Explicit Evaluations of Sums of Sequence Tails

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Abstract In this paper, we use Abel’s summation formula to evaluate several quadratic and cubic sums of the form:

$$F_N(A, B; x) := \sum_{n=1}^{N} (A - A_n)(B - B_n)x^n, \ x \in [-1, 1]$$

and

$$F(A, B, \zeta(r)) := \sum_{n=1}^{\infty} (A - A_n)(B - B_n)(\zeta(r) - \zeta_n(r)),$$

where the sequences $A_n, B_n$ are defined by the finite sums $A_n := \sum_{k=1}^{n} a_k, \ B_n := \sum_{k=1}^{n} b_k \ (a_k, b_k = o(n^{-p}), \Re(p) > 1)$ and $A = \lim_{n \to \infty} A_n, B = \lim_{n \to \infty} B_n, F(A, B; x) = \lim_{n \to \infty} F_n(A, B; x)$. Namely, the sequences $A_n$ and $B_n$ are the partial sums of the convergent series $A$ and $B$, respectively.

We give an explicit formula of $F_n(A, B; x)$ by using the method of Abel’s summation formula. Then we use apply it to obtain a family of identities relating harmonic numbers to multiple zeta values. Furthermore, we also evaluate several other series involving multiple zeta star values. Some interesting (known or new) consequences and illustrative examples are considered.

Keywords Sequence; harmonic number; Abel’s summation formula; Riemann zeta function; multiple zeta value (mzv); multiple zeta star value (mzsv); tail.

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1 Introduction

In recent years there has been significant interest in investigating sums of products of Riemann zeta tails and multiple zeta (star) values. The subject of this paper is studying quadratic and cubic sums

\[ F(A, B; x) := \sum_{n=1}^{\infty} (A - A_n) (B - B_n) x^n, \]  

(1.1)

\[ F(A, B, \zeta(r)) := \sum_{n=1}^{\infty} (A - A_n) (B - B_n) (\zeta(r) - \zeta_n(r)), \]  

(1.2)

are classical harmonic number and generalized harmonic numbers or partial sums of the series (see [1])

\[ \zeta(s) := \sum_{j=1}^{\infty} \frac{1}{j^s}, \Re(s) > 1 \]  

(1.3)

defining the Riemann zeta function, respectively. Here \( \zeta_n(r) \) denotes the harmonic number (also called the partial sum of the Riemann zeta function), which is defined by

\[ \zeta_n(r) := \sum_{j=1}^{n} \frac{1}{j^r}, \quad (r, n \in \mathbb{N} := \{1, 2, 3 \ldots \}), \]  

(1.4)

with \( H_n := \zeta_n(1) \) is the classical harmonic number, and the empty sum \( \zeta_0(r) \) is conventionally understood to be zero. In recent papers [10, 11], O. Furdui, A. Sintămărian and C. Vălean prove some results on sums of products of the tails of the Riemann zeta function, i.e.,

\[ \zeta(s) - \zeta_n(s) = \sum_{j=n+1}^{\infty} \frac{1}{j^s}. \]

In below, we let \( F(A, B) := F(A, B; 1) \). In [10], O. Furdui and A. Sintămărian shown that quadratic series

\[ F(\zeta(s), \zeta(s)) = \sum_{n=1}^{\infty} (\zeta(s) - \zeta_n(s))^2, \quad s \in \mathbb{N} \setminus \{1\} := \{2, 3, \ldots \} \]

can be expressed as a rational linear combination of zeta function values, binomial coefficients and products of two zeta function values. In [11], O. Furdui and C. Vălean proven that the series

\[ F(\zeta(s), \zeta(s+1)) = \sum_{n=1}^{\infty} (\zeta(s) - \zeta_n(s)) (\zeta(s+1) - \zeta_n(s+1)) \quad (s \in \mathbb{N} \setminus \{1\}) \]

can be expressed in terms of Riemann zeta values and a special integral involving a polylogarithms. Here the polylogarithm function is defined as follows

\[ Li_p(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^p}, \Re(p) > 1, \quad |x| \leq 1. \]  

(1.5)
Furthermore, in [14], Hoffman shown a simple general result for the quadratic sum

$$F(\zeta(p), \zeta(q)) = \sum_{n=1}^{\infty} (\zeta(p) - \zeta_n(p)) (\zeta(q) - \zeta_n(q)), \quad p, q \in \mathbb{N} \setminus \{1\}$$

in terms of the multiple zeta values $\zeta(s_1, s_2, \ldots, s_m)$ (for short mzvs) defined by

$$\zeta(s_1, s_2, \ldots, s_m) := \sum_{k_1 > \cdots > k_m \geq 1} \frac{1}{k_1^{s_1} \cdots k_m^{s_m}},$$

(1.6)

where for convergence $s_1 + s_2 + \cdots + s_j > j$ for $j = 1, 2, \ldots, m$. Similarly, the multiple zeta star values $\zeta^*(s_1, s_2, \ldots, s_m)$ (for short mzsvs) defined by

$$\zeta(s_1, s_2, \ldots, s_m) := \sum_{k_1 \geq \cdots \geq k_m \geq 1} \frac{1}{k_1^{s_1} \cdots k_m^{s_m}},$$

(1.7)

where for convergence $s_1 + s_2 + \cdots + s_j > j$ for $j = 1, 2, \ldots, m$. Moreover, Hoffman also provided an explicit evaluation of sums of products of three or more Riemann zeta tails in a closed form in terms of multiple zeta values (see Theorem 4.3 in the reference [14]). The multiple zeta value $\zeta(s_1, s_2, \ldots, s_m)$ is said to have depth $m$ and weight $s_1 + \cdots + s_m$. Multiple zeta values of general depth were introduced independently by Hoffman [12] and D. Zagier [23], but for depth two they were already studied by Euler. In [6], Borwein and Girgensohn gave some evaluation of multiple zeta values with depth three, and they proved that all $\zeta(q, p, r)$ with $r + p + q$ is even or less than or equal to 10 or $r + p + q = 12$ were reducible to zeta values and double zeta values. Further, Eie and Wei [7] proved that for positive integer $p > 1$, all mzvs $\zeta(p, p, 1, 1)$ can be expressed in terms of mzvs of depth $\leq 3$, and gave explicit formulas. There are also a lot of contributions on multiple zeta values in the last two decades, see [4, 5, 12, 13, 22-24] and references therein.

