FINITE EXTENSION OF GROUP WITH INFINITE CONJUGACY CLASSES

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Abstract. We give a characterization of the group property of being with infinite conjugacy classes (or icc, i.e. $\neq 1$ and of which all conjugacy classes except 1 are infinite) for finite extensions of group.

Introduction

A group is said to be with infinite conjugacy classes (or icc) if it is non trivial, and if all its conjugacy classes except $\{1\}$ are infinite. This property is motivated by the theory of Von Neumann algebra, since for any group $\Gamma$, a necessary and sufficient condition for its Von Neumann algebra to be a type $\text{II}_1$ factor is that $\Gamma$ be icc (cf. [ROIV]).

The property of being icc has been characterized in several classes of groups: 3-manifolds and PD(3) groups in [HP], groups acting on Bass-Serre trees in [Cô] and wreath product of groups in [Pr]. We will focus here on groups defined by a finite extension or containing a proper finite index subgroup.

Towards this direction particular results are already known. In [HP] have been proved the following results:

– Let $G$ be defined by a finite exact sequence:

\[ 1 \rightarrow K \rightarrow G \rightarrow \mathbb{Z}_2 \rightarrow 1 \]

Then $G$ is icc if and only if $K$ is icc and $G \neq K \times \mathbb{Z}_2$. It is easily seen that the condition can be rephrased as $K$ icc and the natural homomorphism $Q \rightarrow \text{Out}(K)$ is injective.

– Let $G$ be a finite extension of $K$, with $G$ torsion-free. Then $G$ is icc if and only if $K$ is icc.

We give generalizations of these results. We propose the following characterization of finite extensions of groups with infinite conjugacy classes. A refined and more general version will also be given in §2.

Main theorem. Let $G$ be a group defined by a finite extension:

\[ 1 \rightarrow K \rightarrow G \rightarrow Q \text{ (finite)} \rightarrow 1 \]

Then $G$ is icc if and only if $K$ is icc and the natural homomorphism $\theta : Q \rightarrow \text{Out}(K)$ is injective.

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1. Proof of the main result

Let us first fix some notations: if $G$ is a group and $u,v$ are element of $G$, then $u^v$ is the element of $G$ defined by $u^v = v^{-1}uv$. If $H$ is a subgroup of $G$, then $u^H = \{ u^v \mid v \in H \}$; in particular $u^G$ denote the conjugacy class of $u$ in $G$. Note that the cardinal of $u^G$ equals the index of $Z_G(u)$ in $G$ so that $G \neq 1$ is icc if and only if for any $u \neq 1 \in G$, $Z_G(u)$ has an infinite index in $G$.

Let $\pi : G \longrightarrow Aut(K)$ be the homomorphism defined by $\pi(g)(k) = k^g$ for any $k \in K$. It makes the following diagram commute:

$$
\begin{array}{cccc}
1 & \longrightarrow & K & \longrightarrow & G & \longrightarrow & Q & \longrightarrow & 1 \\
\pi & \downarrow & & & & \phi & \downarrow & & \\
1 & \longrightarrow & Inn(K) & \longrightarrow & Aut(K) & \longrightarrow & Out(K) & \longrightarrow & 1
\end{array}
$$

Proof of the main theorem. We proceed in several steps.

Step 1 : $G$ icc $\implies$ $K$ icc.

Suppose $K$ is not icc, that is there exists $u \neq 1$ in $K$ such that $Z_K(u)$ has a finite index in $K$. Since $K$ has finite index in $G$, $Z_K(u)$ has a finite index in $G$, and hence $Z_G(u)$ which contains $Z_K(u)$ also has a finite index in $G$, so that $G$ is not icc.

Step 2 : $G$ icc $\implies$ $\theta$ injective.

Suppose that $\theta : Q \longrightarrow Out(K)$ is non injective, that is there exists $g \in G \setminus K$ and $h \in K$ such that $\forall k \in K$, $g^{-1}kg = h^{-1}kh$. Then $K$ is contained in $Z_G(gh^{-1})$ and hence $Z_G(gh^{-1})$ has a finite index in $G$, so that $G$ is not icc.

Step 3 : $G$ not icc $\implies$ $K$ not icc or $\theta$ non injective.

We can suppose $G \neq 1$ in $G$ such that $Z_G(g)$ has a finite index in $G$, it must exist since $G \neq 1$ is not icc. If $g$ lies in $K$ then $Z_K(g) = Z_G(g) \cap K$ has a finite index in $K$, so that $K$ is not icc. So suppose in the following that $g$ lies in $G \setminus K$. Let $H = Z_G(g) \cap K$, it is a finite index subgroup of $K$, and let $k_0, k_1, \ldots, k_n$ be a set of representative of $K$ mod $H$. Let $N = \bigcap_{i=1}^n k_i^{-1}Hk_i$, the normalized of $H$ in $K$; $N$ is a finite index normal subgroup of $K$ lying in $Z_G(g)$. Consider the centralizer $Z_K(N)$ of $N$ in $K$. If $Z_K(N) \neq 1$ let $u$ be a non trivial element of $Z_K(N)$. Since $u^N$ is a singleton and $N$ has a finite index in $K$ it follows that $u^K$ is finite so that $K$ is not icc. If $Z_K(N) = 1$, once have been noted that $\pi(g)$ restricted to $N$ is the identity, the following lemma applies to show that $\pi(g)$ is the identity of $K$, so that $\theta$ is non injective.

