EIGENVALUE DISTRIBUTION OF LARGE WEIGHTED BIPARTITE RANDOM GRAPHS

V. Vengerovsky, Institute for Low Temperature Physics, Ukraine

Abstract

We study eigenvalue distribution of the adjacency matrix $A^{(N,p,\alpha)}$ of weighted random bipartite graphs $\Gamma = \Gamma_{N,p}$. We assume that the graphs have $N$ vertices, the ratio of parts is $\frac{\alpha}{1-\alpha}$ and the average number of edges attached to one vertex is $\alpha \cdot p$ or $(1-\alpha) \cdot p$. To each edge of the graph $e_{ij}$ we assign a weight given by a random variable $a_{ij}$ with all moments finite.

We consider the moments of normalized eigenvalue counting measure $\sigma_{N,p,\alpha}$ of $A^{(N,p,\alpha)}$. The weak convergence in probability of normalized eigenvalue counting measures is proved.

1 Introduction

The spectral theory of graphs is an actively developing field of mathematics involving a variety of methods and deep results (see the monographs [4, 5, 10]). Given a graph with $N$ vertices, one can associate with it many different matrices, but the most studied are the adjacency matrix and the Laplacian matrix of the graph. Commonly, the set of $N$ eigenvalues of the adjacency matrix is referred to as the spectrum of the graph. In these studies, the dimension of the matrix $N$ is usually regarded as a fixed parameter. The spectra of infinite graphs is considered in certain particular cases of graphs having one or another regular structure (see for example [12]).

Another large class of graphs, where the limiting transition $N \to \infty$ provides a natural approximation is represented by random graphs [2, 11]. In this branch, geometrical and topological properties of graphs (e.g. number of connected components, size of maximal connected component) are studied for immense number of random graph ensembles. One of the classes of the prime reference is the binomial random graph originating by P. Erdős (see, e.g. [11]). Given a number $p_N \in (0,1)$, this family of graphs $G(N,p_N)$ is defined by taking as $\Omega$ the set of all graphs on $N$ vertices with the probability

$$P(G) = p_N^{e(G)}(1-p_N)^{N \choose 2} - e(G),$$

(1.1)

where $e(G)$ is the number of edges of $G$. Most of the random graphs studies are devoted to the cases where $p_N \to 0$ as $N \to \infty$.

Intersection of these two branches of the theory of graphs contains the spectral theory of random graphs that is still poorly explored. However, a number of powerful tools can be employed here, because the ensemble of random symmetric $N \times N$ adjacency matrices $A_N$ is a particular representative of the random matrix theory, where the limiting transition $N \to \infty$ is intensively studied during half of century since the pioneering works by E. Wigner [17]. Initiated by theoretical physics applications, the spectral theory of random matrices has revealed deep nontrivial links with many fields of mathematics.

Spectral properties of random matrices corresponding to (1.1) were examined in the limit $N \to \infty$ both in numerical and theoretical physics studies [6, 7, 8, 14, 15, 16]. There are two major asymptotic regimes: $p_N \gg 1/N$ and $p_N = O(1/N)$ and corresponding models can be
called the \textit{dilute random matrices} and \textit{sparse random matrices}, respectively. The first studies of spectral properties of sparse and dilute random matrices in the physical literature are related with the works \cite{15, 10, 14}, where equations for the limiting density of states of sparse random matrices were derived. In papers \cite{14} and \cite{9} a number of important results on the universality of the correlation functions and the Anderson localization transition were obtained. But all these results were obtained with non rigorous replica and supersymmetry methods.

On mathematical level of rigour, the eigenvalue distribution of dilute random matrices was studied in \cite{15}. It was shown that the normalized eigenvalue counting function of

\[
\frac{1}{\sqrt{Np_N}} A_{N,p_N}
\]  

converges in the limit $N, p_N \to \infty$ to the distribution of explicit form known as the semicircle, or Wigner law \cite{17}. The moments of this distribution verify well-known recurrent relation for the Catalan numbers and can be found explicitly. Therefore one can say that the dilute random matrices represent explicitly solvable model (see also \cite{15, 16}).

In the series of papers \cite{1, 3, 2} and in \cite{13}, the adjacency matrix and the Laplace matrix of random graphs (1.1) with \( p_N = pN \) were studied. It was shown that this sparse random matrix ensemble can also be viewed as the explicitly solvable model.

In the present paper we consider a bipartite analogue of large sparse random graph. This article is a modification of one part of \cite{13} for this case.

2 Main results

We can introduce the randomly weighted adjacency matrix of random bipartite graphs. Let \( \Xi = \{ a_{ij}, \ i \leq j, \ i, j \in \mathbb{N} \} \) be the set of jointly independent identically distributed (i.i.d.) random variables determined on the same probability space and possessing the moments

\[
Ea_{ij}^k = X_k < \infty \quad \forall \ i, j, k \in \mathbb{N},
\]

where \( E \) denotes the mathematical expectation corresponding to \( \Xi \). We set \( a_{ji} = a_{ij} \) for \( i \leq j \).

