Guarantees for Greedy Maximization of Non-submodular Functions with Applications

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Abstract
We investigate the performance of the GREEDY algorithm for cardinality constrained maximization of non-submodular nondecreasing set functions. While there are strong theoretical guarantees on the performance of GREEDY for maximizing submodular functions, there are few guarantees for non-submodular ones. However, GREEDY enjoys strong empirical performance for many important non-submodular functions, e.g., the Bayesian A-optimality objective in experimental design. We prove theoretical guarantees supporting the empirical performance. Our guarantees are characterized by the (generalized) submodularity ratio $\gamma$ and the (generalized) curvature $\alpha$. In particular, we prove that GREEDY enjoys a tight approximation guarantee of $\frac{1}{\alpha}(1 - e^{-\gamma \alpha})$ for cardinality constrained maximization. In addition, we bound the submodularity ratio and curvature for several important real-world objectives, e.g., the Bayesian A-optimality objective, the determinantal function of a square submatrix and certain linear programs with combinatorial constraints. We experimentally validate our theoretical findings for several real-world applications.

1. Introduction
Consider the important problems of experimental design and sparse modeling. In experimental design, the goal is to select a set of experiments to perform such that some statistical criterion is optimized, e.g., the variance of certain parameter estimates is minimized. This problem arises naturally in domains where performing experiments is costly. In sparse modeling, the task is to identify sparse representations of signals, enabling interpretability and robustness in high-dimensional statistical problems—properties that are crucial in modern data analysis.

These problems are naturally cast as subset selection problems such that a set function $F(S)$ over a $K$-cardinality constraint is maximized, i.e.,

$$\max_{S \subseteq V, |S| \leq K} F(S),$$

where $V = \{v_1, \ldots, v_n\}$ is the ground set. Frequently, the GREEDY algorithm (Alg. 1) is used to (approximately) solve (P). For the case that $F(S)$ is a monotone non-decreasing submodular set function, the GREEDY algorithm enjoys the multiplicative approximation guarantee of $\left(1 - \frac{1}{e}\right)$ (Nemhauser et al., 1978; Sviridenko, 2004; Vondrak, 2008; Krause & Golovin, 2012). This constant factor can be improved by refining the characterization of the objective using the curvature (Conforti & Cornu´ejols, 1984; Vondrak, 2010; Iyer et al., 2013), which informally quantifies how close a submodular function is to being modular (i.e., $F(S)$ and $-F(S)$ are submodular).

However, for many applications, including experimental design and sparse Gaussian processes (Lawrence et al., 2003), $F(S)$ is in general not submodular (Krause et al., 2008) and the above guarantee does not hold. In practice,

Algorithm 1: The GREEDY Algorithm

\begin{itemize}
  \item [1] $S^0 \leftarrow \emptyset$;
  \item [2] for $t = 1, \ldots, K$ do
    \begin{itemize}
      \item [3] $v^* \leftarrow \arg \max_{v \in V \setminus S^{t-1}} F(S^{t-1} \cup \{v\}) - F(S^{t-1})$;
      \item [4] $S^t \leftarrow S^{t-1} \cup \{v^*\}$;
    \end{itemize}
  \end{itemize}

Output: $S^K$

\footnote{$F(\cdot)$ is monotone nondecreasing if $\forall A \subseteq V, v \in V, F(A \cup \{v\}) \geq F(A)$. $F(\cdot)$ is submodular iff. it satisfies the diminishing returns property $F(A \cup \{v\}) - F(A) \geq F(B \cup \{v\}) - F(B)$ for all $A \subseteq B \subseteq V \setminus \{v\}$. Assume w.l.o.g. that $F(\cdot)$ is normalized, i.e. $F(\emptyset) = 0$.}
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Our main contributions.

- We prove the first tight constant-factor approximation guarantees for GREEDY on maximizing non-submodular nondecreasing set functions s.t. a cardinality constraint, characterized by a novel combination of the (generalized) notions of submodularity ratio $\gamma$ and curvature $\alpha$. Our proof techniques are different from previous proofs of GREEDY.

- Our theory implies the first guarantees for several important objectives, including Bayesian A-optimality in experimental design, the determinantal function of a square submatrix and maximization of LPs with combinatorial constraints, by theoretically bounding parameters $(\gamma, \alpha)$ for them. Furthermore, we obtain improved approximation ratios for the subset selection problem using the $R^2$ objective.

- Lastly, we experimentally validate our theory on several real-world applications. It is worth noting that for the Bayesian A-optimality objective, GREEDY generates comparable solutions as the classically used semidefinite programming (SDP) based method, but is usually two orders of magnitude faster.

**Notation.** We use boldface letters, e.g., $x$, to represent vectors, and capital boldface letters, e.g., $A$, to denote matrices. $x_i$ is the $i$th entry of the vector $x$. We refer to $\mathcal{V} = \{v_1, \ldots, v_n\}$ as the ground set. We use $f(\cdot)$ to denote a continuous function, and $F(\cdot)$ to represent a set function. $\text{supp}(x) := \{i \in \mathcal{V} \mid x_i \neq 0\}$ is the support set of the vector $x$, and $[n] := \{1, \ldots, n\}$ for an integer $n \geq 1$. We denote the marginal gain of a set $\Omega \subseteq \mathcal{V}$ in context of a set $S \subseteq \mathcal{V}$ as $\rho_\Omega(S) := F(\Omega \cup S) - F(S)$. For $v \in \mathcal{V}$, we use the shorthand $\rho_v(S)$ for $\rho_{\{v\}}(S)$.

2. Generalized Submodularity Ratio and Curvature

In this section we provide the submodularity ratio and curvature for general, not necessarily submodular functions, they are natural extensions of the classical ones. Let $S^0 = \emptyset$, $S^t = \{j_1, \ldots, j_t\}$, $t = 1, \ldots, K$ be the successive sets chosen by GREEDY. For brevity, let $\rho_t := \rho_j(S^{t-1})$ be the marginal gain in step $t$.

**Definition 1** (Generalized submodularity ratio). The submodularity ratio of a non-negative set function $F(\cdot)$ is the largest scalar $\gamma$ s.t.

$$\sum_{\omega \in \Omega \setminus S} \rho_\omega(S) \geq \gamma \rho_\Omega(S), \forall \Omega, S \subseteq \mathcal{V}.$$  

The greedy submodularity ratio is the largest scalar $\gamma^G$ s.t.

$$\sum_{\omega \in \Omega \setminus S^t} \rho_\omega(S^t) \geq \gamma^G \rho_\Omega(S^t), \forall |\Omega| = K, t = 0, \ldots, K - 1.$$  

It is easy to see that $\gamma^G \geq \gamma$. The submodularity ratio measures to what extent $F(\cdot)$ has submodular properties. We make the following observations:

**Remark 1.** For a nondecreasing function $F(\cdot)$, it holds a) $\gamma, \gamma^G \in [0, 1]$; b) $F(\cdot)$ is submodular iff $\gamma = 1$.

- Recently, Sviridenko et al. (2015) present a notion of curvature for monotone non-submodular functions. We show in Appendix C the details of all these notions and the relations to ours. Additionally, we prove in Remark 3 of Appendix C.2 that our combination of (generalized) curvature and submodularity ratio is more expressive than that of Sviridenko et al. (2015) in characterizing the maximization of problem (P) using standard GREEDY.
Definition 2 (Generalized curvature). The curvature of a non-negative function $F(\cdot)$ is the smallest scalar $\alpha$ s.t.,
\[
\rho_i(S \setminus \{i\} \cup \Omega) \geq (1 - \alpha)\rho_i(S \setminus \{i\}), \forall \Omega, S \subseteq \mathcal{V}, i \in S \setminus \Omega.
\]
The greedy curvature is the smallest scalar $\alpha^G \geq 0$ s.t.,
\[
\rho_j(S^{t-1} \cup \Omega) \geq (1 - \alpha^G)\rho_j(S^{t-1}), \\
\forall \Omega : |\Omega| = K, i : j_i \in S^{K-1} \setminus \Omega.
\]
When $K = n$ or $1, S^{K-1} \setminus \Omega = \emptyset$, it is natural to define $\alpha^G = 0$. It is easy to observe that $\alpha^G \leq \alpha$. Note the classical total curvature is $\alpha_{\text{total}} := 1 - \min_{i \in \mathcal{V}} \frac{\rho_i(\emptyset)}{\rho_i(\{i\})}$.

Remark 2. For a nondecreasing function $F(\cdot)$, it holds: 
a) $\alpha, \alpha^G \in [0, 1]$; b) $F(\cdot)$ is supermodular iff $\alpha = 0$; c) If $F(\cdot)$ is submodular, then $\alpha^G \leq \alpha = \alpha_{\text{total}}$.

So for a submodular function, our notion of curvature is consistent with $\alpha_{\text{total}}$. While it is worth noting that $\alpha^G$ usually characterizes the problem better than $\alpha_{\text{total}}$, as will be validated in Section 5.

3. Approximation Guarantee

In this section, we present our main theoretical result in Theorem 1, providing approximation guarantees of the Greedy algorithm for maximizing nondecreasing (possibly) non-submodular functions characterized by the (generalized) submodularity ratio and curvature. Note that both versions of the submodularity ratio and curvature apply in the proof. For brevity, we use $\gamma$ and $\alpha$ to refer to any of these versions in the sequel. In Section 3.3 we prove tightness of our approximation guarantees. All omitted proofs are given in Appendix B.

Theorem 1. Let $F(\cdot)$ be a non-negative nondecreasing set function with submodularity ratio $\gamma \in [0, 1]$ and curvature $\alpha \in [0, 1]$. The Greedy algorithm enjoys the following approximation guarantee for problem (P):
\[
F(S^K) \geq \alpha^{-1}(1 - e^{-\alpha \gamma})F(\Omega^*),
\]
where $\Omega^*$ is the optimal solution of (P) and $S^K$ the output of the Greedy algorithm.\(^3\)

3.1. Interpreting Theorem 1

Before proving the theorem, we want to give the reader an intuition of the results and show how our results extend the guarantees for Greedy to a larger class of functions. For the case $\alpha = 0$ (i.e., $F(\cdot)$ is supermodular), the approximation guarantee is $\lim_{\alpha \to 0} \frac{1}{\alpha}(1 - e^{-\alpha \gamma}) = \gamma$. When $\gamma = 1$, (i.e., $F(\cdot)$ is submodular), the guarantee is $\alpha^{-1}(1 - e^{-\alpha})$. For the case $\alpha = 1$, we have a guarantee of $1 - e^{-\gamma}$. We plot the constant-factor approximation guarantees for different values of $\gamma$ and $\alpha$ in Fig. 1. One interesting phenomenon is that $\gamma$ and $\alpha$ play different roles: Looking at $\gamma = 0$, the approximation factor is always 0, independent of the value $\alpha$ takes. In contrast, for $\alpha = 0$, the approximation guarantee is $1 - e^{-\gamma}$. This can be interpreted as the curvature boosting the guarantees.

3.2. Proof of Theorem 1

Proof overview. Given a fixed ground set $\mathcal{V}$, let us denote all problem instances of maximizing a non-negative nondecreasing function $F(\cdot)$ s.t. $K$-cardinality constraint $\max_{|S| \leq K} F(\cdot)$ to be $P_{K, \alpha, \gamma}$, where $F(\cdot)$ is parametrized by submodularity ratio $\gamma$ and curvature $\alpha$. Let $P_{\Omega^*, S^K}$ be the instance with optimal solution $\Omega^* \cap S^K$. We group all problem instances $P_{K, \alpha, \gamma}$ according to the set $\Omega^* \cap S^K := \{l_1 = j_{m_1}, l_2 = j_{m_2}, \ldots, l_s = j_{m_s}\}$, where $j_{m_1}, \ldots, j_{m_s}$ are consistent with the order of greedy selection. Let us denote the problem instances with $\Omega^* \cap S^K = \{l_1, \ldots, l_s\}$ as the group $P_{K, \alpha, \gamma}(\{l_1, \ldots, l_s\})$.

The main idea of the proof is to investigate the worst-case approximation ratio of each group of the problem instances $P_{K, \alpha, \gamma}(\{l_1, \ldots, l_s\}), \forall \{l_1, \ldots, l_s\} \subseteq S^K$. We do this by constructing LPs based on the properties of the problem instances. By studying the structures of these LPs, we will prove that the worst-case approximation ratio of all problem instances occurs when $\Omega^* \cap S^K = \emptyset$. Thus the desired approximation guarantee corresponds to the worst-case approximation ratio of $P_{K, \alpha, \gamma}(\emptyset)$.
**The proof.** When $\gamma = 0$ or $F(\Omega^*) = 0$, (1) holds naturally. In the following, let $\gamma \in (0, 1]$ and $F(\Omega^*) > 0$. First, we present Lemma 1, which will be used to construct the LPs.

