ANOTHER SUPER-IDENTITY EQUIVALENT TO THE HOM-MALCEV SUPER-IDENTITY

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Abstract. In a Hom-superalgebra a super-identity, equivalent to the Hom-Malcev super-identity, is found.

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1. Introduction

The theory of Hom-algebras originated from Hom-Lie algebras introduced by J.T. Hartwig, D. Larsson, and S.D. Silvestrov in [5] in the study of quasi-deformations of Lie algebras of vector fields, including q-deformations of Witt algebras and Virasoro algebras. As generalization of Hom-Lie algebras, Hom-Malcev algebras are introduced [13] to study Hom-alternative algebras [8]. A $\mathbb{Z}_2$-graded version of Hom-Lie (resp. Hom-Malcev) algebra called Hom-Lie (resp. Hom-Malcev) superalgebra are introduced [2] (resp. [10], [1]) as generalization of Lie (resp. Malcev) superalgebras. Recall that a Malcev superalgebra is a non-associative superalgebra $A$ with a super skewsymmetric multiplication $\cdot$ (i.e, $xy = -(1)^{\bar{x}\bar{y}}yx$) such that the Malcev super-identity

$$2tJ_A(x, y, z) = J_A(t, x, yz) + (-1)^{\bar{x}(\bar{y} + \bar{z})}J_A(t, y, zx)$$

(1)

$$(-1)^{\bar{z}(\bar{x} + \bar{y})}J_A(t, z, xy)$$

is satisfied for all homogeneous elements $x, y, z, t$ in the superspace $A$, where

$$J_A(x, y, z) = ((xy)z + (-1)^{\bar{x}(\bar{y} + \bar{z})}(yz)x + (1)^{\bar{z}(\bar{x} + \bar{y})}(zx)y$$

(2)

is the super-Jacobian. In particular, Lie superalgebras are examples of Malcev superalgebras. Malcev superalgebras play an important role in the geometry of smooth loops.

Some twisting of the Malcev super-identity (1) along any even algebra self-map $\alpha$ of $A$ gives rise to the notion of a Hom-Malcev superalgebra $(A, [\cdot], \alpha)$ ([1]; see definitions in section 2). Properties and constructions of Hom-Malcev superalgebras, as well as the relationships between these Hom-superalgebras and Hom-alternative or Hom-Jordan superalgebras are investigated in [1]. In particular, it is shown that a Malcev superalgebra can be twisted into a Hom-Malcev superalgebra and that Hom-alternative superalgebras are Hom-Malcev super-admissible. In [1], as for Malcev algebras (see [11], [12]), Hom-Malcev algebras [13], [6] and Malcev superalgebra[1], equivalent defining identities of a Hom-Malcev superalgebra are given. In this note, we mention another super identity in a Hom-Malcev superalgebra that is equivalent to the ones found in [1]. Specifically, we shall prove the following
Theorem 1.1. Let \((M = M_0 \oplus M_1, \cdot, \alpha)\) be a Hom-Malcev superalgebra. Then the super identity
\[
\tilde{J}_M(wx, \alpha(y), \alpha(z)) = \alpha^2(w)\tilde{J}_M(x, y, z) + (-1)^{\tilde{y}(\tilde{y}+\tilde{z})}\tilde{J}_M(w, y, z)\alpha^2(x) \\
-2(-1)^{(\tilde{y}+\tilde{z})(\tilde{z}+\tilde{w})}\tilde{J}_M(yz, \alpha(w), \alpha(x))
\] (3)
holds for all homogeneous elements \(w, x, y, z\) in \(M\).

