Multi-dimensional extensions of the Hegselmann-Krause model

Giulia De Pasquale and Maria Elena Valcher

Abstract—In this paper we consider two multi-dimensional Hegselmann-Krause (HK) models for opinion dynamics. The two models describe how individuals adjust their opinions on multiple topics, based on the influence of their peers. The models differ in the criterion according to which individuals decide whom they want to be influenced from. In the average-based model individuals compare their average opinions on the various topics with those of the other individuals, and interact only with those individuals whose average opinions lie within a confidence interval. For this model we show that the agents’ opinions reach consensus/clustering if and only if their average opinions do so. In the uniform affinity model agents compare their opinions on each single topic and influence each other only if, topic-wise, such opinions do not differ more than a given tolerance. We identify conditions under which the uniform affinity model enjoys the order-preservation property topic-wise and we prove that the global range of opinions (and hence the range of opinions on each single topic) are non-increasing.

I. INTRODUCTION

Social sciences, psychology, economy and control engineering are research areas with a strong interest in understanding and describing opinion dynamics in social networks.

As a consequence, many models of opinion dynamics have been proposed along time [1], [5], [13], [14]. A common objective, in this context, is to understand when reaching a consensus, as a consequence of complex interactions among the agents in the network, is possible [11]. Consensus is an active research topic in many fields [2], [22]. It is about the achievement of an agreement or of a common goal by agents in a network. However, there are contexts in which the reaching of a consensus is either not desirable or does not represent a realistic scenario. This is the case also when dealing with social contexts, e.g., political elections, surveys. It is in these contexts that the disagreement phenomenon, along with consensus, becomes of interest [11].

A sociological model that considers both consensus and disagreement is the one known in the literature as Hegselmann-Krause (HK) model [13]. The HK dynamics evolves under a bounded-confidence mechanism. Confidence intervals are expressed as a function of the gap between pairs of agents’ opinions. Since only agents whose opinions are close enough interact, the model represents a mathematical abstraction of confirmation bias [7], i.e., the natural human propensity to search for and welcome information that supports prior beliefs.

The scalar (i.e., single topic) HK model has been studied both in continuous [17] and discrete time [21]. When all agents adopt the same confidence interval the model is said to be homogeneous. Otherwise it is called heterogeneous [12]. If the lower and upper thresholds of acceptance of other agents’ opinions are the same the model is called symmetric. It is asymmetric otherwise [3]. For the multi-dimensional case, where agents are asked to express their opinion on a pre-fixed and finite number of topics, we mention the non-exhaustive list of works [3], [10], [11], [18]. In [10] the authors focus on the investigation of the termination time of the dynamics. The analysis is based on Lyapunov arguments and a polynomial upper bound for the case in which the connectivity of the network maintains some specific structure is provided. The work in [18] focuses on the homogeneous multidimensional HK model, and assumes that confidence intervals are expressed in terms of vector norms. Stability properties of the model are investigated and the finite time convergence of the dynamics is proved. The results are valid regardless of the choice of the norm into play. In [11] the evolution of the HK model under various assumptions is studied. First the termination time of the synchronous HK model in arbitrary finite dimensions is analyzed and shown to be independent of the dimension of the opinion vectors. The convergence speed of the dynamics is related to the eigenvalues of the adjacency matrix of the connectivity graph. A game-theoretic approach to the study of the asynchronous model is employed and some results on the time of convergence of this variant are provided. Finally, in the heterogeneous case a necessary condition for the termination time to be finite is provided. The work [3] studies the case in which the confidence bound is determined through the Euclidean norm. The work focuses on the bounds on the convergence time of the system. The paper also investigates a noisy version of the model.

In this paper we consider two multi-dimensional HK models for opinion dynamics: the average-based model and the uniform affinity model. In the average-based mode that, to the best of our knowledge, has not been considered before in the literature, individuals compare their average opinions on the various topics with those of the other individuals, and interact only with those individuals whose average opinions lie within a confidence interval. This models fits to contexts in which the topics into play are somehow related, so that the agents are unlikely to assume far apart opinions about the various topics. For this model we provide a new proof for the contractivity of the range of opinions, and show that the agents’ opinions reach consensus/clustering if and only if their average opinions do so. The uniform affinity model is a special instance of the multi-dimensional HK model investigated in [3], [11], [10], [18], where we specifically...
adopt the $\ell_\infty$-norm. In other words, agents compare their opinions on each single topic and influence each other only if, topic-wise, such opinions do not differ more than a given tolerance. We identify conditions under which the uniform affinity model enjoys the order-preservation property topic-wise and we prove that the global range of opinions (and hence the range of opinions on each single topic) are non-increasing.

