General divisor function inequalities and the third cumulant

Zarathustra Brady

Abstract

We extend a lower bound of Munshi on sums over divisors of a number \( n \) which are less than a fixed power of \( n \) from the squarefree case to the general case. In the process we prove a lower bound on the entropy of a geometric distribution with finite support, as well as a lower bound on the probability that a random variable is less than its mean given that it satisfies a natural condition related to its third cumulant.

1 Introduction

We consider the following problem: for which \( \beta, \delta, s \) does the inequality

\[
\tau(n)^s \ll \beta, \delta, s \sum_{d|n} \tau(d)^\beta \nabla
\]

hold for all positive integers \( n \)? Munshi [4] solves this problem for \( s = 1 \) and \( n \) squarefree: if \( 0 < \delta \leq \frac{1}{2} \) and

\[
\beta > \frac{1 - H(\delta)}{\delta},
\]

where

\[
H(\delta) = \delta \log_2 \left( \frac{1}{\delta} \right) + (1 - \delta) \log_2 \left( \frac{1 - \delta}{1 - \delta} \right),
\]

then inequality (1) holds for squarefree \( n \). In fact, Munshi’s argument easily generalizes to any \( s \) if we require that

\[
\beta > \frac{s - H(\delta)}{\delta},
\]

and this is best possible by the same reasoning as in [4].

In this paper we generalize Munshi’s argument to arbitrary natural numbers \( n \). Our main result, proved in section 4, is the following.

Theorem 3. If \( 0 < \delta \leq \frac{1}{2} \), \( \beta, s \geq 0 \) satisfy

\[
\beta > \frac{s - H(\delta)}{\delta},
\]

then

\[
\tau(n)^s \ll \beta, \delta, s \sum_{d|n} \tau(d)^\beta.
\]
The main new idea in the proof is to sample divisors $d$ of $n$ from a probability distribution having high entropy, while keeping the average value of $\log(d)$ small. A crucial ingredient in the proof is the following entropy inequality, which is proved in section 3.

**Corollary** 3. If $X$ is geometrically distributed on $\{0, \ldots, m\}$ with mean $\delta m$ then

$$H(X) \geq \log_2(m + 1)H(\delta),$$

and the inequality is strict if $m > 1$ and $\delta \notin \{0, \frac{1}{2}, 1\}$.

In the process of proving our main result, we also prove a variation on a related inequality due to Soundararajan. Suppose that $\delta$ is a real number between 0 and 1, and define $c(\delta)$ to be the largest real number such that, for any squarefree number $n$, we have the inequality

$$\sum_{d|n} \delta^{\omega(d)}(1 - \delta)^{\omega(n/d)} \geq c(\delta). \quad (2)$$

Taking $n$ to have $k$ prime factors that are sufficiently close in size, we see that

$$\delta < \frac{1}{k} \implies c(\delta) \leq (1 - \delta)^k.$$

Soundararajan has shown in \cite{6} (with different notation - his $A(t)$ is our $c(1/(1 + t))$, and his $B(t)$ is our $1 - c(t/(1 + t))$) the following recursive inequalities:

$$c\left(\frac{\delta}{1 + \delta}\right) \geq \frac{c(\delta)}{1 + c(\delta)},$$

$$c\left(\frac{1}{2 - \delta}\right) \geq \frac{c(\delta)}{1 + c(\delta)}.$$

Using these together with the obvious bound $c(1) = 1$, he shows that $c(1 - 1/k) = 1/k$ for $k \in \mathbb{N}$, and that if $\delta$ is rational with continued fraction $[a_0, a_1, \ldots, a_r]$ then

$$c(\delta) \geq \frac{1}{a_0 + \cdots + a_r}.$$

**Definition 1.** For any integers $n \geq k$, define $g(n, k)$ by

$$g(n, k) = \min_{a_1 + \cdots + a_n = 0} \left| \{S \subseteq \{1, \ldots, n\} | |S| = k, \sum_{i \in S} a_i \geq 0\} \right|.$$
Proposition 1. If $\delta = \frac{a}{b}$, then

$$c(\delta) \geq \frac{g(b,a)}{(b)}.$$ 

In particular, if $g(b,a) = \binom{b-1}{a-1}$ then $c(\delta) \geq \delta$. If the stronger version of the MMS conjecture proposed in [3] holds, then $c(\delta) \geq \delta$ for all $\delta \leq (1 - \delta)^3$.

