THE DICHOTOMY PROPERTY OF $SL_2(R)$-A SHORT NOTE

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Abstract. A recent paper by Polterovich, Shalom and Shem-Tov has shown that non-discrete, conjugation invariant norms on arithmetic Chevalley groups of higher rank give rise to very restricted topologies. Namely, such topologies always have profinite norm-completions. In this note, we sketch an argument showing that this also holds for $SL_2(R)$ for $R$ a ring of algebraic integers with infinitely many units.

1. Introduction

The preprint [5] by Polterovich, Shalom and Shem-Tov studies (among other things) the topologies induced by conjugation-invariant norms on $SL_n(R)$ for $n \geq 3$ and rings $R$ for which $SL_n(R)$ has a certain finiteness property, namely that there is a natural number $L(R, n)$ such that each element of $SL_n(R)$ can be written as a product of at most $L(R, n)$ elementary matrices. This property is called bounded elementary generation. They show in this case that the following theorem holds:

Theorem 1.1. [5, Theorem 1.8] Let $R$ be an integral domain, $n \geq 3$ and assume that $SL_n(R)$ is boundedly elementary generated and that each non-zero ideal of $R$ has finite index. Then each conjugation-invariant norm on a finite index subgroup of $SL_n(R)$ is either discrete or has a profinite norm-completion.

A group satisfying the conclusion of the theorem is said to satisfy the dichotomy property. In any case, there are quite many rings of interest that satisfy the assumption of the theorem, for example rings of integers in global and local fields. The assumption of $n \geq 3$ in Theorem 1.1 is also quite common in the study of conjugation-invariant norms on Chevalley groups. This is mostly due to the fact that the normal subgroup structure of $SL_2(R)$ is more complicated than the one of $SL_n(R)$ for $n \geq 3$. Namely, the normal subgroups of the latter are essentially parametrized by ideals of the ring $R$ (or more precisely so-called admissible pairs of ideals as shown by Abe [1]), whereas normal subgroups of $SL_2(R)$ are given by more complicated subgroups of $(R, +)$ called radices as proven by Costa and Keller [2]. However, as seen in our preprint [6], it is often possible to generalize results of the study of conjugation invariant norms from $n \geq 3$ to $n = 2$ by introducing additional assumptions on the existence of a lot of units in the ring. In this context, [5] raises the following question:

Conjecture 1.2. Let $p$ be a prime in $\mathbb{Z}$. Does $SL_2(\mathbb{Z}[1/p])$ satisfy the dichotomy property?

We will answer the question in Conjecture 1.2 affirmatively with the following more general result:
Theorem 1.3. Let $R$ be the ring of $S$-algebraic integers in a number field such that $R$ has infinitely many units. Then every finite index subgroup of $\text{SL}_2(R)$ has the dichotomy property.

The proof is almost identical to the proof of [5, Theorem 1.8] itself and only requires a small modification in a technical lemma.

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2. Basic definitions and notions

Definition 2.1. Let $G$ be a group with neutral element $1_G$. A conjugation-invariant norm $\| \cdot \| : G \to \mathbb{R}_{\geq 0}$ is a function satisfying the properties

\[ \forall a \in G : \|a\| = 0 \iff a = 1_G, \]
\[ \forall a \in G : \|a\| = \|a^{-1}\|, \]
\[ \forall a, b \in G : \|ab\| \leq \|a\| + \|b\| \text{ and} \]
\[ \forall a, b \in G : \|aba^{-1}\| = \|a\| \]

Further, we recall the following two concepts from [5]:

Definition 2.2. A group $G$ is said to satisfy the dichotomy property, if each non-discrete norm $\| \cdot \|$ on $G$ has a profinite norm-completion. If $G$ is itself a topological group, then $G$ is called norm-complete, if each non-discrete norm $\| \cdot \|$ on $G$ induces the topology of $G$ as a topological group.

Let us next recall the definition of $\text{SL}_2$:

Definition 2.3. Let $R$ be a commutative ring with 1. Then $\text{SL}_2(R) := \{ A \in R^{2 \times 2} \mid a_{11}a_{22} - a_{12}a_{21} = 1 \}$.

Obviously for any commutative ring $R$ and any $x \in R$, the matrices

\[ E_{12}(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \text{ and } E_{21}(x) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \]

are elements of $\text{SL}_2(R)$. They are called elementary matrices and we denote the set of elementary matrices $\{ E_{12}(x), E_{21}(x) \mid x \in R \}$ by $\text{EL}$. The subgroup of $\text{SL}_2(R)$ generated by $\text{EL}$ is denoted by $E(2, R)$. Furthermore, for an ideal $I \subseteq R$, we denote by $E(2, R, I)$ the normal subgroup of $E(2, R)$ generated by the $E(2, R)$-conjugates of elements of $\{ E_{12}(x) \mid x \in I \}$. Additionally, for an ideal $I \subseteq R$, the subgroups $E_{12}(I)$ and $E_{21}(I)$ of $E(2, R)$ are defined as $E_{12}(I) := \{ E_{12}(x) \mid x \in I \}$ and $E_{21}(I) := \{ E_{21}(x) \mid x \in I \}$. Further, for a unit $u \in R^\ast$ the element

\[ h(u) = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \]
is also an element of $\text{SL}_2(R)$.

