de Sitter space and extremal surfaces for spheres

K. Narayan

Chennai Mathematical Institute,
SIPCOT IT Park, Siruseri 603103, India.

Abstract
Following arXiv:1501.03019 [hep-th], we study de Sitter space and spherical subregions on a constant boundary Euclidean time slice of the future boundary in the Poincare slicing. We show that as in that case, complex extremal surfaces exist here as well: for even boundary dimensions, we isolate the universal coefficient of the logarithmically divergent term in the area of these surfaces. There are parallels with analytic continuation of the Ryu-Takayanagi expressions for holographic entanglement entropy in AdS/CFT. We then study the free energy of the dual Euclidean CFT on a sphere holographically using the dS/CFT dictionary with a dual de Sitter space in global coordinates, and a classical approximation for the wavefunction of the universe. For even dimensions, we again isolate the coefficient of the logarithmically divergent term which is expected to be related to the conformal anomaly. We find agreement including numerical factors between these coefficients.
1 Introduction

Generalizations of gauge/gravity duality [1 2 3 4] to de Sitter space, or $dS/CFT$ [5 6 7], involve a hypothetical dual Euclidean CFT on the future timelike infinity $\mathcal{I}^+$ boundary. The late time wavefunction of the universe with appropriate boundary conditions is equated with the partition function of the dual CFT. Further work on $dS/CFT$ including higher spin realizations appears in e.g. [8 9 10 11 12 13 14 15].

Ideas pertaining to entanglement entropy have been of great interest in recent times. In $AdS/CFT$, the Ryu-Takayanagi prescription [16 17] (see [18 19] for reviews) maps entanglement entropy of a field theory subsystem to the area (in Planck units) of a bulk minimal surface (more generally extremal surface [20]) anchored at the subsystem interface and dipping into the bulk, in the gravity approximation. Similar ideas were explored in [21] in de Sitter space with a view to exploring entanglement entropy in the dual CFT with $dS/CFT$ in mind. For strip-shaped subregions on a constant Euclidean time slice of the future boundary, it was found that the area of certain complex extremal surfaces has structural resemblance with entanglement entropy of dual Euclidean CFTs (reviewed in sec. 2). The coefficients of the leading divergent “area law” terms in $dS_{d+1}$ resemble the central charges $\mathcal{C}_d \sim i^{1-d} \frac{R_{d+1}}{G_{d+1}}$ of the CFT$_d$s appearing in the $\langle TT \rangle$ correlators in [7]. The areas of these surfaces obtained thus essentially amount to analytic continuation from the Ryu-Takayanagi expressions for holographic entanglement entropy in $AdS/CFT$. Note that the areas are in general not real-valued or positive definite and are distinct from the entanglement entropy of bulk fields in de Sitter space e.g. [22].

Towards exploring this further, we study spherical subregions in this paper. As in [21], similar complex extremal surfaces can be shown to exist in this case too (sec. 3). The area of these surfaces exhibits a similar leading area law divergence as well as subleading terms: for $dS_{d+1}$ with even $d$, this includes a logarithmically divergent term whose coefficient is analogous to the “universal” terms in $AdS/CFT$ related to the conformal anomaly [17]. In the present case also, we expect this anomaly to arise in the free energy of the CFT on a curved space. With this in mind, we then calculate in the present case (sec. 4), the free energy of the Euclidean CFT on a sphere holographically using the $dS/CFT$ dictionary $Z_{CFT} = \Psi$ [7] with an auxiliary de Sitter space in global coordinates (whose constant time slices are spheres). In the classical regime, we approximate the wavefunction of the universe $\Psi$ in terms of the bulk action $S$ of this auxiliary de Sitter space: this gives $-F = \log Z_{CFT} = \log \Psi \sim iS$. We find precise agreement between the coefficients of the logarithmic terms in the complex extremal surfaces and those in the free energy via the wavefunction of the universe, including numerical factors. Since the coefficient of the logarithmic term in the free energy is related to the trace anomaly for any CFT, this supports the idea that the area of these complex extremal surfaces encodes entanglement entropy of the dual Euclidean CFT in $dS/CFT$, as we discuss in sec. 5.
2 Reviewing de Sitter extremal surfaces: strips

Here we review the study \cite{21} of bulk de Sitter extremal surfaces anchored over strip-shaped subregions on the future boundary $I^+$ in the Poincare slicing. de Sitter space $dS_{d+1}$ in the Poincare slicing or planar coordinate foliation is given by the metric

$$ds^2 = \frac{R^2_{dS}}{\tau^2} (-d\tau^2 + dw^2 + d\sigma_{d-1}^2), \quad (1)$$

where half of the spacetime, e.g. the upper patch, has $I^+$ at $\tau = 0$ and a coordinate horizon at $\tau = -\infty$. This may be obtained by analytic continuation of a Poincare slicing of $AdS$,

$$z \rightarrow -i\tau, \quad R_{AdS} \rightarrow -iR_{dS}, \quad t \rightarrow -iw, \quad (2)$$

where $w$ is akin to boundary Euclidean time, continued from time in $AdS$ (with $z$ the bulk coordinate). The dual Euclidean CFT is taken as living on the future $\tau = 0$ boundary $I^+$. We assume translation invariance with respect to the boundary Euclidean time direction $w$, and consider a subregion on a $w = const$ slice of $I^+$. One might imagine that tracing out the complement of this subregion then gives entropy in some sense stemming from the information lost. In the bulk, we study de Sitter extremal surfaces on the $w = const$ slice, analogous to the Ryu-Takayanagi prescription in $AdS/CFT$. Operationally these extremal surfaces begin at the interface of the subsystem (or subregion) and dip into the bulk time direction.