In general, the alternating multiple zeta values and multiple zeta star values are defined by

$$\zeta(s) \equiv \zeta(s_1, \ldots, s_m) := \sum_{k_1 > \cdots > k_m \geq 1} \prod_{j=1}^{m} n_j^{-|s_j|} sgn(s_j) k_j,$$

(1.8)

$$\zeta^*(s) \equiv \zeta^*(s_1, \ldots, s_m) := \sum_{k_1 \geq \cdots \geq k_m \geq 1} \prod_{j=1}^{m} n_j^{-|s_j|} sgn(s_j) k_j,$$

(1.9)

where $s := \{s_1, \ldots, s_m\} \in (\mathbb{Z}^*)^m$ ($\mathbb{Z}^* := \{\pm 1, \pm 2, \ldots\}$) with $s_1 \neq 1$, and

$$sgn(s_j) := \begin{cases} 1, & s_j > 0, \\ -1, & s_j < 0. \end{cases}$$

Throughout the paper we will use the notation $\bar{p}$ to denote a negative entry $s_j = -p$ ($p$ is a positive integer). For example,

$$\zeta(\bar{s}_1, s_2) = \zeta(-s_1, s_2), \quad \zeta^*(\bar{s}_1, s_2, \bar{s}_3) = \zeta^*(-s_1, s_2, -s_3).$$

We call $l(s) := m$ and $|s| := \sum_{j=1}^{k} |s_j|$ the depth and the weight of (1.8) and (1.9), respectively. For convenience we let $\zeta(\emptyset) = \zeta^*(\emptyset) = 1$ and $\{s_1, \ldots, s_j\}_d$ the set formed by repeating the composition $(s_1, \ldots, s_j)$ $d$ times.
Furthermore, for convenience, we define the generalized multiple harmonic and star sums by

\[
\zeta_n(s) \equiv \zeta_n(s_1, \ldots, s_m) := \sum_{n \geq k_1 \geq \cdots \geq k_m \geq 1} \prod_{j=1}^m n_j^{-|s_j|} \text{sgn}(s_j) k_j,
\]

\[
\zeta_n^*(s) \equiv \zeta_n^*(s_1, \ldots, s_m) := \sum_{n \geq k_1 \geq \cdots \geq k_m \geq 1} \prod_{j=1}^m n_j^{-|s_j|} \text{sgn}(s_j) k_j,
\]

which are also called the partial sums of the multiple zeta and zeta star values, respectively. Here \( s := \{s_1, \ldots, s_m\} \in (\mathbb{Z}^*)^m \).

In this paper, we discuss the analytic representations of \( F(A, B; x) \) and \( F(A, B, \zeta(r)) \). The results which we present here can be seen as an extension of Hoffman’s, Furdui’s and Vălean’s work. Many results of this paper can be expressed as a rational linear combination of products of zeta values and polylogarithms. Using certain integral representations of series, we can prove that the cubic series

\[
F(\zeta(p), \zeta(q), \zeta(r)) = \sum_{n=1}^{\infty} (\zeta(p) - \zeta_n(p)) (\zeta(q) - \zeta_n(q)) (\zeta(r) - \zeta_n(r)) \quad (p, q, r \in \mathbb{N} \setminus \{1\})
\]

can be expressed in terms of multiple zeta values or multiple zeta star values with depth \( \leq 3 \). Furthermore, we also give some evaluation of sums of products of alternating Riemann zeta Tails. For example, we show that

\[
\sum_{n=1}^{\infty} (\zeta(m) - \zeta_n(m)) (\zeta(p) - \zeta_n(p)) \nonumber
\]

\[
= -\zeta^*(\bar{p}, m - 1) - \zeta^*(\bar{m}, p - 1) - \zeta(m) \zeta(p) - \zeta(m + p - 1), \quad (1.10)
\]

where \( \tilde{\zeta}(s) \) and \( \tilde{\zeta}_n(s) \) denote the alternating Riemann zeta function and its partial sum (or called the alternating harmonic number), respectively, which are defined by

\[
\tilde{\zeta}(s) := \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j^s}, \quad \tilde{\zeta}_n(s) := \sum_{j=1}^{n} \frac{(-1)^{j-1}}{j^s}, \quad n \in \mathbb{N}, \Re(s) \geq 1.
\]

From above definitions, we know that \( \tilde{\zeta}(s) = -\zeta(\bar{s}) \). Hence, the formula (1.10) can be rewritten as

\[
F(\zeta(m), \tilde{\zeta}(p)) = \zeta(m) \zeta(p) + \zeta(m + p - 1) - \zeta^*(\bar{p}, m - 1) - \zeta^*(\bar{m}, p - 1), \quad (1.11)
\]

Similarly, by using the above notations, we have the relations

\[
\zeta^*(m, p) = \sum_{n=1}^{\infty} \frac{\zeta_n(p)}{n^m} = \sum_{n=1}^{\infty} \frac{\zeta_n(p)}{n^m} (-1)^{n-1},
\]

\[
\zeta^*(\bar{m}, \bar{p}) = -\sum_{n=1}^{\infty} \frac{\zeta_n(p)}{n^m} (-1)^{n-1}, \zeta^*(m, \bar{p}) = -\sum_{n=1}^{\infty} \frac{\zeta_n(p)}{n^m}.
\]

The evaluation of linear sums (or double sums)

\[
\sum_{n=1}^{\infty} \frac{\zeta_n(p)}{n^m}, \sum_{n=1}^{\infty} \frac{\zeta_n(p)}{n^m} (-1)^{n-1}, \sum_{n=1}^{\infty} \frac{\zeta_n(p)}{n^m} (-1)^{n-1}
\]

and

\[
\sum_{n=1}^{\infty} \frac{\zeta_n(p)}{n^m}
\]
Therefore, we can deduce that

\[
\sum_{n=1}^{\infty} \frac{\zeta_n(1)}{n^3} = \frac{7}{4} \zeta(3) \ln 2 - \frac{5}{16} \zeta(4),
\]

\[
\sum_{n=1}^{\infty} \frac{H_n}{n^3} (-1)^{n-1} = -2 \text{Li}_4 \left( \frac{1}{2} \right) + \frac{11}{4} \zeta(4) + \frac{1}{2} \zeta(2) \ln^2 2 - \frac{1}{12} \ln^4 2 - \frac{7}{4} \zeta(3) \ln 2,
\]

\[
\sum_{n=1}^{\infty} \frac{\zeta_n(1)}{n^3} (-1)^{n-1} = \frac{3}{2} \zeta(4) + \frac{1}{2} \zeta(2) \ln^2 2 - \frac{1}{12} \ln^4 2 - 2 \text{Li}_4 \left( \frac{1}{2} \right),
\]

\[
\sum_{n=1}^{\infty} \frac{\zeta_n(2)}{n^2} (-1)^{n-1} = -\frac{51}{16} \zeta(4) + 4 \text{Li}_4 \left( \frac{1}{2} \right) + \frac{7}{2} \zeta(3) \ln 2 - \zeta(2) \ln^2 2 + \frac{1}{6} \ln^4 2.
\]

Furthermore, we consider the nested sum

\[
\sum_{n=1}^{\infty} \frac{y^n}{n^m} \left( \sum_{k=1}^{n} x^k \right), \quad x, y \in [-1, 1], \quad m, p \in \mathbb{N}.
\]

By taking the sum over complementary pairs of summation indices, we obtain a simple reflection formula

\[
\sum_{n=1}^{\infty} \frac{y^n}{n^m} \left( \sum_{k=1}^{n} x^k \right) + \sum_{n=1}^{\infty} \frac{x^n}{n^p} \left( \sum_{k=1}^{n} y^k \right) = \text{Li}_p \left( x \right) \text{Li}_m \left( y \right) + \text{Li}_{p+m} \left( xy \right).
\]

