MODELS OF DAMPED OSCILLATORS IN QUANTUM MECHANICS

RICARDO CORDERO-SOTO, ERWIN SUAZO, AND SERGEI K. SUSLOV

ABSTRACT. We consider several models of the damped oscillators in nonrelativistic quantum mechanics in a framework of a general approach to the dynamics of the time-dependent Schrödinger equation with variable quadratic Hamiltonians. The Green functions are explicitly found in terms of elementary functions and the corresponding gauge transformations are discussed. The factorization technique is applied to the case of a shifted harmonic oscillator. The time-evolution of the expectation values of the energy related operators is determined for two models of the quantum damped oscillators under consideration.

1. An Introduction

We continue an investigation of the one-dimensional Schrödinger equations with variable quadratic Hamiltonians of the form

\[
i \frac{\partial \psi}{\partial t} = -a(t) \frac{\partial^2 \psi}{\partial x^2} + b(t) x^2 \psi - i \left( c(t) x \frac{\partial \psi}{\partial x} + d(t) \psi \right),
\]

where \(a(t), b(t), c(t),\) and \(d(t)\) are real-valued functions of time \(t\) only; see Refs. [8], [9], [21], [22], [24], [32], [33], and [34] for a general approach and currently known explicit solutions. Here we discuss elementary cases related to the models of damped oscillators. The corresponding Green functions, or Feynman’s propagators, can be found as follows [8], [33]:

\[
\psi = G(x, y, t) = \frac{1}{\sqrt{2\pi i \mu(t)}} e^{i(\alpha(t)x^2 + \beta(t)xy + \gamma(t)y^2)},
\]

where

\[
\alpha(t) = \frac{1}{4a(t)} \frac{\mu'(t)}{\mu(t)} - \frac{d(t)}{2a(t)},
\]

\[
\beta(t) = -\frac{h(t)}{\mu(t)}, \quad h(t) = \exp \left( - \int_0^t (c(\tau) - 2d(\tau)) \, d\tau \right),
\]

\[
\gamma(t) = \frac{a(t) h^2(t)}{\mu(t) \mu'(t)} + \frac{d(0)}{2a(0)} - 4 \int_0^t \frac{a(\tau) \sigma(\tau) h^2(\tau)}{(\mu'(\tau))^2} \, d\tau,
\]

and the function \(\mu(t)\) satisfies the characteristic equation

\[
\mu'' - \tau(t) \mu' + 4\sigma(t) \mu = 0
\]

Date: June 6, 2009.
1991 Mathematics Subject Classification. Primary 81Q05, 35C05. Secondary 42A38.

Key words and phrases. The time-dependent Schrödinger equation, Cauchy initial value problem, Green function, propagator, gauge transformation, damped oscillator, factorization method.
with
\[ \tau (t) = \frac{a'}{a} - 2c + 4d, \quad \sigma (t) = ab - cd + d' \left( \frac{a'}{a} - \frac{d'}{d} \right) \] subject to the initial data
\[ \mu (0) = 0, \quad \mu'(0) = 2a (0) \neq 0. \] The corresponding Hamiltonian structure is discussed in Ref. [9].

The simple harmonic oscillator is of interest in many advanced quantum problems [15], [20], [25], and [31]. The forced harmonic oscillator was originally considered by Richard Feynman in his path integrals approach to the nonrelativistic quantum mechanics [11], [12], [13], [14], and [15]; see also [22]. Its special and limiting cases were discussed by many authors; see Refs. [6], [16], [18], [23], [25], [36] for the simple harmonic oscillator and Refs. [1], [7], [17], [26], [30] for the particle in a constant external field and references therein.

The damped oscillations are well-investigated in classical mechanics; see, for example, Refs. [5] and [19]. Although discussion of a quantum damped oscillator is usually missing in the standard classical textbooks [20], [25], and [31] among others, we believe that the models presented here have a significant value from the pedagogical and mathematical points of view. For instance, one of these models was crucial for our understanding of a “hidden” symmetry of the quadratic propagators in Ref. [9].

The paper is organized as follows. In section 2 we introduce two models of the damped oscillator and derive their propagators following the method of Ref. [8]. The corresponding gauge transformations are discussed in section 3. The next section is concerned with the separation of the variables for a related model of a “shifted” linear harmonic oscillator. The factorization technique is applied to this oscillator in section 5. The time evolution of the expectation values of the energy related operators is determined for these quantum damped oscillators in section 6. One more model of the damped oscillator with a variable quadratic Hamiltonian is introduced in section 7. The last section contains some remarks on the momentum representation.

### 2. The First Two Models

For the time-dependent Schrödinger equation:
\[ i \frac{\partial \psi}{\partial t} = \frac{\omega_0}{2} \left( -\frac{\partial^2 \psi}{\partial x^2} + x^2 \psi \right) + i \lambda \left( x \frac{\partial \psi}{\partial x} + \psi \right) \] with \( a = b = \omega_0/2 \) and \( c = d = -\lambda \), the characteristic equation takes the form of the classical equation of motion for the damped oscillator [5], [19]:
\[ \mu'' + 2\lambda \mu' + \omega_0^2 \mu = 0, \] whose suitable solution is as follows
\[ \mu = \frac{\omega_0}{\omega} e^{-\lambda t} \sin \omega t, \quad \omega = \sqrt{\omega_0^2 - \lambda^2} > 0. \] The corresponding propagator is given by
\[ G(x, y, t) = \sqrt{\frac{\omega e^{\lambda t}}{2\pi i\omega_0 \sin \omega t}} \exp \left( \frac{i\omega}{2\omega_0 \sin \omega t} \left( (x^2 + y^2) \cos \omega t - 2xy \right) \right) \]
\[ \times \exp \left( \frac{i\lambda}{2\omega_0} (x^2 - y^2) \right) \]. (2.4)