**Lemma 1.** Let $K$ be a group, $N$ a normal subgroup of $K$ and $\varphi \in Aut(K)$ which is the identity once restricted to $N$. If $Z_K(N) = 1$ then $\varphi$ is the identity on $K$.

Proof. Let $k \in K$, then for any $h \in N$,

$$khh^{-1} = \varphi(khh^{-1}) = \varphi(k)h\varphi(k^{-1})$$

so that $k^{-1}\varphi(k) \in Z_K(N)$. Hence if $Z_K(N) = 1$, for any $k \in K$, $\varphi(k) = k$. \qed
2. Finite index subgroups in icc groups

Let $H$ be a subgroup of the group $G$; the normalized $N(H)$ of $H$ is defined to be $N(H) = \bigcap_{g \in G} H^g$. It is a normal subgroup both in $H$ and $G$, and $N(H) = H$ exactly when $H$ is normal in $G$. If $H$ has a finite index in $G$, let $g_1, g_2, \ldots, g_n$ be a finite set of representatives of $G \mod H$, then $N(H) = \bigcap_{i=1}^n H^{g_i}$, and $N(H)$ has a finite index both in $H$ and in $G$. Pay attention that the normalizeR has nothing to do with the normalizeR.

Denote by $\pi$ the homomorphism $\pi : G \to \text{Aut}(N(H))$ defined by $\forall g \in G, k \in N(H), \pi(g)(k) = k^g$.

**Theorem 1.** Let $G$ be a group, $H$ a finite index subgroup of $G$ and $N(H)$ the normalized of $H$. Then $G$ is not icc if and only if at least one of the conditions (i), (ii) or (iii) is satisfied:

(i) $H$ is not icc

(ii) $\exists g \neq 1 \in G \setminus H$ with finite order such that $\pi(g)$ is the identity,

(iii) $\exists g \neq 1 \in G \setminus H$ with finite order such that $\pi(g)$ is inner.

Moreover on the one hand (ii) $\implies$ (iii) and on the other (iii) $\implies$ (i) or (ii), so that:

$G$ is not icc $\iff$ (i) or (ii) $\iff$ (i) or (iii).

**Proof of theorem 1.** We proceed in several steps:

Step 1. (i) $\implies$ $G$ not icc.
Since $H$ has a finite index in $G$, if $H$ is not icc then $G$ is clearly also not icc.

Step 2. (ii) $\implies$ $G$ not icc.
If $\exists g \neq 1 \in G \setminus H$ with finite order $n$ such that $\pi(g)$ is the identity, then $G$ contains $N(H) \times \mathbb{Z}_n$ which is obviously not icc and has a finite index in $G$, so that $G$ is not icc.

Step 3. (iii) $\implies$ (i) or (ii).
Let $g \neq 1 \in G \setminus H$ be a finite order element such that $\exists k \in N(H), \forall h \in H, \pi(g)(h) = h^k$. Then $\pi(gk^{-1})$ is the identity on $N(H)$. Let $n \in \mathbb{Z}$ be the order of $g$, then $(gk^{-1})^n$ lies in the center of $N(H)$. Hence either $N(H)$ has a non trivial center and condition (i) follows, or $(gk^{-1})^n = 1$ so that condition (ii) is satisfied.

Step 4. $G$ not icc $\implies$ (i) or (ii).
Suppose $G$ is not icc; $G$ is a finite extension of $N(H)$. Either $H$ is not icc or $\exists g \in G \setminus H$ with $g^G$ finite. For some $n > 1$, $g^a \in N(H)$; if $g^a \neq 1$ then $N(H)$ is not icc and it follows that condition (i) is satisfied. So suppose in the following that $g^a = 1$. The same argument as in the step 3 of the proof of the main theorem—with $N(H)$ instead of $K$—shows that either $N(H)$ is not icc, so that condition (i) is satisfied, or $\pi(g)$ is the identity, so that condition (ii) is satisfied.  

**Corollaire 1.** $G$ is icc if and only if $H$ is icc and $G \not\cong N(H) \times \mathbb{Z}_n$.

**Corollaire 2.** If $G \setminus H$ contains no torsion element (in particular when $G$ is torsion-free) then $G$ is icc if and only if $H$ is icc.
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