqrt{6}(2135484 + 3378672k + 2386233k^2 + 869670k^3 + 165765k^4 + 15660k^5 + 580k^6), \]
\[ n_2 \equiv -3\sqrt{2}\left(3138364 + 31369356k + 2654483k^2 + 10463418k^3 + 2064411k^4 + 197748k^5 + 7324k^6\right), \]
\[ d \equiv 5(3 + 2k)(15 + 2k)(90 + 117k + 13k^2)(954 + 549k + 61k^2). \]  
(3.37)

The OPEs between the spin-2 and spin-\( \frac{3}{2} \) currents, \( T(z) \) and \( G(z) \), with the quasi-primary fields in (3.35) can be found in Appendices D and E, as described before. The structure of the third- and second-order poles can be determined in a straightforward manner. In the first-order term, the first two descendant terms are known and three unknown coefficients \( c_{uoto}, c_{uogp} \) and \( c_{uogp'} \) are introduced. The next step is to determine how to obtain these \( k \)-dependent coefficients. Because the remaining term can be characterized by the primary field of spin-\( \frac{11}{2} \), there should only be the second-order and first-order terms when the OPE between the currents \( T(z) \) or \( G(z) \) and the first-order pole is calculated after subtracting the above five terms including two derivative terms. This condition fixes the above unknown three coefficients, which are given in (3.36) and (3.37), leaving the spin-\( \frac{11}{2} \) primary current, which is given by

\[ O_{\frac{11}{2}}(z) = \frac{n_1}{d_1} f^{abc} \psi^a \psi^b \partial^4 \psi^c (z) + \text{other 22096 terms}, \]
\[ n_1 \equiv 3\sqrt{2}(-1 + k)k(1 + k)(9 + k)(9 + 2k) \times \left(-23328 - 31338k - 12843k^2 - 1370k^3 + 53k^4 + 10k^5\right), \]
\[ d_1 \equiv 10(3 + k)(6 + k)(3 + 2k)(15 + 2k)^2\sqrt{18 + 9k + k^2}(90 + 117k + 13k^2) \times \left(954 + 549k + 61k^2\right), \]  
(3.38)

which generalizes the previous expression (2.56) for an infinite \( k \).

The OPE between the spin-\( \frac{5}{2} \) current (3.6) and spin-\( \frac{9}{2} \) current (3.26), corresponding to the previous result (2.57), for the finite \( k \) can be summarized by

\[ U(z) O_{\frac{9}{2}}(w) = \frac{1}{(z - w)^3} \left[ c_{uop} P^uu_{\frac{9}{2}} - c_{uop'} P^ww_{\frac{9}{2}} \right](w) \]

37
The structure constants in (3.39) can be expressed as

\[
+ \frac{1}{(z-w)^{2}} \left[ \frac{1}{4} c_{uop} \partial P_{4u}^u - \frac{1}{6} c_{uop'} \partial P_{4w}^w + c_{uogo} \left( GO_{\frac{7}{2}} + \frac{1}{4\sqrt{6}} \partial P_{4u}^u - \frac{\sqrt{6}}{4} \partial P_{4w}^w \right) \right] (w) + \dotsb. \tag{3.39}
\]

The structure constants in (3.39) can be expressed as

\[
c_{uop} = \frac{4\sqrt{2}(10368 + 5562k + 2967k^2 + 522k^3 + 29k^4)}{5(3+k)(6+k)(90 + 117k + 13k^2)},
\]

\[
c_{uop'} = \frac{8\sqrt{6}(6966 + 12069k + 7254k^2 + 1314k^3 + 73k^4)}{5(3+k)(6+k)(90 + 117k + 13k^2)},
\]

\[
c_{uogo} = \frac{792(1+k)(8+k)}{5(90 + 117k + 13k^2)}, \quad c_{uotp} = \frac{n_1}{d}, \quad c_{uotp'} = \frac{n_2}{d},
\]

\[
c_{uogo'} = -\frac{12(1+k)(8+k)(378 + 33k + 37k^2)(1818 + 873k + 97k^2)}{25(-6 + 9k + k^2)(90 + 117k + 13k^2)(954 + 549k + 61k^2)},
\]

\[
c_{uogo''} = -\frac{21(9 + 2k)^2(66 + 45k + 5k^2)(-42 + 99k + 11k^2)}{5(3 + 2k)(15 + 2k)(90 + 117k + 13k^2)(366 + 207k + 23k^2)}, \tag{3.40}
\]

where the numerators and denominator in (3.40) of the fourth and fifth constants are given by

\[
n_1 \equiv 28\sqrt{2}\left(830045232 + 8372710800k + 18353902752k^2 + 1904004212k^3 + 11562105645k^4
\]
\[+ 4408598988k^5 + 1070601270k^6 + 164251116k^7 + 15373601k^8 + 800820k^9 + 17796k^{10}\right),
\]
\[
n_2 \equiv -7\sqrt{6}(9794995632 + 61010651520k + 137265126552k^2 + 158181370992k^3
\]
\[+ 105657645375k^4 + 42990128568k^5 + 10861840470k^6 + 1705114836k^7
\]
\[+ 161571871k^8 + 8459820k^9 + 187996k^{10}),
\]
\[
d \equiv 5(3 + 2k)(15 + 2k)(-6 + 9k + k^2)(90 + 117k + 13k^2)(366 + 207k + 23k^2)
\]
\[\times (954 + 549k + 61k^2). \tag{3.41}
\]

The third-order pole in the right hand side can be determined easily. The coefficient for the quasi-primary field in the second-order pole can be fixed without difficulty. The next step
is to determine how the four quasi-primary fields and single primary field can exist in the last first-order term. Four unknown coefficients can be introduced because the derivative terms are fixed completely. The procedure done for the infinite \( k \) case can be used. The four coefficients can be determined. Finally, after subtracting these four quasi-primary fields correctly, the following new spin-6 primary field remains

\[
P_6(z) = \frac{n_1}{d} \psi^a \partial^2 \psi^a(z) + \text{other lower order derivative terms},
\]

\[n_1 \equiv (-1 + k)k(1 + k)(9 + k)(9 + 2k)(-1365527808 - 3366282888k - 3098773908k^2 - 121090014k^3 - 58275207k^4 + 110774898k^5 + 38348106k^6 + 5639106k^7 + 398009k^8 + 11050k^9),
\]

\[d \equiv 75\sqrt{6}(3 + k)(6 + k)^2(3 + 2k)(15 + 2k)^2(-6 + 9k + k^2)(90 + 117k + 13k^2)
\]

\[\times (366 + 207k + 23k^2)(954 + 549k + 61k^2).
\]

As in the previous case, the factor \((k - 1)\) is contained in this \(\psi^a(z)\)-dependent spin-6 current.

Similarly, following OPE between the spin-3 current (3.5) and the spin-4 current (3.18), corresponding to the previous result (2.59), can be derived:

\[
W(z) O_{4'}(w) = \frac{1}{(z-w)^3} \left[ -c_{wop} P_{4'}^{uu} + c_{wop'} P_{4'}^{ww} \right] (w)
\]

\[+ \frac{1}{(z-w)^2} \left[ -\frac{3}{8} c_{wop} \partial P_{4'}^{uu} + \frac{3}{8} c_{wop'} \partial P_{4'}^{ww} + c_{wogo} \left( GO_{4'}^{\perp} + \frac{1}{4\sqrt{6}} \partial P_{4'}^{uu} - \frac{\sqrt{6}}{4} \partial P_{4'}^{ww} \right) \right] (w)
\]

\[+ \frac{1}{(z-w)} \left[ -\frac{1}{12} c_{wop} \partial^2 P_{4'}^{uu} + \frac{1}{12} c_{wop'} \partial^2 P_{4'}^{ww} + \frac{2}{5} c_{wogo} \partial \left( GO_{4'}^{\perp} + \frac{1}{4\sqrt{6}} \partial P_{4'}^{uu} - \frac{\sqrt{6}}{4} \partial P_{4'}^{ww} \right) \right]
\]

\[-c_{wotp} \left( TP_{4'}^{uu} - \frac{1}{6} \partial^2 P_{4'}^{uu} \right) - c_{wotp'} \left( TP_{4'}^{ww} - \frac{1}{6} \partial^2 P_{4'}^{ww} \right)
\]

\[-c_{wogo} \left( G\partial O_{4'}^{\perp} - \frac{7}{3} \partial^2 P_{4'}^{uu} + \frac{1}{9\sqrt{6}} \partial^2 P_{4'}^{ww} - \frac{1}{3} \sqrt{2} \partial^2 P_{4'}^{ww} \right)
\]

\[-c_{wogo'} \left( GO_{4'}^{\perp} - \frac{2}{63} \sqrt{2} \partial^2 P_{4'}^{uu} + \frac{1}{21\sqrt{6}} \partial^2 P_{4'}^{ww} \right) + P_6(w) \right] (w) + \cdots.
\]

The unknown coefficients in (3.43) are fixed by

\[
c_{wop} = \frac{8(-36 + 369k + 41k^2)}{15(90 + 117k + 13k^2)},
\]

\[
c_{wop'} = \frac{8(2106 + 6129k + 3759k^2 + 684k^3 + 38k^4)}{5(3 + k)(6 + k)(90 + 117k + 13k^2)},
\]

\[
c_{wogo} = \frac{132\sqrt{6}(1 + k)(8 + k)}{5(90 + 117k + 13k^2)}, \quad c_{wotp} = \frac{n_1}{d}, \quad c_{wotp'} = \frac{n_2}{d},
\]

39
\[ c_{wogo'} = -\frac{64\sqrt{6}(1+k)(8+k)(369+234k+26k^2)(378+333k+37k^2)}{175(-6+9k+k^2)(90+117k+13k^2)(954+549k+61k^2)}, \]
\[ c_{wogo''} = \frac{6\sqrt{6}(1+k)(8+k)(9+2k)^2(-42+99k+11k^2)}{5(3+2k)(15+2k)(90+117k+13k^2)(366+207k+23k^2)}, \]
\[ n_1 \equiv -\frac{32}{3}(3+k)(6+k)(-109854468 - 570043980k - 583026525k^2 - 175058712k^3 + 15297117k^4 + 2280966k^5 + 6336k^6), \]
\[ n_2 \equiv -4(42504047568 + 157399614720k + 224326680552k^2 + 165246862320k^3 + 70324852617k^4 + 18329683752k^5 + 334279692k^6 + 24334537k^7 + 24772k^8 + 1114740k^9 + 24772k^{10}), \]

where \( d \) in (3.44) is the same that in (3.41). To arrive at the final result (3.43), the OPEs between the spin-2 and spin-\( \frac{3}{2} \) currents with the quasi-primary fields appearing in (3.43) should be obtained. These can be found in Appendices D and E. The structure constants on the two spin-4 fields and its descendant fields are fixed completely. The nontrivial second-order pole can be determined in a similar manner. Based on the results for the second-order pole, the correct three derivatives can be obtained terms in the first-order singular terms and the coefficients appearing in the quasi-primary fields can be determined. After extracting these four quasi-primary fields from the first-order pole, the following new spin-6 primary field can be derived

\[ P_6'(z) = n_1 d \psi^a \partial^5 \psi^a(z) + \text{other lower order derivative terms}, \]
\[ n_1 \equiv 2(-1+k)k(1+k)(9+k)(9+2k)(-114528816 - 144714192k - 9500652k^2 + 6197249k^3 + 35947494k^4 + 56069k^5 + 1210k^6), \]
\[ d \equiv 225(3+k)(6+k)(3+2k)(15+2k)(-6+9k+k^2)(90+117k+13k^2)(366+207k+23k^2)(954+549k+61k^2). \]

The following spin-6 current, which is a superpartner of the spin-\( \frac{11}{2} \) current (3.38), together with (3.42) and (3.45) can be constructed

\[ O_6(z) = -P_6(z) + \sqrt{6}P_6'(z), \]

which is the same as (2.61).