For a strip-shaped subregion on $I^+$ (with width say along $x$), parametrizing the spatial part in (1) as $d\sigma_{d-1}^2 = \sum_{i=1}^{d-1} dx_i^2$, the $dS_{d+1}$ area functional on a $w = const$ slice is

$$S_{dS} = \frac{R^{d-1}_{dS}V_{d-2}}{4G_{d+1}} \int \frac{d\tau}{\tau^{d-1}} \sqrt{\left(\frac{dx}{d\tau}\right)^2 - 1}, \quad \dot{x}^2 = \frac{-A^2\tau^{2d-2}}{1 - A^2\tau^{2d-2}}, \quad (3)$$

with $\frac{dx}{d\tau} \equiv \dot{x}$, and the constant $A^2$ is the conserved quantity obtained in the extremization. For $dS_4$, taking $A^2 > 0$ makes this a complex surface (complex geodesics and surfaces have also appeared in e.g. \cite{23, 24}). Requiring that the width $\Delta x = l$ be real-valued suggests that $\tau$ takes imaginary values, parametrized as $\tau = iT$ with $T$ real. Also, $x(\tau) \sim \pm iAT^3 + x(0)$ as $\tau \sim 0$, so that $x(\tau)$ representing a CFT spatial direction is real-valued only if $\tau$ is pure imaginary. There is now a turning point $\tau_* = \frac{i}{A}$ which is the “deepest” location this (smooth) complex surface dips upto in the bulk (with $|\dot{x}| \rightarrow \infty$). This surface $x(\tau)$ does not correspond to any real bulk

\footnotetext{Taking $A^2 < 0$ gives $\dot{x}^2 = \frac{|A|^2\tau^4}{1 + |A|^2\tau^2}$ which for real $\tau$ shows the absence of a turning point where $\dot{x} \rightarrow \infty$. The area decreases as $A^2$ increases: as $A^2 \rightarrow \infty$, these real surfaces become null with $\dot{x}^2 \rightarrow 1$. They are analogous to restrictions (to a boundary Euclidean time slice) of past lightcone wedges of the boundary subregion, with vanishing area, and no bearing on entanglement. Taking $A^2 = 0$ gives disconnected surfaces $x(\tau) = const$, again with no turning point.}
$dS_4$ subregion. For even $d$, similar complex surface saddle points of the area functional arise taking $A^2 < 0$ and similar paths $\tau = i\tau$. The area of these extremal surfaces gives

$$S_{dS} = -i\frac{R_{dS}^{d-1}}{4G_{d+1}} V_{d-2} \int_{\tau_{UV}}^{\tau} \frac{d\tau}{\tau^{d-1}} \frac{1}{\sqrt{1 - (-1)^{d-1} A^2 \tau^{2d-2}}} = i^{1-d} \frac{R_{dS}^{d-1}}{2G_{d+1}} V_{d-2} \left( \frac{1}{c^{d-2}} - c_d \frac{1}{l^{d-2}} \right),$$

(4)

where $\tau_{UV} = i\epsilon$ and $\tau_* = il$, and the integral is as in AdS (with corresponding constant $c_d$). Note that here we have used the relation $\tau_{UV} = i\epsilon$ for the ultraviolet cutoff in the dual Euclidean field theory suggested by previous investigations in $dS/CFT$ (see e.g. \[7, 8, 9, 13\]) with time evolution mapping to renormalization group flow.

$S_{dS}$ in (4) bears structural resemblance to entanglement entropy in a dual CFT with central charge $C_d \sim i^{1-d} \frac{R_{dS}^{d-1}}{G_{d+1}}$. The first term $S_{dS}^{UV} \sim i^{1-d} \frac{R_{dS}^{d-1}}{G_{d+1}} V_{d-2}$ resembles an area law divergence \[25, 26\], proportional to the area of the interface between the subregion and the environment, in units of the ultraviolet cutoff. It appears independent of the shape of the subregion, expanding (3) and assuming that $\dot{x}$ is small near the boundary $\tau_{UV}$. Written as $C_d \frac{V_{d-2}}{V_{d-3}}$, we see that it is also proportional to the central charge $C_d \sim i^{1-d} \frac{R_{dS}^{d-1}}{G_{d+1}}$ representing the number of degrees of freedom in the dual (non-unitary) CFT: these arose in the $\langle TT \rangle$ correlators obtained in \[7\]. In $dS_4$, the central charge $C \sim -\frac{R_{dS}}{G_4}$ is real and negative, while in $dS_3, dS_5$, it is imaginary.

The second term is a finite cutoff-independent piece. Unlike $dS_4$, note that $S_{dS}$ in $dS_{d+1}$ with even $d$ is not real-valued: e.g. in $dS_3$, we obtain $S_{dS} \sim -i\frac{R_{dS}}{G_3} \log \frac{1}{\epsilon}$ while in $dS_5$, we have $S_{dS} \sim i\frac{R_{dS}}{G_5} V_2 \left( \frac{1}{\epsilon^2} - c_1 \frac{1}{\tau^2} \right)$. It is interesting to note that a replica calculation of entanglement entropy in a free 3d $Sp(N)$ theory for the half-plane \[27\] gives behaviour similar to the leading area law divergence here (although the $Sp(N)$ theory is dual to the higher spin $dS_4$ theory \[9\] and it is unclear if geometric objects such as extremal surfaces are of relevance).

