The group approach to AdS space propagators

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ABSTRACT

We show that AdS two-point functions can be obtained by connecting two points in the interior of AdS space with one point on its boundary by a dual pair of Dobrev’s boundary-to-bulk intertwiners and integrating over the boundary point.
1 Introduction

Though AdS field theory is a classical subject in field theory the appearance of the AdS/CFT correspondence [1] has revived interest in this subject considerably. In most investigations of AdS fields it is tacitly assumed that such a theory is based on a lagrangian with interactions which are treated by a perturbative expansion. In a seminal paper Fronsdal [2] showed that massless higher spin (tensor) fields can be defined and possess a vanishing double trace. In a long series of articles M. A. Vasiliev [3] has studied an interacting theory of infinitely many tensor fields of all ranks, which is invariant under a generalized gauge symmetry and perturbative with respect to a small curvature parameter. These interactions, including gravity, are only possible on AdS space.

The AdS/CFT correspondence maps an AdS field theory with higher spin gauge fields holographically on a conformal field theory on Minkowski space containing a tensor current source for each gauge field. Several such conformal field theories are known: The critical $O(N)$ sigma model at large $N$ in $d$ spacetime dimensions, where $2 < d < 4$, or the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in 4 dimensions and small ’t Hooft coupling $\lambda$. The first class of models has been proposed as a candidate for such a correspondence by Klebanov and Polyakov [4]. In these models all rank $l$ currents have been constructed by operator product expansion and their anomalous dimensions were calculated in [5, 6], with the result that they are all nonzero for tensor rank $l \neq 2$. In addition these models possess a scalar field of approximate conformal dimension 2, independent of $d$, which is termed “auxiliary” or “Lagrangian multiplier” field. Its 3- and 4-point functions are known [5, 6] and serve as a source of detailed information such as the coupling constants between two scalar fields and a current [7]. As a whole they offer themselves as a test object for the AdS/CFT correspondence as proposed by Klebanov and Polyakov. To construct an AdS field candidate for this purpose and to test the correspondence one has to compute the respective 3 and 4-point functions. As a first step in this direction we have developed an algorithm for the derivation of two-point functions in the AdS theory (“bulk-to-bulk propagators”) for all traceless symmetric tensor fields. In a Lagrangian setting they represent only one irreducible component of a tensor field. However, whether the AdS field theory is Lagrangian or, such as conformal field theory, based on a set of fundamental fields, a skeleton expansion and representation theory need not be answered at the start of the investigation. With this article we hope to put representation theory in the right position.

Propagators of symmetric tensor fields in AdS are known for the ranks $l = 0$ [8], $l = 1$ [9, 10] and $l = 2$ [11, 12, 10]. They have been derived from field equations and the requirement of a specific asymptotic behaviour on the boundary of AdS space. The $l = 2$ field has a nonvanishing trace. Our approach is based on representation theory and the use of intertwiners constructed by Dobrev [13], which are bulk-to-boundary propagators. It applies to all kinds of tensor fields,
but for comparison we have treated the cases of symmetric traceless tensors of ranks $l \in \{0, 1, 2\}$ explicitly.

The basic notions of representation theory are presented in section 2. In section 3 we evaluate the convolution of two scalar bulk-to-boundary propagators by integration over the boundary. It is the crucial constructive element for any bulk-to-bulk propagator. These integrals are first presented in the form of Appell’s $F_4$ functions of two variables. However, as proved by Allen and Jacobson [9], there must be a representation of these functions in terms of the geodesic distance alone, thereby replacing the $F_4$ function by functions of a single variable. This is proved to be possible indeed, the resulting functions are Legendre functions of the second kind, which are two-parameter Gaussian hypergeometric functions. The propagators are then expressed in terms of rank $l$ monomials of basic bitensors. In an appendix the technical problem of extracting traces from these monomials of bitensors is studied. Due to the small number of such bitensors the extraction leads to an overdetermined system of linear equations (for $l \geq 4$), but the excessive equations can be shown to be linearly dependent on the others, thereby resulting in a unique solution.

2 The setting

2.1 Euclidean conformal field theory

We consider a euclidean conformal field theory in $d$ space(time) dimensions. The isometry group is then given by the “conformal group” $G = SO(d + 1, 1)$. The fields in this theory are characterized by their transformation behaviour given by a representation $\chi$ of $G$, which is induced from the subgroups of euclidean rotations $M = SO(d)$ and the behaviour under dilatations, i.e.

$$
\begin{align*}
\phi(\vec{x}) & \mapsto r^\Delta \phi(r \vec{x}), \quad r \in \mathbb{R}_+ , \\
\phi(\vec{x}) & \mapsto D^\nu(m) \phi(m^{-1} \vec{x}), \quad m \in M.
\end{align*}
$$

Here, $\Delta$ is the conformal dimension of the field and $D^\nu(m)$ is the $\nu$ representation matrix of $m \in M$. This determines the representation $\chi$ of $G$, thus we will write $\chi = [\nu, \Delta]$. Moreover, let us denote by $\tilde{\chi} = [\tilde{\nu}, d - \Delta]$ the quantum numbers of the “shadow field”, which is obtained by exchanging the $M$ representation $\nu$ by its mirror image $\tilde{\nu}$ and the conformal weight $\Delta$ by $d - \Delta$. Since the dimension of the shadow field appears very often, it is convenient to introduce the notation

$$
\lambda := d - \Delta.
$$

For generic conformal dimension $\Delta$, the representations $\chi$ and $\tilde{\chi}$ are equivalent, i.e. there exists an invertible operator

$$
G_\chi : C_\tilde{\chi} \rightarrow C_\chi,
$$

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which maps the representation spaces to each other and which commutes with the action of $G$. Such an operator is called intertwining operator for $C\tilde{\chi}$ and $C\chi$. It is well known that the two-point function of a conformal field with quantum numbers $\chi$ agrees with this intertwiner $G\chi$. The functional form of a two-point function with quantum numbers $\chi$ itself is fixed by conformal covariance to

$$G\chi(\vec{x}) = \frac{\gamma_\chi}{(\vec{x})^\Delta} D^\nu(r(\vec{x}))$$

$$r(\vec{x}) = \left(\frac{2x_i x_j}{\vec{x}^2} - \delta_{ij}\right), \quad i, j = 1, \ldots, d.$$  

(4)

In this formula $\gamma_\chi$ is a normalization constant, whose value is not important for our purposes. For later use, let us write down the explicit form of a propagator of a symmetric traceless tensor field $t^{(l)}$ of conformal dimension $\Delta$. In this case we can write down a generating function:

$$(\vec{x})^\Delta \sum_{l \geq 0} \rho^l \sum_{(r)(\sigma)} \langle t_{(r)}^{(l)}(\vec{x}) t_{(\sigma)}^{(l)}(0) \rangle^\sigma \vec{a}_{(r)} \vec{b}_{(\sigma)} = \left(1 - 2\rho(\vec{a}, r(\vec{x})\vec{b}) + \rho^2 \vec{a}^2 \vec{b}^2\right)^{1-\mu}$$

$$= \sum_{l \geq 0} \rho^l (\langle\vec{a}\mid\vec{b}\rangle)^l C_{\mu-1}^l \left(\frac{(\vec{a}, r(\vec{x})\vec{b})}{\langle\vec{a}\mid\vec{b}\rangle}\right),$$  

(5)

where $\vec{a}, \vec{b} \in \mathbb{R}^d$ and $C_{\mu-1}^l$ is the Gegenbauer polynomial of degree $l$ with parameter $\mu - 1$. Here we use the convenient abbreviation $\mu = d/2$, which will be applied throughout this work.

2.2 AdS/CFT correspondence as representation equivalence

On the other hand we consider a field theory on a $d + 1$-dimensional Anti-de-Sitter space, which is given by a classical action. The AdS/CFT correspondence connects this field theory to a theory living on the boundary of the AdS space. The isometry group of $d + 1$-dimensional AdS space is $G = SO(d + 1, 1)$, thus matching the isometry groups of both theories implies that the boundary theory must be conformal. By an identification of generating functionals we obtain a prescription for constructing $d$-dimensional correlation functions, which in turn define a conformal field theory. The procedure may be sketched as follows: We take a set of vertices in the AdS theory, connect some of them by bulk-to-bulk propagators and the remaining vertices to the boundary by bulk-to-boundary propagators [14]. These two kinds of propagators can be obtained by solving the free equations of motion derived from the action of the AdS theory. Let us write down the actual form of the scalar bulk-to-boundary operator, which constructs an AdS scalar field of mass $m$ from a conformal scalar of dimension $\Delta$. The mass
m is linked to $\Delta$ by $m^2 = \Delta(\Delta - d)$:

$$K_{\Delta}(x_1, \vec{x}_2) = \left( \frac{x_{10}}{x_{12}} \right)^\lambda,$$

where $x_1 = (x_{10}, \vec{x}_1) \in \mathbb{R}_+ \times \mathbb{R}^d$, $\vec{x}_2 \in \mathbb{R}^d$, and we introduced the notation

$$x^2_{12} = |x_1 - \vec{x}_2|^2 = x^2_{10} + (\vec{x}_1 - \vec{x}_2)^2. \quad (7)$$