Given \( 0 < p \leq N \), let us define the family \( D^{(p)}_N = \{ d_{ij}^{(N,p)}, \ i \leq j, \ i, j \in \mathbb{I} \} \) of jointly independent random variables

\[
d_{ij}^{(N,p)} = \begin{cases} 1, & \text{with probability } p/N, \\ 0, & \text{with probability } 1 - p/N, \end{cases}
\]

We determine \( d_{ji} = d_{ij} \) and assume that \( A^{(p)}_N \) is independent from \( \Xi \).

Let \( \alpha \in (0, 1) \), define \( I_{\alpha,N} = \mathbb{I}_{\lfloor \alpha \cdot N \rfloor} \), where \( \lfloor \cdot \rfloor \) is a floor function. Now one can consider the real symmetric \( N \times N \) matrix \( A^{(N,p,\alpha)}(\omega) \):

\[
\left[ A^{(N,p,\alpha)} \right]_{ij} = \begin{cases} a_{ij} \cdot d_{ij}^{(N,p)}, & \text{if } (i \in I_{\alpha,N} \land j \notin I_{\alpha,N}) \lor (i \notin I_{\alpha,N} \land j \in I_{\alpha,N}), \\ 0, & \text{otherwise} \end{cases}
\]

that has \( N \) real eigenvalues \( \lambda_1^{(N,p,\alpha)} \leq \lambda_2^{(N,p,\alpha)} \leq \ldots \leq \lambda_N^{(N,p,\alpha)} \).

The normalized eigenvalue counting function (or integrated density of states (IDS)) of \( A^{(N,p,\alpha)} \) is determined by the formula

\[
\sigma \left( \lambda; A^{(N,p,\alpha)} \right) = \frac{\sharp \{ j : \lambda_j^{(N,p,\alpha)} < \lambda \} }{N}.
\]
Theorem 1. Under condition
\[ X_{2m} \leq (C \cdot m)^{2m}, m \in \mathbb{N} \]  \hspace{1cm} (2.4)
the measure \( \sigma (\lambda; A^{(N,p,\alpha)}) \) weak converges in probability to nonrandom measure \( \sigma_{p,\alpha} \)
\[ \sigma (\cdot; A^{(N,p,\alpha)}) \to \sigma_{p,\alpha}, \quad N \to \infty, \]  \hspace{1cm} (2.5)
which can be uniquely determine by the moments
\[ \int \lambda^s d\sigma_{p,\alpha} = \begin{cases} m^{(p,\alpha)}_k \sum_{i=0}^k (S^{(1)}(k,i) + S^{(2)}(k,i)), & \text{if } s = 2k, \\ 0, & \text{if } s = 2k - 1, \end{cases} \]  \hspace{1cm} (2.6)
where numbers \( S^{(1)}(k,i), S^{(2)}(k,i) \) can be found from the following system of recurrent relations
\[ S^{(1)}(l,r) = p \sum_{f=1}^r \left( \frac{r-1}{f-1} \right) \cdot X_{2f} \sum_{u=0}^{l-r} S^{(1)}(l-u,f-r) \cdot \sum_{v=0}^{u} \left( \frac{f+v-1}{f-1} \right) \cdot S^{(2)}(u,v) \]  \hspace{1cm} (2.7)
\[ S^{(2)}(l,r) = p \sum_{f=1}^r \left( \frac{r-1}{f-1} \right) \cdot X_{2f} \sum_{u=0}^{l-r} S^{(2)}(l-u,f-r) \cdot \sum_{v=0}^{u} \left( \frac{f+v-1}{f-1} \right) \cdot S^{(1)}(u,v) \]  \hspace{1cm} (2.8)
with the initial conditions
\[ S^{(1)}(l,0) = \alpha \cdot \delta_{l,0}, \quad S^{(2)}(l,0) = (1-\alpha) \cdot \delta_{l,0}. \]  \hspace{1cm} (2.9)

The following denotations are used:
\[ M_k^{(N,p,\alpha)} = \int \lambda^k d\sigma (\lambda; A^{(N,p,\alpha)}), \quad M_k^{(N,p,\alpha)} = E M_k^{(N,p,\alpha)}, \]
\[ C_{k,m}^{(N,p,\alpha)} = E \left\{ M_k^{(N,p,\alpha)} M_m^{(N,p,\alpha)} \right\} - E \left\{ M_k^{(N,p,\alpha)} \right\} \cdot E \left\{ M_m^{(N,p,\alpha)} \right\}. \]

Theorem 1 is a corollary of Theorem 2.

