Photon decay in a CPT-violating extension of quantum electrodynamics

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Abstract

We consider the process of photon decay in quantum electrodynamics with a CPT-violating Chern–Simons-like term added to the action. For a simplified model with only the quadratic Maxwell and Chern–Simons-like terms and the quartic Euler–Heisenberg term, we obtain a nonvanishing probability for the decay of a particular photon state into three others.

Key words: CPT violation, Quantum electrodynamics, Radiative corrections

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1 Introduction

The propagation of light in Maxwell theory with an extra CPT-violating Lorentz-noninvariant Chern–Simons-like term in the action has been studied both classically [1] and quantum mechanically [2]. The Maxwell term and the Chern–Simons-like term are quadratic in the photon field. These terms combined lead to birefringence, even in empty space. Additional (Lorentz-invariant) higher-order photonic terms in the action could perhaps produce other effects such as the decay of photons. (See also Ref. [3] for a discussion of the decay of a massive Dirac fermion in theories with spontaneous Lorentz and CPT violation.)

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The purpose of this paper is to study the process of photon decay in detail, expanding on the brief remark in Section 6 of Ref. [2]. In order to concentrate on the essentials, we keep the theory as simple as possible. In fact, the theory we start from is just quantum electrodynamics—the theory of photons, electrons and positrons (see, e.g., Refs. [4,5]). It is, of course, known that the quantum effects of the electron-positron field lead to a quartic coupling of the photons (giving, e.g., light-by-light scattering), which is described by the quartic part of the so-called Euler–Heisenberg Lagrangian [6].

The photon model considered in this article has only the Maxwell and quartic Euler–Heisenberg terms, together with a hypothetical spacelike Chern–Simons-like term which breaks Lorentz and CPT invariance. For the moment, the precise origin of this Chern–Simons-like term can be left open, but at least one possible mechanism has been identified [7,8] (see Ref. [9] for a review). A more extensive discussion of the possible origin and consequences of the Chern–Simons-like term can be found in Ref. [2], which also contains further references.

The outline of our paper is as follows. The model is presented in Section 2, together with some basic facts on the polarization states of the “photons.” The Lorentz noninvariance of the model allows for photon decay and the relevant kinematics is discussed briefly in Section 3. (Some technical details are relegated to Appendix A). The matrix element for a particular decay channel is then calculated in Section 4 and the corresponding partial decay width for a photon at rest is given in Section 5. (Numerical results for the phase space integral are presented in Appendix B). For this particular case, there is, in principle, one other decay channel available, but it does not contribute as shown in Section 6 with details relegated to Appendix C. The total decay rate of the particular photon state at rest is discussed in Section 7. The corresponding mean life time is expected to be large but finite.

2 Model

The Lagrangian density of the model considered in this paper is

\[ \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1, \quad (2.1) \]

where the free part \( \mathcal{L}_0 \) consists of the usual Maxwell term [4,5] and an additional Chern–Simons-like term [1,2],

\[ \mathcal{L}_0 = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{1}{4} m \epsilon_{\mu \nu \rho \sigma} \eta^{\mu} A^\nu F_{\rho \sigma}. \quad (2.2) \]

Our conventions are \( (g_{\mu \nu}) = \text{diag}(1,-1,-1,-1) \) and \( \epsilon_{0123} = 1 \), together with \( \hbar = c = 1 \).

The Chern–Simons-like term in Eq. (2.2) has a mass parameter \( m \). (Experimentally, there are tight constraints [1,10] on this mass, \( m \lesssim 10^{-33} \text{ eV} \), as...
will be discussed further in Section 7.) The Chern–Simons-like term contains, in addition, a purely spatial “four-vector”

\[(\eta^\mu) = (0, \hat{\eta}^1, \hat{\eta}^2, \hat{\eta}^3)\],

(2.3)
in terms of a “three-vector” \(\hat{\eta}\) of unit length, \(|\hat{\eta}|^2 = 1\). The parameters \(\eta^\mu\) are fixed once and for all, hence the quotation marks. In fact, the condition of having fixed parameters (coupling constants) \(\eta^\mu\) in the Lagrangian \(L_0\) effectively selects a class of preferred inertial frames; cf. Refs. [1,2,3].

In the following, we will also write the Maxwell field strength \(F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu\) in terms of the electric and magnetic fields, \(E_k \equiv F_{k0}\) and \(B_k \equiv \epsilon_{klm} F_{lm}/2\) with \(\epsilon_{123} = 1\). If the gauge needs to be fixed, we will use the radiation (Coulomb) gauge \([4,5,11]\),

\[\vec{\partial} \cdot \vec{A} = 0\].

(2.4)

Further details on the photon propagation and the microcausality of the free theory (2.2)–(2.4) can be found in Ref. [2].

The interaction term \(L_1\) in Eq. (2.1) is taken to be the quartic Euler–Heisenberg Lagrangian [6]

\[L_1 = \frac{2 \alpha^2}{45 M^4} \left[ \left( \frac{1}{2} F_{\mu\nu} F^{\mu\nu} \right)^2 + 7 \left( \frac{1}{8} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \right)^2 \right]\]

\[= \frac{2 \alpha^2}{45 M^4} \left[ (|\vec{E}|^2 - |\vec{B}|^2)^2 + 7 (\vec{E} \cdot \vec{B})^2 \right],\]

(2.5)

where \(M\) is a mass-scale and \(\alpha^2\) a dimensionless coupling constant. For quantum electrodynamics with electron mass \(M \approx 511\) keV and fine-structure constant \(\alpha \approx 1/137\), precisely this term appears in the one-loop effective gauge field action (the corresponding Feynman diagrams have a single electron loop with four external photon lines). The quartic Euler–Heisenberg term in the effective action is relevant for photon energies much less than \(M\), which will be the case for the process discussed in the present paper (with photon energies less or equal to \(m\)). For further details on this effective action, see, e.g., Refs. [11,12,13] and references therein.

