Algorithmic constructions of relative train track maps and CTs

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Abstract

Building on [BH92, BFH00], we proved in [FH11] that every element $\psi$ of the outer automorphism group of a finite rank free group is represented by a particularly useful relative train track map. In the case that $\psi$ is rotationless (every outer automorphism has a rotationless power), we showed that there is a type of relative train track map, called a CT, satisfying additional properties. The main result of this paper is that the constructions of these relative train tracks can be made algorithmic. A key step in our argument is proving that it is algorithmic to check if an inclusion $\mathcal{F} \subseteq \mathcal{F}'$ of $\phi$-invariant free factor systems is reduced.

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1 Introduction

An automorphism $\Phi$ of the rank $n$ free group $F_n$ is typically represented by giving its effect on a basis of $F_n$. Equivalently, if we identify the edges of the rose $R_n$ (the graph with one vertex $*$ and $n$ edges) with basis elements of $F_n$, then $\Phi$ may be represented as a self homotopy equivalence of $R_n$ preserving $*$. In this paper, we are interested in outer automorphisms, that is we are interested in elements of the quotient

$$\text{Out}(F_n) := \text{Aut}(F_n)/\text{Inn}(F_n)$$

of the automorphism group $\text{Aut}(F_n)$ of $F_n$ by its subgroup $\text{Inn}(F_n)$ of inner automorphisms. Since outer automorphisms are defined only up to conjugation, the base point loses its special role and $\phi \in \text{Out}(F_n)$ is typically represented as a self homotopy equivalence of $R_n$ that is not required to fix $*$. More generally it is also typical to take advantage of the flexibility gained by representing $\phi$ as a self homotopy equivalence $f : G \to G$ of a marked graph $G$, i.e. a graph $G$ equipped with a homotopy equivalence $\mu : R_n \to G$. The marking identifies the fundamental group of $G$ with $F_n$, but only up to conjugation. In an analogy with linear maps, representing $\phi$ as $f : G \to G$ corresponds to writing a linear map in terms of a particular basis.
In [BH92], Bestvina-Handel showed that every element of $\text{Out}(F_n)$ has a representation as a relative train track map, that is a representation $f : G \to G$ as above but with strong properties. In the analogy with linear maps, a relative train track map corresponds to a normal form. [BH92] goes on to use relative train track maps to solve the Scott conjecture: the rank of the fixed subgroup of an element of $\text{Aut}(F_n)$ is at most $n$. Further applications spurred the further development of the theory of relative train tracks. See for example, [BFH00] where improved relative train tracks (IRTs) were used to show that $\text{Out}(F_n)$ satisfies the Tits alternative or [FH09] where completely split relative train tracks (CTs) were used to classify abelian subgroups of $\text{Out}(F_n)$.

CTs are relative train tracks that were designed to satisfy the properties that have proven most useful (to us) for investigating elements of $\text{Out}(F_n)$. For the definition, see [FH11, Definition 4.7]. Not every $\phi \in \text{Out}(F_n)$ is represented by a CT, but all rotationless (see Definition 3.1) elements are. This is not a big restriction since there is a specific $M > 0$ depending only on $n$ (see Corollary 3.10) such that $\phi^M$ is rotationless. The main result of this paper is that it is algorithmic to find a CT for a rotationless $\phi \in \text{Out}(F_n)$. We believe that CTs will be of general use in approaching algorithmic questions about $\text{Out}(F_n)$. Verifying that CTs can be constructed algorithmically is a first step in that process.

**Theorem 1.1.** Suppose that $\phi \in \text{Out}(F_n)$ is rotationless and that $C$ is a nested sequence of $\phi$-invariant free factor systems. Then there is an algorithm that produces a CT $f : G \to G$ that represents $\phi$ and whose filtration $\mathcal{F}$ realizes $C$.

In the course of the proof of Theorem 1.1, we show that it is algorithmic to find other objects that are useful in the analysis of a CT, e.g. the set of indivisible (periodic) Nielsen paths (see Section 2.1 for the definition) and the set of Nielsen classes of principal vertices (Definition 3.3). We algorithmically construct a finite graph $S(f)$ with an immersion $S(f) \to G$ that records all fixed1 conjugacy classes of $\phi$ in the sense that a closed circuit $\sigma \subset G$ is Nielsen iff $\sigma$ lifts to $S(f)$. More generally, we record all non-repelling fixed points in $\partial F_n$ of principal representatives of $\phi$ in the sense that there is a finite-type graph $S_N(f)$ (a finite graph $S_0$ union finitely many rays) with and immersion to $G$ and with the property that a ray in $G$ represents a non-repelling fixed point iff an end of the ray lifts to $S_N(f)$. If we subdivide $S_N(f)$ at the pre-image of the vertex set of $G$ then the edges of $S_N(f)$ are labeled by the edges of $G$. The graph $S_N(f)$ is recursively constructible in that the finite graph $S_0$ can be found algorithmically and, for each ray $R$, there is a given finite edge path $\sigma_R$ in $S_0 \cap R$ so that $\sigma_R \cdot f_#(\sigma_R) \cdot f_#^2(\sigma_R) \cdots$ lifts to $R_E$. (Again, see Section 2.1 for notation.)

The proof of Theorem 1.1 is in Section 6. It relies heavily on our paper [FH11] and, more specifically, on the proof of Theorem 4.28 of [FH11] which states that every rotationless outer automorphism of $F_n$ is represented by a CT. We strongly

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1A periodic conjugacy class of a rotationless outer automorphism is fixed [FH11, Lemma 3.30].
recommend that the reader have a copy of [FH11] handy while reading the current one. We will also rely on [FH11] for complete references. Much of the proof of [FH11, Theorem 4.28] is already algorithmic and the main work in the current paper is to make algorithmic the parts of that proof that are not explicitly algorithmic. In fact, there are only three places in the proof of Theorem 4.28 where non-algorithmic arguments are given. The first has to do with relative train track maps for general elements of $\text{Out}(F_n)$. In this case there is an underlying algorithm to the arguments and we make it explicit in Sections 2 and 3.

The second part of the proof to be made algorithmic involves checking whether the filtration by invariant free factor systems induced by the given filtration by invariant subgraphs is reduced. This requires new arguments and is carried out in Section 4. The main result of Section 4 is captured in Corollary 1.2 below, which we believe to be of independent interest. Although we present this result as a corollary of Theorem 1.1, a stand-alone proof could be given using the methods in Section 4; see in particular Proposition 4.9. These methods are in turn key ingredients in the proof of Theorem 1.1.

Recall that if $F_1 \sqsubseteq F_2$ are $\psi$-invariant free factor systems then $\psi$ is said to be fully irreducible relative to $F_1 \sqsubseteq F_2$ if for all $k \geq 1$ there are no $\psi^k$-invariant free factor systems properly contained between $F_1$ and $F_2$. Equivalently, if $\phi$ is a rotationless iterate of $\psi$ then there are no $\phi$-invariant free factor systems properly contained between $F_1$ and $F_2$; see [FH11, Lemma 3.30].

**Corollary 1.2.** There is an algorithm with input an irreducible $\psi \in \text{Out}(F_n)$ and $\psi$-invariant free factor systems $F_1 \sqsubseteq F_2$ and output YES or NO depending on whether or not $\psi$ is fully irreducible relative to $F_1 \sqsubseteq F_2$. In the case that $\psi$ is not fully irreducible, $k \geq 1$ and a $\psi^k$-invariant free factor system that is properly contained between $F_1$ and $F_2$ are found.

**Proof.** Construct a CT $f : G \to G$ with filtration $\emptyset = G_0 \subset G_1 \subset \ldots \subset G_N = G$ for a rotationless $\phi = \psi^M$ with $M$ as in Corollary 3.10 in which $F_1 \sqsubseteq F_2$ are realized by core subgraphs $G_r \subset G_t$. Then, by defining property (Filtration) of a CT, $\psi$ is fully irreducible relative to $F_1 \sqsubseteq F_2$ if there are no core subgraphs $G_s$ properly contained between $G_r$ and $G_t$. If there is such a subgraph then its associated free factor system is properly contained between $F_1$ and $F_2$. \qed

**Remark 1.3.** In the special case that $F_1 = \emptyset$, Corollary 1.2 is an algorithm for checking if $\psi$ is fully irreducible. Our algorithm in this special case is different from the ones given in [Kap14] and [CMP].

The third and final part of the proof of [FH11, Theorem 4.28] needed to be made algorithmic requires a new fixed point result (Lemma 5.5) allowing us to properly attach the terminal endpoints of NEG edges; this takes place in Section 5.

We include some sample applications of Theorem 1.1 – most already known. In Proposition 7.7, we give another proof of the result of Bogopolski and Maslakova [BM]
that it is algorithmic to compute $\text{Fix}(\Phi)$ for $\Phi \in \text{Aut}(F_n)$. The proof of Proposition 7.7 is in Section 7.

We end this introduction with one more application; we reprove a result of Ilya Kapovich [Kap00] that it is algorithmic to tell if a given $\phi \in \text{Out}(F_n)$ is hyperbolic. The Kapovich result is stronger in that $\phi$ is only assumed to be an injective endomorphism. After reviewing relevant definitions, we recall in Proposition 1.5 some conditions equivalent to $\phi$ being hyperbolic. The Kapovich result is then reproved in Corollary 1.6.

**Definition 1.4.** An outer automorphism $\phi \in \text{Out}(F_n)$ is *hyperbolic* if for some $N > 0$ and $\lambda > 1$,

$$\lambda \|\alpha\| \leq \max\{\|\phi^N(\alpha)\|, \|\phi^{-N}(\alpha)\|\}$$

for all non-trivial conjugacy classes $\alpha$ in $F_n$. We say $\phi$ is *atoroidal* if $\phi$ has no non-trivial periodic conjugacy classes. $\Phi \in \text{Aut}(F_n)$ is *hyperbolic* if for some $N > 0$ and $\lambda > 1$,

$$\lambda |a| \leq \max\{|\Phi^N(a)|, |\Phi^{-N}(a)|\}$$

for all non-trivial $a$ in $F_n$. Here $\| \cdot \|$ and $| \cdot |$ denote respectively reduced word length and word length with respect to a fixed basis for $F_n$. The *mapping torus* $M_\Phi$ of $\Phi \in \text{Aut}(F_n)$ is the group with presentation $\langle F_n, t \mid t \Phi(a)t^{-1} = \Phi(a) \text{ for each } a \in F_n \rangle$.

**Proposition 1.5.** Let $\Phi \in \text{Aut}(F_n)$ represent $\phi \in \text{Out}(F_n)$. The following are equivalent.

1. $\phi$ is atoroidal.
2. $\phi$ is hyperbolic.
3. $\Phi$ is hyperbolic.
4. $M_\Phi$ is hyperbolic.

*Proof.* (1) $\implies$ (2) is Brinkmann [Bri00]. We give a different proof in the appendix of the current paper. (2) $\implies$ (3) follows from the proof of Theorem 5.1 of [BFH97]. (3) $\implies$ (4) is a consequence of the first corollary of Section 5 of [BF92]. (4) $\implies$ (1) since otherwise $M_\phi$ contains $\mathbb{Z}^2$.

**Corollary 1.6** ([Kap00]; see also [Dah, page 2]). *There is an algorithm with input $\phi \in \text{Out}(F_n)$ that outputs YES or NO depending on whether or not $\phi$ is hyperbolic.*

*Proof.* Construct a CT $f : G \to G$ for a $\phi^M$ with $M$ as in Corollary 3.10. By Proposition 1.5, $\phi$ is hyperbolic iff $\phi$ is atoroidal. By Lemma 7.4(2), $\phi$ is atoroidal if and only if $S(f)$ has no circuits and this can be checked algorithmically; see Remark 7.3.
2 Relative train track maps in the general case

In this section we revisit three existence theorems for relative train track maps representing arbitrary $\phi \in \text{Out}(F_n)$, rotationless or not. The original statements of these results did not mention algorithms and the original proofs did not emphasize their algorithmic natures. In this section we give the algorithmic versions of two of these results; see Theorem 2.1 and Lemma 2.2. The third existence result that we revisit is [FH11, Theorem 2.19] which is in the current paper is reproduced as Theorem 2.4. We will need the algorithmic version of Theorem 2.4 and its proof is postponed until Section 3.3 because the proof depends on consequences of Theorem 2.4 established in Section 3.

Rather than cut and paste arguments from [BH92], [BFH00] and [FH11] into this paper, we will point the reader to specific sections in those papers and explain how they fit together to give the desired results. In some cases, we refer to arguments that occur in lemmas whose hypotheses are not satisfied in our current context. Nonetheless the arguments that we refer to will apply.

2.1 Some standard notation and definitions

In this section we recall the basic definitions of relative train track theory, assuming that the reader has some familiarity with this material. Complete details can be found in any of [BH92, Sections 1 and 5], [BFH00, Sections 2 and 3], [FH11, Section 2] and [HMb, Section 1].

Identify $F_n$ with $\pi_1(R_n,*)$ where the rose $R_n$ is the graph with one vertex $*$ and $n$ edges. A graph is core if it is the union of its immersed circuits. A marked graph is a finite core graph $G$ equipped with a homotopy equivalence $\mu : R_n \to G$ called the marking by which we identify $\text{Out}(\pi_1(G))$ with $\text{Out}(\pi_1(R_n))$ and hence with $\text{Out}(F_n)$. In this way, a homotopy equivalence $f : G \to G$ determines an element $\phi \in \text{Out}(F_n)$; we say that $f : G \to G$ represents $\phi$. Conversely, each $\phi \in \text{Out}(F_n)$ is represented (non-uniquely) by a homotopy equivalence of any marked graph $G$. Unless otherwise stated, we assume that the restriction of $f$ to each edge of $G$ is an immersion.

We denote the conjugacy class of a subgroup $A$ of $F_n$ by $[A]$. If $A_1 \ast \ldots \ast A_m$ is a free factor of $F_n$ then $\{[A_1], \ldots, [A_m]\}$ is a free factor system and each $[A_i]$ is a component of that free factor system. For any marked graph $G$ and subgraph $\tilde{C}$ with components $C_1, \ldots, C_m$, the fundamental group of each $C_i$ determines a well defined conjugacy class that we denote $[C_i]$ and $[C] := \{[C_1], \ldots, [C_m]\}$ is a free factor system. We say that $C \subset G$ realizes $[C]$. Every free factor system is realized by some subgraph of some marked graph. There is a partial order on free factor systems defined by $\{[A_1], \ldots, [A_k]\} \sqsubseteq \{[B_1], \ldots, [B_l]\}$ if each $A_i$ is conjugate to a subgroup of some $B_j$.

More generally, a subgroup system is a finite collection $\{[A_1], \ldots, [A_k]\}$ of distinct conjugacy classes of finitely generated non-trivial subgroups of $F_n$ with the property that $A_i$ is not conjugate into $A_j$ if $i \neq j$. 
The conjugacy class \([c]\) of \(c \in F_n\) is *carried by* the subgroup system \(\{[A_1], \ldots, [A_k]\}\) if \(c\) is conjugate to an element of some \(A_i\). Equivalently, the fixed points for the action of a conjugate of \(c\) on \(\partial F_n\) are contained in some \(\partial A_i\). More generally if \(P, Q \in \partial A_i\), then the \(F_n\)-orbit of the line with endpoints \(P\) and \(Q\) is *carried by* \(\{[A_1], \ldots, [A_k]\}\). By [BFH00, Lemma 2.6.5], for each collection of conjugacy classes of elements and \(F_n\)-orbits of lines there is a unique minimal (with respect to the above partial order) free factor system that carries them all. If that minimal free factor system is \(\{[F_n]\}\) then we say that the collection *fills* it. In this context, we will treat a subgroup as the collection of conjugacy classes that it carries. In particular, it makes sense to talk about a subgroup system filling.

A *filtration* of a marked graph \(G\) is an increasing sequence of subgraphs \(\emptyset = G_0 \subset G_1 \subset \ldots \subset G_N = G\). The \(r\)th stratum \(H_r\) is the subgraph whose edges are contained in \(G_r\) but not \(G_{r-1}\). A homotopy equivalence \(f : G \to G\) preserves the filtration if \(f(G_r) \subset G_r\) for each \(G_r\). Assuming this to be the case and that the edges in \(H_r\) have been ordered, the transition matrix \(M_r\) associated to \(H_r\) is the square matrix with one row and column for each edge of \(H_r\) and whose \(ij\)th coordinate is the number of times that the \(f\)-image of the \(i\)th edge of \(H_r\) crosses (in either direction) the \(j\)th edge of \(H_r\). After enlarging the filtration if necessary, we may assume that each \(M_r\) is either the zero matrix or irreducible; we say that \(H_r\) is a zero stratum or an irreducible stratum respectively. In the irreducible case, each \(M_i\) has a Perron-Frobenius eigenvalue \(\lambda_i \geq 1\). The stratum \(H_r\) is \(EG\) if \(\lambda_r > 1\) and is \(NEG\) if \(\lambda_r = 1\).

A (finite, infinite or bi-infinite) path in \(G\) is an immersion \(\sigma : J \to G\) defined on an interval \(J\) such that the image of each end of \(J\) crosses infinitely many edges of \(G\); equivalently, \(\sigma\) lifts to a proper map into the universal cover \(\tilde{G}\). Subdividing at the full pre-image of the set \(V\) of vertices of \(G\) we view \(J\) as a simplicial complex and \(\sigma\) as a map that takes edges to edges and is one-to-one on the interior of each edge. In this way we view \(\sigma\) as an edge path; i.e. a concatenation of edges of \(G\), where we allow the first and last to be partial edges if the endpoints are not at vertices. We do not distinguish between paths that have the same associated edge path and we often identify a path with its associated edge path. A path has *height* \(r\) if it is contained in \(G_r\) but not \(G_{r-1}\).

If \(\sigma\) is a path in \(G\) and \(f : G \to G\) is a homotopy equivalence then \(f(\sigma)\) is homotopic, rel endpoints if any, to a path that we denote \(f_{\#}(\sigma)\). If paths \(\sigma_1\) and \(\sigma_2\) can be concatenated then we denote the concatenation by \(\sigma_1 \sigma_2\). A decomposition \(\sigma = \ldots \sigma_1 \sigma_2 \ldots \sigma_m \ldots\) of a path into subpaths is a splitting if \(f_{\#}(\sigma) = \ldots f^k_{\#} (\sigma_1) f^k_{\#} (\sigma_2) \ldots f^k_{\#} (\sigma_m) \ldots\) for all \(k \geq 1\); in this case we usually write \(\sigma = \ldots \sigma_1 \cdot \sigma_2 \cdot \ldots \cdot \sigma_m \ldots\).

If \(f^k_{\#}(\sigma) = \sigma\) for some \(k \geq 1\) then \(\sigma\) is a *periodic Nielsen path*. If \(k = 1\) then \(\sigma\) is a *Nielsen path*. If a (periodic) Nielsen path \(\sigma\) can not be written as the concatenation of non-trivial (periodic) Nielsen subpaths then it is *indivisible*. Two points in \(\text{Fix}(f)\) are in the same *Nielsen class* if they bound a Nielsen path.

A *direction* at a point \(x \in G\) is a germ of finite paths with initial vertex \(x\). If \(x\)
is not a vertex then there are two directions at $x$. Otherwise there is one direction for each oriented edge based at $x$ and we identify the direction with the oriented edge. A homotopy equivalence $f : G \to G$ induces a map $Df$ from directions at $x$ to directions at $f(x)$. A turn at $x$ is an unordered pair $(d_1, d_2)$ of directions based at $x$; it is degenerate if $d_1 = d_2$ and non-degenerate otherwise. A turn is illegal if its image under some iterate of $Df$ is degenerate and is legal otherwise. If $\sigma = \ldots E_i E_{i+1} \ldots$ and if each turn $(\tilde{E}_i, E_{i+1})$ is legal then $\sigma$ is legal. Here $\tilde{E}_i$ denotes $E_i$ with the opposite orientation. We sometimes also use the exponent -1 to indicate the inverse of a path. If $\sigma$ has height $r$ then $\sigma$ is $r$-legal if each turn $(\tilde{E}_i, E_{i+1})$ for which both $E_i$ and $E_{i+1}$ are edges in $H_r$ is legal.

The homotopy equivalence $f : G \to G$ is a relative train track map [BH92, page 38] if the following conditions hold for every EG stratum $H_r$.

(RTT-i) $Df$ maps directions in $H_r$ to directions in $H_r$.

(RTT-ii) If $\sigma \subset G_{r-1}$ is a non-trivial path with endpoints in $H_r \cap G_{r-1}$ then $f_\#(\sigma)$ is a non-trivial path with endpoints in $H_r \cap G_{r-1}$.

(RTT-iii) If $\sigma \subset G_r$ is legal then $f_\#(\sigma)$ is $r$-legal.

2.2 Algorithmic proofs

The following theorem is modeled on [BH92, Theorem 5.12] which proves the existence of relative train track maps that satisfy an additional condition called stability. This property is algorithmically built into CTs at a later stage of the argument (see [FH11, Step 1 (EG Nielsen Paths) in Section 4.5] so we do not need it here. See also [DV96].

Theorem 2.1. There is an algorithm that produces for each $\phi \in \Out(F_n)$ a relative train track map $f : G \to G$ representing $\phi$.

Proof. Suppose that $f : G \to G$ and $\emptyset = G_0 \subset G_1 \subset \ldots \subset G_N = G$ are a topological representative and filtration representing $\phi$. For each EG stratum $H_r$, let $\lambda_r \geq 1$ be the Perron-Frobenious eigenvalue of the transition matrix $M_r$ associated to $H_r$. Let $\Lambda(f)$ be the set of (not necessarily distinct) $\lambda_r$’s associated to EG strata $H_r$ of $f : G \to G$, listed in non-increasing order. Say that $f : G \to G$ is bounded if there are at most $3n - 3$ exponentially growing strata $H_r$ and if each $\lambda_r$ is the Perron-Frobenius eigenvalue of some irreducible matrix with at most $3n - 3$ rows and columns. Note that if each vertex of $G$ has valence at least 3 then $f : G \to G$ is bounded because $G$ has at most $3n - 3$ edges. Note also that if the set of all possible $\Lambda(f)$’s is ordered lexicographically, then any strictly decreasing sequence of $\Lambda(f)$’s associated to bounded $f$’s is finite.

