Symmetric random matrices and the Pfaff lattice

M. Adler*    P. van Moerbeke†

August 24, 1998

Table of contents:
0. Introduction
1. Borel decomposition and the 2-Toda lattice
2. Two-Toda \(\tau\)-functions and Pfaffian \(\tilde{\tau}\)-functions
3. The Pfaffian Toda lattice and skew-orthogonal polynomials
4. The \((s = -t)\)-reduction of the Virasoro vector fields
5. A representation of the Pfaffian \(\tilde{\tau}\)-function as a symmetric matrix integral
6. String equations and Virasoro constraints
7. Virasoro constraints with boundary terms
8. Inductive Painlevé equations for Gaussian and Laguerre ensembles
9. Appendix

The statistics governing the spectrum of Hermitean random matrix ensembles is intimately related to the standard Toda lattice [1]. The joint probabilities for the spectrum of coupled Hermitean matrices are intimately

*Department of Mathematics, Brandeis University, Waltham, Mass 02454, USA. E-mail: adler@math.brandeis.edu. The support of a National Science Foundation grant # DMS-9503246 is gratefully acknowledged.

†Department of Mathematics, Université de Louvain, 1348 Louvain-la-Neuve, Belgium and Brandeis University, Waltham, Mass 02454, USA. E-mail: vanmoerbeke@geom.ucl.ac.be and @math.brandeis.edu. The support of a National Science Foundation grant # DMS-9503246, a Nato, a FNRS and a Francqui Foundation grant is gratefully acknowledged.
related to the two-Toda lattice \cite{2,3}. When the size of the matrices tend to \( \infty \), the probabilities involved in the bulk and edge scaling limits relate to the Korteweg-de Vries equation; see \cite{7,8}. In his doctoral dissertation, H. Peng \cite{14} shows, based on Mehta’s \cite{12} pioneering work on the subject, that the symmetric matrix models and the statistics of the spectrum of symmetric matrix ensembles is governed by a peculiar reduction of the 2-Toda lattice.

In each of these instances, the connection with the integrable system is made by inserting in the probabilities above a “time”-dependent exponential; in all our previous work, we observed that the expressions thus obtained form a ratio of \( \tau \)-functions for the associated integrable system.

What is the integrable system related to integrals over symmetric matrices? The purpose of this paper is to show that Peng’s reduction of the 2-Toda system does not lead to vectors of \( \tau \)-functions, as he conjectured, but rather to a new vector of function, which we shall call “Pfaffian \( \tau \)-functions”; the vector actually lives in a deeper stratum for the Birkhoff decomposition; neither do the individual Pfaffian \( \tau \)-function obey the KP hierarchy, nor do they satisfy the standard Toda bilinear equations. Instead, it is shown that \( \tilde{\tau} \) satisfies a hierarchy of partial differential equations, different from the KP-hierarchy (section 2 and \cite{1}), and that, as a whole, the vector of \( \tilde{\tau} \)'s gives rise to a Lax pair on matrices (section 3), which we call the “Pfaffian Toda Lattice” We show the system satisfies Virasoro constraints (section 6 and 7), reminiscent of the classical ones, as Peng’s work suggests, leading to inductive equations for the spectral probabilities (section 8).

Consider the hierarchy of equations

\[
\frac{\partial m_{\infty}}{\partial t_n} = \Lambda^n m_{\infty}, \quad \frac{\partial m_{\infty}}{\partial s_n} = -m_{\infty} \Lambda^{\top n}, \quad n = 1, 2, ..., \tag{0.1}
\]

on bi-infinite matrices \( m_{\infty} \) for skew-symmetric initial condition \( m_{\infty}(0,0) \), where the matrix \( \Lambda = (\delta_{i,j-1})_{i,j \in \mathbb{Z}} \) is the shift matrix. Then Borel decomposing \( m_\infty(t,s) = S_1^{-1} S_2, \) for \( t, s \in \mathbb{C}_\infty \), into lower- and upper-triangular matrices \( S_1(t,s) \) and \( S_2(t,s) \), leads to a two-Toda system for \( L_1 := S_1 \Lambda S^{-1} \) and \( L_2 = S_2 \Lambda^\top S^{-1}_2 \), which maintains the form \( m_\infty(t,s) = -m_\infty(-s,-t)^\top \). Its \( \tau \)-functions are given by

\[
\tau_n(t,s) = \det m_n(t,s). \tag{0.2}
\]

If the initial matrix \( m_{\infty} \) is semi-infinite (i.e., \( \tau_0 = 1 \), then

\[
\tau_n(t,s) = (-1)^n \tau_n(-s,-t). \tag{0.2}
\]
When \( s \to -t \), formula (0.3) shows that in the limit the odd \( \tau \)-functions vanish, whereas the even \( \tau \)-functions are determinants of skew-symmetric matrices. Thus, we are led to considering Pfaffians:

\[
\tilde{\tau}_n(t) := \tau_n(t, -t)^{1/2} = \left( \det m_n(t, -t) \right)^{1/2}, \quad \text{for even } n \geq 0.
\]

which satisfy a hierarchy of partial differential equations \([9]\), reminiscent of the KP-hierarchy, but with the right hand side not equal to zero:

\[
\left( p_{k+4}(\tilde{\partial}) - \frac{1}{2} \frac{\partial^2}{\partial t_1 \partial t_{k+3}} \right) \tilde{\tau}_{2n} \circ \tilde{\tau}_{2n} = p_k(\tilde{\partial}) \tilde{\tau}_{2n+2} \circ \tilde{\tau}_{2n-2}
\]

(0.4)

\(k, n = 0, 1, 2, \ldots \) . For \( k = 0 \), equations (0.4) can be viewed as an expression for \( \tilde{\tau}_{2n+2} \) in terms of prior \( \tilde{\tau} \)'s.

The “Pfaffian \( \tilde{\tau} \)-functions”, themselves not \( \tau \)-functions, tie up remarkably with the 2-Toda \( \tau \)-functions, as follows:

\[
\tau_{2n}(t, -t - [\alpha]) = \tilde{\tau}_{2n}(t)\tilde{\tau}_{2n}(t + [\alpha])
\]

(0.5)

\[
\tau_{2n+1}(t, -t - [\alpha]) = -\alpha \tilde{\tau}_{2n}(t)\tilde{\tau}_{2n+2}(t + [\alpha]).
\]

When \( \alpha \to 0 \), we are approaching a deep stratum in the Borel decomposition of \( m_\infty \), where every odd \( \tau \)-function vanishes and hence the usual decomposition fails . When we approach the stratum according to to \( [\alpha] \to 0 \), then one finds the formulae (0.5) above, established in \([9]\). In particular, the odd \( \tau \)-functions \( \tau_{2n+1}(t, -t - [\alpha]) \) approach zero linearly in \( \alpha \). Equations (0.5) are crucial in deriving bilinear relations from the the 2-Toda lattice equations for Pfaffian \( \tilde{\tau} \)-functions (mentioned in Theorem 2.2). They give rise to “skew-orthogonal polynomials”, to “Pfaffian wave functions” and ultimately to Lax-type evolution equations on matrices \( L \), given by the AKS-theorem and to be explained in section 3:

\[
\frac{\partial L}{\partial t_n} = \left[ -\frac{1}{2} (L^n)_d - (L^n)_- + J(L^n)_+ J, L \right].
\]

Picking the matrix

\[
m_n(t, s) := (\mu_{ij}(t, s))_{0 \leq i, j \leq n-1}
\]

\[
p_{\infty} := \sum_{i=1}^\infty t^i z^i = \sum_{i=0}^\infty p_i(t) z^i, \quad \tilde{\partial} = \left( \frac{\partial}{\partial t_1}, \frac{1}{2} \frac{\partial}{\partial t_2}, \frac{1}{3} \frac{\partial}{\partial t_3}, \ldots \right)
\]

3[\alpha] := (\alpha, \alpha^2, \alpha^3, \ldots)

3
of moments of the special type\(^4\), for a subset \(E \subset \mathbb{R}\),

\[
\mu_{k,\ell}(t, s) = \int \int_{E^2} x^k y^\ell e^{V(x) + V(y) + \sum_{i=1}^{\infty} (t_i x_i^k - s_i y_i^\ell) \varepsilon(x - y)} dx dy,
\]

yields an example of \(m_\infty\) satisfying the hierarchy of equations (0.1). Then we have

\[
\tau_\ell(t, s) = \det m_\ell(t, s) = \int \int_{E^2} \prod_{k=1}^{\ell} \left( e^{V(x_k) + V(y_k) + \sum_{i=1}^{\infty} (t_i x_i^k - s_i y_i^\ell) \varepsilon(x_k - y_k)} \right) \Delta_\ell(x) \Delta_\ell(y) d\vec{x} d\vec{y},
\]

and, with Peng \(\cite{14}\),

\[
\tilde{\tau}_{2n}(t) = \sqrt{\tau_{2n}(t, -t)} = \int_{S_{2n}(E)} e^{Tr(V(X) + \sum_{i=1}^{\infty} t_i X_i)} dX,
\]

for the Haar measure \(dX\) on symmetric matrices and

\[
S_{2n}(E) := \{2n \times 2n \text{ symmetric matrices } X \text{ with spectrum } \in E\}.
\]

Assuming a potential \(V\) and a disjoint union \(E\) of the form,

\[
V'(z) := \frac{g}{f} = \sum_{i}^\infty \frac{b_i z^i}{a_i z^i}, \quad E = \bigcup_{i=1}^{r} [c_{2i-1}, c_{2i}] \subset \mathbb{R},
\]

the integrals (0.9) satisfy the following Virasoro constraints:

\[
\left( \sum_{i=1}^{2r} c_i^{k+1} f(c_i) \frac{\partial}{\partial c_i} - \sum_{\ell=0}^{\infty} \left( \frac{a_{\ell}}{2} J_{k+\ell,N}^{(2)} + b_{\ell} J_{k+\ell+1,N}^{(1)} \right) \right) \tilde{\tau}_N(t) = 0
\]

for all \(k \geq -1\), and even \(N \geq 0\), where\(^5\)

\[
J_{k}^{(1)} = \left( J_{k,n}^{(1)} \right)_{n \geq 0} = \left( J_{k}^{(1)} + n J_{k}^{(0)} \right)_{n \geq 0},
\]

\(^4\varepsilon(x) := \text{sign}(x)\).

\(^5\)in terms of the customary Virasoro generators in \(t_1, t_2, \ldots\):

\[
J_{n}^{(0)} = \delta_{n0}, \quad J_{n}^{(1)} = \frac{\partial}{\partial t_n} + (-n) t_{-n}, \quad J_{0}^{(1)} = 0,
\]

\[
J_{n}^{(2)} = \sum_{i+j=n} : J_{i}^{(1)} J_{j}^{(1)} : = \sum_{i+j=n} \frac{\partial^2}{\partial t_i \partial t_j} + 2 \sum_{i+j=n} i t_i \frac{\partial}{\partial t_j} + \sum_{i-j=n} (i t_i)(j t_j),
\]
\[
J_k^{(2)} = (J_{k,n}^{(2)})_{n \geq 0} = (J_k^{(2)} + (2n + k + 1)J_k^{(1)} + n(n + 1)J_k^{(0)})_{n \geq 0}.
\]

(0.12)

As we have seen happen in all such cases, the “boundary” and “time” parts decouple! In particular, when \( e^{V(z)} \to 0 \) sufficiently fast at the boundary of the set \( E \), the boundary differential operator is absent. This is also the case, when \( E = \mathbb{R} \) and the integral (0.9) makes sense.

What information does the integrable theory provide about the probabilities

\[
P_{2n}(t, E) := \frac{\int_{S_{2n}(E)} e^{\text{Tr}(V(X) + \sum_{i=1}^{\infty} t_i X^i)} dX}{\int_{S_{2n}(\mathbb{R})} e^{\text{Tr}(V(X) + \sum_{i=1}^{\infty} t_i X^i)} dX},
\]

(0.13)

after setting \( t = 0 \)? Upon expressing \( \tilde{\tau} \)'s partial derivatives in \( t \) at \( t = 0 \) as partial derivatives in the \( c_i \), by means of the Virasoro constraints (0.11), and setting them in the non-linear equations (0.4) for \( \tilde{\tau} \), we find partial differential equations in the \( c_i \) for the \( \tilde{\tau} \) and thus for the probabilities (0.13), at \( t = 0 \). Expressing partial derivatives in \( t \) at \( t = 0 \) in terms of partial derivatives in the \( c_i \) will only be possible, when the potentials satisfy sufficiently strong conditions.

For example, for even \( N \), the probability

\[
P_{N+2}(0, E) = \frac{\int_{S_{N+2}(E)} e^{-\text{Tr}X^2} dX}{\int_{S_{N+2}(\mathbb{R})} e^{-\text{Tr}X^2} dX} \quad \text{(Gaussian integral)}
\]

(0.14)

is expressed in terms of \( P_N(0, E) \) and \( P_{N-2}(0, E) \) and the non-commutative operators

\[
\mathcal{D}_k = \sum_{i=1}^{2r} c_{i+1}^{k+1} \frac{\partial}{\partial c_i},
\]

(0.15)

as follows:

\[
192 b_N \frac{P_{N+2}P_{N-2}}{P_N^2} - 12N(N-1) =
\]

\[
(D_{-1}^4 + 8(2N-1)D_{-1}^2 + 12D_0^2 + 24D_0 - 16D_{-1}D_1) \log P_N + 6(D_{-1}^2 \log P_N)^2.
\]

(0.16)
When the set $E = [c, \infty]$, then $P_{N+2}(0, E)$ is expressed in terms of $G := \frac{\partial}{\partial c} \log P_N(0, c)$, as follows:

$$192b_N \frac{P_{N+2}P_{N-2}}{P_N^2} - 12N(N-1) = G''' + 6G'^2 - 4\left(c^2 - 2(2N-1)\right)G' + 4cG,$$

where the differential operator appearing on the right hand side of (0.17) is reminiscent of the Painlevé IV equation.

In section 8, we also work out the PDE’s for the probabilities for the Laguerre ensemble:

$$P_{N+2}(0, E) = \frac{\int_{S_{N+2}(E)} e^{-\text{Tr}(X - \alpha \log X)} dX}{\int_{S_{N+2}(\mathbb{R}^+)} e^{-\text{Tr}(X - \alpha \log X)} dX}.$$  \hspace{1cm} \text{(Laguerre integral) (0.18)}

The paper contains two methods for obtaining the Virasoro constraints (0.11). The most conceptual one is to establish string relations. Indeed the Borel decomposition $m_\infty = S_1^{-1}S_2$ leads to so-called (monic) string-orthogonal polynomials [3]:

$$p^{(1)}(z) =: S_1 \chi(z) \quad \text{and} \quad p^{(2)}(z) =: h(S_2^{-1})^\top \chi(z),$$

satisfying orthogonality relations with respect to an inner product:

$$\langle p_n^{(1)}(z), p_m^{(2)}(z) \rangle = \delta_{m,n} h_n,$$

determined by the weight appearing in formula (0.8). They also play an important role in developing the skew-orthogonal polynomials, alluded to earlier.