In the next section, we prove that $\delta \leq \frac{1}{2} \implies c(\delta) \geq \frac{1}{2e^{3/2}}$.

In fact, we prove the following stronger claim.

Corollary 1. Let $X_1, ..., X_n$ be independent random variables supported on $\mathbb{N}$ such that for each $i$ the function $k \mapsto \mathbb{P}[X_i = k]$ is decreasing, and let $w_1, ..., w_n \geq 0$. Let $X = \sum_{i=1}^{n} w_i X_i$. Then

$$\mathbb{P}[X \leq \mathbb{E}[X]] \geq \frac{1}{2e^{3/2}}.$$ 

2 Lower bound on the probability that a random variable is less than its mean

The arguments in this section are inspired by a MathOverflow post of fedja [2], which used Bernstein’s trick in a similar way to solve a closely related problem.

Consider the following property which a random variable $X$ might have:

$$\forall t \geq 0 \quad \frac{d^3}{dt^3} \log (\mathbb{E}[e^{-tX}]) \leq 0.$$ 

(P)

If independent random variables $X_1, ..., X_n$ all have property (P), and if $w_1, ..., w_n \geq 0$, then the random variable $X = \sum_i w_i X_i$ also has property (P) by Bernstein’s trick.

Note that when $t = 0$, property (P) says that the third cumulant of $X$, $\mathbb{E}[(X - \mathbb{E}[X])^3]$, is at least zero.

Theorem 1. If a random variable $X$ has property (P), then

$$\mathbb{P}[X \leq \mathbb{E}[X]] \geq \frac{1}{2e^{3/2}}.$$ 

Proof. Let $Y = \mathbb{E}[X] - X$, and define the function $g : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ by

$$g(t) = \log (\mathbb{E}[e^{ty}]).$$

Property (P) says that $g''(t)$ is decreasing for $t \geq 0$. Since $\mathbb{E}[Y] = 0$, we also have $g'(0) = 0$, and so for any $t \geq 0$ we have

$$tg''(t) \leq \int_0^t g''(x) dx = g'(t).$$

Integrating this we see that $tg'(t) \leq 2g(t)$. The inequality $tg''(t) \leq g'(t)$ is easily seen to be equivalent to

$$\frac{\mathbb{E}[(tY)^2 e^{tY}]}{\mathbb{E}[tYe^{tY}]} \leq \frac{\mathbb{E}[tYe^{tY}]}{\mathbb{E}[e^{tY}]} + 1.$$
If $g$ is not identically 0 then we can find $t$ such that $tg'(t) = 1$, or equivalently such that $\mathbb{E}[tYe^t] = \mathbb{E}[e^t] = e^{g(t)}$. For this $t$ we have

$$\mathbb{E} \left[ Y (8 - 3tY) e^{tY} \right] \geq (8 - 6)e^{g(t)} \geq 2e^{1/2}.$$ 

The function $p(x) = x(8 - 3x)e^x$ has $p(x) \leq 0$ for $x \leq 0$ and $p(x) \leq p(2) = 4e^2$ for all $x$, so by Markov’s inequality

$$\mathbb{P}[X \leq \mathbb{E}[X]] = \mathbb{P}[Y \geq 0] \geq \frac{1}{2e^{3/2}}.$$  

**Theorem 2.** Suppose that the random variable $X$ is supported on $\mathbb{N}$, with $\mathbb{P}[X = k]$ a decreasing function of $k$. Then $X$ satisfies property $\text{[P]}$.