Indeed, directly from (1.3)–(1.4):

\[ \alpha (t) = \frac{\omega \cos \omega t + \lambda \sin \omega t}{2\omega_0 \sin \omega t}, \quad \beta (t) = -\frac{\omega}{\omega_0 \sin \omega t}. \] (2.5)

The integral in (1.5) can be evaluated with the help of a familiar antiderivative

\[ \int \frac{dt}{(A \cos t + B \sin t)^2} = \frac{\sin t}{A \cos t + B \sin t} + C. \] (2.6)

It gives

\[ \gamma (t) = \frac{\omega \cos \omega t - \lambda \sin \omega t}{2\omega_0 \sin \omega t} \] (2.7)

with the help of the following identity

\[ \omega^2 - \omega_0^2 \sin^2 \omega t = \omega^2 \cos^2 \omega t - \lambda^2 \sin^2 \omega t \] (2.8)

and the propagator (2.4) is verified. A “hidden” symmetry of this propagator is discussed in Ref. [9].

The time-evolution of the squared norm of the wave function is given by

\[ \|\psi (x, t)\|^2 = \int_{-\infty}^{\infty} |\psi (x, t)|^2 \ dx = e^{\lambda t} \|\psi (x, 0)\|^2. \] (2.9)

It is derived in section 6 among other things.

In a similar fashion, the time-dependent Schrödinger equation of the form

\[ i \frac{\partial \psi}{\partial t} = \frac{\omega_0}{2} (-\frac{\partial^2 \psi}{\partial x^2} + x^2 \psi) + i\lambda x \frac{\partial \psi}{\partial x} \] (2.10)

with \( a = b = \omega_0/2 \) and \( c = -\lambda, \ d = 0 \), has the characteristic equation

\[ \mu'' - 2\lambda \mu' + \omega_0^2 \mu = 0 \] (2.11)

with the solution

\[ \mu = \frac{\omega_0}{\omega} e^{\lambda t} \sin \omega t, \quad \omega = \sqrt{\omega_0^2 - \lambda^2} > 0. \] (2.12)

The corresponding propagator is given by

\[ G (x, y, t) = \sqrt{\frac{\omega e^{-\frac{\lambda t}{2}}}{2\pi i\omega_0 \sin \omega t}} \exp \left( \frac{i\omega}{2\omega_0 \sin \omega t} ((x^2 + y^2) \cos \omega t - 2xy) \right) \times \exp \left( \frac{i\lambda}{2\omega_0} (x^2 - y^2) \right) \] (2.13)

and the evolution of the squared norm is as follows

\[ \|\psi (x, t)\|^2 = e^{-\lambda t} \|\psi (x, 0)\|^2. \] (2.14)

The solution of the Cauchy initial value problem

\[ i \frac{\partial \psi}{\partial t} = H \psi, \quad \psi (x, 0) = \chi (x) \] (2.15)
for our models (2.1) and (2.10) is given by the superposition principle in an integral form
\[ \psi(x,t) = \int_{-\infty}^{\infty} G(x,y,t) \chi(y) \, dy \]
(2.16) for a suitable initial function \( \chi \) on \( \mathbb{R} \); a rigorous proof is given in Ref. [33].

3. The Gauge Transformations

The time-dependent Schrödinger equation
\[ i\frac{\partial \psi}{\partial t} = \left( \frac{\omega_0}{2} (p-A)^2 + U + (p-A)V + W(p-A) \right) \psi, \]
(3.1)
where \( p = i^{-1}\partial/\partial x \) is the linear momentum operator and \( A = A(x,t), \ U = U(x,t), \ V = V(x,t), \ W = W(x,t) \) are real-valued functions, with the help of the gauge transformation
\[ \psi = e^{-if(x,t)}\psi' \]
(3.2)
can be transformed into a similar form
\[ i\frac{\partial \psi'}{\partial t} = \left( \frac{\omega_0}{2} (p-A')^2 + U' + (p-A')V' + W'(p-A') \right) \psi' \]
(3.3)
with the new vector and scalar potentials given by
\[ A' = A + \frac{\partial f}{\partial x}, \quad U' = U - \frac{\partial f}{\partial t}, \quad V' = V, \quad W' = W. \]
(3.4)
Here we consider the one-dimensional case only and may think of \( f \) as being an arbitrary complex-valued differentiable function. See Refs. [20] and [25] for discussion of the traditional case, when \( V = W \equiv 0 \).

