Singular Vectors of $\mathcal{W}$ Algebras via DS Reduction of $A_2^{(1)}$

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Abstract

The BRST quantisation of the Drinfeld - Sokolov reduction applied to the case of $A_2^{(1)}$ is explored to construct in an unified and systematic way the general singular vectors in $\mathcal{W}_3$ and $\mathcal{W}_3^{(2)}$ Verma modules. The construction relies on the use of proper quantum analogues of the classical DS gauge fixing transformations. Furthermore the stability groups $\mathcal{W}^{(\eta)}$ of the highest weights of the $\mathcal{W}$ - Verma modules play an important role in the proof of the BRST equivalence of the Malikov-Feigin-Fuks singular vectors and the $\mathcal{W}$ algebra ones. The resulting singular vectors are essentially classified by the affine Weyl group $\tilde{W}$ modulo $\mathcal{W}^{(\eta)}$.

This is a detailed presentation of the results announced in a recent paper of the authors.

March 1994

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1. INTRODUCTION.

The Hamiltonian reduction of affine Lie algebras (the Drinfeld-Sokolov (DS) reduction) is a powerful method for obtaining and analysing \( \mathcal{W} \) algebras. Its classical aspects are well understood \cite{1}, \cite{2}, \cite{3}, \cite{4}, \cite{5} – starting from a classical analogue of a Kac-Moody (KM) algebra \( \hat{g} \) one imposes on the raising operators constraints specified by a \( sl(2) \) embedding in \( g \). The factorisation over the gauge group generated by the (first class) constraints yields a set of invariant differential polynomials of the affine generators. These are the generating currents of the \( \mathcal{W} \) (Poisson) algebra.

The quantisation of the DS reduction is usually done in a BRST framework and was originally carried out using free field realisations for the affine algebras and Fock modules \cite{6}, \cite{7}, \cite{8}, \cite{9}. The free field representation of KM and \( \mathcal{W} \) algebras is quite involved in general and the representation theory of the general \( \mathcal{W} \) algebras is in a less developed stage.

Recently de Boer and Tjin \cite{10,11} proposed a different BRST scheme for the reduction of an affine Lie algebra based, as in the classical case, on \( sl(2) \) embeddings and not relying on the use of free fields. (A free field realisation of the resulting \( \mathcal{W} \)-algebra appears nevertheless naturally.) This approach is suitable for the reduction of Verma module representations. The description of the irreducible highest weight representations through Verma modules has a direct relevance for the underlying Conformal Field Theory models. It provides information on the characters \cite{12}, \cite{13}, and, accounting for the explicit form of the Verma module singular vectors – eventually on the differential and algebraic equations for the correlation functions. Thus the further development of the approach in \cite{11} would allow to carry over the reduction procedure to the full conformal field theories, as has been done in \cite{14} for the simplest case \( \hat{sl}(2) \to \text{Virasoro algebra.} \)

The description of the singular vectors of Kac-Moody Verma modules is transparent and relatively simple \cite{15}, \cite{16}. Thus it would be desirable to have a method reducing these singular vectors to their \( \mathcal{W} \) algebra counterparts. This has been done in the simplest case of \( A_1^{(1)} \) in \cite{17}, recovering the general Virasoro Verma modules singular vectors in the form proposed by Kent \cite{18}. The idea underlying the derivation in \cite{17} (see also the earlier versions in \cite{14}, \cite{19}) is to exploit a kind of a quantum analogue of the classical Drinfeld-Sokolov gauge fixing transformation, taken in an arbitrary representation. In the classical case it amounts (constraining the raising generator density \( e(z) = 1 \)) to the mapping of the lowering generator \( f(z) \) into the Virasoro stress-energy tensor,

\[
 f(z) \to \frac{1}{\nu} T(z) = f(z) + \frac{1}{\nu} T^{(ff)}(z),
\]

(1.1)

The Heisenberg subalgebra density \( h(z)/2 \) serves as a gauge parameter.

The quantum BRST invariant Virasoro stress-energy tensor \( T(z) = \sum z^{-n-2} L_n \) is given by a formula analogous to that in (1.1), shifting \( \hat{h}(z) \) by a ghost bilinear combination \( \hat{\hat{h}}(z) = h(z) + (bc)(z) \), adding normal products and identifying the constant \( 1/\nu - 2 \) with the value \( k \) of the \( A_1^{(1)} \) level. A quantum analogue of the transformation (1.1) is provided by the operator

\[
 \mathcal{R} = 1 + \frac{\hat{\hat{h}}_{-1}}{2} e_0 + \frac{1}{2} \left( \left( \frac{\hat{h}_{-1}}{2} \right)^2 - \frac{\hat{h}_{-2}}{2} \right) e_0^2 + \ldots
\]

(1.2)

\[
 \mathcal{R} = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\partial^k}{\partial u^k} \left( \exp \int_0^u \frac{\hat{h}_{(-)}(-u')}{2} du' \right) \bigg|_{u=0} e_0^k, \quad \hat{h}_{(-)}(u) \equiv \sum_{n=1}^{\infty} u^{n-1} \hat{h}_{-n},
\]
which has the following properties:

- It leaves invariant the Kac-Moody singular vectors,
- maps horizontal vectors into the kernel of the BRST operator,

i.e., Q kills states of the form $R(f_0)^nV_\lambda$, $n$ - non-negative integer, $V_\lambda$ - a Kac-Moody highest weight state. Moreover and most important
- $R$ intertwines Kac-Moody and Virasoro generators,

more precisely, for any vector $V$ annihilated by all positive grade generators as well as by the ghost zero mode $c_0$, one has

$$
R f_0 V = \frac{1}{\nu} \sum_{p=0} L_{-p-1} R (-e_0)^p V .
$$

These properties together with the quantum constraint allow to recover all the Virasoro singular vectors starting from proper $\hat{sl}(2)$ ones.

Generalising the above approach to the $\hat{sl}(3)$ case, we have outlined in the short letter [20] a program for the construction of the general singular vectors in Verma modules of the related $\mathcal{W}$ algebras. This work provides a detailed presentation of the results announced in [20].

The paper is organised as follows. Section 2 contains notation conventions and a summary of known facts about the reducibility conditions of affine Kac-Moody Verma modules [15], [16]. More details on the Malikov-Feigin-Fuks (MFF) $A^{(1)}_2$ singular vectors and an algorithm transforming Malikov-Feigin-Fuks monomials into ordinary integer-powers polynomials are given in Appendix A. In section 3 we recall the basics about the Zamolodchikov (Z) [21] algebra $\mathcal{W}_3$ and the Polyakov-Bershadsky (PB) [22], [9] $\mathcal{W}_3^{(2)}$ algebras, following the realisation of [11].

Section 4 is devoted to the introduction of the quantum gauge operators relevant for the two algebras and the study of their properties. Appendix B contains the technical details of the derivation of the basic intertwining relations. For comparison with the classical cases we refer the reader to [20] and the original works [3], [4]. In Section 5 the quantum gauge transformations are used to reduce the MFF $A^{(1)}_2$ singular vectors to the singular vectors of $\mathcal{W}_3^{(2)}$ Verma modules. Section 6 deals with the same problem in the case of $\mathcal{W}_3$, starting with the subclass of singular vectors first obtained by Bowcock and Watts [23], following the fusion method of [24]. Appendix C contains some details of the argumentation leading to the general $\mathcal{W}_3$ singular vectors. The symmetry groups of the eigenvalues of the zero modes subalgebras play in both cases essential role, determining the way the quantum constraints are accounted for in the reduction of the MFF vectors. Since we start from the MFF vectors, the resulting $\mathcal{W}_3$ singular vectors are in a form analogous to the one proposed by Kent [18] for the Virasoro case. In Appendix D we describe on an example how one can obtain explicit expressions for the singular vectors starting from Kent type of expressions. We end with a discussion of open problems.

2. NOTATION AND SUMMARY OF KNOWN FACTS.

2.1. Singular vectors of $A^{(1)}_2$ Verma modules

The positive roots of $\tilde{g} = A_2$ consist of the simple roots $\alpha^1$, $\alpha^2$ and the highest root $\alpha^3 \equiv \alpha^1 + \alpha^2$. There is a nondegenerate form $(\cdot, \cdot)$ on the space spanned by the roots. The Cartan matrix $C^{ij} = \langle \alpha^i, \alpha^j \rangle$, $i, j = 1, 2$ and its inverse $C_{ij}$ in the case of $A_2$ are

$$
(C^{ij}) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad (C_{ij}) = \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} .
$$

To each positive/negative root $\pm \alpha^i$ corresponds a raising/lowering operator $e^i / f^i$, respectively. To each simple root corresponds a Cartan generator $h^i, i = 1, 2, \alpha^i(h^j) = C^{ij}$. The commutation
relations of $A_2$ are
\[ [e^i, f^j] = \delta_{ij} h^i, \quad [h^i, h^j] = 0, \quad i = 1, 2, \quad (2.2a) \]
\[ [h^i, e^j] = \langle \alpha^i, \alpha^j \rangle e^j, \quad [h^i, f^j] = -\langle \alpha^i, \alpha^j \rangle f^j, \]
and setting
\[ e^3 = [e^1, e^2], \quad f^3 = [f^2, f^1], \quad (2.2b) \]
the other commutation relations are determined by the Jacobi identity. The quadratic form on the Cartan algebra can be extended to a nondegenerate symmetric form on the whole algebra by
\[ \langle h^i, h^j \rangle = C^{ij}, \quad i, j = 1, 2 \quad (\epsilon^a, f^a) = 1, \quad a = 1, 2, 3, \quad (2.3) \]
the others being zero.

Given the commutators $[X, Y]$ of the elements $X, Y$ of a semisimple Lie algebra $\tilde{g}$ and the Cartan - Killing form $\langle X, Y \rangle$ the affine algebra $g$ associated to $\tilde{g}$ has commutation relations
\[ [X_n, Y_m] = ([X, Y])_{n+m} + k n \delta_{n+m, 0} \langle X, Y \rangle \quad (2.4) \]
where $k$ is a central element. The bilinear form is extended to $g$ by $\langle X_n, Y_m \rangle = \delta_{n+m, 0} \langle X, Y \rangle$ and trivially to $k$, i.e., $\langle X_n, k \rangle = 0$. The algebra $g$ admits the decomposition $g = n_+ \oplus h \oplus n_-$, where $h = h \oplus \mathcal{C}k$ is the Cartan subalgebra and the subalgebra $n_+$ ($n_-$) is generated, via multiple commutators, by
\[ e_0 = f_3, e_1 = e_0, e_2 = e_0^2, \quad (f_0 = e_{-1}, f_1 = f_0^1, f_2 = f_0^2) \]
respectively. These are the raising (lowering) operators corresponding to the simple roots $\alpha^0, \alpha^1, \alpha^2$. Denote $h^0 = [e_0, f_0] = k - h_0^3, h^i = h_0^i, i = 1, 2$, $h_0^3 = h_0^1 + h_0^2$. The Cartan matrix of the horizontal (zero mode) subalgebra $\tilde{g}$ extends to $\langle \alpha^0, \alpha^j \rangle = \alpha^0(h^j) = \alpha^j(h^0) = 2\delta_{j0} - \delta_{j1} - \delta_{j2}.$

The set of real positive roots is
\[ \Delta_{+, \text{real}} = \{ \alpha^i, \pm \alpha^i + n\delta : \quad i = 1, 2, 3; \quad n \in \mathbb{N} \} , \quad (2.5) \]
where $\delta \equiv \alpha^0 + \alpha^3$. The positive roots consist of the positive real roots and the positive degenerate roots $\{ n \delta : \quad n \in \mathbb{N} \}$ which have zero norm.

Denote by $\lambda_0$ the element of $h^*$, dual to $k$, i.e., $\lambda_0(k) = 1$, $\lambda_0(h^i) = 0, i = 1, 2$ and let $\langle \lambda_0, \alpha^j \rangle = \lambda_0(h^j), j = 0, 1, 2$, $\langle \lambda_0, \lambda_0 \rangle = 0$. It is convenient to enlarge the Cartan subalgebra $h$ by adjoining a derivation $d$, $[d, X_m] = m X_m$ and $[d, k] = 0$. The bilinear form on $g$ is extended to a nondegenerate one by requiring $\langle d, k \rangle = 1$, $\langle d, X_n \rangle = 0$, while $\delta$ provides the dual to $d$, i.e., $\delta(d) = 1$, and $\lambda_0(d) = 0, \alpha^i(d) = 0, i = 1, 2$.

We will consider weights $\lambda \in (h \oplus \mathcal{C}d)^*$
\[ (\lambda + \rho)(h^j) = M^j = (\lambda + \rho, \alpha^j), j = 0, 1, 2, \]
\[ M^j \in \mathcal{C}, \quad M^0 + M^1 + M^2 = k + 3 \neq 0, \quad (2.6) \]
where $\rho(h^j) = \langle \rho, \alpha^j \rangle = 1, j = 0, 1, 2$, and $k = \lambda(k) \in \mathcal{C}$, is the level. The projection of $\lambda$ to $\tilde{h}^*$ will be denoted by $\lambda$. We shall not keep track of the $\delta$ - components $\lambda(d)$, if not necessary, i.e., the weights will be characterised by $\{ \lambda, \nu = 1/(k + 3) \} \leftrightarrow \{ M^1, M^2, \nu \}$. 

3
For a real root $\beta$ the shifted action of the Weyl reflection $w_\beta$ on a weight $\lambda$ is
\[ w_\beta \cdot \lambda = w_\beta (\lambda + \rho) - \rho = \lambda - \langle \lambda + \rho, \beta \rangle \beta. \tag{2.7} \]
The affine Weyl group $W$ is generated by the three simple reflections $w_i \equiv w_{\alpha_i}$, $i = 0, 1, 2$.

A Verma module $V_\lambda$ of highest weight $\lambda$ is freely generated by the negative subalgebra $n_-$ acting on the highest weight vector $V_\lambda$ which is a vector of weight $\lambda$, i.e., $h^j V_\lambda = \lambda(h^j) V_\lambda$, $j = 0, 1, 2$, and $V_\lambda$ is singular, i.e., it is annihilated by the raising operators (it is sufficient if it is annihilated by $e_i$, $i = 0, 1, 2$).

Restricting to the case $\langle \lambda + \rho, \beta \rangle = k + 3 \neq 0$ the Kac-Kazhdan theorem states that a Verma module of highest weight $\lambda$ is reducible iff
\[ \langle \lambda + \rho, \beta \rangle = m \in \mathbb{N} \quad \text{for some} \quad \beta \in \Delta_{+, \text{real}} \tag{2.8} \]
and in this case it contains a Verma submodule of highest weight $w_\beta \cdot \lambda = \lambda - m \beta$.

If $\beta$ of the Kac-Kazhdan theorem is a simple root then one immediately has an explicit expression for the singular vector generating the submodule. Indeed, the following simple fact is true. If $V_\lambda$ is a singular vector of weight $\lambda$ such that $\langle \lambda + \rho, \alpha^i \rangle \in \mathbb{N}$ then
\[ V_{w_{sj} \lambda} \equiv (f_j)^{\langle \lambda + \rho, \alpha^i \rangle} V_\lambda, \quad j = 0, 1, 2, \tag{2.9} \]
is a singular vector. If $i \neq j$ the generators $e_i$ annihilate $V_{w_{sj} \lambda}$ because they commute with $f_j$. While for $i = j$ one has $e_j (f_j)^p V_\lambda = p (\lambda + \rho, \alpha^j) - p) (f_j)^{p-1} V_\lambda$ and thus $V_{w_{sj} \lambda}$ is annihilated by $e_j$ also.

Now we turn to the case of a general positive real root $\beta$. The Weyl reflection $w_\beta$ can be written as a product of simple reflections $w_\beta = w_{s_1} \cdots w_{s_2} w_{s_1}$, $i_t = 0, 1, 2$ (see below for explicit expressions). Consider the sequence of vectors
\[ V_{w_{s_p} w_{s_{p-1}} \cdots w_{s_1} \lambda} = (f_{s_p})^{\langle w_{s_p-1} \cdots w_{s_1}, \lambda + \rho, \alpha^{s_p} \rangle} V_{w_{s_{p-1}} \cdots w_{s_1} \lambda}, \tag{2.10} \]
the last element of this sequence being the singular vector of weight $w_\beta \cdot \lambda$
\[ V_{w_\beta \lambda} = P_{\beta, \lambda} V_\lambda \quad \text{with} \quad P_{\beta, \lambda} \equiv (f_{s_p})^{\langle w_{s_{p-1}} \cdots w_{s_1}, \lambda + \rho, \alpha^{s_p} \rangle} \cdots (f_{s_2})^{\langle w_{s_1}, \lambda + \rho, \alpha^{s_2} \rangle} (f_{s_1})^{\langle \lambda + \rho, \alpha^{s_1} \rangle}. \tag{2.11} \]
If all $\langle w_{s_p-1} \cdots w_{s_1}, \lambda + \rho, \alpha^{s_p} \rangle \in \mathbb{N}$ then all the vectors of the sequence (2.10) are singular vectors. In fact a much stronger statement is true. We can relax all these integrality conditions retaining only the Kac-Kazhdan condition $\langle \lambda + \rho, \beta \rangle \in \mathbb{N}$. Even though the generators are taken to, in general, complex powers and the vectors (2.10) for $1 < p < s$ are only formal expressions, the vector defined by (2.11) makes sense as a vector in the Verma module. This is the content of the Malikov-Feigin-Fuks theorem [16]. In a recent paper [25] a rigorous definition of complex powers of the shift operators is given and in particular of the vectors in (2.10). In Appendix A we work out in more details some of the Malikov-Feigin-Fuks vectors.