**Proof.** Expanding property $\text{[P]}$, it becomes

$$\mathbb{E}[X^3e^{-tX}]\mathbb{E}[e^{-tX}]^2 + 2\mathbb{E}[Xe^{-tX}]^3 \geq 3\mathbb{E}[X^2e^{-tX}]\mathbb{E}[Xe^{-tX}]\mathbb{E}[e^{-tX}].$$

Setting $a_k = \mathbb{P}[X = k]$ and $x = e^{-t}$ we get the polynomial inequality

$$\sum_{i,j,k} a_ia_ja_kx^{i+j+k}(i^3 + 2ijk - 3i^2j) \geq 0,$$

which we need to check for $a_0 \geq a_1 \geq \cdots \geq 0$ and $1 \geq x \geq 0$. The left hand side of the above is equal to

$$\sum_{i<j} a_ia_jx^{i+j}(a_ix^i-a_jx^j)(j-i)^3 + \sum_{i<j} a_ia_ja_kx^{i+j+k}(a_jx^j-a_i+k-j)(i+k-2j)(j+k-2i)(2k-i-j),$$

which is obviously nonnegative. 

**Corollary 1.** Let $X_1, \ldots, X_n$ be independent random variables supported on $\mathbb{N}$ such that for each $i$ the function $k \mapsto \mathbb{P}[X_i = k]$ is decreasing, and let $w_1, \ldots, w_n \geq 0$. Let $X = \sum_{i=1}^n w_i X_i$. Then $\mathbb{P}[X \leq \mathbb{E}[X]] \geq \frac{1}{2e^{3/2}}$.

**Corollary 2.** Let $\delta \leq \frac{1}{2}$, and let $f : \mathbb{N} \to [0, \infty)$ be a nonnegative multiplicative function such that for every prime $p$ we have $\frac{f(p)}{f(p)+1} \leq \delta$. Then for any squarefree number $n$ we have

$$\sum_{d|n, d \leq n^{\delta}} f(d) \geq \frac{1}{2e^{3/2}} \sum_{d|n} f(d).$$

### 3 A lower bound for the entropies of certain probability distributions

Let $X$ be a random variable supported the set $\{0, \ldots, m\}$, with probability distribution $\rho = (\rho_0, \ldots, \rho_m)$. We define the entropy of $X$ to be

$$H(X) = \sum_{i=0}^m \rho_i \log_2 \left( \frac{1}{\rho_i} \right).$$
In the next section, we will need the existence of a random variable $X$ as above with $\mathbb{E}[X] = \delta m$ given and $H(X)$ large. It’s a well-known fact that the optimal choice of $X$ will be geometrically distributed. Unfortunately the entropy of a geometric distribution on a finite set, as a function of the mean, is quite complicated and directly proving a lower bound is rather difficult. Instead, we will inductively construct probability distributions which are simpler to analyze and still have sufficiently large entropy.

**Lemma 1.** For every $m \geq 1$ and every $0 \leq \delta \leq \frac{1}{2}$ there is a random variable $X$ supported on the set $\{0, \ldots, m\}$ which has mean $\delta m$ and entropy satisfying the inequality

$$H(X) \geq \log_2(m+1)H(\delta).$$

*Proof.* It’s enough to prove this for $0 < \delta \leq \frac{1}{2}$. We proceed by induction on $m$. The case $m = 1$ is immediate. For $m > 1$, we let $Y$ be a random variable on the set $\{0, \ldots, m-1\}$ with mean $\delta(m-1)$, satisfying $H(Y) \geq \log_2(m)H(\delta)$. Define $X$ to be 0 with probability $\frac{1-\delta}{m\delta + 1 - \delta}$, and to be $1 + Y$ with probability $\frac{m\delta}{m\delta + 1 - \delta}$. Then

$$\mathbb{E}[X] = \frac{m\delta}{m\delta + 1 - \delta} (1 + \mathbb{E}[Y]) = \frac{m\delta}{m\delta + 1 - \delta} (1 + \delta(m-1)) = \delta m,$$

and

$$H(X) = H\left(\frac{1 - \delta}{m\delta + 1 - \delta}\right) + \frac{m\delta}{m\delta + 1 - \delta}H(Y) \geq H\left(\frac{1 - \delta}{m\delta + 1 - \delta}\right) + \frac{m\delta}{m\delta + 1 - \delta}\log_2(m)H(\delta).$$

It suffices to show that the right hand side of the above is at least $\log_2(m+1)H(\delta)$.

