Geodesic acoustic modes in a fluid model of tokamak plasma: the effects of finite beta and collisionality

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Abstract

Starting from the Braginskii equations, relevant for the tokamak edge region, a complete set of nonlinear equations for the geodesic acoustic modes (GAM) has been derived which includes collisionality, plasma beta and external sources of particle, momentum and heat. Local linear analysis shows that the GAM frequency increases with collisionality at low radial wave number $k_r$ and decreases at high $k_r$. GAM frequency also decreases with plasma beta. Radial profiles of GAM frequency for two Tore Supra shots, which were part of a collisionality scan, are compared with these calculations. A discrepancy between experiment and theory is observed, which seems to be explained by a finite $k_r$ for the GAM when flux surface averaged density $n$ and temperature $T$ are assumed to vanish. It is shown that this agreement is incidental and self-consistent inclusion of $(n)$ and $(T)$ responses enhances the disagreement more with $k_r$. So the discrepancy between the linear GAM calculation and experiment, (which also persist for more ‘complete’ linear models such as gyrokinetics) can probably not be resolved by simply adding a finite $k_r$.

Keywords: geodesic acoustic mode, tokamak, drift waves

(Some figures may appear in colour only in the online journal)

I. Introduction

Common wisdom in fusion plasma science is that the transport of heat and particles in tokamaks is largely due to micro-turbulence driven by background gradients of density, temperature, momentum etc. The turbulence saturates via mode coupling, and in particular by interactions with self-generated large scale flow structures such as zonal flows \cite{1}, and in some cases, especially near the edge, with geodesic acoustic modes (GAMs) \cite{2–4}. GAMs are an important class of oscillating zonal flows that appear due to toroidal geometry (i.e. due to geodesic curvature), and are easily observable in tokamak experiments due to their finite frequency. They are usually classified as an $m = n = 0$ perturbation in potential coupled with an $m = 1, n = 0$ perturbation in density or pressure, where $m$ and $n$ are the poloidal and the toroidal mode numbers respectively. GAMs are linearly damped unless fast particles are present \cite{5–11}. Otherwise they are excited by nonlinear processes like turbulent reynolds stresses \cite{12–18}, poloidally asymmetric particle fluxes \cite{19} and heat fluxes \cite{20}. Due to their finite frequency (usually a few kHz), distinct from that of broadband turbulence, GAMs are easier to detect, and thus have been observed on several tokamaks such as the ASDEX Upgrade ( AUG) \cite{21} using Doppler backscattering (DBS), TEXTOR \cite{22} using O-mode correlation reflectometer, and DIID \cite{23} using beam emission spectroscopy (BES). As of today, GAMS are observed in the majority of the tokamaks in the world, including recent observations of GAMs.
in Tore Supra [24] using a DBS system. The common aspect of these measurements is that the GAMs are most prominent in the edge region, right inside the last closed flux surface, and extend into the near edge region (sometimes called the ‘no man’s land’ due to a seemingly systematic discrepancy between simulation and experiment [25]). While gyrokinetics is accepted widely as the most general formulation for strongly magnetised plasmas of tokamak fusion devices, the applicability of gyrokinetic versus fluids models is still somewhat open to debate in the edge region. For most existing tokamaks, as one goes from the core to the edge, the collisions start to play a role, and the parallel connection length increases (since the safety factor q increases), dissipative drift waves, or resistive ballooning modes, start to become important, therefore the validity of a fluid description including the effects of collisions may actually be justified [26, 27].

In this spirit, here we will develop a simple two fluid model for the description of the GAM, using Braginskii equations [28, 29] within a drift expansion, in order to include the effects of collisions. In particular, we include equations of continuity, momentum and heat for ions and electrons (i.e. using a generalized Ohm’s law for electrons) coupled with the Ampère’s law. The formulation allows us to include $v_q \chi_k$, $T_i$ and $T_e$ perturbations of the GAM in a full set of nonlinear equations, which can be linearized and solved to obtain the GAM frequency including the effects of collisions and finite $\beta$, and finite radial mode number $k_r$.

The computed frequency is then compared with the radial profile of GAM frequency that is observed in Tore Supra during a collisionality scan (assuming $k_r \approx 0$). There is an apparent, systematic discrepancy between the theory and the experiment, which seems to be explained when a finite $k_r$ is introduced for the GAM calculation assuming flux surface averaged density ($n$), ion temperature ($T_i$) and flux surface averaged electron temperature ($T_e$) to be zero. However we believe that this agreement is incidental since it breaks down when higher harmonics ($m = 2$ etc) are included in the calculation [30, 31]. The agreement also breaks down on self consistent inclusion of $(n)$, $(T_e,l_e)$ responses on the GAM dispersion for $m = 1$. This indicates that the discrepancy between the linear GAM calculation and the experiment, which persist also for more ‘complete’ linear models such as gyrokinetics, is probably significant and can not be resolved by simply adding a finite $k_r$.

