Island in Charged Black Holes

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Abstract

We study the information problem in the eternal black hole with charges on a doubly-holographic model in general dimensions, where the charged black hole on a Planck brane is coupled to the bath on the conformal boundary. For a brane with weak tension, its backreaction to the bulk is negligible and the entropy of the radiation is dominated by the entanglement entropy of the matter. We analytically calculate the entropy of the radiation and obtain the Page-like curve with the presence of an island on the brane. The Page time is evaluated as well. For a brane with strong tension, we obtain the numerical solution with backreaction in four-dimensional space time and find the quantum extremal surface at $t = 0$. It turns out that the geometric entropy related to the area term on the brane postpones the Page time.

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1 Introduction

The black hole information paradox is originated from the problem of whether the information falling into the black hole evolutes in a unitary fashion. During the evaporation of a black hole, the information originated from the collapsing star appears to conflict with the nearly thermal spectrum of Hawking radiation by semi-classical approximation. One possible explanation comes from quantum information theory. If we assume the unitary and chaotic evaporation of a black hole, then the von Neumann entropy of the radiation may be described by the Page curve, which claims that the entropy will increase until the Page time and then decrease again [1–3], which is in contrast to Hawking’s earlier calculation in which the entropy will keep growing until the black hole totally evaporated [4]. Later, a further debate is raised about whether the ingoing Hawking radiation is burned up at the horizon owing to the “Monogamy of entanglement”, which is known as AMPS firewall paradox [5]. In order to avoid the emergence of firewall, the interior of black hole is suggested to be part of the radiation system through an extra geometric connection (see “ER=EPR” conjecture [6]).

Recently AdS/CFT correspondence brings new breakthrough on the understanding of the Page curve from semi-classical gravity and sheds light on how the interior of the black hole is projected to the radiation [7–9]. In this context, the black hole in AdS spacetime is coupled to a flat space, where the latter is considered as a thermal bath.

Motivated by the Ryu-Takayanagi formula and its generalization [10, 11], it has been proposed that the fine-grained entropy of a system can be calculated by the quantum extremal surface (QES) [12]. When applied to the evaporation of black holes, in order to recover the
Page curve of Hawking radiation, it is found that an island should appear in the gravity region such that the fine-grained entropy of the radiation is determined by the formula of quantum extremal island \[ 13 \]

\[
S_{R} = \min_{I} \left\{ \text{ext} \left[ S_{\text{eff}}[R \cup I] + \frac{\text{Area}[\partial I]}{4G_N} \right] \right\},
\]

where \( R \) represents the radiation system, while the first term is the entanglement entropy of region \( R \cup I \), and the second term is the geometrical entropy from classical gravity. The von Neumann entropy \( S \) is obtained by the standard process: extremizing over all possible islands \( I \), and then taking the minimum of all extremal values.

A doubly-holographic model has been considered in \[ 13, 14 \], in which the matter field in 2D black hole geometry is a holographic CFT which enjoys the \( \text{AdS}_3/\text{CFT}_2 \) correspondence. By virtue of this the first term in (1) can be calculated according to the ordinary HRT formula in \( \text{AdS}_3/\text{CFT}_2 \). At early time, the minimal configuration contains no island and the increase of the entropy is mainly contributed from the first term in (1) as the accumulation of the Hawking particle pairs. But later, with the emergence of the island \( I \), the QES undergoes a phase transition such that a new configuration with island gives rise to smaller entropy. Meanwhile, owing to the shrinking of the black hole, the decrease of the entropy at late time is dominated by the second term in (1). As a result, the whole process can be described by the Page curve and the transition time of the configuration determines the Page time. Moreover, the fact that the island \( I \) is contained in the entanglement wedge of the radiation realizes the ER=EPR conjecture.

A similar information paradox occurs when black holes and flat baths are in equilibrium. Exchanging Hawking modes entangles black holes and baths, but the entanglement entropy should be upper bounded by the twice of the black hole entropy according to information theory. The authors in \[ 15 \] consider the 2D eternal black hole-bath system when the whole holographic system is dual to Hartle-Hawking state. Its island extends outside the horizon and the degrees of freedom (d.o.f.) on the island are encoded in the radiation system by a geometric connection \[ 15 \]. A similar conclusion was made independently by only considering the d.o.f. of radiation \[ 16, 17 \].

In \[ 18 \], a nontrivial setup of the doubly-holographic model in higher dimensions was established, where the lower dimensional gravity is replaced by a Planck brane with Neumann boundary conditions on it \[ 19 \]. The solution at \( t = 0 \) is obtained with the DeTurck trick and it was demonstrated that the islands exist in higher dimensions, and the main results in \[ 8, 13, 15 \] can be extended to higher dimensional case as well. However, compared to the work in literature, the geometric entropy contributed by the lower dimensional gravity, was not taken into account in the discussion \[ 18 \]. Therefore, it is reasonable to ask whether the contribution from the geometric entropy can be ignored. Moreover, in \[ 20 \], the solution without the
geometric entropy is considered in the weak-tension limit at $t = 0$, where the difference of entropy between two configurations might be negative. This peculiar phenomenon implies that the Page time is always $t = 0$, which of course calls for a proper understanding. In addition, since the DeTurck trick does not apply to time-dependent case \cite{21, 22}, in general the standard Page-like curve in higher dimensions is difficult to obtain.

