COMPLETE STRUCTURE OF $Z_n$ YUKAWA COUPLINGS

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Abstract

We give the complete twisted Yukawa couplings for all the $Z_n$ orbifold constructions in the most general case, i.e. when orbifold deformations are considered. This includes a certain number of tasks. Namely, determination of the allowed couplings, calculation of the explicit dependence of the Yukawa couplings values on the moduli expectation values (i.e. the parameters determining the size and shape of the compactified space), etc. The final expressions are completely explicit, which allows a counting of the different Yukawa couplings for each orbifold (with and without deformations). This knowledge is crucial to determine the phenomenological viability of the different schemes, since it is directly related to the fermion mass hierarchy. Other facts concerning the phenomenological profile of $Z_n$ orbifolds are also discussed, e.g. the existence of non–diagonal entries in the fermion mass matrices, which is related to a non–trivial structure of the Kobayashi–Maskawa matrix. Finally some theoretical results are given, e.g. the no–participation of (1,2) moduli in twisted Yukawa couplings. Likewise, (1,1) moduli associated with fixed tori which are involved in the Yukawa coupling, do not affect the value of the coupling.
1 Introduction and brief review

In the last few years an enormous effort has been made in order to establish the connection between string theories \[1\] (especially \(E_8 \times E_8\) heterotic string \[2\]) and low energy physics. Different schemes for constructing classical string vacua have arisen during this time. Using these schemes it has been possible to build up four–dimensional strings that resemble the Standard Model in many aspects, e.g. \(SU(3) \times SU(2) \times U(1)\) gauge group and three generations of particles with the correct representations \[3–6\]. In spite of these achievements there remain many pending questions. In particular there is a large number of classical vacuum states, which reduces the predictive power of the theory. At present there are no dynamical criteria to prefer a particular vacuum, so the best we can do is to study their phenomenological characteristics in order to select the viable vacua. In this respect, orbifold compactifications \[7\] have proved to be very interesting four–dimensional string constructions since they can pass successfully a certain number of low energy tests \[8\]. However, not all the experimental constraints have been used in order to probe the phenomenological potential of orbifolds. The best example of this is the observed structure of fermion masses and mixing angles \[9\].

Concerning the last point, a crucial ingredient in order to relate theory and observation is the complete knowledge of the theoretical Yukawa couplings. This knowledge includes a certain number of aspects:

i) Physical states that enter the couplings.

ii) Allowed couplings.

iii) Numerical values of the Yukawa couplings and dependence of these values on the physical parameters that define the string vacuum (e.g. the size of the compactified space).

iv) Number of different Yukawa couplings and phenomenological viability of the scheme (\textit{i.e.} fitting of the observed pattern of fermion masses and mixing angles by the theoretical Yukawa couplings).

In this sense only for prime Abelian orbifolds, \textit{i.e.} \(Z_3\) and \(Z_7\) are the Yukawa couplings completely known \[10, 11\]. For the other orbifolds points i) and ii) have recently been studied in ref.\[12\]. General expressions of \(Z_n\) Yukawa couplings have been determined in ref.\[13\]. However, although very useful, they are not explicit enough to lucidate points iii) and iv), specially when deformations of the compactified space are considered.
Undoubtedly, a better knowledge of the Yukawa couplings is of utmost importance to select or discard explicit string constructions with a highly non-trivial test. This is the main purpose of this paper, \textit{i.e.} to answer points i), ii), iii), iv) for \textit{all} the $Z_n$ orbifolds. Besides the phenomenological motivation, there are strong theoretical reasons to completely determine the Yukawa couplings. In particular it is the only way to know the moduli dependence of the matter Lagrangian and, in consequence, the superpotential. This allows the examination of the properties of the action under target-space modular transformations (e.g. $R \rightarrow 1/R$) \cite{14}. It is also necessary in order to discuss supersymmetry breaking dynamics \cite{13} and cosmological implications (note that moduli play the role of Brans-Dicke fields in four dimensions) of these theories.

Let us review briefly $Z_n$ orbifold constructions. A $Z_n$ orbifold is constructed by dividing $R^6$ by a six-dimensional lattice $\Lambda$ modded by some $Z_n$ symmetry, called the point group $P$. The space group $S$ is defined as $S = \Lambda \times P$, \textit{i.e.} $S = \{ (\gamma, u); \gamma \in P, u \in \Lambda \}$. The requirement of having $N = 1$ supersymmetry in four dimensions and the absence of tachyons restrict the number of possible point groups \cite{7}. The complete list is given in the first two columns of Table 1, where the so-called twist $\theta$ (\textit{i.e.} the generator of $P$) is represented in an orthogonal complex basis of the six-dimensional space. $\Lambda$ must be chosen so that $\theta$ acts crystallographically on it. If the realization of $\theta$ on the lattice coincides with the Coxeter element of a rank-six Lie algebra root lattice, the orbifold is of the Coxeter type. A list of Coxeter orbifolds, taken from ref. \cite{16}, is given in the third column of Table 1. Some additional examples of Coxeter orbifolds can be found in ref. \cite{12}. The lattice of the $Z_8$–II orbifold, $SO(5) \times SO(8)$, corresponds in fact to a generalized Coxeter orbifold where the Coxeter element has been multiplied by an outer automorphism. Non–Coxeter orbifolds can also be constructed. An example of a non–Coxeter orbifold (the $Z_4$ one with $[SO(4)]^3$ lattice) is studied in Section 3. The total number of possible lattices associated with each $Z_n$ orbifold can be found in ref. \cite{17}. As will become clear in the text, some properties of the Yukawa couplings for a particular $Z_n$ orbifold depend on the lattice chosen, while others do not.

It is important to point out that in a string orbifold construction the lattice $\Lambda$ can get deformations compatible with the point group \cite{9, 11}. These degrees of freedom correspond to the untwisted moduli surviving compactification. Deformations play an important role on the value of the Yukawa couplings.

We are interested in the couplings between twisted fields (the untwisted sector has already been studied \cite{10} and is physically less interesting \cite{9}). As we will see, these couplings present a very rich range, which is extremely attractive as the geometrical origin of the observed variety of fermion masses \cite{10, 18, 9}. A twisted string satisfies
\(x(\sigma = 2\pi) = gx(\sigma = 0)\) as the boundary condition, where \(g\) is an element of the space group whose point group component is non-trivial. Owing to the boundary condition a twisted string is attached to a fixed point of \(g\). Physical twisted fields are associated with conjugation classes of the space group rather than with particular elements. For example \(\{hgh^{-1}, h \in S\}\) is the conjugation class of \(g\). For prime orbifolds conjugation classes are in one-to-one correspondence with the fixed points of \(P\). For non-prime orbifolds the situation is a bit more involved since two different fixed points under \(\theta^n\) may be connected by \(\theta^m\), \(m < n\). Then both of them correspond to the same conjugation class.

The paper is organized as follows. In Section 2 we expound the various steps necessary to obtain the final spectrum of Yukawa couplings for each orbifold, taking the \(Z_6-I\) as a guide example. These steps include: determination of the geometrical structure of the orbifold and deformation parameters, physical states, calculation of explicit Yukawa couplings, and counting of different couplings. Section 3 is devoted to a comparative study of the \([SO(4)]^3\) and \([SU(4)]^2 Z_4\) orbifolds. This shows which properties of the couplings depend on the lattice chosen and which do not. Furthermore, the \([SO(4)]^3\) case provides an example of a non-Coxeter orbifold. Besides this, the \(Z_4\) orbifold allows one to see the physical meaning of a \((1,2)\) modulus (absent in the \(Z_6-I\) case) and its effect in the Yukawa coupling values. The complete results for the rest of \(Z_n\) orbifolds are given in Appendix 1 and summarized in Table 1.

## 2 The method

Several steps are necessary in order to obtain the final spectrum of Yukawa couplings for each orbifold. We explain these steps in the present section, taking the \(Z_6-I\) orbifold as an illustrative example. The reason for this choice is that prime orbifolds \((Z_3\) and \(Z_7)\) have already been studied in depth in references \([10, 9, 11]\). A complete exposition of the method followed here can be found in ref. \([11]\). It is, however, convenient to discuss the present example in some detail, since non-prime orbifolds exhibit certain features which are absent in the prime ones.

### 2.1 Geometrical structure

The twist \(\theta\) of the \(Z_6-I\) orbifold has the form (see Table 1)

\[
\theta = \text{diag}(e^{i\alpha}, e^{i\alpha}, e^{-2i\alpha}), \quad \alpha = \frac{2\pi}{6}
\]
in the complex orthogonal basis \{(\bar{e}_1, \bar{e}_2), (\bar{e}_3, \bar{e}_4), (\bar{e}_5, \bar{e}_6)\}. Very often it is more suitable to work in the lattice basis \{e_1, ..., e_6\}, which in this case is simply a set of simple roots of \(G_2^2 \times SU(3)\). Then \(\theta\) acts as the Coxeter element

\[
\begin{align*}
\theta e_1 &= -e_1 - e_2, & \theta e_2 &= 3e_1 + 2e_2, \\
\theta e_3 &= -e_3 - e_4, & \theta e_4 &= 3e_3 + 2e_4, \\
\theta e_5 &= e_6, & \theta e_6 &= -e_5 - e_6.
\end{align*}
\]

Note that we have labelled as \(e_5, e_6\) the simple roots of \(SU(3)\). As mentioned above \(\Lambda\) can get deformations compatible with the point group. These degrees of freedom correspond to the Hermitian part of the five untwisted \((1,1)\) moduli surviving compactification \(N_{11}, N_{22}, N_{33}, N_{12}, N_{21}\), where \(N_{ij} = |i \rangle >_R \otimes \alpha_{ij}^{-1} |0 \rangle >_L; \quad |0 \rangle >_L (|0 \rangle >_R)\) being the left (right) vacuum, \(\alpha_L\) is an oscillator operator and \(i (j)\) is a holomorphic (antiholomorphic) index. Note that under a deformation the actuation of \(\theta\) on the lattice basis, eq. (2), remains the same. Then \(P\) invariance imposes the following relations:

\[
\begin{align*}
|e_2| &= \sqrt{3} |e_1|, & |e_4| &= \sqrt{3} |e_3|, & |e_5| &= |e_6|, \\
\alpha_{12} &= -\sqrt{3}/2, & \alpha_{34} &= -\sqrt{3}/2, & \alpha_{56} &= -1/2, \\
\alpha_{24} &= \alpha_{13}, & \alpha_{23} &= -\sqrt{3} \alpha_{13} - \alpha_{14}, & \alpha_{ij} &= 0 & i = 1, 2, 3, 4 & j = 5, 6
\end{align*}
\]

where \(\alpha_{ij} = \cos \theta_{ij}\) and \(e_i e_j = |e_i| |e_j| \cos \theta_{ij}\). Therefore we can take the 5 deformation degrees of freedom

\[
R_i = |e_i|; \quad i = 1, 3, 5
\]

\[
\alpha_{13}, \alpha_{14}
\]

\(R_i\) are global scales of the three sublattices \((G_2, G_2, SU(3))\), and for \(\alpha_{13} = \alpha_{14} = 0\) we recover the rigid \(G_2 \times G_2 \times SU(3)\) lattice. The connection between the lattice basis \((e_1, ..., e_6)\) and the orthogonal basis \((\tilde{e}_1, ..., \tilde{e}_6)\) in which \(\theta\) takes the form \([\bar{1}]\) is given by

\[
\begin{align*}
e_i &= A_1 [\cos(\varphi_1^i)\tilde{e}_1 + \sin(\varphi_1^i)\tilde{e}_2] + B_1 [\cos(\varphi_2^i)\tilde{e}_3 + \sin(\varphi_2^i)\tilde{e}_4] \\
e_{i+1} &= -\sqrt{3}[A_1 (\sin(\pi/3 - \varphi_1^i)\tilde{e}_1 + \cos(\pi/3 - \varphi_1^i)\tilde{e}_2) + B_1 (\sin(\pi/3 - \varphi_2^i)\tilde{e}_3 + \cos(\pi/3 - \varphi_2^i)\tilde{e}_4)] \\
e_j &= R_5 [\cos((j - 5)\pi/3 + \phi)\tilde{e}_5 + \sin((j - 5)\pi/3 + \phi)\tilde{e}_6]; \quad i = 1, 3 & j = 5, 6
\end{align*}
\]

where \(\varphi_1^1, \varphi_1^2, \varphi_2^1, \varphi_2^2, \phi\) are arbitrary angles that are irrelevant for our results and

\[
\begin{align*}
A_1 &= R_1 \sqrt{2} [\Delta_1 \pm \Delta_3]^{1/2} & B_1 &= R_1 \sqrt{2} [\Delta_2 \mp \Delta_3]^{1/2} \\
A_3 &= k_2 R_3 \sqrt{2} [\Delta_1 \pm \Delta_3]^{-1/2} & B_3 &= k_1 R_3 \sqrt{2} [\Delta_2 \mp \Delta_3]^{-1/2}
\end{align*}
\]
with
\[ k_i = 2R_1R_3(-1)^i[\alpha_{13}\sin(\frac{\pi}{3} + \varphi_1 - \varphi_3^i) + \alpha_{14}\cos(\varphi_1 - \varphi_3^i)][\sin(\varphi_1^i - \varphi_2^i - \varphi_1^i + \varphi_3^i)]^{-1} \quad i = 1, 2 \]
\[ \Delta_1 = 1 + k_2^2 - k_1^2 \quad \Delta_2 = 1 - k_2^2 + k_1^2 \quad \Delta_3 = [(1 - k_1^2 - k_2^2)^2 - 4k_1^2k_2^2]^{1/2}. \]

Let us consider now the fixed points under the action of the point group. \( f_n \) is a fixed point under \( \theta^n \) if it satisfies \( f_n = \theta^n f_n + u, \ u \in \Lambda \). As \( Z_6-I \) is a non-prime orbifold, a point fixed by \( \theta^n \) \( (n \neq 1) \) may be not fixed by \( \theta^m \) \( (m \neq n) \). Consequently, the fixed points under \( \theta, \theta^2 \) and \( \theta^3 \) must be considered separately \( (\theta^4, \theta^5 \) are simply the antitwists of \( \theta^2 \) and \( \theta \)). It is easy to check from (2) that there are three different fixed points under \( \theta \). Working in the lattice basis their coordinates (up to lattice translations) are

\[ f_1^{(1)} = g_1^{(0)} \otimes g_1^{(0)} \otimes \hat{g}_1^{(0)}, \]
\[ f_1^{(2)} = g_1^{(0)} \otimes g_1^{(0)} \otimes \hat{g}_1^{(1)}, \]
\[ f_1^{(3)} = g_1^{(0)} \otimes g_1^{(0)} \otimes \hat{g}_1^{(2)}. \]  

with
\[ g_1^{(0)} = (0, 0), \quad \hat{g}_1^{(0)} = (0, 0), \quad g_1^{(1)} = (\frac{2}{3}, \frac{2}{3}), \quad \hat{g}_1^{(1)} = (\frac{2}{3}, \frac{1}{3}). \]  

Similarly under \( \theta^2 \) there are 27 fixed points. 12 of them are connected to the others by a \( \theta \) rotation

\[ f_2^{(1)} = g_2^{(0)} \otimes g_2^{(0)} \otimes \hat{g}_2^{(0)}, \]
\[ f_2^{(2)} = g_2^{(0)} \otimes g_2^{(0)} \otimes \hat{g}_2^{(1)}, \]
\[ f_2^{(3)} = g_2^{(0)} \otimes g_2^{(0)} \otimes \hat{g}_2^{(2)}, \]
\[ f_2^{(4)} = g_2^{(0)} \otimes g_2^{(1)} \otimes \hat{g}_2^{(0)} \sim g_2^{(0)} \otimes g_2^{(2)} \otimes \hat{g}_2^{(1)}, \]
\[ f_2^{(5)} = g_2^{(0)} \otimes g_2^{(1)} \otimes \hat{g}_2^{(0)} \sim g_2^{(0)} \otimes g_2^{(2)} \otimes \hat{g}_2^{(1)}, \]
\[ f_2^{(6)} = g_2^{(0)} \otimes g_2^{(1)} \otimes \hat{g}_2^{(1)} \sim g_2^{(0)} \otimes g_2^{(2)} \otimes \hat{g}_2^{(2)}, \]
\[ f_2^{(7)} = g_2^{(1)} \otimes g_2^{(0)} \otimes \hat{g}_2^{(0)} \sim g_2^{(2)} \otimes g_2^{(0)} \otimes \hat{g}_2^{(0)}, \]
\[ f_2^{(8)} = g_2^{(1)} \otimes g_2^{(0)} \otimes \hat{g}_2^{(1)} \sim g_2^{(2)} \otimes g_2^{(0)} \otimes \hat{g}_2^{(1)} \sim g_2^{(0)} \otimes g_2^{(1)} \otimes \hat{g}_2^{(1)}, \]
\[ f_2^{(9)} = g_2^{(2)} \otimes g_2^{(0)} \otimes \hat{g}_2^{(0)} \sim g_2^{(2)} \otimes g_2^{(1)} \otimes \hat{g}_2^{(1)}, \]
\[ f_2^{(10)} = g_2^{(1)} \otimes g_2^{(0)} \otimes \hat{g}_2^{(0)} \sim g_2^{(2)} \otimes g_2^{(1)} \otimes \hat{g}_2^{(0)} \sim g_2^{(2)} \otimes g_2^{(2)} \otimes \hat{g}_2^{(0)} \sim g_2^{(0)} \otimes g_2^{(2)} \otimes \hat{g}_2^{(0)} \sim g_2^{(0)} \otimes g_2^{(2)} \otimes \hat{g}_2^{(2)} \sim g_2^{(0)} \otimes g_2^{(1)} \otimes \hat{g}_2^{(1)} \sim g_2^{(2)} \otimes g_2^{(1)} \otimes \hat{g}_2^{(1)} \sim g_2^{(2)} \otimes g_2^{(2)} \otimes \hat{g}_2^{(1)} \sim g_2^{(2)} \otimes g_2^{(2)} \otimes \hat{g}_2^{(2)} . \]  

with
\[ g_2^{(0)} = (0, 0), \quad g_2^{(1)} = (0, \frac{1}{3}), \quad g_2^{(2)} = (0, \frac{2}{3}), \]
\[ \hat{g}_2^{(0)} = (0, 0), \quad \hat{g}_2^{(1)} = (\frac{1}{3}, \frac{2}{3}), \quad \hat{g}_2^{(2)} = (\frac{2}{3}, \frac{1}{3}). \]  

Consequently, there are 15 conjugation classes under \( \theta^2 \). Finally, under \( \theta^3 \), there are 16 fixed tori that are the direct product of 16 fixed points in the sublattice \((e_1, ..., e_4)\) by
the 2–torus defined by the sublattice \((e_5, e_6)\). (Notice that \(\theta^3\) is trivial in the \(SU(3)\) root lattice.) 15 of these fixed points are connected between themselves by \(\theta\) rotations

\[
\begin{align*}
  f_3^{(1)} &= g_3^{(0)} \otimes g_3^{(0)}, \\
  f_3^{(2)} &= g_3^{(1)} \otimes g_3^{(0)} \sim g_3^{(2)} \otimes g_3^{(0)} \sim g_3^{(3)} \otimes g_3^{(0)}, \\
  f_3^{(3)} &= g_3^{(0)} \otimes g_3^{(1)} \sim g_3^{(0)} \otimes g_3^{(2)} \sim g_3^{(0)} \otimes g_3^{(3)}, \\
  f_3^{(4)} &= g_3^{(1)} \otimes g_3^{(1)} \sim g_3^{(2)} \otimes g_3^{(2)} \sim g_3^{(3)} \otimes g_3^{(3)}, \\
  f_3^{(5)} &= g_3^{(2)} \otimes g_3^{(1)} \sim g_3^{(3)} \otimes g_3^{(2)} \sim g_3^{(1)} \otimes g_3^{(3)}, \\
  f_3^{(6)} &= g_3^{(3)} \otimes g_3^{(1)} \sim g_3^{(1)} \otimes g_3^{(2)} \sim g_3^{(2)} \otimes g_3^{(3)},
\end{align*}
\]

with

\[
\begin{align*}
  g_3^{(0)} &= (0, 0), & g_3^{(1)} &= (0, \frac{1}{2}), & g_3^{(2)} &= (\frac{1}{2}, 0), & g_3^{(3)} &= (\frac{1}{2}, \frac{1}{2}).
\end{align*}
\]

The direct product under the \((e_5, e_6)\) torus has been understood. Consequently, there are 6 conjugation classes under \(\theta^3\). A similar analysis for other orbifolds can be found in Appendix 1.

### 2.2 Physical states

The next step is to determine which are the physical states. These must be invariant under a total \(Z_6\) transformation which, besides the twist \(\theta\) in the 6–dimensional space, includes a \(Z_6\) gauge transformation, usually represented by a shift \(V^I\) (the so–called embedding) on \(\Lambda_{E_8 \times E_8}\) and a shift \(v^t\) on \(\Lambda_{SO(10)}\). Accordingly one has to construct for each \(\theta^k\) sector linear combinations of states, associated with \(\theta^k\) fixed points, that are eigenstates of \(\theta\) \[19, 12\]. If \(f_k\) is a fixed point of \(\theta^k\) such that \(l\) is the smallest number giving \(\theta^l f_k = f_k + u, \ u \in \Lambda\), then the eigenstates of \(\theta\) have the form

\[
\begin{align*}
  |f_k\rangle &= e^{-i\gamma}|\theta f_k\rangle + \ldots + e^{-i(l-1)\gamma}|\theta^{l-1} f_k\rangle \\
  \gamma &= \frac{2\pi p}{l}, & p &= 1, 2, \ldots, l
\end{align*}
\]

with eigenvalue \(e^{-i\gamma}\) (obviously, if \(k = 1\), then \(l = 1\) and eq. \(12\) is trivial). Under a \(Z_6\) transformation the complete state gets a phase \[19, 12\]

\[
\Delta(k, e^{i\gamma}) = \exp\left\{2\pi i (-\frac{1}{2} k (\sum_I (V^I)^2 - \sum_t (v^t)^2) + \sum_I (P^I + kV^I) V^I - \sum_t (p^I + kv^I)v^t)\right\} \exp\{i\gamma\},
\]

where \(p^I\) is the NSR part momentum put on the \(SO(8)\) weight lattice and \(P^I\) is the transverse 8–dim. momentum \((E_8 \times E_8\) root momentum) fulfilling the right–mover and left–mover massless conditions respectively. Then \(\Delta(k, e^{i\gamma}) = 1\) for physical states.
Let us apply this to the $E_8 \times E_8$ heterotic string compactified on the $Z_6$–I orbifold with $V^I = \frac{1}{6}(1, 1, -2, 0, ..., 0)(0, ..., 0)$, i.e. the standard embedding. The unbroken gauge group is $(E_6 \times SU(2) \times U(1)) \times E_8$. In the $\theta$ sector there are three physical states transforming as $(27)$’s of $E_6$ corresponding to $|f_1^{(1)}>, |f_2^{(2)}>, |f_3^{(3)}>$ respectively (see eq. (8)). In the $\theta^2$ sector we can construct 27 eigenstates of $\theta$ (see eq. (8))

$$|f_2^{(1)}>, |f_2^{(2)}>, |f_3^{(3)}>, \{ |f_2^{(j)}> + e^{-i\gamma}|\theta f_2^{(j)}> \} \quad j = 4, ..., 15 \quad \gamma = \pi, 2\pi. \quad (14)$$

After some algebra, only the symmetric combinations survive (i.e. $\Delta(k, e^{i\gamma}) = 1$ for them), giving rise to 15 $(27)$’s under $E_6$. Similarly in the $\theta^3$ sector we can construct 16 eigenstates of $\theta$ (see eq. (10))

$$|f_3^{(1)}>, \{ |f_3^{(j)}> + e^{-i\gamma}|\theta f_3^{(j)}> + e^{-i2\gamma}|\theta^2 f_3^{(j)}> \} \quad j = 2, ..., 6 \quad \gamma = \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi. \quad (15)$$

In this case there survive 6 $(27)$’s, corresponding to the symmetric combinations, and 5 $(\overline{27})$’s, corresponding to the $\gamma = 2\pi/3$ combinations. We have performed a similar analysis for each orbifold.

