MODULAR TECHNIQUES FOR NONCOMMUTATIVE GRAßNER BASES

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Abstract. In this note, we extend modular techniques for computing Gröbner bases from the commutative setting to the vast class of noncommutative $G$-algebras. As in the commutative case, an effective verification test is only known to us in the graded case. In the general case, our algorithm is probabilistic in the sense that the resulting Gröbner basis can only be expected to generate the given ideal, with high probability. We have implemented our algorithm in the computer algebra system SINGULAR and give timings to compare its performance with that of other instances of Buchberger’s algorithm, testing examples from $D$-module theory as well as classical benchmark examples. A particular feature of the modular algorithm is that it allows parallel runs.

1. Introduction

That the concept of Gröbner bases and Buchberger’s algorithm for computing these bases can be extended from the commutative to the noncommutative setting was remarked in the late 1980s in the case of Weyl algebras [8], [27], with particular emphasis on the computational treatment of $D$-modules.

Around the same time, Apel [2] introduced a much more general class of algebras, called $G$-algebras, which are well-suited for Gröbner basis methods (see [20, 22] for some details). These algebras are defined over a field, and are also known as algebras of solvable type [16, 23, 18], or as PBW-algebras [7, 12]. They include the Weyl algebras together with a variety of other important algebras, such as universal enveloping algebras of finite dimensional Lie algebras, many quantum algebras (including coordinate rings of quantum affine planes and algebras of quantum matrices as well as some quantized enveloping algebras of Lie algebras), and numerous algebras formed by common linear partial functional operators.

The use of Gröbner bases allows, more generally, the computational treatment of $GR$-algebras, which are factor algebras of $G$-algebras by two-sided ideals, and which include Clifford algebras (in particular, exterior algebras) and a number of quantum algebras, such as quantum general and quantum special linear groups.

Over the rationals, modular methods not only enable us to avoid intermediate coefficient swell, but also provide a way of introducing parallelism into our computations. In the context of Gröbner bases, this means to reduce

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the given ideal modulo several primes, compute a Gröbner basis for each reduced ideal, and use Chinese remaindering and rational reconstruction to find the desired Gröbner basis over $\mathbb{Q}$. See [3, 10, 15] for the commutative case.

In this paper, we extend the modular Gröbner basis algorithm to $G$-algebras. In a final verification step, the algorithm checks that the result $G$ is indeed a left or right Gröbner basis, and that the left or right ideal generated by $G$ contains the left or right ideal we started with. In the graded case, this guarantees the equality of the two ideals and, thus, that $G$ is a Gröbner basis for the given ideal. In the general case, we can only expect equality, with high probability. Alternatively, we may apply the algorithm in a homogenized situation, and dehomogenize the result (if this is computationally feasible).

The paper is organized as follows: In Section 2 we recall the definition and basic properties of $G$-algebras. Then we address gradings, filtrations and the homogenization of $G$-algebras, and Gröbner bases. In Section 3 we present our modular algorithm and discuss the final verification test for the graded case. In Section 4, based on our implementation in the computer algebra system SINGULAR [9, 13], we compare the performance of the modular algorithm with that of other variants of Buchberger’s algorithm. In Section 5, we conclude the paper with final remarks.

2. Preliminaries

In this section, we introduce some of the terminology used in this paper.

2.1. Basic Notation. We work over a field $K$. Given a finite set of indeterminates $x := \{x_1, \ldots, x_n\}$, we write $\langle x \rangle := \langle x_1, \ldots, x_n \rangle$ for the free monoid on $x$, and consider the corresponding monoid algebra $K\langle x \rangle := K\langle x_1, \ldots, x_n \rangle$, that is, the free associative $K$-algebra generated by $\langle x \rangle$.

A monomial in $x_1, \ldots, x_n$ is an element of $\langle x \rangle$, that is, a word in the finite alphabet $x$. A standard monomial is a monomial of type $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. A standard term in $K\langle x \rangle$ is an element of $K$ times a standard monomial. A standard polynomial in $K\langle x \rangle$ is a sum $f = \sum c_\alpha x^\alpha$ of finitely many nonzero standard terms involving distinct standard monomials. The Newton diagram of such an $f$ is

$$N(f) = \{\alpha \in \mathbb{N}^n \mid c_\alpha \neq 0\}.$$  

By convention, the zero element is a standard polynomial, with $N(0) = \emptyset$.

To give a partial ordering $>$ on the set of standard monomials means to give a partial ordering $>$ on $\mathbb{N}^n$. We only consider orderings which are total, are such that $\alpha > \beta$ implies $\alpha + \gamma > \beta + \gamma$, for all $\alpha, \beta, \gamma \in \mathbb{N}^n$, and are well-orderings. By abuse of language, we then say that $>$ is a monomial ordering on $\langle x \rangle$. Given $>$, it makes sense to speak of the leading exponent $\exp(f) = \exp_>(f)$ of a standard polynomial $f \neq 0$. The lexicographic and degree reverse lexicographic orderings are defined as usual. Similarly for block orderings. We write $e_i = \exp(x_i) = (0, \ldots, 0, 1, 0, \ldots, 0)$, for all $i$.

Given $\omega \in \mathbb{N}^n$, the $\omega$-weighted degree of a standard monomial $x^\alpha$ is

$$|\alpha|_\omega = \langle \omega, \alpha \rangle = \omega_1 \alpha_1 + \cdots + \omega_n \alpha_n,$$  

where $\langle \omega, \alpha \rangle = \sum \omega_i \alpha_i$. 


while that of a standard polynomial \( f \neq 0 \) is
\[
\deg_\omega(f) = \max\{||\alpha||_\omega \mid \alpha \in \mathcal{N}(f)\}.
\]
We say that \( f \) is \( \omega \)-homogeneous (of degree \( \deg_\omega(f) \)) if \( ||\alpha||_\omega = \deg_\omega(f) \) for all \( \alpha \in \mathcal{N}(f) \).

By convention, the zero element is considered to be \( \omega \)-homogeneous of each degree \( d \in \mathbb{N} \), and we write \( \deg_\omega(0) = -\infty \). For any monomial ordering \( > \) on \( \langle x \rangle \), we obtain a new monomial ordering \( >_\omega \) on \( \langle x \rangle \) by setting
\[
\alpha >_\omega \beta \iff \begin{cases} 
||\alpha||_\omega > ||\beta||_\omega \\
\text{or} \\
||\alpha||_\omega = ||\beta||_\omega \text{ and } \alpha > \beta.
\end{cases}
\]

2.2. \( G \)-algebras. Each finitely presented associative \( K \)-algebra \( A \) is isomorphic to a factor algebra of type \( K \langle x_1, \ldots, x_n \rangle/J \), for some \( n \) and some two-sided ideal \( J \subset K \langle x_1, \ldots, x_n \rangle \). If \( J \) is generated by a set of two-sided generators \( g_1, \ldots, g_r \), then we say that \( A \) is generated by \( x_1, \ldots, x_n \) subject to the relations \( g_1, \ldots, g_r \), and write
\[
A = K \langle x_1, \ldots, x_n \mid g_k = 0, \ 1 \leq k \leq r \rangle.
\]

We say that \( A \) as has a Poincaré-Birkhoff-Witt (PBW) basis if the standard monomials in \( K \langle x_1, \ldots, x_n \rangle \) represent a \( K \)-basis for \( A \). In this case, every element \( f \in A \) has a unique representation
\[
f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x^\alpha,
\]
where we abuse our notation by denoting the class of the standard monomial \( x^\alpha \) in \( A \) also by \( x^\alpha \). We then refer to the \( c_\alpha \) as the coefficients of \( f \) and let the Newton diagram \( \mathcal{N}(f) \), the \( \omega \)-weighted degree \( \deg_\omega(f) \), and the property of being \( \omega \)-homogeneous be defined as above. Similarly for the leading exponent \( \exp(f) = \exp_\omega(f) \) with respect to a monomial ordering \( > \) on \( \langle x \rangle \) if \( f \neq 0 \).