**Lemma 1.** For any $\Omega \subseteq \mathcal{V}$ with $|\Omega| = K$ and any $t \in \{0, \ldots, K - 1\}$, let $w^t := |S^t \cap \Omega|$. It holds that

$$\alpha \sum_{i,j \in S^t \setminus \Omega} \rho_i + \sum_{i,j \in S^t \cap \Omega} \rho_i \gamma^{-1}(K - w^t) \rho_{t+1} \geq F(\Omega).$$

We now specify the constructing of the LPs: For any problem instance $P_{\Omega^*,S^K} \in \mathcal{P}_{K,\alpha,\gamma}(\{l_1, \ldots, l_s\})$, we know that $F(S^K) = \sum_{i=1}^K \rho_i$ (telescoping sum). Hence, the approximation ratio is $\frac{F(S^K)}{F(\Omega^*)} = \sum_i \frac{\rho_i}{F(\Omega^*)}$, which we denoted as $R(\{l_1, \ldots, l_s\}) = \sum_i \frac{\rho_i}{F(\Omega^*)}$. Define $x_t := \frac{\rho_t}{F(\Omega^*)}, t \in [K]$. Since $F$ is nondecreasing, $x_1 \geq 0$. Plugging $\Omega = \Omega^*$ into Lemma 1, and considering $t = 0, \ldots, K - 1$, we have in total $K$ constraints over the variables $x_t$, which constitute the constraints of the LP. So the worst-case approximation ratio of the group $\mathcal{P}_{K,\alpha,\gamma}(\{l_1, \ldots, l_s\})$ is:

$$\mathcal{R}(\{l_1, \ldots, l_s\}) = \min \sum_{i=1}^K x_t, \text{ s.t. } x_t \geq 0 \text{ and,}$$

| row (0) | $x_1$ | $x_2$ | $\cdots$ | $x_K$ |
|---------|-------|-------|----------|-------|
| row (1) | $\alpha$ | $\gamma$ | $\cdots$ | $0$ |
| row $(l_i - 1)$ | $\alpha$ | $\alpha$ | $\cdots$ | $\alpha$ |
| row $(l_i - 2)$ | $\alpha$ | $\alpha$ | $\cdots$ | $1$ |
| row $(l_i - 1)$ | $\alpha$ | $\alpha$ | $\cdots$ | $1$ |
| row $(K - 1)$ | $\alpha$ | $\alpha$ | $\cdots$ | $1$ |

The following Lemma presents the key structure of the constructed LPs, which will be used to deduce the relation between the LPs of different problem instance groups $\mathcal{P}_{K,\alpha,\gamma}(\Omega^* \cap S^K), \mathcal{P}(\Omega^* \cap S^K) \subseteq S^K$.

**Lemma 2.** Assume that the optimal solution of the constructed LP is $x^* \in \mathbb{R}^K$ and that $s = |\Omega^* \cap S^K| \geq 1$. For all $1 \leq r \leq s$ it holds that $x^*_q \leq x^*_{q+1}$, where $q = l_r$.

**Proof sketch of Lemma 2.** Assume by virtue of creating a contradiction that $x^*_q > x^*_{q+1}$. We can always create a new feasible solution $y^* \in \mathbb{R}^K$ by decreasing $x^*_q$ by some $\epsilon > 0$, while increasing all the $x^*_{q+1}$ to $x^*_K$ by some proper values, s.t. $y^*$ has smaller LP objective value. Specifically, we define $y^*_q$ as: for $k = 1, \ldots, q - 1, y^*_k := x^*_k$, $y^*_q := x^*_q - \epsilon$; for $k = q + 1, \ldots, K$, $y^*_k := x^*_k + \epsilon_k$ where $\epsilon_k$s are defined recursively as $\epsilon_{k+1} = \frac{\epsilon_k}{\gamma}$, and $\epsilon_{q+u} = \epsilon_{q+u} \frac{K - r - u + 1 - \gamma}{K - r - u}, 1 \leq u \leq K - q - 1$.

**Claim 1.** a) The new solution $y^*$ is feasible; b) All of the rows in (2) are still feasible for $y^*$.

After that the change of the LP objective is

$$\Delta_{LP} = -\epsilon + \epsilon_{q+1} + \epsilon_{q+2} + \ldots + \epsilon_K.$$

One can prove that the LP objective decreases:

**Claim 2.** For all $K \geq 1$, $1 \leq r \leq q < K$, it holds that $\Delta_{LP} \leq 0, \forall r \in (0, 1]$. Equality is achieved when $r = 0$ and $\gamma = 1$.

Therefore we reach the contradiction that $x^*$ is an optimal solution of the constructed LP.

Given Lemma 2, we prove in the following Lemma, which, formally, states that the worst-case approximation ratio of all problem instances occurs when $\Omega^* \cap S^K = \emptyset$.

**Lemma 3.** For all $\{l_1, \ldots, l_s\} \subseteq S^K$, it holds that

$$\mathcal{R}(\{l_1, \ldots, l_s\}) \geq \mathcal{R}(\emptyset) = \frac{1}{\alpha} \left[ 1 - \left( \frac{K - \alpha \gamma}{K} \right)^K \right].$$

So the greedy solution has objective $F(S^K) \geq \frac{1}{\alpha} \left[ 1 - \left( \frac{K - \alpha \gamma}{K} \right)^K \right] F(\Omega^*)$.

**3.3. Tightness result**

We demonstrate that the approximation guarantee in Theorem 1 is tight, i.e., for every submodularity ratio $\gamma$ and every curvature $\alpha$, there exist set functions that achieve the bound exactly.

Assume the ground set $\mathcal{V}$ contains the elements in $S := \{l_1, \ldots, l_K\}$ and the elements in $\Omega := \{\omega_1, \ldots, \omega_K\} (S \cap \Omega = 0)$ and $n - 2K$ dummy elements. The objective function we are going to construct will not depend on these dummy elements, i.e., the objective value of a set does not change if dummy elements are removed from or added to that set. Consequently, the dummy elements will not affect the submodularity ratio and the curvature. For the constants $\alpha \in [0, 1], \gamma \in (0, 1]$, we define the objective function as,

$$F(T) := f(\Omega \cap T) - \alpha \gamma \sum_{i,j \in S \cap T} \rho_i + \sum_{i,j \in S \cap T} \rho_i, \quad (3)$$

where $\rho_i := \frac{1}{K} \left( \frac{K - \alpha \gamma}{K} \right)^i, i \in [K], f(x) = \frac{\gamma^{-1} - 1}{K - \gamma^{-1}} x^2 + \frac{K - \gamma^{-1} - 1}{K - \gamma^{-1}} x$. Note that $f(x)$ is convex nondecreasing over $[0, K]$, and that $f(0) = 0, f(1) = 1, f(K) = K/\gamma$. It is clear that $F(\emptyset) = 0$ and $F(\cdot)$ is monotone nondecreasing.

The following lemma shows that it is generally non-submodular and non-supermodular.
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Lemma 4. For the objective in (3): a) When \( \alpha = 0 \), it is supermodular; b) When \( \gamma = 1 \), it is submodular; c) \( F(T) \) has submodularity ratio \( \gamma \) and curvature \( \alpha \).

Considering the problem of \( \max_{|T| \leq K} F(T) \), we claim that the GREEDY algorithm may output \( S \). This can be proved by induction. One can see that \( \rho_{T_i}(0) = \rho_T = \rho_{\omega}(0) \), so GREEDY can choose \( T_i \) in the first step. Assume in step \( t-1 \) GREEDY has chosen \( S^{t-1} = \{T_1, \ldots, T_{t-1}\} \), one can verify that the marginal gains coincide, i.e., \( \rho_{T_j}(S^{t-1}) = \rho_T = \rho_{\omega}(S^{t-1}) \). However, the optimal solution is actually \( \Omega \) with function value as \( F(\Omega) = \frac{1}{2} \). So the approximation ratio is \( \frac{F(S)}{F(\Omega)} = \frac{1}{\alpha} \left[ 1 - \left( \frac{K - \alpha \gamma}{K} \right)^{K-1} \right] \), which matches our approximation guarantee in Theorem 1.

4. Applications

We consider several important real-world applications and their corresponding objective functions. We show that the submodularity ratio and the curvature of these functions can be bounded and, hence, the approximation guarantees from our theoretical results are applicable.

4.1. Bayesian A-optimality in experimental design

In Bayesian experimental design (Chaloner & Verdinelli, 1995), the goal is to select a set of experiments to perform s.t. some statistical criterion is optimized, e.g., the variance of certain parameter estimates is minimized. Krause et al. (2008) investigated several criteria for this purpose, amongst others the Bayesian A-optimality criterion. This criterion is used to maximally reduce the variance in the posterior distribution over the parameters. In general the criterion is not submodular.\(^4\)

Formally, assume there are \( n \) experimental stimuli \( \{x_1, \ldots, x_n\} \), each \( x_i \in \mathbb{R}^d \), they constitute the data matrix \( X \in \mathbb{R}^{d \times n} \). Let us arrange a set \( S \subseteq \mathcal{V} \) of stimuli as a matrix \( X_S := [x_{v_1}, \ldots, x_{v_s}] \in \mathbb{R}^{d \times |S|} \). \( \theta \in \mathbb{R}^d \) be the parameter vector in the linear model \( y_S = X_S^\top \theta + w \), where \( w \) is the Gaussian noise with zero mean and variance \( \sigma^2 \), i.e., \( w \sim \mathcal{N}(0, \sigma^2 I) \), and \( y_S \) is the vector of dependent variables. Suppose the prior takes the form of an isotropic Gaussian, i.e., \( \theta \sim \mathcal{N}(0, \Lambda^{-1}) \). Then,

\[
\begin{bmatrix}
  y_S \\
  \theta
\end{bmatrix} \sim \mathcal{N}(0, \Sigma), \quad \Sigma = \begin{bmatrix}
  \sigma^2 I + X_S^\top \Lambda^{-1} X_S & X_S^\top \\
  \Lambda^{-1} & \Lambda^{-1}
\end{bmatrix}.
\]

This implies that \( \Sigma_{\theta|y_S} := \Sigma_{\theta|y_S} = (\Lambda + \sigma^{-2} X_S X_S^\top)^{-1} \). The \( \Lambda \)-optimality objective is defined as,

\[
F_\Lambda(S) := \text{tr}(\Sigma_{\theta}) - \text{tr}(\Sigma_{\theta|y_S}) = \text{tr}(\Lambda^{-1}) - \text{tr}(\Lambda + \sigma^{-2} X_S X_S^\top)^{-1}) \quad (4)
\]

The following Proposition gives bounds on the submodularity ratio and curvature of (4).

Proposition 1. Assume normalized stimuli, i.e., \( \|x_i\| = 1, \forall i \in \mathcal{V} \). Let the spectral norm of \( X \) be \( \|X\| \).

Then, a) The objective in (4) is monotone nondecreasing. b) Its submodularity ratio \( \gamma \) can be lower bounded by \( \frac{\beta^2}{\|X\|^2(\beta^2 + \sigma^{-2} \|X\|^2)} \), and its curvature \( \alpha \) can be upper bounded by \( 1 - \frac{\beta^2}{\|X\|^2(\beta^2 + \sigma^{-2} \|X\|^2)} \).

4.2. The determinantal function

The determinantal function of a square submatrix is widely used in many areas, e.g., in determinantal point processes (DPP) (Kulesza et al., 2012) and active set selection for sparse Gaussian processes. We consider monotone nondecreasing determinantal functions in this work, which, for example, appear in the problem of active set selection for sparse Gaussian processes. Assume \( \Sigma \) is the covariance matrix parameterized by a positive definite kernel. In the Informative Vector Machine (Lawrence et al., 2003), the information gain of a chosen subset of points is \( \frac{1}{2} \log F(S) \), where

\[
F(S) := \det(\mathbf{I} + \sigma^{-2} \Sigma_S) \quad (5)
\]

where \( \sigma \) is the noise variance in the Gaussian process regression model, \( \Sigma_S \) is the square submatrix indexed by \( S \). Although \( \log F(S) \) is submodular (Krause & Guestrin, 2005), \( F(S) \) is in general not submodular. The approximation guarantee of GREEDY for maximizing \( \log F(S) \) does not translate to a guarantee for maximizing \( F(S) \). The following Proposition characterizes (5).

Proposition 2. a) \( F(S) \) in (5) is supermodular, its curvature is 0; b) Let the eigenvalues of \( A := \mathbf{I} + \sigma^{-2} \Sigma \) be \( \lambda_1 \geq \cdots \geq \lambda_n > 1 \). The greedy submodularity ratio of \( F(S) \) can be lower bounded by \( \frac{K (\lambda_1 - 1)}{\prod_{i=1}^d \lambda_i - 1} \).

4.3. LP with combinatorial constraints

LPs with combinatorial constraints appear frequently in practice. Consider the following example: Suppose that \( \mathcal{V} \) is the set of all products a company can produce. Given budget constraints on the raw materials needed, companies consider the LP \( \max_{x \in P(d, \mathcal{A})} \), where \( d \) is the vector of profits for the individual products and where \( P \) is a polytope representing the continuous constraints. The above LP can be used to assess the profit maximizing production plan. Usually the company needs to consider combinatorial constraints as well. For instance, the company has at most \( K \) production lines, thus they have to select a subset of \( K \) products to produce.

Often these kind of problems can be formalized as

\(^4\)Krause et al. (2008, Section 8.4) provide a counter example.

