We work out the relation between automorphic forms on $SO(2+s,2,\mathbb{Z})$ and gauge one-loop corrections of heterotic $K3\times T^2$ string compactifications for the cases $s = 0, 1$. We find that one-loop gauge corrections of any orbifold limit of $K3$ can always be expressed by their instanton numbers and generic automorphic forms. These functions classify also one-loop gauge thresholds of $N=1$ $(0,2)$ heterotic compactifications based on toroidal orbifolds $T^6/\mathbb{Z}_\nu$. We compare these results with the gauge couplings of $M$–theory compactified on $S^1/\mathbb{Z}_2 \times T^6/\mathbb{Z}_\nu$ using Witten’s Calabi–Yau strong coupling expansion.
1. Introduction

Within the last years, many perturbative calculations have been accumulated in heterotic $K3 \times T^2$ compactifications, whose scalar field sector of $N=2$ vector multiplets contains, besides the generic $S, T, U$–moduli, which describe the dilaton, the size and shape of the torus $T^2$, respectively, in addition Wilson line moduli. The latter parametrize non–trivial gauge background fields w.r.t. to the internal torus $T^2$. These results –summarized for the two derivative couplings in a holomorphic prepotential $H$ and two other functions– lead to a control of all perturbative one–loop corrections in the gauge sector $[1][2][3][4][5]$, in the gravitational sector $[3][4][6][7][8]$ and the one–loop Kähler corrections via the prepotential $[9][10][11]$. $N=2$ heterotic–type-II duality $[12]$ links the heterotic prepotential, given as sum over trilogarithms $[3]$ with the type-II prepotential, given as weighted instanton expansions $[13]$. The main evidence for the equivalence of a pair with a rank three gauge group was the appearance of the $j$–function, the automorphic function of the perturbative duality group of the heterotic side, in the functional dependence of the CY couplings at a certain boundary of the CY moduli space, which has been identified with the weak coupling limit of the heterotic string. The appearance of automorphic functions of subgroups of $SL(2, \mathbb{Z})$, the typical dependence of the perturbative heterotic couplings, is a general phenomenon in CY spaces of a special fibration structure which has been realized in $[14]$. Moreover it was demonstrated there, that this K3 fibration structure implies the appearance of automorphic functions of modular groups of more variables, a mathematically surprising fact which was subsequently explored in $[16]$. In this way type-II heterotic duality imposes surprising relations of CY mirror maps to automorphic functions of heterotic duality groups, e.g. $SO(2+s,2,\mathbb{Z})$. Usually on the heterotic side, couplings are calculated as power series in $exp(2\pi i S)$. These powers control the non–perturbative contributions coming from space–time instantons and their coefficients themselves are automorphic functions under the perturbative duality group $SO(2+s,2,\mathbb{Z})$ including exchange symmetries $[16][17]$. However, if the type-II heterotic duality provides information about CY periods in the perturbative heterotic regime, the opposite can be said about the strongly coupled phase. In $[14][18][19]$ heterotic–type-II duality was used to derive the non–perturbative duality group mixing the dilaton with the other moduli. In particular in this way one obtains the generalized automorphic functions which reduce to the perturbative heterotic ones at a special boundary of the CY moduli space. This limit corresponds

---

1 Some articles reporting $K3$ dynamics in the context of string–duality are the refs. $[15]$. 
to the large base–space limit of the K3–fibration on the type-II side \[20\]. In this limit
the automorphic functions on the heterotic side carry the information about the K3–fiber
of the type-II side, i.e. in particular the type of singularity on the K3. The simplest
cases are the two or three moduli cases with the perturbative duality groups \(SO(2,2,\mathbb{Z})\)
and \(SO(3,2,\mathbb{Z})\), respectively. Generic formulae may also be given for the higher groups
\(SO(2+s,2,\mathbb{Z})\), \(s \geq 2\). Whereas these symmetries often also appear in N=1 string vacua
(see e.g. \[21\]), not much is known about the underlying modular functions, which unify
N=1 amplitudes in a similar way as described above for the N=2 case. It is one aim of
this paper to work out such a correspondence for the gauge couplings of a class of N=1
string vacua. Due to the similarity of the N=1 and N=2 target space duality groups and
their underlying modular functions one might even guess, that again the relevant physics
can be traced back to \(K3 \times T^2\) dynamics. Therefore, the aim of this paper is twofold:
We find a quite extensive and unifying description of N=2, \(d = 4\) gauge threshold cor-
rections in terms of basic modular functions and spectrum dependent quantities. Second,
we derive general expressions –given as \(SO(2+s,2,\mathbb{Z})\) modular functions– for the one–
loop corrections to the gauge couplings in N=1, \(d = 4\) theories with (0,2) superconformal
symmetry on the world–sheet, realized as toroidal orbifolds. These are singular limits
of Calabi–Yau manifolds (CYMs). Although this represents a specialization to a certain
class of string compactifications, this limit allows us to extract concrete results about (0,2)
compactifications and the full one–loop gauge couplings, including gauge group dependent
and independent contributions. In contrast, for a smooth (2,2) CYM only the difference
of the \(E_6\) and \(E_8'\) one–loop corrections are known (given by the topological index). The
spectrum of toroidal orbifolds with N=1 space–time supersymmetry can be arranged into
N=1, N=2 and N=4 multiplets, respectively, depending on how their field representations
are twisted along the world–sheet torus. However, it is only the N=2 part, which gives
rise to moduli dependent perturbative gauge corrections. Therefore, all the moduli depen-
dence is encoded in \(K3 \times T^2\) dynamics and our study of \(K3 \times T^2\) gauge couplings may be
used, to classify the N=1 couplings. In particular this means, that these couplings may
be expressed by generic modular functions together with the instanton numbers of the
underlying \(K3 \times T^2\) compactifications, which represent N=2 subsectors of the full N=1
orbifold. Our main results are presented in eqs. (4.4) and (4.5). This description allows
us to recover the topological nature of these couplings. In particular, we are able to trace
back their origin to Green–Schwarz terms in ten dimensions or gauge couplings of eleven
dimensional M–theory. We give this link for N=1 orbifolds with both (2,2) and (0,2)
superconformal world–sheet symmetry. This link provides also generic methods for the
dimensional reduction on orbifolds, which is useful for further investigations in $M$–theory
phenomenology.

2. Gauge and gravitational one–loop corrections in heterotic $K3 \times T^2$ compactifications

We consider heterotic $K3 \times T^2$ compactifications, where the instantons are embed-
ded into $H \times H'$ subgroups$^2$ of $E_8 \times E_8$. Their instanton numbers, fulfill $n^{(1)} + n^{(2)} = \chi_{K3}, \chi_{K3} = 24$. Therefore we define $n^{(1)} := 12 - n, n^{(2)} := 12 + n$ referring to $H, H'$, respectively. This expresses the well–known fact, that for K3 compactifications, the in-
stanton numbers have to add up to 24, following from $\int_{K3} dH = 0$, which guarantees a
global well–defined 3–form $H$ on $K3$ and the Bianchi identity $dH = \text{tr}R^2 - v_a \text{tr}F^2_a$. The
remaining unbroken gauge group (the commutant $G'$ of $H'$) is denoted by $G \times G'$. In
general, such vacua have $(0, 4)$ world supersymmetry and N=2 space–time supersym-
metry in $d = 4$. The (new) supersymmetric index $Z(q, \bar{q})$ and variants of it are the basic
objects for string–amplitudes, which are obtained from it after taking the relevant order
in the fields $R$ and $F$ and integrating over its modular invariant part $[23]$. For the case of
vanishing Wilson lines ($s = 0$) and adjoint scalars the (new) supersymmetric index $Z(q, \bar{q})$
factorizes ($q = e^{2\pi i \tau}$)

$$Z(q, \bar{q}) = \eta^{-2}(q)\text{tr}_R \left[ q^{L_0 - \frac{c}{24} - \frac{c}{24}(F_L + F_R)} e^{\pi i (F_L + F_R)} \right]_{(c, \bar{c}) = (22, 9)} = Z_{K3}(q)Z_{2,2}(q, \bar{q}),$$

$$Z_{2,2}(q, \bar{q}, T, U) = \sum_{m_i, n_i} e^{2\pi i \tau(m_1n_1 + m_2n_2)} e^{-\frac{\pi i}{24} |TU n_2^2 + Tn_1^2 - Um_1 + m_2|^2},$$

(2.1)

into a holomorphic $K3$–part $Z_{K3}(q)$ and a generic lattice sum $Z_{2,2}$. In the cases under
consideration, $[2.1]$ refers to the point in the Coulomb branch, where the full gauge group
$(G, G')$ is present. In general $[s \neq 0; \text{cf. eq. (2.5)}]$, the supersymmetric index depends on
the topology of the manifold, e.g. $\chi_{K3}$ and the topology of the gauge bundle, e.g. $n^{(1)}, n^{(2)}$.
As a consequence it does not change under deformations of the hypermultiplet moduli
space. Thus we may do some change in the hypermultiplet moduli space by (un)Higgsing

$^2$ Also combinations $U(1) \times H$ of Abelian and non–Abelian backgrounds are possible. In that
case, the gauge group is of the form $U(1) \times G$ $[22]$. 

3
or moving in the instanton moduli space. Both effects result in a change of the gauge groups \((G, G')\). This way we may very easily move from models with standard instanton embedding \((SU(2) \text{ bundle in one } E_8)\) to non–standard embeddings, if compatible with the index. However, the net number of vector- and hypermultiplets, \(N_H - N_V = 240\) and the instanton numbers \(n^{(1)}, n^{(2)}\) do not change in perturbation theory. There are restrictions on the possibility of maximal Higgsing away \((G, G')\), which depend on the number \(n\): For \((n^{(1)}, n^{(2)}) = (24, 0)\), i.e. \(n = -12\) the second \(E_8\) cannot be broken at all, i.e. \(G' = E_8\). For \(n = 0, 1, 2\) complete Higgsing is possible. On the other hand, e.g. for \(-n = 3, 4, 6, 8,\) there are too few instantons or too less matter in the second \(E_8\) to break it completely, thus ending with the terminal gauge groups \(G = SU(3), SO(8), E_6, E_7\), respectively. In the cases \(-n = 9, 10, 11,\) i.e. \(n^{(2)} < 4\), the instantons on the second \(E_8\) are not stable and become small, because \(D\)–terms in six dimensions do not allow them to acquire a finite size. But then they also cannot break \(E_8\), thus \(G' = E_8\). The small instanton dynamics corresponds to a tensionless non–critical string theory in \(d = 6\) with \(E_8\) chiral algebra.

The index \((2.1)\) is worked out in \([3]\) for the case of \(SU(2)–\text{bundles in the } E_8\)'s with instanton numbers \(n^{(1)}, n^{(2)}\):

\[
Z_{K3}(q) = -2 \left[ \frac{n^{(1)}}{24} \frac{E_6 E_4}{\eta^{24}} + \frac{n^{(2)}}{24} \frac{E_4 E_6}{\eta^{24}} \right] = -2 \frac{E_4 E_6}{\eta^{24}}.
\]  

The relevant objects appearing in the one–loop corrections \(\Delta_a\) to the gauge kinetic term \(k_a g_{\text{string}} F^a_{\mu\nu} F^{a\mu\nu}\) in the low–energy effective action \([27]\)

\[
\Delta_a = \int \frac{d^2 \tau}{\tau_2} [B_a(\tau, \bar{\tau}) - b_a]
\]  

for the gauge couplings \(G_a = G, G'\) and the indices \((2.2)\) are

\[
B_G(\tau, \bar{\tau}) = -\frac{1}{12} \left[ \left( E_2 - \frac{3}{\pi \tau_2} \right) \frac{E_4 E_6}{\eta^{24}} - \frac{n^{(1)}}{24} \frac{E_4^3}{\eta^{24}} - \frac{n^{(2)}}{24} \frac{E_6^2}{\eta^{24}} \right] \mathbb{Z}_{2,2},
\]

\[
B_{G'}(\tau, \bar{\tau}) = -\frac{1}{12} \left[ \left( E_2 - \frac{3}{\pi \tau_2} \right) \frac{E_4 E_6}{\eta^{24}} - \frac{n^{(1)}}{24} \frac{E_6^2}{\eta^{24}} - \frac{n^{(2)}}{24} \frac{E_4^3}{\eta^{24}} \right] \mathbb{Z}_{2,2},
\]

3 In fact, Higgsing and changing the gauge bundle are on the same footing.

4 This follows from cancellation of the \(R^4\) anomaly in six dimensions. In six dimensions: \(N_H - N_V = 244\) for the models we are considering, i.e. with one tensor multiplet \(N_T = 1\).

5 Actually, in an orbifold limit of \(K3\), which is smoothly connected to \(K3\) by blowing up. This limit corresponds to going to special points in the hypermultiplet moduli space.
respectively for the case \(s = 0\). Due to the unique form of the \(K3\) supersymmetric index these functions are unique for all kinds of gauge threshold corrections without Wilson lines and lead to quite generic expressions for the one–loop corrections \([28] [3] [29]\). On the other hand, more general instanton backgrounds give rise to a much wider class of gauge threshold corrections than considered in \([28]\). The modular functions \(B_a\) appearing in \((2.4)\) are just descendents of the genus \(Z_{K3} (2.1)\). This means, that they are obtained from it by a \(q\)–derivative, which leads to the \((F^n)^2\)–part.

In the following we want to include Wilson lines. Wilson lines will allow us to read off the different instanton numbers \(n^{(1)}, n^{(2)}\) of a \(K3\) supersymmetric index with \(SU(2)\)–bundles. In the case with one Wilson line modulus \(V\) w.r.t. \(T^2\) \((s = 1)\) in an \(SU(2)\) subgroup of the second \(E_8^r\) the supersymmetric index \((2.1)\) takes the form \([7]\)

\[
Z(q, \bar{q}) = Z_{K3} \otimes Z_{3,2}(q, \bar{q}) = -2 \left( \frac{n^{(1)}}{24} \frac{E_6 E_{4,1}}{\eta^{24}} + \frac{n^{(2)}}{24} \frac{E_4 E_{6,1}}{\eta^{24}} \right) \otimes Z_{3,2}(q, \bar{q}) , \tag{2.5}
\]

with instanton numbers \((n^{(1)}, n^{(2)}) = (12 - n, 12 + n)\) w.r.t. \(SU(2)\)–bundles in both \(E_8\) and:

\[
Z_{3,2}(q, \bar{q}, T, U, V) = \sum_{m, n, k} e^{2\pi i q (\frac{1}{4} k^2 + m_1 n_1 + m_2 n_2)} e^{-2\pi \tau_2 |p_R|^2} \text{ with } Y = -\frac{1}{4} [(T - \bar{T})(U - \bar{U}) - (V - \bar{V})^2] = T_2 U_2 - V_2^2 \tag{2.6}
\]

\[
p_R = \frac{1}{\sqrt{2Y}} [(T U - V^2) n^2 + T n^1 - U m_1 + m_2 + kV] . \tag{2.6}
\]

In that case, the functions \((2.4)\) change to \((s = 1)\):

\[
B_G = -\frac{1}{12} \left[ \left( E_2 - \frac{3}{\pi \tau_2} \right) \left( \frac{n^{(1)}}{24} \frac{E_{4,1} E_6}{\eta^{24}} + \frac{n^{(2)}}{24} \frac{E_4 E_{6,1}}{\eta^{24}} \right) - \frac{n^{(1)}}{24} \frac{E_4^2 E_{4,1}}{\eta^{24}} - \frac{n^{(2)}}{24} \frac{E_6 E_{6,1}}{\eta^{24}} \right] \otimes Z_{3,2} \tag{2.7}
\]

\[
B_{G'} = -\frac{1}{12} \left[ \left( E_2 - \frac{3}{\pi \tau_2} \right) \left( \frac{n^{(1)}}{24} \frac{E_{4,1} E_6}{\eta^{24}} + \frac{n^{(2)}}{24} \frac{E_4 E_{6,1}}{\eta^{24}} \right) - \frac{n^{(1)}}{24} \frac{E_6 E_{6,1}}{\eta^{24}} - \frac{n^{(2)}}{24} \frac{E_{4,1} E_4^2}{\eta^{24}} \right] \otimes Z_{3,2} . \tag{2.7}
\]

The mixing of \(Z_{3,2}\) with \(Z_{K3}\) by the Wilson line \(V\) is formally denoted by the product \(\otimes\) and explained in appendix A. The \(G, G'\)–beta function coefficients may be determined to:

\[
b_G^{N=2} = 12 - 6n , \quad \quad b_{G'}^{N=2} = \begin{cases} 12 + 6n , & V = 0 \\ 12 + 4n , & V \neq 0 \end{cases} . \tag{2.8}
\]

Of course, the residual group \(G'\) and thus \(b_{G'}^{N=2}\) depend on the choice for the Wilson line. Statements made earlier (for \(s = 0\)) about changes in the hypermultiplet moduli
space remain valid. Since we want at least one $SU(2)$ gauge group in $G'$, in the course of Higgsing, the Wilson line modulus $V$ corresponding to the $U(1)$ Cartan subalgebra remains a flat direction.