Moreover, we note the “vectorial” bulk-to-boundary propagator, which transfers a $d$-dimensional vector field of conformal dimension $\Delta$ to a $d + 1$-dimensional AdS-vector field of mass $m$

$$K_{\Delta}^{(1)}(a, \vec{b}; x_1, \vec{x}_2) = \left( \frac{x^\lambda_{10}}{x^\lambda_{12}} \right) \frac{2(a, x_{12})(\vec{b}, \vec{x}_{12})}{x^2_{12}} - (\vec{a}, \vec{b}) \quad (8)$$

with the relation

$$m^2 = \Delta(\Delta - d) + (d - 1). \quad (9)$$

In (8) we contracted the external indices of the $d + 1$- and $d$-dimensional vectors with $a \in \mathbb{R}^{d+1}$ and $\vec{b} \in \mathbb{R}^d$, respectively, via the $d + 1$- and $d$-dimensional standard scalar products. The construction of the bulk-to-boundary propagator, which connects a symmetric traceless second rank tensor field of conformal dimension $\Delta$ with a symmetric traceless second rank tensor field on $d + 1$-dimensional AdS is simple and can easily be generalized to any rank:

$$K_{\Delta}^{(2)}(a, \vec{b}; x_1, \vec{x}_2) = \left( K_{\Delta}^{(1)}(a, \vec{b}; x_1, \vec{x}_2) \right)^2 - \frac{\vec{b}^2}{d} \times \text{trace with respect to } \vec{b}$$

$$= \frac{x^\lambda_{10}}{(x^\lambda_{12})^2} \left\{ \left( \frac{2(a, x_{12})(\vec{b}, \vec{x}_{12})}{x^2_{12}} - (\vec{a}, \vec{b}) \right)^2 - \frac{\vec{b}^2}{d} \left( \frac{2(a, x_{12})\vec{x}_{12}}{x^2_{12}} - \vec{a} \right)^2 \right\}, \quad (10)$$

where we contracted the external indices with $a \in \mathbb{R}^{d+1}$ and $\vec{b} \in \mathbb{R}^d$, as in (8). After a quick computation one agrees that the trace of this bulk-to-boundary propagator with respect to $a$ vanishes, too.

As shown by Dobrev [13], these bulk-to-boundary operators can be given an interpretation as intertwining operators, and they can even be constructed by this property. They map the conformal representations $C_\chi$ to representations $\hat{C}^\tau$ induced from the maximal compact subgroup $K = SO(d + 1) \subset G$, where $\tau$ is an irreducible representation of $K$ containing the irrep $\nu$ from $\chi = [\nu, \Delta]$ of $M$. Now $\hat{C}^\tau$ is neither uniquely determined (there are infinitely many different irreps of $K$ containing $\nu$) nor irreducible. The lack of uniqueness can be cured by the choice of some “minimal” irrep of $K$. To obtain irreducibility, one has to impose a constraint on the behaviour of the functions in $\hat{C}^\tau$ for $x_0 \to 0$. The
resulting representation $\hat{C}_\chi$ turns out to be irreducible for generic $\Delta$, and the bulk-to-boundary operators are then the integral kernels of the intertwiners

$$K_\chi^\tau : C_\chi \rightarrow \hat{C}_\chi.$$

Dobrev’s group theoretical arguments apply for all elementary irreps $\chi$ of $G$ and all irreps $\tau$ induced from $K$, therefore they do not depend on any actions or field equations.

After all these preliminaries, let us describe the idea of constructing AdS two-point functions, where we restrict on symmetric tensor representations of $M$ and $K$, so that the respective propagators are labeled by their tensor rank $l$ and the parameter $\Delta$. It is simple to convolute the conformal intertwiner $G_{d-\Delta}^{(l)}$ with the bulk-to-boundary operator $K_\Delta^{(l)}$ and check that we obtain a bulk-to-boundary propagator for a conformal field of dimension $d - \Delta$. This begs the following question: If we take a further bulk-to-boundary propagator $K_\Delta^{(l)}$ of dimension $\Delta$ and convolute it with the remaining $d$-dimensional leg of the resulting $K_{d-\Delta}$, do we obtain a scalar bulk-to-bulk propagator $W^{(l)}_{\Delta}$? We show by explicit construction that this guess is almost correct, except that a certain doubling occurs. Each two-point function $W^{(l)}_{\Delta}$ in AdS obtained this way comes along with its shadow partner $W^{(l)}_{d-\Delta}$. This seems natural from the conformal point of view, since in any CFT exchange the shadow of any exchanged field appears, because the shadow irreps are equivalent to the original ones.

### 2.3 Geometric properties of AdS propagators

Let us mention the following geometric aspects of the two-point functions: The two-point function $\langle t^{(l)}(x)t^{(l)}(x') \rangle$ transforms under a coordinate transformation as a tensor of rank $l$ at the points $x$ and $x'$, i.e. the two-point function is a “bitensor”. Thanks to the maximal symmetry of the (Euclidean) AdS space we have the theorem [19] that every bitensor can be expressed in terms of the metric and three fundamental bitensors, which are obtained by differentiating the geodesic distance $\mu(x, x')$ of two points $x, x' \in \text{AdS}$. The first two fundamental bitensors are the two tangent unit vectors along the geodesic

$$n_\nu = D_\nu \mu(x, x') \quad \text{and} \quad n_{\nu'} = D_{\nu'} \mu(x, x'),$$

where the unprimed/primed indices denote tangent space indices at $x$ and $x'$, respectively, and $D$ denotes covariant derivatives. The third fundamental bitensor is the parallel transporter along the geodesic connecting $x$ and $x'$. We choose to work in Poincaré coordinates $x = (x_0, x_1, \ldots, x_d) = (x_0, \vec{x}) \in \mathbb{R}_+ \times \mathbb{R}^d$ for the AdS space, in which the geodesic distance and the fundamental bitensors are

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\[\text{This is already proved in general in [13].}\]
expressed as

\[ \zeta = \cosh(\mu(x, x')) = \frac{x_0^2 + x_0'^2 + (\vec{x} - \vec{x}')^2}{2x_0x_0'}; \]

\[ n_\nu = \frac{\partial_\nu \zeta}{\sqrt{\zeta^2 - 1}}, \]

\[ g_{\nu\nu'} = \partial_\nu \partial_{\nu'} \zeta + \frac{\partial_\nu \zeta \partial_{\nu'} \zeta}{\zeta + 1}. \]

(13)

We note that the tangent vectors are proportional to the first derivative of \( \zeta \) and the parallel transporter is essentially given by two derivatives of \( \zeta \), one with respect to each variable. Moreover, it turns out to be convenient to contract these bitensors with tangent vectors \( a \in T_x\text{AdS}, c \in T_{x'}\text{AdS} \); we denote this by the \( d+1 \)-dimensional scalar product \( \langle \cdots \rangle \). Then we arrive at the following algebraic basis of maximally symmetric bitensors

\[ I_1 := \langle a, \partial \rangle \langle c, \partial' \rangle \zeta \]

\[ I_{2a} := \langle a, \partial \zeta \rangle \]

\[ I_{2c} := \langle c, \partial' \zeta \rangle \]

\[ I_2 := I_{2a}I_{2b} \]

and

\[ a^2 = \sum_{\nu=0}^d a_\nu^2, \quad c^2 = \sum_{\nu=0}^d c_\nu^2 \quad \text{for } l \geq 2. \]

(14)

In our construction we have to perform the splitting \( (a_0, a_1, \ldots, a_d) = (a_0, \vec{a}) \) and the same for \( c \). The invariants then acquire the form

\[ \langle a, \partial \rangle \langle c, \partial' \rangle \zeta = \frac{\langle \vec{a}, \vec{c} \rangle}{x_0x_0'} - \frac{a_0c_0}{x_0x_0'} \zeta - \frac{c_0}{x_0} \langle a, \partial \zeta \rangle - \frac{a_0}{x_0} \langle c, \partial' \zeta \rangle, \]

(15)

\[ \langle a, \partial \zeta \rangle = \langle \vec{a}, \vec{\partial} \zeta \rangle + a_0 \left( \frac{1}{x_0} - \frac{\zeta}{x_0} \right), \]

(16)

\[ \langle c, \partial' \zeta \rangle = \langle \vec{c}, \vec{\partial}' \zeta \rangle + c_0 \left( \frac{1}{x_0} - \frac{\zeta}{x_0} \right). \]

(17)

We are interested in propagators for symmetric traceless tensors of rank \( l \). In order to subtract traces we need further bitensors

\[ I_3 := \frac{a^2}{x_0^2} I_{2c}^2 + \frac{c^2}{x_0^2} I_{2a}^2 \]

\[ I_4 := \frac{a^2c^2}{x_0^2x_0'^2}. \]

(18)

Since taking traces on products of invariant bitensors (14) results in products of invariants with at least one factor \( I_3 \) or \( I_4 \), we call the latter ones “trace terms”.