Making the change of variables $x = \frac{1-\delta}{m\delta}$, we just need to show

$$H\left(\frac{1}{x+1}\right) \geq \left(\log_2(m+1) - \frac{\log_2(m)}{x+1}\right)H\left(\frac{1}{mx + 1}\right)$$

for real numbers $m, x$ satisfying $m \geq 1$ and $mx \geq 1$. Since we clearly have equality when $m = 1$ or $m = \frac{1}{2}$, it is enough to show that the right hand side is a decreasing function of $m$. Taking the derivative with respect to $m$, using the identity $H'(\delta) = \log_2\left(\frac{1-\delta}{\delta}\right)$, we see that we just need to check

$$\left(\log_2(m+1) - \frac{\log_2(m)}{x+1}\right)\frac{x\log_2(mx)}{(mx+1)^2} \geq \log_2(e)\left(\frac{1}{m+1} - \frac{1}{m+mx}\right)H\left(\frac{1}{mx+1}\right).$$

Changing variables back to $m, \delta$ and rearranging, this becomes

$$(1-\delta)\left(1 + \frac{1}{m}\right)\log_2(m+1) + \delta(m+1)\log_2\left(1 + \frac{1}{m}\right) \geq \frac{\log_2(e)(1-2\delta)H(\delta)}{\delta(1-\delta)\log_2\left(\frac{1}{\delta}\right)}.$$

From $(1-\delta) \geq \delta$ and $m \geq 1$ we easily deduce that the left hand side is at least 2, so it is enough to prove the single variable inequality

$$2\delta(1-\delta)\frac{\log(1-\delta) - \log(\delta)}{(1-\delta) - \delta} \geq H(\delta),$$

where the logarithms on the left hand side are taken to the base $e$. We leave this inequality as an exercise for the reader. 

\[\square\]
Corollary 3. If \( X \) is geometrically distributed on \( \{0, \ldots, m\} \) with mean \( \delta m \) then
\[
H(X) \geq \log_2(m+1)H(\delta),
\]
and the inequality is strict if \( m > 1 \) and \( \delta \notin \{0, \frac{1}{2}, 1\} \).

Proof. This follows from the previous lemma together with the well-known fact that a geometric distribution has the maximum entropy among all distributions on a finite set which have a given mean. \( \square \)

Remark 1. In the case \( m + 1 = 2^k \) we can give a much simpler proof of Lemma 1. Let \( B_0, \ldots, B_{k-1} \) be i.i.d. random variables which are each 0 with probability \( 1 - \delta \) and 1 with probability \( \delta \). Then if we take
\[
X = \sum_{i=0}^{k-1} 2^i B_i,
\]
we have \( \mathbb{E}[X] = \delta m \) and \( H(X) = kH(\delta) \). This probability distribution corresponds to a trick used by Wolke in [8].

4 Divisor sum inequalities

Theorem 3. If \( 0 < \delta \leq \frac{1}{2}, \beta, s \geq 0 \) satisfy
\[
\beta > \frac{s - H(\delta)}{\delta},
\]
then
\[
\tau(n)^s \ll_{\beta, \delta, s} \sum_{d|n} \tau(d)^\beta.
\]

Proof. Choose a number \( M \) such that for all \( m \geq M \) we have
\[
\beta > \frac{s - \left\lfloor \frac{\log_2(m+1)}{\log_2(m+1)} \right\rfloor H(\delta)}{\delta}.
\]

Write \( n = \prod_i p_i^{m_i} \). We define a collection of independent random variables \( X_i, X_i \) taking values in \( \{0, \ldots, m_i\} \), as follows. If \( m_i < M \), we take \( X_i \) to be geometrically distributed with mean \( \delta m_i \). If \( m_i \geq M \), choose \( k \) such that \( 2^k - 1 \leq m_i < 2^{k+1} - 1 \), and let \( B_0, \ldots, B_{k-1} \) be \( k \) i.i.d. random variables which are each 0 with probability \( 1 - \delta \) and 1 with probability \( \delta \). Set
\[
X_i = \left( \sum_{j=0}^{k-2} 2^j B_j \right) + (m_i + 1 - 2^{k-1})B_{k-1}.
\]

Finally, we define a random variable \( D \) dividing \( n \) by \( D = \prod_i p_i^{X_i} \).