The remainder of this paper is organized as follows. The complete set of nonlinear electromagnetic equations with collisionality are obtained in section II from the drift reduction of Braginskii equations. The fully nonlinear equations for GAMs are obtained in section III by taking appropriate flux surface averages of the drift reduced electron and ion equations. Linear GAM dispersion properties are obtained in section III A and comparison with experimental data are presented in section III B. Finally the paper is concluded in section IV.

II. Nonlinear model equations

In order to formulate the nonlinear theory of electromagnetic geodesic acoustic modes (GAMs), we start with the simple two fluid Braginskii equations [28, 29], where we keep the following: (i) the non adiabatic electron response with $\delta T_e \neq 0$, and the electron-ion collisionality $\nu_{ei}$. The model equations for GAMs are then derived from the density, momentum and temperature equations for each species $j (= i, e)$.

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i v_i) = 0 \quad (1)$$

$$m_i n_i \left( \frac{\partial v_i}{\partial t} + v_i \cdot \nabla \right) = -\nabla p_i - \nabla \cdot \pi_i + \epsilon \left( E + \frac{1}{c} v_i \times B \right) + \tilde{R}_{ie} \quad (2)$$

$$0 = -\nabla p_e - \epsilon \left( E + \frac{1}{c} v_e \times B \right) + \tilde{R}_{et} \quad (3)$$

$$\frac{3}{2} n_i \left( \frac{\partial}{\partial t} + v_i \cdot \nabla \right) T_j + p_j \nabla \cdot v_j = -\nabla \cdot \tilde{q}_j \quad (4)$$

Here the mass of the electron is neglected and $E = \nabla \phi - (1/c) \partial A / \partial t$. The collisional momentum transfer term is given by $R_{ei} = -R_{ie} = \nu_{ei} f_{ij} - 0.71 n_i \nabla T_i$, where $f_{ij} = \eta_{ei} n_i \nu_{ei}$, and $U$ is the relative velocity between species $j$ and $i$. The thermal conductivities for electrons are given by $\kappa_{ie} = 3.16 n_i T_i / m_i \nu_{ei}$, $\kappa_{ie} = 4.66 n_i T_i / m_i \omega_{ce}^2$ and for ions $\kappa_{ij} = 3.9 n_j T_j / m_j \nu_{ij}$, $\kappa_{ij} = 2 n_j T_j / m_j \omega_{ci}^2$. The diamagnetic heat flux is taken as $\tilde{q}_j = \frac{5}{2} n_i / m_i \tilde{b} \times \nabla T_j$.

In order to develop a drift expansion, we consider ion and electron perpendicular drift velocities in the low frequency regime ($\omega \ll \omega_{ci}, \omega_{ei}$) and finite radial mode number $k_r$.

$$\tilde{v}_E = (c/\beta)^2 \tilde{B} \times \nabla \phi \quad (6)$$

$$\tilde{v}_{eta} = (c/e \nu_{ei} B^2) \tilde{B} \times \nabla \phi \quad (7)$$

$$\tilde{v}_{eta} = -(c/e \nu_{ei} B^2) \tilde{B} \times \nabla \phi \quad (8)$$

$$\tilde{v}_{pe} = \left( \frac{c}{B \omega_{pe}} \frac{\partial}{\partial t} + (\tilde{v}_E + \tilde{v}_{eta}) \cdot \nabla \right) \tilde{\phi} \quad (9)$$

For the equilibrium scale lengths that are larger than the perturbation scales (i.e. $k \ll L$), we can separate the equilibrium ($f_0$) and the fluctuating parts ($\delta f$) in the above set of equations as $f = f_0 + \delta f$. The complete set of resulting reduced nonlinear equations for the perturbations ($\delta f$) are provided in the next subsection which is written in the following normalization scheme. The space time scales are normalized as $r = r / \rho_i$, $\tilde{v}_i = L_i \tilde{v}_i / L_i$ $t = t / \alpha_i$. The field quantities are normalized to their mixing length levels: $\phi = (e \phi / T_i)(L_i / \rho_i)$, $n_i = (\bar{n}_i / n_0)(L_i / \rho_i)$, $\nu_{ij} = (\nu_{ij} c_s)(L_i / \rho_i)$, $p_j = (\delta p_j / P_0)(L_i / \rho_i)$.
\( \dot{A}_f = (2L_{e,c}/\beta_r \lambda_e)(e \dot{A}_f/T_{e0}) \). The remaining dimensionless parameters are: \( \eta_i = L_{n_i}/L_{T_{e0}}, \quad \beta = \tau_1(1 + \eta_i), \quad \tau_\parallel = T_\parallel n_i/\nu_i, \quad L_{\eta_i} = L_{n_i}/L_{T_{e0}} \), and the instantaneous parallel derivative. The electron collision frequency is calculated from \( \nu_i = nZ^2 \ln \lambda/(1.09 \times 10^6 \lambda^{1/2}) \) where \( \ln \lambda = 15.2 - \log(\eta_i/10^{04}) + \log(T_{e0}) \) [32]. \( \nu_i = e_i \omega_{ci} \) is the ion sound frequency. The nonlinearities in the following equations originate mainly from the \( E \times B \) drift nonlinearity and the nonlinearity due to perpendicular magnetic field line bending effect through the expressions for \( E_\parallel \) and the instantaneous parallel derivative.