Theorem 2. Assuming conditions (2.4),
(i) Correlators are vanished in the limit
\[ C_{k,m}^{(N,p,\alpha)} \leq C(k,m,p,\alpha) \cdot N, \quad N \to \infty \quad \forall \ k, m \in \mathbb{N}. \]  \hspace{1cm} (2.10)
(ii) The limit of \( s \)-th moment exists for all \( s \in \mathbb{N} \)
\[ \lim_{N \to \infty} M_s^{(N,p,\alpha)} = \begin{cases} \sum_{i=0}^k (S^{(1)}(k,i) + S^{(2)}(k,i)), & \text{if } s = 2k, \\ 0, & \text{if } s = 2k - 1, \end{cases} \]  \hspace{1cm} (2.11)
where numbers \( S^{(1)}(k,i), S^{(2)}(k,i) \) are determined by the system of recurrent relations (2.7) - (2.8) with the initial conditions (2.9).
(iii) The limiting moments \( \left\{ m_k^{(p,\alpha)} \right\}_{k=1}^\infty \) obey Carleman’s condition
\[ \sum_{k=1}^\infty \frac{1}{2^k m_k^{(p,\alpha)}} = \infty \]  \hspace{1cm} (2.12)
3 Proof of Theorem 1

3.1 Walks and contributions

Using independence of families \( \Xi \) and \( \Lambda^{(p)}_N \), we have

\[
M_k^{(N,p)} = \mathbb{E}\{X^k d_{A_1,N,p,a} \} = \mathbb{E}\left( \frac{1}{N} \sum_{i=1}^{N} \{A_i^{(N,p,a)}\}^k \right) = \frac{1}{N} \mathbb{E}\left( \text{Tr}[A^{(N,p,a)}]^k \right) = \frac{1}{N} \sum_{j_1=1}^{N} \sum_{j_2=1}^{N} \cdots \sum_{j_k=1}^{N} \mathbb{E}\left(A_{j_1,j_2}^{(N,p,a)} A_{j_2,j_3}^{(N,p,a)} \cdots A_{j_k,j_1}^{(N,p,a)} \right) = \frac{1}{N} \sum_{j_1=1}^{N} \sum_{j_2=1}^{N} \cdots \sum_{j_k=1}^{N} \mathbb{E}(a_{j_1,j_2} a_{j_2,j_3} \cdots a_{j_k,j_1}).
\]

Thus we can write

\[
\mathbb{E}\left(d_{j_1,j_2}^{(N,p,a)} d_{j_2,j_3}^{(N,p,a)} \cdots d_{j_k,j_1}^{(N,p,a)} \right) \cdot \xi_{j_1,j_2}^{(N,a)} \cdot \xi_{j_2,j_3}^{(N,a)} \cdots \xi_{j_k,j_1}^{(N,a)} = 1, \quad \text{if} \quad (i \in I_{a,N} \land j \notin I_{a,N}) \lor (i \notin I_{a,N} \land j \in I_{a,N}),
\]

\[
\xi_{ij}^{(N,a)} = \begin{cases} 1, & \text{if} \quad (i \in I_{a,N} \land j \notin I_{a,N}) \lor (i \notin I_{a,N} \land j \in I_{a,N}), \\ 0, & \text{otherwise}. 
\end{cases}
\]

Consider \( W_k^{(N)} \) the set of closed walks of \( k \) steps over the set \( \bar{T}_N \):

\[
W_k^{(N)} = \{ w = (w_1, w_2, \cdots, w_k, w_{k+1} = w_1) : \forall i \in \bar{T}_k \land w_i \in \bar{T}_N \}.
\]

For \( w \in W_k^{(N)} \) let us denote \( a(w) = \prod_{i=1}^{k} a_{w_i, w_{i+1}}, d^{(N,p)}(w) = \prod_{i=1}^{k} d_{w_i, w_{i+1}}^{(N,p)} \) and \( \xi^{(N,a)}(w) = \prod_{i=1}^{k} \xi_{w_i, w_{i+1}}^{(N,a)} \). Then we have

\[
M_k^{(N,p)} = \frac{1}{N} \sum_{w \in W_k^{(N)}} \mathbb{E}d^{(N,p)}(w) \cdot \mathbb{E}d^{(N,p)}(w) \cdot \xi^{(N,a)}(w).
\]

Let \( w \in W_k^{(N)} \) and \( f, g \in \bar{T}_N \). Denote by \( n_w(f,g) \) the number of steps \( f \to g \) and \( g \to f \):

\[
n_w(f,g) = \# \{ i \in \bar{T}_k : (w_i = f \land w_{i+1} = g) \lor (w_i = g \land w_{i+1} = f) \}.
\]

