Magnetic Trapping of Neutral Particles: A Study of a Physically Realistic Model.

S. Gov* and S. Shtrikman†
The Department of Electronics,
Weizmann Institute of Science,
Rehovot 76100, Israel

H. Thomas
The Department of Physics and Astronomy,
University of Basel,
CH-4056 Basel, Switzerland

Abstract

Recently, we developed a method for calculating the lifetime of the particle in the special situation where there is no potential barrier, as a first step in our efforts to understand the quantum-mechanics of magnetic traps. The toy model that was used in this study was physically unrealistic because the magnetic field did not obey Laplace’s equation. Here, we study, both classically and quantum-mechanically, the problem of a neutral particle with spin $S$, mass $m$ and magnetic moment $\mu$, moving in two-dimensions in an inhomogeneous physically realistic magnetic field given by

$$B = B'_\perp (x \hat{x} - y \hat{y}) + B_0 \hat{z}.$$ 

*Also with the Center for Technological Education Holon, 52 Golomb St., P.O.B 305, Holon 58102, Israel.

†Also with the Department of Physics, University of California, San Diego, La Jolla, 92093 CA, USA.
We identify

\[ K \equiv \sqrt{\frac{S^2 (B'_\perp)^2}{\mu m B_0^3}}, \]

which is the ratio between the precessional frequency of the particle and its vibration frequency, as the relevant parameter of the problem.

Classically, we find that when \( \mu \) is antiparallel to \( \mathbf{B} \), the particle is trapped provided that \( K < \frac{\sqrt{4/27}}{27}. \) We also find that viscous friction, be it translational or precessional, destabilizes the system.

Quantum-mechanically, we study the problem of spin \( S = \hbar/2 \) particle in the same field. Treating \( K \) as a small parameter for the perturbation from the adiabatic Hamiltonian, we find that the lifetime \( T_{\text{esc}} \) of the particle in its trapped ground-state is

\[ T_{\text{esc}} = \frac{T_{\text{vib}}}{128\pi^2} \exp\left[ \frac{2}{K} \right], \]

where \( T_{\text{vib}} = 2\pi \sqrt{mB_0/\mu (B'_\perp)^2} \) is the classical period of the particle when placed in the adiabatic potential \( V = \mu |\mathbf{B}|. \)

1 Introduction.

1.1 Magnetic traps for neutral particles.

Recently there has been rapid progress in techniques for trapping samples of neutral atoms at elevated densities and extremely low temperatures. The development of magnetic and optical traps for atoms has proceeded in parallel in recent years, in order to attain higher densities and lower temperatures [1, 2, 3, 4, 5]. We should note here that neutral traps have been around much longer than their realizations for neutral atoms might suggest, and the seminal papers for neutral trapping as applied to neutrons and plasmas date from the sixties and seventies. Many of these papers are referenced by the authors of Refs. [1, 2, 3]. In this paper we concentrate on the study of magnetic traps. Such traps exploit the interaction of the magnetic moment of the atom with the inhomogeneous magnetic field to provide spatial confinement.

Microscopic particles are not the only candidates for magnetic traps. In fact, a vivid demonstration of trapping large scale objects is the hovering magnetic top [3, 7]. This ingenious magnetic device, which hovers in
mid-air for about 2 minutes, has been studied recently by several authors [10, 11, 12, 13, 14].

1.2 Qualitative description.

The physical mechanism underlying the operation of magnetic traps is the adiabatic principle. The common way to describe their operation is in terms of classical mechanics: As the particle is released into the trap, its magnetic moment points antiparallel to the direction of the magnetic field. While inside the trap, the particle experiences lateral oscillations $\omega_{\text{vib}}$ which are slow compared to its precession $\omega_{\text{prec}}$. Under this condition the spin of the particle may be considered as experiencing a slowly rotating magnetic field. Thus, the spin precesses around the local direction of the magnetic field $\mathbf{B}$ (adiabatic approximation) and, on the average, its magnetic moment $\mu$ points antiparallel to the local magnetic field lines. Hence, the magnetic energy, which is normally given by $-\mu \cdot \mathbf{B}$, is now given (for small precession angle) by $\mu |\mathbf{B}|$. Thus, the overall effective potential seen by the particle is

$$V_{\text{eff}} \simeq \mu |\mathbf{B}|.$$  \hspace{1cm} (1)

In the adiabatic approximation, the spin degree of freedom is rigidly coupled to the translational degrees of freedom, and is already incorporated in Eq. (1). Thus, under the adiabatic approximation, the particle may be considered as having only translational degrees of freedom. When the strength of the magnetic field possesses a minimum, the effective potential becomes attractive near that minimum and the whole apparatus acts as a trap.

As mentioned above, the adiabatic approximation holds whenever $\omega_{\text{prec}} \gg \omega_{\text{vib}}$. As $\omega_{\text{prec}}$ is inversely proportional to the spin, this inequality can be satisfied provided that the spin of the particle is small enough. If, on the other hand, the spin of the particle is too large, it cannot respond fast enough to the changes of the direction of the magnetic field. In this limit $\omega_{\text{prec}} \ll \omega_{\text{vib}}$, the spin has to be considered as fixed in space and, according to Earnshaw’s theorem [15], becomes unstable against translations. Note also that $\omega_{\text{prec}}$ is proportional to the field $|\mathbf{B}|$. To prevent $\omega_{\text{prec}}$ of becoming too small, resulting in spin-flips (Majorana transitions), most magneto-static traps include a bias field, so that the effective potential $V_{\text{eff}}$ possesses a nonvanishing minimum.
1.3 The purpose and structure of this paper.

The discussion of magnetic traps in the literature is, almost entirely, done in terms of classical mechanics. In microscopic systems, however, quantum effects become dominant, and in these cases quantum mechanics is suited for the description of the trap [16]. An even more interesting issue is the understanding of how the classical and quantum descriptions of a given system are related.

As a first step in our efforts to understand the quantum mechanics of magnetic traps, we recently developed a method for calculating the lifetime of the particle in the special situation where there is no potential barrier[17]. The toy model that was used in this study consisted of a particle with spin, having only a single translational degree of freedom, in the presence of a 1D inhomogeneous magnetic field. We found that the trapped state of the particle decays with a lifetime given by \( \sim \frac{1}{\sqrt{K\omega_{\text{vib}}}} \exp \left( \frac{2}{K} \right) \) where \( K = \frac{\omega_{\text{vib}}}{\omega_{\text{prec}}} \). The field that was used in this model was not divergenceless, and in this sense, the model is unrealistic. The next step, presented in this paper, is to study, both classically and quantum-mechanically, the case of a particle with spin, having two translational degrees of freedom, in the presence of a physically realistic (i.e. divergenceless) inhomogeneous magnetic field. This model is reminiscent of an Ioffe-Pritchard trap[2, 18], but without the axial translational degree of freedom. We neglect the effect of interactions between the particles in the trap and so we analyze the dynamics of a single particle inside the trap.