Such models has also been used for edge turbulence simulations in the references [33–37].

**II.A. Electron response**

When the drift expansion is considered in toroidal geometry, the electron continuity equation for density perturbation takes the form:

\[
\frac{\partial n_e}{\partial t} + \frac{1}{r} \frac{\partial}{\partial \theta} \left( r n_e \frac{\partial \phi}{\partial \theta} \right) - e_n \left( \cos \theta \frac{\partial}{\partial \theta} + \sin \theta \frac{\partial}{\partial r} \right) \left( \phi - n_e - \tau_{\phi} \right) = 0
\]

\(-\nabla^2 (\phi - n_e) = [\phi, n_e] - \frac{\beta}{2} [A_1, A_1 - v_\parallel]. \quad (10)\)

These and the following equations have been derived assuming large aspect ratio circular flux surfaces. The second and third terms in the above equation results from the \( E \times B \) convection of equilibrium density, the sum of divergence of \( E \times B \) diamagnetic drifts due to inhomogeneous magnetic fields of the tokamak, respectively. The first term on the right hand is the \( E \times B \) convective nonlinearity and the second term results from parallel derivative nonlinearity due to perpendicular magnetic fluctuations. In the limit \( \omega \ll k \xi_e \), the perturbed parallel momentum equation for electrons reads:

\[
\nu_i \dot{A}_f = -\nabla^2 (\phi - n_e - 1.71 \tau_{\phi}) - \frac{\beta}{2} \left[ \frac{\partial}{\partial t} + (1 + 1.71 \eta_i) \frac{\partial}{\partial \theta} \right] A_1
\]

\[+ \frac{\beta}{2} [A_1, \phi - n_e - 1.71 \tau_{\phi}] \quad (11)\]

Similarly, the electron temperature perturbation equation, in the same expansion, becomes:

\[
\frac{\partial}{\partial t} \left[ T_e - \frac{2}{3} n_e \right] + \frac{5}{2} \left[ \cos \theta \frac{\partial}{\partial \theta} + \sin \theta \frac{\partial}{\partial r} \right] T_e
\]

\[+ \left( \eta_i - \frac{2}{3} \right) \frac{1}{\nu_i} \nabla^2 T_e = \left. \phi, T_e - \frac{2}{3} n_e \right|_{T_e}
\]

\[+ \frac{1.07}{\nu_i} \nabla^2 T_e = \left. \phi, T_e - \frac{2}{3} n_e \right|_{T_e}
\]

\[+ \eta_i \beta \frac{\partial A_1}{\partial \theta} - \frac{\beta}{2} [A_1, T_e]
\]

\[= -\nabla^2 (\phi - n_e - 1.71 \tau_{\phi}) - \frac{\beta}{2} \left[ \frac{\partial}{\partial t} + (1 + 1.71 \eta_i) \frac{\partial}{\partial \theta} \right] A_1
\]

\[+ \frac{\beta}{2} [A_1, \phi - n_e - 1.71 \tau_{\phi}] \quad (11)\]

The first term on the right hand side is the \( E \times B \) convective nonlinearity and the second term results from parallel derivative. This equation has been derived using the continuity equation for \( \nabla \cdot \dot{\nu} \) and the diamagnetic heat flux cancellation.

Finally, the \( J_i \) is related to the parallel vector potential \( A_\parallel \) via the Ampère law:

\[J_i = -\nabla^2 A_\parallel. \quad (13)\]

**II.B. Ion response**

The equations for the ion dynamics can similarly be obtained from the two fluid Braginskii equations using the drift expansion, with the aforementioned assumptions. The resulting equation for the continuity of ions take the form:

\[\frac{\partial n_i}{\partial t} + \frac{1}{r} \frac{\partial}{\partial \theta} \left( r n_i \frac{\partial \phi}{\partial \theta} \right) - \rho_i \left( \cos \theta \left( \frac{\partial}{\partial \theta} + \sin \theta \frac{\partial}{\partial r} \right) \right) \phi = 0\]

\[= \left. \phi, n_i \right|_{n_i} + \nabla \cdot \left[ \phi + \rho_i \nabla \phi \right] + \frac{\beta}{2} [A_3, \phi]. \quad (14)\]

Similar to the electron continuity equation, the second and third terms correspond to the \( E \times B \) convection of the background density and the effects of inhomogeneous magnetic field respectively, while the fourth term comes from the divergence of the polarization drift. The first term on the right hand side is the \( E \times B \) convective nonlinearity, the second term is the polarization nonlinearity, and the third term results from parallel derivative nonlinearity due to perpendicular magnetic fluctuations. Adding the electron momentum equation to the ion momentum equation and then using the electron temperature equation, one obtains the parallel ion velocity perturbation equation:

\[\frac{\partial v_{\parallel i}}{\partial t} = 2\tau_{\phi,i} \left( \cos \theta \frac{\partial}{\partial \theta} + \sin \theta \frac{\partial}{\partial r} \right) v_{\parallel i}
\]

\[+ \nabla \cdot \left[ \tau_{\phi,i} + \tau_{\parallel} n_i + \tau_{\perp} + \tau_{\phi}\right] - \frac{\beta}{2} (\tau_{\parallel} (1 + \eta_i) + 1 + \eta_i)
\]

\[\times \left[ \frac{\partial A_1}{\partial \theta} \right] = \left. \phi, v_{\parallel i} \right|_{v_{\parallel i}} + \frac{\beta}{2} [A_3, v_{\parallel i}] + n_i \nabla^2 (\rho_i + R_i). \quad (15)\]

Here, the first term on the right hand side is the \( E \times B \) convective nonlinearity, the second term results from parallel magnetic fluctuation induced parallel derivative nonlinearity, and the third term is the parallel acceleration term.

