Thermodynamic uncertainty relation for open quantum systems

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Related article:
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Thermodynamic uncertainty relation (TUR)

Relation between fluctuation and entropy production [Barato & Seifert, PRL, 2015]

\[
\frac{\text{Var}[\phi]}{\langle \phi \rangle^2} \geq \frac{2}{\sigma}
\]

where \( \sigma \) is entropy production.

Recently, quantum TURs have been studied

- [Erker et al., PRX, 2017], [Brandner et al., PRL, 2018], [Carollo et al., PRL, 2018], [Liu et al., PRE, 2019], [Guarnieri et al., PRR, 2019], [Saryal et al., PRE, 2019], etc

Still, quantum TURs are in a very early stage

- Many studies obtained case-by-case bounds

I will present a quantum TUR valid for general open quantum dynamics
TUR in open quantum systems

\[ \Psi(T) = U|\psi\rangle \otimes |0\rangle = \sum_{m=0}^{M-1} V_m |\psi\rangle \otimes |m\rangle, \quad V_m \equiv \langle m|U|0\rangle \]

\[ \rho(T) = \text{Tr}_E [\Psi(T)\Psi(T)^\dagger] = \sum_{m=0}^{M-1} V_m \rho V_m^\dagger \]

Environment basis: \{\{|0\rangle, |1\rangle, \ldots, |M-1\rangle\}\}
We assume that $g(0) = 0$

As long as this condition is met, $g(m)$ can return any real number

The initial state of $E$ was assumed to be $|0\rangle$. Therefore, when the state of the environment after the interaction is $|0\rangle$, the environment remains unchanged before and after the interaction.
Then we find the following bound for the coefficient of variation of $g(m)$:

$$\frac{\text{Var}[g(m)]}{\langle g(m) \rangle^2} \geq \frac{1}{\Xi}$$

$$\Xi = \text{Tr}_S \left[ \left( V_0^\dagger V_0 \right)^{-1} \rho \right] - 1$$  \hspace{1cm} V_0 \equiv \langle 0|U|0 \rangle

$\Xi$ corresponds to the dynamical activity in classical Markov processes.

This relation holds for
- any open quantum systems as long as $V_0^\dagger V_0 > 0$
- any observable $g(m)$ with $g(0) = 0$
- any initial density operator $\rho$ in $S$
Application: continuous measurement

Consider a Lindblad equation defined by
\[ \frac{d\rho}{dt} = -i[H, \rho] + \sum_c \left[ L_c \rho L_c^\dagger - \frac{1}{2} \{ L_c^\dagger L_c \rho + \rho L_c^\dagger L_c \} \right] \]

where \( L_c \) is a jump operator.

The Lindblad equation renders the dynamics when we do not measure the environment.

On measuring the environment, the Lindblad equation is unraveled to yield a stochastic dynamics conditioned on a measurement record.

Stochastic trajectory is described by a stochastic Schrödinger equation.
Quantum trajectory

\[ d\rho = -i[H, \rho]dt + \sum_c \left[ \rho \text{Tr}[L_c\rho L_c^+] - \frac{1}{2}\{L_c^+L_c, \rho\} \right] + \sum_c \left[ \frac{L_c\rho L_c^+}{\text{Tr}[L_c\rho L_c^+]} - \rho \right] dN \]

Taking average w.r.t. measurement records

This dynamics is a solution of the Lindblad equation

\[ \rho_{ee}(t) \equiv \langle e|\rho(t)|e \rangle \]
Continuous measurement

- The interval $[0, T]$ is divided into $N$ equipartitioned intervals.
- The environmental orthonormal basis is $|m_{N-1}, \ldots, m_0\rangle$.
- $|m_k\rangle$ interacts with $S$ within the time interval $[t_k, t_{k+1}]$ via a unitary operator $U_{t_k}$.
One-step time evolution is

\[ \rho(t + \Delta t) = \sum_c X_c \rho(t) X_c^\dagger \]

where

\[ X_0 \equiv e^{-i\Delta t H} \sqrt{I_S - \Delta t \sum_c L_c^\dagger L_c} \quad \text{(no detection)} \]

\[ X_c \equiv e^{-i\Delta t H} \sqrt{\Delta t L_c} \quad \text{(detection of } c^{\text{th} \text{ event})} \]