Recently, inspired by the doubly-holographic model, a novel framework called wedge holography was established \cite{23}, which manifests an elegant relation between the Newton constant in higher dimensional gravity and that in lower dimensions. This relation implies that the lower-dimensional Newton constant becomes fairly large for the case of weak tension \cite{23, 24}, which may give us a hint to investigate above problems.

Therefore, in this paper our purpose are twofold. Firstly, we intend to extend the analysis in \cite{18} to charged black holes in higher dimensions, but more importantly, we will analytically obtain the Page-like curve of charged black holes in the weak-tension limit where the backreaction of the brane can be ignored. Secondly, we will take the geometric entropy in \cite{43} into account due to the non-zero tension on the brane \cite{15} and find more reasonable solutions of the entropy difference.

The paper is organized as follows. In section \ref{sec:2} we consider the charged matters on the
boundary and build up the doubly-holographic model. In section 3 we explore the Page-like curve in the weak-tension limit in general dimensions and analyze the evolution at different Hawking temperature. In section 4 we investigate the effects of charge on the island in the back-reacted spacetime, and then discuss how the geometric entropy in (43) affects Page time. Our conclusions and discussions are given in section 5.

2 The doubly-holographic Setup

In this section, we will present the general setup for the island within the charged eternal black hole. We consider a $d$-dimensional charged eternal black hole $\mathcal{B}$ in $AdS_d$ coupled to two flat spacetimes $\mathcal{R}$ (bath) on each side, with the strongly coupled conformal matter living in the bulk, as shown in Fig. 1(a). On each side, the black hole corresponds to the region with $\sigma > 0$ and the bath corresponds to the region with $\sigma < 0$. Moreover, at $\sigma = 0$, we glue the conformal boundary of the $AdS_d$ and flat spacetime together and impose the transparent boundary condition on the matter sector. With a finite chemical potential $\mu$, the matter and the black holes carry charges.

There are two possible descriptions of this combined system. The first is the full quantum mechanical description, where $d$-dimensional black holes together with the matter sectors are dual to $(d-1)$-dimensional quantum mechanical systems (QM) at points $\sigma = 0$ on the conformal boundary. While the second description is called the doubly-holographic setup. The matter sector is dual to a $(d+1)$-dimensional spacetime and the $d$-dimensional black hole $\mathcal{B}$ is described by a Planck brane in the bulk, as shown in Fig. 1(b). In this paper, we will adopt the second description, namely the doubly-holographic setup.

We consider the action of the $(d+1)$-dimensional bulk as

$$I = \frac{1}{16\pi G_N^{(d+1)}} \left[ \int d^{d+1}x \sqrt{-g} \left( R + \frac{d(d-1)}{L^2} \right) + 2 \int_{\mathcal{B}} d^d x \sqrt{-h} \left( K - \alpha \right) - \int d^{d+1} x \sqrt{-g} \frac{1}{2} F^2 \right].$$

Here the parameter $\alpha$ is proportional to the tension on the brane $\mathcal{B}$ and will be fixed later. The electromagnetic curvature is $F = dA$. Taking the variation of the action, we obtain the equations of motion as

$$R_{\mu\nu} + \frac{d}{L^2} g_{\mu\nu} = \left( T_{\mu\nu} - \frac{T}{d-1} g_{\mu\nu} \right), \quad \text{with} \quad T_{\mu\nu} = F_{\mu a} F_{\nu}^a - \frac{1}{4} F^2 g_{\mu\nu},$$

$$\nabla_{\mu} F^{\mu\nu} = 0,$$

where $T$ is the trace of energy-stress tensor $T_{\mu\nu}$. 


2.1 The Planck brane

In AdS/CFT setup with infinite volume, the $(d+1)$-dimensional bulk is asymptotic to $AdS_{d+1}$ which in Poincaré coordinates is described by

$$ds^2 = \frac{1}{z^2} \left(-dt^2 + dz^2 + dw_1^2 + \sum_{i=1}^{d-2} dw_i^2\right),$$

with the conformal boundary at $z = 0$. Let $\theta$ denote the angle between the Planck brane and the conformal boundary as shown in Fig. 2. Then the Planck brane $\mathcal{B}$ can be described by the hypersurface

$$z + \omega \tan \theta = 0,$$

near the boundary. One should cut the bulk on the brane $\mathcal{B}$ and restrict it in the region with $z + \omega \tan \theta < 0$ [19, 25–27]. We will also impose this constraint deep into the bulk and find its back-reaction to the geometry.

As for the boundary term in (2), we impose a Neumann boundary condition on the Planck brane which is

$$K_{ij} - Kh_{ij} + \alpha h_{ij} = 0,$$

where $h_{ij}$ is the metric on the brane $\mathcal{B}$. The parameter $\alpha$ is fixed to be a constant $\alpha = (d - 1) \cos \theta$ by solving (7) near the boundary to concrete the tension on the brane, which is $T = \alpha$.

In addition to (7), we also impose the Neumann boundary condition for gauge field $A_\mu$ on the brane $\mathcal{B}$ [28–30], which is

$$n^\mu F_{\mu\nu} h^{\nu}_i = 0,$$

where $n_\mu$ is the normal vector to the brane and $i$ denotes the coordinates along the brane.

