HOMOLOGICAL ALGEBRA OF NOVIKOV-SHUBIN INVARIANTS
AND MORSE INEQUALITIES

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Abstract. It is shown in this paper that the topological phenomenon "zero in the continuous spectrum", discovered by S.P. Novikov and M.A. Shubin, can be explained in terms of a homology theory on the category of finite polyhedra with values in certain abelian category. This approach implies homotopy invariance of the Novikov-Shubin invariants. Its main advantage is that it allows to use the standard homological techniques, such as spectral sequences, derived functors, universal coefficients etc., while studying the Novikov-Shubin invariants. It also leads to some new quantitative invariants, measuring the Novikov-Shubin phenomenon in a different way, which are used in the present paper in order to strengthen the Morse type inequalities of S.P. Novikov and M.A. Shubin [NS1].

§0. Introduction

This paper suggests a conceptually new approach, which unites the $L^2$ cohomology theory and the Novikov-Shubin invariants. It is shown here that these theories are two different parts of a unique cohomology theory with values in an abelian category. This abelian category, denoted $E(A)$, contains the familiar additive category of Hilbertian modules over a von Neumann algebra $A$ as a full subcategory of projectives. An important abelian subcategory of $E(A)$ is formed by torsion virtual Hilbertian modules. It turns out that any object of $E(A)$ has canonically defined torsion and projective parts and coincides with their direct sum. The von Neumann dimension is an invariant of the projective part; similarly, the Novikov-Shubin number is an invariant of the torsion part.

There are natural homology and cohomology theories with values in the abelian category $E(A)$. I denote these theories by $H_i(X, M)$ and $H^i(X, M)$ correspondingly and call extended $L^2$ homology and cohomology. Here $X$ is a CW complex having finitely many cells in every dimension, and $M$ is a Hilbertian $(A - \pi)$- or $(\pi - A)$-bimodule (cf. §6 below), and $\pi = \pi_1(X)$ is the fundamental group of $X$. These theories are homotopy invariant. The projective part of $H^i(X, M)$ coincides with

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the usual reduced $L^2$ cohomology. The Novikov-Shubin invariants of the torsion part of $\mathcal{H}^i(X,M)$ coincide with the Novikov-Shubin invariants of the complex $X$. This gives a conceptually transparent proof of homotopy invariance of both von Neumann Betti numbers (proven by J.Dodziuk [D]) and of the Novikov-Shubin invariants (proven by M.Gromov and M.Shubin [GS]).

Simple examples of torsion virtual Hilbertian modules show that they can be not isomorphic but have equal Novikov-Shubin invariants.

I introduce here a new numerical invariant of torsion objects, which is independent on the Novikov-Shubin number; it is called the minimal number of generators.

As the main application, new Morse type inequalities for the numbers of critical points of a function on compact manifold are found. These inequalities give quantitative information even in the cases when all von Neumann Betti numbers vanish. Note, that Morse inequalities using the von Neumann dimensions of the reduced $L^2$ cohomology were obtained by S.P.Novikov and M.A.Shubin in [NS]. A more complete exposition will appear in [Sh], where a general philosophy of model operator is developed.

The most important advantage of the suggested in this paper new approach to the Novikov-Shubin invariants consists, perhaps, in the fact that this approach allows to apply the standard techniques of homological algebra (such as spectral sequences, derived functors, etc.) to studying the Novikov-Shubin invariants. Some results in this direction are included in this paper. For example, a universal coefficients theorem, and a Poincaré duality theorem, are proven here. We also study the group homology and cohomology with values in the extended abelian category $\mathcal{E}(\mathcal{A})$. It is shown that in the most general case the extended $L^2$ homology and cohomology can be expressed through these group cohomology by means of some Cartan-Eilenberg type spectral sequences; the initial $E^2$-term of these spectral sequences depends only on the homology of the universal covering, considered as modules over the group ring. This allows to conclude, for example, that the Novikov-Shubin invariants of a space with free fundamental group depend only on the homology modules of the universal covering.

Before the present work has started, M.A.Shubin had shown to me a simple argument (due to M.Gromov) which proves that nontriviality of the Novikov-Shubin number implies existence of at least one critical point. M.A.Shubin also suggested to me the problem of finding quantitative estimates. This problem was the main motivation for the present work. I am also thankful to W.Lück who sent to me preprint of [LL] before publication; it was very influential. Finally, I would like to mention interesting and very helpful discussions of different parts of this work with M.Gromov, J.Levine, V.Mathai and S.Weinberger.

This paper distributed as a preprint of the Tel Aviv University in July 1995; a short announcement was published in [Fa].

Wolfgang Lück, in his recent preprint "Hilbert modules and modules over finite von Neumann algebras and applications to $L^2$-invariants" (Mainz, December 1995) suggested a different (more algebraic) approach to studying $L^2$-invariants. He also discussed some interesting examples and applications, and showed that his approach is in fact equivalent to the method of this paper.
§1. The category of Hilbertian modules over a von Neumann algebra

This section briefly describes a modification of the standard category of Hilbert modules over a finite von Neumann algebra. A more detailed exposition of this material can be found in [CFM].

1.1. Let $\mathcal{A}$ be a finite von Neumann algebra with a fixed finite, normal, and faithful trace $\tau: \mathcal{A} \to \mathbb{C}$. The involution in $\mathcal{A}$ will be denoted $\ast$. By $\ell^2(\mathcal{A})$ we denote the completion of $\mathcal{A}$ with respect to the scalar product $\langle a, b \rangle = \tau(b^*a)$, for $a, b \in \mathcal{A}$.

Recall that a Hilbert module over $\mathcal{A}$ is a Hilbert space $M$ together with a continuous left $\mathcal{A}$-module structure such that there exists an isometric $\mathcal{A}$-linear embedding of $M$ into $\ell^2(\mathcal{A}) \otimes H$, for some Hilbert space $H$. Note that this embedding is not part of the structure. A Hilbert module $M$ is finitely generated if it admits an embedding $M \to \ell^2(\mathcal{A}) \otimes H$ as above with finite dimensional $H$.

Any Hilbert module, being a Hilbert space, has a particular scalar product. In this paper we wish to consider a weaker notion obtained from Hilbert module by forgetting the scalar product but preserving its topology and the $\mathcal{A}$-action.

1.2. Definition. A Hilbertian module is a topological vector space $M$ with continuous left $\mathcal{A}$-action such that there exists a scalar product $\langle \cdot, \cdot \rangle$ on $M$ which generates the topology of $M$ and such that $M$ together with $\langle \cdot, \cdot \rangle$ and with the $\mathcal{A}$-action is a Hilbert module. In particular, the involution on $\mathcal{A}$ is compatible with the involution on the space of bounded linear operators on $M$ determined by the scalar product $\langle \cdot, \cdot \rangle$.

If $M$ is a Hilbertian module, then any scalar product $\langle \cdot, \cdot \rangle$ on $M$ with the above properties will be called admissible.

1.3. It is easy to see that given a Hilbertian module, all different choices of admissible scalar products on it produce isomorphic Hilbert modules, cf. [CFM]. The situation here is similar to the case of finite dimensional vector spaces: any choice of a scalar product on a vector space produces an isomorphic Euclidean vector space.

Using the above mentioned fact, we may define finitely generated Hilbertian modules as those for which the corresponding Hilbert modules (obtained by a choice of an admissible scalar product) are finitely generated, cf. 1.1 above.

Similarly, we may correctly define the von Neumann dimension $\text{dim}_\tau(M)$ of a Hilbertian module $M$ as the von Neumann dimension of the Hilbert module obtained by a choice of an admissible scalar product on $M$.

1.4. Let us denote by $\mathcal{H}(\mathcal{A})$ the category, whose objects are finitely generated Hilbertian modules over $\mathcal{A}$ and whose morphisms are continuous linear maps commuting with the action of the algebra $\mathcal{A}$. Obviously, $\mathcal{H}(\mathcal{A})$ is an additive category.

Note that the category $\mathcal{H}(\mathcal{A})$ depends on the choice of the trace $\tau$ in an essential way, although the trace $\tau$ does not appear in the notation $\mathcal{H}(\mathcal{A})$.

Given a morphism $f: M \to N$ in the category $\mathcal{H}(\mathcal{A})$, the set-theoretic kernel $\ker(f) \subset M$ of $f$ has naturally a structure of finitely generated Hilbertian module over $\mathcal{A}$; it coincides with the kernel of $f$ in sense of the category theory. On the contrary, the set-theoretic image $\text{im}(f) \subset N$ is not in general closed and so it is not a Hilbertian module; the categorical image of the morphism $f$ coincides with the closure of the set-theoretic image $\text{cl}(\text{im}(f)) \subset N$. 
Let’s emphasize that the symbols \( \text{im} \) and \( \ker \) will always denote the set-theoretic notions. The symbols \( \text{Im} \), \( \text{Ker} \), \( \text{Coim} \), \( \text{Coker} \) (starting with the capital letters) will denote the corresponding notions of the category theory, cf., for example, [Gr], [F].

1.5. Note the following well-known important property of the category \( \mathcal{H}(A) \). If

\[
0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0
\]

is a sequence of objects and morphisms of the category \( \mathcal{H}(A) \), which is exact, considered as a sequence of abelian groups (i.e. if \( f \) is injective, \( g \) is surjective, and \( \text{im}(f) = \ker(g) \)), then this sequence splits in the category \( \mathcal{H}(A) \).

1.6. Duality. There is a notion of duality in \( \mathcal{H}(A) \). Given a Hilbertian module \( M \), consider the set \( M^* \) of all anti-linear continuous functionals on \( M \). Introduce the following action of \( A \) on \( M^* \): if \( \phi \in M^* \) and \( \lambda \in A \) then \( (\lambda \cdot \phi)(m) = \phi(\lambda^* \cdot m) \) for all \( m \in M \). Here \( \lambda^* \) is defined by the involution \( * \) of \( A \). In particular, this defines an action of \( \mathbb{C} \subset A \) on \( M^* \).

Note that, the dual of \( \ell^2(A) \) is canonically isomorphic to \( \ell^2(A) \). If \( M \) is an arbitrary Hilbertian module then \( M^* \) is also a Hilbertian module, which is isomorphic to \( M \), but not canonically. Any choice of an admissible scalar product on \( M \) gives an isomorphism between \( M \) and \( M^* \). Note that \( M \) is finitely generated if and only if \( M^* \) is.

Duality is a contravariant functor: if \( f : M \to N \) is a morphism of Hilbertian modules then \( f^* : N^* \to M^* \) is also a morphism of Hilbertian modules. Duality is also involutive: \( M^{**} \) is canonically isomorphic to \( M \). The canonical isomorphism \( M \to M^{**} \) is given by \( m \mapsto (f \mapsto \overline{f(m)}) \), where \( m \in M \), and \( f \in M^* \). Here the bar denotes the complex conjugation.

\section*{§2. The Extended abelian category}

In this section an abelian category, containing the category of Hilbertian modules \( \mathcal{H}(A) \), is described.

\( A \) denotes in this section a finite von Neumann algebra supplied with a fixed finite, normal, and faithful trace \( \tau \).

2.1. The category \( \mathcal{H}(A) \) of Hilbertian modules over the von Neumann algebra \( A \) is not an abelian category. In fact, the condition AB1 of [Gr] is satisfied (any morphism has a kernel and a cokernel) but the condition AB2 in general does not hold. Recall that the condition AB2 requires that the canonical map \( \text{Coim}(f) \to \text{Im}(f) \) be an isomorphism. Given a morphism \( f : M \to N \) in the category \( \mathcal{H}(A) \), its coimage \( \text{Coim}(f) \) can be represented by the Hilbertian module \( M/\ker(f) \), while (as it was mentioned above in 1.4) \( \text{Im}(f) = \overline{\text{cl}(\text{im}(f))} \). Thus, the canonical map \( \text{Coim}(f) \to \text{Im}(f) \) is a continuous linear map, which is injective and the set of its values is dense. Such map clearly may be not invertible in \( \mathcal{H}(A) \); one easily constructs the corresponding examples.

A few possible approaches to the general problem of representing a given additive category in abelian categories were discussed by P. Freyd in [F]. One of the constructions, described in [F], gives the following abelian category \( \mathcal{E}(A) \), which we will call the extended category of Hilbertian modules. Note that the category \( \mathcal{E}(A) \) depends on the choice of the trace \( \tau \) although \( \tau \) does not appear explicitly in the notation.
2.2. Definition. An object of the category \( \mathcal{E}(A, \tau) \) is defined as a morphism \((\alpha : A' \to A)\) in the category \( \mathcal{H}(A) \). Recall that here \( A' \) and \( A \) are finitely generated Hilbertian modules over the von Neumann algebra \( A \) and \( \alpha \) is a continuous linear map commuting with the action of the algebra \( A \).