The free Lagrangian \(L_0\) in the radiation gauge gives rise to two modes of plane-wave solutions with polarization vectors \(\vec{\epsilon}_\pm(\vec{k})\) and dispersion relations

\[\omega_\pm(\vec{k}) = \left( k^2 + \frac{1}{2} m^2 \pm \frac{1}{2} m \sqrt{m^2 + 4k^2 \cos^2 \theta} \right)^{1/2},\]

(2.6)

with \(k \equiv |\vec{k}|\) and \(k \cos \theta \equiv \vec{k} \cdot \hat{\eta}\). For generic wave vectors \(\vec{k}\), the unit vectors

\[\hat{\xi}_1 \equiv (\hat{\eta} - \cos \theta \vec{k})/\sin \theta, \quad \hat{\xi}_2 \equiv (\vec{k} \times \hat{\eta})/\sin \theta, \quad \hat{k} \equiv \vec{k}/|\vec{k}|,\]

(2.7)

form an orthonormal tripod with positive orientation, \(\hat{\xi}_1 \times \hat{\xi}_2 = \hat{k}\). The polarization vectors of the two modes are then given by
\[ \bar{\epsilon}_\pm(\vec{k}) = \frac{1}{\sqrt{2\Gamma(\Gamma \pm m)}} \left[ 2\omega_\pm \cos \theta \hat{\xi}_1 \mp i(\Gamma \pm m) \hat{\xi}_2 \right], \quad (2.8) \]

with

\[ \Gamma = \Gamma(\vec{k}) \equiv \sqrt{m^2 + 4k^2 \cos^2 \theta}. \quad (2.9) \]

As usual, the free quantized photon field can be defined in terms of creation and annihilation operators,

\[ \vec{A}(x) = \int \frac{d^3k}{(2\pi)^{3/2}} \left( \frac{1}{\sqrt{2\omega_+}} \left[ a_+(\vec{k}) \bar{\epsilon}_+(\vec{k}) \exp\left(-i\omega_+ t + i\vec{k} \cdot \vec{x} \right) + \text{H.c.} \right] 
+ \frac{1}{\sqrt{2\omega_-}} \left[ a_-(-\vec{k}) \bar{\epsilon}_-(\vec{k}) \exp\left(-i\omega_- t + i\vec{k} \cdot \vec{x} \right) + \text{H.c.} \right] \right). \quad (2.10) \]

The creation and annihilation operators have standard commutation relations,

\[ [a_s(\vec{k}), a_t(\vec{l})] = [a_s^\dagger(\vec{k}), a_t^\dagger(\vec{l})] = 0, \quad [a_s(\vec{k}), a_t^\dagger(\vec{l})] = \delta_{s,t} \delta_3(\vec{k} - \vec{l}). \quad (2.11) \]

for \( s, t \in \{+, -\} \). The Fock space of photon states is then readily constructed \[11,12,13\], starting from the vacuum state \( |0\rangle \) with the property that \( a_s(\vec{k}) |0\rangle = 0 \) for any \((\vec{k}, s)\).

The free electric and magnetic field operators can be expressed in the same way as Eq. (2.10), replacing the polarization vectors \( \bar{\epsilon}_\pm \) there by

\[ \vec{E}_\pm(\vec{k}) \equiv i\omega_\pm \bar{\epsilon}_\pm + m \left( \bar{\epsilon}_\pm \cdot (\hat{k} \times \hat{\eta}) \right) \hat{k} = \frac{i}{\sqrt{2\Gamma(\Gamma \pm m)}} \times \left( 2\omega_\pm \cos \theta \hat{\xi}_1 \mp i\omega_\pm(\Gamma \pm m) \hat{\xi}_2 \mp m \sin \theta(\Gamma \pm m) \hat{k} \right) \quad (2.12) \]

for \( \vec{E} \) and by

\[ \vec{B}_\pm(\vec{k}) \equiv i\vec{k} \times \bar{\epsilon}_\pm = \frac{i\vec{k}}{\sqrt{2\Gamma(\Gamma \pm m)}} \left( 2\omega_\pm \cos \theta \hat{\xi}_2 \pm i(\Gamma \pm m) \hat{\xi}_1 \right) \quad (2.13) \]

for \( \vec{B} \).

Before continuing with the quantized theory, it may be useful to consider the polarizations that result from Eqs. (2.12) and (2.13) for the classical electric and magnetic fields of a plane wave propagating \textit{in vacuo} (see also Ref. \[9\]).

For \( \theta = 0 \) (i.e., a wave vector \( \vec{k} \) in the preferred direction \( \hat{\eta} \)), the + and - modes are circularly polarized. But the vectors \( \hat{\xi}_1 \) and \( \hat{\xi}_2 \) from Eq. (2.7) are not well-defined at \( \theta = 0 \). Instead, they may be defined by a limiting procedure, where the limit \( k_\perp \downarrow 0 \) is performed for a wave vector \( \vec{k} = k_\perp \hat{e}_\perp + k_\parallel \hat{\eta} \). Here, \( \hat{e}_\perp \) is an arbitrary unit vector perpendicular to \( \hat{\eta} \). The resulting vectors are \( \hat{\xi}_1 = -\hat{e}_\perp \) and \( \hat{\xi}_2 = \hat{e}_\perp \times \hat{\eta} \). Consequently, the + mode is right-handed (\( R \), negative helicity) and the - mode is left-handed (\( L \), positive helicity).
For $0 < \theta < \pi/2$, both modes are elliptically polarized and the helicities remain the same ($R/L$ for $+/−$). As long as $k \cos \theta \gg m$, the modes are essentially circularly polarized. In the opposite limit $k \cos \theta \ll m$, both modes approach linear polarizations.

For $\theta = \pi/2$, the two modes behave somewhat differently. The $−$ mode becomes exactly linearly polarized, with $\vec{f}_− \sim i\omega_−\hat{\xi}_1 \perp \hat{k}$. But the electric field for the $+$ mode has also a longitudinal component, $\vec{f}_+ \sim i\omega_+\hat{\xi}_2 + mk$. This longitudinal component is, however, relatively unimportant for frequencies $\omega_+ \gg m$.