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For a symmetric traceless tensor of rank \( l \) we obtain a basis labelled by the pairs \( \{ (l_1, l_2) \} \) with \( l_1 + l_2 = l \):

\[
I^{l_1}_1 I^{l_2}_2 - \sum A^{(l_1, l_2)}_{m_1 m_2 n_1 n_2} I^{m_1}_1 I^{m_2}_2 I^{n_1}_3 I^{n_2}_4,
\]

(19)

where the sum is restricted to

\[
m_1 + m_2 + 2(n_1 + n_2) = l, \quad n_1 + n_2 > 0.
\]

(20)

The coefficients \( A^{(l_1, l_2)}_{m_1 m_2 n_1 n_2} \) can be determined by requiring (19) to be harmonic with respect to the (naive) Laplacian \( \Delta_a = \sum \frac{\partial}{\partial a} \frac{\partial}{\partial a} \). The solution of this removing of traces will be presented in the appendix.

3 Convolution integrals as hypergeometric functions

Now we do the actual computations. First we calculate the integral

\[
A_{\alpha_1, \alpha_2}(x_1, x_3) := \int d^d x_2 \left( \frac{x_{10}}{x_{10}^2 + |\vec{x}_{12}|^2} \right)^{\alpha_1} \left( \frac{x_{30}}{x_{30}^2 + |\vec{x}_{32}|^2} \right)^{\alpha_2},
\]

(21)

where \( \alpha_1, \alpha_2 \) are two real parameters, which are chosen in such a way to ensure convergence of the integral but are independent otherwise. After introducing a Feynman parameter for the two denominators the resulting \( d \)-dimensional integral is easy and we get

\[
A_{\alpha_1, \alpha_2}(x_1, x_3) = \pi^\mu \frac{\Gamma(\alpha_1 + \alpha_2 - \mu)}{\Gamma(\alpha_1) \Gamma(\alpha_2)} x_{10}^{\alpha_1} x_{30}^{\alpha_2} \int_0^1 dt t^{\alpha_1 - 1}(1 - t)^{\alpha_2 - 1} \left[ t(1 - t)x_{13}^2 + tx_{10}^2 + (1 - t)x_{30}^2 \right]^{-(\alpha_1 + \alpha_2 - \mu)}. \]

(22)

Now we extract \( tx_{10}^2 + (1 - t)x_{30}^2 \) out of \( [\cdots] \) in (22) and choose the coordinates \( x_1, x_3 \) in such a way that the resulting expression in \( [\cdots] \) can be expanded in a binomial series. The case of arbitrary values of the coordinates are afterwards obtained by analytical continuation. Then the Feynman parameter can be integrated and we obtain

\[
A_{\alpha_1, \alpha_2}(x_1, x_3) = \pi^\mu \frac{\Gamma(\alpha_1 + \alpha_2 - \mu)}{\Gamma(\alpha_1) \Gamma(\alpha_2)} x_{10}^{\alpha_1} x_{30}^{\alpha_2} \sum_{k \geq 0} \frac{(\alpha_1 + \alpha_2 - \mu)_k}{k!} \frac{\Gamma(\alpha_1 + k) \Gamma(\alpha_2 + k)}{\Gamma(\alpha_1 + \alpha_2 + 2k)} \sigma^{-k} F \left[ \begin{array}{c} \alpha_1 + \alpha_2 - \mu + k, \alpha_1 + k \\ \alpha_1 + \alpha_2 + 2k \end{array} ; 1 - \frac{1}{\rho} \right], \]

(23)
where we introduced the abbreviations
\[ \sigma := \frac{x^2_{13}}{x^2_{10}}, \quad \rho := \frac{x^2_{30}}{x^2_{10}}. \quad (24) \]

Next we apply an analytical continuation formula for the gaussian hypergeometric function (formula 9.132.1 of \[16\]) to transform the argument \(1 - \rho^{-1}\) to \(\rho\) and obtain as a result two hypergeometric functions:

\[
A_{\alpha_1, \alpha_2}(x_1, x_3) = \frac{\pi^{\mu}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \frac{x^{\alpha_2}_{30}}{x^{\alpha_1}_{10}} \sum_{k \geq 0} \frac{\sigma^k}{k!} \left\{ \rho^{\alpha_2 - \mu} \Gamma(\mu - \alpha_2) \Gamma(\alpha_1 + \alpha_2 - \mu + k) \frac{\Gamma(\alpha_2 + k)}{\Gamma(\mu + k)} F\left[ \frac{\alpha_1 + \alpha_2 - \mu + k, \alpha_2 + k}{\alpha_2 - \mu + 1} ; \rho \right] \\
+ \Gamma(\alpha_2 - \mu) \Gamma(\alpha_1 + k) F\left[ \frac{\alpha_1 + k, \mu + k}{1 + \mu - \alpha_2} ; \rho \right] \right\}. \quad (25)
\]

The first term will be called “direct term” and the second will be called “shadow term”.

In the sequel all appearing propagators can be expressed as linear combinations of \(A_{\alpha_1, \alpha_2}\) and derivatives thereof. In these applications the two parameters \(\alpha_1\) and \(\alpha_2\) are not independent but fulfill an equation

\[ \alpha_1 + \alpha_2 = d + q, \quad (26) \]

where \(q\) is an integer. One can check that it suffices to use only one of the two summands in \(A_{\alpha_1, \alpha_2}\), because the other one is obtained by substituting the parameter \(\lambda\) by the shadow parameter \(\Delta\). Therefore we insert

\[ \alpha_1 = \lambda - r, \quad \alpha_2 = \Delta - s, \quad \text{with } \lambda + \Delta = d \text{ and } r, s \in \mathbb{Z}, \]

project onto the (say) first term in (25) and thus define

\[
\Phi_{r,s}(x_1, x_3) := \int d^d x_2 \left( \frac{x^2_{10}}{x^2_{10} + |\vec{x}_{12}|^2} \right)^{\lambda - r} \left( \frac{x^2_{30}}{x^2_{30} + |\vec{x}_{32}|^2} \right)^{\Delta - s} \left. \right|_{\text{direct term}}
\]

\[
= \frac{\pi^{\mu} \Gamma(\mu - \Delta + s)}{\Gamma(\lambda - r)\Gamma(\Delta - s)} \frac{x^{\Delta - s}_{30}}{x^{\lambda - r}_{10}} \rho^{\Delta - \mu - s} \sum_{k \geq 0} \frac{\sigma^k}{k!} \frac{\Gamma(\Delta - s + k)}{\Gamma(\mu - r - s + k)} \Gamma(\mu - r - s + k) F\left[ \frac{\mu - r - s + k, \Delta - s + k}{\Delta - s - \mu + 1} ; \rho \right]. \quad (28)
\]

The series of hypergeometric functions may be summed up in terms of Appell’s \(F_4\)-function (see 9.18 of \[16\] and references therein):

\[
\Phi_{r,s}(x_1, x_3) = \frac{\pi^{\mu} \Gamma(\mu - \Delta + s)}{\Gamma(\lambda - r)\Gamma(\Delta - s)} \rho^{\Delta - s} \frac{\Gamma(\Delta - s + k)}{\Gamma(\mu - r - s + k)} F_4(\Delta - s, \mu - r - s, \Delta - s - \mu + 1, \mu; \rho, \sigma) \quad (29)
\]
Let us note the action of a $d$-dimensional Laplacian on $\Phi_{r,s}$

\[
(\vec{\partial}, \vec{\partial}) \Phi_{r,s} = \frac{\pi^d \Gamma(\mu - \Delta + s)}{\Gamma(\lambda - \mu) \Gamma(\Delta - s)} \frac{d}{d x_3} \frac{(\Delta - s)!}{x_3^{\Delta - s}} \rho^{\Delta - s - \mu} \left( - \frac{4}{x_1^2} \right) \sum_{k \geq 0} \frac{x_{30}^k}{\Gamma(k)} \Gamma(\mu - r - s + k + 1) \\
\left( \frac{\Delta}{\Gamma(\Delta - s + k + 1)} \right) F\left[ \frac{\Delta - s + 1}{2}, \frac{\Delta - s + 2}{2} \Delta - s + t + 1 ; \zeta^{-2} \right],
\]

where $\vec{\partial} = (\frac{\partial}{\partial x_i}), i = 1, \ldots, d$. This can be written again in terms of $\Phi_{r,s}$

\[
(\vec{\partial}, \vec{\partial}) \Phi_{r,s} = \frac{4}{x_3} (\Delta - s)(\Delta - s - \mu + 1) \Phi_{r,s-1} - 4(\Delta - s)_{2\Phi_{r,s-2}}.
\]

