NO-ARBITRAGE SYMMETRIES∗

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Abstract The no-arbitrage property is widely accepted to be a centerpiece of modern financial mathematics and could be considered to be a financial law applicable to a large class of (idealized) markets. This paper addresses the following basic question: can one characterize the class of transformations that leave the law of no-arbitrage invariant? We provide a geometric formalization of this question in a non probabilistic setting of discrete time-the so-called trajectorial models. The paper then characterizes, in a local sense, the no-arbitrage symmetries and illustrates their meaning with a detailed example. Our context makes the result available to the stochastic setting as a special case.

Key words No arbitrage symmetry; convexity preserving maps; non-probabilistic markets

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1 Introduction

The principle of no-arbitrage plays a fundamental role in modern financial mathematics; see [11] and references therein (we mostly restrict our comments and developments to a discrete time setting). In plain language, the assumption of no-arbitrage means that risky asset models should rule out a priori the possibility of making a profit without taking on any risk. This hypothesis implies a pricing methodology based on martingale stochastic processes; this is the risk-neutral valuation ([2]), and as such plays the role of a financial law. The empirical validity of this law has been studied ([14]) and even if arbitrage opportunities may be available, they are

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believed to be rare, short lived and hard to profit from. One could compare the notion of no-arbitrage to a physical law (such as the principle of inertia) that applies in idealized conditions, and think of it as a fundamental financial law applicable to a large class of (idealized) markets. With this point of view in mind, and as a preliminary step, we pose the question: can we characterize the class of transformations that leave the law of no-arbitrage invariant? This is much akin to Galilean/Lorentz transformations leaving the class of inertial frames invariant. Therefore, we look for no-arbitrage preserving transformations (referred to also as no-arbitrage symmetries) mapping a set of financial events into another set of financial events with the property that the no-arbitrage property holds for both classes of events. We are also interested in providing a financial interpretation to such a set of symmetries (much like Galilean/Lorentz transformations having a physical interpretation) and exploring some financial implications.

The financial events mentioned in the previous paragraph have to be linked to financial transactions, as it is the latter that fall under the scope of the no-arbitrage principle. In most situations, each such transaction involves two goods, \( X \) and \( Y \), and a price \( X_Y(t) \), so that \( X = X_Y(t) Y \). \( X_Y(t) \) is the integer number of units of asset \( Y \) required to purchase one unit of asset \( X \). In terms of dimensional units, \( [X_Y(t)] = [X]/[Y] \), and the reference asset \( Y \) is called the (chosen) numeraire ([19]). This discussion suggests that an analysis of the no-arbitrage principle could be done in terms of prices and numeraires (numeraire free approaches are also possible), and that is the way we proceed in the paper.

The setting in which we precisely pose and answer the above raised question is a set of sequences of multidimensional prices that evolve in discrete time. This type of set is called a trajectorial model (for a set of risky assets); in our investigation there is no need to assume any probability structure on such a set. In this way we can work on a more general setting than the (discrete time) stochastic framework and the question we study becomes a natural one unencumbered by unnecessary additional structure. We also briefly indicate how our main results, characterizing no-arbitrage transformations, apply to the stochastic setting as a special case.

The above-mentioned trajectorial framework has been developed in [9] and [6] (see also [10]) for the 1-dimensional case with a 0% interest rate bank account as numeraire and is here extended to the \( d \)-dimensional case and a general numeraire (but we restrict ourselves to finite time, as opposed to unbounded or infinite discrete time as in [9] and [10], respectively). It then follows that the global notion of no-arbitrage, i.e. involving several time steps, can be reduced to the one step notion of no-arbitrage. This is a classical reduction in discrete and finite time and allows us to concentrate our efforts on the local notion (i.e. involving one step into the future) of no-arbitrage.

We can now be more precise about our search for no-arbitrage preserving transformations.

**Definition (Disperse sets)** Consider a set \( E \subseteq \mathbb{R}^d \); \( E \) is called disperse if for each \( h \in \mathbb{R}^d \), \( [h \cdot Y = 0 \ \forall \ Y \in E] \) or \( \inf_{Y \in E} h \cdot Y < 0 \ \& \ \sup_{Y \in E} h \cdot Y > 0 \), where \( h \cdot Y \) represents the euclidean inner product.

We prove in this paper (by means of Proposition 4.2, Proposition 3.5 and Definition 3.6) that the notion of a set being disperse is equivalent to the risky assets (one step time evolution) obeying the no-arbitrage principle. Therefore, our original question on no-arbitrage preserving transformations turns into the challenge of characterizing the set of transformations that leave
the disperse property invariant. We are then in the context of a modern view of a geometry where we study the set of transformations that leave certain properties of a space of points invariant. Lemma 4.7 shows, at the geometric level, how disperse sets are transformed under Strict Inverse Convexity Preserving (SICP) maps. Theorem 5.4 and Corollary 5.6 make use of this connection relying on the financial interpretation, where coordinates are discounted in units of a numeraire.

We work on a self contained framework for financial markets centered on a set of (multi-dimensional) trajectories modeling a collection of assets with the possibility that any of them could play the role of a numeraire asset. No probability measures, filtrations, cardinality or topological assumptions are required of the trajectory set. The approach singles out local trajectory properties that can be used to consistently build an associated option price theory. The paper does not pursue this latter possibility, as we focus on symmetry transformations (option pricing developments are in [9] for the 1-dimensional case). Such trajectory properties have already made their appearance in the stochastic literature ([1, 5, 12]). To relate to the well-established stochastic approach ([11]), the reader could think that our paper concentrates on financial developments that only depend on the support of a given stochastic process independently of any possible probability distribution. Some connections with the stochastic setting are developed in [9]; the reference [10] provides a first mathematical step to extend some martingale notions from the standard setting to a trajectorial setting.

The subject of the paper is the study of fundamental symmetry transformations associated to the no-arbitrage principle in a non-probabilistic setting. Results are obtained with minimal assumptions and, in this way, provide a wider financial context for their availability.

We now describe the contents of the paper. Section 2 introduces the setting, which is centered on a trajectory space. Section 3 studies the notions of no-arbitrage and 0-neutrality (a weakening of no-arbitrage) in trajectory based markets. Subsection 3.1 introduces local conditions (i.e. properties that are conditioned on a given state of affairs and involve one step into the future) which are necessary and sufficient to establish no-arbitrage and 0-neutrality. These local conditions play a role analogous to that of the martingale condition in stochastic markets. Section 4 develops purely geometric results in \( \mathbb{R}^d \), independent of any financial setting, that form the backbone for deriving the set of no-arbitrage and 0-neutral symmetries. Section 5 characterizes two classes of transformations, one preserving the local no-arbitrage property and the other class preserving the 0-neutral property. In particular, we prove that a change of numeraire belongs to both such classes of transformations. The uncovered transformations can then be considered to be symmetries satisfied by price relationships. This latter point of view is carefully developed in an example in Section 6. Appendix A.1 provides proofs for results in the main body of the paper. Appendix A.2 develops some results on convex analysis that we rely upon. Finally, it should be noted that we use the words arbitrage-free and no-arbitrage interchangeably.

2 General Trajectorial Setting

We introduce the mathematical setting of a dynamic financial market with a finite number of assets whose initial prices are known. Uncertainty of future prices is given by a set of
multidimensional sequences that we call trajectories. The trading strategies are given by portfolios that will be successively re-adjusted, taking into account the information available at each stage. The present paper is essentially self-contained, though we extend work presented in [6] and [9]. The latter reference presents a non-probabilistic, one-dimensional, discrete time setting to price European options. The reference [6] provides examples and a computational algorithm to evaluate price bounds for European options. A detailed discussion and justification of why a trajectorial modeling approach is worth studying is presented in [9] as well as in [6]. The present paper extends the setting from those two papers to the multidimensional case.

There is empirical evidence suggesting that liquid markets do not allow for arbitrage opportunities. Therefore, and from a modeling point of view, the no-arbitrage principle assumes that market models should not contain arbitrage strategies. The no-arbitrage assumption allows one to develop a theory constraining relative prices. We remark in passing that under the weaker condition of 0-neutrality (see Definition 3.2 as well as [9]), it is possible to obtain well-defined price bounds for European options.

More precisely, we consider a market with \( d + 1 \) assets that evolve in a fixed time interval \([0, T]\). The model will be discrete in the sense that the trading instances are indexed by integer numbers. Given \( s_0 = (s_0^0, s_1^0, \ldots, s_d^0) \in \mathbb{R}^{d+1} \), as initial prices of assets \( S^k \), we will denote by \( S \) a sequence taking values in \( \mathbb{R}^{d+1} \) such that

\[
S_i = (S_0^i, S_1^i, \ldots, S_d^i) \quad \text{with} \quad S_0 = s_0.
\]

A portfolio will be a sequence of functions defined on the trajectory sets which we will denote by

\[
\Phi = (H^0, H) = \{(H^0_i, H^1_i, \ldots, H^d_i)\}_{i \geq 0}.
\]

Each coordinate \( H^j_i \), \( 0 \leq j \leq d \) represents the portfolio holdings at stage \( i \) for the \( j \)-th asset with \( |H^j| = 1_{S^j} \) (a unit of asset \( S^j \)). The asset values and the invested amounts can take values in general subsets of the real numbers.

The portfolio re-balancing stages may be triggered by arbitrary events of the market without the need to be directly associated with time. To incorporate this greater degree of generality we will add a new source of uncertainty to the trajectories’ coordinates (these additional coordinates are relevant when constructing specific models); we will denote them by \( W = \{W_i\}_{i \geq 0} \), where the \( W_i \) can be vector valued and take values in arbitrary sets. In financial terms, this new variable can represent any observable value of interest, such as volume of transactions, time, quadratic variation of trajectories, etc., as in [8].

In case one intends to price financial derivatives in the proposed setting, we add a finite time horizon \( T \). We will use a positive integer \( m \) to indicate the stage at which the trajectory reaches the time \( T \). This new variable plays a key role in calculating the fair price interval for options, although it does not intervene in the market properties.

**Definition 2.1** (Trajectory set) Consider \( \Sigma = \{\Sigma_i\} \) to be a given family of subsets of \( \mathbb{R}^{d+1}, \Omega = \{\Omega_i\} \) a family of sets and \( \Theta \subseteq \mathbb{N} \). For given \( s_0 \in \mathbb{R}^{d+1} \) and \( w_0 \in \Omega_0 \), a trajectory based set \( S \) is a subset of

\[
S_\infty(s_0, w_0) \equiv \{S = (S_i, W_i, m)_{i \geq 0} : S_i \in \Sigma_i, W_i \in \Omega_i, m \in \Theta \},
\]

such that \((S_0, W_0) = (s_0, w_0)\). The elements of \( S \) will be called trajectories.
It is important to note that if $\tilde{S} = \{(\tilde{S}_i, \tilde{W}_i, \tilde{m}_i)\}$ and $\tilde{S} = \{(\tilde{S}_i, \tilde{W}_i, \tilde{m}_i)\}$ are two trajectories, $\tilde{S}_i$ could unfold at a different time than $\hat{S}_i$. That is, the index $i$ will be associated with portfolio re-balance stages but they will not necessarily be associated to (uniform) time. It is only assumed that the stage $i + 1$ occurs temporarily after the stage $i$.

We define $M : S \rightarrow \mathbb{N}$ as the projection on the third coordinate of $S$; that is, $M(S) = m$. The results and properties that appear in this section only involve the first coordinate $S_i$, nonetheless, we will continue using the notation that includes the coordinates $W_i$ for consistency.

To build an adequate market model, we are going to require that any portfolio be non-anticipative. The non-anticipativity of the portfolios expresses the fact that investments must be made at the beginning of each period, so that they cannot anticipate specific future price changes.

**Definition 2.2 (Portfolio)** Letting $S$ be a trajectory set, a portfolio $\Phi$ is a sequence of (pairs of) functions $\Phi \equiv \{(H^0,S)\}_{i=0}^\infty$ with $H^0 : S \rightarrow \mathbb{R}$ and $H_i : S \rightarrow \mathbb{R}^d$ such that for all $S, S' \in S$, with $S_i = S$, for all $0 \leq i \leq k$, where $k < \min\{M(S), M(S')\}$, we have that $\Phi_k(S) = \Phi_k(S')$.

For a portfolio $\Phi$, $H^0_i(S)$ represents the number of units held for the $j$-th asset during the period between $i$ and $i + 1$. Therefore, $H^0_i(S) S^i_j$ is the value invested in the $j$-th asset at stage $i$, while $H^0_i(S) S^i_{i+1}$ is the value just before rebalancing at the end of the period. Thus, the total value of the portfolio $\Phi$ at the beginning of the period $i$ is

$$H^0_i(S) S^0_i + H_i(S) \cdot S_i \equiv H^0_i(S) S^0_i + \sum_{j=1}^d H^0_i(S) S^i_j,$$

and at the end of the period, the value of $\Phi$ will change to

$$H^0_i(S) S^0_{i+1} + H_i(S) \cdot S_{i+1} = H^0_i(S) S^0_{i+1} + \sum_{j=1}^d H^0_i(S) S^i_{i+1}.$$

In the next re-balancing, the investor will invest $\Phi_{i+1}$; in general, $H^0_{i+1}(S) S^0_{i+1} + H_{i+1}(S) \cdot S_{i+1}$ may be different from $H^0_i(S) S^0_{i+1} + H_i(S) \cdot S_{i+1}$. In this latter case, it follows that some units of the assets were added or removed, without replacement, from the portfolio. However, this situation is precluded for many applications. For example, if the goal is to look for a “fair” price for a certain financial contract, this value should be the minimum necessary to cover the obligations generated by the contract, that is, any injection or withdrawal of money will affect this property. This reasoning justifies the use of the following concept:

**Definition 2.3 (Self-financing portfolio)** A portfolio $\Phi$ is called self-financing if, for all $S \in S$ and $i \geq 0$,

$$H^0_i(S) S^0_{i+1} + H_i(S) \cdot S_{i+1} = H^0_i(S) S^0_{i+1} + H_{i+1}(S) \cdot S_{i+1}. \quad (2.1)$$

The self-financing property means that the portfolio is re-balanced in such a way that its value is preserved. From this property it is clear that the accumulated gains and losses resulting from price fluctuations are the only sources of variation of the portfolio; in other words,

$$H^0_i(S) S^0_k + H_k(S) \cdot S_k = H^0_0 S^0_0 + H_0 \cdot S_0 + \sum_{i=0}^{k-1} (H^0_i(S) \Delta_i S^0 + H_i(S) \cdot \Delta_i S).$$
for \( k \geq 0 \), where \( \Delta_i S^0 = S^0_{i+1} - S^0_i \) and \( \Delta_i S = S_{i+1} - S_i \). The value \( H^0_0 S^0_0 + H_0 \cdot S_0 \) represents the initial investment corresponding to the portfolio coordinate \( \Phi_0 \).