For $\pi/2 < \theta < \pi$, both modes are again elliptically polarized, but the helicities have changed ($L/R$ for $+/−$). For $\theta = \pi$, finally, both modes are circularly polarized, with helicities opposite to the $\theta = 0$ case.

We now return to the quantized theory. The energy-momentum “tensor” of the free theory is given by

$$\Theta^{\mu\nu} = \frac{1}{4} g^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma} − F^{\mu\rho} F^{\nu\sigma} g_{\rho\sigma} − \frac{m}{4} \eta^\nu e^{\mu\rho\sigma\tau} A_\rho F_{\sigma\tau}. \quad (2.14)$$

This tensor is conserved, $\partial_\mu \Theta^{\mu\nu} = 0$, as may be checked with the help of the equations of motion,

$$\partial_\mu F^{\mu\nu} = \frac{m}{2} e^{\nu\rho\sigma\tau} \eta_\rho F_{\sigma\tau}. \quad (2.15)$$

The energy-momentum tensor (2.14) is, however, not symmetric, as long as $m$ is nonzero.

For the case of a purely spatial Chern–Simons “four-vector” $\eta^\mu$, the corresponding field energy and momentum are

$$\mathcal{E} = \int d^3x \Theta^{00} = \int d^3x \frac{1}{2} \left( |\vec{E}|^2 + |\vec{B}|^2 \right), \quad (2.16)$$

$$\mathcal{P}^i = \int d^3x \Theta^{0i} = \int d^3x \left[ (\vec{E} \times \vec{B})^i + \frac{m}{2} \tilde{\eta}^i (\vec{B} \cdot \vec{A}) \right]. \quad (2.17)$$

These operators in terms of the quantized free photon field must, of course, be normal-ordered [11,12,13]. One finds after some algebra

$$: \mathcal{E} : = \int d^3k \left[ \omega_+(\vec{k}) a_+^\dagger(\vec{k}) a_+(\vec{k}) + \omega_-(\vec{k}) a_-^\dagger(\vec{k}) a_-(\vec{k}) \right], \quad (2.18)$$

$$: \vec{P} : = \int d^3k \left[ \vec{k} a_+^\dagger(\vec{k}) a_+(\vec{k}) + \vec{k} a_-^\dagger(\vec{k}) a_-(\vec{k}) \right]. \quad (2.19)$$

The energy eigenvalues, in particular, are nonnegative for the functions $\omega_\pm$ as given by Eq. (2.6).

Note, finally, that the Lagrangian (2.2), with fixed parameters $\eta^\mu = (0, \tilde{\eta})$, is translation invariant but not rotation invariant. This implies that energy-momentum is conserved but not angular momentum.
3 Kinematics of photon decay

In this section, we study the kinematics for the decay of one initial photon with momentum $\vec{q}$ and energy $\omega_{\pm}(\vec{q})$ into three final photons with energies $\omega_{\pm}(\vec{k}_i)$, $i = 1, 2, 3$, and total momentum $\vec{k}_1 + \vec{k}_2 + \vec{k}_3 = \vec{q}$. The theory considered has been given in Section 2. In a first reading, it is possible to skip ahead to the last paragraph of this section, which summarizes the results.

For the usual Lorentz-invariant case ($m = 0$), the total energy of the final state is minimal when all final momenta $\vec{k}_i$ are parallel to the initial momentum $\vec{q}$. The reason is that adding perpendicular momenta, $\vec{k}_i \rightarrow \vec{k}_i + \Delta \vec{k}_i$ with $\Delta \vec{k}_i \cdot \hat{\eta} = 0$ and $\sum_i \Delta \vec{k}_i = 0$, increases the final energy, since $\omega(\vec{k}_i + \Delta \vec{k}_i) > \omega(\vec{k}_i)$.

The same does not hold in our case ($m \neq 0$). For infinitesimal $\Delta \vec{k}$ with $\Delta \vec{k} \cdot \hat{\eta} = 0$, we find

$$\omega_{\pm}(\vec{k} + \Delta \vec{k}) \approx \omega_{\pm}(\vec{k}) \pm \frac{m}{\omega_{\pm} \Gamma} \left( \vec{k} \cdot \hat{\eta} \right) \left( \Delta \vec{k} \cdot \hat{\eta} \right), \quad (3.1)$$

which can be larger or smaller than $\omega_{\pm}(\vec{k})$. This may then result in the lowering of the total energy of a given 3-photon final state, which can be shown as follows.

Assume that we start from a configuration where the final momenta $\vec{k}_i$, $i = 1, 2, 3$, are parallel to the initial momentum $\vec{q}$, which has $\vec{q} \cdot \hat{\eta} \neq 0$ and $\vec{q} \times \hat{\eta} \neq 0$. Take the final particles 1 and 2 to have the same polarization (+ or −) and leave the polarization of the final particle 3 unspecified (±). Then add infinitesimal perpendicular momenta $\Delta \vec{k}_i$ to the final momenta $\vec{k}_i$ and choose $\Delta \vec{k}_1 = -\Delta \vec{k}_2$ and $\Delta \vec{k}_3 = 0$, so that overall momentum conservation is maintained. According to Eq. (3.1), one of the first two final energies is increased and the other decreased (the third final energy is, of course, unchanged). Moreover, the total final energy for the $(- - \pm)$ case is decreased if the energy corresponding to the smaller momentum is decreased (for example, if $|\vec{k}_1| < |\vec{k}_2|$ then $|\Delta \omega_-(\vec{k}_1)| > |\Delta \omega_-(\vec{k}_2)|$). The same result is obtained for the $(+ + \pm)$ case, as long as $|\vec{k}_j \cdot \hat{\eta}| > m/\sqrt{2}$ for $j = 1, 2$.

