Massive quasi-normal mode

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Abstract. This paper purposes to study quasi-normal modes due to massive scalar fields. We, in particular, investigate the dependence of QNM frequencies on the field mass. By this research, we find that there are quasi-normal modes with arbitrarily long life when the field mass has special values. It is also found that QNM can disappear when the field mass exceed these values.

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1. Introduction

Quasi-normal mode (QNM) is one of the important and exciting themes in the black hole physics. QNMs represent the behaviours of fields on a black hole spacetime under a certain condition, imposed both at the event horizon of the black hole and at the asymptotic infinity of the spacetime. This means that QNMs can be strongly affected by the curvature of the spacetime. In other words, we can expect that the property of QNMs strongly reflects the property of the spacetime, and so we can grasp the property of the spacetime by studying on QNMs.

The existence of QNMs was mentioned first by Vishveshwara[11] in the context of scattering problem by black holes. Later, people began to perceive that QNMs were important sources of gravitational waves, and many researchers developed various methods to calculate QNM frequencies [2], [3], [4], [5], [6], [7], [8], [9]. Recently, some researchers have pointed out that the asymptotic behaviour of the distribution of QNMs relates to the quantum gravitational theory [10], [11]. Thus, we recognise that researches on properties of QNMs will further increase its importance.

Our motivation of this article is to study properties of a black hole spacetime using QNMs as a mathematical tool. Almost all studies on QNMs discuss only massless fields such as electromagnetic waves and gravitational waves, since these studies mainly focuses on phenomenological significances. Therefore, there are few studies on QNMs by massive fields (hereafter, we call them massive QNMs). One of the studies related to massive QNMs is a work by Simone and Will [12], which investigates massive QNMs on black hole spacetime using WKB method. They study the dependency of QNM...
frequencies on the mass of the field, but their discussion is restricted to narrow range of
the field mass due to the restriction required by the WKB method. Hence, their analysis
cannot fully reveal the dependency of QNM frequencies on the field mass. Another work
on massive QNM is a study by Konoplya [13]. He study massive QNMs on a charged
black hole background and consider QNMs due to massive and charged fields. His paper
mainly focuses on correlation between QNMs and the charge of the field and the black
hole. Therefore, the dependency of QNMs on the field mass is not argued, though his
article investigate almost same situation that we will study in this article.

Hence, we will study behaviours of QNMs to the field mass in detail. For this
purpose, we first consider a model in which we can derive QNM frequencies analytically.
Through this investigation we find that there is a singular phenomenon that QNMs may
disappear when the field mass becomes sufficiently large. Second, we study QNMs on
the Reisner-Nordström black hole spacetime by a numerical method. As a result, we
confirmed that the singular phenomenon observed in the model can also be observed in
the real black hole case.

This article consists of the following contents. In section 2, we study a correlation
between QNM frequencies and the field mass in a model which can be solved analytically.
In section 3, we examine the real black hole case with a numerical method. In
section 4, we discuss the result obtained in section 3. In section 5, we summarise our
investigation. Additionally, Appendix A shows the review of the calculation algorithm
used in section 3.

2. Analytical model

In this section, we study the dependence of massive QNM frequencies on the field mass
by using an analytically solvable model.

Generally, in order for QNMs to exist, it is necessary that the background has a
potential that can trap waves. Then, the trapped waves will exude out gradually. This
phenomenon is exactly QNMs. If the asymptotic values of the potential are the same at
both side of the potential peak, the waves will be the massless QNMs. If the asymptotic
values are different, they will be massive QNMs.

Let us consider the following model:

\[
\left[ \frac{d^2}{dx_*^2} + \nu^2 - V(x_*) \right] Z(x_*) = 0 \tag{1a}
\]

\[
V = \frac{1}{4M^2} u(1 - u) + m^2 u \tag{1b}
\]

\[
u = \frac{1}{2} \left( 1 + \tanh \frac{x_*}{2L} \right). \tag{1c}
\]

Here, the parameters in the above equations cannot have any physical meanings, but we
can consider some correspondences to the black hole parameters. \( M \) and \( m \) correspond
to the black hole mass and field mass, respectively. \( L \) determines the width of the
peak and this is dependent on \( M \) in the black hole case. The coordinate variable \( x_* \),
corresponds to the so-called “tortoise” coordinates. The potential of this model (1b) is shown in figure 1. For comparison with a real black hole one, we also show the potential for a Schwarzschild black hole. Because both potentials are very similar, we can also expect that the behaviours of QNMs will resemble in both cases.