### 2.3 Allowed Yukawa couplings

Let us now analyse the allowed Yukawa couplings between physical states. A twisted string associated with a fixed point $f$ and a rotation $\theta^j$ is closed due to the action of $g = (\theta^j, (I - \theta^j)f)$, so the corresponding conjugation class is given by $\{(\theta^k, u)(\theta^j, (I - \theta^j)f)(\theta^{-k}, -\theta^{-k}u)\}$, with $k \in \mathbb{Z}$, $u \in \Lambda$. After some algebra, the general expression of the conjugation class of $g$ is

$$\left( \theta^j, (I - \theta^j) \left[ (f + \Lambda) \cup (\theta f + \Lambda) \cup ... \cup (\theta^{j-1} f + \Lambda) \right] \right). \quad (16)$$

The set of translations $(1 - \theta^j) \left\{ \cup (\theta^k f + \Lambda), k = 0, ..., j - 1 \right\}$ is called the coset associated with $\theta^j$ and $f$ (note that the cosets associated with $f$ and $\theta^k f$ are obviously the same). For a trilinear coupling of twisted fields $T_1 T_2 T_3$ to be allowed, the product of the respective conjugation classes should contain the identity. This implies, in particular, that the product of the three point group elements $\theta^{i_1} \theta^{i_2} \theta^{i_3}$ should be 1 (this is the so–called point group selection rule). For the $Z_6$–I orbifold this implies that only $\theta \theta^2 \theta^3$, $\theta^2 \theta^2 \theta^2$ and $\theta \theta^4 \theta^4$ couplings have to be considered. A straightforward application of the H–momentum conservation \cite{10, 20, 21} shows that the $\theta \theta^4 \theta^4$ couplings are also forbidden. Furthermore for the $\theta \theta^2 \theta^3$ couplings we must require

$$(I, 0) \in (\theta, (I - \theta)(f_1 + \Lambda)) \left( \theta^2, (I - \theta^2) \left[ (f_2 + \Lambda) \cup (\theta f_2 + \Lambda) \right] \right) \left( \theta^3, (I - \theta^3) \left[ (f_3 + \Lambda) \cup (\theta f_3 + \Lambda) \cup (\theta^2 f_3 + \Lambda) \right] \right), \quad (17)$$
which leads to the so-called space group selection rule for the coupling $\theta\theta^2\theta^3$ in the $Z_6$–I orbifold

$$f_1 + (I + \theta)f_2 - (I + \theta + \theta^2)f_3 \in \Lambda.$$  

(18)

It should be noticed that if the space group selection rule (18) is satisfied for three fixed points $f_1, f_2, f_3$, then it is also satisfied for $\theta^{k_1}f_1, \theta^{k_2}f_2, \theta^{k_3}f_3$, and, consequently, for all the physical combinations $\sum_{k_1} e^{-ik_1\gamma_1} |\theta^{k_1}f_1|, \sum_{k_2} e^{-ik_2\gamma_2} |\theta^{k_2}f_2|, \sum_{k_3} e^{-ik_3\gamma_3} |\theta^{k_3}f_3|$, see eq. [12]. For the case at hand, i.e. the $Z_6$–I orbifold, one can consider 270 kinds of couplings $f_1^{(j_1)}f_2^{(j_2)}f_3^{(j_3)}$ of the $\theta\theta^2\theta^3$ type, from which only 90 are allowed: those for which the $e_5, e_6$ components (i.e. the $SU(3)$ sublattice projection) of $f_1 - f_2$ are vanishing. So, if we write the fixed points

\[
\begin{align*}
  f_1 &= g_1^{(0)} \otimes g_1^{(0)} \otimes g_1^{(k_1)} \\
  f_2 &= g_2^{(i_2)} \otimes g_2^{(j_2)} \otimes \hat{g}_2^{(k_2)} \\
  f_3 &= g_3^{(i_3)} \otimes g_3^{(j_3)} \otimes [\alpha(e_5) + \beta(e_6)]
\end{align*}
\]

(k_1, k_2, i_2, j_2 = 0, 1, 2, \\
  i_3, j_3 = 0, 1, 2, 3, \\
  \alpha, \beta \in R.
\]  

(19)

the selection rule is

$$k_1 = k_2.$$  

(20)

At this point it is important to stress the following fact: for two given $f_1, f_2$, the third fixed point $f_3$ (corresponding to $\theta^3$) is not determined uniquely \[^1\]. We say that the space group selection rule is not diagonal. From the physical point of view this is extremely important, since it allows for non–diagonal fermion mass matrices and, hence, a non–trivial Kobayashi–Maskawa matrix. This feature is absent for prime orbifolds. On the other hand the space selection rule for the $\theta^2\theta^2\theta^2$ couplings simply reads

$$f_1 + f_2 + f_3 \in \Lambda$$  

(21)

which is diagonal. Note, however, that in this case the selection rule can be satisfied by some representatives of the conjugation classes and not by others. In this case there are 3375 couplings to consider, from which only 369 are allowed. These are

\[
\begin{align*}
  i_1 + i_2 + i_3 &= 0 \\
  j_1 + j_2 + j_3 &= 0 \\
  k_1 + k_2 + k_3 &= 0
\end{align*}
\]  

mod. 3.  

(22)

denoting

\[
\begin{align*}
  f_1 &= g_2^{(i_1)} \otimes g_2^{(j_1)} \otimes \hat{g}_2^{(k_1)} \\
  f_2 &= g_2^{(i_2)} \otimes g_2^{(j_2)} \otimes \hat{g}_2^{(k_2)} \\
  f_3 &= g_2^{(i_3)} \otimes g_2^{(j_3)} \otimes \hat{g}_2^{(k_3)}
\end{align*}
\]

(i_1, i_2, i_3 = 0, 1, 2 \\
  j_1, j_2, j_3 = 0, 1, 2 \\
  k_1, k_2, k_3 = 0, 1, 2.
\]  

(23)

\[^1\]This also happens for given $f_1, f_3$ but not for $f_2, f_3.$
Let us finally note that the product of the $\theta$ eigenvalues of the physical combinations of fixed points involved in the coupling should be one, otherwise the coupling is vanishing. For example, in the $Z_6$–I orbifold the following $\theta^2\theta^3$ coupling

$$\left| f_1^{(j_i)} \right> \left( \left| f_2^{(j_2)} \right> + \theta \left| f_2^{(j_2)} \right> \right) \left( \left| f_3^{(j_3)} \right> + e^{-\frac{2\pi i}{3}} \theta f_3^{(j_3)} \right) \left( \left| f_3^{(j_3)} \right> + e^{-\frac{4\pi i}{3}} \theta^2 f_3^{(j_3)} \right)$$

is forbidden on these grounds, since, due to $\theta$ invariance, it is equal to

$$\left| f_1^{(j_i)} \right> \left( \left| f_2^{(j_2)} \right> + \theta \left| f_2^{(j_2)} \right> \right) \left| f_3^{(j_3)} \right> \left( 1 + e^{-\frac{2\pi i}{3}} + e^{-\frac{4\pi i}{3}} \right) = 0. \quad (25)$$

This result can also be obtained for the standard embedding case from gauge invariance since the state considered in the $\theta^3$ sector corresponds to a $\Theta(27)$, while the others are $(2\overline{7})$‘s. In consequence all the couplings to be considered in the $Z_6$–I involve symmetric combinations of fixed points (i.e. $\theta$ eigenvalue = 1) exclusively. We have performed the previous analysis for all the $Z_n$ orbifolds. In all cases only couplings between symmetric combinations of fixed points survive. We do not really know what is the fundamental principle behind this rule (if any), but it has important consequences. For instance, in ref. [13] it was suggested that the phases of the non–zero $\theta$ eigenvalue states (see eq. (12)) could be the geometrical origin of the phases of the Kobayashi–Maskawa matrix. Clearly, the present rule excludes this possibility.

### 2.4 Calculation of Yukawa couplings

We are interested in couplings of the type $\psi\psi\phi$ (i.e. fermion–fermion–boson). A trilinear string scattering amplitude is given by the correlator $V_1(z_1)V_2(z_2)V_3(z_3)$ of the vertex operators creating the corresponding states. Complete expressions for the vertex operators of the fields under consideration can be found in refs. [10, 19]. As has been pointed out [10] the non–vanishing Yukawa couplings are essentially given by the bosonic twist correlator $\sigma_1(z_1)\sigma_2(z_2)\sigma_3(z_3)$, where $\sigma_i$ represents a twist field creating the appropriate twisted ground state. According to subsection 2.2 $\sigma$ fields for physical states are, in general, linear combinations of $\sigma$ fields associated with specific rotations and fixed points, say $\sigma_{\theta^i,f}$. For example, for a physical state in the $\theta^j$ sector whose twist part is given by

$$\sum_{k=0}^{l-1} e^{-ik\gamma} |\theta^k f >$$

( the meaning of $\gamma$ and $l$ is given in eq. (12) ) the corresponding twist field is simply

$$\sum_{k=0}^{l-1} e^{-ik\gamma} \sigma_{\theta^j,\theta^k f}.$$  

According to the result of the previous subsection only symmetric combinations ($\gamma = 2\pi$) are relevant for trilinear couplings, so the correlator $\langle \sigma_1(z_1)\sigma_2(z_2)\sigma_3(z_3) \rangle$ associated with a $\theta^{j_1}\theta^{j_2}\theta^{j_3}$ will take the form

$$\langle \sigma_1(z_1)\sigma_2(z_2)\sigma_3(z_3) \rangle =$$

$$\left( \sqrt{l_1 l_2 l_3} \right)^{-1} \sum_{k_1=0}^{l_1-1} \sum_{k_2=0}^{l_2-1} \sum_{k_3=0}^{l_3-1} \langle \sigma_{\theta^{j_1},\theta^{k_1} f(1)} (z_1) \sigma_{\theta^{j_2},\theta^{k_2} f(2)} (z_2) \sigma_{\theta^{j_3},\theta^{k_3} f(3)} (z_3) \rangle, \quad (26)$$
where the square root is a normalization factor. The correlation functions on the right-hand side are evaluated following standard lines \[10\]. They are defined by

\[
\langle \sigma_{\theta_j,1} f^{(1)}(z_1) \sigma_{\theta_k,2} f^{(2)}(z_2) \sigma_{\theta_l,3} f^{(3)}(z_3) \rangle = \int DX \ e^{-S} \sigma_{\theta_j,1} f^{(1)}(z_1) \sigma_{\theta_k,2} f^{(2)}(z_2) \sigma_{\theta_l,3} f^{(3)}(z_3) .
\]

(27)

Owing to the Gaussian character of the action

\[
S = \frac{1}{4\pi} \int d^2 z (\partial X \bar{\partial} X + \bar{\partial} X \partial X) ,
\]

(28)

where \( X = X_1 + iX_2 \) and a sum over the three complex coordinates is understood, the scattering amplitude can be separated into a classical and a quantum part \[10\]

\[
Z = Z_{\text{qu}} \sum_{\langle X_{cl} \rangle} \exp(-S_{cl}) .
\]

(29)

The quantum contribution represents a global factor for all the couplings with the same \( \theta_j \theta_k \theta_l \) pattern in a given orbifold; so the physical information mostly resides in the classical contribution. Eventually, a total normalization factor, which depends on the size of the compactified space, has to be determined with the help of the four–point correlation function. The final task lies in writing the couplings in terms of the physically significant parameters, i.e. those that parametrize the size and shape of the orbifold.

Let us consider, for the sake of definiteness, a \( \theta_1^2 \theta_2^3 \) coupling in our guide example, the \( Z_6 \text{-I} \) orbifold. The classical contribution, see eq. \((29)\), to a \( \langle \sigma_{\theta_2} \sigma_3 \sigma_{\theta_3} \rangle \) correlator has been determined in references \[11, 13\], so we escape here the details of the calculation.

The result is that the contribution of the classical (instantonic) solutions to the classical action, eq. \((28)\), for a three–point correlation on the sphere, is

\[
S_{cl}^i = \frac{1}{4\pi} \left| \sin(2\pi k_i/N) \right| \left| \sin(3\pi k_i/N) \right| \left| \sin(\pi k_i/N) \right| |v_i|^2 \left( f_2 - f_3 + \Lambda \right)_\perp
\]

(30)

where \( i = 1, 2, 3 \) denotes the corresponding complex coordinate (and thus the projection over the associated \( z \)-plane), the fields \( X^i \) are twisted by \( \exp(2\pi k_i/N) \) \( (k_1 = 1, k_2 = 1, k_3 = 2 \text{ and } N = 6 \text{ for the } Z_6 \text{-I orbifold}) \) and \( (f_2 - f_3 + \Lambda)_\perp \) selects only \( (f_2 - f_3 + \Lambda) \) shifts that are orthogonal to the invariant plane (this means that we can choose \( (f_2 - f_3)_{i=3} = 0 \) and \( \Lambda = \Lambda_{G_2 \times G_2} \)). Several comments are in order here. First it is clear that the \( i = 3 \) plane \( (i.e. \text{ the invariant plane}) \) does not contribute to the classical action, and the coupling for \( i = 3 \) behaves much as an untwisted one. In fact in the invariant plane the three strings must be attached to the same fixed point, \( i.e. \) \((f_1)_{i=3} = (f_2)_{i=3} = (f_3)_{i=3} \). These facts are
general for all the couplings, in any orbifold, when fixed tori are involved. Second, the $v$–coset in (30) does not depend on $f_1$ since in the calculation of $X_c(z)$, $z_1$ has been sent to infinity by using $SL(2, C)$ invariance. We call this the 2–3 picture (for more details see ref. [11]). Equation (30) can be expressed in the 1–2 and 1–3 pictures as well

$$S^{i}_{cl}(1-2) = \frac{1}{16\pi} \frac{|\sin(\pi k_i/N)|}{|\sin(2\pi k_i/N)| |\sin(3\pi k_i/N)|} |v_i^{(12)}|^2$$

$$(I - \theta^2)(f_1 - f_2 + \Lambda_{12})_\perp,$$ (31)

$$S^{i}_{cl}(1-3) = \frac{1}{16\pi} \frac{|\sin(\pi k_i/N)|}{|\sin(2\pi k_i/N)| |\sin(3\pi k_i/N)|} |v_i^{(13)}|^2$$

$$(I - \theta^3)(f_1 - f_3 + \Lambda_{13})_\perp,$$ (32)

where

$$\Lambda_{12} = (I + \theta + \theta^2)\Lambda + \omega, \quad \Lambda_{13} = (I + \theta)\Lambda + \omega,$$

$$\omega = (I + \theta + \theta^2)f_3 - (\theta + \theta^2)f_2 - f_1.$$ (33)

We can check by using the space group selection rule (18) that there is a one–to–one correspondence between $S_{cl}(1-2), S_{cl}(1-3), S_{cl}(2-3)$. The 2–3 picture, eq. (30) is the most convenient one since $\Lambda_{12}, \Lambda_{13}$ are subsets of the original lattice $\Lambda$. Furthermore $S_{cl}(1-2)$ and $S_{cl}(1-3)$ depend on the three fixed points considered $f_1, f_2, f_3$; while $S_{cl}(2-3)$ depends only on $f_2, f_3$. [2]

We can now write the complete form of the correlator using eqs. (29,30)

$$\langle \sigma_\theta \sigma_\theta \sigma_\theta \rangle = N \sqrt{l_2 l_3} \sum_{u \in \Lambda_{\perp}} \exp \left\{ -\frac{1}{2\pi} \sin(\frac{\pi}{3}) \left[ (f_{23} + u)^2 + (f_{23} + u)^2 \right] \right\},$$ (34)

where $(f_{23} + u)_i$ is the $i$–plane projection of $(f_2 - f_3 + u), \Lambda_{\perp} = \Lambda_{G_2 \times G_2}$, and $N$ is the properly normalized quantum part [13]

$$N = \sqrt{V_{\perp}} \frac{1}{2\pi} \frac{\Gamma(\frac{5}{3})\Gamma(\frac{2}{3})}{\Gamma(\frac{2}{3})\Gamma(\frac{1}{3})},$$ (35)

with $V_{\perp}$ the volume of the $G_2 \times G_2$ unit cell. General expressions for the couplings similar to eq. (34) can be found in ref. [13] for all the $Z_n$ orbifolds. We have performed the calculation in all the cases, checking that the results of the mentioned reference are correct.

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2This difference can be understood recalling that for $f_2, f_3$ given, the space group selection rule (18) determines $f_1$ uniquely, which does not hold for the other two possibilities.
Expression (34) is not explicit enough for most purposes. For example it does not allow examination of the transformation properties of the Yukawa couplings under target–space modular transformations (e.g. $R \rightarrow 1/R$). From a phenomenological point of view eq. (34) does not exhibit the dependence of the value of the coupling on physical quantities, i.e. those that parametrize the size and shape of the compactified space. In fact eq. (34) is not even good enough to calculate the final value of the coupling numerically, especially when deformations are considered. The key point in order to do this is to write $(f_{23} + u_i)$ in terms of $(e_1, \ldots, e_6)$, i.e. the lattice basis. This can be done with the help of the results of subsection 2.1, see eq. (5). Then the correlator (34) appears as an explicit function of the deformation parameters of the compactified space. Substituting the resulting expression in (26) we obtain the final Yukawa coupling, which can be written in a quite compact way

$$C_{\theta\theta_2\theta_3} = N \sqrt{l_2 l_3} \sum_{\vec{u} \in \mathbb{Z}^4} \exp \left[ -\frac{\sqrt{3}}{4\pi}(\vec{f}_{23} + \vec{u})^\top M(\vec{f}_{23} + \vec{u}) \right]$$

$$= \sqrt{V_\perp} \sqrt{l_2 l_3} \frac{\Gamma(\frac{2}{3}) \Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3}) \Gamma(\frac{1}{3})} \vartheta \left( \begin{array}{c} \vec{f}_{23} \\ 0 \end{array} \right) [0, \Omega] \quad [0, \Omega]$$

where $\vec{f}_{23}$ represents the first four components of $(f_2 - f_3)$ (i.e. those corresponding to the $G_2 \times G_2$ sublattice basis $(e_1, \ldots, e_4)$), and

$$\vartheta \left( \begin{array}{c} \vec{f}_{23} \\ 0 \end{array} \right) = \sum_{\vec{u} \in \mathbb{Z}^4} \exp \left[ i\pi(\vec{f}_{23} + \vec{u})^\top \Omega(\vec{f}_{23} + \vec{u}) \right] \quad \Omega = \frac{i\sqrt{3}}{4\pi^2} M$$

with

$$\Omega = \frac{i\sqrt{3}}{4\pi^2} \left( \begin{array}{ccc} R_1^2 & -\frac{3}{2}R_1^2 & R_1 R_3 \alpha_{13} \\ -\frac{3}{2}R_1^2 & 3R_1^2 & R_3(3\alpha_{13} + \sqrt{3}\alpha_{14}) \\ R_1 R_3 \alpha_{13} & -R_1 R_3(3\alpha_{13} + \sqrt{3}\alpha_{14}) & \sqrt{3}R_1 R_3 \alpha_{14} \\ \sqrt{3}R_1 R_3 \alpha_{13} & 3R_1 R_3 \alpha_{13} & \sqrt{3}R_1 R_3 \alpha_{14} \\ R_1^2 & -\frac{3}{2}R_1^2 & R_3^2 \\ \frac{3}{2}R_1^2 & 3R_3^2 & \frac{3}{2}R_3^2 \end{array} \right)$$

(38)

where the deformation parameters $R_i^2$, $\alpha_{ij}$ have been defined in eqs. (3, 4).

It is worthwhile to have a look at eq. (36) to realize which are the physical quantities on which the value of the coupling depends. First $C_{\theta\theta_2\theta_3}$ depends on the relative positions in the lattice of the relevant fixed points to which the physical fields are attached. This information is condensed in $\vec{f}_{23}$. Second $C_{\theta\theta_2\theta_3}$ depends on the size and shape of the compactified space, which is reflected in the orbifold compactification parameters $(R_i^2, \alpha_{13}, \alpha_{14})$ appearing in $\Omega$ and (implicitly) in $V_\perp$. Note that both pieces of information appear in a completely distinguishable way from each other in eq. (36). Notice also
that the deformation parameter $R^2_5$ does not appear in $\Omega$. This is due to the fact that $R_5$ parametrizes the size of the $i = 3$ sublattice, i.e. the fixed torus, and we have learnt that for $i = 3$ the coupling is equivalent to an untwisted one. This is a general fact for all the orbifold couplings in which fixed tori are involved (e.g. it does not occur for the $\theta^2\theta^2\theta^2$ coupling of the $Z_6$–I orbifold, see below). We say that $R_5$ is not an effective deformation parameter for the $\theta^2\theta^2\theta^2$ couplings. The number of effective deformation parameters (4 in this case) is physically relevant since it is strongly related to the number of different Yukawa couplings and their corresponding sizes.

For the other twisted coupling $\theta^2\theta^2\theta^2$ in the $Z_6$–I orbifold, the expression of the coupling can be calculated in the same way as in the $\theta\theta\theta$ case, and is given by

$$C_{\theta^2\theta^2\theta^2} = F(l_1, l_2, l_3) N \sum_{v \in (f_3 - f_2 + \Lambda)} \exp\left[-\frac{\sqrt{3}}{8\pi} |v|^2\right]$$

$$= F(l_1, l_2, l_3) N \sum_{\vec{u} \in \mathbb{Z}^6} \exp\left[-\frac{\sqrt{3}}{8\pi} (\vec{f}_{23} + \vec{u})^\top M(\vec{f}_{23} + \vec{u})\right]$$

$$= F(l_1, l_2, l_3) N \theta \left[ \begin{array}{c} \vec{f}_{23} \\ 0 \end{array} \right] [0, \Omega],$$

where $F = 1$ for $l_1 = 1$ or $l_2 = 1$ or $l_3 = 1$ and $F = \frac{1}{\sqrt{2}}$ for $l_1 = l_2 = l_3 = 2$. $l_i$ is the number of elements in the conjugation class associated with $f_i$, see eq. (12). $\vec{f}_{23}$ represents the components of $(f_2 - f_3)$ in the lattice basis $(e_1, ..., e_6)$. The global normalization factor and the $\Omega$ matrix are given by

$$N = \sqrt{V \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{3}{4})}}$$

$$\Omega = i \frac{\sqrt{3}}{8\pi^2} \left( \begin{array}{cccccc} a & -\frac{3}{2}a & b & c & 0 & 0 \\ -\frac{3}{2}a & 3a & -3b - c & 3b & 0 & 0 \\ b & -3b - c & d & -\frac{3}{2}d & 0 & 0 \\ c & 3b & -\frac{3}{2}d & 3d & 0 & 0 \\ 0 & 0 & 0 & 0 & e & -\frac{1}{2}e \\ 0 & 0 & 0 & 0 & -\frac{1}{2}e & e \end{array} \right)$$

Clearly the number of effective parameters is 5.

We have performed a similar analysis for all the trilinear twisted couplings in all the $Z_n$ orbifolds, giving the number of effective deformation parameters in each case. The results are expounded in Appendix 1.
2.5 Accidental symmetries and the number of different couplings

We are now ready to count the number of different couplings that appear in each orbifold. From the physical point of view this is one of the most relevant questions about a string construction, since it is directly related to the possibility of reproducing the observed pattern of fermion masses and mixing angles. Unfortunately, this task is probably the most tedious part of the work presented here. Again, we expound in some detail the analysis for the two possible couplings in the $Z_6$–I orbifold. Let us begin with the twisted coupling $\theta^2\theta'^3$.

The corresponding results for other orbifolds can be found in Appendix 1.

The first point is that the $\Omega$ matrix appearing in the Jacobi theta function of the coupling, eqs. (36–38), is universal for all the $\theta^2\theta'^3$ couplings. This means that the differences between the Yukawa couplings come exclusively from the sum in $(f_{23} + u)$ in the classical part of the correlation. Hence, two couplings

$$C \sim \vartheta \begin{bmatrix} f_{23} \\ 0 \end{bmatrix} \Omega \quad \text{and} \quad C' \sim \vartheta \begin{bmatrix} f_{23}' \\ 0 \end{bmatrix} \Omega$$

will have the same value if there exists an integer unimodular transformation $U$ (i.e. $U \in GL(4, \mathbb{Z})$, $|U| = \pm 1$) such that

$$U^\top \Omega U = \Omega \quad \text{(42)}$$
$$U f_{23} = f_{23}' + \vec{v}, \quad \vec{v} \in \mathbb{Z}^4 \quad \text{(43)}$$

Then, if the previous equations are true for some $U$, there is a one–to–one correspondence between the terms of the series defining $\vartheta \begin{bmatrix} f_{23} \\ 0 \end{bmatrix}$, see eq. (37), and those of $\vartheta \begin{bmatrix} f_{23}' \\ 0 \end{bmatrix}$. So we have to look for $U$–matrices satisfying (43). There is a set of $U$–matrices that always fulfil (43). These are $\{I, -I\}$ and $\{\theta^n, n \in \mathbb{Z}\}$. To check the latter, note that when the sum (37) is expressed in the complex orthogonal basis, the exponent takes a diagonal form

$$\sum_{\vec{u} \in \Lambda_{\perp}} \exp \left\{ a_1(f_{23} + u)^2_i + a_2(f_{23} + u)^2_2 \right\}$$

as can be seen from (34). Then the terms multiplying the coefficients $a_i$ are unchanged under $\theta^n$ twists, since these correspond to make rotations in each $i$–plane. This argument is always valid because the factorization (44) is a consequence of the fact that the classical contributions can be computed in each $i$–plane separately. In addition to the group generated by $\{-I, \theta\}$, there can be ”accidental $U$–symmetries” leaving $\Omega$ unchanged in
eq. (13). Some of these symmetries can be spontaneously broken when deformations are taken into account. After inspection it turns out that, for the case at hand, these accidental symmetries are generated by

\[ U_1 = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & 1 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}, \quad U_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \] (45)

(when expressed in the complex orthogonal basis) plus products of these matrices by \{-I, \theta\}. \( U_1, U_2, U_3 \) are broken when deformations are considered.

