Vassiliev Invariants for Links from Chern-Simons Perturbation Theory

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ABSTRACT

The general structure of the perturbative expansion of the vacuum expectation value of a product of Wilson-loop operators is analyzed in the context of Chern-Simons gauge theory. Wilson loops are opened into Wilson lines in order to unravel the algebraic structure encoded in the group factors of the perturbative series expansion. In the process a factorization theorem is proved for Wilson lines. Wilson lines are then closed back into Wilson loops and new link invariants of finite type are defined. Integral expressions for these invariants are presented for the first three primitive ones of lower degree in the case of two-component links. In addition, explicit numerical results are obtained for all two-component links of no more than six crossings up to degree four.

MIT-CTP-2547
USC-FT-30-96
hep-th/9607030

June 1996

1This work is supported in part by funds provided by the U.S.A. DOE under cooperative research agreement #DE-FC02-94ER40818 and by the DGICYT of Spain under grant PB93-0344.
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1 Introduction

The complete classification of knots and links embedded in three dimensional manifolds is still an open problem. Apart from the classical results of Alexander, Reidemeister and others (see, for example, [1, 2]), we have now the polynomial invariants of Jones [3] and its generalizations [4, 5, 6]. Unlike the classical Alexander polynomial, these polynomials are able to distinguish knots or links from their mirror images. However, it is still not known if they separate knots.

All these new invariants are strongly rooted in ideas and methods of Quantum Field Theory or Statistical Mechanics. By using Yang-Baxter models, it is possible to define the Jones polynomial and its relatives, which can also be described in terms of quantum groups [8]. On the other hand, the formalism of Chern-Simons gauge theory leads in a natural way to all these new invariants, each of them corresponding to a choice of the gauge group [7]. The main observable in Chern-Simons theory is the Wilson-loop operator, which for a given gauge group depends only on the knot class of the loop. A non-perturbative evaluation of the vacuum expectation value of this operator leads directly to the above mentioned polynomial invariants.

A different set of invariants are the Vassiliev invariants. These were first proposed in [9, 10] to classify knot types. To each knot corresponds an infinite sequence of rational numbers which have to satisfy some consistency conditions in order to be knot class invariants. This infinite sequence is divided into finite subsequences, which form vector spaces. Each subsequence is indexed by a positive integer called its order. The number of independent elements in each finite subsequence is called the dimension of the space of Vassiliev invariants at that order.

An axiomatic definition of these invariants was formulated in [11, 12], in terms of inductive relations for singular knots. This approach is best suited to show the relation to other knot invariants based on quantum groups or in Chern-Simons gauge theory [13, 11, 12, 14, 15]. Several works have been performed to analyze Vassiliev invariants in both frameworks [16, 17, 18, 19, 20]. In [16, 17] it was shown that Vassiliev invariants can be understood in terms of representations of chord diagrams without isolated chords modulo the so called 4T relations (weight systems), and that using semi-simple Lie algebras weight systems can be constructed. It was also shown in [17], using Kontsevich’s representation for Vassiliev invariants [21], that the space of weight systems is the same as the space of Vassiliev invariants. In [18] it was argued that these representations are precisely the ones underlying quantum-group or Chern-Simons invariants.

The connection of Vassiliev invariants to Chern-Simons theory shows up through a perturbative evaluation of the vacuum expectation value of the Wilson-loop operators, in the sense of ordinary perturbative Quantum Field Theory. We observed in [19] that the generalization of the integral or geometrical knot invariant first proposed in [22] and further analyzed in [14], as well as the invariant itself, are Vassiliev invariants. These invariants arise naturally in the perturbative analysis of the Wilson loop. In [19] we proposed an organization of those geometrical invariants and we described a procedure for their calculation from known polynomial knot invariants. This procedure has been applied to obtain Vassiliev knot invariants up to order six for all prime knots up to six crossings [19] and for all torus
knots [23]. These geometrical invariants have also been studied by Bott and Taubes [24] using a different approach. The relation of this approach to the one in [19] has been studied recently in [25].

The Vassiliev invariants of a given knot form an algebra in the sense that the product of two invariants of orders $i$ and $j$ is an invariant of order $i+j$. Therefore the set of independent Vassiliev invariants at a given order can be divided into two subsets: those that are products of invariants of lower orders (composite invariants), and those that are not (primitive invariants). In [26] we called this property “factorization”, and showed how it can be exploited to express the vacuum expectation value of the Wilson-loop operator associated with the knot as an exponential whose argument includes only primitive invariants. This was accomplished by choosing a particular kind of basis of group factors that we called “canonical”.

The aim of this paper is to extend the formalism in [19, 26] to two-component links. A straightforward application of the formalism of canonical bases, though feasible, would not be satisfactory due to the fact that in the case of links there is not a simple algebraic structure among group factors similar to the one present in the case of knots. A similar algebraic structure appears, however, when open links are considered. In the resulting framework a factorization theorem as the one presented for knots in [26] holds. With the help of this theorem finite type link invariants are constructed after introducing a closing operation. As in the case of knots, these invariants are expressed in term of multidimensional path integrals along the loops corresponding to the different components of the link.

It is by now well known that Vassiliev invariants, which were originally defined for knots, can be also defined for other objects as links, string links, braids, tangles, etc. These objects can be regarded as classes of embeddings of one-dimensional objects in a three-dimensional space modulo some kind of isotopy. Either from quantum groups or from Chern-Simons gauge theory, isotopy invariants can be constructed. These are formal Laurent polynomials in a parameter $q$. The coefficients of the expansions of these invariants in power series of $x$, being $q = e^x$, are Vassiliev invariants [15]. Perturbative Chern-Simons gauge theory provides a way to construct geometrical or path integral expressions for the resulting Vassiliev invariants. This fact should hold for any of the objects quoted above. We will concentrate in this work in the case of links but generalizations should be carried out for other cases.

The paper is organized as follows. Section 2 contains an elementary exposition of Chern-Simons quantum field theory, along with the definition of Wilson-loop and Wilson-line operators and some nomenclature. In section 3 we introduce the general structure of the perturbative expansion of a Wilson-line operator for two lines, and the definition of canonical bases. Section 4 contains a group-theoretical result that, though simple, pervades the rest of this work. In section 5 we present the Master Equation, which is the key to the Theorem of Factorization; this theorem encodes the consequences of our having chosen a canonical basis to express the perturbative expansion. In section 6 we define the opening and closing operation, and analyze the invariants so obtained. Explicit integral expressions for these invariants are presented in section 7 up to order four. These are computed for all two-component links of no more than six crossings in section 8. Finally, in section 9 we state our conclusions. Appendices A, B and C, contain details on our group-theoretical conventions and lists of the polynomial invariants used in section 8.
Chern-Simons theory

In this section we will describe known results on Chern-Simons perturbation theory. We do not attempt here to provide a derivation of these results. This can be studied in previous works \cite{19, 22, 27}. What we will do is to point out the salient features of the analysis of Chern-Simons gauge theory in the framework of perturbation theory and to summarize the set of rules which comes out of that analysis. These rules are known as Feynman rules and they can be neatly described in terms of Feynman diagrams. The aim of this section is therefore to provide the necessary framework so that the reader could write down the contribution to the vacuum expectation value of a product of Wilson-line operators at any order in perturbation theory.

We will restrict ourselves to the case in which the three-dimensional manifold is $\mathbb{R}^3$ and the gauge group is a semi-simple compact Lie group $G$. Let $A$ be a $G$-connection. The action of the theory is the integral over $\mathbb{R}^3$ of the Chern-Simons form:

$$S_k(A) = \frac{k}{4\pi} \int_{\mathbb{R}^3} \text{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A), \quad (2.1)$$

where $\text{Tr}$ denotes the trace in the fundamental representation of $G$ and $k$ is a real parameter. As pointed out in \cite{22, 27, 28} there are three problems in the analysis of Chern-Simons gauge theory from the point of view of perturbation theory. First, the theory based on the action (2.1) has a gauge symmetry which has to be fixed. Invariance under gauge transformations which are not connected to the identity implies certain quantization conditions for the parameter $k$ \cite{23}. From the point of view of perturbation theory this condition is not important and we will take $g = \sqrt{4\pi/k}$ as the expansion parameter of the resulting perturbative power series. We will choose as gauge-fixing the Landau gauge considered in \cite{27}. This gauge has the advantage of being covariant and free of infrared divergences.

The second problem that one has to face in the perturbative analysis of Chern-Simons gauge theory is the presence of ultraviolet divergences. This implies that the theory has to be regularized. As described in \cite{22, 27} the theory does not have to be renormalized. Once the theory is regularized and the regulator is removed each of the terms in the perturbative expansion becomes finite. This means that no dimensionful parameter is needed to describe the theory at the quantum level. Different regularizations lead to different perturbative expansions which, however, are related by a redefinition of the parameter $k$. We will choose the regularization proposed in \cite{27} and elaborated for higher loops in \cite{30}. The salient feature of this regularization is that higher-loop contributions to the two and three-point functions account for a shift in $k$: $k \rightarrow k - C_A$, being $C_A$ the quadratic Casimir in the adjoint representation of $G$, so one can disregard them from the perturbative expansion and take as expansion parameter $g = \sqrt{4\pi/(k - C_A)}$.

Finally, one has to cure the intrinsic ambiguity appearing when products of operators are evaluated at the same point. As shown in \cite{1, 22} this problem can be solved without spoiling the topological nature of the theory. However, in the process of fixing the ambiguity in this way one is forced to introduce an integer which will be identified with the framing associated to some of the observables. To be more precise in the description of this effect we need first to introduce the types of operators which are present in Chern-Simons gauge
theory. We will do this next and we postpone the discussion on the role played by this third problem to the end of the next subsection.

The basic gauge invariant operators of Chern-Simons gauge theory which lead to topological invariants are Wilson-loop operators. There are also graph operators but these will not be considered in this paper. Wilson loops are labeled by a loop $C$ embedded in $\mathbb{R}^3$ and a representation $R$ of the gauge group $G$, and correspond to the holonomy around the loop $C$ of the gauge connection $A$:

$$W_R(C, G) = \text{Tr} \left[ P_R \exp \oint A \right].$$

(2.2)

In this equation $P_R$ denotes path-ordered and the fact that $A$ must be considered in the representation $R$: $A = A^a T^a_R$, $a = 1, \ldots, \dim(G)$, the generators of $G$ in the representation $R$. In Chern-Simons theory one considers vacuum expectation values of products of Wilson-line operators:

$$\langle W_{R_1}(C_1, G) W_{R_2}(C_2, G) \ldots W_{R_n}(C_n, G) \rangle = \frac{1}{Z_k} \int [DA] \ W_{R_1}(C_1, G) W_{R_2}(C_2, G) \ldots W_{R_n}(C_n, G) e^{iS_k(A)},$$

(2.3)

where $Z_k$ is the partition function:

$$Z_k = \int [DA] e^{iS_k(A)}.$$  

(2.4)

As shown in [7] the quantity (2.3) is a link invariant associated to a colored $n$-component link whose $j$-component, $C_j$, carries the representation $R_j$, $j = 1, \ldots, n$.

In this paper we will be considering also operators which are not gauge invariant but gauge covariant. If instead of a loop in (2.2) one considers a line with fixed end points, the resulting operator, which we will call Wilson-line operator, is gauge covariant. As shown in [7] these operators also lead to interesting quantities from a topological point of view. For specific choices of the three-manifold they are related to conformal blocks [7, 33] and are invariant under certain deformations of the lines involved. We will label Wilson-line operators in the following way:

$$F_{R}(P, Q, L, G)^{i} = \left[ P_R \exp \int_{P}^{Q} A \right]^{i},$$

(2.5)

where $P$ and $Q$ denote the two fixed end points of the line $L$, and $i$ and $j$ run respectively over the representation $R$ and its conjugate $\bar{R}$. The object of interest in Chern-Simons gauge theory is the product of $n$ Wilson-line operators,

$$\langle F_{R_{1}}(P_{1}, Q_{1}, L_{1}, G) F_{R_{2}}(P_{2}, Q_{2}, L_{2}, G) \ldots F_{R_{n}}(P_{n}, Q_{n}, L_{n}, G) \rangle = \frac{1}{Z_k} \int [DA] \ F_{R_{1}}(P_{1}, Q_{1}, L_{1}, G) F_{R_{2}}(P_{2}, Q_{2}, L_{2}, G) \ldots F_{R_{n}}(P_{n}, Q_{n}, L_{n}, G) e^{iS_k(A)}.$$  

(2.6)

As shown in [7, 33] for some specific choice of manifold and gauge fixing this quantity is related to a conformal block on $S^2$ with $2n$ marked points.
2.1 Perturbative analysis and Feynman Rules

The vacuum expectation values (2.3) and (2.6) can be calculated non-perturbatively by splitting the three manifold (\(R^3\) in our case) into two three manifolds with boundary, in which a WZW theory with sources is induced \([7, 24, 33, 34]\). We shall not follow that approach, but rather analyze the same objects within the framework of perturbation theory. The main arguments have been explained before in \([19, 26, 23]\), where the reader is referred to for details.

To fix ideas we shall outline the perturbative analysis of the vacuum expectation value of the Wilson-loop operator (2.2). Simply stated, this vacuum expectation value is evaluated as a formal power series in the variable \(x = ig^2/2\) after rescaling the gauge field \(A \to gA\) both in the action (2.1) and in the Wilson-loop operator. A useful parametrization of that power series is \([19]\):

\[
\langle W_R(C, G) \rangle = d(R) \sum_{i=0}^{\infty} \sum_{j=1}^{d_i} \alpha^i_j(C) r_{ij}(G) x^i, \tag{2.7}
\]

where the symbols \(\alpha^i_j(C)\) are combinations of path integrals of some kernels along the loop \(C\) and over \(R^3\), and the \(r_{ij}\) are traces of products of generators of the Lie algebra associated with the gauge group \(G\). The index \(i\) is called the “order” in perturbation theory, and \(j\) labels independent contributions to a given order, being \(d_i\) the number of these at order \(i\). In (2.7) \(d(R)\) denotes the dimension of the representation \(R\).

