Origin of Intrinsic Josephson Coupling in the Cuprates and Its Relation to Order Parameter Symmetry: An Incoherent Hopping Model

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Abstract

Experiments on the cuprate superconductors demonstrate that these materials may be viewed as a stack of Josephson junctions along the direction normal to the CuO$_2$ planes (the c-axis). In this paper, we present a model which describes this intrinsic Josephson coupling in terms of incoherent quasiparticle hopping along the c-axis arising from wave-function overlap, impurity-assisted hopping, and boson-assisted hopping. We use this model to compute the magnitude and temperature $T$ dependence of the resulting Josephson critical current $j_c(T)$ for s- and d-wave superconductors. Contrary to other approaches, d-wave pairing in this model is compatible with an intrinsic Josephson effect at all hole concentrations and leads to $j_c(T) \propto T$ at low $T$. By parameterizing our theory with c-axis resistivity data from YBa$_2$Cu$_3$O$_{7-\delta}$ (YBCO), we estimate $j_c(T)$ for optimally doped and underdoped members of this family. $j_c(T)$ can be measured either directly or indirectly through microwave penetration depth experiments, and current measurements on Bi$_2$Sr$_2$CaCu$_2$O$_8$.
and La$_{2-x}$Sr$_x$CuO$_4$ are found to be consistent with $s$-wave pairing and the dominance of assisted hopping processes. The situation in YBCO is still unclear, but our estimates suggest that further experiments on this compound would be of great help in elucidating the validity of our model in general and the pairing symmetry in particular.

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I. INTRODUCTION

While much of the early work on the cuprates has focused on the copper oxide planes, recent experimental and theoretical efforts are beginning to address the $c$-axis properties. In large part, this interest is motivated both by the availability of large, high-quality single crystals and by the realization that a full understanding of the superconductivity cannot be reached until the role of the third direction is established. Indeed, the third dimension is essential for the high critical temperatures in some theories.

While a complete theoretical description of the $c$-axis properties of the cuprates is clearly desirable, a full understanding of these properties is hampered by their unusual experimental signatures. In the normal state, for example, the resistivity along the $c$-axis ($\rho_c$) is quite generally metallic in the optimally doped materials and becomes semiconducting when the hole doping is reduced, while the in-plane resistivity ($\rho_{ab}$) always shows metallic behavior. At first sight, the metallic behavior of both $\rho_{ab}$ and $\rho_c$ in the optimally doped compounds is understandable within the conventional Bloch-Boltzmann theory of transport; however, the semiconducting temperature dependence of $\rho_c$ with metallic $\rho_{ab}$ in the underdoped samples is difficult to reconcile with this picture. In the superconducting state, also, measurements of hysteretic current-voltage curves, Fraunhofer-like $c$-axis critical currents as a function of in-plane magnetic field, and emission of microwave radiation at ac Josephson effect frequencies strongly indicate that the cuprates can be viewed as a stack of SIS Josephson junctions along the $c$-axis. This situation is not at all consistent with the properties of conventional superconductors.

One picture which may account for these anomalous properties is based on incoherent $c$-axis transport. In conventional metals, the presence of the crystal lattice leads to the formation of three-dimensional Bloch waves. The basic postulate of the incoherent approach is that Bloch waves in the $c$-direction do not form due to the layered structure, intrinsic disorder, and/or strong electronic correlations in the cuprates. This situation is not equivalent to localization of the quasiparticles in the $c$-direction, though, because they may still hop
incoherently between the CuO$_2$ layers. Working under this assumption, we will develop a theory which explains the unusual temperature-dependence of $\rho_c$ in the normal state and the intrinsic Josephson effect in the superconducting state and which may also shed light on the question of the order parameter symmetry in these systems.

The specific model we adopt was first articulated by Rojo and Levin$^8$ and asserts that the CuO$_2$ layers can be thought of as independent 2D Fermi liquids weakly coupled along the $c$-axis by three processes: direct inter-layer quasiparticle hopping,$^{2,5}$ hopping assisted through static disorder,$^7$ and hopping assisted through dynamic, boson-mediated (e.g., phonon-mediated) scattering.$^{2,8}$ The semiconducting temperature dependence of $\rho_c$ is then associated with the reduction of the dynamic inter-planar scattering as the temperature is lowered. The $c$-axis conduction is therefore analogous to that of a very dirty, but nevertheless metallic, system.$^8$

We will show that all three of these processes give rise to Josephson coupling, although their relative importance depends on the material and on the order parameter symmetry. It should be no surprise that all three mechanisms produce Josephson coupling between the layers; direct and impurity-assisted hopping processes formally resemble the tunneling processes considered in conventional SIS junctions,$^{12-15}$ and boson-assisted hopping has been known to contribute to tunneling in superconducting junctions in both the quasiparticle$^1$ and Josephson$^{14}$ channels for some time. In the cuprates, interest in these mechanisms has been piqued by the interpretation of experiments involving the $c$-axis transport$^1$ and flux quantization in superconducting rings.$^{17,18}$ This interest is reflected in a rapidly developing theoretical literature. In particular, we mention related results on intrinsic Josephson coupling in the cuprates by Graf $et~al.$$^{19}$ which came to our attention after completion of this work. As in this paper, these authors employ an incoherent hopping model to describe the inter-planar coupling in the cuprates. We differ in that Graf $et~al.$ consider only impurity-assisted hopping while we also include direct and boson-assisted processes. As a result, we find a non-zero $d$-wave Josephson critical current in fully oxygenated YBCO without the need to introduce an anisotropic impurity scattering rate. In addition, because the present
approach can describe the temperature dependence of the normal-state transport, we have been able to compute the full temperature dependence of the critical currents.