Moreover, in any super anticommutative Hom-superalgebra \((M = M_0 \oplus M_1, \cdot, \alpha)\), the super identity (3) is equivalent to the Hom-Malcev super identity
\[
2\alpha^2(t)\tilde{J}_M(x, y, z) = \tilde{J}_M(\alpha(t), \alpha(x), yz) + (-1)^{\tilde{y}(\tilde{y}+\tilde{z})}\tilde{J}_M(\alpha(t), \alpha(y), zx)
\] (4)
\[
(-1)^{\tilde{z}(\tilde{z}+\tilde{y})}\tilde{J}_M(\alpha(t), \alpha(z), xy)
\]
or equivalently
\[
\tilde{J}_M(\alpha(y), \alpha(z), wx) + (-1)^{\tilde{y}(\tilde{y}+\tilde{z})}\tilde{J}_M(\alpha(w), \alpha(z), yx)
\] (5)
\[
= +(-1)^{\tilde{y}(\tilde{y}+\tilde{z})}\tilde{J}_M(y, z, x)\alpha^2(w) + (-1)^{(\tilde{y}+\tilde{z})(\tilde{z}+\tilde{w})}\tilde{J}_M(w, z, x)\alpha^2(y)
\]
for all homogeneous elements \(x, y, z, t\) in the superspace \(M\). (See [1] for equivalence between (4), (5) and another super identity).

Observe that when \(\alpha = Id\) (the identity map) in [1], then (4) is [1] i.e. the Hom-Malcev superalgebra \((M = M_0 \oplus M_1, \cdot, \alpha)\) reduces to the Malcev superalgebra \((M = M_0 \oplus M_1, \cdot)\) (see [1]). Also if all the homogeneous elements are in \(M_0\), (i.e are even) then (3) becomes
\[
\tilde{J}_M(wx, \alpha(y), \alpha(z)) = \alpha^2(w)\tilde{J}_M(x, y, z) + \tilde{J}_M(w, y, z)\alpha^2(x) \\
-2\tilde{J}_M(yz, \alpha(w), \alpha(x))
\]
which is an equivalent identity to Hom-Malcev identity [13] found in [6] i.e \((M = M_0 \oplus M_1, \cdot, \alpha)\) reduces to Hom-Malcev algebras \((M_0, \cdot, \alpha)\).

In section 2 some instrumental lemmas are proved. Some results in these lemmas are a kind of the \(\mathbb{Z}_2\)-graded version of similar results found in [6] in case of Hom-Malcev algebras. The section 3 is devoted to the proof of the theorem.

2. ON HOM-SUPERALGEBRAS

Throughout this paper, \(\mathbb{K}\) is an algebraically closed field of characteristic 0 and \(M\) is a linear super-espace over \(\mathbb{K}\). In this section we recall useful notions on Hom-Lie superalgebras [2] as well as the one of a Hom-Malcev superalgebra [11]. The main result of this section (Lemma 2.8) proves that the \(\mathbb{Z}_2\)-graded version of the identity (2.4) of Lemma 2.7 [6] which is the Hom-version of the identity (6) of [7] holds in any Hom-Malcev superalgebra.

Now let \(M\) be a linear superspace over \(\mathbb{K}\) that is a \(\mathbb{Z}_2\)-graded linear space with a direct sum \(M = M_0 \oplus M_1\). The element of \(M_j, j \in \mathbb{Z}_2\), are said to be homogeneous of parity \(j\). The parity of a homogeneous element \(x\) is denoted by \(\tilde{x}\). In the sequel, we will denote by \(\mathcal{H}(M)\) the set of all homogeneous elements of \(M\).

Definition 2.1. [1] A Hom-superalgebra is a triple \((M, \mu, \alpha)\) in which \(M\) is a \(\mathbb{K}\)-super-module, \(\mu : M \otimes \mathbb{K} \rightarrow M\) is an even bilinear map, and \(\alpha : M \rightarrow M\) is an even linear map such that \(\alpha \circ \mu = \mu \circ \alpha^{\otimes 2}\) (multiplicativity)

Definition 2.2. [2] Let \((M, \cdot, \alpha)\) be a Hom-superalgebra.

1. The Hom-super-Jacobiens is the trilinear map \(\tilde{J}_M\) defined as
\[
\tilde{J}_M(x, y, z) = (xy)\alpha(z) + (-1)^{(\tilde{y}+\tilde{z})}(yz)\alpha(x) + (-1)^{(\tilde{z}+\tilde{y})}(zx)\alpha(y)
\] (6)
for all $x, y, z \in \mathcal{H}(M)$.