We will mention below some examples of strategies that will be used later.

**Example 2.4** 1. For the null portfolio \( \Phi = 0 \),

\[
0(S) = \{(0, 0)\}_{i \geq 0} \text{ for all } S \in S,
\]

where \( 0 \) is the null vector of \( \mathbb{R}^d \).

2. Setting \( h \in \mathbb{R} \) and \( h \in \mathbb{R}^d \), we will define a constant portfolio \( \Phi = h \) by

\[
h(S) = \{(h, h)\}_{i \geq 0} \text{ for all } S \in S.
\]

3. Setting \( \Phi = \{(H^0_0, H_i)\}_{i \geq 0} \) as a self-financing portfolio, we will denote by \( -\Phi \) the sequence of functions \( \{(-H^0_0, -H_i)\}_{i \geq 0} \). It is easy to see that \( -\Phi \) is a self-financing portfolio.

4. Setting \( \tilde{\Phi} = \{(\tilde{H}^0_0, \tilde{H}_i)\}_{i \geq 0} \) and \( \hat{\Phi} = \{(\hat{H}^0_0, \hat{H}_i)\}_{i \geq 0} \) as two portfolios, we define \( \Phi \equiv \tilde{\Phi} + \hat{\Phi} \) to be the sequence

\[
\Phi = \{\hat{H}^0_0 + \tilde{H}^0_0, \hat{H}_i + \tilde{H}_i\}_{i \geq 0}.
\]

### 2.1 Numeraire

To be definite, we will consider that the real numbers \( S^k_i \) express the price of asset \( S^k \) in a common currency, a unit of which we denote generically by \$. That is, in terms of dimensions, \([S^k_i] = \$/1_{S^k} \), where \( 1_{S^k} \) is one unit of asset \( S^k \) (it is well known that an algebra of dimensions is available through dimensional analysis as in \[20\]). Notice that \([H^k_i] = 1_{S^k} \). On the other hand, for financial reasons, it is important to work with an arbitrary reference asset; this is achieved by taking a reference asset as numeraire. For example, in some cases it is useful to select the value of a bank account as numeraire.

To this end, we will assume from here onwards that \( S^0_i > 0 \) for all \( i \geq 0 \). This hypothesis will allow us to use \( S^0 \) as numeraire. For each \( S \in S \), we will build a sequence of relative prices \( X(S) = \{(X(S_i), W_i, m_i)\}_{i \geq 0} \), where \( X : D \subseteq \mathbb{R}^{d+1} \rightarrow \mathbb{R}^d \) is a perspective function defined by

\[
X(s) \equiv \left( \frac{s^1}{s^0}, \ldots, \frac{s^d}{s^0} \right), \quad D \equiv \{s = (s^0, \ldots, s^d) \in \mathbb{R}^{d+1} : s^0 > 0 \}.
\]  

(2.2)

The numerical value of \( X^j(S_i) \) (i.e. stripped from its units) is the number of units of the asset \( S^0 \), now the numeraire, which are required to acquire one unit of the \( S^j \) asset.

**Remark 2.5** Notice the above definition of \( X \) singles out \( s^0 \), but of course any other coordinate could be used (relying on the 0-component simplifies the notation). In fact, and for more generality, one could replace \( s^0 \) by a linear map \( B(s) > 0 \) on \( s^k > 0 \). We do not pursue here this possibility but our results will apply to such a numeraire by just moving to a new trajectory market with \( s^0 = B(s) \).

Given \( S \in S \) and \( k \geq 0 \), we will denote by \( V^\Phi_k(S) \) the relative value of the portfolio \( \Phi \in \mathcal{H} \) given by

\[
V^\Phi_k(S) \equiv H^0_0(S) + H_k(S) \cdot X(S_k).
\]

Clearly \( V^\Phi_k(S) = \frac{\Phi_k(S) \cdot S_k}{s^0_k} \), so \( V^\Phi_k(S) \) can be interpreted as the value of the portfolio at the beginning of the stage \( k \) expressed in units of the numeraire. In addition, \( G^\Phi_k(S) \) will denote
the profits generated up to the stage \( k \) associated with \( \Phi \in \mathcal{H} \) for a trajectory \( S \in \mathcal{S} \); that is,

\[
G^\Phi_k(S) \equiv \sum_{i=0}^{k-1} H_i(S) \cdot \Delta_i X(S) \quad \text{for} \quad k \geq 0 \quad \text{where} \quad \Delta_i X(S) = X(S_{i+1}) - X(S_i).
\]  

(2.3)

\( G^\Phi_k(S) \) reflects, in terms of the numeraire, the net gains accumulated by the portfolio \( \Phi \) at the beginning of the \( k \)-th stage.

A self-financing portfolio for a path \( S \in \mathcal{S} \) will also be self-financing for the \( X(S) \) sequence.

**Proposition 2.6**  Let \( \mathcal{S} \) be a space of trajectories, and let \( \Phi \) be a portfolio on \( \mathcal{S} \). Then the following statements are equivalent:

1. \( \Phi \) is self-financing.
2. \( H^0_{i-1}(S) + H_{i-1}(S) \cdot X(S_i) = H^0_{i}(S) + H_{i}(S) \cdot X(S_i) \) for all \( S \in \mathcal{S} \) and \( i \geq 0 \).
3. \( V^\Phi_k(S) = V^\Phi_0 + G^\Phi_k(S) = H^0_0 + H_0 \cdot X(S_0) + \sum_{i=0}^{k-1} H_i(S) \cdot \Delta_i X(S) \) for all \( k \geq 0 \).

**Proof**  Note that Proposition 5.7 of [11] is valid even in cases where we do not have a market indexed by pre-set time stages. Therefore the same idea used in that result applies to our setting.

\( \square \)

**Remark 2.7**  From the previous Proposition, we know that the \( H^0 \) component of a self-financed portfolio \( \Phi \) satisfies

\[
H^0_k(S) - H^0_{k-1}(S) = -(H_k(S) - H_{k-1}(S)) \cdot X(S_k).
\]

(2.4)

Given that

\[
H^0_0 = V^\Phi_0 - H_0 \cdot X(S_0),
\]

(2.5)

the sequence \( H^0 \) is completely determined by the initial investment \( V^\Phi_0 \) and \( H \) by means of the previous equations.

**Remark 2.8**  For a given set of portfolios \( \mathcal{H} \), in virtue of Remark 2.7 and display (2.3) (which depends on \( \Phi = (H^0, H) \), just through \( H \)) we will set the definition

\[
\mathcal{H}_S \equiv \{ H : (H^0, H) \in \mathcal{H} \}
\]

(2.6)

for later use.

**Definition 2.9**  (Trajectory market)  Given \( s_0 \in \mathbb{R}^{d+1}, w_0 \in \Omega_0 \), a trajectory based set \( \mathcal{S} \subseteq \mathcal{S}_{\infty}(s_0, w_0) \) and a portfolio set \( \mathcal{H} \), we say that \( \mathcal{M} = \mathcal{S} \times \mathcal{H} \) is a trajectory based market if it satisfies the following properties:

1. For each \( S \in \mathcal{S} \), the coordinate \( S^0_i > 0 \) for all \( i \geq 0 \).
2. All \( \Phi \in \mathcal{H} \) are self-financing and \( \Phi = 0 \) belongs to \( \mathcal{H} \).
3. For each \( (S, \Phi) \in \mathcal{M} \) there exists \( N_\Phi(S) \in \mathbb{N} \) such that \( \Phi_k(S) = \Phi_{N_\Phi(S)}(S) = 0 \) for all \( k \geq N_\Phi(S) \).

We will say that the market is semi-bounded if, for each \( \Phi \in \mathcal{H} \), there is \( N_\Phi \in \mathbb{N} \) such that \( N_\Phi(S) \leq n_\Phi \) for all \( S \in \mathcal{S} \) and that it is \( n \)-bounded, for \( n \in \mathbb{N} \), if \( N_\Phi(S) \leq M(S) \leq n \) for each pair \( (S, \Phi) \in \mathcal{M} \). A portfolio set \( \mathcal{H} \) obeying items (2) and (3) above will be called admissible.

The third property of the previous definition states that the adjustments of the portfolio \( \Phi \) for a trajectory \( S \) will end at, or before, the stage \( N_\Phi(S) - 1 \), which means that the portfolio was liquidated on, or before, the period \( N_\Phi(S) \). In this case, the corresponding portfolio will be called liquidated.
The above setting incorporates, as a special case, a discrete time stochastic model. Given a process \( Y = \{ Y_i = (Y_0^i, \ldots, Y_d^i) \}_{i \geq 0} \) on a probability space \( (\Omega, P) \) with filtration \( F = \{ F_i \}_{i \geq 0} \) and \( F_0 \) being trivial, \( Y_1^i : \Omega \to \mathbb{R}, Y_1^i \in F_i \). We can then define \( \mathcal{S} = \{ S = \{(S_i \equiv Y_i(\omega))\}_{i \geq 0} : \) for some \( \omega \in \Omega \}. \) One can also define a set of trajectories \( \mathcal{S} \) by means of a sequence of admissible stopping times \( \tau = \{ \tau_i \}_{i \geq 0} : S \in \mathcal{S} \), so \( S \in \mathcal{S} \), so \( S = \{ (S_i \equiv Y_{\tau_i}(\omega)) \}_{i \geq 0} \) for some \( \omega \in \Omega \). Another way to proceed is to use a given collection of such sequences of stopping times; for the details of this we refer to Section 6 in [9].

### 3 Arbitrage and 0-Neutrality

A model for common market situations should not allow for investors that are able to generate a profit in a transaction without any risk/possibility of losing money. Such an investment opportunity is called an arbitrage opportunity.

**Definition 3.1 (Arbitrage opportunity)** Given a trajectory based market \( \mathcal{M} = S \times \mathcal{H} \), \( \Phi \in \mathcal{H} \) is an arbitrage opportunity if

- \( \forall S \in \mathcal{S}, V_{N\Phi}^\Phi(S) \geq V_0^\Phi \);
- \( \exists S^* \in \mathcal{S} \) such that \( V_{N\Phi}^\Phi(S^*) > V_0^\Phi \).

We say that \( \mathcal{M} \) is arbitrage-free if \( \mathcal{H} \) does not contain arbitrage opportunities.

The particular case \( S_0^i = 1 \), for all \( i \), gives \( X(S_i) = (S_1^i, \ldots, S_d^i) \); i.e. the original currency is the asset \( S_0^i \) and is being used as numeraire, and so \( [S_0^i] = \$/\$. Currency, if included as a traded asset and in the presence of a riskless bank account with non-zero interest rates, will lead to an arbitrage as per Definition 3.1. That is, currency, under the mentioned conditions, will be banned as a traded asset whenever we assume a no arbitrage market (as well as a 0-neutral market). Notice the relevant discussion in [19] about arbitrage and non-arbitrage assets, currency being an arbitrage asset (in contrast to an interest bearing money market account).

Our use of an arbitrary value for \( V_0^\Phi \) in the definition of an arbitrage opportunity is non-standard; textbook definitions require that \( V_0^\Phi \leq 0 \) (see [11]). One can see that the existence of an arbitrage as per Definition 3.1 is equivalent to the existence of an arbitrage portfolio \( \hat{\Phi} \) with \( V_0^{\hat{\Phi}} = 0 \), and so proving the equivalence of our definition and the standard definition. This equivalence allows us also to show that Definition 3.1 is invariant under a change of numeraire and so the latter transformation will be a no arbitrage symmetry according to our definitions; this we show explicitly by a different argument in Corollary 5.7.

The arbitrage-free condition is sufficient for the model to provide fair option prices (a well known result in the classical financial literature.) One can relax the arbitrage free criteria to the requirement that the largest of the minimum possible gains that can be obtained by means of the strategies available in the market is 0. This notion was originally presented in [3] (as equivalent with arbitrage-free) and then formally defined and clarified for the case of a single risky asset in [9] and [6].

**Definition 3.2 (0-neutral market)** Let \( \mathcal{M} = S \times \mathcal{H} \) be a trajectory based market. We say that \( \mathcal{M} \) is 0-neutral if

\[
\sup_{\Phi \in \mathcal{H}} \left\{ \inf_{S \in \mathcal{S}} G_{N\Phi}^\Phi(S) \right\} = \sup_{\Phi \in \mathcal{H}} \left\{ \inf_{S \in \mathcal{S}} \left[ \sum_{i=0}^{N^\Phi(S)-1} H_i(S) \cdot \Delta_i X(S) \right] \right\} = 0.
\]
In [9] it is shown that this property is also sufficient to obtain a pricing interval for financial
derivatives. The next proposition shows that 0-neutrality is weaker than arbitrage-free.

