Non-Abelian Josephson effect in spinor Bose-Einstein condensates

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Abstract. We investigate the non-Abelian Josephson effect in F=2 spinor Bose-Einstein condensates with double optical traps. We propose a real physical system which incorporates the non-Abelian Josephson effect and has very different density and spin tunneling characteristics compared with the Abelian case. We calculate the frequencies of the pseudo Goldstone modes in different phases between two traps, which are the crucial feature of the non-Abelian Josephson effect. We also give an experimental protocol to observe this novel effect in future experiments.

1. Introduction
The Josephson effect is a quantum tunneling phenomenon which occurs when a pair of superconducting or superfluid systems are weakly linked by some kind of physical barrier. Beginning with Josephson’s original paper in 1962 [1], the Josephson effect has become a paradigm of the phase coherence manifestation in a macroscopic quantum system. With the rapid experimental progress in cold atom physics, the Josephson junction has been realized for trapped Bose-Einstein condensates (BEC) of $^{87}$Rb [2] and $^{23}$Na [3]. However, most of the extensive studies about this effect have focussed on the Abelian case so far, in terms of a junction of two systems with spontaneously broken Abelian symmetry [4, 5]. There are also some kinds of Josephson-type effect without junctions, such as the spin mixing process of a spin-1 condensate in a single well discussed as in Ref. [6].

Recently, F. Esposito et al. formulated the Josephson effect in a field theoretic language and provided a straightforward generalization to the non-abelian case [7]. The non-Abelian Josephson effect emerges in a junction of two weakly coupled systems with spontaneously broken non-Abelian symmetries, which often involves multi-component order parameters. The non-Abelian nature of the symmetry will induce more than one kind of tunneling mode in the Josephson effect. These different tunneling modes can be characterized by the excitation of so-called pseudo Goldstone bosons which have small but finite masses [7]. The emergence of pseudo Goldstone bosons is a consequence of the explicit symmetry breaking term due to the coupling between the two condensates. Theoretically speaking, the non-Abelian Josephson effect is ubiquitous in nature, covering many topics from particle physics to condensed matter physics. For example, it may be realized in the superfluid $^3$He Josephson weak link [8], high density phases of QCD [9] and the artificial non-Abelian gauge field induced by nonlinear optics [10]. In Ref [7], the authors gave some samples in a relativistic quantum field system which involves...
$O(N)$ symmetry in an N-component scalar field. However as far as we know, there have been no specific experimental constructions until now.

In this review, we provide an experimental protocol to observe this novel effect in future experiments. To our knowledge, this effect has not been explicitly spelled out in any real physical system. In order to generalize to the non-Abelian junction in experiments, we should consider a system with a multi-component order parameter which has a non-Abelian symmetry in the ground state. In contrast to the situation in a magnetic trap, the spin of alkali atoms is essentially free in an optical trap [12–14]. This spinor nature properly provides the scenario of our non-Abelian symmetry construction. We now briefly introduce the system consisting of a spinor atomic BEC in a double-well optical trap. Although the dynamical tunneling properties of spin-1 and “pseudo spin-1/2” bosonic systems have been calculated [15–19], the essence of the non-Abelian effect has not been captured yet. In the present paper, we focus on the pseudo Goldstone modes due to explicit non-Abelian symmetry breaking, which is at the heart of the Josephson effect. For a concrete construction, we propose a spin-2 BEC in double optical traps. A spin-2 system has possible advantages, compared with a spin-1 system, in the sense that the symmetry properties are much richer to explore the non-Abelian effect. We should note that in a spin-1 system some elements of the symmetry group are hidden in the ground state. For example, in the polar state of a spin-1 system, the symmetry group of the ground state is $U(1) \times S(2)$ which is a subgroup of the symmetry group of the Hamiltonian $U(1) \times SO(3)$ [14]. The reason this happens is that in a low spin system, some rotation in the symmetry group of the Hamiltonian will leave the ground state unchanged and does not contribute to Goldstone modes. In contrast, in the antiferromagnetic and cyclic phase of a spin-2 condensate, the full $U(1) \times SO(3)$ symmetry is preserved in the ground state configuration, except for some specific values of the parameter [20].