\( G \)-algebras are obtained by imposing specific commutation relations:

**Definition 1.** A \( G \)-algebra over \( K \) is a factor algebra of type
\[
A = K \langle x_1, \ldots, x_n \mid x_j x_i = c_{ij} \cdot x_i x_j + d_{ij}, \ 1 \leq i < j \leq n \rangle,
\]
where the \( c_{ij} \in K \) are nonzero scalars and the \( d_{ij} \in K \langle x_1, \ldots, x_n \rangle \) are standard polynomials such that the following two conditions hold:

- There exists a monomial ordering \( > \) on \( \langle x_1, \ldots, x_n \rangle \) such that
  \[
d_{ij} = 0 \quad \text{or} \quad \epsilon_i + \epsilon_j > \exp(d_{ij}) \quad \text{for all} \quad 1 \leq i < j \leq n.
\]

Every such ordering is called admissible for \( A \).

- For all \( 1 \leq i < j < k \leq n \), the elements
  \[
c_{ik}c_{jk} \cdot d_{ij} x_k - x_k d_{ij} + c_{jk} \cdot x_j d_{ik} - c_{ij} \cdot d_{ik} x_j + d_{jk} x_i - c_{ij} c_{ik} \cdot x_i d_{jk}
\]
  reduce to zero with respect to the relations of \( A \).

**Example 2.** The \( n \)th Weyl algebra over \( K \) is the \( G \)-algebra
\[
D_n(K) = K \langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \mid \partial_i x_i = x_i \partial_i + 1, \partial_i x_j = x_j \partial_i \quad \text{for} \quad i \neq j \rangle,
\]
where we tacitly assume that \( x_j x_i = x_i x_j \) and \( \partial_i \partial_i = \partial_i \partial_j \) for all \( i, j \). Note that any monomial ordering on \( \langle x, \partial \rangle \) is admissible for \( D_n(K) \).
Example 3. The $r$th shift algebra over $K$ is the $G$-algebra

$$S_r(K) = \langle s_1, \ldots, s_r, t_1, \ldots, t_r \mid t_j s_k = s_k t_j - \delta_{jk} t_j \rangle,$$

where we tacitly assume that $s_j s_i = s_i s_j$ and $t_j t_i = t_i t_j$ for all $i, j$. Note that any monomial ordering on $(s, t)$ is admissible for $S_r(K)$.

Example 4. If $A = K\langle x_1, \ldots, x_n \mid C_A \rangle$ and $B = K\langle y_1, \ldots, y_m \mid C_B \rangle$ are $G$-algebras, then their tensor product $A \otimes_K B$ is a $G$-algebra as well,

$$A \otimes_K B = K\langle x_1, \ldots, x_n, y_1, \ldots, y_m \mid C_A, C_B, y_j x_i = x_i y_j \rangle.$$

Note that if $>_A$ and $>_B$ are admissible orderings for $A$ and $B$, respectively, then the block ordering $(>_A,>_B)$ is admissible for $A \otimes_K B$.

$G$-algebras enjoy structural properties which are reminiscent of those of commutative polynomial rings (see [6, 17, 19, 20] for definitions and proofs).

Proposition 5. Let $A = K\langle x_1, \ldots, x_n \mid C \rangle$ be a $G$-algebra. Then:

- $A$ has a PBW-basis;
- $A$ is left and right Noetherian domain;
- the Gel’fand-Kirillov dimension of $A$ over $K$ is equal to $n$;
- the global homological dimension of $A$ is at most $n$;
- the generalized Krull dimension of $A$ is at most $n$;
- $A$ is Auslander regular and Cohen-Macaulay.

Computing in a $G$-algebra rather than in a commutative polynomial ring means to additionally apply the commutation relations in the definition of the $G$-algebra. Over the rationals, this typically leads to even more coefficient swell. The following example illustrates this point and indicates, thus, the particular importance of modular methods for computations in $G$-algebras over $\mathbb{Q}$. With regard to notation, $[a + b]^n$ stands for writing out the right hand side of the binomial formula in its commutative version.

Example 6 [21]. Suppose that $K$ is a field of characteristic zero.

- In the Weyl algebra $D_1(K) = K\langle x, \partial \mid \partial x = x \partial + 1 \rangle$, we have

$$ (x + \partial)^n = [x + \partial]^n + \sum_{k=0}^{n-2} \sum_{j=0}^{n-k-2} \binom{n}{j} \binom{n-j}{k} g(n-j-k)x^k\partial^j, $$

where $g(n) = (n-1)!!$ if $n$ is even, and $g(n) = 0$ otherwise.

- In the shift algebra $S_1(K) = K\langle s, t \mid ts = st - t \rangle$, we have

$$ (s + t)^n = [s + t]^n + \sum_{k=0}^{n-1} \sum_{j=0}^{n-k-1} (-1)^{n+k+j} \binom{n}{k} S(n-k, j)s^k t^j, $$

where the $S(n, k)$ denote the Stirling numbers of the second kind.

In what follows, we will summarize some results on $G$-algebras, left or right ideals in $G$-algebras, and left or right Gröbner bases for these ideals. For simplicity of the presentation, we will focus on the case of left ideals.
If \( T \subset A \) is any subset of the \( G \)-algebra \( A \), the notation \( \langle T \rangle = A(T) \) will always refer to the left ideal of \( A \) generated by \( T \).

2.3. **Graded \( G \)-algebras.** In this subsection, we consider a \( G \)-algebra
\[
A = K\langle x_1, \ldots, x_n \mid x_jx_i = c_{ij} \cdot x_i x_j + d_{ij}, \ 1 \leq i < j \leq n \rangle
\]
such that the \( c_{ij} \cdot x_i x_j + d_{ij} \) are \( \omega \)-homogeneous for some weight vector \( \omega \neq 0 \in \mathbb{N}^n \). That is,
\[
(1) \quad \omega_i + \omega_j = |\alpha|_{\omega} \text{ for } 1 \leq i < j \leq n \text{ and all } \alpha \in \mathcal{N}(d_{ij})..
\]
Then \( A \) is **graded** with respect to the \( \omega \)-weighted degree,
\[
A = \bigoplus_{d \geq 0} A_d, \quad \text{with} \quad A_d = \{ f \in A \mid f \text{ \( \omega \)-homogeneous for all } \alpha \},
\]
and where \( A_d A_e \subset A_{d+e} \) for all \( d, e \)
(see [7, Chapter 4, Section 6] for a detailed discussion). In this case, a left ideal \( I \subset A \) is **graded** if it inherits the grading:
\[
I = \bigoplus_{d \geq 0} I_d = \bigoplus_{d \geq 0} (A_d \cap I) \subset A.
\]
Equivalently, \( I \) is generated by (finitely many) \( \omega \)-homogeneous elements.

If \( \omega \in \mathbb{N}^n \), the \( A_d \) are finite-dimensional \( K \)-vector spaces, with \( A_0 = K \). We may then talk about the **Hilbert function**
\[
H_I : \mathbb{N} \to \mathbb{N}, \quad d \mapsto \dim_K I_d
\]
of every graded left ideal \( I \subset A \).

2.4. **Filtrations and Homogenization.** Given any \( G \)-algebra
\[
A = K\langle x_1, \ldots, x_n \mid x_jx_i = c_{ij} \cdot x_i x_j + d_{ij}, \ 1 \leq i < j \leq n \rangle,
\]
a weight vector as in the previous subsection may not exist (consider the Weyl and shift algebras). In contrast, there is always an \( \omega \in \mathbb{N}^n \) satisfying
\[
(2) \quad \omega_i + \omega_j > |\alpha|_{\omega} \text{ for } 1 \leq i < j \leq n \text{ and all } \alpha \in \mathcal{N}(d_{ij})
\]
(see [7, Chapter 3, Section 1]). In practical terms, such an \( \omega \) can be found by solving the following linear programming problem:
\[
\text{minimize } \sum_{i=1}^{n} \omega_i \text{ subject to }
\]
\[
\bullet \ \omega_i + \omega_j > |\alpha|_{\omega} \text{ for } 1 \leq i < j \leq n \text{ and all } \alpha \in \mathcal{N}(d_{ij}),
\]
\[
\bullet \ \omega_1, \ldots, \omega_n > 0.
\]

**Example 7.** For both the Weyl algebra \( D_n(K) \) and the shift algebra \( S_r(K) \), any \( \omega \in \mathbb{N}^n \) will do.

