Perturbative quantum gravity in Batalin-Vilkovisky formalism

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In this Letter we consider the perturbative quantum gravity on the super-manifold which remains invariant under absolutely anticommuting BRST and anti-BRST transformations. In addition to that the theory posses one more symmetry known as shift symmetry. The BRST invariant Batalin-Vilkovisky (BV) action for perturbative quantum gravity is realized as a translation in Grassmann coordinate. Further we show that the quantum master equation of the BV quantization method at one-loop order can be translated to have a superfield structure for the action. However, the BRST as well as anti-BRST invariant BV action is constructed in superspace with the help of two Grassmann coordinates.

I. INTRODUCTION

The gauge theories are one of the basic ingredients in the search for a description of the fundamental interactions involving elementary particles. Gauge invariance of such theories is translated at the quantum level into the fermionic rigid BRST invariance [1,2] and is important in the proof of unitarity, renormalizability and other aspects of field theories [1–4]. The perturbative quantum gravity as a gauge theory is a subject of current research with different aspects [5–7]. Such models of gravity were initially studied as attempts to unify gravity with electromagnetism [8]. These are studied also due to their relevance in string theory [9].

On the other hand BV formulation is known to be one of the most powerful method of quantization for different gauge field theories, supergravity theories and topological field theories in Lagrangian formulation [3,4,12–17]. The BRST and the anti-BRST symmetries for perturbative quantum gravity in four dimensional flat spacetime have been studied by many people [18–20] and their work has been summarized by N. Nakanishi and I. Ojima [21]. The BRST symmetry in two dimensional curved spacetime has been thoroughly studied [22–24]. The BRST and the anti-BRST symmetries for topological quantum gravity in curved spacetime have also been studied [25,26]. However, a superspace formalism for the BV action of different gauge theories has been studied recently [15,27–29] and we will generalize the such results in the case of perturbative quantum gravity in curved spacetime.

In this Letter we analyse the BRST as well as anti-BRST transformations for perturbative quantum gravity which are absolutely anticommuting in nature. Further it is shown that the Lagrangian density for such theory also remains invariant under shift symmetry transformations. The BRST and anti-BRST transformations collectively with shift symmetry transformations are, known as extended BRST and extended anti-BRST transformations respectively, constructed extensively. In this formulation it is shown that the antifields in BV formulation get proper identification naturally through using equations of motion for auxiliary fields. The superspace formulation of extended BRST invariant perturbative quantum theory is constructed with help of one Grassmann coordinate. The superspace description of quantum master equation at one-loop order is also analysed. However to develop the superspace formulation for both extended BRST as well as anti-BRST invariant perturbative quantum theory we need two Grassmann coordinates.

The plan of the Letter is as follows. In section II, we study the preliminaries about the perturbative quantum gravity. The extended BRST invariant theory of gravity is constructed in section III. In section IV, we establish a superspace formulation for such an extended BRST invariant theory. The section V, is devoted to study the superspace description of quantum master equation involved BV technique. The extended anti-BRST transformation is explored in the section VI. In section VII, we construct the both

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extended BRST and anti-BRST transformations for such theory in superspace. In the last section we discuss the results and draw the conclusions.

II. THE PERTURBATIVE QUANTUM GRAVITY: PRELIMINARILY

We start with the Lagrangian density for pure gravity given by

\[ \mathcal{L} = \sqrt{g}(R - 2\lambda), \]

where \( \lambda \) is a cosmological constant and the units are chosen such that \( 16\pi G = 1 \).

In perturbative gravity one writes the full metric \( g_{\alpha\beta}^f \) in terms of a fixed background metric \( g_{\alpha\beta} \) and small perturbations around it. The small perturbation around the fixed background metric, denoted by \( h_{\alpha\beta} \), is considered as a field that is to be quantized. So, we can write

\[ g_{\alpha\beta}^f = g_{\alpha\beta} + h_{\alpha\beta}. \]

