Blowup of $C^2$ Solutions for the Euler Equations and Euler-Poisson Equations in $R^N$

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Abstract
In this paper, we use integration method to show that there is no existence of global $C^2$ solution with compact support, to the pressureless Euler-Poisson equations with attractive forces in $R^N$. And the similar result can be shown, provided that the uniformly bounded functional:

$$\int_{\Omega(t)} K\gamma(\gamma - 1)\rho^{\gamma-2}(\nabla\rho)^2\,dx + \int_{\Omega(t)} K\gamma \rho^{\gamma-1}\Delta\rho\,dx + \epsilon \geq -\delta\alpha(N)M,$$

(1)

where $M$ is the mass of the solutions and $|\Omega|$ is the fixed volume of $\Omega(t)$.

On the other hand, our differentiation method provides a simpler proof to show the blowup result in "D. H. Chae and E. Tadmor, On the Finite Time Blow-up of the Euler-Poisson Equations in $R^N$, Commun. Math. Sci. 6 (2008), no. 3, 785–789."

Key Words: Euler Equations, Euler-Poisson Equations, Blowup, Repulsive Forces, Attractive Forces, $C^2$ Solutions

1 Introduction
The Euler ($\delta = 0$)/ Euler-Poisson ($\delta = \pm 1$) equations can be written in the following form:

$$\begin{cases}
\rho_t + \nabla \cdot (\rho u) = 0, \\
(\rho u)_t + \nabla \cdot (\rho u \otimes u) + \nabla P = \delta \rho \nabla \Phi, \\
\Delta \Phi(t, x) = \alpha(N)\rho,
\end{cases}$$

(2)

where $\alpha(N)$ is a constant related to the unit ball in $R^N$: $\alpha(1) = 1$, $\alpha(2) = 2\pi$ and $\alpha(3) = 4\pi$. And as usual, $\rho = \rho(t, x)$ and $u = u(t, x) \in R^N$ are the density and the velocity respectively. $P = P(\rho)$ is the pressure function. In the above systems, the self-gravitational potential field $\Phi = \Phi(t, x)$ is determined by the density $\rho$ itself, through the Poisson equation (2). For $N = 3$, the equations (2) are the classical (non-relativistic) descriptions of a galaxy, in astrophysics. See [3] and [7], for details about the systems. The $\gamma$-law can be applied on the pressure term $P(\rho)$, i.e.

$$P(\rho) = K\rho^\gamma,$$

(3)

which is a common hypothesis. If the parameter $K > 0$, we call the system with pressure; if $K = 0$, we call it pressureless. The constant $\gamma = c_P/c_v \geq 1$, where $c_P$, $c_v$ are the specific heats per unit

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mass under constant pressure and constant volume respectively, is the ratio of the specific heats, that is, the adiabatic exponent in the equation (3). In particular, the fluid is called isothermal if \( \gamma = 1 \). If the parameter \( K > 0 \), we call the system with pressure; if \( K = 0 \), we call it pressureless. On the other hand, the Poisson equation (2) can be solved as

\[
\Phi(t, x) = \int_{R^N} G(x - y) \rho(t, y) dy,
\]

where \( G \) is the Green’s function for the Poisson equation in the \( N \)-dimensional spaces defined by

\[
G(x) = \begin{cases} 
|x|, & N = 1; \\
\log |x|, & N = 2; \\
\frac{-1}{|x|^{N-2}}, & N \geq 3.
\end{cases}
\]

When \( \delta = 1 \), the system is the compressible Euler equations with repulsive forces. The equation (2) is the Poisson equation through which the potential with repulsive forces is determined by the density distribution of the electrons. In this case, the equations can be viewed as a semiconductor model. See [4], [18] for details about the system. When \( \delta = -1 \), the system can model the self-gravitating fluid, such as gaseous stars. Besides, the evolution of the cosmology can be modelled by the dust distribution without pressure term. That describes the stellar systems of collisionless and gravitational \( n \)-body systems [11]. And the pressureless Euler-Poisson equations can be derived from the Vlasov-Poisson-Boltzmann model with the zero mean free path [13].