Paper organization: Section II introduces some notation and preliminaries definitions and results. Section III to VI investigate the opinion ranges and the steady state behavior of the average-based HK model. Finally, Section VII introduces the uniform affinity model and investigates some conditions that ensure the order preservation of the opinions on each single topic.

II. NOTATION AND PRELIMINARIES

In the following, $\mathbb{R}_{\geq 0}$ denotes the set of nonnegative real numbers. We let $\mathbb{1}_N$ and $0_N$ denote the $N$-dimensional vectors of all ones and all zeros, respectively. The symbol $e_i$ denotes the $i$-th vector of the canonical basis of $\mathbb{R}^N$, where $N$ will be clear from the context. Given a set $S$, we denote by $|S|$ its cardinality. Given $k \geq 2$ square matrices $F_1, F_2, \ldots, F_k$, the symbol $\text{blockdiag}\{F_1, F_2, \ldots, F_k\}$ denotes the block diagonal matrix having $F_i$ as $i$th diagonal block.

Consider the complete undirected graph with $N$ nodes $G = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, 2, \ldots, N\}$ and $\mathcal{E} = (\mathcal{V} \times \mathcal{V}) \setminus \{(i, i) : i \in \mathcal{V}\}$. If $|\mathcal{E}| = m$, we let $C_N \in [-1, 0, 1]^{N \times m}$ denote its oriented incidence matrix [6], defined as follows. For every vertex $h \in \mathcal{V}$ and every edge $e = (i, j) \in \mathcal{E}$, we have

$$[C_N]_{h,e} = \begin{cases} 1, & h = i; \\ -1, & h = j; \\ 0, & \text{otherwise}. \end{cases}$$

Given a matrix $X \in \mathbb{R}^{N \times N}$, we denote by $X_{i\cdot}$ its $i$-th row of $X$, by $X_{\cdot j}$ its $j$-th column, and by $X_{ij}$ its $(i,j)$-th entry.

**Definition 1** (Vector $\ell_\infty$-norm). Given $x \in \mathbb{R}^N$, the $\ell_\infty$-norm of $x$ is $\|x\|_\infty = \max_i|x_i|$.

**Definition 2** (Seminorms). A function $\|\cdot\| : \mathbb{R}^N \to \mathbb{R}_{\geq 0}$ is a seminorm on $\mathbb{R}^N$ if it satisfies the following properties:

- (homogeneity): $\|ax\| = |a| \|x\|$, $\forall x \in \mathbb{R}^N$ and $a \in \mathbb{R}$;
- (subadditivity): $\|x + y\| \leq \|x\| + \|y\|$, $\forall x, y \in \mathbb{R}^N$.

**Definition 3.** ($\ell_\infty$ weighted seminorm [7]) Let $\|\cdot\|_\infty : \mathbb{R}^k \to \mathbb{R}_{\geq 0}$ be the $\ell_\infty$-norm on $\mathbb{R}^k$ and let $R \in \mathbb{R}^{k \times N}$. The $\ell_\infty$-weighted seminorm on $\mathbb{R}^N$ associated with the $\ell_\infty$-norm on $\mathbb{R}^k$ is $\|x\|_{\ell_\infty,R} := \|Rx\|_\infty$, $\forall x \in \mathbb{R}^N$.

**Example 4** ($C_N^A$-weighted seminorm [8]). Given a vector $x \in \mathbb{R}^N$ and the oriented incidence matrix $C_N \in \mathbb{R}^{N \times m}$, the $C_N^A$-weighted seminorm of $x$ associated with the $\ell_\infty$-norm is

$$\|x\|_{\ell_\infty,C_N^A} = \max_{(i,j) \in \mathcal{E}} |x_i - x_j| = \max_{i,j} |x_i - x_j|.$$

**Lemma 5** (Preliminary lemma). Given a vector $x \in \mathbb{R}^N$ and a row stochastic matrix $A \in \mathbb{R}^{N \times N}$,

$$\|Ax\|_{\ell_\infty,C_N^A} \leq \|A\|_{\ell_\infty,C_N^A} \|x\|_{\ell_\infty,C_N^A},$$

where

$$\|A\|_{\ell_\infty,C_N^A} := \max_{\|x\|_{\ell_\infty,C_N^A} = 1} \|Ax\|_{\ell_\infty,C_N^A}$$

is the $C_N^A$-weighted, $\ell_\infty$ induced seminorm of $A$.

**Proof:** Upon noticing that $A(\ker C_N^A) \subseteq \ker C_N^A$, the result follows from Lemma 14 in [8] and the conditional sub-multiplicativity property of the semi norms [15].