**Proposition 3.3** Let $\mathcal{M} = \mathcal{S} \times \mathcal{H}$ be an arbitrage-free trajectory based market. Then $\mathcal{M}$ is 0-neutral.

**Proof** We are going to prove the proposition by contraposition. Note that if $\mathcal{M} = \mathcal{S} \times \mathcal{H}$ is a trajectory based market $0 \in \mathcal{H}$, then it is always true that

$$\sup_{\Phi \in \mathcal{H}} \left\{ \inf_{S \in \mathcal{S}} G_{N_0}^\Phi (S) \right\} \geq 0.$$ 

That is, if $\mathcal{M}$ is not 0-neutral, there exists a portfolio $\Phi$ such that

$$\inf_{S \in \mathcal{S}} G_{N_0}^\Phi (S) > 0.$$ 

Thus $G_{N_0}^\Phi (S) > 0$ for all $S \in \mathcal{S}$. Then

$$V_{N_0}^\Phi (S) = V_0^\Phi + G_{N_0}^\Phi (S) > V_0^\Phi$$

for all $S \in \mathcal{S}$. Then $\Phi$ is an arbitrage portfolio. □

It is clear how to generate simple examples of 0-neutral markets which contain arbitrage
(see [9] and [6]). Following [17, Section 1.2], it is possible to define other properties of the
market, closely related to the 0-neutral and arbitrage-free, namely dominant portfolios and the
law of one price. Under the appropriate hypotheses, the following chain of implications for a
trajectory based market holds:

**Arbitrage-free $\Rightarrow$ No dominant portfolios $\Rightarrow$ 0-neutral $\Rightarrow$ Law of one price.**

At this point, we have introduced enough properties of multidimensional trajectory markets
in order to address our goal of characterizing no-arbitrage symmetry transformations.

### 3.1 Relationships Between Local and Global Properties

From the definitions, it is not clear how to construct arbitrage-free or 0-neutral markets. For
the case of semi-bounded markets, one can obtain necessary and sufficient conditions only by
involving local properties of the trajectory set, implying trajectorial markets that are arbitrage-
free (or 0-neutral). Such characterizations play an analogous role to the equivalence of no
arbitrage stochastic markets and the possibility to equivalently modify the stochastic process
into a martingale process. In fact, in the arbitrage-free case, the local trajectorial conditions
 correspond to a probability free notion of a martingale sequence (see [10]). We will use these
characterizations to pose and answer our opening question on no-arbitrage preserving transfor-
mations.

At the $k$-th stage, the information about the future available to investors is that $S$ is an
element of the set

$$\mathcal{S}_{(S,k)} \equiv \{ S' \in \mathcal{S} : S'_i = S_i, 0 \leq i \leq k \text{ and } M(S') > k \} \subseteq \mathcal{S}.$$ 

We will call the pair $(S, k)$ a node and will refer to the set $\mathcal{S}_{(S,k)}$ as a trajectory set conditioned
at the node $(S, k)$. The future information contained in $\tilde{S} \in \mathcal{S}_{(S,k)}$ depends on the past only
through $S_0, \ldots, S_k$. The multiple number of trajectories emanating from a node reflects the
non-deterministic nature of the assets’ time evolution. As trajectories unfold more coordinates
become available, and so the investors increase their knowledge about possible future scenarios. This is expressed mathematically as

$$S(S, k') \subseteq S(S, k)$$

for $k' > k$. The following notation will also be used:

$$\Delta X(S(S, k)) = \{ \Delta k X(S') : S' \in S(S, k) \} \subseteq \mathbb{R}^d.$$  

\(\Delta_k X(S') = X(S_{k+1}) - X(S_k)\) has been introduced before.

We will refer to as local any property relative to a node \((S, k)\) and only involving elements of \(\Delta X(S(S, k))\).

The definitions below are the local counterpart of those of arbitrage-free and 0-neutral for the whole market. We are then going to derive the global properties from the local ones.

**Definition 3.4 (Local notions)** Given a trajectory based market \(M = S \times \mathcal{H}\), let \(S \in S\) and \(k \geq 0\). Then

1. \((S, k)\) is called an arbitrage-free node with respect to \(\mathcal{H}\) if

$$[\mathcal{H}(S) \cdot \Delta_k X(S') = 0 \forall S' \in S(S, k)] \text{ or } \left[ \inf_{S' \in S(S, k)} \mathcal{H}(S) \cdot \Delta_k X(S') < 0 \right]$$

for all \(H \in \mathcal{H}_S\) (the latter as in (2.6)).

2. \((S, k)\) is called a 0-neutral node with respect to \(\mathcal{H}\) if, for all \(H \in \mathcal{H}_S\),

$$\inf_{S' \in S(S, k)} \mathcal{H}(S) \cdot \Delta_k X(S') \leq 0.$$

\(M\) is called locally arbitrage-free (0-neutral) if each \((S, k)\) is an arbitrage-free (0-neutral) node w.r.t. \(\mathcal{H}\). A node that is not arbitrage-free w.r.t. \(\mathcal{H}\), will be called an arbitrage node w.r.t. \(\mathcal{H}\).

Notice that an arbitrage-free node w.r.t. \(\mathcal{H}\) is always 0-neutral w.r.t. \(\mathcal{H}\). Clearly, there are natural examples of nodes which are 0-neutral w.r.t. \(\mathcal{H}\), but no arbitrage-free w.r.t. \(\mathcal{H}\) (hence these are arbitrage nodes). It is then of interest to indicate that there are results ([9]) that justify option prices obtained for general 0-neutral markets (in particular, these markets may contain 0-neutral nodes which are arbitrage nodes w.r.t. \(\mathcal{H}\)).

Admittedly, attaching the qualifier “w.r.t. \(\mathcal{H}\)” to some of the above notions does not play a substantial role in our paper. In fact, Proposition 3.5 below provides sufficient conditions on trajectory nodes that imply that those nodes are arbitrage-free (0-neutral) w.r.t. any (admissible) \(\mathcal{H}\).

The conclusions in Proposition 3.5 below are consequences of characterizations given by Propositions 4.2 and 4.4 in Subsection 4.1.

Results and notions from convex analysis that we will rely upon throughout the rest of the paper are detailed in Appendix A.2.

**Proposition 3.5** Given a trajectory set \(S\), consider a node \((S, k)\). Then

1. if

$$0 \in \text{ri} \left( \text{co} \left( \Delta X(S(S, k)) \right) \right),$$

\((S, k)\) is an arbitrage-free node w.r.t. any (admissible) \(\mathcal{H}\);

2. if

$$0 \in \text{cl} \left( \text{co} \left( \Delta X(S(S, k)) \right) \right),$$

\((S, k)\) is a 0-neutral node w.r.t. any (admissible) \(\mathcal{H}\).
In accordance with these results, we introduce the following notions which will play a crucial role throughout the remainder of the paper:

**Definition 3.6 (H-Independent local properties)** A node \((S, k)\) is called arbitrage-free if (3.1) is satisfied; it is called 0-neutral if (3.2) is satisfied. We call \(S\) locally arbitrage-free (locally 0-neutral) if every node \((S, k)\) is arbitrage-free (0-neutral).

**Remark 3.7** The above definitions rely on a numeraire (through the perspective function \(X\)). We will show in Section 5 that once the properties hold for one numeraire, they hold for any numeraire.

Therefore, if \(S\) is locally arbitrage-free (locally 0-neutral), then \(M = S \times H\) is locally arbitrage-free (locally 0-neutral) for any (admissible) \(H\).

**Remark 3.8** Condition (3.1) appears in the stochastic literature as equivalent to one step arbitrage-free markets ([4, Lemma 3.42], [7, Prop 3.3.4], [11, Cor 1.50], [13]).

The local notions in Definition 3.6 allow us to ensure global conditions on a trajectory based market. In particular, the results in the rest of this section characterize an arbitrage-free market (0-neutral) by means of arbitrage-free (0-neutral) nodes w.r.t. \(H\).

**Theorem 3.9** (No arbitrage: local implies global) If \(M = S \times H\) is locally arbitrage-free (as per Definition 3.4) and semi-bounded, then \(M\) is arbitrage-free (as per Definition 3.1); see Proof of Theorem 3.9 in Appendix A.1.

In order to establish a converse to Theorem 3.9, consider \(\xi \in \mathbb{R}^d, S \in S\) and \(k \geq 0\), and let us define the function \(\xi_i^{(S,k)} : S \rightarrow \mathbb{R}^d\), for any \(i \geq 0\), by

\[
\xi_i^{(S,k)}(S') = \begin{cases} 
\xi & \text{if } i = k \text{ and } S' \in S_{(S,k)} \\
0 & \text{otherwise}.
\end{cases}
\]

Given \(V_0\), we can obtain from equations (2.4) and (2.5) a sequence of functions \(\{\xi_i^0\}_{i \geq 0}\) in such a way that the sequence

\[
\Xi^{(S,k)} = \{(\xi_i^0, \xi_i^{(S,k)})\}_{i \geq 0}
\]

is self-financing. Also, defining \(N_{\Xi^{(S,k)}}(S') = k + 1\) for all \(S' \in S\), it is easy to see that \(\Xi^{(S,k)}\) is a portfolio. We will call this type of portfolio a restricted portfolio at the node \((S, k)\).

**Proposition 3.10** (No arbitrage: global implies Local) If \(M = S \times H\) is arbitrage free and the restricted portfolios belong to \(H\), then \(S\) is locally arbitrage-free (as per Definition 3.6). In particular, \(M\) is locally arbitrage-free; see Proof of Proposition 3.10 in Appendix A.1.

We now carry out a similar analysis for the notion of 0-neutral. The following Theorem shows that a trajectory based market will be 0-neutral if it is locally 0-neutral:

**Theorem 3.11** (0-neutral: local implies global) Let \(M = S \times H\) be a semi-bounded trajectory market. If \(M\) is locally 0-neutral (as per Definition 3.4), then \(M\) is 0-neutral (as per Definition 3.2); see Proof of Theorem 3.11 in Appendix A.1.

**Proposition 3.12** (0-neutral: global implies local) Let \(M = S \times H\) be a 0-neutral trajectory market such that the restricted portfolios belong to \(H\). Then, any node \((S, k)\) is a 0-neutral node (in particular, \((S, k)\) is 0-neutral with respect to \(H\)); see Proof of Proposition 3.12 in Appendix A.1.
4 Geometric Characterizations

We develop geometric characterizations for the local notions introduced in the previous section. Definition 4.1 below is a stronger version of Definition 3.4 that dispenses of the qualifier “w.r.t. \(H\)” present in the latter definition.

4.1 Local geometric characterizations

**Definition 4.1** (Disperse and 0-neutral sets) Consider a set \(E \subseteq \mathbb{R}^d\); \(E\) is called disperse, if, for each \(h \in \mathbb{R}^d\),

\[
[h \cdot y = 0 \ \forall \ y \in E] \text{ or } \left[ \inf_{y \in E} h \cdot y < 0 \land \sup_{y \in E} h \cdot y > 0 \right].
\]

\[ (4.1) \]

\(E\) is called 0-neutral if, for each \(h \in \mathbb{R}^d\),

\[
\left[ \inf_{y \in E} h \cdot y \leq 0 \land \sup_{y \in E} h \cdot y \geq 0 \right].
\]

\[ (4.2) \]

Notice that (4.2) is equivalent to just requiring the validity of one of the two inequalities appearing in the conjunction in (4.2). Similarly, (4.1) is equivalent to \([h \cdot y = 0 \ \forall \ y \in E]\) or \([\inf_{y \in E} h \cdot y < 0]\) (this later inequality could be replaced by \([\sup_{y \in E} h \cdot y > 0]\)). We have written Definition 4.1 in its present form for emphasis.

**Proposition 4.2** Let \(E \subseteq \mathbb{R}^d\). We have that

\(E\) is disperse if and only if \(0 \in \text{ri}(\text{co}(E))\).

**Proof** Assume first that \(E\) is disperse. In order to proceed to deduce a contradiction, we assume that \(0 \notin \text{ri}(\text{co}(E))\); by the separation Theorem A.11, there exists \(\xi \in \mathbb{R}^d\) such that

- \(\xi \cdot x \geq 0\) for all \(x \in \text{ri}(\text{co}(E))\), and
- \(\xi \cdot x^* > 0\) for some \(x^* \in \text{ri}(\text{co}(E))\).

Then, by means of Proposition A.3, it follows that, for all \(x \in \text{ri}(\text{co}(E))\),

\[
(ax + (1 - \alpha) y) \in \text{ri}(\text{co}(E)) \text{ for all } y \in E \text{ and } \alpha \in (0, 1].
\]

Therefore, \(\xi \cdot (ax + (1 - \alpha) y) = a\xi \cdot x + (1 - \alpha)\xi \cdot y \geq 0\) for all \(y \in E\) and \(\alpha \in (0, 1]\). It then follows that \(\xi \cdot y \geq 0\) for all \(y \in E\). This contradicts the fact that \(E\) is disperse.