We present our calculations as follows. In the next Section, we describe our model and the approximations used in our calculations. For later reference, we will also discuss the normal-state $c$-axis conductivity $\sigma_c$ and derive expressions for the contributions to this conductivity by the three processes. We then move on to the superconducting state and derive the general forms of the Josephson critical current $j_c$ which arise from each inter-layer hopping mechanism. The direct and impurity-assisted critical currents are found to have familiar forms, but the boson-assisted contribution gives rise to a novel expression. Finally, we quantify our model by parameterizing experimental resistivity data in the $\text{YBa}_2\text{Cu}_3\text{O}_{7-\delta}$ (YBCO) system and using the estimated parameters to predict $j_c(T)$ for different oxygen stoichiometries and pairing symmetries. Since $j_c$ can be measured directly or inferred from microwave penetration depth experiments, our predictions may be compared to experiment. Based on these calculations, we conclude that low-temperature measurements of $j_c(T)$ should be able to distinguish $s$- from $d$-wave pairing regardless of oxygen stoichiometry. Moreover, we find that the $c$-axis penetration depth ratio should have either a 3D BCS character if the pairing is $d$-wave or a 3D BCS character which crosses over as a function of hole doping to a 2D Ambegaokar-Baratoff character if the pairing is $s$-wave. Although experiments have not converged in the YBCO system, we discuss the implications of our model for the pairing symmetry in the other cuprates in the last subsection. A preliminary report of these results has been published elsewhere.

II. INCOHERENT HOPPING MODEL

A. Description and Assumptions

We begin by writing the Hamiltonian for the electronic system in terms of the Hamiltonians of the individual CuO$_2$ layers $H_m$ and an inter-layer coupling Hamiltonian $H_\perp$: 
Here, the term “layer” denotes a composite object consisting of all the CuO$_2$ planes within a single unit cell, so the transfer Hamiltonian $H_\perp$ takes quasiparticles from one unit cell to another. Motivated by Ref. 8, we take $H_\perp = A + A^\dagger$, with

$$A = \sum_{im\sigma} (t_\perp + V_{im} + \sum_j g_{i-j,m}\phi_{jm}) c_{i,m+1,\sigma}^\dagger c_{im\sigma}$$  \hspace{1cm} (1)

and $c_{im\sigma}$ a quasiparticle annihilation operator for site $i$ in layer $m$ and spin projection $\sigma$. The terms in $H_\perp$ represent a direct hopping arising from wave-function overlap ($t_\perp$), an impurity-assisted hopping due to static disorder of strength $V_{im}$, and a boson-assisted hopping represented by the field $\phi_{im}$ with coupling strength $g_{im}$. The total Hamiltonian is then the electronic Hamiltonian added to the Hamiltonian governing the dynamics of the $\phi$ field: $H_{\text{tot}} = H_{\text{el}} + H_\phi$.

In studying the properties of this model, we make two assumptions; the first relates to how we include incoherence in the model and the second to how we view the intra-layer dynamics. Microscopically, incoherence may arise from strong intra-layer scattering, a non-Fermi-liquid ground state within the layers, dynamic inter-layer scattering, or strong electromagnetic fluctuations. We do not attempt to construct a fully microscopic theory of this incoherence, but simulate it by taking $H_\perp$ to be a small perturbation on $H_m$ and calculating its effect within second order perturbation theory. This approach is similar in spirit to that of Ioffe et al. and should yield the correct features of a more complete theory. In order to perform quantitative calculations within this model, we also assume that the normal state of $H_m$ for each layer $m$ is a Fermi liquid and that the superconducting state may be described within Migdal-Eliashberg theory.

From these two assumptions, the procedures outlined in the rest of this paper may be applied to any type of inter-layer disorder $V_{im}$ or inelastic scattering $\phi_{im}$. For concreteness, however, we specify these parameters. First, we take the disorder to be delta-function correlated: $\overline{V_{im}} = 0$ and $\overline{V_{im}V_{jn}} = V^2 \delta_{ij}\delta_{mn}$, where the overbar denotes the usual average over impurities. Moreover, we take the electron-inter-layer-boson coupling to be structureless ($g_{im} = g$) and the bosons themselves to be represented by an Einstein spectrum of frequency
$\Omega_0$. These choices do not affect the character of our results in $s$-wave superconductors, but they do affect the critical currents in $d$-wave superconductors in a way which will be discussed in Section III.

When it is necessary to include intra-layer scattering processes in our calculations, we do so through a temperature-dependent scattering rate $\hbar/\tau_{ab}(T)$ which is linear in $T$ at optimal doping, at least down to the critical temperature $T_c$. In addition, all wave-vector integrals are performed by the usual technique of restricting the wave vectors to the Fermi surface, taking the density of states to be constant, and then integrating over the remaining energy dependence. This approximation is appropriate if the Fermi level is far from any structure in the density of states (e.g., Van Hove singularities) and the other energy scales in the problem are much less than the band width. We assume these conditions hold here.

B. Normal-State Conductivity

To illustrate the application of our model and for future reference, we compute in this subsection the normal-state $c$-axis conductivity within the incoherent hopping model outlined above. By assumption, this calculation is to be carried out to second order in $H_\perp$ and thus becomes formally identical to computing the conductivity of an NIN tunnel junction, with $H_\perp$ playing the role of the transfer Hamiltonian.$^{15}$ Applying the standard theory of tunneling,$^{15}$ we compute the correlation function $X(\tau) = -\langle T_\tau [A(\tau)A^\dagger(0)] \rangle$ with $A$ defined by Eq. (1) and with the $\tau$-dependence of $A$ determined by $\sum_m H_m + H_\phi$. The diagrams used in this calculation are shown in Fig. 1; note that vertex corrections to $X(\tau)$ are of higher order in the inter-layer hopping amplitudes and so are neglected. Analytically continuing the Fourier transform of $X(\tau)$ to real frequencies gives the $c$-axis current density produced by a voltage $V$ across nearest neighbor layers as $j_c(V) = -(2e/\hbar a^2)\text{Im}X(eV)$, from which the conductivity $\sigma_c = c(\partial j_c(V)/\partial V)|_{V=0}$ is obtained. (Throughout this paper, $a$ ($c$) denotes the intra- (inter-) layer unit cell dimension). The resulting conductivity is the sum of three terms, each corresponding to one of the inter-layer hopping processes: $\sigma_c = \sigma_c^{\text{direct}} + \sigma_c^{\text{imp}} + \sigma_c^{\text{inel}}$. 