The structure of this paper is as follows: In Sec.(4) we start by defining the system we study, together with useful parameters that will be used throughout this paper. Next, we carry out a classical analysis of the problem in Sec.(3). Here, we find two stationary solutions for the particle inside the trap. One of them corresponds to a state whose spin is parallel to the direction of the magnetic field whereas the other one corresponds to a state whose spin is antiparallel to that direction. When considering the dynamical stability of these solutions, we find that only the antiparallel stationary solution is stable. We also study the same problem but with viscous friction added, and arrive at the result that friction destabilizes the system. In Sec.(4) we reconsider the problem, from a quantum-mechanical point of view. Here, we also find states that refer to parallel and antiparallel orientations of the spin, one of them being bound while the other one unbounded. In this case, however, these two states are coupled due to the inhomogeneity of the field and
we move on to calculate the lifetime of the bound state. Finally, in Sec.(3) we compare the results of the classical analysis with these of the quantum analysis and comment on their implications to practical magnetic traps.

2 Description of the problem.

We consider a particle of mass $m$, magnetic moment $\mu$ and intrinsic spin $S$ (aligned with $\mu$) moving in an inhomogeneous magnetic field $B$ given by

$$B = B_0 \hat{z} + B_\perp (x \hat{x} - y \hat{y}) .$$

This field possesses a nonzero minimum of amplitude at the origin, which is the essential part of the trap. The Hamiltonian for this system is

$$H = \frac{p^2}{2m} - \mu \cdot B$$

where $p$ is the momentum of the particle.

The Hamiltonian is invariant under a group of operations consisting of a rotation of position space about the $z$-axis by an arbitrary angle $\gamma$ combined with a rotation of spin space about the $S_z$-axis by the opposite angle $-\gamma$. Since the generators of these two rotations are the $z$-components of orbital angular momentum $L_z = xp_y - yp_x$ and of spin angular momentum $S_z$, respectively, this symmetry gives rise to a constant of motion,

$$\Lambda = L_z - S_z = \text{const.}$$

Since the magnetic field $B$ does not depend on $z$, the motion along the $z$-direction is trivial. Therefore, we restrict ourselves to studying the motion in the $(x, y)$-plane.

We define $\omega_{\text{prec}}$ as the precessional frequency of the particle when it is at the origin $(x = 0, y = 0)$. Since at that point the magnetic field is $B = B_0 \hat{z}$ we find that

$$\omega_{\text{prec}} \equiv \frac{\mu B_0}{S} .$$

Next, we define $\omega_{\text{vib}}$ as the small-amplitude vibrational frequency of the particle when it is placed in the adiabatic potential field given by

$$V(x) = \mu |B(x)| = \mu B_0 \left(1 + \frac{1}{2} \left(\frac{B'}{B_0}\right)^2 (x^2 + y^2) + O(x^4, x^2 y^2, y^4)\right) .$$
For this potential we have

\[ k_x = k_y = \frac{\partial^2 V}{\partial x^2} \bigg|_{a_x = 0} = \mu (B'_\perp)^2 / B_0, \]

and therefore

\[ \omega_{vib} \equiv \sqrt{\frac{k_x}{m}} = \sqrt{\frac{(B'_\perp)^2 \mu}{mB_0}}. \]

We also define the ratio between \( \omega_{vib} \) and \( \omega_{prec} \),

\[ K \equiv \frac{\omega_{vib}}{\omega_{prec}} = \sqrt{\frac{S^2(B'_\perp)^2}{mB_0^3}}. \]

This will be our ‘measure of adiabaticity’. It is clear that as \( K \) becomes smaller and smaller, the adiabatic approximation becomes more and more accurate. Note that when the bias field \( B_0 \) vanishes, \( K \) becomes infinite, and the adiabatic approximation fails. We will later show that, under this condition, the system become *unstable* against spin flips, which is in agreement with our discussion at the beginning. This shows that the introduction of the bias field \( B_0 \), is *essential* to the operation of the trap with regard to spin-flips. Note also that \( K \) is the only possibility to form a non-dimensional quantity (up to an arbitrary power) out of the parameters of the system. The value of \( K \) therefore, completely determines the behavior of the system.

3 Classical analysis.

3.1 The stationary solutions.

We denote by \( \mathbf{n} \) a unit vector in the direction of the spin (and the magnetic moment). Thus, the equations of motion for the center of mass of the particle are

\[ m \frac{d^2 x}{dt^2} = \mu \frac{\partial}{\partial x} (\mathbf{n} \cdot \mathbf{B}) \]

\[ m \frac{d^2 y}{dt^2} = \mu \frac{\partial}{\partial y} (\mathbf{n} \cdot \mathbf{B}), \]
and the evolution of its spin is determined by

\[ S \frac{d\hat{n}}{dt} = \mu \hat{n} \times \mathbf{B}. \tag{9} \]

It is straightforward to check that the quantity \( \Lambda = L_z - S_z \) is indeed conserved.

The two equilibria solutions to Eqs. (8) and (9) are

\[ \hat{n}(t) = \mp \hat{z} \]  

with

\[ x(t) = 0 \]
\[ y(t) = 0 \]

representing a motionless particle at the origin with its magnetic moment (and spin) pointing antiparallel (\( \hat{n}(t) = -\hat{z} \)) to the direction of the field at that point and a similar solution but with the magnetic moment pointing parallel to the direction of the field (\( \hat{n}(t) = +\hat{z} \)).

### 3.2 Stability of the solutions.

To check the stability of these solutions we now add first-order perturbations. We set

\[ \hat{n}(t) = \mp \hat{z} + \epsilon_x(t)\hat{x} + \epsilon_y(t)\hat{y} \]
\[ x(t) = 0 + \delta x(t) \]
\[ y(t) = 0 + \delta y(t), \]  

(note that, to first order, the perturbation \( \delta \hat{n} = \epsilon_x(t)\hat{x} + \epsilon_y(t)\hat{y} \) is taken to be orthogonal to the value of \( \hat{n} \) for the stationary solution \( \hat{n}_0 = \mp \hat{z} \), since \( \hat{n} \) is a unit vector) substitute these in Eqs. (8) and (9), and retain only first-order terms. We find that the resulting equations for \( \delta x(t) \), \( \delta y(t) \), \( \epsilon_x(t) \) and \( \epsilon_y(t) \)
\[
\frac{d^2 \delta x}{dt^2} = \frac{\mu B'_\perp}{m} \epsilon_x, \\
\frac{d^2 \delta y}{dt^2} = -\frac{\mu B'_\perp}{m} \epsilon_y, \\
\frac{d\epsilon_x}{dt} = \frac{\mu}{S} (\mp B'_\perp \delta y + B_0 \epsilon_y), \\
\frac{d\epsilon_y}{dt} = \frac{\mu}{S} (\mp B'_\perp \delta x - B_0 \epsilon_x). 
\]

The normal modes of the system transform as the irreducible representations of the symmetry group. The 4-dimensional linear space spanned by the deviations \((\delta x, \delta y, \epsilon_x, \epsilon_y)\) from the stationary state carries the irreducible representations \(\Gamma_+\) with characters \(e^{-i\gamma}\) and \(\Gamma_-\) with characters \(e^{+i\gamma}\), and may thus be decomposed into the two 2-dimensional invariant subspaces transforming as \(\Gamma_+\) and \(\Gamma_-\), respectively. These subspaces are spanned by the circular position coordinates and precessional spin coordinates

\[
\Gamma_+ : (\rho_+ = \delta x + i \delta y, \epsilon_- = \epsilon_x - i \epsilon_y); \\
\Gamma_- : (\rho_- = \delta x - i \delta y, \epsilon_+ = \epsilon_x + i \epsilon_y). 
\]

Thus, the normal modes consist of a circular motion in the \((x, y)\)-plane coupled to a precession of the spin vector in the opposite sense.