The $\beta$–function coefficients (2.8) may be viewed (for $V = 0$) in the anomaly–polynomials $I_m$ and/or gauge kinetic terms in six dimensions (without additional tensors, which relax the factorization condition) [22][30][31][32][33]

\[ I_8 = I_4 I_4 = (\text{tr} R^2 - v_G \text{tr} F^2 - v_{G'} \text{tr} F^2_{G'}) \left( \frac{n^{(1)} - 12}{12} \text{tr} F^2 - v_{G'} \frac{n^{(2)} - 12}{12} \text{tr} F^2_{G'} \right) , \]

with the Kac–Moody levels $v_a = 2, 1, \frac{1}{3}, \frac{1}{6}, \frac{1}{30}$ for $a = SU(N), SO(2N), E_6, E_7, E_8$, respectively.

The first piece of (2.7) is universal and appears also in the gravitational one–loop correction

\[ \Delta_{\text{grav.}} = -\frac{1}{96\pi^2} \int \frac{d^2\tau}{\tau^2} \left[ (-2\hat{E}_2) \left( \frac{n^{(1)} E_{4,1} E_6}{24\eta^{24}} + \frac{n^{(2)} E_{4,1} E_6}{24\eta^{24}} \right) \otimes Z_{3,2} - b^{4d,N=2}_{\text{grav.}} \right] , \]

with the gravitational $\beta$–function coefficient:

\[ b^{4d,N=2}_{\text{grav.}} = 48 + 2(N'_H - N'_V) = \begin{cases} 528 , & V = 0 \\ 468 - 24n , & V \neq 0 \end{cases} . \]

The two cases differ by $2(N'_H - N'_V) = 60 + 24n$, with:

\[ N'_V = c_n(-1) = 2 , \quad N'_H = -c_n(-1/4) = 32 + 12n . \]

The numbers $c_n(N)$ are the coefficients in the expansion of the $K3$ index in (2.5) (cf. also appendix A)

\[ \hat{Z}_{K3}(q) = 2 \left( \frac{n^{(1)} E_{4,1} E_6}{24\eta^{24}} + \frac{n^{(2)} E_{4,1} E_6}{24\eta^{24}} \right) = \sum_{N \in \mathbb{Z} \frac{1}{4}} c_n(N) q^N . \]

3. **Gauge threshold corrections in orbifold limits of $K3 \times T^2$**

This section is devoted to a detailed analysis of the supersymmetric index (2.1) for $K3$ orbifolds. However, since the $K3$ index may be calculated in an orbifold limit, we non–freely acting discrete twists $\mathbb{Z}_\nu$ on the $T^4$–torus. We do not consider additional shifts in $T^2$. 

\[ \]
also obtain results for the $K3$ index, itself. Different orbifold limits of $K3$, correspond to
different points in the $K3$ moduli space. The orbifold limits are classified by their choice
of $T^4$–twists $\mathbb{Z}_\nu$ and accompanied shifts $(\gamma, \bar{\gamma})$ in the gauge degrees of freedom. It is one
of the main results of this section to show, that different orbifold limits of $K3$, i.e. different
sets of $(\nu, \gamma, \bar{\gamma})$ or different points in the $K3$–moduli space lead to the same supersymmetric
index as long as their instanton numbers $n$ are the same. For $K3$ vacua with $SU(2)$ bundles,
it is given by (2.2) for vanishing Wilson lines and (2.5) for the case with one Wilson line $V$.
In particular, models of one Higgs chain, which have the same orbifold limit, have the same
genus. For concreteness we only consider two cases: Vanishing Wilson lines $V_i = 0$, i.e. in
this case we sit at that point in the vector moduli space, where the (non) Abelian gauge
group is fully established. Then on the typeIIA side, Higgsing (with fundamental charged
matter) corresponds to extremal transitions between topological distinct CY vacua \cite{33} \cite{34}.
The second case with one non–vanishing Wilson line may be obtained from the first one
by going to the Coulomb branch of an $SU(2)$ subgroup of the full group. The instanton
numbers can be related to the shifts $(\gamma, \bar{\gamma})$ (cf. section 3.6). An immediate consequence
is that similar unifying statements hold for all kinds of physical amplitudes, which are
given by the supersymmetric index and only depend on the vector multiplets, e.g. gauge
and gravitational threshold corrections. However, threshold corrections to a gauge group,
which exist only in the orbifold limit, cannot be similarly unified (cf. section 3.4). One has
to be careful in the choice of non–vanishing Wilson lines and shifts $\gamma, \bar{\gamma}$: Only Wilson lines $V_i$ w.r.t. to the $K3$ gauge groups $G, G'$ represent directions in the smooth $K3$, which are
independent on the hypermultiplet moduli space. Nonvanishing Wilson lines $V_i \neq 0$ mean,
that we stick to a region of the vector multiplet moduli space, where the corresponding
$SU(2)$ gauge symmetry is in the Coulomb phase. On the other hand, when we go back
to the $K3$ limit, we move in the hypermultiplet moduli space and would Higgs away this
$U(1)$, if it belonged to the $K3$–bundle. In that case, the result does depend on the specific
orbifold limit. The non–trivial instanton background $H, H'$ of an orbifold may change
(however not the instanton numbers) when blowing up to a smooth $K3$. E.g. (as we will
see later) there is the $\mathbb{Z}_3$ orbifold with gauge group $U(1)^2 \times SO(14)^2$ and instanton numbers
$(12, 12)$. It has $U(1)$ instanton backgrounds, since the gauge group has rank 16. On the
other hand, since it does not appear in the list of \cite{22}, which shows all possible Abelian

\footnote{This was demonstrated for standard–embedding orbifolds in \cite{4}, whereas we also turn to
non–standard–embedding}
backgrounds of $K3$, we conclude that the $U(1)$–bundles convert to non–Abelian ones in the course of blowing up the $K3$. This is also manifest in the form (2.3) of the $K3$ elliptic genus (cf. table 2). Let us also mention, that an orbifold cannot always be blown up to a smooth $K3$ manifold. This happens, whenever there are not enough mass less oscillator modes, which serve as blowing up operators (see e.g. [35] for examples).

The supersymmetric index for an orbifold of twist order $\nu$ and gauge shifts $(\gamma, \tilde{\gamma})$ takes for generic points in the Coulomb moduli space $(T, U, V_i)$ the form [36]:

$$Z(q, \bar{q}) = -\frac{1}{4} \eta^{-20}(\tau) \sum_{(a, b)} \frac{1}{\nu} Z_{K3}^{(a, b)}(q) Z_{18,2}^{(a, b)}(q, \bar{q}) ,$$

(3.1)

with the corresponding twisted partition functions $(a, b = 0, \ldots, \nu - 1)$:

$$Z_{K3}^{(a, b)}(q) = k_{(a, b)} q^{-\frac{a^2}{2}} \eta^2(\tau) \Theta_1^{-2} \left( \frac{a}{\nu} + \frac{b}{\nu}, \tau \right)$$

$$Z_{18,2}^{(a, b)}(q, \bar{q}, T, U, V_i) = e^{-2\pi i \frac{ab}{\nu} (\gamma^2 + \tilde{\gamma}^2)} \sum_{p \in \Gamma_{18,2} + \frac{p}{\nu} (\gamma + \tilde{\gamma})} e^{2\pi i \frac{b}{\nu} p(\gamma + \tilde{\gamma})} q^{\frac{a}{2} p^2} \bar{q}^{\frac{b}{2} p^2} .$$

(3.2)

We introduced:

$$\theta \left[ \begin{array}{c} a \\ b \end{array} \right] (\tau) = \sum_{k \in \mathbb{Z}} q^{\frac{1}{2} (k+\frac{1}{2}a)^2} e^{i\pi(k+\frac{1}{2}a)b}$$

$$\Theta_1(z, \tau) = i \sum_{k \in \mathbb{Z}} (-1)^k q^{\frac{1}{2} (k-\frac{1}{2})^2} e^{2\pi i z(k-\frac{1}{2})} .$$

(3.3)

At the special points $V_i = 0$ the lattice sum $Z_{18,2}^{(a, b)}$ factorizes into

$$Z_{18,2}^{(a, b)} = Z_{2,2}^{(a, b)}(q, \bar{q}) e^{-2\pi i \frac{ab}{\nu} (\gamma^2 + \tilde{\gamma}^2)} Z_{E_8}^{(a, b)}(q) Z_{E'_8}^{(a, b)}(q) ,$$

with

$$Z_{E_8}^{(a, b)}(q) = \frac{1}{2} \sum_{\alpha, \beta} e^{-i\pi \beta \nu} \sum_{I=1}^{8} \gamma_I^{\alpha} \prod_{I=1}^{8} \theta^{\alpha + 2\frac{a}{\nu} \gamma_I}$$

(3.4)

and an analog expression for $Z_{E'_8}^{(a, b)}$. For the case with one non–vanishing Wilson line $V := V_{16} \neq 0$, the second lattice function becomes

$$Z_{E'_8}^{(a, b)}(q, y) = \frac{1}{2} \sum_{\alpha, \beta} e^{-i\pi \beta \nu} \sum_{I=1}^{8} \gamma_I^{\alpha} \prod_{I=1}^{6} \theta^{\alpha + 2\frac{a}{\nu} \gamma_I}$$

(3.5)

with the Jacobi functions defined in appendix A. For the models we are interested in, the explicit expression for the functions $B_A$ in (2.3) is easily derived from (3.4) and (3.5).
The charge $Q_A$ insertion is accomplished by the respective $q$–derivative on the $\theta$–function corresponding to the relevant $U(1)$–charge. Following [1], we define:
\[
-\frac{1}{4} 1 e^{-2\pi i \frac{ab}{2} (\gamma^2 + \bar{\gamma}^2)} \eta^{-20}(\tau) Z_{K^3}^{(a,b)}(\tau) =: \sum_{m \geq -1} c_A^{(a,b)}(m) q^m. \tag{3.6}
\]

With these definitions the $\beta$–function coefficients (2.8), which ensure that (2.3) remains IR–finite, are determined to be:
\[
b_A^{N=2} \equiv \lim_{\tau_2 \to \infty} B_A(\tau, \bar{\tau}) = \sum_{n_i} \sum_{(a,b)} e^{2\pi i \frac{1}{2} [n + \frac{6}{7} (\gamma + \bar{\gamma})](\gamma + \bar{\gamma})} \left( \frac{n_i Q_i}{Q_A^2} \right) c_A^{(a,b)} \left( -\frac{1}{2} n_i^2 \right). \tag{3.7}
\]

### 3.1. Automorphic forms and gauge threshold corrections

The threshold corrections $\Delta_a$ can be split into three pieces [37]:
\[
\Delta_a = b_a \Delta - G^{(1)} + \sigma. \tag{3.8}
\]

The first term depends on the gauge group under consideration and it is entirely given in terms of $SO(2 + s, 2, \mathbf{Z})$ modular functions. It is that piece\(^8\), which gives rise to automorphic forms. A prominent example is the Dedekind $\eta$–function for the case $s = 0$. With the corresponding logarithmic singularity arising from the Kaluza–Klein states becoming massless at $T \to i\infty$ it reads\(^9\) [28]:
\[
\Delta = -\ln[\kappa (-iT + i\bar{T})(-iU + i\bar{U})] \frac{|\eta(T)|^4 |\eta(U)|^4}{|\eta(\tau)|^4}, \tag{3.9}
\]

with the regularization constant $\kappa = \frac{8\pi}{3\sqrt{3}} e^{1 - \gamma_E}$ and $\gamma_E$ being the Euler–Mascheroni constant. The correction $G^{(1)}$ is the one–loop correction to the Kähler potential [39]. Because of supersymmetry it also appears in the integral (2.3). Finally, $\sigma$ summarizes additional moduli–dependent corrections. They are the subject of [40] for the case without Wilson lines and of [37] when including Wilson lines. Therefore, to isolate the automorphic form $\Delta$ in (3.8), we focus (in this subsection) on the difference of two distinct gauge groups. The integral (2.3) can be formally evaluated [4] and gives for the difference of two gauge groups $G_A$ and $G_A'$
\[
\Delta := \frac{\Delta_A - \Delta_A'}{b_A^{N=2} - b_A'^{N=2}} = -\ln(Y \kappa) - 4\mathrm{Re} \left\{ \sum_{\alpha > 0} \ln e^{\pi i \rho y} \left( 1 - e^{2\pi i \rho y} \right) \frac{d_A^{(a)} - d_A'^{(a)}}{b_A^{(a)} - b_A'^{(a)}} \right\}, \tag{3.10}
\]

---

\(^8\) In the following we call $\Delta$ automorphic form, although it is $\Delta + \ln(Y \kappa)$, which constitutes an automorphic form.

\(^9\) Essential modifications arise in the case of $SL(2, \mathbf{Z})$ subgroups [38].
with
\[ d_A(\alpha) = \frac{(n_iQ_i)^2}{Q_i^2} \sum_{a,b} e^{2\pi i b [n + \frac{1}{2} (\gamma + \tilde{\gamma})] (\gamma + \tilde{\gamma})} c_A^{(a,b)} (kl - \frac{1}{2} n_i^2) , \tag{3.11} \]
and \( \alpha = (k, l, n_i), Y = -\frac{1}{4} [(T - \bar{T})(U - \bar{U}) - \sum_{i=1}^{16} (V_i - \bar{V}_i)^2] \) and \( \alpha y = kT + lU + n_iV_i. \)

The gauge group dependent numbers \( \tilde{d}_A \) appearing in the vector \( \rho = \frac{1}{4(b_n^A = 2 - b_n^{A'})} (\tilde{d}_A - \tilde{d}_{A'}) \) may be looked in \([4]\). The sum \( \alpha > 0 \) runs over all positive lattice vectors \( (i) \) \( k > 0, l \in Z, n_i \in Z, (ii) \) \( k = 0, l > 0, n_i \in Z, (iii) \) \( k, l = 0, n_i > 0. \) The expression \((3.10)\) seems to depend on the orbifold twist \( \nu \), the underlying gauge embeddings \( (\gamma, \tilde{\gamma}) \) and finally on the two gauge groups between which the difference is taken. In the following we want to demonstrate that this is an artifact. In fact, we will see (for the case \( s = 0, 1) , \) that the r.h.s. of \((3.11)\) gives rise to one generic automorphic form (or certain linear combinations) of \( SO(2 + s, s, Z) \), being independent of the orbifold details.