Still, it is worthwhile to study the case of parallel (collinear) final momenta in some detail. The frequencies $\omega_{\pm}$ may then be treated as functions of scalar variables, $\omega_{\pm} = \omega_{\pm}(k_i)$ for $k_i \equiv |\vec{k}_i|$. Assuming both $k_1$ and $k_2$ to be nonzero, one finds for the $-$ state that

$$\omega_-(k_1 + k_2) > \omega_-(k_1) + \omega_-(k_2), \quad (3.2)$$

except for the special case of $\vec{k}_1 \cdot \hat{\eta} = \vec{k}_2 \cdot \hat{\eta} = 0$, for which Eq. (3.2) becomes an equality. As a consequence, the decay $- \rightarrow - - - -$ is allowed kinematically. This also implies that the decay $+ \rightarrow - - - -$ is allowed, because $\omega_+(q) > \omega_-(q)$.

For the $+$ state, on the other hand, the following inequality holds:

$$\omega_+(k_1 + k_2) < \omega_+(k_1) + \omega_+(k_2), \quad (3.3)$$
again assuming nonzero $k_1$ and $k_2$. In this case, no immediate conclusions can be drawn from the inequality, because of relation (3.1). The inequalities (3.2) and (3.3) are proven in Appendix A.

Another kinematically allowed decay is $+ \rightarrow ++--$. This can be shown by establishing the following inequality for appropriate parallel momenta:

$$\omega_+(k_1 + k_2 + k_3) > \omega_+(k_1) + \omega_-(k_2) + \omega_-(k_3),$$

with the notation $k_i \equiv |\vec{k}_i|$. Unlike the inequalities (3.2) and (3.3), inequality (3.4) does not hold for arbitrary values of the $k_i$. Instead, some restrictions must be imposed on the momenta.

Concretely, assume that $k_2 \ll m$, $k_2 \ll k_1$ and $k_3 \ll m$, $k_3 \ll k_1$. Expanding the left-hand side of Eq. (3.4) up to first order in $k_2 + k_3$ and the right-hand side up to first order in $k_2$ and $k_3$, we find

$$\omega_+^{-1}(k_1) \left(k_1 + \frac{2mk_1 \cos^2 \theta}{\sqrt{m^2 + 4k_1^2 \cos^2 \theta}}\right) (k_2 + k_3) > \sin \theta (k_2 + k_3).$$

Defining $x \equiv m/k_1$, this inequality can be written as

$$1 + \frac{2x \cos^2 \theta}{\sqrt{x^2 + 4 \cos^2 \theta}} > \sin \theta \left(1 + \frac{x^2}{2} + \frac{x}{2} \sqrt{x^2 + 4 \cos^2 \theta}\right)^{1/2}.$$

Both sides of the last inequality are manifestly positive. Squaring both sides and re-arranging them somewhat, we arrive at

$$-x^2 + 2 \cos^2 \theta + \frac{8x^2 \cos^4 \theta}{x^2 + 4 \cos^2 \theta} > x \frac{x^2 - 4 \cos^2 \theta}{\sqrt{x^2 + 4 \cos^2 \theta}}.$$

This final inequality certainly holds if the left-hand side is larger than zero and the right-hand side smaller than zero. Writing $b \equiv 4 \cos^2 \theta$, the condition for the left-hand side gives

$$x^2 < b (b - 1)/4 + (b/4) \sqrt{b^2 - 2b + 9}$$

and the condition for the right-hand side

$$x^2 < b.$$
inequality holds also if the final momenta deviate slightly from collinearity. Therefore, our restrictions are restrictions to certain \textit{regions} in phase space and not restrictions to submanifolds of measure zero. As a consequence, the decay $+ \rightarrow + - -$ is allowed kinematically.

Altogether, we have shown that the three decay channels $- \rightarrow - - -$, $+ \rightarrow - - -$ and $+ \rightarrow + - -$ are allowed kinematically. For the five other decay channels, we have not been able to find allowed regions in phase space, either analytically or numerically. These allowed regions in phase space perhaps do not exist, but this remains to be proven.

4 \textbf{Matrix element} $+ \rightarrow - - -$

We now calculate the matrix element for the decay of a $+$ polarization mode into three $-$ modes. To lowest order, there are two contributions to this matrix element from the quartic Euler–Heisenberg Lagrangian. The first term in Eq. (2.5) gives the following contribution:

$$
\langle 0 | a_+ (q) \rangle = \int d^4 x \left( |\bar{E}(x)|^2 - |\bar{B}(x)|^2 \right)^2 : a_+^\dagger (k_1) a_+^\dagger (k_2) a_+^\dagger (k_3) |0\rangle = 
$$

$$
- \frac{1}{2 \pi^2} \delta^3(q - k_1 - k_2 - k_3) \times \left[ \omega_+ (q) \omega_- (k_1) \omega_- (k_2) \right]^{-1/2} \times \left[ f_+ (k_1) \cdot f_+ (k_2) - b_- (k_1) \cdot b_- (k_2) \right] \left[ f_+^\dagger (k_3) \cdot b_-^\dagger (q) - b_- (k_3) \cdot b_-^\dagger (q) \right] + 
$$

$$
\left[ f_+ (k_1) \cdot f_+ (k_2) - b_- (k_1) \cdot b_- (k_2) \right] \left[ f_+^\dagger (k_3) \cdot b_-^\dagger (q) - b_- (k_3) \cdot b_-^\dagger (q) \right] + 
$$

\begin{equation}
(4.1)
\end{equation}

The second term in Eq. (2.5) gives, with different combinatorics,

$$
\langle 0 | a_+ (q) \rangle = \int d^4 x \left( \bar{E}(x) \cdot \bar{B}(x) \right)^2 : a_+^\dagger (k_1) a_+^\dagger (k_2) a_+^\dagger (k_3) |0\rangle = 
$$

$$
- \frac{1}{2 \pi^2} \delta^3(q - k_1 - k_2 - k_3) \times \left[ \omega_+ (q) \omega_- (k_1) \omega_- (k_2) \right]^{-1/2} \times 7/4 \times 
$$

$$
\left[ f_+ (k_1) \cdot b_- (k_2) + f_+ (k_2) \cdot b_- (k_1) \right] \left[ f_+^\dagger (q) \cdot b_- (k_3) + f_- (k_3) \cdot b_-^\dagger (q) \right] + 
$$

$$
\left[ f_+ (k_1) \cdot b_- (k_3) + f_+ (k_3) \cdot b_- (k_1) \right] \left[ f_+^\dagger (q) \cdot b_- (k_2) + f_- (k_2) \cdot b_-^\dagger (q) \right] + 
$$

\begin{equation}
(4.2)
\end{equation}
Unfortunately, the expressions involving the polarization vectors in Eqs. (4.1)–(4.2) are rather complicated and one would like to simplify them by using some type of small $m$ expansion. But such a procedure appears to be quite difficult and, in this paper, we keep the full expressions.