Each term in the expansion (2.7) can be conveniently represented as a Feynman diagram like the ones depicted in Fig. 1. These diagrams are constructed from the lines and vertices described in Fig. 2. Each type of line corresponds to a kernel (or “propagator”), and each vertex to an integration; these correspondences are the Feynman rules.

In Chern-Simons gauge theory the Feynman rules associated with the lines and vertices of Fig. 2 are:

\[
D^{ab}_{\mu\nu}(x-y) = \frac{i}{4\pi} \epsilon_{\mu\nu\rho} \frac{(x-y)^\rho}{|x-y|^3} \delta^{ab},
\]

\[
D^{ab}(x-y) = \frac{i}{4\pi} \frac{1}{|x-y|} \delta^{ab},
\]

\[
V^{\mu\nu\rho}_{abc}(x) = -ig f_{abc}^{\mu\nu\rho} \int_{\mathbb{R}^3} d^3x,
\]

\[
V^\mu_{abc}(x) = ig f_{abc} \partial^\mu \int_{\mathbb{R}^3} d^3\omega,
\]
\[ V_{i,a}^j(x) = g \left( T_a^{(R)} \right)_{ij} \int dx. \] (2.8)

The argument \( x \) in \( V_{i,a}^j(x) \) is a point on the Wilson line, and the integration runs over a segment of the Wilson line limited by the two nearest insertions of the same vertex. The arrow on the last diagram of Fig. 2 indicates the orientation of the Wilson line. Since the structure constants of a semisimple Lie group can be chosen to be totally antisymmetric, there is no need to assign orientation to the internal three-vertices. The gauge lines need not be oriented either because the adjoint representation is real. With the help of the Feynman rules we can evaluate the vacuum expectation value of any Wilson line operator to any order in perturbation theory once we have drawn all the diagrams that contribute at that order.

The simplest non-trivial diagram would consist in a gauge propagator with both endpoints attached to the Wilson loop; this corresponds to the term \( i = 1 \) in (2.7). It is also the simplest way to introduce the third difficulty mentioned above. Owing to the integration along the Wilson loop, the two endpoints of the propagator will eventually get together (or “collapse” in the terminology of [19]). Although the apparently divergent integral is finite [22], it turns out that it is not invariant under small deformations of the Wilson loop, \( i.e. \) it is not a topological invariant. The solution to this problem was proposed in [7] as attaching a “framing” to the Wilson loop; this framing is another loop defined by a small normal vector field along the Wilson loop. Attaching one of the endpoints of the propagator to the Wilson loop and the other to the framing the integration is well-defined and corresponds to computing the linking number of the framing around the Wilson loop.

The only diagrams that perceive the existence of the framing are those with collapsible propagators; these are propagators whose endpoints may get together without crossing over other propagators [28, 19, 29]. It has been shown in those references that the contribution of the collapsible diagrams factorizes in an exponential, in total agreement with
non-perturbative calculations \cite{7}. The framing, however, is of no topological relevance and its contribution must therefore be discarded. We shall dispose of the framing by not including collapsible diagrams in the expansion \cite{2.7}.

### 2.2 Wilson Line Operator for Two Open Lines

The aim of this work is to construct an approach to define numerical invariants for links. We will describe in this paper the case of two-component links in full detail. The generalization for links with an arbitrary number of components will be presented elsewhere. As described in the introduction we shall analyze first the vacuum expectation value of the product of two Wilson-line operators. Let us attach different irreducible representations $\lambda$ and $\mu$ of the gauge group $G$ to each line $L_1$ and $L_2$:

$$
\mathcal{F}_\lambda(P_1, Q_1, L_1, G)_{i_1}^{j_1} \mathcal{F}_\mu(P_2, Q_2, L_2, G)_{i_2}^{j_2} = P_\lambda \exp \left\{ \int_{L_1} A \right\}_{i_1}^{j_1} P_\mu \exp \left\{ \int_{L_2} A \right\}_{i_2}^{j_2}
$$

(2.9)

where, say, $P_\lambda$ has the usual meaning of path-ordering, and the gauge field entering in the corresponding exponential is $A(x) = A^a(x)(T^a_\lambda)_{ij}$. The Greek index in the generators $(T^a_\lambda)_{ij}^k$ designates the representation in which they are defined; if it is $\lambda$, the indices $i, j$ run respectively over the representations $\lambda$ and its conjugate $\bar{\lambda}$. When not strictly necessary these indices, and other two coming from $\mu$, will not be written explicitly in the Wilson-loop operator in order not to clutter the notation. For the same reason we shall omit all the arguments $P$, $Q$, and $G$, and let $\lambda$ or $\mu$ name both the representation of the gauge group defined in each line and, implicitly, the gauge group itself.

Before we take a deeper look inside the general structure of the perturbative expansion of the vacuum expectation value of our operator,

$$
\langle \mathcal{F}_\lambda(L_1) \mathcal{F}_\mu(L_2) \rangle,
$$

(2.10)

let us introduce some vocabulary related to the full set of Feynman diagrams coming out of this expansion. These diagrams are built up with the propagators and three-vertices described in the previous subsection. For the case of two open lines these diagrams are trivalent graphs with two distinguished lines which will be called Wilson lines, carrying the representations $\lambda$ and $\mu$. The other lines correspond to propagators and are called internal lines. We shall refer to the set of internal lines of a given diagram as the Feynman graph. In order to classify the different types of diagrams, we will use the following definitions:

- **subdiagram**: a specific subset of propagators in a given diagram.
- **connected (sub)diagram**: we will say that a (sub)diagram is a connected (sub)diagram if it is possible to go from one propagator to another without ever having to go through any of the two Wilson lines.
- **disconnected diagram**: the previous description is not possible. The diagram will be made of some connected subdiagrams.
- **non-overlapping subdiagrams**: we say that two subdiagrams are non-overlapping if starting, say by the upper points of the two Wilson-lines, we can move along them meeting all the
legs of one subdiagram first, and all the legs of the other in the second place. Here, “legs” means the propagators directly attached to the Wilson lines.

**self-interaction subdiagram**: a subdiagram living only in either of the two Wilson lines.

**interaction subdiagram**: a subdiagram connecting the two Wilson lines.

**standard diagram**: a diagram either connected or made of non-overlapping connected sub-diagrams.

In Fig. 3 the diagram a is connected while the others are disconnected, containing sub-diagrams which are connected. Diagram a contains only one connected subdiagram which coincides with itself. In c and d the connected subdiagrams do not overlap, while in b they do. Diagrams a, b and c contain interaction and self-interaction subdiagrams while d only self-interaction ones. All the subdiagrams in a, c and d are non-overlapping.
3 General structure of the perturbative expansion

As pointed out in the introduction, our aim is to construct a framework to obtain finite type invariants for links. We will consider first the case of open two-component links and therefore we will analyze the perturbative series expansion of the operator (2.9). Recall that our main interest is to analyze two-component closed links, but their group factors do not satisfy a simple algebra and one must first analyze the case of open links.

The strategy for analyzing open links is analogous to the approach described in [27]. The perturbative series expansion we are studying now corresponds to the vacuum expectation value of the operator (2.3). All the possible Feynman diagrams which can be constructed from the Feynman rules enter in this expansion. Once we have excluded loop contributions from the two- and three-point functions and collapsible propagators, as was argued above, the perturbative expansion can be written as a generalization of (2.7):

$$\langle F_\lambda(L_1)F_\mu(L_2) \rangle = \sum_{i=0}^{\infty} \sum_{j=1}^{D_i} A_i^{j}(L_1, L_2) R_{ij}(\lambda, \mu) x^i,$$

(3.1)

where $L_1$ and $L_2$ are the two open lines. The first line is coloured by the representation $\lambda$, and the second by the representation $\mu$. The factors $A_i^{j}$ and $R_{ij}$ in (3.1) incorporate all the dependence dictated from the Feynman rules apart from the dependence on $k$ which is contained in $x$. Of the two factors, $R_{ij}$ and $A_i^{j}$, the first one contains all the group-theoretical dependence, while the second all the geometrical dependence. The quantity $D_i$ denotes the number of independent group structures $R_{ij}$ which appear at order $i$.

Let us define properly the objects $R_{ij}$ and $A_i^{j}$. $R_{ij}$ is the product of two tensors. One comes from the product of generators in the first line, and the other from the product of generators in the second line. Each of these tensors has two indices corresponding to each endpoint of the open line and a given number of indices in the adjoint representation of $G$; these are common to both tensors, and are contracted. Therefore, the tensor $R_{ij}$ has four indices: $i_1, j_1$ for $L_1$ and $i_2, j_2$ for $L_2$.

$$R_{ij}(\lambda, \mu) \rightarrow \left[R_{ij}(\lambda, \mu)\right]_{i_1}^{j_1} {j_2}^{i_2}.$$

(3.2)

On the other hand, $A_i^{j}(L_1, L_2)$ is an integral over the two lines which depends on four fixed points, the endpoints of the two Wilson lines,

$$A_i^{j} \rightarrow A_i^{j}(L_1, P_1, Q_1; L_2, P_2, Q_2),$$

(3.3)

being $P_1$ and $Q_1$ respectively the endpoints of the first line, and $P_2$ and $Q_2$ those of the second line. As an example, the group and geometrical factors of the diagram in Fig. 4 are:

$$[R_{ij}]^{j_1} {j_2}^{i_2} = f^{bcd}(T_a^{(\lambda)})_{i_1}^m (T_b^{(\mu)})^n (T_c^{(\lambda)})_{n}^m (T_d^{(\lambda)})_{i_2}^p (T_d^{(\mu)})^p$$

(3.4)

$$A_i^{j}(L_1, P_1, Q_1; L_2, P_2, Q_2) = \frac{1}{32} \int_{P_1}^{Q_1} d\bar{x}_3 \int_{P_2}^{Q_2} d\bar{y}_2 \int_{P_1}^{Q_1} d\bar{x}_1 \int_{P_2}^{Q_2} d\bar{y}_1 \int_{P_1}^{Q_1} d^{3}\omega \int_{P_1}^{Q_1} d^{3}\omega \int_{P_1}^{Q_1} d^{3}\omega \int_{P_1}^{Q_1} d^{3}\omega$$

(3.5)
where,
\[
\Delta_{\mu\nu}(x - y) = \frac{1}{\pi} \epsilon_{\mu\rho\nu} \frac{(x - y)^\rho}{|x - y|^3}.
\] (3.6)

Notice that in defining group and geometrical factors the overall normalization can be chosen arbitrarily. We will use a convention in which group factors are taken to be the ones dictated by the Feynman rules without any additional numerical factors. Once the group factor has been fixed and the expansion parameter \( x \) extracted, the corresponding geometrical factor contains the rest of the ingredients dictated by the Feynman rules. This fixes completely the normalization ambiguity, but there is still some implicit dependence on the group-theoretical conventions used. The best convention to avoid ambiguities is to fix the values of the resulting primitive finite type invariant for a given link. For the case of knots it was noticed in \([19, 23]\) that there seems to exist a choice, at least up to order six, such that all the invariants are integer-valued. For the case of links, as we will observe in sect. 7, it is not clear from the small amount of invariants which we present if there exist a natural normalization such that all the primitive invariants are integer-valued.

### 3.1 Canonical bases for two-component links

In the general expansion (3.1) there are many possible choices of independent groups factors \( R_{ij} \). Given all Feynman diagrams contributing to a given order in perturbation theory, some of the resulting group factors might be linear combinations of others due to the relations among the generators \( T^{(R)}_a \) and the structure constants \( f_{abc} \) of semi-simple groups. From a diagrammatic point of view these relations are the so-called STU and IHX relations \([17]\). A complete set of independent group factors at each order in perturbation theory will be called a “basis” of group factors \([19, 20]\).

The group factors entering (3.1) are the elements of a given basis. Each of these elements is represented by a Feynman diagram, of which we are only considering the group factor. This representation is of practical importance because it allows an index-free visual display of the elements of the basis. Besides, it simplifies considerably the tasks of calculating them for specific gauge groups and of deciding when a given group factor is (in)dependent of others. In this respect, we should have to indicate when, given a Feynman diagram, we are just considering its group factor and when we are only interested in its geometrical factor.
In order to avoid a cumbersome notation we will not make this difference. It will be always clear from the context which case we are referring to.

Many choices of independent diagrams are possible. Each possible set of group factors $R_{ij}$ represents a basis. In order to study these bases we establish first Proposition 1 which follows trivially from the group-theoretical properties of the group factors:

**Proposition 1:** The group factor of a standard diagram $R_{ij}$ is the tensor product of the group factors of its subdiagrams.

For a choice of orientation for the Wilson lines as the one shown in Fig. 4, the product is taken in the following way:

$$R_{ij}^{j_1j_2} = R_{ij}^{(1)}_{j_1j_2} R_{ij}^{(2)}_{j_1j_2}$$  \hspace{1cm} (3.7)

where $R_{ij}^{(1)}$ and $R_{ij}^{(2)}$ are the group factors of two subdiagrams in $R_{ij}$.

There are two simple but far-reaching facts about the basis $R_{ij}$ which we summarize in Propositions 2 and 3.

**Proposition 2:** It is always possible to choose a basis such that the $R_{ij}$ come from standard diagrams.

**Proposition 3:** The $R_{ij}$ which are tensor products can be chosen as tensor products of connected $R_{ij}$’s of lower orders.

These propositions follow from a simple fact. Using STU relations it is always possible to trade in a disconnected diagram overlapping subdiagrams by connected diagrams and disconnected diagrams containing non-overlapping subdiagrams. A basis where these propositions hold will be called canonical. One can easily see that a canonical basis shows the feature that a connected $R_{ij}$ begets a whole family of group factors of higher orders, in which it enters as a subdiagram.