For the 6 types of positive real roots $\beta$ of $A_2^{(1)}$ one has the following decomposition of $w_\beta$ into simple reflections [26]:
\begin{align*}
\beta &= \beta_{n+, 1} = n \delta + \alpha^1 \quad w_{\beta_{n+, 1}} = w_{1(0201)^n}, \quad n = 0, 1, 2, \ldots, \tag{2.12a} \\
\beta_{n+, 2} &= n \delta + \alpha^2 \quad w_{\beta_{n+, 2}} = w_{2(0102)^n}, \quad n = 0, 1, 2, \ldots, \tag{2.12b} \\
\beta_{n+, 3} &= n \delta + \alpha^3 \quad w_{\beta_{n+, 3}} = w_{121(0121)^n}, \quad n = 0, 1, 2, \ldots, \tag{2.12c} \\
\beta_{n-, 1} &= n \delta - \alpha^1 \quad w_{\beta_{n-, 1}} = w_{020(1020)^{n-1}}, \quad n = 1, 2, 3, \ldots, \tag{2.12d} \\
\beta_{n-, 2} &= n \delta - \alpha^2 \quad w_{\beta_{n-, 2}} = w_{010(2010)^{n-1}}, \quad n = 1, 2, 3, \ldots, \tag{2.12e} \\
\beta_{n-, 3} &= n \delta - \alpha^3 \quad w_{\beta_{n-, 3}} = w_{0(1210)^{n-1}}, \quad n = 1, 2, 3, \ldots, \tag{2.12f} 
\end{align*}
where for short we denote $w_{ij(kl...)} = w_i w_j (w_k w_l...)^n \ldots$

One can have also singular vectors arising from compositions $w_{\beta_1} \ldots w_{\beta_s}$ of Weyl reflections, where $\beta_1, \ldots, \beta_s$ are real positive roots, like e.g., $V_{w_{\beta_2} w_{\beta_1} \cdot \lambda} = P_{\beta_2; w_{\beta_1} \cdot \lambda} P_{\beta_1; \lambda} V_{\lambda}$, if the Kac-Kazhdan condition (8) is satisfied for the root $\beta_1$ and then (by the weight $w_{\beta_1} \cdot \lambda$) for the root $\beta_2$. Such a vector generates a submodule of weight $w_{\beta_2} w_{\beta_1} \cdot \lambda$ which is embedded in the submodule of weight $w_{\beta_1} \cdot \lambda$. Here we will not study in detail the Verma module embedding patterns so let us give as an illustration only the horizontal singular vectors. Thus if both $M^i \in \mathbb{N}$, $i = 1, 2$, besides the singular vectors (9) we have also

$$V_{w_1 w_2 \cdot \lambda} \equiv (f_0^1)^{M_1} (f_0^2)^{M_2} V_{\lambda}, \quad V_{w_2 w_1 \cdot \lambda} \equiv (f_0^2)^{M_1} (f_0^1)^{M_2} V_{\lambda}, \quad V_{w_2 w_1 w_2 \cdot \lambda} \equiv (f_0^1)^{M_1} (f_0^2)^{M_2} V_{w_1 w_2 \cdot \lambda},$$

(2.13),

where $M_3 = M^1 + M^2$. The last vector in (2.13) corresponds to the Weyl reflection $w_\beta = w_1 w_2 w_1 = w_2 w_1 w_2$ with $\beta = \alpha^1 + \alpha^2$ and, being a particular case of (2.11), it provides also a Malikov-Feigin-Fuks type singular vector under the weaker conditions $M^3 \in \mathbb{N}, M^1, M^2 - \text{arbitrary.}$

2.2. Chiral algebra fields.

It is standard in Conformal Field Theory to consider fields

$$A(z) = \sum_{n \in \mathbb{Z}} A_n z^{-n-\Delta_A}$$

(2.14)

as generating functions of the modes $A_n$. The (anti)commutation relations of the modes are equivalent to the singular part of the operator product expansion (OPE). In particular the KM commutation relations (2.4) can be rewritten as (the dots indicate the regular terms)

$$X(z) Y(w) = \frac{k (X,Y)}{(z-w)^2} + \frac{[X,Y](w)}{z-w} + \ldots$$

(2.15)

with the assumption that all the currents $X(z)$ have $\Delta_X = 1$.

The normal product of two fields $(A B)(z)$ can be defined as the zero order term in the expansion of $A$ and $B$. In modes this is equivalent to

$$(A B)_n = \sum_{m \leq -\Delta_A} A_m B_{n-m} + (-1)^{AB} \sum_{m > -\Delta_A} B_{n-m} A_m$$

(2.16)

where $(-1)^{AB}$ is $-1$ if both $A$ and $B$ are fermionic and 1 otherwise. The so defined normal order product is neither commutative nor associative – for rules to work with it see [27]. Since the technique of operator product expansions is well known and widely used, moreover a MATHEMATICA code is available [28], we will skip all explicit computations with normal products needed throughout this work. Let us only write down the standard formula

$$[A_m, B_n] = \sum_{k \geq 1} \binom{m + \Delta_A - 1}{k - 1} (AB)^{(k)}_{m+n},$$

(2.17)

where as usual $(AB)^{(k)}$ are the coefficients of the OPE $A(z) B(w) = \sum_{k \leq \text{finite}} \frac{(AB)^{(k)}(w)}{(z-w)^k}$.  

5
The reduced theories are obtained by imposing constraints on some of the raising operator currents \( e^a(z) \). In the BRST formalism one needs for each constraint a pair of fermion ghost fields \( b^a, c^a \) having operator product expansions

\[
b^a(z) c^b(w) = \frac{\delta_{ab}}{z-w} + \ldots.
\]

(2.18)

and \( \Delta_{b^a} + \Delta_{c^a} = 1 \). We choose \( \Delta_{b^a} = 0, \Delta_{c^a} = 1 \).

3. The \( \mathcal{W} \) ALGEBRAS from REDUCTION of \( A_2^{(1)} \).

3.1. The \( \mathcal{W}_3 \) algebra.

In this subsection we will consider the reduction leading to the \( \mathcal{W}_3 \) algebra of Zamolodchikov [21] which is associated to the so called principal embedding. The corresponding (classical) constraints are

\[
e^1(z) = e^2(z) = 1, \quad e^3(z) = 0.
\]

(3.1)

Following [8], [11] let us introduce the “hatted” generators

\[
\hat{X}^a = X^a + f_3^a (b^β c_α),
\]

(3.2a)

where the summation indices \( α, β \) correspond to the constrained generators \( e^α, α = 1, 2, 3 \). Explicitly

\[
\hat{f}^1 = f^1 + (b^2 c^3), \quad \hat{e}^1 = e^1 + (b^3 c^2),
\]

\[
\hat{f}^2 = f^2 - (b^1 c^3), \quad \hat{e}^2 = e^2 - (b^3 c^1),
\]

\[
\hat{f}^3 = f^3, \quad \hat{e}^3 = e^3,
\]

\[
C_{ij} \hat{h}^j = C_{ij} h^j + (b^i c^j) + (b^j c^i), \quad i = 1, 2,
\]

(summation over \( j = 1, 2 \) assumed).

The BRST charge implementing the constraints (3.1) is

\[
Q = \oint_{C_0} \frac{dz}{2\pi i} \left( \sum_{α=1}^3 c^α \hat{e}^α + (b^3(c^1 c^2)) - c^1 - c^2 \right)(z)
\]

(3.3)

\[
= \sum_{α=1}^3 (c^α c^α)_{-1} - (b^3(c^1 c^2))_{-1} - c^1 - c^2.
\]

The OPE’s among the fields \( \{\hat{f}, \hat{h}\} \) is the same as among the corresponding unhatted ones with the only difference that \( k \) is shifted to \( k + 3 \), i.e.,

\[
\hat{h}^i(z) \hat{h}^j(w) = \frac{C_{ij}}{ν} \frac{1}{(z-w)^2}, \quad ν = \frac{1}{k+3}.
\]

(3.5)

Consider the fields

\[
\frac{1}{ν} T^{(ff)}(z) = \frac{1}{2} C_{ij} (\hat{h}^i \hat{h}^j)(z) + \left( \frac{1}{ν} - 1 \right) \partial \hat{h}^3(z),
\]

\[
6α_ν W^{(ff)}(z) = 2 C_{1i} C_{2j} (\hat{h}^i \hat{h}^j)(z)
\]

(3.6)

\[
+ 3 \left( \frac{1}{ν} - 1 \right) \left( C_{1i} (\hat{h}^i \partial \hat{h}^1) - C_{2i} (\hat{h}^i \partial \hat{h}^2) \right)(z) + \left( \frac{1}{ν} - 1 \right)^2 \partial^2 (\hat{h}^1 - \hat{h}^2)(z),
\]
where \( a_\nu = \epsilon \nu^{-3/2} \), \( \epsilon = \pm 1 \); we shall choose \( \epsilon = -1 \).

Making the identification

\[
\partial \phi_1(z) = \sqrt{-\nu} \tilde{h}^3(z), \quad \partial \phi_2(z) = \sqrt{-\nu} \frac{1}{3} (\tilde{h}^1 - \tilde{h}^2)(z)
\]

the fields \( T^{(ff)}, W^{(ff)} \) acquire the form of the free field realization of the \( W_3 \) algebra of Fateev and Zamolodchikov [29].

The BRST invariant currents are expressed through \( T^{(ff)} \) and \( W^{(ff)} \) according to

\[
\frac{1}{\nu} T(z) = \hat{f}_1(z) + \hat{f}_2(z) + \frac{1}{\nu} T^{(ff)}(z),
\]

\[
a_\nu W(z) = \left( \hat{f}_3 + \frac{1}{2} \left( \frac{1}{\nu} - 1 \right) \partial(\hat{f}_1 - \hat{f}_2) + C_{2i}(\hat{h}^i \hat{f}_1) - C_{1i}(\hat{h}^i \hat{f}_2) \right)(z) + a_\nu W^{(ff)}(z),
\]

or, in modes,

\[
\frac{1}{\nu} L_n = \hat{f}_{n+1}^i + \hat{f}_{n+1}^j + \frac{1}{\nu} L_n^{(ff)} = \hat{f}_{n+1}^i + \hat{f}_{n+1}^j + \frac{C_{ij}}{2}(\hat{h}^i \hat{h}^j)_n + \left( \frac{1}{\nu} - 1 \right)(\partial \hat{h}^3)_n,
\]

\[
a_\nu W_n = \hat{f}_{n+2}^i + \frac{1}{2} \left( \frac{1}{\nu} - 1 \right) \left( \partial \hat{f}_1 - \partial \hat{f}_2 \right)_n + C_{2i}(\hat{h}^i \hat{f}_1)_n - C_{1i}(\hat{h}^i \hat{f}_2)_n + a_\nu W_n^{(ff)},
\]

\[
a_\nu W_n^{(ff)} = C_{1i}C_{2j}(C_{1i} - C_{2i})(\hat{h}^i \hat{h}^j)_n
\]

\[
+ \frac{1}{2} \left( \frac{1}{\nu} - 1 \right) \left( C_{1i}(\hat{h}^i \partial \hat{h}^1) - C_{2i}(\hat{h}^i \partial \hat{h}^2) \right)_n + \frac{1}{6} \left( \frac{1}{\nu} - 1 \right)^2 (\partial^2 \hat{h}^1 - \partial^2 \hat{h}^2)_n.
\]

These expressions were computed in [11], solving the tic-tac-toe equations of the BRST double complex. The algebra of \( T \) and \( W \) (which is identical to that of \( T^{(ff)}, W^{(ff)} \)) is

\[
T(z) T(w) = \frac{c_\nu/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \ldots,
\]

\[
T(z) W(w) = \frac{3W(w)}{(z-w)^2} + \frac{\partial W(w)}{(z-w)} + \ldots
\]

\[
\beta^2 W(z) W(w) = \frac{c_\nu/3}{(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{\partial T(w)}{(z-w)^3} + \frac{\left( \frac{2\beta^2}{3} \Lambda + \frac{3}{10} \partial^2 T \right)(w)}{(z-w)^2}
\]

\[
+ \frac{\left( \frac{\beta^2}{3} \partial \Lambda + \frac{1}{15} \partial^3 T \right)(w)}{(z-w)} + \ldots,
\]

where

\[
\Lambda = (TT) - \frac{3}{10} \beta^2 T, \quad \beta = \sqrt{\frac{48}{22 + 5c_\nu}}
\]
and the conformal anomaly is
\[ c_\nu = 50 - 24 \left( \nu + \frac{1}{\nu} \right). \tag{3.9} \]

### 3.2. The Polyakov-Bershadsky algebra.

In the case of \( sl(3) \) there is only one \( sl(2) \) embedding besides the principal one. This nonprincipal embedding gives the Polyakov-Bershadsky algebra \( \mathcal{W}_3^{(2)} \).

The BRST operator is
\[ Q^{(PB)} = \oint_{C_\alpha} \frac{dz}{2\pi i} \left( (e^3 c^3) + (e^2 c^2) - c^3 \right) (z) = (e^3 c^3)_{-1} + (e^2 c^2)_{-1} - c^3_0. \tag{3.10} \]

This operator corresponds to the classical, first class constraints
\[ e^2(z) = 0, \quad e^3(z) = 1. \tag{3.11} \]

In the classical consideration the third condition \( e^1(z) = 0 \) is achieved by a choice of a gauge fixing \( e^1_\text{gauged} = 0 \). This is a standard way of dealing with second class constraints [5].

Following the general scheme of [11] introduce the hatted quantities (now \( \alpha, \beta \) in (3.2a) run over \( 2, 3 \)):
\[ \hat{e}^1 = e^1 + (b^3 c^2), \quad \hat{e}^2 = e^2, \quad \hat{e}^3 = e^3, \]
\[ \hat{f}^1 = f^1 + (b^2 c^3), \quad \hat{f}^2 = f^2, \quad \hat{f}^3 = f^3, \]
\[ \hat{h}^1 = h^1 - (b^2 c^2) + (b^3 c^3), \]
\[ \hat{h}^2 = h^2 + 2(b^2 c^2) + (b^3 c^3). \tag{3.12} \]

The reduced, BRST invariant generators [11] are:
\[ H = \frac{1}{3}(\hat{h}^2 - \hat{h}^1), \]
\[ G^+ = \hat{f}^1, \]
\[ G^- = \hat{f}^2 + (\hat{e}^1 \hat{h}^2) + \left( \frac{1}{\nu} - 2 \right) \partial \hat{e}^1, \]
\[ \frac{1}{\nu} T = \hat{f}^3 + (\hat{e}^1 \hat{f}^1) + \frac{1}{\nu} T^{(ff)} 
\]
\[ = \hat{f}^3 + (\hat{e}^1 \hat{f}^1) + \frac{1}{2} C_{ij} (\hat{h}^i \hat{h}^j) + \left( \frac{1}{\nu} - 2 \right) \frac{\partial \hat{h}^3}{2}. \]
One readily checks that they close the $W_3^{(2)}$ algebra (we use interchangeably $k$ and $1/\nu = k+3$):

$$H(z) H(w) = \frac{1}{3} \frac{2k+3}{(z-w)^2} + \ldots$$

$$H(z) G^\pm(w) = \frac{G^\pm(w)}{z-w} + \ldots$$

$$T(z) H(w) = \frac{H(w)}{(z-w)^2} + \frac{\partial H}{z-w} + \ldots$$

$$T(z) G^\pm(w) = 3(\frac{G^\pm(w)}{2(z-w)^2} + \frac{\partial G^\pm(w)}{z-w} + \ldots$$

$$G^+(z) G^-(w) = \frac{(k+1)(2k+3)}{(z-w)^2} + \frac{3(k+1)}{(z-w)^2} H(w)$$

\hspace{1cm} + \frac{1}{z-w} \left( 3(H^2)(w) - (k+3) T(w) + \frac{3}{2}(k+1) \partial H(w) \right) + \ldots$$

$$G^\pm(z) G^\pm(w) = 0 + \ldots$$

and $T(z)$ closes a Virasoro subalgebra with central charge

$$c_\nu = 25 - 6\nu - \frac{24}{\nu}.$$ (3.15)

The initial dimensions of $G^\pm$ inherited from the Kac–Moody algebra are $\Delta_{G^+} = 1$, $\Delta_{G^-} = 2$. (These dimensions were used in [20].) Following the more symmetric standard convention let us use instead half integer modes arising in an expansion with $\Delta_{G^\pm} = 3/2$ – which are also the scale dimensions with respect to $T$. Hence

$$H_n = \frac{1}{3} (\hat h^2_n - \hat h^1_n)$$

$$G^+_{n-\frac{1}{2}} = \hat f^1_n$$

$$G^-_{n-\frac{1}{2}} = \hat f^2_n + (\hat e^1 \hat h^2)_{n-1} - \left( \frac{1}{\nu} - 2 \right) n \hat e^1_{n-1}$$

$$\frac{1}{\nu} L_n = \hat f^3_{n+1} + (\hat e^1 \hat f^1)_n + \frac{1}{2} C_{ij} (\hat h^i \hat h^j)_n - \frac{1}{2} \left( \frac{1}{\nu} - 2 \right) (n+1) \hat h^3_n,$$

and one recovers (the Neveu–Schwarz sector of) the standard $W_3^{(2)}$ mode-algebra.

### 4. Quantum Gauge Transformations.

We will consider modules $\Omega_\lambda$ that are tensor products of a $A_2^{(1)}$ Verma module $V_\lambda$ and a ghost Fock module. To simplify notation we will avoid indicating explicitly tensor products, thus assuming that the highest weight vector $V_\lambda$ of $V_\lambda$ is annihilated also by the positive modes of all $b^i(z)$ and the nonnegative modes of $c^i(z)$. Clearly any singular vector in the module of $A_2^{(1)}$ built on $V_\lambda$ is a singular vector in $\Omega_\lambda$ and furthermore we can use equivalently the hatted counterparts.
of the three generating elements of \( \mathfrak{n}_- \) to build these vectors, since \((b^i c^j)_\alpha V_\lambda = 0 \) (cf. (3.2a)). Therefore with \( V_{w,\lambda} \) we denote the singular vectors of both \( \Omega \) and \( V_\lambda \).

The BRST operators (3.3) and (3.10), for the two reductions respectively, annihilate any singular vector \( V_{w,\lambda} \) (including the highest weight states). The generators of the \( \mathcal{W}_3 \) and \( \mathcal{W}_3^{(2)} \) algebras have been expressed in (3.7) and (3.13), respectively, through the \( \widehat{\mathfrak{sl}}(3) \) currents and the respective ghosts. From these expressions (or better from their mode versions (3.7b) and (3.16)) it is immediate that the Kac-Moody singular vectors \( V_{w,\lambda} \) are annihilated by all positive modes of the (respective) \( \mathcal{W} \) algebra. Also the zero modes of the \( \mathcal{W} \) algebra generators reduce on such vectors to some polynomial of the zero modes \( h_0^i \) of the Cartan currents and hence to a number, i.e., \( V_{w,\lambda} \) are eigenvectors of the \( \mathcal{W} \) algebra zero modes (see (5.3,4), (6.2,3) below for explicit expressions). Hence in particular we can identify the Kac-Moody highest weight state \( V_\lambda \) with the highest weight state of a Verma module of the corresponding \( \mathcal{W} \) algebra.

