Quantum heat transfer between non-linearly-coupled bosonic and fermionic baths: an exactly solvable model

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The evolution of a quantum system towards thermal equilibrium is usually studied by approximate methods, which have their limits of validity and should be checked against analytically solvable models. In this paper, we propose an analytically solvable model to investigate the heat transfer between a bosonic bath and a fermionic bath which are non-linearly coupled to each other. The bosonic bath consists of an infinite collection of non-interacting bosonic modes, while the fermionic bath is represented by a chain of interacting fermions with nearest-neighbor interactions. We compare behaviors of the temperature-dependent heat current $J_T$ and temperature-independent heat current $J_{TI}$ for different bath configurations. With respect to the bath spectrum, $J_T$ decays exponentially for Lorentz-Drude type bath, which is the same as the conventional approximations. On the other hand, the decay rate is $1/t^3$ for Ohmic type and $1/t$ for white noise, which doesn’t have conventional counterparts. For the temperature-independent current $J_{TI}$, the decay rate is divergent for the Lorentz-Drude type bath, $1/t^4$ for the Ohmic bath, and $1/t$ for the white noise. When further considering the dynamics of the fermionic chain, the current will be modulated based on the envelope from the bath. As an example, for a bosonic bath with Ohmic spectrum, when the fermionic chain is uniformly-coupled, we have $J_T \propto 1/t^6$ and $J_{TI} \propto 1/t^5$. Remarkably, for perfect state transfer (PST) couplings, there always exists an oscillating quantum heat current $J_{TI}$. Moreover, it is interesting that $J_T$ is proportional to $(N-1)^{1/2}$ at certain times for PST couplings under Lorentz-Drude or Ohmic bath.

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I. INTRODUCTION

Dissipative phenomena in open systems\textsuperscript{1,2} can give rise to a variety of interesting physical scenarios and have been under extensive study across many different fields such as quantum optics, many-body physics, and quantum information sciences. Open systems are notoriously difficult to deal with exactly due to the complexity of the quantum reservoir, whose Hilbert space can be prohibitively large. Such systems are usually tackled with a system-plus-reservoir approach where one treats the composite system as a whole, and later traces out the reservoir degrees of freedom to study the reduced dynamical behaviors of the system under consideration. One widely-used way to model the quantum reservoir is to treat it as non-interacting harmonic oscillators\textsuperscript{3,4}. The dynamics of two Brownian particles in a common reservoir have been studied\textsuperscript{5} as well as its thermal equilibrium properties\textsuperscript{6}. Using a quantum Langevin description, a system-reservoir model is proposed\textsuperscript{7} and a quantum current is observed which is dependent on various parameters of external noise. Spin-boson model also provides a clear physical picture for exploring quantum dissipation effects. This model includes an impurity two-level system (TLS) coupled to a thermal reservoir and displays a rich phase diagram in the equilibrium regime\textsuperscript{7,8}. A generalized non-equilibrium polaron-transformed Redfield equation with an auxiliary counting field was developed recently to study the full counting statistics of quantum heat transfer in a driven non-equilibrium spin-boson model\textsuperscript{9}. For a subsystem which constitutes of two interacting spins, this situation effectively corresponds to a subsystem unharmonically coupled to a bosonic bath, allowing to introduce nonlinear effects\textsuperscript{10}. Exact dynamics of interacting TLS immersed in separate thermal reservoirs or within a common bath has also been studied\textsuperscript{11}. For the device design, such as in molecular devices, people often need to consider the scaling of heat current with system size and time in order to prevent the devices from disintegrating\textsuperscript{12,13}, because excess heat build-up during operation may cause device disintegrating.

However, most theoretical investigations of how a quantum system reaches thermal equilibrium use the approximation methods, such as quantum master equations\textsuperscript{14,15}, Born-Oppenheimer methods\textsuperscript{16}, etc. Typically, such methods only provide numerical results, hindering a direct picture of the microscopic processes in-
II. MODEL