\(^5\)By Weyl’s inequality, a naive upper bound is \( \|X\| \leq \sqrt{n} \).
\[ \max_{\mathcal{P}, \mathcal{P} \subseteq \mathcal{I}} \langle d, x \rangle, \text{ where } \mathcal{I} \text{ is the independent set of the combinatorial structure. Hence, a natural auxiliary set function is,} \]
\[ F(S) := \max_{\mathcal{P} \subseteq \mathcal{I}} \langle d, x \rangle, \forall S \subseteq \mathcal{V} \quad (6) \]

Let \( \mathcal{P} = \{x \in \mathbb{R}^n | 0 \leq x \leq \tilde{u}, A x \leq b, u \in \mathbb{R}_+^n, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m \} \). In general \( F(S) \) in (6) is non-submodular, two counter examples are presented in Appendix D.3. Upper bounding the curvature is equivalent to lower bounding \( F(S) - F(S \setminus \{i\}) \), which can be 0 in the worst case. However, the submodularity ratio can be lower bounded by a non-zero scalar.

**Proposition 3.** a) \( F(S) \) in (6) is a normalized nondecreasing set function. b) With regular non-degeneracy assumptions (details in Appendix D.3.2), its submodularity ratio can be lower bounded by \( \gamma > 0 \).

### 4.4. More applications

There are more real-world applications that can benefit from the theory in this work, for instance, subset selection using the \( R^2 \) objective, sparse modeling and the budget allocation problem with combinatorial constraints. Details on them are deferred to Appendix G.

### 5. Experimental Results

We empirically validated the approximation guarantees characterized by the submodularity ratio and the curvature for several applications. Since it is too time consuming to calculate the full versions of \( \alpha \) and \( \gamma \) using exhaustive search, we only calculated the greedy versions \( (\alpha^G, \gamma^G) \). All averaged results are from 20 repeated experiments.6

#### 5.1. Experimental design

We considered the Bayesian A-optimality objective for both synthetic and real-world data. In all experiments, we normalized the data points to have unit \( \ell_2 \)-norm.

**Real-world results:** We used the Boston Housing Data.7 The dataset has 14 features (e.g., crime rate, property tax rates, etc.) and 516 samples. To be able to quickly calculate the parameters and optimal solution by exhaustive search, the first \( n = 14 \) samples were used. As a baseline, we used an SDP-based algorithm (abbreviated as SDP, details are available in Appendix E). Results are shown in Fig. 2 for varying values of \( K \). In Fig. 2a we can observe that both GREEDY and SDP computes near-optimal solutions. From Fig. 2b we can see that the greedy submodularity ratio \( \gamma^G \)

\[ \text{Figure 2: Results on the Boston Housing data.} \]

#### 5.2. Synthetic results:

We generated random observations from a multivariate Gaussian distribution with different correlations. To be able to assess the ground truth, we used \( n = 12 \) samples with \( d = 6 \) features. Fig. 3 shows the results with correlation 0.2 (first column) and 0.6 (second column), respectively: The first row shows the average objective values over the optimal value with error bars, and the second row shows the parameters. One can observe that GREEDY always obtains near-optimal solutions and that these solutions are roughly comparable with those obtained by the SDP. The classical curvature \( \alpha^\text{total} \) is always close to 1, while \( \alpha^G \) takes smaller values, and \( \gamma^G \) takes values close to 1, thus characterize the performance of GREEDY better.

**Medium-scale synthetic experiments:** To compare the
runtime of SDP and GREEDY, we considered medium-scale datasets (we cannot report results on larger datasets because of the huge computational demands of the SDP). Fig. 4 shows the objective value achieved by GREEDY and SDP for different numbers of features \( d \) and numbers of samples \( n \), as well as the correlations. We can observe that GREEDY computes solutions that are on par or superior to those of SDP. In Table 1 we summarize the runtime of GREEDY and SDP for different values of \( d \) and \( n \), for correlation 0.5. Furthermore, we show the ratio of runtimes of the two algorithms. We can observe that GREEDY is usually two orders of magnitude faster than SDP.

Table 1: Runtime in seconds of GREEDY and SDP. The last row shows the ratio of runtimes of SDP and GREEDY.

| \( d \) | \( n \) | GREEDY | SDP | "SDP/GREEDY" |
|---|---|---|---|---|
| 60 | 80 | 0.278 | 95.2 | 341.7 |
| 40 | 112 | 0.360 | 115.2 | 319.9 |
| 64 | 128 | 0.765 | 205.4 | 268.7 |
| 100 | 200 | 4.666 | 1741.2 | 373.2 |
| 120 | 250 | 10.56 | 3883.5 | 367.7 |

5.2. LPs with combinatorial constraints

We generated synthetic LPs as follows: Firstly, we generated the matrix \( A \in \mathbb{R}^{m \times n}, A_{ij} \in [0, 1] \) by drawing all entries independently from a uniform distribution on \([0, 1]\). We set \( b = d = 1 \), and set \( \alpha = 1 \). The first row of Fig. 5 plots the optimal LP objective (calculated using exhaustive search) and the LP objective returned by GREEDY. The second row shows the curvature and submodularity ratio. The first column (Fig. 5a) presents the results for \( n = 6, m = 20 \), while the second column (Fig. 5b) presents that for \( n = 8, m = 30 \). Note the greedy submodularity ratio takes values between \( \sim 0.15 \) and 1, and that the curvature is close to the worst-case value of 1. These observations are consistent with the theory in Section 4.3.

5.3. Determinantal functions maximization

We experimented with synthetic and real-world data: For synthetic data, we generated random covariance matrices \( \Sigma \in \mathbb{R}^{n \times n} \) with uniformly distributed eigenvalues in \([0, 1]\). We set \( n = 10, \sigma = 1.5 \). In Fig. 6 (left) we plot the optimal determinantal objective value and the value achieved by GREEDY. Fig. 6 (right) traces the greedy submodularity ratio \( \gamma^G \). Since the determinantal objective is supermodular, so the approximation guarantee equals to \( \gamma^G \). We can see that \( \gamma^G \) gives a reasonable prediction of the behavior of GREEDY. For real-world data, we considered an active set selection task on the CIFAR-10\(^8 \) dataset. The first \( n = 12 \) images in the test set were used to calculate the covariance matrix with an squared exponential kernel \( k(x_i, x_j) = \exp(-\|x_i - x_j\|^2/h^2) \), \( h \) was set to be 1). The results in Fig. 7 shows that we get similar results as with the synthetic data.

5.4. Subset selection using the \( R^2 \) objective

For details on this task please refer to Das & Kempe (2011) or Appendix G.1. We did synthetic experiments to illustrate that our theory can give a refined explanation of the performance of GREEDY. We generate random observations from a multivariate standard Gaussian distribution

\[ \text{https://www.cs.toronto.edu/~kriz/cifar.html} \]
with different correlations. We used \( n = 10 \) features and \( m = 100 \) observations. The target regression coefficients \( \alpha \in \mathbb{R}^n \) was generated as a random vector with uniformly distributed entries in \([0, 1]\). Standard Gaussian noise was added to generate the observation of predictor variable \( Z \).

The results are shown in Fig. 8, with first column showing the results with correlation as 0.05, the second column with correlation as 0.5. One can see that the mean of the predictions and submodularity ratio take values in \((0, 1)\), which can be used to give improved approximation bounds for Greedy.

6. Related Work

In this section we briefly discuss various notions of non-submodularity and the optimization of non-submodular functions. Further details are provided in Appendix F.

Submodularity ratio and curvature. Curvature is typically defined for submodular functions. Sviridenko et al. (2015) present a notion of curvature for monotone non-submodular functions. Appendix C provides details of that notion and relates it to our definition. Yoshida (2016) prove an improved approximation ratio for knapsack-constrained maximization of submodular functions with bounded curvature. The submodularity ratio (Das & Kempe, 2011) is a quantity characterizing how close a function is to being submodular.

Approximate submodularity. Krause et al. (2008) define approximately submodular functions with parameter \( \epsilon \geq 0 \) as those functions \( F \) that satisfy an approximate diminishing returns property, i.e., \( \forall A \subseteq B \subseteq V \setminus v \) it holds that \( \rho_v(A) \geq \rho_v(B) - \epsilon \). Greedy yields a solution with objective \( F(S^G) \geq (1 - \epsilon^{-1})F(\Omega^*) - K\epsilon \), for maximizing a monotone \( F \) s.t. a \( K \)-cardinality constraint. Du et al. (2008) study the greedy maximization of non-submodular potential functions with restricted submodularity and shifted submodularity. Restricted submodularity refers to functions which are submodular only over some collection of subsets of \( V \), and shifted submodularity can be viewed as a special case of the approximate diminishing returns as defined above. Recently, Horel & Singer (2016) study \( \epsilon \)-approximately submodular functions, which are related to their research on “noisy” submodular functions. A function \( F(\cdot) \) is \( \epsilon \)-approximately submodular if there exists a submodular function \( G \) s.t. \((1 - \epsilon)G(S) \leq F(S) \leq (1 + \epsilon)G(S), \forall S \subseteq V \).

Weak submodularity. Borodin et al. (2014) study weakly submodular functions, i.e., monotone, normalized functions \( F(\cdot) \) s.t. for any \( S, T \), it holds \( |T|F(S) + |S|F(T) \geq |S \cap T|F(S \cup T) + |S \cup T|F(S \cap T) \). For a function \( F(\cdot) \), we show in Remark 4 that the following two facts do not imply each other: i) \( F(\cdot) \) is weakly submodular; ii) The submodularity ratio of \( F(\cdot) \) is strictly larger than 0, and its curvature is strictly smaller than 1.

Other notions of non-submodularity. Feige & Izsak (2013) introduce the supermodular degree as a complexity measure for set functions. They show that a greedy algorithm for the welfare maximization problem enjoys an approximation guarantee increasing linearly with the supermodular degree. Zhou & Spanos (2016) use the submodularity index to characterize the performance of the RandomGreedy algorithm (Buchbinder et al., 2014) for maximizing a non-monotone function. Further discussions are in Appendix F.

Optimization of non-submodular functions. The submodular-supermodular procedure has been proposed to minimize the difference of two submodular functions (Narasimhan & Bilmes, 2005; Iyer & Bilmes, 2012). Jegelka & Bilmes (2011) present the problem of minimizing “cooperative cut”, which is generally non-submodular, and propose efficient algorithms to optimize it. Kawahara et al. (2015) analyze unconstrained minimization of the sum of a submodular function and tree-structured supermodular function. Bai et al. (2016) investigate the minimization of the ratio of two submodular functions, which can be solved with bounded approximation factor.
7. Conclusion

In this work we analyzed the guarantees for greedy maximization of generally non-submodular nondecreasing set functions. By considering the (generalized) curvature $\alpha$ and submodularity ratio $\gamma$ for generic set functions, we prove the first tight approximation bounds in terms of these definitions for greedily maximizing nondecreasing set functions. These approximation bounds significantly enlarge the domain where GREEDY has guarantees. Furthermore, we theoretically bounded the parameters $\alpha$ and $\gamma$ for several non-trivial applications, and validate our theory in various experiments.

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Appendix

A. Organization of the appendix

Appendix B presents the proofs for our approximation guarantees and its tightness for the Greedy algorithm.

Appendix C provides details on existing notions of curvature and submodularity ratio, and relates it to the notions in this paper.

Appendix D presents detailed proofs for bounding the generalized submodularity ratio and curvature for various applications.

Appendix E gives details on the classical SDP formulation of the Bayesian A-optimality objective.

Appendix F contains omitted proofs in the related work in Section 6.

Appendix G provides information on more applications, including sparse modeling with strongly convex loss functions, subset selection using the $R^2$ objective and optimal budget allocation with combinatorial constraints.

Appendix H provides more experimental results.

B. Proof of the approximation guarantees and tightness results (Section 2 and Section 3 in the main text)

B.1. Proof of remarks in Section 2

Proofs of Remark 1.

a) Because $F$ is nondecreasing, and $\gamma, \gamma^G$ are defined as the largest scalars, $\gamma, \gamma^G \geq 0$. At the same time, both $\gamma$ and $\gamma^G$ can be at most 1 because the conditions in Definition 1 also have to hold for the case that $\Omega \setminus S$ (respectively $\Omega \setminus S^i$) is a singleton.

b) “$\Rightarrow$”:

Let $\Omega \setminus S = \{\omega_1, \ldots, \omega_k\}$, $k \geq 1$. Submodularity implies $\sum_{i=1}^k \rho_{\omega_i}(S) \geq \rho_{\Omega}(S)$. Hence, $\gamma$ can take the largest value 1.

“$\Leftarrow$”:

$\gamma = 1$ implies that (setting $\Omega \setminus S = \{\omega_i, \omega_j\}$), for all $\omega_j, \omega_i \in V \setminus S$, it holds that $F(\{\omega_i\} \cup S) + F(\{\omega_j\} \cup S) \geq F(\{\omega_i, \omega_j\} \cup S) + F(S)$, which is an equivalent way to define submodularity (Bach et al., 2013, Proposition 2.3).