Usually the Euler-Poisson equations can be rewritten in the scalar form:

\[
\left\{ \begin{array}{l}
\frac{\partial \rho}{\partial t} + \sum_{k=1}^{N} u_k \frac{\partial \rho}{\partial x_k} + \rho \sum_{k=1}^{N} \frac{\partial u_k}{\partial x_k} = 0, \\
\rho \left( \frac{\partial u_i}{\partial t} + \sum_{k=1}^{N} u_k \frac{\partial u_i}{\partial x_k} \right) + \frac{\partial \rho}{\partial x_i} = \delta \rho \frac{\partial \Phi}{\partial x_i}, \text{ for } i = 1, 2, ..., N.
\end{array} \right.
\]

For the construction of the analytical solutions for the system with pressure, the interested readers may see [7], [14], [19], [9], [5], [24], [25], [27] and [28]. The results for local existence theories can be founded in [19], [2] and [12]. The analysis of stabilities for the systems may be referred in [23], [1], [15], [10], [20], [21], [22], [8], [9], [25], [16], [26], [5] and [6].

Recently, Chae and Tadmor [5] showed the finite time blowup, for the pressureless Euler-Poisson equations with attractive forces, under the initial condition,

\[
S := \{ a \in R^N | \rho_0(a) > 0, \ \Omega(a) = 0, \ \nabla \cdot u(0, x(0)) < 0 \} \neq \phi.
\]

On the other hand, in [26], we have the blowup results if the solutions with compact support under the condition,

\[
2 \int_{\Omega(t)} (\rho |u|^2 + 2P) dx < M^2 - \epsilon,
\]

where \( M \) is the mass of the solution.

In this article, the alternative approaches are adopted to show that there is no global existence of \( C^2 \) solutions for the system, [6], with compact support without the condition (7):

**Theorem 1** For the pressureless Euler-Poisson equations with attractive forces \( (\delta = -1) \), (7), there do not exist global \( C^2 \) solutions \((\rho, u)\) with compact support. For the systems with pressure, for \( \gamma > 1 \), the above result is also true provided that the uniformly bounded functional:

\[
\int_{\Omega(t)} K \gamma \left[ (\gamma - 1) \rho^{\gamma-2} (\nabla \rho)^2 + \rho^{\gamma-1} \Delta \rho \right] dx + \epsilon \geq -\delta \alpha(N) M,
\]

where \( \epsilon \) is an arbitrary small positive constant, \( M \) is the mass of the solution and \( |\Omega| \) is the fixed volume of \( \Omega(t) \).
2 Integration Method

In this section, we present the proof of Theorem 1.

Proof. First, we show that the \( \rho(t, x(t; x)) \) preserves its positive nature as the mass equation (6) can be converted to be

\[
\frac{D\rho}{Dt} + \rho \nabla \cdot u = 0,
\]

with the material derivative,

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + (u \cdot \nabla).
\]

Integrate the equation (10):

\[
\rho(t, x) = \rho_0(x_0(0, x_0)) \exp \left( - \int_0^t \nabla \cdot u(t, x(t; 0, x_0)) \, dt \right) > 0,
\]

for \( \rho_0(x_0(0, x_0)) > 0 \), along the characteristic curve.

For the pressureless Euler-Poisson equations with attractive forces (\( \delta = -1 \)), we divide \( \rho \) to the momentum equation (6) to have

\[
u_t + u \cdot \nabla u = -\nabla \Phi,
\]

And we take divergence to the above equation:

\[
\nabla (u_t + u \cdot \nabla u) = -\Delta \Phi.
\]

By taking integration to the above equation, we have,

\[
\int_{\Omega(t)} \nabla \left( u_t + u \cdot \nabla u \right) dx = -\int_{\Omega(t)} \alpha(N) \rho dx,
\]

\[
\int_{\Omega(t)} \nabla (u_t + u \cdot \nabla u) dx = -\alpha(N) M,
\]

\[
\int_{\Omega(t)} \left[ \sum_{i=1}^N u_{ix_i} + \sum_{i=1}^N u_i \left( \sum_{k=1}^N u_{kx_k} \right) + \sum_{i=1}^N u_{ix_i}^2 \right] \, dx = -\alpha(N) M.
\]

It becomes along the characteristic curve:

\[
\int_{\mathbb{R}^d} \frac{D}{Dt} \nabla \cdot u(t, x(t; x)) \, dx + \int_{\Omega(t)} |\nabla \cdot u(t, x(t; x))|^2 \, dx \leq -\alpha(N) M.
\]