The Lagrangian density given in Eq. (1) remains invariant under following gauge transformation

\[ \delta_{\Lambda} h_{\alpha\beta} = \nabla_{\alpha} \Lambda_{\beta} + \nabla_{\beta} \Lambda_{\alpha} + \mathcal{L}(\Lambda) h_{\alpha\beta}, \]

where the Lie derivative for \( h_{\alpha\beta} \) is given by

\[ \mathcal{L}(\Lambda) h_{\alpha\beta} = \Lambda^c \nabla_c h_{\alpha\beta} + h_{ac} \nabla_b \Lambda^c + h_{cb} \nabla_a \Lambda^c, \]

and therefore the theory for perturbative quantum gravity has some redundant degrees of freedom. These unphysical degrees of freedom give rise to constraints in the canonical quantization and divergences in the partition function in the path integral quantization. In order to remove the redundancy in degrees of freedom we need to fix the gauge by putting following gauge-fixing condition

\[ G[h]_a = (\nabla^b h_{ab} - k \nabla_a h) = 0, \]

where \( k \neq 1 \). For \( k = 1 \) the conjugate momentum corresponding to \( h_{00} \) vanishes and therefore the partition function diverges again. To ensure the unitarity a Faddeev-Popov ghost term is also needed.

The gauge-fixing term (\( \mathcal{L}_{gf} \)) and ghost term (\( \mathcal{L}_{gh} \)) corresponding to the above gauge-fixing condition are given by

\[ \mathcal{L}_{gf} = \sqrt{g} \left[ i b^a (\nabla^b h_{ab} - k \nabla_a h) \right], \]

\[ \mathcal{L}_{gh} = i \sqrt{g^c} \delta^b_c \nabla^c [\nabla_a c_b + \nabla_b c_a - 2kg_{ab} \nabla_c e^c + \mathcal{L}(\Lambda) h_{ab} - kg_{ab} g^{cd} \mathcal{L}(\Lambda) h_{cd}], \]

\[ \mathcal{L} = \mathcal{L} + \mathcal{L}_{gf} + \mathcal{L}_{gh}, \]

where \( M_{ab} = i \nabla_c [\delta^b_c \nabla_a + g_{ab} \nabla_c - 2kg_{ac} \nabla_b + \nabla_b h^c_a - h_{ab} \nabla^c - h^c_b \nabla_a - kg_{ab} g^{cf} (\nabla_b h_{cf} + h_{eb} \nabla_f + h_{fb} \nabla_e)]. \]

Now, the complete Lagrangian density is given by

\[ \mathcal{L}_C = \mathcal{L} + \mathcal{L}_{gf} + \mathcal{L}_{gh}, \]

which remains invariant under following BRST transformations,

\[ s h_{\alpha\beta} = -(\nabla_{\alpha} c_{\beta} + \nabla_{\beta} c_{\alpha} + \mathcal{L}(\Lambda) h_{\alpha\beta} ), \quad sc^a = -c_b \nabla^b c^a, \quad \bar{s}c^a = b^a, \quad s h^a = 0. \]

This Lagrangian density is also invariant under the following anti-BRST transformations,

\[ \bar{s} h_{\alpha\beta} = -(\nabla_{\alpha} \bar{c}_{\beta} + \nabla_{\beta} \bar{c}_{\alpha} + \mathcal{L}(\bar{\Lambda}) h_{\alpha\beta} ), \quad \bar{s}c^a = -\bar{c}_b \nabla^b c^a, \quad \bar{s}c^a = -b^a, \quad \bar{s} h^a = 0. \]
It is easy to check that the above BRST and anti-BRST transformations are nilpotent as well as absolutely anticommuting in nature i.e.

\[ s^2 = 0, \quad s^2 = 0, \quad s\bar{s} + \bar{s}s = 0. \]  

(11)

Now, we express the gauge-fixing and ghost part of the complete Lagrangian density as follows,

\[ \mathcal{L}_g = \mathcal{L}_{gf} + \mathcal{L}_{gh}, \]

\[ = i\sqrt{g}[\bar{c}^a(\nabla b h_{ab} - k\nabla_a h)], \]

\[ = -i\sqrt{g}[\bar{c}^a(\nabla b h_{ab} - k\nabla_a h)], \]

\[ = -\frac{1}{2}i\sqrt{g}(h_{ab}^b h_{ab}), \]

\[ = \frac{1}{2}i\sqrt{g}(h_{ab}^b h_{ab}). \]

(12)

In the BV formalism, the gauge-fixing and ghost part of the Lagrangian density is generally expressed in terms of BRST variation of a gauge-fixed fermion. It is straightforward to write the \( \mathcal{L}_g \) given in Eq. (12) in terms of gauge-fixed fermion \( \Psi \) as

\[ \mathcal{L}_g = s\Psi, \]

(13)

where the expression for \( \Psi \) is

\[ \Psi = i\sqrt{g}[\bar{c}^a(\nabla b h_{ab} - k\nabla_a h)]. \]

(14)

Now, we will analyse the extended BRST transformation for the perturbative quantum gravity.