We have
\[
\mathbb{E}[\log(D)] = \sum_i \mathbb{E}[X_i] \log(p_i) = \delta \log(n),
\]

so by Corollary 1 we have

\[ P[D \leq n^\delta] \geq \frac{1}{2e^{3/2}}. \]

Setting \( P_n(d) = P[D = d] \), this can be written as

\[ 1 \leq 2e^{3/2} \sum_{d|n, \ d \leq n^\delta} P_n(d). \]

By Hölder’s inequality, for any \( t > 0 \) we have

\[
\sum_{d|n, \ d \leq n^\delta} P_n(d) \leq \left( \sum_{d|n, \ d \leq n^\delta} \tau(d)^\beta \right)^{-\frac{1}{t+1}} \left( \sum_{d|n, \ d \leq n^\delta} P_n(d)^{\frac{t+1}{t}} \tau(d)^{-\frac{\beta}{t}} \right)^{\frac{t}{t+1}}.
\]

Combining the last two inequalities, we see that

\[
\left( \sum_{d|n, \ d \leq n^\delta} P_n(d)^{\frac{t+1}{t}} \tau(d)^{-\frac{\beta}{t}} \right)^{-t} \leq \left( 2e^{3/2} \right)^{t+1} \sum_{d|n, \ d \leq n^\delta} \tau(d)^\beta.
\]

To finish, we just need to choose \( t \) large enough that the left hand side of the above is at least \( \tau(n)^s \). Since the left hand side is a multiplicative function of \( n \), we can restrict to the case \( n = p^m \), with just a single probability distribution \( X \) on the possible exponents \( \{0, \ldots, m\} \). Write \( \rho_m(x) \) for \( P_{p^m}(p^x) = P[X = x] \). Then we just need to choose \( t \) large enough to make the inequality

\[ (m + 1)^s \leq \left( \sum_{x=0}^{m} \rho_m(x)^{\frac{t+1}{t}} (x + 1)^{-\frac{\beta}{t}} \right)^{-t} \]

hold for all \( m \geq 1 \). We have

\[
\lim_{t \to \infty} \left( \sum_{x=0}^{m} \rho(x)^{\frac{t+1}{t}} \left( \frac{x + 1}{\rho(x)} \right)^{-\frac{1}{t}} \right)^{-t} = \prod_{x=0}^{m} \left( \frac{x + 1}{\rho(x)} \right)^{\rho(x)} = 2^{H(X) + \beta \mathbb{E} \log_2(X+1)}.
\]

Since \( \log_2 \) is a concave function, we have

\[
\mathbb{E} \log_2(X + 1) \geq \mathbb{E} \left[ \frac{X}{m} \log_2(m + 1) + \left( 1 - \frac{X}{m} \right) \log_2(1) \right] = \delta \log_2(m + 1).
\]

Thus, by the assumption on \( \beta \) and Corollary 3 we can find a \( t_0 \) such that for any \( t \geq t_0 \) and any \( m < M \) inequality (3) is satisfied. For \( m \geq M \), we use the easy inequality

\[
\left( \sum_{x=0}^{m} \rho(x)^{\frac{t+1}{t}} (x + 1)^{-\frac{\beta}{t}} \right)^{-t} \geq \left( (1 - \delta)^{\frac{t+1}{t}} + \delta^{\frac{t+1}{t}} 2^{-\frac{\beta}{t}} \right)^{-t \left[ \log_2(m+1) \right]},
\]

which follows from the fact that for any \( x, x + 1 \) is at least \( 2^B \), where \( B \) is the number of 1s in the binary representation of \( x \). Thus if we take \( t \) large enough to make

\[
\left( (1 - \delta)^{\frac{t+1}{t}} + \delta^{\frac{t+1}{t}} 2^{-\frac{\beta}{t}} \right)^{-t}
\]

sufficiently close to \( 2^{H(\delta) + \beta \delta} \), then inequality (3) will be satisfied for \( m \geq M \) as well. \( \square \)
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