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The first term in $\sigma_c$ results from the direct hopping characterized by $t_\perp$ and can be written

$$\sigma_{\text{direct}}^c = \sigma_0 N_0 t^2_{\perp} \left( \frac{\tau_{ab}}{\pi \hbar} \right),$$

(2)

where $N_0$ is the density of states per unit cell per spin at the Fermi surface and $\sigma_0 = (4\pi e^2/\hbar)(c/a^2)$. Up to factors of order unity, this conductivity reproduces the results of Kumar and Jayannavar and Ioffe et al. Note that $\sigma_{\text{direct}}^c$ is proportional to the intra-layer lifetime $\tau_{ab}$, just as in the usual Drude expression for the conductivity, despite the fact that the conductivity is viewed as a tunneling process. To understand this surprising result, recall that the Kubo formula for the conductivity in 3D metals leads to a particle-hole bubble with group velocity factors at the vertices, while these vertices become the tunneling matrix element $T_{k'k}$ for tunneling calculations. The Kubo conductivity thus has a different form than the tunneling conductivity in 3D materials. For the 2D metals comprising the layers in our model, however, the tunneling matrix element for direct hopping is $T_{k'k} = t_\perp \delta_{k'k}$, and so vertices are diagonal in 2D wave vector. Consequently, the Kubo and tunneling formalisms give the same results.

The second and third terms in $\sigma_c$ result from the assisted processes and are found to be

$$\sigma_{\text{imp}}^c = \sigma_0 (N_0 V)^2$$

(3)

for impurity-assisted hopping and

$$\sigma_{\text{inel}}^c = \sigma_0 (N_0 g)^2 \frac{\hbar \Omega_0/2k_B T}{\sinh^2 (\hbar \Omega_0/2k_B T)}$$

(4)

for boson-assisted hopping. Observe that the impurity-assisted hopping conductivity is the usual NIN tunneling conductivity. Also note that the procedure in Ref. 8 of simulating the presence of boson-assisted hopping in $\sigma_c$ by allowing the impurity-assisted hopping amplitude to become a function of temperature is microscopically justified by the more complete calculations which yield Eqs. (3)-(4).

The form of the conductivity given by Eqs. (2)-(4) provides a simple explanation for the anomalous normal-state resistivity in the cuprates. At optimal stoichiometry, we expect the
layers to be strongly coupled and nearly coherent, implying that the direct hopping should dominate \( \sigma_c \). Consequently, \( \rho_c \propto \rho_{ab} \) as in conventional Bloch-Boltzmann transport.\(^2\) Off optimal stoichiometry, the direct inter-layer coupling may weaken relative to the assisted processes, making the material increasingly incoherent. While this change would not strongly affect \( \rho_{ab} \), \( \rho_c \) would now be dominated by the assisted hopping, the inelastic component of which would freeze out at low \( T \) [cf. Eq. (4)]. Thus, as \( T \) decreases, the net inter-layer hopping amplitude would be reduced and \( \rho_c \) would increase, as observed experimentally. These qualitative statements can be made quantitative through detailed fits of the normal-state resistivity to the Rojo-Levin model.\(^2\)\(^3\)

**III. JOSEPHSON CRITICAL CURRENT**

The incoherent hopping model described above views the inter-layer transport as an NIN tunneling process. In the superconducting state, this formulation of the transport immediately produces an intrinsic Josephson effect between the layers. In this Section, we compute the temperature-dependent critical current associated with each inter-layer hopping process to determine the general features of the total \( j_c(T) \). In the next Section, we will apply these theoretical results to the cuprates and see what can be revealed about the order parameter symmetry and the relative importance of the inter-layer hopping mechanisms.

**A. Formalism**

The technique for calculating the Josephson critical current in an SIS junction is well known and is closely related to the corresponding NIN calculation. One simply uses the same diagrams from the NIN computation and replaces the normal Green’s functions with anomalous or Gor’kov Green’s functions.\(^4\)\(^5\) Specifically, we define the correlation function \( \Phi(\tau) = -\langle T_\tau [A(\tau)A(0)] \rangle \) and evaluate it using the diagrams in Fig. [I]. The Fourier transform of the resulting expression is then analytically continued to real frequencies to obtain the Josephson current density produced by an applied voltage \( V \):
\[ j_J(t) = -(2e/\hbar a^2) \text{Im} \left[ e^{-2iVt/\hbar} \Phi(eV) \right] \]. Setting \( V = 0 \) and adjusting the relative phase of the order parameters on different layers to maximize \( j_J(0) \), we obtain the Josephson critical current as the sum of three components:

\[ j_c(T) = j_{c}^{\text{direct}}(T) + j_{c}^{\text{imp}}(T) + j_{c}^{\text{inel}}(T). \] (5)

In principle, these critical currents depend not only on the inter-layer hopping mechanism but also on the order parameter symmetry.\(^{28}\)

Although the Josephson critical current and the conductivity are computed from the same set of diagrams, there are significant differences. The most obvious is that the propagators used in computing the diagrams are different; specifically, the phases of the order parameters enter into the Josephson calculation in a fundamental way and affect how the imaginary part of the correlation function \( \Phi \) is taken. Moreover, the conductivity is the derivative of the correlation function \( X \) while the Josephson current is the correlation function \( \Phi \) itself. We therefore cannot expect similar behavior from these two quantities. In particular, we will see that the boson-assisted hopping does not contribute to the conductivity at low \( T \) but \textit{does} contribute to the Josephson current.

In evaluating the correlation function \( \Phi \), we should ideally solve the full Eliashberg equations\(^{23}\) for the layered system and use the results of that calculation to construct the anomalous propagators present in \( \Phi \). This approach would naturally include all the effects of elastic and inelastic scattering which would naively have a large impact on the Josephson coupling. As we will argue below, however, this procedure is not necessary, since inter-layer scattering effects are of higher order in the inter-layer hopping amplitudes and the intra-layer scattering may be incorporated into a BCS-like gap function. As we shall see, this approach is a natural consequence of our model and is supported by the long history of experimental and theoretical work on the Josephson effect in SIS junctions.\(^{14,18}\)

In performing the reduction from the full Eliashberg equations, we first note that all quantities in our model are by construction computed to second order in the inter-layer hopping amplitudes. This approximation is reasonable in the superconducting state as long
as the hopping amplitudes are smaller than the maximum of the gap function, as we expect them to be. Since the vertices in the diagrams for $\Phi$ are each first order in the hopping amplitudes [cf. Fig. 1], the complete second-order calculation involves propagators for only single, isolated layers. In particular, there are no vertex corrections to $\Phi$ and any loss of Josephson coupling due to inelastic inter-layer hopping comes in at higher order in the inter-layer hopping amplitudes.