Figure 2: A simple setup of Randall-Sundrum brane [19]. Here the Planck brane is anchored on the conformal boundary at $(z, w) = (0, 0)$ and penetrates into the bulk with an angle $\theta$. 
2.2 The Quantum Extremal Surface

The von Neumann entropy of radiation $\mathcal{R}$ in (1) is measured by the QES in Fig. 1(a). According to the doubly holographic prescription, the entanglement entropy of the $d$-dimensional matter is measured by the ordinary HRT surface and the geometric entropy is measured by the area term on the brane $\mathcal{B}$, namely

$$S_R = \min_I \left[ \frac{\text{Area}(\gamma_{\mathcal{I}\cup \mathcal{R}})}{4G_N^{(d+1)}} + \frac{\text{Area}(\partial I)}{4G_N^{(d)}} \right],$$

(9)

where $\gamma_{\mathcal{I}\cup \mathcal{R}}$ is the HRT surface sharing the boundary with $\mathcal{I} \cup \mathcal{R}$, as illustrated in Fig. 1(b).

Consider a HRT surface anchor at $w = w_b$ on the boundary as shown in Fig. 1(b). It measures the entanglement between the combined system at $[0, w_b]$, which consists of the quantum mechanical system (QMS) at $w = 0$ and part of the bath at $0 < w \leq w_b$, and the remaining bath system. For simplicity, hereafter, we call the combined system at $[0, w_b]$ as the black hole system, while the remaining bath as the radiation system.

Generally, one will find two candidates of QES, each of which corresponds to a HRT surface $\gamma_{\mathcal{I}\cup \mathcal{R}}$ in the bulk, as shown in Fig. 1. One candidate is a trivial surface $\gamma_{tr}$ anchored on the left and right baths, and the island $\mathcal{I}$ is absent; the other is a surface $\gamma_{pl}$ anchored on the Planck brane $\mathcal{B}$ with non-trivial island $\mathcal{I}$ on the brane. The emergence of island $\mathcal{I}$ keeps the von Neumann entropy (1) from divergence after the Page time $[15]$.

In the context of wedge holographic gravity, the relation of $d$-dimensional and $(d + 1)$-dimensional Newton constants was firstly obtained in $[23]$, and then was generalized in $[24]$. In the wedge holography construction $[24]$, for a specific class of solutions, the relation of the Newton constants becomes

$$\frac{1}{G_N^{(d)}} = \frac{1}{G_N^{(d+1)}} \int_0^\rho \cosh^{d-2}(x)dx. \quad (10)$$

Here, $g_{\mu\nu}$ denotes $(d + 1)$-dimensional metric of the background geometry with $0 \leq x \leq \rho$, while $h_{ij}$ denotes $d$-dimensional metric, which matches the induced metric on the brane $\mathcal{B}$ at $x = \rho$.

In this paper, the solutions we are concerned with are not included in above class, but we propose that two Newton constants could be related by (10), with the tension $T = (d - 1) \tanh \rho$. Therefore, in the weak-tension limit $\rho \to 0$, the lower-dimensional Newton constant $G_N^{(d)}$ is fairly large compared to the higher-dimensional one $G_N^{(d+1)}$. While in the large-tension limit, the relation becomes $L^{d-2}/G_N^{(d)} \gg L^{d-1}/G_N^{(d+1)}$ with the increase of $\rho$, which returns to the case as discussed in $[13]$. 

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Figure 3: (a): For \(\{d, \mu, W_b\} = \{3, 1/2, 1\}\), the two candidates \(\gamma_{tr}\) and \(\gamma_{pl}\) in the weak-tension limit. The black dot near the horizon \(z = 1\) represents the additional geometric entropy. (b): The trivial surface \(\gamma_{tr}\) colored in rose gold at time \(t\). Here \(z_{\text{max}}\) is the turning point with \(z'|_{z_{\text{max}}} = 0\).

### 3 Entropy without back-reaction

Page curve or Page-like curve plays a vital role in understanding the information paradox. In this section, we will explore the Page-like curve in the weak-tension limit, where the lower-dimensional Newton constant \(G_N^{(d)}\) is fairly large compared to the higher-dimensional one \(G_N^{(d+1)}\). In such a limit, the geometric entropy of the black hole is fairly small compared to the von Neumann entropy of the black hole, and the Planck brane \(B\) can be treated as a probe. That is to say, its backreaction to the background geometry is ignored.

Define \(W = z \cot \theta + w\) such that the Planck brane \(B\) is located at \(W = 0\) and we only take part of the manifold with \(W \geq 0\) into account. In the weak-tension limit \(\theta \to \pi/2\), the brane \(B\) applies negligible back-reaction to the background and the geometry can be regarded as the planar RN-AdS\(_{d+1}\), which is

\[
\begin{align*}
\frac{ds^2}{z^2} &= -f(z)dt^2 + \frac{dz^2}{f(z)} + (dW - \cot \theta dz)^2 + \sum_{i=1}^{d-2} dw_i^2, \\
A &= \mu \left(1 - z^{d-2}\right) dt, \\
f(z) &= 1 - \left(1 + \frac{d-2}{d-1} \mu^2\right) z^d + \frac{d-2}{d-1} \mu^2 z^{2d-2},
\end{align*}
\]

where the horizon is located at \(z = 1\) and \(\mu\) is the chemical potential of the system on the
boundary. The Hawking temperature is fixed to be

\[ T_h = \frac{d - d^2 + 4\mu^2 - 4d\mu^2 + d^2\mu^2}{4\pi - 4d\pi}. \]  

From (9), the von Neumann entropy of radiation is determined by the minimum

\[ S_R = 2 \min (S_{il}, S_{tr}). \]  

Here

\[ S_{il} = S_{pl} + \frac{\text{Area}_{pl}^{(d)}}{4G_N^{(d)}}. \]

\[ S_{pl} \] denotes the entanglement entropy of region \( R \cup \mathcal{I} \), while \( S_{tr} \) is the von Neumann entropy of the black hole on one side.