For \( V_{w,\lambda} \) viewed as Kac-Moody singular vectors we have explicit expressions \( V_{w,\lambda} = \mathcal{P} V_\lambda \), where \( \mathcal{P} \) are the Malikov-Feigin-Fuks monomials (or equivalently ordinary polynomials) of the Kac-Moody generators. Thus a Kac-Moody singular vector will determine a \( \mathcal{W} \) algebra singular vector if it can be expressed, eventually up to \( Q \) exact terms, entirely by polynomials \( \mathcal{S} \) of \( \mathcal{W} \) algebra generators, i.e., \( \mathcal{P} V_\lambda = (\mathcal{S} + Q \text{ exact terms}) V_\lambda \). This problem, on the other hand, is far from trivial.

### 4.1. Quantum gauge transformation for \( \mathcal{W}_3 \).

Now trying to “invert” the expression (3.7b) for the negative mode \( \mathcal{W}_3 \) generators one gets in the simplest case (no summation in \( i \))

\[
(M^3 - 1 - \frac{1}{\nu}) \left( f_0^i + (M^i - 1) C_{ij} \hat{h}_{-1}^j \right) V_\lambda = \left( (\delta_{i1} - \delta_{i2}) a_\nu W_{-1} + \frac{1}{2\nu} L_{-1} \left( 2C_{ij} M^j - 1 - \frac{1}{\nu} \right) \right) V_\lambda, \quad i = 1, 2.
\]

(4.1)

Comparing with (2.9) \((i = 1, 2)\) one finds that for weights \( \lambda \) such that \( M^1 = 1 \) (or \( M^2 = 1 \)) the l.h.s. of the above equalities become proportional to the simplest Kac-Moody singular vectors. One can expect that there exists in general a map intertwining the Kac-Moody and \( \mathcal{W}_3 \) singular vectors. In principle it could be found using the Kac-Moody commutation relations, systematically “inverting” (3.7b) and getting rid at the end of all Heisenberg subalgebra depending terms. However this becomes soon technically rather messy. An idea how to find such an intertwining map is actually suggested by the classical Drinfeld-Sokolov approach.

The classical counterparts of the generators (3.7) emerge [1] as differential polynomials of the (classical) fields \( f^a(z), h^i(z) \), and their derivatives, invariant under the gauge symmetry generated by the constraints. Their explicit expressions can be recovered by a gauge fixing transformation of the constrained system – the so called [3] “highest weight” Drinfeld-Sokolov gauge. These reduced classical fields can be quantised by replacing \( f^a(z), h^i(z) \rightarrow \hat{f}^a(z), \hat{h}^i(z) \), normal ordering and identifying an intrinsic parameter with \( k + 2 \). This recovers the r.h.s. of (3.7) with (3.6) taken into account, i.e., the result of [11]. As in the \( \mathfrak{sl}(2) \) case discussed in the Introduction, we have to look for a proper quantum analogue of this gauge transformation, generated by the positive subalgebra \( \mathfrak{n}_+ \), with gauge parameters depending on \( h^i(z), f^a(z) \) and their derivatives. The simplest singular vectors of \( \mathfrak{sl}(3) \) corresponding to the simple roots \( \alpha^1, \alpha^2 \), are given by powers of root vectors \( f_0^i \), \( i = 1, 2 \), acting on the highest weight state (cf. (2.9)). Furthermore the states \( (f_0^i)^t V_\lambda (= (f_0^i)^t V_\lambda) \),
or \((f_0^i)^t V_\lambda = (f_0^i)^t V_\lambda\), are annihilated by \(e_0^3\) and \(e_0^2\), or \(e_0^1\) and \(e_0^1\), respectively. Thus we need not a quantum analogue of the full original Drinfeld-Sokolov gauge transformation, but rather a suitable “\(sl(2)\)”- type transformation generated by the simple-root vectors \(e_0^1\), or \(e_0^2\), respectively.

So consider the following operators, generalising straightforwardly the operator \((1.2)\),

\[
\mathcal{R}^{(i)}(u) = \circ \exp C_{ij} \int_0^u du' \dot{h}^j_{(-)}(-u') \circ, \quad i = 1, 2, \quad \dot{h}^j_{(-)}(u) = \sum_{n=1}^\infty u^{n-1} \dot{h}^j_{-n}, \quad (4.2)
\]

\[
\mathcal{R}^{(i)}(u) = \sum_{k=0}^{\infty} \mathcal{R}^{(i)}_{-k} u^k.
\]

The parameter \(u\) will be identified with some generators, e.g., with \(\hat{e}_0^i\). The \(\circ \circ \) indicate that if \(u\) and \(C_{ij} \dot{h}^j\) do not commute, then in the expansion of the exponent the \(u\)’s should come to the right of the modes of \(C_{ij} \dot{h}^j\). For a recursive definition of \(\mathcal{R}^{(i)}\) see Appendix B. \(^1\)

Now we will describe the analogues for \(\tilde{sl}(3)\) of the basic properties of the gauge transformations

\[
\mathcal{R}^{(i)} \equiv \mathcal{R}^{(i)}(\hat{e}_0^i) \quad i = 1, 2,
\]

which we discussed in the Introduction. The choice \(u = e_0^i\) or \(u = \hat{e}_0^i\), \(i = 1, 2\) is irrelevant for the properties described below since \((\hat{b}^3 \hat{c}^j)_{0} \) annihilate all states in \(\Omega_\lambda\) created only by Kac-Moody generators, so we will use equally both. The first property

- \(\mathcal{R}^{(i)}\) leave invariant all Kac-Moody singular vectors –

follows from \(\mathcal{R}^{(i)}_0 = 1\) and the choice \(u = \hat{e}_0^i\). Denote \(V^{(i)}_t \equiv (f_0^i)^t V_\lambda\). For each of the simple roots \(\alpha^i, i = 1, 2\), these vectors form a \(sl(2)\) Verma module. The second property says

- \(\mathcal{R}^{(i)}\) maps the \(sl(2)\) Verma module corresponding to \(\alpha^i\) into the kernel of the BRST operator, i.e.,

\[
Q \mathcal{R}^{(i)}(f_0^i)^t V_\lambda = 0, \quad \text{any } t = 0, 1, 2 \ldots . \quad (4.3)
\]

The proof is straightforward: from the OPEs it follows

\[
[Q, C_{ij} \dot{h}^j_{-n}] = -\dot{c}^j_{-n}, \quad [Q, \hat{e}^i_{0}] = 0, \quad i = 1, 2,
\]

hence one proves inductively

\[
[Q, \mathcal{R}^{(i)}_{-k}] = -\mathcal{R}^{(i)}_{-k+1} \dot{c}^i_{-1},
\]

or,

\[
[Q, \mathcal{R}^{(i)}] = -\mathcal{R}^{(i)} \dot{c}^i_{-1} \hat{c}_0^i. \quad (4.4)
\]

Since \(V^{(i)}_t \equiv (f_0^i)^t V_\lambda\) is annihilated by all positive mode generators, by \(\hat{e}_0^3\) and \(c_0^a, a = 1, 2, 3\), and \(\hat{e}_0^i V^{(j)}_t = 0\) for \(i \neq j\), we have

\[
Q V^{(i)}_t = c^i_{-1} \hat{e}_0^i V^{(i)}_t
\]

which combined with (4.4) proves (4.3). Finally,

- the gauge transformations intertwine Kac-Moody and \(W\) algebra generators,

\(^1\) Note that the coefficients in the expansion of \(\mathcal{R}^{(i)}(u)\) in powers of \(u\) represent elementary Schur polynomials.
\[ R^{(i)} \left( h_0^2 + 2 - \frac{1}{\nu} \right) f_0^i V = \frac{1}{\nu} \sum_{p=1}^{\infty} \left( (\delta_{i1} - \delta_{i2}) a_{ij} - \frac{1}{\nu} \right) \left( 1 + \frac{1}{\nu} C_{ij} \right) (\partial L)_{-p} \]

\[ - \frac{C_{ij}}{\nu} L_{-p} R^{(i)} + L_{-p} R^{(i)} C_{ij} h_0^j \right) (-e_0^i)^{p-1} V . \]

where \( V \) is any vector annihilated by all positive mode generators and by \( e_0^a, a = 1, 2, 3 \). Also we have \( R^{(i)} V^{(j)}_t = V^{(j)}_t \), if \( i \neq j \).

The detailed (rather lengthy) proof of (4.5) is presented in Appendix B. In Section 6 this key relation will be used to transform Kac-Moody singular vectors into \( \mathcal{W}_3 \) algebra ones. The second property (4.3) suggests that the states \{\( R^{(i)} (f_0^i)^t V_\lambda, t = 0, 1, 2, \ldots \} \) are actually elements in the universal enveloping (negative mode) subalgebra of \( \mathcal{W}_3 \).

Note that, as in the \( sl(2) \) case \([17]\), one can reformulate (4.5) identifying the parameter \( u \) with the raising operators \( t^{+i}, i = 1, 2 \), of an auxiliary \( sl(3) \) algebra \( \{t^a\} \), instead of \( \hat{e}_0^i \).

### 4.2. Quantum gauge transformation for the Polyakov-Bershadsky algebra.

Now we turn to “inverting” (3.16), i.e., expressing the Kac-Moody generators in terms of \( \mathcal{W}_3^{(2)} \) generators, at least in the Malikov-Feigin-Fuks monomials. Again the key role is played by the projections \( R^{(i)} \) along simple root directions \( \alpha^i, i = 1, 2 \), of the respective “quantum gauge transformation”. The operators \( R \) have properties analogous to the ones discussed in the introduction and in the previous subsection, only now things are much simpler. Of course, the remarks at the beginning of subsection 4.1 apply in the present case also.

Since \( G^+_1 = f_0^1 \), the corresponding gauge transformation \( R^{(1)} \) is (as in the classical case) just an identity. For the second (horizontal) direction introduce the operator:

\[ R^{(2)}(u) = \exp \hat{e}_{-1}^1 u \equiv \sum_{n=0}^{\infty} \frac{1}{n!} (\hat{e}_{-1}^1)^n u^n. \]

Choosing the generator of the gauge transformation to be \( u = e_0^2 \) (the relevant properties of \( R \) are unaltered if we choose \( u = \hat{e}_0^2 \)) we set

\[ R^{(2)} \equiv R^{(2)}(e_0^2). \]

The first property of \( R \) is obvious. As before denote \( V^{(2)}_t = (f_0^2)^t V_\lambda \). From \([\hat{e}_{-1}^1, Q^{(PB)}] = e_{-1}^2, [\hat{e}_{-1}^1, e_{-1}^2] = 0, \) and \([e_{-1}^2, Q^{(PB)}] = 0 \) we get

\[ [R^{(2)}, Q^{(PB)}] = R^{(2)} e_{-1}^2 e_{-1}^2, \]

which combined with \( Q^{(PB)} V^{(2)}_t = e_{-1}^2 e_{-1}^2 V^{(2)}_t \) gives the second property of the gauge transformation:

\[ Q^{(PB)} R^{(2)} V^{(2)}_t = 0. \]

Finally the gauge transformation \( R^{(2)} \) has the intertwining property

\[ R^{(2)} f_0^2 V^{(2)}_t = G_{-2} R^{(2)} V^{(2)}_t. \]
The proof is straightforward. Notice that from the definition (3.16) of $G_{-\frac{1}{2}}$ and the commutation relations we get $G_{-\frac{1}{2}} \mathcal{R}^{(2)} V_{t}^{(2)} = (f_{0}^{2} + e_{-1}h_{0}^{2}) \mathcal{R}^{(2)} V_{t}^{(2)}$. Using the Kac-Moody commutation relations we have furthermore $(f_{0}^{2} + e_{-1}h_{0}^{2}) \mathcal{R}^{(2)} = \mathcal{R}^{(2)} f_{0}^{2}$ and (4.8) follows.

Repeating (4.8) until $\mathcal{R}^{(2)}$ reaches $V_{\lambda}$ we obtain
\begin{equation}
\mathcal{R}^{(2)} (f_{0}^{2})^{t+1} V_{\lambda} = (G_{-\frac{1}{2}})^{t+1} V_{\lambda} .
\end{equation}

5. SINGULAR VECTORS of $\mathcal{W}^{(2)}_{3}$ VERMA MODULES.

Having the quantum gauge transformations we turn now to the description of the $\mathcal{W}$ algebras singular vectors. In this section we will deal with the simpler case of $\mathcal{W}^{(2)}_{3}$. The strategy is the same as in the Virasoro case [17] – for a general Malikov-Feigin-Fuks monomial use the quantum constraint
\begin{equation}
f_{0}(\equiv e_{-1}^{3}) = 1 + \{ Q^{(PB)} , b_{0}^{3} \}
\end{equation}
in the direction of the affine simple root $\alpha^{0}$, while in the directions of the non-affine simple roots use the quantum gauge transformations.

The remarks and notational convention from the beginning of section 4 hold. More explicitly we have
\begin{equation}
B_{n} V_{w, \lambda} = 0 \quad \text{for} \quad B_{n} = L_{n} , H_{n} , \quad G_{n-\frac{1}{2}}^{\pm} \quad \text{or} \quad G_{n-\frac{1}{2}}^{\pm} , \quad n \in \mathbb{N} ,
\end{equation}
i.e., the positive modes of the $\mathcal{W}^{(2)}_{3}$ generators (see (3.16)) annihilate any Kac-Moody singular vector $V_{w, \lambda}$. These singular vectors are also eigenvectors of the zero modes
\begin{equation}
L_{0} V_{w, \lambda} = h_{w, \lambda}^{(PB)} V_{w, \lambda} , \quad H_{0} V_{w, \lambda} = q_{w, \lambda} V_{w, \lambda} ,
\end{equation}
with
\begin{equation}
h_{\lambda}^{(PB)} = \frac{\nu}{2} (\bar{\lambda}, \lambda - \kappa^{(PB)} \rho) , \quad q_{\lambda} = (C_{2j} - C_{1j}) (\bar{\lambda}, \alpha^{j}) , \quad \kappa^{(PB)} = \frac{1}{\nu} - 2 .
\end{equation}

In particular we can identify $V_{\lambda}$ with the highest weight state $| q_{\lambda}, h_{\lambda}^{(PB)} \rangle$ of a $\mathcal{W}^{(2)}_{3}$ Verma module. In fact the correspondence between Kac-Moody singular vectors and $\mathcal{W}^{(2)}_{3}$ ones is not 1 to 1. Indeed, the $\mathcal{W}^{(2)}_{3}$ algebra weights (5.4) are invariant under the shifted action of $w_{0}$, i.e.,
\begin{equation}
h_{\lambda}^{(PB)} = h_{w_{0} \lambda}^{(PB)} , \quad q_{\lambda} = q_{w_{0} \lambda} ,
\end{equation}
or, equivalently – under a $\frac{1}{2} \kappa^{(PB)}$-shifted action on the projected weights $\overline{\lambda}$ of the reflection in the $\alpha^{3}$ direction, $\overline{\lambda} - \frac{1}{2} \kappa^{(PB)} \overline{\rho} = w_{0} \alpha^{3} (\overline{\lambda} - \frac{1}{2} \kappa^{(PB)} \overline{\rho}) , \quad \overline{\lambda} = \overline{w_{0} \lambda}$. Since a Verma module is determined uniquely by the highest weights we have to identify $| q_{\lambda}, h_{\lambda}^{(PB)} \rangle$ with the pair of Kac-Moody highest weight vectors $V_{\lambda}$ and $V_{w_{0} \lambda}$.

Now let $\lambda$ be such that it satisfies the Kac-Kazhdan condition (2.8) with respect to the affine root $\alpha_{0}$, i.e. $\langle \lambda + \rho, \alpha^{0} \rangle = m \in \mathbb{N}$. The Kac-Moody singular vector in (2.9) corresponding to $\alpha^{0}$, does not produce nontrivial singular vectors in the $\mathcal{W}^{(2)}_{3}$ Verma module. The reason is that $V_{w_{0} \lambda}$ is cohomologically equivalent to the highest weight state $V_{\lambda}$, i.e.,
\begin{equation}
V_{w_{0} \lambda} = (1 + Q^{(PB)} \cdots ) V_{\lambda} \simeq V_{\lambda}
\end{equation}
(With \(\simeq\) we denote equality up to \(Q\)-exact terms and note that since (5.2) holds for both \(V_\lambda\) and \(V_{w_0,\lambda}\) it holds for the \(Q\)-exact terms as well.) This is consistent with the symmetry (5.5) of the zero modes eigenvalues.

The invariance subgroup \(W^{(n)} = \{1, w_0\}\) of the Weyl group \(W\) is directly connected with the defining vector of the \(sl(2)\) embedding, in the case under consideration this is \(\eta = \alpha^3/2\), and the (nontrivial) constraint, here (5.1). In the \(W_3^{(2)}\) case one is constraining to 1 the current of the \(\alpha^3\) root vector, or in modes \(e_1^2 \simeq 1\) and note that \(e_1^2\) is the root vector of \((-\delta + \alpha^3) = -\alpha_0\).