The final OPE, which is more involved, can be calculated

\[
W(z) O_6^\omega(w) = \frac{1}{(z-w)^2} c_{wogo} O_{2\omega z} + \frac{1}{(z-w)^3} \left[ \frac{2}{7} c_{wogo} \partial O_{2\omega z} - c_{wogo'} O_{2\omega z} \right] (w) \\
+ \frac{1}{(z-w)^2} \left[ \frac{3}{56} c_{wogo} \partial^2 O_{\omega z} - \frac{1}{3} c_{wogo'} \partial O_{\omega z} + c_{wogo''} O_{\omega z} \right]
\]
The structure constants in (3.47) are given by

\[
+c_{\text{woo}} \left( TO_{2} - \frac{3}{16} \partial^{2}O_{2} \right) + c_{\text{wogp}} \left( GP_{4}^{uu} - \frac{4\sqrt{6}}{9} \partial O_{2} - \frac{\sqrt{6}}{56} \partial^{2}O_{2} \right)
\]

\[
-c_{\text{wogp}} \left( GP_{4}^{ww} - \frac{2}{9} \sqrt{\frac{2}{3}} \partial O_{2} - \frac{1}{14} \sqrt{\frac{2}{3}} \partial^{2}O_{2} \right) \right] (w)
\]

\[
+ \frac{1}{(z - w)} \left[ \frac{1}{126} c_{\text{woo}} \partial^{3}O_{2} - \frac{1}{15} c_{\text{woo}}' \partial^{2}O_{2} + \frac{4}{11} c_{\text{woo}}'' \partial O_{2} \right]
\]

\[
+ \frac{4}{11} c_{\text{woto}} \partial \left( TO_{2} - \frac{3}{16} \partial^{2}O_{2} \right) + \frac{4}{11} c_{\text{wogp}} \partial \left( GP_{4}^{uu} - \frac{4\sqrt{6}}{9} \partial O_{2} - \frac{\sqrt{6}}{56} \partial^{2}O_{2} \right)
\]

\[
- \frac{4}{11} c_{\text{wogp}}' \partial \left( GP_{4}^{ww} - \frac{2}{9} \sqrt{\frac{2}{3}} \partial O_{2} - \frac{1}{14} \sqrt{\frac{2}{3}} \partial^{2}O_{2} \right)
\]

\[
- c_{\text{woto}} \left( TO_{2} - \frac{3}{20} \partial^{2}O_{2} \right) - c_{\text{woto}}' \left( T \partial O_{2} - \frac{7}{4} \partial TO_{2} - \frac{1}{9} \partial^{3}O_{2} \right)
\]

\[
- c_{\text{wogp}}' \left( GP_{4}^{uu} - \frac{8}{3} \partial GP_{4}^{uu} - \frac{\sqrt{6}}{5} \partial^{2}O_{2} - \frac{2}{63} \sqrt{\frac{2}{3}} \partial^{3}O_{2} \right)
\]

\[
+ c_{\text{wogp}}'' \left( GP_{4}^{ww} - \frac{8}{3} \partial GP_{4}^{ww} - \frac{1}{5\sqrt{6}} \partial^{2}O_{2} - \frac{8}{189} \sqrt{\frac{2}{3}} \partial^{3}O_{2} \right) + \frac{2}{11} \sqrt{\frac{2}{3}} O_{11} \right] (w)
\]

\[+ \cdots. \]

(3.47)

The structure constants in (3.47) are given by

\[
c_{\text{woo}} = \frac{12\sqrt{6}(1+k)(8+k)(378 + 333k + 37k^2)}{5(3+k)(6+k)(90 + 117k + 13k^2)},
\]

\[
c_{\text{woo}'} = \frac{2\sqrt{\frac{2}{3}}(9 + 2k)^2(-42 + 99k + 11k^2)}{(3+k)(6+k)(90 + 117k + 13k^2)}, \quad c_{\text{woo}''} = \frac{1}{\sqrt{6}},
\]

\[
c_{\text{woto}} = \frac{216\sqrt{6}(1+k)(8+k)(1354 + 801k + 89k^2)}{5(90 + 117k + 13k^2)(954 + 549k + 61k^2)}, \quad c_{\text{wogp}} = \frac{n_1}{d}, \quad c_{\text{wogp}'} = \frac{n_2}{d},
\]

\[
c_{\text{woto}'} = \frac{56\sqrt{\frac{2}{3}}(9 + 2k)^2(-42 + 99k + 11k^2)}{5(90 + 117k + 13k^2)(366 + 207k + 23k^2)},
\]

\[
c_{\text{woto}''} = \frac{64\sqrt{\frac{2}{3}}(1+k)(8+k)(378 + 333k + 37k^2)}{55(-6 + 9k + k^2)(90 + 117k + 13k^2)},
\]

\[
c_{\text{wogp}'} = -\frac{56(152280 + 426492k + 372846k^2 + 151785k^3 + 30505k^4 + 2943k^5 + 109k^6)}{55(-6 + 9k + k^2)(90 + 117k + 13k^2)(366 + 207k + 23k^2)},
\]

\[
c_{\text{wogp}''} = \frac{84(629532 + 163630k + 1520823k^2 + 647478k^3 + 133171k^4 + 12960k^5 + 480k^6)}{55(-6 + 9k + k^2)(90 + 117k + 13k^2)(366 + 207k + 23k^2)},
\]

\[
n_1 \equiv 4(24802524 + 60056964k + 54194319k^2 + 2272930k^3 + 4640895k^4
\]

\[+ 450468k^5 + 1668k^6),
\]

\[
n_2 \equiv -3(195174684 + 484795692k + 441009171k^2 + 185289498k^3 + 37865451k^4
\]


How does one obtain a complete set of structure constants? In principle, the procedure used for the infinite \( k \) case can be followed. On the other hand, the first-order pole is rather complicated. First of all, the first-order singular terms should be calculated completely but this is time consuming. After that, the derivative terms appearing in the first-order terms are known from the higher-order singular terms. This leaves four quasi-primary fields and one additional primary field of spin-\( \frac{13}{2} \), which is the last element of the higher spin current in the list (2.20).

One way to determine the unknown \( k \)-dependent coefficient functions and explicit form the spin-\( \frac{13}{2} \) is to write down the spin-\( \frac{13}{2} \) current using its superpartner \( O_{\nu}^{\prime}(z) \). The current \( O_{\nu}^{\prime}(z) \) can be determined, which is similar to (3.46) because \( P_{6}(z) \) and \( P_{\nu}^{\prime}(z) \) can be obtained from (3.39) and (3.43)

\[
O_{\nu}^{\prime}(z) = P_{6}(z) + \frac{7}{2} \sqrt{\frac{3}{2} P_{\nu}^{\prime}(z)},
\]  

which is the same as (2.64). By equating the first-order pole from the WZW currents to the sum of the derivative terms, quasi-primary fields with four unknown coefficients and spin-\( \frac{13}{2} \) current, the four unknown coefficients can be fixed, as expressed in (3.48). The last spin-\( \frac{13}{2} \) current, which is a superpartner of (3.49), can be given by

\[
O_{\frac{13}{2}}^{\prime}(z) = \frac{n}{d} f^{abc} \psi^{a} \psi^{b} \partial^{c} \psi(z) + \text{other lower order derivative terms},
\]

\[
n \equiv 16 \sqrt{2} (-1 + k) (1 + k) (9 + k) (9 + 2k)^2 (-1998 - 123k + 573k^2 + 103k^3 + 5k^4),
\]

\[
d \equiv 75 (3 + k) (6 + k) (15 + 2k) (-6 + 9k + k^2) \sqrt{18 + 9k + k^2} (90 + 117k + 13k^2)
\]

\[
\times (366 + 207k + 23k^2).
\]

The OPEs between the spin-2 and spin-\( \frac{3}{2} \) currents with the seven quasi-primary fields appearing in (3.47) can be found in Appendices D and E.

Therefore, the higher spin currents are given by (3.6), (3.5), (3.11), (3.12), (3.18), (3.19), (3.21), footnote 16, (3.20), (3.31), (3.38), (3.40), (3.49), and (3.50), and some of the OPEs between them are computed. In these calculations, the infinite \( k \) case in previous section is crucial because the algebraic structure in the OPEs is common to each other. Although all the singular terms in the OPEs can be obtained by defining OPE between the current \( \psi^{a}(z) \) and current \( K^{a}(z) \), it is difficult to express those singular items in terms of the quasi-primary fields and higher spin currents. Note that the quasi-primary fields can be expressed as those
higher spin currents and the stress energy tensor with its superpartner. The right hand side of the remaining OPEs not considered in this study should contain only the known primary currents (and their composite fields that can be either quasi-primary fields or primary fields) in the list (2.20).

4 Conclusions and outlook

In this paper we have found concrete expressions for the higher spin currents in the list of (2.20) of the \( c \leq 4 \) model in terms of the WZW currents \( \psi^a(z) \) and \( K^a(z) \). They satisfy the following superfusion rules

\[
\begin{align*}
[\hat{W}] [\hat{W}] &= [\hat{I}] + [\hat{O}_2] + [\hat{O}_4], \\
[\hat{W}] [\hat{O}_2] &= [\hat{W}] + [\hat{O}_4] + [\hat{O}_4'], \\
[\hat{W}] [\hat{O}_4] &= [\hat{O}_2] + [\hat{O}_4] + [\hat{O}_6].
\end{align*}
\] (4.1)

All the coefficients in the OPEs (4.1) are fixed. In the third superfusion rule, some of the algebraic structures of the first one occur. The remaining OPEs were not calculated. According to the observation of [21], they will, in general, take the form \([\hat{I}] + [\hat{W}] + [\hat{O}_2] + [\hat{O}_4] + [\hat{O}_4'] + [\hat{O}_4''] + [\hat{O}_6],\) in the right hand side. Finding a concrete expression for a quasi-primary field of some given spin in terms of 16 currents is a nontrivial task. For the most complicated OPE between the spin-\( \frac{13}{2} \) current and itself, the singular terms have a 13-th order pole, \( \cdots \), second-order pole, and first-order pole. The highest quasi-primary field of spin-12 can then appear in the first-order singular term and should be written in terms of the known higher spin currents. Possibly formula (B.4) will be helpful in finding the structure of this quasi-primary field.

An interesting direction is to generalize the coset discussed in this paper to the case of a general \( N \),

\[
\frac{\hat{S}\hat{U}(N)_k \oplus \hat{S}\hat{U}(N)_N}{\hat{S}\hat{U}(N)_{k+N}}, \quad c = \frac{(N^2 - 1)}{2} \left[ 1 - \frac{2N^2}{(k + N)(k + 2N)} \right] < \frac{(N^2 - 1)}{2}. \] (4.2)

This will be a supersymmetric extension of \( W_N \) algebra. As pointed out in reference [20], the lowest model in the series of coset models \((k = 1 \) or \( c = \frac{(-1-N)(1+3N)}{2(1+2N)} \) has “minimal” super \( W_N \) algebra where there are supercurrents of spins \( \frac{3}{2}, \frac{5}{2}, \cdots, (N - \frac{1}{2}) \). For the general \( k > 1 \), extra additional currents should appear. As in the present paper, it is better to look at the \( k \rightarrow \infty \) model (or \( c = \frac{(N^2 - 1)}{2} \) fermion model in the adjoint representation of \( SU(N) \)) first because it contains all the algebraic structures and is simpler than the more general
$c < \frac{(N^2-1)}{2}$ model. All the OPEs in sections 2 and 3 should be generalized to the OPEs with $N$-dependence. In the context of minimal model holography \[1\] \[2\], the correct normalizations should be calculated for the higher spin currents with spins greater than 3. In the original paper \[21\], the character technique was used to generate the complete currents in $c = 4$ eight fermion model. Generalizing this for an arbitrary $N$ would be interesting study. As a first step, it will also be useful to consider the $N = 4$ case.

A study of the most general coset models by $\hat{SU}(N)_{k} \oplus \hat{SU}(N)_{l}$ with a central charge would be interesting. For this particular case ($l = N$), this model reduces to the above model \[12\]. For the higher spin 3,4 currents, the explicit construction was calculated as mentioned in the introduction. Obtaining other higher spin currents explicitly, e.g. the spin-5 Casimir operator, is an open problem.

In reference \[46\], the coset model was based on the group $SO(N)$ with a given central charge. Therefore, more study will be needed to determine if the present result can be applied to the different minimal model.

The following describes some partial results in CFT that hold for the general $N$. The spin-2 and spin-$\frac{3}{2}$ currents can be generalized without difficulty from \[3.2\] and \[3.4\], respectively. To match with the convention of reference \[16\], the spin-1 currents are rescaled together using rescaled structure constants, the second order pole for the OPE $J^a(z) J^b(w)$ given in \[2.14\] has $-N \delta^{ab}$, and the one for the OPE $K^a(z) K^b(w)$ given in \[3.1\] has $-k \delta^{ab}$. The relative $N$-dependent coefficients for $T(z)$ and $G(z)$ are determined by calculating the OPEs $T(z) T(w)$ and $G(z) G(w)$ completely to satisfy \[2.3\] and \[2.2\], respectively, with the central charge \[4.2\]. For the spin-3 current, the formula can be obtained from reference \[15\] (or the OPE $W(z) W(w)$, with four unknown coefficients, can be used explicitly as in \[15\] or the regularity of the spin-3 current with the diagonal spin-1 current and the primary condition of spin-3 current under the stress energy tensor can be applied \[16\]). For its superpartner, $U(z)$, there are several ways to determine the complete relative $N$-dependent coefficients. The OPE between $G(z)$ and $W(w)$ can be used and $U(w)$ can be read off from \[2.7\] (or the OPE $U(z) U(w)$ calculated to fix the coefficients).

The two $N = 1$ supermultiplets with complete $N$-dependent coefficients are listed as follows.

\[
T(z) = -\frac{1}{4N} J^a J^a(z) - \frac{1}{2(k+N)} K^a K^a(z) + \frac{1}{2(k+2N)} (J^a + K^a)(J^a + K^a)(z),
\]

\[
G(z) = -\frac{\sqrt{2k}}{3\sqrt{N(k+N)(k+2N)}} \psi^a \left( J^a - \frac{3N}{k} K^a \right)(z),
\]
\[
W(z) = \frac{i \epsilon^{abc}}{6N(N + k)(2N + k)\sqrt{6(N + 2k)(5N + 2k)(N^2 - 4)}} [k(k + N)(2k + N)J^a J^b J^c \\
- 6N(k + N)(2k + N)J^a J^b K^c + 18N^2(k + N)J^a K^b K^c - 6N^3 K^a K^b K^c] (z),
\]
\[
U(z) = \frac{i \epsilon^{abc}}{\sqrt{50N(k + N)(k + 2N)(2k + 5N)(-4 + N^2)}} [k(2k + N)\psi^a J^b J^c \\
- 5N(2k + N)\psi^a J^b K^c + 10N^2 \psi^a K^b K^c] (z).
\] (4.3)

These reduce to the previous results when \(N = 3\) up to the overall constants (due to the different normalizations). Note that there are half-integer (higher) spin currents. How can these \(N\)-dependent expressions be interpreted? The zero modes for these currents (in particular, the higher spin currents) can be analyzed in a similar to that in references [17, 16], which will lead to three-point functions (the generalization of [18]) with the scalars for all values of the \(\text{t Hooft}^{'}\) coupling in the large \((N, k)\) \(\text{t Hooft}^{'}\) limit.