Firstly we consider the surface \( \gamma_{pl} \) ended on the Planck brane as the QES. We may introduce two different parameterizations in different intervals, just as performed in [18]. In \((W, z)\) plane, for the curve anchored between point \((W_b, 0)\) and \((W_c, Z_c)\) (Fig. 3(a)), we introduce \( W = W(z) \), while for the curve anchored between \((W_c, z_c)\) and \((0, z_b)\), we introduce \( z = z(W) \) instead, with \( z'(W_c) = W'(z_c)^{-1} \). Therefore, the corresponding lagrangian and area term are given as

\[ S_{pl} = \frac{L^{d-1} \Omega_{d-2}}{4G_N^{(d+1)}} \left( \int_0^{z_c} \frac{dz}{z^{d-1}} \sqrt{f(z) \left( \cot(\theta) - W'(z) \right)^2 + 1} \right) + \int_0^{W_c} \frac{dW}{z(W)^{d-1}} \left( \frac{f(z(W)) \left( \cot(\theta) z'(W) - 1 \right)^2 + z'(W)^2}{f(z(W))} \right), \]

\[ \text{Area}_{pl}^{(d)} = \frac{L^{d-2} \Omega_{d-2}}{z(0)^{d-2}}. \]

Here \( \Omega_{d-2} \) is the volume of the \((d - 2)\) relevant spatial directions and \( W_b \) is the location where \( \gamma_{pl} \) is anchored on the conformal boundary \( z = 0 \), as shown in Fig. 3(a).

Next we consider the trivial surface \( \gamma_{tr} \) penetrated the horizon as the QES. We express the area functional in the Eddington-Finkelstein coordinates and the entanglement entropy \( S_{tr} \) becomes

\[ S_{tr} = \frac{L^{d-1} \Omega_{d-2}}{4G_N^{(d+1)}} \int \frac{d\lambda}{z(\lambda)^{d-1}} \sqrt{-v'(\lambda) \left[ f(z(\lambda))v'(\lambda) + 2z'(\lambda) \right]}, \]

where \( \lambda \) is the intrinsic parameter of \( \gamma_{tr} \) (Fig. 3(b)) and

\[ v = t - \int \frac{dz}{f(z)}. \]
For now, we assume that $S_{tr} < S_{il}$ (which is not always correct as discussed in the following) and hence, $\gamma_{tr}$ is the genuine HRT surface at $t = 0$. As time passes by, $S_{il}$ maintains its value due to the static geometry, while $S_{tr}$ keeps growing due to the growth of the black hole interior in the $(d+1)$-dimensional ambient geometry [31], which also leads to the Hawking curve in the entropy of radiation.

But the Hawking curve contradicts with the subadditivity of von Neumann entropy. Since the total system is in a pure state, we have $S_R = S_R \leq 2S_{il}$, where $S_R$ is the von Neumann entropy of black hole system. In other words, at some moment, the growing $S_{tr}(t)$ must reach $S_{il}$ and after that the genuine HRT surface becomes $\gamma_{pl}$. $S_R$ stops growing at the transition, which determines the Page time $t_P$.

We are interested in the growth of the entropy with the time, thus we define $\Delta S(t) := S_R(t) - S_R(0)$, which is free from UV divergence. It behaves as

$$\Delta S = \begin{cases} 
2S_{tr}(t) - 2S_{tr}(0), & t < t_P \\
S_0, & t \geq t_P 
\end{cases}$$

where

$$S_0 = 2S_{il} - 2S_{tr}(0).$$

As shown in Fig. 4 for small endpoint $W_b$, the difference between two candidates is $S_0 < 0$, which was firstly obtained in [20] with zero tension. At the first sight this looks peculiar, and our understanding on this result is given as follows. Recall that for zero tension at $\theta = \pi/2$, the Newton constant $G_N^{(d)}$ is divergent, such that the geometric entropy term
vanishes and the von Neumann entropy of the black hole is totally contributed from the entanglement of matter fields at \((0, W_b)\). Therefore, for small endpoint \(W_b\), the negativity of the entropy difference \(S_0\) implies that the black hole system lacks d.o.f. to be entangled with the radiation and no process of the accumulation of the Hawking particle pairs in such cases, which leads to the saturation of the entropy at the beginning.

While for large endpoint \(W_b\) together with a non-zero but weak tension on the brane, the entropy difference is \(S_0 > 0\). In these cases, the backreaction of the brane to the background can be ignored due to the weak tension, and we expect that a Page-like curve may occur, which actually portrays the evolution that the von Neumann entropy of the black hole is mainly contributed by the internal entanglement of the matter fields, as well as a little contribution from the geometric entropy. We will focus on these cases and explore its evolution behavior in the next subsection.