Indeed, after introducing the \((\rho_\pm, \epsilon_\mp)\)-coordinates into Eqs.(12), this set of four equations decomposes into one pair of equations for \((\rho_+, \epsilon_-)\) and another pair for \((\rho_-, \epsilon_+)\). We now look for oscillatory (stable) solutions of these equations and set

\[
\rho_\pm = \rho_{\pm,0} e^{-i\omega t}, \quad \epsilon_\pm = \epsilon_{\pm,0} e^{-i\omega t}. 
\]

This yields the algebraic equations

\[
\Gamma_+ : \begin{pmatrix}
\omega^2 & \omega_{\text{vib}} B_0/B_1 \\
\pm i \omega_{\text{prec}} B'_1/B_0 & i(\omega + \omega_{\text{prec}})
\end{pmatrix}
\begin{pmatrix}
\rho_{+,0} \\
\epsilon_{-,0}
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}, \\
\Gamma_- : \begin{pmatrix}
\omega^2 & \omega_{\text{vib}} B_0/B_1 \\
\mp i \omega_{\text{prec}} B'_1/B_0 & i(\omega - \omega_{\text{prec}})
\end{pmatrix}
\begin{pmatrix}
\rho_{-,0} \\
\epsilon_{+,0}
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}. 
\]
These equations have non-trivial solutions whenever the determinant of either of the two matrices vanishes. This yields the secular equations

\begin{align}
\Gamma_+ : \quad & K \left( \frac{\omega}{\omega_{\text{vib}}} \right)^3 + \left( \frac{\omega}{\omega_{\text{vib}}} \right)^2 \pm 1 = 0, \\
\Gamma_- : \quad & K \left( \frac{\omega}{\omega_{\text{vib}}} \right)^3 - \left( \frac{\omega}{\omega_{\text{vib}}} \right)^2 \pm 1 = 0,
\end{align}

which determine the eigenfrequencies \( \omega \) of the various modes. Since the system has three degrees of freedom, we expect to have three normal modes. Indeed, when \( \omega \) is a solution of the first equation, then \(-\omega\) is a solution of the second equation. We define the mode frequencies in Eq.(18) to be positive (or, in the case of complex \( \omega \), to have positive real part); the negative \( \omega \)-values are needed to construct real solutions. Then, the \( \Gamma_+ \)-modes describe vibrational motions turning counter-clockwise coupled to spin precessions turning clockwise, i.e., opposite to the natural spin precession, and the \( \Gamma_- \)-modes describe vibrational motions turning clockwise coupled to spin precessions turning counter-clockwise, i.e., in the same sense as the natural spin precession.

When the lower sign is taken in Eqs.(11), corresponding to a spin parallel to the magnetic field, \( \hat{n} = +\hat{z} \), we find two \( \Gamma_+ \)-modes with complex-conjugate mode frequencies. Thus, one of the mode frequencies possesses a positive imaginary part, indicating that the spin-up state is unstable for any value of \( K \). We therefore concentrate on the stationary state with the spin antiparallel to the magnetic field, \( \hat{n} = -\hat{z} \), corresponding to the upper sign in Eqs.(11). For the spin-down state we find one \( \Gamma_- \)-mode for any value of \( K \), and for \( K \) smaller than a critical value \( K_c = \sqrt{4/27} \) two \( \Gamma_- \)-modes, with real frequencies, as is shown in Fig.(1).

For \( K \to 0 \), the \( \Gamma_+ \)-mode, which goes like \( \omega \simeq \omega_{\text{vib}} (1 - K/2) \), and the slower of the \( \Gamma_- \)-modes, which behaves as \( \omega \simeq \omega_{\text{vib}} (1 + K/2) \), become degenerate with frequency \( \omega = \omega_{\text{vib}} \), corresponding to two linear independent purely vibrational modes. With increasing \( K \), the coupling between translational motion and spin motion lifts the degeneracy and gives rise to an increasing spin component, which leads to a decrease of the \( \Gamma_+ \) mode frequency and an increase of the slow \( \Gamma_- \) mode frequency. The fastest of the \( \Gamma_- \)-modes, on the other hand, which for \( K \to 0 \) is a pure spin precession mode with frequency \( \omega = \omega_{\text{vib}}/K = \omega_{\text{prec}} \), acquires with increasing \( K \) an increasing vibrational component, leading to a decrease of the mode frequency.
Figure 1: Real and imaginary parts of the mode frequencies as a function of \( K \).
At $K = K_c$, it becomes degenerate with the slower $\Gamma_-$-mode, with mode frequency $\omega = \sqrt{3} \omega_{\text{vib}}$. For $K > K_c$, the two $\Gamma_-$-modes have complex-conjugate frequencies. Fig. (1) shows the real and imaginary parts of the frequencies $\omega$ of the three modes as a function of $K$.

The dependence of the modes on the parameter $K$ is shown more explicitly by the form of the eigenvectors. The general form of these is given by

$$
\begin{pmatrix}
\rho_{\pm,0} \\
\epsilon_{\mp,0}
\end{pmatrix}
= \begin{pmatrix}
B_0 \omega_{\text{vib}} \\
B_\perp^{\prime} \omega \\
-\omega \\
\omega_{\text{vib}}
\end{pmatrix} A_{\pm},
$$

where $A_{\pm}$ are dimensionless amplitude parameters.

3.3 The excitation energy of the modes.

The excitation energy of a given mode $\xi$ is defined as the difference between the energy of the mode and the energy of the stationary state,

$$
\xi = -\mu \hat{n} \cdot \mathbf{B} + \frac{1}{2} m \left[ \left( \frac{d\delta x}{dt} \right)^2 + \left( \frac{d\delta y}{dt} \right)^2 \right] - \mu B_0.
$$

(21)

Note that the energy contains bilinear terms in the coordinates and hence, one cannot neglect the $\hat{z}$-component of the spin. Instead, one must set

$$
\hat{n} \cdot \hat{z} = -\sqrt{1 - (\epsilon_x^2 + \epsilon_y^2)} \simeq - \left(1 - \frac{1}{2} (\epsilon_x^2 + \epsilon_y^2)\right).
$$

Thus, the correct expression of the energy for small amplitudes is

$$
\xi \simeq -\mu \left[ \frac{1}{2} (\epsilon_x^2 + \epsilon_y^2) B_0 + B_\perp^{\prime} (\delta x \epsilon_x - \delta y \epsilon_y) \right] + \frac{1}{2} m \left[ \left( \frac{d\delta x}{dt} \right)^2 + \left( \frac{d\delta y}{dt} \right)^2 \right].
$$

(22)

The deviations $\delta x(t), \delta y(t), \epsilon_x(t), \epsilon_y(t)$ from the stationary state belonging to the normal modes are given in real form by

$$
\delta x(t) = \frac{1}{2} \rho_{\pm,0} e^{-i\omega t} + \text{c.c.}, \quad \delta y(t) = \pm \frac{1}{2i} \rho_{\pm,0} e^{-i\omega t} + \text{c.c.},
$$