Using the ion continuity equation for \( \nabla \cdot \dot{\nu} \) and the ion diamagnetic heat flux cancellation the ion temperature perturbation equation becomes:

\[\frac{\partial}{\partial t} \left[ T_i - \frac{2}{3} n_i \right] + \frac{5}{2} \left[ \cos \theta \frac{\partial}{\partial \theta} + \sin \theta \frac{\partial}{\partial r} \right] T_i
\]

\[+ \left( \eta_i - \frac{2}{3} \right) \frac{1}{\nu_i} \nabla^2 T_i = \left. \phi, T_i - \frac{2}{3} n_i \right|_{T_i}
\]

\[+ \eta_i \beta \frac{\partial A_1}{\partial \theta} - \frac{\beta}{2} [A_1, T_i]
\]

\[= -\nabla^2 (\phi - n_i - 1.71 \tau_{\phi}) - \frac{\beta}{2} \left[ \frac{\partial}{\partial t} + (1 + 1.71 \eta_i) \frac{\partial}{\partial \theta} \right] A_1
\]

\[+ \frac{\beta}{2} [A_1, \phi - n_i - 1.71 \tau_{\phi}] \quad (11)\]

Adding the electron and ion continuity equations after multiplying by the respective charges and then assuming quasi
neutrality for the perturbations results in the plasma vorticity equation:
\[
\left(\frac{\partial}{\partial t} - \frac{K}{r} \frac{\partial}{\partial \theta}\right) \nabla^2 \phi + \epsilon_n \left(\cos \theta \frac{\partial}{\partial r} \frac{\partial}{\partial \theta} + \sin \theta \frac{\partial}{\partial r}\right) \left[\phi_{ri} + \tau T_i\right] + n_c + T_i \nabla^2 \phi = -\nabla \cdot \left[\phi + p_i \nabla \varphi_i \right] + \beta \left[A_{hir}, \nabla^2 A_{hi}\right]
\]
where the \(\beta\) dependent nonlinear term comes from the perpendicular magnetic perturbations.

The set of equations presented above for ions and electrons, provide a full drift-Braginskii system that can be used to describe the GAM oscillations including corrections due to finite \(\beta\) and collisionality, which may be relevant for the edge and near edge regions of tokamaks where GAMs have traditionally been observed.

### III. Geodesic acoustic mode

GAMs are low poloidal mode number \((m)\) axisymmetric fluctuations, that are supported by the geodesic component of the equilibrium magnetic curvature in tokamaks. In order to derive the set of equations that can be used to describe them, we start by taking the flux surface average of the vorticity equation (17):
\[
\frac{\partial}{\partial t} \nabla^2 \phi + \epsilon_n \left(\cos \theta \frac{\partial}{\partial r} \frac{\partial}{\partial \theta} + \sin \theta \frac{\partial}{\partial r}\right) \left[\phi_{ri} + \tau T_i\right] + n_c + T_i \nabla^2 \phi = -\nabla \cdot \left[\phi + p_i \nabla \varphi_i \right] + \beta \left[A_{hir}, \nabla^2 A_{hi}\right].
\]

This shows that the flux surface averaged potential is linearly coupled to the \(m = 1\) mode and temperature perturbations in the form of flux surface averaged \((n \sin \theta), (T_i \sin \theta)\) and \((T_e \sin \theta)\) perturbations. This coupling happens due to the geodesic curvature. The effect of normal curvature for \(m = 1\) mode is of the order of \(\rho_i r \sim \rho_i a \sim 10^{-3}\) and hence can be neglected. The nonlinear terms on the right hand side constitute the flux surface averaged poloidal momentum flux/Reynolds stress and Maxwell stress and act as turbulent source/sink of vorticity. Equation (18) should be supplemented by the equation for \((n \sin \theta)\), which can be obtained by multiplying the ion continuity equation by \(\sin \theta\) followed by flux surface averaging:
\[
\frac{\partial}{\partial t} \left[n \sin \theta \right] - \frac{\partial}{\partial r} \nabla^2 \left(n \sin \theta \right) - \frac{\epsilon_n}{2q} \nabla \cdot \left(\phi + n \sin \theta \right) \frac{\partial}{\partial r} \left(\phi + n \sin \theta \right) - \frac{\beta}{2} \left[A_{hir}, \nabla^2 A_{hi}\right] + \beta \left[A_{hir}, \nabla^2 A_{hi}\right]
\]