### 3.1 Time evolution of the QES

For large endpoints \(W_b\), we have \(S_0 > 0\) and the growth rate of \(\Delta S(t)\) can be obtained similarly with the method applied in [32]. Since the integrand of (19) does not depend explicitly on \(v\), one can derive a conserved quantity as

\[
C = \frac{\frac{f(z)v' + z'}{z'^{d-1} \sqrt{-v'[f(z)v' + 2z']}},}{(22)}
\]
It is also noticed that the integral shown in (19) is invariant under the reparametrization, hence the integrand can be chosen freely as

$$\sqrt{-v'[f(z)v'+2z']} = z^{d-1}. \tag{23}$$

Substituting (22) and (23) into (19), we have

$$\frac{d}{dt}S_{tr} = L^{d-1}\Omega_{d-2} \frac{\sqrt{-f(z_{\text{max}})}}{4G_{N}^{(d+1)}} z_{\text{max}}^{d-1}, \tag{24}$$

$$t = \int_{0}^{z_{\text{max}}} \frac{dz}{f(z)\sqrt{f(z) + C^{2}z^{2d-2}}}. \tag{25}$$

Here $z_{\text{max}}$ is the turning point of trivial surface $\gamma_{tr}$ as shown in Fig. 3(b), and the relation between $z_{\text{max}}$ and the conserved quantity $C$ is given by

$$f(z_{\text{max}}) + C^{2}z_{\text{max}}^{2d-2} = 0. \tag{26}$$

At later time, the trivial extremal surface $\gamma_{tr}$ tends to surround a special extremal slice $z = z_{M}$, as shown in [31]. Define

$$F(z) := \frac{\sqrt{-f(z)}}{z^{d-1}}, \tag{27}$$

we find that $C^{2} = F(z_{\text{max}})^{2}$ keeps growing until meeting the extremum at $z_{\text{max}} = z_{M}$, where we have

$$F'(z_{M}) = (1 - d)z_{M}^{-d} \sqrt{-f(z_{M})} - \frac{z_{M}^{-d} f'(z_{M})}{2\sqrt{-f(z_{M})}} = 0. \tag{28}$$

By solving (28), we finally obtain the evolution of the entropy as

$$\lim_{t \to \infty} \frac{d}{dt}S_{tr} = K F(z_{M}), \quad K := \frac{L^{d-1}\Omega_{d-2}}{4G_{N}^{(d+1)}}. \tag{29}$$

Next we discuss the growth rate of entropy for the neutral case and the charged case separately.

• The Neutral Case: For $\mu = 0$, substituting $f(z) = 1 - z^{d}$ into (28), we have the growth rate of entanglement entropy at late time as

$$\lim_{t \to \infty} \frac{d}{dt}S_{tr} = c_{d} K T_{h}^{d-1}, \quad K := \frac{L^{d-1}\Omega_{d-2}}{4G_{N}^{(d+1)}}, \tag{30}$$

where $c_{d} = 2^{d+1} - \frac{1}{2} \pi^{d-1} 1^{d-2}(1 + d)(-2 + d) \frac{d-2}{2}$. For $d = 2$, $c_{d} = 2\pi$ and the growth rate is proportional to Hawking temperature of the black hole, which is exactly
Figure 6: For \( \{d, W_b, L, \theta\} = \{3, 1, 1, 2\pi/5\} \), the Page-like curves at different Hawking temperature are shown in the figure, which are labeled by different rose gold curves. The dashed blue curve represents the Page time \( t_\mu = t_P \mu \) of each Page-like curve. In the plot, The Newton constant is fixed by substituting \( T = (d - 1) \cos \theta = (d - 1) \tanh \rho \) into (10). Therefore, we can set \( \frac{L^2}{4G_N^2} = 1, \frac{\rho}{4G_N^2} \approx 0.32 \) and \( \tilde{S}(t) := \frac{1}{\Delta w_1} S(t) \).

in agreement with the result in [15,31].

Due to the exchanging of Hawking modes, the entanglement entropy of radiation grows linearly during most of time at a rate proportional to \( T_{d-1}^{-1} \). If there was no island, the entanglement entropy would keep growing and finally exceeding the maximal entropy the black hole system allowed to contain. It would be an information paradox similar to the version of evaporating black hole. The formation of quantum extremal island at Page time resolves this paradox, since some d.o.f. of black hole system are encoded in the radiation and the growth will saturate (Fig. 5).

• The Charged Case: Recall that in general dimensions, when turning on the chemical potential, the blackening factor becomes

\[
f(z) = 1 - \left( 1 + \frac{d-2}{d-1} \mu^2 \right) z^d + \frac{d-2}{d-1} \mu^2 z^{2d-2}. \tag{31}
\]

The late time behavior of entanglement entropy with \( d \geq 3 \) is obtained by substituting
the above equation into [28], which is

\[
\lim_{t \to \infty} \frac{d}{dt} S_{\text{tr}} = L^d \Omega_{d-2} \frac{2(d-1)^2}{4G_N^{(d+1)}} \left( \frac{d-2}{(d-2)(d-1 + (d-2)\mu^2)} \right)^{\frac{1-d}{d}} \right)^{\frac{(d-1)}{d} \mu^2}.
\] (32)

Specifically, for \( d = 3 \), we have

\[
\lim_{t \to \infty} \frac{d}{dt} S_{\text{tr}} = \frac{L^2 \Delta w_1}{16G_N^{(4)}} \sqrt{3(2 + \mu^2)^3} - 8\mu^2.
\] (33)

Here \( \Delta w_1 \) is the integral along \( w_1 \) direction. Since the saturation occurs approximately at \( S_0 \), the Page time is obtained as

\[
t_P \approx \frac{4G_N^{(4)}}{L^2 \Delta w_1} \frac{4S_0}{\sqrt{3(2 + \mu^2)^3} - 8\mu^2}.
\] (34)

Some concrete cases are shown in Fig. 6: in the high temperature limit \( T_h/\mu \to \infty \), the Page-like curve recovers the neutral case as mentioned above, while in the low temperature limit \( T_h/\mu \to 0 \), it becomes the extremal case.

