Crack onset at the spherical-inclusion matrix interface. Application of a coupled stress and energy criterion
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Abstract
The problem of crack initiation at the interface of a spherical inclusion embedded in an infinite matrix subjected to a remote tension is studied by the means of the coupled criterion of the finite fracture mechanics. This allows us to predict the critical tension necessary to initiate the crack at the interface as a function of the elastic properties of the particle and matrix, the particle size and the main parameters defining the strength and fracture properties of the interface. The coupled criterion assumes that a crack of a finite length appears when the next two criteria are simultaneously fulfilled: on the one hand a stress criterion which imposes a condition over the stresses at points where the crack initiates and on the other hand the crack onset must be energetically allowed. The coupled criterion is applied with the aid of an axisymmetric boundary element method code. The numerical model shows a strong influence of the spherical-inclusion radius on the critical remote tension.

Keywords: composites, crack initiation, interface crack, spherical inclusion, size effect, finite fracture mechanics

1. Introduction
Composites reinforced with spherical particles present some advantages in comparison with those reinforced with long fibers, e.g., easier tailoring, isotropy (if anisotropy is not particularly wanted) and cost effectiveness. Fracture behavior of these composites are very linked to their microstructure since failure initiates at or in the vicinity of the particle-matrix interface. In particular, when spherical particles are stiffer and the composite is subjected to uniaxial tension under static loading, first microdebonds and microvoids, respectively, initiate at and close to an interface pole as reported by [25]. A wide variety of works has been presented dealing with this problem, see [25] for more references.

The present work aims to develop a theoretical model for the prediction of crack initiation at the particle-matrix interface when the matrix is subjected to a remote tension $\sigma^\infty$ by means of the coupled criterion of Finite Fracture...
Mechanics (FFM). This criterion assumes that a crack of a finite length appears when the two next conditions are fulfilled simultaneously: stresses at the points where the crack will appear exceed a critical value (stress criterion) and the onset of the crack is energetically allowed (energy criterion).

The problem under study is represented in Fig. 1. Initially the spherical particle is perfectly adhered to the surrounding matrix. A monotonically-increasing remote tension $\sigma_\infty$ is applied at the matrix up to a critical value $\sigma_\infty^c$ for which a debond of angle $\Delta \theta_c > 0$ appears. The situations prior and after the onset are axisymmetric. Hence, in what follows the problem is studied as an axisymmetric problem and every angle is defined from the axisymmetry axis. By way of example, the model is applied to an alumina-titanium composite, recently proposed for human implants. Both inclusion and matrix are modeled as linear elastic isotropic with Young’s modulus $E_1 = 400$ GPa and Poisson’s ratio $\nu_1 = 0.25$ for the alumina inclusion and $E_2 = 116$ GPa and $\nu_2 = 0.32$ for the titanium matrix.

First, both stress and energy criteria of the coupled criterion will be applied separately in Sections ?? and ?? and subsequently combined in Section ??.

2. Stress criterion

The stress criterion used here requires that the normal interface tractions $\sigma$ at every point defined by an angle $\theta$ where the crack will appear have to be equal or exceed the interface tensile strength $\sigma_c$,

$$\sigma(\theta) \geq \sigma_c \quad \forall \theta \in [0, \Delta \theta].$$

In view of this condition, it is necessary to know $\sigma(\theta)$ at the interface before the debond onset. Although [?] deduced an analytical solution of this problem, stresses are obtained here by the BEM code used in the next section to solve the problem with a debond, for which no analytical solution is available. This BEM code was developed by [?] and is used here since it allows us to study, with a high accuracy, axisymmetric problems of interface cracks with a possible contact zone at the crack tip, features necessary in next section.

The BEM model with linear elements used here is shown in Fig. 1. Note that a half of the geometry is modeled due to the symmetry. The matrix is represented by a square with side 133 times larger than the inclusion radius $a$. The boundary element mesh is uniform at the interface and the outer matrix boundary. At the lower horizontal edge, where the symmetry conditions are applied, the mesh is not uniform, the element length decreasing when approaching the interface following a geometric series with ratio $1.2$. The extreme length of this edge matches the element length at the interface at one extreme and the element length at the outer matrix boundary at the other extreme. A uniform remote tension $\sigma_\infty$ is applied at the upper horizontal edge. Finally, the whole interface is defined as perfectly adhered.

Fig. 1 shows the normal, $\sigma$, and tangential, $\tau$, tractions at the interface computed by the BEM model. As $\sigma(\theta)$ has a maximum at $\theta = 0^\circ$ and decreases for increasing $\theta \in [0^\circ, 90^\circ]$, the condition (??) is fulfilled for the whole interval.
perfect bonding at the interface

axisymmetry axis

Fig. 2. A BEM axisymmetric model used to compute the interface tractions at the perfectly bonded inclusion-matrix interface.

