Linearized coherent states for Hamiltonian systems with two equidistant ladder spectra

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Abstract. A simple way to construct exactly solvable Hamiltonians whose spectra contain two equidistant ladders, one finite and another infinite, appears when applying supersymmetric quantum mechanics to the harmonic oscillator. Some of those supersymmetric partners have third order differential ladder operators, although the order of the transformation is higher than one. In this work the linearized coherent states for these specific Hamiltonians are studied. To each SUSY partner Hamiltonian corresponds two families of linearized coherent states: one inside the subspace associated with the isospectral part of the spectrum and another one in the finite subspace generated by the states inserted through the SUSY technique.

1. Introduction

In the beginning of quantum mechanics, Erwin Schrödinger was interested in finding a connection between the new science and classical mechanics [1]. With this interest in mind, a set of quantum states that restored what he considered as the classical behavior for the position operator of a quantum system was developed. In the last fifty years, the study of these states has taken an important place in quantum physics, since they can be used to establish an essential link between quantum and semi-classical behaviors of a system [2–8]. The name given to them is coherent states (CS), and it was coined by Glauber when studying electromagnetic correlation functions [9, 10].

For the harmonic oscillator, the standard CS are expressed as

\[ \psi_z = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \phi_n, \]  

where $\phi_n$ are the normalized eigenstates of the harmonic oscillator Hamiltonian with eigenvalues $E_n = n + 1/2$ (in dimensionless units) and $z \in \mathbb{C}$. Some important properties of the standard coherent states are the following: they are eigenstates of the standard annihilation operator $a$; they appear by acting the displacement operator $D(z) = \exp (z a^+ - z^* a^-)$ onto the ground state; they minimize the Heisenberg uncertainty relation for the position and momentum operators; the resolution of the identity operator is satisfied, namely,

\[ 1 = \frac{1}{\pi} \int_{\mathbb{C}} |z \rangle \langle z| \, dz. \]
These four properties have been used as definitions to look for the coherent states for systems different from the harmonic oscillator.

In this work we will study Hamiltonians with two equidistant ladder spectra, one finite and another infinite, which are obtained through supersymmetric quantum mechanics (SUSY QM) departing from the harmonic oscillator. These Hamiltonians possess third order differential ladder operator, and through them we will obtain two sets of coherent states, one inside the subspace generated by the eigenfunctions associated with the infinite ladder and another one having as basis the eigenfunctions of the finite ladder. The two sets of CS will be generated as displaced versions of the corresponding extremal states, and it will be shown that for both it is valid the resolution of the identity in the respective subspace.

The structure of this paper is the following: in section 2 the construction of Hamiltonians whose spectra consist of two equidistant ladders is done. In the third section the linearized coherent states for these systems are obtained and some of their properties will be studied. The section 4 contains the calculation of some important mean values in the limit of the harmonic oscillator. Our conclusions shall be presented in the last section.

2. Supersymmetric quantum mechanics and harmonic oscillator

The supersymmetric quantum mechanics, which is closely related to the factorization method and the intertwining technique, is a procedure to find exactly solvable Hamiltonians. It departs from a given Hamiltonian $H_0$, whose orthogonal set of eigenvectors is known, and looks for another Hamiltonian $H_k$, whose eigenfunctions are going to be obtained. Both Hamiltonians will have a similar spectrum, differing at most in a finite number of energy levels that can be chosen at will. A more detailed information about this method can be found in [11–15].

The two intertwined Hamiltonians take the form

$$H_0 = -\frac{1}{2} \frac{d^2}{dx^2} + V_0(x), \quad H_k = -\frac{1}{2} \frac{d^2}{dx^2} + V_k(x).$$

(3)

In order to apply this technique, let us suppose the existence of a $k$-th order differential operator $B_k^\dagger$ that intertwines the previous Schrödinger Hamiltonians in the way

$$H_k B_k^\dagger = B_k^\dagger H_0.$$  

(4)

Since $B_k^\dagger$ is a $k$-th order differential operator, this version of the technique is referred as $k$-SUSY. In the standard terminology it is also said that the potentials $V_0(x)$ and $V_k(x)$ are SUSY partners.