Proof of Remark 2.

a) “If $F(\cdot)$ is nondecreasing, then $\alpha, \alpha^G \in [0, 1]$”;

When $\Omega = \emptyset$, $\alpha$ is at least 0. From the definition, $\alpha^G \geq 0$. Since $F$ is nondecreasing, $\rho_i(S \setminus \{i\} \cup \Omega) \geq 0$ (respectively, $\rho_j(S^{i-1} \cup \Omega) \geq 0$), and we defined $\alpha, \alpha^G$ to be the smallest scalar, it must hold that $\alpha, \alpha^G \leq 1$.

b) “For a nondecreasing function $F(\cdot)$, $F(\cdot)$ is supermodular iff $\alpha = 0$”;

“$\Rightarrow$”:

If $F$ is supermodular, it always holds that $\rho_i(S \setminus \{i\} \cup \Omega) \geq \rho_i(S \setminus \{i\})$, combined with the fact that $\alpha$ is at least 0, we know that $\alpha$ must be 0.

“$\Leftarrow$”:

One can observe that $\alpha = 0$ is equivalent to $-F(\cdot)$ satisfying the diminishing returns property, which is equivalent to $F(\cdot)$ being supermodular.

c) “If $F(\cdot)$ is nondecreasing submodular, then $\alpha^G \leq \alpha = \alpha^{\text{total}}$.”
Since it always holds that $\alpha^G \leq \alpha$, we only need to prove that $\alpha = \alpha^{\text{total}}$. W.l.o.g., assume $\rho_i(S \setminus \{i\}) > 0$. Then,

$$1 - \alpha = \min_{\Omega, S \subseteq V, i \in S \setminus \Omega} \frac{\rho_i(S \setminus \{i\} \cup \Omega)}{\rho_i(S \setminus \{i\})} = \min_{S \subseteq V, i \in S} \frac{\rho_i(V \setminus \{i\})}{\rho_i(S \setminus \{i\})} \quad \text{(diminishing returns, and taking } \Omega = V \setminus \{i\})$$

$$= \min_{i \in V} \frac{\rho_i(\emptyset)}{\rho_i(\{i\})} \quad \text{(diminishing returns, and taking } S = \{i\})$$

$$= 1 - \alpha^{\text{total}}.$$  

So it holds that $\alpha^G \leq \alpha = \alpha^{\text{total}}$.

**B.2. Proof of Lemma 1**

*Proof of Lemma 1.* The proof needs the definitions of generalized curvature, submodularity ratio, and the selection rule of the GREEDY algorithm.

Firstly let us prove the following identity,

$$F(\Omega \cup S^t) = F(\Omega) + \sum_{i \in S^t \setminus \Omega} \rho_j(\Omega \cup S^{i-1}). \tag{7}$$

We know that $S^t = \{j_1, \ldots, j_t\}$, w.l.o.g., let $S^t \setminus \Omega = \{j_a, \ldots, j_a, j_b, \ldots, j_t\}$, $a \leq b - 1$, and $S^t \cap \Omega = \{j_{a+1}, \ldots, j_{b-1}\}$. By telescoping we know that,

$$\sum_{i=1}^{a} \rho_j(\Omega \cup S^{i-1}) = F(\Omega \cup S^a) - F(\Omega) \quad \text{and}$$

$$\sum_{i=b}^{t} \rho_j(\Omega \cup S^{i-1}) = F(\Omega \cup S^t) - F(\Omega \cup S^{b-1}).$$

Because $S^t \cap \Omega = \{j_{a+1}, \ldots, j_{b-1}\}$, it holds that $F(\Omega \cup S^{b-1}) = F(\Omega \cup S^n)$, so $\sum_{i=j_1}^{a} \rho_j(\Omega \cup S^{i-1}) = \sum_{i=1}^{a} \rho_j(\Omega \cup S^{i-1}) + \sum_{i=b}^{t} \rho_j(\Omega \cup S^{i-1}) = F(\Omega \cup S^t) - F(\Omega)$, which proves (7).

From the definition of the submodularity ratio,

$$F(\Omega \cup S^t) \leq F(S^t) + \frac{1}{\gamma} \sum_{\omega \in \Omega \setminus S^t} \rho_\omega(S^t). \tag{8}$$

From the definition of curvature (for the greedy curvature, since it holds for $S^{K-1}$, it must also hold for $S^t \subseteq S^{K-1}$), we have,

$$\sum_{i,j \in S^t \setminus \Omega} \rho_j(\Omega \cup S^{i-1}) \geq (1 - \alpha) \sum_{i,j \in S^t \setminus \Omega} \rho_j(S^{i-1}). \tag{9}$$

Combining (7) to (9), we have,

$$F(\Omega) \leq  \alpha \sum_{i,j \in S^t \setminus \Omega} \rho_i + F(S^t) - \sum_{i,j \in S^t \setminus \Omega} \rho_i + \frac{1}{\gamma} \sum_{\omega \in \Omega \setminus S^t} \rho_\omega(S^t)$$

$$= \alpha \sum_{i,j \in S^t \setminus \Omega} \rho_i + \sum_{i \in S^t \setminus \Omega} \rho_i + \frac{1}{\gamma} \sum_{\omega \in \Omega \setminus S^t} \rho_\omega(S^t)$$

$$\leq \alpha \sum_{i,j \in S^t \setminus \Omega} \rho_i + \sum_{i \in S^t \setminus \Omega} \rho_i + \gamma^{-1}(K - w^t) \rho_{t+1},$$

where the last inequality is because of the selection rule of the GREEDY algorithm ($\rho_\omega(S^t) \leq \rho_{t+1}, \forall \omega$). 

$\blacksquare$
B.3. Proof of Claim 1

Since the proof heavily relies on the structure of the constructed LPs, we restate it here:

The worst-case approximation ratio of the group $\mathcal{P}_{K,\alpha,\gamma}(\{l_1, \cdots, l_s\})$ is

$$R(\{l_1, \cdots, l_s\}) = \min \sum_{i=1}^{K} x_i, \text{ s.t. } x_i \geq 0, i \in [K]$$

\[
\begin{array}{c|cccc}
\text{row (0)} & K/\gamma & \alpha & K/\gamma & \cdots \\
\text{row (1)} & \alpha & \alpha & \cdots & K/\gamma \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\text{row } (l_1 - 1) & \alpha & \alpha & \cdots & (K - 1)/\gamma \\
\text{row } (l_2 - 1) & \alpha & \alpha & \cdots & 1 & \frac{K-r}{\gamma} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\text{row } (q - 1) & \alpha & \alpha & \cdots & 1 & \frac{K-r}{\gamma} \\
\text{row } (K - 1) & \alpha & \alpha & \cdots & 1 & 1 & \alpha & \cdots & 1 & \cdots & \frac{K-s}{\gamma} \\
\end{array}
\]

\[
\begin{pmatrix}
1 \\
x_1 \\
x_2 \\
\vdots \\
x_{l_1} \\
x_{l_2} \\
\vdots \\
x_{l_q - 1} \\
x_{q} \\
x_K
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_{l_1} \\
x_{l_2} \\
\vdots \\
x_{l_q - 1} \\
x_{q} \\
x_K
\end{pmatrix}
\begin{pmatrix}
1 \\
1 \\
\vdots \\
1 \\
1 \\
\vdots \\
1 \\
1 \\
\vdots \\
1
\end{pmatrix}
\]

For notational simplicity, w.l.o.g., assume that $j_i = i, i \in [K]$.

**Proof of Claim 1.**

Note that the $(q-1)^{th}$ and $q^{th}$ rows in (2) are

\[\alpha x_1^* + \cdots + \alpha(1)x_{q-1}^* + \frac{K-r+1}{\gamma} x_q^* \geq 1\]  

\[\alpha x_1^* + \cdots + \alpha(1)x_{q-1}^* + x_q^* + \frac{K-r}{\gamma} x_{q+1}^* \geq 1\]  

(10) is not tight since the L.H.S. of (10) minus the L.H.S. of (11) amounts to,

$$x_q^* \frac{K-r+1}{\gamma} - x_{q+1}^* \frac{K-r}{\gamma} > x_q^* \frac{1}{\gamma} \geq 0.$$  

Thus after decreasing $x_q^*$ by $\epsilon = \frac{K-r}{K-r+1} (x_q^* - x_{q+1}^*) > 0$, (10) is still feasible.

After increasing $x_{q+1}^*$ by $\epsilon_{q+1} = \epsilon \frac{\gamma}{K-r}$, the $q^{th}$ row in (2) is feasible since the change in its L.H.S. is $-\epsilon + \epsilon = 0$.

**a)** It is easy to see that $y_q^* \geq 0$ since the only decreased entry is the $q^{th}$ entry, and one can easily see that $y_q^* = x_q^* - \epsilon \geq 0$.

**b)** For the rows 0 to $q - 2$ in (2), there is no change, so they are still feasible.

For the rows $q + 1$ to $K$ in (2), let us prove by induction.

For the base case, consider the $(q + 1)^{th}$ row in (2), it can be either,

\[\alpha x_1^* + \cdots + \alpha(1)x_{q-1}^* + x_q^* + x_{q+1}^* + \frac{K-r-1}{\gamma} x_{q+2}^* \geq 1\]

or

\[\alpha x_1^* + \cdots + \alpha(1)x_{q-1}^* + x_q^* + \alpha x_{q+1}^* + \frac{K-r}{\gamma} x_{q+2}^* \geq 1\]

It can be easily verified that the $(q + 1)^{th}$ row in (2) is still feasible in both the above two situations. Let us use $\Delta_{q+u}$ to denote the change of L.H.S. of the $(q + u + 1)^{th}$ row after applying the changes.
For the inductive step, assume that the claim holds for \(u = u\)', i.e., the \((q + u')\)th row in (2) is feasible or \(\Delta_{q+u'} \geq 0\). The \((q + u')\)th row is,

\[
\left( \ldots \text{same as } (q + u')^{th} \text{ row} \right) + \frac{K - r - v}{\gamma}x_{q+u'+1} \geq 1
\]

where \(0 \leq v \leq u'\) is some integer dependent on the structure of (2), but not affect the final analysis. Then the \((q + u' + 1)\)th row can be either,

\[
\left( \ldots \text{same as } (q + u')^{th} \text{ row} \right) + x_{q+u'+1}' + \frac{K - r - v - 1}{\gamma}x_{q+u'+2}' \geq 1 \quad \text{(case 1)}
\]

or

\[
\left( \ldots \text{same as } (q + u')^{th} \text{ row} \right) + \alpha x_{q+u'+1}' + \frac{K - r - v}{\gamma}x_{q+u'+2}' \geq 1 \quad \text{(case 2)}
\]

In (case 1), the L.H.S. of \((q + u')^{th}\) row minus the L.H.S. of \((q + u)\)th row is \(\frac{K - r - v - 1}{\gamma}x_{q+u'+2}' - \frac{K - r - v - \gamma}{\gamma}x_{q+u'+1}'\), so

\[
\Delta_{q+u'+1} - \Delta_{q+u} = \frac{K - r - v}{\gamma} \epsilon_{q+u'+2}' - \frac{K - r - v - \gamma}{\gamma} \epsilon_{q+u'+1}'
\]

\[
= \left[ \frac{K - r - v - 1}{\gamma} - \frac{K - r - u' - \gamma}{\gamma} \right] \epsilon_{q+u'+1}'
\]

\[
= \left[ \frac{(K - r - v - 1) - (K - r - u' - \gamma)}{\gamma} \right] \epsilon_{q+u'+1}'
\]

\[
\geq 0 \quad \text{(since } 0 \leq v \leq u').
\]

so the \((q + u' + 1)\)th row is still feasible.

In (case 2), the L.H.S. of \((q + u')^{th}\) row minus the L.H.S. of \((q + u)\)th row is \(\frac{K - r - v}{\gamma}x_{q+u'+2}' - \frac{K - r - v - \gamma}{\gamma}(x_{q+u'+1}' - \alpha)\), so

\[
\Delta_{q+u'+1} - \Delta_{q+u} = \frac{K - r - v}{\gamma} \epsilon_{q+u'+2}' - \frac{K - r - v - \gamma}{\gamma} \epsilon_{q+u'+1}'
\]

\[
\geq \frac{K - r - v}{\gamma} (\epsilon_{q+u'+2}' - \epsilon_{q+u'+1}' \text{ (since } \alpha \geq 0))
\]

so the \((q + u' + 1)\)th row is feasible. Thus we finish proving the claim.

\[\square\]

**B.4. Proof of Claim 2**

**Proof of Claim 2.** The change of the LP objective is

\[
\Delta_{LP} = -\epsilon + \epsilon_{q+1} + \epsilon_{q+2} + \cdots + \epsilon_{K}
\]

\[
= \epsilon[-1 + \frac{\gamma}{K - r} + \frac{\gamma}{K - r} \cdot \frac{K - r - \gamma}{K - r - 1} + \cdots + \frac{\gamma}{K - r} \cdot \frac{K - r - \gamma}{K - r - 1} \cdots \frac{K - r - m + 2 - \gamma}{K - r - m + 1}],
\]

where inside the bracket there are \(m = K - q\) items except for the "\(-1\)". For notational simplicity, let the sum inside the bracket to be,

\[
h_\gamma(\gamma) := -1 + \frac{\gamma}{K - r} + \frac{\gamma}{K - r} \cdot \frac{K - r - \gamma}{K - r - 1} + \cdots + \frac{\gamma}{K - r} \cdot \frac{K - r - \gamma}{K - r - 1} \cdots \frac{K - r - m + 2 - \gamma}{K - r - m + 1}.
\]  

(12)
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First of all, since \( K - r \geq K - q = m \), we have that

\[
h_r(\gamma) \leq h_{r=q}(\gamma) = -1 + \frac{\gamma}{m} + \frac{\gamma}{m} \cdot \frac{m - \gamma}{m - 1} + \ldots + \frac{\gamma}{m} \cdot \frac{m - \gamma \cdot 2 - \gamma}{2}.
\]

(13)

Let us merge the items in Eq. (13) from left to right one by one,

\[
h_{r=q}(\gamma) = -1 + \frac{\gamma}{m} + \frac{\gamma}{m} \cdot \frac{m - \gamma}{m - 1} + \ldots + \frac{\gamma}{m} \cdot \frac{m - \gamma \cdot 2 - \gamma}{2}
\]

\[
= -\frac{m - \gamma}{m} + \frac{\gamma}{m} \cdot \frac{m - \gamma}{m - 1} + \ldots + \frac{\gamma}{m} \cdot \frac{m - \gamma \cdot 2 - \gamma}{2}
\]

\[
= -\frac{m - \gamma}{m} + \frac{m - \gamma}{m - 1} + \ldots + \frac{\gamma}{m} \cdot \frac{m - \gamma \cdot 2 - \gamma}{2}
\]

\[
= -\frac{(m - \gamma)(m - \gamma - 1) \ldots (2 - \gamma)(1 - \gamma)}{m(m - 1) \ldots 2 \cdot 1}
\]

setting \( \gamma \) to be 1

Then \( h_r(\gamma) \leq 0, \forall \gamma \in [0, 1] \). And it is easy to see that the equality holds if \( r = q \) and \( \gamma = 1 \).