Solving for the propagator is facilitated by observing that the critical current depends only on the weak-coupling propagator with the strong-coupling gap function $\Delta(k, i\omega_n) (= \phi(k, i\omega_n)/Z(k, i\omega_n)$ in the Nambu notation\(^{23}\)) in the approximation where wave vectors are restricted to the Fermi surface and the energy integrals are extended to infinity. Hence, all intra-layer elastic and inelastic scattering can be incorporated into our model by computing their effect on the gap function. Actually performing this computation for the cuprates is complicated by the presence of the unusual intra-layer inelastic scattering, which presumably gives rise to the linear-in-$T$ resistivity. Since the origin of this scattering is presently controversial, we are forced to take a phenomenological approach at this point.

Empirically, the cuprates exhibit the generic features of BCS superconductors, although with quantitative differences in, for example, $2\Delta/k_BT_c$. We therefore adopt the conventional picture and compute $j_c$ using the BCS form for the gap function. Anderson’s theorem\(^{29}\) in combination with the experimental fact that even very dirty tunnel junctions exhibit Ambegaokar-Baratoff behavior, suggests that this approach should be reasonable in the $s$-wave case. For $d$-wave superconductors, the effect of disorder and scattering is more subtle\(^{30}\) so our results should only be taken qualitatively.

We conclude this subsection by recapitulating some of the assumptions which underlie the calculations which follow. First and foremost, we have simulated the effects of incoherent inter-layer transport through second-order perturbation theory. This treatment does not address the details of how the incoherence arises in the first place, so further work will be required in order to obtain a fully self-consistent description. We have also neglected the detailed structure of the cuprates within a unit cell (multiple CuO$_2$ planes, CuO chains,
etc.) and have treated the intra-layer scattering mechanisms approximately; in particular, we have neglected the possibility of anisotropic scattering, which may be important in $d$-wave superconductors. Additionally, we have not attempted a fully self-consistent description of the superconducting state, but have relied on BCS theory. As we have argued, these assumptions are well justified. We therefore expect that the results which follow will not be strongly modified in more sophisticated treatments.

B. Results

From Eq. (5), the total Josephson critical current is the sum of three terms, each one corresponding to an inter-layer hopping process. Following the preceding discussion, we calculate in this subsection the critical current from each inter-layer process using the diagrams in Fig. 1 and a BCS gap function.

For direct hopping, this procedure gives

$$j_c^{\text{direct}}(T) = \frac{2e}{\hbar a^2} N_0 t_{\perp}^2 \int_{-\omega_D}^{\omega_D} dc \frac{\Delta^2_k}{E_k^2} \left[ \frac{\tanh (E_k/2k_B T)}{2E_k} + \frac{\partial f(E_k)}{\partial E_k} \right],$$

where $E_k = \sqrt{\epsilon^2 + \Delta_k^2}$, $f(x)$ is the Fermi distribution function, and the angle brackets denote an average over the Fermi surface. In the limit of zero-temperature, $j_c^{\text{direct}}(0) = (2e/\hbar a^2) N_0 t_{\perp}^2$ for either $s$- or $d$-wave symmetry, independently of the magnitude of $\Delta_0(0)$. By contrast, $j_c(0) \propto \Delta_0(0)$ in the Ambegaokar-Baratoff formula. As a function of temperature, $j_c^{\text{direct}}(T)/j_c^{\text{direct}}(0)$ varies exactly as the in-plane BCS superfluid density, which is plotted in Fig. 2 for both $s$- and $d$-wave pairing. At first sight, it is surprising that an incoherent treatment of the direct interlayer hopping yields the same temperature dependence as the (coherent) in-plane transport. However, as in the normal state, the reason behind this behavior derives from the fact that the vertices in the tunneling diagram are diagonal in 2D wave vector and so give an effectively 3D result (cf. the discussion after Eq. (2)). We also point out that $j_c^{\text{direct}}(T)$ depends strongly on the pairing symmetry at low $T$, being nearly independent of $T$ for an isotropic $s$-wave gap but proportional to $T$ for a clean $d$-wave gap, as might be expected from the presence of nodes in $\Delta_k$. 

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Impurity-assisted inter-layer hopping gives rise to a contribution to the critical current which has the Ambegaokar-Baratoff form if the order parameter is s-wave,\[^{13}\]

\[
j_{c}^{\text{imp}}(T) = \frac{2e}{\hbar a^2} (\pi N_0 V)^2 \Delta_0(T) \tanh \left( \frac{\Delta_0(T)}{2k_B T} \right)
\]

[cf. the solid line in Fig. 2], and vanishes by symmetry if the order parameter is d-wave and the impurity scattering is isotropic. The assumption of isotropic scattering is crucial if \(j_{c}^{\text{imp}}(T)\) is to vanish for d-wave superconductors; if the scattering is anisotropic, then the critical current is in general finite.\[^{19}\]

One generally expects impurity scattering to be mainly isotropic, so \(j_{c}^{\text{imp}}(T)\) is probably small in d-wave superconductors, but detailed calculations of the scattering matrix element in these systems are required to make a quantitative estimate.