Consider two baths \((H_{LB} \text{ and } H_{RB})\) connected by a central system \(H_S\) (see Fig. 1). The central system \(S\) consists of two interacting fermions. The left bath \(LB\) is modeled as a collection of non-interacting bosonic modes \(\alpha=1,2,\ldots\), maintained at a fixed temperature \(T = \beta^{-1}\), with \(k_B = 1\). The right bath \(RB\) is modeled as a fermionic bath, represented by a one-dimensional chain with nearest-neighbor interactions. The total Hamiltonian is given by

\[
H = H_S + H_{LB} + H_{RB} + V_L + V_R, \tag{1}
\]

where \(V_L (V_R)\) is the interaction between the left (right) bath and the central system. Treating the central system and the fermionic right bath as a whole, we denote \(H_{ch} = H_S + H_{RB} + V_R\) and rearranged the indices so that the interacting fermions in central system is labeled 1 and 2 and the sites in the fermion bath are labeled 3 through \(N\),

\[
H_{ch} = H_S + H_{RB} + V_R = -\sum_{i=1}^{N-1} \tau_i (c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i), \tag{2}
\]

where \(H_S = -\tau_0(c_1^\dagger c_2 + c_2^\dagger c_1)\), \(H_{RB} = -\sum_{i=3}^{N-1} \tau_i (c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i)\), \(V_R = -\tau_2(c_3^\dagger c_1 + c_1^\dagger c_3)\), \(c_i^\dagger\) is standard fermionic creation operator of electron spin, and \(\tau_i\) is the coupling constant between the nearest neighbor sites. Additionally, we take \(\tau_i > 0\) which corresponds to ferromagnetic couplings throughout. \(H_{ch}\) describes a chain of interacting spin-less fermions, which can be mapped to a one-dimensional XY chain under Jordan-Wigner transformation. For simplicity, the total Hamiltonian \(H\) can be written as

\[
H = H_{ch} + H_B + V, \tag{3}
\]

where \(H_B = H_{LB} = \sum_{\alpha} \omega_\alpha b_\alpha^\dagger b_\alpha\) is the bosonic bath’s Hamiltonian. \(b_\alpha^\dagger\) is the bosonic creation operator of phonons, and \(\omega_\alpha\) is the frequency of the \(\alpha\)-th mode. We will discuss the heat transfer between the two baths next. The heat current will be zero after some time \(t\), at which point thermal equilibrium is reached.

A specific magnon-phonon interaction \(V = V_L\) can be introduced by the so-called dressing transformation \(W^\dagger (H_{ch} + H_B)W\) with a displacement operator \(W = \exp[\sum_\alpha (\Gamma_\alpha b_\alpha^\dagger - \Gamma_\alpha^* b_\alpha)]\), where \(n_i = c_i^\dagger c_i\) is the number operator. Clearly, it is possible to introduce different types of interactions using different types of dressing transformations \([14]\). The exact form of the interaction operator is now given by

\[
V = \sum_\alpha \omega_\alpha (\Gamma_\alpha^* b_\alpha + \Gamma_\alpha b_\alpha^\dagger + |\Gamma_\alpha|^2) n_1 + \tau_1 (c_1^\dagger c_2 e^{i\Gamma_\alpha b_\alpha^\dagger - \Gamma_\alpha^* b_\alpha} - 1 + h.c.). \tag{4}
\]

It describes the interaction between the phonons and magnons. \(\Gamma_\alpha\) parameterizes the system-bath coupling strength, and is assumed to be a complex number.

III. THE CALCULATION OF ENERGY EXPECTATION VALUE AND HEAT CURRENT

In this section, we will give a general analysis of the heat current between the bosonic bath and the composite system of the fermionic bath plus the central system. The heat current operator can be defined as \([14]\)

\[
J = i[V, H_B]. \tag{5}
\]

The expectation value of the energy current is given by

\[
J(t) = Tr[\rho_0 J], \tag{6}
\]

where \(\rho_0 = \rho_B \otimes \rho_{ch}\) is the initial density operator of the whole system and \(J = e^{-iHt} \hat{J}(0)e^{iHt}\). Equivalently, the heat current may be rewritten as

\[
J(t) = \frac{\partial \langle H_B(t) \rangle}{\partial t} \tag{7}
\]
where \((H_B(t)) = Tr(\rho_B H_B(t))\) is the energy expectation value of the bath, where \(H_B = \text{e}^{iHt}\hat{H}_B(0)\text{e}^{-iHt}\). The bosonic bath at temperature \(T\) is modeled as a canonical ensemble with distribution \(\rho_B = \exp(-\beta H_B)/Tr[\exp(-\beta H_B)]\). It can be readily shown that

\[ U_0^\dagger b_a^\dagger b_a U_0 = b_a^\dagger e^{i\omega_a t}, \]