The choice of a canonical basis allows us to classify the group factors into three different types:

$$R_{ij}(\lambda, \mu) = \{ r_{ij}(\lambda), r_{ij}(\mu), s_{ij}(\lambda, \mu), \text{mixed} \}$$  \hspace{1cm} (3.8)

The first two sets $r_{ij}(\lambda)$ and $r_{ij}(\mu)$ are group factors corresponding to diagrams made out of non-overlapping connected self-interaction subdiagrams. Depending on their arguments ($\lambda$ or $\mu$) they are attached to either one of the two Wilson lines. The third set ($s_{ij}(\lambda, \mu)$) contains group factors which correspond to diagrams made out of non-overlapping connected interaction subdiagrams. Finally, the fourth set contains diagrams with both non-overlapping connected interaction and self-interaction subdiagrams. A general mixed diagram is shown in Fig. 5.
4 Group-theoretical considerations

Having adopted a canonical basis for our expansion (3.1), we now show that the product (3.7) of group factors corresponding to non-overlapping subdiagrams is commutative. This property is essential in order to prove the Factorization Theorem below. Let us consider an arbitrary group factor as the one shown in Fig. 6, which represents a general connected interaction diagram; the dashed zone may include any Feynman graph. For this object we shall adopt a dual notation: it will be denoted by $U_{i_1 j_1 i_2 j_2}$ if we wish to indicate explicitly its representation indices; if not, it will be $U(\lambda, \mu)$. Before studying its properties let us recall some facts from group theory.

Let $i_1, j_1, k_1, \ldots$ and $i_2, j_2, k_2, \ldots$ be indices for the unitary irreducible representations (“irreps”) $\lambda$ and $\mu$ of the compact semisimple Lie group $G$. The product $\Psi^i_\lambda \Psi^{i_2}_{\mu}$ of two vectors corresponding to these two irreps decomposes in a Clebsch-Gordan (CG) sum of vectors $\Psi^i_\rho$ where $\rho \subset \lambda \otimes \mu$:

$$\Psi^i_\lambda \Psi^{i_2}_{\mu} = \sum_{\rho \subset \lambda \otimes \mu} \sum_{i_1 i_2 i_3 = 1} d(\rho) \left( \begin{array}{c} \lambda \\ i_1 \\ i_2 \\ i_3 \\ \rho \end{array} \right) \left( \begin{array}{c} \mu \\ j_1 \\ j_2 \\ j_3 \\ \rho \end{array} \right) \Psi^i_\rho.$$

(4.1)

This notation is abstract in the sense that the indices $i_1, i_2, \ldots$ may be, in a concrete case, composite indices. The quantity $d(\rho)$ represents the dimension of the irrep $\rho$.

The CG coefficients satisfy the completeness and orthogonality relations:

$$\sum_{\rho \subset \lambda \otimes \mu} \sum_{i_1 i_2 i_3 = 1} \left( \begin{array}{c} \lambda \\ i_1 \\ i_2 \\ i_3 \\ \rho \end{array} \right)^* \left( \begin{array}{c} \lambda \\ j_1 \\ j_2 \\ j_3 \\ \rho \end{array} \right) = \delta_{i_1 j_1} \delta_{i_2 j_2},$$

$$\sum_{i_1 i_2 i_3 = 1} \sum_{\rho \subset \lambda \otimes \mu} \left( \begin{array}{c} \lambda \\ i_1 \\ i_2 \\ i_3 \\ \rho' \end{array} \right)^* \left( \begin{array}{c} \lambda \\ i_1 \\ i_2 \\ i_3 \\ \rho \end{array} \right) = \delta_{\rho' \rho} \delta_{i_3}. \quad (4.2)$$

From now on we shall assume that repeated representation indices (the Latin indices) are summed over. Now we insert the completeness relation in the tensor described in Fig. 6.
Figure 6: An invariant tensor $U(\lambda, \mu)$.

One finds,

$$U_{i_1 i_2}^{j_1 j_2} = \sum_{\rho, \rho' \subset \lambda \otimes \mu} U_{k_1 k_2}^{l_1 l_2} \left( \lambda \mu \rho \right) \left( \lambda \mu \rho' \right)^* \left( \lambda \mu \rho \right) \left( \lambda \mu \rho' \right)^* \left( \lambda \mu \rho \right)^* \left( \lambda \mu \rho' \right)^* . \quad (4.3)$$

In our investigation of the group factors associated to Feynman diagrams, the tensors $U_{i_1 i_2}^{j_1 j_2}$ are constructed out of generators of the Lie algebra $(T_i(R)^j_k)$ contracted with structure constants $f_{abc}$, Killing-Cartan metrics $\delta^{ab}$ or unit matrices $\delta_{ij}$. A tensor such constructed is necessarily an invariant tensor [35]. As the CG coefficients are also invariant tensors, we can apply Schur's lemma to

$$U_{k_1 k_2}^{l_1 l_2} \left( \lambda \mu \rho \right) \left( \lambda \mu \rho' \right)^* \left( \lambda \mu \rho \right) \left( \lambda \mu \rho' \right)^* = U(\lambda \mu \rho) \delta_{\rho \rho'} \delta^{m} m', \quad (4.4)$$

since it is an invariant tensor with only two free indices corresponding to the representations $\rho$ and $\rho'$. Given that both $\rho$ and $\rho'$ are irreps, Schur’s lemma implies that

$$U_{k_1 k_2}^{l_1 l_2} \left( \lambda \mu \rho \right) \left( \lambda \mu \rho' \right)^* = U(\lambda \mu \rho) \delta_{\rho \rho'} \delta^{m} m', \quad (4.5)$$

where,

$$U(\lambda \mu \rho) = \frac{1}{d(\rho)} U_{k_1 k_2}^{l_1 l_2} \left( \lambda \mu \rho \right) \left( \lambda \mu \rho \right)^* . \quad (4.6)$$

Inserting (4.5) in (4.3) we arrive at a variant of the Wigner-Eckart theorem:

$$U_{i_1 i_2}^{j_1 j_2} = \sum_{\rho \subset \lambda \otimes \mu} U(\lambda \mu \rho) \left( \lambda \mu \rho \right) \left( \lambda \mu \rho \right)^* . \quad (4.7)$$

The analogy with the Wigner-Eckart comes from the fact that the free indices factorize in each term of the CG sum in a structure independent of the tensor; all the information relative to the tensor is summarized in the scalars $U(\lambda \mu \rho)$.

We turn now to a more complicated diagram, which consists of two connected non-overlapping subdiagrams as depicted in Fig. 7. It can be written as the product of two tensors of the type just considered:

$$U_{i_1 i_2}^{j_1 j_2} V_{j_1 j_2}^{p_1 p_2} . \quad (4.8)$$
Applying our result (4.7) and the relations (4.2), it follows that

\[ U_{j_1 j_2}^i 1^i_1 2^i_2 \cdot V_{p_1 p_2}^j 1^p_1 2^p_2 = \sum_{\rho \subset \lambda \otimes \mu} U(\lambda \mu \rho) V(\lambda \mu \rho) \left( \begin{array}{c} \lambda \mu \rho \\ i_1 i_2 m \end{array} \right) \left( \begin{array}{c} \lambda \mu \rho \\ p_1 p_2 m \end{array} \right). \] (4.9)

We wish to emphasize that all free indices have been separated for each irrep \( \rho \subset \lambda \otimes \mu \) in a factor independent of the structure of the tensors \( U \) or \( V \). The information relative to \( U \) and \( V \) is encoded in the scalars \( U(\lambda \mu \rho) V(\lambda \mu \rho) \). From this formula it is clear that the product of connected non-overlapping subdiagrams is commutative in the sense that the order of the subdiagrams is irrelevant for the group factor of the diagram,

\[ U_{j_1 j_2}^i 1^i_1 2^i_2 \cdot V_{p_1 p_2}^j 1^p_1 2^p_2 = V_{j_1 j_2}^i 1^i_1 2^i_2 \cdot U_{p_1 p_2}^j 1^p_1 2^p_2 \] (4.10)

This result can be generalized to diagrams with an arbitrary number of non-overlapping subdiagrams: the group factor of such a diagram does not depend on the order of the subdiagrams.

There is one further question regarding the tensors \( U \) that will be relevant in what follows. Let us allow the gauge group to be the product of two compact simple Lie groups: \( G \times G' \). The corresponding irreps will be denoted by \( \lambda \lambda' \). The generalization of the tensor \( U(\lambda, \mu) \) is \( U(\lambda \lambda', \mu \mu') \). If the diagram that represents the tensor \( U \) consists of several connected non-overlapping subdiagrams \( U^{(p)} \) with \( p = 1, \ldots, N \), it is easy to see that

\[ U(\lambda \lambda', \mu \mu') = \prod_{p=1}^N \left( U^{(p)}(\lambda, \mu) I^{\lambda'} I^{\mu'} + I^{\lambda} I^{\mu} U^{(p)}(\lambda', \mu') \right), \] (4.11)

where \( I^R \) denotes the \( d(R) \)-dimensional identity matrix. Moreover, as a result of (4.10), whenever two of these \( U^{(p)} \) corresponding to the same representations \( \lambda \) and \( \mu \) are multiplied, we need not care about the order in which they appear.
5 The Master Equation

In this section we shall demonstrate that the expansion (5.1) is a product of three factors: two of them subsume all the information relative to each of the lines separately, while the third one encodes their “linkedness”. This will be made precise in the next subsection.

A general element \( R_{ij} \) of a canonical basis would look like the diagram in Fig. 5. The subdiagrams denoted by \( r(\lambda), r(\mu) \) and \( s(\lambda, \mu) \) need not be connected; they may contain subdiagrams.

Let a given \( R_{ij} \) be composed of \( p_{ij} \) subdiagrams of type \( r \) on one Wilson line, \( q_{ij} \) subdiagrams of type \( r \) on the other Wilson line and \( t_{ij} \) subdiagrams of type \( s \); all these subdiagrams must be connected and non-overlapping. We can write symbolically:

\[
R_{ij}(\lambda, \mu) = \left\{ r_{ij}^{(p)}(\lambda), r_{ij}^{(q)}(\mu), s_{ij}^{(t)}(\lambda, \mu) \right\},
\]

where \( p = 1, \ldots, p_{ij}, q = 1, \ldots, q_{ij} \) and \( t = 1, \ldots, t_{ij} \). The indices \( ij \) of the \( r \)'s and \( s \) in (5.1) do not denote the order \( i \) in perturbation theory or the element \( j \) of the basis at each order as would be usual. Rather, they are a reminder that the subdiagrams are part of the whole diagram \( R_{ij} \). The order in perturbation theory of \( r_{ij}^{(p)} \) or \( s_{ij}^{(t)} \) will be denoted by \( \mathcal{O}(ij, p) \) and \( \mathcal{O}(ij, t) \) respectively. If the gauge group is simple, it holds that

\[
R_{ij}(\lambda, \mu) = \prod_{p=1}^{p_{ij}} r_{ij}^{(p)}(\lambda) \prod_{q=1}^{q_{ij}} r_{ij}^{(q)}(\mu) \prod_{t=1}^{t_{ij}} s_{ij}^{(t)}(\lambda, \mu). \tag{5.2}
\]

Let the gauge group be the product \( G \times G' \) as by the end of the preceding section. The generalization of (5.2) is

\[
R_{ij}(\lambda\lambda', \mu\mu') = \prod_{p=1}^{p_{ij}} \left( r_{ij}^{(p)}(\lambda) I_{ij}^{\lambda} + I_{ij}^{\lambda} r_{ij}^{(p)}(\lambda') \right) \prod_{q=1}^{q_{ij}} \left( r_{ij}^{(q)}(\mu) I_{ij}^{\mu} + I_{ij}^{\mu} r_{ij}^{(q)}(\mu') \right) \prod_{t=1}^{t_{ij}} \left( s_{ij}^{(t)}(\lambda, \mu) I_{ij}^{\lambda} I_{ij}^{\mu} + I_{ij}^{\lambda} I_{ij}^{\mu} s_{ij}^{(t)}(\lambda', \mu') \right) \tag{5.3}
\]

The last ingredient we need is the following identity, which follows from the definition of the Wilson line operator:

\[
\langle F_{\lambda\lambda'}(L_1) F_{\mu\mu'}(L_2) \rangle = \langle F_\lambda(L_1) F_\mu(L_2) \rangle \langle F_{\lambda'}(L_1) F_{\mu'}(L_2) \rangle,
\]