Here, the term representing \(E \times B\) convection of equilibrium density gradient is again dropped due to the fact that it is of the order of \(\rho_i r \sim \rho_i a \sim 10^{-3}\). Equation (19) shows a linear coupling, this time with the parallel velocity fluctuation in the form of the flux surface averaged quantity \(\langle v_i \cos \theta \rangle\). As is usually the case for GAMs, it is assumed that \(\left[(\phi, n, T_i) \sin^2 \theta\right] = \left[(\phi), (n), (T_i)\right](\sin^2 \theta)\), that is couplings to \(m = 2\) and higher harmonics are ignored. In fact, the retention of \(m = 2\) demands for the equations for \(m = 3\) and so on, which continues up to infinity. This closure problem, and its possible resolution will be discussed in a future publication [31]. The nonlinear terms on the right hand side acting as source/sink are poloidally asymmetric turbulent particle flux, poloidal momentum flux (i.e. Reynolds stress), and the electromagnetic component of parallel momentum flux. An asymmetry in the external particle source may also act as a source for the \((n \sin \theta)\) component and therefore the GAM. The equation for \(\langle v_i \cos \theta \rangle\) can be obtained by multiplying the parallel ion velocity equation (15) by \(\cos \theta\) followed by flux surface averaging:
\[
\frac{\partial}{\partial t} \left\langle v_i \cos \theta \right\rangle + \frac{\epsilon_n}{2q} \left[(1 + \tau)(n \sin \theta) + \tau (T_i \sin \theta) + \langle T_i \sin \theta \rangle\right] = -\left\langle (\phi, v_i \sin \theta)\right\rangle + \beta \left[A_{hir}, p_i + p_r\right] \cos \theta
\]
\[
= \left\langle (n \nabla (p_i + p_r)) \cos \theta \right\rangle + \left\langle (S_i n \cos \theta)\right\rangle
\]

where the normal curvature term and second harmonic terms like \(\langle v_i \sin 2\theta \rangle\) are again dropped. The various nonlinear terms on right hand side of the above equation can be identified as follows: The first term coming from the \(E \times B\) convective nonlinearity is the \(\cos \theta\) weighted, flux surface averaged, divergence of the parallel velocity flux. The second term is the electromagnetic analog due to perpendicular magnetic perturbation. The third term is the flux surface average of the turbulent parallel acceleration weighted by \(\cos \theta\). This term survives only when there is a \(k_\theta\) symmetry breaking mechanism present [38], which breaks the dipolar structure of acceleration in \(\theta\). The last term is the \(\theta\) symmetric part of the external velocity/momentum source. The equation for \(\langle T_i \sin \theta \rangle\), which appears in equations (18) and (20) can be obtained by multiplying the ion temperature perturbation equation by \(\sin \theta\) followed by flux surface averaging:
\[
\frac{\partial}{\partial t} \left\langle T_i \sin \theta \right\rangle - \frac{2}{3} \left\langle n \sin \theta \right\rangle \frac{5 \epsilon_n}{3} = \frac{5 \epsilon_n}{3} \nabla \cdot \left\langle T_i \sin \theta \right\rangle
\]
\[
= -\left\langle (\phi, T_i - \frac{2}{3} n \sin \theta) \right\rangle + \left\langle S_i n \sin \theta \right\rangle.
\]
The first nonlinear term on the right hand side is the divergence of heat flux minus particle flux weighted by $\sin \theta$. The second term is the poloidally asymmetric part of external heating.

Multiplying parallel electron velocity equation by $\cos \theta$ and then taking the flux surface average gives:

$$\left( \nu \nabla^2 \theta - \frac{\beta}{2} \frac{\partial}{\partial \theta} \right) \langle A_i \cos \theta \rangle$$

$$= \frac{e_n}{2q} \left[ \langle \phi \sin \theta \rangle - \langle n \sin \theta \rangle - 1.71 \langle T_e \sin \theta \rangle \right]$$

$$= \frac{\beta}{2} \left[ \langle A_{i\parallel} \phi - n - 1.71T_i \rangle \cos \theta \right].$$

The electromagnetic nonlinear term on the right hand side comes from the perpendicular magnetic perturbation from the parallel derivative. Finally the equation for $\langle \phi \sin \theta \rangle$ is obtained by multiplying the vorticity equation by $\sin \theta$ and then taking the flux surface average:

$$\frac{\partial}{\partial t} \nabla^2 \langle \phi \sin \theta \rangle - \frac{e_n}{2q} \nabla^2 \langle A_i \cos \theta \rangle + \frac{\varepsilon_n}{2} \nabla \left[ (1 + \tau_i)(n) + \tau_e(T_e) + \tau_i(T_i) \right]$$

$$= - \left[ \nabla \cdot \left[ (\phi + p_i, \nabla, \phi) \sin \theta \right] + \frac{\beta}{2} \left[ \langle A_{i\parallel} \nabla^2 A_i \rangle \sin \theta \right].$$