It can be shown that the new potential has the form

$$V_k(x) = V_0(x) - \frac{d^2}{dx^2} \ln W [u(x, \epsilon_0), \ldots, u(x, \epsilon_{k-1})],$$

(5)

where the $k$ functions $u(x, \epsilon_j), \ j = 0, \ldots, k - 1$ are known as transformations functions and fulfill the initial Schrödinger equation

$$H_0 u(x, \epsilon) = \epsilon u(x, \epsilon).$$

(6)

The parameters $\epsilon_j$ are called factorization energies, and they do not necessarily belong to the spectrum of $H_0$, i.e., the transformations functions do not always satisfy the same boundary conditions as the eigenfunctions of $H_0$ do.
The intertwining operator can be found directly from the intertwining relation (4); for the purposes of this paper it is enough to write down its formal expression:

$$B_k^\dagger = \frac{1}{W(u_{\epsilon_0}, u_{\epsilon_1}, \ldots, u_{\epsilon_{k-1}})} \begin{vmatrix} u_{\epsilon_0} & u_{\epsilon_1} & \ldots & 1 \\ u_{\epsilon_0}' & u_{\epsilon_1}' & \ldots & d/dx \\ \vdots & \vdots & \ddots & \vdots \\ u_{\epsilon_0}^{(k)} & u_{\epsilon_1}^{(k)} & \ldots & d^{(k)}/dx^{(k)} \end{vmatrix}. \quad (7)$$

The intertwining relation (4) ensures that if $\phi_n$ is an eigenfunction of $H_0$ with eigenvalue $E_n$ then $\psi^k_n \propto B_k^\dagger \phi_n$ will be an eigenfunction of $H_k$ with the same eigenvalue. One could ask now if $\{\psi^k_n, \ n = 0, 1, 2, \ldots\}$ is a complete orthogonal set. In order to answer, let us suppose the existence of other vectors $\psi \phi_j^k$ which are orthogonal to each one of the previous set, i.e.,

$$\langle \psi^k_j | \psi^k_n \rangle \propto \langle \psi^k_j | B_k^\dagger \phi_n \rangle = \langle B_j \psi_j^k | \phi_n \rangle = 0 \ \ \forall \ n. \quad (8)$$

Since $\{\phi_n, \ n = 0, 1, 2, \ldots\}$ is a complete orthogonal set, it turns out that,

$$B_k \psi_j^k = 0, \quad (9)$$

i.e., in order to complete the orthogonal set of eigenfunctions of $H_k$ we need to analyze also the kernel of $B_k \equiv (B_k^\dagger)^\dagger$. Since $B_k$ is a $k$-th order differential operator, its kernel is a subspace of dimension at most $k$. The different choices of the transformation functions give families of Hamiltonians whose spectra differ from the original one in a finite numbers of energy levels.

In order to generate exactly solvable Hamiltonians whose spectra are composed of two equidistant ladders, we will apply this technique to the harmonic oscillator; it turns out that one of the ladders is infinite, composed of the same energy levels of the harmonic oscillator; the second ladder is a finite one, formed by the new levels inserted by the SUSY technique.

To perform the transformation we need to find the general solution $u(x, \epsilon)$ of the Schrödinger equation for the potential $V_0(x) = x^2/2$ with a factorization energy $\epsilon$, which is given by

$$u(x, \epsilon) = e^{-x^2/2} \left[ 1F_1 \left( \frac{1-2\epsilon}{4}, \frac{1}{2}, x^2 \right) + 2\nu x \frac{\Gamma(\frac{3-2\epsilon}{4})}{\Gamma(1-\frac{2\epsilon}{4})} 1F_1 \left( \frac{3-2\epsilon}{4}, \frac{3}{2}, x^2 \right) \right], \quad (10)$$

where $\nu$ is a real arbitrary constant and $1F_1(a, c, x)$ is the confluent hypergeometric function. Suppose that $\epsilon \leq E_0$, then it is known that for $|\nu| < 1$ the solution will not have zeroes but for $|\nu| > 1$ it will have one zero in some point on the real line. In order to generate a non singular potential $V_k(x)$ with $k$ new levels, the factorization energies have to be chosen as $\epsilon_0 < \epsilon_1 < \cdots < \epsilon_{k-1} < E_0$ with $|\nu_k-j| < 1$ for $j$ odd and $|\nu_k-j| > 1$ for $j$ even, $j = 1, 2, \ldots, k$.