Inserting the expansion (5.1) in each of the factors in (5.4) we arrive at the Master Equation:

\[
\sum_{i=0}^{\infty} \sum_{j=1}^{D_i} A_{ij}^{L_1, L_2} \prod_{p=1}^{p_{ij}} \left( r_{ij}^{(p)}(\lambda) I_{ij}^{\lambda} x_{ij}^{\mathcal{O}(ij,p)} + I_{ij}^{\lambda} r_{ij}^{(p)}(\lambda') x_{ij}^{\mathcal{O}(ij,p)} \right) \tag{5.5}
\]
\[
\times \prod_{q=1}^{q_{ij}} \left( r_{ij}^{(q)}(\mu) I_{ij}^{\mu} x_{ij}^{\mathcal{O}(ij,q)} + I_{ij}^{\mu} r_{ij}^{(q)}(\mu') x_{ij}^{\mathcal{O}(ij,q)} \right) \]
\[
\times \prod_{t=1}^{t_{ij}} \left( s_{ij}^{(t)}(\lambda, \mu) I_{ij}^{\lambda} I_{ij}^{\mu} x_{ij}^{\mathcal{O}(ij,t)} + I_{ij}^{\lambda} I_{ij}^{\mu} s_{ij}^{(t)}(\lambda', \mu') x_{ij}^{\mathcal{O}(ij,t)} \right)
\]
composed of connected non-overlapping subdiagrams of types \( r \) and \( s \) respectively, and \( T \) classes of subdiagrams \( t \). The index after the semicolon labels the “class” of the subdiagram. For concreteness we shall say that there are \( P \) classes of subdiagrams \( r \)'s by \( \alpha \) and \( Q \) classes of subdiagrams \( s \)'s by \( \gamma \), and \( T \) classes of subdiagrams \( t \)'s. Let \( A_i^j(L_1, L_2) \) be the geometric factor associated to a group factor \( R_{ij}(\lambda, \mu) \) composed of connected non-overlapping subdiagrams of types \( r \) and \( s \) as in (5.1). Let

\[
A_i^j(L_1, L_2) = \left( \sum_{k=0}^{\infty} \sum_{l=1}^{D_k} A_k^l(L_1, L_2) \prod_{p=1}^{p_{kl}} r_{kl}^{(p)}(\lambda)x^{O(kl,p)} \prod_{q=1}^{q_{kl}} r_{kl}^{(q)}(\mu)x^{O(kl,q)} \prod_{t=1}^{t_{kl}} s_{kl}^{(t)}(\lambda, \mu)x^{O(kl,t)} \right) \times \left( \sum_{m=0}^{\infty} \sum_{n=1}^{D_m} A_m^n(L_1, L_2) \prod_{p=1}^{p_{mn}} r_{mn}^{(p)}(\lambda)x^{O(mn,p)} \prod_{q=1}^{q_{mn}} r_{mn}^{(q)}(\mu)x^{O(mn,q)} \prod_{t=1}^{t_{mn}} s_{mn}^{(t)}(\lambda, \mu)x^{O(mn,t)} \right).
\]

and similar relations for the \( r_{ij}^{(q)}(\mu) \) (changing also the \( p \)'s by \( q \)'s) and for the \( s_{ij}^{(t)}(\lambda, \mu) \) (changing the \( p \)'s by \( t \)'s). Equation (5.6) is an enumeration of the possible identical subdiagrams of type \( r \) attached to the Wilson line that carries the representation \( \lambda \), and similar enumerations hold for the subdiagrams type \( r \) in the other Wilson lines and for the subdiagrams type \( s \). The index after the semicolon labels the “class” of the subdiagram. For concreteness we shall say that there are \( P \) classes of subdiagrams \( r(\lambda) \), \( Q \) classes of subdiagrams \( r(\mu) \), and \( T \) classes of subdiagrams \( s(\lambda, \mu) \).

Let \( \alpha_{ij}^u(L_1) \) and \( \alpha_{kl}^v(L_2) \) be the geometric factor corresponding to \( r_{ij;u}(\lambda) \) and \( r_{kl;v}(\mu) \) respectively, and \( \gamma_{mn;w}(L_1, L_2) \) the geometric factor corresponding to \( s_{mn;w}(\lambda, \mu) \).

We can now express in a compact way the relations between different geometric factors stemming from the master equation:

\[
A_i^j(L_1, L_2) = P \prod_{u=1}^{P} \frac{1}{p_{ij}^{(u)}} \left( \alpha_{ij}^u(L_1) \right)^{p_{ij}^{(u)}} \prod_{v=1}^{Q} \frac{1}{q_{ij}^{(v)}} \left( \alpha_{ij;v}(L_2) \right)^{q_{ij}^{(v)}} \prod_{w=1}^{T} \frac{1}{t_{ij}^{(w)}} \left( \gamma_{ij;w}(L_1, L_2) \right)^{t_{ij}^{(w)}}.
\]

This equation is the generalization of the corresponding result for a single closed loop [26], and it holds only for canonical bases. Following the strategy described in [26] it is easy to conclude that Eq. (5.7) implies that the vacuum expectation value of the Wilson line operator (2.9) is the product of three exponentials:

**Factorization Theorem**

\[
\langle \mathcal{F}_\lambda(L_1) \mathcal{F}_\mu(L_2) \rangle = \langle \mathcal{F}_\lambda(L_1) \rangle \langle \mathcal{F}_\mu(L_2) \rangle \langle \mathcal{L}_{\lambda,\mu}(L_1, L_2) \rangle,
\]

where \( \mathcal{F}_\lambda(L_1) \) and \( \mathcal{F}_\mu(L_2) \) are the Wilson line operators, \( \mathcal{L}_{\lambda,\mu}(L_1, L_2) \) is the product of three exponentials.
where,

\[
\langle F_\lambda(L_1) \rangle = \exp \left\{ \sum_{i=0}^{\infty} \sum_{j=1}^{\hat{d}_i} \alpha_i^c j(L_1)r_{ij}^c(\lambda)x^i \right\}
\]

\[
\langle F_\mu(L_2) \rangle = \exp \left\{ \sum_{i=0}^{\infty} \sum_{j=1}^{\hat{d}_i} \alpha_i^c j(L_2)r_{ij}^c(\mu)x^i \right\}
\]

(5.9)

\[
\langle L_{\lambda,\mu}(L_1, L_2) \rangle = \exp \left\{ \sum_{i=0}^{\infty} \sum_{j=1}^{\hat{d}_i} \gamma_i^c j(L_1, L_2)s_{ij}^c(\lambda, \mu)x^i \right\}.
\]

In this last equation we are restoring the original meaning of the symbols \(r_{ij}\) and \(\alpha_i^j\): the index \(i\) denotes the order in perturbation theory, and \(j\) labels different independent contributions at a given order \(i\). Similarly for the indices in \(s_{ij}\) and \(\gamma_i^j\). We have added the superindex \(c\) to denote that only the connected elements of the canonical basis and their corresponding geometric factors must appear in (5.8); the expansion of the exponentials generate all the rest of the diagrams. The numbers \(\hat{d}_i\) and \(\hat{\delta}_i\) stand for the number of independent group factors \(r\) or \(s\) at order \(i\) corresponding to connected diagrams. These group factors will be called primitive group factors.
6 Opening and closing Wilson loops

Our primary aim was to obtain the Vassiliev invariants for two closed links. We begun studying the more general case of open links because there we were able to find a complete set of relations defining the algebra of the open geometric factors (eq. 5.7). The property of the $R_{ij}$ stated in Proposition 1, that is, the fact that the $R_{ij}$ of a diagram made of non-overlapping connected subdiagrams is the commutative tensor product of the group factors of its subdiagrams, was fundamental to obtain them. But this property is lacking for closed links.

At this point we have reached the central idea of this paper. Given that we cannot apply directly the formalism of [26] to closed links, we must “open” them, apply the results of the preceding sections, “close” the lines back to the original link and see how things change. We shall clarify this idea in the rest of this section.

There are many inequivalent ways of closing an open link. Given a two-component open link it is always possible to close its lines to end up with any arbitrary two-component closed link. Therefore we must define the relation between open and closed links more precisely if it is to be of any use. Let us suppose that we have a two-component closed link $L$; the most natural prescription is the following:

1. Select a point on each loop; call these two points $P$ and $P'$.
2. Eliminate a small segment of each loop, starting on the selected point. We now have a two-component open link which we shall call $\dot{L}$. Let the end points be $P$, $P + \vec{\epsilon}$ for one line and $P'$, $P' + \vec{\epsilon}'$ for the other, where $\vec{\epsilon}$ and $\vec{\epsilon}'$ are elements of $\mathbb{R}^3$ with very small components. The endpoints carry also representation indices. Let these be $i$, $j$ for one line and $i'$, $j'$ for the other.
3. Apply the formalism described in the previous sections to $\dot{L}$.
4. Let $\vec{\epsilon} \to 0$, $\vec{\epsilon}' \to 0$ and contract $i$, $j$ with $\delta_{j}^{i}$ and $i'$, $j'$ with $\delta_{j'}^{i'}$.

The first step is justified by the fact that the object of interest is the vacuum expectation value of the Wilson line of a closed two-component link in the framework of Chern-Simons gauge theory, which we know to be a link invariant:

$$\langle W_{\lambda}(C_{1}, G) W_{\mu}(C_{2}, G) \rangle = \langle \text{Tr}_{\lambda} \left( P \exp \oint_{C_{1}} A \right) \text{Tr}_{\mu} \left( P \exp \oint_{C_{2}} A \right) \rangle,$$

(6.1)

where $C_{1}$ and $C_{2}$ are the two linked loops. In order to define the line integrals in (6.1) we have to parametrize each loop, and this already introduces a selected point on each of them. These selected points can be chosen to coincide with $P$ and $P'$ in step 1.

Steps 1 and 2 are roughly represented in Fig. 8. The open link $\dot{L}$ is an auxiliary entity which enables us to extract the geometric factors $\gamma_{i}^{j}$ described in the previous section. Step 2 creates a $\dot{L}$ as similar to $L$ as possible. This step is similar to a point-splitting regularization in that the small vectors $\vec{\epsilon}$ and $\vec{\epsilon}'$ must be small enough to avoid “forgetting” the original shape of the closed link.
Step 4 defines the closing operation. This operation should be understood as a generalized trace which operates both on the representation indices $i, j, i', j'$ and on the coordinates of the endpoints of the open lines.

The $\gamma^j_i$ of the open link are not topological invariants, but rather depend on the shape of the open link even if we let the four endpoints fixed. They also depend on $\vec{e}$ and $\vec{e}'$ through their dependence on the endpoints of the open lines. The relevance of closing is that the $\gamma^j_i$ become topological invariants of the link in the limit $\vec{e} \to 0, \vec{e}' \to 0$.

These four steps describe our approach to defining numerical invariants of closed links. In what follows we shall analyze the effect of the closing operation on the geometric factors $\gamma^j_i$. It will always be understood that the open links have been generated from closed links as described above.

The perturbative expansion of (6.1) can be written as:

$$\langle W_\lambda(C_1, G)W_\mu(C_2, G) \rangle = d(\lambda)d(\mu) \sum_{i=0}^{\infty} \sum_{j=1}^{D_i} A^j_i(C_1, C_2) R_{ij}(\lambda, \mu)x^i$$

(6.2)

The notation is the same as in (3.1), except that now we are dealing with closed links. From now on, to distinguish between objects pertaining to open lines from objects pertaining to loops, we shall denote the former with a dot above them. The $R_{ij}$ are no longer tensors but numbers, because now we have to take the trace over the Wilson lines, and the $A^j_i$ represent integrals over two closed loops which can be shown not to depend on the parametrization (so they do not depend on the choice of $P$ and $P'$ described in Step 1 above).

Of course the dimension $D_i$ will be in general different from $\tilde{D_i}$ in (3.1). More precisely, it will be lower because taking the traces nullifies some group factors or makes some of them identical. Also, if some group factors were dependent when the Wilson line was open the
The trace will never render them independent. The closing operation is such that it transforms the \( \hat{R}_{ij} \) and \( \hat{A}_i^j \) of open lines in the corresponding factors of two oriented loops. Formally:

\[
\hat{R}_{ij} \longrightarrow R_{ij}, \\
\hat{A}_i^j \longrightarrow A_i^j.
\]

We will denote this operation by \( C \); it is pictorially represented for an example in Fig. 9 (a). Although we are applying \( C \) only to the independent group and geometrical factors, it obviously affects the whole set of Feynman diagrams. With respect to the group factors, the closing operation consists of taking the traces over each and all tensors corresponding to both Wilson lines.

\[
R_{ij}(G) = C[(\hat{R}_{ij})_{i_1 i_2}] := (\hat{R}_{ij})_{i_1 i_2}^i
\]

For the diagram in Fig. 9 (a) we have:

\[
(T_a^{(\lambda)}T_b^{(\lambda)})_{i_1}^{j_1} (T_a^{(\mu)}T_a^{(\mu)})_{i_2}^{j_2} \rightarrow \text{Tr}(T_a^{(\lambda)}T_b^{(\lambda)})\text{Tr}(T_a^{(\lambda)}T_a^{(\lambda)})
\]

As for the geometrical factors, closing means identifying the endpoints of each line, so that we finish with integrals over closed loops:

\[
A_i^j(C_1, C_2) = C[\hat{A}_i^j(L_1, P_1, Q_1, L_2, P_2, Q_2)]
\]

For example in Fig. 9 (a) one has:

\[
\int^{Q_1} d\bar{x}_2 \int^{x_2} d\bar{x}_1 \int^{Q_2} d\bar{y}_2 \int^{y_2} d\bar{y}_1 f(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2) \rightarrow \int d\bar{x}_2 \int^{x_2} d\bar{x}_1 \int d\bar{y}_2 \int^{y_2} d\bar{y}_1 f(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2)
\]

where \( f(\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2) \) is the corresponding integrand.

Let us analyze the example shown in Fig. 9 to observe that Proposition 1, which was introduced for open Wilson lines, does not hold for Wilson loops. For open Wilson lines we have:

\[
[s_{21}]_{i_1 i_2}^{j_1 j_2} = (T_a^{(\lambda)})_{i_1}^{m} (T_b^{(\lambda)})_{i_2}^{n} (T_a^{(\mu)})_{j_1}^{m} (T_b^{(\mu)})_{j_2}^{n} = (T_a^{(\lambda)})_{i_1}^{m} (T_a^{(\mu)})_{i_2}^{n} (T_b^{(\lambda)})_{j_1}^{m} (T_b^{(\mu)})_{j_2}^{n}
\]

but for loops:

\[
[s_{21}]_{i_1 i_2}^{j_1 j_2} = \text{Tr}(T_b^{(\lambda)}T_a^{(\lambda)})\text{Tr}(T_a^{(\mu)}T_b^{(\mu)}) \neq \text{Tr}(T_a^{(\lambda)})\text{Tr}(T_a^{(\mu)})\text{Tr}(T_b^{(\lambda)})\text{Tr}(T_b^{(\lambda)})
\]

Our next task is to apply this closing operation to the results of the previous chapter and find out the relations holding in the case of loops.