The above equations are complemented by the transport like equations for $\langle n_e \rangle$ and $\langle T_{e,i} \rangle$

$$\frac{\partial}{\partial t} \langle n \rangle - \varepsilon_n \frac{\partial}{\partial \theta} \left[ \langle \phi \sin \theta \rangle - \langle n \sin \theta \rangle - \langle T_e \sin \theta \rangle \right] = \langle \dot{\psi}, n_i \rangle$$

$$- \frac{\beta}{2} \langle [A_{i\parallel}, J_i - v_{i\parallel}] \rangle + \langle S_n \rangle$$

$$\frac{\partial}{\partial t} \left[ \frac{2}{3} \langle n \rangle \right] - \frac{5e_n}{3} \nabla \langle T_e \sin \theta \rangle$$

$$= - \left[ \phi, T_e - \frac{2}{3} n \right] + \langle S_{n \phi} \rangle$$

(26)

$$\frac{\partial}{\partial t} \left[ \frac{2}{3} \langle n \rangle \right] + \frac{5e_n}{3} \nabla \langle T_e \sin \theta \rangle$$

$$= - \left[ \phi, T_e - \frac{2}{3} n \right] + \langle S_{n \phi} \rangle.$$  

(27)

The equations (25)–(27) differ from the usual transport equations by the presence of a curvature term. This is due to the fact that these equations are obtained on taking flux surface average of the drift reduced equations (10), (12) and (16) respectively where as the regular transport equations are obtained by taking the flux surface average of the starting fluid equations (1)–(4). As a consequence $\langle n_e \rangle$ and $\langle T_{e,i} \rangle$ also vary in GAM time scale rather than on slower transport time scale.

The electron density and temperature equations arise because of the finite beta extension which demands non-adiabatic electron response. In electrostatic case with adiabatic electron response one still needs to keep the equation for $\langle T_i \rangle$ to be consistent. Many of the previous papers except [39] on GAM happened to miss this somehow. Retention of this equation leads to another lower frequency branch with frequency going to zero when $k_r \to 0$ which is distinctly different from the standard GAM whose frequency remains non-zero when $k_r \to 0$.

In the above equations $S_n$, $S_T$ and $S_e$ represents particle, heat and momentum sources respectively. This system of equations can be used to describe the complete nonlinear dynamics of GAMs using a reduced drift-Braginskii description including the effects of finite $\beta$ and collisionality. It is evident from the above equations that the GAM does contain $m = 1$ electromagnetic component which is in contrast to the previous [40, 41] but in line with the [42, 43]. However these calculations can not explain the dominance of $m = 2$ electromagnetic perturbation as observed in some experiment and simulations [44, 45] as the above equations are terminated at $m = 1$. Extension to $m = 2$ and beyond is differ from future. The linearized form of the above equations can be written as a linear vector equation for the GAM state vector $G$

$$\frac{\partial G}{\partial t} = MG$$  

(28)

where $M$ is a coupling matrix with elements depending on $k_r$, $\tau_i$, and $q$. The matrix $M$ is provided in the Appendix A by equation (A.1). Up to $m = 1$ the GAM state vector $G$ is made of

$$G = (\langle n \rangle, \langle \phi \rangle, \langle T_e \rangle, \langle T_i \rangle, \langle n \sin \theta \rangle, \langle \phi \sin \theta \rangle, \langle A_{i\parallel} \cos \theta \rangle, \langle v_{i\parallel} \cos \theta \rangle, \langle T_e \sin \theta \rangle, \langle T_{i\parallel} \sin \theta \rangle).$$

(29)

and $M$ is a $10 \times 10$ matrix. In the $\nu \to 0$ but $\beta \neq 0$ limit it is straightforward to see that $\langle T_e \sin \theta \rangle = 0$ and hence $\langle T_e \rangle = \langle n \rangle$. Hence the GAM state vector will become

$$G = (\langle n \rangle, \langle \phi \rangle, \langle T_e \rangle, \langle n \sin \theta \rangle, \langle \phi \sin \theta \rangle, \langle A_{i\parallel} \cos \theta \rangle, \langle v_{i\parallel} \cos \theta \rangle, \langle T_e \sin \theta \rangle)$$

(30)

and $M$ becomes a $8 \times 8$ matrix. In the limit $\nu \to 0$ and $\beta \to 0$ from the above equations it follows that $\langle n \sin \theta \rangle = \langle \phi \sin \theta \rangle = \langle A_{i\parallel} \cos \theta \rangle = \langle T_e \sin \theta \rangle = \langle n \rangle = \langle T_e \rangle = 0$. Hence the GAM state vector reduces to

$$G = (\langle \phi \rangle, \langle T_e \rangle, \langle n \sin \theta \rangle, \langle v_{i\parallel} \cos \theta \rangle, \langle T_{i\parallel} \sin \theta \rangle)$$

(31)

and $M$ becomes a $5 \times 5$ matrix.

III.A. Linear GAM dispersion

Linearizing the above set of equations (18)–(24), taking Fourier transforms and neglecting $\langle n \rangle$, $\langle T_{e,i} \rangle$, one obtains the following dispersion relation
to the equation (34) of $n_i$ is given by...