Acting with $z$ and $\partial/\partial z$ on the semi-infinite vectors $p^{(i)}(z)$ leads to semi-infinite matrices $L_i$ and $M_i$ respectively, for $i = 1, 2$. Assuming the potential $V$ of the form (0.10) and a sufficiently fast decay to 0 of $e^{V(z)}$ at the boundary of its support, we have that the orthogonality of the polynomials leads to “string equations” for $k \geq -1$:

$$6\text{setting } h_n = \frac{\det m_{n+1}}{\det m_n} = \frac{\tau_{n+1}}{\tau_n}$$

6
\documentclass{article}
\usepackage{amsmath}
\usepackage{amsfonts}
\usepackage{amssymb}

\begin{document}

\section{Borel decomposition and the 2-Toda lattice}

In \cite{1,2}, we considered the following differential equations for the bi-infinite or semi-infinite moment matrix $m_\infty$

\begin{equation}
\frac{\partial m_\infty}{\partial t_n} = \Lambda^n m_\infty, \quad \frac{\partial m_\infty}{\partial s_n} = -m_\infty \Lambda^{\top n}, \quad n = 1, 2, \ldots, \tag{1.1}
\end{equation}

where the matrix $\Lambda = (\delta_{i,j-1})_{i,j \in \mathbb{Z}}$ is the shift matrix; then (1.1) has the following solutions:

\begin{equation}
m_\infty(t, s) = e^{\sum t_n \Lambda^n} m_\infty(0, 0) e^{-\sum s_n \Lambda^\top n} \tag{1.2}
\end{equation}

in terms of some initial condition $m_\infty(0, 0)$.

Consider the Borel decomposition $m_\infty = S_1^{-1} S_2$, for

\begin{align*}
S_1 \in G_- &= \{ \text{lower-triangular invertible matrices, with 1’s on the diagonal} \} \\
S_2 \in G_+ &= \{ \text{upper-triangular invertible matrices} \},
\end{align*}

with corresponding Lie algebras $g_-, g_+$; then setting $L_1 := S_1 \Lambda S_2^{-1}$, $L_2 := S_2 \Lambda^\top S_1^{-1}$, we find

\begin{align*}
S_1 \frac{\partial m_\infty}{\partial t_n} S_2^{-1} &= S_1(S_1^{-1} S_2) S_2^{-1} = -\dot{S}_1 S_1 + \dot{S}_2 S_2^{-1} \in g_- + g_+ \\
S_1 \Lambda^n m_\infty S_2^{-1} &= S_1 \Lambda^n S_1^{-1} = L_1^n = (L_1^n)_- + (L_1^n)_+ \in g_- + g_+;
\end{align*}

the uniqueness of the decomposition $g_- + g_+$ leads to

\begin{align*}
-\frac{\partial S_1}{\partial t_n} S_1^{-1} &= (L_1^n)_-, \quad \frac{\partial S_2}{\partial t_n} S_2^{-1} = (L_1^n)_+ \tag{0.1}.
\end{align*}

Similarly setting $L_2 = S_2 \Lambda^\top S_2^{-1}$, we find

\begin{align*}
-\frac{\partial S_1}{\partial s_n} S_1^{-1} &= -(L_2^n)_-, \quad \frac{\partial S_2}{\partial s_n} S_2^{-1} = -(L_2^n)_+ \tag{0.2}.
\end{align*}

\end{document}
This leads to the 2-Toda equations for $S_1, S_2$ and $L_1, L_2$:

$$
\frac{\partial S_{1,2}}{\partial t_n} = \mp (L_1^n)^+ S_{1,2}, \quad \frac{\partial S_{1,2}}{\partial s_n} = \pm (L_2^n)^- S_{1,2} \quad (1.3)
$$

$$
\frac{\partial L_i}{\partial t_n} = [(L_1^n)^+, L_i], \quad \frac{\partial L_i}{\partial s_n} = [(L_2^n)^-, L_i], \quad i = 1, 2, \ldots \quad (1.4)
$$

By 2-Toda theory \[4\] the problem is solved in terms of a sequence of tau-function

$$
\tau_n(t, s) = \det m_{n}(t, s),
$$

with $m_{n}(t, s)$ defined in (0.6). Notice that in the bi-infinite case ($n \in \mathbb{Z}$):

$$
m_{n}(t, s) := (\mu_{ij}(t, s))_{-\infty < i, j \leq n-1},
$$

semi-infinite case ($n \geq 0$):

$$
m_{n}(t, s) := (\mu_{ij}(t, s))_{0 \leq i, j \leq n-1}, \quad \text{with } \tau_0 = 1. \quad (1.5)
$$

The two pairs of wave functions $\Psi = (\Psi_1, \Psi_2)$ and $\Psi^* = (\Psi^*_1, \Psi^*_2)$ defined by

$$
\Psi_1(t, s, z) = e^{\sum_{i} t_i z^i} S_1 \chi(z), \quad \Psi^*_1(t, s, z) = e^{-\sum_{i} t_i z^i} (S_1^\top)^{-1} \chi(z^{-1})
$$

$$
\Psi_2(t, s, z) = e^{\sum_{i} s_i z^{-i}} S_2 \chi(z), \quad \Psi^*_2(t, s, z) = e^{-\sum_{i} s_i z^{-i}} (S_2^\top)^{-1} \chi(z^{-1}) \quad (1.6)
$$

satisfy

$$
L \Psi = (z, z^{-1}) \Psi, \quad L^* \Psi^* = (z, z^{-1}) \Psi^*,
$$

and

$$
\begin{cases}
\frac{\partial}{\partial t_n} \Psi = ((L_1^n)^+, (L_1^n)^+) \Psi \\
\frac{\partial}{\partial s_n} \Psi = ((L_2^n)^-, (L_2^n)^-) \Psi \\
\frac{\partial}{\partial t_n} \Psi^* = -((L_1^n)^+, (L_1^n)^+) \Psi^* \\
\frac{\partial}{\partial s_n} \Psi^* = -((L_2^n)^-, (L_2^n)^-) \Psi^*.
\end{cases} \quad (1.7)
$$

Also define the wave operators

$$
W_1 := S_1 e^{\sum_{i} t_i \Lambda^i} \quad \text{and} \quad W_2 := S_2 e^{\sum_{i} s_i \Lambda^\top i}, \quad (1.8)
$$
and the operators \( M_i \) and \( M_i^* \):

\[
M = (M_1, M_2) := (W_1\varepsilon W_1^{-1}, W_2\varepsilon^* W_2^{-1})
\]

\[
M^* = (M_1^*, M_2^*) = (-M_1^T + L_1^{-1}, -M_2^T + L_2^{-1}).
\]

(1.9)

The operators \( L, L^*, M \) and \( M^* \) satisfy, in view of (1.1):

\[
L\Psi = (z, z^{-1})\Psi,
\]

\[
M\Psi = \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial (z^{-1})} \right) \Psi,
\]

\[
[L, M] = (1, 1),
\]

(1.10)

In [16], with a slight notational modification [6] the wave functions are shown to have the following \( \tau \)-function representation:

\[
\Psi_1(t, s; z) = \left( \frac{\tau_n(t - [z^{-1}], s) - \sum_1^\infty t_i z^i}{\tau_n(t, s)} \right)_{n \in \mathbb{Z}}
\]

\[
\Psi_2(t, s; z) = \left( \frac{\tau_{n+1}(t, s - [z]) - \sum_1^\infty s_i z^{-i}}{\tau_{n+1}(t, s)} \right)_{n \in \mathbb{Z}}
\]

\[
\Psi_1^*(t, s; z) = \left( \frac{\tau_{n+1}(t + [z^{-1}], s) - \sum_1^\infty t_i z^i}{\tau_{n+1}(t, s)} \right)_{n \in \mathbb{Z}}
\]

\[
\Psi_2^*(t, s; z) = \left( \frac{\tau_n(t, s + [z]) - \sum_1^\infty s_i z^{-i}}{\tau_n(t, s)} \right)_{n \in \mathbb{Z}},
\]

(1.11)

with the following bilinear identities satisfied for the wave and adjoint wave functions \( \Psi \) and \( \Psi^* \), for all \( m, n \in \mathbb{Z} \) (bi-infinite) and \( m, n \geq 0 \) (semi-infinite) and \( t, s, t', s' \in \mathbb{C}^\infty \):

\[
\oint_{z=\infty} \Psi_{1n}(t, s, z)\Psi_{1m}^*(t', s', z) \frac{dz}{2\pi i z} = \oint_{z=0} \Psi_{2n}(t, s, z)\Psi_{2m}^*(t', s', z) \frac{dz}{2\pi i z}.
\]

(1.12)

The \( \tau \)-functions\[7\] satisfy the following bilinear identities:

\[
\oint_{z=\infty} \tau_n(t - [z^{-1}], s)\tau_{m+1}(t' + [z^{-1}], s')e^{\sum_1^\infty (t_i-t'_i) z^i}z^{n-m-1}dz
\]

\[7\]\( \varepsilon \) and \( \varepsilon^* \) are infinite matrices such that \( \varepsilon \chi(z) = \frac{\partial}{\partial z} \chi(z) \), and \( \varepsilon^* \chi(z) = \frac{\partial}{\partial z} \chi(z) \).

\[8\]The first contour runs clockwise about a small neighborhood of \( z = \infty \), while the second runs counter-clockwise about \( z = 0 \).
they characterize the 2-Toda lattice $\tau$-functions. Note (1.6) and (1.11) yield

$$\left( S_2 \right) = \text{diag}(..., \frac{\tau_{n+1}(t, s)}{\tau_n(t, s)}, ...):= h(t, s).$$

The symmetry vector fields $Y_N$ acting on $\Psi$ and $L$,

$$Y_{M_1^\alpha L_i^\beta} (\Psi_1, \Psi_2) := (-1)^{i-1} \left( -(M_1^\alpha L_i^\beta)_- \Psi_1, (M_1^\alpha L_i^\beta)_+ \Psi_2 \right),$$

$$Y_{M_2^\alpha L_i^\beta} (L_1, L_2) := (-1)^{i-1} \left( \left[ -(M_1^\alpha L_i^\beta)_-, L_1 \right], \left[ (M_1^\alpha L_i^\beta)_+, L_2 \right] \right)$$

for $i = 1, 2$ and $\alpha, \beta \in \mathbb{Z}, \alpha \geq 0$, lift to an action on $\tau$, according to the Adler-Shiota-van Moerbeke formula [5, 6]:

**Proposition 1.1** For $n, k \in \mathbb{Z}, n \geq 0$, the symmetry vector fields $Y_{M_i^n L_i^{n+k}}$, $(i = 1, 2)$ acting on $\Psi$ lead to the correspondences

$$- \frac{\left( (M_1^n L_1^{n+k})_- \Psi_1 \right)_m}{\Psi_1,m} = \frac{1}{n+1} \left( e^{-\eta} - 1 \right) \frac{W^{(n+1)}_{m,k} (\tau_m)}{\tau_m},$$

$$\frac{\left( (M_1^n L_1^{n+k})_+ \Psi_2 \right)_m}{\Psi_2,m} = \frac{1}{n+1} \left( e^{-\eta} - 1 \right) \frac{W^{(n+1)}_{m+1,k} (\tau_{m+1})}{\tau_{m+1}} - \frac{W^{(n+1)}_{m,k} (\tau_m)}{\tau_m},$$

$$\frac{\left( (M_2^n L_2^{n+k})_- \Psi_1 \right)_m}{\Psi_1,m} = \frac{1}{n+1} \left( e^{-\eta} - 1 \right) \frac{\tilde{W}^{(n+1)}_{m-1,k} (\tau_m)}{\tau_m},$$

$$\frac{\left( (M_2^n L_2^{n+k})_+ \Psi_2 \right)_m}{\Psi_2,m} = \frac{1}{n+1} \left( e^{-\eta} - 1 \right) \frac{\tilde{W}^{(n+1)}_{m,k} (\tau_{m+1})}{\tau_{m+1}} - \frac{\tilde{W}^{(n+1)}_{m-1,k} (\tau_m)}{\tau_m},$$

(1.15)

where

$$\eta = \sum_{i=1}^\infty z^i \frac{\partial}{\partial t_i} \quad \text{and} \quad \tilde{\eta} = \sum_{i=1}^\infty z^i \frac{\partial}{\partial s_i},$$

so that

$$e^{a\eta + b\tilde{\eta}} f(t, s) = f(t + a[z^{-1}], s + b[z])$$
In Proposition 1.1, the $W$-generators take on the following form in terms of the customary $W$-generators

$$W^{(k)}_{n,\ell} = \sum_{j=0}^{k} \binom{n}{j} (k-j) W^{(k-j)}_{\ell} \quad \text{and} \quad \tilde{W}^{(k)}_{n,\ell} = W^{(k)}_{-n,\ell} \big|_{t \to s}. \quad (1.16)$$

We shall only need the $W^{(k)}_{n,\ell}$-generators for $0 \leq k \leq 2$:

$W^{(0)}_{n} = \delta_{n,0}, \quad W^{(1)}_{n} = J^{(1)}_{n} \quad \text{and} \quad W^{(2)}_{n} = J^{(2)}_{n} - (n+1)J^{(1)}_{n}, \quad n \in \mathbb{Z} \quad (1.17)$
and

$$W^{(1)}_{m,i} = W^{(1)}_{i} + mW^{(0)}_{i} = J^{(1)}_{i} + m\delta_{i0} \quad \text{and} \quad W^{(2)}_{m,i} = W^{(2)}_{i} + 2mW^{(1)}_{i} + m(m-1)W^{(0)}_{i} = J^{(2)}_{i} + (2m-i-1)J^{(1)}_{i} + m(m-1)\delta_{i0}, \quad (1.18)$$

expressed in terms of the Virasoro generators $J$ (see footnote 5). The corresponding expression $\tilde{W}^{(k)}_{n,\ell}$ can be read off from the above, using (1.16), with $J^{(k)}_{n}$ replaced by $\tilde{J}^{(k)}_{n} = J^{(k)}_{n} \big|_{t \to s}$.