Because \( V_0 \equiv \langle 0 | U | 0 \rangle \) corresponds to “no jump events” during \([0, T]\), it is given by \( V_0 = \lim_{N \to \infty} X_0^N \)
TUR for continuous measurement

\( V_0 \) can be computed via Trotter product formula as follows

\[ V_0 = e^{-T(iH + \frac{1}{2} \sum_c L_c^\dagger L_c)} \]

Therefore, a quantum TUR becomes

\[ \frac{\text{Var}[g(m)]}{\langle g(m) \rangle^2} \geq \frac{1}{\Xi} \]

\[ \Xi = \text{Tr}_S \left[ e^{T(iH + \frac{1}{2} \sum_c L_c^\dagger L_c)} e^{T(-iH + \frac{1}{2} \sum_c L_c^\dagger L_c)} \right] - 1 \]

This relation holds for any Lindblad dynamics (time-independent \( H \) and \( L_c \)) and for any initial density operator

\( \Xi \) reduces to the dynamical activity in classical Markov processes in a particular limit
Effect of quantumness

- When we emulate classical Markov processes with the Lindblad equation, 
\[ [H, \sum_c L_c^\dagger L_c] = 0 \] holds. In this case, \( \Xi \) reduces to
\[ \Xi_{\text{CL}} = \text{Tr}_S \left[ e^{T \sum_c L_c^\dagger L_c} \right] \]

- When \( T \ll 1 \), we have
\[ \Xi = \Xi_{\text{CL}} + \frac{1}{2} T^2 \chi + O(T^3) \]
where \( \chi \equiv i \sum_c \text{Tr}_S \left[ [H, L_c^\dagger L_c] \rho \right] \).

- When \( \chi > 0 \), the system gains a precision enhancement due to the quantumness.

- For a particular model, \( \chi \) corresponds to non-diagonal elements in density operators.
The Lindblad equation is invariant under the following transformation,
\[ H \rightarrow H - \frac{i}{2} (\zeta^* L - \zeta L^+) \]
\[ L \rightarrow L + \zeta \]
where \( \zeta \) is an arbitrary complex parameter.

Unravelling with different \( \zeta \) corresponds to different continuous measurement

Both quantum trajectories reduce to the same dynamics on average
Effect of measurement

- Under this transformation, $\Xi$ is
  \[\Xi = e^{|\zeta|^2T} \text{Tr}_S \left[ e^{T(iH + \frac{1}{2}L^\dagger L + \zeta^* L)} e^{T(-iH + \frac{1}{2}L^\dagger L + \zeta L^\dagger)} \right] - 1\]

- Therefore, for $|\zeta| \to \infty$, $\Xi \sim e^{|\zeta|^2T}$

- The lower bound of the quantum TUR can be arbitrary small by employing a continuous measurement with large $|\zeta|$.

- Measurements can be a thermodynamics resource. It is possible to extract work from single reservoir without feedback [Yi et al., PRE, 2017].
Classical limit and dynamical activity

- For classical Markov processes with transition rate $\gamma_{ji}(t)$ (from $i$ to $j$) with the initial probability distribution $P_i$, $\Xi$ becomes

$$\frac{\text{Var}[g(m)]}{\langle g(m) \rangle^2} \geq \frac{1}{\Xi}, \quad \Xi = \sum_i P_i \exp \left[ \sum_{j \neq i} \int_0^T dt \, \gamma_{ji}(t) \right] - 1$$

- This relation holds for any time-dependent Markov chains
- For $T \ll 1$, we have

$$\Xi \approx T \sum_i \sum_{j \neq i} P_i \gamma_{ji}(t)$$

which is the dynamical activity in classical Markov processes

- Dynamical activity plays important roles in classical Markov processes [Shiraishi et al., PRL, 2018], [Garrahan, PRE, 2017]
Numerical verifications

Random quantum channel and random $g(m)$

The same as (b). $\Xi$ is replaced with $\Xi_{CL}$.