For $f : G \to G$ to be a relative train track map, each EG stratum $H_r$ must satisfy conditions (RTT-i), (RTT-ii) and (RTT-iii) listed in Section 2.1. It is a finite process
to check if these conditions hold and so is a finite process to check if a given \( f : G \to G \) is a relative train track map.

To begin the train track algorithm, choose any bounded \( f : G \to G \) representing \( \phi \). For example, choose an automorphism \( \Phi : F_n \to F_n \) representing \( \phi \) and let \( f : G \to G \) be the corresponding homotopy equivalence of the rose \( R_n \). It suffices to show that if \( f : G \to G \) is not a relative train track map then we can construct a bounded topological representative \( \bar{f} : \bar{G} \to \bar{G} \) with \( \Lambda(\bar{f}) < \Lambda(f) \).

If \( H_r \) satisfies (RTT-i) but not (RTT-iii) then there is a, not necessarily bounded, topological representative \( f' : G' \to G' \) such that \( \Lambda(f') < \Lambda(f) \). An algorithm to construct \( f' : G' \to G' \) is contained in the proof of [BH92, Lemma 5.9]. An algorithm that takes \( f' : G' \to G' \) as input and produces a bounded topological representative \( f'' : G'' \to G'' \) such that \( \Lambda(f'') < \Lambda(f) \) is contained in the proof of [BH92, Lemma 5.5]. It therefore suffices to have an algorithm that takes a bounded \( f'' : G'' \to G'' \) as input and produces a bounded \( f^* : G^* \to G^* \) such that \( \Lambda(f^*) \leq \Lambda(f'') \) and such that each EG stratum of \( f^* : G^* \to G^* \) satisfies (RTT-i) and (RTT-ii). Such an algorithm is described explicitly on pages 42 - 45 of [BH92] using the operations ‘core subdivision’ and ‘collapsing an inessential connecting path’.

**Corollary 2.2.** There is an algorithm that takes \( \phi \in \text{Out}(F_n) \) and a nested sequence \( \mathcal{C} = F_1 \sqsubseteq F_2 \sqsubseteq \cdots \sqsubseteq F_m \) of \( \phi \)-invariant free factor systems as input and produces a relative train track map \( f : G \to G \) and filtration \( \emptyset = G_0 \subset G_1 \subset \cdots \subset G_N = G \) representing \( \phi \) and such that for each \( F_i \) there exists \( G_j \) satisfying \( F_i = [G_j] \).

**Proof.** The proof of this corollary is explicitly contained in the proof of [BFH00, Lemma 2.6.7] (even though the statement that lemma is weaker in that it assumes that \( \mathcal{C} \) is a single free factor system). The first step of the proof of the lemma is to inductively construct a topological representative \( f : G \to G \) and filtration \( \emptyset = G_0 \subset G_1 \subset \cdots \subset G_N = G \) representing \( \phi \) such that for each \( F_i \) there exists \( G_j \) satisfying \( F_i = [G_j] \). Then one applies the relative train track algorithm of Theorem 2.1 to promote \( f : G \to G \) to a relative train track map.

The third existence theorem that needs discussion is [FH11, Theorem 2.19]. We first recall some notation that is used in its statement.

**Notation 2.3.** If \( u < r \) and

1. \( H_u \) is irreducible;
2. \( H_r \) is EG and each component of \( G_r \) is non-contractible; and
3. for each \( u < i < r \), \( H_i \) is a zero stratum that is a component of \( G_{r-1} \) and each vertex of \( H_i \) has valence at least two in \( G_r \),

then we say that each \( H_i \) is *enveloped by* \( H_r \) and write \( H_r^z = \bigcup_{k=u+1}^r H_k \).
Theorem 2.4 ([FH11, Theorem 2.19]). For each $\phi \in \text{Out}(F_n)$ there is a relative train track map $f : G \to G$ and a filtration $\emptyset = G_0 \subset G_1 \subset \ldots \subset G_N = G$ representing $\phi$ and satisfying the following properties.

(V) The endpoints of all indivisible periodic Nielsen paths are vertices.

(P) If a stratum $H_m \subset \text{Per}(f)$ is a forest then there exists a filtration element $G_j$ such $[G_j] \neq [G_l \cup H_m]$ for any $G_l$.

(Z) Each zero stratum $H_i$ is enveloped by an EG stratum $H_r$. Each vertex in $H_i$ is contained in $H_r$ and has link contained in $H_i \cup H_r$.

(NEG) The terminal endpoint of an edge in a non-periodic NEG stratum $H_i$ is periodic and is contained in a filtration element of height less than $i$ that is its own core.

(F) The core of each filtration element is a filtration element.

Moreover, if $\mathcal{C}$ is a nested sequence of $\phi$-invariant free factor systems then we may choose $f : G \to G$ so that for each $F_i \in \mathcal{C}$ there exists $G_j$ satisfying $F = [G_j]$.

The proof that $f : G \to G$ as in Theorem 2.4 can be constructed algorithmically is contained in Section 3.3

3 Rotationless iterates

Every element of $\text{Out}(F_n)$ has a rotationless iterate. Corollary 3.10 below gives an explicit bound on the size of the iterate; see also [FH11, Lemma 4.42] for a proof that such a bound exists.

3.1 Principal automorphisms, principal points and more on markings

In this subsection we discuss markings in more detail and recall definitions and results from [FH11, Section 3]. We assume throughout this subsection that $f : G \to G$ is a relative train track map representing $\phi \in \text{Out}(F_n)$.

Markings are used to translate the geometric properties of $f : G \to G$ into algebraic properties of $\phi$. In this paper, we will focus on the geometric properties of the homotopy equivalences and only bring in markings at the last minute when necessary. Some details on the material presented in this section can be found in [FH11, Section 2.3].

Recall that the rose $R_n$ denotes the rose with vertex $*$ and that we have once and for all identified $\pi_1(R_n, *)$ with $F_n$. A lift $\bar{*} \in \tilde{R}_n$ of $*$ to the universal cover $\tilde{R}_n$ determines an isomorphism $J_{\bar{*}}$ from $F_n = \pi_1(R_n, *)$ to the group $\mathcal{T}(\tilde{R}_n)$ of covering
translations of $\tilde{R}_n$ given by $[\gamma]$ maps to the covering translation $T$ of $\tilde{R}_n$ that takes $\hat{x}$ to the terminal endpoint of the lift of $\gamma$ with initial endpoint $\hat{x}$.

Let $G$ be finite graph equipped with a marking $\mu : R_n \to G$. Denoting $\mu(\ast)$ by $\ast$, $\mu : (R_n, \ast) \to (G, \ast)$ induces an isomorphism $\mu_\# : \pi_1(R_n, \ast) \to \pi_1(G, \ast)$ that identifies $F_n$ with $\pi_1(G, \ast)$. Fix a lift $\hat{x}$ of $\ast$ to $\hat{G}$. The lift $\hat{\mu} : (\tilde{R}_n, \ast) \to (\hat{G}, \hat{x})$ determines a homeomorphism $\partial \hat{\mu} : \partial \tilde{R}_n \to \partial \hat{G}$ of Gromov boundaries. In this way $\partial F_n$, $\partial \tilde{R}_n$, and $\partial \hat{G}$ are all identified. Since covering translations are determined by their action on Gromov boundaries, there is an induced identification of $T(\tilde{R}_n)$ with $T(\hat{G})$. For any $v \in G$ and lift $\hat{v} \in \hat{G}$, there is an induced isomorphism $J_{\hat{v}} : \pi_1(G, v) \to T(\hat{G})$ defined exactly as $J_{\hat{v}}$. It is straightforward to check that $\mu_\# = J_{\hat{v}}^{-1} J_{\hat{x}} : \pi_1(R_n, \ast) \to \pi_1(G, \ast)$.

We also have an identification of automorphisms representing $\phi \in \text{Out}(F_n)$ with lifts $\tilde{f} : \tilde{G} \to \hat{G}$ of $f : G \to G$ given by $\Phi \leftrightarrow \tilde{f}$ if the actions of $\Phi$ and $\tilde{f}$ on $\partial F_n$ agree, i.e. if $\partial \Phi = \partial \tilde{f}$. We usually specify $\tilde{f}$ by specifying $\tilde{f}(\hat{x})$ or equivalently by specifying the path $\tilde{\rho} = [\hat{x}, \tilde{f}(\hat{x})]$ or its image $\rho$ in $G$. We say that $\Phi$ or $\tilde{f}$ is determined by $\tilde{f}(\hat{x})$, $\tilde{\rho}$, or $\rho$. The action of $\Phi$ on $\pi_1(G, \ast)$ is given by $\gamma \mapsto f(\gamma)^\rho := \rho f(\gamma) \rho$. If $\tilde{f}$ is determined by $\rho$ and $\tilde{f}'$ is determined by $\rho'$ then $\Phi' = i, \Phi$ where $\Phi \leftrightarrow \tilde{f}, \tilde{f}' \leftrightarrow \tilde{f}'$, and $\gamma \in F_n$ is represented by the loop $\rho \tilde{\gamma}$. Working in the universal cover $\hat{G}$ is algorithmic in the sense that we can always compute the action of $\tilde{f}$ on arbitrarily large balls (in the graph metric) around $\hat{x}$. In particular, given $\Phi$ we may algorithmically find $\tilde{f}$ with $\Phi \leftrightarrow \tilde{f}$ and vice versa. If $\tilde{f}$ fixes $\hat{v} \in \hat{G}$ then $\tilde{f} \leftrightarrow \Phi$ for $\Phi$ determined by $\rho$ where $\tilde{\rho} = \tilde{\eta} \tilde{f}(\tilde{\eta}^{-1})$ and $\tilde{\eta} = [\hat{x}, \hat{v}]$.

Automorphisms $\Phi_1, \Phi_2 \in \text{Aut}(F_n)$ are isogredient if $\Phi_1 = i_a \Phi_2 i_a^{-1}$ for some inner automorphism $i_a$. Lifts $\tilde{f}_1$ and $\tilde{f}_2$ of $f$ are isogredient if the corresponding automorphisms are isogredient. That is, $\tilde{f}_1$ and $\tilde{f}_2$ are isogredient if there exists a covering translation $T$ of $\hat{G}$ such that $\tilde{f}_2 = T \tilde{f}_1 T^{-1}$. The set of attracting laminations for $\phi \in \text{Out}(F_n)$ is denoted $\mathcal{L}(\phi)$; see [BFH00, Section 3].

**Definition 3.1.** [FH11, Definition 3.1] For $\Phi \in \text{Aut}(F_n)$ representing $\phi$, let $\text{Fix}_N(\partial \Phi) \subset \text{Fix}(\partial \Phi)$ be the set of non-repelling fixed points of $\partial \Phi$. We say that $\Phi$ is a principal automorphism and write $\Phi \in P(\phi)$ if either of the following hold.

- $\text{Fix}_N(\partial \Phi)$ contains at least three points.
- $\text{Fix}_N(\partial \Phi)$ is a two point set that is neither the set of fixed points for the action of some non-trivial $a \in F_n$ on $\partial F_n$ nor the set of endpoints of a lift of a generic leaf of an element of $\mathcal{L}(\phi)$.

The lift $\tilde{f} : \tilde{G} \to \hat{G}$ corresponding to $\Phi$ is a principal lift.

If $\Phi \in P(\phi)$ and $k > 1$ then $\text{Fix}_N(\partial \Phi) \subset \text{Fix}_N(\partial \Phi^k)$ and $\Phi^k \in P(\phi^k)$. It may be that the injection $\Phi \mapsto \Phi^k$ of $P(\phi)$ into $P(\phi^k)$ is not surjective. It may also be that $\text{Fix}_N(\partial \Phi^k)$ properly contains $\text{Fix}_N(\partial \Phi)$ for some principal $\Phi$ and some $k > 1$. If neither of these happen then we say that $\phi$ is forward rotationless. For a formal definition, see [FH11, Definition 3.13].
Remark 3.2. It is becoming common usage to suppress the word “forward” in “forward rotationless” and we will follow that convention in this paper. So, when we say that $\phi \in \text{Aut}(F_n)$ is rotationless, we mean that $\phi$ is forward rotationless. This convention was followed in the recent work of Handel-Mosher [HMb, HMc, HMe, HMd]. Be aware though that the term “rotationless” has a slightly different meaning in [FH09].

By [FH11, Corollary 3.17], $\text{Fix}(\tilde{f}) \neq \emptyset$ for each principal lift $\tilde{f}$. The projected image of $\text{Fix}(\tilde{f})$ is exactly a Nielsen class in $\text{Fix}(f)$ and a pair of principal lifts are isogredient if and only if they determine the same Nielsen class of $\text{Fix}(f)$ [FH11, Lemma 3.8].

Definition 3.3. We say that $x \in \text{Per}(f)$ is principal if neither of the following conditions are satisfied.

- $x$ is not an endpoint of a non-trivial periodic Nielsen path and there are exactly two periodic directions at $x$, both of which are contained in the same EG stratum.

- $x$ is contained in a component $C$ of $\text{Per}(f)$ that is topologically a circle and each point in $C$ has exactly two periodic directions.

If each principal periodic vertex is fixed and if each periodic direction based at a principal periodic vertex is fixed then we say that $f$ is rotationless.

Remark 3.4. By definition, a point is principal with respect to $f$ if and only if it is principal with respect to $f^k$ for all $k \geq 1$.

Remark 3.5. Definition 3.3 is a corrected version of Definition 3.18 of [FH11] in which ‘$x$ is not an endpoint of a non-trivial Nielsen path’ in the first item of Definition 3.3 is replaced with the inequivalent condition ‘$x$ is the only point in its Nielsen class’. Our thanks to Lee Mosher who pointed this out to us. Fortunately, the definition we give here and not the one given in [FH11] is the one that is actually used in [FH11] so no further corrections to [FH11] are necessary.

We are mostly interested in the case of a CT, where characterizations of principal points are simpler. The next lemma gives two.

Lemma 3.6. Suppose $f : G \to G$ is a CT.

(1) A point $x \in \text{Per}(f)$ is principal iff $x \in \text{Fix}(f)$ and the following condition is not satisfied.

- $x$ is not an endpoint of a non-trivial Nielsen path and there are exactly two periodic directions at $x$, both of which are contained in the same EG-stratum.

(2) The following are equivalent for a point $x \in \text{Fix}(f)$. Let $\tilde{f} : \tilde{G} \to \tilde{G}$ be a lift of $f$ fixing a lift $\tilde{x}$ of $x$. 

(a) $x$ is principal.
(b) $\tilde{f}$ is principal.
(c) $\text{Fix}_N(\partial \tilde{f}^2)$ is not the set of endpoints of a generic leaf of an element of $\mathcal{L}(\phi)$.

Proof. (1): Periodic Nielsen paths in a CT are fixed [FH11, Lemma 4.13] and so the bulleted item in the lemma is equivalent to the first bulleted item of the Definition 3.3. By definition, periodic edges of a CT are fixed and the endpoints of fixed edges are principal. Therefore the second item in Definition 3.3 never holds. To complete the proof it remains to show that all principal points of $f$ are fixed. This holds for vertices because CTs are rotationless. If $x$ is a periodic but not fixed point in the interior of an edge then (by definition of a CT) that edge must be in an $EG$ stratum and so $x$ is not principal.

(2): By [FH11, Corollaries 3.22 and 3.27], (2a) and (2b) are equivalent. If $\tilde{f}$ is principal, then by definition of rotationless and principal, $\text{Fix}_N(\partial \tilde{f}) = \text{Fix}_N(\partial \tilde{f}^2)$ is not contained in the set of endpoints of a generic leaf. We see (2b) implies (2c). If $x$ is not principal for $f$ then the bulleted item in (1) holds. In particular there are exactly two periodic directions at $x$, both of which are in the same $EG$-stratum. By [FH11, Lemma 2.13], $\text{Fix}_N(\partial \tilde{f}^2)$ contains the set of endpoints of a generic leaf of an element of $\mathcal{L}(\phi)$. By Remark 3.4, $x$ is not principal for $f^2$, and so $\tilde{f}^2$ is not a principal lift. Hence $|\text{Fix}_N(\partial \tilde{f}^2)| < 3$. We conclude (2c) implies (2a). □

3.2 A sufficient condition to be rotationless and a uniform bound

Before turning to Lemma 3.8, which gives a sufficient condition for an outer automorphism to be rotationless, we recall the connection between edges in a CT and attractors in $\text{Fix}_N(\partial \Phi)$.

Suppose that $\phi \in \text{Out}(F_n)$ is rotationless. For each $\Phi \in P(\phi)$, let $\text{Fix}_+(\partial \Phi)$ be the set of points in $\text{Fix}_N(\partial \Phi)$ that are attractors for the action for $\partial \Phi$ on $\partial F_n$. Define $\text{Fix}_+(\partial \phi) = (\cup \{\text{Fix}_+(\partial \Phi) \mid \Phi \in P(\phi)\})/F_n = \{[P] \in \partial F_n/F_n \mid P \in \text{Fix}_+(\partial \Phi), \Phi \in P(\phi)\}$ where $[P]$ denotes the $F_n$-orbit of $P$.

Given a CT $f : G \to G$ representing $\phi$, let $\mathcal{E}$ be the set of oriented, non-fixed, and non-linear, oriented edges in $G$ whose initial vertex is principal and whose initial direction is fixed by $Df$. For each $E \in \mathcal{E}$, there is a path $u$ such that $f_k^\#(E) = E \cdot u \cdot f_{\#}(u) \cdot \ldots \cdot f_{\#(k-1)}(u)$ for all $k \geq 1$ and such that $|f_{\#(k)}(u)| \to \infty$ with $k$. The union of the increasing sequence $E \subset f(E) \subset f_{\#}^2(E) \subset \ldots$ of paths in $G$ is a ray $R_E$. Each lift of $R_E$ to the universal cover of $G$ has a well-defined terminal endpoint in $\partial F_n$ and so $R_E$ determines an $F_n$-orbit $[R_E]$ in $\partial F_n$. 

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Lemma 3.7. Suppose that $f : G \to G$ is a CT representing $\phi$. Then $E \mapsto [R_E]$ defines a surjection from $\mathcal{E}$ to $\text{Fix}_+ (\partial \phi)$.

Proof. Suppose that $x$ is the initial endpoint of $E \in \mathcal{E}$, that $\tilde{f} : \tilde{G} \to \tilde{G}$ fixes a lift $\tilde{x}$ of $x$ and that $\tilde{R}_E$ is the lift of $R_E$ that begins at $\tilde{x}$. Lemma 3.6(2) implies that $\tilde{f}$ is a principal lift and [FH11, Lemma 4.36(1)] implies that $\tilde{R}_E$ converges to a point $P \in \text{Fix}_N (\partial \Phi)$ where $\Phi$ is the principal automorphism corresponding to $\tilde{f}$. Since $|f^k(u)| \to \infty$, it follows [GJLL98, Proposition I.1] that $P \in \text{Fix}_+ (\partial \Phi)$. This proves that $[R_E] \in \text{Fix}_+ (\partial \Phi)$. [FH11, Lemma 4.36(2)] implies that $E \mapsto [R_E]$ is surjective.

Lemma 3.8. Suppose that $\theta \in \text{Out}(F_n)$ acts trivially on $H_1(F_n; \mathbb{Z}/2\mathbb{Z})$ and induces the trivial permutation on $\text{Fix}_+ (\partial \phi)$ for some (any) rotationless iterate $\phi = \theta^k$ of $\theta$. Then $\theta$ is rotationless.

Proof. We show below that

(*) For any $\Phi \in P(\phi)$, there is $\Theta \in P(\theta)$ with the property that $\text{Fix}_N (\partial \Phi) \subset \text{Fix}_N (\partial \Theta)$.

To see why this is sufficient to prove the lemma, let $\Theta_k \in P(\theta^k)$ for some $k \geq 1$. Since $\theta^{kL} = \phi^k$, $\Theta_k^L \in P(\phi^k)$. Since $\phi$ is rotationless, there exists $\Phi \in P(\phi)$ such that $\Theta_k^L = \Phi^k$ and $\text{Fix}_N (\partial \Phi) = \text{Fix}_N (\partial \Phi^k) = \text{Fix}_N (\partial \Theta_k^L)$. By (*), there is $\Theta \in P(\theta)$ such that $\text{Fix}_N (\partial \Theta_k) \subset \text{Fix}_N (\partial \Theta_k^L) = \text{Fix}_N (\partial \Phi) \subset \text{Fix}_N (\partial \Theta) \subset \text{Fix}_N (\partial \Theta^k)$.

It follows that $\Theta_k = \Theta^k$. We have now seen that $P(\theta) \to P(\theta^k)$ given by $\Theta \mapsto \Theta^k$ is surjective. By [FH11, Definition 3.13 and Remark 3.14], to show that $\theta$ is rotationless it remains to show that $\text{Fix}_N (\partial \Theta^k) = \text{Fix}_N (\partial \Theta)$ for all $\Theta \in P(\theta)$ and $k \geq 1$. This follows from the above displayed sequence of inclusions by taking $\Theta_k := \Theta^k$.

We now turn to the proof of (*). Set $\mathbb{F} := \text{Fix}(\Phi)$. We claim that there exists $\Theta$ representing $\theta$ such that $\mathbb{F} \subset \text{Fix}(\Theta)$. If the rank of $\mathbb{F}$ is $< 2$ then this follows from [HMc, Theorem 4.1], which implies that $\theta$ fixes each conjugacy class that is fixed by $\phi$ and in particular fixes each conjugacy class represented by an element of $\text{Fix}(\Phi)$.

Suppose then that $\mathbb{F}$ has rank $\geq 2$. We recall two facts.

- Each element of $F_n$ is fixed by only finitely many elements of $P(\phi)$ and the root-free ones that are fixed by at least two such automorphisms determine only finitely many conjugacy classes; see [FH11, Lemma 4.40].