An interesting special case of the gauge transformation related to this paper is given by
\[ A = 0, \quad U = \frac{\omega_0}{2} x^2, \quad V = -\lambda x, \quad W = 0, \quad f = \frac{i\lambda t}{2}, \]
(3.5)
\[ A' = 0, \quad U' = \frac{\omega_0}{2} x^2 - \frac{i\lambda}{2}, \quad V' = -\lambda x, \quad W' = 0, \]
(3.6)
when the new Hamiltonian is
\[ H' = \frac{\omega_0}{2} (p-A')^2 + U' + pV' \]
(3.7)
\[ = \frac{\omega_0}{2} \left( -\frac{\partial^2}{\partial x^2} + x^2 \right) + i\frac{\lambda}{2} \left( 2x \frac{\partial}{\partial x} + 1 \right), \]
and equation (2.1) takes the form
\[ i\frac{\partial \psi}{\partial t} = \frac{\omega_0}{2} \left( -\frac{\partial^2 \psi}{\partial x^2} + x^2 \psi \right) + \frac{i\lambda}{2} \left( 2x \frac{\partial \psi}{\partial x} + \psi \right). \]
(3.8)

The corresponding Green function is given by
\[ G(x,y,t) = \sqrt{\frac{\omega}{2\pi i\omega_0 \sin \omega t}} \exp \left( \frac{i\omega}{2\omega_0 \sin \omega t} ((x^2 + y^2) \cos \omega t - 2xy) \right) \]
\[ \times \exp \left( \frac{i\lambda}{2\omega_0} (x^2 - y^2) \right), \quad \omega = \sqrt{\omega_0^2 - \lambda^2} > 0 \]
(3.9)
and the norm of the wave function is conserved with time. This can be established once again directly from our equations (1.2)–(1.8). We leave the details to the reader. A traditional method of separation of the variables and using the Mehler formula for Hermite polynomials is discussed in the next section. The factorization technique is applied to this Hamiltonian in section 5.

Equation (3.8), in turn, admits another local gauge transformation:

\[ A = 0, \quad U = \frac{\omega_0}{2} x^2, \quad V = W = -\frac{\lambda x}{2}, \quad f = -\frac{\lambda x^2}{2\omega_0}, \quad (3.10) \]

\[ A' = -\frac{\lambda x}{\omega_0}, \quad U' = \frac{\omega_0}{2} x^2, \quad V' = W' = -\frac{\lambda x}{2} \quad (3.11) \]

and the Hamiltonian becomes

\[ H' = \frac{\omega_0}{2} (p - A')^2 + U' + (p - A') V' + W' (p - A') \]

\[ = \frac{\omega_0}{2} \left( p + \frac{\lambda x}{\omega_0} \right)^2 + \frac{\omega_0}{2} x^2 \]

\[ + \left( p + \frac{\lambda x}{\omega_0} \right) \left( -\frac{\lambda x}{\omega_0} \right) + \left( -\frac{\lambda x}{\omega_0} \right) \left( p + \frac{\lambda x}{\omega_0} \right) \]

\[ = \frac{\omega_0}{2} p^2 + \frac{\omega_0^2 - \lambda^2}{2\omega_0} x^2. \quad (3.12) \]

As a result, equation (3.8) takes the form of equation for the harmonic oscillator:

\[ i \frac{\partial \psi}{\partial t} = \frac{\omega_0}{2} \left( -\frac{\partial^2 \psi}{\partial x^2} + \frac{\omega_0^2}{\omega_0^2} x^2 \psi \right), \quad \omega^2 = \omega_0^2 - \lambda^2 > 0 \quad (3.13) \]

and can be solved, once again, by the traditional method of separation of the variables or by the factorization technique.

4. SEPARATION OF VARIABLES FOR A SHIFTED HARMONIC OSCILLATOR

We shall refer to the case (3.8) as one of a shifted linear harmonic oscillator. The Ansatz

\[ \psi (x, t) = e^{-iE t} \varphi (x) \quad (4.1) \]

in the time-dependent Schrödinger equation results in the stationary Schrödinger equation

\[ H \varphi = E \varphi \quad (4.2) \]

with the Hamiltonian (3.7). The last equation, namely,

\[ -\varphi'' + x^2 \varphi + \frac{i\lambda}{\omega_0} (2x \varphi' + \varphi) = \frac{2E}{\omega_0} \varphi, \quad (4.3) \]

with the help of the substitution

\[ \varphi = \exp \left( \frac{i\lambda x^2}{2\omega_0} \right) u (x) \quad (4.4) \]

is reduced to the following equation

\[ -u'' + \frac{\omega_0^2}{\omega_0^2} x^2 u = \frac{2E}{\omega_0} u. \quad (4.5) \]
The change of the variable

\[ u(x) = v(\xi), \quad x = \xi \sqrt{\frac{\omega_0}{\omega}} \]  

(4.6)
gives us the stationary Schrödinger equation for the simple harmonic oscillator [20], [25], [28], [31]:

\[ v'' + (2\varepsilon - \xi^2) v = 0 \]  

(4.7)

with \( \varepsilon = E/\omega \), whose eigenfunctions are given in terms of the Hermite polynomials as follows

\[ v_n = C_n e^{-\xi^2/2} H_n(\xi), \]  

(4.8)

and the corresponding eigenvalues are

\[ \varepsilon_n = n + \frac{1}{2}, \quad E_n = \omega \left( n + \frac{1}{2} \right) \quad (n = 0, 1, 2, \ldots) . \]  

(4.9)

Thus the normalized wave functions of our shifted oscillator (3.8) are given by

\[ \psi_n(x,t) = e^{-i\omega(n+1/2)t} \varphi_n(x), \]  

(4.10)

where

\[ \varphi_n(x) = C_n \exp \left( \frac{i\lambda x^2}{2\omega_0} \right) e^{-\xi^2/2} H_n(\xi), \quad \xi = \sqrt{\frac{\omega}{\omega_0}} x \]  

(4.11)

and

\[ |C_n|^2 = \frac{\sqrt{\omega}}{\omega_0 \sqrt{\pi} 2^n n!} \]  

(4.12)
in view of the orthogonality relation

\[ \int_{-\infty}^{\infty} \varphi_n^*(x) \varphi_m(x) \, dx = \delta_{nm}. \]  

(4.13)

We use the star for complex conjugate.