Regarding the symmetry behind the currents in (4.3), it is natural to ask how the OPEs between these currents arise. From expressions (4.3), the following nontrivial OPEs between the two lowest higher spin currents, \(W(z)\) and \(U(z)\), are expected in the more general coset model [12]:

\[
W(z) W(w) = \frac{1}{(z - w)^6} \left[ \frac{c}{3} + \frac{1}{(z - w)^4} 2T(w) + \frac{1}{(z - w)^3} \partial T(w) \right. \\
+ \frac{1}{(z - w)^2} \left[ \left( \frac{3}{10} \right) \partial^2 T + \frac{32}{22 + 5c} \left( T^2 - \frac{3}{10} \partial^2 T \right) + P^{ww}_4 + P^{ww}_4 \right] (w) \\
+ \frac{1}{(z - w)} \left[ \left( \frac{1}{15} \right) \partial^3 T + \left( \frac{1}{2} \right) \frac{32}{22 + 5c} \partial \left( T^2 - \frac{3}{10} \partial^2 T \right) \right. \\
+ \left( \frac{1}{2} \right) \partial P^{ww}_4 + \left( \frac{1}{2} \right) \partial P^{ww}_4 \right] (w) + \cdots,
\]

\[
W(z) U(w) = \frac{1}{(z - w)^4} \left[ \frac{3}{\sqrt{6}} G(w) + \frac{1}{(z - w)^3} \left( \frac{2}{3} \right) \frac{3}{\sqrt{6}} \partial G(w) \right. \\
+ \frac{1}{(z - w)^2} \left[ \left( \frac{1}{4} \right) \frac{3}{\sqrt{6}} \partial^2 G + \frac{11\sqrt{6}}{4c + 21} \left( GT - \frac{1}{8} \partial^2 G \right) + O_\frac{3}{2} \right] (w) \\
+ \frac{1}{(z - w)} \left[ \left( \frac{1}{15} \right) \frac{3}{\sqrt{6}} \partial^3 G + \left( \frac{4}{7} \right) \frac{11\sqrt{6}}{4c + 21} \partial \left( GT - \frac{1}{8} \partial^2 G \right) + \left( \frac{4}{7} \right) \partial O_\frac{3}{2} \right. \\
+ \frac{12\sqrt{6}}{7(10c - 7)} \left( \frac{4}{3} T \partial G - G \partial T - \frac{4}{15} \partial^3 G \right) + O_\frac{3}{2} \right] (w) + \cdots,
\]

\[
U(z) U(w) = \frac{1}{(z - w)^5} \left[ \frac{2c}{5} + \frac{1}{(z - w)^3} 2T(w) + \frac{1}{(z - w)^2} \partial T(w) \\
+ \frac{1}{(z - w)} \left[ \left( \frac{3}{10} \right) \partial^2 T + \frac{27}{22 + 5c} \left( T^2 - \frac{3}{10} \partial^2 T \right) + P^{ww}_4 + P^{ww}_4 \right] (w)
\]

45
where the two primary fields \( W_4 \) corresponding to (3.13) and (3.8) can be expressed as

\[
P_{ww}^4(z) = \frac{8(10 - 7c)}{(4c + 21)(10c - 7)} \left[ -\frac{7}{10} \partial^2 T + \frac{17}{22 + 5c} \left( T^2 - \frac{3}{10} \partial^2 T \right) + G\partial G \right](z),
\]

\[
P_{wu}^4(z) = -\frac{3(2c - 83)}{(4c + 21)(10c - 7)} \left[ -\frac{7}{10} \partial^2 T + \frac{17}{22 + 5c} \left( T^2 - \frac{3}{10} \partial^2 T \right) + G\partial G \right](z).
\]

The \( k \)-dependent coefficient functions appearing in (3.13), (3.10) and (3.7) are simply replaced with the central charge (3.3) and generalized to the \( N \)-dependent central charge (1.2). (1.4) has no self-coupling constant. In other words, there are no \( W(w) \)-terms in the OPE \( W(z) W(w) \) and \( U(w) \)-terms in the OPE \( U(z) U(w) \). This allows only \( c \)-dependent coefficient functions to be observed. The classical \( c \to \infty \) limit suggests that the \( \partial^2 T(w), \partial^3 T(w), \partial^2 G(w), \partial^3 G(w) \)-terms with \( c \)-dependent coefficients on the right hand side of (4.4) vanish and the only \( G\partial G(z) \) term in (4.4) survives in this classical limit.

Note that the OPE \( W(z) W(w) \) contains a composite current \( G\partial G(w) \) (even at the classical level) due to the \( N = 1 \) supersymmetry. Furthermore, it has another spin-4 current. This is a new feature compared to the bosonic theory [27, 24]. In other words, compared to the standard Zamolodchikov’s \( W_3 \) algebra, there are two additional primary fields, \( P_{ww}^4(w) \) and \( P_{wu}^4(w) \). Compared to the bosonic extended algebra [24], there is an additional primary field \( P_{ww}^4(w) \). The OPEs (4.4) hold for any \( N \), and are exact and complete expressions except that the Casimir operators of spin \( s \) with \( s = \frac{7}{2}, 4, 4, \frac{9}{2} \) for general \( N \) are unknown. On the other hand, they can be generated by \( W(z) \) and \( U(z) \) in (4.3) in addition to \( T(z) \) and \( G(z) \).

The unknown four currents (so far the explicit forms in terms of \( \psi^a(z) \) and \( K^a(z) \) for general \( N \) are not known), \( P_{ww}^4(w), O_{\frac{7}{2}}(w), O_{\frac{9}{2}}(w) \), and \( P_{wu}^4(w) \) on the right hand side of (4.4) can be found, in principle, by calculating the OPEs between the explicit expressions (1.3). The spin-\( \frac{7}{2} \) current, \( O_{\frac{7}{2}}(z) \), is expected to contain the following nonderivative composite operators by realizing the possible combinations from the currents \( W(z) \) and \( U(w) \), given in (4.3):

\[
d^{abc}d^{cde}\psi^a J^b J^d J^e, \quad d^{abc}d^{cde}\psi^a J^b J^d K^e, \quad d^{abc}d^{cde}\psi^a J^d J^e K^b, \quad d^{abc}d^{cde}\psi^a J^b K^d K^e, \\
d^{abc}d^{cde}\psi^a J^d K^b K^e, \quad d^{abc}d^{cde}\psi^a K^b K^d K^e.
\]

Of course, there are different types of derivative terms due to normal ordering in the composite fields. Note that the quantity, \( f^{abc}\psi^b J^c(z) \), is zero due to (2.13). The complete relative \( N \)-dependent coefficients can be determined by calculating the corresponding OPE \( W(z) U(w) \) and extracting the structure of the second-order pole carefully. Each \( d \) symbol in (4.6) comes
from the currents, $W(z)$ and $U(w)$, respectively. Similarly, the spin $\frac{2}{2}$ current can be obtained from the first-order pole of this OPE. For the spin-4 current, $P_{ww}^4(z)$, the field contents can be obtained from the previous result \cite{16} and the other terms in the second order pole in the OPE $W(z)W(w)$. Finally, another spin-4 current, $P_{uu}^4(z)$, can be determined from the OPE of $U(z)U(w)$ explicitly. A further study should determine these currents.

What happens with the self-coupling term? The OPE of $O_4(z)O_4(w)$ has a self coupling term $O_4(w)$ and one direct way to see this $N$-dependent self coupling constant is to calculate this OPE explicitly (eventhough this will be very complicated). A further study should determine other OPEs for general $N$ not considered in this study.

One application of these results is the ability to analyze a large $k$ limit with $N$ fixed ($c = 4$ model) similar to that in reference \cite{50}. The OPEs are summarized by \eqref{4.1}. The asymptotic symmetry of the higher spin $AdS_3$ gravity (at the quantum level) can be summarized from the two dimensional CFT results obtained thus far. The OPEs \eqref{4.4} with \eqref{4.5} should provide asymptotic symmetry algebra in the $AdS_3$ bulk theory at both the classical and quantum levels. The three-point functions from the CFT computations should correspond to the three-point functions in the $AdS_3$ bulk theory. A further study should examine the corresponding bulk theory computations. See also recent papers \cite{51, 52}. The coset model is a $\mathcal{N} = 1$ version of the coset model studied in reference \cite{18}. Therefore, the bulk theory would have higher spin gauge symmetry in $AdS_3$ string theory. This is because the central charge \eqref{4.2} in this coset model is proportional to $N^2$ rather than $N$. The algebra described is larger than the conventional $W_N$ algebra: the existence of half-integer spin currents. See also relevant studies \cite{11, 13}, where the $\mathcal{N} = 2$ minimal model holography can be determined using asymptotic symmetry algebra.

**Acknowledgments**

CA would like to thank his previous collaborators K. Schoutens and A. Sevrin for the work of \cite{21}. This study was supported by the Mid-career Researcher Program through the National Research Foundation of Korea (NRF) grant funded by the Korean government (MEST) (No. 2012-045385). CA acknowledges warm hospitality from the School of Liberal Arts (and Institute of Convergence Fundamental Studies), Seoul National University of Science and Technology.
Appendix A  The coefficients appearing in the descendant fields of quasi-primary or primary fields (1.3)

The introduction presented the OPE between the two quasi-primary fields (including the primary fields), where the coefficient functions that depend on the spins, $h_i, h_j, h_k$ and the number of derivatives $n$, are given by

\[
A_{i,j,k,n} = \frac{1}{n!} \frac{\Gamma(h_i - h_j + h_k + n)}{\Gamma(h_i - h_j + h_k)} \frac{\Gamma(2h_k)}{\Gamma(2h_k + n)} = \frac{1}{n!} \prod_{x=0}^{n-1} \frac{(h_i - h_j + h_k + x)}{(2h_k + x)}. \tag{A.1}
\]

Here, they are written in terms of the Pochhammer symbol because sometimes the denominator in the original expression can have a zero values. To avoid this feature, the ratio of two Gamma functions should be expressed in terms of the Pochhammer symbol as shown in (A.1).

Using the definition of the coefficients (A.1), those vanishing and nonvanishing coefficients appearing in the Sections 2, and 3, should be checked as follows:

\[
\begin{align*}
A_{2,2,2,1} &= \frac{1}{2}, & A_{2,\frac{3}{2},\frac{3}{2},1} &= \frac{2}{3}, & A_{2,3,\frac{3}{2},1} &= \frac{1}{5}, & A_{\frac{3}{2},4,\frac{3}{2},1} &= \frac{1}{7}, \\
A_{\frac{3}{2},\frac{3}{2},4,1} &= \frac{1}{8}, & A_{3,3,2,1} &= \frac{1}{2}, & A_{3,3,2,2} &= \frac{3}{20}, & A_{3,3,2,3} &= \frac{1}{30}, \\
A_{3,3,3,1} &= \frac{1}{2}, & A_{3,\frac{5}{2},\frac{3}{2},1} &= \frac{2}{3}, & A_{\frac{5}{2},\frac{5}{2},2,2} &= \frac{1}{4}, & A_{\frac{5}{2},\frac{3}{2},3,1} &= \frac{1}{15}, \\
A_{3,\frac{5}{2},\frac{5}{2},1} &= \frac{4}{7}, & A_{\frac{5}{2},\frac{3}{2},2,1} &= \frac{1}{2}, & A_{\frac{5}{2},\frac{5}{2},2,2} &= \frac{3}{20}, & A_{\frac{5}{2},\frac{5}{2},3,1} &= \frac{1}{3}, \\
A_{\frac{5}{2},\frac{5}{2},3,2} &= \frac{1}{14}, & A_{\frac{5}{2},\frac{3}{2},4,1} &= \frac{3}{8}, & A_{\frac{5}{2},4,\frac{3}{2},1} &= \frac{1}{5}, & A_{\frac{5}{2},4,\frac{5}{2},2} &= \frac{1}{30}, \\
A_{\frac{5}{2},4,\frac{5}{2},3} &= \frac{1}{210}, & A_{\frac{5}{2},4,\frac{5}{2},4} &= \frac{1}{3}, & A_{2,\frac{7}{2},\frac{7}{2},1} &= -\frac{1}{5}, & A_{\frac{7}{2},\frac{7}{2},3,1} &= -\frac{1}{6}, \\
A_{3,\frac{7}{2},\frac{7}{2},1} &= \frac{2}{5}, & A_{3,\frac{7}{2},\frac{7}{2},2} &= \frac{1}{10}, & A_{3,\frac{7}{2},\frac{7}{2},3} &= \frac{2}{105}, & A_{3,\frac{7}{2},\frac{7}{2},4,1} &= \frac{4}{9}, \\
A_{3,4,3,1} &= \frac{1}{3}, & A_{3,4,3,2} &= \frac{1}{14}, & A_{3,4,3,3} &= \frac{1}{84}, & A_{3,4,4,1} &= \frac{3}{8}, \\
A_{3,4,4,2} &= \frac{1}{12}, & A_{3,4,5,1} &= \frac{2}{5}, & A_{\frac{5}{2},\frac{5}{2},\frac{3}{2},1} &= -\frac{1}{5}, & A_{2,6,3,1} &= -\frac{1}{6}, \\
A_{\frac{5}{2},6,\frac{7}{2},1} &= -\frac{2}{5}, & A_{\frac{5}{2},6,\frac{7}{2},2} &= \frac{1}{30}, & A_{\frac{5}{2},4,\frac{7}{2},1} &= \frac{2}{7}, & A_{\frac{5}{2},4,\frac{7}{2},2} &= \frac{3}{56}, \\
A_{\frac{5}{2},4,\frac{7}{2},3} &= \frac{1}{3}, & A_{\frac{5}{2},4,\frac{7}{2},4} &= \frac{1}{4}, & A_{\frac{5}{2},4,\frac{7}{2},2} &= \frac{1}{24}, & A_{\frac{5}{2},4,\frac{7}{2},5,1} &= \frac{3}{10}, \\
A_{3,4,4,1} &= \frac{3}{8}, & A_{3,4,4,2} &= \frac{1}{12}, & A_{3,4,5,1} &= \frac{2}{5}, & A_{3,4,\frac{7}{2},1} &= \frac{2}{7}, \\
A_{3,4,4,1} &= \frac{3}{56}, & A_{3,\frac{7}{2},\frac{7}{2},3} &= \frac{1}{126}, & A_{3,\frac{7}{2},\frac{7}{2},1} &= \frac{1}{3}, & A_{3,\frac{7}{2},\frac{7}{2},2} &= \frac{1}{15},
\end{align*}
\]
\[ A_{3,\frac{21}{2},1} = \frac{4}{11}. \]  