The probability for decay of a single $+$ state into three $-$ states can be calculated by integrating the square of the amplitude over the final momenta. The result will, however, be a function of $|\vec{q}|$ and $\hat{q} \cdot \hat{n}$, for given $m$ and $M$. For this reason, we turn to a simpler problem in the next section, namely the decay of a $+$ state at rest.

5 Partial decay width for $+ \rightarrow - - -$

The decay probability for $+ \rightarrow - - -$ will be evaluated for the case of vanishing initial momentum, $\vec{q} = 0$. The energy $\omega_+(\vec{q})$ for $\vec{q} = 0$ equals $m$ and the initial “photon” behaves like a massive particle at rest. The partial decay width is then obtained by squaring the matrix element of the previous section and integrating over the final momenta $\vec{k}_i$, for $i = 1 \ldots 3$. A combinatorial factor $1/3! = 1/6$ must be inserted because of the three identical particles in the final state.

Purely on dimensional grounds, we can write the partial decay width to order $\alpha^4$ as

$$\Gamma_{+ \rightarrow - - -}^{(4)} = \frac{1}{512 \pi^5} \kappa \left( \frac{2 \alpha^2}{45} \right)^2 \frac{m^9}{M^8}, \quad (5.1)$$

with a single number $\kappa \geq 0$ to be determined.

The relevant phase space integral is defined as follows:

$$I_{- - -} \equiv \int d^3 k_1 \, d^3 k_2 \, d^3 k_3 \, \delta^3(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) \, \delta[\omega_-(\vec{k}_1) + \omega_-(\vec{k}_2) + \omega_-(\vec{k}_3) - m]$$

$$\times \left[ \omega_+(0) \omega_-(\vec{k}_1) \omega_-(\vec{k}_2) \omega_-(\vec{k}_3) \right]^{-1} g(\vec{k}_1, \vec{k}_2, \vec{k}_3), \quad (5.2)$$

where the nonnegative function $g$ depends on the electric and magnetic polarization vectors. Specifically, the factor $g$ is given by the absolute value squared of the terms in large brackets in Eqs. (4.1) and (4.2) for $\vec{q} = 0$,

$$g(\vec{k}_1, \vec{k}_2, \vec{k}_3) = \left| \left[ \left[ \vec{f}_-(\vec{k}_1) \cdot \vec{b}_-(\vec{k}_2) - \vec{b}_-(\vec{k}_1) \cdot \vec{b}_-(\vec{k}_2) \right] \vec{f}_-(\vec{k}_3) \cdot \vec{f}_+(0) + \cdots \right] + \frac{7}{4} \right|^2$$

$$\times \left( \left[ \vec{f}_-(\vec{k}_1) \cdot \vec{b}_-(\vec{k}_2) + \vec{f}_-(\vec{k}_2) \cdot \vec{b}_-(\vec{k}_1) \right] \vec{f}_+(0) \cdot \vec{b}_-(\vec{k}_3) + \cdots \right). \quad (5.3)$$

The expressions for $\vec{f}_+(0)$ and $\vec{b}_+(0)$ will be given in the next section. Note that only the mass-scale $m$ appears in the integral (5.2), so that $I_{- - -} \propto m^9$. 

9
A numerical calculation shows that the integrand of Eq. (5.2) is strictly positive over a finite region of phase space (see Appendix B). For the numerical constant $\kappa$ in the partial decay width (5.1), this implies

$$\kappa = \left( \frac{1}{3!} \right) \left( \frac{1}{2\pi^2} \right)^2 I_{--} / m^9 > 0,$$

which is the main result of the present paper.

6 Partial decay width for $+ \rightarrow + --$

The decay of an initial $+$ polarization state is in general more complicated than the decay of an initial $-$ state. But for initial momentum $\vec{q} = 0$ a major simplification occurs: the second decay channel $+ \rightarrow + --$ no longer contributes, as will be demonstrated in this section.

The matrix element for $+ \rightarrow + --$ decay follows from Eqs. (4.1) and (4.2) by replacing all $-$ labels that occur in conjunction with $\vec{k}_1$ by $+$ labels. The energy delta function, for example, becomes

$$\delta[\omega_+ (\vec{q}) - \omega_+ (\vec{k}_1) - \omega_- (\vec{k}_2) - \omega_- (\vec{k}_3)].$$

The energy conservation condition for the decay of a $+$ photon at rest $(\vec{q} = 0)$ now takes the form

$$\left( \omega_+ (\vec{k}_1) - m \right) + \omega_- (\vec{k}_2) + \omega_- (\vec{k}_3) = 0.$$

This relation holds only if all three final momenta $\vec{k}_i$ are zero, because each of the three terms in the above equation is positive semi-definite and zero only for vanishing momentum. The conditions $\vec{k}_i = 0$, for $i = 1, 2, 3$, are restrictions to a lower-dimensional submanifold in phase space and have, therefore, measure zero. The resulting phase space integrals will lead to a zero contribution to the decay width, unless certain singularities of the phase-space integrand appear in the limit $\vec{k}_i \rightarrow 0$. In the following, we will show that such infrared singularities are absent.