## 2 Two-Toda $\tau$-functions and Pfaffian $\tilde{\tau}$-functions

In this section we state the properties of the 2-Toda lattice, associated with an initial skew-symmetric bi-infinite matrix $m_{\infty}(0, 0)$, which the reader can find in [6]. When the matrix $m_{\infty}(0, 0)$ is semi-infinite, the $\tau$-functions $\tau_{n}(t, s)$ have the property

$$\tau_{n}(t, s) = (-1)^{n}\tau_{n}(-s, -t). \quad (2.1)$$

**Theorem 2.1** [6] *If the initial matrix $m_{\infty}(0, 0)$ is skew-symmetric, then, under the 2-Toda flow, $m_{\infty}(t, s)$ evolves as follows:*

$$m_{\infty}(t, s) = -m_{\infty}(-s, -t)^{\top}. \quad (2.2)$$

Moreover,

$$h^{-1}S_{1}(t, s) = -S_{2}^{\top-1}(-s, -t) \quad \text{and} \quad h^{-1}S_{2}(t, s) = S_{1}^{\top-1}(-s, -t),$$
$$h^{-1}\Psi_{1}(t, s, z) = -\Psi_{2}^{*}(-s, -t, z^{-1}) \quad \text{and} \quad h^{-1}\Psi_{2}(t, s, z) = \Psi_{1}^{*}(-s, -t, z^{-1}),$$

11
\[ L_1(t,s) = hL_2^\top h^{-1}(-s,-t) \quad \text{and} \quad L_2(t,s) = hL_1^\top h^{-1}(-s,-t), \]
with \( h(-s,-t) = -h(t,s). \)
\( \text{(2.3)} \)

Finally in the semi-infinite case
\[ \tau_n(-s,-t) = (-1)^n \tau_n(t,s). \]
\( \text{(2.4)} \)

For a skew-symmetric semi-infinite initial matrix \( m_\infty(0,0) \), relation (2.2) guarantees the skew-symmetry of \( m_\infty(t,-t) \). Therefore the odd \( \tau \)-functions vanish and the even ones have a natural square root, the Pfaffian \( \tilde{\tau}_{2n}(t) \):
\[ \tau_{2n+1}(t,-t) = 0, \quad \tau_{2n}(t,-t) =: \tilde{\tau}_{2n}^2(t), \]
where the Pfaffian, together with its sign specification, is determined by the formula:
\[ \tilde{\tau}_{2n}(t)dx_0 \wedge dx_1 \wedge \ldots \wedge dx_{2n-1} := pf(\mu_{ij}(t, -t)dx_i \wedge dx_j)^n. \]
\( \text{(2.6)} \)

**Theorem 2.2** \[ 9 \] For a semi-infinite, skew-symmetric initial condition \( m_\infty(0,0) \), the 2-Toda \( \tau \)-function \( \tau(t,s) \) and the Pfaffians \( \tilde{\tau}(t) \) are related by
\[ \tau_{2n}(t,-t - [\alpha]) = \tau_{2n}(t)\tilde{\tau}_{2n}(t + [\alpha]) \]
\[ \tau_{2n+1}(t,-t - [\alpha]) = -\alpha \tau_{2n}(t)\tilde{\tau}_{2n+2}(t + [\alpha]) \]
or alternatively
\[ \tau_{2n}(t - [\alpha],-t) = \tau_{2n}(t - [\alpha])\tilde{\tau}_{2n}(t) \]
\[ \tau_{2n+1}(t - [\alpha],-t) = -\alpha \tau_{2n}(t - [\alpha])\tilde{\tau}_{2n+2}(t). \]
\( \text{(2.7)} \)

The \( \tilde{\tau} \)-functions satisfy the bilinear relations
\[ \int_{z=\infty} \tilde{\tau}_{2m+2}(t' + [z^{-1}])e^{\sum_{i=0}^\infty (t_i - t'_i)z^{-i}} z^{2n-2m-2}dz \]
\[ + \int_{z=0} \tilde{\tau}_{2n+2}(t + [z])\tilde{\tau}_{2m}(t' - [z])e^{\sum_{i=0}^\infty (t'_i - t_i)z^{-i}} z^{2n-2m}dz = 0. \]
\( \text{(2.8)} \)
The next theorem states that the Pfaffian $\bar{\tau}$-functions satisfy differential Fay identities, but also hierarchy of equations, whose left hand side is reminiscent of the KP-equation, but augmented with a non-zero right hand side.

**Theorem 2.3** [9] For semi-infinite, skew-symmetric initial condition $m_\infty(0,0)$, the functions $\tilde{\tau}_{2n}(t)$ satisfy the following “differential Fay identity”

\[
\{ \tilde{\tau}_{2n}(t-[u]), \tilde{\tau}_{2n}(t-[v]) \} 
+ (u^{-1} - v^{-1})(\tilde{\tau}_{2n}(t-[u])\tilde{\tau}_{2n}(t-[v]) - \tilde{\tau}_{2n}(t-\lfloor u \rfloor)\tilde{\tau}_{2n}(t-\lfloor u \rfloor - [v])) 
= uv(u-v)\tilde{\tau}_{2n-2}(t-[u]-[v])\tilde{\tau}_{2n+2}(t),
\]

(2.9)

and Hirota type bilinear equations, always involving nearest neighbours:

\[
\left( p_{k+4}(\bar{\partial}) - \frac{1}{2} \frac{\partial^2}{\partial t_1 \partial t_{k+3}} \right) \tilde{\tau}_{2n} \circ \tilde{\tau}_{2n} = p_k(\bar{\partial}) \tilde{\tau}_{2n+2} \circ \tilde{\tau}_{2n-2}
\]

(2.10)

\(k, n = 0, 1, 2, \ldots\).

3 The Pfaffian Toda lattice and skew-orthogonal polynomials

Consider the $(t, s)$-dependent inner product

\[
\langle f, g \rangle = \int \int_{\mathbb{R}^2} f(x)g(y)e^{\sum_{i=1}^{\infty}(t_i x_i - s_i y_i)} F(x,y)dx \, dy, \quad t, s \in \mathbb{C}^\infty,
\]

with regard to the skew-symmetric weight $F(x,y)$,

\[
F(y, x) = -F(x, y),
\]

(3.1)

and the moment matrix

\[
m_n(t, s) := (\mu_{k\ell}(t, s))_{0 \leq k, \ell \leq n-1} = \left( (x_k, y_\ell) \right)_{0 \leq k, \ell \leq n-1}.
\]

(3.2)

\[10\{f, g\} = f'g - fg', \text{ where }' = \partial/\partial t_1.\]
Define
\[
\tau_n(t, s) := \det m_n(t, s)
\]
\[
= \int \int_{(R^2)^n} \prod_{k=1}^n \left( e^{\sum_{i=1}^\infty (t_k x_i - s_i y_i)} F(x_k, y_k) \right) \Delta_n(x) \Delta_n(y) \, dx \, dy.
\]

(3.3)

The proof that the two expressions (3.3) for \(\tau_n\) are identical, is based on an identity involving Vandermonde determinants and can be found in [2].

We are exactly in the framework of section 2; indeed, on the one hand, the moments \(\mu_{ij}\) in (3.2) satisfy the equations
\[
\frac{\partial \mu_{ij}}{\partial t_k} = \mu_{i+k, j} \quad \text{and} \quad \frac{\partial \mu_{ij}}{\partial s_k} = -\mu_{i, j+k},
\]
and so \(m := m_\infty\) satisfies (0.1), and \((\tau_n(t, s))_{n \geq 0}\) is a 2-Toda \(\tau\)-vector.

On the other hand, the skew-symmetry (3.1) of \(F\) implies \(\mu_{ij}(0, 0) = -\mu_{ji}(0, 0)\), and so the skewness of \(m_\infty(0, 0)\); so, by theorem 2.1, we have
\[
\mu_{ij}(s, t) = -\mu_{ji}(-s, -t).
\]

Therefore also, \(m_\infty, S_1, S_2, \Psi_1, \Psi_2, h \) and \(\tau\) have the properties mentioned in theorem 2.1, and
\[
\tilde{\tau}_{2n}(t) = (\det m_{2n}(t, -t))^{1/2} = pf(0, ..., 2n - 1)
\]

(3.5)
satisfies the relations of Theorem 2.2 and the non-linear hierarchy, mentioned in Theorem 2.3.

We now construct the vector of “wave functions” \(\tilde{\Psi} = (\tilde{\Psi}_k)_{k \geq 0}\), containing the functions \(p := (p_k)_{k \geq 0}\):
\[
\tilde{\Psi}_n(t, z) := e^{\sum t_i z^i} \tilde{p}_{2n}(t, z) := e^{\sum t_i z^i} \tilde{z}^{2n} \frac{\tilde{\tau}_{2n}(t - [z^{-1}])}{\tilde{\tau}_{2n}(t)}
\]
\[
\tilde{\Psi}_{n+1}(t, z) := e^{\sum t_i z^i} \tilde{p}_{2n+1}(t, z) := e^{\sum t_i z^i} \frac{\left( z + \frac{\partial}{\partial t_1} \right) z^{2n} \tilde{\tau}_{2n}(t - [z^{-1}])}{\tilde{\tau}_{2n}(t)}
\]

(3.6)
and the diagonal matrix

\[ \bar{h} = \text{diag}(\bar{h}_0, \bar{h}_0, \bar{h}_1, \bar{h}_1, \ldots) \text{ with } \bar{h}_n := \frac{\bar{\tau}_{2n+2}(t)}{\bar{\tau}_{2n}(t)}. \] (3.7)

Consider the Lie algebra of semi-infinite matrices, whose elements are $2 \times 2$ matrices, instead of scalars. The 0th band consists of $2 \times 2$ matrices along the diagonal, whereas the first band contains $2 \times 2$ matrices just above the 0th band, etc... This leads to a gradation of matrices

\[ D = \sum_i D_i, \quad D_+ = \sum_{i>0} D_i, \quad D_- = \sum_{i<0} D_i, \]

giving rise to the matrix decomposition

\[ A = A_- + A_0 + A_+. \]

For future use, define the matrix

\[ J := \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & 1 & \\ & & -1 & 0 & \\ & & & & \end{pmatrix}, \quad \text{with } J^2 = -I. \] (3.8)

Theorem 3.1 The $\tilde{p}_k(t, z)$ are monic polynomials\footnote{with the understanding that $pf(\text{odd set}) = 0$ and $pf(0, \ldots, \hat{k}, \ldots, \hat{2n}, 2n + 1) = -pf(0, \ldots, 2n - 1)$ for $k = 2n + 1$.} in $z$:

\[ \tilde{p}_{2n}(t, z) = \frac{1}{\bar{\tau}(t)} \sum_{0 \leq k \leq 2n} (-z)^k pf(0, \ldots, \hat{k}, \ldots, 2n, 2n + 1) \]

\[ = z^{2n} + \sum_{0 \leq k \leq 2n-1} (-z)^k p f(0, \ldots, \hat{k}, \ldots, 2n) \frac{pf(0, \ldots, 2n - 1)}{pf(0, \ldots, 2n - 1)} \]

\[ \tilde{p}_{2n+1}(t, z) = \frac{1}{\bar{\tau}(t)} \sum_{0 \leq k \leq 2n+1} (-z)^k pf(0, \ldots, \hat{k}, \ldots, \hat{2n}, 2n + 1) \]

\[ = z^{2n+1} + \sum_{0 \leq k \leq 2n-1} (-z)^k p f(0, \ldots, \hat{k}, \ldots, \hat{2n}, 2n + 1) \frac{pf(0, \ldots, 2n - 1)}{pf(0, \ldots, 2n - 1)}, \] (3.9)

which are skew-orthogonal with respect to the inner product $\langle \quad, \quad \rangle|_{s=-t}$,

\[ \langle \tilde{p}_{2n}(z), \tilde{p}_{2n+1}(z) \rangle|_{s=-t} = -\langle \tilde{p}_{2n+1}(z), \tilde{p}_{2n}(z) \rangle|_{s=-t} = \frac{\bar{\tau}_{2n+2}(t)}{\bar{\tau}_{2n}(t)} = \bar{h}_n \]

or otherwise,

\[ \langle \tilde{p}_i(z), \tilde{p}_j(z) \rangle|_{s=-t} = 0 \]
i.e.,

\[(\langle \tilde{p}_i, \tilde{p}_j \rangle)_{0 \leq i, j < \infty} = J\tilde{h} =: \tilde{J}.\]  \hspace{1cm} (3.10)

The polynomials \( \tilde{p}_{2n}(z) \) give rise to the semi-infinite matrix \( \tilde{P} \) of coefficients of the polynomials \( \tilde{p}_{2n}(z) \); i.e.,

\[\tilde{P}\chi(z) = \tilde{p}(z),\]  \hspace{1cm} (3.11)

which, in view of (3.9), has the form \( \tilde{P} \in I + D_{-} \), taking into account the precise definition of \( D_{-} \). We now dress up the shift \( \Lambda \) by means of \( \tilde{P} \), yielding two matrices:

\[\tilde{L} := \tilde{P}\Lambda\tilde{P}^{-1} \quad \text{and} \quad L := \tilde{h}^{-1/2}\tilde{L}\tilde{h}^{1/2}.\]  \hspace{1cm} (3.12)

Define the projection

\[\mathcal{H} : D \rightarrow D_0 \qquad \chi \mapsto \tilde{A}_d := \chi + JA^\top J = \text{diag}(a_{00} + a_{11}, a_{00} + a_{11}, a_{22} + a_{33}, a_{22} + a_{33}, \ldots).\]

With this notation, we now state:

**Theorem 3.2 (Pfaffian Toda Lattice)** The matrices \( \tilde{L} \) and \( L \) satisfy the equations for \( n \geq 1 \),

\[\frac{\partial \tilde{L}}{\partial t_n} = \left[ - (\tilde{L}^n)_- + \tilde{J}(\tilde{L}^n)^\top \tilde{J}, \tilde{L} \right]\]  \hspace{1cm} (3.13)

and

\[\frac{\partial L}{\partial t_n} = \left[ - \frac{1}{2} (L^n)_d - (L^n)_- + J(L^n)^\top J, L \right].\]  \hspace{1cm} (3.14)

4 **The \((s = -t)\)-reduction of the Virasoro vector fields**

In this section we explain, as a remarkable feature, how the Virasoro vector fields for 2-Toda behave under the reduction \( s = -t \).
Proposition 4.1 $\tau_{2n}(t,s)$ satisfies the following identity, near $s = -t$, for $i = 1, 2$:

$$
(J^{(i)} + (-1)^i \tilde{J}^{(i)}) \tau_{2n}(t,s)|_{s=-t} = 2\sqrt{\tau_{2n}(t,-t)} \tilde{J}^{(i)} \left( \sqrt{\tau_{2n}(t,-t)} \right),
$$

(4.1)

where $\tilde{J}^{(i)}$ on the right hand side is the same operator $J^{(i)}$, but with partials $\partial/\partial t_i$ replaced by total derivatives $d/dt_i$.