Conversely, assume that \(0 \in \text{ri}(\text{co}(E))\). We may assume that there exists \(\hat{y} \in E\) such that \(h \cdot \hat{y} \neq 0\). It is enough to establish that \(\inf_{y \in E} h \cdot y < 0\); we may then assume that there exists \(y^* \in E\) such that \(h \cdot y^* > 0\). As \(y^* \in \text{co}(E)\) and \(0 \in \text{ri}(\text{co}(E))\), it follows from Proposition A.5 in Appendix A.2 that there exists \(\epsilon > 0\) such that \(-\epsilon y^* \in \text{co}(E)\). Then it follows from Theorem A.13 that there exists \(y^{(1)}, \ldots, y^{(d+1)} \in E\) such that

\[
-\epsilon y^* = \lambda_1 y^{(1)} + \cdots + \lambda_{d+1} y^{(d+1)} \text{ with } \sum_{i=1}^{d+1} \lambda_i = 1, \quad \lambda_i \geq 0.
\]

Then

\[
0 > -\epsilon (h \cdot y^*) = \sum_{i=1}^{d+1} \lambda_i (h \cdot y^{(i)}).
\]

Therefore, there must be some \(1 \leq j \leq d + 1\) such that \(h \cdot y^{(j)} < 0\), and then \(\inf_{y \in E} h \cdot y < 0\). \(\square\)

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Remark 4.3 Notice that \((S, k)\) is arbitrage free if and only if \(\Delta X(S(k, k))\) is a disperse set.

Similarly to Proposition 4.2, the following result characterizes the 0-neutral property of \(E\):

**Proposition 4.4** Let \(E \subseteq \mathbb{R}^d\). We have that
\[
E \text{ is 0-neutral if and only if } 0 \in \text{cl}(\text{co}(E)).
\]

**Proof** Assume that \(E\) is 0-neutral and \(0 \notin \text{cl}(\text{co}(E))\). Since the closure of a convex set is a convex set (Proposition A.4) and closed, by Theorem A.11 in Appendix A.2, it follows that there exists \(\xi \in \mathbb{R}^d\) such that \(\inf_{y \in E} \xi \cdot y > 0\), which contradicts our hypothesis.

Assume now that \(0 \in \text{cl}(\text{co}(E))\). It is enough to show that \(\inf_{y \in E} h \cdot y \leq 0\) for any \(h \in \mathbb{R}^d\). To proceed by contradiction, assume that there exists \(h \in \mathbb{R}^d\) and \(\epsilon > 0\) such that \(\epsilon < \inf_{y \in E} h \cdot y \leq h \cdot y\) for all \(y \in E\) (otherwise we are done). From our hypothesis, there exists a sequence \(\{x_j\}_{j=1}^\infty \subseteq \text{co}(D)\) such that \(x_j \to 0\) as \(j \to \infty\). By Theorem A.13 in Appendix A.2, for each \(x_j\) there exists \(y^{(1,j)}, \ldots, y^{(d+1,j)} \in E\) such that
\[
x_j = \lambda^j_1 y^{(1,j)} + \cdots + \lambda^j_{d+1} y^{(d+1,j)} \quad \text{with} \quad \sum_{i=1}^{d+1} \lambda^j_i = 1, \quad \lambda^j_i \geq 0.
\]

Then
\[
0 = h \cdot 0 = h \cdot \left( \lim_{j \to \infty} x_j \right) = \lim_{j \to \infty} (h \cdot x_j)
= \lim_{j \to \infty} \sum_{i=1}^{d+1} \lambda^j_i (h \cdot y^{(i,j)}) \geq \lim_{j \to \infty} \sum_{i=1}^{d+1} \lambda^j_i \epsilon = \epsilon,
\]
which is a contradiction, which concludes the proof. \(\Box\)

**Lemma 4.5** below uses the following notation: for \(E \subseteq \mathbb{R}^d\) and \(x_0 \in \mathbb{R}^d\), let
\[
E - x_0 \equiv \{ x - x_0 : x \in E \} \subseteq \mathbb{R}^d.
\]

Since the translation \(t_{x_0} : \mathbb{R}^d \to \mathbb{R}^d\), given by \(t_{x_0}(x) = x - x_0\), is of the form (A.2) in Appendix A.2, and is a homeomorphism, we have the following lemma:

**Lemma 4.5**
\[
0 \in \text{ri} (\text{co}(E - x_0)) \quad \text{if and only if} \quad x_0 \in \text{ri} (\text{co}(E)).
\]
\[
0 \in \text{cl} (\text{co}(E - x_0)) \quad \text{if and only if} \quad x_0 \in \text{cl} (\text{co}(E)).
\]

### 4.2 Convexity preserving maps

In order to identify transformations that preserve no-arbitrage (0-neutrality), and in view of Proposition 4.2 (Proposition 4.4) and Lemma 4.5, we first look for transformations \(F : \mathbb{R}^d \to \mathbb{R}^d\) preserving relative interiors or closures of convex sets in \(\mathbb{R}^d\).

The notions introduced below are expanded in Appendix A.2, where we also provide due references and introduce related definitions and further results.
Definition 4.6 (Strict Inversely Convexity Preserving (SICP)) Let $V$ and $V'$ be real linear spaces, and $C \subset V$ a nonempty convex subset. A map $g : C \to V'$ is called strict inversely convexity preserving (SICP) if
\[ g((x,y)) \subseteq (g(x),g(y)) \quad \text{for all } x,y \in C, \]
where $(x,y) = \{tx + (1-t)y : 0 < t < 1\}$ (with a similar definition for $[x,y]$, see Appendix A.2). Moreover, $g$ is said to preserve segments strictly if equality holds in (4.3).

The next lemma provides no-arbitrage preserving conditions on a transformation $F$ that are equivalent to the SICP property.

Lemma 4.7 Let $C \subseteq \mathbb{R}^d$ be a convex set and $F : C \to \mathbb{R}^d$. The following three statements are equivalent:
1. $F$ is a SICP map.
2. For any $E \subset C$,
\[ x \in \text{ri}(\text{co}(E)) \implies F(x) \in \text{ri}(\text{co}(F(E))). \] (4.4)
3. For any $E \subset C$ and $x \in E$,
\[ \text{If } (E-x) \text{ is a disperse set, then } (F(E)-F(x)) \text{ is a disperse set.} \]

Proof (1 $\implies$ 2) Let $x \in \text{ri}(\text{co}(E))$ and $b' \in \text{co}(F(E))$. Assume first that $F(C)$ is contained in a straight line. Fix $b \in \text{co}(E)$, from Corollary A.6, there exists $a \in \text{co}(E)$ such that $x \in (a,b)$. Then by our hypothesis on $F$ and Proposition A.7, $F(a), F(b) \in F(\text{co}(E)) \subset \text{co}(F(E))$, and
\[ F(x) \in F((a,b)) \subset (F(a), F(b)). \]
Now, since $\text{co}(F(E))$ is a segment, because it is contained in a straight line, it follows that if $b' \in (F(x), F(b))$, or $F(b) \in (F(x), b')$, then $F(x) \in (F(a), b') \subset \text{co}(F(E))$. On the other hand, $F(x) \in (F(b), b') \subset \text{co}(F(E))$. Thus, in any case, by Corollary A.6, $F(x) \in \text{ri}(\text{co}(F(E)))$.

If $F(C)$ is not contained in a straight line, by Theorem A.10, $F$ preserves segments strictly and $\text{co}(F(E)) = F(\text{co}(E))$, so $b' = F(b)$ with $b \in \text{co}(E)$.

As before, there exists $a \in \text{co}(E)$ such that $x \in (a,b)$. Then $F(a) \in \text{co}(F(E))$ and $F(x) \in F((a,b)) = (F(a), F(b))$, which also leads to $F(x) \in \text{ri}(\text{co}(F(E)))$.

(2 $\implies$ 1) Considering the case $E = \{a,b\}$,
\[ (a,b) = \text{ri}(\text{co}(E)) \quad \text{and} \quad (F(a), F(b)) = \text{ri}(\text{co}(F(E))). \]
Now, from our hypothesis $F(x) \in (F(a), F(b))$ for any $x \in (a,b)$, and therefore $F$ is a SICP map.

Finally, the equivalence (2 $\iff$ 3) follows by making repeated use of Proposition 4.2 and Lemma 4.5. \qed

Remark 4.8 Observe that the implication (2 $\implies$ 1) in the previous Lemma is still valid if (4.4) holds just for two point sets $E = \{a,b\} \subset C$.

In a fashion analogous to the relationship between SICP maps and no-arbitrage nodes, we look for a suitable class of maps that preserve 0-neutral nodes. $F$ preserves $\text{cl}(\text{co}(E))$ if, for $x \in [a,b] = \text{cl}(\text{co}([a,b]))$, it holds that
\[ F(x) \in \text{cl}(\text{co}([F(a), F(b)])) = [F(a), F(b)]. \]
That is, $F$ needs to be inversely convexity preserving (see Proposition A.7 in Appendix A.2). However this condition on its own is not sufficient (see the next Lemma and Example 4.10).

**Lemma 4.9** Let $C \subseteq \mathbb{R}^d$ be a convex set, let $E \subset C$ and let $F : C \rightarrow \mathbb{R}^d$ be a continuous inversely convexity preserving map. If $x_0 \in \text{cl(co}(E))$, then $F(x_0) \in \text{cl(co}(F(E)))$.

**Proof** By the continuity of $F$, for all $E \subseteq C$ it holds that $F(\text{cl}(E)) \subseteq \text{cl}(F(E))$. Furthermore, since $F$ is a continuously convexity preserving map, $F(\text{co}(E)) \subseteq \text{co}(F(E))$. Thus, since $x_0 \in \text{cl(co}(E))$, $F(x_0) \in F(\text{cl(co}(F(E))) \subseteq \text{cl}(F(\text{co}(E))) \subseteq \text{cl(co}(F(E)))$. □

**Example 4.10** The hypothesis of continuity in Lemma 4.9 cannot be removed; to see this, consider that $F : \mathbb{R} \rightarrow \mathbb{R}$, given by

$$F(x) = \begin{cases} x & \text{if } x \leq 0 \\ x + 1 & \text{if } x > 0, \end{cases}$$

is inversely convexity preserving, though not continuous and $0 \in \text{cl}(\{0, 1]\}$, but that $F(0) = 0 \notin \{1, 2\} = \text{cl(co}(\{0, 1]\})$.

### 4.3 Induced transformations

As indicated in Section 2.1, we have taken a standard view in which the original sequence $S_i$ is given in a currency numeraire and then the sequence $X(S_i)$ is given in another (arbitrary) numeraire. Since we look for transformations between trajectories of financial markets that preserve their local properties, we will be dealing with two associated functions, $f$ and $F$, the former acting on $S_i$ and the latter on $X(S_i)$. Thus, we will have $\mathbb{R}^{d+1} \xrightarrow{L} \mathbb{R}^{d+1}$ and $\mathbb{R}^d \xrightarrow{F} \mathbb{R}^d$. One could proceed differently and develop an approach which abstracts away this multiplicity; nonetheless, we have decided to proceed the way we do, as in practice that is how data is usually presented. This decision makes our results more readily applicable, albeit at the price of some complications.

Let $X$ and $X'$ be the perspective functions over $\mathbb{R}^{d+1}$ and $\mathbb{R}^{d+1}$, respectively, as defined in (2.2). Since local properties are based on properties of discounted values, the function $f$ should induce a function $F : \mathbb{R}^d \rightarrow \mathbb{R}^d$ in such a way that the following diagram commutes:

$$\begin{array}{ccc}
\text{dom } X & \xrightarrow{f} & \text{dom } X' \\
\downarrow & & \downarrow \\
\text{Im } X & \xrightarrow{F} & \text{Im } X'
\end{array}$$

that is, for $s \in \text{dom } X$, $F(X(s)) = X'(f(s))$. Therefore, if $X(s) = X(\tilde{s})$ for $s, \tilde{s} \in \text{dom } X$, we will require that $X'(f(s)) = X'(f(\tilde{s}))$, which gives a condition on $f$, as we describe next.

Assuming that $s = (s^0, \ldots, s^d)$, $\tilde{s} = (\tilde{s}^0, \ldots, \tilde{s}^d)$,

$$\left(\frac{s^1}{s^0}, \ldots, \frac{s^d}{s^0}\right) = X(s) = X(\tilde{s}) = \left(\frac{\tilde{s}^1}{\tilde{s}^0}, \ldots, \frac{\tilde{s}^d}{\tilde{s}^0}\right) \Leftrightarrow s = \frac{s^0}{s^0} \tilde{s},$$

from whence $f$ needs to satisfy

$$f(\lambda s) = \mu_{\lambda, s} f(s), \quad \lambda, \mu_{\lambda, s} > 0, \quad \text{for any } s \in \mathbb{R}^{d+1}.$$  

We will require (4.6) in our main result, Theorem 5.4 (as well as in Theorem 5.8).
Lemma 4.7 will be used in Corollary 5.6 to establish that the SICP property for the induced map $F$ is necessary and sufficient for $f$ to preserve the no-arbitrage property of any given node in any locally arbitrage-free trajectory set. Lemma 4.11 item 1, below, provides sufficient conditions on $f$ to establish the SICP property of $F$; on the other hand, Example 4.12 shows that the assumptions on $f$, while being sufficient, are not necessary.

**Lemma 4.11** Let $f : \text{dom } X \to \text{dom } X'$ be a function satisfying (4.6). Then there exists a unique map $F : \text{Im } X \to \text{Im } X'$ which makes commutative the diagram (4.5). Moreover,

1. if $f$ is (strict) inversely convexity preserving, then $F$ is (strict) inversely convexity preserving;
2. $F$ is continuous if and only if $f$ is continuous.

**Proof** For all $x \in \text{Im } X$, there exists $s \in \text{dom } X$ such that $X(s) = x$. The only way to define $F$ is then $F(x) = X'(f(s))$ for all $x \in \mathbb{R}^d$, and it is well-defined by condition (4.6).