Then

\[
\mathbb{E}d^{(N,p)}(w) = \prod_{f=1}^{N} \prod_{g=1}^{N} X_{n_w(f,g)}.
\]

Given \( w \in W_k^{(N)} \), let us define the sets \( V_w = \bigcup_{i=1}^{k} \{w_i\} \) and \( E_w = \bigcup_{i=1}^{k} \{(w_i, w_{i+1})\} \), where \( (w_i, w_{i+1}) \) is a non-ordered pair. It is easy to see that \( G_w = (V_w, E_w) \) is a simple non-oriented graph and the walk \( w \) covers the graph \( G_w \). Let us call \( G_w \) the skeleton of walk \( w \). We denote by \( n_w(e) \) the number of passages of the edge \( e \) by the walk \( w \) in direct and inverse directions. For \( (w_j, w_{j+1}) = e_j \in E_w \) let us denote \( a_{e_j} = a_{w_j, w_{j+1}} = a_{w_{j+1}, w_j} \). Then we obtain

\[
\mathbb{E}a(w) = \prod_{e \in E_w} \mathbb{E}a^{n_w(e)} = \prod_{e \in E_w} X_{n_w(e)}.
\]

Similarly we can write

\[
\mathbb{E}d^{(N,p)}(w) = \prod_{e \in E_w} \mathbb{E}\left([d_e^{(N,p)}]^{n_w(e)}\right) = \prod_{e \in E_w} \frac{p}{N}.
\]
Then, we can rewrite (3.2) in the form

\[ M_k^{(N,p)} = \frac{1}{\theta} \sum_{w \in W_k^{(N)}} \xi^{(N,\alpha)}(w) \cdot \prod_{\varepsilon \in E_w} \frac{p^{\varepsilon}}{\theta} X_{n_{w}(\varepsilon)} = \]

\[ = \sum_{w \in W_k^{(N)}} \xi^{(N,\alpha)}(w) \cdot \left( \frac{p_{\varepsilon}}{N^{|E_w|+1}} \prod_{\varepsilon \in E_w} X_{n_{w}(\varepsilon)} \right) = \sum_{w \in W_k^{(N)}} \theta(w), \quad (3.3) \]

where $\theta(w)$ is the contribution of the walk $w$ to the mathematical expectation of the corresponding moment. To perform the limiting transition $N \to \infty$ it is natural to separate $W_k^{(N)}$ into classes of equivalence. Walks $w^{(1)}$ and $w^{(2)}$ are equivalent $w^{(1)} \sim w^{(2)}$, if and only if there exists a bijection $\phi$ between the sets of vertices $V_{w^{(1)}}$ and $V_{w^{(2)}}$ such that for $i = 1, 2, \ldots, k$ $w^{(2)}_i = \phi(w^{(1)}_i)$

$w^{(1)} \sim w^{(2)} \iff \exists \phi : V_{w^{(1)}} \to V_{w^{(2)}} : \forall i \in [1,k+1] \ w^{(2)}_i = \phi(w^{(1)}_i)$

Last condition requires that every vertex and it image be in same component. It’s essential for further computations because contribution of walk equals zero in the case when origin and end of some step are in the same component. Let us denote by $[w]$ the class of equivalence of walk $w$ and by $C_k^{(N)}$ the set of such classes. It is obvious that if two walks $w^{(1)}$ and $w^{(2)}$ are equivalent then their contributions are equal.

$w^{(1)} \sim w^{(2)} \implies \theta(w^{(1)}) = \theta(w^{(2)})$

Cardinality of the class of equivalence $[w]$ is equal the number of all mappings $\phi : V_w \to \overline{\mathbb{N}}$ such that $\phi(V_{1,w}) \subset I_{\alpha,N}$ and $\phi(V_{2,w}) \subset \overline{\mathbb{N}} \setminus I_{\alpha,N}$, where $V_{1,w} = V_w \cap I_{\alpha,N}$ and $V_{2,w} = V_w \setminus I_{\alpha,N}$, i.e.\ $[\alpha \cdot N] \cdot ([\alpha \cdot N] - 1) \cdot \ldots \cdot ([\alpha \cdot N] - |V_{1,w}| + 1) \cdot (N - |V_{2,w}|) \cdot (N - |\alpha \cdot N| - |V_{1,w}| - 1) \cdot \ldots \cdot (N - |\alpha \cdot N| - |V_{2,w}| + 1)$.