Let us pursue this idea further. For concreteness we will specialize to the one Wilson line \( V_{16} := V \) case. By looking at the perturbative duality symmetry \( SO(3, 2, Z) \) and at the singularity structure in the moduli space of \((2.3)\), the gauge group–dependent part \( b_n^{N=2} \Delta \) of threshold corrections \( \Delta_a \) involving one Wilson line modulus could be derived in \([4]\). Two cases of physical gauge couplings are relevant \([4]\). In the first case, no (under the considered gauge group) charged particles become massless for \( V \to 0 \) and the form of these thresholds is given by\([4]\).

\[ \Delta = -\frac{1}{12} \ln(\kappa Y)^{12} |\chi_{12}|^2 . \tag{3.12} \]

In the second case, some particles, charged under the gauge group under consideration, become massless for \( V \to 0 \). This means that the effective one–loop correction develops a logarithmic singularity in this limit, since those particles, which run in the loop and become massless, have been integrated out. The form of these thresholds is given by

\[ \Delta = -\frac{1}{10} \ln(\kappa Y)^{10} |\chi_{10}|^2 . \tag{3.13} \]

Not any universal contribution are included in these functions. Both thresholds are entirely due to the gauge group dependent part of the charge insertion \( Q_a \) in \((2.3)\). I.e. they may be determined by considering a difference \( \Delta \) of two gauge groups thresholds \( \Delta_{G_a}, \Delta_{G_a'} \).

\[ \text{See e.g. the appendix of [37] and [4] for further information about Siegel forms. The relevance of Siegel modular forms in the context of string one–loop corrections was first observed in [4].} \]
the first case, a difference involving two gauge groups, which are not enhanced at special points in the moduli space. In the second case, two gauge groups, which are both enhanced at $V \to 0$. The appearance of the $SO(3,2,\mathbb{Z})$ automorphic form $\chi_{12}$ in (3.12) is plausible as it gives the correct result (3.9) in the limit $V \to 0$, due to $\chi_{12}|_{V=0} = \eta^{24}(T)\eta^{24}(U)$. Moreover, the expression $\Delta_A - \Delta_A'$ can be expanded in powers of $V$ (cf. appendix C)

$$\Delta \equiv \frac{\Delta_A - \Delta_A'}{b_A^{N=2} - b_{A'}^{N=2}} = -\ln(\kappa Y) - \frac{1}{12}|\eta^{24}(T)\eta^{24}(U)|^2|1 + 12V^2\partial_T \ln \eta^2(T)\partial_U \ln \eta^2(U)|^2 + \ldots$$

$$= -\frac{1}{12} \ln(\kappa Y)^{12}|\chi_{12}|^2,$$

(3.14)

which agrees with the lowest order of (3.13). The form of (3.13) ensures the correct logarithmic behaviour at $V \to 0$ ($\chi_{10} \to V^2\eta^{24}(T)\eta^{24}(U)$). Finally, in (3.10) one may also consider the case of two gauge groups, where one gets enhanced for $V \to 0$ and the other does not:

$$\Delta = -\frac{1}{12} \ln(\kappa Y)^{12}|\chi_{12}|^2 + \left[1 - \frac{b_A^{N=2}(V = 0)}{b_{A'}^{N=2}(V \neq 0)}\right] \ln \left|\frac{\chi_{10}^{1/2}}{\chi_{12}^{5/12}}\right|^2.$$

(3.15)

This represents a case, where automorphic forms show up in a linear combination, with a gauge group dependent factor. This dependence may be eliminated if one considers a second pair of gauge groups.

In the next subsection we will see that differences of gauge threshold corrections (3.10) of any $K3$ orbifold always take the form (3.13) or (3.15). This result is quite intriguing, since (3.10) depends on the specific gauge embedding $(\gamma, \tilde{\gamma})$ of the orbifold and its spectrum, whereas (3.13) and (3.15) are model–independent. All model dependence goes into the $\beta$–function coefficients, which in (3.8) appear just as prefactors. We are not able to reproduce (3.12) from (3.10). We would have to choose a pair of two non–singular couplings. However, looking at (3.18), such a pair only exists for the cases $(iii)$ and $(v)$. They have the same $\beta$–function coefficients, a fact which formula (3.10) does not allow. We believe, that this failure is no accident, since it would lead to a product formula for $\chi_{12}$, whereas its divisors do not have any simple form [41].

### 3.2. Standard orbifold limits of $K3 \times T^2$

In this section we work out (2.3),(2.5) and in particular (3.10) for standard and non–standard orbifolds $K3$ orbifold limits. Since the result (3.9) holds for all types of orbifolds of $K3 \times T^2$, i.e. in the case of both standard and non–standard embedding of the twist into
the gauge degrees of freedom, we expect this to hold also for the case when one non-trivial gauge background field, i.e. one Wilson line $V$ is turned on. It is one of the aim of this section to obtain general expressions for $\Delta$ in those cases.

In the following we consider the standard-embeddings

\[ \gamma^I = (1, -1, 0, 0, 0, 0, 0, 0), \quad \tilde{\gamma}^I = (0, 0, 0, 0, 0, 0, 0, 0), \quad \nu = 2, 3, 4, 6, \]  

with the N=2 gauge groups $SU(2) \times E_7 \times E_8$ for $\nu = 2$ ($b_{SU(2)}^{N=2} = b_{E_7}^{N=2} = 84$, $b_{E_8}^{N=2} = -60$) and $U(1) \times E_7 \times E_8$ for the others. In these cases eq. (3.1) simplifies drastically. In fact, it reduces to the form (2.5) with $(n^{(1)}, n^{(2)}) = (24, 0)$. Moreover the threshold corrections take the form (2.4) and (2.7) with the same instanton numbers.

3.3. Non-standard orbifold limits of $K3 \times T^2$

For concreteness, let us discuss seven cases of non-standard embeddings.

\[
\begin{array}{|c|c|c|c|}
\hline
\nu & \gamma & \tilde{\gamma} & \text{perturbative gauge group} \\
\hline
i & 2 & (1,0,0,0,0,0,0,0) & (2,0,0,0,0,0,0,0) & SU(2) \times E_7 \times SO(16)' \\
ii & 3 & (1,0,0,0,0,0,0,0) & (2,1,0,0,0,0,0,0) & U(1) \times E_7 \times SU(3)' \times E_6' \\
iii & 3 & (2,0,0,0,0,0,0,0) & (2,0,0,0,0,0,0,0) & U(1) \times SO(14) \times U(1)' \times SO(14)' \\
iv & 3 & (2,1,1,0,0,0,0,0) & (2,1,1,1,0,0,0,0) & SU(3) \times E_6 \times SU(9)' \\
v & 4 & (2,2,2,0,0,0,0,0) & (3,1,1,1,1,0,0,0) & SU(4) \times SO(10) \times SU(2)' \times SU(8)' \\
vii & 4 & (1,1,1,3,0,0,0,0) & (1,1,-2,0,0,0,0,0) & SU(4) \times SO(10) \times U(1)' \times SU(2)' \times E_6' \\
vii & 6 & (1,1,1,1,4,0,0,0) & (1,1,1,1,1,-5,0,0) & U(1) \times SU(4) \times SU(5) \times SU(2)' \times SU(3)' \times SU(6)' \\
\hline
\end{array}
\]

Table 1: Examples of non-standard orbifold limits of heterotic $K3 \times T^2$ compactifications ($\nu, \gamma, \tilde{\gamma}$) and their perturbative gauge group.

Models (ii), (vi), (vii) correspond to the $r = 10, 8, 4$ chains, respectively discussed in [12]. Actually in total, there are 2 different embeddings for $Z_2$, 5 for $Z_3$, 12 for $Z_4$ (table 3) and 61 for $Z_6$ (table 8).

With the relation

\[ b_{n}^{N=2} = 2\text{Tr}_{H}(Q_{n}^{2}) - 2\text{Tr}_{V}(Q_{n}^{2}), \quad (3.17) \]

\[ ^{11} \text{For all N=2 orbifolds (see the following tables) we expanded (3.1) in a power series in } q \text{ and } y \text{ and found agreement with (2.3) up to an arbitrary high order.} \]
where $Q_a$ is any generator of the group $G_a$, we determine the following $N=2$ $\beta$-function coefficients

\begin{align}
(i) \quad b_{E_7}^{N=2} &= -12 , \quad b_{SO(16)'}^{N=2} = \begin{cases} 36 , & V = 0 \\ 28 , & V \neq 0 \end{cases} \\
(ii) \quad b_{E_7}^{N=2} &= -24 , \quad b_{E_6}^{N=2} = \begin{cases} 48 , & V = 0 \\ 36 , & V \neq 0 \end{cases} \\
(iii) \quad b_{SO(14)}^{N=2} &= 12 , \quad b_{SO(14)'}^{N=2} = 12 \\
(v) \quad b_{SO(10)}^{N=2} &= 12 , \quad b_{SU(8)'}^{N=2} = 12 \\
(vi) \quad b_{SO(10)}^{N=2} &= 36 , \quad b_{E_6}^{N=2} = \begin{cases} -12 , & V = 0 \\ -4 , & V \neq 0 \end{cases} \\
(vii) \quad b_{SU(5)}^{N=2} &= 24 , \quad b_{SU(6)'}^{N=2} = \begin{cases} 0 , & V = 0 \\ 4 , & V \neq 0 \end{cases}
\end{align}

respectively. Our results can be easily converted to other embeddings. Interestingly, an explicit calculation shows that in all seven cases the supersymmetric index (3.1) reduces to the expression (2.5) with instanton numbers $(n^{(1)}, n^{(2)})$ referring to the $SU(2)$ bundles in $E_8 \times E_8$. This is also the case for the other orbifold limits of $K3$ (cf. e.g. table 3 for all $T^4/Z_4$ and appendix D for all $T^4/Z_6$ orbifolds).

| $(n^{(1)}, n^{(2)})$ | $n$ |
|-------------------|-----|
| i (8,16)          | 4   |
| ii (6,18)         | 6   |
| iii (12,12)       | 0   |
| iv (9,15)         | 3   |
| v (12,12)         | 0   |
| vi (16,8)         | −4  |
| vii (14,10)       | −2  |

Table 2: The instanton numbers $(n^{(1)}, n^{(2)})$ of the previous examples.

Of course, this is quite remarkable as in (3.1) we are summing over all various twisted sectors. Altogether this results in (2.5). On the other hand, this fact may be understood from the point of view of modular functions: The expression (3.1) is given by a modular function of weight $−2$ and a certain pole structure dictated by the states becoming massless.

\[12\] In the case of non-vanishing Wilson line $V \neq 0$, generators $Q_a$, which do not survive the Wilson line projection, are excluded. In that case, the $\beta$-function coefficient refers to the surviving (smaller) gauge group.
for $\tau \to -i\infty$ (tachyon) and in the IR $\tau \to i\infty$. This fixes the form of (3.14). Therefore we conclude, that (2.5) and (2.7) are the most general expressions for the index and gauge threshold corrections in orbifold limits of $K3$, respectively.

| $\gamma$ | $\bar{\gamma}$ | Perturbative gauge group | $(n^{(1)}, n^{(2)})$ |
|----------|----------------|--------------------------|------------------|
| (1,1,0,0,0,0,0,0) | (0,0,0,0,0,0,0,0) | $U(1) \times E_7 \times E_8'$ | (24,0) |
| (1,1,0,0,0,0,0,0) | (2,2,0,0,0,0,0,0) | $U(1) \times E_7 \times SU(2)' \times E_7'$ | (12,12) |
| (1,1,0,0,0,0,0,0) | (4,0,0,0,0,0,0,0) | $U(1) \times E_7 \times SO(16)'$ | (16,8) |
| (1,1,0,0,0,0,0,0) | (1,1,1,1,1,1,1,1) | $U(1) \times E_7 \times U(1)' \times SU(8)'$ | (6,18) |
| (2,1,1,0,0,0,0,0) | (2,0,0,0,0,0,0,0) | $U(1) \times SU(2) \times E_6 \times U(1)' \times SO(14)'$ | (12,12) |
| (2,1,1,0,0,0,0,0) | (2,2,2,0,0,0,0,0) | $U(1) \times SU(2) \times E_6 \times SU(4)' \times SO(10)'$ | (8,16) |
| (3,1,0,0,0,0,0,0) | (0,0,0,0,0,0,0,0) | $U(1) \times SU(2) \times SO(12) \times E_8'$ | (24,0) |
| (3,1,0,0,0,0,0,0) | (2,0,0,0,0,0,0,0) | $U(1) \times SU(2) \times SO(12) \times SU(2)' \times E_7'$ | (20,4) |
| (3,1,0,0,0,0,0,0) | (4,0,0,0,0,0,0,0) | $U(1) \times SU(2) \times SO(12) \times SO(16)'$ | (16,8) |
| (3,1,1,1,1,1,1,0) | (2,2,2,0,0,0,0,0) | $SU(2) \times SU(8) \times SU(4)' \times SO(10)'$ | (12,12) |
| (3,1,1,1,1,1,0,0) | (2,0,0,0,0,0,0,0) | $SU(2) \times SU(8) \times U(1)' \times SO(14)'$ | (12,12) |
| (1,1,1,1,0,0,0,0,1) | (3,1,0,0,0,0,0,0) | $U(1) \times SU(8) \times U(1)' \times SU(2)' \times SO(12)'$ | (14,10) |

Table 3: All twelve $T^4/Z_4$-orbifolds with gauge twist $(\gamma, \bar{\gamma})$: Their perturbative gauge group and instanton numbers $(n^{(1)}, n^{(2)})$ w.r.t. $SU(2)$-bundles.

As we have seen, the function $B_a$ in (2.3) is given by (2.7) with the topological numbers, given in the tables 2,3 and 8, respectively. The $\tau$–integral in (2.3) can be done guided by $\mathbb{E}_8$. For the difference $\Delta_G - \Delta_{G'}$ of gauge threshold corrections, involving a non–singular gauge group $G$ and a singular one $G'$, we obtain the proposed form (3.13): In that case,

$$B_G - B_{G'} = -\frac{n}{2} \mathcal{Z}_{K3}(q, y) \otimes \mathcal{Z}_{3,2},$$

(3.19)

with the $K3$ elliptic genus $\mathcal{Z}_{K3}(q, y)$ introduced eq. (A.5). Thus $\mathbb{E}_8$:

$$\Delta_G - \Delta_{G'} = n \ln(\kappa Y)^{10}|\chi_{10}|^2.$$

(3.20)

The other expression (3.14) appears in the next subsection.

3.4. Enhanced gauge group threshold corrections

Since in the case of $K3$ compactifications the manifold has no isometries, the gauge group $(G, G')$ derives from $E_8 \times E_8$ (and the bundle structure), only. However, at special points in the moduli space, like e.g. the orbifold points of $K3$, additional gauge group
factors \((H, H')\) appear. The maximal possible gauge group is \(E_8 \times E_7 \times SU(2)^5\). One feature (cf. previous sections) for threshold corrections w.r.t. \((G, G')\) is that, they may be equal for different choices of \((\nu, \gamma, \tilde{\gamma})\) as long as they have the same instanton numbers \((n^{(1)}, n^{(2)})\). On the other hand, this statement no longer holds for the enhanced gauge group \((H, H')\) thresholds, which in general depend on the chosen orbifold limit. Which gauge group arises at the orbifold limit, depends on the twist embedding. Therefore, threshold corrections w.r.t. those gauge groups depend on the specific form of the shift vector \((\gamma, \tilde{\gamma})\). As a consequence they are not expressible in terms of \((n^{(1)}, n^{(2)})\) alone. Eq. (2.3) then takes the form

\[
B_{H, H'} = \frac{-1}{12} \left[ \left( E_2 - \frac{3}{\pi \tau_2} \right) \left( \frac{n^{(1)}}{24} E_{4,1} E_6 + \frac{n^{(2)}}{24} E_{4,1} E_{6,1} \right) + \frac{d_1}{24} E_4^2 E_{4,1} - \frac{d_2}{24} E_6 E_{6,1} \right] \otimes Z_{3,2},
\]

with the additional coefficients \((d_1, d_2)\). For the examples of table 1 we determined these coefficients:

| \(H, H'\) | \((d_1, d_2)\) | \(d\) |
|---|---|---|
| i | \(SU(2)\) | \((40, -16)\) | -28 |
| ii | \(U(1)\) | \((30, -6)\) | -18 |
| iii | \(SU(3)'\) | \((18, 6)\) | -6 |
| iv | \(U(1)\) | \((28, -4)\) | -16 |
| v | \(U(1)'\) | \((28, -4)\) | -16 |
| vi | \(SU(4)\) | \((20, 4)\) | -8 |
| vii | \(SU(4)'\) | \((16, 8)\) | -4 |
| viii | \(SU(2)'\) | \((28, -4)\) | -16 |
| ix | \(U(1)'\) | \((20, 4)\) | -8 |

**Table 4:** Details \((d_1, d_2)\) for enhanced gauge group \(H, H'\) threshold corrections.

Interestingly, for all cases \(d_1 + d_2 = 24\) and it is convenient to introduce \(d\) with \(d_1 = 12 - d\), \(d_2 = 12 + d\). For the \(\beta\)-functions we obtain:

\[
b_{H, H'}^{N=2} = \begin{cases} 
12 - 6d & , \ V = 0 \\
12 - n - 5d & , \ V \neq 0 
\end{cases}.
\]

\(^{13}\) This requires also a choice of special points in the vector multiplet moduli space \((T, U)\).
For the examples of table 4:

(i) \[ b_{SU(2)}^{N=2} = \begin{cases} 180, & V = 0 \\ 148, & V \neq 0 \end{cases} \]

(ii) \[ b_{U(1)}^{N=2} = \begin{cases} 120, & V = 0 \\ 96, & V \neq 0 \end{cases}, \quad b_{SU(3)'}^{N=2} = \begin{cases} 48, & V = 0 \\ 36, & V \neq 0 \end{cases} \]