For the neutral case, the entanglement grows rapidly and saturates at the highest level. With the decrease of Hawking temperature, the entanglement grows tardily and finally saturates at a lower level. While for the extremal case, one can easily read from (33), the entanglement entropy never grows.

Since the entanglement between the black hole and radiation system is built up by the exchanging of Hawking modes before the Page time \( t_P \), the phenomenon that entropy increases rapidly at higher temperatures indicates that the higher the Hawking temperature is, the higher the rate of exchanging is.

### 4 Entropy with back-reaction

In this section we will consider the QES in the presence of island when the back-reaction of the Plank brane to the bulk is taken into account for \( d = 3 \). We will apply the DeTurck trick to handle the static equations of motion and find the numerical solution via spectrum method. Nonetheless, the time dependence in the higher-dimensional geometry is hard to explore using DeTurck trick. Therefore, in this section we will just derive the solution at \( t = 0 \) with the geometric entropy [43] and investigate the effects of the charge on the island.

We will also discuss how the geometric entropy affects the Page time.
4.1 The metric ansatz

We introduce the Deturck method \[22\] to numerically solve the background in the presence of the Planck brane in this subsection, and the numerical results for the QES over such backgrounds will be presented in next subsection. Instead of solving (3) directly, we solve the so-called Einstein-DeTurck equation, which is

\[
R_{\mu\nu} + 3 g_{\mu\nu} = \left( T_{\mu\nu} - \frac{T}{2} g_{\mu\nu} \right) + \nabla_{(\mu} \xi_{\nu)},
\]

(35)

where

\[
\xi^\mu := [\Gamma_{\mu\nu\sigma}^\mu(g) - \Gamma_{\mu\nu\sigma}^\mu(\bar{g})] g^{\nu\sigma}
\]

is the DeTurck vector and \(\bar{g}\) is the reference metric, which is required to satisfy the same boundary conditions as \(g\) only on Dirichlet boundaries, but not on Neumann boundaries \[18\].

Now we introduce the metric ansatz and the boundary conditions in the doubly-holographic setup. For \(W \to \infty\), the ambient geometry is asymptotic to 4\(D\) planar RN-AdS black \[11\] with \(d = 3\). Furthermore, for numerical convenience we define two new coordinates in the same way as applied in \[18\], which are

\[
x = \frac{W}{1 + W} \quad \text{and} \quad y = \sqrt{1 - z}.
\]

(36)

The domain of the first coordinate \(x\) is compact, while the second coordinate \(y\) keeps the metric from divergence at the outer horizon \(y_+ = 0\).

When the back-reaction of the Planck brane is taken into account, the translational symmetry along \(x\) direction is broken. Therefore, the most general ansatz of the background is

\[
ds^2 = \frac{L^2}{(1 - y^2)^2} \left[ -y^2 P(y) Q_1 dt^2 + \frac{4Q_2}{P(y)} dy^2 + \frac{Q_4}{(1 - x)^4} \left( dx + 2y(1 - x)^2 Q_3 dy \right)^2 + Q_5 dw_1^2 \right]
\]

(37)

\[
A = y^2 \psi dt,
\]

(38)

\[
P(y) = 2 - y^2 + (1 - y^2)^2 - \frac{1}{2}(1 - y^2)^3 \mu^2.
\]

(39)

where \(\{Q_1, Q_2, Q_3, Q_4, Q_5, \psi\}\) are the functions of \((x, y)\). All the boundary conditions are listed in Tab. \[1\]. Moreover, the boundary conditions at the horizon \(y = 0\) also imply that \(Q_1(x, 0) = Q_2(x, 0)\), which fixes the temperature of the black hole \[14\] as

\[
T_h = \frac{6 - \mu^2}{8\pi}.
\]

(40)

The reference metric \(\bar{g}\) is given by \(Q_1 = Q_2 = Q_4 = Q_5 = 1\) and \(Q_3 = \cot \theta\). In this
In this subsection, all free parameters will be fixed for numerical analysis. To regularize the UV divergence of entropy, we should introduce a UV cut-off. Since $\gamma_{pl}$ and $\gamma_{tr}$ share the same asymptotic behavior, their difference should be independent of the choice of the cut-off given that it is small enough. We find that $y_\epsilon = 1 - 1/100$ is good enough for numerical calculation.

For the trivial surface $\gamma_{tr}$ at $t = 0$, we introduce the parameterization $x = x(y)$, which leads to the corresponding lagrangian

$$S_{\gamma_{tr}} := S_{tr} = \frac{L^2}{4G_N^{(4)}} \int_0^1 dy \frac{1}{(1 - y^2)^2} \left[ Q_5 \left( \frac{4Q_2}{P(y)} + \frac{Q_4(2y(x(y) - 1)^2Q_3 + x'(y))^2}{(x(y) - 1)^4} \right) \right].$$

For numerical convenience, we have set $L^2/(4G_N^{(4)}) = 1$ in the following discussion.