So we have that \( \Delta_{LP} = c \cdot h_r(\gamma) \leq 0 \), where the equality is achieved at “boundary” situation \( (r = q \) and \( \gamma = 1) \). \( \square \)

B.5. Proof of Lemma 3

Proof of Lemma 3.

a) Firstly let us prove that \( R(\{l_1, \ldots, l_s\}) \geq R(\emptyset) \).

The high level idea is to change the structure of the constraint matrix in the LP associated with \( \{l_1, \ldots, l_s\} \), such that in each change, the optimal LP objective value \( R \) never increases.

To better explain the proof, let us state the setup first of all. Let us call the elements inside the set \( \Omega^* \cap S^K = \{l_1 = j_{m_1}, l_2 = j_{m_2}, \ldots, l_s = j_{m_s}\} \) the “common elements”, the elements outside of it are called the “non-common” elements. For the common elements, two elements \( l_i, l_j \) being “adjacent” means that \( l_i + 1 = l_j \). Mapping to the constraint matrix (2), it means that the corresponding columns (column \( l_i \) and column \( l_j \)) are adjacent with each other. So we also call the corresponding columns “common columns”.

We prove part a) of Lemma 3 by two steps: In the first step, we try to make all of the common elements inside \( \{l_1, l_2, \ldots, l_s\} \) to be adjacent with each other; In the second step, we get rid of the common columns in the constraint matrix from left to right, one by one. Specifically,

Step 1. Assume that some elements inside \( \{l_1, l_2, \ldots, l_s\} \) are not adjacent, like the example in (2). Suppose that \( l_r \) and \( l_{r+1} \) are not adjacent, which means \( l_r + 1 < l_{r+1} \). Denote \( p = l_r \) for notational simplicity. Let us use \( A \) to represent the constraint matrix in the constructed LP associated with \( \{l_1, l_2, \ldots, l_{r-1}, l_r, l_{r+1}, \ldots, l_s\} \), let \( A' \) represent the constraint matrix associated with \( \{l_1, l_2, \ldots, l_{r-1}, l_r + 1, l_{r+1}, \ldots, l_s\} \). Notice that \( l_r + 1 \) is a non-common element for \( A \), but a common element for \( A' \). Furthermore \( A \) and \( A' \) only differ by columns \( p \) and \( p + 1 = l_r + 1 \). One can verify that any solution \( x \in R^m_+ \) that satisfies \( A' \cdot x \geq 1 \) also satisfies \( A' \cdot x \geq 1 \), which implies that,

\[
R(\{l_1, l_2, \ldots, l_{r-1}, l_r, l_{r+1}, \ldots, l_s\}) \geq R(\{l_1, l_2, \ldots, l_{r-1}, l_r + 1, l_{r+1}, \ldots, l_s\}).
\]

(14)

The change from \( \{l_1, l_2, \ldots, l_{r-1}, l_r, l_{r+1}, \ldots, l_s\} \) to \( \{l_1, l_2, \ldots, l_{r-1}, l_r + 1, l_{r+1}, \ldots, l_s\} \) is essentially to swap the roles of one originally non-common element \( l_r + 1 \) and the common element \( l_r \). Repeatedly applying this operation for all \( 1 \leq r \leq s - 1 \) such that \( l_r + 1 < l_{r+1} \), we can get that,

\[
R(\{l_1, l_2, \ldots, l_{r-1}, l_r, l_{r+1}, \ldots, l_s\}) \geq R(\{l_s - s + 1, l_s - s + 2, \ldots, l_s - 1, l_s\}).
\]

(15)

Now the \( s \) common elements inside \( \{l_s - s + 1, l_s - s + 2, \ldots, l_s - 1, l_s\} \) are adjacent with each other.
Step 2. Let $B$ be the constraint matrix associated with $\{l_s - s + 1, l_s - s + 2, \cdots, l_s - 1, l_s\}$, and $B'$ be the constraint matrix associated with $\{l_s - s + 2, \cdots, l_s - 1, l_s\}$. $B$ and $B'$ differ in the columns from $l_s - s + 1$ to the end. Suppose the vector $x^*$ is the optimal solution of $R(\{l_s - s + 1, l_s - s + 2, \cdots, l_s - 1, l_s\})$. According to Lemma 2 we know that $x_{l_s+1}^* \geq x_{l_s+2}^* \geq \cdots \geq x_{l_s}^* \geq x_{l_s+1}^*$. So one can easily verify that it must hold that $B'x^* \geq 1$, which implies
\[
R(\{l_s - s + 1, l_s - s + 2, \cdots, l_s - 1, l_s\}) \geq R(\{l_s - s + 2, \cdots, l_s - 1, l_s\}).
\] (16)

Apply this process repeatedly $s$ times, one can reach that $R(\{l_s - s + 1, l_s - s + 2, \cdots, l_s - 1, l_s\}) \geq R(\emptyset)$.

Combining step 1 and step 2, we prove part a) of Lemma 3.

b) Then let us prove that $R(\emptyset) = \frac{1}{\alpha} \left[ 1 - \left( \frac{K - \alpha \gamma}{K} \right)^K \right]$.

The constructed LP associated with $R(\emptyset)$ is,
\[
R(\emptyset) = \min \sum_{i=1}^{K} x_i
\]
subject to the constraints that,
\[
x_i \geq 0, \forall i = 1, \cdots, K
\]
and
\[
\begin{bmatrix}
\frac{K}{\gamma} & \frac{K}{\gamma} & \cdots & \frac{K}{\gamma} \\
\alpha & \frac{K}{\gamma} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\alpha & \alpha & \cdots & \alpha & \frac{K}{\gamma} \\
\alpha & \alpha & \cdots & \alpha & \alpha & \frac{K}{\gamma} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha & \alpha & \cdots & \alpha & \alpha & \cdots & \alpha & \frac{K}{\gamma} \\
\alpha & \alpha & \cdots & \alpha & \alpha & \cdots & \alpha & \cdots & \alpha & \frac{K}{\gamma} \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_a \\
x_b \\
\vdots \\
x_d \\
x_K
\end{bmatrix}
\geq \begin{bmatrix} 1 \\
1 \\
\vdots \\
1 \\
1 \\
\vdots \\
1 \\
1
\end{bmatrix}
\] (17)

One can observe that the vector $y \in \mathbb{R}_+^K$ such that $y_i = \frac{\gamma}{K} \left( \frac{K - \alpha \gamma}{K} \right)^{i-1}, i = 1, \cdots, K$ satisfies all the constraints and every row in (17) is tight, hence $y$ is the optimal solution. So
\[
R(\emptyset) = \sum_{i=1}^{K} y_i = \frac{1}{\alpha} \left[ 1 - \left( \frac{K - \alpha \gamma}{K} \right)^K \right].
\] □

B.6. Proof of Lemma 4 for the tightness result

Proof of Lemma 4.

a) “When $\alpha = 0$, it is supermodular”:

It is easy to see that $\rho_i = 1/K, i \in \{K\}$. Since $f(\cdot)$ is convex, it can be easily verified that $F(\cdot)$ is supermodular.

b) “When $\gamma = 1$, it is submodular”:

Now $f(x) = x$. Assume there are $T_1 \subseteq T_2 \subseteq V, t \in V \setminus T_2$. Let $T_1 = S'_1 \cup \Omega'_1, T_2 = S'_2 \cup \Omega'_2$, so it holds $S'_1 \subseteq S'_2, \Omega'_1 \subseteq \Omega'_2$. Now there are two situations:
1) When \( t = j_i \in S \). There is,
\[
\rho_{j_i}(T_1) = \left[ 1 - \frac{\alpha \gamma}{K} f(|\Omega'_i|) \right] \rho_i
\]
\[
\rho_{j_i}(T_2) = \left[ 1 - \frac{\alpha \gamma}{K} f(|\Omega''_i|) \right] \rho_i
\]
Because \( f(\cdot) \) is nondecreasing, so it holds \( \rho_{j_i}(T_1) \geq \rho_{j_i}(T_2) \).

2) When \( t = \omega_i \in \Omega \). It reads,
\[
\rho_{\omega_i}(T_1) = \frac{1}{K} \left[ 1 - \alpha \gamma \sum_{j_i \in S'_1} \rho_i \right]
\]
\[
\rho_{\omega_i}(T_2) = \frac{1}{K} \left[ 1 - \alpha \gamma \sum_{j_i \in S'_2} \rho_i \right]
\]
Because \( S'_1 \subseteq S'_2 \), so \( \rho_{\omega_i}(T_1) \geq \rho_{\omega_i}(T_2) \).

The above two situations prove the submodularity of \( F(T) \) when \( \gamma = 1 \).

e) \( "F(T)" \) has submodularity ratio as \( \gamma \) and curvature as \( \alpha " \).

Let us assume \( T = A \cup B \) and \( T' = A' \cup B' \) are two disjoint sets \( (T \cap T' = \emptyset) \), where \( A \) and \( A' \) are subsets of \( S \) while \( B \) and \( B' \) are subsets of \( \Omega \).

First of all, for the submodularity ratio, assume w.l.o.g. that \( \rho_{T'}(T) > 0 \), so the submodularity ratio is \( \gamma = \min_{T,T'} \frac{\sum_{i \in T'} \rho_i(T)}{\rho_{T'}(T)} \).

One can see that,
\[
\rho_{T'}(T) = F(T' \cup T) - F(T)
\]
\[
= \frac{f(|B \cup B'|) - f(|B|)}{K} (1 - \alpha \gamma \sum_{j_i \in A} \rho_i) + \left[ 1 - \frac{\alpha \gamma}{K} f(|B \cup B'|) \right] \sum_{j_i \in A'} \rho_i
\]
and
\[
\sum_{i \in T'} \rho_i(T) = \sum_{\omega_i \in B'} \rho_{\omega_i}(T) + \sum_{j_i \in A'} \rho_{j_i}(T)
\]
\[
= |B'| \frac{f(|B|) + 1 - f(|B'|)}{K} \left( 1 - \alpha \gamma \sum_{j_i \in A} \rho_i \right) + \left[ 1 - \frac{\alpha \gamma}{K} f(|B|) \right] \sum_{j_i \in A'} \rho_i
\]

Because \( f(|B|) \leq f(|B \cup B'|) \), so one has \( \left[ 1 - \frac{\alpha \gamma}{K} f(|B \cup B'|) \right] \sum_{j_i \in A'} \rho_i \leq \left[ 1 - \frac{\alpha \gamma}{K} f(|B|) \right] \sum_{j_i \in A'} \rho_i \), the equality is achieved when \( B' = \emptyset \) or \( A' = \emptyset \). Therefore,
\[
\frac{\sum_{i \in T'} \rho_i(T)}{\rho_{T'}(T)} = \frac{|B'| \frac{f(|B|) + 1 - f(|B'|)}{K} \left( 1 - \alpha \gamma \sum_{j_i \in A} \rho_i \right) + \left[ 1 - \frac{\alpha \gamma}{K} f(|B|) \right] \sum_{j_i \in A'} \rho_i}{\frac{f(|B \cup B'|) - f(|B|)}{K} \left( 1 - \alpha \gamma \sum_{j_i \in A} \rho_i \right) + \left[ 1 - \frac{\alpha \gamma}{K} f(|B \cup B'|) \right] \sum_{j_i \in A'} \rho_i}
\]
\[
\geq \frac{|B'| \left( f(|B|) + 1 - f(|B'|) \right) \left( 1 - \alpha \gamma \sum_{j_i \in A} \rho_i \right) + \left[ 1 - \frac{\alpha \gamma}{K} f(|B|) \right] \sum_{j_i \in A'} \rho_i}{f(|B \cup B'|) - f(|B|)}
\]
\[
= \frac{|B'| \left( f(|B|) + 1 - f(|B'|) \right)}{f(|B \cup B'|) - f(|B|)} \left( 1 - \alpha \gamma \sum_{j_i \in A} \rho_i \right) \leq \frac{f(|B \cup B'|) - f(|B|)}{f(|B \cup B'|) - f(|B|)} \left( 1 - \alpha \gamma \sum_{j_i \in A} \rho_i \right).
\]

Where (18) comes from the fact: \( f(\cdot) \) is convex and nondecreasing in \([0, K]\), then
\[
\frac{|B'| \left( f(|B|) + 1 - f(|B'|) \right)}{K} \left( 1 - \alpha \gamma \sum_{j_i \in A} \rho_i \right) \leq \frac{f(|B \cup B'|) - f(|B|)}{K} \left( 1 - \alpha \gamma \sum_{j_i \in A} \rho_i \right).
\]
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Now to continue with (18), one can verify that by setting $B = \emptyset, B' = \Omega$, the minimum of (18) is achieved as $\gamma$, thus proving the submodularity ratio to be $\gamma$.