The spectrum of the corresponding Hamiltonian $H_k$ will be

$$\text{Sp} \{H_k\} = \{\epsilon_0, \epsilon_1, \ldots, \epsilon_{k-1}, E_0, E_1, E_2, \ldots\} \quad (11)$$

It is important to define the following operators

$$L_k^+ = B_k^\dagger a^+ B_k, \quad L_k^- = B_k^\dagger a^- B_k, \quad (12)$$

where $a^\pm$ are the standard creation and annihilation operators of the harmonic oscillator. These are $(2k + 1)$-th order differential operators that obey the following commutation relations

$$[H_k, L_k^\pm] = \pm L_k^\pm, \quad (13)$$
i.e. they are ladder operators for the Hamiltonian $H_k$.

Since the eigenfunctions associated with the inserted levels belong to the Kernel of $B_k$, they cannot be reached by applying $L_k^\pm$ onto vectors which are outside this subspace. Thus, the Hilbert space $\mathcal{H}$ is naturally separated in two orthogonal subspaces, one generated by the vectors $\psi_{k0}^H$ (called $\mathcal{H}_{iso}$) and other generated by the eigenfunctions associated with the new levels (named $\mathcal{H}_{new}$).

There are some special cases when third order differential ladder operators for $H_k$ can be found, which will connect now the eigenfunctions of $H_k$ belonging to $\mathcal{H}_{new}$. These specific cases are of major interest of this work, and they must fulfill the conditions contained in the following theorem:

**Factorization theorem**

Suppose that the $k$-th order SUSY partner $H_k$ of the harmonic oscillator Hamiltonian $H_0$ is generated by $k$ transformation functions $u(x,\epsilon_j), j = 0, \ldots, k - 1$, which are connected by the standard annihilation operator in the way:

$$u(x,\epsilon_{k-j-1}) = (a^-)^j u(x,\epsilon_{k-1}), \quad \epsilon_{k-j-1} = \epsilon_{k-1} - j,$$

where $u(x,\epsilon_{k-1})$ is a nodeless solution of the stationary Schrödinger equation associated to $H_0$, given by equation (10) with $\epsilon_{k-1} < 1/2$ and $|\nu_{k-1}| < 1$. Therefore, the natural ladder operator $L_k^+ \equiv B_k^1 a^+ B_k$ of $H_k$, which is of $(2k+1)$-th order, can be factorized in the form

$$L_k^+ = P_{k-1}(H_k)l_k^+, \quad (15)$$

where $P_{k-1}(H_k) = (H_k - \epsilon_1) \cdots (H_k - \epsilon_{k-1})$ is a polynomial of degree $k - 1$ in $H_k$, $l_k^+$ is a third-order differential ladder operator such that $[H_k, l_k^+] = l_k^+$ and

$$l_k^+ l_k^- = \left( H_k - \frac{1}{2} \right) (H_k - \epsilon_0)(H_k - \epsilon_{k-1} - 1). \quad (16)$$

The proof of this theorem can be found in [16] (see also [17]). It implies that a subset of Hamiltonians $H_k$, beside having their natural $(2k+1)$-th order differential ladder operators also have third order ones. These SUSY partner Hamiltonians have an equidistant ladder with $k$ steps below the ground state energy $E_0 = 1/2$ whose separation is the same as for the harmonic oscillator. Also, the transformation functions are not longer arbitrary: once the first one $u(x,\epsilon_{k-1})$ is chosen, all the others become fixed by the theorem. As a result, the only parameters that can be varied are the number of levels $k$ to be inserted, the gap $E_0 - \epsilon_{k-1}$ between the ladders (with the restriction $\epsilon_{k-1} < E_0$), and the real parameter $\nu_{k-1}$ of the transformation function $u(x,\epsilon_{k-1})$ (with $|\nu_{k-1}| < 1$).