(2) $(M, \cdot, \alpha)$ is called Hom-Lie superalgebra if “$\cdot$” is super anticommutative (i.e $xy = -(1)^{\bar{x} \bar{y}}yx$) and $\tilde{J}_M(x, y, z) = 0$ for all $x, y, z \in \mathcal{H}(M)$.

**Definition 2.3.** [1] A Hom-Malcev superalgebra is a super anticommutative Hom-superalgebra $(M, \cdot, \alpha)$ such that

\[
2\alpha^2(t) \tilde{J}_M(x, y, z) = \tilde{J}_M(\alpha(t), \alpha(x), yz) + (1)^{\bar{y} + \bar{z}} \tilde{J}_M(\alpha(t), \alpha(y), zx)
\]

\[
(1)^{\bar{z} + \bar{y}} \tilde{J}_M(\alpha(t), \alpha(z), xy)
\]

(7)

(see (4) above) for all $x, y, z \in \mathcal{H}(M)$.

**Remark 2.4.** Observe that any Hom-Lie superalgebra is a Hom-Malcev superalgebra since the Hom-Lie superidentity implies the one of Hom-Malcev

We have the following

**Lemma 2.5.** In any anticommutative Hom-superalgebra $(A = A_0 \oplus A_1, \cdot, \alpha)$, the following holds

(i) $\tilde{J}_A(x, y, z)$ is super skew-symmetric in its three variables that is

$\tilde{J}_A(x, y, z) = -(1)^{\bar{x} \bar{y}} \tilde{J}_A(y, x, z) = -(1)^{\bar{y} \bar{z}} \tilde{J}_A(x, z, y) = -(1)^{\bar{x} \bar{y} + \bar{y} \bar{z} + \bar{z} \bar{x}} \tilde{J}_A(z, y, x)$

(ii) $\alpha^2(w) \tilde{J}_A(x, y, z) - (1)^{\bar{w} + \bar{x} + \bar{y} + \bar{z} + \bar{w}} \alpha^2(x) \tilde{J}_A(y, z, w) + (1)^{\bar{y} + \bar{z} + \bar{w}} \alpha^2(y) \tilde{J}_A(z, w, x)$

\[
- (1)^{\bar{z} + \bar{y} + \bar{w}} \alpha^2(z) \tilde{J}_A(w, x, y)
\]

\[
= \tilde{J}_A(wx, \alpha(y), \alpha(z)) + (1)^{\bar{y} + \bar{z} + \bar{w}} \tilde{J}_A(yz, \alpha(w), \alpha(x)) + (1)^{\bar{y} + \bar{z} + \bar{w}} \tilde{J}_A(wy, \alpha(z), \alpha(x))
\]

\[
+ (1)^{\bar{z} + \bar{y} + \bar{w}} \tilde{J}_A(zx, \alpha(w), \alpha(y)) - (1)^{\bar{y} + \bar{z} + \bar{w}} \tilde{J}_A(zw, \alpha(x), \alpha(y))
\]

\[
- (1)^{\bar{y} + \bar{z} + \bar{w}} \tilde{J}_A(xy, \alpha(z), \alpha(w))
\]

for all $x, y, z \in \mathcal{H}(A)$.

**Proof** The super skew-symmetry of $\tilde{J}_A(x, y, z)$ in the homogeneous elements $w, x, y, z$ in $A$ follows from the super skew-symmetry of the operation “$\cdot$”. Expanding the expression in the left-hand side of (ii) and then rearranging terms, we get by the super skew-symmetry of “$\cdot$”

\[
\alpha^2(w) \tilde{J}_A(x, y, z) - (1)^{\bar{w} + \bar{x} + \bar{y} + \bar{z} + \bar{w}} \alpha^2(x) \tilde{J}_A(y, z, w) + (1)^{\bar{y} + \bar{z} + \bar{w}} \alpha^2(y) \tilde{J}_A(z, w, x)
\]