**Theorem 6** (Expression for the $C_N^A$-weighted, $\ell_\infty$ induced seminorm [8]). For a row stochastic matrix $A \in \mathbb{R}^{N \times N}$,

$$\|A\|_{\ell_\infty,C_N^A} = 1 - \min_{ij} \sum_{k=1}^N \min\{A_{ik}, A_{jk}\}. \quad (1)$$

III. THE AVERAGE-BASED (MULTI-DIMENSIONAL) HK MODEL

In this section we introduce a multi-dimensional extension of the HK model in which agents compare their (scalar) average opinions on a set of topics, rather than (the vectors representing) their specific opinions topic by topic. This model is suitable to describe the situation when the opinions that an agent has on the different topics are not too far apart, as it happens, for instance, when the topics are related and homogeneous. When so, the average value of the opinions seems a good representative of the opinion vector and hence a reasonable quantity to decide about the proximity of the agents’ opinions.

Given a group of $N \geq 2$ agents and $m \geq 2$ (related) topics, we let $X_{ij}(t)$ denote the opinion that agent $i$ has about the topic $j$ at the time $t$. The average opinion that the agent $i$ has about the $m$ topics at the time instant $t$ is given by

$$\bar{x}_i(t) = \frac{1}{m} \sum_{j=1}^m X_{ij}(t)$$

and in vector form

$$\bar{x}(t) = \frac{1}{m} X(t) e_m. \quad (2)$$

We assume that the opinion that the $i$-th agent has on topic $j$ at time $t+1$ is influenced only by the opinions at time $t$ on that same topic of agents whose average opinion about the $m$ topics is not too far from agent $i$’s average opinion at time $t$. Specifically, given a certain confidence threshold $\varepsilon > 0$, we define the set of neighbors (or influencers) of the agent $i$ at the time instant $t$ as a function of the average opinions of the agents, namely as:

$$N^\text{ave}_{i}(\bar{x}(t)) = \{k \in \{1, \ldots, N\} : |\bar{x}_k(t) - \bar{x}_i(t)| \leq \varepsilon\}. \quad (3)$$

Accordingly, by adopting a notation similar to the one in [19], the influence matrix $\Phi^\text{ave}_{i}(\bar{x}(t)) \in \{0, 1\}^{N \times N}$ of this average-based HK model is defined as

$$\Phi^\text{ave}_{ik}(\bar{x}(t)) = \begin{cases} 1, & \text{if } k \in N^\text{ave}_{i}(\bar{x}(t)); \\ 0, & \text{otherwise}. \end{cases} \quad (4)$$
Upon defining the matrix
\[
D_{\text{ave}}(\bar{x}(t)) := \text{blockdiag}\{|N_{\text{ave}}^1(\bar{x}(t))|, \ldots, |N_{\text{ave}}^N(\bar{x}(t))|\},
\]
the opinion matrix \(X(t)\) evolves over time as
\[
X(t + 1) = A_{\text{ave}}(\bar{x}(t))X(t),
\tag{5}
\]
where \(A_{\text{ave}}(\bar{x}(t)) := D_{\text{ave}}(\bar{x}(t))^{-1}\Phi_{\text{ave}}(\bar{x}(t))\) is well-posed since \(D_{\text{ave}}(\bar{x}(t))\) is nonsingular and as a consequence of the fact that \(i \in N_{\text{ave}}^i(\bar{x}(t))\) (and hence \(|N_{\text{ave}}^i(\bar{x}(t))| \geq 1\) \(\forall i \in \{1, \ldots, N\}, \forall t \geq 0\). Equation (5) component-wise reads as
\[
X_{ij}(t + 1) = \frac{1}{|N_{\text{ave}}^i(\bar{x}(t))|} \sum_{k=1}^{n} \Phi_{tk}^{\text{ave}}(\bar{x}(t))X_{kj}(t).\tag{6}
\]

IV. AVERAGE-BASED HK MODEL: MAIN DEFINITIONS

In this section we introduce some fundamental definitions for the average-based HK model that will be used in the following.

**Definition 7** (Consensus for average-based HK model). The average-based HK model
\[
X(t + 1) = A_{\text{ave}}(\bar{x}(t))X(t),\tag{7}
\]
with \(X(0) \in \mathbb{R}^{N \times m}\), is said to reach consensus if
\[
\lim_{t \to \infty} X(t) = 1_N c^\top, \quad \exists c \in \mathbb{R}^m.\tag{9}
\]

**Definition 8** (Clustering for average-based HK model). The average-based HK model (7)-(8) reaches clustering if there exists a partitioning of the agents \(V_1, V_2, \ldots, V_d\) \((V_i \cap V_j = \emptyset\) for \(i \neq j\), and \(\cup_{i=1}^d V_i = \{1, \ldots, N\}\) such that \(\forall i, k \in V_t, \ell \in \{1, \ldots, d\}\),
\[
\lim_{t \to \infty} X_{ik}(t) = \lim_{t \to \infty} X_{kj}(t)\tag{10}
\]
and \(\forall i \in V_t, \forall k \in V_p, \ell \neq p,\)
\[
\lim_{t \to \infty} X_{ik}(t) \neq \lim_{t \to \infty} X_{kj}(t).\tag{11}
\]