### III. THE BV FORMALISM WITH EXTENDED BRST TRANSFORMATION

In this section we study the extended BRST transformations for perturbative quantum gravity in the BV context. To do so we extend the BV action by shifting the all fields as follows

\[ \tilde{\mathcal{L}}_g = \mathcal{L}_g(h_{ab} - \tilde{h}_{ab}, c^a - \tilde{c}^a, \bar{c}^a - \tilde{\bar{c}}^a, b^a - \tilde{b}^a). \]

(15)

The above Lagrangian density remains invariant under following extended BRST transformations,

\[ sh_{ab} = \psi_{ab}, \quad \tilde{s}h_{ab} = \tilde{\psi}_{ab} + (\nabla_a c_b - \tilde{\nabla}_a \tilde{c}_b + \nabla_b c_a - \tilde{\nabla}_b \tilde{c}_a + \mathcal{L}(c)h_{ab} - \mathcal{L}(\tilde{c})\tilde{h}_{ab}), \]

\[ sc^a = \rho^a, \quad \tilde{s}c^a = \rho^a + c_b \nabla_b c^a - \tilde{\nabla}_b \tilde{c}_a, \quad \tilde{sc}^a = B^a, \quad sc^a = \tilde{B}^a - b^a + \tilde{b}^a, \]

\[ sb^a = \chi^a, \quad \tilde{sb}^a = \tilde{\chi}^a, \quad \tilde{sc}[\psi_{ab}, \rho^a, B^a, \chi^a] = 0, \]

(16)

where the fields \( \psi_{ab}, \rho^a, B^a \) and \( \chi^a \) are the associated ghost for the shift symmetric fields. Further we need to introduce the antighost fields \( h^*_{ab}, c^*_a, \bar{c}^*_a \) and \( b^*_a \) having opposite statistics to that of the respective fields. The nilpotent BRST transformation for the antighost fields are given by

\[ sh^*_{ab} = -l_{ab}, \quad sc^*_{a} = -m_a, \quad sc^*_{a} = -n_a, \quad sb^*_{a} = -r_a, \quad s[l_{ab}, m_a, n_a, r_a] = 0, \]

(17)

where \( l_{ab}, m_a, n_a \) and \( r_a \) are the Nakanishi-Lautrup type auxiliary fields.

If we gauge fix the shift symmetry such that all the tilde fields vanish, then, of course, we recover our original theory. To achieve this we choose the following shifted gauge-fixed Lagrangian density

\[ \tilde{\mathcal{L}}_g = -l_{ab}\tilde{h}^{ab} + h^{ab}(\psi_{ab} + \nabla_a c_b - \tilde{\nabla}_a \tilde{c}_b + \nabla_b c_a - \tilde{\nabla}_b \tilde{c}_a + \mathcal{L}(c)h_{ab} - \mathcal{L}(\tilde{c})\tilde{h}_{ab}) \]

\[ + m_a c^a - c_a(B^a - b^a + \tilde{b}^a) - n_a \bar{c}^a + \bar{c}^a(\rho^a + c_b \nabla_b c^a - \tilde{\nabla}_b \tilde{c}_a) \]

\[ + r_a \tilde{b}^a + b^*_a \chi^a, \]

(18)
which remains invariant under the extended BRST symmetry transformations given in Eqs. (19) and (17).