Solution of the initial value problem (2.15) can be found by the superposition principle in the form

\[ \psi(x,t) = \sum_{n=0}^{\infty} c_n \psi_n(x,t), \]  

(4.14)

where

\[ \psi(x,0) = \chi(x) = \sum_{n=0}^{\infty} c_n \varphi_n(x) \]  

(4.15)

and

\[ c_n = \int_{-\infty}^{\infty} \varphi_n^*(y) \chi(y) \, dy \]  

(4.16)
in view of the orthogonality property (4.13). Substituting (4.16) into (4.14) and changing the order of the summation and integration, one gets

\[ \psi(x,t) = \int_{-\infty}^{\infty} G(x,y,t) \chi(y) \, dy, \]  

(4.17)

where the Green function is given as the eigenfunction expansion:

\[ G(x,y,t) = \sum_{n=0}^{\infty} e^{-i\omega(n+1/2)t} \varphi_n(x) \varphi_n^*(y). \]  

(4.18)
This infinite series is summable with the help of the Poisson kernel for the Hermite polynomials (Mehler’s formula) \[29\]
\[
\sum_{n=0}^{\infty} \frac{H_n(x) H_n(y)}{2^n n!} r^n = \frac{1}{\sqrt{1-r^2}} \exp \left( \frac{2xyr - (x^2 + y^2) r^2}{1-r^2} \right), \quad |r| < 1.
\]

The result is given, of course, by equation \[3.9\].

5. The Factorization Method for Shifted Harmonic Oscillator

It is worth applying the well-known factorization technique (see, for example, \[2\], \[3\], \[4\], \[10\] and \[25\]) to the Hamiltonian \(3.7\). The corresponding ladder operators can be found in the forms
\[
a = (\alpha + i\beta) x + \gamma \frac{\partial}{\partial x},
\]
\[
a^\dagger = (\alpha - i\beta) x - \gamma \frac{\partial}{\partial x},
\]
where \(\alpha, \beta\) and \(\gamma\) are real numbers to be determined as follows. One gets
\[
aa^\dagger \psi = (\alpha^2 + \beta^2) x^2 \psi + (\alpha - i\beta) \gamma \psi - 2i\beta \gamma x \frac{\partial \psi}{\partial x} - \gamma^2 \frac{\partial^2 \psi}{\partial x^2},
\]
\[
a^\dagger a \psi = (\alpha^2 + \beta^2) x^2 \psi - (\alpha + i\beta) \gamma \psi - 2i\beta \gamma x \frac{\partial \psi}{\partial x} - \gamma^2 \frac{\partial^2 \psi}{\partial x^2},
\]
whence
\[
(aa^\dagger - a^\dagger a) \psi = 2\alpha \gamma \psi
\]
and
\[
\frac{1}{2} (aa^\dagger + a^\dagger a) \psi = -\gamma^2 \frac{\partial^2 \psi}{\partial x^2} + (\alpha^2 + \beta^2) x^2 \psi - i\beta \gamma \left( 2x \frac{\partial \psi}{\partial x} + \psi \right).
\]
The canonical commutation relation occurs and the Hamiltonian \(3.7\) takes the standard form:
\[
H = \frac{\omega}{2} (aa^\dagger + a^\dagger a),
\]
if
\[
2\alpha \gamma = 1, \quad \omega (\alpha^2 + \beta^2) = \omega \gamma^2 = \frac{1}{2} \omega_0, \quad \omega \beta \gamma = -\frac{1}{2} \lambda.
\]
The relation \(\omega_0^2 = \omega^2 + \lambda^2\), which defines the new oscillator frequency, holds. As a result, the explicit form of the annihilation and creation operators is given by
\[
\sqrt{2} a = \left( \sqrt{\frac{\omega}{\omega_0}} - \frac{i\lambda}{\sqrt{\omega_0^{\omega}}} \right) x + \sqrt{\frac{\omega}{\omega_0}} \frac{\partial}{\partial x},
\]
\[
\sqrt{2} a^\dagger = \left( \sqrt{\frac{\omega}{\omega_0}} + \frac{i\lambda}{\sqrt{\omega_0^{\omega}}} \right) x - \sqrt{\frac{\omega}{\omega_0}} \frac{\partial}{\partial x}.
\]
The special case \(\lambda = 0\) and \(\omega = \omega_0\) gives a traditional form of these operators. The oscillator spectrum \(4.9\) and the corresponding stationary wave functions \(4.11\) can be obtain now in a standard way by using the Heisenberg–Weyl algebra of the raising and lowering operators.
6. Dynamics of Energy Related Expectation Values

The expectation value of an operator $A$ in quantum mechanics is given by the formula

$$\langle A \rangle = \int_{-\infty}^{\infty} \psi^*(x) A \psi(x) \, dx,$$

(6.1)

where the wave function satisfies the time-dependent Schrödinger equation

$$i \frac{\partial \psi}{\partial t} = H \psi.$$  

(6.2)

The time derivative of this expectation value can be written as

$$i \frac{d}{dt} \langle A \rangle = i \left( \frac{\partial A}{\partial t} \right) + \langle AH - H^\dagger A \rangle,$$

(6.3)

where $H^\dagger$ is the Hermitian adjoint of the Hamiltonian operator $H$. Our formula is a simple extension of the well-known expression [20], [25], [31] to the case of a nonself-adjoint Hamiltonian.