### 6.1 Factorizing out the part corresponding to knots

The aim of this subsection is to show that the factorization of the disconnected contributions that was obtained for Wilson lines in (5.8) can be partially extended to the case of loops.
Figure 9: Closed group factors do not factorize.

Owing to the different algebraic structure of the group factors, a total factorization cannot be achieved. We shall show, however, that the individual contributions from each of the Wilson loops do factorize. Our starting point is the factorization theorem contained in eq. (5.8). In that equation the terms $\langle F_{\lambda}(L_1) \rangle$ and $\langle F_{\mu}(L_2) \rangle$ are diagonal in the group indices and therefore proportional to the identity matrix in their respective representations $\lambda$ and $\mu$. This means that after closing, (5.8) becomes:

$$\langle W_{\lambda}(C_1, G)W_{\mu}(C_2, G) \rangle = \langle W_{\lambda}(C_1, G) \rangle \langle W_{\mu}(C_2, G) \rangle \langle Z_{\lambda\mu}(C_1, C_2, G) \rangle$$

(6.10)

where $W_{\lambda}(C_1, G)$ is the Wilson-loop operator in (2.2) and $\langle Z_{\lambda\mu}(C_1, C_2, G) \rangle$ is the quantity which results of applying the closing operation to $\langle L_{\lambda\mu}(L_1, L_2, G) \rangle$ in (5.8). The loops resulting from the closing of the lines $L_1$ and $L_2$ have been denoted by $C_1$ and $C_2$ respectively. This last part, which we have been calling interaction part, is the pure link contribution of the perturbative series. The meaning of equation (6.10) is, on the one hand, that the contribution from the knot invariant associated to each loop factorizes from the full vacuum expectation value (2.3), on the other hand, that one has well defined intrinsic numerical link invariants. These link invariants are indeed the geometrical factors originated from $\gamma_{i}^{c_{j}}(L_1, L_2)$ in (5.8) after performing the closing operation. Our next task is to describe the properties of these link invariants.
6.2 Numerical link invariants

Let us begin rewriting the last term in (5.10) with our new notation in which quantities for open Wilson lines are denoted with dots on top:

\[
\langle \mathcal{L}_{\lambda,\mu}(L_1, L_2) \rangle = \exp \left\{ \sum_{i=0}^{\infty} \sum_{j=1}^{\delta_i} \gamma_{ij}^c(L_1, L_2) s_{ij}^c(\lambda, \mu) x^i \right\}.
\]

(6.11)

Recall that the sum is over the primitive elements of the algebra. The closing operation does not preserve the product of the group factors present in (6.11):

\[
C[\hat{s}_{ij} \hat{s}_{kl}] \neq C[\hat{s}_{ij}]C[\hat{s}_{kl}],
\]

(6.12)

where \(\hat{s}_{ij}\) and \(\hat{s}_{kl}\) might or might not be primitive. This implies that the result of applying the closing operation to (6.11) has to be analyzed order by order. Notice also that in this operation we are losing information and therefore one expects a lower number of independent group factors. As we will observe below, it is still possible to have a notion of primitiveness for the resulting link invariants.

We introduce for the quantity \(\langle Z_{\lambda\mu}(C_1, C_2, G) \rangle\) in (6.10) the following perturbative series expansion:

\[
\langle Z_{\lambda,\mu}(C_1, C_2, G) \rangle = \exp \left\{ \sum_{i=0}^{\infty} \sum_{j=1}^{\delta_i} \gamma_{ij}^c(C_1, C_2) s_{ij}(\lambda, \mu, G) x^i \right\}.
\]

(6.13)

where \(\delta_i\) is the number of independent closed group factors. Notice that we have restored the dependence of \(s_{ij}\) on the group \(G\).

Closed geometrical factors will not simply be the closure of open ones. If two or more group factors become proportional after the closing operation, we will choose one of them as the independent one; the closed geometric factor will be the sum of these geometric factors with the appropriate factors. Once the independent group factors \(s_{ij}(\lambda, \mu, G)\) have been chosen, their corresponding geometrical factors are linear combinations of the open ones.

Using the relations (5.7) of the algebra of the open geometrical factors, we can find out the algebra of the new invariants for two-component links. Notice that in general we may not have pure diagonal relations like in (5.7), due to the presence of the linear relations of the type relating link invariants to closed geometrical factors. It might exists, however, a suitable choice of basis in which most of the relations would be diagonal. This happens at least up to order four.

The invariance of the geometrical factors \(\gamma_{ij}^c\) follows from the general properties of Chern-Simons gauge theory. Being the vacuum expectation value invariant, each term in the perturbative power series expansion is also invariant. If the contributions appearing in each term are organized in terms of the independent group factors, the coefficient of each independent group factor is invariant. These coefficients are precisely the geometrical factors \(\gamma_{ij}^c\). A simple application of the theorems in [11, 12] shows that these geometrical factors are indeed Vassiliev invariants or numerical invariants of finite type. The arguments are the same as the ones presented in [19].

In the next chapter we will construct the link invariants introduced in this section up to order four, and we will compute them for all two-component links of no more than six crossings.
7 Explicit results up to order four

The aim of this section is to apply the results of the previous sections to obtain explicit expressions for the numerical link invariants \( \gamma_i^j(C_1, C_2) \) in (3.13) up to order four in perturbation theory. We will begin analyzing first the situation corresponding to open lines. Then we will carry out the closing operation to achieve our goal.

Our first task is to choose a basis for the \( \dot{s}_{ij} (\lambda, \mu, \otimes_k G_k) \) in (3.1), with \( G_k \) simple. Among them we will select the primitive group factors, which will be the ones entering (5.11). The procedure is the following, write all possible group factors at each order, use STU and IHX to find relations among them, and extract a set of independent ones. Up to order four this computation is rather simple but it gets complicated as the order is increased. The result is depicted in Fig. 10. Among all those group factors there are only one primitive group factor at first and second orders, \( \dot{s}_{11} \) and \( \dot{s}_{22} \), two at order three, \( \dot{s}_{33} \) and \( \dot{s}_{34} \), and five at order four, \( \dot{s}_{46}, \dot{s}_{47}, \dot{s}_{48}, \dot{s}_{49} \), and \( \dot{s}_{4,10} \). Using this analysis we can then use (5.7) to write the \( \dot{\gamma}^i_j \) in terms of the primitive elements of the associated geometrical factors. One finds

\[
\dot{\gamma}^1_1 = \frac{\dot{\gamma}^1_1^n}{n!}, \quad n = 1 \ldots 4, \\
\dot{\gamma}^2_n = \frac{\dot{\gamma}^1_1^{n-2}}{(n-2)!} \dot{\gamma}_2^2, \quad n = 3, 4, \\
\dot{\gamma}^3_4 = \frac{\dot{\gamma}^2_2}{2!}, \quad \dot{\gamma}_4^4 = \dot{\gamma}_1^1 \dot{\gamma}_3^3, \quad \dot{\gamma}_4^5 = \dot{\gamma}_1^1 \dot{\gamma}_3^4.
\]

(7.1)

Notice that the primitive elements are the geometrical factors associated to the connected and independent \( \dot{s}_{ij} \). The fact that \( G = \otimes_k G_k \) is semi-simple affects directly the dimension \( \dot{\delta}_i \). If we were working with a simple group not all the diagrams pictured in Fig. 10 would be independent. For example, \( \dot{s}_{43} \) and \( \dot{s}_{44} \) would be proportional. In general, the dimension \( \dot{\delta}_i \) would be lower for the case of simple groups. This has implications for the link invariants obtained after closing and therefore one can affirm that, as in the case of knots, simple Lie algebras are not enough to determine all invariants. As expected from the factorization theorem, using relations (7.1) we can write the perturbative series of \( \langle \mathcal{L}_{\lambda, \mu}(L_1, L_2) \rangle \) up to order four in the following form:

\[
1 + x\dot{\gamma}_1^1 \dot{s}_{11} + x^2 \left[ \frac{(\dot{\gamma}_1^1)^2}{2!} \dot{s}_{11}^2 + \dot{\gamma}_2^1 \dot{s}_{12}^2 \right] + x^3 \left[ \frac{(\dot{\gamma}_1^1)^3}{3!} \dot{s}_{11}^3 + \dot{\gamma}_1^1 \dot{\gamma}_2^1 \dot{s}_{11} \dot{s}_{12} + \dot{\gamma}_1^3 \dot{s}_{33} + \dot{\gamma}_1^4 \dot{s}_{34} \right] + x^4 \left[ \frac{(\dot{\gamma}_1^1)^4}{4!} \dot{s}_{11}^4 + \frac{(\dot{\gamma}_1^1)^2}{2!} \dot{\gamma}_2^1 \dot{s}_{11} \dot{s}_{12} + \dot{\gamma}_2^2 \dot{s}_{22}^2 + \dot{\gamma}_1^1 \dot{\gamma}_3^3 \dot{s}_{11} \dot{s}_{33} + \dot{\gamma}_1^1 \dot{\gamma}_3^4 \dot{s}_{11} \dot{s}_{34} + \sum_{j=6}^{10} \dot{\gamma}_4^j \dot{s}_{4j} \right] + O(x^5).
\]

(7.2)

Our next task is to apply the closing operation to this expression. Recall that the open group factors are depicted in Fig. 10. We will carry out this analysis order by order, describing in detail the choices made.

i) Order one. In this case one finds that the resulting closed group factor vanishes unless we extend the scope of Lie groups under consideration beyond the semi-simple case. Indeed, one has:

\[
s_{11} = C[\dot{s}_{11}] \neq 0 \iff G = \otimes_k G_k \otimes U(1)
\]

(7.3)
Figure 10: Basis up to order four.
Nothing is lost restricting ourselves to the semi-simple case since the geometrical factor $\gamma_1^1$, which is multiplying $s_{11}$ in the perturbative series, appears at higher orders. We will maintain the gauge group $G$ to be semi-simple and therefore $\delta_1 = 0$. For these groups there is not linear term in the perturbative series expansion (7.3). Recall that for knots linear terms appear only if a non-trivial framing is attached to the knot. Thus according to the factorization formula (6.10), if the group is semi-simple and the two components of the link are taken in the standard framing ($n = 0$), no linear term in $x$ would appear in the expansion of any polynomial link invariant.

**ii) Order two.** In this case one finds,

\[ C[\dot{s}_{22}] = 0, \]
\[ s_{21} = C[\dot{s}_{11} \dot{s}_{11}], \]

and therefore one has $\delta_2 = 1$. Notice that in this case $C[\dot{s}_{22}] = 0$ even if the gauge group had a $U(1)$ factor. In (7.4) and in similar equations below the product of group factors inside the squared brackets has to be understood as a tensor product.

**iii) Order three.** In this case one finds the following set of relations:

\[ s_{31} = C[\dot{s}_{11} \dot{s}_{11} \dot{s}_{11}], \]
\[ s_{32} = C[\dot{s}_{11} \dot{s}_{22}], \]
\[ C[\dot{s}_{33}] = 0, \quad C[\dot{s}_{34}] = 2s_{32}, \]

so that $\delta_3 = 2$.

**iv) Order four.** At this order the number of relations increases notably:

\[ s_{41} = C[\dot{s}_{11} \dot{s}_{11} \dot{s}_{11} \dot{s}_{11}], \]
\[ s_{42} = C[\dot{s}_{11} \dot{s}_{11} \dot{s}_{22}], \]
\[ s_{43} = C[\dot{s}_{22} \dot{s}_{22}], \]
\[ C[\dot{s}_{11} \dot{s}_{33}] = -s_{43}, \quad C[\dot{s}_{11} \dot{s}_{34}] = s_{42}, \]
\[ C[\dot{s}_{46}] = 0, \quad C[\dot{s}_{47}] = s_{43}, \]
\[ C[\dot{s}_{48}] = 2s_{43}, \quad C[\dot{s}_{49}] = 4s_{43}, \quad C[\dot{s}_{4,10}] = 0. \]

The corresponding dimension is $\delta_4 = 3$.

Notice that, especially at order four, the dimensions have decreased significantly with respect to the case of open lines. The independent group factors that we have chosen are pictured in Fig. [11]. This choice plus the relations (7.1) lead to the following expressions for the geometrical factors:

\[ \gamma_1^1 = C[\dot{\gamma}_1^1], \]
\[ \gamma_2^1 = \frac{C[\dot{\gamma}_1^1]^2}{2!}, \]
\[ \gamma_3^1 = \frac{C[\dot{\gamma}_1^1]^3}{3!}, \quad \gamma_3^2 = C[\dot{\gamma}_1^1]C[\dot{\gamma}_2^2] + 2C[\dot{\gamma}_3^4], \]
Figure 11: Basis of group factors up to order four.

\[
\gamma_4^1 = \frac{C[\gamma_1^1]^4}{4!}, \quad \gamma_4^2 = \frac{C[\gamma_1^1]^2}{2!}C[\gamma_2^2] + C[\gamma_1^1]C[\gamma_3^4], \\
\gamma_4^3 = \frac{C[\gamma_2^2]^2}{2!} - C[\gamma_1^1]C[\gamma_3^3] + C[\gamma_4^7] + 2C[\gamma_4^8] + 4C[\gamma_4^9].
\] (7.7)

From these equations one can read the algebra of invariants for two-component links:

\[
\gamma_2^1 = \frac{(\gamma_1^1)^2}{2!}, \\
\gamma_3^1 = \frac{(\gamma_1^1)^3}{3!}, \\
\gamma_4^1 = \frac{(\gamma_1^1)^4}{4!}, \quad \gamma_4^2 = \frac{\gamma_1^1\gamma_3^2}{2},
\] (7.8)

so we have three primitive invariants up to this order: \(\gamma_1^1, \gamma_3^2\) and \(\gamma_4^3\). Using this results we can finally write the power series expansion for \(\langle Z_{\lambda,\mu}(C_1, C_2, G) \rangle\) in (6.13) up to order four:

\[
1 + x^2\left[\frac{(\gamma_1^1)^2}{2!}s_{21}\right] + x^3\left[\frac{(\gamma_1^1)^3}{3!}s_{31} + \gamma_3^2 s_{32}\right] + x^4\left[\frac{(\gamma_1^1)^4}{4!}s_{41} + \frac{\gamma_1^1\gamma_3^2}{2}s_{42} + \gamma_4^3 s_{43}\right],
\] (7.9)

where each of the quantities entering this expression are given in (7.4), (7.5), (7.6) and (7.7).