While there is structural resemblance with entanglement entropy, there are questions. Since these are bulk complex extremal surfaces, changing the sign in the square root branch in (3) introduces an overall $\pm i$ factor. Fixing this in (4) as $-i$ makes the leading divergence to be of the form of the area law $C_d \frac{V_{d-2}}{V_{d-3}}$ with $C_d$ the central charges in \[7\]. This is also corroborated by analytic continuation from the Ryu-Takayanagi expressions for holographic entanglement entropy in AdS/CFT. While this is suggestive, it would be useful to explore this further with a view to associating these complex extremal surfaces and corresponding area with entanglement entropy in $dS/CFT$.

With a view to understanding this better, we now study spherical subregions and the corresponding complex extremal surfaces. For $dS_{d+1}$ with even $d$, there is a term in the area with logarithmic dependence on the cutoff $\epsilon$ whose coefficient can be compared with that obtained from the conformal anomaly appearing in the free energy of the CFT on a sphere holographically using $dS/CFT$. We obtain agreement between both sides: this vindicates the signs we have used above in defining these complex surfaces, and analytic continuation.
3 Spherical extremal surfaces in de Sitter space

Building on \[21\] for strip-shaped subregions, here we consider spherical subregions on the boundary \(\mathcal{I}^+\), with radius \(l\) parametrized as \(0 \leq r \leq l\). Since we are interested in spherical entangling surfaces, we will parametrize \(d\sigma_{d-1}^2\) in \([11]\) in polar coordinates. Then the \(w = \text{const}\) surface (i.e. a constant boundary Euclidean time surface) is a bulk \(d\)-dim subspace with metric

\[
d s^2 = \frac{R_{ds}^2}{\tau^2} \left( -d \tau^2 + dr^2 + r^2 d\Omega_{d-2}^2 \right).
\]

(5)

The bulk surface on the \(w = \text{const}\) slice bounding this subregion and dipping into the \(\tau\)-direction is bulk codim-2: let us parametrize this as \(r = r(\tau)\). Its area functional in Planck units is

\[
S_{ds} = \frac{1}{4G_{d+1}} \int \prod_{i=1}^{d-2} \frac{R_{ds}d\Omega_i}{\tau} \frac{R_{ds}}{\tau} \sqrt{dr^2 - d\tau^2} = \frac{R_{ds}^{-1} \Omega_{d-2}}{4G_{d+1}} \int \frac{d\tau}{\tau^{d-2}} \sqrt{\left( \frac{dr}{d\tau} \right)^2 - 1}.
\]

(6)

The variational equation of motion for an extremum \(\frac{\partial}{\partial \tau} \left( \frac{\partial \mathcal{L}}{\partial r} \right) = \frac{\partial \mathcal{L}}{\partial r}\) gives

\[
\frac{\partial}{\partial \tau} \left( \frac{\tau^{d-2}}{\sqrt{\tau^2 - 1}} \dot{r} \right) = \frac{d-2}{\tau^{d-1}} r^{d-3} \sqrt{\tau^2 - 1},
\]

(7)

where \(\frac{dr}{d\tau} \equiv \dot{r}\). It can be seen that

\[
r(\tau) = \sqrt{l^2 + \tau^2}
\]

is an extremal surface that solves \([11]\), thus extremizing \(S_{ds}\), using

\[
\dot{r} = \frac{\tau}{\sqrt{l^2 + \tau^2}}, \quad \dot{r}^2 - 1 = \frac{-l^2}{l^2 + \tau^2}.
\]

(9)

Unlike the strip case, there are no parameters for the surface in this case.

We see from \([8]\), \([9]\), that for \(\tau\) real, there is no bulk turning point where \(\frac{dr}{d\tau} \to \infty\) with the surface turning around smoothly: instead the surface asymptotically approaches \(r^2 \to \tau^2\). Furthermore this surface has \(r(\tau) \geq l\) whereas all interior points within the subregion satisfy \(0 \leq r \leq l\) with \(r \to l\) near \(\tau \to 0\), so that this surface bends “outwards” from the subregion boundary. This is a real timelike surface\(^2\) with \(\dot{r} \leq 1\) and is unsatisfactory for our present purposes.

\(^2\) It is then more natural to consider, rather than \([9]\), the area as \(S_{ds} = \frac{R_{ds}^{-1} \Omega_{d-2}}{4G_{d+1}} \int \frac{d\tau}{\tau^{d-2}} \sqrt{1 - \tau^2}\) which is then real-valued. Since the surface does not “end” at any finite \(\tau\), we consider the whole \(\tau\)-range and obtain \(S_{ds} = \frac{R_{ds}^{-1} \Omega_{d-2}}{4G_{d+1}} \int_{\tau_{UV}}^{\infty} \frac{d\tau}{\tau^{d-2}} \sqrt{1 - \tau^2}\). Taking \(\tau_{UV} = \epsilon\), this can be evaluated giving \(S_{ds} = \frac{R_{ds}^{-1}}{2G_{d+1}} \log \frac{1}{\epsilon}\) [for \(dS_3\)], \(S_{ds} = \frac{R_{ds}^{-1} A_1}{4G_{d+1} \epsilon\tau}\) [for \(dS_4\), with \(A_1 = 2\pi l\) the interface area of the spherical subregion], and \(S_{ds} = \frac{R_{ds}^{-1} A_2}{2G_{d+1} \epsilon\tau} + \frac{R_{ds}^{-1}}{2G_{d+1}} \log \frac{1}{\epsilon}\) [for \(dS_5\), with \(A_2 = 4\pi l^2\) the sphere interface area]. Note that there are no cutoff-independent pieces here since those contributions die at \(\tau \to \infty\).
This suggests that we consider imaginary $\tau$ parametrized as $\tau = iT$ with $T$ real, as for the strip case. Thus (8) becomes