(A.2)

Appendix B  The OPE between the stress energy tensor and the quasiprimary or primary fields in \( c = 4 \) model

To determine if a conformal field is quasi-primary field, the OPE between the stress energy tensor \( T(z) \) and a field \( \Phi(w) \) should be calculated and the vanishing of third-order pole in the OPE \( T(z) \Phi(w) \) should be checked. All the quasi-primary fields (where there are three primary fields) in sections 2 and 3 are listed as follows

\[
T(z) \left( TT - \frac{3}{10} \partial^2 T \right) (w) = \frac{1}{(z-w)^4} \frac{42}{5} T(w) + \mathcal{O}((z-w)^{-2}),
\]

\[
T(z) \left( GT - \frac{1}{8} \partial^2 G \right) (w) = \frac{1}{(z-w)^4} \frac{37}{8} G(w) + \mathcal{O}((z-w)^{-2}),
\]

\[
T(z) \left( G\partial T - \frac{4}{3} T\partial G + \frac{4}{15} \partial^3 G \right) (w) = \frac{1}{(z-w)^5} \frac{33}{5} G(w)
\]

\[
+ \frac{1}{(z-w)^4} (-\frac{1}{3}) \frac{33}{5} \partial G(w) + \mathcal{O}((z-w)^{-2}),
\]

\[
T(z) \left( GU - \frac{\sqrt{6}}{3} \partial W \right) (w) = \mathcal{O}((z-w)^{-2}),
\]

\[
T(z) \left( TW - \frac{3}{14} \partial^2 W \right) (w) = \frac{1}{(z-w)^4} \frac{71}{7} W(w) + \mathcal{O}((z-w)^{-2}),
\]

\[
T(z) \left( G\partial U - \frac{5}{3} \partial G U - \frac{\sqrt{6}}{7} \partial^2 W \right) (w) = -\frac{1}{(z-w)^4} \frac{124}{7} \sqrt{\frac{2}{3}} W(w) + \mathcal{O}((z-w)^{-2}),
\]

\[
T(z) \left( GW - \frac{1}{6\sqrt{6}} \partial^2 U \right) (w) = \frac{1}{(z-w)^4} 5\sqrt{\frac{2}{3}} U(w) + \mathcal{O}((z-w)^{-2}),
\]

\[
T(z) \left( TU - \frac{1}{4} \partial^2 U \right) (w) = \frac{1}{(z-w)^4} \frac{33}{4} U(w) + \mathcal{O}((z-w)^{-2}),
\]

\[
T(z) \left( TO_{\frac{z}{2}} - \frac{3}{16} \partial^2 O_{\frac{z}{2}} \right) (w) = \frac{1}{(z-w)^4} \frac{193}{16} O_{\frac{z}{2}}(w) + \mathcal{O}((z-w)^{-2}),
\]

\[
T(z) \left( GP_{4}^{uu} - \frac{4\sqrt{6}}{9} \partial O_{\frac{z}{2}} - \frac{\sqrt{6}}{56} \partial^2 O_{\frac{z}{2}} \right) (w) = \frac{1}{(z-w)^4} \frac{17}{4} \sqrt{\frac{3}{2}} O_{\frac{z}{2}}(w) + \mathcal{O}((z-w)^{-2}),
\]

\[
T(z) \left( GP_{4}^{ww} - \frac{2}{3} \partial O_{\frac{z}{2}} - \frac{1}{14} \sqrt{\frac{2}{3}} \partial^2 O_{\frac{z}{2}} \right) (w) = \frac{1}{(z-w)^4} \frac{17}{4} \sqrt{\frac{3}{2}} O_{\frac{z}{2}}(w) + \mathcal{O}((z-w)^{-2}),
\]

\[
T(z) \left( GO_{\frac{z}{2}} + \frac{1}{4\sqrt{6}} \partial P_{4}^{uu} - \frac{\sqrt{6}}{4} \partial P_{4}^{ww} \right) (w) = +\mathcal{O}((z-w)^{-2}),
\]
$$T(z) \left( TP_{uu}^{w} - \frac{1}{6} \partial^2 P_{uu}^{w} \right) (w) = \frac{1}{(z-w)^4} 14 P_{uu}^{w}(w) + \mathcal{O}((z-w)^{-2}),$$

$$T(z) \left( TP_{ww}^{uw} - \frac{1}{6} \partial^2 P_{ww}^{uw} \right) (w) = \frac{1}{(z-w)^4} 14 P_{ww}^{uw}(w) + \mathcal{O}((z-w)^{-2}),$$

$$T(z) \left( G \partial O_{zz}^{w} - \frac{7}{3} \partial G O_{zz}^{w} + \frac{1}{9 \sqrt{6}} \partial^2 P_{uu}^{w} - \frac{1}{3} \sqrt{ \frac{2}{3} } \partial^2 P_{ww}^{uw} \right) (w) =$$

$$\frac{1}{(z-w)^4} \left[ \frac{25}{3 \sqrt{6}} P_{uu}^{w} - 25 \sqrt{ \frac{2}{3} } P_{ww}^{uw} \right] (w) + \mathcal{O}((z-w)^{-2}),$$

$$T(z) \left( GO_{zz}^{w} - \frac{2}{63} \sqrt{ \frac{2}{3} } \partial^2 P_{uu}^{w} + \frac{1}{21 \sqrt{6}} \partial^2 P_{ww}^{uw} \right) (w) =$$

$$\frac{1}{(z-w)^4} \left[ \frac{104}{21} \sqrt{ \frac{2}{3} } P_{uu}^{w} - \frac{26}{7} \sqrt{ \frac{2}{3} } P_{ww}^{uw} \right] (w) + \mathcal{O}((z-w)^{-2}),$$

$$T(z) \left( T O_{zz}^{w} - \frac{3}{20} \partial^2 O_{zz}^{w} \right) (w) = \frac{1}{(z-w)^4} \frac{319}{20} O_{zz}^{w}(w) + \mathcal{O}((z-w)^{-2}),$$

$$T(z) \left( T \partial O_{zz}^{w} - \frac{7}{4} \partial T O_{zz}^{w} - \frac{1}{9} \partial^3 O_{zz}^{w} \right) (w) = - \frac{1}{(z-w)^5} \frac{301}{12} O_{zz}^{w}(w)$$

$$- \frac{1}{(z-w)^4} \left( - \frac{1}{7} \right) \frac{301}{12} \partial O_{zz}^{w}(w) + \mathcal{O}((z-w)^{-2}),$$

$$T(z) \left( G \partial P_{uu}^{w} - \frac{8}{3} \partial G P_{uu}^{w} - \frac{\sqrt{6}}{5} \partial^2 O_{zz}^{w} - \frac{2}{63} \sqrt{ \frac{2}{3} } \partial^3 O_{zz}^{w} \right) (w) =$$

$$- \frac{1}{(z-w)^5} \frac{50}{3} \sqrt{ \frac{2}{3} } O_{zz}^{w}(w) + \frac{1}{(z-w)^4} \left[ \left( - \frac{1}{7} \right) \frac{50}{3} \sqrt{ \frac{2}{3} } \partial O_{zz}^{w} - \frac{286}{25} \sqrt{ \frac{2}{3} } O_{zz}^{w} \right] (w) + \mathcal{O}((z-w)^{-2}),$$

$$T(z) \left( G \partial P_{ww}^{uw} - \frac{8}{3} \partial G P_{ww}^{uw} - \frac{1}{5 \sqrt{6}} \partial^2 O_{zz}^{w} - \frac{8}{189} \sqrt{ \frac{2}{3} } \partial^3 O_{zz}^{w} \right) (w) =$$

$$- \frac{1}{(z-w)^5} \frac{200}{9} \sqrt{ \frac{2}{3} } O_{zz}^{w}(w) + \frac{1}{(z-w)^4} \left[ \left( - \frac{1}{7} \right) \frac{200}{9} \sqrt{ \frac{2}{3} } \partial O_{zz}^{w} - \frac{143}{15} \sqrt{ \frac{2}{3} } O_{zz}^{w} \right] (w)$$

$$+ \mathcal{O}((z-w)^{-2}),$$

$$T(z) \left( T \partial U - \frac{5}{4} \partial T U - \frac{1}{7} \partial^3 U \right) (w) = - \frac{1}{(z-w)^5} \frac{345}{28} U(w) - \frac{1}{(z-w)^4} \left( - \frac{1}{5} \right) \frac{345}{28} \partial U(w)$$

$$+ \mathcal{O}((z-w)^{-2}),$$

$$T(z) \left( G \partial W - 2 \partial G W - \frac{1}{21 \sqrt{3}} \partial^3 U \right) (w) = - \frac{1}{(z-w)^5} \frac{55}{7} \sqrt{ \frac{2}{3} } U(w)$$

$$- \frac{1}{(z-w)^4} \left( - \frac{1}{5} \right) \frac{55}{7} \sqrt{ \frac{2}{3} } \partial U(w) + \mathcal{O}((z-w)^{-2}),$$
\[ T(z) \left( GO_{\frac{3}{2}} - \frac{2}{9} \partial O_{\frac{3}{2}} \right)(w) = +O((z - w)^{-2}), \]
\[ T(z) \left( G \partial^2 U - 4 \partial G \partial U + \frac{5}{2} \partial^2 GU - \frac{1}{2 \sqrt{6}} \partial^3 W \right)(w) = \frac{1}{(z - w)^5} 9 \sqrt{6} W(w) \]
\[ + \frac{1}{(z - w)^4} \left[ \left( -\frac{1}{6} \right) 9 \sqrt{6} \partial W + \frac{75}{2} \left( GU - \frac{\sqrt{6}}{3} \partial W \right) \right](w) + O((z - w)^{-2}), \]
\[ T(z) \left( T \partial W - \frac{3}{2} \partial TW - \frac{1}{8} \partial^3 W \right)(w) = - \frac{1}{(z - w)^5} 18 W(w) - \frac{1}{(z - w)^4} (-\frac{1}{6}) 18 \partial W(w) + O((z - w)^{-2}), \]
\[ T(z) \left( TO_{\frac{1}{2}} - \frac{1}{6} \partial^2 O_{\frac{1}{2}} \right)(w) = \frac{1}{(z - w)^4} 14 O_{\frac{1}{2}}(w) + O((z - w)^{-2}), \]
\[ T(z) \left( GO_{\frac{1}{3}} - \frac{1}{9} \partial^2 O_{\frac{1}{3}} \right)(w) = \frac{1}{(z - w)^4} \frac{52}{3} O_{\frac{1}{3}}(w) + O((z - w)^{-2}). \] (B.1)

(B.1) has no third-order pole. The vanishing and nonvanishing coefficients appearing in (B.1) can be checked using the definition of the coefficients (A.1) as follows:

\[ A_{2,4,2,1} = 0, \quad A_{2,\frac{2}{3},\frac{2}{3},1} = 0, \quad A_{2,\frac{2}{3},\frac{2}{3},2} = -\frac{1}{3}, \quad A_{2,\frac{2}{3},\frac{2}{3},2} = 0, \]
\[ A_{2,\frac{2}{3},\frac{2}{3},1} = 0, \quad A_{2,5,3,1} = 0, \quad A_{2,\frac{1}{2},\frac{1}{2},2} = -\frac{1}{5}, \quad A_{2,\frac{1}{2},\frac{1}{2},2} = 0, \]
\[ A_{2,\frac{1}{2},\frac{1}{2},1} = 0, \quad A_{2,6,3,1} = -\frac{1}{6}, \quad A_{2,6,3,2} = 0, \quad A_{2,6,4,1} = 0, \]
\[ A_{2,\frac{1}{2},\frac{1}{2},2} = -\frac{1}{7}, \quad A_{2,\frac{1}{2},\frac{1}{2},2} = 0, \quad A_{2,\frac{1}{2},\frac{1}{2},1} = 0. \] (B.2)

For example, the first OPE in (B.1) has no third-order singular term. This can be realized by the disappearance of \( A_{2,4,2,1} \) in (B.2). In other words, the disappearance of the third-order singular term can be understood from the explicit WZW currents and this can be confirmed from (1.3).

Note that the following quasi-primary fields \( \Phi_i \) can be derived from (B.1)

\( \left( T \Phi_i - \frac{3}{2(2 h_i + 1)} \partial^2 \Phi_i \right)(z). \) (B.3)

The relative coefficient in (B.3) is fixed from the definition of quasi-primary condition. Therefore the other quasi-primary fields that contain the derivatives in the quadratic normal ordered product should be considered.

In reference [26], any quasi-primary field can be written in terms of quadratic part and linear part

\[ \sum_{r=0}^{n} B_{i,j,n,r} \partial^r N(\Phi_j, \partial^{n-r} \Phi_i) + \sum_{k: h_i + h_j - h_k \geq 1} C_{ijk} C_{i,j,k,n} \partial^{h_i + h_j - h_k + n} \Phi_k. \] (B.4)
When $n = 0$, the first term in (B.4) does not contain any derivatives. Each coefficient functions are introduced as follows:

$$
B_{i,j,n,r} = (-1)^r \frac{\Gamma(n+1)}{\Gamma(r+1)(n-r+1)} \frac{\Gamma(2h_i + n)}{\Gamma(2h_i + 2h_j + 2n - r - 1)} \frac{\Gamma(r+1)\Gamma(2h_i + 2h_j + 2n - r - 1)}{\Gamma(2h_i + 2h_j + 2n - 1)} ,
$$

$$
C_{i,j,k,n} = (-1)^n \frac{\Gamma(h_i + h_j - h_k + n)}{\Gamma(h_i + h_j - h_k)(n+1)} \frac{\Gamma(h_i + h_j + 2h_j + n - 1)}{\Gamma(h_i + h_j + 2h_j + 2n - 1)} \frac{\Gamma(n+1)}{\Gamma(2h_i + n)} \frac{\Gamma(h_i + h_j - h_k)(n+1)}{\Gamma(h_i + h_j + h_k - 1)} \frac{1}{(h_i + h_j + h_k + n - 1)} \times \frac{\Gamma(h_i + h_j - h_k + n + 1)}{\Gamma(h_i - h_j + h_k)} \frac{\Gamma(h_i + h_j + h_k - 1)}{\Gamma(h_i + h_j + h_k)} .
$$

(B.5)

Their binomial symbols can be rewritten simply as Gamma functions. Although $B_{i,j,n,r}$ and $C_{i,j,k,n}$ can be simplified further, they maintain their present form. The structure constant, $C_{ijk}$, is the same as that in (1.3) where fields $\Phi_i(z)$ and $\Phi_j(w)$ appear in the first quadratic part in (B.4). This suggests that the complete structure of quasi-primary field is determined only if the OPE $\Phi_i(z) \Phi_j(w)$ is known because the second term contains the above structure constant $C_{ijk}$. Otherwise, this structure constant is not known and this unknown structure constant is fixed by the Jacobi identities when the Jacobi identities mentioned before are used.