Continuous measurement in two-level atom driven by laser field. Observable is the number of emitted photon.

Quantum walk and random $g(m)$
Numerical verifications

Random quantum channel and random \( g(m) \)

The same as (b). \( \Xi \) is replaced with \( \Xi_{CL} \).

Points lower than \( 1/\Xi_{CL} \) is a signature of precision enhancement due to quantumness.

Quantum walk and random \( g(m) \)
Quantum Cramér-Rao inequality

- Bound on statistical estimator in quantum systems

\[
\text{Var}\left[\hat{\theta}\right] \geq \frac{1}{\mathcal{F}_Q(\theta)}
\]

\[
\frac{\text{Var}\left[\hat{\Theta}(\theta)\right]}{(\partial_\theta \langle \hat{\Theta} \rangle)^2} \geq \frac{1}{\mathcal{F}_Q(\theta)}
\]

Typical scenario

- Initial state
  - Pure state \(|\psi\rangle\)
- Time evolution
  - Unitary \(U_\theta\)
- Final state
  - \(U_\theta|\psi\rangle\)
- System evolution
- Measurement
- Classical estimation
Quantum Cramér-Rao bound

- Quantum Cramér-Rao bound has been applied to obtain quantum uncertainty relations
  - Robertson uncertainty relation, Quantum speed limit

- Classical Cramér-Rao inequality has been applied to obtain classical thermodynamic uncertainty relations
  - [Hasegawa et al., PRE, 2019], [Dechant, JPA, 2019], [Ito et al. PRX, in press]

- It is much harder to find quantum Fisher information than in classical cases
Quantum Fisher information

- Quantum Fisher information is
  \[ F_Q(\theta) = \max_{\mathcal{M}} F_C(\theta; \mathcal{M}) \]
  where \( \mathcal{M} \) is POVM and \( F_C \) is a classical Fisher information.

- Therefore, quantum Cramér-Rao inequality is satisfied for any quantum measurements (POVMs).

- Quantum Fisher information is calculated by
  \[ F_Q(\theta) = \text{Tr}[\mathcal{L}^2 \rho] \]
  where \( \mathcal{L} \) is known as symmetric logarithm derivative.

- In general, \( \mathcal{L} \) is difficult to obtain.
Quantum Fisher information

[Escher et al, Nat. Phys., 2011] showed that quantum Fisher information is upper bounded by

\[ F_Q(\theta) \leq 4[\langle \psi | H_1(\theta) | \psi \rangle - \langle \psi | H_2(\theta) | \psi \rangle^2] \]

where

\[ H_1(\theta) = \sum_{m=0}^{M-1} \frac{\partial V_m^\dagger(\theta)}{\partial \theta} \frac{\partial V_m(\theta)}{\partial \theta}, \quad H_2(\theta) = i \sum_{m=0}^{M-1} \frac{\partial V_m^\dagger(\theta)}{\partial \theta} V_m(\theta) \]
To derive the main result, we consider the following parametrization
\[ V_m(\theta) = e^{\theta/2} V_m \quad (1 \leq m \leq M - 1) \]

We cannot freely parametrize \( V_0(\theta) \) due to the completeness relation
\[ \sum_{m=0}^{M-1} V_m^\dagger(\theta)V_m(\theta) = I \]

Any \( V_0(\theta) \) satisfying the completeness relation can be represented by
\[ V_0(\theta) = Y \sqrt{\left( I - \sum_{m=1}^{M-1} V_m^\dagger(\theta)V_m(\theta) \right)} = Y \sqrt{\left( I - e^{\theta} \sum_{m=1}^{M-1} V_m^\dagger V_m \right)} \]
where \( Y \) is a unitary operator.
**Derivation**

- Using these parametrization, QFI is upper bounded by
  \[ F_Q(\theta = 0) \leq \langle \psi \middle| (V_0^\dagger V_0)^{-1} \middle| \psi \rangle - 1 \]