Finally, the boson-assisted hopping processes also contribute to the Josephson critical current. Direct calculation of the relevant diagram yields

\[
j_{c}^{\text{inel}} = \frac{\pi e}{\hbar a^2} N_0 \lambda \Delta_0^2(0) I_{\text{inel}}(T),
\]

where \((\hbar = k_B = 1)\)

\[
I_{\text{inel}}(T) = \frac{2}{\pi} \frac{\Delta_0^2(T)}{\Delta_0^2(0)} \int_0^{2\pi \omega_D} \frac{dE}{E} \int_0^{2\pi \omega_D} \frac{dE'}{E'} \times \left\{ \left[ f(-E') - f(E) \right] \frac{2\Omega_0}{(E' + E)^2 - \Omega_0^2} \left[ (E' + E) \coth \left( \frac{\Omega_0}{2T} \right) - \Omega_0 \coth \left( \frac{E' + E}{2T} \right) \right] \right\} + \left[ f(E') - f(E) \right] \frac{2\Omega_0}{(E' - E)^2 - \Omega_0^2} \left[ (E' - E) \coth \left( \frac{\Omega_0}{2T} \right) - \Omega_0 \coth \left( \frac{E' - E}{2T} \right) \right],
\]

\(^{(9)}\)

\(\Omega_0\) is the Einstein phonon frequency, \(\lambda = 2N_0 g^2/\Omega_0\), and s-wave pairing is assumed. For d-wave pairing and structureless electron-boson coupling, the boson-assisted Josephson current vanishes.\[^{13}\]

The \(T\) and \(\Omega_0\) dependence of \(I_{\text{inel}}(T)\) are illustrated in Fig. 3 for a 100 K superconductor. This figure shows several interesting features of \(I_{\text{inel}}(T)\). First, we see that this function is approximately a non-zero constant at low \(T\), and the magnitude of this constant grows with \(\Omega_0\). Second, \(I_{\text{inel}}(T)\) resembles the Ambegaokar-Baratoff result for \(\Omega_0 > \Delta_0(0)\), but exhibits a peak when \(\Omega_0 < \Delta_0(0)\). Finally, near \(T_c\), \(I_{\text{inel}}(T)\) shows a linear dependence on \(T_c - T\).
We may explain these features simply. First, the Josephson current does not vanish at zero temperature because the zero-point energy of the bosons allows the creation of virtual bosons to mediate the inter-layer Cooper pair tunneling. Since the zero-point energy is proportional to $\Omega_0$, the increase of $I_{\text{inel}}(0)$ with $\Omega_0$ is expected. Second, when $\Omega_0 > \Delta_0(0)$, $\Delta_0(0)$ is the lowest energy scale and controls the integral, so $I_{\text{inel}}(T)$ is structureless. On the other hand, when $\Omega_0 < \Delta_0(0)$, we see from Eq. (6) that the $\Omega_0 \coth(\Omega_0/2T)$ terms dominate the integral and lead to an increase in $I_{\text{inel}}(T)$ with increasing $T$. At larger $T$, the decrease in $\Delta_0(T)$ compensates for this rise and eventually brings $I_{\text{inel}}(T)$ to zero at $T_c$. Third, the behavior near $T_c$ results from the fact that $j_{\text{inel}}^c(T) \propto \Delta_0^2(T)$ as $\Delta_0(T)$ goes to zero.

The different forms for the critical currents discussed in this Section give rise to the possibility that measurements of $j_c(T)$ or the associated penetration depth may shed some light on the order parameter symmetry and the relative importance of the inter-layer hopping mechanisms in the cuprates. We have demonstrated that the effects of pairing symmetry in this simple model are quite strong at low temperatures: $j_c(T) \propto T$ at low $T$ if the pairing is $d$-wave, and $j_c(T)$ is exponentially close to $j_c(0)$ if it is isotropic $s$-wave. This conclusion follows from the fact that $j_c^{\text{direct}}(T) \propto T$ and $j_c^{\text{imp}}(T) = j_c^{\text{inel}}(T) = 0$ for $d$-wave pairing, while all contributions are exponentially close to their $T = 0$ values for isotropic $s$-wave pairing. Even in a more sophisticated model that includes anisotropy in the impurity- and boson-assisted scattering processes, the power-law dependence of $j_c(T)$ at low $T$ for $d$-wave pairing should persist due to the presence of the nodes in the gap function. Consequently, the low-temperature behavior of the $c$-axis critical current may be an important probe of the order parameter symmetry. In addition, we have seen that each inter-layer hopping process yields a $j_c$ with a distinct temperature dependence. Looking at the temperature dependence of the total $j_c(T)$ may therefore provide some clues about the relative importance of the different inter-layer hopping processes. The goal of the next section is to make a quantitative estimate of the critical currents for optimally doped and de-oxygenated YBCO and see if these expectations are realistic.
IV. RELATION TO EXPERIMENT

The model we have presented builds on a natural connection between the normal- and superconducting-state properties. In this Section, we further exploit this connection by using experimental results on the normal-state properties of optimally doped and de-oxygenated YBCO to estimate $j_c(T)$ for these compounds. In the next two subsections, we describe the method for estimating the parameters in our model and extract the zero-temperature critical currents and penetration depths. We will use straightforward back-of-the-envelope reasoning in making these estimates, but the magnitudes of these estimates are confirmed by more detailed fits to normal-state transport in this model. In the succeeding subsection, we discuss the temperature dependence of the resulting critical currents and determine whether questions relating to the order parameter symmetry and the inter-layer hopping mechanism can be addressed in measurements of $j_c(T)$. Finally, we interpret recent experimental measurements based on the results of our calculations.

A. Optimally Doped YBCO

Experimentally, the resistivity in optimally doped ($T_c = 90$ K) YBCO is roughly linear in temperature in both the in-plane and $c$-axis directions. Within the incoherent hopping model, we can attribute this behavior to purely direct hopping between the layers: from Eq. (2), the intra-layer scattering rate is directly reflected in the $c$-axis resistivity, implying that $\rho_{ab} \propto \rho_c$. In the superconducting state, this situation corresponds to $j_c(T) \approx j_c^{\text{direct}}(T)$. To estimate the magnitude of this critical current, we observe that the band structure $c$-axis plasma frequency

$$
(\hbar \omega_c)^2 = \frac{4\pi e^2}{a^2 c} \sum_{k\sigma} \delta(\epsilon_k) \left( \frac{\partial \epsilon_k}{\partial k_z} \right)^2 \approx 16\pi e^2 c N_0 t_{\perp}^2.
$$

The $T = 0$ limit of Eq. (3) then yields

$$
j_c^{\text{direct}}(0) \approx \frac{(\hbar \omega_c)^2}{8\pi e \hbar c}.
$$

(10)
Optical measurements give $\hbar \omega_c = 400$ meV from which we deduce $j_c^{\text{direct}}(0) \approx 9 \times 10^7$ A cm$^{-2}$.