### 3. Linearized coherent states

Once we have identified the Hamiltonians $H_k$ whose spectra are composed of two equidistant ladder and used the factorization theorem to obtain the corresponding third order differential ladder operators, we proceed now to do a linearization process by defining new ladder operators that have a simpler action on the eigenvectors of $H_k$. With the last ladder operators we will find two sets of coherent states, one inside $\mathcal{H}_{iso}$ and other in $\mathcal{H}_{new}$.

The linearized ladder operators $\ell_k^\pm$ are defined departing from the third order ones $l_k^\pm$ as

$$\ell_k^+ \equiv \sigma(H_k) l_k^+, \quad \ell_k^- \equiv \sigma(H_k + 1) l_k^-,$$

with

$$\sigma(H_k) = \left( (H_k - \epsilon_0)(H_k - \epsilon_0 - k) \right)^{-1/2}, \quad (18)$$
where we will take the positive square root. Their action onto the basis of $H_{iso}$ and $H_{new}$ will be given below, and it will help us to make the coherent state construction.

Note that this kind of linearization process has been applied previously to the generic SUSY partners of the harmonic oscillator in order to obtain several sets of CS by using different annihilation operators [18–21].

3.1. Coherent states in the subspace $H_{iso}$

The action of the new ladder operators $\ell_+^k$ on the eigenfunctions of $H_k$ in the subspace $H_{iso}$ is given by

$$\ell_-^k \psi_n^k = \sqrt{n} \psi_{n-1}^k, \quad \ell_+^k \psi_n^k = \sqrt{n+1} \psi_{n+1}^k.$$  

Furthermore, we can easily show that the operators $\{\ell_-^k, \ell_+^k, H_k\}$ obey the following commutation rules inside $H_{iso}$:

$$[\ell_-^k, \ell_+^k] = 1 | H_{iso}, \quad [H_k, \ell_\pm^k] = \pm \ell_\pm^k.$$

Equations (20) mean that the linearized ladder operators satisfy a Heisenberg-Weyl algebra on $H_{iso}$.

Now, let us generate a set of linearized CS using an analogue of the displacement operator, defined as

$$\mathcal{D}(z) = \exp \left( -\frac{1}{2} |z|^2 \right) \exp \left( z \ell_+^k \right) \exp \left( -z^\dagger \ell_-^k \right).$$

Then, the CS are constructed by applying the last operator onto the extremal state $\psi_0^k$, which leads to

$$\psi_{z_{iso}}^k = \mathcal{D}(z) \psi_0^k = \exp \left( -\frac{|z|^2}{2} \right) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \psi_n^k,$$

i.e., we obtain a similar expression as for the CS of the harmonic oscillator of equation (1). The difference rely in the states which are involved in the summation: for the harmonic oscillator they are the eigenstates of $H_0$, while here they are the eigenstates of $H_k$ in $H_{iso}$. Note that the coherent states of equation (22) are already normalized.

Following the same procedure as for the harmonic oscillator [22], it can be shown that for the family of coherent states $\psi_{z_{iso}}^k$, it is valid the standard resolution of the identity inside this subspace

$$\frac{1}{\pi} \int \langle z_{iso}^+ | z_{iso}^- \rangle \, d \text{Re}(z) \, d \text{Im}(z) = 1 | H_{iso}.$$  