Then we can rewrite (3.3) in the form

\[ M_k^{(N,p)} = \sum_{w \in W_k^{(N)}} \xi^{(N,\alpha)}(w) \cdot \left( \frac{p_{\varepsilon}}{N^{|E_w|+1}} \prod_{\varepsilon \in E_w} X_{n_{w}(\varepsilon)} \right) = \]

\[ = \sum_{[w] \in CW_k^{(N)}} \xi^{(N,\alpha)}(w) \prod_{\varepsilon \in E_w} X_{n_{w}(\varepsilon)} \left( \frac{[\alpha \cdot N] \cdot ([\alpha \cdot N] - 1) \cdot \ldots \cdot ([\alpha \cdot N] - |V_{1,w}| + 1)}{N^{|E_w|+1} \cdot p^{-|E_w|}} \right) \]

\[ (N - [\alpha \cdot N]) \cdot (N - [\alpha \cdot N] - 1) \cdot \ldots \cdot (N - [\alpha \cdot N] - |V_{2,w}| + 1) = \sum_{[w] \in CW_k^{(N)}} \hat{\theta}([w]). \quad (3.4) \]

In second line of (3.4) for every class $[w]$ we choose arbitrary walk $w$ corresponding to this class of equivalence.

### 3.2 Minimal and essential walks

Class of walks $[w]$ of $CW_k^{(N)}$ has at most k vertices. Hence, $CW_k^{(1)} \subset CW_k^{(2)} \subset \ldots \subset CW_k^{(k-\min(\alpha,1-\alpha)-1)} = CW_k^{(k-\min(\alpha,1-\alpha)-1)+1} = \ldots$. It is natural to denote $CW_k = CW_k^{(\min(\alpha,1-\alpha)-1)}$. Then (3.4) can be written as

\[ m_k^{(p)} = \lim_{N \to \infty} \sum_{[w] \in CW_k} \xi^{(N,\alpha)}(w) \cdot \alpha^{V_{1,w}} \cdot (1 - \alpha)^{|V_{1,w}| - |V_{2,w}|} \left( \frac{N^{|V_{2,w}| - |E_w| - 1}}{\prod_{\varepsilon \in E_w} \frac{X_{n_{w}(\varepsilon)}}{p^{-1}}} \right). \quad (3.5) \]
The set $CW_k$ is finite. Regarding this and (3.3), we conclude that the class $[w]$ has non-vanishing contribution, if $|V_w| - |E_w| - 1 \geq 0$ and $w$ is a bipartite walk through the complete bipartite graph $K_{I_{n,N},I_{n,N}}$. But for each simple connected graph $G = (V, E)$ $|V_w| \leq |E_w| + 1$, and the equality takes place if and only if the graph $G$ is a tree.

It is convenient to use a notion of minimal walk.

**Definition 1.** The walk $w$ is a minimal walk, if $w_1$ (the root of walk) has the number 1 and the number of each new vertex is equal to zero.

Let us denote the set of all minimal walks of $W^{(N)}_k$ by $MW^{(N)}_k$.

**Example 1.** The sequences $(1,2,1,2,3,1,2,1,4,2,1,4,3,1)$ and $(1,2,3,2,4,2,3,2,1,2,4,1,5,1)$ represent the minimal walks.

**Definition 2.** The minimal walk $w$ that has a tree as a skeleton is an essential walk.

Let us denote the set of all essential walks of $W^{(N)}_k$ by $EW^{(N)}_k$. Therefore we can rewrite (3.5) in the form

$$m_k^{(p)} = \sum_{w \in EW_k} (\theta_1(w) + \theta_2(w)),$$

where

$$\theta_1(w) = \alpha^{\beta(w)} \cdot (1 - \alpha)^{|V_w| - \beta(w)} \left( \prod_{e \in E_w} \left( p \cdot X_{n_w(e)} \right) \right),$$

$$\theta_2(w) = (1 - \alpha)^{\beta(w)} \cdot \alpha^{|V_w| - \beta(w)} \left( \prod_{e \in E_w} \left( p \cdot X_{n_w(e)} \right) \right),$$

where $\beta(w)$ is a number of such vertices $v$ that the distance between $v$ and the first vertex $w_1$ are even. The number of passages of each edge $e$ belonging to the essential walk $w$ is even. Hence, the limiting mathematical expectation $m_k^{(p)}$ depends only on the even moments of random variable of $\alpha$. It is clear that the limiting mathematical expectation $\lim_{N \to \infty} M_2^{(N,p,\alpha)}$ is equal to zero.