Then for the curvature, for any $t \in T = A \cup B$, we want to lower bound $\frac{\rho_{T}(T \setminus \{t\} \cup T')}{\rho_{T}(T \setminus \{t\})}$. There are two situations:

1) When $t = j_{i} \in A$, we have

$$\frac{\rho_{j_{i}}(T \setminus \{j_{i}\} \cup T')}{\rho_{j_{i}}(T \setminus \{j_{i}\})} = \frac{[1 - \frac{\alpha \gamma}{K} f(|B \cup B'|)] \rho_{i}}{[1 - \frac{\alpha \gamma}{K} f(|B|)] \rho_{i}} = 1 - \frac{\frac{\alpha \gamma}{K} f(|B \cup B'|)}{1 - \frac{\alpha \gamma}{K} f(|B|)}. \quad (19)$$

Since $f(\cdot)$ is convex and nondecreasing in $[0, K]$, it is easy to see that the minimum of (19) is achieved when $B = \emptyset, B' = \Omega$ as $1 - \alpha$.

2) When $t = \omega_{i} \in B$, we have,

$$\frac{\rho_{\omega_{i}}(T \setminus \{\omega_{i}\} \cup T')}{\rho_{\omega_{i}}(T \setminus \{\omega_{i}\})} = \frac{f(|B \cup B'|) - f(|B|)}{f(|B|) - f(|B|)} \frac{[1 - \alpha \gamma \sum_{i \in A \cup A'} \rho_{i}]}{[1 - \alpha \gamma \sum_{i \in A} \rho_{i}]} \geq \frac{1 - \alpha \gamma \sum_{i \in A \cup A'} \rho_{i}}{1 - \alpha \gamma \sum_{i \in A} \rho_{i}} = \frac{1 - \alpha + \alpha - \alpha \gamma \sum_{i \in A \cup A'} \rho_{i}}{1 - \alpha \gamma \sum_{i \in A} \rho_{i}} \quad (20)$$

where (20) is because $f(\cdot)$ is convex and nondecreasing in $[0, K]$.

Since $\alpha - \alpha \gamma \sum_{i \in A \cup A'} \rho_{i} \geq 0$ and $-\alpha \gamma \sum_{i \in A} \rho_{i} \leq 0$, continuing with (21) we have,

$$\frac{\rho_{\omega_{i}}(T' \setminus \{\omega_{i}\} \cup T)}{\rho_{\omega_{i}}(T \setminus \{\omega_{i}\})} \geq 1 - \alpha. \quad \square$$

The above two situations jointly proves that the objective in (3) has curvature $\alpha$.

C. Existing notions of curvature and submodularity ratio

In this section we firstly discuss the details of existing notions of curvature and submodularity ratio, then we present the relations to the notions in this paper.

C.1. Classical curvature of submodular functions

Curvature of submodular functions measures how close a submodular set function is to being modular, and has been used to prove improved theoretical results for constrained submodular minimization and learning of submodular functions (Iyer et al., 2013). Earlier, it has been used to tighten bounds for submodular maximization subject to a cardinality constraint (Conforti & Cornuéjols, 1984) or a matroid constraint (Vondrák, 2010).

**Definition 3** (Curvature of submodular functions (Conforti & Cornuéjols, 1984; Vondrák, 2010; Iyer et al., 2013)). The total curvature $\kappa_{F}$ (which we term as $\kappa_{\text{total}}$ in the main text) of a submodular function $F$ and the curvature $\kappa_{F}(S)$ w.r.t. a set $S \subseteq V$ are defined as,

$$\kappa_{F} := 1 - \min_{j \in V} \frac{\rho_{j}(V \setminus \{j\})}{\rho_{j}(\emptyset)} \quad \text{and} \quad \kappa_{F}(S) := 1 - \min_{j \in S} \frac{\rho_{j}(S \setminus \{j\})}{\rho_{j}(\emptyset)},$$

respectively. Assume without loss of generality that $F(\{j\}) > 0, \forall j \in V$. One can observe that $\kappa_{F}(S) \leq \kappa_{F}$. A modular function has curvature $\kappa_{F} = 0$, and a matroid rank function has maximal curvature $\kappa_{F} = 1$. Vondrák (2010) also defines
the relaxed notion of curvature, which is called curvature with respect to the optimum to be the smaller scalar \( \tilde{\kappa}_F(S) \) s.t.

\[
\rho_T(S) + \sum_{j \in S \cup T} \rho_j(S \cup T \setminus \{j\}) \geq (1 - \tilde{\kappa}_F(S))\rho_T(\emptyset), \forall T \subseteq V.
\] (22)

Iyer et al. (2013) propose two new notions of curvature, which are,

\[
\hat{\kappa}_F(S) := 1 - \min_{T \subseteq V} \frac{\rho_T(S) + \sum_{j \in S \cup T} \rho_j(S \cup T \setminus \{j\})}{\rho_T(\emptyset)},
\]

\[
\hat{\kappa}_F(S) := 1 - \frac{\sum_{j \in S} \rho_j(S \setminus \{j\})}{\sum_{j \in S} \rho_j(\emptyset)}.
\]

Iyer et al. (2013) show that for submodular functions, it holds that \( \hat{\kappa}_F(S) \leq \kappa_F(S) \leq \tilde{\kappa}_F(S) \leq \kappa_F \).

C.2. Curvature of non-submodular functions and comparison to the notions in Section 2

Sviridenko et al. (2015) present a new notion of curvature for monotone set functions. We show how it is related to our notion of curvature in Definition 2. We also show that our approximation factors using the combination of curvature and submodularity ratio characterize the performance of GREEDY for problem (P) better.

Specifically, for a nondecreasing function \( F \), Sviridenko et al. (2015, Section 8) define the curvature \( c \) as

\[
c = 1 - \min_{j \in V} \min_{A, B \in V \setminus \{j\}} \frac{\rho_j(A)}{\rho_j(B)}.
\] (Sviridenko et al., 2015, Theorem 8.1) show that for maximizing a nondecreasing function with bounded curvature \( c \in [0, 1] \), under a matroid constraint, GREEDY enjoys an approximation guarantee of \( (1 - c) \), and it is tight in terms of the definition of \( c \) in (23). The following remark discusses the relation to our definition of curvature.

**Remark 3.** For a nondecreasing function \( F \), it holds: a) \( c \) in (23) is always larger than the notion of curvature \( \alpha \) in Definition 2, i.e., \( c \geq \alpha \); b) For the GREEDY algorithm, there exists a class of functions for which the approximation guarantee characterized by \( c \) (which is \( 1 - c \)) is strictly smaller than the approximation guarantee characterized by the combination of \( \alpha \) and \( \gamma \) (which is \( \alpha^{-1}(1 - e^{-\alpha\gamma}) \) according to Theorem 1).

**Proof of Remark 3.**

**a)** Note that the definition of curvature in Definition 2 is equivalent to the smallest scalar \( \alpha \) such that

\[
\forall j \in V, B \subseteq A \in V \setminus \{j\}, \rho_j(A) \geq (1 - \alpha)\rho_j(B).
\]

Now it is easy to see that \( c \geq \alpha \).

**b)** Consider the class of functions in our tightness result in (3). From Lemma 4 we know that its curvature is \( \alpha \) and submodularity ratio is \( \gamma \). So its curvature \( c \) in (23) must be greater than or equal to \( \alpha \). Note that the approximation guarantee characterized by \( c \) is \( 1 - c \leq 1 - \alpha \). Taking \( \alpha = 1 \) in (3), the approximation guarantee of Sviridenko et al. (2015) is 0. While our approximation guarantee is \( \gamma \), for any \( \gamma \in (0, 1] \), our approximation guarantee is strictly higher than \( 1 - c \). \( \Box \)

**Submodularity ratio.** Informally, the submodularity ratio quantifies how close a set function is to being submodular (Das & Kempe, 2011).

**Definition 4** (Submodularity ratio from Das & Kempe (2011)). Let \( S, L \subseteq V \) be two disjoint sets and let \( F \) be a non-negative nondecreasing set function. The submodularity ratio of a set \( \Omega \) w.r.t. an integer \( K \) is given by,

\[
\gamma_{\Omega, K} := \min_{L \in \Omega} \min_{S: L \cap S = \emptyset, |S| \leq K} \frac{\sum_{j \in L} \rho_j(S)}{\rho_L(S)}.
\]
C.3. Relations to the notions in this work

- Our curvature and submodularity ratio are natural extensions of the classical ones for monotone submodular functions.

- Note that classical notions of curvature measure how close a submodular set function is to being modular. The notions of (generalized) curvature in Definition 2 measures how close a set function is to being supermodular.

- Our combinations of (generalized) curvature and submodularity ratio gives tight approximation guarantees for GREEDY, and this combination is more expressive than the curvature by Sviridenko et al. (2015), as shown in Remark 3.

D. Proofs for bounding parameters of applications

D.1. Proving Proposition 1

Proof of Proposition 1.

The following proof strategy considers the spectral parameters of the matrix $X_{S}X_{S}^{\top}$. For brevity, let us write $B = \Lambda + \sigma^{-2}X_{S}X_{S}^{\top}$. The matrix $B$ is symmetric positive definite matrix, thus can be factorized as $B = PD^{-1}P^{-1}$.

Let the eigenvalues of $X_{S}X_{S}^{\top}$ be $\lambda_{1}(S) \geq \cdots \geq \lambda_{d}(S) \geq 0$, where we use the notation that $\lambda_{i}(S) := \lambda_{i}(X_{S}X_{S}^{\top}), \forall i \in [d]$. Then the eigenvalues of $B$ are $\beta_{i}^{2} + \sigma^{-2}\lambda_{i}(S), i \in [d]$. One can easily see that $B^{-1} = PD^{-1}P^{-1}$, and $\text{tr}(B^{-1}) = \sum_{i=1}^{d} \frac{1}{\beta_{i}^{2} + \sigma^{-2}\lambda_{i}(S)}$.

Let the singular values of $X_{S}$ be $\sigma_{1}(X_{S}) \geq \cdots \geq \sigma_{q}(X_{S})$, where $q \leq \min\{d, |S|\}$. For notational simplicity, when $|S| < d$, we still use the notation $\sigma_{i}(X_{S}) = 0, i = q + 1, \cdots, d$ to represent the zeros values. It is easy to see that $\sigma_{i}^{2}(X_{S}) = \lambda_{i}(S), i = 1, \cdots, d$.

Monotonicity. It can be easily seen that $F_{A}(\emptyset) = 0$. To prove that $F_{A}(S)$ is monotone nondecreasing, one just needs to show that $\forall \omega \in V \setminus S$, it holds that $F(\{\omega\} \cup S) - F(S) \geq 0$. One can see that,

$$F(\{\omega\} \cup S) - F(S) = \sum_{i=1}^{d} \frac{1}{\beta_{i}^{2} + \sigma^{-2}\sigma_{i}^{2}(X_{S})} - \sum_{j=1}^{d} \frac{1}{\beta_{j}^{2} + \sigma^{-2}\sigma_{j}^{2}(X_{S \cup \{\omega\}})} \geq 0 \quad (\text{Cauchy interlacing inequality of singular values}).$$

Bounding parameters. Let us restate the assumptions: 1) The data points are normalized, i.e., $||x_{i}|| = 1, \forall i \in V$; 2) Spectral norm of the data matrix (the largest singular value) $||X|| = \sigma_{\text{max}}(X)$ is finite (i.e., $||X|| < \infty$). This assumption naturally holds because of Weyl’s inequality, it is easy to see that $||X|| \leq \sqrt{n}$.

Bounding submodularity ratio: For the submodularity ratio, we need to lower bound $\frac{\sum_{\omega \in \Omega \setminus S} \rho_{\omega}(S)}{\rho_{\emptyset}(S)} = \frac{\sum_{\omega \in \Omega \setminus S} F(\{\omega\} \cup S) - F(S)}{F(\Omega \setminus S) - F(S)}$.