3.2. Coherent states in the subspace $H_{new}$

It is straightforward to calculate now the actions of the linearized ladder operators $\ell_\pm^k$ onto the basis vectors $\left\{ \psi_{\epsilon_j}^k, j = 0, \ldots, k-1 \right\}$ of $H_{new}$:

$$\ell_-^k \psi_{\epsilon_j}^k = (1 - \delta_{j,k-1}) \sqrt{\epsilon_{j+1} - E_0} \psi_{\epsilon_{j+1}}^k, \quad \ell_+^k \psi_{\epsilon_j}^k = (1 - \delta_{j,0}) \sqrt{\epsilon_j - E_0} \psi_{\epsilon_{j-1}}^k,$$

which come from the definition of $\ell_\pm^k$ in equations (17). The first term on the right hand side of equations (24) represent the fact that $\ell_-^k$ annihilates the eigenstate $\psi_0^k$, while $\ell_+^k$ annihilates $\psi_{\epsilon_{k-1}}^k$. Despite the fact that the operators $\ell_\pm^k$ do not satisfy the Heisenberg-Weyl algebra in the subspace $H_{new}$, we will use the same displacement operator of equation (21), and we will apply it now to the ground state $\psi_0^k$ in order to obtain

$$\psi_{z_{new}}^k = C_z \mathcal{D}(z) \psi_0^k = C_z \sum_{j=0}^{k-1} \frac{(iz)^j}{j!} \sqrt{\frac{1}{\Gamma(E_0 - \epsilon_0 - j)}} \psi_{\epsilon_j}^k.$$

(5)
A direct calculation leads now to the following mean values:

\[ H = 4.1. \] Subspace relation for a coherent state can be evaluated.

On the other hand, for \( k > 0 \) the complete Hilbert space and we recover the mean values for the standard coherent states. \( \psi \) with \( U \) if the parameters of the transformation function details see the appendix.

\[ l_k^- = a^- (H_k - 1/2), \quad l_k^+ = (H_k - 1/2) a^+. \]

It can be shown that this set satisfies the resolution of the identity in the subspace \( \mathcal{H}_{\text{new}} \), namely,

\[ \int_\mathbb{C} |z^k_{\text{new}} \rangle \langle z^k_{\text{new}} | \mu(z) dz = 1 | \mathcal{H}_{\text{new}} , \]

where \( \mu(z) \) is a positive definite function given by

\[ \mu(z) = \frac{\Gamma^2(E_0 + 1 - \epsilon_0)}{\pi C^2} U(E_0 + 1 - \epsilon_0, 1; |z|^2) , \]

with \( U(a, c; z) \) being the logarithmic solution of the confluent hypergeometric equation (for more details see the appendix).

4. Harmonic oscillator limit

If the parameters of the transformation function \( u(x, \epsilon_{k-1}) \) are chosen as \( \epsilon_{k-1} = -1/2 \) and \( \nu_{k-1} = 0 \), the SUSY partner potential becomes \( V_k(x) = x^2/2 - k \), i.e. we obtain again the harmonic oscillator but with the energy origin displaced. The corresponding third order ladder operators are now

\[ l_k^- = a^- (H_k - 1/2), \quad l_k^+ = (H_k - 1/2) a^+. \]

It can be seen that \( l_k^- \) annihilates the vectors \( \psi_k^0 = \phi_k \) and \( \psi_{\epsilon_k-1} = \phi_{k-1} \), as expected. The operators \( X \) and \( P \) can be expressed in terms of the standard creation and annihilation operators and with that expressions the Heisenberg uncertainty relation for a coherent state can be evaluated.

4.1. Subspace \( \mathcal{H}_{\text{iso}} \)

A direct calculation leads now to the followings mean values:

\[ \langle X \rangle_z = \sqrt{2} \text{Re}(z) e^{-|z|^2} \sum_{n=0}^{\infty} \frac{|z|^{2n}}{n!} \sqrt{\frac{n + 1 + k}{n + 1}} , \]

\[ \langle P \rangle_z = \sqrt{2} \text{Im}(z) e^{-|z|^2} \sum_{n=0}^{\infty} \frac{|z|^{2n}}{n!} \sqrt{\frac{n + 1 + k}{n + 1}} , \]

\[ \langle X^2 \rangle_z = |z|^2 + k + 1/2 + \text{Re}(z^2) e^{-|z|^2} \sum_{n=0}^{\infty} \frac{|z|^{2n}}{n!} \sqrt{\frac{(n + 1 + k)(n + 2 + k)}{(n + 1)(n + 2)}} , \]

\[ \langle P^2 \rangle_z = |z|^2 + k + 1/2 - \text{Re}(z^2) e^{-|z|^2} \sum_{n=0}^{\infty} \frac{|z|^{2n}}{n!} \sqrt{\frac{(n + 1 + k)(n + 2 + k)}{(n + 1)(n + 2)}} . \]