Suppose now that an \( \omega \in \mathbb{N}^n \) satisfying \( (2) \) is given, and fix a monomial ordering \( > \) on \( \langle x \rangle \). Then the induced ordering \( >_{\omega} \) is admissible for \( A \). Furthermore, we get a filtration of \( A \), the \( \omega \)-**filtration**, if we set
\[
E_d^\omega A = \{ f \in A \mid \deg_{\omega}(f) \leq d \} \text{ for all } d \in \mathbb{N}.
\]
The Rees algebra corresponding to this filtration is the subalgebra

\[ R^\omega A = \bigoplus_{d \geq 0} (F_d^\omega A) t^d \subset A[t] = A \otimes K [t]. \]

Note that each element \( F \in (F_d^\omega A) t^d \) has the form

\[ F = \sum_{|\alpha| \omega \leq d} c_\alpha x^\alpha t^d = \sum_{|\alpha| \omega \leq d} c_\alpha \tilde{\omega}^d - |\alpha| \omega, \]

where

\[ \tilde{x}_i = x_i t^{\omega_i} \text{ and } \tilde{\omega} = \tilde{\omega}_1 \cdots \tilde{\omega}_n. \]

So \( F \) is \( \tilde{\omega} \)-homogeneous of degree \( d \), where \( \tilde{\omega} = (\omega, 1) \in \mathbb{N}^{n+1} \). In particular, if we set

\[ \tilde{d}_{ij} = d_{ij} t^{\omega_i + \omega_j} = \sum_{\alpha \in \mathbb{N}(d_{ij})} c_{ij}(\alpha) x^\alpha t^{\omega_i + \omega_j} = \sum_{\alpha \in \mathbb{N}(d_{ij})} c_{ij}(\alpha) \tilde{\omega}^d - |\alpha| \omega, \]

then the \( c_{ij} \cdot x_i x_j + \tilde{d}_{ij} \) are \( \tilde{\omega} \)-homogeneous of degree \( \omega_i + \omega_j \). We introduce a monomial ordering \( >^h_\omega \) on \( (\tilde{x}, t) \) by setting

\[ (\alpha, d) >^h_\omega (\beta, e) \iff \begin{cases} |(\alpha, d)|_\omega > |(\beta, e)|_\omega \\
 or \\
 |(\alpha, d)|_\omega = |(\beta, e)|_\omega \text{ and } \alpha >_\omega \beta. \end{cases} \]

Then we have:

**Theorem 8.** In the situation above, the Rees algebra \( R^\omega A \) is a graded \( G \)-algebra,

\[ R^\omega A = \bigoplus_{d \geq 0} (F_d^\omega A) t^d = K\langle \tilde{x}_1, \ldots, \tilde{x}_n, t \mid \tilde{C}\rangle, \]

with commutation relations

\[ \tilde{C} : \tilde{x}_j \tilde{x}_i = c_{ij} \cdot \tilde{x}_i \tilde{x}_j + \tilde{d}_{ij}, \ t \tilde{x}_i = \tilde{x}_i t, \]

and admissible ordering \( >^h_\omega \).

**Proof.** Clear from the discussion above. \( \square \)

**Definition 9** (Homogenization and Dehomogenization). In the situation above, the homogenization of an element \( f \in A \) is the \( \tilde{\omega} \)-homogeneous element \( f^h = ft^d \in R^\omega A \), where \( d = \deg_\omega(f) \). The homogenization of a left ideal \( I \subset A \) is the graded left ideal

\[ I^h = \langle f^h \mid f \in I \rangle = \bigoplus_{d \geq 0} (F_d^\omega A \cap I) t^d \subset R^\omega A. \]

The dehomogenization of an element \( F \in (F_d^\omega A) t^d \) is the element \( F \mid_{t=1} \in A \).

Note that if \( I \) is given by generators \( f_1, \ldots, f_r \), then

\[ I^h = \langle f_1^h, \ldots, f_r^h \rangle : t^\infty. \]

(3)
2.5. **Gröbner Bases in G-Algebras.** The concept of Gröbner bases extends from commutative polynomial rings to G-algebras. We give a brief account of this, referring to [7, 11, 19] for details and proofs.

To begin with, recall that a nonempty subset \( E \subset \mathbb{N}^n \) is called a monoid ideal if \( E + \mathbb{N}^n = E \). Dickson’s lemma tells us that each such \( E \) is finitely generated: There exist \( \alpha_1, \ldots, \alpha_s \in \mathbb{N}^n \) such that \( E = \bigcup_{i=1}^s (\alpha_i + \mathbb{N}^n) \).

Given a G-algebra
\[
A = K\langle x_1, \ldots, x_n \mid x_jx_i = c_{ij} x_i x_j + d_{ij}, 1 \leq i < j \leq n \rangle,
\]
an admissible ordering \( > \) for \( A \), and a subset \( I \subset A \), we set
\[
\exp(I) = \{ \exp(f) \mid f \in I \setminus \{0\} \}.
\]

Note that if \( I \) is a nonzero left ideal of \( A \), then \( \exp(I) \) is a monoid ideal of \( \mathbb{N}^n \). Moreover, Macaulay’s classical result on factor rings of commutative polynomial rings extends as follows:

**Remark 10.** Let \( I \) be a nonzero left ideal of \( A \). Then the standard monomials \( x^\alpha, \alpha \in \mathbb{N}^n \setminus \exp(I) \), represent a \( K \)-vector space basis for \( A/I \) (see, for example, [24] Proposition 9.1).

**Definition 11.** Let \( I \) be a nonzero left ideal of \( A \). A (left) Gröbner basis for \( I \) (with respect to \( > \)) is a finite subset \( G = \{ g_1, \ldots, g_s \} \subset I \setminus \{0\} \) such that
\[
\exp(I) = \bigcup_{i=1}^s (\exp(g_i) + \mathbb{N}^n).
\]

A finite subset \( G \subset A \setminus \{0\} \) is a left Gröbner basis if it is a Gröbner basis for the left ideal \( \langle G \rangle \subset A \) it generates.

As in the commutative case, every nonzero left ideal \( I \subset A \) has a Gröbner basis, and every such basis generates \( I \) as a left ideal. Furthermore, we have the concepts of left normal forms (LNF) (that is, left division with remainder) and left S-polynomials (LSP) in G-algebras. Based on this, there are adapted versions of Buchberger’s criterion and, thus, Buchberger’s algorithm which allow us to characterize and compute left Gröbner bases, respectively:

**Theorem 12** (Buchberger’s Criterion). Let \( G = \{ g_1, \ldots, g_s \} \) be a finite subset of \( A \setminus \{0\} \). Then \( G \) is a left Gröbner basis iff
\[
\text{LNF}(\text{LSP}(g_i, g_j)) = 0 \quad \text{for} \quad 1 \leq i < j \leq s.
\]

The following result will be useful for establishing our modular algorithm.

**Corollary 13** (Finite Determinacy of Gröbner Bases). Let \( I \) be a nonzero left ideal of \( A \), and let \( G = \{ g_1, \ldots, g_s \} \) be a Gröbner basis for \( I \) with respect to \( > \). There exists a finite set \( F \) of standard monomials such that if \( >_1 \) is any admissible ordering for \( A \) which coincides with \( > \) on \( F \), then:

1. \( \exp_>(g_i) = \exp_{>_1}(g_i) \) for all \( g_i \in G \).
2. \( G \) is a Gröbner basis for \( I \) also with respect to \( >_1 \).