(23)
\[ \epsilon_x(t) = \frac{1}{2} \epsilon_\parallel 0 e^{-i\omega t} + \text{c.c.}, \quad \epsilon_y(t) = \mp \frac{1}{2} \epsilon_\parallel 0 e^{-i\omega t} + \text{c.c.} \]  

(24)

With the help of Eq. (20), one obtains

\[ \xi = \mu B_0 \frac{3 \omega_{\text{vib}}^2 - \omega^2}{\omega_{\text{vib}}^2} |A_\pm|^2. \]  

(25)

From this result we conclude that for \( 0 < K < \sqrt{4/27} \), the excitation energy of the vibrational modes, for which \( \omega^2 < 3 \omega_{\text{vib}}^2 \), is positive while the excitation energy of the precessional mode, satisfying \( \omega^2 > 3 \omega_{\text{vib}}^2 \), is always negative. At the point \( K = \sqrt{4/27} \), where the clockwise vibrational mode and the precessional mode coalesce, the excitation energy vanishes. We will further refer to these observations in the following section.

### 3.4 The effect of viscous friction.

When friction is introduced into the system, the equations of motion become

\[ m \frac{d^2 x}{dt^2} = \mu \frac{\partial}{\partial x} (\hat{n} \cdot B) - r_t \frac{dx}{dt}, \]

(26)

\[ m \frac{d^2 y}{dt^2} = \mu \frac{\partial}{\partial y} (\hat{n} \cdot B) - r_t \frac{dy}{dt}, \]

and

\[ S \frac{d\hat{n}}{dt} = \mu \hat{n} \times B - r_p \hat{n} \times \frac{d\hat{n}}{dt}, \]  

(27)

where \( r_t \) and \( r_p \) are translational and precessional friction coefficients, respectively. The second term in the right-hand side of Eq. (27) is the spin-damping contributed by the change in the direction of the spin from \( \hat{n} \) to \( \hat{n} + d\hat{n} \). Since, by definition, \( \hat{n} \) is a unit vector, it must point perpendicular to \( d\hat{n} \). Thus, \( \Omega_\perp = |d\hat{n}/dt| \) is the angular velocity associated with the change of \( \hat{n} \). Since the direction of \( \Omega_\perp \) must be perpendicular to both \( d\hat{n} \) and \( \hat{n} \) we form the cross product \( \Omega_\perp = \hat{n} \times (d\hat{n}/dt) \) which incorporates both the correct value and the right direction. Multiplying \( \Omega_\perp \) by \( r_p \) yields the spin-damping term.

To first order in \( r_r \) and \( r_t \) the secular equation in this case is given by

\[ 0 = -K^2 \omega_n^6 + \omega_n^4 - 2 \omega_n^2 + 1 + 2iK \omega_n^5 r_p \frac{B_0}{S} - 2iK^3 \omega_n^5 r_t \frac{B_0}{S} \left( \frac{B_0}{B_\perp} \right)^2 \]  

\[ + 2iK \omega_n^3 r_t \frac{B_0}{S} \left( \frac{B_0}{B_\perp} \right)^2, \]  

(28)
where we defined
\[ \omega_n \equiv \frac{\omega}{\omega_{vib}} \]
to make the expression simple. Let \( \omega_{n,0} \) be the eigenfrequencies \( \omega_n \) of the frictionless problem, given by Eq.(19). When adding small friction to the problem, the eigenfrequencies will change by a small amount \( \delta \omega_n \). We find an approximate expression for \( \delta \omega_n \) by expanding Eq.(28) around \( \omega_{n,0} \) to first order in \( \delta \omega_n \) and making use of Eq.(19). This gives
\[
\delta \omega_n = \frac{iK}{S} \left( \frac{r_p \omega_{n,0}^4 + r_t (B_0/B'_0)^2}{\omega_{n,0}^2 - 3} \right) + \mathcal{O} (r_t^2, r_t r_p, r_p^2).
\]

Eq.(29) has an interesting consequence: The numerator in Eq.(29) is positive for all three modes while the denominator is negative for the two vibrational modes and positive for the precessional mode. We therefore conclude that friction, either translational or precessional, stabilizes the vibrational modes and, simultaneously, destabilizes the precessional mode. The system all together becomes of course, unstable.

The fact that spin damping leads to an exponential growth of the precessional mode is no surprise in view of its negative excitation energy. Also, the exponential decay of the vibrational modes due to translational friction is to be expected on account of their positive excitation energy. What is important is the fact that due to the coupling between translation and precession, translational friction causes an exponential growth of the precessional mode, with a growth time which, compared to the effect of spin damping, is smaller by a factor of \( r_t K^2 S^2 / \mu m r_p B_0^2 \) in the limit of small \( K \).

4 Quantum-mechanical analysis.

4.1 The Hamiltonian and its diagonalized form.

In this section we consider the problem of a neutral particle with spin half (\( S = \hbar / 2 \)) in a 2D inhomogeneous magnetic field from a quantum-mechanical point of view. Unlike the classical analysis, in which the derivation was valid for any value of the adiabaticity parameter \( K \), we concentrate here on the behavior of the system when \( K \) is small. We choose to analyze the case of a spin half particle because this case already shows the essentials of the
quantum-mechanical problem. Note also that, quantum mechanically, the magnetic moment $\mu$ and the spin $S$ of a particle are related by

$$\mu = \gamma S,$$

where $\gamma$ is the gyromagnetic ratio of the particle. Setting $\mu = \gamma S$ and $S = \hbar/2$ in Eq.(7) gives

$$K = \sqrt{\frac{\hbar(B'_0)^2}{2\gamma m B_0^3}}.$$

Now, it is convenient to express the spatial dependence of the magnetic field in polar coordinates $r = \sqrt{x^2 + y^2}$ and $\phi = \arctan(y/x)$. We also denote by $B$ the amplitude of $\mathbf{B}$, by $\theta$-its direction with respect to the $\hat{z}$ axis and by $\varphi$ the angle between the projection of $\mathbf{B}$ onto the $x$-$y$ plane and the $\hat{x}$-axis. Thus, Eq.(2) is rewritten as

$$\mathbf{B} = B [\sin \theta \cos \varphi \hat{x} + \sin \theta \sin \varphi \hat{y} + \cos \theta \hat{z}] \quad (30)$$

where

$$B = B_0 \sqrt{1 + \left(\frac{B'_0}{B_0}\right)^2 r^2}, \quad (31)$$

$$\theta = \arctan \left(\frac{B'_0}{B_0} r\right),$$

$$\varphi = \arctan \left(\frac{B_y}{B_x}\right) = - \arctan \left(\frac{y}{x}\right) = -\phi.$$

Thus, $B$ and $\theta$ depends only on $r$ whereas $\varphi$ depends (linearly) only on $\phi$.