We apply formula (6.3) to the Hamiltonian

$$H = \frac{\omega_0}{2} \left( p^2 + x^2 \right) - \lambda px,$$

(6.4)

in equation (2.1). A few examples will follow. In the case of the identity operator $A = 1$, one gets

$$AH - H^\dagger A = \lambda (xp - px) = i\lambda$$

(6.5)

by the Heisenberg commutation relation

$$[x, p] = xp - px = i.$$  

(6.6)

As a result,

$$\frac{d}{dt} \|\psi\|^2 = \lambda \|\psi\|^2,$$

(6.7)

and time-evolution of the squared norm of the wave function for our model of the damped quantum oscillator is given by equation (2.9).

In a similar fashion, if $A = H$, then

$$H^2 - H^\dagger H = \left( H - H^\dagger \right) H = i\lambda H,$$

(6.8)

and

$$\frac{d}{dt} \langle H \rangle = \lambda \langle H \rangle, \quad \langle H \rangle = \langle H \rangle_0 e^{\lambda t}.$$  

(6.9)

Moreover,

$$\frac{d}{dt} \langle H^n \rangle = \lambda \langle H^n \rangle, \quad \langle H^n \rangle = \langle H^n \rangle_0 e^{\lambda t} \quad (n = 0, 1, 2, ...),$$

(6.10)

which unifies the both of the previous cases.

For the mechanical energy operator $E$ defined by

$$E = H_0 = \frac{\omega_0}{2} \left( p^2 + x^2 \right),$$

(6.11)

so that

$$H = H_0 - \lambda px,$$

(6.12)
we obtain

\[ H_0 H - H^\dagger H_0 = \lambda (xpH_0 - H_0 px) \] (6.13)

\[ = \frac{\lambda \omega_0}{2} ([x, p^2] - x [x, p] x) \]

\[ = \frac{i \lambda \omega_0}{2} (3p^2 - x^2). \]

Introducing the kinetic and potential energy operators in a usual manner,

\[ T = \frac{\omega_0}{2} p^2, \quad U = \frac{\omega_0}{2} x^2, \quad E = T + U, \] (6.14)

one arrives at the following estimates

\[ \frac{d}{dt} (e^{\lambda t} \langle E \rangle) = 4 \lambda e^{\lambda t} \langle T \rangle \geq 0, \] (6.15)

\[ \frac{d}{dt} (e^{-3\lambda t} \langle E \rangle) = -4 \lambda e^{3\lambda t} \langle U \rangle \leq 0 \] (6.16)

for the expectation value of the mechanical energy of the damped oscillator under consideration.

In a similar fashion, one can choose \( A = p^2, \ A = x^2 \) and \( A = px + xp \), respectively, in order to obtain the following system:

\[ \frac{d}{dt} \langle p^2 \rangle = 3\lambda \langle p^2 \rangle - \omega_0 \langle px + xp \rangle, \]

\[ \frac{d}{dt} \langle x^2 \rangle = -\lambda \langle x^2 \rangle + \omega_0 \langle px + xp \rangle, \] (6.17)

\[ \frac{d}{dt} \langle px + xp \rangle = 2\omega_0 (\langle p^2 \rangle - \langle x^2 \rangle) + \lambda \langle px + xp \rangle. \]

Indeed,

\[ p^2 H - H^\dagger p^2 = \frac{\omega_0}{2} [p^2, x^2] + \lambda [x, p^3] \] (6.18)

\[ = 3i\lambda p^2 - i\omega_0 (px + xp), \]

\[ x^2 H - H^\dagger x^2 = \frac{\omega_0}{2} [x^2, p^2] - \lambda x [x, p] x \] (6.19)

\[ = i\omega_0 (px + xp) - i\lambda x^2, \]

and

\[ (px + xp) H - H^\dagger (px + xp) \]

\[ = \frac{\omega_0}{2} ([p, x^3] + [x, p^3]) \]

\[ + \frac{\omega_0}{2} (p [x, p] p - x [x, p] x) \]

\[ + \lambda ((xp)^2 - (px)^2) \]

\[ = 2i\omega_0 (p^2 - x^2) + i\lambda (px + xp), \]

which results in (6.17).
The system can be solved explicitly, thus providing the complete dynamics of these expectation values. The eigenvalues are given by $r_0 = \lambda$, $r_\pm = \lambda \pm 2i\omega$ and the corresponding linearly independent eigenvectors are

$$x_0 = \begin{pmatrix} \omega_0 \\ \omega_0 \\ 2\lambda \end{pmatrix}, \quad x_\pm = \begin{pmatrix} (\lambda \pm i\omega)^2 \\ \omega_0^2 \\ 2\omega_0 (\lambda \pm i\omega) \end{pmatrix}$$  

with the determinant

$$\begin{vmatrix} \omega_0 & (\lambda + i\omega)^2 & (\lambda - i\omega)^2 \\ \omega_0 & \omega_0^2 & \omega_0^2 \\ 2\lambda & 2\omega_0 (\lambda + i\omega) & 2\omega_0 (\lambda - i\omega) \end{vmatrix} = -8i\omega_0^2 \omega^3 \neq 0.$$  