**Proof.** Let \( F \) be the set of all standard monomials of all elements of \( A \) occurring during the reduction process of the LSP\((g_i, g_j)\) to zero modulo \( G \) with respect to \( > \). Then for any admissible ordering \( >_1 \) for \( A \) which coincides with \( > \) on \( F \), the LSP\((g_i, g_j)\) also reduce to zero modulo \( G \) with respect to \( >_1 \). The result follows from Theorem [12] \( \square \).
The notion of a reduced left Gröbner basis is analogous to that in the commutative case, every left ideal of $A$ has a uniquely determined such basis, and this basis can be computed by a variant of Buchberger’s algorithm.

If $A$ is graded with respect to some weight vector $\omega$ as in Subsection 2.3 and $>$ is an admissible ordering for $A$, then the induced ordering $>_\omega$ is also admissible for $A$. When computing a Gröbner basis with respect to $>_\omega$, starting from $\omega$-homogeneous elements, Buchberger’s algorithm will return Gröbner basis elements which are $\omega$-homogeneous as well. In particular, reduced Gröbner bases consist of $\omega$-homogeneous elements.

With regard to homogenizing and dehomogenizing Gröbner bases, the proposition known from the commutative case extends as follows:

**Proposition 14.** In the situation of Theorem 8, let $I \subset A$ be a nonzero left ideal, and let $I^h \subset R^\omega A$ be its homogenization. Then:

1. If $G = \{g_1, \ldots, g_r\}$ is a Gröbner basis for $I$ with respect to $>_\omega$, then $G^h = \{g_1^h, \ldots, g_r^h\}$ is a Gröbner basis for $I^h$ with respect to $>_h$ which consists of $\tilde{\omega}$-homogeneous elements.
2. Conversely, if $G = \{G_1, \ldots, G_s\}$ is a Gröbner basis for $I^h$ with respect to $>_h$ which consists of $\tilde{\omega}$-homogeneous elements, then $G |_{t=1} = \{G_1 |_{t=1}, \ldots, G_s |_{t=1}\}$ is a Gröbner basis for $I$ with respect to $>_\omega$.

Part (1) of the proposition will be of theoretical use for establishing our modular algorithm. From a practical point of view, as already pointed out, we may wish to verify the correctness of the Gröbner basis returned by the modular algorithm by working in a homogenized situation (provided this is computationally feasible). One possible approach for this is to compute $I^h$ using formula (3), and apply part (2) of the proposition. We describe a second approach which does not require to compute the saturation and is more flexible with regard to monomial orderings. For this, given $>$ and a vector $\omega \in \mathbb{N}_+^2$ as in Subsection 2.4 consider the monomial ordering $>_h$ on $\langle \tilde{x}, t \rangle$ defined by

$$(\alpha, d) >_h (\beta, e) \iff \begin{cases} |(\alpha, d)|_\omega > |(\beta, e)|_\omega \\
\text{or} \\
|((\alpha, d)|_\omega = |(\beta, e)|_\omega \text{ and } \alpha > \beta .
\end{cases}$$

This ordering is admissible for $R^\omega A$ and we have:

**Proposition 15.** In the situation above, let $I = \langle f_1, \ldots, f_r \rangle \subset A$ be a nonzero left ideal, and let $J = \langle f_1^h, \ldots, f_r^h \rangle \subset R^\omega A$. If $G$ is a Gröbner basis for $J$ with respect to $>_h$ which consists of $\tilde{\omega}$-homogeneous elements, then $G |_{t=1}$ is a Gröbner basis for $I$ with respect to $>$. 

**Proof.** Since dehomogenizing $J$ gives us back $I$, we have $G |_{t=1} \subset I$.

Let $f \in I$ be any nonzero element. Then $f^h \in I^h$, and we conclude from formula (3) that there is an integer $e \in \mathbb{N}$ such that $f^h t^e \in J$. Since $G$ is a Gröbner basis for $J$ with respect to $>_h$, there is an element $G \in G$ such that \( \exp_>_h (f^h t^e) = \exp_>_h (G) + (\beta, \tilde{e}) \) for some $\beta \in \mathbb{N}^n$ and some $\tilde{e} \in \mathbb{N}$. Then $\exp_>_ (f) = \exp_> (G |_{t=1}) + \beta$, which proves the proposition. \( \square \)
In Subsection 4.1, we will consider particular instances of computing Bernstein-Sato polynomials to compare the performance of our modular algorithm with that of other versions of Buchberger’s algorithm. Such computations require the elimination of variables.

**Definition 16.** Fix a subset \( \sigma \subset \{1, \ldots, n\} \), write \( x_\sigma \) for the set of variables \( x_i \) with \( i \in \sigma \), and let \( A_\sigma \) be the \( K \)-linear subspace of \( A \) which is generated by the standard monomials in \( \langle x_\sigma \rangle \). An elimination ordering for \( x \setminus x_\sigma \) is an admissible ordering for \( A \) such that

\[
f \in A \setminus \{0\}, \ x^{\exp(f)} \in \langle x_\sigma \rangle \implies f \in A_\sigma.
\]

Suppose now that an elimination ordering > as above exists. Then \( d_{ij} \in A_\sigma \) for each pair of indices \( 1 \leq i < j \leq n \) with \( i, j \in \sigma \). Furthermore, \( A_\sigma \) is a subalgebra of \( A \) with admissible ordering >\(_\sigma\), where >\(_\sigma\) is the restriction of > to the set of standard monomials in \( \langle x_\sigma \rangle \). Finally, if \( I \subset A \) is a nonzero left ideal, and \( \mathcal{G} \) is a Gröbner basis for \( I \) with respect to >, then \( \mathcal{G} \cap A_\sigma \) is a Gröbner basis for \( I \cap A_\sigma \) with respect to >\(_\sigma\).

Note that in general, an elimination ordering for \( x \setminus x_\sigma \) may not exist. In practical terms, the question is whether the following linear programming problem has a solution (see [11]):

\[
\text{minimize } \sum_{i=1}^{n} \omega_i \text{ subject to }
\]

- \( \omega_i + \omega_j \geq |\alpha|_\omega \) for \( 1 \leq i < j \leq n \) and all \( \alpha \in N(d_{ij}) \),
- \( \omega_i = 0 \) for \( i \in \sigma \) and \( \omega_i > 0 \) for \( i \in \{1, \ldots, n\} \setminus \sigma \).

If there is a solution \( \omega \), and >\(_\omega\) is any admissible ordering for \( A \), then >\(_\omega\) is an elimination ordering for \( x \setminus x_\sigma \). For computing Bernstein-Sato polynomials as discussed in Subsection 4.1 appropriate block orderings will do.

**Remark 17.** The definition of a right Gröbner basis is completely analogous to that of left Gröbner basis, while a two-sided Gröbner basis is a left Gröbner basis \( \mathcal{G} \) satisfying \( A(\mathcal{G}) = A(\mathcal{G})A \). Having implemented means for computing left Gröbner bases, right and two-sided Gröbner bases are obtained by computing left Gröbner bases in the opposite algebra \( A^{opp} \) and the enveloping algebra \( A^{env} = A \otimes_K A^{opp} \), respectively. See [19, 12] for details.

Rather than restricting ourselves just to \( G \)-algebras, we should finally point out that the use of Gröbner bases as discussed above allows for an effective computationally treatment of a more general class of algebras:

**Remark 18 (GR-Algebras).** A \( GR \)-algebra is the quotient \( A/J \) of a \( G \)-algebra \( A \) by a two-sided ideal \( J \subset A \). Having implemented \( A \) in a computer algebra system such as Singular, we can implement \( A/J \) by computing a two-sided Gröbner basis for \( J \).

3. **A Modular Gröbner Basis Algorithm for \( G \)-Algebras**

In this section, we extend the modular Gröbner bases algorithm from commutative polynomial rings [3, 10, 15] to \( G \)-algebras. As before, we focus on left Gröbner bases. By Remark 17 however, the algorithm presented
below also gives modular ways of computing right and two-sided Gröbner bases of ideals (and modules).