- $\mathbb{F}$ is its own normalizer in $F_n$. Proof: Since $\mathbb{F}$ has rank $> 1$, we can choose $1 \neq x \in \mathbb{F}$ that is not fixed by any element of $P(\phi)$ other than $\Phi$. If $y \in F_n \setminus \mathbb{F}$ and $yxy^{-1} \in \mathbb{F}$ then $x$ is fixed by both $\Phi$ and $i_y \Phi i_y^{-1} \in P(\phi)$. Since $y \not\in \mathbb{F}$, we have $\Phi(y) \neq i_y \Phi i_y(y)$, a contradiction.


We may therefore choose a basis \( \{ b_j \} \) for \( F \) consisting of elements that are not fixed by any other element of \( P(\phi) \). Applying [HMc, Theorem 4.1] again, choose an automorphism \( \Theta_j \) representing \( \theta \) and fixing \( b_j \). The automorphism \( \Theta_j \Phi \Theta_j^{-1} \) fixes \( b_j \) by construction and belongs to \( P(\phi) \) by [FH09, Lemma 2.6] and the fact that \( \theta \) and \( \phi \) commute. By uniqueness, \( \Theta_j \Phi \Theta_j^{-1} = \Phi \) and so \( \Theta_j \) commutes with \( \Phi \). In particular, \( \Theta_j \) preserves \( F \). Since \( F \) is its own normalizer, the outer automorphism \( \theta|F \) of \( F \) determined by \( \Theta_j \) is independent of \( j \). It follows that \( \theta|F \) acts trivially on \( H_1(F; \mathbb{Z}/3\mathbb{Z}) \). Since \( \theta^L|F \) is the identity and since the kernel of natural map \( \text{Out}(F_n) \to H_1(F_n; \mathbb{Z}/3\mathbb{Z}) \) is torsion-free\(^2\), \( \theta|F \) is the identity and the claim is proved.

Since \( \theta \) acts trivially on \( \text{Fix}_+(\partial \phi) \), each \( Q \in \text{Fix}_+(\partial \phi) \) is fixed by some \( \Theta_Q \) representing \( \theta \). Since \( \Theta_Q \) and \( \Phi \) both fix \( Q \) and represent \( \phi \) we have \( \Theta_Q^L = \Phi \). As above, \( \Theta_Q \) commutes with \( \Phi \) and so preserves \( F \) and \( \text{Fix}_+(\partial \phi) \). For any other \( Q' \in \text{Fix}_+(\partial \phi) \) we have \( \Theta_Q = i_a \Theta_{Q'} \) for some \( a \in F_n \). Since both \( \Theta_Q \) and \( \Theta_Q' \) preserve \( F \), \( i_a \) does as well and so \( a \in F \).

It suffices to show that \( \Theta_Q \) is independent of \( Q \) and \( F \subset \text{Fix}(\Theta_Q) \). This is obvious if \( F \) is trivial. If \( F \) has rank one then \( F = \text{Fix}(\Theta_Q) \) and \( Q' = \Phi(Q') = \Theta_Q^L(Q') = i_a^LQ' \), which implies that \( a \) must be trivial and we are done. If \( F \) has rank \( \geq 2 \) then there is a unique \( \Theta \) such that \( F \subset \text{Fix}(\Theta) \) and \( \Phi \) is the only automorphism representing \( \phi \) such that \( F \subset \text{Fix}(\Phi) \). Thus \( \Theta^L = \Phi \). There exists \( b \in F \) such that \( \Theta_Q = i_b \Theta \). We have \( \Phi = \Theta_Q^L = i_b^L \Theta^L = i_b^L \Phi \) so \( b \) is trivial and the proof is complete. \( \square \)

To apply Lemma 3.8 we will need a bound on the cardinality of \( \text{Fix}_+(\partial \phi) \).

**Lemma 3.9.** If \( \phi \in \text{Out}(F_n) \) is rotationless then \( |\text{Fix}_+(\partial \phi)| \leq 15(n - 1) \).

**Proof.** By Lemma 3.7 it suffices to show that the cardinality of the image of \( E \mapsto [R_E] \) is bounded by \( 15(n - 1) \).

There are at most \( 6(n - 1) \) oriented edges based at natural vertices. Some of these are not fixed and so do not contribute to \( E \). For example, if \( E \) is NEG then (Lemma 4.21 of [FH11]) the terminal vertex of \( E \) is natural and the direction determined by \( E \) is not fixed. Similarly, if \( (E_1, E_2) \) is an illegal turn of EG height then the basepoint for this turn is natural and either \( E_1 \) or \( E_2 \) determines a non-fixed direction. It follows that \( 6(n - 1) \) is an upper bound for the sum of the number of edges in \( E \) that are based at natural vertices plus the number of non-fixed NEG edges plus the number of EG stratum \( H_r \) with an illegal turn of height \( r \).

It remains to account for those EG edges \( E \in E \) that are based at valence two vertices \( v \). By our previous estimate there are at most \( 6(n - 1) \) such \( v \) with the other edge incident to \( v \) being non-fixed NEG. The only other possibility is that both edges incident to \( v \) are EG. If the edges were in different strata, say \( H_r \) and \( H_{r'} \) with \( r < r' \) then \( v \) would have valence \( \geq 2 \) in \( G_r \) (because \( G_r \) is a core subgraph), a contradiction.

\(^2\)This follows from the standard fact that the kernel of the natural map \( GL_n(\mathbb{Z}) \to GL_n(\mathbb{Z}/3\mathbb{Z}) \) is torsion-free and the result of Buamslag-Taylor [BT68] that the kernel of the natural map \( \text{Out}(F_n) \to GL_n(\mathbb{Z}) \) is torsion-free.
Thus both edges belong to the same stratum \( H_r \). Since \( v \) is a principal vertex, it must be an endpoint of a Nielsen path \( \rho \) of height \( r \). There are at most four edges incident to valence two vertices at the endpoints of \( \rho \) and these determine at most three points in \( \text{Fix}_+(\partial \phi) \) because the two directions pointing into \( \rho \) determine the same point in \( \text{Fix}_+(\partial \phi) \). There are at most four edges incident to valence two vertices at the endpoints of \( \rho \) and these determine at most three points in \( \text{Fix}_+(\partial \phi) \) because the two directions pointing into \( \rho \) determine the same point in \( \text{Fix}_+(\partial \phi) \). There is at most one such \( \rho \) for each EG stratum \( H_r \) and \( \rho \) has an illegal turn of height \( r \) so our initial bound of \( 6(n - 1) \) counted each \( \rho \) once; we now have to count it two more times. In passing from the highest core \( G_s \) with \( s < r \) to \( G_r \), at least two natural edges are added. It follows that the number of EG strata is \( \leq \frac{3}{2}(n - 1) \). The total count then is \( 6(n - 1) + 6(n - 1) + 3(n - 1) = 15(n - 1) \).

**Corollary 3.10.** Let \( h(n) = |GL(\mathbb{Z}/3/\mathbb{Z}, n)| = 3^{(n^2 - 1)} \), let \( g(m) \) be Landau’s function, the maximum order of an element in the symmetric group \( S_m \), and let \( K_n = h(n) \cdot g(15(n - 1)) \). If \( \theta \in \text{Out}(F_n) \) then \( \theta^{K_n} \) is rotationless.

**Proof.** Lemma 3.9 implies that \( \theta^{(15(n - 1))} \) induces the trivial permutation of \( \text{Fix}_+(\partial \theta) \) and hence that \( \theta^{K_n} = (\theta^{(15(n - 1))}h(n)) \) satisfies the hypotheses of Lemma 3.8.

### 3.3 Algorithmic Proof of Theorem 2.4

We review the proof of the existence of \( f : G \to G \) for \( \theta \) as given in [FH11, Theorem 2.19], altering it slightly to make it algorithmic.

If \( \mathcal{C} \) is not specified, take it to be the single free factor system \([F_n]\). Apply Corollary 2.2 to construct a relative train track map \( f : G \to G \) and \( \emptyset = G_0 \subset G_1 \subset \ldots \subset G_N = G \) such that for each \( F_i \in \mathcal{C} \) there exists \( G_j \) satisfying \( F = [G_j] \). The modifications necessary to arrange all the properties but (V) are explicitly described in the original proof. These steps come after (V) has been established in that proof but make no use of (V) so there is no harm in our switching the order in which properties are arranged. For notational simplicity we continue to refer to the relative train track map as \( f : G \to G \) even though it has been modified to satisfy all the properties except possibly (V).

For \( K_n \) as in Corollary 3.10, \( \theta^{K_n} \) is rotationless. Subdivide \( f : G \to G \) at the (finite) set \( S \) of isolated points in \( \text{Fix}(f^{K_n}) \) that are not already vertices; these occur only in EG edges \( E \) and are in one to one correspondence with the occurrences of \( E \) or \( \hat{E} \) in the edge path \( f^{K_n}_\mathcal{C}(E) \). We claim that property (V) is satisfied. If not, then arrange (V) by a further finite ([FH11, Lemma 2.12]) subdivision. [FH11, Proposition 3.29] and [FH11, Lemma 3.28] imply that every periodic Nielsen path of \( f : G \to G \) (after the further subdivision) has period at most \( K_n \). But then \( S \) contains the endpoints of all indivisible periodic Nielsen paths after all and no further subdivision was necessary. Since subdivision at \( S \) is algorithmic, we are done.
4 Reducibility

Given a relative train track map \( f : G \to G \) and filtration \( \emptyset = G_0 \subset G_1 \subset \ldots \subset G_N = G \) representing \( \theta \in \text{Out}(F_n) \), let \( \emptyset = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_K \) be the increasing sequence of distinct \( \theta \)-invariant free factor systems that are realized by the \( G_i \)'s. Assuming that \( f : G \to G \) satisfies property (F) of Theorem 2.4, \( \mathcal{F}_i \) is realized by a unique core filtration element for each \( i \geq 1 \) and \( \mathcal{F}_0 \) is realized by \( G_0 \). If \( \mathcal{F} \) is a \( \theta \)-invariant free factor system that is properly contained between \( \mathcal{F}_i \) and \( \mathcal{F}_{i+1} \) then we say that \( \mathcal{F} \) is a reduction for \( \mathcal{F}_i \subset \mathcal{F}_{i+1} \) with respect to \( \theta \); if there is no such \( \mathcal{F} \) then \( \mathcal{F}_i \subset \mathcal{F}_{i+1} \) is reduced with respect to \( \theta \). If each \( \mathcal{F}_i \subset \mathcal{F}_{i+1} \) is reduced with respect to \( \theta \) then we say that \( f : G \to G \) is reduced.

The main results of this subsection are Proposition 4.9 and Lemma 4.12. The former, which assumes that \( f : G \to G \) satisfies the conclusions of Theorem 2.4, provides an algorithm in the EG case for deciding if \( \mathcal{F}_i \subset \mathcal{F}_{i+1} \) is reduced and for finding a reduction if there is one. The latter has stronger requirements and easily leads to an algorithm that handles the NEG case. We save the final details of that algorithm for Section 6.

4.1 The EG case

Recall ([BFH00, Section 2] or [HMb, Fact 1.3]) that a pair of free factor systems \( \mathcal{F}^1 \) and \( \mathcal{F}^2 \) has a well-defined meet \( \mathcal{F}^1 \wedge \mathcal{F}^2 \) characterized by \([A] \in \mathcal{F}^1 \wedge \mathcal{F}^2 \) if and only if there exists subgroups \( A^1, A^2 \) such that \([A'] \in \mathcal{F}^1 \) and \( A^1 \cap A^2 = A \).

Let \( \mathcal{B} \) be the basis of \( F_n \) corresponding to the edges of \( R_n \) (see Section 2.1). If \( A \) is a finitely generated subgroup of \( F_n \) then the Stallings graph \( R_A \) of the conjugacy class \([A] \) of \( A \) is the core of the cover of \( R_n \) corresponding to \( A \). There is an immersion \( R_A \to R_n \) and if we subdivide \( R_A \) at the pre-image of the vertex of \( R_n \) then we view the edges of \( R_A \) as labeled by their image edges in \( R_n \) and hence by elements of \( \mathcal{B} \). The complexity of \( A \) is the number of edges in (subdivided) \( R_A \). Stallings graphs are generalized in Section 7.2 and more discussion can be found there.

**Lemma 4.1.** Given free factor systems \( \mathcal{F}^1 \) and \( \mathcal{F}^2 \) one can algorithmically construct \( \mathcal{F}^1 \wedge \mathcal{F}^2 \).

*Proof.* We may assume without loss that \( \mathcal{F}^1 = \{[A]\} \) and \( \mathcal{F}^2 = \{[B]\} \) for given subgroups \( A, B \). According to Stallings [Sta83, Theorem 5.5 and Section 5.7(b)], the conjugacy classes of the intersections of \( A \) with conjugates of \( B \) are all represented by components of the pullback of the diagram \( R_A \to R \leftarrow R_B \). \( \square \)

**Lemma 4.2.** Given a finite set \( \{a_i\} \) of elements of \( F_n \) and a finite set \( \{A_j\} \) of finitely generated subgroups of \( F_n \) there is an algorithm that finds the unique minimal free factor system that carries each \([a_i]\) and each conjugacy class carried by some \([A_j]\).
Proof. By replacing $[a_i]$ by $\langle a_i \rangle$, we may assume that finite set $\{a_i\}$ is empty. The complexity of $A = \{[A_j]\}$ is the sum of the complexities of $[A_j]$. By Gersten [Ger84], there is an algorithm to find $\Theta \in \text{Aut}(F_n)$ so that $\Theta(A) = \{[\Theta(A_j)]\}$ has minimal complexity in the orbit of $A$ under the action of $\text{Aut}(F_n)$. Let $P$ be the finest partition of $B$ such that the labels of each Stallings graph $R_{\Theta(A_j)}$ are contained in some element of $P$. The free factor system $F(P)$ determined by $P$ is the minimal free factor system carrying $\Theta(A)$; see [DF05, Lemma 9.19]. Hence $\Theta^{-1}(F(P))$ is the minimal free factor system carrying $A$.

Corollary 4.3. Suppose that $\phi \in \text{Out}(F_n)$, that $F^1$ is a proper free factor system and that $F^0 \subset F^1$ is a (possibly trivial) $\phi$-invariant free factor system. Then there is an algorithm that decides if there is a $\phi$-invariant free factor system $F \subset F^1$ that properly contains $F^0$ and that finds such an $F$ if one exists.

Proof. First check if $F^1$ is $\phi$-invariant or if $F^1 = F^0$. If the latter is true the output of the algorithm is NO. If the latter is false and the former is true the output is YES. If neither is true apply Lemma 4.1 to compute $F^1 \land \phi(F^1)$ which is properly contained in $F^1$, contains $F^0$ and contains every $\phi$-invariant free factor system that is contained in $F^1$. Repeat these steps with $F^1 \land \phi(F^1)$ replacing $F^1$. Since there is a uniform bound to the length of a strictly decreasing sequence of free factor systems [HMB, Fact 1.3] the process stops after finitely many steps.

The following lemma is used in Step 1 of the proof of Proposition 4.9. The definition of a path being weakly attracted to $\Lambda^+$ appears as [BFH00, Definition 4.2.3].

Lemma 4.4. Suppose that $f : G \to G$ is a relative train track map and that $H_r$ is an $EG$ stratum with associated attracting lamination $\Lambda^+$. Then there is a computable constant $C$ such that if $\sigma_0 \subset G_r$ is an $r$-legal path that crosses at least $C$ edges in $H_r$ then every path $\sigma \subset G_r$ that contains $\sigma_0$ as a subpath is weakly attracted to $\Lambda^+$.  

Proof. Choose $l$ so that the $f^l$ image of each edge in $H_r$ crosses at least two edges in $H_r$. It is shown in the proof of [BFH00, Lemma 4.2.2] (see also [BFH00, Corollary 4.2.4]) that $C = 4lC_0 + 1$ satisfies the conclusions of our lemma for any constant $C_0$ that is greater than or equal to the bounded cancellation constant for $f$. Since the latter can be computed [BFH97, Lemma 3.1] using only the transition matrix for $f$, we are done.

Lemma 4.5. Suppose that $f : G \to G$ and $\emptyset = G_0 \subset G_1 \subset \ldots \subset G_N = G$ are a relative train track map and filtration representing a rotationless $\phi$ and satisfying the conclusions of Theorem 2.4. Suppose further that $H_r$ is an $EG$ stratum. Then every indivisible periodic Nielsen path with height $r$ has period one and there is an algorithm that finds them all.
**Remark 4.6.** For a more efficient method than the one described in the proof see [HMa, Section 3.4].

**Proof.** Suppose that $\rho$ is an indivisible periodic Nielsen path of height $r$. Proposition 3.29 and Lemma 3.28 of [FH11] imply that $\rho$ has period 1 and property (V) of Theorem 2.4 implies that the endpoints of $\sigma$ are vertices. By [BH92, Lemma 5.11], $\rho$ decomposes as a concatenation $\rho = \alpha\beta^{-1}$ of $r$-legal edge paths whose initial and terminal edges are in $H_r$. Let $\alpha_0$ and $\beta_0$ be the initial edges of $\alpha$ and $\beta$ respectively. By Lemma 4.4 we can bound the number of $H_r$ edges crossed by $\alpha$ and $\beta$ by some positive constant $C$. Since $H_r$ is an EG stratum we can choose $k$ so that $f^k(\rho)$ crosses more than $C$ edges in $H_r$ for each edge $E$ in $H_r$. Since $\rho = f^k(\rho)$ is obtained from $f^k(\alpha)f^k(\beta^{-1})$ by canceling edges at the juncture point and since no edges in $H_r$ are cancelled when $f^k(\alpha)$ and $f^k(\beta)$ are tightened to $f^k(\alpha)$ and $f^k(\beta)$, $\alpha \subset f^k(\alpha_0)$ and $\beta \subset f^k(\beta_0)$. In particular, we can compute an upper bound for the number of edges crossed by $\rho$, reducing us to testing a finite set of paths to decide which are indivisible Nielsen paths.

**Lemma 4.7.** Suppose that $f : G \to G$ and $\emptyset = G_0 \subset G_1 \subset \ldots \subset G_N = G$ are a relative train track map and filtration representing a rotationless $\phi$ and satisfying the conclusions of Theorem 2.4. Suppose also that $H_N$ is an EG stratum with attracting lamination $\Lambda^+$, and that $[G_u] \cup \Lambda^+$ fills where $G_u = G \setminus H_N$. Then the following are satisfied.

1. There is a unique subgroup system $A$ such that a conjugacy class $[a]$ is not weakly attracted to $\Lambda^+$ if and only if $[a]$ is carried by an element of $A$.

2. A circuit $\sigma \subset G$ represents an element of $A$ if and only if $\sigma$ splits as a concatenation of subpaths each of which is either contained in $G_u$ or is an indivisible Nielsen path of height $N$.

3. There is a proper free factor system $F' \supset [G_u]$ such that one of the following holds.

   a. $A = F'$

   b. $A$ fills and $A = F' \cup \{[A]\}$ where $[A]$ has rank one.

Moreover, $[G_u] \subseteq [G_N]$ is reduced with respect to $\phi$ if and only if $F' = [G_u]$.

**Remark 4.8.** $G_u$ is not necessarily a core subgraph. It deformation retracts to a core subgraph $G_s$ and is obtained from $G_s$ by adding NEG edges with terminal endpoints in $G_s$. We use $G_u$ in this lemma rather than $G_s$ because indivisible Nielsen paths of height $N$ can have endpoints at the valence one vertices of $G_u$. 19
Proof. The existence of $\mathcal{A}$ as in (1) follows from [BFH00, Theorem 6.1]; see also [HMe, Definitions 1.2]. Uniqueness of $\mathcal{A}$ follows from the fact [HMe, Proposition 1.4(1)] that $\mathcal{A}$ is a vertex group subsystem and is hence [HMc, Lemma 3.1] determined by the conjugacy classes that it carries. For the rest of this proof we take (1) to be the defining property of $\mathcal{A}$. Note that $\mathcal{A}$ depends only on $\Lambda^+$ and $\phi$ and not on the choice of $f : G \to G$. In particular, $\mathcal{A}$ is $\phi$-invariant.

If a circuit $\sigma \subseteq G$ splits into subpaths that are either contained in $G_u$ or are Nielsen paths of height $N$ then the number of $H_N$ edges in $f^k_\#(\sigma)$ is independent of $k$ and $\sigma$ is not weakly attracted to $\Lambda^+$. This proves the if direction of (2).

The only if direction of (2) is more work. [BFH00, Lemmas 4.2.6 and 2.5.1] and Lemma 4.5 imply that there exists $k \geq 1$ such that $f^k_\#(\sigma)$ splits into subpaths that are either contained in $G_{N-1}$, are indivisible Nielsen paths of height $N$ or are edges of height $N$. Assuming that $\sigma$, and hence $f^k_\#(\sigma)$, is not weakly attracted to $\Lambda^+$, [BFH00, Corollary 4.2.4] implies that no term in this splitting is an edge of height $N$. If $f^k_\#(\sigma) \subseteq G_u$ then $\sigma \subseteq G_u$ and we are done. If $f^k_\#(\sigma)$ is a closed Nielsen path of height $N$ then $\sigma$ and $f^k_\#(\sigma)$ have the same $f^k_\#$-image and so are equal. In particular, $\sigma$ is a Nielsen path of height $N$. In the remaining case, there is a splitting

$$f^k_\#(\sigma) = \mu_1 \cdot \nu_1 \cdot \mu_2 \cdot \nu_2 \cdots \mu_m \cdot \nu_m$$

into subpaths $\mu_i \subseteq G_u$ and Nielsen paths $\nu_i$ of height $N$. Since the endpoints of each $\mu_i$ are fixed by $f$ and since the restriction of $f$ to each $f$-invariant component of $G_u$ is a homotopy equivalence, there exist paths $\mu'_i \subseteq G_u$ with the same endpoints as $\mu_i$ such that $f^k_\#(\mu'_i) = \mu_i$. Letting

$$\sigma' = \mu'_1 \cdot \nu_1 \cdot \mu'_2 \cdot \nu_2 \cdots \mu'_m \cdot \nu_m$$

we have $f^k_\#(\sigma') = f^k_\#(\sigma)$ and hence $\sigma' = \sigma$. In particular, $\sigma$ splits into subpaths of $G_u$ and indivisible Nielsen paths of height $N$. This completes the proof of (2).