Note that their normal ordered product used in this paper is different from the one in reference [5]. In other words [5],

$$
N(\Phi_j, \partial^{n-r} \Phi_i)(z) = (\partial^{n-r} \Phi_i \Phi_j)(z).
$$

(B.6)

Once formula (B.4) is used, the convention (B.6) should be applied to the expression of the quasi-primary field. The quasi-primary field $\left(TG - \frac{4}{3}T\partial G + \frac{4}{15}\partial^3 G\right)(z)$ can be obtained from (B.1). How can this be observed from (B.4)? Consider $\Phi_i(z) = T(z), \Phi_j(z) = G(z)$. $T(z) G(w) = \frac{1}{(z-w)^2}G(w) + \frac{1}{(z-w)^3}\partial G(w) + \cdots$. From this, the field, $\Phi_k(z) = G(z)$, with $C_{ijk} = \frac{3}{2}$. From (B.5), $B_{2,\frac{4}{3},1,0} = 1, B_{2,\frac{4}{3},1,1} = -\frac{4}{7}$ and $C_{2,\frac{4}{3},\frac{4}{3},1} = \frac{8}{105}$. $\frac{3}{7}N(TG\partial T)(z) - \frac{4}{7}N(\partial GT)(z) + \frac{4}{35}\partial^3 G(z)$ can be obtained by substituting these numerical values in (B.4). On the other hand, from (B.6), $\frac{3}{7} \left(\partial TG - \frac{4}{3}T\partial G + \frac{4}{15}\partial^3 G\right)(z)$. Furthermore, $\frac{3}{7} \left(G\partial T - \frac{4}{3}T\partial G + \frac{4}{15}\partial^3 G\right)(z)$, which is proportional to the quasi-primary field can be obtained using the relation $G\partial T(z) = \partial TG(z)$. Similar analysis and checks can be performed to determine if all the quasi-primary fields appearing in this paper can be read off from the general formula.

All the quasi-primary fields can be checked using (B.4). The relevant coefficient functions that appear in the quasi-primary fields in (B.1) are listed as follows:

$$
B_{2,\frac{4}{3},1,0} = 1, \quad B_{2,\frac{4}{3},1,1} = -\frac{4}{7}, \quad B_{\frac{4}{3},\frac{4}{3},1,0} = 1, \quad B_{\frac{4}{3},\frac{4}{3},1,1} = -\frac{5}{8}.
$$

52
\begin{align*}
B_{\frac{7}{3},\frac{4}{3},1,0} & = 1, & B_{\frac{7}{3},\frac{4}{3},1,1} & = -\frac{7}{10}, & B_{\frac{7}{3},2,1,0} & = 1, & B_{\frac{7}{3},2,1,1} & = -\frac{7}{11},
B_{\frac{4}{3},\frac{2}{3},1,0} & = 1, & B_{\frac{4}{3},\frac{2}{3},1,1} & = -\frac{8}{11}, & B_{\frac{4}{3},2,1,0} & = 1, & B_{\frac{4}{3},2,1,1} & = -\frac{5}{9},
B_{\frac{3}{3},\frac{1}{3},1,0} & = 1, & B_{\frac{3}{3},\frac{1}{3},1,1} & = -\frac{2}{3}, & B_{\frac{3}{3},2,0,0} & = 1, & B_{\frac{3}{3},\frac{2}{3},2,1} & = -\frac{6}{5},
B_{\frac{2}{3},\frac{1}{3},2,2} & = \frac{1}{3}, & B_{\frac{2}{3},2,1,0} & = 1, & B_{3,2,1,1} & = -\frac{3}{5}, & B_{\frac{2}{3},\frac{2}{3},0,0} & = 1,
B_{\frac{1}{3},3,0,0} & = 1, & B_{\frac{1}{3},4,0,0} & = 1, & B_{\frac{1}{3},\frac{3}{3},0,0} & = 1, & B_{\frac{1}{3},\frac{2}{3},0,0} & = 1, \tag{B.7}
\end{align*}
and
\begin{align*}
C_{\frac{2}{3},\frac{2}{3},1} & = \frac{8}{105}, & C_{\frac{2}{3},\frac{3}{3},1} & = \frac{5}{28}, & C_{\frac{2}{3},\frac{4}{3},1} & = \frac{7}{30}, & C_{\frac{2}{3},\frac{5}{3},1} & = \frac{10}{99},
C_{\frac{4}{3},\frac{2}{3},1} & = \frac{16}{99}, & C_{\frac{4}{3},\frac{3}{3},1} & = \frac{14}{55}, & C_{\frac{4}{3},\frac{4}{3},1} & = \frac{4}{63}, & C_{\frac{4}{3},\frac{5}{3},1} & = \frac{8}{63},
C_{\frac{5}{3},\frac{1}{3},2} & = -\frac{1}{18}, & C_{\frac{5}{3},2,3,1} & = \frac{1}{12}, & C_{\frac{5}{3},\frac{3}{3},3,0} & = -\frac{1}{3}, & C_{\frac{5}{3},\frac{4}{3},3,0} & = -\frac{1}{30},
C_{\frac{6}{3},\frac{2}{3},0} & = -\frac{1}{56}, & C_{\frac{6}{3},\frac{4}{3},0} & = -\frac{2}{9}, & C_{\frac{6}{3},\frac{5}{3},0} & = -\frac{1}{4}, & C_{\frac{6}{3},\frac{7}{3},0} & = -\frac{1}{72}. \tag{B.8}
\end{align*}

The previous relation \((B.3)\) can be obtained because \(B_{\frac{2}{3},i,0,0} = 1, C_{2,2,i,0} = -\frac{3}{2h_{i}(2h_{i}+1)}\) and \(C_{2ii} = h_{i}\) (the second-order pole of OPE \(\Phi_{i}(z) T(w)\) is given by \(\frac{1}{(z-w)^{2}} h_{i} \Phi_{i}(w)\) and there is also first-order singular term).

What happens when the spin-\(\frac{3}{2}\) current \(G(z)\) is combined with any primary field \(\Phi_{i}(z)\) of spin-\(h_{i}\)? As done in \((B.7)\) and \((B.8)\), \(B_{\frac{2}{3},i,0,0} = 1, C_{\frac{2}{3},i,i-\frac{1}{2},0} = -\frac{1}{4h_{i}(h_{i}-\frac{1}{2})}\) and \(C_{2,2,i-\frac{1}{2}} = 2(h_{i}-\frac{1}{2})\) (when the second-order pole of OPE \(\Phi_{i}(z) G(w)\) is given by \(\frac{1}{(z-w)^{2}} \Phi_{i}(z) \Phi_{i-\frac{1}{2}}(w)\) and there exists a first-order singular term), which is similar to \((B.3)\). Furthermore, when the first-order pole of OPE \(\Phi_{i}(z) G(w)\) is given by \(\frac{1}{(z-w)^{2}} \Phi_{i}(z) \Phi_{i-\frac{1}{2}}(w)\), \(C_{2,2,i-\frac{1}{2}} = 1\) with \(B_{\frac{2}{3},i,0,0} = 1\) and \(C_{\frac{2}{3},i,0,0} = -\frac{1}{h_{i}(h_{i}+\frac{1}{2})}\). From these two cases, the following can be derived
\begin{equation}
\left( G \Phi_{i} - \frac{1}{2h_{i}} \partial^{2} \Phi_{i-\frac{1}{2}} \right) (z), \quad \text{or} \quad \left( G \Phi_{i} - \frac{1}{(h_{i}+\frac{1}{2})} \partial \Phi_{i+\frac{1}{2}} \right) (z). \tag{B.9}
\end{equation}
Therefore, all the quasi-primary fields containing \(T(z)\) or \(G(z)\) in this paper can be classified by \((B.3)\) and \((B.9)\).

The mixed form of \((B.9)\) should be used for the case, \(\Phi_{i}(z) = P_{i}^{uu}(z)\) or \(\Phi_{i}(z) = P_{i}^{ww}(z)\). Note that the explicit OPEs are given in footnote \(^{10}\) The original expression \((B.5)\) should be used when the field, \(\Phi_{i}(z)\) or \(\Phi_{j}(w)\), does not contain \(T(z)\) or \(G(z)\). This will occur when the OPEs between the higher spin currents not considered in this paper are calculated.
Appendix C  The OPE between the spin-$\frac{3}{2}$ current and the quasiprimary or primary field fields in $c = 4$ model

For the correct superpartner in the given superfield, it is important to know the OPE between the spin-$\frac{3}{2}$ current and arbitrary quasi-primary fields

\[
G(z) \left( TT - \frac{3}{10} \partial^2 T \right)(w) = \frac{1}{(z-w)^4} \left[ \frac{51}{20} G(w) + \frac{1}{(z-w)^3} \left( -\frac{1}{3} \frac{51}{20} \partial G(w) + \mathcal{O}((z-w)^{-2}) \right) \right],
\]

\[
G(z) \left( GT - \frac{1}{8} \partial^2 G \right)(w) = \frac{1}{(z-w)^3} \left[ \frac{37}{6} T(w) + \mathcal{O}((z-w)^{-2}) \right],
\]

\[
G(z) \left( G \partial T - \frac{4}{3} T \partial G + \frac{4}{15} \partial^3 G \right)(w) = -\frac{1}{(z-w)^4} \left[ \frac{44}{5} T(w) \right] - \frac{1}{(z-w)^3} \left( -\frac{1}{3} \frac{44}{5} \partial T(w) + \mathcal{O}((z-w)^{-2}) \right),
\]

\[
G(z) \left( GU - \frac{\sqrt{6}}{3} \partial W \right)(w) = \frac{1}{(z-w)^3} \left[ \frac{13}{3} U(w) + \mathcal{O}((z-w)^{-2}) \right],
\]

\[
G(z) \left( TW - \frac{3}{14} \partial^2 W \right)(w) = \frac{1}{(z-w)^4} \left[ \frac{155}{14 \sqrt{6}} U(w) + \frac{1}{(z-w)^3} \frac{155}{14 \sqrt{6}} \partial U(w) + \mathcal{O}((z-w)^{-2}) \right],
\]

\[
G(z) \left( G \partial U - \frac{5}{3} \partial GU - \frac{\sqrt{6}}{7} \partial^2 W \right)(w) = -\frac{1}{(z-w)^4} \left[ \frac{335}{21} U(w) \right] - \frac{1}{(z-w)^3} \left( -\frac{1}{5} \frac{335}{21} \partial U(w) + \mathcal{O}((z-w)^{-2}) \right),
\]

\[
G(z) \left( GW - \frac{1}{6 \sqrt{6}} \partial^2 U \right)(w) = \frac{1}{(z-w)^3} \left[ \frac{25}{3} W(w) + \mathcal{O}((z-w)^{-2}) \right],
\]

\[
G(z) \left( TU - \frac{1}{4} \partial^2 U \right)(w) = \frac{1}{(z-w)^3} \left[ 2 \sqrt{6} W(w) + \mathcal{O}((z-w)^{-2}) \right],
\]

\[
G(z) \left( TO_\phi - \frac{3}{16} \partial^2 O_\phi \right)(w) = \frac{1}{(z-w)^3} \left[ \frac{17}{8 \sqrt{6}} P_{uu} + \frac{17}{4} \sqrt{\frac{3}{2}} P_{ww} \right] (w) + \mathcal{O}((z-w)^{-2}),
\]

\[
G(z) \left( GP_{uu} - \frac{4 \sqrt{6}}{9} \partial O_\phi - \frac{\sqrt{6}}{56} \partial^2 O_\phi \right)(w) = \frac{1}{(z-w)^3} \left[ \frac{239}{36} P_{uu} + \frac{17}{6} P_{ww} \right] (w) + \mathcal{O}((z-w)^{-2}),
\]

\[
G(z) \left( GP_{ww} - \frac{2}{9} \sqrt{\frac{7}{3}} \partial O_\phi - \frac{1}{14} \sqrt{\frac{2}{3}} \partial^2 O_\phi \right)(w) =
\]