(iii) \[ b_{U(1)}^{N=2} = \begin{cases} 108, & V = 0 \\ 92, & V \neq 0 \end{cases}, \quad b_{U(1)'}^{N=2} = \begin{cases} 108, & V = 0 \\ 92, & V \neq 0 \end{cases} \]

(v) \[ b_{SU(4)}^{N=2} = \begin{cases} 60, & V = 0 \\ 52, & V \neq 0 \end{cases} \]

(vi) \[ b_{SU(4)}^{N=2} = 36, \quad b_{U(1)'}^{N=2} = \begin{cases} 60, & V = 0 \\ 56, & V \neq 0 \end{cases}, \quad b_{SU(2)'}^{N=2} = \begin{cases} 108, & V = 0 \\ 96, & V \neq 0 \end{cases} \]

3.5. Differential equation for the N=2 prepotential

For the cases discussed in [3] it was shown that the one–loop correction \( h_n \) to the prepotential of the underlying N=2 theory fulfils a second order differential equation. Also in the cases at hand we can derive a differential equation for the one loop correction \( h_n \) to the N=2 prepotential \( H_n \)

\[
H_n(S, T, U, V) = S(TU - V^2) + h_n(T, U, V) + O(e^{-2\pi i S}), \tag{3.24}
\]

with [3]

\[
h_n(T, U, V) = -\frac{i}{2\pi} d_n(T, U, V) - \frac{1}{(2\pi)^4} \sum_{(k,l,b) > 0} c_n \left( kl - \frac{1}{4} b^2 \right) Li_3(e^{2\pi i (kT + lU + bV)}) + \text{const.} \tag{3.25}
\]

and

\[
d_n(T, U, V) = \frac{1}{3} U^3 + \left( \frac{4}{3} + n \right) V^3 - \left( 1 + \frac{n}{2} \right) U V^2 - \frac{n}{2} T V^2. \tag{3.26}
\]

The coefficients \( c_n \) refer to the \( K3 \) genus \( \tilde{Z}_{K3}(q) \) given in [2.5] and its coefficients [2.13]. There are ambiguities for the cubic polynomial (3.26) due to the fact that the holomorphic prepotential is only fixed up to quadratic pieces in the homogeneous coordinates \( \hat{X}^I \). These quadratic pieces include e.g. cubics in \( V \). On the other hand, this ambiguity can be fixed when comparing the prepotential with the corresponding one of the typeII theory, which leads to the form (3.24) [4]. For the differential equation we find

\[
\text{Re} \left\{ \frac{16\pi^2}{5} (\partial_T \partial_U - \frac{1}{4} \partial_V^2) h_n \right\} - G^{(1)}_{N=2} = A_{\alpha}^{n,d} + \frac{16}{5} \text{Re} \left[ \ln \Psi_{n,d}(T, U, V) \right] + b_{\alpha,n,d}^{N=2} \ln Y, \tag{3.27}
\]
with
\[ G_{N=2}^{(1)} = \frac{8\pi^2}{Y} \text{Re} \left\{ h_n - \sum_{y=T,U,V} \text{Im}(y) \frac{\partial}{\partial y} h_n \right\} . \] (3.28)

The above differential equation holds for general gauge threshold corrections \( \Delta^{(n,d)} \) (2.3) which are given by (3.21) for some \( n, d \). Of course, \( d = \pm n \) leads to (2.7), whereas \( d \neq \pm n \) describes the cases in table 4. The holomorphic function \( \Psi_{n,d} \) is (3.21):
\[ \Psi_{n,d}(T,U,V) = \frac{\Delta_{10}^{1/2}}{\Delta_{35}^{1/2}} . \] (3.29)

From that differential equation and (3.8) one immediately obtains an expression for \( \sigma \):
\[ \sigma_n(T,U,V) = \text{Re} \left\{ \frac{16\pi^2}{5} (\partial_T \partial_U - \frac{1}{4} \partial_V^2) h_n - \frac{4}{5\pi} \ln \frac{\Delta_{35}}{\Delta_{10}^{1/2}} \chi_{12}^{5/4n-5/2} \right\} . \] (3.30)

All our investigations concern the so-called \( A \)-models [44]. These are heterotic \( K3 \times T^2 \) compactifications where the \( K3 \) of the typeIIA dual CYM (which is a \( K3 \) fibration) itself a fibration with \( E_8 \)-torus fiber. In particular this means, that on the Kähler modulus \( T \) of the heterotic torus \( T^2 \), to be identified with the Kähler modulus of the torus fiber, the full \( SL(2,\mathbb{Z})_T \) (to be embedded into \( SO(2,2+s,\mathbb{Z}) \) is realized [44][46]. The \( B, C \)-models have fiber tori \( E_7, E_6 \), respectively. In these cases, the heterotic duality group of the \( T \)-modulus is only a subgroup of \( SL(2,\mathbb{Z})_T \) and e.g. for \( s = 1 \) the results of the previous sections involve \( Sp(4,\mathbb{Z})_T \) subgroups [45][8].

After Higgsing completely, the models with \( n = 0,1,2 \) become the so-called \( STU \)-models or one step before –with an \( SU(2) \) gauge group in the Coulomb phase– the \( STUV \)-models. Their duality to typeIIA CYM, which are elliptic fibrations over the Hirzebruch surfaces \( F_n \) with
\[ c_n(0) = \chi(X_{F_n}) = \begin{cases} -480 & , \ V = 0 \\ -420 + 24n & , \ V \neq 0 \end{cases} , \] (3.31)
and \( h_{(1,1)} = N_V - 1 = 4 \) and \( h_{(2,1)} = N_H - 1 = 214 - 12n \), has been checked in [4]. The models with \( n \neq 0,1,2 \) have some terminal gauge groups after Higgsing completely (cf. also section 2) and again are related to CYM being elliptically fibered over \( F_n \). The CY prepotential and the gravitational coupling, specialized to the sublocus of their Kähler moduli space, where this gauge group is fully established (or up to an \( SU(2) \) factor, which is in the Coulomb phase in the case of one non-vanishing Wilson line), can be linked to our results (3.23). Dictated by the Jacobi functions, at such a point the instanton expansion of the CY prepotential will arrange w.r.t. to \( SU(2) \) representations like (2.3).

\[ \text{The case } n = 12, d = \mp12 \text{ was discussed in [37].} \]
3.6. Geometric interpretation

In the previous sections we have seen, that the instanton numbers $n^{(1)}, n^{(2)}$ and the relation $n^{(1)} + n^{(2)} = 24$ are important ingredients entering the threshold results. See e.g. (2.7) and (3.21). On the other hand, all the information about an orbifold, e.g. the massless spectrum and the instanton numbers are encoded in $\nu$ and the shifts $\gamma^I, \tilde{\gamma}^I$. Therefore in this section we want to express $n^{(1)}, n^{(2)}$ in terms of $\nu, \gamma^I, \tilde{\gamma}^I$. We know of two ways, to find this relation. The first one uses the supersymmetric index (2.1) (cf. the previous sections and tables 2, 3 and 8). I.e. we write the index (3.1) in such a way (2.5), that we are able to read off the instanton numbers. This fixes the instanton numbers $n^{(1)}, n^{(2)}$ completely. The second way uses results about small instantons at $Z_\nu$ orbifold singularities [46]. The individual instanton numbers w.r.t. $E_8 \times E'_8$ at an orbifold fixed point $f_\alpha$ with gauge twists $\gamma_\alpha, \tilde{\gamma}_\alpha$ of twist order $\nu_\alpha$ are given by [46][35]:

$$n^{(1)}_\alpha = k^{(1)}_\alpha + \sum_{j=0}^{\nu_\alpha-1} j(\nu_\alpha - j) \frac{1}{4 \nu_\alpha} w_j = k^{(1)}_\alpha + \frac{1}{2 \nu_\alpha} \sum_{I=1}^{8} \gamma^I_\alpha(\nu_\alpha - \gamma^I_\alpha),$$

$$n^{(2)}_\alpha = k^{(2)}_\alpha + \sum_{j=0}^{\nu_\alpha-1} j(\nu_\alpha - j) \frac{1}{4 \nu_\alpha} \tilde{w}_j = k^{(2)}_\alpha + \frac{1}{2 \nu_\alpha} \sum_{I=1}^{8} \tilde{\gamma}^I_\alpha(\nu_\alpha - \tilde{\gamma}^I_\alpha).$$

(3.32)

Here $k^{(1)}_\alpha, k^{(2)}_\alpha$ are arbitrary integers, which are already present for instantons in flat Euclidean $\mathbb{R}^4$ and $w_\mu$ is the number of orbifold twist eigenvalues $\exp(2\pi i \mu/\nu_\alpha)$ in $\gamma_\alpha$. To finally obtain $n^{(1)}, n^{(2)}$ one has to sum over all possible fixed point of order $\nu_\alpha = 0, \ldots, \nu - 1$. Of course, $n^{(1)} + n^{(2)} = 24$, which relates the small instanton physics at the orbifold fixed point to the $K3$–geometry. At one fixed point $f_\alpha$ a similar equation holds, which relates $n^{(1)}_\alpha + n^{(2)}_\alpha$ to the Euler number of the $ALE$ space (see section 5 for more discussions). This allows to fix one of the constants $k^{(1)}_\alpha, k^{(2)}_\alpha$. Therefore, only for $K3$ compactification with $SO(32)$ gauge group, the constants $k^{(1)}_\alpha, k^{(2)}_\alpha$ may be fixed (cf. the discussions in [33]). Besides, in the case of Abelian bundles one may impose additional equations, relating these numbers to the number of twisted matter fields charged under these $U(1)$’s, which eventually fix these numbers [22]. However, in the case of non–Abelian instanton backgrounds, the method which uses the supersymmetric index, seems to be more restrictive. It allows us to fix the numbers $k^{(1)}_\alpha, k^{(2)}_\alpha$ completely (see also the tables 2, 3 and 8).
4. Gauge threshold corrections in N=1 (0,2) orbifold compactifications

A generic feature of gauge couplings in N=1, d = 4 string vacua is their dependence on scalars of chiral multiplets. The latter describe the universal dilaton $S$ and the moduli $T, U, V_i$ arising from the internal compactification. At tree–level the gauge couplings of all gauge group $G_a$ factors are given by the string–coupling, which is determined by the vev of the dilaton field $S$. In effective string theory, this relation is modified by a mixing between the dilaton $S$ and the other moduli, described by the non–harmonic function $G^{(1)}$:

$$g_{\text{string}}^{-2} = \frac{-iS + i\overline{S}}{2} + \frac{1}{16\pi^2} G^{(1)}(T, U, V_i). \quad (4.1)$$

$G^{(1)}$ is the one–loop correction to the Kähler potential [39]. In addition, one has to take into account string threshold corrections originating from string modes with masses above the string scale $M_{\text{string}}$, which have been integrated out. They (effectively) split the couplings at the string scale, i.e.

$$g_a^{-2}(\mu)\big|_{\mu = M_{\text{string}}} = k_a g_{\text{string}}^{-2} + \frac{1}{16\pi^2} \Delta_a. \quad (4.2)$$

Whereas the Kähler potential $G$ receives contributions beyond one–loop, which are (so far) not under control, the harmonic (Wilsonian) part of $\Delta_a$ is completely determined already at one–loop thanks to non–renormalization theorems for the gauge kinetic function $f$. On the other hand, the non–harmonic part of $\Delta_a$ is expected to have the opposite sign of $G^{(1)}$. Thus with (4.1) it has no influence on the physical coupling (4.2) – at least at one–loop [39] [37]. However, it does affect the precise relation of $M_{\text{Planck}}$ and $M_{\text{string}}$ at one–loop [37]:

$$M_{\text{Planck}}^2 = \left[ \text{Im}(S) + \frac{1}{16\pi^2} G^{(1)}(T, U, V_i) \right] M_{\text{string}}^2. \quad (4.3)$$

In this section we want to obtain generic results for the N=1 gauge thresholds $\Delta_a$, focusing for concreteness, on N=1 toroidal orbifolds [41]. In these cases $\Delta_a$ receives only moduli–dependent contributions from N=2 subsectors [28]. Therefore, to obtain their analytic form, many results from the previous sections may be borrowed. For concreteness, let us discuss six N=1 examples.
Table 5: Examples of N=1, d = 4 heterotic $Z_\nu$ orbifolds with (2,2) or (0,2) world–sheet supersymmetry.

The $Z_2 \times Z_2$ models have the internal twists $\theta_1 = (-1, -1, +1)$, $\theta_2 = (-1, +1, -1)$, the $Z_4$ has $\theta = \frac{1}{4}(1, 1, -2)$ and the $Z_6 - II$ orbifold has the twist $\theta = \frac{1}{6}(1, 2, -3)$. Models (I), (V) have standard embeddings of the twist into the gauge group, thus allowing for (2,2) world–sheet supersymmetry. On the other hand, models (II), (III), (IV), (VI) are orbifolds with non–standard twist embeddings with only (0, 2) world–sheet supersymmetry. Since only their N=2 sectors give rise to a modulus dependence of $\Delta_a$, let us investigate these sectors and give their relations to the previous sections.

| Orbifold | 1st Plane | 2nd Plane | 3rd Plane |
|----------|-----------|-----------|-----------|
| I        | $Z_{2st}$, $n_1 = -12, \nu_1 = 2$ | $Z_{2st}$, $n_2 = -12, \nu_2 = 2$ | $Z_{2st}$, $n_3 = -12, \nu_3 = 2$ |
| II       | $Z_{2nst}(i)$, $n_1 = 4, \nu_1 = 2$ | $Z_{2nst}(i)$, $n_2 = 4, \nu_2 = 2$ | $Z_{2nst}(i)$, $n_3 = 4, \nu_3 = 2$ |
| III      | $Z_{2nst}(i), n_1 = 4, \nu_1 = 2$ | $Z_{2nst}(i), n_2 = 4, \nu_2 = 2$ | $Z_{2nst}(i), n_3 = 4, \nu_3 = 2$ |
| IV       | $Z_{2st}$, $n_2 = -12, \nu_3 = 2$ | $Z_{3st}$, $n_3 = -12, \nu_3 = 3$ |
| V        | $Z_{2nst}(i), n_2 = 4, \nu_3 = 2$ | $Z_{3nst}(ii), n_3 = 6, \nu_3 = 3$ |
| VI       | $Z_{2nst}(i), n_2 = 4, \nu_3 = 2$ | $Z_{3nst}(ii), n_3 = 6, \nu_3 = 3$ |

Table 6: Twist invariant planes and $K3 \times T^2$ details $(\nu_i, n_i)$ of the previous examples.

Here $\nu_i$ is the twist order of the N=2 subsector, which leaves invariant the $i$th plane with moduli fields $T^i, U^i$. If one plane $i$ does not give rise to an N=2 sector, we just take $\nu_i = 0$ in all the following sums. In particular, this is the case for $\nu = prime$. Besides, in the cases $\nu_1/\nu \neq 1, \frac{1}{2}$ the $U^i$–modulus is frozen. The spectra of these models have been worked out in the appendix of [48]. Also cancellation of anomalies, produced by triangle graphs involving the Kähler and sigma–model connection, have been discussed there. In particular, $G^{(1)}_{i,N=1} = 0$, whenever $\nu_i/\nu = 1, \frac{1}{2}$, however $G^{(1)}_{i,N=1} \neq 0$, if $\nu_i/\nu \neq 1, \frac{1}{2}$. Since the string–modes running in the loop arrange in N=2 multiplets [28], the N=2 $\beta$–function
coefficients of the underlying N=2 (sub)theory (2.8) will reappear in the calculations. The latter can be expressed by the N=1 Kähler and sigma–model anomaly coefficients \(\alpha^i_{G_A,G_{A'}}\) referring to the N=1 gauge group \(G_A,G_{A'}\) [18].