Therefore, we can deduce that

\[
\sum_{n=1}^{\infty} \frac{\zeta_n(2)}{n^2} (-1)^{n-1} = \frac{13}{16} \zeta(4),
\]

\[
\sum_{n=1}^{\infty} \frac{\zeta_n(3)}{n} (-1)^{n-1} = \frac{19}{16} \zeta(4) - \frac{3}{4} \zeta(3) \ln 2,
\]

\[
\sum_{n=1}^{\infty} \frac{\zeta_n(2)}{n^2} = \frac{85}{16} \zeta(4) - 4 \text{Li}_4 \left( \frac{1}{2} \right) + \zeta(2) \ln^2 2 - \frac{1}{6} \ln^4 2 - \frac{7}{2} \zeta(3) \ln 2,
\]

\[
\sum_{n=1}^{\infty} \frac{\zeta_n(3)}{n} (-1)^{n-1} = 2 \text{Li}_4 \left( \frac{1}{2} \right) + \frac{3}{4} \zeta(3) \ln 2 + \frac{1}{12} \ln^4 2 - \frac{1}{2} \zeta(4) - \frac{1}{2} \zeta(2) \ln^2 2.
\]

Some results for sums of harmonic numbers or alternating harmonic numbers may be seen in the works of [2, 3, 8, 15–20] and references therein.

The main results of this paper are the following theorems.

**Theorem 1.1** If \(a_k, b_k = o(n^{-p}), R(p) > 1\) and \(x \in [-1, 1]\), then

\[
(1 - x) F_n (A, B; x) = (1 - x) \sum_{k=1}^{n} (A - A_k) (B - B_k) x^k
\]

\[
= ABx - (A - A_n) (B - B_n) x^{n+1} - \left( \sum_{k=1}^{n} a_k x^k \right) (B - B_n) - \left( \sum_{k=1}^{n} b_k x^k \right) (A - A_n)
\]
In this section, we will use Abel’s summation formula to prove Theorem 2.1.

### 2.1 Proof of Theorem 1.1

From the definition of function $F_n(A, B; x)$, we have

$$F_n(A, B; x) = \sum_{k=1}^{n} (A - A_k) (B - B_k) x^k$$

$$= (A - a_1) (B - b_1) x + \sum_{k=1}^{n-1} (A - A_{k+1}) (B - B_{k+1}) x^{k+1}$$

$$= (A - a_1) (B - b_1) x + \sum_{k=1}^{n-1} (A - A_k - a_{k+1}) (B - B_k - b_{k+1}) x^{k+1}$$

$$= (A - a_1) (B - b_1) x + \sum_{k=1}^{n-1} (A - A_k) (B - B_k) x^{k+1}$$

$$- \sum_{k=1}^{n-1} \{a_{k+1} (B - B_k) + b_{k+1} (A - A_k)\} x^{k+1} + \sum_{k=1}^{n-1} a_{k+1} b_{k+1} x^{k+1}$$

$$= (A - a_1) (B - b_1) x + \sum_{k=1}^{n} (A - A_k) (B - B_k) x^{k+1} - (A - A_n) (B - B_n) x^{n+1}$$

### Theorem 1.2

If $a_k, b_k = o(n^{-p}), \Re(p) > 1$ and $r \in \mathbb{N} \setminus \{1\}$, then

$$F(A, B, \zeta(r)) = \sum_{k=1}^{\infty} (A - A_k) (B - B_k) (\zeta(r) - \zeta_k(r))$$

$$= \zeta(r) \left( \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} ka_k \right) b_n + \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} kb_k \right) a_n - \sum_{n=1}^{\infty} n a_n b_n - AB \right)$$

$$- \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} a_k (k \zeta_k(r) - \zeta_k(r-1)) \right) b_n - \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} b_k (k \zeta_k(r) - \zeta_k(r-1)) \right) a_n$$

$$+ \sum_{n=1}^{\infty} a_n b_n (n \zeta_n(r) - \zeta_n(r-1)).$$

**2 Proofs of Main Theorems**

In this section, we will use Abel’s summation formula to prove Theorem 1.1 and 1.2. The Abel’s summation by parts formula states that if $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ are two sequences of real numbers and $A_n = \sum_{k=1}^{n} a_k$, then

$$\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} + \sum_{k=1}^{n} A_k (b_k - b_{k+1}).$$

Firstly, we are ready to prove Theorem 1.1.

### 2.1 Proof of Theorem 1.1

From the definition of function $F_n(A, B; x)$, we have

$$F_n(A, B; x) = \sum_{k=1}^{n} (A - A_k) (B - B_k) x^k$$

$$= (A - a_1) (B - b_1) x + \sum_{k=1}^{n-1} (A - A_{k+1}) (B - B_{k+1}) x^{k+1}$$

$$= (A - a_1) (B - b_1) x + \sum_{k=1}^{n-1} (A - A_k - a_{k+1}) (B - B_k - b_{k+1}) x^{k+1}$$

$$= (A - a_1) (B - b_1) x + \sum_{k=1}^{n-1} (A - A_k) (B - B_k) x^{k+1}$$

$$- \sum_{k=1}^{n-1} \{a_{k+1} (B - B_k) + b_{k+1} (A - A_k)\} x^{k+1} + \sum_{k=1}^{n-1} a_{k+1} b_{k+1} x^{k+1}$$

$$= (A - a_1) (B - b_1) x + \sum_{k=1}^{n} (A - A_k) (B - B_k) x^{k+1} - (A - A_n) (B - B_n) x^{n+1}$$
Taking by a direct calculation, we obtain the proof of Theorem 2.2

\[
\sum_{k=1}^{n} \{a_k (B - B_k) + b_k (A - A_k)\} x^k - \sum_{k=1}^{n} a_k b_k x^k.
\]

Then by using the Abel’s summation formula, we can find that

\[
\sum_{k=1}^{n} \{a_k (B - B_k)\} x^k = \left(\sum_{k=1}^{n} a_k x^k\right) (B - B_n) + \sum_{k=1}^{n} \left(\sum_{i=1}^{k} a_i x^i\right) b_k - \sum_{k=1}^{n} a_k b_k x^k,
\]

\[
\sum_{k=1}^{n} \{b_k (A - A_k)\} x^k = \left(\sum_{k=1}^{n} b_k x^k\right) (A - A_n) + \sum_{k=1}^{n} \left(\sum_{i=1}^{k} b_i x^i\right) a_k - \sum_{k=1}^{n} a_k b_k x^k.
\]

Hence, substituting formulas (2.2) and (2.3) into (2.1) respectively, we deduce the (1.12). Thus, the proof of Theorem 1.1 is finished.