Now, two Yukawa couplings \( C, C' \) are equal (in the non–deformed case) if \( f_{23}^\prime \) and \( f_{23} \) are connected as in (13) by one of the \( U \)–matrices mentioned above. The analysis has to be performed for the \( 90 \theta^2 \theta^3 \) allowed couplings, see subsection 2.3. The result is that for the rigid \( G_2 \times G_2 \times SU(3) \) lattice (i.e. \( R_1^2 = R_2^2 = R_3^2 \), \( \alpha_{13} = \alpha_{14} = 0 \)) there are 10 different couplings, corresponding to the following set of \( f_{23}^\prime \) shifts (in \( G_2 \times G_2 \))

\[ l_3 = 1 \quad l_2 = 1 : \ (0,0) \otimes (0,0), \]
\[ l_3 = 3 \quad l_2 = 1 : \ (0,0) \otimes (0,\frac{1}{2}), \ (0,\frac{1}{2}) \otimes (0,\frac{1}{2}), \]
\[ l_3 = 1 \quad l_2 = 2 : \ (0,0) \otimes (0,\frac{1}{3}), \ (0,\frac{1}{3}) \otimes (0,\frac{1}{3}), \]
\[ l_3 = 3 \quad l_2 = 2 \begin{cases} (0,0) \otimes (0,\frac{1}{6}), & (0,\frac{1}{6}) \otimes (0,\frac{1}{6}), & (0,\frac{1}{6}) \otimes (0,\frac{1}{6}), \\ (0,\frac{1}{3}) \otimes (0,\frac{1}{3}), & (0,\frac{1}{3}) \otimes (0,\frac{1}{3}) \end{cases} \] (46)

The meaning of \( l_i \) and its influence in the couplings are given in eqs. (12),(20). With deformations the symmetry of the \( \Omega \) matrix is smaller, as explained above, and it turns out to be 30 different couplings

\[ l_3 = 1 \quad l_2 = 1 : \ (0,0) \otimes (0,0), \]
\[ l_3 = 1 \quad l_2 = 2 : \ (0,0) \otimes (0,\frac{1}{3}), \ (0,\frac{1}{3}) \otimes (0,0), \ (0,\frac{1}{3}) \otimes (0,\frac{1}{3}), \ (0,\frac{1}{3}) \otimes (0,\frac{2}{3}), \]
\[ l_3 = 3 \quad l_2 = 1 \begin{cases} (0,0) \otimes (0,\frac{1}{2}), & (0,\frac{1}{2}) \otimes (0,0), & (0,\frac{1}{2}) \otimes (0,\frac{1}{2}), \\ (0,\frac{1}{3}) \otimes (\frac{1}{2},\frac{1}{2}), & (0,\frac{1}{3}) \otimes (\frac{1}{2},\frac{1}{2}), \end{cases} \]
\[ l_3 = 3 \quad l_2 = 2 \begin{cases} (0,0) \otimes (0,\frac{1}{6}), & (0,\frac{1}{6}) \otimes (0,0), & (0,\frac{1}{6}) \otimes (0,\frac{1}{6}), & (0,\frac{1}{6}) \otimes (0,\frac{5}{6}), \\ (0,\frac{1}{3}) \otimes (\frac{1}{2},\frac{1}{2}), & (0,\frac{1}{3}) \otimes (\frac{1}{2},\frac{1}{2}), & (0,\frac{1}{3}) \otimes (\frac{1}{2},\frac{1}{2}), & (0,\frac{1}{3}) \otimes (\frac{1}{2},\frac{1}{2}), \end{cases} \] (47)

The absolute and relative size of these 30 couplings obviously depend on the value of the deformation parameters, as reflected in eqs. (36),(38).

Performing an analysis similar to the \( \theta^2 \theta^3 \) case, we find out the number of inequivalent shifts for the \( \theta^2 \theta^2 \theta^3 \) coupling. For the non–deformed case there are 8 different couplings, namely
\[ f_{23} = \begin{cases} l_1 = 1 \text{ or } l_2 = 1 \text{ or } l_3 = 1 & \begin{cases} g_2^{(0)} \otimes g_2^{(0)} \otimes \hat{g}_2^{(0)}, & g_2^{(0)} \otimes \hat{g}_2^{(1)}, & g_2^{(0)} \otimes \hat{g}_2^{(0)}; \\
 g_2^{(0)} \otimes \hat{g}_2^{(1)} \otimes \hat{g}_2^{(0)}, & \hat{g}_2^{(0)} \otimes \hat{g}_2^{(1)} \otimes \hat{g}_2^{(0)}, & g_2^{(1)} \otimes \hat{g}_2^{(1)} \otimes \hat{g}_2^{(1)}. \end{cases} \\
 l_1 = l_2 = l_3 = 2 & \begin{cases} g_2^{(0)} \otimes g_2^{(0)} \otimes \hat{g}_2^{(0)}, & g_2^{(0)} \otimes g_2^{(0)} \otimes \hat{g}_2^{(1)}. \end{cases} \end{cases} \]

(48)

For deformations the number is increased to 12 different couplings given by the following shifts

\[ f_{23} = \begin{cases} l_1 = 1 \text{ or } l_2 = 1 \text{ or } l_3 = 1 & \begin{cases} g_2^{(0)} \otimes g_2^{(0)} \otimes \hat{g}_2^{(0)}, & g_2^{(0)} \otimes \hat{g}_2^{(1)}, & g_2^{(0)} \otimes \hat{g}_2^{(0)}; \\
 g_2^{(0)} \otimes \hat{g}_2^{(1)} \otimes \hat{g}_2^{(0)}, & \hat{g}_2^{(0)} \otimes \hat{g}_2^{(1)} \otimes \hat{g}_2^{(0)}, & g_2^{(1)} \otimes \hat{g}_2^{(1)} \otimes \hat{g}_2^{(1)}. \end{cases} \\
 l_1 = l_2 = l_3 = 2 & \begin{cases} g_2^{(0)} \otimes g_2^{(0)} \otimes \hat{g}_2^{(0)}, & g_2^{(0)} \otimes g_2^{(0)} \otimes \hat{g}_2^{(1)}. \end{cases} \end{cases} \]

(49)

We have performed a similar analysis for all the \( Z_n \) orbifolds, the results are in Appendix 1. In all cases we have checked by computer that the number of different Yukawa couplings is correct.

3 A comparative study of the \([SO(4)]^3\) and \([SU(4)]^2\) \( \mathbb{Z}_4 \) orbifolds

Although most of the aspects of orbifold Yukawa couplings have been adequately illustrated in the previous section by the \( Z_6 \)–I example, there are still some interesting features that can be exhibited in the framework of a \( \mathbb{Z}_4 \) orbifold. In particular we will see the physical meaning of a \((1,2)\) modulus (absent in the \( Z_6 \)–I case) and its effect in the Yukawa coupling values. Furthermore, the comparison of the Yukawa couplings of a \( \mathbb{Z}_4 \) orbifold, formulated in an \([SO(4)]^3\) lattice, with those of a \( \mathbb{Z}_4 \) orbifold, formulated in an \([SU(4)]^2\) lattice, will show us which properties of the couplings depend on the chosen lattice and which do not. Moreover, the \([SO(4)]^3\) case provides an example of a non–Coxeter orbifold. The twist of a \( \mathbb{Z}_4 \) orbifold in an orthogonal complex basis has the form (see Table 1)

\[
\theta = \text{diag}(e^{i\alpha}, e^{i\alpha}, e^{-2i\alpha}), \quad \alpha = \frac{2\pi}{4}.
\]

(50)

Again, the lattice \( \Lambda \) can get deformations compatible with the twist \( \theta \). These degrees of freedom correspond to the Hermitian part of the five \((1,1)\) moduli surviving compactification, \( N_{11}, \ N_{22}, \ N_{33}, \ N_{12}, \ N_{21} \) with \( N_{ij} = |i > R \otimes \alpha_{jL}^{-1}|0 > L, \) and the \((1,2)\) modulus
$N_{33} = |3 >_R \otimes \alpha_{3L}^{-1}|0 >_L$. (Notice that no $N_{ij}$ moduli appeared in the $Z_6$–I case.) Un-twisted moduli can be easily expressed in terms of $g_{mn}$, $b_{mn}$ (m, n = 1,...,6), i.e. the internal metric and torsion respectively. It is easy to check, however, that $N_{33}$ contains only $g_{55}$ degrees of freedom, more precisely $(g_{55} - g_{66})$ and $g_{56}$. Therefore both Re($N_{33}$) and Im($N_{33}$) correspond to deformation parameters. In order to see what these parameters are, let us choose first an $[SO(4)]^3$ root lattice, with basis $(e_1,...,e_6)$, as a lattice on which the twist $\theta$, see eq. (50), acts crystallographically as

$$\begin{align*}
\theta e_1 &= e_2, \quad \theta e_3 = e_4, \quad \theta e_5 = -e_5, \\
\theta e_2 &= -e_1, \quad \theta e_4 = -e_3, \quad \theta e_6 = -e_6.
\end{align*}$$

Then, as in subsection 2.1, $P$ invariance impose the following relations

$$\begin{align*}
|e_1| &= |e_2|, & |e_3| &= |e_4|, \\
\alpha_{ij} &= 0 & i = 1, 2, 3, 4 & j = 5, 6 & \alpha_{14} &= -\alpha_{23}, \\
\alpha_{12} &= \alpha_{34} = 0, & \alpha_{13} &= \alpha_{24}.
\end{align*}$$

where $\alpha_{ij} = \cos \theta_{ij}$ and $e_i e_j = |e_i| |e_j| \cos \theta_{ij}$. Therefore we can take the seven deformation degrees of freedom as

$$\begin{align*}
R_i &= |e_i| & i = 1, 3, 5, 6, \\
\alpha_{13}, \alpha_{14}, \alpha_{56}.
\end{align*}$$

Now it is easy to see that the two deformation parameters coming from $N_{33}$ correspond to a variation of the relative size of $|e_5|$ and $|e_6|$ and to the $\theta_{56}$ angle; thus allowing for a rhomboidal-like lattice from the original third $SO(4)$ sublattice. It is remarkable however that, as will be seen shortly, the deformation parameters coming from $N_{33}$ are not involved in the Yukawa couplings.

Let us briefly summarize the main results of the $[SO(4)]^3 Z_4$ orbifold. They have been obtained performing an analysis similar to that followed in the previous section for the $Z_6$–I one. There are 16 fixed points under $\theta$ in this orbifold, given by

$$f^{(ijk)}_1 = g^{(i)}_1 \otimes g^{(j)}_1 \otimes g^{(k)}_1; \quad i, j = 0, 2; \quad k = 0, 1, 2, 3$$

where

$$g^{(0)}_1 = (0, 0), \quad g^{(1)}_1 = (\frac{1}{2}, 0), \quad g^{(2)}_1 = (\frac{1}{2}, \frac{1}{2}), \quad g^{(3)}_1 = (0, \frac{1}{2}).$$

Under $\theta$ each fixed point is associated with a conjugation class in a one–to–one correspondence. Under $\theta^2$ there are 16 fixed tori that are the product of 16 fixed points in the
sublattice \((e_1, e_2, e_3, e_4)\) by the 2–torus defined by the sublattice \((e_5, e_6)\) (with or without deformations). Six of these 16 fixed points are connected to the others through \(\theta\) rotations. The fixed points are (in the first two \(SO(4)\)’s)

\[
f^{(ij)}_2 = g^{(i)}_2 \otimes g^{(j)}_2; \quad i, j = 0, 1, 2, 3
\]

with \(g^{(i)}_2 = g^{(i)}_1\). And we can see that

\[
g^{(0)}_2 \otimes g^{(1)}_2 \sim g^{(0)}_2 \otimes g^{(3)}_2, \quad g^{(1)}_2 \otimes g^{(0)}_2 \sim g^{(3)}_2 \otimes g^{(0)}_2,
\]

\[
g^{(2)}_2 \otimes g^{(1)}_2 \sim g^{(2)}_2 \otimes g^{(3)}_2, \quad g^{(1)}_2 \otimes g^{(2)}_2 \sim g^{(3)}_2 \otimes g^{(2)}_2,
\]

\[
g^{(1)}_2 \otimes g^{(2)}_2 \sim g^{(3)}_2 \otimes g^{(3)}_2, \quad g^{(1)}_2 \otimes g^{(3)}_2 \sim g^{(3)}_2 \otimes g^{(2)}_2.
\]

Note that in the two first \(SO(4)\)’s \(g^{(0)}_2\) and \(g^{(2)}_2\) are fixed points under \(\theta\) while \(\theta g^{(1)}_2 \rightarrow g^{(3)}_2\). Consequently there are 10 \(\theta^2\) conjugation classes and, as was explained in subsection 2.3, only symmetric combinations of fixed points (for the conjugation classes with more than one fixed point) take part in the Yukawa couplings. For this orbifold all the twisted couplings are of the \(\theta\theta\theta^2\) type and the selection rule reads

\[
f_1 + f_2 - (I + \theta)f_3 \in \Lambda,
\]

where \(f_3\) is the \(\theta^2\) fixed point. Denoting the fixed points by

\[
\begin{align*}
f_1 &= g_1^{(i_1)} \otimes g_1^{(j_1)} \otimes g_1^{(k_1)}, \quad &i_1, i_2, j_1, j_2 = 0, 2, \\
f_2 &= g_1^{(i_2)} \otimes g_1^{(j_2)} \otimes g_1^{(k_2)}, \quad &k_1, k_2, i_3, j_3 = 0, 1, 2, 3 \\
f_3 &= g_2^{(i_3)} \otimes g_2^{(j_3)} \otimes [\alpha(e_5) + \beta(e_6)] \quad &\alpha, \beta \in R,
\end{align*}
\]

see eqs. \((54-55)\), the selection rule is simply

\[
\begin{align*}
i_1 + i_2 + 2i_3 &= 0 \\
j_1 + j_2 + 2j_3 &= 0 \\
k_1 &= k_2
\end{align*} \mod 4.
\]

The number of allowed couplings is 160. It is clear now that the third \(SO(4)\) lattice always enters in the couplings as the fixed torus associated with the \(\theta^2\) field. Then the coupling in this invariant plane is of the untwisted type and, consequently, the deformation parameters for the third \(SO(4)\) sublattice (i.e. \(R_5, R_6, \alpha_{56}\); see eq. \((53)\)) do not affect the value of the coupling. Two of these parameters are precisely those coming from \(N_{33}\). Remarkably enough we have checked that this is a general property for all the orbifolds: (1,2) moduli are not involved in the expressions of the Yukawa couplings. It looks as though there is a selection rule (unknown to us) forbidding this kind of dependences.
For the case where \( f_3 \) is also a \( \theta \) fixed point, the value of the coupling in the 2–3 picture is

\[
C_{\theta \theta \theta}^2 = N \sum_{v \in (f_2 - f_3 + \Lambda)_\perp} \exp \left[ -\frac{1}{4\pi} (|v_1|^2 + |v_2|^2) \right]
= N \sum_{v \in (f_2 - f_3 + \Lambda)_\perp} \exp \left[ -\frac{1}{4\pi} \vec{v}^\top M \vec{v} \right]
= N \vartheta \begin{bmatrix} \vec{f}_{23} \\ 0 \end{bmatrix} [0, \Omega],
\]

where \((f_2 - f_3 + \Lambda)_\perp\) selects only \((f_2 - f_3 + \Lambda)\) shifts that are orthogonal to the invariant sublattice, \textit{i.e.} the third \(SO(4)\) lattice. Thus, \((f_2 - f_3 + \Lambda)_\perp\) has non-zero components in the first two \(SO(4)'s\) only. Similarly \(\vec{f}_{23}\) represents the four components of \((f_2 - f_3)\) in the basis \((e_1, ..., e_4)\) of the first \(SO(4)\) lattices. Finally

\[
N = \sqrt{V_\perp \frac{r^2(\frac{3}{4})}{r^2(\frac{1}{4})}}
\]

\[
M = (-4\pi^2 i)\Omega = \begin{pmatrix} R_1^2 & 0 & R_1 R_3 \alpha_{13} & R_1 R_3 \alpha_{14} \\ 0 & R_1^2 & -R_1 R_3 \alpha_{14} & R_1 R_3 \alpha_{13} \\ R_1 R_3 \alpha_{13} & -R_1 R_3 \alpha_{14} & R_3^2 & 0 \\ R_1 R_3 \alpha_{14} & R_1 R_3 \alpha_{13} & 0 & R_3^2 \end{pmatrix}
\]

where \(V_\perp\) is the volume of the unit cell of the first two \(SO(4) \times SO(4)\) sublattice orthogonal to the invariant plane.

If \(f_3\) is not fixed by \(\theta\), see eq. \((60)\), the result is exactly the same but multiplying \(C_{\theta \theta \theta}^2\) by \(\sqrt{2}\). Clearly the number of effective deformation parameters is 4. The number of \textit{different} Yukawa couplings, from the 160 allowed ones, is 6 (without deformations), corresponding to

\[
\vec{f}_{23} = \begin{cases} \text{l}_3 = 1 & : & g_1^{(0)} \otimes g_1^{(0)}, & g_1^{(2)} \otimes g_1^{(0)}, & g_1^{(2)} \otimes g_1^{(2)}, \\ \text{l}_3 = 2 & : & g_2^{(0)} \otimes g_2^{(1)}, & g_2^{(2)} \otimes g_2^{(1)}, & g_2^{(1)} \otimes g_2^{(1)} \end{cases}
\]

and 10 (when deformations are considered), namely

\[
\vec{f}_{23} = \begin{cases} \text{l}_3 = 1 & : & g_1^{(0)} \otimes g_1^{(0)}, & g_1^{(2)} \otimes g_1^{(0)}, & g_1^{(0)} \otimes g_1^{(2)}, & g_1^{(2)} \otimes g_1^{(2)}, \\ \text{l}_3 = 2 & : & g_2^{(0)} \otimes g_2^{(1)}, & g_2^{(2)} \otimes g_2^{(1)}, & g_2^{(1)} \otimes g_2^{(1)}, & g_2^{(1)} \otimes g_2^{(0)} \end{cases}
\]
We would like to compare all the previous results with those of the Z₄ orbifold based on a Coxeter twist acting on an [SU(4)]² root lattice. This will illustrate what aspects of the orbifold dynamics are independent of the chosen lattice and what aspects do not. Furthermore, for the [SU(4)]² Z₄ orbifold, the lattice cannot be decomposed as the direct product of an invariant sublattice under θ² times an orthogonal sublattice, as happened in the [SO(4)]³ case. This peculiarity, which is shared by other orbifolds, introduces some additional complications which we would like to show. The Coxeter element in the [SU(4)]² root lattice is of the form

\[ \theta e_1 = e_2, \quad \theta e_2 = e_3, \quad \theta e_3 = -e_1 - e_2 - e_3, \]
\[ \theta e_4 = e_5, \quad \theta e_5 = e_6, \quad \theta e_6 = -e_4 - e_5 - e_6. \] (63)

The 7 deformation parameters coming from \((N_{11}, N_{22}, N_{33}, N_{12}, N_{21}, N_{33})\) are

\[ R_i = |e_i|, \quad i = 1, 4 \]
\[ \alpha_{12}, \alpha_{14}, \alpha_{15}, \alpha_{16}, \alpha_{45}, \] (64)

where \((e_1, e_2, e_3)\) is the basis of the first SU(4), and \((e_4, e_5, e_6)\) the basis of the second one. Equation (64) should be compared with eq. (53), i.e. its analogue in the [SO(4)]³ case. Clearly the geometrical interpretation of the deformation parameters is different for each one. Other parameters of the SU(4)² lattice are related to the previous ones by

\[ |e_1| = |e_2| = |e_3|, \quad |e_4| = |e_5| = |e_6|, \]
\[ \alpha_{23} = \alpha_{12}, \quad \alpha_{34} = \alpha_{16}, \]
\[ \alpha_{13} = -1 - 2\alpha_{12}, \quad \alpha_{24} = \alpha_{35} = -\alpha_{14} - \alpha_{15} - \alpha_{16}, \]
\[ \alpha_{25} = \alpha_{36} = \alpha_{14}, \quad \alpha_{56} = \alpha_{45}, \]
\[ \alpha_{26} = \alpha_{15}, \quad \alpha_{46} = -1 - 2\alpha_{45}. \] (65)

It is important to point out that the [SU(4)]² lattice cannot be consistently deformed into an [SO(4)]³ one. To see this, note that the invariant sublattice under the action of the Coxeter element (63) is generated by \((e_1 + e_3)\) and \((e_4 + e_6)\). If such a deformation existed, these vectors could be identified with the basis of the invariant SO(4) sublattice in the [SO(4)]³ case. Now, it can be shown that we cannot construct a basis of [SU(4)]² with \((e_1 + e_3)\), \((e_4 + e_6)\) and four additional lattice vectors orthogonal to these (with or without deformations). In fact, it is easy to check that the [SU(4)]² Coxeter element (63) has the same form as the twist θ of the [SO(4)]³ case, i.e. eq. (53), when acting in the following set of lattice vectors

\[ \tilde{e}_1 = e_1 + e_2, \quad \tilde{e}_3 = e_1 + e_3, \quad \tilde{e}_5 = e_5 + e_6, \]
\[ \tilde{e}_2 = e_2 + e_3, \quad \tilde{e}_4 = e_4 + e_5, \quad \tilde{e}_6 = e_4 + e_6. \] (66)
Notice that when deformations are included, \((\tilde{e}_1, \tilde{e}_2, \tilde{e}_4, \tilde{e}_5)\) remain orthogonal to \((\tilde{e}_3, \tilde{e}_6)\). Actually, \((\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4, \tilde{e}_5, \tilde{e}_6)\) generate an \([SO(4)]^3\) sublattice of the \([SU(4)]^2\) lattice but they are not a basis of the whole lattice. Anyway the \(\tilde{e}_i\) will be of help below. The number of fixed points is the same in both cases. For the case at hand, \([SU(4)]^2\), there are 16 fixed points under \(\theta\) which can be expressed as

\[ f_1^{(ij)} = g_1^{(i)} \otimes g_1^{(j)} ; \quad i, j = 0, 1, 2, 3 \]  

(67)

with

\[ g_1^{(0)} = (0, 0, 0), \quad g_1^{(1)} = (\frac{1}{2}, \frac{1}{2}, \frac{3}{2}), \quad g_1^{(2)} = (\frac{1}{2}, 0, \frac{1}{2}), \quad g_1^{(3)} = (\frac{3}{2}, \frac{1}{2}, \frac{1}{2}). \]

Under \(\theta^2\) there is a fixed torus generated by \((e_1 + e_3)\) and \((e_4 + e_6)\). Then we can form 16 fixed tori as products of this fixed torus by the following 16 \(\theta^2\) fixed points (six of them connected to the others by \(\theta\) rotations)

\[
\begin{align*}
&g_2^{(0)} \otimes g_2^{(0)}, \quad g_2^{(2)} \otimes g_2^{(2)}, \quad g_2^{(0)} \otimes g_2^{(2)}, \quad g_2^{(2)} \otimes g_2^{(0)}, \\
&g_2^{(0)} \otimes g_2^{(1)} \sim g_2^{(0)} \otimes g_2^{(3)}, \quad g_2^{(2)} \otimes g_2^{(1)} \sim g_2^{(2)} \otimes g_2^{(3)}, \\
&g_2^{(1)} \otimes g_2^{(0)} \sim g_2^{(3)} \otimes g_2^{(0)}, \quad g_2^{(1)} \otimes g_2^{(2)} \sim g_2^{(3)} \otimes g_2^{(2)}, \\
&g_2^{(1)} \otimes g_2^{(1)} \sim g_2^{(3)} \otimes g_2^{(3)}, \quad g_2^{(3)} \otimes g_2^{(1)} \sim g_2^{(3)} \otimes g_2^{(1)}
\end{align*}
\]  

(68)

where

\[ g_2^{(0)} = (0, 0, 0), \quad g_2^{(1)} = \frac{1}{2}(1, 1, 0), \quad g_2^{(2)} = \frac{1}{2}(1, 0, 1), \quad g_2^{(3)} = \frac{1}{2}(0, 1, 1). \]

Note that \(g_2^{(0)}\) and \(g_2^{(2)}\) are fixed under \(\theta\) but \(g_2^{(1)}\) and \(g_2^{(3)}\) are connected by a \(\theta\) rotation, \(\theta g_2^{(1)} \rightarrow g_2^{(3)}\). As in the \([SO(4)]^3\) case there are 10 conjugation classes. The space group selection rule also has the same form

\[ f_1 + f_2 - (I + \theta)f_3 \in \Lambda. \]  

(69)

Denoting the fixed points by

\[
\begin{align*}
f_1 &= g_1^{(i_1)} \otimes g_1^{(j_1)}, \quad f_2 = g_1^{(i_2)} \otimes g_1^{(j_2)}, \quad f_3 = [g_1^{(i_3)} \otimes g_1^{(j_3)}] \otimes [\alpha(e_1 + e_3) + \beta(e_4 + e_6)], \\
i_1, i_2, i_3, j_1, j_2, j_3 &= 0, 1, 2, 3, \quad \alpha, \beta \in \mathbb{R}
\end{align*}
\]  

(70)

eq. (69) can be expressed as

\[
\begin{align*}
i_1 + i_2 + 2i_3 &= 0 \\
j_1 + j_2 + 2j_3 &= 0
\end{align*}
\]

mod. 4.  