For \( k = 0 \), i.e., if we do not make any SUSY transformation at all, the subspace \( \mathcal{H}_{\text{iso}} \) becomes the complete Hilbert space and we recover the mean values for the standard coherent states. On the other hand, for \( k > 0 \) it turns out that \( \langle \Delta X \rangle_z \langle \Delta P \rangle_z \) approaches 1/2 for large \( |z| \) (see an example in figure 1). In addition, for \( z = 0 \) it is obtained that \( \psi_z = \psi_k^0 = \phi_k \), i.e., the coherent state becomes the \( k \)-th excited state of the harmonic oscillator, for which \( \Delta X \Delta P = k + 1/2 \).
Figure 1. Heisenberg uncertainty relation for the linearized CS (considering just the first 100 terms) in the harmonic oscillator limit with $k = 3$ inside the subspace $\mathcal{H}_{\text{iso}}$.

4.2. Subspace $\mathcal{H}_{\text{new}}$

For this finite subspace, the corresponding mean values are now

$$\langle X \rangle_z = \sqrt{2} |C_z|^2 \text{Re}(iz) \sum_{j=0}^{k-2} \frac{|z|^{2j}}{(j!)^2 (k-1-j)!} \frac{1}{\sqrt{(j+1)(k-1-j)}}$$  \hspace{1cm} (34)

$$\langle P \rangle_z = \sqrt{2} |C_z|^2 \text{Im}(iz) \sum_{j=0}^{k-2} \frac{|z|^{2j}}{(j!)^2 (k-1-j)!} \frac{1}{\sqrt{(j+1)(k-1-j)}}$$  \hspace{1cm} (35)

$$\langle X^2 \rangle_z = |C_z|^2 \sum_{j=1}^{k-1} \frac{|z|^{2j}}{(j!)^2 (k-1-j)!} \frac{j}{2} + \frac{1}{2}$$

$$- |C_z|^2 \text{Re}(z^2) \sum_{j=0}^{k-3} \frac{|z|^{2j}}{(j!)^2 (k-3-j)!} \frac{1}{\sqrt{(j+1)(j+2)(k-1-j)(k-2-j)}}$$  \hspace{1cm} (36)

$$\langle P^2 \rangle_z = |C_z|^2 \sum_{j=1}^{k-1} \frac{|z|^{2j}}{(j!)^2 (k-1-j)!} \frac{j}{2} + \frac{1}{2}$$

$$+ |C_z|^2 \text{Re}(z^2) \sum_{j=0}^{k-3} \frac{|z|^{2j}}{(j!)^2 (k-3-j)!} \frac{1}{\sqrt{(j+1)(j+2)(k-1-j)(k-2-j)}}$$  \hspace{1cm} (37)

In figure 2 it can be seen an example of the Heisenberg uncertainty relation as a function of $z$. For $z = 0$ the corresponding CS reduces to the ground state, minimizing then the Heisenberg uncertainty relation; for large $|z|$ it approaches to $k - 1/2$, which is the value of the Heisenberg uncertainty relation for the $(k - 1)$-th excited state, precisely the one with the highest energy inside this subspace.
Figure 2. Heisenberg uncertainty relation for the linearized CS in the harmonic oscillator limit with $k = 3$ inside the subspace $\mathcal{H}_{new}$.