54
\[
\frac{1}{(z - w)^3} \left[ -\frac{17}{27} P_{uu}^w + \frac{98}{9} P_{ww}^w \right] (w) + O((z - w)^{-2}),
\]
\[
G(z) \left( GO_z^w + \frac{3}{4 \sqrt{6}} \partial P_{uu}^z - \frac{\sqrt{6}}{4} \partial P_{ww}^z \right) (w) = \frac{1}{(z - w)^3} \frac{37}{6} O_z^w (w) + O((z - w)^{-2}),
\]
\[
G(z) \left( TP_{uu}^z - \frac{1}{6} \partial^2 P_{uu}^z \right) (w) = \frac{1}{(z - w)^4} \frac{5 \sqrt{3}}{2} O_z^w (w)
+ \frac{1}{(z - w)^3} \left[ \left( -\frac{1}{7} \right) 5 \sqrt{3} \partial O_z^w + \frac{13}{3 \sqrt{6}} \partial O_z^w \right] (w) + O((z - w)^{-2}),
\]
\[
G(z) \left( TP_{ww}^z - \frac{1}{6} \partial^2 P_{ww}^z \right) (w) = \frac{1}{(z - w)^4} \frac{10 \sqrt{3}}{2} O_z^w (w)
+ \frac{1}{(z - w)^3} \left[ \left( -\frac{1}{7} \right) 10 \sqrt{3} \partial O_z^z + \frac{13}{3 \sqrt{6}} \partial O_z^z \right] (w) + O((z - w)^{-2}),
\]
\[
G(z) \left( G \partial O_z^z - \frac{7}{3} G \partial G O_z^z + \frac{1}{9 \sqrt{6}} \partial^2 P_{uu}^z - \frac{1}{3 \sqrt{3}} \partial^2 P_{ww}^z \right) (w) =
- \frac{1}{(z - w)^4} \frac{77}{3} O_z^z (w) - \frac{1}{(z - w)^3} \left( -\frac{1}{7} \right) 77 \partial O_z^z (w) + O((z - w)^{-2}),
\]
\[
G(z) \left( GO_z^y - \frac{2}{63} \sqrt{3} \partial^2 P_{uu}^z + \frac{1}{21 \sqrt{6}} \partial^2 P_{ww}^z \right) (w) = \frac{1}{(z - w)^4} \frac{103}{9} O_z^z (w) + O((z - w)^{-2}),
\]
\[
G(z) \left( TO_z^y - \frac{3}{20} \partial^3 O_z^z \right) (w) = \frac{1}{(z - w)^4} \left[ \frac{208}{3 \sqrt{6}} - \frac{52}{35} \partial P_{uu}^z - \frac{25}{6 \sqrt{6}} P_{ww}^z \right] (w)
+ \frac{1}{(z - w)^3} \left( -\frac{1}{8} \right) \partial \left[ \frac{208}{3 \sqrt{6}} P_{uu}^z - \frac{52}{35} \partial P_{ww}^z \right] (w) + O((z - w)^{-2}),
\]
\[
G(z) \left( T \partial O_z^z - \frac{7}{4} \partial T O_z^z - \frac{1}{9} \partial^3 O_z^z \right) (w) = \frac{1}{(z - w)^4} \left[ \frac{25}{6 \sqrt{6}} P_{uu}^z - \frac{25}{6 \sqrt{6}} P_{ww}^z \right] (w)
+ \frac{1}{(z - w)^3} \left[ \left( -\frac{1}{8} \right) \frac{25}{6 \sqrt{6}} \partial P_{uu}^z - \left( -\frac{1}{8} \right) \frac{25}{6 \sqrt{6}} \partial P_{ww}^z \right.
\]
\[
\left. - \frac{21}{4} \left( GO_z^z + \frac{1}{4 \sqrt{6}} \partial P_{uu}^z - \frac{\sqrt{6}}{4} \partial P_{ww}^z \right) \right] (w) + O((z - w)^{-2}),
\]
\[
G(z) \left( G \partial P_{uu}^z - \frac{8}{3} G \partial G P_{uu}^z - \frac{\sqrt{6}}{5} \partial^2 O_z^z - \frac{2}{63} \sqrt{3} \partial^3 O_z^z \right) (w) =
\frac{1}{(z - w)^4} \left[ \frac{1444}{45} P_{uu}^z + \frac{56}{15} P_{ww}^z \right] (w)
+ \frac{1}{(z - w)^3} \left[ \left( -\frac{1}{8} \right) \frac{1444}{45} \partial P_{uu}^z + \left( -\frac{1}{8} \right) \frac{56}{15} \partial P_{ww}^z \right.
\]
\[
- \frac{2 \sqrt{6}}{4} \left( GO_z^z + \frac{1}{4 \sqrt{6}} \partial P_{uu}^z - \frac{\sqrt{6}}{4} \partial P_{ww}^z \right) \right] (w) + O((z - w)^{-2}),
\]
\[
G(z) \left( G \partial P_{\mu \nu} - \frac{8}{3} \partial G P_{\mu \nu} - \frac{1}{15} \partial^2 O_{\frac{3}{2}} - \frac{8}{189} \sqrt{\frac{2}{3}} \partial^3 O_{\frac{3}{2}} \right) (w) = \]
\[
\frac{1}{(z - w)^4} \left[ -\frac{112}{135} P_{\mu \nu} - \frac{112}{45} P_{\mu \nu} \right] (w) + \frac{1}{(z - w)^3} \left[ -\left( -\frac{1}{8} \right) \frac{112}{135} \partial P_{\mu \nu} - \left( -\frac{1}{8} \right) \frac{1192}{45} \partial P_{\mu \nu} \right.
\]
\[
- \frac{8}{3} \left( G O_{\frac{3}{2}} + \frac{1}{4 \sqrt{6}} \partial^2 P_{\mu \nu} - \frac{\sqrt{6}}{4} \partial^2 P_{\mu \nu} \right) \right] (w) + \mathcal{O}((z - w)^{-2}),
\]
\[
G(z) \left( T \partial U - \frac{5}{4} T \partial U - \frac{1}{7} \partial^3 U \right) (w) = - \frac{1}{(z - w)^4} \frac{33}{7} \frac{\sqrt{3}}{2} W(w)
\]
\[
+ \frac{1}{(z - w)^3} \left[ -\left( -\frac{1}{6} \right) \frac{33}{7} \frac{\sqrt{3}}{2} \partial W - \frac{15}{4} \left( G U - \frac{\sqrt{6}}{3} \partial W \right) \right] (w) + \mathcal{O}((z - w)^{-2}),
\]
\[
G(z) \left( G \partial W - 2 \partial G W - \frac{1}{21} \sqrt{\frac{2}{3}} \partial^3 U \right) (w) = - \frac{1}{(z - w)^4} \frac{116}{7} W(w)
\]
\[
+ \frac{1}{(z - w)^3} \left[ -\left( -\frac{1}{6} \right) \frac{116}{7} \partial W - \frac{5}{\sqrt{3}} \left( G U - \frac{\sqrt{6}}{3} \partial W \right) \right] (w) + \mathcal{O}((z - w)^{-2}),
\]
\[
G(z) \left( GO_{\nu} - \frac{2}{9} \partial O_{\frac{5}{2}} \right) (w) = \frac{1}{(z - w)^3} \frac{64}{9} O_{\nu} (w) + \mathcal{O}((z - w)^{-2}),
\]
\[
G(z) \left( G \partial^2 U - 4 \partial G \partial U + \frac{5}{2} \partial^2 G U - \frac{1}{2 \sqrt{6}} \partial^3 W \right) (w) = \frac{1}{(z - w)^5} 85 U(w)
\]
\[
+ \frac{1}{(z - w)^4} \left( -\frac{2}{5} \right) 85 \partial U(w)
\]
\[
+ \frac{1}{(z - w)^3} \left[ \frac{1}{30} 85 \partial^2 U - 2 \sqrt{6} \left( G W - \frac{1}{6 \sqrt{6}} \partial^2 U \right) + 10 \left( TU - \frac{1}{4} \partial^2 U \right) \right] (w) + \mathcal{O}((z - w)^{-2}),
\]
\[
G(z) \left( T \partial W - \frac{3}{2} \partial T W - \frac{1}{8} \partial^3 W \right) (w) = - \frac{1}{(z - w)^6} \frac{5}{2} \frac{\sqrt{3}}{2} U(w) - \frac{1}{(z - w)^4} \left( \frac{2}{5} \right) \frac{5}{2} \frac{\sqrt{3}}{2} \partial U(w)
\]
\[
+ \frac{1}{(z - w)^3} \left[ -\frac{1}{30} \frac{5}{2} \frac{\sqrt{3}}{2} \partial^2 U - \frac{9}{2} \left( G W - \frac{1}{6 \sqrt{6}} \partial^2 U \right) + 5 \frac{2}{3} \left( TU - \frac{1}{4} \partial^2 U \right) \right] (w)
\]
\[
+ \mathcal{O}((z - w)^{-2}),
\]
\[
G(z) \left( TO_{\nu} - \frac{1}{6} \partial^2 O_{\nu} \right) (w) = \frac{1}{(z - w)^3} \frac{13}{6} O_{\nu}(w) + \mathcal{O}((z - w)^{-2}),
\]
\[
G(z) \left( GO_{\nu} - \frac{1}{9} \partial^2 O_{\nu} \right) (w) = \frac{1}{(z - w)^3} \frac{103}{9} O_{\nu}(w) + \mathcal{O}((z - w)^{-2}).
\]

(C.1)

Using the definition of the coefficients (A.1), those vanishing and nonvanishing coefficients appearing in (C.1) can be checked as follows:

\[
A_{\frac{1}{2},\frac{1}{2},\frac{1}{2},1} = -\frac{1}{3}, \quad A_{\frac{1}{2},\frac{1}{2},\frac{1}{2},2} = 0, \quad A_{\frac{1}{2},\frac{1}{2},2,1} = -\frac{1}{4}, \quad A_{\frac{3}{2},\frac{1}{2},\frac{1}{2},1} = -\frac{1}{5},
\]

56
\[ A_{\frac{1}{2}, \frac{11}{2}, 3, 1} = -\frac{1}{6}, \quad A_{\frac{3}{2}, 6, \frac{7}{2}, 1} = -\frac{2}{5}, \quad A_{\frac{5}{2}, 6, \frac{9}{2}, 2} = \frac{1}{30}, \quad A_{\frac{7}{2}, 6, \frac{11}{2}, 1} = -\frac{1}{7}, \]

\[ A_{\frac{9}{2}, \frac{13}{2}, 4, 1} = -\frac{1}{8}. \]

(C.2)

For example, the first OPE in (C.1) has a third-order singular term with a coefficient \(-\frac{1}{3}\), which coincides with the value, \( A_{\frac{1}{2}, 4, \frac{7}{2}, 1} = -\frac{1}{3} \).

Appendix D  The OPE between the stress energy tensor and the quasi-primary or primary fields in the \( c < 4 \) coset model

In Appendices B and C, the central charge \( c \) was fixed to \( c = 4 \). Now this section considers the OPEs for general central charge \( c < 4 \). The OPEs are given as follows:

\[
T(z) \left( T T - \frac{3}{10} \partial^2 T \right)(w) = \frac{1}{(z-w)^4} \left[ \frac{6(66 + 63k + 7k^2)}{5(3+k)(6+k)} \right] T(w) + \mathcal{O}((z-w)^{-2}),
\]

\[
G \partial T - \frac{1}{8} \partial^2 G \right)(w) = \frac{1}{(z-w)^4} \left[ \frac{378 + 333k + 37k^2}{8(3+k)(6+k)} \right] G(w) + \mathcal{O}((z-w)^{-2}),
\]

\[
T(z) \left( T \partial G + \frac{4}{15} \partial^3 G \right)(w) = \frac{1}{(z-w)^5} \left[ \frac{3(-42 + 99k + 11k^2)}{5(3+k)(6+k)} \right] G(w) + \mathcal{O}((z-w)^{-2}),
\]

\[
+ \frac{1}{(z-w)^4} \left[ \frac{3(-42 + 99k + 11k^2)}{5(3+k)(6+k)} \right] \partial G(w) + \mathcal{O}((z-w)^{-2}),
\]

\[
T(z) \left( G U - \frac{\sqrt{2}}{3} \partial W \right)(w) = \mathcal{O}((z-w)^{-2}),
\]

\[
T(z) \left( T W - \frac{3}{14} \partial^2 W \right)(w) = \frac{1}{(z-w)^4} \left[ \frac{1026 + 639k + 71k^2}{7(3+k)(6+k)} \right] W(w) + \mathcal{O}((z-w)^{-2}),
\]

\[
T(z) \left( G \partial U - \frac{5}{3} \partial G U - \frac{\sqrt{2}}{7} \partial^2 W \right)(w) = -\frac{1}{(z-w)^4} \frac{124}{7} \sqrt{\frac{2}{3}} W(w) + \mathcal{O}((z-w)^{-2}),
\]

\[
T(z) \left( G W - \frac{1}{6\sqrt{6}} \partial^2 U \right)(w) = \frac{1}{(z-w)^4} \frac{1}{5} \sqrt{\frac{2}{3}} U(w) + \mathcal{O}((z-w)^{-2}),
\]

\[
T(z) \left( T U - \frac{1}{4} \partial^2 U \right)(w) = \frac{1}{(z-w)^4} \left[ \frac{3(150 + 99k + 11k^2)}{4(3+k)(6+k)} \right] U(w) + \mathcal{O}((z-w)^{-2}),
\]

\[
T(z) \left( T O_{\frac{3}{2}}^z - \frac{3}{16} \partial^2 O_{\frac{3}{2}}^z \right)(w) = \frac{1}{(z-w)^4} \left[ \frac{2898 + 1737k + 193k^2}{16(3+k)(6+k)} \right] O_{\frac{3}{2}}^z(w) + \mathcal{O}((z-w)^{-2}),
\]

\[
T(z) \left( GP_{uu}^r - \frac{4\sqrt{6}}{9} \partial O_{\frac{3}{2}} - \frac{\sqrt{6}}{56} \partial^2 O_{\frac{3}{2}} \right)(w) = \frac{1}{(z-w)^4} \frac{17}{4} \sqrt{\frac{3}{2}} O_{\frac{3}{2}}^z(w) + \mathcal{O}((z-w)^{-2}),
\]
\[
T(z) \left( GP_{wv}^w - \frac{2}{9} \sqrt{\frac{2}{3}} \partial O_{\frac{w}{3}} - \frac{1}{14} \sqrt{\frac{2}{3}} \partial^2 O_{\frac{w}{3}} \right) (w) = \frac{1}{(z-w)^4} \frac{17}{\sqrt{6}} O_{\frac{w}{3}}(w) + \mathcal{O}((z-w)^{-2}),
\]