Given a pair of objects \( X = (\alpha : A' \to A) \) and \( Y = (\beta : B' \to B) \) of \( \mathcal{E}(A) \), a morphism \( X \to Y \) in the category \( \mathcal{E}(A) \) is an equivalence class of morphisms \( f : A \to B \) of category \( \mathcal{H}(A) \) such that \( f \circ \alpha = \beta \circ \gamma \) for some morphism \( \gamma : A' \to B' \) in \( \mathcal{H}(A) \). Two morphisms \( f : A \to B \) and \( f' : A \to B \) of \( \mathcal{H}(A) \) represent identical morphisms of \( \mathcal{E}(A) \) \( X \to Y \) iff \( f - f' = \beta \circ F \) for some morphism \( F : A \to B' \) of category \( \mathcal{H}(A) \). This defines an equivalence relation. The morphism \( X \to Y \), represented by \( f : A \to B \) well be denoted \([f] : (\alpha : A' \to A) \to (\beta : B' \to B)\) or \([f] : X \to Y\).

Sometimes we will say that \([f]\) is represented by a diagram

\[
\begin{array}{c}
(A' \xrightarrow{\alpha} A) \\
\downarrow f \\
(B' \xrightarrow{\beta} B)
\end{array}
\]

this emphasizes existence of the morphism \( g : A' \to B' \) making the diagram commutative.

The composition of morphisms is defined as the composition of the corresponding morphisms \( f \) in the category \( \mathcal{H}(A) \); this unambiguously defines a composition law for morphisms. The identity morphism is \([\text{id}_A]\). Clearly, \( \mathcal{E}(A) \) is an additive category.

Objects of \( \mathcal{E}(A) \) will be called virtual Hilbertian modules.

It follows from the work of P.Freyd [F], that the extended category \( \mathcal{E}(A) \) is an abelian category.

Having in mind our further purposes in this paper, we are going to sketch an independent proof.

First, we have to be able to compute kernels and cokernels of morphisms in \( \mathcal{E}(A) \).

2.3. Proposition. Suppose that \([f] : (\alpha : A' \to A) \to (\beta : B' \to B)\) is a morphism in the extended category of Hilbertian modules \( \mathcal{E}(A) \). Then its kernel is represented by

\[
[k] : (\gamma : P' \to P) \to (\alpha : A' \to A),
\]

where

\[
\begin{array}{c}
P \xrightarrow{k} A \\
\downarrow f \\
B' \xrightarrow{\beta} B
\end{array}
\quad \text{and} \quad
\begin{array}{c}
P' \xrightarrow{k'} A' \\
\downarrow f \circ \alpha \\
B' \xrightarrow{\beta} B
\end{array}
\]

are the pullbacks of the diagrams
\[
\begin{array}{c}
\begin{array}{c}
A \\
\downarrow f
\end{array} \\
B' \xrightarrow{\beta} B
\end{array}
\quad
\begin{array}{c}
\begin{array}{c}
A' \\
\downarrow f \circ \alpha
\end{array} \\
B' \xrightarrow{\beta} B
\end{array}
\end{array}
\]

(2)

correspondingly, and \( \gamma : P' \to P \) is the canonical map, induced by the obvious map of the right diagram (2) into the left one.

The cokernel of the above morphism \([f]\) is represented by

\[
[id_B] : (\beta : B' \to B) \to ((\beta, -f) : B' \oplus A \to B).
\]

Proof. It follows by checking straightforwardly the definitions. We will leave this proof to the reader. □

Thus, we obtain that any morphism of \( \mathcal{E}(A) \) has a kernel and a cokernel and so condition AB1 of [Gr], §1.4 is satisfied.

Using Proposition 2.3, we may compute explicitly the coimage \( \text{Coim}([f]) \) (which is defined as the cokernel of the kernel) and the image \( \text{Im}([f]) \) (which is defined as the kernel of the cokernel); the result in both cases coincides with

\[
[f] : (k : P \to A) \to (\beta : B' \to B)
\]

where \( P \) and \( k \) are the same as in Proposition 2.3.

This shows that the condition AB2 from the definition of abelian categories, cf. [Gr], chapter 1, is satisfied. Hence \( \mathcal{E}(A) \) is an abelian category.

2.4. Excision. Note that two very different maps \((\alpha : A' \to A)\) and \((\beta : B' \to B)\) may represent isomorphic objects of category \( \mathcal{E}(A) \). In order to clarify this question, consider the following situation.

Suppose that \((\alpha : A' \to A)\) is an object of \( \mathcal{E}(A) \) and \( P \subset A' \) is a closed \( A \)-submodule such that its image \( \alpha(P) \subset A \) is also closed. Let \( B' \) and \( B \) denote the factor-modules \( B' = A'/P \) and \( B = A/\alpha(P) \). The map \( \alpha \) induces the obvious map \( \beta : B' \to B \). We claim now that the obtained object \((\beta : B' \to B)\) of \( \mathcal{E}(A) \) is isomorphic to the initial \((\alpha : A' \to A)\).

The passage from \((\alpha : A' \to A)\) to \((\beta : B' \to B)\) described above, will be called excision with respect to \( P \). We will also say that \((\alpha : A' \to A)\) is an enlargement of \((\beta : B' \to B)\).

Observe first that our claim is immediate in the case \( \alpha(P) = 0 \).

It follows that we may always make an excision with respect to the kernel of the map \( \alpha \) and so any object of \( \mathcal{E}(A) \) can be represented by \((\alpha : A' \to A)\) with an injective morphism \( \alpha \).

Suppose now that \((\beta : B' \to B)\) is obtained from \((\alpha : A' \to A)\) with injective \( \alpha \) by excision with respect to \( P \subset A \). Then the following sequence in \( \mathcal{H}(A) \)

\[
0 \to A' \xrightarrow{\left[\begin{array}{c} f' \\ \alpha \end{array}\right]} B' \oplus A \xrightarrow{(-\beta,-f)} B \to 0
\]
is exact. Here \( f' : A' \rightarrow A'/P = B' \) and \( f : A \rightarrow A/P = B \) are the canonical projections. Hence this sequence splits (by 1.5). Consider a splitting of the above sequence

\[
0 \leftarrow A' \xrightarrow{(\delta', \sigma')} B' \oplus A \xleftarrow{[\delta]} B \leftarrow 0
\]

Then the relations

\[
f \circ \alpha = \beta \circ f' \quad \text{and} \quad \delta \circ \beta = \alpha \circ \delta'
\]

show that \([f] : (\alpha : A' \rightarrow A) \rightarrow (\beta : B' \rightarrow B)\) and \([\delta] : (\beta : B' \rightarrow B) \rightarrow (\alpha : A' \rightarrow A)\) are morphisms of \(E(A)\), and the relations

\[
- \beta \circ \sigma + f \circ \delta = 1_B \quad \text{and} \quad \alpha \circ \sigma' + \delta \circ f = 1_A
\]

show that the above morphisms \([f]\) and \([\delta]\) are mutually inverse. This completes the proof.

Note that a morphism \((\alpha : A' \rightarrow A)\) represents a null object of \(E(A)\) (which can be characterized as an object with the property that its identity morphism coincides with the zero morphism) if and only if \(\alpha\) is surjective.

2.5. Monomorphisms and epimorphisms. Let \([f] : (\alpha : A' \rightarrow A) \rightarrow (\beta : B' \rightarrow B)\) be a morphism of \(E(A)\). It is easy to see (using Proposition 2.3) that \([f]\) is a monomorphism of the category \(E(A)\) if and only if

\[
\alpha(A') \supset f^{-1}(\beta(B'))
\]

and \([f]\) is an epimorphism of the category \(E(A)\) if and only if

\[
B = \beta(B') + f(A)
\]

In particular, any morphism \(f : A \rightarrow B\) of \(\mathcal{H}(A)\) determines a morphism \([f] : (0 \rightarrow A) \rightarrow (0 \rightarrow B)\) in \(E(A)\) and \(f\) is injective (respectively, surjective) as a morphism vector spaces, if and only if the morphism \([f]\) is a monomorphism (respectively, epimorphism).

The following remark will be useful later.

2.6. Lemma. Given a monomorphism in \(E(A)\)

\[
[f] : (\alpha : A' \rightarrow A) \rightarrow (\beta : B' \rightarrow B),
\]

one can perform an excision on \((\alpha : A' \rightarrow A)\) such that the same monomorphism will be represented by a diagram

\[
\begin{array}{ccc}
(C' & \xrightarrow{\gamma} & C) \\
\downarrow{g} & & \\
(B' & \xrightarrow{\beta} & B)
\end{array}
\]

with injective \(g\).

Proof. Let \(P'\) denote \(\ker f \subset A\) and let \(P = \alpha^{-1}(P')\). Then \(\alpha(P) = P'\) (by virtue of (3)) is closed and so we can perform an excision with respect to \(P\). This completes the proof. \(\Box\)

We can now strengthen Lemma 2.6:
2.7. Corollary. (1) For any monomorphism $\mathcal{X} \to \mathcal{Y}$ in $\mathcal{E}(A)$, one can perform a sequence of excisions and enlargements on $\mathcal{X}$ and on $\mathcal{Y}$ such that the given monomorphism will be represented by a diagram

$$
\begin{array}{ccc}
(A') & \xrightarrow{\alpha} & A \\
\downarrow h & & \downarrow f \\
(B') & \xrightarrow{\beta} & B
\end{array}
$$

(6)

with $\alpha, \beta, f$ monomorphisms and $h$ isomorphism. (Note, that $h$ is determined uniquely in this situation by $\alpha, \beta, f$.) Conversely, any morphism of $\mathcal{E}(A)$ represented by such diagram is a monomorphism.

(2) For any epimorphism $\mathcal{X} \to \mathcal{Y}$ in $\mathcal{E}(A)$ one can perform a sequence of excisions and enlargements on $\mathcal{X}$ and on $\mathcal{Y}$ such that the given epimorphism will be represented by a diagram (6) as above with $\alpha, \beta$, and $h$ injective and $f$ bijective. (Here again $h$ is determined uniquely by $\alpha, \beta$ and $f$.) Conversely, any such diagram represents an epimorphism in $\mathcal{E}(A)$.

Proof. (1) First we represent a given monomorphism by a diagram (6) with $\beta$ injective and $h$ surjective; this can easily be arranged. Then we perform excision with respect to $\ker \alpha$ and then with respect to $\ker(f)$ (as explained in 2.6). This will give us a diagram representing the given morphism having the desired properties. The converse statement follows from 2.5.

(2) First, represent a given epimorphism by a diagram of form (6) with $\beta$ injective and $f$ surjective. This can be done starting from an arbitrary representation with injective $\beta$ by considering the diagram

$$
\begin{array}{ccc}
A' \oplus B' & \xrightarrow{\begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}} & A \oplus B' \\
\downarrow [h,1] & & \downarrow [f,\beta] \\
B' & \xrightarrow{\beta} & B.
\end{array}
$$

Assuming that this has been arranged, we perform a similar enlargement such that the image of the new map $\alpha$ would contain the kernel of $f$. Now consider $\ker(f) = P'$ and $\alpha^{-1}(P') = P$. Performing excision with respect to $P$ (using the arguments similar to those of Lemma 2.6) we obtain a diagram representing the given epimorphism which has $\beta$ injective, $f$ bijective and $\alpha$ injective. Then $h$ is injective, and the result follows. The converse statement follows obviously from 2.5. □

2.8. Embedding of $\mathcal{H}(A)$ into $\mathcal{E}(A)$. Given a Hilbertian module $A$, consider the zero morphism $(0 \to A)$ as an object of $\mathcal{E}(A)$. Any morphism $f: A \to B$ in $\mathcal{H}(A)$ determines the morphism $[f]: (0 \to A) \to (0 \to B)$ in $\mathcal{E}(A)$; conversely, any morphism in $\mathcal{E}(A)$

$$(0 \to A) \to (0 \to B)$$
determines uniquely a morphism of \( \mathcal{H}(A) \) between \( A \) and \( B \). This shows that we have a functor
\[
\mathfrak{F} : \mathcal{H}(A) \to \mathcal{E}(A),
\]
and this functor is a full embedding.

Let us show that an object \( X \in \text{ob}(\mathcal{E}(A)) \) is isomorphic to a Hilbertian module in \( \mathcal{E}(A) \) if and only if \( X \) is projective.

Note that it is not true that any projective \( X \in \text{ob}(\mathcal{E}(A)) \) is a Hilbertian module, i.e. comes from \( \mathcal{H}(A) \).

First we will show that any Hilbertian module \((0 \to X)\) is projective in \( \mathcal{E}(A) \). Suppose that we have a diagram
\[
\begin{array}{ccc}
A' & \longrightarrow & A \\
\downarrow f & & \downarrow g \\
B' & \longrightarrow & B \\
\end{array}
\]
with morphism \([f]\) being epimorphism (as a morphism of \( \mathcal{E}(A) \)); this means that \( B = \beta(B') + f(A) \) by 2.5. Then we have an exact sequence in \( \mathcal{H}(A) \)
\[
B' \oplus A \overset{(-\beta, f)}{\longrightarrow} B \to 0,
\]
and so the map \( g : X \to B \) can be lifted into \( B' \oplus A \) (by 1.5), i.e. there exists a morphism
\[
\left[ \begin{array}{c} \sigma \\ \delta \end{array} \right] : X \to B' \oplus A,
\]
such that
\[
\left[ \begin{array}{c} \sigma \\ \delta \end{array} \right] \circ (-\beta, f) = 1_X.
\]
This shows that \([f] \circ [\delta] = [g]\) in category \( \mathcal{E}(A) \). and so \((0 \to X)\) is projective.