### 2.2 Proof of Theorem 1.2

Now we are ready to prove Theorem 1.2. Multiplying (1.12) by \(\frac{\ln^{r-1} x}{(1 - x)^2}\) and integrating over the interval \((0,1)\), and using the following elementary integral identities

\[
\int_{0}^{1} x^{k} \ln^{r-1} x \frac{1}{1 - x} \, dx = (-1)^{r-1} (r - 1)! \left(\zeta (r) - \zeta_k (r)\right), \quad r \in \mathbb{N} \setminus \{1\},
\]

\[
\int_{0}^{1} x^{k} \ln^{r-1} x \frac{1}{(1 - x)^2} \, dx = (-1)^{r-1} (r - 1)! \left(\zeta (r - 1) - \zeta_k (r - 1) - k \zeta (r) + k \zeta_k (r)\right), \quad r \in \mathbb{N} \setminus \{1, 2\},
\]

by a direct calculation, we obtain

\[
F (A, B, \zeta (r)) = AB \left(\zeta (r - 1) - \zeta (r)\right)
+ \sum_{k=1}^{\infty} a_k b_k \left(\zeta (r - 1) - \zeta_k (r - 1) - k \zeta (r) + k \zeta_k (r)\right)
- \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} a_k \left(\zeta (r - 1) - \zeta_k (r - 1) - k \zeta (r) + k \zeta_k (r)\right)\right) b_n
- \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} b_k \left(\zeta (r - 1) - \zeta_k (r - 1) - k \zeta (r) + k \zeta_k (r)\right)\right) a_n.
\]

Taking \(x = 1\) in (1.12), we get

\[
\sum_{k=1}^{n} A_k b_k + \sum_{k=1}^{n} B_k a_k = \sum_{k=1}^{n} a_k b_k + A_n B_n.
\]
Letting $n \to \infty$, then
\[
\sum_{k=1}^{\infty} A_k b_k + \sum_{k=1}^{\infty} B_k a_k = \sum_{k=1}^{\infty} a_k b_k + AB.
\]

Substituting (2.6) into (2.4), we know that when $r \in \mathbb{N} \setminus \{1, 2\}$, the equality (1.13) holds.

Similarly, multiplying (1.12) by $\frac{\ln x}{(1 - x)^2}$ and integrating over the interval $(0, t)$, then letting $t \to 1$, we can prove that if $r = 2$, the formula (1.2) is also true. The proof of Theorem 1.2 is finished.

## 3 Some Results on Zeta Tails

In this section, we will give some closed form of sums of sequences tails.

Firstly, multiplying $(1-x)^{-1}$ in both sides of (1.12), then letting $x \to 1$ and using L'Hopital’s rule, we deduce that
\[
\sum_{k=1}^{n} (A - A_k) (B - B_k) = F_n (A, B; 1) = \lim_{x \to 1} \frac{(1 - x) F_n (A, B; x)}{1 - x}
\]
\[
= \sum_{k=1}^{n} \left( \sum_{i=1}^{k} i a_i \right) b_k + \sum_{k=1}^{n} \left( \sum_{i=1}^{k} i b_i \right) a_k + \left( \sum_{k=1}^{n} k a_k \right) (B - B_n) + \left( \sum_{k=1}^{n} k b_k \right) (A - A_n)
\]
\[
+ (n + 1) (A - A_n) (B - B_n) - \left( \sum_{k=1}^{n} k a_k b_k \right) - AB.
\]

Letting $n \to \infty$ in (3.1), we arrive at the conclusion that
\[
\sum_{n=1}^{\infty} (A - A_n) (B - B_n) = \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} k a_k \right) b_n + \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} k b_k \right) a_n - \sum_{n=1}^{\infty} n a_n b_n - AB.
\]

Putting $(a_n, b_n) = \left( \frac{1}{n^m}, \frac{1}{n^p} \right)$, $(\frac{\ln n}{n^m}, \frac{\ln n}{n^p})$ in (3.2), and using the following relations
\[
\zeta^* (m, p) = \zeta (m, p) + \zeta (m + p),
\]
\[
\zeta^* (\bar{m}, \bar{p}) = \zeta (\bar{m}, \bar{p}) + \zeta (m + p),
\]
\[
\zeta^* (\bar{m}, p) = \zeta (\bar{m}, p) + \zeta (m + \bar{p}),
\]
\[
\zeta^* (m, \bar{p}) = \zeta (m, \bar{p}) + \zeta (m + \bar{p}),
\]
we obtain the following identities
\[
F (\zeta (m), \zeta (p)) = \sum_{n=1}^{\infty} (\zeta (m) - \zeta_n (m)) (\zeta (p) - \zeta_n (p))
\]
\[
= \zeta (p, m - 1) + \zeta (m, p - 1) + \zeta (m + p - 1) - \zeta (m) \zeta (p),
\]
\[
F (\zeta (\bar{m}), \zeta (\bar{p})) = \sum_{n=1}^{\infty} (\zeta (\bar{m}) - \zeta_n (\bar{m})) (\zeta (\bar{p}) - \zeta_n (\bar{p}))
\]
\[
\begin{align*}
&= \zeta (\bar{m}, p - 1) + \zeta (\bar{p}, m - 1) + \zeta (m + p - 1) - \zeta (m) \zeta (\bar{p}), \quad (3.4) \\
F(\zeta(m), \zeta(\bar{p})) &= \sum_{n=1}^{\infty} (\zeta(m) - \zeta_n(m)) (\zeta(\bar{p}) - \zeta_n(\bar{p})) \\
&= \zeta (m, p - 1) + \zeta (\bar{p}, m - 1) + \zeta (m + p - 1) - \zeta (m) \zeta (\bar{p}). \quad (3.5)
\end{align*}
\]

Similarly, in the case \((a_n, b_n) = \left(\frac{1}{n^m}, \frac{1}{n^p}\right)\), Theorem 1.2 gives

\[
\begin{align*}
F(\zeta(m), \zeta(p), \zeta(r)) &= \sum_{n=1}^{\infty} (\zeta(m) - \zeta_n(m)) (\zeta(p) - \zeta_n(p)) (\zeta(r) - \zeta_n(r)) \\
&= \zeta(r) \{\zeta^*(p, m - 1) + \zeta^*(m, p - 1) - \zeta (m + p - 1) - \zeta (m) \zeta (p)\} \\
&\quad + \{\zeta^*(m + p - 1, r) - \zeta^*(m + p, r - 1)\} \\
&\quad - \{\zeta^*(p, m - 1, r) - \zeta^*(p, m, r - 1)\} \\
&\quad - \{\zeta^*(m, p - 1, r) - \zeta^*(m, p, r - 1)\}. \quad (3.6)
\end{align*}
\]

By the definition of multiple zeta function and multiple zeta-star function, the following relation is easily derived

\[
\zeta^*(q, p, r) = \zeta (q, p, r) + \zeta (q, p + r) + \zeta (q + p, r) + \zeta (q + p + r). \quad (3.7)
\]

Hence, by a direct calculation, we can rewrite (3.7) as

\[
\begin{align*}
F(\zeta(m), \zeta(p), \zeta(r)) &= \zeta (p, m, r - 1) + \zeta (p, r, m - 1) + \zeta (m, p, r - 1) + \zeta (m, r, p - 1) \\
&\quad + \zeta (r, p, m - 1) + \zeta (r, m, p - 1) + \zeta (m + p, r - 1) + \zeta (m + p, r - 1) \\
&\quad + \zeta (m + r, p - 1) + \zeta (p, m + r - 1) + \zeta (m, p + r - 1) \\
&\quad + \zeta (r, p + m - 1) + \zeta (m + p + r - 1) - \zeta (p) \zeta (m) \zeta (r). \quad (3.8)
\end{align*}
\]

This formula was also given by Hoffman in [14].