(71)

(Note that in spite of eqs. (69), (71) being formally identical with (56), (58) the meaning of the vectors implicitly involved is quite different.) The number of allowed couplings is again the same, 160. The Yukawa coupling, if \(f_3\) is fixed by \(\theta\), is given by
\( C_{\theta \theta \theta} = \bar{N} \sum_{v \in (f_2-f_3+\Lambda)_{\perp}} \exp[-\frac{1}{4\pi} \bar{v}^T \bar{M} \bar{v}]. \) (72)

As usual, the arrows denote components in the lattice basis \((e_1, ... e_6)\). The subscript \(\perp\) means that only \(v\) shifts orthogonal to the invariant plane (defined by \((e_1 + e_3)\) and \((e_4 + e_6)\)) have to be considered. If \(f_3\) is not fixed by \(\theta\) the previous expression has to be multiplied by a \(\sqrt{2}\) factor. \(\bar{N}\) and \(\bar{M}\) are given by

\[
\bar{N} = \sqrt{V} \frac{1}{2\pi} \frac{\Gamma^2(\frac{3}{4})}{\Gamma^2(\frac{1}{4})}
\]

\[
\bar{M} = \begin{pmatrix}
  a & b & -a - 2b & e & f & g \\
  b & a & b & -e - f - g & e & f \\
  -a - 2b & b & a & g & -e - f - g & e \\
  e & -e - f - g & g & c & d & -c - 2d \\
  f & e & -e - f - g & d & c & d \\
  g & f & e & -c - 2d & d & c
\end{pmatrix}
\]

\[a = R_1^2 \quad b = R_1^2 \alpha_{12} \quad c = R_4^2 \quad d = R_4^2 \alpha_{45} \quad e = R_1 R_4 \alpha_{14} \quad f = R_1 R_4 \alpha_{15} \quad g = R_1 R_4 \alpha_{16}.\] (73)

where \(V_{\perp}\) is the volume of the sublattice orthogonal to the invariant plane (see below).

By addition of lattice vectors we can always choose \(f_2\) and \(f_3\) in (72) such that \(f_2 - f_3\) is orthogonal to the invariant plane. Then \(f_2 - f_3\) can be expressed in the ”basis” (66) as

\[f_2 - f_3 = x_1 \hat{e}_1 + x_2 \hat{e}_2 + x_3 \hat{e}_4 + x_5 \hat{e}_5.\] (74)

We can check that \(x_i = 0, \frac{1}{2}\) (up to lattice vectors) for all the choices of \(f_2, f_3\). However it is amusing to see that many of the possibilities are in fact equivalent. Consider, for definiteness, the case \(f_2 - f_3 = 0, i.e. x_i = 0\) in (74). Now we can add to \(f_2 - f_3\) any shift contained in the invariant plane,

\[f_2 - f_3 = \alpha (e_1 + e_3) + \beta (e_4 + e_6), \quad \alpha, \beta \in R.\] (75)

Demanding \(v = f_2 - f_3 + \Lambda\) to be orthogonal to the invariant plane we find constraints for \(\alpha\) and \(\beta\). A shift \(\sum_i a_i e_i\) is orthogonal to the invariant plane if it satisfies the condition \(a_1 - a_2 + a_3 = 0\) and \(a_4 - a_5 + a_6 = 0\) (with or without deformations). Then

\[v = \alpha (e_1 + e_3) + \beta (e_4 + e_6) + \sum_{i=1}^{6} n_i e_i, \quad n_i \in Z\] (76)
is orthogonal if
\[(\alpha, \beta) = (0, 0), \ (0, \frac{1}{2}), \ (\frac{1}{2}, 0), \ (\frac{1}{2}, \frac{1}{2}),\] (77)
up to lattice vectors. Then \(v\) can be expressed as
\[v = (n_1 + \alpha)\vec{e}_1 + (n_3 + \alpha)\vec{e}_2 + (n_4 + \beta)\vec{e}_4 + (n_6 + \beta)\vec{e}_5.\] (78)
Therefore we have to sum up four possibilities for \((x_1, x_2, x_4, x_5)\), namely
\[(0, 0, 0, 0), \ (\frac{1}{2}, \frac{1}{2}, 0, 0), \ (0, 0, 1, 0), \ (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}).\] (79)
This is characteristic of the lattices that cannot be decomposed as the direct product of an invariant sublattice times an orthogonal sublattice. In particular it did not happen in the \([SO(4)]^3\) lattice. In order to write the coupling we have to add to each case in (79) lattice vectors orthogonal to the invariant plane, \textit{i.e.} of the form
\[u_\perp = n_1\vec{e}_1 + n_2\vec{e}_2 + n_4\vec{e}_4 + n_5\vec{e}_5,\] (80)
as is reflected in (78). Now we can express the coupling (72), which contained a \(6\ \times\ 6\ M\) matrix, as a sum of four \(\vartheta\) functions defined in the four-dimensional lattice \((\vec{e}_1, \vec{e}_2, \vec{e}_4, \vec{e}_5)\)
\[C_{\theta\theta\vartheta} = N \sum_{f_23} \sum_{\vec{e} \in (f_23 + \Lambda_\perp)} \exp[-\frac{1}{4\pi} \vec{v}^\top M' \vec{v}] = N \sum_{f_23} \vartheta \left[ \begin{array}{c} \vec{f}_{23} \\ 0 \end{array} \right] [0, \Omega'] \] (81)
where \(\vec{v}\) and \(\vec{f}_{23}\) are the components in \((\vec{e}_1, \vec{e}_2, \vec{e}_4, \vec{e}_5)\) of \(v\) and \((f_2 - f_3)\) respectively. \(\vec{f}_{23}\) runs over the possibilities displayed in (79), and \(\Omega'\) is given by
\[
\Omega' = i\frac{1}{4\pi^2} M' = i\frac{1}{4\pi^2} \left( \begin{array}{ccc} a & 0 & \bar{b} \\ 0 & a & -\bar{c} \\ \bar{b} & -\bar{c} & \bar{d} \end{array} \right) \begin{array}{c} \bar{a} \\ \bar{b} \\ \bar{c} \end{array} = 2R_1^2(1 + \alpha_{12}) \begin{array}{c} \bar{a} \\ \bar{b} \\ \bar{c} \end{array} = R_1R_4(\alpha_{14} - \alpha_{16}) \begin{array}{c} \bar{a} \\ \bar{b} \\ \bar{c} \end{array} = R_1R_4(\alpha_{14} + 2\alpha_{15} + \alpha_{16}) \begin{array}{c} \bar{a} \\ \bar{b} \\ \bar{c} \end{array} = 2R_4^2(1 + \alpha_{45}). \] (82)
Note that there are 4 effective deformation parameters, as in the \([SO(4)]^3\) case. Besides (74), there are three other inequivalent possibilities for \(f_3 - f_2\), namely
\[
\{ (0, 0, 0, \frac{1}{2}), \ (0, 0, \frac{1}{2}, 0), \ (\frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}), \ (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0) \}, \{ (0, \frac{1}{2}, 0, 0), \ (0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), \ (\frac{1}{2}, 0, 0, 0), \ (\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}) \}, \{ (0, \frac{1}{2}, 0, \frac{1}{2}), \ (0, \frac{1}{2}, \frac{1}{2}, 0), \ (\frac{1}{2}, 0, 0, \frac{1}{2}), \ (\frac{1}{2}, 0, \frac{1}{2}, 0) \}. \] (83)
Taking into account that the coupling gets a factor $\sqrt{2}$ if $f_3$ is not fixed by $\theta$, this gives 8 different Yukawa couplings when deformations are considered, and 6 without deformations (the two first possibilities in (83) are equal). This differs from the $[SO(4)]^3$ case, where there were 10 and 6 respectively. Note that the matrix $\Omega'$, eq. (82), appearing in the coupling is formally identical to that of $[SO(4)]^3$, eq. (60). However, as we have seen, the structure of possible shifts is very different. In any case the number of effective deformation parameters is the same for both cases.

4 Conclusions

We have calculated the complete twisted Yukawa couplings for all the $Z_n$ orbifold constructions in the most general case, i.e. when deformations of the compactified space are considered. This includes a certain number of tasks. Namely, determination of the allowed couplings, calculation of the explicit dependence of the Yukawa couplings values on the moduli expectation values (i.e. the parameters determining the size and shape of the compactified space), etc. Some progress in this direction has recently been made but without arriving at such explicit expressions as those given in this paper. This is an essential ingredient in order to relate theory and observation. In particular it allows a counting of the different Yukawa couplings for each orbifold (with and without deformations), which is crucial to determine the phenomenological viability of the different schemes, since it is directly related to the fermion mass hierarchy. In this sense some orbifolds (e.g. $Z_3$, $Z_4$, $Z_6$–I, $Z_8$–I, $Z_{12}$–I) have much better phenomenological prospects than others (e.g. $Z_7$, $Z_6$–II, $Z_8$–II, $Z_{12}$–II). The results for the whole set of Coxeter orbifolds are summarized in Table 1. Other facts concerning the phenomenological profile of $Z_n$ orbifolds are also discussed, e.g. the existence of non–diagonal entries in the fermion mass matrices, which is related to a non–trivial structure of the Kobayashi–Maskawa matrix. In this sense non–prime orbifolds are favoured over prime ones which do not have off–diagonal entries in the mass matrices at this fundamental level.

The results of this paper give the precise form in which moduli fields are coupled to twisted matter. This is essential in order to study in detail other important issues. Namely, the supersymmetry breaking mechanism by gaugino condensation (in which the moduli develop an additional non–perturbative superpotential), and cosmological implications (note that the moduli are also coupled to gravity in a Jordan-Brans-Dicke–like way). The level of explicitness given in the paper is also necessary for more theoretical matters (e.g. the study of the transformation properties of the Yukawa couplings under target–space modular transformations like $R \to 1/R$). Concerning the last aspect we
have found some appealing results, such as the fact that (1,2) moduli never appear in the expressions of the Yukawa couplings. Likewise, (1,1) moduli associated with fixed tori which are involved in the Yukawa coupling, do not affect the value of the coupling. It is worth noticing that the above mentioned moduli are precisely the only ones which contribute to the string loop corrections to gauge coupling constants \[^{22}\].

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APPENDIX 1

We follow a notation as compact as possible. The precise meaning of all the concepts appearing here is explained in detail in the text for the $Z_6$ and $Z_4$ examples.

**ORBIFOLD $Z_3$**

**Twist**\[ \theta = \text{diag}(e^{i\alpha}, e^{i\alpha}, e^{-2i\alpha}), \quad \alpha = \frac{2\pi}{3} \]

**Lattice** \[ [SU(3)]^3 \]

**Coxeter element**
\[ \theta e_i = e_{i+1}, \quad \theta e_{i+1} = -e_i - e_{i+1}, \quad i = 1, 3, 5 \]

**Deformation parameters**

**Relations**
\[ |e_i|^2 = |e_{i+1}|^2, \quad \alpha_{i,i+1} = -\frac{1}{2}, \quad \alpha_{i,j} = \alpha_{i+1,j+1}, \]
\[ \alpha_{i,j} + \alpha_{i,j+1} + \alpha_{i+1,j} = 0, \quad i, j = 1, 3, 5 \quad i < j \]
\[ \alpha_{ij} \equiv \cos(\theta_{ij}) \]

**Degrees of freedom (9)**
\[ R_i = |e_i|, \quad \alpha_{i,j}, \quad \alpha_{i,j+1}, \quad i, j = 1, 3, 5 \quad i < j \]

**Lattice basis ($e_i$) in terms of orthogonal basis ($\tilde{e}_i$)**

Not necessary in this case.

**Fixed points of $\theta$ (27)**
\[ f_1^{(ijk)} = g_1^{(i)} \otimes g_1^{(j)} \otimes g_1^{(k)}, \quad i, j, k = 0, 1, 2, \]
\[ g_1^{(0)} = (0, 0), \quad g_1^{(1)} = \left( \frac{1}{3}, \frac{2}{3} \right), \quad g_1^{(2)} = \left( \frac{2}{3}, \frac{1}{3} \right) \]

**Coupling $\theta \theta \theta$**

**Selection rule**
\[ f_1 + f_2 + f_3 \in \Lambda \]

**Denoting**
\[ f_1 = g_1^{(i_1)} \otimes g_1^{(j_1)} \otimes g_1^{(k_1)}, \quad i_1, i_2, i_3 = 0, 1, 2 \]
\[ f_2 = g_1^{(j_2)} \otimes g_1^{(j_2)} \otimes g_1^{(k_2)}, \quad j_1, j_2, j_3 = 0, 1, 2 \]
\[ f_3 = g_1^{(j_3)} \otimes g_1^{(j_3)} \otimes g_1^{(k_3)}, \quad k_1, k_2, k_3 = 0, 1, 2 \]
the selection rule reads
\[
\begin{align*}
  i_1 + i_2 + i_3 &= 0 \\
  j_1 + j_2 + j_3 &= 0 \\
  k_1 + k_2 + k_3 &= 0
\end{align*}
\] \mod 3

Number of allowed couplings: 729

Expression of the coupling
\[
\begin{align*}
  C_{\theta\theta\theta} &= N \sum_{v \in (\mathbb{R}^3 - f_3 + f_3 + \Lambda)} \exp\left[ -\frac{1}{4\pi} \sin\left(\frac{2\pi}{3}\right) |v|^2 \right] \\
  &= N \sum_{\bar{v} \in \mathbb{Z}^3} \exp\left[ -\frac{\sqrt{3}}{8\pi} (\vec{f}_{23} + \bar{v})^\top M (\vec{f}_{23} + \bar{v}) \right] \\
  &= N \vartheta \left[ \begin{array}{c} \vec{f}_{23} \\ 0 \end{array} \right] [0, \Omega]
\end{align*}
\]

with
\[
\begin{align*}
  \Omega &= i \frac{\sqrt{3}}{8\pi^2} M, \quad N = \sqrt{V_L} \frac{3^{3/4}}{8\pi^{3 \over 4}} \frac{\Gamma^6(2)}{\Gamma^3(\frac{3}{4})} \\
  \begin{pmatrix}
    R_1^2 & -R_2^2 & R_1 R_3 \alpha_{13} & R_1 R_3 \alpha_{14} & R_1 R_5 \alpha_{15} & R_1 R_5 \alpha_{16} \\
    -R_2^2 & R_1^2 & R_1 R_3 \alpha_{23} & R_1 R_3 \alpha_{13} & R_1 R_5 \alpha_{25} & R_1 R_5 \alpha_{15} \\
    R_1 R_3 \alpha_{13} & R_1 R_3 \alpha_{23} & R_3^2 & -R_3^2 & R_3 R_5 \alpha_{35} & R_3 R_5 \alpha_{36} \\
    R_1 R_3 \alpha_{14} & R_1 R_3 \alpha_{13} & -R_3^2 & R_3^2 & R_3 R_5 \alpha_{45} & R_3 R_5 \alpha_{35} \\
    R_1 R_5 \alpha_{15} & R_1 R_5 \alpha_{25} & R_3 R_5 \alpha_{35} & R_3 R_5 \alpha_{45} & R_5^2 & -R_5^2 \\
    R_1 R_5 \alpha_{16} & R_1 R_5 \alpha_{15} & R_3 R_5 \alpha_{36} & R_3 R_5 \alpha_{35} & -R_5^2 & R_5^2
  \end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
  \alpha_{23} &= - (\alpha_{13} + \alpha_{14}) , \quad \alpha_{25} = - (\alpha_{15} + \alpha_{16}) , \quad \alpha_{45} = - (\alpha_{35} + \alpha_{36})
\end{align*}
\]

Number of effective parameters: 9

Number of different couplings without deformations: 4
corresponding to the following \vec{f}_{23} shifts
\[
\vec{f}_{23} = g_1^{(0)} \otimes g_1^{(0)} \otimes g_1^{(0)} , \quad g_1^{(1)} \otimes g_1^{(0)} \otimes g_1^{(0)} , \quad g_1^{(1)} \otimes g_1^{(1)} \otimes g_1^{(0)} , \quad g_1^{(1)} \otimes g_1^{(1)} \otimes g_1^{(1)} 
\]

Number of different couplings with deformations: 14
corresponding to the following \vec{f}_{23} shifts
\[
\vec{f}_{23} = \begin{cases}
  g_1^{(0)} \otimes g_1^{(0)} \otimes g_1^{(0)} , & g_1^{(1)} \otimes g_1^{(0)} \otimes g_1^{(0)} , & g_1^{(1)} \otimes g_1^{(1)} \otimes g_1^{(0)} , & g_1^{(1)} \otimes g_1^{(1)} \otimes g_1^{(1)} , \\
  g_1^{(1)} \otimes g_1^{(1)} \otimes g_1^{(0)} , & g_1^{(1)} \otimes g_1^{(0)} \otimes g_1^{(0)} , & g_1^{(1)} \otimes g_1^{(1)} \otimes g_1^{(1)} , & g_1^{(1)} \otimes g_1^{(1)} \otimes g_1^{(2)} , \\
  g_1^{(1)} \otimes g_1^{(1)} \otimes g_1^{(2)} , & g_1^{(1)} \otimes g_1^{(1)} \otimes g_1^{(2)} , & g_1^{(1)} \otimes g_1^{(1)} \otimes g_1^{(1)} , & g_1^{(1)} \otimes g_1^{(1)} \otimes g_1^{(2)} , \\
  g_1^{(1)} \otimes g_1^{(2)} \otimes g_1^{(2)} , & g_1^{(1)} \otimes g_1^{(2)} \otimes g_1^{(1)} , & g_1^{(1)} \otimes g_1^{(2)} \otimes g_1^{(1)} 
\end{cases}
\]
**ORBIFOLD $Z_4$**
See Section 3

**ORBIFOLD $Z_6-I$**
See Section 2

**ORBIFOLD $Z_6-II$**

Twist $\theta = \text{diag}(e^{i\alpha}, e^{2i\alpha}, e^{-3i\alpha})$, \( \alpha = \frac{2\pi}{6} \)

Lattice $SU(6) \otimes SU(2)$

Coxeter element

$$\theta e_i = e_{i+1}, \hspace{1em} i = 1, \ldots, 4, \hspace{1em} \theta e_5 = -e_1 - e_2 - e_3 - e_4 - e_5, \hspace{1em} \theta e_6 = -e_6$$

Deformation parameters

Relations

$$|e_1| = |e_2| = |e_3| = |e_4| = |e_5|, \hspace{1em} \alpha_{12} = \alpha_{23} = \alpha_{34} = \alpha_{45} = -\frac{1}{2}(1 + \alpha_{14} + 2\alpha_{15}),$$

$$\alpha_{15} = \alpha_{13} = \alpha_{24} = \alpha_{35}, \hspace{1em} \alpha_{16} = -\alpha_{26} = \alpha_{36} = -\alpha_{46} = \alpha_{56},$$

$$\alpha_{14} = \alpha_{25}$$

$$\alpha_{ij} \equiv \cos(\theta_{ij})$$

Degrees of freedom (5)

$$R_1 = |e_1|, \hspace{1em} R_6 = |e_6|, \hspace{1em} \alpha_{14}, \hspace{1em} \alpha_{15}, \hspace{1em} \alpha_{16}$$

Lattice basis ($e_i$) in terms of orthogonal basis ($\tilde{e}_i$)

$$e_i = \sum_{j=1,3,5} A_j [\cos(\varphi_j + (i-1)b_j\alpha)\tilde{e}_j + \sin(\varphi_j + (i-1)b_j\alpha)\tilde{e}_{j+1}] \hspace{1em} i = 1, \ldots, 5$$

$$e_6 = R_6 [\cos(\varphi_3 + \Delta)\tilde{e}_5 + \sin(\varphi_3 + \Delta)\tilde{e}_6]$$

with $\alpha = \frac{\pi}{3}$ and $b_1 = 1, b_2 = 2, b_3 = 3$

$$\cos(\Delta) = \frac{\sqrt{3}a_{16}}{\sqrt{1 + 2a_{15}}}, \hspace{1em} A_1 = \frac{R_1}{\sqrt{6}} \sqrt{1 - 3a_{14} - 4a_{15}}, \hspace{1em} A_3 = \frac{R_1}{\sqrt{2}} \sqrt{1 + a_{14}}, \hspace{1em} A_5 = \frac{R_1}{\sqrt{3}} \sqrt{1 + 2a_{15}}$$

$\varphi_1, \varphi_2, \varphi_3$ are free parameters.
Fixed points of \( \theta \) (12)

\[
f_1^{(ij)} = g^{(i)}_1 \otimes g^{(j)}_1, \quad i = 0, 1, \ldots, 5, \quad j = 0, 1
\]

\[
g^{(0)}_1 = (0, 0, 0, 0, 0), \quad g^{(1)}_1 = \frac{1}{6}(5, 4, 3, 2, 1), \quad g^{(2)}_1 = \frac{1}{6}(4, 8, 6, 4, 2),
\]

\[
g^{(3)}_1 = \frac{1}{6}(3, 6, 9, 6, 3), \quad g^{(4)}_1 = \frac{1}{6}(2, 4, 6, 8, 4), \quad g^{(5)}_1 = \frac{1}{6}(1, 2, 3, 4, 5),
\]

\[
g^{(0)}_1 = (0), \quad g^{(1)}_1 = \left(\frac{1}{2}\right)
\]

Fixed points of \( \theta^2 \) (9)

Fixed torus: \( \alpha(e_1 + e_3 + e_5) + \beta(e_6) \), \( \alpha, \beta \in R \)

\[
f_2^{(i)} = g^{(i)}_2 \otimes [\alpha(e_1 + e_3 + e_5) + \beta(e_6)], \quad i = 0, 1, \ldots, 8, \quad \alpha, \beta \in R
\]

\[
g^{(0)}_2 = (0, 0, 0, 0, 0), \quad g^{(1)}_2 = \frac{1}{3}(0, 1, 1, 2, 2), \quad g^{(2)}_2 = \frac{1}{3}(0, 2, 2, 1, 1),
\]

\[
g^{(3)}_2 = \frac{1}{6}(1, 0, -2, 0, 1), \quad g^{(4)}_2 = \frac{1}{3}(1, 1, 2, 2, 0), \quad g^{(5)}_2 = \frac{1}{3}(2, 2, 1, 1, 0),
\]

\[
g^{(6)}_2 = \frac{1}{3}(-1, 0, 2, 0, -1), \quad g^{(7)}_2 = \frac{1}{3}(2, -2, 0, 2, -2), \quad g^{(8)}_2 = \frac{1}{3}(1, -1, 0, 1, -1)
\]

Note that \( \theta : g^{(1)}_2 \to g^{(4)}_2, g^{(2)}_2 \to g^{(5)}_2, \quad \theta : g^{(3)}_2 \to g^{(6)}_2 \)

Number of conjugation classes: 6

Fixed points of \( \theta^3 \) (14)