(8)

and

\[ U_0^\dagger c_l^\dagger c_l U_0 = c_l^\dagger(t) = \sum_l f_{l,l}^* c_l^\dagger, \]

(9)

where \(U_0 = e^{-i(H_B+H_S)t}\) and \(f_{l,l}\) is the transition amplitude of an excitation (the \(1\) site) from site \(l\) to site \(l\) in the chain. For simplicity, we restrict the fermion chain to have only one excitation. Denoting \(\ket{l}\) as the state where the one fermion excitation is at site \(l\), we write the initial state of the fermion chain as \(\ket{\Psi(0)} = \frac{1}{\sqrt{a}} \sum_{l=1}^a \ket{l}\), where the \(1\) excitation is restricted to the first \(a\) sites. Then, \(\langle H_B(t) \rangle\) can be expressed as (neglecting the time-independent part)

\[ \langle H_B(t) \rangle = \langle H_B(t) \rangle_T + \langle H_B(t) \rangle_{TI}, \]

(10)

where \(\langle H_B(t) \rangle_T\) is the temperature dependent part

\[
\langle H_B(t) \rangle_T = \frac{1}{a} \sum_{\alpha} \omega_\alpha |\Gamma_\alpha|^2 \langle D(\Gamma) \rangle_{eq} \{2 \coth \frac{\beta \omega_\alpha}{2} \sin \omega_\alpha t \text{Im}[F(t)] + 2(1 - \cos \omega_\alpha t) \text{Re}[F(t)]\},
\]

(11)

and \(\langle H_B(t) \rangle_{TI}\) is the temperature independent part

\[
\langle H_B(t) \rangle_{TI} = \frac{1}{a} \sum_{\alpha} \omega_\alpha |\Gamma_\alpha|^2 [(1 - 2 \cos \omega_\alpha t) |f_{11}|^2 + G(t)],
\]

(12)

where \(\langle D(\Gamma) \rangle_{eq} = \exp(-\frac{1}{2} \sum \alpha |\Gamma_\alpha|^2 \coth \frac{\beta \omega_\alpha}{2})\) is the expectation value of the displacement operator in the thermal equilibrium state, and \(F(t), G(t)\) indicate the dynamics of the chain and depend on the initial state of the chain. In the following sections, we will discuss the behavior of the heat current for different initial states, different bath spectra for the bosonic bath and different coupling configurations for the fermion bath.

\[ J = J_T + J_{TI}, \]

(13)

where

\[
J_T = \frac{2}{a} \sum_{\alpha} \omega_\alpha |\Gamma_\alpha|^2 \langle D(\Gamma) \rangle_{eq} \{\coth \frac{\beta \omega_\alpha}{2} \sin \omega_\alpha t \text{Im}(F(t)) + \sin \omega_\alpha t dt \text{Re}(F(t)) + (1 - \cos \omega_\alpha t) dt \text{Re}(F(t)/dt)\},
\]

(14)

is the temperature dependent current, and

\[
J_{TI} = \frac{1}{a} \sum_{\alpha} \omega_\alpha |\Gamma_\alpha|^2 |2 \omega_\alpha \sin \omega_\alpha t |f_{11}|^2 + (1 - \cos \omega_\alpha t) dt |f_{11}|^2 dt + dG(t)/dt dt,
\]

(15)

is the temperature independent current.

(i) \(\ket{\Psi(0)} = \ket{1}\). In this case, \(a = 1, F(t) = 0, G(t) = 0\), and \(J_T = 0\). Therefore, the current is independent of the temperature and represents a pure quantum current. We also note that if the initial state is prepared as \(\ket{0}\) or \(\ket{1}\), i.e., the first site is an arbitrary pure state \(a |0\rangle + \beta |1\rangle\) and all other states at site \(0\), the state on site one can not be transferred to other sites under the spin chain dynamics.

(ii) \(\ket{\Psi(0)} = \frac{1}{\sqrt{N}} (\ket{1} + \ket{2} + \ldots + \ket{N})\), now \(a = N, F(t) = f_{11}^\dagger f_{11}, G(t) = \sum_{l,m=2}^N f_{l,m}^\dagger f_{1,m}\).

(iii) \(\ket{\Psi(0)} = \frac{1}{\sqrt{2}} (\ket{1} + \ket{2}), \quad a = 2, F(t) = f_{11}^\dagger f_{12}, G(t) = |f_{1,2}|^2\).