### 3.3 First edge decomposition of essential walks

Let us start with necessary definitions. The first vertex $w_1 = 1$ of the essential walk $w$ is called the root of the walk. We denote it by $\rho$. Let us denote the second vertex $w_2 = 2$ of the essential walk $w$ by $\nu$. We denote by $l$ the half of walk’s length and by $r$ the number of steps of $w$ starting from root $\rho$. In this subsection we derive the recurrent relations by splitting of the walk (or of the tree) into two parts. To describe this procedure, it is convenient to consider the set of the essential walks of length $2l$ such that they have $r$ steps starting from the root $\rho$. We denote this set by $\Lambda(l, r)$. One can see that this description is exact, in the sense that it is minimal and gives complete description of the walks we need. Denote by $S^{(1)}(l, r), S^{(2)}(l, r)$ the sum of contributions of the walk of $\Lambda(l, r)$ with weights $\theta_1$ and $\theta_2$ respectively. Let us remove the edge $(\rho, \nu) = (1,2)$ from $G_w$ and denote by $G_w^-$ the graph obtained. The graph $G_w^-$ has two components. Denote the component that contains the vertex $\nu$ by $G_2$ and the component containing the root $\rho$ by $G_1$. Add the edge $(\rho, \nu)$ to the edge set of the tree $G_2$. Denote the result of this operation by $G_2$. Denote by $u$ the half of the walk’s length over the tree $G_2$ and by $f$ the number of steps $(\rho, \nu)$ in the walk $w$. It is clear that the following inequalities hold for all essential walks (excepting the walk of length zero) $1 \leq f \leq r, r + u \leq l$. Let us denote by $\Lambda_1(l, r, u, f)$ the set of the essential walks with fixed parameters $l, r, u, f$ and by $S_1^{(1)}(l, r, u, f)$ ($S_1^{(2)}(l, r, u, f)$) the sum of contributions of the walks of $\Lambda_1(l, r, u, f)$ with weight $\theta_1$ ($\theta_2$). Denote
by $\Lambda_2(l, r)$ the set of the essential walks of $\Lambda(l, r)$ such that their skeleton has only one edge attached the root $\rho$. Also we denote by $S_1^{(1)}(l, r)$ and $S_2^{(1)}(l, r)$ the sum of weights $\theta_1$ and $\theta_2$ respectively of the walk of $\Lambda_2(l, r)$. Now we can formulate the first lemma of decomposition. It allows express $S^{(1)}, S^{(2)}$ as functions of the $S_1^{(1)}, S_2^{(1)}, S_2^{(2)}$.

**Lemma 1** (First decomposition lemma). The following relation holds

\[
S^{(1)}(l, r) = \sum_{f=1}^{r-1} \sum_{u=0}^{r-f-1} S_1^{(1)}(l, r, u, f),
\]

\[
S^{(2)}(l, r) = \sum_{f=1}^{r-1} \sum_{u=0}^{r-f-1} S_2^{(1)}(l, r, u, f),
\]

where

\[
S_1^{(1)}(l, r, u, f) = \alpha^{-1} \cdot \left(\frac{r-1}{f-1}\right) \cdot S_2^{(1)}(f + u, f) \cdot S^{(1)}(l - u - f, r - f),
\]

\[
S_2^{(1)}(l, r, u, f) = (1 - \alpha)^{-1} \cdot \left(\frac{r-1}{f-1}\right) \cdot S_2^{(1)}(f + u, f) \cdot S^{(2)}(l - u - f, r - f).
\]

**Proof.** The first two equalities are obvious. The last two equalities follow from the bijection

\[
\Lambda_1(l, r, u, f) \xrightarrow{bij} \Lambda_2(f + u, f) \times \Lambda(l - u - f, r - f) \times \Theta_1(r, f),
\]

where $\Theta_1(r, f)$ is the set of sequences of 0 and 1 of length $r$ such that there are exactly $f$ symbols 1 in the sequence and the first symbol is 1.

Let us construct this mapping $F$. Regarding one particular essential walk $w$ of $\Lambda_1(l, r, u, f)$, we consider the first edge $e_1$ of the graph $G_w$ and separate $w$ in two parts, the left and the right ones with respect to this edge $e_1$. Then we add a special code that determines the transitions from the left part to the right one and back through the root $\rho$. Obviously these two parts are walks, but not necessary minimal walks. Then we minimize these walks. This decomposition is constructed by the following algorithm. We run over $w$ and simultaneously draw the left part, the right part, and code. If the current step belongs to $G_1$, we add it to the first part, otherwise we add this step to the second part. The code is constructed as follows. Each time the walk leaves the root the sequence is enlarged by one symbol. If current step is $\rho \rightarrow \nu$ and "0" otherwise, this symbol is "1". It is clear that the first element of the sequence is "1", the number of signs "1" is equal to $f$, and the full length of the sequence is $r$. Now we minimize the left and the right parts. Thus, we have constructed the decomposition of the essential walk $w$ and the mapping $F$. The weight $\theta_1(w)(\theta_2(w))$ of the essential walk is multiplicative with respect to edges and vertices. In factors $S_2^{(1)}(f + u, f), S^{(1)}(l - u - f, r - f)$ we twice count multiplier corresponding to the root, so we need add factor $\alpha^{-1}$ in (3.9).