The proof is based on identities, involving skew-symmetric matrices and Pfaffians. Indeed to a skew-symmetric matrix $A_{2n-1}$ augmented with an arbitrary row and column

$$
M = \begin{pmatrix}
A_{2n-1} & x_0 \\
-\ y_0 & \cdots & -y_{2n-2} & x_{2n-2} \\
& & & & z
\end{pmatrix},
$$

we associate, in a natural way, skew-symmetric matrices

$$
A = \begin{pmatrix}
A_{2n-1} & x_0 \\
& \vdots \\
& x_{2n-2} \\
& & 0
\end{pmatrix},
B = \begin{pmatrix}
A_{2n-1} & y_0 \\
& \vdots \\
& y_{2n-2} \\
& & 0
\end{pmatrix}.
$$

Similarly, to a skew-symmetric matrix $A_{2n-2}$ augmented with two arbitrary rows and columns

$$
N = \begin{pmatrix}
A_{2n-2} & x_0 & y_0 \\
-\ u_0 & \cdots & -u_{2n-3} & -u_{2n-2} & y_{2n-2} \\
-\ v_0 & \cdots & -v_{2n-3} & -v_{2n-2} & x_{2n-1} & -v_{2n-1}
\end{pmatrix},
$$

17
we associate the four skew-symmetric matrices

$$C = \begin{pmatrix}
A_{2n-2} & x_0 & v_0 \\
\vdots & \vdots & \vdots \\
x_{2n-3} & v_{2n-3} & 0 \\
* & 0 & -x_{2n-1} \\
x_{2n-1} & 0 & 0
\end{pmatrix}, \quad D = \begin{pmatrix}
A_{2n-2} & u_0 & y_0 \\
\vdots & \vdots & \vdots \\
x_{2n-3} & u_{2n-3} & 0 \\
* & 0 & -y_{2n-2} \\
x_{2n-1} & 0 & 0
\end{pmatrix},$$

$$E = \begin{pmatrix}
A_{2n-2} & x_0 & u_0 \\
\vdots & \vdots & \vdots \\
x_{2n-3} & u_{2n-3} & 0 \\
* & 0 & u_{2n-2} \\
x_{2n-1} & 0 & 0
\end{pmatrix}, \quad F = \begin{pmatrix}
A_{2n-2} & v_0 & y_0 \\
\vdots & \vdots & \vdots \\
x_{2n-3} & y_{2n-3} & 0 \\
* & 0 & -v_{2n-1} \\
x_{2n-1} & 0 & 0
\end{pmatrix}.$$

**Lemma 4.2** Given the matrices $M$ and $N$ above, we have

$$\det M = pf(A)pf(B), \quad \det N = pf(C)pf(D) - pf(E)pf(F).$$

In the proof of proposition 4.1, we shall use the following symbolic notation for the Pfaffian of $\mu$:

$$pf(0, \ldots, 2n-1) := \sqrt{\det (\mu_{ij})_{0 \leq i, j \leq 2n-1}},$$

with the sign specification mentioned in (2.6).

**Proof of Proposition 4.1:** Since

$$\tau_{2n}(t, s) = \tau_{2n}(-s, -t),$$

we have

$$\left( \frac{\partial}{\partial t_i} + \frac{\partial}{\partial s_i} \right) \tau_{2n} \big|_{s=-t} = 0, \quad \left( \frac{\partial^2}{\partial s_i \partial t_j} - \frac{\partial^2}{\partial s_j \partial t_i} \right) \tau_{2n} \big|_{s=-t} = 0. \quad (4.2)$$

Since $J_{\ell}^{(2)}$ for $\ell \geq -1$ consists of two parts, a first order one and a second order one for $\ell \geq 2$, let us first dispose of the first one:
\[
\sum_{i=1}^{\infty} \left( it_i \frac{\partial}{\partial t_{i+\ell}} + is_i \frac{\partial}{\partial s_{i+\ell}} \right) \tau_{2n}(t, s) \bigg|_{s=-t} = \\
\sum_{i=1}^{\infty} \left( it_i \left( \frac{\partial}{\partial t_{i+\ell}} - \frac{\partial}{\partial s_{i+\ell}} \right) \right) \tau_{2n}(t, s) \bigg|_{s=-t} = \\
\sum_{i=1}^{\infty} it_i \frac{d}{dt_{i+\ell}} \tau_{2n}(t, -t) = \\
2 \sqrt{\tau_{2n}(t, -t)} \left( \sum_{i=1}^{\infty} \frac{it_i \partial}{\partial t_{i+\ell}} \right) \sqrt{\tau_{2n}(t, -t)}. \quad \text{(4.3)}
\]

To deal with the second order part of \( J^{(2)}_\ell \), with \( \ell \geq 2 \) and setting \( \tau_{2n}(t, s) = f^2(t, s) \), we compute:

\[
\sum_{i+j=k} \left\{ \left( \frac{\partial^2}{\partial t_i \partial t_j} + \frac{\partial^2}{\partial s_i \partial s_j} \right) \tau_{2n}(t, s) \right\}_{s=-t} \bigg|_{s=-t} = -2 \tau_{2n}(t, -t)^{1/2} \frac{d^2}{dt_{i+\ell} dt_j} \tau_{2n}(t, -t)^{1/2}
\]

\[
= \sum_{i+j=k} \left\{ \left( \frac{\partial^2}{\partial t_i \partial t_j} + \frac{\partial^2}{\partial s_i \partial s_j} \right) f^2(t, s) \right\}_{s=-t} \bigg|_{s=-t} - 2f(t, -t) \frac{d^2}{dt_{i+\ell} dt_j} f(t, -t)
\]

\[
= 2 \sum_{i+j=k} \left\{ \frac{\partial f}{\partial t_i} \frac{\partial f}{\partial t_j} + \frac{\partial f}{\partial s_i} \frac{\partial f}{\partial s_j} + f \frac{\partial^2 f}{\partial t_i \partial t_j} + f \frac{\partial^2 f}{\partial s_i \partial s_j} - f \left( \frac{\partial}{\partial t_j} - \frac{\partial}{\partial s_j} \right) \left( \frac{\partial}{\partial t_i} - \frac{\partial}{\partial s_i} \right) f \right\}_{s=-t}
\]

\[
= 2 \sum_{i+j=k} \left\{ \frac{\partial f}{\partial t_i} \frac{\partial f}{\partial t_j} + \frac{\partial f}{\partial s_i} \frac{\partial f}{\partial s_j} + f \frac{\partial^2 f}{\partial t_i \partial t_j} + f \frac{\partial^2 f}{\partial s_i \partial s_j} \right\}_{s=-t}
\]

\[
= \frac{1}{2} \sum_{i+j=k} \left\{ \frac{4}{\partial t_i} - \frac{4}{\partial s_i} \right\} f \left( \frac{\partial}{\partial t_j} - \frac{\partial}{\partial s_j} \right) f + 4 \frac{\partial^2}{\partial s_i} \left( f \frac{\partial f}{\partial t_j} \right) + 4 \frac{\partial^2}{\partial t_i} \left( f \frac{\partial f}{\partial s_j} \right) \bigg|_{s=-t}
\]

\[
= \frac{1}{2f^2} \sum_{i+j=k} \left\{ \frac{\partial f^2}{\partial t_i} \frac{\partial f}{\partial t_j} f^2 + \frac{\partial^2 f^2}{\partial s_i \partial t_j} f^2 + 2f^2 \left( \frac{\partial^2}{\partial s_i \partial t_j} + \frac{\partial^2}{\partial t_i \partial s_j} \right) f^2 \right\}_{s=-t}
\]

\[
= \frac{2}{f^2} \sum_{i+j=k} \left\{ \frac{\partial^2 f^2}{\partial t_i \partial t_j} + f^2 \frac{\partial^2 f^2}{\partial t_i \partial s_j} \right\}_{s=-t} \text{ using (4.2)}
\]
\[\sum_{i+j=k} \left\{ \frac{\partial \tau_{2n}}{\partial t_i} \frac{\partial \tau_{2n}}{\partial t_j} + \frac{\partial^2 \tau_{2n}}{\partial t_i \partial s_j} \right\}_{s=-t}, \text{ using } f^2 = \tau_{2n}(t, s) = 0 \] (4.4)

The vanishing of this last expression, is based on the argument below, using (3.4). Indeed the action of \(\frac{\partial}{\partial t_i}\) (respectively, \(\frac{\partial}{\partial s_j}\)) on the determinant of the moment matrix \(m_{2n}\) amounts to a sum (over \(0 \leq k \leq 2n - 1\)) of determinants of the same matrices, but with the \(k\)th row (respectively, the \(k\)th column) replaced by \((\mu_{k+i,0}, \ldots, \mu_{k+i,2n-1})\) (respectively, by \((\mu_0, \ell+j, \ldots, \mu_{2n-1}, \ell+j)^\top\)). Thus, the matrices in the sum are matrices of size \(2n - 1\), which are skew-symmetric except for an additional (arbitrary) row and column. Note that, when \(0 \leq k + i \leq 2n\), the corresponding matrix has zero determinant. So, using the first relation of Lemma 4.2, one finds\(^{[2]}\):

\[\frac{\partial \tau_{2n}}{\partial t_i} \bigg|_{s=-t} = \sum_{\ell} pf(0, \ldots, 2n-1) pf(0, \ldots, \ell \mapsto \ell + i, \ldots, 2n-1)\]

and hence,

\[\frac{\partial \tau_{2n}}{\partial t_i} \frac{\partial \tau_{2n}}{\partial t_j} \bigg|_{s=-t} = \sum_{\ell,m} pf(0, \ldots, m \mapsto m+j, \ldots, 2n-1) pf(0, \ldots, \ell \mapsto \ell + i, \ldots, 2n-1)\] (4.5)

Similarly, the second derivative \(\partial^2 / \partial s_i \partial t_j\) amounts to a sum of determinants (over \(0 \leq m, \ell \leq 2n - 1\)) of skew-symmetric matrices, except that the \(\ell\)th row and \(m\)th column got replaced by the \(\ell + i\)th row and \(m + j\)th column respectively. So, all in all, we get a sum of determinants of the second type in Lemma 4.2, thus leading to

\(^{[2]} pf(0, \ldots, \ell \mapsto \ell + i, \ldots, n-1)\) denotes the Pfaffian of the skew-symmetric matrix \(m_{\infty}(t, -t)\), with the \(\ell\)th column replaced by the \(\ell + i\)th column of \(m_{\infty}(t, -t)\).
\[ -\frac{\partial^2 \tau_{2n}}{\partial t_i \partial s_j} \bigg|_{s=-t} \]
\[ = \sum_{\ell,m} \det(\ell\text{th row} \mapsto (\ell + i)\text{th row}, \ m\text{th column} \mapsto (m + j)\text{th column}) \]
\[ = \sum_{\ell,m} \{ pf(\ldots, m \mapsto m, \ldots, \ell \mapsto \ell, \ldots)pf(\ldots, m \mapsto m + j, \ldots, \ell \mapsto \ell + i, \ldots) \]
\[ + pf(\ldots, m \mapsto m, \ldots, \ell \mapsto m + j, \ldots)pf(\ldots, m \mapsto \ell + i, \ldots, \ell \mapsto \ell, \ldots) \} \]
\[ = \sum_{\ell,m} \{ pf(0, \ldots, n - 1)pf(0, \ldots, m \mapsto m + j, \ldots, \ell \mapsto \ell + i, \ldots, 2n - 1) \]
\[ + pf(0, \ldots, \ell \mapsto m + j, \ldots, 2n - 1)pf(0, \ldots, m \mapsto \ell + i, \ldots, 2n - 1) \}. \]

(4.6)

Therefore, summing both contributions (4.5) and (4.6), one finds:
\[ - \sum_{i+j=k} \left\{ \frac{\partial \tau_{2n}}{\partial t_i} \frac{\partial \tau_{2n}}{\partial t_j} \right\} \bigg|_{s=-t} \]
\[ = \sum_{\ell,m,i+j=k} pf(0, \ldots, m \mapsto m + j, \ldots, \ell \mapsto \ell + i, \ldots, 2n - 1)pf(0, \ldots, 2n - 1) \]
\[ + \sum_{\ell,m,i+j=k} \left\{ pf(0, \ldots, \ell \mapsto m + j, \ldots, 2n - 1)pf(0, \ldots, m \mapsto \ell + i, \ldots, 2n - 1) \right. \]
\[ - pf(0, \ldots, m \mapsto m + j, \ldots, n - 1)pf(0, \ldots, \ell \mapsto \ell + i, \ldots, 2n - 1) \right\} \]
\[ (4.7) \]

The expression above consists of two sums; we now show each of the sums vanish separately. The first sum vanishes, because it is a sum of zero pairs\[^{13}\]
\[ pf(\ldots, m \mapsto m + j, \ldots, \ell \mapsto \ell + i, \ldots) + pf(\ldots, m \mapsto m + j', \ldots, \ell \mapsto \ell + i', \ldots) = 0, \]
upon picking \( m + j' = \ell + i, \ \ell + i' = m + j \), thus respecting the requirement \( i + j = i' + j' = k \). The argument is similar for the second sum in (4.7).

\[^{13}\]A Pfaffian flips sign, upon permuting two indices.
5 A representation of the Pfaffian \( \tilde{\tau} \)-function as a symmetric matrix integral

In this section we consider skew-symmetric weights (3.1) of the special form:

\[
F(x, y) := e^{V(x)} + e^{V(y)} I_E(x) I_E(y) \varepsilon(x - y),
\]

for a union of intervals \( E \subset \mathbb{R} \). Here we give a matrix representation, due to Peng [14], based on arguments of Mehta [12], for \( \tilde{\tau}_{2n} = \tau_{2n}(t, -t)^{1/2} \) in terms of the set

\[
S_{2n}(E) := \{ 2n \times 2n \text{ symmetric matrices } X \text{ with spectrum } \varepsilon \in E \}.
\]

**Theorem 5.1**

If

\[
\tau_{\ell}(t, s) = \det(\mu_{ij}(t, s))_{0 \leq ij \leq \ell - 1}
\]

\[
= \int \int_{E^{2\ell}} \prod_{k=1}^{\ell} \left( e^{V(x_k) + V(y_k) + \sum_{i=1}^{\infty} (t_i x_k^i - s_i y_k^i) \varepsilon(x_k - y_k) } \right) \Delta_{\ell}(x) \Delta_{\ell}(y) dx dy,
\]

then

\[
\tilde{\tau}_{2n}(t) = \sqrt{\tau_{2n}(t, -t)} = \int_{S_{2n}(E)} e^{\text{Tr}(V(X) + \sum_{i=1}^{\infty} t_i X^i)} dX,
\]

where \( dX \) is Haar measure on symmetric matrices.