The time-independent Schrödinger equation for this system is

$$\left[ -\frac{\hbar^2}{2m} \nabla^2 - \mu B (\sin \theta \cos \varphi \hat{\sigma}_x + \sin \theta \sin \varphi \hat{\sigma}_y + \cos \theta \hat{\sigma}_z) \right] \Psi'(r, \phi) = E \Psi'(r, \phi) \quad (32)$$

where $\hat{\sigma}_x$, $\hat{\sigma}_y$ and $\hat{\sigma}_z$ are the Pauli matrices given by

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

14
$E$ is the eigenenergy and $\Psi'$ is the two-components spinor

$$\Psi' = \begin{pmatrix} \psi'_\uparrow(r, \phi) \\ \psi'_\downarrow(r, \phi) \end{pmatrix}. \quad (33)$$

In matrix form Eq.(32) becomes

$$(H_K + H_M) \begin{pmatrix} \psi'_\uparrow(r, \phi) \\ \psi'_\downarrow(r, \phi) \end{pmatrix} = E \begin{pmatrix} \psi'_\uparrow(r, \phi) \\ \psi'_\downarrow(r, \phi) \end{pmatrix} \quad (34)$$

where $H_K$ and $H_M$, given by

$$H_K \equiv -\frac{\hbar^2}{2m} \nabla^2 \quad (35)$$

$$H_M \equiv -\mu B \begin{pmatrix} \cos \theta & \sin \theta e^{-i\varphi} \\ \sin \theta e^{i\varphi} & -\cos \theta \end{pmatrix},$$

are the kinetic part and the magnetic part of the Hamiltonian $H$, respectively.

In order to diagonalize the magnetic part of the Hamiltonian, we make a local \textit{passive} transformation of coordinates on the wavefunction such that the spinor is expressed in a new coordinate system whose $\hat{z}$ axis coincides with the direction of the magnetic field at the point $(r, \phi)$. We denote by $R(r, \phi)$ the required transformation and set $\Psi = R\Psi'$. Thus, $\Psi$ represent \textit{the same} direction of the spin as before the transformation but using the \textit{new} coordinate system. The Hamiltonian in this newly defined system is clearly given by $RHR^{-1}$. In the case of the magnetic field given in Eqs.(30) and (31) the required transformation is accomplished by using the three Euler angles: First, we perform a rotation through an angle $\varphi$ around the $\hat{z}$ axis. Second, we make a rotation through an angle $\theta$ around the \textit{new} position of the $\hat{y}$ axis. At the end of this process the new $\hat{z}$ axis coincide with the direction of the magnetic field. Now the value of the last Euler angle, which is a rotation around the new $\hat{z}$ axis, has no effect on this axis. For simplicity we choose this angle to be 0. Thus, the representation of the complete transformation for spin half particle is given by

$$R = \exp \left[ i \frac{\theta}{2} \hat{\sigma}_y \right] \exp \left[ i \frac{\varphi}{2} \hat{\sigma}_z \right],$$

while its inverse is given by

$$R^{-1} = \exp \left[ -i \frac{\varphi}{2} \hat{\sigma}_z \right] \exp \left[ -i \frac{\theta}{2} \hat{\sigma}_y \right].$$
It is easily verified that the transformation indeed diagonalizes the magnetic part of the Hamiltonian as

\[ RH_M R^{-1} = -\mu B \hat{\sigma}_z. \]

For the kinetic part we find, after some algebra, that

\[
RH_K R^{-1} = -\frac{\hbar^2}{2m} \left[ \begin{array}{c}
\nabla^2 - \frac{1}{4} \left( \frac{d\theta}{dr} \right)^2 - \frac{1}{4r^2} + i r^2 \hat{\sigma}_x \cos \theta \frac{\partial}{\partial \phi} \\
+ \frac{i}{r^2} \hat{\sigma}_x \cos \theta \frac{\partial}{\partial \phi} \\
\end{array} \right] - \frac{i}{r^2} \hat{\sigma}_y \sin \theta \frac{\partial}{\partial \phi} \]

Thus, the Hamiltonian of the system in the rotated frame may be written as

\[ H = H_{\text{diag}} + H_{\text{int}} \] (36)

where

\[
H_{\text{diag}} = -\frac{\hbar^2}{2m} \left[ \nabla^2 - \frac{1}{4} \left( \frac{d\theta}{dr} \right)^2 - \frac{1}{4r^2} + \frac{i}{r^2} \hat{\sigma}_x \cos \theta \frac{\partial}{\partial \phi} \right] - \mu B \hat{\sigma}_z \] (37)

\[
H_{\text{int}} = -\frac{\hbar^2}{2m} \left[ -i \hat{\sigma}_y \left( \frac{d\theta}{dr} \right) \frac{\partial}{\partial r} + \frac{1}{2} \frac{d^2 \theta}{dr^2} + \frac{1}{2r} \frac{d \theta}{dr} \right] - \frac{i}{r^2} \hat{\sigma}_x \sin \theta \frac{\partial}{\partial \phi} \right].
\]

The first part of the Hamiltonian \( H_{\text{diag}} \) is diagonal. It contains the kinetic part \( \sim \nabla^2 \), a term whose form is \( \mp \mu B \) which is to be identified as the adiabatic effective potential and the terms \( \sim 1/r^2, i r^{-2} \hat{\sigma}_x \partial/\partial \phi \) which appear due to the rotation. The second part of the Hamiltonian \( H_{\text{int}} \) contains only non-diagonal components. These will be shown to be of order \( O(K) \) and hence may be regarded as a small perturbation. We proceed to find the eigenstates of \( H_{\text{diag}} \).
4.2 Stationary states of $H_{\text{diag}}$. 

Since $H_{\text{diag}}$ is diagonal, the two spin states of the wavefunction are decoupled. We then seek a solution of the form

$$\Psi_\downarrow = \begin{pmatrix} 0 \\ \psi_\downarrow(r, \phi) \end{pmatrix} ; \quad E = E_\downarrow,$$

referred to as the spin-down state, and another solution

$$\Psi_\uparrow = \begin{pmatrix} \psi_\uparrow(r, \phi) \\ 0 \end{pmatrix} ; \quad E = E_\uparrow,$$

which we call the spin-up state.

The equation for the non-vanishing component of the spin-down state is given by

$$\left\{ -\frac{\hbar^2}{2m} \left[ \nabla^2 - \frac{1}{4} \left( \frac{d \theta}{dr} \right)^2 - \frac{1}{4r^2} - \frac{i}{r^2} \cos \theta \frac{\partial}{\partial \phi} \right] + \mu B \right\} \psi_\downarrow = E_\downarrow \psi_\downarrow,$$

whereas the equation for the non-vanishing component of the spin-up state is

$$\left\{ -\frac{\hbar^2}{2m} \left[ \nabla^2 - \frac{1}{4} \left( \frac{d \theta}{dr} \right)^2 - \frac{1}{4r^2} + \frac{i}{r^2} \cos \theta \frac{\partial}{\partial \phi} \right] - \mu B \right\} \psi_\uparrow = E_\uparrow \psi_\uparrow.$$