**Definition 9** (Range of opinions on a specific topic). Given the average-based HK model (7)-(8), the range of opinions on topic \(j\), at the time instant \(t\), is defined as
\[
\nu_j(X(t)) = \max_{i,k \in \{1, \ldots, N\}} |X_{ij}(t) - X_{kj}(t)|.\tag{12}
\]

**Remark 10.** Note that \(\nu_j(X(t)) = \|X_{\cdot j}(t)\|_{\infty, c_N^\top}\).

V. AVERAGE-BASED HK MODEL: OPINION RANGES

In this section we explore the monotonicity properties of the range of opinions defined in the previous section. As we will see, the average-based HK model preserves several nice properties of the scalar HK model [13].

**Proposition 11** (Range of opinions on topic). Given the average-based HK model (7)-(8), for every choice of \(X(0) \in \mathbb{R}^{N \times m}\), the range of opinions on a specific topic \(\{\nu_j(X(t))\}_{t \geq 0}, j \in \{1, \ldots, m\}\), is a non-increasing sequence.

**Proof:** The proof follows from the fact that each column of \(X(t)\) in (7) updates according to the equation
\[
X_{\cdot j}(t + 1) = A(\bar{x}(t))X_{\cdot j}(t).\tag{13}
\]
where \(A(\bar{x}(t))\) is row stochastic.

**Remark 12.** By the same reasoning adopted to prove the previous result we can claim that \(\forall i \in \{1, \ldots, N\}, j \in \{1, \ldots, m\}\) and \(t \geq 0\), one has \(X_{ij}(t) \in [\min_k X_{kj}(0), \max_k X_{kj}(0)]\). Consequently, if consensus is reached and we assume \(c = [c_1 \ldots c_m]^\top\), then
\[
\min_k X_{ki}(0) \leq c_i \leq \max_k X_{ki}(0).\tag{14}
\]

In the following proposition we provide an alternative proof for the rate of contractivity of the range of opinions in the average-based HK model (see Remark 14, below).

**Proposition 13** (Range of opinions). Given the average-based HK model (7)-(8), for every choice of \(X(0) \in \mathbb{R}^{N \times m}\), the range of opinions on a specific topic \(\{\nu_j(X(t))\}_{t \geq 0}, j \in \{1, \ldots, m\}\), satisfies
\[
\nu_j(X(t + 1)) \leq \gamma(\bar{x}(t))\nu_j(X(t)),\tag{15}
\]
where
\[
\gamma(\bar{x}(t)) := 1 - \min_{i \neq k} \sum_{j=1}^{N} \min\{A_{ik}(\bar{x}(t)), A_{kj}(\bar{x}(t))\}.\tag{16}
\]

**Proof:** Consider (13), where \(A(\bar{x}(t))\) is row stochastic. From Remark 10 and the submultiplicativity property of the induced matrix seminorms, we get
\[
\nu_j(X(t + 1)) = \|X_{\cdot j}(t + 1)\|_{\infty, c_N^\top}
= \|A(\bar{x}(t))X_{\cdot j}(t)\|_{\infty, c_N^\top} \leq \|A(\bar{x}(t))\|_{\infty, c_N^\top}\|X_{\cdot j}(t)\|_{\infty, c_N^\top}
= \left(1 - \min_{i \neq k} \sum_{j=1}^{N} \min\{A_{ik}(\bar{x}(t)), A_{kj}(\bar{x}(t))\}\right)\nu_j(X(t)),
\]
where the inequality follows from Lemma 5, while the last identity from Theorem 6.

**Remark 14.** The result in Proposition 13 has also been proven in Lemma 1 in [16], and Theorem 5.2 and Lemma 5.1 [20] for the scalar case.

Opinions of the agents on each topic do not enjoy any order preservation property. So, even if \(\bar{x}_j(t) \leq \bar{x}_j(t)\), nothing can be said about \(X_{ik}(t)\) and \(X_{jk}(t)\) for specific values of \(k \in \{1, \ldots, m\}\).