In layered materials, this critical current would also be related to the $c$-axis penetration depth measured in microwave experiments: $\lambda_c = \sqrt{\frac{\hbar c^2}{8\pi e c j_c^{\text{direct}}}}$, where $c$ is the speed of light. This relation gives $\lambda_c \approx 0.5 \mu$m for optimally doped YBCO, which is in reasonable accord with independent measurements of this quantity. In addition, Graf et al. estimate $\lambda_c \approx 0.9 \mu$m for optimally doped YBCO, purely disorder-assisted hopping, and $s$-wave pairing, which is also in good agreement with our (independent) result. Assuming a density of states of $3$ eV$^{-1}$, we estimate $t_\perp \approx 10$ meV. These results are summarized in the first column of Table I.

**B. De-Oxygenated YBCO**

To estimate the model parameters in the de-oxygenated case is more complicated, since we expect all inter-layer hopping processes to contribute to the critical current approximately equally. In particular, the boson-assisted hopping contribution should be significant in order to account for the upturn in $\rho_c$ at low temperatures. We begin by examining experimental measurements of the $c$-axis conductivity in de-oxygenated samples with $T_c = 40$ K and find that $\sigma_c \approx 1 \Omega^{-1}$ cm$^{-1}$ near $T_c$ and $\sigma_c$ is approximately linear in $T$ at high temperatures with a slope $(d\sigma_c/dT)_{\text{high T}} \approx 0.018$ $\Omega^{-1}$ cm$^{-1}$ K$^{-1}$. These asymptotic results are expected within our model, as can be seen from Eqs. (2)-(4) (and see below).

It is reasonable to assume (and detailed fits demonstrate) that the boson-assisted component of the normal-state conductivity has frozen out for $T \sim T_c$, so we partition the remaining low-temperature conductivity equally between the direct and disorder-assisted processes. In addition, the lattice parameters for YBCO give $\sigma_0 = 0.24 \mu\Omega^{-1}$ cm$^{-1}$ in Eqs. (2)-(4). Combining these facts with Eq. (3) immediately gives $N_0V \approx 1.4 \times 10^{-3}$ and $j_c^{\text{imp}}(0) = 3.7 \times 10^4$ A cm$^{-2}$. To extract the magnitude of the direct hopping, we note that optical measurements give $\hbar / \tau_{ab} = 2\pi\lambda_{ab} k_B T$ with $\lambda_{ab} = 0.2$ to 0.4. Taking $\lambda_{ab} =$
0.3, one obtains $\hbar/\tau_{ab} \approx 6$ meV at $T_c = 40$ K. Inserting this result into Eq. (2) produces $N_0 t_{\perp}^2 \approx 4 \times 10^{-5}$ meV or $t_{\perp} \approx 0.1$ meV if $N_0 = 3 \text{ eV}^{-1}$. From $N_0 t_{\perp}^2$ and $\sigma_0$, we obtain $j_{c, \text{direct}}(0) \approx 1.4 \times 10^4$ A cm$^{-2}$.

At high temperatures, Eqs. (2)-(4) can be written

$$
\sigma_c \sim \sigma_0 \left( \frac{N_0 t_{\perp}^2}{2\pi^2 \lambda_{ab} k_B T} + (N_0 V)^2 + \lambda N_0 k_B T \right),
$$

from which we obtain

$$
J_{c, \text{inel}}(T) \approx \frac{1}{4eC} \left. \frac{d\sigma_c}{dT} \right|_{\text{high}T} \Delta^2(0) I_{\text{inel}}(T).
$$

To compute this term, we also need the optical phonon frequency $\Omega_0$. Recent $c$-axis-polarized Raman experiments indicate that the 500 cm$^{-1}$ (720 K) O(4) phonon is important in interlayer hopping, and so we take $\hbar \Omega_0/k_B = 720$ K. Since $\hbar \Omega_0 >> \Delta_0(0)$, Fig. 3 indicates that $J_{c, \text{inel}}(T)$ will monotonically decrease with $T$. Numerical computation of $I_{\text{inel}}(0)$ then gives $J_{c, \text{inel}}(0) = 12.5 \times 10^4$ A cm$^{-2}$.

The estimated parameters for de-oxygenated YBCO are summarized in the second column of Table 1. Observe that $t_{\perp}$ has been reduced by two orders of magnitude from its fully oxygenated value: $t_{\perp} \approx 10$ meV in $T_c = 90$ K compounds, but $t_{\perp} \approx 0.1$ meV in $T_c = 40$ K compounds. This strong variation in $t_{\perp}$ with doping is consistent with an exponentially decreasing wave-function overlap induced by the increasing $c$-axis unit cell dimension and by disorder and vacancies in the chains. Empirically, the $c$-axis resistivity increases by two orders of magnitude from optimally doped to de-oxygenated samples, which would also naively correspond to at least an order-of-magnitude decrease in $t_{\perp}$. The data in Table 1 also indicate that $J_{c, \text{inel}}(0) : J_{c, \text{imp}}(0) : J_{c, \text{direct}}(0) \approx 9:2.6:1$, indicating that the Josephson current is dominated by the assisted processes in the de-oxygenated samples. This result is quite reasonable, given the importance of this term in determining the semiconducting temperature dependence of $\rho_c(T)$. As a final check to the consistency of our theory, we note that $t_{\perp} \sim T_c$ in optimally doped compounds, indicating weak incoherence, but $t_{\perp} << T_c$ in the de-oxygenated compounds, which puts these compounds firmly in the incoherent regime.
C. Estimated $j_c(T)$

Using the parameters derived in the preceding two subsections in Eqs. (5)-(8), we can compute the total $j_c(T)$ in both optimally doped and de-oxygenated YBCO for both $s$- and $d$-wave pairing. The results of this calculation are shown in Fig. 4. For the optimally doped case [Fig. 4(a)], we see that the low-temperature $j_c(T)$ for $d$-wave pairing can be clearly distinguished from $j_c(T)$ for $s$-wave pairing, although $j_c(0)$ is independent of the pairing symmetry. In addition, the critical currents for either pairing symmetry are distinct from the conventional Ambegaokar-Baratoff result, and this difference should be observable. Specifically, the direct hopping dominates $j_c(T)$ in the fully oxygenated case, so the observed critical current should have the same temperature dependence as the in-plane superfluid density. Experimentally, this means that $\lambda_{ab}^2(0)/\lambda_{ab}^2(T)$ should equal $\lambda_{c}^2(0)/\lambda_{c}^2(T)$ in fully oxygenated YBCO.