Fixed torus: \( \alpha(e_1 + e_4) + \beta(e_2 + e_5) \), \( \alpha, \beta \in R \)

\[
f_3^{(ij)} = g^{(i)}_3 \otimes g^{(j)}_3 \otimes [\alpha(e_1 + e_4) + \beta(e_2 + e_5)], \quad i = 0, 1, \ldots, 7, \quad j = 0, 1, \quad \alpha, \beta \in R
\]

\[
g^{(0)}_3 = (0, 0, 0, 0, 0), \quad g^{(1)}_3 = \frac{1}{2}(1, 1, 1, 0, 0), \quad g^{(2)}_3 = \frac{1}{2}(1, 1, 0, -1, -1),
\]

\[
g^{(3)}_3 = \frac{1}{2}(0, 1, 1, 1, 0), \quad g^{(4)}_3 = \frac{1}{2}(1, 0, 0, 1, 0), \quad g^{(5)}_3 = \frac{1}{2}(0, 0, 1, 1, 1),
\]

\[
g^{(6)}_3 = \frac{1}{2}(0, 1, 0, 0, 1), \quad g^{(7)}_3 = \frac{1}{2}(1, 0, 1, 1, 0), \quad g^{(8)}_3 = \frac{1}{2}(1, 0, 1, 0, 1),
\]

\[
g^{(0)}_3 = (0), \quad g^{(1)}_3 = \left(\frac{1}{2}\right)
\]

Note that in the SU(6) lattice \( \theta : g^{(1)}_3 \to g^{(3)}_3 \to g^{(5)}_3 \) and \( \theta : g^{(2)}_3 \to g^{(4)}_3 \to g^{(6)}_3 \)

Number of conjugation classes: 8

Coupling \( \theta \theta^2 \theta^3 \)

Selection rule

\[
f_1 + (I + \theta) f_2 - (I + \theta + \theta^2) f_3 \in \Lambda
\]
Denoting

\[
\begin{align*}
    f_1 &= g_1^{(i_1)} \otimes \hat{g}_1^{(j_1)} \\
    f_2 &= g_2^{(i_2)} \otimes [\alpha(e_1 + e_3 + e_5) + \beta(e_6)] \\
    f_3 &= g_3^{(i_3)} \otimes [\gamma(e_1 + e_4) + \delta(e_2 + e_5)]
\end{align*}
\]

the selection rule reads

\[
\begin{align*}
    i_1 = 0, 1, \ldots, 5, & \quad i_1 = 0, 1, \ldots, 5, \\
    j_1, j_3 = 0, 1, \quad i_2 = 0, 1, \ldots, 8, & \quad i_3 = 0, 1, \ldots, 7, \\
    \alpha, \beta, \gamma, \delta \in \mathbb{R}
\end{align*}
\]

Number of allowed couplings: 48

Expression of the coupling

\[
C_{gg\theta^3} = \sqrt{l_2 l_3} N \sum_{v \in (f_3 - f_2 + \Lambda)_{\perp}} \exp[-\sqrt{3}/4\pi |v_1|^2]
\]

where \(l_i\) is the number of elements in the \(f_i\) conjugation class and \((f_3 - f_2 + \Lambda)_{\perp}\) denotes elements orthogonal to the two invariant planes

\[
(f_3 - f_2 + \Lambda)_{\perp} = \sum_{i=1}^{6} (2h_i^i + n_i^i) \frac{1}{2} e_1 + e_2 + e_3 + \frac{1}{2} e_4 + (2h_i^i + n_i^i) (-e_1 - e_2 + e_4 + e_5)
\]

where denoting \(\vec{f}_{23}^1 \equiv (h_1^i, h_2^i)\), \(\vec{f}_{23}^2\) is always

\[
\begin{align*}
    \vec{f}_{23}^1 &= (0, 0) & \vec{f}_{23}^2 &= (0, \frac{1}{2}) & \vec{f}_{23}^3 &= (\frac{1}{3}, \frac{1}{3}) \\
    \vec{f}_{23}^4 &= (\frac{1}{3}, \frac{5}{6}) & \vec{f}_{23}^5 &= (\frac{2}{3}, \frac{2}{3}) & \vec{f}_{23}^6 &= (\frac{5}{3}, \frac{5}{6})
\end{align*}
\]

with \(n_1^i, n_2^i \in \mathbb{Z}\). The coupling takes the final form

\[
C_{gg\theta^3} = \sqrt{l_2 l_3} N \sum_i \sum_{u \in \mathbb{Z}^2} \exp[-\sqrt{3}/4\pi (\vec{f}_{23}^i + \vec{u})^\top M (\vec{f}_{23}^i + \vec{u})]
\]

with

\[
\Omega = i \sqrt{\frac{3}{4\pi^2}} M = i \sqrt{\frac{3}{2\pi^2}} \rho_1^2 \left(1 - 3\alpha_{14} - 4\alpha_{15}\right) \begin{pmatrix} 1/4 & -1/4 \\ -1/4 & 1 \end{pmatrix}, \quad N = \sqrt{V_\perp} \frac{1}{2\pi} \sqrt{\frac{\Gamma(\frac{5}{6})\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})\Gamma(\frac{1}{6})}}
\]

30
with $V_\perp$ the volume of the unit cell generated by $\{\frac{1}{2}e_1 + e_2 + e_3 + \frac{1}{2}e_4, e_1 + e_2 - e_4 - e_5\}$

Number of effective parameters: 1
Number of different couplings without deformations: 4
Number of different couplings with deformations: 4

### Coupling $\theta \theta \theta^4$

**Selection rule**

$$f_1 + f_2 - (I + \theta)f_3 \in \Lambda$$

Denoting

$$f_1 = g_1^{(i_1)} \otimes \hat{g}_1^{(j_1)}$$
$$f_2 = g_1^{(i_2)} \otimes \hat{g}_1^{(j_2)}$$
$$f_3 = g_2^{(i_3)} \otimes [\alpha(e_1 + e_3 + e_5) + \beta(e_6)]$$

the selection rule reads

$$i_1 + i_2 + 4i_3 = 0$$
$$j_1 = j_2$$

mod. 6

Number of allowed couplings: 72

**Expression of the coupling**

$$C_{\theta \theta \theta^4} = \sqrt{\frac{3}{8\pi}} N \sum_{v \in (f_3 - f_2 + \Lambda)_{\perp}} \exp[-\frac{\sqrt{3}}{8\pi} (|v_1|^2 + |v_2|^2)]$$

where $(f_3 - f_2 + \Lambda)_{\perp}$ denotes that the coset elements must be orthogonal to the $(e_1 + e_3 + e_5, e_6)$ plane

$$(f_3 - f_2 + \Lambda)_{\perp} = \sum_{i=1}^{3} [(h_i^1 + n_i^1)(e_1 - e_3) + (h_i^2 + n_i^2)(e_2 + e_3) + (h_i^3 + n_i^3)(e_3 + e_4) + (h_i^4 + n_i^4)(e_5 - e_3)]$$

where denoting $\vec{f}_{23} = (h_1^i, ..., h_4^i)$ there are two possible triplets of values for $\vec{f}_{23}$ depending on the values of $f_2, f_3$

$$\vec{f}_{23} = (0, 0, 0, 0) \quad \vec{f}_{23} = \left(\frac{1}{3}, 0, 0, \frac{1}{3}\right) \quad \vec{f}_{23} = \left(\frac{2}{3}, 0, 0, \frac{2}{3}\right)$$

and

$$\vec{f}_{23} = \left(\frac{1}{2}, \frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right) \quad \vec{f}_{23} = \left(\frac{2}{3}, \frac{1}{3}, 0, 0\right) \quad \vec{f}_{23} = \left(0, \frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$
with \( n_1^i, n_2^i, n_3^i, n_4^i \in \mathbb{Z} \). Finally the coupling takes the form

\[
C_{\theta \theta \theta} = \sqrt{l_3} N \sum_i \sum_{\vec{u} \in \mathbb{Z}^4} \exp\left[ -\sqrt{3} \frac{1}{8\pi} (\vec{f}_{23} + \vec{u})^\top M (\vec{f}_{23} + \vec{u}) \right]
\]

\[
= \sqrt{l_3} N \sum_i \theta \left[ f_{23} \quad 0 \right] [0, \Omega]
\]

with

\[
N = \sqrt{V_\perp} \frac{1}{2\pi} \sqrt{\frac{\Gamma(\frac{5}{6}) \Gamma(\frac{7}{6})}{\Gamma(\frac{1}{6}) \Gamma(\frac{1}{5})}} \quad \Omega = i \sqrt{\frac{3}{8\pi^2}} M
\]

\( V_\perp \) is the unit cell volume of the sublattice orthogonal to the invariant plane

\[
\Omega = i \sqrt{\frac{3}{8\pi^2}} \left( \begin{array}{cccc}
2a & -a & \frac{a+c-2b}{2} & -a \\
-a & b & c & \frac{a+c-2b}{2} \\
\frac{a+c-2b}{2} & c & b & -a \\
-a & \frac{a+c-2b}{2} & -a & 2a \\
\end{array} \right)
\]

\( a = R_1^2(1 - \alpha_{15}) \)
\( b = R_1^2(1 - \alpha_{14} - 2\alpha_{15}) \)
\( c = R_1^2(\alpha_{14} + \alpha_{15}) \)

Number of effective parameters: 3
Number of different couplings without deformations: 4
Number of different couplings with deformations: 4

**ORBIFOLD \( Z_7 \)**

Twist \( \theta = \text{diag}(e^{i\alpha}, e^{2i\alpha}, e^{-3i\alpha}) \), \( \alpha = \frac{2\pi}{7} \)

Lattice \( SU(7) \)

Coxeter element

\( \theta e_i = e_{i+1}, \quad i = 1, \ldots, 5, \quad \theta e_6 = -e_1 - e_2 - e_3 - e_4 - e_5 - e_6 \)

Deformation parameters

Relations

\[
|e_1| = |e_2| = |e_3| = |e_4| = |e_5| = |e_6|, \quad \alpha_{12} = \alpha_{23} = \alpha_{34} = \alpha_{45} = \alpha_{56},
\]

\( \alpha_{13} = \alpha_{24} = \alpha_{35} = \alpha_{46} = \alpha_{16}, \quad \alpha_{14} = \alpha_{25} = \alpha_{36} = \alpha_{15} = \alpha_{26} = -\frac{1}{2} - \alpha_{12} - \alpha_{13} \)

\( \alpha_{ij} \equiv \cos(\theta_{ij}) \)

Degrees of freedom (3)

\( R = |e_1|, \quad \alpha_{12}, \quad \alpha_{13} \)
Lattice basis \((e_i)\) in terms of orthogonal basis \((\tilde{e}_i)\)

\[
e_i = \sum_{j=1,3,5} R_j [\cos((i-1)b_j \alpha + \varphi_j)\tilde{e}_j + \sin((i-1)b_j \alpha + \varphi_j)\tilde{e}_{j+1}] \quad i = 1,...,6
\]

with \(\alpha = \frac{2\pi}{7}\) and \(b_1 = 1, b_3 = 2, b_5 = 4\)

\[
R_j = R^2_1[\alpha_1 (\alpha_2^2 - \alpha_1^2) + \alpha_3 (\alpha_5^2 - \alpha_3^2) + \frac{1}{7} \alpha_1^2]
R_2 = R^2_2[\alpha_1 (\alpha_2^2 - \alpha_1^2) + \alpha_3 (\alpha_5^2 - \alpha_3^2) + \frac{1}{7} \alpha_1^2]
R_5 = R^2_2[\alpha_1 (\alpha_2^2 - \alpha_1^2) + \alpha_3 (\alpha_5^2 - \alpha_3^2) + \frac{1}{7} \alpha_1^2]
\]

\[
\alpha_i^2 = \frac{4}{7}[1 - \cos(b_i \alpha)] \quad i = 1,3,5
\]

\(\varphi_1, \varphi_2, \varphi_3\) are free parameters.

Fixed points of \(\theta\) (7)

\[
f^{(0)}_1 = (0,0,0,0,0,0), \quad f^{(1)}_1 = \frac{1}{7}(6,5,4,3,2,1), \quad f^{(2)}_1 = \frac{1}{7}(5,3,1,6,4,2), \quad f^{(3)}_1 = \frac{1}{7}(4,1,5,2,6,3), \quad f^{(4)}_1 = \frac{1}{7}(3,6,2,5,1,4), \quad f^{(5)}_1 = \frac{1}{7}(2,4,6,1,3,5), \quad f^{(6)}_1 = \frac{1}{7}(1,2,3,4,5,6)
\]

Coupling \(\theta \theta^2 \theta^4\)

Selection rule

\[
f_1 + 2f_2 - 3f_3 \in \Lambda
\]

Denoting

\[
\begin{align*}
f_1 &= f_1^{(i_1)} \\
\vdots \\
f_3 &= f_1^{(i_3)}
\end{align*}
\]

\(i_1, i_2, i_3 = 0,1,...,6\)

the selection rule reads

\[
i_1 + 2i_2 - 3i_3 = 0 \mod. 7
\]

Number of allowed couplings: 49

Expression of the coupling:

\[
C_{\theta \theta^2 \theta^4} = N \sum_{v \in (f_3 - f_2 + \Lambda)} \exp \left[ -\frac{1}{4\pi} \sin(\alpha) \sin(2\alpha) \sin(3\alpha) \left( \frac{|v_1|^2}{\sin^2(3\alpha)} + \frac{|v_2|^2}{\sin^2(\alpha)} + \frac{|v_3|^2}{\sin^2(2\alpha)} \right) \right]
\]

\[
= N \sum_{\tilde{u} \in Z^6} \exp \left[ -\frac{1}{4\pi} (f_{23} + \tilde{u})^\top M (f_{23} + \tilde{u}) \right]
\]

\[
= N \vartheta \left[ \begin{array}{c} f_{23} \\
0 \end{array} \right] [0, \Omega]
\]

33
\[ \Omega = i \frac{1}{4\pi^2} M \quad N = \sqrt{V_\Lambda} \left[ \frac{1}{2\pi} \right]^{3/2} \left[ \frac{\Gamma(\frac{3}{7}) \Gamma(\frac{5}{7}) \Gamma(\frac{6}{7})}{\Gamma(\frac{1}{7}) \Gamma(\frac{2}{7}) \Gamma(\frac{3}{7})} \right]^{3/2} \]

\[ \Omega = i \frac{1}{4\pi^2} \sin(\alpha) \sin(2\alpha) \sin(3\alpha) \]

\[
\begin{pmatrix}
    a & b & c & d & d & c \\
    b & a & b & c & d & d \\
    c & b & a & b & c & d \\
    d & d & c & b & a & d \\
    c & d & c & b & a & d
\end{pmatrix}
\]

\[ a = \frac{R_1^2 \cos(\alpha)}{\sin^2(\alpha)} + \frac{R_2^2 \cos(2\alpha)}{\sin^2(\alpha)} + \frac{R_3^2 \cos(3\alpha)}{\sin^2(2\alpha)} \]

\[ b = \frac{R_1^2 \cos(\alpha)}{\sin^2(3\alpha)} + \frac{R_2^2 \cos(2\alpha)}{\sin^2(3\alpha)} + \frac{R_3^2 \cos(3\alpha)}{\sin^2(2\alpha)} \]

\[ c = \frac{R_1^2 \cos(2\alpha)}{\sin^2(3\alpha)} + \frac{R_2^2 \cos(3\alpha)}{\sin^2(3\alpha)} + \frac{R_3^2 \cos(\alpha)}{\sin^2(2\alpha)} \]

\[ d = \frac{R_1^2 \cos(3\alpha)}{\sin^2(3\alpha)} + \frac{R_2^2 \cos(\alpha)}{\sin^2(3\alpha)} + \frac{R_3^2 \cos(2\alpha)}{\sin^2(2\alpha)} \]

Number of effective parameters: 3
Number of different couplings without deformations: 2
  corresponding to the following \( f_{23} \) shifts

\[ f_{23} = \{ f_1^{(0)}, f_1^{(1)} \} \]

Number of different couplings with deformations: 4
  corresponding to the following \( f_{23} \) shifts

\[ f_{23} = \{ f_1^{(0)}, f_1^{(1)}, f_1^{(2)}, f_1^{(3)} \} \]

**ORBIFOLD Z\(_8\)-I**

Twist \( \theta = \text{diag}(e^{i\alpha}, e^{2i\alpha}, e^{-3i\alpha}) \), \( \alpha = \frac{2\pi}{8} \)

Lattice \( SO(5) \otimes SO(9) \)

Coxeter element

\[
\begin{align*}
\theta e_1 &= e_1 + 2e_2, \\
\theta e_2 &= -e_1 - e_2, \\
\theta e_3 &= e_4, \\
\theta e_4 &= e_5, \\
\theta e_5 &= e_3 + e_4 + e_5 + 2e_6, \\
\theta e_6 &= -e_3 - e_4 - e_5 - e_6
\end{align*}
\]

Deformation parameters

Relations

\[
\begin{align*}
|e_1| &= \sqrt{2} |e_2|, \\
|e_3| &= |e_4| = |e_5|, \\
\alpha_{12} &= -\frac{1}{\sqrt{2}}, \\
\alpha_{35} &= 0, \\
\alpha_{36} &= \alpha_{46}, \\
\alpha_{36} &= \frac{1}{2\alpha_{56}} - \alpha_{56}, \\
\alpha_{ij} &= 0 \quad i = 1, 2 \quad j = 3, 4, 5, 6
\end{align*}
\]

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\[ \alpha_{ij} \equiv \cos(\theta_{ij}) \]

Degrees of freedom (3)

\[ R_1 = |e_1|, \quad R_3 = |e_3|, \quad \alpha_{56} \]

Lattice basis \((e_i)\) in terms of orthogonal basis \((\hat{e}_i)\)

\[
e_1 = \frac{R_4}{2} \left[ \left( 2 + \sqrt{2} \right)^{1/2} \cos(\varphi_1) + (2 - \sqrt{2})^{1/2} \sin(\varphi_1) \right] \hat{e}_1 + \left( 2 + \sqrt{2} \right)^{1/2} \sin(\varphi_1) + (2 - \sqrt{2})^{1/2} \cos(\varphi_1) \right] \hat{e}_2 \\
e_2 = \frac{R_4}{2} \left[ \left( 2 + \sqrt{2} \right)^{1/2} \cos(\varphi_1) + (2 - \sqrt{2})^{1/2} \sin(\varphi_1) \right] \hat{e}_1 + \left( 2 + \sqrt{2} \right)^{1/2} \sin(\varphi_1) + (2 - \sqrt{2})^{1/2} \cos(\varphi_1) \right] \hat{e}_2 \\
e_3 = A[\cos(\varphi_2) \hat{e}_3 + \sin(\varphi_2) \hat{e}_4] + B[\cos(\varphi_3) \hat{e}_5 + \sin(\varphi_3) \hat{e}_6] \\
e_4 = A[\cos(\alpha + \varphi_2) \hat{e}_3 + \sin(\alpha + \varphi_2) \hat{e}_4] - B[\cos(\alpha + \varphi_3) \hat{e}_5 + \sin(\alpha + \varphi_3) \hat{e}_6] \\
e_5 = -A[\sin(\varphi_2) \hat{e}_3 - \cos(\varphi_2) \hat{e}_4] - B[\sin(\varphi_3) \hat{e}_5 - \cos(\varphi_3) \hat{e}_6] \\
e_6 = \frac{A}{2} \left[ -\cos(\varphi_2) + \sin(\varphi_2) - 2 \cos(\varphi_2) \cos(\alpha) \right] \hat{e}_3 + \\
+ \frac{4}{2} \left[ -\cos(\varphi_2) - \sin(\varphi_2) - 2 \sin(\varphi_2) \cos(\alpha) \right] \hat{e}_4 + \\
+ \frac{A}{2} \left[ -\cos(\varphi_3) + \sin(\varphi_3) + 2 \cos(\varphi_3) \cos(\alpha) \right] \hat{e}_5 + \\
+ \frac{4}{2} \left[ -\cos(\varphi_3) - \sin(\varphi_3) + 2 \sin(\varphi_3) \cos(\alpha) \right] \hat{e}_6
\]

with \(\alpha = \frac{2\pi}{8}\) and

\[ A = R_3 \left[ \frac{1 + \sqrt{2}}{2} - \frac{1}{4\sqrt{2}a_{56}} \right]^{1/2}, \quad B = R_3 \left[ \frac{1 - \sqrt{2}}{2} + \frac{1}{4\sqrt{2}a_{56}} \right]^{1/2} \]

\(\varphi_1, \varphi_2, \varphi_3\) are free parameters.

Fixed points of \(\theta\) (4)

\[ f_1^{(ij)} = g_1^{(i)} \otimes g_1^{(j)} \quad i = 0, 1, \quad j = 0, 1 \]

\[ g_1^{(0)} = (0, 0), \quad g_1^{(1)} = \frac{1}{2}(1, 0), \quad \hat{g}_1^{(0)} = (0, 0, 0, 0), \quad \hat{g}_1^{(1)} = \frac{1}{2}(1, 0, 1, 0) \]

Fixed points of \(\theta^2\) (16)

\[ f_2^{(ij)} = g_2^{(i)} \otimes g_2^{(j)} \quad i, j = 0, 1, 2, 3 \]

\[ g_2^{(0)} = (0, 0), \quad g_2^{(1)} = \frac{1}{2}(0, 1), \quad g_2^{(2)} = \frac{1}{2}(1, 0), \quad g_2^{(3)} = \frac{1}{2}(1, 1), \]

\[ \hat{g}_2^{(0)} = (0, 0, 0, 0), \quad \hat{g}_2^{(1)} = \frac{1}{2}(0, 1, 1, 0), \quad \hat{g}_2^{(2)} = \frac{1}{2}(1, 0, 1, 0), \quad \hat{g}_2^{(3)} = \frac{1}{2}(1, 1, 0, 0) \]
Note that $\theta : g_2^{(1)} \rightarrow g_2^{(3)}$ and $\theta : \hat{g}_2^{(1)} \rightarrow \hat{g}_2^{(3)}$.