\[
- (1)^{\bar{z} + \bar{y} + \bar{w}} \alpha^2(z) \tilde{J}_A(w, x, y)
\]

\[
= (wx \cdot \alpha(y)) \cdot \alpha^2(z) + (1)^{\bar{w} + \bar{z} + \bar{w}} \alpha^2(y)
\]

\[
+ (1)^{\bar{y} + \bar{z} + \bar{w}} [(yz \cdot \alpha(w)) \cdot \alpha^2(x) + (1)^{\bar{y} + \bar{z} + \bar{w}} \alpha^2(w)]
\]

\[
+ (1)^{\bar{y} + \bar{z} + \bar{w}} [(wy \cdot \alpha(z)) \alpha^2(z) + (1)^{\bar{y} + \bar{z} + \bar{w}} \alpha^2(z)]
\]

\[
+ (1)^{\bar{y} + \bar{z} + \bar{w}} [(z \cdot \alpha(w)) \alpha^2(y) + (1)^{\bar{y} + \bar{z} + \bar{w}} \alpha^2(y) - (1)^{\bar{y} + \bar{z} + \bar{w}} \alpha^2(x)]
\]

\[
- (1)^{\bar{y} + \bar{z} + \bar{w}} [(z \cdot \alpha(x)) \alpha^2(y) + (1)^{\bar{y} + \bar{z} + \bar{w}} \alpha^2(x) - (1)^{\bar{y} + \bar{z} + \bar{w}} \alpha^2(z)]
\]

Next, adding and subtracting $\alpha^2(w) \alpha(x) \alpha(y) \cdot \alpha(wx)$

\[
\alpha^2(w) \alpha(x) \alpha(y) \cdot \alpha(wx)
\]

\[
\alpha^2(w) \alpha(x) \alpha(y) \cdot \alpha(wx)
\]

\[
\alpha^2(w) \alpha(x) \alpha(y) \cdot \alpha(wx)
\]

\[
\alpha^2(w) \alpha(x) \alpha(y) \cdot \alpha(wx)
\]

\[
\alpha^2(w) \alpha(x) \alpha(y) \cdot \alpha(wx)
\]

and $\alpha^2(w) \alpha(x) \alpha(y) \cdot \alpha(wx)$ in the first (resp. second, third, fourth, fifth, and sixth) line of the right-hand side expression in the last equality above, we come to the equality (ii)
of the lemma.

In a Hom-Malcev superalgebra \((M = M_0 \oplus M_1, \cdot, \alpha)\) we define the multilinear map \(G\) by

\[
G(w, x, y, z) = \bar{J}_M(wx, \alpha(y), \alpha(z)) - (-1)^{\bar{w} \bar{y}} \alpha^2(x) \bar{J}_M(w, y, z) - (-1)^{\bar{w} \bar{y} \bar{z} \bar{w}} \bar{J}_M(x, y, z) \alpha^2(w)
\]

(8)

**Lemma 2.6.** In a Hom-Malcev superalgebra \((M = M_0 \oplus M_1, \cdot, \alpha)\) the function \(G(w, x, y, z)\) defined by (8) is super skew-symmetric in its four variables that is

\[
G(w, x, y, z) = (-1)^{\bar{w} \bar{y}} G(x, w, y, z)
\]

\[
G(w, x, z, y) = (-1)^{\bar{w} \bar{y}} G(w, x, y, z)
\]

\[
G(w, x, y, z) = (-1)^{\bar{w} \bar{y}} G(w, x, y, z)
\]

\[
G(w, x, y, z) = (-1)^{\bar{w} \bar{y} \bar{z} \bar{w}} G(y, x, w, z)
\]

\[
G(w, x, y, z) = (-1)^{\bar{w} \bar{y} \bar{z} \bar{w}} G(z, x, y, w)
\]

for all \(x, y, z, w \in H(M)\).