Conversely, suppose that \((\alpha : A' \to A)\) represents a projective object of \( \mathcal{E}(A) \). We may assume that \( \alpha \) is an injective as a map (by 2.4). The diagram
\[
\begin{array}{ccc}
0 & \longrightarrow & A \\
\downarrow \text{id} & & \downarrow \text{id} \\
A' & \longrightarrow & A \\
\end{array}
\]
represents an epimorphism in \( \mathcal{E}(A) \) and thus there exists a lifting. It is represented by a diagram
\[
\begin{array}{ccc}
0 & \longrightarrow & A \\
\downarrow f & & \downarrow f \\
A' & \longrightarrow & A, \\
\end{array}
\]
where \( f : A \to A \) is a morphism in \( \mathcal{H}(A) \) such that
\[
f \circ \alpha = 0 \quad \text{and} \quad f = 1_A - \alpha \circ g
\]
for some \( g : A \to A' \). It follows that the kernel of \( f \) coincides with the \( \text{im}(\alpha) \).

Thus we may make an excision with respect to \( P = A' \) and so the original object
\((\alpha : A' \to A)\) isomorphic to \((0 \to A/\alpha(A'))\). This completes the proof.

Note additionally, that given an object \((\alpha : A' \to A)\) of \( \mathcal{E}(A) \), its projective
resolution can be constructed as follows:
\[
0 \to (0 \to A') \xrightarrow{[\alpha]} (0 \to A) \xrightarrow{[\text{id}]} (\alpha : A' \to A) \to 0
\]
(assuming that \( \alpha \) is an injective map, cf. 2.4). Thus we conclude that any object
of \( \mathcal{E}(A) \) admits a projective resolution of length two. Hence the homological dimension
of \( \mathcal{E}(A) \) equals to one.

2.9. Example. Consider a chain complex in the abelian category \( \mathcal{E}(A) \)
\[
\cdots \to C_{i+1} \xrightarrow{\partial} C_i \xrightarrow{\partial} C_{i-1} \to \cdots,
\]
consisting of projective Hilbertian modules (note that it can be equivalently consid-
ered as a chain complex in the additive category \( \mathcal{H}(A) \)). Its \( i \)-dimensional homology
in the extended abelian category \( \mathcal{E}(A) \) is represented by the morphism
\[
H_i(C_\ast) = (\partial : C_{i+1} \to Z_i)
\]
where \( Z_i \) is the submodule of cycles, \( Z_i = \ker[\partial : C_i \to C_{i-1}] \).

It is an easy exercise, based on Proposition 2.3.

§3. Torsion subcategory \( \mathcal{T}(A) \)

Now we are going to describe another full subcategory \( \mathcal{T}(A) \) of \( \mathcal{E}(A) \) which is
in some sense complementary to \( \mathcal{H}(A) \). We will call it the torsion subcategory and
denote \( \mathcal{T}(A) \); its objects will be called torsion virtual Hilbertian modules.

3.1. Definition. A virtual Hilbertian module \((\alpha : A' \to A)\) of \( \mathcal{E}(A) \) will be called
torsion if \( A = \text{cl}(\alpha(A')) \).

Equivalently, a virtual Hilbertian module \( \mathcal{X} \) is torsion if and only if it admits
no non-trivial morphisms \( \mathcal{X} \to P \) in projective objects \( P \) of \( \mathcal{E}(A) \). The definition
in this form clearly does not depend on a particular representation of the virtual
Hilbertian module and depends only on its isomorphism class in \( \mathcal{E}(A) \).

Another equivalent definition says: a virtual Hilbertian module represented by
\((\alpha : A' \to A)\) is torsion iff
\[
\dim_r(A) = \dim_r(A') + \dim_r(\ker \alpha).
\]

\( \mathcal{T}(A) \) will denote the full subcategory of \( \mathcal{E}(A) \) generated by torsion virtual Hilber-
tian modules.

We will first mention some formal properties of the torsion subcategory.
3.2. Proposition. Given an exact sequence

$$0 \to \mathcal{X}' \to \mathcal{X} \to \mathcal{X}'' \to 0$$

of objects and morphisms of category $\mathcal{E}(A)$, the middle object $\mathcal{X}$ is torsion if and only if both $\mathcal{X}'$ and $\mathcal{X}''$ are torsion.

Proof. Suppose first that both $\mathcal{X}'$ and $\mathcal{X}''$ are torsion. Then any morphism $\mathcal{X} \to P$, where $P$ is projective, vanishes on $\mathcal{X}'$ and so it can be factorized through a morphism $\mathcal{X}'' \to P$ which also must vanish, since $\mathcal{X}''$ is torsion.

Assume now that $\mathcal{X}$ is torsion. Then clearly $\mathcal{X}''$ is torsion and we are left to show that a subobject of a torsion module is torsion. Suppose that

$$[f] : (\alpha : A' \to A) \to (\beta : B' \to B) \quad (10)$$

represents a monomorphism in $\mathcal{E}(A)$ with torsion object $(\beta : B' \to B)$. As was shown in 2.4 and in 2.6, we may assume that all morphisms $\alpha, \beta, f$ are injective. Also, $\alpha(A') \supset f^{-1}(\beta(B'))$, cf. 2.5. Therefore we obtain

$$\dim_\tau(A') \geq \dim_\tau(B') = \dim_\tau(B) \geq \dim_\tau(A).$$

Thus the subobject $(\alpha : A' \to A)$ is also torsion. □

3.3. We will see now that any object of the extended category $\mathcal{E}(A)$ determines canonically a pair of objects, a torsion and a projective.

Let $\mathcal{X} = (\alpha : A' \to A)$ be an object of $\mathcal{E}(A)$. Denote

$$T(\mathcal{X}) = (\alpha : A' \to \text{cl}(\text{im}(\alpha))) \quad \text{and} \quad P(\mathcal{X}) = (0 \to A/\text{cl}(\text{im}(\alpha))).$$

$T(\mathcal{X})$ is clearly a subobject of $\mathcal{X}$ (by 2.7.1), and $P(\mathcal{X})$ is a factor-object of $\mathcal{X}$ (by 2.7.2). It is easy to see that we have a short exact sequence

$$0 \to T(\mathcal{X}) \to \mathcal{X} \to P(\mathcal{X}) \to 0 \quad (11)$$

in $\mathcal{E}(A)$. We will say that $T(\mathcal{X})$ is the torsion part of $\mathcal{X}$ and that $P(\mathcal{X})$ is the projective part of $\mathcal{X}$. Note that the exact sequence (11) splits (since $P(\mathcal{X})$ is projective) but the splitting is not canonical.

3.4. Given a morphism $f : \mathcal{X} \to \mathcal{Y}$ in the category $\mathcal{E}(A)$, consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T(\mathcal{X}) & \stackrel{i}{\longrightarrow} & \mathcal{X} & \stackrel{j}{\longrightarrow} & P(\mathcal{X}) & \longrightarrow & 0 \\
& & f' \downarrow & & f \downarrow & & f'' \downarrow & \\
0 & \longrightarrow & T(\mathcal{Y}) & \stackrel{i'}{\longrightarrow} & \mathcal{Y} & \stackrel{j'}{\longrightarrow} & P(\mathcal{Y}) & \longrightarrow & 0.
\end{array}$$

The morphism $j' \circ f \circ i$ is zero and so there exists a unique morphism $f' : T(\mathcal{X}) \to T(\mathcal{Y})$ such that $f \circ i = i' \circ f'$. Thus, we obtain, that any morphism of $\mathcal{E}(A)$ maps the torsion part of $\mathcal{X}$ into the torsion part of $\mathcal{Y}$.

Similarly, morphism $f$ above uniquely determines a morphism $f'' : P(\mathcal{X}) \to P(\mathcal{Y})$ between the projective parts.

We conclude that there are defined two covariant functors

$$T : \mathcal{E}(A) \to T(A) \quad \text{and} \quad P : \mathcal{E}(A) \to H(A),$$

which we will call the torsion and the projective part, respectively.
3.5. Given an exact sequence

\[ 0 \to X' \to X \to X'' \to 0 \quad (12) \]

in \( \mathcal{E}(A) \), it determines the exact sequence

\[ 0 \to T(X') \to T(X) \to T(X'') \quad (13) \]

of the torsion parts. Thus, the functor of torsion part is left exact.

Note, that the similar sequence

\[ 0 \to P(X') \to P(X) \to P(X'') \to 0 \quad (14) \]

for projective parts may be not exact in the middle term. But this sequence is always weakly exact, i.e. it is exact if considered as a sequence in the original additive category \( \mathcal{H}(A) \).

It turns out that the homology of the sequences (13) and (14) coincide. We will formulate this as the following proposition.

3.6. Proposition. Suppose that

\[ 0 \to X' \to X \to X'' \to 0 \]

is an exact sequence in \( \mathcal{E}(A) \) and let \( \mathcal{H} \) be defined by the exact sequence

\[ 0 \to T(X') \to T(X) \to T(X'') \to \mathcal{H} \to 0 \]

Then the homology of the complex (14) of the projective parts in the middle term is isomorphic to \( \mathcal{H} \).

Proof. Consider the original exact sequence as a chain complex \( C_* \) and let \( T_* \) denotes its subcomplex formed by the torsion parts. Similarly, let \( P_* \) denote the factor-complex formed by the projective parts. Then we have an exact sequence of chain complexes in \( \mathcal{E}(A) \)

\[ 0 \to T_* \to C_* \to P_* \to 0. \]

From the long exact sequence for homology we obtain the isomorphism \( H_i(P_*) \to H_{i-1}(T_*) \) and in particular \( H_1(P_*) \simeq H_0(T_*) = \mathcal{H}. \)

3.7. Example. Consider a chain complex in \( \mathcal{E}(A) \)

\[ \cdots \to C_{i+1} \xrightarrow{\partial} C_i \xrightarrow{\partial} C_{i-1} \to \cdots \quad (15) \]

consisting of projective Hilbertian modules. As we have seen in Example 2.9, its \( i \)-dimensional homology in the extended abelian category \( \mathcal{E}(A) \) is represented by the morphism

\[ H_i(C_*) = (\partial : C_{i+1} \to Z_i), \quad (16) \]
where $Z_i$ is the submodule of cycles. The projective part of the homology is the following Hilbertian module

$$P(H_i(C_*) = Z_i/\text{cl}(\text{im}([\partial : C_{i+1} \to C_i])), \quad (17)$$

which coincides with the definition of reduced $L^2$ homology of $C_*$. Observe, that the full homology $H_i(C_*)$, as an object of $E(A)$, may have a non-trivial torsion part

$$T(H_i(C_*)) = (\partial : C_{i+1} \to \text{cl}(\text{im}([\partial : C_{i+1} \to C_i]))). \quad (18)$$

Note also that $T(H_i(C_*))$ coincides with the torsion of the virtual Hilbertian module $(\partial : C_{i+1} \to C_i)$:

$$T(H_i(C_*)) = T(\partial : C_{i+1} \to C_i) \quad (19)$$

We will see in the sequel that the torsion part of the homology of $C_*$ determines completely the Novikov-Shubin invariants.

### 3.8. Duality for torsion objects

It turns out that there are two different notions of duality in the extended abelian category $E(A)$, one duality for projective objects and another duality for torsion objects. The duality for projective objects in $E(A)$ coincides essentially with the duality in $H(A)$, described in 1.6. Now we will construct duality for torsion objects.

Let $\mathcal{X} = (\alpha : A' \to A)$ be a torsion object represented by an injective morphism $\alpha$. Define the dual torsion Hilbertian module $\varepsilon(\mathcal{X})$ by

$$\varepsilon(\mathcal{X}) = (\alpha^* : A^* \to (A')^*),$$

where all $\alpha^*$, $A^*$ and $(A')^*$ are defined as explained in subsection 1.6.

Suppose now that we have two torsion objects $\mathcal{X} = (\alpha : A' \to A)$ and $\mathcal{Y} = (\beta : B' \to B)$ with injective $\alpha$ and $\beta$ and let $[f] : \mathcal{X} \to \mathcal{Y}$ be a morphism represented by a diagram

$$\begin{array}{ccc}
(A' & \xrightarrow{\alpha} & A) \\
\downarrow & & \downarrow f \\
(B' & \xrightarrow{\beta} & B).
\end{array} \quad (20)$$

According to definition 2.2, there exists a morphism $h : A' \to B'$ making this diagram commutative; this $h$ is in fact unique, because of injectivity of $\beta$. We define the dual of $[f]$ as the morphism

$$\varepsilon([f]) = [h^*] : \varepsilon(\mathcal{Y}) \to \varepsilon(\mathcal{X}). \quad (21)$$

It is represented by the diagram

$$\begin{array}{ccc}
(B^* & \xrightarrow{\beta^*} & B'^* \\
\downarrow h^* & & \downarrow \\
(A^* & \xrightarrow{\alpha^*} & A'^*).
\end{array}$$
We have to check correctness of this definition. If \( F : A \to B' \) is an arbitrary morphism, then the morphism \( f' = f + \beta \circ F \) represents the same morphism \([f] = [f']\) in \( \mathcal{E}(A) \). Then the corresponding morphism \( h' \) is \( h' = h + F \circ \alpha \) and thus

\[
h'' = h^* + \alpha^* \circ F^*,
\]

which means that \( h^* \) and \( h'' \) represent the same morphism in \( \mathcal{E}(A) \).