Now let us turn to the simplest Kac-Moody singular vectors corresponding to the simple reflections \(w_i, i = 1, 2\). Let \(M_i \equiv \langle\lambda + \rho, \alpha_i\rangle \in \mathcal{N}\) for \(i = 1\) or 2, then the corresponding KM singular vectors are given in (2.9). Using (3.16), the iterated relation (4.9), and the fact that \(\mathcal{R}\) keeps singular vectors invariant, we obtain

\[
V_{w_1,\lambda} = (f_0^1)^{M_1} V_\lambda = (G^+_{-\frac{1}{2}})^{M_1} V_\lambda ,
\]

or,

\[
V_{w_2,\lambda} = (f_0^2)^{M_2} V_\lambda = (G^-_{-\frac{1}{2}})^{M_2} V_\lambda .
\]

The other horizontal KM singular vectors (2.13) are represented similarly by compositions of the operators in the r.h.s. of (5.7), e.g., starting from the first vector in (2.13) and using the properties of the gauge transformation, equivalently (5.7), the following chain leads us to an expression for \(V_{w_1w_2,\lambda}\) entirely in terms of \(W_3^{(2)}\) algebra generators:

\[
V_{w_1w_2,\lambda} = (f_0^1)^{M_1} V_{w_2,\lambda} = (G^+_{-\frac{1}{2}})^{M_1} (G^+_{-\frac{1}{2}})^{M_2} V_{w_2,\lambda} = (G^+_{-\frac{1}{2}})^{M_3} (G^-_{-\frac{1}{2}})^{M_2} V_\lambda .
\]

Next we consider a general Kac-Moody singular vector represented by a Malikov-Feigin-Fuks monomial \(V_{w,\lambda} = \mathcal{P}_{\beta,\lambda} V_\lambda\) (see Appendix A for explicit expressions). The procedure of expressing \(\mathcal{P}_{\beta,\lambda}\) in terms of \(W_3^{(2)}\) algebra generators, modulo \(Q\) exact terms, consists of moving from left to right and exploiting the properties of the BRST operator \(Q^{(PB)}\) and the gauge operator \(\mathcal{R}\). How \(\mathcal{R}\) transforms the factors corresponding to simple non-affine reflections is clear from the above example, so consider a factor corresponding to \(w_0\). Let \(w_\beta = w' w_0 w''\) for some \(w', w'' \in W\) and assume all factors corresponding to \(w'\) have been expressed in terms of \(W_3^{(2)}\) algebra generators, more precisely – \(G^+_{-\frac{1}{2}}\)'s. From the quantum constraint (5.1), or equivalently from (5.6), we have

\[
V_{w_\beta,\lambda} \simeq \underbrace{(e_3^3)^{(w'' \cdot \lambda + \rho, \alpha_0)} V_{w'',\lambda}}_{\mathcal{W} \text{ generators}} \underbrace{(1 + Q^{(PB)} A) V_{w'',\lambda}}_{\mathcal{W} \text{ generators}} ,
\]

for some \(A\). Because \(Q^{(PB)}\) commutes with the \(W_3^{(2)}\) algebra generators it can be pulled to the left producing a \(Q\) exact contribution. The end result is very simple –

\[
\text{in a Malikov-Feigin-Fuks monomial substitute}\ \{f_0^1, f_0^2, e_3^3\} \text{ with}\ \{G^+_{-\frac{1}{2}}, G^-_{-\frac{1}{2}}, 1\}
\]

As an illustration consider \(\lambda\) such that \(M_1 = m_1 - \frac{m_{i-1}}{2}\), with \(m_1, m'_1 \in \mathcal{N}\) and take the Kac-Kazhdan root \(\beta = (m'_1 - 1)\delta + \alpha^1\). Then starting from the explicit expression for the MFF vector written in (A.1) of Appendix A one obtains

\[
V_{w_\beta,\lambda} \simeq (G^+_{-\frac{1}{2}})^{m_1} (G^-_{-\frac{1}{2}})^{-\frac{m_{i-1}}{2}} \cdots (G^+_{-\frac{1}{2}})^{m_1 - \frac{m_{i-3}}{2}} (G^-_{-\frac{1}{2}})^{m_1 - \frac{m_{i-2}}{2}} (G^+_{-\frac{1}{2}})^{M_1} V_\lambda .
\]
Using (5.2-4) one sees that (5.11) provides a singular vector in the $W_3^{(2)}$ Verma module built on $|q_\lambda, h^{(PB)}_\lambda\rangle \equiv V_\lambda$ (whose $L_0$ level is $m_1(2m'_1 - 1)/2$).

The symmetry (5.5) under the two element group $\{1, w_0\}$ implies that the vector (5.11) can be alternatively obtained starting from the KM singular vector $V_{w_\beta w_\lambda} = V_{w_0 w_\beta w_\lambda}$ in the KM Verma module $\Omega_{w_0 w_\lambda}$, where $\beta' = m'_1\delta - \alpha^2$. Indeed, from the explicit expressions for $P_\beta w_\lambda$ (compare (A.1) and (A.3) - with “1” and “2” interchanged), the rule (5.10) and (5.5) we obtain

$$V_{w_\beta w_\lambda} \simeq V_{w_\beta' w_\lambda'}, \quad \text{with} \quad \beta' = w_0(\beta), \quad \lambda' = w_0 \cdot \lambda,$$

(5.12)

which extends (5.6).

The other two series, corresponding to the roots $\beta = (m'_2 - 1)\delta + \alpha^2$, $m'_2 \in \mathbb{N}$, and $\beta = (m'_3 - 2)\delta + \alpha^3$, $m'_3 - 1 \in \mathbb{N}$, for which $M^2 = m_2 - \frac{m'_2 - 1}{\nu}$, $m_2 \in \mathbb{N}$, and $M^3 = m_3 - \frac{m'_3 - 2}{\nu}$, $m_3 \in \mathbb{N}$, respectively, are obtained in exactly the same way, starting from the explicit expressions (A.3), (A.4) for the Malikov-Feigin-Fuks vectors. As in (5.12) these $W_3^{(2)}$ singular vectors admit alternative derivations using that $w_0(\beta_{n, +2}) = \beta_{n+1, -1}$ and $w_0(\beta_{n, +3}) = \beta_{n+2, -3}$ respectively.

The question arises to give meaning to formal expressions like (5.11) and proving rigorously that the r.h.s. is indeed a $W_3^{(2)}$ singular vector. Rather than repeating the argumentation from [16] we will describe an algorithm which transforms expressions like (5.11) into ordinary polynomials of $W_3^{(2)}$ algebra generators acting on the h.w. state. First note that the commutation relation

$$G_{-p - \frac{1}{2}}^+(G_{-\frac{1}{2}}^-)^s G_{-p - \frac{1}{2}}^- - s (G_{-\frac{1}{2}}^+)^{s-1} \left(3(HH) + \frac{3}{2} \partial H - (k + 3) T\right)_{-p-\frac{1}{2}}$$

$$- s(s-1)(G_{-\frac{1}{2}}^+)^{s-2} \left(3(G^+H) + (k + 3) \partial G^+\right)_{-p-\frac{1}{2}} \quad (5.13)$$

(and the analogous one with $G^\pm$ reverted) can be analytically continued in $s$. (The other necessary commutation relations do not involve non-linearities and thus are simpler and are left to the interested reader.) To obtain (5.13) one has to use (2.17).

Next, transforming a Malikov-Feigin-Fuks monomial by the procedure (5.10), results in a monomial having a property analogous to (A.5), i.e., in the middle there is a generator to an integer power, it is surrounded by a generator with powers adding to an integer and so on. Thus the algorithm is (in complete analogy with the one, described at the end of Appendix A, for Malikov-Feigin-Fuks vectors) – start from the middle and using relations like (5.13) move outwards.

Let us see it explicitly in the case $m_1 = 1$, $m'_1 = 2$, i.e.

$$\beta = \delta + \alpha^1, \quad M^1 = 1 - \frac{1}{\nu}.$$

(5.14)

Then (5.11) reduces to

$$V_{w_\beta w_\lambda} \simeq (G_{-\frac{1}{2}}^+)^{1 + \frac{1}{2}} G_{-\frac{1}{2}}^- (G_{-\frac{1}{2}}^+)^{1 - \frac{1}{2}} V_\lambda.$$  

(5.15)

The general formula (5.13) simplifies when applied on a singular vector, in particular

$$G_{-\frac{1}{2}}^-(G_{-\frac{1}{2}}^+)^s V_\lambda = (G_{-\frac{1}{2}}^-)^{s-2} \left[s(k + 3) G_{-\frac{1}{2}}^- L_{-1} - 3s(2q_\lambda + s - 1) G_{-\frac{1}{2}}^- H_{-1}ight.$$

$$- s(s-1)(3q_\lambda + 2s - 1) G_{-\frac{1}{2}}^- + (G_{-\frac{1}{2}}^+)^2 G_{-\frac{1}{2}}^+\right] V_\lambda,$$

(5.16)

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and setting $s = 1 - \frac{1}{\nu} = -k - 2$, for the r.h.s. of (5.15), we get
\begin{equation}
V_{w,\beta,\lambda} \simeq \left[ -(k + 3)(k + 2) G^+_{-\frac{1}{2}} L_{-1} + 3(k + 2)(2q_\lambda - k - 3) G^+_{-\frac{1}{2}} H_{-1} 
\right.
\end{equation}
\begin{equation}
\left. - (k + 3)(k + 2)(3q_\lambda - 2k - 5) G^+_{-\frac{1}{2}} + (G^+_{-\frac{1}{2}})^2 G^-_{-\frac{1}{2}} \right] V_\lambda.
\end{equation}

One can independently check (5.2), i.e., that the above expression is indeed a $\mathcal{W}_3^{(2)}$ algebra singular vector. For example a rather lengthy computation gives
\begin{equation}
\left( M^1 - 1 + \frac{1}{\nu} \right) (M^2 - \frac{2}{\nu} + 1) \left[ - (k + 3)(k + 2) L_{-1} \right.
\end{equation}
\begin{equation}
\left. + 2(k + 2)(M^2 - M^1) - 3(k + 1)(k + 4) H_{-1} + 2G^+_{-\frac{1}{2}} G^-_{-\frac{1}{2}} \right] V_\lambda.
\end{equation}

We remark that the two factors $(M^1 - 1 + \frac{1}{\nu})(M^2 - \frac{2}{\nu} + 1)$ appear in all the r.h.s.’s of $B_n V_{w,\beta,\lambda}$, $n \in \mathbb{N}$. The first factor reflects the Kac-Kazhdan condition (5.14), while the second one comes from the Kac-Kazhdan condition $M^2 = \frac{2}{\nu} - 1$ related to the highest weight $w_0 \cdot \lambda$ and $\beta' = w_0(\beta) = 2\delta - \alpha^2$ in agreement with (5.12).

Having the singular vectors one can analyse the reducibility structure of $\mathcal{W}_3^{(2)}$ Verma modules for different values of the central charge $c_\nu$. It is clear that it will be very similar to the structure of the KM Verma modules, modulo the identifications accounting for the symmetry (5.5). Hence also the $\mathcal{W}_3^{(2)}$ characters can be found following the standard procedures, or directly from the $A_2^{(1)}$ characters, as done in [13] for the case of the principal embeddings.

6. SINGULAR VECTORS of $\mathcal{W}_3$ VERMA MODULES.

6.1. The basic $\mathcal{W}_3$ algebra singular vectors.

While the derivation of the $\mathcal{W}_3^{(2)}$ algebra singular vectors is apparently quite analogous to the one in the $\widehat{sl}(2)$ case [17], the case $\mathcal{W}_3$ is more involved. We begin with describing how the quantum gauge transformations of section 4.1. can be used to recover the simplest series of $\mathcal{W}_3$ singular vectors obtained first by Bowcock and Watts [23].

Let $V_{w,\lambda}$ be some $\widehat{sl}(3)$ singular vector in $\Omega_\lambda$ of weight $w \cdot \lambda$. As already discussed in the beginning of section 4 all Kac-Moody singular vectors are annihilated by the positive modes of the reduced $\mathcal{W}$ generators (3.7b), i.e.,
\begin{equation}
L_n V_{w,\lambda} = L_n^{(ff)} V_{w,\lambda} = 0, \quad W_n V_{w,\lambda} = W_n^{(ff)} V_{w,\lambda} = 0, \quad n \in \mathbb{N}.
\end{equation}

For the zero modes one obtains
\begin{equation}
L_0 V_{w,\lambda} = L_0^{(ff)} V_{w,\lambda} = h^{(2)}_{w,\lambda} V_{w,\lambda}, \quad W_0 V_{w,\lambda} = W_0^{(ff)} V_{w,\lambda} = h^{(3)}_{w,\lambda} V_{w,\lambda},
\end{equation}
where
\begin{equation}
h^{(2)}_{\lambda} = \frac{\nu}{2} C_{ij} \lambda(h^0_{ij} \lambda - 2\kappa\rho)(h^0_{ij} \lambda - 2\kappa\rho) = \frac{\nu}{2} (\lambda, \lambda - 2\kappa\rho), \quad \kappa = \frac{1}{\nu} - 1,
\end{equation}
\begin{equation}
a_\nu h^{(3)}_{\lambda} = C_{1i} C_{2j} (C_{1l} - C_{2l}) (\lambda - \kappa\rho, \alpha^i) (\lambda - \kappa\rho, \alpha^i) (\lambda, \alpha^i).
\end{equation}
We shall start with the simplest subseries of Kac-Moody singular vectors given in (2.9) for \( j = 1, 2 \). We will describe how the key relation (4.5) transforms a Kac-Moody singular vector of this type into a \( W_3 \) algebra one. Take \( V = V_{t-1}^{(i)} \equiv (f_0^i)^t V_\lambda \). Then (4.5) can be rewritten as

\[
\mathcal{R}^{(i)} V_{t-1}^{(i)} = \sum_{p=1}^{t-1} \mathcal{L}_{t-t-p}^{(i)} \mathcal{R}^{(i)} V_{t-p}^{(i)}
\]  

(6.4)

where we have denoted

\[
\mathcal{L}_{t-t-p}^{(i)} = \frac{\prod_{l=1}^{p-1} (l-t)(M^1 + l-t)}{\nu (M^3 - \frac{1}{\nu} - t)}.
\]  

(6.5)

\[
\cdot \left( (\delta_{i1} - \delta_{i2}) a_\nu \nu W_{-p} + \left( C_{ij} M^j - 1 - t + p - \frac{2}{3\nu} \right) L_{-p} - \left( \frac{1}{2} + \frac{1}{6\nu} \right) (\partial L)_{-p} \right).
\]

Iterating (6.5) until we reach the singular vector \( V_\lambda \) we obtain

\[
\mathcal{R}^{(i)} (f_0^i)^t V_{t} = \mathcal{O}_\lambda^{(i;t)} V_{\lambda}
\]  

(6.6)

where according to (4.3) the r.h.s. of (6.6) is in the kernel of the operator \( Q \). Explicitly it reads

\[
\mathcal{O}_\lambda^{(i;t)} \equiv \mathcal{O}_{\lambda}^{(i;M^1,M^2;\nu)} = \sum_{k=1}^{t} \sum_{\{p_1\}_{n=1}^{k-1}} \mathcal{L}_{t-p_k-1}^{(i)} \mathcal{L}_{p_k-1,p_k-2}^{(i)} \ldots \mathcal{L}_{p_1,0}^{(i)},
\]  

(6.7)

where the second sum is over all \( \{t > p_{k-1} > \ldots > p_1 > 0\} \).

Denote for short \( \mathcal{O}_\lambda^{(i)} = \mathcal{O}_\lambda^{(i;M^i)} \). Using (6.6), the property that \( \mathcal{R}^{(i)} \) leaves singular vectors invariant and (6.1,2), which in particular implies that we can identify \( V_\lambda \) with the highest weight state \( |h_\lambda^{(2)}, h_\lambda^{(3)}\rangle \) of a \( W_3 \) algebra Verma module, we obtain:

*The Kac-Moody singular vectors (2.9) \((i = 1, 2)\) are equal to \( W_3 \) algebra singular vectors\n
\[
V_{w_1,\lambda} = (f_0^i)^{M^1} V_{\lambda} = \mathcal{O}_{\lambda}^{(i)} V_{\lambda}.
\]  

(6.8)

Furthermore, this is true for all horizontal Kac-Moody singular vectors (2.13),

\[
V_{w_1 w_2,\lambda} = (f_0^1)^{M^1} (f_0^2)^{M^2} V_{\lambda} = \mathcal{O}_{w_2,\lambda}^{(1)} V_{w_2,\lambda} = \mathcal{O}_{w_2,\lambda}^{(2)} V_{\lambda},
\]

\[
V_{w_2 w_1,\lambda} = (f_0^2)^{M^2} (f_0^1)^{M^1} V_{\lambda} = \mathcal{O}_{w_1,\lambda}^{(1)} V_{\lambda},
\]  

(6.9)

\[
V_{w_2 w_1 w_2,\lambda} = (f_0^1)^{M^1} (f_0^2)^{M^2} (f_0^1)^{M^2} V_{\lambda} = \mathcal{O}_{w_1 w_2,\lambda}^{(1)} V_{w_2,\lambda} = \mathcal{O}_{w_1 w_2,\lambda}^{(2)} V_{\lambda}.
\]

Let us recall that in the first and the second equalities in (6.9) both \( M^1 \) and \( M^2 \) are nonnegative integers, while in the third one it is sufficient that \( M^3 = M^1 + M^2 \) is a nonnegative integer. Thus already here one encounters the need for a proper analytic continuation of the basic vectors in (6.8).
Thus we recover the result of [23] obtained by a different method – namely, by a generalisation of the fusion method of [24] exploited in the Virasoro case. Note that in view of (6.1,2) the identification (6.8), (6.9), makes the proof of singularity of the $W_3$ states in the right hand sides straightforward in our approach.

The states (6.6,7) for a given $i$ and any $t = 0, 1, 2, \ldots$, provide the basis for a matrix reformulation, in analogy with the one in [24], of the $W_3$ singular vectors of the type considered up to now.

6.2. The general $W_3$ algebra singular vectors.

We turn to a systematic description of the general $W_3$ singular vectors following the strategy outlined in [20].

The Kac-Moody singular vectors in (2.9), corresponding to the affine root $\alpha^0$, produce now $Q$-exact terms, i.e.,

$$V_{w_0, \lambda} \simeq 0,$$

because of the quantum constraint

$$f_0 = \{ Q, b_0^i \}$$

(recall that $f_0 = \hat{e}_3 - 1$) and the properties of the BRST charge $Q$. Thus we cannot exploit exactly the same mechanism as in the case of the $W_3^{(2)}$ Verma modules in order to construct the general $W_3$ singular vectors. On the other hand the modes of the currents which are now constrained to $1$, i.e.,

$$\hat{e}_i = 1 + \{ Q, b_0^i \}, \quad i = 1, 2,$$

do not appear explicitly in the Malikov-Feigin-Fuks monomials. Yet one notices that the affine reflections $w_\beta$ in (2.12), underlying the construction of the MFF vectors, contain in their decompositions elements of the type $w_0 w_1 w_0 = w_{\delta - \alpha^2}$ and $w_0 w_2 w_0 = w_{\delta - \alpha^1}$. The root vectors corresponding to these reflections are exactly the modes constrained in (6.12). Furthermore these elements of the affine Weyl group generate the finite subgroup $\overline{W}^{(\rho)}$, which keeps invariant (under the shifted action on $\lambda$) the zero mode eigenvalues (6.3), i.e.,

$$h_\lambda^{(2)} = h_{w_\lambda}^{(2)}, \quad h_\lambda^{(3)} = h_{w_\lambda}^{(3)}, \quad w \in \overline{W}^{(\rho)}.$$  

This is equivalent to the well known invariance under a $\kappa$-shifted action $\lambda - \kappa \rho = w_\lambda - \kappa \rho$ of the finite Weyl group $\overline{W}$ on the projected weights.