5. Concluding remarks
In this work we have studied the linearized coherent states for the $k$-SUSY partners of the harmonic oscillator which posses third order differential ladder operators. These operators were first linearized, and then the coherent states were calculated by applying the corresponding displacement operator onto the extremal states. As a result, two families of coherent states were generated: one inside the subspace isospectral to the harmonic oscillator and another one in the subspace generated by the eigenfunctions associated to the inserted energy levels. Both families of coherent states admit a resolution of the identity in the corresponding subspace. Since the harmonic oscillator is a shape invariance potential, then the Heisenberg uncertainty relation for the operators $X$ and $P$ can be simply analyzed in the case when the $k$-SUSY partner potential becomes the initial one.

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Appendix
In order to guarantee the resolution of the identity in the subspace $\mathcal{H}_{new}$, i.e.,

$$\int_{\mathcal{C}} \langle \zeta^k \rangle \langle \zeta^k | \mu(z) \rangle dz = 1 \mid_{\mathcal{H}_{new}},$$

we have to find a positive definite function $\mu(z)$ which will make valid equation (38). We follow the work of Sixdeniers and Penson [23], where they have solved a similar problem. If we substitute the expression for $|\zeta^k_{new}\rangle$ given by equation (25), write $z$ in polar coordinates and integrate the angular one, we obtain

$$1 \mid_{\mathcal{H}_{new}} = 2\pi \sum_{j=0}^{k-1} \frac{\langle \epsilon_j \rangle \langle \epsilon_j \rangle}{(j!)^2 \Gamma(E_0 - \epsilon_0 - j)} \int_0^{\infty} C^2 z^{2j+1} \mu(r) dr.$$
In order to simplify the last equation, let us introduce the function $f(r) = \pi C^2 \mu(r)$, then let us make a change of variable $r^2 = x$ and of index $j = s - 1$ in such a way that the following equation must be fulfilled

$$\int_0^\infty x^{s-1} f(x) dx = \Gamma^2(s) \Gamma(E_0 + 1 - \epsilon_0 - s) \equiv M[f(x); s]. \quad (40)$$

Now we need to find the inverse Mellin transform $M^{-1}[\Gamma^2(s) \Gamma(E_0 + 1 - \epsilon_0 - s); x]$. It is possible to find several inverse Mellin transforms in tables, for example in Erdélyi’s book [24]. In this case, the function $f(x)$ turns out to be a Meijer $G$ function with $m = 2$, $n = 1$, $p = 1$, $q = 2$, $a_1 = \epsilon_0 - E_0$, $b_1 = b_2 = 0$, i.e.,

$$f(r) = G^{21}_{12} \left( \begin{array}{c} \epsilon_0 - E_0 \\ 0,0 \end{array} \right | r^2 \right). \quad (41)$$

Even more, in Erdélyi’s book on transcendental functions [25] one can find some expressions for the Meijer $G$ function in terms of other special functions, in particular of the Whittaker function $W_{\kappa,\mu}(z)$, which in turn can be written in terms of the logarithmic solution of the confluent hypergeometric equation $U(a, c; z)$ [26]. Then we have

$$f(r) = \Gamma^2(E_0 + 1 - \epsilon_0) U(E_0 + 1 - \epsilon_0, 1; r^2). \quad (42)$$

We still need to prove the positiveness of $\mu(z)$. To accomplish this, we use the integral representation of $U(a, c; z)$:

$$U(a, c; z) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-zt} t^{a-1} (1 + t)^{b-a-1} dt, \quad (43)$$

and thus the measure $\mu(r)$ can be written as

$$\mu(r) = \frac{\Gamma(E_0 + 1 - \epsilon_0)}{\pi C^2} \int_0^\infty e^{-r^2 t} t^{E_0 - \epsilon_0}(1 + t)^{\epsilon_0 - E_0 - 1} dt. \quad (44)$$

Besides, taking into account that $E_0 > \epsilon_0$, and that the domain of $r$ and $t$ is $[0, \infty)$ we can conclude that we have found, at least, one positive definite measure, i.e, this CS admit a resolution of the identity in the subspace $\mathcal{H}_{\text{new}}$.

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