We introduce Legendre functions of the second kind and write them in terms of gaussian hypergeometric functions

\[
\Lambda_{s,t}(\zeta) := \Gamma(\Delta - s) 2^{-(\Delta - s)} \zeta^{-(\Delta - s)} F\left[ \frac{\Delta - s}{2}, \frac{\Delta - s + 1}{2} \Delta - s + t + \mu + 1 ; \zeta^{-2} \right],
\]

where $s, t \in \mathbb{Z}$ and the AdS invariant variable $\zeta$ is defined by (13) with $x = x_1$ and $x' = x_3$. We note the derivative of $\Lambda_{r,s}$ with respect to $\zeta$.

\[
\frac{d}{d \zeta} \Lambda_{s,t}(\zeta) = - (\Delta - s) \Gamma(\Delta - s) 2^{-(\Delta - s)} \zeta^{-(\Delta - s + 1)} F\left[ \frac{\Delta - s + 1}{2}, \frac{\Delta - s + 2}{2} \Delta - s + t + \mu + 1 ; \zeta^{-2} \right] = -2 \Lambda_{s-1,t-1}(\zeta).
\]

Using this equation we get for the action of the $d$ dimensional Laplacian $(\vec{\partial}, \vec{\partial})$

\[
(\vec{\partial}, \vec{\partial}) \Lambda_{s,t}(\zeta) = 4 \Lambda_{s-2,t-2}(\zeta) (\vec{\partial} \zeta, \vec{\partial} \zeta) - 2 \Lambda_{s-1,t-1}(\zeta) (\vec{\partial} \zeta, \vec{\partial} \zeta) \zeta.
\]

Moreover, we note the following two identities, which are simple consequences of eqns. 9.137, 6 and 12 in [16], respectively

\[
\Lambda_{s,t}(\zeta) = \Lambda_{s,t+1}(\zeta) + \frac{1}{(\Delta - \mu - s + t + 1)_2} \Lambda_{s,t-1}(\zeta),
\]

\[
\zeta \Lambda_{s-1,t-1}(\zeta) = \frac{\Delta - s}{2} \Lambda_{s,t}(\zeta) + \frac{1}{(\Delta - \mu - s + t + 1)} \Lambda_{s,t-1}(\zeta).
\]

Now we want to express certain linear combinations and derivatives of the functions $\Phi_{r,s}$ in terms of the $\Lambda_{s,t}$. This can be established with the help of the following two formulae. They both hold in the case $r + s = m$:

\[
\sum_{j=0}^{m} \binom{m}{j} \frac{(\lambda - r)_j}{(\Delta - s - j)_j} \frac{1}{x_3^{m-j} x_1^j} \Phi_{r-j,s+j}(x_1, x_3) = \frac{\pi^m \Gamma(\mu - \Delta + s)}{\Gamma(\lambda - \mu) \Gamma(\Delta - s)} \Lambda_{s+m,m}(\zeta)
\]
\[
(\bar{\partial}, \partial)^m \Phi_{r,s}(x_1, x_3) = \pi^\mu \frac{\Gamma(\mu - \Delta + s)}{\Gamma(\lambda - r) \Gamma(\Delta - s)} \left( \frac{-4}{x_{30}} \right)^m \sum_{j=0}^{m} \binom{m}{j} \frac{(-1)^j \rho^{j/2}}{(\Delta - \mu - s + 1)_j} \Lambda_{s-m-j,-m}(\zeta). \quad (37)
\]

The proof of (36) proceeds by induction, so let \( m = 0 \), to formulate the start of the induction. We have \( r = -s \), and in this case the \( F_4 \)-function of (29) can be summed up in terms of a gaussian hypergeometric function (eq. 9.182.8 of [16]), which after some algebra with the coordinates reads:

\[
\Phi_{r,s}(x_1, x_3) = \pi^\mu \frac{\Gamma(\mu - \Delta + s)}{\Gamma(\lambda - r) \Gamma(\Delta - s)} \rho^{\frac{\Delta + s}{2}} \Gamma(\Delta - s) F_4(\Delta - s, \mu, \Delta - s - \mu + 1, \mu; \rho, \sigma) = \pi^\mu \frac{\Gamma(\mu - \Delta + s)}{\Gamma(\lambda - r) \Gamma(\Delta - s)} \Lambda_{s,0}(\zeta), \quad (38)
\]

thus showing the start of the induction of the proof of our formula. For the inductive step, we take a look at the left hand side of (38) for \( m + 1 \) and use \( \binom{m+1}{j} = \binom{m}{j-1} + \binom{m}{j} \) to obtain

\[
\sum_{j=0}^{m+1} \left[ \binom{m}{j-1} + \binom{m}{j} \right] \frac{(\lambda - r)_j}{(\Delta - s - j)_j} \frac{1}{x_{30}^{m+1-j} x_{10}^j} \Phi_{r-s+j}(x_1, x_3)
\]

\[
= \sum_{j=0}^{m} \binom{m}{j} \frac{(\lambda - r)_j}{(s - \Delta + 1)_j} \frac{(-1)^j}{x_{30}^{m-j} x_{10}^j} \left[ -\frac{\lambda - r + j}{s - \Delta + j + 1} \frac{1}{x_{10}} \Phi_{r-j-1,s+j+1}(x_1, x_3) + \frac{1}{x_{30}} \Phi_{r-j,s+j}(x_1, x_3) \right]. \quad (39)
\]

With formula 9.137,17 of [16], the first summand in \([\cdots]\) of (39) can be written
as

\[
\sum_{k \geq 0} \sigma^k \frac{\Gamma(\mu - r - s + k) \Gamma(\Delta - s - j - 1 + k)}{k! \Gamma(\mu + k)} F\left[ \frac{\mu - r - s + k, \Delta - s - j - 1 + k}{\Delta - \mu - s - j} ; \rho \right]
\]

\[
= \frac{\pi \mu \Gamma(\mu - \Delta + s + j)}{\Gamma(\Delta - s - j)} \sum_{k \geq 0} \sigma^k \frac{\Gamma(\mu - r - s + k)}{k! \Gamma(\mu + k)} \Gamma(\Delta - s - j + k) \left\{ -F\left[ \frac{\mu - r - s + k, \Delta - s - j + k}{\Delta - \mu - s - j + 1} ; \rho \right] + \frac{\mu - 1 + k}{\Delta - s - j + 1} F\left[ \frac{\mu - r - s + k, \Delta - s - j - 1 + k}{\Delta - \mu - s - j + 1} ; \rho \right] \right\}
\]

(40)

The first term in \{ \cdots \} in (40) together with the prefactors and the sum cancels the second term in [ \cdots ] in (39), therefore we obtain

\[
\text{LHS of (36)} \bigg|_{m+1} = \sum_{j=0}^{m} \binom{m}{j} \frac{(r' - r)_j}{(\Delta')_j} \frac{(-1)^j}{x_{30}^{m-j} x_{10}^{\Delta' - s - j}} \frac{\pi}{\Gamma(\Delta') \Gamma(\Delta - s)} \Phi_{r', s + j}(x_1, x_3) \bigg|_{\Delta', \lambda', \mu'}
\]

(41)

where we have set \Delta' = \Delta - 1, \lambda' = \lambda - 1, \mu' = \mu - 1 and \( r' = r - 1 \), and the function \( \Phi \) is to be understood with the unprimed parameters replaced by the primed ones. Now we recognize that \( r' + s = m \), thus the sum can by done by induction hypothesis, giving

\[
\text{LHS of (36)} \bigg|_{m+1} = \frac{-\pi}{s - \Delta'} \frac{\pi \mu \Gamma(\mu - \Delta' + s)}{\Gamma(\Delta' - s)} \Lambda_{s+m, m}(\zeta) \bigg|_{\Delta', \lambda', \mu'}
\]

\[
= \frac{\pi \mu \Gamma(\mu - \Delta + s)}{\Gamma(\lambda - r) \Gamma(\Delta - s)} \Lambda_{s+m+1, m+1}(\zeta),
\]

(42)

where we used the definition (32) of \( \Lambda_{s,t} \). This proves (36).

Using (41), the proof of (47) can also be done by induction, where the start of the induction is the same as for (36), but in this case the direct proof is even simpler. To this end we apply \( m \) times the Laplacian \( (\bar{\partial}, \bar{\partial}) \) on the series in (28)

\[
(\bar{\partial}, \bar{\partial})^m \Phi_{r,s}(x_1, x_3) = \frac{\pi \mu \Gamma(\mu - \Delta + s)}{\Gamma(\lambda - r) \Gamma(\Delta - r)} \frac{x_{10}^{\Delta - s}}{x_{30}^{r' - r}} \rho^{\Delta - s} \left( \frac{4}{x_{10}^2} \right)^n \sum_{k \geq 0} \sigma^k \frac{\Gamma(\Delta - s + k + n)}{k!} F\left[ \frac{\mu + k, \Delta - s + k + n}{\Delta - \mu - s + 1} ; \rho \right].
\]

(43)
Now we apply $m$ times the formula

$$F[a,b,c,z] = F[a+1,b,c,z] - \frac{b}{c}zF[a+1,b+1,c+1,z], \quad (44)$$

see eq. 9.137,12 of [16], and obtain a sum of series of hypergeometric functions, of which each series can be summed with 9.182,8 of [16], to give directly (36).