In the limit $\nu \to 0$ and $\beta \to 0$ the above GAM dispersion relation becomes:

$$\omega^2(1 + k_T^2) = \left( \frac{\varepsilon_n}{2} + \frac{\varepsilon_n}{2q} \right) \left( 1 + \frac{5\nu}{3} \right)$$

consistent with the basic GAM frequency as obtained by many authors (e.g. [20, 39]). However it is slightly different in the temperature ratio dependence as obtained from the gyrokinetic calculations [46–48] due to anisotropic temperature perturbations.

In contrast, (32) includes the effects of finite $\beta$ and finite collisionality. Notice that these effects appear together with $k_i$ in the above dispersion relation, implying that for $k_i = 0$, they can actually be neglected. We now explore some of the characteristics of the GAM frequency, especially its scalings with $\beta$, $\nu$ and $k_i$ via the numerical solution of the dispersion relation (32). Figure 1 shows the dispersion properties of GAM. Without collisionality the frequency decreases with $k_i$ monotonically and frequency also decreases with $\beta$ at any $k_i$. This behavior is also consistent with the gyrokinetic calculation in [41, 43]. At finite collisionality, $\nu = 0.1$, the GAM frequency shifts up. The amount of up-shift depends on $k_i$ at any given $\beta$. At low values of $\beta$ the up-shift in frequency is more towards $k_i \to 0$ and $k_i \to 1$ then when $k_i \to 0.5$. Whereas at higher $\beta$ values ($>0.02$) the up-shift in GAM frequency is noticeable only beyond $k_i = 0.4$.

The radial group velocity at any $\beta$ and $k_i$ is either zero or negative when $\nu = 0$. Whereas at finite $\nu$ the radial group velocity may either be negative, zero or positive depending on the values of $\beta$, $\nu$ and $k_i$.

Self consistent inclusion of flux surface averaged temperature leads to the following modification of GAM dispersion in collisionless electrostatic limit

$$\omega^2(1 + k_T^2) = R_{th} \sin \theta + \left( \frac{\varepsilon_n}{2} + \frac{\varepsilon_n}{2q} \right) \left( 1 + r_T + r_{Tth} \sin \theta \sin \phi \right)$$

where the response $R_{Tth} \sin \phi \sin \theta$ of $(T \sin \theta)$ to $(n \sin \theta)$

$$R_{Tth} \sin \phi \sin \theta = \frac{2/3}{1 - 2R_{Tth} \sin \theta}$$

and the response $R_{th} \sin \theta$ of $(T)$ to $(n \sin \theta)$ is given by

$$R_{th} \sin \theta = \frac{\varepsilon_n}{2} R_{Tth} \sin \phi \sin \theta$$

where $R_{Tth} \sin \theta$ represents the response of $(T)$ to $(T \sin \theta)$

$$R_{Tth} \sin \theta = \frac{1}{\omega} \frac{5\varepsilon_n}{3} k_T.$$
III.B. Comparison with experiment

We compare the theoretical GAM frequency as given by the dispersion relations (32) and (35) and the with the experimental GAM frequency observed in the Tore Supra tokamak for two different values of collisionality (i.e. shots #45494 and #45511) [51]. Equilibrium profiles of density and temperature are shown in figure 4. The temperature ratio $\tau_i$ is greater than 1 towards the edge but less than 1 towards the core. The density remains almost the same towards the edge in these two discharges. Radial profiles of collisionality and beta, which are calculated using these equilibrium profiles, are shown in figure 5.

The radial profiles of experimental GAM frequencies are compared against the theoretical values as obtained from equations (32), (33) and (35) and from Sugama’s formula [46] for $k_r = 0$ are shown in figure 6. Notice that our theoretical frequency compares well with Sugama’s for both the shots but experimental frequencies are lower than both. The frequency increases inward from the edge, which is consistent with increase of temperature, but the absolute values of experimental frequencies are about 50% below the theoretical values, consistently. We discuss below if this is due to finite $k_r$.

However, the experimental frequency is higher in low collisionality shots than in high collisionality shots. This might give the impression that GAM frequency goes down with increasing $k_r$ at any radius as shown in figures 7 and 8. However it goes up with $k_r$ at the edge since collisionality is felt stronger at shorter scale lengths. The frequency values computed by assuming a finite $k_r$ intersect with experimental observations for different wave numbers at different radii. In order to match with experimental observations, the radial wavenumber of the GAM has to increase with radius. The overlapping region of $k_r$ for high collisionality shots is [0.7, 1.5] and for low collisionality shots is [1.3, 1.7].

However we think that such a profile of $k_r$ is rather unrealistic and is probably not the explanation of the observed discrepancy. The reason being that on self-consistent inclusion of $\langle n \rangle$ and $\langle T_{ei} \rangle$ responses through equation (35) breaks the monotonically decreasing behavior of GAM frequency on $k_r$. Another important reason for this conclusion is that, in fact when other harmonics of the GAMs are considered (i.e. $m = 2$, $m = 3$ etc), which are linearly coupled to $m = 1$, the $k_r$ dependence may be observed to change substantially. The gap between experiment and theory increases with $k_r$ at high $k_r$ on self-consistent inclusion of $\langle n \rangle$ and $\langle T_{ei} \rangle$ responses at any radius as can be seen in figures 9 and 10. Other possible sources of GAM frequency reduction are effects of impurities, geometrical shaping, and equilibrium temperature anisotropy. But it has been shown in another recent experimental paper that in Tore Supra the impurities hardly lower the GAM frequency by 10%. It is also shown that geometrical shaping, impurities

Figure 2. GAM frequencies versus $k_r$ with effect of $\langle T_i \rangle$ for $\nu = 0, \beta = 0$. 
...and temperature anisotropy do not add up to reproduce the experimental GAM frequency [53].