Let us see that $F$ is a SICP map if $f$ is assumed to satisfy that property. Fix $\hat{x}, \hat{x} \in \text{Im } X$ and let $x \in \text{Im } X$ such that $x = \alpha \hat{x} + (1 - \alpha)\hat{x}$ with $0 < \alpha < 1$. Then, there exists $\hat{s}, \hat{s} \in \text{dom } X$ such that $X(\hat{s}) = \hat{x}$ and $X(\hat{s}) = \hat{x}$. Moreover, since $X$ is a strict segment preserving map (Theorem A.9), there exists $\beta \in (0, 1)$ such that

$$x = X(\beta \hat{s} + (1 - \beta)\hat{s}).$$

Then, since $X'$ is strict segment preserving,

$$F(x) = X'(f(\beta \hat{s} + (1 - \beta)\hat{s})) \in (X'(f(\hat{s})), X'(f(\hat{s}))) = (F(\hat{x}), F(\hat{x})).$$

The proof for the inversely convexity preserving case is similar. This gives item 1.

For item 2, observe that the perspective functions $X, X'$ are continuous and open. The last assertion follows because if $Q$ is an open cube in $\mathbb{R}^d$ and $a^0 < b^0$ are positive real numbers, then

$$X((a^0, b^0) \times Q) = \bigcup_{r \in (a^0, b^0)} \frac{1}{r}Q$$

is open in $\mathbb{R}^d$. Thus, by composition, $F$ is continuous if and only if $f$ is continuous. \qed

**Example 4.12** The converse in item 1 of Lemma 4.11 is not valid. Consider $f : \{(x, y, z) \in \mathbb{R}^3 : x > 0\} \to \{(x, y, z) \in \mathbb{R}^3 : x > 0\}$ given by

$$f(x, y, z) = \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}, \frac{z}{x^2 + y^2}\right).$$

The induced function $F$ is the identity function on $\mathbb{R}^2$ and $f$ is not inversely convexity preserving because $(1, 0, 0) \in \{(1, -1, 0), (1, 1, 0)\}$, but

$$f(1, 0, 0) = (1, 0, 0) \notin (f(1, -1, 0), f(1, 1, 0)) = ((1/2, -1/2, 0), (1/2, 1/2, 0)).$$

Under the additional hypothesis that the Im $F$ contains a nondegenerate triangle, the next Theorem characterizes those $f : \text{dom } X \to \text{dom } X'$ inducing a SICP map $F$. That hypothesis on Im $F$ is equivalent to Im $F$ not being contained in a straight line. As a complement, Lemma 4.14 shows that this last condition on Im $F$ holds if and only if Im $f$ is not contained in a 2 dimensional subspace.

The appearance of “0” in $\frac{f^k(s)}{L^k(s)}$ below merely reflects our arbitrary choice of $S^0$ as numeraire; choosing $S^k$ as numeraire will result in the appearance of $\frac{f^k(s)}{L^k(s)}$ in the next result (see Section 6 for an example).
Theorem 4.13  Assume that \( f : \text{dom } X \rightarrow \text{dom } X' \) satisfying (4.6) induces a SICP map \( F \) such that \( \text{Im } F \) is not contained in a straight line. Then

\[
f(s) = \frac{f^0(s)}{L^0(s)} L(s),
\]

with \( f^0 \) (its first coordinate function) satisfying (4.6), \( L : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^{d+1} \), a linear map, with \( L^0, f^0 > 0 \) on \( s^0 > 0 \), \( (L^0 \text{ the first coordinate of } L) \).

Conversely, if \( f \) has the form (4.7) and satisfies the properties listed after the formula display, then it induces a SICP map \( F \).

**Proof**  Let us consider first that \( \text{dom } X = \{ s \in \mathbb{R}^{d+1} : s^0 > 0 \} \), so \( \text{dom } F = \{ x \in \mathbb{R}^d : x^i > 0 \forall i \} \) is convex. Then, by Theorem A.10,

\[
X'(f(s)) = F(X(s)) = \frac{A^1(X(s)) + b^1, \ldots, A^d(X(s)) + b^d}{B(X(s)) + c}
\]

It then follows that, for \( 1 \leq i \leq d' \),

\[
\frac{f^i(s)}{f^0(s)} = \frac{A^i(X(s)) + b^i}{B(X(s)) + c} = \frac{a_{i,0} s^0 + \cdots + a_{i,d} s^d + b^i}{B^1 s^0 + \cdots + B^d s^d + c},
\]

which can be written as

\[
f^i(s) = f^0(s) \frac{L^i(s)}{L^0(s)}, \quad \text{with}
\]

\[
L^i(s) = b^i s^0 + a_{i,1} s^1 + \cdots + a_{i,d} s^d, \quad \text{and} \quad L^0(s) = c s^0 + B^1 s^1 + \cdots + B^d s^d.
\]

From this, defining \( L(s) = (L^0(s), L^1(s), \ldots, L^d(s)) \), (4.7) holds with the expected conditions, since \( f^0 \) satisfies (4.6) because \( f \) does, as do both \( f^0, L^0 > 0 \) on \( s^0 > 0 \).

Assume now that \( \text{dom } X = \{ s \in \mathbb{R}^{d+1} : s^0 > 0 \} \), which implies that \( \text{dom } F = \mathbb{R}^d \), so by [16, Cor 1], (4.8) can be written with \( B(x) + c \equiv 1 \). Consequently, (4.7) holds, with \( L^0(s) = s^0 \).

Conversely, if \( f \) has the form (4.7) with the required conditions, then it satisfies (4.6), because \( f^0, L, L^0 \) satisfy (4.6), by hypothesis and linearity, respectively, and consequently, by Lemma 4.11, there exists \( F \) such that \( F(X(s)) = X'(f(s)) \). Let us now show that \( F \) is SICP:

\[
X'(f(s)) = \left( \frac{L^1(s)}{L^0(s)}, \ldots, \frac{L^d(s)}{L^0(s)} \right),
\]

where, for \( 1 \leq i \leq d' \),

\[
\frac{L^i(s)}{L^0(s)} = \frac{a_{i,0} s^0 + \cdots + a_{i,d} s^d}{a_{0,0} s^0 + \cdots + a_{0,d} s^d} = \frac{a_{i,0} + a_{i,1} x^1 + \cdots + a_{i,d} x^d}{a_{0,0} + a_{0,1} x^1 + \cdots + a_{0,d} x^d} = \frac{a_{i,0} + A^i(X(s))}{a_{0,0} + B(x)}.
\]

In the last expression, \( A^i(x) = a_{i,1} x^1 + \cdots + a_{i,d} x^d \), and \( B(x) = a_{0,1} x^1 + \cdots + a_{0,d} x^d \). Defining \( A = (A^1, \ldots, A^d) \) and \( b = (a_{1,0}, \ldots, a_{d,0}) \), it follows that

\[
F(x) = \frac{b + A(x)}{a_{0,0} + B(x)}.
\]

which is SICP, by Theorem A.9. \( \Box \)

**Lemma 4.14**  Assume that \( f : \text{dom } X \rightarrow \text{dom } X' \) is a function satisfying (4.6) and that \( F \) is the induced function as in Lemma 4.11. Then, \( \text{Im } F \) is contained in a straight line if and only if \( \text{Im } f \) is contained in a 2-dimensional subspace.
Assuming that $f(s) = (y^0, \ldots, y^{d'})$,
\[ F(X(s)) = X'(f(s)) = \frac{1}{y^0}(y^1, \ldots, y^{d'}). \]

It follows that
\[ \operatorname{Im} F = \{ z \in \mathbb{R}^{d'} : (1, z) \in \lambda(\operatorname{Im} f), \text{ for some } \lambda > 0 \}. \]

If $\operatorname{Im} f \subset \pi$, a 2-dimensional subspace, then $\operatorname{Im} F \subset \{ z \in \mathbb{R}^{d'} : (1, z) \in \pi \}$, and this set is contained in the straight line $\pi \cap \{ y^0 = 1 \} \subset \mathbb{R}^{d'+1}$.

Conversely, assume that there exist $s^1, s^2, s^3 \in \text{dom} X$ such that $f(s^1), f(s^2), f(s^3)$ are l.i. Since $\operatorname{Im} F$ is contained in a straight line, it follows that there exists $\alpha \in \mathbb{R}$ such that
\[ F(X(s^3)) = \alpha F(X(s^1)) + (1 - \alpha) F(X(s^2)) = \alpha X'(f(s^1)) + (1 - \alpha) X'(f(s^2)), \]
which leads to the contradiction $f(s^3) = \frac{f''(s^3)}{f'(s^3)} \alpha f(s^1) + \frac{f''(s^3)}{f'(s^3)} (1 - \alpha) f(s^2)$. \hfill $\Box$

5 No Arbitrage Invariance

This section studies a class of transformations that do not change a given node’s local properties of being arbitrage-free (this latter notion as per Definition 3.6). We also provide an explicit characterization for such symmetry transformations; this is achieved under a general and weak condition restricting their ranges.

As a special case, we will prove that the no-arbitrage property is unchanged under a change of numeraire. We also describe similar results that apply for the property of 0-neutrality, and therefore need to also pursue some developments that apply to this concept as well. In general, the class of transformations studied should represent symmetries obeyed by any type of functional relationship among assets’ prices resulting from no arbitrage considerations. In particular, if prices $S$ satisfy a $h(S) = 0$ relation, one then expects $h(S') = 0$ where $S$ and $S'$ are related by a no-arbitrage symmetry as per Definition 5.1 below. This fact is illustrated with an example in Section 6.

Let $S$ and $S'$ be trajectory sets with $d + 1$ assets and $d' + 1$ assets, respectively. A transformation of $S$ onto $S'$ will be given by a function $f : \mathbb{R}^{d+1} \to \mathbb{R}^{d'+1}$, which will be called a trajectory transformation. That is, to a trajectory $S = (S, W, m) \in S$ corresponds a trajectory $S' = (S', W', m') \in S'$, where $S_k' = f(S_k)$, $k \geq 0$ and $W', m'$ are transformed in consequence. For instance, if $W$ represents the quadratic variation of the logarithm of the assets’ prices, then
\[ W'_k = \sum_{i=0}^{k-1} (\log f(S_{i+1}) - \log f(S_i))^2. \]

This example illustrates a case when $W'$ can be obtained from $S'$. In other cases, when this is not possible, $W'$ and $m'$ should be prescribed, but how this is actually done does not affect the developments in the present section.

Recall from Definition 3.4 that local conditions are based on properties of the increment set $\Delta X(S(s, k))$, where $(S, k)$ is a node of the market model. This set is totally determined by the values taken by the trajectories in the stage $k + 1$ and the value of $S_k$. To make this fact explicit, for each node $(S, k)$, we introduce a notation for the set of reachable prices:
\[ \Sigma_k(S) \equiv \{ \hat{S}_{k+1} : \hat{S} = (\hat{S}, \hat{W}, \hat{m}) \in S(s, k) \} \subseteq \mathbb{R}^{d+1}. \]
Definition 5.1 (No-Arbitrage Symmetry (NAS) Transformation) A trajectory transformation $f$, as introduced above, which leaves invariant the arbitrage-free property (0-neutral property), as per Definition 3.6, of any locally arbitrage-free (locally 0-neutral) trajectory set $S$ with $d + 1$ assets, will be called a no-arbitrage symmetry (0-neutral symmetry).

More specifically, $f : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^{d'} + 1$, is a NAS if and only if

$$X(S_k) \in \text{ri}(\text{co}(X(\Sigma_k(S)))) \implies X'(f(S_k)) \in \text{ri}(\text{co}(X'(f(\Sigma_k(S))))),$$

(5.1)

and this holds for any possible trajectory set $S$ that is locally arbitrage-free (0-neutral).

Therefore, if the node $(S, k)$ is arbitrage-free, so too will be $(S', k)$ if $f$ is a no-arbitrage symmetry (similarly for a 0-neutral symmetry), and this is required to hold for any trajectory set $S$ that is locally arbitrage-free. This remark also shows that the composition of no-arbitrage symmetries (0-neutral symmetries) is a no-arbitrage symmetry (0-neutral symmetry). We may refer to either type of symmetry as NAS (No-Arbitrage Symmetries) when there is no need to be specific.

Remark 5.2 The above notions depend on a choice of numeraire through Definition 3.6, but we will prove in Corollary 5.7 that a symmetry transformation remains as such under a numeraire change. Of course, the interest is in general symmetry transformations $f : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^{d'} + 1$ that behave so for any possible node in any possible trajectory set (with corresponding dimension $d$ and satisfying the arbitrage-free/0-neutral property), as we have required in our definition.

The next proposition (which follows from Proposition 3.5 in Section 3.1 and Lemma 4.5 in Section 4.2) shows that local conditions can be rewritten in terms of the set $\Sigma_k(S)$.

Proposition 5.3 Given a trajectory based set $S$, $S = \{ (S_i, W_i, m) \}_{i \geq 0} \in S$ and an integer $k \geq 0$,

1. the node $(S, k)$ is arbitrage-free if and only if $X(S_k) \in \text{ri}(\text{co}(X(\Sigma_k(S))))$;
2. the node $(S, k)$ is 0-neutral if and only if $X(S_k) \in \text{cl}(\text{co}(X(\Sigma_k(S))))$.

Theorem 5.4 (Arbitrage-Free Invariance) Assume $f : \text{dom} \ X \rightarrow \text{dom} \ X'$ to be a map satisfying (4.6), and that the function $F$, induced by Lemma 4.11, is SICP. Given a trajectory set $S$, let $S \in S$ and $k \geq 0$. If $(S, k)$ is an arbitrage-free node, then $(S', k)$, where $S'_i = f(S_i)$ $i \geq 0$, is an arbitrage-free node in the transformed trajectory set $S'$; that is,

$$X'(f(S_k)) = F(X(S_k)) \in \text{ri}(\text{co}(F[X(\Sigma_k(S)]))) = \text{ri}(\text{co}(X'(f(\Sigma_k(S))))),$$

and so $f$ is a no-arbitrage symmetry.