The main statement of (3) follows from [HMe, Remark: The case of a top stratum in Section 1.1] (which applies because there is a CT representing $\phi$ in which $\Lambda^+$ corresponds to the highest stratum and $[G_u]$ is realized by a core filtration element). Since $\phi$ preserves $\mathcal{A}$, it acts periodically on the components of $\mathcal{A}$. [HMc, Theorem 4.1] therefore implies that $\phi$ preserves each rank one component of $\mathcal{A}$ and so also preserves $\mathcal{F}'$. If $\mathcal{F}' \neq [G_u]$ then $\mathcal{F}'$ is a reduction for $[G_u] \subseteq [G_N]$. This proves the only if direction of the moreover statement.

Suppose then that $\mathcal{F}' = [G_u]$. If either (a) or (b) holds then any free factor system $\mathcal{F}$ that properly contains $[G_u]$ carries a conjugacy class not carried by $\mathcal{A}$. Item (1) implies that $\mathcal{F}$ carries a conjugacy class that is weakly attracted to $\Lambda^+$. If $\mathcal{F}$ is $\phi$-invariant then $\mathcal{F}$ carries $\Lambda^+$ in addition to containing $G_u$ and so is improper. This completes the proof of the if direction of the moreover statement.

The next proposition shows how to reduce a relative train track map satisfying the conclusions of Theorem 2.4. One way to create an unreduced example is to identify
a pair of distinct fixed points in a stratum $H_r$ of a CT where $H_r$ is both highest and EG.

**Proposition 4.9.** Suppose that $f : G \to G$ and $\emptyset = G_0 \subset G_1 \subset \ldots \subset G_N = G$ are a relative train track map and filtration representing a rotationless $\phi$ and satisfying the conclusions of Theorem 2.4. Suppose also that $H_r$ is an EG stratum and that $G_s$ is the highest core filtration element below $G_r$. Then there is an algorithm to decide if $[G_s] \sqsubset [G_r]$ is reduced and if it is not to find a reduction.

**Proof.** By [FH11, Lemma 3.30], the non-contractible components of $G_r$ are $f$-invariant; in particular $H_r$ is contained in a single component of $G_r$. By restricting to this component we may assume that $H_r$ is the top stratum and hence that $r = N$.

Let $\Lambda^+ \in \mathcal{L}(\phi)$ be the lamination associated to $H_N$. By [BFH00, Lemma 3.2.4] there exists $\Lambda^- \in \mathcal{L}(\phi^{-1})$ such that the smallest free factor system that carries $\Lambda^+$ is the same as the smallest free factor system that carries $\Lambda^-$ and we denote this by $\mathcal{F}_\Lambda$. It follows that the realizations of $\Lambda^+$ and $\Lambda^-$ in any marked graph cross the same set of edges in that graph. It also follows that the smallest free factor system that carries $[G_s]$ and $\Lambda^+$ is the same as the smallest free factor system that carries $[G_s]$ and $\Lambda^-$ and we denote this by $\mathcal{F}_{s,\Lambda}$. Since both $[G_s]$ and $\Lambda^\pm$ are $\phi$-invariant, [BFH00, Corollary 2.6.5] implies that $\mathcal{F}_{s,\Lambda}$ is $\phi$-invariant.

Choose constants as follows.

- $C_E = 6(n - 1)$ is the maximal number of oriented natural edges in a marked graph of rank $n$.
- $M = 2n$; if $\mathcal{F}_0 \sqsubset \mathcal{F}_1 \sqsubset \ldots \sqsubset \mathcal{F}_p$ is an increasing nested sequence of free factor systems of $F_n$ then $p \leq M - 1$. (This follows by induction on the rank $n$ and the observation that if $\mathcal{F}_p = \{F_n\}$ with $n > 1$ then there exists $\mathcal{F}$ such that $\mathcal{F}_{p-1} \sqsubset \mathcal{F} \sqsubset \mathcal{F}_p$ and $\mathcal{F}$ consists of a pair of conjugacy classes whose ranks add to $n - 1$.
- $C_0 = C + 2$ where $C$ satisfies the conclusion of Lemma 4.4.

**Step 1: An existence result when $\mathcal{F}_{s,\Lambda}$ is proper:** Consider the set $\mathcal{K}$ of marked graphs $K$ containing a (possibly empty) core subgraph $K_0$ and equipped with a marking preserving homotopy equivalence $p : K \to G$ such that

- (a) $p \mid K_0 : K_0 \to G_s$ is a homeomorphism (If $s = 0$ then $G_s = K_0 = \emptyset$).
- (b) the restriction of $p$ to each natural edge is either an immersion or constant.
- (c) there is at most one natural edge on which $p$ is constant.

Each natural edge $E$ of $K \in \mathcal{K}$ is labeled by $p(E)$, thought of as a (possibly trivial) edge path in $G$. We do not distinguish between two elements of $\mathcal{K}$ if there is a label
preserving homeomorphism between them. The length \(|E|\) of a natural edge \(E\) is the number of edges in \(p(E)\) and the total length \(|K|\) of \(K\) is the sum of the length of its natural edges. The number of \(H_N\) edges in \(p(E)\) is denoted \(|E|_N\). Let \(\Lambda^+_K\) and \(\Lambda^-_K\) be the realizations of \(\Lambda^+\) and \(\Lambda^-\) in \(K\). As observed above, the set of edges crossed by \(\Lambda^+\) is the same as the set of edges crossed by \(\Lambda^-\). We denote this common core subgraph by \(K_{\Lambda}\).

We claim that if \(F_{s,\Lambda}\) is proper then there exists an element \(K \in \mathcal{K}\) with the following properties.

1. \(|E|_N \leq C_0\) for all natural edges \(E\) of \(K\).
2. The restriction of \(p\) to a leaf of \(\Lambda^+_K\) is an immersion.
3. \(K_0 \cup K_{\Lambda}\) is a proper core subgraph.

Note that \(K_0 \cup K_{\Lambda}\) is a core subgraph because it is a union of core subgraphs so the content of (3) is that \(K_0 \cup K_{\Lambda}\) is proper. Our proof of the claim makes use of an idea from the proof of Proposition 3.4 of [HMD].

Assuming that \(F_{s,\Lambda}\) is proper, there exists a marked graph \(K\) with proper core subgraphs \(K_0 \subset K_1 \subset K\) and a marking preserving homotopy equivalence \(p : K \to G\) satisfying (a) and (b) and \([K_1] = F_{s,\Lambda}\). The last property implies that \(K_1 = K_0 \cup K_{\Lambda}\) so (3) is satisfied. If there is at least one natural edge in \(K_1\) on which \(p\) is an immersion then collapse each natural edge in \(K\) on which \(p\) is constant to a point. Otherwise, collapse each natural edge in \(K_1\) on which \(p\) is constant and all but one natural edge in \(G \setminus K_1\) on which \(p\) is constant to a point. The resulting marked graph, which we continue to denote \(K\), still satisfies (a) and (b) because \(p|K_0\) is injective and so \(K_0\) is unaffected by the collapsing and (c) is now satisfied so \(K \in \mathcal{K}\). Replacing \(K_1\) by its image under the collapse, it is still true that \(K_1 = K_0 \cup K_{\Lambda}\) is a proper core subgraph and now the restriction of \(p\) to each natural edge of \(K_1\) is an immersion.

(1) If \(p\) is not an immersion then there is a pair of natural edges \(E_1, E_2\) in \(K_1\) with the same initial vertex and such that the edge paths \(p(E_1)\) and \(p(E_2)\) have the same first edge, say \(e\). Folding the initial segments of \(E_1\) and \(E_2\) that map to \(e\) produces an element \(K' \in \mathcal{K}\) with subgraph \(K'_0\) satisfying (a) and such that \(|K'| < |K|\). Note that (3) is still satisfied because \(K'_0 \cup K'_{\Lambda}\) is contained in the image \(K'_1 \subset K'\) and \([K_1] = [K'_1]\). Replacing \(K\) with \(K'\) and repeating this finitely many times, we may assume that \(p|K_1\) is an immersion and hence that \(p\) restricts to an immersion on leaves of \(\Lambda^+_K\) and \(\Lambda^-_K\). In particular, (2) is satisfied. If (1) is satisfied then the proof of the claim is complete.

Suppose then that there is a natural edge \(E\) of \(K\) such that \(|E|_r > C_0 = C + 2\). By [BFH00, Theorem 6.0.1], a leaf of \(\Lambda^-\) is not weakly attracted to a generic leaf of \(\Lambda^+\). Lemma 4.4 therefore implies that at least one of the two laminations \(\Lambda^+_K\) and \(\Lambda^-_K\) does not cross \(E\). It follows that \(F_{\Lambda} \subset [K \setminus E]\) and hence that \(E\) is contained in the complement of \(K_{\Lambda}\). Since \(E\) is obviously in the complement of \(K_0\), we have that \(E\)
is contained in the complement of $K_0 \cup K_\Lambda$. If there is a natural edge on which $p$ is constant, it must be in the complement of $K_0 \cup K_\Lambda \cup E$ and we collapse it to a point. As above, the resulting marked graph is still in $\mathcal{K}$ and (2) and (3) are still satisfied. We may now assume that $p$ is an immersion on each natural edge of $K$. The map $p : K \to G$ is not a homeomorphism so there is at least one pair of edges that can be folded. Perform the fold and carry the $K_0 \subset K$ notation to the new marked graph. It is still true that $K \in \mathcal{K}$ and that (2) holds. Folding reduces $|E|_r$ by at most 2 so there is still an edge with $|E|_r > C$. Arguing as above we see that $E$ is contained in the complement of $K_0 \cup K_\Lambda \cup E$ and we collapse it to a point. As above, the resulting marked graph is still in $\mathcal{K}$ and (2) and (3) are still satisfied.

We may now assume that $p$ is an immersion on each natural edge of $K$. The map $p : K \to G$ is not a homeomorphism so there is at least one pair of edges that can be folded. Perform the fold and carry the $K_0 \subset K$ notation to the new marked graph. It is still true that $K \in \mathcal{K}$ and that (2) holds. Folding reduces $|E|_r$ by at most 2 so there is still an edge with $|E|_r > C$. Arguing as above we see that $E$ is contained in the complement of $K_0 \cup K_\Lambda$ so (3) is still satisfied. If (1) is satisfied then the proof of the claim is complete. Otherwise perform another fold. Conditions (2) and (3) are satisfied so check condition (1) again. Folding reduces $|K|$ so after finitely many folds, (1) is satisfied and the claim is proved.

**Step 2: Part 1 of the algorithm** : In this step we present an algorithm that either finds a reduction for $[G_s] \sqsubseteq [G_r]$ or concludes that $F_s, \Lambda$ is improper, i.e. $\mathcal{F}_{s,\Lambda} = \{[F_n]\}$. In the former case we are done. In the latter case we move on to the second part of the algorithm in which we either find a reduction for $[G_s] \sqsubseteq [G_r]$ or we conclude that $[G_s] \sqsubseteq [G_r]$ is irreducible.

(A1) Choose a generic leaf $\gamma \subset G$ of $\Lambda^+$. One way to do this is to choose an edge $e$ of $H_r$ and $k > 0$ so that at least one occurrence of $e$ in the edge path $f^k(e)$ is neither the first nor last $H_r$ edge. Then $e \subset f^k(e) \subset f^{2k}(e) \subset \ldots$ and the union of these paths is a generic leaf of $\Lambda^+$ by [BFH00, Corollary 3.1.11 and Lemma 3.1.15].

(A2) Let $C_1 = 1$. Choose a subpath $\gamma_1$ of $\gamma$ that crosses at least

$$(C_E + 1)C_0 + C_E(C_1 + C_0) + 2C_0$$

edges in $H_r$ and let $L_1$ be the number of edges in $\gamma_1$. (The choice of these constants will be clarified in step 3.)

(A3) Enumerate all core graphs $J$ of rank $< n$ satisfying: there is a core subgraph $J_0 \subset J$ and a map $p : J \to G$ such that the restriction of $p$ to each natural edge is an immersion onto a path of length at most $L_1$ and such that $p \mid J_0 : J_0 \to G_s$ is a homeomorphism. Label the natural edges of $J$ by their $p$ images. We do not distinguish between labeled graphs that differ by a label preserving homeomorphism so there are only finitely many $J$ and we consider them one at a time. If $\sigma \subset J$ is a path with endpoints at natural vertices and if $p$ restricts to an immersion on $\sigma$ then we let $|\sigma|$ be the number of edges crossed by $p(\sigma)$ and $|\sigma|_r$ the number of $H_r$ edges crossed by $p(\sigma)$.

Let $p_\#$ be the homomorphism induced by $p : J \to G$ on fundamental groups. By Lemma 4.2 we can decide if $p_\#$ is an isomorphism to a free factor system $[J]$ of $F_n$. If not, then move on to the next candidate. If yes, then apply Lemma 4.3

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with \( \mathcal{F}_0 = [J_0] = [G_s] \) and \( \mathcal{F}_1 = [J] \). If this produces a \( \phi \)-invariant \( \mathcal{F} \subset [J] \)
that properly contains \([G_s]\) then we have found a reduction and the algorithm stops. Otherwise we know that \([J]\) does not contain such an \( \mathcal{F} \). In particular 
\([J]\) does not contain \( \mathcal{F}_{s,\Lambda} \) and so does not carry \( \Lambda^+ \). Choose a finite subpath 
\( \gamma_{1,J} \subset \gamma \subset G \) that does not lift to \( J \).

By [BFH00, Lemma 3.1.10(4)] there exists an edge \( E \) in \( H_r \) and \( k \geq 1 \) such that 
\( \gamma_{1,J} \) is a subpath of \( \tilde{f}_\#^k(E) \). By [BFH00, Lemma 3.1.8(3)] there is a computable 
\( k_0 \) so that for any edge \( E' \) in \( H_r \) and any \( l \geq k + k_0 \), \( \tilde{f}_\#^k(E) \), and hence \( \gamma_{1,J} \),
is a subpath of \( f_\#^l(E') \). Finally, by [BFH00, Lemma 3.1.10(3)] there exists
computable \( C_{2,J} > 0 \) so that if \( \sigma \) is a subpath of \( \gamma \) that crosses \( \geq C_{2,J} \) edges of 
\( H_r \) then \( \sigma \) contains some \( f_\#^l(E') \), and hence contains \( \gamma_{1,J} \), as a subpath and so
does not lift into \( J \). Now move on to the next candidate.

At the end of the process we have either found a reduction and the algorithm stops or we have found a constant 
\( C_2 = \max\{C_{2,J}\} \) so that if \( \sigma \) is a subpath of \( \gamma \) that crosses at least \( C_2 \) edges of \( H_r \) then \( \sigma \) does not lift into any \( J \). Choose a subpath \( \gamma_2 \) of \( \gamma \) that crosses at least
\[(C_E + 1)C_0 + C_E(C_2 + C_0) + 2C_0\]
edges of \( H_r \) and let \( L_2 \) be the number of edges crossed by \( \gamma_2 \).

**(A4)** Repeat (A3) replacing \( L_1 \) with \( L_2 \). At the end of the process we have either found a reduction and the algorithm stops or we have found a constant \( C_3 \) so
that if \( \sigma \) is a subpath of \( \gamma \) that crosses at least \( C_3 \) edges of \( H_r \) then \( \sigma \) does not lift into any \( J \). Choose a subpath \( \gamma_3 \) of \( \gamma \) that crosses at least
\[(C_E + 1)C_0 + C_E(C_3 + C_0) + 2C_0\]
edges of \( H_r \) and let \( L_3 \) be the number of edges crossed by \( \gamma_3 \).

**(A5)** Iterate this process up to \( M \) times. If after \( M \) iterations the algorithm has not
found a reduction and stopped, then stop and conclude that \( \mathcal{F}_{s,\Lambda} = \{[F_n]\} \).

**Step 3: Justifying part 1 of the algorithm:** In this step we verify that if \( \mathcal{F}_{s,\Lambda} \) is
proper then the above algorithm finds a reduction in at most \( M \) steps.

Choose any \( K \in \mathcal{K} \) satisfying (1) - (3) and let \( K(L_1) \) be the subgraph of \( K \)
consisting of natural edges \( E \) with \(|E| \leq L_1 \). Lift the path \( \gamma_1 \) into \( K_\Lambda \). After
removing initial and terminal subpaths contained in single natural edges of \( K_\Lambda \), we
have a natural (in \( K \)) edge path \( \gamma_{1,K} \subset \Lambda_\Lambda^+ \subset K_\Lambda \) that projects onto all of \( \gamma_1 \) except
perhaps initial and terminal segments that cross at most \( C_0 \) edges of \( H_r \). Thus
\( \gamma_{1,K} \subset K(L_1) \) and 
\[ |\gamma_{1,K}|_r \geq (C_E + 1)C_0 + C_E(C_1 + C_0) \]
Combining this inequality with (1), we see that $\gamma_{1,K}$ has a subpath that decomposes as

$$\alpha_1\beta_1\alpha_2 \ldots \beta_{C_E}\alpha_{C_E+1}$$

where each $\alpha_i$ is a single natural edge and each $\beta_i$ is a natural subpath whose image under $p$ crosses at least $C_1$ edges in $H_r$. Since there are at most $C_E$ oriented natural edges in $K$, it follows that $\gamma_{1,K}$ contains a natural subpath that begins and ends with the same oriented natural edge and whose $p$-image crosses at least $C_1 = 1$ edges in $H_r$. This proves that there is a circuit in $K(L_1)$ that is not in $K_0$ and hence that $[K(L_1)]$ properly contains $[K_0]$. By (3), $K_0 \cup K_0$ is proper and so if $K_0 \cup K_0 \subset K(L_1)$ then $K_0 \cup K_0$ occurs as a $J$ in the first iteration of the process and Lemma 4.3 finds a reduction in the first iteration of (A3) because $F_{s,A} \subset [K_0 \cup K_0]$.

If no reduction is found in the first iteration then proceed to the second iteration as described in (A4). By the same reasoning, there is a natural edge path $\gamma_{2,K} \subset K(L_2)$ such that

$$|\gamma_{2,K}|_r \geq (C_E + 1)C_0 + C_E(C_2 + C_0)$$

and there is a subpath of $\gamma_{1,K}$ that decomposes as

$$\alpha_1\beta_1\alpha_2 \ldots \beta_{C_E}\alpha_{C_E+1}$$

where each $\alpha_i$ is a single natural edge and each $\beta_i$ is a natural subpath whose image under $p$ crosses at least $C_2$ edges in $H_r$. It follows that there is a circuit in $K(L_2)$ that is not in $K(L_1)$ and hence that $[K(L_2)]$ properly contains $[K(L_1)]$. If $K_0 \cup K_0 \subset K(L_2)$ then $K_0 \cup K_0$ occurs as a $J$ in the second iteration of the process and our algorithm finds a reduction.

Continuing on, the iteration either produces a reduction within $M$ steps or produces a properly nested sequence

$$[K_0] \subset [K(L_1)] \subset [K(L_2)] \subset \ldots \subset [K(L_M)]$$

of free factors. Since the latter contradicts the definition of $M$, a reduction must have been found.

**Step 4: Part 2 of the algorithm:** In this part of the algorithm we assume that $F_{s,A}$ is improper. The filtration element $G_u = G_N \setminus H_N^2$ deformation retracts to $G_s$, see Remark 4.8. In particular, $[G_u] = [G_s]$.

Given a circuit $\sigma \subset G$ [resp. a subgroup $A$] let $F_{u,\sigma}$ [resp. $F_{u,A}$] be the smallest free factor system that carries $[\sigma]$ [resp. every conjugacy class in $[A]$] and every conjugacy class in $[G_u]$. By Lemma 4.2, $F_{u,\sigma}$ and $F_{u,A}$ can be algorithmically determined.

By Lemma 4.5, the set $P$ of indivisible periodic Nielsen paths with height $r$ is finite, can be determined algorithmically and each element of $P$ has period one. For notational convenience we assume that $P$ is closed under orientation reversal. Let $\Sigma$ be the set of circuits in $G$ that split into a concatenation of paths in $G_u$ and elements
of $P$. Lemma 4.7 implies that $\Sigma$ is the set of circuits that are not weakly attracted to $\Lambda^+$ and that the conjugacy classes determined by $\Sigma$ are exactly those carried by the subgroup system $A$ of that lemma.

We consider several cases, each of which can be checked by inspection of $G_u$ and the elements of $P$. If every circuit in $\Sigma$ is contained in $G_u$ then $A = [G_u]$ and $H_N$ is reduced by Lemma 4.7(3).

As a second case, suppose that there is a non-trivial path $\mu \subset G$ with endpoints $x, y \in G_u$ such that $\mu$ is homotopic rel endpoints to a concatenation of elements of $P$ and so is a Nielsen path of height $N$. Let $[B]$ be the conjugacy class of the subgroup of $F_n$ represented by closed paths based at $x$ that decompose as a concatenation of subpaths, each of which is either $\mu, \mu^{-1}$ or a path in $G_u$ with endpoints in $\{x, y\}$. Then $[B]$ is $\phi$-invariant, has rank $\geq 2$, and each conjugacy class in $[B]$ is represented by a circuit $\sigma \in \Sigma$. By Lemma 4.7(3) there is a proper free factor system that carries $[G_u]$ and the conjugacy class of each element of $B$. It follows that $F_{u,B}$ is proper. Since both $[G_u]$ and $[B]$ are $\phi$-invariant, $F_{u,B}$ is $\phi$-invariant by [BFH00, Corollary 2.6.5] and we have found a reduction of $H_N$.