3. PROOF OF THE MAIN RESULT

We first introduce the needed concept of $R$ containing a large number of units:

**Definition 3.1.** Let $R$ be a commutative ring with 1 such that for each $c \in R - \{0\}$, there is a unit $u \in R$ such that $u - 1 \in c^2R$ and $u^8 - 1 \neq 0$. Then we call $R$ a ring with many units.

**Remark 3.2.** Note, that this is a similar but still slightly different property to the concept of rings with many units introduced in [6].

This enables us to formulate:

**Theorem 3.3.** Let $R$ be an integral domain with many units such that $E(2, R)$ is boundedly generated by elementary matrices and such that each non-zero ideal of $R$ has finite index in $R$. Then each finite index subgroup $H$ of $E(2, R)$ has the dichotomy property.

To prove this, we need the following modified version of [5, Lemma 2.3]:

**Lemma 3.4.** Let $R$ be as in Theorem 3.3. Let $\| \cdot \|$ further be the restriction of a non-discrete conjugation-invariant norm on $E(2, R, I)$ to the elementary subgroup $E_{12}(I)$. Then the norm completion $E_{12}((I))$ of $E_{12}(I)$ with respect to $\| \cdot \|$ is profinite.

To prove this we need the following technical lemma, which in turn is a modified version of [5, Lemma A9]:

**Lemma 3.5.** Let $R$ be an integral domain, $c \in R$ non-zero and $u \in R$ a unit such that $u - 1 \in c^2R$. Assume further that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an element of $\text{SL}_2(R)$. Then each element of $E_{12}(cR)$ is a product of at most four $E(2, R, cR)$-conjugates of $A$ and $A^{-1}$.

**Proof.** As $u - 1$ is an element of the ideal $cR$ by assumption, so is $u^4 - 1$. Hence choose $x \in R$ with $u^4 - 1 = cx$ and set $t := ax$. But note that as $u - 1 \in c^2R$, there is an $y \in R$ such that $u - 1 = c^2y$ holds. Hence

$$cx = u^4 - 1 = (u - 1) \cdot (u^3 + u^2 + u + 1) = c^2y \cdot (u^3 + u^2 + u + 1)$$

and so $x = cy \cdot (u^3 + u^2 + u + 1) \in cR$. Thus $t$ is an element of $cR$. Then the matrix $Y := E_{12}(t)A^{-1}E_{12}(-t)h(u^2)Ah(u^{-2})$ is a product of two $E(2, R, cR)$-conjugates of $A$ and $A^{-1}$ and has the form

$$Y = \begin{pmatrix} u^{-4} & q \\ 0 & u^4 \end{pmatrix}$$
for some \( q \in cR \). But then note for \( p \in R \) that
\[
E_{12}((u^{-4} - u^4)(p + q)) = Y \cdot \begin{pmatrix} u^4 & p \\ 0 & u^{-4} \end{pmatrix} \cdot Y^{-1} \cdot \begin{pmatrix} u^{-4} & -p \\ 0 & u^4 \end{pmatrix}
\]
Hence choosing \( p := -q + z \) for \( z \in cR \), we obtain that \( E_{12}((u^4 - u^{-4})z) \) is a product of four \( E(2, R, cR) \)-conjugates of \( A \) and \( A^{-1} \).

**Remark 3.6.** Implicit in this proof is the claim that \( h(u) \) is an element of \( E(2, R, cR) \). We will not prove this, but it follows from a short calculation with the standard decomposition of \( h(u) \) into elementary matrices.

Using this lemma, we can prove Lemma 3.4.

**Proof.** For each ideal \( J \) of \( R \) contained in \( I \) consider the closure \( U_J \) of \( E_{12}(J) \) in \( \overline{E_{12}(I)} \). As a non-zero ideal \( J \) in \( R \) has finite index, this implies that the closed subgroup \( U_J \) has finite index in \( \overline{E_{12}(I)} \). Thus \( U_J \) is also open in \( \overline{E_{12}(I)} \). Hence to show that \( \overline{E_{12}(I)} \) is profinite, it suffices to show that the identity in \( \overline{E_{12}(I)} \) has a neighborhood basis of subgroups of the form \( U_J \). First, note that \( \| \cdot \| \) is the restriction of a non-discrete norm on \( E(2, R, I) \), which we will also denote by \( \| \cdot \| \). Thus for each \( \epsilon > 0 \), there is an \( A \in E(2, R, I) \) with \( \| A \| \leq \epsilon / 4 \). As there are only finitely many scalar matrices in \( E(2, R) \), we may by possibly choosing a smaller \( \epsilon \) or by considering a suitable conjugate or commutator of \( A \) assume that \( A \) has the \((2, 1)\)-entry \( c \) with \( c \neq 0 \). But \( R \) is a ring with many units, so we may choose a unit \( u \) in \( R \) with \( u^8 - 1 \neq 0 \) and \( u \equiv 1 \) mod \( c^2R \). According to Lemma 3.3, the subgroup \( E_{12}((u^8 - 1)cR) \) is then contained in the \( 4 \cdot \| A \| \) ball around \( E_2 \) as \( c \) is an element of \( I \). Hence \( J_\epsilon := (u^8 - 1)cR \) is contained in \( I \) and \( E_{12}(J_\epsilon) \) is contained in the \( \epsilon \)-ball around \( E_2 \). But this implies that \( U_{J_\epsilon} \) is a subgroup of \( \overline{E_{12}(I)} \) contained in the \( \epsilon \)-ball around \( E_2 \).