Fix a $G$-algebra $A = \mathbb{Q}(x_1, \ldots, x_n \mid C)$ whose commutation relations

$$C : x_j x_i = c_{ij} \cdot x_i x_j + d_{ij}, \quad 1 \leq i < j \leq n,$$

involve integer coefficients only, and a monomial ordering on $\langle x \rangle$ which is admissible for $A$. Write $A_0$ for the subring of $A$ formed by the elements with integer coefficients. That is, $A_0$ is obtained from the free associative $\mathbb{Z}$-algebra $\mathbb{Z}(x_1, \ldots, x_n)$ by imposing the commutation relations $C$:

$$A_0 = \mathbb{Z}(x_1, \ldots, x_n \mid C).$$

Similarly, if $N \geq 2$ is an integer which

(4)

does neither divide any $c_{ij}$ nor any coefficient of any $d_{ij}$,
then write $A_N = \mathbb{Z}/NZ(x_1, \ldots, x_n \mid C_N)$, where $C_N$ is obtained from $C$ by reducing the $c_{ij}$ and the coefficients of the $d_{ij}$ modulo $N$. Note that if $p$ is a prime satisfying (4), then $A_p$ is a $G$-algebra over the finite field $\mathbb{F}_p$.

If $\frac{a}{b} \in \mathbb{Q}$ with $\gcd(a, b) = 1$ and $\gcd(b, N) = 1$, set

$$\left(\frac{a}{b}\right)_N := (a + NZ)(b + NZ)^{-1} \in \mathbb{Z}/NZ.$$

If $f \in A$ is an element such that $N$ is coprime to the denominator of any coefficient of $f$, then its reduction modulo $N$ is the element $f_N \in A_N$ obtained by mapping each coefficient $c$ of $f$ to $c_N$. If $\mathcal{H} = \{h_1, \ldots, h_t\} \subset A$ is a set of elements such that $N$ is coprime to the denominator of any coefficient of a coefficient of any $h_i$, set $\mathcal{H}_N = \{(h_1)_N, \ldots, (h_t)_N\} \subset A_N$.

Let $I \subset A$ be a nonzero left ideal. We will explain how to compute a left Gröbner basis for $I$ using modular methods. For this, we write

$$I_0 = I \cap A_0 \quad \text{and} \quad I_N = \{f_N \mid f \in I_0\} \subset A_N,$$

and call $I_N$ the reduction of $I$ modulo $N$. We will rely on the following result:

**Lemma 19.** With notation as above, fix a set of generators $f_1, \ldots, f_r$ for $I$ with coefficients in $\mathbb{Z}$. Then for all but finitely many primes $p$, the reduction $I_p$ is generated by the reductions of the $f_j$. That is, the ideal

$$\tilde{I}_p = A_p \langle (f_1)_p, \ldots, (f_r)_p \rangle$$

coincides with $I_p$ for all but finitely many primes $p$.

**Proof.** Let $g_1, \ldots, g_s$ be a set of generators for the left ideal $I_0 = I \cap A_0$. Then each $g_i$ has a representation of type $g_i = \sum_{j=1}^r c_{ij} f_j$, with elements $c_{ij} \in A$. Clearing denominators in the coefficients of the $c_{ij}$, we get a non-zero integer $d$ such that $d \cdot g_i \in A_0 \langle f_1, \ldots, f_r \rangle$ for all $i$. That is,

$$I_0 = A_0 \langle f_1, \ldots, f_r \rangle : d.$$

Then $I_p = \tilde{I}_p$ for each prime $p$ which does not divide $d$. The result follows.
Otherwise, reconstruction as introduced in [5] does not fulfil condition (4). If condition (4) is fulfilled, we work with the left ideal I_p rather than with \( I_p \). The finitely many primes \( p \) for which \( I_p \) and \( \bar{I}_p \) differ will not influence the final result if we use error tolerant rational reconstruction as introduced in [4] and discussed below.

In the following discussion, for simplicity of the presentation, we will ignore the primes which do not fulfil condition (4). We will write \( \bar{G} \) for the reduced Gröbner basis of \( I \), and \( G(p) \) for that of \( I_p \). The basic idea of the modular Gröbner basis algorithm is then as follows: First, choose a finite set of primes \( \mathcal{P} \) and compute \( G(p) \) for each \( p \in \mathcal{P} \). Second, lift the \( G(p) \) coefficientwise to a set of elements \( G \subset A \). We then expect that \( G \) is a Gröbner basis which coincides with our target Gröbner basis \( G(0) \).

The lifting process consists of two steps. First, use Chinese remaindering to lift the \( G(p) \subset A_p \) to a set of elements \( G(N) \subset A_N \), with \( N := \prod_{p \in \mathcal{P}} p^\omega \). Second, compute a set of elements \( G \subset A \) by lifting the coefficients occurring in \( G(N) \) to rational coefficients. Here, to identify Gröbner basis elements corresponding to each other, we require that \( \text{exp}(G(p)) = \text{exp}(G(q)) \) for all \( p, q \in \mathcal{P} \). This leads to the second condition in the definition below:

**Definition 21.** With notation as above, a prime \( p \) is called lucky if

- \((L1)\) \( I_p = \bar{I}_p \) and
- \((L2)\) \( \text{exp}(G(0)) = \text{exp}(G(p)) \).

Otherwise, \( p \) is called unlucky.

**Lemma 22.** The set of unlucky primes is finite.

**Proof.** By Lemma [19] \( I_p = \bar{I}_p \) for all but finitely many primes \( p \). Given such a \( p \), we have \( \text{exp}(G(0)) = \text{exp}(G(p)) \) if \( p \) does not divide the denominator of any coefficient of any element of \( A \) occurring when testing whether \( G(0) \) is a Gröbner basis using Buchberger’s criterion. The result follows. \( \square \)

**Lemma 23.** If \( p \) is a prime satisfying condition \((L2)\), then \( G(0)_p = G(p) \).

**Proof.** We proceed as in the commutative setting. First of all, the graded case can be handled as in [3] Theorems 5.12 and 6.2. Next, we reduce the general case to the graded case by adapting the proof of [15] Theorem 2.4. For this, let \( F(0) \) and \( F(p) \) be finite sets of standard monomials obtained by applying Corollary [13] to the Gröbner bases \( G(0) \) and \( G(p) \), respectively. Then apply [14] Lemma 1.2.11 to the set

\[ F = F(0) \cup F(p) \cup \{ x_i x_j \mid i < j \} \cup \{ x^\alpha \mid \alpha \in N(d_{ij}) \text{ for some } d_{ij} \} \]

to get a vector \( \omega \in \mathbb{N}^n_{\geq 1} \) such that, for all \( x^\alpha, x^\beta \in F \), we have \( x^\alpha \succ x^\beta \) iff \( x^\alpha \succ_\omega x^\beta \). Then, in particular, \( \succ_\omega \) is admissible for \( A \). Our choice of \( F(0) \) gives that \( \text{exp}_{\succ}(G(0)) = \text{exp}_{\succ_\omega}(G(0)) \) and that \( G(0) \subset A \) is the reduced Gröbner basis for \( I \) also with respect to \( \succ_\omega \). Similarly, \( \text{exp}_{\succ}(G(p)) = \text{exp}_{\succ_\omega}(G(p)) \) and \( G(p) \subset A_p \) is the reduced Gröbner basis for \( \bar{I}_p \) also with respect to \( \succ_\omega \). Passing to the Rees algebra \( R^\omega A \), it follows from Proposition [14.1] that \( G(0)^h \) and \( G(p)^h \) are the reduced Gröbner bases for \( I^h \) and \( \bar{I}_p^h \), respectively, with \( \text{exp}_{\succ_\omega}(G(0)^h) = \text{exp}_{\succ_\omega}(G(p)^h) \). Since the result holds in
the graded case, we conclude that \((G(0)^h)_p = (G(0)^h)_p = G(p)^h \) and, thus, that \(G(0)^h_p = G(p)^h \).