Notice that although there is no natural notion of primitiveness for the group factors entering the power series expansion of \(\langle Z_{\lambda,\mu}(C_1, C_2, G) \rangle\) in (6.13), we have obtained one for the geometrical factors from Eq. (5.7). This last equation is a consequence of the master equation (5.6) and therefore of the property of factorization of vacuum expectation values in Chern-Simons gauge theory.

In the rest of this section we provide the explicit integral expressions of the primitive invariants \(\gamma_1^1, \gamma_3^2\) and \(\gamma_4^3\). There are two forms to face this computation. The first one
consists in writing down the relations for the geometrical factors contained in (7.7). The second form does not use relations (7.7) and one just writes down the general form of the contribution in terms of Feynman diagrams for a product of Wilson loops at a given order, organizing the expression so obtained in terms of independent group factors. In this second approach one should obtain relations (7.8) as a consequence. Indeed, we have verified that the adequate parts of the contributions factorize confirming predictions (7.8) from the factorization theorem. We will write down the expressions for the primitives \( \gamma_{11}^{1} \), \( \gamma_{23}^{1} \), \( \gamma_{34}^{1} \), using the second approach. Of course, for \( \gamma_{11}^{1} \) both approaches lead to the same expression. For \( \gamma_{23}^{1} \) and \( \gamma_{34}^{1} \), however, the expressions are rather different. Its equivalence is one more prediction of the factorization theorem. Though one could think that the first approach would lead to simpler expressions, it turns out that this is not the case. The integral expressions for \( \dot{\gamma}_{34}^{4} \), \( \dot{\gamma}_{74}^{8} \), \( \dot{\gamma}_{84}^{4} \) and \( \dot{\gamma}_{94}^{4} \) are rather long as compared to the ones obtained in the second approach.

In order to obtain the expressions for \( \gamma_{11}^{1} \), \( \gamma_{23}^{1} \) and \( \gamma_{34}^{1} \) one must take into account that each term which contributes is made up of the geometrical factor of a diagram whose group factor depends on the corresponding independent factor, multiplied by the constant that determines this dependence. The diagrams entering in \( \gamma_{11}^{1} \), \( \gamma_{23}^{1} \) and \( \gamma_{34}^{1} \) are pictured in Figs. 12, 13 and 14 respectively. All integration paths are taken anticlockwise. Before writing down these expressions, we will introduce some notation that will simplify them considerably. (The number of lines needed to write the integral \( \gamma_{23}^{1} \) is big, but still reasonable. But for \( \gamma_{34}^{1} \) the number of pages will certainly be unreasonable). We shall write the multiple integral over the first Wilson loop as,

\[
\oint dx_n^\mu \int^{x_n} d x_{n-1}^\mu \cdots \int^{x_2} d x_1^\mu \equiv \oint_{1<2<\cdots<n} dx,
\]

while the one over the second loop,

\[
\oint dy_m^\nu \int^{y_m} d y_{m-1} \cdots \int^{y_2} d y_1 \equiv \oint_{1<2<\cdots<m} dy.
\]

In these equations \( n \) and \( m \) label the number of points over the first and second Wilson loops, respectively. Also, the variable \( x \) will always correspond to points attached to the first loop, and the variable \( y \) to the second. Internal vertices will be labelled by variables \( \omega_i \). One finds four types of propagators,

\[
\begin{align*}
 p(x_i, y_j) &= \Delta_{\mu_i \nu_j} (x_i - y_j), & p(\omega_i, \omega_j) &= \Delta_{\theta_i \theta_j} (\omega_i - \omega_j), \\
 p(x_i, \omega_j) &= \Delta_{\mu_i \theta_j} (x_i - \omega_j), & p(\omega_i, y_j) &= \Delta_{\theta_i \nu_j} (\omega_i - y_j),
\end{align*}
\]

depending if they connect points on the Wilson lines, two internal vertices, or one point on a Wilson line and the other on an internal vertex. In (7.12) \( \Delta_{\mu \nu}(x-y) \) is the quantity defined in (3.6).

From the Feynman rules, and taking into account that one has to extract a factor \( x_i \) at order \( i \) where \( x = ig^2/2 \), one finds the following set of effective rules for the computation of the geometrical factors associated to the diagrams shown in Figs. 12, 13 and 14:

- One \( p(, \) as in (7.12) for each internal line, whose variables are fixed by its end points, with a factor \( \frac{1}{4} \) for each.
- Two path ordered integrals, one over the \( n \) points on the first loop, and the other over the \( m \) on the second.
- A factor \( \epsilon^{\alpha_i \beta_i \gamma_i} = \epsilon_i \) and a three-dimensional integral \( \int d^3 \omega_i \), for each three-vertex, with the lines in the vertex ordered counterclockwise. The \( p(, ) \) attached to the vertex will be written in this order so that one can keep track of it.
- A factor \( 2^i \), where \( i \) is the order in perturbation theory.
- Finally, there might be a multiplicative factor coming from the relation between the group factor of a given diagram and the one chosen as independent.

Notice that in these rules there is not a factor of the form \( i \) to some power as one could naively expect form the Feynman rules. The reason for this is that the resulting power of \( i \) is always absorbed in the parameter \( x \) at each order with no sign left.

The integral expresion for \( \gamma_1^1 \) is easily obtained applying these rules to the diagram shown in Fig. 12:

\[
\gamma_1^1 = \frac{1}{2} \oint dx \oint dy \ p(x, y).
\] (7.13)

This expression is twice the integral defining the linking number of the two components of the link under consideration. Therefore, our first numerical link invariant turns out to be one of the classical link invariants. At higher orders, however, the numerical invariants that we present are new.

For the only primitive invariant at order three (recall that there are not primitive invariants at order two), \( \gamma_3^2 \), the corresponding diagrams are depicted in Fig. 13. In writing their corresponding integrals it is important to take into account the following two observations. First, notice that if the whole diagram but the two Wilson loops is not symmetric under a reflection around its medium vertical axis, the diagram obtained reflecting the graph also contributes. Second, each diagram represents a whole class of contributions: all the distinct contributions which can be obtained permuting the order in which the legs are attached to
the Wilson loops. One finds:

\[
\gamma_3^2 = \frac{1}{8} \int_{1<2<3} dx \int_{1<2<3} dy \left[ p(x_1, y_2)p(x_2, y_1)p(x_3, y_1) + p(x_1, y_1)p(x_2, y_3)p(x_3, y_2) + p(x_1, y_3)p(x_2, y_2)p(x_3, y_1) \right] \\
+ \frac{1}{32} \int_{1<2<3} dx \int_{1<2<3} dy \int d^3 \omega \epsilon \left[ p(x_2, \omega)p(x_3, \omega)p(x_1, \omega)p(\omega, y_1) + p(x_1, y_1)p(x_2, \omega)p(x_3, \omega)p(\omega, y_2) + p(x_1, \omega)p(x_2, \omega)p(x_3, \omega)p(\omega, y_1) + p(x_1, y_2)p(x_2, \omega)p(x_3, \omega)p(\omega, y_1) \right] \\
- \frac{1}{32} \int_{1<2} dx \int_{1<2} dy \int d^3 \omega \epsilon \left[ p(x_1, \omega)p(\omega, y_3)p(x_2, y_1)p(x_2, y_2) + p(x_1, y_1)p(x_2, \omega)p(\omega, y_3)p(x_2, y_2) + p(x_1, \omega)p(x_2, \omega)p(\omega, y_3)p(x_2, y_2) + p(x_1, y_2)p(x_2, \omega)p(x_2, y_1)p(x_2, y_2) \right] \\
+ \frac{1}{64} \int_{1<2} dx \int_{1<2} dy \int d^3 \omega_1 \int d^3 \omega_2 \epsilon \epsilon_2 \left[ p(x_1, \omega_1)p(\omega_1, \omega_2)p(x_2, \omega_2)p(\omega_2, y_2)p(\omega_1, y_1) + p(x_1, \omega_2)p(x_2, \omega_1)p(\omega_2, y_2)p(\omega_1, \omega_2)p(\omega_1, y_1) \right] \\
- \frac{1}{8} \int_{1<2<3<4} dx \int_{1<2} dy \left[ p(x_1, x_3)p(x_2, y_1)p(x_4, y_2) + p(x_1, y_1)p(x_2, x_4)p(x_3, y_2) + p(x_1, x_3)p(x_2, y_2)p(x_4, y_1) + p(x_1, y_2)p(x_2, x_4)p(x_3, y_1) \right] \\
- \frac{1}{8} \int_{1<2} dx \int_{1<2<3<4} dy \left[ p(x_1, y_2)p(x_2, y_3)p(y_1, y_3) + p(x_1, y_1)p(x_2, y_3)p(y_2, y_4) + p(x_1, y_4)p(x_2, y_2)p(y_1, y_3) + p(x_1, y_3)p(x_2, y_1)p(y_2, y_4) \right].
\] (7.14)

In (7.14) we have written the integrands in such a way that the relative minus signs between the different terms in a given integral, which may arise from the group factor dependence of some of the permutations, are reabsorbed by the order in which propagators are written. The overall sign before each integral is taken to be the one associated to the choice of permutation drawn in the figures.

Before writing the integral expression for \( \gamma_4^3 \) we are going to introduce a new notation which notably simplifies our formulae. We will make the substitution:

\[
p(x_i, y_j) \rightarrow \theta_{i,j}, \quad p(\omega_i, y_j) \rightarrow \theta_{i,j},
\] (7.15)
in such a way that indices with a bar above them label the variables $y$, while the others label the variables $x$ or $\omega$. There is no confusion between them as the indices from the $\omega$ variables will appear repeated three times. Of course, the $\theta$’s will be written preserving the order of the $p(.,.)$’s, which is that of the propagators entering in each three-vertex. Given a topology, for example any of the diagrams pictured in Fig. 13, our integral has a term for each of the permutations whose group factor gives a contribution to the invariant. These permutations are realized in a given order of the variables $\{x_1, \ldots , x_n; y_1, \ldots , y_m; \omega_1, \ldots , \omega_t\}$ in the integrand. One can always choose one of them as a reference, and make a change of variables in the others, applying the inverse of the given permutation in each case. We will end up with a sum of integrals with the same integrand but different domains of integration:

$$\oint_{C} dx \oint_{C} dy \int d\omega_1 \cdots d\omega_t \sum_{\sigma} f(\sigma(x_1 \cdots \omega_t))$$

$$= \sum_{\sigma^{-1}} \oint_{C} dx \oint_{C} dy \int d\omega_{\sigma^{-1}(1)} \cdots d\omega_{\sigma^{-1}(t)} f(x_1, \ldots , \omega_t) \quad (7.16)$$

We will write this integral in a more compact form, defining the domain of integration in the following way

$$\int_{C_{n,m}} f = \frac{1}{N_d} \sum_{\sigma^{-1}} \oint_{C_{\sigma^{-1}(1) \cdots \sigma^{-1}(n)}} dx \oint_{C_{\sigma^{-1}(1) \cdots \sigma^{-1}(m)}} dy \int d\omega_{\sigma^{-1}(1)} \cdots \int d\omega_{\sigma^{-1}(t)} f$$

$$\quad (7.17)$$

where $N_d$ is the number of different domains. So $C_{n,m}$ represents a sum of integrals of dimensions $n + m + 3t$, where the value of $t$ is read from the number of repeated indices in the integrand, which is a product of the $\theta$’s defined in (7.15). Using this notation we can rewrite the invariant $\gamma_3^2$ in (7.14) as:

$$\gamma_3^2 = \frac{3}{8} \int_{C_{3,3}} \theta_{1,2} \theta_{2,1} \theta_{3,3} + \frac{3}{16} \int_{C_{3,2}} \theta_{2,4} \theta_{1,4} \theta_{3,2} \theta_{4,1} - \frac{3}{16} \int_{C_{2,3}} \theta_{1,4} \theta_{4,2} \theta_{4,1} \theta_{2,3}$$

$$+ \frac{1}{32} \int_{C_{2,2}} \theta_{1,3} \theta_{3,4} \theta_{2,4} \theta_{4,2} \theta_{2,3} - \frac{1}{2} \int_{C_{4,2}} \theta_{1,3} \theta_{2,1} \theta_{4,2} - \frac{1}{2} \int_{C_{3,4}} \theta_{1,2} \theta_{2,4} \theta_{1,3}$$

$$\quad (7.18)$$

Let us now write down the integral expression for $\gamma_3^3$. Diagrams contributing to it, up to permutations, are pictured in Fig. 14. For a non symmetric Feynman graph, the reflected diagram also contributes, although we have not drawn it. Notice that at this order we encounter the first diagram coming from the ghost sector, which is the one with dashed lines in the figure. The last three diagrams are considered different topologies, because of the different relative position between the interaction and the loop subdiagrams.

As for the ghost diagram, we follow similar conventions: $\hat{\theta}$ stands for the ghost propagator. The three-vertex between the ghosts and gauge fields introduces, besides the three dimensional integration, a derivative in one of the variables in the ghost propagator. Recall that this vertex is not antisymmetric, so the order in which propagators are written is not important.

$$\hat{\theta}_{i,j} = \partial_i D(x_i - y_j) \quad (7.19)$$
Figure 14: Diagrams contributing to $\gamma_4^2$. 
where $D(x_i - y_j)$ is the geometrical part of the ghost propagator written in (2.8). The order of the subindices in $\delta_{i,j}$ keeps track of the variable on which the derivative acts.