One can now prove Theorem 3.3 in essentially the same way as [5, Theorem 1.8]:

**Proof.** We first prove the dichotomy property in the case of \( H = E(2, R) \). So let \( \| \cdot \| \) be a non-discrete, conjugation-invariant norm on \( E(2, R) \) and let \( G \) be the norm-completion of \( E(2, R) \) with respect to \( \| \cdot \| \). Further, let \( U_1 \) and \( U_2 \) be the closures of \( E_{12}(R) \) and \( E_{21}(R) \) within \( G \) respectively. Then by assumption the group \( E(2, R) \) is boundedly generated by its elementary subgroups. Thus \( G \) is boundedly generated by its subgroups \( U_1, U_2 \) and so there is a natural number \( N(R) \) such that \( G = U_{i_1} \cdot U_{i_2} \cdots U_{i_{N(R)}} \) holds for \( i_1, \ldots, i_{N(R)} \in \{1, 2\} \). But according to Lemma 3.3, the subgroups \( U_1, U_2 \) of \( G \) are profinite. But it was observed in the proof of [5, Theorem 1.8(i)], that if a topological group \( G \) is a set-theoretic product of finitely many subgroups profinite in the relative topology, then \( G \) itself is profinite. This finishes the proof for \( E_2(R) \). The general case works the same way as in [5] as well: Let \( H \) be a finite index subgroup of \( E(2, R) \). Then following precisely the line of arguments from [5] and the already shown case \( H = E(2, R) \), one reduces the dichotomy proof for \( H \) to the claim that for a non-zero ideal \( I \) and a conjugation-invariant, non-discrete norm \( \| \cdot \| \) on \( E(2, R, I) \), said norm restricts to norms on \( E_{12}(I) \) (and \( E_{21}(I) \)) having profinite norm-completions. But this is precisely the claim of Lemma 3.4. 

\[ \square \]
If $R$ is an integral domain containing a unit $v$ of infinite order and each of its non-zero ideals has finite index, then it has many units: Namely, let $c \in R - \{0\}$ be given. Assuming wlg that $c$ is not a unit in $R$, we note that $R/c^2 R$ is finite. Then consider $\pi := v + c^2 R$. This element must have finite order in $(R/c^2 R)^*$ and so there is a $k > 0$ such that $\pi^k = 1 + c^2 R$. Note that $u := v^k$ has infinite order and so $u^8$ can not be $1$. Also $u - 1 \in c^2 R$ by choice of $u$. This argument applies for example for rings of $S$-algebraic integers with infinitely many units as they have units of infinite order according to Dirichlet’s Theorem \cite[Corollary 11.7]{Neukirch}. Additionally $\text{SL}_2(R)$ has bounded elementary generation \cite[Theorem 1.1]{Morgan}. Thus Theorem \ref{sl2} implies Theorem \ref{main}. Additionally, we note the following version of \cite[Theorem 1.12]{Polterovich}: 

\textbf{Theorem 3.7.} Let $R$ be a compact metrizable ring with many units such that each of its non-zero ideals has finite index and such that $R$ is also an integral domain. Then $\text{SL}_2(R)$ equipped with the relative topology from $R^{2 \times 2}$ as well as any of its finite index subgroups are norm-complete. 

The proof is virtually identical to the proof of Theorem \ref{sl2} above, so we will skip it. We do however want to note, that this yields a proof of \cite[Theorem 3.6]{Polterovich}, which does not require the rather technical \cite[Lemma 3.7]{Polterovich}. 

4. Closing remarks 

To round out this short note, we note that one can prove a generalization of \cite[Theorem 1.8]{Polterovich} also for all other split Chevalley groups besides the $\text{SL}_n$. The main difference is not the statement, which would be the same, but that a bit of additional care has to be taken in the proof: Namely, the root subgroups associated to short and long roots require slightly different approaches to prove the appropriate form of Lemma \ref{lemma3}. Also the exceptional groups $\text{Sp}_4$ and $G_2$ will require different strategies in case the ring $R$ in question has the bad characteristics 2 or 3. Ultimately though, all the split cases are very similar; the more interesting cases are likely those arising from more complicated algebraic groups. 

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