### 4 Results for the propagators

#### 4.1 The scalar case

Now that we have all these formulae at hand, the calculation of the scalar bulk-to-bulk propagator is ultra-simple. We convolute a bulk-to-boundary propagator of dimension $\Delta$ with a bulk-to-boundary propagator of dimension $\lambda$ “along the boundary”, i.e. integrate over the $d$ dimensional boundary variable and get with \( (38) \) for $s = 0$

$$A_{\lambda,\Delta}(x_1, x_3) = \Phi_{0,0}(x_1, x_3) + \{\Delta \leftrightarrow \lambda\}$$

$$= \frac{\pi^\mu \Gamma(\mu - \Delta)}{\Gamma(\lambda)\Gamma(\Delta)} A_{0,0}(\zeta) + \{\Delta \leftrightarrow \lambda\}. \quad (45)$$

This is up to normalization just the AdS two-point function of a field with parameter $\lambda$ plus the two-point function of a field with parameter $\Delta$. Thus we see that in the computation of the AdS two-point function by intertwining the legs of a conformal two-point function we automatically obtain as an artefact of the technique the two-point function of the field with the shadow parameter.

#### 4.2 The vector case

Now we consider the convolution of two vector bulk-to-boundary propagators, i.e. we consider the integral

$$A^{(1)}_{\lambda,\Delta}(x_1, x_3) = \int d^d x_2 K^{(1)}_{\lambda}(a, \vec{\nabla}_b; x_{12}) K^{(1)}_{\Delta}(\vec{b}, c; x_{23})$$

$$= \int d^d x_2 \frac{x_1^{\lambda-1}}{(x_{12}^2)^\lambda} \left( \frac{2(a, x_{12})(\vec{x}_{12}, \vec{\nabla}_b)}{x_{12}^2} - (\vec{a}, \vec{\nabla}_b) \right)$$

$$\frac{x_2^{\Delta-1}}{(x_{23}^2)^\Delta} \left( \frac{2(b, \vec{x}_{23})(x_{23}, c)}{x_{23}^2} - (\vec{b}, \vec{c}) \right) \quad (46)$$

The idea of calculating this integral is to write each bulk to boundary operator as a differential operator with respect to the exterior variables acting on a linear
combination of scalar bulk-to-boundary like terms. We then get

\[ A_{\lambda, \Delta}^{(1)}(x_1, x_3) = \int d^2x_2 \left\{ -a_0 x_{10}^{\lambda} \left( \bar{\nabla}_b \Phi_{r1} \right) x_{12}^{-2\lambda} + \frac{1 - \lambda}{\lambda} x_{10}^{\lambda-1} (\bar{\nabla}_b \Phi_{r1}) x_{12}^{-2\lambda} + \frac{x_{10}^{\lambda-1}}{2(\lambda - 1)^2} (\bar{\nabla}_b \Phi_{r1}) x_{12}^{-2(\lambda-1)} \right\} \left\{ -c_0 x_{30}^{\Delta} (\bar{\nabla}_b \Phi_{r3}) x_{23}^{-2\Delta} + \frac{1 - \Delta}{\Delta} x_{30}^{\Delta-1} (\bar{\nabla}_b \Phi_{r3}) x_{23}^{-2\Delta} + \frac{x_{30}^{\Delta-1}}{2(\Delta - 1)^2} (\bar{\nabla}_b \Phi_{r3}) x_{23}^{-2(\lambda-1)} \right\}. \] (47)

Since all derivatives act on variables which are not integrated, we can take them in front of the integrals and perform these in terms of the functions \( \Phi_{r,s} \), where we restrict on the direct term. The \( \Phi_{r,s} \) are functions of \( x_{13} \), thus we can use \( \vec{\partial} := \vec{\partial}_1 = -\vec{\partial}_3 \). The \( \vec{b} \) are contracted and we are left with

\[ A_{\lambda, \Delta}^{(1)} \bigg|_{\text{direct}} = A_{\lambda, \Delta}^{(1)} \bigg|_1 + A_{\lambda, \Delta}^{(1)} \bigg|_{a_0} + A_{\lambda, \Delta}^{(1)} \bigg|_{c_0} + A_{\lambda, \Delta}^{(1)} \bigg|_{a_0 c_0}, \] (48)

where we expanded in powers of \( a_0 \) and \( c_0 \). The first term is given by

\[ A_{\lambda, \Delta}^{(1)}(x_1, x_3) \bigg|_1 = (\bar{\nabla}_b \Phi_{r1}) (\bar{\nabla}_b \Phi_{r3}) = \frac{(1 - \lambda)(1 - \Delta)}{\lambda \Delta} \frac{1}{x_{10} x_{30}} \Phi_{0,0}(x_1, x_3) \]

\[ + (\bar{\nabla}_b \Phi_{r1})(\bar{\nabla}_b \Phi_{r3}) \left[ \frac{1}{2(\lambda - 1)^2} \Phi_{1,1}(x_1, x_3) + \frac{1 - \lambda}{2(\Delta - 1)^2} \Phi_{0,1}(x_1, x_3) \right]. \] (49)

Now we observe that the first line can be presented directly as a Legendre function by \( 36 \). For the term proportional to \( (\bar{\nabla}_b \Phi_{r1})(\bar{\nabla}_b \Phi_{r3}) \) we use \( 31 \) on the first summand in \([\cdots]\), and find that we can apply \( 36 \) on the result

\[ A_{\lambda, \Delta}^{(1)}(x_1, x_3) \bigg|_1 = (\bar{\nabla}_b \Phi_{r1})(\bar{\nabla}_b \Phi_{r3}) = \frac{(1 - \lambda)(1 - \Delta)}{\lambda \Delta} \frac{1}{x_{10} x_{30}} \Phi_{0,0}(x_1, x_3) \]

\[ + (\bar{\nabla}_b \Phi_{r1})(\bar{\nabla}_b \Phi_{r3}) \left[ \frac{1}{2(\lambda - 1)^2} \Phi_{1,1}(x_1, x_3) + \frac{1 - \lambda}{2(\Delta - 1)^2} \Phi_{0,1}(x_1, x_3) \right] \]

\[ = \frac{\pi^\mu \Gamma(\mu - \Delta)}{\Gamma(\Lambda + 1)\Gamma(\Delta + 1)} \left\{ -\frac{1}{2(\lambda - 1)^2} \Phi_{1,1}(x_1, x_3) + \frac{1 - \lambda}{2(\Delta - 1)^2} \Phi_{0,1}(x_1, x_3) \right\}. \] (50)

The second term in \( 48 \) is given by

\[ A_{\lambda, \Delta}^{(1)} \bigg|_{a_0} = a_0(\bar{\nabla}_b \Phi_{r3}) \left\{ -\frac{1}{2(\Delta - 1)^2} (\bar{\nabla}_b \Phi_{r3}) \Phi_{0,1} - \frac{1 - \Delta}{\lambda \Delta} \Phi_{0,0} \right\}. \] (51)

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We observe that both terms can be summed up by (37), to result in
\begin{align*}
A^{(1)}_{\lambda,\Delta}(x_{10}, x_{30})|_{a_0} &= \frac{\pi^\mu \Gamma(\mu - \Delta)}{\Gamma(\lambda + 1)\Gamma(\Delta + 1)} a_0(\vec{c}, \vec{\partial}) \\
& \quad \left\{ \frac{\lambda - 1}{x_{30}} \Lambda_{0,0}(\zeta) + \frac{2}{x_{30} \mu - \Delta - 1} \Lambda_{-2,-1}(\zeta) + \frac{2}{x_{10}} \Lambda_{-1,-1}(\zeta) \right\}. 
\end{align*}
(52)

In a similar manner we obtain for the third term in (48)
\begin{align*}
A^{(1)}_{\lambda,\Delta}(x_{10}, x_{30})|_{c_0} &= -c_0(\vec{a}, \vec{\partial}) \left\{ -\frac{1}{2(\lambda - 1)^2} (\vec{\partial}, \vec{\partial}) \Phi_{1,0} - \frac{1}{x_{10} \lambda \Delta} \Phi_{0,0} \right\} \\
& = -\frac{\pi^\mu \Gamma(\mu - \Delta)}{\Gamma(\lambda + 1)\Gamma(\Delta + 1)} c_0(\vec{a}, \vec{\partial}) \left\{ \frac{\lambda - 1}{x_{10}} \Lambda_{0,0}(\zeta) \\
& \quad + \frac{1}{x_{10} \mu - \Delta - 1} \Lambda_{-2,-1}(\zeta) + \frac{2}{x_{30}} \Lambda_{-1,-1}(\zeta) \right\}. 
\end{align*}
(53)