Number of conjugation classes: 10

Fixed points of $\theta^3$ (4)
The same as for $\theta$.

\[ f_3^{(ij)} = g_3^{(i)} \otimes \hat{g}_3^{(j)}, \quad i = 0, 1, \quad j = 0, 1 \]

\[ g_3^{(0)} = (0, 0), \quad g_3^{(1)} = \frac{1}{2}(1, 0), \quad \hat{g}_3^{(0)} = (0, 0, 0, 0), \quad \hat{g}_3^{(1)} = \frac{1}{2}(1, 0, 1, 0) \]

Fixed points of $\theta^4$ (16)
Fixed torus: $\alpha(e_1) + \beta(e_2), \quad \alpha, \beta \in R$

\[ f_4^{(i)} = [\alpha(e_1) + \beta(e_2)] \otimes \hat{g}_4^{(i)}, \quad i = 0, 1, ..., 15, \quad \alpha, \beta \in R \]

\[ g_4^{(0)} = (0, 0, 0, 0), \quad \hat{g}_4^{(1)} = \frac{1}{2}(1, 0, 1, 1), \quad \hat{g}_4^{(2)} = \frac{1}{2}(1, 0, 0, 0), \quad \hat{g}_4^{(3)} = \frac{1}{2}(1, 0, 0, 1), \]

\[ g_4^{(4)} = \frac{1}{2}(1, 1, 0, 0), \quad \hat{g}_4^{(5)} = \frac{1}{2}(0, 0, 0, 1), \quad \hat{g}_4^{(6)} = \frac{1}{2}(0, 1, 0, 0), \quad \hat{g}_4^{(7)} = \frac{1}{2}(0, 0, 1, 1), \]

\[ g_4^{(8)} = \frac{1}{2}(1, 0, 1, 0), \quad \hat{g}_4^{(9)} = \frac{1}{2}(1, 1, 0, 1), \quad \hat{g}_4^{(10)} = \frac{1}{2}(0, 0, 1, 0), \quad \hat{g}_4^{(11)} = \frac{1}{2}(0, 1, 0, 1), \]

\[ g_4^{(12)} = \frac{1}{2}(0, 1, 1, 0), \quad \hat{g}_4^{(13)} = \frac{1}{2}(0, 1, 1, 1), \quad \hat{g}_4^{(14)} = \frac{1}{2}(1, 1, 1, 0), \quad \hat{g}_4^{(15)} = \frac{1}{2}(1, 1, 1, 1) \]

Note that in the $SO(9)$ lattice

\[ \theta : \hat{g}_4^{(4)} \rightarrow \hat{g}_4^{(12)}, \quad \theta : \hat{g}_4^{(2)} \rightarrow \hat{g}_4^{(6)} \rightarrow \hat{g}_4^{(10)} \rightarrow \hat{g}_4^{(14)}, \quad \theta : \hat{g}_4^{(5)} \rightarrow \hat{g}_4^{(7)} \rightarrow \hat{g}_4^{(13)} \rightarrow \hat{g}_4^{(15)} \]

Number of conjugation classes: 6

**Coupling $\theta^2\theta^2\theta^4$**

Selection rule

\[ f_1 + f_2 - (I + \theta^2)f_3 \in \Lambda \]

Denoting

\[ f_1 = g_2^{(i_1)} \otimes \hat{g}_2^{(j_1)} \quad \text{for} \quad i_1, i_2, j_1, j_2 = 0, 1, 2, 3, \]

\[ f_2 = g_2^{(i_2)} \otimes \hat{g}_2^{(j_2)} \quad \text{for} \quad j_3 = 0, 1, ..., 15, \]

\[ f_3 = [\alpha(e_1) + \beta(e_2)] \otimes \hat{g}_4^{(j_3)} \quad \text{for} \quad \alpha, \beta \in R \]

the selection rule reads

\[ i_1 = i_2 \]

\[ j_1 + (-1)^{j_3+1}j_2 = j_3 \quad \text{mod. 4} \]
Number of allowed couplings: 84

Expression of the coupling

\[ C_{\theta f_2 f_3} = \frac{F(l_1, l_2, l_3)}{2} N \left\{ \sum_{v \in (f_3 - f_2 + \Lambda)_\perp} \exp \left[ -\frac{1}{4\pi} |v|^2 \right] + \sum_{v \in (\theta f_3 - f_2 + \Lambda)_\perp} \exp \left[ -\frac{1}{4\pi} |v|^2 \right] \right\} \]

where \((f_3 - f_2 + \Lambda)_\perp\) denotes that only coset elements belonging to \(SO(9)\) lattice are considered; \(f_{23} = f_2 - f_3\), \(f'_{23} = \theta f_2 - f_3\), the arrows denote components in the \(SO(9)\) lattice. \(l_i\) is the number of elements in the \(f_i\) conjugation class (in all the cases, except \(l_1, l_2, l_3 = 2\), \(f_{23} = f'_{23}\)). Finally the values of \(F(l_1, l_2, l_3)\) are

- \(l_1 = l_2 = l_3 = 1\) : \(F = 1\)
- \(l_1 = l_2 = 1\), \(l_3 = 2\) : \(F = \sqrt{2}\)
- \(l_1 = l_2 = 2\), \(l_3 = 1\) : \(F = 1\)
- \(l_1 = l_2 = 2\), \(l_3 = 4\) : \(F = \sqrt{2}\)
- \(l_1 = 1\), \(l_2 = 2\), \(l_3 = 4\) : \(F = \sqrt{2}\)

\[ N = \sqrt{V_\perp \frac{1}{2\pi} \frac{\Gamma^2 \left( \frac{3}{4} \right)}{\Gamma^2 \left( \frac{1}{4} \right)}} \]
\[ \Omega = \frac{i}{4\pi^2} \left( \begin{array}{cccc} a & b & 0 & c \\ b & a & b & c \\ 0 & b & a & d \\ c & c & d & e \end{array} \right) \]

where \(V_\perp\) is the volume of the \(SO(9)\) lattice

Number of effective parameters: 2

Number of different couplings without deformations: 8

corresponding to the following \(\tilde{f}_{23}\) shifts

\[ \tilde{f}_{23} = \begin{cases} 
F = 1 & \left\{ (0, 0, 0, 0), \left(\frac{1}{2}, 0, \frac{1}{2}, 0\right), \left(\frac{1}{2}, \frac{1}{2}, 0, 0\right), \right\} \\
F = \sqrt{2} & \left\{ (\frac{1}{2}, \frac{1}{2}, 0, 0), \left(\frac{1}{2}, 0, 0, \frac{1}{2}\right), \right\} \\
F = 2 & \left\{ (0, 0, \frac{1}{2}, \frac{1}{2}), \left(0, 0, 0, 0\right) \cup \left(\frac{1}{2}, 0, \frac{1}{2}, 0\right), \right\} 
\end{cases} \]

Number of different couplings with deformations: 9
corresponding to the following $\vec{f}_{23}$ shifts

$$
\begin{align*}
\vec{f}_{23} &= \begin{cases} 
F = 1 & \{(0,0,0,0), \ (1/2,0,1/2,0), \\
(1/2, 1/2, 0, 0), \ (0, 1/2, 0, 0)\}, \\
F = \sqrt{2} & \{(1/2, 1/2, 0, 0), \ (1/2, 0, 0, 1/2), \\
(0, 0, 1/2, 1/2), \ (0, 0, 0, 0) \cup (1/2, 0, 1/2, 0)\}, \\
F = 2 & (1/2, 0, 0, 0) 
\end{cases}
\end{align*}
$$

### Coupling $\theta^2 \theta^5$

Selection rule

$$
f_1 + (I + \theta) f_2 - (I + \theta^2) f_3 \in \Lambda
$$

Denoting

$$
\begin{align*}
\hat{f}_1 &= g_1^{(i_1)} \otimes \hat{g}_1^{(j_1)} \\
\hat{f}_2 &= g_2^{(i_2)} \otimes \hat{g}_2^{(j_2)} \\
\hat{f}_3 &= g_1^{(i_3)} \otimes \hat{g}_1^{(j_3)}
\end{align*}
$$

the selection rule reads

$$
\begin{align*}
i_1 + i_2 + i_3 &= 0 \\
 j_1 + j_2 + j_3 &= 0
\end{align*} \ mod. \ 2
$$

Number of allowed couplings: 40

Expression of the coupling

$$
C_{\theta^2 \theta^5} = \sqrt{l_2} N \sum_{v \in (f_3-f_2+\Lambda)_+} \exp\left[-\frac{1}{4\pi}\left(\frac{\sqrt{2}+1}{\sqrt{2}}|v_1|^2 + |v_2|^2 + \frac{\sqrt{2}-1}{\sqrt{2}}|v_3|^2\right)\right]
$$

$$
= \sqrt{l_2} N \sum_{\vec{u} \in \mathbb{Z}^6} \exp\left[-\frac{1}{4\pi} (\vec{f}_{23} + \vec{u})^\top M (\vec{f}_{23} + \vec{u})\right]
$$

$$
= \sqrt{l_2} N \ \mathcal{U} \left[ \begin{array}{c} \vec{f}_{23} \\
0 \end{array} \right] [0, \Omega]
$$

with $l_2$ the number of elements in the $f_2$ conjugation class

$$
\Omega = \frac{1}{4\pi^2} M \ \mathcal{N} = \sqrt{V_\Lambda} \left[ \begin{array}{c} \Gamma(7/8) \Gamma(3/8) \Gamma^2(3/4) \\
\Gamma(3/8) \Gamma(3/8) \Gamma^2(3/4) \end{array} \right]^3/2
$$

\[38\]
\[
\Omega = i \frac{1}{4\pi^2} \begin{pmatrix}
a & -a & 0 & 0 & 0 & 0 \\
-a & 2a & 0 & 0 & 0 & 0 \\
0 & 0 & b & c & 0 & e \\
0 & 0 & c & b & c & d \\
0 & 0 & 0 & c & b & e \\
0 & 0 & e & d & e & f
\end{pmatrix}
\]

Number of effective parameters: 3

Number of different couplings without deformations: 8

\[
l_2 = 1 \begin{cases}
g_2^{(0)} \otimes g_2^{(0)}, & g_2^{(0)} \otimes g_2^{(2)}, & g_2^{(2)} \otimes g_2^{(0)}, & g_2^{(2)} \otimes g_2^{(2)}, \\
g_2^{(0)} \otimes g_2^{(1)}, & g_2^{(1)} \otimes g_2^{(0)}, & g_2^{(1)} \otimes g_2^{(1)}, & g_2^{(1)} \otimes g_2^{(1)}
\end{cases}
\]

\[
l_2 = 2 \begin{cases}
g_2^{(0)} \otimes g_2^{(0)}, & g_2^{(0)} \otimes g_2^{(2)}, & g_2^{(2)} \otimes g_2^{(0)}, & g_2^{(2)} \otimes g_2^{(2)}, \\
g_2^{(0)} \otimes g_2^{(1)}, & g_2^{(1)} \otimes g_2^{(0)}, & g_2^{(1)} \otimes g_2^{(1)}, & g_2^{(1)} \otimes g_2^{(1)}
\end{cases}
\]

Number of different couplings with deformations: 9

\[
l_2 = 1 \begin{cases}
g_2^{(0)} \otimes g_2^{(0)}, & g_2^{(0)} \otimes g_2^{(2)}, & g_2^{(2)} \otimes g_2^{(0)}, & g_2^{(2)} \otimes g_2^{(2)}, \\
g_2^{(0)} \otimes g_2^{(1)}, & g_2^{(1)} \otimes g_2^{(0)}, & g_2^{(1)} \otimes g_2^{(1)}, & g_2^{(1)} \otimes g_2^{(1)}
\end{cases}
\]

\[
l_2 = 2 \begin{cases}
g_2^{(0)} \otimes g_2^{(0)}, & g_2^{(0)} \otimes g_2^{(2)}, & g_2^{(2)} \otimes g_2^{(0)}, & g_2^{(2)} \otimes g_2^{(2)}, \\
g_2^{(0)} \otimes g_2^{(1)}, & g_2^{(1)} \otimes g_2^{(0)}, & g_2^{(1)} \otimes g_2^{(1)}, & g_2^{(1)} \otimes g_2^{(1)}
\end{cases}
\]

**ORBITFOLD $Z_8$--II**

Twist $\theta = \text{diag}(e^{i\alpha}, e^{3i\alpha}, e^{-4i\alpha})$, $\alpha = \frac{2\pi}{8}$

Lattice $SO(4) \otimes SO(8)$

Twist in the lattice basis

\[
\theta e_1 = -e_1, \quad \theta e_2 = -e_2, \quad \theta e_3 = e_4 + e_5, \quad \theta e_4 = e_3 + e_4 + e_6, \quad \theta e_5 = -e_3 - e_4 - e_5 - e_6, \quad \theta e_6 = -e_3 - e_4
\]

Deformation parameters

Relations

\[
|e_3| = |e_5|, \quad |e_4| = \frac{1}{\sqrt{2}}\sqrt{|e_3|^2 + |e_6|^2}, \quad \alpha_{35} = 0, \quad \alpha_{36} = \frac{|e_4|^2 - |e_6|^2}{2|e_3||e_6|}, \quad \alpha_{45} = \frac{1}{\sqrt{2}}\sqrt{\frac{|e_3|^2 - |e_6|^2}{|e_3||e_6|}}, \quad \alpha_{46} = \frac{1}{\sqrt{2}}\sqrt{\frac{|e_3|^2 + |e_6|^2}{|e_6|}}, \quad \alpha_{ij} = 0 \quad i = 1, 2 \quad j = 3, 4, 5, 6
\]
\[ \alpha_{ij} \equiv \cos(\theta_{ij}) \]

Degrees of freedom (5)

\[ R_1 = |e_1|, \ R_2 = |e_2|, \ R_3 = |e_3|, \ R_6 = |e_6|, \ \alpha_{12} \]

Lattice basis (\(e_i\)) in terms of orthogonal basis (\(\tilde{e}_i\))

\[
\begin{align*}
    e_1 &= R_1[\sin(\varphi_1 + \theta_{12})\tilde{e}_1 + \cos(\varphi_1 + \theta_{12})\tilde{e}_2] \\
    e_2 &= R_2[\sin(\varphi_1)\tilde{e}_1 + \cos(\varphi_1)\tilde{e}_2] \\
    e_3 &= A[\cos(\varphi_2)\tilde{e}_3 + \sin(\varphi_2)\tilde{e}_4] + \rho_2[\cos(\varphi_3)\tilde{e}_5 + \sin(\varphi_3)\tilde{e}_6] \\
    e_4 &= \frac{A}{\sqrt{2}}[(\cos(\varphi_2) + (1 + \sqrt{2})\sin(\varphi_2))\tilde{e}_3 + (-(1 + \sqrt{2})\cos(\varphi_2) + \sin(\varphi_2))\tilde{e}_4] - \rho_2[\cos(\varphi_3) - (1 - \sqrt{2})\sin(\varphi_3)]\tilde{e}_5 + ((1 - \sqrt{2})\cos(\varphi_3) + \sin(\varphi_3))\tilde{e}_6 \\
    e_5 &= -A[\sin(\varphi_2)\tilde{e}_3 - \cos(\varphi_2)\tilde{e}_4] + \rho_2[\sin(\varphi_3)\tilde{e}_5 - \cos(\varphi_3)\tilde{e}_6] \\
    e_6 &= -(1 + \sqrt{2})A[\cos(\varphi_2)\tilde{e}_3 + \sin(\varphi_2)\tilde{e}_4] + (\sqrt{2} - 1)\rho_2[\cos(\varphi_3)\tilde{e}_5 + \sin(\varphi_3)\tilde{e}_6]
\end{align*}
\]

\[ A = \frac{R_5}{\sqrt{3}} \left[ \left( \frac{R_6}{R_5} \right)^2 - (1 - \sqrt{2})^2 \right]^{1/2}, \ \rho_2 = \frac{R_6}{2\sqrt{3}} \left[ \left( \frac{R_4}{R_6} \right)^2 (1 + \sqrt{2})^2 - 1 \right]^{1/2} \]

\(\varphi_1, \varphi_2, \varphi_3\) are free parameters.

Fixed points of \(\theta\) (8)

\[ f_1^{(ij)} = g_1^{(i)} \otimes \hat{g}_1^{(j)}, \ i = 0, 1, 2, 3, \ j = 0, 1 \]

\[ g_1^{(0)} = (0, 0), \ g_1^{(1)} = \frac{1}{2}(1, 0), \ g_1^{(2)} = \frac{1}{2}(1, 1), \ g_1^{(3)} = \frac{1}{2}(0, 1) \]

Fixed points of \(\theta^2\) (4)

Fixed torus: \(\alpha(e_1) + \beta(e_2), \ \alpha, \ \beta \in R\)

\[ f_2^{(i)} = [\alpha(e_1) + \beta(e_2)] \otimes \hat{g}_2^{(i)}, \ i = 0, 1, 2, 3, \ \alpha, \beta \in R \]

\[ \hat{g}_2^{(0)} = (0, 0, 0, 0), \ \hat{g}_2^{(1)} = \frac{1}{2}(1, 0, 1, 0), \ \hat{g}_2^{(2)} = \frac{1}{2}(0, 0, 1, 1), \ \hat{g}_2^{(3)} = \frac{1}{2}(1, 0, 0, 1) \]

Note that in the SO(8) lattice \(\theta : \hat{g}_2^{(1)} \rightarrow \hat{g}_2^{(3)}\).

Number of conjugation classes: 3

Fixed points of \(\theta^3\) (8)
The same as for \( \theta \).

\[
 f_3^{(ij)} = g_3^{(i)} \otimes \hat{g}_3^{(j)}, \quad i = 0, 1, 2, 3, \quad j = 0, 1
\]

\[
 g_3^{(0)} = (0, 0), \quad g_3^{(1)} = \frac{1}{2}(1, 0), \quad g_3^{(2)} = \frac{1}{4}(1, 1), \quad g_3^{(3)} = \frac{1}{2}(0, 1)
\]

**Fixed points of \( \theta^4 \) (16)**

Fixed torus: \( \alpha(e_1) + \beta(e_2) \), \( \alpha, \beta \in R \)

\[
 f_4^{(i)} = [\alpha(e_1) + \beta(e_2)] \otimes \hat{g}_4^{(i)}, \quad j = 0, 1, ..., 15, \quad \alpha, \beta \in R
\]

\[
 \hat{g}_4^{(0)} = (0, 0, 0, 0), \quad \hat{g}_4^{(1)} = \frac{1}{2}(1, 0, 0, 0), \quad \hat{g}_4^{(2)} = \frac{1}{2}(0, 1, 0, 0), \quad \hat{g}_4^{(3)} = \frac{1}{4}(0, 0, 0, 1), \\
 \hat{g}_4^{(4)} = \frac{1}{2}(1, 0, 1, 0), \quad \hat{g}_4^{(5)} = \frac{1}{2}(0, 1, 1, 0), \quad \hat{g}_4^{(6)} = \frac{1}{2}(1, 1, 0, 1), \quad \hat{g}_4^{(7)} = \frac{1}{4}(1, 1, 0, 0), \\
 \hat{g}_4^{(8)} = \frac{1}{2}(0, 0, 1, 1), \quad \hat{g}_4^{(9)} = \frac{1}{2}(0, 1, 0, 1), \quad \hat{g}_4^{(10)} = \frac{1}{2}(0, 1, 1, 1), \quad \hat{g}_4^{(11)} = \frac{1}{2}(1, 0, 1, 1), \\
 \hat{g}_4^{(12)} = \frac{1}{2}(1, 0, 0, 1), \quad \hat{g}_4^{(13)} = \frac{1}{2}(1, 1, 1, 1), \quad \hat{g}_4^{(14)} = \frac{1}{2}(1, 1, 1, 0), \quad \hat{g}_4^{(15)} = \frac{1}{2}(0, 1, 0, 0)
\]

Note that in the \( SO(8) \) lattice

\[
 \theta : \hat{g}_4^{(4)} \to \hat{g}_4^{(12)}, \quad \theta : \hat{g}_4^{(1)} \to \hat{g}_4^{(5)} \to \hat{g}_4^{(9)} \to \hat{g}_4^{(13)}, \\
 \theta : \hat{g}_4^{(2)} \to \hat{g}_4^{(6)} \to \hat{g}_4^{(10)} \to \hat{g}_4^{(14)}, \quad \theta : \hat{g}_4^{(3)} \to \hat{g}_4^{(7)} \to \hat{g}_4^{(11)} \to \hat{g}_4^{(15)}
\]

Number of conjugation classes: 6

**Coupling \( \theta \theta \theta \)**

Selection rule

\[
 f_1 + f_2 - (I + \theta)f_3 \in \Lambda
\]

Denoting

\[
 f_1 = g_1^{(i_1)} \otimes \hat{g}_1^{(j_1)} \quad \left\{ \begin{array}{l} i_1, i_2, j_3 = 0, 1, 2, 3, \end{array} \right. \\
 f_2 = g_1^{(i_2)} \otimes \hat{g}_1^{(j_2)} \quad \left\{ \begin{array}{l} j_1, j_2 = 0, 1, \end{array} \right. \\
 f_3 = [\alpha(e_1) + \beta(e_2)] \otimes \hat{g}_2^{(j_3)} \quad \left\{ \begin{array}{l} \alpha, \beta \in R, \end{array} \right. \\
\]

the selection rule reads

\[
 i_1 = i_2 \\
 j_1 + j_2 + j_3 = 0 \quad \left\{ \begin{array}{l} \text{mod. 2} \end{array} \right.
\]

Number of allowed couplings: 24

Expression of the coupling
\[ C_{\theta^3 \theta^4} = \sqrt{l_3} N \sum_{v \in (f_3 - f_2 + \Lambda)_{\perp}} \exp[-\frac{\sqrt{2}}{8\pi} (|v_2|^2 + |v_3|^2)] \]

\[ = \sqrt{l_3} N \sum_{\vec{u} \in \mathbb{Z}^4} \exp[-\frac{\sqrt{2}}{8\pi} (\vec{f}_{23} + \vec{w})^\top M (\vec{f}_{23} + \vec{w})] \]

\[ = \sqrt{l_3} N \vartheta \left[ \begin{array}{c} f_{23} \\ 0 \end{array} \right] [0, \Omega] \]

where the expression \((f_3 - f_2 + \Lambda)_{\perp}\) indicates that the coset elements must belong to \(SO(8)\) and \(\vec{f}_{23}\) is the restriction of \(f_{23}\) to \(SO(8)\), \(l_3\) denotes the number of elements in the \(f_3\) conjugation class, and the arrows denote the components in the \(SO(8)\) lattice.

\[ N = \sqrt{V_{\perp}} \frac{1}{2\pi} \frac{\Gamma(\frac{7}{8})\Gamma(\frac{5}{8})}{\Gamma(\frac{1}{8})\Gamma(\frac{3}{8})}, \quad \Omega = i \frac{\sqrt{2}}{8\pi^2} M \]

\[ \Omega = \left( \begin{array}{ccc} a & b & 0 & c \\ b & d & e & -\frac{d}{2} \\ 0 & e & a & 0 \\ c & -\frac{d}{2} & 0 & f \end{array} \right) \]

\[ a = R_3^2 \]

\[ b = -\frac{3}{4} R_3^2 + \frac{1}{2} R_6^2 \]

\[ c = \frac{1}{2} R_3^2 - \frac{1}{2} R_6^2 \]

\[ d = \frac{1}{2} R_3^2 + \frac{1}{2} R_6^2 \]

\[ e = -\frac{3}{4} R_6^2 + \frac{1}{2} R_3^2 \]

\[ f = R_6^2 \]

where \(V_{\perp}\) is the volume of the \(SO(8)\) lattice

Number of effective parameters: 2

Number of different couplings without deformations: 3

Number of different couplings with deformations: 3

Corresponding to the following \(\vec{f}_{23}\) shifts

\[ \vec{f}_{23} = \left[ \begin{array}{c} l_3 = 1 \\ g_2^{(0)} \\ l_3 = 2 \\ g_2^{(1)} \end{array} \right] \]

\[ \text{Coupling } \theta^3 \theta^4 \]

Selection rule

\[ f_1 + f_2 - (I + \theta + \theta^2 + \theta^3) f_3 \in \Lambda \]
Denoting

\[ f_1 = g_1^{(i_1)} \otimes \hat{g}_1^{(j_1)} \]
\[ f_2 = g_1^{(i_2)} \otimes \hat{g}_1^{(j_2)} \]
\[ f_3 = [\alpha(e_1) + \beta(e_2)] \otimes \hat{g}_1^{(j_3)} \]

\[
\begin{align*}
  i_1, i_2 &= 0, 1, 2, 3, \\
  j_1, j_2 &= 0, 1, \\
  j_3 &= 0, 1, \ldots, 15, \\
  \alpha, \beta &\in R
\end{align*}
\]

the selection rule reads

\[
\begin{align*}
  i_1 &= i_2, \\
  j_1 + j_2 + j_3 &= 0
\end{align*}
\]  \mod 2

Number of allowed couplings: 48

Expression of the coupling

\[
C_{\theta \theta \theta} = \sqrt{l_3} \frac{N}{v \in (f_3 - f_2 + \Lambda) \perp} \exp\left[ -\frac{1}{4\pi} \left( ((\sqrt{2} + 1)|v_2|^2 + (\sqrt{2} - 1)|v_3|^2) \right) \right]
\]

\[
= \sqrt{l_3} \frac{N}{\bar{u} \in \mathbb{Z}^4} \exp\left[ -\frac{1}{4\pi} \left( \bar{f}_{23} + \bar{u} \right)^\top M (\bar{f}_{23} + \bar{u}) \right]
\]

\[
= \sqrt{l_3} \frac{N}{\bar{u}} \begin{bmatrix} \bar{f}_{23} \\ 0 \end{bmatrix} [0, \Omega]
\]

where \((f_3 - f_2 + \Lambda) \perp\) indicates that the coset elements must belong to \(SO(8)\), and \(V_\perp\) is the volume of the \(SO(8)\) lattice

\[
N = \sqrt{V_\perp} \frac{1}{2\pi} \frac{\Gamma(\frac{7}{8})\Gamma(\frac{5}{8})}{\Gamma(\frac{3}{8})\Gamma(\frac{1}{8})} \Omega = i \frac{1}{4\pi^2} M
\]

\[
\Omega = i \frac{1}{4\pi^2} \begin{pmatrix} a & b & 0 & e \\ b & c & d & d \\ 0 & d & a & 0 \\ e & d & 0 & f \end{pmatrix}
\]

\[
\begin{align*}
  a &= [(\sqrt{2} + 1)A^2 + (\sqrt{2} - 1)B^2] \\
  b &= \frac{1}{\sqrt{2}}[(\sqrt{2} + 1)A^2 - (\sqrt{2} - 1)B^2] \\
  c &= [(\sqrt{2} + 2)(\sqrt{2} + 1)A^2 - (\sqrt{2} - 1)(\sqrt{2} - 2)B^2] \\
  d &= -\frac{1}{\sqrt{2}}[(\sqrt{2} + 1)^2A^2 + (\sqrt{2} - 1)^2B^2] \\
  e &= -[(\sqrt{2} + 1)^2A^2 - (\sqrt{2} - 1)^2B^2] \\
  f &= [(\sqrt{2} + 1)^3A^2 + (\sqrt{2} - 1)^3B^2]
\end{align*}
\]

Number of effective parameters: 2

Number of different couplings without deformations: 6

Number of different couplings with deformations: 6
corresponding to the following $\vec{f}_{23}$ shifts

$$
\vec{f}_{23} = \begin{cases}
l_3 = 1 & \begin{cases} g_{4}^{(0)}, \\
g_{4}^{(8)}, \\
g_{4}^{(4)}, \\
g_{4}^{(1)}, \\
g_{4}^{(2)}, \\
g_{4}^{(3)}
\end{cases} \\
l_3 = 2 & g_{4}^{(4)} \\
l_3 = 4 & g_{4}^{(3)}
\end{cases}
$$