This model can be solved analytically as follows. First, by changing the coordinate $x_*$ to $u$, (1a) can be rewritten into the following form,

$$\left[ u(1-u) \frac{d^2}{du^2} + (1-2u) \frac{d}{du} + \frac{L^2 u^2}{u(1-u)} - \frac{L^2}{4M^2} - \frac{m^2 L^2}{1-u} \right] Z(u) = 0. \quad (2)$$

Note that the infinity of $x_* = \pm \infty$ corresponds to $u = 1 (+\infty)$ and 0 ($-\infty$). These two points $u = 0$ and 1 are the regular singularities of the differential equation (2). Then, the behaviour of $Z(u)$ around these two singularities is given by

$$Z(u) \sim u^a (1-u)^b,$$  \hspace{1cm} (3)
where $a^2 = -L^2\nu^2$, $b^2 = -L^2(\nu^2 - m^2)$. Therefore, by setting
\[ Z(u) = u^a(1 - u)^b g(u) \] (4)
and substituting (4) into (2), we obtain a differential equation for $g(u)$ as follows,
\[
\left[ u(1 - u) \frac{d^2}{du^2} + \{(1 + 2a) - 2(1 + a + b)u\} \frac{d}{du} \\
+ \left\{ -a - b - 2ab - L^2m^2 - \frac{L^2}{4M^2} + 2L^2\nu^2 \right\} \right] g(u) = 0. \] (5)
This is exactly the hyper-geometric equation and the solution of this equation can be given by the hyper-geometric function $F(\alpha, \beta, \gamma; u)$. In consequence, we find the solution of (2) as follows,
\[ Z(u) = u^a(1 - u)^b F(\alpha, \beta, \gamma; u), \] (6a)
\[ \alpha, \beta = \frac{1}{2} + a + b \pm \sqrt{1 - \left(\frac{L}{M}\right)^2}, \] (6b)
\[ \gamma = 1 + 2a. \] (6c)

Now, let us give our attention to the following relation
\[ u^a(1 - u)^b F(\alpha, \beta, \gamma; u) \]
\[ = \frac{\Gamma(\gamma - \alpha - \beta)\Gamma(\gamma)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)} u^a(1 - u)^b F(\alpha, \beta, 1 + \alpha + \beta - \gamma; 1 - u) \]
\[ + \frac{\Gamma(\gamma)\Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha)\Gamma(\beta)} u^a(1 - u)^{-b} F(\gamma - \alpha, \gamma - \beta, 1 - \alpha - \beta + \gamma; 1 - u). \] (7)

According to (7), we can analytically determine QNM frequencies from the behaviour of the solution around $u = 0$ and 1, because QNM is defined by the boundary conditions at $x^* = \pm\infty$, i.e. $u = 0, 1$. The boundary condition is that: QNMs are

(i) ingoing waves at $x^* \sim -\infty$ ($u = 0$),
(ii) outgoing waves at $x^* \sim +\infty$ ($u = 1$), and
(iii) damping modes ‡.

From the first condition at $u = 0$, we have to choose the sign of $a$ as,
\[ a = -iLv. \] (8)

The second condition at $u = 1$ means
\[ \frac{\Gamma(\gamma)\Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha)\Gamma(\beta)} = 0, \] (9)
because $(1 - u)^{-b}$ represents the incoming waves at $u = 1$ when we set $b = +iL\sqrt{\nu^2 - m^2}$ §. Therefore, QNM frequencies can be determined by,
\[ \alpha, \beta = -n \ (n = 0, 1, \cdots), \] (10)
‡ We suppose the time dependency is $\exp(-i\nu t)$
§ The branch of $\sqrt{z}$ is $-\pi/2 < \arg\sqrt{z} \leq \pi/2$. 
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According to (9), by solving this equation for \( \nu \), we obtain

\[
\nu = -i \frac{(2n + 1)(4n^2 + 4n + \mu^2 - 4\mu^2\epsilon^2)}{4L(4n^2 + 4n + \mu^2)} \pm \frac{(4n^2 + 4n + \mu^2 + 4\mu^2\epsilon^2)}{4L(4n^2 + 4n + \mu^2)} \left(i\sqrt{1 - \mu^2}\right),
\]

where we set \( L = \mu M \) and \( \epsilon = m M \). When \( \mu \leq 1 \), \( \nu \) is pure imaginary. However, \( \nu \) becomes a complex number if \( \mu > 1 \). In this case, the first term in the right hand side of (11) represents the damping part and the second term does the oscillation part. Consequently, the damping coefficient of QNMs approaches to zero as a quadratic function of \( \epsilon \). However, \( \nu \) never becomes a real number because the branch of square root changes discontinuously when QNM frequencies get across the real axis. According to this result, we can conclude that there exist QNMs with arbitrarily long decaying time when \( m \) changes appropriately. In other words, we find an interesting fact that there is a kind of resonance mode in a limiting situation and QNMs can disappear when \( m \) exceeds a certain value.