**Proof** Since the group \(S_4 = \{t_{12}, t_{23}, t_{34}\}\), where \(t_{ij}\) are transpositions between \(i\) and \(j\), it suffices to prove

\[
G(w, x, y, z) = -(-1)^{\bar{w} \bar{y}} G(x, w, y, z)
\]

\[
G(w, x, y, z) = -(-1)^{\bar{w} \bar{y}} G(w, x, y, z)
\]

\[
G(w, x, y, z) = -(-1)^{\bar{w} \bar{y} \bar{z} \bar{w}} G(y, x, w, z)
\]

\[
G(w, x, y, z) = -(-1)^{\bar{w} \bar{y} \bar{z} \bar{w}} G(z, x, y, w)
\]

Then we have

\[
G(w, x, y, z) = -(-1)^{\bar{w} \bar{y}} G(w, x, y, z)
\]

(9)

The following lemma is a consequence of the definition of \(G(w, x, y, z)\) and the super skew-symmetry of \(\bar{J}_M(t, u, v)\) and \(G(w, x, y, z)\).

**Lemma 2.7.** Let \((M = M_0 \oplus M_1, \cdot, \alpha)\) be a Hom-Malcev superalgebra. Then

\[
\bar{J}_M(wx, \alpha(y), \alpha(z)) + (-1)^{\bar{w} \bar{y} \bar{z} \bar{w}} \bar{J}_M(xy, \alpha(z), \alpha(w)) + (-1)^{\bar{w} \bar{y} \bar{z} \bar{w}} \bar{J}_M(xz, \alpha(x), \alpha(y)) = 0
\]

(10)

for all \(x, y, z, w \in H(M)\).
Lemma 2.8. Then (9) says that $g$ is a super-identity equivalent to the Hom-Malcev super-identity $5$.

Proof From the definition of $G(w, x, y, z)$ (see (8)) we have

$$
\tilde{J}_M(wx, \alpha(y), \alpha(z)) = G(w, x, y, z) + (-1)^{\tilde{\omega}(\tilde{g}+\tilde{z})}\tilde{J}_M(w, y, z) $$

First, again from the expression of $G(w, x, y, z)$, we get (9).

Next, from the expression of $G(w, x, y, z)$,

$$
\tilde{J}_M(wx, \alpha(y), \alpha(z)) + (-1)^{\tilde{\omega}(\tilde{g}+\tilde{z})}\tilde{J}_M(zy, \alpha(w), \alpha(x)) = [G(w, x, y, z) + (-1)^{\tilde{\omega}(\tilde{g}+\tilde{z})}\tilde{J}_M(w, y, z) + (-1)^{\tilde{\omega}(\tilde{g}+\tilde{z})}\tilde{J}_M(zy, \alpha(w), \alpha(x))] $$

so that we get (10).

From Lemma and Lemma, we get the following expression of $G(w, x, y, z)$.

Lemma 2.8. Let $(M = M_0 \oplus M_1, \cdot, \alpha)$ be a Hom-Malcev superalgebra. Then

$$
G(w, x, y, z) = 2[\tilde{J}_M(wx, \alpha(y), \alpha(z)) + (-1)^{\tilde{\omega}(\tilde{g}+\tilde{z})}\tilde{J}_M(zy, \alpha(w), \alpha(x))] $$

for all $x, y, z, w \in \mathcal{H}(M)$.

Proof Set

$$
g(w, x, y, z) = \tilde{J}_M(wx, \alpha(y), \alpha(z)) + (-1)^{\tilde{\omega}(\tilde{g}+\tilde{z})}\tilde{J}_M(xy, \alpha(z), \alpha(w)) $$

Then (9) says that $g(w, x, y, z) = 0$ for all $w, x, y, z$ in $A$. Now, by adding $g(w, x, y, z) - (-1)^{\tilde{\omega}(\tilde{g}+\tilde{z})}g(x, w, y, z)$ to the right-hand side of Lemma (ii), we get

$$
\alpha^2(w)\tilde{J}_M(x, y, z) - (-1)^{\tilde{\omega}(\tilde{g}+\tilde{z})}\alpha^2(x)\tilde{J}_M(y, z, w) $$

$$
\tilde{J}_M(wx, \alpha(y), \alpha(z)) + (-1)^{\tilde{\omega}(\tilde{g}+\tilde{z})}\tilde{J}_M(zy, \alpha(w), \alpha(x)) $$