For the de-oxygenated case [Fig. 4(b)], we again find a clear difference between the critical currents for $s$- and $d$-wave pairing at low temperature (although it is not obvious from the figure). However, $j_c(0)$ now depends strongly on the pairing symmetry since the assisted processes are significant and contribute to the $s$-wave $j_c$ but not to the $d$-wave $j_c$. Additionally, while $d$-wave pairing yields a $j_c(T)$ very different from the Ambegaokar-Baratoff result, the $s$-wave critical current deviates from this form only very slightly. Thus, if the pairing is $s$-wave, it is unlikely that a measurement of $j_c(T)$ will enable one to deduce the relative importance of the inter-layer hopping mechanisms.

The analysis of these figures allows us to draw two general conclusions. First, the order parameter symmetry should be indicated by the low-temperature behavior of the measured critical current regardless of oxygen content, being a power-law for $d$-wave and exponentially close to a constant for isotropic $s$-wave. This result is robust and independent of the parameters estimated in the preceding two subsections. On the other hand, the reduction in magnitude of the critical current from $s$- to $d$-wave pairing in de-oxygenated samples depends sensitively on the model parameters, so it is unlikely that the pairing symmetry
will be indicated by an experimental measurement of $j_c(0)$, unless the observed magnitude is exceedingly small (indicating $d$-wave pairing). Second, the presence of incoherent hopping in $\lambda_c$ should be revealed by the observation of either a 3D BCS character if the pairing is $d$-wave or a 3D BCS character that crosses over as a function of hole doping to a 2D Ambegaokar-Baratoff character if the pairing is $s$-wave. It is unlikely, however, that these measurements will in isolation allow one to disentangle the relative contributions of the different inter-layer hopping processes in the de-oxygenated samples.

D. Comparison to Experiment

Our model can be compared to direct experimental measurements of $j_c(T)$ or to indirect measurements of this quantity through, for example, microwave penetration depth measurements. In the latter experiments, the penetration depth $\lambda_c$ is related to the critical current $j_c$ by the relation $j_c(T)/j_c(0) = \lambda_c^2(0)/\lambda_c^2(T)$. Given the results of the preceding subsection, we should ideally compare our theoretical estimates to measurements of the critical current or penetration depth in YBCO. Unfortunately, the experimental situation is not yet resolved in this cuprate: some penetration depth measurements show an SNS-like $\lambda_c^2(0)/\lambda_c^2(T)$ which is incompatible with our model, while others yield more conventional curves. However, we may still examine measurements on other cuprates and look for some of the generic features of our model.

The most thoroughly studied cuprate with regard to intrinsic Josephson junction effects is Bi$_2$Sr$_2$CaCu$_2$O$_8$ (BSCCO). Direct critical current measurements are available for this material that show Ambegaokar-Baratoff behavior in oxygen-annealed samples and a temperature-independent $j_c(T)$ at low $T$ that falls rapidly to zero near $T_c$ in argon-annealed samples. Within our model, the Ambegaokar-Baratoff-like $j_c(T)$ in the oxygen-annealed samples suggests $s$-wave pairing and a significant contribution to $j_c$ from assisted hopping, which is analogous to de-oxygenated YBCO [cf. Fig. 4(b)]. The approximately constant $j_c(T)$ in the argon-annealed samples is also indicative of $s$-wave pairing, but the rapid re-
duction of \( j_c \) near \( T_c \) is difficult to explain within our model.

In addition to BSCCO, recent measurements of the \( c \)-axis penetration depth on La\(_{2-x}\)Sr\(_x\)CuO\(_4\) (LSCO) have been reported and interpreted within an intrinsic Josephson effect picture.\(^2\) As with oxygen-annealed BSCCO, LSCO shows an approximately Ambegaokar-Baratoff \( j_c(T) \), which we interpret as evidence for significant assisted inter-layer hopping as well as \( s \)-wave pairing. However, the data do not extend to very low temperatures, so additional experimental measurements in this compound (and in BSCCO) would be illuminating.

V. CONCLUSIONS

In this paper, we have considered an incoherent hopping model that reproduces the features of the \( c \)-axis resistivity in both optimally doped and underdoped cuprates. We have assumed that there are three processes which transport quasiparticles from one CuO\(_2\) layer to another: a direct hopping induced by wave-function overlap, an impurity-assisted hopping due to disorder, and a boson-assisted hopping induced by, for example, modulation of the inter-layer distance by phonons. By performing calculations to second order in the inter-layer hopping amplitudes, we have shown that this model yields an intrinsic Josephson effect in the superconducting state, and we have analyzed the contributions to the Josephson critical current arising from these three processes. We find a robust dependence of the low-temperature \( c \)-axis critical current on the order parameter symmetry which should be reflected in experimental measurements of \( j_c(T) \) or the \( c \)-axis microwave penetration depth ratio \( \lambda_c^2(0)/\lambda_c^2(T) \).

This conclusion is supported by the application of our model to YBCO, where we have estimated the magnitude and temperature dependence of \( j_c \) and find a clear-cut signature of the pairing symmetry in the low-temperature behavior of \( j_c(T) \) for both fully oxygenated and de-oxygenated compounds. Additionally, our model predicts that the \( c \)-axis penetration depth ratio should have either a 3D BCS character if the pairing is \( d \)-wave or a 3D BCS
character which crosses over as a function of hole doping to a 2D Ambegaokar-Baratoff character if the pairing is \( s \)-wave. Measurements on BSCCO and LSCO suggest that the pairing in these materials is \( s \)-wave and the inter-layer hopping has a significant assisted component, but direct comparison of our estimates with YBCO is not possible because the data have not yet converged. Further experimental work on both fully oxygenated and de-oxygenated YBCO is clearly required and would provide a stringent test of our theory.