The final case is that there are no paths $\mu \subset G$ as in the second case and there is at least one element $\sigma \in \Sigma$ that is not contained in $G_u$. Each such $\sigma$ is homotopic to a concatenation of elements of $P$. In particular, the conjugacy class determined by $\sigma$ is $\phi$-invariant and so $F_{u,\sigma}$ is $\phi$-invariant. Choose one such $\sigma$ and check if $F_{u,\sigma}$ is proper. If it is then we have found a reduction of $H_N$ and we are done so suppose that it is not. Lemma 4.7(3) implies that $[\sigma]$ is carried by a rank one component $[A]$ of $A$ and that $A = F' \cup [A]$ for some $\phi$-invariant free factor system $F'$. If there exists an element $\sigma' \in \Sigma$ that is not carried by $[G_u]$ and such that $\sigma$ and $\sigma'$ are not multiples of the same root free circuit then $[\sigma']$ is not carried by $[A]$ so $F_{u,\sigma'}$ is a reduction. Otherwise, $F' = [G_u]$ and $H_r$ is reduced. This completes the second step in the algorithm and so also the proof of the proposition. \hfill \Box

### 4.2 The NEG case

We now consider reducibility for NEG strata, beginning with a pair of examples.

**Example 4.10.** Suppose that $f : G \to G$ is a homotopy equivalence with filtration $\emptyset = G_0 \subset G_1 \subset \ldots \subset G_N = G$ representing $\phi \in \text{Out}(F_n)$ and that $G_{r+2} = G_r \cup E_{r+1} \cup E_{r+2}$ where $G_r$ is a connected core subgraph and where $H_{r+1} = E_{r+1}$ and $H_{r+2} = E_{r+2}$ are oriented edges with a common initial vertex not in $G_r$ and a common terminal endpoint in $G_r$. Suppose also that $f(E_{r+1}) = E_{r+1}u$ and $f(E_{r+2}) = E_{r+2}u$ for some closed non-trivial path $u \subset G_r$ and that $f \mid G_r$ is a CT. Then the (NEG Nielsen Paths) property of $f \mid G_{r+2}$ fails. The CT algorithm corrects this (see Lemma 5.5) by discovering that $E_{r+2}E_{r+1}$ is a Nielsen path and then sliding the terminal endpoint of $E_{r+2}$ along $E_{r+1}$. In other words, $E_{r+2}$ is replaced by a fixed loop $E_{r+2}'$ based at the initial endpoint of $E_{r+1}$. Note that while establishing (NEG Nielsen Paths) for
f \mid G_r$, we have discovered a reduction of $[G_r] \sqsubset [G_{r+2}]$. Namely, the $\phi$-invariant free factor system $\{[G_r],[E_{r+2}']\}$ is properly contained between $[G_{r+1}] = [G_r]$ and $[G_{r+2}]$.

**Example 4.11.** Suppose that $f : G \to G$ is a homotopy equivalence with filtration $\emptyset = G_0 \subset G_1 \subset \ldots \subset G_N = G$ representing $\phi \in \text{Out}(F_n)$, that $G_s$ is a core subgraph, that $H_s = \{E_s\}$ is an NEG stratum, that $C$ is the component of $G_s$ that contains $H_s$ and that $f \mid C$ satisfies all the properties of a CT except that $[G_r] \sqsubset [G_{r+1}]$ is not reduced. The CT algorithm corrects this in stages. First, a Nielsen path $\sigma \subset G_r$ connecting $y$ to $x$ is found. Then the terminal end of $E_{r+1}$ is slid along $\sigma$ so that its new terminal endpoint is $x$. Thus $E_{r+1}$ is replaced by a fixed edge $E_{r+1}'$ with both endpoints at $x$. Finally, $x$ is blown up to a fixed edge $E_{r+2}'$ with $E_{r+1}'$ attached at the ‘new’ endpoint of $E_{r+2}'$ and the remaining edges in the link of $x$ still attached at $x$. Both $[G_r] \sqsubset [G_{r+1}]$ and $[G_{r+1}] \sqsubset [G_{r+2}]$ are reduced.

**Lemma 4.12.** Suppose that $f : G \to G$ is a homotopy equivalence with filtration $\emptyset = G_0 \subset G_1 \subset \ldots \subset G_N = G$ representing $\phi \in \text{Out}(F_n)$, that $G_s$ is a core subgraph, that $H_s = \{E_s\}$ is an NEG stratum, that $C$ is the component of $G_s$ that contains $H_s$ and that $f \mid C$ satisfies all the properties of a CT except that $[C \setminus E_s] \sqsubset [C]$ is reducible. Then $E_s$ is fixed and its terminal endpoint is connected to its initial endpoint by a Nielsen path $\beta \subset C \setminus E_s$. In particular, $E_s\beta$ is a basis element and $\{[G_{s-1}], [E_s\beta]\}$ is a $\phi$-invariant free factor system that is properly contained between $[G_{s-1}]$ and $[G_s]$.

**Proof.** By restricting to $C$, we may assume that $G = C = G_s$ and hence that $G_{s-1}$ has either a single component of rank $n-1$ or two components whose ranks add to $n$. Since $[C \setminus E_s] \sqsubset [C]$ is reducible, there is a marked graph $K$ with distinct proper core subgraphs $K_1 \subset K_2 \subset K$ such that $[K_1] = [G_{s-1}]$ and such that $[K_2]$ is $\phi$-invariant. From 

$$-\chi(K_1) \leq -\chi(K_2) < -\chi(K) = -\chi(K_1) + 1$$

it follows that $K_2$ is obtained from $K_1$ by adding a disjoint loop $\alpha$ and that $K \setminus K_2$ is an edge $E$. In particular, $K_1$ is connected.

We claim that the circuit $\sigma \subset G$ representing $[\alpha]$ crosses $E_s$ exactly once. The marked graph $K'$ obtained from $K$ by collapsing the components of a maximal forest is a rose with $\alpha$ as one of its edges. The marked graph $G'$ obtained from $G$ by collapsing the components of a maximal forest in $G_{s-1}$ is a rose with $E_s$ as one of its edges. Moreover $[K' \setminus \alpha] = [K_1] = [G_{s-1}] = [G' \setminus E_s]$. Let $h : K' \to G'$ be a homotopy equivalence that respects markings and that restricts to an immersion on each edge.

Then $h(K' \setminus \alpha) = G' \setminus E_s$ and it suffices to show that $h(\alpha)$ crosses $E_s$ exactly once. This follows from [BFH00, Corollary 3.2.2].

Having verified the claim, we can now complete the proof of the lemma. Since $[K_2]$ is $\phi$-invariant and $\alpha$ is a component of $K_2$, $\alpha$ determines a $\phi$-invariant conjugacy class. It follows that $\sigma$ decomposes as a concatenation of indivisible Nielsen paths.
and fixed edges. Since $\sigma$ crosses $E_s$ exactly once, the (NEG Nielsen Paths) property of $f : G \rightarrow G$ implies that $E_s$ is a fixed edge. Thus $\sigma$ decomposes as a circuit into $E_s \beta$ where $\beta$ is a Nielsen path in $G_{s-1}$. 

5 Sliding NEG edges

The key step for arranging that an NEG edge has good properties under iteration is to slide the terminal endpoint of the edge into an optimal position in the lower filtration element that contains it. This is carried out in [BFH00, Proposition 5.4.3]. In this section we make the algorithmic arguments needed to replace the non-algorithmic parts of the original proof.

5.1 Completely Split Rays

Recall from [FH11, Definition 4.4] that a splitting $\sigma = \sigma_1 \cdot \sigma_2 \ldots$ is a complete splitting if each $\sigma_i$ is either a single edge in an irreducible stratum, an indivisible Nielsen path, an exceptional path or a maximal subpath in a zero stratum (with some additional features that we will not recall here.) A finite path or circuit has at most one complete splitting. The first item of our next lemma states that the same is true for rays.

Recall from Section 2.1 that a path $\sigma$ is thought of as an edge path $\sigma = E_0 E_1 \ldots$. The initial and terminal endpoints, $w_i$ and $w_{i+1}$ of $E_i$ are the vertices of $\sigma$. We view the set $W$ of vertices of a path as being ordered by their subscripts. A decomposition of $\sigma$ into subpaths is specified by a subset of $W$; if $w_i$ and $w_j$ are consecutive elements of the subset then $E_i \ldots E_{j-1}$ is a term in the decomposition. If the decomposition is a splitting then we refer to these vertices as splitting vertices. A similar definition holds for circuits.

Lemma 5.1. Suppose that $f : G \rightarrow G$ is a CT, that $R \subset G$ is a completely split ray and that $R_0$ is a subray of $R$ that has a complete splitting. Then

(1) The complete splitting of $R$ is unique.

(2) Let $v$ be the first splitting vertex for $R$ that is contained in $R_0$ (when the edge path $R_0$ is viewed as a subpath of the edge path $R$). Then each splitting vertex $w$ for $R_0$ that comes after $v$ (in the ordering of splitting vertices of $R_0$) is a splitting vertex for $R$.

Proof. The second item implies the first by taking $R_0 = R$ so we need only prove the second. Let $\mu_0$ be the term in the complete splitting of $R_0$ whose initial vertex is $w$. If $\mu_0$ is either an indivisible Nielsen path or an exceptional path then the interior of $\mu_0$ is an increasing union of pre-trivial paths by [FH11, Remark 4.2 and Lemma 2.11(2)] and so by [FH11, Lemma 4.11(2)] is contained in a single term $\mu$ of the complete splitting of $R$. Obviously $\mu$ is not a single edge and is not contained in a zero stratum.
so it must be either an indivisible Nielsen path or an exceptional path. Since \( v \) is the initial endpoint of some term in the complete splitting of \( R \) and \( w \) comes after \( v \), it follows that \( v \) is not contained in the interior of \( \mu \) and so \( \mu \subset R_0 \). The symmetric argument therefore applies to show that \( \mu \) is contained in a term of the complete splitting of \( R_0 \) and hence that \( \mu = \mu_0 \) as desired. If \( \mu_0 \) is either a single edge or is a maximal subpath in a zero stratum and \( \mu_0 \) is not a term in the complete splitting of \( R \) then \( \mu_0 \) is properly contained in a term \( \mu \) of \( R \) that is an indivisible Nielsen path or an exceptional path. But this violates the hard splitting property [FH11, Lemma 4.11(2)] for the complete splitting of \( R_0 \) (applied to its finite completely split subpaths) and the fact that the interior of \( \mu \) is the increasing union of pre-trivial paths. Thus \( \mu_0 \) is a term in the complete splitting of \( R \) and we are done. 

\[ \text{Definitions 5.2.} \] Suppose that \( f : G \to G \) is a CT, that \( x \in G \), that \( \sigma \subset G \) is a non-trivial completely split path connecting \( x \) to \( f(x) \) and that the turn at \( f(x) \) determined by \( \bar{\sigma} \) and \( f_\#(\sigma) \) is legal. The ray \( R = \sigma \cdot f_\#(\sigma) \cdot f_\#^2(\sigma) \cdot \ldots \) satisfies \( f_\#(R) \subset R \) and the given splitting of \( R \) has a refinement that is a complete splitting by [FH11, Lemma 4.11]; we say that \( R \) is generated by \( \sigma \).

If \( E \) is a non-fixed edge of \( G \) whose initial direction is fixed then \( f(E) = E \cdot \sigma \) for some \( \sigma \) as above. The ray \( R_E = E \cdot \sigma \cdot f_\#(\sigma) \cdot f_\#^2(\sigma) \cdot \ldots \) is the eigenray determined by \( E \). Note that we are not requiring that the initial vertex of \( f \) be principal (as we did in Section 3.2) or that \( E \) is non-linear and NEG so we are using the term eigenray a little more generally than is sometimes the case. We will need this inclusiveness in the proof of Lemma 5.5. For the same reason we assume that each isolated fixed point for \( f \) is a vertex.

\[ \text{Lemma 5.3.} \] Suppose that \( f : G \to G \) is a CT and that \( \sigma \) and \( \sigma' \) are completely split non-Nielsen paths generating rays \( R \) and \( R' \) respectively. Then there is an algorithm to decide if the rays \( R \) and \( R' \) have a common terminal subray and if so to find initial subpaths \( \tau \subset R \) and \( \tau' \subset R' \) that terminate at splitting vertices of \( R \) and \( R' \) respectively and whose complementary terminal subrays are equal. Equivalently we find splitting vertices \( v \in R \) and \( v' \in R' \) such that terminal subrays of \( R \) and \( R' \) initiating at \( v \) and \( v' \) are equal (as edge paths).

\[ \text{Proof.} \] Let \( \mathcal{V} = \{ v_0, v_1, \ldots \} \) be the set of splitting vertices for \( R \) ordered so that \( v_{i-1} \) and \( v_i \) are the endpoints of the \( i \)th term in the complete splitting. By construction, \( f(\mathcal{V}) \subset \mathcal{V} \). For each \( i \geq 0 \), let \( \sigma_i \subset R \) be the path connecting \( v_i \) to \( f(v_i) \) and let \( \ell_i = |\sigma_i| \) be the number of edges in \( \sigma_i \); in particular, \( \sigma = \sigma_0 \). Note that \( \sigma_i \) generates the terminal subray of \( R \) that begins with \( v_i \). Define \( V', \sigma'_j \) and \( \ell'_j \) similarly using \( \sigma' \) and \( R' \) in place of \( \sigma \) and \( R \). It is obvious that \( R \) and \( R' \) have a common terminal subray if \( \sigma_i = \sigma'_j \) for some \( i \) and \( j \). (Namely, the subrays of \( R \) and \( R' \) initiating at \( v_i \in \mathcal{V}' \) and \( v'_j \in \mathcal{V}' \) respectively.) The converse follows from Lemma 5.1. Our goal then is to either find \( i \) and \( j \) such that \( \sigma_i = \sigma'_j \) or to conclude that no such \( i \) and \( j \) exist.
Let $r$ [resp $r'$] be the maximal height of a term in the complete splitting of $\sigma$ [resp. $\sigma'$] that is not a Nielsen path. Since $f(\sigma)$ and $\sigma$ have a common endpoint, $\sigma$ is not entirely contained in a zero stratum. Thus any term $\tau$ in the complete splitting of $\sigma$ that is contained in a zero stratum is adjacent to a term that intersects the EG stratum that envelopes $\tau$ (Notation 2.3). Since this adjacent edge has at least non-fixed endpoint, it is neither an exceptional path nor a Nielsen path so must be a single edge in that EG stratum. We conclude that $H_r$ is not a zero stratum. It follows that if $\mu$ is any height $r$ term in the complete splitting of $\sigma$ and if $\mu$ is not a Nielsen path then the length of $f_\#(\mu)$ goes to infinity with $i$; we say that $\mu$ is growing. Note that $r$ is the maximal height of a growing term in the complete splitting of any $\sigma_i$ and similarly for $r'$ and $\sigma'_j$. Note also that for any given $L > 0$ one can find, by inspection, $M > 0$ such that $\ell_i, \ell'_j > L$ for all $i, j \geq M$.

The first step in the algorithm is to check if $r = r'$. If yes, then move on to step two. If not then there do not exist $i$ and $j$ such that $\sigma_i = \sigma'_j$ so the algorithm stops and outputs NO.

We may now assume that $r = r'$. If $H_r$ is NEG define $K = 1$. Otherwise $H_r$ is EG and we choose $K$ so that for each edge $E$ of $H_r$, $f^K_\#(E)$ contains at least $C$ edges of $H_r$ where $C$ is the constant of Lemma 4.4. Now define $I$ to be the number of terms in the complete splitting of $\sigma \cdot f_\#(\sigma) \cdot \ldots \cdot f^K_\#(\sigma)$ or equivalently so that $i \leq I$ if and only if $\tilde{v}_i \in \tilde{\sigma} \cdot \tilde{f}_\#(\tilde{\sigma}) \cdot \ldots \cdot \tilde{f}^K_\#(\tilde{\sigma})$. Define $J$ to be the number of terms in the complete splitting of $\sigma' \cdot f_\#(\sigma') \cdot \ldots \cdot f^K_\#(\sigma')$ or equivalently so that $j \leq J$ if and only if $\tilde{v}'_j \in \tilde{\sigma}' \cdot \tilde{f}_\#(\tilde{\sigma}') \cdot \ldots \cdot \tilde{f}^K_\#(\tilde{\sigma}')$.

Fix $j$. To check if $\sigma_j' = \sigma_i$ for some $i$ we need only consider $i < M$ where $\ell_i > \ell'_j$ for all $i \geq M$. The symmetric argument implies that for any fixed $i$ we can check if $\sigma_i = \sigma'_j$ for some $j$. The second and final step of the algorithm is to decide if there exists $i \leq I$ such that $\sigma_i = \sigma'_j$ for some $j$ or if there exists $j \leq J$ such that $\sigma_i = \sigma'_j$ for some $i$. If not then the algorithm outputs NO. If yes then the algorithm outputs YES and $v_i, v'_j$.

It remains to prove that if $R$ and $R'$ have a common subray then the algorithm outputs YES in the second step. Suppose that $R = \rho R''$ and $R' = \rho' R''$ where $\rho \subset R$ and $\rho' \subset R'$ are finite and $R''$ is a maximal common subray. Lift $R = \rho R'' \subset G$ to $\tilde{R} = \tilde{\rho} \tilde{R}'' \subset \tilde{G}$ and the initial segment $\sigma \subset R$ to an initial segment $\tilde{\sigma} \subset \tilde{R}$. Let $\tilde{f} : \tilde{G} \to \tilde{G}$ be the lift of $f$ that takes the initial endpoint of $\tilde{\sigma}$ to the terminal endpoint of $\tilde{\sigma}$ and note that $\tilde{R} = \tilde{\sigma} \cdot \tilde{f}_\#(\tilde{\sigma}) \cdot \tilde{f}^2_\#(\tilde{\sigma}) \cdot \ldots$. Since $\sigma$ is not a Nielsen path, $|f^K_\#(\sigma)| \to \infty$. It follows that the terminal endpoint $P \in \partial F_\eta$ of $\tilde{R}$ is an attractor for the action of $\partial \tilde{f}$ and so is not fixed by any covering translation. In particular, $\tilde{f}$ is the only lift of $f$ that fixes $P$. Lift $R' = \rho' R''$ to $\tilde{R}' = \tilde{\rho}' \tilde{R}''$ and $\sigma'$ to an initial segment $\tilde{\sigma}'$ of $\tilde{R}'$. The uniqueness of $\tilde{f}$ implies that $\tilde{R}' = \tilde{\sigma}' \cdot \tilde{f}_\#(\tilde{\sigma}') \cdot \tilde{f}^2_\#(\tilde{\sigma}') \cdot \ldots$. Let $\tilde{E}''$ be the first height $r$ edge crossed by $\tilde{R}''$. There exist unique $k, k' \geq 0$ such that $\tilde{E}''$ is crossed by $\tilde{f}_\#^k(\tilde{\sigma})$ and by $\tilde{f}_{k'}^k(\tilde{\sigma}')$. We may assume without loss that $k' \geq k$. By Lemma 5.1, it suffices to show that $k \leq K$ or equivalently that $\rho \subset \tilde{\sigma} \cdot \tilde{f}_\#(\tilde{\sigma}) \cdot \ldots \cdot \tilde{f}^K(\tilde{\sigma})$. 

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If \( H_r \) is NEG then by the basic splitting property of NEG edges [BFH00, Lemma 4.1.4] there is a unique height \( r \) edge whose image under \( \tilde{f}^k \) crosses \( \tilde{E}' \). Since both \( \tilde{\sigma} \) and \( \tilde{f}^k_{\#}(\tilde{\sigma}') \) cross such an edge, their intersection is non-empty. It follows that \( \tilde{\rho} \subset \tilde{\sigma}_1 \) so we are done.

If \( H_* \) is EG then there exist \( \tilde{x} \in \tilde{\sigma} \) and \( \tilde{x}' \in \tilde{f}^k_{\#}(\tilde{\sigma}') \) such that \( \tilde{f}^k(\tilde{x}) = \tilde{f}^k(\tilde{x}') \).

The path \( \tilde{\tau} \) from \( \tilde{x} \) to \( \tilde{x}' \) decomposes as a concatenation of subpaths \( \tilde{\alpha}\tilde{\beta}^{-1} \) where \( \tilde{\alpha} \subset \tilde{\rho} \) and \( \tilde{\beta} \subset \tilde{\rho}' \). By construction \( \tilde{f}^k_{\#}(\tilde{\tau}) \) is trivial so in particular \( \tau \) is not weakly attracted to \( \Lambda^+ \). Lemma 4.4 implies that \( \tilde{\alpha} \) does not cross \( C \) edges that project to \( H_r \) and so does not contain \( \tilde{f}^k_{\#}(E) \) for any edge \( E \) that projects to \( H_r \). Since \( \tilde{\sigma} \) crosses such an \( E \) it follows that \( \tilde{\alpha} \) and hence \( \tilde{\rho} \) is contained in \( \tilde{\sigma} \cdot \tilde{f}^k_{\#}(\tilde{\sigma}) \cdot \ldots \cdot \tilde{f}^k_{\#}(\tilde{\sigma}) \) as desired. \( \square \)

The next lemma will be needed in Section 9.

**Lemma 5.4.** Suppose that \( f : G \to G \) is a CT and \( \tilde{f} : \tilde{G} \to \tilde{G} \) is a lift of \( f \). If \( \tilde{E} \) is a lift of \( E \in \mathcal{E}_f \) and if the initial endpoint of \( \tilde{E} \) is contained in \( \text{Fix}(\tilde{f}) \) then the lift \( \tilde{R}_E \) of \( R_E \) that begins with \( \tilde{E} \) converges to a point in \( \text{Fix}_+(\partial \tilde{f}) \).

**Proof.** \( \tilde{R}_E \) converges to some \( P \in \text{Fix}_N(\partial \tilde{f}) \) by [FH11, Lemma 4.36(1)]. Since \( E \) is not linear, \( u_E \) is not a Nielsen path and hence not a periodic Nielsen path. The length of \( f^k_{\#}(u_E) \) therefore goes to infinity with \( k \). [GJLL98, Proposition I.1] implies that \( P \in \text{Fix}_+(\partial \tilde{f}) \).