Now, it is straightforward to check that using equations of motion of auxiliary fields, $l_a, m_a, n_a, r_a$, all the tilde fields disappear from the above expression. The extended Lagrangian density $\tilde{L}_g$ then remains with the following form:

$$\tilde{L}_g = -h^{ab}_{\alpha}(\psi_{ab} + \nabla_a c_b + \nabla_b c_a + \mathcal{L}(c) h_{ab})$$

$$- c_a^*(B^a - b^a) + c_b^*(c_b^b c^a) + b^a_{\alpha} \chi^a.$$  \hspace{1cm} (19)

The gauge-fixed fermion $\Psi$ depends only on the original fields, then a general gauge-fixing Lagrangian density for perturbative quantum gravity with original BRST symmetry will have the following form:

$$L_g = s \Psi = s h_{ab} \frac{\delta \Psi}{\delta h_{ab}} + s c_a \frac{\delta \Psi}{\delta c_a} + s b_a \frac{\delta \Psi}{\delta b_a},$$

$$= -\frac{\delta \Psi}{\delta h_{ab}} \psi_{ab} + \frac{\delta \Psi}{\delta c_a} \rho_a - \frac{\delta \Psi}{\delta b_a} \chi_a.$$ \hspace{1cm} (20)

Now, the total Lagrangian density $L_T = L_0 + L_g + \tilde{L}_g$ is then given by

$$L_T = \sqrt{g}(R - 2\lambda) - h^{ab}_{\alpha}(\nabla_a c_b + \nabla_b c_a + \mathcal{L}(c) h_{ab}) - c_a^* b^a + c_b^*(c_b^b c^a)$$

$$- \left( h^{ab}_{\alpha} + \frac{\delta h_{ab}}{\delta h_{ab}} \right) \psi_{ab} + \left( c_a^* + \frac{\delta \Psi}{\delta c_a} \right) \rho_a - \left( c_a^* + \frac{\delta \Psi}{\delta b_a} \right) B_a + \left( b_a^* + \frac{\delta \Psi}{\delta b_a} \right) \chi_a. \hspace{1cm} (21)$$

where we have used the Eqs. (11), (13) and (20). Integration over ghost fields associated with the shift symmetry leads to the following identification

$$h^{ab}_{\alpha} = -\frac{\delta \Psi}{\delta h_{ab}}, \hspace{0.5cm} c_a^* = \frac{\delta \Psi}{\delta c_a}, \hspace{0.5cm} b_a^* = \frac{\delta \Psi}{\delta b_a}. \hspace{1cm} (22)$$

For the gauge-fixed fermion given in Eq. (14), the antifields associated with the theory are identified as

$$h^{ab}_{\alpha} = i\sqrt{g}(\nabla_b c_a - k_{ab} c^c c^c), \hspace{0.5cm} c_a^* = 0, \hspace{0.5cm} b_a^* = 0.$$ \hspace{1cm} (23)

With these identification the total Lagrangian density reduces to the original theory for perturbative quantum gravity. Now, we are able to write the gauge-fixing part of the total Lagrangian density in terms of the BRST variation of a generalized gauge-fixed fermion, i.e.

$$L_g + \tilde{L}_g = s \left( h^{ab}_{\alpha} \tilde{h}_{ab} - c_a^* \tilde{c}^a + c_b^* \tilde{c}^b - b_a^* \tilde{b}^a \right). \hspace{1cm} (24)$$

As expected the ghost number of $(h^{ab}_{\alpha} \tilde{h}_{ab} - c_a^* \tilde{c}^a + c_b^* \tilde{c}^b - b_a^* \tilde{b}^a)$ is equal to $-1$. In the next section, we construct the superspace formulation of extended BRST invariant perturbative quantum gravity theory.

IV. EXTENDED BRST INVARIANT SUPERSPACE FORMULATION AT CLASSICAL LEVEL

In Ref. [15], a superspace formulation for the shifted field approach to the Batalin-Vilkovisky action at the classical level was presented for the case of the higher form gauge theory. Here we will review this formulation, presenting it in a general way for the perturbative quantum gravity.