2. Ground state structure of a spin-2 BEC

Let us consider a system of a homogeneous spin-2 Bose gas with an s-wave interaction. This system can be described by the following mean field free energy:

$$F(\psi) = \frac{1}{2} \langle \psi^\dagger \psi \rangle^2 + c_1 \langle f \rangle^2 + \frac{c_2}{5} \langle \Theta \rangle^2 - \mu \langle \psi \rangle$$

(1)

where $\psi = (\psi_2, \psi_1, \psi_0, \psi_{-1}, \psi_{-2})^T$ is the order parameter of the system, $c_0, c_1$ and $c_2$ are interaction strengths related to the scattering length in different spin channels, $\langle f \rangle = \langle \psi^\dagger f \psi \rangle$ is the mean value of the spin operator and $\Theta = \sum_{a=-2}^{2} (-1)^a \psi_a \psi_{-a}$ represents a single pair of identical spin-2 particles. The ground state configuration can be determined by minimizing this free energy. There are several distinct phases in this system [21, 22]. We will discuss these phases under zero magnetic field and analyze the symmetry and low-lying excitation spectrum of each phase.

2.1. Ferromagnetic phases

When $c_1 < 0$ and $c_1 - c_2/20 < 0$, two kinds of ferromagnetic phases are energetically favoured. The corresponding ground state configurations are given by $\psi = \sqrt{n}e^{i\theta}(1, 0, 0, 0, 0)$ or $\psi = \sqrt{ne^{i\theta}}(0, 1, 0, 0, 0)$, where $n = \mu/(c_0 + 4c_1)$ is the particle density and $\theta$ is an arbitrary global phase. It is obvious that these ground states have a $U(1)$ symmetry which leads to only one massless Goldstone mode. Therefore, two uncoupled systems have a $U(1) \otimes U(1)$ symmetry. This symmetry will softly break into a $U(1)$ diagonal symmetry when a weak coupling is applied. This pattern of symmetry breaking corresponds to an Abelian Josephson effect. The low-lying excitation spectrum of this state has been derived as $\omega_k = \sqrt{\epsilon_k(\epsilon_k + 2g_4)}$ [21], where $\epsilon_k = k^2$ and $g_4 = c_0 + 4c_1$. We should note that this Goldstone mode will break into two modes when coupling is applied: one zero energy mode and one pseudo Goldstone mode. This pseudo
Goldstone mode has a finite but small gap and leads to a density mode fluctuation in d.c. Josephson current.

![Figure 1](image_url)  
**Figure 1.** Experimental schematic of a spin-2 Bose gas trapped in a double well with chemical potentials $\mu_L$ of left and $\mu_R$ of right trap, which initially satisfy $\mu_L = \mu_R$. To drive the Josephson effect, we add a small distortion $\delta\mu$ to $\mu_R$.

2.2. Antiferromagnetic phase
In the absence of a magnetic field, there is only one kind of antiferromagnetic phase when $c_2 < 0$ and $c_1 - c_2/20 > 0$ are satisfied. The corresponding ground state configuration is degenerate with respect to five continuous variables which lead to five massless Goldstone modes [22]. Four of them correspond to the $U(1) \times SO(3)$ symmetry of the free energy. The extra degeneracy besides the $U(1) \times SO(3)$ symmetry is correct only on the mean field level and will be removed when the quantum fluctuation is considered. In this paper, we will work at the mean field level and maintain all five Goldstone modes. The effect of quantum fluctuation will be included in future work.