5.2 Finding a Fixed Point

A CT is a relative train track map \( f : G \to G \) and filtration satisfying a list of nine properties. One usually thinks of \( G \) as being a core subgraph but in certain induction arguments it is necessary to consider the restriction of \( f \) to a (not necessarily core) component of a filtration element. With this in mind we say that a homotopy equivalence of a finite graph \( f : G \to G \) is a CT if it satisfies the relative train track properties and the nine properties of a CT.

We state our next result in terms of a lift \( \tilde{f} : \tilde{G} \to \tilde{G} \) of a CT \( f : G \to G \) and fixed points for \( \tilde{f} \). We could just as easily have stated it in terms of finite paths \( \rho \subset G \) as described in Section 3.1 but it seems more natural to work with lifts.

Following Definition 5.2 we say that a completely split path \( \tilde{\sigma} \subset \tilde{G} \) generates a completely split ray \( \tilde{R} \) if \( \tilde{R} = \tilde{\sigma} \cdot \tilde{f}^1_{\#}(\tilde{\sigma}) \cdot \tilde{f}^2_{\#}(\tilde{\sigma}) \cdot \ldots \). As we have seen, \( \tilde{f}^k_{\#}(\tilde{R}) \subset \tilde{R} \) and \( \tilde{f} \) maps the set of splitting vertices for \( \tilde{R} \) into itself.

**Lemma 5.5.** There is an algorithm that takes a lift \( \tilde{f} : \tilde{G} \to \tilde{G} \) of a CT \( f : G \to G \) as input and determines if \( \text{Fix}(\tilde{f}) \) is non-empty. If it is non-empty then the output of the algorithm is an element of \( \text{Fix}(\tilde{f}) \). If it is empty then the output of the algorithm is a completely split path \( \tilde{\sigma} \subset \tilde{G} \) that generates a completely split ray \( \tilde{R} \). Moreover, if \( \text{Fix}(\tilde{f}) = \emptyset \) and if the projection \( \sigma \) of \( \tilde{\sigma} \) is not a Nielsen path then \( \text{Fix}_N(\partial \tilde{f}) = \{ P \} \) where \( P \) is the endpoint of \( \tilde{R} \) and \( P \) is not the endpoint of an axis of a covering translation.
Proof. We dispense with the moreover statement first. Suppose that \( \text{Fix}(\tilde{f}) = \emptyset \) and that \( \sigma \) is not a Nielsen path. Then \( |f^k_\#(\sigma)| \to \infty \) and [GJL98, Proposition I.1] implies that the terminal endpoint of \( \tilde{R} = \tilde{\sigma} \cdot \tilde{f}_\#(\tilde{\sigma}) \cdot \tilde{f}^2_\#(\tilde{\sigma}) \cdot \ldots \), which is evidently fixed by \( \partial \tilde{f} \), is an attractor for the action of \( \partial \tilde{f} \), is contained in \( \text{Fix}_N(\partial \tilde{f}) \) and is not the endpoint of an axis of a covering translation. Since \( \text{Fix}(\tilde{f}) = \emptyset \), [FH11, Corollary 3.16] implies that \( P \) is the only attractor in \( \text{Fix}_N(\partial \tilde{f}) \). If there were another point in \( \text{Fix}_N(\partial \tilde{f}) \) then it would be the endpoint of the axis of a covering translation that commuted with \( \tilde{f} \) and the translates of \( P \) would be additional attractors in \( \text{Fix}_N(\partial \tilde{f}) \). Thus \( \text{Fix}_N(\partial \tilde{f}) = \{ P \} \).

We now turn to the algorithm. Following the proof of [BFH00, Proposition 5.4.3], we say that for each non-fixed vertex \( \tilde{v} \in \tilde{G} \), the initial edge of the path from \( \tilde{v} \) to \( \tilde{f}(\tilde{v}) \) is **preferred** by \( \tilde{v} \). If both \( \tilde{E} \) and \( \tilde{E}^{-1} \) are preferred by their initial vertices then some sub-interval of \( \tilde{E} \) is mapped over itself by \( \tilde{f} \) and so contains a fixed point.

Consider the following (possibly infinite) method for finding either a fixed point in \( \tilde{G} \) or a ray whose terminal endpoint is an element of \( \text{Fix}_N(\partial \tilde{f}) \). Choose any vertex \( \tilde{v}_0 \). If \( \tilde{v}_0 \) is not fixed, let \( \tilde{E}_0 \) be the edge preferred by \( \tilde{v} \). If \( \tilde{E}_0^{-1} \) is preferred by the terminal vertex of \( \tilde{E}_0 \) then \( \tilde{E}_0 \) contains a fixed point that we can find by inspection (Section 3.3). Otherwise, let \( \tilde{E}_1 \) be the edge preferred by the terminal vertex of \( \tilde{E}_0 \). Repeat this to either find a fixed point in \( \tilde{E}_1 \) or define \( \tilde{E}_2 \) and so on. If this process does not terminate by finding a fixed point then the ray \( \tilde{R}_{\tilde{v}_0} = \tilde{E}_1 \tilde{E}_2 \ldots \) that it produces converges to a point in \( \partial \tilde{E}_n \) that is evidently fixed and not repelling so is contained in \( \text{Fix}_N(\partial \tilde{f}) \). For each \( m \geq 0 \), let \( \tilde{\sigma}_m \) be the path connecting the initial endpoint of \( \tilde{E}_m \) to its \( \tilde{f} \)-image.

**Step 1 of the algorithm:** Modify the above process by stopping not only if \( E_m \) contains a fixed point but also if \( \tilde{\sigma}_m \) is completely split and the turn between \( \tilde{\sigma}_m \) and \( \tilde{f}_\#(\tilde{\sigma}_m) \) is legal.

To see that this modified process stops in finite time, it suffices to show that if the original process produces a ray \( \tilde{R}_{\tilde{v}_0} \) then at least one of the \( \tilde{\sigma}_m \)'s has the desired properties. We verify this by following (and tweaking) the proof of [BFH00, Proposition 5.4.3].

Consider the subsequence \( \{ \tilde{v}_i \} \) of the set of vertices of \( \tilde{R}_{\tilde{v}_0} \) starting with \( \tilde{v}_0 \) and inductively defined by letting \( p \geq i \) be the largest integer such that the closest point to \( \tilde{f}(\tilde{v}_i) \) in \( E_0 E_1 \ldots E_p \) is the terminal endpoint of \( E_p \) and then taking \( v_{i+1} \) to be the terminal endpoint of \( E_p \). Equivalently, \( v_{i+1} \) is the nearest point in \( \tilde{R}_{\tilde{v}_0} \) to \( \tilde{f}(\tilde{v}_i) \).

Letting \([\tilde{v}_i, \tilde{v}_{i+1}] \) be the path connecting \( \tilde{v}_i \) to \( \tilde{v}_{i+1} \), the key property of the \( \tilde{v}_i \)'s is

\[
\tilde{f}_\#([\tilde{v}_i, \tilde{v}_{i+1}]) \supset [\tilde{v}_{i+1}, \tilde{v}_{i+2}]
\]

For \( m \geq 1 \), define

\[
\tilde{Y}_m = \{ \tilde{y} \in [\tilde{v}_0, \tilde{v}_1] : \tilde{f}^i(\tilde{y}) \in [\tilde{v}_i, \tilde{v}_{i+1}] \ \forall \ 1 \leq i \leq m \}
\]
The obvious induction argument shows that $\tilde{f}(\tilde{Y}_m) = [\tilde{v}_m, \tilde{v}_{m+1}]$ and in particular that $\tilde{Y}_m$ is non-empty. The $Y'_m$'s are a nested sequence of closed subsets of $[\tilde{v}_0, \tilde{v}_1]$ and so their intersection $\cap_{m=0}^\infty \tilde{Y}_m$ is non-empty. Each element of $\cap_{m=0}^\infty \tilde{Y}_m$ is contained in $\tilde{X} = \{ \tilde{x} : \{ \tilde{x}, \tilde{f}(\tilde{x}), \tilde{f}^2(\tilde{x}), \ldots \} \text{ is an ordered sequence of } \tilde{R}_{v_0} \}$

In the first two paragraphs on page 68 of [BFH00] it is shown that $\tilde{X} \subset \tilde{R}_{v_0}$ contains a vertex $\tilde{v}$ that is the initial vertex of an irreducible edge. For sufficiently large $k$, the paths $\tilde{\mu} := [\tilde{f}^k(\tilde{v}), \tilde{f}^{k-1}(\tilde{v})]$ and $\tilde{\nu} := [\tilde{f}^{k+1}(\tilde{v}), \tilde{f}^{k+2}(\tilde{v})]$ are completely split by [FH11, Lemma 4.25]. Moreover, the initial directions of $\mu^{-1}$ and $\nu$ are periodic by $Df$. Since $\tilde{v} \in \tilde{X}$, these directions are distinct and so the turn they define is legal. Letting $\tilde{f}^k(\tilde{v})$ be the initial vertex of $E_m$, we have found the desired $\tilde{\sigma}_m$. (The proof of [BFH00, Proposition 5.4.3] allows the possibility of subdividing at an endpoint of a periodic Nielsen path; in our context, these points are already fixed vertices so no subdivision is required.) This completes the proof that the first part of our algorithm stops in finite time.

If the first step of the algorithm produces a fixed point we are done and the algorithm stops. Suppose then that the first step produces a path $\tilde{\sigma} = \tilde{\sigma}_m$ as above. Let $P \in \text{Fix}_N(\partial \tilde{f})$ be the terminal endpoint of the ray $R = \tilde{\sigma} \cdot \tilde{f}_k(\tilde{\sigma}) \cdot \ldots$ generated by $\tilde{\sigma}$. The hard splitting property [FH11, Lemma 4.11(2)] implies that $R$ is fixed point free. If $\text{Fix}(\tilde{f}) \neq \emptyset$ then there exists a ray $\tilde{R}'$ with initial endpoint $\tilde{z} \in \text{Fix}(\tilde{f})$, terminal endpoint $P$ and with interior disjoint from $\text{Fix}(\tilde{f})$. [FH11, Lemma 3.16] implies that the initial edge $\tilde{E}$ of $\tilde{R}'$ determines a fixed direction. Let $R_E$ be the eigenray determined by $E$ (Definition 5.2), let $\tilde{R}_E$ be the lift of $R_E$ with initial edge $\tilde{E}$ and let $P' \in \text{Fix}_N(\partial \tilde{f})$ be the terminal endpoint of $\tilde{R}_E$. Since the interior of $\tilde{R}_E$ is disjoint from $\text{Fix}(\tilde{f})$, a second application of [FH11, Lemma 3.16] implies that $P' = P$. In other words $\tilde{R}$ and $\tilde{R}_E = \tilde{R}'$ have a common terminal subray.

If $\sigma$ is a Nielsen path then $E$ is a linear edge, $f(E) = Ew^d$ for some root-free Nielsen path $w$ that forms a circuit, $R_E = Ew^\infty$ and $\tilde{R}$ is a ray in a line $\tilde{L}$ that projects to $w$. It follows that that the terminal endpoint of $\tilde{E}$ is contained in $\tilde{L}$. The root-free covering translation that preserves $L$ commutes with $\tilde{f}$. After translating by some iterate of this covering translation we may assume that the terminal endpoint of $\tilde{E}$ is contained in any chosen lift $\tilde{w}$ of $w$ in $\tilde{R}$. This analysis justifies the next two steps of the algorithm.

**Step 2 of the algorithm:** Check if $\sigma$ is a Nielsen path. If it is not, go to step 3. If it is, then the algorithm ends as follows. Consider the finite set of points that are the initial vertex of a linear edge $\tilde{E}$ with terminal endpoint in $\tilde{\sigma}$. If an element of this set is contained in $\text{Fix}(\tilde{f})$ then that point is the output of the algorithm. Otherwise conclude that $\text{Fix}(\tilde{f}) = \emptyset$.

We may now assume that $\sigma$ is not a Nielsen path and hence that $P$ is not fixed by any covering translation. It follows that $\tilde{f}$ is the only lift of $f$ that fixes $P$ and hence that $\tilde{f}$ fixes the initial endpoint of any eigenray that converges to $P$. 33
Step 3 of the algorithm: Apply Lemma 5.3 to the set of eigenrays (Definition 5.2) for \( f \), one by one, to decide if there is an eigenray \( R_E \) that shares a terminal end with \( R \). If there is no such eigenray, then \( \text{Fix}(\tilde{f}) = \emptyset \). Otherwise, we have an edge \( E \), its eigenray \( R_E \) and decompositions \( R = \tau R'' \) and \( R_E = \tau' R'' \) for some ray \( R'' \). Let \( \tau \) be the lift of \( \tau \) that begins with \( \tilde{\sigma} \) and let \( \tilde{\tau}' \) be the lift of \( \tau' \) that shares a terminal endpoint with \( \tilde{\tau} \). Equivalently, \( \tilde{R} = \tilde{\tau} \tilde{R}'' \) and \( \tilde{R}_E = \tilde{\tau}' \tilde{R}'' \). Then the initial endpoint of \( \tilde{\tau}' \) is fixed by \( \tilde{f} \). \( \square \)

6 Proof of Theorem 1.1

We follow the proof of the existence theorem for \( f : G \to G \) and \( \emptyset = G_0 \subset G_1 \subset \ldots \subset G_N = G \) given in [FH11, Theorem 4.28] filling in the pieces that were not algorithmic in the original proof. That proof begins by assuming that \( \mathcal{C} \) is maximal which guarantees that any filtration that realizes \( \mathcal{C} \) will be reduced. We cannot make this assumption so will arrange part of this now, leaving one step for the end.

Start by applying [FH11, Theorem 2.19] to produce a relative train track map \( f : G \to G \) and filtration \( \emptyset = G_0 \subset G_1 \subset \ldots \subset G_N = G \) that realizes \( \mathcal{C} \). Enlarge \( \mathcal{C} \) if necessary so that it contains each \([G_i] \). If \( H_r \) is an EG stratum then we determine if \( H_r \) is reduced, and find an intermediate \( \phi \)-invariant free factor system if it is not, by applying Proposition 4.9. If it is reduced, no action is required. If it is not then extend \( \mathcal{C} \) by adding this new free factor system and reapply [FH11, Theorem 2.19].

After finitely many such steps we may assume that each EG stratum is reduced. It may be at this point that some NEG statum is not reduced. This will be corrected during the induction step that handles NEG strata. In the meantime, this causes no problem because the rest of the algorithm makes no use of NEG strata being reduced.

The proof of [FH11, Theorem 4.28] is divided into six steps. Steps 1, 3, 4 and 6 are algorithmic. (The proof of [FH11, Lemma 4.32] incorrectly refers to [FH11, Lemma 4.17]; it should instead refer to [BH92, Lemma 3.9].) Step 2 is an application of [FH11, Theorem 2.19], which we have made algorithmic in Lemma 2.4.

It remains to consider Step 5, which is the induction step for NEG edges. The situation is that \( H_s \) is a single NEG edge \( E_s \), that \( f \mid G_{s-1} \) satisfies all the properties of a CT and that \( f(E_s) = E_su_s \) for some path \( u_s \subset G_{s-1} \). Let \( \mathcal{C} \) be the component of \( G_{s-1} \) that contains \( u_s \). Choose a lift \( \tilde{E}_s \) of \( E_s \), let \( \tilde{f} : \tilde{G} \to \tilde{G} \) be the lift of \( f \) that fixes the initial endpoint of \( \tilde{E}_s \), let \( \Gamma \subset \tilde{G} \) be the component of the full pre-image of \( \mathcal{C} \) that contains the terminal endpoint of \( \tilde{E}_s \) and let \( h = \tilde{f} \mid \Gamma : \Gamma \to \Gamma \). Apply Lemma 5.5 to either find a fixed point for \( h \) or to conclude that \( h \) is fixed point free and find a completely split path \( \tilde{\sigma} \) that generates a completely split path \( \tilde{R} \). With this in hand, Step 5 becomes algorithmic. At the end of this step, \( f \mid G_s \) satisfies all the properties of a CT except perhaps that \( H_s \) is not reduced. If \( E_s \) is not a fixed edge then \( H_s \) is reduced by Lemma 4.12. If \( E_s \) is fixed, check (Lemma 4.5 and (NEG Nielsen Paths)) if there is a Nielsen path \( \beta \subset G_{s-1} \) connecting the terminal endpoint of \( E_s \) to the initial endpoint of \( E_s \). If not, then \( H_s \) is reduced by Lemma 4.12. If
yes, then $E_s\beta$ is a closed Nielsen path and the free factor system $\{[G_s]_{-1}, [E_s\beta]\}$ is properly contained between $[G_s]_{-1}$ and $[G_s]$. In this case, we can enlarge $C$ and start all over. Since there is a uniform bound to the number of distinct terms in $C$ this process terminates in finitely many steps. (More practically, one can simply modify $f: G \to G$ as described in Example 4.11.) \qed

\section{Finding $\text{Fix}(\Phi)$}

The goal of this section is to give another proof of the result of Bogopoliski-Maslakova [BM] that there is an algorithm that, given $\Phi \in \text{Aut}(F_n)$, computes $\text{Fix}(\Phi)$.

\subsection{The periodic case}

In this section, we examine the special case that $\Phi$ is periodic. The analysis in this section will parallel that of the general case.

Recall from Section 3.1 that if $G$ is a marked graph and $\tilde{v} \in \tilde{G}$ is a lift to the universal cover of $v \in G$ then there is an isomorphism $J_\tilde{v}: \pi_1(G,v) \to T(\tilde{G})$ given by mapping $[\gamma]$ to the covering translation $T$ of $G$ that takes $\tilde{v}$ to the terminal endpoint of the lift $\tilde{\gamma}$ of $\gamma$ with initial endpoint $\tilde{v}$.

For $\tilde{f}: \tilde{G} \to \tilde{G}$ a lift of $f: G \to G$, we denote the subgroup of $T(\tilde{G})$ consisting of covering translations that commute with $\tilde{f}$ by $Z_T(\tilde{f})$.

\lemma{7.1} Suppose that $h: G \to G$ is a periodic homeomorphism of a marked graph $G$ and that $\tilde{h}: \tilde{G} \to \tilde{G}$ is a periodic lift of $h$ to the universal cover $\tilde{G}$. Then

1. $\text{Fix}(\tilde{h}) \neq \emptyset$ and a point $\tilde{v} \in \text{Fix}(\tilde{f})$ can be found algorithmically.

2. If $\tilde{v} \in \text{Fix}(\tilde{h})$ projects to $v \in \Sigma$ then $J_\tilde{v}(\pi_1(\text{Fix}(h), v)) = Z_T(\tilde{h})$.

\proof After subdividing if necessary we may assume that $h$, and hence $\tilde{h}$, pointwise fixes each edge that it preserves. We are now in the setting of Bass-Serre theory and we use its language. Note that $\tilde{h}$ is not hyperbolic, for otherwise $\tilde{h}$ has infinite order. Hence $\tilde{h}$ is elliptic; equivalently $\text{Fix}(\tilde{h}) \neq \emptyset$. It is algorithmic to find a fixed point $\tilde{v}$. Indeed, if $\tilde{x} \in \tilde{\Sigma}$ then the midpoint of $[\tilde{x}, \tilde{h}(\tilde{x})]$ is fixed. This completes the proof of (1).

For (2), let $\text{Fix}(\tilde{h})$ denote the subtree of $\tilde{\Sigma}$ consisting of $\tilde{h}$-fixed edges. Given $[\gamma] \in \pi_1(\text{Fix}(h), v)$, let $\tilde{\gamma}$ be the lift of $\gamma$ that begins at $\tilde{v}$ and let $\tilde{w}$ be the terminal endpoint of $\tilde{\gamma}$. Then $T = J_\tilde{v}(\gamma)$ is the covering translation that carries $\tilde{v}$ to $\tilde{w}$. Since $\tilde{h}$ fixes $\tilde{v}$ and $\gamma \subseteq \text{Fix}(h)$, $\tilde{h}$ fixes $\tilde{w}$. In particular, $T \circ \tilde{h}(\tilde{v}) = T(\tilde{w}) = \tilde{w} = \tilde{h}(\tilde{w}) = \tilde{h} \circ T(\tilde{v})$ and so $T$ and $\tilde{h}$ commute. We see that $J_\tilde{v}(\pi_1(\text{Fix}(h), v))$ is contained in $Z_T(\tilde{h})$. To see surjectivity, let $T \in Z_T(\tilde{h})$. Then $T(\tilde{v}) \in \text{Fix}(\tilde{h})$. Since $\text{Fix}(\tilde{h})$ is a tree, $[\tilde{v}, T(\tilde{v})]$ descends to a closed path in $\text{Fix}(\tilde{h})$ based at $v$. \qed

We could not find a reference for the following result so we have included a proof.

\begin{center}
35
\end{center}
Lemma 7.2. There is an algorithm that, given periodic $\Phi \in \text{Aut}(F_n)$, computes $\text{Fix}(\Phi)$.

Proof. Let $\phi \in \text{Out}(F_n)$ be represented by $\Phi$. Since the only periodic automorphisms of $\mathbb{Z}$ are the identity and $x \mapsto -x$, we may assume that $n \geq 2$. The relative train track algorithm of [BH92] (see Theorem 2.1) finds a periodic homeomorphism $h : G \to G$ of a marked graph representing $\phi$.