To write a superspace formalism of the extended BRST invariant theory we consider a superspace with coordinates $(x^a, \theta)$ where $\theta$ is fermionic coordinate. In such a superspace the “superconnection” 2-form can be written as

$$\omega^{(2)} = \frac{1}{2!} H_{ab}(x, \theta)(dx^a \wedge dx^b) + C_a(x, \theta)(dx^a \wedge d\theta), \hspace{1cm} (25)$$
where $d$ is an exterior derivative and is defined as $d = dx^a \nabla_a + d \theta \partial_\theta$. The requirement for super curvature (field strength $F^{(3)} = d \omega^{(2)}$) to vanish along the $\theta$ direction restricts the component of the superfields to have following form

$$
\mathcal{H}_{ab}(x, \theta) = h_{ab}(x) + \theta(s h_{ab}) = h_{ab}(x) + \theta \psi_{ab},
$$

$$
\mathcal{C}_a(x, \theta) = c_a(x) + \theta(s c_a) = c_a(x) + \theta \rho_a. 
$$

(26)

In this formalism, the antighosts and auxiliary field have to be introduced as additional superfields of the form

$$
\tilde{\mathcal{C}}_a(x, \theta) = \tilde{c}_a(x) + \theta(s \tilde{c}_a) = \tilde{c}_a(x) + \theta \rho_a + \theta B_a = \tilde{c}_a(x) + \theta B_a - b_a.
$$

$$
\tilde{\mathcal{B}}_a(x, \theta) = \tilde{b}_a(x) + \theta(s \tilde{b}_a) = \tilde{b}_a(x) + \theta \chi_a.
$$

(27)

Similarly, we define all the superfields corresponding to the shifted fields involved in extended action as

$$
\mathcal{H}_{ab}^*(x, \theta) = h_{ab}^*(x) + \theta(s h_{ab}^*) = h_{ab}^*(x) - \theta l_{ab},
$$

$$
\mathcal{C}_a^*(x, \theta) = c_a^*(x) + \theta(s c_a^*) = c_a^*(x) - \theta m_a,
$$

$$
\mathcal{B}_a^*(x, \theta) = b_a^*(x) + \theta(s b_a^*) = b_a^*(x) - \theta r_a.
$$

(28)

Using Eqs. (28) and (29) we are able to write the following expressions

$$
\frac{\delta}{\delta \theta} \mathcal{H}_{ab}^* \mathcal{H}^{ab} = -l_{ab} \psi_{ab} - h_{ab}^* \psi_{ab} + \nabla_a c_b - \nabla_b c_a - \nabla_b \tilde{c}_a
$$

$$
+ \mathcal{L}_{(c)} h_{ab} - \mathcal{L}_{(c)} \tilde{h}_{ab},
$$

$$
\frac{\delta}{\delta \theta} \mathcal{C}_a \tilde{\mathcal{C}}^a = -n_a \tilde{c}_a + \tilde{c}_a \rho_a + c_a \nabla^b c_a - \tilde{c}_a \nabla^b \tilde{c}_a - \nabla^b \tilde{c}_a
$$

$$
- \delta \frac{\delta}{\delta \theta} \mathcal{B}_a \tilde{\mathcal{B}}^a = r_a \tilde{b}_a + b_a \chi_a.
$$

(30)

Here we notice that the gauge-fixed Lagrangian density for shift symmetry given in Eq. (13) can be written in the superspace formulation as

$$
\hat{\mathcal{L}}_g = \frac{\delta}{\delta \theta} \left[ \mathcal{H}_{ab}^* \mathcal{H}^{ab} + \mathcal{C}_a^* \tilde{c}_a - \mathcal{C}_a^* \tilde{c}_a - \mathcal{B}_a^* \tilde{b}_a \right].
$$

(31)

Being the $\theta$ component of superfields the extended Lagrangian density $\hat{\mathcal{L}}_g$ remains invariant under the extended BRST transformation. If the gauge-fixing fermion depends only on the original fields, then one can define the fermionic superfield $\Gamma$ as

$$
\Gamma = \Psi + \theta s \Psi,
$$

$$
= \Psi + \theta \left[ -\frac{\delta \Psi}{\delta h_{ab}} \psi_{ab} + \frac{\delta \Psi}{\delta c_a} \rho_a + \frac{\delta \Psi}{\delta c_a} B_a - \frac{\delta \Psi}{\delta b_a} \chi_a \right].
$$

(32)
With these realization the original gauge-fixing Lagrangian density $L_g$ in the superspace formalism can be expressed as

$$L_g = \frac{\delta \Gamma}{\delta \theta}.$$  

(33)

Further we notice that invariance of $L_g$ under the extended BRST transformation is assured as it is $\theta$ component of super gauge-fixing fermion.