2.3. Cyclic phase
When $c_1 > 0$ and $c_2 > 0$, the cyclic phase is energetically favoured. The ground state configuration is given by $\psi = \sqrt{n}(\psi_{2}^{\theta_2}, 0, \psi_{0}^{\theta_0}, 0, \psi_{-2}^{\theta_2})$ where $n = \mu/c_0$ is the particle density and the global phases $\theta_2$ and $\theta_0$ satisfy $\theta_2 + \theta_{-2} - 2\theta_0 = \pi$ [22, 23]. Using the Schwinger boson representation [20], one can see that the ground state of the cyclic phase is mapped to a tetrahedron on the unit sphere. Therefore, the ground state of the cyclic phase has a full $U(1) \times SO(3)$ degeneracy and leads to four Goldstone modes. We will show that these modes also lead to the non-Abelian Josephson effect.

3. Non-Abelian Josephson effect and pseudo-Goldstone modes
We will analyze the Josephson effect of a spin-2 BEC system in a double-well optical trap, as shown in Fig. 1. We assume that the energy barrier between the two wells is strong enough so that the coupling between the Bose gas in each well is very weak and the overlap of the ground state wave functions in the left and right well (which we denote by $\varphi_L(x)$ and $\varphi_R(x)$) can be safely neglected. We will also use the single mode approximation which means we take the same mode function for all five spin components; this is a widely used approximation and is valid when the spin interaction is symmetric. Under these assumptions, the system can be described by the following potential

$$V_{\text{couple}} = F(\psi_L) + F(\psi_R) - J(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L),$$

where $\psi_L$ and $\psi_R$ are the order parameters of the Bose system in the left and right wells, respectively, and $J$ is the coupling parameter. It should be noted that we have taken the same
chemical potential for the Bose gas in the right and left well, because we will only be interested in the d.c. Josephson effect which captures the essence of the non-Abelian symmetry breaking as simply as possible. Using the dynamic equations of the condensate \( i \frac{d}{dt} \psi_{aL} = \frac{\delta V_{\text{couple}}}{\delta \psi_{aL}} \) and \( i \frac{d}{dt} \psi_{aR} = \frac{\delta V_{\text{couple}}}{\delta \psi_{aR}} \), we can derive the equation of motion of the Josephson current in each phase and analyze the pseudo Goldstone modes. However, as we have mentioned above, the symmetry of the ground state in the ferromagnetic phase only leads to an Abelian Josephson effect which is not of interest in the present paper. Therefore, we will just analyze the antiferromagnetic phase and cyclic phase which are important realizations of the non-Abelian Josephson effect.

3.1. Antiferromagnetic phase
To obtain the pseudo Goldstone modes, we need to linearize the equation of motion around the ground state \( \psi_L = \psi_R = \sqrt{n}(1, 0, 0, 0, 1) \). By minimizing the potential \( V_{\text{couple}} \) under the above antiferromagnetic ground state, we get \( n = (\mu + J)/(c_0 + c_2/5) \). With this ground state configuration, we can obtain the linearized equations of the fluctuations in this phase and analyze the excitation spectrum.

The \( m = 0 \) mode.
The equation of motion of the \( m = 0 \) mode is given as
\[
i \frac{d}{dt} \phi_0L = (-\frac{c_2}{5}n + J)\phi_0L + \frac{c_2}{5}n\phi_0^*L - J\phi_0R,
\]
and a similar equation for \( \phi_0R \). Since this mode is decoupled from the others, it corresponds to an Abelian Josephson current. By solving equation (3), we obtain the eigenenergies of this mode, one is zero corresponding to a massless Goldstone mode and the other is \( \omega_0 = 2\sqrt{J(J + \frac{2c_2n}{5})} \) corresponding to a pseudo Goldstone mode.