We check that this one is the same as (52) after interchanging $a \leftrightarrow c$ and $x_1 \leftrightarrow x_3$, up to the sign. This must be the case, because the derivative $\vec{\partial} = \vec{\partial}_3$ acting on $\vec{x}_{13}$ equals $-\vec{\partial}_3$, therefore the sum of (52) and (53) is symmetric under the above permutation, which is evident from the defining integral.\(^3\) Finally, the fourth term in (48) is given by the Laplacian acting on a scalar bulk-to-bulk propagator
\begin{align*}
A^{(1)}_{\lambda,\Delta}(x_{10}, x_{30})|_{a_0c_0} &= \frac{1}{\lambda \Delta} (\vec{\partial}, \vec{\partial}) \Phi_{0,0} = \frac{\pi^\mu \Gamma(\mu - \Delta)}{\Gamma(\lambda + 1)\Gamma(\Delta + 1)} a_0 c_0(\vec{\partial}, \vec{\partial}) \Lambda_{0,0}(\zeta). 
\end{align*}
(54)

Let us now write the vector two-point function in terms of the fundamental bitensors (13). The two fundamental bitensors for tensor rank 1 are $\langle a, \partial \zeta \rangle \langle c, \partial' \zeta \rangle$ and $\langle a, \partial \rangle \langle c, \partial' \zeta \rangle$, where the unprimed variables correspond to $x_1$ and the primed ones to $x_3$. We perform the differentiations of the Legendre functions $\Lambda_{s,t}$ in (50, 52, 53) with (33), e.g. let us look at (50)
\begin{align*}
A^{(1)}_{\lambda,\Delta}(x_{10}, x_{30})|_{a_0} &= \frac{\pi^\mu \Gamma(\mu - \Delta)}{\Gamma(\lambda + 1)\Gamma(\Delta + 1)} \left\{ \\
& \quad 2 \left[ (\lambda - 1) \Lambda_{-1,-1}(\zeta) + \frac{2}{\mu - \Delta - 1} \Lambda_{-3,-2}(\zeta) \right] (\vec{a}, \vec{\partial' \zeta})(\vec{c}, \vec{\partial' \zeta}) \\
& \quad + \left[ \Delta(\lambda - 1) \Lambda_{0,0}(\zeta) + \frac{2}{\mu - \Delta - 1} \Lambda_{-2,-1}(\zeta) \right] (\vec{a}, \vec{\zeta}) (\vec{c}, \vec{\zeta}) \right\}, 
\end{align*}
(55)
add the various contributions up, solve (15) for the $d$ dimensional quantities, and insert them into the sum. We then obtain for the complete vectorial two-point

\(^3\)Strictly speaking, it may happen that a possible asymmetric part is cancelled with the asymmetric part in the shadow term. However, from the comparison with the vector propagator in the literature we know that this assumption is indeed justified.
function

\[ A^{(1)}_{\lambda, \Delta}^{\text{direct}} = \frac{\pi^\mu \Gamma(\mu - \Delta)}{\Gamma(\lambda + 1) \Gamma(\Delta + 1)} \left\{ 2 \left[ (\lambda - 1) \Lambda_{-1,-1}(\zeta) + \frac{2}{\mu - \Delta - 1} \Lambda_{-3,-2}(\zeta) \right] I_2 \right. \]

\[ \left. - \left[ \Delta(\lambda - 1) \Lambda_{0,0}(\zeta) + \frac{2}{\mu - \Delta - 1} \Lambda_{-2,-1}(\zeta) \right] I_1 \right\}. \quad (56) \]

All other terms vanish, as can be checked with the identities (35). Comparing (55) with (56) one notes, that the result is completely determined by the “principal contribution”, i.e. terms of maximal degree in \( \vec{a} \) and \( \vec{c} \). This is clear, because when we substitute the \( d \) dimensional expressions composed of \( \vec{a} \) and \( \vec{c} \) by the \( d + 1 \) dimensional invariants, we add components with positive degree in \( a_0 \) or \( c_0 \), which must cancel the contributions from (52-54), due to symmetry. Thus we only have to determine the principal contributions.

**4.3 The symmetric traceless tensor of rank 2**

In this case the complete calculations are unappealingly lengthy, but with our experience gained from the vectorial case we know how to shorten the route. In contrast to the case of the vector we now have to take the subtraction of traces into account. First we write down the bulk to boundary propagator

\[ K^{(2)}_\lambda(a, \vec{\nabla} b; x_{12}) = \frac{x_{10}^{\lambda - 2}}{(x_{12}^4)^{\lambda}} \left\{ \frac{\left( 2 \langle a, x_{12} \rangle (\vec{x}_{12}, \vec{b}) \right)^2}{x_{12}^2} - \frac{\vec{b}^2}{d} \left( \frac{2 \langle a, x_{12} \rangle \vec{x}_{12} - \vec{a} \right)^2 \right\} \]

\[ = \frac{x_{10}^{\lambda - 2}}{(x_{12}^4)^{\lambda}} \left\{ 4 \frac{(\vec{a}, \vec{x}_{12})^2 (\vec{b}, \vec{x}_{12})^2}{x_{12}^4} - 4 \frac{(\vec{a}, \vec{b}) (\vec{a}, \vec{x}_{12}) (\vec{b}, \vec{x}_{12})}{x_{12}^2} + (\vec{a}, \vec{b})^2 \right. \]

\[ \left. - \frac{\vec{b}^2}{d} \left( \vec{a}^2 - 4x_{10}^2 \frac{(\vec{a}, \vec{x}_{12})^2}{x_{12}^4} \right) + \text{terms of positive order in } a_0, c_0 \right\}, \quad (57) \]

which is traceless with respect to the bulk and the boundary variables. As before we write down the integral for the bulk-to-bulk propagator, restrict to the direct terms and expand into powers of \( a_0 \) and \( c_0 \)

\[ 2 A^{(2)}_{\lambda, \Delta}(x_1, x_3) \Big|_{\text{direct}} = \int d^d x_2 K^{(2)}_\lambda(a, \vec{\nabla} b; x_{12}) K^{(2)}_\Delta(\vec{b}; x_{23}) \Big|_{\text{direct}} \]

\[ = A^{(2)}_{\lambda, \Delta}(x_1, x_3) \Big|_{\text{p.c.}} + \text{terms of positive order in } a_0, c_0, \quad (58) \]

where \( \text{p.c.} \) denotes the principal contribution, which is the only one we have to take into account. The factor 2 on the left hand side comes from the two contractions of \( \vec{\nabla} b \) with \( b \).

The calculation of \( A^{(2)}_{\lambda, \Delta} \text{p.c.} \) is similar to the case of the vector propagator, i.e express it in terms of derivatives of \( \Phi_{r,s} \) and manipulate the resulting expressions.
with the formula (31) until we can write the sum of derivatives of the \( \Phi_{r,s} \) functions in terms of the Legendre functions \( \Lambda_{s,t} \) with (36) and (37). We decompose \( A_{\lambda,\Delta}^{(2)} \big|_{p.c.} \) into summands of different numbers of derivatives:

\[
A_{\lambda,\Delta}^{(2)} \big|_{p.c.} = A_{\lambda,\Delta}^{(2)}[(\bar{a}, \bar{b})^2(\bar{c}, \bar{d})^2] + A_{\lambda,\Delta}^{(2)}[(\bar{a}, \bar{d})(\bar{c}, \bar{d})(\bar{a}, \bar{c})] + A_{\lambda,\Delta}^{(2)}[(\bar{a}, \bar{c})^2] \\
+ A_{\lambda,\Delta}^{(2)}[(\bar{a}, \bar{d})^2(\bar{c}, \bar{d})] + A_{\lambda,\Delta}^{(2)}[a^2(\bar{c}, \bar{d})^2] + A_{\lambda,\Delta}^{(2)}[\bar{a}^2(\bar{c}, \bar{d})^2].
\] (59)

Before turning to the calculation, note that the invariant bitensors we have to expect for the symmetric tensor of rank 2 are \( I_1^2, I_1 I_2, I_2^2, \) and \( I_3, I_4 \) for the removal of the traces of the former invariants. The terms in the second line of (59) contain \( a^2 \) and \( c^2 \) (after adding appropriate terms with \( a_0 \) and \( c_0 \)), so they remove the traces of the terms in the first line and can consequently be computed from them. Therefore it is sufficient to consider only the terms in the first line.