Proof We know from Lemma 4.11 that there exists $F : \text{dom} \ X \rightarrow \text{dom} \ X'$ given by $F(x) = X'(f(s))$, where $s \in \text{dom} \ X$ such that $X(s) = x$. Thus, since, by hypothesis, it is a SICP map, from Proposition 5.3 and Lemma 4.7, it follows that $F(X(S_k)) \in \text{ri}(\text{co}(F[X(\Sigma_k(S)])))$. □

Remark 5.5 By Lemma 4.11 item 1, if $f$ satisfies (4.6) and it is SICP, then the induced $F$ satisfies the hypothesis of Theorem 5.4. Also notice that if $\text{Im}F$ contains a nondegenerate triangle, by Theorem 4.13, $f$ is of the form given by (4.7).

Corollary 5.6 (Explicit Characterization) Assume that $f : \text{dom} \ X \rightarrow \text{dom} \ X'$ satisfies (4.6). We then have the following:
1. If \( f \) is a no-arbitrage symmetry (as per Definition 5.1), then \( F \) (as appearing in Lemma 4.11) is SICP. Moreover, if \( \text{Im} f \) is not contained in a 2-dimensional subspace, then \( f \) is characterized by expression (4.7).

2. Conversely, if \( f \) has the form (4.7), then it is a no-arbitrage symmetry.

**Proof** We recall that (4.6) assures us of the existence of the induced function \( F \) as in Lemma 4.11. Assuming that \( f \) is a no-arbitrage symmetry for trajectory sets with \( d + 1 \) assets, then, for any node \((S, k)\) from a locally arbitrage-free trajectory set \( S \), by Proposition 5.3, \( F \) verifies that \( F((X(S_k))) \in \text{ri} (\text{co}(X(\Sigma_k(S)))) \). Since for any \( a, b \in \mathbb{R}^d \) and \( x \in (a, b) = \text{ri} (\text{co} \{a, b\}) \), there exists a binary trajectory set with a node \((S, k)\) such that \( \text{ri} (\text{co}(X(\Sigma_k(S)))) = \text{ri} (\text{co} \{a, b\}) \) and \( X(S_k) = x \), it then follows from Lemma 4.7 (see Remark 4.8) that \( F \) is SICP. Moreover, if \( \text{Im} f \) is not contained in a 2-dimensional subspace, Lemma 4.14 implies that \( \text{Im} F \) is not contained in a straight line. Finally by Theorem 4.13, \( f \) takes the form (4.7). This proves 1.

For the converse, if \( f \) has the form (4.7), the converse of Theorem 4.13 implies that the induced function \( F \) is SICP. Thus by Lemma 4.7 and Theorem 5.4, \( f \) is a no-arbitrage symmetry. \( \square \)

Observe that the composition of no-arbitrage symmetries of the form (4.7) is again of this form.

A transformation of interest in financial terms is the one that changes the market model’s numeraire. Let us take \( S \) with \( S_k^0, S_k^1 > 0 \), for all \( k \geq 0 \) and all \( S \in S \). For the purposes of the next proposition, let \( S' \) be defined by \( S' \in S' \) if and only if \( S_k' = f(S_k) \) for some \( S \in S \), where \( f(s_0, s_1, s_2, \ldots, s_d) \equiv (s_1, s_0, s_2, \ldots, s_d) \).

**Corollary 5.7** (Change of Numeraire is a NAS) Consider \( S \) to be a trajectory set such that \( S_k^0, S_k^1 > 0 \) for all \( k \geq 0 \) and all \( S \in S \), and let \( S' \) and \( f \) be as introduced above. Then, if \( S \) is locally arbitrage-free, it follows that \( S' \) is locally arbitrage-free as well.

**Proof** Since \( f \) is a linear map, it follows from Theorem A.9 in Appendix A.2 that it is a strict segment preserving map (in particular it is a SICP map). From Remark 5.5 we see that Theorem 5.4 applies to our particular \( f \). Therefore, \( f \) is a NAS, and so (5.1) implies that \( S' \) is locally arbitrage-free. \( \square \)

Our next goal is to find market transformations \( f \) that preserve 0-neutral nodes (i.e. 0-neutral symmetries). From Lemma 4.9 we know that the induced transformation \( F \) needs to be continuous and inversely convexity preserving in order to preserve the closure of convex sets. The following Theorem shows that these conditions are sufficient to obtain a 0-neutral symmetry, as per Definition 5.1:

**Theorem 5.8** (0-Neutral Invariance) Let \( f : \text{dom} \ X \to \text{dom} \ X' \) be a continuous map satisfying (4.6) and the function \( F \), induced by Lemma 4.11, is inversely convexity preserving. Given a trajectory set \( S \), let \( S \in S \) and \( k \geq 0 \). If \((S, k)\) is a 0-neutral node, then \((S', k)\), where \( S_i' = f(S_i) \) \( i \geq 0 \) is a 0-neutral node in the transformed trajectory set \( S' \), i.e.,

\[
X'(f(S_k)) = F(X(S_k)) \in \text{cl}(\text{co}(X(\Sigma_k(S)))) = \text{cl}(\text{co}(X'[f(\Sigma_k(S)]))).
\]

so \( f \) is a 0-neutral symmetry.

**Proof** We know from Lemma 4.11 in Section 4.2 that there exists a continuous map...
\[ F : \mathbb{R}^d \rightarrow \mathbb{R}^d, \] given by \( F(x) = X'(f(s)) \), where \( s \in \text{dom} \ X \) such that \( X(s) = x \). Moreover, by hypothesis, this map is inversely convexity preserving.

Thus, since, by hypothesis, \( X(S_k) \in \text{cl}\left(\text{co}\left(X(\Sigma_k(S))\right)\right) \), from Lemma 4.9 in Section 4.2, it follows that \( X'(f(S_k)) \in \text{cl}\left(\text{co}\left(X'(f(\Sigma_k(S)))\right)\right) \).

By the converse of Theorem 4.13, if \( f \) is given as in the expression (4.7), with the prescribed conditions, then the induced function \( F \) has the expression (4.9). Therefore, it is also inversely convexity preserving and continuous by Theorem A.9, so \( f \) preserves 0-neutral nodes and hence, it is a 0-neutral symmetry.

In the 0-neutral market definition, the selection of an explicit numeraire is required. The proof of the next result is analogous to Corollary 5.7, and so it is omitted.

**Corollary 5.9** (Change of Numeraire is a 0-Neutral Symmetry) Consider \( S \) to be a trajectory set such that \( S^0_k, S^1_k > 0 \) for all \( k \geq 0 \) and all \( S \in S \), and let \( S' \) and \( f \) be as in Corollary 5.7. Then, if \( S \) is locally 0-neutral, it follows that \( S' \) is locally 0-neutral as well.

### 6 Example

We will provide a slightly non-traditional development on the call-put parity relationship. This is a simple relation among prices of certain assets; it is derived in many textbooks and can be obtained through a no-arbitrage based proof. We will derive it under the weaker hypothesis of 0-neutrality and relate the relationship to NAS (No-Arbitrage Symmetries). Our main point of revisiting the call-put parity is that it will allow us to provide an explicit example of NAS (besides a change of numeraire) as well as to illustrate their meaning in this context.

#### 6.1 Call-Put Parity Under 0-Neutrality

Consider an arbitrary time evolution of four assets \( S_t \equiv (C_t, P_t, Y_t, B_t), 0 \leq t \leq T \). We require that

\[
C_T = (Y_T - B_T)_+, P_T = (B_T - Y_T)_+, \text{ and } B_T = K \text{ where } K \text{ is a constant.}
\]

That is to say that \( C \) is a European call written on asset \( Y \), with strike \( K \) and expiration \( T \). It is similar for the European put \( P \). \( B \) is a bond. Clearly, \( (C_T - P_T - Y_T + B_T) = 0 \), which can be thought of as a boundary condition. Under an appropriate no-arbitrage assumption the call-put parity is the following result [15, Cor 1.4.2]:

\[
(C_t - P_t - Y_t + B_t) = 0, \quad \forall \quad 0 \leq t \leq T.
\]

That is, under the said conditions, no-arbitrage constrains the evolution of the four assets according to (6.1).

We will add details on dimensions that are neglected in the above formulation; dispensing with units/dimensions is standard in the literature but making them explicit is relevant to our philosophy as a change of units should be a NAS (but we do not explore this view in the paper). We will insert appropriate dimensions/units whenever relevant but switch (or alternate) to suppressing units (as usual) whenever the relevant dimensions have been made clear. We write \( Z = (Z)[Z] \) where \( (Z) \) is the (dimensionless) numerical value and \( [Z] \) are the dimensional units of the variable \( Z \).
We will have \([C_i] = \frac{1}{1_C}\), which requires (see below) the insertion of a dimensional constant \(a\) with units \([a] = \frac{1}{1_C}\) with \((a)\) representing the number of shares associated to a call option. We will take \((a) = 1\), but an arbitrary value of \((a)\) will have the effect of multiplying the call-put parity by \((a)\) (usually, in practice \((a) = 100\)). Thus, \((a)\) represents the number of shares per call contract; this is not an artificial insertion as it is a feature of traded options. Similarly, \(P_i\) will contain a dimensional constant \(b\) with \([b] = \frac{1}{1_P}\) with \((b)\) representing the number of shares associated to a put option; we will take \((b) = (a) = 1\) in order to derive the put-call parity relationship (as indicated, one can multiply the resulting expression by an arbitrary dimensionless number \((a)\)).

In order to provide a derivation of (6.1) under 0-neutrality, we first express the above setting in our trajectorial framework. The above formulation is in continuous time but we consider this to be a nonessential point (as we argue below). We will work with trajectories of the form \(S_i = (S_i, t_i, m) = (S_i^0, S_i^1, S_i^2, S_i^3, t_i, m) = (C_i, P_i, Y_i, B_i, t_i, m)\), where \(0 = t_0 < t_1 < \ldots < t_m = T\). Clearly, the times \(t_i\) are trajectory dependent; as a particular case, we could take \(t_i = \frac{i}{S}\), \(0 \leq i \leq M\) for a given constant \(M\). Given that the argument will apply to any trajectory set with these coordinates, we can approximate any arbitrary time \(t\) by taking \(M\) larger.

Let \(S\) denote any 0-neutral trajectory set with the above introduced coordinates that obeys

\[
S_{M(S)}^0 = C_{M(S)} = a \left( S_{M(S)}^2 - [K] S_{M(S)}^3 \right)_+ + a \left( Y_{M(S)} - K \frac{1}{1_B} \right)_+, \\
S_{M(S)}^1 = P_{M(S)} = b \left( [K] S_{M(S)}^3 - S_{M(S)}^2 \right)_+ + b \left( K \frac{1}{1_B} - Y_{M(S)} \right)_+, \\
S_{M(S)}^3 = B_{M(S)} = (K) \frac{1}{1_B} \text{ for all } S. K \text{ is a dimensional constant with } [K] = \frac{1}{1_S};
\]

\(K\) represents the number of bond units per share and so \(K \frac{1}{1_B}\) is the strike price. Thus, we have \([C_i] = [a] \frac{1}{1_C}, [P_i] = [b] \frac{1}{1_P}, [Y_i] = \frac{1}{1_Y}\) and \([B_i] = \frac{1}{1_B}\). Moreover, assume \(M(S) = m\) to be a stopping time in the sense that if \(S_k = S_k\) for all \(0 \leq k \leq M(S)\), then \(M(S') = M(S)\). Finally, we also assume that \(t_{M(S)} = T\). Such an \(S\) will be called admissible.

The previous call-put parity is now written with units and taking \((a) = (b):\)

\[
\beta(S_i) \equiv (1_C C_i - 1_P P_i - 1_Y Y_i + 1_B B_i) = 0, \quad \forall \ 0 \leq i \leq M. \tag{6.2}
\]

This shows that no-arbitrage, under the said conditions, constrains the evolution of the four assets accordingly to (6.2). (6.2) holds if and only if

\[
1_B \pi(X(S_i)) \equiv 1_B \left[ \frac{C_i}{B_i} - \left( \frac{P_i}{B_i} \right)_+ \left( \frac{Y_i}{B_i} + 1 \right) \right] = 0,
\]

where, as defined before,

\[
X(S_{M(S)}) = \left( \frac{S_{M(S)}^0}{S_{M(S)}^3}, \frac{S_{M(S)}^1}{S_{M(S)}^3}, \frac{S_{M(S)}^2}{S_{M(S)}^3}, \frac{S_{M(S)}^3}{S_{M(S)}^3} \right) = \left( \frac{C_{M(S)}}{B_{M(S)}}, \frac{P_{M(S)}}{B_{M(S)}}, \frac{Y_{M(S)}}{B_{M(S)}} \right)
\]

(notice that we are abusing the notation by using \(S^3\) as numeraire instead of the usual \(S^0\)).

To establish (6.2), we will return now to the usual practice of supressing the units. In particular, in the proof below, \(X(S_i)\) will be interpreted as the coordinates without the dimensions, i.e. \((\frac{C_i}{B_i}, \frac{P_i}{B_i}, \frac{Y_i}{B_i})\).