\( \square \)

Error tolerant rational reconstruction as introduced in [5] makes use of Gaussian reduction. If applied as discussed in what follows, the finitely many primes not satisfying condition (L1) will not influence the final result. We start with a definition which reflects that we rely on Gaussian reduction:

**Definition 24 ([5]).** If \( \mathcal{P} \) is a finite set of primes, set

\[
N' = \prod_{p \in \mathcal{P} \text{ lucky}} p \quad \text{and} \quad M = \prod_{p \in \mathcal{P} \text{ unlucky}} p.
\]

Then \( \mathcal{P} \) is called sufficiently large if

\[
N' > (a^2 + b^2) \cdot M
\]

for any coefficient \( \frac{a}{b} \) of any element of \( G(0) \) (assume \( \gcd(a, b) = 1 \)).

**Lemma 25.** If \( \mathcal{P} \) is a sufficiently large set of primes satisfying condition (L2), then the reduced Gröbner bases \( G(p) \), \( p \in \mathcal{P} \), lift via Chinese remaindering and error tolerant rational reconstruction to the reduced Gröbner basis \( G(0) \).

**Proof.** Since all primes in \( \mathcal{P} \) satisfy condition (L2), Lemma 23 gives \( G(0)^h_p = G(p)^h \) for each \( p \in \mathcal{P} \). Since \( \mathcal{P} \) is sufficiently large, the result follows as in the proof of [5, Lemma 5.6] from [5, Lemma 4.3].

Lemma 22 guarantees, in particular, that a sufficiently large set \( \mathcal{P} \) of primes satisfying condition (L2) exists. So from a theoretical point of view, the idea of finding \( G(0) \) is now as follows: Consider such a set \( \mathcal{P} \), compute the reduced Gröbner bases \( G(p) \), \( p \in \mathcal{P} \), and lift the results to \( G(0) \).

¿From a practical point of view, however, we face the problem that condition (L2) can only be checked a posteriori. On the other hand, as already pointed out, we need that the \( G(p) \), \( p \in \mathcal{P} \), have the same set of leading monomials in order to identify corresponding Gröbner basis elements in the lifting process. To remedy this situation, we suggest to proceed in a randomized way: First, fix an integer \( t \geq 1 \) and choose a set of \( t \) primes \( \mathcal{P} \) at random. Second, compute \( G\mathcal{P} = \{G(p) | p \in \mathcal{P}\} \), and use a majority vote:

**DELETEBYMAJORITYVOTE:** Define an equivalence relation on \( \mathcal{P} \) by setting \( p \sim q :\iff \exp(G(p)) = \exp(G(q)) \). Then replace \( \mathcal{P} \) by an equivalence class of largest cardinality\(^2\) and change \( G\mathcal{P} \) accordingly.

Now, all \( G(p) \), \( p \in \mathcal{P} \), have the same set of leading monomials. Hence, we can apply the error tolerant lifting algorithm to the coefficients of the Gröbner bases in \( G\mathcal{P} \). If this algorithm returns false at some point, we enlarge the set \( \mathcal{P} \) by \( t \) primes not used so far, and repeat the whole process. Otherwise, the lifting yields a set of elements \( G \subset A \). Furthermore, if \( \mathcal{P} \) is sufficiently large, all primes in \( \mathcal{P} \) satisfy condition (L2). Since we cannot check, however, whether \( \mathcal{P} \) is sufficiently large, we include a final (partial)

\(^2\)We have to use a weighted cardinality count: when enlarging \( \mathcal{P} \), the total weight of the elements already present must be strictly smaller than the total weight of the new elements. Otherwise, though highly unlikely in practical terms, it may happen that only unlucky primes are accumulated.
verification step in characteristic zero as discussed below. Since this test is particularly expensive if \( \mathcal{G} \neq \mathcal{G}(0) \), we first perform a test in positive characteristic in order to increase our chances that the two sets are equal:

**pTest:** Randomly choose a prime \( p \notin \mathcal{P} \) which does neither divide the numerator nor the denominator of any coefficient occurring in any element of \( \mathcal{G} \). Return `true` if \( \mathcal{G}_p = \mathcal{G}(p) \), and `false` otherwise.

If `pTest` returns `false`, then \( \mathcal{P} \) is not sufficiently large (or the extra prime chosen in `pTest` is unlucky). In this case, we enlarge \( \mathcal{P} \) as above and repeat the process. If `pTest` returns `true`, however, then most likely \( \mathcal{G} = \mathcal{G}(0) \). In this case, we verify at least that \( \mathcal{G} \) is a left Gröbner basis, and that the left ideal \( \langle \mathcal{G} \rangle \) generated by \( \mathcal{G} \) contains the given left ideal \( I \) (in the graded case discussed below, these two conditions actually guarantee that \( \langle \mathcal{G} \rangle = I \)). If the (partial) verification fails, we again enlarge \( \mathcal{P} \) and repeat the process.

We summarize this approach in Algorithm 1 (as before, we ignore the primes which do not fulfil condition (4)).

**Algorithm 1** Modular Gröbner Basis Algorithm

**Input:** A nonzero left ideal \( I \subset A \) given by finitely many generators, and an admissible monomial ordering for \( A \).

**Output:** A subset \( \mathcal{G} \subset A \) which is expected to be a Gröbner basis for \( I \) (in the graded case, \( \mathcal{G} \) is guaranteed to be such a Gröbner basis).

1. choose a set \( \mathcal{P} \) of random primes
2. \( \mathcal{GP} = \emptyset \)
3. loop
   4. for \( p \in \mathcal{P} \) do
   5. compute \( \mathcal{G}(p) \subset A_p \)
   6. \( \mathcal{GP} = \mathcal{GP} \cup \mathcal{G}(p) \)
   7. \((\mathcal{P}, \mathcal{GP}) = \) deleteByMajorityVote\((\mathcal{P}, \mathcal{GP})\)
   8. lift the Gröbner bases in \( \mathcal{GP} \) to \( \mathcal{G} \subset A \) via Chinese remaindering and error tolerant rational reconstruction
   9. if the lifting succeeds and `pTest`\((I, G, P)\) then
   10. if \( \mathcal{G} \) is a Gröbner basis for \( \langle \mathcal{G} \rangle \) then
   11. if \( I \subset \langle \mathcal{G} \rangle \) then
   12. return \( \mathcal{G} \)
   13. enlarge \( \mathcal{P} \) with primes not used so far

Now, we address the graded case. We suppose that there is an \( \omega \in \mathbb{N}_{\geq 1}^n \) such that \( A \) and \( I \subset A \) are graded with respect to the \( \omega \)-weighted degree as in Subsection 2.3 and that \( I \) is given by \( \omega \)-homogeneous generators. We use the index \( d \) to indicate the graded pieces of \( A \) and \( I \) of degree \( d \). Similarly for the other rings and ideals considered in this section (such as \( A_0, I_0 \), and \( \tilde{I}_p \)), which all inherit the grading.

We proceed by considering Hilbert functions as in Arnold’s paper [3] which handles the commutative case.

**Lemma 26.** With notation and assumptions as above, let \( p \) be a prime. Then \( H_{G}(p)(d) \leq H_I(d) \) for each \( d \in \mathbb{N} \).
Proof. Fix a degree \(d \in \mathbb{N}\). We must show that \(\dim_{\mathbb{Z}/p\mathbb{Z}}(I_p)_d \leq \dim_{\mathbb{Q}}I_d\). For this, first note that \((I_0)_d\) is a free \(\mathbb{Z}\)-submodule of \((A_0)_d\) of finite rank. Let \(B = \{b_1, \ldots, b_m\}\) be a \(\mathbb{Z}\)-basis for \((I_0)_d\). Then note that for each \(f \in I\), there is an integer \(a \in \mathbb{Z}\) such that \(a \cdot f \in I_0\). This implies that \(B\) is also a \(\mathbb{Q}\)-basis for \(I_d\), so that \(\dim_{\mathbb{Q}}I_d = \text{rank}_{\mathbb{Z}}(I_0)_d\). Furthermore, the reduction \(B_p\) still generates the \(\mathbb{Z}/p\mathbb{Z}\)-vector space \(I_p(d)\). Hence, \(\dim_{\mathbb{Z}/p\mathbb{Z}}(I_p)_d \leq \text{rank}_{\mathbb{Z}}(I_0)_d\).