**Example 2.** For $w = (1, 2, 1, 2, 3, 2, 1, 4, 1, 2, 5, 2, 1, 4, 6, 4, 1, 2, 5, 2, 3, 2, 3, 2, 1, 4, 1)$ the left part, the right one, and the code are $(1, 2, 1, 2, 3, 2, 1, 4, 1, 2, 5, 2, 1, 4, 6, 4, 1, 2, 5, 2, 3, 2, 3, 2, 1), (1, 2, 1, 2, 3, 2, 1, 2, 1), (1, 1, 0, 1, 0, 1, 0)$, respectively.

Let us denote the left part by $(w^{(f)})$ and the right part by $(w^{(s)})$. These parts are really walks with the root $\rho$. For each edge $e$ in the tree $G_2$ the number of passages of $e$ of the essential
walk \( w \) is equal to the corresponding number of passages of \( e \) of the left part \((w^{(l)})\). Also for each edge \( e \) belonging to the tree \( G_1 \) the number of passages of \( e \) of essential walk \( w \) is equal to the corresponding number of passages of \( e \) of the right part \((w^{(r)})\). The weight of the essential walk is multiplicative with respect to edges. Then the weight of the essential walk \( w \) is equal to the product of weights of left and right parts. The walk of zero length has unit weight. Combining this with (3.11), we obtain

\[
S^{(1)}_1(l, r, u, f) = \alpha^{-1} \cdot |\Theta_1(r, f)| \cdot S^{(1)}_2(f + u, f) \cdot S^{(1)}(l - u - f, r - f),
\]

(3.12)

\[
S^{(2)}(l, r, u, f) = (1 - \alpha)^{-1} \cdot |\Theta_1(r, f)| \cdot S^{(2)}(f + u, f) \cdot S^{(2)}(l - u - f, r - f).
\]

(3.13)

Taking into account that \( |\Theta_1(r, f)| = (f^{-1}) \), we derive from \((3.12), (3.13)\) \((3.9), (3.10)\).

Now let us prove that for any given elements \( w^{(f)} \) of \( \Lambda_2(f + u, f) \), \( w^{(s)} \) of \( \Lambda(l - u - f, r - f) \), and the sequence \( \theta \in \Theta_1(r, f) \), one can construct one and only one element \( w \) of \( \Lambda_1(l, r, u, f) \). We do this with the following gathering algorithm. We go along either \( w^{(f)} \) or \( w^{(s)} \) and simultaneously draw the walk \( w \). The switch from \( w^{(f)} \) to \( w^{(s)} \) and back is governed by the code sequence \( \theta \). In fact, this procedure is inverse to the decomposition procedure described above up to the fact that \( w^{(s)} \) is minimal. This difficulty can be easily resolved for example by coloring vertices of \( w^{(f)} \) and \( w^{(s)} \) in red and blue colors respectively. Certainly, the common root of \( w^{(f)} \) and \( w^{(s)} \) has only one color. To illustrate the gathering procedures we give the following example.

**Example 3.** For \( w^{(f)} = (1, 2, 1, 2, 3, 2, 1, 2, 4, 2, 1, 2, 4, 2, 1, 2, 3, 2, 3, 2, 1, 2, 1, 2, 3, 2, 1, 2, 1, 2, 3, 2, 1, 2, 1) \), \( w^{(s)} = (1, 2, 1, 2, 3, 2, 1, 2, 1, 2, 3, 2, 1, 2, 1, 2, 3, 2, 1, 2, 1, 2, 3, 2, 1, 2, 1, 2, 3, 2, 1, 2, 1) \), \( \theta = (1, 1, 0, 1, 0, 1, 0) \) the gathering procedure gives \( w = (1, 2, 1, 2, 3, 2, 1, 4, 1, 2, 5, 2, 1, 4, 6, 4, 1, 2, 5, 2, 3, 2, 3, 2, 1, 4, 1, 2, 5, 2, 3, 2, 3, 2, 1, 4, 1) \).

It is clear that the decomposition and gathering are injective mappings. Their domains are finite sets, and therefore the corresponding mapping (3.11) is bijective. This completes the proof of Lemma 1.

To formulate Lemma 2, let us give necessary definitions. We denote by \( v \) the number of steps starting from the vertex \( \nu \) excepting the steps \( \nu \rightarrow \rho \) and by \( \Lambda_3(u + f, f, v) \) the set of essential walks of \( \Lambda_2(u + f, f) \) with fixed parameter \( v \). Also we denote by \( S^{(1)}_3(u + f, f, v) \) \((S^{(2)}_3(u + f, f, v))\) the sum of weights \( \theta_1(\theta_2) \) of walks of \( \Lambda_3(u + f, f, v) \). Let us denote by \( G_{1,2} \) the graph consisting of only one edge \((\rho, \nu)\) and by \( \Lambda_4(f) \) the set of essential walks of length \( 2f \) such that their skeleton coincides with the graph \( G_{1,2} \). It is clear that \( \Lambda_4(f) \) consists of the only one walk \((1,2,1,2,\ldots,2,1)\) of weight \( \frac{X_{2f}}{p^f} \). The previous lemma allows us to express \( S^{(1)}_2, S^{(2)}_2 \) as functions of \( S^{(1)}, S^{(2)} \). The next lemma allows to express \( S^{(1)}_2, S^{(2)}_2 \) as functions of \( S^{(1)}, S^{(2)} \). Thus, two lemmas allow us to express \( S^{(1)}, S^{(2)} \) as functions of \( S^{(1)}, S^{(2)} \).