\[
T(z) \left( GO_{\frac{w}{3}} + \frac{1}{4\sqrt{6}} \partial P_{wv}^{uu} - \sqrt{\frac{7}{3}} \partial P_{wv}^{wv} \right) (w) = +\mathcal{O}((z-w)^{-2}),
\]

\[
T(z) \left( TP_{wv}^{uu} - \frac{1}{6} \partial^2 P_{wv}^{wv} \right) (w) = \frac{1}{(z-w)^4} \left[ \frac{2(108 + 63k + 7k^2)}{(3+k)(6+k)} \right] P_{wv}^{uu}(w) + \mathcal{O}((z-w)^{-2}),
\]

\[
T(z) \left( TP_{wv}^{wv} - \frac{1}{6} \partial^2 P_{wv}^{wv} \right) (w) = \frac{1}{(z-w)^4} \left[ \frac{2(108 + 63k + 7k^2)}{(3+k)(6+k)} \right] P_{wv}^{wv}(w) + \mathcal{O}((z-w)^{-2}),
\]

\[
T(z) \left( G\partial O_{\frac{w}{3}} - \frac{7}{4} \partial GO_{\frac{w}{3}} + \frac{1}{9\sqrt{6}} \partial^2 P_{wv}^{wu} - \frac{1}{3} \sqrt{\frac{2}{3}} \partial^2 P_{wv}^{wv} \right) (w) =
\]

\[
\frac{1}{(z-w)^4} \left[ \frac{25}{3\sqrt{6}} P_{wv}^{uu} - 25 \sqrt{\frac{2}{3}} P_{wv}^{wv} \right] (w) + \mathcal{O}((z-w)^{-2}),
\]

\[
T(z) \left( GO_{\frac{w}{3}} - \frac{2}{63} \sqrt{\frac{2}{3}} \partial^2 P_{wv}^{wu} + \frac{1}{21\sqrt{6}} \partial^2 P_{wv}^{wv} \right) (w) =
\]

\[
\frac{1}{(z-w)^4} \left[ \frac{104}{21} \sqrt{\frac{2}{3}} P_{wv}^{uu} - \frac{26}{7} \sqrt{\frac{2}{3}} P_{wv}^{wv} \right] (w) + \mathcal{O}((z-w)^{-2}),
\]

\[
T(z) \left( TO_{\frac{w}{3}} - \frac{3}{20} \partial^2 O_{\frac{w}{3}} \right) (w) = \frac{1}{(z-w)^4} \left[ \frac{(5022 + 2871k + 319k^2)}{20(3+k)(6+k)} \right] O_{\frac{w}{3}}(w) + \mathcal{O}((z-w)^{-2}),
\]

\[
T(z) \left( T\partial O_{\frac{w}{3}} - \frac{7}{4} \partial TO_{\frac{w}{3}} - \frac{1}{9} \partial^3 O_{\frac{w}{3}} \right) (w) = -\frac{1}{(z-w)^5} \left[ \frac{7(342 + 387k + 43k^2)}{12(3+k)(6+k)} \right] O_{\frac{w}{3}}(w)
\]

\[
-\frac{1}{(z-w)^4} \left[ \frac{1}{7} \right] \left[ \frac{7(342 + 387k + 43k^2)}{12(3+k)(6+k)} \right] \partial O_{\frac{w}{3}}(w) + \mathcal{O}((z-w)^{-2}),
\]

\[
T(z) \left( G\partial P_{wv}^{wu} - \frac{8}{3} \partial GP_{wv}^{wu} - \sqrt{\frac{6}{5}} \partial^2 O_{\frac{w}{3}} - \frac{2}{63} \sqrt{\frac{2}{3}} \partial^3 O_{\frac{w}{3}} \right) (w) =
\]

\[
-\frac{1}{(z-w)^5} \frac{50}{3} \sqrt{\frac{2}{3}} O_{\frac{w}{3}}(w) + \frac{1}{(z-w)^4} \left[ \left[ \left( \frac{1}{7} \right) \frac{50}{3} \sqrt{\frac{2}{3}} \right] \partial O_{\frac{w}{3}} - \frac{286}{5} \sqrt{\frac{2}{3}} O_{\frac{w}{3}} \right] (w) + \mathcal{O}((z-w)^{-2}),
\]

\[
T(z) \left( G\partial P_{wv}^{wv} - \frac{8}{3} \partial GP_{wv}^{wv} - \frac{1}{5\sqrt{6}} \partial^2 O_{\frac{w}{3}} - \frac{8}{189} \sqrt{\frac{2}{3}} \partial^3 O_{\frac{w}{3}} \right) (w) =
\]

\[
-\frac{1}{(z-w)^5} \frac{200}{9} \sqrt{\frac{2}{3}} O_{\frac{w}{3}}(w) + \frac{1}{(z-w)^4} \left[ \left[ \left( \frac{1}{7} \right) \frac{200}{9} \sqrt{\frac{2}{3}} \right] \partial O_{\frac{w}{3}} - \frac{143}{15} \sqrt{\frac{2}{3}} O_{\frac{w}{3}} \right] (w) + \mathcal{O}((z-w)^{-2}),
\]

\[
T(z) \left( T\partial U - \frac{5}{4} \partial TU - \frac{1}{7} \partial^3 U \right) (w) = -\frac{1}{(z-w)^5} \left[ \frac{15(78 + 207k + 23k^2)}{28(3+k)(6+k)} \right] U(w)
\]

\[
-\frac{1}{(z-w)^4} \left[ \frac{1}{5} \right] \left[ \frac{15(78 + 207k + 23k^2)}{28(3+k)(6+k)} \right] \partial U(w) + \mathcal{O}((z-w)^{-2}),
\]

58
\[
T(z) \left( G\partial W - 2\partial GW - \frac{1}{21}\sqrt{\frac{2}{3}}\partial^2 U \right)(w) = -\frac{1}{(z-w)^5} \frac{55}{7} \sqrt{\frac{2}{3}} U(w)
\]
\[
-\frac{1}{(z-w)^4} \left( \frac{1}{5} \right) \partial U(w) + \mathcal{O}((z-w)^{-2}),
\]
\[
T(z) \left( GO_{\nu} - \frac{2}{9} \partial O_{\nu} \right)(w) = +\mathcal{O}((z-w)^{-2}),
\]
\[
T(z) \left( G\partial^2 U - 4\partial G\partial U + \frac{5}{2} \partial^2 GU - \frac{1}{2\sqrt{6}} \partial^3 W \right)(w) = \frac{1}{(z-w)^5} 9\sqrt{6} W(w)
\]
\[
+ \frac{1}{(z-w)^4} \left[ -3\sqrt{\frac{3}{2}} \partial W + 75 \left( GU - \frac{\sqrt{6}}{3} \partial W \right) \right](w) + \mathcal{O}((z-w)^{-2}),
\]
\[
T(z) \left( T\partial W - \frac{3}{2} \partial TW - \frac{1}{8} \partial^2 W \right)(w) = -\frac{1}{(z-w)^5} \left[ \frac{18(6+9k+k^2)}{(3+k)(6+k)} \right] W(w)
\]
\[
-\frac{1}{(z-w)^4} \left( -\frac{1}{6} \right) \frac{18(6+9k+k^2)}{(3+k)(6+k)} \partial W(w) + \mathcal{O}((z-w)^{-2}),
\]
\[
T(z) \left( TO_{\nu} - \frac{1}{6} \partial^2 O_{\nu} \right)(w) = \frac{1}{(z-w)^4} \left[ \frac{2(108+63k+7k^2)}{(3+k)(6+k)} \right] O_{\nu}(w) + \mathcal{O}((z-w)^{-2}),
\]
\[
T(z) \left( GO_{\nu} - \frac{1}{9} \partial^2 O_{\nu} \right)(w) = \frac{1}{(z-w)^4} \left[ \frac{52}{3} O_{\nu}(w) + \mathcal{O}((z-w)^{-2}). \right.
\]

(D.1)

Of course, after the \( k \rightarrow \infty \) limit, the above OPEs (D.1) become OPEs (B.1). The \( k \)-dependence in the quasi-primary fields that contain the \( T(w) \) can be derived because the OPE between \( T(z) \) and \( T(w) \) contains the central charge. On the other hand, the OPEs between the \( T(z) \) and quasi-primary fields which do not have \( T(w) \) in their expression are the same as those for the \( c = 4 \) model. In this case, the previous coefficients (B.2) hold.

**Appendix E** The OPE between the spin-\( \frac{3}{2} \) current and the quasi-primary or primary fields in the \( c < 4 \) coset model

Similarly, the general central charge and OPEs are given by

\[
G(z) \left( TT - \frac{3}{10} \partial^2 T \right)(w) = \frac{1}{(z-w)^4} \left[ \frac{51}{20} G(w) + \frac{1}{(z-w)^3} \left( \frac{1}{3} \right) \frac{51}{20} \partial G(w) + \mathcal{O}((z-w)^{-2}),
\]
\[
G(z) \left( GT - \frac{1}{8} \partial^2 G \right)(w) = \frac{1}{(z-w)^3} \left[ \frac{378 + 333k + 37k^2}{6(3+k)(6+k)} \right] T(w) + \mathcal{O}((z-w)^{-2}),
\]
\[
G(z) \left( G\partial T - \frac{4}{3} T\partial G + \frac{4}{15} \partial^2 G \right)(w) = -\frac{1}{(z-w)^4} \left[ \frac{4(-42 + 99k + 11k^2)}{5(3+k)(6+k)} \right] T(w)
\]

59
\[-\frac{1}{(z-w)^3} \left( -\frac{1}{4} \left[ \frac{4(-42 + 99k + 11k^2)}{5(3+k)(6+k)} \right] \partial T(w) + \mathcal{O}((z-w)^{-2}), \right.\]

\[G(z) \left( GU - \frac{\sqrt{6} \partial W}{3} \right)(w) = \frac{1}{(z-w)^3} \left[ \frac{90 + 117k + 13k^2}{3(3+k)(6+k)} \right] U(w) + \mathcal{O}((z-w)^{-2}),\]

\[G(z) \left( TW - \frac{3}{14} \partial^2 W \right)(w) = \frac{1}{(z-w)^4} \frac{155}{14\sqrt{6}} U(w) + \mathcal{O}((z-w)^{-2}),\]

\[G(z) \left( G\partial U - \frac{5}{3} \partial GU - \frac{\sqrt{6}}{7} \partial^2 W \right)(w) = \frac{1}{(z-w)^3} \left[ \frac{5(198 + 603k + 67k^2)}{21(3+k)(6+k)} \right] U(w) + \mathcal{O}((z-w)^{-2}),\]

\[G(z) \left( GW - \frac{1}{6\sqrt{6}} \partial^2 U \right)(w) = \frac{1}{(z-w)^3} \left[ \frac{306 + 225k + 25k^2}{3(3+k)(6+k)} \right] W(w) + \mathcal{O}((z-w)^{-2}),\]

\[G(z) \left( TU - \frac{1}{4} \partial^2 U \right)(w) = \frac{1}{(z-w)^3} 2\sqrt{6} W(w) + \mathcal{O}((z-w)^{-2}),\]

\[G(z) \left( TO_{\frac{1}{2}} - \frac{3}{8} \partial^2 O_{\frac{1}{2}} \right)(w) = \frac{1}{(z-w)^3} \left[ -\frac{17}{24\sqrt{6}} P_{4w}^{uu} + \frac{17}{4} \frac{\sqrt{3}}{2} P_{4w}^{uw} \right] (w) + \mathcal{O}((z-w)^{-2}),\]

\[G(z) \left( GP_{4w}^{uu} - \frac{4\sqrt{6}}{9} \partial O_{\frac{1}{2}} + \frac{\sqrt{6}}{56} \partial^2 O_{\frac{1}{2}} \right)(w) = \frac{1}{(z-w)^3} \left[ \frac{(2574 + 2151k + 239k^2)}{36(3+k)(6+k)} P_{4w}^{uu} + \frac{17}{6} P_{4w}^{uw} \right] (w) + \mathcal{O}((z-w)^{-2}),\]

\[G(z) \left( GP_{4w}^{uw} - \frac{2\sqrt{2}}{9} \partial O_{\frac{1}{2}} - \frac{1}{14} \frac{2\sqrt{3}}{3} \partial^2 O_{\frac{1}{2}} \right)(w) = \frac{1}{(z-w)^3} \left[ -\frac{17}{27} P_{4w}^{uw} + \frac{2(666 + 441k + 49k^2)}{9(3+k)(6+k)} P_{4w}^{uw} \right] (w) + \mathcal{O}((z-w)^{-2}),\]

\[G(z) \left( GO_{\frac{1}{2}} + \frac{4}{4\sqrt{6}} \partial P_{4w}^{uu} - \frac{\sqrt{6}}{4} \partial P_{4w}^{uw} \right)(w) = \frac{1}{(z-w)^3} \left[ \frac{378 + 333k + 37k^2}{6(3+k)(6+k)} \right] O_{\frac{1}{2}}(w) + \mathcal{O}((z-w)^{-2}),\]

\[G(z) \left( TP_{4w}^{uu} - \frac{1}{6} \partial^2 P_{4w}^{uu} \right)(w) = \frac{1}{(z-w)^4} \frac{\sqrt{3}}{2} O_{\frac{1}{2}}(w) + \mathcal{O}((z-w)^{-2}),\]