**Proof.** Upon setting \( V(x, t) := V(x) + \sum_0^\infty t_i x^i \), and

\[
F_i(x) := \int_{-\infty}^x y^i e^{V(x, t)} I_E(y) dy, \quad \text{and} \quad G_i(x) := x^i e^{V(x, t)} I_E(x),
\]

we notice the following representation for the moment:

\[
\mu_{ij}(t, -t) = \int \int_{\mathbb{R}^2} x^i y^j e^{V(x, t)} + e^{V(y, t)} I_E(x) I_E(y) dx dy
\]

\[
= \int \int_{x \geq y} (x^i y^j - x^j y^i) e^{V(x, t) + V(y, t)} I_E(x) I_E(y) dx dy
\]

\[
= \int_E (F_j(x) G_i(x) - F_i(x) G_j(x)) dx.
\]

Applying the spectral theorem to the symmetric matrix \( X = O^T \text{diag}(x_1, ..., x_{2n}) O \), with \( O \in SO(2n) \), we find

\[
dX = |\Delta_{2n}(x)| dx_1 ... dx_{2n} dO.
\]

Upon integrating the orthogonal group, one finds:
\[ \int_{S_n(E)} e^{TrV(X,t)}dX \]

\[ = c_{2n} \int_{\mathbb{R}^{2n}} \prod_{i=1}^{2n} (e^{V(x_i,t)}I_E(x_i)dx_i)|\Delta_{2n}(x)| \]

\[ = c'_{2n} \int_{-\infty x_1 < x_2 < \ldots x_{2n} < \infty} \prod_{k=1}^{n} (dx_k e^{V(x_{2k},t)}I_E(x_{2k})) \]

\[ \det \left( F_i(x_2) x_2^i \quad F_i(x_4) - F_i(x_2) \quad x_4^i \quad \ldots \quad F_i(x_{2n}) - F_i(x_{2n-2}) \quad x_{2n}^i \right)_{0 \leq i \leq 2n-1} \]

\[ = c'_{2n} \int_{-\infty x_1 < x_2 < \ldots x_{2n} < \infty} \prod_{k=1}^{n} (dx_k e^{V(x_{2k},t)}I_E(x_{2k})) \]

\[ \det \left( F_i(x_2) x_2^i \quad F_i(x_4) x_4^i \quad \ldots \quad F_i(x_{2n}) x_{2n}^i \right)_{0 \leq i \leq 2n-1} \]

\[ = c'_{2n} \int_{-\infty x_1 < x_2 < \ldots x_n < \infty} \prod_{i=1}^{n} dx_i \]

\[ \det \left( F_i(x_1) \quad G_i(x_1) \quad \ldots \quad F_i(x_n) \quad G_i(x_n) \right)_{0 \leq i \leq 2n-1} \]

\[ = \frac{c_{2n}}{n!} \int_{\mathbb{R}^n} \prod_{i=1}^{n} dx_i \]

\[ \det \left( F_i(x_1) \quad G_i(x_1) \quad \ldots \quad F_i(x_n) \quad G_i(x_n) \right)_{0 \leq i \leq 2n-1} \]

\[ \text{upon permuting the } x_i, \]

\[ = c'_{2n} \left( \det \left( G_i(x)F_j(x) - F_i(x)G_j(x) \right)_{0 \leq i, j \leq n-1} \right)^{1/2}, \]

\[ \text{using de Bruijn's Lemma [12], p.446,} \]

\[ = c'_{2n} \det \left( \mu(t, -t) \right)_{0 \leq i, j \leq n-1}^{1/2} \text{ using (5.2)} \]

\[ = c_{2n} \tilde{\tau}_{2n}(t), \]

\[ \text{using (2.5).} \]
6 String equations and Virasoro constraints

In this section, we consider the moments (3.2), with regard to the skew-symmetric weight
\[ F(x, y) := e^{V(x)+V(y)}\varepsilon(x - y), \] (6.1)
assuming the following form for the potential \( V \), as in (0.10):
\[ V'(z) = g = \sum_{i=0}^{\infty} b_i z^i \sum_{i=0}^{\infty} a_i z^i, \] (6.2)
with \( e^{V(z)} \) decaying to 0 fast enough at the boundary of its support.

According to [2, 4], the semi-infinite matrices
\[ S_1 = S_1(t, s) \in D_{-\infty,0} \quad \text{and} \quad S_2 = S_2(t, s) \in D_{0,\infty} \]
in the Borel decomposition \( m_\infty = S^{-1}_1 S_2 \) of the semi-infinite moment matrix \( m_\infty \) lead to string-orthogonal (monic) polynomials
\[ p^{(1)}(z) =: S_1 \chi(z) \quad \text{and} \quad p^{(2)}(z) =: h(S_2^{-1})^\top \chi(z), \] (6.3)
satisfying the orthogonality relations
\[ \langle p^{(1)}_n, p^{(2)}_m \rangle = \delta_{n,m} h_n \]
for the inner product
\[ \langle f, g \rangle = \int_{\mathbb{R}^2} dy dz \varepsilon(y - z) e^{V(y)+V(z)} + \sum_{i=1}^{\infty} (t_i y^i - s_i z^i) f(y) g(z). \] (6.4)

Besides \( L_1 \in D_{-\infty,1} \) and \( L_2 \in D_{-1,\infty} \), we also define \( Q_1, Q_2 \in D_{-\infty,-1} \), as follows
\[ \begin{align*}
(i) & \quad z p^{(1)}_n(z) = \sum_{\ell \leq n+1} (L_1)_{n\ell} p^{(1)}_\ell(z) \quad z p^{(2)}_n(z) = \sum_{\ell \leq n+1} (h L_2^\top h^{-1})_{n\ell} p^{(2)}_\ell(z) \\
(ii) & \quad \frac{\partial}{\partial z} p^{(1)}_n(z) = \sum_{\ell \leq n-1} (Q_1)_{n\ell} P^{(1)}_\ell(z) \quad \frac{\partial}{\partial z} p^{(2)}_n(z) = \sum_{\ell \leq n-1} (Q_2)_{n\ell} P^{(2)}_\ell(z).
\end{align*} \] (6.5)

Finally, setting for the sake of this section,
\[ V_t(x) := \sum_{i=1}^{\infty} t_i x^i \quad \text{and} \quad V_s(z) := \sum_{i=1}^{\infty} s_i y^i, \]
we define matrices
\[ M_1 := Q_1 + \frac{\partial V}{\partial y}(L_1), \quad M_2 := -hQ_2 \hbar^{-1} + \frac{\partial V}{\partial z}(L_2) + L_2^{-1}, \quad (6.6) \]
which are shown to be compatible with the definition of the \( M_i \) in (1.9). We now state the two main theorems of this section:

**Theorem 6.1** ("String equations"). The semi-infinite matrices \( L_i \) and \( M_i \) satisfy the following matrix identities in terms of \( V' = g/f \), for all \( k \geq -1 \):

\[
M_1 L_{k+1}^1 + L_{k+1}^2 f(L_1) - M_2 L_{k+1}^2 f(L_2) + L_{k+1}^2 g(L_1) + L_{k+1}^2 g(L_2) + (L_{k+1}^1 f(L_1))' + L_{k+1}^2 f(L_2) = 0. \quad (6.7)
\]

The proof of this theorem will be postponed until later in this section.

This fact, together with the ASV-correspondence (Proposition 1.1) and proposition 5.1, leads at once to the constraints for the 2-Toda \( \tau \)-functions and the Pfaffian \( \tilde{\tau} \)-functions.

**Theorem 6.2** ("Virasoro constraints"). The multiple integrals \((\tau_0 = 1)\)

\[
\tau_n(t, s) = \det (\mu_{ij}(t, s))_{0 \leq i, j \leq n-1} = \int \int_{\mathbb{R}^{2n}} \prod_{k=1}^{n} \left( e^{V(x_k) + V(y_k) + \sum_{i=1}^{\infty} (t_i x_k - s_i y_k)} \varphi(x_k - y_k) \right) \Delta_n(x) \Delta_n(y) d\vec{x} d\vec{y}
\]
form a \( \tau \)-vector for the 2-Toda lattice and satisfy the following Virasoro constraints for all \( k \geq -1 \) and \( n \geq 0 \):

\[
\sum_{i \geq 0} \left\{ \frac{a_k}{2} \left( J_{i+k}^{(2)} + \tilde{J}_{i+k}^{(2)} + (2n + i + k + 1)(J_{i+k}^{(1)} - \tilde{J}_{i+k}^{(1)}) + 2n(n + 1)J_{i+k}^{(0)} \right) \right\} \tau_n = 0. \quad (6.8)
\]

The Pfaffian

\[
\tilde{\tau}_N(t) = \tau_N(t, -t)^{1/2} = \int_{\mathbb{R}^N} \left( e^{V(x_k) + \sum_{i} t_i x_k} \right) |\Delta_N(x)| d\vec{x}, \quad N \text{ even},
\]
satisfies the Pfaffian-KP hierarchy (2.9) and (2.10), together with the following Virasoro constraints, for all \( k \geq -1 \) and even \( N \geq 0 \):

\[
\sum_{\ell=0}^{\infty} \left( \frac{a_{k}^{(2)} J_{k+\ell,N}^{(2)} + b_{k} J_{k+\ell+1,N}^{(1)}}{2} \right) \tilde{\tau}_{N}(t) = 0.\quad (6.9)
\]

Before proceeding with the proof of theorems 6.1 and 6.2, we must show that the matrices \( L_{i} \) and \( M_{i} \), obtained by (6.5) and (6.6), coincide with the ones defined in the general theory, as in (1.9).

**Lemma 6.3** The string-orthogonal polynomials relate to the wave vectors \( \Psi_{1} \) and \( \Psi_{2}^{*} \) as follows:

\[
\Psi_{1} := e^{\sum t_{k} z^{k} p^{(1)}(z)} = e^{\sum t_{k} z^{k} S_{1} \chi(z)}, \quad \Psi_{2}^{*} := e^{-\sum s_{k} z^{-k} h^{-1} p^{(2)}(z^{-1})} = e^{-\sum s_{k} z^{-k} (S_{2}^{-1})^{T} \chi(z^{-1})}, \quad (6.10)
\]

whereas the matrices \( L_{1}, L_{2}^{\top}, M_{1}, M_{2}^{\top} = (L_{2}^{-1} - M_{2})^{\top} \) satisfy the desired relations

\[
(L_{1}, L_{2}^{\top})(\Psi_{1}, \Psi_{2}^{*}) = (z, z^{-1})(\Psi_{1}, \Psi_{2}^{*}) \quad \text{and} \quad (M_{1}, M_{2}^{\top})(\Psi_{1}, \Psi_{2}^{*}) := \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial z^{-1}} \right)(\Psi_{1}, \Psi_{2}^{*}). \quad (6.11)
\]

**Proof:** Indeed, \( \Psi_{1} \) and \( \Psi_{2}^{*} \), defined in (6.10), have the correct asymptotics. Also, we check, using (6.5), that

\[
z \Psi_{1} = e^{\sum t_{k} z^{k} p^{(1)}(z)} = e^{\sum t_{k} z^{k} L_{1} p^{(1)}} = L_{1} \Psi_{1},
\]

and

\[
z^{-1} \Psi_{2}^{*}(z) = e^{-\sum s_{k} z^{-k} h^{-1} z^{-1} p^{(2)}(z^{-1})} = e^{-\sum s_{k} z^{-k} L_{2}^{\top} h^{-1} p^{(2)}(z^{-1})} = L_{2}^{\top} \Psi_{2}^{*},
\]

leading to the first formula (6.11). From the second formula for \( \Psi_{1} \) and \( \Psi_{2}^{*} \), one shows that

\[
L_{1} = S_{1} A S_{1}^{-1} \quad \text{and} \quad L_{2} = S_{2} A^{\top} S_{2}^{-1}.
\]
To prove the second formula (6.11), observe, from (6.5), that for $\Psi_1 = \Psi_1(t, s; z)$

$$M_1 \Psi_1 = \frac{\partial \Psi_1}{\partial z} = \frac{\partial}{\partial z} \left( e^{\Sigma t_k z^k} p^{(1)}(z) \right)$$

$$= \sum_{k \geq 1} k t_k z^{k-1} \Psi_1 + e^{\Sigma t_k z^k} \frac{\partial}{\partial z} p^{(1)}(z)$$

$$= \left( \frac{\partial V_t}{\partial z} (L_1) + Q_1 \right) \Psi_1,$$

and similarly that for $\Psi^*_2 = \Psi^*_2(t, s; z)$,

$$M^*_2 \Psi^*_2 = \frac{\partial \Psi^*_2}{\partial z^{-1}} = \frac{\partial}{\partial z^{-1}} \left( e^{-\sum s_k z^{-k}} h^{-1} p^{(2)}(z^{-1}) \right)$$

$$= - \sum k s_k z^{-k+1} \Psi^*_2 + e^{-\sum s_k z^{-k}} h^{-1} \frac{\partial}{\partial z^{-1}} p^{(2)}(z^{-1})$$

$$= \left( - \frac{\partial V_s (L^*_2)}{\partial z} + h^{-1} Q_2 h \right) \Psi^*_2,$$

concluding the proof of Lemma 6.3.

**Proof of Theorem 6.1.** Using

$$\frac{\partial}{\partial y} \varepsilon(y - z) = 2 \delta(y - z),$$

setting $V_t(x) = \sum t_i x^i$, and using the hypothesis that $e^V$ vanishes fast enough at the boundary of its support, we compute, at first,

$$0 = \int_{\mathbb{R}} dy \frac{\partial}{\partial y} y^k f(y) \left\{ \left( \int_{\mathbb{R}} dz \, \varepsilon(y - z) e^{V(y)} \right) e^{V(z)} p^{(2)}(z) \right\} e^{V(y)} + V_t(y) p^{(1)}(y)$$

$$= \int_{\mathbb{R}} dy \left( \int_{\mathbb{R}} dz \, \varepsilon(y - z) e^{V(z)} \right) e^{V(y)} + V_t(y) \left\{ (V(y) + V_t(y)) f(y) y^k + (y^k f(y))' \right\} p^{(1)}(y) + p^{(1)}(y) y^k f(y)$$

$$+ 2 \int_{\mathbb{R}^2} e^{V(y) + V(z)} + V_t(y - z) y^k f(y) p^{(1)}(y) p^{(2)}(z) \delta(y - z) dy dz$$

We imagine doing the calculation for all $t_i$ and $s_j$ vanishing beyond $t_{2k}$ and $s_{2k}$ and letting the latter be strictly negative and positive respectively.
Adding the two expressions yields the matrix identity
\[
\{(Q_1 + V'_t(L_1))L^k_1 f(L_1) + g(L_1)L^k_1 + (L^k_1 f(L_1))'\} h
+ h\{(Q_2 - V'_s(L_2))L^k_2 f(L_2) + g(L_2)L^k_2 + (L^k_2 f(L_2))'\}^T = 0. \tag{6.12}
\]
Upon shifting $k \to k + 1$, upon using
\[ h^{-1} L_2 h = L_2^\top, \quad Q_1 + V'_t(L_1) = M_1, \quad (h^{-1} Q_2 h)^\top - V'_s(L_2) = L_2^{-1} - M_2, \]
we have that identity (6.12) leads to
\[
\begin{aligned}
M_1 L_1^{k+1} f(L_1) + g(L_1) L_1^{k+1} + \left( L_1^{k+1} f(L_1) \right)' \\
+ L_2^{k+1} f(L_2)(L_2^{-1} - M_2) + L_2^{k+1} g(L_2) + \left( L_2^{k+1} f(L_2) \right)' = 0;
\end{aligned}
\]
Finally, the fact that for any function $F(z)$, (since $[L_2, M_2] = I$)
\[ F(L_2) M_2 = M_2 F(L_2) + F'(L_2), \]
leads to the identity, announced in theorem 6.1.