By denoting

\[
H := H(t, x) = \int_{\Omega(t)} \sum_{i=1}^N u_{ix_i}(t, x(t; x)) \, dx = \int_{\Omega(t)} \nabla \cdot u(t, x(t; x)) \, dx,
\]

and with the Cauchy-Schwarz inequality,

\[
\left| \int_{\Omega(t)} \nabla \cdot u(t, x(t; x)) \, 1 \, dx \right| \leq \left( \int_{\Omega(t)} |\nabla \cdot u(t, x(t; x))|^2 \, dx \right)^{1/2} \left( \int_{\Omega(t)} 1 \, dx \right)^{1/2},
\]

\[
\left| \int_{\Omega(t)} \nabla \cdot u(t, x(t; x)) \, dx \right| \leq \left( \int_{\Omega(t)} |\nabla \cdot u(t, x(t; x))|^2 \, dx \right)^{1/2},
\]

\[
\frac{H(t)^2}{|\Omega|} \leq \int_{\Omega(t)} |\nabla \cdot u(t, x(t; x))|^2 \, dx,
\]

respectively.
where $|\Omega|$ is the fixed volume of $\Omega(t)$, we have

$$\frac{DH}{Dt} + \frac{H^2}{|\Omega|} \leq -\alpha(N)M, \quad (23)$$

$$H(t) \leq -\sqrt{|\Omega|} \alpha(N)M \tan \left( \sqrt{\frac{\alpha(N)M}{|\Omega|} t} - \tan^{-1} \left( \sqrt{\frac{1}{|\Omega|} \frac{1}{\alpha(N)M} H_0} \right) \right), \quad (24)$$

$$H(T) \leq -\infty, \quad (25)$$

with the finite time $T > 0$, such that

$$T = \sqrt{\frac{|\Omega|}{\alpha(N)M}} \tan^{-1} \left( \sqrt{\frac{1}{|\Omega|} \frac{1}{\alpha(N)M} H_0} + \frac{\pi}{2} \right). \quad (26)$$

Therefore, for any $C^2$ solutions with compact support, they blow up before $T = \sqrt{\frac{|\Omega|}{\alpha(N)M}} \pi$.

On the other hand, for the systems with pressure ($\gamma > 1$), we divide $\rho$ to the momentum equation to have,

$$\nabla (u_t + u \cdot \nabla u) + K \gamma \rho^{\gamma - 2} (\nabla \rho)^2 + K \gamma \rho^{\gamma - 1} \Delta \rho = -\delta \partial \Phi/\partial x_i, \quad (28)$$

Take differentiation to the momentum equation for $\gamma > 1$:

$$\nabla (u_t + u \cdot \nabla u) + K \gamma (\gamma - 1) \rho^{\gamma - 2} (\nabla \rho)^2 + K \gamma \rho^{\gamma - 1} \Delta \rho = -\delta \alpha(N) \rho, \quad (29)$$

with the Poisson equation. By taking integration to the above equation, we have,

$$\int_{\Omega(t)} \nabla (u_t + u \cdot \nabla u) dx + \int_{\Omega(t)} K \gamma (\gamma - 1) \rho^{\gamma - 2} (\nabla \rho)^2 dx + \int_{\Omega(t)} K \gamma \rho^{\gamma - 1} \Delta \rho dx = -\delta \alpha(N) \rho dx, \quad (30)$$

$$\int_{\Omega(t)} \nabla (u_t + u \cdot \nabla u) dx + \int_{\Omega(t)} K \gamma (\gamma - 1) \rho^{\gamma - 2} (\nabla \rho)^2 dx + \int_{\Omega(t)} K \gamma \rho^{\gamma - 1} \Delta \rho dx = -\delta \alpha(N) M, \quad (31)$$

$$\int_{\Omega(t)} \left[ \sum_{i=1}^{N} u_{it}, + \sum_{i=1}^{N} u_i \left( \sum_{k=1}^{N} u_{kx_i} x_k \right) + \sum_{i=1}^{N} u_{ix_i}^2 \right] dx \leq -\epsilon, \quad (32)$$

with the required condition. Then for any $H(0)$, we have:

$$\frac{DH(t)}{Dt} + \frac{H(t)^2}{|\Omega|} \leq -\epsilon, \quad (33)$$

$$H(T) \leq -\infty, \quad (34)$$

with a finite time $T$.