+ $(-1)^{\tilde{\omega}(\tilde{g}+\tilde{z})}\tilde{J}_M(wx, \alpha(y), \alpha(z)) + (-1)^{\tilde{\omega}(\tilde{g}+\tilde{z})}\tilde{J}_M(xy, \alpha(z), \alpha(w))$

Then (9) says that $g(w, x, y, z) = 0$ for all $w, x, y, z$ in $A$. Now, by adding $g(w, x, y, z) - (-1)^{\tilde{\omega}(\tilde{g}+\tilde{z})}g(x, w, y, z)$ to the right-hand side of Lemma (ii), we get

$$
\tilde{J}_M(wx, \alpha(y), \alpha(z)) + (-1)^{\tilde{\omega}(\tilde{g}+\tilde{z})}\tilde{J}_M(zy, \alpha(w), \alpha(x)) $$

+ $(-1)^{\tilde{\omega}(\tilde{g}+\tilde{z})}\tilde{J}_M(wx, \alpha(y), \alpha(z)) + (-1)^{\tilde{\omega}(\tilde{g}+\tilde{z})}\tilde{J}_M(xy, \alpha(z), \alpha(w))$

+ $(-1)^{\tilde{\omega}(\tilde{g}+\tilde{z})}\tilde{J}_M(yz, \alpha(w), \alpha(x)) + (-1)^{\tilde{\omega}(\tilde{g}+\tilde{z})}\tilde{J}_M(zw, \alpha(x), \alpha(y))$
From (14), we get
\[ i.e \]
\[-(1)^{2\bar{a}} \tilde{J}_M (wx, \alpha(y), \alpha(z)) - (1)^{2(\bar{a}+\bar{g}+\bar{z})+2\bar{a}} \tilde{J}_M (wy, \alpha, \alpha(x)) \]
\[-(1)^{(\bar{g}+\bar{z})(\bar{w}+\bar{x})+\bar{a}} \tilde{J}_M (yz, \alpha(x), \alpha(w)) - (1)^{2(\bar{a}+\bar{g}+\bar{w})+\bar{a}} \tilde{J}_M (xz, \alpha(w), \alpha(y)) \]
\[ = 3\tilde{J}_M (wx, \alpha(y), \alpha(z)) + 3(1)^{(\bar{g}+\bar{z})(\bar{w}+\bar{x})} \tilde{J}_M (yz, \alpha(w), \alpha(x)) \]
\[ i.e \]
\[ \alpha^2 (w) \tilde{J}_M (x, y, z) - (1)^{\bar{w}(\bar{a}+\bar{g}+\bar{z})} \alpha^2 (x) \tilde{J}_M (y, z, w) \]
\[ + (1)^{(\bar{g}+\bar{z})(\bar{a}+\bar{w})} \alpha^2 (y) \tilde{J}_M (z, w, x) - (1)^{2(\bar{a}+\bar{g}+\bar{w})} \alpha^2 (z) \tilde{J}_M (w, x, y) \]
\[ = 3\tilde{J}_M (wx, \alpha(y), \alpha(z)) + 3(1)^{(\bar{g}+\bar{z})(\bar{w}+\bar{x})} \tilde{J}_M (yz, \alpha(w), \alpha(x)) \]
Next, adding (11) and (12) together, we get
\[ 2G (w, x, y, z) - \alpha^2 (w) \tilde{J}_M (x, y, z) + (1)^{2\bar{a}} \alpha^2 (x) \tilde{J}_M (w, y, z) \]
\[ -(1)^{(\bar{g}+\bar{z})(\bar{a}+\bar{w})} \alpha^2 (y) \tilde{J}_M (z, w, x) - (1)^{2(\bar{a}+\bar{g}+\bar{w})} \alpha^2 (z) \tilde{J}_M (w, x, y) \]
\[ + \alpha^2 (w) \tilde{J}_M (x, y, z) - (1)^{\bar{w}(\bar{a}+\bar{g}+\bar{z})} \alpha^2 (x) \tilde{J}_M (y, z, w) \]
\[ + (1)^{(\bar{g}+\bar{z})(\bar{a}+\bar{w})} \alpha^2 (y) \tilde{J}_M (z, w, x) - (1)^{2(\bar{a}+\bar{g}+\bar{w})} \alpha^2 (z) \tilde{J}_M (w, x, y) \]
\[ = 3\tilde{J}_M (wx, \alpha(y), \alpha(z)) + 3(1)^{(\bar{g}+\bar{z})(\bar{w}+\bar{x})} \tilde{J}_M (yz, \alpha(w), \alpha(x)) \]
\[ i.e \]
\[ 2G (w, x, y, z) = 4\tilde{J}_M (wx, \alpha(y), \alpha(z)) + 4(1)^{(\bar{g}+\bar{z})(\bar{w}+\bar{x})} \tilde{J}_M (yz, \alpha(w), \alpha(x)) \]
and (11) follows.