In total we get for the N=1 threshold corrections \(\Delta_a\) (cf. (3.8))

\[
\Delta_{G_A} = - \sum_{i=1,2,3} \frac{\nu_i}{\nu} \left\{ (12 - 6n_i) \ln(\kappa T_2 U_2) |\eta(T^i)\eta(U^i)|^4 + k_A \sigma_n(T^i,U^i) + k_A G^{(1)}_{i,N=2} \right\} \\
- k_A G^{(1)}_{i,N=1} + \text{const.} ,
\]

\[
\Delta_{G_{A'}} = - \sum_{i=1,2,3} \frac{\nu_i}{\nu} \left\{ (12 + 6n_i) \ln(\kappa T_2 U_2) |\eta(T^i)\eta(U^i)|^4 + k_{A'} \sigma_n(T^i,U^i) + k_{A'} G^{(1)}_{i,N=2} \right\} \\
- k_{A'} G^{(1)}_{i,N=1} + \text{const.} .
\]

(4.4)

for vanishing Wilson lines \(V_i\) (\(s = 0\)). With one Wilson–line \(V := V_{16}\) switched on (\(s = 1\)) we find:

\[
\Delta_{G_A} = - \sum_{i=1,2,3} \frac{\nu_i}{\nu} \left\{ \frac{12 - 6n_i}{12} \ln(\kappa Y_i)^{12}|\chi_{12}|^2 + k_A \sigma_n(T^i,U^i,V) + k_A G^{(1)}_{i,N=2} \right\} \\
- k_A G^{(1)}_{i,N=1} + \text{const.} ,
\]

\[
\Delta_{G_{A'}} = - \sum_{i=1,2,3} \frac{\nu_i}{\nu} \left\{ \frac{12 + 4n_i}{12} \ln(\kappa Y_i)^{12}|\chi_{12}|^2 + 2n_i \ln \left| \frac{\Lambda^{1/2}_{10}}{\Lambda^{1/2}_{5/12}} \right|^2 + k_{A'} \sigma_n(T^i,U^i,V) + k_{A'} G^{(1)}_{i,N=2} \right\} \\
- k_{A'} G^{(1)}_{i,N=1} + \text{const.} .
\]

(4.5)

The second piece in \(\Delta_{G_{A'}}\) accounts for the subthreshold effect which is caused by particles becoming massless for \(V \to 0\). In this case both \(G_{A'}\) and the N=2 gauge group \(G'\) are enhanced (cf. also (2.8) and (3.18)).

We see, that \(\Delta_{G_A}, \Delta_{G_{A'}}\) are given by \(SO(2 + s, 2, \mathbb{Z})\) modular functions depending on the Kähler \(T^i\) and complex structure moduli \(U^i\) of the N=1 compactification and some topological data. The latter are the instanton numbers \(n_i\), which refer to the individual \(N = 2\) subsectors, described by \(K3 \times T^2\) dynamics (cf. table 6). It has already been stressed in [37], that, in contrast to certain statements made in the past, the harmonic piece (3.30) \(\sigma_n\) in (1.4) and (4.3) is of fundamental importance to recover the correct decompactification limits to \(d = 6\) (cf. also (2.9)) and \(d = 10\) dimensions (cf. next section).

Our results (4.4) and (4.5) hold quite general for N=1 orbifolds. In practice one only has to read the information \(n_i\) about their N=2 subsectors \(\nu_i\) from the tables 2,3 and 8. We may also go opposite: For given \(n_i\), i.e. gauge twist \((\gamma, \tilde{\gamma})\), construct an N=1 (modular
invariant) $\mathbb{Z}_\nu$ orbifold with twist $(\theta, \frac{1}{\nu} \Gamma, \frac{1}{\nu} \tilde{\Gamma})$, whose N=2 subsector has the gauge twist $\frac{1}{\nu_i} (\gamma, \tilde{\gamma})$. Modular invariance (see. e.g. [49])

$$\sum_{I=1}^{8} \Gamma_I - \sum_{I=1}^{8} \tilde{\Gamma}_I = 0 \mod 2 \quad (4.6)$$

$$\sum_{i=1}^{3} \theta_i^2 - \sum_{I=1}^{8} \Gamma_I^2 - \sum_{I=1}^{8} \tilde{\Gamma}_I^2 = 0 \mod 2\nu$$

is quite restrictive and may rule out a lot of combinations. Nevertheless, it is e.g. possible, to find N=1 orbifolds with $n_i = 0$: This happens, when the N=2 subsector, described by K3 dynamics has an equal number of instantons in both $E_8$–factors, i.e. $(n_{i}^{(1)}, n_{i}^{(2)}) = (12, 12)$. In these cases, there is no one–loop correction to the gauge kinetic function\[13\] (cf. section 5.1):

$$f_{G_A, G_A'} = -i S + \mathcal{O}(e^{8\pi^2 i S}) \quad (4.7)$$

The gauge group dependent part of the one–loop threshold correction $b_{G}^{N=2} \triangle$ cancels against the group independent part $\sigma$. These models look like N=4 models \[50\], which do not have any perturbative corrections to the (two–derivative) gauge couplings due to the lack of enough fermionic contractions in a correlation function with two gauge bosons. However, our models have N=1 space–time supersymmetry (with some N=2 subsector structure) and any two gauge boson correlator must not vanish due to supersymmetry arguments. On the other hand, these models do have moduli dependent wave–function renormalizations or one–loop corrections to the Kähler potential $G^{(1)}$. Let us give a list of these orbifolds, since they might be of some phenomenological use:

---

\[16\] This is also true for prime orbifolds, which do not possess any twist invariant planes.
\( \mathbb{Z}_\nu: \theta \)

| \( \mathbb{Z}_8-I: \frac{1}{3}(1,3,-4) \) | \( (1,1,0,0,0,0,0,0) \) | \( (2,2,0,0,0,0,0,0) \) | \( U(1) \times E_7 \times U(1) \times E_7' \) |
|---|---|---|---|
| | \( (2,1,1,0,0,0,0,0) \) | \( (2,0,0,0,0,0,0,0) \) | \( U(1) \times SU(2) \times SO(16) \times SO(16)' \) |
| \( \mathbb{Z}_{12-I}: \frac{1}{12}(1,4,-5) \) | \( (3,1,1,1,1,1,0,0) \) | \( (2,2,2,0,0,0,0,0) \) | \( U(1)^2 \times SU(2) \times SU(10) \times SU(4) \times SO(16)' \) |
| \( \mathbb{Z}_{12-II}: \frac{1}{12}(1,5,-6) \) | \( (3,2,1,0,0,0,0,0) \) | \( (4,2,2,0,0,0,0,0) \) | \( U(1)^2 \times E_6 \times U(1) \times SU(2) \times E_6' \) |
| | \( (3,1,0,0,0,0,0,0) \) | \( (2,0,0,0,0,0,0,0) \) | \( U(1)^2 \times SO(12) \times SO(16)' \) |
| | \( \frac{1}{2}(7,1,1,1,1,1,-1) \) & \( (3,3,1,1,1,1,-1) \) & \( U(1)^2 \times E_6 \times SU(4) \times SO(16)' \) |

#### Table 7: \( \mathbb{N}=1 \mathbb{Z}_\nu \) orbifolds with vanishing perturbative corrections to the gauge kinetic function \( f \): Their twists \( (\theta, \frac{1}{3} \Gamma, \frac{1}{3} \bar{\Gamma}) \) and \( \mathbb{N}=1 \) gauge groups.

The \( \mathbb{Z}_8-I \) models have an invariant plane for \( \theta^2 \) of \( \mathbb{Z}_4 \) (non-standard) \( K3 \)-orbifold structure with instanton number \( n_3 = 0 \). The one \( \mathbb{Z}_{12-I} \) example has a fixed plane for \( \theta^3 \), thus producing also a \( \mathbb{Z}_4 \) \( K3 \)-orbifold with \( n_2 = 0 \). Finally, the three \( \mathbb{Z}_{12-II} \) cases have an invariant plane for \( \theta^2 \) of \( \mathbb{Z}_6 \) \( K3 \)-orbifold structure with \( n_3 = 0 \). Thus, there are no harmonic one-loop corrections to \( (1,2) \).

5. **M–theory origin of \( d=4 \) gauge couplings**

In this section we want to discuss the relation\(^\text{17}\) of our \( \mathbb{N}=1 \) gauge threshold results (4.4) and (4.5) to the strongly coupled heterotic string in ten dimensions, which is described by M–theory compactified on \( S^1/\mathbb{Z}_2 \) [53]. This question has been raised in [54] and worked out for standard–embedding in [37].

5.1. **Gauge kinetic function in \( \mathbb{N}=1, d=4 \) weakly coupled heterotic string theory**

The relevant object to link the four-dimensional one-loop corrections to the strong coupling expansion of M–theory is the gauge kinetic function \( f \) of the gauge groups \( G_A, G_{A'} \), in which the findings of the previous sections are summarized:

\[
 f_{G_A G_{A'}}(S, T^i, U^i, V) = -iS + \sum_{i=1,2,3} \frac{\nu_i}{2} \left( \frac{1}{5} (\partial_{T^i} \partial_{U^i} - \frac{1}{4} \partial_{T^i}^2) h_{n_i} - \frac{1}{5\pi^2} \ln \Psi_{n_i}(T^i, U^i, V) \right) + \mathcal{O}(e^{2\pi i S}). 
\]

(5.1)

\(^{17}\) Our \( K3 \) gauge threshold results (cf. sections 2 and 3) can also be related to M–theory compactified on \( S^1/\mathbb{Z}_2 \times K3 \) (see also [51, 52]).
After the rescalings $S \rightarrow 4\pi S$, $f \rightarrow 4\pi f$, which corresponds to the dilaton choice $S = \frac{\theta_0}{2\pi} + i \frac{4\pi}{g_{\text{string}}}$ the large $T^i$–expansion\(^{18}\) (large Kähler moduli) of (5.1) becomes (using $\Delta_{35} \rightarrow e^{4\pi iT}$, $\Delta_{10} \rightarrow e^{2\pi iT}$):

$$f_{G_A} = -iS - i \sum_{i=1,2,3} \frac{\nu_i}{\nu}\left(\frac{b^{N=2}_{G_i}}{12} - 1\right)T^i + O(e^{8\pi^2 iS}) = -iS + i \sum_{i=1,2,3} \frac{\nu_i}{2} \frac{n_i}{\nu} T^i + O(e^{8\pi^2 iS}) ,$$

$$f_{G_A'} = -iS - i \sum_{i=1,2,3} \frac{\nu_i}{\nu}\left(\frac{b'^{N=2}_{G_i}}{12} - 1\right)T^i + O(e^{8\pi^2 iS}) = -iS - i \sum_{i=1,2,3} \frac{\nu_i}{2} \frac{n_i}{\nu} T^i + O(e^{8\pi^2 iS}) .$$

These expressions may be directly identified with the $f$–functions, which arise upon dimensional reduction of the weekly–coupled ten–dimensional heterotic string. This holds—at least in this limit— for generic $n$, as in this reduction $n$ enters only as the instanton number of the gauge bundle in the Bianchi identity. Therefore, from the ten–dimensional viewpoint, the form of the gauge kinetic function (5.1) in four dimensions is dictated by the Green–Schwarz anomaly cancellation in ten dimensions, together with target–space duality\(^{55,37}\). As a remark, let us mention that for the compactifications we have considered, i.e. N=1 orbifolds with N=2 sectors, which are described by $K3 \times T^2$ dynamics, (5.1) can be also deduced from the relevant Green–Schwarz anomaly cancellation terms in six dimensions (2.9), since in (5.2) each N=2 subsector may be thought as a decompactification limit $T^i \rightarrow \infty$ to six dimensions.

For (2,2) Calabi–Yau compactifications $X$, there exists a relation of one–loop gauge threshold corrections to the (large Kähler modulus expansion thereof) topological index $F_1$\(^{56}\). The identity $\Delta_{E_6} - \Delta_{E_6'} \rightarrow 12 F_1$ allows us, to write for the large radius expansion of the gauge kinetic functions\(^{27}\):

$$f_{E_6'} - f_{E_6} \rightarrow 2 \sum_{i=1}^{h_{(1,1)}} t^i \int_X J_i \wedge c_2(R) .$$

(5.3)

Here $J_i$ is a basis for the Kähler class $H^{(1,1)}$ and $c_2$ is the second Chern class of the Calabi–Yau threefold $X$. The same limit (5.3) appears after a dimensional reduction of the ten–dimensional Green–Schwarz term specializing to the difference of the $E_8$, $E_6$

\(^{18}\) For $n_i = -12$ and restricting the sum to $i = 3$, we recover the results of \(^{37}\).
axionic couplings. Applied\footnote{To the orbifolds with standard–embedding (e.g. $I$ and $V$ of table 5), we get e.g.} to the orbifolds with standard–embedding (e.g. $I$ and $V$ of \tableref{5}), we get e.g.
\[
\hat{f}_{E_8} - \hat{f}_{E_6} \longrightarrow - \frac{1}{2} \chi_{K3} \sum_{i=1,2,3} \frac{\nu_i}{\nu} T^i = - \frac{1}{4} \chi(T^6/\mathbb{Z}_\nu) \sum_{i=1,2,3} t^i ,
\]
with the CY moduli $t^i = \frac{\nu_i}{\nu} T^i$ and the CY–Euler number\footnote{The relation (5.3) keeps its validity in the orbifold limit, although some CY moduli $t^i$ are frozen at finite values in the Kähler moduli space.} $\chi(X) = 48$ of the underlying $\mathbb{N}=1$ orbifold $X$. The prime means, that we only sum over such moduli, which appear from $\mathbb{N}=2$ subsectors. This limit is in agreement with (5.2) for $n_i = -12$. In that case it is straightforward to work out the integral (cf. below for the more general case).

5.2. M–theory on $S^1/\mathbb{Z}_2 \times T^6/\mathbb{Z}_\nu$

In \cite{54,37} it was argued that \eqref{5.1} encodes for standard–embedding the strong coupling expansion (an expansion in the eleven dimensional gravitational coupling constant $\kappa^2 := \kappa_{11}^2$) of M–theory on $S^1/\mathbb{Z}_2$ compactified on a CYM $X$. I.e. a perturbative heterotic gauge threshold calculation (as performed in the previous section and summarized in eq. \eqref{5.1}) gives the gauge couplings of M–theory on $S^1/\mathbb{Z}_2$ compactified on this CYM. To zeroth order in $\kappa^2$ the relative sizes of the CY and $S^1$ are not relevant and the expansion of the strongly coupled heterotic string theory gives the same effective action in four dimensions as the dimensional reduction of the weakly coupled ten–dimensional heterotic string. Moreover, at higher orders in $\kappa$ their four–dimensional effective actions take the same analytic form and thus cannot be distinguished from each other. In this section we want to discuss the case with non–standard embedding, since it leads to realistic string vacua \cite{57}. We will see, how \eqref{5.2} arises from $M$–theory compactification.

The $G_A, G_{A'}$ gauge fields live on the two nine–branes. After compactification on the CYM $X$ their coupling is given by
\[
g_{G_A,G_{A'}}^{\hat{\kappa}^2} = \frac{2V_X}{(4\pi)^{5/3} \kappa^{4/3}} ,
\]
\footnote{For all toroidal $\mathbb{Z}_\nu$ orbifolds $X$ we have $\chi(T^6/\mathbb{Z}_\nu) = 48$, except $\chi(T^6/\mathbb{Z}_3) = 72$. However a $\mathbb{Z}_3$ orbifold does not give rise to moduli dependent thresholds $\nu_i = 0$ and we may safely introduce $\chi(X) = 48$ in (5.4).} to order $\hat{\kappa}^{2/3}$ relative to the bulk. Here, $V_X$ is the Calabi–Yau volume at the boundaries $x_{11} = 0$ and $x_{11} = \pi \rho$, respectively. Corrections coming from interactions to the bulk,
start at order $\kappa^{4/3}$ and modify this relation. This results in a variation of the CY volume $V$ over the interval $x_{11}$. Therefore, to determine the two gauge couplings $g_{G_A}$ and $g_{G_A'}$, we need an expression for the two volumina $V(0)$ and $V(\pi \rho)$ at the two fixed points. Here $\rho$ is the radius of the eleventh dimension $S^1$. In Wittens linear approximation their difference is given by [51]

$$V(\pi \rho) - V(0) = 2\pi^2 \rho \left( \frac{\kappa}{4\pi} \right)^{2/3} \int_X \omega \wedge \frac{\text{tr} F \wedge F - \frac{1}{2} \text{tr} R \wedge R}{8\pi^2}. \quad (5.6)$$

The r.h.s. is an integral over the CYM $X$ to be worked out at the boundary $x_{11} = 0$. In particular this means, that the gauge fields (in the following denoted by $F^{(1)}$) refer to the gauge group $G_A$, which lives on the wall $x_{11} = 0$. Fortunately, the r.h.s. is independent on $x_{11}$ in the linear approximation. This means that the CY moduli entering there are $x_{11}$–independent and the whole integral describes a generic topological coupling on the CYM $X$. Nonetheless, the interpretation of the CY moduli appearing in the four–dimensional low–energy effective action as chiral fields is different for $M$–theory on $S^1/\mathbb{Z}_2 \times X$ and 10d heterotic string on $X$. Since the former are coming from an eleven-dimensional theory, they describe five–dimensional fields, which have to be averaged. To the order we are considering, this averaging means, that all CY moduli fields (more precisely: their non–axionic parts) refer to metric scalars $g^{11}_{MN}$ in the middle $x_{11} = \frac{1}{2} \pi \rho$ of $S^1/\mathbb{Z}_2$. More details about this identification can be found in [58][59]. Besides, further aspects have been analyzed in a burst of recent papers [58][59][60], (however, all of them dealing with standard embedding).