Thus we can identify the highest weight state $| h_\lambda^{(2)}, h_\lambda^{(3)} \rangle$ of a $W_3$ algebra Verma module with the set of Kac-Moody highest weight states $\{ V_{w_\lambda}; w \in \overline{W}^{(\rho)} \}$.

Explicitly the group $\overline{W}^{(\rho)}$ acts on the weights according to

$$w_{010} \cdot \lambda + \rho = \left( M^3 - \frac{1}{\nu}, -M^2 + \frac{2}{\nu} \right),$$

$$w_{020} \cdot \lambda + \rho = \left( -M^1 + \frac{2}{\nu}, M^3 - \frac{1}{\nu} \right),$$

$$w_{0210} \cdot \lambda + \rho = \left( -M^3 + \frac{3}{\nu}, M^1 \right).$$
\[
\begin{align*}
\overline{w_{0120} \cdot \lambda + \rho} &= \left( M^2, -M^3 + \frac{3}{\nu} \right), \\
\overline{w_{01210} \cdot \lambda + \rho} &= \left( \frac{2}{\nu} - M^2, \frac{2}{\nu} - M^1 \right).
\end{align*}
\]

(6.14d) (6.14e)

Now let us start with the Kac-Moody singular vectors \( V_{w_{\beta} \cdot \lambda} \) for \( \beta = \delta - \alpha^1 \), or \( \beta = \delta - \alpha^2 \), which occur in the modules \( \Omega_{\lambda} \) when \( \lambda \) satisfy the Kac-Kazhdan condition (2.8), i.e., for \( M^1 = -m_1 + \frac{1}{\nu}, m_1 \in \mathbb{N}, \) or \( M^2 = -m_2 + \frac{1}{\nu}, m_2 \in \mathbb{N}, \) respectively. The singular vectors \( V_{w_{\beta-a^i} \cdot \lambda} \) are cohomologically equivalent to the corresponding highest weight states \( V_\lambda \), i.e.,

\[
V_{w_{\beta-a^i} \cdot \lambda} \simeq A_i (\tilde{e}^i_{1})^{m_i} V_\lambda \simeq A_i V_\lambda, \quad i = 1, 2
\]

(6.15)

\[
A_i = \frac{\Gamma((\lambda + \rho, \delta - \alpha^i - \alpha^0))}{\Gamma((\lambda + \rho, -\alpha^0))}.
\]

The first statement in this chain follows, using the Kac-Moody and ghost algebra commutation relations, from the explicit form of the corresponding MFF vectors, i.e., (cf. (2.11) or (A.3))

\[
V_{w_{\beta_1} \cdot \lambda_1} = f_0^{M^2} f_2^{m_1} f_0^{m_1-M^2} V_{\lambda_1}, \quad V_{w_{\beta_2} \cdot \lambda_2} = f_0^{M^1} f_1^{m_2} f_0^{m_2-M^1} V_{\lambda_2},
\]

and the quantum constraint (6.11), while the second – from the quantum constraints (6.12) (see Appendix C for details). Quite similarly one shows that the Kac-Moody singular vectors occurring for \( M^3 = -m_3 + \frac{2}{\nu}, m_3 \in \mathbb{N}, \) and related to the root \( \beta = 2\delta - \alpha^3 = w_0(\alpha^3), (w_\beta = w_{01210} \in \overline{W}(\rho)) \) reproduce trivially the highest weight states in the corresponding \( \mathcal{W}_3 \) Verma modules.

The relations (6.15) provide the basic tool which has to be used along with the elementary singular vectors (6.8) to construct the general singular vectors of the \( \mathcal{W}_3 \) modules. Indeed recalling that each of the subfactors in the general MFF monomial represents formally a singular vector we can start as in the previous section to replace from the left the powers of the simple roots vectors \( f_i, i = 1, 2 \) with the corresponding operator (6.8), while the triples corresponding to the reflections \( w_{010} \) and \( w_{020} \) can be simply “deleted”, more precisely substituted by the numbers \( A_i \) according to (6.15), reflecting the implementation of the constraints \( \tilde{e}^{a^i} \simeq 1, i = 1, 2. \)

Let us first illustrate this on the simple example corresponding to the root \( \beta = \beta_{1,+1} = \delta + \alpha^1, \)

\[
w_\beta = w_1 w_0 w_2 w_0 w_1.
\]

According to the general formula (A.1) in Appendix A the Kac-Moody singular vector is given explicitly by

\[
V_{w_{\beta} \cdot \lambda} = (f_0^{1})^{m_1+\frac{1}{\nu}} (e_{-1}^3)^{m_1-M^2} (f_0^2)^{m_1} (e_{-1}^2)^{M^2} (f_0^1)^{m_1-M^1} V_\lambda
\]

(6.16a)

\[
= (f_0^{1})^{(w_{0201} \cdot \lambda + \rho, \alpha^3)} V_{w_{0201} \cdot \lambda},
\]

where \( m_1 \in \mathbb{N}. \)

First we apply (6.6) to the left most power of \( f_0^1 \) and then, accounting for (6.15) we get rid of the central triple of Kac-Moody generators replacing it by identity modulo cohomologically trivial terms, (to be more precise, by identity times the numerical constants \( A_i \) – cohomological equalities up to numerical coefficients we will denote by \( \sim \)) i.e.,

\[
V_{w_{0201} \cdot \lambda} = (e_{-1}^3)^{m_1-M^2} (f_0^2)^{m_1} (e_{-1}^2)^{M^2} V_{w_1 \cdot \lambda} \sim V_{w_1 \cdot \lambda}.
\]

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Using once again (6.8), the properties of $Q$ and the BRST invariance of the $W_3$ generators (3.7) (namely $Q$ annihilates Kac-Moody singular vectors and commutes with $O$) one obtains

$$V_{w_{10201}:\lambda} \sim \mathcal{O}_{w_{10201}:\lambda}^{(1)} \mathcal{O}_{\lambda}^{(1)} V_{\lambda}. \quad (6.16b)$$

One has $h_{w_{10201}:\lambda}^{(2)} = h_{\lambda}^{(2)} + 2m_1$, i.e., using (6.1.2) one finds that the vector (6.16b) provides a singular vector at $L_0$ level $2m_1$ of the $W_3$ Verma module over $V_\lambda$ identified with $|h_{\lambda}^{(2)} , h_{\lambda}^{(3)} \rangle$.

In getting (6.16b) we have worked formally assuming that a proper analytic continuation of the basic operators (6.8) exists. One can expand $\mathcal{O}_\lambda^{(i)}$ as a series in decreasing powers of $W_{-1}$ (though at first it may seem more natural to expand in powers of $L_{-1}$ this is not possible due to the term $L_{-p} R^{(i)} C_{ij} h_0^i$ in (4.5))

$$\mathcal{O}_\lambda^{(i)} = \frac{\Gamma((\lambda + \rho_i - \alpha^0 - \alpha^i))}{\Gamma((\lambda + \rho_i - \alpha^0))} \sum_{k=0}^{\infty} W_{-1}^{M_i - k} \sum_{n_1, \ldots, n_t} c_{k;n_1,\ldots,n_t}^{(i)} W_{-n_1} \cdots W_{-n_t} L_{-n_1} \cdots L_{-n_t}, \quad (6.17)$$

where the second sum is over positive sets of integers $\{2 \leq m_1 \leq \ldots \leq m_s\}$ and $\{n_1 \leq \ldots \leq n_t\}$ such that $\sum_i m_i + \sum_j n_j = k$. The coefficients are polynomials in the parameters characterising the weight, i.e., $M_1, M_2$, and $\frac{1}{\nu}$, so (dropping the overall constant) one can continue analytically (6.17) to an infinite series in the spirit of Kent [18]. The first coefficient is (up to a sign) $c_0^{(i)} = 1$ (absorbing here for simplicity the normalisation $a_\nu$ in the current $W(z)$). Because of the similarity with the case in [18] we shall not carry out the analog of the full analysis performed there. Instead in Appendix D we show how the universal coefficients in this expansion can be computed, giving them explicitly up to $k = 2$, which is enough to write down the vector in example (6.16b) with $m_1 = 1$ in a standard, integer powers form.

Following the same steps, as in the example above, one obtains in general starting from (A.1), (A.3), respectively, the following expressions for the $W_3$ singular vectors corresponding to the roots (2.12a,b)

$$V_{w_{\beta}:\lambda} \simeq \mathcal{S}^{(1)}_{w_{\beta}:\lambda} V_{\lambda} \equiv N_1 O_{(w_{0201})^{n'_1-1},\lambda}^{(1)} \cdots O_{w_{10201}:\lambda}^{(1)} V_{\lambda}, \quad (6.18)$$

if $M_1 = m_1 - \frac{m'_1 - 1}{\nu}$, $m_1, m'_1 \in \mathcal{N}$, $\beta = (m'_1 - 1) \delta + \alpha^1$,

$$V_{w_{\beta}:\lambda} \simeq \mathcal{S}^{(2)}_{w_{\beta}:\lambda} V_{\lambda} \equiv N_2 O_{(w_{0102})^{m''_2-1},\lambda}^{(2)} \cdots O_{w_{0102}:\lambda}^{(2)} V_{\lambda}, \quad (6.19)$$

if $M_2 = m_2 - \frac{m''_2 - 1}{\nu}$, $m_2, m'_2 \in \mathcal{N}$, $\beta = (m'_2 - 1) \delta + \alpha^2$,

where $N_i$ are numerical constants (see Appendix C). Similarly in the case (2.12c) one obtains

$$V_{w_{\beta}:\lambda} \simeq \mathcal{S}^{(3)}_{w_{\beta}:\lambda} V_{\lambda} \equiv N_3 O_{(w_{2010121})^{n''_3-1},\lambda}^{(1)} O_{w_{1020121}:\lambda}^{(2)} \cdots O_{w_{1020121}:\lambda}^{(2)} O_{w_{102121}:\lambda}^{(2)} O_{\lambda}^{(1)} V_{\lambda}, \quad (6.20)$$

if $M_3 = m_3 - \frac{n}{\nu}$, $n = m'_3 - 2, m'_3 - 1, m_3 \in \mathcal{N}$, $\beta = (m'_3 - 2) \delta + \alpha^3$.

According to (6.1.3) these states (6.18-20) are annihilated by the positive modes of the $W_3$ algebra currents, while their $L_0$ level is $m_1, m_1, m'_2$, and $m_3, m'_3$, respectively, since $h_{w_{\beta_{n,+i}}}^{(2)} = h_{\lambda}^{(2)} + (\lambda + \rho, \beta_{n,+i}) (n + (\rho, \alpha^i))$. Formally also any subfactor of these expressions obtained by deleting from the left the first, the first two, etc., operators $O$, is a singular vector.
Since the representations (2.12) for the affine Weyl reflections are not unique, there are various equivalent ways of representing the Kac-Moody and hence the $\mathcal{W}_3$ singular vectors. For example, the vector in (6.20) for $n$-even can be also represented starting from (A.2) as

$$V_{w_3\cdot \lambda} \simeq S_{w_3\cdot \lambda} V_{\lambda} \equiv N''_{\lambda} O_{w_{(0121)}^{(2)}\cdot \lambda}^{(3)} \cdots O_{w_{(0121)}^{(1)}\cdot \lambda}^{(3)} V_{\lambda}, \quad (6.21a)$$

where

$$O_{w_{(0121)}^{(1)}\cdot \lambda}^{(3)} = O_{w_{21(0121)}^{(2)}\cdot \lambda}^{(1)} O^{(2)}_{w_{(0121)}^{(1)}\cdot \lambda} O^{(1)}_{w_{(0121)}^{(2)}\cdot \lambda}. \quad (6.21b)$$

Furthermore exploiting the symmetry (6.13) the same vectors (6.18-20) can be recovered starting from the MFF vectors corresponding to the Weyl reflections $w_\beta$, with $\beta$ as in (2.12d), (2.12e), or (2.12f), respectively, i.e., $\beta = (m_1 + 1)\delta - \alpha_1, \beta = (m_2 + 1)\delta - \alpha_2, \text{ or } \beta = (m_3 + 2)\delta - \alpha_3$; in the $\hat{sl}(3)$ modules $\Omega_{w_{020}\cdot \lambda}, \Omega_{w_{010}\cdot \lambda}$, or, $\Omega_{w_{01210}\cdot \lambda}$, respectively, since (cf. also (6.14b), (6.14a), or, (6.14e) respectively)

$$\langle w_{020} \cdot \lambda + \rho, \beta_{n+2,-1} \rangle = m_1 = \langle \lambda + \rho, \beta_{n+1,1} \rangle,$n \in \mathbb{N}, \text{ we have enumerated all } \mathcal{W}_3 \text{ singular vectors resulting from Kac-Moody ones. However the vectors (6.18-20) are not all independent. Indeed the symmetry (6.13) implies furthermore that for } M^1 = m - \frac{n}{2}, m, n + 1 \in \mathbb{N}, \text{ the three types of vectors (6.18), (6.19), (6.20), provide different realisations for one and the same (up to an overall constant) singular vector in the } \mathcal{W}_3 \text{ module, after identifying the Kac-Moody h.w. states with the } \mathcal{W}_3 \text{ one, more precisely, (cf. (6.14d) and (6.14a) and note that the symmetry group intertwines the 6 types of real positive roots in (2.12))}

$$S_{w_{\beta_{n+2,-2}} \cdot \lambda} V_{\lambda} \sim S_{w_{\beta_{n+2,-2}} \cdot \lambda} V_{w_{010} \cdot \lambda}, \quad (6.22b)$$

Apart from the case $\beta = 3\delta - \alpha^3, \frac{2}{\nu} - M^3 = m_3 \in \mathbb{N}$, we have enumerated all $\mathcal{W}_3$ singular vectors resulting from Kac-Moody ones. Indeed the symmetry (6.13) implies furthermore that for $M^1 = m - \frac{n}{2}, m, n + 1 \in \mathbb{N}$, the three types of vectors (6.18), (6.19), (6.20), provide different realisations for one and the same (up to an overall constant) singular vector in the $\mathcal{W}_3$ module, after identifying the Kac-Moody h.w. states with the $\mathcal{W}_3$ one, more precisely, (cf. (6.14d) and (6.14a) and note that the symmetry group intertwines the 6 types of real positive roots in (2.12))

$$S_{w_{\beta_{n+4,-3}} \cdot \lambda} V_{\lambda} \sim S_{w_{\beta_{n+4,-3}} \cdot \lambda} V_{w_{01210} \cdot \lambda}. \quad (6.22c)$$

(For $n = 0$ only the first relation in (6.23) holds and the two vectors reduce to the elementary singular vectors (6.8)). Similar relations hold for $M^2 = m - \frac{n}{2}$, if $m, n \in \mathbb{N}$, starting with the vector in (6.19), and for $M^3 = m - \frac{n}{2}$, if $m, n + 1 \in \mathbb{N}$, starting with the vector in (6.20). Finally for $\beta = 3\delta - \alpha^3$ and $\frac{2}{\nu} - M^3 = m_3 \in \mathbb{N}$ (the only case not covered by the above) the singular vectors are simply given by $O_{w_{0210} \cdot \lambda}^{(1)} V_{w_{0210} \cdot \lambda}.$

Combining (6.22) and (6.23) for $\lambda$ satisfying, say the condition in (6.18) ($n \in \mathbb{N}$), we can write

$$S_{w_\beta \cdot \lambda} V_{\lambda} \sim S_{w_\beta \cdot \lambda} V_{\lambda'}, \quad \lambda' = w \cdot \lambda, \quad \beta' = w(\beta), \quad w \in \overline{W}(\rho), \quad (6.24)$$
where we have used the notation $S_{w_{ij},-\varepsilon,\lambda} := S_{w_{ij},-\varepsilon,\lambda}^{(i)}$. This implies that (at least for generic weights, see below) the basic singular vectors of the type in (6.18) are enough to reproduce all $\mathcal{W}_3$ singular vectors. Let us finally remark that the relations (6.23) lead to relations between the basic operators $O_{\lambda}^{(i)}$ for $i = 1, 2$. The simplest one corresponds to the first relation in (6.23), i.e.,

$$O_{w_{0210}}^{(2)} = \frac{\Gamma((\lambda + \rho, 2\delta - \alpha^3))}{\Gamma((\lambda + \rho, 2\delta - \alpha^3 - \delta))} O_{\lambda}^{(1)}$$

and comparing the coefficients in the expansion (6.17) we check that indeed this relation is true up to the calculated order (see Appendix D). ²

Once we insert the explicit form (6.17) of the operators $O_{\lambda}^{(i)}$ into (6.18-20), we get the overall constants $N_i$, $i = 1, 2, 3$, given by the $N_i$, $i = 1, 2, 3$ of Eqs. (C.14) and (C.15), times the appropriate products of the factors appearing in the r.h.s. of (6.17). The result is (up to signs):

$$N_i = \frac{\Gamma((\lambda + \rho, \alpha^3 - \delta) - m_i)}{\Gamma((\lambda + \rho, \alpha^3 - \delta))}, \quad \text{for} \quad M^i = m_i - \frac{n_i}{\nu}, \quad n_i \equiv m_i' - 1, \quad i = 1, 2; \quad (6.25a)$$

$$N_3 = \frac{\Gamma((\lambda + \rho, \alpha^3 - \delta) - 2m_3)}{\Gamma((\lambda + \rho, \alpha^3 - \delta))}, \quad \text{for} \quad M^3 = m_3 - \frac{n_3}{\nu}, \quad n_3 \equiv m_3' - 2. \quad (6.25b)$$

In other words $N_j$ is the overall constant in (6.18-20) when the coefficient of the highest power of $W_{-1}$ is normalized to 1, e.g., in (6.18) we have $S_{w_{ij},-\varepsilon,\lambda}^{(1)} V_{\lambda} = N_i \left( W_{-1}^{m_i} + \ldots \right)$.