We recall some notation from Section 3.1. The marking homotopy equivalence is $\mu : (R_n, *) \to (G, \star)$. Via a lift of $\star$ to $\tilde{\star} \in \tilde{G}$ we have an identification of $T(\tilde{G})$ with $F_n$, an isomorphism $J_\tilde{\star} : \pi_1(G, \star) \to T(\tilde{G})$ and a lift $\tilde{h} : \tilde{G} \to \tilde{G}$ that can be found algorithmically and that corresponds to $\Phi$ in a sense made precise in Section 3.1. The key points for us are that $\tilde{h}$ is periodic and that $\mathbb{Z}T(\tilde{h})$ and $\text{Fix}(\Phi)$ are equal when viewed as subgroups of $T(\tilde{G})$, see [FH11, Lemma 2.1]. Since $F_n$ has been identified via $\mu#$ with $\pi_1(G, \star)$, our goal is to find $J_{\tilde{\star}}^{-1}Z_T(\tilde{h}) < \pi_1(G, \star)$. By Lemma 7.1 we can algorithmically find an element $\tilde{v} \in \text{Fix}(\tilde{h})$. Moreover, letting $v \in G$ be the projection of $\tilde{v}$ and $H := \pi_1(\text{Fix}(h), v) < \pi_1(G, v)$, we have $J_v(H) = Z_T(\tilde{h})$. Let $\tilde{\eta}$ be the path in $\tilde{G}$ from $\tilde{\star}$ to $\tilde{v}$ and let $\eta \subset G$ be its projection. A quick chase through the definitions shows that $J_{\tilde{\star}}^{-1}J_v : \pi_1(G, v) \to \pi_1(G, \star)$ is defined by $[\gamma] \to [\eta\gamma\eta^{-1}]$. Thus $\text{Fix}(\Phi)$ is identified with $H^\eta < \pi_1(G, \star)$.

\section{A $G$-graph of Nielsen paths}

For the remainder of Section 7 we assume that $f : G \to G$ is a CT.

Let $\Sigma$ be a (not necessarily connected) graph. It is often useful to work in the Stallings category of graphs labeled by $\Sigma$ or $\Sigma$-graphs, i.e. an object is a graph $H$ with a cellular immersion $H \to \Sigma$ and a morphism from $H \to \Sigma$ to $H' \to \Sigma$ is a cellular immersion $H \to H'$ making the following diagram commute:

$$
\begin{array}{ccc}
H & \longrightarrow & H' \\
\downarrow & & \downarrow \\
\Sigma & & \\
\end{array}
$$

The map to $\Sigma$ is often suppressed. In this section we assume that $H$ is finite but in Section 9 we allow $H$ to be infinite. If we give $H$ the $CW$-structure whose vertex set is the preimage of the vertex set of $\Sigma$, then the resulting edges of $H$ (often called edgelets) are labeled by their image edges in $\Sigma$ and we refer to the oriented edges of $\Sigma$ as the set of labels. An edge-path is labeled by its sequence of oriented edges. The core of $H$ is the $\Sigma$-graph that is the union of all immersed circles in $H$. $H$ is core if it is its own core.

$\Sigma$-graphs are useful because the immersion $H \to \Sigma$ induces an injection on the level of $\pi_1$ and so $H$ is a geometric realization of the conjugacy class of $\pi_1(\Sigma)$ determined by the image of $\pi_1(H)$. Our goal in this section is to construct a $G$-graph whose
components realize all possible values of \([\text{Fix}(\Phi)]\) as \(\Phi\) varies over all automorphisms whose corresponding lifts \(\tilde{f} : \tilde{G} \to \tilde{G}\) have non-empty fixed point set.

We precede the construction with a quick review of Nielsen paths in a CT. Since Nielsen paths with endpoints at vertices split as products of fixed edges and indivisible Nielsen paths, \([\text{FH}11, \text{Lemma 4.25}]\), we focus on indivisible Nielsen paths. There are only two sources of indivisible Nielsen paths \(\mu\). If \(E \in \text{Lin}(f)\), then \(f(E) = Ew^d\) for some twist path \(w_E\) and some \(d \neq 0\) and \(Ew^kE = \text{an indivisible Nielsen path for } k \neq 0\). By the (NEG Nielsen Paths) property of a CT, all indivisible Nielsen paths of NEG-height have this form. To \(E\) we associate a \(G\)-graph \(Y_{\mu_E}\) which is a lollipop. Specifically, \(Y_{\mu_E}\) is the union of an edge labeled \(E\) and a circle labeled \(w_E\). Note that each \(Ew^kE \subset G\) lifts to a path in \(Y_{\mu_E}\) with both endpoints at the initial vertex of the edge labeled \(E\). The other possibility is an indivisible Nielsen path \(\mu\) of \(EG\)-height, say \(r\). In this case, \(\mu\) and \(\bar{\mu}\) are the only indivisible Nielsen paths of height \(r\) and the initial edges of \(\mu\) and \(\bar{\mu}\) are distinct edges of \(H_r\) \([\text{FH}11, \text{Lemma 4.19}]\).

We construct a \(G\)-graph \(S(f)\) as follows. Start with the subgraph \(S_1\) of \(G\) consisting of all vertices in \(\text{Fix}(f)\) and all fixed edges. For each \(E \in \text{Lin}(f)\), attach the lollipop \(Y_{\mu_E}\) to \(S_1(f)\) at the initial vertex of \(E\) thought of as a vertex in \(S_1(f)\). For each \(EG\) stratum with an indivisible Nielsen path of that height, choose one \(\mu\) of that height (there are only two and they differ by orientation) and attach an edge, say \(E_\mu\), labeled by \(\mu\) to \(S_1\) with initial and terminal endpoints equal to those of \(\mu\). This completes the construction of the graph \(S(f)\). There is a natural map \(h : S(f) \to G\) given by inclusion on \(S_1(f)\) and by the \(G\)-graph structures on each \(Y_{\mu_E}\) and \(E_\mu\). By construction, \(h\) is an immersion away from the attaching points in \(S_1(f)\) and is a local homeomorphism at each vertex in \(S_1(f)\). Thus \(S(f)\) is a \(G\)-graph.

For each vertex \(v \in \text{Fix}(f)\), define \(S(f, v)\) to be the component of \(S(f)\) containing the component of \(S_1(f)\) that contains \(v\). We abuse notation slightly by denoting the unique lift of \(v\) in \(S_1(f)\) by \(v\). It is possible that \(S(f, v)\) is not core.

Figure 1: A CT \(f : G \to G\) given by \(a \mapsto ab, b \mapsto bab\) and the graph \(S(f)\). \(S_1\) is the unique vertex of \(G\). \(S(f)\) is the closed Nielsen path \(\mu = abab\).

Remark 7.3. Since the construction of \(f : G \to G\), finding the set of indivisible Nielsen paths of \(EG\) height, and finding \(\text{Fix}(f)\) are all algorithmic, it follows that the constructions of \(S(f)\) and \(S(f, v)\) are algorithmic.

Lemma 7.4. (1) A path \(\sigma \subset G\) with endpoints at vertices lifts to a necessarily unique path \(\hat{\sigma} \subset S(f)\) with endpoints in \(S_1(f)\) if and only if \(\sigma\) is a Nielsen
Figure 2: A CT $h : H \to H$ given by $a \mapsto a, b \mapsto ba^2, c \mapsto ca, d \mapsto db$ and the graph $S(h)$. $S_1$ is the loop $a$ and the common initial vertex of $c$ and $d$. Add the lollipop associated to $b$ to the former component and the lollipop associated to $c$ to the latter to make $S(h)$.

(2) If $f$ represents $\phi \in \text{Out}(F_n)$ then non-trivial $\phi$-periodic (equivalently $\phi$-fixed) conjugacy classes in $F_n$ are characterized as those classes represented by circuits in $S(f)$.

Proof. (1): Since CTs are by definition completely split, a non-degenerate path $\sigma \subset G$ with endpoints at vertices is a Nielsen path if and only if $\sigma$ factors as $\sigma_1 \cdots \sigma_N$ where each $\sigma_i$ is either a fixed edge or an indivisible Nielsen path. By the construction of $S(f)$, the initial endpoint $v$ of $\sigma_1$ lifts uniquely to a vertex in $S_1(f) \subset S(f)$, $\sigma$ lifts uniquely to a path in $S(f)$ starting at $v$ and the terminal endpoint of this lift is a vertex in $S_1(f)$.

(2): By construction, every circuit in $S(f)$ projects in $G$ to a concatenation of fixed edges and indivisible Nielsen paths. Conversely, again using the fact that $f$ is completely split, every $\phi$-periodic conjugacy class is represented by a circuit $\sigma \subset G$ that splits as a product of fixed edges and indivisible Nielsen paths. By (1), $\sigma$ lifts to $S(f)$.

7.3 $\text{Fix}(\Phi)$ for rotationless $\Phi$

In this section we compute $\text{Fix}(\Phi)$ for rotationless $\Phi \in \text{Aut}(F_n)$. We begin with the analog of Lemma 7.1(2). Recall that the unique lift of $v \in \text{Fix}(f)$ in $S_1(f,v)$ is denoted $\tilde{v}$.

Lemma 7.5. For each vertex $v \in \text{Fix}(f)$ and lift $\tilde{v} \in \tilde{G}$ let $\tilde{J}_{\tilde{v}} = J_{\tilde{v}} h_{\#} : \pi_1(S(f,v),v) \to \mathcal{T}(\tilde{G})$ where $h_{\#} : \pi_1(S(f,v),v) \to \pi_1(G,v)$ is induced by the immersion $h : S(f,v) \to G$. Let $\tilde{f} : \tilde{G} \to \tilde{G}$ be the lift of $f$ that fixes $\tilde{v}$. Then $\tilde{J}_{\tilde{v}}$ is injective and has image equal to $Z_{\tilde{G}}(\tilde{f})$.

Proof. $\tilde{J}_{\tilde{v}}$ is injective because $h_{\#}$ is injective and $J_{\tilde{v}}$ is an isomorphism.

If $\gamma \subset S(f,v)$ is a closed path based at $\tilde{v}$ then $\gamma := h(\tilde{\gamma}) \subset G$ is a closed Nielsen path based at $v$ by Lemma 7.5. Lift $\gamma$ to a Nielsen path $\tilde{\gamma} \subset \tilde{G}$ for $\tilde{f}$ with initial
Lemma 7.6. There is an algorithm that, given rotationless $\Phi \in \text{Aut}(F_n)$, computes $\text{Fix}(\Phi)$.

Proof. Let $f : G \to G$ be a CT representing the element $\phi \in \text{Out}(F_n)$ determined by $\Phi$. The setup is similar to that of Lemma 7.2. The marking homotopy equivalence is $\mu : (R_n,\star) \to (G,\star)$. Via a lift of $\star$ to $\hat{\star} \in \hat{G}$ we have an identification of $\mathcal{T}(\hat{G})$ with $F_n$, an isomorphism $J_* : \pi_1(G,\star) \to \mathcal{T}(\hat{G})$ and a lift $\hat{f} : \hat{G} \to \hat{G}$ that can be found algorithmically and that corresponds to $\Phi$ in a sense made precise in Section 3.1. The key point for us is that $Z_{\mathcal{T}}(\hat{f})$ and $\text{Fix}(\Phi)$ are equal when viewed as subgroups of $\mathcal{T}(\hat{G})$, see [FH11, Lemma 2.1].

Apply Lemma 5.5 to decide if $\text{Fix}(\hat{f}) = \emptyset$.

If $\text{Fix}(\hat{f}) \neq \emptyset$ then Lemma 5.5 finds $\hat{v} \in \text{Fix}(\hat{f})$. Let $v \in G$ be the projection of $\hat{v}$, let $\eta \subset G$ be the projection of the path $\hat{\eta} \subset \hat{G}$ from $\hat{\star}$ to $\hat{v}$, let $h : S(f,v) \to G$ be the immersion given by the $G$-structure and let $\hat{v}$ be a lift of $v$ to $S(f,v)$. Define $H = h_{\#}(\pi_1(S(f,v),v)) < \pi_1(G,v)$. Arguing as in Lemma 7.2, with Lemma 7.1(2) replaced by Lemma 7.5 we conclude that $\text{Fix}(\Phi)$ is identified with $H^n < \pi_1(G,\star)$. Since $F_n$ has been identified via $\mu_{\#}$ with $\pi_1(G,\star)$, we are done.

If $\text{Fix}(\hat{f}) = \emptyset$ then $\hat{f}$ is not principal [FH11, Corollary 3.17] and so $\text{Rank}(\text{Fix}(\Phi)) \leq 1$ [FH11, Remark 3.3] and $\text{Fix}_N(\partial \hat{f}) = \text{Fix}_N(\partial \Phi)$ has at most two points. Lemma 5.5 finds a completely split path $\hat{\sigma}$ that generates a ray $\hat{R}$. There are two subcases. If the projected image $\sigma \subset G$ is not a Nielsen path then $|f_{\#}^k(\sigma)| \to \infty$ and [GJLL98, Proposition 1.1] implies that the terminal endpoint $P$ of $\hat{R} = \hat{\sigma} \cdot f_{\#}(\hat{\sigma}) \cdot f_{\#}^2(\hat{\sigma}) \cdot \ldots$, which is evidently fixed by $\partial \hat{f}$, is an attractor for the action of $\partial \hat{f}$, is contained in $\text{Fix}_N(\partial \hat{f})$ and is not the endpoint of an axis of a covering translation. If $Z_{\mathcal{T}}(\hat{f})$ contains a non-trivial element $T$ then the $T$-orbit of $P$ would be an infinite set in $\text{Fix}_N(\partial \hat{f})$. This contradiction shows that $Z_{\mathcal{T}}(\hat{f})$, and hence $\text{Fix}(\Phi)$, is trivial. The remaining subcase is that $\sigma$ is a Nielsen path. Let $\hat{v}$ and $\hat{w}$ be the initial and terminal endpoints of $\hat{\sigma}$ respectively and let $T$ be the covering translation that carries $\hat{v}$ to $\hat{w}$. Since $\hat{f}_{\#}(\hat{\sigma})$ is the unique lift of $\sigma$ with initial endpoint at $\hat{w}$, we have $\hat{f}(\hat{v}) = T(\hat{v}) = \hat{w}$ and $\hat{f}(\hat{w}) = T(\hat{w})$. Thus $\hat{f}T(\hat{v}) = T\hat{f}(\hat{v})$ and $T \in Z_{\mathcal{T}}(\hat{f})$. Equivalently $J_{\hat{v}}([\sigma]) \in Z_{\mathcal{T}}(\hat{f})$. Letting $H < \pi_1(G,v)$ be the primitive cyclic subgroup that contains $[\sigma]$, we have $J_{\hat{v}}(H) = Z_{\mathcal{T}}(\hat{f})$ and the usual argument shows that $\text{Fix}(\Phi)$ is identified with $H^n < \pi_1(G,\star)$. \qed
7.4 The general case

**Proposition 7.7** (Bolgopolski-Maslakova [BM]). There is an algorithm that, given \( \Phi \in \text{Aut}(F_n) \), computes \( \text{Fix}(\Phi) \).

**Proof.** The case of rotationless \( \Phi \) was handled in Lemma 7.6.

Suppose then that \( \Phi \) is not rotationless. In Corollary 3.10 we computed \( M \) so that \( \Phi^M \) is rotationless. So quoting Lemma 7.6 again, \( \text{Fix}(\Phi^M) \) can be computed algorithmically. We are reduced to finding the fixed subgroup of the periodic action of \( \Phi \) on \( \text{Fix}(\Phi^M) \). More generally, we are reduced to finding \( \text{Fix}(\Phi) \) for a finite order \( \Phi \in \text{Aut}(F_n) \), and this is the content of Lemma 7.2. \( \square \)

8 \( S(f) \) and \( \text{Fix}(\phi) \)

Let \( f : G \to G \) be a CT representing \( \phi \in \text{Out}(F_n) \). In this section we characterize the components of \( S(f) \) and relate \( S(f) \) to the invariant \( \text{Fix}(\phi) \).

Recall two facts from the review in Section 3.1:

- If \( \tilde{f} \) is a principal lift of \( f \) then \( \text{Fix}(\tilde{f}) \) is non-empty.
- If \( \tilde{f}_1, \tilde{f}_2 \) are principal lifts of \( f \) then \( \tilde{f}_1 \) and \( \tilde{f}_2 \) are isogredient iff \( \text{Fix}(\tilde{f}_1) \) and \( \text{Fix}(\tilde{f}_2) \) have equal projections in \( G \).

By the definition of a CT, endpoints of indivisible Nielsen paths are vertices and so Lemma 3.6(1) implies that, for principal \( \tilde{f} \), \( \text{Fix}(\tilde{f}) \) consists of vertices and edges. It follows that there are only finitely many equivalence classes of principal lifts of \( f \) and that there is a 1-1 correspondence between isogredience classes \( [\tilde{f}] \) of principal lifts \( \tilde{f} \) of \( f \) and Nielsen classes \( [v] \) of principal vertices \( v \) in \( G \) given by \( [\tilde{f}] \leftrightarrow [v] \) iff \( \tilde{f} \) fixes some lift of \( v \). It is algorithmic to tell if a vertex of \( G \) is principal and to find its Nielsen class (Lemmas 3.6(1) and 7.4).

By construction, for principal vertices \( v \) and \( v' \), \( S(f, v) = S(f, [v]) = S([\tilde{f}]) \) where \( [v] \leftrightarrow [\tilde{f}] \) and see that \( S(f) = \sqcup_{\tilde{f}} S([\tilde{f}]) = \sqcup_{v} S([v]) \) where \( [v] \) runs over Nielsen classes of principal vertices in \( G \) and \( [\tilde{f}] \) runs over isogredience classes of principal lifts of \( f \).

If \( \tilde{f} \leftrightarrow \Phi \) and \( \tilde{f}' \leftrightarrow \Phi' \) are isogredient then \( \text{Fix}(\Phi) \) and \( \text{Fix}(\Phi') \) are conjugate and \( Z_G(\tilde{f}) \) and \( Z_G(\tilde{f}') \) are conjugate. Hence, the isogredience classes of \( \Phi \) and \( \tilde{f} \) determine a conjugacy class \( \text{Fix}([\Phi]) \) of subgroup of \( F_n \) and conjugacy class \( Z_G([\tilde{f}]) \) of subgroup of \( T(\tilde{G}) \). The sets \( \text{Fix}([\Phi]) \) and \( Z_G([\tilde{f}]) \) correspond under the identifications of Section 3.1; we denote this by \( \text{Fix}([\Phi]) \leftrightarrow Z_G([\tilde{f}]) \). The invariant \( \text{Fix}(\phi) \) of \( \phi \) is defined to be \( \sqcup_{[\Phi]} \text{Fix}([\Phi]) \) where \( [\Phi] \) runs over isogredience classes of principal lifts of \( \phi \). We have that \( S(f) \) represents \( \text{Fix}(\phi) \) in the sense that, under our identifications, the image in \( G \) of \( S(f, v) \) represents the conjugacy class \( \text{Fix}([\Phi]) \leftrightarrow Z_G([\tilde{f}]) \) where \( \tilde{f} \) is a lift of \( f \) that fixes a lift of \( v \) and \( \tilde{f} \leftrightarrow \Phi \).
9 A Stallings graph for $\text{Fix}_N(\partial \phi)$

Given rotationless $\phi \in \text{Out}(F_n)$ there is an invariant $\text{Fix}_N(\partial \phi) := \sqcup \text{Fix}_N(\partial \Phi)$ where $\Phi$ runs over isogredience classes of principal lifts $\Phi$ of $\phi$ and $\text{Fix}_N(\partial \Phi)$ is defined as follows. Recall that $\text{Fix}_N(\partial \Phi)$ denotes the subset of $\partial F_n$ of non-repelling fixed points of the action of $\partial \Phi$. If $\Phi$ and $\Phi'$ are isogredient then $\text{Fix}_N(\partial \Phi)$ and $\text{Fix}_N(\partial \Phi')$ differ by the action of an inner automorphism. The set of translates of $\text{Fix}_N(\partial \Phi)$ by inner automorphisms is denoted $\text{Fix}_N(\partial [\Phi])$. In the main theorem of [FH11], rotationless outer automorphisms are characterized in terms of two invariants, one qualitative and the other quantitative. $\text{Fix}_N(\partial \phi)$ is precisely our qualitative invariant.

As usual, throughout Section 9, $f : G \to G$ will denote a CT for $\phi$. The goal of this section is to describe a $G$-graph representing $\text{Fix}_N(\partial \phi)$.

9.1 Abstract Stallings graph

In terms of $f$, we will define two $G$-graphs; one definition is abstract and the other constructive. In Section 9.3, we will see these two $G$-graphs are naturally isomorphic.

In this section, we give the abstract definition. If $\tilde{f}$ is principal then by definition $|\text{Fix}_N(\partial \tilde{f})| \geq 2$ and the convex hull $\tilde{\text{AS}}(\tilde{f})$ of $\text{Fix}_N(\partial \tilde{f})$ is defined, that is, $\tilde{\text{AS}}(\tilde{f})$ is the union of all lines in $\tilde{G}$ with both endpoints in $\text{Fix}_N(\partial \tilde{f})$.

$\text{Stab}_{\tilde{G}}(\tilde{\text{AS}}(\tilde{f}))$ denotes the group of covering transformations of $\tilde{G}$ leaving $\tilde{\text{AS}}(\tilde{f})$ (or equivalently $\text{Fix}_N(\partial \tilde{f})$) invariant. $\text{AS}(\tilde{f})$ (for Abstract Stallings) denotes the $G$-graph $\tilde{\text{AS}}(\tilde{f})/\text{Stab}_{\tilde{G}}(\tilde{\text{AS}}(\tilde{f}))$.

If $\tilde{f}'$ and $\tilde{f}$ are isogredient then $\text{AS}(\tilde{f}')$ and $\text{AS}(\tilde{f})$ are naturally isomorphic. Indeed, suppose $\tilde{f}' = T \tilde{f} T^{-1}$ for some covering transformation $T$ then $\partial T$ takes $\text{Fix}_N(\partial \tilde{f})$ to $\text{Fix}_N(\partial \tilde{f}')$ and so $T : \text{AS}(\tilde{f}) \to \text{AS}(\tilde{f}')$ descends to an isomorphism $\text{AS}(\tilde{f}) \to \text{AS}(\tilde{f}')$. To see the independence of the choice of $T$, suppose $T'$ is another covering transformation such that $\tilde{f}' = T' \tilde{f} T'^{-1}$. Then $T'^{-1} T$ commutes with $\tilde{f}$ and so $T'$ and $T$ descend to the same map.