![Figure 2. Frequencies of the pseudo Goldstone modes as a function of coupling parameter \( J \) for the case of the antiferromagnetic phase. All frequencies are proportional to \( \sqrt{J} \) when \( J \) approaches zero and the dependence on \( J \) becomes linear when \( J \) is large compared to the interaction energy.](image)

The \( m = \pm 1 \) coupled modes.
The fluctuations of \( m = \pm 1 \) spin components are coupled in the following equations
\[
i \frac{d}{dt} \phi_{L} = [n(c_1 - \frac{c_2}{5}) + J]\phi_{1L} + n(c_1 - \frac{c_2}{5})\phi_{-1L}^* - J\phi_{1R},
\]
\[
i \frac{d}{dt} \phi_{-1L} = [n(c_1 - \frac{c_2}{5}) + J]\phi_{-1L} + n(c_1 - \frac{c_2}{5})\phi_{1L}^* - J\phi_{-1R},
\]
and a similar set of equations for \( \phi_{\pm 1R} \). The solution involves two pseudo Goldstone modes with the same gap \( \omega_{\pm 1} = 2\sqrt{n(c_1 - \frac{c_2}{5})J + J^2} \).
The $m = \pm 2$ coupled modes.

The equation of motion of $m = \pm 2$ spin components are coupled as follows

\[
i \frac{d}{dt} \phi_{2L} = \left[ \frac{n}{2} (c_0 + 4c_1) + J (\phi_{2L} + \phi^*_{2L}) + \frac{n}{2} (c_0 - 4c_1 + \frac{2}{5} c_2) (\phi_{-2L} + \phi^*_{-2L}) - J \phi_{2R} \right], \quad (5)
\]

\[
i \frac{d}{dt} \phi_{-2L} = \left[ \frac{n}{2} (c_0 + 4c_1) + J (\phi_{-2L} + \phi^*_{-2L}) + \frac{n}{2} (c_0 - 4c_1 + \frac{2}{5} c_2) (\phi_{2L} + \phi^*_{2L}) - J \phi_{-2R} \right], \quad (6)
\]

and a similar set of equations for $\phi_{\pm 2R}$. By solving the above equations, we obtain two pseudo Goldstone modes with energy gap $\omega_{\pm 2}^{(1)} = 2 \sqrt{n(c_0 + \frac{c_2}{2}) J + J^2}$ and $\omega_{\pm 2}^{(2)} = 2 \sqrt{n(c_1 - \frac{c_2}{2}) J + J^2}$.

We can see that there are in total five pseudo Goldstone modes with four different gaps in this phase which is consistent with our analysis of the ground state degeneracy. Figure 2 shows the dependence of the above frequencies on coupling parameter $J$. Recently, a polar behavior has been observed in the F=2 ground state of the $^{87}$Rb condensate [24]. We expect that the above modes of fluctuations can be observed in this system in future experiments. In the case of the $^{87}$Rb system, the values of the interaction strengths under typical experimental conditions are given as [24]: $c_{1n} : 0 - 10 \text{ nK}$, $c_{2n} : 0 - 0.2 \text{ nK}$ and $c_{0n}$ about $150 \text{ nK}$. According to the weak coupling limit, we assume that the coupling parameter $J$ is much smaller than the interaction energy of the condensate and given as about $0.1 \text{ nK}$. Under these conditions, we can obtain the frequencies of the fluctuation related to the antiferromagnetic phase, which is of order 100 Hz. The measurement of fluctuations on this characteristic time scale (about 10 ms) is accessible in current experiments.

### Figure 3. Frequencies of pseudo Goldstone modes as a function of coupling parameter $J$ for the case of the cyclic phase. The dependence on $J$ is similar to the case in the antiferromagnetic phase. However, there are two degenerate modes $\omega_{0,\pm 2}^{(2)}$ and $\omega_{\pm 1}$ in this case, which originate from the two degenerate Bogoliubov modes in the uncoupled system.

#### 3.2. Cyclic phase

Following the same procedure as in the antiferromagnetic phase, the pseudo Goldstone modes for the cyclic phase can be obtained. We will not list the detailed equations of motion but just give the result in this section.

The $m = \pm 2, 0$ coupled modes.

Each Goldstone mode in the corresponding uncoupled system [21] breaks into one massless mode and one pseudo Goldstone mode. Since there are two Goldstone modes in the uncoupled system, they induce two pseudo Goldstone modes with energy $\omega_{0,\pm 2}^{(1)} = 2 \sqrt{J^2 + c_{0n} J}$ and $\omega_{0,\pm 2}^{(2)} = 2 \sqrt{J^2 + 2c_{1n} J}$.