3. Proof
Relaying on the lemmas of section 2, we are now in position to prove the theorem.

Proof of theorem First we establish the identity (3) in a Hom-Malcev superalgebra. We may write (3) in an equivalent form:
\[ (\tilde{J}_M^{\text{SM}})(wx, \alpha(y), \alpha(z)) = (1)^{2\bar{a}} \alpha^2 (x) \tilde{J}_M (w, y, z) + (1)^{(\bar{a}+\bar{g}+\bar{z})} \tilde{J}_M (x, y, z) \alpha^2 (w) + G (w, x, y, z) \]
Now in (13), replace \( G (w, x, y, z) \) with its expression from (11) to get
\[ \tilde{J}_M (wx, \alpha(y), \alpha(z)) = (1)^{2\bar{a}} \alpha^2 (x) \tilde{J}_M (w, y, z) + (1)^{(\bar{a}+\bar{g}+\bar{z})} \tilde{J}_M (x, y, z) \alpha^2 (w) \]
\[ + 2(1)^{(\bar{g}+\bar{z})(\bar{w}+\bar{x})} \tilde{J}_M (yz, \alpha(w), \alpha(x)) \]
which leads to (3). Now, we proceed to prove the equivalence of (5) with (3) in a supercommutative Hom-Malcev superalgebra. First assume (5). Then Lemmas 2.2, 2.3, 2.4 and 2.5 imply that (3) holds in any Hom-Malcev superalgebra. Conversely, assume (3) that is
\[ \tilde{J}_M (\alpha(y), \alpha(z), wx) = (1)^{(\bar{a}+\bar{g}+\bar{z})} \alpha^2 (w) \tilde{J}_M (y, z, x) + \tilde{J}_M (y, z, w) \alpha^2 (x) \]
\[ - 2(-1)^{(\bar{g}+\bar{z})(\bar{a}+\bar{w})} \tilde{J}_M (\alpha(w), \alpha(x), yz) \]
From (14), we get
\[ \tilde{J}_M (\alpha(y), \alpha(z), wx) + (1)^{(\bar{a}+\bar{g}+\bar{z})} \tilde{J}_M (\alpha(w), \alpha(z), wx) \]
\[ = \left[ (1)^{(\bar{a}+\bar{g}+\bar{z})} \alpha^2 (w) \tilde{J}_M (y, z, x) + \tilde{J}_M (y, w, x) \alpha^2 (x) \right] \]
\[ - 2(-1)^{(\bar{g}+\bar{z})(\bar{a}+\bar{w})} \tilde{J}_M (\alpha(w), \alpha(x), yz) + (-1)^{(\bar{a}+\bar{g}+\bar{z})} \tilde{J}_M (w, x, y) \alpha^2 (y) \]
\[ + \tilde{J}_M (w, z, y) \alpha^2 (x) - 2(-1)^{(\bar{a}+\bar{g}+\bar{z})} \tilde{J}_M (\alpha(y), \alpha(x), wz) \]
\[ = - (1)^{(\bar{a}+\bar{g}+\bar{z})} \tilde{J}_M (w, z, x) \alpha^2 (y) - (1)^{(\bar{a}+\bar{g}+\bar{z})+2\bar{a}} \tilde{J}_M (w, z, x) \alpha^2 (y) \]
\[ - 2(-1)^{(\bar{g}+\bar{z})(\bar{a}+\bar{w})} \tilde{J}_M (\alpha(w), \alpha(x), yz) + (-1)^{(\bar{a}+\bar{g}+\bar{z})} \tilde{J}_M (\alpha(y), \alpha(x), wz) \]
\[\tilde{J}_M(\alpha(y), \alpha(z), wx) + (-1)^{\tilde{y} + \tilde{w}} \tilde{J}_M(\alpha(w), \alpha(z), yx)\]
\[= -(-1)^{\tilde{y} + \tilde{w}} \tilde{J}_M(y, z, x)\alpha^2(w) - (-1)^{\tilde{y}(\tilde{z} + \tilde{w} + \tilde{x}) + \tilde{z} \tilde{w}} \tilde{J}_M(w, z, x)\alpha^2(y)\]
\[= -2(-1)^{\tilde{y}(\tilde{z} + \tilde{w} + \tilde{x}) + \tilde{z} \tilde{w}}[\tilde{J}_M(\alpha(w), \alpha(x), yz) + (-1)^{\tilde{y} + \tilde{w} + \tilde{x}} \tilde{J}_M(\alpha(y), \alpha(x), wz)]\]