Fig. 3. (a) Normal and tangential tractions, $\sigma$ and $\tau$, at the inclusion-matrix interface. Points represent the values computed by the BEM model and the solid line is the interpolation used in the subsequent calculations. (b) Stress criterion: Minimum remote tension necessary for the debond onset as a function of the debond angle $\Delta \theta$ for alumina/titanium.

$[0, \Delta \theta]$ if it is for the angle of debond at onset $\Delta \theta$, giving the final expression of the stress criterion

$$\frac{\sigma^\infty}{\sigma_c} \geq s(\Delta \theta) = \frac{1}{\sigma(\Delta \theta)/\sigma^\infty}. \quad (2)$$

Fig. ?? plots the function $s(\Delta \theta)$ which defines the minimum required $\sigma^\infty$ for the onset as a function of $\Delta \theta$. The required $\sigma^\infty$ increases with $\Delta \theta$ according to this criterion.

3. Energy criterion

The energy criterion imposes that the energetic balance between the states prior and after the onset must fulfill the first law of Thermodynamics,

$$\Delta \Pi(\sigma^\infty, \Delta \theta) + \Delta E_k(\sigma^\infty, \Delta \theta) + \Delta \Gamma(\sigma^\infty, \Delta \theta) = 0, \quad (3)$$

where $\Delta \Pi$ and $\Delta E_k$ are the change in potential elastic and kinetic energy, respectively. $\Delta \Gamma$ is the dissipated energy during the onset. $\Delta \Pi$ can be obtained by integrating the energy release rate (ERR) $G$ for an “instantaneous debond” $\theta_d$ ranging from $0^\circ$ to $\Delta \theta$. $\Delta E_k \geq 0$ since the initial state is assumed to be static. Analogously to $\Delta \Pi$, the value of $\Delta \Gamma$ is approximated by integrating the interface fracture toughness $G_c$ in $[0, \Delta \theta]$, $G_c$ being assumed to be possibly dependent on the angle $\psi$ of interface tractions prior to the onset (obtained in Section ??), where $\tan \psi = \tau/\sigma$, according to a justification by [? ?]. We write $G_c(\psi) = G_{1c}G_c(\psi)$, where $G_{1c}$ is the fracture toughness associated to the opening mode I ($\psi = 0^\circ$), and $G_c$ is a dimensionless function measuring the influence of $|\psi|$. Thus, (??) is rewritten as

$$\int_0^{\Delta \theta} G(\theta_d)r(\theta_d)d\theta_d \geq G_{1c} \int_0^{\Delta \theta} \tilde{G}_c(\psi(\theta))r(\theta)d\theta, \quad (4)$$
where \( r(\theta) \) is the distance to the axisymmetry axis. In the following, \( \hat{G}_c \) is approximated by a phenomenological law due to [? ], \( \hat{G}_c(\psi) = 1 + \tan^2(1 - \lambda)\psi, \lambda \in [0, 1] \).

To impose the criterion in (??) it is necessary to compute the ERR \( G(\theta_d) \). The amount of computations can be reduced by a dimensional analysis of \( G \), defining a dimensionless ERR \( \hat{G} \) by

\[
G(\sigma^\infty, a, \theta_d, E_1, E_2, \nu_1, \nu_2) = \left(\frac{\sigma^\infty}{\sigma_c}\right)^2 \frac{a}{E_2} \hat{G}(\theta_d/E_2, \sigma_c/E_1, \nu_1, \nu_2). \tag{5}
\]

Values of \( \hat{G} \) are obtained by BEM models as in the example shown in Fig. ???. This model is similar to that used in Section ?? but with the full geometry simulated because of the asymmetric debond configuration. In addition, since a strong stress gradient is expected near the crack tip, the mesh is refined near it. Thus, the angle of a boundary element at the interface goes from \( 0.0001^\circ \), at the crack tip, to \( 2^\circ \), far from it. The interface part corresponding to the crack lips is considered as possible frictionless contact zone and the rest as perfectly bonded. A BEM model is generated for every value of \( \theta_d = 1^\circ, 2^\circ, \ldots, 88^\circ, 89^\circ \). Finally, VCCT is applied to the results obtained taking the virtual angle of crack closure \( \delta\theta_d = 0.5^\circ \). Fig. ?? shows the results for \( \hat{G} \) obtained.

Fig. 4. (a) \( \hat{G} \) values extracted from the BEM results for alumina/titanium. (b) Example of a BEM axisymmetric model used to compute the ERR of a crack at the inclusion-matrix interface.