We want to work out\footnote{Recently, the relation eq. (5.6) has been discussed in [61] for one specific CY example and in [62] with the emphasis on the axionic symmetries of the Kähler moduli $t^i$.} (5.6) for the models we have considered in section 4, i.e. in particular for instanton non–standard embeddings. Since we are compactifying on a Calabi–Yau manifold, $R_{ik} = 0$ and $F^{(a)}_{ij} = 0$, and the only non–vanishing components of $\text{tr} R \wedge R$ and $\text{tr} F \wedge F$ come from the combinations $R_{ijkl}$ and $F_{i}^{(1)} F_{k}^{(1)}$, respectively [83], we may expand the fields

$$\text{tr} R \wedge R = \chi i d^i, \quad \chi i = \int_{C_i} \text{tr} R \wedge R, \quad (5.7)$$

$$\text{tr} F^{(1)} \wedge F^{(1)} = n_i^{(1)} d^i, \quad n_i^{(1)} = \int_{C_i} \text{tr} F^{(1)} \wedge F^{(1)}$$
w.r.t. a basis $d^i$ of harmonic $(2,2)$–forms $(i = 1,\ldots,h_{(2,2)})$. Of course, from Poincaré duality $h_{(2,2)} = h_{(1,1)}$. The corresponding 4–cycles $C_i$ are chosen to fulfil the following intersection properties
\[
\int_{C_i} d^j = \delta^j_i, \quad \int_X J_i \wedge d^j = \delta^j_i,
\]
where $J_i$ is a basis of $(1,1)$–forms to which the Kähler moduli $t^i (i = 1,\ldots,h_{(1,1)})$ are associated. Besides we expand the Kähler form $\omega$ of $X \omega = t^i J_i$. In the orbifold limit of a CYM all the instantons are stuck at the fixed points. Therefore, the 4–cycle integrals (5.7) receive contributions only from orbifold singularities (of (complex) codimension 2), since these are the only sources for curvature and places for non–trivial gauge connections. Thus we have to consider all 4–cycle $C_{f_\alpha}$ integrals around fixed points $f_\alpha$ of codimension 2 rather than 3. These are precisely the N=2 sectors of the N=1 orbifold under consideration and they can be described by K3 dynamics. Locally at these points, the manifold is replaced by an $ALE$–space with $A_\nu$–type singularity. These are asymptotic locally flat non–compact spaces. They have the Euler number $\chi_{ALE_\nu} = \frac{\nu^2-1}{\nu}$. Thus, in (5.7) we obtain the coefficients:
\[
\begin{align*}
n^{(1)}_{\alpha} &= \int_{C_{f_\alpha}} \text{tr}F^{(1)} \wedge F^{(1)} = k^{(1)}_{\alpha} + \frac{1}{2} \frac{1}{\nu_\alpha} \sum_{I=1}^{8} \gamma^I_\alpha (\nu_\alpha - \gamma^I_\alpha), \\
\chi_{ALE_\nu_{\alpha}} &= \int_{C_{f_\alpha}} \text{tr}R \wedge R = \frac{\nu^2_\alpha - 1}{\nu_\alpha}.
\end{align*}
\]
Here $n^{(1)}_{\alpha}$ is the individual gauge instanton number (3.32) at the fixed point $f_\alpha$ (which is supposed to have twist order $\nu_{\alpha}$) and the curvature singularity contributes $\chi_{ALE_\nu_{\alpha}}$ to the gravitational instanton contribution. Let us mention the identity $\int_{C_{f_\alpha}} dH = n^{(1)}_{\alpha} + n^{(2)}_{\alpha} + \chi_{ALE_\nu_{\alpha}} = 0$, which expresses local charge cancellation at the fixed point $f_\alpha$. In addition, after (5.8), to get a non–zero wedge product, the Kähler $J_i$ form has to lie in the remaining orthogonal one (complex) dimensional plane. This is the plane $T^2_i$, left invariant under the orbifold twist, with Kähler modulus $t^i \equiv \frac{\nu}{\nu_\nu} T^i$.

In total, summing up all source contributions (5.9) at codimension 2 fixed points and noting the fact (which holds for every $t^i$, which has a set of codimension 2 fixed points $f_\alpha$)
\[
\sum_k N_k \chi_{ALE_k} = \chi_{K3} = n^{(1)}_{i} + n^{(2)}_{i},
\]

22 Since prime orbifolds ($\nu = \text{prime}$) have no codimension 2 fixed points, the integral in (5.6) vanishes. In those cases, there are also no perturbative one–loop corrections, i.e. $\nu_i = 0$ in (4.4) and (4.5).
where $N_k$ is the number of (dimension 1) fix–planes of order $k$, we derive for (5.6):

$$V(\pi\rho) - V(0) = 2\pi^2 \rho \left(\frac{\kappa}{4\pi}\right)^{2/3} \sum_i \frac{\nu_i}{\nu} T^i \left[n_i^{(1)} - \frac{1}{2}(n_i^{(1)} + n_i^{(2)})\right]$$

$$= -2\pi^2 \rho \left(\frac{\kappa}{4\pi}\right)^{2/3} \sum_i \frac{\nu_i}{\nu} n_i T^i .$$

(5.11)

To compare with (5.2) we have to translate the eleven dimensional scales $\kappa_{11}$ and $R_{11} \equiv \pi\rho$ to ten dimensional heterotic string quantities [54]:

$$\frac{\rho}{\kappa_{11}^{2/3}} = \frac{1}{2^{7/3} \pi^{8/3}} \frac{1}{\alpha'} .$$

(5.12)

With (5.11) and (5.12), the difference of the two gauge couplings $\alpha_{G_A}$ and $\alpha_{G_A'}$ becomes

$$g_{G_A'}^{-2} - g_{G_A}^{-2} = -\frac{1}{32 \pi^3} \frac{1}{\alpha'} \sum_i \frac{\nu_i}{\nu} n_i T^i ,$$

(5.13)

which agrees with (5.2) up to numerical constants. This leads to the generalization of (5.3) to non–standard embedding orbifolds:

$$f_{G_A'} - f_{G_A} \rightarrow - \sum_i \frac{\nu_i}{\nu} n_i T^i .$$

(5.14)

Tracing back (5.14) to its origin (5.6) we conjecture the large radius expansion of the holomorphic index (5.3) for (0,2)–compactifications:

$$f_{G_A'} - f_{G_A} \rightarrow 2 \sum_{i=1}^{h_{(1,1)}} t^i \int_{X} J_i \wedge \left(1 + F \wedge F - \frac{1}{2} R \wedge R \right) .$$

(5.15)

Let us make some final remarks: The techniques, developed in this section for the dimensional reduction on (0,2) orbifolds, may be also used to extract other (than harmonic gauge coupling) terms in the four dimensional effective action. In particular, we find it interesting to trace back the origin of the non–harmonic coupling $G^{(1)}$, appearing in eqs. (4.4) and (4.5) to ten or eleven dimensions. On the other hand, in four dimensions it is due to IR–effects and its large radius behaviour (maximally $logT$) is quite different than

---

23 Concretely: $\nu = 2$: $T^4/Z_2$, $N_2 = 16$; $\nu = 3$: $T^4/Z_3$, $N_3 = 9$; $\nu = 4$: $T^4/Z_4$, $N_4 = 4$, $N_2 = 6$; $\nu = 6$: $T^4/Z_6$, $N_6 = 1$, $N_3 = 4$, $N_2 = 5$.

24 Of course, for the orbifold examples in table 7 we get $V(\pi\rho) - V(0) = 0$. This allows for equal couplings at both boundaries.
that of the gauge couplings. Therefore, we cannot obtain it in the limit described above, which gives order $T$ effects in the effective 4$d$ action. Moreover, it would be interesting to determine the expression (5.13) (and eventually eqs. (1.4) and (4.5)) from $F$–theory on a fourfold by considering 7–brane exchange interactions. A similar treatment has been accomplished in $d = 8$ with $F$–theory on $K3$ [65]. There, four–point couplings $R^4$ and $F^4$ could be calculated by means of 7–brane exchanges. However, the $d = 4$ case is more involved because of the complicated bundle structure on the fourfold and 7–branes.

5.3. Including NS 5–branes

In [32], the possibility of adding NS 5–branes into the space–time was considered. This then may be considered as additional source term in the Bianchi identity for the 4–form $G$ [51]. Again, this effect may be studied in N=1 non–perturbative orbifold constructions [33]. These orbifolds are non–modular invariant at the perturbative level. That means that they have gauge and/or gravitational anomalies at the perturbative level. However, non–perturbative effects, like additional 5–branes [31][32], render the theories consistent. Usually, they may have N=2 subsectors, whose non–perturbative formulation may be traced back25 to known N=1, $d = 6$ smooth $K3$ dynamics of tensionless strings, small instantons or 5–branes, compactified on the torus $T^2$. In M–theory, where these effects are described by NS 5–branes approaching one of the 9–branes, the characteristic length is their relative distance $<\Phi>$ to one 9–brane. The field $\Phi$ is a real scalar of a tensor multiplet in $d = 6$, N=1. After torus compactification it becomes a scalar of an N=2, $d = 4$ vector multiplet, whose gauge field show the coupling $25$

$$\text{Re}(U) F_{\mu\nu} F^{\mu\nu}.$$  

(5.16)

Physical quantities are expanded w.r.t. $1/u \sim e^{2\pi^2 i(T_1 + i<\Phi>T_2)}$, accounting for the instantons, which are strings of tension $<\Phi>$, wrapped around the torus $T^2$. In particular, this gives e.g. SW–like expansions (in $u$) for the gauge coupling (5.16). Together with the usual (conventional) perturbative expansion26 w.r.t. the dilaton $S$, in $d = 4$ we are then left with two expansions, valid in different regimes of the moduli space. On the other

25 Provided there are enough massless blow up modes.

26 I.e. $S \to \infty$, which is clearly not the right limit, when one wants to take into account effects of the NS 5–branes. This limit puts together the two 9–branes and we lose the effects of the 5–branes, which were in between them.
hand, the rôle of a perturbative string threshold correction in the sense of sects. 2 and 3 as a perturbative expansion in the dilaton field $S$ does no longer make sense, since we are dealing with an anomalous or non–modular invariant theory. The perturbative part of the partition function alone is not sufficient to consider, since it is not one–loop modular invariant and we do not know its non–perturbative extension rendering modular invariance. However, modular invariance is a quite important ingredient in string–perturbation theory. In fact, the results of sections 2–4 heavily rely on the modular invariance of the partition function and we are not allowed to apply them for those kinds of models, although a naive guess might urge us, to just insert in (2.4) instanton numbers $(n_i^{(1)}, n_i^{(2)})$, fulfilling $n_i^{(1)} + n_i^{(2)} \neq 24$. In (5.2) this would lead to an asymmetric $T$–dependence.

However, for the twisted sectors of the N=1 non–perturbative orbifold we do not have any description. Less is known about the non–perturbative effects, which are supposed to reinforce modular invariance or anomaly freedom. It is believed that the analog of small instanton dynamics in N=1, $d = 6$ is played by chirality change in N=1, $d = 4$ [66].

Let us draw one conclusion (just from considering the N=2 subsectors of the non–perturbative orbifolds): The coupling (5.14) shows a quite different structure than what one expects in ordinary string perturbation theory, where the dilaton $S$ controls all tree–level couplings (4.1). We have different expansions for the gauge couplings, valid in different regions of the moduli space. So far lacking a complete (non–perturbative) heterotic description, which eventually puts $S$ and the moduli $T, U, V$ on the same footing. This is naturally provided by $F$–theory compactifications, which will certainly lead to quite new concepts in string phenomenology [67].

Acknowledgements: I wish to thank G.L. Cardoso, J.–P. Derendinger, and K. Intriligator for interesting discussions. Moreover, I thank Z. Lalak, W. Lerche, H.P. Nilles, B. Pioline, and especially P. Mayr for helpful discussions. The major part of this work was carried out at Neuchâtel University with the support of the Swiss National Science Foundation, the European Commission TMR programme ERBFMRX–CT96–0045, and OFES no. 95.0856.
Appendix A. Jacobi functions

The coupling of the genus (2.5) to one Wilson line in (2.6) is described by Jacobi forms \([2][6]\). A Jacobi form (for more details see [68]) of weight \(s\) and index \(m\) enjoys

\[
\begin{align*}
    f_{s,m}(a\tau + b, c\tau + d, z) &= (c\tau + d)^s e^{\pi i \frac{m z^2}{c\tau + d}} f_{s,m}(\tau, z), \\
    f_{s,m}(\tau, z + \lambda\tau + \mu) &= e^{-\pi i m (\lambda^2 \tau + 2\lambda z)} f_{s,m}(\tau, z),
\end{align*}
\]

for \((a\ b\ c\ d) \in SL(2,\mathbb{Z})\) and \(\lambda, \mu \in \mathbb{Z}\). Prominent examples (for index 1) are the Jacobi \(\theta\)-functions \((y = e^{2\pi i z})\):

\[
\begin{align*}
    \theta_{(\alpha\beta)}(q, y) &= \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(n + \frac{1}{2}\alpha)^2} e^{\pi i (n + \frac{1}{2}\alpha) y} \frac{1}{y^{\frac{1}{2}(n + \frac{1}{2})}}.
\end{align*}
\]

Explicitly,

\[
\begin{align*}
    \theta_{(1\ 1)}(q, y) &= \theta_1(q, y) = i \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2}(n + \frac{1}{2})^2} y^{n + \frac{1}{2}} \\
    \theta_{(1\ 0)}(q, y) &= \theta_2(q, y) = \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(n + \frac{1}{2})^2} y^{n + \frac{1}{2}} \\
    \theta_{(0\ 0)}(q, y) &= \theta_3(q, y) = \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}n^2} y^{n} \\
    \theta_{(0\ 1)}(q, y) &= \theta_4(q, y) = \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2}n^2} y^{n}.
\end{align*}
\]

The ring of Jacobi forms of index 1 is generated by the Jacobi–Eisenstein functions

\[
\begin{align*}
    E_{4,1}(q, y) &= \frac{1}{2} [\theta_2(q, y)^2 \theta_2^6 + \theta_3(q, y)^2 \theta_3^6 + \theta_4(q, y)^2 \theta_4^6], \\
    E_{6,1}(q, y) &= \frac{1}{2} [\theta_2^6 \theta_4(q, y)^2 (\theta_2^4 + \theta_3^4) + \theta_3^6 \theta_4(q, y)^2 (\theta_2^4 - \theta_3^4) - \theta_2^6 \theta_2(q, y)^2 (\theta_2^4 + \theta_3^4)].
\end{align*}
\]

with \(\theta_1 = \theta_{[1\ 1]}, \theta_2 = \theta_{[1\ 0]}, \theta_3 = \theta_{[0\ 0]}\) and \(\theta_4 = \theta_{[0\ 1]}\). The K3 elliptic genus \([36]\) is a Jacobi form of weight 0 and index 1:

\[
Z_{K3}(q, y) = \frac{1}{\eta^{24}} \left( \frac{E_2^2 E_{4,1}(q, y) - E_6 E_{6,1}(q, y)}{\eta^{24}} \right).
\]

Any Jacobi form \(f_{s,1}(q, y)\) of index 1 can be decomposed as

\[
f_{s,1}(q, y) = f_{s,1}^{even}(q) \theta_{even}(q, y) + f_{s,1}^{odd}(q) \theta_{odd}(q, y),
\]
with \( \theta_{\text{even}}(\tau, z) = \theta_3(2\tau, 2z) \) and \( \theta_{\text{odd}}(\tau, z) = \theta_2(2\tau, 2z) \). Finally, we define

\[
\hat{f}_{s,1}(q) = f_{s,1}^{\text{even}}(q) + f_{s,1}^{\text{odd}}(q) .
\] (A.7)

Applying (A.6) for \( E_{4,1}(q, y) \) provides (for \( y = 1, z = 0 \)) the decomposition of one \( E_8 \) gauge factor into \( E_7 \times A_1 \)

\[
E_4 = E_{4,1}(q, 1) = E_{4,1}^{\text{even}}(q)\theta_3(2\tau) + E_{4,1}^{\text{odd}}(\tau)\theta_2(2\tau) ,
\] (A.8)

with the \( E_7 \)-characters \( Z_{133} = E_{4,1}^{\text{even}} \) and \( Z_{56} = E_{4,1}^{\text{odd}} \) and \( A_1 \)-characters \( Z_{A_1} = \theta_3(2\tau) \) and \( Z_{A_1} = \theta_2(2\tau) \). Then, the coupling of one Wilson to the Narain lattice sum \( Z_{3,2} \) and to one \( E_8 \)-gauge group factor (or more generally to the remaining part of the index) can be described by the Jacobi form \( E_{4,1}(q) \) (or \( f_{s,1}(q) \))

\[
f_{s,1}(q, y) \otimes Z_{3,2}(q, \bar{q}) := \sum_{k=\text{even}} \sum_{m_i, n_i} e^{2\pi i \tau (\frac{1}{4}k^2 + m_1 n_1 + m_2 n_2)} e^{-2\pi \tau |p_R|^2} f_{s,1}^{\text{even}}(q) + \sum_{k=\text{odd}} \sum_{m_i, n_i} e^{2\pi i \tau (\frac{1}{4}k^2 + m_1 n_1 + m_2 n_2)} e^{-2\pi \tau |p_R|^2} f_{s,1}^{\text{odd}}(q) ,
\] (A.9)

with \( p_R \) defined in eq. (2.6).