IV. Conclusions

Using the reduced Braginskii equations under the drift approximation, a set of nonlinear electromagnetic equations retaining plasma beta and electron ion collisionality were obtained. Appropriate flux surface averaging were applied on the resulting set of equations in order to derive a fully nonlinear set of equations for the GAMs. This approach clearly shows that the GAM perturbations consist of \( \langle \phi \rangle \), \( \langle \phi \sin \theta \rangle \), \( \langle n \sin \theta \rangle \), \( \langle T_s \sin \theta \rangle \), \( \langle v_\parallel \cos \theta \rangle \), \( \langle A_\parallel \cos \theta \rangle \) up to the first poloidal harmonic. These equations can be used for studying both the nonlinear drive and the linear oscillation of the GAM in a collisional, electromagnetic model of the plasma edge. However, one needs a better closure [52, 54, 55] in order to describe the collisionless GAM damping, which can in principle be included in the current model. Note however that initial studies in Tore Supra suggest that GAM damping near the edge region is mainly collisional as well.

In linearizing these equations, a general linear dispersion relation is obtained, which contains the effects of electron ion collisionality and plasma beta. The following results were obtained without \( \langle n \rangle \) and \( \langle T_{ei} \rangle \). At zero collisionality and beta, the GAM frequency monotonically decreases with \( k_r \). At finite beta the GAM frequency shifts down preserving the monotonically decreasing nature with \( k_r \). At finite collisionality and low beta the frequency is shifted up, decreasing in \( k_r \) at low \( k_r \) and increasing at high \( k_r \). The up-shift in frequency with collisionality is more prominent at low and high \( k_r \) than the down shift at high \( k_r \). The GAM frequencies may be tailored to match the theoretical values by assuming finite \( k_r \) values in the range \( k_r \in [0.7, 1.5] \) for the high collisionality shots and \( k_r \in [1.3, 1.7] \) for the low collisionality shot. However since these values are rather large, and since the \( k_r \) dependence is mainly a feature of taking only the first poloidal harmonic of GAM, it is argued that \( \text{finite } k_r \text{ effects} \) probably does not explain the observed discrepancy. Also the self-consistent accounting of \( \langle n \rangle \) and \( \langle T_{ei} \rangle \) responses makes the experiment-theory disagreement even worst on increasing \( k_r \). Note also that while the experimental GAM frequencies are lower at high collisionality than at low collisionality, this observed trend is due to the change in temperature profiles and not due to the change in collisionality and plasma beta. This implies that in order to scan the collisionality dependence of the GAM frequency, one has to keep the temperature constant. Since these shots were part of a collisionality scan for the confinement, it was the other dimensionless variables, such as \( \rho_\parallel \) etc. who were kept constant.

These results leave the question of the discrepancy between the GAM frequency measured in Tore Supra and the theoretical predictions. Notice that the discrepancy is equally important if one uses a more complex gyrokinetic formula, which...
is shown in figure 6. We think that future work should include higher GAM harmonics in a similar fluid model of the edge, which may hopefully resolve this discrepancy.

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The matrix $M$ in the equation (28) is

$$M = (M_0; M_0; M_T; M_{\sin \theta}; M_{\cos \theta}; M_{\sin \theta}; M_{\cos \theta})$$

(A.1)

where

$$M_0 = (0, 0, 0, 0, -i\varepsilon_0 k_r, i\varepsilon_0 k_t, 0, 0, 0, 0, i\varepsilon_0 k_t)$$

(A.2)

$$M_\theta = (0, 0, 0, 0, i\varepsilon_\theta(1 + \gamma)/k_r, 0, 0, 0, i\varepsilon_\theta/k_r, i\varepsilon_\theta/k_t)$$

(A.3)

$$M_T = (0, 0, 0, 0, -i2\varepsilon_0 k_r/k_t, i2\varepsilon_0 k_t/k_r, 0, 0, i5\varepsilon_0 k_r/k_t, -i2\varepsilon_0 k_t/k_r)$$

(A.4)

$$M_{\sin \theta} = (0, 0, 0, 0, -i2\varepsilon_{\sin \theta} k_r/k_t, i2\varepsilon_{\sin \theta} k_t/k_r, 0, 0, 0, -i7\varepsilon_{\sin \theta} k_r/k_t)$$

(A.5)

$$M_{\cos \theta} = (-i\varepsilon_0 k_t/2, i\varepsilon_0 k_t/2, 0, -i\varepsilon_0 k_t/2, 0, 0, -i\varepsilon_0 k_t^2/2q, i\varepsilon_0/2q, 0, 0)$$

(A.6)

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