The $m = \pm 1$ coupled modes.
In the uncoupled system, there are two massless Goldstone modes with the same energy [21]. By solving the equation of motions for this mode, it is found that each of them leads to a pseudo Goldstone mode with a gap \( \omega_{\pm 1} = 2\sqrt{J^2 + 2c_1 nJ} \). There are four pseudo Goldstone modes in the cyclic phase which is consistent with the previous analysis on the symmetry of this phase. These kinds of fluctuations in cyclic phase are expected to be realized in a condensate of \(^{85}\text{Rb}\) atoms [22]. Based on current estimates of scattering lengths, the value of interaction strengths under typical experimental condition are given as: \( c_1 n : 0 - 20 \text{ nK}, c_2 n : 0 - 0.6 \text{ nK} \) and \( c_0 n \) about 600 nK. Under these conditions, we can also estimate the d.c. Josephson frequencies in the cyclic phase which are about 100-300 Hz. The dependence of the above frequencies on the coupling parameter \( J \) is shown in Fig. 3.

4. Experimental signatures of Non-Abelian Josephson effect
The experimental set up of a spin-2 Bose gas trapped in a double well is illustrated in Fig. 1. The d.c. non-Abelian Josephson current can be detected with the following steps. The first step is to initiate a density oscillation in the system. This can be realized by slightly changing the depth of one well, which will cause a small imbalance in chemical potential (\( \mu_R \rightarrow \mu_R + \delta\mu \)) between the two wells, and then tuning it back. The next step is to detect the time dependence of the particle numbers in the different spin components. Such kind of detection can be realized by first spatially separating different spin component with a Stern-Gerlach method during time of flight after switching off the trapping potential. Then, the number of atoms in each spin component will be related to the respective spatial density distributions which can be evaluated by the absorption imaging method. Following the above steps, one can measure the density oscillations in each spin component which are coupled together due to the non-Abelian symmetry of the system. A measurement of the dependence of the oscillation frequencies on \( J \) can be realized by varying the barrier between the two wells and repeating the above measurement.

In mean field theory, the condensates of \(^{87}\text{Rb}\) atom in F=2 state are predicted to be polar \((c_1 - c_2/20 > 0 \text{ and } c_2 < 0)\), but close to the boundary of the cyclic phase \((c_1 > 0 \text{ and } c_2 > 0)\) [22]. Furthermore, polar behaviour in the F=2 ground state of the \(^{87}\text{Rb}\) condensate has been observed in recent experiment [24]. As a result, we expect that the pseudo Goldstone modes of the antiferromagnetic phase could be observed in experiments. As we have mentioned, the values of interaction strengths under typical experimental conditions are given as [24]: \( c_1 n : 0 - 10 \text{ nK}, c_2 n : 0 - 0.2 \text{ nK} \) and \( c_0 n \) about 150 nK, which leads to the fluctuation time scale of about 10 ms in this system. On this time scale, the measurement we proposed above is completely accessible within recent experimental techniques in F=2 spinor Bose-Einstein condensates of the \(^{87}\text{Rb}\) system [24, 27]. In order to observe this dynamical oscillation clearly in an experiment, the temperature should be lower than the gap of the pseudo Goldstone modes, which is about 1-10 nK. Although there is still no such kind of measurement performed in a system with cyclic phase, we expect that it will be realized in a condensate of \(^{85}\text{Rb}\) atoms in the near future [22].

In summary, we reveal a novel Josephson effect in spin-2 Bose system which involves non-Abelian symmetry and propose an experimental protocol to realize the so called non-Abelian Josephson effect in this system. It is found that the frequencies of the pseudo Goldstone modes are not only related to the coupling parameter but also to the interaction strength, which is a nonlinear effect due to the spin dependent interaction. These results are of particular significance for exploring the new features of the non-Abelian Josephson effect which are very distinct from the Abelian case.

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