The kind of multiplicative factors entering in this term are of the same nature as before, i.e., $2^i$ for order $i$ perturbation theory, $1/4$ for each propagator, and the factors coming from the group factor dependence and the number of different domains (or diagrams contributing to that topology). There is also an extra factor 2 coming from the fact that the two orientations of the ghost loop must be taken into account. Factors of $i$ in diagrams involving ghosts are also reabsorbed in the perturbation parameter with no sign left.

Using the rules described above our expression for the invariant $\gamma_i^3$ is the following:

$$
\gamma_i^3 = \frac{1}{4} \int_{C_{4,4}} \theta_{1,2} \theta_{2,1} \theta_{3,4} \theta_{4,3} + \frac{7}{64} \int_{C_{4,3}} \theta_{1,2} \theta_{2,1} \theta_{3,5} \theta_{5,3} \theta_{4,5} + \frac{7}{64} \int_{C_{4,4}} \theta_{1,2} \theta_{2,1} \theta_{3,5} \theta_{4,5} \theta_{5,3} \\
+ \frac{1}{64} \int_{C_{4,2}} \theta_{1,5} \theta_{5,1} \theta_{2,5} \theta_{3,6} \theta_{6,2} \theta_{4,6} + \frac{1}{64} \int_{C_{4,2}} \theta_{5,1} \theta_{5,2} \theta_{1,5} \theta_{6,3} \theta_{6,4} \theta_{1,6} - \frac{9}{256} \int_{C_{3,3}} \theta_{1,4} \theta_{4,1} \theta_{4,2} \theta_{2,5} \theta_{5,3} \theta_{5,3} \\
- \frac{9}{256} \int_{C_{3,3}} \theta_{1,1} \theta_{2,5} \theta_{5,6} \theta_{5,3} \theta_{6,2} \theta_{6,1} + \frac{3}{64} \int_{C_{4,2}} \theta_{1,1} \theta_{2,5} \theta_{5,2} \theta_{5,6} \theta_{6,4} \theta_{6,3} + \frac{3}{64} \int_{C_{4,2}} \theta_{1,1} \theta_{2,5} \theta_{5,2} \theta_{5,6} \theta_{6,3} \theta_{6,4} \\
+ \frac{1}{512} \int_{C_{2,2}} \theta_{1,3} \theta_{3,4} \theta_{3,6} \theta_{5,6} \theta_{6,2} \theta_{4,1} \theta_{4,5} \theta_{5,2} + \frac{1}{64} \int_{C_{3,2}} \theta_{1,4} \theta_{4,1} \theta_{4,5} \theta_{5,2} \theta_{5,6} \theta_{6,3} \theta_{6,2} \\
+ \frac{1}{64} \int_{C_{3,2}} \theta_{1,4} \theta_{4,1} \theta_{4,5} \theta_{5,6} \theta_{5,2} \theta_{6,2} \theta_{6,3} + \frac{1}{32} \int_{C_{3,3}} \theta_{1,4} \theta_{4,2} \theta_{4,5} \theta_{2,1} \theta_{5,3} \theta_{5,3} \\
+ \frac{3}{512} \int_{C_{3,2}} \theta_{1,4} \theta_{4,1} \theta_{4,5} \theta_{5,6} \theta_{5,2} \theta_{6,2} \theta_{6,3} - \frac{3}{512} \int_{C_{3,2}} \theta_{1,4} \theta_{4,1} \theta_{4,5} \theta_{5,2} \theta_{5,6} \theta_{6,3} \theta_{6,2} \\
- \frac{15}{32} \int_{C_{5,2}} \theta_{1,3} \theta_{2,1} \theta_{4,6} \theta_{6,2} \theta_{5} - \frac{15}{32} \int_{C_{5,2}} \theta_{3,1} \theta_{3,1} \theta_{6,3} \theta_{6,5} \theta_{2,6} + \frac{3}{16} \int_{C_{4,3}} \theta_{1,1} \theta_{2,4} \theta_{3,5} \theta_{5,2} \theta_{5,3} \\
+ \frac{3}{16} \int_{C_{3,4}} \theta_{1,1} \theta_{2,3} \theta_{3,5} \theta_{5,2} + \frac{1}{2} \int_{C_{4,4}} \theta_{1,1} \theta_{2,4} \theta_{3,3} \theta_{2,4} - \frac{15}{16} \int_{C_{5,3}} \theta_{1,3} \theta_{2,1} \theta_{4,3} \theta_{5,2} \\
- \frac{15}{16} \int_{C_{5,3}} \theta_{1,3} \theta_{2,1} \theta_{5,6} \theta_{5,2} \theta_{6,3} \theta_{2,4} - \frac{1}{32} \int_{C_{4,2}} \theta_{1,5} \theta_{5,1} \theta_{5,6} \theta_{6,2} \theta_{6,3} \theta_{2,4} \\
- \frac{1}{32} \int_{C_{2,4}} \theta_{1,5} \theta_{5,1} \theta_{5,6} \theta_{6,2} \theta_{2,4} + \frac{3}{8} \int_{C_{6,2}} \theta_{1,3} \theta_{2,1} \theta_{4,6} \theta_{5,2} + \frac{3}{8} \int_{C_{6,2}} \theta_{1,3} \theta_{1,2} \theta_{4,6} \theta_{2,5} \\
- \frac{5}{32} \int_{C_{5,2}} \theta_{2,1} \theta_{4,2} \theta_{1,6} \theta_{5,6} \theta_{6,5} - \frac{5}{32} \int_{C_{2,5}} \theta_{1,2} \theta_{2,4} \theta_{6,1} \theta_{5,6} \theta_{3,6} + \frac{3}{4} \int_{C_{6,2}} \theta_{1,4} \theta_{2,1} \theta_{3,6} \theta_{5,2} \\
+ \frac{3}{4} \int_{C_{6,2}} \theta_{1,4} \theta_{4,2} \theta_{3,6} \theta_{2,5} + \frac{1}{2} \int_{C_{6,2}} \theta_{1,4} \theta_{2,6} \theta_{1,3} \theta_{5,2} + \frac{1}{2} \int_{C_{2,6}} \theta_{1,4} \theta_{2,6} \theta_{1,3} \theta_{2,5} \\
+ \frac{1}{256} \int_{C_{2,2}} \theta_{1,3} \theta_{2,4} \theta_{1,6} \theta_{2,5} \theta_{3,4} \theta_{4,5} \theta_{5,6} \theta_{6,3}, \quad (7.20)
$$

where the last term stands for the ghost diagram contribution.
8 Numerical link invariants for links up to six crossings

In this section we present the results of a numerical computation of the Vassiliev invariants for links up to six crossings up to order four. Although we know integral expressions for the $\gamma^2_i$, we will not evaluate them, as we have a shorter way to proceed. The computation will be carried out using information of the l.h.s. of (6.2) coming from the polynomial invariants for links. Given a link, we may use the polynomials defined for different Lie groups and representations, and compare them with the r.h.s. of (6.2). As all the group dependence is encoded in the $R_{ij}$, and the $\gamma^2_i$ only depend on the link under consideration, we end with a set of linear equations for them. As in [19] we could consider the following cases: $SO(N)$ in its fundamental representation (Kauffman polynomial [5]), $SU(N)$ in its fundamental representation (HOMFLY polynomial [4]), $SU(2)_j$ in an arbitrary spin $j$ representation (Jones and Akutsu-Wadati polynomials [3, 8, 34, 36]), and $SU(N) \times SO(N)$ also in the fundamental representation. All these invariants are known and can be collected from the literature (for example, [37] and [34]). At the order in perturbation theory considered in this work, however, it is enough to consider the HOMFLY and the Kauffman polynomials for the links under study. They are listed in Appendix C.

The structure of the computation to be carried out is as follows. Once the polynomial invariant corresponding to the l.h.s. of (6.2) is collected, one replaces the variable $q$ by $e^x$ and expands in powers of $e^x$. For the case considered here we only need the expansion up to order four. The coefficients of $e^x$ are either polynomials in $N$ or polynomials in $j$. On the other hand, in the r.h.s. of (6.2), the group factors are the ones given diagrammatically in Fig. 11; their explicit expressions are written in Appendix A. Again, these group factors are polynomials in $N$ or $j$. Comparing both sides of (6.2) leads to a series of linear equations for the geometrical factors $\gamma^2_i$ that determine uniquely all the $\gamma^2_i$ up to order four. These values are listed in the following table. One can check that the algebraic relations in (7.8) hold, and that the $\gamma^2_i$ are all rational numbers. The notation used to label links is the same one as in [37]. The big number represents the number of crossings, the superscript the number of link components, and the subscript refers just to a standard ordering of links with a given number of crossings. The sign stands for the relative orientation between the two components: it is + when the linking number is taken positive, and − otherwise. If there is no sign, one of the links is obtained from the other by reversing the orientation of space. While the invariants of even order remain unchanged after reversing the orientation of space, the ones of order odd change sign.
Recall that only $\gamma_1^1$, $\gamma_3^2$ and $\gamma_4^3$ are primitive invariants. The results presented in the table are consistent with our interpretation of the invariant $\gamma_1^1$ as a twice the linking number of the two components. It is not clear, however, if the other two, $\gamma_3^2$ and $\gamma_4^3$ are related to some known numerical invariants.

| Link | $\gamma_1^1$ | $\gamma_2^1$ | $\gamma_3^1$ | $\gamma_3^2$ | $\gamma_4^1$ | $\gamma_4^2$ | $\gamma_4^3$ |
|------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|
| $2^2_1$ | 2 | 2 | 4/3 | 2/3 | 2/3 | 2/3 | 0 |
| $4^2_{1+}$ | 4 | 8 | 32/3 | 28/3 | 32/3 | 56/3 | 8 |
| $4^2_{1-}$ | -4 | 8 | -32/3 | -4/3 | 32/3 | 8/3 | 0 |
| $5^2_1$ | 0 | 0 | 0 | -8 | 0 | 0 | -8 |
| $6^2_{1+}$ | 6 | 18 | 36 | 34 | 54 | 102 | 48 |
| $6^2_{1-}$ | -6 | 18 | -36 | -2 | 54 | 6 | 0 |
| $6^2_2$ | 6 | 18 | -36 | -18 | 54 | 54 | 16 |
| $6^2_{3+}$ | 4 | 8 | 32/3 | 52/3 | 32/3 | 104/3 | 32 |
| $6^2_{3-}$ | -4 | 8 | -32/3 | 20/3 | 32/3 | -40/3 | 8 |
9 Conclusions

In this paper we have presented a generalization of the approach introduced in [19] for knots to the case of two-component links. There is a very fundamental difference between both approaches since the simple algebraic structure for group factors in the case of knots is not present for links. There is, however, a rather similar algebraic structure if one considers open links. This fact leads us to introduce an opening and closing operation and to deal first with the analysis of the algebraic structure for open links. After showing the commutative nature of the tensor product involved in that algebraic structure we are able to prove a factorization theorem similar to the one first introduced in [26]. This theorem allows, on the one hand, to factorize the dependence on numerical knot invariants corresponding to each of the components, and therefore to isolate the pure link contributions, on the other hand, to define a notion of primitiveness for these pure link invariants. After closing, the relations derived from the factorization theorem play a fundamental role to obtain relations among numerical invariants for links, and a notion of primitiveness is also defined for these invariants. The number \( \hat{\delta}_i \) of primitive numerical link invariants at order \( i \) are 1, 0, 1 and 1, for \( i = 1, \ldots, 4 \) respectively. The first one can be identified with twice the linking number of the two components of the link.

Using the theorems in [11, 12] we have been able to argue that the geometrical factors \( \gamma^j_i \) are Vassiliev invariants or numerical invariants of finite type. Our work provides explicit integral expressions for some of the simplest cases. We have not been able to identify the invariants \( \gamma_3^2 \) and \( \gamma_4^3 \) with known ones.

We have used the methods first introduced in [19] to compute invariants up to order four for some simple links. It turns out that up to this order it is enough to use the information derived from the HOMFLY and the Kauffman polynomials. We have obtained a set of rational numbers which are consistent with (7.8), which is a direct consequence of the factorization theorem. It is not clear from our results that a natural normalization which makes all the primitive invariants integer-valued (as in the case of knot invariants [19, 23]) exists.

This work opens a variety of investigations. First, one should try to compute higher-order invariants. The cases studied in this work are the simplest ones and should be considered as examples to implement our program for numerical invariants. Second, the approach should be generalized to \( n \)-component links. In particular, one should study how the factorization theorem generalizes in that case and how many classes of invariants can be defined. Third, one should analyze the properties of the \( \hat{\gamma}_i^j \). These quantities are not ambient isotopy invariants since they are gauge dependent but they might be interesting in some gauges if one is able to show that they preserve some form of isotopy. This seems to be the case as shown in [40] and the resulting invariants might be related to string link invariants [11]. Finally, the invariants presented in this paper should be also regarded from the approach proposed in [24]. We plan to report on these and some other issues related to Vassiliev invariants in future work.
In this appendix we present a summary of our group-theoretical conventions. We choose the generators of the Lie algebra $A$ to be antihermitian such that

$$[T^a, T^b] = -f^{ab}_{\ c}T^c,$$  \hspace{1cm} (A.1)

where $f^{ab}_{\ c}$ are the structure constants. These satisfy the Jacobi identity,

$$f^{ab}_{\ e}f^{ec}_{\ d} + f^{cb}_{\ e}f^{ae}_{\ d} + f^{ac}_{\ e}f^{be}_{\ d} = 0$$ \hspace{1cm} (A.2)

The generators are normalized in such a way that for the fundamental representation,

$$\text{Tr}(T^a T^b) = -\frac{1}{2} \delta_{ab},$$ \hspace{1cm} (A.3)

where $\delta_{ab}$ is the Kronecker delta. This can always be done for compact semisimple Lie algebras which is the case considered in this paper.