This disappearance is due to the relation of the height of the peak of the potential with the field mass. Therefore, we can expect that the same phenomenon will occur in the black hole case. Thus, we will investigate the massive QNMs on the black hole spacetime in the next section.

3. Massive QNM on black hole spacetime

In this section, we consider QNMs of a massive scalar field on a spherical black hole spacetime. In the black hole case, we must tackle the problem in a numerical way because it is difficult to study it fully analytically.

The metric of a spherical symmetric spacetime is given by

\[
ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega^2.
\]

In this article, we consider the Reisner-Nordström black hole as the background spacetime, so the metric function \( f(r) \) is given by:

\[
f(x) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} = 1 - \frac{2}{x} + \frac{q^2}{x^2},
\]

where \( M \) is the black hole mass and \( Q \) is the black hole charge. We also introduce the normalised variables \( x \) and \( q \) as below,

\[
x = \frac{r}{M}, \quad (14a)
\]

\[
q = \frac{Q}{M}, \quad (14b)
\]

respectively.

In this spacetime, the radial equation for a scalar field in term of the normalised variables is given by

\[
\left[ \frac{d^2}{dx^2} + \frac{df}{dx} \frac{d}{dx} + \left( \frac{\nu^2}{f^2} - \frac{l(l + 1)}{x^2 f} - \frac{m^2}{f} - \frac{df}{dx} \frac{d}{xf} \right) \right] Z(x) = 0,
\]

(15)
where \( \nu \) and \( \epsilon \) are normalised frequency and mass of the scalar field and they are related to original frequency \( \omega \) and mass \( m \) by

\[
\nu = M\omega, \quad (16a)
\]
\[
\epsilon = mM, \quad (16b)
\]
respectively.

In order to find the potential for the scalar field, let us introduce a so-called tortoise coordinate \( x_* \) by

\[
\frac{dx_*}{dx} = \frac{1}{f}, \quad (17)
\]
and we obtain the following equation,

\[
\left[ \frac{d^2}{dx_*^2} + \nu^2 - V(x_*) \right] Z = 0, \quad (18)
\]

\( \parallel \) The original field equation is \((\Box - m^2)\phi = 0\), and \( \omega \) is introduced by variable separation, \( \phi \sim e^{-i\omega t} Z(r)Y_{lm}(\theta, \varphi) \).
where the potential $V(x_*)$ is given by

$$V(x_*) = \left( l(l+1) + \epsilon^2 + \frac{df}{dx} \right) \times f(x).$$  \hspace{1cm} (19)

Note that $x$ in the right hand side of (19) is a function of $x_*$ by (17). The profiles of this potential in some cases are shown in figure 3 and figure 4. Figure 2 in section 2 also shows the potential for comparison with the potential of the analytic model.

In order to calculate the QNM frequencies, we solve (15) numerically. We consider the case of $|q| < 1$ in this article, because the mathematical structure of (15) with $q = 1$ is apparently different from the case of $|q| < 1$. The reason of excluding the case of $|q| > 1$ is that there is no horizon in this case and then the central singularity is naked. In this case, QNMs cannot be defined. Although the Schwarzschild case, $q = 0$, seems to be different case, we verify that there is no difference between the two cases in our results: (i) $q = 0$ and (ii) $q \to 0$.

The numerical method which we use in calculating QNM frequencies is the continued fraction method developed by Leaver[3]. In Appendix A, we will briefly review this method for our case. In this article, we examine only $l = 0$ and calculate QNM frequencies from $\epsilon = 0$ to around 0.4 in several cases of $q = 0$ (Schwarzschild case), 0.25, 0.5, 0.75, 0.95 and 0.99. The results are shown in figure 5, figure 6, figure 7 and figure 8. In these figures, $\nu_I$ and $\nu_R$ mean the imaginary and real part of the QNM frequencies, respectively.