The general solution of the system (6.17) can be obtained in a complex form as follows

$$\begin{pmatrix} \langle p^2 \rangle \\ \langle x^2 \rangle \\ \langle px + xp \rangle \end{pmatrix} = C_0 e^{\lambda t} \begin{pmatrix} \omega_0 \\ \omega_0 \\ 2\lambda \end{pmatrix} + C_+ e^{(\lambda + 2i\omega)t} \begin{pmatrix} (\lambda + i\omega)^2 \\ \omega_0^2 \\ 2\omega_0 (\lambda + i\omega) \end{pmatrix} + C_- e^{(\lambda - 2i\omega)t} \begin{pmatrix} (\lambda - i\omega)^2 \\ \omega_0^2 \\ 2\omega_0 (\lambda - i\omega) \end{pmatrix},$$

where $C_0$ and $C_\pm$ are constants. The corresponding solution of the initial value problem is given by

$$\begin{pmatrix} \langle p^2 \rangle \\ \langle x^2 \rangle \\ \langle px + xp \rangle \end{pmatrix} = \frac{1}{2\omega^2} \left( \omega_0 \left( \langle p^2 \rangle_0 + \langle x^2 \rangle_0 \right) - \lambda \langle px + xp \rangle_0 \right) e^{\lambda t} \begin{pmatrix} \omega_0 \\ \omega_0 \\ 2\lambda \end{pmatrix} + \frac{1}{2\omega^2} \left( \frac{\omega - \lambda^2}{\omega_0^2} \langle x^2 \rangle_0 - \langle p^2 \rangle_0 \right) \times e^{\lambda t} \begin{pmatrix} (\lambda^2 - \omega^2) \cos 2\omega t - 2\lambda \omega \sin 2\omega t \\ \omega_0^2 \cos 2\omega t \\ 2\lambda \omega_0 \cos 2\omega t - 2\omega_0 \sin 2\omega t \end{pmatrix} + \frac{1}{2\omega_0^2} \left( \langle px + xp \rangle_0 - \frac{2\lambda}{\omega_0} \langle x^2 \rangle_0 \right) \times e^{\lambda t} \begin{pmatrix} 2\lambda \omega \cos 2\omega t + (\lambda^2 - \omega^2) \sin 2\omega t \\ \omega_0^2 \sin 2\omega t \\ 2\omega_0 \omega \cos 2\omega t + 2\lambda \omega_0 \sin 2\omega t \end{pmatrix}.$$

The case of the second Hamiltonian:

$$H = \frac{\omega_0}{2} (p^2 + x^2) - \lambda xp,$$

which is the Hermitian adjoint of the Hamiltonian (6.24), is similar. Here

$$H^{n+1} - H^\dagger H^n = (H - H^\dagger) H^n = \lambda [p, x] H^n = -i\lambda H^n$$

and

$$\frac{d}{dt} \langle H^n \rangle = -\lambda \langle H^n \rangle, \quad \langle H^n \rangle = \langle H^n \rangle_0 e^{-\lambda t} \quad (n = 0, 1, 2, \ldots).$$
Moreover,

\[
p^2 H - H^\dagger p^2 = \frac{\omega_0}{2} [p^2, x^2] + \lambda p [x, p] p \tag{6.27}
\]

\[
= i\lambda p^2 - i\omega_0 (px + xp),
\]

\[
x^2 H - H^\dagger x^2 = \frac{\omega_0}{2} [x^2, p^2] + \lambda [p, x^3] \tag{6.28}
\]

\[
= -3i\lambda x^2 + i\omega_0 (px + xp),
\]

\[
(px + xp) H - H^\dagger (px + xp) \tag{6.29}
\]

\[
= \frac{\omega_0}{2} \left( [p, x^3] + [x, p^3] \right)
\]

\[
+ \frac{\omega_0}{2} (p [x, p] p - x [x, p] x)
\]

\[
- \lambda ((xp)^2 - (px)^2)
\]

\[
= 2i\omega_0 (p^2 - x^2) - i\lambda (px + xp),
\]

and the corresponding system has the form

\[
\frac{d}{dt} \langle p^2 \rangle = \lambda \langle p^2 \rangle - \omega_0 \langle px + xp \rangle,
\]

\[
\frac{d}{dt} \langle x^2 \rangle = -3\lambda \langle x^2 \rangle + \omega_0 \langle px + xp \rangle, \tag{6.30}
\]

\[
\frac{d}{dt} \langle px + xp \rangle = 2\omega_0 (\langle p^2 \rangle - \langle x^2 \rangle) - \lambda \langle px + xp \rangle.
\]

The change \( p \leftrightarrow x, \lambda \rightarrow -\lambda, \omega_0 \rightarrow -\omega_0 \) transforms this system back into (6.17). This observation allows us to obtain solution of the initial value problem from the previous solution given by (6.24). We leave the details to the reader.