Note that for any torsion object \( \mathcal{X} \) the dual torsion object \( e(\mathcal{X}) \) is isomorphic to \( \mathcal{X} \), but not canonically. This follows from the existence of self-adjoint representation, cf. 4.1.

Clearly,

\[
e(e(\mathcal{X})) \simeq \mathcal{X},
\]

and this isomorphism is canonical.

3.9. Theorem (Universal coefficients). Let \( C_* \) be a projective chain complex in \( \mathcal{E}(A) \)

\[
C_* : \cdots \to C_{i+1} \xrightarrow{\partial} C_i \xrightarrow{\partial} C_{i-1} \to \cdots,
\]

and let \( C^* \) denote the dual cochain complex

\[
C^* : \cdots \to C^*_{i-1} \xrightarrow{\partial^*} C^*_i \xrightarrow{\partial^*} C^*_{i+1} \to \cdots,
\]

where the duality is understood as in 1.6. Consider homology of chain complex \( C_* \) and cohomology of cochain complex \( C^* \) in the extended abelian category \( \mathcal{E}(A) \). Then the projective part of the \( i \)-dimensional cohomology \( \mathcal{H}^i(C^*) \) is canonically dual (in the sense of 1.6) to the projective part of the \( i \)-dimensional homology \( \mathcal{H}_i(C) \), i.e.

\[
P(\mathcal{H}^i(C^*)) \simeq (P(\mathcal{H}_i(C)))^*.
\]

Moreover, the torsion part of the \( i \)-dimensional cohomology \( \mathcal{H}^i(C^*) \) is canonically dual (in the sense of 3.8) to torsion part of the \((i-1)\)-dimensional homology \( \mathcal{H}_{i-1}(C) \), i.e.

\[
T(\mathcal{H}^i(C^*)) \simeq e(T(\mathcal{H}_{i-1}(C))).
\]

Proof. As usual, let’s denote the space of boundaries by \( B_i = \text{im}[d : C_{i+1} \to C_i] \), and the space of cycles by \( Z_i = \ker[d : C_i \to C_{i-1}] \). Let \( H_i \) denote a complement to \( \overline{B}_i \) inside \( Z_i \) and let \( X_i \) be a complement to \( Z_i \) inside \( C_i \). Thus, we have the decompositions

\[
C_i = \overline{B}_i \oplus H_i \oplus X_i, \quad Z_i = \overline{B}_i \oplus H_i
\]

and the differential \( \partial \) restricts to an injective morphism \( \alpha : X_i \to \overline{B}_{i-1} \) with dense image. The extended \( L^2 \) homology of \( C \) is represented (cf. 3.7) by

\[
\mathcal{H}_i(C) = (\partial : C_{i+1} \to Z_i) \simeq \\
(\partial : X_{i+1} \to Z_i) \simeq \\
(\alpha : X_{i+1} \to \overline{B}_i) \oplus H_i
\]
Thus, the projective part of $H_i(C)$ is $H_i$ and the torsion part is $(\alpha : X_{i+1} \rightarrow B_i)$.

Consider now the cochain complex $C^*$. We have

$$C^*_i = B^*_i \oplus H^*_i \oplus X^*_i$$

and $\partial^* : C^*_i \rightarrow C^*_{i+1}$ vanishes on $H^*_i \oplus X^*_i$ and it is injective on $B^*_i$. Thus, we obtain

$$H^i(C^*) = (\partial^* : C^*_i \rightarrow (\ker : d^* : C^*_i \rightarrow C^*_{i+1})) \cong (\partial^* : B^*_i \rightarrow X^*_i) \cong (\alpha^* : B^*_i \rightarrow X^*_i) \oplus H^*_i.$$

Hence, $P(H^i(C^*)) = H^*_i$ and $T(H^i(C^*)) = (\alpha^* : B^*_i \rightarrow X^*_i)$, which is equal to $e(T(H^i-1(C)))$, according to the definition above (cf. 3.8). □

§ 4. Density functions and the Novikov-Shubin invariants

Von Neumann dimension is a natural numerical invariant of projective virtual Hilbertian modules. In this section we discuss a numerical invariant of torsion Hilbertian modules, known as the Novikov-Shubin invariant. It was introduced by S.P. Novikov and M.A. Shubin in [NS] and then studied by M. Gromov and M.A. Shubin [GS], and also by J. Lott and W. Lück [LL]; it was considered as an invariant of Hilbert chain complexes. Our point view in this section is slightly different; we consider the Novikov-Shubin number (or, more generally, the equivalence class of the spectral density function) as an invariant of torsion virtual Hilbertian modules.

We will see later (cf. § 5) that the Novikov-Shubin invariant do not determine the isomorphism type of torsion Hilbertian module. We will also construct some other numerical invariant in the sequel (cf. § 7).

4.1. First we observe that the polar decomposition theorem (cf., for example, [Di], appendix 3) implies that any torsion virtual Hilbertian module $\mathcal{X}$ over $A$ admits a self-adjoint representation, i.e. representation of the form

$$\mathcal{X} = (\alpha : A \rightarrow A),$$

where $A$ is a finitely generated Hilbert module over $A$ (with a fixed admissible scalar product) and $\alpha$ is a self-adjoint positive operator ($\alpha^* = \alpha$, $\alpha > 0$), commuting with the action of the von Neumann algebra $A$.

Since we will use this fact a few times in this paper, we will present a complete proof.

Let $\mathcal{X} = (\beta : A' \rightarrow A)$ be an arbitrary representation of a torsion virtual Hilbertian module $\mathcal{X}$ with $\beta$ injective. Then the image of $\beta$ is dense in $A$. Consider arbitrary admissible scalar products $\langle , \rangle_A$ and $\langle , \rangle_{A'}$ on $A$ and $A'$ correspondingly (cf. 1.2). This choice determines uniquely a continuous self-adjoint positive operator $S : A' \rightarrow A'$ such that

$$\langle \beta(x), \beta(y) \rangle_A = \langle Sx, y \rangle_{A'}$$

for all $x, y \in A'$. It follows that this operator $S$ is injective and commutes with the action of the von Neumann algebra $A$; the image of $S$ is dense.
Let \( T = S^{1/2} : A' \to A' \) be the positive square root of \( S \). Then \( T \) is injective with dense image and commuting with the action of \( A \). We obtain that for all \( x, y \in A' \)
\[
\langle \beta(x), \beta(y) \rangle_A = \langle Tx, Ty \rangle_{A'},
\]
and thus the map \( U = \beta \circ T^{-1} \) is an isometry \( T(A') \to \text{im}(\beta) \); this isometry can be uniquely extended to an isometry \( U : A' \to A \). We obtain that the torsion virtual Hilbertian module
\[
(\alpha = \beta \circ U^{-1} : A \to A)
\]
is isomorphic to \((\beta : A' \to A)\) and thus, it gives a self-adjoint representation of \( \mathcal{X} \).

The above argument also proves the following useful fact:

4.2. Corollary. If \((\beta : A' \to A)\) is a representation of a torsion virtual Hilbertian module by an injective morphism \( \beta \) then the Hilbertian modules \( A \) and \( A' \) are isomorphic.

We may apply the spectral theorem to (27) to get the representation
\[
\alpha = \int_0^\infty \lambda dE_\lambda
\]
where \( E_\lambda \) are the self-adjoint projectors which commute with \( \alpha \) and with the action of the algebra \( A \).

4.3. Given a torsion Hilbertian module \( \mathcal{X} \), it admits a self-adjoint representations of the form \((\alpha : A \to A)\) with \( \dim_r(A) \) being arbitrarily small.

Proof. Start from an arbitrary self-adjoint representation \( \mathcal{X} = (\alpha : A \to A) \). For any \( \epsilon > 0 \) there exists \( \delta > 0 \) such that \( \dim_r(E_\delta A) < \epsilon \). Denote \( A_\delta = E_\delta A \) and let \( \alpha_\delta : A_\delta \to A_\delta \) acts as \( \alpha \). Then we have
\[
(\alpha : A \to A) \simeq (\alpha_\delta : A_\delta \to A_\delta)
\]
are isomorphic as objects of \( \mathcal{E}(A) \) (since one is obtained from the other by excision (cf. 2.4) with respect to the submodule \((1 - E_\delta)A \subset A\). \( \square \)

4.4. Suppose that a torsion Hilbertian module \( \mathcal{X} \) in a self-adjoint representation \( \mathcal{X} = (\alpha : A \to A) \) is given. Denote
\[
F(\lambda) = \dim_r(E_\lambda A),
\]
where \( E_\lambda \) is the spectral projector determined by (28). Clearly, \( F(\lambda) \) is a monotone non-decreasing right continuous function, defined for \( \lambda \geq 0 \); note that \( F(\lambda) > 0 \) for \( \lambda > 0 \) and \( F(0) = 0 \). \( F(\lambda) \) will be called the spectral density function of the torsion module \( \mathcal{X} \). This notion was introduced by S.P. Novikov and M.A. Shubin [NS].

S.P. Novikov and M.A. Shubin [NS] also found an appropriate equivalence relation between the spectral density functions, called dilatational equivalence, cf. also [GS], and [LL]. It turns out that up to this equivalence relation the spectral density function is an invariant of a torsion Hilbertian module.

Let’s recall this notion. Two spectral density functions \( F(\lambda) \) and \( G(\lambda) \) are called dilatationally equivalent if there exist constants \( C > 1 \) and \( \epsilon > 0 \) such that
\[
G(C^{-1}\lambda) \leq F(\lambda) \leq G(C\lambda)
\]
holds for all \( \lambda \in [0, \epsilon] \).
4.5. Proposition. The spectral density function, considered up to dilatational equivalence, is an invariant of isomorphism class in \( \mathcal{E}(A) \) of a torsion Hilbertian module.

Proof. First observe that the equivalence class of spectral density function \( F(\lambda) \) of torsion Hilbertian module \( \mathcal{X} = (\alpha : A' \to A) \) does not depend on the choice of admissible scalar products on \( A \) and on \( A' \) which are used in the definition of \( F(\lambda) \). This easy follows from results of §1 of [LL].

Now, considering only injective representations of torsion modules (which does not restrict generality by §2.4), we observe that morphisms between \( (\alpha : A' \to A) \) and \( (\beta : B' \to B) \) in the abelian category \( \mathcal{E}(A) \) (according to definition 2.2) coincide with homotopy classes of morphisms between \( (\alpha : A' \to A) \) and \( (\beta : B' \to B) \) (which are now considered as short chain complexes). Thus, isomorphism in \( \mathcal{E}(A) \) corresponds to homotopy equivalence of the chain complexes. Hence to complete the proof we may use the result of Gromov and Shubin [GS], corollary 2.6, which states equivalence of the spectral density functions of chain homotopy equivalent chain complexes. \( \square \)

4.6. Definition. The Novikov-Shubin invariant of a non-trivial torsion Hilbertian module \( \mathcal{X} \) is defined as

\[
ns(\mathcal{X}) = \lim_{\lambda \to 0^+} \inf \frac{\ln F(\lambda)}{\ln \lambda} \in [0, \infty]
\] (29)

4.7. Observe, that if \( \mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2 \) is direct sum of two torsion objects, then the spectral density function \( F(\lambda) \) of \( \mathcal{X} \) is the sum

\[
F(\lambda) = F_1(\lambda) + F_2(\lambda)
\] (30)

of the spectral density functions \( F_1(\lambda) \) and \( F_2(\lambda) \) of \( \mathcal{X}_1 \) and \( \mathcal{X}_2 \), respectively. It follows that

\[
ns(\mathcal{X}_1 \oplus \mathcal{X}_2) = \min\{ns(\mathcal{X}_1), ns(\mathcal{X}_2)\}. \quad (31)
\]

4.8. The results of [LL] seem to suggest that a more convenient numerical invariant of torsion Hilbertian modules \( \mathcal{X} \) is given by

\[
c(\mathcal{X}) = ns(\mathcal{X})^{-1}.
\]

I will call this number the capacity of \( \mathcal{X} \).