**ORBIFOLD $Z_{12}-I$**

**Twist** $\theta = \text{diag}(e^{i\alpha}, e^{4i\alpha}, e^{-5i\alpha})$, $\alpha = \frac{2\pi}{12}$

**Lattice** $SU(3) \otimes F_4$

**Coxeter element**

$$
\theta e_1 = e_2, \quad \theta e_2 = -e_1 - e_2, \quad \theta e_3 = e_4, \\
\theta e_4 = e_3 + e_4 + 2e_5, \quad \theta e_5 = e_6, \quad \theta e_6 = -e_3 - e_4 - e_5 - e_6
$$

**Deformation parameters**

**Relations**

$$
|e_1| = |e_2|, \quad |e_3| = |e_4| = \sqrt{2}|e_5| = \sqrt{2}|e_6|, \quad \alpha_{32} = -\frac{1}{2}, \quad \alpha_{45} = \frac{1}{\sqrt{2}}, \\
\alpha_{34} = \alpha_{56}, \quad \alpha_{35} = \alpha_{46} = \alpha_{36}, \quad \alpha_{35} = -\frac{1}{\sqrt{2}}[1 + 2\alpha_{34}], \quad \alpha_{ij} = 0 \quad i = 1, 2, \quad j = 1, 2, 3, 4
$$

$$
\alpha_{ij} \equiv \cos(\theta_{ij})
$$

**Degrees of freedom (3)**

$$
R_1 = |e_1|, \quad R_3 = |e_3|, \quad \alpha_{34}
$$

**Lattice basis** $(e_i)$ in terms of orthogonal basis $(\tilde{e}_i)$

$$
e_1 = R_1 \cos(\phi_1)\tilde{e}_1 + R_1 \sin(\phi_1)\tilde{e}_2 \\
e_2 = R_1 \cos(\phi_1 + \alpha)\tilde{e}_1 + R_1 \sin(\phi_1 + \alpha)\tilde{e}_2 \\
e_3 = A \cos(\phi_2)\tilde{e}_3 + A \sin(\phi_2)\tilde{e}_4 + B \cos(\phi_3)\tilde{e}_5 + B \cos(\phi_3)\tilde{e}_6 \\
e_4 = A \cos(\phi_2 + \beta)\tilde{e}_3 + A \sin(\phi_2 + \beta)\tilde{e}_4 + B \cos(\phi_3 + 7\beta)\tilde{e}_5 + B \cos(\phi_3 + 7\beta)\tilde{e}_6 \\
e_5 = \frac{A}{\sqrt{2}}[-\sin(\phi_2 + \frac{5}{2}\beta)\tilde{e}_3 + \cos(\phi_2 + \frac{5}{2}\beta)\tilde{e}_4] + \frac{B}{\sqrt{2}}[\sin(\phi_3 + \frac{11}{2}\beta)\tilde{e}_5 - \cos(\phi_3 + \frac{11}{2}\beta)\tilde{e}_6] \\
e_6 = \frac{A}{\sqrt{2}}[-\sin(\phi_2 + \frac{7}{2}\beta)\tilde{e}_3 + \cos(\phi_2 + \frac{7}{2}\beta)\tilde{e}_4] + \frac{B}{\sqrt{2}}[\sin(\phi_3 + \frac{1}{2}\beta)\tilde{e}_5 - \cos(\phi_3 + \frac{1}{2}\beta)\tilde{e}_6]
$$
Fixed points of $\theta$ (3)

$$f^{(i)}_1 = g^{(i)}_1 \otimes \hat{g}^{(0)}_1, \quad i = 0, 1, 2$$

$$g^{(0)}_1 = (0, 0), \quad g^{(1)}_1 = \frac{1}{3}(1, 2), \quad g^{(2)}_1 = \frac{1}{3}(2, 1), \quad \hat{g}^{(0)}_1 = (0, 0, 0, 0)$$

Fixed points of $\theta^2$ (3)

The same as for $\theta$

$$f^{(i)}_2 = g^{(i)}_2 \otimes \hat{g}^{(0)}_2, \quad i = 0, 1, 2$$

$$g^{(0)}_2 = (0, 0), \quad g^{(1)}_2 = \frac{1}{3}(1, 2), \quad g^{(2)}_2 = \frac{1}{3}(2, 1), \quad \hat{g}^{(0)}_2 = (0, 0, 0, 0)$$

Fixed points of $\theta^3$ (4)

Fixed torus: $\alpha(e_1) + \beta(e_2), \quad \alpha, \beta \in R$

$$f^{(i)}_3 = [\alpha(e_1) + \beta(e_2)] \otimes \hat{g}^{(i)}_3, \quad i = 0, 1, 2, 3, \quad \alpha, \beta \in R$$

$$\hat{g}^{(0)}_3 = (0, 0, 0, 0), \quad \hat{g}^{(1)}_3 = \frac{1}{2}(1, 0, 0, 0), \quad \hat{g}^{(2)}_3 = \frac{1}{2}(0, 1, 0, 0), \quad \hat{g}^{(3)}_3 = \frac{1}{2}(1, 1, 0, 0)$$

Note that in the $F_4$ lattice $\theta: \hat{g}^{(1)}_3 \rightarrow \hat{g}^{(2)}_3 \rightarrow \hat{g}^{(3)}_3$

Number of conjugation classes: 2

Fixed points of $\theta^4$ (27)

$$f^{(i)(j)}_4 = g^{(i)}_4 \otimes \hat{g}^{(j)}_4, \quad i = 0, 1, 2, \quad j = 0, 1, ..., 8$$

$$g^{(0)}_4 = (0, 0), \quad g^{(1)}_4 = \frac{1}{3}(1, 2), \quad g^{(2)}_4 = \frac{1}{3}(2, 1), \quad \hat{g}^{(0)}_4 = (0, 0, 0, 0),$$

$$\hat{g}^{(1)}_4 = \frac{1}{3}(2, 1, 2, 0), \quad g^{(2)}_4 = \frac{1}{3}(2, 2, 0, 2), \quad g^{(3)}_4 = \frac{1}{3}(1, 0, 2, 2), \quad \hat{g}^{(4)}_4 = \frac{1}{3}(0, 2, 2, 1),$$

$$\hat{g}^{(5)}_4 = \frac{1}{3}(1, 2, 1, 0), \quad \hat{g}^{(6)}_4 = \frac{1}{3}(1, 1, 0, 1), \quad \hat{g}^{(7)}_4 = \frac{1}{3}(2, 0, 1, 1), \quad \hat{g}^{(8)}_4 = \frac{1}{3}(0, 1, 1, 2)$$

Note that in the $F_4$ lattice $\theta: \hat{g}^{(1)}_4 \rightarrow \hat{g}^{(3)}_4 \rightarrow \hat{g}^{(5)}_4 \rightarrow \hat{g}^{(7)}_4$ and $\theta: \hat{g}^{(2)}_4 \rightarrow \hat{g}^{(4)}_4 \rightarrow \hat{g}^{(6)}_4 \rightarrow \hat{g}^{(8)}_4$
θ

Fixed points of θ^5 (3)

The same as for θ

\[ f_5^{(i)} = g_5^{(i)} \otimes \hat{g}_5^{(0)}, \quad i = 0, 1, 2 \]

\[ g_5^{(0)} = (0, 0), \quad g_5^{(1)} = \frac{1}{3}(1, 2), \quad g_5^{(2)} = \frac{1}{3}(2, 1), \quad \hat{g}_5^{(0)} = (0, 0, 0, 0) \]

Fixed points of θ^6 (16)

Fixed torus: α(e_1) + β(e_2), \quad α, β ∈ R

\[ f_6^{(i)} = [α(e_1) + β(e_2)] \otimes \hat{g}_6^{(i)}, \quad i = 0, 1, ..., 15, \quad α, β ∈ R \]

\[ \begin{align*}
    \hat{g}_6^{(0)} &= (0, 0, 0, 0), \quad \hat{g}_6^{(1)} = \frac{1}{2}(1, 1, 1, 1), \quad \hat{g}_6^{(2)} = \frac{1}{2}(0, 0, 0, 1), \quad \hat{g}_6^{(3)} = \frac{1}{2}(0, 0, 1, 0), \\
    \hat{g}_6^{(4)} &= \frac{1}{2}(1, 0, 0, 0), \quad \hat{g}_6^{(5)} = \frac{1}{2}(0, 0, 1, 1), \quad \hat{g}_6^{(6)} = \frac{1}{2}(0, 1, 0, 1), \quad \hat{g}_6^{(7)} = \frac{1}{2}(1, 0, 1, 0), \\
    \hat{g}_6^{(8)} &= \frac{1}{2}(0, 1, 0, 0), \quad \hat{g}_6^{(9)} = \frac{1}{2}(0, 1, 1, 1), \quad \hat{g}_6^{(10)} = \frac{1}{2}(1, 1, 0, 1), \quad \hat{g}_6^{(11)} = \frac{1}{2}(0, 1, 1, 0), \\
    \hat{g}_6^{(12)} &= \frac{1}{2}(1, 1, 0, 0), \quad \hat{g}_6^{(13)} = \frac{1}{2}(1, 0, 1, 1), \quad \hat{g}_6^{(14)} = \frac{1}{2}(1, 0, 0, 1), \quad \hat{g}_6^{(15)} = \frac{1}{2}(1, 1, 1, 0) 
\end{align*} \]

Note that in the F_4 lattice θ : \hat{g}_6^{(3)} → \hat{g}_6^{(2)} → \hat{g}_6^{(1)} → \hat{g}_6^{(11)} → \hat{g}_6^{(10)} → \hat{g}_6^{(9)},

\[ \theta : \hat{g}_6^{(7)} → \hat{g}_6^{(6)} → \hat{g}_6^{(5)} → \hat{g}_6^{(15)} → \hat{g}_6^{(14)} → \hat{g}_6^{(13)} \]

and θ : \hat{g}_6^{(4)} → \hat{g}_6^{(8)} → \hat{g}_6^{(12)}

Number of conjugation classes: 4

Coupling θθ^2θ^6

Selection rule

\[ f_1 + (I + θ)f_2 - (I + θ^2)f_3 \in \Lambda \]

Denoting

\[
\begin{align*}
    f_1 &= g_1^{(i_1)} \otimes \hat{g}_1^{(0)} \\
    f_2 &= g_2^{(i_2)} \otimes \hat{g}_2^{(0)} \\
    f_3 &= [α(e_1) + β(e_2)] \otimes \hat{g}_3^{(j_3)}
\end{align*}
\]

\[ \begin{align*}
    i_1, i_2, j_3 &= 0, 1, 2, 3, \\
    α, β &\in R
\end{align*} \]

the selection rule reads

\[ i_1 = i_2 \]

Number of allowed couplings: 6

Expression of the coupling
\[ C_{\theta_2 \theta_3} = \sqrt{l_3} N \sum_{v \in (f_3 - f_2 + \Lambda)_\perp} \exp\left[ -\frac{1}{4\pi} \sin\left( \frac{\pi}{6} \sin\left( \frac{\pi}{4} \right) \left( \frac{|v_2|^2}{\sin\left( \frac{\pi}{12} \right)} \right) + \frac{|v_3|^2}{\cos\left( \frac{\pi}{12} \right)} \right] \]

\[ = \sqrt{l_3} N \sum_{v \in \mathbb{Z}^4} \exp\left[ -\frac{\sqrt{2}}{4\pi} (\vec{f}_{23} + \vec{u})^\top M (\vec{f}_{23} + \vec{u}) \right] \]

\[ = \sqrt{l_3} N \vartheta \begin{bmatrix} f_{23} \\ 0 \end{bmatrix} [0, \Omega] \]

where \((f_3 - f_2 + \Lambda)_\perp\) indicates that the coset elements must belong to \(F_4\), \(l_3\) is the number of elements in the \(f_3\) conjugation class, the arrows denote components in the \(F_4\) lattice, and \(V_\perp\) is the volume of the \(F_4\) lattice unit cell.

\[ N = \sqrt{V_\perp} \frac{1}{2\pi} \frac{\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{1}{6}\right)} \left[ \frac{\Gamma\left(\frac{11}{12}\right) \Gamma\left(\frac{5}{12}\right)}{\Gamma\left(\frac{7}{12}\right) \Gamma\left(\frac{5}{12}\right)} \right]^{1/2}, \quad \Omega = i \frac{\sqrt{2}}{4\pi^2} M \]

\[ \Omega = i \frac{\sqrt{2}}{4\pi^2} \begin{pmatrix} a & \frac{b\sqrt{3}}{2} & -b \sqrt{2} & -b \\ \frac{b\sqrt{3}}{2} & a & -b \frac{\sqrt{2}}{2} & -b \\ -b \sqrt{2} & -a \frac{\sqrt{2}}{2} & a & 0 \\ -b \frac{\sqrt{3}}{2} & \frac{b\sqrt{3}}{2} & 0 & \frac{a}{2} \end{pmatrix} \]

\[ a = [A^2 \cos(\frac{\pi}{12}) + B^2 \sin(\frac{\pi}{12})] \]

\[ b = [A^2 \cos(\frac{\pi}{12}) - B^2 \sin(\frac{\pi}{12})] \]

Number of effective parameters: 2
Number of different couplings without deformations: 2
Number of different couplings with deformations: 2
corresponding to the following \(f_{23}\) shifts

\[ \vec{f}_{23} = \hat{g}_3^{(0)}, \hat{g}_3^{(1)} \]

Note that this coupling is the same as \(\theta^2 \theta^3 \theta^7\)

**Coupling \(\theta \theta^4 \theta^7\)**

**Selection rule**

\[ f_1 + (I + \theta + \theta^2 + \theta^3) f_2 - (I + \theta + \theta^2 + \theta^3 + \theta^4) f_3 \in \Lambda \]

Denoting

\[
\begin{align*}
\begin{align*}
f_1 &= g_{1}^{(i_1)} \otimes \hat{g}_1^{(0)} \\
f_2 &= g_{4}^{(i_2)} \otimes \hat{g}_4^{(j_2)} \\
f_3 &= g_{5}^{(i_3)} \otimes \hat{g}_5^{(0)}
\end{align*}
\end{align*}
\]

\[ i_1, i_2, i_3 = 0, 1, 2, \]

\[ j_2 = 0, 1, \ldots, 8 \]

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the selection rule reads

\[ i_1 + i_2 + i_3 = 0 \mod 3 \]

Number of allowed couplings: 6

Expression of the coupling

\[
C_{\vartheta \vartheta \vartheta} = \sqrt{l_2} N \sum_{v \in (f_3 - f_2 + \Lambda)} \exp\left[-\frac{1}{4\pi} \left(\frac{\sqrt{3}}{2}|v_1|^2 + 2\sqrt{3}\left(\frac{|v_2|^2}{\cos^2(\frac{5\pi}{12})} + \frac{|v_3|^2}{\sin^2(\frac{5\pi}{12})}\right)\right] \\
= \sqrt{l_2} N \sum_{\vec{u} \in \mathbb{Z}^6} \exp\left[-\frac{\sqrt{3}}{8\pi} (\vec{f}_{23} + \vec{u})^\top M (\vec{f}_{23} + \vec{u})\right] \\
= \sqrt{l_2} N \Omega \left[ \begin{array}{c} \vec{f}_{23} \\ 0 \end{array} \right] [0, \Omega]
\]

\[ l_2 \] is the number of elements in the \( f_2 \) conjugation class, and the arrows denote components in the \( SU(3) \otimes F_4 \) lattice

\[
N = \sqrt{V_\Lambda} \frac{3^{1/4}}{4\pi^2} \frac{\Gamma^3(\frac{2}{3})}{\Gamma^2(\frac{1}{3})} \frac{\Gamma(\frac{11}{12})\Gamma(\frac{5}{12})}{\Gamma(\frac{1}{12})\Gamma(\frac{7}{12})} , \quad \Omega = i \frac{\sqrt{2}}{4\pi^2} M
\]

\[
\Omega = \frac{i \sqrt{3}}{8\pi^2} \begin{pmatrix}
R_1^2 & -R_2^2/2 & 0 & 0 & 0 & 0 \\
-R_2^2/2 & R_1^2 & 0 & 0 & 0 & 0 \\
0 & 0 & a & \frac{bx}{\sqrt{2}} & -b & -b \\
0 & 0 & \frac{bx}{\sqrt{2}} & a & -a/2 & \frac{by}{\sqrt{2}} \\
0 & 0 & -b/\sqrt{2} & -a/2 & \frac{a}{2} & \frac{bx}{4} \\
0 & 0 & -b/\sqrt{2} & \frac{bx}{4} & \frac{by}{\sqrt{2}} & \frac{a}{2}
\end{pmatrix}
\]

Number of effective parameters: 3

Number of different couplings without deformations: 4

Number of different couplings with deformations: 4

corresponding to the following \( \vec{f}_{23} \) shifts

\[
\vec{f}_{23} = \begin{cases}
\begin{array}{c}
l_2 = 1 \quad \{ g_4^{(0)} \otimes g_4^{(0)}, g_4^{(1)} \otimes g_4^{(0)} \}
\end{array}
\end{cases}
\begin{cases}
\begin{array}{c}
l_2 = 2 \quad \{ g_4^{(0)} \otimes g_4^{(1)}, g_4^{(1)} \otimes g_4^{(1)} \}
\end{array}
\end{cases}
\]

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Coupling $\theta^2 \theta^4 \theta^6$

Selection rule

$$f_1 + (I + \theta^2)f_2 - (I + \theta^2 + \theta^4)f_3 \in \Lambda$$

Denoting

$$f_1 = g_{i_1}^{(i_1)} \otimes \hat{g}_2^{(0)} \quad \text{for} \quad i_1, i_2 = 0, 1, 2,$$

$$f_2 = g_{j_2}^{(j_2)} \otimes \hat{g}_4^{(j_2)} \quad \text{for} \quad j_2 = 0, 1, \ldots, 8,$$

$$f_3 = [\alpha(e_1) + \beta(e_2)] \otimes \hat{g}_6^{(j_3)} \quad \text{for} \quad j_3 = 0, 1, \ldots, 15,$$

$$\alpha, \beta \in R$$

the selection rule reads

$$i_1 = i_2 ,$$

Number of allowed couplings: 36

Expression of the coupling (in all the cases except $l_3 = 6, l_2 = 4$)

$$C_{\theta^2 \theta^4 \theta^6} = N \sqrt{l_2 l_3} \sum_{v \in (f_{23} + \Lambda)_\perp} \exp\left[-\frac{1}{4\pi} \frac{\sin\left(\frac{\pi}{3}\right)}{\sin\left(\frac{\pi}{6}\right)} |v|^2\right]$$

$$= N \sqrt{l_2 l_3} \sum_{\vec{u} \in \mathbb{Z}^4} \exp\left[-\frac{\sqrt{3}}{4\pi} (\vec{f}_{23} + \vec{u})^\top M(\vec{f}_{23} + \vec{u})\right]$$

$$= N \sqrt{l_2 l_3} \vartheta \left[\begin{array}{c} \vec{f}_{23} \\ 0 \end{array}\right] [0, \Omega]$$

where $f_{23} = f_2 - f_3$, $\vec{f}_{23}$ is the restriction of $f_{23}$ to the $F_4$ lattice; $(f_{23} + \Lambda)_\perp$ indicates that the coset must belong to $F_4$ and $l_i$ is the number of elements in the $f_i$ conjugation class. $V_\perp$ is the volume of the $F_4$ unit cell

In the $l_3 = 6, l_2 = 4$ case

$$C_{\theta^2 \theta^4 \theta^6} = N \sqrt{6} \sum_{v \in (f_2 - f_3 + \Lambda)_\perp \cup (\theta f_2 - f_3 + \Lambda)_\perp} \exp\left[-\frac{1}{4\pi} \frac{\sin\left(\frac{\pi}{3}\right)}{\sin\left(\frac{\pi}{6}\right)} |v|^2\right]$$

$$= N \sqrt{6} \left\{ \vartheta \left[\begin{array}{c} \vec{f}_{23} \\ 0 \end{array}\right] [0, \Omega] + \vartheta \left[\begin{array}{c} \vec{f}_{23}^\prime \\ 0 \end{array}\right] [0, \Omega] \right\}$$

$$N = \sqrt{V_\perp} \frac{1}{2\pi} \left[\frac{\Gamma\left(\frac{2}{3}\right) \Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{1}{3}\right) \Gamma\left(\frac{2}{3}\right)}\right], \quad \Omega = i \frac{\sqrt{3}}{4\pi^2} M$$
Coupling \( \theta \)

\[
\Omega = \frac{i \sqrt{3}}{4 \pi^2 R_3^2} \begin{pmatrix}
1 & \alpha_{34} & -\frac{1}{4}[1 + 2\alpha_{34}] & -\frac{1}{4}[1 + 2\alpha_{34}]
\alpha_{34} & 1 & -\frac{1}{2} & -\frac{1}{4}[1 + 2\alpha_{34}]
-\frac{1}{4}[1 + 2\alpha_{34}] & -\frac{1}{2} & \frac{1}{2} & \frac{\alpha_{34}}{2}
-\frac{1}{4}[1 + 2\alpha_{34}] & -\frac{1}{4}[1 + 2\alpha_{34}] & \frac{\alpha_{34}}{2} & \frac{1}{2}
\end{pmatrix}
\]

Number of effective parameters: 2
Number of different couplings without deformations: 7
Number of different couplings with deformations: 7

corresponding to the following \( \vec{f}_{23} \) shifts

\[
\vec{f}_{23} = \begin{cases}
l_2 = l_3 = 1 & (0, 0, 0, 0), \\
l_2 = 1 l_3 = 3 & (\frac{1}{2}, 0, 0, 0), \\
l_2 = 1 l_3 = 6 & (0, 0, \frac{1}{2}, 0), \\
l_2 = 4 l_3 = 1 & (\frac{2}{3}, \frac{1}{3}, \frac{2}{3}, 0), \\
l_2 = 4 l_3 = 3 & (\frac{1}{6}, \frac{1}{2}, \frac{1}{2}, 0), \\
l_2 = 4 l_3 = 6 & (\frac{2}{3}, \frac{1}{3}, \frac{1}{6}, 0) \cup (\frac{2}{3}, \frac{1}{3}, \frac{2}{3}, \frac{1}{2}) \cup (\frac{1}{6}, \frac{1}{2}, \frac{1}{2}, 0) \cup (\frac{2}{3}, \frac{5}{6}, \frac{2}{3}, \frac{1}{2})
\end{cases}
\]

Coupling \( \theta^4 \theta^4 \theta^4 \)

Selection rule

\[
f_1 + f_2 + f_3 \in \Lambda
\]

Denoting

\[
f_1 = \tilde{g}_4^{(i_1)} \otimes \tilde{g}_4^{(j_1)},
\]

\[
f_2 = \tilde{g}_4^{(i_2)} \otimes \tilde{g}_4^{(j_2)},
\]

\[
f_3 = \tilde{g}_4^{(i_3)} \otimes \tilde{g}_4^{(j_3)}
\]

\[
i_1, i_2, i_3 = 0, 1, 2,
\]

\[
j_1, j_2, j_3 = 0, 1, ..., 8,
\]

The selection rule reads

\[
i_1 + i_2 + i_3 = 0 \mod 3
\]

\[
j_{\sigma(1)} = 0 \quad j_{\sigma(2)}, j_{\sigma(3)} \not= 0 \quad j_{\sigma(2)} - j_{\sigma(3)} = 4
\]

\[
j_{\sigma(1)} \text{ even } j_{\sigma(2)}, j_{\sigma(3)} \text{ odd } \quad j_{\sigma(3)} - 5 = j_{\sigma(2)} + 1 = j_{\sigma(1)}
\]

\[
j_{\sigma(1)} \text{ odd } j_{\sigma(2)}, j_{\sigma(3)} \text{ even } \quad j_{\sigma(3)} - 7 = j_{\sigma(2)} - 5 = j_{\sigma(1)}
\]

\[
j_{\sigma(1)}, j_{\sigma(2)}, j_{\sigma(3)} \text{ odd or even } \quad j_{\sigma(3)} = j_{\sigma(2)} = j_{\sigma(1)}
\]

\[
\mod 8
\]

\[
\sigma \equiv \text{ permutation of } \{1, 2, 3\}
\]

Number of allowed couplings: 189
Expression of the coupling
\[ C_{\theta^{4} \theta^{4}} = N \sum_{v \in (f_{3} - f_{2} + A)} \exp \left[ -\frac{\sqrt{3}}{8\pi} |v|^2 \right] \]

\[ = N \sum_{\vec{u} \in \mathbb{Z}^6} \exp \left[ -\frac{\sqrt{3}}{8\pi} (\vec{f}_{23} + \vec{u})^\top M (\vec{f}_{23} + \vec{u}) \right] \]

where \( F = 1 \) except for the case \( f_{1} = f_{2} = f_{3} \) and \( l_{1} = l_{2} = l_{3} = 4 \) in which \( F = \frac{1}{2} \)

\[ \Omega = i \frac{\sqrt{3}}{8\pi} M \quad N = \sqrt{V_{\Lambda}} \frac{3^{1/4}}{4\pi^2} \left[ \frac{\Gamma(5/2)}{\Gamma(4/6)} \right] \]

\[ \Omega = \begin{pmatrix} R_{3}^{2} & -\frac{R_{3}^{2}}{2} & 0 & 0 & 0 & 0 \\
-\frac{R_{3}^{2}}{2} & R_{3}^{2} & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{R_{3}^{2}}{2} & R_{3}^{2} \alpha_{34} & -\frac{R_{3}^{2} \alpha_{34} + 2\alpha_{34}}{4} & -\frac{R_{3}^{2} \alpha_{34} + 2\alpha_{34}}{4} \\
0 & 0 & -\frac{R_{3}^{2} \alpha_{34} + 2\alpha_{34}}{4} & -\frac{R_{3}^{2}}{2} & -\frac{R_{3}^{2}}{2} & R_{3}^{2} \alpha_{34} \\
0 & 0 & R_{3}^{2} \alpha_{34} & -\frac{R_{3}^{2}}{2} & -\frac{R_{3}^{2}}{2} & R_{3}^{2} \alpha_{34} \\
0 & 0 & R_{3}^{2} & 0 & 0 & 0 \end{pmatrix} \]

Number of effective parameters: 3
Number of different couplings without deformations: 6
Number of different couplings with deformations: 6

\[ \vec{f}_{23} = \begin{cases} F = \frac{1}{2} \\
F = 1 \end{cases} \begin{cases} (0, 0, 0, 0, 0, 0), \\
(0, 0, 0, 0, 0, 0), \\
(1, 0, 0, 0, 0, 0), \\
(0, 1, 0, 0, 0, 0), \\
(0, 0, 1, 0, 0, 0), \\
(0, 0, 0, 1, 0, 0), \\
(0, 0, 0, 0, 1, 0), \\
(0, 0, 0, 0, 0, 1) \end{cases} \]

\[ \textbf{ORBIFOLD } Z_{12} - \text{II} \]

Twist \( \theta = \text{diag}(e^{i\alpha}, e^{5i\alpha}, e^{-6i\alpha}) \), \( \alpha = \frac{2\pi}{12} \)
Lattice \( SO(4) \otimes F_{4} \)
Coxeter element
\[ \theta e_1 = -e_1, \quad \theta e_2 = -e_2, \quad \theta e_3 = e_4, \]
\[ \theta e_4 = e_3 + e_4 + 2e_5, \quad \theta e_5 = e_6, \quad \theta e_6 = -e_3 - e_4 - e_5 - e_6 \]

Deformation parameters

Relations
\[ |e_3| = |e_4| = \sqrt{2}|e_5| = \sqrt{2}|e_6|, \quad \alpha_{34} = -\frac{1}{\sqrt{2}}, \quad \alpha_{35} = \alpha_{46} = \alpha_{36}, \quad \alpha_{35} = -\frac{1}{2\sqrt{2}}[1 + 2\alpha_{34}] \]
\[ \alpha_{ij} = \cos(\theta_{ij}) \]

Degrees of freedom (5)
\[ R_1 = |e_1|, \quad R_2 = |e_2|, \quad R_3 = |e_3|, \quad \alpha_{12}, \quad \alpha_{34} \]

Lattice basis \((e_i)\) in terms of orthogonal basis \((\tilde{e}_i)\)

Not necessary in this case.