The case of a general variable quadratic Hamiltonian of the form

\[
H = a(t) p^2 + b(t) x^2 + c(t) px + d(t) xp,
\]

where \( a(t), b(t), c(t), d(t) \) are real-valued functions of time only, is considered in a similar fashion. One gets

\[
H^{n+1} - H^\dagger H^n = (H - H^\dagger) H^n = (c - d) [p, x] H^n = i\lambda (d - c) H^n \tag{6.32}
\]

and

\[
\frac{d}{dt} \langle H^n \rangle = (d(t) - c(t)) \langle H^n \rangle, \quad \langle H^n \rangle = \langle H^n \rangle_0 \exp \left( \int_0^t (d(\tau) - c(\tau)) \ d\tau \right). \tag{6.33}
\]

Moreover,

\[
p^2 H - H^\dagger p^2 = b [p^2, x^2] + c [p^3, x] + dp [p, x] p \tag{6.34}
\]

\[
= -i (3c + d) p^2 - 2ib (px + xp),
\]

\[
x^2 H - H^\dagger x^2 = a [x^2, p^2] + cx [x, p] x + d [x^3, p] \tag{6.35}
\]

\[
= i (3d + c) x^2 + 2ia (px + xp),
\]

\[
(px + xp) H - H^\dagger (px + xp) \tag{6.36}
\]
\begin{align*}
&= a \left( [x, p^3] + p [x, p] p \right) \\
&\quad + b \left( [p, x^3] + x [p, x] x \right) \\
&\quad + (c - d) \left( (px)^2 - (xp)^2 \right) \\
&= 4iap^2 - 4ibx^2 - i (c - d) (px + xp),
\end{align*}

and the corresponding system has the form
\begin{align*}
\frac{d}{dt} \langle p^2 \rangle &= - (3c + d) \langle p^2 \rangle - 2b \langle px + xp \rangle, \\
\frac{d}{dt} \langle x^2 \rangle &= (c + 3d) \langle x^2 \rangle + 2a \langle px + xp \rangle, \\
\frac{d}{dt} \langle px + xp \rangle &= 4a \left( \langle p^2 \rangle - 4b \langle x^2 \rangle \right) + (d - c) \langle px + xp \rangle.
\end{align*}

We have used the familiar identities
\begin{align*}
[x, p] &= i, \quad (xp)^2 - (px)^2 = i (px + xp), \\
[x^2, p^2] &= 2i (px + xp), \quad [x, p^3] = 3ip^2, \quad [x^3, p] = 3ix^2
\end{align*}

once again.

7. The Third Model

For the time-dependent Schrödinger equation with variable quadratic Hamiltonian:
\begin{equation}
\frac{i}{\hbar} \frac{\partial \psi}{\partial t} = \frac{\hbar^2}{2} \left( -e^{-2\lambda t} \frac{\partial^2 \psi}{\partial x^2} + e^{2\lambda t} x^2 \psi \right), \tag{7.1}
\end{equation}
where \( a = (\omega_0/2) e^{-2\lambda t} \), \( b = (\omega_0/2) e^{2\lambda t} \) and \( c = d = 0 \), the characteristic equation takes the form (2.2) with the same solution (2.3). The corresponding propagator has the form (1.2) with
\begin{align*}
\alpha (t) &= \frac{\omega \cos \omega t - \lambda \sin \omega t}{2\omega_0 \sin \omega t} e^{2\lambda t}, \\
\beta (t) &= -\frac{\omega}{\omega_0 \sin \omega t} e^{\lambda t}, \\
\gamma (t) &= \frac{\omega \cos \omega t + \lambda \sin \omega t}{2\omega_0 \sin \omega t}.
\end{align*}

This can be derived directly from equations (1.2)–(1.8) with the help of identity (2.8). We leave the details to the reader. It is worth noting that equation (7.1) can be obtain by introducing a variable unit of length \( x \rightarrow xe^{\lambda t} \) in the Hamiltonian of the linear oscillator.

8. Momentum Representation

The time-dependent Schrödinger equations for the damped oscillators are also solved in the momentum representation. One can easily verify that under the Fourier transform our first Hamiltonian (6.4) takes the form of the second Hamiltonian (6.25) with \( \lambda \rightarrow -\lambda \) and visa versa (see, for example, Ref. [9] for more details). Moreover, the inverses of the corresponding time evolution operators are obtained by the time reversal. Therefore, all identities of the commutative evolution
diagram introduced in Ref. [9] for the modified oscillators are also valid for the quantum damped oscillators under consideration. We leave further details to the reader.

Acknowledgment. We thank Professor Carlos Castillo-Chávez for support, valuable discussions and encouragement.