**Lemma 2** (Second decomposition lemma).

\[
S^{(1)}_2(f + u, f) = \sum_{v=0}^{u} S^{(1)}_3(f + u, f, v)
\]

(3.14)

\[
S^{(2)}_2(f + u, f) = \sum_{v=0}^{u} S^{(2)}_3(f + u, f, v)
\]

(3.15)

\[
S^{(1)}_3(f + u, f, v) = \alpha \cdot \left( \frac{f + v - 1}{f - 1} \right) \cdot \frac{X_{2f}}{p^f} \cdot S^{(2)}(u, v)
\]

(3.16)
\[ S_3^{(2)}(f + u, f, v) = (1 - \alpha) \cdot \left( \frac{f + v - 1}{f - 1} \right) \cdot \frac{X_2}{p^{1/2}} \cdot S^{(1)}(u, v) \] (3.17)

The first two equalities are trivial, the second two follow from the bijection

\[ \Lambda_3(f + u, f, v) \xrightarrow{\text{bij}} \Lambda(u, v) \times \Lambda_4(f) \times \Theta_2(f + v, f), \] (3.18)

where \( \Theta_2(f + v, f) \) is the set of sequences of 0 and 1 of length \( f + v \) such that there are exactly \( f \) symbols 1 in the sequence and last symbol of it is 1. The proof is analogous to the proof of the first decomposition lemma. The factor \( \alpha \) in (3.16) is a contribution of the root in the weight.

Combining these two decomposition lemmas and changing the order of summation, we get the recurrent relations (2.7)-(2.8) with the initial conditions (2.9).

References

[1] M.Bauer and O.Golinelli. Random incidence matrices: spectral density at zero energy, Saclay preprint T00/087; [cond-mat/0006472]
[2] B. Bollobas *Random Graphs* Acad. Press (1985)
[3] M.Bauer and O.Golinelli. Random incidedence matrices: moments and spectral density, J.Stat. Phys. **103**, 301-336, 2001
[4] Fan R.K. Chung, *Spectral Graph Theory* AMS (1997)
[5] D.M. Cvetković, M.Doob, and H.Sachs. *Spectra of Graphs*, Acad. Press (1980)
[6] S.N. Evangelou. Quantum percolation and the Anderson transition in dilute systems, *Phys. Rev. B* **27** (1983) 1397-1400
[7] S.N. Evangelou and E.N. Economou. Spectral density singularities, level statistics, and localization in sparse random matrices, *Phys. Rev. Lett.* **68** (1992) 361-364
[8] S.N. Evangelou. A numerical study of sparse random matrices, *J. Stat. Phys.* **69** (1992) 361-383
[9] Y.V. Fyodorov, A.D. Mirlin. Strong eigenfunction correlations near the Anderson localization transition. [arXiv:cond-mat/9612218v1]
[10] Ch. Godzil, G. Royle, *Algebraic Graph Theory*. Springer-Verlag, New York (2001)
[11] S. Janson, T. Łuczak, A. Rucinski, *Random Graphs*. John Wiley & Sons, Inc. New York (2000)
[12] D. Jacobson, S.D. Miller, I. Rivin, and Z. Rudnick. Eigenvalue spacing for regular graphs, in: *Emerging applications of number theory*. Ed. D.A. Hejhal et al. Springer-Verlag (1999)
[13] Khorunzhy O., Shecherbina M., and Vengerovsky V. Eigenvalue distribution of large weighted random graphs, J. Math. Phys. **45** N.4: (2004), 1648-1672.
[14] A.D. Mirlin, Y.V. Fyodorov. Universality of the level correlation function of sparse random matrices, J.Phys.A:Math.Jen. **24**, (1991), 2273-2286.
[15] G.J. Rodgers and A.J. Bray. Density of states of a sparse random matrix, Phys.Rev.B **37**, (1988), 3557-3562.
[16] G.J. Rodgers and C. De Dominicis. Density of states of sparse random matrices, J.Phys.A:Math.Jen. **23**, (1990), 1567-1566.
[17] E.P. Wigner. On the distribution of the roots of certain symmetric matrices, Ann.Math. **67**: (1958), 325-327.