$$r^2 = t^2 - T^2 \implies r_{\text{min}} = 0 \text{ at the turning point } \tau_\ast = it \implies \Delta r = t . \quad (10)$$

Now $r(\tau)$ maps each point on the surface directly to a corresponding real-valued spatial location within the subregion in the dual CFT. We require that the boundary subregion radial parameter $r$ be real-valued in (8) since this represents a spatial direction in the CFT: this excludes more general paths in complex $\tau$-space. The range of $\tau$ is now restricted, and the subregion size given by $\Delta r \equiv r_{\text{max}} - r_{\text{min}}$ is bounded. (Perhaps more general complex paths and surfaces exist if both $r, \tau$ are complexified.)

Thus using (8), (9), (10), the area (6) in $dS_{d+1}$ becomes

$$S_{dS} = \frac{R_{dS}^{d-1} \Omega_{d-2}}{4G_{d+1}} \int_{\tau_{UV}}^{\tau_\ast} \frac{d\tau}{\tau^{d-1}} (it^2 + \frac{\tau^2}{2})^{(d-3)/2} . \quad (11)$$

The integration is along the path $\tau = iT$, with $\tau_{UV} = ie$ and $\tau_\ast = it$. The leading divergence here is of the form of the area law (for $d > 2$), given by

$$S_{dS}^{\text{div}} = \frac{i}{d-2} \frac{R_{dS}^{d-1} \Omega_{d-2}}{4G_{d+1}} \frac{l^{d-2}}{\tau_{UV}^{d-2}} = \frac{i^{1-d}}{d-2} \frac{R_{dS}^{d-1} A_{d-2}}{4G_{d+1}} \frac{A_{d-2}}{\tau_{UV}^{d-2}} , \quad (12)$$

with $A_{d-2} \equiv l^{d-2} \Omega_{d-2}$ the interface area. There is an overal $\pm$ sign ambiguity in the choice of the square root branch in (6), (11), which we have fixed to be $+$ in (12), as in the strip subregions reviewed earlier. This sign corresponds to choosing $\sqrt{-t^2} = -it$ in (11). As for the strip [21], the leading divergence here has the form $C_d \frac{A_{d-2}}{\tau_{d-2}}$ with $C_d \sim in^{-d} \frac{R_{dS}^{d-1}}{G_{d+1}}$ of the form appearing in the $\langle TT \rangle$ correlators in [7]. We also see that analytic continuation using (2) from the leading area law divergence from the Ryu-Takayanagi expression in $AdS/CFT$ gives

$$\frac{R_{dS}^{d-1} \Omega_{d-2}}{4G_{d+1}} \frac{A_{d-2}}{\tau_{d-2}} \rightarrow \frac{i^{1-d} R_{dS}^{d-1} l^{d-2} \Omega_{d-2}}{d-2 4G_{d+1}} \frac{A_{d-2}}{\tau_{UV}^{d-2}}$$

which is the sign above.

There are subleading terms: e.g. for $dS_4$, the area (11) gives

$$S_{dS} = \frac{R_{dS}^2 \Omega_4}{4G_4} (it) \int_{\tau_{UV}}^{\tau_\ast} \frac{d\tau}{\tau^2} = \frac{-\pi R_{dS}^2}{2G_4} \left( t \frac{l}{\epsilon} - 1 \right) . \quad (13)$$

The finite constant cutoff-independent piece $\frac{\pi R_{dS}^2}{2G_4}$ is a universal term. For $d$ even, one of the subleading terms is the universal logarithmic term. Expanding (11), this logarithmic term can be seen to be

$$-i \left( \frac{d-3}{d-2} \right) 4 \frac{\Omega_{d-2}}{G_{d+1}} \frac{R_{dS}^{d-1}}{G_{d+1}} \log \frac{t}{\epsilon} , \quad (14)$$

where $\binom{t}{k}$ is the (generalized) binomial coefficient of the $x^k$-term in the expansion of $(1+x)^r$ and $\Omega_d = \frac{2\pi^{(d+1)/2}}{\Gamma((d+1)/2)}$ is the $d$-dim sphere volume. The argument in the logarithmic term is obtained
as \( \frac{\tau}{\tau_U} = \frac{u}{u} \). Explicitly, the coefficients (14) for \( dS_3, dS_5, dS_7 \) are

\[
- \frac{i R_{dS}}{2G_3} \left[ dS_3 \right], \quad - \frac{i \pi R_{dS}^4}{2G_5} \left[ dS_5 \right], \quad - \frac{i \pi^2 R_{dS}^5}{4G_7} \left[ dS_7 \right].
\]  

(15)

For \( dS_3 \), the area contains only the logarithmic term and the coefficient can be calculated directly from (6), (11). These resemble those arising in the \( \langle TT \rangle \) correlators in [7], except that the numerical factors are unambiguously fixed here.