As for the surface $\gamma_{pl}$ at $t = 0$, we also introduce two different parameterizations in different intervals. In $(x, y)$ plane, for the curve anchored between point $(x_b, 1)$ and $(x_c, y_c)$ (Fig. 8(a)), we introduce $x = x(y)$ just as the parameterization of the trivial surface $\gamma_{tr}$, while for the curve anchored between $(x_c, y_c)$ and $(0, y_b)$, we introduce $y = y(x)$ instead, with $y'(x_c) = x'(y_c)^{-1}$. Therefore, the corresponding lagrangian and area term are given as

$$S_{\gamma_{pl}} := \frac{S_{pl}}{\Delta w_1}$$

$$= \int_{y_b}^{1} \frac{dy}{(1 - y^2)^2} \left[ Q_5 \left( \frac{4Q_2}{P(y)} + \frac{Q_4(2y(x(y) - 1)^2Q_3 + x'(y))^2}{(x(y) - 1)^4} \right) \right] + \int_{0}^{x_c} dx \frac{dy}{(y(x)^2 - 1)^2}$$

$$\left[ Q_5 \left( \frac{4Q_2y'(x)^2}{P(y(x))} + Q_4 \left( 4Q_3^2y(x)^2y'(x)^2 + \frac{4Q_3y(x)y'(x)}{(x - 1)^2} + \frac{1}{(x - 1)^4} \right) \right) \right].$$

$$\tilde{A}(y_b) := \frac{\text{Area}_{\gamma_{pl}}}{\Delta w_1} = \frac{L\sqrt{Q_5}}{1 - y_b^2}, \quad y_b := y(0).$$
Figure 7: The charge density $\rho(W)$ in the case for $\mu = 1/2$ and $\theta = \pi/4$.

The difference between two solutions associated with HRT surface $\gamma_{pl}$ and $\gamma_{tr}$ at $t = 0$ is

$$\tilde{S}_0 = \text{ext}_{y_b} \left( \tilde{S}_{\gamma_{pl}}(y_b) + \frac{\tilde{A}(y_b)}{4G_N^3} \right) - \tilde{S}_{\gamma_{tr}},$$

(44)

which also represents the saturation value of entropy after Page time as mentioned in the previous section. Notice that for numerical convenience, we also set $L/(4G_N^3) = 1$ in the following discussion.

The numerical result for the difference between two candidates is shown in Fig. 8(b). It indicates the trivial surface $\gamma_{tr}$ always dominates at $t = 0$ and $\tilde{S}_0$ is obtained when the extremal value is reached. As Hawking radiation continues, the entanglement entropy keeps growing until Page time $t_p$. Then, the surface $\gamma_{pl}$ dominates at $t = t_p$ and entanglement saturates due to the phase transition.

After Page time, the island solution dominates and the island $I$ emerges in the entanglement wedge of the radiation. Therefore, the d.o.f. on the island can be encoded in the radiation by quantum error correction process.

By varying the endpoint $x_b$, we illustrate different scenarios black hole for the d.o.f. encoded in the radiation in Fig. 9(a). In detail, the solution with $x_b = 0$ measures the entanglement between the QMS at $x = 0$ and its complementary, namely the whole bath system. While for $x_b > 0$, the island solutions measure the entanglement between QMS at $x = 0$ together with part of the bath system at $0 < x \leq x_b$ and its complementary, namely, the remaining bath system at $x > x_b$.

Firstly, we find that in most cases, the island will stretch out of the horizon ($y_b > 0$), indicating that besides the interior, the region near the exterior of the horizon will also be encoded in the radiation by the entanglement wedge. This result is consistent with that in [15] and reflects the spirit of the ER=EPR proposal, which suggests that two distant systems are connected by some geometric structure. Furthermore, the boundary of island approaches the horizon when $x_b \to 1$, which indicates that the whole exterior is encoded in
Figure 8: (a): Two candidates of HRT surface colored in rose gold (trivial surface $\gamma_{tr}$) and blue (island surface $\gamma_{pl}$) are anchored at $x = x_b$, while the half of island at $t = 0$ is colored in red. (b): For $\mu = 1$ and $x_b = 1/100$, the difference between two candidates is plotted when varying $y_b$, where $\bar{S}_0 \approx 4.42716$.

the combined system, namely QMS plus baths. While for $x_b \to 0$, the boundary of island approaches $y_b \approx 0.52796$, which indicates that only the region near the horizon ($0 < y < y_b$) is likely to be encoded in the pure baths without QMS, while the region near the boundary ($0 < y < y_b|x_b\to0$) is going to be encoded in a system that contains the QMS at $x = 0$.

Moreover, the saturated entropy $\bar{S}_0/\mu$ increases with Hawking temperature $T_h/\mu$, which manifests that the growth of temperature allows more entanglement to be built up between the black hole and radiation system as shown in Fig. 9(b).

In addition, the saturated entropy also increases with $x_b$ (Fig. 9(c)). As discussed above, the island solutions with $x_b > 0$ measure the quantum entanglement between the QMS together with part of the bath and the remaining. Since the bath system is of infinite size, the remaining system is also infinite. Therefore, the upper bound of the entanglement only depends on the size of the bath system we take into account, namely, the choice of $x_b$. As a result, the more d.o.f. of the bath system we take into account (besides those of the QMS), the larger the entanglement entropy is.