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Specifically, we take $\Delta(k, i\omega_n) = \Delta_k \theta(\omega_D - |\omega_n|)$, where $\Delta_k = \Delta_0(T) \eta_k$ and $\eta_k = 1$ for $s$-wave and $\cos 2\phi$ for $d$-wave symmetry. Here, $\phi$ is the angular coordinate along the 2D Fermi surface (assumed circular), and the temperature dependence of $\Delta_0$ is determined from the BCS equation. In the numerical calculations, we take $\omega_D = 1000$ K.

Note that Eq. (6) contains a term mistakenly omitted in Ref. 26.

The disappearance of the $d$-wave $j_{c\text{inel}}(T)$ is a consequence of the simple model used and would generally not apply if structure in the electron-boson coupling $g$ or dispersion of the inter-layer bosons were included. As with the impurity-assisted term, detailed calculations are required in order to make a quantitative estimate of the residual critical current in $d$-wave superconductors, but we expect that its magnitude would be small.

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FIGURES

FIG. 1. Feynman diagrams used to compute the c-axis conductivity and Josephson critical current within the incoherent hopping model. The solid dots at the vertices represent the direct inter-layer hopping amplitude $t_\perp$, the dashed line with a cross represents the averaged inter-layer disorder, and the wavy line represents the propagator for boson-assisted hopping. The two electronic Green’s functions in each diagram correspond to nearest-neighbor layers; normal-state propagators are used to compute the conductivity while Gor’kov propagators are used in the superconducting state. See text for details.

FIG. 2. Normalized contributions to the c-axis critical current $j_c(T)/j_c(0)$ as a function of the reduced temperature $T/T_c$ from various inter-layer hopping processes and pairing symmetries. Shown are critical currents for $d$-wave pairing and direct hopping ($j_c^{\text{direct}}(T)/j_c^{\text{direct}}(0)$ from Eq. (6), dashed line), $s$-wave pairing and direct hopping ($j_c^{\text{direct}}(T)/j_c^{\text{direct}}(0)$ from Eq. (3), dot-dashed line), and $s$-wave pairing and impurity-assisted hopping ($j_c^{\text{imp}}(T)/j_c^{\text{imp}}(0)$ from Eq. (7), solid line). These results are identical to, respectively, the in-plane BCS superfluid density for $d$-wave (dashed line) and $s$-wave pairing (dot-dashed line) and the critical current in a macroscopic SIS tunnel junction with $s$-wave pairing obtained within the Ambegaokar-Baratoff (AB) theory\cite{13} (solid line). Note that the $d$-wave impurity-assisted hopping critical current vanishes for isotropic scattering.

FIG. 3. Normalized boson-assisted c-axis critical current $I_{\text{inel}}(T)$ [Eq. (9)] as a function of temperature $T$ for Einstein phonons with frequencies $\hbar \Omega_0/k_B$ (from top to bottom) of 800 K, 400 K, 200 K, 100 K, 50 K, and 25 K with fixed coupling constant $\lambda$ (solid lines; see text for details). The temperature dependence of the gap function used in this calculation is determined from the $s$-wave BCS equation with a Debye temperature of 1000 K and a coupling constant fixed so that the critical temperature is 100 K. These choices yield $\Delta_0(0) = 178$ K. For comparison, the Ambegaokar-Baratoff (AB) result for an SIS Josephson junction\cite{13} normalized to the largest $I_{\text{inel}}(0)$ is also shown (dot-dashed line).
FIG. 4. Estimated c-axis Josephson critical current $j_c(T)$ obtained within the incoherent hopping model as a function of temperature $T$ for (a) optimally doped ($T_c = 90$ K) and (b) de-oxygenated ($T_c = 40$ K) YBa$_2$Cu$_3$O$_{7-\delta}$ with $s$-wave (dot-dashed line) and $d$-wave (dashed line) pairing. For comparison, the Ambegaokar-Baratoff result normalized to the largest $j_c(0)$ is shown as a solid line. The parameters used to generate these curves are discussed in the text.
TABLES

TABLE I. Estimates of the direct hopping amplitude $t_{\perp}$, the zero-temperature critical currents, and the zero-temperature penetration depths for fully oxygenated ($T_c = 90$ K) and de-oxygenated ($T_c = 40$ K) YBa$_2$Cu$_3$O$_{7-\delta}$ within the incoherent hopping model. The critical currents are presented for the direct ($j_c^{\text{direct}}(0)$), impurity-assisted ($j_c^{\text{imp}}(0)$), and boson-assisted ($j_c^{\text{inel}}(0)$) inter-layer hopping processes along with the total critical currents ($j_c^s(0)$, $j_c^d(0)$) and corresponding penetration depths ($\lambda_c^s(0)$, $\lambda_c^d(0)$) for s- and d-wave pairing. See text for details.

| Parameter                  | Optimally Doped | De-oxygenated |
|----------------------------|-----------------|---------------|
| $t_{\perp}$ (meV)          | 10              | 0.1           |
| $j_c^{\text{direct}}(0)$ (A cm$^{-2}$) | $9 \times 10^7$ | $1.4 \times 10^4$ |
| $j_c^{\text{imp}}(0)$ (A cm$^{-2}$) | $\sim 0$       | $3.7 \times 10^4$ |
| $j_c^{\text{inel}}(0)$ (A cm$^{-2}$) | $\sim 0$       | $12.5 \times 10^4$ |
| $j_c^s(0)$ (A cm$^{-2}$)    | $9 \times 10^7$ | $17.6 \times 10^4$ |
| $j_c^d(0)$ (A cm$^{-2}$)    | $9 \times 10^7$ | $1.4 \times 10^4$ |
| $\lambda_c^s(0)$ (µm)      | 0.5             | 11            |
| $\lambda_c^d(0)$ (µm)      | 0.5             | 40            |
The graph shows the relationship between $j_c(T)$ and $T$ for different types of wave: Ambegaokar-Baratoff, s-wave, and d-wave. The y-axis represents $j_c(T) \times 10^4$ in A cm$^{-2}$, and the x-axis represents temperature $T$ in K. The graph indicates a decreasing trend as temperature increases for all wave types.