V. THE QUANTUM MASTER EQUATION OF PERTURBATIVE QUANTUM GRAVITY IN SUPERSPACE

In this section we first investigate the BRST variation of the quantum action in the standard BV quantization method. Then we will analyse the corresponding behavior in the superspace approach with one Grassmannian coordinate. For this purpose we first define the vacuum functional for perturbative quantum gravity with ordinary field in BV formulation as

$$Z_\Psi = \int \prod D\Phi \exp \left[ \frac{i}{\hbar} W \left( \Phi, \Phi^* = \frac{\partial \Psi}{\partial \Phi} \right) \right],$$  

(34)

where $\Phi$ and $\Phi^*$ are the generic notation for all fields and corresponding antifields involved in the theory. However, $W$ is the extended action corresponding to the Lagrangian density given in Eq. (8).

The condition of gauge independence of generating functional is translated into the so-called quantum master equation given by

$$\frac{1}{2} (W, W) = i\hbar \Delta W,$$  

(35)

where the antibracket $(F,G)$ and operator $\Delta$ are defined as

$$(X,Y) = \frac{\partial_r F}{\partial \Phi} \frac{\partial_l G}{\partial \Phi^*} - \frac{\partial_r F}{\partial \Phi} \frac{\partial_l G}{\partial \Phi^*},$$
$$\Delta = \frac{\partial_r}{\partial \Phi} \frac{\partial_l}{\partial \Phi^*}.$$  

(36)

The quantum action can be extended up to the one-loop order correction as

$$W(\Phi, \Phi^*) = S_C(\Phi, \Phi^*) + \hbar M_1(\Phi, \Phi^*),$$  

(37)

where $S_C$ is the complete action for Lagrangian density given in Eq. (8) and $M_1$ appears from nontrivial measure factors.

The behavior of $W$ with respect to BRST transformations can be given by

$$sW = i\hbar \Delta W.$$  

(38)

For non-anomalous gauge theory up to first-order correction $M_1$ does not depend on antifields. In this situation, the BRST transformations of the complete action $S_C$ and $M_1$ are given by

$$sS_C = 0, \quad sM_1 = i\Delta S_C.$$  

(39)

Now we apply the $\Delta$ operator on total action (along with shifted fields) as

$$\Delta S_C = \Delta S_T = \frac{\partial_r}{\partial \Phi} \frac{\partial_l}{\partial \Phi^*} S_T,$$  

(40)
here Φ and Φ* includes all the fields, shifted fields, ghosts and corresponding antighosts fields. Therefore, at one-loop order, we must build up a superfield
\[ M_1 = M_1 + \theta i \Delta S_T. \]  
(41)

However, the extended action in the superspace will have following form
\[ W = W + \theta i \hbar \Delta W. \]  
(42)

The Δ operator in the superspace with one Grassmann coordinate can be defined as
\[ \tilde{\Delta} = \partial_r \partial_{\Phi(x,\theta)} \partial_l \partial_{\Phi^*(x,\theta)}. \]  
(43)

The quantum master equation in superspace is simply described by
\[ \frac{\partial}{\partial \theta} W = i \hbar \tilde{\Delta} W. \]  
(44)

Here we notice that the role of generator of BRST transformation is essentially played by the differentiation with respect to θ. Therefore, enlarging the configuration space with the variable θ, we are equipping it with Grassmannian translations that reproduce the effect of the antibrackets given in Eq. (35).