Similarly for $N'_{j}$, $j = 1, 2, 3$ – the overall constants of the singular vectors corresponding to the roots $n_j \delta - \alpha_j$ (to be used in the r.h. sides of (6.22a-c) properly transforming the weights), we have the same expressions as in (6.25) with $m_j$, $n_j$ replaced by $-m_j$, $-n_j$. For $M^i = -m_i + \frac{1}{\nu}, \quad i = 1, 2$, $N'_i$ reproduce the constants $A_i$ in the BRST relations (6.15).

**6.3. The Drinfeld-Sokolov reduction and the $\mathcal{W}_3$ pseudo - Verma modules.**

The description of the singular vectors in the $\mathcal{W}_3$ Verma modules which we obtained exploiting the quantum hamiltonian reduction is qualitatively in agreement with the formulae for the corresponding Kac determinant [12], [30], [31]. In particular the $\mathcal{W}_3$ singular vectors for $\lambda$ such that $M^3 = m_3 + \frac{1}{\nu}$, $m_3 \in \mathbb{N}$, – a condition which does not originate from a KK one – are recovered exploiting the symmetry (6.12) from the vectors

$$S_{w_{ij},-\varepsilon,\lambda}^{(1)} V_{w_{0210}}^{(i)} \sim S_{w_{ij},-\varepsilon,\lambda}^{(2)} V_{w_{020}}^{(2)}$$

(cf. (6.23)).

What remains to be understood however, is the exact meaning of the expressions for the vectors described. Indeed, up to now we have worked formally and we have neglected the possible singularities of the numerical constants (6.25) in the explicit expressions for the singular vectors. The rigorous treatment of these singularities requires further investigation – here we will only sketch the implications in some simple examples.

² It is not so straightforward in general to show explicitly this coincidence, thus the statement relies on the assumption that at a given weight (6.3) there is at most one singular vector in a $\mathcal{W}_3$ Verma module.
Let us start with the simplest case, when there is an overall constant $1/\epsilon \to \infty$ in the basic singular vector (6.8) (cf. (6.17)). Since the l.h.s. of (6.8) is finite, this singularity implies that the $W_3$ singular vector vanishes identically when we express the $W_3$ generators back in terms of the Kac-Moody ones – and simultaneously identify the vacuum state in the $W_3$ module with a Kac-Moody one. In particular a relative infinite constant might appear in the relations (6.22), (6.23), so that some of the representatives of a $W_3$ vector reproduce it directly, while some provide it only after taking the residue of the analytically continued expression. E.g., consider the example $M^1 = m \in \mathbb{N}$, $M^2 = \frac{1}{\nu}$, i.e., $M^3 = m + \frac{1}{\nu}$, and assume that $1/\nu$ is not an integer in the interval $[1, m]$. According to (6.17) the overall constant in the first vector in (6.23) is infinite while in the second it is finite. Thus this latter vector has to be used to represent the $W_3$ singular vector.

The case $M^1 = m = m' + \frac{1}{\nu}$, $M^2 = \frac{1}{\nu}$, where $1/\nu$ is an integer in the interval $[1, m]$, in this example is somewhat more exceptional, since neither of the first two vectors in (6.23) reproduces directly a well defined expression for the $W_3$ singular vector. Consider the simplest of these examples – $(M_1, M_2) = (\frac{1}{\nu}, \frac{1}{\nu})$. For the simple roots $\alpha^i, i = 1, 2$ we have two different singular vectors in the given KM module, which have the same eigenvalues $\{h^{(2)}_\lambda + m, 0\}$ of the zero modes $L_0, W_0$.

Let us for simplicity illustrate this degenerate case by the simplest value $m = 1$, i.e., by the example in (4.1) for $\frac{1}{\nu} = 1$. In both linear combinations in (4.1) the overall constant $M^3 = 1 - \frac{1}{\nu}$ in the l.h.s. vanishes along with the constants in front of $L_{-1}$, while the vector $W_{-1}V_\lambda$ is identically zero (cf. (3.7b)). Thus we can extract at most one finite nontrivial linear combination of the two vectors, namely $(f^3_0 + f^3_0)V_\lambda = L_{-1}V_\lambda$, while the state $(f^3_0 - f^3_0)V_\lambda$, although belonging to the kernel of $Q$, cannot be expressed in terms of $W_3$ generators. The only state selected $L_{-1}V_\lambda$, gives rise to a $W_3$ (factor) module of weight $\{h^{(2)}_\lambda', h^{(3)}_\lambda\}$, where accounting for the symmetry (6.13,14) $\lambda'$ can be taken to be $\overline{\lambda'} = (2, 2)$, Furthermore the composite singular vectors in (6.9) (there are three here since both $M^1$ and $M^2$ are integers – $V_{u_{12}, \lambda}, V_{w_{21}, \lambda}$ and $V_{w_{121}, \lambda}$) will be affected – namely they have to be built using the finite linear combination. (Note that they do not involve further singularities and in particular the weights of the first two vectors $V_{u_{12}, \lambda}$ and $V_{w_{21}, \lambda}$ lie on different orbits of the symmetry group.)

There is another alternative of extracting finite expressions from the two linear combinations in the r.h.s. of (4.1). Indeed, no more assuming the original identification (3.7b) of the $W_3$ generators with elements in the Kac-Moody enveloping algebra one can continue analytically in the parameter $\frac{1}{\nu} = 1 + \epsilon$. Then by taking proper limits one can get from the initial singular linear combinations a pair of states,

$$
\lim_{\epsilon \to 0} \frac{\epsilon}{\epsilon} S^{(1)}_\epsilon = W_{-1},
$$

$$
\lim_{\epsilon \to 0} \frac{\partial}{\partial \epsilon} (\epsilon S^{(1)}_\epsilon) = \frac{1}{2} L_{-1},
$$

(6.26)

where

$$S^{(i)}_\epsilon = \frac{1}{\epsilon} (\delta_{i1} - \delta_{i2})W_{-1} + \frac{1}{2} L_{-1}, \quad \frac{1}{\nu} = 1 + \epsilon, \quad M^1 = M^2 = 1,$$

neglecting the terms of order $\epsilon$ or higher. (There is an arbitrariness in the second state, namely one can add $b W_{-1}$ with an arbitrary constant $b$; we have dropped such a finite term in $S^{(1)}_\epsilon$.) The positive mode $W_3$ generators annihilate both states of the pair, while (6.2) implies that $W_0 S^{(i)}_\epsilon V_\lambda = \epsilon (\delta_{i1} - \delta_{i2})S^{(i)}_\epsilon V_\lambda$, up to higher orders in $\epsilon$, and hence $W_0$ acts inhomogeneously on the pair $\{W_{-1}V_\lambda, L_{-1}V_\lambda\}$, i.e., the would be two singular vectors combine and provide an indecomposable representation of the subalgebra $\{L_0, W_0\}$. Thus the pair $\{W_{-1}V_\lambda, L_{-1}V_\lambda\}$, provides the vacuum state of a pseudo-Verma module of $W_3$ (see [12], [32]). Imposing the invariant
condition $W_{-1} \equiv 0$ one recovers the factor module built on the state $L_{-1} V_{\lambda}$. (The first state in (6.26) can be alternatively recovered directly by the representative in the r.h.s. of (6.22a), since the corresponding overall constant $N^i_1$, determined after (6.25b), is finite and nonzero.)

One can expect that what happens in this example is rather general. There may be more than one nontrivial singular vectors in a given Kac-Moody module, such that their weights $\lambda$ lie on one and the same orbit of the invariance group, and such that both reduced expressions have overall singularities. More generally, identifying the KM highest weight states lying on an orbit of the stability group, the formal doubling of the singular vectors in the resulting overall singularities. More generally, identifying the KM highest weight states lying on an orbit of the invariance group, and such that both reduced expressions have

reduction leading to a

W

a formalism which is a step towards an understanding of the Verma module resolutions of the

Verma modules singular vectors.

The first important ingredient used is the stability group $W^{(\eta)}$ of the eigenvalues of the zero modes subalgebras, i.e., of the weights of the $W$ algebra. This is a finite subgroup of the affine Weyl group, specific for the given $sl(2)$ embedding. The $W$ algebra singular vectors are governed by $W$ modulo $W^{(\eta)}$, much in the same way as the affine Weyl group classifies the Kac-Moody
singular vectors and gives their embedding pattern. This is demonstrated rather constructively in our procedure. Indeed the initial (powers of) root vectors \( \{f_0^i, i = 1, 2, \ldots , r; e_{-1}\} \), (\( \theta \)-the highest root) in the general Malikov-Feigin-Fuks monomial, corresponding to the generators of the affine Weyl group \( \{w_\alpha, i = 1, 2, \ldots , r; w_{\delta - \theta}\} \) are regrouped to arrive at a subset of \( \{f_0^i, i = 1, 2, \ldots , r\} \) and \( \{e_{-1}^i, \alpha \in I(\eta)\} \), the latter corresponding to the generators \( \{w_{\delta - \alpha}, \alpha \in I(\eta)\} \) of the symmetry group. Here \( I(\eta) \) is the set of roots of \( g \) for which \( e^\alpha = 1 \) (classically). This suggests the form of the stability group in general, as it is confirmed, e.g., by the case of the principal embedding for \( g = sl(n) \) (see also [33] for the case of finite \( W \) algebras).

Secondly, we have exploited a quantum gauge transformation, playing a role quite analogous to the classical Drinfeld-Sokolov gauge fixing transformations. Because we have used the Malikov-Feigin-Fuks description of Kac-Moody singular vectors it was sufficient to have the projections \( R(i) \) of the gauge transformations along the directions of the simple roots \( \alpha^i, i = 1, \ldots , r \). The classical analogs might give a clue for finding these quantum operators in general. What seems however still not quite satisfactory is the rather technically involved derivation of the intertwining relation in the principal embedding case (see Appendix B), which we are not able at present to simplify considerably. This reflects the intrinsic asymmetry of our approach – as compared with the fusion method initiated in [24], in treating the pairs of labels \( (m_i, m_i') \) in the Kac-Kazhdan conditions. A more ambitious problem would be to find the general quantum gauge transformation, a multi series involving also the negative modes of the raising currents \( e^{\alpha_i}, i = 1, 2 \), which we expect to reproduce the \( \mathcal{W} \) algebra singular vectors directly. This is an open problem even for the case \( g = sl(2) \). A related problem is the adaptation in the standard BRST framework and the generalisation of the algorithm in [19] for constructing the general singular vectors, which starts from a simplified expression, BRST equivalent to the initial KM singular vector, and in particular makes unnecessary the use of Kent – type expressions.

While in the example of \( \mathcal{W}_3 \) there are complications due to the appearance of indecomposable pairs of singular vectors at some values of the parameters \( M^1, M^2, \) and \( \nu \), in the “less constrained” case of the Polyakov-Bershadsky algebra there is a striking similarity of the expressions for the singular vectors with those for the initial affine algebra. In this case the stabilising group is \( \{1, w_0 = w_{\delta - \theta}\} \) and the embedding pattern and hence the characters are expected to follow in a straightforward manner from the corresponding information about the Kac-Moody algebra.

There are other possible applications of the results found here. In the case of \( \hat{\mathfrak{sl}}(2) \) we have a full understanding also of the reduction of the general correlation functions and the reduction of the Knizhnik-Zamolodchikov equation (combined with the algebraic equations resulting from the Malikov-Feigin-Fuks vectors) to the null-decoupling equations of Belavin-Polyakov-Zamolodchikov [14], [19]. It would be interesting to generalise these results for the \( \mathcal{W} \) algebras related to \( \hat{\mathfrak{sl}}(3) \). In particular it is expected that differential equations originating from the Knizhnik-Zamolodchikov equations can be found accounting also for the \( \mathcal{W} \)- algebra singular vectors. The simplest differential equations corresponding to the fundamental representations of \( sl(3) \) were considered in [34], [35].

**Appendix A. Malikov-Feigin-Fuks vectors of \( A_2^{(1)} \).**

In this appendix we describe in more details the Malikov-Feigin-Fuks vectors for \( A_2^{(1)} \). As usual we parametrize a highest weight \( \lambda \) by \( M^1 = \lambda + \rho, \alpha^i \), \( i = 1, 2 \). For the six types of positive roots \( \beta \) of \( A_2^{(1)} \) using the decomposition (2.12) of \( w_\beta \) into simple reflections one gets from (2.11) the following expressions.

For \( M^1 = m - \frac{m_0}{\rho}, m, n \in \mathbb{N}, M^2 \) arbitrary and \( \beta = (n-1) \delta + \alpha^1 \) one has \( w_\beta = w_1(w_{020})^{n-1} \)
and the Malikov-Feigin-Fuks vector is

\[ V_{w, \beta} = f_1^{A_{n-1}} f_1^{D_{n-2}} f_2^{C_{n-2}} f_1^{B_{n-2}} f_1^{A_{n-1}} \ldots f_0^{D_0} f_2^{C_0} f_0^{B_0} f_1^{A_0} V_\lambda, \]  

(A.1)

where

\[ A_p \equiv \langle w_{(0201)}^p \cdot \lambda + \rho, \alpha^1 \rangle = M^1 + \frac{2p}{\nu}, \]

\[ B_p \equiv \langle w_{(0201)}^p \cdot \lambda + \rho, \alpha^0 \rangle = -M^2 + \frac{p + 1}{\nu}, \]

\[ C_p \equiv \langle w_{01(0201)}^p \cdot \lambda + \rho, \alpha^2 \rangle = M^1 + \frac{2p + 1}{\nu}, \]

\[ D_p \equiv \langle w_{21(0201)}^p \cdot \lambda + \rho, \alpha^0 \rangle = M^1 + M^2 + \frac{p}{\nu}. \]

For \( M^1 \) arbitrary, \( M^2 = m - \frac{n - 1}{\nu}, m, n \in \mathbb{N} \), and \( \beta = (n - 1) \delta + \alpha^2 \) one has \( w_\beta = w_2(w_{0102})^{n-1} \) and the Malikov-Feigin-Fuks vector has the same form as the one above except that “1” and “2” are interchanged.

For \( M^1 + M^2 = m - \frac{n - 1}{\nu}, m, n \in \mathbb{N} \) and \( \beta = (n - 1) \delta + \alpha^1 + \alpha^2 \) one has \( w_\beta = w_{121}(w_{0121})^{n-1} \) and the Malikov-Feigin-Fuks vector is

\[ V_{w, \beta} = f_1^{C_{n-1}} f_2^{B_{n-1}} f_1^{A_{n-1}} f_1^{D_{n-2}} f_2^{C_{n-2}} f_1^{B_{n-2}} f_1^{A_{n-2}} \ldots f_0^{D_0} f_1^{C_0} f_2^{B_0} f_1^{A_0} V_\lambda, \]  

(A.2)

where

\[ A_p \equiv \langle w_{(0121)}^p \cdot \lambda + \rho, \alpha^1 \rangle = M^1 + \frac{p}{\nu}, \]

\[ B_p \equiv \langle w_{(0121)}^p \cdot \lambda + \rho, \alpha^2 \rangle = M^1 + M^2 + \frac{2p}{\nu}, \]

\[ C_p \equiv \langle w_{21(0121)}^p \cdot \lambda + \rho, \alpha^1 \rangle = M^2 + \frac{p}{\nu}, \]

\[ D_p \equiv \langle w_{121(0121)}^p \cdot \lambda + \rho, \alpha^0 \rangle = M^1 + M^2 + \frac{2p + 1}{\nu}. \]

For \( M^1 = -m + \frac{n}{\nu}, m, n \in \mathbb{N}, M^2 \) arbitrary and \( \beta = n \delta - \alpha^1 \) one has \( w_\beta = w_{020}(w_{1020})^{n-1} \) and the Malikov-Feigin-Fuks vector is

\[ V_{w, \beta} = f_0^{C_{n-1}} f_2^{B_{n-1}} f_0^{A_{n-1}} f_1^{D_{n-2}} f_2^{C_{n-2}} f_1^{B_{n-2}} f_1^{A_{n-2}} \ldots f_0^{D_0} f_0^{C_0} f_0^{B_0} f_1^{A_0} V_\lambda, \]  

(A.3)

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where

\[ A_p \equiv \langle w_{(1020)^p} \cdot \lambda + \rho, \alpha^0 \rangle = -M^1 - M^2 + \frac{p + 1}{\nu}, \]

\[ B_p \equiv \langle w_{0(1020)^p} \cdot \lambda + \rho, \alpha^2 \rangle = -M^1 + \frac{2p + 1}{\nu}, \]

\[ C_p \equiv \langle w_{20(1020)^p} \cdot \lambda + \rho, \alpha^0 \rangle = M^2 + \frac{p}{\nu}, \]

\[ D_p \equiv \langle w_{020(1020)^p} \cdot \lambda + \rho, \alpha^1 \rangle = -M^1 + \frac{2p + 2}{\nu}. \]

For \( M^1 \) arbitrary, \( M^2 = -m + \frac{n}{\nu}, m, n \in \mathbb{N} \), and \( \beta = n \delta - \alpha^2 \) one has \( w_\beta = w_{010}(w_{210})^{n-1} \) and the Malikov-Feigin-Fuks vector has the same form as the one above except that “1” and “2” are interchanged.