\(\square\) Springer
6.2 Proof of call-put parity

Let \( \Pi \equiv \{ x \in \mathbb{R}^3 : \pi(x^1, x^2, x^3) = x^1 - x^2 - x^3 + 1 = 0 \} \). Consider an admissible trajectory set as described above; according to Proposition 5.2 item 2,
\[
X(S_{M(S)}-1) \in \text{cl}(\text{co}(X(\Sigma_{M(S)}-1(S)))).
\]
Clearly, \( \text{cl}(\text{co}(X(\Sigma_{M(S)}-1(S)))) \subseteq \Pi \), and therefore, \( \pi(X(S_{M(S)}-1)) = 0 \). Continuing the argument by induction, we obtain that
\[
\pi(X(S_i)) = \left[ \left( \frac{C_i}{B_i} \right) - \left( \frac{P_i}{B_i} \right) - \left( \frac{Y_i}{B_i} \right) + 1 \right] = 0 \text{ for all } 0 \leq i \leq M(S),
\]
which is our version of the call-put parity. The result is here established solely under the hypothesis of 0-neutrality that is weaker than the no-arbitrage assumption.

6.3 An example of a NAS

Let us introduce the following transformation:
\[
C_i \rightarrow C'_i = \frac{P_i}{Y_i}, \quad P_i \rightarrow P'_i = \frac{C_i}{Y_i B_i}, \quad Y_i \rightarrow Y'_i = \frac{1}{Y_i}, \quad B_i \rightarrow B'_i = \frac{1}{B_i}.
\]
Therefore
\[
(C_i, P_i, Y_i, B_i) \rightarrow (C'_i, P'_i, Y'_i, B'_i) = \left( \frac{1}{Y_i B_i} \right)(P_i, C_i, B_i, Y_i).
\]
We then have (we are disregarding dimensional constants with numerical value 1)
\[
C'_{M(S)} = (Y'_{M(S)} - B'_{M(S)}) = \frac{1}{Y_{M(S)}} - \frac{1}{K};
\]
\[
P'_{M(S)} = (B'_{M(S)} - Y'_{M(S)}) = \frac{1}{Y_{M(S)}} - \frac{1}{K}.
\]

In financial terms, the transformed variables \( C'_i \) and \( P'_i \) are prices of call and put options, respectively, but now depend on the price of the same asset \( Y_i \) but expressed in terms of shares per currency unit. This is not equivalent to using \( Y \) as the numeraire.

Notice that
\[
C'_i - P'_i - Y'_i + B'_i = \frac{1}{Y_i B_i}(P_i - C_i - B_i + Y_i) = 0.
\]

In fact, we will argue that \( \rightarrow \) is indeed a NAS. We change notation to touch base with the formal notation in the paper, letting \( s \rightarrow s' \) be given by \( s' = f(s) \), where, with the notation \( s \equiv (s^0, s^1, s^2, s^3) \), \( f(s^0, s^1, s^2, s^3) = \frac{(s^0 + s^1 + s^3)}{s^2} \), and we notice that if \( L(s^0, s^1, s^2, s^3) \) is a linear function, we obtain \( f(s) = \frac{f(s)}{L(s)} L(s) \). Thus \( f \) has the form (4.7) and by Corollary 5.6 item 2, \( f \) is a no-arbitrage symmetry. In fact, \( f \) preserves 0-neutrality as well, and this follows from the converse of Theorem 4.13 and Theorem 5.8.

6.4 Call-put parity under a no-arbitrage symmetry

Let us now see the effect on the call-put parity relation after applying a no-arbitrage symmetry. Towards this end, consider \( f \) to be a no-arbitrage symmetry satisfying (4.6) and such that \( \text{Im} f \) is not contained in a 2-dimensional subspace. From Corollary 5.6, we have that
\[
f(S^0_i, \ldots, S^4_i) = f(S_i) = \frac{f^4(S_i)}{L^4(S_i)} L(S_i),
\]

\( \odot \) Springer
where \( f^3, L^3, L \) are as in Theorem 4.13. Then
\[
F(X(S_i)) = F\left( \begin{pmatrix} S_1^0 & S_1^1 & S_1^2 \\ S_2^0 & S_2^1 & S_2^2 \\ S_3^0 & S_3^1 & S_3^2 \end{pmatrix} \right) = \left( \begin{pmatrix} C'_i & P'_i & Y'_i \\ B'_i & B'_i & B'_i \end{pmatrix} \right) = \left( \begin{pmatrix} L^0(S_i) & L^1(S_i) & L^2(S_i) \\ L^3(S_i) & L^3(S_i) & L^3(S_i) \end{pmatrix} \right).
\]
All in all, we will then take (with some abuse of notation)
\[
F(x) = \frac{A(x) + b}{B(x) + c} \quad \text{where} \quad (B(x) + c) > 0,
\]
with \( A : \mathbb{R}^3 \to \mathbb{R}^3 \) and \( B : \mathbb{R}^3 \to \mathbb{R} \) both being linear transformations (notice that we have reproduced computations from Theorem 4.13).

Before proceeding to a computation we need to impose that the boundary condition behaves as follows:
\[
C'_T = (Y'_T - B'_T)_+, P'_T = (B'_T - Y'_T)_+.
\] (6.3)
This is to say that the corresponding transformed price coordinates are prices of a call and a put on the transformed asset. Such an imposition is necessary for the derivation to follow, and prescribes that the boundary condition is invariant under \( F \).

We briefly sketch an argument establishing that
\[
\pi(F(X(S_i))) = \frac{a_F}{B(x) + c} \pi(X(S_i)),
\] (6.4)
where \( a_F \equiv (a_{1,1} - a_{2,1} - a_{3,1} - a_{4,1}) \) and \( a_{j,k} \) are the matrix coordinates of a matrix representation of \( A \). The relationship (6.4) makes it immediately apparent that \( \pi(X(S_i)) = 0 \) implies \( \pi(F(X(S_i))) = 0 \), and hence reflects the notion of symmetry embodied by \( F \). The implication \( \pi(X(S_i)) = 0 \implies \pi(F(X(S_i))) = 0 \) is known to us without recourse to (6.4); this is so because \( f \) is a no-arbitrage symmetry and so is a 0-neutral symmetry, and given that \( S \) is assumed to be 0-neutral, so then will \( S' \) be (this trajectory set is obtained from \( S \) by acting with \( f \) on the trajectories \( S \in S \)).

Given that \( \pi \) is linear, it is enough to consider the case \( F(x) = A(x) \) and to establish the existence of \( a_F \) such that \( \pi(F(X(S_i))) = a_F \pi(X(S_i)) \).

To start, subtracting the two equations in (6.3) we obtain that
\[
(a_{1,1} - a_{2,1})C_T + (a_{1,2} - a_{2,2})P_T + (a_{1,3} - a_{2,3})Y_T + (a_{1,4} - a_{2,4})B_T
= (a_{3,1} - a_{4,1})C_T + (a_{3,2} - a_{4,2})P_T + (a_{3,3} - a_{4,3})Y_T + (a_{3,4} - a_{4,4})B_T.
\] (6.5)
It turns out that in order to establish \( \pi(F(X(S_i))) = a_F \pi(X(S_i)) \) we will only need to obtain some relationships among the matrix entries \( a_{i,j} \). For reasons of space we only sketch the derivations which follow from (6.5). First, let \( Y_T > B_T \), and equating coefficients of \( Y_T \) (equating coefficients of variables does require some minimal assumptions on \( Y_T \) and \( B_T \) which we do not make explicit) we obtain that
\[
(a_{1,1} - a_{2,1}) + (a_{1,3} - a_{2,3}) = (a_{3,1} - a_{4,1}) + (a_{3,3} - a_{4,3});
\] (6.6)
a similar relation is obtained for the coefficients of \( B_T \). Two more analogous relationships among coefficients are obtained from the case \( Y_T < B_T \). The said relationships allow us to evaluate as follows:
\[
\pi(F(X(S_i))) = \pi(A(S_i))
= a_{1,1}C_i + a_{1,2}P_i + a_{1,3}S_i + a_{1,4}B_i - a_{2,1}C_i - a_{2,2}P_i - a_{2,3}S_i - a_{2,4}B_i
\]
\[-a_{3,1}C_i - a_{3,2}P_i - a_{3,3}S_i - a_{4,1}B_i + a_{4,2}P_i + a_{4,3}S_i + a_{4,4}B_i \]
\[= (a_{1,1} - a_{2,1} - a_{3,1} + a_{4,1})C_i + (a_{1,2} - a_{2,2} - a_{3,2} + a_{4,2})P_i \]
\[+ (a_{1,3} - a_{2,3} - a_{3,3} + a_{4,3})S_i + (a_{1,4} - a_{2,4} - a_{3,4} + a_{4,4})B_i \]
\[= a_F \pi(X(S_i)). \]

7 Conclusion

The paper poses and solves the following basic question: what transformations, acting on financial events, leave the no-arbitrage property invariant? Such transformations are called no-arbitrage symmetries (NAS) and are interpreted as mapping financial events to financial events. We make use of results from convex analysis and a general non-probabilistic framework to characterize and provide explicit expressions for the NAS. We take advantage of a formulation of arbitrage free markets (as per Section 4) in terms of geometric assumptions of the trajectories in discrete time. The problem formulation naturally provides the characterization, in a local sense, of no-arbitrage preserving transformations.

The transformed variables, i.e. the output values of NAS, do require an interpretation, as the original setting is abstract and general. For instance, in Section 6 we have to impose that boundary conditions should also be invariant under NAS, and in so doing we required that two of the transformed variables acted as call and put options on the two remaining transformed variables. From such a general point of view, we think that the results of applying a NAS to financial events are admissible prices for financial events, but the latter will require an interpretation that will depend on the context and the specific NAS under consideration.

Appendix

A.1 Results and proofs from section 3

The following simple characterization of 0-neutral markets is used in one of our results:

**Proposition A.1** A trajectory based market \( \mathcal{M} = \mathcal{S} \times \mathcal{H} \) is 0-neutral if and only if, for each \( \Phi \in \mathcal{H} \) and \( \epsilon > 0 \), there exist \( S^\epsilon \in \mathcal{S} \) such that

\[
\sum_{i=0}^{N_\Phi(S^\epsilon)-1} H_i(S^\epsilon) \cdot \Delta_i X(S^\epsilon) < \epsilon. \tag{A.1}
\]

**Proof** Suppose first that \( \mathcal{M} \) is 0-neutral. From the definition it follows that, for any \( \epsilon > 0 \),

\[
\inf_{S^\epsilon \in \mathcal{S}} \left[ \sum_{i=0}^{N_\Phi(S^\epsilon)-1} H_i(S^\epsilon) \cdot \Delta_i X(S^\epsilon) \right] \leq 0 < \epsilon
\]

for all \( \Phi \in \mathcal{H} \). Then, for each \( \Phi \), there exists \( S^\Phi \in \mathcal{S} \) such that

\[
\sum_{i=k}^{N_\Phi(S^\Phi)-1} H_i(S^\Phi) \cdot \Delta_i X(S^\Phi) < \epsilon
\]

for any \( \epsilon > 0 \). Thus we have proved the necessary condition.
For the sufficient condition, fix $\epsilon > 0$. Then, by hypothesis, for each $\Phi \in \mathcal{H}$, there is $S^c \in S$ such that
\[
\sum_{i=0}^{N_{\Phi}(S')-1} H_i(S') \cdot \Delta_i X(S') < \epsilon.
\]
Then, for each $\Phi \in \mathcal{H}$,
\[
\inf_{S' \in S} \left[ \sum_{i=0}^{N_{\Phi}(S')-1} H_i(S') \cdot \Delta_i X(S') \right] < \epsilon.
\]
Since $\epsilon > 0$ was chosen arbitrarily, it follows that
\[
\inf_{S' \in S} \left[ \sum_{i=0}^{N_{\Phi}(S')-1} H_i(S') \cdot \Delta_i X(S') \right] \leq 0
\]
for all $\Phi \in \mathcal{H}$. Therefore, since $0 \in \mathcal{H}$, we conclude that $\mathcal{M}$ is 0-neutral.

**Proof of Theorem 3.9** Assume that $\mathcal{M}$ is locally arbitrage-free and semi-bounded and fix $\Phi \in \mathcal{H}$ once and for all. If, for all nodes $(S, k)$, $H_k(S) \cdot \Delta_k X(S) = 0$ holds for all $S' \in S_{(S, k)}$, then
\[
G_{N_{\Phi}}^\Phi(S) = \sum_{i=0}^{N_{\Phi}(S)-1} H_i(S) \cdot \Delta_i X(S) = 0
\]
for all $S \in S$, and so
\[
V_{N_{\Phi}}^\Phi(S) = V_0^\Phi + G_{N_{\Phi}}^\Phi(S) = V_0^\Phi, \ \forall S \in S,
\]
therefore $\Phi$ is not an arbitrage opportunity.

We may then assume that there exists a trajectory $S^{(0)} \in S$ and an integer $k \geq 0$ such that, at the node $(S^{(0)}, k)$, $H_k(S^{(0)}) \cdot \Delta_k X(S) \neq 0$ for some $S \in S_{(S^{(0)}, k)}$.

Then, by Definition 3.4, 1., it is possible to choose $k_1, 0 \leq k_1 \leq k$ as the smallest integer such that, for $0 \leq j < k_1$, $H_j(S) \cdot \Delta_j X(S) = 0$ for all $S \in S_{(S^{(0)}, j)}$, and there exists $S^{(1)} \in (S^{(0)}, k_1)$ such that
\[
\sum_{i=0}^{k_1} H_i(S^{(1)}) \cdot \Delta_i X(S^{(1)}) < 0.
\]
Consider the case when, for all $k_1 < k \leq N_{\Phi}(S^{(1)})$, $H_k(S^{(1)}) \cdot \Delta_k X(S^{(1)}) = 0$ holds (such case we label (*)). Then,
\[
G_{N_{\Phi}}^\Phi(S^{(1)}) = \sum_{i=0}^{N_{\Phi}(S^{(1)})-1} H_i(S^{(1)}) \cdot \Delta_i X(S^{(1)}) < 0,
\]
and under condition (*) we have then established that $\Phi$ is not an arbitrage opportunity.