For the numerator, we have,

$$\sum_{\omega \in \Omega \setminus S} F(\{\omega\} \cup S) - F(S) = \sum_{\omega \in \Omega \setminus S} \left[ \sum_{i=1}^{d} \frac{1}{\beta_{i}^{2} + \sigma^{-2}\sigma_{i}^{2}(X_{S})} - \sum_{j=1}^{d} \frac{1}{\beta_{j}^{2} + \sigma^{-2}\sigma_{j}^{2}(X_{S \cup \{\omega\}})} \right]$$

$$= \sum_{\omega \in \Omega \setminus S} \sum_{i=1}^{d} \frac{\sigma^{-2}\sigma_{i}^{2}(X_{S \cup \{\omega\}}) - \sigma_{i}^{2}(X_{S})}{(\beta_{i}^{2} + \sigma^{-2}\sigma_{i}^{2}(X_{S}))((\beta_{i}^{2} + \sigma^{-2}\sigma_{i}^{2}(X_{S})) - \sigma_{i}^{2}(X_{S}))} \geq (\beta^{2} + \sigma^{-2}\sigma_{\text{max}}^{2}(X))^{-2} \sum_{\omega \in \Omega \setminus S} \sum_{i=1}^{d} \sigma^{-2}[\sigma_{i}^{2}(X_{S \cup \{\omega\}}) - \sigma_{i}^{2}(X_{S})] \geq (\beta^{2} + \sigma^{-2}||X||^{-2})^{-2} \sum_{\omega \in \Omega \setminus S} \sum_{i=1}^{d} \sigma^{-2}[\lambda_{i}(S \cup \{\omega\}) - \lambda_{i}(S)]$$
Combining Equations (26) and (27) we can get,

\[
= (\beta^2 + \sigma^{-2}\|X\|^2)^{-2} \sum_{\omega \in \Omega \setminus S} \sigma^{-2} \left[ tr(X_{S \cup \{\omega\}}X_{S \cup \{\omega\}}^T) - tr(X_SX_S^T) \right]
\]

\[
= (\beta^2 + \sigma^{-2}\|X\|^2)^{-2} \sum_{\omega \in \Omega \setminus S} \sigma^{-2} \left[ tr(x_\omega x_\omega^T) - tr(X_SX_S^T) \right]
\]

\[
= (\beta^2 + \sigma^{-2}\|X\|^2)^{-2} \sum_{\omega \in \Omega \setminus S} \sigma^{-2} tr(x_\omega x_\omega^T) \quad \text{(linearity of the trace)}
\]

\[
= (\beta^2 + \sigma^{-2}\|X\|^2)^{-2} \sum_{\omega \in \Omega \setminus S} \sigma^{-2} \|x_\omega\|^2
\]

\[
= \sigma^{-2}(\beta^2 + \sigma^{-2}\|X\|^2)^{-2}|\Omega \setminus S| \quad \text{(normalization of the data points)} \tag{24}
\]

For the denominator, one has,

\[
F(\Omega \cup S) - F(S) = \sum_{i=1}^d \frac{1}{\beta^2 + \sigma^{-2}\sigma_i^2(X_S)} - \sum_{j=1}^d \frac{1}{\beta^2 + \sigma^{-2}\sigma_j^2(X_{S \cup \{\omega\}})}
\]

\[
\leq \sum_{i=1}^d \frac{1}{\beta^2 + \sigma^{-2}\sigma_i^2(X_S)} - \frac{|\Omega \setminus S|}{\beta^2 + \sigma^{-2}\sigma_j^2(X_{S \cup \{\omega\}})} \quad \text{(interlacing inequality of singular values)}
\]

\[
\leq |\Omega \setminus S| \left(\frac{1}{\beta^2} - \frac{1}{\beta^2 + \sigma^{-2}\|X\|^2} \right)
\]

\[
= |\Omega \setminus S| \frac{\sigma^{-2}\|X\|^2}{\beta^2(\beta^2 + \sigma^{-2}\|X\|^2)} \tag{25}
\]

Combining Eq. (24) and Eq. (25) it reads,

\[
\frac{\sum_{\omega \in \Omega \setminus S} F(\omega \cup S) - F(S)}{F(\Omega \cup S) - F(S)} \geq \frac{|\Omega \setminus S|(1 + \|X\|^2)^{-2}}{|\Omega \setminus S| \frac{\sigma^{-2}\|X\|^2}{\beta^2(\beta^2 + \sigma^{-2}\|X\|^2)}}
\]

\[
= \frac{\beta^2}{\|X\|^2(\beta^2 + \sigma^{-2}\|X\|^2)} \quad =: \gamma_0 > 0.
\]

Bounding curvature. We want to lower bound \(1 - \alpha\), which corresponds to lower bounding \(\frac{F(S \cup \Omega) - F(S \setminus \{i\} \cup \Omega)}{F(S) - F(S \setminus \{i\})}\). For the numerator, one has,

\[
F(S \cup \Omega) - F(S \setminus \{i\} \cup \Omega) = \sum_{i=1}^d \frac{1}{\beta^2 + \sigma^{-2}\sigma_i^2(X_{S \setminus \{i\} \cup \Omega})} - \sum_{j=1}^d \frac{1}{\beta^2 + \sigma^{-2}\sigma_j^2(X_{S \cup \{\omega\}})}
\]

\[
\geq \sigma^{-2}(\beta^2 + \sigma^{-2}\|X\|^2)^{-2} \quad \text{(similar derivation as in Eq. (24))} \tag{26}
\]

For the denominator, one has,

\[
F(S) - F(S \setminus \{i\}) = \sum_{i=1}^d \frac{1}{\beta^2 + \sigma^{-2}\sigma_i^2(X_{S \setminus \{i\}})} - \sum_{j=1}^d \frac{1}{\beta^2 + \sigma^{-2}\sigma_j^2(X_S)}
\]

\[
\leq \frac{1}{\beta^2 + \sigma^{-2}\sigma_i^2(X_{S \setminus \{i\}})} - \frac{1}{\beta^2 + \sigma^{-2}\sigma_j^2(X_S)} \quad \text{(Cauchy interlacing inequality)}
\]

\[
\leq \frac{\sigma^{-2}\|X\|^2}{\beta^2(\beta^2 + \sigma^{-2}\|X\|^2)} \tag{27}
\]

Combining Equations (26) and (27) we can get,

\[
\frac{F(S \cup \Omega) - F(S \setminus \{i\} \cup \Omega)}{F(S) - F(S \setminus \{i\})} \geq \frac{\beta^2}{\|X\|^2(\beta^2 + \sigma^{-2}\|X\|^2)}
\]
D.2. Proofs for determinantal functions of square submatrix

Proof of Proposition 2.

a) We want to prove that $F(\cdot)$ is supermodular. Assume that $A \subseteq B \subseteq \mathcal{V}$ and $i \in \mathcal{V} \setminus B$, then

$$
\rho_i(A) = \det(\mathbf{I} + \sigma^{-2} \Sigma_{A \cup \{i\}}) - \det(\mathbf{I} + \sigma^{-2} \Sigma_A)
\leq \sum_{S \subseteq A} \det((\sigma^{-2} \Sigma)_{S \cup \{i\}})
\leq \sum_{S \subseteq B} \det((\sigma^{-2} \Sigma)_{S \cup \{i\}})
= \det(\mathbf{I} + \sigma^{-2} \Sigma_{B \cup \{i\}}) - \det(\mathbf{I} + \sigma^{-2} \Sigma_B)
= \rho_i(B).
$$

which proves that $F(\cdot)$ is supermodular.

b) We want to lower bound $\frac{\sum_{\omega \in \Omega \setminus S} \rho_i(S)}{\rho_i(S)} = \frac{\sum_{\omega \in \Omega \setminus S} F(\omega \cup S) - F(S)}{F(\Omega \cup S) - F(S)}$.

For the numerator, one has,

$$
\sum_{\omega \in \Omega \setminus S} F(\omega \cup S) - F(S) = \sum_{\omega \in \Omega \setminus S} \left|_{S \cup \{\omega\}} \prod_{i=1}^{\lfloor |S_{\cup \{\omega\}}| \rfloor} \lambda_i(A_{S \cup \{\omega\}}) - \prod_{j=1}^{\lfloor |S| \rfloor} \lambda_j(A_S) \right|
= \sum_{\omega \in \Omega \setminus S} \lambda_i(A_{S \cup \{\omega\}}) \left|_{S \cup \{\omega\}} \prod_{i=1}^{\lfloor |S| \rfloor} \lambda_i(A_S) - \prod_{j=1}^{\lfloor |S| \rfloor} \lambda_j(A_S) \right|
\geq \sum_{\omega \in \Omega \setminus S} \lambda_i(A_{S \cup \{\omega\}}) \left|_{S \cup \{\omega\}} \prod_{i=1}^{\lfloor |S| \rfloor} \lambda_i(A_S) - \prod_{j=1}^{\lfloor |S| \rfloor} \lambda_j(A_S) \right| \quad \text{(Cauchy interlacing inequalities)}
= \sum_{\omega \in \Omega \setminus S} (\lambda_i(A_{S \cup \{\omega\}}) - 1) \prod_{i=1}^{\lfloor |S| \rfloor} \lambda_i(A_S). \quad \text{(28)}
$$

For the denominator, it holds,

$$
F(\Omega \cup S) - F(S) = \prod_{i=1}^{\lfloor |\Omega \cup S| \rfloor} \lambda_i(A_{\Omega \cup S}) \prod_{j=1}^{\lfloor |\Omega \setminus S| \rfloor} \lambda_j(A_{\Omega \setminus S}) - \prod_{i=1}^{\lfloor |\Omega \setminus S| \rfloor} \lambda_i(A_{\Omega \setminus S})
\leq \left( \prod_{j=1}^{\lfloor |\Omega \setminus S| \rfloor} \lambda_j(A_{\Omega \setminus S} \setminus \mathcal{A}_i) - 1 \right) \prod_{i=1}^{\lfloor |\Omega \setminus S| \rfloor} \lambda_i(A_S) \quad \text{(Cauchy interlacing inequalities)}. \quad \text{(29)}
$$

Combining (28) and (29) it reads,

$$
\frac{\sum_{\omega \in \Omega \setminus S} F(\omega \cup S) - F(S)}{F(\Omega \cup S) - F(S)} \geq \frac{\sum_{\omega \in \Omega \setminus S} (\lambda_i(A_{S \cup \{\omega\}}) - 1) \prod_{i=1}^{\lfloor |S| \rfloor} \lambda_i(A_S)}{\left( \prod_{j=1}^{\lfloor |\Omega \setminus S| \rfloor} \lambda_j(A_{\Omega \setminus S} \setminus \mathcal{A}_i) - 1 \right) \prod_{i=1}^{\lfloor |\Omega \setminus S| \rfloor} \lambda_i(A_S)}
= \frac{\sum_{\omega \in \Omega \setminus S} (\lambda_i(A_{S \cup \{\omega\}}) - 1)}{\left( \prod_{j=1}^{\lfloor |\Omega \setminus S| \rfloor} \lambda_j(A_{\Omega \setminus S} \setminus \mathcal{A}_i) - 1 \right)}.
$$
where the last inequality comes from that $|\Omega \setminus S| \leq K$.

\[ \geq \frac{K(\lambda_m - 1)}{\prod_{j=1}^{K} \lambda_j - 1}, \]

D.3. LP with combinatorial constraints

D.3.1. Two examples where $F(S)$ is non-submodular

1. Considering the following LP:

\[
\begin{align*}
\text{max} & \quad 4x_1 + x_2 + 4x_3 \\
\text{s.t.} & \quad 2x_1 + x_2 + x_2 + 2x_3 \leq 2 \\
& \quad x_2 + 2x_3 \leq 2 \\
& \quad x_1, x_2, x_3 \geq 0.
\end{align*}
\]

(30)

For this LP, one can easily see that $F(\{1, 2\}) = 4, F(\{2\}) = 2, F(\{1, 2, 3\}) = 8, F(\{2, 3\}) = 4$, thus $F(\{1, 2\}) - F(\{2\}) < F(\{1, 2, 3\}) - F(\{2, 3\})$, which shows $F$ is non-submodular.

2. Considering the following LP:

\[
\begin{align*}
\text{max} & \quad 10x_1 + 12x_2 + 12x_3 \\
\text{s.t.} & \quad 2x_1 + 2x_2 + x_3 \leq 20 \\
& \quad 2x_1 + 2x_2 + x_3 \leq 20 \\
& \quad 2x_2 + x_3 \leq 20 \\
& \quad x_1, x_2, x_3 \geq 0.
\end{align*}
\]

(31)

For this LP, one can see that $F(\{1, 2\}) = 120, F(\{2\}) = 120, F(\{1, 2, 3\}) = 136, F(\{2, 3\}) = 120$, thus $F(\{1, 2\}) - F(\{2\}) < F(\{1, 2, 3\}) - F(\{2, 3\})$. But this one has degenerate basic feasible solutions.