Figures 5 and 6 show the dependence of the QNM frequencies on the mass of the scalar field. It is obvious from the figures that the imaginary part of the QNMs on the black hole spacetime approaches to 0 as well as the analytical model in the last section. This tendency is more clear in figure 7. Figure 7 shows the trajectories of the QNM frequencies on the $\nu$-plane when the field mass $\epsilon$ changes while keeping $q$ constant. From this figure, we can find that the trajectories of the QNM frequencies are almost linear, and we can guess the frequencies when QNMs disappear. We call the modes with these real frequencies as quasi-resonance modes (QRM). According to the linearity of the QNM trajectories, we can calculate the dependence of QRM frequencies on the field mass $\epsilon$ with the following fitting function,

$$\nu_I = k(\nu_R - \nu_{QRM}).$$  \hspace{1cm} (20)

The behaviour of this $\nu_{QRM}$ to $\epsilon$ is plotted in figure 8.

We will give more detailed discussion on our results in the next section.

4. Discussion

In this article, we have studied massive QNMs for the two cases, the analytically solvable model and the black hole. We can find the same phenomenon, that the life of QNMs becomes longer as the field mass increases, in both cases. From this fact, we can conclude that there exists a QNM, oscillating without damping for an arbitrary period. This extreme mode may be regarded as a kind of resonance mode (after this fact, we named
Figure 5. Dependence of $\text{Im}(\nu)$ to the field mass $\epsilon$ for the cases of $q = 0.0, 0.25, 0.50$ and $0.75$.

Figure 6. Dependence of $\text{Im}(\nu)$ to the field mass $\epsilon$ for the cases of $q = 0.95$ and $0.99$.

this mode quasi-resonance mode, QRM, in the last section). Now, we propose a physical picture of the disappearance of QNM and the existence of QRMs.

First, let us discuss the case of the analytic model. We consider only $n = 0$ case for simplicity. By setting $n = 0$ in (11), we obtain

$$\nu = -i \frac{1 - 4\epsilon^2}{4L} \pm \frac{1 + 4\epsilon^2}{4L} \left(i \sqrt{1 - \mu^2}\right).$$

(21)

What is noticed from (21) is that QNM frequencies are almost determined by the width $L$ of the potential peak when $\epsilon \sim 0$, and they turn out to depend on $\epsilon^2$ as $\epsilon$ increases. Since the energy of QNMs is proportional to the real part of $\nu$, the larger the energy of QNMs is, the gentler the peak of the potential becomes. Then the peak will vanish at $\epsilon = 1/2$. According to the WKB analysis of QNM by Schutz and Will [4], QNMs can be regarded as the waves trapped by the peak of the potential. Therefore, in their picture, vanishing of the peak is equivalent to the potential being unable to trap any waves. This means that QNMs disappear.
Figure 7. Trajectory of QNM frequencies in $\nu$-plane.

Figure 8. Dependence of QRM frequencies on the charge of the black hole.

Figure 9. Fitting by a parabola $\nu_1 = c_0 + c_2 \epsilon^2$. The solid line shows the QNM trajectory of $q = 0$ and the dashing line shows the fitting curve.
Next, let us consider the black hole case in this picture. In the black hole case, the peak of the potential vanishes at, for example, \( \epsilon = 0.25 \) for \( q = 0 \) and \( \epsilon \sim 0.256 \) for \( q = 0.5 \). On the other hand, figure 3 and figure 6 indicate disappearance of QNMs approximately at \( \epsilon = 0.4 \sim 0.5 \). Thus, the above picture seems to be realized approximately. Because the QNM frequencies are complex numbers, it is unreasonable to think that the disappearance of QNMs perfectly coincides with vanishing of the peak of the real potential in general. Hence, it is natural to consider that the coincidence in the analytic model is accidental. However, the behaviours of QNMs to \( m \) in both cases resemble each other very well. Figure 4 shows that the trajectory of QNMs can be fitted by a parabola very well. That is, our analytic model imitates the black hole case precisely. Therefore, since the analytic model predicts the disappearance of QNMs, it is undeniable that there exist QRM in the case of black hole as well as the case of analytic model, and that QNMs will disappear as the field mass exceed a certain value.

We also examine the effect of the black hole charge. We can see the behaviours of QNMs and QRM when we change the black hole charge, \( q \), in figure 7 and figure 8. These figures indicate that QNM and QRM frequencies tend to get larger when \( q \) increases. This can be explained by the potential profiles. According to figure 3 and 4 when \( q \) increases, the peak of the potential becomes higher and the value of \( \epsilon \) at vanishing point of the peak becomes larger. Consequently, QNMs can exist at larger \( \epsilon \) and QRM frequencies get higher. These facts, too, support our picture of the disappearance of QNMs.