Note that the analytical expression of the heat current above is obtained without approximations. In the next section, we will discuss case (iii) as an illustrative example.

V. SPECTRUM DISTRIBUTION OF THE BOSONIC BATH

When the chain-bath couplings are weak \((\Gamma_\alpha/\tau_\alpha \rightarrow 0)\), which corresponds to the Markovian limit, the displacement operator expectation value in the thermal equilibrium can be approximated its the first order term \(\exp(-\frac{1}{2} \sum \alpha |\Gamma_\alpha|^2 \coth \frac{\beta \omega_\alpha}{2}) \approx 1\). Additionally, in the high temperature \(T\), or low frequency limit \(\omega_\alpha \rightarrow 0\), the hyperbolic cotangent \(\coth \frac{\beta \omega_\alpha}{2} \rightarrow \infty\), so the dominant term will be the first two terms in Eq (13). Using the Taylor expansion for the hyperbolic cotangent \(\coth x = \sum_{n=1}^{\infty} \frac{\omega_n x^{2n-1}}{(2n)!}\) (where \(0 < |x| < \pi\) and \(B_n\) is the \(n\)th Bernoulli number) and taking the first order for \(x \rightarrow 0\), the current in Eq (13) can be further reduced to
The envelope for the currents is $1/t$. Note that for $J_{T1}(t)$ the current depends on the chain’s dynamics only in the long time limit.

VI. EFFECTS OF THE FERMION BATH CONFIGURATION

Now we reveal how different configurations of the fermion bath can affect the heat current. The transition amplitude $f_{m,m'}$ depends on the types of couplings in the chain. First we discuss the perfect state transfer (PST) couplings $\tau_k = 2\tau\sqrt{k(N-k)/N^2}$. The transition amplitude reads

$$f_{m,m'}(t) = \exp\left[\frac{\pi}{2}(m-m')d_{m,m'}^d(2\tau t)\right],$$

where $d_{m,m'}^d(2\tau t)$ is the Wigner d matrix $^{22}$. The indices of the site number of a 1-dimensional chain can be mapped onto the magnetic quantum numbers $m$ of the total angular momentum $l$, such that $l = \frac{N-1}{2}, m = -\frac{N-1}{2} + k - 1$, where $k$ is the site number $^{24}$. Using this relation, we obtain $f_{1,1}(t) = [\cos(\tau t)]^{N-1}, f_{1,2}(t) = i\sqrt{N-1}\sin(\tau t)[\cos(\tau t)]^{N-2}, f_{1,N}(t) = (-1)^N\exp[i\pi(N-1)][\sin(\tau t)]^{N-1}$.

For the Lorentz-Drude spectrum, from Eq (17), the current is given by

$$J_T = e^{-\omega_4t}\left\{\frac{\Omega}{t}\sin(\Omega t)\text{Im}(F(t))\right\} \left\{\frac{1 - \cos(\Omega t)}{t} d\text{Im}(F(t))dt\right\} + \frac{\Omega^2}{t} d(\sin(\Omega t)\text{Im}(F(t))dt)$$

$$J_{T1} = \frac{\Omega^2}{t} d(\sin(\Omega t)\text{Im}(F(t))dt).$$
\[
\frac{\tau}{4}(N - 1)[\cos 2\tau t(\cos \tau t)]^{2N-6} - (2N - 6) \sin \tau t \sin 2\tau t(\cos \tau t)^{2N-8} - 2\tau(N - 1) \sin \tau t[\cos \tau t]^{2(N-1)-1}.
\]

Then there always exists an oscillation quantum current \(J_{TT}\) for PST couplings with an Ohmic bath.

For uniform couplings \(\tau_i = \tau/2\), the transition amplitudes \(f_{j,l}\) from site \(j\) to \(l\) is,

\[
f_{j,l} = \frac{2}{N + 1} \sum_{m=1}^{N} \sin(q_m j) \sin(q_m l)e^{iE_m t}, \quad (25)
\]

where \(q_m = \pi m/(N + 1), E_m = -\tau \cos q_m\).

Note that the transition probability \(f_{1,1}(t)\) is real when \(l\) is odd, and imaginary when \(l\) is even. That is a typical odd-even effect and it is a universal properties for finite systems \([24]\). The more evident effects will be displayed for smaller \(N\). Clearly when \(N = 2\), \(\Re(f_{1,1,f_{1,2}}) = 0\).