In (1.12), letting \(n \to \infty\), then multiplying it by \(\frac{\ln^{r-1} x}{x(1 - x)} \ (r \in \mathbb{N} \setminus \{1\})\) and integrating over the interval \((0,1)\), we conclude that

\[
\sum_{n=1}^{\infty} \frac{(A - A_n) (B - B_n)}{n^r} = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} a_k \zeta_{k-1}(r)\right) b_n + \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} b_k \zeta_{k-1}(r)\right) a_n \\
- \sum_{n=1}^{\infty} a_n b_n \zeta_{n-1}(r). \quad (3.9)
\]

In fact, it is easily shown that the formula (3.9) is also true when \(r = 1\).

Taking \((a_n, b_n) = \left(\frac{1}{n^m}, \frac{1}{n^p}\right)\) in (3.9) and using the relation (3.7), we get

\[
\sum_{n=1}^{\infty} \frac{(\zeta(m) - \zeta_n(m)) (\zeta(p) - \zeta_n(p))}{n^r} = \zeta (m, p, r) + \zeta (p, m, r) + \zeta (m + p, r). \quad (3.10)
\]
Moreover, by a simple calculation, we derive the following identity

\[ \sum_{n=1}^{\infty} \frac{\zeta_n(m) \zeta_n(p)}{n^r} = \zeta^*(r, m, p) + \zeta^*(r, p, m) - \zeta^*(r, m + p) \quad (r > 1, m, p \geq 1). \]

Hence, we give the general formula

\[ \lim_{n \to \infty} \sum_{k=1}^{n} \frac{\zeta_2(k) \zeta_2(k)}{n^r} = \zeta^*(r, m, p) + \zeta^*(r, p, m) - \zeta^*(r, m + p) \quad (r > 1, m, p \geq 1). \]

Moreover, by a simple calculation, we derive the following identity

\[ \sum_{n=1}^{\infty} \frac{\zeta_n(p) \zeta_n(r)}{n^m} = \zeta^*(p + m, r) + \zeta(p) \zeta^*(m, r) - \zeta^*(p, m, r), \quad p, m \in \mathbb{N} \setminus \{1\}, r \in \mathbb{N}. \quad (3.11) \]

Substituting (3.11) into (3.10), we obtain the symmetry identity

\[ \sum_{n=1}^{\infty} \left\{ \frac{\zeta_n(m) \zeta_n(p)}{n^r} + \frac{\zeta_n(p) \zeta_n(r)}{n^m} - \frac{\zeta_n(m) \zeta_n(r)}{n^p} \right\} = \zeta^*(m + p, r) + \zeta(p) \zeta(m) \zeta(r) + \zeta(p + m, r), \quad p, m, r \in \mathbb{N}/\{1\}. \quad (3.12) \]

From formula (3.11) and references [6, 8, 19, 21, 22], we know that the quadratic sums

\[ S_{mp,r} := \sum_{n=1}^{\infty} \frac{\zeta_n(m) \zeta_n(p)}{n^r} \]

can be evaluated in terms of zeta values and double zeta (star) values in the following cases: $m = p = 1$, $m = p = r$ and $m + p + r$ is even or less than or equal to $m + p + r \leq 10$. For example, from [19, 21] we have

\[ \sum_{n=1}^{\infty} \frac{H_n \zeta_n(3)}{n^3} = \frac{83}{8} \zeta(7) + \frac{1}{4} \zeta(3) \zeta(4) - \frac{11}{2} \zeta(2) \zeta(5), \]

\[ \sum_{n=1}^{\infty} \frac{H_n \zeta_n(2)}{n^5} = -\frac{343}{48} \zeta(8) + 12 \zeta(3) \zeta(5) - \frac{5}{2} \zeta(2) \zeta^2(3) - \frac{3}{4} \zeta^*(6, 2), \]

\[ \sum_{n=1}^{\infty} \frac{\zeta_n^2(2)}{n^4} = 11 \zeta^*(6, 2) + \frac{457}{18} \zeta(8) + 6 \zeta(2) \zeta^2(3) - 40 \zeta(3) \zeta(5), \]

\[ \sum_{n=1}^{\infty} \frac{\zeta_n^2(2) \zeta_n(3)}{n^2} = -\frac{617}{72} \zeta(9) + \zeta^3(3) + \frac{91}{8} \zeta(2) \zeta(7) - \frac{17}{4} \zeta(4) \zeta(5) - \frac{329}{84} \zeta(3) \zeta(6), \]

\[ \sum_{n=1}^{\infty} \frac{\zeta_n^2(2)}{n^6} = \frac{2697}{40} \zeta(10) - 41 \zeta^2(5) - 63 \zeta(3) \zeta(7) + 16 \zeta(2) \zeta(3) \zeta(5) + 4 \zeta^2(3) \zeta(4), \]

\[ + \frac{23}{2} \zeta^*(8, 2) + 2 \zeta(2) \zeta^*(6, 2). \]

Similarly, from (1.12), and by a similar as method in the proofs of (3.3)-(3.5), we deduce the following results

\[ \sum_{n=1}^{\infty} \frac{\zeta(n) - \zeta_2(n)}{1} = 1 \]
Then, by using the definition of harmonic numbers, it is easily seen that
\[
\sum_{n=1}^{\infty} (\zeta^*(p + 1, p) - \zeta^*_n(p + 1, p)) (\zeta^*(q + 1, q) - \zeta^*_n(q + 1, q))
\]
\[
= \frac{1}{2} \sum_{n=1}^{\infty} \left\{ \frac{\zeta^3_n(p)}{n^{q+1}} + \frac{\zeta^2_n(p) \zeta_n(2p) \zeta_n(q)}{n^{p+1}} \right\}
\]
\[- \sum_{n=1}^{\infty} \frac{\zeta_n(p) \zeta_n(q)}{n^{p+q+1}} - \left( \sum_{n=1}^{\infty} \frac{\zeta_n(p)}{n^{p+1}} \right) \left( \sum_{n=1}^{\infty} \frac{\zeta_n(q)}{n^{q+1}} \right),
\]
\[
\sum_{n=1}^{\infty} (\zeta^*(p + 1, p) - \zeta^*_n(p + 1, p)) (\zeta^*(p + 2, p) - \zeta^*_n(p + 2, p))
\]
\[
= \frac{1}{2} \sum_{n=1}^{\infty} \left\{ \frac{\zeta^3_n(p)}{n^{p+2}} + \frac{\zeta^2_n(p) \zeta_n(2p) \zeta_n(q)}{n^{p+1}} \right\} + \frac{1}{2} \left( \sum_{n=1}^{\infty} \frac{\zeta_n(p)}{n^{p+1}} \right)^2
\]
\[- \left( \sum_{n=1}^{\infty} \frac{\zeta_n(p)}{n^{p+1}} \right) \left( \sum_{n=1}^{\infty} \frac{\zeta_n(p)}{n^{p+2}} \right).
\]

Next, we give a Theorem.