**Proof of Theorem 6.2:** Using the representation (6.2) of $V'(z)$, one obtains from (6.7) that
\[
\sum_{i \geq 0} a_i \left( M_1 L_1^{k+i+1} - M_2 L_2^{k+i+1} + (i + k + 1) L_1^{i+k} + L_2^{i+k} \right)
+ \sum_{i \geq 0} b_i (L_1^{i+k+1} + L_2^{i+k+1}) = 0.
\]
We now apply proposition 1.1. The vanishing of the matrix expression above implies obviously that the $(-)$ and $(+)$ parts vanish as well, so that acting respectively on the wave vectors $\Psi_1$ and $\Psi_2$ lead to the vanishing of the four right hand sides of (1.15) in proposition 1.1, for the corresponding combination of $W$’s. Therefore we have
\[
L_{k,m} \tau_m := \sum_{i \geq 0} \left\{ a_i (W^{(2)}_{m,k+i} + \tilde{W}^{(2)}_{m-1,k+i} + 2(i + k + 1) W^{(1)}_{m,i+k} - 2 \tilde{W}^{(1)}_{m-1,i+k}) \\
+ 2b_i (W^{(1)}_{m,k+i+1} - W^{(1)}_{m-1,k+i+1}) \right\} \tau_m
= c_k \tau_m;
\]
the point is that $c_k$ is independent of $t$, using the first and third relations of proposition 1.1, and independent of $s$ and $n$ using the second and fourth relations. Finally, in view of the relations (1.18), we have
\[ \mathcal{L}_{k,m}\tau_m = \left\{ \sum_{i \geq 0} a_i \left( J_{i+k}^{(2)} + J_{i+k}^{(2)} + (2m - i - k - 1)J_{i+k}^{(1)} \right) \\
+ (2(1 - m) - i - k - 1)J_{i+k}^{(1)} + 2m(m - 1)\delta_{i,k,0} \right\} \tau_m \\
+ 2\sum_{i \geq 0} a_i \left( (i + k + 1)(J_{i+k}^{(1)} + m\delta_{i+k,0} - (J_{i+k}^{(1)} + (1 - m)\delta_{i+k,0}) \right) \\
+ 2\sum_{i \geq 0} b_i \left( (J_{i+k+1}^{(1)} + m\delta_{i+k+1,0}) - (J_{i+k+1}^{(1)} + (1 - m)\delta_{i+k+1,0}) \right) \right\} \tau_m \\
= \left\{ \sum_{i \geq 0} a_i \left( J_{i+k}^{(2)} + J_{i+k}^{(2)} + (2m - i + k + 1)(J_{i+k}^{(1)} - J_{i+k}^{(1)}) + (2m(m + 1) - 2)\delta_{i+k,0} \right) \\
+ 2\sum_{i \geq 0} b_i \left( (J_{i+k+1}^{(1)} - J_{i+k+1}^{(1)}) + (2m - 1)\delta_{i+k+1,0} \right) \right\} \tau_m. \]

Since \( c_k \) is independent of \( m \) and \( \tau_0 = 1 \), and since most of \( \mathcal{L}_{k,m} \) vanishes, when acting on a constant, we have

\[ \frac{\mathcal{L}_{k,m}\tau_m}{\tau_m} = \frac{\mathcal{L}_{k,0}\tau_0}{\tau_0} = -2\sum_{i \geq 0} (a_i\delta_{i+k,0} + b_i\delta_{i+k+1,0}), \]

and so

\[ \left( \mathcal{L}_{k,m} + 2\sum_{i \geq 0} (a_i\delta_{i+k,0} + b_i\delta_{i+k+1,0}) \right) \tau_m = 0, \]

yielding the identity (6.8). The proof of the Virasoro constraints

\[ \sum_{i \geq 0} \left\{ \frac{a_i}{2} \left( J_{i+k}^{(2)} + (2n + i + k + 1)J_{i+k}^{(1)} + n(n + 1)J_{i+k}^{(0)} \right) \right\} \bar{\tau}_n(t) = 0. \]

for \( \bar{\tau}(t) \) follows at once from (6.8) and proposition 4.1, from which (6.9) follows, using the notation (0.12). \[ \blacksquare \]
7 Virasoro constraints with boundary terms

As before, consider the matrix integral over symmetric matrices
\[
\tilde{\tau}_{2n}(t, E) = \int_{\mathcal{S}_{2n}(E)} e^{tr(V(X) + \sum_{i=1}^{\infty} t_i X_i)} dX, \tag{7.1}
\]
integrated over the space \(\mathcal{S}_{2n}(E)\) of symmetric matrices with spectrum in \(E \subset \mathbb{R}\), where
\[
E = \text{disjoint union } \bigcup_{i=1}^{r} [c_{2i-1}, c_{2i}] \text{ and } V'(z) = \frac{g}{f} = \frac{\sum b_i z^i}{\sum a_i z^i}. \tag{7.2}
\]
The purpose of this section is to show that the integral (7.1) satisfies Virasoro constraints, with an extra-contribution coming from the boundary of \(E\). When \(E = \mathbb{R}\), one recovers the equations of Theorem 6.2, without the boundary contribution. The method here hinges on the explicit integral representation of \(\tilde{\tau}\) in terms of an integral over symmetric matrices, whereas the string equation method does not use that representation, but rather reveals the rich mathematical structures behind the \(\tilde{\tau}\)-functions.

Remember the definition of the Virasoro vectors given as in (0.12) and the notation \(V(x, t)\) of section 5:
\[
J_k^{(1)} = (J_{k,n}^{(1)})_{n \geq 0} = (J_k^{(1)} + nJ_k^{(0)})_{n \geq 0},
\]
\[
J_k^{(2)} = (J_{k,n}^{(2)})_{n \geq 0} = (J_k^{(2)} + (2n + k + 1)J_k^{(1)} + n(n + 1)J_k^{(0)})_{n \geq 0}.
\]

Instead, we shall consider a slight generalization of these Virasoro vectors:
\[
J_k^{(1)} = (J_{k,n}^{(1)})_{n \geq 0} = (J_k^{(1)} + nJ_k^{(0)})_{n \geq 0},
\]
\[
J_k^{(2)} = (J_{k,n}^{(2)})_{n \geq 0} = (\beta J_k^{(2)} + (2n\beta + (2 - \beta)(k + 1))J_k^{(1)} + n(\beta n + 2 - \beta)J_k^{(0)})_{n \geq 0}.
\]
with
\[
J_k^{(0)} := \delta_{k0},
\]
\[
J_k^{(1)} := \frac{\partial}{\partial t_k} + \frac{1}{\beta}(-k)t_{-k}, \quad J_0^{(1)} = 0,
\]
\[
J_k^{(2)} := \sum_{i+j=n} \frac{\partial^2}{\partial t_i \partial t_j} + \frac{2}{\beta} \sum_{-i+j=n} it_i \frac{\partial}{\partial t_j}.
\]
Theorem 7.1 Given the potential $V$ as in (7.2), the integrals $\tilde{\tau}_N(t, E)$ satisfy the following Virasoro constraints
\[
\left( \sum_{i=1}^{2r} c_i f(c_i) \frac{\partial}{\partial c_i} - \sum_{\ell=0}^{\infty} \left( \frac{a_{-\ell}}{2} \mathbf{J}_{k+N}^{(2)} + b_{\ell} \mathbf{J}_{k+N+1}^{(1)} \right) \right) \tilde{\tau}_N(t, E) = 0 \quad (7.3)
\]
for all $k \geq -1$, and even $N \geq 0$.

Lemma 7.2 Given a symmetric $N \times N$ matrix $X$, the following variational formula holds:
\[
\frac{d}{d\varepsilon} \left. \left( X + \varepsilon f(X)X^{k+1} \right) e^{tr V(X+\varepsilon f(X)X^{k+1},t)} \right|_{\varepsilon=0} = \sum_{\ell=0}^{\infty} \left( \frac{a_{-\ell}}{2} \mathbf{J}_{k+N}^{(2)} + b_{\ell} \mathbf{J}_{k+N+1}^{(1)} \right) dX e^{tr V(X,t)}. \quad (7.4)
\]

Proof: At first, note that, in view of (5.4),
\[
dX e^{tr V(X,t)} = |\Delta_N(x)|^\beta e^{\sum_{k=1}^{N}(V(x_k)+\sum_{i=1}^{\infty} t_i x_i^k)dx_1...dx_N dO} \quad (7.5)
\]
and the map $X \mapsto X + \varepsilon f(X)X^{k+1}$ induces a map on the $x_1, ..., x_N$:
\[
x_i \mapsto x_i + \varepsilon f(x_i)x_i^{k+1}. \quad (7.6)
\]
Also, observe the following two relations for $k \geq 0$:
\[
\left( \frac{1}{2} \sum_{i,j=0} \frac{\partial^2}{\partial t_i \partial t_j} - \frac{N}{2} \delta_{k,0} \right) e^{tr V(X)}
= \left( \sum_{1 \leq a < \beta \leq N} x_a^i x_\beta^j + \frac{k-1}{2} \sum_{1 \leq \alpha \leq N} x_\alpha^k \right) e^{tr V(X)}, \quad (7.7)
\]
\[
\left( \frac{\partial}{\partial t_k} + N \delta_{k,0} \right) e^{tr V(X)} = \left( \sum_{1 \leq \alpha \leq N} x_\alpha^k \right) e^{tr V(X)} \quad (7.8)
\]
So, the point now is to compute the $\varepsilon$-derivative
\[
\frac{d}{d\varepsilon} \left. \left( |\Delta_N(x)|^\beta e^{\sum_{k=1}^{N}(V(x_k)+\sum_{i=1}^{\infty} t_i x_i^k)dx_1...dx_N} \right) \right|_{\varepsilon=0}, \quad (7.9)
\]
which consists of three contributions:
part 1:

\[
\frac{\partial}{\partial \varepsilon} |\Delta(x + \varepsilon f(x)x^{k+1})|^{\beta} \bigg|_{\varepsilon=0}
\]

\[
= \beta |\Delta(x)|^\beta \sum_{1 \leq \alpha < \gamma \leq N} \frac{\partial}{\partial \varepsilon} \log \left( |x_\alpha - x_\gamma + \varepsilon(f(x_\alpha)x^{k+1}_\alpha - f(x_\gamma)x^{k+1}_\gamma)| \right) \bigg|_{\varepsilon=0}
\]

\[
= \beta |\Delta(x)|^\beta \sum_{1 \leq \alpha < \gamma \leq N} \frac{f(x_\alpha)x^{k+1}_\alpha - f(x_\gamma)x^{k+1}_\gamma}{x_\alpha - x_\gamma}
\]

\[
= \beta |\Delta(x)|^\beta \sum_{\ell=0}^\infty a_\ell \sum_{1 \leq \alpha < \gamma \leq N} x^{k+\ell+1}_\alpha - x^{k+\ell+1}_\gamma
\]

\[
\left( \sum_{i+j+k+\ell \neq i,j>0} x_i^a x_j^b \right) (N-1) \sum_{1 \leq \alpha \leq N} x^{k+\ell+1}_\alpha - x^{k+\ell+1}_\gamma
\]

\[
= \beta e^{-tr V(X,t)} |\Delta(x)|^\beta \sum_{\ell=0}^\infty a_\ell \left( \sum_{i+j+k+\ell \neq i,j>0} x_i^a x_j^b \right) (N-1) \sum_{1 \leq \alpha \leq N} x^{k+\ell+1}_\alpha - x^{k+\ell+1}_\gamma
\]

\[
\times \left( N - \frac{k + \ell + 1}{2} \right) \left( \frac{\partial}{\partial t_i} + N \delta_{k+i,0} \right) - \frac{N}{2} \delta_{k+i,0} e^{tr V(X,t)}
\]

\[
= \beta e^{-tr V(X,t)} |\Delta(x)|^\beta \sum_{\ell=0}^\infty a_\ell \left( \sum_{i+j+k+\ell \neq i,j>0} \frac{\partial^2}{\partial t_i \partial t_j} \right) + \left( N - \frac{k + \ell + 1}{2} \right) \left( \frac{\partial}{\partial t_k+\ell} + N \delta_{k+\ell,0} \right) - \frac{N}{2} \delta_{k+i,0} e^{tr V(X,t)}
\]

(7.10)

part 2:

\[
\frac{\partial}{\partial \varepsilon} \prod_1^N d(x_\alpha + \varepsilon f(x_\alpha)x^{k+1}_\alpha) \bigg|_{\varepsilon=0}
\]

\[
= \sum_{1}^{N} \left( f'(x_\alpha)x^{k+1}_\alpha + (k+1)f(x_\alpha)x^{k}_\alpha \right) \prod_1^N dx_i
\]

\[
= \sum_{\ell=0}^{\infty} (\ell + k+1)a_\ell \sum_{\alpha=1}^{N} x^{k+\ell+1}_\alpha \prod_1^N dx_i
\]
\[ e^{-\text{tr} V(X,t)} \sum_{\ell=0}^{\infty} (\ell + k + 1) a_\ell \left( \frac{\partial}{\partial t_{k+\ell}} + N \delta_{k+\ell,0} \right) e^{\text{tr} V(X,t)} \prod_{i=1}^{N} dx_i, \]  

(7.11)

part 3:

\[ \frac{\partial}{\partial \varepsilon} \prod_{\alpha=1}^{N} \exp \left( V \left( x_\alpha + \varepsilon f(x_\alpha) x_\alpha^{k+1} \right) + \sum_{i=1}^{\infty} t_i \sum_{\alpha=1}^{N} \left( x_\alpha + \varepsilon f(x_\alpha) x_\alpha^{k+1} \right) \right) \bigg|_{\varepsilon=0} \]

\[ = \left( \sum_{\alpha=1}^{N} V'(x_\alpha) f(x_\alpha) x_\alpha^{k+1} + \sum_{i=1}^{\infty} t_i \sum_{\alpha=1}^{N} f(x_\alpha) x_\alpha^{k+i} \right) e^{\text{tr} V(X,t)} \]

\[ = \left( \sum_{\ell=0}^{\infty} b_\ell \sum_{\alpha=1}^{N} x_\alpha^{k+\ell+1} + \sum_{i=1}^{\infty} a_i t_i \sum_{\alpha=1}^{N} x_\alpha^{k+i+\ell} \right) e^{\text{tr} V(X,t)} \]

\[ = \left( \sum_{\ell=0}^{\infty} b_\ell \left( \frac{\partial}{\partial t_{k+\ell+1}} + N \delta_{k+\ell+1,0} \right) \right. \]

\[ \left. + \sum_{i=1}^{\infty} a_i \sum_{i=1}^{\infty} t_i \left( \frac{\partial}{\partial t_{i+k+\ell}} + N \delta_{i+k+\ell,0} \right) \right) e^{\text{tr} V(X,t)}. \]  

(7.12)

As mentioned, for knowing (7.9), we must add up the three contributions (7.10), (7.11) and (7.12), resulting in

\[ \frac{\partial}{\partial \varepsilon} d(X + \varepsilon f(X) X^{k+1}) e^{\text{tr} V(X+\varepsilon f(X) X^{k+1},t)} \bigg|_{\varepsilon=0} \]

\[ = \sum_{\ell=0}^{\infty} \frac{a_\ell}{2} \left( \sum_{i,j \geq 1} \beta \frac{\partial^2}{\partial t_i \partial t_j} + \sum_{i \geq 1} t_i \frac{\partial}{\partial t_{i+k+\ell}} \right) \]

\[ +(2\beta N + (2 - \beta) k + 1) \left( \frac{\partial}{\partial t_{k+\ell}} + \frac{t_1}{\beta} \delta_{k+\ell,-1} \right) \]

\[ + N(\beta N - \beta + 2) \delta_{k+\ell,0} \right) \sum_{\ell=0}^{\infty} b_\ell \left( \frac{\partial}{\partial t_{k+\ell+1}} + N \delta_{k+\ell+1,0} \right) \bigg|_{\varepsilon=0} dX e^{\text{tr} V(X,t)} \]

\[ = \sum_{\ell=0}^{\infty} \frac{a_\ell}{2} \left( J^{(2)}_{k+\ell} + (2\beta N + (2 - \beta) k + 1) J^{(1)}_{k+\ell} + N(\beta N - \beta + 2) \delta_{k+\ell,0} \right) \]

\[ + \sum_{\ell=0}^{\infty} b_\ell \left( J^{(1)}_{k+\ell+1} + N \delta_{k+\ell+1,0} \right) dX e^{\text{tr} V(X,t)}, \]
ending the proof of lemma 7.2.