Then formula (31) can be rewritten as

\[
c(\mathcal{X}_1 \oplus \mathcal{X}_2) = \max\{c(\mathcal{X}_1), c(\mathcal{X}_2)\} \quad (32)
\]

4.9. Proposition. For a short exact sequence of torsion virtual Hilbertian modules

\[
0 \to \mathcal{X}_1 \to \mathcal{X} \to \mathcal{X}_2 \to 0
\]

holds

\[
\max\{c(\mathcal{X}_1), c(\mathcal{X}_2)\} \leq c(\mathcal{X}) \leq c(\mathcal{X}_1) + c(\mathcal{X}_2). \quad (33)
\]
**Proof.** Using Corollary 2.7, we may represent the inclusion \( X_1 \rightarrow X \) by a commutative diagram

\[
\begin{array}{ccc}
(A' & \xrightarrow{\alpha} & A) \\
\downarrow_{h} & & \downarrow_{f} \\
(B' & \xrightarrow{\beta} & B)
\end{array}
\]

with \( h \) isomorphism and \( \alpha, \beta, f \) monomorphisms with dense images. By Corollary 4.2 all Hilbertian modules \( A', A, B', B \) are isomorphic. Thus, we may represent the inclusion \( X_1 \rightarrow X \) by a commutative diagram of the form

\[
\begin{array}{ccc}
(C & \xrightarrow{\alpha} & C) = X_1 \\
\downarrow_{\text{id}} & & \downarrow_{\gamma} \\
(C' & \xrightarrow{\beta} & C) = X
\end{array}
\]

where \( \alpha, \beta, \gamma \) are injective morphisms \( C \rightarrow C \) with dense images. Here \( \beta = \gamma \circ \alpha \).

Applying Proposition 2.3, we obtain that the cokernel of the map \( X_1 \rightarrow X \) can be represented by \( X_2 = (\gamma : C \rightarrow C) \).

Now our statement follows from Lemma 1.12 of [LL], which establishes inequalities between the Novikov-Shubin numbers of the maps \( \alpha, \gamma \) and \( \beta = \gamma \circ \alpha \). \( \square \)

Note that Theorem 2.3 of [LL] follows from our Propositions 3.6 and 4.9.

**4.10. Corollary.** Fix a number \( \nu \in [0, \infty) \) and let \( \varell_{\nu}(A) \) denote the full subcategory of \( \varcal{E}(A) \) whose class of objects constitute all torsion Hilbertian modules \( X \) over \( A \) with capacity less or equal to \( \nu \). Then \( \varell_{\nu}(A) \) is an abelian category.

Note that \( \varell_{\infty}(A) = \varell(A) \) and \( \varell_{0}(A) \) is, generally, not empty. The categories \( \varell_{\nu}(A) \) form a chain of abelian subcategories.

**§5. Some examples**

Here we will discuss some interesting examples demonstrating the properties of torsion Hilbertian modules.

**5.1.** Consider the group ring \( \mathbb{C}[\mathbb{Z}] \) of the infinite cyclic group \( \mathbb{Z} \). The group \( \mathbb{Z} \), lying in \( \mathbb{C}[\mathbb{Z}] \), forms an orthonormal base of a scalar product \( \langle \cdot, \cdot \rangle \) on \( \mathbb{C}[\mathbb{Z}] \); completing this scalar product, we obtain a Hilbert space \( \ell^2(\mathbb{Z}) \) with the action of the group ring \( \mathbb{C}[\mathbb{Z}] \). The von Neumann algebra \( A = \varcal{N}(\mathbb{Z}) \) is the commutant of this action. The von Neumann trace on \( A \) is given by

\[
\tau(a) = \langle a \cdot 1, 1 \rangle,
\]

where \( 1 \in \ell^2(\mathbb{Z}) \) is the unit of the group ring \( \mathbb{C}[\mathbb{Z}] \) considered as an element of \( \ell^2(\mathbb{Z}) \).

We are going to compute examples in the extended abelian category \( \varcal{E}(A) \).

Using Fourier decompositions, one may identify \( \ell^2(\mathbb{Z}) \) with the space of square integrable function on the circle \( S^1 \). Then the group ring \( \mathbb{C}[\mathbb{Z}] \) will be identified with the space of Laurent polynomials acting on \( L^2(S^1) \) by multiplication. The von Neumann algebra \( A = \varcal{N}(\mathbb{Z}) \) will be identified with the space \( L^\infty(S^1) \) of essentially
bounded complex functions, acting by multiplication on $L^2(S^1)$. The von Neumann trace $\tau$ on $A$ coincides then with the Lebesgue integral:

$$\tau(f) = \int_{S^1} f(z)dz \quad \text{for} \quad f \in L^\infty(S^1).$$

Here $z$ denotes the coordinate along the circle, $z = \exp(i\theta)$.

**5.2.** Fix a point $z_0 = \exp(i\theta_0)$ on the unit circle and a positive number $\nu > 0$. Consider the function

$$f(z) = |z - z_0|^{\nu} \in L^\infty(S^1).$$

It determines the linear bounded self-adjoint operator

$$\alpha : L^2(S^1) \to L^2(S^1)$$

given by the multiplication on $f$, i.e. $\alpha(\phi) = f\phi$. Obviously, $\alpha$ commutes with the action of the von Neumann algebra $A = L^\infty(S^1)$. This operator is injective, but it is not invertible, since the function $|z - z_0|^{-\nu}$ is not essentially bounded. Denote by $\mathcal{X}_{\nu,\theta_0}$ the following torsion Hilbertian module

$$\mathcal{X}_{\nu,\theta_0} = (\alpha : L^2(S^1) \to L^2(S^1)). \quad (34)$$

**5.3.** Computing the spectral density function $F(\lambda)$ of $\mathcal{X}_{\nu,\theta_0}$, we obtain (using the definition in 4.4 and also the remarks in 5.1):

$$F(\lambda) = \mu\{z \in S^1; |z - z_0|^{\nu} < \lambda\}$$

$$= \mu\{z \in S^1; |z - z_0|^2 < \lambda^{2/\nu}\} \quad (35)$$

$$= \pi^{-1}\cos^{-1}(1 - 1/2\lambda^{2/\nu}).$$

Here $\mu$ denotes the Lebesgue measure. Using the fact that

$$\lim_{x \to 0} \frac{\cos^{-1}(1-x)}{\sqrt{2x}} = 1,$$

we obtain that the Novikov-Shubin invariant of $\mathcal{X}_{\nu,\theta_0}$ equals

$$\text{ns}(\mathcal{X}_{\nu,\theta_0}) = \frac{1}{\nu}, \quad (36)$$

and for the capacity of $\mathcal{X}_{\nu,\theta_0}$ we obtain

$$\epsilon(\mathcal{X}_{\nu,\theta_0}) = \nu. \quad (37)$$
5.4. Observe that the spectral density function of $\mathcal{X}_{\nu,\theta_0}$ depends only on $\nu$ and does not depend on the angle $\theta_0$.

Let us prove that for different angles $\theta_0 \neq \theta_1$ the torsion Hilbertian modules $\mathcal{X}_{\nu,\theta_0}$ and $\mathcal{X}_{\nu,\theta_1}$ are not isomorphic in $\mathcal{E}(A)$.

In fact, we will now show that for $\theta_0 \neq \theta_1$ any morphism

$$\mathcal{X}_{\nu,\theta_0} \rightarrow \mathcal{X}_{\nu,\theta_1}$$

is zero.

It is clear that any morphism $\mathcal{X}_{\nu,\theta_0} \rightarrow \mathcal{X}_{\nu,\theta_1}$ can be presented by a commutative diagram

$$\begin{array}{ccc}
L^2(S^1) & \xrightarrow{|z-z_0|^\nu} & L^2(S^1) \\
\downarrow m_g & & \downarrow m_f \\
L^2(S^1) & \xrightarrow{|z-z_1|^\nu} & L^2(S^1)
\end{array}$$

(38)

where $z_1 = \exp(i\theta_1)$ and $m_f$ and $m_g$ are operators of multiplication by some functions $f, g \in L^\infty(S^1)$. Commutativity of this diagram means that

$$|z-z_0|^\nu f(z) = g(z)|z-z_1|^\nu,$$

which can be rewritten as

$$\frac{g(z)}{|z-z_0|^\nu} = \frac{f(z)}{|z-z_1|^\nu} \equiv h(z). \quad (39)$$

Now we see that the function $h(z)$ is essentially bounded, $h \in L^\infty(S^1)$, since from the representation as the first fraction in (39) we obtain that $h(z)$ is essentially bounded everywhere except a neighbourhood of $z_0$ and, similarly, from representation as the second fraction in (39) we obtain that $h(z)$ is essentially bounded everywhere except a neighbourhood of $z_1$.

The multiplication by $h$ defines a morphism $m_h : L^2(S^1) \rightarrow L^2(S^1)$ and $m_f = m_h \cdot |z-z_1|^\nu$. Now, by the definition of morphisms of the category $\mathcal{E}(A)$ (cf. 2.2) we get that the morphism $\mathcal{X}_{\nu,\theta_0} \rightarrow \mathcal{X}_{\nu,\theta_1}$ under consideration, vanishes.

5.5. Corollary. The spectral density function and the Novikov-Shubin invariant $\text{ns}(\mathcal{X})$ do not determine the isomorphism type of a torsion Hilbertian module $\mathcal{X}$.

The constructed example suggests the notion of "local Novikov-Shubin invariants" which would measure the spectral density of the $\theta$-"local part" of a torsion Hilbertian module for any angle $\theta$. We are going to discuss this subject in another place.

5.6. Observe also that for equal angle $\theta_0$ and different values of $\nu$ the torsion modules $\mathcal{X}_{\nu,\theta_0}$ do admit nontrivial maps. For example, let’s mention that for $\nu' > \nu$ there is an exact sequence of torsion modules

$$0 \rightarrow \mathcal{X}_{\nu,\theta_0} \rightarrow \mathcal{X}_{\nu',\theta_0} \rightarrow \mathcal{X}_{\nu-\nu',\theta_0} \rightarrow 0$$

and this sequence does not split (since we have $c(\mathcal{X}_{\nu',\theta_0}) = c(\mathcal{X}_{\nu,\theta_0}) + c(\mathcal{X}_{\nu-\nu',\theta_0})$, cf. 4.7).

This gives an example of a torsion module containing a chain of submodules having continuum members.
§6. Extended $L^2$ homology and cohomology of cell complexes

Here we will define cohomological functors from the category of polyhedra to the extended abelian category $\mathcal{E}(A)$; we will call them extended $L^2$ homology and cohomology. We will see that they are homotopy invariant. The projective part of the extended $L^2$ cohomology coincides with the reduced $L^2$ cohomology; the torsion part of the extended $L^2$ cohomology determines the Novikov-Shubin invariants. Thus, this general homotopy invariant construction gives a very transparent (and short!) proof of homotopy invariance of both, the reduced $L^2$-cohomology (the fact, proven by J. Dodziuk in [D]), and also homotopy invariance of the Novikov-Shubin invariants (established by M. Gromov and M. A. Shubin in [GS]).

Note, that the isomorphism type of the extended $L^2$ cohomology is not determined by the von Neumann Betti numbers and the Novikov-Shubin invariants, as examples in §5 show.

We prove in this section the universal coefficients theorem expressing the extended $L^2$ homology through cohomology and vice versa. We also prove the version of Poincaré duality.

6.1. Let $X$ be a connected CW complex having finitely many cells in each dimension, and let $\pi = \pi_1(X)$ be its fundamental group. We denote by $C_*(\widetilde{X})$ the chain complex of the universal covering $\widetilde{X}$ of $X$ generated by the lifts of cells of $X$. It is a complex of free left $\mathbb{C}[\pi]$-modules.

Let $A$ be a finite von Neumann algebra with a fixed finite, normal, and faithful trace $\tau$ and let $M$ be a finitely generated Hilbertian module over $A$ (cf. §1). Suppose that a representation

$$\rho : \pi \rightarrow \mathcal{B}_A(M)$$

is given; here $\mathcal{B}_A(M)$ denotes the commutant of $M$, i.e. the space of all linear bounded operators $M \rightarrow M$ commuting with the action of $A$. We will say that any such $M$, carrying the above structures, is a $(A - \pi)$-Hilbertian bimodule.

The chain complex

$$M \otimes_\pi C_*(\widetilde{X})$$

can be considered as a projective complex in the extended abelian category $\mathcal{E}(A)$. Denote by

$$\mathcal{H}_i(X, M), \quad i = 0, 1, 2 \ldots$$

the homology of this complex; it is an object of $\mathcal{E}(A)$. We will call $\mathcal{H}_i(X, M)$ the extended $L^2$ homology of $X$ with coefficients in $M$.

Observe, that as an object of the extended category, $\mathcal{H}_i(X, M)$ is a direct sum of its projective and torsion parts. Using the definition (cf. 3.3), we see that the projective part $P(\mathcal{H}_i(X, M))$ equals to the reduced $L^2$ homology (defined as the factor-module of the cycles of the complex $M \otimes_\pi C_*(\widetilde{X})$ divided by the closure of the submodule of boundaries). The torsion part of $\mathcal{H}_i(X, M)$ is isomorphic to the torsion part of the following virtual Hilbertian module

$$(d_{i+1} : M \otimes_\pi C_{i+1}(\widetilde{X}) \rightarrow M \otimes_\pi C_i(\widetilde{X})),$$

(cf. 3.7).
To define extended $L^2$ cohomology, we will assume that $M$ satisfies the similar conditions as above, with the following modifications: the von Neumann algebra $\mathcal{A}$ acts on $M$ from the right and the group $\pi$ acts on $M$ from the left, so that $M$ is a $(\pi - \mathcal{A})$-Hilbertian bimodule. Then we form the complex $\text{Hom}_\pi(C_*(\bar{X}), M)$. The von Neumann algebra $\mathcal{A}$ acts on this complex from the right; let us transform this right action into left action by using the involution of $\mathcal{A}$, i.e. by the rule $\lambda \cdot x = x \cdot \lambda^*$. After this has been done, we obtain a projective cochain complex in the extended category $\mathcal{E}(\mathcal{A})$ which we denote $\text{Hom}_\pi(C_*(\mathcal{A}), M)$. Let

$$\mathcal{H}^i(X, M), \quad i = 0, 1, 2, \ldots,$$

(41)
denote the cohomology (understood in $\mathcal{E}(\mathcal{A})$) of this cochain complex; these virtual Hilbertian modules will be called extended $L^2$ cohomology of $X$ with coefficients in $M$.