**Theorem 3.1** Let \( m, p \geq 2 \) be integers, then we have
\[
\sum_{n=1}^{\infty} \frac{\zeta(m) \zeta(p) - \zeta(p) \zeta(m)}{n} = \zeta(p) \zeta(m, 1) - \zeta(m) \zeta(p, 1).
\]

Next, we will use two different methods to prove the theorem above.

**Proof.** 1. First, we give a complicated proof. To prove the identity (3.16), we construct the following power function
\[
y := \sum_{n=1}^{\infty} \{H_n \zeta_n(m) - \zeta_n(m + 1)\} x^{n-1}, \ x \in (-1, 1).
\]

Then, by using the definition of harmonic numbers, it is easily seen that
\[
y = \sum_{n=1}^{\infty} \left\{ \left( H_n + \frac{1}{n+1} \right) \left( \zeta_n(m) + \frac{1}{(n+1)^m} \right) - \left( \zeta_n(m + 1) + \frac{1}{(n+1)^{m+1}} \right) \right\} x^n
\]
\[
= \sum_{n=1}^{\infty} \left\{ H_n \zeta_n(m) - \zeta_n(m + 1) + \frac{H_n}{(n+1)^m} + \frac{\zeta_n(m)}{n+1} \right\} x^n.
\]

Therefore, from (3.17), we obtain the following relation
\[
\sum_{n=1}^{\infty} \{H_n \zeta_n(m) - \zeta_n(m + 1)\} x^{n-1} = \sum_{n=1}^{\infty} \left\{ \frac{H_n}{(n+1)^m} + \frac{\zeta_n(m)}{n+1} \right\} \frac{x^n}{1-x}.
\]
Thus, combining (3.18) with \( \ln^{p-1} x \) and integrating over \((0,1)\), we conclude that

\[
\sum_{n=1}^{\infty} \frac{H_n \zeta_n (m) - \zeta_n (m + 1)}{n^p} = \sum_{n=1}^{\infty} \left\{ \frac{H_n}{(n+1)^m} + \frac{\zeta_n (m)}{n+1} \right\} \{ \zeta (p) - \zeta (p) \}. \tag{3.19}
\]

Hence, by a direct calculation, the result is

\[
\sum_{n=1}^{\infty} \left\{ \frac{H_n \zeta_n (m)}{n^p} + \frac{H_n \zeta_n (p)}{n^m} \right\} = \zeta (p) \sum_{n=1}^{\infty} \frac{H_n}{n^p} + \sum_{n=1}^{\infty} \frac{H_n}{n^p+1} + \sum_{n=1}^{\infty} \frac{\zeta_n (m)}{n^p+1} - \sum_{n=1}^{\infty} \frac{\zeta_n (m + 1)}{n^p} + \sum_{n=1}^{\infty} \frac{\zeta_n (m)}{n} \{ \zeta (p) - \zeta (p) \}. \tag{3.20}
\]

Changing \( m, p \) to \( p, m \) in above equation, we can get

\[
\sum_{n=1}^{\infty} \left\{ \frac{H_n \zeta_n (p)}{n^m} + \frac{H_n \zeta_n (m)}{n^p} \right\} = \zeta (m) \sum_{n=1}^{\infty} \frac{H_n}{n^m} + \sum_{n=1}^{\infty} \frac{H_n}{n^{m+p}} + \sum_{n=1}^{\infty} \frac{\zeta_n (p)}{n^{m+p}} - \sum_{n=1}^{\infty} \frac{\zeta_n (p + 1)}{n^m} + \sum_{n=1}^{\infty} \frac{\zeta_n (p)}{n} \{ \zeta (m) - \zeta (m) \}. \tag{3.21}
\]

Thus, combining (3.20) with (3.21), and using the simple relation

\[
\sum_{n=1}^{\infty} \frac{H_n}{n^m} = \zeta (m, 1) + \zeta (m + 1),
\]

we may deduce the desired result. The proof of Theorem 3.1 is finished.

\( \square \)

**Proof.** 2. In fact, one can prove a more general result without using the above complicated way. Writing \( \zeta_{>n} \) for \( \zeta (n) - \zeta (n) \), one has

\[
\sum_{n=1}^{\infty} \frac{\zeta (m) \zeta_n (p) - \zeta (p) \zeta_n (m)}{n^k} = \sum_{n=1}^{\infty} \frac{(\zeta_n (m) + \zeta_{>n} (m)) \zeta_n (p) - (\zeta_n (p) + \zeta_{>n} (p)) \zeta_n (m)}{n^k} = \sum_{n=1}^{\infty} \frac{\zeta_{>n} (m) \zeta_n (p) - \zeta_{>n} (p) \zeta_n (m)}{n^k} = \zeta (m, k, p) + \zeta (m, k + p) - \zeta (p, k, m) - \zeta (p, k + m)
\]

for any positive integer \( k \). One the other hand, by the well-known “harmonic product” (also called the “stuffle product”) of multiple zeta values \([13]\), we have

\[
\zeta (p) \zeta (m, k) = \zeta (p, m, k) + \zeta (p + m, k) + \zeta (m, p, k) + \zeta (m, p + k) + \zeta (m, k, p).
\]

Similarly expanding out \( \zeta (m) \zeta (p, k) \) and cancelling gives

\[
\sum_{n=1}^{\infty} \frac{\zeta (m) \zeta_n (p) - \zeta (p) \zeta_n (m)}{n^k} = \zeta (p) \zeta (m, k) - \zeta (m) \zeta (p, k). \tag{3.22}
\]
It is clear that (3.16) is immediate corollary of (3.22). □

Next, we continue to establish some relations between sums of zeta tails and multiple zeta (star) values.

\[
\sum_{n=1}^{\infty} \frac{\zeta(m) \zeta(p) - \zeta_n(m) \zeta_n(p)}{n} = \zeta(p) \sum_{n=1}^{\infty} \frac{\zeta(m) - \zeta_n(m)}{n} + \sum_{n=1}^{\infty} \frac{\zeta_n(m)(\zeta(p) - \zeta_n(p))}{n}, \quad m, p \in \mathbb{N} \setminus \{1\}. \tag{3.23}
\]

In [11], O. Furdui and C. Vălean gave the formula

\[
\sum_{n=1}^{\infty} \frac{\zeta(m) - \zeta_n(m)}{n} = \zeta(m, 1). \tag{3.24}
\]

Hence, combining (3.20), (3.23) and (3.24), we have the following corollary.