The phase-space integral in spherical polar coordinates $k_i, \theta_i, \varphi_i, \text{ for } i = 1, 2, 3$, takes the form

$$I_{++-} \equiv \int \delta^3 (\vec{k}_1 + \vec{k}_2 + \vec{k}_3) \delta[\omega_+(\vec{k}_1) + \omega_- (\vec{k}_2) + \omega_- (\vec{k}_3) - m] \ h(k_i, \theta_i, \varphi_i)$$

$$\times \frac{k_2^2 k_3^2 \sin \theta_2 \sin \theta_3 \ d^3 k_1 \ d k_2 \ d k_3 \ d \theta_2 \ d \theta_3 \ d \varphi_2 \ d \varphi_3}{\omega_+ (0) \omega_+ (\vec{k}_1) \omega_- (\vec{k}_2) \omega_- (\vec{k}_3)},$$

where the nonnegative function $h(k_i, \theta_i, \varphi_i)$ depends on the electric and magnetic polarization vectors $\vec{f}_\pm (\vec{k}_1)$ and $\vec{b}_\pm (\vec{k}_i)$. Using the bounds $\omega_+ (\vec{k}) \geq m$ and $\omega_- (\vec{k}) \geq \vec{k} \sin \theta$ in the denominator and integrating over the momentum $\vec{k}_1$, we find for the phase-space integral

\[
I_{++-} \equiv \int d^3 \vec{k}_2 \ d^3 \vec{k}_3 \ \delta[\omega_+(\vec{k}_1) + \omega_- (\vec{k}_2) + \omega_- (\vec{k}_3) - m] \ h(k_2, \theta_2, \varphi_2) \ h(k_3, \theta_3, \varphi_3)
\times \frac{1}{\omega_+ (0) \omega_+ (\vec{k}_1) \omega_- (\vec{k}_2) \omega_- (\vec{k}_3)},
\]
0 \leq I_{++-} \leq \int \delta[\omega_+(\vec{k}_2 + \vec{k}_3) + \omega_-(\vec{k}_2) + \omega_-(\vec{k}_3) - m] h(k_j, \theta_j, \varphi_j) \frac{k_2 k_3 dk_2 dk_3 d\theta_2 d\theta_3 d\varphi_2 d\varphi_3}{m^2}, \quad (6.4)

where \( h \) now depends on the polarization vectors \( \vec{f}_\pm(\vec{k}_j) \) and \( \vec{b}_\pm(\vec{k}_j) \), for \( j = 2, 3 \), and the vectors \( \vec{f}_\pm(-\vec{k}_2 - \vec{k}_3) \) and \( \vec{b}_\pm(-\vec{k}_2 - \vec{k}_3) \).

The integral in Eq. (6.4) has two potential sources of infrared singularities, the first of which is the factor \( h(k_j, \theta_j, \varphi_j) \). But it is not difficult to see that \( h(k_j, \theta_j, \varphi_j) \) is nonsingular in the limit \( k_j \to 0 \). As \( h \) depends only on the electric and magnetic polarization vectors, it suffices to demonstrate that these vectors are nonsingular in the infrared. Writing \( \vec{k} = k_\perp \hat{e}_\perp + k_\parallel \hat{\eta} \) with \( \hat{e}_\perp \) an arbitrary unit vector orthogonal to \( \hat{\eta} \), it can indeed be shown that

\[
\lim_{k \to 0} \vec{f}_+(\vec{k}) = im(\hat{e}_\perp - i\hat{e}_\perp \times \hat{\eta})
\]  

and

\[
\lim_{k \to 0} \vec{f}_-(\vec{k}) = \lim_{k \to 0} \vec{b}_+(\vec{k}) = \lim_{k \to 0} \vec{b}_-(\vec{k}) = 0. \quad (6.6)
\]

The second potential source of infrared singularities in the integral (6.4) is the energy delta function itself, if singularities of the type \( \int dk \delta(k^2) \) are produced. The proof that this does not happen is somewhat involved and is relegated to Appendix C.

Altogether, we find that the \( + \to + -- \) channel does not contribute to \(+ \) decay at vanishing initial momentum, \( \vec{q} = 0 \), so that

\[
\Gamma_{++ \to ++-}^{(4)} = 0. \quad (6.7)
\]

For \( \vec{q} = 0 \), no other channels contribute, besides the two channels already considered. The reason is energy conservation (cf. Section 2): the energy of an initial \(+ \) state at rest is \( m \), whereas the energies of the final states \( ++- \) and \( +++ \) are at least \( 2m \) and \( 3m \), respectively.

7 Total decay rate of a \(+ \) photon at rest

In this paper, we have studied the decay of photons in a relatively simple model, for which the Lagrangian (2.1) contains only the usual Maxwell and Euler–Heisenberg terms, together with a CPT-violating Chern–Simons-like term. The two mass parameters of the model are \( m \), which enters the Chern–Simons-like term (2.2) linearly, and \( M \), whose fourth power enters the Euler–Heisenberg term (2.5) inversely. The Euler–Heisenberg term has, in addition, an overall coupling constant \( \alpha^2 \).

The photon of this model has two polarization states, labeled \( \pm \), with energies and momenta given by Eqs. (2.6), (2.18) and (2.19). The total decay
rate to order $\alpha^4$ of a $+$ state at rest follows from the sum of the partial decay widths (5.1) and (6.7),

$$
\Gamma^{(4)}_+ = \frac{1}{512 \pi^5} \kappa \left( \frac{2 \alpha^2}{45} \right)^2 \frac{m^9}{M^8},
$$

(7.1)

with a constant $\kappa \geq 0$. A numerical calculation shows that $\kappa$ is nonvanishing and, most likely, not vastly different from 1. [See Eq. (5.4) and Appendix B.]

As explained in Section 2, the model considered is part of the effective gauge field action of quantum electrodynamics with an additional Chern–Simons-like term. An order of magnitude estimate for the photon mean lifetime can then be obtained by writing

$$
\Gamma^{(4)}_+ = \kappa \left[ 3 \times 10^{336} \text{ yr} \right]^{-1} \left( \frac{\alpha}{1/137} \right)^4 \left( \frac{511 \text{ keV}}{M} \right)^8 \left( \frac{m}{10^{-33} \text{ eV}} \right)^9,
$$

(7.2)

where for $\alpha$ and $M$ the values of the fine-structure constant and the electron mass have been inserted and for $m$ the experimental upper limit from polarization measurements on distant radio galaxies (see Refs. [1,10] and references therein). For $m \sim 10^{-35} \text{ eV}$ as might be expected from the CPT anomaly [7], the photon lifetime would be larger by a factor of $10^{18}$.