Let us start with the calculation of the first term in (59):

\[
A_{\lambda,\Delta}^{(2)}(0, 0; x_1, x_3) = \left(\frac{2(\bar{d}, \bar{d})^2}{2^4(\lambda - 2)_4(\Delta - 2)_4} \Phi_{2,2}(x_1, x_3) + \frac{1}{x_{30}} \frac{(6 - 4\Delta)(\bar{d}, \bar{d})}{2^3(\lambda - 2)_4(\Delta - 1)_3} \Phi_{2,1}(x_1, x_3) \right) \\
+ \frac{1}{x_{30}^2} \frac{(\Delta - 1)_2}{2(\lambda - 2)_4(\Delta - 1)} \Phi_{2,0}(x_1, x_3) + \frac{1}{x_{10}} \frac{(6 - 4\lambda)(\bar{d}, \bar{d})}{2^3(\lambda - 1)_3(\Delta - 2)_4} \Phi_{1,2}(x_1, x_3) \\
+ \frac{(\lambda - 2)(\Delta - 2) - 1 + \mu}{(x_{10} x_{30}) (\lambda - 1)_3(\Delta - 1)_3} \Phi_{1,1}(x_1, x_3) + \frac{1}{x_{10}^2} \frac{d - 2(\lambda - 1)}{2(\lambda - 2)_4} \Phi_{0,2}(x_1, x_3) \\
+ \frac{1}{2d(\lambda - 2)_4(\Delta)_2} \Phi_{2,0}(x_1, x_3) + \frac{1}{x_{10} d(\lambda - 1)_3(\Delta)_2} \Phi_{1,0}(x_1, x_3) \right) \\
= 2 \frac{\pi^d \Gamma(\mu - \Delta)}{\Gamma(\lambda + 2) \Gamma(\Delta + 2)} (\bar{a}, \bar{d})^2(\bar{c}, \bar{d})^2 \left\{ \frac{1}{(\mu - \Delta - 2)_2} (1 - \frac{1}{d}) \Lambda_{-2,0}(\zeta) \right. \\
+ \frac{(\lambda - 1)}{\mu - \Delta - 1} \left(1 - \frac{1}{d}\right) \Lambda_{0,1}(\zeta) + \frac{1}{2} \frac{1}{\lambda - 1_2} \Lambda_{2,2}(\zeta) \right\}. (60)
\]
The second term in (59) is given by

\[
A^{(2)}_{\lambda,\Delta}(x_1, x_3)\bigg|_{(\bar{a}, \bar{b})(\bar{c}, \bar{d})(\bar{a}, \bar{c})} = (\bar{a}, \bar{b})(\bar{c}, \bar{d})(\bar{a}, \bar{c}) \left\{ \begin{array}{c}
- \frac{1}{x_{10}^2} \frac{2(\lambda - 1)(\Delta - 1)}{2} \Phi_{1,0}(x_1, x_3) - \frac{1}{x_{10}^2 x_{30}^2} \frac{2(\lambda - 1)(\Delta - 1)}{2} \Phi_{0,1}(x_1, x_3) \\
+ \frac{1}{x_{10} x_{30}} \frac{\Gamma(\Delta - 1)}{(\lambda - 1)(\Delta - 1)\Phi_{1,1}(x_1, x_3)} \end{array} \right\}
\]

\[
= 2 \frac{\pi^{\mu}\Gamma(\mu - \Delta)}{\Gamma(\lambda + 2)\Gamma(\Delta + 2)} (\bar{a}, \bar{\partial})(\bar{c}, \bar{\partial})(\bar{a}, \bar{c}) \frac{1 - \Delta}{x_{10} x_{30}} \left\{ (\lambda - 1)2\Lambda_{1,1}(\zeta) + 2 \frac{\lambda - 1}{\mu - \Delta - 1}\Lambda_{-1,0}(\zeta) \right\}. \tag{61}
\]

Finally, the third summand in (59) is

\[
A^{(2)}_{\lambda,\Delta}(x_1, x_3)\bigg|_{(\bar{a}, \bar{c})^2} = (\bar{a}, \bar{c})^2 \frac{2}{x_{10}^2 x_{30}^2} \frac{(\lambda - 1)(\Delta - 1)}{2} \Phi_{0,0}(x_1, x_3)
\]

\[
= 2 \frac{\pi^{\mu}\Gamma(\mu - \Delta)}{\Gamma(\lambda + 2)\Gamma(\Delta + 2)} (\bar{a}, \bar{c})^2 \frac{2(\lambda - 1)(\Delta - 1)}{x_{10}^2 x_{30}^2} \Lambda_{0,0}(\zeta). \tag{62}
\]

Now it is straightforward to write down the full propagator for a symmetric traceless tensor field of rank 2. We perform the differentiations in (60) and (61), add them up together with (62) and sort them with respect to the invariants \(I_1^2, I_2^2,\) and \(I_1 I_2.\) After some not very labourious algebra we then find the propagator of a symmetric traceless tensor field of rank 2

\[
A^{(2)}_{\lambda,\Delta}(x_1, x_3)_{\text{direct}} = \frac{\pi^{\mu}\Gamma(\mu - \Delta)}{\Gamma(\lambda + 2)\Gamma(\Delta + 2)} \left\{ \begin{array}{c}
\left[ 2^{\delta - 1} \frac{\lambda \Delta + d - 1}{\mu - \Delta - 1} \Lambda_{-2,2}(\zeta) \right] I_1^2 + \left[ -2^{\delta - 1} \frac{\lambda \Delta + d - 1}{\mu - \Delta - 1} \Lambda_{-2,2}(\zeta) \right] I_1 I_2 + \left[ 2^{\delta - 1} \frac{\lambda \Delta + d - 1}{\mu - \Delta - 1} \Lambda_{-2,2}(\zeta) \right] I_2^2 \\
- \text{traces} \end{array} \right\}. \tag{63}
\]

**Acknowledgements**

This work is supported in part by the German Volkswagenstiftung. The work of R. M. was supported by DFG (Deutsche Forschungsgemeinschaft).
Appendix: The trace terms

Consistency of the group approach to traceless symmetric tensor fields on AdS space requires that the bitensor propagators can be made traceless using only the basis of geometric bitensors $I_1, I_2, I_3$ and $I_4$ from [14,18]. We apply the Laplacian $\Delta_a$ to equation (19) and express the result as a linear combination of the independent monomials

$$I_1^{m_1} I_2^{m_2} I_3^{n_3} I_4^{n_4} R_{1,2}$$

with

$$R_1 := \frac{1}{z^0} I_2^2, \quad R_2 := \frac{1}{a^2} I_4$$

In fact, the basic formula is

$$\frac{1}{2} \Delta_a I_1^{m_1} I_2^{m_2} I_3^{n_3} I_4^{n_4} = \left( \frac{m_1}{2} \right) (R_1 + R_2) I_1^{m_1-2} I_2^{m_2} I_3^{n_3} I_4^{n_4}$$

$$+ m_1 m_2 \zeta R_1 I_1^{m_1-1} I_2^{m_2-1} I_3^{n_3} I_4^{n_4} + \left( \frac{m_2}{2} \right) (\zeta^2 - 1) R_1 I_1^{m_1} I_2^{m_2-2} I_3^{n_3} I_4^{n_4}$$

$$+ \left[ 4 \left( \frac{n_1}{2} \right) (R_1 + (\zeta^2 - 1) R_2) + n_1 ((d+1) R_1 + (\zeta^2 - 1) R_2) \right] I_1^{m_1} I_2^{m_2} I_3^{n_3-1} I_4^{n_4}$$

$$+ \left[ 4 \left( \frac{n_2}{2} \right) R_2 + n_2 (d+1) R_2 + 2 (m_1 + m_2 + 2 n_1) n_2 R_2 \right] I_1^{m_1} I_2^{m_2} I_3^{n_3} I_4^{n_4-1}$$

$$+ \left[ 4 \left( \frac{n_1}{2} \right) R_2 I_1^{m_1} I_2^{m_2+2} I_3^{n_3-2} I_4^{n_4} - 4 \left( \frac{n_1}{2} \right) (\zeta^2 - 1) R_1 I_1^{m_1} I_2^{m_2} I_3^{n_3-2} I_4^{n_4+1} \right.$$

$$+ 2 m_1 n_1 \zeta R_2 I_1^{m_1-1} I_2^{m_2+1} I_3^{n_3-1} I_4^{n_4}$$

In the case $l = 2$, we have $(l_1, l_2) \in \{(2,0), (1,1), (0,2)\}$ and we calculate the matrix $V$ representing $\frac{1}{2} \Delta_a$ in the bases $\{I_1^2, I_2^2\}$ and $\{R_1, R_2\}$ respectively. We obtain

$$R_1 \begin{bmatrix} (2,0) & (1,1) & (0,2) \\ 1 & \zeta & \zeta^2 - 1 \end{bmatrix}$$

where we indicated the respective basis elements by the labels above each column to the left of the rows. For the second term in (19) we need the matrix $M$ representing $\frac{1}{2} \Delta_a$ on $\text{span}(I_3, I_4)$, and the image is again $\text{span}(R_1, R_2)$. This must be the case, because otherwise there were no solution to (19). We get from (66)

$$\mathcal{M} = \begin{bmatrix} d + 1 & 0 \\ \zeta^2 - 1 & d + 1 \end{bmatrix}$$
and collect the unknown coefficients in a matrix
\[ A = \begin{bmatrix} A_{0010}^{(2,0)} & A_{0010}^{(1,1)} & A_{0010}^{(0,2)} \\ A_{0001}^{(2,0)} & A_{0001}^{(1,1)} & A_{0001}^{(0,2)} \end{bmatrix}. \] (69)