Switching simultaneous \(y\) and \(w\) then \(x\) and \(z\) in (15), we get
\[\tilde{J}_M(\alpha(w), \alpha(x), yz) + (-1)^{\tilde{y} + \tilde{w} + \tilde{x}} \tilde{J}_M(\alpha(y), \alpha(x), wz)\]
\[= -(-1)^{\tilde{y} + \tilde{w}} \tilde{J}_M(w, x, z)\alpha^2(y) - (-1)^{\tilde{y}(\tilde{z} + \tilde{w} + \tilde{x}) + \tilde{z} \tilde{w}} \tilde{J}_M(y, x, z)\alpha^2(w)\]
\[= -2(-1)^{\tilde{y}(\tilde{z} + \tilde{w} + \tilde{x}) + \tilde{z} \tilde{w}}[\tilde{J}_M(\alpha(y), \alpha(z), wx) + (-1)^{\tilde{y} + \tilde{w} + \tilde{x}} \tilde{J}_M(\alpha(w), \alpha(z), yx)]\]

Now, replacing (16) in (15) and using the super skew-symmetry of \(\tilde{J}_M\), we get
\[\tilde{J}_M(\alpha(y), \alpha(z), wx) + (-1)^{\tilde{y} + \tilde{w}} \tilde{J}_M(\alpha(w), \alpha(z), yx)\]
\[= -3(-1)^{\tilde{y} + \tilde{w}} \tilde{J}_M(y, z, x)\alpha^2(w) - 3(-1)^{\tilde{y}(\tilde{z} + \tilde{w} + \tilde{x}) + \tilde{z} \tilde{w}} \tilde{J}_M(w, z, x)\alpha^2(y)\]
\[+ 4\tilde{J}_M(\alpha(y), \alpha(z), wx) + 4(-1)^{\tilde{y} + \tilde{w} + \tilde{x}} \tilde{J}_M(\alpha(w), \alpha(z), yx)\]

i.e
\[-3\tilde{J}_M(\alpha(y), \alpha(z), wx) - 3(-1)^{\tilde{y} + \tilde{w}} \tilde{J}_M(\alpha(w), \alpha(z), yx)\]
\[= -3(-1)^{\tilde{y} + \tilde{w}} \tilde{J}_M(y, z, x)\alpha^2(w) - 3(-1)^{\tilde{y}(\tilde{z} + \tilde{w} + \tilde{x}) + \tilde{z} \tilde{w}} \tilde{J}_M(w, z, x)\alpha^2(y)\]

and therefore (5) follows.

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