**Corollary 3.2** For positive integers \(m > 1\) and \(p > 1\), then we have

\[
\sum_{n=1}^{\infty} \frac{\zeta(m) \zeta(p) - \zeta_n(m) \zeta_n(p)}{n} = \sum_{n=1}^{\infty} \left\{ \frac{H_n \zeta_n(m)}{n^p} + \frac{H_n \zeta_n(p)}{n^m} \right\} + \zeta^*(p, m + 1) - \zeta(p) \zeta(m + 1) - \zeta^*(p + m, 1) - \zeta^*(p + 1, m). \tag{3.25}
\]

Similarly, noting that

\[
\sum_{n=1}^{\infty} \frac{\zeta_n(m) \zeta(p) \zeta(q) - \zeta(m) \zeta_n(p) \zeta_n(q)}{n} = \zeta(m) \sum_{n=1}^{\infty} \frac{\zeta(p) \zeta(q) - \zeta_n(p) \zeta_n(q)}{n} - \zeta(p) \zeta(q) \sum_{n=1}^{\infty} \frac{\zeta(m) - \zeta_n(m)}{n}, \quad m, p, q \in \mathbb{N} \setminus \{1\},
\]

we obtain the following corollary.

**Corollary 3.3** For positive integers \(m, p, q > 1\), then we have

\[
\sum_{n=1}^{\infty} \frac{\zeta_n(m) \zeta(p) \zeta(q) - \zeta(m) \zeta_n(p) \zeta_n(q)}{n} = \zeta(m) \left\{ \sum_{n=1}^{\infty} \left( \frac{H_n \zeta_n(q)}{n^p} + \frac{H_n \zeta_n(p)}{n^q} \right) + \zeta^*(p, q + 1) \right\} - \zeta(p) \zeta(q) \zeta(m, 1). \tag{3.26}
\]

Furthermore, by considering the following power series function

\[
y = \sum_{n=1}^{\infty} \left( H_n \zeta_n(p) \zeta_n(q) - \zeta_n(p + q + 1) \right) x^{n-1}, \quad x \in (-1, 1),
\]

and proceeding in a similar fashion to evaluation of the Theorem 3.1, we can give the following Theorem.
Theorem 3.4 For positive integers \( p, m > 1 \) and \( q > 0 \), we have

\[
\sum_{n=1}^{\infty} \left( \zeta (m) \zeta_n (p) \zeta_n (q) - \zeta (p) \zeta_n (m) \zeta_n (q) \right) = \zeta (p) \left\{ \sum_{n=1}^{\infty} \left( \frac{H_n \zeta_n (m)}{n^q} + \frac{H_n \zeta_n (q)}{n^m} \right) - \zeta^* (q + 1, m) - \zeta^* (m + q, 1) - \zeta^* (m + 1, q) \right\} - \zeta (m) \left\{ \sum_{n=1}^{\infty} \left( \frac{H_n \zeta_n (q)}{n^p} + \frac{H_n \zeta_n (p)}{n^q} \right) - \zeta^* (q + 1, p) - \zeta^* (p + q, 1) - \zeta^* (p + 1, q) \right\} + \zeta (m + q + 1) \zeta (p) - \zeta (p + q + 1) \zeta (m).
\]

On the other hand, we note that

\[
\sum_{n=1}^{\infty} \left( \zeta_n (m) \left( \zeta (p) - \zeta_n (p) \right) \right) = \zeta (p, 1, m) + \zeta (p, m + 1). \tag{3.27}
\]

The relations (3.20) and (3.27) yield the following result:

\[
\sum_{n=1}^{\infty} \left\{ \frac{H_n \zeta_n (m)}{n^p} + \frac{H_n \zeta_n (p)}{n^m} \right\} = \zeta (p) \zeta (m, 1) + \zeta (p) \zeta (m + 1) + \zeta (p + m, 1) + \zeta (p + 1, m) + \zeta (p + m + 1) + \zeta (p, 1, m). \tag{3.28}
\]

Therefore, by using the above relation (3.28), Corollary 3.2, Corollary 3.3 and Theorem 3.4 can be written as follows.

Corollary 3.5 For positive integers \( m > 1 \) and \( p > 1 \), then the following relations hold:

\[
\sum_{n=1}^{\infty} \left( \zeta (m) \zeta_n (p) - \zeta_n (m) \zeta_n (p) \right) = \zeta (p) \zeta (m, 1) + \zeta (p, 1, m) + \zeta (p, m + 1). \tag{3.29}
\]

\[
\sum_{n=1}^{\infty} \left( \zeta_n (m) \zeta_n (p) \zeta (q) - \zeta (m) \zeta_n (p) \zeta_n (q) \right) = \zeta (m) \left[ \zeta (p) \zeta (q, 1) + \zeta (p, 1, q) + \zeta (p, q + 1) \right] - \zeta (p) \zeta (q) \zeta (m, 1), \tag{3.30}
\]

\[
\sum_{n=1}^{\infty} \left( \zeta (m) \zeta_n (p) \zeta_n (q) - \zeta (p) \zeta_n (m) \zeta_n (q) \right) = \zeta (p) \left[ \zeta (m, 1, q) + \zeta (m, q + 1) \right] - \zeta (m) \left[ \zeta (p, 1, q) + \zeta (p, q + 1) \right]. \tag{3.31}
\]

In fact, applying the same arguments as in the proof of (3.22), the above formulas (3.29)-(3.31) can be also proved by using the method of stuffle product.

We now close this section with a final theorem.
Theorem 3.6 For positive integers $m, p, k > 1$, then the following relations hold:

$$
\sum_{n=1}^{\infty} \frac{\zeta(n) \zeta(p) - \zeta_n(m) \zeta_n(p)}{n^k} = \zeta(p) \zeta(m, k) + \zeta(p, k, m) + \zeta(p, m + k).
$$

(3.32)

$$
\sum_{n=1}^{\infty} \frac{\zeta_n(m) \zeta(p) \zeta(q) - \zeta(m) \zeta_n(p) \zeta_n(q)}{n^k} = \zeta(m) [\zeta(p \zeta(q, k) + \zeta(p, k, q) + \zeta(p, q + k)] - \zeta(p) \zeta(q) \zeta(m, k).
$$

(3.33)

$$
\sum_{n=1}^{\infty} \frac{\zeta(m) \zeta_n(p) \zeta_n(q) - \zeta(p) \zeta_n(m) \zeta_n(q)}{n^k} = \zeta(p) [\zeta(m, k, q) + \zeta(m, q + k)] - \zeta(m) [\zeta(p, k, q) + \zeta(p, q + k)].
$$

(3.34)

Proof. On the one hand, for any integers $m, p, k > 1$, we note that the series on the left hand side of formulas (3.32)-(3.34) can be written as

$$
\sum_{n=1}^{\infty} \frac{\zeta(m) \zeta(p) - \zeta_n(m) \zeta_n(p)}{n^k} = \zeta(m) \zeta(p) \zeta(k) - \sum_{n=1}^{\infty} \frac{\zeta_n(m) \zeta_n(p)}{n^k},
$$

(3.35)

$$
\sum_{n=1}^{\infty} \frac{\zeta_n(m) \zeta(p) \zeta(q) - \zeta(m) \zeta_n(p) \zeta_n(q)}{n^k} = \zeta(p) \zeta(q) \zeta(k, m) - \zeta(m) \sum_{n=1}^{\infty} \frac{\zeta_n(p) \zeta_n(q)}{n^k},
$$

(3.36)

$$
\sum_{n=1}^{\infty} \frac{\zeta(m) \zeta_n(p) \zeta_n(q) - \zeta(p) \zeta_n(m) \zeta_n(q)}{n^k} = \zeta(m) \sum_{n=1}^{\infty} \frac{\zeta_n(p) \zeta_n(q)}{n^k} - \zeta(p) \sum_{n=1}^{\infty} \frac{\zeta_n(m) \zeta_n(q)}{n^k}.
$$

(3.37)

On the other hand, from (3.7) and (3.11), we readily find that

$$
\sum_{n=1}^{\infty} \frac{\zeta_n(m) \zeta_n(p)}{n^k} = \zeta(p) \zeta(m) \zeta(k) - \zeta(p) \zeta(m, k) - \zeta(p, k, m) - \zeta(p, k + m).
$$

(3.38)

Then, substituting (3.38) into (3.35)-(3.37), by a simple calculation, we may easily deduce the desired results.