\[G(z) \left( TP_{4w}^{uw} - \frac{1}{6} \partial^2 P_{4w}^{uw} \right)(w) = \frac{1}{(z-w)^4} 10 \frac{\sqrt{3}}{2} O_{\frac{1}{2}}(w) + \mathcal{O}((z-w)^{-2}).\]
\[
\begin{align*}
G(z) & \left( \frac{7}{3} \partial \partial O_{\frac{1}{2}} z - \frac{1}{2} \partial ^{2} P_{uu}^{w} - \frac{1}{2} \partial ^{2} P_{ww}^{w} \right) (w) = \\
& - \frac{1}{(z-w)^4} \left[ \frac{7(54 + 99k + 11k^2)}{3(3+k)(6+k)} \right] \partial O_{\frac{1}{2}} (w) - \frac{1}{(z-w)^3} \left[ \frac{7(54 + 99k + 11k^2)}{3(3+k)(6+k)} \right] \partial O_{\frac{1}{2}} (w) \\
+ & \mathcal{O}((z-w)^{-2}), \\
G(z) & \left( \frac{1}{20} \partial ^{2} O_{\frac{1}{2}} z - \frac{1}{2} \partial O_{\frac{1}{2}} z \right) (w) = \frac{1}{(z-w)^4} \left[ \frac{208}{35} \frac{\sqrt{2}}{3} P_{uu}^{w} - \frac{52}{35} \frac{\sqrt{6}}{\sqrt{6}} P_{ww}^{w} \right] (w) \\
+ & \mathcal{O}((z-w)^{-2}), \\
G(z) & \left( \frac{7}{4} \partial O_{\frac{1}{2}} z - \frac{1}{9} \partial ^{3} O_{\frac{1}{2}} z \right) (w) = \frac{1}{(z-w)^4} \left[ \frac{25}{6\sqrt{6}} P_{uu}^{w} - \frac{25}{25} P_{ww}^{w} \right] (w) \\
+ & \mathcal{O}((z-w)^{-2}), \\
G(z) & \left( \frac{4}{3} \partial G P_{\frac{1}{2}} z - \frac{\sqrt{6}}{5} \partial ^{2} P_{uu}^{w} - \frac{2}{63} \frac{\sqrt{2}}{3} \partial ^{3} O_{\frac{1}{2}} z \right) (w) = \\
\begin{align*}
& + \frac{1}{(z-w)^4} \left[ \frac{4(2178 + 3249k + 361k^2)}{45(3+k)(6+k)} P_{uu}^{w} + \frac{56}{15} P_{ww}^{w} \right] (w) \\
+ & \frac{1}{(z-w)^3} \left[ \frac{1}{3} \frac{4(2178 + 3249k + 361k^2)}{45(3+k)(6+k)} \partial P_{uu}^{w} + \frac{1}{15} \frac{56}{15} \partial P_{ww}^{w} \\
- & 2\sqrt{6} \left( \frac{1}{4\sqrt{6}} \partial P_{uu}^{w} - \frac{\sqrt{6}}{4} \partial P_{ww}^{w} \right) \right] (w) + \mathcal{O}((z-w)^{-2}), \\
G(z) & \left( \frac{8}{3} \partial G P_{\frac{1}{2}} z - \frac{1}{5\sqrt{6}} \partial ^{2} O_{\frac{1}{2}} z \right) (w) = \\
\begin{align*}
& + \frac{1}{(z-w)^4} \left[ \frac{112}{135} P_{uu}^{w} - \frac{8(522 + 1341k + 149k^2)}{45(3+k)(6+k)} P_{ww}^{w} \right] (w) \\
+ & \frac{1}{(z-w)^3} \left[ \frac{1}{8} \frac{112}{135} \partial P_{uu}^{w} - \frac{1}{8} \frac{8(522 + 1341k + 149k^2)}{45(3+k)(6+k)} \partial P_{ww}^{w} \right] (w)
\end{align*}
\end{align*}
\]
Therefore, in (E.1), the \( k \)-dependence for the quasi-primary fields contain \( G(w) \), whereas the \( k \)-dependence is not observed in the remaining quasi-primary fields. This is obvious because the OPE between \( G(z) \) and \( G(w) \) has an explicit \( c \)-dependence from (2.2). Steps can be taken to check that the coefficients (C.2) hold in this case.
References

[1] M. R. Gaberdiel and R. Gopakumar, “An $AdS_3$ Dual for Minimal Model CFTs,” Phys. Rev. D 83, 066007 (2011) [arXiv:1011.2986 [hep-th]].

[2] M. R. Gaberdiel and R. Gopakumar, “Triality in Minimal Model Holography,” JHEP 1207, 127 (2012) [arXiv:1205.2472 [hep-th]].

[3] I. R. Klebanov and A. M. Polyakov, “AdS dual of the critical O(N) vector model,” Phys. Lett. B 550, 213 (2002) [hep-th/0210114].

[4] C. Ahn, “The Large N ’t Hooft Limit of Coset Minimal Models,” JHEP 1110, 125 (2011) [arXiv:1106.0351 [hep-th]].

[5] M. R. Gaberdiel and C. Vollenweider, “Minimal Model Holography for SO(2N),” JHEP 1108, 104 (2011) [arXiv:1106.2634 [hep-th]].

[6] S. F. Prokushkin and M. A. Vasiliev, “Higher spin gauge interactions for massive matter fields in 3-D AdS space-time,” Nucl. Phys. B 545, 385 (1999) [hep-th/9806236].

[7] T. Creutzig, Y. Hikida and P. B. Ronne, “Higher spin $AdS_3$ supergravity and its dual CFT,” JHEP 1202, 109 (2012) [arXiv:1111.2139 [hep-th]].

[8] C. Candu and M. R. Gaberdiel, “Supersymmetric holography on $AdS_3$,” arXiv:1203.1939 [hep-th].

[9] M. Henneaux, G. Lucena Gomez, J. Park and S. -J. Rey, “Super- W(infinity) Asymptotic Symmetry of Higher-Spin $AdS_3$ Supergravity,” JHEP 1206, 037 (2012) [arXiv:1203.5152 [hep-th]].

[10] K. Hanaki and C. Peng, “Symmetries of Holographic Super-Minimal Models,” arXiv:1203.5768 [hep-th].

[11] C. Ahn, “The Large N ’t Hooft Limit of Kazama-Suzuki Model,” JHEP 1208, 047 (2012) [arXiv:1206.0054 [hep-th]].

[12] C. Candu and M. R. Gaberdiel, “Duality in N=2 minimal model holography,” arXiv:1207.6646 [hep-th].

[13] C. Ahn, “The Operator Product Expansion of the Lowest Higher Spin Current at Finite N,” JHEP 1301, 041 (2013) [arXiv:1208.0058 [hep-th]].
[14] T. Creutzig, Y. Hikida and P. B. Ronne, “N=1 supersymmetric higher spin holo-
graphy on AdS$_3$,” [arXiv:1209.5404] [hep-th].

[15] F. A. Bais, P. Bouwknegt, M. Surridge and K. Schoutens, “Coset Construction for Ex-
tended Virasoro Algebras,” Nucl. Phys. B 304, 371 (1988).

[16] C. Ahn, “The Coset Spin-4 Casimir Operator and Its Three-Point Functions with 
Scalars,” JHEP 1202, 027 (2012) [arXiv:1111.0091] [hep-th].

[17] M. R. Gaberdiel and R. Gopakumar, “Minimal Model Holography,” [arXiv:1207.6697] 
[hep-th].

[18] R. Gopakumar, A. Hashimoto, I. R. Klebanov, S. Sachdev and K. Schoutens, “Strange 
Metals in One Spatial Dimension,” Phys. Rev. D 86, 066003 (2012) [arXiv:1206.4719] 
[hep-th]].

[19] M. R. Douglas, “G / H Conformal Field Theory,” CALT-68-1453.

[20] K. Schoutens and A. Sevrin, “Minimal superW(N) algebras in coset conformal field 
theories,” Phys. Lett. B 258, 134 (1991).

[21] C. Ahn, K. Schoutens and A. Sevrin, “The full structure of the super W(3) algebra,” 
Int. J. Mod. Phys. A 6, 3467 (1991).

[22] K. Hornfeck and E. Ragoucy, “A Coset Construction For The Super W(3) Algebra,” 
Nucl. Phys. B 340, 225 (1990).

[23] T. Inami, Y. Matsuo and I. Yamanaka, “Extended Conformal Algebras With N=1 Su-
persymmetry,” Phys. Lett. B 215, 701 (1988).

[24] F. A. Bais, P. Bouwknegt, M. Surridge and K. Schoutens, “Extensions of the Virasoro 
Algebra Constructed from Kac-Moody Algebras Using Higher Order Casimir Invariants,” 
Nucl. Phys. B 304, 348 (1988).

[25] P. Bowcock, “Quasi-primary Fields And Associativity Of Chiral Algebras,” Nucl. Phys. 
B 356, 367 (1991).

[26] R. Blumenhagen, M. Flohr, A. Kliem, W. Nahm, A. Recknagel and R. Varnhagen, “W 
algebras with two and three generators,” Nucl. Phys. B 361, 255 (1991).

[27] P. Bouwknegt and K. Schoutens, “W symmetry in conformal field theory,” Phys. Rept. 
223, 183 (1993) [hep-th/9210010].
[28] W. Nahm, “Algebras of two-dimensional chiral fields and their classification,” In *Islamabad 1989, Proceedings, Mathematical physics* 283-300. (see Conference Index)

[29] W. Nahm, “Chiral algebras of two-dimensional chiral field theories and their normal ordered products,” In *Trieste 1989, Proceedings, Recent developments in conformal field theories* 81-84. (see Conference Index)

[30] M. Ammon, M. Gutperle, P. Kraus and E. Perlmutter, “Black holes in three dimensional higher spin gravity: A review,” arXiv:1208.5182 [hep-th].

[31] E. Perlmutter, T. Prochazka and J. Raeymaekers, “The semiclassical limit of $W_N$ CFTs and Vasiliev theory,” arXiv:1210.8452 [hep-th].

[32] J. R. David, M. Ferlaino and S. P. Kumar, “Thermodynamics of higher spin black holes in 3D,” arXiv:1210.0284 [hep-th].

[33] T. Creutzig and A. R. Linshaw, “The super $W_{1+\infty}$ algebra with integral central charge,” arXiv:1209.6032 [math.QA].

[34] B. Chen, J. Long and Y. -n. Wang, “Black holes in Truncated Higher Spin AdS$_3$ Gravity,” JHEP 1212, 052 (2012) [arXiv:1209.6185 [hep-th]].

[35] M. Taronna, “Higher-Spin Interactions: three-point functions and beyond,” arXiv:1209.5755 [hep-th].

[36] S. Banerjee, A. Castro, S. Hellerman, E. Hijano, A. Lepage-Jutier, A. Maloney and S. Shenker, “Smoothed Transitions in Higher Spin AdS Gravity,” arXiv:1209.5396 [hep-th].

[37] P. Kraus and E. Perlmutter, “Probing higher spin black holes,” arXiv:1209.4937 [hep-th].

[38] S. Lal and B. Sahoo, “Holographic Renormalisation for the Spin-3 Theory and the (A)dS3/CFT2 correspondence,” JHEP 1301, 004 (2013) [arXiv:1209.4804 [hep-th]].

[39] H. Afshar, M. Gary, D. Grumiller, R. Rashkov and M. Riegler, “Non-AdS holography in 3-dimensional higher spin gravity - General recipe and example,” JHEP 1211, 099 (2012) [arXiv:1209.2860 [hep-th]].

[40] I. Fujisawa and R. Nakayama, “Second-Order Formalism for 3D Spin-3 Gravity,” Class. Quant. Grav. 30, 035003 (2013) [arXiv:1209.0894 [hep-th]].
[41] S. Fredenhagen and C. Restuccia, “The geometry of the limit of N=2 minimal models,” J. Phys. A 46, 045402 (2013) [arXiv:1208.6136 [hep-th]].

[42] H. S. Tan, “Exploring Three-dimensional Higher-Spin Supergravity based on sl(N -N -1) Chern-Simons theories,” JHEP 1211, 063 (2012) [arXiv:1208.2277 [hep-th]].

[43] A. Campoleoni, S. Fredenhagen, S. Pfenninger and S. Theisen, “Towards metric-like higher-spin gauge theories in three dimensions,” [arXiv:1208.1851 [hep-th]].

[44] K. Thielemans, “A Mathematica package for computing operator product expansions,” Int. J. Mod. Phys. C 2, 787 (1991).

[45] D. Friedan, E. J. Martinec and S. H. Shenker, “Conformal Invariance, Supersymmetry and String Theory,” Nucl. Phys. B 271, 93 (1986).

[46] C. Ahn, “The Primary Spin-4 Casimir Operators in the Holographic SO(N) Coset Minimal Models,” JHEP 1205, 040 (2012) [arXiv:1202.0074 [hep-th]].

[47] M. R. Gaberdiel and T. Hartman, “Symmetries of Holographic Minimal Models,” JHEP 1105, 031 (2011) [arXiv:1101.2910 [hep-th]].

[48] M. R. Gaberdiel, R. Gopakumar, T. Hartman and S. Raju, “Partition Functions of Holographic Minimal Models,” JHEP 1108, 077 (2011) [arXiv:1106.1897 [hep-th]].

[49] P. Bowcock and G. M. T. Watts, “On the classification of quantum W algebras,” Nucl. Phys. B 379, 63 (1992) [hep-th/9111062].

[50] M. R. Gaberdiel and P. Suchanek, “Limits of Minimal Models and Continuous Orbifolds,” JHEP 1203, 104 (2012) [arXiv:1112.1708 [hep-th]].

[51] T. Creutzig, Y. Hikida and P. B. Ronne, “Three point functions in higher spin AdS\textsubscript{3} supergravity,” JHEP 1301, 171 (2013) [arXiv:1211.2237 [hep-th]].

[52] H. Moradi and K. Zoubos, “Three-Point Functions in N=2 Higher-Spin Holography,” arXiv:1211.2239 [hep-th].

[53] K. Thielemans, “An Algorithmic approach to operator product expansions, W algebras and W strings,” hep-th/9506159.

[54] R. Blumenhagen, W. Eholzer, A. Honecker, K. Hornfeck and R. Hubel, “Coset realization of unifying W algebras,” Int. J. Mod. Phys. A 10, 2367 (1995) [hep-th/9406203].