For \( M^1 + M^2 = -m + \frac{n}{\nu}, m, n \in \mathbb{N} \) and \( \beta = n \delta - \alpha^1 - \alpha^2 \) one has \( w_\beta = w_{0}(w_{1210})^{n-1} \) and the Malikov-Feigin-Fuks vector is

\[ V_{w_\beta \cdot \lambda} = f_0^{A_{n-1}} f_1^{D_{n-2}} f_2^{C_{n-2}} f_1^{B_{n-2}} f_0^{A_{n-2}} \ldots f_1^{D_0} f_2^{C_0} f_1^{B_0} f_0^{A_0} V_\lambda, \quad (A.4) \]

where

\[ A_p \equiv \langle w_{(1210)^p} \cdot \lambda + \rho, \alpha^0 \rangle = -M^1 - M^2 + \frac{2p + 1}{\nu}, \]

\[ B_p \equiv \langle w_{0(1210)^p} \cdot \lambda + \rho, \alpha^1 \rangle = -M^1 + \frac{p + 1}{\nu}, \]

\[ C_p \equiv \langle w_{10(1210)^p} \cdot \lambda + \rho, \alpha^2 \rangle = -M^1 - M^2 + \frac{2p + 2}{\nu}, \]

\[ D_p \equiv \langle w_{210(1210)^p} \cdot \lambda + \rho, \alpha^1 \rangle = -M^1 + \frac{p + 1}{\nu}. \]

Now we describe an algorithm that transforms a Malikov-Feigin-Fuks monomial, in which the generators are raised in general to complex powers and thus a priori is not an element of the universal enveloping algebra, into an ordinary polynomial of elements of \( n_+ \) and thus an element of the universal enveloping algebra. First note that all the above monomials have the form

\[ \ldots f_{i_2}^{\gamma_2} f_{i_1}^{\gamma_1} f_{i_0}^{\gamma_0} f_{i_1}^{\gamma_1-1} f_{i_2}^{\gamma_2-1} \ldots \quad (A.5) \]

where \( f_{i_0}^{\gamma_0} \) is the middle term (a Weyl reflection \( w_\beta \) always decomposes into an odd number of simple reflections) and moreover \( \gamma_0, \gamma_i + \gamma_{-i}, \) are positive integers. Next note that it is straightforward to continue analitically in \( \gamma \) commutation formulas of the sort

\[ [f_0^\gamma, f_0^{\lambda}] = -\gamma e_{-1}^2 f_0^{\gamma-1} \quad (A.6) \]

or more generally

\[ f_0^\gamma (f_0^1)^n = \sum_{k=0}^n (-1)^k \binom{n}{k} f_0^{\gamma^1 (f_0^1)^{n-k} (e_{-1}^2)^k f_0^{\gamma-k}}. \quad (A.7) \]
Thus the algorithm is, using the above and their analogs, start from the middle and move outwards. For example take (A.1) with \( n - 1 = 2r \) even. In this case \( i_0 = 1, i_1 = 0, i_2 = 2, i_3 = 0, i_4 = 1, \) etc., and \( \gamma_0 = A_r = m, \gamma_1 = B_r = -M^2 + \frac{r+1}{\nu}, \gamma_{-1} = D_r = m + M^2 - \frac{r+1}{\nu}, \) etc. Thus using (A.7) we transform the three middle terms of (A.5) into an ordinary polynomial. Using the analogs of (A.7) we can continue moving outwards.

Appendix B. Derivation of the basic relation (4.5)

In this appendix we will derive the key relation (4.5).

On both sides of (4.5) we have a vector \( V \) such that

\[
e_n^i V = f_n^i V = h_n^i V = b_n^i V = c_n^i V = c_0^i V = 0 \quad \forall n \geq 1, i = 1, 2, 3, \tag{B.1}
\]

and thus \( L_n V = L_n^{(\text{ff})} V = 0 \) for \( n \geq 1 \) and \( e_n^i V = e_1^i V, f_n^i V = f_1^i V, h_n^i V = h_1^i V. \)

To be more concrete we will consider the intertwining property of the gauge transformation projected on the “1st” direction, i.e., \( \mathcal{R}^{(i)} \) with \( i = 1. \) For notational simplicity set

\[
H(z) = C_1^i \hat{h}^i(z). \tag{B.2}
\]

and skip the superscript of \( \mathcal{R} \), i.e., our gauge transformation (4.2) is

\[
\mathcal{R}(u) = \exp(\int H(-)(-u) \, du) = \sum_{n=0}^{\infty} \mathcal{R}_{-n} u^n. \tag{B.3}
\]

Though we should set \( u = e_0^1 \) for the moment let us view it as a scalar parameter. Taking derivatives in \( u \) we immediately obtain

\[
n \mathcal{R}_{-n} = \sum_{k=0}^{n-1} (-1)^{n-k+1} \mathcal{R}_{-k} H_{-n+k},
\]

\[
n(n-1) \mathcal{R}_{-n} = \sum_{k=0}^{n-2} (-1)^{n-k+1} \mathcal{R}_{-k} (\partial H - H \circ H)_{-n+k}, \tag{B.4}
\]

\[
n(n-1)(n-2) \mathcal{R}_{-n} = \sum_{k=0}^{n-3} (-1)^{n-k+1} \mathcal{R}_{-k} \left( (H^3) - \frac{3}{2} (\partial H^2) + \partial^2 H \right)_{-n+k},
\]

where for short \( \circ \) indicates the corresponding normal product in which only negative mode generators are retained.

In fact the modes \( \mathcal{R}_{-n} \) of the gauge transformation can be defined recursively by \( \mathcal{R}_0 = 1 \) and the first of the equalities (B.4). Then the other equalities are obtained by induction.

First we will give some motivation for what we will be doing below. In the \( sl(2) \) case the derivation of the corresponding key relation can be divided roughly into two steps: 1) one commutes the powers of the gauge generator \( (e_0^1)^n \) through the zero mode of the lowering operator \( f_0 \) producing factors \( n \) and \( n(n-1) \) which by (B.4) are transformed into first and second powers of the Heisenberg field and thus one recovers the free field (ff) part the stress energy tensor; 2) commuting the modes \( \mathcal{R}_{-n} \) through \( f_0 \) one obtains lower modes of \( f \) which together with the
(ff) piece give the reduced energy momentum tensor. In the \( sl(3) \) case we will follow analogous steps. Since in the example we are considering we want to move along the “1st simple direction” it is natural to take a combination of the reduced generators \( W \) and \( T \) (2.7) which eliminates \( f^2 \).

Denote
\[
\mathcal{X}(z) = \left( \frac{1}{\nu} (H T) + \frac{1}{2\nu} (\frac{1}{\nu} - 1) \partial T + a_\nu W \right)(z). \tag{B.5}
\]

From (3.6) and (3.7) we have
\[
\mathcal{X}(z) - \mathcal{X}^{(\nu)}(z) = \left( (h^3 f^1) + (\frac{1}{\nu} - 1) \partial f^1 + f^3 \right)(z). \tag{B.6}
\]

with
\[
\mathcal{X}^{(\nu)}(z) = \left( (H (H^2)) + \frac{3}{2} \left( \frac{1}{\nu} - 1 \right) (\partial H^2) + \left( \frac{1}{\nu} - 1 \right)^2 \partial^2 H \right)(z). \tag{B.7}
\]

The fact that \( \mathcal{X}^{(\nu)}(z) \) depends only on the Heisenberg field \( H(z) \) is a good sign and we can hope to recover it from derivatives of (B.3). Taking \( \oint dz z \) of the r.h.s. of (B.6) one obtains \( (h^3 f^1)_0 + (\frac{1}{\nu} - 1)(\partial f^1)_0 + (f^3)_1 \) which applied to a vector \( V \) with properties (B.1) gives
\[
(h^3 + 2 - \frac{1}{\nu}) f^1_0 V. \tag{B.8}
\]

This will be our starting point, namely we apply \( \mathcal{R} \) to (B.8) and move it to the right. There is another motivation for starting with (B.8) (i.e., not replacing \( h^3_0 \) with the corresponding numerical eigenvalue) – in the \( sl(3) \) case the (ff) part of the reduced generators is third order in the Heisenberg field so extending the \( sl(2) \) case we will need factors \( n, n(n-1) \) and \( n(n-1)(n-2) \) which could be obtained from the commutation of \((e^1_0)^n\) through the quadratic (in the KM generators) expression of (B.8).

Now we start with step 1) of the calculation. One immediately gets
\[
[(e^1_0)^n, (h^3_0 + 2 - \frac{1}{\nu}) f^1_0] = -n f^1_0 (e^1_0)^n \tag{B.9}
\]

\[= - \left\{ -n(n-1)(n-2) + (3H_0 - \frac{1}{\nu}) n(n-1) - (\hat{h}^3_0 + 1 - \frac{1}{\nu}) h^1_0 n \right\} (e^1_0)^{n-1}. \]

First consider the expression in curly brackets in the r.h.s. of (B.9). The terms with \( n, n(n-1) \) and \( n(n-1)(n-2) \) effectively give first, second and third derivative of \( \mathcal{R} \), i.e., formulae (B.4). Moreover when applied on \( V \) with the property (B.1) we have
\[
(H^3) V = \left( (H^3) + 3H_0(H^2) + 3(H_0)^2 H \right) V, \tag{B.10}
\]
\[
(\partial H^2) V = \left( (\partial H^2) + 2H_0 \partial H - 2H_0 H \right) V, \tag{B.10}
\]
\[
(H^2) V = \left( (H^2) + 2H_0 H \right) V, \tag{B.10}
\]

and after some algebra we obtain
\[
\mathcal{R}_{-n} [(e^1_0)^n, (h^3_0 + 2 - \frac{1}{\nu}) f^1_0] V = -n \mathcal{R}_{-n} f^1_0 (e^1_0)^n V + \sum_k (-1)^{n-k-1} \mathcal{R}_{-k} \mathcal{Y}_{-n+k} (e^1_0)^{n-1}, V \tag{B.11}
\]
\[\mathcal{Y}_{-n} = \left( (H^3) - \frac{3}{2} (\partial H^2) + \partial^2 H \right)_{-n}\]  

\[-3(H_0)^2 H_{-n} - \frac{1}{\nu} (H)_{-n} + \frac{1}{\nu} \partial H_{-n} - \frac{1}{\nu} h_{0} H_{-n} + \hat{h}_{0}^{3} (\hat{h}_{0}^{1} + \frac{1}{\nu} - 1) H_{-n} .\]  

Until now we have moved only \( u = e_{0}^{1} \) to the right while the modes \( R_{-n} \) are still to the left of the Heisenberg field. To move also the modes we will need for mulae of the type

\[\sum_{n=0}^{\infty} R_{-n} \sum_{p=1}^{\infty} (H^2)_{-p} u^{n+p-1} = \sum_{p=1}^{\infty} \left( (H^2) + 2 \frac{C_{11}}{\nu} \partial H \right)_{-p} u^{p-1} R ,\]

\[\sum_{n=0}^{\infty} R_{-n} \sum_{p=1}^{\infty} (\partial H^2)_{-p} u^{n+p-1} = \sum_{p=1}^{\infty} \left( (\partial H^2) - 2 \frac{C_{11}}{\nu} \partial H + \frac{C_{11}}{\nu} \partial^2 H \right)_{-p} u^{p-1} R ,\]

\[\sum_{n=0}^{\infty} R_{-n} \sum_{p=1}^{\infty} (H^3)_{-p} u^{n+p-1} = \sum_{p=1}^{\infty} \left( (H^3) + 3 \frac{C_{11}}{\nu} (\partial H^2) + 3 \frac{C_{11}}{\nu} (H^2) \right)_{-p} u^{p-1} R ,\]

which are proved by induction using the recursive definition of \( R \) and

\[[R_{-n}, H_{m}] = \frac{C_{11}}{\nu} \sum_{k=0}^{n-1} (-1)^{n-k} \delta_{k,n-m} R_{-k} .\]

while the last formula is derived by induction from \([H_{m}, H_{n}] = \frac{C_{11}}{\nu} m \delta_{m,-n} .\]

Before proceeding we prepare another formula – (B.17). First from the definition of \( L^{(ff)} \)

\[[L_{n}^{(ff)}, H_{-m}] = m H_{n-m} - \left( \frac{1}{\nu} - 1 \right) m(m + 1) \delta_{n,m} \]

from where, again by induction, follows

\[[L_{m}^{(ff)}, R_{-n}] = \sum_{k=0}^{n-1} (-1)^{n-k+1} R_{-k} H_{m-n+k} \]

\[+ \sum_{k=0}^{n-1} \left( (m - 1) \frac{C_{12}}{\nu} + (m + 1) \left( \frac{1}{\nu} - 1 \right) \right) (-1)^{n-k} \delta_{k,n-m} R_{-k} .\]
Exploiting the above and the fact that positive modes of \( L^{(ff)} \) annihilate \( V \), see (B.1), we obtain

\[
- \sum_{n=1}^{\infty} \left( \sum_{k=0}^{\infty} H_{-n-k} L^{(ff)}_k \right) u^{n-1} \mathcal{R} \, V = \mathcal{R} \sum_{n=1}^{\infty} \left( \frac{1}{\nu} (H^2) + \frac{1}{2\nu} (\partial H^2) + \frac{1}{\nu} \left( \frac{1}{2} - \frac{C_{11}}{\nu} \right) \partial^2 H \right)
\]

\[
+ \frac{2}{\nu} \left( 1 - \frac{1}{\nu} \right) \partial H + \hat{h}_0^3 \left( \hat{h}_0^1 + \frac{1}{\nu} - 1 \right) H - \frac{1}{\nu} H_0 H - 3(H_0)^2 H \right)_{-n} u^{n-1} V .
\]  

(B.17)

With the help of (B.13) and (B.17) we can move the modes \( \mathcal{R}_{-n} \) to the right of the Heisenberg fields in the second term in the r.h.s. of (B.11)

\[
\sum_{n=0}^{\infty} \mathcal{R}_{-n} \sum_{p=0}^{\infty} \mathcal{Y}_{-p} (-e_0^1)^{p-1}(e_0^1)^n V = \sum_{p=0}^{\infty} \mathcal{Z}_{-p} \mathcal{R} (-e_0^1)^{p-1} V \quad (B.18)
\]

where

\[
\mathcal{Z}_{-n} = \left( (H^3) + \frac{3}{2} \left( \frac{1}{\nu} - 1 \right) (\partial H^2) + \left( \frac{1}{\nu} - 1 \right)^2 \partial^2 H \right)_{-n}
\]

\[
+ \frac{1}{\nu} \left( \frac{1}{2} \partial^2 H + \partial H \right)_{-n} - \frac{1}{\nu} \sum_{k=0}^{\infty} H_{-n-k} L^{(ff)}_k .
\]  

(B.19)

Let us recapitulate the result till now – we have

\[
\mathcal{R} \left( \hat{h}_0^3 + 2 - \frac{1}{\nu} \right) f_0^1 V = \sum_{n=0}^{\infty} \mathcal{R}_{-n} \left( \hat{h}_0^3 + 2 - \frac{1}{\nu} - n \right) f_0^1 (e_0^1)^n V + \sum_{p=0}^{\infty} \mathcal{Z}_{-n} \mathcal{R} (-e_0^1)^{p-1} V . \quad (B.20)
\]

The first term in (B.19) is exactly \( X^{(ff)} \) but note that in the expressions \( X(z) \) and \( X^{(ff)}(z) \) we have the products \( (HT)(z) \) and \( (HT^{(ff)})(z) \) which, when \( \mathcal{R} \) is moved to the right, should not survive but instead should be expressible in terms of \( T \) and \( T^{(ff)} \). Thus let us rewrite \( (HT^{(ff)})(z) \)

\[
(HL^{(ff)})_{-t} = \sum_{k=0}^{t-1} L^{(ff)}_{-t+k} H_{-k} - \left( \frac{1}{2} \partial^2 H + \partial H \right)_{-t} + \sum_{k=0}^{\infty} H_{-t-k} L^{(ff)}_k + \sum_{k=0}^{\infty} L^{(ff)}_{-t-k} H_k .
\]  

(B.21)

With the help of (B.14) we obtain

\[
\sum_{p=0}^{\infty} \left( \sum_{k=1}^{\infty} L^{(ff)}_{-p-k} H_k \right) \mathcal{R} u^{p-1} V = - \sum_{p=0}^{\infty} \frac{C_{11}}{\nu} \left( L^{(ff)} + \partial L^{(ff)} \right)_{-p} \mathcal{R} u^{p-1} V
\]

and substituting all into \( Z \) we find

\[
\sum_{n=0}^{\infty} \mathcal{Z}_{-n} \mathcal{R} (-e_0^1)^{n-1} V = \sum_{n=0}^{\infty} \left( a_\nu \mathcal{W}_{-n}^{(ff)} - \frac{C_{11}}{\nu^2} L_{-n}^{(ff)} - \frac{1}{2} \left( 1 + \frac{1}{\nu} (2C_{11} - 1) \right) (\partial L^{(ff)})_{-n}
\]

\[
+ \sum_{k=0}^{n-1} L^{(ff)}_{-n-k} H_{-k} \right) \mathcal{R} (-e_0^1)^{n-1} V .
\]  

(B.23)

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The last term of the above, using \([(-e_0^1)^p, H_0] = -p(-e_0^1)^{p-1}\), can be rewritten as

\[
\sum_{n=0}^{\infty} \left( \sum_{k=0}^{n-1} L_{-n+k}^{(ff)} H_{-k} \right) \mathcal{R} (-e_0^1)^{n-1} V = \sum_{n=0}^{\infty} L_{-n}^{(ff)} \mathcal{R} H_0 (-e_0^1)^{n-1} V.
\]  

(B.24)

Recapitulating once again the sum with \(Z\) in (B.20) exactly recovers the free field part of the key intertwining formula (4.5) and thus we have completed step 1) of our calculation.