4.3 The contribution of the geometric entropy

Notice that in the model with $2d/3d$ duality [13], one requires $1 \ll c \ll \text{Area}/(4G_N^{(2)})$ to ensure that it works in the semiclassical limit in 2d as well as a large-radius dual in 3d, where $c$ is the central charge of CFT$_2$. A similar relation in higher dimensions is $L^{d-2}/G_N^{(d)} \gg L^{d-1}/G_N^{(d+1)}$.  

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According to this relation, it is obvious that after Page time $t \geq t_P$, we have

$$\Delta S(t) = S_0 \approx 2 \frac{\text{Area}^{(d)}_{\text{pl}}}{4G^{(3)}_N} \approx 2S_{BH},$$

(45)

where $S_{BH}$ is the Bekenstein-Hawking entropy of the $d$-dimensional black hole. Nevertheless, as mentioned in Sec. 2.2, we find that (45) is only valid for $\theta \to 0$, where the geometric entropy contributes the most part in (21) and (44).\footnote{We define the contribution of the geometric entropy as the proportion of Area$/G^{(3)}_N$ in (44).}

In the case of weak-tension limit, we find $S_0 < 0$ for small endpoints $x = x_b$ (or equivalently, $W = W_b$), which leads to the saturation of the entropy at the beginning. As the growth of the tension, the geometric entropy becomes significant. Specifically, in the case of $d = 3$ and $\theta = \pi/4$, one has the relation $L/G^{(3)}_N = L^2/G^{(4)}_N$. It turns out that the geometric entropy becomes significant in total $S_0$ and hence can not be ignored as illustrated in Fig. 9(b) and Fig. 9(c). Compared with the result in [18], we find that taking the geometric entropy into account will not change the sign of $S_0$ for the case $\mu = 0$ and $\theta = \pi/4$, but it will increase the Page time considerably as well as the entropy after saturation due to the finite tension on the brane indeed.

5 Conclusions and Discussions

In this paper, we have investigated the black hole information paradox in the doubly-holographic setup for charged black holes in general dimensions. In this setup, the holographic structure provides a natural framework to study the thermal properties of the black hole and the quantum entanglement between the bulk and the boundary. We have shown that for a charged black hole in general dimensions, the saturated entropy $S_0$ is governed by the tension $\mu$ and the angular momentum $\theta$. In particular, we have derived the explicit expression for $S_0$ and found that it is proportional to the geometric entropy of the black hole. This result is consistent with the holographic principle, which states that the information content of a black hole is encoded on its boundary. Moreover, we have studied the effect of the tension on the entropy and found that the geometric entropy becomes significant as the tension increases. This is particularly evident in the case of weak-tension limit, where the geometric entropy contributes the most part in the total entropy.

In conclusion, our work provides a deeper understanding of the holographic structure of charged black holes in general dimensions. The results we have obtained not only confirm the holographic principle but also shed light on the role of the geometric entropy in the black hole information paradox. Further studies in this direction are expected to reveal more insights into the fundamental nature of quantum gravity.
graphic dual of a two-sided black hole is in equilibrium with baths such that the geometry keeps stationary under the evaporation. The Newton constants in different dimensions are related by (10), which indicates that the Newton constant in the lower dimension becomes fairly large compared to the higher-dimensional one in the weak-tension limit. By analytical analysis, we have found that for small endpoints of the HRT surface \((W_b \to 0)\), the system has no evolution and the entropy saturates at the beginning, because of the lack of d.o.f. for the black hole system to be entangled with the radiation, while for large endpoint \(W_b\), the evolution of the von Neumann entropy in the weak-tension limit still obeys the Page-like curve. Specifically, for the neutral case in general \((d + 1)\) dimensions, the entropy grows linearly at the rate proportional to \(T_h^{d-1}\) due to the evaporation and finally saturates at a constant which depends on the size of the radiation system. In 4\(D\) charged case, we have plotted the Page-like curves under different temperatures. At high temperatures, the black hole seems to “evaporate” more rapidly than the cold one and will build more entanglement with the radiation system. While for the near-extremal black hole, the evaporating process seems to be frozen, since the exchanging of Hawking modes takes place extremely slowly.

Moreover, the stationary solution with backreaction is obtained by the standard DeTurck trick. For the entanglement entropy, the more d.o.f. of baths we take into account, the more entanglement will be built up and the less region near the black hole will be encoded in the remaining baths\(^2\). Similarly, with the increase of Hawking temperature, more entanglement will be built up between the black hole and the radiation system. In addition, the contribution from the geometric entropy to the von Neumann entropy is also discussed. Compared to the result obtained in [18], the Page time is postponed distinctly due to the sufficient geometric entropy.

We have obtained the Page-like curve in the weak-tension limit in which the backreaction of the Planck brane to the bulk can be ignored. Notice that in these cases, the final saturated entropy is not equal to two times of the Bekenstein-Hawking entropy as in the standard process. The reason is that in the weak-tension limit, the d.o.f. for the black hole system are fairly few and the entanglement is mainly from the matter fields. Hence, it is very desirable to investigate the time evolution of the entanglement entropy with backreaction, which involves in the dynamics of black holes, thus beyond the DeTurck method. Furthermore, beyond the simple model with doubly-holographic setup, the information paradox in high-dimensional evaporating black holes is more complicated and difficult to describe. Therefore, it is quite interesting to develop new methods to explore the evaporation of black holes in holographic approach.

\[^2\]A similar result is obtained in literature [15].
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