We first note that the vector of the average opinions \(\bar{x}(t)\) in (2) obeys the dynamics
\[
\bar{x}(t + 1) = \frac{1}{m}X(t + 1)1_m = \frac{1}{m}A_{\text{ave}}(\bar{x}(t))X(t)1_m
= A_{\text{ave}}(\bar{x}(t))\bar{x}(t),\tag{16}
\]
and hence it follows a scalar HK model. Upon a reordering of the agents, so that \(\bar{x}_1(0) \leq \bar{x}_2(0) \leq \cdots \leq \bar{x}_N(0)\) then, see Proposition 1 in [4], we can guarantee that \(\bar{x}_1(t) \leq \bar{x}_2(t) \leq \cdots \leq \bar{x}_N(t), \forall t \geq 0\). Also, by Theorem 1 in [4],
each sequence \( \{\bar{x}_i(t)\}_{t\geq 0}, i \in \{1, \ldots, m\} \), converges to a limit in finite time. Moreover, if \( \bar{x}^* := \lim_{t \to \infty} \bar{x}(t), \forall i \in \{1, \ldots, N - 1\} \) either \( \bar{x}_i^* = \bar{x}_{i+1}^* \) or \(|\bar{x}_{i+1}^* - \bar{x}_i^*| > \varepsilon \), namely the steady state average opinions reach either consensus or clustering.

**Remark 15.** As proved in [9] consensus is reached if and only if the sequence \( \bar{x}(t) \) is an \( \varepsilon \)-chain for all \( t \geq 0 \), by this meaning that, assuming the initial ordering \( \bar{x}_1(0) \leq \bar{x}_2(0) \leq \ldots \leq \bar{x}_N(0) \), then we have \(|\bar{x}_{i+1}(t) - \bar{x}_i(t)| \leq \varepsilon \), for every \( i \in \{1, \ldots, N - 1\} \) and \( t \geq 0 \).

We have the following result.

**Theorem 16 (Steady state of average-based HK model).**
Given the average-based HK model (7)-(8), for every choice of \( X(0) \in \mathbb{R}^{N \times m} \) the systems dynamics reaches a steady state configuration in a finite number of steps. Moreover, the average-based HK model reaches consensus (clustering) if and only if the HK model describing the evolution of the average opinions reaches consensus (clustering).

**Proof:** Suppose, w.l.o.g., to permute the rows of \( X(0) \) (namely the agents’ order) so that \( \bar{x}_1(0) \leq \bar{x}_2(0) \leq \cdots \leq \bar{x}_N(0) \). As previously recalled, there exists \( t^* \geq 0 \) such that
\[
\bar{x}(t) = \frac{1}{m} X(t)1_m = \bar{x}^* \in \mathbb{R}^N, \quad \forall t \geq t^*,
\]
and hence
\[
X(t + 1) = A^{ave}(\bar{x}^*) X(t).
\]
If the average opinions reach consensus, namely \( \bar{x}^* = e^* 1_N \), then \( A^{ave}(\bar{x}^*) = \frac{1}{N} 1_N 1_N^\top \). Consequently, \( \forall t \geq t^* \)
\[
X(t + 1) = \frac{1}{N} 1_N 1_N^\top X(t).
\]
Since \( X(t^* + 1) = 1_N [\bar{m}_1(t^*), \ldots, \bar{m}_m(t^*)] \), where
\[
\bar{m}_j(t^*) := \frac{1}{N} \sum_{i=1}^N X_{ij}(t^*)
\]
represents the average opinion of the agents on the \( j \)-th topic at time instant \( t^* \), and \( 1_N [\bar{m}_1(t^*), \ldots, \bar{m}_m(t^*)] = A^{ave}(\bar{x}^*) 1_N [\bar{m}_1(t^*), \ldots, \bar{m}_m(t^*)] \), it follows that
\[
X(t) = 1_N [\bar{m}_1(t^*), \ldots, \bar{m}_m(t^*)], \quad \forall t \geq t^* + 1,
\]
and hence the punctual opinions of the agents on the topics reach consensus.

If the average opinions clusterize into \( d \) disjoint clusters, namely \( \bar{x}^* = [c_1^\top 1_{n_1} n_1 \cdots c_d^\top 1_{n_d} n_d]^\top \), with \( c_i^\top \in \mathbb{R} \) and \(|c_i^\top - c_{i+1}^\top| > \varepsilon, \forall i \in \{1, \ldots, d - 1\} \), then
\[
D^{ave}(\bar{x}^*) = \text{blockdiag}\{n_1 1_{n_1} n_1 \cdots n_d 1_{n_d} n_d\}
\]
\[
\Phi^{ave}(\bar{x}^*) = \text{blockdiag}\{1_{n_1} 1_{n_1} n_1 \cdots 1_{n_d} n_d\}
\]
and for all \( t \geq t^* \)
\[
X(t + 1) = \text{blockdiag}\left\{\frac{1}{n_1} 1_{n_1} 1_{n_1}, \ldots, \frac{1}{n_d} 1_{n_d} n_d\right\} X(t).
\]
Consequently, \( X(t^* + 1) = \text{blockdiag}\{1_{n_1}, \ldots, 1_{n_d}\} M(t^*) \)
with \( M(t^*) \in \mathbb{R}^{d \times m} \)
\[
eq \sum_{i \in I_i} e_i^\top X(t^*),
\]
where \( I_i = \{n_1 + \cdots + n_{i-1} + 1, \ldots, n_1 + \cdots + n_{i-1} + n_i\} \) is the set of agents in the \( i \)-th cluster. The \( j \)-th entry of the row vector \( e_i^\top M(t^*) \in \mathbb{R}^{1 \times m} \) represents the average opinion on the \( j \)-th topic of the agents in the \( i \)-th cluster. Moreover,
\[
X(t) = \text{blockdiag}\{1_{n_1}, \ldots, 1_{n_d}\} M(t^*), \quad \forall t \geq t^* + 1.
\]
Therefore, if the average opinions clusterize at the time instant \( t^* \) then, from \( t^* + 1 \) onward, the punctual opinions clusterize as well by maintaining the same partition, in \( d \) clusters, as the average opinions of the agents over the \( m \) topics. The converse parts are trivially true.