- We next evaluate \( \partial_\theta \langle g(m) \rangle_\theta \) in quantum Cramér-Rao inequality

- Since we have assumed that \( g(0) = 0 \), a complicated scaling dependence of \( V_0(\theta) \) on \( \theta \) can be ignored

\[
\langle g(m) \rangle_\theta = \sum_{m=0}^{M-1} \langle \psi \middle| V_m^\dagger(\theta)V_m(\theta)\middle| \psi \rangle g(m)
\]

\[
= \sum_{m=1}^{M-1} \langle \psi \middle| V_m^\dagger(\theta)V_m(\theta)\middle| \psi \rangle g(m)
\]

\[
= e^{\theta} \langle g(m) \rangle_{\theta=0}
\]
Derivation

- Substituting into these equality to the quantum Cramér-Rao inequality, we obtain the main result

\[
\frac{\text{Var}[g(m)]}{\langle g(m) \rangle^2} \geq \frac{1}{\langle \psi | (V_0^+ V_0)^{-1} | \psi \rangle - 1}
\]

- The main result also holds for any initial mixed states $\rho$ through the purification

\[
\frac{\text{Var}[g(m)]}{\langle g(m) \rangle^2} \geq \frac{1}{\text{Tr} \left[ (V_0^+ V_0)^{-1} \rho \right] - 1}
\]
**Conclusion**

- TUR in open quantum systems is obtained
  - Quantum dynamics = Joint unitary evolution on principal and environment systems
  - Observable = Projective measurement on the environment
  - Thermodynamic cost = Quantum analogue of dynamical activity

- Effects of quantumness on precision
  - Measurements improve the precision
  - Non-commutativeness improves the precision
Questions?
Quantum walk

- The quantum walk is defined on the chirality space spanned by \{\ket{R}, \ket{L}\} and the position space spanned by \{\ket{n}\}

- One step evolution is operated via a unitary operator

\[
\mathcal{U} = \mathcal{S}(\mathcal{C} \otimes \mathbb{I}_E)
\]

where

\[
\mathcal{C} \equiv \frac{\ket{R}\langle R| + \ket{R}\langle L| + \ket{L}\langle R| - \ket{L}\langle L|}{\sqrt{2}}
\]

\[
\mathcal{S} = \sum_n \left[ \ket{R}\langle R| \otimes \ket{n+1}\langle n| + \ket{L}\langle L| \otimes \ket{n-1}\langle n| \right]
\]
Quantum walk

Chirality space

\[ S |R\rangle \]

\[ t_0 = 0 \]
\[ t_1 \]
\[ t_2 \]
\[ t_3 = T \]

Position space

\[ E |0\rangle \]

\[ \mathcal{U} \]

\[ \mathcal{U} \]

\[ \mathcal{U} \]

---

(a)

\[
\frac{|R\rangle + i |L\rangle}{\sqrt{2}} \otimes |0\rangle
\]

\[ p(n, t) \]

0.4

0.2

0.1

0.05

0.0

-10

0

10

n

(b)

\[
\frac{|R\rangle + i |L\rangle}{\sqrt{2}} \otimes |0\rangle
\]

\[ |R\rangle \otimes |0\rangle \]

\[ |R\rangle \otimes |0\rangle \]

0.15

0.1

0.05

0.0

-100

0

100

n
Quantum walk

- The amplitudes after $t$ steps were known.
- Therefore, dynamical activity $\Xi$ after $t$ steps can be calculated analytically as:

$$
\Xi = \begin{cases} 
2^{2u+1} \left( \frac{u}{2} \right)^{-2} - 1 & u \in \text{even} \\
2^{2u-1} \left( \frac{u-1}{2} \right)^{-2} - 1 & u \in \text{odd}
\end{cases}
$$

where $u \equiv \frac{t}{2}$

- By using Stirling approximation,

$$
\Xi \sim \pi u
$$
Quantum walk

- $\Xi$ linearly depends on the number of steps.
- This is in contrast to the classical case where $\Xi$ exponentially depends on time.