Therefore, for any $C^2$ solutions with compact support, they blow up on or before a finite time $T$.

This completes the proof.
3 Differentiation Method

Alternatively, we may take differentiation to the momentum equations (6) to have the same result of in [5]:

Proposition 2 Suppose $(\rho, u)$ are $C^2$ solutions for the pressureless $(\delta = 0)$ Euler-Poisson equations in $\mathbb{R}^N$, (7), the solutions blow up before $T = -1/H_0$, with the initial velocity at some non-vacuum point:

$$H_0 = \nabla \cdot u(0, x_0) < 0.$$  \hspace{1cm} (35)

Proof. For $\rho_0(x_0(0, x_0)) > 0$, along the characteristic curve, we have:

$$u_t + u \cdot \nabla u = -\nabla \Phi,$$  \hspace{1cm} (36)

$$\nabla (u_t + u \cdot \nabla u) = -\Delta \Phi,$$  \hspace{1cm} (37)

$$\sum_{i=1}^N u_{ix_i} + \sum_{i=1}^N u_i \left( \sum_{k=1}^N u_{kx_i} x_i \right) + \frac{1}{N} \left( \sum_{i=1}^N u_{ix_i} \right)^2 + \alpha(N) \rho = 0,$$  \hspace{1cm} (38)

with the required condition (35). By defining $H = H(t) := \sum_{i=1}^N u_{ix_i}(t, x(t))$, we have

$$\frac{DH}{Dt} + \frac{1}{N} H^2 + \alpha(N) \rho = 0,$$  \hspace{1cm} (39)

with the Poisson equation (2). It becomes

$$\frac{DH}{Dt} \leq -\frac{1}{N} H^2,$$  \hspace{1cm} (40)

$$H(t) \leq \frac{N \sum_{i=1}^N u_{ix_i}(0; (0, x_0))}{N + \sum_{i=1}^N u_{ix_i}(0; (0, x_0))} \to -\infty,$$  \hspace{1cm} (41)

as $t \to -N/ \left[ \sum_{i=1}^N u_{ix_i}(0; (0, x_0)) \right] := -N/ \left[ \sum_{i=1}^N u_{ix_i}(0, x_0) \right]$ and $\sum_{i=1}^N u_{ix_i}(0; (0, x_0)) < 0$, for some point $x_0$. Therefore, we show that there does not exist global $C^2$ solutions, with the initial velocity at some point,

$$H_0 = \nabla \cdot u(0, x_0) < 0.$$  \hspace{1cm} (42)

The proof is completed. \hspace{1cm} ■

Remark 3 The above method provides a simpler way to show the same inequality in [2],

$$\frac{D \text{div} u}{Dt} \leq -\frac{1}{N} (\text{div} u)^2,$$  \hspace{1cm} (43)

without the analysis of spectral dynamics.

4 Discussion

Makino, Ukai, Kawashima initially defined the "Tame" solutions (the solutions to the pressureless Euler equations) [20]. Then Makino and Perthame considered the Tame solutions for the system with gravitational forces [21]. After that Perthame studied the 3-dimensional pressureless system with repulsive forces [22]. All of the above results rely on the solutions with radial symmetry:

$$u_t + uu_r = \frac{\alpha(N) \delta}{r} \int_0^r \rho(t, s)s^{N-1} ds,$$  \hspace{1cm} (44)
where $r := \left( \sum_{i=1}^{N} x_i^2 \right)^{1/2}$ is the radius. And the Emden ordinary differential equations were
deduced on the boundary point of the solutions with compact support:

$$ \frac{D^2 R}{Dt^2} = \frac{\delta M}{R^{N-1}}, \quad R(0, R_0) = R_0 \geq 0, \quad \dot{R}(0, R_0) = 0, \quad (45) $$

where $\frac{dR}{dt} := u$ and $M$ is the mass of the solutions, along the characteristic curve. They showed
the blowup results for the $C^1$ solutions of the system \[ [4] \]. And recently, Chae and Tadmor \[ [5] \]
obtain the blowup result, which does not require the solutions in radial symmetry. However, all
the above results concern about the pressureless cases with the external forces only. This article
has shed new light on the situations with the pressure term. In particular, it answers some cases
for the Euler equations or the Euler Poisson equations. However, the condition \[ [9] \] in our theorem
is too restricted. An refinement for it is expected in the future work.

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