We conclude that \(\dim_{\mathbb{Z}/p\mathbb{Z}}(I_p)_d \leq \dim_{\mathbb{Q}}I_d\), as claimed. \(\Box\)

We can now prove:

**Theorem 27** (Final Verification, Graded Case). With notation and assumptions as above, suppose that

1. \(\exp(G) = \exp(G(p))\) for some prime \(p\),
2. \(G\) is a Gröbner basis, and
3. \(I \subset \langle G \rangle\).

Then \(G\) is a Gröbner basis for \(I\).

Proof. The result will follow from the second assumption once we show that \(I = \langle G \rangle\). Since \(I \subset \langle G \rangle\) by the third assumption, it suffices to show that \(H_1(d) = H_{\langle G \rangle}(d)\) for all \(d \in \mathbb{N}\). This, in turn, holds since we have

\[
H_1(d) \leq H_{\langle G \rangle}(d) = H_{I_p}(d) \leq H_{I_p}(d) \leq H_1(d)
\]

for each \(d\) and each prime \(p\) satisfying the first assumption. Indeed, the first and second inequality are clear since \(I \subset \langle G \rangle\) and \(I_p \subset I_p\), respectively; the equality follows from the first assumption (see Remark 10); the third inequality has been established in Lemma 26. \(\Box\)

**Remark 28.** Note that in all non-graded examples where we could check the output of Algorithm 7 by computing the desired Gröbner basis also directly over \(\mathbb{Q}\), the result was indeed correct.

4. **Timings**

We have implemented our modular algorithm for computing Gröbner bases in G-algebras over \(\mathbb{Q}\) in the subsystem PLURAL [13, 22] of the computer algebra system SINGULAR [9]. This system offers two variants of Buchberger’s algorithm which within PLURAL are adapted to the noncommutative case: While the std command refers to the default version of Buchberger’s algorithm in SINGULAR, the ideas behind slimgb aim at keeping elements short with small coefficients.

In this section, we compare the performance of the modular algorithm with that of std and slimgb applied directly over the rationals. In the tables below, when referring to the modular algorithm, we write modular std respectively modular slimgb to indicate which version of Buchberger’s algorithm is used for the mod \(p\) computations.

We have carried out the computations on a Dell PowerEdge R720 with two Intel(R) Xeon(R) CPU E5-2690 @ 2.90GHz, 20 MB Cache, 16 Cores, 32 Threads, 192 GB RAM with a Linux operating system (Gentoo).

In the tables, we abbreviate seconds, minutes, hours as s,m,h and threads as thr. The symbol \(\infty\) indicates that the computation did not finish within 25 days or was halted since it consumed more than 100 GB of memory.
4.1. Examples From D-Module Theory Involving the Weyl Algebra.

We consider families of ideals which are computationally challenging and of interest in the context of D-modules, specifically in the context of Bernstein-Sato polynomials.

4.1.1. The Setup. Let $K$ be a field of characteristic zero, and consider a non-constant polynomial $f \in K[x_1, \ldots, x_n]$. Write $D_n(K) = K[x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \mid \partial_i x_i = x_i \partial_i + 1, \partial_i x_j = x_j \partial_i \text{ for } i \neq j]$, for the $n$-th Weyl algebra as in Example 2, let $s$ be an extra variable, and set $K[x]_f[s] = K[x_1, \ldots, x_n]_f \otimes_K K[s]$ and $D_n(K)[s] = D_n(K) \otimes_K K[s]$. Let $K[x]_f[s]f^s$ stand for the free $K[x]_f[s]$-module of rank one generated by the symbol $f^s$. This is a left $D_n(K)[s]$-module with the action of a vector field $\theta$ being defined by the formula $\theta \cdot f^s = \theta(f^s) = s\theta(f)f^{-1}f^s$ and the product rule. Consider the left annihilator $\text{Ann}_{D_n(K)[s]}(f^s) \subset D_n(K)[s]$. The Bernstein-Sato polynomial $b_f \in K[s]$ is the nonzero monic polynomial of smallest degree such that there exists an operator $P \in D_n(K)[s]$ with

$$b_f - P \cdot f \in \text{Ann}_{D_n(K)[s]}(f^s).$$

Put differently, $b_f$ is defined to be the monic generator of the ideal

$$\left(\text{Ann}_{D_n(K)[s]}(f^s) + D_n(K)[s] \langle f \rangle \right) \cap K[s]$$

which, by a result of Bernstein [4], is nonzero. More generally, given polynomials $f_1, \ldots, f_r \in K[x_1, \ldots, x_n]$ and extra variables $s = s_1, \ldots, s_r$, consider the symbol $f^s = f_1^{s_1} \cdots f_r^{s_r}$. Then the analogous construction yields the Bernstein-Sato ideal

$$B_{f_1, \ldots, f_r}(s) = \left(\text{Ann}_{D_n(K)[s]}(f^s) + D_n(K)[s] \langle f_1 \cdots f_r \rangle \right) \cap K[s],$$

which is nonzero by a result of Sabbah [20].

4.1.2. Computing the Annihilator. There are several algorithms for computing $\text{Ann}_{D_n(K)[s]}(f^s)$ (see [1]). For our tests here, we use the method of Briançon and Maisonobe which can be described as follows: Consider the $r$th shift algebra

$$_rS_r(K) = \langle s_1, \ldots, s_r, t_1, \ldots, t_r \mid t_j s_k = s_k t_j - \delta_{jk} t_j \rangle$$

as in Example 3, the tensor product

$$A = D_n(K) \otimes _rS_r(K),$$

and the left ideal

$$I = \left\langle s_j + f_j t_j, \sum_{k=1}^r \frac{\partial f_k}{\partial x_i} t_k + \partial_i \mid 1 \leq j \leq r, 1 \leq i \leq n \right\rangle \subset A.$$

Then $\text{Ann}_{D_n(K)[s]}(f^s) = I \cap D_n(K)[s]$. Hence, the annihilator is obtained by computing a left Gröbner basis for $I$ with respect to an elimination ordering for $t_1, \ldots, t_r$. 
4.1.3. Computing the Bernstein-Sato Ideal. By its very definition, the Bernstein-Sato ideal and, thus, the Bernstein-Sato polynomial if \( r = 1 \) can be found by computing a left Gröbner basis for

\[
\text{Ann}_{D_n(K)[s]}(f^r) + D_n(K)[s] \langle f_1 \cdots f_r \rangle
\]

with respect to an elimination ordering for \( x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \).

Remark 29. There are more effective ways of computing Bernstein-Sato polynomials. The method described above, however, allows one to compute Bernstein-Sato ideals in general. See [1] for more details.

4.1.4. Explicit Examples. We focus on the computation of Bernstein-Sato polynomials as outlined above (the case \( r = 1 \)). In all examples presented in what follows, the time for computing the annihilator in 4.1.2 is negligible. We will therefore only list the time needed for the elimination step in 4.1.3. Here, we use the block ordering obtained by composing the respective degree reverse lexicographical orderings.

Example 30 (Reiffen\((p,q)\), [25]). We consider the family of polynomials

\[
x^p + y^q + xy^{q-1} \in \mathbb{Q}[x, y], \text{ where } q \geq p + 1,
\]

and, correspondingly, the second Weyl algebra \( D_2(\mathbb{Q}) \).

|      | std | slimgb | modular slimgb |         |         |         |         |
|------|-----|--------|----------------|---------|---------|---------|---------|
|      |     |        | 1 thr | 2 thr | 4 thr | 8 thr | 16 thr |
| Reiffen\((5,6)\) | \(\infty\) | 63.86 h | 12.25 m | 7.21 m | 4.7 m | 3.45 m | 2.6 m |
| Reiffen\((6,7)\) | \(\infty\) | \(\infty\) | 10.43 h | 6.03 h | 4.65 h | 4.24 h | 3.54 h |
| Reiffen\((7,8)\) | \(\infty\) | \(\infty\) | 336.25 h | 212.24 h | 170 h | 146 h | 118 h |

For more insight, we also give timings for running our algorithm without the final tests which check whether \( \mathcal{G} \) is a left Gröbner basis and whether \( I \subset \langle \mathcal{G} \rangle \) (see the discussion in Section 3). We use just one thread.