Note that \(-i \frac{R_{dS}}{G_{d+1}} \) under the analytic continuation (2) becomes \((-1)^\frac{d-1}{2} \frac{R_{dS}}{G_{d+1}} \) which we recall arises in the coefficient of the logarithmic term in the \( AdS \) case for spherical surfaces [17] (and the numerical factors also corroborate). This coefficient is proportional to the \( a \) central charge appearing in the trace anomaly of the CFT on a sphere (for even \( d \)).

This suggests that in \( dS/CFT \), the coefficients of the logarithmic term for these complex extremal surfaces are the analogs of these \( a \)-type central charges in the Einstein gravity approximation. These coefficients match with those in the logarithmically divergent terms in the CFT free energy evaluated using \( dS/CFT \) as we discuss now.

4 \( \Psi \sim e^{iS}, \) CFT on sphere and conformal anomaly

For what follows, it is useful to recall the \( dS/CFT \) correspondence for de Sitter space. A version of \( dS/CFT \) [3, 6, 7] states that quantum gravity in de Sitter space is dual to a Euclidean CFT living on the boundary \( I^+ \). More specifically, the CFT partition function with specified sources \( \phi_{i0}(\vec{x}) \) coupled to operators \( O_i \) is identified with the bulk wavefunction of the universe as a functional of the boundary values of the fields dual to \( O_i \) given by \( \phi_{i0}(\vec{x}) \). In the classical regime this becomes

\[
Z_{CFT} = \Psi[\phi_{i0}(\vec{x})] \sim e^{iS_{cl}[\phi_{i0}]} \]

where we need to impose regularity conditions on the past cosmological horizon \( \tau \to -\infty \). e.g. scalar modes satisfy \( \phi_k(\tau) \sim e^{ik\tau} \), which are Hartle-Hawking (or Bunch-Davies) initial conditions. Operationally, certain \( dS/CFT \) observables can be obtained by analytic continuation (2) from \( AdS \) (see e.g. [7], as well as [8]).

For even dimensions \( d \), the free energy of the CFT, \( \Psi \) on a sphere is expected to contain a logarithmic divergence whose coefficient is related to the integrated conformal anomaly of the CFT. Since the (nonunitary) CFT here is that dual to de Sitter space, this can be calculated holographically using the \( dS/CFT \) dictionary \( Z_{CFT} = \Psi [7] \) with an auxiliary de Sitter space in global coordinates whose constant time slices are spheres. In the classical regime, we approximate the Hartle-Hawking wavefunction of the universe \( \Psi \) in terms of the bulk action \( S \) of this auxiliary de Sitter space: this gives

\(-F = \log Z_{CFT} = \log \Psi \sim iS \). We can then calculate the coefficient of the logarithmic term in the classical approximation.
The spatial slices are $d$-spheres, with minimum radius $R_{dS}$ at $t = 0$. This is a solution to Einstein gravity $R_{MN} = \frac{d}{R_{dS}} g_{MN}$ with cosmological constant $\Lambda = \frac{d(d-1)}{2R_{dS}^2}$. The on-shell bulk action is

$$S = \frac{1}{16\pi G_{d+1}} \int d^{d+1}x \sqrt{-g} (R - 2\Lambda) = \frac{1}{16\pi G_{d+1}} \int dt d^d\Omega_d R_{dS}^d \left( \cosh \frac{t}{R_{dS}} \right)^d \frac{2d}{R_{dS}^2},$$

(17)

where $R - 2\Lambda = \frac{d(d-1)-d(d-1)}{R_{dS}}$ and $\Omega_d$ the $d$-dim sphere volume. We have suppressed writing the surface terms and counterterms for cancelling the leading divergences in this action since the logarithmic term we are interested in arises solely from the bulk action: this is motivated by similar arguments in AdS/CFT (see e.g. [28][29][30][31] and the review [34]). This gives

$$S = \frac{2d}{16\pi G_{d+1}} R_{dS}^{d-1} \int \frac{dt}{R_{dS}} \left( \cosh \frac{t}{R_{dS}} \right)^d \frac{R_{dS}^{d-1}}{16\pi G_{d+1}} \frac{2d}{2d} \int d\left( \frac{t}{R_{dS}} \right) e^{dt/R_{dS}} \left( 1 + e^{-2t/R_{dS}} \right)^d.$$

With $\tau = 2R_{dS} e^{-t/R_{dS}}$, the metric (16) at asymptotically late times becomes of Poincare form $ds^2 \sim \frac{R_{dS}^2}{\tau^2} (-dt^2 + R_{dS}^2 d\Omega_d^2)$. The dual CFT ultraviolet cutoff is defined as $\tau_{UV} = i\epsilon$ so it is useful to write the bulk action by redefining $y = e^{t/R_{dS}} = \frac{2R_{dS}}{\tau}$, we have

$$S = \frac{R_{dS}^{d-1}}{16\pi G_{d+1}} \frac{2d}{2d} \int y^{UV} dy y^{d-1} \left( 1 + \frac{1}{y^2} \right)^d.$$

(19)

The upper limit of integration here is at large $t$ i.e. the future cutoff $\tau_{UV}$. The lower limit will not be important in what follows as long as some regularity conditions are satisfied: it could be taken as the $t = 0$ slice where the sphere $S^d$ shrinks to minimal size (and the spacetime could be taken to be glued onto a Euclidean half-sphere as in the Hartle-Hawking prescription [32]).