Introducing (??) into (??), after some algebraic manipulation and normalizing with \( \sigma_c \), the condition for the energy criterion is written as,

\[
\frac{\sigma^\infty}{\sigma_c} \geq \gamma \sqrt{g(\Delta\theta)} \geq \frac{1}{\sigma_c} \sqrt{G_{1c} E_2 a} \sqrt{\frac{1}{\gamma} \int_0^{\Delta\theta} \hat{G}(\theta_d) \sin(\theta_d) d\theta_d} \left(\int_0^{\Delta\theta} \hat{G}_c(\theta) \sin(\theta) d\theta\right) \sqrt{g(\Delta\theta)}, \tag{6}
\]

where \( \gamma \) is a dimensionless brittleness number and \( g(\Delta\theta) \) is a dimensionless function representing the ratio of the dissipated to released energy, cf. [? ]. This condition is plotted in Fig. ???. Note that, on the contrary to the stress criterion, the function governing the energy criterion decreases up to a minimum value at \( \Delta\theta = \Delta\theta_{\text{min}}^E \), thus, the required value of \( \sigma^\infty \), allowing an onset from the energy point of view, decreases with increasing value of \( \Delta\theta (\leq \Delta\theta_{\text{min}}^E) \).

4. Coupled criterion

According to a hypothesis by [? ], the critical remote tension for the onset \( \sigma_c^\infty \) is given by the minimum value of \( \sigma^\infty \) for which both criteria (??) and (???) are simultaneously fulfilled, i.e.

\[
\frac{\sigma_c^\infty}{\sigma_c} = \min_{\Delta\theta} \left(\max\left(s(\Delta\theta), \gamma \sqrt{g(\Delta\theta)}\right)\right). \tag{7}
\]
Fig. 5. Energy criterion: Minimum remote tension necessary for the debond onset as a function of the debond angle $\Delta \theta$ for alumina/titanium and $\lambda = 0.3$.

The form in which both criteria are combined depends strongly on the value of $\gamma$, see Fig. ??, two scenarios being possible,

- **Scenario A**: Curves representing both criteria intersect for $\Delta \theta \leq \Delta \theta_{\text{min}}$. In this case the critical values for debond $\Delta \theta_c$ and $\sigma^\infty_c$ are given by the intersection point.
- **Scenario B**: Curves do not intersect for $\Delta \theta \leq \Delta \theta_{\text{min}}$. Then, $\Delta \theta_c$ and $\sigma^\infty_c$ are given by the minimum of the energy criterion.

Both scenarios are separated by a threshold value of $\gamma_{\text{th}}$ which corresponds to the value of $\gamma$ for which both curves intersect at $\Delta \theta = \Delta \theta_{\text{min}}$.

Fig. 6. Examples of how both criteria combine for two values of $\gamma = 0.5$ and 2 corresponding, respectively, to scenarios A and B, for alumina/titanium and $\lambda = 0.3$.

5. Results and discussion

A wide variety of parametric studies can be carried out by using the present model, but, for the sake of brevity, only the main results will be highlighted here.

The critical remote tension $\sigma^\infty_c$ as a function of $\gamma$ is plotted in Fig. ?? This figure shows that $\sigma^\infty_c$ increases with $\gamma$ going from a brittle configuration for low $\gamma$, where $\sigma^\infty_c$ is almost independent of $\gamma$, to a tough situation for large $\gamma$, where it is a linear function of $\gamma$.

Since $\gamma$ depends on the sphere radius $a$, $\sigma^\infty_c$ can be expressed as a function of $a$, see Fig. ?? This figure shows a strong size effect, predicting that the particle-matrix system strength increases drastically for small spheres. Note that analogously to $\gamma_{\text{th}}$ a threshold value is defined for the radius as $a_{\text{th}}$. 
Critical angle of debond $\Delta \theta_c$ is plotted in Fig. ?? along with the arrest angle $\theta_a$ computed using the LEFM immediately after the onset for a fixed value of the remote tension $\sigma_c^\infty$. Note that whereas tough configurations correspond to a fixed $\Delta \theta_c$ with no unstable growth, in the brittle limit a large unstable growth is predicted after the debond onset.

6. Concluding remarks

A theoretical model, based on the coupled criterion of FFM, has been developed to predict the crack onset at the spherical-inclusion matrix interface. The value of the debond angle after the onset and the critical remote tension originating an onset are obtained as a function of the main problem parameters.

The model predicts a strong size effect, which has a strong influence on the critical remote tension for the debond onset. In addition, the sphere size modifies the fracture behavior, going from brittle to tough for large and small spheres respectively.

The present work is the first step in the study of the failure mechanism associated to the particle debonding in a composite. Subsequent steps are debond propagation, kinking out of the interface and coalescence between matrix cracks.

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