Appendix B. Orbifold details

In this section we give the phase matrix \( k_{(a,b)} \), which appears in the \( K3 \) index (3.2).

B.1. \( Z_2 \) orbifold

\[
k_{(a,b)} = 64 \begin{pmatrix}
0 & 1 \\
1 & e^{-\pi i \frac{1}{16}(2-\gamma^2)}
\end{pmatrix}
\] (B.1)

B.2. \( Z_3 \) orbifold

\[
k_{(a,b)} = 36 \begin{pmatrix}
0 & 1 & 1 \\
1 & e^{-\frac{1}{6} \pi i (2-\gamma^2)} & e^{-\frac{1}{6} \pi i (2-\gamma^2)} \\
1 & e^{-\frac{2}{3} \pi i (2-\gamma^2)} & e^{-\frac{2}{3} \pi i (2-\gamma^2)}
\end{pmatrix}
\] (B.2)

B.3. \( Z_4 \) orbifold

\[
k_{(a,b)} = 16 \begin{pmatrix}
0 & 1 & 4 & 1 \\
1 & e^{-\pi i \frac{1}{16}(2-\gamma^2)} & e^{-\pi i \frac{1}{16}(2-\gamma^2)} & e^{-\pi i \frac{1}{16}(2-\gamma^2)} \\
4 & e^{\pi i \frac{1}{16}(2-\gamma^2)} & 4e^{-\pi i \frac{1}{16}(2-\gamma^2)} & e^{\pi i \frac{1}{16}(2-\gamma^2)} \\
1 & e^{\pi i \frac{3}{16}(2-\gamma^2)} & e^{\pi i \frac{3}{16}(2-\gamma^2)} & e^{-\pi i \frac{3}{16}(2-\gamma^2)}
\end{pmatrix}
\] (B.3)
B.4. $Z_6$ orbifold

\[ k(a,b) = 4 \begin{pmatrix} 0 & 1 & 9 & 16 & 9 & 1 \\ 1 & e^{\frac{1}{3}\pi i\Gamma} & e^{\frac{2}{3}\pi i\Gamma} & e^{\pi i\Gamma} & e^{\frac{4}{3}\pi i\Gamma} & e^{\frac{5}{3}\pi i\Gamma} \\ 9 & e^{\frac{2}{3}\pi i\Gamma} & 9 e^{\frac{2}{3}\pi i\Gamma} & 1 & 9 e^{-\frac{2}{3}\pi i\Gamma} & e^{-\frac{4}{3}\pi i\Gamma} \\ 16 & e^{\pi i\Gamma} & 16 e^{\pi i\Gamma} & 1 & 16 e^{-\pi i\Gamma} & e^{-\frac{8}{3}\pi i\Gamma} \\ 9 & e^{-\frac{2}{3}\pi i\Gamma} & 9 e^{-\frac{2}{3}\pi i\Gamma} & 1 & 9 e^{\frac{4}{3}\pi i\Gamma} & e^{\frac{8}{3}\pi i\Gamma} \\ 1 & e^{-\frac{2}{3}\pi i\Gamma} & e^{-\frac{2}{3}\pi i\Gamma} & e^{-\pi i\Gamma} & e^{-\frac{4}{3}\pi i\Gamma} & e^{\frac{2}{3}\pi i\Gamma} \end{pmatrix} \]  

with $12\Gamma := \gamma^2 + \tilde{\gamma}^2 - 2$.

Appendix C. Lowest Expansion of $B_A$

In this part we derive the lowest $V$–expansion in (3.14). We consider the integral (2.3) with the integrand (2.7) w.r.t. to a gauge group $G_A$, which is not enhanced at any point in the moduli space, except $T \to i\infty$. It is easy to show

\[ \frac{\partial \Delta_A}{\partial V} \bigg|_{V=0} = 0 , \]  

due to a possible relabelling of the quantum numbers appearing in the sum of $Z$. We use the identity $(p_R,0 := p_R|_{V=0})$

\[ \frac{\partial^2}{\partial V^2} Z_{3,2} \bigg|_{V=0} = \frac{2i}{-iU + i\bar{U}} \frac{\partial}{\partial T} Z_{2,2} + \frac{2i}{-iT + i\bar{T}} \frac{\partial}{\partial U} Z_{2,2} + \frac{8\pi^2 \tau_2^2}{(-iT + i\bar{T})(-iU + i\bar{U})} \bar{p}_{R,0}^2 \left( k^2 - \frac{1}{2\pi \tau_2} \right) Z_{2,2} \]  

(C.2)

to perform the integrand of (2.3) leading to

\[ \frac{\partial^2}{\partial V^2} \Delta_A \bigg|_{V=0} = \frac{2i}{-iU + i\bar{U}} \frac{\partial}{\partial T} \Delta_A + \frac{2i}{-iT + i\bar{T}} \frac{\partial}{\partial U} \Delta_A + \frac{8\pi^2 \tau_2^2}{(-iT + i\bar{T})(-iU + i\bar{U})} \bar{Z}_{2,2} \bar{p}_{R,0}^2 \left( k^2 - \frac{1}{2\pi \tau_2} \right) \]  

(C.3)

Here $\Delta_A$ denotes the threshold correction for vanishing Wilson line. Let us denote the last integral by $R$. Then we perform a manipulation similar to the one introduced in [69]. With

\[ (-iU + i\bar{U})^2 \frac{\partial}{\partial T} \left[ \sum \frac{\bar{p}_{L,0}^2}{(-iU + i\bar{U})} q^\frac{1}{2}|p_L|^2 q^\frac{1}{2}|p_R|^2 \right] = \frac{T - T}{\pi \tau_2} \sum \partial_T [\tau_2 \partial_T Z] , \]  

(C.4)
we calculate $\partial_U R$

$$\partial_U R = \frac{-8\pi i}{(-iU + iU)^2} \int \frac{d^2 \tau}{\tau_2} \sum_k \tau_2 \partial_{\tau} (\tau_2 \partial_T Z_0) \left( k^2 - \frac{1}{2\pi \tau_2} \right) C(\tau)$$

$$= \frac{-2}{(-iU + iU)^2} \frac{\partial \tilde{\Delta}_A}{\partial T}$$

(C.5)

where the last eq. follows after a partial integration. After a duality respecting integration we find:

$$R = -2 \frac{\partial \tilde{\Delta}_A}{\partial U} \frac{\partial \tilde{\Delta}_A}{\partial T}.$$  

(C.6)

Therefore we have

$$\frac{\partial^2 \Delta_A}{\partial V^2} \bigg|_{V=0} = 2 \frac{\tilde{\Delta}_{A,T}}{U-U} + 2 \frac{\tilde{\Delta}_{A,U}-2\tilde{\Delta}_{A,U}\tilde{\Delta}_{A,T}}{T-T}$$

(C.7)

and finally:

$$\frac{\partial^2 \Delta_A}{\partial V^2} \bigg|_{V=0} = \frac{8\eta'(T)\eta'(U)}{\eta(T)\eta(U)} - \frac{2}{(T-T)(U-U)}.$$  

(C.8)

A similar result is obtained for $\frac{\partial \Delta_A}{\partial V^2} \bigg|_{V=0}$. However one may show

$$\frac{\partial^2 \Delta_A}{\partial V \partial V} \bigg|_{V=0} = \frac{2b_A^{N=2}}{(-iT+iT)(-iU+iU)},$$

verifying the integrability condition of $[69]$.

Appendix D. $Z_6$–orbifold limits of $K3$ and their instanton numbers
| $\gamma$          | $\delta$          | perturbative gauge group | $(n^{(1)}, n^{(2)})$ |
|-------------------|-------------------|--------------------------|----------------------|
| (1,1,0,0,0,0,0,0) | (0,0,0,0,0,0,0,0) | $U(1) \times E_7 \times E_8'$ | (24,0)               |
| (3,2,1,0,0,0,0,0) | (0,0,0,0,0,0,0,0) | $U(1)^2 \times E_6 \times E_8'$ | (24,0)               |
| (5,1,0,0,0,0,0,0) | (0,0,0,0,0,0,0,0) | $U(1) \times SU(2) \times SO(12) \times E_8'$ | (24,0)               |
| $\frac{1}{2}(7,1,1,1,1,1,-1)$ | (0,0,0,0,0,0,0,0) | $U(1) \times SU(2) \times SU(7) \times E_8'$ | (24,0)               |
| $\frac{1}{2}(9,3,3,1,1,1,1)$ | (0,0,0,0,0,0,0,0) | $U(1) \times SU(3) \times SU(6) \times E_8'$ | (24,0)               |
| (2,2,0,0,0,0,0,0) | (3,3,0,0,0,0,0,0) | $U(1) \times SU(2) \times SU(2)^j \times E_8'$ | (12,12)              |
| (3,1,1,1,1,1,1) | (3,3,0,0,0,0,0,0) | $SU(9) \times SU(2)^j \times E_8'$ | (14,10)              |
| (1,1,1,1,1,1,1,−1) | (3,3,0,0,0,0,0,0) | $U(1) \times SU(8) \times SU(2)^j \times E_7'$ | (18,6)               |
| (5,1,1,1,1,1,1,−1) | (3,3,0,0,0,0,0,0) | $U(1) \times SU(8) \times SU(2)^j \times E_7'$ | (20,4)               |
| (2,2,2,2,0,0,0,0) | (3,3,0,0,0,0,0,0) | $U(1) \times SU(4)^j \times SU(2)^j \times E_7'$ | (20,4)               |
| (4,2,2,0,0,0,0,0) | (1,1,0,0,0,0,0,0) | $SU(3) \times E_6 \times U(1)^j \times E_7'$ | (12,12)              |
| (6,0,0,0,0,0,0,0) | (1,1,0,0,0,0,0,0) | $SO(16) \times U(1)^j \times E_7'$ | (8,16)               |
| (2,2,2,0,0,0,0,0) | (1,1,0,0,0,0,0,0) | $U(1) \times SU(3) \times SO(10) \times U(1)^j \times E_7'$ | (18,6)               |
| (3,3,1,1,1,1,1,−1) | (1,1,0,0,0,0,0,0) | $U(1) \times SU(2)^j \times SU(2)^j \times E_7'$ | (18,6)               |
| (2,2,1,0,0,0,0,0) | (2,2,0,0,0,0,0,0) | $U(1) \times SU(2)^j \times SU(6) \times U(1)^j \times E_7'$ | (18,6)               |
| (4,1,1,0,0,0,0,0) | (2,2,0,0,0,0,0,0) | $U(1)^2 \times SU(2)^j \times SU(2)^j \times E_7'$ | (20,4)               |
| (5,1,1,1,1,1,0,0) | (2,2,0,0,0,0,0,0) | $SU(2)^j \times SU(3)^j \times SU(6) \times U(1)^j \times E_7'$ | (20,4)               |
| (3,2,1,0,0,0,0,0) | (4,2,2,0,0,0,0,0) | $U(1)^2 \times E_6 \times SU(3)^j \times U(1)^j \times E_7'$ | (12,12)              |
| (5,1,0,0,0,0,0,0) | (4,2,2,0,0,0,0,0) | $U(1)^2 \times E_6 \times SU(3)^j \times U(1)^j \times E_7'$ | (14,10)              |
| $\frac{1}{2}(7,1,1,1,1,1,1,−1)$ | (4,2,2,0,0,0,0,0) | $U(1) \times SU(2)^j \times SU(2)^j \times E_7'$ | (17,7)               |
| $\frac{1}{2}(9,3,3,1,1,1,1)$ | (4,2,2,0,0,0,0,0) | $U(1)^2 \times E_6 \times SU(3)^j \times U(1)^j \times E_7'$ | (18,6)               |
| (4,2,0,0,0,0,0,0) | (2,2,1,0,0,0,0,0) | $U(1)^2 \times SU(2)^j \times SU(2)^j \times E_6'$ | (12,12)              |
| (5,1,1,1,1,1,1,1) | (2,2,1,0,0,0,0,0) | $SU(9) \times U(1)^j \times SU(2)^j \times E_6'$ | (11,13)              |
| (1,1,1,1,1,1,1,−1) | (2,2,1,0,0,0,0,0) | $U(1) \times SU(8)^j \times SU(2)^j \times U(1)^j \times E_6'$ | (15,9)               |
| (5,1,1,1,1,1,1,−1) | (2,2,1,0,0,0,0,0) | $U(1) \times SU(8)^j \times SU(2)^j \times U(1)^j \times E_6'$ | (11,13)              |
| (2,2,2,2,2,0,0,0) | (2,2,1,0,0,0,0,0) | $U(1)^2 \times SU(4)^j \times SU(2)^j \times E_6'$ | (16,8)               |
| (6,0,0,0,0,0,0,0) | (3,2,1,0,0,0,0,0) | $SO(16) \times U(1)^j^2 \times E_6'$ | (8,16)               |
| (2,2,2,0,0,0,0,0) | (3,2,1,0,0,0,0,0) | $U(1)^2 \times SU(3)^j \times SU(10)^j \times U(1)^j \times E_6'$ | (16,8)               |
| (3,3,1,1,1,1,1,−1) | (3,2,1,0,0,0,0,0) | $U(1)^2 \times SU(2)^j \times SU(6) \times U(1)^j^2 \times E_6'$ | (17,7)               |
| (5,1,0,0,0,0,0,0) | (6,0,0,0,0,0,0,0) | $U(1)^2 \times SU(2)^j \times SU(10)^j \times SO(16)'$ | (16,8)               |
| $\gamma$ | $\tilde{\gamma}$ | perturbative gauge group | $(n^{(1)}, n^{(2)})$ |
|-------|---------|----------------------|----------------|
| $\frac{1}{2}(7,1,1,1,1,1,-1)$ | $(6,0,0,0,0,0,0,0)$ | $U(1) \times SU(2) \times SU(7) \times SO(16)'$ | $(16,8)$ |
| $\frac{1}{2}(9,3,1,1,1,1,1)$ | $(6,0,0,0,0,0,0,0)$ | $U(1) \times SU(3) \times SU(6) \times SO(16)'$ | $(16,8)$ |
| $(3,1,0,0,0,0,0,0)$ | $(2,0,0,0,0,0,0,0)$ | $U(1)^2 \times SO(12) \times U(1)' \times SO(14)'$ | $(12,12)$ |
| $(3,3,2,0,0,0,0,0)$ | $(2,0,0,0,0,0,0,0)$ | $U(1) \times SU(2)^2 \times SO(10) \times U(1)' \times SO(14)'$ | $(12,12)$ |
| $\frac{1}{2}(9,1,1,1,1,1,1)$ | $(2,0,0,0,0,0,0,0)$ | $U(1)^2 \times SU(7) \times U(1)' \times SO(14)'$ | $(12,12)$ |
| $(3,1,0,0,0,0,0,0)$ | $(4,0,0,0,0,0,0,0)$ | $U(1)^2 \times SO(12) \times U(1)' \times SO(14)'$ | $(16,8)$ |
| $(3,3,2,0,0,0,0,0)$ | $(4,0,0,0,0,0,0,0)$ | $U(1) \times SU(2)^2 \times SO(10) \times U(1)' \times SO(14)'$ | $(16,8)$ |
| $\frac{1}{2}(9,1,1,1,1,1,1)$ | $(4,0,0,0,0,0,0,0)$ | $U(1)^2 \times SU(7) \times U(1)' \times SO(14)'$ | $(16,8)$ |
| $(4,1,1,0,0,0,0,0)$ | $(4,2,0,0,0,0,0,0)$ | $U(1)^2 \times SU(2) \times SO(10) \times U(1)' \times SU(2)' \times SO(12)'$ | $(14,10)$ |
| $(5,1,1,1,1,1,1,0)$ | $(4,2,0,0,0,0,0,0)$ | $SU(2) \times SU(3) \times SU(6) \times U(1)' \times SU(2)' \times SO(12)'$ | $(14,10)$ |
| $(2,2,2,0,0,0,0,0)$ | $(5,1,0,0,0,0,0,0)$ | $U(1) \times SU(3) \times SO(10) \times U(1)' \times SU(2)' \times SO(12)'$ | $(16,8)$ |
| $(3,3,1,1,1,1,1,1)$ | $(5,1,0,0,0,0,0,0)$ | $(5,3,1,1,1,1,1)$ | $(16,8)$ |
| $(5,1,1,1,0,0,0,0)$ | $(3,1,0,0,0,0,0,0)$ | $(3,1,1,1,1,1,1,1)$ | $(14,10)$ |
| $(3,3,1,1,1,1,1,1)$ | $(3,1,0,0,0,0,0,0)$ | $(2,2,2,2,0,0,0,0)$ | $(12,12)$ |
| $\frac{1}{2}(7,1,1,1,1,1,1,1)$ | $(2,2,2,0,0,0,0,0)$ | $(9,3,1,1,1,1,1,1)$ | $(13,11)$ |
| $(5,1,1,1,0,0,0,0)$ | $(3,3,2,0,0,0,0,0)$ | $(5,1,1,1,1,1,1,1)$ | $(14,10)$ |
| $(3,1,1,1,1,1,1,1)$ | $(3,3,2,0,0,0,0,0)$ | $(5,1,1,1,1,1,1,1)$ | $(14,10)$ |
| $(5,1,1,1,1,1,1,1)$ | $(4,1,1,0,0,0,0,0)$ | $(5,1,1,1,1,1,1,1)$ | $(10,14)$ |
| $(1,1,1,1,1,1,1,-1)$ | $(4,1,1,0,0,0,0,0)$ | $(5,1,1,1,1,1,1,1)$ | $(14,10)$ |
| $(5,1,1,1,1,1,1,1,-1)$ | $(4,1,1,0,0,0,0,0)$ | $(5,1,1,1,1,1,1,1)$ | $(11,13)$ |
| $(2,2,2,2,0,0,0,0)$ | $(4,1,1,0,0,0,0,0)$ | $(5,1,1,1,1,1,1,1)$ | $(15,9)$ |
| $\frac{1}{2}(9,1,1,1,1,1,1,1)$ | $(5,1,1,0,0,0,0,0)$ | $(5,1,1,1,1,1,1,1)$ | $(11,13)$ |
| $(5,1,1,1,1,1,1,1)$ | $(5,1,1,1,0,0,0,0)$ | $(1,1,1,1,1,1,1,1)$ | $(10,12)$ |
| $(1,1,1,1,1,1,1,1,-1)$ | $(5,1,1,1,1,1,0,0)$ | $(1,1,1,1,1,1,1,1)$ | $(4,20)$ |
| $(5,1,1,1,1,1,1,1,-1)$ | $(5,1,1,1,1,1,0,0)$ | $(5,1,1,1,1,1,1,1)$ | $(1,23)$ |
| $(2,2,2,2,0,0,0,0)$ | $(5,1,1,1,1,1,0,0)$ | $(2,2,2,2,0,0,0,0)$ | $(4,20)$ |
| $\frac{1}{2}(7,1,1,1,1,1,1,1,-1)$ | $(3,3,1,1,1,1,1,1)$ | $(2,2,2,2,0,0,0,0)$ | $(12,12)$ |
| $(3,3,1,1,1,1,1,1)$ | $(3,3,1,1,1,1,1,1)$ | $(9,3,1,1,1,1,1)$ | $(12,12)$ |