Again, the projective part $P(\mathcal{H}^i(X, M))$ of the extended cohomology coincides with the reduced $L^2$ cohomology, which uses the closure of the space of coboundaries in its definition.

There are two natural constructions leading to extended $L^2$ homology and cohomology.

6.2. Example. Given a connected polyhedron $X$, having finitely many cells in each dimension, with $\pi = \pi_1(X)$, consider the $L^2$-completion of the group ring $M = \ell^2(\pi)$ with the right action of $\pi$ (right regular representation) and with the left action of the von Neumann algebra $\mathcal{N}(\pi)$ of $\pi$. Recall that the von Neumann algebra $\mathcal{N}(\pi)$ consists of all bounded linear maps $\ell^2(\pi) \to \ell^2(\pi)$ commuting with the action of $\pi$ from the right. Thus, we may form the extended $L^2$ homology $\mathcal{H}^i(X, \ell^2(\pi))$ (viewing $\ell^2(\pi)$ as a $(\mathcal{N}(\pi) - \pi)$-Hilbertian bimodule) and the extended $L^2$ cohomology $\mathcal{H}^i(X, \ell^2(\pi))$ (if we will consider $\ell^2(\pi)$ as a $(\pi - \mathcal{N}(\pi))$-Hilbertian bimodule).

6.3. Example. A slightly more general construction consists in the following. Suppose that $X$ is a connected polyhedron with finitely many cells in each dimension, supplied with a finite dimensional representation $\pi = \pi_1(X) \to \text{End}_\mathbb{C}(V)$. Consider $V$ as a right $\mathbb{C}[\pi]$-module via this representation. Then the tensor product $\ell^2(\pi) \otimes_\mathbb{C} V$ has a natural $(\mathcal{N}(\pi) - \pi)$-Hilbertian bimodule structure, where the right $\pi$-action is the diagonal one. Thus, we may consider the extended homology of $X$ with coefficients in $M = \ell^2(\pi) \otimes_\mathbb{C} V$. This is essentially the previous construction, twisted by a flat finite dimensional bundle $V$ over $X$. Similar constructions were studied by I.M. Singer in [S].

6.4. Theorem. (Homotopy invariance) Let $X$ and $Y$ be two connected cell complexes having finitely many cells in each dimension, and let $f : X \to Y$ be a homotopy equivalence. Identify the fundamental groups of $X$ and $Y$ via the induced isomorphism $f_* : \pi = \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$, where $x_0 \in X$ is the base point. Then for any $(\mathcal{A} - \pi)$-Hilbertian bimodule $M$ (cf. 6.1), the homotopy equivalence $f$ induces natural isomorphisms of virtual Hilbertian modules of extended $L^2$ homology

$$f_* : \mathcal{H}_i(X, M) \to \mathcal{H}_i(Y, M), \quad i = 0, 1, 2, \ldots.$$
Similarly, for any \((\pi - A)\)-Hilbertian bimodule \(M\), the homotopy equivalence \(f\) induces natural isomorphisms of virtual Hilbertian modules of extended \(L^2\) cohomology

\[
f^* : \mathcal{H}^i(Y, M) \to \mathcal{H}^i(X, M), \quad i = 0, 1, 2, \ldots
\]

**Proof.** The induced chain map \(f^* : C^*(\tilde{X}) \to C^*(\tilde{Y})\) is a chain homotopy equivalence. Applying the functor \(M\otimes_{\pi}\), we obtain that the chain map id \(\otimes f^* : M \otimes_{\pi} C^*(\tilde{X}) \to M \otimes_{\pi} C^*(\tilde{Y})\) is a chain homotopy equivalence between chain complexes in abelian category \(\mathcal{E}(\mathcal{A})\). Then the induced map on the homology is an isomorphism.

Similarly one gets the result concerning cohomology. \(\square\)

Isomorphism of Hilbertian modules induces isomorphism of their projective and torsion parts. Thus, the above theorem essentially states that both the projective and torsion parts of the extended \(L^2\) homology and cohomology are homotopy invariant. For the projective part this gives the result of J.Dodziuk [D]. For the torsion part this fact combined with Proposition 4.5 implies homotopy invariance of the Novikov-Shubin numbers and the spectral density functions (cf. [GS]).

**6.5. Corollary (Homotopy invariance of Novikov-Shubin invariants).** Let \(X\) be a connected polyhedron having finitely many cells in each dimension, with fundamental group \(\pi\), and let \(M\) be a Hilbertian \((A - \pi)\)-bimodule (cf. 6.1). Then for any \(i = 0, 1, 2, \ldots\) the spectral density function (considered up to dilatational equivalence) and the Novikov-Shubin number of the torsion submodule of

\[
(id \otimes d : M \otimes_{\pi} C_{i+1}(\tilde{X}) \to M \otimes_{\pi} C_i(\tilde{X}))
\]

are homotopy invariants of \(X\).

Note, that the original definition of the Novikov-Shubin numbers, given in [NS], uses the spectral density functions of the Laplacians, acting on smooth forms. A.Efremov [E] proved equivalence of the above ”analytic” definition to the ”combinatorial” definition which uses the spectral density functions of the Laplacians acting on the chain complexes, constructed by the cell decomposition of \(X\). J.Lott and W.Lück [LL] showed that instead of the Laplacian, one may consider the boundary homomorphism itself.

Comparing our notations with notations of J.Lott and W.Lück in [LL], we note the that there is a shift of dimensions: number ns\((T(\mathcal{H}_i(C)))\) in notations of [LL] is \(\alpha_{i+1}(C)\).

The extended \(L^2\) homology and cohomology determine each other as the following theorem shows:

**6.6. Theorem (Universal coefficients).** Let \(X\) be a polyhedron with finitely many simplexes in each dimension, and let \(M\) be a Hilbertian \((\pi - A)\)-bimodule. Then the dual module \(M^*\) is naturally defined as Hilbertian \((\pi - A)\)-bimodule and there are canonical isomorphisms for the projective and torsion parts of the extended \(L^2\) homology and cohomology

\[
P(\mathcal{H}_i(X, M^*)) \simeq P(\mathcal{H}_i(X, M))^*.
\]
This follows from Theorem 3.9 applied to the projective chain complex $C = M \otimes \pi C_*(\tilde{X})$. Then the dual cochain complex is $C^* = \text{Hom}_\pi(C_*(X), M^*)$ and the statement follows. □

There is also Poincaré duality relating the extended $L^2$ homology and cohomology of complementary dimensions. Here is the precise statement.

6.7. Theorem (Poincaré duality). Let $X$ be an $n$-dimensional closed PL manifold (or, more generally, Poncaré complex) and let $w : \pi = \pi_1(X) \to \{+1, -1\}$ denote the first Stifel-Whitney class of $X$. Let $M$ be a Hilbertian $(\mathcal{A} - \pi)$-bimodule. Denote by $M^w$ the following $(\pi - \mathcal{A})$-Hilbertian bimodule structure on $M$: the right action of $\pi$ on $M$ is given by $g \cdot m = w(g)m \cdot g^{-1}$ where $g \in \pi$, $m \in M$, and the left action of $\mathcal{A}$ on $M$ is given by $m \cdot a = a^* \cdot m$ for $a \in \mathcal{A}$, $m \in M$. Then there exists a natural isomorphism of virtual Hilbertian modules

$$\mathcal{H}_i(X, M) \to \mathcal{H}^{n-i}(X, M^w), \quad i = 0, 1, 2, \ldots, n. \quad (45)$$

Proof. Denote by $C$ the chain complex of the universal covering $\tilde{X}$ of $X$, $C = C_*(\tilde{X})$. It is a chain complex of free finitely generated left $\mathbb{Z}[\pi]$-modules. Let $C^*$ be the following cochain complex $C^* = \text{Hom}_\pi(C, \mathbb{Z}[\pi])$ of free finitely generated right $\mathbb{Z}[\pi]$-modules. Let $(C^*)^D$ denote the chain complex of left $\mathbb{Z}[\pi]$-modules, which is obtained from $C^*$, first, by enumeration of the dimensions in the opposite way: $(C^*)^D = (C^*)^{n-i}$ and, secondly, by transforming the right action of $\pi$ into a left action by using the following involution of the group ring $\mathbb{Z}[\pi]: g \mapsto w(g)g^{-1}$ for $g \in \pi$.

The Poincaré duality theorem (cf. for example, [M]) states that there exists natural chain homotopy equivalence $C \to (C^*)^D$. Tensoring it with $M$, we obtain a chain homotopy equivalence

$$M \otimes \pi C \to M \otimes \pi (C^*)^D$$

between projective chain complexes in $\mathcal{E}(\mathcal{A})$. The $i$-dimensional extended $L^2$ homology of the last chain complex can be identified with $\mathcal{H}^{n-i}(X, M^w)$. This implies the theorem. □

Combining this theorem with Theorem 6.6 we obtain:

6.8. Corollary. If $X$ as an $n$-dimensional closed PL manifold (or Poincaré complex) with first Stifel-Whitney class $w$ then for arbitrary Hilbertian $(\mathcal{A} - \pi)$-bimodule $M$ there are natural isomorphisms for the projective and torsion parts

$$P(\mathcal{H}_i(X, M)) \simeq P(\mathcal{H}_{n-i}(X, M^{*w})), \quad i = 0, 1, 2, \ldots, n, \quad (46)$$

$$T(\mathcal{H}_i(X, M)) \simeq \varepsilon(T(\mathcal{H}_{n-i-1}(X, M^{*w}))), \quad i = 0, 1, 2, \ldots, n - 1. \quad (47)$$

The first isomorphism uses the duality for projective objects (cf. 1.6), while the second isomorphism uses the duality for torsion objects (cf. 3.8). Here $M^{*w}$ denotes the Hilbertian $(\mathcal{A} - \pi)$-bimodule which is obtained from the Hilbertian $(\pi - \mathcal{A})$-bimodule $M^*$ by using the involution of $\mathcal{A}$ and the involution on $\pi$ determined by $w$ (cf. above).
6.9. Note that $T(\mathcal{H}_n(X, M)) = 0$ for any $n$-dimensional polyhedron $X$.

6.10. Another observation: $\mathcal{H}^0(X, M) = 0$ for any Hilbertian $(\pi - \mathcal{A})$-bimodule $M$ satisfying the condition: there is no $m \in M$, $m \neq 0$, such that $g \cdot m = m$ for any $g \in \pi$. In fact, it is easy to see that $\mathcal{H}^0(X, M)$ has no torsion and it is isomorphic to

$$\{m \in M; gm = m \text{ for all } g \in \pi\}$$

which has natural structure of Hilbertian module. It is not difficult to find examples when the last module is nonzero.

It follows, in particular, that $\mathcal{H}^0(X, \ell^2(\pi)) = 0$ if $\pi$ is infinite.

On the contrary, the extended $L^2$ homology in dimension zero $\mathcal{H}_0(X, M)$ has non-trivial torsion part in many cases. More precisely, theorem of R.Brooks [B] states that $\mathcal{H}_0(X, \ell^2(\pi))$ has nontrivial torsion part if and only if the fundamental group $\pi$ is amenable.

§7. Minimal number of generators

In this section we will introduce and study a new numerical invariant of Hilbertian modules, which we call the minimal number of generators. Considered on projective modules, it is comparable with the von Neumann dimension, although it is always integral and may be much larger, than the dimension. Unlike the von Neumann dimension, this invariant is also nontrivial on torsion modules. We show in this section (by computing examples) that this new invariant is independent from the Novikov-Shubin invariant.

We will apply this invariant in the next section to the Morse theory.

We denote by $\mathcal{A}$ a fixed finite von Neumann algebra supplied with a finite, normal and faithful trace $\tau$. We consider categories $\mathcal{H}(\mathcal{A})$ and $\mathcal{E}(\mathcal{A})$, introduced in §1 and §2. Recall, that they actually depend on $\tau$ as well, although in our notation it is suppressed.

7.1. First, observe that the Hilbertian module $\ell^2(\mathcal{A})$ is a fixed generator of the category $\mathcal{E}(\mathcal{A})$ in the sense of the category theory [F]. It means, in this particular case, that any object $\mathcal{X}$ of category $\mathcal{E}(\mathcal{A})$ is factor-object of a finite direct sum of the form $\oplus \ell^2(\mathcal{A})$.

Recall that a Hilbertian module is called free if it is a direct sum of several copies of $\ell^2(\mathcal{A})$. A chain complex is called free if it consists of free modules.