We now show that in the limit of small $K$ we can neglect the term $\sim (d\theta/dr)^2$ in both Eq. (40) and Eq. (41): We compare the order of magnitude of the term $\mu B$ to that of the term $\hbar^2 (d\theta/dr)^2 / 8m$. Using Eq. (31) it can be easily shown that the maximum value of $d\theta/dr$ is $B'_\perp / B_0$ whereas the minimum value of $\mu B$ is $\mu B_0$. Thus,

$$\frac{\mu B_{\text{min}}}{\left( \frac{\hbar^2}{8m} \left( \frac{d \theta}{dr} \right)^2 \right)_{\text{max}}} = \frac{8\mu m B_0^3}{(B'_\perp)^2 \hbar^2} = \frac{2}{K^2},$$

and hence we can neglect the term $\sim (d\theta/dr)^2$ when $K$ is small. Furthermore, as we are interested in the solutions near the origin we replace the $\cos \theta$ term by its zeroth-order approximation around $r = 0$. We will justify this
approximation later. Under these approximations, Eqs.(40) and (41) simplify to

\[
\left\{-\frac{\hbar^2}{2m} \left[ \nabla^2 - \frac{1}{4r^2} - \frac{i}{r^2} \frac{\partial}{\partial \phi} \right] + \mu B \right\} \psi_\downarrow = E_\downarrow \psi_\downarrow \tag{42}
\]

and

\[
\left\{-\frac{\hbar^2}{2m} \left[ \nabla^2 - \frac{1}{4r^2} + \frac{i}{r^2} \frac{\partial}{\partial \phi} \right] - \mu B \right\} \psi_\uparrow = E_\uparrow \psi_\uparrow. \tag{43}
\]

The approximate solutions of these equations is outlined in the next two subsections.

4.2.1 Stationary spin-down states.

Eq.(42) represents a particle in a symmetric attractive potential. If the extent of the wave function is small enough we can expand \(B\) in Eq.(31) to second order in \(r\)

\[
B \simeq B_0 \left[ 1 + \frac{1}{2} \left( \frac{B'_\perp r}{B_0} \right)^2 \right] + \mathcal{O}(r^4), \tag{44}
\]

and apply the well-known solution of the harmonic oscillator\[20\] in two dimensions. Under this approximation, Eq.(42) becomes

\[
\left[ -\frac{\hbar^2}{2m} \left( \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} - \frac{1}{r^2} \left( \frac{i}{\partial \phi} + \frac{1}{2} \right)^2 \right) + \frac{\mu (B'_\perp)^2 r^2}{2B_0} \right] \psi_\downarrow = (E_\downarrow - \mu B_0) \psi_\downarrow. \tag{45}
\]

We seek a solution whose form is

\[
\psi_\downarrow(r,\phi) = f(r)e^{i\nu \phi} \tag{46}
\]

and then the equation satisfied by \(f(r)\) is

\[
-\frac{\hbar^2}{2m} \left[ \frac{1}{r} \frac{df}{dr} + \frac{d^2 f}{dr^2} - \frac{f}{r^2} \left( \nu - \frac{1}{2} \right)^2 \right] + \frac{\mu (B'_\perp)^2 r^2 f}{2B_0} = (E_\downarrow - \mu B_0) f. \tag{47}
\]
which is an eigenvalue problem for \( f \). The smallest eigenvalue for this problem is obtained by setting

\[
\nu = \frac{1}{2},
\]

for which the eigenfunction \( f \) is

\[
f (r) = D \exp \left( -\sqrt{\frac{\mu m (B'_\perp)^2}{4\hbar^2 B}} r^2 \right) = D \exp \left( -\frac{1}{4K} \left( \frac{B'_\perp}{B_0} \right)^2 r^2 \right).
\]

Thus, under the harmonic oscillator approximation the down-part of the spin-down state is

\[
\psi_\downarrow = \frac{B'_\perp}{B_0 \sqrt{2\pi K}} \exp \left( -\frac{1}{4K} \left( \frac{B'_\perp}{B_0} \right)^2 r^2 \right) e^{i\phi/2}.
\] (48)

where the normalization constant \( D \) has been calculated by demanding that

\[
\int_0^\infty r dr \int_0^{2\pi} d\phi |\psi_\downarrow|^2 = 1,
\]

using the definite integral

\[
\int_0^\infty re^{-ar^2} dr = \frac{1}{2a}.
\]

Note that the extent of this wave function over which it changes appreciably is given by

\[
\Delta r_\downarrow \sim \sqrt{K \frac{B_0}{B'_\perp}},
\] (49)

whereas the extent over which \( \mu B \) changes significantly (see Eq.(31)) is

\[
\Delta r_{\mu B} \sim \frac{B_0}{B'_\perp}.
\] (50)
Thus, the ratio between these two length scales is
\[
\frac{\Delta r_\downarrow}{\Delta r_{\mu B}} \sim \sqrt{K}.
\] (51)

We therefore conclude that when \( K \) is small enough, the harmonic approximation is justified. Note also that \( \Delta r_{\mu B} \) is also the typical length of \( \cos \theta \). This shows that the substitution of \( \cos \theta \) in Eq.(10) by 1 is also justified.

The wave function \( \psi_\downarrow \), given by Eq.(48), then represents the lowest possible bound state for this system. This state corresponds to a \textit{trapped} particle. The energy of this state is clearly
\[
E_\downarrow = \mu B_0 + 2 \left( \frac{\hbar}{2} \omega_{\text{vib}} \right) = \mu B_0 (1 + 2K) \simeq \mu B_0,
\] (52)
while its full spinor representation is
\[
\Psi_\downarrow = \begin{pmatrix} 0 \\ \frac{B'_\perp}{B_0 \sqrt{2\pi K}} \exp \left[ -\frac{1}{4K} \left( \frac{B'_\perp r}{B_0} \right)^2 \right] e^{i\phi/2} \end{pmatrix}.
\] (53)

### 4.2.2 Stationary spin-up states.

Eq.(13) describes a particle in a repulsive potential. It corresponds to an unbounded state representing an \textit{untrapped} particle. In this case there is a continuum of states, each with its own energy. As we are interested in non-radiative decay, we focus on finding a solution with an energy which is \textit{equal} to the energy found for the trapped state, that is
\[
E_\uparrow = E_\downarrow \simeq \mu B_0.
\] (54)

When evaluating the lifetime in the next section, we compute the matrix element of \( H_{\text{int}} \) between the states \( \psi_\uparrow \) and \( \psi_\downarrow \). Thus, most of the contribution to this integral comes from the region in \( r \) where \( \psi_\downarrow \) is substantial. According to Eq.(51), \( \mu B \) changes very little in this range and, as a first approximation, we may take \( \cos \theta \simeq 1 \) and the potential in this region as \textit{uniform},
\[
\mu B \simeq \mu B_0
\] (55)
in Eq.(13). We now set a solution whose form is
\[
\psi_\uparrow(r, \phi) = g(r) e^{i\gamma \phi}.
\]
Substituting this, together with Eqs. (54) and (55) into Eq. (43) gives

\[-\frac{\hbar^2}{2m} \left( \frac{1}{r} \frac{dg}{dr} + \frac{d^2g}{dr^2} - \frac{g}{r^2} \left( \gamma + \frac{1}{2} \right)^2 \right) = 2\mu B_0 g,\]

whose non-singular solution is

\[g(r) = J_{\gamma+1/2} \left( \frac{B'_r r}{B_0 K} \right),\]

where \( J_\alpha(x) \) is the Bessel function of the first kind of order \( \alpha \).