The expansion of this action has a logarithmic term for $d$ even. Now the wavefunction of the universe in the classical approximation is $\Psi = e^{iS}$ and the free energy is $-F \equiv \log Z = \log \Psi \equiv iS$ [7]. Thus the logarithmic term in the free energy can be found by expanding the action (19), which arises as

$$iS = \ldots + i \left( \frac{d}{2} \right) 2d \frac{\Omega_d R_{dS}^{d-1}}{16\pi 2d G_{d+1}} \log \frac{R_{dS}}{\epsilon} + \ldots = -i \left( \frac{d}{2} \right) 2d \frac{\Omega_d R_{dS}^{d-1}}{16\pi 2d G_{d+1}} \log \epsilon + \ldots ,$$

(20)

where the bulk cutoff is related to the CFT cutoff as $y_{UV} = \frac{2R_{dS}}{\tau_{UV}} = \frac{2R_{dS}}{\epsilon}$, and $\binom{d}{k}$ is the binomial coefficient. In Euclidean $AdS_{d+1}$ with metric $ds^2 = dz^2 + R^2 \sinh^2 \frac{z}{R} d\Omega_d^2$ (the expected gravity dual for a conventional unitary Euclidean CFT), a similar calculation yields the anomaly coefficient as is well known [28][29][30][31]: the bulk Euclidean $AdS$ action ($Z \sim e^{-S_{EAdS}}$)
continues as \(-S^{E AdS} = \frac{1}{16\pi G_{d+1}} \int d^d x \sqrt{g} (R + 2|\Lambda|) \to iS^{dS}\) and the \(-\frac{R_{d-1}^{d-1}}{G_{d+1}}\) factor is seen to continue to \((-1)^{\frac{d-1}{2}} \frac{R_{d-1}^{d-1}}{G_{d+1}}\) in \(E AdS\).

The coefficient of this logarithmic term in \([20]\) in the free energy via \(\Psi\) can be seen to be the same as that in the logarithmic term \([14]\) in the complex spherical extremal surfaces. They appear to be the analogs of the \(a\)-type central charges in \(dS/CFT\).

Further light is shed on this calculation in light of \([33]\). A conformal mapping was used there to transform the entanglement entropy of a spherical subsystem in \(AdS/CFT\) to the thermal entropy of the CFT in the static patch of an auxiliary de Sitter space. For \(AdS\), this allows a precise comparison with the coefficient of the logarithmic term appearing in the extremal surface area. These coefficients are related to the conformal anomaly and the central charge \(a\) (which is also \(c\)) \([17]\) in the Einstein gravity approximation (see also \([31, 33]\) for higher derivative theories).

It would appear that a similar argument is at play here modulo some caveats (below). The CFT in this case, intrinsically Euclidean, lives on the flat Euclidean space on the future boundary \(I^+\) of de Sitter space and is nonunitary. We use a conformal mapping to transform this flat \(d\)-dim Euclidean space \(ds_E^2 = dt_E^2 + dr^2 + r^2 d\Omega_{d-2}^2\) to a sphere: this is given by the coordinate transformation

\[
t_E = l \frac{\cos \theta \sin \frac{\phi}{2}}{1 + \cos \theta \cos \frac{\phi}{2}} , \quad r = l \frac{\sin \theta}{1 + \cos \theta \cos \frac{\phi}{2}} : \quad ds_E^2 = l^2 \left[ \cos^2 \theta d\rho^2 + l^2 (d\theta^2 + \sin^2 \theta d\Omega_{d-2}^2) \right] , \quad \Omega = \frac{1}{1 + \cos \theta \cos \frac{\phi}{2}} . \tag{21}\]

Removing the conformal factor \(\Omega^2\), this space becomes \(d\tilde{s}^2 = \cos^2 \theta d\rho^2 + l^2 (d\theta^2 + \sin^2 \theta d\Omega_{d-2}^2)\). Demanding that this space be smooth, we must avoid a conical singularity at \(\theta = \frac{\pi}{2}\): then the coordinate \(\rho\) must be taken to be periodic with period \(\Delta \rho = 2\pi l\). This space can then be seen to be a \(d\)-sphere \(d\tilde{s}^2 = l^2 (d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \sin^2 \theta_1 \sin^2 \theta_2 d\Omega_{d-2}^2)\), using a coordinate transformation \(\sin \theta = \sin \theta_1 \sin \theta_2\), \(\tan \frac{\rho}{l} = \cos \theta_2 \tan \theta_1\).