5. Summary

Let us summarise our results. First, we studied massive QNMs in the analytical model, and derived QNM frequencies analytically. By this analysis, we found that QNMs will disappear when the field mass increases and exceeds a certain value. Secondly, we examined the dependence of QNM frequencies on the field mass by numerical method in the case of black hole. There, we again found the disappearance of QNMs as we did in the analytic model case. We also studied the dependence of QNM frequencies on the charge of the black hole. Based on these examinations, we proposed a picture about the disappearance of QNMs.

Appendix A. Continued fraction method

In the appendix, we briefly review the continued fraction method, which is first developed by Leaver[3].

First, let \( x_{\pm} \) be the solution of \( f(x) = 0 \) in (15). Then, the points \( x = 0, x_{-}, x_{+} \) are regular singularities and the point \( x = \infty \) is an irregular singularity of the differential equation (15). Note also that \( x = x_{-} \) corresponds to the inner horizon of the black hole and \( x = x_{+} \) corresponds to the outer horizon. QNMs on a Reisner-Nordström black hole spacetime exist in regions connected to the spatial infinity of the spacetime.
These regions are shown as $\Lambda$, $\Lambda'$ in the conformal diagram (figure A1). Therefore, the boundary conditions for QNMs on this spacetime are imposed at $x = x_+$ and at $x = \infty$.

Let us expand the solution of (15) as follows,

$$Z(u) = e^{iu} u^\rho (x - x_-)^b x \sum_{n=1}^\infty a_n u^n,$$

where

$$u = \frac{x - x_+}{x - x_-},$$
$$\rho^2 = \chi^2 - \epsilon^2,$$
$$\rho = -i\chi \frac{x_+^2}{x_+ - x_-},$$
$$b = -1 + \frac{i}{2\nu}(\nu^2 + \chi^2)(x_+ + x_-).$$

Substituting (A.1) into (15), we obtain the following recursion relation,

$$A_1 a_1 + B_1 a_0 = 0,$$
$$A_n a_n + B_n a_{n-1} + C_n a_{n-2} = 0,$$ for $n = 2, 3, \cdots$

where

$$A_n = n^2(x_+ - x_+) + 2inx_+^2\chi,$$
$$B_n = -2n^2(x_+ - x_+)$$
$$+ n \times \left\{ i(x_+ - x_+) \left[ \frac{\chi^2}{\nu}(x_- + x_+) - \nu(x_- - 3x_+) \right] + 2(x_- - x_+ - 2ix_+^2\chi) \right\}$$
$$+ \left\{ -l(l+1) + 1 \right\}(x_- - x_+).$$
$$\begin{align*}
+ \left[ \nu^2(x_- - x_+) + 2i\chi - (x_- + 3x_+)\chi^2 \right] x_+^2 \\
- \frac{i}{2}(x_- - x_+ - 2ix_+^2\chi) \left[ \frac{\chi^2}{\nu} (x_- + x_+) - \nu(x_- - 3x_+) \right] \right \}, \quad (A.5)
\end{align*}$$

\[ C_n = n^2(x_- - x_+) \]

$$\begin{align*}
+ n \times \left[ -i\nu + \frac{\chi^2}{\nu} \right] (x_-^2 - x_+^2) - 2(x_- - x_+ - ix_+^2\chi) \\
+ \left\{ -\frac{1}{4} \left( \nu^2 + \frac{\chi^2}{\nu^2} \right) (x_-^2 - x_+^2) (x_- + x_+) \\
+ i \left( \nu + \frac{\chi^2}{\nu} \right) (x_- + x_+) (x_- - x_+ - ix_+^2\chi) \\
+ \left[ (x_- - x_+) - 2ix_+^2\chi + \frac{1}{2} (x_- + x_+) (x_-^2 + 3x_+^2)\chi^2 \right] \right \}. \quad (A.6)
\end{align*}$$

Using these coefficients \( A_n, B_n \) and \( C_n \), QNM frequencies are given by the vanishing point of the following continued fraction equation,

\[ 0 = B_1 - \frac{A_1C_2 A_2C_3 \ldots A_{n-1}C_n}{B_2 - \frac{B_3 - \ldots}{B_n - \ldots}}. \quad (A.7) \]

For practical calculation, we must choose \( n \) for the needed accuracy and solve the algebraic equation of finite but very large degrees.

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