VI. EXTENDED ANTI-BRST TRANSFORMATION

Let us next generalize the anti-BRST transformation given in Eq. (10). For that purpose we construct the extended BRST transformation under which the extended Lagrangian density remains invariant as follows
\[
\begin{align*}
\bar{s}h_{ab} &= h_{ab}, \quad \bar{s}l_{ab} = 0, \quad \bar{s}m_a = m_a, \quad \bar{s}n_a = 0, \quad \bar{s}r_a = 0, \\
\bar{s}c^a &= c^a, \quad \bar{s}c_a = c_a^*, \quad \bar{s}c^* = c^a, \quad \bar{s}\tilde{c}^a = \tilde{c}_a^*, \quad \bar{s}\tilde{c}_a = \tilde{c}_a, \quad \bar{s}k_a = k_a, \quad \bar{s}k_a^* = k_a^*, \quad \bar{s}l_{ab} = 0, \quad \bar{s}m_a = 0, \quad \bar{s}n_a = 0, \quad \bar{s}r_a = 0.
\end{align*}
\]  
(45)

We note that the above anti-BRST transformation absolutely anticommute with the extended BRST transformation given in Eq. (10), i.e. \{s, s\} = 0.

The anti-gauge-fixing fermion \( \bar{\Psi} \) (gauge-fixing fermion in case of anti-BRST transformation) for such theory is defined as
\[ \bar{\Psi} = -i \sqrt{\gamma} c^a (\nabla^b h_{ab} - k \nabla_a h). \]  
(46)

The gauge-fixing as well as the ghost part of the Lagrangian density can be written in terms of anti-BRST variation of \( \bar{\Psi} \) as
\[ \mathcal{L}_g = \bar{s} \bar{\Psi}. \]  
(47)

The ghost fields associated with the shift symmetry have the following extended anti-BRST transformations,
\[
\begin{align*}
\bar{s}a_{ab} &= l_{ab}, \quad \bar{s}a_a = m_a, \quad \bar{s}a \mathbb{B}_a = n_a, \quad \bar{s}a a_a = r_a, \\
\bar{s}l_{ab} &= 0, \quad \bar{s}m_a = 0, \quad \bar{s}n_a = 0, \quad \bar{s}r_a = 0.
\end{align*}
\]  
(48)

The transformations given in Eqs. (45) and (48) are the extended anti-BRST transformations under which the Lagrangian density with shifted fields remains invariant.
VII. EXTENDED BRST AND ANTI-BRST INVARIANT SUPERSPACE FORMULATION

The extended BRST and anti-BRST invariant Lagrangian density is written in superspace with the help of two Grassmannian coordinates $\theta$ and $\bar{\theta}$. All the superfields in this superspace are the function of coordinates $(x^a, \theta, \bar{\theta})$. The “super connection” 2-form $(\omega^{(2)})$ is defined in this case as

$$\omega^{(2)} = \frac{1}{2!} H_{ab}(x^a, \theta, \bar{\theta}) (dx^a \wedge dx^b) + C_{a}(x, \theta, \bar{\theta}) (dx^a \wedge d\theta) + \bar{C}_{\dot{a}}(x, \theta, \bar{\theta}) (dx^a \wedge d\bar{\theta}).$$

(49)

Here the exterior derivative has the following form $d = dx^a \nabla_a + d\theta \nabla_\theta + d\bar{\theta} \nabla_{\bar{\theta}}$.

Requiring the field strength to vanish along all extra directions $\theta$ and $\bar{\theta}$ determines the superfields to have the following forms

$$H_{ab}(x^a, \theta, \bar{\theta}) = h_{ab}(x^a) + \theta \psi_{ab} + \bar{\theta} [h_{ab}^* - (\nabla_a \tilde{c}_b - \tilde{\nabla}_a \tilde{c}_b + \tilde{\nabla}_b \tilde{c}_a - \tilde{\nabla}_b \tilde{c}_a)]$$

$$+ \frac{1}{2!} L^c h_{ab} - L^c \tilde{h}_{ab})] + \theta \bar{\theta} l_{ab} - \nabla_a \tilde{b}_b + \nabla_b \tilde{b}_a$$

$$- \nabla_b \bar{a}_a - s(L_c h_{ab}) + s(L_c \tilde{h}_{ab})],$$

$$C_a(x^a, \theta, \bar{\theta}) = c_a(x^a) + \theta [\rho_a + \tilde{c}_a^\ast - b_a + \tilde{b}_a] + \theta \bar{\theta} m_a,$$

$$\tilde{C}_a(x^a, \theta, \bar{\theta}) = \tilde{c}_a(x^a) + \theta [\bar{\rho}_a - \bar{c}_a^\ast - \bar{c}_b \bar{\nabla}_b \bar{c}_a - \bar{\nabla}_b \bar{c}_a] + \theta \bar{\theta} m_a,$$

$$\bar{C}_a(x^a, \theta, \bar{\theta}) = \bar{c}_a(x^a) + \bar{\theta} [C_a - b_a + \bar{b}_a]$$

$$+ \bar{\theta} \bar{\theta} n_a + \bar{\theta} \bar{\theta} r_a.$$}

(50)