The very large photon lifetime as indicated by Eq. (7.2) is perhaps not of direct relevance, at least for the current epoch in the history of the universe. There remain, however, fundamental questions about the formulation of this particular CPT-violating extension of quantum electrodynamics if certain photon states are no longer absolutely stable (cf. Ref. [14]).

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A Energy inequalities

In this appendix, we prove the energy inequalities (3.2) and (3.3) for parallel momenta $\vec{k}_1 \parallel \vec{k}_2$. [The $\omega_-$ case has the additional conditions $\vec{k}_j \cdot \vec{\eta} \neq 0$, for $j = 1, 2$.] It is, in fact, not difficult to verify that these inequalities hold for sufficiently small $k_j \equiv |\vec{k}_j| \ll m$. For the general case, we give a proof by contradiction.

Assume that the inequalities do not hold for all values of $k_1$ and $k_2$. Then there must exist values for $k_1$ and $k_2$ with

$$
\omega(k_1 + k_2) = \omega(k_1) + \omega(k_2).
$$

(A.1)
These special values of $k_1$ and $k_2$ could, of course, be different for the $+$ and the $-$ case. But the proof is analogous in both cases and we treat both cases at once, writing $\omega$ for either $\omega_+$ or $\omega_-$. The equality (A.1) can also be written as

$$v_{ph}(k_1 + k_2) = \frac{k_1}{k_1 + k_2} v_{ph}(k_1) + \frac{k_2}{k_1 + k_2} v_{ph}(k_2), \quad (A.2)$$

where $v_{ph}(k) = \omega(k)/k$ is the absolute value of the phase velocity. Equation (A.2), now, holds only if one of the following three conditions is met:

(i) $v_{ph}(k_1) = v_{ph}(k_2) = v_{ph}(k_1 + k_2),$

(ii) $v_{ph}(k_1) > v_{ph}(k_1 + k_2)$ and $v_{ph}(k_2) < v_{ph}(k_1 + k_2),$

(iii) $v_{ph}(k_1) < v_{ph}(k_1 + k_2)$ and $v_{ph}(k_2) > v_{ph}(k_1 + k_2). \quad (A.3)$

Each of these conditions implies that $v_{ph}(k)$ has an extremum somewhere in the momentum interval $[\min(k_1, k_2), k_1 + k_2]$. But this conclusion contradicts the simple observation that, for both $\omega_+$ and $\omega_-$, the derivative of the phase velocity $dv_{ph}/dk$ is nonzero for all positive values of $k$. [The observation requires for the $\omega_-$ case the additional condition $\vec{k} \cdot \hat{\eta} \neq 0$.] This then implies that the assumption above Eq. (A.1) must be incorrect and that the inequalities (3.2) and (3.3) hold for all positive values of $k_1$ and $k_2$, as stated in the main text.

B Numerical result for $I_{--}$

In this appendix, we report on a numerical calculation of the integral $I_{--}$, as defined by Eq. (5.2). A dimensionless quantity $I$ is obtained by setting $I \equiv I_{--}/m^9$. We can be relatively brief in describing our results since the calculation of a decay rate is well-known (see, e.g., Sections 3.6 and 3.7 of Ref. [15]). We proceed in four steps.

First, the integral over $\vec{k}_3$ is performed and the remaining six integration variables are taken to be the following dimensionless spherical coordinates:

$$k_1 \equiv |\vec{k}_1|, \quad k_2 \equiv |\vec{k}_2|, \quad \theta_1, \theta_2, \varphi_\pm \equiv \varphi_2 \pm \varphi_1, \quad (B.1)$$

with polar angles $\theta_1, \theta_2 \in [0, \pi]$ defined with respect to an axis in the preferred direction $\hat{\eta}$ and azimuthal angles $\varphi_1, \varphi_2 \in [0, 2\pi]$ in the plane orthogonal to this axis. The angles $\varphi_+$ and $\varphi_-$ can be taken to run over $[0, 4\pi]$ and $[-\pi, +\pi]$, respectively.

Second, the energy delta function effectively sets $\varphi_-$ to a fixed value,

$$\varphi_- = (1 - 2n) \arccos \chi, \quad (B.2)$$

for a known function $\chi = \chi(k_1, \theta_1, k_2, \theta_2)$ and integer $n = 0$ or 1. Also, there are the constraints that both $|\chi|$ and $\omega_-(k_1, \theta_1) + \omega_-(k_2, \theta_2)$ must be less
than 1, which can be implemented by introducing appropriate step functions $\Theta(1 - |\chi|)$ and $\Theta[1 - \omega_-(k_1, \theta_1) - \omega_-(k_2, \theta_2)]$ into the integrand.

Third, the resulting integral has the following structure:

$$I = \int_0^\infty dk_1 dk_2 \int_0^\pi d\theta_1 d\theta_2 \int_0^{4\pi} d\varphi_+ \frac{1}{2} \sum_{n=0}^1 \mathcal{I} |\varphi_=| = (1 - 2n) \arccos \chi; \vec{k}_3 = -\vec{k}_1 - \vec{k}_2,$$ (B.3)

with the integrand

$$\mathcal{I} = [k_1/\omega_-(k_1, \theta_1)] [k_2/\omega_-(k_2, \theta_2)] \Theta[1 - \omega_-(k_1, \theta_1) - \omega_-(k_2, \theta_2)]$$

$$\times \Theta (1 - |\chi(k_1, \theta_1, k_2, \theta_2)|) \sin \varphi_- |^{-1} g(\vec{k}_1, \vec{k}_2, \vec{k}_3),$$(B.4)

where $\omega_-$ and $g$ are dimensionless functions (i.e., the expressions of the main text with $m \equiv 1$). The integral (B.3) is complicated, but its integrand is still nonnegative; cf. the definition (5.3).

Fourth, a numerical calculation with Mathematica [16] shows the integrand of Eq. (B.3) to be independent of $\varphi_+$ and $n$. This effectively reduces the integral to a four-dimensional one and a numerical estimate gives

$$I \equiv I_- = m^9 \approx 0.2.$$ (B.5)

From this estimate, one obtains the result (5.4) quoted in the main text.