We note that all the four terms of \( H_{int} \) does not operate on the \( \phi \) co-ordinate. This is a consequence of the fact that \( L_z - S_z \) (where \( L_z \) is the \( z \)-component of the orbital angular momentum and \( S_z = \hbar \sigma_z / 2 \) is the \( z \)-component of the spin) is conserved. Hence, in order to have a non-vanishing matrix element between the up-state and the down-state, they must have the same \( \phi \)-dependence. Thus, \( \gamma = \nu = 1/2 \) and as a result

\[\psi_\uparrow = C J_1 \left( \frac{B'_r r}{B_0 K} \right) e^{i\phi/2}, \tag{56}\]

with

\[\Psi_\uparrow = \left( C J_1 \left( \frac{B'_r r}{B_0 K} \right) e^{i\phi/2} \begin{pmatrix} B_0 \\ 0 \end{pmatrix} \right) . \tag{57}\]

where \( C \) is the normalization constant which is chosen to be real.

The wave function given in Eq. (56) is oscillatory. It has a period of about

\[\Delta r_\uparrow \sim K \frac{B_0}{B'_r} \tag{58}\]

near the origin. Comparing it to \( \Delta r_\downarrow \) given in Eq. (49), we find that

\[\frac{\Delta r_\uparrow}{\Delta r_\downarrow} \sim \sqrt{K}, \tag{59}\]

which shows that, for \( K \ll 1 \), the wavefunction \( \psi_\uparrow \) executes many oscillations in the region where \( \psi_\downarrow \) is appreciable.
4.3 The lifetime.

To evaluate the lifetime $T_{\text{esc}}$ of the particle in its trapped state, which is the average time it takes for the particle to escape, we calculate the transition rate from the bound state given by Eq.(53), to the unbounded state Eq.(57), according to Fermi’s golden rule\cite{21}. Thus,

$$\frac{1}{T_{\text{esc}}} = \frac{2\pi}{\hbar} |H_{↓,↑}|^2 g(E_{↑})$$  \hspace{1cm} (60)

where

$$H_{↓,↑} = \int_0^{\infty} r dr \int_0^{2\pi} d\phi \psi_{↑}^* H_{\text{int}} \psi_{↓}$$  \hspace{1cm} (61)

is the matrix element of $H_{\text{int}}$ Eq.(37) between $\psi_{↓}$ and $\psi_{↑}$, and $g(E_{↑})$ is the density of the final states at energy $E_{↑}$.

The integrand in Eq.(61) consists of a product of three elements: The function $\psi_{↓}^*$ whose ‘width’ is about $\Delta r_{↓}$ (given in Eq.(49)) around the origin, An operator consisting of four $\theta$-dependent terms whose extent around the origin $\Delta r_{\mu B}$ is roughly $\sqrt{1/K}$ larger than $\Delta r_{↓}$ and the function $\psi_{↑}$ which is an oscillatory function with a characteristic period near the origin $\Delta r_{↑}$ which is $\sqrt{K}$ smaller than $\Delta r_{↓}$. This suggests that we can approximate the integral in Eq.(61) by substituting $\sin \theta$, $d\theta/dr$ and $d^2\theta/dr^2$ by their value at $r = 0$,

$$\sin \theta \simeq \frac{B_{↓}}{B_0} r$$  \hspace{1cm} (62)

$$\frac{d\theta}{dr} \simeq \frac{\gamma B_{↓}}{B_0}$$

$$\frac{d^2\theta}{dr^2} \simeq 0$$

Substituting Eqs.(62), (48) and (56) into Eq.(61) gives

$$H_{↓,↑} \simeq -\sqrt{\pi} \hbar^2 \frac{\gamma B_{↓}}{mB_0} C \sqrt{\frac{2}{K}} \exp \left[-\frac{1}{K}\right]$$  \hspace{1cm} (63)
where we have used the definite integral
\begin{equation}
\int_0^{+\infty} r^2 J_1 (br) e^{-ar^2} dr = \frac{b}{4a^2} \exp \left[ - \frac{b^2}{4a} \right].
\end{equation}

When Eq. (63) is substituted into Eq. (60) the term \( C^2 g(E) \) appears. This term can be calculated by temporarily introducing suitable boundary conditions: Assume that the system is bounded by an infinite potential wall at \( r = R \), the radius \( R \) being large compared to \( \Delta r \) yet small when compared to \( \Delta r_{\mu B} \). In this case the uniform potential approximation holds for all \( r < R \), and the wave function has the form
\[
\psi^\uparrow (r, \phi) = C g(r) e^{i\phi/2},
\]
where the radial part \( g(r) \) satisfies the Schrödinger equation
\[
-\frac{\hbar^2}{2m} \left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{dg}{dr} - \frac{g}{r^2} \right] = (\mu_B + E) g.
\]
This equation has as non-singular solutions the Bessel functions of order 1,
\[
g(r) = J_1(kr) \quad \text{where} \quad k^2 = \frac{2m}{\hbar^2} (E + \mu_B),
\]
with eigenvalues \( k = k_n \) determined by the boundary condition. From \( R \gg r \) it follows that \( kR \gg 1 \) such that \( J_1(kR) \) may be approximated by the first term of its asymptotic expansion. Therefore, the boundary condition reads
\[
J_1(kR) = \sqrt{\frac{2}{\pi kR}} \cos \left( kR - \frac{3\pi}{4} \right) = 0.
\]
This yields the eigenvalues \( k_n = (n + 1/4)\pi/R \). The density of states on the \( k \)-axis is thus given by \( dN/dk = R/\pi \), from which one obtains the density of states on the energy axis at \( E = \mu_B \)
\[
\rho(E = \mu_B) = \frac{dN}{dE} = \frac{mR}{\pi \hbar^2 k} = \frac{1}{2\pi} \frac{m}{\hbar^2 \mu_B} R.
\]
\[\tag{65}\]
The constant \( C \) is determined by the normalization condition
\[
\int_0^R [\psi^\uparrow]^2 r dr d\phi = 2\pi C^2 \int_0^R [J_1(kr)]^2 r dr = 1,
\]
\[\tag{64}\]
which gives
\[ \int_0^R [J_1(kr)]^2 r \, dr = \frac{1}{2} R^2 [J_2(kR)]^2. \]

In the asymptotic region \( kR \gg 1 \), the function \( J_2(kR) \) takes the values \( \pm \sqrt{2/(\pi kR)} \) at the zeros of \( J_1(kR) \). This gives
\[ C = \frac{k}{2R} \]
and therefore
\[ C^2 \rho(E = \mu B_0) = \frac{m}{2\pi\hbar^2}. \]  

Finally, using Eqs. (67) and (63) inside Eq. (60) gives
\[ T_{\text{esc}} = \frac{1}{64\pi\omega_{\text{vib}}} \exp \left[ \frac{2}{K} \right] = \frac{T_{\text{vib}}}{128\pi^2} \exp \left[ \frac{2}{K} \right], \]
where \( T_{\text{vib}} = 2\pi/\omega_{\text{vib}} \) is the period of classical oscillations inside the trap.

5 Discussion.

Summarizing all we have found we conclude that the problem we have studied has three important time scales: The shortest time scale is \( T_{\text{prec}} \), which is the time required for one precession of the spin around the axis of the local magnetic field. The intermediate time scale is \( T_{\text{vib}} = T_{\text{prec}}/K \), which is the time required to complete one cycle of the center of mass around the center of the trap. These two time scales appear both in the classical and the quantum-mechanical analysis. The longest time scale (provided \( K \) is small) \( T_{\text{esc}} \), which is not present at the classical problem, is the time it takes for the particle to escape from the trap.