Fixed points of \(\theta\) \((4)\)
\[ f^{(i)} = g^{(i)} \otimes \hat{g}^{(0)} \quad 0, 1, 2, 3 \]
\[ g^{(0)} = (0, 0), \quad g^{(1)} = \frac{1}{2}(0, 1), \quad g^{(2)} = \frac{1}{2}(1, 1) \]
\[ g^{(3)} = \frac{1}{2}(1, 0), \quad \hat{g}^{(0)} = (0, 0, 0) \]

Fixed points of \(\theta^2\) \((1)\)
Fixed torus: \(\alpha(e_1) + \beta(e_2)\), \(\alpha, \beta \in R\)
\[ f^{(2)} = [\alpha(e_1) + \beta(e_2)] \otimes \hat{g}_2^{(0)}, \quad \alpha, \beta \in R \]
\[ \hat{g}_2^{(0)} = (0, 0, 0, 0) \]

Fixed points of \(\theta^3\) \((16)\)
\[ f^{(ij)} = g^{(i)} \otimes \hat{g}_3^{(j)}, \quad i, j = 0, 1, 2, 3 \]
\[ g^{(0)} = (0, 0), \quad g^{(1)} = \frac{1}{2}(0, 1), \quad g^{(2)} = \frac{1}{2}(1, 1), \quad g^{(3)} = \frac{1}{2}(1, 0) \]
\[ \hat{g}_3^{(0)} = (0, 0, 0, 0), \quad \hat{g}_3^{(1)} = \frac{1}{2}(1, 0, 0, 0), \quad \hat{g}_3^{(2)} = \frac{1}{2}(1, 1, 0, 0), \quad \hat{g}_3^{(3)} = \frac{1}{2}(0, 1, 0, 0) \]

Note that, in \(F_4\), \(\theta : \hat{g}_3^{(1)} \rightarrow \hat{g}_3^{(3)} \rightarrow \hat{g}_3^{(2)}\)
Number of conjugation classes: 8

Fixed points of $\theta^4$ (9)

Fixed torus: $\alpha(e_1) + \beta(e_2)\, ,\, \alpha, \beta \in \mathcal{R}$

$$f_4^{(i)} = [\alpha(e_1) + \beta(e_2)] \otimes \hat{g}_4^{(i)}, \, i = 0, 1, ..., 8$$

$$\hat{g}_4^{(0)} = (0, 0, 0, 0), \quad \hat{g}_4^{(1)} = \frac{1}{3}(2, 1, 2, 0), \quad \hat{g}_4^{(2)} = \frac{1}{3}(2, 2, 0, 2), \quad \hat{g}_4^{(3)} = \frac{1}{3}(1, 0, 2, 2), \quad \hat{g}_4^{(4)} = \frac{1}{3}(0, 2, 2, 1), \quad \hat{g}_4^{(5)} = \frac{1}{3}(1, 1, 2, 0), \quad \hat{g}_4^{(6)} = \frac{1}{3}(1, 1, 0, 1), \quad \hat{g}_4^{(7)} = \frac{1}{3}(2, 0, 1, 1), \quad \hat{g}_4^{(8)} = \frac{1}{3}(0, 1, 1, 2)$$

Note that, in $F_4$, $\theta: \hat{g}_4^{(1)} \rightarrow \hat{g}_4^{(3)} \rightarrow \hat{g}_4^{(5)} \rightarrow \hat{g}_4^{(7)}$ and $\theta: \hat{g}_4^{(2)} \rightarrow \hat{g}_4^{(4)} \rightarrow \hat{g}_4^{(6)} \rightarrow \hat{g}_4^{(8)}$

Number of conjugation classes: 3

Fixed points of $\theta^5$ (4)

The same as for $\theta$.

$$f_5^{(i)} = g_5^{(i)} \otimes \hat{g}_5^{(0)}, \quad i = 0, 1, 2, 3$$

$$g_5^{(0)} = (0, 0), \quad g_5^{(1)} = \frac{1}{3}(0, 1), \quad g_5^{(2)} = \frac{1}{3}(1, 1), \quad g_5^{(3)} = \frac{1}{3}(1, 0), \quad g_5^{(0)} = (0, 0, 0, 0)$$

Fixed points of $\theta^6$ (16)

Fixed torus: $\alpha(e_1) + \beta(e_2)\, ,\, \alpha, \beta \in \mathcal{R}$

$$f_6^{(i)} = [\alpha(e_1) + \beta(e_2)] \otimes \hat{g}_6^{(i)}, \, i = 0, 1, ..., 15, \, \alpha, \beta \in \mathcal{R}$$

$$\hat{g}_6^{(0)} = (0, 0, 0, 0), \quad \hat{g}_6^{(1)} = \frac{1}{3}(1, 1, 1, 1), \quad \hat{g}_6^{(2)} = \frac{1}{3}(0, 0, 0, 1), \quad \hat{g}_6^{(3)} = \frac{1}{3}(0, 0, 1, 0), \quad \hat{g}_6^{(4)} = \frac{1}{2}(1, 0, 0, 0), \quad \hat{g}_6^{(5)} = \frac{1}{2}(0, 0, 1, 1), \quad \hat{g}_6^{(6)} = \frac{1}{2}(0, 1, 0, 1), \quad \hat{g}_6^{(7)} = \frac{1}{2}(1, 0, 1, 0), \quad \hat{g}_6^{(8)} = \frac{1}{2}(0, 1, 0, 0), \quad \hat{g}_6^{(9)} = \frac{1}{2}(0, 1, 1, 1), \quad \hat{g}_6^{(10)} = \frac{1}{2}(1, 1, 0, 1), \quad \hat{g}_6^{(11)} = \frac{1}{2}(0, 1, 1, 0), \quad \hat{g}_6^{(12)} = \frac{1}{2}(1, 1, 0, 0), \quad \hat{g}_6^{(13)} = \frac{1}{2}(1, 0, 1, 1), \quad \hat{g}_6^{(14)} = \frac{1}{2}(1, 0, 0, 1), \quad \hat{g}_6^{(15)} = \frac{1}{2}(1, 1, 1, 0)$$

Note that in $F_4$, $\theta: \hat{g}_6^{(3)} \rightarrow \hat{g}_6^{(2)} \rightarrow \hat{g}_6^{(1)} \rightarrow \hat{g}_6^{(11)} \rightarrow \hat{g}_6^{(10)} \rightarrow \hat{g}_6^{(9)} \rightarrow \hat{g}_6^{(8)} \rightarrow \hat{g}_6^{(6)} \rightarrow \hat{g}_6^{(5)} \rightarrow \hat{g}_6^{(15)} \rightarrow \hat{g}_6^{(14)} \rightarrow \hat{g}_6^{(13)} \rightarrow \hat{g}_6^{(12)} \rightarrow \hat{g}_6^{(4)} \rightarrow \hat{g}_6^{(8)} \rightarrow \hat{g}_6^{(12)}$

Number of conjugation classes: 4

Coupling $\theta \theta \theta^{10}$

Selection rule

$$f_1 + f_2 - (I + \theta) f_3 \in \Lambda$$
Denoting
\[
\begin{align*}
  f_1 &= g^{(1)}_1 \otimes \hat{g}^{(0)}_1 \\
  f_2 &= g^{(1)}_2 \otimes \hat{g}^{(0)}_1 \\
  f_3 &= [\alpha(e_1) + \beta(e_2)] \otimes \hat{g}^{(0)}_2
\end{align*}
\]
\[
\left\{ \begin{array}{c}
i_1, i_2 = 0, 1, 2, 3, \alpha, \beta \in R
\end{array} \right.
\]
The selection rule reads
\[
i_1 = i_2,
\]
Number of allowed couplings: 4
Expression of the coupling
\[
C_{\theta\theta\theta^{10}} = N \sum_{v \in (f_3 - f_2 + \Lambda)_\perp} \exp\left[-\frac{1}{4\pi} \sin\left(\frac{\pi}{6} |v|^2\right)\right]
\]
\[
= N \sum_{\vec{u} \in \mathbb{Z}^4} \exp\left[-\frac{1}{4\pi} \sin\left(\frac{\pi}{6} (\vec{f}_2 - \vec{u})^\top M (\vec{f}_2 + \vec{u})\right)\right]
\]
\[
= N \vartheta \left[ \begin{array}{c} \vec{f}_2 \\
0 \end{array} \right] [0, \Omega]
\]
where \(\vec{f}_2\) is the restriction of \(f_2\) to the \(F_4\) lattice, \((f_3 - f_2 + \Lambda)_\perp\) indicates that the coset elements must belong to \(F_4\) and \(V_\perp\) is the volume of the \(F_4\) unit cell. In all cases \(\vec{f}_2 = 0\). Finally
\[
\Omega = i \frac{1}{4\pi^2} \sin\left(\frac{\pi}{6}\right) M, \quad N = \sqrt{V_\perp} \frac{1}{2\pi} \left[ \frac{\Gamma\left(\frac{11}{12}\right)\Gamma\left(\frac{7}{12}\right)}{\Gamma\left(\frac{11}{12}\right)\Gamma\left(\frac{5}{12}\right)} \right]
\]
\[
\Omega = i \frac{1}{4\pi^2} \sin\left(\frac{\pi}{6}\right) R_3^2 \left( \begin{array}{ccc} 1 & \alpha_{34} & -\frac{1}{4}[1 + 2\alpha_{34}] \\
\alpha_{34} & 1 & -\frac{1}{2} \\
-\frac{1}{4}[1 + 2\alpha_{34}] & -\frac{1}{2} & \alpha_{34}\frac{2}{2} \end{array} \right)
\]
Number of effective parameters: 2
Number of different couplings without deformations: 1
Number of different couplings with deformations: 1
Note that this coupling is the same as \(\theta^2\theta^5\theta^5\)

**Coupling \(\theta\theta^3\theta^8\)**
Selection rule

\[ f_1 + (I + \theta + \theta^2)f_2 - (I + \theta + \theta^2 + \theta^3)f_3 \in \Lambda \]

Denoting

\[
\begin{aligned}
    f_1 &= g_{1}^{(i_1)} \otimes \hat{g}_1^{(0)} \\
    f_2 &= g_{3}^{(j_2)} \otimes \hat{g}_3^{(j_2)} \\
    f_3 &= [\alpha(e_1) + \beta(e_2)] \otimes \hat{g}_4^{(j_3)}
\end{aligned}
\]

\(i_1, i_2, j_2 = 0, 1, 2, 3\),

\(j_3 = 0, 1, ..., 8\),

\(\alpha, \beta \in \mathbb{R}\)

the selection rule reads

\[ i_1 = i_2 \]

Number of allowed couplings: 6

Expression of the coupling

\[
C_{\theta\theta^3\theta^8} = N \sqrt{l_2 l_3} \sum_{v \in (f_3 - f_2 + \Lambda)_\perp} \exp\left(-\frac{1}{4\pi} \frac{\sin(\frac{\pi}{4}) \sin(\frac{\pi}{12})}{\sin(\frac{\pi}{12})} |v|^2 \right)
\]

\[
= N \sqrt{l_2 l_3} \sum_{u \in \mathbb{Z}^4} \exp\left(-\frac{1}{4\pi} \frac{\sin(\frac{\pi}{4}) \sin(\frac{\pi}{12})}{\sin(\frac{\pi}{12})} (\vec{f}_{23} + \vec{u})^\top M (\vec{f}_{23} + \vec{u}) \right)
\]

\[
= N \sqrt{l_2 l_3} \nu \begin{bmatrix} \vec{f}_{23} \\ 0 \end{bmatrix} [0, \Omega]
\]

with the same notation as in the previous coupling, \(l_i\) is the number of elements of the \(f_i\) conjugation class and \(V_\perp\) is the volume of the \(F_4\) unit cell. Finally

\[
\Omega = i \frac{1}{4\pi^2} \frac{\sin(\frac{\pi}{4}) \sin(\frac{\pi}{12})}{\sin(\frac{\pi}{12})} M , \quad N = \sqrt{V_\perp} \frac{1}{2\pi} \frac{\Gamma(\frac{3}{4}) \Gamma(\frac{11}{12}) \Gamma(\frac{7}{12})}{\Gamma(\frac{7}{12}) \Gamma(\frac{11}{12})} \]

\[
\Omega = i \frac{1}{4\pi^2} \frac{\sin(\frac{\pi}{4}) \sin(\frac{\pi}{12})}{\sin(\frac{\pi}{12})} R_3^2 \begin{pmatrix}
    1 & \alpha_{34} & -\frac{1}{4}[1 + 2\alpha_{34}] & -\frac{1}{4}[1 + 2\alpha_{34}] \\
    \alpha_{34} & 1 & -\frac{1}{2} & -\frac{1}{4}[1 + 2\alpha_{34}] \\
    -\frac{1}{4}[1 + 2\alpha_{34}] & -\frac{1}{2} & 1 & \frac{\alpha_{34}}{2} \\
    -\frac{1}{4}[1 + 2\alpha_{34}] & -\frac{1}{4}[1 + 2\alpha_{34}] & \frac{\alpha_{34}}{2} & 1
\end{pmatrix}
\]

Number of effective parameters: 2

Number of different couplings without deformations: 4

Number of different couplings with deformations: 4

corresponding to the following \(\vec{f}_{23}\) shifts
\[ \bar{\mathbf{f}}_{23} = \begin{cases} 
\quad l_2 = 1 \quad l_3 = 1 & \begin{pmatrix} 0, 0, 0, 0 \end{pmatrix}, \\
\quad l_2 = 3 \quad l_3 = 1 & \begin{pmatrix} \frac{1}{2}, 0, 0, 0 \end{pmatrix}, \\
\quad l_2 = 1 \quad l_3 = 4 & \begin{pmatrix} \frac{2}{3}, \frac{2}{3}, 0, 0 \end{pmatrix}, \\
\quad l_2 = 3 \quad l_3 = 4 & \begin{pmatrix} \frac{1}{6}, \frac{1}{6}, 0, \frac{2}{3} \end{pmatrix} \end{cases} \]

**Coupling \( \theta^3 \theta^3 \theta^6 \)

**Selection rule**

\[ f_1 + f_2 - (I + \theta^3) f_3 \in \Lambda \]

Denoting

\[
\begin{align*}
  f_1 &= g_3^{(i_1)} \otimes g_3^{(j_1)} \\
  f_2 &= g_3^{(i_2)} \otimes g_3^{(j_2)} \\
  f_3 &= [\alpha(e_1) + \beta(e_2)] \otimes g_6^{(j_3)}
\end{align*}
\]

the selection rule reads

\[
\begin{align*}
  i_1 &= i_2 \\
  j_1 + (-1)^{(j_3+1)} j_2 &= j_3 \quad \text{mod. 4}
\end{align*}
\]

**Number of allowed couplings:** 56

**Expression of the coupling**

In all the cases, except for the case \( l_1 = l_2 = l_3 = 3 \)

\[
C_{\theta^3 \theta^3 \theta^6} = N F(l_1, l_2, l_3) \sum_{v \in (f_3 - f_2 + \Lambda)} \exp[-\frac{1}{4\pi} |v|^2] \exp[-\frac{1}{4\pi} (\bar{\mathbf{f}}_{23} + \bar{u})^\top M(\bar{\mathbf{f}}_{23} + \bar{u})]
\]

with the same notation as in the previous coupling. \( l_i \) is the number of elements of the \( f_i \) conjugation class. \( F(l_1, l_2, l_3) \) is given by

\[
\begin{align*}
  l_1 = l_2 = l_3 &= 1 \quad \text{and} \quad l_1 = l_2 = 3 \quad l_3 = 1 : \quad F = 1 \\
  l_1 = l_2 = 1 \quad l_3 = 3 : \quad F = \sqrt{3} \\
  l_1 = 1(3) \quad l_2 = 3(1) \quad l_3 = 6 : \quad F = \sqrt{2} \\
  l_1 = l_2 = 3 \quad l_3 = 6 : \quad F = 2\sqrt{2}
\end{align*}
\]

In the case \( l_1 = l_2 = l_3 = 3 \)
\[ C_{\theta_3 \bar{\theta}_3 \bar{\theta}_3} = N \frac{1}{\sqrt{3}} \sum_{\nu \in \mathbb{L}_u} \exp \left[ -\frac{1}{4\pi} |\nu|^2 \right] \]

\[ = N \frac{1}{\sqrt{3}} \left\{ \vartheta \begin{bmatrix} \vec{\bar{f}}_{23}^\nu \\
0 \end{bmatrix} [0, \Omega] + \vartheta \begin{bmatrix} \vec{f}_{23}^\nu \\
0 \end{bmatrix} [0, \Omega] + \vartheta \begin{bmatrix} \vec{f}_{23}^\nu \\
0 \end{bmatrix} [0, \Omega] \right\} \]

\[ f'_{23} = \theta f_2 - f_3 \text{ and } f''_{23} = \theta^2 f_2 - f_3 \]

\[ \Omega = i \frac{1}{4\pi^2} M \quad N = \sqrt{V_{\perp}} \frac{1}{2\pi} \left[ \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \right]^2 \]

\[ V_{\perp} \text{ is the volume of the } F_4 \text{ unit cell} \]

\[ \Omega = i \frac{1}{4\pi^2} R_{23}^2 \begin{pmatrix}
1 & \alpha_{34} & -\frac{1}{4}[1 + 2\alpha_{34}] & -\frac{1}{4}[1 + 2\alpha_{34}] \\
\alpha_{34} & 1 & -\frac{1}{2} & -\frac{1}{4}[1 + 2\alpha_{34}] \\
-\frac{1}{4}[1 + 2\alpha_{34}] & -\frac{1}{2} & \frac{1}{2} & \frac{\alpha_{34}}{2} \\
-\frac{1}{4}[1 + 2\alpha_{34}] & -\frac{1}{4}[1 + 2\alpha_{34}] & \frac{\alpha_{34}}{2} & \frac{1}{2}
\end{pmatrix} \]

Number of effective parameters: 2
Number of different couplings without deformations: 6
Number of different couplings with deformations: 6

corresponding to the following \( \vec{f}_{23} \) shifts

\[ \vec{f}_{23} = \begin{cases} 
  l_1 = l_2 = l_3 = 1 & (0, 0, 0), \\
l_1 = l_2 = 3 l_3 = 1 & \left( \frac{1}{2}, 0, 0, 0 \right), \\
l_1 = l_2 = 1 l_3 = 3 & \left( \frac{1}{2}, 0, 0, 0 \right), \\
l_1 = l_2 = l_3 = 3 & (0, 0, 0, 0) \cup \left( \frac{1}{2}, 0, 0, 0 \right) \cup (0, \frac{1}{2}, 0, 0), \\
l_1 = 1(3) l_2 = 3(1) l_3 = 6 & (0, \frac{1}{2}, \frac{1}{2}, 0), \\
l_1 = l_2 = 3 l_3 = 6 & \left( \frac{1}{2}, 0, \frac{1}{2}, 0 \right)
\end{cases} \]
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| Orb. | Twist $\theta$ | Lattice | $\#DP$ | Coupling | $\#AC$ | $\#EDP$ | $\#DCR$ | $\#DCD$ |
|-----|----------------|---------|--------|----------|--------|---------|---------|---------|
| $Z_3$ | $(1,1,-2)/3$ | $SU(3)^3$ | 9 | $\theta^3\theta$ | 729 | 9 | 4 | 14 |
| $Z_4$ | $(1,1,-2)/4$ | $SU(4)^2$ | 7 | $\theta^2\theta^2$ | 160 | 4 | 6 | 10 |
| $Z_6-I$ | $(1,1,-2)/6$ | $G_2 \times SU(3)$ | 5 | $\theta^2\theta^3$ | 90 | 4 | 10 | 30 |
| $Z_6-II$ | $(1,2,-3)/6$ | $SU(6) \times SU(2)$ | 5 | $\theta^2\theta^3$ | 48 | 4 | 4 | 4 |
| $Z_7$ | $(1,2,-3)/7$ | $SU(7)$ | 3 | $\theta^2\theta^4$ | 49 | 3 | 2 | 4 |
| $Z_8-I$ | $(1,2,-3)/8$ | $SO(5) \times SO(9)$ | 3 | $\theta^2\theta^4$ | 84 | 2 | 8 | 9 |
| $Z_8-II$ | $(1,3,-4)/8$ | $SO(4) \times SO(8)$ | 5 | $\theta^3\theta^4$ | 24 | 2 | 3 | 3 |
| $Z_{12}-I$ | $(1,4,-5)/12$ | $SU(3) \times F_4$ | 3 | $\theta^2\theta^9$ | 6 | 2 | 2 | 2 |
| $Z_{12}-II$ | $(1,5,-6)/12$ | $SO(4) \times F_4$ | 5 | $\theta^3\theta^8$ | 4 | 2 | 1 | 1 |

Table 1: Characteristics of twisted Yukawa couplings for $Z_n$ Coxeter orbifolds (the non–Coxeter $Z_4$ one with $SO(4)^3$ lattice is given for comparison). The twist $\theta$ is specified by the three $c_i$ parameters (one for each complex plane rotation) appearing in $\theta = \exp(\sum c_i J_i)$. $\#DP \equiv \#$ of deformation parameters, $\#AC \equiv \#$ of allowed couplings, $\#EDP \equiv \#$ of effective deformation parameters, $\#DCR \equiv \#$ of different Yukawa couplings for the non–deformed (rigid) orbifold, $\#DCD \equiv \#$ of different Yukawa couplings when deformations are considered.

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