When \(N\) is infinite, the transition amplitude can be calculated as \(f_{1,1} = \frac{1}{2}[J_0(\tau t) + J_2(\tau t)]\), where \(J_n(t)\) is the Bessel function of the first kind. When \(\tau > 1\) and \(n = 0, 1, 2, ...\)

\[
f_{1,1} = \begin{aligned}
\frac{1}{\tau^l} J_l(\tau t), & \quad l = 4n + 1 \\
\frac{-1}{\tau^l} J_l(\tau t), & \quad l = 4n + 2 \\
\frac{1}{\tau^l} J_l(\tau t), & \quad l = 4n + 3 \\
\frac{1}{\tau^l} J_l(\tau t), & \quad l = 4n + 4
\end{aligned} \quad (26)
\]

From the expression of \(f_{1,1}\) and \(f_{1,2}\), we can see that for both PST and uniform couplings, \(f_{1,1}\)s real and \(f_{1,2}\) is imaginary. Then \(\Re[F(t)] = 0(F(t) = f_{1,1,f_{1,2}})\) in Eq \([14]\). Thus even if we do not consider the high temperature or low frequency, the last two terms can be neglected for PST or uniform couplings. Using Ohmic spectrum for the bosonic bath as an example,

\[
J_T \approx \frac{2\pi T |\Gamma|^2}{\omega_c} \left\{ \frac{2J_1 J_3 - J_2^2 J_0 - J_2}{4\pi^2 t^6} \right\} + \frac{6J_1 J_2}{\pi^2 t^6}, \quad (27)
\]

\[
J_{TT} \approx \frac{\pi |\Gamma|^2}{2} t^6 \left[ f_{1,1}(t) \right]^2 + \frac{6}{\omega_c t^4} d(|f_{1,1}(t)|^2)/dt
\]

\[
+ \omega_c^3 d(|f_{1,2}(t)|^2)/dt]. \quad (28)
\]

When \(t \rightarrow \infty, J_n(t) \approx \sqrt{\frac{2}{\pi t}} \cos(t - \frac{n\pi}{2} - \frac{\pi}{4})\), so \(J_T \propto 1/t^6 = (1/t^3)^2\), where the bath spectrum and chain’s dynamics both contribute a factor of \(1/t^3\). On the other hand, \(J_{TT} \propto 1/t^3\) only, which corresponds to the dissipation in the uniform chain. The ratio of \(J_T(t)/J_{TT}(t) \approx \frac{T \tan(\pi t - \frac{\pi}{2})}{2\pi \omega_c t^3}\) is proportional to temperature \(T\), couplings intensity \(1/\tau\), cutoff frequency \(1/\omega_c\), and time \(1/t^3\) modulated by a tan function.

VII. CONCLUSIONS

We analytically calculate the heat current in a hybrid quantum system where a bosonic bath and a fermionic bath are connected by a central unit. The central unit and the fermionic bath are taken as a subsystem of a one-dimensional fermion chain. We discuss the heat transfer between the bosonic bath and the rest of the system. For our system, only an initial entanglement state in the chain can induce a temperature dependent heat current. The heat current \(J(t)\) depends on both the bosonic bath spectrum and the coupling mechanisms within the fermion chain. With respect to the effects of the bath spectrum, \(J_T\) will decrease to zero, with an exponential decay for Lorentz-Drude type which is in accordance with the conventional Markovian approximation. On the other hand, it is proportional to \(1/t^3\) for Ohmic spectrum, and \(1/t\) for white noise. For \(J_{TT}\), the effect of the bath spectrum becomes divergent, \(1/t^4\) and \(1/t\), respectively. When different coupling configurations in the fermion chain are introduced, the envelope of the heat currents will be modulated. For PST couplings, the oscillation is governed by a periodical function and for uniform couplings it is governed by the Bessel function of the first kind. Additionally, for PST couplings with a Lorentz-Drude or Ohmic bosonic bath, \(J_T\) is found to be proportional to \((N - 1)^{1/2}\) at certain times. This behavior can be interpreted as larger baths have the capacity to absorb more energy. It is also interesting to note that for PST couplings with an Ohmic bosonic bath, there always exist an oscillating quantum current \(J_{TT}\). For uniform couplings with an Ohmic bath, \(J \propto 1/t^6\) and \(J_{TT} \propto 1/t^3\) in the big \(N\) and long \(t\) limit. To the best of our knowledge, our work is the first to give an exact analytical expression for the heat current in hybrid quantum structures.

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