It is clear that the identities (3.32)-(3.34) can be proved by using the method of stuffle product.

4 Some Examples

In section 3, we prove many interesting results involving zeta tails. In this section, we apply these results to give various specific cases. Some illustrative examples follow.

$$
\sum_{n=1}^{\infty} \frac{(\zeta(2) - \zeta_n(2))^2}{n} = 5\zeta(2) \zeta(3) - 9\zeta(5),
$$

15
\[
\sum_{n=1}^{\infty} (\zeta(2) - \zeta_n(2))^2 = 3\zeta(2) \ln 2 - \frac{9}{4} \zeta(3) - \frac{5}{8} \zeta(4),
\]
\[
\sum_{n=1}^{\infty} \frac{(\zeta(2) - \zeta_n(2)) (\zeta(3) - \zeta_n(3))}{n} = \zeta^2(3) - \frac{61}{48} \zeta(6),
\]
\[
\sum_{n=1}^{\infty} (\zeta(2) - \zeta_n(2))^3 = 9\zeta(2) \zeta(3) - \frac{35}{8} \zeta(6) - \frac{25}{2} \zeta(5),
\]
\[
\sum_{n=1}^{\infty} (\zeta^*(2,1) - \zeta^*_n(2,1))^2 = \frac{15}{2} \zeta(5) + 3\zeta(2) \zeta(3) - 4\zeta^2(3),
\]
\[
\sum_{n=1}^{\infty} \frac{\zeta^2(3) - \zeta_n^2(3)}{n} = -\frac{73}{4} \zeta(7) + \frac{3}{2} \zeta(3) \zeta(4) + 10\zeta(2) \zeta(5),
\]
\[
\sum_{n=1}^{\infty} (\zeta(2) - \zeta_n(2)) (\zeta(2) - \zeta_n(2)) = \frac{3}{2} \zeta(2) \ln 2 - \frac{5}{4} \zeta(4) - \frac{3}{8} \zeta(3),
\]
\[
\sum_{n=1}^{\infty} (\zeta(2) - \zeta_n(2)) (\zeta(3) - \zeta_n(3)) (-1)^{n-1} = \frac{9}{16} \zeta(2) \zeta(3) - \frac{31}{32} \zeta(5),
\]
\[
\sum_{n=1}^{\infty} (\zeta(2) - \zeta_n(2))^2 (\zeta(3) - \zeta_n(3)) = \frac{7}{6} \zeta(6) + \frac{3}{2} \zeta^2(3) - \frac{5}{2} \zeta(3) \zeta(4),
\]
\[
\sum_{n=1}^{\infty} (\zeta^*(2,1) - \zeta^*_n(2,1)) (\zeta^*(3,1) - \zeta^*_n(3,1)) = -\frac{1}{3} \zeta(6) + 3\zeta^2(3) - \frac{5}{2} \zeta(3) \zeta(4),
\]
\[
\sum_{n=1}^{\infty} (\zeta(2) - \zeta_n(2))^2(-1)^{n-1} = 4\text{Li}_4 \left(\frac{1}{2}\right) - \frac{29}{8} \zeta(4) - \zeta(2) \ln^2 2 + \frac{1}{6} \ln^4 2 + \frac{7}{2} \zeta(3) \ln 2,
\]
\[
\sum_{n=1}^{\infty} \frac{\zeta_n(2) \zeta^2(3) - \zeta(2) \zeta_n^2(3)}{n} = -\frac{73}{4} \zeta(2) \zeta(7) + \frac{21}{8} \zeta(3) \zeta(6) + 25\zeta(4) \zeta(5) - \zeta^3(3),
\]
\[
\sum_{n=1}^{\infty} (\zeta(2) - \zeta_n(2)) (\zeta(3) - \zeta_n(3)) = \frac{21}{16} \zeta(4) + \frac{1}{2} \zeta(2) \ln^2 2 - \frac{1}{12} \ln^4 2 - 2\text{Li}_4 \left(\frac{1}{2}\right) - \frac{3}{8} \zeta(2) \zeta(3),
\]
\[
\sum_{n=1}^{\infty} \frac{\zeta^2(p) - \zeta_n^2(p)}{n^k} = \zeta(p) \zeta(p, k) + \zeta(p, k, p) + \zeta(p, k + p) \quad (k \geq 1, p > 1),
\]
\[
\sum_{n=1}^{\infty} (\zeta(p) - \zeta_n(p))^2 = 2\zeta(p) \zeta(p, p - 1) + \zeta(2p - 1) - \zeta^2(p) \quad (p > 1).
\]

These identities can be obtained from the main theorems and corollaries which are presented in the paper. Some above results are already in the literature, e.g., the fourth, ninth and the last one equations appear as examples of Theorem 4.1 in [14].

In fact, by using the method of this paper, it is possible to evaluate other sums involving zeta tails. For example, we have
\[
\sum_{n=1}^{\infty} \frac{\zeta(m) \zeta_n(p) - \zeta(p) \zeta_n(m)}{n^k} = \zeta(p) \zeta(m, k) - \zeta(m) \zeta(p, k),
\]
\[
\sum_{n=1}^{\infty} \frac{\zeta(m) \zeta_n(p) - \zeta(p) \zeta_n(m)}{n^k} = \zeta(p) \zeta(m, k) - \zeta(n) \zeta(p, k),
\]

\[
\sum_{n=1}^{\infty} \frac{\zeta^2(p) - \zeta_n^2(p)}{n^k} = \zeta(p) \zeta(p, k) + \zeta(p, k, p) + \zeta(p, k + p),
\]

\[
\sum_{n=1}^{\infty} \frac{\zeta(m) \zeta(p) - \zeta_n(m) \zeta_n(p)}{n^k} = \zeta(p) \zeta(m, k) + \zeta(p, k, m) + \zeta(p, m + k),
\]

\[
\sum_{n=1}^{\infty} \frac{\zeta(m) \zeta(p) - \zeta_n(m) \zeta_n(p)}{n^k} = \zeta(p) \zeta(m, k) + \zeta(p, k, m) + \zeta(p, m + k).
\]

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