**Proof of Theorem 7.1:** The change of integration variable

\[ X \mapsto Y = X + \varepsilon f(X)X^{k+1} \]  

(7.13)

in the matrix integral

\[ \int_{S_{2n}(E)} e^{Tr V(X,t)} dX \]

leaves the integral invariant, but it induces a change of limits of integration, given by the inverse of the map (7.14); namely the \( c_i \)'s in \( E = \bigcup_1^r [c_{2i-1}, c_{2i}] \), get mapped as follows

\[ c_i \mapsto c_i - \varepsilon f(c_i)c_i^{k+1} + O(\varepsilon^2). \]

Therefore, setting

\[ E^\varepsilon = \bigcup_1^r [c_{2i-1} - \varepsilon f(c_{2i-1})c_{2i-1}^{k+1} + O(\varepsilon^2), c_{2i} - \varepsilon f(c_{2i})c_{2i}^{k+1} + O(\varepsilon^2)], \]

we find, using (7.4) and the fundamental theorem of calculus,

\[ 0 = \frac{\partial}{\partial \varepsilon} \int_{E^\varepsilon}^r |\Delta_2n(x + \varepsilon f(X)x^{k+1})| \prod_{i=1}^{2n} e^{V(x_i + \varepsilon f(x_i)x_i^{k+1}, t)} d(x_i + \varepsilon f(x_i)x_i^{k+1}) \]

\[ = \left( -\sum_{i=1}^{2r} c_i^{k+1} f(c_i) \frac{\partial}{\partial c_i} + \sum_{i=0}^{\infty} \left( \frac{a_i}{2} J^{(2)}_{k+\ell, 2n} + b_i J^{(1)}_{k+\ell+1, 2n} \right) \right) \tilde{\tau}_2n(t, E), \]

ending the proof of Theorem 7.1.

\[ \boxed{\text{8 Inductive equations for Gaussian and Laguerre ensembles}} \]

Consider a time-dependent probability density

\[ \frac{e^{V(z) + \sum_{i=1}^{\infty} t_i z^i} dz}{\int_F e^{V(z) + \sum_{i=1}^{\infty} t_i z^i} dz} \]  

35
on an interval $F \subset \mathbb{R}$, with $e^{V(z)}$ decaying fast enough to 0 at the boundary of $F$. The aim of this section is to find an inductive expression, given the disjoint union $E = \bigcup_i [c_{2i-1}, c_{2i}] \subset F$, for the probability

$$\text{Prob (spectrum } X \in E) := P_{N+2}(t, E) = \frac{\int_{S_{N+2}(E)} e^{Tr(V(X)+\sum \iota_i X_i)} dX}{\int_{S_{N+2}(F)} e^{Tr(V(X)+\sum \iota_i X_i)} dX} = \frac{\tilde{\tau}_{N+2}(t, E)}{\tilde{\tau}_{N+2}(t, F)}, \quad (8.1)$$

after setting $t = 0$, in terms of $P_{N}(0, E)$ and $P_{N-2}(0, E)$. It hinges on turning on the time $t$ in $P_{N}(0, E)$ as in (8.1) and to combine the Virasoro relations for $\tilde{\tau}_N(t, E)$, as obtained in Theorem 7.1,

$$\left(\sum_{i=1}^{2r} c_i^{k+1} f(c_i) \frac{\partial}{\partial c_i} - \sum_{\ell=0}^{\infty} \left(\frac{\alpha_{\ell}}{2} J_{k+\ell, N}^{(2)} + b_{\ell} J_{k+\ell+1}^{(1)} \right)\right) \tilde{\tau}_N(t, E) = 0, \quad V' = \frac{g}{f}, \quad (8.2)$$

with the non-linear hierarchy of PDE’s (0.4) for $k = 0$, but expressed in terms of $\log \tilde{\tau}_N(t, E)$:

$$\left(\frac{\partial^4}{\partial t_1^4} + 3 \frac{\partial^2}{\partial t_2^2} - 4 \frac{\partial^2}{\partial t_1 \partial t_3}\right) \log \tilde{\tau}_N + 6 \left(\frac{\partial^2}{\partial t_1^2} \log \tilde{\tau}_N\right)^2 = 12 \frac{\tilde{\tau}_{N-2} \tilde{\tau}_{N+2}}{\tilde{\tau}_N^2}. \quad (8.3)$$

Given $E \subset F$, define the non-commutative differential operators

$$\mathcal{D}_k = \sum_{i=1}^{2r} c_i^{k+1} \frac{\partial}{\partial c_i}, \quad \text{for } k \geq -1. \quad (8.4)$$

We now state:

**Theorem 8.1 (Gaussian ensemble)** For even $N$, the probability

$$P_{N+2}(0, E) = \frac{\int_{S_{N+2}(E)} e^{-TrX^2} dX}{\int_{S_{N+2}(R)} e^{-TrX^2} dX} \quad (8.5)$$

is expressed in terms of $P_{N-2}(0, E)$ and the non-commutative operators $\mathcal{D}_k$, acting on $P_{N}(0, E)$, as follows:
When the set \(E = [c, \infty]\), then \(P_{N+2}(0, E)\) is expressed in terms of \(G(c) := \frac{\partial}{\partial c} \log P_N(0, [c, \infty])\), as follows:

\[
192b_N \frac{P_{N+2}P_{N-2}}{P_N^2} - 12N(N-1) = G''' + 6G' - 4(c^2 - 2(2N-1))G' + 4cG. \tag{8.7}
\]

Remark: The differential operator appearing on the right hand side of (8.7) is reminiscent of the Painlevé IV equation. In (8.7), the constant \(b_N\) takes on the following form:

\[
b_N^{-1} = \frac{\left(\int_{S_N(R)} e^{-TrX^2}\right)^2}{\int_{S_{N-2}(R)} e^{-TrX^2} \int_{S_{N+2}(R)} e^{-TrX^2}}
\]

Proof: Setting \(V = -z^2\), we have \(V' = -2z =: g/f\), with \(g = -2z, f = 1\), and so:

\[
\begin{align*}
b_0 &= 0, \quad b_1 = -2, \quad \text{all other } b_i = 0 \\
a_0 &= 1, \quad a_1 = 0, \quad \text{all other } a_i = 0,
\end{align*}
\]

With these data, the equations (8.2) become for \(k \geq -1\):

\[
\mathcal{D}_k \tilde{\tau}_N = \frac{1}{2} \left( J_k^{(2)} + (2N + k + 1)J_k^{(1)} + N(N+1)J_k^{(0)} - 4J_{k+2}^{(1)} \right) \tilde{\tau}_N.
\]

In particular for \(k = -1, 0, 1\), the function \(F(t, E) := \log \tilde{\tau}_N(t, E)\) satisfies

\[
\begin{align*}
\mathcal{D}_{-1} F &= \left( \sum_{i \geq 2} it_i \frac{\partial}{\partial t_{i-1}} - 2 \frac{\partial}{\partial t_1} \right) F + Nt_1 \\
\mathcal{D}_0 F &= \left( \sum_{i \geq 1} it_i \frac{\partial}{\partial t_i} - 2 \frac{\partial}{\partial t_2} \right) F + \frac{N(N+1)}{2} \\
\mathcal{D}_1 F &= \left( \sum_{i \geq 1} it_i \frac{\partial}{\partial t_{i+1}} - 2 \frac{\partial}{\partial t_3} + (N+1) \frac{\partial}{\partial t_1} \right) F. \tag{8.8}
\end{align*}
\]
Upon taking linear combinations of (8.8), in order to get as leading term the partials $\partial/\partial t_i$, and upon setting

$$B_1 = -\frac{1}{2}D_{-1}, \quad B_2 = -\frac{1}{2}D_0, \quad B_3 = -\frac{1}{2} \left( D_1 + \frac{N+1}{2}D_{-1} \right),$$

(8.9)

one finds:

$$B_1 F = \left( \frac{\partial}{\partial t_1} - \frac{1}{2} \sum_{i \geq 2} it_i \frac{\partial}{\partial t_{i-1}} \right) F - \frac{Nt_1}{2},$$

$$B_2 F = \left( \frac{\partial}{\partial t_2} - \frac{1}{2} \sum_{i \geq 1} it_i \frac{\partial}{\partial t_i} \right) F - \frac{N(N+1)}{4},$$

$$B_3 F = \left( \frac{\partial}{\partial t_3} - \frac{1}{2} \sum_{i \geq 1} it_i \frac{\partial}{\partial t_{i+1}} - \frac{N+1}{4} \sum_{i \geq 2} it_i \frac{\partial}{\partial t_{i-1}} \right) F - \frac{N(N+1)}{4} t_1.$$

These expressions have the precise form (9.1) in the appendix with:

$$\left\{ \begin{array}{l}
\gamma_{1,-1} = -\frac{1}{2}, \quad \gamma_{1,0} = \gamma_1 = 0, \quad \delta_1 = -\frac{N}{2} \\
\gamma_{2,-1} = 0, \quad \gamma_{2,0} = -\frac{1}{2}, \quad \gamma_{2,1} = 0, \quad \gamma_2 = -\frac{N(N+1)}{4}, \quad \delta_2 = 0 \\
\gamma_{3,-1} = -\frac{N+1}{4}, \quad \gamma_{3,0} = 0, \quad \gamma_{3,1} = -\frac{1}{2}, \quad \gamma_{3,2} = \gamma_3 = 0, \quad \delta_3 = -\frac{N(N+1)}{4}.
\end{array} \right.$$

Expressing the partial derivatives of $F(t, c) := \log \tilde{\tau}_N(t, c)$ with respect to the $t_i$'s at $t = 0$, in terms of the $B_i F$ and the $B_i B_j F$, as in (9.3), and setting those into equation (8.3) leads to:

$$(B_1^2 + 6NB_1^2 + 3B_2^2 - 3B_2 - 4B_1B_3) F + 6(B_1^2 F)^2 + \frac{3}{4} N(N-1) = 12 \frac{\tau_{N-2} \tau_{N+2}}{\tau_N^2}.$$  

(8.10)

(8.11)

Substituting the expressions (8.9) into (8.11) yields:

$$(D_{-1}^1 + 8(2N-1)D_{-1}^2 + 12D_0^2 + 24D_0 - 16D_{-1}D_1) F + 6(D_{-1}^2 F)^2 + 12N(N-1) = 192 \frac{\tau_{N-2} \tau_{N+2}}{\tau_N^2},$$

which is (8.6). Note that since $F$ appears on the left hand side, always preceded by a differential operator $D_k$ and since $\tau_N(t, R)$ is independent of $c$, we may set $F = \log \tilde{\tau}_N(0, E)$ instead, in the expression above. On the right hand side, we substitute $\tilde{\tau}_N(0, E) = P_N(0, E)\tilde{\tau}_N(0, R)$, thus establishing (8.6). When the set $E = [c, \infty]$, equation (8.7) follows at once from (8.6).
Theorem 8.2 (Laguerre ensemble) For even $N$ and $c = (c_1, ..., c_{2r})$, the probability

$$P_{N+2}(0, E) = \frac{\int_{S_{N+2}(E)} e^{-\text{Tr}(X - \alpha \log X)} dX}{\int_{S_{N+2}(\mathbb{R}^+)} e^{-\text{Tr}(X - \alpha \log X)} dX},$$

(8.12)
is expressed in terms of $P_{N-2}$ and the non-commutative operators $D_k$, acting on $P_N$, as follows:

$$12b_N \frac{P_{N+2}(0, E)P_{N-2}(0, E)}{P_N^2(0, E)} - \frac{3}{4} N(N - 1)(N + 2\alpha)(N + 2\alpha + 1)$$

$$= \left( D_0^4 - 4D_0^2 + (3(N + \alpha)^2 - (2\alpha - 1)(2\alpha + 3))D_0^2 - 3(N^2 + 2\alpha N - \alpha)D_0 \right.$$  

$$+ 3D_1^2 - 3(N + \alpha)D_1 + 2(N + \alpha)D_0D_1 + 6D_2 - 4D_0D_2 \big) \log P_N$$

$$+ 3(D_0 \log P_N)^2 - 8(D_0 \log P_N)(D_0^2 \log P_N) + 6(D_0^2 \log P_N)^2.$$  

(8.13)

When the set $E = [c, \infty]$, then $P_{N+2}(0, E)$ is expressed in terms of $G := \frac{\partial}{\partial c} \log P_N(0, E)$, as follows:

$$12b_N \frac{P_{N+2}(0, E)P_{N-2}(0, E)}{cP_N^2(0, E)} - \frac{3}{4c} N(N - 1)(N + 2\alpha)(N + 2\alpha + 1)$$

$$= c^3 G''' + 2c^2G'' + c \left( 3N^2 + 2cN + 6\alpha N - (c - \alpha)^2 - 4\alpha - 2 \right) G'$$

$$+ \left( cN + \alpha c - \alpha^2 - \alpha \right) G + c \left( 6c^2G'^2 + 4cG'G + G^2 \right)$$

(8.14)

Remark: The right hand side of (8.14) is the expression appearing in Painlevé V. In (8.13) and (8.14), the constant $b_N$ takes on the following form:

$$b_N^{-1} = \frac{\left( \int_{S_{N}(\mathbb{R}^+)} e^{-\text{Tr}(X - \alpha \log X)} \right)^2}{\int_{S_{N-2}(\mathbb{R}^+)} e^{-\text{Tr}(X - \alpha \log X)} \int_{S_{N+2}(\mathbb{R}^+)} e^{-\text{Tr}(X - \alpha \log X)}}$$

Proof: Setting $e^V(z) = z^\alpha e^{-z}$, we have $V' = \frac{\alpha - z}{z}$, and so $g = \alpha - z$ and $f = z$, with

$$b_0 = \alpha, \quad b_1 = -1, \quad \text{all other } b_i = 0$$

$$a_0 = 0, \quad a_1 = 1, \quad \text{all other } a_i = 0.$$

The equations (8.2) become for $k \geq -1$:

39
\[ \mathcal{D}_{k+1} \tilde{\tau}_N = \left( \frac{1}{2} J_{k+1}^{(2)} + \frac{1}{2} (2N + k + 2\alpha + 2) J_{k+1}^{(1)} + \frac{1}{2} N(N + 2\alpha + 1) J_{k+1}^{(0)} - J_{k+2}^{(1)} \right) \tilde{\tau}_n, \]

and so for \( k = -1, 0, 1 \), the function \( F := \log \tilde{\tau}_N \) satisfies the following equations:

\[
\begin{align*}
\mathcal{D}_0 F &= \left( -\frac{\partial}{\partial t_1} + \sum_{i \geq 1} i t_i \frac{\partial}{\partial t_i} \right) F + \frac{1}{2} N(N + 2\alpha + 1) \\
\mathcal{D}_1 F &= \left( -\frac{\partial}{\partial t_2} + (N + \alpha + 1) \frac{\partial}{\partial t_1} + \sum_{i \geq 1} i t_i \frac{\partial}{\partial t_{i+1}} \right) F \\
\mathcal{D}_2 F &= \left( -\frac{\partial}{\partial t_3} + \left( N + \alpha + \frac{3}{2} \right) \frac{\partial}{\partial t_2} + \sum_{i \geq 1} i t_i \frac{\partial}{\partial t_{i+2}} + \frac{1}{2} \frac{\partial^2}{\partial t_1^2} \right) F \\
&\quad + \frac{1}{2} \left( \frac{\partial}{\partial t_1} \log F \right)^2.
\end{align*}
\]

Upon taking appropriate linear combinations of the equations above and setting

\[
\begin{align*}
B_1 &= -\mathcal{D}_0 \\
B_2 &= -\mathcal{D}_1 - (N + \alpha + 1) \mathcal{D}_0 \\
B_3 &= -\mathcal{D}_2 - \left( N + \alpha + \frac{3}{2} \right) \mathcal{D}_1 - \left( N + \alpha + \frac{3}{2} \right)(N + \alpha + 1) \mathcal{D}_0,
\end{align*}
\]

one finds

\[
\begin{align*}
B_1 F &= \left( \frac{\partial}{\partial t_1} - \sum_{i \geq 1} i t_i \frac{\partial}{\partial t_i} \right) F - \frac{N}{2} (N + 2\alpha + 1) \\
B_2 F &= \left( \frac{\partial}{\partial t_2} - (N + \alpha + 1) \sum_{i \geq 1} i t_i \frac{\partial}{\partial t_i} - \sum_{i \geq 1} i t_i \frac{\partial}{\partial t_{i+1}} \right) F \\
&\quad - \frac{N}{2} (N + \alpha + 1)(N + 2\alpha + 1) \\
B_3 F &= \left( \frac{\partial}{\partial t_3} - \sum_{i \geq 1} i t_i \frac{\partial}{\partial t_{i+2}} - \left( N + \alpha + \frac{3}{2} \right) (N + \alpha + 1) \sum_{i \geq 1} i t_i \frac{\partial}{\partial t_i} \right)
\end{align*}
\]
\[- \left( N + \alpha + \frac{3}{2} \right) \sum_{i \geq 1} it_i \frac{\partial}{\partial t_{i+1}} \right) \right) F
\]
\[- \frac{1}{2} \left( \frac{\partial^2 F}{\partial t_i^2} + \left( \frac{\partial F}{\partial t_1} \right)^2 \right) - \frac{N}{2} (N + 1 + \alpha) (N + 1 + 2\alpha) \left( N + \frac{3}{2} + \alpha \right).\]

Then referring to the appendix, one sets \( \delta_1 = \delta_2 = \delta_3 = 0 \), and

\[\gamma_{1,-1} = 0, \gamma_{1,0} = -1, \gamma_1 = -\frac{N}{2} (N + 1 + 2\alpha),\]
\[\gamma_{2,-1} = 0, \gamma_{2,0} = -(N + \alpha + 1), \gamma_{2,1} = -1, \gamma_2 = -\frac{N}{2} (N + \alpha + 1)(N + 2\alpha + 1),\]
\[\gamma_{3,-1} = 0, \gamma_{3,0} = -(N + \alpha + 1)(N + \alpha + \frac{3}{2}), \gamma_{3,1} = -(N + \alpha + \frac{3}{2}), \gamma_{3,2} = -1,\]
\[\gamma_3 = -\frac{N}{2} (N + \alpha + \frac{3}{2})(N + \alpha + 1)(N + 2\alpha + 1),\]

to find

\[
\begin{aligned}
&\left( B_1^4 + 4B_1^3 + 2(N^2 + 2(2\alpha + 1)N + 3)B_1^2 + 3(N^2 + (2\alpha + 1)N + 1)B_1 \\
&+ 3B_2^2 + 6(N + \alpha + 1)B_2 - 6B_3 - 4B_1B_3 \right) \log \tilde{\tau}_N \\
&+ 3(B_1 \log \tilde{\tau}_N)^2 + 6(B_1^2 \log \tau_N)^2 + 8(B_1 \log \tilde{\tau}_N)(B_1^2 \log \tilde{\tau}_N) \\
&= -\frac{3}{4} N(N - 1)(N + 2\alpha)(N + 2\alpha + 1) + 12 \frac{\tilde{\tau}_{N-2} \tilde{\tau}_{N+2}}{\tilde{\tau}_N^2}.
\end{aligned}
\]

Expressing the \( B_i \) in terms of the \( D_i \) according to (8.15), one finds:

\[
\begin{aligned}
&\left( D_0^4 - 4D_0^3 + (3(N + \alpha)^2 - (2\alpha - 1)(2\alpha + 3))D_0^2 - 3(N^2 + 2\alpha N - \alpha)D_0 \\
&+ 3D_1^2 - 3(N + \alpha)D_1 + 2(N + \alpha)D_0D_1 + 6D_2 - 4D_0D_2 \right) \log \tilde{\tau}_N \\
&+ 3(D_0 \log \tilde{\tau}_N)^2 - 8(D_0 \log \tilde{\tau}_N)(D_0^2 \log \tilde{\tau}_N) + 6(D_0^2 \log \tilde{\tau}_N)^2 \\
&= -\frac{3}{4} N(N - 1)(N + 2\alpha)(N + 2\alpha + 1) + 12 \frac{\tilde{\tau}_{N-2} \tilde{\tau}_{N+2}}{\tilde{\tau}_N^2}.
\end{aligned}
\]

which amounts to equation (8.13), upon using the same kind of argument as in the Gaussian ensemble. Specializing to the case of a single semi-infinite interval, yields equation (8.14).  \( \blacksquare \)

\[^{15}\left[ D_0, D_1 \right] = D_1, \left[ D_0, D_2 \right] = 2D_2\]
9 Appendix

Given first order operators $B_1, B_2, B_3$ in $c = (c_1, ..., c_{2r}) \in \mathbb{R}^{2r}$ and a function $F(t, c)$, with $t \in C^\infty$. Let $F$ satisfy the following partial differential equations in $t$ and $c$:

$$B_k F = \frac{\partial F}{\partial t_k} + \sum_{-1 \leq j < k} \gamma_{kj} V_j(F) + \gamma_k + \delta_k t_1, \quad k = 1, 2, 3,$$

(9.1)

with

$$V_j(F) = \sum_{i,i+j \geq 1} i^n t_i \frac{\partial F}{\partial t_{i+j}} + \frac{1}{2} \delta_{2,j} \left( \frac{\partial^2 F}{\partial t_1^2} + \left( \frac{\partial F}{\partial t_1} \right)^2 \right), \quad -1 \leq j \leq 2.$$

(9.2)

Then, at $t = 0$, one computes

$$\left. \frac{\partial F}{\partial t_1} \right|_{t=0} = B_1 F - \gamma_1$$

$$\left. \frac{\partial^2 F}{\partial t_1^2} \right|_{t=0} = \left( B_1^2 - \gamma_1 B_1 \right) F + \gamma_{10} \gamma_1 - \delta_1$$

$$\left. \frac{\partial^3 F}{\partial t_1^3} \right|_{t=0} = \left( B_1^3 - 3 \gamma_{10} B_1^2 + 2 \gamma_1^2 B_1 \right) F + 2 \gamma_{10} (\delta_1 - \gamma_1 \gamma_{10})$$

$$\left. \frac{\partial^4 F}{\partial t_1^4} \right|_{t=0} = \left( B_1^4 - 6 \gamma_{10} B_1^3 + 11 \gamma_{10}^2 B_1^2 - 6 \gamma_{10}^3 B_1 \right) F - 6 \gamma_{10}^2 (\delta_1 - \gamma_1 \gamma_{10})$$

$$\left. \frac{\partial F}{\partial t_2} \right|_{t=0} = B_2 F - \gamma_2$$

$$\left. \frac{\partial^2 F}{\partial t_2^2} \right|_{t=0} = \left( B_2^2 - 2 \gamma_{20} B_2 + \gamma_{21} \gamma_{32} B_1^2 - (2 \gamma_1 + \gamma_{10}) \gamma_{21} \gamma_{32} + 2 \gamma_{2,-1} B_1 \right.$$

$$-2 \gamma_{21} B_3 \big) F + \gamma_{21} \gamma_{32} (B_1 F)^2$$

$$+ \gamma_{21} \gamma_{32} (\gamma_1^2 + \gamma_{10} \gamma_1 - \delta_1) + 2 (\gamma_{21} \gamma_3 + \gamma_{20} \gamma_2 + \gamma_{1} \gamma_{2,-1})$$

$$\left. \frac{\partial F}{\partial t_3} \right|_{t=0} = \left( B_3 - \frac{\gamma_{32}}{2} B_1^2 + \frac{\gamma_{32}}{2} (2 \gamma_1 + \gamma_{10}) B_1 \right) F - \frac{\gamma_{32}}{2} (B_1 F)^2$$

$$+ \frac{\gamma_{32}}{2} (\delta_1 - \gamma_1 \gamma_{10} - \gamma_1^2) - \gamma_3$$

42
Indeed, the method consists of expressing \( \frac{\partial F}{\partial t_k} \bigg|_{t=0} \) in terms of \( B_k f \bigg|_{t=0} \), using (9.1). Second derivatives are obtained by acting on \( B_k F \) with \( B_\ell \), by noting that \( B_\ell \) commutes with all \( t \)-derivatives, by using the equation for \( B_\ell F \); and by setting in the end \( t = 0 \):

\[
B_\ell B_k F = B_\ell \frac{\partial F}{\partial t_k} + \sum_{-1 \leq j < k} \gamma_{kj} B_\ell (V_j(F))
\]

\[
= \left( \frac{\partial}{\partial t_k} + \sum_{-1 \leq j < k} \gamma_{kj} V_j \right) B_\ell (F), \quad \text{provided } V_j(F) \text{ does not contain non-linear terms}
\]

\[
= \left( \frac{\partial}{\partial t_k} + \sum_{-1 \leq j < k} \gamma_{kj} V_j \right) \left( \frac{\partial F}{\partial t_\ell} + \sum_{-1 \leq j < \ell} \gamma_{\ell j} V_j(F) + \delta_{\ell t_1} \right)
\]

\[
= \frac{\partial^2 F}{\partial t_k \partial t_\ell} + \text{lower-weight terms}.
\]

When the non-linear term is present, it is taken care as follows:

\[
B_\ell \left( \frac{\partial F}{\partial t_1} \right)^2 = 2 \frac{\partial F}{\partial t_1} B_\ell \frac{\partial F}{\partial t_1}
\]

\[
= 2 \frac{\partial F}{\partial t_1} \frac{\partial}{\partial t_1} B_\ell F
\]

\[
= 2 \frac{\partial F}{\partial t_1} \frac{\partial}{\partial t_1} \left( \frac{\partial F}{\partial t_\ell} + \sum_{-1 \leq j < \ell} \gamma_{\ell j} V_j(F) + \gamma_\ell + \delta_{\ell t_1} \right)
\]

\[
= 2 \frac{\partial F}{\partial t_1} \left( \frac{\partial^2 F}{\partial t_\ell \partial t_1} + \sum_{-1 \leq j < \ell} \gamma_{\ell j} V_j \left( \frac{\partial F}{\partial t_1} \right) + \delta_\ell \right);
\]

higher derivatives are obtained in the same way.
References

[1] M. Adler and P. van Moerbeke: *Matrix integrals, Toda symmetries, Virasoro constraints and orthogonal polynomials*, Duke Math. J., 80 (3), 863–911 (1995).

[2] M. Adler and P. van Moerbeke: *String orthogonal Polynomials, String Equations and two-Toda Symmetries*, Comm. Pure and Appl. Math., L, 241–290 (1997).

[3] M. Adler and P. van Moerbeke: *The spectrum of coupled random matrices*, Annals of Math. (1999).

[4] M. Adler and P. van Moerbeke: *Group factorization, moment matrices and 2-Toda lattices*, Intern. Math. Research Notices, 12, 555-572 (1997).

[5] M. Adler, T. Shiota and P. van Moerbeke: *From the $w_\infty$-algebra to its central extension: a $\tau$-function approach*, Physics Letters A, 194, 33–43 (1994).

[6] M. Adler, T. Shiota and P. van Moerbeke: *A Lax pair representation for the Vertex operator and the central extension*, Commun. Math. Phys. 171, 547–588 (1995).

[7] M. Adler, T. Shiota and P. van Moerbeke: *Random matrices, vertex operators and the Virasoro algebra*, Phys. Lett. A208, 67–78 (1995).

[8] M. Adler, T. Shiota and P. van Moerbeke: *Random matrices, Virasoro algebras and non-commutative KP*, Duke Math. J. (1998).

[9] M. Adler, T. Shiota and P. van Moerbeke: *Pfaffian $\tau$-functions*, to appear (1998).

[10] E. Brézin, H. Neuberger: *Multicritical points of unoriented random surfaces*, Nuclear Physics B, 350, 513-553 (1991).

[11] L. Dickey: Soliton equations and integrable systems, World Scientific (1991).

[12] M.L. Mehta: Random matrices, 2nd ed. Boston: Acad. Press, 1991.
[13] M.L. Mehta: *A non-linear differential equation and a Fredholm determinant*, J. de Phys I, 2, 1721–1729 (1992)

[14] H. Peng: *The spectrum of random matrices for symmetric ensembles*, Brandeis dissertation (1997).

[15] C.A. Tracy and H. Widom: *On orthogonal and symplectic matrix ensembles*, Comm. Math. Phys. 177, 103–130 (1996).

[16] K. Ueno and K. Takasaki: *Toda Lattice Hierarchy*, Adv. Studies in Pure Math. 4, 1–95 (1984).

[17] van Moerbeke, P.: *The spectrum of random matrices and integrable systems*, Group 21, Physical applications and Mathematical aspects of Geometry, Groups and Algebras, Vol.II, 835-852, Eds.:H.-D. Doebner, W. Scherer, C. Schulte, World scientific, Singapore, 1997.