Whereas the classical analysis yields an upper bound of \( K = \sqrt{4/27} \) for trapping to occur, no such sharp bound exists in the quantum-mechanical analysis. This is related to the fact that one cannot associate an effective potential well with a finite barrier with the system.

As an example, we apply our results to the case of a neutron and an atom trapped with a field \( B_0 = 100 \) Oe and \( B_0/B'_\perp = 10 \) cm. These parameters correspond to typical traps used in Bose-Einstein condensation.
experiments\footnote{23, 24, 25, 26}. The results, being correct to within an order of magnitude, are outlined in the following table.

|        | $B_0 = 100$ Oe | $B_0/B'_\perp = 10$cm |
|--------|---------------|------------------------|
|        | Neutron       | Atom                   |
| $m_{\text{gr}}$ | $\sim 10^{-25}$ | $\sim 10^{-22}$        |
| $\mu_{\text{emu}}$ | $\sim 10^{-23}$ | $\sim 10^{-20}$        |
| $K$     | $\sim 10^{-5}$  | $\sim 10^{-8}$         |
| $T_{\text{prec sec}}$ | $\sim 10^{-6}$ | $\sim 10^{-9}$        |
| $T_{\text{vib sec}}$ | $\sim 10^{-1}$ | $\sim 10^{-1}$        |
| $T_{\text{esc sec}}$ | $\sim 10^{(10^5)}$ | $\sim 10^{(10^8)}$ |

We note that in both cases $K$ is much smaller than 1. Also, the calculated lifetime of the particle in the trap is extremely large, suggesting that the particle (either neutron or atom) is tightly trapped in this field.

The problem studied in this paper deals with a spin $1/2$ particle. Though this fact has little influence on the solution of the classical problem, the extension to higher spin values complicates the analysis of the quantum-mechanical problem. In this case one has to deal with a $(2S+1)$-component spinor, and the interaction Hamiltonian does no longer connect the $(-S)$-state to the $(+S)$-state, but only to the $(+S)$ and $(+S+2)$ states which for $S \geq 5/2$ will still be trapped.

\section*{References}

[1] A. L. Migdall, J. V. Prodan, W. D. Phillips, T. H. Bergeman and H. J. Metcalf, “First observation of magnetically trapped neutral atoms”, \textit{Phys. Rev. Lett.}, \textbf{54} (24), 2596-2599 (1985).

[2] T. Bergeman, G. Erez and H. J. Metcalf, “Magnetostatic trapping fields for neutral atoms”, \textit{Phys. Rev. A.}, \textbf{35} (4), 1535-1546 (1987).

[3] V. S. Bagnato, G. P. Lasyatis, A. G. Martin, E. L. Raab, R. N. Ahmad-Bitar and D. E. Pritchard, “Continuous stopping and trapping of neutral atoms”, \textit{Phys. Rev. Lett.}, \textbf{58} (21), 2194-2197 (1987).
[4] W. Petrich, M. H. Anderson, J. R. Ensher and E. A. Cornell, “Stable, tightly confining magnetic trap for evaporative cooling of neutral atoms”, Phys. Rev. Lett., 74 (17), 3352-3355 (1995).

[5] M. O. Mewes, M. R. Andrews, N. J. Van-Druten, D. M. Kurn, D. S. Durfee, W. Ketterle, “Bose-Einstein condensation in a tightly confining DC magnetic trap”, Phys. Rev. Lett., 77(3), 416-419 (1996).

[6] The Levitron is available from ‘Fascinations’, 18964 Des Moines Way South, Seattle, WA 98148.

[7] The U-CAS is available from Masudaya International Inc., 6-4, Kuramae, 2-Chome, Taito-Ku, Tokyo, 111 Japan.

[8] R. Harrigan, U.S. Patent Number: 4,382,245, Date of Patent: May 3, 1983.

[9] Hones et al., U.S. Patent Number: 5,404,062, Date of Patent: Apr. 4, 1995.

[10] R. Edge, “Levitation using only permanent magnets”, Phys. Teach. 33, 252-253 (1995) and “Corrections to the levitation paper”, ibid. 34, 329 (1996).

[11] M. V. Berry, Proc. R. Soc. Lond. A 452, 1207-1220 (1996).

[12] S. Gov and S. Shtrikman, Proc. of the 19th IEEE Conv. in Israel, 184-187 (1996).

[13] M. D. Simon, L. O. Hefflinger and S. L. Ridgway, Am. J. Phys. 65 (4), 286-292 (1997).

[14] S. Gov, S. Shtrikman and H. Thomas, Los-Alamos E-Print Archive, http://xxx.lanl.gov/physics/9803020 (1998).

[15] S. Earnshaw, Trans. Cambridge Philos. Soc. 7, 97-112 (1842).

[16] The quantized motion of atoms in a quadrupole magnetic trap has been studied numerically by T. H. Bergeman, P. McNicholl, J. Kycia, H. Metcalf and N. L. Balazs, “Quantized motion of atoms in a quadrupole magnetostatic trap”, J. Opt. Soc. Am. B, 6 (11), 2249 (1989). Here, we use an analytic method to find the lifetime of the particle in such a trap.
[17] S. Gov, S. Shtrikman and H. Thomas, *Los-Alamos E-Print Archive*, [http://xxx.lanl.gov/physics/9808007](http://xxx.lanl.gov/physics/9808007), Submitted to *Am. J. Phys.*

[18] J. D. Weinstein and K. G. Libbrecht, *Phys. Rev. A.*, 52 (5), 4004-4008 (1995).

[19] L. D. Landau and E. M. Lifshitz, Quantum Mechanics (Pergamon Press), 3rd ed., pp. 213-214.

[20] ‘Quantum Mechanics’ by E. Merzbacher, *John Wiley & Sons.*, 2nd Ed., Ch. 5, Sec. 3, 57-61.

[21] ‘Quantum Mechanics’ by E. Merzbacher, *John Wiley & Sons.*, 2nd Ed., Ch. 18, Sec. 8, 475-481.

[22] This integral may be found in ‘Table of Integrals, Series, and Products’ by I. S. Gradshteyn and I. M. Ryzhik, *Academic Press*, 5th Ed., 6.521.1, pp. 697. Note that this integral make explicit use of the fact that \( J_1 (kR) = 0 \).

[23] M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman and E. A. Cornell, “Observation of Bose-Einstein condensation in a dilute atomic vapor”, *Science* 269, 198 (1995).

[24] K. B. Davis, M. O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurm and W. Ketterle, “Bose-Einstein condensation in a gas of Sodium atoms”, *Phys. Rev. Lett.* 75, 3969 (1995).

[25] C. C. Bradley, C. A. Sackett, J. J. Tollett and R. G. Hulet, “Evidence of Bose-Einstein condensation in an atomic gas with attractive interactions”, *Phys. Rev. Lett.* 75, 1687 (1995); *ibid* 79, 1170 (1997).

[26] E. A. Cornell and C. E. Wiemann, “The Bose-Einstein condensate”, *Sci. Am.* 278, 26-31 (1998).