The value (B.5) is to be considered preliminary. More work is needed to obtain an accurate result, both analytically (e.g., to make the independence of the azimuthal coordinate $\varphi_+$ manifest) and numerically (e.g., to sample phase space efficiently).

C Analytic result for $I_{+-}$

In this appendix, we demonstrate the vanishing of the integral $I_{+-}$ as defined by Eq. (6.3). We start with two preliminary steps. First, we introduce the following representation for the energy delta function:

$$\delta(\omega) = \lim_{a \to 0} \frac{1}{\sqrt{\pi} a} \exp \left(-\omega^2/a^2\right), \quad a > 0.$$ (C.1)

Second, we replace the nonnegative function $h(k_j, \theta_j, \varphi_j)$ in the integral of Eq. (6.4) by the following bound:

$$h(k_j, \theta_j, \varphi_j) \leq \sum_{l,n=0}^4 H(l, n) k_2^l k_3^n,$$ (C.2)

where $H(l, n)$ are appropriate nonnegative numbers. This bound may be understood from the observation that the factor $h(k_j, \theta_j, \varphi_j)$ is the absolute square of a sum of terms of the type
\[
[\vec{v}_1(\vec{k}_2) \cdot \vec{v}_2(\vec{k}_3)] [\vec{v}_3(-\vec{k}_2 - \vec{k}_3) \cdot \vec{v}_4(0)],
\]
(C.3)

where each vector \( \vec{v}_i \), for \( i = 1, \ldots, 4 \), is either an electric polarization vector \( \vec{f}_\pm \) or a magnetic polarization vector \( \vec{b}_\pm \), as defined by Eqs. (2.12) and (2.13).

Using the energy bounds \( \omega_+(\vec{k}_j) \leq k_j + m \) and \( \omega_-(\vec{k}_j) \leq k_j \), each vector \( \vec{v}_i(\vec{k}_j) \) can be shown to obey the following inequality:

\[
|\vec{v}_i(\vec{k}_j)| \leq |\vec{v}_{0,i}(\vec{k}_j) + k_j\vec{v}_{1,i}(\vec{k}_j)|,
\]
(C.4)

where \( \vec{v}_{0,i} \) and \( \vec{v}_{1,i} \) are uniformly bounded vectors (i.e., they never exceed a certain length). In the product (C.3), then, the momenta \( \vec{k}_2 \) and \( \vec{k}_3 \) show up at most quadratically, so that no powers higher than four occur in the absolute square of a sum of such terms. This explains the bound (C.2).

With these two steps, the bound (6.4) becomes

\[
0 \leq I_{+-+-} \leq \lim_{a \to 0} \frac{1}{a\sqrt{\pi m^2}} \int_0^\infty dk_2 dk_3 \int_0^\pi d\theta_2 d\theta_3 \int_0^{2\pi} d\varphi_2 d\varphi_3 \sum_{l,n=0}^4 H(l,n) \times k_2^{l+1} k_3^{n+1} \exp\left(-\left[\omega_+(\vec{k}_2 + \vec{k}_3) - m + \omega_-(\vec{k}_2) + \omega_-(\vec{k}_3)\right]^2/a^2\right),
\]
(C.5)

Now change the momentum variables \( k_j \) to \( k_j/a \) and use the estimates \( \omega_+ \geq m \) and \( [\omega_-(k_2) + \omega_-(k_3)]^2 \geq \omega_2(k_2) + \omega_2(k_3) \). This gives

\[
0 \leq I_{+-+-} \leq \lim_{a \to 0} \sum_{l,n=0}^4 \frac{a^{3+n+l}}{\sqrt{\pi} m^2} \int dk_2 dk_3 d\theta_2 d\theta_3 d\varphi_2 d\varphi_3 H(l,n) k_2^{l+1} k_3^{n+1} \times \exp\left(-k_2^2 - \frac{m_2^2}{2} + \frac{m_2^2}{2} \sqrt{1 + 4 \cos^2 \theta_2 k_2^2/m_2^2} \right.
\]
\[
\left.-k_3^2 - \frac{m_3^2}{2} + \frac{m_3^2}{2} \sqrt{1 + 4 \cos^2 \theta_3 k_3^2/m_3^2}\right),
\]
(C.6)

with \( m_a \equiv m/a \). Since \( \cos^2 \theta_j \leq 1 \), we finally arrive at

\[
0 \leq I_{+-+-} \leq \lim_{m_a \to \infty} \sum_{l,n=0}^4 H(l,n) \frac{m_1^{1+l+n}}{\sqrt{\pi} m_3^{3+l+n}} \int_0^\pi d\theta_2 d\theta_3 \int_0^{2\pi} d\varphi_2 d\varphi_3 I_l I_n,
\]
(C.7)

in terms of the dimensionless integrals

\[
I_l \equiv \int_0^\infty dk \ k^{l+1} \exp\left(-k^2 - \frac{m_a^2}{2} + \frac{m_a^2}{2} \sqrt{1 + 4 k^2/m_a^2}\right).
\]
(C.8)

The momentum integrals \( I_l \), for \( l = 0, \ldots, 4 \), can be evaluated analytically with the help of MATHEMATICA [16], but we are only interested in their
asymptotic behavior for $m_a \to \infty$. Making the change of variables $y = k^2/m_a$ and Taylor expanding the square root in Eq. (C.8), we obtain

$$I_l \sim c_l m_a^{1+l/2}, \quad \text{for} \quad m_a \to \infty,$$

(C.9)

with positive coefficients $c_l$.

Inserting the asymptotic results for $I_l$ and $I_n$ into the bound (C.7), we get ($\lambda$ is a positive constant)

$$0 \leq I_{+-} \leq \lambda \lim_{m_a \to \infty} \sum_{l,n=0}^4 H(l, n) m_a^{1+l+n} c_l c_n m_a^{-1-l/2-n/2} = 0,$$

(C.10)

so that

$$I_{+-} = 0,$$

(C.11)

which implies Eq. (6.7) in the main text.

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