Table 8: All 60 $T^4/\mathbb{Z}_6$–orbifolds with gauge twist $(\gamma, \tilde{\gamma})$: Their perturbative gauge group and instanton numbers $(n^{(1)}, n^{(2)})$ w.r.t. $SU(2)$–bundles.
References

[1] P. Mayr and S. Stieberger, Phys. Lett. B 355 (1995) 107.
[2] T. Kawai, Phys. Lett. B 372 (1996) 59.
[3] J.A. Harvey and G. Moore, Nucl. Phys. B 463 (1996) 315.
[4] M. Henningson and G. Moore, Nucl. Phys. B 482 (1996) 187.
[5] S. Stieberger, One–loop corrections and gauge coupling unification in superstring theory, Ph.D. thesis Munich May 1995, TUM–HEP–220/95.
[6] T. Kawai, Phys. Lett. B 397 (1997) 51.
[7] G.L. Cardoso, G. Curio, and D. Lüst, Nucl. Phys. B 491 (1997) 147.
[8] G.L. Cardoso, hep–th/9612200.
[9] I. Antoniadis, S. Ferrara, E. Gava, K. Narain and T. Taylor, Nucl. Phys. B 447 (1995) 35;
B. de Wit, V. Kaplunovsky, J. Louis and D. Lüst, Nucl. Phys. B 451 (1995) 53.
[10] K. Förger and S. Stieberger, Nucl. Phys. B 514 (1998) 135.
[11] C. Kokorelis, hep–th/9802099.
[12] S. Kachru and C. Vafa, Nucl. Phys. B 450 (1995) 69.
[13] P. Candelas, X. de la Ossa, A. Font, S. Katz and D. Morrison, Nucl. Phys. B 416 (1994) 481;
S. Hosono, A. Klemm, S. Theisen and S.T. Yau, Comm. Math. Phys. 167 (1995) 301;
Nucl. Phys. B 433 (1995) 501.
[14] A. Klemm, W. Lerche and P. Mayr, Phys. Lett. B 357 (1995) 313.
[15] P. Aspinwall, hep–th/9611137;
A. Klemm, hep–th/9705131;
L.E. Ibáñez and A.M. Uranga, hep–th/9707073;
D. Lüst, hep–th/9803072.
[16] B. Lian and S.-T. Yau, hep–th/9507151, hep–th/9507153.
[17] M. Henningson and G. Moore, Nucl. Phys. B 472 (1996) 518;
P. Berglund, M. Henningson and N. Wyllard, Nucl.Phys. B 503 (1997) 256.
[18] S. Kachru, A. Klemm, W. Lerche, P. Mayr and C. Vafa, Nucl. Phys. B 459 (1996) 537.
[19] I. Antoniadis and H. Partouche, Nucl. Phys. B 460 (1996) 470.
[20] P. Aspinwall and J. Louis, Phys. Lett. B 369 (1996) 233
[21] S. Ferrara and S. Theisen, Moduli spaces, effective actions and duality symmetry in string compactifications, CERN-TH-5652-90, 3rd Hellenic Summer School, Corfu, Greece, Sep 13-23, 1989.
[22] M. Green, J.H. Schwarz and P. West, Nucl. Phys. B 254 (1985) 327.
[23] A. Schellekens and N. Warner, Nucl. Phys. B 287 (1987) 317;
E. Witten, Comm. Math. Phys. 109 (1987) 525;
W. Lerche, B.E.W. Nilsson, A.N. Schellekens and N.P. Warner, *Nucl. Phys.* B 299 (1988) 91;
W. Lerche, *Nucl. Phys.* B 308 (1988) 102;
W. Lerche, B.E.W. Nilsson and A.N. Schellekens, *Nucl. Phys.* B 289 (1987) 609;
W. Lerche, A.N. Schellekens and N.P. Warner, *Phys. Rept.* 177 (1989) 1.

[24] E. Witten, *Nucl. Phys.* B 460 (1996) 541.

[25] O.J. Ganor and A. Hanany, *Nucl. Phys.* B 474 (1996) 122;
O.J. Ganor, *Nucl. Phys.* B 488 (1997) 223;
O.J. Ganor, D.R. Morrison and N. Seiberg, *Nucl. Phys.* B 487 (1997) 93.

[26] A. Klemm, P. Mayr and C. Vafa, [hep-th/9607013];
W. Lerche, P. Mayr and N.P. Warner, *Nucl. Phys.* B 499 (1997) 125.

[27] V. Kaplunovsky, *Nucl. Phys.* B 307 (1988) 145 and hep-th/920570.

[28] L. Dixon, V. Kaplunovsky and J. Louis, *Nucl. Phys.* B 355 (1991) 649.

[29] E. Kiritsis, C. Kounnas, P.M. Petropoulos and J. Rizos, *Nucl. Phys.* B 483 (1997) 141.

[30] J. Erler, *J. Math. Phys.* 35 (1994) 1819.

[31] N. Seiberg and E. Witten, *Nucl. Phys.* B 471 (1996) 121.

[32] M. Duff, R. Minasian and E. Witten, *Nucl. Phys.* B 465 (1996) 413.

[33] P. Berglund, S. Katz, A. Klemm and P. Mayr, *Nucl. Phys.* B 483 (1997) 209.

[34] J. Louis, J. Sonnenschein, S. Theisen and S. Yankielowicz, *Nucl. Phys.* B 480 (1996) 185.

[35] G. Aldazabal, A. Font, L.E. Ibáñez, A.M. Uranga and G. Violero, *Nucl. Phys.* B 519 (1998) 239.

[36] T. Eguchi, H. Ooguri, A. Taormina and S. Yang, *Nucl. Phys.* B 315 (1989) 193;
T. Kawai, Y. Yamada and S. Yang, *Nucl. Phys.* B 414 (1994) 191.

[37] H.P. Nilles and S. Stieberger, *Nucl. Phys.* B 499 (1997) 3.

[38] P. Mayr and S. Stieberger, *Nucl. Phys.* B 407 (1993) 725;
D. Bailin, A. Love, W. Sabra and S. Thomas, Mod. Phys. Lett. A9 (1994) 67; A10 (1995) 337.

[39] J.P. Derendinger, S. Ferrara, C. Kounnas and F. Zwirner, *Nucl. Phys.* B 372 (1992) 145;
I. Antoniadis, E. Gava, K.S. Narain and T.R. Taylor, *Nucl. Phys.* B 407 (1993) 706.

[40] E. Kiritsis and C. Kounnas, *Nucl. Phys.* B 442 (1995) 472; *Nucl. Phys. Proc. Suppl.* 45BC (1996) 207;
P.M. Petropoulos and J. Rizos, Phys. Lett. B 374 (1996) 49.

[41] R.E. Borcherds, Invent. Math. 120 (1995) 161.

[42] G. Aldazabal, A. Font, L.E. Ibáñez and F. Quevedo, *Nucl. Phys.* B 461 (1996) 85.

[43] J.H. Schwarz, *Phys. Lett.* B 371 (1996) 223.
[44] P. Candelas and A. Font, *Nucl. Phys.* B 511 (1998) 295;  
G. Aldazabal, A. Font, L.E. Ibáñez and A.M. Uranga, *Nucl. Phys.* B 492 (1997) 119.  
[45] V.A. Gritsenko and V.A. Nikulin, alg–geo/9611028.  
[46] K. Intriligator, *Nucl. Phys.* B 496 (1997) 177;  
J.D. Blum and K. Intriligator, *Nucl. Phys.* B 506 (1997) 199.  
[47] L. Dixon, J.A. Harvey, C. Vafa and E. Witten, *Nucl. Phys.* B 261 (1985) 678;  
*Nucl. Phys.* B 274 (1986) 285;  
L.E. Ibáñez, H. P. Nilles and F. Quevedo, *Nucl. Phys.* B 301 (1988) 157.  
[48] V. Kaplunovsky and J. Louis, *Nucl. Phys.* B 444 (1995) 191.  
[49] Y. Katsuki, Y. Kawamura, T. Kobayashi, N. Ohtsubo, Y. Ono and K. Tanioka, *Tables of $\mathbb{Z}_N$ orbifold models*, Kanazawa DPKU–8904.  
[50] E. Kiritsis, C. Kounnas, P.M. Petropoulos and J. Rizos *Phys. Lett.* B 385 (1996) 87.  
[51] E. Witten, *Nucl. Phys.* B 471 (1996) 135.  
[52] N. Wyllard, *JHEP* 4 (1998) 009.  
[53] P. Horava and E. Witten, *Nucl. Phys.* B 460 (1996) 506;  
*Nucl. Phys.* B 475 (1996) 94.  
[54] T. Banks and M. Dine, *Nucl. Phys.* B 479 (1996) 173.  
[55] J.–P. Derendinger, L.E. Ibáñez and H.P. Nilles, *Nucl. Phys.* B 267 (1986) 365;  
L.E. Ibáñez and H.P. Nilles, *Phys. Lett.* B 169 (1986) 354;  
K. Choi and J.E. Kim, *Phys. Lett.* B 165 (1985) 71.  
[56] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Nucl. Phys.* B 405 (1993) 279;  
*Comm. Math. Phys.* 165 (1994) 311.  
[57] See e.g.: P. Mayr and S. Stieberger, *Low energy properties of (0,2) compactifications*, hep-th/9412190, 28th International Symposium on Particle Theory, Wendisch–Rietz, Germany, 30 Aug - 3 Sep 1994.  
[58] H.P. Nilles, M. Olechowski and M. Yamaguchi, *Phys. Lett.* B 415 (1997) 24; hep-th/9801030.  
[59] A. Lukas, B. A. Ovrut and D. Waldram, hep-th/9801087, *Phys. Rev.* D 57 (1998) 7529.  
A. Lukas, B. A. Ovrut, K.S. Stelle and D. Waldram, hep-th/9806051.  
[60] I. Antoniadis and M. Quirós, *Phys. Lett.* B 392 (1997) 61;  
T. Li, J. L. Lopez and D. V. Nanopoulos, *Mod. Phys. Lett.* A12 (1997) 2647;  
E. Dudas and C. Grojean, *Nucl. Phys.* B 507 (1997) 553;  
E. Dudas and J. Mourad, *Phys. Lett.* B 400 (1997) 71;  
I. Antoniadis and M. Quirós, *Phys. Lett.* B 416 (1998) 327;  
Z. Lalak and S. Thomas, *Nucl. Phys.* B 515 (1998) 55;  
E. Dudas, *Phys. Lett.* B 416 (1998) 309;  
J. Ellis, A. E. Faraggi and D. V. Nanopoulos, *Phys. Lett.* B 419 (1998) 123;  
K. Choi, H. B. Kim and C. Muñoz, hep-th/9711158.
T. Li, hep-th/9801123;
M. Faux, hep-th/9801204;
P. Majumdar and S. SenGupta, hep-th/9802111;
D. Bailin, G.V. Kraniotis and A. Love, hep-ph/9803274;
J. Ellis, Z. Lalak, S. Pokorski and W. Pokorski, hep-ph/9805377.

[61] K. Benakli, hep–th/9805181.
[62] K. Choi, Phys. Rev. D 56 (1997) 6588.
[63] M.B. Green, J.H. Schwarz and E. Witten, Superstring theory, Volume 2, Cambridge 1987.
[64] T. Eguchi, P.B. Gilkey and A.J. Hanson, Phys. Rept. 66 (1980) 213.
[65] W. Lerche and S. Stieberger, Prepotential, mirror map and F–theory on K3, hep–th/9804176.
[66] S. Kachru and E. Silverstein, Nucl. Phys. B 504 (1997) 272.
[67] V. Kaplunovsky and J. Louis, Phys. Lett. B 417 (1998) 45.
[68] M. Eichler and D. Zagier, The theory of Jacobi forms, Birkhäuser 1985.
[69] I. Antoniadis, E. Gava, K. Narain and T. Taylor, Nucl. Phys. B 432 (1994) 187.