If $[\tilde{f}]$ is an isogredience class then $\text{AS}([\tilde{f}])$ denotes the class of naturally isomorphic $G$-graphs just defined. In light of the bijection $[\tilde{f}] \leftrightarrow [v]$ between isogredience classes of principal lifts of $\tilde{f}$ and Nielsen classes of principal vertices of $G$, we will use $\text{AS}(f, [v])$ interchangeably with $\text{AS}([\tilde{f}])$. Finally $\text{AS}(f) := \sqcup [\tilde{f}] \text{AS}([\tilde{f}])$ or equivalently $\sqcup [v] \text{AS}(f, [v])$ where $[\tilde{f}]$ runs over isogredience classes of principal lifts and $[v]$ runs over Nielsen classes of principal vertices in $G$. As noted in Section 8, these are finite unions.

9.2 Constructive Stallings graph

In this section we define a second $G$-graph, $\text{CS}(f)$. In Section 9.3 we will verify that $\text{AS}(f)$ and $\text{CS}(f)$ are naturally isomorphic $G$-graphs. $\text{CS}(f)$ will be constructed by adding a third stage to the construction of $S(f)$. 
Stage 3. For each $E \in \mathcal{E}_f$, attach $R_E$ to $S(f)$ by identifying the initial endpoint of $R_E$ with the initial endpoint of $E$ thought of as a vertex in $S_1$. The result maps to $G$ and $CS(f)$ is the result of folding this map. An equivalent description of $CS(f)$ is to attach $R_E$ for each $E \in \mathcal{E}_f$ that is not an initial edge of a Nielsen path of $EG$-height and to attach $Y_\mu$ to $S(f)$ along $\mu$ for each Nielsen path $\mu$ of $EG$-height.

Remark 9.1. From the definition of $CS(f)$ it is clear that $CS(f)$ has finite type. Given that the construction of $S(f)$ is algorithmic and that any initial segment of a ray $R_E$ of prescribed length can be explicitly computed, it follows that there is an algorithm that, given $d > 0$, constructs the $d$-neighborhood (in the graph metric) of $S(f)$ in $CS(f)$. The construction of $R_E = E \cdot f_\#(E) \cdot f_\#^2(E) \cdots$ is recursive.

Figure 3: $CS(f)$ for $f$ as in Figure 1. $CS(f)$ is $Y_\mu$ with its two endpoints identified.

Figure 4: $CS(h)$ for $h$ as in Figure 2. $CS(f)$ is obtained from $S(f)$ by adding the eigenray $dbb\ldots$.

By construction of $CS(f)$, its components are in a 1-1 correspondence with the components of $S(f)$. That is, components of $CS(f)$ are in a bijective correspondence with Nielsen classes $[v]$ of principal vertices $v$ in $G$. Let $CS(f,[v])$ denote the component of $CS(f)$ containing $v$. We have $CS(f) = \sqcup_{[v]} CS(f,[v])$ where $[v]$ runs over Nielsen classes of principal vertices in $G$.

9.3 Properties of $AS(f)$ and $CS(f)$

In this section we record some properties of $AS(f)$ and $CS(f)$ including the fact that $AS(f)$ and $CS(f)$ are naturally isomorphic $G$-graphs. Recall from the construction of $CS(f)$ that its principal vertices correspond bijectively to principal vertices in $G$. We use the same name (usually $v$) for corresponding principal vertices. We defined the term core of a $\Sigma$-graph $H$ in Section 7.2. The weak core of $H$ is the union of all
immersed lines in $H$. $H$ is \textit{weakly core} if it is its own weak core. We say that $H$ has \textit{finite type} if it is the union of a finite graph and finitely many rays.

(1) $CS(f)$ has finite type and all vertices have valence $\geq 2$. In particular, $CS(f)$ is weakly core.

\textit{Proof.} By construction, $CS(f)$ has finite type and its non-principal vertices have valence either two or three. By [FH11, Lemma 4.14], a principal vertex $v$ has at least two fixed directions in $G$, each corresponding to a fixed edge, an edge in $\text{Lin}(f)$, or an edge in $\mathcal{E}_f$. By construction, each of these edges contributes a direction at $v$ in $CS(f)$. Thus all principal vertices, and hence all vertices, have valence at least two.

(2) No component of $AS(f)$ is a circle, an axis of $\tilde{G}$, or a generic leaf of an attracting lamination of $f$.

\textit{Proof.} The universal cover of a component of $AS(f)$ has the form $\tilde{AS}(\tilde{f})$ for principal $\tilde{f}$. Since $\tilde{AS}(\tilde{f})$ is the convex hull of $\text{Fix}_N(\partial \tilde{f})$, Property (2) follows from the definition of principal.

(3) Let $v \in G$ be principal and let $\tilde{f} : \tilde{G} \to \tilde{G}$ be the lift of $f$ fixing a lift $\tilde{v}$ of $v$. Pick a lift $\tilde{v}$ of $v$ to the universal cover $\tilde{CS}(f, [v])$. The image under the embedding\textsuperscript{3} $\tilde{CS}(f, [v]) \to \tilde{G}$ taking $\tilde{v}$ to $\tilde{v}$ is $\tilde{AS}(\tilde{f})$.

\textit{Proof.} By definition, $\tilde{AS}(\tilde{f})$ is the convex hull in $\tilde{G}$ of $\text{Fix}_N(\partial \tilde{f})$. Since $CS(f)$ is weakly core it is enough to show that the set of ends of $\tilde{CS}(f, [v])$ is $\text{Fix}_N(\partial \tilde{f})$. We first show the set of ends is contained in $\text{Fix}_N(\partial \tilde{f})$. By construction a ray in $CS(f)$ with initial vertex $v$ is either:

(a) a concatenation of fixed edges and indivisible Nielsen paths (in which case the lift to $\tilde{CS}(f, [v])$ converges to a point in the closure of $\text{Fix}(\tilde{f})$);

(b) a concatenation of fixed edges and indivisible Nielsen paths followed by the ray $E \cdot u_\infty^\infty$ for some $E \in \text{Lin}(f)$ (in which case the lift again converges to a point in the closure of $\text{Fix}(\tilde{f})$); or

(c) a concatenation of fixed edges and indivisible Nielsen paths followed by $R_E$ for some $E \in \mathcal{E}_f$ (in which case the lift converges to an isolated point of $\text{Fix}_N(\partial \tilde{f})$ by Lemma 5.4).

\textsuperscript{3}since $CS(f, [v])$ is a $G$-graph.
For the other containment, let \( \tilde{R} = [\tilde{v}, P] \) be a ray in \( \tilde{G} \) with initial vertex \( \tilde{v} \) and an end \( P \) in \( \text{Fix}_N(\partial f) \). There are two cases. If \( P \) is isolated then \( \tilde{R} \) has the form as in (c) by [FH11, Lemma 3.21]. Otherwise there are points \( \tilde{x}_0, \tilde{x}_1, \ldots \) in \( \tilde{G} \) converging to \( P \) that are fixed by \( \tilde{f} \). If one can choose such points in \( \tilde{R} \) then we are in case (a); if not, then we may assume that \( \tilde{x}_1 \) is last fixed point in \( \tilde{R} \). The path \( \tilde{\alpha}_k \) from \( \tilde{x}_1 \) to \( \tilde{x}_k \) is a concatenation of subpaths \( \tilde{\rho}_1 \ldots \tilde{\rho}_m \) each of which is either a fixed edge or an indivisible Nielsen path. Since \( \tilde{x}_1 \) is the last fixed point in \( \tilde{R} \), \( \tilde{\rho}_1 \) contains all of \( \tilde{\alpha}_k \cap \tilde{R} \). Since fixed edges and indivisible Nielsen paths of EG-height have uniformly bounded length, it follows that for all sufficiently large \( k \) there exists \( E \in \text{Lin}(f) \) such that \( \tilde{\rho}_1 = \tilde{E}u_E^{\ast}\tilde{E} \). Since this holds for all large \( k \), \( \tilde{R} \) has the form in (b). This completes the proof of (3).

(4) \( AS(f) \) and \( CS(f) \) are naturally isomorphic \( G \)-graphs.

Proof. We established a bijective correspondence between components of \( CS(f) \) and \( AS(f) \) and we will show that corresponding components are naturally isomorphic. More precisely, in terms of the notation of Property (3) we will show that the isomorphism \( \tilde{CS}(f, [v]) \rightarrow \tilde{AS}(\tilde{f}) \) descends to an isomorphism

\[
CS(f, [v]) = \tilde{CS}(f, [v]) / \text{Stab} \rightarrow \tilde{AS}(\tilde{f}) / \text{Stab}_{\tilde{G}}(\tilde{AS}(\tilde{f})) = AS(\tilde{f}) \tag{1}
\]

where \( \text{Stab} \) denotes the group of covering transformations of \( \tilde{CS}(f, [v]) \) \( \rightarrow \) \( CS(f, [v]) \). Since elements of \( \text{Stab} \) preserve labels, the isomorphism \( \tilde{CS}(f, [v]) \rightarrow \tilde{AS}(\tilde{f}) \) conjugates an element of \( \text{Stab} \) to an element of \( \text{Stab}_{\tilde{G}}(\tilde{AS}(\tilde{f})) \). Thus we have an injective homomorphism \( \text{Stab} \rightarrow \text{Stab}_{\tilde{G}}(\tilde{AS}(\tilde{f})) \). In particular, the quotient map (1) is well-defined.

To see that (1) is an isomorphism, it remains to show that our homomorphism is surjective. Let \( T \in \text{Stab}_{\tilde{G}}(\tilde{AS}(\tilde{f})) \). By [FH11, Lemma 2.1], \( T \) commutes with \( \tilde{f} \) and so \( T(\tilde{v}) \) is also \( \tilde{f} \)-fixed and \( [\tilde{v}, T(\tilde{v})] \) descends to a closed Nielsen path in \( G \). We saw (Lemma 7.4 and Section 9.2) that this Nielsen path lifts to a closed Nielsen path in \( CS(f, [v]) \) based at \( v \), and so determines an element of \( \text{Stab} \) mapping to \( T \).

Definition 9.2. The Stallings graph of \( \text{Fix}_N(\partial \phi) \) in terms of \( f \), denoted \( S_N(f) \), is the isomorphism class \( AS(f) \) (or equivalently \( CS(f) \)). Its components are parametrized by Nielsen classes of principal vertices of \( G \) (or equivalently isogredience classes of principal lifts of \( f \)).

\[ S_N(f) = \bigsqcup_{[v]} S_N(f, [v]) = \bigsqcup_{[\tilde{f}]} S_N(\tilde{f}) \]
Appendix: Hyperbolic and atoroidal automorphisms

In this appendix we use CTs to reprove the result of Brinkmann that was quoted in the proof of Corollary 1.6.

**Lemma** ([Bri00]). Suppose that no conjugacy class in $F_n$ is fixed by an iterate of $\phi$. Then $\phi$ is hyperbolic.

**Proof.** Suppose that $\Lambda^\pm_1, \ldots, \Lambda^\pm_m$ are the lamination pairs for $\phi$. After replacing $\phi$ by an iterate, we may assume that $\phi$ and $\phi^{-1}$ are rotationless. Since $\phi$ does not fix any conjugacy classes, each $\Lambda^\pm_i$ is non-geometric, and so ([HMe, Theorem F]; see also [BFH00, Definition 5.1.4 and Theorem 6.0.1]) there is a $\phi$-invariant free factor system $A_i$ for which the following are equivalent for each conjugacy class $[a]$ in $F_n$.

- $[a]$ is not weakly attracted to $\Lambda^+_i$ under iteration by $\phi$.
- $[a]$ is not weakly attracted to $\Lambda^-_i$ under iteration by $\phi^{-1}$.
- $[a]$ is carried by $A_i$.

In particular, $A_i$ does not carry $\Lambda^\pm_i$.

We claim that no line is carried by every $A_i$. If this failed, then one could choose free factors $A_i$ such that $[A_i]$ is a component of $A_i$ and such that $A := A_1 \cap \ldots \cap A_m$ is non-trivial. Thus $A$ is a non-trivial free factor that does not carry any $\Lambda^\pm_i$. There is a CT representing $\phi$ in which $[A]$ is represented by a filtration element. The lowest stratum cannot be EG and so must be a fixed loop. But that contradicts the assumption that there are no $\phi$-invariant conjugacy classes and so completes the proof of the claim.

Choose CTs $f : G \to G$ representing $\phi$ and $f' : G' \to G'$ representing $\phi^{-1}$. Let $\lambda_i$ the expansion factor ([BFH00, Definition 3.3.2]) for $\phi$ with respect to $\Lambda^+_i$, $\mu_i$ the expansion factor for $\phi^{-1}$ with respect to $\Lambda^-_i$ and $\lambda = \min\{\lambda_i, \mu_i\} > 1$.

Let $H_i \subset G$ be the stratum corresponding to $\Lambda^+_i$. By [BFH00, Lemma 4.2.2] there exists a subpath $\delta_i \subset G$ of $\Lambda^+_i$ (any subpath of $\Lambda^+_i$ that crosses sufficiently many edges of $H_i$ will do) with the following property: if $\sigma \subset G$ is a circuit or path and $\sigma_0 \subset \sigma$ is a copy of $\delta_i$ or its inverse $\bar{\delta}_i$, then there is a splitting of $\sigma$ in which one of the terms is an edge of $H_i$ contained in $\sigma_0$. Let $B^+(\sigma)$ be the maximum number of disjoint subpaths of $\sigma$ that are copies of some $\delta_i$ or $\bar{\delta}_i$. Then $\sigma$ has a splitting in which $B^+(\sigma)$ terms are single edges in EG strata. It follows that

$$|f^k_\#(\sigma)| > CB^+(\sigma)\lambda^k$$

for all $k \geq 1$ where $C$ is a positive constant and $|\cdot|$ denotes length.

Define $\delta'_i \subset G'$ and $B^-(\sigma')$ symmetrically replacing $\Lambda^+_i$ with $\Lambda^-_i$ with $f' : G' \to G'$. Each path or circuit $\sigma' \subset G'$ has a splitting in which $B^-(\sigma')$ terms
are single edges in EG strata of $f' : G' \to G'$. After decreasing $C$ if necessary, we have

$$|(f')^k(\sigma')| > C B^-(\sigma') \lambda^k$$

for all $k \geq 1$.

Let $h : G \to G'$ be a homotopy equivalence that preserves markings. After replacing $\phi$ (and hence $f$ and $f'$) by an iterate, we may assume that for each $i$ the neighborhood of $\Lambda_i^+$ consisting of lines that contain either $\delta_i$ or $\tilde{\delta}_i$ as a subpath is mapped into itself by $\phi$ and that the neighborhood of $\Lambda_i^-$ determined by $\delta'_i$ is mapped into itself by $\phi^{-1}$. By [HMe, Theorem H] there is a positive integer $N_i$ so that if $\beta \subset G$ is a line that is not carried by $A_i$, then either $f^k_\#(\delta_i)$ contains a copy of $\delta'_i$ or $\delta_i$ for all $k \geq 0$ or $f^k_\#(\beta)$ contains a copy of $\delta_i$ or $\tilde{\delta}_i$ for all $k \geq N_i$. In other words $B^+(f^k_\#(\beta)) + B^-(f^k_\#(\beta)) \geq 1$ for all $k \geq 0$.

Let $N = \max\{N_i\}$. We claim that there exists a positive integer $L$ so that if $\sigma \subset G$ has length at least $L$ then $B^+(f^N(\sigma)) + B^-(f^N_\#(\sigma)) \geq 1$. Indeed, if this fails then there exists $L_j \to \infty$ and paths $\sigma_j \subset G$ with length $\geq L_j$ such that for all $1 \leq i \leq m$

- $h_\#(\sigma_j)$ does not contain $\delta'_i$ or $\tilde{\delta}_i$ as a subpath.
- $f^N(\sigma_j)$ does not contain $\delta_i$ or $\tilde{\delta}_i$ as a subpath.

By focusing on the ‘middle’ of each $\sigma_j$ and passing to a subsequence we may assume that there are subpaths $\beta_j \subset \sigma_j$ such that $\beta_1 \subset \beta_2 \subset \ldots$ is an increasing sequence of paths whose union is a line $\beta \subset G$. As verified above, there exists $1 \leq i \leq m$ such that $A_i$ does not carry $\beta$. By [HMb, Lemma 1.6] $h_\#(\beta_j) \subset h_\#(\beta_{j+1}) \cap h_\#(\sigma_j)$ and $h_\#(\beta)$ is the union of the $h_\#(\beta_j)$’s. It follows that $h_\#(\beta)$ does not contain $\delta'_i$ or $\tilde{\delta}_i$ as a subpath and so $f^N_\#(\beta)$ must contain a copy of $\delta_i$ or $\tilde{\delta}_i$. But then $f^N_\#(\beta_j)$ contains a copy of $\delta_i$ or $\tilde{\delta}_i$ for all sufficiently large $j$ and hence $f^N_\#(\beta_{j'})$ contains a copy of $\delta_i$ or $\tilde{\delta}_i$ for all sufficiently large $j'$. This contradiction completes the proof of the claim.

We are now ready to complete the proof. After replacing $\phi$ with $\phi^N$, we may assume that $N = 1$. Given a circuit $\sigma \subset G$ with $|\sigma| \geq 2L$ divide it into $[|\sigma|/L]$ subpaths $\sigma_i$ of length at least $L$ where $[x]$ is the greatest integer function. Applying the preceding claim and the fact that the $f_\#(\sigma_i)$’s are disjoint subpaths of $f_\#(\sigma)$ we have

$$\max\{B^+(f_\#(\sigma)), B^-(\sigma')\} \geq \frac{|\sigma|}{2L}$$

In conjunction with the above displayed inequalities this completes the proof of the lemma if $|\sigma| \geq 2L$. As there are only finitely many remaining $\tilde{\sigma}$ and none of these is fixed by an iterate of $\phi$ we are done. \hfill \square
References

[BF92] M. Bestvina and M. Feighn, A combination theorem for negatively curved groups, J. Differential Geom. 35 (1992), no. 1, 85–101. MR 93d:53053

[BFH97] M. Bestvina, M. Feighn, and M. Handel, Laminations, trees, and irreducible automorphisms of free groups, Geom. Funct. Anal. 7 (1997), no. 2, 215–244. MR 98c:20045

[BFH00] Mladen Bestvina, Mark Feighn, and Michael Handel, The Tits alternative for $\text{Out}(F_n)$. I. Dynamics of exponentially-growing automorphisms, Ann. of Math. (2) 151 (2000), no. 2, 517–623. MR 1 765 705

[BH92] Mladen Bestvina and Michael Handel, Train tracks and automorphisms of free groups, Ann. of Math. (2) 135 (1992), no. 1, 1–51.

[BM] O. Bogopolski and O. Maslakova, An efficient algorithm for finding a basis of the fixed point subgroup of an automorphism of a free group, arXiv:1204.6728v6.

[Bri00] P. Brinkmann, Hyperbolic automorphisms of free groups, Geom. Funct. Anal. 10 (2000), no. 5, 1071–1089. MR 1800064 (2001m:20061)

[BT68] Gilbert Baumslag and Tekla Taylor, The centre of groups with one defining relator, Math. Ann. 175 (1968), 315–319. MR 0222144 (36 #5196)

[CMP] M. Clay, J. Mangahas, and A. Pettet, An algorithm to detect full irreducibility by bounding the volume of periodic free factors, arXiv:1402.7342.

[Dah] F. Dahmani, On suspensions, and conjugacy of hyperbolic automorphisms (and of a few more), arXiv:1307.2108v3, to appear in TAMS.

[DF05] Guo-An Diao and Mark Feighn, The Grushko decomposition of a finite graph of finite rank free groups: an algorithm, Geom. Topol. 9 (2005), 1835–1880 (electronic). MR 2175158 (2006i:20045)

[DV96] Warren Dicks and Enric Ventura, The group fixed by a family of injective endomorphisms of a free group, Contemporary Mathematics, vol. 195, American Mathematical Society, Providence, RI, 1996. MR 1385923 (97h:20030)

[FH09] Mark Feighn and Michael Handel, Abelian subgroups of $\text{Out}(F_n)$, Geom. Topol. 13 (2009), no. 3, 1657–1727. MR MR2496054

[FH11] ———, The recognition theorem for $\text{Out}(F_n)$, Groups Geom. Dyn. 5 (2011), no. 1, 39–106. MR 2763779 (2012b:20061)
[Ger84] S. M. Gersten, *On Whitehead’s algorithm*, Bull. Amer. Math. Soc. (N.S.) **10** (1984), no. 2, 281–284. MR 85g:20051

[GJLL98] Damien Gaboriau, Andre Jaeger, Gilbert Levitt, and Martin Lustig, *An index for counting fixed points of automorphisms of free groups*, Duke Math. J. **93** (1998), no. 3, 425–452.

[HMa] M. Handel and L. Mosher, *Axes in Outer Space*, to appear in Memoirs of the AMS.

[HMb] M. Handel and L. Mosher, *Subgroup decomposition in Out($F_n$) part I: Geometric models*, preprint.

[HMc] ———, *Subgroup decomposition in Out($F_n$) part II: A relative Kolchin theorem*, preprint.

[HMd] ———, *Subgroup decomposition in Out($F_n$) part II: Relatively irreducible subgroups*, preprint.

[HMe] ———, *Subgroup decomposition in Out($F_n$) part III: Weak attraction theorem*, preprint.

[Kap00] Ilya Kapovich, *Mapping tori of endomorphisms of free groups*, Comm. Algebra **28** (2000), no. 6, 2895–2917. MR 1757436 (2001c:20098)

[Kap14] ———, *Algorithmic detectability of iwip automorphisms*, Bull. Lond. Math. Soc. **46** (2014), no. 2, 279–290. MR 3194747

[Sta83] J. Stallings, *Topology of finite graphs*, Inv. Math. **71** (1983), 551–565.