The free energy \(F\) of the Euclidean CFT on this sphere is expected to exhibit a logarithmically divergent term (in even dimensions) whose coefficient is related to the conformal anomaly. In general, we have an expansion \(-F_{CFT} = \log Z_{CFT} = (\text{non-universal terms}) + a \log \epsilon + (\text{finite})\), with \(\epsilon\) the ultraviolet cutoff. The CFT energy-momentum tensor \([7]\) is defined as \(T_{ij} = \frac{2}{\sqrt{h}} \frac{\delta Z_{CFT}}{\delta h^{ij}} = \frac{2}{\sqrt{h}} \frac{\delta \Psi}{\delta h^{ij}}\) which becomes \(T_{ij} \sim \frac{2}{\sqrt{h}} \frac{\delta (-F_{CFT})}{\delta h^{ij}} \sim i \frac{2}{\sqrt{h}} \frac{\delta \tilde{s}}{\delta h^{ij}}\) in the classical approximation for \(\Psi\). Under an infinitesimal conformal transformation \(h_{ij} \to (1 + 2\delta \lambda) h_{\mu \nu}\), we have \(\frac{\delta F_{CFT}}{\delta \lambda} = \int d^d x \sqrt{h} (T^k_k) + (\text{div})\), which is the integrated trace anomaly. Due to conformal invariance, this must be equivalent to simply shifting the ultraviolet cutoff \(\epsilon \to (1 - \delta \lambda) \epsilon\). This gives the coefficient \(a = \int (T^k_k)\). This argument does not appear to require unitarity of the conformal field theory. We have calculated this free energy holographically.
assuming the nonunitary CFT has a de Sitter gravity dual and using the $dS/CFT$ dictionary $Z_{CFT} = \Psi$ [7] with an auxiliary de Sitter space in global coordinates (where constant time slices are spheres). As we have seen, we find agreement with the coefficients of the logarithmic terms in the complex extremal surfaces earlier. The fact that these coefficients are pure imaginary is expected from the $i$ in the relation $-F \sim iS_{bulk}$.

## 5 Discussion

We have studied complex extremal surfaces for spherical subregions on a constant boundary Euclidean time slice of the future boundary of de Sitter space, building on [21]. For even boundary dimensions, there is a logarithmically divergent term in the area of these surfaces whose coefficient is a universal term. Comparing this with a corresponding coefficient (related to the integrated conformal anomaly) in a logarithmically divergent term in the free energy of the dual Euclidean CFT on a sphere using the $dS/CFT$ dictionary for a dual de Sitter space in global coordinates in a classical approximation for the wavefunction $\Psi \sim e^{iS}$, we find agreement including numerical factors. This coefficient is of the form $-i\nu_d \frac{R_{d-1}}{G_{d+1}}$ where $\nu_d$ is a real positive numerical factor. Perhaps this agreement is not surprising since both sides in this Einstein gravity approximation effectively amount to analytic continuation from the $AdS$ case (where there is agreement), but it shows consistency between the two sides in the present case.

These investigations and those in [21] support the idea that the areas of these complex extremal surfaces encode entanglement entropy of the dual Euclidean CFT in $dS/CFT$. It also suggests that in $dS/CFT$, the coefficients of the logarithmic terms for these complex extremal surfaces are perhaps the analogs of the $a$-type central charges. Relatedly it may be interesting to study analogs of [36] in the de Sitter case. It is clear however that all our calculations are in the bulk and so cannot clearly pinpoint the interpretation of entanglement entropy. (As an aside however, in the 2d CFT dual to $dS_3$ with central charge $-i\frac{R_{3-1}}{G_3}$, a replica calculation of entanglement entropy [37, 17] appears to give $\sim -i\frac{R_{3-1}}{G_3} \log \frac{l}{\epsilon}$ under certain assumptions, most importantly the existence of twist sector ground states.) In general the notion of entanglement entropy requires certain basic assumptions on the CFT Hilbert space, most importantly the existence of a ground state. The dual CFT for the pure de Sitter theory here in general has complex conformal dimensions (the higher spin $dS/CFT$ of [9] is better-behaved in this regard, but geometric extremal surfaces may not be of relevance in this case). Also the CFT is intrinsically Euclidean, with no notion of time evolution (while the dual bulk time direction emerges). Thus the use of a conformal transformation along the lines of [33] to map the reduced density matrix to e.g. a thermal one appears more delicate, in such nonunitary CFTs. It would appear that this CFT entanglement entropy, assuming it exists, encodes CFT correlations.
and is thus likely, if only indirectly, to also encode bulk de Sitter expectation values which have intricate connections to the dual CFT correlation functions [7]. These issues would be interesting to explore further.

Acknowledgements: It is a pleasure to thank Shamik Banerjee for several useful discussions and initial collaboration. This work is partially supported by a grant to CMI from the Infosys Foundation.

References

[1] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. 2, 231 (1998) [Int. J. Theor. Phys. 38, 1113 (1999)] [arXiv:hep-th/9711200].

[2] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from non-critical string theory,” Phys. Lett. B 428, 105 (1998) [arXiv:hep-th/9802109].

[3] E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. 2, 253 (1998) [arXiv:hep-th/9802150].

[4] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, “Large N field theories, string theory and gravity,” Phys. Rept. 323, 183 (2000) [arXiv:hep-th/9905111].

[5] A. Strominger, “The dS / CFT correspondence,” JHEP 0110, 034 (2001) [hep-th/0106113].

[6] E. Witten, “Quantum gravity in de Sitter space,” [hep-th/0106109].

[7] J. M. Maldacena, “Non-Gaussian features of primordial fluctuations in single field inflationary models,” JHEP 0305, 013 (2003), [astro-ph/0210603].

[8] D. Harlow and D. Stanford, “Operator Dictionaries and Wave Functions in AdS/CFT and dS/CFT,” [arXiv:1104.2621 [hep-th]].

[9] D. Anninos, T. Hartman and A. Strominger, “Higher Spin Realization of the dS/CFT Correspondence,” [arXiv:1108.5735 [hep-th]].