Step 2) of the calculation basically involves moving the modes \(\mathcal{R}_{-n}\) through \(f^1_0\) in the r.h.s. of (B.20) and for this purpose we need

\[
\mathcal{R}_{-n} f^1_0 = \sum_{k=0}^{n} (-1)^{n+k} f^1_{-n+k} \mathcal{R}_{-k}
\]  

(B.25)

which follows from the Kac-Moody commutation relations and induction. Using (B.24), (B.4a) and several resumptions for the first term in the r.h.s. of (B.20) we obtain

\[
\sum_{n=0}^{\infty} \mathcal{R}_{-n} \left( \hat{h}_0^3 + 2 - \frac{1}{\nu} - n \right) f^1_0 u^n V = \sum_{n=0}^{\infty} \left( \left( \hat{h}_0^3 + 1 - \frac{1}{\nu} \right) \hat{f}_{-n}^1 - (\partial f^1)_{-n} + (\hat{f}^1 \circ H)_{-n} \right) \mathcal{R} (-u)^n V
\]  

(B.26)

Equation (B.24) holds also for \(T\), i.e.,

\[
\sum_{n=0}^{\infty} \left( (L - L^{(ff)}) \circ H \right)_{-n} \mathcal{R} u^{n-1} V = \sum_{n=0}^{\infty} \left( (L_{-n} - L^{(ff)}_{-n}) \mathcal{R} H_0 u^{n-1} V
\]  

(B.27)

so we rewrite \(\nu(f^1 \circ H)_{-t} = (T \circ H)_{-t-1} - (T^{(ff)} \circ H)_{-t-1} - \nu(f^2 \circ H)_{-t}\) and use

\[
\sum_{t=1}^{\infty} (\hat{f}^2 \circ H)_{-t} \mathcal{R} u^t V = \sum_{t=1}^{\infty} \left( (H \hat{f}^2)_{-t} - \hat{f}^2_{-t} H_0 - \frac{C_{11}}{\nu}(\hat{f}^2 + \partial \hat{f}^2)_{-t} \right) \mathcal{R} u^t V.
\]  

(B.28)

Next note that

\[
0 = \mathcal{R} \hat{f}_1^3 V = \sum_{n=0}^{\infty} \hat{f}_{-n+1}^3 \mathcal{R} u^n V,
\]

\[
0 = \mathcal{R} \sum_{n=1}^{\infty} C_{2j}(\hat{h}_n^j \hat{f}_n^1 + \hat{f}_n^1 \hat{h}_n^j) V
\]

\[
= \sum_{t=0}^{\infty} \left( \sum_{n=1}^{\infty} C_{2j}(\hat{h}_n^j \hat{f}_{t+n}^1 + \hat{f}_{n}^1 \hat{h}_{-t+n}^j) + \frac{C_{12}}{\nu}(\partial \hat{f}^1 + \hat{f}^1)_{-t} \right) \mathcal{R} (-u)^t V
\]

because by the property (B.1) any positive mode generator annihilates the vector \(V\).

Putting all this together we get for (B.26):

\[
\sum_{n=0}^{\infty} \mathcal{R}_{-n} \left( \hat{h}_0^3 + 2 - \frac{1}{\nu} - n \right) f^1_0 u^n V =
\]

\[
= \sum_{t=0}^{\infty} \left( \hat{f}_{t+2}^3 + (\hat{h}_0^3 + 1) \hat{f}_{t+1}^1 - (1 - \frac{1}{\nu})(\partial \hat{f}^1)_{-t+1} - \frac{C_{11}}{\nu}(\hat{f}^1 + \hat{f}^2 + \partial \hat{f}^1 + \partial \hat{f}^2)_{-t+1} \right)
\]

\[
+ (C_{2j}(\hat{h}^j \hat{f}^1)_{-t+1} - C_{1j}(\hat{h}^j \hat{f}^2))_{-t+1} - C_{2j} \hat{f}_{-t+1}^1 \hat{h}_{t}^j + C_{1j} \hat{f}_{-t+1}^2 \hat{h}_{t}^j) \mathcal{R}(-u)^t V.
\]

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With this we have completed step 2) and thus the derivation of (4.5).

Appendix C. Derivation of the cohomology equivalence (6.15)

In this appendix we will demonstrate how in a Malikov-Feigin-Fuks monomial the factors corresponding to $w_{020} = w_{k_2-\alpha_1}^{\frac{1}{2}}$ and $w_{010} = w_{k_2-\alpha_2}^{\frac{1}{2}}$ are cohomologically equivalent to powers of the corresponding root vectors, i.e., $\hat{e}_{1-1}^{\frac{1}{2}}$ and $\hat{e}_{2-1}^{\frac{1}{2}}$, respectively, and hence, accounting for the constraints, these factors can be set equal to a constant modulo $Q$-exact terms.

For the generators of $\mathfrak{n}_-$ we use the notation $f_i$, $i$ runs over the simple roots of the affine algebra. In particular $f_0 = e_{3-1}$ and $f_2 = f_2^0$. From the commutation relations we have

$$f_0^1 + A f_2 f_0^{-A} = f_0 f_2 + A e_{-1}^{1}. \quad (C.1)$$

Using the quantum constraint $f_0 = \{Q, b_0^3\}$ we get

$$f_0 f_2 = b_0^3 \left[ (c^2 \tilde{h}_{2})_{-1} + (c^3 e_{1})_{-1} \right] + \{Q, b_0^3 f_2\}. \quad (C.2)$$

As a first example consider (C.1) applied to a singular vector $V_\mu$ such that $\langle \alpha^2, \mu \rangle = A$, i.e.,

$$b_0^2 V_\mu = A V_\mu. \quad (C.3)$$

Then

$$f_0^1 + A f_2 f_0^{-A} V_\mu \simeq A \left( e_{-1}^{1} + (b_0^3 c_{-1}^2) \right) V_\mu = A e_{-1}^{1} V_\mu \quad (C.4)$$

and substituting the constraint $e_{-1}^{1} = 1 + \{Q, b_0^3\}$ we get

$$f_0^1 + A f_2 f_0^{-A} V_\mu \simeq A V_\mu \quad (C.5)$$

where $\simeq$ denotes equality modulo $Q$ exact terms.

As a second example consider $f_0^{2+1} f_2^2 f_0^{-A}$ applied on a singular vector $V_\mu$ such that

$$\langle \alpha^2, \mu \rangle = A + 1. \quad (C.6)$$

Using twice (C.1) we can write

$$f_0^{2+1} f_2^2 f_0^{-A} = f_0 f_2 \left( f_0 f_2 + (2A + 1)e_{-1}^{1} \right) + A(A + 1) (e_{-1}^{1})^2. \quad (C.7)$$

For the first term on the r.h.s. of the above using (C.2), the constraint for $f_0$, and (C.6) we get

$$f_0 f_2 \left( f_0 f_2 + (2A + 1)e_{-1}^{1} \right) V_\mu = (b_0^3 (c^2 \tilde{h}_{2})_{-1} + \{Q, b_0^3 f_2\}) \left( f_0 f_2 + (2A + 1)e_{-1}^{1} \right) V_\mu$$

$$\simeq A (b_0^2 c_{-1}^2) \left( -[f_2, f_0] + (2A + 1)e_{-1}^{1} \right) V_\mu \quad (C.8)$$

and thus

$$f_0^{2+1} f_2^2 f_0^{-A} V_\mu \simeq A(A + 1) \left( 2 (b_0^3 c_{-1}^2) e_{-1}^{1} + (e_{-1}^{1})^2 \right) V_\mu = A(A + 1) (e_{-1}^{1})^2 V_\mu. \quad (C.9)$$

and substituting the constraint for $e_{-1}^{1}$ we finally have

$$f_0^{2+1} f_2^2 f_0^{-A} V_\mu \simeq A(A + 1) V_\mu \quad (C.10)$$
More generally if $V_\mu$ is such that

$$\langle \alpha^2, \mu \rangle = (A + p - 1) \quad \text{(C.11)}$$

then

$$f_0^{p+A} f_2^p f_0^{-A} V_\mu \simeq \frac{\Gamma(A + p)}{\Gamma(A)} V_\mu \quad \text{(C.12)}$$

We will extend the last formula for arbitrary $p$, not necessarily integer. Now consider $u_{020}$ applied to $w_{1(020)}^t \cdot \lambda$. Thus we are in the situation of (C.12) with

$$\mu = w_{1(020)}^t \cdot \lambda, \quad A = M^2 - \frac{t+1}{\nu}, \quad p = M^1 + \frac{2t+1}{\nu},$$

and

$$h^2(w_{1(020)}^t \cdot \lambda) = M^1 + M^2 + \frac{t}{\nu} - 1,$$

thus the condition (C.11) is satisfied and we get

$$V_{u_{020}w_{1(020)}^t \cdot \lambda} \simeq \frac{\Gamma(M^1 + M^2 + \frac{t}{\nu})}{\Gamma(M^2 - \frac{t+1}{\nu})} V_{w_{1(020)}^t \cdot \lambda} \quad \text{(C.13)}$$

Repeating this step consecutively for $t = n - 1, n - 2, \ldots, 0$ one effectively “wipes away” all groups of Kac-Moody generators in (A.1) corresponding to the reflections $w_{020}$. The powers of $f_1$ are replaced by the operators (6.7) as in (6.8) and altogether this reproduces the explicit expression (6.18) of the singular vector. Collecting the numerical factors from (C.13) for a singular vector we get, e.g., for the coefficient $N_1$ in (6.18)

$$N_1 = \prod_{t=1}^{m'_1-1} \frac{\Gamma((\lambda + \rho, \alpha^3 + (t-1)\delta))}{\Gamma((\lambda + \rho, \alpha^2 - t\delta))} \quad \text{(C.14)}$$

The same formula with $\alpha^2, m'_1$ replaced by $\alpha^1, m'_2$ reproduces up to a sign the constant $N_2$ in (6.19). The constant $N'_1$ (which appear in the case $M^1 = -m_1 + \frac{m'_1-1}{\nu}$) is provided by the same expression (C.14) with $\alpha^2$ and $\alpha^3$ interchanged, while $N'_2$ (for $M^2 = -m_2 + \frac{m'_2-1}{\nu}$) is obtained from the corresponding expression of $N_2$ with $\alpha^1$ and $\alpha^3$ interchanged.

To obtain (6.20) one has to use that the reflection in (2.12c) can be also rewritten as $w_\beta = w_{12}w_{1(012)}^t w_1 = w_{12}w_{0(012)}^t w_1$. Then one has to transform the corresponding MFF monomial as above. The resulting overall constant $N_3$ in (6.20) reads (again up to a sign)

$$N_3 = \prod_{t=1}^{m'_2-2} \frac{\Gamma((\lambda + \rho, \alpha^2 + (t-1)\delta))}{\Gamma((\lambda + \rho, -\alpha^1 - t\delta))} \quad \text{(C.15)}$$

while $N'_3$ (for $M^3 = -m_3 + \frac{m'_2-2}{\nu}$) is obtained from (C.15) interchanging $\alpha^2$ and $-\alpha^1$.

**Appendix D. Analytic continuation of $O_\lambda$ and an explicit example of a singular vector.**

In this appendix we will illustrate on a concrete example how one can work with the formal expressions (6.18-20) and get explicit results.
Consider the expansion (6.17) of $O^{(i)}_\lambda$ in decreasing powers of $W^{-1}$, keeping terms up to $k = 2$ (for notational clarity we will do the case $i = 1$; for short in this appendix we absorb the normalization $a_\nu$ into the current $W(z)$.)

$$O^{(1)} = K \left( (W^{-1})^p + A (W^{-1})^{p-1} L_{-1} + (W^{-1})^{p-2} (B L_{-2}^2 + C L_{-2} + D W_{-2}) + \ldots \right) \quad (D.1)$$

The dots stand for terms of lower order in $W^{-1}$. For short denote the parameters $(M^1, M^2)$ of the weight $\lambda$ by $(p, q)$. The overall normalization is

$$K(p, q) = (-1)^p \frac{\Gamma(q - \frac{1}{\nu})}{\Gamma(p + q - \frac{1}{\nu})} \quad (D.2)$$

while the coefficients $A, B, \text{etc.}$ are polynomials in the parameters:

$$A(p, q) = \frac{p(p + 2q - \frac{3}{\nu})}{6\nu}$$

$$B(p, q) = \frac{p(p - 1) \left( 4q^2 + (4p - \frac{12}{\nu})q + p^2 - (\frac{5}{p} + 3)p + 9k^2 - 3 \right)}{72\nu^2}$$

$$C(p, q) = \frac{p(p^2 - 1) \left( -10q^2 + (\frac{30}{\nu} - 10p)q - 4p^2 + \frac{15}{\nu}p - \frac{20}{\nu^2} + 6 \right)}{180\nu}$$

$$D(p, q) = \frac{p(p - 1)(p + \frac{1}{\nu} + 1)(\frac{3}{\nu} - p - 2q)}{12}$$

etc.

The derivation of (D.3) is straightforward though lengthy. One starts from the definition (6.4) assuming that $M^1 = p$ is an integer

$$(-1)^{M^1} O^{(1)}_\lambda = {{\cal L}^{(1)}_{p,p-1}} \cdots {{\cal L}^{(1)}_{2,1}} {{\cal L}^{(1)}_{1,0}} + \sum_{q=2}^{p} \underbrace{{{\cal L}^{(1)}_{p,p-1}} \cdots {{\cal L}^{(1)}_{q+1,q}}} \underbrace{{{\cal L}^{(1)}_{q,q-2}} \cdots {{\cal L}^{(1)}_{q-2,q-3}} \cdots {{\cal L}^{(1)}_{1,0}}} + \ldots \quad (D.4)$$

To obtain the expansion up to the order displayed in (D.1) we need only the two terms in (D.4) and the commutator $[L_{-1}, W_{-1}] = -W_{-2}$. We also make multiple use of the formula

$$\sum_{q=0}^{p} \binom{q}{k} = \binom{p + 1}{k + 1} \quad (D.5)$$

Since the coefficients (D.3) are polynomials in $p$ we can use the expansion of $O^{(1)}$ in powers of $W_{-1}$ to define it for arbitrary complex $M^1 = p$. Of course, in analogy with [18] we have to enlarge the algebra to contain arbitrary powers of $W_{-1}$.

Next we have to consider compositions $\tilde{O} O$, where $\tilde{O}$ has the expansion (D.1) as $O$ but with coefficients $\tilde{A} = A(\tilde{p}, \tilde{q}), \tilde{B} = B(\tilde{p}, \tilde{q})$, etc. Up to the same order we get (again we need only
\[ L_{-1}, W_{-1} = -W_{-2} \]
\[ \mathcal{O} \mathcal{O} = \mathcal{K} K (W_{-1})^{\bar{p}+p-2}. \]

\[ \cdot \left( W_{-1}^2 + (A + \bar{A}) W_{-1} L_{-1} + (B + \bar{B} + A\bar{A}) L_{-1}^2 + (C + \bar{C}) L_{-1} - 2 + (D + \bar{D} - p\bar{A}) W_{-2} + \cdots \right) \quad (D.6) \]

Now we turn to the simplest example with \( \lambda \) such that \( M^1 = 1 - \frac{1}{\nu}, (M^2 = q \) arbitrary). The singular vector corresponding to \( \beta = \delta + \alpha^1 \) is

\[ \mathcal{O}^{(1)}_{\nu_0/201} \mathcal{O}^{(1)}_\lambda V_\lambda. \quad (D.7) \]

In (D.6) we have to set \( p = 1 - \frac{1}{\nu}, \bar{p} = 1 + \frac{1}{\nu} \) and \( \bar{q} = q - \frac{1}{\nu} \) obtaining

\[ A + \bar{A} = \frac{1}{3\nu}(1 - \frac{4}{\nu^2} + 2q) \]
\[ B + \bar{B} + A\bar{A} = \frac{4q^2 + (4 - 16q) + \frac{16q}{\nu^2} - \frac{8}{\nu^2} + 1}{36 \nu^2} \quad (D.8) \]
\[ C + \bar{C} = \frac{-q^2 + (\frac{1}{\nu} - 1)q - \frac{4}{\nu} + \frac{2}{\nu}}{3 \nu^3} \]
\[ D + \bar{D} - p\bar{A} = \frac{(\frac{3}{\nu} + 1)(\frac{2}{\nu} - 1 - 2q)}{6 \nu} \]

Moreover the full analysis shows that the other coefficients vanish, i.e., the dots in the r.h.s. of (D.6) for these particular parameters can be removed, and we get a finite polynomial expression with integer powers of the \( W \) algebra generators for the \( \mathcal{P}_{\beta, \lambda} \).

In this particular example one can obtain the singular vector in a much simpler way and the above exercise becomes merely a check that our algorithm is correct. The class of weights \( \lambda \) such that \( M^1 = m - \frac{1}{\nu}, (M^2 = m - \frac{1}{\nu}) \), for some \( m = 0, 1, 2, \ldots \), maps into the class of weights integral along the first (or second) root under the following generalization of the duality transformation of [14]

\[ \nu \mapsto \bar{\nu} = \frac{1}{\nu}, \quad M^1 \mapsto \bar{M}^1 = 1 - \nu(M^2 - 1), \quad M^2 \mapsto \bar{M}^2 = 1 - \nu(M^1 - 1). \quad (D.9) \]

In our particular example (where \( m = 1 \)) we have

\[ \bar{M}^1 = 1 + \nu - \nu M^2, \quad \bar{M}^2 = 2 \quad (D.10) \]

and the singular vector (D.7) is equal up to an overall constant to \( \mathcal{O}^{(2)}_\lambda V_\lambda \). Recall that for \( \bar{\lambda} \) with (D.10) the operator \( \mathcal{O}^{(2)}_\lambda \) is an ordinary polynomial in the generators \( W_{-n} \) and \( L_{-n} \) and no analytic continuation is necessary, namely

\[ \mathcal{O}^{(2)}_\lambda = \bar{\mathcal{O}}^{(2)}_{2,1} \mathcal{O}^{(2)}_{1,0} + \bar{\mathcal{O}}^{(2)}_{2,0} \quad (D.11) \]

where for short \( \bar{\mathcal{O}} \) stands for \( \mathcal{L} \) with \( \bar{\lambda} \).
Acknowledgements

We are indebted to L. Bonora and especially to V.K. Dobrev for useful discussions and to A. Koubek and C.J. Zhu for patient help with MATHEMATICA. V.B.P. acknowledges the hospitality of the Arnold Sommerfeld Institute for Mathematical Physics, TU Clausthal. A.Ch.G and V.B.P. acknowledge the financial support and the hospitality of INFN, Sezione di Trieste and SISSA, Trieste. A.Ch.G. was supported also by the Humboldt foundation.

This work was supported in part by the Bulgarian Foundation for Fundamental Research under contract $\phi - 11 - 91$.

Note Added

After this work was essentially written the paper [36] appeared in hep-th. The main result of this paper is to give a rigorous meaning of the analytic continuation of the expressions $C^{(i)}_\lambda$ to complex $\langle \lambda + \rho, \alpha \rangle$ and hence of the general $W_3$ singular vectors. In this respect it is complementary to our main emphasis. Nevertheless we have left the explicit example in appendix D which demonstrates that the method of Kent is algorithmic and can be used for concrete calculations just as the fusion method of [24].

In [36] there is also a proof of the important fact that there is exactly one $W_3$ algebra singular vector for every weight – something we have only conjectured.

Finally we should point out that though the author of [36] argues that the Verma modules with pseudo-singular vectors occurring are exhausted by the weights $\frac{1}{\nu} = m = M^1 = M^2, m \in \mathbb{N}$, in fact there are other possibilities, demonstrated for the case $\nu = 1$ already in [32].

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