Then (19) takes the form of a matrix equation
\[ V = MA, \] (70)
which is easy to solve thanks to the triangular shape of \( M \):
\[ A = M^{-1}V = \frac{1}{d + 1} \begin{bmatrix} 1 & -\zeta & \zeta^2 - 1 \\ 3(1 - \frac{\zeta^2 - 1}{d+3}) & -2 \zeta \frac{\zeta^2 - 1}{d+3} & \zeta^2 - 1 & 0 \\ 0 & 1 & 2 \zeta & 3(\zeta^2 - 1) \\ -\frac{6\zeta}{d+3} & 1 - \frac{7\zeta^2 - 3}{d+3} & -8 \zeta \frac{\zeta^2 - 1}{d+3} & -9(\zeta^2 - 1)^2 \end{bmatrix}. \] (71)

In the case \( l = 3 \) we also get a triangular 4 \( \times \) 4 matrix \( M \) acting on the trace terms and the matrix \( A \) of unknowns comes out as
\[
\begin{bmatrix}
A_{1010}^{(30)} & A_{1010}^{(21)} & A_{1010}^{(12)} & A_{1010}^{(03)} \\
A_{0101}^{(30)} & A_{0101}^{(21)} & A_{0101}^{(12)} & A_{0101}^{(03)} \\
A_{0110}^{(30)} & A_{0110}^{(21)} & A_{0110}^{(12)} & A_{0110}^{(03)} \\
A_{0101}^{(30)} & A_{0101}^{(21)} & A_{0101}^{(12)} & A_{0101}^{(03)}
\end{bmatrix}
= \frac{1}{d + 3} \begin{bmatrix}
3 & 2\zeta & \zeta^2 - 1 & 0 \\
3(1 - \frac{\zeta^2 - 1}{d+3}) & -2 \zeta \frac{\zeta^2 - 1}{d+3} & \zeta^2 - 1 & 0 \\
0 & 1 & 2 \zeta & 3(\zeta^2 - 1) \\
-\frac{6\zeta}{d+3} & 1 - \frac{7\zeta^2 - 3}{d+3} & -8 \zeta \frac{\zeta^2 - 1}{d+3} & -9(\zeta^2 - 1)^2
\end{bmatrix}. \] (72)

In the case \( l = 4 \) a dramatic change occurs since the matrix \( M \) representing \( \frac{1}{2} \Delta_a \) on the rank 4 trace terms becomes a 9 \( \times \) 10 rectangular matrix, i.e. there are 9 trace terms to cancel the traces of the 10 invariants of rank 4, thereby leading to 10 equations for the unknown entries of \( A \). In general, for \( l = 2p \) even, the number of trace terms and the number of invariants start to differ at \( p \geq 2 \):

\[
\text{Number of trace terms} = p\left(\frac{1}{3}p^2 + \frac{3}{2}p + \frac{1}{6}\right)
\]
\[
\text{Number of equations} = p(p + 1)\left(\frac{2}{3}p + \frac{1}{3}\right). \] (73)

Thus the set of equations for \( A \) contains
\[ p\left(\frac{1}{3}p^2 - \frac{1}{2}p + \frac{1}{6}\right) \] (74)
linearly dependent equations. We compute the linear system for \( l = 4 \) and show thereby that there is 1 linearly dependent equation. The matrix \( V \) representing \( \frac{1}{2} \Delta_a \) on
\[ \text{span}\{I_1^1, I_1^3 I_2, I_1^2 I_2^2, I_1 I_2^3, I_2^4\} \] (75)
is calculated by \((66)\) to result in a \(10 \times 5\) matrix, which decomposes into

\[
\mathcal{V} = \begin{bmatrix} \mathcal{V}_0 \\ 0 \end{bmatrix}, \quad (76)
\]

where

\[
\mathcal{V}_0 = \begin{bmatrix} 6 & 3\zeta & \zeta^2 - 1 & 0 & 0 \\
6 & 0 & 0 & 0 & 0 \\
0 & 3 & 4\zeta & 3(\zeta^2 - 1) & 0 \\
0 & 3 & 0 & 0 & 0 \\
0 & 0 & 1 & 3\zeta & \zeta^2 - 1 \\
0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad (77)
\]

The basis in the preimage of \(\frac{1}{\Delta} \Delta_a\) is ordered as in \((75)\), and the 10 basis elements \(\{I_iI_jR_k\}\) in the image are ordered lexicographically, i.e. \(I_2^2R_1, I_3^2R_2, \ldots, I_4R_2\). By the simple transformation

\[
A^{(l_1l_2)}_{0201} \mapsto A^{(l_1l_2)}_{0201} + \frac{4}{d + 5} A^{(l_1l_2)}_{0020} \quad (78)
\]

we can bring the system of equations into the block form

\[
\begin{bmatrix} \mathcal{V}_0 \\ 0 \end{bmatrix} = \begin{bmatrix} \mathcal{M}_0 & 0 \\ \mathcal{B} & \mathcal{M}_1 \end{bmatrix} \begin{bmatrix} \mathcal{A}_0 \\ \mathcal{A}_1 \end{bmatrix}, \quad (79)
\]

where the 4-tupels \((m_1, m_2, n_1, n_2)\) are ordered as \(\ldots, (1101), (1110), (2010), (2001), (0210), (0201), (0020), (0011), (0002)\). \(\ldots\)

Then \(\mathcal{M}_0\) has triangular shape

\[
\mathcal{M}_0 = \begin{bmatrix} d + 5 & 0 & 0 & 0 & 0 \\
\zeta^2 - 1 & d + 5 & 0 & 0 & 0 \\
0 & 0 & d + 5 & 0 & 0 \\
4\zeta & 0 & 3(\zeta^2 - 1) & d + 5 & 0 \\
0 & 0 & 0 & 0 & d + 5 \\
0 & 0 & 2\zeta & 0 & 5(\zeta^2 - 1) \end{bmatrix}, \quad (81)
\]

thus \(\mathcal{M}_0\) can be inverted easily to give

\[
\mathcal{A}_0 = \frac{1}{d + 5} \begin{bmatrix} 6 & -3\zeta & \zeta^2 - 1 & 0 & 0 \\
6(1 - \frac{\zeta^2 - 1}{d + 5}) & -3\zeta^2 - 1 & \zeta^2 - 1 & 0 & 0 \\
-\frac{24\zeta}{d + 5} & 3(1 - \frac{7\zeta^2 - 3}{d + 5}) & 3(\zeta^2 - 1) & 0 & 0 \\
0 & -\frac{6\zeta}{d + 5} & 1 - \frac{13\zeta^2 - 5}{d + 5} & -21\zeta^2 - \frac{1}{d + 5} & -30(\zeta^2 - 1) \end{bmatrix} \quad (82)
\]
Finally it remains to solve
\[ -BA_0 = M_1 A_1. \] (83)

We manipulate both sides the following way: Multiplying the first row with \( \zeta((\zeta^2 - 1), d+5) \), the second with \(-1 + \frac{1}{d+5}\) and adding both to the third row results in a zero row, which we skip on both sides of (83). We thus get the transformed equations
\[ M_1 \mapsto \widetilde{M}_1, \quad BA_0 \mapsto \widetilde{BA}_0 \] (84)

with
\[
\widetilde{M}_1 = \begin{bmatrix}
2(d+3) & 0 & 0 \\
6(\zeta^2 - 1) & d+5 & 0 \\
0 & (\zeta^2 - 1) & 2(d+3)
\end{bmatrix}
\] (85)

and
\[
\widetilde{BA}_0 = \frac{1}{d+5} \begin{bmatrix}
6 & 6\zeta & 2(3\zeta^2 - 1) & 6(\zeta^2 - 1) & 6(\zeta^2 - 1)^2 \\
6 & 3\zeta & (\zeta^2 - 1) & 0 & 0 \\
6(1 - \frac{\zeta^2 - 1}{d+5}) & -3\zeta^2 - 1 & -\frac{(\zeta^2 - 1)^2}{d+5} & 0 & 0
\end{bmatrix}
\] (86)

We evaluate then
\[ A_1 = -\widetilde{M}_1^{-1} \widetilde{BA}_0 \] (87)

and get
\[
A_1 = -\frac{1}{d+5} \begin{bmatrix}
\frac{3}{d+3} & \frac{3}{d+3} & \frac{3}{d+3} \\
\frac{d+5}{d+3} \left(1 - \frac{3(\zeta^2 - 1)}{(d+3)(d+5)}\right) & \frac{3}{d+3} \left(1 - \frac{6(\zeta^2 - 1)}{(d+3)(d+5)}\right) & \frac{3}{d+3} \left(1 - \frac{6(\zeta^2 - 1)}{(d+3)(d+5)}\right) \\
\frac{3\zeta^2 - 1}{d+5} & \frac{3\zeta(\zeta^2 - 1)}{(d+3)(d+5)} & \frac{3\zeta(\zeta^2 - 1)}{(d+3)(d+5)}
\end{bmatrix}
\] (88)

At the end, we have to invert the transformation (78).

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