Using the above expressions the Lagrangian density $\tilde{L}_g$ given in Eq. (18) can be written as

$$\tilde{L}_g = \frac{1}{2!} \frac{\delta}{\delta \theta} \frac{\delta}{\delta \bar{\theta}} \left[ H_{ab} \tilde{H}^{ab} + \tilde{C}_{\dot{a}} \tilde{L}^{\dot{a}} - \tilde{C}_{\dot{a}} \tilde{L}^{\dot{a}} - \tilde{B}_{\dot{a}} \tilde{B}^{\dot{a}} \right].$$

(51)

Being the $\theta \bar{\theta}$ component of the superfields the $\tilde{L}_g$ remains invariant under extended BRST as well as extended anti-BRST transformations. Furthermore we define the super gauge-fixed fermion as

$$\Gamma(x, \theta, \bar{\theta}) = \Psi + \theta (s \Psi) + \bar{\theta} (s \bar{\Psi}) + \theta \bar{\theta} (s s \Psi),$$

(52)

to express the $L_g$ as $\frac{1}{4} [\Gamma(x, \theta, \bar{\theta})]$. The $\theta \bar{\theta}$ component of $\Gamma(x, \theta, \bar{\theta})$ vanishes due to equations of motion in the theories having both BRST and anti-BRST invariance.

The complete gauge-fixing and ghost part of the Lagrangian density which is invariant under both extended BRST and extended anti-BRST transformations can therefore be written as

$$L_g + \tilde{L}_g = - \frac{1}{2!} \frac{\delta}{\delta \theta} \frac{\delta}{\delta \bar{\theta}} \left[ H_{ab} \tilde{H}^{ab} + \tilde{C}_{\dot{a}} \tilde{L}^{\dot{a}} - \tilde{C}_{\dot{a}} \tilde{L}^{\dot{a}} - \tilde{B}_{\dot{a}} \tilde{B}^{\dot{a}} \right] + \frac{\delta}{\delta \bar{\theta}} [\Gamma(x, \theta, \bar{\theta})].$$

(53)

Using equations of motion for auxiliary fields and the ghost fields associated with shift symmetry the antifields can be calculated. With these antifields the original gauge-fixed Lagrangian density in BV formulation can be recovered.

Here we note that the one can generalize the quantum master equation of BV formalism in superspace for two Grassmannian coordinates following the section V.
VIII. CONCLUSIONS

Although the superspace formulation for gauge theory have been done, we have generalized it for perturbative quantum gravity. The BV formulation represents a very powerful framework for the quantization of the most general gauge theories where the gauge algebra is open or closed. We have considered the perturbative quantum gravity in such framework. In such formalism one extends the configuration space by introducing antifields corresponding to all the original fields present in the theory. These antifields get identification with the functional derivative of a gauge fixing fermion with respect to the corresponding fields. We have analysed the extended BRST and anti-BRST transformations (including the shift symmetry transformations) for this theory where we have shown that the antifields get its proper identification automatically using equations of motion of auxiliary fields. Further, it has been shown that to write the superspace formulation of extended BRST invariant theory one needs one fermionic coordinate. We have shown that the master equation of the BV formalism can be represented as the requirement of a superspace structure for the quantum action. The quantum master equation at one-loop order has realized as a translation in $\theta$ variable. However, for both extended BRST and anti-BRST invariant theory the superspace description need two Grassmannian coordinates. In BV formalism antifields dependent terms (quantum corrections) of master equation must satisfy the consistency conditions proposed by Wess-Zumino as the theory becomes anomalous. The study of anomalies with antifields included in the action is based on the Zinn-Justin version [30] of BV master equation. It will be interesting to analyse the perturbative theory of quantum gravity in superspace for higher order in $\hbar$ where the quantum corrections depend on antifields.

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