The following result shows that if the maximum gap between the average opinions does not change when moving from time \( t \) to time \( t + 1 \), then the same maximum gap remains at all subsequent times, thus showing that if such gap is nonzero then consensus is not reached.

**Proposition 17.** Consider the average-based HK model (7)-(8).
If at some time \( t \geq 0 \) one gets
\[
\max_{i \in \{1, \ldots, N\}} |\bar{x}_i(t) - \bar{x}_{i_j}(t)| = \max_{i \in \{1, \ldots, N\}} |\bar{x}_i(t + 1) - \bar{x}_{i_j}(t + 1)|
\]
then
\[
\max_{i \in \{1, \ldots, N\}} |\bar{x}_i(t + 1) - \bar{x}_{i_j}(t + 1)| = \max_{i \in \{1, \ldots, N\}} |\bar{x}_i(t + 2) - \bar{x}_{i_j}(t + 2)|.
\]
(23)
Therefore, if the quantity in (22) is positive, then the average-based HK model (7)-(8) does not achieve consensus.

**Proof:** Assume, without loss of generality, that \( \bar{x}_1(t) \leq \bar{x}_2(t) \leq \cdots \leq \bar{x}_N(t) \), then one has \( \max_{i \in \{1, \ldots, N\}} |\bar{x}_i(t) - \bar{x}_{i_j}(t)| = \bar{x}_1(t) - \bar{x}_{i_j}(t) \). Since \( \bar{x}_1(t + 1) \geq \bar{x}_2(t) \) and \( \bar{x}_1(t + 1) \leq \bar{x}_N(t) \), then (22) implies \( \bar{x}_1(t + 1) = \bar{x}_2(t) \) and \( \bar{x}_N(t + 1) = \bar{x}_N(t) \), that easily leads to (23). Since the sequence of average opinions does not reach consensus, neither does the sequence \( \{X(t)\}_{t \geq 0} \).

VI. THE UNIFORM AFFINITY MODEL

The multi-dimensional HK model investigated in [11], [10], [18] has a structure similar to the one we explored in the previous sections, however it adopts as a criterion to define the opinion proximity the distance (induced by the norm) between the opinion vectors of the agents. Specifically, it is assumed that the neighbours of agent \( i \) at time \( t \) are
\[
N_i(X(t)) = \{k \in \{1, \ldots, N\} : \|X_{ik}(t)^\top - X_{ik}(t)^\top\| \leq \varepsilon\},
\]
where \( \varepsilon > 0 \) is the confidence threshold and \( \|\cdot\| \) denotes an arbitrary norm. Accordingly, the influence matrix \( \Phi \in \{0, 1\}^{N \times N} \) at time \( t \) is the one whose \((i, k)\)-th entry is
\[
\Phi_{ik}(X(t)) = \begin{cases} 1, & \text{if } k \in N_i(X(t)); \\ 0, & \text{otherwise}. \end{cases}
\]
(24)
Since the norm is formally defined for column vectors, while \( X_{ik}(t) \) and \( X_{ik}(t) \) are row vectors, we moved to their transposed versions.
Upon defining the matrix $D(X(t)) := \text{blockdiag}\{|N_1(X(t))|, \ldots, |N_N(X(t))|\}$ the opinion matrix $X(t)$ evolves over time as

$$X(t+1) = A(X(t))X(t),$$

where

$$A(X(t)) := D(X(t))^{-1}\Phi(X(t))$$

is well-posed ($D(X(t))$ is nonsingular) and row stochastic. In the references [10], [11], [18] the main focus has been on proving that the multi-dimensional HK model (25), with the row stochastic matrix $A(X(t))$ defined as above, (for any choice of the norm $\|\cdot\|$) converges to a steady-state solution in a finite number of steps, and on providing an upper bound on the termination time (see, in particular, [11]). The interesting aspect is that the termination time is independent of the number $m$ of topics. See Figure 1 for an example of an uniform affinity model with $N = 10$ agents and $m = 2$ topics that reaches consensus.