The generators $T^a$ in the adjoint representation coincide with the structure constants,

$$(T^a)^b_{\ c} = f^{ab}_{\ c} \hspace{1cm} \text{(adjoint representation).}$$ \hspace{1cm} (A.4)

The quadratic Casimir in the adjoint representation, $C_A$, is defined as

$$f^{ad}_{\ c}f^{bc}_{\ d} = C_A \delta^{ab}.$$ \hspace{1cm} (A.5)

The value of $C_A$ for the groups $SU(N)$ and $SO(N)$ is $-N$ and $-\frac{1}{2}(N-2)$ respectively. The Killing metric is chosen to be the identity matrix and therefore one can lower and raise group indices freely. For the case under consideration $f^{abc}$ is totally antisymmetric.

The convention chosen in (A.1) seems unusual but it is the most convenient when the Wilson line is defined as in (2.2). If we had chosen $if^{abc}$ instead of $-f^{abc}$, the exponential of the Wilson line would have had $ig$ instead of $g$. Our convention also introduces a $-1$ in the gauge vertex (2.8)

Our aim is to calculate the independent group structures appearing in the perturbation series expansion of a two-component link (6.2), for some given representations of the Lie algebra. For the factorized knot part, they correspond to the Casimirs of that algebra, and where already worked out in [19]. Here we will restrict ourselves to the pure link factors. They can be thought as generalized Casimirs, living in the tensor product of two representations of the enveloping algebra, as they general form is a product of two traces over invariant tensors:

$$s = \text{Tr}_{R_1}(f^{abc}T^{(R_1)}_dT^{(R_1)}_e\ldots)\text{Tr}_{R_2}(f^{qrs}T^{(R_2)}_aT^{(R_2)}_v\ldots)$$ \hspace{1cm} (A.6)

where all the indices are repeated and summed over. The evaluation procedure is similar to the case for knots [38, 39, 19]. First, we get rid of the structure constants using the commutation relations, so we end with a sum of terms with the form (A.6) but made only of products of generators. Secondly, we need some expression for the tensor product of generators with the same index in arbitrary representations.

$$(T^{(R_1)}_a)^j_i \otimes (T^{(R_2)}_a)^l_k$$ \hspace{1cm} (A.7)
These group-theoretical objects are called projection operators. They are explicitly known for every classical Lie group except $E_8$ when both, $R_1$ and $R_2$, stand for the fundamental representation. In the case of $SU(N)$ the projection operator is:

$$
(T_a)_i^j (T_a)_k^l = -\frac{1}{2} \left( \delta_i^j \delta_k^l - \frac{1}{N} \delta_i^j \delta_k^l \right),
$$

(A.8)

while for $SO(N)$,

$$
(T_a)_i^j (T_a)_k^l = -\frac{1}{4} \left( \delta_k^j \delta_i^l - \delta_i^j \delta_k^l \right).
$$

(A.9)

Similar identities can be read from [39] for other groups. This solves the problem of calculating the group factors in the fundamental representation of these groups.

Higher representations can be introduced as properly symmetrized products of fundamental representations. These products span the representation ring of any compact Lie group. Extensions of (A.8) and (A.9) can be found, which would enable us to evaluate the analog of (A.7) in these more involved cases.

Using these rules we have computed all the independent group factors $s_{ij}$ up to order four for the simple Lie algebras which have been used in this paper. They correspond to the fundamental representations of $SU(N)$ and $SO(N)$. The general form of the independent group factors pictured in Fig. 11 are:

$$
\begin{align*}
  s_{21} &= \frac{1}{d_1 d_2} \text{Tr}_{R_1}(T_a T_b) \text{Tr}_{R_2}(T_a T_b) \\
  s_{31} &= \frac{1}{d_1 d_2} \text{Tr}_{R_1}(T_b T_c T_a) \text{Tr}_{R_2}(T_a T_b T_c) \\
  s_{32} &= -\frac{1}{d_1 d_2} f^{dea} \text{Tr}_{R_1}(T_b T_c T_d) \text{Tr}_{R_2}(T_a T_b) \\
  s_{41} &= \frac{1}{d_1 d_2} \text{Tr}_{R_1}(T_c T_d T_a T_b) \text{Tr}_{R_2}(T_a T_b T_c T_d) \\
  s_{42} &= -\frac{1}{d_1 d_2} f^{eda} \text{Tr}_{R_1}(T_b T_c T_d T_e) \text{Tr}_{R_2}(T_a T_b T_c) \\
  s_{43} &= \frac{1}{d_1 d_2} f^{ecd} f^{eag} \text{Tr}_{R_1}(T_c T_d T_e T_g) \text{Tr}_{R_2}(T_a T_b) 
\end{align*}
$$

(A.10)

where we have omitted the superscripts on the generators which label the representation. In these expressions $d_1$ and $d_2$ are the dimensions of representations $R_1$ and $R_2$. Their values in the fundamental representations are contained in the following list:

$$
\begin{align*}
  SU(N)_f : s_{21} &= \frac{1}{4N^2} (N^2 - 1) \\
  s_{31} &= \frac{1}{4N^3} (N^2 - 1) \\
  s_{32} &= -\frac{1}{8N} (N^2 - 1) \\
  s_{41} &= \frac{1}{16N^4} (N^2 - 1)(N^2 + 3)
\end{align*}
$$
\[ s_{42} = -\frac{1}{8N^2}(N^2 - 1) \]
\[ s_{43} = \frac{1}{16}(N^2 - 1) \]

\[ SO(N)_f : s_{21} = \frac{1}{8N}(N - 1) \]
\[ s_{31} = \frac{1}{64N}(N - 1)(N - 2) \]
\[ s_{32} = -\frac{1}{32N}(N - 1)(N - 2) \]
\[ s_{41} = \frac{1}{256N}(N - 1)(4 - 3N + N^2) \]
\[ s_{42} = -\frac{1}{256N}(N - 1)(N - 2)^2 \]
\[ s_{43} = \frac{1}{128N}(N - 1)(N - 2)^2 \]  
(A.12)
APPENDIX B

In this appendix we present some elementary facts about semisimple Lie algebras relevant to the analysis of group factors in the perturbative expansion.

Consider the diagram depicted on the left of Fig. 15. It is possible to reduce its group factor by means of the totally antisymmetry of the structure constants $f_{abc}$ at the cost of introducing $C_A$:

$$ f_{abc} T_b T_c = \frac{1}{2} f_{abc}[T_b, T_c] = -\frac{1}{2} f_{abc} f_{cde} T_d = -\frac{1}{2} C_A T_a. $$  \hspace{1cm} (B.1)

There is a point which may be worth commenting. It was already pointed out in the Appendix B of [19] for the case of knots. For an arbitrary semisimple Lie algebra $A = \bigoplus_{k=1}^n A_k$ in an arbitrary representation, the Wilson line can be imagined as consisting of $n$ Wilson lines each one corresponding to one of the simple Lie algebras $A_k$ in its respective representation. Therefore, a connected diagram can be regarded as a sum of similar diagrams where in each term, the legs on the first Wilson line are attached to some component of it, and the legs on the second Wilson line to that same component but in the second Wilson line. The sum runs over all these components.

When a diagram is made of some connected subdiagrams, its group factor will be a sum over all the possible ways of attaching the connected subdiagrams to the $n$ components of the Wilson lines.

As a consequence, the group factors associated to direct products of Lie algebras cannot be directly identified with the group factors written in (A.10). For a Lie group $\otimes_k G_k \otimes U(1)$ with $G_k$ simple, the group factors associated with the diagrams in figure (11) will be:

$$ s_{21} = \sum_k s_{21}(G_k) + s_{11}^2 $$

$$ s_{31} = \sum_k s_{31}(G_k) + 3 s_{11} \sum_k s_{21}(G_k) + s_{11}^3 $$

$$ s_{32} = \sum_k s_{32}(G_k) $$

$$ s_{41} = \sum_k s_{41}(G_k) + 4 s_{11} \sum_k s_{31}(G_k) + 6 \sum_k \sum_{l<k} s_{21}(G_k) s_{21}(G_l) + 6 s_{11}^2 \sum_k s_{21}(G_k) + s_{11}^4 $$

$$ s_{42} = \sum_k s_{42}(G_k) + 2 s_{11} \sum_k s_{32}(G_k) $$

$$ s_{43} = \sum_k s_{43}(G_k) $$  \hspace{1cm} (B.2)

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Of course, if we consider semisimple Lie algebras only, the terms with a $s_{11}$ factor will not be present.

For a general diagram, once it has been drawn, its group factor is found as follows. First one has to obtain its decomposition in a sum over all the possible ways of attaching the subdiagrams to the $n$ components of the Wilson line, as explained in the previous paragraph. Then consider a term of the sum, reduce all the “fishtails” of the diagram chosen by means of (B.1). Then, the group factor of this diagram has to be calculated separately and written in terms of the basis elements we had chosen by repeated use of the commutation relations. Repeat this procedure for each term. The result is the group factor of the diagram we begun with. Following these steps for all diagrams present in the sum, we are done. This procedure is similar, but somehow alternative, to the one presented in the text, where we begun with open Wilson lines, worked out their group factor and then closed the lines to loops. The steps carried out to calculate the group factor for an open general diagram are exactly the ones explained in [19] for the case of knots.
APPENDIX C

In this appendix we list the link polynomials which have been used in this work to compute the numerical invariants introduced in this work up to order four for links up to six crossings. They were collected from [37], after taking care of our orientation convention.

For $SU(N)$ we have the following list for the HOMFLY polynomials:

\[ 2_1^2: z^{-1}(t^{-3} - t^{-1}) - zt^{-1} \]
\[ 4_{1+}^2: z^{-1}(t^{-5} - t^{-3}) + z(t^{-5} - 3t^{-3}) - z^2t^{-3} \]
\[ 4_1^2: z^{-1}(t^5 - t^3) - z(t^3 - t) \]
\[ 5_{1+}^2: z^{-1}(t - t^{-1}) + z(t^{-3} - 2t^{-1} + t) - z^3t^{-1} \]
\[ 6_{1+}^2: z^{-1}(t^{-5} - t^{-7}) + z(6t^{-5} - 3t^{-7}) + z^3(5t^{-5} - t^{-7}) + z^5t^{-5} \]
\[ 6_1^2: z^{-1}(-t^{-5} - t^{-7}) - z(t^5 + t^3 + t) \]
\[ 6_2^2: z^{-1}(-t^{-5} + t^{-7}) + z(t^7 - 2t^5 - 2t^3) - z^3(t^5 - t^3) \]
\[ 6_2^2: z^{-1}(t^{-3} - t^{-5}) + z(2t^{-3} - t^{-5} - t^{-7}) + z^3(-t^{-3} + t^{-5}) \]
\[ 6_3^2: z^{-1}(-t^{-3} - t^{-5}) + z(-2t^3 + t^{-1} - t) + z^3t \]

where $z = q^{1/2} - q^{-1/2}$ and $t = q^{-N/2}$.

And for $SO(N)$ we have the following list for the Kauffman polynomials:

\[ 2_1^2: z^{-1}(a^3 - a) + a^2 + z(a^3 - a) \]
\[ 4_{1+}^2: z^{-1}(a^5 - a^3) + a^4 + z(a^6 + 2a^5 - 3a^3) + z^2(a^4 - a^6) + z^3(a^5 - a^3) \]
\[ 4_1^2: z^{-1}(a^5 - a^3) - a^4 + z(3a^5 - 2a^3 - a^{-1}) + z^2(-a^{-4} + a^{-2}) + z^3(a^5 - a^3) \]
\[ 5_1^2: z^{-1}(a^{-1} - a) - 1 + z(2a^{-1} - 4a + 2a^3) - z^2(a^4 + 1) \]
\[ + z^3(a^{-1} - 3a + 2a^3) + z^4(1 - a^2) \]
\[ 6_{1+}^2: z^{-1}(-a^5 - a^7) - a^{-6} + z(6a^{-7} - 4a^{-5} - a^{-3} - a^{-1}) \]
\[ + z^2(-3a^{-6} + 2a^{-4} + a^{-2}) + z^3(5a^{-7} - 4a^{-5} - a^{-3}) - z^4(a^{-6} + a^{-4}) + z^5(-a^{-5} + a^{-7}) \]
\[ 6_1^2: z^{-1}(a^5 - a^7) - a^6 + z(6a^5 - 4a^7 - a^9 - a^{11}) + z^2(-3a^6 + 2a^8 - a^9 + a^{10}) \]
\[ + z^3(5a^5 - 4a^7 - a^9) + z(-a^6 + a^8) + z^5(a^5 - a^7) \]
\[ 6_2^2: z^{-1}(-a^{-5} - a^{-7}) - a^{-6} + z(2a^{-9} + 3a^{-7} - 3a^{-5} - 2a^{-3}) + z^2(-2a^{-6} + a^{-8} + a^{-4}) \]
\[ + z^3(a^{-9} - a^{-7} - 2a^{-5} - a^{-3}) + z^4(a^{-4} - 2a^{-6} - a^{-8}) + z^5(-a^{-5} + a^{-7}) \]
\[ 6_{3+}^3 : z^{-1}(a^3 - a^5) - a^4 + z(2a^3 - a^5 - a^9) + z^2(-3a^6 + 3a^8) + z^3(a^3 - a^9) + z^4(a^4 - 3a^6 + 2a^8) + z^5(a^5 - a^7) \]

\[ 6_{3-}^3 : z^{-1}(a^{-5} - a^{-3}) - a^{-4} + z(2a^{-5} - a^{-3} - a) + z^2(-3a^{-2} + 3) + z^3(a^{-5} - a) + z^4(a^{-4} - a^{-2} + 2) + z^5(a^{-1} - a^{-3}) \]

where \( z = q^{1/4} - q^{-1/4} \) and \( a = q^{N-1/4} \).
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