D.3.2. Proving Proposition 3

To prove Proposition 3, we first need to present the setup. The LP corresponding to $F(S)$ is,

\[
\begin{align*}
\text{max} & \quad \langle d_S, x_S \rangle \\
\text{s.t.} & \quad A_Sx_S \leq b \\
& \quad x_S \geq 0.
\end{align*}
\]

(32)

where the columns of $A_S \in \mathbb{R}^{m \times |S|}$ is the columns of $A$ indexed by the set $S$. $x_S$ (respectively, $d_S$) is the subvector of $x$ (respectively, $d$) indexed by $S$. To apply the optimality condition of a LP in the standard form, let us change $(LP_S)$ to be the following standard LP by introducing the slack variable $\xi \in \mathbb{R}^m$,

\[
\begin{align*}
\text{max} & \quad \langle d_S, x_S \rangle \\
\text{s.t.} & \quad A_Sx_S + I_m \xi = b \\
& \quad x_S \geq 0, \xi \geq 0.
\end{align*}
\]

(33)

where $c_S := -d_S$. Let us denote $\bar{A} := [A_S, I_m] \in \mathbb{R}^{m \times (|S| + m)}, \bar{x} := [x_S^T, \xi^T]^T$.

Let $(x^{(S)}, \xi^{(S)})$ denote the optimal solution of $(LP_S^*)$. The corresponding basis of of $(LP_S^*)$ is $B^{(S)}$, which is a subset of $V \cup \{\xi_1, \cdots, \xi_m\}$, and $|B^{(S)}| = m$.

According to Bertsimas & Tsitsiklis (1997, Chapter 3.1), the optimality condition for $(LP_S^*)$ is: Given a basic feasible solution $(x, \xi)$ with the basis as $B$, the reduced cost is $c_j = c_j - c_{B^-1}A_B^T A_{-j}$. 1) If $(x, \xi)$ is optimal and non-degenerate, then $c_j \geq 0, \forall j$; 2) If $c_j \geq 0, \forall j$, then $(x, \xi)$ is optimal.

Proof of Proposition 3. First of all, let us detail the non-degeneracy assumption.
Non-degeneracy assumption: The basic feasible solutions of the corresponding LP in standard form ($LP^*_S$) is non-degenerate ∀$S \subseteq \mathcal{V}$.

a) It is easy to see that $F(\emptyset) = 0$, and $F(S)$ is nondecreasing.

b) For the submodularity ratio, we want to lower bound $\frac{\sum_{\omega \in \Lambda(S) \setminus \Omega} \rho(\omega)}{\Omega(S)}$. There could be in total four situations:

1) $\sum_{\omega \in \Omega(S)} \rho(\omega) = 0$ but $\Omega(S) > 0$. We will prove that this situation cannot happen, or in the other words, $\sum_{\omega \in \Omega(S)} F(\{\omega\} \cup S) - F(S) = 0$ implies that $F(\Omega \cup S) - F(S) = 0$ as well.

First of all, since $F(S)$ is nondecreasing, so $F(\{\omega\} \cup S) - F(S) = 0$, $\forall \omega$. We know that $(x^{(S)}, \xi(S))$ is the optimal solution of ($LP^*_S$), and $(x^{(S)}, \xi(S))$ is a basic feasible solution of ($LP^*_{S \cup \{\omega\}}$), so $(x^{(S)}, \xi(S))$ is also the optimal solution of ($LP^*_{S \cup \{\omega\}}$). Since ($LP^*_{S \cup \{\omega\}}$) is non-degenerate, according to the optimality condition, the reduced cost of $x_\omega$; $\bar{c}_\omega$ must be greater than or equal zero.

Now we know that $\bar{c}_\omega \geq 0, \forall \omega \in \Omega(S)$, and $(x^{(S)}, \xi(S))$ is a basic feasible solution of ($LP^*_{S \cup \{\omega\}}$) as well, again using the optimality condition, we know that $(x^{(S)}, \xi(S))$ is optimal for ($LP^*_{S \cup \{\omega\}}$). So $F(\Omega \cup S) - F(S) = 0$.

2) $\sum_{\omega \in \Omega(S)} \rho(\omega) = 0$ and $\Omega(S) = 0$. The submodularity ratio is 1 in this situation.

3) $\sum_{\omega \in \Omega(S)} \rho(\omega) > 0$ and $\Omega(S) = 0$. This can be ignored since we want a lower bound.

4) $\sum_{\omega \in \Omega(S)} \rho(\omega) > 0$ and $\Omega(S) > 0$. This situation gives the lower bound:

$$\frac{\sum_{\omega \in \Omega(S)} \rho(\omega)}{\Omega(S)} \geq \frac{\max_{\omega \in \Omega(S)} \rho(\omega)}{F(\mathcal{V})} \geq \frac{\min_{\rho(\omega) > 0, \omega \in \mathcal{V}, \omega \not\in \Omega(S)} \rho(\omega)}{F(\mathcal{V})} =: \gamma_0 > 0.$$ 

□

E. Details about SDP formulation of Bayesian A-optimality

The SDP formulation used in this paper is consistent with that from Boyd & Vandenberghe (2004, Chapter 7.5) and Krause et al. (2008). To make this work self-contained, we present the details here.

First of all, maximizing the Bayesian A-optimality objective is equivalent to,

$$\min_{S \subseteq \mathcal{V}, |S| \leq K} \text{tr}((\Lambda + \sigma^{-2}X_S X^\top_S)^{-1})$$ (34)

By introducing binary variables $m_j, j \in [n]$, (34) is equivalent to,

$$\min \text{tr}((\Lambda + \sigma^{-2} \sum_{j=1}^n m_j x_j x_j^\top)^{-1})$$ (35)

s.t. $m_j \in \{0, 1\}, j \in [n], m_1 + \cdots + m_n \leq K$

A good relaxation is (relax the variables $\lambda_j = m_j/K, j \in [n]$),

$$\min \text{tr}((\Lambda + \sigma^{-2} \sum_{j=1}^n \lambda_j x_j x_j^\top)^{-1})$$ (36)

s.t. $\lambda \in \mathbb{R}^n_+, 1^\top \lambda = 1.$

According to the (extended) Schur complement lemma, the relaxed formulation (36) is equivalent to the following SDP problem,

$$\min_{u \in \mathbb{R}^n} 1^\top u$$ (37)
For argument b): Let us take a minimum cardinality function with $k = 3.11$ in Borodin et al. (2014). From 1). The submodularity ratio can be lower bounded by $n_F$. For argument a): Let $\lambda$ be a weakly submodular function. 

Proof of Remark 4.

For argument a): Let $F(S) := |S|^4$, $S \subseteq V$, which is a supermodular function, so the curvature is 0 (upper-bounded away from 1). The submodularity ratio can be lower bounded by $n^{-5}$. But it is not weakly submodular according to Proposition 3.11 in Borodin et al. (2014).

For argument b): Let us take a minimum cardinality function with $k = 2$, i.e., $F(S) = B > 0$ iff. $|S| \geq 2$, otherwise $F(S) = 0$. According to Proposition 3.5 in Borodin et al. (2014), it is weakly submodular, but it is easy to see that its submodularity ratio is $0$. 

More on submodularity index. It is defined as (equivalent to that in Zhou & Spanos (2016)):

$$\min_\Omega, \min_{S \subseteq \Omega, |S| \leq K} \left( \sum_{\omega \in \Omega \setminus S} \rho_\omega(S) - \rho_{\Omega}(S) \right)$$

which is closely related to the submodularity ratio by Das & Kempe (2011).

Note that our analysis of the Greedy algorithm, which considers a novel combination of the (generalized) submodular ratio and curvature, is different from the classical analysis. Furthermore, it provides stronger bounds for the maximization of monotone submodular functions as long as the the curvature is upper-bounded away from 1.

G. More applications

G.1. Subset selection using the $R^2$ objective

Subset selection aims to estimate a predictor variable $Z$ using linear regression on a small subset from the set of observation variables $V = \{X_1, \ldots, X_n\}$. Let $C$ be the covariance matrix among the observation variables $\{X_1, \ldots, X_n\}$. We use $b$ to denote the covariances between $Z$ and the $X_i$, with entries $b_i = \text{Cov}(Z, X_i)$. Assuming there are $m$ observations, let us arrange the data of all the observation variables to be a design matrix $X \in \mathbb{R}^{m \times n}$, with each column representing the observations of one variable. Given a budget parameter $K$, subset selection tries to find a set $S \subseteq V$ of at most $K$ elements, and a linear predictor $Z' = \sum_{i \in S} \alpha_i X_i = X' \alpha_S$, in order to maximize the squared multiple correlation (Johnson et al., 2002), $R_{Z,S}^2 = \frac{\text{Var}(Z) - \text{Var}(Z'|Z)}{\text{Var}(Z)}$, it measures the fraction of variance of $Z$ explained by variables in $S$. Assume $Z$ is normalized to have variance 1, and it is well-known that the optimal regression coefficients is $\alpha_S = (C_S)^{-1}b_S$, so the $R^2$ objective can be formulated as,

$$F(S) := R_{Z,S}^2 = b_S^T (C_S)^{-1} b_S, S \subseteq V.$$  \hspace{1cm} (38)

Das & Kempe (2011) show that the submodularity ratio of $F$ in (38) can be lower bounded by $\lambda_{\min}(C)$, which is the smallest eigenvalue of $C$. The theoretical results in this work suggests that the approximation guarantees for maximizing $F$ in (38) can be further improved by analyzing the curvature parameters. The experimental results in Section 5.4 demonstrates that it is promising to upper bound the curvature parameters of (38) (possibly with regular assumptions).
Guarantees for Greedy Maximization of Non-submodular Functions with Applications

G.2. Sparse modeling with strongly convex loss functions

Sparse modeling aims to build a model with a small subset of at most $K$ features, out of in total $n$ features. Let $f(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ to be the loss function, the corresponding objective is to,

$$\min f(x) \text{ s.t. } |\text{supp}(x)| \leq K.$$  

Assume that $f(x)$ is $m$-strongly convex and has Lipschitz continuous gradient with parameter $L$, which is equivalent to say that $g(x) := -f(x)$ is $m$-strongly concave and has $L$-Lipschitz continuous gradient. Then for all $x, y \in \text{dom}(f)$ it holds,

$$\frac{m}{2} \|y - x\|^2 \leq -g(y) + g(x) + \langle \nabla g(x), y - x \rangle \leq \frac{L}{2} \|y - x\|^2. \quad (39)$$  

In solving this problem, the GREEDY algorithm maximizes the corresponding auxiliary set function,

$$F(S) := \max_{\text{supp}(x) \subseteq S} g(x), S \subseteq [n] \quad (40)$$

Elenberg et al. (2016) analyzed the approximation guarantees of GREEDY by bounding the submodularity ratio of $F(S)$. Specifically,

Lemma 5 (Paraphrasing Theorem 1 in Elenberg et al. (2016)). The submodularity ratio of $F(S)$ in (40) is lower bounded by $\frac{m}{2L}$.

By further bounding the curvature parameters of the auxiliary set function in (40), one can get improved approximation guarantees according to our theoretical findings.

G.3. Optimal budget allocation with combinatorial constraints

Optimal budget allocation (Soma et al., 2014) is a special case of the influence maximization problem, it aims to distribute the budget (e.g., space of an inline advertisement, or time for a TV advertisement) among the customers, and to maximize the expected influence on the potential customers. A concrete application is for the search marketing advertiser bidding the budget among the customers, and to maximize their influence on the customers, while respecting various continuous and combinatorial constraints. For the continuous constraints, for instance, each vendor has a specified budget limit for advertising, and the ad space associated with each search keyword can not be too large. These continuous constraints can be formulated as a convex set $P$. For combinatorial constraints, each vendor needs to obey the Internet regulations of sensitive search keywords in his country, so the search engine company can only choose a subset of “legal” keywords for a specific vendor. The combinatorial constraints can be arranged as a matroid $\mathcal{M} = (\mathcal{V}, \mathcal{I})$. Hence the problem in general can be formulated as,

$$\max_{x \in P \text{ and supp}(x) \subseteq \mathcal{I}} g(x),$$

where $g(x)$ is the total influence modeled by a DR-submodular function. For one of its possible forms, one can refer to Bian et al. (2016). The GREEDY algorithm solves this problem by maximizing the following auxiliary set function $F(S)$ while respecting the combinatorial constraints,

$$\max_{S \in \mathcal{I}} F(S), \text{ where } F(S) := \max_{\text{supp}(x) \subseteq S, x \in P} g(x). \quad (41)$$  

By studying the submodularity ratio and curvature parameters of $F(S)$ in (41), one could obtain theoretical guarantees of the GREEDY algorithm according to Theorem 1 in this work.

H. More experimental results

H.1. Bayesian A-optimality experiments

We put the results on a specific randomly generated dataset, to illustrate what does the proved bounds looks like. In the synthetic experiments we generate random observations from a multivariate Gaussian distribution with correlation
Figure 9: Function value, parameters and approximation bounds of experimental design on synthetic data. Correlation: 0.5

as 0.5. Fig. 9 shows the results (function value, parameters and approximation bounds) for one randomly generated data set with $d = 6$ features and $n = 12$ observations. Specifically, Fig. 9c traces the two approximation bounds from Theorem 1 (and Lemma 3): one curve shows the constant-factor bound $\frac{1}{\alpha}(1 - e^{-\alpha \gamma})$ and the other the $K$-dependent bound $\frac{1}{\alpha} \left[ 1 - \left( \frac{K - \alpha \gamma}{K} \right)^K \right]$. We observe that both bounds give reasonable predictions of the performance of GREEDY.