In this section we want to explore some monotonicity properties of the previous model by considering specifically the case when the norm is the $\ell_\infty$-norm. This means that $N_i(X(t)) = \{j : \max_{k \in \{1, \ldots, m\}} |X_{ik}(t) - X_{jk}(t)| \leq \varepsilon\}$ so, in order for two agents to influence each other, their opinions must be close topic-wise. This model is in line with the spirit of bounded-confidence even in contexts in which agents take different positions about the various topics. We will refer to the multi-dimensional HK model with $\ell_\infty$-norm (25) as the uniform affinity model.

We first prove that if we consider the range of opinions on a specific topic $k$ at time $t$ and we consider the largest of such values over all the possible topics, then such a quantity is non-increasing over time.

**Proposition 18 (Range of opinions in uniform affinity HK model).** For the uniform affinity model, the quantity

$$\nu(X(t)) := \max_{i,j \in \{1, \ldots, N\}} \max_{k \in \{1, \ldots, m\}} |X_{ik}(t) - X_{jk}(t)| = \max_{k \in \{1, \ldots, m\}} \nu_k(X(t))$$

is non-increasing over time, namely $\nu(X(0)) \geq \nu(X(1)) \geq \nu(X(2)) \geq \ldots$.

**Proof:** We first observe that for all $k \in \{1, \ldots, m\}$

$$\nu(X(t)) \geq \max_{i,j \in \{1, \ldots, N\}} \max_{u \in \{1, \ldots, m\}} |X_{uk}(t) - X_{uk}(t)| = \max_{i,j} \left| X_{ik}(t) - X_{jk}(t) \right| = \max_{i,j} (X_{ik}(t) - X_{jk}(t))$$

for some specific $i, j, u, l$. For all $i, j \in \{1, \ldots, N\}$ and $k \in \{1, \ldots, m\}$

$$|X_{ik}(t+1) - X_{jk}(t+1)| = \left| \sum_{d \in N_i(t)} \frac{1}{|N_i(t)|} X_{ik}(t) - \sum_{d \in N_j(t)} \frac{1}{|N_j(t)|} X_{dk}(t) \right| \leq \max_{t} X_{ik}(t) - \min_{t} X_{ik}(t) = |X_{uk}(t) - X_{uk}(t)| = X_{uk}(t) - X_{uk}(t) \leq \nu(X(t)).$$

Since this is true for all $i, j \in \{1, \ldots, N\}$ and for all $k \in \{1, \ldots, m\}$, then it is also true that $\nu(X(t+1)) = \max_{i,j \in \{1, \ldots, N\}} \left| X_{ik}(t+1) - X_{jk}(t+1) \right| \leq \nu(X(t))$.

A consequence of Proposition 18 is that the opinion gap on each single topic (see definition 9) is non-increasing too.

Differently from what happens with the standard HK model (and partly with the average-based HK model), there is no way to introduce a meaningful total ordering in $\mathbb{R}^m$ and hence in the set of all agents’ opinions. In fact, in general, the ordering is different on each topic, and condition $|X_{ik}(t) - X_{jk}(t)| \leq \varepsilon$ for some specific $k$ does not ensure that $i$ and $j$ are neighbours. So, it may happen that $X_{ik}(t) < X_{jk}(t)$, but at the subsequent time step $X_{ik}(t+1) > X_{jk}(t+1)$. See, for instance, Figure 1, where moving from $t = 0$ to $t = 1$ the opinions of agents 1 and 5 on topic 2 swap (even if eventually they all converge to a consensus).

**Proposition 19 (Sufficient condition for one-step order preservation).** Consider the uniform affinity model and suppose that at some time $t \geq 0$ one has that for every $i, j \in \{1, \ldots, N\}$ condition $j \notin N_i(t)$ implies

$$|X_{ik}(t) - X_{jk}(t)| > \varepsilon, \quad \forall k \in \{1, \ldots, m\}. \quad (27)$$

If for every $k \in \{1, \ldots, m\}$ we sort the agents’ opinions (the order specifically depending on $k$) so that $X_{i_1,k}(t) \leq X_{i_2,k}(t) \leq \cdots X_{i_N,k}(t)$, then the same opinion ordering on that topic is preserved at $t+1$, i.e., $X_{i_1,k}(t+1) \leq X_{i_2,k}(t+1) \leq \cdots \leq X_{i_N,k}(t+1)$.