Otherwise, i.e., when the case (*) does not hold, we proceed by induction. Assume that for $i \geq 1$ we obtained the strictly increasing sequence of non negative integers $(k_j)_{j=1}^i$ and $S^{(j)} \in S_{(S^{(j-1)}, k_j)}$, $1 \leq j \leq i$ such that, for $k_{j-1} < k < k_j$, $(k_0 = 0)$, $H_k(S^{(j)}) \cdot \Delta_k X(S^{(j)}) = 0$ and $H_{k_j}(S^{(j)}) \cdot \Delta_{k_j} X(S^{(j)}) < 0$. In particular,
\[
\sum_{j=0}^{k_1} H_j(S^i) \cdot \Delta_j X(S^i) < 0.
\]

\[\square\] Springer
The same argument that we used for the node \((S_1, k_1)\) above, but now applied to \((S^{(i)}, k_i)\), and with the inductive hypothesis, gives the logical alternatives:

a) \(\Phi\) is not an arbitrage opportunity by condition (*)

b) the inductive hypothesis holds for \(i + 1\).

Due to our hypothesis that \(\mathcal{M}\) is semi-bounded and that \(\Phi\) is fixed, we remark that the alternative b) becomes, eventually, empty, and so the alternative a) holds for \(i\) large enough. Since \(\Phi\) is arbitrary, \(\mathcal{M}\) is arbitrage free.

\textbf{Proof of Proposition 3.10} We proceed by contrapositive. Assume that \(\mathcal{S}\) is not locally arbitrage-free. Therefore, there is a node \((\mathcal{S}, k)\) which is not arbitrage-free, i.e., by Proposition 4.2 (in subsection 4.1), \(\Delta X(\mathcal{S}_{(s,k)})\) is disperse, so there exists \(\xi \in \mathbb{R}^d\) such that

- \(\xi \cdot \Delta_k X(S') \geq 0\) for all \(S' \in \mathcal{S}_{(s,k)}\),
- \(\exists S' \in \mathcal{S}_{(s,k)}\) such that \(\xi \cdot \Delta_k X(S') > 0\).

Since, by hypothesis, \(\Xi(S) = 1\), it follows from Proposition 2.6 that

\[
V_{\Xi(S)^{\mathcal{K}}}(S') = \xi \cdot \Delta_k X(S') \geq 0
\]

for all \(S' \in \mathcal{S}\), and there exists \(S^* \in \mathcal{S}\) such that

\[
V_{\Xi(S)^{\mathcal{K}}}(S^*) = \xi \cdot \Delta_k X(S^*) > 0.
\]

Therefore, \(\Xi(S)^{\mathcal{K}}\) is an arbitrage opportunity.

\textbf{Proof of Theorem 3.11} Fix \(\Phi \in \mathcal{H}\) and \(\epsilon > 0\). We are going to show that there exists \(S' \in \mathcal{S}\) such that (A.1) holds.

Fix \(\mathcal{S} \in \mathcal{S}\). Given that \((\mathcal{S}, 0)\) is a 0-neutral node w.r.t. \(\mathcal{H}\), it follows that there exists \(S^{(1)} = S = S_{(0,0)}\) such that \(H_0(\mathcal{S}) \cdot \Delta_0 X(S^{(1)}) < \frac{\epsilon}{2}\).

Then, if \(N_\Phi(S^{(1)}) = 1\),

\[
\sum_{i=0}^{N_\Phi(S^{(1)})-1} H_i(S^{(1)}) \cdot \Delta_i X(S^{(1)}) < \frac{\epsilon}{2} < \epsilon.
\]

If \(N_\Phi(S^{(1)}) > 1\), in the same way as before, we can choose a finite sequence \((S^{(j)})_{j=1}^n\) with \(n \leq n_\Phi\) such that, for \(2 \leq j \leq n\),

\[
S^{(j)} \in \mathcal{S}_{(S^{(j-1)}, j-1)} \text{ and } \sum_{i=0}^{j-1} H_i(S^{(j)}) \cdot \Delta_i X(S^{(j)}) < \sum_{i=1}^{j} \frac{\epsilon}{2^i} < \epsilon.
\]

Since \(\mathcal{M}\) is semi-bounded, there exists \(0 \leq n \leq n_\Phi\) such that

\[
\sum_{i=0}^{N_\Phi(S^{(n)})-1} H_i(S^{(n)}) \cdot \Delta_i X(S^{(n)}) < \sum_{i=0}^{n} \frac{\epsilon}{2^i} < \epsilon.
\]

Thus, (A.1) holds with \(S' = S^{(n)}\). Thus, since \(\Phi \in \mathcal{H}\) was chosen arbitrarily, it follows from Proposition A.1 that \(\mathcal{M}\) is 0-neutral.

\textbf{Proof of Proposition 3.12} Supposing that \(\mathcal{M}\) is 0-neutral but that some node \((\mathcal{S}, k)\) is not 0-neutral, it then follows from Proposition 4.4 (in subsection 4.1) that there exists \(\xi \in \mathbb{R}^d\) satisfying

\[
\inf_{S' \in \mathcal{S}_{(s,k)}} \xi \cdot \Delta_k X(S') > 0 \text{ for all } S' \in \mathcal{S}_{(s,k)}.
\]
By hypothesis, \( \Xi^{(S,k)} \in H \) (see the definition preceding Proposition 3.10). Then

\[
\inf_{S' \in S(S,k)} \left[ \sum_{i=k}^{N_{\Xi^{(S,k)}}(S')-1} \Xi^{(S,k)}(S') \cdot \Delta_i X(S') \right] > 0,
\]

which is a contradiction. Therefore \((S,k)\) is a 0-neutral node. \( \square \)

### A.2 Convex analysis

For \( x, y \in \mathbb{R}^d \), we define the closed segment \([x, y]\) and the open segment \((x, y)\) by

\[
[x, y] \equiv \{tx + (1-t)y : 0 \leq t \leq 1\} \quad \text{and} \quad (x, y) \equiv \{tx + (1-t)y : 0 < t < 1\}.
\]

To begin, let us remember the notion of relative interior, which will be very important in the characterizations of local properties.

**Definition A.2** (Relative interior) Let \( E \subset \mathbb{R}^d \) be a convex set. The relative interior of \( E \), that we will denote by \( \text{ri}(E) \), is the interior of the set relative to its affine hull, that is,

\[
\text{ri}(E) = \{x \in E : B(x, r) \cap \text{aff} E \subseteq E \text{ for some } r > 0\}.
\]

The following property relates the notions of closure and relative interior:

**Proposition A.3** ([18, Theorem 6.1]) Let \( E \subset \mathbb{R}^d \) be a non-empty convex set. Then, for each \( x \in \text{ri}(E) \), \( \alpha x + (1-\alpha)y \in \text{ri}(E) \) for all \( y \in \text{cl}(E) \) and for all \( \alpha \in (0,1] \).

The Proposition that follows describes one of the most important properties of the closure and the relative interior of convex sets.

**Proposition A.4** Let \( E \subset \mathbb{R}^d \) be a convex set. Then \( \text{cl}(E) \) and \( \text{ri}(E) \) are convex sets.

The following characterizations of the relative interior for convex sets are useful:

**Proposition A.5** ([18, Corollary 6.4.1]) Let \( E \subset \mathbb{R}^d \) be a convex set. Then the relative interior of \( E \) is the set of all points \( x \in E \) such that, for all \( y \in E \), there exist some \( \epsilon > 0 \) with

\[
x - \epsilon (y - x) \in E.
\]

**Corollary A.6** Let \( E \subset \mathbb{R}^d \) be a convex set. Then: \( x \in \text{ri}(E) \) if and only if, for any \( b \in E \), there exists \( a \in E \) such that \( x \in (a,b) \).

**Proof** From Proposition A.5, if \( x \in \text{ri}(E) \), for \( b \in E \) there exists some \( \epsilon > 0 \) with

\[
a = x - \epsilon (b-x) \in E, \quad \text{so} \quad x = \frac{1}{1+\epsilon}a + \frac{\epsilon}{1+\epsilon}b \in (a,b).
\]

Conversely, if \( x = ta + (1-t)b \), with \( t \in (0,1) \), then \( x - \frac{1}{1+\epsilon}(b-x) = a \in E \). \( \square \)

In the next results we will describe some operations that preserve convexity. These operations are helpful in determining or establishing when a set is convex. Given a map \( g : \mathbb{R}^d \rightarrow \mathbb{R}^{d'} \), we are going to present two properties of the preservation of convex sets by \( g \) introduced in [16].

We say that \( g \) preserves convexity if \( g(E) \) is convex for all convex subsets \( E \subseteq \mathbb{R}^d \). Analogously, we say that \( g^{-1} \) preserves convexity, or that \( g \) is inversely convexity preserving, if \( g^{-1}(E') \) is convex whenever \( E' \) is a convex subset of \( g(\mathbb{R}^{d'}) \). The following results are then immediately apparent:

**Proposition A.7** Letting \( g : \mathbb{R}^d \rightarrow \mathbb{R}^{d'} \),

1. \( g \) preserves convexity if and only if \( [g(x), g(y)] \subseteq g([x, y]) \) for all \( x, y \in \mathbb{R}^d \);
2. \( g \) is inversely convexity preserving if and only if \( g([x, y]) \subseteq [g(x), g(y)] \) for all \( x, y \in \mathbb{R}^d \).
Note that it follows from the previous Proposition that a convexity preserving function which is, at the same time, inversely convexity preserving, satisfies \( [g(x), g(y)] = g([x, y]) \) for all \( x, y \in \mathbb{R}^d \). This motivates the following definition:

**Definition A.8 (Segment preserving)** We say that a map \( g : \mathbb{R}^d \to \mathbb{R}^d' \) preserves segments if \( [g(x), g(y)] = g([x, y]) \) for all \( x, y \in \mathbb{R}^d \). This motivates the following definition:

Then, \( g \) preserves segments if and only if \( g \) preserves convexity and preserves convexity inversely. Clearly, if \( g \) preserves segments strictly, then it preserves segments; the converse, however, may not be valid.

The obvious candidates for being functions that preserve segments strictly are the affine functions. Recall that a function \( g : \mathbb{R}^d \to \mathbb{R}^d' \) is affine if it is the sum of a linear function plus a constant, that is:

\[
g(x) = Ax + b,
\]

where \( A \in \mathbb{R}^{d \times d'} \) and \( b \in \mathbb{R}^{d'} \). There is a larger class of functions which also preserve segments strictly.

**Theorem A.9 ([16, Theorem 1])** Let \( A : \mathbb{R}^d \to \mathbb{R}^{d'} \) and \( B : \mathbb{R}^d \to \mathbb{R} \) be linear functions and \( b \in \mathbb{R}^{d'} \) and \( c \in \mathbb{R} \). Let \( D = \{ x \in \mathbb{R}^d : B(x) + c > 0 \} \). Then: \( g : D \to \mathbb{R}^{d'} \), given by

\[
g(x) = \frac{A(x) + b}{B(x) + c}, \quad (A.2)
\]

preserves segments strictly.

Consider the function \( X : \mathbb{R}^{d+1} \to \mathbb{R}^d \) with dom \( X = \{ x \in \mathbb{R} : x > 0 \} \times \mathbb{R}^d \) defined in (2.2) by

\[
X(x) = \frac{1}{x_0}(x^1, x^2, \ldots, x^d).
\]

This function, called a perspective function, scales or normalizes vectors, so the first component is one, and then it drops the first component, since it has the form (A.2), then it preserves segments strictly.

The following result is key to our analysis:

**Theorem A.10 ([16, Theorem 2])** Let \( C \subset V \) be a nonempty convex subset and let \( g : C \to V' \) be a SICP function such that \( \text{Im} \ g \) contains a nondegenerate triangle. Then, there exist \( A : V \to V' \) and \( B : V \to \mathbb{R} \) linear functions, \( b \in V' \) and \( c \in \mathbb{R} \), such that

\[
B(x) + c > 0 \quad \text{for } x \in C,
\]

and

\[
g(x) = \frac{A(x) + b}{B(x) + c}. \quad (A.3)
\]

Moreover, by [16, Theorem 1], \( g \) preserves segments strictly; this latter notion means that equality holds in (4.3).

We will present below the Separation Theorem that we use to prove Proposition 4.2.

**Theorem A.11 ([11, Proposition A.1])** Suppose that \( E \subset \mathbb{R}^d \) is a non-empty convex set such that \( 0 \notin E \). Then, there exists \( a \in \mathbb{R}^d \) such that \( a \cdot x \geq 0 \) for all \( x \in E \), and \( a \cdot x_0 > 0 \) for at least one \( x_0 \in E \). Furthermore, if \( \inf_{x \in E} \| x \|_d > 0 \), then we can find \( a \in \mathbb{R}^d \) such that

\[
\inf_{x \in E} |a \cdot x| > 0.
\]

Next we will define the convex hull of a set.
**Definition A.12** (Convex hull) The convex hull of a set $E \subset \mathbb{R}^d$, that we will denote by $co(E)$, is the smallest convex set containing $E$.

One of the most important characterizations of the convex hull is the Carathéodory Theorem.

**Theorem A.13** (Carathéodory theorem) Let $E \subset \mathbb{R}^d$. Then,

$$co(E) = \left\{ \sum_{i=1}^{d+1} \lambda_i x_i : x_i \in E, \lambda_i \geq 0, \sum_{i=1}^{d+1} \lambda_i = 1 \right\}.$$

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