We arrive at the following definition.

7.2. Definition. Let $\mathcal{X}$ be a virtual Hilbertian module. We will denote by $\mu(\mathcal{X})$ the minimal integer $\mu$ such that there exists an epimorphism of the direct sum of $\mu$ copies of $\ell^2(\mathcal{A})$ onto $\mathcal{X}$. We will call $\mu(\mathcal{X})$ the minimal number of generators of $\mathcal{X}$.

Obviously, $\mu(\mathcal{X}) = 0$ if and only if $\mathcal{X} = 0$.

If $\mathcal{X}$ is projective, then $\mu(\mathcal{X}) \geq \dim_\tau(\mathcal{X})$.

We also have the following property

$$\max\{\mu(\mathcal{X}), \mu(\mathcal{Y})\} \leq \mu(\mathcal{X} \oplus \mathcal{Y}) \leq \mu(\mathcal{X}) + \mu(\mathcal{Y}).$$

(48)

Examples below show that both extreme cases allowed by this inequality can be realized.
7.3. Example. Let $\mathcal{A} = \mathcal{N}(\mathbb{Z})$ be the von Neumann algebra of the infinite cyclic group, cf. 5.1. Let $\mathcal{X}_{\nu,\theta}$ be the torsion module constructed in 5.2. Then obviously
\[ \mu(\mathcal{X}_{\nu,\theta}) = 1. \] (49)

7.4. Example. Here we will present a more interesting example. Fix an arbitrary $\nu > 0$ and an angle $\theta$. Let $\mathcal{X}$ denote the direct sum of $n$ copies of $\mathcal{X}_{\nu,\theta}$.

Claim. The minimal number of generators of $\mathcal{X}$ equals to $n$.

Proof. Let $F$ denote the direct sum of $n$ copies of $\ell^2(\mathcal{A})$. Obviously, $\mathcal{X}$ has a representation $\mathcal{X} = (\alpha : F \to F)$, where $\alpha$ is given by a diagonal $n \times n$-matrix with the functions $|z - z_0|^{\nu}$ standing along the diagonal (here $z_0 = \exp(i\theta)$). We identify the von Neumann algebra $\mathcal{N}(\mathbb{Z})$ with $L^\infty(S^1)$ and $\ell^2(S^1)$ with $L^2(S^1)$, cf. §5. $z$ denotes the coordinate along the circle.

We have to show that there is no epimorphism $[f] : F' \to \mathcal{X}$ if $F'$ is a direct sum of $m$ copies of $\ell^2(\mathcal{A})$ with $m < n$. Suppose that such epimorphism exists; it is then represented by a diagram

\[ \begin{array}{ccc}
F' & \xrightarrow{f} & F \\
\downarrow & & \downarrow \\
\mathcal{X} & = & (\alpha : F \longrightarrow F). 
\end{array} \]

By 2.5, we obtain that the morphism $[\alpha, f] : F \oplus F' \to F$ is an epimorphism in $\mathcal{H}(\mathcal{A})$. Then there exists a splitting (cf. 1.5)

\[ F \oplus F' \xleftarrow{\begin{bmatrix} \beta \\ \gamma \end{bmatrix}} F, \]

so that
\[ \alpha \beta + f \gamma = \text{id}_F. \] (50)

Note that the morphisms $\alpha$, $\beta$, $\gamma$, $f$ are represented by rectangular matrices with entries in the von Neumann algebra $\mathcal{A} = L^\infty(S^1)$. Denote by $\chi : L^\infty(S^1) \to \mathbb{C}$ any multiplicative homomorphism, which has the following property: for any continuous function $\phi \in C(S^1) \subset L^\infty(S^1)$, one has $\chi(\phi) = \phi(z_0)$. Such multiplicative functional exists by virtue of §12 of [GRS].

Now, applying $\chi$ to the matrix equation (50), we obtain (since $\chi(\alpha) = 0$) the equality
\[ \chi(f)\chi(\gamma) = 1_{n \times n}. \] (51)

In the last equation, $\chi(f)$ is a $n \times m$ complex matrix and $\chi(\gamma)$ is an $m \times n$-matrix; (51) is impossible, if $m < n$. □

7.5. Example. Suppose now that the angles $\theta_1$ and $\theta_2$ are different. Let $\nu$ and $\nu'$ be two arbitrary positive numbers.
Claim.

\[ \mu(\mathcal{X}_{\nu,\theta_1} \oplus \mathcal{X}_{\nu',\theta_2}) = 1. \]

To prove this it is enough to show that the diagram

\[
\begin{array}{c}
L^2(S^1) \\
\alpha \downarrow \\
\beta \alpha \\
\end{array}
\]

represents an epimorphism in category \( E(A) \). Here \( \alpha \) denotes the operator of multiplication by \(|z - z_1|^\nu\), with \( z_1 = \exp(i\theta_1) \), and \( \beta \) denotes the operator of multiplication by \(|z - z_2|^\nu'\), with \( z_2 = \exp(i\theta_2) \). Decompose the circle into the union of two intervals, such that one of the intervals contains the point \( z_1 \) in its interior, and the other interval contains \( z_2 \) in its interior. Let \( \chi_1(z) \) be the characteristic function of the interval containing \( z_1 \), and let \( \chi_2(z) \) be the characteristic function of the other interval. Then we have, \( \chi_1(z) + \chi_2(z) = 1 \) for all \( z \) (except for two end points of the intervals) and \( \chi_j^2 = \chi_j \), where \( j = 1, 2 \).

To show that the above diagram represents an epimorphism, we may use the criterion 2.5. Thus, we have to show that, given arbitrary pair \((\phi, \psi) \in L^2(S^1) \oplus L^2(S^1)\), there exist a pair \((a, b) \in L^2(S^1) \oplus L^2(S^1)\) and a function \( c \in L^2(S^1) \) such that

\[
\begin{bmatrix}
\alpha 0 \\
0 \beta
\end{bmatrix}
\begin{bmatrix}
a \\
b
\end{bmatrix}
+
\begin{bmatrix}
\beta \\
\alpha
\end{bmatrix}
\cdot
\begin{bmatrix}
c
\end{bmatrix}
\]

This can be achieved by setting

\[
\begin{aligned}
a &= \alpha^{-1} \chi_2 \phi + \beta \alpha^{-2} \chi_2 \psi \\
b &= \beta^{-1} \chi_1 \phi + \alpha \beta^{-2} \chi_1 \psi \\
c &= \beta^{-1} \chi_1 \phi + \alpha^{-1} \chi_2 \psi
\end{aligned}
\]

This proves our statement.

7.6. Comparing the numerical invariants \( ns(\mathcal{X}) \) and \( \mu(\mathcal{X}) \), we conclude that they are independent. In fact, the sum of \( n \) copies of \( \mathcal{X}_{\nu,\theta} \) has capacity \( \nu \) and its minimal number of generators is \( n \). Thus, any pair of numbers \( \nu \in \mathbb{R}^+, n \in \mathbb{N} \) can be realized.

§8. Application: Morse inequalities

In the original papers of S.P.Novikov and M.A.Shubin [NS], [NS1] (cf. also [Sh]) it was shown how one may use the notion of dimension in the sense of von Neumann, in order to improve the classical Morse inequalities for the numbers of critical points of functions on compact manifolds. We are going to show in this section that the phenomenon responsible for the Novikov-Shubin invariants (namely, measuring the "rate" of zero being in the continuous spectrum), which was discovered in the same papers [NS], [NS1] of S.P.Novikov and M.A.Shubin, may also be used to further strengthening the Morse inequalities. More precisely, we will use the numerical invariant \( \mu(\mathcal{X}) \), introduced in the previous section, and prove a version of Morse inequalities, which involves this new invariant applied to the extended \( L^2 \) homology. This approach extracts a quantitative information from the torsion part of the extended \( L^2 \) homology, as well.
8.1. Theorem. Let $\mathcal{A}$ be a finite von Neumann algebra supplied with a finite normal and faithful trace $\tau$, and let
\[ C : \cdots \to C_{i+1} \to C_i \to C_{i-1} \to \cdots \] (53)
be a free chain complex (cf. 7.1) in $\mathcal{E}(\mathcal{A})$. Then for any integer $i$ the following inequality holds:
\[ \dim_\tau(C_i) \geq \mu[\mathcal{H}_i(C) \oplus T(\mathcal{H}_{i-1}(C))]. \] (54)

Proof. If $Z_i$ denotes the submodule of $i$-dimensional cycles, then we can find a splitting
\[ C_i = Z_i \oplus X_i. \]
Let $f : C_i \to X_i$ and $f' : C_i \to Z_i$ denote the projections. The restriction of the boundary homomorphism onto $X_i$ determines an injective bounded linear map $\alpha : X_i \to B_{i-1}$ with dense image. Thus we obtain
\[ T(\mathcal{H}_{i-1}(C)) = (\alpha : X_i \to B_{i-1}). \]
By Corollary 4.2 we get that there exists an isomorphisms of Hilbertian modules $g : X_i \to B_{i-1}$. Now, the map
\[ [f' \ g f] : C_i \to Z_i \oplus B_{i-1} \]
is an epimorphism. Composing it with the obvious epimorphism
\[ Z_i \oplus B_{i-1} \to \mathcal{H}_i(C) \oplus T(\mathcal{H}_{i-1}(C)) = (C_{i+1} \to Z_i) \oplus (X_i \to B_{i-1}), \]
we obtain an epimorphism from free module $C_i$ onto the last module. This shows that
\[ \mu[\mathcal{H}_i(C) \oplus T(\mathcal{H}_{i-1}(C))] \leq \dim_\tau(C_i), \]
and completes the proof. ☐

8.2. Theorem (Morse type inequalities). Let $X$ be a closed manifold and let $f : X \to \mathbb{R}$ be a non-degenerate Morse function on $X$. Suppose that a finite dimensional representation $\pi \to \text{End}_\mathbb{C}(V)$ of the fundamental group $\pi = \pi_1(X)$ is given, and let $M$ denote the Hilbertian $(N(\pi) - \pi)$-bimodule $\ell^2(\pi) \otimes_\mathbb{C} V$, constructed in 6.3. Then for the Morse numbers $m_i(f)$ of critical points of $f$ of index $i$ the following inequalities hold:
\[ m_i(f) \geq (\dim_\mathbb{C} V)^{-1} \cdot \mu[\mathcal{H}_i(X, M) \oplus T(\mathcal{H}_{i-1}(X, M))], \quad i = 0, 1, 2, \ldots \] (55)

Proof. The theorem follows by applying the previous Theorem 8.1 to the chain complex $M \otimes_\pi C_*(X)$, where $C_*(X)$ is constructed by means of the cell decomposition of $X$, determined by the Morse function $f$. ☐

8.3. Example. Consider the simplest possible example: let $X$ be the circle $S^1$. In this case all von Neumann Betti numbers vanish and so the Morse type inequalities of Novikov and Shubin [NS], [NS1] do not predict critical points.

Let us apply Theorem 8.2 with $V$ the trivial one-dimensional representation. Then the only non-vanishing extended homology is $\mathcal{H}_0(X, M) = X_{\nu, \theta}$ (using the notation introduced in 5.2), where $\nu = 1$ and $\theta = 0$. Applying Theorem 8.2 we obtain (cf. 7.3) that any Morse function on the circle must have at least one minimum and at least one maximum!
§9. Group homology and cohomology with values in $E(A)$

We are going to define in this section some homological functors which associate to a representation of a discrete group a sequence of objects of the extended abelian category $E(A)$. The projective part of these functors coincides with the reduced $L^2$ cohomology, studied, by J.Cheeger and M.Gromov in [CG]. We denote these new functors $\mathcal{Tor}_q^\pi(N, M)$ and $\mathcal{Ext}_q^\pi(N, M)$ since their construction is similar to building the usual Tor and Ext functors.

Our main goal in this section is to express the extended $L^2$ homology and cohomology of a cell complex $X$ (defined in §6), through the homology of its universal covering of $X$; it is done with the aid of $\mathcal{Tor}$ and $\mathcal{Ext}$ functors and a spectral sequence in $E(A)$.

9.1. Let $A$ be a finite von Neumann algebra supplied with a finite, normal, and faithful trace $\tau$.

Let $\pi$ be a discrete group.

Let $M$ be a Hilbertian $(A-\pi)$-bimodule (cf. 6.1), and let $N$ be a left $C[\pi]$-module, having a free $C[\pi]$-resolution

$$\cdots \rightarrow C_{q+1} \rightarrow C_q \rightarrow C_{q-1} \rightarrow \cdots \rightarrow C_0 \overset{\varepsilon}{\rightarrow} N \rightarrow 0$$

such that $C_q$ is finitely generated over $C[\pi]$ in each dimension $q$. Applying functor $M \otimes \pi$ to (56), we obtain the following projective chain complex in $E(A)$

$$\cdots \rightarrow M \otimes \pi C_{q+1} \rightarrow M \otimes \pi C_q \rightarrow M \otimes \pi C_{q-1} \rightarrow \cdots$$

We denote by

$$\mathcal{Tor}_q^\pi(M, N)$$

its $q$-dimensional homology in $E(A)$. Since any two free resolutions of $N$ are homotopy equivalent, we obtain that $\mathcal{Tor}_q^\pi(M, N)$ is correctly defined and is a covariant functor of $N$.