**Proof:** Let $k$ be arbitrary in $\{1, \ldots, m\}$ and consider $i_h \in \{i_1, \ldots, i_{N-1}\}$. We preliminarily observe that if $i_{h+1} \notin N_{i_h}(t)$, assumption (27) implies that $N_{i_h}(t) \cap N_{i_{h+1}}(t) = \emptyset$, $N_{i_h}(t) \cap N_{i_{h+1}}(t) \subseteq \{i_1, \ldots, i_h\}; N_{i_{h+1}}(t) \cap N_{i_h}(t) \subseteq \{i_{h+1}, \ldots, i_N\}$. On the other hand, if $i_{h+1} \in N_{i_h}(t)$, then $N_{i_h}(t) \cap N_{i_{h+1}}(t) \subseteq \{i_1, \ldots, i_h, i_{h+1}\}$ and $N_{i_{h+1}}(t) \cap N_{i_h}(t) \subseteq \{i_{h+2}, \ldots, i_N\}$. So, we can define

$$\Delta_{hk}(t) := \frac{1}{|N_{i_h}(t) \cap N_{i_{h+1}}(t)|} \sum_{\ell \in N_{i_h}(t) \cap N_{i_{h+1}}(t)} X_{\ell k}(t),$$

$$\tilde{X}_{i_{h+1}k}(t) := \frac{1}{|N_{i_{h+1}}(t) \setminus N_{i_h}(t)|} \sum_{\ell \in N_{i_{h+1}}(t) \setminus N_{i_h}(t)} X_{\ell k}(t),$$

and get

$$X_{i_hk}(t+1) = \frac{|N_{i_h}(t) \cap N_{i_{h+1}}(t)|}{|N_{i_h}(t)|} \Delta_{hk}(t) + \frac{|N_{i_h}(t) \setminus N_{i_{h+1}}(t)|}{|N_{i_h}(t)|} \tilde{X}_{i_{h+1}k}(t).$$
and similarly
\[
X_{i_{h+1,k}}(t + 1) = \frac{|N_{i_{h+1}}(t) \cap N_{i_{h,k}}(t)|}{|N_{i_{h,k}}(t)|} \Delta_{i_{h,k}}(t) + \frac{|N_{i_{h+1}}(t) \setminus N_{i_{h,k}}(t)|}{|N_{i_{h,k}}(t)|} \tilde{X}_{i_{h,k}}(t).
\]

Since \(\tilde{X}_{i_{h+1,k}}(t) \geq \Delta_{i_{h,k}}(t) \geq \tilde{X}_{i_{h,k}}(t)\) if \(i_h\) and \(i_{h+1}\) are neighbours, while \(\Delta_{i_{h,k}}(t) = 0\) and \(\tilde{X}_{i_{h+1,k}}(t) > \tilde{X}_{i_{h,k}}(t)\) if \(i_h\) and \(i_{h+1}\) are not neighbours, it follows that \(\tilde{X}_{i_{h+1,k}}(t + 1) \geq \tilde{X}_{i_{h,k}}(t + 1)\).

The reasoning behind the previous result can be extended to a different situation when the agents’ opinions at some time \(t\) are ordered so that \(X_{i_{h,k}}(t) \leq X_{i_{k,2k}}(t) \leq \cdots \leq X_{i_{N_k,k}}(t)\), for every \(k \in \{1, \ldots, m\}\). When so, such ordering is preserved at all subsequent time instants. This is based on the fact that if \(i < j\) and \(i\) and \(j\) are not neighbours, then \(\tilde{X}_{i_{h,k}}(t) - X_{i_{k,t}}(t) > \varepsilon\). But this implies that for every \(p < q\) one has \(X_{i_{h,k}}(t) - X_{i_{k,t}}(t) > \varepsilon\), and for every \(q > j\) one has \(X_{i_{h,k}}(t) - X_{i_{k,t}}(t) > \varepsilon\). Consequently, \(N_i(t) \cap \{j, j+1, \ldots, N_k\} = \emptyset\) and similarly \(N_j(t) \cap \{1, 2, \ldots, i\} = \emptyset\). Conversely, if \(i < j\) and \(i\) and \(j\) are neighbours, then \(N_i(t) \cap N_j(t) \supseteq \{i, i+1, \ldots, j\}\). Based on these comments, the proof of the following result can be easily obtained by mimicking the proof of Proposition 19.

**Proposition 20** (Sufficient condition for order preservation). Consider the uniform affinity model. If at some time \(t \geq 0\) one has \(X_{i_{1,k}}(t) \leq X_{i_{2,k}}(t) \leq \cdots \leq X_{i_{N_k,k}}(t)\), for every topic \(k \in \{1, \ldots, m\}\), then it is also true that \(X_{i_{1,k}}(t + \tau) \leq X_{i_{2,k}}(t + \tau) \leq \cdots \leq X_{i_{N_k,k}}(t + \tau)\), for every \(\tau \geq 0\) and every topic \(k \in \{1, \ldots, m\}\).

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