References

[1] G. P. Arrighini and N. L. Durante, More on the quantum propagator of a particle in a linear potential, Am. J. Phys. 64 (1996) # 8, 1036–1041.
[2] R. Askey and S. K. Suslov, The $q$-harmonic oscillator and an analogue of the Charlier polynomials, J. Phys. A 26 (1993) # 15, L693–L698.
[3] R. Askey and S. K. Suslov, The $q$-harmonic oscillator and the Al-Salam and Carlitz polynomials, Lett. Math. Phys. 29 (1993) # 2, 123–132; arXiv:math/9307207v1 [math. CA] 9 Jul 1993.
[4] N. M. Atakishiyev and S. K. Suslov, Difference analogues of the harmonic oscillator, Theoret. and Math. Phys. 85 (1990) #1, 1055–1062.
[5] H. Bateman, Partial Differential Equations of Mathematical Physics, Dover, New York, 1944.
[6] L. A. Beauregard, Propagators in nonrelativistic quantum mechanics, Am. J. Phys. 34 (1966), 324–332.
[7] L. S. Brown and Y. Zhang, Path integral for the motion of a particle in a linear potential, Am. J. Phys. 62 (1994) # 8, 806–808.
[8] R. Cordero-Soto, R. M. Lopez, E. Suazo, and S. K. Suslov, Propagator of a charged particle with a spin in uniform magnetic and perpendicular electric fields, Lett. Math. Phys. 84 (2008) #2–3, 159–178.
[9] R. Cordero-Soto and S. K. Suslov, The time inversion for modified oscillators, arXiv:0808.3149v9 [math-ph] 8 Mar 2009.
[10] Shi-H. Dong, Factorization Method in Quantum Mechanics, Springer–Verlag, Dordrecht, 2007.
[11] R. P. Feynman, The Principle of Least Action in Quantum Mechanics, Ph. D. thesis, Princeton University, 1942; reprinted in: “Feynman’s Thesis – A New Approach to Quantum Theory”, (L. M. Brown, Editor), World Scientific Publishers, Singapore, 2005, pp. 1–69.
[12] R. P. Feynman, Space-time approach to non-relativistic quantum mechanics, Rev. Mod. Phys. 20 (1948) # 2, 367–387; reprinted in: “Feynman’s Thesis – A New Approach to Quantum Theory”, (L. M. Brown, Editor), World Scientific Publishers, Singapore, 2005, pp. 71–112.
[13] R. P. Feynman, The theory of positrons, Phys. Rev. 76 (1949) # 6, 749–759.
[14] R. P. Feynman, Space-time approach to quantum electrodynamics, Phys. Rev. 76 (1949) # 6, 769–789.
[15] R. P. Feynman and A. R. Hibbs, Quantum Mechanics and Path Integrals, McGraw–Hill, New York, 1965.
[16] K. Gottfried and T.-M. Yan, Quantum Mechanics: Fundamentals, second edition, Springer–Verlag, Berlin, New York, 2003.
[17] B. R. Holstein, The linear potential propagator, Am. J. Phys. 65 (1997) #5, 414–418.
[18] B. R. Holstein, The harmonic oscillator propagator, Am. J. Phys. 67 (1998) #7, 583–589.
[19] L. D. Landau and E. M. Lifshitz, Mechanics, Pergamon Press, Oxford, 1976.
[20] L. D. Landau and E. M. Lifshitz, Quantum Mechanics: Nonrelativistic Theory, Pergamon Press, Oxford, 1977.
[21] N. Lanfear and S. K. Suslov, The time-dependent Schrödinger equation, Riccati equation and Airy functions, arXiv:0903.3608v5 [math-ph] 22 Apr 2009.
[22] R. M. Lopez and S. K. Suslov, The Cauchy problem for a forced harmonic oscillator, arXiv:0707.1902v8 [math-ph] 27 Dec 2007.
[23] V. P. Maslov and M. V. Fedoriuk, Semiclassical Approximation in Quantum Mechanics, Reidel, Dordrecht, Boston, 1981.
[24] M. Meiler, R. Cordero-Soto, and S. K. Suslov, Solution of the Cauchy problem for a time-dependent Schrödinger equation, J. Math. Phys. 49 (2008) #7, 072102: 1–27; published on line 9 July 2008, URL: http://link.aip.org/link/?JMP/49/072102; see also arXiv: 0711.0559v4 [math-ph] 5 Dec 2007.
[25] E. Merzbacher, Quantum Mechanics, third edition, John Wiley & Sons, New York, 1998.
[26] P. Nardone, Heisenberg picture in quantum mechanics and linear evolutionary systems, Am. J. Phys. 61 (1993) #3, 232–237.
A. F. Nikiforov, S. K. Suslov, and V. B. Uvarov, Classical Orthogonal Polynomials of a Discrete Variable, Springer–Verlag, Berlin, New York, 1991.

A. F. Nikiforov and V. B. Uvarov, Special Functions of Mathematical Physics, Birkhäuser, Basel, Boston, 1988.

E. D. Rainville, Special Functions, The Macmillan Company, New York, 1960.

R. W. Robinett, Quantum mechanical time-development operator for the uniformly accelerated particle, Am. J. Phys. 64 (1996) #6, 803–808.

L. I. Schiff, Quantum Mechanics, third edition, McGraw-Hill, New York, 1968.

E. Suazo, and S. K. Suslov, An integral form of the nonlinear Schrödinger equation with variable coefficients, arXiv:0805.0633v2 [math-ph] 19 May 2008.

E. Suazo and S. K. Suslov, Cauchy problem for Schrödinger equation with variable quadratic Hamiltonians, under preparation.

E. Suazo, S. K. Suslov, and J. M. Vega, The Riccati differential equation and a diffusion-type equation, arXiv:0807.4349v4 [math-ph] 8 Aug 2008.

S. K. Suslov and B. Trey, The Hahn polynomials in the nonrelativistic and relativistic Coulomb problems, J. Math. Phys. 49 (2008) #1, 012104: 1–51; published on line 22 January 2008, URL: http://link.aip.org/link/?JMP/49/012104

N. S. Thomber and E. F. Taylor, Propagator for the simple harmonic oscillator, Am. J. Phys. 66 (1998) #11, 1022–1024.

Mathematical and Computational Modeling Center, Arizona State University, Tempe, AZ 85287–1804, U.S.A.

E-mail address: ricardojavier81@gmail.com

School of Mathematical and Statistical Sciences, Mathematical and Computational Modeling Center, Arizona State University, Tempe, AZ 85287–1804, U.S.A.

E-mail address: suazo@mathpost.la.asu.edu

School of Mathematical and Statistical Sciences, Mathematical and Computational Modeling Center, Arizona State University, Tempe, AZ 85287–1804, U.S.A.

E-mail address: sks@asu.edu

URL: http://hahn.la.asu.edu/~suslov/index.html