Note, that if $N$ is finitely generated free over $C[\pi]$ then the tensor product $M \otimes \pi N$ is well defined as a projective object of $E(A)$, and $\mathcal{Tor}_q^\pi(M, N) \simeq M \otimes \pi N$, and $\mathcal{Tor}_p^\pi(M, N) = 0$ for $p \geq 1$.

9.2. Example. Suppose that $\pi = \mathbb{Z}$ and $N = \mathbb{Z}$ with the trivial $\pi$-action. Then

$$0 \rightarrow \mathbb{C}[\mathbb{Z}] \xrightarrow{z-1} \mathbb{C}[\mathbb{Z}] \rightarrow \mathbb{Z} \rightarrow 0$$

is a free resolution of $\mathbb{Z}$ and thus we obtain

$$\mathcal{Tor}_0^\pi(\ell^2(\mathbb{Z}), \mathbb{Z}) = \mathcal{X}_{1,0}$$

(in the notations of 5.2) and

$$\mathcal{Tor}_p^\pi(\ell^2(\mathbb{Z}), \mathbb{Z}) = 0$$

for all $p \geq 1$. 
To define a similar cohomological notion, suppose that $M$ is a Hilbertian $(\pi - A)$-bimodule and that $N$ is (as before) a left $\mathbb{C}[\pi]$-module, having free resolution (56), which is finitely generated in every dimension. Form the cochain complex $\text{Hom}_\pi(C_*, M)$; it is a cochain complex of right $A$-Hilbertian modules. We may convert it into a cochain complex of left $A$-Hilbertian modules by using the involution of $A$; the resulting complex let’s denote by $\text{Hom}_\pi(C_*, M)$. Now, define

$$\mathcal{E}xt^q_\pi(N, M) = H^q(\text{Hom}_\pi(C_*, M)).$$

(60)

It is a contravariant functor of $N$.

9.4. Example. Suppose that the $\mathbb{C}[\pi]$-module $N = \mathbb{Z}$ with the trivial $\pi$-action admits a free resolution which is finitely generated in every dimension. Then the above construction produces the group cohomology with values in $\mathcal{E}(A)$:

$$\mathcal{H}^p(\pi, M) = \mathcal{E}xt^p_\pi(M, \mathbb{Z}).$$

(61)

Note that it can be understood also in the framework of construction of subsection 6.1 as $\mathcal{H}^p(K(\pi, 1), M)$, where $K(\pi, 1)$ denotes the Eilenberg-MacLane space. In the case, when $M = \ell^2(\pi)$, the projective part of the cohomology $P(\mathcal{H}^p(\pi, M))$ is denoted in [CG] by $\mathcal{H}^p_{(2)}(\pi)$.

9.5. Using theorem 6.6 we get the following duality relations:

$$P(\mathcal{E}xt^i_\pi(N, M)) \simeq P(\mathcal{T}or^i_\pi(M^*, N))^*$$

(62)

and

$$T(\mathcal{E}xt^i_\pi(N, M)) \simeq c(T(\mathcal{T}or^i_\pi-1(M^*, N))).$$

(63)

Here $M$ is an arbitrary Hilbertian $(\pi - A)$-bimodule.

9.6. Corollary. The module $\mathcal{E}xt^0_\pi(N, M)$ is always projective and it is isomorphic to $\text{Hom}_\pi(N, M)$.

The following theorem is one of the main result of this section. It establishes a relation between the homology of the universal covering, considered as a module over the group ring of the fundamental group, and the extended $L^2$ homology and cohomology.

9.7. Theorem. Let $X$ be a finite cell complex with fundamental group $\pi = \pi_1(X)$ and let $\tilde{X}$ denote the universal cover of $X$. Suppose that the homology modules $H_q(\tilde{X}, \mathbb{C})$, considered as left $\mathbb{C}[\pi]$-modules, have free resolutions with finitely generated $\mathbb{C}[\pi]$-modules of chains in all dimensions. Suppose that $M$ is a Hilbertian $(A - \pi)$-bimodule. Then there exists a spectral sequence in the abelian category $\mathcal{E}(A)$ with the initial $E^2$-term

$$E^2_{p,q} = \mathcal{T}or^\pi_q(M, H_p(\tilde{X})) \Rightarrow \mathcal{H}_{p+q}(X, M).$$

(64)

The limit term $E^\infty_{p,q}$ coincides with $E^r_{p,q}$ for some large $r$.

Proof. First, we will construct a special Cartan-Eilenberg resolution $D_{**} \to C_*$ of the chain complex $C_* = C_*(\tilde{X})$, cf. [W], §5.7. We will need this resolution to
satisfy some properties, additional to those, mentioned in [W]. Here $D_{pq}$ is a double complex, with two differentials $d^h$ (horizontal) and $d^v$ (vertical). It is required to satisfy the following conditions:

- $D_{pq}$ is free and finitely generated over $\mathbb{C}[\pi]$ for any pair $p, q$;
- $D_{pq} = 0$ if $p < 0$ or if $p > \dim X$;
- For any number $p$, the modules of boundaries $B_p(D_{**}, d^h)$, the module of cycles $Z_p(D_{**}, d^h)$, and the module of homology $H_p(D_{**}, d^h)$ of the double complex with respect to the horizontal differential $d^h$, are free finitely generated resolutions of $B_p(C)$, $Z_p(C)$, $H_p(C)$, respectively.

Such Cartan-Eilenberg resolution can be constructed inductively using the Horseshoe lemma (cf. [W], 2.2.8) similarly to proof of Lemma 5.7.2 in [W]; in fact we use a version the Horseshoe lemma, where instead of projective resolutions we deal with free and finitely generated ones.

If we are given resolutions of $B_p(C)$ and $H_p(C)$, then, using the Horseshoe lemma applied to the exact sequence

$$0 \to B_p(C) \to Z_p(C) \to H_p(C) \to 0,$$

we obtain a resolution of $Z_p(C)$; the resolution of $Z_p(C)$ produces a resolution of $B_{p-1}(C)$ via the exact sequence

$$0 \to Z_p(C) \to C_p \to B_{p-1}(C) \to 0.$$ (65)

And, at last, using exact sequence (65) again and the constructed resolutions of $Z_p(C)$ and of $B_{p-1}(C)$, we obtain a resolution of $C_p$. It gives a column $D_{p,*}$ of the Cartan-Eilenberg resolution, standing above $C_p$. We apply this procedure inductively, starting from the maximal dimension $p = \dim(X)$ and moving downwards.

Suppose now that the Cartan-Eilenberg resolution $D_{**}$ with the above properties has been constructed. Consider the following double complex $M \otimes_{\pi} D_{pq}$ as a double complex in the abelian category $\mathcal{E}(A)$. Computing first homology of this double complex with respect to vertical differential, we obtain that that nontrivial homology will appear only on the row $q = 0$ and the homology at point $(p, 0)$ is $M \otimes_{\pi} C_p$. Thus we obtain that the homology of the total complex $\text{Tot}_*(D_{**})$ is precisely the extended $L^2$ homology $\mathcal{H}_*(X, M)$.

On the other hand, if we compute first homology of the double complex $M \otimes_{\pi} D_{pq}$ with respect to the horizontal differential $d^h$ and then with respect to the vertical $d^v$, we will get on place $(p, q)$ the following virtual Hilbertian module

$$E_{p,q}^2 = \text{Tor}_q^\pi(M, H_p(\tilde{X})).$$

This gives the desired spectral sequence.

It clearly stabilizes after a finite number of steps. □.

There is also a cohomological version of Theorem 9.7.

9.8. Theorem. Let $X$ be a finite cell complex with fundamental group $\pi = \pi_1(X)$ and let $\tilde{X}$ denote the universal cover. Suppose that the homology modules $H_p(\tilde{X}, \mathbb{C})$
(considered as left \( \mathbb{C}[\pi]\)-modules) have free resolutions with finitely generated modules in all dimensions. Let \( M \) be a Hilbertian \((\pi - \mathcal{A})\)-bimodule. Then there exists a spectral sequence in the abelian category \( \mathcal{E}(\mathcal{A}) \) with the initial \( E_2 \)-term
\[
E_2^{p,q} = \mathcal{E}xt_q^p(H_p(\tilde{X}), M) \Rightarrow \mathcal{H}^{p+q}(X, M). 
\] (66)

The limit term \( E_\infty^{p,q} \) coincides with \( E_r^{p,q} \) for some large \( r \).

**Proof.** It is similar to Proof of Theorem 9.7. □

9.9. As an application, consider the case when the fundamental group \( \pi = \pi_1(X) \) is free. Note, that the group ring \( \mathbb{C}[\pi] \) of the free group is a FIR (cf. [C]) and it is coherent, cf. [C], page 554. This implies that all \( \text{Tor}_q^\pi \) with \( q > 1 \) vanish and for any finite polyhedron \( X \) with \( \pi_1(X) = \pi \) the homology groups of the universal covering \( H_p(\tilde{X}) \) are finitely presented as \( \mathbb{C}[\pi]\)-modules (so Theorems 9.7 and 9.8 can always be applied). Applying Theorem 9.7, we obtain the following exact sequence
\[
0 \to \text{Tor}_1^\pi(M, H_{p-1}(\tilde{X})) \to \mathcal{H}_p(X, M) \to \text{Tor}_0^\pi(M, H_p(\tilde{X})) \to 0. 
\] (67)

Observe, that the module on the left is projective.

9.10. **Proposition.** Exact sequence (67) splits.

**Proof.** Denote by \( C \) the chain complex \( C_*(\tilde{X}) \) and by \( B_p, Z_p \) and \( H_p \) the boundaries, cycles, and the homology of this chain complex, respectively. Since \( \mathbb{C}[\pi] \) has homological dimension one, any submodule of a free module is free. Thus, we have the free resolution
\[
0 \to B_p \to Z_p \to H_p \to 0 
\] (68)
and also, there is a splitting
\[
C_p = Z_p \oplus X_p
\]
where \( X_p \) is a free submodule such that the boundary homomorphism maps it isomorphically onto \( B_{p-1} \). Thus, we obtain
\[
\text{Ker}[M \otimes_\pi C_p \to M \otimes_\pi C_{p-1}] = M \otimes_\pi Z_p \oplus \text{Ker}[M \otimes_\pi X_p \to M \otimes_\pi Z_{p-1}]. \quad (69)
\]

Clearly, \( \text{Ker}[M \otimes_\pi X_p \to M \otimes_\pi Z_{p-1}] \) is isomorphic to \( \text{Tor}_1^\pi(M, H_{p-1}(\tilde{X})) \) according to the definition 9.1. By example 2.9, the extended \( L^2 \) homology \( \mathcal{H}_p(X, M) \) is represented by the following morphisms of \( \mathcal{E}(\mathcal{A}) \):
\[
(M \otimes_\pi C_{p+1} \to \text{Ker}[M \otimes_\pi C_p \to M \otimes_\pi C_{p-1}]) \simeq \\
(M \otimes_\pi C_{p+1} \to M \otimes_\pi Z_p) \oplus \text{Tor}_1^\pi(M, H_{p-1}(\tilde{X})) \simeq \\
\text{Tor}_0^\pi(M, H_p(\tilde{X})) \oplus \text{Tor}_1^\pi(M, H_{p-1}(\tilde{X})).
\]

This completes the proof. □
9.11. Corollary. If the fundamental group \( \pi = \pi_1(X) \) is free, then for any \( p \) the torsion part of the extended \( L^2 \) homology in dimension \( p \) coincides with the torsion part of \( \text{Tor}_0^\pi(M, H_p(\tilde{X})) \) and, in particular, it depends only on the \( \mathbb{C}[[\pi]] \) homology module \( H_p(\tilde{X}) \) of the universal covering \( \tilde{X} \). Thus, the Novikov-Shubin invariant of \( H_p(X, M) \) depends only on \( H_p(\tilde{X}) \). \( \square \)

Thus, in the case of the free fundamental group the Novikov-Shubin invariants depend only on the homology modules of the universal covering \( H_p(\tilde{X}) \).

An example, computed by the author jointly with J.Hillman, shows that for \( \pi = \mathbb{Z}^2 \) the Novikov-Shubin invariants are not functions of homology of the universal covering (unlike the case of the free group) and depend also on the \( k \)-invariants.

A typical application of the spectral sequence of Theorem 9.7 consists in getting estimates from above for the capacity of the extended \( L^2 \) homology. The following statement yields an example.

9.12. Theorem. Suppose that under the conditions of Theorem 9.7 it is known that all Hilbertian modules

\[
E^2_{p,q} = \text{Tor}_q^\pi(M, H_p(\tilde{X}))
\]

are torsion and have capacity less or equal than some \( \nu \in [0, \infty) \). Then the extended \( L^2 \) homology \( H_*(X, M) \) is also torsion and its capacity is less or equal than \( \nu \).

Proof. It follows from Theorem 9.7 and Corollary 4.10. \( \square \)

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