We see that for Reiffen\((5,6)\), Reiffen\((6,7)\), and Reiffen\((7,8)\), the final tests take about 17%, 37%, and 40% of the total computing time, respectively.

Example 31. We consider the following polynomials with rational coefficients,

\[
f = xy^5z + y^6 + x^5z + x^4y,
g = xy^6z + y^7 + x^6z + x^5y,
h = (x - z)xyz(-x + y)(y + z),
cusp(p,q) = x^p - y^q, \text{ where } \gcd(p,q) = 1,
\]
and, correspondingly, the third and second Weyl algebras over \( \mathbb{Q} \), respectively.

\[
\begin{array}{|c|c|c|c|}
\hline
\text{symbol} & \text{std} & \text{slimgb} & \text{modular slimgb} \\
\hline
f & \infty & 3.93 \text{ h} & 3.59 \text{ h} \\
g & \infty & \infty & 284.46 \text{ h} \\
h & \infty & \infty & 19.19 \text{ h} \\
cusp(9,8) & \infty & 2.00 \text{ s} & 30.81 \text{ s} \\
cusp(10,9) & \infty & 4.53 \text{ h} & 3.17 \text{ h} \\
cusp(11,7) & \infty & 2.06 \text{ s} & 2.18 \text{ m} \\
cusp(11,8) & \infty & 3.17 \text{ h} & 1.97 \text{ h} \\
cusp(12,7) & \infty & 9.53 \text{ s} & 1.04 \text{ m} \\
cusp(13,7) & \infty & 1.21 \text{ h} & 40.32 \text{ m} \\
\hline
\end{array}
\]

We observe that for the smaller examples such as cusp(9,8), cusp(11,7), and cusp(12,7), the \textit{slimgb} version of Buchberger’s algorithm is superior due to the overhead of the modular algorithm.

Considering the substitution homomorphism \( D_n(K)[s] \rightarrow D_n(K), \ s \mapsto -1 \), it easily follows from Equation 5 that the Bernstein-Sato polynomial \( b_f(s) \) is divisible by \( s + 1 \) (recall that we suppose that \( f \) is non-constant). The polynomial \( \frac{b_f(s)}{s + 1} \in K[s] \) is sometimes called the \textit{reduced Bernstein-Sato polynomial}. It is easy to see that the following holds:

\[
(6) \quad \left\langle \frac{b_f(s)}{s + 1} \right\rangle = \left( \text{Ann}_{D_n(\mathbb{Q})[s]}(f^s) + \left\langle f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle \right) \cap K[s].
\]

Computing the Bernstein-Sato polynomial via this equation may be considerably faster than using the method described earlier: Compare the timings for cusp(13,7) in the tables above and below.

\textbf{Example 32.} Equation (6) allows us to compute the Bernstein-Sato polynomials in some of the more involved cusp\((p, q)\) instances:
Example 33. We consider the polynomial
\[ f = (x^4 + y^4)(w^2 + z^2)(x + z) \in \mathbb{Q}[w, x, y, z], \]
and compute the Bernstein-Sato polynomial \( b_f \) using Equation (6):

For the polynomial \( g = (x^5 + y^5)(w^2 + z^2)(x + z) \in \mathbb{Q}[w, x, y, z], \)
already the Gröbner basis computation over \( \mathbb{F}_p \), for just one randomly selected Singular prime \( p \), takes 240 hours. The direct computation over \( \mathbb{Q} \) using \( \text{std} \) and \( \text{slimgb} \) runs out of memory.

4.2. Some Well-Known Benchmark Examples.

Example 34. We consider the quasi-commutative graded \( \mathbb{Q} \)-algebra
\[ A = \mathbb{Q}(x_1, \ldots, x_n \mid x_jx_i = 2x_ix_j, \, 1 \leq i < j \leq n) \]

\[ \text{std} \quad \text{slimgb} \quad \text{modular slimgb} \]
\begin{array}{|c|c|c|c|}
\hline
\text{cusp(13,7)} & \infty & 3.94 \text{ m} & 1.21 \text{ m} \\
\text{cusp(13,8)} & \infty & \infty & 2.21 \text{ m} \\
\text{cusp(13,9)} & \infty & \infty & 5.67 \text{ m} \\
\text{cusp(13,10)} & \infty & \infty & 9.43 \text{ m} \\
\text{cusp(13,11)} & \infty & \infty & 18.71 \text{ m} \\
\text{cusp(13,12)} & \infty & \infty & 27.25 \text{ m} \\
\text{cusp(14,9)} & \infty & \infty & 7.83 \text{ m} \\
\text{cusp(14,11)} & \infty & \infty & 27.08 \text{ m} \\
\text{cusp(14,13)} & \infty & \infty & 1.16 \text{ h} \\
\text{cusp(15,7)} & \infty & 2.74 \text{ h} & 2.15 \text{ m} \\
\text{cusp(15,8)} & \infty & \infty & 4.00 \text{ m} \\
\text{cusp(15,11)} & \infty & \infty & 36.01 \text{ m} \\
\text{cusp(15,13)} & \infty & \infty & 1.56 \text{ h} \\
\text{cusp(17,13)} & \infty & \infty & 3.23 \text{ h} \\
\text{cusp(19,13)} & \infty & \infty & 6.12 \text{ h} \\
\text{cusp(19,17)} & \infty & \infty & 29.06 \text{ h} \\
\hline
\end{array}

Example 34. We consider the quasi-commutative graded \( \mathbb{Q} \)-algebra
\[ A = \mathbb{Q}(x_1, \ldots, x_n \mid x_jx_i = 2x_ix_j, \, 1 \leq i < j \leq n) \]
|                | slimgb | modular slimgb |
|----------------|--------|----------------|
|                | 1 thr  | 2 thr  | 4 thr  | 8 thr  | 16 thr |
| cyclic(7)      | 11.24 m | 27.66 s | 16.13 s | 9.66 s | 7.81 s | 6.64 s |
| cyclic(8)      | 55.28 h | 2.51 h  | 1.21 h  | 34.65 m | 27.64 m | 17.13 m |
| katsura(9)     | 4.49 m  | 1.51 m  | 49.27 s | 30.60 s | 21.77 s | 16.28 s |
| katsura(10)    | 10.65 h | 26.83 m | 14.54 m | 8.59 m  | 3.53 m  | 3.38 m  |
| katsura(11)    | 199.71 h| 4.32 h  | 2.76 h  | 1.59 h  | 46.48 m | 24.52 m |
| katsura(12)    | infinity | 13.78 h | 7.68 h  | 4.40 h  | 2.34 h  | 1.46 h  |
| katsura(13)    | infinity | 50.14 h | 32.33 h | 17.74 h | 10.72 h | 5.80 h  |
| reimer(4)      | 14.62 s | 3.14 s  | 2.69 s  | 1.99 s  | 1.58 s  | 1.48 s  |
| reimer(5)      | 29.07 h | 2.59 h  | 1.57 h  | 58.47 m | 26.33 m | 18.04 m |
| eco(15)        | 25.93 h | 9.40 h  | 5.77 h  | 3.54 h  | 2.55 h  | 1.83 h  |

4.3. A Remark on the Number of Primes.

Remark 35. The efficiency of our algorithm depends, in particular, on the number of modular Gröbner basis computations before the lifting and testing steps. In our implementation, this is the smallest multiple of the number of available threads which is greater than or equal to 20.

5. Conclusion

In this paper, we have introduced modular techniques for the computation of Gröbner bases in $G$-algebras defined over $\mathbb{Q}$. On the theoretical side, we have shown that the final verification test for graded ideals, which is well-known from the commutative case, also works in the noncommutative setting. On the practical side, we have implemented our modular algorithm in the subsystem PLURAL of SINGULAR and have demonstrated that the new algorithm is typically superior to the non-modular versions of Buchberger’s algorithm in PLURAL.

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