[10] G. S. Ng and A. Strominger, “State/Operator Correspondence in Higher-Spin dS/CFT,” Class. Quant. Grav. 30, 104002 (2013) [arXiv:1204.1057 [hep-th]].

[11] D. Das, S. R. Das, A. Jevicki and Q. Ye, “Bi-local Construction of Sp(2N)/dS Higher Spin Correspondence,” JHEP 1301, 107 (2013) [arXiv:1205.5776 [hep-th]].

[12] D. Anninos, F. Denef and D. Harlow, “The Wave Function of Vasiliev’s Universe - A Few Slices Thereof,” Phys. Rev. D 88, 084049 (2013) [arXiv:1207.5517 [hep-th]].

[13] D. Das, S. R. Das and G. Mandal, “Double Trace Flows and Holographic RG in dS/CFT correspondence,” [arXiv:1306.0330 [hep-th]].
[14] S. Banerjee, A. Belin, S. Hellerman, A. Lepage-Jutier, A. Maloney, j. j. Radievi and S. Shenker, “Topology of Future Infinity in dS/CFT,” JHEP 1311, 026 (2013) [arXiv:1306.6629 [hep-th]].

[15] D. Das, S. R. Das and K. Narayan, “dS/CFT at uniform energy density and a de Sitter 'bluewall',” JHEP 1404, 116 (2014) [arXiv:1312.1625 [hep-th]].

[16] S. Ryu and T. Takayanagi, “Holographic derivation of entanglement entropy from AdS/CFT,” Phys. Rev. Lett. 96, 181602 (2006) [hep-th/0603001].

[17] S. Ryu and T. Takayanagi, “Aspects of Holographic Entanglement Entropy,” JHEP 0608, 045 (2006) [hep-th/0605073].

[18] T. Nishioka, S. Ryu and T. Takayanagi, “Holographic Entanglement Entropy: An Overview,” J. Phys. A 42 (2009) 504008;

[19] T. Takayanagi, “Entanglement Entropy from a Holographic Viewpoint,” Class. Quant. Grav. 29 (2012) 153001 [arXiv:1204.2450 [gr-qc]].

[20] V. E. Hubeny, M. Rangamani and T. Takayanagi, “A Covariant holographic entanglement entropy proposal,” JHEP 0707 (2007) 062 [arXiv:0705.0016 [hep-th]].

[21] K. Narayan, “de Sitter extremal surfaces,” arXiv:1501.03019 [hep-th].

[22] J. Maldacena and G. L. Pimentel, “Entanglement entropy in de Sitter space,” JHEP 1302, 038 (2013) [arXiv:1210.7244 [hep-th]].

[23] L. Fidkowski, V. Hubeny, M. Kleban and S. Shenker, “The Black hole singularity in AdS / CFT,” JHEP 0402, 014 (2004) [hep-th/0306170].

[24] S. Fischetti, D. Marolf and A. Wall, “A paucity of bulk entangling surfaces: AdS wormholes with de Sitter interiors,” arXiv:1409.6754 [hep-th].

[25] L. Bombelli, R. K. Koul, J. Lee and R. D. Sorkin, “A Quantum Source of Entropy for Black Holes,” Phys. Rev. D 34 (1986) 373.

[26] M. Srednicki, “Entropy and area,” Phys. Rev. Lett. 71 (1993) 666 [hep-th/9303048].

[27] Y. Sato, “Comments on Entanglement Entropy in the dS/CFT Correspondence,” Phys. Rev. D 91, no. 8, 086009 (2015) [arXiv:1501.04903 [hep-th]].

[28] M. Henningson, K. Skenderis, “The Holographic Weyl anomaly,” JHEP 9807, 023(1998) [hep-th/9806087].

[29] M. Henningson and K. Skenderis, “Holography and the Weyl anomaly,” Fortsch. Phys. 48, 125 (2000) [hep-th/9812032].

[30] V. Balasubramanian and P. Kraus, “A stress tensor for anti-de Sitter gravity,” Commun. Math. Phys. 208, 413 (1999) [arXiv:hep-th/9902121].

[31] S. de Haro, S. N. Solodukhin and K. Skenderis, “Holographic reconstruction of spacetime and renormalization in the AdS/CFT correspondence,” Commun. Math. Phys. 217, 595 (2001) [arXiv:hep-th/0002230].
[32] J. B. Hartle and S. W. Hawking, “Wave Function of the Universe,” Phys. Rev. D 28, 2960 (1983).

[33] H. Casini, M. Huerta and R. C. Myers, “Towards a derivation of holographic entanglement entropy,” JHEP 1105, 036 (2011) [arXiv:1102.0440 [hep-th]].

[34] R. C. Myers and A. Sinha, “Seeing a c-theorem with holography,” Phys. Rev. D 82, 046006 (2010) [arXiv:1006.1263 [hep-th]].

[35] R. C. Myers and A. Sinha, “Holographic c-theorems in arbitrary dimensions,” JHEP 1101, 125 (2011) [arXiv:1011.5819 [hep-th]].

[36] A. Lewkowycz and J. Maldacena, “Generalized gravitational entropy,” JHEP 1308, 090 (2013) [arXiv:1304.4926 [hep-th]].

[37] P. Calabrese and J. L. Cardy, “Entanglement entropy and quantum field theory,” J. Stat. Mech. 0406, P06002 (2004) [hep-th/0405152].